A NOTE ON FAILURE OF ENERGY REVERSAL FOR CLASSICAL FRACTIONAL SINGULAR INTEGRALS

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Abstract. For $0 \leq \alpha < n$ we demonstrate the failure of energy reversal for the vector of $\alpha$-fractional Riesz transforms, and more generally for the vector of all $\alpha$-fractional convolution singular integrals having a kernel with vanishing integral on every great circle of the sphere.

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1. Introduction

To set notation we recall a special case of Theorem 1 from our paper [SaShUr], using notation from that paper.

Theorem 1. Suppose that $\sigma$ and $\omega$ are locally finite positive Borel measures in $\mathbb{R}^n$ with no common point masses, and assume the finiteness of the $\alpha$-energy condition constant

$$\langle \mathcal{E}_\alpha \rangle^2 \equiv \sup_{Q=\cup Q_r, Q_r \in \mathcal{D}^n} \left\{ \frac{1}{|J|} \sum_{r=1}^{\infty} \sum_{J \in \mathcal{M}_{\mathfrak{r}, \mathfrak{e} deep}(Q_r)} \left( \mathcal{P}^{\alpha}(J, 1Q\sigma) \right) \frac{1}{|J|^\frac{\alpha}{n}} \right\}^2 \| p_{\text{subgood}, \omega} x \|_{L^2(\omega)}^2$$

+ $\sup_{\ell \geq 0} \frac{1}{|J|} \sum_{J \in \mathcal{M}_{\mathfrak{r}, \mathfrak{e} deep}(Q)} \left( \mathcal{P}^{\alpha}(J, 1Q\sigma) \right) \frac{1}{|J|^\frac{\alpha}{n}} \| p_{\text{subgood}, \omega} x \|_{L^2(\omega)}^2$,

and its dual, uniformly over all dyadic grids $\mathcal{D}^n$, and where the goodness parameters $\mathfrak{r}$ and $\mathfrak{e}$ implicit in the definition of $\mathcal{M}_{\mathfrak{r}, \mathfrak{e} deep}(K)$ are fixed sufficiently large and small respectively depending on $n$ and $\alpha$. Let $T^{\alpha}$ be a standard strongly elliptic $\alpha$-fractional Calderón-Zygmund operator in Euclidean space $\mathbb{R}^n$. Then $T^{\alpha}$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if and only if the $A^\alpha_2$ condition

$$\mathcal{A}^{\alpha}_2 \equiv \sup_{Q \in \mathcal{Q}^n} \mathcal{P}^{\alpha}(Q, \sigma) \frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}} < \infty$$

References

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and its dual hold, the cube testing conditions

\[(1.2) \quad \int_Q |T^\alpha (1_Q\sigma)|^2 \omega \leq T^2_T \int_Q d\sigma \quad \text{and} \quad \int_Q |(T^\alpha)^* (1_Q\omega)|^2 \leq T^2_T \int_Q d\omega,\]

hold for all cubes \(Q\) in \(\mathbb{R}^n\), and the weak boundedness property for \(T^\alpha\) holds:

\[\left| \int_Q T^\alpha (1_Q\sigma) d\omega \right| \leq WB P_{T^\alpha} \sqrt{|Q|} |Q|^{\alpha},\]

for all cubes \(Q, Q'\) with \(1 \leq \frac{|Q|^\frac{\alpha}{2}}{|Q'|^{\frac{\alpha}{2}}} \leq C,\)

and either \(Q \subset 3Q' \setminus Q'\) or \(Q' \subset 3Q \setminus Q\).

In [SaShUr3] we used Theorem 1 to prove the \(T1\) theorem for the vector of Riesz transforms in \(\mathbb{R}^n\) in the special case when one of the measures \(\sigma, \omega\) is supported on a line in \(\mathbb{R}^n\). The key to that proof was proving control of the above energy constants \(E_\alpha\) and \(E^*_\alpha\) in terms of the constants in the hypotheses (1.1) and (1.2). A number of attempts have been made by us and others (see e.g. earlier versions of [SaShUr] and [LaWi]) to prove such control of various different energy conditions by invoking an energy reversal for the Riesz transforms and similar operators - see (2.4) below - but all of these attempts have been met with failure. The purpose of this short note is to show first that this short note is to show first that \(E_\alpha\) is false, not only for the vector of fractional Riesz transforms in the plane when \(0 \leq \alpha < 2\), but also for the vectors of classical \(\alpha\)-fractional singular integrals in the plane,

\[T^\alpha_M \equiv \{ T\Omega : \Omega \in \mathcal{P}_M \}, \quad \mathcal{P}_M \equiv \{ \cos n\theta, \sin n\theta \}_{n=1}^M,\]

where \(T^\alpha_M\) has convolution kernel \(\frac{\Omega(z)}{|z|^{n-\alpha}} = \frac{\Omega(\theta)}{|\theta|^{n-\alpha}}\) and \(0 \leq \alpha < 2\). The linear space \(\mathcal{L}_M\) of trigonometric polynomials with vanishing mean and degree at most \(M\) is spanned by the monomials \(\mathcal{P}_M\), and so we also obtain the failure of energy reversal for the infinite vector \(T^\alpha_M \equiv \{ T\Omega : \Omega \in \mathcal{L}_M \}\). A standard limiting argument applied to the proof below extends this failure to all sufficiently smooth \(\Omega(\theta)\) with vanishing mean on the circle. Finally, we embed an analogue of the planar measure constructed below into Euclidean space \(\mathbb{R}^n\) in order to obtain the failure of energy reversal for any vector of classical convolution Calderón-Zygmund operators with odd kernel in \(\mathbb{R}^n\) - and more generally for kernels \(\frac{\Omega(x)}{|x|^{n-\alpha}}\) where \(\Omega\) has vanishing integral on every great circle in the sphere \(\mathbb{S}^{n-1}\). A key to our proof is the positivity of the determinants \(\frac{\Gamma(z)^2}{(z - i\xi_j)(z + i\xi_j)}\) for all \(n \geq 1\). See also [LaWi] for related results regarding fractional Riesz transforms in higher dimensions. We thank Michael Lacey for pointing out to us that the 1-fractional Riesz transform \(R^1\sigma(z) = \int_{\mathbb{R}} \frac{\xi - z}{|x|^n} d\sigma(\xi)\) of the unit circle measure \(\sigma\) vanishes identically for \(z\) inside the unit disk. Indeed, \(R^1\sigma\) is the gradient of the planar Newtonian potential \(N\sigma(z) = \int_{\mathbb{R}} \ln |z - \xi| d\sigma(\xi)\), and \(N\sigma\) is constant inside the disk.
2. Failure of reversal of energy

Recall the energy $E(J, \omega)$ of $\omega$ on a cube $J,$

$$E(J, \omega)^2 = \frac{1}{|J|} \int \int \frac{1}{|J|^2} \left| \frac{x - z}{|J|^2} \right|^2 d\omega(x) d\omega(z) = \frac{1}{|J|} \int \left| x - \frac{\omega x}{|J|^2} \right|^2 d\omega(x).$$

Define its associated coordinate energies $E_j(J, \omega)$ by

$$E_j(J, \omega)^2 = \frac{1}{|J|} \int \int \left| \frac{x^j - z^j}{|J|^2} \right|^2 d\omega(x) d\omega(z), \quad j = 1, 2, ..., n,$$

and the rotations $E_{\mathcal{R}}(J, \omega)$ of the coordinate energies by a rotation $\mathcal{R} \in SO(n),$ which we refer to as partial energies,

$$E_{\mathcal{R}}^j(J, \omega)^2 = \frac{1}{|J|} \int \int \left| \frac{x^j - z^j_{\mathcal{R}}}{|J|^2} \right|^2 d\omega(x) d\omega(z), \quad j = 1, 2, ..., n,$$

where for $\mathcal{R} \in SO(n),$ $x_{\mathcal{R}} = \left( x^1_{\mathcal{R}}, ..., x^n_{\mathcal{R}} \right) = \mathcal{R}(x)_{j=1}^n = \mathcal{R}x.$ Set $E_{\mathcal{R}}(J, \omega)^2 = E_{\mathcal{R}}^1(J, \omega)^2 + ... + E_{\mathcal{R}}^n(J, \omega)^2.$ We have the following elementary computations.

**Lemma 1.** For $\mathcal{R} \in SO(n)$ we have

$$E_{\mathcal{R}}(J, \omega)^2 = E_{\mathcal{R}}^1(J, \omega)^2 + ... + E_{\mathcal{R}}^n(J, \omega)^2 = E(J, \omega)^2. \quad (2.1)$$

More generally, if $\mathcal{R} = \{ \mathcal{R}_j \}_{j=1}^n \subset SO(n)$ is a collection of rotations such that the matrix $M_{\mathcal{R}} = \begin{bmatrix} \mathcal{R}_1e^1 \\ \vdots \\ \mathcal{R}_ne^1 \end{bmatrix}$ with rows $\mathcal{R}_\ell e^1$ is nonsingular, then

$$E(J, \omega)^2 \leq \frac{1}{\epsilon_{\mathcal{R}}} \sum_{\ell=1}^n E_{\mathcal{R}_\ell}^1(J, \omega)^2, \quad (2.2)$$

where $\epsilon_{\mathcal{R}}$ is the least eigenvalue of $M_{\mathcal{R}}^* M_{\mathcal{R}}.$

**Proof.** We have

$$|x^1_{\mathcal{R}} - z^1_{\mathcal{R}}|^2 + ... + |x^n_{\mathcal{R}} - z^n_{\mathcal{R}}|^2 = |\mathcal{R}(x - z)|^2 = |x - z|^2 = |x^1 - z^1|^2 + ... + |x^n - z^n|^2,$$

so that

$$E_{\mathcal{R}}(J, \omega)^2 = E_{\mathcal{R}}^1(J, \omega)^2 + ... + E_{\mathcal{R}}^n(J, \omega)^2 = E^1(J, \omega)^2 + ... + E^n(J, \omega)^2 = E(J, \omega)^2.$$

More generally, if $M_{\mathcal{R}}^\ell$ denotes the $\ell^{th}$ row of the matrix $M_{\mathcal{R}},$ we have

$$\epsilon_{\mathcal{R}} |x - z|^2 \leq (x - z)^{tr} M_{\mathcal{R}}^* M_{\mathcal{R}} (x - z) = \sum_{\ell=1}^n |\mathcal{R}^\ell e^1 \cdot (x - z)|^2,$$
so that
\[
\epsilon \|E(J, \omega)\|^2 = \left(\frac{1}{|J|} \right)^2 \int_J \int_J \epsilon \|x - z\|^2 \, d\omega(x) \, d\omega(z) \\
\leq \left(\frac{1}{|J|} \right)^2 \int_J \int_J \left\{ \sum_{\ell=1}^n \left| R_\ell e^1 \cdot (x - z) \right|^2 \right\} \, d\omega(x) \, d\omega(z) \\
= \sum_{\ell=1}^n E_{R_\ell} (J, \omega)^2.
\]

The point of the estimate (2.2) is that it could hopefully be used to help obtain a reversal of energy for a vector transform \(T^{n, \alpha} = \{T^{n, \alpha}_\ell\}_{\ell=1}^{N}\), where the convolution kernel \(K^{n, \alpha}_\ell(w)\) of the operator \(T^{n, \alpha}_\ell\) has the form
\[
K^{n, \alpha}_\ell(w) = \frac{\Omega_\ell^\alpha \left( \frac{w}{w|w|} \right)}{|w|^{n-\alpha}},
\]
and where \(\Omega_\ell^\alpha\) is smooth on the sphere \(S^{n-1}\). We refer to the operator \(T^{n, \alpha}_\ell\) as an \(\alpha\)-fractional convolution Calderón-Zygmund operator. If in addition we require that \(\Omega_\ell^\alpha\) has vanishing integral on the sphere \(S^{n-1}\), we refer to \(T^{n, \alpha}_\ell\) as a classical \(\alpha\)-fractional Calderón-Zygmund operator.

However, we now dash this hope, at least for the most familiar singular operators in the plane, in a spectacular way. A vector \(T^{\alpha} = \{T^{\alpha}_\ell\}_{\ell=1}^{N}\) of \(\alpha\)-fractional transforms in Euclidean space \(\mathbb{R}^n\) satisfies a strong reversal of \(\omega\)-energy on a cube \(J\) if there is a positive constant \(C_0\) such that for all \(\gamma \geq 2\) sufficiently large and for all positive measures \(\mu\) supported outside \(\gamma J\), we have the inequality
\[
E(J, \omega)^2 \, P^{\alpha}(J, \mu)^2 \leq C_0 \, \mathbb{E}_J^{\omega(z)} \mathbb{E}_J^{\omega(z)} \left| T^{\alpha}_\ell \mu(z) - T^{\alpha}_\ell \mu(z) \right|^2.
\]

We show that (2.4) is false by stating and proving a variant of Lemma 9 in \cite{SaShUr2}.

**Lemma 2** (Failure of Reverse Energy). Suppose that \(J\) is a square in the plane \(\mathbb{R}^2\), \(0 \leq \alpha < 2, \gamma > 2\) and that \(R^{\alpha} = \{R^{\alpha}_\ell\}_{\ell=1}^{2}\) is the vector of \(\alpha\)-fractional Riesz transforms in the plane \(\mathbb{R}^2\) with kernels \(K^{\alpha}_\ell(w) = \frac{\Omega_\ell^{\alpha \gamma} \left( \frac{w}{w|w|} \right)}{|w|^{2-\gamma}}\) and \(\Omega_\ell^{\alpha \gamma} \left( \frac{w}{w|w|} \right) = \frac{w_\ell}{|w|}\). Finally suppose that \(C_0 > 0\) is given. For \(\gamma\) sufficiently large, there exists a positive measure \(\mu\) on \(\mathbb{R}^2\) supported outside \(\gamma J\) and depending only on \(\alpha\) and \(\gamma\), such that the strong reversal of energy inequality (2.4) fails. Moreover, we can choose \(\mu\) as above so that in addition, for any \(M \geq 1\), the strong reversal of energy inequality (2.4) fails for the vector \(T^{\alpha}_M\).

As a corollary of the proof of this lemma we easily obtain an extension to higher dimensions by simply embedding an appropriate planar measure into Euclidean space \(\mathbb{R}^n\).

**Corollary 1** (of the proof of Lemma 2). Suppose that \(J\) is a cube in \(\mathbb{R}^n\), \(0 \leq \alpha < n, \gamma > 2\) and suppose that \(C_0 > 0\) is given. For \(\gamma\) sufficiently large, there exists a positive measure \(\mu\) on \(\mathbb{R}^n\) supported outside \(\gamma J\) and depending only on \(\alpha\) and \(\gamma\), such that the strong reversal of energy inequality (2.4) fails for any vector
\( T_\alpha = \{ T_\alpha^\ell \}_{\ell=1}^N \) of \( \alpha \)-fractional smooth Calderón-Zygmund operators in \( \mathbb{R}^n \) with kernels \( K_\alpha^\ell (w) = \frac{\Omega_\ell (\frac{w}{|w|^2})}{|w|^2}, \) where \( \Omega_\ell \) has vanishing integral on every great circle in the sphere \( S^{n-1} \) - in particular this holds if each \( K_\alpha^\ell \) is odd.

Proof of Lemma \ref{lem:riesz} for the Riesz transform vector. Let \( \varepsilon > 0 \). We let \( \Omega_\ell (\frac{w}{|w|^2}) \) be an arbitrary standard kernel for the moment. With \( K_\alpha^\ell (x,y) = K_\alpha^\ell (x-y) \) we have

\[
T_\alpha^\ell \mu (x) = \int K_\alpha^\ell (x-y) d\mu (y) = \int \frac{\Omega_\ell (x-y)}{|y-x|^{n-\alpha}} d\mu (y)
\]

\[
= \int \{ K_\alpha^\ell (cJ-y) + (x-cJ) \cdot \nabla K_\alpha^\ell (cJ-y) \} d\mu (y) + E_{\ell,x},
\]

and so

\[
T_\alpha^\ell \mu (x) - T_\alpha^\ell \mu (z) = \int \{ (x-z) \cdot \nabla K_\alpha^\ell (cJ-y) \} d\mu (y) + [E_{\ell,x} - E_{\ell,z}^\alpha]
\]

\[
\Rightarrow \Lambda_\alpha + [E_{\ell,x} - E_{\ell,z}^\alpha],
\]

where if \( \gamma > 2 \) is sufficiently large,

\[
(2.5) \quad |E_{\ell,x}^\alpha - E_{\ell,z}^\alpha| \leq C_{\gamma} \frac{P_{\alpha} (J, \mu)}{|J|^\frac{4}{\gamma}} |x-z| \leq \varepsilon \frac{P_{\alpha} (J, \mu)}{|J|^\frac{4}{\gamma}} |x-z|.
\]

The point of this inequality (2.5) is that it permits the replacement of the difference \( T_\alpha^\ell \mu (x) - T_\alpha^\ell \mu (z) \) in (2.4) by the linear part \( \Lambda_\alpha \) of the Taylor expansion of the kernel \( K_\alpha^\ell \).

Now we make the choice

\[
\begin{align*}
\Omega_\ell (w) &= \Omega (\theta_\ell (w)); \\
\theta_\ell (w) &= \tan^{-1} \left( \frac{(-1)^\ell w^{\ell'}}{w^\ell} \right), \quad 1 \leq \ell \leq 2,
\end{align*}
\]

where \( w^{\ell'} \) denotes the coordinate variable other than \( w^\ell \), i.e. \( \ell + \ell' = 3 \). Thus \( \theta_1 \) is the usual angular coordinate on the circle and \( \theta_2 = \theta_1 + \frac{\pi}{2} \). We now use

\[
\nabla |w|^{\alpha-2} = \left( \frac{\partial}{\partial w^\ell} \left( (w^1)^2 + (w^2)^2 \right)^{\frac{\alpha-2}{2}}, \frac{\partial}{\partial w^{\ell'}} \left( (w^1)^2 + (w^2)^2 \right)^{\frac{\alpha-2}{2}} \right)
\]

\[
= \frac{\alpha-2}{2} \left( (w^1)^2 + (w^2)^2 \right)^{\frac{\alpha-2}{2} - 1} 2w
\]

\[
= (\alpha-2) |w|^{\alpha-4} w.
\]

and

\[
\frac{\partial}{\partial w^{\ell'}} \tan^{-1} \frac{w^{\ell'}}{w^\ell} = \frac{1}{1 + \left( \frac{w^{\ell'}}{w^\ell} \right)^2} \frac{-w^{\ell'}}{|w|^2},
\]

\[
\frac{\partial}{\partial w^\ell} \tan^{-1} \frac{w^{\ell'}}{w^\ell} = \frac{1}{1 + \left( \frac{w^{\ell'}}{w^\ell} \right)^2} \frac{1}{w^{\ell'}} = \frac{w^\ell}{|w|^2}.
\]
to calculate that the gradient of the convolution kernel

\[ K^\alpha_\ell (w) = \frac{\Omega_\ell (w)}{|w|^{1-\alpha}} = \frac{\Omega (\theta_\ell (w))}{|w|^{2-\alpha}} = \frac{\Omega \left( \tan^{-1} \frac{w^\ell}{w^\perp} \right)}{|w|^{2-\alpha}}, \]

is given by,

\[
\nabla K^\alpha_\ell (w) = \nabla \left( \frac{\Omega_\ell (w)}{|w|^{2-\alpha}} \right) = \Omega (\theta_\ell (w)) \nabla |w|^{\alpha-2} + |w|^{\alpha-2} \Omega' (\theta_\ell (w)) \nabla \theta_\ell = \frac{(\alpha - 2) \Omega (\theta_\ell (w)) w + \Omega' (\theta_\ell (w)) w^\perp}{|w|^{1-\alpha}}.
\]

Thus the linear part \( \Lambda^\alpha_\ell \) in the Taylor expansion of \( T^\alpha_{\ell \mu} \) is given by

\[
\Lambda^\alpha_\ell = (x - z) \cdot \int \nabla K^\alpha_\ell (c_j - y) \, d\mu (y) \equiv (x - z) \cdot Z^\alpha_{\ell \mu} (c_j; \mu),
\]

where

\[
Z^\alpha_{\ell \mu} (c_j; \mu) = \int_{\mathbb{R}^2} \left\{ (\alpha - 2) \Omega (\theta_\ell (c_j - y)) (c_j - y - \Omega' (\theta_\ell (c_j - y)) (c_j - y)^\perp \right\} d\mu (y)
\]

\[
= \int_{w \in \mathbb{S}^1} \left\{ (\alpha - 2) \Omega (\theta_\ell (w)) w^1 - \Omega' (\theta_\ell (w)) w^2 \right\} e^1 d\Psi_\mu (w)
\]

\[
+ \int_{w \in \mathbb{S}^1} \left\{ (\alpha - 2) \Omega (\theta_\ell (w)) w^2 + \Omega' (\theta_\ell (w)) w^1 \right\} e^2 d\Psi_\mu (w),
\]

and \( e^\ell \) is the coordinate vector with a 1 in the \( \ell^{th} \) position. Here the measure \( \Psi_\mu \) is an essentially arbitrary positive finite measure on the circle \( \mathbb{S}^1 \) given formally by

\[
d\Psi_\mu (w) = \int_0^\infty r^{\alpha-3} d\mu_w (r) = \int_0^\infty r^{\alpha-3} d\mu (rw), \quad w \in \mathbb{S}^1.
\]

We use,

\[
\tan \theta_\ell (w) = \frac{(-1)^\ell w^\perp}{w^\ell},
\]

\[
\csc \theta_\ell (w) = (-1)^\ell \sqrt{1 + \cot^2 \theta_\ell (w)} = (-1)^\ell \sqrt{1 + \left( \frac{w^\ell}{w^\perp} \right)^2} = \frac{|w|}{(-1)^\ell w^\ell},
\]

\[
\sin \theta_\ell (w) = \frac{(-1)^\ell w^\ell}{|w|} \quad \text{and} \quad \cos \theta_\ell (w) = \frac{w^\ell}{|w|},
\]

for \( w \neq 0 \), to obtain

\[
Z^\alpha_{\ell \mu} (c_j; \mu) = \int_{\mathbb{S}^1} \{ (\alpha - 2) \Omega (\theta_1 (w)) \cos \theta_1 (w) - \Omega' (\theta_1 (w)) \sin \theta_1 (w) \} e^1 d\Psi_\mu
\]

\[
+ \int_{\mathbb{S}^1} \{ (\alpha - 2) \Omega (\theta_1 (w)) \sin \theta_1 (w) + \Omega' (\theta_1 (w)) \cos \theta_1 (w) \} e^2 d\Psi_\mu
\]

\[
\equiv \int_{\mathbb{S}^1} \{ A^1_\alpha (\theta_1 (w)) e^1 + B^1_\alpha (\theta_1 (w)) e^2 \} d\Psi_\mu,
\]
and
\[ Z_{\Omega t}^{\alpha} (c; \mu) = \int_{S^1} \{-(\alpha - 2) \Omega (\theta_2 (w)) \sin \theta_2 (w) - \Omega' (\theta_2 (w)) \cos \theta_2 (w)\} e^1 d\Psi_\mu \\
+ \int_{S^1} \int \{-(\alpha - 2) \Omega (\theta_2 (w)) \cos \theta_2 (w) - \Omega' (\theta_2 (w)) \sin \theta_2 (w)\} e^2 d\Psi_\mu \\
= \int_{S^1} \{ A_\alpha^2 (\theta_2 (w)) e^1 + B_\alpha^2 (\theta_2 (w)) e^2\} d\Psi_\mu ,
\]
with
\[
\begin{align*}
A_\alpha^1 (t) &= (\alpha - 2) \Omega (t) \cos t - \Omega' (t) \sin t = B_\alpha^2 (t), \\
B_\alpha^1 (t) &= (\alpha - 2) \Omega (t) \sin t + \Omega' (t) \cos t = -A_\alpha^2 (t).
\end{align*}
\]

Now we show below in (2.11) that a necessary condition for reversal of energy on \( J \) is that the span of the pair of vectors \( \{Z_{\Omega t}^{\alpha} (c; \mu)\}_{\alpha=1}^2 \) is all of \( \mathbb{R}^2 \):
\[
\text{Span} \left\{Z_{\Omega t}^{\alpha} (c; \mu)\right\}_{\alpha=1}^2 = \mathbb{R}^2.
\]
So it suffices to show the failure of (2.7), i.e. that \( Z_{\Omega t}^{\alpha} (c; \mu) \) and \( Z_{\Omega t}^{\alpha} (c; \mu) \) are parallel.

At this point we take \( \ell = 1 \) and set \( \theta = \theta_1 (w) \) so that we obtain
\[
\begin{align*}
A_\alpha (\theta) &= A_\alpha^1 (\theta_1 (w)) = (\alpha - 2) \Omega (\theta) \cos \theta - \Omega' (\theta) \sin \theta, \\
B_\alpha (\theta) &= B_\alpha^1 (\theta_1 (w)) = (\alpha - 2) \Omega (\theta) \sin \theta + \Omega' (\theta) \cos \theta.
\end{align*}
\]
In the case \( \alpha = 1 \) these coefficients are perfect derivatives,
\[
\begin{align*}
A_1 (\theta) &= -\Omega (\theta) \cos \theta - \Omega' (\theta) \sin \theta = -[\Omega (\theta) \sin \theta]', \\
B_1 (\theta) &= -\Omega (\theta) \sin \theta + \Omega' (\theta) \cos \theta = -[\Omega (\theta) \cos \theta]',
\end{align*}
\]
and so have vanishing integral on the circle. Thus with the choice \( d\Psi_\mu (\theta) = d\theta \) we have
\[
Z_{\Omega} (c; \mu) = \int_{S^1} \{ A_1 (\theta) e^1 + B_1 (\theta) e^2\} d\theta = 0
\]
the zero vector, for every choice of differentiable \( \Omega \) on the circle.

In the case \( 0 \leq \alpha < 2 \) with \( \alpha \neq 1 \), it is no longer possible to find a nontrivial measure \( \mu \) so that \( Z_{\Omega t}^{\alpha} (c; \mu) \) vanishes for all differentiable \( \Omega \), but we will see that we can always find a positive measure \( \mu \) such that the vectors \( Z_{\Omega t}^{\alpha} (c; \mu) \) and \( Z_{\Omega t}^{\alpha} (c; \mu) \) are parallel for the choice \( \Omega (\theta) = \cos \theta \) that corresponds to the vector of Riesz transforms.

Indeed, in the special case that \( \Omega (t) = \cos t \), and recalling that \( \theta_2 (w) = \theta_1 (w) + \frac{\pi}{2} = \theta + \frac{\pi}{2} \), we have
\[
\begin{align*}
A_1^1 (\theta_1 (w)) &= A_1^1 (\theta) = (\alpha - 2) \cos^2 \theta + \sin^2 \theta; \\
B_1^1 (\theta_1 (w)) &= B_1^1 (\theta) = (\alpha - 3) \cos \theta \sin \theta; \\
A_\alpha^2 (\theta_2 (w)) &= -B_\alpha^1 \left( \theta + \frac{\pi}{2} \right) = -\left(\alpha - 3\right) \cos \left( \theta + \frac{\pi}{2} \right) \sin \left( \theta + \frac{\pi}{2} \right) \\
&= (\alpha - 3) \cos \theta \sin \theta; \\
B_\alpha^2 (\theta_2 (w)) &= A_\alpha^1 \left( \theta + \frac{\pi}{2} \right) = (\alpha - 2) \cos^2 \left( \theta + \frac{\pi}{2} \right) + \sin^2 \left( \theta + \frac{\pi}{2} \right) \\
&= (\alpha - 2) \sin^2 \theta + \cos^2 \theta.
\end{align*}
\]
Thus we also have

\[
Z_{\Omega_1}^\alpha (c_j; \mu) = \int_{S^1} \left\{ A_\alpha^1 (\theta_1 (w)) e^1 + B_\alpha^1 (\theta_1 (w)) e^2 \right\} d\Psi_\mu
= \int_{S^1} \left\{ \left[ (\alpha - 2) \cos^2 \theta + \sin^2 \theta \right] e^1 \right\} d\Psi_\mu
= \left\{ \int_{S^1} \left[ (\alpha - 2) \cos^2 \theta + \sin^2 \theta \right] d\Psi_\mu \right\} e^1 + \left\{ \int_{S^1} \left[ (\alpha - 3) \cos \theta \sin \theta \right] d\Psi_\mu \right\} e^2
\]

and

\[
Z_{\Omega_2}^\alpha (c_j; \mu) = \int_{S^1} \left\{ A_\alpha^2 (\theta_2 (w)) e^1 + B_\alpha^2 (\theta_2 (w)) e^2 \right\} d\Psi_\mu
= \int_{S^1} \left\{ \left[ (\alpha - 3) \cos \theta \sin \theta \right] e^1 \right\} d\Psi_\mu
= \left\{ \int_{S^1} \left[ (\alpha - 3) \cos \theta \sin \theta \right] d\Psi_\mu \right\} e^1 + \left\{ \int_{S^1} \left[ (\alpha - 2) \sin^2 \theta + \cos^2 \theta \right] d\Psi_\mu \right\} e^2.
\]

Using

\[
(2.9) \quad (\alpha - 2) \cos^2 \theta + \sin^2 \theta = (\alpha - 3) \cos^2 \theta + 1,
(\alpha - 2) \sin^2 \theta + \cos^2 \theta = (\alpha - 3) \sin^2 \theta + 1,
\sin \theta \cos \theta = \frac{1}{2} \sin 2\theta, \cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \sin^2 \theta = \frac{1 - \cos 2\theta}{2},
\]

we see that

\[
(\alpha - 2) \cos^2 \theta + \sin^2 \theta = (\alpha - 3) \frac{1 + \cos 2\theta}{2} + 1 = \frac{\alpha - 3}{2} \cos 2\theta + \frac{\alpha - 1}{2},
(\alpha - 2) \sin^2 \theta + \cos^2 \theta = (\alpha - 3) \frac{1 - \cos 2\theta}{2} + 1 = -\frac{\alpha - 3}{2} \cos 2\theta + \frac{\alpha - 1}{2},
(\alpha - 3) \cos \theta \sin \theta = \frac{\alpha - 3}{2} \sin 2\theta.
\]

Plugging these formulas into those for \(Z_{\Omega_1}^\alpha (c_j; \mu)\) and \(Z_{\Omega_2}^\alpha (c_j; \mu)\) we obtain

\[
\det \begin{bmatrix}
Z_{\Omega_1}^\alpha (c_j; \mu) \\
Z_{\Omega_2}^\alpha (c_j; \mu)
\end{bmatrix}
= \det \left[ \int_{S^1} \left[ \frac{\alpha - 3}{2} \cos 2\theta + \frac{\alpha - 1}{2} \right] d\Psi_\mu, \int_{S^1} \left[ \frac{\alpha - 3}{2} \sin 2\theta \right] d\Psi_\mu \right]
= \left( \frac{\alpha - 3}{2} \right)^2 \left( \int_{S^1} \cos 2\theta d\Psi_\mu \right)^2 - \left( \frac{\alpha - 1}{2} \right)^2 \left( \int_{S^1} \sin 2\theta d\Psi_\mu \right)^2.
\]

Thus \(\det \begin{bmatrix}
Z_{\Omega_1}^\alpha (c_j; \mu) \\
Z_{\Omega_2}^\alpha (c_j; \mu)
\end{bmatrix} = 0\) if and only if the length of the vector

\[
\frac{\alpha - 3}{2} \left( \int_{S^1} \cos 2\theta d\Psi_\mu \right)
\]
equals $|\alpha - 1|/\Psi_\mu$, i.e.
\begin{equation}
(2.10) \quad \| \left( \int G^\alpha \cos 2\theta d\Psi_\mu \right) \| = |\alpha - 1|/\Psi_\mu \|
\end{equation}

To construct a positive probability measure $d\Psi_\mu$ on the circle that satisfies (2.10), we first observe that if $d\Psi_\mu = \delta_0$ is the unit point mass at 0, then
\[ \left\| \left( \int G^\alpha \cos 2\theta d\Psi_\mu \right) \right\| = \left\| \left( \int G^\alpha \right) \right\| = \| \Psi_\mu \|, \]
and since $|\alpha - 1| < |\alpha - 3|$ for all $0 \leq \alpha < 2$, we have
\[ \left\| \left( \int G^\alpha \cos 2\theta d\Psi_\mu \right) \right\| > |\alpha - 1|/|\alpha - 3| \| \Psi_\mu \|, \]
in this case. On the other hand, if $d\Psi_\mu (\theta) = \frac{1}{2\pi} d\theta$ is normalized Lebesgue measure on the circle, we have
\[ \left\| \left( \int G^\alpha \cos 2\theta d\Psi_\mu \right) \right\| = \left\| \left( 0 \right) \right\| = 0 < |\alpha - 1|/|\alpha - 3| \| \Psi_\mu \|. \]

It is now easy to see that there is a convex combination $d\Psi_\mu = (1 - \lambda) \delta_0 + \lambda \frac{1}{2\pi} d\theta$ such that (2.10) holds. Thus (2.7) fails, and we now show that energy reversal fails.

In fact, we may assume that both $Z_{\Omega_0_{\ell_1}} (c_\ell; \mu)$ and $Z_{\Omega_0_{\ell_1}} (c_\ell; \mu)$ are parallel to the coordinate vector $e_2$, and in this case we will see that we can reverse at most the coordinate energy $E^2 (J, \omega)$, defined above by
\begin{align*}
E^2 (J, \omega)^2 &\equiv \frac{1}{|J|} \frac{1}{|J|} \int \int \left| \frac{1}{|J|^{1/2}} \left( x^2 - z^2 \right) \right|^2 \omega (x) \omega (z), \\
\int &\int \left( T_\alpha \mu (x) - T_\alpha \mu (z) \right) \bigg| \omega (x) \omega (z) \bigg|, \\
&\leq \sum_{\ell=1}^2 \int \int \left| P_\alpha (J, \mu) \frac{Z_{\Omega_{\ell_1}} (c_\ell; \mu)}{|J|^{1/2}} \right|^2 \omega (x) \omega (z) \\
&+ C \sum_{\ell=1}^2 \int \int \frac{\epsilon P_\alpha (J, \mu)}{|J|^{1/2}} \left| x - z \right|^2 \omega (x) \omega (z) \\
&\leq E^2 (J, \omega)^2 P_\alpha (J, \mu)^2 + C \epsilon^2 E (J, \omega)^2 P_\alpha (J, \mu)^2 \\
&\leq \frac{1}{10} C_0 E (J, \omega)^2 P_\alpha (J, \mu)^2, 
\end{align*}

provided we choose $\gamma$ so large that $C \epsilon^2 \leq \frac{1}{10} C_0$ and provided we choose $\omega$ so that $E^2 (J, \omega) = 0$ but $E (J, \omega) > 0$. This completes the proof of the first assertion in Lemma 2. \qed
Remark 1. The condition (2.10) must be invariant under rotations, i.e. invariant under replacing \( \theta \) by \( \theta - \phi \) for any constant \( \phi \), and this is easily seen using (2.9) above:
\[
\left( \int_{S^1} \cos 2(\theta - \phi) \, d\Psi_\mu \right) = \left( \cos 2\phi \int_{S^1} \cos 2\theta d\Psi_\mu + \sin 2\phi \int_{S^1} \sin 2\theta d\Psi_\mu \right) = \cos 2\phi \left( \int_{S^1} \cos 2\theta d\Psi_\mu \right) - \sin 2\phi \left( \int_{S^1} \sin 2\theta d\Psi_\mu \right),
\]
which has length independent of \( \phi \).

Remark 2. The above proof shows that for each \( t \in \mathbb{R} \), the convolution kernel
\[
\Phi_{\alpha, t}(x, y) = \frac{x \cos t + y \sin t}{(x^2 + y^2)^{3/2}},
\]
in the plane with coordinates \((x, y)\), \(x, y \in \mathbb{R}\), and the probability measure \(d\mu_\alpha\) supported on the circle \(S^1 = [0, 2\pi)\) given by
\[
d\mu_\alpha(\theta) = \frac{1}{|\alpha - 3|} \delta_0(\theta) + \frac{|\alpha - 3| - |\alpha - 1|}{2\pi} d\theta,
\]
satisfy the property that \( \text{grad } (\Phi_{\alpha, t} * \mu_\alpha)(0, 0) \) points in the same direction for all \( t \). A direct calculation shows that
\[
\text{grad } (\Phi_{\alpha, t} * \mu_\alpha)(0, 0) = (\alpha - 1) \left\{ \begin{array}{ll}
[\cos t, 0] & \text{for } 0 \leq \alpha < 1 \\
[0, \sin t] & \text{for } 1 < \alpha < 2 .
\end{array} \right.
\]
Indeed, if for \( \theta \in \mathbb{R} \) we define \( \Phi^\theta_{\alpha, t} \) to be the convolution of \( \Phi_{\alpha, t} \) with the unit point mass \( \delta_{e^{i\theta}} \) at \( e^{i\theta} \) in the circle,
\[
\Phi^\theta_{\alpha, t}(x, y) = (\Phi_{\alpha, t} * \delta_{e^{i\theta}})(x, y) = \frac{(x - \cos \theta) \cos t + (y - \sin \theta) \sin t}{(x - \cos \theta)^2 + (y - \sin \theta)^2}.
\]
then we have
\[
\text{grad } \Phi^\theta_{\alpha, t}(x, y) = \left[ \left( \frac{\partial}{\partial x} \Phi^\theta_{\alpha, t} \right)(x, y), \left( \frac{\partial}{\partial y} \Phi^\theta_{\alpha, t} \right)(x, y) \right]
\]
\[
= \frac{[\cos t, \sin t]}{(x - \cos \theta)^2 + (y - \sin \theta)^2} + \frac{3 - \alpha}{2} (x - \cos \theta) \cos t + (y - \sin \theta) \sin t \frac{[2(x - \cos \theta), 2(y - \sin \theta)]}{(x - \cos \theta)^2 + (y - \sin \theta)^2},
\]
and when \((x, y) = (0, 0)\) we get
\[
\text{grad } \Phi^\theta_{\alpha, t}(0, 0) = [\cos t, \sin t] - (3 - \alpha) \{\cos \theta \cos t + \sin \theta \sin t\} [\cos \theta, \sin \theta].
\]
Thus we have
\[
\text{grad } \Phi^0_{\alpha}(0, 0) = \text{grad } \Phi^0_1(0, 0) = [\cos t, \sin t] - (3 - \alpha) \cos t [1, 0]
\]
\[
= [- (2 - \alpha) \cos t, \sin t],
\]
and
\[
\text{grad} \left( \Phi_t * \frac{d\theta}{2\pi} \right) (0, 0) = \text{grad} \int_0^{2\pi} (\Phi_t * \delta_{\alpha,\omega}) (0, 0) \frac{d\theta}{2\pi} \\
= \left[ \cos t, \sin t \right] - \frac{3 - \alpha}{2} \left[ \cos t, \sin t \right] \\
= \left[ \frac{\alpha - 1}{2} \cos t, \frac{\alpha - 1}{2} \sin t \right].
\]

Thus
\[
(3 - \alpha) \text{grad} (\Phi_{\alpha,t} * \mu_{\alpha}) (0, 0) \\
= |\alpha - 1| |- (2 - \alpha) \cos t, \sin t| + (|\alpha - 3| - |\alpha - 1|) \left[ \frac{\alpha - 1}{2} \cos t, \frac{\alpha - 1}{2} \sin t \right] \\
= \left\{ -(2 - \alpha) |\alpha - 1| + (|\alpha - 3| - |\alpha - 1|) \frac{\alpha - 1}{2} \right\} \cos t, \left\{ |\alpha - 1| + (|\alpha - 3| - |\alpha - 1|) \frac{\alpha - 1}{2} \right\} \sin t \right].
\]

Now for \(0 \leq \alpha < 1\) we get
\[
|\alpha - 1| + (|\alpha - 3| - |\alpha - 1|) \frac{\alpha - 1}{2} = 1 - \alpha + 2 \frac{\alpha - 1}{2} = 0,
\]
and
\[
- (2 - \alpha) |\alpha - 1| + (|\alpha - 3| - |\alpha - 1|) \frac{\alpha - 1}{2} = (\alpha - 1) (3 - \alpha).
\]

For \(1 < \alpha < 2\) we get
\[
|\alpha - 1| + (|\alpha - 3| - |\alpha - 1|) \frac{\alpha - 1}{2} = (\alpha - 1) (3 - \alpha),
\]
and
\[
- (2 - \alpha) |\alpha - 1| + (|\alpha - 3| - |\alpha - 1|) \frac{\alpha - 1}{2} = 0.
\]

Proof of Lemma 2 for the vector of trig polynomials. Recall that with \(\theta = \theta_1 (w)\) we obtain
\[
A_\alpha (\theta) = (\alpha - 2) \Omega (\theta) \cos \theta - \Omega' (\theta) \sin \theta \\
B_\alpha (\theta) = (\alpha - 2) \Omega (\theta) \sin \theta + \Omega' (\theta) \cos \theta.
\]

Thus we have
\[
A_\alpha (\theta) = \{ (\alpha - 2) \Omega (\theta) + i \Omega' (\theta) \} \{ \cos \theta + i \sin \theta \} \\
- i \{ (\alpha - 2) \Omega (\theta) \sin \theta + \Omega' (\theta) \cos \theta \} \\
= \{ (\alpha - 2) \Omega (\theta) + i \Omega' (\theta) \} \{ \cos \theta + i \sin \theta \} - i B_\alpha (\theta),
\]
and so
\[
\{ (\alpha - 2) \Omega (\theta) + i \Omega' (\theta) \} \{ \cos \theta + i \sin \theta \} = A_\alpha (\theta) + i B_\alpha (\theta).
\]

This shows that in complex notation,
\[
Z_{\alpha}^{\pm} (c_j; \mu) = \int_{S^1} \{ A_\alpha (\theta) + i B_\alpha (\theta) \} d\Psi \mu \\
= \int_{S^1} \{ (\alpha - 2) \Omega (\theta) + i \Omega' (\theta) \} \{ \cos \theta + i \sin \theta \} d\Psi \mu \\
= \int_{S^1} \Omega_\alpha (\theta) e^{i \theta} d\Psi \mu.
\]
where

\[ \Omega_{\alpha} (\theta) \equiv (\alpha - 2) \Omega (\theta) + i \Omega' (\theta). \]

Recall the product formulas

\[
\begin{align*}
2 \cos A \cos B &= \cos (A - B) + \cos (A + B); \\
2 \sin A \sin B &= \cos (A - B) - \cos (A + B); \\
2 \sin A \cos B &= \sin (A - B) + \sin (A + B).
\end{align*}
\]

In the special case that \( \Omega_{1}^{k} (t) = \cos kt \) we thus have

\[
\begin{align*}
A_{\alpha} (\theta) &= (\alpha - 2) \cos k\theta \cos \theta + k \sin k\theta \sin \theta \\
&= (\alpha - 2) \frac{1}{2} [\cos (k - 1) \theta + \cos (k + 1) \theta] \\
&\quad + k \frac{1}{2} [\cos (k - 1) \theta - \cos (k + 1) \theta] \\
&= \left\{ \frac{\alpha + k}{2} - 1 \right\} \cos (k - 1) \theta + \left\{ \frac{\alpha - k}{2} - 1 \right\} \cos (k + 1) \theta; \\
B_{\alpha} (\theta) &= (\alpha - 2) \cos k\theta \sin \theta - k \sin k\theta \cos \theta \\
&= (\alpha - 2) \frac{1}{2} [- \sin (k - 1) \theta + \sin (k + 1) \theta] \\
&\quad - k \frac{1}{2} [\sin (k - 1) \theta + \sin (k + 1) \theta] \\
&= - \left\{ \frac{\alpha + k}{2} - 1 \right\} \sin (k - 1) \theta + \left\{ \frac{\alpha - k}{2} - 1 \right\} \sin (k + 1) \theta,
\end{align*}
\]

and so

\[
\begin{align*}
Z_{\Omega_{1}^{k}}^\alpha (c, J; \mu) &= \int_{S^{3}} \left\{ A_{\alpha} (\theta) e^{1} + B_{\alpha} (\theta) e^{2} \right\} d\Psi_{\mu} \\
&= \int_{S^{3}} \left[ \left\{ \frac{\alpha + k}{2} - 1 \right\} \cos (k - 1) \theta + \left\{ \frac{\alpha - k}{2} - 1 \right\} \cos (k + 1) \theta \right] d\Psi_{\mu} e^{1} \\
&\quad + \int_{S^{3}} \left[ - \left\{ \frac{\alpha + k}{2} - 1 \right\} \sin (k - 1) \theta + \left\{ \frac{\alpha - k}{2} - 1 \right\} \sin (k + 1) \theta \right] d\Psi_{\mu} e^{2} \\
&= \left\{ \frac{\alpha + k - 2}{2} \right\} \int_{S^{3}} \left( \frac{\cos (k - 1) \theta}{\sin (k - 1) \theta} \right) d\Psi_{\mu} \\
&\quad + \left\{ \frac{\alpha - k - 2}{2} \right\} \int_{S^{3}} \left( \frac{\cos (k + 1) \theta}{\sin (k + 1) \theta} \right) d\Psi_{\mu} \\
&= \left\{ \frac{\alpha + k - 2}{2} \right\} \int_{S^{3}} \left( e^{-i(k-1)\theta} + \frac{\alpha - k - 2}{2} \right) e^{i(k+1)\theta} d\Psi_{\mu} \\
&\quad - \left\{ \frac{\alpha + k - 2}{2} \right\} \int_{S^{3}} \left( \frac{1}{\Psi_{\mu}} (k - 1) + \frac{\alpha - k - 2}{2} \right) \Psi_{\mu} (k + 1) d\Psi_{\mu}.
\end{align*}
\]
Next we take $\Omega^k_2 (\theta) = \sin k \theta$ so that

\[
A_\alpha (\theta) = (\alpha - 2) \sin k \theta \cos \theta - k \cos \theta \sin \theta
\]
\[
= (\alpha - 2) \frac{1}{2} [\sin (k - 1) \theta + \sin (k + 1) \theta]
\]
\[
- k \frac{1}{2} [- \sin (k - 1) \theta + \sin (k + 1) \theta]
\]
\[
= \left\{ \frac{\alpha + k}{2} - 1 \right\} \sin (k - 1) \theta + \left\{ \frac{\alpha - k}{2} - 1 \right\} \sin (k + 1) \theta;
\]
\[
B_\alpha (\theta) = (\alpha - 2) \sin k \theta \sin \theta + k \cos k \theta \cos \theta
\]
\[
= (\alpha - 2) \frac{1}{2} [\cos (k - 1) \theta - \cos (k + 1) \theta]
\]
\[
+ k \frac{1}{2} [\cos (k - 1) \theta + \cos (k + 1) \theta]
\]
\[
= \left\{ \frac{\alpha + k}{2} - 1 \right\} \cos (k - 1) \theta - \left\{ \frac{\alpha - k}{2} - 1 \right\} \cos (k + 1) \theta.
\]

Thus with $\Omega^k_2 (\theta) = \sin k \theta$ we obtain

\[
Z^\alpha_{\Omega^k_2} (c; J; \mu) = \int_{S^1} \left\{ A_\alpha (\theta) e^1 + B_\alpha (\theta) e^2 \right\} d\Psi_\mu
\]
\[
= \int_{S^1} \left\{ \frac{\alpha + k}{2} - 1 \right\} \sin (k - 1) \theta + \left\{ \frac{\alpha - k}{2} - 1 \right\} \sin (k + 1) \theta \right\} d\Psi_\mu e^1
\]
\[
+ \int_{S^1} \left\{ \frac{\alpha + k}{2} - 1 \right\} \cos (k - 1) \theta - \left\{ \frac{\alpha - k}{2} - 1 \right\} \cos (k + 1) \theta \right\} d\Psi_\mu e^2
\]
\[
= \left\{ \frac{\alpha + k - 2}{2} \right\} \int_{S^1} \sin (k - 1) \theta \cos (k + 1) \theta \right\} d\Psi_\mu
\]
\[
+ \left\{ \frac{\alpha - k - 2}{2} \right\} \int_{S^1} \cos (k - 1) \theta - \cos (k + 1) \theta \right\} d\Psi_\mu
\]
\[
= \int_{S^1} \left\{ \left( \frac{\alpha + k - 2}{2} \right) ie^{-i(k-1)\theta} - \left( \frac{\alpha - k - 2}{2} \right) ie^{i(k+1)\theta} \right\} d\Psi_\mu
\]
\[
= i \left( \frac{\alpha + k - 2}{2} \right) \Psi_\mu (k - 1) - i \left( \frac{\alpha - k - 2}{2} \right) \Psi_\mu (k + 1).
\]

Altogether we have

\[
(2.1) Z^\alpha_{\Omega^k_1} (c; J; \mu) = \left( \frac{\alpha + k - 2}{2} \right) \Psi_\mu (k - 1) + \left( \frac{\alpha - k - 2}{2} \right) \Psi_\mu (k + 1);
\]
\[
Z^\alpha_{\Omega^k_2} (c; J; \mu) = i \left[ \left( \frac{\alpha + k - 2}{2} \right) \Psi_\mu (k - 1) - \left( \frac{\alpha - k - 2}{2} \right) \Psi_\mu (k + 1) \right].
\]
Thus \( \det \left[ \begin{pmatrix} Z_{\Omega_1}^\alpha (c_j; \mu) \\ Z_{\Omega_2}^\alpha (c_j; \mu) \end{pmatrix} \right] \) is the imaginary part of \( Z_{\Omega_1}^\alpha (c_j; \mu) Z_{\Omega_2}^\alpha (c_j; \mu) \), which is 

\[
-1 \text{ times the real part of } \left\{ \left( \alpha + k - \frac{2}{2} \right) \overline{\Psi_\mu (k - 1)} + \left( \alpha - k - \frac{2}{2} \right) \overline{\Psi_\mu (k + 1)} \right\} 
\times \left\{ \left( \alpha + k - \frac{2}{2} \right) \overline{\Psi_\mu (k - 1)} - \left( \alpha - k - \frac{2}{2} \right) \overline{\Psi_\mu (k + 1)} \right\} 
= \left( \alpha + k - \frac{2}{2} \right)^2 \left| \overline{\Psi_\mu (k - 1)} \right|^2 - \left( \alpha - k - \frac{2}{2} \right)^2 \left| \overline{\Psi_\mu (k + 1)} \right|^2 
+ \Re \left[ \left( \alpha + k - \frac{2}{2} \right) \left( \alpha - k - \frac{2}{2} \right) \left( \overline{\Psi_\mu (k + 1)} \overline{\Psi_\mu (k - 1)} - \overline{\Psi_\mu (k - 1)} \overline{\Psi_\mu (k + 1)} \right) \right] 
= \left( \alpha + k - \frac{2}{2} \right)^2 \left| \overline{\Psi_\mu (k - 1)} \right|^2 - \left( \alpha - k - \frac{2}{2} \right)^2 \left| \overline{\Psi_\mu (k + 1)} \right|^2 ,
\]

since \( \overline{\Psi_\mu (k + 1)} \overline{\Psi_\mu (k - 1)} - \overline{\Psi_\mu (k - 1)} \overline{\Psi_\mu (k + 1)} \) is pure imaginary. We conclude that

\[
(2.13) \quad \det \left[ \begin{pmatrix} Z_{\Omega_1}^\alpha (c_j; \mu) \\ Z_{\Omega_2}^\alpha (c_j; \mu) \end{pmatrix} \right] = 0 \iff \left| \overline{\Psi_\mu (k + 1)} \right| = \left| \frac{\alpha + k - \frac{2}{2}}{\alpha - k - \frac{2}{2}} \right| \left| \overline{\Psi_\mu (k - 1)} \right| , \quad \text{all } k.
\]

We also have that \( \det \left[ \begin{pmatrix} Z_{\Omega_1}^\alpha (c_j; \mu) \\ Z_{\Omega_2}^\alpha (c_j; \mu) \end{pmatrix} \right] \) is the imaginary part of \( Z_{\Omega_1}^\alpha (c_j; \mu) Z_{\Omega_2}^\alpha (c_j; \mu) \), i.e. the imaginary part of

\[
\left\{ \left( \alpha + k - \frac{2}{2} \right) \overline{\Psi_\mu (k - 1)} + \left( \alpha - k - \frac{2}{2} \right) \overline{\Psi_\mu (k + 1)} \right\} 
\times \left\{ \left( \alpha + \ell - \frac{2}{2} \right) \overline{\Psi_\mu (k + 1)} + \left( \alpha - \ell - \frac{2}{2} \right) \overline{\Psi_\mu (k + 3)} \right\} .
\]

If we now suppose that \( \overline{\Psi_\mu (n)} \) is real for all \( n \), then \( Z_{\Omega_1}^\alpha (c_j; \mu) \) is real for all \( k \), and it follows that

\[
(2.14) \quad \det \left[ \begin{pmatrix} Z_{\Omega_1}^\alpha (c_j; \mu) \\ Z_{\Omega_2}^\alpha (c_j; \mu) \end{pmatrix} \right] = \Im \left( Z_{\Omega_1}^\alpha (c_j; \mu) Z_{\Omega_2}^\alpha (c_j; \mu) \right) = 0 , \quad \text{all } k, \ell .
\]

We are now ready to construct the measure \( \mu \) with an appropriate density \( \Psi_\mu \). In the case \( 1 \leq \alpha < 2 \) there is a choice of density that is easy to prove positive, and we give that first. Then we give a density for all cases \( 0 \leq \alpha < 2 \), but that is much harder to prove positive. Finally we give a particularly simple proof for the case \( \alpha = 0 \).

**Construction of a density in the case** \( 1 \leq \alpha < 2 \):

Define a density \( \Psi (\theta) \) by

\[
\Psi (\theta) = 1 + 2 \sum_{n=1}^{\infty} b_n \cos (2n\theta) = 1 + \sum_{n=1}^{\infty} b_n \left( e^{i2n\theta} + e^{-i2n\theta} \right) ,
\]
where
\[ b_n = \frac{\alpha + (2n - 3)\alpha + (2n - 5)\alpha + 3\alpha + 1\alpha - 1}{\alpha - (2n + 1)\alpha - (2n - 1)\alpha - 7\alpha - 5\alpha - 3} \]
with \( a_n = \alpha(n) = a_n a_{n-1} \cdots a_1, \quad n \geq 1; \)
\[ \Phi (\theta) = \frac{\alpha + (2n - 3)\alpha + (2n - 5)\alpha + 3\alpha + 1\alpha - 1}{\alpha - (2n + 1)\alpha - (2n - 1)\alpha - 7\alpha - 5\alpha - 3} \]
if \( x = 2 - \alpha. \)

Then we have
\[ \Psi (2n) = b_n = \Psi (-2n), \quad n \geq 1, \]
\[ \Psi (k) = 0 \text{ if } k \text{ is odd,} \]
and in particular that \( \left| \Psi (k + 1) \right| = \left| \frac{\alpha + k - 2}{\alpha - k - 1} \right| \left| \Psi (k - 1) \right| \) for all \( k \geq 1. \) Now choose a measure \( \mu \) giving rise to the density \( \Psi. \) In the case \( 1 < \alpha < 2 \) we have \( \left| \frac{\alpha + k - 2}{\alpha - k - 1} \right| = -\frac{\alpha + k - 2}{\alpha - k - 1} \) for \( k \geq 1, \) and so from (2.12) we actually obtain that \( Z_{\Omega k}^\alpha (c_j; \mu) = 0 \) for all \( k \geq 1, \) and that \( Z_{\Omega k}^\alpha (c_j; \mu) \) is imaginary for all \( k \geq 1. \) Thus all of the vectors \( \{Z_{\Omega k}^\alpha (c_j; \mu); Z_{\Omega k}^\alpha (c_j; \mu)\}_{k=1}^\infty \) are multiples of the unit vector \((0, 1)\) in the plane (it is the failure of such a conclusion for the case \( 0 < \alpha < 1 \) that forces a different construction below).

We must now show that the density \( \Psi (\theta) \) is nonnegative. We have \( \Psi (\theta) = \Phi (2\theta) \) where \( \Phi (0) = 1 \) and
\[ \Phi (n) = \Phi (-n) = b_n = a_n a_{n-1} \cdots a_1, \quad n \geq 1. \]

We claim that the nonnegative sequence \( \{1, b_1, b_2, \ldots\} \) is convex for \( 0 < x \leq 2, \) and has limit 0 as \( n \to \infty. \) With this established, the density \( \Phi \) is a positive sum of Féjer kernels, and hence \( \Phi (\theta) \geq 0. \) Since \( a_n = \frac{2n+1}{2n+1+x} = 1 - \frac{2x}{2n+1+x} \) and \( \sum_{n=1}^\infty \frac{2x}{2n+1+x} = \infty, \) we see that \( \lim_{n \to \infty} b_n = \prod_{n=1}^\infty \left( 1 - \frac{2x}{2n+1+x} \right) = 0. \) To see the convexity we note that
\[ b_{n+1} + b_{n-1} - 2b_n = a_{n+1} a_n [a_{n-1} \cdots a_2 a_1] + [a_{n-1} \cdots a_2 a_1] - 2a_n [a_{n-1} \cdots a_2 a_1] \]
\[ = [a_{n+1} a_n + 1 - 2a_n] [a_{n-1} \cdots a_2 a_1] \]
is positive if and only if \( a_{n+1} a_n + 1 - 2a_n \) is positive. But for \( n \geq 2 \) and \( 0 < x \leq 2, \) we have \( a_n = \frac{2n+1-x}{2n+1+x} \) and so
\[ a_{n+1} a_n + 1 - 2a_n = \frac{(a_{n+1} - 2) a_n + 1}{(2n+1-x)} \]
\[ = \frac{2n+1-x}{2n+1+x} - 2 \frac{2n+1-x}{2n+1+x} + 1 \]
\[ = \frac{2n+1-x}{2n+1+x} - 2 \frac{2n+1-x}{2n+1+x} + 1 \]
\[ = \frac{(2n+1+x)(2n+1-x)(2n+1-x)}{(2n+1+x)(2n+1+x)} \]
\[ = \frac{4x^2 + 4x}{(2n+1+x)(2n-1+x)} > 0. \]
This calculation is valid also when \( n = 1 \) and \( 0 < x \leq 1 \), so it remains to consider only the case \( n = 1 \) and \( 1 \leq x \leq 2 \). But then we have \( a_1 = \frac{4}{3 + x} \) and so

\[
a_2 a_1 + 1 - 2a_1 = (a_2 - 2) a_1 + 1 = \left( \frac{3 - x}{3 + x} - 2 \right) \frac{x - 1}{1 + x} + 1 = \frac{6 - 2x}{3 + x} > 0.
\]

**Construction of a density in the general case \( 0 \leq \alpha < 2 \):**

This time we modify the definition of our density to be

\[
\hat{\Psi}(\theta) = 1 + 2 \sum_{n=1}^{\infty} b_n \cos(2n\theta) = 1 + \sum_{n=1}^{\infty} b_n \left\{ e^{i2n\theta} + e^{-i2n\theta} \right\},
\]

where

\[
b_n = \frac{\alpha + (2n - 3) \alpha + (2n - 5) \alpha + 3 \alpha + 1 \alpha - 1}{\alpha - (2n + 1) \alpha - (2n - 1) \alpha - 7 \alpha - 5 \alpha - 3} = a_n a_{n-1} \ldots a_2 a_1,
\]

\( n \geq 1; \)

where \( a_n = \frac{\alpha + (2n - 3)}{\alpha - (2n + 1)} = -\frac{2n - 1 - x}{2n - 1 + x} \) if \( x = 2 - \alpha \).

Then we have

\[
\hat{\Psi}(2n) = b_n = \hat{\Psi}(-2n), \quad 1 \leq n \leq N,
\]

\[
\hat{\Psi}(k) = 0 \text{ if } k \text{ is odd},
\]

and in particular, if \( \bar{\mu} \) is chosen to give rise to the density \( \hat{\Psi} \), then from equation (2.12) we obtain that \( Z_{(\alpha_k^1)}^{\alpha} (c_j; \bar{\mu}) = 0 \) for all \( k \geq 1 \), and that \( Z_{(\alpha_k^1)}^{\alpha} (c_j; \bar{\mu}) \) is real for all \( k \geq 1 \). Thus all of the vectors \( \{ Z_{(\alpha_k^1)}^{\alpha} (c_j; \bar{\mu}) \} \}_{k=1}^{\infty} \) are multiples of the unit vector \((1, 0)\) in the plane.

Finally, we must show that the density \( \hat{\Psi}(\theta) \) is positive. Now

\[
\hat{\Psi}(2n) = b_n = a_n a_{n-1} \ldots a_2 a_1,
\]

and so by Böchner’s theorem (more precisely Herglotz’s theorem in this application - see e.g. Rudin [Rud] for an extension to locally compact abelian groups), it suffices to check that the following matrices are positive semidefinite for \( n \geq 2 \):

\[
B_n = \begin{pmatrix}
\hat{\Psi}(0) & \hat{\Psi}(2) & \hat{\Psi}(4) & \cdots & \hat{\Psi}(2n) \\
\hat{\Psi}(2) & \hat{\Psi}(0) & \hat{\Psi}(2) & \cdots & \hat{\Psi}(2n-2) \\
\hat{\Psi}(4) & \hat{\Psi}(2) & \hat{\Psi}(0) & \cdots & \hat{\Psi}(2n-4) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\hat{\Psi}(2n) & \hat{\Psi}(2n-2) & \hat{\Psi}(2n-4) & \cdots & \hat{\Psi}(0)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & a_1 & a_2 a_1 & \cdots & a_n a_1 \\
a_1 & 1 & a_1 & \cdots & a_{n-1} a_1 \\
a_2 a_1 & a_1 & 1 & \cdots & a_{n-2} a_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_n a_1 & a_{n-1} a_1 & a_{n-2} a_1 & \cdots & 1
\end{pmatrix}
\]
Since $a_n = -\frac{2n-1-x}{2n-1+x}$, the matrix $B_n$ is

\[
B_n(x) = \begin{bmatrix}
1 & -\frac{1-x}{1+x} & \frac{3-x}{3+x} & \frac{1-x}{1+x} & \cdots & \cdots & (-1)^{n+1} \frac{(2n-3-x)}{(2n-3)+x} & \frac{3-x}{3+x} & \frac{1-x}{1+x} \\
-\frac{1-x}{1+x} & 1 & -\frac{1-x}{1+x} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{3-x}{3+x} & -\frac{1-x}{1+x} & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
(-1)^{n+1} \frac{(2n-3-x)}{(2n-3)+x} & \frac{3-x}{3+x} & \frac{1-x}{1+x} & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
\end{bmatrix},
\]

and a standard reduction in matrix theory shows that it is enough to show that $\det B_n(x) \geq 0$ for all $n \geq 2$.

In the appendix below, we prove that these determinants satisfy the recursion formula

\[
\det B_{n+1}(x) = 2^{2n} \frac{n! (n-1+x) (n-2+x) \cdots (x)}{[(2n-1+x) (2n-3+x) \cdots (1+x)]^2}, \quad n \geq 1.
\]

From this recursion we immediately obtain that for $x > 0$, the determinants $\det B_n(x)$ and $\det B_{n+1}(x)$ have the same sign. Then since $\det B_1(x) = 1$, induction shows that

\[
\det B_n(x) > 0 \quad \text{for all } x > 0, \quad n \geq 1.
\]

This completes the proof that the matrices $B_n$ are positive definite for all $n \geq 1$ and $x > 0$, and hence that the density $\tilde{\Psi}$ is positive. We note that this completes the proof of Lemma 2 for all $0 \leq \alpha < 2$.

**Construction of the density in the case $\alpha = 0$:**

The case $\alpha = 0$ corresponds to the usual singular integrals in the plane, and for this case there is an especially simple proof of the nonnegativity of the density $\tilde{\Psi}$. We simply note that the density $\tilde{\Psi}$ is nonnegative by taking absolute values inside the sum,

\[
\tilde{\Psi}(\theta) = 1 + 2 \sum_{n=1}^{\infty} b_n \cos (2n\theta) \geq 1 - 2 \sum_{n=1}^{\infty} |b_n|,
\]

and then calculating that

\[
|b_n| = |a_n a_{n-1} \cdots a_2 a_1| = \frac{(2n-3) (2n-5) \cdots 3 1 1}{(2n+1) (2n-1) \cdots 7 5 3} = \frac{1}{(2n+1) (2n-1)} = \frac{1}{2} \left( \frac{1}{2n-1} - \frac{1}{2n+1} \right),
\]

hence

\[
\sum_{n=1}^{\infty} |b_n| = \sum_{n=1}^{\infty} \frac{1}{2} \left( \frac{1}{2n-1} - \frac{1}{2n+1} \right) = \frac{1}{2}.
\]

Now we show how to adapt the above proof to prove Corollary 1.
Proof of Corollary[7] First we note that if $\Omega$ is sufficiently smooth with vanishing integral on the circle, then it is an absolutely convergent sum of the trig functions $\cos n\theta$ and $\sin n\theta$ for $n \geq 1$. Thus a standard limiting argument extends the above failure of energy reversal to any finite vector of such $\Omega$. Now embed the measure $\tilde{\mu}$ with density $\Psi$ constructed above into Euclidean space $\mathbb{R}^n$ via the embedding $\mathbb{R}^2 \ni (x_1, x_2) \mapsto (x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$. Here we are letting the parameter $x = n - \alpha$ lie in the interval $(0, n]$. Then the above proof shows that for cubes $J$ with center $c_J \in \mathbb{R}^2 \times \{0\}$, the gradients $Z^0_n (c_J; \tilde{\mu})$ of the kernels $\Omega$ have their planar projections parallel to $(1, 0)$, and hence all the gradients $Z^0_n (c_J; \tilde{\mu})$ are perpendicular to the fixed direction $(0, 1, 0, \ldots, 0)$ in $\mathbb{R}^n$. As a consequence, reversal of energy fails in $J$ for the measure $\tilde{\mu}$, and it remains only to show that the density $\Psi$ is positive. But this is implied by the positivity of $\det B_n (x)$ for $x \in (0, n]$, which follows from the recursion (2.16) and the fact that $\det B_n (x) = 1 > 0$. □

3. Appendix

We can rewrite the recursion (2.16) above as

$$\frac{\det B_{n+1} (x)}{\det B_n (x)} = \frac{\Omega^n_n (x-1)}{[\Omega^n_{n+1} (x)]^2}, \quad n \geq 1,$$

where for any positive integer $n$ and real number $a$ we define the combinatorial coefficient

$$\Omega^n_n (a) = \frac{(n + a) (n - 1 + a) \ldots (1 + a)}{(n) (n - 1) \ldots (1)}.$$

We now prove the recursion formula (3.1) using the well known block determinant formula

$$\det \begin{bmatrix} B & c \\ r & a \end{bmatrix} = a \det B - r [\text{cof } B]^\text{tr} c = \det B \{a - rB^{-1}c\},$$

where $B$ is an $n \times n$ matrix and $r$ and $c$ are $n$-dimensional row and column vectors respectively. Here $[\text{cof } B]^\text{tr}$ denotes the transposed cofactor matrix of $B$ and the inverse of $B$ is given by $B^{-1} = \frac{1}{\det B} [\text{cof } B]^\text{tr}$. If we apply this with $B = B_n (x)$ and $\begin{bmatrix} B & c \\ r & a \end{bmatrix} = B_{n+1} (x)$ we get

$$\det B_{n+1} (x) = \det \begin{bmatrix} B_n (x) & c^n (x) \\ r_n (x) & 1 \end{bmatrix} = \det B_n (x) \left\{1 - r_n (x) B_n (x)^{-1} c^n (x)\right\},$$

where $r_n (x)$ denotes the $n$-dimensional row vector consisting of the first $n$ entries of the bottom row of $B_{n+1} (x)$, and similarly $c^n (x)$ denotes the $n$-dimensional column vector consisting of the first $n$ entries of the rightmost column of $B_{n+1} (x)$. Note also that $r_n (x)$ and $c^n (x)$ are transposes of each other.

Motivated by computer algebra calculations, we define the column vector

$$v^n (x) \equiv (-1)^{n-1} \left[(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \Gamma_k \left(\frac{x-1}{2}\right)\right]_{k=0}^{n-1},$$

where

$$\Gamma_k^n (a) \equiv \frac{\Gamma (k + a + 1) \Gamma (n - k + a)}{\Gamma (n + a + 1) \Gamma (a)} = \frac{(k + a) \ldots (a)}{(n + a) \ldots (n - k + a)}.$$
Lemma 3. For \( n \geq 1 \) we have
\[
B_n(x)^{-1} c^n(x) = v^n(x).
\]

Proof. It suffices to show the vector identity
\[
B_n(x) v^n(x) = c^n(x), \quad n \geq 1,
\]
and to prove this we will use the well known fact that an \( n^{th} \) order difference of a polynomial of degree less than \( n \) vanishes. More specifically the polynomial in question will be
\[
P_{n-1}(s) \equiv \frac{\Gamma(n-1+s)}{\Gamma(s)} = (n-1+s) \cdots (1+s) s.
\]
Indeed,
\[
v^n(x) = \left[ (-1)^k \binom{n}{n-1-k} \Gamma(n-k+\frac{x-1}{2}) \Gamma(1+k+\frac{x-1}{2}) \right]_{k=0}^{n-1}
\]
\[
= \left[ (-1)^k \binom{n}{k+1} \frac{\Gamma(n-k+\frac{x-1}{2}) \Gamma(1+k+\frac{x-1}{2})}{\Gamma(n+1+\frac{x-1}{2}) \Gamma(\frac{x-1}{2})} \right]_{k=0}^{n-1}
\]
\[
= \left[ (-1)^k \binom{n}{k} \frac{\Gamma(n-k+z) \Gamma(k+1+z)}{\Gamma(n+z) \Gamma(-1+z)} \right]_{k=0}^{n-1},
\]
where
\[
z = \frac{x-1}{2} + 1 = \frac{x+1}{2}.
\]
Now we use
\[
\frac{(x-1)(x-3)(x-5)\cdots(x-(2n-1))}{(x+1)(x+3)(x+5)\cdots(x+(2n-1))}
\]
\[
= \frac{(\frac{x-1}{2}) (\frac{x-1}{2} - 1) (\frac{x-1}{2} - 2) \cdots (\frac{x-1}{2} - (n-1))}{(\frac{x-1}{2} + 1) (\frac{x-1}{2} + 2) (\frac{x-1}{2} + 3) \cdots (\frac{x-1}{2} + n)}
\]
\[
= \frac{\Gamma(\frac{x-1}{2} + 1) \Gamma(\frac{x-1}{2} + n)}{\Gamma(\frac{x-1}{2} + (n-1)) \Gamma(\frac{x-1}{2} + n)}
\]
\[
= \frac{\Gamma^2(z)}{\Gamma(z-n) \Gamma(z+n)}
\]
to obtain that
\[
B_n(x) = \left[ \frac{\Gamma^2(z)}{\Gamma(z-j-i) \Gamma(z+j-i)} \right]_{i,j=1}^{n}
\]
Thus the first row of \( B_n(x) \) is
\[
\begin{pmatrix}
1 & \frac{\Gamma^2(z)}{\Gamma(z-1) \Gamma(z+1)} & \cdots & \frac{\Gamma^2(z)}{\Gamma(z-n) \Gamma(z+n)}
\end{pmatrix}
\]
\[
= \left[ \frac{\Gamma^2(z)}{\Gamma(z-(k-1)) \Gamma(z+(k-1))} \right]_{k=1}^{n}
\]
Thus we get
\[
\left[ \frac{\Gamma (z)^2}{\Gamma (z - (k - 1)) \Gamma (z + (k - 1))} \right]_{k=1}^{n} \cdot v^n (x)
\]
\[= - \sum_{k=1}^{n} (-1)^k \left( \begin{array}{c} n \\ k \end{array} \right) \frac{\Gamma (n - k + z) \Gamma (k - 1 + z)}{\Gamma (n + z) \Gamma (-1 + z)} \frac{\Gamma (z)^2}{\Gamma (z - (k - 1)) \Gamma (z + (k - 1))}
\]
\[= - \sum_{k=1}^{n} (-1)^k \left( \begin{array}{c} n \\ k \end{array} \right) \frac{\Gamma (z)^2}{\Gamma (z + n) \Gamma (z - 1)} \sum_{k=1}^{n} (-1)^k \left( \begin{array}{c} n \\ k \end{array} \right) \{(z - k + n - 1) \ldots (z - k + 1)\}
\]
\[= - \frac{\Gamma (z)^2}{\Gamma (z + n) \Gamma (z - 1)} \sum_{k=1}^{n} (-1)^k \left( \begin{array}{c} n \\ k \end{array} \right) P^n_z (k)
\]
where \(P^n_z (w) = (z - w + n - 1) \ldots (z - w + 1)\) is a polynomial of degree \(n - 1\). Now recall that if \(\triangle f \equiv f (1) - f (0)\) is the unit difference operator at 0, then
\[
\triangle^n f = \sum_{k=0}^{n} (-1)^k \left( \begin{array}{c} n \\ k \end{array} \right) f (k)
\]
Thus we have
\[
\sum_{k=0}^{n} (-1)^k \left( \begin{array}{c} n \\ k \end{array} \right) P^n_z (k) = \triangle^n P^n_z = 0
\]
since \(P^n_z\) has degree less than \(n\), and so
\[
\left[ \frac{\Gamma (z)^2}{\Gamma (z - (k - 1)) \Gamma (z + (k - 1))} \right]_{k=1}^{n} \cdot v^n (x)
\]
\[= \frac{\Gamma (z)^2}{\Gamma (z + n) \Gamma (z - 1)} (z + n - 1) \ldots (z + 1)
\]
\[= \frac{\Gamma (z)^2}{\Gamma (z + 1) \Gamma (z - 1)}
\]
which is the first component of \(c^n (x)\) as required. A similar argument proves the equality of the remaining components, and this completes the proof of Lemma 3. □

**Lemma 4.** For \(n \geq 1\) we have
\[
1 - r_n (x) \cdot v_n (x) = \frac{\Omega^n_n (x - 1)}{[\Omega^n_n (\frac{x-1}{2})]^2}.
\]

**Proof.** Again, this is an application of the fact that an \(n^{th}\) order difference of a polynomial of degree less than \(n\) vanishes, but a bit more complicated. Recall that
\[
\Omega^n_n (a) \equiv \frac{(n + a)(n - 1 + a) \ldots (1 + a)}{(n)(n - 1) \ldots (1)} = \frac{\Gamma (n + 1 + a)}{\Gamma (1 + a) n!}.
\]
so that we have
\[
\frac{\Omega_n^2 (x-1)}{[\Omega_n (\frac{x}{2})]^2} = n! \frac{(n + x - 1)(n - 1 + x - 1) \ldots (1 + x - 1)}{(n + \frac{x-1}{2})(n - 1 + \frac{x-1}{2}) \ldots (1 + \frac{x-1}{2})^2} = \frac{\Gamma(n + 1) \Gamma(n + x) \Gamma(1 + \frac{x-1}{2})^2}{\Gamma(x) \Gamma(n + 1 + \frac{x-1}{2})^2}.
\]

We also have
\[
v^n(x) = \left[ (-1)^{k+1} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{\Gamma(n-k+\frac{x-1}{2}) \Gamma(1+k+\frac{x-1}{2})}{\Gamma(n+1+\frac{x-1}{2})} \right]_{k=0}^{n-1}
\]
and from (2.15), we have
\[
r_n(x) = \left[ (-1)^n \frac{(2n-1-x)}{(2n-1+x)} \ldots \frac{3-x}{3+x} \ldots \frac{3-x}{3+x} \ldots -\frac{1-x}{1+x} \right]_{k=0}^{n-1}
\]
and hence dividing all factors top and bottom by 2, we get
\[
r_n(x) = \left[ \frac{-k + \frac{x-1}{2}}{k+1 + \frac{x-1}{2}} \ldots -1 + \frac{x-1}{2} \ldots 1 + \frac{x-1}{2} \ldots \frac{1-x}{1+x} \right]_{k=0}^{n-1}
\]
Thus our identity to be proved is
\[
\sum_{k=0}^{n-1} (-1)^k \left( \begin{array}{c} n \\ k+1 \end{array} \right) \frac{\Gamma(1+\frac{x-1}{2})^2}{\Gamma(n-k+\frac{x-1}{2}) \Gamma(k+2+\frac{x-1}{2})} \times \frac{\Gamma(n-k+\frac{x-1}{2}) \Gamma(k+1+\frac{x-1}{2})}{\Gamma(n+1+\frac{x-1}{2})} = 1 - \frac{\Gamma(n + 1) \Gamma(n + x) \Gamma(1 + \frac{x-1}{2})^2}{\Gamma(x) \Gamma(n + 1 + \frac{x-1}{2})^2}.
\]
If we set \( z = 1 + \frac{w-1}{2} \) then this identity becomes
\[
\sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \frac{\Gamma(z)^2}{\Gamma(-k-1+z)\Gamma(k+1+z)} \frac{\Gamma(n-k-1+z)\Gamma(k+z)}{\Gamma(n+z)\Gamma(-1+z)} = 1 - \frac{\Gamma(n+1)\Gamma(n-1+2z)\Gamma(z)^2}{\Gamma(-1+2z)\Gamma(n+z)^2},
\]
and if we replace \( k \) by \( k - 1 \) we get
\[
\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{\Gamma(z)^2}{\Gamma(-k+z)\Gamma(k+z)} \frac{\Gamma(n-k+z)\Gamma(k-1+z)}{\Gamma(n+z)\Gamma(-1+z)} = 1 - \frac{\Gamma(n+1)\Gamma(n-1+2z)\Gamma(z)^2}{\Gamma(-1+2z)\Gamma(n+z)^2},
\]
Note that the term \( k = 0 \) in the sum on the left would be \(-1\), so that we can subtract \(1\) from both sides, and then multiply by \(-1\) to get
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(z+n-k)\Gamma(z+k-1)}{\Gamma(z-k)\Gamma(z+k)} = \frac{\Gamma(n+1)\Gamma(z-1)\Gamma(2z+n-1)}{\Gamma(z+n)\Gamma(2z-1)},
\]
which is equivalent to
\[[3.6] \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(z+n-k)\Gamma(z+k-1)}{\Gamma(z-k)\Gamma(z+k)} = \frac{\Gamma(n+1)\Gamma(z-1)\Gamma(2z+n-1)}{\Gamma(z+n)\Gamma(2z-1)},\]
We now use
\[
\frac{\Gamma(s+m+1)}{\Gamma(s)} = (s+m)(s+m-1)...(s+1)s
\]
to rewrite \[[3.6] \] as
\[[3.7] \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(z+n-k-1)...(z-k)}{(z+k-1)} = n!(2z+n-2)...(2z)(2z-1)\frac{(z+n-1)...(z)(z-1)}{(z+n+1)...(z)(z-1)},\]
Denote the left and right hand sides of \[[3.7] \] by \( LHS_n(z) \) and \( RHS_n(z) \) respectively. Then the left hand side \( LHS_n(z) \) of \[[3.7] \] is
\[
LHS_n(z) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(z+n-k-1)...(z+1-k)([z+k-1]-[2k-1])}{\Gamma(z+k-1)}
\]
\[
= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (z+n-k-1)...(z+1-k)
\]
\[
+ \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \frac{(z+n-k-1)...(z+1-k)}{(z+k-1)}(2k-1),
\]
where the first sum on the right hand side above vanishes since it is an \(n\)th order difference of the polynomial
\[
P(w) \equiv (z+n-w-1)...(z+1-w)
of degree \( n - 1 \). Thus we have
\[
LHS_n(z) = \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \frac{(z+n-k-1)\ldots(z+2-k)(\lfloor z+k-1 \rfloor - [2k-2])}{(z+k-1)} (2k-1)
\]
\[
= \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} (z+n-k-1)\ldots(z+2-k)(2k-1)
\]
\[
+ \sum_{k=0}^{n} (-1)^{k+2} \binom{n}{k} \frac{(z+n-k-1)\ldots(z+2-k)}{(z+k-1)} (2k-2)(2k-1),
\]
where the first sum on the right hand side above vanishes since it is an \( n^\text{th} \) order difference of the polynomial
\[
P(w) \equiv (z+n-w-1)\ldots(z+2-w)(2w-1)
\]
of degree \( n - 1 \). Continuing in this way we get
\[
LHS_n(z) = \sum_{k=0}^{n} (-1)^{k+n} \binom{n}{k} \frac{1}{(z+k-1)} (2k-n)(2k-2)(2k-1).
\]
Now the right hand side \( RHS_n(z) \) of (3.7) is a quotient of a polynomial of degree \( n \) by a polynomial of degree \( n + 1 \), and so has a partial fraction decomposition of the form
\[
RHS_n(z) = n! \frac{(2z+n-2)\ldots(2z)(2z-1)}{(z+n-1)\ldots(z)(z-1)} = \sum_{k=0}^{n} A_k \frac{1}{z+k-1},
\]
for uniquely determined coefficients \( A_0, \ldots, A_n \). So the proof of (3.7) has been reduced to proving the identity,
\[
A_k = (-1)^{k+n} \binom{n}{k} (2k-n)(2k-2)(2k-1).
\]
Now \( A_k \) is the residue of the meromorphic function \( RHS_n(z) \) at \( z = -(k-1) \), hence using the notation \( (z+k-1) \) to indicate that the factor \( (z+k-1) \) is missing, we get
\[
A_k = \text{res}(RHS_n(z); -(k-1))
\]
\[
= n! \frac{(2z+n-2)\ldots(2z)(2z-1)}{(z+n-1)\ldots(z+k-1)(z+k-2)\ldots(z)(z-1)} \bigg|_{z=-(k-1)}
\]
\[
= n! \frac{(2[1-k]+n-2)\ldots(2[1-k]+2)(2[1-k]-1)(1-k)\ldots((1-k)(1-k)-1)}{(1-k)\ldots(1-k)(1)(0)(-1)^k(1)\ldots(k-1)(k)}
\]
\[
= (-1)^{n-k} \frac{n!}{(n-k)!k!} (2k-n)(2k-2)(2k-1),
\]
which proves (3.8). This completes the proof of Lemma 4.

The proof of our claimed recursion (3.1) is now completed by combining Lemmas 3 and 4 with (3.3).


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