Information Content of Elementary Systems as a Physical Principle

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Quantum physics has remarkable characteristics such as quantum correlations, uncertainty relations, no cloning, which make for an interpretative and conceptual gap between the classical and the quantum world. To provide more unified framework the generalized probabilistic theories were formulated. Recently, it turned out that such theories include so called "postquantum" ones which share many of the typical quantum characteristics but predict supraquantum effects such as correlations stronger than quantum ones. As a result we reveal even more dramatic gap between classical/quantum and post-quantum world. Therefore it is imperative to search for information principles characterizing physical theories. In recent years, different principles has been proposed, however all of the principles considered so far has been correlation ones. Here, we introduce an elementary system information content principle (ICP) whose basic ingredient is the phenomenon of Heisenberg uncertainty. The principle states that the amount of non-redundant information which may be extracted from a given single system is bounded by a perfectly decodable information content of the system. We show that this new principle is respected by classical and quantum theories and is violated by hidden variable theories as well as post-quantum ones: p-theory and polygon theories. Remarkably, ICP is sometimes more sensitive than Tsirelson's bound: it allows to rule out even some theories which do not violate Tsirelson's bound. Especially the ability of the ICP to rule out some specific hidden variable bit theories makes it to our knowledge unique among the informational principles known so far. The elementary system character of ICP suggests that it might be one of the foundational bricks of Nature.

I. INTRODUCTION

We understand mathematics of quantum mechanics quite well but we have problem with understanding quantum mechanics itself. This is notoriously manifested by the variety of interpretations of quantum mechanics. One of the reasons is the way the postulates of quantum mechanics are expressed: they refer to highly abstract mathematical terms without clear physical meaning. This drives physicists to look for an alternative way of telling quantum mechanics.

The problem was attacked on different levels. There were attempts of deriving whole quantum theory from more intuitive axioms and, more recently, focusing on informational perspective. On the other hand an effort was made to derive some principles characterizing physical theories. In recent years, different principles has been proposed, however all of the principles considered so far has been correlation ones. As a result we reveal even more dramatic gap between classical/quantum and post-quantum world. Therefore it is imperative to search for information principles characterizing physical theories. In recent years, different principles has been proposed, however all of the principles considered so far has been correlation ones. Here, we introduce an elementary system information content principle (ICP) whose basic ingredient is the phenomenon of Heisenberg uncertainty. The principle states that the amount of non-redundant information which may be extracted from a given single system is bounded by a perfectly decodable information content of the system. We show that this new principle is respected by classical and quantum theories and is violated by hidden variable theories as well as post-quantum ones: p-theory and polygon theories. Remarkably, ICP is sometimes more sensitive than Tsirelson's bound: it allows to rule out even some theories which do not violate Tsirelson's bound. Especially the ability of the ICP to rule out some specific hidden variable bit theories makes it to our knowledge unique among the informational principles known so far. The elementary system character of ICP suggests that it might be one of the foundational bricks of Nature.

Second, the existing approach refers to compound system, so that the proposed principles do not apply to an elementary system. In contrast, the principle we are going to propose concerns properties of elementary system rather than being another correlation principle. We thus want to refer to linkage between information theory and any of its physical implementations, i.e. the question: what are the informational properties of elementary physical system, if we exploit it as an information carrier or, on other words, physical implementation of an information unit? In fact, the origins of quantum information theory lied in the single physical system (called later quantum bit) which was a polarisation of photon exploited in the concept of quantum money and famous Bennett-Brassard protocol. As a single information unit, quantum bit (qbit) has unique informational properties including presence of superposition of standard bit.
states. Can those properties be physically stronger than quantum mechanics allows? If so, to what extent? Which properties should not appear in physical theories?

The third issue that we want to be present in our approach, is to include the phenomenon of Heisenberg uncertainty as a basic ingredient of the proposed principle. So far the role of the uncertainty for information principles were used to provide constraints for correlations [27, 29]. Yet, uncertainty appears already on the level of single system, and therefore it should emerge naturally in the context of elementary system.

Here we propose a principle that possesses the three desired features: excluding specific local hidden variable theories (those with EHV-bit), referring to an elementary system and involving uncertainty principle. Namely we provide a constraint that ties together (i) the amount of non-redundant information which extracted from the system by the set of complementary observables and (ii) systems’ informational content understood in terms of maximal number of bits that may be encoded in the system in perfectly decodable way. We call this constraint information content principle (ICP). It is obeyed by quantum and classical theories but is violated by post-quantum theories [17–20]. Yet, uncertainty appears already on the level of single system, and therefore it should emerge naturally in the context of elementary system.

If applied to classical theory ICP reflects the fact that there is basically one type of information and different fine grained observables available for classical discrete system disclose the same information with respect to the irrelevant outcome relabeling. Information extracted by one observable is redundant with respect to the information extracted by another one. On the contrary, in quantum mechanics, there are much different "species" of information which is reflected by the presence of incompatible observables. However only one type of information may be completely present in the system which is connected to entropic uncertainty relations. On the one hand, only one observable from the complementary set may be measured perfectly. On the other hand, two observables which are "less incompatible" than complementary, reveal information which is redundant. ICP give the trade off between how much of information may be extracted and how redundant the information is.

ICP provides attractive explanation for the minimal amount of uncertainty presents in the physical theory (particular it refers to entropic uncertainty relations [31]). We discuss this on the ground of generalized probabilistic theories (GPT) which provides common framework capable to express classical, quantum and post quantum theories [17, 20].

We show via examples from GPT that uncertainty relations less restrictive than quantum mechanical violate ICP. In this way we gave alternative answer to the question posed in [28, 29] concerning the strength on uncertainty relations in quantum theory. Since uncertainty relations limit maximum recovery probability of random access code (RAC), as a side effect we get some insight on the bounds for quantum 2 → 1 random access code. Interestingly, in some cases ICP is more sensitive than Tsirelson’s bound since it allows to discriminate theories which do not allow for correlations stronger than Tsirelson’s bound.

We discuss also the relation of ICP with Hall’s information exclusion principle [30].

II. GENERALIZED PROBABILISTIC THEORIES.

A generalized probabilistic theory consists of a convex state space \( \Omega \subseteq \mathbb{R}^n \) i.e. the set of admissible states the system may be prepared in, and the set of measurements \( \mathcal{M} \). Measurement outcome is represented by effect \( e \) which is linear map \( e : \Omega \to [0, 1] \). \( e(\omega) \) is the probability of outcome \( e \) when the measurement is performed on the system in state \( \omega \). The special effect is the unit effect \( u \) such that for every \( \omega \in \Omega \) there is \( u(\omega) = 1 \) (here we consider only normalized states). The measurement is the set of effects \( \{e_i\} \) summing up to unit effect \( u \).

The state of the system is entirely determined by the probabilities \( p(a|x) \) it assigns to the outcomes \( a \) of every measurement \( x \). However there exist subset of measurements called fiducial measurements \( \mathcal{F} \subseteq \mathcal{M} \) which is enough to describe the state.

Particular examples of systems which may be expressed in terms of GPT (see Fig. 1) are: classical bit, qbit and sbit (square-bit). The last one, sometimes called gbit for generalized bit, is the building block of PR-box [20]. If the set of measurement is not reach enough, one may obtain a classical system with hidden variables. For example, EHV-bit (for essentially hidden variable bit) consists of two classical bits. One of two observables may be measured on system giving access to different bit. After measurement, information from the complementary bit is unavoidably lost. This property reflects lack of fine grained observable in hidden variable theory. More sophisticated example of hidden variable theory may be found in [36].

For given GPT, we may ask for maximal number of states that can be perfectly distinguished in a single-shot measurement [36, 38]. We will call this value observed system dimension and denote it by \( d \). In terms of GPT, we look for the biggest set \( \{\omega_i\} \in \Omega \) such that there exist set of effects \( \{e_j\} \) which obey \( e_j(\omega_i) = \delta_{ij} \). The set of states \( \{\omega_i\} \) together with set of effects \( \{e_j\} \) may be interpreted as a maximal classical subsystem of GPT.

III. INFORMATION CONTENT PRINCIPLE

We consider single system \( \mathcal{S} \) with the set of observables \( \mathcal{M} = \{O_1, O_2, \ldots, O_n\} \) which belongs to theory \( \mathcal{T} \). State of the system \( \mathcal{S} \) and theory \( \mathcal{T} \) is understood in terms of GPT. In particular system \( \mathcal{S} \) may be a classical bit, qbit, sbit, hbit.
another words how much outcome of
information in the information extracted by observables or in
from the set
possible outcome of
where
tem
S
observable) and we call it extractable information
non-redundant information (i.e. not repeated by another
more concrete shape using entropy as a particular mea-
content of a given single system
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in perfectly decodable way
mal number of bits that may be encoded in the system
X
formation. This is reflected in perfect correlations of
states that amount of "non-redundant" information
in our situation which may be extracted from the system
A
proposed in this paper, this difference is not relevant.
FIG. 1: Elementary systems in exemplary GPT from the perspec-
tive of two distinguished dichotomic observables X and
Z. (a) bit, (b) hbit, (c) qbit and (d) sbit. Qbit differs from
the sbit and hbit by the amount of uncertainty. Classical bit
admits no uncertainty, however
and
Z
state space of sbit and hbit observed from perspective of
two observables may seem to be equivalent. What is missing
on that picture is the issue of decomposition into pure states.
Any mixed state of hbit admits a unique decomposition as a
mixture of pure states. This is not the case for sbit where
we can observe some reminiscence of quantum freedom of de-
composition. However, from the point of view of the principle
proposed in this paper, this difference is not relevant.

Now we postulate information content principle which
states that amount of "non-redundant" information
which may be extracted from the system
S
is bounded
by its information content
I_C(S).
By the information content of a given single system
S
we mean the maxi-
mal number of bits that may be encoded in the system
in perfectly decodable way
[2].

It is the function of the number
d
of different symbols that can subject to the
prefect readout (observed dimension).

We claim that this principle is valid for any physical
theory.

ICP is exemplified by the inequality in general, sym-
bolic form:

\[ \sum_{O_i \in \mathcal{M}} I_G(O_i : S) - R(\mathcal{M}, S) = I_E(\mathcal{M}, S) \leq I_C(S), \tag{1} \]

where
I_G
is information gain about the state of the system
S
obtained by the measurement of the observable
O_i
from the set
\mathcal{M},
R
measures amount of redundant information in the information extracted by observables or in
another words how much outcome of
O_i
tells us about possible outcome of
O_2.
The difference
I_G - R
quantifies non-redundant information (i.e. not repeated by another observables) and we call it extractable information
I_E.

Below, we will put general formulation of
ICP
in
more concrete shape using entropy as a particular mea-
sure of information. We will show that the principle in
that form holds for classical and quantum theory (more
generally for every theory where entropy fulfills some spe-
cific conditions but is violated by unphysical theories, cf.
[12]). Here we consider, for simplicity, only the case with
two observables
M = \{X, Z\}. The scenario is depicted
in FIG. 2. For more general settings see SEC. A.

![FIG. 1](image1)

![FIG. 2](image2)
where $I(X : A), I(Z : B), I(A : B)$ are classical mutual information ($I(A : B) = H(A) - H(A|B)$) and $d$ is an observed system dimension.

We will proceed in the spirit of [12]: we will show that [3] may be proved with some abstract entropies $\mathcal{H}$ fulfilling set of axioms. In particular Shannon and von Neuman entropies satisfy these axioms (cf. [21]). We assume that (i) mutual information $I$ is defined as $I(S : F) = \mathcal{H}(S) - \mathcal{H}(S|F)$; (ii) entropy of the system is bounded by $\mathcal{H}(S) \leq \log_2 d$; (iii) conditional entropy of any system correlated with classical one is non-negative $\mathcal{H}(S|C) \geq 0$ where $C$ is classical system; (iv) strong superadditivity $\mathcal{H}(S|A) + \mathcal{H}(S|B) \leq \mathcal{H}(S|AB) + \mathcal{H}(S)$; (v) information processing inequality for measurement $I(S : A) \geq I(X : A)$ where $X$ denotes measurement outcome.

Here we use (i), (ii) and (iii) to obtain upper bound for mutual information between system $S$ and $AB$ for a state $\rho_{SAB}$:

\[ I(S : AB) = \mathcal{H}(S) - \mathcal{H}(S|AB) \leq \mathcal{H}(S) \leq \log_2 d. \tag{4} \]

Using chain rule for mutual information (which stem from (i)), we get

\[ I(S : AB) = I(S : A) + I(S : B|A) \tag{7} \]
\[ I(S : AB) = I(S : B|A) + I(A : B). \tag{8} \]

Putting this together with strong superadditivity and information processing inequality for measurements, we obtain

\[ I(S : AB) \geq I(S : A) + I(S : B) - I(A : B) \tag{9} \]
\[ I(S : AB) \geq I(X : A) + I(Z : B) - I(A : B). \tag{10} \]

In this way we proved [3]. We will show violation of [4] in two types of theories: (i) post-quantum theories represented here by sbit; (ii) incomplete theories where as example we consider hbit. Moreover, in Sec. [14] we will show violation for two families post-quantum theories. To show violation for sbit and hbit, we evaluate (3) on the state:

\[ \omega_{SAB}^{\text{sbit}} = \sum_{i,j=0}^{1} \frac{1}{4} \omega_{i,j}^S \otimes \sigma_i^A \otimes \sigma_j^B, \tag{12} \]

where $\omega_{i,j}^S$ is the sbit or hbit in the state that the outcome $i, j$ after measuring $X, Z$ respectively is certain (i.e. $p(a = i|x = X) = 1$ and $p(a = j|x = Z) = 1$). Information encoded in the observables is completely uncorrelated i.e $I(A : B) = 0$ for $\omega_{SAB}^{\text{sbit}}$. Since there is no uncertainty in the system, information encoded in each observable might be recover completely hence $I(X : A) = I(Z : B) = 1$. Taking that together we obtain violation of [3] since $I(X : A) + I(Z : B) - I(A : B) = 2 > 1$. At this point it is worth to notice that in the case of hbit, violation comes from the fact, that observed dimension $d$ is different from what we could call "intrinsic" system dimension: the observables available in theory have two outputs while the internal state of system is described by 2 classical bits. The theory is incomplete because of lack of fine grained observable that could access full information available in the system.

At this point we back to the issue of different "species" of information present in the elementary system. As we said in the introduction, in classical theories there is only one type of information. Let's take how ICP helps us in quantitative expression of this statement for classical 1-bit system with two observables $X$ and $Z$. For this purpose we consider system $\rho_{SAB}$ which maximize $I(X : A) + I(Z : B)$ under condition that $I(X : A) = I(Z : B)$. We put these constraints to ensure that information is encoded fairly in each observable and to prevent artificial situation where $\sigma^A$ or $\sigma^B$ is uncorrelated (or weakly correlated) with $\rho^S$. What we get is $I(X : A) + I(Z : B) = 2$. However this information is redundant in big part since $I(A : B) = 1$. Extractable (non-redundant) information for that setup is only $I(X : A) + I(Z : B) - I(A : B) = 1$ what agrees with ICP. It reflects impossibility of classical RAC and is connected with the fact that the two observables above are necessarily equivalent to each other.

What happens when we put in this setup qbit and complementary observables $X$ and $Z$? If we refer to quantum RAC, we will obtain $I(X : A) + I(Z : B) \approx 0.798$ with $I(A : B) = 0$. Therefore two independent "species" of information may be present in one qubit. Because of the uncertainty relations, these two "species" of information disturb each other. It is visible even more if one "species" of information is present completely ($I(X : A) = 1$). In that case, due to ICP, there is not place for information from second species ($I(Z : B) = 0$). If we rotate observable $Z$ towards $X$, we observe that $I(X : A) + I(Z : B)$ grows up together with $I(A : B)$. The observables disclose more information, however the information is more redundant. Extractable information cannot exceed the bound given by ICP. Here we saw that quantum mechanics may fulfill ICP in different ways than classical one.

Interestingly, if we release uncertainty principle, two different "species" of information may be encoded in the system perfectly. We have seen this for sbit where $I(X : A) + I(Z : B) = 2$ and $I(A : B) = 0$.

IV. VIOLATIONS OF ICP IN GENERAL PROBABILISTIC THEORIES

In this section we analyze how ICP is broken in two families of theories that originate from GPT. We start with theories introduced in [27] (we call them p-theories). These theories violate the uncertainty relation for anti-commuting observables [28] and they were originally de-
developed to study how Tsirelson’s bound for the CHSH inequality emerges from the uncertainty relation. Here we show that violation of the uncertainty relation not only leads to violation of Tsirelson’s bound but also to violation of ICP. On the ground of these theories, we can also discuss bound on maximum recovery probability of quantum 2 → 1 RAC. In the latter part we move to polygon theories \[13\]. Interestingly, in this case ICP is more sensitive than Tsirelson’s bound since it allows to discriminate theories which do not allow for correlations stronger than Tsirelson’s bound.

Normalized states of \(p\)-theories and polygon theories are real vectors \(\omega \in \mathbb{R}^2\). Maximal number of perfectly distinguishable states satisfies \(d \leq 3\). Equality holds if and only if the states space is a simplex \[22\]. Therefore any non-classical theory has \(d \leq 2\) so that the information contents of the system used in ICP satisfies \(I_C \leq 1\).

### A. Violation uncertainty relation for anti-commuting observables

We consider the \(p\)-theory with two dichotomic observables \(X\) and \(Z\). Admissible states fulfill uncertainty relation:

\[
s^p_x + s^p_z \leq 1,
\]

where \(p \in [2, \infty)\) is parameter of the theory and \(s_x = p(a = +|x = X) - p(a = -|x = X)\) is the mean value of observable \(X\) measured on the system in state \(\psi\) (analogically for \(s_z\) and \(Z\)). It is straightforward to see that \(13\) is uncertainty relations since it bounds the probability that the state has well defined outcome of each observable. \(p\)-theory is a simplified version of model discussed in \[27\], while we only deal with the case of 2 observables available in the elementary system however our results may be easy generalized to the case of 3 observables \(X, Y, Z\).

The set of admissible states for \(p = 2\) correspond to the set of states from the great circle of the Bloch ball (in case of 3 observables, the set of admissible states becomes full Bloch ball and relation of type \[13\] define state space of single qubit). On the other hand, for \(p \to \infty\) we approach to state space of qubit. Therefore increase of \(p\) leads to relaxation of the uncertainty relation.

Now we show, that each theory with \(p > 2\) violates ICP. For that purpose, (i) we show that there exist a state \(\psi_{++}\) with entropic uncertainty small enough (i.e. \(H(X)_{\psi_{++}} + H(Z)_{\psi_{++}} < 1\)); (ii) then by symmetry of the state space we construct state \(\rho^{SAB}\) which we use to prove violation.

Let us parameterize by \(s_x\) states \(\psi\), that saturate \[13\]. For simplicity we assume that \(s_x > 0\). Due to \[13\], we have \(s_x = \sqrt{1 - s_x^p}\). For \(s_x = 1\), the outcome of the observable \(X\) is certain but we have no knowledge on the outcome of observable \(Z\). As \(s_x\) decrease, the knowledge on the outcome of \(Z\) increase by the cost of certainty of the outcome of \(X\). Rate of this exchange depends on the uncertainty relation and interestingly for \(p > 2\), some states near to \(s_x = 1\) have entropic uncertainty smaller than in the quantum case. Precisely, we show in SEC. \[B\] that there exist \(\delta_x\) that any state with \(1 - \delta_x < s_x < 1\), fulfill:

\[
H(X)_{\psi} + H(Z)_{\psi} < 1.
\]

We take any state \(\psi_{++}\) that 
\(H(X) + H(Z) = \bar{H} < 1\). The state is described by \((\bar{s}_x, \bar{s}_z)\). By the symmetry of \(13\) and \[14\], we know that states \(\psi_{++}, \psi_{+-}, \psi_{-+}\) obtained from \(\psi_{++}\) by negation of proper parameter are also admissible and have the same entropic uncertainty \(\bar{H}\). This allow us to construct the state:

\[
\rho^{SAB} = \frac{1}{4} \sum_{i,j \in \{-, +\}} \psi^S_{ij} \otimes i^A \otimes j^B.
\]

It is easy to observe that outcome of \(X\) and \(Z\) for reduced state \(\frac{1}{4} \sum_{i,j \in \{-, +\}} \psi^S_{ij}\) is completely random. Hence we may write:

\[
I(X : A) + I(Z : B) = H(X) + H(Z)
\]

\[
- \sum_{i,j \in \{-, +\}} \frac{1}{4} (H(X)_{\psi_{i,j}} + H(Z)_{\psi_{i,j}})
\]

\[
= 2 - \bar{H}
\]

\[
> 1.
\]

Since, in addition, from \[15\] we have \(I(A : B) = 0\) the above shows the expected violation and finishes the proof.

Interestingly, relaxation of uncertainty relation also lead to increase of maximum recovery probability for \(2 \rightarrow 1\) RAC and it reads \(p_{rec} = (1/2)^{1/p}\). Therefore, excluding \(p\)-theories with \(p > 2\), ICP puts bound on the \(p_{rec}\).

### B. Polygon theories

Polygon theories (parameterized by \(n\) was developed in \[18\] to study connection between the strength of non-local correlations and the structure of the state spaces of individual systems. They may be viewed as a progressive relaxation of superposition principle (c.f. relaxation of uncertainty relation in \(p\)-theories) moving from quantum case \(n \to \infty\) to sbit (\(n = 4\)) and classical trit (\(n = 3\)). Relaxation of superposition principle means that more restriction are putted on the way the states can be superposed.

The proof of violation of ICP by polygon theories is quite technical and base mostly on construction of state \(\rho^{SAB}\) with proper measurement entropies, therefore we moved it to SEC. \[C\] and here we discuss only the results. ICP violation in polygon theories connects as in previous case with uncertainty relations. It is easy to see especially for even \(n\). We notice that for \(n = 4m + 2\),
where \( m \) is integer, non-complementary observables are measured. In case of odd \( n \), role of uncertainty is less obvious because of asymmetry of the state \( \rho^{SAB} \).

Correlations obtained in models with odd \( n \) do not violate Tsirelson’s bound \(^{18}\). It means that this class of theories cannot be separated from the quantum theory due to standard argumentation \(^{12}\). Since nonlocality is tightly connected with uncertainty relations we may look for explanation of encounter limits in impossibility of steering maximally certain states \(^{29}\). Violation of ICP suggest more direct connection of ICP and uncertainty relations.

Very recent results on the classical information transmission in polygon theories \(^{28}\) provide some more insight into this issue. It turns out that polygon theories with odd \( n \) allow for communication of more than 1-bit per elementary system in Holevo sense (i.e. in asymptotic limit). It means that \(^1\) is violated even in one observable setup when non-pure measurement is performed. Therefore our result for odd \( n \) may be viewed as a simple consequence of the fact, that Holevo like capacity exceeds number of bits which may be encoded in the system in perfect decodable way. This is contrary to what we observe for classical and quantum systems. For even \( n \), Holevo like capacity of elementary system is 1-bit. It emphasizes the advantage of multiosservable approach over Holevo like in discrimination of post-quantum theories. It is interesting if for odd \( n \), information content for multiple observables may exceed Holevo limit.

V. CONCLUSIONS

We have identified elementary system information content principle, which is respected by classical and quantum theories and is violated by incomplete classical theories as well as postquantum ones: \( p \)-theory and poligon theories. The heart of the principle is uncertainty relation. Remarkably, the principle can rule out some theories that do not violate Tsirelson bound, which therefore are not detected by information causality and local orthogonality. Indeed, ICP removes theories admitting the existence of essentially hidden variable bits (EHV-bits) theory - which, as can be seen independently, must obey Tsirelson bound.

Note that the central property of the theory is that some part of the encoded classical information, despite its objective existence can not be observed experimentally. This particular feature is to some extend similar to that of the most popular variants of the de Broglie-Bohm theory where the hidden trajectories objectively do exist but can not be observed (they only manifest in the observed quantities indirectly through average quantities). The significant difference is that in case of EHV-bits some freedom is left to the observer, namely the freedom to decide what is to be operational and can be observed and what, as to say, should play the role of the - nonoperational from the point of view of the observer - Bohmian trajectory. However the possibility of our principle to rule out the existence of hidden variable that is observationally nonoperational represents, to our knowledge, its unique ability. No other principles, at least in their explicit forms, seem to share that advantage.

ICP brings to mind the Information Exclusion Principle \( (\text{IEP})^{30}\). It is important to understand the difference between these two concepts. In the dynamical scenario (see FIG. 2), ICP focus on the information which may be decoded by Bob (the receiver) from the system prepared by Alice (the sender) about her classical registers. Contrary, IEP quantifies how much information Alice has in advance on the outcome of randomly chosen measurement which will be performed on mixed state correlated with her classical register. Therefore ICP and IEP views on communication process from two opposite perspectives on communication process from two opposite perspectives. This difference situates ICP closer to Holevo information while IEP may be interpreted in terms of uncertainty relations with classical side information \(^{33}\) (cf. uncertainty with quantum side information \(^{34}\)). Interestingly, those two concepts coincide for complementary observables and lead to the same values.

It is an open issue to explore relations between our principle and information causality. Indeed, the derivations share some similarities. It seems that the principle may serve to deconstruct information causality into local part and some additional ingredient concerning correlations.

Finally the natural issue is to find the correlation variant of the present principle, which will be a subject of separate study. In any case, we believe that the present principle may be an important tool for analysis of the forthcoming theories or yet to be discovered, like expected quantum gravity or possible future dark matter/energy descriptions. In fact, it may be used to quickly identify the information features of elementary system description within any new mathematical formalism built to explain the properties of the Nature.

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\[ ^{1} \text{A. Einstein, B. Podolsky, N. Rosen, Can quantum-mechanical description of physical reality be considered complete?, Phys. Rev. 47, 777 (1935).} \]
Appendix A: Information Content Principle for multiple observables

We prove ICP for the setup with n-observables $\{X_i\}$. For the classically correlated state (cf. [3])

$$\rho^{S_{A_1},...,A_n} = \sum_{i_1,...,i_n} \rho^{S}_{i_1,...,i_n} \otimes \sigma^{A_1}_{i_1} \otimes \ldots \otimes \sigma^{A_n}_{i_n},$$

(A1)

where $\rho_S$ represents system $S$ belonging to considered theory and $\{\sigma^{A_i}_{i}\}$ denote classical registers, we show that holds

$$\sum_i I(X_i : A_i) - I(A_1 : \ldots : A_n) \leq \log_2 d.$$ 

(A2)

$d$ is observed system dimension (cf. [3]) and $I(A_1 : \ldots : A_n) = \sum_i H(A_i) - H(A_1, \ldots, H_n)$ is multivariable mutual information. Upper bound $I(S : A_1, \ldots, A_n) \leq \log_2 d$ comes in exactly the same way as in (3) hence we omit this part of proof and focus on LHS of (A2).

We start using chain rule to write:

$$I(S : A_1, \ldots, A_n) = I(S : A_1) + I(S : A_2 | A_1) + \ldots + I(S : A_n | A_1, \ldots, A_{n-1}).$$

We use chain rule once again to express express conditional mutual information in the form:

$$I(S : A_2 | A_1) = I(A_1, S : A_2) - I(A_1 : A_2)$$

$$I(S : A_3 | A_1, A_2) = I(A_1, A_2, S : A_3) - I(A_1, A_2 : A_3)$$

$$\ldots$$

$$I(S : A_n | A_1, \ldots, A_{n-1}) = I(A_1, \ldots, A_{n-1}, S : A_n) - I(A_1, \ldots, A_{n-1} : A_n).$$

Combining these together with strong superadditivity we
get:
\[ \mathcal{I}(S : A_1, \ldots, A_n) \geq \sum_i I(X_i : A_i) - I(A_1 : \ldots : A_n). \]  

That finishes the proof.

Appendix B: Existence of states with small entropic uncertainty

This section contains technical part where we prove that for \( p > 2 \) there exist state \( \psi \) in \( p \)-theory that
\[ H(X) + H(Z) < 1. \]  

For parametrization of state \( \psi \) by \( s_x \), entropies of measurements take a form \( H(X) = H(\frac{s_x}{2} + 1), H(Z) = H(\frac{s_x}{2} + 1) \) and are bounded in the following way:
\[ H(\frac{s_x}{2} + 1) \leq (\frac{1}{2})^{1+\epsilon}, \text{ for } \epsilon > 0 \text{ and } \frac{1}{2s_x} \leq \epsilon \text{ and } H(\frac{s_x}{2} + 1) \leq 1 - (\frac{s_x}{2})^2. \]  

This allows us to rewrite condition \( B1 \) as:
\[ \left(1 - \frac{s_x}{2}\right)^{1+\epsilon} \leq \frac{1}{4} (1 - s_x^2)^{2/p}. \]  

Now let us observe that:
\[ \left. \left(1 - \frac{s_x}{2}\right)^{1+\epsilon} \right|_{s_x=1} = 1/4 (1 - s_x^2)^{2/p} \right|_{s_x=1} = 0, \]  

and for \((1+\epsilon)p > 2:\)
\[ \lim_{s_x \to 1} (1 - s_x^2)^{1+\epsilon} = \infty. \]  

It means that LHS of \( B2 \) converge to 0 faster than RHS as \( s_x \to 1 \). Since both sides of \( B2 \) are positive, it implies that \( B2 \) holds for \( 1 - \delta_x < s_x < 1 \) with \( \delta_x \) small enough.

Appendix C: States violating ICP in polygon theories

We start with short description of polygon theories mainly following B3. For more details see original paper.

State space \( \Omega \) of a single system in polygon theory is a regular polygon with \( n \) vertices. For fixed \( n \), \( \Omega \) is represented as a convex hull of \( n \) pure states \( \{ \omega_i \}_{i=1}^n \):
\[ \omega_i = \left( \frac{r_n \cos \left( \frac{2\pi}{n} i \right)}{r_n \sin \left( \frac{2\pi}{n} i \right)} \right) \in \mathbb{R}^3, \]  

where \( r_n = 1/\sqrt{\cos(\pi/n)} \).

The set of effects is the convex hull of the unit effect, zero effect and the extreme effects. The unit effect has form:
\[ u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \]  

Extreme effects for even \( n \) are given by:
\[ e_i = \frac{1}{2} \left( \frac{r_n \cos \left( \frac{2\pi}{n} i \right)}{r_n \sin \left( \frac{2\pi}{n} i \right)} \right), \]  

and for odd \( n \) in slightly different form:
\[ e_i = \frac{1}{2} + \frac{r_n \cos \left( \frac{2\pi}{n} i \right)}{r_n \sin \left( \frac{2\pi}{n} i \right)}, e' = u - e_i \in \mathbb{R}^3. \]  

\( e(\omega) = e \cdot \omega \) is the Euclidean inner product of the vectors representing the effect and the state.

Now we are in position to construct states which violate Information Content principle in the polygon theories. We will consider separately the case of even and odd \( n \).

For even \( n \) we use the state
\[ \rho_{SA} = \frac{1}{4} \left( \omega_1 ^S \otimes \sigma_0 ^A \otimes \sigma_0 ^B + \omega_2 ^S \otimes \sigma_0 ^A \otimes \sigma_1 ^B + \omega_3 ^S \sigma_2 ^A \otimes \sigma_1 ^B + \omega_4 ^S \sigma_2 ^B \right) \]  

along with measurement \( X \) and \( Z \) given by effects \( \{ e_2, u-e_2 \} \) \( \{ e_{n/4} ^2, u-e_{n/4} ^2 \} \). Respectively it is easy to see that \( I(A : B) = 0 \) since each combination \( \sigma_1 ^A \otimes \sigma_2 ^B \) occurs with the same probability \( 1/4 \). To calculate \( I(X : A) \) and \( I(X : A) \) we need conditional entropy of measurement outcome which may be obtained from probability of given effects for particular state (i.e. \( e_j(\omega_i) \)). For \( I(X : A) \) the probabilities are \( e_2(\omega_1) = e_2(\omega_2) = 1 \) and \( e_2(\omega_{n/2+1}) = e_2(\omega_{n/2+2}) = 0, \) hence \( I(X : A) = 1 \). For \( I(Z : B) \), straightforward calculations lead to:
\[ p(Z = 0 | B = 0) = \frac{1}{2} \left( 1 + \sin \left( \frac{2\pi}{n} \left( \frac{1}{2} \right) \right) \tan \left( \frac{\pi}{n} \right) \right). \]
It shows that $I(Z : B) > 0$, hence the violation of the Information Content principle was proved.

For odd $n$ we use the state

$$
\rho^{SAB} = \frac{1}{4} \left( \omega_1^S \otimes \sigma_0^A \otimes \sigma_0^B \\
+ \omega_1^S \otimes \sigma_0^A \otimes \sigma_t^B \\
+ \omega_{[n/2]+1}^S \otimes \sigma_1^A \otimes \sigma_0^B \\
+ \omega_{[n/2]+2}^S \otimes \sigma_1^A \otimes \sigma_t^B \right). \tag{C6}
$$

In this case measurement $X$ and $Z$ are given by effects $\{e_1, u - e_1\}$ and $\{e_{[n/4]+1}, u - e_{[n/4]+1}\}$ respectively. Once again we have that $I(A : B) = 0$ and $I(X : A) = 1$. Formulas for $p(Z = 0|B = 0)$ and $p(Z = 1|B = 1)$ are more complicated:

$$
p(Z = 0|B = 0) = \frac{1}{4} \left( 2 \cos \left( \frac{\pi}{n} \right) + \cos \left( 2 \pi \frac{[n/4]}{n} \right) \right) + (C7) \\
\cos \left( 2 \pi \frac{[n/4] - \lfloor n/2 \rfloor}{n} \right) \sec \left( \frac{\pi}{2n} \right)^2
$$

$$
p(Z = 1|B = 1) = \frac{1}{4} \left( 2 - \cos \left( 2 \pi \frac{[n/4]}{n} \right) \right) - (C8) \\
\cos \left( 2 \pi \frac{[n/4] - \lfloor n/2 \rfloor - 1}{n} \right) \sec \left( \frac{\pi}{2n} \right)^2.
$$