PROPERTY $T$ OF REDUCED $C^*$-CROSSED PRODUCTS
BY DISCRETE GROUPS

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ABSTRACT. We generalize the main result of [6] and show that if $G$ is an amenable discrete group with an action $\alpha$ on a finite nuclear unital $C^*$-algebra $A$ such that the reduced crossed product $A\rtimes_{\alpha,r}G$ has property $T$, then $G$ is finite and $A$ is finite dimensional. As an application, an infinite discrete group $H$ is non-amenable if and only if the uniform Roe algebra $C^*_u(H)$ has property $T$.

1. Introduction

Property $T$ for unital $C^*$-algebras was introduced by Bekka in [1] and was studied by different people (see e.g. [2, 6, 8, 9]). In particular, it was shown by Kamalov in [6] that

if $G$ is a discrete amenable group acting on a commutative unital $C^*$-algebra $A$ such that the crossed product has property $T$, then $G$ is finite and $A$ is finite dimensional.

The aims of this paper is to extend this result to the case of finite nuclear unital $C^*$-algebras, and to give an application of this result. As expected, a result of Brown in [2] is one of our main tools.

2. The main results

Throughout this article, $G$ is a discrete group acting on a unital $C^*$-algebra $A$ through an action $\alpha$ (by automorphisms).

Let $T(A)$ be the set of all tracial states on $A$. For any $\tau \in T(A)$, we denote by $\pi_\tau : A \rightarrow B(\mathcal{H}_\tau)$ the GNS representation corresponding to $\tau$ and by $\xi_\tau$ a norm one cyclic vector in $\mathcal{H}_\tau$ with

$$\tau(a) = \langle \pi_\tau(a)\xi_\tau, \xi_\tau \rangle \quad (a \in A).$$

Recall that $A$ is said to be finite if $T(A)$ separates points of $A_+$ ([4, Theorem 3.4]). We also recall from [1, Remark 2] that if $T(A) = \emptyset$, then $A$ has property $T$.

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We use $T_\alpha(A)$ to denote the set of all $\alpha$-invariant tracial states on $A$, and recall that $A$ is said to be $\alpha$-finite if $T_\alpha(A)$ separates points of $A_+$ (see [3, Theorem 8.1]). We also denote by $A \rtimes_{\alpha,r} G$ the reduced crossed product of $\alpha$, and identify $A \subseteq A \rtimes_{\alpha,r} G$ as well as $G \subseteq A \rtimes_{\alpha,r} G$ through their canonical embeddings.

Let us first give the following well-known facts. Since we cannot find precise references for them, we present their simple arguments here.

**Lemma 1.** (a) $T(A \rtimes_{\alpha,r} G) \neq \emptyset$ if and only if $T_\alpha(A) \neq \emptyset$.

(b) If $A$ is $\alpha$-finite, then $A \rtimes_{\alpha,r} G$ is finite.

(c) If $G$ is amenable and $T(A) \neq \emptyset$, then $T_\alpha(A) \neq \emptyset$.

**Proof:** Let us denote $B := A \rtimes_{\alpha,r} G$, and consider $E : B \to A$ to be the canonical conditional expectation (see e.g. [3, Proposition 4.1.9]).

(a) If $\sigma \in T(B)$, then $\sigma(\alpha_t(a)) = \sigma(tat^{-1}) = \sigma(a)$ ($a \in A; t \in G$), which means that $\sigma|_A \in T_\alpha(A)$. Conversely, for any $\tau \in T_\alpha(A)$ and any $x = \sum_{s \in G} a_s s$ with $a_s = 0$ except for a finite number of $s$, one has

$$\tau(E(x^*x)) = \tau\left(\sum_{r \in G} \alpha_{r^{-1}}(a_r^*a_r)\right) = \tau\left(\sum_{r \in G} a_r a_r^*\right) = \tau(E(xx^*)).$$ 

Hence, $\tau \circ E$ belongs to $T(B)$, because it is continuous.

(b) As $E$ is faithful, we know that $B$ is a Hilbert $A$-module under the $A$-valued inner product

$$\langle x, y \rangle_A := E(x^*y) \quad (x, y \in B).$$

Moreover, for any $\tau \in T_\alpha(A)$, if $\pi^B_\tau$ is the cocvainal isotype of $B$ on the Hilbert space $B \otimes_{\pi_\tau} \mathcal{H}_\tau$ (see e.g. [7, Proposition 4.5] for its definition; note that we identify a Hilbert $C$-module with a Hilbert space by considering the conjugation of the inner product), then $(B \otimes_{\pi_\tau} \mathcal{H}_\tau, \pi^B_\tau)$ coincides with $(\mathcal{H}_\tau \otimes E, \pi_\tau \circ E)$ (observe that $1 \otimes \xi_\tau$ is a cyclic vector for $\pi^B_\tau$ with the state defined by $1 \otimes \xi_\tau$ being $\tau \circ E$).

Let $(\mathcal{H}_0, \pi_0) := \bigoplus_{\tau \in T_\alpha(A)} (\mathcal{H}_\tau, \pi_\tau)$. Since $A$ is $\alpha$-finite, one knows that $\pi_0$ is faithful. It is easy to verify that the representation $\pi^B_0$ of $B$ on $B \otimes_{\pi_0} \mathcal{H}_0$ induced by $\pi_0$ is also faithful, and that $\pi^B_0$ coincides with $\bigoplus_{\tau \in T_\alpha(A)} \pi^B_\tau$. Consequently, $\bigoplus_{\tau \in T_\alpha(A)} (\mathcal{H}_\tau \otimes E, \pi_\tau \circ E)$ is faithful, which means that $\{\tau \circ E : \tau \in T_\alpha(A)\}$ (which is a subset of $T(B)$ by the argument of part (a)) separates points of $B_+$.

(c) Note that $T(A)$ is a non-empty weak*-compact convex subset of $A^*$ and $\alpha$ induces an action of $G$ on $T(A)$ by continuous affine maps. Day’s fixed point theorem (see [3, Theorem 1]) produces a fixed point $\tau_0 \in T(A)$ for this action. Obviously, $\tau_0 \in T_\alpha(A)$. \qed

We warn the readers that part (c) of the above is not true for non-unital $C^*$-algebras.
Our main theorem concerns with the situation when $A \rtimes_{\alpha,r} G$ is nuclear and has property $T$. In this situation, [2, Theorem 5.1] tells us that $A \rtimes_{\alpha,r} G$ is a direct sum of a finite dimensional $C^*$-algebra and a nuclear $C^*$-algebra with no tracial state (note that although all $C^*$-algebras in [2] are assumed to be separable, [2, Theorem 5.1] is true in the non-separable case because one can use [3, Theorem 6.2.7] to replace [2, Theorem 4.2]). The following theorem implies that if $G$ is infinite, then we arrive at one of the extreme that the whole reduced crossed product has no tracial state. This proposition, together with its proof, is a main ingredient in the argument for our main theorem.

**Proposition 2.** Let $G$ be an infinite discrete group acting on a unital $C^*$-algebra $A$ through an action $\alpha$. If $A \rtimes_{\alpha,r} G$ is nuclear and has property $T$, then $T(A \rtimes_{\alpha,r} G) = \emptyset$.

**Proof:** Let $I_\alpha := \bigcap_{\tau \in T_\alpha(A)} \ker \pi_\tau$ and $A_\alpha := A/I_\alpha$. Suppose on contrary that $T(A \rtimes_{\alpha,r} G) \neq \emptyset$. Then $I_\alpha \neq A$ because of Lemma [1(a)].

As $\ker \pi_\tau = \{ x \in A : \tau(x^*x) = 0 \}$ ($\tau \in T(A)$), we know that $I_\alpha$ is $\alpha$-invariant, and hence $\alpha$ produces an action $\beta$ of $G$ on $A_\alpha$. Moreover, every element in $T_\alpha(A)$ induces an element in $T_\beta(A_\alpha)$, which gives the $\beta$-finiteness of $A_\alpha$.

Since $A_\alpha \rtimes_{\beta,r} G$ is a quotient $C^*$-algebra of $A \rtimes_{\alpha,r} G$, the hypothesis implies $A_\alpha \rtimes_{\beta,r} G$ to be nuclear and having property $T$. Therefore, [2, Theorem 5.1] tells us that $A_\alpha \rtimes_{\beta,r} G = C \oplus D$, where $C$ is finite dimensional and $T(D) = \emptyset$. However, the finiteness of $A_\alpha \rtimes_{\beta,r} G$ (which follows from Lemma [1(b)]) tells us that $D = (0)$. Consequently, $A_\alpha \rtimes_{\beta,r} G$ is a non-zero finite dimensional $C^*$-algebra, which contradicts the fact that $G$ is infinite. \[\square\]

The following is our main theorem which concerns with the other extreme. More precisely, what we obtained is a situation (which include the one in [6]) under which the reduced crossed product is finite dimensional.

Notice that the finiteness assumption of $A$ is indispensible. In fact, if $A$ is the direct sum of $\mathbb{C}$ with a nuclear unital $C^*$-algebra having no tracial state, then $A$ has a tracial state (but is not finite), and the reduced crossed product of the trivial action of a finite group on $A$ is nuclear and has property $T$. We will see at the end of this article that one cannot weaken the amenability assumption of $G$ neither.

**Theorem 3.** Let $G$ be an amenable discrete group and $A$ be a finite nuclear unital $C^*$-algebra. If there is an action $\alpha$ of $G$ on $A$ such that $A \rtimes_{\alpha,r} G$ has property $T$, then $G$ is finite and $A$ is finite dimensional.

**Proof:** Set $I_\alpha := \bigcap_{\tau \in T_\alpha(A)} \ker \pi_\tau$ and $A_\alpha := A/I_\alpha$. Denote $B := A \rtimes_{\alpha,r} G$. The finiteness assumption of $A$ and Lemma [1(c)] imply that $I_\alpha \neq A$ and that $T(B) \neq \emptyset$ (see also Lemma [1(a)]). Hence, $G$ is finite (by...
Proposition 2. Moreover, the argument of Proposition 2 tells us that \( I_\alpha \) is \( \alpha \)-invariant and \( B_\alpha := A_\alpha \rtimes G \) is finite dimensional. Therefore, it suffices to show that \( I_\alpha = \{0\} \).

Suppose on the contrary that \( I_\alpha \neq \{0\} \). By [2, Theorem 5.1], we know that \( B \cong B_0 \oplus B_1 \), where \( B_0 \) is finite dimensional and \( T(B_1) = \emptyset \). Thus, \( I_\alpha \rtimes r G = J_0 \oplus J_1 \), with \( J_k \) being a closed ideal of \( B_k \) for \( k \in \{0,1\} \). The short exact sequence
\[
0 \to I_\alpha \to A \to A_\alpha \to 0,
\]
induces a short exact sequence concerning their full crossed products, which coincide with the reduced crossed products because \( G \) is amenable. From this, we obtain
\[
B_\alpha = B/(I_\alpha \rtimes G) = B_0/J_0 \oplus B_1/J_1.
\]
Hence, \( B_1/J_1 \) is a quotient \( C^* \)-algebra of the finite dimensional \( C^* \)-algebra \( B_\alpha \), which implies \( J_1 = B_1 \) (otherwise, \( B_1 \) will have a tracial state). Consequently, \( B_\alpha \cong B_0/J_0 \), or equivalently, \( B_0 \cong B_\alpha \oplus J_0 \) (as \( B_0 \) is finite dimensional). This gives
\[
B \cong B_\alpha \oplus J_0 \oplus B_1 = B_\alpha \oplus (I_\alpha \rtimes r G).
\]
Thus, \( I_\alpha \rtimes r G \) is unital and so is \( I_\alpha \) (but its identity may not be the identity of \( A \)).

Now, by the finiteness assumption of \( A \), one knows that \( T(I_\alpha) \neq \emptyset \), and Lemma 1(c) produces an element \( \tau \in T_\alpha(I_\alpha) \). Let \( \Phi : A \to I_\alpha \) be the canonical \( G \)-equivariant \( * \)-epimorphism, and define
\[
\tau'(a) := \langle \pi_\tau(\Phi(a))\xi_\tau, \xi_\tau \rangle \quad (a \in A).
\]
Then \( \tau' \in T_\alpha(A) \) and \( \tau'|_{I_\alpha} = \tau \). However, the existence of \( \tau' \) contradicts the definition of \( I_\alpha \). \( \square \)

**Corollary 4.** Let \( G \) be an infinite discrete group and \( \alpha_G \) be the left translation action of \( G \) on \( \ell^\infty(G) \). The following are equivalent.

1. \( G \) is non-amenable.
2. \( \ell^\infty(G) \rtimes r G \) does not have a tracial state.
3. \( \ell^\infty(G) \rtimes r G \) has strong property \( T \) (see [5]).
4. \( \ell^\infty(G) \rtimes r G \) has property \( T \).
5. There is a finite nuclear unital \( C^* \)-algebra \( A \) and an action \( \alpha \) of \( G \) on \( A \) such that \( A \rtimes r G \) has property \( T \).

**Proof:** If \( G \) is non-amenable, then \( T_{ac}(\ell^\infty(G)) = \emptyset \) and Lemma 1(a) tells us that Statement (2) holds. On the other hand, if \( \ell^\infty(G) \rtimes r G \) does not have a tracial state, then [5] Proposition 5.2 gives Statement (3). Moreover, a strong property \( T \) \( C^* \)-algebra clearly have property \( T \). Finally, suppose that \( A \rtimes r G \) has property \( T \) but \( G \) is amenable. Then Theorem 3 produces the contradiction that \( G \) is finite. \( \square \)
The following comparison of Corollary 4 with the main result of Ozawa in [10] (see also Theorem 5.1.6 and Proposition 5.1.3 of [3]) may be worth mentioning:

a discrete $G$ is exact if and only if $\ell^\infty(G) \rtimes_{\alpha} G$ is nuclear
(or equivalently, the action $\alpha$ is amenable).

This result tells us that one cannot weaken the amenability assumption of $G$ in Theorem 3 to an amenable action $\alpha$ with $A \rtimes_{\alpha} G$ being nuclear, since if $G$ is an infinite exact non-amenable group, the action of $G$ on $\ell^\infty(G)$ is amenable, and the reduced crossed product has property $T$ and is nuclear.

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