Compactification of Drinfeld Moduli Spaces as Moduli Spaces of $A$-Reciprocal Maps and Consequences for Drinfeld Modular Forms

Richard Pink

Department of Mathematics
ETH Zürich
8092 Zürich
Switzerland
pink@math.ethz.ch

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In memory of David Goss

Abstract

We construct a compactification of the moduli space of Drinfeld modules of rank $r$ and level $N$ as a moduli space of $A$-reciprocal maps. This is closely related to the Satake compactification, but not exactly the same. The construction involves some technical assumptions on $N$ that are satisfied for a cofinal set of ideals $N$. In the special case $A = \mathbb{F}_q[t]$ and $N = (t^n)$ we obtain a presentation for the graded ideal of Drinfeld cusp forms of level $N$ and all weights and can deduce a dimension formula for the space of cusp forms of any weight. We expect the same results in general, but the proof will require more ideas.

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Introduction

Consider an admissible coefficient ring $A$ over $\mathbb{F}_q$ with field of quotients $F$ and a non-zero proper ideal $N$ of $A$. Let $M_{A,N}^r$ denote the fine moduli space of Drinfeld $A$-modules of rank $r \geq 1$ in generic characteristic with a full level $N$-structure. This is an irreducible smooth affine algebraic variety of finite type and dimension $r - 1$ over $F$. Let $\overline{M}_{A,N}^r$ be its Satake compactification according to [14].

$\mathbb{F}_q$-reciprocal maps: First consider the special case $A = \mathbb{F}_q[t]$ and $N = (t)$. Here $M_{A,N}^r$ is the base change from $\mathbb{F}_q$ of the open subscheme $\Omega_V$ of $\mathbb{P}^{\frac{r-1}{2}}_{\mathbb{F}_q}$ obtained by removing all $\mathbb{F}_q$-rational hyperplanes. Moreover $\overline{M}_{A,N}^r$ is the base change from $\mathbb{F}_q$ of a certain compactification $Q_V$ of $\Omega_V$ constructed in [15]. (For $r \geq 3$ this $Q_V$ is not isomorphic to the tautological compactification $\mathbb{P}^{\frac{r-1}{2}}_{\mathbb{F}_q}$.) The construction comprises an explicit presentation of the projective coordinate ring $R_V$ of $Q_V$ and hence of the ring of Drinfeld modular forms of level $N$ and all weights. Here $Q_V$ is obtained by giving a simple construction of $R_V$, but it is also a fine moduli scheme, as follows.

Let $V$ be an $\mathbb{F}_q$-vector space of dimension $r$ and set $\hat{V} := V \setminus \{0\}$. For any $\mathbb{F}_q$-algebra $R$ a fiberwise invertible $\mathbb{F}_q$-reciprocal map $\rho: \hat{V} \to R^\times$ is any map of the form $v \mapsto \lambda(v)^{-1}$ for an $\mathbb{F}_q$-linear map $\lambda: V \to R$ satisfying $\lambda(\hat{V}) \subset R^\times$. Any such map is characterized by the equations:

(a) $\rho(v) \cdot \rho(w) = \rho(v + w) \cdot (\rho(v) + \rho(w))$ for all $v, w \in \hat{V}$ with $v + w \in \hat{V}$, and

(b) $\alpha \rho(\alpha v) = \rho(v)$ for all $\alpha \in \mathbb{F}_q^\times$ and $v \in \hat{V}$.

A general $\mathbb{F}_q$-reciprocal map $\rho: \hat{V} \to R$ is defined simply as any map with values in $R$ which satisfies the same equations. This notion is globalized to maps $\hat{V} \to \mathcal{L}(S)$ with values in an invertible sheaf $\mathcal{L}$ on a scheme $S$ over $\mathbb{F}_q$ in Definition [14.1]. Then $Q_V$ becomes a fine moduli scheme for isomorphism classes of pairs $(\mathcal{L}, \rho)$ consisting of an invertible sheaf $\mathcal{L}$ and a fiberwise non-zero $\mathbb{F}_q$-reciprocal map $\rho$.

$A$-reciprocal maps: The goal of the article at hand is to generalize this theory to produce a compactification of $M_{A,N}^r$ for general $A$ and $N$. For this we consider the finite $A$-module $V_N^r := (N^{-1}/A)^{\oplus r}$. To any Drinfeld $A$-module $\varphi: A \to R[\tau]$ of rank $r$ and any level $N$-structure $\lambda: V_N^r \xrightarrow{\sim} \varphi[N]$ we associate the fiberwise invertible $\mathbb{F}_q$-reciprocal map $\rho: \hat{V}_N^r \to R^\times$, $v \mapsto \lambda(v)^{-1}$. Our first job was to find a useful additional identity satisfied by $\rho$ which reflects the action of $A$ on $V_N^r$. This problem had a surprisingly simple solution. Namely, consider the set $\text{Div}(N) := \{a \in A \mid N \subset (a)\}$ of divisors of $N$, and for any $a \in \text{Div}(N)$ consider the $A$-submodule $V_a^r := (Aa^{-1}/A)^{\oplus r} \subset V_N^r$. Then by Proposition [2.5.3] we have the additional identity:

(b') $a \rho(\alpha v) = \sum_{v' \in V_a^r} \rho(v - v')$ for all $a \in \text{Div}(N)$ and $v \in V_N^r \setminus V_a^r$.

We therefore define a general $A$-reciprocal map $\rho: \hat{V}_N^r \to R$ as any map with values in $R$ that satisfies the conditions (a) and (b').
Then we must show that any fiberwise invertible $A$-reciprocal map $\hat{V}_N^r \to R^s$ arises from a unique pair $(\varphi, \lambda)$ as above. We achieve this in Proposition 2.5.3 under certain technical conditions on the level $N$ that are collated in Assumption 2.2.2. The main requirement is that $\text{Div}(N)$ generates $A$ as an $\mathbb{F}_q$-algebra, while the other assumptions appear for technical reasons and can perhaps be discarded. By Proposition 2.2.5 the assumptions are satisfied for a cofinal set of non-zero ideals $N$. For the following we assume that they hold for $N$.

Next the notion of $A$-reciprocal maps is globalized to maps $\hat{V}_N^r \to \mathcal{L}(S)$ with values in an invertible sheaf $\mathcal{L}$ on a scheme $S$ over $\mathbb{F}_q$ in Definition 2.6.1. By standard arguments there is a fine moduli scheme $Q_{A,V_N}$ of isomorphism classes of pairs $(\mathcal{L}, \rho)$ consisting of an invertible sheaf $\mathcal{L}$ and a fiberwise non-zero $A$-reciprocal map $\rho$, and $Q_{A,V_N}$ is projective over $F$. It also contains an open subscheme $\Omega_{A,V_N}$ which is a fine moduli scheme of isomorphism classes of fiberwise invertible $A$-reciprocal maps and therefore naturally isomorphic to $M_{A,N}^r$. Thus $Q_{A,V_N}$ constitutes a natural compactification of $M_{A,N}^r$ as a fine moduli scheme.

**Relation with the Satake compactification:** We show that $Q_{A,V_N}$ shares many properties with the Satake compactification $\overline{M}_{A,N}^r$. For instance, in Theorem 2.7.6 we prove that the open subscheme $M_{A,N}^r \cong \Omega_{A,V_N}$ is dense in $Q_{A,V_N}$. In Theorem 2.7.7 we show that $Q_{A,V_N}$ is stratified by finitely many locally closed subschemes $\Omega_W$ which are indexed by all non-zero free $A/N$-submodules $W \subset V_N^r$ and are isomorphic to $M_{A,N}^r$ for $1 \leq s \leq r$. In Theorem 2.9.7 we prove that the Satake compactification $\overline{M}_{A,N}^r$ is the normalization of $Q_{A,V_N}$ in the function field of $M_{A,N}^r$, and in Proposition 2.9.6 we show that the natural morphism $\pi: \overline{M}_{A,N}^r \to Q_{A,V_N}$ is finite and surjective.

However, a computation of Härerli [12, Prop. 7.13, Cor. 7.28] implies that in general distinct points from the Satake compactification are identified with each other in $Q_{A,V_N}$. Nevertheless, I expect that this is the only difference. More precisely, following Härerli it will be possible to say precisely which points are identified in $Q_{A,V_N}$. My fond hope is then that $Q_{A,V_N}$ is simply the quotient of $\overline{M}_{A,N}^r$ by the resulting equivalence relation on the underlying topological space.

**The projective coordinate ring:** Most of our constructions are done in the projective coordinate ring $R_{A,V_N}$ underlying $Q_{A,V_N}$. This ring is given by an explicit presentation in Construction 2.3.3. The open subscheme $M_{A,N}^r \cong \Omega_{A,V_N}$ corresponds to a certain localization $R_{A,V_N}^s$ of $R_{A,V_N}$ which is a regular graded integral domain. We expect that $R_{A,V_N}$ is itself an integral domain and that the natural homomorphism $R_{A,V_N} \to R_{A,V_N}^s$ is injective, but are not yet able to prove this in general.

But let $R_{A,V_N}^{\text{norm}}$ denote the integral closure of $R_{A,V_N}$ in $R_{A,V_N}^s$. By the above-mentioned result on the Satake compactification this is the projective coordinate ring of $\overline{M}_{A,N}^r$. Also, let $I_{A,V_N}^{\text{norm}} \subset R_{A,V_N}^{\text{norm}}$ denote the graded ideal of the reduced boundary $(\overline{M}_{A,N}^r \setminus M_{A,N}^r)_{\text{red}}$. Then in Theorem 2.11.3 we deduce that $R_{A,V_N}^{\text{norm}}$ is the ring of Drinfeld modular forms of level $N$ and all weights and that $I_{A,V_N}^{\text{norm}}$ is the ideal of all cusp forms therein.

Moreover, in (2.10.1) we construct a certain reduced ideal $I_{A,V_N} \subset R_{A,V_N}$ such that $I_{A,V_N}^{\text{norm}}$ is the radical of the associated ideal $I_{A,V_N} \cdot R_{A,V_N}^{\text{norm}}$. The fond hope expressed above corresponds to the expectation that the natural map $I_{A,V_N} \to I_{A,V_N}^{\text{norm}}$ is an isomorphism. Given the explicit presentation of the ring $I_{A,V_N}$ this would provide an explicit presentation
of the ideal of all cusp forms. As a consequence this might lead to a dimension formula for spaces of cusp forms.

A special case: I would not bother writing all this up without more positive results in some new cases. Assume that $A = \mathbb{F}_q[t]$ and $N = (t^n)$ for some $n \geq 1$. In Theorem 3.3.3 we then prove that $R_{A,V,N}$ is an integral domain and injects into $RS_{A,V,N}$. In Theorem 3.3.3 we show that $R_{A,V,N}$ and hence $Q_{A,V,N}$ is Cohen-Macaulay. It is therefore reasonable to expect that $P_{A,V,N}^\text{form}$ and $\mathcal{M}_{A,N}$ are Cohen-Macaulay as well and that the same holds for general $A$ and $N$. In Theorem 3.4.2 we prove that the natural map $I_{A,V,N} \rightarrow I_{A,V,N}^\text{norm}$ is an isomorphism in this special case. In Subsection 3.5 we deduce a simple dimension formula for the space of Drinfeld cusp forms associated to any arithmetic subgroup $\Gamma < \text{SL}_r(\mathbb{F}_q[t])$ satisfying $\Gamma(t) < \Gamma < \Gamma_1(t)$.

The methods to attain these results are partly a refinement of methods from [15]. There we had already considered a maximal unipotent subgroup $U < \text{Aut}_{\mathbb{F}_q}(V)$ and shown that the ring of $U$-invariants $R_V^U$ is isomorphic to a polynomial ring in dimensions $(V)$ variables over $\mathbb{F}_q$ and that $R_V$ is a free $R_V^U$-module with an explicit basis. In our special case we again consider a maximal subgroup $U < \text{GL}_r(\mathbb{F}_q[t]/(t^n))$ of $q$-power order, prove that the ring of invariants $R_{A,V,N}^U$ is isomorphic to a polynomial ring in $r$ variables over $F$, and show that $R_{A,V,N}$ is a free module over $R_{A,V,N}^U$ with an explicit basis. The method works, because $U$ is a group of $q$-power order acting on an $\mathbb{F}_q$-vector space, because we can compute its invariants in $RS_{A,V,N}$, and because the respective module that we wish to describe happens to be a free module over the group ring $F[U]$. Unfortunately this method only succeeds in the case $N = (t^n)$, and proving similar results in the general case will require additional ideas.

Outlook: In addition to the expectations mentioned above one can ask which form a dimension formula for Drinfeld cusp forms might take in general. Based on the results from Theorem 3.5.6 in our special case, we can surmise that for any fine congruence subgroup $\Gamma < \text{SL}_r(A)$, the space $S_d(\Gamma)$ of cusp forms of weight $d$ and level $\Gamma$ has dimension

$$c(A,r) \cdot [\text{SL}_r(A) : \Gamma] \cdot \binom{d-1}{r-1},$$

where the constant $c(A,r)$ depends only on $A$ and $r$. This constant might involve the class number of $A$ and/or be related to a version of the Tamagawa number of $\text{SL}_r(A)$ or $\text{GL}_r(A)$, as Gekeler suggests. Moreover, for any two fine congruence subgroups $\Gamma \triangleleft \Gamma' < \text{SL}_r(A)$, the space $S_d(\Gamma)$ should be a free module over the group ring $\mathbb{F}_q[\Gamma'/\Gamma]$.

In another direction one may ask for a dimension formula for modular forms instead of cusp forms. In [15] Thm. 4.1 and [14] Thm. 8.4 we already gave such a formula in the case $A = \mathbb{F}_q[t]$ for any subgroup $\Gamma$ satisfying $\Gamma(t) < \Gamma < \Gamma_1(t)$. An explicit presentation of the ring $P_{A,V,N}^\text{form}$ would probably yield a dimension formula in general. Note that for rank $r = 2$ and sufficiently large weight a dimension formula for modular forms was already given by Gekeler [7] §6, and a formula for cusp forms can be obtained in the same way.

Reflection on the notion of $A$-reciprocal maps: The ad hoc definition of $A$-reciprocal maps and their study in H¨ aberli’s thesis [12] §8.2 was an important encouragement for me. But I consider it as provisional and the new definition proposed in this
paper as more useful. One can check that our conditions imply his by Proposition 1.3.4 (b) and Proposition 2.4.4 and that both definitions lead to moduli schemes with the same underlying reduced subscheme. But the conditions from [12, Def. 8.14] are homogeneous of high degree and will therefore introduce an excess of nilpotent elements in the local rings at the boundary. By contrast, the new relations (b') above are homogeneous of degree 1 and cannot be outdone in regard to their degree or their elegance.

Nevertheless, I am not yet sure that the new definition is quite final. It is still open which assumptions on the level \( N \) are really necessary, and perhaps the definition should be augmented in order to reduce them.

One should also ask whether a variant of the definition of \( A \)-reciprocal maps might yield a fine moduli scheme that is isomorphic to the Satake compactification \( \overline{M}_{A,N} \). With the results on the ideal of the boundary \( I_{A,V,N} \) in our special case we seem to be almost there, and we would need just a little more data to distinguish different points at the boundary. Can one discover another property of reciprocal maps that helps to achieve this?

**Relation with other work:** In a recent manuscript Gekeler [9] pursues similar goals with a different approach. For simplicity he restricts himself to the case \( A = \mathbb{F}_q[t] \). Let \( \mathbb{C}_\infty \) denote the completion of the algebraic closure of the field \( \mathbb{F}_q((t^{-1})) \), let \( \Omega^r \) be the Drinfeld period domain of rank \( r \) over \( \mathbb{C}_\infty \), and let \( \Gamma(N) < \text{SL}_r(\mathbb{F}_q[t]) \) be the principal congruence subgroup of level \( N \). Then \( \Gamma(N)\backslash \Omega^r \) is one of finitely many irreducible components of \( M'_{A,N}(\mathbb{C}_\infty) \). Let \( \text{Mod}(N) \) be the ring of analytic modular forms of level \( N \) and all weights, so that Proj(\( \text{Mod}(N) \)) is the Satake compactification of \( \Gamma(N)\backslash \Omega^r \), that is, the corresponding irreducible component of \( \overline{M}_{A,N}(\mathbb{C}_\infty) \).

Gekeler proposes to consider the \( \mathbb{C}_\infty \)-subalgebra \( \text{Eis}(N) \) of \( \text{Mod}(N) \) that is generated by all Eisenstein series of weight 1 and to view Proj(\( \text{Eis}(N) \)) as a natural compactification of \( \Gamma(N)\backslash \Omega^r \). In [9, Cor. 7.6] he proves that the natural morphism \( \pi: \text{Proj}(\text{Mod}(N)) \to \text{Proj}(\text{Eis}(N)) \) is bijective. He hopes that \( \text{Mod}(N) \) is equal to or at least very close to \( \text{Eis}(N) \), but unfortunately has no methods to decide that for rank \( r > 2 \).

To see the relation with our approach recall that \( R_{A,V,N}^{\text{norm}} \) is the ring of algebraic Drinfeld modular forms of level \( N \) and all weights; hence \( R_{A,V,N}^{\text{norm}} \otimes_F \mathbb{C}_\infty \) is isomorphic to a finite direct sum of copies of \( \text{Mod}(N) \) for all irreducible components of \( M'_{A,N}(\mathbb{C}_\infty) \). Also observe that the generators \( \frac{1}{y} \otimes 1 \) of our ring \( R_{A,V_N} \) from Construction 2.3.3 represent the reciprocals of all non-zero \( \tilde{N} \)-torsion points of the universal Drinfeld module over \( M'_{A,N} \); hence they correspond to all Eisenstein series of weight 1 for the group \( \Gamma(N) \), for instance by [3 (15.4)]. Thus the image of the homomorphism \( R_{A,V,N} \otimes_F \mathbb{C}_\infty \to R_{A,V,N}^{\text{norm}} \otimes_F \mathbb{C}_\infty \) followed by the projection to any one factor \( \text{Mod}(N) \) is precisely the subring \( \text{Eis}(N) \). In the special case \( N = (t^n) \) one can hope that by combining our results on the ideal of the boundary \( I_{A,V,N} \) with the bijectivity of the morphism \( \text{Proj}(\text{Mod}(N)) \to \text{Proj}(\text{Eis}(N)) \) one can deduce that \( \text{Mod}(N) = \text{Eis}(N) \) in this case.

**Structure of the paper:** The article is composed of three major sections. Section I is devoted to \( \mathbb{F}_q \)-reciprocal maps and can be viewed as a continuation of the article [15]. For clarity I now call \( \mathbb{F}_q \)-reciprocal maps what we simply called reciprocal maps in [15]. We cover some additional topics with applications to \( A \)-reciprocal maps. In Subsection 1.2 we
discuss the functoriality of $\mathbb{F}_q$-reciprocal maps under homomorphisms of finite dimensional $\mathbb{F}_q$-vector spaces. In Subsection 1.3 we deduce some nice formulas for $\mathbb{F}_q$-reciprocal maps, which motivated the above condition (b') for $A$-reciprocal maps, and which are crucial for everything that follows. In Subsections 1.5 and 1.6 we collect some technical results for later use, and in Subsection 1.7 we give an explicit description of the ideal of the boundary. The arguments in the last two sections are simpler versions of central arguments from Section 3; it should be helpful for the reader to study them here first.

Section 2 contains all general definitions and results concerning $A$-reciprocal maps and their moduli schemes, and Section 3 contains our results in the special case $A = \mathbb{F}_q[t]$ and $N = (t^n)$. The most notable content of these sections was already summarized above.

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1 $\mathbb{F}_q$-reciprocal maps

1.1 Basic $\mathbb{F}_q$-reciprocal maps

We begin by reviewing some basic constructions from [15, §1 and §7]. Throughout this article we fix a finite field $\mathbb{F}_q$ of order $q$. For any $\mathbb{F}_q$-vector space $V$ we abbreviate $\hat{V} := V \setminus \{0\}$. Consider a finite dimensional $\mathbb{F}_q$-vector space $V$ and a commutative $\mathbb{F}_q$-algebra $R$.

Definition 1.1.1 A map $\rho: \hat{V} \to R$ is called $\mathbb{F}_q$-reciprocal if

(a) $\rho(v) \cdot \rho(w) = \rho(v + w) \cdot (\rho(v) + \rho(w))$ for all $v, w \in \hat{V}$ with $v + w \in \hat{V}$, and

(b) $\alpha \rho(\alpha v) = \rho(v)$ for all $\alpha \in \mathbb{F}_q^\times$ and $v \in \hat{V}$.

Definition 1.1.2 An $\mathbb{F}_q$-reciprocal map $\rho: \hat{V} \to R$ is called

(a) fiberwise non-zero if for every $p \in \text{Spec}(R)$ there exists $v \in \hat{V}$ with $\rho(v) \notin p$.

(b) fiberwise invertible if for every $p \in \text{Spec}(R)$ and every $v \in \hat{V}$ we have $\rho(v) \notin p$. Equivalently: If for every $v \in \hat{V}$ we have $\rho(v) \in R^\times$.

Proposition 1.1.3 For any $\mathbb{F}_q$-linear map $\lambda: V \to R$ satisfying $\lambda(\hat{V}) \subset R^\times$, the map $\rho: \hat{V} \to R, v \mapsto \lambda(v)^{-1}$, is a fiberwise invertible $\mathbb{F}_q$-reciprocal map, and any fiberwise invertible $\mathbb{F}_q$-reciprocal map $\hat{V} \to R$ arises in this way.

Construction 1.1.4 We set

$S_V :=$ the symmetric algebra of $V$ over $\mathbb{F}_q$,
$K_V :=$ the field of quotients of $S_V$,
$R_V :=$ the $\mathbb{F}_q$-subalgebra of $K_V$ generated by the elements $\frac{1}{v}$ for all $v \in \hat{V}$,
$RS_V :=$ the $\mathbb{F}_q$-subalgebra of $K_V$ generated by $R_V$ and $S_V$.

Note that these are all integral domains, and $RS_V$ is the localization of $R_V$ obtained by inverting the elements $\frac{1}{v}$ for all $v \in \hat{V}$. For any basis $X_1, \ldots, X_r$ of $V$, the field $K_V$ becomes the field of rational functions $\mathbb{F}_q(X_1, \ldots, X_r)$ and $R_V$ becomes the $\mathbb{F}_q$-subalgebra generated by the elements $(\alpha_1 X_1 + \ldots + \alpha_r X_r)^{-1}$ for all $(\alpha_1, \ldots, \alpha_r) \in \mathbb{F}_q^r \setminus \{(0, \ldots, 0)\}$.

By [15, §1 and Thm. 7.12] we have:

Theorem 1.1.5 (a) The map

$$\rho_{\text{univ}}: \hat{V} \to R_V, \ v \mapsto \frac{1}{v}$$

is $\mathbb{F}_q$-reciprocal.

(b) For any $\mathbb{F}_q$-algebra $R$ and any $\mathbb{F}_q$-reciprocal map $\rho: \hat{V} \to R$ there exists a unique $\mathbb{F}_q$-algebra homomorphism $f: R_V \to R$ such that $\rho = f \circ \rho_{\text{univ}}$.

(c) This $f$ extends to a ring homomorphism $RS_V \to R$ if and only if $\rho$ is fiberwise invertible.
1.2 Functionality

Consider a short exact sequence of finite dimensional \( \mathbb{F}_q \)-vector spaces

\[
0 \rightarrow V' \xrightarrow{i} V \xrightarrow{p} V'' \rightarrow 0.
\]

**Proposition-Definition 1.2.2** For any \( \mathbb{F}_q \)-reciprocal map \( \rho: \breve{V} \rightarrow \mathbb{R} \) the map

\[
i^* \rho := \rho \circ i: \breve{V}' \rightarrow \mathbb{R}, \ v' \mapsto \rho(i(v'))
\]

is \( \mathbb{F}_q \)-reciprocal. We call it the pullback of \( \rho \) under \( i \).

**Proof.** Clear from Definition 1.1.1, the injectivity ensuring that \( i \) induces a map \( \breve{V}' \rightarrow \breve{V} \).

**Proposition-Definition 1.2.3** For any \( \mathbb{F}_q \)-reciprocal map \( \rho: \breve{V}' \rightarrow \mathbb{R} \) the map

\[
i_* \rho: \breve{V} \rightarrow \mathbb{R}, \ v \mapsto \begin{cases} 
\rho(v) & \text{if } v = i(v') \text{ for } v' \in \breve{V}', \\
0 & \text{if } v \notin i(\breve{V}'),
\end{cases}
\]

is \( \mathbb{F}_q \)-reciprocal. We call it the extension by zero or the pushforward of \( \rho \) under \( i \).

**Proof.** Clearly \( i_* \rho \) satisfies the condition 1.1.1 (b). It also satisfies 1.1.1 (a) whenever \( v, w, v + w \) lie in \( i(\breve{V}') \). In all other cases at least two of \( v, w, v + w \) lie in \( V \setminus i(\breve{V}') \); hence at least two of the values \( (i_* \rho)(v), (i_* \rho)(w), (i_* \rho)(v + w) \) are zero, and the equality in 1.1.1 (a) for \( i_* \rho \) holds trivially. Thus the extension by zero is \( \mathbb{F}_q \)-reciprocal.

**Proposition 1.2.4** (a) The functor \( i^* \) is represented by an injective \( \mathbb{F}_q \)-algebra homomorphism \( \varepsilon_i: R_{V'} \rightarrow R_V \) that sends \( \frac{1}{v'} \) to \( \frac{1}{i(v')} \) for all \( v' \in \breve{V}' \).

(b) The functor \( i_* \) is represented by a surjective \( \mathbb{F}_q \)-algebra homomorphism \( \pi_i: R_V \twoheadrightarrow R_{V'} \) that sends \( \frac{1}{i(v')} \) to \( \frac{1}{v} \) for all \( v' \in \breve{V}' \) and \( \frac{1}{v} \) to 0 for all \( v \in V \setminus i(\breve{V}') \).

(c) The kernel of \( \pi_i \) is generated by the elements \( \frac{1}{v} \) for all \( v \in V \setminus i(\breve{V}') \).

**Proof.** Let \( \rho_{V'}^{\text{univ}}: \breve{V} \rightarrow R_V \) and \( \rho_{V'}^{\text{univ}}: \breve{V}' \rightarrow R_{V'} \) denote the respective universal \( \mathbb{F}_q \)-reciprocal maps. Then \( \varepsilon_i \) and \( \pi_i \) are obtained from the universal property of \( (R_{V'}, \rho_{V'}^{\text{univ}}) \) and \( (R_V, \rho_V^{\text{univ}}) \) as the unique \( \mathbb{F}_q \)-algebra homomorphisms making the following diagrams commute:

\[
\begin{array}{ccc}
\breve{V}' & \xrightarrow{i^* \rho_{V'}^{\text{univ}}} & R_V \\
\varepsilon_i & \downarrow & \\
R_{V'} & \xrightarrow{\rho_{V'}^{\text{univ}}} & R_V,
\end{array}
\quad
\begin{array}{ccc}
\breve{V} & \xrightarrow{i_* \rho_{V'}^{\text{univ}}} & R_{V'} \\
\pi_i & \downarrow & \\
R_V & \xrightarrow{\rho_V^{\text{univ}}} & R_{V'}.
\end{array}
\]
By construction they represent the functors \( i^* \) and \( i_* \) and are given on the generators as stated. Since \( \varphi^* i_* \rho_{V''}^{\text{univ}} = \rho_{V'}^{\text{univ}} \), the universal property of \((R_{V''}, \rho_{V''}^{\text{univ}})\) implies that \( \pi_i \circ \varepsilon_i = \text{id}_{R_{V'}^*} \); hence \( \varepsilon_i \) is injective and \( \pi_i \) is surjective. This proves (a) and (b).

For (c) let \( J \subset R_V \) denote the ideal generated by the elements \( \frac{1}{v} \) for all \( v \in V \setminus i(V') \). Then the factor ring \( R_V / J \) represents the functor of all \( \mathbb{F}_q \)-reciprocal maps on \( \hat{V} \) which are identically zero on \( V \setminus i(V') \). But these are precisely the extensions by zero of \( \mathbb{F}_q \)-reciprocal maps on \( \hat{V}' \); hence this functor is already represented by \( R_{V'} \). It follows that \( \pi_i \) induces an isomorphism \( R_{V}/J \to R_{V'} \); proving (c).

**Proposition-Definition 1.2.5** For any \( \mathbb{F}_q \)-reciprocal map \( \rho: \hat{V} \to R \) the map
\[
p_* \rho: \hat{V}'' \to R, \quad v'' \mapsto \sum_{v \in p^*(v'')} \rho(v)
\]
is \( \mathbb{F}_q \)-reciprocal. We call it the pushforward of \( \rho \) under \( p \).

**Proof.** Setting \( U := \text{Ker}(p) \), the condition 1.1.1 (a) for \( p_* \rho \) is equivalent to the formula
\[
\sum_{u \in U} \rho(v + u) \cdot \sum_{u' \in U} \rho(w + u') = \sum_{u \in U} \rho(v + w + u) \cdot \sum_{u' \in U} (\rho(v + u') + \rho(w + u'))
\]
for all \( v, w \in V \setminus U \) with \( v + w \in V \setminus U \). This equation follows from the condition 1.1.1 (a) for \( \rho \) by rearranging and reindexing the sums. Also, condition 1.1.1 (b) for \( p_* \rho \) immediately follows from that for \( \rho \). Thus \( p_* \rho \) is \( \mathbb{F}_q \)-reciprocal.

**Proposition 1.2.6** The functor \( p_* \) is represented by an injective \( \mathbb{F}_q \)-algebra homomorphism \( \varepsilon_p: R_{V''} \to R_V \) that sends \( \frac{1}{v''} \) to \( \frac{1}{v''} \sum_{v \in p^{-1}(v'')} \frac{1}{v} \) for all \( v'' \in V'' \).

**Proof.** Let \( \rho_{V''}^{\text{univ}}: \hat{V}'' \to R_{V''} \) denote the universal \( \mathbb{F}_q \)-reciprocal map. Then \( \varepsilon_p \) is obtained from the universal property of \((R_{V''}, \rho_{V''}^{\text{univ}})\) as the unique \( \mathbb{F}_q \)-algebra homomorphisms making the following diagram commute:

\[
\begin{array}{ccc}
\hat{V}'' & \xrightarrow{\rho_{V''}^{\text{univ}}} & R_{V''} \\
\downarrow{p_* \rho_{V''}^{\text{univ}}} & & \downarrow{\varepsilon_p} \\
R_V & &
\end{array}
\]

By construction this represents the functor \( p_* \) and is given on the generators as stated. To finish choose a homomorphism \( j: V'' \to V \) such that \( p \circ j = \text{id}_{V''} \). Then the defining formulas in Propositions 1.2.3 and 1.2.5 show that \( p_* j_* \rho_{V''}^{\text{univ}} = \rho_{V''}^{\text{univ}} \). Thus the universal property of \((R_{V''}, \rho_{V''}^{\text{univ}})\) implies that \( \pi_j \circ \varepsilon_p = \text{id}_{R_{V''}} \); hence \( \varepsilon_p \) is injective, and we are done.

**Remark 1.2.7** Factoring an arbitrary homomorphism of finite dimensional \( \mathbb{F}_q \)-vector spaces as \( f = i \circ p \) for a surjection \( p \) and an injection \( i \), one can define the pushforward under \( f \) by \( f_* := i_* \circ p_* \). This is actually given by the same formula as in Proposition 1.2.5 for \( f \) in place of \( p \).
1.3 Some nice formulas

As before let \( R \) be a commutative \( \mathbb{F}_q \)-algebra. Let \( R[\tau] \) denote the ring of \( \mathbb{F}_q \)-linear polynomials over \( R \), that is, of polynomials of the form \( f(X) = \sum_{i \geq 0} u_i X^q^i \) with all \( u_i \in R \). Setting \( \tau(X) := X^q \), we write such a polynomial in the shorter form \( f = \sum_{i \geq 0} u_i \tau^i \). The multiplication in \( R[\tau] \) is defined as composition \( f \circ g \), and the identity element 1 of \( R[\tau] \) is the polynomial \( \tau^0 = X \). For any \( u \in R \) we have \( \tau \circ u = u^q \circ \tau \); so in general this ring is non-commutative. For any \( f = \sum_{i \geq 0} u_i \tau^i \in R[\tau] \) we have \( df := \frac{d}{dX} f(X) = u_0 \), and the map \( d: R[\tau] \to R \) is an \( \mathbb{F}_q \)-algebra homomorphism.

For the rest of this subsection we fix an \( \mathbb{F}_q \)-reciprocal map \( \rho: \check{\check{V}} \to R \). To \( \rho \) we associate the polynomial

\[
e_{\rho}(X) := X \cdot \prod_{v \in \check{\check{V}}} (1 - \rho(v)X) \in R[X].\tag{1.3.1}
\]

Proposition 1.3.2 We have \( e_{\rho} \in R[\tau] \).

Proof. By Theorem 1.1.5 it suffices to prove this for the universal \( \mathbb{F}_q \)-reciprocal map \( \rho^{\text{univ}}: \check{\check{V}} \to R_V \). Since \( R_V \) is an integral domain, it then suffices to prove the statement over the quotient field of \( R_V \), where \( \rho^{\text{univ}} \) becomes fiberwise invertible. In view of Proposition 1.1.3 it thus suffices to prove the statement for the map \( \rho = (\lambda | \check{\check{V}})^{-1} \) associated to any injective \( \mathbb{F}_q \)-linear map \( \lambda: V \hookrightarrow k \) for any field \( k \) over \( \mathbb{F}_q \). In that case

\[
e_{\rho}(X) = X \cdot \prod_{v \in \check{\check{V}}} \left( 1 - \frac{X}{\lambda(v)} \right),
\]

is the exponential function associated to the subgroup \( \lambda(V) \subset k \), and the statement follows from [11, Cor. 1.2.2]. \( \square \)

Proposition 1.3.3 We have the following identities in the ring \( R[X, e_{\rho}(X)^{-1}] \):

\[
(a) \quad \frac{1}{e_{\rho}(X)} = \frac{1}{X} + \sum_{v \in \check{\check{V}}} \frac{-\rho(v)}{1 - \rho(v)X}.
\]

\[
(b) \quad \prod_{v \in \check{\check{V}}} \frac{1}{1 - \rho(v)X} = \frac{X}{e_{\rho}(X)} = -\sum_{v \in \check{\check{V}}} \frac{1}{1 - \rho(v)X} \quad \text{if } V \neq 0.
\]

Proof. (Compare Goss [11, §6].) Since \( e_{\rho}(X) \) is an \( \mathbb{F}_q \)-linear polynomial with linear term \( X \), we have \( \frac{d}{dX} e_{\rho}(X) = 1 \). Applying the logarithmic derivative to the formula (1.3.1) thus yields the equation (a). Multiplying it by \( X \) implies that

\[
\frac{X}{e_{\rho}(X)} = 1 + \sum_{v \in \check{\check{V}}} \frac{1 - \rho(v)X - 1}{1 - \rho(v)X} = |V| + \sum_{v \in \check{\check{V}}} \frac{-1}{1 - \rho(v)X}.
\]
Since \( q \) divides \(|V|\) if \( V \neq 0 \), this implies (b). \(\square\)

Now we return to the short exact sequence \([1.2.4]\). For simplicity we assume that \( i \) is the inclusion of an \( \mathbb{F}_q \)-subspace \( V' \hookrightarrow V \).

**Proposition 1.3.4** For any \( v \in V \setminus V' \) we have

\[(a) \quad \left( \sum_{v' \in V'} \rho(v - v') \right) \cdot \left( \prod_{v' \in V'} \rho(v') \right) = \prod_{v' \in V'} \rho(v - v'). \]

\[(b) \quad \left( \sum_{v' \in V'} \rho(v - v') \right) \cdot \left( \prod_{v' \in V'} (\rho(v) - \rho(v')) \right) = \rho(v)^{|V'|}. \]

\[(c) \quad \left( \sum_{v' \in V'} \rho(v - v') \right) \cdot e_{i^*\rho}\left( \frac{1}{\rho(v)} \right) = 1 \text{ if } \rho(v) \in R^\times. \]

**Proof.** As in the proof of Proposition \([1.3.2]\) showing these equations reduces to the case that \( \rho = (\lambda|V')^{-1} \) for an injective \( \mathbb{F}_q \)-linear map \( \lambda: V \hookrightarrow k \) to a field \( k \). Applying Proposition \([1.3.3]\) (a) to \( i^*\rho \) in place of \( \rho \) then shows that

\[(1.3.5) \quad \frac{1}{e_{i^*\rho}(X)} = \frac{1}{X} + \sum_{v' \in V'} \frac{-\rho(v')}{1 - \rho(v)X} = \frac{1}{X} + \sum_{v' \in V'} \frac{1}{\lambda(v') - X} = \sum_{v' \in V'} \frac{1}{X - \lambda(v')}.

For any \( v \in V \setminus V' \) we have

\[(1.3.6) \quad e_{i^*\rho}(\lambda(v)) = \lambda(v) \cdot \prod_{v' \in V'} \left(1 - \frac{\lambda(v)}{\lambda(v')}\right) = \lambda(v) \cdot \prod_{v' \in V'} \frac{\lambda(v') - \lambda(v)}{\lambda(v')},

where all factors are non-zero by the injectivity of \( \lambda \). Thus by multiplying the formula \([1.3.5]\) by \( e_{i^*\rho}(X) \), substituting \( X = \lambda(v) \), and using the additivity of \( \lambda \) we deduce that

\[(1.3.7) \quad 1 = \left( \sum_{v' \in V'} \frac{1}{\lambda(v) - \lambda(v')} \right) \cdot e_{i^*\rho}(\lambda(v)) = \left( \sum_{v' \in V'} \rho(v - v') \right) \cdot e_{i^*\rho}\left( \frac{1}{\rho(v)} \right),

proving (c). Also, by \([1.3.6]\), the additivity of \( \lambda \), and the fact that \((-1)^{q-1} = 1 \) in \( \mathbb{F}_q \) we have

\[e_{i^*\rho}\left( \frac{1}{\rho(v)} \right) = e_{i^*\rho}(\lambda(v)) = (-1)^{|V'|} \cdot \prod_{v' \in V'} \frac{\lambda(v - v')}{\lambda(v')} = \prod_{v' \in V'} \rho(v - v'),

which together with \([1.3.7]\) proves (a). Finally, we can rewrite \([1.3.6]\) also in the form

\[e_{i^*\rho}\left( \frac{1}{\rho(v)} \right) = \frac{1}{\rho(v)} \cdot \prod_{v' \in V'} \left(1 - \frac{\rho(v')}{\rho(v)}\right) = \frac{1}{\rho(v)^{|V'|}} \cdot \prod_{v' \in V'} (\rho(v) - \rho(v')).

Combined with \([1.3.7]\) this proves (b). \(\square\)
**Explanation 1.3.8** It is well-known that an injective $\mathbb{F}_q$-linear map $\lambda: V \hookrightarrow k$ induces an injective $\mathbb{F}_q$-linear map $p_\lambda: V'' \hookrightarrow k$ by the formula $(p_\lambda)(p(v)) := e_{i^*(\lambda|\hat{V})^{-1}}(v)$. The formula in Proposition 1.3.2 (c) translates this equation into the surprisingly simple formula for the reciprocals

\[(1.3.9) \frac{1}{(p_\lambda)(p(v))} = \sum_{v' \in V'} \frac{1}{\lambda(v - v')}\]

for all $v \in V \setminus V'$. The fact that the right hand side is a polynomial in the values of $(\lambda|\hat{V})^{-1}$ is central for everything that follows in this article. It motivated both the quotient construction for reciprocal maps in Definition 1.2.5 and the definition of $A$-reciprocal maps in Definition 2.3.1.

**Proposition 1.3.10** (a) If $\rho$ is fiberwise invertible, then so is $p_*\rho$.

(b) If for every $p \in \text{Spec}(R)$ there exists $v \in V \setminus V'$ with $\rho(v) \notin p$, then $p_*\rho$ is fiberwise non-zero.

**Proof.** For any $p \in \text{Spec}(R)$ and any $v \in V \setminus V'$ with $\rho(v) \notin p$, Proposition 1.3.4 (b) and the definition of $p_*\rho$ imply that $(p_*\rho)(p(v)) \notin p$. By applying the universal quantifier $\forall$ or the existential quantifier $\exists$ to $v$ we obtain the respective result.  

**Proposition 1.3.11** We have the following identities in $R[\tau]$:

(a) $e_\rho \circ u = u \circ e_{up}$ for any $u \in R$.

(b) $e_\rho = e_{p_*\rho} \circ e_{i^*\rho}$.

**Proof.** (a) follows by direct computation from 1.3.1. To establish (b) we may, as in the proof of Proposition 1.3.2, reduce ourselves to the case that $\rho = (\lambda|\hat{V})^{-1}$ for an injective $\mathbb{F}_q$-linear map $\lambda: V \hookrightarrow k$ to a field $k$. Since $e_{i^*\rho}$ is an $\mathbb{F}_q$-linear polynomial with the precise set of zeros $\lambda(i(V'))$, there is a unique injective $\mathbb{F}_q$-linear map $\lambda": V'' \hookrightarrow k$ satisfying $\lambda''(p(v)) = e_{i^*\rho}(\lambda(v))$ for all $v \in V$. But by Proposition 1.3.4 (c) and the definition of $p_*\rho$ we have $(p_*\rho)(p(v)) = e_{i^*\rho}(\lambda(v))^{-1}$ for all $v \in V \setminus V'$. Thus $p_*\rho = (\lambda''|\hat{V}'')^{-1}$. Now the formula (b) is precisely that in [3] (1.12)].

### 1.4 General $\mathbb{F}_q$-reciprocal maps

Now we globalize the concept of $\mathbb{F}_q$-reciprocal maps following [15, §7]. For this we assume that $V \neq 0$. Let $S$ be a scheme over $\mathbb{F}_q$, let $\mathcal{L}$ be an invertible sheaf on $S$, and let $\mathcal{L}(S)$ denote the space of global sections of $\mathcal{L}$. For any section $\ell \in \mathcal{L}(S)$ and any point $s \in S$ we let $\ell(s) \in \mathcal{L} \otimes_{\mathcal{O}_S} k(s)$ denote the value of $\ell$ over the residue field $k(s)$ of $s$. The (tensor) product of sections $\ell_1, \ldots, \ell_n \in \mathcal{L}(S)$ is a section $\ell_1 \cdots \ell_n \in \mathcal{L}^\otimes(S)$, and the inverse of a nowhere vanishing section $\ell \in \mathcal{L}(S)$ is a section $\ell^{-1} \in \mathcal{L}^\vee(S)$. 

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**Definition 1.4.1** A map $\rho: \hat{V} \to \mathcal{L}(S)$ is called $\mathbb{F}_q$-reciprocal if

(a) $\rho(v) \cdot \rho(w) = \rho(v + w) \cdot (\rho(v) + \rho(w))$ in $\mathcal{L}^S(S)$ for all $v, w \in \hat{V}$ with $v + w \in \hat{V}$, and

(b) $\alpha \rho(\alpha v) = \rho(v)$ for all $\alpha \in \mathbb{F}_q^\times$ and $v \in \hat{V}$.

**Definition 1.4.2** An $\mathbb{F}_q$-reciprocal map $\rho: \hat{V} \to \mathcal{L}(S)$ is called

(a) fiberwise non-zero if for every point $s \in S$ there exists $v \in \hat{V}$ with $\rho(v)(s) \neq 0$.

(b) fiberwise invertible if for every point $s \in S$ and every $v \in \hat{V}$ we have $\rho(v)(s) \neq 0$.

**Remark 1.4.3** When $S = \text{Spec}(R)$ and $\mathcal{L} = \mathcal{O}_X$, Definitions 1.4.1 and 1.4.2 agree precisely with Definitions 1.1.1 and 1.1.2. Conversely, for an arbitrary invertible sheaf $\mathcal{L}$ consider a covering of $S$ by open affines $U_i = \text{Spec}(R_i)$ and an isomorphism $f_i: \mathcal{O}_S|U_i \cong \mathcal{L}|U_i$ for each $i$. Giving an $\mathbb{F}_q$-reciprocal map $\rho: \hat{V} \to \mathcal{L}(S)$ is then equivalent to giving $\mathbb{F}_q$-reciprocal maps $\rho_i: \hat{V} \to \mathcal{O}_S(U_i)$ for all $i$ such that $f_i \circ \rho_i = f_j \circ \rho_j$ over $U_i \cap U_j$ for all $i, j$.

Thus all the results from Subsection 1.3 have direct analogues in this more general setting. We also obtain a moduli space, as follows.

**Construction 1.4.4** We endow the rings $R_{V,d}$ and $RS_{V,d}$ from Construction 1.1.3 with the unique $\mathbb{Z}$-grading for which the elements $\frac{1}{v}$ are homogeneous of degree 1 for all $v \in \hat{V}$. For any integer $d$ let $R_{V,d}$ and $RS_{V,d}$ denote the respective homogenous parts of degree $d$. By construction $R_{V}$ is generated over $\mathbb{F}_q$ by its homogeneous part of degree 1. Thus

$$Q_V := \text{Proj}(R_V)$$

is a projective scheme over $\mathbb{F}_q$ endowed with a natural very ample invertible sheaf $\mathcal{O}(1)$ and a natural homomorphism $R_{V,d} \to \mathcal{O}(d)(Q_V)$ for all $d \in \mathbb{Z}$. In fact this is an isomorphism by [15, Cor. 5.4].

Since $R_V$ is an integral domain and we have now assumed $V \neq 0$, we have $R_V \neq \mathbb{F}_q$ and $Q_V$ is an integral scheme. Also, since $RS_V$ is the localization of $R_V$ obtained by inverting a non-empty finite set of homogeneous elements of degree 1, the scheme

$$\Omega_V := \text{Proj}(RS_V) \cong \text{Spec}(RS_{V,0})$$

is an affine open dense subscheme of $Q_V$.

**Definition 1.4.5** Consider two pairs $(\mathcal{L}, \rho)$ and $(\mathcal{L}', \rho')$ consisting of an invertible sheaf and an $\mathbb{F}_q$-reciprocal map. An isomorphism of invertible sheaves $f: \mathcal{L} \cong \mathcal{L}'$ satisfying $\rho' = f \circ \rho$ is called an isomorphism $(\mathcal{L}, \rho) \cong (\mathcal{L}', \rho')$. If there exists such an isomorphism, the pairs $(\mathcal{L}, \rho)$ and $(\mathcal{L}', \rho')$ are called isomorphic.

If $\rho$ or $\rho'$ is fiberwise non-zero, there exists at most one isomorphism $(\mathcal{L}, \rho) \cong (\mathcal{L}', \rho')$. Thus the isomorphism classes of such pairs form a well-posed moduli problem. By [15, Thm. 7.10 and Prop. 7.11] we have:
Theorem 1.4.6  (a) The map 
\[ \rho_{\text{univ}} : \hat{V} \to R_{V,1} = \mathcal{O}(1)(Q_V), \; v \mapsto \frac{1}{v} \]
is \( \mathbb{F}_q \)-reciprocal and fiberwise non-zero.

(b) For any scheme \( S \) over \( \mathbb{F}_q \), any invertible sheaf \( \mathcal{L} \) on \( S \), and any fiberwise non-zero \( \mathbb{F}_q \)-reciprocal map \( \rho : \hat{V} \to \mathcal{L}(S) \) there exists a unique morphism \( f : S \to Q_V \) over \( \mathbb{F}_q \) such that \( (\mathcal{L}, \rho) \cong f^*(\mathcal{O}(1), \rho_{\text{univ}}) \).

(c) This \( f \) factors through \( \Omega_V \) if and only if \( \rho \) is fiberwise invertible.

We end this subsection by discussing the effect of the functors from Subsection 1.2.

Consider the exact sequence (1.2.1).

Proposition 1.4.7  (a) The homomorphism \( \pi_i : R_V \to R_{V'} \) from Proposition 1.2.4 (b) induces a closed embedding \( \varepsilon_i : Q_{V'} \hookrightarrow Q_V \), whose image is defined by the equations \( \rho_{\text{univ}}(v) = 0 \) for all \( v \in V \setminus i(V') \).

(b) Consider any fiberwise non-zero \( \mathbb{F}_q \)-reciprocal map \( \rho : \hat{V} \to \mathcal{L}(S) \) over a scheme \( S \) over \( \mathbb{F}_q \). Then the associated morphism \( S \to Q_V \) factors through \( \varepsilon_i \) if and only if \( \rho = i_* \rho' \) for an \( \mathbb{F}_q \)-reciprocal map \( \rho' : \hat{V'} \to \mathcal{L}(S) \).

Proof. The description in Proposition 1.2.4 shows that \( \pi_i \) is a surjective graded \( \mathbb{F}_q \)-algebra homomorphism whose kernel is generated by the elements \( \frac{1}{v} \) for all \( v \in V \setminus i(V') \). This directly implies (a). Part (b) follows as in the proof of Proposition 1.2.4 (c). \( \square \)

Remark 1.4.8 Let \( X \) be the closed subscheme of \( Q_V \) that is defined by the equations \( \rho_{\text{univ}}(i(v')) = 0 \) for all \( v' \in \hat{V}' \). Then the pullback \( i^* \rho_{\text{univ}} \) is fiberwise non-zero over \( Q_V \setminus X \); hence by the universal property of \( Q_V \), it corresponds to a morphism \( Q_V \setminus X \to Q_{V'} \). In fact, this is the morphism induced by the graded \( \mathbb{F}_q \)-algebra homomorphism \( \varepsilon_i : R_V \hookrightarrow R_{V'} \) from Proposition 1.2.4 (a). By the same argument as in the proof of Proposition 1.2.4 the morphism \( Q_V \setminus X \to Q_{V'} \) is a left inverse of the embedding \( \varepsilon_i : Q_{V'} \hookrightarrow Q_V \) above.

Remark 1.4.9 Using the description of \( \varepsilon_i(Q_{V'}) \) in Proposition 1.4.7 (a), Proposition 1.3.10 (b) implies that the pushforward \( p_* \rho_{\text{univ}} \) is fiberwise non-zero over \( Q_V \setminus \varepsilon_i(Q_{V'}) \). By the universal property of \( Q_{V''} \) it therefore corresponds to a morphism \( Q_V \setminus \varepsilon_i(Q_{V'}) \to Q_{V''} \). In fact, this is the morphism induced by the graded \( \mathbb{F}_q \)-algebra homomorphism \( \varepsilon_p : R_{V''} \hookrightarrow R_V \) from Proposition 1.2.6.
1.5 Description of the ring

For later use we recall the description of $R_V$ from Section 2 of [15]. Choose a basis $X_1, \ldots, X_r$ of $V$, and for each $0 \leq k \leq r$ consider the subspace $V_k := \mathbb{F}_q X_1 + \cdots + \mathbb{F}_q X_k$. For every $1 \leq k \leq r$ consider the finite subsets

\begin{align*}
\Delta_k &:= \left\{ \frac{1}{X_k + w} \mid w \in \hat{V}_{k-1} \right\} \cup \{1\} \quad \text{and} \\
E_k &:= \left\{ \frac{1}{X_k + w} \mid w \in V_{k-1} \right\}
\end{align*}

of $R_V$, each of cardinality $|V_{k-1}| = q^{k-1}$. Note that the $\Delta_k$ differs from $E_k$ only in that the element $\frac{1}{X_k}$ is replaced by 1. Observe that by unique factorization for polynomials in the variables $X_1, \ldots, X_r$ we have bijective maps

$$
\Delta_1 \times \ldots \times \Delta_r \sim \Delta_1 \cdots \Delta_r := \left\{ e_1 \cdots e_r \mid \forall k: e_k \in \Delta_k \right\} \subset R_V \\
E_1 \times \ldots \times E_r \sim E_1 \cdots E_r := \left\{ e_1 \cdots e_r \mid \forall k: e_k \in E_k \right\} \subset R_V.
$$

Let $U$ be the subgroup of $\text{Aut}_{\mathbb{F}_q}(V)$ which sends each $X_k$ to an element of the coset $X_k + V_{k-1}$. In the given basis this corresponds to the subgroup of all upper triangular matrices in $\text{GL}_r(\mathbb{F}_q)$ with all diagonal entries 1. Then $U$ permutes each set $E_k$ transitively (but it does not act on $\Delta_k$). Moreover, giving an element $g \in U$ is equivalent to giving the images $g(\frac{1}{X_k}) \in E_k$ for all $k$, which can be chosen independently; hence $U$ acts freely transitively on $E_1 \cdots E_r$. It also follows that for each $1 \leq k \leq r$ the element

$$
f_k := \sum_{e_k \in E_k} e_k = \sum_{w \in V_{k-1}} \frac{1}{X_k + w} \in R_V.
$$

is fixed by $U$. By [15] Thm. 2.7, Thm. 2.11 we have:

**Theorem 1.5.4**

(a) The elements $f_1, \ldots, f_r$ are algebraically independent over $\mathbb{F}_q$.

(b) The ring of $U$-invariants is $R^U_V = \mathbb{F}_q[f_1, \ldots, f_r]$.

(c) The ring $R_V$ is a free module over $R^U_V$ with basis $\Delta_1 \cdots \Delta_r$.

We will also need the following fact:

**Proposition 1.5.5** We have $RS_V = R_V[f_1^{-1}, \ldots, f_r^{-1}]$.

**Proof.** For any $1 \leq k \leq r$ we apply Proposition [1.3.4] (a) to $v := X_k$ and $V' := V_{k-1}$ and the universal reciprocal map $\rho_{\text{univ}}: \hat{V} \to R_V$, $v \mapsto \frac{1}{v}$. By the definition (1.5.3) of $f_k$ we obtain that

$$
f_k \cdot \prod_{v' \in V_{k-1}} \frac{1}{v'} = \left( \sum_{v' \in V_{k-1}} \frac{1}{X_k - v'} \right) \cdot \left( \prod_{v' \in \hat{V}_{k-1}} \frac{1}{v'} \right) = \prod_{v' \in V_{k-1}} \frac{1}{X_k - v'}
$$

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in $R_V$. Here all the factors except $f_k$ are already invertible in $RS_V$; hence $f_k$ is also invertible in $RS_V$. This proves the inclusion “$\supset$”.

For the inclusion “$\subset$” it suffices to show that for all $1 \leq k \leq r$ and all $v' \in V_{k-1}$ the element $\frac{1}{x_k - v'}$ is invertible in $R_V[f_1^{-1}, \ldots, f_r^{-1}]$. We will achieve this by induction on $k$. Suppose that we already know it for all values smaller than $k$. Then in particular the element $\frac{1}{v'}$ is invertible in $R_V[f_1^{-1}, \ldots, f_r^{-1}]$ for any $v' \in V_{k-1}$. Thus the left hand side of \((1.5.6)\) is invertible in $R_V[f_1^{-1}, \ldots, f_r^{-1}]$; hence so is the right hand side, and therefore also each factor on the right hand side, as desired. □

1.6 A partial boundary

We will need a variant of Theorem 1.5.4 that concerns a partial boundary of $\Omega_V$ in $Q_V$. Fix an arbitrary $\mathbb{F}_q$-subspace $V' \subset V$. We assume that the basis of $V$ in the preceding subsection was chosen such that $V' = V_s$ for some $0 \leq s \leq r$. Consider the following ideal of $R_V$:

\[ J_s := \left\{ \frac{1}{v'} \left| v' \in V_s \right. \right\}. \]

We want to give an explicit description of the factor ring $R_V/J_s$ (which happens to be the projective coordinate ring of the closed subscheme $X \subset Q_V$ used in Remark 1.4.8). For this we consider the submodule

\[ M_s := \bigoplus_{e \in \Delta_{s+1} \cdots \Delta_r} \mathbb{F}_q[f_{s+1}, \ldots, f_r] \cdot e \subset R_V, \]

which by Theorem 1.5.4 is free with the indicated basis over $\mathbb{F}_q[f_{s+1}, \ldots, f_r]$.

**Lemma 1.6.3** The submodule $M_s$ is $U$-invariant and $M_s^U = \mathbb{F}_q[f_{s+1}, \ldots, f_r]$.

**Proof.** First fix any $s + 1 \leq k \leq r$ and observe that $\mathbb{F}_q[f_k] = \mathbb{F}_q \oplus \mathbb{F}_q[f_k] : f_k$. Also note that the set $E_k$ from (1.5.2) is obtained from $\Delta_k$ on replacing the element $1 \in \Delta_k$ by the element $\frac{1}{X_k} = f_k - \sum_{1 \neq e \in \Delta_k} e_k$. Combining these facts we see that

\[ \bigoplus_{e \in \Delta_k} \mathbb{F}_q[f_k] \cdot e_k = \mathbb{F}_q \oplus \mathbb{F}_q[f_k] \cdot f_k \oplus \bigoplus_{1 \neq e \in \Delta_k} \mathbb{F}_q[f_k] \cdot e_k = \mathbb{F}_q \oplus \bigoplus_{e_k \in E_k} \mathbb{F}_q[f_k] \cdot e_k. \]

Taking the tensor product over $s + 1 \leq k \leq r$ and setting $E_I := \prod_{k \in I} E_k$, we deduce that

\[ M_s = \bigoplus_{e \in \Delta_{s+1} \cdots \Delta_r} \mathbb{F}_q[f_{s+1}, \ldots, f_r] \cdot e \]

\[ = \bigoplus_{I \subset \{s+1, \ldots, r\}} \bigoplus_{e \in E_I} \mathbb{F}_q[f_k|_{k \in I}] \cdot e. \]

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As \( U \) permutes each \( E_k \) and fixes each \( f_k \), it therefore acts on \( M_s \). Since \( U \) acts transitively on \( E_1 \cdots E_r \), it also acts transitively on \( E_I \) for each subset \( I \subset \{s+1, \ldots, r\} \). The fact that \( f_k = \sum_{e_k \in E_k} e_k \) and the above description of \( M_s \) thus imply that

\[
M^U_s = \bigoplus_{I \subset \{s+1, \ldots, r\}} \mathbb{F}_q[f_k|_{k \in I}] \cdot \sum_{e \in E_I} e = \bigoplus_{I \subset \{s+1, \ldots, r\}} \mathbb{F}_q[f_k|_{k \in I}] \cdot \prod_{k \in I} f_k = \mathbb{F}_q[f_{s+1}, \ldots, f_r],
\]

as desired. \( \square \)

**Lemma 1.6.4** Let \( j: V''_s \hookrightarrow V \) denote the inclusion of the subspace that is generated by \( X_{s+1}, \ldots, X_r \). Then the associated homomorphism \( \pi_j: R_V \to R_{V''_s} \) from Proposition 1.2.4 (b) restricts to an embedding

\[
\mathbb{F}_q[f_{s+1}, \ldots, f_r] \hookrightarrow R_{V''_s}.
\]

**Proof.** For each \( s+1 \leq k \leq r \) we have \( f_k = \sum_{w \in V_{k-1}} \frac{1}{X_k + w} \), where \( X_k + w \in V''_s \) if and only if \( w \in V''_s \). Thus

\[
\pi_j(f_k) = \sum_{w \in V_{k-1}} \pi_j(\frac{1}{X_k + w}) = \sum_{w \in V''_s \cap V_{k-1}} \frac{1}{X_k + w} \in R_{V''_s}.
\]

The elements \( \pi_j(f_{s+1}), \ldots, \pi_j(f_r) \in R_{V''_s} \) therefore play the same role for the space \( V''_s \) with the basis \( X_{s+1}, \ldots, X_r \) as the elements \( f_1, \ldots, f_r \) play for the space \( V \) with the basis \( X_1, \ldots, X_r \). By Theorem 1.5.4 (a) they are therefore algebraically independent over \( \mathbb{F}_q \). This means precisely that the restriction of \( \pi_j \) to \( \mathbb{F}_q[f_{s+1}, \ldots, f_r] \) is injective. \( \square \)

**Theorem 1.6.5** The projection \( \pi: R_V \to R_V/J_s \) induces an isomorphism \( M_s \cong R_V/J_s \).

**Proof.** Recall from Theorem 1.5.4 (c) that \( R_V \) is generated by \( \Delta_1 \cdots \Delta_r \) as a module over \( \mathbb{F}_q[f_1, \ldots, f_r] \). For any \( 1 \leq k \leq s \), the definition of \( J_s \) together with (1.5.1) shows that any element of \( \Delta_k \setminus \{1\} \) lies in \( J_s \), and with (1.5.3) it shows that \( f_k \in J_s \). Thus \( R_V/J_s \) is already generated by \( \pi(\Delta_{s+1} \cdots \Delta_r) \) as a module over \( \mathbb{F}_q[f_{s+1}, \ldots, f_r] \). In other words the induced map \( M_s \to R_V/J_s \) is surjective.

It remains to show that this map is injective, or equivalently that its kernel \( M_s \cap J_s \) is zero. For this observe that \( U \) stabilizes \( V' \) and therefore acts on \( J_s \). By Lemma 1.6.3 it therefore also acts on \( M_s \cap J_s \). Since \( U \) is a finite group of \( q \)-power order acting on the \( \mathbb{F}_q \)-vector space \( M_s \cap J_s \), we have \( M_s \cap J_s = 0 \) if and only if \( M^U_s \cap J_s = (M_s \cap J_s)^U = 0 \). We are therefore reduced to showing that \( \pi|M^U_s \) is injective.

But by the definition of \( J_s \) the map \( \pi_j: R_V \to R_{V''_s} \) from Lemma 1.6.4 factors through \( \pi: R_V \to R_V/J_s \). Thus it suffices to show that \( \pi_j|M^U_s \) is injective. This is now guaranteed by combining Lemmas 1.6.3 and 1.6.4. \( \square \)
1.7 The ideal of the boundary

For each $\mathbb{F}_q$-subspace $V' \subset V$ consider the embedding map $i_{V'}: V' \hookrightarrow V$ and the associated ring homomorphism $\pi_{i_{V'}}: R_V \to R_{V'}$ from [12.4] (b). In this subsection we are interested in the ideal

\begin{equation}
I_V := \bigcap_{0 \neq V' \subsetneq V} \ker(\pi_{i_{V'}}) \subset R_V.
\end{equation}

Since each $\pi_{i_{V'}}$ is a homomorphism from $R_V$ to an integral domain, this is a reduced ideal, and by construction it is graded. The associated closed subscheme of $Q_V$ is the reduced subscheme at the boundary

\begin{equation}
\partial \Omega_V := (Q_V \setminus \Omega_V)^{\text{red}} = \bigcup_{0 \neq V' \subsetneq V} \varepsilon_{i_{V'}}(Q_{V'}).
\end{equation}

The following results give explicit generators for $I_V$ in analogy to Theorem [15,4] (c). We knew them at the time of writing [15] and included them only as an exercise [15, Ex. 8.8] in order to shorten the paper. (Caution: The present ideal $I_V$ is not the ideal $I_V$ from [15, §6].)

**Theorem 1.7.3**  
(a) The ideal $I_V$ is a free module over $R^U_V$ with basis $E_1 \cdots E_r$.

(b) The ideal $I_V$ is a free module over the group ring $\mathbb{F}_q[U]$.

**Proof.** By [15, Lemma 2.10] the set $E_1 \cdots E_r$ is the basis of a free $R^U_V$-submodule of $R_V$. Denoting this submodule by $M$, it follows that $M$ is a free module over $\mathbb{F}_q[U]$. It remains to show that $M = I_V$.

For this note first that for any element $e_1 \cdots e_r \in E_1 \cdots E_r$, the reciprocals $e_1^{-1}, \ldots, e_r^{-1}$ form a basis of $V$. Thus for any $\mathbb{F}_q$-subspace $0 \neq V' \subset V$; at least one of them lies in $V \setminus V'$. By the description of $\pi_{i_{V'}}$ in Proposition [12.4] (b) it follows that $e_1 \cdots e_r \in \ker(\pi_{i_{V'}})$. Varying $V'$ this shows that $e_1 \cdots e_r \in I_V$, and varying $e_1 \cdots e_r$ then implies that $M \subset I_V$.

Next observe that $0 \to M \to I_V \to I_V/M \to 0$ is a short exact sequence of $\mathbb{F}_q[U]$-modules. Since $M$ is a free $\mathbb{F}_q[U]$-module, taking $U$-invariants yields a short exact sequence $0 \to M^U \to I^U_V \to (I_V/M)^U \to H^1(U, M) = 0$. Also, since $U$ is a finite group of $q$-power order acting on the $\mathbb{F}_q$-vector space $I_V/M$, we have $I_V/M = 0$ if and only if $(I_V/M)^U = 0$. To prove that $M = I_V$, by the short exact sequence it is therefore enough to prove that $M^U = I^U_V$.

As the given basis $E_1 \cdots E_r$ of $M$ over $R^U_V$ is a single free orbit under $U$, the submodule $M^U$ is the free $R^U_V$-module generated by the element $\sum E_1 \cdots E_r = f_1 \cdots f_r$. In other words it is the principal ideal of $R^U_V$ generated by $f_1 \cdots f_r$. Since $R^U_V = \mathbb{F}_q[f_1, \ldots, f_r]$ with algebraically independent $f_1, \ldots, f_r$, this ideal is the intersection of the ideals $R^U_V \cdot f_k$ for all $1 \leq k \leq r$. Thus it suffices to prove that $I^U_V \subset R^U_V \cdot f_k$ for every fixed $1 \leq k \leq r$. 

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To achieve this consider the subspace $V'$ of codimension 1 that is generated by all $X_j$ except $X_k$. Then $\pi_{i,V'}(f_k) = 0$, while for each $1 \leq j \leq r$ with $j \neq k$ we have

$$\pi_{i,V'}(f_j) = \sum_{w \in V' \cap V_{j-1}} \frac{1}{X_j + w} \in R_{V'}.$$ 

Thus by Theorem 1.5.4 (a) for $V'$ in place of $V$, the elements $\pi_{i,V'}(f_j)$ for $j \neq k$ are algebraically independent over $\mathbb{F}_q$, so the kernel of the homomorphism $\pi_{i,V'}|R_{V'}^U$ is the ideal $R_{V'}^U \cdot f_k$. By construction $I_V^U = I_V \cap R_{V'}^U$ is contained in this kernel; hence we are done. □

Corollary 1.7.4 The ideal $I_V$ is generated by the elements $\frac{1}{v_1 \cdots v_r}$ for all bases $v_1, \ldots, v_r$ of $V$.

Proof. For any basis $v_1, \ldots, v_r$ of $V$, setting $X_i := v_i$ shows that $\frac{1}{v_1 \cdots v_r} \in I_V$ by Theorem 1.7.3. Conversely, Theorem 1.7.3 shows that $I_V$ is generated by elements of this form. □

Corollary 1.7.5 The ideal sheaf of $\partial \Omega_V \subset Q_V$ is in general not locally principal.

Proof. Consider any subspace $0 \neq V' \subseteq V$ and choose a complement $V'' \subset V$. Then by [15, Prop. 8.3], there exist open subschemes $U$ and $U'$ and an isomorphism between them making the commutative diagram:

$$\Omega_{V'} \xrightarrow{\varepsilon_{i,V''}} U \subset Q_V$$

Moreover, the construction in [15] shows that this isomorphism induces an isomorphism

$$\Omega_V \cap U \xrightarrow{i} U \subset Q_V$$

Thus near the stratum $\varepsilon_{i,V''}(\Omega_{V''}) \subset Q_V$, the open embedding $\Omega_V \hookrightarrow Q_V$ is isomorphic to the open embedding $\Omega_{V'} \times \text{Spec}(RS_{V''}) \hookrightarrow \Omega_{V'} \times \text{Spec}(R_{V''})$. This isomorphism identifies the ideal sheaf of the boundary $\partial \Omega_V \subset Q_V$ with the pullback from $\text{Spec}(R_{V''})$ of the ideal sheaf associated to $I_{V''} \subset R_{V''}$. If $\dim_{\mathbb{F}_q}(V'') \geq 2$, Corollary 1.7.4 shows that the graded ideal $I_{V''}$ is not principal. Hence the associated ideal sheaf on $\text{Spec}(R_{V''})$ is not locally principal at the apex $\{0\}$ of the cone $\text{Spec}(R_{V''})$. □
2 \ A\text{-reciprocal maps}

2.1 Reminders on Drinfeld modules

In this subsection we briefly recall various notions concerning Drinfeld modules (see for instance [14 \S 3]).

Throughout we fix an admissible coefficient ring $A$ containing $\mathbb{F}_q$ and set $F := \text{Quot}(A)$. The degree of an element $a \in A$ is the number $\deg_A(a) := \dim_{\mathbb{F}_q}(A/(a))$ if $a \neq 0$, respectively $-\infty$ if $a = 0$. Consider any commutative $F$-algebra $R$. By a slight abuse of notation we denote the image in $R$ of an element $a \in A$ again by $a$. A standard Drinfeld $A$-module of rank $r \geq 1$ over $R$ is an $\mathbb{F}_q$-algebra homomorphism $\varphi : A \to R[\tau]$, $a \mapsto \varphi_a$ satisfying for every $a \in A \setminus \{0\}$:

(a) $\varphi_a = \sum_{i=0}^{r \deg_A(a)} \varphi_{a,i} \tau^i$ with $\varphi_{a,r \deg_A(a)} \in R^*$, and

(b) $d\varphi_a = \varphi_{a,0} = a$.

As $R$ is an $F$-algebra, condition (b) means that $\varphi$ has generic characteristic.

More generally consider any scheme $S$ over $F$. For any line bundle $E$ on $S$ let $\text{End}_{\mathbb{F}_q}(E)$ denote the ring of $\mathbb{F}_q$-linear endomorphisms of the commutative group scheme underlying $E$. Any trivialization of line bundles $\mathcal{G}_a \to E|U$ over an open affine subscheme $U = \text{Spec}(R) \subset S$ induces an isomorphism $R[\tau] \simeq \text{End}_{\mathbb{F}_q}(E|U)$. A general Drinfeld $A$-module of rank $r \geq 1$ over $S$ is an $\mathbb{F}_q$-algebra homomorphism $\varphi : A \to \text{End}_{\mathbb{F}_q}(E)$, $a \mapsto \varphi_a$ which for any trivialization of $E$ over an open affine subscheme becomes a standard Drinfeld $A$-module of rank $r$.

For any such $(E, \varphi)$ and any non-zero ideal $N \subset A$, the intersection $\varphi[N] := \bigcap_{a \in N} \text{Ker}(\varphi_a)$ is an $A$-module subscheme of $E$ that is finite étale over $S$ and whose sections over any geometric point of $S$ form a free $A/N$-module of rank $r$. Consider the $A$-module $V_N^r := (N^{-1}/A)^{gr}$, where $N^{-1} \subset F$ denotes the inverse fractional ideal of $N$. This is also a free $A/N$-module of rank $r$. A level $N$-structure on $\varphi$ is an $A$-module homomorphism $\lambda : V_N^r \to \varphi[N](S)$ which induces an isomorphism in every fiber. Observe that for any two non-zero ideals $N \subset N' \subset A$, we have a natural inclusion $V_N^r \subset V_N^r$; hence any level $N$-structure restricts to a level $N'$-structure. In particular we can apply this when $N' = (a)$ for $a \in A \setminus \{0\}$, in which case we abbreviate $V_a^r := V_{(a)}^r$.

An isomorphism of triples $(E, \varphi, \lambda) \xrightarrow{\sim} (E', \varphi', \lambda')$ as above is an isomorphism of line bundles that is compatible with $\varphi$ and $\lambda$. From now on we assume that $N$ is a proper ideal of $A$. Then there exists at most one isomorphism $E \xrightarrow{\sim} E'$ that is compatible with the level $N$-structures $\lambda$ and $\lambda'$, and so the isomorphism classes of such triples form a well-posed moduli problem. By [3 \S 5] there is a fine moduli scheme in the following sense:

**Theorem 2.1.1** There is a scheme $M'_{A,N}$ over $F$ and a triple $(E^{\text{univ}}, \varphi^{\text{univ}}, \lambda^{\text{univ}})$ as above over $M'_{A,N}$ such that:

(a) For any scheme $S$ over $F$ and any triple $(E, \varphi, \lambda)$ as above over $S$, there exists a unique morphism $f : S \to M'_{A,N}$ over $F$ such that $(E, \varphi, \lambda) \cong f^*(E^{\text{univ}}, \varphi^{\text{univ}}, \lambda^{\text{univ}})$. 

(b) This $M_{A,N}^{r}$ is an irreducible smooth affine algebraic variety of finite type and dimension $r - 1$ over $F$.

For a less canonical description of this moduli problem fix any $v_0 \in \tilde{V}_{N}^{r}$. Then any triple $(E, \varphi, \lambda)$ as above determines an isomorphism of line bundles $\mathbb{G}_{a,S} \cong E$, $u \mapsto u \cdot \lambda(v_0)$. Giving an isomorphism class of triples $(E, \varphi, \lambda)$ is therefore equivalent to giving a standard Drinfeld $A$-module $\varphi' : A \rightarrow O_{S}(\tau)$ of rank $r$ with a level $N$-structure $\lambda' : V_{N}^{r} \rightarrow \varphi'[N](S)$ such that $\lambda'(v_0) = 1$.

2.2 Conditions on the level

Let $A$ be as in Subsection 2.1. For any non-zero ideal $N \subset A$ consider the set of divisors of $N$

\[(2.2.1) \quad \text{Div}(N) := \{a \in A \mid N \subset (a)\}.
\]

We are interested in a non-zero proper ideal $N \subset A$ satisfying the following conditions:

**Assumption 2.2.2** There exists a subset $D \subset \text{Div}(N)$ with the properties:

(a) For any element $a \in A \setminus \{0\}$ there exists an element $b \in A \setminus \{0\}$ which is a product of elements of $D$ such that $\deg_{A}(a - b) < \deg_{A}(a)$.

(b) For any distinct $a, b \in D$ we have $ab \in \text{Div}(N)$.

(c) Any element of $\text{Div}(N)$ is a product of elements in $D$.

(d) $N$ is principal.

**Remark 2.2.3** Condition (a) implies that $\text{Div}(N)$ generates $A$ as an $\mathbb{F}_{q}$-algebra. One might hope that this consequence be enough to deduce all our results, but at present all conditions (a–d) are used in technical arguments. Condition (a) is used to show Proposition 2.9.3, conditions (b) and (c) to prove Proposition 2.4.4, and condition (d) for Proposition 2.5.4.

**Remark 2.2.4** In the case $A = \mathbb{F}_{q}[t]$ Assumption 2.2.2 holds for any non-zero ideal $N \subset (t - \alpha)$ for any $\alpha \in \mathbb{F}_{q}$, where $D$ consists of $\mathbb{F}_{q}^\times$ and all monic irreducible divisors of $N$.

**Proposition 2.2.5** For any maximal ideal $p$ and any ideal $a$ not contained in $p$, there exists an ideal $N$ contained in $a$, but not in $p$, which satisfies Assumption 2.2.2

**Proof.** For any integer $d \geq 0$ abbreviate $A_{\leq d} := \{a \in A \mid \deg_{A}(a) \leq d\}$. By Riemann-Roch there exists an integer $d > 0$ such that for every element $a \in A \setminus A_{\leq d}$ there exists an element $b \in A$ which is a product of elements of $A_{\leq d}$ such that $\deg_{A}(a - b) < \deg_{A}(a)$. Note that any element $c \in A_{\leq d} \cap p$ that is a factor of $b$ is non-constant, so we can replace $c$
by \(c + 1\) in the product defining \(b\) without destroying the property \(\deg_A(a - b) < \deg_A(a)\). Thus the set \(A_{\leq d} \setminus p\) has the property of \(D\) in (a), and all its elements are non-zero.

Choose an element \(a_0 \in A \setminus p\), and let \(b \subset A\) be the principal ideal generated by \(a_0 \cdot \prod_{a \in A_{\leq d} \setminus p} a\). Then \(b\) is contained in \(a\), but not in \(p\). Let \(h\) be the class number of \(A\). We claim that the ideal \(N := b^{2h}\) has the desired properties.

Indeed, by construction \(N\) is contained in \(a\), but not in \(p\). Also, since \(b\) is principal, so is \(N\); hence condition (d) holds.

Next let \(p_1, \ldots, p_m\) be the distinct prime factors of \(b\). Let \(D\) be the set of all \(a \in A \setminus \{0\}\) for which \((a) = p_1^{e_1} \cdots p_m^{e_m}\) with all \(e_i \leq h\). Then, in particular, for each \(i\) there exists an element \(b_i \in D\) with \((b_i) = p_i^{e_i}\). It follows that every non-zero element of \(A\) whose prime ideal factorization contains no primes other than \(p_1, \ldots, p_m\) is a product of elements of \(D\). In particular this therefore holds for all divisors of \(N\), establishing condition (c).

Moreover, any element of \(D\) is a divisor of \(b^h\); hence any product of two elements of \(D\) is a divisor of \(N\), proving condition (b).

Finally, every element of \(A_{\leq d} \setminus p\) is a divisor of \(N\) and therefore a product of elements of \(D\). Thus \(D\) satisfies the condition (a), and we are done.

For our later purposes, Proposition 2.2.5 guarantees that there are sufficiently many non-zero ideals \(N\) satisfying Assumption 2.2.2.

### 2.3 Basic \(A\)-reciprocal maps

For the following we fix an ideal \(N \subset A\) satisfying Assumption 2.2.2. We also fix an integer \(r > 0\) and define \(V^r_N := (N^{-1}/A)^{\oplus r}\) and \(V^r_a := (a^{-1}A/A)^{\oplus r}\) as in Subsection 2.1. Then for any \(a \in \Div(N)\) we have \(V^r_a \subset V^r_N\). Consider any \(F\)-algebra \(R\).

**Definition 2.3.1** A map \(\rho: \hat{V}^r_N \to R\) is called \(A\)-reciprocal if

1. \(\rho(v) \cdot \rho(w) = \rho(v + w) \cdot (\rho(v) + \rho(w))\) for all \(v, w \in \hat{V}^r_N\) with \(v + w \in \hat{V}^r_N\), and
2. \(a \rho(\alpha v) = \sum_{v' \in V^r_N} \rho(v - v')\) for all \(a \in \Div(N)\) and \(v \in \hat{V}^r_N \setminus V^r_a\).

**Proposition 2.3.2** Any \(A\)-reciprocal map is \(\mathbb{F}_q\)-reciprocal.

**Proof.** Every \(\alpha \in \mathbb{F}_q^\times\) lies in \(\Div(N)\) and satisfies \(V^r_\alpha = \{0\}\).

**Construction 2.3.3** Let \(R_{A,V^r_N}\) denote the factor ring of \(R_{V^r_N} \otimes_{\mathbb{F}_q} F\) modulo the ideal

\[
\left\{ \frac{1}{av} \otimes a - \sum_{v' \in V^r_N} \frac{1}{v - v'} \otimes 1 \mid \begin{array}{l}
\text{all } a \in \Div(N), \\
\text{all } v \in V^r_N \setminus V^r_a
\end{array} \right\}.
\]

For any element \(f \in R_{V^r_N} \otimes_{\mathbb{F}_q} F\) let \([f]\) denote its image in \(R_{A,V^r_N}\). Let \(RS_{A,V^r_N}\) be the localization of \(R_{A,V^r_N}\) obtained by inverting the elements \(\left\{ \frac{1}{v} \otimes 1 \right\}\) for all \(v \in V^r_N\).
Theorem 2.3.4  (a) The map \\
\[ \rho^{\text{univ}}: \hat{V}_N^r \to R_{A,V_N^r}, \ v \mapsto \left[ \frac{1}{v} \otimes 1 \right] \]

is \( A \)-reciprocal.

(b) For any \( F \)-algebra \( R \) and any \( A \)-reciprocal map \( \rho: \hat{V}_N^r \to R \) there exists a unique \( F \)-algebra homomorphism \( f: R_{A,V_N^r} \to R \) such that \( \rho = f \circ \rho^{\text{univ}} \).

(c) This \( f \) extends to a ring homomorphism \( RS_{A,V_N^r} \to R \) if and only if \( \rho \) is fiberwise invertible.

Proof. Theorem 1.1.5 (a) implies that \( \rho^{\text{univ}} \) is \( \mathbb{F}_q \)-reciprocal. It is \( A \)-reciprocal, because in the construction of \( R_{A,V_N^r} \) we have divided out precisely the relations corresponding to 2.3.1 (b) that make an \( \mathbb{F}_q \)-reciprocal map an \( A \)-reciprocal map. This proves (a).

In the situation of (b), the \( \mathbb{F}_q \)-reciprocal map underlying \( \rho \) already corresponds to a unique \( \mathbb{F}_q \)-algebra homomorphism \( f': R_{V_N^r} \to R \) such that \( \rho = f' \circ \rho^{\text{univ}} \) by Theorem 1.1.5 (b). As \( R \) is an \( F \)-algebra, this \( f' \) corresponds to a unique \( F \)-algebra homomorphism \( f'': R_{V_N^r} \otimes_{\mathbb{F}_q} F \to R \), which sends \( \frac{1}{v} \otimes 1 \) to \( \rho(v) \) for all \( v \in \hat{V}_N^r \). The condition 2.3.1 (b) for \( \rho \) then implies that the ideal in Construction 2.3.3 lies in the kernel of \( f'' \). Thus \( f'' \) factors through the desired \( f \), proving (b).

Assertion (c) then follows directly from the construction of \( RS_{A,V_N^r} \). \( \square \)

Again we finish this subsection by addressing functoriality:

Proposition 2.3.5 Consider any \( 1 \leq s \leq r \) and any injective \( A \)-linear map \( i: V_N^s \hookrightarrow V_N^r \). Then for any \( A \)-reciprocal map \( \rho: \hat{V}_N^r \to R \) the extension by zero \( i_*\rho: \hat{V}_N^s \to R \) from Proposition 1.2.3 is \( A \)-reciprocal.

Proof. By Proposition 1.2.3 the map \( i_*\rho \) is already \( \mathbb{F}_q \)-reciprocal. Next take any \( a \in \text{Div}(N) \) and any \( v \in V_N^s \setminus V_a^r \). Suppose first that \( v = i(w) \) for some \( w \in \hat{V}_N^{s} \). Then \( w \not\in V_a^r \). Moreover, for any \( v' \in V_a^r \) we have \( v - v' \in i(V_N^s) \) if and only if \( v' \in i(V_a^s) \). By the condition 2.3.1 (b) for \( \rho \) and the definition of \( i_*\rho \), we deduce that 

\[ a \cdot i_*\rho(av) = \rho(aw) = \sum_{w' \in V_a^s} \rho(w - w') = \sum_{v' \in V_a^r} i_*\rho(v - v'). \]

Next we observe that the far left and right sides of this equation depend only on the coset \( v + V_a^r \). Thus the total equation also holds if \( v \in i(V_N^s) + V_a^r \). Finally suppose that \( v \not\in i(V_N^s) + V_a^r \). Then for all \( v' \in V_a^r \) we have \( v - v' \not\in i(V_N^s) \) and hence \( i_*\rho(v - v') = 0 \). On the other hand, since \( i(V_N^s) \) is a direct summand of \( V_N^r \) as an \( A \)-module, the assumption \( v \not\in i(V_N^s) + V_a^r \) implies that \( av \not\in i(V_N^s) \). Thus \( i_*\rho(av) = 0 \) as well, and the total equation holds trivially in this case. Together this proves that \( i_*\rho \) satisfies the condition 2.3.1 (b). Therefore \( i_*\rho \) is \( A \)-reciprocal. \( \square \)
Proposition 2.3.6  (a) The functor $i_*$ on $A$-reciprocal maps is represented by a surjective $F$-algebra homomorphism $\pi_i: R_{A,V}^s \to R_{A,V}^r$ that sends $[\frac{1}{v'} \otimes 1]$ to $[\frac{1}{v} \otimes 1]$ for all $v' \in \hat{V}_N^s$ and $[\frac{1}{v} \otimes 1]$ to $0$ for all $v \in V_N^r \setminus i(V_N^s)$.

(b) The kernel of $\pi_i$ is generated by the elements $[\frac{1}{v} \otimes 1]$ for all $v \in V_N^r \setminus i(V_N^s)$.

Proof. Precisely analogous to the proof of Proposition 1.2.4 (b) and (c). □

2.4 The associated ring homomorphism

Keeping the notation of the preceding subsection, we now fix an $A$-reciprocal map $\rho: \hat{V}_N^r \to R$. For any $a \in \text{Div}(N)$ consider the polynomial

$$\varphi_a(X) := a \cdot X \cdot \prod_{v \in V_a^r} (1 - \rho(v)X) \in R[X].$$

Comparison with (1.3.1) shows that $\varphi_a = a \circ e_{\rho|\hat{V}_a^r}$; hence $\varphi_a \in R[\tau]$ by Proposition 1.3.2.

Lemma 2.4.2 For any $a, b \in \text{Div}(N)$ with $ab \in \text{Div}(N)$ we have $\varphi_a \circ \varphi_b = \varphi_{ab}$.

Proof. By assumption we have a short exact sequence

$$0 \longrightarrow V_b^r \xrightarrow{i} V_{ab}^r \xrightarrow{p} V_a^r \longrightarrow 0,$$

where $i$ denotes the inclusion $V_b^r \hookrightarrow V_{ab}^r$ and $p$ denotes multiplication by $b$. By the definition 1.2.5 of $p_*$ and the condition 2.3.1 (b), for every $v \in V_{ab}^r \setminus V_b^r$ we have

$$p_*(\rho|\hat{V}_{ab}^r)(bv) = \sum_{v' \in V_b^r} \rho(v - v') = b\rho(bv).$$

In other words we have $p_*(\rho|\hat{V}_{ab}^r) = b \cdot \rho|\hat{V}_a^r$. Using Proposition 1.3.11 we find that

$$\varphi_a \circ \varphi_b = a \circ e_{\rho|\hat{V}_a^r} \circ b \circ e_{\rho|\hat{V}_b^r}$$

$$= a \circ b \circ e_{\rho|\hat{V}_a^r} \circ e_{\rho|\hat{V}_b^r}$$

$$= ab \circ e_{p_*(\rho|\hat{V}_{ab}^r)} \circ e_{i^*(\rho|\hat{V}_{ab}^r)}$$

$$= ab \circ e_{\rho|\hat{V}_{ab}^r}$$

$$= \varphi_{ab},$$

as desired. □

Lemma 2.4.3 Any element of $R[\tau]\tau$ which commutes with $\varphi_a$ for some non-constant $a \in \text{Div}(N)$ is zero.
Proof. Suppose that there is a non-zero $\eta \in R[\tau]$ which commutes with $\varphi_a$. Write
$\eta = u\tau^i + (\text{higher terms in } \tau)$ with $u \in R \setminus \{0\}$ and $i \geq 1$. By the construction (2.4.1) we have $\varphi_a = a + (\text{higher terms in } \tau)$. Thus

$$0 = \eta \circ \varphi_a - \varphi_a \circ \eta = u\tau^i \circ a - a \circ u\tau^i + (\text{higher terms in } \tau) = u(a^q - a)\tau^i + (\text{higher terms in } \tau),$$

and hence $u(a^q - a) = 0$. But since $a$ is a transcendental element of $F$ and $i \geq 1$, we have $a^q - a \in F^\times$. Thus we conclude that $u = 0$, contrary to the assumption. \qed

Proposition 2.4.4 The map $\text{Div}(N) \to R[\tau], a \mapsto \varphi_a$ extends to a unique $\mathbb{F}_q$-algebra homomorphism $\varphi: A \to R[\tau], a \mapsto \varphi_a$, which satisfies $d\varphi_a = a$ for all $a \in A$.

Proof. Let $D = \{a_1, \ldots, a_n\}$ be the subset of $\text{Div}(N)$ from Assumption 2.2.2. Since $D$ generates $A$ as an $\mathbb{F}_q$-algebra, we have a surjective $\mathbb{F}_q$-algebra homomorphism

$$\Pi: \mathbb{F}_q[Y_1, \ldots, Y_n] \to A, \quad P(1, \ldots, a_n).$$

Next, for any two distinct elements $a_i, a_j$ of $D$ we have $a_ia_j \in \text{Div}(N)$ by assumption. Thus by Lemma 2.4.2 we have $\varphi_{a_i} \circ \varphi_{a_j} = \varphi_{a_ia_j} = \varphi_{a_j} \circ \varphi_{a_i}$. In other words, the elements $\varphi_{a_i} \in R[\tau]$ all commute with each other; hence we also have an $\mathbb{F}_q$-algebra homomorphism

$$\Phi: \mathbb{F}_q[Y_1, \ldots, Y_n] \to R[\tau], \quad P \mapsto P(\varphi_{a_1}, \ldots, \varphi_{a_n}).$$

Since each $\varphi_{a_i} = a_i + (\text{higher terms in } \tau)$, for any polynomial $P \in \mathbb{F}_q[Y_1, \ldots, Y_n]$ we have $P(\varphi_{a_1}, \ldots, \varphi_{a_n}) = P(a_1, \ldots, a_n) + (\text{higher terms in } \tau)$. In particular, for any $P \in \text{Ker}(\Pi)$ we have $P(\varphi_{a_1}, \ldots, \varphi_{a_n}) \in R[\tau]$. But at least one $a_i$ is non-constant, and since the associated $\varphi_{a_i}$ commutes with all $\varphi_{a_j}$, it commutes with $P(\varphi_{a_1}, \ldots, \varphi_{a_n})$. Using Lemma 2.4.3 we therefore deduce that $P(\varphi_{a_1}, \ldots, \varphi_{a_n}) = 0$. Varying $P$ this shows that $\text{Ker}(\Pi) \subset \text{Ker}(\Phi)$, which implies that $\Phi = \psi \circ \Pi$ for a unique $\mathbb{F}_q$-algebra homomorphism $\psi: A \to R[\tau]$.

By construction this algebra homomorphism satisfies $\psi_a = \varphi_a$ for all $a \in D$. By Assumption 2.2.2 (c) and Lemma 2.4.2 the same equality then follows for all $a \in \text{Div}(N)$. This proves that the desired extension $\varphi$ exists. The uniqueness follows from the fact that $D$ generates $A$ as an $\mathbb{F}_q$-algebra.

Finally, by (2.4.1) the formula $d\varphi_a = a$ holds for all $a \in \text{Div}(N)$. Since $\varphi$ is an $\mathbb{F}_q$-algebra homomorphism, the same follows for all $a \in A$. \qed

2.5 Constant rank

In this subsection we consider an $A$-reciprocal map $\rho: \hat{V}_N^\tau \to R$ satisfying the condition

$$\forall v \in \hat{V}_N^\tau: \quad \rho(v) = 0 \text{ or } \rho(v) \in R^\times.$$
Lemma 2.5.2 The subset

\[ W := \{0\} \cup \{ v \in V^r_N \mid \rho(v) \in R^\times \} \]

is an A-submodule of \( V^r_N \).

**Proof.** Consider any \( v, w \in W \). If one or more of \( v, w, v + w \) is zero, we directly see that \( v + w \in \{ v, w, 0 \} \subset W \). Otherwise we have \( \rho(v), \rho(w) \in R^\times \) by construction. Thus by Definition 2.3.1 (a) we have \( \rho(v + w) \cdot (\rho(v) + \rho(w)) = \rho(v) \cdot \rho(w) \in R^\times \); hence \( \rho(v + w) \in R^\times \) and so \( v + w \in W \). Together this shows that \( W + W \subset W \).

Next consider any \( a \in \text{Div}(N) \) and any \( v \in W \). If \( v \in V^r_a \), we have \( av = 0 \in W \). Otherwise we have \( \rho(v) \in R^\times \) by construction, and so by Definition 2.3.1 (b) and Proposition 1.3.4 (b) we deduce that

\[ ap(av) \cdot \prod_{v' \in V^r_a} (\rho(v) - \rho(v')) = \rho(v)|_{V^r_a} \in R^\times. \]

Thus \( \rho(av) \in R^\times \) and therefore \( av \in W \). This shows that \( aW \subset W \) for all \( a \in \text{Div}(N) \).

Since \( F^\times_q \subset \text{Div}(N) \), this implies that \( W \) is an \( F_q \)-subspace of \( V^r_N \). As \( \text{Div}(N) \) generates \( A \) as an \( F_q \)-algebra, it is then also an \( A \)-submodule. \( \square \)

Lemma 2.5.3 The map

\[ \lambda: W \to R, \quad v \mapsto \begin{cases} 0 & \text{if } v = 0, \\ \rho(v)^{-1} & \text{if } v \neq 0, \end{cases} \]

is additive and satisfies \( \lambda(av) = \varphi_a(\lambda(v)) \) for all \( v \in W \) and \( a \in A \).

**Proof.** Since \( \rho \) is \( F_q \)-reciprocal by Proposition 2.3.2 Proposition 1.1.3 implies that \( \lambda \) is \( F_q \)-linear. In particular it is additive.

Next consider any \( a \in \text{Div}(N) \) and any \( v \in W \). If \( v = 0 \), we also have \( av = 0 \) and hence \( \lambda(av) = 0 = \varphi_a(0) = \varphi_a(\lambda(v)) \). If \( v \in V^r_a \setminus \{0\} \), we still have \( av = 0 \) and hence \( \lambda(av) = 0 \). But from (2.4.1) we then obtain that

\[ \varphi_a(\lambda(v)) = a \cdot \lambda(v) \cdot \prod_{v' \in V^r_a} (1 - \rho(v')\lambda(v)), \]

where the factor \( 1 - \rho(v)\lambda(v) \) associated to \( v' = v \) is zero. Thus again we find that \( \lambda(av) = 0 = \varphi_a(\lambda(v)) \). Suppose now that \( v \notin V^r_a \). Then by combining the formulas in (2.4.1) and Definition 2.3.1 (b) and Proposition 1.3.4 (c) we deduce that

\[ \rho(av) \cdot \varphi_a(\lambda(v)) = ap(av) \cdot e_{\rho|_{V^r_a}}(\lambda(v)) = 1. \]

Thus again we find that \( \lambda(av) = \varphi_a(\lambda(v)) \).

As this formula holds for all \( a \in \text{Div}(N) \), which generate \( A \) as an \( F_q \)-algebra, and each \( \varphi_a \) is also \( F_q \)-linear, the formula then follows for all \( a \in A \). \( \square \)
Proposition 2.5.4  (a) If $W$ is zero, we have $\varphi_a = a$ in $R[\tau]$ for all $a \in A$.

(b) If $W$ is non-zero, it is a free $A/N$-module of some rank $1 \leq s \leq r$, and $\varphi$ is a standard Drinfeld $A$-module of rank $s$, and for any isomorphism $i: V^s_N \rightarrow W$ the map $\lambda \circ i$ is a level $N$-structure of $\varphi$.

Proof. Assume first that $W = 0$. Then for any $a \in \text{Div}(N)$ the definition (2.4.1) of $\varphi_a$ shows that $\varphi_a(X) = aX$ and hence $\varphi_a = a$ in $R[\tau]$. As $\text{Div}(N)$ generates $A$ as an $\mathbb{F}_q$-algebra, the equality $\varphi_a = a$ then holds for all $a \in A$.

Assume now that $W \neq 0$. The desired assertions hold trivially if $R = 0$, so we may also assume that $R \neq 0$. By Assumption 2.2.2(d) we have $N = (a)$ for some non-constant element $a \in \text{Div}(A)$. The definition (2.4.1) of $\varphi_a$ and the assumption (2.5.1) imply that
\[
\varphi_a(X) = a \cdot X \cdot \prod_{v \in W} \left(1 - \frac{X}{\lambda(v)}\right)
\]
with highest non-zero coefficient in $R^s$. As $W \neq 0$, we have $\varphi_a \notin R$. By general theory (for instance [5, §2]) it thus follows that $\varphi$ is a Drinfeld $A$-module of some constant rank $s \geq 1$ and that $W$ is a free $A/N$-module of rank $s$. Since $W \subset V^r_N \cong (A/N)^{\oplus r}$, this implies that $s \leq r$. Finally, since $\lambda \circ i$ is $A$-linear and injective, it is a level $N$-structure of $\varphi$. \qed

We also have the following converse of Proposition 2.5.4.

Proposition 2.5.5 Let $\varphi: A \rightarrow R[\tau]$ be a standard Drinfeld $A$-module of rank $1 \leq s \leq r$ with a level $N$-structure $\lambda': V^r_N \rightarrow R$, and let $i: V^s_N \rightarrow V^r_N$ be any injective $A$-linear map. Then the map
\[
\rho: V^r_N \rightarrow R, \ v \mapsto \begin{cases} 
\lambda'(w)^{-1} & \text{if } v = i(w) \text{ for } w \in V^s_N, \\
0 & \text{if } v \notin i(V^s_N),
\end{cases}
\]
is $A$-reciprocal and satisfies condition (2.5.1), and we have $(W, \lambda \circ i, \varphi) = (i(V^s_N), \lambda', \varphi')$.

Proof. Suppose first that $s = r$. Then $\rho$ is $\mathbb{F}_q$-reciprocal by Proposition 1.1.3. Next take any $a \in \text{Div}(N)$. The fact that $\lambda'$ is a level $N$-structure of $\varphi'$ and the assumption $d\varphi_a = a$ imply that
\[
(2.5.6) \quad \varphi'_a(X) = a \cdot X \cdot \prod_{v' \in V^r_a} \left(1 - \frac{X}{\lambda'(v')}\right) = a \cdot X \cdot \prod_{v' \in V^r_a} (1 - \rho(v')X).
\]
For any $v \in V^r_N \setminus V^r_a$, the $A$-linearity of $\lambda'$ combined with (2.5.6) shows that
\[
\lambda'(av) = \varphi'_a(\lambda'(v)) = a \cdot \lambda'(v) \cdot \prod_{v' \in V^r_a} (1 - \rho(v')\lambda'(v)).
\]
Applying Proposition 1.3.4 (c) to the $F_q$-reciprocal map $\rho$ and the subspace $V_r^r$ thus implies that
\[
\left( \sum_{v' \in V_r^r} \rho(v - v') \right) \cdot \frac{\lambda'(av)}{a} = 1
\]
and hence
\[
a \cdot \rho(av) = \sum_{v' \in V_r^r} \rho(v - v').
\]
Thus $\rho$ satisfies the condition 2.3.1 (b) and is therefore $A$-reciprocal. By construction $\rho$ is fiberwise invertible, so it satisfies condition (2.5.1), and by Lemmas 2.5.2 and 2.5.3 we have $W = V_N^r$ and $\lambda = \lambda'$. Lastly, for all $a \in \text{Div}(N)$ we have $\varphi_a = \varphi'_a$ by (2.4.1) and (2.5.6). As $\text{Div}(N)$ generates $A$ as an $F_q$-algebra, this implies that $\varphi_a = \varphi'_a$ for all $a \in A$, and we are done.

In the general case the same arguments with $s$ in place of $r$ yield a fiberwise invertible $A$-reciprocal map $\rho': \hat{V}_N^r \to R$ which returns $\lambda'$ and $\varphi'$. The map $\rho$ in question is then simply the extension by zero of $\rho'$ via $i$; hence it is $A$-reciprocal by Proposition 2.3.5. By construction this $\rho$ satisfies condition (2.5.1), and by Lemmas 2.5.2 and 2.5.3 we have $W = i(V_N^r)$ and $\lambda \circ i = \lambda'$. Finally, the formula (2.4.1) shows that $\varphi_a$ does not change under extension by zero for $a \in \text{Div}(N)$. As $\text{Div}(N)$ generates $A$ as an $F_q$-algebra, this implies that $\varphi_a = \varphi'_a$ for all $a \in A$, and we are done. □

### 2.6 General $A$-reciprocal maps

We keep $A$ and $F = \text{Quot}(A)$ and $N$ as in Subsection 2.3. Let $S$ be a scheme over $F$ and $\mathcal{L}$ an invertible sheaf on $S$.

**Definition 2.6.1** A map $\rho: \hat{V}_N^r \to \mathcal{L}(S)$ is called $A$-reciprocal if

(a) $\rho(v) \cdot \rho(w) = \rho(v + w) \cdot (\rho(v) + \rho(w))$ in $\mathcal{L} \otimes \mathcal{L}$ for all $v, w \in \hat{V}_N^r$ with $v + w \in \hat{V}_N^r$,

and

(b) $a \rho(av) = \sum_{v' \in V_a^r} \rho(v - v')$ for all $a \in \text{Div}(N)$ and $v \in V_N^r \setminus V_a^r$.

**Remark 2.6.2** When $S = \text{Spec}(R)$ and $\mathcal{L} = \mathcal{O}_X$, Definition 2.6.1 agrees precisely with Definition 2.3.1. As in Remark 1.4.3, giving a general $A$-reciprocal map $\hat{V}_N^r \to \mathcal{L}(S)$ reduces to giving compatible $A$-reciprocal maps $\hat{V}_N^r \to R_i$ for a suitable covering of $S$ by open affines $U_i = \text{Spec}(R_i)$.

**Construction 2.6.3** The given grading on $R_N$ induces a grading on $R_N \otimes_{\mathbb{F}_q} F$, and all the generators of the ideal in Construction 2.3.3 are homogeneous of degree 1. Thus the factor ring $R_{A,V_N}$ inherits a unique grading, and so does $RS_{A,V_N}$. For any integer $d$ let $R_{A,V_N,d}$ and $RS_{A,V_N,d}$ denote the respective homogeneous parts of degree $d$. By construction $R_{A,V_N}$ is generated over $F$ by its homogeneous part of degree 1. Thus

$$Q_{A,V_N} := \text{Proj}(R_{A,V_N})$$
is a projective scheme over $F$ endowed with a natural very ample invertible sheaf $\mathcal{O}(1)$ and a natural homomorphism $R_{A,V_N,n} \to \mathcal{O}(n)(Q_{A,V_N})$ for all $n \in \mathbb{Z}$. Note that it comes with a closed embedding

$$Q_{A,V_N} \hookrightarrow Q_{V_N} \times_{\text{Spec}(\mathbb{F}_q)} \text{Spec}(F).$$

Also, since $RS_{A,V_N}$ is the localization of $R_{A,V_N}$ obtained by inverting a non-empty finite set of elements of degree 1, the scheme

$$\Omega_{A,V_N} := \text{Proj}(RS_{A,V_N}) \cong \text{Spec}(RS_{A,V_N,0})$$

is an affine open subscheme of $Q_{A,V_N}$.

As in Subsection 1.4 for any two pairs $(\mathcal{L}, \rho)$ and $(\mathcal{L}', \rho')$ consisting of an invertible sheaf and a fiberwise non-zero $A$-reciprocal map, there exists at most one isomorphism $(\mathcal{L}, \rho) \sim (\mathcal{L}', \rho')$. Thus the isomorphism classes of such pairs form a well-posed moduli problem.

**Theorem 2.6.4**

(a) The composite map

$$\rho^\text{univ}: V^r_N \xrightarrow{\frac{1}{v} \otimes 1} R_{A,V_N,1} \to \mathcal{O}(1)(Q_{A,V_N})$$

is $A$-reciprocal and fiberwise non-zero.

(b) For any scheme $S$ over $F$, any invertible sheaf $\mathcal{L}$ on $S$, and any fiberwise non-zero $A$-reciprocal map $\rho: V^r_N \to \mathcal{L}(S)$ there exists a unique morphism $f: S \to Q_{A,V_N}$ over $F$ such that $(\mathcal{L}, \rho) \cong f^*(\mathcal{O}(1), \rho^\text{univ})$.

(c) This $f$ factors through $\Omega_{A,V_N}$ if and only if $\rho$ is fiberwise invertible.

**Proof.** That this $\rho^\text{univ}$ is $A$-reciprocal is a direct consequence of Theorem 2.3.4. That it is fiberwise non-zero follows from the fact that its images $\frac{1}{v} \otimes 1$ generate the augmentation ideal of $R_{A,V_N}$. This proves (a).

In the situation of (b), the $\mathbb{F}_q$-reciprocal map underlying $\rho$ already corresponds to a unique morphism $f': S \to Q_{V_N} = \text{Proj}(R_{V_N})$ such that $(\mathcal{L}, \rho) \cong (f')^*(\mathcal{O}(1), \rho^\text{univ})$ by Theorem 1.4.6 (b). As $S$ is a scheme over $F$, this $f'$ corresponds to a unique morphism $f'': S \to \text{Proj}(R_{V_N} \otimes_{\mathbb{F}_q} F)$ over $F$. The condition 2.6.1 (b) for $\rho$ then implies that this morphism factors through the closed subscheme $Q_{A,V_N} \subseteq \text{Proj}(R_{V_N} \otimes_{\mathbb{F}_q} F)$ defined by the graded ideal in Construction 2.3.3. This yields the desired morphism $f$, proving (b).

Assertion (c) then follows directly from the construction of $\Omega_{A,V_N}$. □

**Theorem 2.6.5** There is a natural isomorphism

$$M'_{A,N} \cong \Omega_{A,V_N}.$$
Proof. First consider an affine scheme \( S = \text{Spec}(R) \) over \( F \). By Propositions 2.5.4 and 2.5.5 in the case \( W = V^r_N \), giving a triple of the form \((G_{a,S}, \varphi, \lambda)\) consisting of a Drinfeld \( A \)-module and a level \( N \)-structure over \( S \) is equivalent to giving a fiberwise invertible \( A \)-reciprocal map \( \rho: \hat{V}^r_N \to R \). By Theorem 2.3.4 this is in turn equivalent to giving an \( F \)-algebra homomorphism \( f: RS_{AV^r_S} \to R \).

Fix any \( v_0 \in \hat{V}^r_N \). Then in the above equivalences we have \( \lambda(v_0) = 1 \) if and only if \( \rho(v_0) = 1 \) if and only if \( f([\frac{1}{v_0} \otimes 1]) = 1 \). Giving a triple \((G_{a,S}, \varphi, \lambda)\) with \( \lambda(v_0) = 1 \) is thus equivalent to giving an \( F \)-algebra homomorphism \( f: RS_{AV^r_S} \to R \) satisfying \( f([\frac{1}{v_0} \otimes 1]) = 1 \). Since \( RS_{AV^r_S} = RS_{AV^r_S,0}[[\frac{1}{v_0} \otimes 1]] \), the latter is equivalent to giving an \( F \)-algebra homomorphism \( RS_{AV^r_S,0} \to R \). That in turn is equivalent to giving a morphism \( S = \text{Spec}(R) \to \text{Spec}(RS_{AV^r_S,0}) = \Omega_{AV^r_S} \) over \( S \).

On the other hand, by the remark following Theorem 2.1.1, giving a triple \((G_{a,S}, \varphi, \lambda)\) with \( \lambda(v_0) = 1 \) is equivalent to giving an isomorphism class of triples \((E, \varphi, \lambda)\). Therefore \( M'_{r,N} \) and \( \Omega_{AV^r_S} \) represent isomorphic functors on the category of affine schemes over \( F \). By gluing affine schemes it follows that \( M'_{r,N} \) and \( \Omega_{AV^r_S} \) represent isomorphic functors on the category of all schemes over \( F \). This isomorphism of functors induces the desired isomorphism \( M'_{r,N} \cong \Omega_{AV^r_S} \). \( \square \)

Corollary 2.6.6 The ring \( RS_{AV^r_S} \) is a regular integral domain.

Proof. Since \( \text{Spec}(RS_{AV^r_S,0}) \cong M'_{r,N} \) is a regular integral scheme, the ring \( RS_{AV^r_S,0} \) is a regular integral domain. Thus so is \( RS_{AV^r_S} \cong RS_{AV^r_S,0}[[\frac{1}{v_0} \otimes 1]] \) for any \( v_0 \in \hat{V}^r_N \). \( \square \)

2.7 Stratification

Proposition 2.7.1 Consider any injective \( A \)-linear map \( i: V^s_N \hookrightarrow V^r_N \) for \( 1 \leq s \leq r \).

(a) The homomorphism \( \pi_i: R_{AV^s_S} \to R_{AV^r_S} \) from Proposition 2.3.6 induces a closed embedding \( \varepsilon_i: Q_{AV^s_S} \hookrightarrow Q_{AV^r_S} \) whose image is defined by the equations \( \rho^\text{univ}(v) = 0 \) for all \( v \in V^r_N \setminus i(V^s_N) \).

(b) Consider any fiberwise non-zero \( A \)-reciprocal map \( \rho: \hat{V}^r_N \to \mathcal{L}(S) \) over a scheme \( S \) over \( F \). Then the associated morphism \( S \to Q_{AV^r_S} \) factors through \( \varepsilon_i \) if and only if \( \rho = i \circ \rho' \) for an \( A \)-reciprocal map \( \rho': \hat{V}^s_N \to \mathcal{L}(S) \).

Proof. The description in Proposition 2.3.6 shows that \( \pi_i \) is a surjective graded \( F \)-algebra homomorphism whose kernel is generated by the elements \( [\frac{1}{v} \otimes 1] \) for all \( v \in V^r_N \setminus i(V^s_N) \). This directly implies (a). Part (b) follows as in the proof of Proposition 1.2.4 (c). \( \square \)

Remark 2.7.2 The image subscheme \( \varepsilon_i(Q_{AV^r_S}) \) depends only on the submodule \( W := i(V^s_N) \), which can be any non-zero free \( A/N \)-submodule of \( V^r_N \). The same holds for \( \varepsilon_i(\Omega_{AV^r_S}) \), which is the locally closed affine subscheme of \( Q_{AV^r_S} \) that is defined by the equations
\[ \rho_{\text{univ}}(v) = 0 \text{ for all } v \in V_N' \setminus i(V_N') \text{ and the inequalities } \rho_{\text{univ}}(v) \neq 0 \text{ for all } v \in i(V_N'). \]

In the following we abbreviate

\[ (2.7.3) \quad \Omega_W := \varepsilon_i(\Omega_{A,V_N^s}). \]

By Theorem 2.6.5 any isomorphism \( i: V_N^s \rightarrow W \) induces a natural isomorphism

\[ (2.7.4) \quad M_{A,N}^s \cong \Omega_W. \]

**Theorem 2.7.5** The subschemes \( \Omega_W \) for all non-zero free \( A/N \)-submodules \( W \subset V_N^r \) are pairwise disjoint and their union is \( Q_{A,V_N^s} \).

**Proof.** (Compare H"{a}blerl [12 Thm. 8.16].) Consider any point of \( Q_{A,V_N^s} \) over a field \( k \). This point can be represented by a non-zero \( A \)-reciprocal map \( \rho: \tilde{V}_N^r \rightarrow k \) which is unique up to multiplication by \( k^x \). Since \( k \) is a field, this map necessarily satisfies the condition \( 2.5.1 \). The subset \( W \) associated to \( \rho \) by Lemma 2.5.2 is then a free \( A/N \)-submodule of some rank \( 1 \leq s \leq r \) by Proposition 2.5.4 and \( \rho \) is the extension by zero of an invertible \( A \)-reciprocal map \( \rho': \tilde{V}_N^s \rightarrow k \) under an isomorphism \( i: V_N^s \cong W \subset V_N^r \), as in Proposition 2.3.5. This \( \rho' \) then corresponds to a point in \( \Omega_{A,V_N^s} \); hence the original point lies in the stratum \( \Omega_W \). This shows that \( Q_{A,V_N^r} \) is the union of all \( \Omega_W \).

Conversely, the construction of \( \Omega_W \) implies that the points of \( \Omega_W \) over a field \( k \) are precisely those whose associated subset from Lemma 2.5.2 is \( W \). Since \( W \) depends only on the equivalence class of \( \rho \) under multiplication by \( k^x \), it is uniquely associated to the point. This shows that the \( \Omega_W \) are pairwise disjoint. \( \square \)

**Theorem 2.7.6** The open subscheme \( \Omega_{A,V_N^s} \) is dense in \( Q_{A,V_N^s} \).

**Proof.** Consider any point of \( Q_{A,V_N^s} \) over a field \( k \). Suppose that it lies in the stratum \( \Omega_W \) and choose an isomorphism \( i: V_N^s \rightarrow W \). Then the point corresponds to a Drinfeld \( A \)-module \( \varphi \) of rank \( s \) over \( k \) with a level \( N \)-structure \( \lambda: V_N^s \rightarrow k \). Set \( R := k[[x]] \) and \( K := k((x)) \) for a new variable \( x \). By Tate uniformization as in [5, §7] we can deform \( \varphi \) to a Drinfeld \( A \)-module \( \tilde{\varphi} \) of rank \( r \) with coefficients in \( R \) which is congruent to \( \varphi \) modulo \( (x) \). After replacing \( R \) and \( K \) by a finite extension we may without loss of generality assume that \( \tilde{\varphi}[N](K) \cong V_N^r \). Reduction modulo \( (x) \) then induces an isomorphism \( \tilde{\varphi}[N](K) \cap R \cong \varphi[N](k) \). Via \( i \) we can therefore extend \( \lambda \) to a level \( N \)-structure \( \tilde{\lambda}: V_N^r \rightarrow K \) of \( \tilde{\varphi} \).

Consider now the associated invertible \( A \)-reciprocal map \( \tilde{\rho} := (\tilde{\lambda}|V_N^r)^{-1} \). By construction it lands in \( R \), and its reduction modulo \( (x) \) is the non-zero \( A \)-reciprocal map corresponding to our given point of \( \Omega_W \). By the modular interpretation of \( Q_{A,V_N^s} \) in Theorem 2.6.4 we therefore obtain a morphism \( \text{Spec}(R) \rightarrow Q_{A,V_N^s} \) which maps the closed point of \( \text{Spec}(R) \) to the given point of \( \Omega_W \) and the generic point to \( \Omega_{A,V_N^s} \). Thus the given point lies in the closure of \( \Omega_{A,V_N^s} \), as desired. \( \square \)

**Corollary 2.7.7** For any non-zero free \( A/N \)-submodule \( W \subset V_N^r \), the closure of \( \Omega_W \) in \( Q_{A,V_N^s} \) is the union of \( \Omega_{W'} \) for all non-zero free \( A/N \)-submodules \( W' \subset W \).
Proof. Choose any isomorphism $i: V_N \cong W$. Then the closure of $\Omega_{A,V_N}$ in $Q_{A,V_N}$ is $Q_{A,V_N}$ by Theorem 2.7.6. Since $\varepsilon_i$ is a closed embedding, it follows that the closure of $\Omega_W = \varepsilon_i(\Omega_{A,V_N})$ in $Q_{A,V_N}$ is $\varepsilon_i(Q_{A,V_N})$. But $Q_{A,V_N}$ is the union of its strata associated to all non-zero free $A/N$-submodules of $V_N$; and $\varepsilon_i$ maps these strata to the strata $\Omega_W$ associated to all non-zero free $A/N$-submodules of $W$. \qed

2.8 Changing the level

In this subsection we consider two non-zero proper ideals $N \subset N' \subset A$ which both satisfy Assumption 2.2.2.

Proposition 2.8.1 For any $A$-reciprocal map $\rho: \hat{V}_N^r \to R$ or $\rho: \hat{V}_N^r \to \mathcal{L}(S)$,

(a) the restriction $\rho|_{\hat{V}_N^r}$ is an $A$-reciprocal map, and

(b) $\rho$ is fiberwise non-zero, resp. invertible, if and only if $\rho|_{\hat{V}_N^r}$ has the same property.

Proof. Part (a) follows directly from Definition 2.3.1, because for any $a \in \text{Div}(N')$ we have $a \in \text{Div}(N)$ and $V_a^r \subset V_N^r \subset V_{N'}^r$. For (b) it suffices to consider an $A$-reciprocal map $\rho: \hat{V}_N^r \to k$ to a field $k$. By Proposition 2.5.4 the submodule $W \subset V_N^r$ from Lemma 2.5.2 is then a free $A/N$-module of some rank $0 \leq s \leq r$. It follows that $W \cap V_N^r$ is a free $A/N'$-module of the same rank $s$. Thus $\rho$ is non-zero if and only if $s \geq 1$ if and only if $\rho|_{\hat{V}_N^r}$ is non-zero, and $\rho$ is invertible if and only if $s = r$ if and only if $\rho|_{\hat{V}_N^r}$ is invertible. \qed

Construction 2.8.2 By the universal property in Theorem 2.3.4 (b), the restriction of $A$-reciprocal maps corresponds to a natural $F$-algebra homomorphism $R_{A,V_N^r} \to R_{A,V_N}$ which sends $\left[\frac{1}{V} \otimes 1\right]$ to $\left[\frac{1}{V} \otimes 1\right]$ for all $v \in \hat{V}_N^r$. By Construction 2.3.3 this induces a commutative diagram of graded $F$-algebras

\[ \begin{array}{ccc} RS_{A,V_N} & \longrightarrow & R_{A,V_N} \\ \uparrow & & \uparrow \\ RS_{A,V_N^r} & \longrightarrow & R_{A,V_N^r} \end{array} \]

(2.8.3)

Applying Proj as in Construction 2.6.3 this yields a commutative diagram of schemes over $F$

\[ \begin{array}{ccc} M_{A,N}^r & \cong & \Omega_{A,V_N^r} \quad \subset \quad Q_{A,V_N^r} \\ \downarrow & & \downarrow \\ M_{A,N'}^r & \cong & \Omega_{A,V_N'} \quad \subset \quad Q_{A,V_N'} \end{array} \]

(2.8.4)

where the vertical morphism on the left is induced by the restriction of a level $N$-structure on a Drinfeld $A$-module to a level $N'$-structure. The other two vertical morphisms represent
the restriction of isomorphism classes of fiberwise invertible, resp. fiberwise non-zero, $A$-reciprocal maps. Moreover, the ‘if’ part of Proposition 2.8.1 (b) implies that this diagram is cartesian.

**Remark 2.8.5** It is known that $M_N^r \to M_{N'}^r$ is a finite étale Galois covering with Galois group $\text{Ker}(\text{GL}_r(A/N) \to \text{GL}_r(A/N'))$. By Corollary 2.6.6 it follows that $RS_{A,V_N^r} \to RS_{A,V_{N'}^r}$ is a finite étale Galois extension of integral domains. In particular it is injective.

We actually expect that all homomorphisms in the diagram (2.8.3) are injective and that $R_{A,V_N^r} \to R_{A,V_{N'}^r}$ is a finite ring extension, but cannot conclude that (yet) at this point.

### 2.9 Satake compactification

In this subsection we relate $Q_{A,V_N^r}$ with the Satake compactification $\overline{M}_{A,N}^r$ of $M_{A,N}^r$. For this we recall the axiomatic characterization of $\overline{M}_{A,N}^r$ and its properties from [14, §§3-5].

As before we fix an integer $r \geq 1$ and consider a commutative $F$-algebra $R$. Following [14, Def. 3.1], a standard generalized Drinfeld $A$-module of rank $\leq r$ over $R$ is an $\mathbb{F}_q$-algebra homomorphism $\varphi : A \to R[\tau], a \mapsto \varphi_ a$ satisfying for every $a \in A \setminus \{0\}$:

(a) $\varphi_a = \sum_{i=0}^{r \deg a(a)} \varphi_{a,i} \tau^i$ and for every $p \in \text{Spec}(R)$ there exists $i > 0$ with $\varphi_{a,i} \not\in p$, and

(b) $d\varphi_a = \varphi_{a,0} = a$.

More generally consider a scheme $S$ over $F$ and a line bundle $E$ on $S$. An arbitrary generalized Drinfeld $A$-module of rank $\leq r$ over $R$ is an $\mathbb{F}_q$-algebra homomorphism $\varphi : A \to \text{End}_{\mathbb{F}_q}(E), a \mapsto \varphi_a$ which for any trivialization of $E$ over an open affine subscheme becomes a standard generalized Drinfeld $A$-module of rank $\leq r$. Then the fiber over any point of $S$ must be a Drinfeld $A$-module of some rank $1 \leq s \leq r$, but this $s$ can vary over $S$.

An isomorphism of generalized Drinfeld $A$-modules is an isomorphism of line bundles that is compatible with $\varphi$. Following [14, Def. 3.9] we call a generalized Drinfeld $A$-module $(E, \varphi)$ over $S$ weakly separating if, for any Drinfeld $A$-module $(E', \varphi')$ over any field $L$ containing $F$, at most finitely many fibers of $(E, \varphi)$ over $L$-valued points of $S$ are isomorphic to $(E', \varphi')$.

Recall from Theorem 2.1.1 that $M_{A,N}^r$ is an integral affine algebraic variety of finite type over $F$. By [14, Def. 4.1], any open embedding $M_{A,N}^r \to \overline{M}_{A,N}^r$ into a normal integral proper algebraic variety over $F$, such that the universal family $(E^{\text{univ}}, \varphi^{\text{univ}})$ on $\overline{M}_{A,N}^r$ extends to a weakly separating generalized Drinfeld $A$-module $(E^{\text{univ}}, \varphi^{\text{univ}})$ over $\overline{M}_{A,N}^r$, is called a Satake compactification of $M_{A,N}^r$. By abuse of terminology we call $(E^{\text{univ}}, \varphi^{\text{univ}})$ the universal family on $\overline{M}_{A,K}^r$. By [14, Thm. 4.2] such a Satake compactification exists, and it together with its universal family is unique up to unique isomorphism. Moreover, let $\hat{\mathcal{M}}$ denote the dual of the relative Lie algebra of $E^{\text{univ}}$, which is an invertible sheaf on $\overline{M}_{A,N}^r$. Then $\overline{M}_{A,N}^r$ is projective over $F$ and $\hat{\mathcal{M}}$ is ample by [14, Thm. 5.3].

Now we return to $A$-reciprocal maps.
Proposition 2.9.1 Consider any $A$-reciprocal map $\rho: \hat{V}_N^r \to \mathcal{L}(S)$ over $S$. Let $E$ be the line bundle on $S$ whose sheaf of sections is the dual $\mathcal{L}_e$. Then there is a unique $\mathbb{F}_q$-algebra homomorphism $\varphi: A \to \text{End}_{\mathbb{F}_q}(E)$ which for any trivialization $G_{a,U} \xrightarrow{\sim} E|U$ over an open affine subscheme $U = \text{Spec}(R) \subset S$ induces the homomorphism $A \to \text{End}_{\mathbb{F}_q}(E|U) \cong R[\tau]$ from Proposition 2.4.4.

Proof. First consider any $a \in \text{Div}(N)$. Then for any open subscheme $U \subset S$, any section $e \in E(U)$, and any $v \in V_N$, the expression $1 - \rho(v)e$ is a well-defined section of $G_a(U) = \mathcal{O}_S(U)$. Thus the formula (2.9.1) globalizes to a morphism $\varphi_a: E \to E$ over $S$. Since locally over $S$ it is $\mathbb{F}_q$-linear, it defines an element of $\text{End}_{\mathbb{F}_q}(E)$. As $A$ is generated by $\text{Div}(N)$, we obtain $\varphi_a \in \text{End}_{\mathbb{F}_q}(E)$ for all $a \in A$.

We apply Proposition 2.9.1 to the universal $A$-reciprocal map $\rho^\text{univ}: A \to \mathcal{O}(1)(Q_{A,V_N^r})$ from Theorem 2.6.4 and obtain a line bundle $E$ on $Q_{A,V_N^r}$ with an $\mathbb{F}_q$-algebra homomorphism

$$\varphi: A \to \text{End}_{\mathbb{F}_q}(E).$$

Proposition 2.9.3 This $\varphi$ is a weakly separating generalized Drinfeld module of rank $\leq r$.

Proof. (Compare H"{a}berli [12, Cor. 8.21].) For any $a \in \text{Div}(N)$ we have $\dim_{\mathbb{F}_q}(V_a^r) = r \dim_{\mathbb{F}_q}(A/(a)) = r \deg_A(a)$; hence the formula (2.9.1) shows that

$$\varphi_a = \sum_{i=0}^{r \deg_A(a)} \varphi_{a,i} \tau^i.$$

In particular this holds for any element $a$ of the set $D$ from Assumption 2.2.2. As the degree is additive in products, the expansion (2.9.4) follows whenever $a$ is a product of elements of $D$. It also holds for $a = 0$, because the empty sum is zero. We claim that it holds for all $a \in A$.

To prove this we use induction on $\deg_A(a)$. Consider any integer $d \geq 0$ and suppose that (2.9.4) holds for all elements $a \in A$ with $\deg_A(a) < d$. Consider an $a \in A$ with $\deg_A(a) = d$ and choose $b$ as in Assumption 2.2.2 (a). Then (2.9.4) holds for $b$ in place of $a$, and we have $\deg_A(a - b) < \deg_A(a)$. Thus (2.9.4) holds for $a - b$ by the induction hypothesis; and so it holds for $a = (a - b) + b$ as well, finishing the induction proof of the claim.

Next, from (2.7.4) we know that $Q_{A,V_N^r}$ is the union of finitely many strata $\Omega_W$ with isomorphisms $\Omega_W \cong \Omega_{A,V_N^r} \cong M_{A,N}^s$ for varying $1 \leq s \leq r$. Moreover, by construction the pullback of the universal $A$-reciprocal map $\rho^\text{univ}$ from $Q_{A,V_N^r}$ to $\Omega_{A,V_N^r}$ is simply the universal $A$-reciprocal map on $\Omega_{A,V_N^r}$. Thus the pullback of the pair $(E, \varphi)$ is simply the Drinfeld $A$-module of rank $s$ over $\Omega_{A,V_N^r}$ that corresponds to the universal $A$-reciprocal map over $\Omega_{A,V_N^r}$. Since $s \geq 1$, this and the above claim show that $\varphi$ satisfies all conditions for a generalized Drinfeld module of rank $\leq r$. Also, transferring $(E, \varphi)$ to $M_{A,N}^s$ yields the universal Drinfeld $A$-module over $M_{A,N}^s$ with the level $N$-structure removed. As any Drinfeld $A$-module over a field possesses only finitely many level $N$-structures, the universal Drinfeld $A$-module over $M_{A,N}^s$ is weakly separating. As we have only finitely many strata altogether, it follows that $(E, \varphi)$ is weakly separating.
Construction 2.9.5 Let $R_{A,V_N}^{\text{norm}}$ denote the integral closure of $R_{A,V_N}$ in $R_{S,A,V_N}$. As $R_{A,V_N}$ is a normal integral domain by Corollary 2.6.6, so is $R_{A,V_N}^{\text{norm}}$ and we have $R_{A,V_N}^{\text{norm}} \otimes_{R_{A,V_N}} R_{S,A,V_N} = R_{S,A,V_N}$. The ring $R_{A,V_N}^{\text{norm}}$ inherits a grading, so that we can define

$$Q_{A,V_N}^{\text{norm}} := \text{Proj}(R_{A,V_N}^{\text{norm}}).$$

The natural isomorphism $M_{A,N}^r \cong \Omega_{A,V_N} \subset Q_{A,V_N}$ from Theorem 2.6.5 yields an open embedding

$$M_{A,N}^r \hookrightarrow Q_{A,V_N}^{\text{norm}}.$$ 

Proposition 2.9.6 The natural morphism $\pi: Q_{A,V_N}^{\text{norm}} \to Q_{A,V_N}$ is finite and surjective.

Proof. Since $R_{A,V_N}$ is an algebra of finite type over a field, the ring $R_{A,V_N}^{\text{norm}}$ is a finite $R_{A,V_N}$-algebra by Noether’s theorem [6, Thm. 4.14]. Thus $\pi$ is a finite morphism. Therefore its image is closed in $Q_{A,V_N}$. Since this image contains $\Omega_{A,V_N}$, which by Theorem 2.7.6 is dense in $Q_{A,V_N}$, it follows that the image of $\pi$ is $Q_{A,V_N}$. Thus $\pi$ is finite and surjective. □

Theorem 2.9.7 The scheme $Q_{A,V_N}^{\text{norm}}$ is the Satake compactification $\overline{M}_{A,N}^r$ of $M_{A,N}^r$.

Proof. (Compare H"{a}berli [12, Cor. 8.22] ) By construction $Q_{A,V_N}^{\text{norm}}$ is a normal integral proper algebraic variety over $F$ which contains $\Omega_{A,V_N} \cong M_{A,N}^r$ as an open subvariety. Moreover, since $(E, \varphi)$ is weakly separating and $\pi$ is finite, the pullback $\pi^*(E, \varphi)$ is a weakly separating generalized Drinfeld $A$-module which extends the universal family $(E_{\text{univ}}, \varphi_{\text{univ}})$ on $M_{A,N}^r$. By the uniqueness part of [14, Thm. 4.2] it follows that $Q_{A,V_N}^{\text{norm}}$ is the Satake compactification of $M_{A,N}^r$. □

Remark 2.9.8 The computation in H"{a}berli [12] Prop. 7.13, Cor. 7.28] implies that the fiber over every geometric point at the boundary consists of $|\text{Pic}(A)| \cdot |(A/N)^\times/A^\times|$ geometric points. Usually $\pi: Q_{A,V_N}^{\text{norm}} \to Q_{A,V_N}$ is therefore not an isomorphism. A fortiori $R_{A,V_N} \to R_{A,V_N}^{\text{norm}}$ is not an isomorphism in general.

2.10 The ideal of the boundary

For any $1 \leq s < r$ and any injective $A$-linear map $i: V_N^s \hookrightarrow V_N^r$ consider the composite ring homomorphism

$$\tilde{\pi}_i: R_{A,V_N} \longrightarrow R_{A,V_N} \longrightarrow R_{S,A,V_N}$$

where $\pi_i$ is the homomorphism from Proposition 2.3.6. Also, let $R_{A,V_N}^{+,d} := \bigoplus_{d>0} R_{A,V_N,d}^+$ denote the augmentation ideal of $R_{A,V_N}$. We are interested in the ideal

$$(2.10.1) \quad I_{A,V_N} := R_{A,V_N}^{+,d} \cap \bigcap_{s,i} \text{Ker}(\tilde{\pi}_i) \subset R_{A,V_N}^{+,d}.$$
Since $RS_{AV_N}$ is an integral domain, this ideal is reduced. By construction it is graded, so it defines a reduced closed subscheme $\partial\Omega_{AV_N}$ of $Q_{AV_N}$. By Theorem 2.7.3 its complement is $\Omega_{AV_N}$; hence

$$\partial\Omega_{AV_N} = (Q_{AV_N} \setminus \Omega_{AV_N})^{\text{red}}.$$  

We are also interested in the ideal

$$I_{\text{norm}}^{AV_N} := \sqrt{I_{AV_N} \cdot R_{\text{norm}}^{AV_N}} \subset R_{\text{norm}}^{AV_N}.$$  

By construction this is a reduced graded ideal of $R_{\text{norm}}^{AV_N}$. Recall from Theorem 2.9.7 that $Q_{\text{norm}}^{AV_N} = M_{rA,N}$ is the Satake compactification of $M_{rA,N}$. Thus the closed subscheme associated to $I_{\text{norm}}^{AV_N}$ is the reduced subscheme at the boundary

$$\partial M_{rA,N} := (M_{rA,N} \setminus M_{rA,N})^{\text{red}}.$$  

We expect that $I_{AV_N} = I_{\text{norm}}^{AV_N}$. In Subsection 3.4 we will prove this in a special case.

### 2.11 Modular forms and cusp forms

Let $O(1)$ denote the pullback to $M_{rA,N}$ of the very ample invertible sheaf $O(1)$ under the morphism $\pi$ from Proposition 2.9.6. As usual, for any quasicoherent sheaf $F$ on $M_{rA,N}$ and any integer $d$ we set $F(d) := F \otimes O(1)^{\otimes d}$. Let $I \subset O_{M_{rA,N}}$ denote the ideal sheaf of the reduced boundary $\partial M_{rA,N}$ from (2.10.4). In other words it is the ideal sheaf associated to the graded ideal $I_{\text{norm}}^{AV_N} \subset R_{AV_N}^{AV_N}$. For any integer $d$ we call

$$\Gamma(M_{rA,N}, O(d))$$  

the space of modular forms and

$$\Gamma(M_{rA,N}, I(d))$$  

the space of cusp forms of rank $r$ and level $N$ and weight $d$.

For any integer $d$ let $I_{AV_N,d}^{\text{norm}} \subset R_{AV_N,d}^{\text{norm}}$ denote the homogenous parts of degree $d$ of $I_{AV_N}^{\text{norm}} \subset R_{AV_N}^{\text{norm}}$.

**Theorem 2.11.3** For any $d \geq 1$ we have natural isomorphisms

$$R_{AV_N,d}^{\text{norm}} \xrightarrow{\sim} \Gamma(M_{rA,N}, O(d))$$  

and

$$I_{AV_N,d}^{\text{norm}} \xrightarrow{\sim} \Gamma(M_{rA,N}, I(d)).$$  

**Proof.** Since $M_{rA,N}$ is normal with graded coordinate ring $R_{AV_N}^{AV_N}$, by [13, Ch.II, Ex. 5.14(a)] (whose proof does not require that $R_{AV_N}^{AV_N}$ be generated by elements of degree 1) the ring $\bigoplus_{d \geq 0} \Gamma(M_{rA,N}, O(d))$ is the integral closure of $R_{AV_N}^{AV_N}$, and hence equal to $R_{AV_N}^{AV_N}$. This yields the first isomorphism, and that in turn directly implies the second.  

\[\square\]
3 The special case \( A = \mathbb{F}_q[t] \) and \( N = (t^n) \)

### 3.1 Setup

Throughout this section we assume that \( A = \mathbb{F}_q[t] \) and \( N = (t^n) \) for some \( n \geq 1 \). Then

\[
\text{Div}(N) = \{ \alpha t^\nu \mid \alpha \in \mathbb{F}_q^\times, 0 \leq \nu \leq n \},
\]

so it satisfies Assumption 2.2.2 with the subset \( D := \mathbb{F}_q^\times \cup \{ t \} \). To reduce notation we fix \( r \geq 1 \) and abbreviate \( V_n := V_n^{t^n} = (t^{-n}A/A)^{\oplus r} \). By induction on \( \nu \) and a short computation we have:

**Lemma 3.1.1** A map \( \rho: V_n \to R \) is \( A \)-reciprocal if and only if it is \( \mathbb{F}_q \)-reciprocal and satisfies

\[
t\rho(tv) = \sum_{v' \in V_1} \rho(v - v')
\]

for all \( v \in V_n \setminus V_1 \).

Next we abbreviate \( \tilde{R}_n := R_{V_n} \otimes_{\mathbb{F}_q} F \) and let \( J_n \) be its ideal from Construction 2.3.3. As a consequence of Lemma 3.1.1 the ideal \( J_n \) is already generated by the relations

\[
\text{Rel}_v := \frac{1}{tv} \otimes t - \sum_{v' \in V_1} \frac{1}{v - v'} \otimes 1
\]

for all \( v \in V_n \setminus V_1 \). We abbreviate the factor ring as \( R_n := \tilde{R}_n/J_n = R_{A,V_n^{t^n}} \) and denote the projection map by \( \pi: \tilde{R}_n \to R_n \). We also abbreviate \( RS_n := RS_{A,V_n^{t^n}} \). The natural action of \( \text{GL}_r(\mathbb{F}_q[t]/(t^n)) \) on \( V_n \) induces an action on \( \tilde{R}_n \) and \( J_n \) and hence on \( R_n \) and \( RS_n \).

Note that in the case \( n = 1 \) there are no relations (3.1.2); hence \( R_1 = \tilde{R}_1 = R_{V_1} \otimes_{\mathbb{F}_q} F \) and \( RS_1 = RS_{V_1} \otimes_{\mathbb{F}_q} F \), and the schemes \( \Omega_1 \subset Q_1 \) are obtained from \( \Omega_{V_1} \subset Q_{V_1} \) by base change from \( \text{Spec}(\mathbb{F}_q) \) to \( \text{Spec}(F) \). In fact, in this case any \( \mathbb{F}_q \)-reciprocal map is already \( A \)-reciprocal by Lemma 3.1.1.

### 3.2 Description of the ring

The goal of this subsection is to give an explicit description of the ring \( R_n \) for arbitrary \( n \). These results are collected in Theorem 3.2.15, but in order to get there, we must first introduce some auxiliary notation.

Let \( b_1, \ldots, b_r \) denote the standard basis of \( \mathbb{F}_q^{\oplus r} \). Then the elements \( X_{k,\nu} := [t^{-\nu}b_k] \) for all \( 1 \leq k \leq r \) and \( 1 \leq \nu \leq n \) form a basis of \( V_n \) over \( \mathbb{F}_q \). We bring these elements in the order

\[
X_{1,1}, X_{2,1}, \ldots, X_{r,1}, X_{1,2}, \ldots, X_{r,2}, X_{1,3}, \ldots, \ldots, X_{r,n-1}, X_{1,n}, \ldots, X_{r,n}.
\]
Then multiplying $X_{k,\nu}$ by $t$ yields 0 if $\nu = 1$, or an earlier element of the list if $\nu > 1$. In particular $X_{1,n}, \ldots, X_{r,n}$ is a basis of $V_n$ over $A_f(t^n)$. For any $(k, \nu)$ we let $V'_{k,\nu}$ denote the $\mathbb{F}_q$-subspace that is generated by all elements occurring strictly before $X_{k,\nu}$. In other words $V'_{k,\nu}$ is the direct sum of $V_{\nu-1} \subset V_n$ and the $\mathbb{F}_q$-subspace generated by $X_{1,\nu}, \ldots, X_{k-1,\nu}$.

For every $1 \leq k \leq r$ we consider the finite subsets

\[(3.2.2) \quad \widetilde{\Delta}_k := \left\{ \frac{1}{X_{k,n} + w} \otimes 1 \ \bigg| \ w \in \tilde{V}'_{k,n} \right\} \cup \{1\} \quad \text{and} \quad (3.2.3) \quad \tilde{E}_k := \left\{ \frac{1}{X_{k,n} + w} \otimes 1 \ \bigg| \ w \in V'_{k,n} \right\}
\]

of $\tilde{R}_n$, each of cardinality $|\tilde{V}'_{k,n}| = q^{(n-1)+k-1}$. Observe that by unique factorization for polynomials in the variables $X_{k,\nu}$ we have bijective maps

\[\tilde{\Delta}_1 \times \cdots \times \tilde{\Delta}_r \sim \tilde{\Delta}_1 \cdots \tilde{\Delta}_r \subset \tilde{R}_n \quad \text{and} \quad \tilde{E}_1 \times \cdots \times \tilde{E}_r \sim \tilde{E}_1 \cdots \tilde{E}_r \subset \tilde{R}_n.\]

Let $U < \text{GL}_r(\mathbb{F}_q[t]/(t^n))$ be the subgroup of matrices which are congruent modulo $(t)$ to an upper triangular matrix with diagonal entries 1. Viewing $V_n$ as a space of column vectors and letting $\text{GL}_r(\mathbb{F}_q[t]/(t^n))$ act on $V_n$ by left multiplication, this is the subgroup of all $g \in \text{GL}_r(\mathbb{F}_q[t]/(t^n))$ such that $g(X_{k,n}) \in X_{k,n} + V'_{k,n}$ for all $1 \leq k \leq r$. In fact, any independent choice of an element of $X_{k,n} + V'_{k,n}$ for all $k$ corresponds to a unique element of $U$. For the induced action on $\tilde{R}_n$ it follows that $U$ acts transitively on $\tilde{E}_k$ and freely transitively on $\tilde{E}_1 \times \cdots \times \tilde{E}_r$. It also follows that for each $1 \leq k \leq r$ the element

\[(3.2.4) \quad \tilde{f}_k := \sum_{e_k \in \tilde{E}_k} e_k = \sum_{w \in V'_{k,n}} \frac{1}{X_{k,n} + w} \otimes 1 \in \tilde{R}_n
\]

is fixed by $U$. By Theorem [1.3.4] and tensoring with $F$ the elements $\tilde{f}_1, \ldots, \tilde{f}_r$ are algebraically independent over $F$ and the elements of $\tilde{\Delta}_1 \cdots \tilde{\Delta}_r$ are linearly independent over the subring $F[\tilde{f}_1, \ldots, \tilde{f}_r]$. We are interested in the submodule

\[(3.2.5) \quad \tilde{M}_n := \bigoplus_{e \in \tilde{\Delta}_1 \cdots \tilde{\Delta}_r} F[\tilde{f}_1, \ldots, \tilde{f}_r] \cdot e \subset \tilde{R}_n.
\]

**Lemma 3.2.6** The projection $\pi$ induces a surjection $\tilde{M}_n \to R_n$.

**Proof.** The assertion is equivalent to $\tilde{R}_n = \tilde{M}_n + J_n$. By the construction of $J_n$ we must therefore show that for every $f \in \tilde{R}_n$ there exist $m \in \tilde{M}_n$ and elements $g_v \in \tilde{R}_n$ such that

\[(3.2.7) \quad f = m + \sum_{v \in V_n \setminus V_1} g_v \cdot \text{Rel}_v.\]
For this recall that $\tilde{R}_n = R_{V_n} \otimes_{\mathbb{F}_q} F$. Also observe that $\tilde{\Delta}_k = \{ e \otimes 1 \mid e \in \Delta'_k \}$ for the subset $\Delta'_k := \{ \frac{1}{X_{s+n}} \mid w \in V'_{s+n} \} \cup \{ 1 \} \subset R_{V_n}$ and that $\tilde{f}_k = f'_k \otimes 1$ with $f'_k := \sum_{w \in V_{s+n}} \frac{1}{X_{s+n}}$.

Thus $\tilde{M}_n = M'_n \otimes_{\mathbb{F}_q} F$ for the submodule $M'_n := \bigoplus_{e \in \Delta'_k} \mathbb{F}_q[f'_1, \ldots, f'_r] \cdot e \subset R_{V_n}$. Then the variable $t \in F = \mathbb{F}_q(t)$ enters into the equation (3.2.8) only through the relations $\text{Rel}_v$. Rescaling these to

$$\text{Rel}'_v := \text{Rel}_v \cdot t^{-1} = \frac{1}{t^v} \otimes 1 - \sum_{v' \in V_1} \frac{1}{v - v'} \otimes t^{-1},$$

it suffices to show that for every $f' \in R_{V_n}$ there exist elements $m \in M'_n \otimes_{\mathbb{F}_q} F$ and $g_v \in R_{V_n} \otimes_{\mathbb{F}_q} F$ such that

$$(3.2.8) \quad f' \otimes 1 = m + \sum_{v \in V_n \setminus V_1} g_v \cdot \text{Rel}'_v.$$  

This is a vector valued inhomogeneous linear equation with coefficients in the ring $\mathbb{F}_q[t^{-1}]$. In terms of any bases of $R_{V_n}$ and $M'_n$ over $\mathbb{F}_q$, it is equivalent to a system of inhomogeneous linear equations with coefficients in $\mathbb{F}_q[t^{-1}]$. Our job is to find a solution in $\mathbb{F}_q(t)$. But a system of inhomogeneous linear equations over a field has a solution over that field if and only if it has a solution over any overfield. It therefore suffices to find a solution over the completion $\mathbb{F}_q((t^{-1}))$. In fact, we will show that there exists a solution in $\mathbb{F}_q[[t^{-1}]]$.

We first find a solution modulo $(t^{-1})$. For this observe that $\text{Rel}'_v \equiv \frac{1}{t^v} \otimes 1$ modulo $(t^{-1})$, and that as $v$ runs through $V_n \setminus V_1$, the element $v' := tv$ runs through $V_{n-1}$. To solve the problem modulo $(t^{-1})$ we must therefore show that for every $f' \in R_{V_n}$ there exist $m' \in M'_n$ and elements $g'_v \in R_{V_n}$ such that

$$(3.2.9) \quad f' = m' + \sum_{v' \in V_{n-1}} g'_v \cdot \frac{1}{v'}.$$  

This is equivalent to saying that $R_{V_n} = M'_n + J'_n$ for the ideal $J'_n \subset R_{V_n}$ that is generated by the elements $\frac{1}{v'}$ for all $v' \in V_{n-1}$, or again to saying that the induced homomorphism $M'_n \to R_{V_n}/J'_n$ is surjective. But that is guaranteed by Theorem 1.6.5 on identifying the data $(V, f_{s+k}, \Delta_{s+k})$ from Subsection 1.5 with the present $(V_n, f'_k, \Delta'_k)$ and the data $(V_s, J_s, M_s)$ from Subsection 1.6 with the present $(V_{n-1}, J'_n, M'_n)$.

Next consider any $i \geq 1$ and suppose that for all $0 \leq j < i$ we already have $m_j \in M'_n$ and $g_{v,j} \in R_{V_n}$ such that $m = \sum_{j=0}^{i-1} m_j \otimes t^{-j}$ and $g_v = \sum_{j=0}^{i-1} g_{v,j} \otimes t^{-j}$ solve the equation (3.2.8) modulo $(t^{-i})$. Then the left hand side minus the right hand side is congruent to $f'_i \otimes t^{-i}$ modulo $(t^{-i-1})$ for some $f'_i \in R_{V_n}$. Solve the equation (3.2.9) with $(f'_i, m_i, g_{v,i})$ in place of $(f', m', g'_v)$. Then the elements $m = \sum_{j=0}^{i} m_j \otimes t^{-j}$ and $g_v = \sum_{j=0}^{i} g_{v,j} \otimes t^{-j}$ solve the equation (3.2.8) modulo $(t^{-i-1})$.

Before passing to the limit we must take care of one more problem. Namely, observe that $R_{V_n}$ and its submodule $M'_n$ are graded and that the relations $\text{Rel}'_v$ are homogeneous of degree 1. We can therefore decompose the equation (3.2.8) into its homogeneous parts.
In particular we may assume that $f'$ is homogeneous of some degree $d$. Then for any solution of (3.2.8) modulo $(t^{-i})$, replacing $m$ and $g_v$ by their homogeneous parts of degree $d$, respectively $d - 1$, yields another solution modulo $(t^{-i})$. In the inductive process above, we can therefore always arrange that $m_i$ and $g_{v,i}$ are homogeneous of degree $d$, respectively $d - 1$. Then they all lie in fixed finite dimensional $\mathbb{F}_q$-subspaces of $M'_n$ and $R_{V_n}$. This then guarantees that $m := \sum_{j=0}^{\infty} m_j \otimes t^{-j}$ lies in $M'_n \otimes_{\mathbb{F}_q} \mathbb{F}_q[[t^{-1}]]$ and $g_v := \sum_{j=0}^{\infty} g_{v,j} \otimes t^{-j}$ lies in $R_{V_n} \otimes_{\mathbb{F}_q} \mathbb{F}_q[[t^{-1}]]$. In other words, we have found a solution of (3.2.8) with coefficients in $\mathbb{F}_q[[t^{-1}]]$, as desired. 

**Lemma 3.2.10** The elements $\pi(f_1), \ldots, \pi(f_r) \in R_n$ are algebraically independent over $\mathbb{F}_q$.

**Proof.** For any $1 \leq k \leq r$ and $1 \leq \nu \leq n$ consider the element

\begin{equation}
(3.2.11) \quad f'_{k,\nu} := \sum_{w \in V'_{k,\nu}} \frac{1}{X_{k,\nu} + w} \in R_{V_n}.
\end{equation}

If $\nu > 1$, we have $V_1 \subset V'_{k,\nu}$, and then the relations $\text{Rel}_{X_{k,\nu} + w}$ from (3.1.2) show that

\begin{align*}
f'_{k,\nu} \otimes 1 &= \sum_{w \in V'_{k,\nu}, \text{mod } V_1} \frac{1}{X_{k,\nu} + w + \nu' \otimes 1} \\
&= \sum_{w \in V'_{k,\nu}, \text{mod } V_1} \frac{1}{tX_{k,\nu} + tw} \otimes t \\
&= \sum_{w' \in V'_{k,\nu-1}} \frac{1}{X_{k,\nu-1} + w' \otimes t} \\
&= f'_{k,\nu-1} \otimes t
\end{align*}

and hence $\pi(f'_{k,\nu} \otimes 1) = \pi(f'_{k,\nu-1} \otimes t)$. By induction on $\nu$ it follows that $\pi(f'_{k,\nu} \otimes 1) = \pi(f'_{k,1} \otimes t^{\nu-1})$ for all $1 \leq \nu \leq n$. In particular we have $\pi(f_k) = \pi(f'_{k,1} \otimes 1)$. It therefore suffices to show that the elements $\pi(f'_{1,1} \otimes 1), \ldots, \pi(f'_{r,1} \otimes 1)$ are algebraically independent over $\mathbb{F}_q$.

For this observe that (2.8.3) yields a commutative diagram

$$
\begin{array}{c}
RS_n \quad \quad \quad R_n \quad \supset \quad \pi(f'_{k,1} \otimes 1) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
RS_{V_1} \otimes_{\mathbb{F}_q} F = RS_1 \quad \quad R_1 = \tilde{R}_1 \supset f'_{k,1} \otimes 1
\end{array}
$$

Here the lower horizontal arrow is injective by Construction 1.1.4 and the left vertical arrow is injective by Remark 2.8.5. Also, the elements $f'_{1,1} \otimes 1, \ldots, f'_{r,1} \otimes 1$ of $\tilde{R}_1 = R_{V_1} \otimes_{\mathbb{F}_q} F$ are algebraically independent over $\mathbb{F}_q$ by Theorem 1.5.4(a). Together this implies that the images of $\pi(f'_{1,1} \otimes 1), \ldots, \pi(f'_{r,1} \otimes 1)$ in $RS_n$ are algebraically independent over $\mathbb{F}_q$. They are therefore themselves algebraically independent over $\mathbb{F}_q$, as desired. \qed
Lemma 3.2.12 The submodule $\bar{M}_n \subset \bar{R}_n$ is stable under $U$ and its $U$-invariants are

$$\bar{M}_n^U = F[\bar{f}_1, \ldots, \bar{f}_r].$$

Proof. For any $1 \leq k \leq r$ the set $\bar{E}_k$ from (3.2.3) is obtained from the set $\bar{\Delta}_k$ on replacing the element $1 \in \bar{\Delta}_k$ by the element $\frac{1}{X_k} = \bar{f}_k - \sum_{1 \neq e_k \in \bar{\Delta}_k} e_k$. By combining this fact with the decomposition $F[\bar{f}_k] = F \oplus F[\bar{f}_k] \cdot \bar{f}_k$ we find that

$$\bigoplus_{e_k \in \bar{\Delta}_k} F[\bar{f}_k] \cdot e_k = F \oplus F[\bar{f}_k] \cdot f_k \bigoplus_{1 \neq e_k \in \bar{\Delta}_k} F[\bar{f}_k] \cdot e_k$$

$$= F \bigoplus_{e_k \in \bar{E}_k} F[\bar{f}_k] \cdot e_k.$$

Taking the tensor product over $1 \leq k \leq r$ and setting $\bar{E}_I := \prod_{k \in I} \bar{E}_k$, we deduce that

(3.2.13) $$\bar{M}_n = \bigoplus_{e \in \bar{\Delta}_1 \cdots \bar{\Delta}_r} F[\bar{f}_1, \ldots, \bar{f}_r] \cdot e = \bigoplus_{I \subset \{1, \ldots, r\}} \bigoplus_{e \in \bar{E}_I} F[\bar{f}_k |_{k \in I}] \cdot e.$$

Since $U$ permutes each $\bar{E}_k$ and fixes each $\bar{f}_k$, this shows that $U$ acts on $\bar{M}_n$. Also, since $U$ acts transitively on $\bar{E}_1 \cdots \bar{E}_r$, it also acts transitively on $\bar{E}_I$ for each subset $I \subset \{1, \ldots, r\}$. The fact that $\bar{f}_k = \sum_{e \in \bar{E}_k} e$ and the above description of $\bar{M}_n$ therefore implies that

$$\bar{M}_n^U = \bigoplus_{I \subset \{1, \ldots, r\}} F[\bar{f}_k |_{k \in I}] \cdot \sum e_{e \in \bar{E}_I}$$

$$= \bigoplus_{I \subset \{1, \ldots, r\}} F[\bar{f}_k |_{k \in I}] \cdot \prod_{k \in I} \bar{f}_k = F[\bar{f}_1, \ldots, \bar{f}_r],$$

as desired. □

Lemma 3.2.14 The projection $\pi$ induces an isomorphism $\bar{M}_n \xrightarrow{\sim} R_n$.

Proof. By Lemma 3.2.6 the induced map $\pi: \bar{M}_n \rightarrow R_n$ is already surjective. It remains to show that it is injective, or equivalently that its kernel $\bar{M}_n \cap J_n$ is zero. For this observe that $\bar{M}_n$ is stable under $U$ by Lemma 3.2.12 and recall that $J_n$ is stable under $U$ by construction. Thus the intersection $\bar{M}_n \cap J_n$ is also stable under $U$. Since $U$ is a finite group of $q$-power order acting on an $\mathbb{F}_q$-vector space, we have $\bar{M}_n \cap J_n = 0$ if and only if $\bar{M}_n^U \cap J_n = (\bar{M}_n \cap J_n)^U = 0$. We are therefore reduced to showing that $\pi | \bar{M}_n^U$ is injective. But by Lemma 3.2.12 we have $\bar{M}_n^U = F[\bar{f}_1, \ldots, \bar{f}_r]$, and Lemma 3.2.10 implies that $\pi | F[\bar{f}_1, \ldots, \bar{f}_r]$ is injective. □

To simplify notation, we now set $f_k := \pi(\bar{f}_k)$ and $\Delta_k := \pi(\bar{\Delta}_k)$ and $E_k := \pi(\bar{E}_k)$ for all $1 \leq k \leq r$ and abbreviate $E_I := \prod_{k \in I} E_k$ for any subset $I \subset \{1, \ldots, r\}$. Then we can summarize the results of this subsection as follows:

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Theorem 3.2.15  
(a) The elements \( f_1, \ldots, f_r \in R_n \) are algebraically independent over \( F \).

(b) Each \( \Delta_k \) and \( E_k \) is a subset of \( R_n \) of cardinality \( q^{(n-1)+k-1} \).

(c) The product induces bijective maps
\[
\Delta_1 \times \cdots \times \Delta_r \sim \Delta_1 \cdots \Delta_r \subset R_n \quad \text{and} \quad E_1 \times \cdots \times E_r \sim E_1 \cdots E_r \subset R_n.
\]

(d) The ring \( R_n \) is a free module over \( \mathbb{F}[f_1, \ldots, f_r] \) with basis \( \Delta_1 \cdots \Delta_r \).

(e) We have \( R_n^U = \mathbb{F}[f_1, \ldots, f_r] \).

(f) As an \( \mathbb{F} \)-vector space \( R_n \) decomposes as
\[
R_n = \bigoplus_{I \subset \{1, \ldots, r\}} \bigoplus_{e \in E_I} \mathbb{F}[f_k | k \in I] \cdot e.
\]

Proof. Assertion (a) is the content of Lemma 3.2.10, and (d) and the parts of (b) and (c) concerning the subsets \( \Delta_k \) follow from Lemma 3.2.14. Part (e) follows directly from Lemmas 3.2.12 and 3.2.14. The remaining assertions follow by combining Lemma 3.2.14 with the decomposition 3.2.13. \( \square \)

3.3 Consequences

Proposition 3.3.1 We have \( RS_n = R_n[f_1^{-1}, \ldots, f_r^{-1}] \).

Proof. By the construction (2.3.3) of \( R_n \) and \( RS_n \) we have \( RS_n = RS_{V_n} \otimes_{RV_n} R_n \). By Proposition 1.5.5 applied to the elements \( X_k, \nu \in V_n \) in the order 3.2.11, the ring \( RS_{V_n} \) is the localization of \( RV_n \) obtained by inverting the elements \( f_{k, \nu} \) from 3.2.11 for all \( k \) and \( \nu \). Thus \( RS_n \) is the localization of \( R_n \) obtained by inverting the elements \( \pi(f_{k, \nu} \otimes 1) \) for all \( k \) and \( \nu \). But by downward induction on \( \nu \), the equation \( \pi(f_{k, \nu} \otimes 1) = \pi(f_{k, \nu-1} \otimes t) \) from the proof of Lemma 3.2.10 implies that
\[
\pi(f_{k, \nu} \otimes 1) = \pi(f'_{k, \nu} \otimes t^{\nu-n}) = \pi(f_{k}) \cdot t^{\nu-n} = f_k \cdot t^{\nu-n}.
\]
for all \( k \) and \( \nu \). Thus \( RS_n \) is the localization of \( R_n \) obtained by inverting the elements \( f_k \) for all \( k \), as desired. \( \square \)

Theorem 3.3.2 The ring \( R_n \) is an integral domain and injects into \( RS_n \).

Proof. Theorem 3.2.15 (d) implies that \( R_n \) injects into \( R_n[f_1^{-1}, \ldots, f_r^{-1}] \); hence by Proposition 3.3.1 it injects into \( RS_n \). Since \( RS_n \) is an integral domain by Corollary 2.6.6, everything follows. \( \square \)
Theorem 3.3.3  The ring $R_n$ is Cohen-Macaulay.

Proof. By Theorem 3.2.15 the ring $R_n$ is free of finite rank over the polynomial ring $F[f_1, \ldots, f_r]$. Thus $R_n$ has Krull dimension $r$ and the elements $f_1, \ldots, f_r$ form a regular sequence in $R_n$ of length $r$. The same then follows for the localization of $R_n$ at the irrelevant maximal ideal $\bigoplus_{d>0} R_{n,d}$; hence this localization is Cohen-Macaulay. Using [1, Cor. 2.2.15] it follows that the graded ring $R_n$ itself is Cohen-Macaulay.

Theorem 3.3.4  For any $1 \leq n' \leq n$ the natural homomorphism $R_{n'} \to R_n$ from Construction 2.8.2 induces an isomorphism from $R_{n'}$ to the subring of invariants in $R_n$ under the kernel of the natural surjection $GL_r(F_q[t]/(t^n)) \to GL_r(F_q[t]/(t^{n'}))$.

Proof. By induction on $n'$ it suffices to consider the case $n' = n - 1$ with $n \geq 2$. Let $H$ denote the subgroup of $GL_r(F_q[t]/(t^n))$ in question. Then $H$ consists of all elements $h \in U$ such that $h(X_{k,n}) \in X_{k,n} + V_1$ for all $1 \leq k \leq r$. In fact, any independent choice of an element of $X_{k,n} + V_1$ for each $k$ corresponds to a unique element of $H$. Thus we may write $H = H_1 \times \ldots \times H_r$, where each $H_k$ permutes the coset $X_{k,n} + V_1$ simply transitively and fixes $X_{k',n}$ for all $k' \neq k$. Each $H_k$ then also acts trivially on $E_k'$ for all $k' \neq k$. It follows that for any subset $I \subset \{1, \ldots, r\}$, the orbits of $H$ on $E_I := \bigcap_{k \in I} E_k$ are simply the products over $k \in I$ of the orbits of $H_k$ on $E_k$. By the relation (3.1.2), the sum over the $H_k$-orbit of an element $\frac{1}{X_{k,n}+w} \otimes 1 \in E_k$ comes out as

$$\sum_{v' \in V_1} \left[ \frac{1}{X_{k,n}+w-v'} \otimes 1 \right] = \left[ \frac{1}{tX_{k,n}+tw} \otimes t \right] = \left[ \frac{1}{X_{k,n-1}+tw} \otimes 1 \right] \cdot t.$$ 

Here $w$ runs through $V_{k,n}$; hence the element $tw$ runs through $tV'_{k,n} = V'_{k,n-1}$. Thus with

$$E'_k := \frac{1}{X_{k,n-1}+w} \otimes 1 \quad w \in V'_{k,n-1}$$

and $E'_I := \pi(E'_I)$ and $E'_I := \bigcap_{k \in I} E'_k$, the description of $R_n$ in Theorem 3.2.15 (f) implies that

$$(3.3.5) \quad R^H_n = \bigoplus_{I \subset \{1, \ldots, r\}} \bigoplus_{e' \in E'_I} F[|f_i|_{i \in I}] \cdot e'.$$

This coincides with the description of $R_{n-1}$ from Theorem 3.2.15 (f) for $n - 1$ in place of $n$. Indeed, we already have $E'_k \subset \tilde{R}_{n-1} \subset \tilde{R}_n$, and the subset $E'_k \subset R_n$ is precisely the image of the corresponding subset of $R_{n-1}$ under the natural homomorphism $R_{n-1} \to R_n$. Moreover, the equation $\pi(f'_{k,n} \otimes 1) = \pi(f'_{k,n-1} \otimes t)$ from the proof of Lemma 3.2.10 implies that the elements $f_k \in R_n$ and $f_k' \in R_{n-1}$ differ only by a factor of $t$. Thus (3.3.5) implies that the natural homomorphism $R_{n-1} \to R_n$ induces an isomorphism $R_{n-1} \cong R^H_n$, as desired. \qed
3.4 The ideal of the boundary

Consider the ideal \( I_n := I_{A,V_{t,n}} \subset R_n \) from (2.10.1). Our first result is entirely analogous to Theorem 1.7.3.

**Theorem 3.4.1**  
(a) The ideal \( I_n \) is a free module over \( R_n^U \) with basis \( E_1 \cdots E_r \).

(b) The ideal \( I_n \) is a free module over the group ring \( F[U] \).

**Proof.** Theorem 3.2.15 implies that \( U \) acts freely transitively on \( E_1 \cdots E_r \) and that \( R_n^U = F[f_1, \ldots, f_r] \). Theorem 3.2.15 (f) thus shows that \( E_1 \cdots E_r \) is the basis of a free \( R_n^U \)-submodule of \( R_n \). Denoting this submodule by \( M \), it follows that \( M \) is a free module over \( F[U] \). It remains to show that \( M = I_n \).

For this note first that for any element \( e_1 \cdots e_r \in E_1 \cdots E_r \), the reciprocals \( e_1^{-1}, \ldots, e_r^{-1} \) form a basis of \( V_n \) over \( \mathbb{F}_q[t]/(t^n) \). Thus for any \( 1 \leq s < r \) and any \( i : V_n^s \hookrightarrow V_n^r = V_n \) as above, at least one of \( e_1^{-1}, \ldots, e_r^{-1} \) lies in \( V_n^r \backslash i(V_n^s) \). By the description of \( \pi_i \) in Proposition 2.3.6 it follows that \( e_1 \cdots e_r \in \text{Ker}(\pi_i) \). Varying \( s \) and \( i \) this shows that \( e_1 \cdots e_r \in I_n \), and varying \( e_1 \cdots e_r \) then implies that \( M \subset I_n \).

Next observe that \( 0 \to M \to I_n \to I_n/M \to 0 \) is a short exact sequence of \( F[U] \)-modules. Since \( M \) is a free \( F[U] \)-module, taking \( U \)-invariants yields a short exact sequence \( 0 \to M_U \to I_n^U \to (I_n/M)^U \to H^1(U,M) \to 0 \). Also, since \( U \) is a finite group of \( q \)-power order acting on the \( F \)-vector space \( I_n/M \) with \( \mathbb{F}_q \subset F \), we have \( I_n/M = 0 \) if and only if \( (I_n/M)^U = 0 \). To prove that \( M = I_n \), by the short exact sequence it is therefore enough to prove that \( M^U = I_n^U \).

As the given basis \( E_1 \cdots E_r \) of \( M \) over \( R_n^U \) is a single free orbit under \( U \), the submodule \( M^U \) is the free \( R_n^U \)-module generated by the element \( \sum E_1 \cdots E_r = f_1 \cdots f_r \). In other words it is the principal ideal of \( R_n^U \) generated by \( f_1 \cdots f_r \). Since \( R_n^U = F[f_1, \ldots, f_r] \) with algebraically independent \( f_1, \ldots, f_r \), this ideal is the intersection of the ideals \( R_n^U \cdot f_k \) for all \( 1 \leq k \leq r \). Thus it suffices to prove that \( I_n^U \subset R_n^U \cdot f_k \) for every fixed \( 1 \leq k \leq r \).

To achieve this suppose first that \( r = 1 \). Then \( I_n \) is the augmentation ideal of \( R_n \) by construction (2.10.1); hence \( I_n^U \) is the augmentation ideal of \( R_n^U = F[f_1] \) and thus equal to \( R_n^U \cdot f_1 \), and we are done.

Otherwise consider the permutation \( \sigma := (k; k+1, \ldots, r) \) and let \( i : V_n^{r-1} \hookrightarrow V_n^r \) denote the \( \mathbb{F}_q[t]/(t^n) \)-linear embedding defined by \( i(X_{j,n}) = X_{\sigma j,n} \) for all \( 1 \leq j \leq r - 1 \). Then the image of \( i \) is the \( \mathbb{F}_q[t]/(t^n) \)-submodule generated by \( X_{k,n} \) for all \( 1 \leq k' \leq r \) with \( k' \neq k \).

For all \( 1 \leq k' \leq r \), the definition of \( f_{k'} \) shows that

\[
\pi_i(f_{k'}) = \sum_{w \in V_{k',n}} \pi_i \left( \frac{1}{X_{k,n} + w} \otimes 1 \right).
\]

In the case \( k' = k \) we have \( X_{k,n} + w \notin i(V_n^{r-1}) \) for all \( w \in V_{k,n} \) and therefore \( \pi_i(f_k) = 0 \). Otherwise we have \( k' = \sigma j \) for some \( 1 \leq j \leq r - 1 \) and the description of \( \pi_i \) in Proposition 2.3.6 implies that

\[
\pi_i(f_{k'}) = \sum_{w' \in i^{-1}(V_{k',n})} \left[ \frac{1}{X_{j,n} + w'} \otimes 1 \right].
\]

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But the definition of \( V_{k',n} \) implies that \( i^{-1}(V_{k',n}) \) is precisely the subspace \( V'_{r,n} \subset V'_{r-1} \) for \( r-1 \) in place of \( r \). Thus the elements \( \pi_i(f_{k'}) \in R_{A,V'_{r-1}} \) for all \( k' \neq k \) are precisely the elements \( f_1, \ldots, f_{r-1} \in R_{A,V'_{r-1}} \). As they are algebraically independent over \( F \) by Theorem \( 3.2.15 \) (a), it follows that the kernel of \( \pi_i[F[f_1, \ldots, f_r]] \) is the ideal \( F[f_1, \ldots, f_r] \cdot f_k \). By construction \( I_n^U = I_n \cap R_n^U \) is contained in this kernel; hence we are done. \( \square \)

Next consider the integral closure \( R_n^{\text{norm}} := R_{A,V'_{r,n}}^{\text{norm}} \) from Construction \( 2.9.5 \) and its ideal \( I_n^{\text{norm}} := I_{A,V'_{r,n}}^{\text{norm}} \) from \( (2.10.3) \).

**Theorem 3.4.2** The homomorphism \( R_n \to R_n^{\text{norm}} \) induces an isomorphism \( I_n \cong I_n^{\text{norm}} \).

**Proof.** Theorem \( 3.3.2 \) implies that \( R_n \) injects into \( R_n^{\text{norm}} \), hence \( I_n \) injects into \( I_n^{\text{norm}} \). We identify it with its image and must then show that \( I_n = I_n^{\text{norm}} \).

For this observe that \( 0 \to I_n \to I_n^{\text{norm}} \to I_n^{\text{norm}}/I_n \to 0 \) is a short exact sequence of \( F[U] \)-modules. Since \( I_n \) is a free \( F[U] \)-module by Theorem \( 3.4.1 \) (b), taking \( U \)-invariants yields a short exact sequence \( 0 \to I_n^U \to (I_n^{\text{norm}})^U \to (I_n^{\text{norm}}/I_n)^U \to H^1(U, I_n) = 0 \). Also, since \( U \) is a finite group of \( q \)-power order acting on the \( F \)-vector space \( I_n^{\text{norm}}/I_n \) with \( \mathbb{F}_q \subset F \), we have \( I_n^{\text{norm}}/I_n = 0 \) if and only if \( (I_n^{\text{norm}}/I_n)^U = 0 \). To prove that \( I_n = I_n^{\text{norm}} \), by the short exact sequence it is therefore enough to prove that \( I_n^U = (I_n^{\text{norm}})^U \).

We will determine both sides of this equation separately. In the proof of Theorem \( 3.4.1 \) we have already seen that \( I_n^U = M^U = R_n^U \cdot f_1 \cdots f_r \). Also, Proposition \( 3.3.1 \) shows that the ideal \( R_n \cdot f_1 \cdots f_r \subset R_n \) defines a closed subscheme of \( Q_{A,V_n} \) whose support is precisely the boundary. This implies that \( I_n = \sqrt{R_n \cdot f_1 \cdots f_r} \) within \( R_n \). By \( (2.10.3) \) we therefore have

\[
I_n^{\text{norm}} = \sqrt{I_n^{\text{norm}} \cdot f_1 \cdots f_r}
\]

within \( R_n^{\text{norm}} \). This in turn implies that

\[
(I_n^{\text{norm}})^U = \sqrt{(I_n^{\text{norm}})^U \cdot f_1 \cdots f_r}
\]

within \( (R_n^{\text{norm}})^U \). But the construction of \( R_n^{\text{norm}} \) implies that \( (R_n^{\text{norm}})^U \) is the integral closure of \( R_n^U \) in \( R S_n^U \). Since \( R_n^U \) is already a regular integral domain by Theorem \( 3.2.15 \) we deduce that \( (R_n^{\text{norm}})^U = R_n^U \). As \( R_n^U \cdot f_1 \cdots f_r \) is already a reduced ideal in \( R_n^U = F[f_1, \ldots, f_r] \), it follows that \( (I_n^{\text{norm}})^U = R_n^U \cdot f_1 \cdots f_r = I_n^U \), as desired. \( \square \)

**Remark 3.4.3** Theorem \( 3.4.2 \) says that the ideals of the boundary in the graded coordinate rings of \( Q_{A,V_n} \) and \( Q_{A,V_n}^{\text{norm}} = \mathcal{M}_{A,t,n} \) are the same. Thus, in a sense \( Q_{A,V_n} \) and \( \mathcal{M}_{A,t,n} \) differ only in the reduced subschemes at the boundary in that some points are identified. More precisely I expect that \( Q_{A,V_n} \) is the quotient of \( \mathcal{M}_{A,t,n} \) by the resulting equivalence relation on the underlying topological space. Compare Remark \( 2.9.8 \).
3.5 Modular forms and cusp forms

For any integer $d$ let $I_{n,d} \subset R_{n,d}$ denote the homogeneous parts of degree $d$ of $I_n \subset R_n$.

**Theorem 3.5.1** For any integer $d \geq 1$ we have

$$\dim_F(R_{n,d}) = \sum_{\emptyset \neq I \subset \{1, \ldots, r\}} \frac{d-1}{|I|-1} \cdot \prod_{k \in I} q^{r(n-1)+k-1}.$$ 

**Proof.** By Theorem 3.2.15 (a), for any subset $I \subset \{1, \ldots, r\}$ the ring $F[f_k|_{k \in I}]$ is isomorphic to a polynomial ring in $|I|$ variables over $F$. Since each $f_k$ is homogeneous of degree 1 and $d \geq 1$, the homogeneous part of $F[f_k|_{k \in I}]$ of degree $d - |I|$ is zero if $I = \emptyset$ and has dimension $\left(\frac{d-1}{|I|-1}\right)$ otherwise. Also recall that each element of $E_k$ is homogeneous of degree 1. The formula follows by combining all this with Theorem 3.2.15 (b) and (c) and (f). \(\square\)

**Remark 3.5.2** Since $R_{n,d} \leftrightarrow R_{n,d}^{\text{norm}}$ is not an isomorphism in general, Theorems 3.5.1 and 2.9.7 together do not yet give us a dimension formula for spaces of modular forms. For cusp forms, on the other hand, we succeed using Theorem 3.4.2.

**Theorem 3.5.3** For any $d \geq 1$ the space $I_{n,d}$ is free module of rank $\binom{d-1}{r-1}$ over the group ring $F[U]$. 

**Proof.** By Theorem 3.2.15 the ring $R_n^U = F[f_1, \ldots, f_r]$ is isomorphic to a polynomial ring in $r$ variables over $F$. Since each $f_k$ is homogeneous of degree 1 and $d \geq 1$, the homogeneous part of $R_n^U$ of degree $d - r$ has dimension $\binom{d-1}{r-1}$. Also recall that each element of $E_k$ is homogeneous of degree 1 and that $U$ acts freely transitively on $E_1 \cdots E_r$ by Theorem 3.2.15 (b) and (c). Thus the formula follows from Theorem 3.4.2. \(\square\)

**Theorem 3.5.4** For any $d \geq 1$ the space of cusp forms $\Gamma(\overline{M}_{r,A,U}, \mathcal{I}(d))$ is a free module of rank $\binom{d-1}{r-1}$ over the group ring $F[U]$.

**Proof.** Combine Theorems 2.9.7 and 2.11.3 and 3.4.2 and 3.5.3. \(\square\)

Finally consider any subgroup $U' \subset U$. Then $U' \backslash \overline{M}_{r,A,U}$ is the Satake compactification of a Drinfeld moduli space of some intermediate level associated to $U'$, and $\Gamma(\overline{M}_{r,A,U}, \mathcal{I}(d))^{U'}$ is the space of cusp forms of weight $d$ of that level. From Theorem 3.5.4 we directly deduce:

**Theorem 3.5.5** For any $d \geq 1$ we have

$$\dim_F \Gamma(\overline{M}_{r,A,U}, \mathcal{I}(d))^{U'} = [U:U'] \cdot \binom{d-1}{r-1}.$$ 

We can also turn this into a dimension formula for analytic cusps forms, as follows. Take the natural homomorphism $\kappa: \text{SL}_r(F_q[t]) \to \text{GL}_r(F_q[t]/(t^n))$ and consider the arithmetic subgroups $\Gamma(t^n) := \ker(\kappa)$ and $\Gamma_1(t) := \kappa^{-1}(U)$ of $\text{SL}_r(F_q[t])$. Consider an arbitrary subgroup $\Gamma(t^n) < \Gamma < \Gamma_1(t)$. Let $C_\infty$ denote the completion of an algebraic closure of the field $F_q((t^{-1}))$. Let $S_d(\Gamma)$ denote the space of analytic cusps forms of rank $r$ and weight $d$ and level $\Gamma$ according to [1, Def. 6.1].
Theorem 3.5.6  For any \( d \geq 1 \) we have
\[
\dim_{C^\infty} \mathcal{S}_d(\Gamma) = [\Gamma_1(t) : \Gamma] \cdot \left( \frac{d-1}{r-1} \right).
\]

Proof. We must identify the analytic cusp forms with algebraic cusp forms as in [2]. For this let \( \hat{A} \cong \prod_p A_p \) denote the profinite completion of \( A \). Let \( K(t^n) \) be the kernel of the natural homomorphism \( \hat{\kappa} : \text{GL}_r(\hat{A}) \to \text{GL}_r(\mathbb{F}_q[t]/(t^n)) \) and set \( K := \Gamma \cdot K(t^n) \). Then \( \hat{\kappa} \) induces isomorphisms \( \Gamma/K(t^n) \cong K/K(t^n) \cong U' \) for a certain subgroup \( U' \) of the group \( U < \text{GL}_r(\mathbb{F}_q[t]/(t^n)) \) from above. In the case \( \Gamma = \Gamma_1(t) \) this subgroup is \( U := \text{SL}_r(\mathbb{F}_q[t]/(t^n))) \), whose index in \( U \) is \( q^{n-1} \). In the general case we therefore have
\[
[U : U'] = q^{n-1} \cdot [\Gamma_1(t) : \Gamma].
\]

Next, the fact that \( \hat{\kappa}(K(t^n)) < \hat{\kappa}(K) < U \) shows that \( K(t^n) \) and \( K \) are fine open compact subgroups in the sense of [14, Def. 1.4]. The associated Drinfeld moduli spaces from [14] are \( \mathcal{M}_{r,A,K}(t^n) = \mathcal{M}_{r,A} \) and \( \mathcal{M}_{r,K} = U' \mathcal{M}_{r,A} \). Moreover \( \det(K) = \det(K(t^n)) \) is the kernel of the natural homomorphism \( \hat{\Lambda}^\times \to (\mathbb{F}_q[t]/(t^n))^\times \). Using [2, Prop. 8.7] this implies that \( \mathcal{M}_{A,K} = \mathcal{M}_{A}(t^n) \). The main point is that [2, Thm. 10.9] yields an isomorphism between algebraic and analytic modular forms
\[
\Gamma(\mathcal{M}_{r,A}(t^n), \mathcal{O}(d)) \otimes_{F_K} C^\infty \cong \mathcal{M}_d(\Gamma(t^n)).
\]
This induces an isomorphism between algebraic and analytic cusp forms
\[
\Gamma(\mathcal{M}_{r,A}(t^n), \mathcal{I}(d)) \otimes_{F_K} C^\infty \cong \mathcal{S}_d(\Gamma(t^n)).
\]
By taking \( U' \)-invariants we obtain an isomorphism
\[
\Gamma(\mathcal{M}_{r,A}(t^n), \mathcal{I}(d))^{U'} \otimes_{F_K} C^\infty \cong \mathcal{S}_d(\Gamma(t^n))^{U'} = \mathcal{S}_d(\Gamma).
\]
Using Theorem 3.5.5 we deduce that
\[
\dim_{C^\infty} \mathcal{S}_d(\Gamma) = \dim_{F_K} \Gamma(\mathcal{M}_{r,A}(t^n), \mathcal{I}(d))^{U'} = [F_K/F]^{-1} \cdot \dim_{F} \Gamma(\mathcal{M}_{r,A}(t^n), \mathcal{I}(d))^{U'} = q^{1-n} \cdot [U : U'] \cdot \left( \frac{d-1}{r-1} \right) = [\Gamma_1(t) : \Gamma] \cdot \left( \frac{d-1}{r-1} \right),
\]
as desired. \( \square \)
References

[1] D. J. Basson, F. Breuer, R. Pink: Analytic Drinfeld Modular Forms of Arbitrary Rank, Part I: Analytic Theory. Preprint May 2018 24p.

[2] D. J. Basson, F. Breuer, R. Pink: Analytic Drinfeld Modular Forms of Arbitrary Rank, Part II: Comparison with Algebraic Theory. Preprint May 2018 29p.

[3] D. J. Basson, F. Breuer, R. Pink: Analytic Drinfeld Modular Forms of Arbitrary Rank, Part III: Examples. Preprint May 2018 30p.

[4] Bruns, W., Herzog, J.: Cohen-Macaulay Rings, Cambridge: Cambridge Univ. Press (1996).

[5] Drinfeld, V. G.: Elliptic modules (Russian), Mat. Sbornik 94 (1974), 594–627 translated in Math. USSR Sbornik 23 (1974), 561–592.

[6] Eisenbud, D.: Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics 150. New York: Springer 1995.

[7] Gekeler, E.-U.: Drinfeld Modular Curves. Lecture Notes in Mathematics 1231, Springer (1986).

[8] Gekeler, E.-U.: On power sums of polynomials over finite fields. J. Number Theory 30 (1988), 11–26.

[9] Gekeler, E.-U.: On Drinfeld modular forms of higher rank IV: Modular forms with level. Manuscript Nov. 2018, https://arxiv.org/abs/1811.09460

[10] Goss, D.: The algebraist’s upper half-plane. Bull. Amer. Math. Soc. 2 (1980), no. 3, 391–415.

[11] Goss, D.: Basic structures in function field arithmetic, Springer-Verlag, 1996.

[12] Häberli, S.: Satake compactification of analytic Drinfeld modular varieties, Ph.D. thesis, ETH Zürich 2018.

[13] Hartshorne, R.: Algebraic Geometry. GTM 52, Springer-Verlag, New York Heidelberg Berlin 1977

[14] Pink, R.: Compactification of Drinfeld modular varieties and Drinfeld modular forms of arbitrary rank. Manuscripta Math., 140 Issue 3-4 (2013), 333–361.

[15] Pink, R., Schieder, S.: Compactification of a period domain associated to the general linear group over a finite field. J. Algebraic Geometry, 23 (2014), no. 2, 201–243.