Critical and off-critical properties of the $XXZ$ chain in external homogeneous and staggered magnetic fields

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Abstract

The phase diagram of the $XXZ$ chain under the influence of homogeneous and staggered magnetic fields is calculated. The model has a rich phase structure with three types of phases. A fully ferromagnetic phase, an antiferromagnetic phase and a massless phase with partial ferromagnetic and antiferromagnetic order. This massless phase has its critical fluctuations governed by a conformal field theory with central charge $c = 1$. When the $\sigma_z$-anisotropy $\Delta$ is zero the model is exactly integrable through a Jordan-Wigner fermionization and our results are analytic. For $\Delta \neq 0$ our analysis are done numerically using lattice sizes up to $M = 20$. Our results shows that for $-1 \geq \Delta \geq \frac{\sqrt{2}}{2}$ the staggered magnetic field perturbation is relevant and for $1 \leq \Delta \leq \frac{\sqrt{2}}{2}$ it is irrelevant. In the region of relevant perturbations we show that the massive continuum field theory associated to the model has the same mass spectrum as the sine-Gordon model.
1 Introduction

The anisotropic Heisenberg model with spin $S = 1/2$, or $XXZ$ chain, is one of the most studied quantum spin system in Statistical Mechanics. Since its exact integrability by Yang and Yang\cite{1} this model is considered as the classical example of success of the Bethe ansatz. The $XXZ$ chain give us the first example of a critical line with critical exponents varying continuously with the anisotropy. More recently \cite{2,3} with the advance of the conformal invariance ideas\cite{4}, the whole operator content of this model was obtained. The critical fluctuations along this critical line is governed by a conformal field theory with conformal anomaly $c = 1$ and moreover their currents satisfy an $U(1)$ Kac-Moody algebra\cite{5}.

The effect of an uniform magnetic field in the phase diagram of the $XXZ$ chain\cite{6} (see Fig. 1) is to extend the critical phase over a finite region delimited by the critical line of Pokrovsky-Talapov (P.T. line in Fig. 1)\cite{7}, where the chain becomes fully ferromagnetic. Finite-size studies of this extended massless phase\cite{8} showed that it is also described by a $c = 1$ theory with critical exponents now depending on the anisotropy and magnetic field. In the antiferromagnetic region ( $\Delta < -1$ in Fig. 1 ), where the model is massive (noncritical), as the magnetic field increases it destroys the massive phase entering again in the massless phase.

Since the uniform magnetic field does not destroy the exact integrability of the quantum chain, the eigenspectra in the whole extended massless phase is exact solvable. This is not the case when we introduce other perturbations, like a thermal perturbation (which is related to a dimerization in the model) or an staggered magnetic field, where exact integrability is lost. Some years ago Den Nijs\cite{9}, by exploring the connection between the 8-vertex model and the Gaussian model, conjectured that a staggered magnetic field could be a relevant or irrelevant perturbation depending upon the anisotropy strength. Consequently in the region where this last perturbation is relevant we should expect an apposite effect as that of the uniform magnetic field, since it will bring the system into a massive phase. Motived by this we study in this paper the phase diagram and critical properties of the $XXZ$ chain when both magnetic fields are present. The $XXZ$ Hamiltonian with these fields is
given by

\[ H(\Delta, h, h_s) = -\frac{1}{2} \sum_{i=1}^{M} \left[ \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z + 2 \left( h + (-1)^i h_s \right) \sigma_i^z \right]; \]

where \( \sigma_i^x, \sigma_i^y \) and \( \sigma_i^z, i = 1, 2, ..., M \) are Pauli matrices attached at the \( M \) sites of the chain, \( \Delta \) is the \( \sigma^z \)-anisotropy parameter and \( h, h_s \) are the uniform and staggered fields, respectively. In this paper we assume periodic boundary conditions in (1), and the lattice size \( M \) as an even number.

Using all the machinery coming from finite-size scaling\[10\] and conformal invariance\[4\], we will calculate the critical lines, exponents as well the masses appearing in the underlying massive field theory around relevant perturbations.

The paper is organized as follows. In Section 2 we review the conformal invariance relations relevant for our purposes and the operator content in the absence of magnetic fields. In Section 3 we consider the cases where \( h_s \neq 0 \). Initially we consider the case where \( \Delta = 0 \) or the \( XY \) chain. This case is special since the eigenspectra can be calculated analytically through a Jordan-Wigner fermionization of the Hamiltonian. The general situation where \( \Delta \neq 0 \) is studied numerically. In Section 4 we conclude with a general discussion of our results.

2 Conformal invariance relations and the operator content when \( h_s = 0 \)

Like most statistical mechanics systems in 1 + 1 dimensions we assume that the Hamiltonian (1) in its massless regime is conformally invariant. Under this assumption for each operator\[4, 11\] \( O_\alpha \) with dimension \( x_\alpha \) and spin \( s_\alpha \) in the operator algebra of the infinite system, there exists an infinite tower of states in the quantum Hamiltonian, in a periodic chain of \( M \) sites, whose energy and momentum as \( M \to \infty \) are given by

\[ E_{j,j'}^\alpha(M) = E_0(M) + \frac{2\pi v}{M} (x_\alpha + j + j') + o(M^{-1}) \]

and

\[ P_{j,j'}^\alpha(M) = \frac{2\pi}{M} (s_\alpha + j - j') ; \]
where $j, j' = 0, 1, \ldots$, $E_0(M)$ is the ground-state energy and $v$ is the velocity of sound that can be determined by the dispersion relation of the spectra or by the difference among energy levels belonging to the same conformal tower.

The conformal anomaly $c$ can be obtained from the finite-size corrections of the ground-state energy. For periodic chains, the ground-state energy behaves asymptotically as \[ E_0(M) \sim e_\infty - \frac{\pi c v}{6M^2} + o(M^{-2}) \]

where $e_\infty$ is the ground-state energy per site in the bulk limit.

The above relations has been applied successfully to a large number of statistical mechanics systems. In the case of the $XXZ$ Hamiltonian with no external fields, $H(\Delta, h = 0, h_s = 0)$ and $-1 \leq \Delta \leq 1$ these relations give us the whole operator content of the underlying field theory \cite{2, 3, 5}. Along the critical line $-1 \leq \Delta \leq 1$ the central charge is $c = 1$ and the anomalous dimensions appearing in the model are given by integer numbers or by

$$x_{n,m} = n^2 x_p + \frac{m^2}{4x_p}$$ (5)

where $x_p = \frac{\pi - \cos^{-1}(-\Delta)}{2\pi}$ and $n$ and $m$ are integers. The dimension $x_{0,1} = x_p$ is the same as that of the polarization operator in the six vertex model. Moreover we can show \cite{5}, by combining the integer dimensions with Eq. (5), the model is governed by a conformal field theory satisfying a larger algebra than the Virasoro conformal one, namely, a $U(1)$ Kac-Moody algebra. This $U(1)$ symmetry comes from the commutation of the Hamiltonian \cite{4} with the $z$ component of the total spin \cite{13}

$$S_z = \frac{1}{2} \sum_i^M \sigma_i^z = n \, .$$ (6)

For a given sector of the Hilbert space where the total spin $S_z = n$ it will corresponds a set of primary operators $O_{n,m}$ with dimensions $x_{n,m}$ ($m = 0, \pm 1, \pm 2, \ldots$) and the number of descendants operators (with dimensions $x_{n,m} + j + j'$, $j, j' \in \mathbb{Z}$) will be given by the product of two $U(1)$ Kac-Moody characters \cite{5}. The operators $O_{n,m}$ can be interpreted, in a Gaussian language \cite{14}, as the operators with vorticity $n$ and spin wave excitation number $m$. The critical exponents are obtained from the dimensions \cite{5}. For
example the exponents $\eta_x$ and $\eta_z$ governing the correlations

$$<\sigma^x(0)\sigma^x(r)> \sim r^{-\eta_x}$$

(7)

$$<\sigma^z(0)\sigma^z(r)> \sim r^{-\eta_z}$$

(8)

are related to the lowest eigenenergies in the sector with $S_z = 1$ and $S_z = 0$, respectively. The exponents are given by $\eta_x = 2x_{0,1} - \pi \cos^{-1}(-\Delta)$ and $\eta_z = \eta_x^{-1}$ for $0 \geq \Delta \geq -1$ and $\eta_z = 2$ for $1 \geq \Delta \geq 0$, since in this last region of anisotropies the lowest dimension above zero is one.

All the above results were obtained for $h_s = h = 0$ and by exploring the exact integrability of the model through the Bethe ansatz, which numerically[2] or analytically[3] produced very precise results. Such integrability is not destroyed by the introduction of the uniform magnetic field and precise results can also be obtained in this case. In the case where $h_s = 0$, $h \neq 0$, as we see in Fig. 1, the massless line extends forming a massless phase. This massless phase is ordered ferromagnetically and for a given value of the anisotropy $\Delta$ the magnetization changes from zero to its maximum value where the system is fully ordered. The points where the magnetization reaches its maximum value form the Pokrovsky-Talapov transition line[7], which separate the massive fully ferromagnetic phase from the massless partially ordered ferromagnetic phase.

The physical mechanism producing the extended massless phase in Fig. 1 is due to the following. From Eqs. (1) and (6) the magnetic field only decreases the eigenenergies in the sectors where $n > 0$. In the critical line $-1 \leq \Delta \leq 1$ and $h = 0$ the ground-state belongs to the sector $n = 0$. Since we have no gap a small but finite positive magnetic field will lead the ground-state into a sector with $n > 0$, producing a partially ordered ferromagnet phase. However the fluctuations around this ordered phase are of the same nature as those appearing in the absence of the magnetic field ($h = 0$). This argument imply that we should expect in the whole massless phase a $c = 1$ Gaussian field theory with the anomalous dimensions like those given in Eq. (3). In fact this fact was observed analytically[8] for $-1 \leq \Delta \leq 1$ and numerically[13] in the whole extended phase. In Refs. [8], although a closed analytic form for the dimensions in the extended massless phase was not derived, the dimensions are shown to be of Gaussian type like in (3), but with $x_p$ now depending on $\Delta$ and $h$. The case where $\Delta = 0$ is special since only in this case the dimensions do not depend on the magnetic field $h$. 

4
The finite-size analysis of the Hamiltonian (1) with a nonzero uniform magnetic field deserves some comments since for a fixed value of \( h \) the ground-state changes sector as the lattice size increase. In order to keep the bulk limit physics (\( M \to \infty \)) we should keep the magnetization per particle fixed in the finite chains. This imply that for a given magnetic field only a sequence of lattice sizes producing the fixed magnetization, related to \( h \), should be used.

The transition line of Pokrovsky-Talapov type in Fig. 1 is obtained by calculating the lowest value of the magnetic field where the ground-state is the ferromagnetic state (sector \( n = M/2 \)). This magnetic field is \( h = 1 - \Delta \) for arbitrary values of the lattice size \( M \).

In the following sections we will calculate the effects where the staggered and the uniform magnetic field are included.

3 Results for \( h_s \neq 0 \)

We now analyze the situation where both magnetic fields \( h, h_s \) are nonzero. The staggered magnetic field for \( h = 0 \) produces an antiferromagnetic ordered ground-state. The ground-state belongs, as when \( h_s = 0 \), to the sector with \( n = 0 \). In this section we are going to study analytically for \( \Delta = 0 \), and numerically for arbitrary \( \Delta \) the nature of the resulting phase when \( h_s \) is introduced. Irrespective of the \( h_s \) value we do expect that for sufficiently high values of \( h \) (\( h >> h_s \)) the same mechanism discussed in last section which changes the ground-state sector will also take place. Consequently we should have a phase transition surface of Pokrovsky-Talapov type where the ground-state becomes the fully ordered ferromagnetic state. The magnetic field \( h(\Delta, h_s) \) at this surface, as in the previous section, is calculated by imposing the ground-state as the fully ordered ferromagnet state. This critical surface

\[
h = h_{cPT}^T = \sqrt{h_s^2 + 1} - \Delta.
\]

is obtained by equating the lowest eigenenergy \( E_1 = -\Delta^2 (M - 4) - h(M - 2) - 2\sqrt{h_s^2 + 1} \), in the sector with one spin down with the ferromagnetic energy (zero spin down) \( E_0 = -(\Delta^2 + h)M \). For \( h > h_{cPT}^T \) the ground-state of \( H(\Delta, h, h_s) \) is the ferromagnetic state. In order to study the cases where \( h < h_{cPT}^T \) we will consider first the special case where \( \Delta = 0 \).
3.1 $\Delta = 0$

In this case the Hamiltonian $H(\Delta, h, h_s)$ can be diagonalized through a Jordan-Wigner transformation \cite{16}. This is done by introducing fermionic operators

$$c_j = \prod_{l=1}^{j} e^{i\pi\sigma^+_l \sigma^-_l} \sigma^-_j, \quad c_j^\dagger = \sigma^+_j \prod_{l=1}^{j} e^{-i\pi\sigma^+_l \sigma^-_l}$$

(10)

where $\sigma^\pm_j = \frac{1}{2}(\sigma^x_j \pm i\sigma^y_j)$ for $j = 1, 2, ..., M$. It is easy to check that these operators satisfy the fermionic algebra

$$\{c_j, c_{j'}^\dagger\} = \delta_{j,j'} \quad \{c_j^\dagger, c_{j'}\} = \{c_j, c_{j'}^\dagger\} = 0.$$  

(11)

In terms of these operators the conserved $U(1)$ charge is given by

$$S_z = \frac{1}{2} \sum_{i=1}^{M} \sigma^z_i = \sum_{i=1}^{M} \sigma^+_i \sigma^-_i - \frac{M}{2} = \sum_{i=1}^{M} c_i^\dagger c_i - \frac{M}{2}$$

(12)

which imply that the Hilbert space associated to $H(0, h, h_s)$ can be separated into block disjoint sectors labelled according to the fermionic number

$$N_F = \sum_{i=1}^{M} c_i^\dagger c_i = \frac{2n + M}{2}.$$  

(13)

For the sake of simplicity let us restrict to the case where $M$ is multiple of 4, so that $N_F$ and $n$ have the same parity. For a given number of fermions $N_F$ the Hamiltonian is given by

$$H(0, h, h_s) = \sum_{i=1}^{M-1} \left[ - (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + (-1)^i h_s \left( 2c_i^\dagger c_i - 1 \right) \right]$$

$$-2nh + e^{i\pi N_F} \left( c_M^\dagger c_1 + c_1^\dagger c_M \right).$$  

(14)

We see that the sectors with even and odd number of fermions are obtained by antiperiodic and periodic boundary conditions respectively.

Since the Hamiltonian (14) is quadratic in fermions operators we can (see Appendix A of ref. \cite{16}), through a linear combination of $c_j, c_j^\dagger$ introduce new fermionic momentum operators $\eta_k, \eta_k^\dagger$ that brings the Hamiltonian into a diagonal form

$$H(0, h, h_s) = \sum_k \left( 2\theta_k \sqrt{h^2 + \cos^2 k} \right) \eta_k^\dagger \eta_k - 2nh$$

(15)
\[ \theta_k = +1 \text{ for } -\frac{\pi}{2} < k \leq \frac{\pi}{2} \text{ and } \theta_k = -1 \text{ otherwise}, \]  
where the sum now runs over the momenta \(-\pi \leq k \leq \pi\) and \(\theta_k\) are signals\[17\].

Due to the boundary condition term in (14) the possible values of momentum \(k\) will depend on the lattice size parity and are given by

\[ (2l + 1)\frac{\pi}{M}, \quad l = -\frac{M}{2}, -\frac{M}{2} + 1, ..., \frac{M}{2} - 2, \frac{M}{2} - 1 \]  
for \(N_F\) even, and

\[ 2l\frac{\pi}{M}, \quad l = -\frac{M}{2}, -\frac{M}{2} + 1, ..., \frac{M}{2} - 1, \frac{M}{2} \]  
for \(N_F\) odd. The eigenenergies in a given sector with \(N_F\) fermions are obtained from the \(\frac{M!}{N_F!} (M - N_F)!\) combinations of fermion energies. The momentum of a given eigenstate is obtained by adding the fermions momenta \(P = \sum_k k\).

For example the lowest eigenenergy in the sector with \(U(1)\) charge \(n\) will be

\[ E_n = 2 \sum_{l=-\frac{M}{2}}^{\frac{M}{2} - 1} \frac{\theta_k}{\sqrt{h_s^2 + \cos^2 \left( (2l + 1)\frac{\pi}{M} \right)}} - 2nh \]

for \(n\) even and

\[ E_n = 2 \sum_{l=-\frac{M}{2}}^{\frac{M}{2} + 1} \frac{\theta_k}{\sqrt{h_s^2 + \cos^2 \left( 2l\frac{\pi}{M} \right)}} - 2nh \]

for \(n\) odd. These energies correspond to zero-momentum states since they come from a symmetric distribution of fermion momenta.

From (13) we know that at \(h = \sqrt{h_s^2 + 1}\) the system undergoes a phase transition to the ordered ferromagnetic state. In order to detect other phase transitions let us calculate the mass gap of \(H(0, h_s, h)\) for small values of
In such cases the ground-state remains in the sector $n = 0$ and the first excited state in the sector $n = 1$. From Eqs. (19) and (20) the mass gap is given by

$$G = \lim_{M \to \infty} (E_1 - E_0) = 2(h_s - h). \quad (21)$$

Therefore, as long as $h_s > h$ the model is massive and ordered antiferromagnetically. At $h = h_s$ the model becomes massless and by the same mechanism we discussed in the last section we expect that as $h$ increases the model stay massless until the Pokrovsky-Talapov line is reached. This intermediate phase has a ferromagnetic order since now the ground-state change sectors as $h$ increases. In Fig. 2 we show the phase diagram at $\Delta = 0$.

**Intermediate phase at $\Delta = 0$**

In order to analyze this phase using finite-size scaling we should keep the magnetization per particle $\mu$ fixed. This is done by choosing a sequence of lattice sizes $M_1, M_2, \ldots$ and sectors $n_1, n_2, \ldots$, such that $\mu = \frac{n_i}{M_i}$, $i = 1, 2, \ldots$ is fixed. For a given lattice size $M_i$ and magnetization $\mu$ the magnetic field that brings the ground-state in the sector where $n_i = \mu M_i$ should be in the range $h_{\text{min}}(M_i, \mu) \leq h \leq h_{\text{max}}(M_i, \mu)$, where $h_{\text{min}}(M_i, \mu) = (E_{n_i} - E_{n_i-1})/2$ and $h_{\text{max}}(M_i, \mu) = (E_{n_i+1} - E_{n_i})/2$. From Eqs. (19) and (20) we obtain in the bulk limit

$$h_{\text{min}}(\infty, \mu) = h_{\text{max}}(\infty, \mu) = h_{\mu} = \sqrt{h_s^2 + \sin^2 \alpha}, \quad (22)$$

where $\mu = \frac{n_i}{M_i}$ and $\alpha = \mu \pi$. Consequently if the applied field has the value $h = h_{\mu}$ all the finite lattice sequence $M_i$ ($i = 1, 2, \ldots$) will have its ground-state in the sector with $U(1)$ charge $n_i = \mu M_i$ ($i = 1, 2, \ldots$). It is simple to verify form Eqs. (19), (20) that for arbitrary values of $0 \leq \mu \leq 1$ the gap vanishes as $M \to \infty$, and consequently this partially ordered phase is critical. Assuming the critical fluctuations in this phase governed by a conformal field theory, the relations (2), (3) and (4) of Sec. 2 enable us to calculate the anomalous dimensions and central charge of this underline field theory.

For fixed value of $\mu$ the finite-size correction of the ground-state energy $E_{\mu M}^{\text{GS}}$ in the asymptotic regime $M \to \infty$ can be obtained by using the Euler-Maclaurin formula[18] in Eq. (19), which give us

$$\frac{E_{\mu M}^{\text{GS}}}{M} = e_\infty - \frac{\pi}{6M^2} \frac{\sin(2\alpha)}{h_{\mu}} + o(M^{-2}), \quad (23)$$
where
\[ e_\infty = -\frac{2}{\pi} \int_0^\infty \sqrt{h_s^2 + \cos^2 t} \, dt - 2h_\mu \mu \] (24)
is the energy per particle in the bulk limit \( M \to \infty \). The low lying excited states with nonzero momentum are calculated by changing the fermions momenta entering in the ground state. From these energies we can derive the dispersion relation and the sound velocity
\[ \zeta = \frac{\sin(2\alpha)}{h_\mu} \] (25)
which changes continuously along the massless phase. By comparing Eqs. (23) and (24) with Eq. (4) we see that the conformal anomaly is \( c = 1 \) for all values of \( \mu \).

From Eq. (2) the anomalous dimensions of the conformal operators are calculated from the finite-size corrections of the mass gap amplitudes. Such corrections, for a certain fermionic configuration, are calculated straightforwardly by using the Euler-Maclaurin formula. For an arbitrary but rational magnetization \( \mu = \frac{p}{q} \) (or magnetic field \( h = \sqrt{h_s^2 + \sin^2 \mu} \)) the lattice sequence with size \( M_i = iq \) \((i = 1, 2, \ldots)\) will have its ground-state in the \( U(1) \) sector \( n_i = \mu M_i = ip \) \((i = 1, 2, \ldots)\). The lowest eigenstate in neighboring sectors where \( n_i = n_i + \bar{n} \) \((\bar{n} = \pm 1, \pm 2, \ldots)\) has zero momenta and will give the mass gap
\[ E_{n_i} - E^{GS}_{n_i} = \frac{2\pi \sqrt{\bar{n}^2}}{M} \bar{n}^2 + o \left( M^{-1} \right) . \] (26)
The excited states in the sectors \( n_i \) \((i = 1, 2, \ldots)\), obtained by the addition (subtraction) of a momenta \( j \frac{2\pi}{M} \) \((j' \frac{2\pi}{M})\) to the fermions with positive (negative) momenta will give us an eigenstate with momentum
\[ P_{n_i}^{j,j'} = \frac{2\pi}{M} (j - j'). \] (27)
and mass gap
\[ E_{n_i}^{j,j'} - E^{GS}_{n_i} = \frac{2\pi \sqrt{\bar{n}^2}}{M} \zeta \left( \frac{\bar{n}^2}{4} + j + j' \right) + o \left( M^{-1} \right) . \] (28)
The relations (2) and (3) enable us to identify the eigenenergies \( E_{n_i}^{j,j'} \) as the conformal tower of the primary operator with dimension \( x_{\bar{n},0} = \frac{1}{4} \bar{n}^2 \).
Other excited states in the sectors where \( n_i = n_i + \bar{n} \) (\( \bar{n} = \pm 1, \pm 2, \ldots \)) are derived from the fermion configuration producing \( E_{n_i,j}' \). These eigenenergies are obtained by replacing fermions with positive (negative) momenta by fermions with negative (positive) momenta. The momenta of such states are

\[
P_{n_i,j',\bar{m}} = 2\pi \mu + \frac{2\pi}{M}(\bar{n}\bar{m} + j - j')
\]

(29)

and the mass gaps are

\[
E_{n_i,j',\bar{m}} - E_{n_i}^{GS} = \frac{2\pi}{M}\zeta \left( \bar{n}^2 x_p + \frac{\bar{m}^2}{4x_p} + j + j' \right) + o(M^{-1})
\]

(30)

where \( x_p \) and \( \bar{n}, \bar{m}, j, j' = 0, \pm 1, \pm 2, \ldots \). From relations (2) and (3) we identify Eq. (30) as the conformal tower of the primary operator with dimension \( x_{\bar{n},\bar{m}} = \left( \frac{\bar{n}^2}{4} + \bar{m}^2 \right) \). Following the discussion of Sec. 2 (see Eq. (5)) we can interpret \( x_{\bar{n},\bar{m}} \) as the dimension of a Gaussian operator \( O_{\bar{n},\bar{m}} \) with vorticity \( \bar{n} \) and spin-wave number \( \bar{m} \). Furthermore, from the degeneracies of the low lying energies in the conformal towers \( \{ E_{n_i,j',\bar{m}} \} \) we verify that their are given by the product of the characters of \( U(1) \) Kac-Moody algebra as discussed in Sec. 2 for the case where \( h_s = 0 \). Therefore the whole massless regime has the same critical nature as in the case where \( h_s = 0 \), being governed by a \( c = 1 \) conformal field theory with operators satisfying a \( U(1) \) Kac-Moody algebra.

Comparing Eqs. (31), (25) and (5) we see that while the sound velocity depend on \( h \) and \( h_s \) the dimensions \( x_{\bar{n},\bar{m}} \) are constants. For example the correlations (7) and (8) are given by

\[
< \sigma^x(0)\sigma^z(r) > \sim r^{-\frac{1}{2}}
\]

(31)

\[
< \sigma^z(0)\sigma^z(r) > -\mu^2 \sim r^{-2}
\]

(32)

where the magnetization per site \( \mu \) is given by

\[
\mu = \frac{1}{\pi} \sin^{-1} \left( \sqrt{h^2 - h_s^2} \right).
\]

(33)

As we shall see this independence of exponents is a particular feature of the case where \( \Delta = 0 \). It is interesting to observe that contrary to standard conformal towers appearing in critical systems, Eq. (23) tell us that
the eigenstates in the conformal tower has a macroscopic momenta as long 
\( \mu \neq 0 \). This is due to the fact that the fluctuations ruling the critical 
behavior are in top of a partially ordered ferromagnet state. Another delicate 
point appearing here\[19\] is that a continuum conformal theory can only be 
construct for \( h = \sqrt{h_s^2 + \sin^2 \mu} \), with \( \mu \) having a rational value, since the lattice sizes and sectors appearing in the finite-size sequences are integer values 
\( \mu = \frac{n_i}{M_i} \) (\( i = 1, 2, ... \)). For \( h \) connected with an irrational value of \( \mu \) we can only obtain a conformal theory by approximating \( \mu \) by a close rational number.

Massive phase at \( \Delta = 0 \)

The antiferromagnetic phase for \( h < h_s \) is massive with a mass gap 
given by \( (21) \). A continuum field theory describing the physics in this massive phase can be obtained in the neighborhood of the perturbing parameter 
\( \delta = h_s - h \geq 0 \). Such field theory will be massive and the masses can be estimated from the finite-size behavior of the eigenspectra. The mass spectrum can be calculated by applying the scheme followed by Sagdeev and Zamolodchikov\[20\] in the study of the Ising model in an external magnetic field. To do such calculations we should initially find the finite-size corrections of the zero-momenta eigenenergies \( E_k(\delta, M) \), \( k = 1, 2, ... \), at the conformal invariant point \( \delta = 0 \). In the case of the Hamiltonian \( (1) \) the results of Ref. \[2\] tell us that such corrections, not only at \( \Delta = 0 \) but for arbitrary values of \( \Delta \) \((-1 \leq \Delta \leq 1), \) are governed mainly by the irrelevant operator with dimension \( \bar{x} = x_{0,2} = 1/x_p \) and the descendent of the identity operator with dimension 4. From Ref. \[2\] we have

\[
E_k(\delta = 0, M) = e_\infty M + \frac{2\pi v}{M} \left( x_k - \frac{c}{12} \right) + a_1 \left( \frac{1}{M} \right)^3 + a_2 \left( \frac{1}{M} \right)^{\bar{x} - 1} + a_3 \left( \frac{1}{M} \right)^{2\bar{x} - 3} + a_4 \left( \frac{1}{M} \right)^{3\bar{x} - 5} + ... \tag{34}
\]

where \( x_k \) is one of the dimensions \( (5) \) associated to \( E_k \) and \( a_1, a_2, ... \) are \( M \)-independent factors. According to the scheme of Ref. \[20\] if the perturbed operator which produces the massive behavior has dimension \( y \) we should calculate the eigenspectra in the asymptotic regime \( \delta \rightarrow 0, \ M \rightarrow \infty \), with

\[
X = \delta^{\frac{1}{2\bar{x} - 7}} M \tag{35}
\]
kept fixed. In this regime (34) is replaced by

$$E_k(\delta, M) = e_\infty M + \delta^{\frac{1}{2-y}} F_k(X) + \delta^{\frac{1}{2-y}(\delta-1)} G_k(X) + \delta^{\frac{1}{2-y}(2\delta-3)} V_k(X) + \delta^{\frac{1}{2-y}(3\delta-5)} H_k(X) + \ldots.$$  (36)

The masses in the massive continuum field theory are obtained from the large-X behavior of the function \[ F_k(X), i.e., \]

$$m_k \sim F_k(X) - F_0(X)$$  (37)

The relations in Eqs. (34)-(37) can be used for arbitrary $\Delta$. In fact once we know $\bar{x} = 1/x_p$, the dimension associated to the off-critical perturbation $y$ as well the generated masses can be calculated from Eqs. (35)-(37).

In the case $\Delta = 0$ the perturbing parameter is $\delta = h_s - h > 0$. By keeping $X = (h_s - h)^{1-y} M$ fixed, with $y$ unknown, we obtain from Eqs. (19) and (20)

$$E_1 - E_0 = 2(h_s - h) -$$

$$\frac{X}{2h_s^{1-y}} \sum_{l=1}^{n/2} \sqrt{h_s^2 + \sin^2 \left[ \frac{2\pi}{X}(2l)h_s^{\frac{1}{1-y}} \right]} - \sqrt{h_s^2 + \sin^2 \left[ \frac{\pi}{X}(2l - 1)h_s^{\frac{1}{1-y}} \right]}$$

which gives for $h_s \to 0$

$$E_1 - E_0 \approx 2(h_s - h) - \frac{\Omega}{X} h_s^{\frac{1}{1-y}}$$  (39)

where $\Omega$ is a constant. Consequently by comparing Eq. (39) with Eqs. (36)-(37) we obtain the dimension $y = 1$ for the perturbing operator and the mass $m_1 = 2$ associated to the first gap.

The gap associated to the lowest eigenenergy in the sectors with $n$ even is obtained from Eq. (19)

$$E_n - E_0 = 4 \sum_{l=1}^{n/2} \sqrt{h_s^2 + \sin^2 \left[ \frac{\pi}{X}(2l - 1)h_s^{\frac{1}{1-y}} \right]}.$$  (40)

Since $n$ is finite, expanding for small values of $h_s$ we get, by comparing with Eq. (36), again the dimension $y = 1$ for the perturbing operator and

$$F_n(X) - F_0(X) = 4 \sum_{l=1}^{n/2} \sqrt{1 + \frac{\pi}{X}(2l - 1)^2}$$  (41)
which gives, for \( X \to \infty \), the masses \( m_n = n^2 = m_1 \). The same result is also obtained in the case where \( n \) is odd. Similar calculations for zero-momentum (modulo \( \pi \)) excited states show us that these states are associated with multiples of the mass \( m_1 \). However our calculations shows the existence of two equal masses in the system: \( m_1 = m_2 = 2 \). This can be seen clearly by a two-fold degeneracy of the first excited state with zero momentum (modulo \( \pi \)) in the ground state sector \( n = 0 \). The levels are related to the threshold energy \( 2m_1 \).

The dimension \( y = 1 \) for the perturbing operator and the Gaussian dimensions given in Eq. (5), with \( x_p = 1/4 \), indicate that this operator is the \( O_{0,1} \) operator (see Sec. 2) with dimension \( y = x_{0,1} = 1 \). This imply that in the region where \( x_{0,1} > 2 \), \( (\Delta < \frac{\sqrt{2}}{2}) \) this perturbation will be irrelevant and the phase stays massless after the introduction of a small \( h_s \).

### 3.2 \( \Delta \neq 0 \)

In this case the analytical diagonalization done for \( \Delta = 0 \) does not work since the Hamiltonian through the Jordan-Wigner transformation \( [10] \) will have a four-body fermionic interaction. Consequently our analysis will be done numerically. The critical surfaces of \( H(\Delta, h, h_s) \), according to finite-size scaling theory (FSS) \( [11] \) can be obtained from the extrapolation \( (M \to \infty) \) of the sequences \( (\Delta^{(M)}, h^{(M)}, h_s^{(M)}) \) \( (M = 2, 4, \ldots) \), obtained by solving the equation

\[
MG_M(\Delta, h, h_s) = (M - 2)G_{M-2}(\Delta, h, h_s)
\]

where \( G_M(\Delta, h, h_s) \) is the gap of the Hamiltonian \( [11] \) with \( M \) sites.

In order to simplify our analysis let us consider initially the plane \( h = 0 \). The critical surface is obtained by solving (12) with \( \Delta \) or \( h_s \) fixed. In Table 1 and 2 we show some of the finite-size sequences for lattice sizes up to \( M = 18 \). In Table 1 \( h_s \) is fixed while in Table 2 \( \Delta \) is fixed. In Fig. 3 we show the curves in the space \( (\Delta, h) \), that solve Eq. (12) for several lattice sizes. We clearly see in the figure that as \( (M \to \infty) \) the curves for \( \Delta < \Delta^* \approx 0.7 \) tend toward \( h_s = 0 \), while for \( \Delta > \Delta^* \) the curves tend toward nonzero values of \( h_s \). The blow up region \( \Delta \approx \Delta^* \) is also included in Fig. 3, where the points we used to draw the continuum curves are also shown. These results clearly indicate that for \( \Delta < \Delta^* \approx 0.7 \) the perturbation introduced by \( h_s \) is relevant producing a massive antiferromagnetic phase, while for \( 1 > \Delta > \Delta^* \)
the model stays massless until it reaches a critical staggered field \( h_s^* = h_s^*(\Delta) \) which depends on the value of \( \Delta \). This is in complete agreement with the predictions we did considering the \( \Delta = 0 \) results, where we associated the operator with dimension \( x_{0,1} = 1/x_p = \frac{x}{2(\pi - \cos^{-1}(\Delta))} \) with the staggered field perturbation. Therefore the perturbation is irrelevant for \( \Delta > \Delta^* = \frac{\sqrt{2}}{2} \approx 0.707 \) and the phase which appears for \( \Delta > \Delta^* \) in Fig. 3 should be a massless antiferromagnetic ordered phase, since \( h_s \neq 0 \) in this phase. This is complete agreement with predictions did in an earlier work by den Nijs [9].

The massless phase (denoted by ML in Fig. 3) is separated from the fully ferromagnetic phase (denoted by FE in Fig. 3) by the Pokrovsky-Talapov line (PT) \( h_s = \sqrt{\Delta^2 - 1} \), obtained from Eq. (9).

We confirmed numerically the massless nature of the ML phase. Using relations (2),(3) and (4) we verified that the whole phase is governed by a \( c = 1 \) Gaussian conformal field theory with dimensions as in (5), with \( x_p \) varying continuously in the whole phase. As a consequence of (3), ratio of dimensions like \( x_{n,0}/x_{1,0} = n^2 \) should be independent of \( x_p \). In order to illustrate this let us consider the ratio \( G_2/G_1 \) between the gaps in the sectors \( n = 2 \) and \( n = 1 \). Since the gaps are related to the dimensions \( x_{2,0} \) and \( x_{1,0} \), as \( (M \to \infty) \) this ratio should tend to the value \( x_{2,0}/x_{1,0} = 4 \). In Fig. 4 we show these ratios as a function of \( \Delta \) for \( M = 20 \) and some values of \( h_s \).

It is clear from this figure that the ratio changes to the expected value of 4 at values of \( \Delta \) which depend on the \( h_s \) value, which is consistent with the results shown in Fig. 3.

Our results also show that the PT line of Fig. 3 has a different behavior than the rest of the Pokrovsky-Talapov surface (3), since only along this line we have a first order phase transition. When \( h = 0 \) by crossing the PT line the ground-state changes from the ferromagnetic sector \( n = M/2 \) to the sector \( n = 0 \) discontinuously. As a consequence the magnetization per lattice point changes discontinuously from 1 to zero, as we move from the FE phase to the ML phase (see Fig. 3). Similarly the staggered magnetization

\[
\langle S^*_z \rangle = \frac{1}{2} \sum_{i=1}^{M} (-)^i \sigma^*_i
\]  

changes discontinuously from zero to a finite value, which depend on \( \Delta \). Also along the surface PT the ground-state wave function which is nondegenerated for \( h \neq 0 \) becomes \( M \)-degenerated when \( h = 0 \). All the lowest states in the sectors with \( n = \frac{-M}{2}, \frac{-M}{2} + 1, \ldots, \frac{M}{2} - 1, \frac{M}{2} \) become degenerated. We can also
show that along this PT curve these degenerated ground-state wave function, for a given sector \( n \), is given exactly by

\[
\Psi_n(\Delta, h_s, h = 0) = \sum_{\{s\}} q^\frac{1}{2} \sum_{i=1}^{M} (-)^{s_i} |s_1, s_2, ..., s_M\rangle
\]

where

\[
\Delta = \sqrt{h_s^2 + 1} = q + \frac{1}{q} , \quad \quad \quad (45)
\]

\( s_i = \pm 1 \) \( (i = 1, 2, ..., M) \) are the eigenvalues in the \( \sigma_z \) basis and the prime in the sum indicates that the configurations \( \{s\} \) are constrained to \( \sum_{i=1}^{M} s_i = n \).

**Massive phase at \( \Delta \neq 0 \)**

Similarly as we did for \( \Delta = 0 \) let us now investigate the continuum massive field theory describing the fluctuations in the massive phase (AF) of Fig. 3. The masses of this continuum theory are obtained from the eigenspectrum of the Hamiltonian in the scaling regime given by Eq. (35) with \( \delta = h_s \) and from our present analysis the dimension \( y \) of the perturbing operator is \( y = x_{0,1} = \frac{\pi}{2(\pi - \cos^{-1}(-\Delta))} \). Using Eqs. (34), (36) and (37) we can calculate the mass ratios of the continuum theory. These are calculated from the asymptotic regime \( M \to \infty \) and \( X \to \infty \) of the finite-size sequences

\[
R_k^{(n)}(X, M) = \frac{F_k^{(n)}(X, M) - F_0^{(0)}(X, M)}{F_1^{(0)}(X, M) - F_0^{(0)}(X, M)} \to \frac{m_k^n}{m_1^0} \quad (46)
\]

The functions \( F_k^{(n)}(X, M) \) are obtained by using in (36) the finite-size sequence of the \( k \) zero-momentum state \( (k = 0, 1, 2, ...) \) in the \( U(1) \) sector \( n \). The associated mass is \( m_k^n \) while the lightest mass, obtained from the lowest eigenenergy in the sector with \( n = \pm 1 \), is \( m_0^{(1)} \). Our results indicate that for \( \Delta \geq 0 \) we have only two equal masses \( M = m_0^{(1)} = m_0^{(-1)} \) and a continuum starting at \( 2M \). For \( \Delta < 0 \) other masses appears depending on the value of \( \Delta \).

In Table 3 we present some of our estimators for the mass ratios. We show for \( M = 20 \) the ratios (46) for some values of \( X \) and \( \Delta \). Our results show the appearance for \( \Delta = -1/2 \) of two masses \( m_1 = m_0^{(1)} \) and \( m_2 = m_1^{(0)} \) with a ratio \( m_2/m_1 = 1.61 \pm 0.03 \). For \( \Delta = -\frac{\sqrt{2}}{2} \) \( (\Delta = -\frac{\sqrt{3}}{2}) \) a third mass appears with the ratios \( m_3/m_1 = m_0^{(2)}/m_0^{(1)} = 2.0 \pm 0.2 \) \((= 1.9 \pm 0.1)\), \( m_2/m_1 = 1.41 \pm 0.01 \) \((= 1.2 \pm 0.1)\).
It is well known that the six-vertex model (or the \(XXZ\) chain) is transformed into the eight-vertex model (or the \(XYZ\) chain) by a thermal perturbation [21]. The spectrum of the transfer matrix of the eight-vertex model was calculated [22] explicitly and is the same as that of the sine-Gordon model [23]. From these results the thermal perturbation in the \(XXZ\) chain will produce the following sine-Gordon masses. For \(\Delta \geq 0\) there will have a mass \(M\) twice degenerated plus a continuum starting at \(2M\). These are the soliton-antisoliton pair. For \(\Delta < 0\), beyond the soliton-antisoliton pair bounded states will appear with masses depending on \(\Delta\),

\[
m_{i+1} = 2m_1 \sin \left( \frac{\pi}{2} \frac{i}{\alpha} \right), \quad i = 1, 2, ..., [\alpha]
\]

(47)

where

\[
\alpha = \frac{3\pi - 4\cos^{-1}(-\Delta)}{\pi}
\]

(48)

and \([\alpha]\) is the integer part of \(\alpha\).

Our results of Table 3 shows that also in the case where the perturbing field is the staggered magnetic field the mass spectrum is the same as the sine-Gordon model given by Eqs. (47) and (48).

4 Conclusions

The results we obtained in last section enable us to obtain a complete schematic phase diagram of the Hamiltonian (1). For simplicity we only draw the phase diagram for \(\Delta \geq 0\) in Fig. 5. The surface OGFHI separates the antiferromagnetic phase AF from the massless phase ML, while the surface AEDCB separates this last phase from the frozen ferromagnetic phase FE. The order parameter characterizing such phases is the magnetization

\[
\overline{M}_h = \left\langle \frac{1}{2} \sum_{i=1}^{M} \sigma_i^z \right\rangle
\]

(49)

and the staggered magnetization

\[
\overline{M}_{hs} = \left\langle \frac{1}{2} \sum_{i=1}^{M} (-)^i \sigma_i^z \right\rangle.
\]

(50)
In phase $\text{FE}$ $\overline{M}_h \neq 0$ and $\overline{M}_{h_s} = 0$, in phase $\text{AF}$ $\overline{M}_h = 0$ and $\overline{M}_{h_s} \neq 0$, while for $h \neq 0$ in phase $\text{ML}$ we have $\overline{M}_h \neq 0$ and $\overline{M}_{h_s} \neq 0$. At $h = 0$ phases $\text{AF}$ and $\text{ML}$ has only antiferromagnetic order $(\overline{M}_h = 0, \overline{M}_{h_s} \neq 0)$, but these phases differs since contrary to phase $\text{AF}$, (or $\text{FE}$) where the correlations have an exponential decay in the massless phase $\text{ML}$ the correlations has a power-law decay.

All the phase transition surfaces of Fig. 5 are of second order except the curve AE obtained when $h = 0$. The surface AEDCB is the Pokrovsky-Talapov surface given by Eq. (9). In this whole surface the ground state is the ordered Heisenberg state with all the spin up, except at the line AE where the ground-state is $M$ degenerated with a wave function given exactly by Eq. (44).

In the entire massless phase our numerical and analytical results show that the critical exponents change continuously with $h, h_s$ and $\Delta$, except at $\Delta = 0$ where the exponents are constant. The critical fluctuations in this phase is governed by a conformal field theory with central charge $c = 1$ and operators satisfying a $U(1)$ Kac-Moody algebra.

Our numerical and analytical analysis clearly indicate that the staggered magnetic field perturbation is associated to the operator with dimension $x_{0,1} = \frac{1}{4\pi} = \frac{\pi}{2(\pi - \cos^{-1}(-\Delta))}$. This confirms an earlier prediction of den Nijs[9] which states that for $\Delta > \sqrt{2}$ this perturbation is irrelevant.

Coming from the massless phase $\text{ML}$ to the antiferromagnetic phase $\text{AF}$, of Fig. 3 the fluctuations will be ruled by a massive continuum field theory. Our analysis of last section indicate that such massive field theory is the sine-Gordon field theory with a mass spectrum that depends on the value of $\Delta$ like in Eq. (47). This is the same mass spectrum obtained by perturbing the $XXZ$ by a thermal field.

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Figure Captions

Figure 1 - Phase diagram of the Hamiltonian $H(\Delta, h, h_s)$ given in Eq. (1) for $h_s = 0$. The different phases and the Pokrovsky-Talapov line are indicated.

Figure 2 - Phase diagram of the Hamiltonian $H(\Delta, h, h_s)$ given in Eq. (1) for $\Delta = 0$. The massless phase is separated from the ferromagnetic phase by the Pokrovsky-Talapov line $h = \sqrt{h_s^2 + 1}$ and from the antiferromagnetic phase by the line $h = h_s$.

Figure 3 - Finite-size estimators of the phase diagram of the Hamiltonian $H(\Delta, h, h_s)$ given in Eq. (1) for $h_s = 0$. The lines are obtained by solving Eq. (42) for lattice size pairs $(M-2, M)$ $(M = 10, 12, ..., 18)$. The antiferromagnetic, massless and ferromagnetic phases are indicated by AF, ML and FE respectively. The Pokrovsky-Talapov line PT is also indicated. The blow up region $0.57 \leq \Delta \leq 0.82$ is also shown with the points used to draw the continuum curves.

Figure 4 - Ratios $G_2/G_1$ between the gaps $G_2$ in sector $n = 2$ and $G_1$ in the sector $n = 1$ of the Hamiltonian $H(\Delta, 0, h_s)$ given in Eq. (1). The ratios are shown as a function of $\Delta$ for lattice size $M = 20$ and some values of $h_s$. In the region where the phase is massless we expect the Gaussian relation $G_2/G_1 \rightarrow x_{2,0}/x_{1,0} = 4$ as $M \rightarrow \infty$.

Figure 5 - Schematic phase diagram of $H(\Delta, h, h_s)$ given in Eq. (1). The phases indicated by FE, ML and AF are the fully ferromagnetic, massless and antiferromagnetic phases, respectively. The Pokrovsky-Talapov surface ABCDE separates the ferromagnetic and massless phases and the surface FGOIH separates this last phase from the antiferromagnetic one. All the phase transition surfaces are of second order excepting the line AE where the transition is first order.
Table Captions

Table 1 - Sequences of estimators for the anisotropy $\Delta$ in the plane $h = 0$, obtained by solving Eq. (42) with $h_s$ kept fixed.

Table 2 - Sequences of estimators for the anisotropy $\Delta$ in the plane $h = 0$, obtained by solving Eq. (42) with $\Delta$ kept fixed.

Table 3 - The mass-ratio estimators $R^{(n)}_k(X, M)$ defined in Eq. (46) for some values of the anisotropy $\Delta$ in the plane $h = 0$. The conjectured values of last line is given by Eq. (47).
Table 1

| $M - 2, M$ | $h = 0.01$ | $h = 0.03$ | $h = 0.05$ | $h = 0.1$ | $h = 0.2$ | $h = 0.5$ | $h = 1.0$ |
|------------|------------|------------|------------|------------|------------|------------|------------|
| 8,10       | -0.4540731 | -0.0159442 | 0.3100008  | 0.5969325  | 0.7614057  | 0.9864091  | 1.3628443  |
| 10,12      | -0.3507805 | 0.2311949  | 0.4642322  | 0.6591962  | 0.7855170  | 0.9930197  | 1.3634265  |
| 12,14      | -0.2052818 | 0.3724661  | 0.5441739  | 0.6931414  | 0.7996157  | 0.9972828  | 1.3660457  |
| 14,16      | -0.0457780 | 0.4570385  | 0.5922408  | 0.7145697  | 0.8088957  | 1.0002445  | 1.3650563  |
| 16,18      | 0.0891170  | 0.5122369  | 0.6242583  | 0.7293916  | 0.8154915  | 1.0024008  | 1.3657592  |

Table 2

| $M - 2, M$ | $\Delta = 0.6$ | $\Delta = 0.65$ | $\Delta = 0.675$ | $\Delta = 0.7$ | $\Delta = 0.725$ | $\Delta = 0.75$ | $\Delta = 0.8$ |
|------------|----------------|----------------|------------------|----------------|------------------|----------------|----------------|
| 8,10       | 0.1010376      | 0.1213905      | 0.1344105        | 0.1499071      | 0.1682486        | 0.1890890      | 0.2420444      |
| 10,12      | 0.0776255      | 0.0958401      | 0.1078859        | 0.1225705      | 0.1404855        | 0.1621757      | 0.2176779      |
| 12,14      | 0.0624300      | 0.0789604      | 0.0901779        | 0.1041465      | 0.1215976        | 0.1432584      | 0.2005065      |
| 14,16      | 0.0518302      | 0.0669150      | 0.0773753        | 0.0906412      | 0.1075649        | 0.1290517      | 0.1875885      |
| 16,18      | 0.0440532      | 0.0578975      | 0.0676763        | 0.0802761      | 0.0966522        | 0.1178787      | 0.1773956      |
Table 3

| X   | $\frac{m_2}{m_1}$ | $\frac{m_3}{m_1}$ | $\frac{m_4}{m_1}$ | $\frac{m_5}{m_1}$ | $\frac{m_3}{m_1}$ | $\frac{m_4}{m_1}$ |
|-----|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 6   | 1.64115           | 1.41970           | 2.20667           | 1.25317           | 2.15167           |
| 10  | 1.61967           | 1.40502           | 2.06304           | 1.18785           | 2.01352           |
| 14  | 1.62223           | 1.40555           | 2.03037           | 1.15271           | 1.99746           |
| Eq.(47) | 1.61803           | 1.41421           | 2.00000           | 1.24698           | 1.94986           |