ON THE EQUATION $m = xyzw$ WITH $x \leq y \leq z \leq w$ IN POSITIVE INTEGERS

MADJID MIRZAVAZIRI AND DANIEL YAQUBI

Abstract. As a well-known enumerative problem, the number of solutions of the equation $m = m_1 + \ldots + m_k$ with $m_1 \leq \ldots \leq m_k$ in positive integers is $\Pi(m, k) = \sum_{i=0}^{k} \Pi(m - k, i)$ and $\Pi$ is called the additive partition function. In this paper, we give a recursive formula for the so-called multiplicative partition function $\mu_1(m, k) :=$ the number of solutions of the equation $m = m_1 \ldots m_k$ with $m_1 \leq \ldots \leq m_k$ in positive integers. In particular, using an elementary proof, we give an explicit formula for the cases $k = 1, 2, 3, 4$.

1. Introduction

Partitioning numbers and sets are two important enumerative problems. These problems can be stated in different ways. Given a positive integer $m$ the total number of ways of writing $m$ as a sum of positive integers is clearly $2^{m-1}$. To see this we can put $m$ balls in a row and we then realize that the ways of writing $m$ as a sum of positive integers correspond to putting some walls between the balls. As we can put walls in $m-1$ places, we have the result. On the other hand, the number of solutions of the equation $m = m_1 \ldots m_k$ in positive integers for a fixed $k$, is equal to the number of ways of putting $k-1$ walls into $m-1$ allowed places; i.e. $\binom{m-1}{k-1}$.

We can also consider the equation $m = m_1 + \ldots + m_k$ in non-negative integers. For a fixed $k$, the number of solutions is $\binom{m+k-1}{k-1}$. This equals to the number of ways of putting $m$ balls and $k-1$ walls in a row. The total number of ways of writing $m$ as a sum of non-negative integers cannot be a good question, since the answer is infinity.

Moreover, we can add some conditions to the problem. For instance, we can think about the number of solutions of the equation $m = m_1 + \ldots + m_k$ with the conditions $\ell_i \leq m_i \leq \ell'_i$ for $i = 1, \ldots, k$. A straightforward application of the Inclusion Exclusion Principle solves this problem.

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On the other hand, if we add the condition $m_1 \leq \ldots \leq m_k$ to our problem then we have a more difficult situation. The answer is denoted by $\Pi(m, k)$ and a recursive argument shows that $\Pi(m, k) = \sum_{i=0}^{k} \Pi(m - k, i)$ with the initial values $\Pi(m, m) = \Pi(m, 1) = \Pi(0, 0) = 1$. For a discussion about these problems see [1].

These considerations motivate us to think about a multiplicative version of the above problems. Various types of this problem can be found in sequences A001055, A034836, A122179, A122180 and A088432 of [7]. We deal with this problem in the present paper and we give a solution for the problem whenever $k = 1, 2, 3$ or 4. A survey on this problem can be found in [4]. The reader is also referred to [3] and [5].

There is a solution to the case $k = 4$ of multiplicative and additive forms of our problem using the Polya Enumeration Theorem. For a complete solutions of these problems see section 7.5 of [2] and for a background material on the Polya method and multiset cycle indices see [6].

2. A Recursive Formula

**Definition 2.1.** Let $m, k$ and $\ell$ be positive integers. We denote the number of solutions of the equation

- $m = m_1 \ldots m_k$ with $m_i \geq \ell$ by $\nu_{\ell}(m, k)$;
- $m = m_1 \ldots m_k$ with $\ell \leq m_1 \leq \ldots \leq m_k$ by $\mu_{\ell}(m, k)$;
- $m = m_1 \ldots m_k$ with $\ell \leq m_1, \ldots, m_k$ and $k \in \mathbb{N}$ by $\nu_k(m)$;
- $m = m_1 \ldots m_k$ with $\ell \leq m_1 \leq \ldots \leq m_k$ and $k \in \mathbb{N}$ by $\mu_k(m)$;
- $m = m_1 \ldots m_k$ with $m_i \geq \ell$ and $\{m_1, \ldots, m_k\} = \{m_1', \ldots, m_r'\}$ and $\beta_j = |\{i : m_i = m_j'\}|$, $1 \leq j \leq r$ by $\nu_r'(m; \beta_1, \ldots, \beta_r)$;
- $m = m_1^{\beta_1} \ldots m_r^{\beta_r}$ with $\beta_i \in \mathbb{N}$, $\beta_1 + \ldots + \beta_r = k$ and $\ell \leq m_1 < \ldots < m_r$ by $\mu_r'(m; \beta_1, \ldots, \beta_r)$.

**Lemma 2.2.** Let $m > 1, k$ be positive integers and $p_1^{\alpha_1} \ldots p_n^{\alpha_n}$ be the prime decomposition of $m$. Then

$$\nu_1(m, k) = \prod_{j=1}^{n} \binom{\alpha_j + k - 1}{k - 1}$$

and

$$\nu_2(m, k) = \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} \prod_{j=1}^{n} \binom{\alpha_j + k - i - 1}{k - i - 1}.$$ 

**Proof.** Suppose that for $1 \leq j \leq n$ we have $\alpha_j$ balls labelled $p_j$ and we want to put these balls into $k$ different cells. There is a one to one correspondence between these situations
and multiplicative partitioning \( m = m_1 \ldots m_k \). In fact we can consider \( m_j \) as the product of the balls in cell \( j \). There are \( \binom{\alpha_j + k - 1}{k - 1} \) ways to put balls labelled \( p_j \). Thus the first part is obvious.

For the second part, let \( E_r \) be the set of all situations in which the cell \( r \) is empty, where \( 1 \leq r \leq k \). Then we have

\[
|E_{r_1} \cap \ldots \cap E_{r_i}| = \prod_{j=1}^{n} \binom{\alpha_j + k - i - 1}{k - i - 1}, \quad 1 \leq i \leq k - 1.
\]

Thus the Inclusion Exclusion Principle implies the result. \( \square \)

**Example 2.3.** We evaluate \( \nu_2(m) \) for \( m = p^\alpha \). Clearly

\[
\nu_1(m, k) = \binom{\alpha + k - 1}{k - 1}
\]

and

\[
\nu_2(m) = \sum_{k=1}^{\alpha} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} \prod_{j=1}^{n} \binom{\alpha_j + k - i - 1}{k - i - 1}.
\]

Suppose that there are \( k \) boys and \( \alpha - 1 \) girls and we want to choose a team consisting of \( k - 1 \) girls. This is obviously equal to \( \binom{\alpha - 1}{k - 1} \).

Let \( A_r \) be the set of all situations in which the boy \( r \) is belonged to the chosen team. Then

\[
|A_{r_1} \cap \ldots \cap A_{r_i}| = \binom{\alpha + k - i - 1}{k - i - 1}, \quad 1 \leq i \leq k - 1.
\]

By the Inclusion Exclusion Principle we therefore have

\[
\nu_2(m) = \sum_{k=1}^{\alpha} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} \prod_{j=1}^{n} \binom{\alpha_j + k - i - 1}{k - i - 1}
\]

\[
= \sum_{k=1}^{\alpha - 1} \binom{k - 1}{\alpha - 1}
\]

\[
= 2^{\alpha - 1}.
\]

**Proposition 2.4.** Let \( m > 1 \) and \( k, \ell \) be positive integers. Suppose that \( \ell^s \) divides \( m \) but \( \ell^{s+1} \) does not divide \( m \). Then

\[
\mu_\ell(m, k) = \sum_{i=\max\{k-s,1\}}^{\min\{k,s\}} \mu_{\ell+1}(m, i).
\]
Proof. Let

\[ E = \{(m_1, \ldots, m_k) : m = m_1 \ldots m_k, \ell \leq m_1 \leq \ldots \leq m_k\} \]

and

\[ E_i = \{(m_1, \ldots, m_k) \in E : m_1 = \ldots = m_i = \ell, m_{i+1} \neq \ell\}, \quad i = 0, 1, \ldots, \min\{k-1, s\}. \]

Then \( E = \bigcup_{i=0}^{\min\{k-1, s\}} E_i \) and the union is disjoint. Thus

\[
\mu_{\ell}(m, k) = \sum_{i=0}^{\min\{k-1, s\}} |E_i| = \sum_{i=0}^{\min\{k-1, s\}} \mu_{\ell+1}(m, k-i) = \sum_{i=\max\{k-s, 1\}}^{\min\{k, s\}} \mu_{\ell+1}(m, i). \]

□

Corollary 2.5. Let \( m > 1 \) and \( k, \ell \) be positive integers. Then

\[
\mu_1(m, k) = \sum_{i=1}^{k} \mu_2(m, i). \]

Proof. Using Proposition 2.4 we have the result, since \( 1^s \mid m \) for each positive integer \( s \). □

The following result is straightforward.

Lemma 2.6. Let \( m, k \) and \( \ell \) be positive integers. Then

i. \( \nu_\ell(m, k) = \sum_{d|m, d|\ell} \nu_\ell(m/d, k-1) \);

ii. \( \mu_\ell(m, k) = \sum_{d|m, d|\ell} \mu_\ell(m/d, k-1) \);

iii. \( \nu_\ell(m, k) = \sum_{\beta_i \in \mathbb{N}, \beta_1 + \ldots + \beta_r = \ell} \mu_\ell(m; \beta_1, \ldots, \beta_r) \);

iv. \( \mu_\ell(m, k) = \sum_{\beta_i \in \mathbb{N}, \beta_1 + \ldots + \beta_r = \ell} \mu_\ell(m; \beta_1, \ldots, \beta_r) \);

v. \( \mu_{\ell'}(m; \beta_1, \ldots, \beta_r) = \frac{k!}{\beta_1! \ldots \beta_r!} \nu_{\ell'}(m; \beta_1, \ldots, \beta_r), \quad \beta_i \in \mathbb{N}, \beta_1 + \ldots + \beta_r = k. \)

3. An Explicit Formula For the Cases \( k = 1, 2, 3, 4 \) and \( \ell = 1, 2 \)

We now aim to give an explicit formula for the cases \( k = 1, 2, 3, 4 \) and \( \ell = 1, 2 \). In the following, we denote the number of natural divisors of \( m \) by \( \tau(m) \). Moreover,

\[
\varepsilon_i(m) = \begin{cases} 1 & \text{if } \sqrt{m} \in \mathbb{N} \\
0 & \text{otherwise} \end{cases} \]

Proposition 3.1. Let \( m > 1 \) be a positive integer. Then

\[
\mu_1(m, 1) = \mu_2(m, 1) = 1, \]

\[
\mu_1(m, 2) = \left\lfloor \frac{\tau(m)}{2} \right\rfloor \quad \text{and} \quad \mu_2(m, 2) = \left\lfloor \frac{\tau(m)}{2} \right\rfloor - 1. \]
We have

\[ \nu(m, 2) = \frac{1}{2} \left( \frac{\tau(m) - 1}{2} \right) + 1 = \left\lceil \frac{\tau(m)}{2} \right\rceil. \]

Using Corollary 2.5, we now have

\[ \mu_2(m, 2) = \mu_1(m, 2) - 1. \]

**Theorem 3.2.** Let \( m > 1 \) be a positive integer and \( p_1^{a_1} \ldots p_n^{a_n} \) be its prime decomposition. Then

\[
\mu_1(m, 3) = \frac{1}{6} \prod_{j=1}^{n} \left( \frac{\alpha_j + 2}{2} \right) + \frac{1}{2} \prod_{j=1}^{n} \left( \frac{\alpha_j + 2}{2} \right) + \frac{\varepsilon_3(m)}{3},
\]

\[
\mu_2(m, 3) = \mu_1(m, 3) - \left\lceil \frac{\tau(m)}{2} \right\rceil.
\]

**Proof.** We have

\[
\nu_1(m, 3) = \nu_1'(m; 1, 1, 1) + \nu_1'(m; 1, 2) + \nu_1'(m; 3) = 6\nu_1'(m; 1, 1, 1) + 3\nu_1'(m; 1, 2) + \mu_1'(m; 3).
\]

We know that \( \mu_1'(m; 1, 2) \) is the number of ways to write \( m \) as \( xy^2 \), where \( x \neq y \). This is equal to the number of \( y \)'s such that \( y^2 \mid m \) minus the number of ways such that \( \frac{m}{y^2} = y \), in which the latter is equal to \( \varepsilon_3(m) \). The number of \( y \)'s such that \( y^2 \mid m \) is \( \prod_{j=1}^{n} \left( \frac{\alpha_j + 2}{2} \right) \). Moreover, \( \mu_1'(m; 3) = \varepsilon_3(m) \). Thus Lemma 2.2 implies

\[
\mu_1'(m; 1, 1, 1) = \frac{1}{6} \left( \nu_1(m, 3) - 3 \left( \prod_{j=1}^{n} \left( \frac{\alpha_j + 2}{2} \right) - \varepsilon_3(m) \right) - \varepsilon_3(m) \right)
\]

\[ = \frac{1}{6} \nu_1(m, 3) - \frac{1}{2} \prod_{j=1}^{n} \left( \frac{\alpha_j + 2}{2} \right) + \frac{1}{3} \varepsilon_3(m)
\]

\[ = \frac{1}{6} \prod_{j=1}^{n} \left( \frac{\alpha_j + 2}{2} \right) - \frac{1}{2} \prod_{j=1}^{n} \left( \frac{\alpha_j + 2}{2} \right) + \frac{\varepsilon_3(m)}{3},
\]

we have

\[
\mu_1(m, 3) = \mu_1'(m; 1, 1, 1) + \mu_1'(m; 1, 2) + \mu_1'(m; 3)
\]

\[ = \frac{1}{6} \prod_{j=1}^{n} \left( \frac{\alpha_j + 2}{2} \right) - \frac{1}{2} \prod_{j=1}^{n} \left( \frac{\alpha_j + 2}{2} \right) + \frac{\varepsilon_3(m)}{3} + \prod_{j=1}^{n} \left( \frac{\alpha_j + 2}{2} \right) - \varepsilon_3(m) + \varepsilon_3(m).
\]

This proves the first part.
For the second part, using Corollary 2.5, we have

$$\mu_2(m, 3) = \mu_1(m, 3) - \mu_2(m, 2) - 1 = \mu_1(m, 3) - \left\lfloor \frac{\tau(m)}{2} \right\rfloor + 1 - 1.$$  

\[ \square \]

Lemma 3.3. Let \( k, \ell > 1 \) be a positive integer and \( p_1^{\alpha_1} \ldots p_n^{\alpha_n} \) be the prime decomposition of \( k \). Then

$$\sum_{d|k} \tau(d) = \prod_{j=1}^{n} \left( \frac{\beta_j + 2}{2} \right)$$

and

$$\sum_{d|k} \varepsilon_\ell(d) = \prod_{j=1}^{n} \left\lfloor \frac{\beta_j + \ell}{\ell} \right\rfloor.$$  

Theorem 3.4. Let \( m > 1 \) be a positive integer and \( p_1^{\alpha_1} \ldots p_n^{\alpha_n} \) be its prime decomposition. then

$$\mu_1(m, 4) = \frac{1}{24} \prod_{i=1}^{n} \left( \frac{\alpha_i + 3}{3} \right) + \frac{1}{3} \prod_{i=1}^{n} \left\lfloor \frac{\alpha_i + 3}{3} \right\rfloor + \frac{1}{4} \prod_{i=1}^{n} \left\lfloor \frac{\alpha_i + 2}{2} \right\rfloor \left( \alpha_i - \left\lfloor \frac{\alpha_i - 2}{2} \right\rfloor \right)$$

$$+ \frac{\varepsilon_2(m)}{4} \prod_{i=1}^{n} \left\lfloor \frac{\alpha_i + 2}{2} \right\rfloor - \frac{\varepsilon_2(m)}{4} \left\lfloor \frac{\tau(\sqrt{m})}{2} \right\rfloor + \frac{3\varepsilon_4(m)}{8}.$$  

Moreover,

$$\mu_2(m, 4) = \mu_1(m, 4) - \mu_1(m, 3).$$  

Proof. We have

$$\nu_1(m, 4) = \nu'_1(m; 1, 1, 1, 1) + \nu'_1(m; 1, 1, 2) + \nu'_1(m; 1, 3) + \nu'_1(m; 2, 2) + \nu'_1(m; 4)$$

$$= 24\mu'_1(m; 1, 1, 1, 1) + 12\mu'_1(m; 1, 1, 2) + 4\mu'_1(m; 1, 3) + 6\mu'_1(m; 2, 2) + \mu'_1(m; 4).$$

We know that \( \mu'_1(m; 1, 1, 2) \) is the number of ways to write \( m \) as \( xyz^2 \), where \( x, y \) and \( z \) are different positive integers. This is equal to the number of \( z \)'s such that \( z^2 | m \) minus the number of ways to write \( m \) as \( xz^3, x^2z^2 \) or \( z^4 \), where \( x \neq z \). The number of \( z \)'s such that \( z^2 | m \) is \( \sum_{z^2|m} \left\lfloor \frac{\tau(z^2)}{2} \right\rfloor \). Thus

$$\mu'_1(m; 1, 1, 2) = \sum_{z^2|m} \left\lfloor \frac{\tau(z^2)}{2} \right\rfloor - \mu'_1(m; 1, 3) - \mu'_1(m; 2, 2) - \varepsilon_4(m).$$
If \( m \) is not a perfect square then \( \tau(m) \) is even. Let \( z = p_1^{\beta_1} \ldots p_n^{\beta_n} \). There exists an index \( i \) such that \( \alpha_i \) is odd and then

\[
\sum_{z \mid m} \left\lfloor \frac{\tau(m) + 1}{2} \right\rfloor = \sum_{z \mid m} \frac{1}{2} \tau(p_1^{\alpha_1-2\beta_1} \ldots p_n^{\alpha_n-2\beta_n}) = \frac{1}{2} \prod_{i=1}^{n} \sum_{0 \leq \beta_i \leq \frac{\alpha_i}{2}} (\alpha_i - 2\beta_i + 1) \]

\[
= \frac{1}{2} \prod_{i=1}^{n} \sum_{0 \leq \beta_i \leq \frac{\alpha_i}{2}} \left( \frac{\alpha_i}{2} - \left\lfloor \frac{\alpha_i}{2} \right\rfloor \right).
\]

Moreover, \( \mu'_1(m; 1, 3) \) is the number of ways to write \( m \) as \( xz^3 \), where \( x \neq z \). This is equal to the number of \( z \)'s such that \( z^3 \mid m \) minus the number of ways to write \( m \) as \( z^4 \). The number of \( z \)'s such that \( z^3 \mid m \) is \( \prod_{j=1}^{n} \left\lfloor \frac{\alpha_j + 3}{3} \right\rfloor \). Whence

\[
\mu'_1(m; 1, 3) = \prod_{j=1}^{n} \left\lfloor \frac{\alpha_j + 3}{3} \right\rfloor - \varepsilon_4(m).
\]

Since \( m \) is not a perfect square, \( \mu'_1(m; 2, 2) = 0 \). Thus Lemma 2.2 implies

\[
\mu'_1(m; 1, 1, 1, 1) = \frac{1}{24} (\nu_1(m, 4) - 12\mu'_1(m; 1, 1, 2) - 4\mu'_1(m; 1, 3) - 6\mu'_1(m; 2, 2) - \varepsilon_4(m))
\]

\[
= \frac{1}{24} \prod_{i=1}^{n} \left( \frac{\alpha_i + 3}{3} \right) - \frac{1}{4} \prod_{i=1}^{n} \left( \frac{\alpha_i + 2}{2} \right) (\alpha_i - \left\lfloor \frac{\alpha_i - 2}{2} \right\rfloor)
\]

\[
+ \frac{1}{3} \mu'_1(m; 1, 3) + \frac{1}{4} \mu'_1(m; 2, 2) + \frac{11}{24} \varepsilon_4(m)
\]

\[
= \frac{1}{24} \prod_{i=1}^{n} \left( \frac{\alpha_i + 3}{3} \right) - \frac{1}{4} \prod_{i=1}^{n} \left( \frac{\alpha_i + 2}{2} \right) (\alpha_i - \left\lfloor \frac{\alpha_i - 2}{2} \right\rfloor) + \frac{1}{3} \mu'_1(m; 1, 3).
\]

Therefore we have

\[
\mu_1(m, 4) = \mu'_1(m; 1, 1, 1, 1) + \mu'_1(m; 1, 1, 2) + \mu'_1(m; 1, 3) + \mu'_1(m; 2, 2) + \mu'_1(m; 4)
\]

\[
= \frac{1}{24} \prod_{i=1}^{n} \left( \frac{\alpha_i + 3}{3} \right) + \frac{1}{3} \prod_{i=1}^{n} \left( \frac{\alpha_i + 3}{3} \right) + \frac{1}{4} \prod_{i=1}^{n} \left( \frac{\alpha_i + 2}{2} \right) (\alpha_i - \left\lfloor \frac{\alpha_i - 2}{2} \right\rfloor).
\]
Now let \( m \) be a perfect square. Then \( \tau(m) \) is odd and we have
\[
\sum_{z^2|m} \left\lfloor \frac{\tau(m)}{2} \right\rfloor = \sum_{z^2|m} \left( \frac{\tau(m)}{2} + \frac{1}{2} \right) 
\]
\[
= \sum_{z^2|m} \frac{1}{2} \tau(p_1^{\alpha_1-2\beta_1} \cdots p_n^{\alpha_n-2\beta_n}) + \frac{1}{2} \cdot \sum_{z^2|m} 1
\]
\[
= \frac{1}{2} \sum_{0 \leq \beta_i \leq \frac{\alpha_i}{2}} \prod_{i=1}^{n} (\alpha_i - 2\beta_i + 1) + \frac{1}{2} \cdot \sum_{0 \leq \beta_i \leq \frac{\alpha_i}{2}} 1
\]
\[
= \frac{1}{2} \prod_{i=1}^{n} \sum_{0 \leq \beta_i \leq \frac{\alpha_i}{2}} (\alpha_i - 2\beta_i + 1) + \frac{1}{2} \prod_{i=1}^{n} (\left\lfloor \frac{\alpha_i+2}{2} \right\rfloor)
\]
Thus
\[
\mu'_1(m; 1, 1, 1) = \frac{1}{24} \prod_{i=1}^{n} \left( \frac{\alpha_i + 3}{3} \right) - \frac{1}{4} \left( \prod_{i=1}^{n} (\frac{\alpha_i + 2}{2}) (\alpha_i - (\frac{\alpha_i - 2}{2})) \right)
\]
\[
+ \frac{1}{2} \prod_{i=1}^{n} (\left\lfloor \frac{\alpha_i + 2}{2} \right\rfloor) + \frac{1}{3} \mu'_1(m; 1, 1, 1) - \frac{1}{4} \mu'_1(m; 2, 2) - \frac{1}{24} \varepsilon_4(m).
\]
Furthermore, \( \mu'_1(m; 2, 2) \) is the number of ways to write \( m \) as \( x^2y^2 = (xy)^2 \), where \( x \neq y \).

If \( m \) is not a perfect square then this number is 0 and if \( m \) is a perfect square then \( \sqrt{m} \in \mathbb{N} \) and thus
\[
\mu'_1(m; 2, 2) = \varepsilon_2(m) \mu'_2(\sqrt{m}; 1, 1) = \varepsilon_2(m) (\mu_2(\sqrt{m}; 2) - \varepsilon_2(\sqrt{m})
\]
\[
= \varepsilon_2(m) \left( \left\lfloor \frac{\tau(\sqrt{m})}{2} \right\rfloor - \varepsilon_4(m) \right)
\]
\[
= \varepsilon_2(m) \left( \left\lfloor \frac{\tau(\sqrt{m})}{2} \right\rfloor - \varepsilon_4(m) \right).
\]
Therefore
\[
\mu_1(m, 4) = \frac{1}{24} \prod_{i=1}^{n} \left( \frac{\alpha_i + 3}{3} \right) + \frac{1}{3} \prod_{i=1}^{n} (\frac{\alpha_i + 3}{3}) + \frac{1}{4} \prod_{i=1}^{n} (\frac{\alpha_i + 2}{2}) (\alpha_i - (\frac{\alpha_i - 2}{2}))
\]
\[
+ \frac{1}{4} \prod_{i=1}^{n} \left( \frac{\alpha_i + 2}{2} \right) - \frac{1}{4} \varepsilon_2(m) \left( \left\lfloor \frac{\tau(\sqrt{m})}{2} \right\rfloor - \varepsilon_2(m) \right) + \frac{3\varepsilon_4(m)}{8}.
\]
This proves the first part. The second part is obvious. □
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Note that if a positive integer $m$ is a prime power, say $m = p^n$, then $\mu_2(m, k) = \Pi(n, k)$, where $\Pi$ is the additive partition function.

**Corollary 3.5.** Let $n$ be a positive integer. Then the number of all solutions of the equation

$$x_1 + x_2 + x_3 = n$$

in $\mathbb{N}$, under the condition $x_1 \leq x_2 \leq x_3$, is

$$\frac{1}{6} \binom{n+2}{2} + \frac{1}{2} \left\lfloor \frac{n+2}{2} \right\rfloor + \frac{\varepsilon_3(p^n)}{3}.$$

**Corollary 3.6.** Let $n$ be a positive integer. Then the number of all solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = n$$

in $\mathbb{N}$, under the condition $x_1 \leq x_2 \leq x_3 \leq x_4$, is

$$\frac{1}{24} \binom{n+3}{3} + \frac{1}{3} \left\lfloor \frac{n+3}{3} \right\rfloor + \frac{1}{4} \left( \left\lfloor \frac{n+2}{2} \right\rfloor (n - \left\lfloor \frac{n-2}{2} \right\rfloor) + \varepsilon_2(p^n) \right)$$

$$+ \frac{\varepsilon_3(p^n)}{4} \left\lfloor \frac{n+2}{2} \right\rfloor - \frac{\varepsilon_2(p^n)}{4} \left\lceil \frac{\tau(\sqrt{p^n})}{2} \right\rceil + \frac{3\varepsilon_4(p^n)}{8}.$$

**Remark 3.7.** We can compute $\mu_1(m, 4)$ using the following Mathematica code.

```mathematica
ClearAll
n = Input["enter your number"]
For[m = 1, n
a = Power[m];
b = xyzw[m, a];
c[m, 1] = b;
End

function a = Power(m1)
B = factor(m1); [n2, n1] = size(B); i = 1; j = 1; a(j) = 1;
while i < n1
    if B(i) == B(i+1)
        a(j) = a(j) + 1;
    else
        j = j + 1;
a(j) = 1;
end
i = i + 1;
```
function b=xyzw(m,a)
    [n2 n1]=size(a);
    b1=1;b2=1;b3=1;
    e2=0;e4=0;
    m1=floor(m^(1/2));
    if m1==(m^(1/2))
        e2=1;
    end
    m1=floor(m^(1/4));
    if m1==m^(1/4)
        e4=1;
    end
    for i=1:n1
        b1=(1/6)*(a(i)+3)*(a(i)+2)*(a(i)+1)*b1;
    end
    for i=1:n1
        b2=floor((a(i)+3)/3)*b2;
    end
    for i=1:n1
        b3=floor((a(i)+2)/2)*(a(i)-floor((a(i)-2)/2))*b3;
    end
    b4=1;
    for i=1:n1
        b4=floor((a(i)+2)/2)*b4;
    end
    b5=1;
    for i=1:n1
        b5=b5*((a(i)/2)+1);
    end
    b=(1/24)*b1+(1/3)*b2+(1/4)*b3+(1/4)*e2*b4-(1/4)*e2*ceil(b5)+(3/8)*e4;
end
ON THE EQUATION $m = xyzw$ WITH $x \leq y \leq z \leq w$ IN POSITIVE INTEGERS

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DEPARTMENT OF PURE MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD, P. O. BOX 1159, MASHHAD 91775, IRAN.

E-mail address: mirzavaziri@gmail.com, danyal_352@yahoo.es