An upper $J$-Hessenberg reduction of a matrix through symplectic Householder transformations

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Abstract

In this paper, we introduce a reduction of a matrix to a condensed form, the upper $J$-Hessenberg form, via elementary symplectic Householder transformations, which are rank-one modification of the identity. Features of the reduction are highlighted. Two variants numerically more stables are then derived. Some numerical experiments are given, showing the efficiency of these variants.

Keywords: Indefinite inner product, structure-preserving eigenproblems, symplectic Householder transformations, $SR$ decomposition, upper $J$-Hessenberg form.

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1. Introduction

Let $A$ be a $2n \times 2n$ real matrix. The $SR$ factorization consists in writing $A$ as a product $SR$, where $S$ is symplectic and $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$ is such that $R_{11}$, $R_{12}$, $R_{22}$ are upper triangular and $R_{21}$ is strictly upper triangular $\mathbb{R}$. This decomposition plays an important role in structure-preserving methods for solving the eigenproblem of a class of structured matrices.

More precisely, the $SR$ decomposition can be interpreted as the analog of the $QR$ decomposition $\mathbb{F}$, when instead of an Euclidean space, one considers a symplectic space: a linear space, equipped with a skew-symmetric inner...
product (see for example [7] and the references therein). The orthogonal group with respect to this indefinite inner product, is called the symplectic group and is unbounded (contrasting with the Euclidean case).

There are two classes of methods for computing the $SR$ decomposition. The first lies in the Gram-Schmidt like algorithms and leads to the symplectic Gram-Schmidt (SGS) algorithms. The second class is constructed from a variety of elementary symplectic transformations. Each choice of such transformations leads to the corresponding $SR$ decomposition. Since these elementary transformations are quite heterogeneous, the $SR$ decomposition is considerably affected by their choice.

Results on numerical aspects of SGS-algorithms can be found for example in [7]. These algorithms and their modified versions are usually involved in structure-preserving Krylov subspace-type methods, for sparse and large structured matrices.

In the literature, the symplectic elementary transformations involved in the $SR$ decomposition can be partitioned in two subsets. The first subset is constituted of two kind of both symplectic and orthogonal transformations introduced in [6, 12] and a third symplectic but non-orthogonal transformations, proposed in [2]. In fact, in [3], it has been shown that $SR$ decomposition of a general matrix could not be carried out by using only the above orthogonal and symplectic transformations. An algorithm, named SRDECO, based on these three transformations was derived in [2].

From linear algebra point of view, the $SR$ decomposition via SRDECO algorithm does not correspond to the analog of Householder $QR$ decomposition, since SRDECO involves transformations which are not elementary rank-one modification of the identity (transvections), see [1, 5].

In [8] a study, based on linear algebra concepts and focusing on the construction of the analog of Householder transformations in a symplectic linear space, has been accomplished. This has led to the second subset of transformations. Such analog transformations, which are rank-one modification of the identity are called symplectic Householder transformations. Their main features have been established, especially the mapping problem has been solved. Then, the analog of Householder $QR$ decomposition in a symplectic linear space has been derived. The algorithm SRSH for computing the $SR$ decomposition, using these symplectic Householder transformations has been then presented in details. Unlike Householder $QR$ decomposition, the new algorithm SRSH involves free parameters and advantages may be taken from this fact. It has been demonstrated how these parameters can be de-
terminated in an optimal way providing an optimal version \cite{9} of the algorithm (SROSH). The error analysis and computational aspects of this algorithm have been studied \cite{10}. Also, recently, a mathematical and numerical equivalence between modified symplectic Gram-Schmidt and Householder SR algorithms (typically SRSH or SROSH) have been established in \cite{11}. Computation aspects and numerical comparisons between SGS and SROSH have clearly showed the superiority of SROSH over SGS and also that SROSH and SRDECO mostly behave quite similarly, except when SRDECO breaks down. In fact, the latter suffers seriously from the eventuality to encounter a fatal breakdown. The algorithm SROSH works well in these cases, and hence seems to be adequate to be used in general, or to be an alternative to cure the breakdowns in SRDECO.

In order to build a $SR$-algorithm (which is a $QR$-like algorithm) for computing the eigenvalues and eigenvectors of a matrix \cite{13}, a reduction of the matrix to an upper $J$-Hessenberg form is crucial. This is due to the fact that the final algorithm we are looking for should have $O(n^3)$ as complexity. In \cite{2}, a reduction of a general matrix to an upper $J$-Hessenberg form is presented, using to this aim, the three symplectic transformations of the above first subset. The algorithm, called JHESS, is based on an adaptation of SRDECO.

In this paper, we focus on the reduction of a general matrix, to an upper $J$-Hessenberg form, using only the symplectic Householder transformations (the second subset above). We show how this reduction can be constructed. The new algorithm, which will be called JHSH algorithm, is based on an adaptation of SRSH algorithm. A variant of JHSH, named JHOSH is then obtained by taking some optimal choice of the free parameters. The JHOSH is numerically better than JHSH. However, the accuracy may be lost, since the transformations involved in are not necessarily orthogonal. This leads us to derive another variant, based in replacing when possible, each symplectic non-orthogonal transformation by another one, which is symplectic and orthogonal. This gives rise to JHMSH algorithm and its variant JHMSH2.

In this work, we restrict ourselves to the construction of such algorithms. Numerical aspects of the new algorithms and new insights on JHESS algorithm (the choice of the free parameters, near breakdows, breakdows, prediction of breakdows, different strategies of curing near breakdows, ...) will be studied separately in a forthcoming paper. Nevertheless, two illustrating numerical examples are given, showing in particular the efficiency of JHMSH and its variant JHMSH2. More precisely, for these examples, the
algorithm JHESS encounter a fatal breakdown, and hence fails to provide any $J$-Hessenberg reduction, while our new algorithms JHMSH, JHMSH2, with a slight modification, perform the $J$-Hessenberg reduction, with a very satisfactory precision for both the errors in the factorization and in the loss of $J$-orthogonality.

The remainder of this paper is organized as follows. Section 2, is devoted to the necessary preliminaries. In the section 3, we show how we obtain the method of reducing a general matrix to an upper $J$-Hessenberg, based only on the symplectic Householder transformations. Also, we present two variants, motivated by the numerical stability. Numerical experiments and comparisons between JHESS and the new JHMSH are given. We conclude in the section 4.

2. Preliminaries

Let $J_{2n}$ (or simply $J$) be the $2n$-by-$2n$ real matrix

$$J_{2n} = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix},$$

(1)

where $0_n$ and $I_n$ stand respectively for $n$-by-$n$ null and identity matrices. The linear space $\mathbb{R}^{2n}$ with the indefinite skew-symmetric inner product

$$(x, y)_J = x^T J y$$

(2)

is called symplectic. For $x, y \in \mathbb{R}^{2n}$, the orthogonality $x \perp' y$ stands for $(x, y)_J = 0$. The symplectic adjoint $x^J$ of a vector $x$, is defined by

$$x^J = x^T J.$$  

(3)

The symplectic adjoint of $M \in \mathbb{R}^{2n \times 2k}$ is defined by

$$M^J = J_{2k}^T M^T J_{2n}.$$  

(4)

A matrix $S \in \mathbb{R}^{2n \times 2k}$ is called symplectic if

$$S^J S = I_{2k}.$$  

(5)

The symplectic group (multiplicative group of square symplectic matrices) is denoted $\mathbb{S}$. A transformation $T$ given by
\[ T = I + cvv' \] where \( c \in \mathbb{R} \), \( v \in \mathbb{R}^\nu \) (with \( \nu \) even), \( c \in \mathbb{R} \), \( v \in \mathbb{R}^\nu \) (with \( \nu \) even), \( (6) \)

is called symplectic Householder transformation \( [8] \). It satisfies

\[ T^J = I - cvv'. \] (7)

The vector \( v \) is called the direction of \( T \).

For \( x, y \in \mathbb{R}^{2n} \), there exists a symplectic Householder transformation \( T \)

such that \( Tx = y \) if \( x = y \) or \( x'y \neq 0 \). When \( x'y \neq 0 \), \( T \) is given by

\[ T = I - \frac{1}{x'y}(y-x)(y-x)'). \]

Moreover, each non null vector \( x \) can be mapped onto any non null vector \( y \) by a product of at most two symplectic Householder transformations \( [8] \). Symplectic Householder transformations are rotations, i.e. \( \det(T) = 1 \) and the symplectic group \( S \) is generated by symplectic Householder transformations. We recall that a matrix \( H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \in \mathbb{R}^{2n \times 2n} \), is upper

\( J \)-Hessenberg when \( H_{11}, H_{21}, H_{22} \) are upper triangular and \( H_{12} \) is upper Hessenberg. \( H \) is called unreduced when \( H_{21} \) is nonsingular and the Hessenberg \( H_{12} \) is unreduced, i.e. the entries of the subdiagonal are all nonzero.

3. Upper \( J \)-Hessenberg reduction via symplectic Householder transformations

3.1. Toward the algorithm

Let \( \{e_1, \ldots, e_{2n}\} \) be the canonical basis of \( \mathbb{R}^{2n} \), \( a \in \mathbb{R}^{2n} \) and \( \rho, \mu, \nu \) be arbitrary scalars. We seek for symplectic Householder transformations \( T_1 \) and \( T_2 \) such that

\[ T_1(a) = \rho e_1, \]

and

\[ T_2(e_1) = e_1, \quad T_2(a) = \mu e_1 + \nu e_{n+1}. \] (9)

The fact that \( T_2 \) is a symplectic isometry yields the necessary condition

\[ (T_2(a))'J(T_2(e_1)) = a'J e_1, \]

which implies \( \nu = a(n+1) \) and \( \mu \) arbitrary. We get
Theorem 1. Let $\rho$, $\mu$ be arbitrary scalars and $\nu = a(n + 1)$. Setting
\[ c_1 = -\frac{1}{\rho a^T e_1}, \quad v_1 = \rho e_1 - a, \quad c_2 = -\frac{1}{a^T(\mu e_1 + \nu e_{n+1})}, \quad v_2 = \mu e_1 + \nu e_{n+1} - a, \]
then
\[ T_1 = I + c_1 v_1 v_1^T \quad \text{respectively} \quad T_2 = I + c_2 v_2 v_2^T \] satisfy (8) (respectively (9)).
(11)

Remark 1. Since the $n + 1$th component of $v_2$ is zero, $T_2$ keeps the $n + 1$th component of $T_2 x$ unchanged, for any $x \in \mathbb{R}^{2n}$. More on the properties of such transformations $T_1$ or $T_2$ can be found in [9, 10].

We also need the following

Theorem 2. Let $v \in \mathbb{R}^{2n}$, with the partition $v = [0^T, u^T, 0^T, w^T]^T$, where $[u, w] \in \mathbb{R}^{(n-i) \times 2}$, for a given $1 \leq i \leq n - 1$ and set $\tilde{v} = [u^T, w^T]^T$. Consider the symplectic transformations $T = I + cvv^T$ and $\tilde{T} = I + c\tilde{v}\tilde{v}^T$. We have
\[ \forall \alpha \in \mathbb{R}^i, \forall \beta \in \mathbb{R}^i, \forall x \in \mathbb{R}^{n-i}, \forall y \in \mathbb{R}^{n-i}, \]
\[ T[\alpha^T, x^T, \beta^T, y^T]^T = [\alpha^T, x^T, \beta^T, y^T]^T, \quad \text{with} \quad [x^T, y^T]^T = \tilde{T}[x^T, y^T]^T. \]

Proof. We have $v^T[\alpha^T, x^T, \beta^T, y^T]^T = u^T y - w^T x = [u^T w^T]J[x^T y^T]^T$. Then
\[ T[\alpha^T, x^T, \beta^T, y^T]^T = [\alpha^T, x^T, \beta^T, y^T]^T + c[0^T, u^T, 0^T, w^T]^T[u^T w^T]J[x^T y^T]^T. \]
We check easily
\[ \frac{x'}{y'} = \begin{bmatrix} x \\ y \end{bmatrix} + c \begin{bmatrix} u \\ w \end{bmatrix} [u^T w^T]J \begin{bmatrix} x \\ y \end{bmatrix} = \tilde{T} \begin{bmatrix} x \\ y \end{bmatrix}, \]
and
\[ T[\alpha^T, 0^T, \beta^T, 0^T]^T = [\alpha^T, 0^T, \beta^T, 0^T]^T. \]

Note that the Theorem 2 remains valid if one takes $T^J$ instead of $T$. This result, with Theorem 1 constitute the main tool on which the $SR$ factorization (based on symplectic Householder transformations) is constructed. We will adapt this tool for reducing a general matrix to an upper $J$-Hessenberg form, based on these symplectic Householder transformations.

3.2. The $J$-Hessenberg reduction: the JHSH algorithm

We explain here the steps of the algorithm by illustrating the general pattern. Let $A = [a_1, \ldots, a_n, a_{n+1}, \ldots, a_{2n}] \in \mathbb{R}^{2n \times 2n}$ be a given matrix and set $A^{(0)} = A$. We will use the notation $A_{(i_1 ; i_2 ; j_1 ; j_2)}$ to denote the submatrix obtained from the matrix $A$ by deleting all rows and columns except rows $i_1$.
until $i_2$ and columns $j_1$ until $j_2$.

1. Choose a symplectic Householder transformation $H_1$ (i.e. $c_1 \in \mathbb{R}$ and $v_1 \in \mathbb{R}^{2n}$), with $H_1 e_1 = e_1$, to zero out entries 2 through $n$ and entries $n+2$ through $2n$ of the first column of $A$. The vector $e_1$ stands for the first canonical vector of $\mathbb{R}^{2n}$. The transformation $H_1$ corresponds to the transformation $T_2$, given in Theorem 1. Set $v_1$ the direction vector of $H_1$. Since $H_1 e_1 = e_1$, we obtain $v_1^T e_1 = v_1^T J e_1 = 0$. Thus the $n + 1$th component of $v_1$ is zero. It follows that for any vector $x$, the $n + 1$th component of $H_1 x$ remains unchanged. The direction $v_1$ of $H_1$ is given by $v_1 = A^{(1)}_{(1,1)} e_1 + a_1(n + 1)e_{n+1} - a_1$, where $A^{(1)}_{(1,1)}$ is an arbitrary given scalar. Notice that we have also $H_1^T e_1 = e_1$, and hence the first column of $H_1$ and $H_1^T$ is $e_1$. Thus, multiplying $A^{(0)}$ on the left by $H_1$ leaves unchanged the $n + 1$th row and creates the desired zeros in the first column. We get

$$A^{(1)} = H_1 A^{(0)} = \begin{bmatrix} A^{(1)}_{(1,1)} & A^{(1)}_{(1,2:n)} & A^{(1)}_{(1,n+1:2n)} \\ 0 & A^{(1)}_{(2:n,2:n)} & A^{(1)}_{(2:n,n+1:2n)} \\ A^{(0)}_{(n+1,1)} & A^{(0)}_{(n+1,2:n)} & A^{(0)}_{(n+1,n+1:2n)} \\ 0 & A^{(1)}_{(n+2:n,2:n)} & A^{(1)}_{(n+2:n,n+1:2n)} \end{bmatrix}.$$ 

The step involves the free parameter $A^{(1)}_{(1,1)}$.

Multiplying $H_1 A^{(0)}$ on the right by $H_1^T$ leaves the first column of $H_1 A^{(0)} H_1^T$ unchanged, and we obtain

$$A^{(1)} = H_1 A^{(0)} H_1^T = \begin{bmatrix} A^{(1)}_{(1,1)} & A^{(1)}_{(1,2:n)} & A^{(1)}_{(1,n+1:2n)} \\ 0 & A^{(1)}_{(2:n,2:n)} & A^{(1)}_{(2:n,n+1:2n)} \\ A^{(0)}_{(n+1,1)} & A^{(0)}_{(n+1,2:n)} & A^{(0)}_{(n+1,n+1:2n)} \\ 0 & A^{(1)}_{(n+2:n,2:n)} & A^{(1)}_{(n+2:n,n+1:2n)} \end{bmatrix}.$$ 

The next step consists in choosing a symplectic Householder $H_2$ to zero out the entries 3 through $n$, the entries $n + 2$ through $2n$ of the $n + 1$th column of $A^{(1)}$. To do this, let $A^{(2)} = \begin{bmatrix} A^{(1)}_{(2:n,2:n)} & A^{(1)}_{(2:n,n+1:2n)} \\ A^{(1)}_{n+2:n,2:n} & A^{(1)}_{n+2:n,n+1:2n} \end{bmatrix}$ be the matrix obtained from $A^{(1)}$ by deleting the first column and the first and the $n + 1$th rows. And let $A^{(2)}_{(2:n,1)} \neq 0$ be an arbitrary given scalar.

We apply $H_2 = I_{2n-2} + c_2 \tilde{v}_2 \tilde{v}_2^T$ given by Theorem 1 with \( \tilde{v}_2 = \begin{bmatrix} u_2 \\ w_2 \end{bmatrix} =
$A^{(2)}(2,n+1)e_1 - \tilde{A}^{(1)}(:, n) \in \mathbb{R}^{2n-2}, \ u_2 \in \mathbb{R}^{n-1}, \ w_2 \in \mathbb{R}^{n-1}$, where $e_1$ stands for the first canonical vector of $\mathbb{R}^{2n-2}$. We obtain

$$\tilde{A}^{(2)} = \tilde{H}_2 \tilde{A}^{(1)} = \begin{bmatrix} A^{(2)}_{(1,2:n)} & A^{(2)}_{(2,n+1)} & A^{(2)}_{(2,n+2:2n)} \\ A^{(2)}_{(3:n,2:n)} & 0 & A^{(2)}_{(3:n,n+2:2n)} \\ A^{(2)}_{(n+2:2n,2:n)} & 0 & A^{(2)}_{(n+2:2n,n+2:2n)} \end{bmatrix}.$$ 

The transformation $\tilde{H}_2$ corresponds to the choice $T_1$ in Theorem 1. Setting $H_2 = I_{2n} + c_2v_2v_2^T$, with $v_2 = \begin{bmatrix} u_2 \\ w_2 \end{bmatrix} \in \mathbb{R}^{2n}$ then $H_2$ is a symplectic Householder transformation. Using Theorem 2 we get

$$A^{(2)} = H_2 A^{(1)} = \begin{bmatrix} A^{(1)}_{(1,1)} & A^{(1)}_{(1,2:n)} & A^{(1)}_{(1,n+1)} & A^{(1)}_{(1,n+2:2n)} \\ 0 & A^{(2)}_{(2,2:n)} & A^{(2)}_{(2,n+1)} & A^{(2)}_{(2,n+2:2n)} \\ 0 & A^{(2)}_{(3:n,2:n)} & 0 & A^{(2)}_{(3:n,n+2:2n)} \\ A^{(0)}_{(n+1,1)} & A^{(2)}_{(n+1,2:n)} & A^{(1)}_{(n+1,n+1)} & A^{(1)}_{(n+1,n+2:2n)} \\ 0 & A^{(2)}_{(n+2:2n,2:n)} & 0 & A^{(2)}_{(n+2:2n,n+2:2n)} \end{bmatrix}. $$

$H_2$ leaves the first and the $n+1$ th rows of $H_2 A^{(1)}$ unchanged. It leaves the first column of $H_2 A^{(1)}$ unchanged, and creates the desired zeros in the column $n+1$.

The multiplication of $H_2 A^{(1)}$ on the right by $H_2^T$ leaves the first and the $n+1$th columns of $H_2 A^{(1)} H_2^T$ unchanged. We obtain

$$A^{(2)} = H_2 A^{(1)} H_2^T = \begin{bmatrix} A^{(1)}_{(1,1)} & A^{(2)}_{(1,2:n)} & A^{(1)}_{(1,n+1)} & A^{(2)}_{(1,n+2:2n)} \\ 0 & A^{(2)}_{(2,2:n)} & A^{(2)}_{(2,n+1)} & A^{(2)}_{(2,n+2:2n)} \\ 0 & A^{(2)}_{(3:n,2:n)} & 0 & A^{(2)}_{(3:n,n+2:2n)} \\ A^{(0)}_{(n+1,1)} & A^{(2)}_{(n+1,2:n)} & A^{(1)}_{(n+1,n+1)} & A^{(2)}_{(n+1,n+2:2n)} \\ 0 & A^{(2)}_{(n+2:2n,2:n)} & 0 & A^{(2)}_{(n+2:2n,n+2:2n)} \end{bmatrix}.$$ 

It is worth noting that $H_2 e_1 = e_1$ and $H_2 e_{n+1} = e_{n+1}$. Thus the first column (respectively the $n+1$th column) of $H_2$ and $H_2^T$ is $e_1$ (respectively $e_{n+1}$).

In the next step, we want to zero out the entries 3 through $n$ and $n + 3$ through $2n$ of the second column of $A^{(2)}$ and the entries 4 through $n$ and $n + 3$
through $2n$ of the column $n+2$ of $A^{(2)}$. Let $\tilde{A}^{(2)}$ be the matrix obtained from $A^{(2)}$ by deleting the first, the $n+1$ rows, and the corresponding columns, i.e. 

$$
\tilde{A}^{(2)} = \begin{bmatrix}
A_{(2,n,2:n)}^{(2)} & A_{(2,n,n+2:2n)}^{(2)} \\
A_{(n+2:2n,2:n)}^{(2)} & A_{(n+2:2n,n+2:2n)}^{(2)}
\end{bmatrix}.
$$

2. We apply now exactly the same two steps of 1., to the new size reduced matrix $\tilde{A}^{(2)}$. In other words, we choose a symplectic Householder transformation $\tilde{H}_3$, which means to compute a vector $\tilde{v}_3 = [u_3^T, w_3^T]^T$ with $u_3 \in \mathbb{R}^{n-1}$, $w_3 \in \mathbb{R}^{n-1}$ and a real $c_3$ such that $\tilde{H}_3 = I + c_3 \tilde{v}_3 \tilde{v}_3^T$ zero out the entries 2 through $n-1$ and the entries $n$ through $2n-2$ of the first column of $\tilde{A}^{(2)}$ with $\tilde{H}_3 e_1 = e_1 \in \mathbb{R}^{2n-2}$. The transformation $\tilde{H}_3$ corresponds to the transformation $T_2$, in Theorem I. The direction vector $\tilde{v}_3$ of $\tilde{H}_3$ is given by $\tilde{v}_3 = A_{(2,2)}^{(3)} e_1 + \tilde{A}^{(2)}(n,1)e_n - \tilde{A}^{(2)}(:,1)$, where $A_{(2,2)}^{(3)}$ is an arbitrary non zero scalar. $\tilde{H}_3$ leaves unchanged the $n$th row of $\tilde{H}_3\tilde{A}^{(2)}$. We get

$$
\tilde{A}^{(3)} = \tilde{H}_3 \tilde{A}^{(2)} = \begin{bmatrix}
A_{(2,2)}^{(3)} & A_{(2,3:n)}^{(3)} & A_{(2,n+2:2n)}^{(3)} \\
0 & A_{(3:n,3:n)}^{(3)} & A_{(3:n,n+2:2n)}^{(3)} \\
A_{(n+2,2)}^{(3)} & A_{(n+2,n+2:2n)}^{(3)} & A_{(n+3:2n,n+2:2n)}^{(3)}
\end{bmatrix}.
$$

Remark that the $n$th component of $\tilde{v}_3$ is zero. Take now $v_3 = [0 \ u_3^T \ 0 \ w_3^T]^T$ and set $H_3 = I + c_3 v_3 v_3^T$. Then $H_3$ is obviously a symplectic Householder transformation of order $2n$. The components 1, $n+1$ and $n+2$ of $v_3$ are equal to zero. Thus $H_3$ leaves the rows 1, $n+1$ and $n+2$ of $H_3 A^{(2)}$ unchanged and satisfy $H_3 e_1 = e_1$, $H_3 e_2 = e_2$ and $H_3 e_{n+1} = e_{n+1}$. Thus $H_3$ leaves the first and the $n+1$th columns of $H_3 A^{(2)}$ unchanged and zero out the entries 3 through $n$ and the entries $n+3$ through $2n$ of the second column.

We have

$$
A^{(3)} = H_3 A^{(2)} = \begin{bmatrix}
A_{(1,1)}^{(1)} & A_{(1,2)}^{(2)} & A_{(1,3:n)}^{(2)} & A_{(1,n+1)}^{(1)} & A_{(1,n+2:2n)}^{(2)} \\
0 & A_{(2,2)}^{(1)} & A_{(2,3:n)}^{(2)} & A_{(2,n+1)}^{(1)} & A_{(2,n+2:2n)}^{(2)} \\
0 & 0 & A_{(3:n,3:n)}^{(3)} & 0 & A_{(3:n,n+2:2n)}^{(3)} \\
A_{(n+1,1)}^{(0)} & A_{(n+1,2)}^{(2)} & A_{(n+1,3:n)}^{(2)} & A_{(n+1,n+1)}^{(1)} & A_{(n+1,n+2:2n)}^{(2)} \\
0 & A_{(n+2,2)}^{(2)} & A_{(n+2,3:n)}^{(2)} & 0 & A_{(n+2,n+2:2n)}^{(2)} \\
0 & 0 & A_{(n+3:2n,3:n)}^{(3)} & 0 & A_{(n+3:2n,n+2:2n)}^{(3)}
\end{bmatrix}.
$$
The transformation $H_3$ leaves the column 1, 2 and $n + 1$ of $H_3 A^{(2)} H_3^T$ unchanged since $H_3^T e_1 = e_1$, $H_3^T e_2 = e_2$ and $H_3^T e_{n+1} = e_{n+1}$. We get

$$A^{(3)} = H_3 A^{(2)} H_3^T = \begin{bmatrix} A^{(1)} & A^{(2)} & A^{(3)} & A^{(4)} & A^{(5)} \\ A^{(6)} & A^{(7)} & A^{(8)} & A^{(9)} & A^{(10)} \\ A^{(11)} & A^{(12)} & A^{(13)} & A^{(14)} & A^{(15)} \end{bmatrix}.$$ 

Now, deleting the rows 1, 2, $n + 1$, $n + 2$ and the columns 1, 2, $n + 1$ of $A^{(3)}$ and setting $\tilde{A}^{(3)} = \begin{bmatrix} A^{(3)}_{(3,3:n)} & A^{(3)}_{(3,3:n+2:2)} \\ A^{(3)}_{(n+3:2:3:n)} & A^{(3)}_{(n+3:2:3:n+2:2)} \end{bmatrix}$, we find $c_4 \in \mathbb{R}$ and $\tilde{v}_4 = \frac{u_4}{w_4}$, with $u_4 \in \mathbb{R}^{n-2}$ and $w_4 \in \mathbb{R}^{n-2}$ such that the action of $\tilde{H}_4 = I + c_4 \tilde{v}_4 v_4^T$ gives

$$\tilde{A}^{(4)} = \tilde{H}_4 \tilde{A}^{(3)} = \begin{bmatrix} A^{(4)}_{(3,3:n)} & A^{(4)}_{(3,3:n+2:2)} \\ A^{(4)}_{(4,3:n)} & A^{(4)}_{(4,3:n+2:2)} \\ A^{(4)}_{(n+3:2:n+3:2:n)} & A^{(4)}_{(n+3:2:n+3:2:n+2:2)} \end{bmatrix}.$$ 

The coefficient $A^{(4)}_{(3,n+2)}$ is an arbitrary chosen scalar. Taking $v_4 = [0 \ 0 \ u_4^T \ 0 \ 0 \ w_4^T]^T$ then the transformation $H_4 = I + c_4 v_4 v_4^T$ leaves unchanged the rows 1, 2, $n + 1$, $n + 2$ and columns 1, 2, and $n + 1$ of $A^{(4)} = H_4 A^{(3)}$ and creates the desired zeros in the column $n + 2$. We obtain

$$A^{(4)} = \begin{bmatrix} A^{(1)} & A^{(2)} & A^{(3)} & A^{(4)} & A^{(5)} \\ A^{(6)} & A^{(7)} & A^{(8)} & A^{(9)} & A^{(10)} \\ A^{(11)} & A^{(12)} & A^{(13)} & A^{(14)} & A^{(15)} \end{bmatrix}.$$
$H^J_1$ leaves unchanged the first, the second, the $n + 1$, $n + 2$ columns of $A(4) = H_4 A(3) H^J_1$ since $H^J_1(e_i) = e_i$ for $i = 1, 2, n + 1, n + 2$. Hence, we get

$$
A(4) = 
\begin{bmatrix}
A^{(1)}_{(1,1)} & A^{(2)}_{(1,2)} & A^{(4)}_{(1,3:n)} & A^{(1)}_{(1,n+1)} & A^{(3)}_{(1,n+2)} & A^{(4)}_{(1,n+3:2n)} \\
\begin{array}{c}
0 \\
0 \\
0 \\
A^{(0)}_{(n+1,1)} \\
0 \\
0
\end{array}
& A^{(2)}_{(2,2)} & A^{(4)}_{(2,3:n)} & A^{(2)}_{(2,n+1)} & A^{(4)}_{(2,n+2)} & A^{(4)}_{(2,n+3:2n)} \\
\begin{array}{c}
0 \\
0 \\
A^{(0)}_{(n+1,2)} \\
A^{(2)}_{(n+2,2)} \\
0 \\
0
\end{array}
& A^{(3)}_{(3,3:n)} & 0 & A^{(3)}_{(3,n+2)} & A^{(4)}_{(3,n+3:2n)} \\
\begin{array}{c}
A^{(n+1,1)} \\
A^{(n+1,2)} \\
0 \\
A^{(n+2,2)} \\
0 \\
A^{(n+3:2n,3:n)}
\end{array}
& 0 & 0 & 0 & A^{(n+2,n+2)} & A^{(4)}_{(n+3:2n,n+3:2n)}
\end{bmatrix}
$$

3. The $j$th step is now clear. It involves two sub-steps. The first consists in finding $H_{2j-1}$, i.e. the scalar $c_{2j-1}$ and the vector $v_{2j-1}$ such that $H_{2j-1} = I + c_{2j-1} v_{2j-1}v_{2j-1}^T$ leaves the rows $1, \ldots, j - 1$, the rows $n + 1, \ldots, n + j$, the columns $1, \ldots, j - 1$, and the columns $n + 1, \ldots, n + j - 1$ of $H_{2j-1} A^{(2j-2)}$ unchanged and zero out the entries $j + 1$ through $n$ and the entries $n + j + 1$ through $2n$ of the $j$th column. The vector $v_{2j-1} \in \mathbb{R}^{2n}$ has the structure $v_{2j-1} = [0^T, u_{2j-1}^T, 0^T, w_{2j-1}^T]^T$, with $u_{2j-1} \in \mathbb{R}^{n-j+1}$, $w_{2j-1} \in \mathbb{R}^{n-j+1}$. The first component of $w_{2j-1}$ is zero. Thus $H_{2j-1} e_i = e_i$ for $i = 1, \ldots, j$ and for $i = n + 1, \ldots, n + j - 1$. The $j$th column $H_{2j-1} A^{(2j-2)}(:, j)$ is transformed as follows

$$
H_{2j-1} A^{(2j-2)}(:, j) = \begin{bmatrix}
A^{(2j-2)}(1 : j - 1, j) \\
A^{(2j-1)}(j, j) \\
0 \\
A^{(2j-2)}(n + 1 : n + j, j) \\
0
\end{bmatrix}
\begin{bmatrix}
\{j - 1\} \\
\{1\} \\
\{n - j\} \\
\{j\} \\
\{n - j\}
\end{bmatrix}
$$

The entry $A^{(2j-1)}(j, j)$ is a free parameter.

The multiplication of $H_{2j-1} A^{(2j-2)}$ on the right by $H^J_{2j-1}$ leaves the columns $1, \ldots, j$, and the columns $n + 1, \ldots, n + j - 1$, of $H_{2j-1} A^{(2j-2)} H^J_{2j-1}$ unchanged. The coefficient $c_{2j-1}$, the vector $v_{2j-1}$ and hence the symplectic transformation $H_{2j-1}$ are simply and explicitly given by Theorem 11. The matrix $A^{(2j-1)} = H_{2j-1} A^{(2j-2)} H^J_{2j-1}$ has the desired form. Let us set $H_{2j-1} = I + c_{2j-1} \check{v}_{2j-1} \check{v}_{2j-1}^T$, $\check{v}_{2j-1} = [u_{2j-1}^T, w_{2j-1}^T]^T$, where $[u_{2j-1}, w_{2j-1}] \in \mathbb{R}^{\alpha_j \times 2}$, with $\alpha_j = n - j + 1$ and $A^{(2j-2)}(:, j)$ the $j$th column of $A^{(2j-2)}$ obtained from $A^{(2j-2)}(:, j)$ by deleting the rows $1, \ldots, j - 1$ and rows $n + 1, \ldots, n + j - 1$. We
obviously obtain \( \tilde{H}_{2j-1} \tilde{A}^{(2j-2)}(\cdot; j) = A^{(2j-1)}(j; j)e_1 + A^{(2j-2)}(n + j; j)e_{\alpha_j+1} \). Here \( e_1 \) and \( e_{\alpha_j+1} \) denote the first and the \( \alpha_j+1 \)th canonical vectors of \( \mathbb{R}^{2\alpha_j} \).

In a similar way, the second sub-step consists in finding \( H_{2j} \), i.e. the scalar \( c_{2j} \) and the vector \( v_{2j} \) such that \( H_{2j} = I + c_{2j} v_{2j} v_{2j}^T \) leaves the rows 1, \ldots, \( j \), the rows \( n + 1, \ldots, n + j \), the columns 1, \ldots, \( j \), and the columns \( n + 1, \ldots, n + j - 1 \) of \( H_{2j} A^{(2j-1)} \) unchanged and zero out the entries \( j + 2 \) through \( n \) and the entries \( n + j + 1 \) through \( 2n \) of the \( n + j \)th column. The vector \( v_{2j} \in \mathbb{R}^{2n} \) has the structure \( v_{2j} = [0^T, u_{2j}^T, 0^T, w_{2j}^T]^T \), with \( u_{2j} \in \mathbb{R}^{n-j} \), \( w_{2j} \in \mathbb{R}^{n-j} \). Thus \( H_{2j} e_i = e_i \) for \( i = 1, \ldots, j \) and for \( i = n + 1, \ldots, n+j \). The \( n+j \)th column of \( H_{2j} A^{(2j-1)}(\cdot; n + j) \) is transformed as follows

\[
H_{2j} A^{(2j-1)}(\cdot; n + j) = \begin{bmatrix}
A^{(2j-1)}(1; j, n + j) \\
A^{(2j)}(j + 1, n + j) \\
0 \\
A^{(2j-1)}(n + 1; n + j, n + j) \\
0
\end{bmatrix}
\begin{cases}
\{ j \} \\
\{ 1 \} \\
\{ n - j - 1 \} \\
\{ j \} \\
\{ n - j \}
\end{cases}.
\]

The entry \( A^{(2j)}(j + 1, n + j) \) is a free parameter.

The multiplication of \( H_{2j} A^{(2j-1)} \) on the right by \( H_{2j}^T \) leaves the columns 1, \ldots, \( j \), and the columns \( n + 1, \ldots, n + j \), of \( H_{2j} A^{(2j-1)} H_{2j}^T \) unchanged. The coefficient \( c_{2j} \), the vector \( v_{2j} \) and hence the symplectic transformation \( H_{2j} \) are explicitly given by Theorem 1. The matrix \( A^{(2j)} = H_{2j} A^{(2j-1)} H_{2j}^T \) has the desired form.

Let us set \( \tilde{H}_{2j} = I + c_{2j} \tilde{v}_{2j} \tilde{v}_{2j}^T \), with \( \tilde{v}_{2j} = [u_{2j}^T, w_{2j}^T]^T \), where \( [u_{2j}, w_{2j}] \in \mathbb{R}^{\beta_j \times 2} \), \( \beta_j = n - j \) and \( \tilde{A}^{(2j-1)}(\cdot; n + j) \) the \( n + j \)th column of \( \tilde{A}^{(2j-1)} \) obtained from \( A^{(2j-1)}(\cdot; n + j) \) by deleting the rows 1, \ldots, \( j \) and rows \( n + 1, \ldots, n + j \). We obviously obtain \( \tilde{H}_{2j} \tilde{A}^{(2j-1)}(\cdot; n + j) = A^{(2j)}(j + 1, n + j)e_1 \). Here \( e_1 \) denotes the first canonical vector of \( \mathbb{R}^{2\beta_j} \).

Thus, it is worth noting that each step \( j \) involves two free parameters \( A^{(2j-1)}(j; j) \) and \( A^{(2j)}(j + 1, n + j) \), and that these parameters are located as highlighted above, in the corresponding symplectic Householder transformations \( H_{2j-1} \) and \( H_{2j} \) (or equivalently \( \tilde{H}_{2j-1} \) and \( \tilde{H}_{2j} \)).

At the last step (the \( n - 1 \)th step), we obtain

\[
H_{2n-2} \ldots H_2 H_1 A(H_{2n-2} \ldots H_2 H_1)^T = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = H \in \mathbb{R}^{2n \times 2n},
\]

with \( H_{11}, H_{21}, H_{22} \) upper triangular and \( H_{12} \) upper Hessenberg. We get \( A = S^T H S \) with \( S = H_{2n-2} \ldots H_1 \). The entries of the diagonal of \( H_{11} \) are the free parameters \( A^{(2j-1)}(j; j) \), i.e. \( H_{11}(j, j) = A^{(2j-1)}(j, j) \) for \( j = 1, \ldots, n \). Also,
The entries of the sub-diagonal of $H_{12}$ are the free parameters $A^{(2j)}(j + 1, n + j)$, i.e. $H_{12}(j + 1, j) = A^{(2j)}(j + 1, n + j)$ for $j = 1, \ldots, n - 1$. We propose here the algorithm in its general version, written in pseudo Matlab code, for computing the reduction of a matrix to the upper $J$-Hessenberg form, via symplectic Householder transformations (JHSH algorithm).

**Algorithm 3.** function $[S,H]=\text{JHSH}(A)$

```matlab
function [S,H]=JHSH(A)

  twon = size(A(:,1)); n = twon/2; S = eye(twon);
  for j = 1 : n - 1
    J = [zeros(n-j+1), eye(n-j+1); -eye(n-j+1), zeros(n-j+1)];
    ro = [j : n, n+j : 2n]; co = [j : n, n+j : 2n];
    [c, v] = sh2(A(ro,j));
    % Updating $A$:
    A(ro,co) = A(ro,co) + c * v * (v' * J * A(ro,co));
    A(:,co) = A(:,co) - (A(:,co) * (c * v)) * v' * J;
    % Updating $S$ (if needed):
    S(ro,2:end) = S(ro,2:end) + c * (v * v') * J * S(ro,2:end);
  end
```

**Algorithm 4.** function $[c, v] = \text{sh1}(a)$

```matlab
function [c, v] = sh1(a)
%compute c and v such that $T_1a = \rho e_1$,
%$\rho$ is a free parameter, and $T_1 = (\text{eye}(\text{twon}) + c * v * v' * J)$;
  twon = length(a); n = twon/2;
  J = [zeros(n), eye(n); -eye(n), zeros(n)];
  choose $\rho$; aux = a(1) - $\rho$;
  if aux == 0
    c = 0; v = zeros(twon,1); %T = eye(twon);
  elseif a(n+1) == 0
```
Algorithm 5. function \([c, v] = \text{sh2}(a)\)
%compute \(c\) and \(v\) such that \(T_2 e_1 = e_1\), and \(T_2 a = \mu e_1 + \nu e_{n+1}\),
%\(\mu\) is a free parameter, and \(T_2 = (\text{eye}(\text{twon}) + c * v * v' * J)\);
twon = length(a); \(n = \text{twon}/2\);
\(J = [\text{zeros}(n), \text{eye}(n); -\text{eye}(n), \text{zeros}(n)];\)
if \(n == 1\)
    \(v = \text{zeros}(\text{twon}, 1); \ c = 0; \ %T = \text{eye}(\text{twon});\)
else
    choose \(\mu;\)
    \(v = a(n+1);\)
    if \(v == 0\)
        display(‘division by zero’)
        return
    else
        \(v = \mu e_1 + \nu e_{n+1} - a, \ c = \frac{1}{a(n+1)(a(1)-\mu)};\)
    end
end

3.3. JHOSH, JHMSH algorithms

From an algebraic point of view, JHSH is the analog in the symplectic case, of the algorithm performing the Hessenberg reduction of a matrix via Householder transformations in the Euclidean case. Recall that JHSH involves two free parameters at each steps, and the involved symplectic Householder transformations are not orthogonal. In the sequel, we show how one can take benefit from these free parameters in some optimal way. In order to get an algorithm numerically stable as possible, the free parameters will be chosen so that the symplectic Householder transformations used in the reduction have minimal norm-2 condition number. The choice of such parameters is as follows [9]:

\[ \text{JHOSH, JHMSH algorithms} \]
Theorem 6. Let \( \{e_1, \ldots, e_{2n}\} \) be the canonical basis of \( \mathbb{R}^{2n} \) and \( a \in \mathbb{R}^{2n} \) given. Take \( \rho = \text{sign}(a(1)) \|a\|_2 \) and \( \mu = a(1) \pm \xi, \nu = a(n+1) \) with \( \xi = \sqrt{\sum_{i=2, i \neq n+1}^{2n} a(i)^2} \). Setting

\[
c_1 = -\frac{1}{\rho a^T e_1}, \quad v_1 = \rho e_1 - a, \quad c_2 = -\frac{1}{a^T (\mu e_1 + \nu e_{n+1})}, \quad v_2 = \mu e_1 + \nu e_{n+1} - a,
\]

then

\[
T_1 = I + c_1 v_1 v_1^T \quad \text{(respectively} \quad T_2 = I + c_2 v_2 v_2^T \quad \text{respectively)},
\]  

(12)

with \( T_1 \) (respectively \( T_2 \)) has the minimal norm-2 condition number.

Proof. See [9].

For these choices of the free parameters, we refer to \( T_1 \) (respectively \( T_2 \)) as the first optimal symplectic Householder (osh1) transformation (respectively the second optimal symplectic Householder osh2) transformation. This optimal version of JHSH is referred to as JHOSH algorithm and is given as follows:

Algorithm 7. function [S,H]=JHOSH(A)

replace in the body of JHSH the sh1 by osh1 and sh2 by osh2.
end.

The pseudo code Matlab of osh1 and osh2 is a follows

Algorithm 8. function [c, v] = osh1(a)

twon = length(a); n = twon/2;
J = [zeros(n), eye(n); -eye(n), zeros(n)];
\( \rho = \text{sign}(a(1)) \|a\|_2; \) aux = a(1) - \( \rho; \)
if aux == 0
c = 0; v = zeros(twon, 1); 9T = eye(twon);
elseif a(n + 1) == 0
display('division by zero');
return
else
v = \( \frac{a}{aux}; \) c = \( \frac{aux^2}{\rho \ast a(n + 1)}; \) v(1) = 1;
9T = (eye(twon) + c \ast v \ast v' \ast J);
end
end
Algorithm 9. function \([c, v] = osh2(a)\)

\[
\text{twon} = \text{length}(u); \ n = \text{twon}/2;
\]

\[
J = [\text{zeros}(n), \text{eye}(n); -\text{eye}(n), \text{zeros}(n)];
\]

\[
\text{if } n == 1
\]

\[
v = \text{zeros}(\text{twon}, 1); \ c = 0; \ %T = \text{eye}(\text{twon});
\]

\[
\text{else}
\]

\[
I = [2 : n, n + 2 : \text{twon}]; \ \xi = \text{norm}(a(I));
\]

\[
\text{if } \xi == 0
\]

\[
v = \text{zeros}(\text{twon}, 1); \ c = 0; \ %T = \text{eye}(\text{twon});
\]

\[
\text{else}
\]

\[
v = a(n + 1);
\]

\[
\text{if } v == 0
\]

\[
\text{display}(\text{’division by zero’})
\]

\[
\text{return}
\]

\[
\text{else}
\]

\[
v = -a/\xi; \ v(1) = 1; \ v(n + 1) = 0; \ c = \xi/v;
\]

\[
%T = (\text{eye}(\text{twon}) + c \times v \times v' \times J);
\]

\[
\text{end}
\]

\[
\text{end}
\]

\[
\text{end}
\]

We have seen that the symplectic Householder transformations used in JHOSH algorithm have minimal norm-2 condition number, and thus numerically, JHOSH presents a significant advantage over JHSH. However, all these symplectic Householder transformations are not orthogonal. It is well known that it is not possible to handle a SR decomposition using only transformations which are both symplectic and orthogonal (see [3]). Nevertheless, we will show that half of them (all the transformations \(H_{2j}\) above) may be replaced by specified transformations which are both orthogonal and symplectic. Furthermore, we will show that the two type of orthogonal and symplectic transformations, introduced by Paige et al. \([6, 12]\) can be used to replace the symplectic transformations \(H_{2j}\), to zero desired components of a vector. The first type is

\[
H(k, w) = \begin{pmatrix}
\text{diag}(I_{k-1}, P) & 0 \\
0 & \text{diag}(I_{k-1}, P)
\end{pmatrix}, \quad (13)
\]

where

\[
P = I - 2ww^T/ww^Tw, \ w \in \mathbb{R}^{n-k+1}.
\]
The transformation $H(k, w)$ is just a direct sum of two ordinary $n$-by-$n$ Householder matrices [14]. We refer to $H(k, w)$ as Van Loan’s Householder transformations. The second type is

$$J(k, c, s) = \begin{pmatrix} C & S \\ -S & C \end{pmatrix},$$

(14)

where $c^2 + s^2 = 1$, and

$$C = \text{diag}(I_{k-1}, c, I_{n-k}),$$

$$S = \text{diag}(0_{k-1}, s, 0_{n-k}).$$

$J(k, c, s)$ is a Givens transformation, which is an ordinary $2n$-by-$2n$ Givens rotation that rotates in planes $(k, k+n)$ [14]. We refer to $J(k, c, s)$ as Van Loan’s Givens rotation. Van Loan’s Householder and Givens transformations are both orthogonal and symplectic. It is worth noting that for $i \neq k$ and $i \neq n+k$, we have $J(k, c, s)e_i = e_i$. Also, we have $J(k, c, s)e_k = ce_k - se_{n+k}$ and $J(k, c, s)e_{n+k} = se_k + ce_{n+k}$. Thus, $J(k, c, s)$ leaves unchanged all the rows of $J(k, c, s)a$ except rows $k$ and $n+k$. It is obvious also that $H(k, w)e_i = e_i$ for $i = 1, \ldots, k-1$ and $i = n+1, \ldots, n+k-1$. The modification of the even sub-steps of JHOSH (or JHSH) algorithm is as follows. Let $A = [a_1, \ldots, a_n, a_{n+1}, \ldots, a_{2n}] \in \mathbb{R}^{2n \times 2n}$ be a given matrix and set $A^{(0)} = A$. The first sub-step is obtained by creating the desired zeros in the first column, via the $H_1$ as above. The updated matrix is $A^{(1)}$. Now, for creating the desired zeros in the column $n+1$ and keeping the first column unchanged, we shall use the Van Loan’s transformations, instead of $H_2$. For $k = n, \ldots, 2$, we compute $J(k, c, s)$ such that a zero is created in position $n+k$ in the $n+1$th column of $J(k, c, s)A^{(1)}$. The first column as well as the already created zeros remain unchanged. The transformation $H(2, w)$ leaves unchanged the first and the $n+1$ columns of the updated matrix $A^{(2)} = H(2, w)A^{(2)}H(2, w)^T$. 

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At the \( j \)th step, the first sub-step is obtained by creating the desired zeros in the \( j \)th column, via the \( H_{2j-1} \) as in JHOSH. The updated matrix is \( A^{(2j-1)} \).

Now, the desired zeros in the column \( n+j \) are created by using the Van Loan’s givens rotations, instead of \( H_{2j} \). For \( k = n, \ldots, j+1 \), we compute \( J(k, c, s) \) such that a zero is created in position \( n+k \) in the \( n+j \)th column of \( J(k, c, s)A^{(2j-1)} \). The columns \( 1, \ldots, j \) and \( n+1, \ldots, n+j-1 \) as well as the already created zeros in the current \( n+j \) column of \( A^{(2j-1)} \) remain unchanged. The columns \( 1, \ldots, j \) and \( n+1, \ldots, n+j \) of \( J(k, c, s)A^{(2j-1)} \) leave unchanged when the latter is multiplied on the right by \( J(k, c, s)^T \). The matrix \( A^{(2j-1)} \) is then updated with \( A^{(2j)} = J(k, c, s)A^{(2j-1)}J(k, c, s)^T \). So the entries at positions \( n+j+1, \ldots, 2n \) in the \( n+j \) column of \( A^{(2j)} \) are zeros.

Now, we compute \( w \) so that the action of Van Loan’s Householder in the product \( H(j, w)A^{(2j)} \) creates zeros in the positions \( j+2, \ldots, n \) in the \( n+j \)th column. The columns \( 1, \ldots, j \) and \( n+1, \ldots, n+j-1 \) as well as the already created zeros in the current \( n+j \) column of \( A^{(2j)} \) remain unchanged. \( H(j, w) \) leaves unchanged the columns \( 1, \ldots, j \) and \( n+1, \ldots, n+j \) of the updated matrix \( A^{(2j)} = H(j, w)A^{(2j)}H(j, w)^T \). We obtain the following algorithm

**Algorithm 10. function** \([S,H]=JHMSH(A)\)

\[
twon = \text{size}(A(:,1)); \ n = twon/2; S = \text{eye}(twon);
for j = 1:n-1
\]
\[
J = [\text{zeros}(n-j+1), \text{eye}(n-j+1), -\text{eye}(n-j+1), \text{zeros}(n-j+1)];
ro = [j:n, n+j:2n]; co = [j:n, n+j:2n];
[c, v] = osh2(A(ro, j));

% Updating A:
A(ro, co) = A(ro, co) + c * v * (v’ * J * A(ro, co));
A(:, co) = A(:, co) - (A(:, co) * (c * v)) * v’ * J;

% Updating S (if needed):
S(:, co) = S(:, co) - c * (v * v’) * J * S(:, co);
for k = 2n : n+j+1,
\[
[c, s] = vlq(k, A(:, n+j));
\]

% Updating A:
\[
\begin{bmatrix}
A(k, co) \\
A(n+k, co)
\end{bmatrix} = \begin{bmatrix}
c & s \\
-s & c
\end{bmatrix} \begin{bmatrix}
A(k, co) \\
A(n+k, co)
\end{bmatrix};
\]
\[
\begin{bmatrix}
A(:, k) \\
A(:, n+k)
\end{bmatrix} = \begin{bmatrix}
c & -s \\
s & c
\end{bmatrix} \begin{bmatrix}
A(:, k) \\
A(:, n+k)
\end{bmatrix};
\]

% Updating S (if needed):
\[
\begin{bmatrix}
S(:, k) & S(:, n + k)
\end{bmatrix} = \begin{bmatrix}
S(:, k) & S(:, n + k)
\end{bmatrix} \begin{bmatrix}
c & -s \\
s & c 
\end{bmatrix};
\]

end

if \( j \leq n - 2 \)

[\( \beta, w \)] = vlh(\( j+1, A(:, n+j) \));

% Updating A:
\( A(j+1 : n, co) = A(j+1 : n, co) - \beta * w * w' * A(j+1 : n, co) \)
\( A(j+1+n : 2n, co) = A(j+1+n : 2n, co) - \beta * w * w' * A(j+1+n : 2n, co) \);
\( A(:, j+1 : n) = A(:, j+1 : n) - \beta * A(:, j+1 : n) * w * w' \);
\( A(:, n+j+1 : 2n) = A(:, n+j+1 : 2n) - \beta * A(:, n+j+1 : 2n) * w * w' \);

% Updating S (if needed):
\( S(:, j+1 : n) = S(:, j+1 : n) - \beta * S(:, j+1 : n) * w * w' \);
\( S(:, n+j+1 : 2n) = S(:, n+j+1 : 2n) - \beta * S(:, n+j+1 : 2n) * w * w' \);

end

end

Algorithm 11. function \([c, s] = vlg(k,a)\)

twon = length(a); n = twon/2;
\[ r = \sqrt{a(k)^2 + a(n+k)^2}; \]
if \( r = 0 \) then \( c = 1 \); \( s = 0 \);
else \( c = \frac{a(k)}{r}; \)
\( s = \frac{a(n+k)}{r}; \)
end

Algorithm 12. function \([\beta, w] = vlh(k,a)\)

twon = length(a); n = twon/2;
% \( w = (w_1, \ldots, w_{n-k+1})^T; \)
\( r1 = \sum_{i=2}^{n-k+1} a(i + k - 1)^2; \)
\[ r = \sqrt{a(k)^2 + r1}; \]
\( w_1 = a(k) + \text{sign}(a(k))r; \)
\( w_i = a(i + k - 1) \) for \( i = 2, \ldots, n - k + 1; \)
\[ r = w_1^2 + r1; \]
\( \beta = \frac{2}{r}; \)
% \( P = I - \beta ww^T; \) \( (H(k,w)a)_i = 0 \) for \( i = k+1, \ldots, n. \)
end
3.4. Numerical experiments

In this work, we restrict ourselves to the algorithmic aspect of J Hessenberg reduction of a matrix, via symplectic Householder transformations. We showed how this reduction may be handled. The reduction process involves free parameters. We outlined how some optimal choice can be done, which gave rise to JHOSH algorithm. The latter uses only symplectic Householder transformations, which are not orthogonal. We succeed to replace half of them by transformations which are both orthogonal and symplectic. This gave rise to JHMSH algorithm, which behaves with satisfactory properties and is better than all the previous ones. Very important questions on numerical aspects as for example the other choices of the free parameters, breakdowns, near breakdowns, different strategies to cure these near breakdowns, and also their early prediction before performing computations which are not necessary, and so on, deserves a detailed study. This will be the focus of a forthcoming paper. Nevertheless, we propose below two significant numerical examples in the following sense: in the literature, to our knowledge, only the JHESS algorithm is used to perform a J-Hessenberg reduction of a matrix, with symplectic transformations. The JHESS belongs to the same class of algorithms as are JHOSH and JHMSH. The figures below compare JHMSH, JHMSH2 (which is a slight modification of JHMSH) and JHESS. The numerical examples show that the later, as presented in [2] meets a fatal breakdown and thus fails for all \( n \geq 3 \), while the JHMSH, JHMSH2, with a slight modification, work up with very satisfactory precision. Let us consider the following matrix

\[
A = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad \text{with } M_{11} = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ & \ddots & \ddots \\ & & 2 & 1 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ & \ddots & \ddots \\ & & 2 & 1 \end{pmatrix},
\]

\[
M_{21} = \begin{pmatrix} 0 & 2 & & \\ 0 & 1 & \ddots & \ddots \\ & \ddots & \ddots & 2 \\ & & 0 & 1 \end{pmatrix} \quad \text{and } M_{22} = \begin{pmatrix} 1 & & \\ 3 & 1 & \ddots \\ & \ddots & \ddots \\ & & 3 & 1 \end{pmatrix}.
\]

Each block \( M_{ij} \) is of size \( n \times n \). We obtain
| $n$ | Loss of $J$-Orthogonality $\|I - S' S\|_2$ | Error of the reduction $\|H - S' AS\|_2$ |
|-----|---------------------------------|---------------------------------|
|     | $JH\text{ESS}$ | $JH\text{MHS}$ | $JH\text{MHS\_2}$ | $JH\text{ESS}$ | $JH\text{MHS}$ | $JH\text{MHS\_2}$ |
| 2   | fails | $2.5168e-16$ | $3.1402e-16$ | fails | $1.0361e-15$ | $1.0262e-15$ |
| 3   | fails | $1.0412e-15$ | $9.7146e-16$ | fails | $1.0623e-14$ | $5.6678e-15$ |
| 4   | fails | $3.1015e-15$ | $3.6572e-15$ | fails | $6.3153e-14$ | $2.9172e-14$ |
| 5   | fails | $2.8250e-14$ | $3.3284e-14$ | fails | $1.4279e-13$ | $6.8545e-14$ |
| 6   | fails | $4.1918e-14$ | $4.3812e-14$ | fails | $2.5845e-13$ | $1.6997e-13$ |
| 7   | fails | $2.0709e-13$ | $1.1965e-13$ | fails | $2.7021e-12$ | $5.7755e-13$ |
| 8   | fails | $1.7497e-12$ | $7.4477e-13$ | fails | $1.0972e-11$ | $3.5435e-12$ |
| 9   | fails | $1.2988e-10$ | $5.8035e-11$ | fails | $1.0461e-09$ | $3.8219e-10$ |
| 10  | fails | $4.8062e-10$ | $1.1476e-10$ | fails | $3.4164e-09$ | $7.1532e-10$ |
| 11  | fails | $6.6942e-10$ | $1.7784e-10$ | fails | $4.7274e-09$ | $5.7041e-10$ |
| 12  | fails | $4.5165e-10$ | $1.7250e-10$ | fails | $1.1306e-08$ | $8.0399e-10$ |
| 13  | fails | $7.9908e-10$ | $2.9785e-10$ | fails | $7.4063e-09$ | $1.7637e-09$ |
| 14  | fails | $7.6406e-10$ | $1.7497e-10$ | fails | $8.3607e-09$ | $1.0158e-09$ |
| 15  | fails | $1.7248e-09$ | $1.9073e-10$ | fails | $1.1932e-08$ | $9.8201e-10$ |
| 16  | fails | $6.9530e-10$ | $1.9133e-10$ | fails | $5.6770e-09$ | $1.1922e-09$ |
| 17  | fails | $1.9515e-09$ | $2.1889e-10$ | fails | $1.4054e-08$ | $1.2598e-09$ |
| 18  | fails | $1.1824e-09$ | $6.2781e-10$ | fails | $1.4967e-07$ | $5.7161e-09$ |
| 19  | fails | $3.6906e-09$ | $2.2293e-10$ | fails | $2.5400e-08$ | $1.4194e-09$ |
| 20  | fails | $2.8172e-09$ | $2.6019e-10$ | fails | $1.2725e-07$ | $2.0413e-09$ |
| 21  | fails | $1.5606e-08$ | $8.6765e-10$ | fails | $2.6936e-07$ | $5.1208e-09$ |
| 22  | fails | $1.0522e-09$ | $2.4081e-10$ | fails | $1.1047e-08$ | $1.9222e-09$ |
| 23  | fails | $3.8242e-09$ | $2.6805e-10$ | fails | $2.1954e-08$ | $1.6025e-09$ |
| 24  | fails | $1.1119e-09$ | $4.8392e-10$ | fails | $5.6800e-08$ | $3.2751e-09$ |
| 25  | fails | $3.9755e-09$ | $4.2710e-10$ | fails | $2.2816e-08$ | $2.6839e-09$ |
| 26  | fails | $1.8132e-09$ | $1.4496e-09$ | fails | $3.2416e-08$ | $1.0678e-08$ |
| 27  | fails | $1.2417e-08$ | $1.1257e-09$ | fails | $1.0768e-07$ | $1.0010e-08$ |
| 28  | fails | $2.2564e-09$ | $1.1255e-09$ | fails | $1.4462e-07$ | $8.2262e-09$ |
| 29  | fails | $3.9904e-08$ | $2.3791e-09$ | fails | $6.3257e-07$ | $4.1958e-08$ |
| 30  | fails | $1.6554e-09$ | $5.4776e-10$ | fails | $5.9380e-08$ | $4.0406e-09$ |

Consider now the Hamiltonian case:
$A = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$, where $M_{11} = \begin{pmatrix} 1 & 1 \\ \vdots & \ddots \\ 2 & 1 \end{pmatrix}$, $M_{12} = \begin{pmatrix} 1 & 2 \\ \vdots & \ddots \\ 2 & 1 \end{pmatrix}$, $M_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 3 \\ \vdots & \ddots \\ 3 & 1 \end{pmatrix}$ and $M_{22} = -M_{11}^T$. We get

| $n$ | Loss of $J$-Orthogonality $\| I - S^T S \|_2$ | Error of the reduction $\| H - S^T A S \|_2$ |
|-----|---------------------------------|---------------------------------|
| JHESS | JHMSH | JHMSH2 | JHESS | JHMSH | JHMSH2 |
| 2 fails | 1.3843e−16 | 2.7756e−17 | fails | 3.4732e−16 | 7.5047e−16 |
| 3 fails | 2.1967e−15 | 4.1153e−15 | fails | 1.4123e−14 | 9.5826e−15 |
| 4 fails | 3.1724e−14 | 1.1623e−14 | fails | 1.0235e−13 | 1.1283e−13 |
| 5 fails | 5.5639e−13 | 4.5393e−13 | fails | 2.2678e−12 | 1.4082e−12 |
| 6 fails | 1.3229e−14 | 3.1824e−14 | fails | 1.6308e−13 | 1.8500e−13 |
| 7 fails | 1.9456e−13 | 2.9018e−13 | fails | 4.2300e−12 | 5.7276e−12 |
| 8 fails | 2.4182e−13 | 9.1255e−14 | fails | 2.6360e−12 | 1.2184e−12 |
| 9 fails | 7.0030e−12 | 4.6008e−12 | fails | 2.8308e−11 | 6.0019e−11 |
| 10 fails | 6.7908e−11 | 1.8421e−11 | fails | 1.8128e−10 | 4.2484e−11 |
| 11 fails | 1.2746e−10 | 3.6111e−11 | fails | 1.2132e−09 | 1.3393e−10 |
| 12 fails | 1.6379e−09 | 1.1448e−10 | fails | 5.6804e−09 | 1.0683e−09 |
| 13 fails | 5.7401e−09 | 1.8386e−09 | fails | 4.3477e−07 | 5.7596e−09 |
| 14 fails | 5.9220e−09 | 2.7826e−09 | fails | 1.1117e−07 | 1.1405e−08 |
| 15 fails | 1.1198e−07 | 1.5282e−08 | fails | 8.4815e−07 | 2.1596e−07 |
| 16 fails | 3.2853e−07 | 1.9260e−07 | fails | 3.6979e−06 | 8.2332e−07 |
| 17 fails | 1.0707e−06 | 1.9526e−07 | fails | 1.4713e−05 | 3.9805e−06 |
| 18 fails | 2.2014e−05 | 2.2887e−05 | fails | 1.3000e−03 | 4.6621e−04 |
| 19 fails | 7.0710e−05 | 2.0118e−05 | fails | 1.5000e−03 | 4.0607e−04 |
| 20 fails | 7.9995e−04 | 4.0086e−05 | fails | 4.1000e−03 | 6.8321e−04 |

4. Conclusion

In this paper, we presented a reduction of a matrix to the upper J-Hessenberg form, based on the symplectic Householder transformations, which are rank-one modification of the Identity. This reduction is the crucial step
for constructing an efficient SR-algorithm. The method is the analog of the reduction of a matrix to Hessenberg form, via Householder transformations, when instead of an Euclidean linear space, one takes a symplectic one. Then the algorithm JHOSH is derived, corresponding to an optimal choice of the free parameters. Furthermore, JHOSH is significantly improved by showing that half of these symplectic Householder transformations may be replaced by Van Loan’s symplectic and orthogonal transformations leading to two variants JHMSH and JHMSH2 which are significantly more stable numerically. The numerical experiments confirm the expected results.

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