\textbf{Su}(1, 2) Algebraic Structure of the \textit{XYZ} Antiferromagnetic Model in Linear Spin-Wave Frame

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Abstract

The \textit{XYZ} antiferromagnetic model in linear spin-wave frame is shown explicitly to have an \textit{su}(1, 2) algebraic structure: the Hamiltonian can be written as a linear function of the \textit{su}(1, 2) algebra generators. Based on it, the energy eigenvalues are obtained by making use of the similar transformations, and the algebraic diagonalization method is investigated. Some numerical solutions are given, and the results indicate that only one group solution could be accepted in physics.

PACS number(s): 03.65.Fd, 05.30.Jp

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I. INTRODUCTION

As is well known, the Heisenberg model is a simple but realistic and extensively studied solid-state system [1, 2]. According to the sign of the interaction intensity $J$, the model can be classified as the ferromagnetic type and the antiferromagnetic one. Based on the interaction intensity along the different space directions, the model can be labelled as the $XXX$, $XXZ$, and $XYZ$ ones. Recently, it has been found that the Heisenberg interaction is not localized in spin system, and it can be realized in quantum dots [3, 4], nuclear spins [5], cavity QED [6, 7]. Thus, the study of this basic model is of interests and wide applications in physical fields. In the investigation of these models, it is a basic task to get the exact solutions [8, 9, 10, 11, 12], i.e., to diagonalize the Hamiltonian. So far, only some special Heisenberg models can be exactly solved, such as $XXX$ antiferromagnetic model [13]. In general, the linear spin-wave [14, 15, 16] approximation is widely applied when we study the Heisenberg models. It has been known that the $XXZ$ model in linear spin-wave frame can be diagonalized by coherent state operators [17, 18] of $su(1, 1)$ algebra. However, for the $XYZ$ model in this frame, the method of the coherent states does not work. Therefore, it is necessary for us to develop another algebraic diagonalization method to get the energy spectrum by using the algebraic structure which the model has.

In this Letter, we review the $su(1, 1)$ coherent states of the $XXZ$ antiferromagnetic model in linear spin-wave frame. Then, the $XYZ$ antiferromagnetic model in linear spin-wave frame can be written as the generators of $su(1, 2)$ algebra. At last, the energy eigenvalues are obtained in terms of the algebraic diagonalization method, and some numerical solutions are given and discussed.

II. $XXZ$ ANTI-FERROMAGNETIC MODEL AND $su(1,1)$ COHERENT STATES

The Hamiltonian of the $XXZ$ antiferromagnetic model reads:

$$H_{XXZ} = -J \sum_{\langle i,j \rangle} (S_i^z S_j^z + S_i^x S_j^x + \eta S_i^y S_j^y) \quad (J < 0),$$

where the notation $\langle i, j \rangle$ denote the nearest neighbor bonds. Starting from the two-sublattice model and Holstein-Primakoff transformation [19]:

$$S_a^z = -s + a^\dagger a, \quad S_b^z = s - b^\dagger b,$$

$$S_a^\dagger = (2s)^{\frac{1}{2}} (1 - a^\dagger a/2s)^{\frac{1}{2}} a, \quad S_a^- = (S_a^\dagger)^\dagger,$$
\[ S_b^\dagger = (2s)^{1/2} b^\dagger (1 - b^\dagger b / 2s)^{1/2}, \quad S_b^- = (S_b^\dagger)^\dagger, \] (2)

where \( a^+ \) and \( a, (b^+, b) \) can be regarded as the creation and annihilation operators of boson on sublattice A (sublattice B), respectively, but the particle numbers \( n_a = a^+ a, n_b = b^+ b \) can’t excel \( 2s \), respectively. Because in low temperature and low excitation condition, \( < a^\dagger a >, < b^\dagger b > \ll s \), the non-linear interaction in Eq. (2) could be reasonable ignored \[20\]. Based on it, transferring the operators into momentum space, we obtain

\[ H_{XXZ} = -2ZsJ(Ns\eta - \sum_k H_k), \] (3)

\[ H_k = \eta(a_k^\dagger a_k + b_k^\dagger b_k) + \gamma_k(a_k b_k + a_k^\dagger b_k^\dagger). \] (4)

Here

\[ \gamma_k = Z^{-1} \sum_R e^{i k \cdot R} = \gamma_{-k}, \] (5)

in which \( R \) is a vector connecting an atom with its nearest neighbor, and the sum runs over the \( Z \) nearest neighbors. \( 2N \) is the total number of the lattices. \( H_k \) can be expressed as the linear combination of the \( su(1, 1) \) algebra generators in the form

\[ H_k = 2\eta E_z^k + \gamma_k(E_+^k + E_-^k), \] (6)

with

\[ E_+^k = a_k^\dagger b_k^\dagger, \quad E_-^k = a_k b_k, \quad E_z^k = \frac{1}{2}(n_k^a + n_k^b + 1), \] (7)

which obey the commutation relations of \( su(1, 1) \) Lie algebra:

\[ [E_+^k, E_-^k] = -2E_z^k, \quad [E_z^k, E_\pm^k] = \pm E_\pm^k. \] (8)

By introducing an \( su(1, 1) \) displacement operator

\[ W(\xi_k) = \exp(\xi_k E_+^k - \xi_k^* E_-^k) \] (9)

with the coherent parameter \( \xi_k = re^{i\theta} \), then we have

\[ W^{-1}(\xi_k)H_k W(\xi_k) = \alpha E_z^k + (\beta E_+^k + \beta^* E_-^k), \] (10)

where

\[ \alpha = 2\eta \cosh 2r + \gamma_k(e^{i\theta} + e^{-i\theta}) \sinh 2r, \] (11)

\[ \beta = \eta e^{i\theta} \sinh 2r + \gamma_k(\cosh^2 r + e^{2i\theta} \sinh^2 r). \] (12)
In order to diagonalize $H_k$, the coefficient before the non-Cartan generators $E^+_k$ and $E^-_k$ of Lie algebra, $\beta$, should be chosen to zero (here we set $\theta = 0$ for simplicity) and this leads to

$$\tanh 2r = -\frac{\gamma_k}{\eta}, \quad \alpha = 2\sqrt{\eta^2 - \gamma^2_k}.$$  \hspace{1cm} (13)

So if we denote

$$|\xi_k\rangle = W(\xi_k)|vac\rangle,$$  \hspace{1cm} (14)

then one has

$$H_k|\xi_k\rangle = (n_a + n_b + 1)\epsilon_k|\xi_k\rangle,$$  \hspace{1cm} (15)

$$\epsilon_k = 2Zs\sqrt{\eta^2 - \gamma^2_k}.$$  \hspace{1cm} (16)

where requiring $|\eta| \geq |\gamma_k|$. The diagonalization of the $XXZ$ antiferromagnetic model in linear spin-wave frame has turned out to be a direct product of $su(1,1)$ coherent states $\bigotimes_k|\xi_k\rangle$. One can see $\epsilon_k$ is the quantum of antiferromagnetic spin-wave, i.e. the dispersion relation. From Eq. (15), it is known that for any $k$, there exist two branches of degenerate antiferromagnetic spin-wave which the quasi-particle numbers are described by $n_a$ and $n_b$, respectively.

### III. $XYZ$ ANTIFERROMAGNETIC MODEL IN LINEAR SPIN-WAVE FRAME WITH THE $su(1,2)$ ALGEBRAIC STRUCTURE

Owing to the different interaction intensity along the different space directions, in general, the Hamiltonian of the $XYZ$ antiferromagnetic model is described by

$$H_{XYZ} = -J \sum_{<i,j>} (\eta_x S^x_i S^x_j + \eta_y S^y_i S^y_j + S^z_i S^z_j)$$

$$+(J < 0, \ \eta_x, \eta_y > 0),$$  \hspace{1cm} (17)

where we have set $\eta_z = 1$. Similar to the former case of the $XXZ$ antiferromagnetic model, the $XYZ$ antiferromagnetic model in linear spin-wave frame is given by the Hamiltonian:

$$H_{XYZ} = 2ZsJ[NS - (\sum_k H_k - 1)],$$  \hspace{1cm} (18)

$$H_k = a_k^\dagger a_k + b_k^\dagger b_k$$

$$+v_k(a_k b_{-k}^\dagger + a_{-k}^\dagger b_k) + \rho_k(a_k b_k + a_k^\dagger b_{-k}^\dagger),$$  \hspace{1cm} (19)

with

$$v_k = \frac{\eta_x - \eta_y}{2} \gamma_k, \quad \rho_k = \frac{\eta_x + \eta_y}{2} \gamma_k.$$  \hspace{1cm} (20)
If we choose

\[ I_+^k = a_k^+ b_{-k}, \quad I_-^k = a_k b_{-k}^+, \quad U_+^k = a_k b_k, \]
\[ V_+^k = b_k^+ b_{-k}, \quad V_-^k = b_k b_{-k}^+, \quad U_-^k = a_k^+ b_k^+, \]
\[ I_3^k = \frac{1}{2}(n_k^a - n_k^b), \]
\[ I_8^k = -\frac{1}{3}(n_k^a + n_k^b - 2n_k^b + 2), \]

(21)

then, one can see they obey the commutation relations of \( su(1, 2) \) Lie algebra (here we omit the momentum sign \( k \)):

\[ [I_3, I_\pm] = \pm I_\pm, \quad [I_+, I_-] = 2I_3, \quad [I_8, I_\alpha] = 0, \quad (\alpha = \pm, 3) \]
\[ [I_3, U_\pm] = \mp \frac{1}{2} U_\pm, \quad [I_8, U_\pm] = \pm U_\pm, \quad [U_+, U_-] = I_3 - \frac{3}{2} I_8, \]
\[ [I_3, V_\pm] = \mp \frac{1}{2} V_\pm, \quad [I_8, V_\pm] = \pm V_\pm, \quad [V_+, V_-] = I_3 + \frac{3}{2} I_8, \]
\[ [I_\pm, U_\mp] = \mp V_\mp, \quad [U_\pm, V_\mp] = \pm I_\mp, \quad [I_\mp, V_\pm] = \pm U_\mp, \]
\[ [I_\pm, U_\mp] = [I_\mp, V_\pm] = [U_\pm, V_\mp] = 0. \]

(22)

From Eqs. (19) and (21), \( H_k \) can be expressed as the linear combination of six generators of Lie algebra \( su(1, 2) \) and processes \( su(1, 2) \) algebraic structure, i.e.,

\[ H_k = I_3^k - \frac{3}{2} I_8^k + \rho_k (I_+^k + I_-^k) + \nu_k (U_+^k + U_-^k). \]

(23)

IV. THE DIAGONALIZATION AND THE EIGENVALUES

If we set the general linear combination form of Lie algebra \( su(1, 2) \) as

\[ H_0 = a I_+ + b I_- + c U_+ + d U_- \]
\[ + e V_+ + f V_- + g I_3 + h I_8, \]

(24)

for \( H_k \) (23), the coefficients in Eq. (24) are:

\[ a = b = \rho_k, \quad c = d = \nu_k, \quad e = f = 0, \]
\[ g = 1, \quad h = -\frac{3}{2}. \]

(25)

Until now, the coherent state operator as Eq. (9) has not been found for the XYZ antiferromagnetic model in linear spin-wave frame. But following the standard Lie algebraic theory [17, 21, 22, 23, 24], if \( H_0 \) is a linear function of the generators of a compact semi-simple Lie
group, it can be transformed into a linear combination of the Cartan operators of the corresponding Lie algebra by

$$H_1 = \mathcal{W}H_0\mathcal{W}^{-1}.$$  

(26)

Here $\mathcal{W} = \prod_i^N \exp(x_i A_i)$ is an element of the group and $\mathcal{W}^{-1}$ denotes the inverse of $\mathcal{W}$, in which $A_i$ ($i = 1, \ldots, N$) is a basis set in Cartan standard form of the semi-simple Lie algebra, and $x_i$ can be set to zero if the corresponding $A_i$ is a Cartan operator. By choosing

$$\mathcal{W} = \exp(x_{31} V_+ \exp(x_{21} I_- \exp(x_{32} U_-)) \exp(x_{12} I_+ \exp(x_{23} U_+ \exp(x_{13} V_-)),$$  

(27)

and letting the coefficients of the non-Cartan operators vanish, while substituting Eqs. (24) (25) (27) into the right-hand side of Eq. (26), we get a complete set of algebraic equations of $x_{ij}$ after lengthy computation:

$$\begin{cases}
-(h + \frac{1}{2}g)x_{13} + (a + dx_{13})x_{23} = 0 \\
c + bx_{13} + (\frac{1}{2}g - h)x_{23} + dx_{23} = 0 \\
a + dx_{13} - (g + dx_{23})x_{12} - bx_{12}^2 = 0,
\end{cases}$$  

(28)

and the Hamiltonian after the transformation of $\mathcal{W}$ becomes diagonal:

$$H_1 = \mathcal{W}H_0\mathcal{W}^{-1} = (g + dx_{23} + 2bx_{12})I_3 + (h - \frac{3}{2}dx_{23})I_8.$$  

(29)

One can see that although operator $\mathcal{W}$ is not unitary, the similar transformation (26) guarantees that the eigenvalues of $H_0$ equal those of $H_1$. This is acceptable for we are only concerned with the eigenvalues. If the total particle number $N = n^a_k + n^{-}^a_k + n^b_k$ ($n^a_k = a_k^\dagger a_k$, $n^{-}_k = b^\dagger_k b^{-}_k$, $n^b_k = b_k^\dagger b_k$), from Eq. (22), $[\Gamma, N] = 0$ ($\Gamma = I_\pm, V_\pm, U_\pm, I_3, I_8$) holds. Hence, supposing the common eigenstates of the Cartan generators $I_3$ and $I_8$ of Lie algebra $su(1, 2)$ are the Fock states $| n^a_k, n^{-}_k, n^b_k >$, i.e., for the commutative set $\{I_3, I_8, N\}$ there exist:

$$I_3 | n^a_k, n^{-}_k, n^b_k >= \frac{1}{2}(n^a_k - n^{-}_k) | n^a_k, n^{-}_k, n^b_k >,$$

$$I_8 | n^a_k, n^{-}_k, n^b_k > = -\frac{1}{3}(n^a_k + n^{-}_k + 2n^b_k + 2) | n^a_k, n^{-}_k, n^b_k >,$$

$$N | n^a_k, n^{-}_k, n^b_k > = (n^a_k + n^{-}_k + n^b_k) | n^a_k, n^{-}_k, n^b_k >.$$  

(30)
From Eqs. (29) (30) it follows the eigenvalue of the Hamiltonian (23):

\[ E = \omega_k^a n_k^a + \omega_k^b n_k^b + \omega_{-k}^b n_{-k}^b + \omega_k^E, \]
\[ \omega_k^a = \left( \frac{1}{2} g - \frac{1}{3} h + bx_{12} + dx_{23} \right), \]
\[ \omega_k^b = \left( -\frac{2}{3} h + dx_{23} \right), \]
\[ \omega_{-k}^b = \left( \frac{1}{2} g - \frac{1}{3} h - bx_{12} \right), \]
\[ \omega_k^E = -\frac{2}{3} h + dx_{23}, \]  

(31)

where the coefficients \( b, d, g, h, \) are given in Eq. (25) and \( x_{ij} \) can be obtained by solving Eq. (28), and \( \omega_k^a, \omega_k^b, \omega_{-k}^b \) are the energies of the three different magnons respectively. In fact, the order of the operators in \( W \) can be chosen arbitrarily, but the coefficients \( x_i \) are strongly dependent on the order. Although any specified order has a solution, a properly chosen order can simplify the procedure to get the \( x_i \). In general, for the Hamiltonian with \( su(n) \) (whose Cartan operators are \( A_{ij} = b_i^+ b_j \)) or the isomorphic algebra \( su(p, q) \) \((p + q = n)\) structure, the transformation operator \( W \) can be chosen as

\[ W = \exp(x_{N1}A_{N1})\exp(x_{N2}A_{N2})\ldots \]
\[ \exp(x_{2N}A_{2N})\exp(x_{1N}A_{1N}), \]

(32)

where the order of the operators of \( \exp(x_{ij}A_{ij}) \) \((i \neq j)\) is arranged according to the roots of \( A_{ij} \) in a decreasing way. For example, the root of \( A_{N1} \) is highest, and that of \( A_{1N} \) is lowest. As a rule of our choice, the right or left \( A_{ij} \) in Eq. (32) is the one that is missing in the Hamiltonian \( H_0 \); the middle operators sequence forms a circle root diagram. With this specification, in our experience the coefficients \( x_{ij} \) are relatively easy to work out. We also choose the form of \( W \) in Eq. (27) as \( \exp(x_{13}V_-)\exp(x_{12}I_+)\exp(x_{23}U_+)\exp(x_{21}I_-)\exp(x_{32}U_-)\exp(x_{31}V_+) \), and it can be proofed they lead to the same eigenvalues.

V. NUMERICAL SOLUTIONS

From the Eq. (23), maybe there have several sets of solutions, which consist the complete set. But the different sets of the solutions cannot be the eigenstate of the Hamiltonian (19) or (23) simultaneously. Only those who possess the physical meaning is the solutions we need. In order to illustrate it, we consider a concrete example of the Hamiltonian (19) or (23) with \( \eta_x = 0.8, \eta_y = 0.5, \gamma_k = 1 \). Then \( v_k = 0.15, \rho_k = 0.65 \). Using maple, one can show that there
are six sets solutions of Eq. (28), in which only one set \( x_{13} = -0.6018692595e - 1, x_{23} = 0.9389946768e - 1, x_{12} = 0.4827143955 \) leads to the positive energy \( \omega^a_k = 1.327849277, \omega^b_k = 1.014084920, \omega^{b*}_k = 0.6862356430 \) of the magnons. It is clear that only this solution is the accepted in physics. Other five sets solutions (with negative energy) are non-physical. This procedure is easily to be taken in solving the Eq. (28) and Eq. (23).

VI. CONCLUSION AND REMARKS

In conclusion, the eigenvalue problems of the \( XYZ \) antiferromagnetic model in linear spin-wave frame is solved from an algebraic point of view. To use this algebraic diagonalization method, first, it is needed to find the algebraic structure of the Hamiltonian and manage to write the Hamiltonian into a linear combination form of the algebraic generators just as Eq. (23). Second, according the particular structure of some Lie algebra, one may looking for the transformation operator. The key is that we can let the coefficients of the non-Cartan operators vanish successfully and get the solvable equations of the parameters through the transformation. Some numerical solutions check our diagonalization method, whose advantage is that the eigenvalues of \( H_0 \) equal those of \( H_1 \) although the Hamiltonian changes. Due to the different interaction intensity along the different space directions for the Heisenberg model, the complication in physical model leads to the enlargement of the algebra structure, such as the \( XXZ \) case to the \( XYZ \) case corresponds to the \( su(1,1) \) algebra to the \( su(1,2) \) one. Of course, the change of the algebra structure brings the different method in diagonalizing the Hamiltonian. It is reasonable to believe that more useful physical applications of the algebraic diagonalization method should be found. It may be possible to extend this case to higher-rank Lie algebras.

Acknowledgments

This work is in part supported by the National Science Foundation of China under Grant No. 10447103, Education Department of Beijing Province and Beihang University.

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