A COMPUTATIONAL CRITERION FOR THE KAC CONJECTURE

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Abstract. We give a criterion for the Kac conjecture asserting that the free term of the polynomial counting the absolutely indecomposable representations of a quiver over a finite field of given dimension coincides with the corresponding root multiplicity of the associated Kac-Moody algebra. Our criterion suits very well for computer tests.

1. Introduction

Let $\Gamma$ be a quiver without loops and let $n$ be its number of vertices. For any $\alpha \in \mathbb{N}^n$, let $m_\alpha(q)$ and $a_\alpha(q)$ be respectively the number of representations and absolutely indecomposable representations of $\Gamma$ over $\mathbb{F}_q$ of dimension $\alpha$. The underlying graph of $\Gamma$ defines a symmetric generalized $n \times n$ Cartan matrix $C$ with

$$c_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -b_{ij} & \text{if } i \neq j, \end{cases}$$

where $b_{ij}$ is the number of edges connecting the vertices $i$ and $j$. Let $g$ be the corresponding Kac-Moody algebra [9]. It was shown by Kac [8] that $m_\alpha(q)$ and $a_\alpha(q)$ are polynomials in $q$ with integer coefficients. Moreover, $a_\alpha$ is nonzero if and only if $\alpha$ is a root of $g$ and $a_\alpha = 1$ if and only if $\alpha$ is a real root. It was conjectured by Kac [8] that

Conjecture 1. $a_\alpha(0)$ is equal to the multiplicity of $\alpha$ in $g$.

Conjecture 2. All coefficients of $a_\alpha$ are nonnegative.

These conjectures were proved by Crawley-Boevey and Van den Bergh [3] in the case when $\alpha$ is indivisible. In the general case, there is a criterion for the first Kac conjecture given by Sevenhant, Van den Bergh and Hua [14, 7] which, however, uses the intrinsic structure of the Kac-Moody algebra and is not very suitable for testing the Kac conjecture on computer. In contrast to the latter we suggest a criterion which is well adapted for computer tests.

For any $\alpha \in \mathbb{Z}^n$, define $ht \alpha$ to be the sum of the coordinates of $\alpha$. Define the $\mathbb{Z}$-valued quadratic form $T$, called the Tits form, on $\mathbb{Z}^n$ by the matrix $\frac{1}{2}C$. Let $P$ be the set of partitions. For any multipartition $\lambda = (\lambda^1, \ldots, \lambda^n) \in P^n$, we define $|\lambda| := (|\lambda^1|, \ldots, |\lambda^n|) \in \mathbb{N}^n$ and $\lambda_k := (\lambda^1_k, \ldots, \lambda^n_k) \in \mathbb{N}^n$ for $k \geq 1$. Define $r_\lambda \in \mathbb{Q}(q)$ by

$$r_\lambda(q) := \prod_{k \geq 1} \prod_{i=1}^{n_k} \frac{q^{T(\lambda_k)}}{\phi_{\lambda_k - \lambda_{k+1}}(q-1)}.$$
Theorem 3. The first Kac conjecture is true if and only if, for any \( \alpha \in \mathbb{N}^n \), either \( r_\alpha(0) = 0 \) or \( T(\alpha) = \text{ht} \alpha \).

Our proof is based on the formulas relating the functions \( r_\alpha \) with the polynomials \( m_\alpha \) and \( a_\alpha \) together with the Peterson recursive formula for the root multiplicities of the Kac-Moody algebra [12].

Denote by \( \Phi \) the set of all irreducible polynomials in \( \mathbb{F}_q[t] \) with the leading coefficient 1, excluding \( t \). Denote by \( \Phi_d(q) \) the number of polynomials in \( \Phi \) having degree \( d \); it is a polynomial in \( q \). The next formula, relating \( r_\lambda \) and \( m_\alpha \), is due to Kac and Stanley [8, p. 90]

\[
m_\alpha(q) = \sum_{\nu : \Phi \to \mathbb{P}^n} \prod_{f \in \Phi} \psi_{\deg f}(r_{\nu(f)}(q)) \]

where \( |\nu| = \sum_{f \in \Phi} \deg f \cdot |\nu(f)| \) and \( \psi_d \) are the Adams operations on \( \mathbb{Q}(q) \) (see Appendix) given by \( \psi_d(f(q)) = f(q^d) \) for \( f \in \mathbb{Q}(q) \). This formula can be simplified using generating functions. Consider the functions

\[
m(q) := \sum_{\alpha \in \mathbb{N}^n} m_\alpha(q)x^\alpha, \quad a(q) := \sum_{\alpha \in \mathbb{N}^n} a_\alpha(q)x^\alpha, \quad r(q) := \sum_{\alpha \in \mathbb{N}^n} r_\alpha(q)x^\alpha
\]

in \( \mathbb{Q}(q)[x_1, \ldots, x_n] \). Then the Kac-Stanley formula can be written in the form (see Lemma 4)

\[
m = \prod_{d \geq 1} \psi_d(r)^{\Phi_d}.
\]

The functions \( m \) and \( a \) are related by the formula (see Lemma 4 and Appendix)

\[
m = \text{Exp}(a).
\]

We give a rather technical proof of this formula in Section 2 although it has an intuitive explanation. Namely, assume for simplicity that for all \( \alpha \in \mathbb{N}^n \) there exists the moduli space \( M_\alpha \) (respectively, \( A_\alpha \)) of representations (respectively, absolutely indecomposable representations) of dimension \( \alpha \) and that they have cellular decompositions. (Note that in general these moduli spaces do not exist). Then the numbers of cells of fixed dimension correspond to the coefficients of the polynomials \( m_\alpha \) (respectively, \( a_\alpha \)). The fact that any representation can be uniquely (up to the permutation of summands) written as a direct sum of indecomposable representations implies

\[
\sum_{\alpha \in \mathbb{N}^n} \#M_\alpha(\mathbb{F}_q)x^\alpha = \prod_{\alpha \in \mathbb{N}^n} (1 + \#S^1A_\alpha(\mathbb{F}_q)x^\alpha + \#S^2A_\alpha(\mathbb{F}_q)x^{2\alpha} + \ldots),
\]

where \( S^\alpha \) denotes the symmetric product. Using the existence of a cellular decomposition of \( A_\alpha \) we obtain \( \#S^\alpha A_\alpha(\mathbb{F}_q) = \sigma_\alpha(a_\alpha)(q) \). It follows

\[
\sum_{\alpha \in \mathbb{N}^n} m_\alpha x^\alpha = \prod_{\alpha \in \mathbb{N}^n} (1 + \sigma_1(a_\alpha)x^\alpha + \sigma_2(a_\alpha)x^{2\alpha} + \ldots) = \prod_{\alpha \in \mathbb{N}^n} \text{Exp}(a_\alpha x^\alpha) = \text{Exp}(a).
\]
The above formulas can be already used for the explicit calculation of the polynomials \(a_\alpha\). There is, however, a direct relation between \(a\) and \(r\) due to Hua (cf. [7, Theorem 4.6])

\[ a(q) = (q - 1) \log(r(q)). \]

Equivalently, this formula can be written as \(m(q) = \text{Pow}(r(q), q - 1)\) and we show in Theorem 6 that it follows easily from the Kac-Stanley formula.

In Section 2 we discuss various relations between the generating functions \(m\), \(a\) and \(r\). In Section 3 we prove the criterion for the first Kac conjecture using these relations together with a Peterson recursive formula for the multiplicities of the Kac-Moody algebra [12]. As the paper of Peterson [12] is unpublished, we include a rather detailed description of his approach. In the Appendix we gather basic definitions concerning the \(\lambda\)-rings. All the computations in the paper were performed using the algebraic combinatorics package “MuPAD-Combinat”.

After this paper was finished I became aware of the preprint [6], where the proof of the first Kac conjecture is announced.

2. Relations between \(m\), \(a\) and \(r\)

The aim of this section is to prove various relations between the generating functions \(m\), \(a\) and \(r\). As a point of departure, we use the Kac-Stanley formula

\[ m_\alpha(q) = \sum_{\nu: \Phi \rightarrow P^n} \prod_{f \in \Phi} \psi_{\deg f}(r_{\nu(f)})(q) \prod_{\nu: |\nu| = \alpha} |\nu| \prod_{f \in \Phi} \psi_{\deg f}(r_{\nu(f)})(q) x_{|\nu|}. \]

We refer to Appendix for the basic definitions concerning \(\lambda\)-rings. In the notation from Introduction, the formula of Kac and Stanley has the form

**Lemma 4.** We have

\[ m = \prod_{d \geq 1} \psi_d(r)^{\Phi_d}. \]

**Proof.** Using the Kac-Stanley formula we get

\[
\sum_{\alpha \in \mathbb{N}^n} m_\alpha(q)x^\alpha = \sum_{\nu: \Phi \rightarrow P^n} \prod_{f \in \Phi} \psi_{\deg f}(r_{\nu(f)})(q)x_{|\nu|} = \sum_{\nu: \Phi \rightarrow P^n} \prod_{f \in \Phi} \psi_{\deg f}(r_{\nu(f)})(q)x_{|\nu(f)|}(q) = \prod_{f \in \Phi} \psi_{\deg f}(r)(q) = \prod_{d \geq 1} \psi_d(r)(q)^{\Phi_d}(q).
\]

□

**Lemma 5.** We have \(m = \text{Exp}(a)\).

**Proof.** Let \(i_\alpha(q)\) denote the number of indecomposable representations (do not confuse them with the absolutely indecomposable representations) of dimension \(\alpha\). Then it holds (see [8, p.91])

\[ i_\alpha(q) = \sum_{d \geq 1} \sum_{k|d} \frac{\mu(k)}{d} a_{\alpha/d}(q^{d/k}), \]

where \(\mu\) is the Möbius function.
where the sum runs over those $d$ that divide the coefficients of $\alpha$. On the other hand, it is clear that

$$\sum_{\alpha} m_\alpha(q) x^\alpha = \prod_{\alpha} \left(1 + x^\alpha + x^{2\alpha} + \ldots\right)^{i_\alpha(q)} = \prod_{\alpha} \left(1 - x^\alpha\right)^{-i_\alpha(q)},$$

so we have to prove that

$$\sum_{\alpha} i_\alpha(q) \log \frac{1}{1 - x^\alpha} = \log(m(q)) = \Psi(a(q)).$$

We have

$$\sum_{\alpha} \sum_{r \geq 1} i_\alpha(q) \frac{x^{r_\alpha}}{r} = \sum_{\alpha} \sum_{r \geq 1} \sum_{d \mid k} \mu(k) \frac{a_{\alpha / d}(q^{d/k}) x^{r_\alpha}}{r}$$

$$= \sum_{\beta = \alpha / d} \sum_{m \geq 1 n \geq 1} a_\beta(q^n) \frac{x^{mn_\beta}}{mn} \sum_{k \mid m} \mu(k) = \sum_{\beta} \sum_{n \geq 1} a_\beta(q^n) \frac{x^{n_\beta}}{n} = \Psi(a(q)).$$

△

**Theorem 6** (Hua’s formula). We have

$$a(q) = (q - 1) \log(r(q))$$

or, equivalently, $m(q) = \text{Pow}(r(q), q - 1)$.

**Proof.** The minimal polynomial of a nonzero element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is contained in $\Phi$ and has degree dividing $n$. Conversely, for any $d \mid n$ and any $f \in \Phi$ of degree $d$ there are $d$ elements in $\mathbb{F}_{q^n}$ having $f$ as a minimal polynomial. This implies $q^n - 1 = \sum_{d \mid n} d \Phi_d(q)$. Applying Lemma 22 we get

$$m(q) = \prod_{d \geq 1} \psi_d(r(q))^{\Phi_d(q)} = \text{Pow}(r(q), q - 1).$$

△

**Remark 7.** The proof of the above theorem can be generalized as follows (cf. [13]). Let $X$ be an algebraic variety over $\mathbb{F}_q$ (i.e., a separated scheme of finite type over $\mathbb{F}_q$) such that there exists a polynomial $p_X$ satisfying $\# X(\mathbb{F}_{q^n}) = p_X(q^n)$ for $n \geq 1$. Let $|X|$ be the set of closed points of $X$ and for any $x \in |X|$ let $\deg x = [k(x) : \mathbb{F}_q]$. Then $\# \{x \in |X| \mid \deg x = d\} = \Phi_d(q)$, where the polynomials $\Phi_d$, $d \geq 1$ are defined by the formula $\sum_{d \mid n} d \Phi_d = \psi_n(p_X)$. Given a function $r \in \mathbb{Q}[q][x_1, \ldots, x_n]$, we have

$$\text{Pow}(r, p_X) = \prod_{d \geq 1} \psi_d(r)^{\Phi_d}.$$  

Considering $q$ as a number of the elements of the base field, we get

$$\text{Pow}(r, p_X)(q) = \prod_{x \in |X|} \psi_{\deg x}(r)(q).$$
3. On the verification of the Kac conjecture

Let $g$ be a Kac-Moody algebra with a generalized Cartan matrix $C$ as in Introduction. Let $h$ be a Cartan subalgebra of $g$ and let $\{\alpha_1, \ldots, \alpha_n\} \subset h^*$ be the simple roots of $g$. Let $(-,-)$ be the standard non-degenerate symmetric bilinear form on $h^*$ (see \cite{9} Section 2.1]). Let $W$ be the Weyl group of $g$ and $\rho \in h^*$ be any element with $(\rho, \alpha_i) = 1$ for all $i = 1, \ldots, n$. We always identify the root lattice (generated by $\alpha_1, \ldots, \alpha_n$) with $\mathbb{Z}^n$. Note that the restriction of the bilinear form $(-,-)$ to this lattice is given by the matrix $C$. In particular, for any $\alpha \in \mathbb{N}^n$ it holds $T(\alpha) = \frac{1}{2}(\alpha, \alpha)$.

The goal of this section is to prove Theorem 3. One direction of the theorem (the “only if” condition) is rather simple. Assume that $a_\alpha(0)$ coincides with the root multiplicity $\text{mult}_\alpha$ for any $\alpha \in \mathbb{N}^n$. Then we deduce from Hua’s formula that $r(q) = \text{Exp}(\sum_{\alpha} a_\alpha q^\alpha)$ and therefore

$$r(0) = \text{Exp}(-a(0)) = \prod_{\alpha} \text{Exp}(x^\alpha)^{-a_\alpha(0)} = \prod_{\alpha} (1 + x^\alpha + x^{2\alpha} + \ldots)^{-a_\alpha(0)} = \prod_{\alpha} (1 - x^\alpha)^{a_\alpha(0)} = \sum_{w \in W} (-1)^{l(w)} x^{\rho_w - \rho_0},$$

where the last equation is the Kac-Weyl denominator formula (see, e.g., \cite{9}, 10.4.4). We note that for any $\alpha \in \mathbb{N}^n$ it holds $T(\alpha) - \text{ht}_\alpha = \frac{1}{2}(\alpha, \alpha) - (\rho, \alpha)$. In particular,

$$T(\rho - w\rho) - \text{ht}(\rho - w\rho) = \frac{1}{2}(\rho - w\rho, \rho - w\rho) - (\rho, \rho - w\rho)$$

$$= \frac{1}{2}((w\rho, w\rho) - (\rho, \rho)) = 0,$$

where the last equality follows from the $W$-invariance of the bilinear form $(-,-)$.

To prove the other direction we will apply the approach of Peterson \cite{12} for a recursive calculation of the root multiplicities. Roughly speaking, one shows in this approach that the generating function $\sum_{\alpha} a_\alpha x^\alpha$ satisfies certain second order differential equation and can be determined by its “boundary values”. We will show that, under the conditions of the proposition, the function $r(0) = \text{Exp}(-a(0))$ also satisfies this differential equation. As it has the same “boundary values” as for the root multiplicities, we obtain $a(0) = \sum_{\alpha} a_\alpha x^\alpha$.

The original paper by Peterson \cite{12} is unpublished. Our references for his approach were \cite{9}, Exercise 11.11 and \cite{10}. First, we define some formal differential operators. Define the derivation $\nabla : \mathbb{C}[x_1, \ldots, x_n] \to h^* [x_1, \ldots, x_n]$ by the formula

$$\nabla f := \sum_{\alpha} f_\alpha x^\alpha,$$

for any $f = \sum_{\alpha} f_\alpha x^\alpha$.

It is easy to see that it satisfies $\nabla(fg) = f \nabla(g) + \nabla(g)f$. Define the symmetric bilinear form $(-,-) : h^*[x_1, \ldots, x_n] \times h^*[x_1, \ldots, x_n] \to \mathbb{C}^*[x_1, \ldots, x_n]$ by the formula

$$(F, G) := \sum_{\alpha, \beta} (F_\alpha, G_\beta) x^{\alpha + \beta},$$
where $F = \sum_{\alpha} F_{\alpha} x^\alpha$ and $G = \sum_{\alpha} G_{\alpha} x^\alpha$ are elements in $\mathfrak{h}^*[x_1, \ldots, x_n]$. For any $\lambda \in \mathfrak{h}^*$, define the operator $\partial_\lambda$ on $\mathbb{C}[x_1, \ldots, x_n]$ by

$$\partial_\lambda f := (\lambda, \nabla f) = \sum_{\alpha} (\lambda, \alpha) f_{\alpha} x^\alpha.$$ 

Let $\lambda_1, \ldots, \lambda_r$ be an orthonormal basis of $\mathfrak{h}^*$. Define the operator $\Delta$ on $\mathbb{C}[x_1, \ldots, x_n]$ by the formula

$$\Delta f := \sum_{i=1}^r \partial_{\lambda_i}^2 f,$$ 

Lemma 8. Let $f = \sum_{\alpha} f_{\alpha} x^\alpha \in \mathbb{C}[x_1, \ldots, x_n]$. Then

1. $\Delta f = \sum_{i=1}^r \partial_{\lambda_i}^2 f = \sum_{\alpha} (\alpha, \alpha) f_{\alpha} x^\alpha$.
2. $\sum_{i=1}^r (\partial_{\lambda_i} f)^2 = (\nabla f, \nabla f)$.

Proof. It is enough to prove the formula for $f = x^\alpha$. But

$$\Delta(x^\alpha) = \sum_{i=1}^r (\lambda_i, \alpha)^2 x^\alpha = (\alpha, \alpha) x^\alpha$$

by the Pythagorean theorem. The second statement is analogous. □

Lemma 9. Let $f \in \mathbb{C}[x_1, \ldots, x_n]^+$ and $g := \exp(f)$. Then

1. $\frac{\Delta g}{g} = \nabla f$.
2. $\frac{\Delta g}{g} = \Delta f + (\nabla f, \nabla f)$.

Proof. From $f = \log(g)$ we obtain $\nabla f = \nabla \log(g) = \frac{\Delta g}{g}$. Let $\partial_i = \partial_{\lambda_i}$. Then $\partial_i \exp(f) = \exp(f) \partial_i f$ and therefore $\partial_i^2 \exp(f) = \exp(f) \partial_i^2 f + \exp(f)(\partial_i f)^2$. It follows

$$\Delta \exp(f) = \sum_{i=1}^r \partial_i^2 f = \exp(f) \sum_{i=1}^r (\partial_i^2 f + (\partial_i f)^2) = \exp(f)(\Delta f + (\nabla f, \nabla f)).$$

□

Let $\pi_\alpha := \text{mult } \alpha$ be the multiplicities of the Kac-Moody algebra $\mathfrak{g}$. Define

$$\Xi := \sum_{\alpha} \pi_\alpha x^\alpha, \Xi := \Psi \Xi \text{ and } \Xi := \text{Exp}(\Xi) = \exp(-\Xi).$$

Lemma 10. It holds

1. $(\Delta - 2\partial_\rho) \Xi = 0$.
2. $(\Delta - 2\partial_\rho) \Xi = (\nabla \Xi, \nabla \Xi)$.

Proof. We note that

$$\Xi = \text{Exp}(\Xi) = \prod_{\alpha \in \mathbb{N}} (1 - x^\alpha)^{\pi_\alpha} = \sum_{\omega \in W} (-1)^{l(w)} x^{\rho - w\rho}.$$

To prove the first formula, we have to show that $(\Delta - 2\partial_\rho) (x^{\rho - w\rho}) = 0$, i.e.,

$$(\rho - w\rho - (\rho - w\rho) - 2(\rho, \rho - w\rho) = 0.$$ But this was already shown in (9). The second formula follows from the first one if we recall that $\Xi = \exp(-\Xi)$ and apply Lemma 9. □
This Lemma implies that for any $\alpha \in \mathbb{N}^n$

$$(\alpha, \alpha - 2\rho)\pi_\alpha = \sum_{\beta + \gamma = \alpha} (\beta, \gamma)\pi_\beta\pi_\gamma.$$  

We recall that $\pi = \Psi \pi$ and therefore $\pi_\alpha = \sum_{n \geq 1} \frac{1}{n}a_{\alpha/n}$, where the sum is over all $n$ dividing the coefficients of $\alpha$. We deduce that

$$(\alpha, \alpha - 2\rho)\pi_\alpha = \sum_{\beta + \gamma = \alpha} (\beta, \gamma)\pi_\beta\pi_\gamma - (\alpha, \alpha - 2\rho)\sum_{n > 1} \frac{1}{n}a_{\alpha/n}$$

and this allows us to calculate the numbers $\pi_\alpha$ inductively by height (note that $(\alpha, \alpha) < 2(\rho, \alpha)$ for any non-simple root by [9, 11.6.1]). The initial values (i.e., values at simple roots) are equal 1.

Assuming that the condition

$$r_\alpha(0) = 0 \text{ or } T(\alpha) = \text{ht } \alpha, \quad \forall \alpha \in \mathbb{N}^n$$

of Theorem 3 is satisfied, we will show that the coefficients of $a(0)$ satisfy the same recursive formula as the coefficients of $\pi$ above. Let $c = \Psi(a)$.

**Lemma 11.** Assume that for any $\alpha \in \mathbb{N}^n$ either $r_\alpha(0) = 0$ or $T(\alpha) = \text{ht } \alpha$. Then

1. $(\Delta - 2\partial_\rho)r(0) = 0$.
2. $(\Delta - 2\partial_\rho)c(0) = (\bigtriangledown c(0), \bigtriangledown c(0))$.

**Proof.** To prove the first formula, we have to show that for any $\alpha \in \mathbb{N}^n$ it holds

$$(\Delta - 2\partial_\rho)(r_\alpha(0)x^\alpha) = ((\alpha, \alpha) - 2(\rho, \alpha))r_\alpha(0)x^\alpha = 0,$$

but this is precisely the condition of the lemma. The second formula follows from the first one by applying Lemma 9. □

**Proof.** Proof of Theorem 3 The “only if” part has already been shown in the beginning of this section. Assume that for any $\alpha \in \mathbb{N}^n$ either $r_\alpha(0) = 0$ or $T(\alpha) = \text{ht } \alpha$. Then it follows from Lemma 11 that

$$(\Delta - 2\partial_\rho)c(0) = (\bigtriangledown c(0), \bigtriangledown c(0))$$

and we obtain, in the same way as above, that the coefficients of $a(0)$ can be inductively determined by the formula

$$(\alpha, \alpha - 2\rho)a_\alpha(0) = \sum_{\beta + \gamma = \alpha} (\beta, \gamma)c_\beta(0)c_\gamma(0) - (\alpha, \alpha - 2\rho)\sum_{n > 1} \frac{1}{n}a_{\alpha/n}(0).$$

Here we note that if $\alpha \in \mathbb{N}^n$ is not a root then $a_\alpha = 0$ by the results of Kac [8] and if $\alpha$ is a non-simple root then $(\alpha, \alpha) < 2(\rho, \alpha)$ as it has already been mentioned above. The initial values (i.e., values at simple roots) equal 1 as before. All this implies that $a(0) = \pi$ and the theorem is proved. □

**Example 12.** Consider a quiver with the underlying graph

```
1 ---- 2
    |    |
    |    |
4 ---- 3
```
Using the formula for $r_\alpha(q)$ given in Introduction, we can write the first terms of $r(0)$ (coefficients by $x^\alpha$ with $\alpha$ smaller than $(3,3,3)$)

$$r(0) = x^{(0,0,0)}(0) - x^{(0,0,0,1)}(0) - x^{(0,1,0,0)}(0) + x^{(1,0,0,0)}(0) + x^{(0,1,0,1)}(0) + x^{(1,0,1,0)}(0) + x^{(0,0,1,2)}(0) + x^{(2,1,0,0)}(0) + x^{(2,0,2,0)}(0) + x^{(0,1,3,0)}(0) + x^{(0,3,1,0)}(0) + x^{(1,0,0,3)}(0) + x^{(3,0,0,1)}(0) - x^{(0,3,1,3)}(0) - x^{(2,1,3,0)}(0) - x^{(3,0,2,1)}(0).$$

It is easy to see that all $\alpha$ with $r_\alpha(0) \neq 0$ satisfy $T(\alpha) = \text{ht}(\alpha)$.

**APPENDIX A. $\lambda$-RINGS**

We follow Getzler [4]. Let $\Lambda$ be the ring of symmetric functions, $e_n$ be the elementary symmetric functions, $h_n$ be the complete symmetric functions, and $p_n$ be the power sums (see, e.g., [11]).

Define the operation $\circ : \Lambda \times \Lambda \to \Lambda$, called plethysm, by the properties

$$p_n \circ f(x_1, \ldots, x_k) = f(x_1^n, \ldots, x_k^n), \quad n \geq 1, \quad f \in \Lambda,$$

$$(- \circ f) : \Lambda \to \Lambda \text{ is a ring homomorphism for any } f \in \Lambda.$$

**Remark 13.** Note that this operation is associative. It is, however, not commutative and not additive in the second argument. For example, $0 \circ 1 = 0$ and $1 \circ 0 = 1$.

**Definition 14.** A pre-$\lambda$-ring is a commutative ring $R$ together with a map $\circ : \Lambda \times R \to R$ such that $(- \circ a) : \Lambda \to R$ is a ring homomorphism for any $a \in R$ and, for $\lambda_n := (e_n \circ -) : R \to R$, it holds $\lambda_1 = \text{Id}_R$ and $\lambda_n(a + b) = \sum_{i=0}^{n} \lambda_i(a) \lambda_{n-i}(b)$.

**Remark 15.** The operations $\lambda_n : R \to R$, $\sigma_n : R \to R$ and $\psi_n : R \to R$ induced, respectively, by $e_n$, $h_n$ and $p_n$ are called $\lambda$-operations, $\sigma$-operations and Adams operations, respectively. To define the pre-$\lambda$-ring structure on $R$ it is enough just to define the $\lambda$-operations or $\sigma$-operations satisfying the corresponding conditions. If $R$ is an algebra over $\mathbb{Q}$ then it suffices to define the Adams operations satisfying $\psi_1 = \text{Id}_R$ and $\psi_n(a + b) = \psi_n(a) + \psi_n(b)$. For simplicity, we will always assume that $R$ is an algebra over $\mathbb{Q}$.

**Definition 16.** A pre-$\lambda$-ring $R$ is called a $\lambda$-ring if it holds

$$f \circ (g \circ a) = (f \circ g) \circ a, \quad f, g \in \Lambda, \quad a \in R,$$

and $\lambda_n(1) = 0$ for any $n \geq 2$.

**Remark 17.** A pre-$\lambda$-ring $R$ is a $\lambda$-ring if and only if $\psi_n(1) = 1$ for every $n \geq 1$ and $\psi_m(\psi_n(a)) = \psi_{mn}(a)$ for any $m, n \geq 1$ and $a \in R$. It can be shown that $\psi_n$ are actually ring homomorphisms.

**Example 18.** The basic example is a $\lambda$-ring with all Adams operations being identities. If $R$ is a $\lambda$-ring, we endow the ring $R[x_1, \ldots, x_r]$ with a $\lambda$-ring structure by defining the Adams operations as

$$\psi_n(a \cdot x^\alpha) = \psi_n(a) \cdot x^{n\alpha}, \quad n \geq 1, \quad a \in R, \quad \alpha \in \mathbb{N}^r.$$

In the same way, we endow the ring of formal power series over $R$ with a $\lambda$-ring structure.
Remark 19. In order to work with infinite sums in the \( \lambda \)-rings, we will assume that they are complete graded \( \lambda \)-rings. A \( \lambda \)-ring \( R \) is called a complete graded \( \lambda \)-ring if it is a complete graded ring \( R = \bigoplus_{n \geq 0} R_n \) and \( \psi_m(R_n) \subset R_{mn} \) for any \( m \geq 1, n \geq 0 \). We define \( R^+ = \bigoplus_{n \geq 1} R_n \). Our main example of a complete graded \( \lambda \)-ring is the ring \( R[[x_1, \ldots, x_r]] \), where \( R \) is a usual \( \lambda \)-ring and the grading is given by
\[
\deg(ax^\alpha) = \text{ht} \alpha = \sum \alpha_i, \quad a \in R, \alpha \in \mathbb{N}^r.
\]

Lemma 20. Let \( R \) be a complete graded \( \lambda \)-ring. The map \( \Psi : R^+ \to R^+, \Psi(f) = \sum_{n \geq 1} \frac{1}{n} \psi_n(f) \) has an inverse \( \Psi^{-1} : R^+ \to R^+ \) given by \( \Psi^{-1}(f) = \sum_{n \geq 1} \psi_n(f) \). We define \( R^+ = \bigoplus_{n \geq 1} R_n \). Our main example of a complete graded \( \lambda \)-ring is the ring \( R[[x_1, \ldots, x_r]] \), where \( R \) is a usual \( \lambda \)-ring and the grading is given by
\[
\deg(ax^\alpha) = \text{ht} \alpha = \sum \alpha_i, \quad a \in R, \alpha \in \mathbb{N}^r.
\]

Lemma 22. Let \( f \in 1 + R^+ \) and \( g \in R^+ \). Define the elements \( g_d \in R, d \geq 1 \) by the formula \( \sum_d d \cdot g_d = \psi_n(g) \). Then we have
\[
\text{Pow}(f, g) = \prod_{d \geq 1} \psi_d(f)^{g_d}.
\]
Proof. After taking logarithms we have to prove
\[ \Psi(g \log(f)) = \sum_{d \geq 1} g_d \psi_d(\log(f)). \]
Let \( h = \log(f) \). Then \( \Psi(h) = \log(f) \) and we have to show
\[ \Psi(gh) = \sum_{d \geq 1} g_d \psi_d(\Psi(h)). \]
We have \( \psi_n(gh) = \psi_n(g)\psi_n(h) = \sum_{d|n} d g_d \psi_n(h) \). This implies
\[ \Psi(gh) = \sum_{d|n} d g_d \frac{\psi_n(h)}{n} = \sum_{d,k \geq 1} g_d \frac{\psi_d \psi_k(h)}{d} = \sum_{d \geq 1} g_d \psi_d(\Psi(h)). \]
\[ \square \]

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