Difference operators via GKLO-type homomorphisms: shuffle approach and application to quantum Q-systems

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Abstract
We present a shuffle realization of the GKLO-type homomorphisms for shifted quantum affine, toroidal, and quiver algebras in the spirit of Feigin and Odesskii (Funktsional. Anal. Prilozhen. 31(3):57–70, 1997), thus generalizing its rational version of Frassek and Tsymbaliuk (Commun. Math. Phys. 392:545–619, 2022) and the type A construction of Finkelberg and Tsymbaliuk (Arnold Math. J. 5(2–3):197–283, 2019). As an application, this allows us to construct large families of commuting and $q$-commuting difference operators, in particular, providing a convenient approach to the $Q$-systems where it proves a conjecture of Di Francesco and Kedem (Commun. Math. Phys. 369(3):867–928, 2019).

Keywords Shuffle algebras · GKLO-type homomorphisms · Quantum Q-systems · Generalized Macdonald operators · Quantum loop algebras

Mathematics Subject Classification 17B37 · 81R10

1 Introduction

1.1 Summary

The key result of this note is the shuffle realization of the GKLO-type homomorphisms from various shifted quantum “loop” algebras to the algebras of (localized) difference operators. We use this to reinterpret the recent results of [4, 5] on the quantum $Q$-systems of type $A$. In the upcoming work, this will be also used as the main technical ingredient to:

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• prove the regularity of certain trigonometric $BCD$-type Lax matrices
  (generalizing the rational counterpart of [13]),
• develop the integral forms of $K$-theoretic Coulomb branches
  (generalizing the $A$-type case of [12]),
• study difference operators arising from large families of $q$-commuting elements in
  quantum affine algebras (generalizing [4] with $\mathfrak{sl}_2$ been replaced by any simple $\mathfrak{g}$).

The GKLO-type homomorphisms for the quantum loop algebras $U_q(L\mathfrak{g})$ were first
introduced in [14] (hence, their acronym). Their analogues for the “shifted” versions
(the shift refers to the fact that Cartan currents $\psi^\pm_{i}(z)$ start not necessarily from $z^0$
modes, while the defining relations are kept unchanged) arise naturally in the recent
study of the quantized Coulomb branches, see [1, 2] and [11], providing algebraic
models for the geometric objects.

On the other hand, the shuffle approach provides a convenient combinatorial model for
the positive and negative subalgebras of such quantum algebras. An essential benefit
of this approach is that it allows to work with various elements of quantum algebras
that are provided by complicated formulas in the original loop generators, making
it hard to work with them directly. In the present note, we focus on the following
cases: quantum affine of any simple $\mathfrak{g}$, quantum toroidal of $\mathfrak{gl}_1$ and $\mathfrak{sl}_n$
($n \geq 3$) with two parameters, and quantum quiver algebras, for which the shuffle realizations were
established in [19], [15, 16], and [18], respectively.

Let $U^>_L$ denote the corresponding positive subalgebra, generated by the loop gener-
ators $\{e_{i,r}\}_{i \in I}^{r \in \mathbb{Z}}$ (here, $I$ denotes a labeling set, while the subscript “$L$” is merely used
to remind of the loop realization, in spirit of [3]) subject to the corresponding defining
relations. Then, one considers an $\mathbb{N}^I$-graded vector space $\mathbb{S} = \bigoplus_{k \in \mathbb{N}^I} \mathbb{S}_k$, with $\mathbb{S}_k$
consisting of multisymmetric rational functions in the variables $\{x_{i,r}\}_{i \in I}^{1 \leq r \leq k_i}$ subject
to rather simple “pole” conditions, equipped with an algebra structure via the shuffle
product $\star$: $\mathbb{S}_k \times \mathbb{S}_\ell \to \mathbb{S}_{k+\ell}$ given by

$$F(\ldots,x_{i,1},\ldots,x_{i,k_i},\ldots) \star G(\ldots,x_{i,1},\ldots,x_{i,\ell_i},\ldots) := \frac{1}{\prod_{i \in I} k_i! \cdot \ell_i!} \times \text{Sym} \left( F(\{x_{i,r}\}_{i \in I}^{1 \leq r \leq k_i}) \cdot G(\{x_{i',r'}\}_{i' \in I}^{k_i' \leq r' \leq k_i'+\ell_i'}) \cdot \prod_{i' \in I} \prod_{r \leq k_i'} \zeta_{i'i} \left( \frac{x_{i',r'}}{x_{i',r'}} \right) \right).$$

The rational $\zeta$-factors are specifically chosen to allow for an algebra embedding

$$\Upsilon: U^>_L \hookrightarrow \mathbb{S} \quad \text{given by} \quad e_{i,r} \mapsto x_{i,1}^r \quad \text{for all} \quad i \in I, r \in \mathbb{Z}. \quad (1)$$

On the other hand, (the restriction of) the aforementioned GKLO-type homomor-
phism

$$\widetilde{\Phi}: U^>_L \longrightarrow \tilde{\mathbb{A}}_\mathfrak{g} \quad (2)$$

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to the algebra $\tilde{A}_a$ of localized difference operators, generated by $\{w_{i,r}, D_{i,r}\}_{i \in I}$ as well as $\{(w_{i,r} - q^m_{i} w_{i,s})^{-1}\}_{r \neq s}$ subject to

$$[w_{i,r}, w_{j,s}] = 0 = [D_{i,r}, D_{j,s}]$$

and

$$D_{i,r} w_{j,s} = q_i^{\delta_{ij} \delta_{rs}} w_{j,s} D_{i,r}$$

for some $q_i$, is explicitly given by specifying $\tilde{\Phi}(\epsilon_{i,r})$, reminiscent of the Gelfand–Tsetlin formulas in type $A$.

Thus, our main construction is the algebra homomorphism

$$\hat{\Phi} : S \rightarrow \tilde{A}_a,$$  \hspace{1cm} (3)

where $\tilde{A}_a$ denotes a localization of $A_a$ at some other elements $w_{i,r} - \gamma w_{j,s}$, given by

$$S \ni E \overset{\hat{\Phi}}{\mapsto} \sum_{m_{i,1}^{(i)} + \ldots + m_{i,n}^{(i)} = k_i} \left( \sum_{1 \leq p \leq m_i^{(i)} \leq a_i} \prod_{i \in I} D_{i,r}^{m_i^{(i)}} \right) \cdot \text{(rational prefactor)} \cdot \prod_{i \in I} D_{i,r}^{m_i^{(i)}} (4)$$

and such that its composition with $\Upsilon$ of (1) recovers $\tilde{\Phi}$ of (2):

$$\tilde{\Phi} = \hat{\Phi} \circ \Upsilon : U_L^\gamma \rightarrow \tilde{A}_a.$$  \hspace{1cm} (5)

In particular, the image of $U_L^\gamma$ under the composition (5) is in the subalgebra $\tilde{A}_a$ of $A_a$. This $\hat{\Phi}$ can be perceived as a trigonometric counterpart of a much older construction from [9].

We want to emphasize that this construction of $\hat{\Phi}$ is a general phenomenon that applies in a much wider setup. However, if one wishes to remain in the realm of quantum algebras, then one needs to restrict $\hat{\Phi}$ to the image of the embedding $\Upsilon$ of (1). The latter is often described by certain “wheel” conditions, see (16, 42, 63, 89) for the cases treated in the present note, which actually constitutes the core of the aforementioned shuffle algebra isomorphisms.

In the simplest case of quantum affine $sl_2$, some of the resulting difference operators can be patched nicely to form a $q$-commuting family satisfying the quantum $Q$-system relations of type $A$. On the other hand, for the case of quantum toroidal $gl_1$, we obtain the famous Macdonald difference operators as well as their generalizations from [5]. Finally, for the case of quantum toroidal $sl_n$, the images of natural commutative subalgebras of the quantum toroidal $U_L^\gamma$ give rise to compelling large families of pairwise commuting difference operators (it is interesting to understand their relation to the recent construction of [20], if any).
1.2 Outline of the paper

The structure of the present paper is the following:

• In Sect. 2, we recall the notion of shifted quantum affine algebras and the GKLO-type homomorphisms $\tilde{\Phi}_{\mu}^{\lambda}Z$ of (9), following [11]. The main result of this section is Theorem 2.8, which provides a shuffle realization of $\tilde{\Phi}_{\mu}^{\lambda}Z$ restricted to the positive and negative subalgebras (actually, extending it to larger algebras $S(g)$ and $S(g)_{\text{op}}$, whose elements are rational functions of (13) that do not necessarily satisfy the wheel conditions (16)). As an application, we construct a natural family of elements in the shifted quantum affine algebras whose $\tilde{\Phi}_{\mu}^{\lambda}Z$-images are given by simple and interesting formulas of Lemma 2.12. In Remark 2.10, we explain the resemblance between our Theorem 2.8 and a much older result [9, Proposition 2].

• In Sect. 3, we generalize the results of Sect. 2 to the context of shifted quantum toroidal algebras of $gl_1$ (depending on two parameters). The main result of this section is Theorem 3.10, providing shuffle realization of the restrictions of the homomorphisms $\tilde{\Phi}_{\mu}^{Z}$ from Proposition 3.4 to the positive and negative subalgebras (again extended to the larger algebras $S$ and $S_{\text{op}}$). In Lemma 3.12, we derive interesting difference operators as the images of (52, 53).

• In Sect. 4, we generalize the results of Sect. 2 to the context of shifted quantum toroidal algebras of $sl_n$ (depending on two parameters). The main result of this section is Theorem 4.8, providing shuffle realization of the restrictions of the homomorphisms $\tilde{\Phi}_{\mu}^{Z}$ from Proposition 4.3 to the positive and negative subalgebras (extended to the larger algebras $S[n]$ and $S[n]_{\text{op}}$). In Lemma 4.10, we get interesting difference operators as the images of (72, 73). In Example 4.11, we use the shuffle descriptions [10, 21, 22] of the Bethe and horizontal Heisenberg subalgebras to construct large commutative families of difference operators.

• In Sect. 5, we generalize the results of Sect. 2 to the context of (shifted) quantum quiver algebras as recently introduced in [18]. The main result of this section is Theorem 5.7, providing shuffle realization of the restrictions of the new GKLO-type homomorphisms from Proposition 5.3 to the positive and negative subalgebras (extended to the larger algebras $S^Q$ and $S^Q_{\text{op}}$), in analogy with Theorems 2.8, 3.10, 4.8.

• In Sect. 6, we present a shuffle interpretation of the quantum $Q$-system of type $A$, thus simplifying proofs of [4, Theorems 2.10, 2.11], see Propositions 6.3, 6.7, 6.8. We also match the difference operators of [4, §6] with those from Sect. 2 in the simplest case of $g = sl_2$, see Lemma 6.12 and Proposition 6.13. Finally, in Lemma 6.15, we explain how the images of the Cartan and negative subalgebras can be expressed via the images of finitely many elements in the positive subalgebra, after a localization at two elements.

• In Sect. 7, we provide a shuffle interpretation of the $(t, q)$-deformed $Q$-system of type $A$ as recently investigated in [5]. In particular, we identify the generalized Macdonald operators (124) of [5] with the elements of Lemma 3.12, see Proposition 7.13.
Theorem 7.14. Let $g$ be a simple Lie algebra, and $\{\alpha^\vee_i\}_{i \in I}$ (resp. $\{\alpha_i\}_{i \in I}$) be the simple roots (resp. simple coroots) of $g$. Let $<,>$ denote the corresponding pairing on the root lattice, and set $d_i := \frac{<\alpha^\vee_i, \alpha_i^\vee>}{2} \in \{1, 2, 3\}$. Let $(c_{ij})_{i,j \in I}$ be the Cartan matrix of $g$, so that $d_ic_{ij} = (\alpha^\vee_i, \alpha^\vee_j) = djc_{ji}$. Let $\Lambda$ be the coweight lattice of $g$, and $\Lambda^+ \subset \Lambda$ be the submonoid of dominant integral weights.

Given coweights $\mu^\pm \in \Lambda$, set $b^\pm = \{b_i^\pm\}_{i \in I} \in \mathbb{Z}^I$ with $b_i^\pm := \alpha^\vee_i(\mu^\pm)$. Following [11, §5(i)], we define the simply connected version of shifted quantum affine algebra, denoted by $U^{sc}_{\mu^+, \mu^-}$ or $U^{sc}_{b^+, b^-}$, as the associative $\mathbb{C}(q)$-algebra generated by $\{e_i, f_i, \psi_{i, \pm b_i^\pm} \mid i \in I\}$ with the following defining relations (for all $i, j \in I$ and $e, e' \in \{\pm\}$):

\[
[\psi^e_i(z), \psi^{e'}_j(w)] = 0, \quad (\psi^e_{i, \mp b_i^\pm} \cdot (\psi^e_{i, \mp b_i^\pm})^{-1} = (\psi^e_{i, \mp b_i^\pm}^{-1} \cdot (\psi^e_{i, \mp b_i^\pm})^{-1}, \quad (U1)
\]

\[
(z - q_i^{c_{ij}}w)e_i(z)e_j(w) = (q_i^{c_{ij}}z - w)e_j(w)e_i(z), \quad (U2)
\]

\[
(q_i^{c_{ij}}z - w)f_i(z)f_j(w) = (z - q_i^{c_{ij}}w)f_j(w)f_i(z), \quad (U3)
\]

\[
(z - q_i^{c_{ij}}w)\psi^e_i(z)e_j(w) = (q_i^{c_{ij}}z - w)e_j(w)\psi^e_i(z), \quad (U4)
\]

\[
(q_i^{c_{ij}}z - w)\psi^e_i(z)f_j(w) = (z - q_i^{c_{ij}}w)f_j(w)\psi^e_i(z), \quad (U5)
\]

\[
[e_i(z), f_j(w)] = \frac{\delta_{ij}}{q_i - q_i^{-1}} \delta \left( \frac{z}{w} \right) \left( \psi^+_i(z) - \psi^-_i(z) \right), \quad (U6)
\]

\[
\text{Sym}_{z_1, \ldots, z_{1-c_{ij}}} \sum_{r=0}^{1-c_{ij}} (-1)^r \left[ \begin{array}{c} 1 - c_{ij} \\ r \end{array} \right] q_i^{1-r} e_i(z_1) \cdots e_i(z_r) e_j(w) e_i(z_{r+1}) \cdots e_i(z_{1-c_{ij}}) = 0, \quad (U7)
\]

\[
\text{Sym}_{z_1, \ldots, z_{1-c_{ij}}} \sum_{r=0}^{1-c_{ij}} (-1)^r \left[ \begin{array}{c} 1 - c_{ij} \\ r \end{array} \right] q_i^{1-r} f_i(z_1) \cdots f_i(z_r) f_j(w) f_i(z_{r+1}) \cdots f_i(z_{1-c_{ij}}) = 0, \quad (U8)
\]

where $q_i := q^{d_i}, [a, b]_x := ab - x \cdot ba, [m]_q := \frac{q^m - q^{-m}}{q - q^{-1}}, [m]_q := \frac{[m]_q}{[1]_q \cdots [r]_q}$, Sym stands for the symmetrization in $z_1, \ldots, z_s$, and the generating series are defined $z_1, \ldots, z_s$. 

This clarifies a shuffle approach in [5] and also establishes [5, Conjecture 1.17], see Theorem 7.14.
as follows:

\[ e_i(z) := \sum_{r \in \mathbb{Z}} e_{i,r} z^{-r}, \quad f_i(z) := \sum_{r \in \mathbb{Z}} f_{i,r} z^{-r}, \]

\[ \psi_{i}^{\pm}(z) := \sum_{r \geq -b_{i}^{\pm}} \psi_{i,\pm r}^{\pm} z^{r}, \quad \delta(z) := \sum_{r \in \mathbb{Z}} z^{r}. \] (6)

Let \( U_{\mu^{+},\mu^{-}}^{\text{sc},<}, U_{\mu^{+},\mu^{-}}^{\text{sc},>, U_{\mu^{0},\mu^{-}}^{\text{sc},0} \) be the \( \mathbb{C}(q) \)-subalgebras of \( U_{\mu^{+},\mu^{-}}^{\text{sc}} \) generated by

\[ \{ f_{i,r} \}_{i \in I}, \{ e_{i,r} \}_{i \in I}, \{ \psi_{i,\pm r}^{\pm} \}_{i \in I}, \{ (\psi_{i,\pm r}^{\pm})^{-1} \}_{i \in I} \] with the defining relations (U3, U8), (U2, U7), and (U1), respectively. In particular, \( U_{\mu^{+},\mu^{-}}^{\text{sc},<}, U_{\mu^{+},\mu^{-}}^{\text{sc},>} \) are independent of \( \mu^{\pm} \in \Lambda \).

The algebras \( U_{\mu^{+},\mu^{-}}^{\text{sc}} \) and \( U_{\mu^{0},\mu^{+}+\mu^{-}}^{\text{sc}} \) are naturally isomorphic for any \( \mu^{\pm} \in \Lambda \), see [11, p. 162]. Therefore, we do not lose generality by considering only \( U_{q}^{(b)} = U_{q}^{\mu} := U_{0,\mu}^{\text{sc}} \) in the rest of this note. The quantum loop algebra \( U_{q}(L\mathfrak{g}) \) is isomorphic to \( U_{0,0}^{\text{sc}} / (\psi_{i,0}^{\pm} \psi_{i,0}^{-} - 1)_{i \in I} \).

### 2.3 GKLO-type homomorphisms

Fix an orientation of the graph \( \text{Dyn}(\mathfrak{g}) \) obtained from the Dynkin diagram of \( \mathfrak{g} \) by replacing all multiple edges by simple ones. The notation \( j \rightarrow i \) (resp. \( j \leftarrow i \)) is to indicate an edge (resp. oriented edge pointing towards \( i \) or \( j \)) between the vertices \( i, j \in \text{Dyn}(\mathfrak{g}) \). We fix a dominant coweight \( \lambda \in \Lambda^{+} \) and a coweight \( \mu \in \Lambda \), such that \( \lambda - \mu = \sum_{i \in I} a_{i} \alpha_{i} \) with \( a_{i} \in \mathbb{N} \). We also fix a sequence of fundamental coweights, such that \( \sum_{k=1}^{N} \omega_{ik} = \lambda \), as well as a sequence \( \mathbb{Z} = (z_{1}, \ldots, z_{N}) \in (\mathbb{C}^{\times})^{N} \).

Consider the associative \( \mathbb{C}(q) \)-algebra \( \hat{A}_{\text{frac}}^{\mu} \) generated by \( \{ D_{i,r}^{\pm}, w_{i,r}^{\pm} \}_{i \in I, 1 \leq r \leq a_{i}} \) subject to

\[
[D_{i,r}, D_{j,s}] = \left[ w_{i,r}^{1/2}, w_{j,s}^{1/2} \right] = 0, \quad D_{i,r}^{\pm} D_{i,r}^{\mp} = w_{i,r}^{\pm 1/2} w_{i,r}^{\mp 1/2} = 1,
\]

\[
D_{i,r} w_{j,s}^{1/2} = q_{i}^{\delta_{ij} \delta_{rs}} w_{j,s}^{1/2} D_{i,r} \] (7)
for all \( i, j \in I, 1 \leq r \leq a_i, 1 \leq s \leq a_j \). Let \( \hat{\mathbb{A}}_{\text{frac}}^q \) be the localization of \( \hat{\mathbb{A}}_{\text{frac}}^q \) by the multiplicative set generated by \( \{ w_{i,r} - q_i^{m} w_{i,s} \}_{i \in I, m \in \mathbb{Z}} \), which obviously satisfies the Ore conditions. We also define:

\[
Z_i(z) := \prod_{1 \leq s \leq N} \left( 1 - \frac{q_i z_s}{z} \right), \quad W_i(z) := \prod_{r=1}^{a_i} \left( 1 - \frac{w_{i,r}}{z} \right), \quad W_{i,r}(z) := \prod_{1 \leq s \leq a_i} \left( 1 - \frac{w_{i,s}}{z} \right).
\]

(8)

The following result has been established in [11, Theorem 7.1] (in the unshifted case \( \mu^+ = \mu^- = 0 \), more precisely for \( U_q(L\mathfrak{g}) \), this result appeared without a proof in [14]):

**Proposition 2.4** [11] There exists a unique \( \mathbb{C}(q) \)-algebra homomorphism

\[
\Phi_{\mu}^\pm: U_q^{\mu} \longrightarrow \hat{\mathbb{A}}_{\text{frac}}^q
\]

such that

\[
e_i(z) \mapsto \frac{-q_i}{1 - q_i z} \prod_{t=1}^{a_i} w_{i,t} \prod_{j=1}^{a_j} \prod_{t=1}^{D_{j,t}} W_{j,t}^{c_{j,t}/2} \cdot \sum_{r=1}^{a_i} \delta \left( \frac{w_{i,r}}{z} \right) Z_i(w_{i,r}) \prod_{j=1}^{a_j} W_j(q_j^{-c_{j,t}-2p} z) D_{i,r}^{-1},
\]

\[
f_i(z) \mapsto \frac{1}{1 - q_i z} \prod_{j=1}^{a_j} w_{i,j} \prod_{t=1}^{D_{i,t}} W_{j,t}^{c_{j,t}/2} \cdot \sum_{r=1}^{a_i} \delta \left( \frac{q_i w_{i,r}}{z} \right) \frac{1}{W_i(w_{i,r})} \prod_{j=1}^{a_j} W_j(q_j^{-c_{j,t}-2p} z) D_{i,r},
\]

\[
\psi_i^\pm(z) \mapsto \prod_{t=1}^{a_i} w_{i,t} \prod_{j=1}^{a_j} \prod_{t=1}^{D_{j,t}} W_{j,t}^{c_{j,t}/2} \cdot \left( \frac{Z_i(z)}{W_i(z) W_i(q_i^{-2p} z)} \prod_{j=1}^{a_j} W_j(q_j^{-c_{j,t}-2p} z) \right)^\pm.
\]

We write \( \gamma(z)^\pm \) for the expansion of a rational function \( \gamma(z) \) in \( z^{\pm 1} \), respectively.

### 2.5 Shuffle algebra realization of the positive and negative subalgebras

According to Proposition 2.2(b), we have algebra isomorphisms for any \( \mu^+, \mu^- \in \Lambda \):

\[
U_q^{\mu^+} \xrightarrow{\sim} U_q^> (L\mathfrak{g}) \quad \text{given by } \quad e_{i,r} \mapsto e_{i,r} \quad \text{for } i \in I, r \in \mathbb{Z},
\]

\[
U_q^{\mu^-} \xrightarrow{\sim} U_q^< (L\mathfrak{g}) \quad \text{given by } \quad f_{i,r} \mapsto f_{i,r} \quad \text{for } i \in I, r \in \mathbb{Z}.
\]

(11)

We also note the algebra isomorphism

\[
U_q^< (L\mathfrak{g}) \xrightarrow{\sim} U_q^< (L\mathfrak{g})^{\text{op}} \quad \text{given by } \quad f_{i,r} \mapsto e_{i,r} \quad \text{for } i \in I, r \in \mathbb{Z},
\]

(12)

where for any algebra \( A \) we use \( A^{\text{op}} \) to denote the algebra with the opposite multiplication.
Consider an $\mathbb{N}^I$-graded $\mathbb{C}(q)$-vector space $S^{(g)} = \bigoplus_{k=(k_i)_{i \in I} \in \mathbb{N}^I} S_k^{(g)}$, with the graded components

$$S_k^{(g)} = \left\{ F = \frac{f((x_i, r)_{i \in I})^{1 \leq r \leq k_i}}{\prod_{i \neq j} \prod_{1 \leq r \leq k_i} (x_i, r - x_j, s)} \bigg| f \in \mathbb{C}[\{x_i, r\}_{i \in I}^{1 \leq r \leq k_i}] S_k \right\},$$

where $S_k := \prod_{i \in I} S(k_i)$ is the product of symmetric groups. We also fix rational functions:

$$\zeta_{ij} \left( \frac{z}{w} \right) = \frac{z - q_i^{-c_{ij}} w}{z - w}, \quad \forall i, j \in I.$$

Let us now introduce the bilinear shuffle product $\star$ on $S^{(g)}$ as follows:

$$F(\ldots, x_i, 1, \ldots, x_i, k_i, \ldots) \star G(\ldots, x_i, 1, \ldots, x_i, l_i, \ldots) := \frac{1}{k! \ell!} \times \text{Sym} \left( F(\{x_i, r\}_{i \in I}^{1 \leq r \leq k_i}) G(\{x_i', r'\}_{i \in I}^{1 \leq r' \leq k_i'}) \prod_{i \in I} \prod_{r \leq k_i} \zeta_{ii} \left( \frac{x_i, r}{x_i', r'} \right) \right).$$

Here, $k! = \prod_{i \in I} k_i!$, while the symmetrization of $f \in \mathbb{C}(\{x_i, 1, \ldots, x_i, m_i\}_{i \in I})$ is defined via:

$$\text{Sym} \left( f(\{x_i, 1, \ldots, x_i, m_i\}_{i \in I}) \right) = \sum_{\sigma_i \in S(m_i)} f(\{x_i, \sigma_i(1), \ldots, x_i, \sigma_i(m_i)\}_{i \in I}).$$

This endows $S^{(g)}$ with a structure of an associative $\mathbb{C}(q)$-algebra with the unit $1 \in S^{(g)}(0, \ldots, 0)$.

We are interested in an $\mathbb{N}^I$-graded $\mathbb{C}(q)$-subspace of $S^{(g)}$ defined by the wheel conditions:

$$F(\{x_i, r\}) \bigg|_{(x_i, 1, x_i, 2, x_i, 3, \ldots, x_i, 1 - c_{ij}) \mapsto (w, wq_i^2, wq_i^4, \ldots, wq_i^{2c_{ij}}), x_j, 1 \mapsto wq_i^{-c_{ij}}} = 0$$

for any connected vertices $i - j$ in Dyn($g$). Let $S^{(g)} \subset S^{(g)}$ denote the subspace of all such elements $F$. It is straightforward to check that $S^{(g)} \subset S^{(g)}$ is $\star$-closed. The resulting algebra $(S^{(g)}, \star)$ is called the (trigonometric Feigin–Odesskii) shuffle algebra of type $g$.

The following result has been recently established in [19, Theorem 1.7]:

**Proposition 2.6** [19] The assignments $e_{i, r} \mapsto x_i^{r'}$ and $f_{i, r} \mapsto x_i^{r}$ for $i \in I, r \in \mathbb{Z}$ give rise to $\mathbb{C}(q)$-algebra isomorphisms:

$$\Upsilon : U_q^>(Lg) \xrightarrow{\sim} S^{(g)} \quad \text{and} \quad \Upsilon : U_q^<(Lg) \xrightarrow{\sim} S^{(g), \text{op}}.$$
2.7 Shuffle algebra realization of the GKLO-type homomorphisms

The main new result of this section is the shuffle algebra interpretation of the homomorphisms $\Phi_{\mu}^{\mathbb{Z}}$. We note that the type A case of this result is due to [12, Theorem 4.11], while its rational counterpart is due to [13, Theorem B.17], where they played crucial roles.

To this end, for any $i \in I$ and $1 \leq r \leq a_i$, we define:

$$Y_{i,r}(z) := \frac{1}{q_i - q_i^{-1}} \prod_{t=1}^{a_i} w_{i,t} \prod_{j=1}^{a_j} w_{j,t}^{c_{ji}/2} \cdot \frac{Z_i(z) \prod_{p=1}^{c_{ji}} W_j(zq_j^{-c_{ji} - 2p})}{w_{i,r}(z)},$$

$$Y'_{i,r}(z) := \frac{1}{1 - q_i^{-1}} \prod_{j=1}^{a_j} w_{j,t}^{c_{ji}/2} \cdot \prod_{p=1}^{c_{ji}} W_j(zq_j^{-c_{ji} - 2p}) / w_{i,r}(z).$$

(18)

Define the $\mathbb{C}(q)$-algebra $\tilde{A}_{\text{frac}}^q$ as the further localization of $\tilde{A}_{\text{frac}}^q$ by the multiplicative set generated by $(w_{i,r} - q^{m} w_{j,s})_{i-j,m \in \mathbb{Z}}$. We note that $\tilde{A}_{\text{frac}}^q$ is naturally embedded into $\tilde{A}_{\text{frac}}^q$. Then, we have the following result:

**Theorem 2.8** (a) The assignment

$$S_{k}^{(g)} \ni E \mapsto \prod_{i \in I} \frac{k_i^{-k_i^2}}{1} \times \frac{\sum_{m_1^{(i)} + \ldots + m_{a_i}^{(i)} = k_i} \prod_{i \in I} w_{i,r}^{c_{ji}/2} \cdot \prod_{p=1}^{c_{ji}} W_j(zq_j^{-c_{ji} - 2p}) / w_{i,r}(z)}{w_{i,r}(z)}$$

(19)

gives rise to the algebra homomorphism

$$\Phi_{\mu}^{\mathbb{Z}} : S^{(g)} \rightarrow \tilde{A}_{\text{frac}}^q.$$  

(20)
Moreover, the composition

\[
U^\mu_> \xrightarrow{(11)} U^> (Lg) \xrightarrow{\Upsilon} S^{(g)} \xrightarrow{\hat{\Phi}_\mu^{\lambda, z}} \hat{A}_\text{trac}'
\]

(21)

coinsides with the restriction of the homomorphism \(\hat{\Phi}_\mu^{\lambda, z}\) of (9) to the subalgebra \(U^\mu_>\) of \(U^\mu\). In particular, the image of \(U^\mu_>\) under the composition (21) is in the subalgebra \(\hat{A}_\text{trac}'\) of \(\hat{A}_\text{trac}\).

(b) The assignment

\[
\mathcal{S}_{(g), \text{op}}(\xi) \ni F \mapsto \sum_{m^{(i)}_i \in \mathbb{N} \forall i \in I} \prod_{i \in I} \prod_{1 \leq r \leq a_i} \prod_{1 \leq p \leq m^{(i)}_r} Y_{i, r}^{a_i \, m^{(i)}_r} \cdot F \left( \left\{ w_{i, r} q^2_{i_1} \right\}_{1 \leq p \leq m^{(i)}_r} \right)
\]

\[
\times \prod_{i \in I} \prod_{1 \leq r \neq r' \leq a_i} 1 \leq p \leq m^{(i)}_r \prod_{1 \leq p \leq 2p_2 \leq m^{(i)}_r} q^{1 - 1}_{i_i} \left( w_{i, r_2} q^2_{i_1} \right) 
\]

\[
\times \prod_{j = i} \prod_{1 \leq r_1 \leq a_i} \prod_{1 \leq p \leq 2p_2 \leq m^{(j)}_r} \xi^{1}_{j_j} \left( w_{j, r_2} q^2_{j_1} \right) \cdot \prod_{i \in I} \prod_{r = 1} a_i \prod_{1 \leq p \leq m^{(i)}_r} D_{i, r}^{m^{(i)}_r}
\]

(22)

gives rise to the algebra homomorphism

\[
\hat{\Phi}_\mu^{\lambda, z} : \mathcal{S}(g), \text{op} \rightarrow \hat{A}_\text{trac}'.
\]

(23)

Moreover, the composition

\[
U^\mu_< \xrightarrow{(11)} U^< (Lg) \xrightarrow{\Upsilon} S^{(g), \text{op}} \xrightarrow{\hat{\Phi}_\mu^{\lambda, z}} \hat{A}_\text{trac}'
\]

(24)

coinsides with the restriction of the homomorphism \(\hat{\Phi}_\mu^{\lambda, z}\) of (9) to the subalgebra \(U^\mu_<\) of \(U^\mu\). In particular, the image of \(U^\mu_<\) under the composition (24) is in the subalgebra \(\hat{A}_\text{trac}'\) of \(\hat{A}_\text{trac}\).

**Proof** (a) Let us denote the right-hand side of (19) by \(\hat{\Phi}_\mu^{\lambda, z}(E)\). A tedious straightforward verification proves \(\hat{\Phi}_\mu^{\lambda, z}(E \ast E') = \hat{\Phi}_\mu^{\lambda, z}(E) \hat{\Phi}_\mu^{\lambda, z}(E')\) for any \(E \in S^{(g)}_\xi, E' \in S^{(g)}_\xi\).
with arbitrary \( k, \xi \in \mathbb{N}^l \). Thus, \( \tilde{\Phi}_{\mu}^{\mathbb{Z}} : \mathbb{S}(q) \rightarrow \widetilde{A}_{\text{frac}}^{q^{'}} \) is a \( \mathbb{C}(q) \)-algebra homomorphism, and clearly the images of \( \{ e_i,r \}_{i \in I} \) under (21) and \( \tilde{\Phi}_{\mu}^{\mathbb{Z}} \) do coincide. This completes our proof of Theorem 2.8(a).

(b) The proof of Theorem 2.8(b) is completely analogous. \( \square \)

Remark 2.9 We note that Theorem 2.8 can actually be used to simplify our proof of Proposition 2.4. Indeed, it immediately implies the compatibility of the assignment \( \tilde{\Phi}_{\mu}^{\mathbb{Z}} \) with the defining relations (U2, U3, U7, U8), while the compatibility with (U1, U4, U5) is easily checked. Thus, it remains only to prove the compatibility with (U6), which is verified by expressing \( \gamma(z)^+ - \gamma(z)^- \) as a sum of delta-functions in a standard way, see [11, Lemma C.1, §C(vi)].

Remark 2.10 The construction (19) is reminiscent of that from [9, Proposition 2] in the elliptic setting. To this end, we consider the \( \mathbb{C}(q) \)-algebra \( \tilde{B}^{(a)}_{\text{frac}} \) generated by \( \{ w_{i,r}, E_{i,r} \}_{i \in I} \), further localized by the multiplicative set generated by \( \{ w_{i,r} - q^{mcij} w_{j,s} \}_{(i,r) \neq (j,s)} \), with:

\[
\begin{align*}
    w_{i,r}w_{j,s} &= w_{j,s}w_{i,r}, \\
    E_{i,r}w_{j,s} &= q_i^{-2\delta_{ij}\delta_{rs}}w_{j,s}E_{i,r}, \\
    E_{i,r}E_{j,s} &= \frac{q_i^{c_{ij}}}{w_{i,r} - q_i^{c_{ij}}}w_{j,s}E_{j,s}E_{i,r}.
\end{align*}
\]

This algebra is equipped with the following homomorphism to the algebra \( \widetilde{A}_{\text{frac}}^{q^{'}} \):

\[
\zeta : \tilde{B}^{(a)}_{\text{frac}} \rightarrow \tilde{A}_{\text{frac}}^{q^{'}} \text{ given by } w_{i,r} \mapsto w_{i,r}, E_{i,r} \mapsto Y_{i,r}(w_{i,r})D_{i,r}^{-1}.
\]

Then:

(a) The restriction of the algebra homomorphism \( \tilde{\Phi}_\mu^{\mathbb{Z}} \) to the positive subalgebra \( U_{\mu}^{\mathbb{Z}} \), identified with \( U_{\mu}^{\mathbb{Z}}(Lg) \) via (11), can be interpreted as a composition of \( \zeta \) from (26) and

\[
\Phi_\mu : U_{\mu}^{\mathbb{Z}}(Lg) \rightarrow \tilde{B}^{(a)}_{\text{frac}} \text{ given by } e_i(z) \mapsto \sum_{r=1}^{a_i} \delta \left( \frac{w_{i,r}}{z} \right) \cdot E_{i,r}.
\]

(b) The homomorphisms \( \Phi_\mu \) of (27) can be obtained from their simplest counterparts with \( a = (0, \ldots, 0, 1, \ldots, 0) \) via the “twisted tensor product”. To this end, for \( a^{(1)}, a^{(2)} \in \mathbb{N}^l \) set \( a^{(12)} := a^{(1)} + a^{(2)} \), and consider the corresponding algebras \( \tilde{B}^{e^{(1)}(a^{(12)})}_{\text{frac}}, \tilde{B}^{e^{(2)},q}_{\text{frac}}, \tilde{B}^{\mu,(2),q}_{\text{frac}} \). Let \( U_{\mu}^{\mathbb{Z}}(Lg) \) be the subalgebra generated by \( \{ e_{i,r}, \psi_{i,-k} \}_{i \in I} \). It is endowed with the formal coproduct:

\[
\Delta : e_i(z) \mapsto e_i(z) \otimes 1 + \psi_i^-(z) \otimes e_i(z), \quad \psi_i^-(z) \mapsto \psi_i^-(z) \otimes \psi_i^-(z).
\]
Following (10), let us extend the algebra homomorphism (27) to \( \Phi_\alpha : U_q^+(Lg) \to \hat{B}_\text{frac}^{(q), q} \). We also consider the algebra embedding \( \iota : \hat{B}_\text{frac}^{(12), q} \hookrightarrow \hat{B}_\text{frac}^{(1), q} \otimes \hat{B}_\text{frac}^{(2), q} \) determined by

\[
\begin{align*}
 w_{i,r} &\mapsto \begin{cases} 
 w^{(1)}_{i,r} & \text{if } r \leq a_i^{(1)} \\
 w^{(2)}_{i,r-a_i^{(1)}} & \text{if } r > a_i^{(1)}
\end{cases}, \\
 E_{i,r} &\mapsto \begin{cases} 
 E^{(1)}_{i,r} & \text{if } r \leq a_i^{(1)} \\
 F^{(1)}_{\alpha_i^{-1}} (\psi^{(2)}_{i,r-a_i^{(1)}}) E^{(2)}_{i,r-a_i^{(1)}} & \text{if } r > a_i^{(1)}
\end{cases}.
\end{align*}
\]

Then, \( \Phi_{\alpha_1 + \alpha_2} : U_q^+(Lg) \to \hat{B}_\text{frac}^{(12), q} \) factors through the composition \( (\Phi_{\alpha_1} \otimes \Phi_{\alpha_2}) \circ \Delta \), that is:

\[
\iota \circ \Phi_{\alpha_1 + \alpha_2} = (\Phi_{\alpha_1} \otimes \Phi_{\alpha_2}) \circ \Delta.
\]

2.11 Special difference operators

For any \( k \in \mathbb{N}^I \) and any multisymmetric Laurent polynomial \( g \in \mathbb{C}(q) \left[ \left( x_{i,r}^{\pm 1} \right)_{i \in I} \right]^{S_k} \), consider the following shuffle element \( \tilde{E}_k(g) \in S_k^{(g)} \):

\[
\tilde{E}_k(g) := \prod_{i \in I} \left\{ k_i - q_i^{2} \left( q_i - q_i^{-1} \right) k_i \right\} \prod_{i \in I} \prod_{1 \leq r < s \leq k_i} \left( x_{i,r} - q_i^{-2} x_{i,s} \right) \cdot g \left( \left( x_{i,r}^{\pm 1} \right)_{i \in I} \right) \frac{\prod_{i \to j} \prod_{r \leq k_i} (x_{j,s} - x_{i,r})}{\prod_{i \in I} \prod_{1 \leq r < s \leq k_i} (x_{i,r} - q_i^{2} x_{i,s})}. 
\]

These elements obviously satisfy the wheel conditions (16), due to the presence of the factor \( \prod_{i \in I} \prod_{1 \leq r < s \leq k_i} (x_{i,r} - q_i^{-2} x_{i,s}) \) and thus can be written as \( \tilde{E}_k(g) = \Upsilon (\tilde{e}_k(g)) \) for unique \( \tilde{e}_k(g) \in U_q^{\mu, \geq} \simeq U_q^+(Lg) \) by Proposition 2.6, so that \( \tilde{E}_k(g) = \Phi_{\mu, \geq}^{(12)} (\tilde{E}_k(g)) = \tilde{F}_k^{(12)} (\tilde{e}_k(g)) \) by Theorem 2.8(a). We also consider \( \tilde{F}_k(g) \in S_k^{(g)} \) defined via:

\[
\tilde{F}_k(g) := \prod_{i \in I} \left\{ k_i - q_i^{2} \left( 1 - q_i^{2} k_i \right) \right\} \prod_{i \in I} \prod_{1 \leq r < s \leq k_i} \left( x_{i,r} - q_i^{-2} x_{i,s} \right) \cdot g \left( \left( x_{i,r}^{\pm 1} \right)_{i \in I} \right) \frac{\prod_{i \to j} \prod_{r \leq k_i} (x_{j,s} - x_{i,r})}{\prod_{i \in I} \prod_{1 \leq r < s \leq k_i} (x_{i,r} - q_i^{2} x_{i,s})}. 
\]

The following result generalizes its type A case established in [12, Proposition 4.12]:

**Lemma 2.12** (a) For \( \tilde{E}_k(g) \in S_k^{(g)} \) given by (30), we have:

\[
\Phi_{\mu, \geq}^{(12)} (\tilde{E}_k(g)) = \prod_{i \in I} \left( \prod_{t = 1}^{a_i} W_{i,t} \right)^{k_i + \sum_{j \in \iota^{-1} \mathbb{N}} \frac{\iota(j)}{2} k_j}
\]

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\[ \times \sum_{J_i \subseteq \{1, \ldots, a_i\}} \prod_{J_i = k_i} \forall i \in I \left( \prod_{j - i} \prod_{r \in J_i} 1^{1 \leq s \leq a_j} \prod_{p=1}^{c_{ji}} - \delta_{s \in J_j} \left(1 - \frac{q_j + 2p}{w_{i,r}} \right) \right) \cdot g \left( \{w_{i,r}\}_{i \in I} \right) \]

\[ \times \prod_{i \in I} \prod_{r \in J_i} Z_i(w_{i,r}) \cdot \prod_{i \in I} \left( \prod_{r \in J_i} w_{i,r} \right)^{k_i - 1 - \sum_{j \rightarrow i} k_j} \cdot \prod_{i \in I} \prod_{r \in J_i} D_i^{-1} \right) . \]  

(b) For \( \tilde{F}_k(g) \in S_k^{(g)} \), given by (31), we have:

\[ \hat{\Phi}_{\mu}^{\lambda, Z}(\tilde{F}_k(g)) = \prod_{i \in I} \left( \prod_{t = 1}^{a_i} w_{i,t} \right) \sum_{j \rightarrow i} \frac{c_{ij}}{Z_i} k_j \]

\[ \times \sum_{J_i \subseteq \{1, \ldots, a_i\}} \prod_{J_i = k_i} \forall i \in I \left( \prod_{j - i} \prod_{r \in J_i} 1^{1 \leq s \leq a_j} \prod_{p=1}^{c_{ji}} - \delta_{s \in J_j} \left(1 - \frac{q_j + 2p}{w_{i,r}} \right) \right) \cdot g \left( \{q_i^2 w_{i,r}\}_{i \in I} \right) \]

\[ \times \prod_{i \in I} \left( \prod_{r \in J_i} w_{i,r} \right)^{k_i - 1 - \sum_{j \rightarrow i} k_j} \cdot \prod_{i \in I} q_i \cdot \prod_{i \in I} \prod_{r \in J_i} D_i \right) . \]  

**Proof** The proof is straightforward and is based on (19, 22). Due to the presence of the factors \( \prod_{r \in J_i} 1^{1 \leq s \leq a_j} \), the summands of (19, 22) with at least one \( m_r^{(i)} > 1 \) do vanish. This explains why the summations over all partitions of \( k_i \) into \( a_i \) nonnegative terms in (19, 22) are replaced by the summations over all cardinality \( k_i \) subsets of \( \{1, \ldots, a_i\} \) in (32, 33).

**Corollary 2.13** If \( k_i > a_i \) for some \( i \in I \), then \( \hat{\Phi}_{\mu}^{\lambda, Z}(\tilde{F}_k(g)) = 0 = \hat{\Phi}_{\mu}^{\lambda, Z}(\tilde{F}_k(g)) \) for all \( g \).

### 3 Generalization to the quantum toroidal \( \mathfrak{gl}_1 \)

The above constructions admit natural generalizations to the case of shifted version of the quantum toroidal algebra \( \tilde{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1) \), related (e.g. via [2]) to the Jordan quiver. We shall state the key results, skipping the proofs when they are similar to those from Sect. 2.

#### 3.1 Shifted quantum toroidal \( \mathfrak{gl}_1 \)

Fix \( q_1, q_2, q_3 \in \mathbb{C}^\times \) that are not roots of unity and satisfy \( q_1 q_2 q_3 = 1 \). For \( b^+, b^- \in \mathbb{Z} \), we define the shifted quantum toroidal algebra of \( \mathfrak{gl}_1 \), denoted by \( \tilde{U}_{b^+, b^-}(\mathfrak{gl}_1) \), to be the associative \( \mathbb{C} \)-algebra generated by \( \{e_r, f_r, \psi_{\pm s}^\pm, (\psi_{\pm b^\pm})^{-1}\}_{r \in \mathbb{Z}} \) with the following defining relations:
\[ \psi^\pm(z), \psi^\mp(w) = 0, \quad \psi^\pm_{\mp\mp} \cdot (\psi^\pm_{\mp\pm})^{-1} = (\psi^\pm_{\mp\pm})^{-1} \cdot \psi^\pm_{\mp\mp} = 1, \]
\[ (z - q_1 w)(z - q_2 w)(z - q_3 w)e(z)e(w) = (q_1 z - w)(q_2 z - w)(q_3 z - w)e(w)e(z), \]
\[ (q_1 z - w)(q_2 z - w)(q_3 z - w)f(z)f(w) = (z - q_1 w)(z - q_2 w)(z - q_3 w)f(w)f(z), \]
\[ (z - q_1 w)(z - q_2 w)(z - q_3 w)\psi^\pm(z)e(w) = (q_1 z - w)(q_2 z - w)(q_3 z - w)e(w)\psi^\pm(z), \]
\[ (q_1 z - w)(q_2 z - w)(q_3 z - w)\psi^\pm(z)f(w) = (z - q_1 w)(z - q_2 w)(z - q_3 w)f(w)\psi^\pm(z), \]
\[ [e(z), f(w)] = \frac{1}{\beta_1} \delta \left( \frac{z}{w} \right) (\psi^+(z) - \psi^-(z)), \]
\[ \text{Sym}_{z_1, z_2, z_3} \frac{z_2}{z_3} [e(z_1), e(z_2), e(z_3)] = 0, \]
\[ \text{Sym}_{z_1, z_2, z_3} \frac{z_2}{z_3} [f(z_1), f(z_2), f(z_3)] = 0, \]

where \( \epsilon, \epsilon' \in \{ \pm \}, \beta_1 = (1 - q_1)(1 - q_2)(1 - q_3) \), and the generating series are defined via:
\[ e(z) := \sum_{r \in \mathbb{Z}} e_r z^{-r}, \quad f(z) := \sum_{r \in \mathbb{Z}} f_r z^{-r}, \quad \psi^\pm(z) := \sum_{r \geq -b^\pm} \psi^\pm_{\pm\mp} z^{-r}. \]

**Remark 3.2**
(a) The original quantum toroidal algebra of \( \mathfrak{gl}_1 \), denoted by \( \hat{U}_{q_1, q_2, q_3} (\mathfrak{gl}_1) \), is isomorphic to \( \hat{U}_{q_1, q_2, q_3}^{(0,0)} (\psi^+_0, \psi^-_0 - 1) \).
(b) We note the \( S(3) \)-symmetry of \( \hat{U}_{q_1, q_2, q_3}^{(b^+, b^-)} \) with respect to the permutations of \( q_1, q_2, q_3 \).

The algebras \( \hat{U}_{q_1, q_2, q_3}^{(b^+, b^-)} \) and \( \hat{U}_{q_1, q_2, q_3}^{(0, b^+ + b^-)} \) are naturally isomorphic for any \( b^\pm \). Hence, we do not lose generality by considering only \( \hat{U}_{q_1, q_2, q_3}^{(0, b)} \), which will be denoted by \( \tilde{U}_{q_1, q_2, q_3}^{(b)} \) for simplicity.

### 3.3 GKLO-type homomorphisms

Fix a pair of integers: \( a \geq 1 \) and \( N \geq 0 \) (following [2, §A(iii)], one can interpret them as \( a = \dim(V) \) and \( N = \dim(W) \) in the Jordan quiver). Let \( \tilde{A}^{q_1} \) be the associative \( \mathbb{C} \)-algebra generated by \( \{ D_r^{\pm 1}, w_r^{\pm 1} \}_{1 \leq r \leq a} \) with the only nontrivial commutator \( D_r w_s = q_1^{\delta_{rs}} w_s D_r \), and let \( \tilde{A}^{q_1} \) be the localization of \( \tilde{A}^{q_1} \) by the multiplicative set generated by \( \{ w_r - q_1^m w_s \}_{1 \leq r < s \leq a} \). We also choose a sequence \( Z = (z_1, \ldots, z_N) \in (\mathbb{C}^*)^N \) and define \( Z(z) := \prod_{k=1}^N \left( 1 - \frac{z_k}{z} \right) \).

Then, we have the following analogue of Proposition 2.4:

**Proposition 3.4** There exists a unique \( \mathbb{C} \)-algebra homomorphism
\[ \Phi^Z_{\alpha}: \hat{U}_{q_1, q_2, q_3}^{(N)} \rightarrow \tilde{A}^{q_1} \]

such that

\[ \hat{U}_{q_1, q_2, q_3}^{(N)} \rightarrow \tilde{A}^{q_1} \]
As before, $\gamma(z)^\pm$ denotes the expansion of a rational function $\gamma(z)$ in $z^{\pm 1}$, respectively.

**Remark 3.5** Due to the $S(3)$-symmetry of $\tilde{U}^{(N)}_{q_1 q_2 q_3}$ (Remark 3.2(b)), we can replace $q_2$ by $q_3$ in (35). Overall, we have six similar homomorphisms: two $\tilde{U}^{(N)}_{q_1 q_2 q_3} \to \tilde{A}^{q_i}$ for each $i = 1, 2, 3$.

**Remark 3.6** In the unshifted case $N = 0$, (34) factors through $\tilde{\Phi}_a: \tilde{U}^{(N)}_{q_1 q_2 q_3}(g_l_1) \to \tilde{A}^{q_i}$ (see Remark 3.2(a)) that maps:

$$
\tilde{\Phi}_a: \quad e_0 \mapsto \frac{-1}{1-q_i} \sum_{r=1}^a w_r \frac{w_r - q_i^{-1} w_i}{w_r - w_i} D_r^{-1}, \quad f_0 \mapsto \frac{1}{1-q_i} \sum_{r=1}^a \prod_{1 \leq s \leq a} w_r - q_i^{-1} w_i D_r, \\
\psi_1^+ \mapsto (1-q_2^{-1})(1-q_3^{-1}) \sum_{r=1}^a w_r, \quad \psi_1^- \mapsto (1-q_2)(1-q_3) \sum_{r=1}^a w_r^{-1}, \quad \psi_0^\pm \mapsto 1.
$$

Let us compare this with [7, Proposition 5.1], where a natural $\tilde{U}^{(N)}_{q_1 q_2 q_3}(g_l_1)$-representation of [7, Lemma 3.7] is interpreted as an algebra homomorphism $\tilde{\Phi}_a: \tilde{U}^{(N)}_{q_1 q_2 q_3}(g_l_1) \to \tilde{A}^{q_i}$ given by (we swap $q_2 \leftrightarrow q_3$ in the formulas of [7]):

$$
\tilde{\Phi}_a: \quad e_0 \mapsto \frac{1}{1-q_i} \sum_{r=1}^a w_r, \quad f_0 \mapsto \frac{-1}{1-q_i} \sum_{r=1}^a w_r^{-1}, \quad \psi_0^\pm \mapsto 1, \\
\psi_1^+ \mapsto (1-q_2)(1-q_3) \sum_{r=1}^a \prod_{s \neq r} w_r - q_2^{-1} w_s D_r, \\
\psi_1^- \mapsto (1-q_2^{-1})(1-q_3^{-1}) \sum_{r=1}^a \prod_{s \neq r} w_r - q_2^{-1} w_s D_r^{-1}.
$$

Both $\tilde{\Phi}_a$ and $\tilde{\Phi}_a$ factor through the central quotient $\tilde{U}^{(N)}_{q_1 q_2 q_3}(g_l_1)/(\psi_0^\pm - 1)$ and the resulting homomorphisms $\tilde{\Phi}_a, \tilde{\Phi}_a: \tilde{U}^{(N)}_{q_1 q_2 q_3}(g_l_1)/(\psi_0^\pm - 1) \to \tilde{A}^{q_i}$ are related via $\tilde{\Phi}_a = \hat{\tilde{\Phi}}_a \circ \hat{\sigma}$, where $\hat{\sigma}$ is an automorphism (a version of the Burban–Schifmann/Miki’s automorphism) of $\tilde{U}^{(N)}_{q_1 q_2 q_3}(g_l_1)/(\psi_0^\pm - 1)$ determined by:

$$
\hat{\sigma}: \psi_1^+ \mapsto \beta_1 f_0, \quad \psi_1^- \mapsto \beta_1 e_0, \quad e_0 \mapsto q_1^{-1} \beta_1^{-1} \psi_1^+, \quad f_0 \mapsto q_1 \beta_1^{-1} \psi_1^-.
$$

### 3.7 Shuffle algebra realization of the positive and negative subalgebras

Similar to (11, 12), we have the following algebra isomorphisms:

$$
\tilde{U}^{(N), >}_{q_1 q_2 q_3} \cong \tilde{U}^{>}_{q_1 q_2 q_3}(g_l_1), \quad \tilde{U}^{(N), <}_{q_1 q_2 q_3} \cong \tilde{U}^{<}_{q_1 q_2 q_3}(g_l_1), \\
\tilde{U}^{<}_{q_1 q_2 q_3}(g_l_1) \cong \tilde{U}^{<}_{q_1 q_2 q_3}(g_l_1)^{op}.
$$
with subalgebras $\tilde{U}_{q_1,q_2,q_3}^{(N),>}, \tilde{U}_{q_1,q_2,q_3}^{(N),<}, \tilde{U}_{q_1,q_2,q_3}^{(N),<} (\mathfrak{gl}_1)$ defined in a self-explaining way.

Consider an $\mathbb{N}$-graded $\mathbb{C}$-vector space $S = \bigoplus_{k \in \mathbb{N}} S_k$, with the graded components

$$S_k = \left\{ F = \frac{f(x_1, \ldots, x_k)}{\prod_{1 \leq r \neq s \leq k} (x_r - x_s)} \mid f \in \mathbb{C} \left[ x_1^{\pm 1}, \ldots, x_k^{\pm 1} \right] S^{(k)} \right\}. \quad (40)$$

We also fix a rational function

$$\zeta \left( \frac{z}{w} \right) = \frac{(z - q_1^{-1}w)(z - q_2^{-1}w)(z - q_3^{-1}w)}{(z - w)^3}. \quad (41)$$

The bilinear shuffle product $\star$ on $S$ is defined completely analogously to (15), thus endowing $S$ with a structure of an associative unital $\mathbb{C}$-algebra. As before, we are interested in an $\mathbb{N}$-graded subspace of $S$ defined by the following wheel conditions:

$$F(x_1, \ldots, x_k) = 0 \text{ once } \begin{cases} x_1, x_2, x_3 \end{cases} = \begin{cases} q_1, q_2, q_3 \end{cases}. \quad (42)$$

Let $S \subset S$ denote the subspace of all such elements $F$, which is easily seen to be $\star$-closed. The resulting shuffle algebra $(S, \star)$ is related to $\tilde{U}_{q_1,q_2,q_3}^{(N),<} (\mathfrak{gl}_1)$ via the following result of [15]:

**Proposition 3.8** [15] The assignments $e_r \mapsto x_r^r$ and $f_r \mapsto x_r^r$ for $r \in \mathbb{Z}$ give rise to $\mathbb{C}$-algebra isomorphisms

$$\Upsilon: \tilde{U}_{q_1,q_2,q_3}^{(N),>} (\mathfrak{gl}_1) \xrightarrow{\sim} S \text{ and } \Upsilon: \tilde{U}_{q_1,q_2,q_3}^{(N),<} (\mathfrak{gl}_1) \xrightarrow{\sim} S^{\text{op}}. \quad (43)$$

### 3.9 Shuffle algebra realization of the GKLO-type homomorphisms

For $1 \leq r \leq a$, we define:

$$Y_r(z) := \frac{-1}{1 - q_1^{-1}} Z(z) \prod_{1 \leq s \leq a} \frac{z - w_s q_2^{-1}}{z - w_s}, \quad Y_r'(z) := \frac{1}{1 - q_1^{-1}} \prod_{1 \leq s \leq a} \frac{z q_1^{-1} - w_s q_2}{z q_1^{-1} - w_s}. \quad (44)$$

We also define

$$\varphi \left( \frac{z}{w} \right) := \frac{\left( q_1^{-1/2} z - q_1^{-1/2} w \right) \left( q_2^{1/2} z - q_2^{-1/2} w \right)}{(z - w)^2}. \quad (45)$$

Let $\mathcal{A}^{q_1,1}$ be the localization of $\mathcal{A}^{q_1}$ by the multiplicative set generated by $\{ w_r - q_1^m q_2 w_s r \neq s \}$. The following is our key result and is proved completely analogously to Theorem 2.8:
Theorem 3.10  (a) The assignment

\[
\mathbb{S}_k \ni E \mapsto \sum_{m_1 + \ldots + m_a = k} \left\{ \prod_{r=1}^{a} \prod_{p=1}^{m_r} Y_r \left( w_r q_1^{-(p-1)} \right) \cdot E \left( \left\{ w_r q_1^{-(p-1)} \right\}_{1 \leq p \leq m_r r \leq a} \right) \right. \\
\times \left. \prod_{1 \leq r \leq a} \prod_{1 \leq p_1 < p_2 \leq m_r} \zeta^{-1} \left( w_r q_1^{-(p_1-1)} / w_r q_1^{-(p_2-1)} \right) \right. \\
\times \left. \prod_{1 \leq r_1 \neq r_2 \leq a} \prod_{1 \leq p_1 \leq m_{r_2}} \varphi^{-1} \left( w_{r_2} q_1^{p_2} / w_{r_1} q_1^{p_1} \right) \cdot \prod_{r=1}^{a} D_r^{m_{r_2}} \right\}
\]  

(46)
gives rise to the algebra homomorphism

\[
\tilde{\Phi}_z^a : \mathbb{S} \rightarrow \tilde{A}^{q_1}.
\]  

(47)

Moreover, the composition

\[
\tilde{U}^{(N),>}_{q_1,q_2,q_3} \xrightarrow{(39)} \tilde{U}^{>}_{q_1,q_2,q_3} (\mathfrak{g}_1) \xrightarrow{\gamma} \mathbb{S} \xrightarrow{\tilde{\Phi}_z^a} \tilde{A}^{q_1}.
\]  

(48)

coincides with the restriction of the homomorphism \( \tilde{\Phi}_z^a \) of (34) to the subalgebra \( \tilde{U}^{(N),>}_{q_1,q_2,q_3} \).

(b) The assignment

\[
\mathbb{S}_k^{\text{op}} \ni F \mapsto \sum_{m_1 + \ldots + m_a = k} \left\{ \prod_{r=1}^{a} \prod_{p=1}^{m_r} Y_r' \left( w_r q_1^{p} \right) \cdot F \left( \left\{ w_r q_1^{p} \right\}_{1 \leq p \leq m_r r \leq a} \right) \right. \\
\times \left. \prod_{1 \leq r \leq a} \prod_{1 \leq p_1 < p_2 \leq m_r} \zeta^{-1} \left( w_r q_1^{p_2} / w_r q_1^{p_1} \right) \right. \\
\times \left. \prod_{1 \leq r_1 \neq r_2 \leq a} \prod_{1 \leq p_1 \leq m_{r_2}} \varphi^{-1} \left( w_{r_2} q_1^{p_2} / w_{r_1} q_1^{p_1} \right) \cdot \prod_{r=1}^{a} D_r^{m_{r_2}} \right\}
\]  

(49)
gives rise to the algebra homomorphism

\[
\tilde{\Phi}_z^a : \mathbb{S}^{\text{op}} \rightarrow \tilde{A}^{q_1}.
\]  

(50)

Moreover, the composition

\[
\tilde{U}^{(N),<}_{q_1,q_2,q_3} \xrightarrow{(39)} \tilde{U}^{<}_{q_1,q_2,q_3} (\mathfrak{g}_1) \xrightarrow{\gamma} \mathbb{S}^{\text{op}} \xrightarrow{\tilde{\Phi}_z^a} \tilde{A}^{q_1}.
\]  

(51)

coincides with the restriction of the homomorphism \( \tilde{\Phi}_z^a \) of (34) to the subalgebra \( \tilde{U}^{(N),<}_{q_1,q_2,q_3} \).
3.11 Special difference operators

For any \( g \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_k^{\pm 1}]^S(k) \), consider the following shuffle elements \( \tilde{E}_k(g) \in S_k \):

\[
\tilde{E}_k(g) := q_3^{k-\frac{k^2}{2}}(q_1^{-1} - 1)^k \cdot \frac{\prod_{1 \leq r \neq s \leq k}(x_r - q_1^{-1}x_s) \cdot g(x_1, \ldots, x_k)}{\prod_{1 \leq r \neq s \leq k}(x_r - x_s)}, \tag{52}
\]

which obviously satisfy the wheel conditions (42). Due to Proposition 3.8, \( \tilde{E}_k(g) = \Upsilon(\tilde{e}_k(g)) \) for unique elements \( \tilde{e}_k(g) \in \tilde{U}_{q_1,q_2,q_3}^{(N)} \simeq \tilde{U}_{q_1,q_2,q_3}(q_1) \), so that \( \Phi^Z_a(\tilde{E}_k(g)) = \tilde{\Phi}^Z_a(\tilde{e}_k(g)) \) by Theorem 3.10(a). We also consider \( \tilde{F}_k(g) \in S_k^{op} \) defined via:

\[
\tilde{F}_k(g) := (q_2/q_1)^{k-\frac{k^2}{2}}(1 - q_1)^k \cdot \frac{\prod_{1 \leq r \neq s \leq k}(x_r - q_1^{-1}x_s) \cdot g(x_1, \ldots, x_k)}{\prod_{1 \leq r \neq s \leq k}(x_r - x_s)}. \tag{53}
\]

The following result is established completely analogously to Lemma 2.12:

**Lemma 3.12** (a) For \( \tilde{E}_k(g) \in S_k \) given by (52), we have:

\[
\Phi^Z_a(\tilde{E}_k(g)) = \sum_{|J|=k \atop J \subset [1, \ldots, a]} \left\{ \prod_{r \in J} \frac{w_r - q_2^{-1}w_s}{w_r - w_s} \cdot \prod_{r \in J} Z(w_r) \cdot g\left(\{w_r\}_{r \in J}\right) \cdot \prod_{r \in J} D_r^{-1} \right\}. \tag{54}
\]

(b) For \( \tilde{F}_k(g) \in S_k^{op} \) given by (53), we have:

\[
\Phi^Z_a(\tilde{F}_k(g)) = \sum_{|J|=k \atop J \subset [1, \ldots, a]} \left\{ \prod_{r \in J} \frac{w_r - q_2w_s}{w_r - w_s} \cdot g\left(q_1\{w_r\}_{r \in J}\right) \cdot \prod_{r \in J} D_r \right\}. \tag{55}
\]

**Example 3.13** For \( N = 0 \) and \( g = 1 \), we recover the famous Macdonald difference operators:

\[
\Phi^Z_a(\tilde{E}_k(1)) = \sum_{|J|=k \atop J \subset [1, \ldots, a]} \left\{ \prod_{r \in J} \frac{w_r - q_2^{-1}w_s}{w_r - w_s} \cdot \prod_{r \in J} D_r^{-1} =: D^k_a(q_1, q_2), \right.
\]

\[
\Phi^Z_a(\tilde{F}_k(1)) = \sum_{|J|=k \atop J \subset [1, \ldots, a]} \left\{ \prod_{r \in J} \frac{w_r - q_2w_s}{w_r - w_s} \cdot \prod_{r \in J} D_r =: D^k_a(q_1^{-1}, q_2^{-1}). \right. \tag{56}
\]

**Remark 3.14** We note that the crucial and rather nontrivial commutativity

\[
\left[ D^k_a(q_1, q_2), D^{k'}_a(q_1, q_2) \right] = 0 \quad \text{for all} \quad 1 \leq k, k' \leq a
\]
thus arises as an immediate consequence of a simple equality \[ \hat{E}_k(1), \hat{E}_{k'}(1) = 0 \] in the shuffle algebra \( S \), see [8, Proposition 2.21].

### 4 Generalization to the quantum toroidal \( sl_n \) (\( n \geq 3 \))

The above constructions admit natural generalizations to the case of shifted version of the quantum toroidal algebra \( \hat{U}_{q,d}(sl_n) \), related (e.g. via [2]) to the cyclic \( n \)-vertex quiver. We shall state the key results, skipping the proofs when they are similar to those from Sect. 2.

#### 4.1 Shifted quantum toroidal \( sl_n \)

For \( n \geq 3 \), consider an index set \( [n] := \{0, 1, \ldots, n-1\} \) (also viewed as a set of residues modulo \( n \)). We define two matrices \((c_{ij})_{i,j \in [n]}\) (the Cartan matrix of \( \hat{sl}_n \)) and \((m_{ij})_{i,j \in [n]}\) via:

\[
    c_{ii} = 2, \quad c_{i,i+1} = -1, \quad m_{i,i+1} = \mp 1, \quad \text{and} \quad c_{ij} = 0 = m_{ij} \quad \text{otherwise.} \quad (57)
\]

Fix \( q, d \in \mathbb{C}^\times \) such that \( q, qd^{\pm 1} \) are not roots of unity. Given \( b^{\pm}_i = \{b^\pm_i\}_{i \in [n]} \in \mathbb{Z}^{[n]} \), we define the **shifted quantum toroidal algebra of** \( sl_n \), denoted by \( \hat{U}_{q,d}(sl_n) \), to be the associative \( \mathbb{C} \)-algebra generated by \( \{e_{i,r}, f_{i,r}, \psi_{i,\pm s_i^+}, (\psi_{i,\mp s_i^+})^{-1}\}_{i \in [n]} \) with the following defining relations (for all \( i, j \in [n] \) and \( \epsilon, \epsilon' \in \{\pm\} \)):

\[
    [\psi^\epsilon_i(z), \psi^{\epsilon'}_j(w)] = 0, \quad \psi_{i,\mp s_i^+}^{\pm}_i = (\psi_{i,\mp b_i^+}^{\pm})^{-1} \cdot \psi_{i,\mp b_i^+}^{\pm} = 1, \quad (T1)
\]

\[
    (d_m^\epsilon_i z - q^c_{ij} d_m^\epsilon_j z - w) e_{i}(z) e_{j}(w) = (q^c_{ij} d_m^\epsilon_i z - w) e_{j}(z) e_{i}(w), \quad (T2)
\]

\[
    (q^c_{ij} d_m^\epsilon_i z - w) f_{i}(z) f_{j}(w) = (d_m^\epsilon_j z - q^c_{ij} w) f_{j}(z) f_{i}(w), \quad (T3)
\]

\[
    (d_m^\epsilon_i z - q^c_{ij} z - w) \psi^\epsilon_i(z) e_{j}(w) = (q^c_{ij} d_m^\epsilon_i z - w) e_{j}(w) \psi^\epsilon_i(z), \quad (T4)
\]

\[
    (q^c_{ij} d_m^\epsilon_i z - w) \psi^\epsilon_i(z) f_{j}(w) = (d_m^\epsilon_j z - q^c_{ij} w) f_{j}(w) \psi^\epsilon_i(z), \quad (T5)
\]

\[
    [e_i(z), f_j(w)] = \frac{\delta_{ij}}{q - q^{-1}} \delta \left( \frac{z}{w} \right) \left( \psi^+_i(z) - \psi^-_i(z) \right), \quad (T6)
\]

\[
    \text{Sym}_{z_1, z_2} \left( e_i(z_1) e_i(z_2) e_{i,\pm 1}(w) - (q + q^{-1}) e_i(z_1) e_{i,\pm 1}(w) e_i(z_2) + e_{i,\pm 1}(w) e_i(z_1) e_i(z_2) \right) = 0, \quad (T7)
\]

\[
    \text{Sym}_{z_1, z_2} \left( f_i(z_1) f_i(z_2) f_{i,\pm 1}(w) - (q + q^{-1}) f_i(z_1) f_{i,\pm 1}(w) f_i(z_2) + f_{i,\pm 1}(w) f_i(z_1) f_i(z_2) \right) = 0, \quad (T8)
\]

where the generating series \( \{e_i(z), f_i(z), \psi^\pm_i(z)\}_{i \in [n]} \) are defined as in (6).

The algebras \( \hat{U}_{q,d,(b^+, b^-)} \) and \( \hat{U}_{q,d,(0, b^+ + b^-)} \) are naturally isomorphic for any \( b^\pm_i \in \mathbb{Z}^{[n]} \). Thus, we do not lose generality by considering only \( \hat{U}_{q,d,(0,b)} \), which will be denoted by \( \hat{U}_{q,d} \) for simplicity. The original quantum toroidal algebra \( \hat{U}_{q,d}(sl_n) \) is isomorphic to \( \hat{U}_{q,d}(0,0) / (\psi^+_1, \psi^-_1, 0 - 1)_{i \in [n]} \).
4.2 GKLO-type homomorphisms

Fix $b \in \mathbb{Z}^{|n|}$ and let $a \in \mathbb{N}^{|n|}$ be such that $N_i := b_i + 2a_i - a_{i-1} - a_{i+1} \geq 0$ for all $i \in [n]$ (in particular, existence of such $a$ forces $\sum_{i \in [n]} b_i \geq 0$). We pick $Z = ([Z_{i,r} \mid 1 \leq r \leq N_i]_{i \in [n]})$ with $Z_{i,r} \in \mathbb{C}^\times$, as well as an orientation of the cyclic quiver $\text{Dyn}(\widehat{\mathfrak{s}l}_n)$ with the vertex set $[n]$ and the vertex $i$ connected to the vertices $i + 1, i - 1$.

We define the $\mathbb{C}$-algebra $\mathcal{A}^q$ as in Sect. 2.3 (note that we omit the subscript “frac” as it is now a $\mathbb{C}$-algebra) and follow the notations (8).

Then, we have the following analogue of Proposition 2.4:

**Proposition 4.3** There exists a unique $\mathbb{C}$-algebra homomorphism

$$\tilde{\Phi}_{\frac{a}{b}} : \tilde{U}_{q,d} \longrightarrow \mathcal{A}^q$$

(58)

such that

$$e_i(z) \mapsto \frac{-q^n}{1 - q^n} \prod_{t=1}^{a_i} w_{i,t} \prod_{j \rightarrow i}^{a_j} \prod_{l=1}^{a_l} w_{j,l}^{-1/2} \sum_{r=1}^{a_i} \left( \frac{w_{i,r}}{z} \right) Z_{i,r} \prod_{j \rightarrow i}^{a_l} W_j \left( q^{-1} d^{m_{i,j}} z \right) D_{i,r}^{-1},$$

$$f_i(z) \mapsto \frac{1}{1 - q^n} \prod_{j \rightarrow i}^{a_j} \prod_{l=1}^{a_l} w_{j,l}^{-1/2} \sum_{r=1}^{a_i} \delta \left( \frac{q^2 w_{i,r}}{z} \right) \frac{1}{W_i(r)} \prod_{j \rightarrow i}^{a_l} W_j \left( q^{-1} d^{m_{i,j}} z \right) D_{i,r},$$

$$\psi_i^\pm(z) \mapsto \prod_{j \rightarrow i}^{a_j} \prod_{l=1}^{a_l} w_{j,l}^{-1/2} \left( \frac{Z_i(z)}{W_i(z)} \prod_{j \rightarrow i}^{a_l} W_j \left( q^{-1} d^{m_{i,j}} z \right) \right)^\pm. \quad (59)$$

As before, $\gamma(z)^\pm$ denotes the expansion of a rational function $\gamma(z)$ in $z^{\pm 1}$, respectively.

**Remark 4.4** We note that the unshifted case $b = 0$ corresponds to $a_0 = a_1 = \ldots = a_{n-1}$.

4.5 Shuffle algebra realization of the positive and negative subalgebras

Similar to (11, 12, 39), we have the following algebra isomorphisms:

$$\tilde{U}_{q,d}^{(b), >} \sim \tilde{U}_{q,d}^>(\mathfrak{s}l_n), \quad \tilde{U}_{q,d}^{(b), <} \sim \tilde{U}_{q,d}^<(\mathfrak{s}l_n), \quad \tilde{U}_{q,d}^<\mathfrak{s}l_n \sim \tilde{U}_{q,d}^>\mathfrak{s}l_n^{op},$$

(60)

with the subalgebras $\tilde{U}_{q,d}^{(b), >}, \tilde{U}_{q,d}^>(\mathfrak{s}l_n), \tilde{U}_{q,d}^{(b), <}, \tilde{U}_{q,d}^<(\mathfrak{s}l_n), \tilde{U}_{q,d}^<\mathfrak{s}l_n$ defined in a self-explaining way.

Consider an $\mathbb{N}^{|n|}$-graded $\mathbb{C}$-vector space $\mathfrak{s}^{[n]} = \bigoplus_{k=(k_i)_{i \in [n]} \in \mathbb{N}^{|n|}} S_k$, with the graded components

$$\mathfrak{s}_k^{[n]} = \left\{ f \mid \begin{array}{l} f \left( \{x_i, r \}_{i \in [n]} \right) \\ \prod_{i \in [n]} \prod_{r \leq k_i} (x_i - x_{i+1, r}) \end{array} \right\}, \quad (61)$$

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where $S_k := \prod_{i \in [n]} S(k_i)$. We also fix rational functions $\{\xi_{ij}(z)\}_{i,j \in [n]}$ via:

\begin{align*}
\xi_{i,i+1} \left( \frac{z}{w} \right) &= \frac{d^{-1} z - q w}{z - w}, \\
\xi_{i,i-1} \left( \frac{z}{w} \right) &= \frac{z - q d^{-1} w}{z - w}, \\
\xi_{ii} \left( \frac{z}{w} \right) &= \frac{z - q^{-2} w}{z - w}, \\
\xi_{ij} \left( \frac{z}{w} \right) &= 1 \text{ if } j \neq i, i \pm 1. \quad (62)
\end{align*}

The bilinear shuffle product $\star$ on $S^{[n]}$ is defined completely analogously to (15), thus endowing $S^{[n]}$ with a structure of an associative unital $\mathbb{C}$-algebra. As before, we are interested in an $\mathbb{N}^{[n]}$-graded subspace of $S^{[n]}$ defined by the following wheel conditions:

\begin{equation}
F \left( \{x_{i,r}\} \right) = 0 \text{ once } x_{i,2} = q^2 x_{i,1} \text{ and } x_{i+\epsilon,1} = q d^{-\epsilon} x_{i,1} \text{ for } i \in [n], \ \epsilon = \pm 1.
\end{equation}

Let $S^{[n]} \subset S^{[n]}$ denote the subspace of all such elements $F$, which is easily seen to be $\star$-closed. The resulting shuffle algebra $(S^{[n]}, \star)$ is related to $\hat{U}_{q,d}(\mathfrak{sl}_n)$ via the following result of [16]:

**Proposition 4.6** [16] The assignments $e_{i,r} \mapsto x_{i,1}^r$ and $f_{i,r} \mapsto x_{i,1}^r$ for $i \in [n], r \in \mathbb{Z}$ give rise to $\mathbb{C}$-algebra isomorphisms

\begin{equation}
\Upsilon : \hat{U}^{\geq}_{q,d}(\mathfrak{sl}_n) \overset{\sim}{\longrightarrow} S^{[n]} \quad \text{and} \quad \Upsilon : \hat{U}^{\leq}_{q,d}(\mathfrak{sl}_n) \overset{\sim}{\longrightarrow} S^{[n], \text{op}}. \quad (64)
\end{equation}

### 4.7 Shuffle algebra realization of the GKLO-type homomorphisms

For any $i \in [n]$ and $1 \leq r \leq a_i$, we define:

\begin{align*}
Y_{i,r}(z) := \frac{1}{q - q^{-1}} \left( \prod_{t=1}^{a_i} W_{i,t} \right) \left( \prod_{j \rightarrow i}^{a_j} W_{j,t}^{-1/2} \cdot \frac{Z_i(z) \prod_{j \rightarrow i} W_{j}(z q^{-1} d_{mij})}{W_{i,r}(z)} \right), \\
Y'_{i,r}(z) := \frac{1}{1 - q^2} \left( \prod_{j \leftarrow i}^{a_j} W_{j,t}^{-1/2} \cdot \prod_{j \leftarrow i}^{a_j} W_{j}(z q^{-1} d_{mij}) W_{i,r}(z q^{-2}) \right). \quad (65)
\end{align*}

Define the $\mathbb{C}$-algebra $\tilde{A}'^q$ as the further localization of $\tilde{A}^q$ by the multiplicative set generated by $\{d_{mij} q^{c_{ij}} W_{i,r} - q^{-2} m W_{j,s}\}_{j \leq i}^{r \leq a_i, s \leq a_j}$. We note that $\tilde{A}'^q$ is naturally embedded into $\tilde{A}'^q'$. The following is our key result and is proved completely analogously to Theorem 2.8:

**Theorem 4.8** (a) The assignment

\[ S^{[n]}_k \ni E \mapsto q^{-\sum_{i \in [n]} (k_i - k_i^2)} \]
The assignment (b) \[ \hat{\Phi}^{\varpi, \mathcal{Z}}_{\mathfrak{b}} : S[n] \rightarrow \tilde{A}q. \]

Moreover, the composition
\[ \hat{U}^{(b, \mathcal{Z})}_{q, d} \sim \hat{U}^{(b)}_{q, d} (\mathfrak{sl}_n) \sim S[n] \xrightarrow{\hat{\Phi}^{\varpi, \mathcal{Z}}_{\mathfrak{b}}} \tilde{A}q. \]

coincides with the restriction of the homomorphism \( \tilde{\Phi}^{\varpi, \mathcal{Z}}_{\mathfrak{b}} \) of (58) to the subalgebra \( \hat{U}^{(b)}_{q, d} \). In particular, the image of \( \hat{U}^{(b, \mathcal{Z})}_{q, d} \) under the composition (68) is in the subalgebra \( \tilde{A}q \) of \( \tilde{A}q. \).

(b) The assignment
\[ S[n, \text{op}] \ni F \mapsto \sum_{m^{(i)} _1 + \ldots + m^{(i)} _n = k_i \atop m^{(i)} _r \in \mathbb{N} \atop i \in \{n\}} \left\{ \prod_{i \in \{n\}} \prod_{r = 1}^{m^{(i)} _i} Y_{i,r}^{(i)} \left( w_{i,r} q^{-2(p-1)} \right) \cdot F \left( \left\{ w_{i,r} q^{2p} \right\} \prod_{i \in \{n\}, 1 \leq r \leq a_i}^{1 \leq p \leq m^{(i)} _r} \right) \cdot \prod_{i \in \{n\}} \prod_{r = 1}^{m^{(i)} _r} \zeta_{ii}^{-1} \left( w_{i,r} q^{2p} \right) \cdot \prod_{i \in \{n\}} \prod_{r = 1}^{m^{(i)} _r} \zeta_{ii}^{-1} \left( w_{i,r} q^{2p} \right) \cdot \prod_{i \in \{n\}} \prod_{r = 1}^{m^{(i)} _r} D_{i,r}^{m^{(i)} _r} \right\} \] (69)
gives rise to the algebra homomorphism

\[ \Phi^{a,Z}_b : S^{[n],\text{op}} \rightarrow \tilde{A}^{q,'}. \]  

(70)

Moreover, the composition

\[ \tilde{U}^{(b),\prec}_q \xrightarrow{\sim} \tilde{U}^{\prec}_q \circ (\Phi_1) \xrightarrow{\sim} S^{[n],\text{op}} \xrightarrow{\Phi^{a,Z}_b} \tilde{A}^{q,'}. \]  

(71)

coincides with the restriction of the homomorphism \( \Phi^{a,Z}_b \) of (58) to the subalgebra \( \tilde{U}^{(b),\prec}_q \). In particular, the image of \( \tilde{U}^{(b),\prec}_q \) under the composition (71) is in the subalgebra \( \tilde{A}^q \) of \( \tilde{A}^{q,'} \).

### 4.9 Special difference operators

For any \( k \in \mathbb{N}^{[n]} \) and any multisymmetric Laurent polynomial \( g \in \mathbb{C}(q) \left[ \{x_i^{\pm 1}\}_{i \in [n]} \right]^{S_k} \), consider the following shuffle elements \( \tilde{E}_k(g) \in S^{[n],\text{op}}_k \):

\[
\tilde{E}_k(g) := \prod_{i \in [n]} \left\{ q^{k_i^2 - k_i} (q - q^{-1})^{k_i} \right\} \cdot \frac{\prod_{i \in [n]} \prod_{1 \leq r \neq s \leq k_i} (x_i, r - q^{-2}x_i, s) \cdot g \left( (x_i, r)_{1 \leq r \leq k_i} \right)}{\prod_{i \to j} \prod_{r \leq k_j} (x_j, s - x_i, r)}
\]

which obviously satisfy the wheel conditions (63). Due to Proposition 4.6, \( \tilde{E}_k(g) = \Upsilon(\tilde{E}_k(g)) \) for unique elements \( \tilde{E}_k(g) \in \tilde{U}^{(b),\succ}_q \simeq \tilde{U}^{\succ}_q(\Phi_1) \), so that \( \Phi^{a,Z}_b(\tilde{E}_k(g)) = \tilde{\Phi}^{a,Z}_b(\tilde{E}_k(g)) \) by Theorem 4.8(a). We also consider \( \tilde{F}_k(g) \in S^{[n],\text{op}}_k \) defined via:

\[
\tilde{F}_k(g) := \prod_{i \in [n]} \left\{ q^{k_i^2 - k_i} (1 - q^{-2})^{k_i} \right\} \cdot \frac{\prod_{i \in [n]} \prod_{1 \leq r \neq s \leq k_i} (x_i, r - q^{-2}x_i, s) \cdot g \left( (x_i, r)_{1 \leq r \leq k_i} \right)}{\prod_{i \to j} \prod_{r \leq k_j} (x_i, r - x_j, s)}
\]

(73)

The following result is established completely analogously to Lemma 2.12:

**Lemma 4.10** (a) For \( \tilde{E}_k(g) \in S^{[n],\text{op}}_k \) given by (72), we have:

\[
\tilde{\Phi}^{a,Z}_b(\tilde{E}_k(g)) = d^\sum_{i \in [n]} k_i k_{i+1} k_i+1 \prod_{i \in [n]} \left( \prod_{t=1}^{a_i} w_{i,t} \right)^{k_i - \frac{1}{2} \sum_{j \neq i} k_j} \times \sum_{J_i \subseteq \{1, \ldots, a_i\} \mid |J_i| = k_i} \prod_{i \in [n]} \prod_{r \in J_i} \left( 1 - \frac{q d^{m,j_i} w_{i,s}}{w_{i,r}} \right) \cdot g \left( |w_{i,r}|_{1 \leq i \leq n} \right)
\]
\[ \times \prod_{i \in [n]} \prod_{r \in J_i} Z_i(w_{i,r}) \cdot \prod_{i \in [n]} \left( \prod_{r \in J_i} w_{i,r} \right)^{k_i - 1 - \sum_{j \neq i} k_j} \cdot \prod_{i \in [n]} \prod_{r \in J_i} D_i^{-1}_{i,r} \]. \tag{74} \]

(b) For \( \tilde{F}_k(g) \in S_k^{[n],\text{op}} \) given by \( (73) \), we have:

\[ \hat{\Phi}_k^{a,z}(\tilde{F}_k(g)) = d \sum_{i \in [n]} k_i k_i + 1 - \delta_{i+1} \cdot q^{-3} \sum_{i \in [n]} k_i k_i + 1 \prod_{i \in [n]} \left( \prod_{r \in J_i} w_{i,r} \right)^{k_i - 1 - \sum_{j \neq i} k_j} \times \sum_{J_i \subset \{1, \ldots, a_i\}} \left\{ \frac{\prod_{j \neq i} \prod_{r \in J_i} \left(1 - \frac{q^{-1} d^{m_i} w_{i,r}}{w_{i,r}}\right)}{\prod_{i \in [n]} \prod_{r \in J_i} \left(1 - \frac{w_{i,r}}{w_{i,r}}\right)} \cdot g \left( (q^z w_{i,r})_{i \in [n]} \right) \right\} \]

\[ \times \prod_{i \in [n]} \left( \prod_{r \in J_i} w_{i,r} \right)^{k_i - 1 - \sum_{j \neq i} k_j} \cdot \prod_{i \in [n]} \prod_{r \in J_i} D_i^{-1}_{i,r} \]. \tag{75} \]

**Example 4.11** Consider the orientation of the cyclic quiver with arrows \( i \to i + 1 (i \in [n]) \).

(a) For \( p \in [n] \) and \( k \geq 1 \), consider the degree \( k = (k, \ldots, k) \in N^n \) elements

\[ \Gamma^0_{p,k} := \tilde{E}_k \left( \prod_{i \in [n]} x_{i,1} \cdots x_{i,k} \right)^{1 + \delta_{0} - \delta_{p}} = q^{n(k^2 - k) (q - q^{-1})^k} \cdot \frac{\prod_{i \in [n]} \prod_{1 \leq r \leq k} (x_{i,r} - q^{-2} x_{i,s}) \cdot \prod_{i \in [n]} \prod_{r=1}^{k} x_{i,r}}{\prod_{i \in [n]} \prod_{1 \leq r, s \leq k} (x_{i,r} - x_{i,s})} \cdot \prod_{r=1}^{k} x_{0,r} \cdot x_{p,r}. \tag{76} \]

Their images under \( \hat{\Phi}_k^{a,z} \) of \( (20) \) vanish if \( k > \min\{a_i\} \) and otherwise are given by:

\[ \hat{\Phi}_k^{a,z}(1_{-p;i}) = \prod_{i \in [n]} \left( \prod_{r \in J_i} w_{i,r} \right)^{k_i - \frac{1}{2} k_i + 1} \times \prod_{j \subset \{1, \ldots, a_i\}} \left\{ \prod_{r \in J_i} \left(1 - \frac{q^{-1} d^{m_i} w_{i,r}}{w_{i,r}}\right) \cdot \frac{\prod_{r \in J_i} w_{i,r}}{w_{i,r}} \right\} \]

\[ \times \prod_{i \in [n]} \prod_{r \in J_i} Z_i(w_{i,r}) \cdot \prod_{i \in [n]} \left( \prod_{r \in J_i} w_{i,r} \right)^{k_i - k_i + 1 + \delta_{0} - \delta_{p}} \cdot \prod_{i \in [n]} \prod_{r \in J_i} D_i^{-1}_{i,r}. \tag{77} \]

Similar to Remark 3.14, the difference operators \( (77) \) pairwise commute, due to the equality \( \left\{ \Gamma^0_{p,k}, \Gamma^0_{p',k'} \right\} = 0 \) in the shuffle algebra \( S^{[n]} \) established in [10, Remark 4.11 (a)] (the limit case of [10, Theorem 3.3], see part (b) below). According to [21, 22], the elements \( \{ \Upsilon^{-1}(\Gamma^0_{p,k}) \}_{p \in [n]}^{k \geq 1} \) generate the “positive half of the horizontal” Heisenberg subalgebra of \( \tilde{U}_{q,d}(\mathfrak{sl}_n) \).
(b) For $\mu \in \mathbb{C}$, $k \geq 1$, and $\underline{s} = (s_0, s_1, \ldots, s_{n-1}) \in (\mathbb{C}^\times)^n$ satisfying $s_0 s_1 \cdots s_{n-1} = 1$, consider:

$$F_k^{\mu}(\underline{s}) := \hat{\Phi}_\underline{s}(\prod_{i \in [n]} \left( s_0 \cdots s_i \prod_{r=1}^k x_{i,r} - \mu \prod_{r=1}^k x_{i+1,r} \right) ) = q^{n(k^2-k)}(q - q^{-1})^{nk}$$

$$\times \prod_{i \in [n]} \prod_{1 \leq r \neq s \leq k} (x_{i,r} - q^{-2}x_{i,s}) \cdot \prod_{i \in [n]} (s_0 \cdots s_i \prod_{r=1}^k x_{i,r} - \mu \prod_{r=1}^k x_{i+1,r})$$

$$\prod_{i \in [n]} \prod_{1 \leq r, s \leq k} (x_{i,r} - x_{i-1,s}).$$

(78)

Their images under $\hat{\Phi}_{\underline{a}}^{a,z}$ of (67) vanish if $k > \min\{a_i\}$ and otherwise are given by:

$$\hat{\Phi}_{\underline{a}}^{a,z}(F_k^{\mu}(\underline{s})) = \prod_{i \in [n]} \left( \prod_{l=1}^{a_i} \right)^{k_i - \frac{1}{2} k_{i+1}}$$

$$\times \sum_{J_i \subset \{1, \ldots, a_i\}} \prod_{|J_i| = k} \prod_{i \in [n]} \prod_{J_i} \left( 1 - \frac{q^{-1} w_{i-1, s}}{w_{i, r}} \right)$$

$$\times \prod_{i \in [n]} \prod_{r \in J_i} Z_i(w_{i,r}) \cdot \prod_{i \in [n]} \prod_{r \in J_i} \left( \sum_{l=1}^{q^{-1} x_{i, r}} \frac{q^{-\mu - 1} w_{i, s}}{w_{i, r}} \right)^{k_i - k_{i-1}}$$

$$\prod_{i \in [n]} \prod_{r \in J_i} D_{i,r}^{\mu}. \quad (79)$$

Similar to part (a), the difference operators (79) pairwise commute, due to the equality $[F_k^{\mu}(\underline{s}), F_k^{\mu'}(\underline{s})] = 0$ in the shuffle algebra $S^{[n]}$ established in [10, Theorem 3.3]. According to [10, Theorem 4.10], we note that the elements $\{ \Upsilon^{-1}(F_k^{\mu}(\underline{s})) \}$ in fact generate the Bethe commutative subalgebra of the “horizontal” quantum affine subalgebra $U_q(\widehat{gl}_n)$ of $\hat{U}_q(\mathfrak{sl}_n)$.

5 Generalization to the quantum quiver algebras

The above constructions admit natural generalizations to the case of quantum algebras associated with quivers as recently introduced in [18] following [17]. We shall state the key results, skipping the proofs when they are similar to those from Sect. 2.

5.1 Shifted quantum algebras associated with quivers

Let $E$ be a finite quiver, with a vertex set $I$ and an edge set $E$ (here, multiple edges and edge loops are allowed). Any edge $e$ of $E$ from a vertex $i \in I$ to a vertex $j \in I$ shall
be written as $e = ij \in E$. We fix $q \in \mathbb{C}^\times$ and equip every edge $e \in E$ with a weight $t_e \in \mathbb{C}^\times$. Furthermore, following [17, 18], we shall make the following assumption (cf. [18, Definition 5.2]):

$$|q| < |t_e| < 1 \quad \text{for all } e \in E. \quad \tag{\dagger}$$

We define rational functions $\{\zeta_{ij}(z)\}_{i,j \in I}$ via:

$$\zeta_{ij} \left( \frac{z}{w} \right) = \left( \frac{zq - 1 - w}{z - w} \right)^{\delta_{ij}} \prod_{e = ij \in E} \left( \frac{1}{t_e} - \frac{z}{w} \right) \prod_{e = ji \in E} \left( 1 - \frac{w t_e}{zq} \right). \quad \tag{80}$$

Let $\overline{E}$ be the “double” of the edge set $E$, i.e. there are two edges $e = ij, e^* = ji \in \overline{E}$ for every $e = ij \in E$. Note the canonical involution $e \leftrightarrow e^*$ on $\overline{E}$ and extend the notation $t_e$ to $\overline{E}$ via:

$$t_e^* := \frac{q}{t_e}. \quad \tag{81}$$

For any $b_i^\pm = \{b_i^\pm\}_{i \in I} \in \mathbb{Z}^I$, we define the shifted quantum quiver algebra, denoted by $U^Q_{Q(b_i^+, b_i^-)}$, to be the associative $\mathbb{C}$-algebra generated by $\{e_{i,r}, f_{i,r}, \psi_{i,\pm}^{\pm}\}_{i \in I} \in \mathbb{Z}$ with the following defining relations (for all $i, j \in I$ and $\epsilon, \epsilon' \in \{\pm\}$):

$$[\psi_{i,\epsilon}(z), \psi_{j,\epsilon'}(w)] = 0, \quad \psi_{i,\pm}^{\pm} \cdot (\psi_{i,\pm}^{\pm})^{-1} = (\psi_{i,\mp}^{\pm})^{-1} \cdot \psi_{i,\mp}^{\pm} = 1,$$

$$\zeta_{ji} \left( \frac{w}{z} \right) e_{i}(z)e_{j}(w) = \zeta_{ij} \left( \frac{z}{w} \right) e_{j}(w)e_{i}(z), \quad \tag{Q1}$$

$$\zeta_{ij} \left( \frac{z}{w} \right) f_{i}(z)f_{j}(w) = \zeta_{ji} \left( \frac{w}{z} \right) f_{j}(w)f_{i}(z), \quad \tag{Q2}$$

$$\zeta_{ji} \left( \frac{w}{z} \right) \psi_{i}^{\epsilon}(z)e_{j}(w) = \zeta_{ij} \left( \frac{z}{w} \right) e_{j}(w)\psi_{i}^{\epsilon}(z), \quad \tag{Q3}$$

$$\zeta_{ij} \left( \frac{z}{w} \right) \psi_{j}^{\epsilon}(z)f_{j}(w) = \zeta_{ji} \left( \frac{w}{z} \right) f_{j}(w)\psi_{j}^{\epsilon}(z), \quad \tag{Q4}$$

$$[e_{i}(z), f_{j}(w)] = \delta_{ij} \delta \left( \frac{z}{w} \right) \left( \psi_{i}^{+}(z) - \psi_{j}^{-}(z) \right), \quad \tag{Q5}$$

and more complicated cubic Serre relations of [18, §5.4] that shall be omitted for brevity. Here, the generating series $\{e_{i}(z), f_{i}(z), \psi_{i}^{\pm}(z)\}_{i \in I}$ are defined as in (6). The original quantum quiver algebra $U^Q_{Q}$ of [18] is isomorphic to $U^Q_{0,0}/(\psi_{i}^{+} \psi_{i}^{-} - 1)_{i \in I}$. 

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5.2 GKLO-type homomorphisms

Fix $a = (a_i)_{i \in I} \subseteq \mathbb{N}^I$, $N = (N_i)_{i \in I} \subseteq \mathbb{N}^I$, and a collection $z = \{z_{i,r}\}_{i \in I}^1 \leq r \leq N_i$ with $z_{i,r} \in \mathbb{C}^\times$. We define $\mathbb{Z}_i(z) := \prod_{r=1}^{N_i} \left(1 - \frac{z_{i,r}}{z}\right)$. Finally, we consider the following particular $b^\pm \in \mathbb{Z}^I$:

$$b_i^+ = \sum_{j \in I} a_j \cdot \# \left\{ e = ij \in E \right\} - a_i, \quad b_i^- = -N_i - \sum_{j \in I} a_j \cdot \# \left\{ e = ji \in E \right\} + a_i. \quad (82)$$

For any $i, j \in I$, we also define constants $\gamma_{ij}^+$, $\gamma_{ij}^-$, $\gamma_{ij}^0$ via:

$$\gamma_{ij}^+ = \sum_{e = ij \in E} \log_q (t_e), \quad \gamma_{ij}^- = -\sum_{e = ji \in E} \log_q (t_e), \quad \gamma_{ij}^0 = \gamma_{ij}^+ + \gamma_{ij}^- \quad (83)$$

Let $\hat{A}^q$ be the associative $\mathbb{C}$-algebra generated by $\{w_{i,r}^\pm, D_{i,r}^\pm \}_{i \in I}^1 \leq r \leq a_i$ satisfying the relations $[w_{i,r}, w_{j,s}] = 0 = [D_{i,r}, D_{j,s}]$ and $D_{i,r}w_{i,r} = q^{-\delta_{ij}\delta_{rs}}w_{i,r}D_{i,r}$. Let $A^q$ be obtained from $\hat{A}^q$ by formally adjoining $((\prod_{r=1}^{a_i} w_{i,r}) \gamma_{ij}^\pm)_{i,j \in I}$ satisfying the relations $w_{t,s} \left(\prod_{r=1}^{a_i} w_{i,r}\right) = (\prod_{r=1}^{a_i} w_{i,r}) w_{t,s}$ and $D_{t,s} \left(\prod_{r=1}^{a_i} w_{i,r}\right) = q^{-\delta_{it}\gamma_{ij}^\pm} \left(\prod_{r=1}^{a_i} w_{i,r}\right)$ for all $i, j, t, s$. We define $\tilde{A}^q$ as the localization of $A^q$ by the multiplicative set generated by $\{w_{i,r} = q^m w_{i,s}m \in \mathbb{Z}\}_{i \in I, r \neq s}$.

Then, we have the following analogue of Proposition 2.4:

**Proposition 5.3** There exists a unique $\mathbb{C}$-algebra homomorphism

$$\tilde{\Phi}^z_A : U_{Q}^{(b^+, b^-)} \longrightarrow \tilde{A}^q$$

for any $a$ and $z$ as above, with $b^\pm \in \mathbb{Z}^I$ defined via (82), such that

$$e_i(z) \mapsto \prod_{j \neq i} \left(\prod_{s=1}^{a_j} w_{j,s}\right)^{\gamma_{ij}^+},$$

$$f_i(z) \mapsto \prod_{j \neq i} \left(\prod_{s=1}^{a_j} w_{j,s}\right)^{\gamma_{ij}^-},$$

$$\psi_i^\pm(z) \mapsto \frac{q^{\pm 1} - 1}{\prod_{m=\pm i} \left\{ \frac{1}{\gamma} (1 - \frac{\gamma}{\pm i}) \right\} \prod_{j \neq i} \left(\prod_{s=1}^{a_j} w_{j,s}\right)^{\gamma_{ij}^0}} \cdot \prod_{j \neq i} \left\{ \prod_{s=1}^{a_j} w_{j,s}\right\}^{\gamma_{ij}^0},$$

$$\prod_{j \in I} \prod_{s=1}^{a_j} \left\{ \prod_{m \neq i} \left\{ \frac{1}{\gamma} (1 - \frac{\gamma}{\pm i}) \cdot \prod_{j \neq i} \left(\prod_{s=1}^{a_j} w_{j,s}\right) \right\}^{\gamma_{ij}^0} \right\} \cdot \left(\prod_{i \in I} \prod_{j \neq i} \left\{ \prod_{m \neq i} \left\{ \frac{1}{\gamma} (1 - \frac{\gamma}{\pm i}) \cdot \prod_{j \neq i} \left(\prod_{s=1}^{a_j} w_{j,s}\right) \right\}^{\gamma_{ij}^0} \right\} \right)^{\pm}.$$
Here, \( e \in E \) and \( \gamma(z)^\pm \) denotes the expansion of a rational function \( \gamma(z) \) in \( z^{\pm 1} \), respectively.

### 5.4 Shuffle algebra realization of the positive and negative subalgebras

Similar to (11, 12, 39, 60), we have the following algebra isomorphisms:

\[
U_Q^{(b^+, b^-), >} \xrightarrow{\sim} U_Q^>, \quad U_Q^{(b^+, b^-), <} \xrightarrow{\sim} U_Q^<, \quad U_Q^< \xrightarrow{\sim} U_Q^{>, \text{op}},
\]

with the subalgebras \( U_Q^{(b^+, b^-), >}, U_Q^>, U_Q^{(b^+, b^-), <}, U_Q^< \) defined in a self-explaining way.

Consider an \( \mathbb{N}^I \)-graded \( \mathbb{C} \)-vector space \( S_Q = \bigoplus_{k=(k_i)_{i \in I} \in \mathbb{N}^I} S_k^Q \), with the graded components

\[
S_k^Q = \left\{ F \in \mathbb{C} \left[ \{ x_i^{\pm 1} \}_{i \in I} \right]^{S_k} \right\}.
\]

Evoking the rational functions of (80), we equip \( S_Q \) with the bilinear shuffle product \( \star \) completely analogously to (15), thus making \( S_Q \) into an associative unital \( \mathbb{C} \)-algebra.

As before, we are interested in an \( \mathbb{N}^I \)-graded subspace of \( S_Q \) defined by the following wheel conditions:

\[
F |_{x_{i,2} = qx_{i,1}} \text{ is divisible by } (x_{j,1} - \gamma x_{i,1})^{b_{ij}(\gamma)}
\]

for any \( \gamma \in \mathbb{C}^\times \) and \( j \in I \), where

\[
b_{ij}(\gamma) = \# \left\{ e = \tilde{ij} \in E \mid t_e = \gamma \right\}.
\]

In particular, as pointed out in [17, 18], if for any \( i, j \in I \) all the weights \( \{ t_e | e = \tilde{ij} \in \overline{E} \} \) are pairwise distinct, then (87) may be written in a more familiar form, cf. (16, 42, 63), as:

\[
F \left( \{ x_i^{r} \} \right) = 0 \quad \text{once } \ x_{i,a} = q t_{e^{-1}} x_{j,b} = q x_{i,c} \n\]

for any edge \( \overline{E} \ni e = \tilde{ij} \) and \( a \neq c \), where \( a \neq b \neq c \) if \( i = j \).

Let \( S_Q^Q \subset S_Q \) denote the subspace of all such elements \( F \), which is easily seen to be \( \star \)-closed. The resulting shuffle algebra \( (S_Q^Q, \star) \) is related to \( U_Q \) via [18, Theorem 5.8]:

**Proposition 5.5** [18] The assignments \( e_{i, r} \mapsto x_{i,1}^r \) and \( f_{i, r} \mapsto x_{i,1}^r \) for \( i \in I, r \in \mathbb{Z} \) give rise to \( \mathbb{C} \)-algebra isomorphisms

\[
\Upsilon : U_Q^> \xrightarrow{\sim} S_Q^Q \quad \text{and} \quad \Upsilon : U_Q^< \xrightarrow{\sim} S_Q^{Q, \text{op}}.
\]
5.6 Shuffle algebra realization of the GKLO-type homomorphisms

For any $i \in I$ and $1 \leq r \leq a_i$, we define:

$$Y_{i,r}(z) := \prod_{j \neq i} \left( \prod_{j=1}^{a_j} \left( \prod_{j=1}^{w_{j,s}} \right) \right)^{\delta_{ij}} \cdot \frac{Z_i(z) \prod_{j \neq i}^{s \leq a_j} \prod_{e=\delta_{ij}} \left( 1 - \frac{z}{w_{j,s}} \right) \prod_{e=\delta_{ij}}^{s \neq r} \left( 1 - \frac{z}{w_{j,s}} \right)}{\prod_{s \neq r} \left( 1 - \frac{z}{w_{j,s}} \right)}.$$

$$Y'_{i,r}(z) := \prod_{j \neq i} \left( \prod_{j=1}^{a_j} \left( \prod_{j=1}^{w_{j,s}} \right) \right)^{\delta_{ij}} \cdot \frac{\prod_{j \neq i}^{s \leq a_j} \prod_{e=\delta_{ij}} \left( 1 - \frac{w_{j,s}q}{zq} \right) \prod_{e=\delta_{ij}}^{s \neq r} \left( 1 - \frac{w_{j,s}q}{zq} \right)}{\prod_{s \neq r} \left( 1 - \frac{w_{j,s}q}{zq} \right)}.$$

(91)

We also define

$$\varphi_{ij}(\frac{z}{w}) = \left( \frac{z-w}{zq-1-w} \right)^{\delta_{ij}} \prod_{e=\delta_{ij}} \left( 1 - \frac{z}{w} \right)^{-1}.$$

(92)

Define the $\mathbb{C}$-algebra $\tilde{\mathcal{A}}^q$ as the further localization of $\tilde{\mathcal{A}}^q$ by the multiplicative set generated by $\{w_{i,r} - t^{-1}_e q^{m_{j,s}}w_{j,s} \}_{r \leq a_j, s \leq a_j, e=\delta_{ij}, m \in \mathbb{Z}}$. We note that $\tilde{\mathcal{A}}^q$ is naturally embedded into $\tilde{\mathcal{A}}^q'$.

The following result is proved completely analogously to Theorem 2.8:

**Theorem 5.7** (a) The assignment

$$S^Q_k \ni E \mapsto \prod_{i \in I} \prod_{e=\delta_{ij}} t_e \frac{z_i - z_j^2}{z_i}$$

$$\times \sum_{m^{(i)}_1 + \ldots + m^{(i)}_{a_i} = k_i} \left\{ \prod_{i \in I} \prod_{r=1}^{p=1} Y_{i,r}(w_{i,r}q^{p-1}) \cdot E \left( \left\{ w_{i,r}q^{p-1} \right\}_{i \in I, 1 \leq r \leq a_i} \right) \right\}$$

$$\times \prod_{i \in I} \prod_{1 \leq r \leq a_i} \prod_{1 \leq p \leq m^{(i)}_{r}} \left( \zeta_{ii}^{-1} \left( w_{i,r}q^{p_{1-1}} / w_{i,r}q^{p_{2-1}} \right) \cdot \prod_{e=\delta_{ij}} t_e \right)$$

$$\times \prod_{i,j \in I} \prod_{1 \leq r_1 \leq a_i} \prod_{1 \leq r_2 \leq a_j} \prod_{1 \leq p_1 \leq m^{(i)}_{r_1}} \prod_{1 \leq p_2 \leq m^{(i)}_{r_2}} \varphi_{ij}(w_{i,r_1}q^{p_{1-1}} / w_{j,r_2}q^{p_{2-1}}) \cdot \prod_{i \in I} \prod_{r=1}^{a_i} D_{i,r}^{-m^{(i)}_{r}}.$$

(93)

gives rise to the algebra homomorphism

$$\tilde{\Phi}^Z_k : S^Q \rightarrow \tilde{\mathcal{A}}^q'.$$

(94)
Moreover, for \( b^\pm \in \mathbb{Z}^I \) defined via (82), the composition

\[
U_Q^{(b_+^-, b^-_+)} \xrightarrow{(85)} U_Q \xrightarrow{\gamma} S^Q \xrightarrow{\widehat{\Phi}_q^Z} \widehat{A}_q',
\]

(95)

coincides with the restriction of the homomorphism \( \widehat{\Phi}_q^Z \) of (84) to the subalgebra \( U_Q^{(b_+^-, b^-_+)} \). In particular, the image of \( U_Q^{(b_+^-, b^-_+)} \) under (95) is in the subalgebra \( \widehat{A}_q' \) of \( \widehat{A}_q \).

(b) The assignment

\[
\mathbb{S}_k^{Q, \text{op}} \ni F \longmapsto \prod_{i \in I} \prod_{e = i \in E} \left( \frac{k_i + 1}{k_i} \right)^{\frac{1}{2}} \sum_{m^{(i)}_1 + \ldots + m^{(i)}_{m_i} = k_i \atop m^{(i)}_r \in \mathbb{N} \ \forall \ i \in I} Y_{i,r} \left( w_{i,r} q^{-p} \right) \cdot F \left( \left( w_{i,r} q^{-p} \right)_{1 \leq p \leq m^{(i)}_r} \right)
\]

\[
\times \prod_{i \in I} \prod_{1 \leq r \leq a_i} \prod_{1 \leq p_1 < p_2 \leq m^{(i)}_r} \left( \epsilon_{ii}^{-1} \left( w_{i,r} q^{-p_2} / w_{i,r} q^{-p_1} \right) \cdot \prod_{e = i \in E} t_e \right)
\]

\[
\times \prod_{i,j \in I} \prod_{1 \leq r_1 \leq a_i} \prod_{1 \leq r_2 \leq a_j} \prod_{1 \leq p_1 \leq m^{(i)}_{r_1}} \psi_{ji} \left( w_{j,r_2} q^{-p_2} / w_{j,r_1} q^{-p_1} \right) \cdot \prod_{i \in I} \prod_{r = 1} a_i \prod_{r = 1} D^{m^{(i)}_{r}}_{i,r}
\]

(96)

gives rise to the algebra homomorphism

\[
\widehat{\Phi}_q^Z : \mathbb{S}_k^{Q, \text{op}} \longrightarrow \widehat{A}_q' .
\]

(97)

Moreover, for \( b^\pm \in \mathbb{Z}^I \) defined via (82), the composition

\[
U_Q^{(b_+^-, b^-_+)} \xrightarrow{(85)} U_Q \xrightarrow{\gamma} S^Q \xrightarrow{\widehat{\Phi}_q^Z} \widehat{A}_q',
\]

(98)

coincides with the restriction of the homomorphism \( \widehat{\Phi}_q^Z \) of (84) to the subalgebra \( U_Q^{(b_+^-, b^-_+)} \). In particular, the image of \( U_Q^{(b_+^-, b^-_+)} \) under (98) is in the subalgebra \( \widehat{A}_q' \) of \( \widehat{A}_q \).

**Remark 5.8** This theorem immediately implies that the assignment of Proposition 5.3 is indeed compatible with the cubic Serre relations of [18, §5.4] which we omitted, cf. Remark 2.9.
6 Relation to quantum $Q$-systems of type $A$

In this section, we explain how the shuffle approach from Sect. 2 in the simplest case of $\mathfrak{g} = \mathfrak{sl}_2$ simplifies some of the tedious arguments of [4] in their study of $A$-type $Q$-systems. We also match their difference operators representing the $M$-system with those of Sect. 2.

6.1 Elements $E_{k,n}$ and $M_{k,n}$ for $\mathfrak{g} = \mathfrak{sl}_2$

For any $k \geq 1$ and $n \in \mathbb{Z}$, consider the elements $E_{k,n} \in S_k = S_k^{(\mathfrak{sl}_2)}$ defined via:

$$E_{k,n}(x_1, \ldots, x_k) := \prod_{1 \leq r \leq k} x_r^n \prod_{1 \leq r \neq s \leq k} (x_r - q^{-2} x_s). \tag{99}$$

The following result identifies these elements with those featuring in [11, (9.2)]:

**Lemma 6.2** The elements $E_{k,n}$ correspond to explicit $q$-commutators in $U_q^\infty (L\mathfrak{sl}_2)$:

$$E_{k,n} = \frac{(-1)^k}{(1 - q^{-2})^{k-1}} \cdot \Upsilon \left([e_0, [e_{n+2}, \cdots, [e_{n+2(k-2)}, e_{n+2(k-1)}]_{q^{-4}} \cdots]_{q^{-2(k-1)}}]_{q^{-2k}} \right), \tag{100}$$

where $[x, y]_{q^r} = xy - q^r \cdot yx$ as before.

**Proof** It suffices to prove (100) for $n = 0$. The proof is by induction on $k \geq 1$, the base case $k = 1$ being obvious. For a step of induction, deducing the $k = \ell + 1$ case of (100) from its validity for $k \leq \ell$, we first note (by direct computations) that $[x^0, E_{\ell,2}]_{q^{-2(\ell+1)}} = x^0 \ast E_{\ell,2} - q^{-2(\ell+1)} E_{\ell,2} \ast x^0 \in S_{\ell+1}$ vanishes under the specialization $x_{\ell+1} = q^2 x_\ell$; hence, it is divisible by the product $\prod_{1 \leq r \neq s \leq \ell+1} (x_r - q^{-2} x_s)$.

As $[x^0, E_{\ell,2}]_{q^{-2(\ell+1)}}$ is a polynomial in $x_1, \ldots, x_{\ell+1}$ of the total degree $\ell(\ell + 1)$, we get:

$$\Upsilon \left([e_0, [e_{2}, \cdots, [e_{2(\ell-1)}, e_{2\ell}]_{q^{-4}} \cdots]_{q^{-2\ell+1}}]_{q^{-2(\ell+1)}} \right) = c_{\ell+1} \cdot E_{\ell+1,0} \tag{101}$$

for some constant $c_{\ell+1}$. To determine this constant, we plug $x_{\ell+1} = t$ into (101), divide both sides by $t^{2\ell}$, and consider the $t \to \infty$ limit to obtain:

$$(-q^{-2})^{\ell} c_{\ell+1} E_{\ell,0} = (-1)^{\ell-1} q^{-2\ell} c_{\ell} \Upsilon \left([e_0, [e_2, \cdots, [e_{2(\ell-2)}, e_{2\ell-2}]_{q^{-4}} \cdots]_{q^{-2\ell+2}}]_{q^{-2\ell}} \right)$$

$$= (-1)^{\ell-1} q^{-2\ell} \frac{c_{\ell}}{c_{\ell-1}} \Upsilon \left([e_0, [e_2, \cdots, [e_{2(\ell-2)}, e_{2\ell-2}]_{q^{-4}} \cdots]_{q^{-2\ell+2}}]_{q^{-2\ell}} \right)$$

$$= (-1)^{\ell-1} q^{-2\ell} \frac{d_{\ell}^{2}}{c_{\ell-1}} E_{\ell,0},$$

where $d_{\ell} := c_{\ell+1} / c_{\ell}$. The result then follows from the fact that $c_{\ell+1} / c_{\ell}$ is a constant determined by the choice of $q$. 

\( \square \) Springer
where we used the induction assumption for \( k = \ell - 1 \) and \( k = \ell \). Combining the resulting equality \( c_{\ell+1} = -\frac{c_1^2}{c_{\ell-1}} \) with \( c_1 = 1 \) and \( c_2 = q^{-2} - 1 \), we get
\[
c_{\ell+1} = (-1)^{\frac{(\ell+1)}{2}} (1 - q^{-2})^\ell.
\]
\[\square\]

Let us now compare this with [4]. To this end, we define \( M_{k,n} \) via [4, (2.23)]:\(^1\)
\[
M_{k,n} := (-1)^{\frac{k(k-1)}{2}} (1 - q^{-2})^{1-k} q^{k(k-1)} \cdot \left[ \cdots \left[ [M_{1,n-k+1}, M_{1,n-k+3}] q^2, M_{1,n-k+5}] q^3, \cdots, M_{1,n+k-1} \right] q^4, \right.
\]
where we identify \( M_{1,n} \) with our \( e_{-n} \) and their parameter \( q \) with our \( q^2 \), in accordance with [4, (2.20)]. Due to (100), we get:
\[
\Upsilon(M_{k,n}) = (-1)^{\frac{k(k-1)}{2}} (1 - q^{-2})^{1-k} q^{k(k-1)} \cdot \Upsilon \left( [e_{-n-k+1}, \cdots, e_{n+k-3}, e_{n+k-1}] q^{-3} \cdots q^{-2n} \right) \]
\[
= q^{k(k-1)} \cdot E_{k,1-k} (x_1, \ldots, x_k) = q^{k(k-1)} \cdot \prod_{1 \leq r \leq k} x_r^{1-k} \prod_{1 \leq r \neq s \leq k} (x_r - q^{-2} x_s).
\]
Thus, the generating series \( m_k(z) := \sum_{n \in \mathbb{Z}} M_{k,n} z^n \) of [4, (2.13)] is identified with:
\[
\Upsilon(m_k(z)) = q^{k(k-1)} \cdot \prod_{1 \leq r \leq k} x_r^{1-k} \prod_{1 \leq r \neq s \leq k} (x_r - q^{-2} x_s) \cdot \delta \left( \frac{x_1 \cdots x_k}{z} \right),
\]
where \( \delta(z) \) is the delta-function of (6). This immediately implies [4, Theorem 2.10] (expressing \( M_{k,n} \) as a noncommutative polynomial in \( M_{1,m} \)'s with coefficients in \( \mathbb{Z}[q, q^{-1}] \)):

**Proposition 6.3** Let \( \Delta_q(u_1, \ldots, u_k) = \prod_{1 \leq r < s \leq k} (1 - q^{u_r / u_s}) \). Then, we have:
\[
m_k(z) = \text{CT}_{u_1, \ldots, u_k} \left( \Delta_q(u_1, \ldots, u_k) m_1(u_1) \cdots m_1(u_k) \delta \left( \frac{u_1 \cdots u_k}{z} \right) \right), \tag{105}
\]
where \( \text{CT}_{u_1, \ldots, u_k} \) denotes the “constant term” \((i.e. u_1^0 \cdots u_k^0 \text{-coefficient}) \) of any series in \( u_r \)'s.

**Proof** Combining the key property \( f(u)\delta(u/z) = f(z)\delta(u/z) \) of the delta-functions (6) with \( \Upsilon(m_1(z)) = \delta(x_1/z) \) and evoking the definition of the shuffle product (15), we obtain:
\[
\Upsilon \left( \Delta_q(u_1, \ldots, u_k) m_1(u_1) \cdots m_1(u_k) \delta \left( \frac{u_1 \cdots u_k}{z} \right) \right) = (-q^2)^{\frac{k(k-1)}{2}}.
\]
\(^1\) There seems to be a sign typo in [4, (2.23)] making it actually incompatible with [4, (2.25)].
Combining this equality with the simple identity
\[ \text{CT}_{u_1, \ldots, u_k} \left\{ \sum_{\mathcal{Y}} \left( \Delta_q(u_1, \ldots, u_k) m_1(u_1) \cdots m_1(u_k) \delta \left( \frac{u_1 \cdots u_k}{z} \right) \right) \right\} = (-1)^{\frac{k(k-1)}{2}} q^{k(k-1)} \prod_{1 \leq r \neq s \leq k} (x_r - q^{-2} x_s) \cdot \delta \left( \frac{x_1 \cdots x_k}{z} \right) \cdot \text{Sym} \left\{ \prod_{1 \leq r \leq s \leq k} \frac{1}{x_r (x_r - x_s)} \right\}. \]

Comparing the constant terms of both sides in the above equality, we get:
\[ \text{CT}_{u_1, \ldots, u_k} \left\{ \sum_{\mathcal{Y}} \left( \Delta_q(u_1, \ldots, u_k) m_1(u_1) \cdots m_1(u_k) \delta \left( \frac{u_1 \cdots u_k}{z} \right) \right) \right\} = (-1)^{\frac{k(k-1)}{2}} q^{k(k-1)} \prod_{1 \leq r \neq s \leq k} (x_r - q^{-2} x_s) \cdot \delta \left( \frac{x_1 \cdots x_k}{z} \right) \cdot \text{Sym} \left\{ \prod_{1 \leq r \leq s \leq k} \frac{1}{x_r (x_r - x_s)} \right\}. \]

Combining this equality with the simple identity
\[ \text{Sym} \left\{ \prod_{1 \leq r < s \leq k} \frac{1}{x_r (x_r - x_s)} \right\} = (-1)^{\frac{k(k-1)}{2}} \prod_{1 \leq r \leq k} x_r^{1-k}, \tag{106} \]
we obtain (105) as a direct consequence of the shuffle realization (104) of \( m_k(z) \).

**Remark 6.4** The equality (106) is equivalent to \( \text{Sym} \left\{ \prod_{1 \leq r < s \leq k} \frac{x_r}{x_r - x_s} \right\} = 1 \), which is nothing but the standard Vandermonde determinant formula.

### 6.5 Verifying the M-system relations through the shuffle algebra

Let us now explain how the shuffle approach also allows to establish the key relations of [4, (2.1, 2.2)] satisfied by \( M_{k,n} \) of (102), thus providing a simple proof of [4, Theorem 2.11].

We start with the following \( q \)-commutativity property:

**Lemma 6.6** (a) For any \( k \geq 1 \) and \( m, n \in \mathbb{Z} \) such that \( -1 \leq m - n \leq 2k - 1 \), we have:
\[ [x^m, E_{k,n}]_{q^{2(m-n-k+1)}} = 0. \tag{107} \]

(b) For any \( k \geq \ell \geq 1 \) and \( a, b \in \mathbb{Z} \) such that \( -1 \leq a - b \leq 2k - 2\ell + 1 \), we have:
\[ [E_{\ell,a}, E_{k,b}]_{q^{2\ell(a-b+k-k)}} = 0. \tag{108} \]

(c) For any \( k \geq 1, n \in \mathbb{Z} \), and a collection \( \epsilon_1, \ldots, \epsilon_{k-1} \in \{0, 1, 2\} \), the following \( 2k \) elements:
\[ E_{k,n}, E_{k,n+1}, E_{k-1,n+\epsilon_1}, E_{k-1,n+\epsilon_1+1}, \ldots, E_{1,n+\epsilon_1+\ldots+\epsilon_{k-1}}, E_{1,n+\epsilon_1+\ldots+\epsilon_k-1+1} \]

pairwise \( q \)-commute and are in the \( \mathcal{Y} \)-image of the subalgebra generated by \( \{ e_r \}_{r=n}^{1-2k-1} \).
Proof (a) It suffices to prove (107) for \( n = 0 \). We note that \([x^m, E_{k,0}]q^{2(m-k+1)} \in S_{k+1}\) vanishes under the specialization \( x_{k+1} = q^2 x_k \), and thus, it is divisible by \( \prod_{1 \leq r \neq s \leq k+1} (x_r - q^{-2} x_s) \). If \( 0 \leq m \leq 2k - 1 \), then \([x^m, E_{k,0}]q^{2(m-k+1)} \) is a polynomial in \( x_1, \ldots, x_{k+1} \) of the total degree \( m + k(k - 1) \). This implies (107) as \( \deg(\prod_{1 \leq r \neq s \leq k+1} (x_r - q^{-2} x_s)) = k(k + 1) > m + k(k - 1) \). If \( m = -1 \), then similarly \( x_1 \cdots x_{k+1} \cdot [x^{-1}, E_{k,0}]q^{-2k} \) is a polynomial in \( x_1, \ldots, x_{k+1} \) of the total degree \( k^2 \) which is divisible by the product \( \prod_{1 \leq r \neq s \leq k+1} (x_r - q^{-2} x_s) \) of the total degree \( k(k + 1) > k^2 \). Therefore, \([x^{-1}, E_{k,0}]q^{-2k} = 0 \) as well.

(b) As \(-1 \leq a - b, a + 2 - b, \ldots, a + 2(\ell - 1) - b \leq 2k - 1 \), (108) is in fact an immediate corollary of (107), due to (100) that can be written as:

\[
E_{\ell,a} = (-1)^{\ell(\ell-1)/2} (1 - q^{-2})^{1-\ell} \times [x^a, [x^{a+2}, \ldots, [x^{a+2(\ell-2)}, x^{a+2(\ell-1)}]_{q^{-4}} \cdots ]_{q^{-2(\ell-1)}}]_{q^{-2\ell}}. \tag{110}
\]

(c) The \( q \)-commutativity part follows from (b), while the second part is a consequence of (110). \( \square \)

As particular cases of (108), we obtain the following equalities:

\[
[E_{k,1}, E_{k,0}]q^{2k} = 0 \quad \text{and} \quad [E_{\ell,k-\ell+\epsilon}, E_{k,0}]q^{2\epsilon} = 0 \quad \text{for} \quad 1 \leq \ell \leq k, \ \epsilon \in \{-1, 0, 1\}.
\]

Since \( \Upsilon(M_{\alpha,n}) \in S_\alpha \) is a multiple of \( E_{\alpha,1-\alpha-n} \), due to (103), we thus recover [4, (2.2)]:

**Proposition 6.7** For any \( \alpha, \beta \in \mathbb{N}, \ n \in \mathbb{Z}, \ \epsilon \in \{0, 1\} \), the elements \( M_{k,n} \) of (102) satisfy:

\[
M_{\alpha,n} M_{\beta,n+\epsilon} = q^{\min(\alpha,\beta)\epsilon} M_{\beta,n+\epsilon} M_{\alpha,n}. \tag{111}
\]

We also have the following result (which together with Proposition 6.7 constitute the content of [4, Theorem 4.18], thus providing a simple proof of [4, Theorem 2.11]):

**Proposition 6.8** The elements (102) satisfy the following \( M \)-system relation [4, (2.1)]:

\[
M_{\alpha,n}^2 - q^\alpha M_{\alpha,n+1} M_{\alpha,n-1} = M_{\alpha+1,n} M_{\alpha-1,n} \quad \text{for any} \quad \alpha \geq 1, \ n \in \mathbb{Z}. \tag{112}
\]

Due to (103), this is a direct consequence of the corresponding relation for \( E_{k,n} \) of (99):

**Lemma 6.9** For any \( k \geq 1 \) and \( n \in \mathbb{Z} \), the following quadratic relation holds in \( S = S^{(a_1)} \):

\[
E_{k,n}^2 - q^{2k} E_{k,n-1} E_{k,n+1} = q^2 E_{k+1,n-1} \star E_{k-1,n+1}. \tag{113}
\]
Proof} It suffices to prove (113) for \( n = 0 \), that is, to show that the shuffle element

\[
E_k' := E_{k,0} \star E_{k,0} - q^{2k} E_{k,-1} \star E_{k,-1} - q^2 E_{k+1,-1} \star E_{k+1,-1} \in S_{2k}
\]  

(114)

vanishes. We prove (114) by induction on \( k \geq 1 \), the base case \( k = 1 \) following immediately from Proposition 6.3 (applied to \( k = 2 \)).

For the step of induction (assuming that (114) holds for all \( k < \ell \)), it suffices to prove

\[
E_{\ell}'(x_1, \ldots, x_{2\ell-2}, y, q^2 y) = 0.
\]

(115)

Indeed, (115) implies that \( x_1 \cdots x_{2\ell} \cdot E_{\ell}'(x_1, \ldots, x_{2\ell}) \) is a polynomial in \( x_1, \ldots, x_{2\ell} \) of the total degree \( 2\ell^2 \) which is divisible by the product \( \prod_{1 \leq r \neq s \leq 2\ell} (x_r - q^{-2} x_s) \) of degree \( 2\ell(2\ell - 1) \). As \( 4\ell^2 - 2\ell > 2\ell^2 \) for \( \ell > 1 \), we thus obtain \( E_{\ell}'(x_1, \ldots, x_{2\ell}) = 0 \) which establishes the step of induction. Finally, the equality (115) follows from the following straightforward computation:

\[
E_{\ell}'(x_1, \ldots, x_{2\ell-2}, y, q^2 y) = (1 + q^{-2})q^{-6(\ell-1)} \\
\times \prod_{r=1}^{2\ell-2} (x_r - q^{-2} y)(x_r - q^4 y) \cdot E_{\ell-1}'(x_1, \ldots, x_{2\ell-2}) = 0
\]

with the latter equality due to the induction hypothesis. \( \square \)

**Remark 6.10** We note that similar shuffle interpretations of the relations (111, 112) were suggested (without a proof) in [5, Lemma 8.5].

### 6.11 Comparison of the difference operators I

Let us now compare the realization of the \( M \)-system by difference operators as presented in [4, §6] with the construction of Sect. 2. To this end, we fix \( r \in \mathbb{N} \) and let \( \mathcal{B}^q_{\text{frac}} \) denote the \( \mathbb{C}(q^{1/2}) \)-algebra generated by \( \{ x_i^{\pm 1}, \Gamma_i^{\pm 1} \}_{i=1}^{r+1} \), being further localized by the multiplicative set generated by \( \{ x_i - q^m x_j \}_{i \neq j}^{m \in \mathbb{Z}} \), with all elements pairwise commuting except for \( \Gamma_i x_i = q x_i \Gamma_i \). Following [4, §6], consider the following series in \( z \) with coefficients in \( \mathcal{B}^q_{\text{frac}} \):

\[
e(z)_{\text{DFK}} = \sum_{i=1}^{r+1} \delta (q^{1/2} x_i z) \prod_{1 \leq j \leq r+1}^{j \neq i} \frac{x_i}{x_i - x_j} \Gamma_i
\]

\[
f(z)_{\text{DFK}} = \sum_{i=1}^{r+1} \delta (q^{-1/2} x_i z) \prod_{1 \leq j \leq r+1}^{j \neq i} \frac{x_i}{x_j - x_i} \Gamma_i^{-1}
\]

\[
\psi^+(z)_{\text{DFK}} = (-q^{-1/2} z)^{r+1} \cdot \prod_{i=1}^{r+1} x_i \cdot \prod_{i=1}^{r+1} (1 - q^{1/2} x_i z)^{-1} (1 - q^{-1/2} x_i z)^{-1}
\]
We shall now identify these currents and those in the construction from Sect. 2 in the special case of $g = \mathfrak{sl}_2$, $\mu = -(2r + 2)\omega$ with $\omega$ being the fundamental coweight of $\mathfrak{sl}_2$, $\lambda = 0$, so that $a = r + 1$. To this end, we identify $\iota: \widehat{A}_\text{frac}^q \sim \widehat{B}_\text{frac}^q$ via

$$
\iota: q \mapsto q^{1/2}, \quad w_i^{\pm 1} \mapsto x_i^{\mp 1/2}, \quad D_i^{\pm 1} \mapsto \Gamma_i^{\mp 1}, \quad 1 \leq i \leq r + 1,
$$

and the corresponding shifted quantum affine algebras $j: U_{-r-1}^{sc} \sim U_{0,-2r-2}^{sc}$ via

$$
j: e(z) \mapsto z^{-r-1}e(z), \quad f(z) \mapsto f(z), \quad \psi^\pm(z) \mapsto z^{-r-1}\psi^\pm(z).
$$

Define the composition:

$$
\Phi_{r+1}^q: U_{-r-1}^{sc} \xrightarrow{j} U_{0,-2r-2}^{sc} \xrightarrow{\Phi_0^{-2r-2}} \widehat{A}_\text{frac}^q \xrightarrow{i} \widehat{B}_\text{frac}^q.
$$

The following is straightforward:

**Lemma 6.12** The currents (116) can be expressed as:

$$
\begin{align*}
\varepsilon(z)^{\text{DFK}} &= (-1)^r(q^{1/2} - q^{-1/2})\Phi_{r+1}(\varepsilon(z)), \\
\eta(z)^{\text{DFK}} &= (1 - q)\Phi_{r+1}(\eta(z)), \\
\psi^+(z)^{\text{DFK}} &= (-1)^{r+1}\Phi_{r+1}(\psi^-(z)), \\
\psi^-(z)^{\text{DFK}} &= (-1)^{r+1}\Phi_{r+1}(\psi^+(z)).
\end{align*}
$$

In particular, this immediately shows that the currents (116) indeed satisfy the relations of [4, (5.7)–(5.11)]. Furthermore, we also immediately obtain [4, (6.1)]:

**Proposition 6.13** Under the assignment $\sum_{n \in \mathbb{Z}} M_{1,n} z^n = m_1(z) \mapsto \varepsilon(q^{-1/2}z)^{\text{DFK}}$, the elements $\{M_{k,n}\}_{k \geq 1}$ of (102) are mapped to:

$$
M_{k,n} \mapsto \sum_{J \subset \{1, \ldots, r+1\}; |J| = k} \prod_{i \in J} x_i \prod_{i \in J} \frac{x_j}{x_i - x_j} \cdot \prod_{i \in J} \Gamma_i.
$$

**Proof** Formula (119) immediately follows by combining Lemma 6.12 with the shuffle realization (103) of the elements $M_{k,n}$ and the shuffle realization of $\Phi_0^{-2r-2}$ from Theorem 2.8(a). $\square$
6.14 Finite set of generators

We shall follow the setup of the previous subsection, that is, \( g = sl_2 \), \( \lambda = 0 \), \( \mu = -(2r + 2)\omega \). The last result of this section explains why it essentially suffices to consider only \( \tilde{\Phi}_0^{0}_{-2r-2}(E_{k,n}) \):

**Lemma 6.15** For any \( n \in \mathbb{Z} \), the \( \mathbb{C}(q) \)-subalgebra of \( \tilde{\mathcal{A}}_q^{\text{frac}} \) generated by \( \{ \tilde{\Phi}_0^{0}_{-2r-2}(e_p) \}_{p=\pm n+1} \) and further localized at \( \{ \tilde{\Phi}_0^{0}_{-2r-2}(\psi^{-1}_{E_{r+1,p}})(\tilde{\mathcal{A}}_q^{\text{frac}}) \}_{p=\pm n+1} \) coincides with all image \( \tilde{\Phi}_0^{0}_{-2r-2}(U_q^{-1}(E_{r+1,p})) \).  

**Proof** Let \( C_n \) denote the \( \mathbb{C}(q) \)-subalgebra of \( \tilde{\mathcal{A}}_q^{\text{frac}} \) generated by the above \( 2r + 4 \) elements. Since the \( \tilde{\Phi}_0^{0}_{-2r-2} \)-images of \( \psi^\pm_s \) are symmetric Laurent polynomials in \( \{ w_{k} \}_{k=1}^{n+1} \), to prove the inclusions \( \tilde{\Phi}_0^{0}_{-2r-2}(\psi^\pm_s) \in C_n \), it suffices to show that the elementary symmetric polynomials \( \{ e_k(w_1,\ldots,w_{r+1}) \}_{k=1}^{r+1} \) belong to \( C_n \). To this end, we define

\[
X^{(\pm)}_{r+1,n} := \sum_{s_0,\ldots,s_t=0,1} \left[ e_{n+2} \right. \left. e_{n+2r-2s+1} \cdots e_{n+2r-2s+1} \right] q^{-4} \cdots q^{-2r} q^{-2r+2}.
\]

(120)

We note that \( X^{(\pm)}_{r+1,n+1} \) are generated by \( \{ e_p \}_{p=\pm n+1}^{n+2r+1} \). It is also clear that

\[
\Upsilon(X^{(\pm)}_{r+1,n}) = \Upsilon(\{ e_n, e_{n+2}, \ldots, e_{n+2r-2}, e_{n+2r+1} \} q^{-4} \cdots q^{-2r} q^{-2r+2}) \cdot e_k(x_1^\pm, \ldots, x_{r+1}^\pm)
\]

with the latter equality due to Lemma 6.2. Applying Lemma 2.12(a), we find:

\[
e_k(w_1,\ldots,w_{r+1}) = (-1)^{r(r+1)/2} (1-q^{-2})^r q^{-2k} \times \tilde{\Phi}_0^{0}_{-2r-2}(\psi^{-1}_{E_{r+1,n}})^{-1} \cdot \tilde{\Phi}_0^{0}_{-2r-2}(X_{r+1,n}^{(\pm)}).
\]

and similarly:

\[
e_k(w_1^1,\ldots,w_{r+1}^1) = (-1)^{r(r+1)/2} (1-q^{-2})^r q^{2k} \times \tilde{\Phi}_0^{0}_{-2r-2}(\psi^{-1}_{E_{r+1,n+1}})^{-1} \cdot \tilde{\Phi}_0^{0}_{-2r-2}(X_{r+1,n+1}^{(\pm)}).
\]

This proves \( e_k(w_1^\pm,\ldots,w_{r+1}^\pm) \in C_n \) for \( k \leq r + 1 \), hence, \( \tilde{\Phi}_0^{0}_{-2r-2}(\psi^\pm_s) \in C_n \) for all possible \( s \).

The inclusions \( \tilde{\Phi}_0^{0}_{-2r-2}(e_p) \in C_n \), for all \( p \in \mathbb{Z} \), follow now by induction from the equalities:

\[
\tilde{\Phi}_0^{0}_{-2r-2}(e_p) = (1-q^2)^{-1} \left[ e_1(w_1^\pm,\ldots,w_{r+1}^\pm), \tilde{\Phi}_0^{0}_{-2r-2}(e_p) \right].
\]
Finally, the inclusions $\tilde{\Phi}_{-2r-2}(f_p) \in C_n$, for all $p \in \mathbb{Z}$, follow from the equality:

$$\tilde{\Phi}_{-2r-2}(f_p) = (-1)^{r+1} q^{-2r-1}(q - q^{-1})^{-2} \times \Phi_{-2r-2}(E_{r+1,-2r-1-p})^{-1} \cdot \Phi_{-2r-2}(E_{r,-2r-p}),$$

whose right-hand side belongs to $C_n$, due to Theorem 2.8(a) and Lemma 6.2. □

7 Relation to $(t, q)$-deformed $Q$-systems of type $A$

In this section, we discuss the $(t, q)$-deformation of the construction and results of Sect. 6.11. In particular, we use the results of Sect. 3 to establish [5, Conjecture 1.17].

7.1 Comparison of the difference operators II

We start by recalling the setup of [5, §3]. To this end, choose two generic complex parameters $q$ and $t = \theta^2$, as well as $N \geq 1$. Define the $\mathbb{C}$-algebra $B^q$ as in Sect. 6.11 with $r + 1 = N$ (the subscript “frac” is omitted as it is now a $\mathbb{C}$-algebra). Following [5, (3.6, 3.10)], consider the following series in $z$ with coefficients in $B^q$:

$$\epsilon_1(z)_{\text{DFK}} = \frac{q^{1/2}}{1 - q} \sum_{i=1}^{N} \delta(q^{1/2} x_i z) \prod_{1 \leq j \leq N, j \neq i} \frac{\theta x_i - \theta^{-1} x_j \Gamma_i}{x_i - x_j},$$

$$f_1(z)_{\text{DFK}} = \frac{q^{-1/2}}{1 - q^{-1}} \sum_{i=1}^{N} \delta(q^{-1/2} x_i z) \prod_{1 \leq j \leq N, j \neq i} \frac{\theta^{-1} x_i - \theta x_j \Gamma_i^{-1}}{x_i - x_j},$$

$$\psi^\pm(z)_{\text{DFK}} = \left( \prod_{i=1}^{N} \frac{(1 - q^{-1/2} tx_i z)(1 - q^{1/2} t^{-1} x_i z)}{(1 - q^{-1/2} x_i z)(1 - q^{1/2} x_i z)} \right)^\mp.$$ (121)

Let us now match these currents to those arising for the quantum toroidal algebra of $\mathfrak{gl}_1$ in Sect. 3. To this end, let us first relate our former parameters to the above ones via:

$$q_1 = q, \quad q_2 = 1/t, \quad q_3 = 1/q_1 q_2 = t/q \quad \text{as well as} \quad N = 0, \quad a = N.$$ (122)

We identify $i: \tilde{A}^{q_1} \overset{\sim}{\rightarrow} B^q$ via $w^\pm_i \mapsto x_i^{\mp 1} q^{\mp 1/2}, \quad D^\pm_i \mapsto \Gamma_i^{\mp 1}$, cf. (117). Define the composition:

$$\Phi_N: \tilde{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1) \overset{i}{\rightarrow} \tilde{A}^{q_1} \overset{\Phi_N}{\rightarrow} B^q.$$ (123)

The following is straightforward:
Lemma 7.2 The currents (121) can be expressed as (recall that $\theta = t^{1/2}$):

\[
\begin{align*}
\epsilon_1(z)_{\text{DFK}} & = q^{-\frac{1}{2}}t^{\frac{1-N}{4}}\Phi_N(e(z)), \\
\phi_1(z)_{\text{DFK}} & = -q^{\frac{1}{2}}t^{\frac{N-1}{4}}\Phi_N(f(z)), \\
\psi^+(z)_{\text{DFK}} & = \Phi_N(\psi^-(z)), \\
\psi^-(z)_{\text{DFK}} & = \Phi_N(\psi^+(z)).
\end{align*}
\]

In particular, this immediately shows that the currents (121) indeed satisfy the defining relations (t1–t8) with the parameters $q_1, q_2, q_3$ as in (122), thus implying [5, Theorem 3.5].

7.3 Generalized Macdonald operators

Following [5, Definition 1.13], for any $1 \leq \alpha \leq N$ and any symmetric Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_\alpha^{\pm 1}]^S(\alpha)$, define the generalized Macdonald operator $A_\alpha(P) \in B^q$ via:

\[
A_\alpha(P) := \frac{1}{\alpha ! \cdot (N - \alpha)!} \cdot \operatorname{Sym}_{x_1, \ldots, x_N} \left( P(x_1, \ldots, x_\alpha) \prod_{1 \leq i < j \leq N} \left( \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j} \cdot \Gamma_1 \cdots \Gamma_\alpha \right) \right).
\]

(124)

In particular, $\iota^{-1}(A_\alpha(1)) \in \tilde{A}^{q_1}$ is a multiple of the Macdonald operator $D_N^q(q_1, q_2)$ from (56).

Remark 7.4 We note that the definition (124) is made in [5] for any symmetric rational function $P \in \mathbb{C}(x_1, \ldots, x_\alpha)^S(\alpha)$. However, some of the key results below seem to fail in this generality, see Remarks 7.6, 7.15.

Following [5, Definition 1.15], we also define the difference operator $B_\alpha(P) \in B^q$ via:

\[
B_\alpha(P) := \frac{1}{\alpha !} \operatorname{CT}_{u_1, \ldots, u_\alpha} \left( P(u_1^{-1}, \ldots, u_\alpha^{-1}) \prod_{1 \leq i < j \leq \alpha} \left( \frac{(u_i - u_j)(u_i - qu_j)}{(u_i - tu_j)(u_i - qt^{-1} u_j)} \delta(u_1) \cdots \delta(u_\alpha) \right) \right).
\]

(125)

where the constant term $\operatorname{CT}_{u_1, \ldots, u_\alpha}$ is defined as in Proposition 6.3, and $\delta(z)$ is defined via:

\[
\delta(z) = \sum_{n \in \mathbb{Z}} D_{1; n^\alpha} := (q^{-1/2} - q^{1/2}) \epsilon_1(q^{-1/2}z)_{\text{DFK}}.
\]

(126)

The above two constructions (124) and (125) are related via [5, Theorem 1.16]:

\[ \text{Springer} \]
Proposition 7.5 [5] For any \( 1 \leq \alpha \leq N \) and \( P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_\alpha^{\pm 1}]^{S(\alpha)} \), we have:
\[
\mathcal{A}_\alpha(P) = \mathcal{B}_\alpha(P).
\] (127)

Remark 7.6 We note that this result is stated in [5] for any \( P \in \mathbb{C}(x_1^{\pm 1}, \ldots, x_\alpha^{\pm 1})^{S(\alpha)} \). However, this does not look true in that generality as \( \mathcal{B}_\alpha(P) \) will involve terms with some powers \( \Gamma_i^{\geq 1} \), unlike \( \mathcal{A}_\alpha(P) \). For one thing, the constant term \( \mathcal{B}_\alpha(P) \) should be treated carefully for rational functions by specifying the region in which they are expanded as series.

7.7 Comparing the shuffle algebras

In order to relate the above construction to our Sect. 3, we shall first clarify the shuffle algebra considered in [5, §7] and its relation to the one from Sect. 3.7. To this end, consider an \( \mathbb{N} \)-graded \( \mathbb{C} \)-vector space \( S_{DFK} = \bigoplus_{k \in \mathbb{N}} S_{DFK}^k \), with the graded components
\[
S_{DFK}^k = \left\{ F = \frac{f(x_1, \ldots, x_k)}{\prod_{1 \leq r \neq s \leq k} (x_r - q^{-1} x_s)} \mid f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_k^{\pm 1}]^{S(k)} \right\}. \tag{128}
\]

We also choose a rational function of [5, §7.1]:
\[
\zeta_{DFK}(x) = \frac{(1 - tx)(1 - qt^{-1}x)}{(1 - x)(1 - qx)}.
\] (129)

The bilinear shuffle product \( \star \) on \( S_{DFK} \) is defined completely analogously to (15), thus making \( S_{DFK} \) into an associative unital \( \mathbb{C} \)-algebra. As before, consider an \( \mathbb{N} \)-graded subspace of \( S_{DFK} \) defined by the same wheel conditions (but now on the numerators appearing in (128)):
\[
f(x_1, \ldots, x_k) = 0 \quad \text{once} \quad \left\{ \frac{x_1}{x_2}, \frac{x_2}{x_3}, \frac{x_3}{x_1} \right\} = \left\{ q, \frac{1}{t}, \frac{t}{q} \right\}.
\] (130)

Let \( S_{DFK} \subset S_{DFK} \) denote the subspace of all such elements \( F \), which is easily seen to be \( \star \)-closed. This construction is related to that of Sect. 3.7 via:

Lemma 7.8 For \( q_1 = q, q_2 = 1/t, q_3 = t/q \) as in (122), the assignment
\[
P(x_1, \ldots, x_k) \mapsto q^{\frac{k(k-1)}{2}} \prod_{1 \leq r \neq s \leq k} \frac{x_r - x_s}{x_r - q^{-1} x_s} \cdot P(x_1^{-1}, \ldots, x_k^{-1})
\] (131)
gives rise to the algebra isomorphism
\[
\eta : S \xrightarrow{\sim} S_{DFK},
\] (132)
which further restricts to the shuffle algebra isomorphism
\[ \eta: S \xrightarrow{\sim} S_{\text{DFK}}. \]  

**Proof** Straightforward. \( \square \)

Combining this with Proposition 3.8, we obtain:

**Corollary 7.9** The assignments \( e_r \mapsto x_1^{-r} \) and \( f_r \mapsto x_1^{-r} \) give rise to \( \mathbb{C} \)-algebra isomorphisms

\[ \tilde{\Upsilon}: \tilde{U}_{q,1/t,q}(\mathfrak{gl}_1) \xrightarrow{\sim} S_{\text{DFK}} \quad \text{and} \quad \tilde{\Upsilon}: \tilde{U}_{q,1/t,q}(\mathfrak{gl}_1) \xrightarrow{\sim} S_{\text{DFK,op}}. \]

**Remark 7.10** In [5], neither pole (128) nor wheel (130) conditions were imposed.

### 7.11 Generalized Macdonald operators via GKLO-type homomorphisms

Now we are finally ready to relate the aforementioned constructions to those of Sect. 3. To this end, for any \( 1 \leq \alpha \leq N \) and \( g \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_\alpha^{\pm 1}]^{S(\alpha)} \), recall \( \tilde{E}_\alpha(g) \in S_\alpha \) defined in (52) with the parameters \( q_1 = q, q_2 = 1/t, q_3 = t/q \) as in (122). The following is straightforward:

**Lemma 7.12** \( \eta(\tilde{E}_\alpha(g)) = t^{\frac{q_2^2}{2}} (q^{-1} - 1)^\alpha \cdot g(x_1^{\frac{q_3}{2}} - 1, \ldots, x_\alpha^{\frac{q_3}{2}} - 1) \in S_\alpha^{\text{DFK}}. \)

Therefore, the span of \( \tilde{E}_\alpha(g) \in S_\alpha \) is matched under (132) with the subspace of all symmetric Laurent polynomials in \( S_{\text{DFK,op}} \), for which the constructions and results of Sect. 7.3 apply. In particular, comparing our Lemma 3.12 with the definition (124), we immediately obtain:

**Proposition 7.13** For any \( 1 \leq \alpha \leq N \) and \( g \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_\alpha^{\pm 1}]^{S(\alpha)} \), we have:

\[ \iota(\Phi_N(\tilde{E}_\alpha(g))) = t^{q_\alpha(N-\alpha)} \cdot A_\alpha(P) \quad \text{with} \]

\[ P(x_1, \ldots, x_\alpha) = g(q^{-1/2}x_1^{\frac{q_3}{2}} - 1, \ldots, q^{-1/2}x_\alpha^{\frac{q_3}{2}} - 1) \]

and the identification \( \iota: \tilde{A}_q \xrightarrow{\sim} B_q \) being defined right after (122).

As an immediate corollary, we obtain the following result:

**Theorem 7.14** All generalized Macdonald operators \( A_\alpha(P) \in B_q \) of (124) can be expressed as polynomials in \( D_{1,n} \)'s of (126).

This establishes [5, Conjecture 1.17] by choosing \( P \) to be a generalized Schur function:

\[ P(x_1, \ldots, x_\alpha) = s_{a_1, \ldots, a_\alpha}(x_1, \ldots, x_\alpha) = \frac{\det(x_{i}^{a_j-a_j})_{1 \leq i,j \leq \alpha}}{\det(x_i^{a_j-a_j})_{1 \leq i,j \leq \alpha}}, \quad a_1, \ldots, a_\alpha \in \mathbb{Z}. \]
Proof (Proof of Theorem 7.14) Due to (135) and the equality \( D_{1;n} = A_1(x^n) \), it suffices to show that \( \tilde{E}_\alpha(g) \in S_\alpha \) can be expressed as a polynomial in \( x^n \in S_1 \). This immediately follows from Proposition 3.8 identifying \( S \) with \( \mathring{\mathcal{U}}_{q,1/t,t/q}(\mathfrak{gl}_1) \), the latter generated by \( e_r = \gamma^{-1}(x^r) \).

\[ \Box \]

Remark 7.15 Interpreting the restriction of GKLO-homomorphism \( \Phi_N : \mathring{\mathcal{U}}_{q,1/t,t/q}(\mathfrak{gl}_1) \to \hat{A}_q \) as \( \Phi_N : S^{DFK} \to B^q \), we thus see that the images of symmetric Laurent polynomials recover the generalized Macdonald operators of (124), while the image of any nonpolynomial \( F \in S^{DFK} \) will necessarily contain terms with at least one \( \Gamma_1^{-1} \), due to our explicit formula (46).

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Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The author states that there is no conflict of interest.

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