CONCENTRATED SOLUTIONS TO FRACTIONAL SCHRÖDINGER EQUATIONS WITH PRESCRIBED $L^2$-NORM

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Abstract. In this work, we investigate the existence and local uniqueness of normalized $k$-peak solutions for the fractional Schrödinger equations with attractive interactions with a class of degenerated trapping potential with non-isolated critical points.

Precisely, applying the finite dimensional reduction method, we first obtain the existence of $k$-peak concentrated solutions and especially describe the relationship between the chemical potential $\mu$ and the attractive interaction $a$. Second, after precise analysis of the concentrated points and the Lagrange multiplier, we prove the local uniqueness of the $k$-peak solutions with prescribed $L^2$-norm, by use of the local Pohozaev identities, the blow-up analysis and the maximum principle associated to the nonlocal operator $(-\Delta)^s$.

To our best knowledge, there is few results on the excited normalized solutions of the fractional Schrödinger equations before this present work. The main difficulty lies in the non-local property of the operator $(-\Delta)^s$. First, it makes the standard comparison argument in the ODE theory invalid to use in our analysis. Second, because of the algebraic decay involving the approximate solutions, the estimates, on the Lagrange multiplier for example, would become more subtle. Moreover, when studying the corresponding harmonic extension problem, several local Pohozaev identities are constructed and we have to estimate several kinds of integrals that never appear in the classic local Schrödinger problems. In addition, throughout our discussion, we need to distingish the different cases of $p - 1 < \frac{4}{N}$, $p - 1 = \frac{4}{N}$, and $p - 1 > \frac{4}{N}$, which are called respectively the mass-subcritical, the mass-critical, and the mass-supercritical case, due to the mass-constraint condition. Another difficulty comes from the influence of the different degenerate rates along different directions at the critical points of the potential $V(x)$.

1. Introduction and main results

In this paper, we consider the following fractional Schrödinger equation

$$
\begin{cases}
(-\Delta)^s u + V(x)u = au^p + \mu u, & x \in \mathbb{R}^N, \\
u \in H^s(\mathbb{R}^N),
\end{cases}
$$

under the mass constraint

$$
\int_{\mathbb{R}^N} u^2(x)dx = 1,
$$

where $s \in (0, 1)$, $p \in (1, 2^*_s - 1)$ with $2^*_s = \frac{2N}{N - 2s}$ is the fractional critical Sobolev exponent.

The fractional Laplacian operator, appearing in many areas including biological modeling, physics and mathematical finances, can be regarded as the infinitesimal generator of a stable
Levy process [1]. For any \(s \in (0,1)\), \((-\Delta)^s\) is the nonlocal operator defined as
\[
(-\Delta)^s u = c(N,s) \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy,
\]
where \text{P.V.} is the principal value and \(c(N,s) = \frac{\pi^{2s+N}}{2^s \Gamma(\frac{N}{2}) \Gamma(-s)}\). One could refer to [12, 22] for more details on the fractional Laplacian operator. Particularly, this nonlocal operator \((-\Delta)^s\) in \(\mathbb{R}^N\) can be expressed as a generalized Dirichlet-to-Neumann map for a certain elliptic boundary value problem with local differential operators defined on the upper half-space \(\mathbb{R}^N_+ = \{(x,t): x \in \mathbb{R}^N, t > 0\}\). Precisely, by [7], for any \(u \in H^s(\mathbb{R}^N)\), set
\[
\tilde{u}(x,t) = \mathcal{P}_s[u] = \int_{\mathbb{R}^N} \mathcal{P}_s(x-z,t)u(z)dz, \quad (x,t) \in \mathbb{R}^{N+1}_+,
\]
where
\[
\mathcal{P}_s(x,t) = \beta(N,s) \frac{t^{2s}}{(|x|^2 + t^2)^{\frac{N+2s}{2}}}
\]
with a constant \(\beta(N,s)\) such that \(\int_{\mathbb{R}^N} \mathcal{P}_s(x,1)dx = 1\). Then \(\tilde{u} \in L^2(t^{1-2s},K)\) for any compact set \(K \subset \mathbb{R}^N_+\), \(\nabla \tilde{u} \in L^2(t^{1-2s},\mathbb{R}^{N+1}_+)\) and \(\tilde{u} \in C^\infty(\mathbb{R}^{N+1}_+)\). Moreover, \(\tilde{u}\) satisfies
\[
\begin{cases}
\text{div}(t^{1-2s} \nabla \tilde{u}) = 0, \quad x \in \mathbb{R}^{N+1}_+,

-\lim_{t \to 0} t^{1-2s} \partial_t \tilde{u}(x,t) = \omega_s (-\Delta)^s u(x), \quad x \in \mathbb{R}^N,
\end{cases}
\]
in the distribution sense, where \(\omega_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)\). Moreover, it holds that
\[
\|\tilde{u}\|_{L^2(t^{1-2s},\mathbb{R}^{N+1}_+)} = \omega_s \|u\|_{H^s}.
\]
Without loss of generality, we may assume \(\omega_s = 1\).

Problems with fractional Laplacian have been extensively studied recently, see for example [2-7, 17-21, 23, 24, 31-36, 38] and the references therein. In particular, the existence of multiple spike solution involving the stable critical points of the potential \(V(x)\) was considered in [15], where the critical points of \(V(x)\) seem to be isolated.

We first recall the well-known results about the ground state of the following equation
\[
(-\Delta)^s u + u = u^p, \quad u > 0, \quad x \in \mathbb{R}^N, \quad u(0) = \max_{x \in \mathbb{R}^N} u(x).
\]  
(1.3)

Let \(N \geq 1, s \in (0,1)\) and \(1 < p < 2^*_s - 1\). Then the following hold (c.f. [18, 19]).

(i) (Uniqueness) The ground state solution \(U \in H^s(\mathbb{R}^N)\) of (1.3) is unique.

(ii) (Symmetry, regularity, and decay) \(U(x)\) is radial, positive, and strictly decreasing in \(|x|\). Moreover, \(U \in H^{2s+1}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)\) and satisfies
\[
\frac{C_1}{1 + |x|^{N+2s}} \leq U(x) \leq \frac{C_2}{1 + |x|^{N+2s}}, \quad \text{for} \quad x \in \mathbb{R}^N,
\]
with some constants \(C_2 \geq C_1 > 0\).

(iii) (Non-degeneracy) The linearized operator \(L_0 = (-\Delta)^s + 1 - pU^{p-1}\) is non-degenerate, i.e. its kernel is given by
\[
\text{ker} L_0 = \text{span}\{\partial_{x_1} U, \partial_{x_2} U, \ldots, \partial_{x_N} U\}.
\]
Moreover, By Lemma C.2 of [19], for \( j = 1, \ldots, N, \partial_{x_j} U \) has the decay estimate
\[
|\partial_{x_j} U| \leq \frac{C}{1 + |x|^{N+2s}}.
\]

Throughout this paper, we assume
\[
N \geq \max\{2s, 2 - 2s\}
\]
and set
\[
a_*= \int_{\mathbb{R}^N} U^2, \quad a_0 \equiv \begin{cases} +\infty, & \text{if } p - 1 < \frac{4s}{N}, \\ (ka_*)^{\frac{p-1}{2}}, & \text{if } p - 1 = \frac{4s}{N}, \\ 0, & \text{if } p - 1 > \frac{4s}{N}. \end{cases}
\]

Our first result is as follows.

**Theorem 1.1.** Suppose that \( u_a \) satisfies (1.1)-(1.2) concentrated at some points as \( a \to a_0 \). Then it holds that \( \mu_a \to -\infty \) as \( a \to a_0 \).

Moreover, if \( p - 1 \leq \frac{4s}{N} \), there holds that
\[
u_a = \left( \frac{-\mu_a}{a} \right)^{\frac{p-1}{2}} \left( \sum_{i=1}^k U \left( \left( -\mu_a \right)^{\frac{1}{2s}} (x - x_{a,i}) + \omega_a \right) \right),
\]
with
\[
\int_{\mathbb{R}^N} \left( -\frac{1}{\mu_a} \right)^{\frac{1}{2}} \left( (\Delta)^{\frac{s}{2}} \omega_a \right)^2 + \omega_a^2 \right) = o \left( \left( -\mu_a \right)^{\frac{N}{2s}} \right).
\]

We call \( u_a \) a k-peak solution of (1.1)–(1.2) if \( u_a \) satisfies (1.5). The problem has been extensively investigated if the critical points of \( V(x) \) are isolated, while few is known otherwise. In this paper, we assume that \( V(x) \) obtains its local minimum or local maximum at a closed \( N - 1 \) dimensional hyper-surface \( \Gamma_i(i = 1, \ldots, k) \), satisfying \( \Gamma_i \cap \Gamma_j = \emptyset \) if \( i \neq j \). Precisely, we assume that

(V). There exist some \( \delta > 0 \) and some \( C^2 \) compact hyper surfaces \( \Gamma_i(i = 1, \ldots, k) \) without boundary, satisfying
\[
V(x) = V_i, \quad \frac{\partial V(x)}{\partial v_i} = 0, \quad \frac{\partial^2 V(x)}{\partial v_i^2} \neq 0, \text{ for any } x \in \Gamma_i \text{ and } i = 1, \ldots, k,
\]
where \( V_i \in \mathbb{R}_i, \nu_i \) is the unit outward normal of \( \Gamma_i \) at \( x \in \Gamma_i \). Moreover, \( V(x) \in C^4(\cup_{i=1}^k W_{\delta,i}) \) with \( W_{\delta,i} := \{ x \in \mathbb{R}^N : \text{dist}(x, \Gamma_i) < \delta \} \).

**Remark 1.2.** We point out that the assumption (V) was first introduced in [30], where they studied the Bose-Einstein condensates. The assumption (V) implies that the potential \( V(x) \) obtains its local minimum or local maximum on the hypersurface \( \Gamma_i \) for \( i = 1, \ldots, k \). If \( \delta > 0 \) is small, we set \( \Gamma_{t,i} = \{ x : V(x) = t \} \cap \Gamma_{\delta,i} \), which consists of two compact hypersurfaces in \( \mathbb{R}^N \) without boundary for \( t \in [V_0, V_0 + \sigma) \) or \( t \in (V_0 - \sigma, V_0] \) provided \( \sigma > 0 \) is small enough. Moreover, the outward unit normal vector \( \nu_{t,i}(x) \) and the \( j \)-th principal tangential unit vector \( \tau_{t,i,j}(x), j = 1, \ldots, N - 1 \) of \( \Gamma_{t,i} \) at \( x \) are Lip-continuous in \( W_{\delta,i} \).

We can apply the Pohozaev identities to show that a \( k \)-peak solution of (1.1)-(1.2) must concentrate at some critical points of \( V(x) \). The following result explains where the concentrated points locate on \( \Gamma = \cup_{i=1}^k \Gamma_i \).
Theorem 1.3. Under the condition (V), if $u_a$ is a k-peak solution of (1.1)–(1.2), concentrating at \( \{ b_1, b_2, \ldots, b_k \} \) with \( b_i \in \Gamma \), \( b_i \neq b_j \) if \( i \neq j \), and \( |x_{a,i} - \Gamma_i| = O((-\mu_a)^{-\frac{1}{k}}) \) as \( a \rightarrow a_0 \), then
\[
(D_{\tau_{i,j}} \Delta V)(b_i) = 0, \quad \text{with } i = 1, \ldots, k \text{ and } j = 1, \ldots, N - 1,
\] where \( \tau_{i,j} \) is the \( j \)-th principal tangential unit vector of \( \Gamma \) at \( b_i \).

To study the converse of Theorem 1.3, we need another non-degenerate condition on the critical point of \( V(x) \). In fact, we define that \( x_0 \in \Gamma_i \) is non-degenerate on \( \Gamma_i \) if there holds that
\[
\frac{\partial^2 V(x_0)}{\partial \nu_i^2} \neq 0, \quad \text{and } \det \left( \frac{\partial^2 \Delta V(x_0)}{\partial \tau_{i,l} \partial \tau_{i,j}} \right)_{1 \leq l,j \leq N-1} \neq 0.
\]

Theorem 1.4. Assume that the condition (V) holds. If \( b_i \in \Gamma \) are non-degenerate critical points of \( V(x) \) on \( \Gamma \) for \( i = 1, \ldots, k \) satisfying (1.6) and \( b_i \neq b_j \) for \( i \neq j \), then (1.1)–(1.2) has a k-peak solution \( u_a \) concentrating at \( b_1, \ldots, b_k \) as \( a \rightarrow a_0 \).

On the other hand, if we assume the function \( \Delta V(x)|_{x \in \Gamma_i} \) has an isolated maximum point \( b \in \Gamma_i \) with some \( i_0 \in \{1, \ldots, k\} \), that is \( \Delta V(x) < \Delta V(b) \) for all \( x \in \Gamma_i \cap (B_\delta(b) \setminus b) \), then we can obtain a k-peak solution concentrating at one point.

Theorem 1.5. Assume (V) and \( \frac{\partial^2 V(x)}{\partial \sigma_{i_0}^2} \neq 0 \) for any \( x \in \Gamma_i \) with some \( i_0 \in \{1, \ldots, k\} \). If \( b \in \Gamma_i \) is an isolated maximum point of \( \Delta V(x)|_{x \in \Gamma_i} \) on \( \Gamma_i \), then for any integer \( k > 0 \), problem (1.1)–(1.2) has a k-peak solution \( u_a \) concentrating at \( b \).

Another main result of this paper is the following local uniqueness result.

Theorem 1.6. Suppose (V), and if further \( N \geq 2s + 4 \). Let \( u^{(1)}_a(x) \) and \( u^{(2)}_a(x) \) be two k-peak solutions of (1.1)–(1.2) concentrating at \( b_1, \ldots, b_k \) with \( b_i \in \Gamma \), and \( b_i \neq b_j \) if \( i \neq j \). If \( b_i \) is non-degenerate, \( i = 1, \ldots, k \), \( \sum_{i=1}^k \Delta V(b_i) \neq 0 \) when \( p - 1 = \frac{4s}{N} \), and
\[
\left( \frac{\partial^2 \Delta V(b_i)}{\partial \tau_{i,l} \partial \tau_{i,j}} \right)_{1 \leq l,j \leq N-1} + \frac{\partial \Delta V(b_i)}{\partial \nu_i} \text{diag}(\kappa_{i,1}, \ldots, \kappa_{i,N-1}), \quad \text{for } i = 1, \ldots, k
\]
is non-singular, where \( \kappa_{i,j} \) is the \( j \)-th principal curvature of \( \Gamma \) at \( b_i \) for \( j = 1, \ldots, N - 1 \), then there exists a small positive number \( \sigma \), such that
\[
u^{(1)}_a(x) \equiv u^{(2)}_a(x)
\]
for all \( a \) with \( 0 < \left| a - (ka) \right|^{\frac{1}{s-1}} \leq \sigma \) if \( p - 1 = \frac{4s}{N} \), or \( 0 < \sigma \) if \( p - 1 > \frac{4s}{N} \), or \( a \geq \frac{1}{\sigma} \) if \( p - 1 < \frac{4s}{N} \).

Remark 1.7. In fact, the result of Theorem 1.6 can hold for more general \( p \) and \( N \). Here, in order to avoiding writing too dispersively, we assume that \( N \geq 2s + 4 \).

As far as we know, there are very few results on the local uniqueness for the fractional Schrödinger equations, especially with the subcritical nonlinearities. In particular, since the Lagrange multiplier \( \mu_a \) in (1.1) depends on the solution \( u_a \), the corresponding linearized operator has changed, which brings more nontrivial analysis, as mentioned in [30]. For the local uniqueness results of peak (or bubbling) solutions, the classical moving plane method is
not available. If \( x_0 \) is a non-degenerate critical point of \( V(x) \), that is, \((D^2 V)\) is non-singular at \( x_0 \), one can prove the local uniqueness of the peak solution concentrating at \( x_0 \) either by counting the local degree of the corresponding reduced finite dimensional problem as in [8, 10, 27], or by using Pohozaev type identities as in [9, 16, 26, 28, 29]. One of the advantage in applying the Pohozaev identities can be found in dealing with the degenerate case (see [9, 16, 26]). In these results, the rate of degeneracy along each direction is the same, although the critical point \( x_0 \) is degenerate. Following [30], under the condition \((V)\), the function \( V(x) \) is non-degenerate along the normal direction \( \nu_i \) of \( \Gamma_i \). But along each tangential direction of \( \Gamma_i \), \( V(x) \) is degenerate. Such non-uniform degeneracy makes the estimates more sophisticated.

At the end of this section, we outline the main idea of the proof and discuss the main difficulties. For the existence result, we first consider the following problem without constraint,

\[
\begin{align*}
\left\{ \begin{array}{ll}
(\Delta)^s w + (\lambda + V(x))w &= w^p, & x \in \mathbb{R}^N, \\
w &\in H^s(\mathbb{R}^N),
\end{array} \right.
\]

where \( \lambda > 0 \) is a large parameter. For large \( \lambda > 0 \), by the standard reduction argument, we could construct various positive solutions concentrating at some stable points of \( V(x) \). Particularly, we can construct positive \( k \)-peak solutions for (1.7) of the form

\[ w_\lambda(x) = \lambda^{\frac{2}{p-1}} \left( \sum_{j=1}^{k} U(\lambda^{\frac{1}{2s}}(x - x_{\lambda,i})) + \omega_\lambda \right), \]

with \( \int_{\mathbb{R}^N} \left( \frac{1}{\lambda} |(\Delta)^s \omega_\lambda|^2 + \omega_\lambda^2 \right) = o\left( \lambda^{\frac{2}{p}} \right) \). Let \( u_\lambda = \frac{w_\lambda}{\left( \int_{\mathbb{R}^N} w_\lambda^2 \right)^{\frac{1}{2}}} \). Then \( \int_{\mathbb{R}^N} u_\lambda^2 = 1 \), and

\[
\left\{ \begin{array}{ll}
(\Delta)^s u_\lambda + (\lambda + V(x))u_\lambda &= a_\lambda u_\lambda^p, & x \in \mathbb{R}^N, \\
u_\lambda &\in H^s(\mathbb{R}^N),
\end{array} \right.
\]

with

\[ a_\lambda = \left( \int_{\mathbb{R}^N} w_\lambda^2 \right)^{\frac{p-1}{2}} = \left( k\lambda^{\frac{2}{p-1}} \right)^{\frac{N}{2s}} \left( a_s + o(1) \right)^{\frac{p-1}{2}}. \]

We note that \( a_\lambda > 0 \). Moreover, as \( \lambda \to +\infty \), \( a_\lambda \to a_0 \), which is defined by (1.4). Therefore, we obtain a concentrated solution with \( k \)-peaks for (1.7) with normalized \( L^2 \)-norm, where \( \mu = -\lambda \) and some suitable \( a_\lambda \). Hence, we are sufficed to answer a converse question that for any \( a > 0 \) close to \( a_0 \), whether one can choose a suitable large \( \lambda = \lambda_a > 0 \), such that (1.1)–(1.2) hold with \( \mu = -\lambda_a \), \( u_a = \frac{w_\lambda}{\left( \int_{\mathbb{R}^N} w_\lambda^2 \right)^{\frac{1}{2}}} \), finally concluding Theorem 1.4. In fact, we solve the problem by discussing the relationship between \( a \) and \( \mu \).

Recently, for the classical BECs problem \((s = 1, p = 3, N = 2, 3)\), a complete description of the solutions \( u_a \) concentrating at \( a_s \) related to problem was given in [30]. Specifically, in this case if \( u_a \) is a solution of problem (1.1)–(1.2) with \( s = 1, p = 3 \) and \( N = 2, 3 \), which is concentrated at some points as \( a \to a_0 \) with \( a_0 \in \mathbb{R} \), then it holds

\[ a_0 = ka_s > 0 \quad \text{and} \quad \mu = \mu_a \to -\infty \quad \text{as} \quad a \to a_0, \]

where \( k \) is the number of peaks of the concentrated solution \( u_a \). Moreover, the existence and local uniqueness of this kind concentrated solutions have been proved in [30]. Unlike the argument used in [30], a non-existence result, which is of great importance to prove the main
result, could not be obtained by a standard comparison argument in ODE theory. We would apply a totally different method—the sliding method—corresponding to the non-local operator to show a counterpart result (Theorem 1.1).

On the other hand, we point out that, due to the non-local property of the operator \((-\Delta)^s\), we cannot construct the local Pohozaev identities directly, which, however, is inevitable when we carry out the blow-up analysis especially in the proof of the local uniqueness. To this end, we are supposed to study the corresponding harmonic extension problem (see section 4 and section 5), and have to estimate several kind of integrals which never appear in the classic local Schrödinger problems. Similar arguments can be found in [25] etc..

Last but not least, throughout our discussion, we need to distinct the different cases of \(p - 1 < \frac{4}{N}\), \(p - 1 = \frac{4}{N}\), and \(p - 1 > \frac{4}{N}\), which are called respectively that the mass-subcritical, the mass-critical, and the mass-supercritical case. Especially compared with the problem without any constraint, in the proof of the local uniqueness result, we have to adopt one more different local Pohozaev identity, associated with the mass-constraint condition.

This paper is organized as follows. In section 2, we prove Theorem 1.1 using the sliding method. Then in section 3, we estimate the Lagrange multiplier \(\mu_a\) in terms of \(a\). The results for the location of the peaks and for the existence of peak solutions are proved in section 4, and the local uniqueness of peak solutions are investigated in section 5.

2. Proof of Theorem 1.1 by a non-existence result

**Lemma 2.1.** Assume that \(P(x)\) satisfies \(P(x) > 1\) in \(B_R(0) \setminus B_r(0)\) for some fixed \(r > 0\) and large \(R > 0\). Then the problem

\[
(-\Delta)^s u = P(x)u, \quad u > 0, \quad \text{in } \mathbb{R}^N
\]

has no solution.

**Proof.** We take \(\lambda_1\) as the first eigenvalue of \((-\Delta)^s\) and \(\phi\) as the associated eigenfunction satisfying

\[
\begin{cases}
(-\Delta)^s \phi = \lambda_1 \phi, & x \in B, \\
\phi = 0, & x \in B^c,
\end{cases}
\]

where \(B \subset B_R(0) \setminus B_r(0)\) is a large ball. Since \(R > 0\) is large enough, it may hold that

\[
\lambda_1 \leq 1 < P(x) \quad \text{in } B.
\]

Now, we set \(w(x) := \max_{x \in B} \frac{\phi(x)}{u(x)} \cdot u(x)\). Obviously, \(w(x) \geq \phi(x)\) and \(w(x_0) = \phi(x_0)\) with \(x_0 \in B\) the maximal point of \(\frac{\phi(x)}{u(x)}\). To get a contradiction, we are sufficed to prove that \(w(x) > \phi(x)\) by use of the maximum principle.

In fact, if we denote \(\gamma = \max_{x \in B} \frac{\phi(x)}{u(x)}\), then there holds that

\[
(-\Delta)^s w = \gamma(-\Delta)^s u = \gamma P(x)u \geq \gamma \lambda_1 u = \lambda_1 \max_{x \in B} \frac{\phi(x)}{u(x)} u \geq \lambda_1 \phi = (-\Delta)^s \phi,
\]

which gives that

\[
(-\Delta)^s w \geq (-\Delta)^s \phi \quad \text{in } B.
\]
On the other hand, \( w \geq \phi \) in \( B^c \) obviously. Moreover, since here \( w \not\equiv \phi \), the strong maximum principle implies that \( w > \phi \) concluding the Lemma by contradiction. \( \square \)

**Remark 2.2.** For the classical BECs problem when \( s = 1, p = 3 \) and \( N = 2, 3 \) in [30], the standard comparison method from the ODE theory is sufficed to show such a non-existence result, which, however, is invalid in the fractional case. We take the sliding method to overcome this obstacle, which, developed by Chen, Li and Zhu [11], would be typically available in the study of some other similar problems when dealing with the fractional operator.

**Proof of Theorem 1.1.** We divide the proofs into following three steps.

Step 1. We first show \( \mu_a \to -\infty \). By contradiction, we first suppose \( |\mu_a| \leq M \). Since \( \int_{\mathbb{R}^N} u_a^2 = 1 \), we could use the argument in [21] to get \( u_a \) is uniformly bounded, which means that \( u_a \) does not blow up. Then if \( \mu_a \to +\infty \), we set \( P(x) = \mu_a - V(x) + au_a^{p-1} \). Since \( u_a \) concentrates at some points, we may assume

\[
au_a^{p-1} > -1 \text{ in } \mathbb{R}^N \setminus B_r(0) \text{ for some } r > 0.
\]

Hence, for any \( R > 0 \), we have

\[
P(x) > 1 \text{ for } x \in B_R(0) \setminus B_r(0),
\]

which gives a contradiction by Lemma 2.1.

Step 2. It holds that \( a > 0 \). If not, we could obtain then

\[
(-\Delta)^s u_a \leq 0.
\]

By the maximum principle corresponding to the fractional operator [21], we find \( u_a \leq 0 \), which is a contradiction.

Step 3. In the case of \( p - 1 \leq \frac{4s}{N} \). Denote \( v_a(x) = (-\mu_a)^{-\frac{1}{p-1}} u_a((-\mu_a)^{-\frac{1}{p-1}} x) \). Then

\[
(-\Delta)^s v_a + \left(1 + \frac{1}{-\mu_a} V((-\mu_a)^{-\frac{1}{p-1}} x)\right) v_a = av_a^p
\]

and \( \int_{\mathbb{R}^N} v_a^2 = (-\mu_a)^{\frac{2N}{4s-N}} \). Therefore, we apply the Moser iteration to get \( v_a \) is bounded uniformly in \( a \). Let \( y_a \) be a maximum point of \( v_a \). Then we could show that \( a \) is bounded from below. Applying the blow-up argument established by [21], we obtain that there exists some integer \( k > 0 \) such that

\[
v_a = \sum_{i=1}^{k} U_{a0}(x - y_{a,i}) + \tilde{\omega}_a,
\]

for some \( y_{a,i} \in \mathbb{R}^N \) with \( |y_{a,i} - y_{a,j}| \to +\infty \) for \( i \neq j \), where \( U_{a0} \) is the unique positive solution of

\[
\begin{cases}
(-\Delta)^s U_{a0} + U_{a0} = a_0 U_{a0}^p, \\
U_{a0}(0) = \max_{x \in \mathbb{R}^N} U_{a0}(x),
\end{cases}
\]

and

\[
\int_{\mathbb{R}^N} \left( ((-\Delta)^s \tilde{\omega}_a)^2 + \tilde{\omega}_a^2 \right) = o(1)
\]
Since \( \int_{\mathbb{R}^N} u_a^2 = 1 \), we have
\[
\int_{\mathbb{R}^N} v_a^2 = (-\mu_a)^{\frac{N}{2} - \frac{2}{p}}.
\]

In view of \( U_a = a_0^{-\frac{1}{p-1}} U \), there holds that
\[
(-\mu_a)^{\frac{N}{2} - \frac{2}{p}} = k a_0^{-\frac{2}{p-1}} a_* + o(1),
\]
implying the conclusion.

\[ \square \]

3. Some estimates by Pohozaev identities

We use Pohozaev identities to estimate \( \mu_a \) with respect to \( a \). Let \( \varepsilon = (-\mu_a)^{-\frac{1}{2s}} \) and \( v_a = (-\mu_a)^{-\frac{1}{p-1}} u_a \). Then (1.1) turns to
\[
\varepsilon^{2s} (-\Delta)^s v_a + (1 + \varepsilon^{2s} V(x)) v_a = v_a^p, \quad v_a \in H^s(\mathbb{R}^N).
\]  

For any \( a > 0 \), we define the norm \( \|v\|_a = \left( \int_{\mathbb{R}^N} \varepsilon^{2s} \left| (-\Delta)^{\frac{s}{2}} v \right|^2 + v^2 \right)^{\frac{1}{2}} \).

Hence, a \( k \)-peak solution of (3.1) has the form of
\[
v_a(x) = \sum_{i=1}^k U_{\varepsilon, x_{a,i}}(x) + \varphi_a(x),
\]
where
\[
U_{\varepsilon, x_{a,i}}(x) = (1 + \varepsilon^{2s} V_i)^{\frac{1}{p-1}} U \left( \frac{(1 + \varepsilon^{2s} V_i)^{\frac{1}{2}} (x - x_{a,i})}{\varepsilon} \right), \quad \|\varphi_a\|_a = O(\varepsilon^{\frac{N}{2}}).
\]

We write then
\[
\varepsilon^{2s} (-\Delta)^s \varphi_a + \left( (1 + \varepsilon^{2s} V(x)) - p \sum_{i=1}^k U_{\varepsilon, x_{a,i}}^{p-1} \right) \varphi_a = N_a(\varphi_a) + l_a(x),
\]
where
\[
l_a(x) = -\varepsilon^{2s} \sum_{i=1}^k (V(x) - V_i) U_{\varepsilon, x_{a,i}} + \left( \sum_{i=1}^k U_{\varepsilon, x_{a,i}} \right)^p - \sum_{i=1}^k U_{\varepsilon, x_{a,i}}^p,
\]
and
\[
N_a(\varphi_a) = \left( \sum_{i=1}^k U_{\varepsilon, x_{a,i}} + \varphi_a \right)^p - \left( \sum_{i=1}^k U_{\varepsilon, x_{a,i}} \right)^p - p \sum_{i=1}^k U_{\varepsilon, x_{a,i}}^{p-1} \varphi_a.
\]  

Set
\[
E_{a, x_{a,i}} := \left\{ u \in H^s(\mathbb{R}^N) : \langle u, \frac{\partial U_{\varepsilon, x_{a,i}}}{\partial x_j} \rangle_a = 0, \quad j = 1, \ldots, N \right\}.
\]

One may move \( x_{a,i} \) such that the perturbing term \( \varphi_a \in \bigcap_{i=1}^k E_{a, x_{a,i}} \).
Let $L_a$ be the bounded linear operator from $H^s(\mathbb{R}^N)$ to itself, with the form of

$$\langle L_a u, v \rangle_a = \int_{\mathbb{R}^N} \varepsilon^{2s}(-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}} v + (1 + \varepsilon^{2s} V(x)) uv - p \sum_{i=1}^{k} U_{\varepsilon,x,a,i}^{p-1} uv. \quad (3.3)$$

It is standard to prove the following result.

**Lemma 3.1.** There exists some constant $c > 0$, such that for all $a$ around $a_0$, it holds then

$$\|L_a u\|_a \geq c \|u\|_a, \quad \text{for all } u \in \bigcap_{i=1}^{k} E_{a,x,a,i}.$$  

**Lemma 3.2.** A $k$-peak solution $v_a$ of (3.1) concentrating at $b_1, \ldots, b_k$ has the following form

$$v_a(x) = \sum_{i=1}^{k} U_{\varepsilon,x,a,i}(x) + \varphi_a(x),$$

with $\varphi_a \in \bigcap_{i=1}^{k} E_{a,x,a,i}$ and

$$\|\varphi_a\|_a = O\left( \left| \sum_{i=1}^{k} (V(x,a,i) - V_i) \right| \varepsilon^{N/2 + 2s} + \left| \sum_{i=1}^{k} \nabla V(x,a,i) \right| \varepsilon^{N/2 + s + 1 + \varepsilon^{N/2 + 2s}} \right)
+ \begin{cases} O\left( \varepsilon^{N/2 + \frac{s}{2}(N+2s)} \right), & \text{if } 1 < p \leq 2, \\ O\left( \varepsilon^{N/2 + (N+2s)} \right), & \text{if } p > 2. \end{cases} \quad (3.4)$$

**Proof.** By calculation, we have for any $v \in H^s(\mathbb{R}^N)$,

$$\langle l_a(x), v \rangle_a = -\varepsilon^{2s} \sum_{i=1}^{k} \int_{\mathbb{R}^N} (V(x) - V_i) U_{\varepsilon,x,a,i} v + \int_{\mathbb{R}^N} \left( \sum_{i=1}^{k} U_{\varepsilon,x,a,i} \right)^p - \sum_{i=1}^{k} U_{\varepsilon,x,a,i}^p \right) v$$

$$= -\varepsilon^{2s} \sum_{i=1}^{k} \int_{\mathbb{R}^N} (V(x) - V(x,a,i)) U_{\varepsilon,x,a,i} v - \varepsilon^{2s} \sum_{i=1}^{k} \int_{\mathbb{R}^N} (V(x,a,i) - V_i) U_{\varepsilon,x,a,i} v$$

$$+ \int_{\mathbb{R}^N} \left( \left( \sum_{i=1}^{k} U_{\varepsilon,x,a,i} \right)^p - \sum_{i=1}^{k} U_{\varepsilon,x,a,i}^p \right) v := -A_1 - A_2 + A_3.$$
Similarly,
\[ A_2 = \varepsilon^{2s} \sum_{i=1}^{k} \int_{\mathbb{R}^N} (V(x_{a,i}) - V_i) U_{\varepsilon,x_{a,i}} v \leq C \varepsilon^{2s+\frac{N}{2}} \sum_{i=1}^{k} |V(x_{a,i}) - V_i| \cdot \|v\|_a. \]

Finally, since
\[ \int_{\mathbb{R}^N} \left( \sum_{i=1}^{k} U_{\varepsilon,x_{a,i}} \right)^p - \sum_{i=1}^{k} U_{\varepsilon,x_{a,i}}^p \right) v = \begin{cases} O\left( \int_{\mathbb{R}^N} \sum_{i \neq j} U_{\varepsilon,x_{a,i}}^{p-1} U_{\varepsilon,x_{a,j}} v \right), & \text{if } p > 2, \\ O\left( \int_{\mathbb{R}^N} \sum_{i \neq j} U_{\varepsilon,x_{a,i}}^{\frac{p}{2}} U_{\varepsilon,x_{a,j}}^\frac{p}{2} v \right), & \text{if } 1 < p \leq 2, \end{cases} \]
then by Lemma A.1, we compute directly that if \( p > 2 \)
\[ \left| \int_{\mathbb{R}^N} \sum_{i \neq j} U_{\varepsilon,x_{a,i}}^{p-1} U_{\varepsilon,x_{a,j}} v \right| \leq C \varepsilon^{\frac{N}{2}} \sum_{i \neq j} \left( \int_{\mathbb{R}^N} \left( 1 + |y - \frac{y_{a,i}}{\varepsilon}| \right)^2 (N+2s)(p-1) \right)^\frac{1}{2} \|v\|_a \]
\[ \leq C \varepsilon^{\frac{N}{2}} \sum_{i \neq j} \left( \int_{\mathbb{R}^N} \left( 1 + |y - \frac{y_{a,i}}{\varepsilon}| \right)^2 (N+2s)(p-1) \right)^\frac{1}{2} \|v\|_a \]
while for \( 1 < p \leq 2 \),
\[ \left| \int_{\mathbb{R}^N} \sum_{i \neq j} U_{\varepsilon,x_{a,i}}^\frac{p}{2} U_{\varepsilon,x_{a,j}}^\frac{p}{2} v \right| \leq C \varepsilon^{\frac{N}{2}} \sum_{i \neq j} \left( \int_{\mathbb{R}^N} \left( 1 + |y - \frac{y_{a,i}}{\varepsilon}| \right)^p (N+2s) \right)^\frac{1}{2} \|v\|_a \]
\[ \leq C \varepsilon^{\frac{N}{2}} \sum_{i \neq j} \left( \int_{\mathbb{R}^N} \left( 1 + |y - \frac{y_{a,i}}{\varepsilon}| \right)^p (N+2s) \right)^\frac{1}{2} \|v\|_a \]
\[ \leq C \varepsilon^{\frac{N}{2}} \sum_{i \neq j} \left| \frac{1}{N+2s} \right| \|v\|_a \leq C \varepsilon^{\frac{N}{2} + \frac{p}{2}(N+2s)} \|v\|_a. \]

To sum up, the result has been proved.

\[ \square \]

**Lemma 3.3.** It holds that
\[ \int_{\mathbb{R}^N} U^2 = \left( \frac{N}{s(p+1)} - \frac{N}{2s} + 1 \right) \int_{\mathbb{R}^N} U^{p+1}. \]

**Proof.** It is directly obtained by the following Pohozaev identities:
\[ \frac{N - 2s}{2} \int_{\mathbb{R}^N} x^{1-2s} |\nabla U|^2 = - \frac{N}{2} \int_{\mathbb{R}^N} U^2 + \frac{N}{p+1} \int_{\mathbb{R}^N} U^{p+1} \]
and
\[ \int_{\mathbb{R}^{N+1}_+} x^{1-2s} |\nabla U|^2 = - \int_{\mathbb{R}^N} U^2 + \int_{\mathbb{R}^N} U^{p+1}. \]

\[ \square \]
Proposition 3.4. There holds that

\[
a^\frac{2}{p-1} = \sum_{i=1}^{k} \left( V_i - \mu \right)^{\frac{2}{p-1} - \frac{N}{2s}} \int_{\mathbb{R}^N} U^2 + O \left( \left\| \sum_{i=1}^{k} (V(x_{a,i}) - V_i) \right\| \left( -\mu \right)^{\frac{2}{p-1} - 1 - \frac{N}{2s}} \right) \\
+ \left\| \sum_{i=1}^{k} \nabla V(x_{a,i}) \right\| \left( -\mu \right)^{\frac{2}{p-1} - 1 - \frac{N}{2s}} \left( x_{a,i} \right) + \left\| \sum_{i
ot= j} \nabla V(x_{a,j}) \right\| \left( -\mu \right)^{\frac{2}{p-1} - 1 - \frac{N}{2s}} \left( x_{a,j} \right)
\]

\[
+ O \left( \sum_{i\not= j} \left( -\mu \right)^{\frac{2}{p-1} - \frac{N}{2s} - 1} \left| x_{a,j} - x_{a,i} \right| \left( N + 2s \right) \right) + \begin{cases} 
O \left( \left( -\mu \right)^{\frac{2}{p-1} - \frac{N}{2s} - \frac{pN}{2s} - \frac{p}{2} \right), & \text{if } 1 < p \leq 2, \\
O \left( \left( -\mu \right)^{\frac{2}{p-1} - \frac{N}{s} - 1} \right), & \text{if } p > 2.
\end{cases}
\]

Proof. Let \( u_a \) be a solution of (1.1)-(1.2). It holds that

\[
1 = \int_{\mathbb{R}^N} |u_a(x)|^2 = \int_{\mathbb{R}^N} \left( \sum_{i=1}^{k} \left( \frac{V_i - \mu}{a} \right)^{\frac{1}{p-1}} U \left( \frac{V_i - \mu}{a} \right)^{\frac{1}{p-1}} U \right)^2 \]

\[
= \sum_{i=1}^{k} \left( \frac{V_i - \mu}{a} \right)^{\frac{1}{p-1} - \frac{N}{2s}} \int_{\mathbb{R}^N} U^2 + \sum_{i=1}^{k} \left( \frac{V_i - \mu}{a} \right)^{\frac{1}{p-1} - \frac{N}{2s}} \int_{\mathbb{R}^N} U(x) \varphi_a \left( \frac{V_i - \mu}{a} \right)^{\frac{1}{p-1}} U \left( \frac{V_i - \mu}{a} \right)^{\frac{1}{p-1}} U \varphi_a \\
+ \frac{(-\mu)^{\frac{2}{p-1} - \frac{N}{2s}}}{a^{\frac{2}{p-1}}} \int_{\mathbb{R}^N} \varphi_a^2 \\
= \sum_{i=1}^{k} \left( \frac{V_i - \mu}{a} \right)^{\frac{1}{p-1} - \frac{N}{2s}} \int_{\mathbb{R}^N} U^2 + \sum_{i=1}^{k} \left( \frac{V_i - \mu}{a} \right)^{\frac{1}{p-1}} O \left( \left\| \varphi_a \right\|_{L^2} \right) \\
+ O \left( \sum_{i\not= j} \left( -\mu \right)^{\frac{2}{p-1} - \frac{N}{2s} \left( V_j - \mu \right)^{\frac{1}{p-1} - \frac{N}{2s} - 1} \left| x_{a,j} - x_{a,i} \right| \left( N + 2s \right) \right) + \frac{(-\mu)^{\frac{2}{p-1} - \frac{N}{2s}}}{a^{\frac{2}{p-1}}} \int_{\mathbb{R}^N} \varphi_a^2 \\
= \sum_{i=1}^{k} \left( \frac{V_i - \mu}{a} \right)^{\frac{1}{p-1} - \frac{N}{2s}} \int_{\mathbb{R}^N} U^2 + \frac{(-\mu)^{\frac{2}{p-1} - \frac{N}{4s}}}{a^{\frac{2}{p-1}}} O \left( \left\| \sum_{i=1}^{k} (V(x_{a,i}) - V_i) \right\| \left( \frac{2}{p-1} \right) \left( N + 2s \right) \right) \\
+ \left\| \sum_{i=1}^{k} \nabla V(x_{a,i}) \left( \frac{2}{p-1} \right) \left( N + 2s \right) + \frac{(-\mu)^{\frac{2}{p-1} - \frac{N}{4s}}}{a^{\frac{2}{p-1}}} \right\| \left( x_{a,j} - x_{a,i} \right) \left( N + 2s \right) \right) + O \left( \frac{(-\mu)^{\frac{2}{p-1} - \frac{N}{4s}}}{a^{\frac{2}{p-1}}} \left( \frac{2}{p-1} \right) \left( N + 2s \right) \right), \\
\]

which gives the result. \( \square \)

4. Existence of the Peak Solutions

4.1. Locating of the peaks.
Let \( v_a \) be a peak solution of (3.1). Following the argument in section 5 of [15], we obtain that the concentrating solution \( v_a \) to (3.1) must satisfy that

\[
\sup_{x \in \mathbb{R}^N} \left( \sum_{i=1}^k \frac{1}{(1 + |x - x_{a,i}|^2 + \frac{s}{4})} \right)^{-1} |v_a(x)| < +\infty, \tag{4.1}
\]

where \( \theta \) is any constant such that \( \theta \in (0, \frac{N}{2} + 2s) \). Now we quote the extension of \( v_a \) and its equation:

\[
\begin{align*}
\left\{ \begin{array}{ll}
\text{div}(t^{1-2s} \nabla \tilde{v}_a) = 0, & x \in \mathbb{R}^{N+1}, \\
-\varepsilon^{2s} \lim_{t \to 0} t^{1-2s} \partial_t \tilde{v}_a(y, t) = v_a^p - (1 + \varepsilon^{2s} V(x)) v_a, & x \in \mathbb{R}^N.
\end{array} \right.
\tag{4.2}
\end{align*}
\]

Denote

\[
B_\rho(x_0) = \{ y \in \mathbb{R}^N : |y - x_0| \leq \rho \} \subseteq \mathbb{R}^N,
\]

\[
B_\rho^+(x_0) = \{ Y = (y, t) : |Y - (x_0, 0)| \leq \rho, t > 0 \} \subseteq \mathbb{R}^{N+1},
\]

\[
\partial' B_\rho^+(x_0) = \{ Y = (y, t) : |Y - (x_0, 0)| \leq \rho, t = 0 \} \subseteq \mathbb{R}^N,
\]

\[
\partial'' B_\rho^+(x_0) = \{ Y = (y, t) : |Y - (x_0, 0)| = \rho, t > 0 \} \subseteq \mathbb{R}^{N+1},
\]

\[
\partial B_\rho^+(x_0) = \partial' B_\rho^+(x_0) \cup \partial'' B_\rho^+(x_0).
\]

Multiplying (4.2) by \( \frac{\partial \tilde{v}_a}{\partial x_j} \) and integrating by parts on \( B_\rho(x_{a,i}) \) (or \( B_\rho^+(x_{a,i}) \)) with some \( \rho > 0 \), we have the following Potheaev identities:

\[
\varepsilon^{2s} \int_{B_\rho(x_{a,i})} \frac{\partial V(x)}{\partial x_j} v_a^2 = -2\varepsilon^{2s} \int_{\partial' B_\rho^+(x_{a,i})} t^{1-2s} \frac{\partial \tilde{v}_a}{\partial \nu} \frac{\partial \tilde{v}_a}{\partial y_j} + \varepsilon^{2s} \int_{\partial'' B_\rho^+(x_{a,i})} t^{1-2s} |\nabla \tilde{v}_a|^2 v_j
\]

\[
+ \int_{\partial B_\rho(x_{a,i})} (1 + \varepsilon^{2s} V(x)) v_a^p v_j - \frac{2}{p+1} \int_{\partial B_\rho(x_{a,i})} v_a^{p+1} v_j. \tag{4.3}
\]

Using Lemma A.2 and Remark A.3, we estimate that for \( (y - x)^2 + t^2 = \rho^2, t > 0, \theta \in (0, \frac{N}{2} + 2s) \),

\[
|\tilde{v}_a(y, t)| \leq C \sum_{i=1}^k \int_{\mathbb{R}^N} \left( \frac{t^{2s}}{\xi - x_{a,i} + t} \right)^{N+2s} \left( 1 + \frac{1}{\xi - x_{a,i}} \right)^{N+2s-\theta} d\xi.
\]

\[
\leq C \left\{ \begin{array}{ll}
\varepsilon^{N+2s-\theta} \sum_{i=1}^k \frac{1}{(1 + |y - x_{a,i}|)^{N+2s-\theta}}, & \theta > 2s, \\
\varepsilon^N \sum_{i=1}^k \frac{1}{(1 + |y - x_{a,i}|)^{N+2s-\theta}}, & \theta < 2s.
\end{array} \right.
\]

Similarly, we could also prove that

\[
|\frac{\partial}{\partial y^j} \tilde{v}_a(y, t)|, \quad |\frac{\partial}{\partial \nu} \tilde{v}_a(y, t)| \leq C \left\{ \begin{array}{ll}
\varepsilon^{N+2s-\theta} \sum_{i=1}^k \frac{1}{(1 + |y - x_{a,i}|)^{N+2s-\theta}}, & \theta > 2s, \\
\varepsilon^N \sum_{i=1}^k \frac{1}{(1 + |y - x_{a,i}|)^{N+2s-\theta}}, & \theta < 2s.
\end{array} \right.
\]
Therefore, we estimate the boundary terms as follows: if $\theta > 2s$

$$-2\varepsilon^2 \int_{\partial^\nu B^+_n(x_{a,i})} t^{1-2s} \frac{\partial \tilde{v}_a}{\partial \nu} \frac{\partial \tilde{v}_a}{\partial y^j} \leq C\varepsilon^2 \int_{\partial^\nu B^+_n(x_{a,i})} t^{1-2s} \left( \sum_{i=1}^k \frac{1}{(1 + |y - x_{a,i}|)^{N+2s-\theta}} \right)^2 \leq C\varepsilon^{2(N+3s-\theta)};$$

and similarly,

$$\varepsilon^2 \int_{\partial^\nu B^+_n(x_{a,i})} t^{1-2s} |\nabla \tilde{v}_a|^2 \nu_j \leq C\varepsilon^{2(N+3s-\theta)};$$

$$\int_{\partial^p B_n(x_{a,i})} (1 + \varepsilon^{2s} V(x)) \tilde{v}_a^2 \nu_j \leq C\varepsilon^{2(N+2s-\theta)};$$

$$\frac{2}{p+1} \int_{\partial^p B_n(x_{a,i})} \tilde{v}_a^{p+1} \nu_j \leq C\varepsilon^{(p+1)(N+2s-\theta)}.$$

While if $\theta < 2s$,

$$-2\varepsilon^2 \int_{\partial^\nu B^+_n(x_{a,i})} t^{1-2s} \frac{\partial \tilde{v}_a}{\partial \nu} \frac{\partial \tilde{v}_a}{\partial y^j} \leq C\varepsilon^2 \int_{\partial^\nu B^+_n(x_{a,i})} t^{1-2s} \left( \sum_{i=1}^k \frac{1}{(1 + |y - x_{a,i}|)^{N+2s-\theta}} \right)^2 \leq C\varepsilon^{2(N+s)};$$

$$\varepsilon^2 \int_{\partial^\nu B^+_n(x_{a,i})} t^{1-2s} |\nabla \tilde{v}_a|^2 \nu_j \leq C\varepsilon^{2(N+s)};$$

$$\int_{\partial^p B_n(x_{a,i})} (1 + \varepsilon^{2s} V(x)) \tilde{v}_a^2 \nu_j \leq C\varepsilon^{2(N+2s-\theta)};$$

and

$$\int_{\partial^p B_n(x_{a,i})} \tilde{v}_a^{p+1} \nu_j \leq C\varepsilon^{(p+1)(N+2s-\theta)}.$$

Hence (4.3) gives then for $\theta \in (0, \frac{N}{2} + 2s)$,

$$\int_{B_n(x_{a,i})} \frac{\partial V(x)}{\partial x_j} \tilde{v}_a^2 = \begin{cases} O(\varepsilon^{2(N+s-\theta)}), & \theta > 2s, \\ O(\varepsilon^{\min\{2N,2(N+s-\theta)\}}), & \theta < 2s, \end{cases}$$

which implies the first necessary condition for the concentrated points $b_i$:

$$\nabla V(b_i) = 0, \text{ for } i = 1, \ldots, k.$$

**Remark 4.1.** In our discussion, since we are sufficed to consider certain small $\theta \in (0, \frac{N}{2} + 2s)$ fixed, then for simplicity, we may as well suppose

$$\theta < s.$$
\[\text{Hence (4.4) turns to be}\]
\[\int_{B_p(x_{a,i})} \frac{\partial V(x)}{\partial x_j} v_a^2 = O(\varepsilon^{2N}). \quad (4.5)\]

**Proof of Theorem 1.3.** In view of \(x_{a,i} \rightarrow b_i \in \Gamma_i\), if \(\Gamma_i\) is a local minimum set of \(V(x)\), there exists some \(t_a \in [V_i, V_i + \theta]\); while if \(\Gamma_i\) is a local maximum set of \(V(x)\), there exists some \(t_a \in [V_i - \theta, V_i]\), such that \(x_{a,i} \in \Gamma_{t_a,i}\).

Let \(H(x) = \langle \nabla V(x), \tau_{a,i} \rangle\) and \(\tau_{a,i}\) be the unit tangential vector of \(\Gamma_{t_a,i}\) at \(x_{a,i}\). Then
\[H(x_{a,i}) = 0.\]

On the one hand, we have the expansion for \(x \in B_p(x_{a,i})\),
\[H(x) = \langle \nabla H(x_{a,i}), x - x_{a,i} \rangle + \frac{1}{2} \langle \nabla^2 H(x_{a,i}), x - x_{a,i}, x - x_{a,i} \rangle + o(|x - x_{a,i}|^2),\]
and then,
\[\int_{B_p(x_{a,i})} H(x)U_{\varepsilon,x_{a,i}}^2(x)\]
\[= -2 \int_{B_p(x_{a,i})} H(x)U_{\varepsilon,x_{a,i}}(x)\varphi_a - \int_{B_p(x_{a,i})} H(x)\varphi_a^2 + O(\varepsilon^{2(N+s-\theta)}) \quad (4.6)\]
\[= O\left(\varepsilon^{N+2s+2}\right)\]
where we used the fact that, by assumption,
\[|V(x_{a,i}) - V_i| = O(\varepsilon)\text{ and }\|\varphi_a\|_a = O(\varepsilon^{N+2s+1}).\]

On the other hand, since \(H(x_{a,i}) = 0\),
\[\int_{B_p(x_{a,i})} H(x)U_{\varepsilon,x_{a,i}}^2(x) = \frac{1}{2N} \varepsilon^{N+2} \Delta H(x_{a,i}) \int_{\mathbb{R}^N} |x|^2 U^2 + O(\varepsilon^{N+4})\]
which, combined with (4.6), gives that
\[\Delta H(x_{a,i}) = O(\varepsilon^{2s}).\]

Hence from the condition (V), we get (1.6) concluding Theorem 1.3. \(\square\)

4.2. Existence of the normalized peak solutions.

For the existence result of problem (1.7) with large \(\lambda > 0\), we set \(\eta = \lambda^{-\frac{1}{2s}}\) and \(w(x) \mapsto \lambda^{\frac{1}{2s}}w(x)\), and (1.7) turns to
\[
\begin{cases}
\eta^{2s}(-\Delta)^s w + (1 + \eta^{2s}V(x))w = w^p, & x \in \mathbb{R}^N, \\
w \in H^s(\mathbb{R}^N),
\end{cases}
\]
Moreover, we have the following results.

We denote \( \eta > 0 \), small, we construct a \( k \)-peak solution \( u_\eta \) to (4.7) concentrating at \( b_1, \ldots, b_k \).

**Proposition 4.2.** There exists an \( \eta_0 > 0 \), such that for any \( \eta \in (0, \eta_0] \), and \( d(z_i, \Gamma_i) = O(\eta) \), there exist \( \omega_{\eta,z} \) with \( z = (z_1, \ldots, z_k) \), such that

\[
\int_{\mathbb{R}^N} (\eta^{2s}(1/2) w_\eta (1/2) \psi + (1 + \eta^{2s} V(x)) w_\eta \psi) = \int_{\mathbb{R}^N} w_\eta \psi, \quad \text{for all } \psi \in F_{\eta,z},
\]

where \( w_\eta(x) = \sum_{i=1}^k U_{\eta,z} + \omega_{\eta,z}(x) \), and

\[
F_{\eta,z} = \left\{ \psi(x) \in H^s(\mathbb{R}^N) : \langle v, \frac{\partial U_{\eta,z}}{\partial x_j} \rangle_\eta = 0, j = 1, \ldots, N, i = 1, \ldots, k \right\}.
\]

Moreover, for \( N \geq \max\{2 - 2s, 2s\} \)

\[
\|\omega_{\eta,z}\|_\eta = O\left( \sum_{i=1}^k |V(z_i) - V_1|^{\frac{N}{2} + 2s} + \sum_{i=1}^k |\nabla V(z_i)|^{\frac{N}{2} + 2s + 1} + \eta^{\frac{N}{2} + 2s + 2} \right) + \left\{ \begin{array}{ll}
O(\eta^{\frac{N}{2} + s(N + 2s)}), & \text{if } 1 < p \leq 2 \\
O(\eta^{\frac{N}{2} + s(N + 2s)}), & \text{if } p > 2
\end{array} \right.
\]

(4.8)

with \( \gamma = \max\{0, 1 - 2s\} \).

**Remark 4.3.** The proof of the existence result is standard, and we omit the details. The last inequality in (4.8) holds because that \( N \geq \max\{2 - 2s, 2s\} \) implies \( \frac{N}{2}(N + 2s) > 2s + \gamma \).

It is standard to know that, to obtain a true solution of (4.7), we need to choose \( z \) such that

\[
- \eta^{2s} \int_{\partial^s B_\rho(x_a,i)} t^{1-2s} \frac{\partial \omega_{\eta,z}}{\partial \nu} \frac{\partial \omega_{\eta,z}}{\partial y^j} + \frac{1}{2} \eta^{2s} \int_{\partial^s B_\rho(x_a,i)} t^{1-2s} |\nabla \omega_{\eta,z}|^2 \nu_j
\]

\[
= - \int_{\partial B_\rho(x_a,i)} (1 + \eta^{2s} V(x)) \omega_{\eta,z} \frac{\partial \omega_{\eta,z}}{\partial y^j} + \int_{\partial B_\rho(x_a,i)} \omega_{\eta,z} \frac{\partial \omega_{\eta,z}}{\partial y^j},
\]

which, similar to the proof of (4.5), is equivalent to

\[
\int_{B_\rho(x_a,i)} \frac{\partial V(x)}{\partial x_j} w_{\eta,z}^2 = O(\eta^{2N}).
\]

We consider \( z = (z_1, \ldots, z_k) \), \( z_i \) close to \( b_i \), \( z_i \in \Gamma_{t,i} \), for some \( t \) close to \( V_1 \). Denote \( \nu_i \) as the unit normal vector of \( \Gamma_{t,i} \) at \( z_i \), while \( \tau_{i,j} \) as the principal directions of \( \Gamma_{t,i} \), with \( i = 1, \ldots, k \), \( j = 1, \ldots, N - 1 \). Then, at \( z_i \), there holds that

\[
D_{\tau_{i,j}} V(z_i) = 0, \quad \text{for } j = 1, \ldots, N - 1, \quad \text{and } |\nabla V(z_i)| = |D_{\nu_i} V(z_i)|.
\]

Moreover, we have the following results.
Lemma 4.4. Under the assumption (V), then
\[
\int_{B_\rho(x_{a,i})} D_{\nu_i} V(x) w_{\eta,z}^2 = O(\eta^{2N})
\]
is equivalent to
\[
D_{\nu_i} V(z_i) = O(\eta^{2s+\gamma}).
\]

Proof. On the one hand,
\[
\int_{B_\rho(x_{a,i})} D_{\nu_i} V(x) U_{\eta,z_i}^2(x) = -2 \int_{B_\rho(x_{a,i})} D_{\nu_i} V(x) U_{\eta,z_i}(x) \omega_{\eta,z} - \int_{B_\rho(x_{a,i})} D_{\nu_i} V(x) \omega_{\eta,z}^2 + O(\eta^{2N})
\]
\[
= O(\eta^{N+2s+\gamma}). \tag{4.9}
\]
On the other hand,
\[
\int_{B_\rho(x_{a,i})} D_{\nu_i} V(x) U_{\eta,z_i}^2(x) = a_i \eta^N D_{\nu_i} V(z_i) + O(\eta^{N+2})
\]
which, combined with (4.9), gives that
\[
D_{\nu_i} V(z_i) = O(\eta^{2s+\gamma}).
\]

Lemma 4.5. Under the assumption (V), then
\[
\int_{B_\rho(x_{a,i})} D_{\tau_i} V(x) w_{\eta,z}^2 = O(\eta^{2N})
\]
is equivalent to
\[
(D_{\tau_i} \Delta) V(z_i) = O\left( \sum_{i=1}^k |V(z_i) - V_i| \eta^{2s-1} + \eta^{2s} \right).
\]

Proof. Let \( H(x) = \langle \nabla V(x), \tau_i \rangle \). On the one hand,
\[
\int_{B_\rho(x_{a,i})} H(x) U_{\eta,z_i}^2(x) = -2 \int_{B_\rho(x_{a,i})} H(x) U_{\eta,z_i}(x) \omega_{\eta,z} - \int_{B_\rho(x_{a,i})} H(x) \omega_{\eta,z}^2 + O(\eta^{2N})
\]
\[
= O\left( (\eta^{N+1} |\nabla H(z_i)| + \eta^{N+2}) \|\omega_{\eta,z}\|_\eta + |\nabla H(z_i)| \|\omega_{\eta,z}\|^2_\eta \right) + O(\eta^{2N}). \tag{4.10}
\]
On the other hand, since \( H(x_{a,i}) = 0 \), we set
\[
B = \frac{1}{N} \int_{\mathbb{R}^N} |x|^2 U^2, \tag{4.11}
\]
and then
\[
\int_{B_\rho(x_{a,i})} H(x) U_{\eta,z_i}^2(x) = \frac{B}{2} \eta^{N+2} \Delta H(z_i) + O(\eta^{N+4})
\]
which, combined with (4.10), gives the result.
Theorem 4.6. For $\lambda > 0$ large, the problem (1.7) has a solution $w_\lambda$ of the form

$$w_\lambda(x) = \sum_{i=1}^{k} \lambda^{i-1} U \left( \lambda^{\frac{i}{2}} (x - x_{\lambda,i}) \right) + \omega_\lambda,$$

where $x_{\lambda,i} \to b_i$ and $\int_{\mathbb{R}^N} \left( \|(-\Delta)^{\frac{\tau}{2}} \omega_\lambda\|^2 + \omega_\lambda^2 \right) \to 0$ as $\lambda \to +\infty$.

Proof. We are sufficed to solve (4.7). From Lemma 4.4 and Lemma 4.5, we know that solving (4.7) is equivalent to solving $d(z_i, \Gamma_i) = O(\eta)$,

$$D_{\nu_i} V(z_i) = O(\eta^{2s+\gamma}), \quad (D_{\tau_i} \Delta) V(z_i) = O\left( \sum_{i=1}^{k} |V(z_i) - V_i|^{2s-1} + \eta^{2s} \right).$$

Now we take $\bar{z}_i \in \Gamma_i$ to be the point such that $z_i - \bar{z}_i = \alpha_i \nu_i$ for some $\alpha_i \in \mathbb{R}$. Then, we have $D_{\nu_i} V(\bar{z}_i) = 0$. Then,

$$D_{\nu_i} V(z_i) = D_{\nu_i} V(z_i) - D_{\nu_i} V(\bar{z}_i) = D_{\nu_i}^2 V(\bar{z}_i) \langle z_i - \bar{z}_i, \nu_i \rangle + O(|z_i - \bar{z}_i|^2).$$

By the non-degenerate assumption, it holds that

$$D_{\nu_i} V(z_i) = O(\eta^{2s+\gamma}) \iff \langle z_i - \bar{z}_i, \nu_i \rangle = O(\eta^{2s+\gamma} + |z_i - \bar{z}_i|^2),$$

which means that

$$|z_i - \bar{z}_i| = O(\eta^{2s+\gamma}) \Rightarrow d(z_i, \Gamma_i) = |z_i - \bar{z}_i| = O(\eta). \quad (4.12)$$

Let $\bar{\tau}_{i,j}$ be the $j$-th tangential unit vector of $\Gamma_i$ at $\bar{z}_i$. By the assumption (V), we have

$$(D_{\bar{\tau}_{i,j}} \Delta) V(z_i) = (D_{\bar{\tau}_{i,j}} \Delta) V(\bar{z}_i) + O(|z_i - \bar{z}_i|) = (D_{\bar{\tau}_{i,j}} \Delta) V(\bar{z}_i) + O(\eta^{2s+\gamma}),$$

and

$$(D_{\bar{\tau}_{i,j}} \Delta) V(\bar{z}_i) = (D_{\bar{\tau}_{i,j}} \Delta) V(\bar{z}_i) - (D_{\bar{\tau}_{i,j,0}} \Delta) V(b_i) = \langle (\nabla T D_{\bar{\tau}_{i,j,0}} \Delta V)(b_i), \bar{z}_i - b_i \rangle + O(|\bar{z}_i - b_i|^2),$$

where $\nabla T_i$ is the tangential gradient on $\Gamma_i$ at $b_i \in \Gamma_i$, and $\bar{\tau}_{i,j,0}$ is the $j$-th tangential unit vector of $\Gamma_i$ at $b_i$. Therefore, we have

$$(D_{\bar{\tau}_{i,j}} \Delta) V(z_i) = O\left( \sum_{i=1}^{k} |V(z_i) - V_i|^{2s-1} + \eta^{2s} \right),$$

which implies that

$$\langle (\nabla T D_{\bar{\tau}_{i,j,0}} \Delta V)(b_i), \bar{z}_i - b_i \rangle = O\left( \sum_{i=1}^{k} |V(z_i) - V_i|^{2s-1} + \eta^{2s} \right) = O(\eta^{2s}). \quad (4.13)$$

Hence, we can solve (4.12) and (4.13) to obtain $z_i = x_{\eta,i}$ with $x_{\eta,i} \to b_i$. 

Now we are in a position to prove Theorem 1.4.
Proof of Theorem 1.4. Let \( w \) be a peak solution obtained in Theorem 4.6. Define \( u_\lambda = w_\lambda / \left( \int_{\mathbb{R}^N} w_\lambda^2 \right)^{\frac{1}{2}} \). Then \( \int_{\mathbb{R}^N} u_\lambda^2 = 1 \) and
\[
(-\Delta)^s u_\lambda + V(x) u_\lambda = a_\lambda u_\lambda^p - \lambda u_\lambda, \quad a_\lambda = \int_{\mathbb{R}^N} w_\lambda^2.
\] (4.14)

Similar to the proof of Proposition 3.4, there holds that
\[
\lambda^{\frac{N}{2s} - \frac{2}{p-1}} \left( \int_{\mathbb{R}^N} w_\lambda^2 \right)^{\frac{p}{p-1}} = ka_s + o(1), \text{ as } a \to a_0
\]
with \( a_0 = 0 \) if \( \frac{N}{2s} - \frac{2}{p-1} > 0 \), \( a_0 = (ka_s)^{\frac{p}{2}} \) if \( \frac{N}{2s} - \frac{2}{p-1} = 0 \), while \( a_0 = +\infty \) if \( \frac{N}{2s} - \frac{2}{p-1} < 0 \).

Take \( \lambda_0 > 0 \) large and \( b_0 = \int_{\mathbb{R}^N} w_{b_0}^2 \). Then, if \( \frac{N}{2s} - \frac{2}{p-1} > 0 \), for any \( a \in (0, b_0) \), by the mean value theorem, there exists some large \( \lambda_a \), such that the solution \( u_a \) to (1.1) with \( \lambda = \lambda_a \) satisfies \( \int_{\mathbb{R}^N} u_a^2 = \lambda \), and for such \( a \), we obtain a \( k \)-peak solution for (1.7), with \( \mu_a = -\lambda_a \).

Similarly, if \( \frac{N}{2s} - \frac{2}{p-1} = 0 \), for any \( a \) between \( b_0 \) and \( (ka_s)^{\frac{p}{2}} \), while if \( \frac{N}{2s} - \frac{2}{p-1} < 0 \), for any \( a > b_0 \) large enough, there exists \( \lambda_a \), such that the solution \( u_a \) to (1.1) with \( \lambda = \lambda_a \) satisfies \( \int_{\mathbb{R}^N} u_a^2 = \lambda \), and for such \( a \), we obtain a \( k \)-peak solution for (1.7), with \( \mu_a = -\lambda_a \).

\[\square\]

4.3. Clustering peak solutions.

In the end of this section, we are concerned with the clustering multi-peak solutions to problem (1.1)–(1.2), and sketch the proof of Theorem 1.5.

The function \( \Delta V(x)|_{x \in \Gamma_1} \) has a minimum point and a maximum point. Let us assume that the function \( \Delta V(x)|_{x \in \Gamma_{i_0}} \) has isolated maximum point \( b \in \Gamma_{i_0} \), that is \( \Delta V(x) < \Delta V(b) \) for all \( x \in \Gamma_{i_0} \cap (B_\delta(b) \setminus b) \). Let \( x_{\eta,j} \to b, j = 1, \ldots, k, \frac{|x_{\eta,j} - x_0|}{\eta} \to +\infty, i \neq j \) as \( \eta \to 0 \). We take
\[
\sum_{j=1}^k U_{\eta,x_{\eta,j}} \text{ as the approximate solution of (4.7) and sufficed to show the following result.}
\]

Proposition 4.7. Assume \( (V) \) and \( \frac{\partial^2 V(x)}{\partial x_i^2} \neq 0 \) for any \( x \in \Gamma_{i_0} \) with some \( i_0 \in \{1, \ldots, k\} \).
If \( b \in \Gamma_{i_0} \) is an isolated maximum point of \( \Delta V(x)|_{x \in \Gamma_{i_0}} \) on \( \Gamma_{i_0} \), then for any integer \( k > 0 \), there exists an \( \eta_0 > 0 \), such that for any \( \eta \in (0, \eta_0) \), problem (4.7) has a solution \( w_\eta \) of the form
\[
w_\eta(x) = \sum_{j=1}^k U_{\eta,x_{\eta,j}} + \omega_\eta,
\]
where \( x_{\eta,j} \to b, j = 1, \ldots, k, \frac{|x_{\eta,j} - x_0|}{\eta} \to +\infty, i \neq j \), and \( \int_{\mathbb{R}^N} (\eta^2 s|(-\Delta)^{\frac{s}{2}} \omega_\eta|^2 + \omega_\eta^2) = o(\eta^N) \) as \( \eta \to 0 \).

Proof. Define the energy functional
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( \eta^2 s|(-\Delta)^{\frac{s}{2}} u|^2 + (1 + \eta^2 s V(x)) u^2 \right) - \frac{1}{p+1} \int_{\mathbb{R}^N} u^{p+1}.
\]
We obtain the energy expansion:

\[
I\left(\sum_{j=1}^{k} U_{\eta,x_{\eta,j}}\right) = (1 + \eta^{2s} V_{i_0})^{-\frac{k}{N}} \left(k E_1 \eta^N + E_2 \eta^{N+2s} \sum_{j=1}^{k} \frac{\partial^2 V(\bar{x}_{\eta,j})}{\partial \nu_{i,j}^2} r_{\eta,j}^2 + E_3 \eta^{N+2s+2} \sum_{j=1}^{k} \Delta V(\bar{x}_{\eta,j})ight)
- \sum_{j>m} (c_0 + o(1)) \eta^N \left(\frac{\eta}{|x_{\eta,m} - x_{\eta,j}|}\right)^{N+2s}
+ O\left(\eta^{N+2s+3} + \eta^N \sum_{j>m} \left(\frac{\eta}{|x_{\eta,m} - x_{\eta,j}|}\right)^{N+2s+\tau} + \eta^{N+2s} \eta^3 r_{\eta,j}^3\right),
\]

where \( E_1, c_0 > 0, l = 2, 3, \)

\[
E_1 = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^N} U^{p+1} > 0 \quad \text{and} \quad r_{\eta,j} = |x_{\eta,j} - \bar{x}_{\eta,j}|,
\]

with \( \bar{x}_{\eta,j} \in \Gamma_{i_0} \) the point such that \( |x_{\eta,j} - \bar{x}_{\eta,j}| = d(x_{\eta,j}, \Gamma_{i_0}) \).

In fact, it is easy to get

\[
I\left(\sum_{j=1}^{k} U_{\eta,x_{\eta,j}}\right) = (1 + \eta^{2s} V_{i_0})^{-\frac{k}{N}} \left(k E_1 \eta^N + E_2 \eta^{N+2s} \sum_{j=1}^{k} (V(x_{\eta,j}) - V(\bar{x}_{\eta,j}))
+ E_3' \eta^{N+2s+2} \sum_{j=1}^{k} \Delta V(x_{\eta,j}) - \sum_{j>m} (c_0 + o(1)) \eta^N \left(\frac{\eta}{|x_{\eta,m} - x_{\eta,j}|}\right)^{N+2s}
+ O\left(\eta^{N+2s+3} + \eta^N \sum_{j>m} \left(\frac{\eta}{|x_{\eta,m} - x_{\eta,j}|}\right)^{N+2s+\tau}\right),
\]

and

\[
V(x_{\eta,j}) = 1 + \frac{\partial^2 V(\bar{x}_{\eta,j})}{\partial \nu_{i,j}^2} r_{\eta,j}^2 + O(r_{\eta,j}^3) \quad \text{and} \quad \Delta V(x_{\eta,j}) = \Delta V(\bar{x}_{\eta,j}) + O(r_{\eta,j}).
\]

So, (4.15) follows from (4.16) and (4.17).

To obtain a solution \( w_{\eta} \) of the form \( \sum_{j=1}^{k} U_{\eta,x_{\eta,j}} + \omega_{\eta} \), we can first carry out the reduction argument as in Proposition 4.2 to obtain \( \omega_{\eta} \), satisfying

\[
\|\omega_{\eta}\|_{\eta} = O\left(\eta^{N+2s+1} + \eta^N \sum_{i \neq j} \left(\frac{\eta}{|x_{\eta,i} - x_{\eta,j}|}\right)^{\min\{\frac{N}{2}, 1\}(N+2s)}\right).
\]

Define

\[
K(x_{\eta,1}, \ldots, x_{\eta,k}) = I\left(U_{\eta,x_{\eta,j}} + \omega_{\eta}\right).
\]
From (4.18), we get the same expansion (4.15) for \( K(x_{\eta,1}, \cdots, x_{\eta,k}) \). Now we set
\[
B = \{ (r, \bar{x}) : r \in (-\delta \eta, \delta \eta), \bar{x} \in B_{\delta}(b_i) \cap \Gamma_i \}.
\]

If \( \Gamma_{i_0} \) is a local maximum set of \( V(x) \) and \( \frac{\partial^2 V(x)}{\partial x_{i_0}^2} < 0 \) for any \( \bar{x} \in \Gamma_{i_0} \), then it is easy to prove that \( K(x_{\eta,1}, \cdots, x_{\eta,k}) \) has a critical point, which is a maximum point of \( K \) in
\[
S_{\eta,k} := \{ (x_{\eta,1}, \cdots, x_{\eta,k}) : x_{\eta,j} \in B, \ |x_{\eta,j} - x_{\eta,m}| > \eta^{|\sigma| - 1}, m \neq j \},
\]
where \( \sigma \in (\frac{1}{2}, \frac{N-1}{N+2s}) \) is some constant such that both \( \sigma > \frac{1}{2} \) and \( (1 - \sigma)(N + 2s) > 2s + 1 \) hold.

If \( \Gamma_{i_0} \) is a local minimum set of \( V(x) \) and \( \frac{\partial^2 V(x)}{\partial x_{i_0}^2} > 0 \) for any \( \bar{x} \in \Gamma_{i_0} \), then
\[
E_2 \frac{\partial^2 V(\bar{x})}{\partial x_{i_0}^2} r^2 + E_3 \eta^2 \Delta V(\bar{x})
\]
has a saddle point \((0, b)\) in \( B \). We can use a topological argument as in [13, 14] to prove that \( K(x_{\eta,1}, \cdots, x_{\eta,k}) \) has a critical point in \( S_{\eta,k} \).

\[ \square \]

5. Local Uniqueness

5.1. Precise estimates by Pohozaev identities.

From Lemma 3.2, we know that a \( k \)-peak solution \( v_a \) to equation (3.1) can be written as
\[
v_a(x) = \sum_{i=1}^{k} U_{\varepsilon,x,a,i}(x) + \varphi_a(x),
\]
where \( |x_{a,i} - b_i| = o(1), d(x_{a,i}, \Gamma_i) = O(\varepsilon), \varepsilon = (-\mu_a)^{-\frac{1}{\nu}} \),
\[
U_{\varepsilon,x,a,i}(x) = (1 + \varepsilon^{2s}V_i)^{\frac{1}{p-1}}U \left( \left( 1 + \varepsilon^{2s}V_i \right)^{\frac{1}{p}} (x - x_{a,i}) \right),
\]
\( \varphi_a \in \bigcap_{i=1}^{k} E_{a,x,a,i} \). For \( N \geq 2s + 2 \), it holds that \( \frac{d}{2}(N + 2s) > 2s + 1 \), which implies then
\[
\|\varphi_a\|_a = \|\varphi_a\|_\varepsilon = O \left( \left| \sum_{i=1}^{k} (V(x_{a,i}) - V_i) \right| \varepsilon^{\frac{N}{2} + 2s} + \left| \sum_{i=1}^{k} \nabla V(x_{a,i}) \right| \varepsilon^{\frac{N}{2} + 2s + 1} + \varepsilon^{\frac{N}{2} + 2s + 1} \right), \quad (5.1)
\]

Moreover, \( x_{a,i} \in \Gamma_{t_a,i} \) with some \( t_a \to V_i \). Denote \( v_{a,i} \) and \( \tau_{a,i,j} \) as the unit normal vector and the principal direction of \( \Gamma_{t_a,i} \) at \( x_{a,i} \) respectively. Then at \( x_{a,i} \), there hold that
\[
D_{\tau_{a,i,j}} V(x_{a,i}) = 0, \quad |\nabla V(x_{a,i})| = |D_{v_{a,i}} V(x_{a,i})|.
\]

Lemma 5.1. Under the assumption \((V)\), it holds that
\[
D_{v_{a,i}} V(x_{a,i}) = O \left( \varepsilon^{\min\{2,2s+1\}} \right).
\]
Proof. Using (4.5), we obtain that
\[ \int_{B_\rho(x_{a,i})} D_{\nu_{a,i}} V(x) v_a^2 = O(\varepsilon^{2N}). \]

On the one hand,
\begin{align*}
\int_{B_\rho(x_{a,i})} D_{\nu_{a,i}} V(x) U^2_{\varepsilon,x_{a,i}}(x) \\
= -2 \int_{B_\rho(x_{a,i})} D_{\nu_{a,i}} V(x) U_{\varepsilon,x_{a,i}}(x) \varphi_a - \int_{B_\rho(x_{a,i})} D_{\nu_{a,i}} V(x) \varphi_a^2 + O(\varepsilon^{2N}) \\
= O\left( \varepsilon^2 |D_{\nu_{a,i}} V(x_{a,i})| \|\varphi_a\|_a + \varepsilon^{N+1} \|\varphi_a\|_a + \|\varphi_a\|^2_a \right) + O(\varepsilon^{2N}) \\
= O\left( \sum_{i=1}^{k} |V(x_{a,i}) - V_i| \varepsilon^{N+2s} + \sum_{i=1}^{k} \nabla V(x_{a,i}) |\varepsilon^{N+2s+1} + \varepsilon^{N+2s+2} \right) = O(\varepsilon^{N+2s+1}). \tag{5.2}
\end{align*}

On the other hand, using the definition (4.11)
\begin{align*}
\int_{B_\rho(x_{a,i})} D_{\nu_{a,i}} V(x) U^2_{\varepsilon,x_{a,i}}(x) \\
= a_* \varepsilon^N \left( 1 + \varepsilon^N V_i \right)^{\frac{s}{s-1}} \int_{B_\rho(x_{a,i})} D_{\nu_{a,i}} V(x_{a,i}) + \frac{B}{2} \varepsilon^{N+2} D_{\nu_{a,i}} V(x_{a,i}) + O(\varepsilon^{N+4}), \tag{5.3}
\end{align*}
which, combined with (5.2), gives that
\[ D_{\nu_{a,i}} V(x_{a,i}) = O(\varepsilon^{\min\{2,2s+1\}}). \]

Now we take \( \bar{x}_{a,i} \in \Gamma_i \) be the point such that \( x_{a,i} - \bar{x}_{a,i} = \alpha_{a,i} \nu_{a,i} \) for some \( \alpha_{a,i} \in \mathbb{R} \).

Lemma 5.2. Under the condition \((V)\), we have that
\begin{align*}
\begin{cases}
\bar{x}_{a,i} - b_i = L_i \varepsilon^2 + O(\varepsilon^{2s+2}), \\
x_{a,i} - \bar{x}_{a,i} = -\frac{B}{2a_*} \Delta D_{\nu_{a,i}} V(b_i) \left( \frac{\partial^2 V(b_i)}{\partial x^2_i} \right)^{-1} \varepsilon^2 + O(\varepsilon^{2s+2}), \tag{5.4}
\end{cases}
\end{align*}
where \( B \) is as in (4.11), \( L_i \) is a vector depending on \( b_i \) and \( i = 1, \ldots, k \).

Proof. From (5.3), and
\begin{align*}
\int_{B_\rho(x_{a,i})} D_{\nu_{a,i}} V(x) U^2_{\varepsilon,x_{a,i}}(x) \\
= -2 \int_{B_\rho(x_{a,i})} D_{\nu_{a,i}} V(x) U_{\varepsilon,x_{a,i}}(x) \varphi_a - \int_{B_\rho(x_{a,i})} D_{\nu_{a,i}} V(x) \varphi_a^2 + O(\varepsilon^{2N}) \\
= O\left( \varepsilon^2 |D_{\nu_{a,i}} V(x_{a,i})| \|\varphi_a\|_a + \varepsilon^{N+1} \|\varphi_a\|_a + \|\varphi_a\|^2_a \right) + O(\varepsilon^{2N}) \\
= O(\varepsilon^{N+2s+2}),
\end{align*}
we obtain that
\[
(a_*+O(\varepsilon^{2s}))Dv_{a,i}V(x_{a,i}) + \frac{B}{2}\varepsilon^2\Delta Dv_{a,i}V(x_{a,i})
\]
\[
= O\left(\varepsilon^{2s+2} + \varepsilon^{2s+1}\sum_{i=1}^k |V(x_{a,i}) - V_i|\right)
\]
\[
= O\left(\varepsilon^{2s+2} + \varepsilon^{2s+1}\sum_{i=1}^k |x_{a,i} - \bar{x}_{a,i}|^2\right).
\]

Since \(\frac{\partial^2 V(b_i)}{\partial \nu^2} \neq 0\), the outward vector \(\nu_{a,i}(x)\) and the tangential unit vector \(\tau_{a,i}(x)\) of \(\Gamma_{a,i}\) at \(x_{a,i}\) are Lip-continuous in \(W_{\delta,i}\), we obtain from the above that
\[
x_{a,i} - \bar{x}_{a,i} = -\frac{B}{2a_*}\Delta Dv_{a,i}V(b_i)\left(\frac{\partial^2 V(b_i)}{\partial \nu^2}\right)^{-1} \varepsilon^2 + O\left(\varepsilon^{2s+2} + \varepsilon^{2s+1}\sum_{i=1}^k |x_{a,i} - \bar{x}_{a,i}|^2\right). \tag{5.5}
\]

From (5.1) and (5.5), for \(\frac{s+1}{N-2s} < p < \frac{N+2s}{N-2s}\) (which can be satisfied by \(N \geq 2s + 4\)),
\[
\|\varphi_a\|_a = O\left(\sum_{i=1}^k |x_{a,i} - \bar{x}_{a,i}|^{\frac{N}{2}+2s} + \varepsilon^{\frac{N}{2}+4s+2} + \varepsilon^{\frac{N}{2}+2s+2}\right) = O\left(\varepsilon^{\frac{N}{2}+2s+2}\right). \tag{5.6}
\]

Let \(H(x) = \langle \nabla V(x), \tau_{a,i}\rangle\). Then for \(N \geq 2s + 4\),
\[
\int_{B_\rho(x_{a,i})} H(x)U^2_{\varepsilon,x_{a,i}}(x)
\]
\[
= -2 \int_{B_\rho(x_{a,i})} H(x)U_{\varepsilon,x_{a,i}}(x)\varphi_a - \int_{B_\rho(x_{a,i})} H(x)\varphi_a^2
\]
\[
= -2 \int_{B_\rho(x_{a,i})} H(x)U_{\varepsilon,x_{a,i}}(x)\varphi_a + O(\|\varphi_a\|^2_\alpha) + O(\varepsilon^{2N})
\]
\[
= -2 \int_{B_\rho(x_{a,i})} \langle \nabla H(x_{a,i}), x - x_{a,i} \rangle U_{\varepsilon,x_{a,i}}(x)\varphi_a + O(\varepsilon^{N+2s+4}) \tag{5.7}
\]

On the one hand, since \(\nabla V = 0\), for \(x \in \Gamma_i\),
\[
\nabla H(x_{a,i}) = \langle \nabla^2 V(x_{a,i}), \tau_{a,i}\rangle = \langle \nabla^2 V(\bar{x}_{a,i}), \tilde{\tau}_{a,i}\rangle + O(|x_{a,i} - \bar{x}_{a,i}|) = O(|x_{a,i} - \bar{x}_{a,i}|), \tag{5.8}
\]
where \(\bar{x}_{a,i} \in \Gamma_i\) is the point such that \(x_{a,i} - \bar{x}_{a,i} = \beta_{a,i}\nu_{a,i}\) for some \(\beta_{a,i} \in \mathbb{R}\) and \(\tilde{\tau}_{a,i}\) is the tangential vector of \(\Gamma_i\) at \(\bar{x}_{a,i} \in \Gamma_i\). Hence,
\[
\int_{B_\rho(x_{a,i})} \langle \nabla H(x_{a,i}), x - x_{a,i} \rangle U_{\varepsilon,x_{a,i}}(x)\varphi_a
\]
\[
= O\left(\varepsilon^{\frac{N}{2}+1}|\nabla H(x_{a,i})| \cdot \|\varphi_a\|_a\right) = O\left(\varepsilon^{N+2s+3}|x_{a,i} - \bar{x}_{a,i}|\right) = O\left(\varepsilon^{N+2s+5}\right).
\]

Then by (5.7) and (5.8), it follows
\[
\int_{B_\rho(x_{a,i})} H(x)U^2_{\varepsilon,x_{a,i}}(x) = O\left(\varepsilon^{N+2s+4}\right). \tag{5.9}
\]

On the other hand, applying the Taylor’s expansion, we obtain that
\[
\int_{B_p(x_{a,i})} H(x)U^2_{\varepsilon,x_{a,i}}(x) = \frac{B\varepsilon^{N+2}}{2} (1 + V_i\varepsilon^{2s})(D_{\tau_{a,i}}\Delta V)(x_{a,i}) \\
+ \frac{H_{\tau_i}\varepsilon^{N+4}}{24} + O(\varepsilon^{N+2s+4}),
\]

where

\[ H_{\tau_i} = \sum_{l=1}^{2} \sum_{m=1}^{2} \frac{\partial^4 H(b_i)}{\partial x_l^2 \partial x_m^2} \int_{\mathbb{R}^N} x_l^2 x_m^2 U^2. \]

Combining (5.9) and (5.10), we obtain that

\[ (D_{\tau_{a,i}}\Delta V)(x_{a,i}) = -\frac{H_{\tau_i}\varepsilon^2}{12B} + O(\varepsilon^{2s+2}). \]

By (5.5), we get

\[ (D_{\tau_{a,i}}\Delta) V(x_{a,i}) = (D_{\tau_{a,i}}\Delta) V(\bar{x}_{a,i}) + \langle A_{\tau_i}, x_{a,i} - \bar{x}_{a,i} \rangle + O(|x_{a,i} - \bar{x}_{a,i}|^2) \]
\[ = (D_{\tau_{a,i}}\Delta) V(\bar{x}_{a,i}) + B_{\tau_i}\varepsilon^2 + O(\varepsilon^{2s+2}), \]

where \( A_{\tau_i} \) is a vector depending on \( b_i \) and \( B_i \) is a constant depending on \( b_i \). Moreover,

\[ (D_{\tau_{a,i}}\Delta) V(\bar{x}_{a,i}) = ((D_{\tau_{a,i}}^2\Delta) V(b_i))(\bar{x}_{a,i} - b_i) + O(|\bar{x}_{a,i} - b_i|^2). \]

Therefore, by (5.11), (5.12) and (5.13), there hold that

\[ ((D_{\tau_{a,i}}^2\Delta) V(b_i))(x_{a,i} - b_i) = -\left(\frac{H_{\tau_i}}{12B} + B_{\tau_{a,i}}\right)\varepsilon^2 + O(\varepsilon^{2s+2}) + O(|\bar{x}_{a,i} - b_i|^2). \]

Since \((D_{\tau_{a,i}}^2\Delta) V(b_i)\) is non-singular, we conclude the proof. \(\square\)

### 5.2. Local uniqueness.

We assume \( \frac{4s+1}{N+2s} < \frac{N+2s}{N-2s} \), which can be satisfied by \( N \geq 2s + 4 \). Set

\[
\delta_a = \begin{cases} 
\left( \frac{a^{-2v}}{ka_s} \right)^N - \frac{1}{p-1}, & p - 1 \neq \frac{4s}{N}, \\
\left( \frac{2s}{(N+2s)B_1} |ka_s - a^{2v-1}| \right)^{\frac{1}{2s+1}}, & p - 1 = \frac{4s}{N},
\end{cases}
\]

where \( B_1 = \frac{1}{N} \sum_{i=1}^{k} \Delta V(b_i) \int_{\mathbb{R}^N} |x|^2 U^2(x) \).

**Proposition 5.3.** Under the assumption (V), there hold that

\[ -\mu_a \delta_a^{2s} = 1 + \gamma_1 \delta_a^{2s} + O(\delta_a^{2s+2}), \]

and

\[ x_{a,i} - b_i = \bar{L}_i \delta_a^{2s} + O(\delta_a^{2s+2}), \quad i = 1, \ldots, k, \]

where \( \gamma_1 \) and the vector \( \bar{L}_i \) are constants.
Proof. (a) Firstly, if \( p - 1 \neq \frac{4s}{N} \), we follow the proof of Proposition 3.4 to find

\[
- \frac{a^{\frac{p-1}{p+1}}}{-\mu_a} - \frac{N}{2s} \int_{\mathbb{R}^N} U^2 + O\left( \sum_{i=1}^{k} (V(x_{a,i}) - V_i)(-\mu_a)^{-1} \right) \\
+ \sum_{i=1}^{k} \nabla V(x_{a,i})(-\mu_a)^{-1}\frac{1}{2} + (-\mu_a)^{-1}\frac{1}{2} + \begin{cases} O\left( (-\mu_a)^{-\frac{N}{2s}} \right), & \text{if } 1 < p \leq 2, \\
O\left( (-\mu_a)^{-\frac{N}{2s}} \right), & \text{if } p > 2, \end{cases} \\
+ O\left( (-\mu_a)^{-\frac{N}{2s}} |x_{a,j} - x_{a,i}|^{-(N+2s)} \right),
\]

which implies that

\[
- \frac{a^{\frac{p-1}{p+1}}}{k\mu_a(-\mu_a)^{\frac{1}{p+1}} - \frac{N}{2s}} = 1 + \sum_{i=1}^{k} \left( \frac{2}{p-1} - \frac{N}{2s} \right) \frac{V_i}{-\mu_a} + O\left( \sum_{i=1}^{k} (V(x_{a,i}) - V_i)(-\mu_a)^{-1} \right) \\
+ \sum_{i=1}^{k} \nabla V(x_{a,i})(-\mu_a)^{-1}\frac{1}{2} + (-\mu_a)^{-1}\frac{1}{2} + \begin{cases} O\left( (-\mu_a)^{-\frac{N}{2s}} \right), & \text{if } 1 < p \leq 2, \\
O\left( (-\mu_a)^{-\frac{N}{2s}} \right), & \text{if } p > 2. \end{cases} \\
+ O\left( (-\mu_a)^{-\frac{N}{2s}} |x_{a,j} - x_{a,i}|^{-(N+2s)} \right).
\]

Set \( \delta_a = \left( \frac{a^{\frac{p-1}{k\mu_a}}}{{\mu_a}} \right)^{\frac{1}{p+1}} \). Then we obtain (5.14). Finally, (5.15) can be obtained by (5.14) and Lemma 5.2.

(b) Secondly, we turn to deal with the case of \( p - 1 = \frac{4s}{N} \). From (5.4), it holds that

\[ x_{a,i} - b_i = \tilde{L}_i(-\mu_a)^{-\frac{1}{2}} + O\left( (-\mu_a)^{-\frac{1}{2}} \right), \]

with some vector \( \tilde{L}_i \). Moreover, in view of (5.6), \( \|\varphi_a\|_a = O\left( \varepsilon^{\frac{N}{2} + 2s + 2} \right). \)

We know that \( u_a \) is the \( k \)-peak solutions of (1.1)-(1.2), and we have the following Pohozaev identity:

\[
- \int_{\partial^n B_p(x_{a,i})} t^{1-2s} \frac{\partial \tilde{u}_a}{\partial \nu} \langle Y - X_{a,i}, \nabla \tilde{u}_a \rangle + \frac{1}{2} \int_{\partial^n B_p(x_{a,i})} t^{1-2s} |\nabla \tilde{u}_a|^2 \langle Y - X_{a,i}, \nu \rangle \\
- \frac{N - 2s}{2} \int_{\partial^n B_p(x_{a,i})} t^{1-2s} \frac{\partial \tilde{u}_a}{\partial \nu} \tilde{u}_a \\
- \int_{\partial B_p(x_{a,i})} \left( \frac{u_a|p+1}{p+1} - \frac{-\mu_a + V(x)}{2} \right) (u_a)^2 \langle y - x_{a,i}, \nu \rangle \\
= \int_{B_p(x_{a,i})} \left( \frac{N - 2s}{2} - \frac{N}{p+1} \right) |u_a|^{p+1} + s(-\mu_a + V(x))(u_a)^2 \\
+ \frac{1}{2} \langle y - x_{a,i}, \nabla V(x) \rangle (u_a)^2,
\]

which can be rewritten as

\[
\int_{B_p(x_{a,i})} sV(x)(u_a)^2 + \frac{1}{2} \langle y - x_{a,i}, \nabla V(x) \rangle (u_a)^2 \\
= \int_{B_p(x_{a,i})} s\mu_a (u_a)^2 + a \left( \frac{N}{p+1} - \frac{N - 2s}{2} \right) |u_a|^{p+1} + \int_{\partial^n B_p(x_{a,i})} W_1 + \int_{\partial B_p(x_{a,i})} W_1,
\]
where
\[ W_1 = -t^{1-2s} \frac{\partial \bar{u}_{a}}{\partial \nu} (Y - X_{a,i}, \nabla \bar{u}_{a}) + \frac{1}{2} t^{1-2s} |\nabla \bar{u}_{a}|^2 (Y - X_{a,i}, \nu) - \frac{N - 2s}{2} t^{1-2s} \frac{\partial \bar{u}_{a}}{\partial \nu} \bar{u}_{a}, \]

\[ W_2 = \left(- \frac{a(u_a)^{p+1}}{p+1} + \frac{(-\mu_a + V(x))(u_a)^2}{2}\right) \langle y - x_{a,i}, \nu \rangle. \]

Recall that
\[ u_a(x) = \sum_{i=1}^{k} \left( \frac{-\mu_a + V_i}{a} \right)^{\frac{1}{p-1}} \left( U(((-\mu_a + V_i)^{\frac{1}{p}}(x - x_{a,i})) + \varphi_a \right). \]

Then,
\[
\begin{aligned}
\int_{B_\rho(x_{a,i})} sV(x)(u_a)^2 + \frac{1}{2} \langle y - x_{a,i}, \nabla V(x) \rangle (u_a)^2 \\
= \left( \frac{-\mu_a + V_i}{a} \right)^{\frac{2}{p-1}} \int_{B_\rho(x_{a,i})} \left( s(V(x) - V_i) + \frac{1}{2} \langle y - x_{a,i}, \nabla V(x) \rangle \right) U^2((-\mu_a + V_i)^{\frac{1}{p}}(x - x_{a,i})) \\
+ \int_{B_\rho(x_{a,i})} u_a^2 + O((-\mu_a)^{-2 - \frac{2}{p}}) \\
= \left( \frac{2}{p} + 1 \right) (-\mu_a + V_i)^{\frac{1}{p}} \Delta V(x_{a,i}) \int_{\mathbb{R}^N} |x|^2 U^2(x) + sV_i \frac{a_s}{a^{p-1}} + O((-\mu_a)^{-\frac{1}{s}} - 1). 
\end{aligned}
\]

Moreover, we calculate more precisely that for some constants \( c_l, l = 1, 2, 3, \)
\[
\Delta V(x_{a,i}) = \Delta V(b_l) + c_1(-\mu_a)^{-\frac{1}{s}} + O((-\mu_a)^{-\frac{1}{s} - 1}). 
\]

Hence,
\[
\begin{aligned}
\int_{B_\rho(x_{a,i})} sV(x)(u_a)^2 + \frac{1}{2} \langle y - x_{a,i}, \nabla V(x) \rangle (u_a)^2 \\
= \left( \frac{2}{p} + 1 \right) (-\mu_a + V_i)^{\frac{1}{p}} \Delta V(b_i) \int_{\mathbb{R}^N} |x|^2 U^2(x) + sV_i \frac{a_s}{a^{p-1}} + c_2(-\mu_a)^{-\frac{1}{s} - 1} + O((-\mu_a)^{-\frac{2}{s} - 1}).
\end{aligned}
\]

Now summing (5.16) from \( i = 1 \) to \( i = k \), since \( \frac{N}{p+1} - \frac{N-2s}{N+2s} = \frac{2s^2}{N+2s} > 0 \), we have
\[
sp_a + \frac{2as^2}{N + 2s} \int_{\mathbb{R}^N} |u_a|^{p+1} \\
= \sum_{i=1}^{k} \left( \frac{2}{p} + 1 \right) (-\mu_a)^{\frac{1}{p}} \Delta V(b_i) \int_{\mathbb{R}^N} |x|^2 U^2(x) + \sum_{i=1}^{k} sV_i \frac{a_s}{a^{p-1}} + c_2(-\mu_a)^{-\frac{1}{s} - 1} + O((-\mu_a)^{-\frac{2}{s} - 1}).
\]

(5.17)
On the other hand, by the orthogonality  \( \int_{\mathbb{R}^N} U^p ((-\mu_a + V_i) \frac{1}{\delta} (x - x_{a,i})) \varphi_a = 0 \) for \( i = 1, \ldots, k \), there holds that

\[
\int_{\mathbb{R}^N} u_{a}^{p+1} = \sum_{i=1}^{k} \left( \frac{-\mu_a + V_i}{\delta} \right)^{\frac{p+1}{p-1}} \left( \frac{-\mu_a + V_i}{\delta} \right)^{-\frac{N}{2}} \int_{\mathbb{R}^N} U^p + (p+1) \int_{\mathbb{R}^N} (U^p ((-\mu_a + V_i) \frac{1}{\delta} (x - x_{a,i})) \varphi_a + O(|\varphi_a|^{p+1}))
\]

\[
= \frac{N + 2s}{2s} a_s \frac{a_s}{a^{p-1}} \sum_{i=1}^{k} \left( -\mu_a + V_i \right)^{\frac{p+1}{p-1}} \frac{N}{2} + O(\mu_a^{\frac{p+1}{p-1}} |\varphi_a|^2)
\]

\[
= \frac{N + 2s}{2s} a_s \frac{a_s}{a^{p-1}} \sum_{i=1}^{k} \left( -\mu_a + V_i \right)^{\frac{p+1}{p-1}} \frac{N}{2} + O((-\mu_a)^{-1-\frac{2}{p}})
\]

which, combined with (5.17), gives then

\[
s(-\mu_a)^{1+\frac{2}{s}} (ka_s - a \frac{2}{p-1}) = \left( \frac{s}{2} + 1 \right) \sum_{i=1}^{k} \Delta V(b_i) \int_{\mathbb{R}^N} |x|^2 u^2(x) + c_3 (-\mu_a)^{-1} + O((-\mu_a)^{-\frac{2}{p}-1}).
\]

If we set \( \delta = \left( \frac{2s}{s+2}\right)^{1/2} |ka_s - a \frac{2}{p-1}| \frac{1}{2(s+1)} \), then we obtain (5.14), and (5.15) can be found by (5.14) and Lemma 5.2.

By a change of variable, the problem (1.1)-(1.2) can be rewritten as

\[
\delta_a^{2s} (-\Delta)^s u + \left( -\mu_a \delta_a^{2s} + \delta_a^{2s} V(x) \right) u = u^p, \quad u \in H^s(\mathbb{R}^N),
\]

\[
\int_{\mathbb{R}^N} u^2 = (a \delta_a^{2s})^{\frac{2}{p-1}}.
\]

Similar to Lemma 3.2, the \( k \)-peak solution of (5.18)-(5.19) concentrating at \( b_1, \ldots, b_k \) can be written as

\[
u = \sum_{i=1}^{k} \tilde{U}_{\delta_a, x_{a,i}} + \varphi_a
\]

with \( |x_{a,i} - b_i| = o(1), \| \varphi_a \|_{\delta_a} = o(\delta_a^{-\frac{N}{2}}) \). Moreover,

\[
\varphi_a \in \bigcap_{l=1}^{k} \tilde{E}_{a, x_{a,l}} := \left\{ \phi \in H^s(\mathbb{R}^N) : \langle \phi, \frac{\partial \tilde{U}_{\delta_a, x_{a,i}}}{\partial x_j} \delta_a \rangle = 0, j = 1, \ldots, N \right\},
\]

where

\[
\tilde{U}_{\delta_a, x_{a,i}}(x) := (1 + (\gamma_1 + V_i) \delta_a^{2s})^{\frac{1}{p-1}} U \left( \frac{1}{\delta_a} (\gamma_1 + V_i) \delta_a^{2s} \frac{1}{\delta_a} (x - x_{a,i}) \right),
\]

\[
\| \phi \|_{\delta_a}^2 = \int_{\mathbb{R}^N} (\delta_a^2 |(-\Delta)^s \phi|^2 + \phi^2).
\]
Now, we write the equation (5.18) as

$\tilde{L}_a(\tilde{\varphi}_a) = N_a(\tilde{\varphi}_a) + \tilde{I}_a(x),$

where

$\tilde{L}_a(\tilde{\varphi}_a) = \delta_a^{2s}(-\Delta)^s \tilde{\varphi}_a + [1 + (-\mu_a \delta_a^{2s} - 1) + \delta_a^{2s} V] \tilde{\varphi}_a - p \sum_{i=1}^{k} \tilde{U}_{\delta_a,x_a,i}^p \tilde{\varphi}_a,$

$N_a, L_a$ are defined by (3.2), (3.3), and

$\tilde{I}_a(x) = (\mu_a \delta_a^{2s} + 1 + \gamma_1 \delta_a^{2s} - \delta_a^{2s} V(x) + V_i \delta_a^{2s}) \sum_{i=1}^{k} \tilde{U}_{\delta_a,x_a,i} + \left( \sum_{i=1}^{k} \tilde{U}_{\delta_a,x_a,i} \right)^p - \sum_{i=1}^{k} \tilde{U}_{\delta_a,x_a,i}^p.$

Lemma 5.4. It holds that

$\|\tilde{\varphi}_a\|_{\delta_a} = O\left(\frac{N}{\delta_a^{2s+2}}\right).$ (5.20)

Proof. Similar to Lemma 3.2, we obtain (5.20) only from proving

$\|\tilde{I}_a\|_{\delta_a} = O\left(\frac{N^s}{\delta_a^{2s+2}} + |\sum_{i=1}^{k} V(x_{a,i})| N^{s-2s+1} + \delta_a^{2s+2}\right) = O\left(\frac{N^{s+1}}{\delta_a^{2s+2}}\right).

\square

Let $u_a^{(1)}$ and $u_a^{(2)}$ be two $k$-peak solutions to (5.18) and (5.19) concentrating at $k$ points $b_1, \ldots, b_k$, of the form

$u_a^{(l)} = \sum_{i=1}^{k} \tilde{U}_{\delta_a,x_a,i}^{(l)} + \tilde{\varphi}_a^{(l)}, \quad \tilde{\varphi}_a^{(l)} \in \bigcap_{i=1}^{k} \tilde{E}_{a,x_a,i}^{(l)}, \quad l = 1, 2.$ (5.21)

Set $\xi_a(x) = \frac{u_a^{(1)} - u_a^{(2)}}{\|u_a^{(1)} - u_a^{(2)}\|_{L^\infty(\mathbb{R}^N)}}$. Then $\|\xi_a\|_{L^\infty(\mathbb{R}^N)} = 1$. We get from (5.18) that

$\delta^{2s}(-\Delta)^s \xi_a + C_a(x) \xi_a = g_a(x),$

where

$C_a(x) = \delta^{2s} V(x) - \delta^{2s} \mu_a^{(1)} - p \int_0^1 (t u_a^{(1)} + (1-t) u_a^{(2)})^{p-1} dt,$

$g_a(x) = \frac{\delta^{2s} \mu_a^{(1)} - \mu_a^{(2)}}{\|u_a^{(1)} - u_a^{(2)}\|_{L^\infty(\mathbb{R}^N)}} u_a^{(2)}.$

Following [15] and (4.1), we can prove that

$|\xi_a(x)| + |\nabla \xi_a(x)| \leq C \sum_{i=1}^{k} \frac{1}{1 + \left| \frac{x - x_{a,i}}{\delta_a} \right|^{N + 2s - \theta'}}.$ (5.22)

Applying blowing-up technique, we set

$\tilde{\xi}_{a,i}(x) = \xi_{a,i} \left( \frac{\delta_a}{P_i} \frac{x - x_{a,i}^{(1)}}{P_i} \right), \quad P_i = 1 + (\gamma_1 + V_i) \delta^{2s}.$
The $\bar{\xi}_{a,i}$ satisfies that
\begin{equation}
(-\Delta)^s \bar{\xi}_{a,i} + \frac{1}{P_i} C_a \left( \frac{\delta_a}{P_i^{2s}} x + x_{a,i}^{(1)} \right) \bar{\xi}_{a,i} = \frac{1}{P_i} d_a \left( \frac{\delta_a}{P_i^{2s}} x + x_{a,i}^{(1)} \right).
\end{equation}

**Lemma 5.5.** For $x \in B_{\delta_a^{-1} P_i^s}(0)$, it holds that
\begin{equation}
\frac{1}{P_i} C_a \left( \frac{\delta_a}{P_i^{2s}} x + x_{a,i}^{(1)} \right) = 1 - pU^{p-1}(x) + O\left( \delta_a^{2s} + \sum_{l=1}^{2} \varphi_a^{(l)} \left( \frac{\delta_a}{P_i^{2s}} x + x_{a,i}^{(1)} \right) \right),
\end{equation}
and
\begin{equation}
\frac{1}{P_i} d_a \left( \frac{\delta_a}{P_i^{2s}} x + x_{a,i}^{(1)} \right) = -\frac{p-1}{ka_s} U(x) \sum_{j=1}^{k} U(x) \xi_{a,j}(x) + O\left( \delta_a^{2s+2} + \sum_{l=1}^{2} \varphi_a^{(l)} \left( \frac{\delta_a}{P_i^{2s}} x + x_{a,i}^{(1)} \right) \right).
\end{equation}

**Proof.** Since
\begin{equation}
C_a \left( \frac{\delta_a}{P_i^{2s}} x + x_{a,i}^{(1)} \right) = \delta_a^{2s} V \left( \frac{\delta_a}{P_i^{2s}} x + x_{a,i}^{(1)} \right) - \delta_a^{2s} \mu_a^{(1)},
\end{equation}
we can obtain (5.24) directly.

From (5.18) and (5.19), for $l = 1, 2$, we get that
\begin{equation}
\mu_a^{(l)} \delta_a^{2s} = \delta_a^{2s} \int_{\mathbb{R}^N} \left( \left( (-\Delta)^s u_a^{(1)} \right)^2 + V(x) \left( u_a^{(1)} \right)^2 \right) - \int_{\mathbb{R}^N} |u_a^{(l)}|^{p+1},
\end{equation}
which implies that
\begin{align}
&\frac{\delta_a^{2s}}{\mu_a^{(1)} - \mu_a^{(2)}} \left( \frac{1}{P_i^{2s}} \right) \left( u_a^{(1)} - u_a^{(2)} \right)_{L^\infty(\mathbb{R}^N)}
= \delta_a^{2s} \int_{\mathbb{R}^N} \left( \left( (-\Delta)^s u_a^{(1)} \right) \cdot \left( (-\Delta)^s u_a^{(2)} \right) \right) \cdot \left( (-\Delta)^s \xi_a \right) + V(x) \left( u_a^{(1)} + u_a^{(2)} \right) \xi_a \\
&- \int_{\mathbb{R}^N} \left| u_a^{(1)} \right|^{p+1} - \left| u_a^{(2)} \right|^{p+1} \left( \mu_a^{(1)} - \mu_a^{(2)} \right) \left( u_a^{(1)} - u_a^{(2)} \right)_{L^\infty(\mathbb{R}^N)}
= \delta_a^{2s} \int_{\mathbb{R}^N} \left( \left( (-\Delta)^s u_a^{(1)} \right) \cdot \left( (-\Delta)^s u_a^{(2)} \right) \right) \cdot \left( (-\Delta)^s \xi_a \right) + V(x) \left( u_a^{(1)} + u_a^{(2)} \right) \xi_a \\
&- \int_{\mathbb{R}^N} \left| u_a^{(1)} \right|^{p+1} - \left| u_a^{(2)} \right|^{p+1} \left( \mu_a^{(1)} - \mu_a^{(2)} \right) \delta_a^{2s} \int_{\mathbb{R}^N} \left( u_a^{(1)} + u_a^{(2)} \right) \xi_a \\
= (\mu_a^{(1)} - \mu_a^{(2)}) \delta_a^{2s} \int_{\mathbb{R}^N} u_a^{(1)} \xi_a + \int_{\mathbb{R}^N} \left( \left| u_a^{(1)} \right|^{p+1} - \left| u_a^{(2)} \right|^{p+1} \right) \xi_a - (p+1) \int_{\mathbb{R}^N} \xi_a \int_{0}^{1} \left( 1 - t \right) u_a^{(2)} \, dt.
\end{align}
Hence, in view of (5.14), (5.15), (5.20), (5.26) gives that
\[
\frac{\delta_a^{2s_i}(\mu_a(1) - \mu_a(2))}{\|u_a^{(1)} - u_a^{(2)}\|_{L^\infty(\mathbb{R}^N)}} = -\frac{p - 1}{ka_x\delta_a^N}\left(\int_{\mathbb{R}^N} (|u_a^{(1)}|^p + |u_a^{(2)}|^p)\xi_a - (p + 1)\xi_a \int_0^1 (tu_a^{(1)} + (1 - t)u_a^{(2)})^p dt\right)
+ O(\delta_a^{2s + 2}),
\]
which implies (5.25).

Since \(\bar{\xi}_{a,i} \leq 1\), we suppose that \(\bar{\xi}_{a,i} \rightarrow \bar{\xi}_i(x)\) in \(C^1_{loc}(\mathbb{R}^N)\). Lemma 5.5 actually shows the following result.

**Lemma 5.6.** There hold that
\[
(-\Delta)^s \bar{\xi}_i + (1 - pU^{p-1})\bar{\xi}_i = -\frac{p - 1}{ka_x}U(x)\sum_{j=1}^k \int_{\mathbb{R}^N} U^p(x)\bar{\xi}_j(x), \quad i = 1, \ldots, k. \tag{5.27}
\]

In order to show \(\bar{\xi}_i = 0\), we write
\[
\bar{\xi}_{a,i}(x) = \sum_{j=0}^N \gamma_{a,i,j}\Psi_j + \bar{\xi}_{a,i}(x), \quad \text{in } H^s(\mathbb{R}^N), \tag{5.28}
\]
where \(\Psi_j (j = 0, 1, \ldots, N)\) are functions in (A.3), \(\bar{\xi}_{a,i} \in \bar{E}\) and
\[
\bar{E} = \{u \in H^s(\mathbb{R}^N), \langle u, \Psi_j \rangle = 0, \text{ for } j = 0, 1, \ldots, N\}.
\]

It is standard to show the following problem.

**Lemma 5.7.** For any \(u \in \bar{E}\), there holds that
\[
\|\bar{L}(u)\| \geq c_0\|u\|,
\]
where \(c_0 > 0\) is a constant, and the linear operator is defined by
\[
\bar{L}(u) = (-\Delta)^s u + (1 - pU^{p-1})u + \frac{p - 1}{a_x}U(x)\int_{\mathbb{R}^N} U^p(x)u(x)dx. \tag{5.29}
\]

**Proposition 5.8.** The \(\bar{\xi}_{a,i}(x)\) defined by (5.28) satisfies that
\[
\|\bar{\xi}_{a,i}\| = O(\delta_a^{2s}). \tag{5.30}
\]

**Proof.** Using (5.23)-(5.27), for any \(\psi \in H^s(\mathbb{R}^N)\) we have by Lemma 5.5 and (5.20) that
\[
\langle \bar{L}(\bar{\xi}_{a,i}), \psi \rangle = \int_{\mathbb{R}^N} \left[(1 - pU^{p-1}) - \frac{1}{P_i} a\left(\frac{\delta_a}{P_i^{2s}}x + x_a^{(1)}\right)\right] \bar{\xi}_{a,i} \psi
\]
\[
+ \int_{\mathbb{R}^N} \left[\frac{1}{P_i} a\left(\frac{\delta_a}{P_i^{2s}}x + x_a^{(1)}\right) + \frac{p - 1}{ka_x}U(x)\sum_{j=1}^k \int_{\mathbb{R}^N} U^p(x)\bar{\xi}_{a,j}(x)\right] \psi
\]
\[
= O(\delta_a^{2s})\|\psi\| + O\left(\int_{\mathbb{R}^N} \sum_{l=1}^2 a^{(l)}(\frac{\delta_a}{P_i^{2s}}x + x_a^{(1)})\|\psi\|\right)
\]
\[
= O(\delta_a^{2s})\|\psi\| + O(\delta^{-\frac{2s}{p}}a^{(l)})\|\psi\| = O(\delta_a^{2s})\|\psi\|,
\]
which implies (5.30) by Lemma 5.7.

\[\square\]
Lemma 5.9. It holds that
\[
\delta_a^{2s} \sum_{i=1}^k \int_{B_r(x_{a,i}^{(1)})} \left( sV(x) + \frac{1}{2} \left< y - x_{a,i}^{(1)}, \nabla V(x) \right> \right) (u_a^{(1)}) + (u_a^{(2)}) \xi_a
\]

\[= (\mu_a^{(1)} - \mu_a^{(2)}) \delta_a^{2s} \int_{\mathbb{R}^N} u_a^{(1)} \xi_a + \int_{\mathbb{R}^N} ((u_a^{(1)})^p + (u_a^{(2)})^p) \xi_a
\]

\[- (p + 1) \int_{\mathbb{R}^N} \int_0^1 (tu_a^{(1)} + (1 - t)u_a^{(2)})^p \xi_a
\]

\[+ \left( \frac{N}{p + 1} - \frac{N - 2s}{2} \right) (p + 1) \int_{B_r(x_{a,i}^{(1)})} \int_0^1 (tu_a^{(1)} + (1 - t)u_a^{(2)})^p \xi_a + O(\delta_a^{2N+2s}).
\]

Proof. Since \( u_{a}^{(l)} \) are the \( k \)-peak solutions of (5.18)–(5.19), we have the following Pohozaev identity:

\[- \delta_a^{2s} \int_{\partial^* B_r(x_{a,i}^{(1)})} t^{1-2s} \frac{\partial u_a^{(l)}}{\partial \nu} \langle Y - X_{a,i}, \nabla u_a^{(l)} \rangle + \frac{\delta_a^{2s}}{2} \int_{\partial^* B_r(x_{a,i}^{(1)})} t^{1-2s} |\nabla u_a^{(l)}|^2 \langle Y - X_{a,i}, \nu \rangle
\]

\[- \int_{\partial B_r(x_{a,i}^{(1)})} \left( \frac{(u_a^{(l)})^{p+1}}{p+1} - \frac{\delta_a^{2s}(-\mu_a^{(l)} + V(x))(u_a^{(l)})^2}{2} \right) \langle y - x_{a,i}, \nu \rangle
\]

\[= \int_{B_r(x_{a,i}^{(1)})} \left( \frac{N - 2s}{2} - \frac{N}{p + 1} \right) (u_a^{(l)})^{p+1} + s\delta_a^{2s}(-\mu_a^{(l)} + V(x))(u_a^{(l)})^2
\]

\[+ \frac{\delta_a^{2s}}{2} \langle y - x_{a,i}^{(l)}, \nabla V(x) \rangle (u_a^{(l)})^2,
\]

which can be rewritten as

\[
\delta_a^{2s} \int_{B_r(x_{a,i}^{(1)})} (sV(x)(u_a^{(l)})^2 + \frac{1}{2} \left< y - x_{a,i}^{(1)}, \nabla V(x) \right> (u_a^{(l)})^2)
\]

\[= \int_{B_r(x_{a,i}^{(1)})} \delta_a^{2s} s\mu_a^{(l)}(u_a^{(l)})^2 + \left( \frac{N}{p + 1} - \frac{N - 2s}{2} \right) (u_a^{(l)})^{p+1}
\]

\[+ \delta_a^{2s} \int_{\partial^* B_r(x_{a,i}^{(1)})} W_1^{(l)} + \delta_a^{2s} \int_{\partial B_r(x_{a,i}^{(1)})} W_2^{(l)},
\]

where

\[
W_1^{(l)} = -t^{1-2s} \frac{\partial u_a^{(l)}}{\partial \nu} \langle Y - X_{a,i}, \nabla u_a^{(l)} \rangle + \frac{1}{2} t^{1-2s} |\nabla u_a^{(l)}|^2 \langle Y - X_{a,i}, \nu \rangle - \frac{N - 2s}{2} t^{1-2s} \frac{\partial u_a^{(l)}}{\partial \nu} \nabla u_a^{(l)},
\]

\[
W_2^{(l)} = \left( -\frac{(u_a^{(l)})^{p+1}}{\delta_a^{2s}(p + 1)} + \frac{(-\mu_a^{(l)} + V(x))(u_a^{(l)})^2}{2} \right) \langle y - x_{a,i}, \nu \rangle.
\]
Then, the above local Pohozaev identity implies that

\[
\begin{align*}
\delta_a^{2s} \int_{B_p(x_a^{(i)}_{a,i})} \left( sV(x) + \frac{1}{2} (y - x_a^{(1)}_{a,i}, \nabla V(x)) \right) (u_a^{(l)})^2 + u_a^{(2)}) \xi_a \\
= \frac{(\mu_a^{(1)} - \mu_a^{(2)}) \delta_a^{2s}}{\|u_a^{(l)} - u_a^{(2)}\|_{L^\infty}} \int_{B_p(x_a^{(i)}_{a,i})} (u_a^{(1)})^2 + \mu_a^{(2)} \delta_a^{2s} \int_{B_p(x_a^{(i)}_{a,i})} (u_a^{(1)} + u_a^{(2)}) \xi_a \\
+ \left( \frac{N}{p + 1} - \frac{N - 2s}{2} \right) (p + 1) \int_{B_p(x_a^{(i)}_{a,i})} \int_0^1 (tu_a^{(1)} + (1 - t)u_a^{(2)})^p \xi_a + J_{a,i,1} + J_{a,i,2},
\end{align*}
\]

where

\[
J_{a,i,1} = \int_{\partial' B_p(x_a^{(i)}_{a,i})} \frac{\delta_a^{2s}(W_1^{(1)} - W_2^{(2)})}{\|u_a^{(l)} - u_a^{(2)}\|_{L^\infty}} \\
= -\delta_a^{2s} \int_{\partial' B_p(x_a^{(i)}_{a,i})} t^{1-2s} \frac{\partial \tilde{u}_a^{(1)}}{\partial \nu} (Y - X_{a,i}, \nabla \tilde{\xi}_a) - \delta_a^{2s} \int_{\partial' B_p(x_a^{(i)}_{a,i})} t^{1-2s} \frac{\partial \tilde{\xi}_a}{\partial \nu} (Y - X_{a,i}, \nabla \tilde{\tilde{u}}_a^{(2)}) \\
+ \frac{\delta_a^{2s}}{2} \int_{\partial' B_p(x_a^{(i)}_{a,i})} t^{1-2s} \nabla (\tilde{u}_a^{(1)} + \tilde{u}_a^{(2)}) \cdot \nabla \tilde{\xi}_a (Y - X_{a,i}, \nu) \\
- \frac{N - 2s}{2} \delta_a^{2s} \int_{\partial' B_p(x_a^{(i)}_{a,i})} t^{1-2s} \left( \frac{\partial \tilde{\xi}_a}{\partial \nu} \tilde{u}_a^{(1)} - \frac{\partial \tilde{u}_a^{(2)}}{\partial \nu} \tilde{\tilde{u}}_a^{(2)} \right) = O(\delta_a^{2s+2N}),
\]

and since \(p + 1 > 2\)

\[
J_{a,i,2} = \int_{\partial B_p(x_a^{(i)}_{a,i})} \frac{\delta_a^{2s}(W_1^{(1)} - W_2^{(2)})}{\|u_a^{(l)} - u_a^{(2)}\|_{L^\infty}} = O(\delta_a^{2s+2(N+2s-\theta) + \delta_a^{(p+1)N+2ps}}) = O(\delta_a^{2s+2(N+2s-\theta)}).
\]

Summing (5.32) from \(i = 1\) to \(i = k\) and considering the decay (4.1) and (5.22),

\[
\begin{align*}
\sum_{i=1}^k \delta_a^{2s} \int_{B_p(x_a^{(i)}_{a,i})} \left( sV(x) + \frac{1}{2} (y - x_a^{(1)}_{a,i}, \nabla V(x)) \right) (u_a^{(l)})^2 + u_a^{(2)}) \xi_a \\
= \frac{(\mu_a^{(1)} - \mu_a^{(2)}) \delta_a^{2s}}{\|u_a^{(l)} - u_a^{(2)}\|_{L^\infty}} \int_{\mathbb{R}^N} (u_a^{(1)})^2 + \mu_a^{(2)} \delta_a^{2s} \int_{\mathbb{R}^N} (u_a^{(1)} + u_a^{(2)}) \xi_a \\
+ \left( \frac{N}{p + 1} - \frac{N - 2s}{2} \right) (p + 1) \int_{B_p(x_a^{(i)}_{a,i})} \int_0^1 (tu_a^{(1)} + (1 - t)u_a^{(2)})^p \xi_a + O(\delta_a^{2N+2s}),
\end{align*}
\]

which, combined with (5.26), gives (5.31), concluding the proof.

\[\square\]

In the following, we first show \(\gamma_{a,i,0} = o(1)\) for \(i = 1, \ldots, k\).

**Lemma 5.10.** It holds that

\(\gamma_{a,i,0} = o(1), \quad i = 1, \ldots, k.\)

**Proof.** From (5.15), (5.20), (5.21) and

\[
u_a^{(2)} \left( \frac{\delta_a}{P_i} x + x_a^{(1)}_{a,i} \right) = P_i U(x) + O \left( \frac{P_i^\frac{1}{2}}{\delta_a} \left\| x_a^{(1)}_{a,i} - x_a^{(2)}_{a,i} \right\| \right) + \varphi_a \left( \frac{\delta_a}{P_i^\frac{1}{2}} x + x_a^{(1)}_{a,i} \right),
\]
we obtain that
\[
\int_{B_p(x_{a,i}^{(1)})} \left( s V(x) + \frac{1}{2} \langle y - x_{a,i}^{(1)}, \nabla V(x) \rangle \right) (u_a^{(l)} + u_a^{(2)}) \xi_a
\]
\[
= \int_{B_p(x_{a,i}^{(1)})} \left( s (V(x) - V_i) + \frac{1}{2} \langle y - x_{a,i}^{(1)}, \nabla V(x) \rangle \right) \left( \sum_{i=1}^{k} (\tilde{U}_{\delta_a,x_{a,i}^{(1)}} + \tilde{U}_{\delta_a,x_{a,i}^{(2)}} + \varphi_a^{(1)} + \varphi_a^{(2)}) \right) \xi_a
\]
\[
+ \int_{B_p(x_{a,i}^{(1)})} s V_i (u_a^{(l)} + u_a^{(2)}) \xi_a
\]
\[
= 2 \int_{B_p(x_{a,i}^{(1)})} \left( s (V(x) - V_i) + \frac{1}{2} \langle y - x_{a,i}^{(1)}, \nabla V(x) \rangle \right) \tilde{U}_{\delta_a,x_{a,i}^{(1)}} \xi_a
\]
\[
+ \int_{B_p(x_{a,i}^{(1)})} s V_i (u_a^{(l)} + u_a^{(2)}) \xi_a + O(\delta_a^{N+2s+2}).
\]
From (5.4) (5.15), (5.28) and (5.30), we obtain that
\[
\int_{B_p(x_{a,i}^{(1)})} (V(x) - V_i) \tilde{U}_{\delta_a,x_{a,i}^{(1)}} \xi_a
\]
\[
= \delta_a^N \int_{\mathbb{R}^N} (V(\frac{\delta_a}{P_{1}^{\frac{1}{2}}} x + x_{a,i}^{(1)}) - V_i) U(x) \left( \sum_{j=0}^{N} \gamma_{a,i,j} \Psi_j \right) + O(\delta_a^{N+2s+2})
\]
\[
= - \frac{B}{2} \Delta V(b_i) \gamma_{a,i,0} \delta_a^{N+2} + O(\delta_a^{N+2s+2} + \delta_a^{N+3}),
\]
where $B$ is the constant as defined in (4.11).

Similarly,
\[
\int_{B_p(x_{a,i}^{(1)})} \langle y - x_{a,i}^{(1)}, \nabla V(x) \rangle \tilde{U}_{\delta_a,x_{a,i}^{(1)}} \xi_a = - \frac{B}{2} \Delta V(b_i) \gamma_{a,i,0} \delta_a^{N+2} + O(\delta_a^{N+2s+2} + \delta_a^{N+3}).
\]

Therefore, it holds
\[
LHS \ of \ (5.31) = - \frac{2s+1}{4} B \Delta V(b_i) \gamma_{a,i,0} \delta_a^{N+2s+2} + O(\delta_a^{N+6s} + \delta_a^{N+2s+2})
\]
\[
+ s \delta_a^{2s} V_i \int_{B_p(x_{a,i}^{(1)})} (u_a^{(l)} + u_a^{(2)}) \xi_a.
\]

Moreover, we have
\[
\int_{B_p(x_{a,i}^{(1)})} (u_a^{(l)} + u_a^{(2)}) \xi_a
\]
\[
= 2 \gamma_{a,i,0} \delta_a^N \int_{\mathbb{R}^N} U(U + \frac{p-1}{2s} x \cdot \nabla U) + O(\|x_{a,i}^{(1)} - x_{a,i}^{(1)}\|_1 \delta_a^{N-1} + \delta_a^N \|\varphi_a^{(2)}\|_1)
\]
\[
= 2 \gamma_{a,i,0} \delta_a^N \left( 1 - \frac{N(p-1)}{4s} \right) \int_{\mathbb{R}^N} U^2 + O(\delta_a^{N+2s+2} + \delta_a^{N+3}),
\]
which, combined with \((5.33)\), gives that
\[
LHS \text{ of } (5.31) = -\frac{2s+1}{4}B\Delta V(b_i)\gamma_{a,i,0}\delta_a^{N+2+2s} + O(\delta_a^{N+6s} + \delta_a^{N+3+2s}) + sV_i\alpha_a\gamma_{a,i,0}(1 - \frac{N(p-1)}{4s})\delta_a^{N+2s}.
\]

On the other hand, for the RHS of \((5.31)\),
\[
\int_{\mathbb{R}^N} ((u_a^{(1)})^p + (u_a^{(2)})^p)\xi_a - (p+1)\int_{\mathbb{R}^N} \int_0^1 (tu_a^{(1)} + (1-t)u_a^{(2)})^p\xi_a
+ \left(\frac{N}{p+1} - \frac{N-2s}{2}\right)(p+1) \int_{B_{\rho}(x_a,1)} \int_0^1 (tu_a^{(1)} + (1-t)u_a^{(2)})^p\xi_a
= \left(N - (p-1) - \frac{(N-2s)(p+1)}{2}\right)\left(1 - \frac{N}{p+1}\right)\gamma_{a,i,0} \int_{\mathbb{R}^N} U^{p+1} + O(\delta_a^{N+2+4s})
= \left(N - (p-1) - \frac{(N-2s)(p+1)}{2}\right)\left(1 - \frac{N(p-1)}{2s(p+1)}\right)\frac{1}{\rho(\rho+1)}d_a^{\gamma_{a,i,0}}
= -(\tilde{\gamma} + o(1))d_a^{\gamma_{a,i,0}}\delta_a^N
\]
with some constant \(\tilde{\gamma} > 0\) since \(N \geq 2s + 2\) and \(p < \frac{N+2s}{N-2s}\). Moreover, by \((5.14),(5.15)\) and \((5.20)\), it holds that
\[
(\mu_a^{(1)} - \mu_a^{(2)})\delta_a^{2s} \int_{\mathbb{R}^N} u_a^{(1)}\xi_a = O(\delta_a^{N+6s}).
\]
To sum up, we finally obtain that
\[
-\frac{2s+1}{4}B\Delta V(b_i)\gamma_{a,i,0}\delta_a^{N+2+2s} + O(\delta_a^{N+6s} + \delta_a^{N+3+2s} + \delta_a^{2N+2s})
+ sV_i\alpha_a\gamma_{a,i,0}(1 - \frac{N(p-1)}{4s})\delta_a^{N+2s} = -(\tilde{\gamma} + o(1))d_a^{\gamma_{a,i,0}}\delta_a^N,
\]
which implies \(\gamma_{a,i,0} = o(1)\).

\[
\text{Lemma 5.11. It holds that } \gamma_{a,i,j} = o(1), \text{ for } i = 1, \ldots, k \text{ and } j = 1, \ldots, N.
\]

\textbf{Proof.} This Lemma can be proved just following that in \([30]\), so we only sketch the main steps as follows.

\textbf{Step 1:} To prove \(\gamma_{a,i,N} = O(\delta_a)\) for \(i = 1, \cdots, k\).

On one hand, using \((4.5)\), we deduce
\[
\int_{B_{\rho}(x_a^{(1)})} \frac{\partial V(x)}{\partial \nu_{a,i}}A_a(x)\xi_a = O(\delta_a^{2N}), \tag{5.34}
\]
where \(\nu_{a,i}\) is the outward unit vector of \(\partial B_{\rho}(x_a^{(1)})\) at \(x\), \(A_a(x) = \sum_{l=1}^2 u_a^{(l)}(x)\).
On the other hand, by (5.15), and the Taylor expansions
\[
\int_{B_d(x^{(1)}_{a,i})} \frac{\partial V(x)}{\partial \tau_{a,i,j}} A_a(x) \xi_a + \int_{\mathbb{R}^N} \nabla \frac{\partial V(x^{(1)}_{a,i})}{\partial \nu_{a,i}} \cdot A_a(x) \xi_a + O(\delta_a^{N+1})
\]
(5.35)
\[
= - \frac{\partial^2 V(x^{(1)}_{a,i})}{\partial \nu_{a,i}^2} \alpha \gamma_{a,i,N} \delta_a^{N+1} + O(\delta_a^{N+1}).
\]
Then (5.34) and (5.35) imply \(\gamma_{a,i,N} = O(\delta_a)\).

**Step 2:** To prove \(\gamma_{a,i,j} = o(1)\) for \(i = 1, \ldots, k\) and \(j = 1, \ldots, N - 1\).

Similar to (5.34), we first have
\[
\int_{B_d(x^{(1)}_{a,i})} \frac{\partial V(x)}{\partial \tau_{a,i,j}} A_a(x) \xi_a = O(\delta_a^{2N}), \quad \text{for } i = 1, \ldots, k, \ j = 1, \ldots, N - 1.
\]
(5.36)
Using suitable rotation, we assume that \(\tau_{a,i,1} = (1, 0, \ldots, 0), \ldots, \tau_{a,i,N-1} = (0, \ldots, 0, 1, 0)\) and \(\nu_{a,i} = (0, \ldots, 0, 1)\). Under the condition (V), we know
\[
\frac{\partial V(\delta_a x + x^{(1)}_{a,i})}{\partial \tau_{a,i,j}} = \delta_a \sum_{l=1}^{N} \frac{\partial^2 V(x^{(1)}_{a,i})}{\partial x_l \partial \tau_{a,i,j}} x_l + \frac{\delta_a^2}{2} \sum_{m=1}^{N} \sum_{l=1}^{N} \frac{\partial^3 V(x^{(1)}_{a,i})}{\partial x_m \partial x_l \partial \tau_{a,i,j}} x_m x_l
\]
\[
+ \frac{\delta_a^3}{6} \sum_{s=1}^{N} \sum_{m=1}^{N} \sum_{l=1}^{N} \frac{\partial^4 V(x^{(1)}_{a,i})}{\partial x_s \partial x_l \partial x_m \partial \tau_{a,i,j}} x_s x_l x_m + o(\delta_a^3 | x |^3), \quad \text{in } B_{d\delta_a}(0).
\]
(5.37)
By (1.6), (5.4), (5.20), (5.28), and the symmetry of \(\varphi_j(x)\), we find, for \(j = 1, \ldots, N - 1\),
\[
\sum_{m=1}^{N} \sum_{l=1}^{N} \frac{\partial^3 V(x^{(1)}_{a,i})}{\partial x_m \partial x_l \partial \tau_{a,i,j}} \int_{B_{d\delta_a}(0)} A_a(\delta_a x + x^{(1)}_{a,i}) \bar{\zeta}_{a,i} x_m x_l
\]
\[
= B \gamma_{a,i,0} \frac{\partial \Delta V(x^{(1)}_{a,i})}{\partial \tau_{a,i,j}} + O(\delta_a^2) = O(|x^{(1)}_{a,i} - b_i|) + O(\delta_a^2) = O(\delta_a^{2k}).
\]
Also from Step 1, we get
\[
\sum_{s=1}^{N} \sum_{m=1}^{N} \sum_{l=1}^{N} \frac{\partial^4 V(x^{(1)}_{a,i})}{\partial x_s \partial x_l \partial x_m \partial \tau_{a,i,j}} \int_{B_{d\delta_a}(0)} A_a(\delta_a x + x^{(1)}_{a,i}) \bar{\zeta}_{a,i} x_s x_l x_m
\]
\[
= - 3B \sum_{q=1}^{N-1} \frac{\partial^2 \Delta V(b_i)}{\partial \tau_{a,i,j} \partial \tau_{a,i,j}} \gamma_{a,i,q} + o(1).
\]
By (A.2), we estimate
\[
\frac{\partial^2 V(x^{(1)}_{a,i})}{\partial x_l \partial \tau_{a,i,j}} = - \frac{\partial V(x^{(1)}_{a,i})}{\partial \nu_{a,i}} \kappa_{i,l} (x^{(1)}_{a,i}) \delta_{lj}, \quad l, j = 1, \ldots, N - 1.
\]
Since $\frac{\partial V(x^{(1)}_a)}{\partial \nu_a} = 0$, from (5.4), we find
\[
\frac{\partial^2 V(x^{(1)}_{a,i})}{\partial x_l \partial \tau_{a,i,j}} = \frac{B}{2a_s} \frac{\partial \Delta V(b_i)}{\partial \nu_i} \delta_a^{2} \kappa_{i,l}(b_i) \delta_{ij} + o(\delta_a^{2}).
\] (5.38)

Therefore from (5.28), (5.20) and (5.38), we get
\[
\sum_{l=1}^{N} \frac{\partial^2 V(x^{(1)}_{a,i})}{\partial x_l \partial \tau_{a,i,j}} \int_{B_{\delta_a}(0)} A_a(\delta_a x + x^{(1)}_{a,i}) \xi_{a,i} x_l = -\frac{B}{2} \frac{\partial \Delta V(b_i)}{\partial \nu_i} \delta_a^{2} \kappa_{i,l}(b_i) \gamma_{a,i,j} + o(\delta_a^{2}).
\] (5.39)

Combining (5.37)–(5.39), we obtain
\[
\int_{B_{\rho}(x^{(1)}_{a,i})} \frac{\partial V(x)}{\partial \tau_{a,i,j}} A_a(x) \xi_a = -\frac{B}{2} \frac{\partial \Delta V(b_i)}{\partial \nu_i} \delta_a^{N+3} \kappa_{i,j}(b_i) \gamma_{a,i,j} + \left(\sum_{l=1}^{N-1} \frac{\partial^2 \Delta V(b_i)}{\partial \tau_{i,l} \partial \tau_{i,j}} \gamma_{a,i,l}\right) \delta_a^{N+3} + o(\delta_a^{N+3}).
\] (5.40)

From (5.36) and (5.40), we find
\[
\frac{\partial \Delta V(b_i)}{\partial \nu_i} \kappa_{i,j}(b_i) \gamma_{a,i,j} + \left(\sum_{l=1}^{N-1} \frac{\partial^2 \Delta V(b_i)}{\partial \tau_{i,l} \partial \tau_{i,j}} \gamma_{a,i,l}\right) = o(1),
\]
which implies $\gamma_{a,i,j} = o(1)$ for $i = 1, \ldots, k$ and $j = 1, \ldots, N - 1$. \hfill \Box

Now we are ready to prove Theorem 1.6.

**Proof of Theorem 1.6:** First, for large fixed $R$, (5.24) and (5.25) give
\[
C_a(x) \geq \frac{1}{2}, \quad |g_a(x)| \leq C \sum_{i=1}^{k} \frac{\delta_a^{N+2s}}{|x - x^{(1)}_{a,i}|^{N+2s}}, \quad x \in \mathbb{R}^N \setminus \bigcup_{i=1}^{k} B_{\delta_a}(x^{(1)}_{a,i}).
\]

Using the comparison principle associated to the nonlocal operator $(-\Delta)^s$, we get
\[
\xi_a(x) = o(1), \quad x \in \mathbb{R}^N \setminus \bigcup_{i=1}^{k} B_{\delta_a}(x^{(1)}_{a,i}).
\]

On the other hand, it follows from (5.28), (5.30), Lemmas 5.10 and 5.11 that
\[
\xi_a(x) = o(1), \quad x \in \bigcup_{i=1}^{k} B_{\delta_a}(x^{(1)}_{a,i}).
\]

This is in contradiction with $\|\xi_a\|_{L^\infty(\mathbb{R}^N)} = 1$. So $u_{a}^{(1)}(x) \equiv u_{a}^{(2)}(x)$ for $a \to a_0$. \hfill \Box

**Appendix**

A. Basic estimates

**Lemma A.1** (c.f. [37]). Let $\alpha, \beta \geq 1$ be two constants. For any $0 < \delta \leq \min\{\alpha, \beta\}$, and $x_1 \neq x_2$, it holds that
\[
\frac{1}{(1 + |y - x_1|)^{\alpha}(1 + |y - x_2|)^{\beta}} \leq \frac{C}{|x_1 - x_2|^\delta}\left(\frac{1}{(1 + |y - x_1|)^{\alpha+\beta-\delta}} + \frac{1}{(1 + |y - x_2|)^{\alpha+\beta-\delta}}\right).
\]
Lemma A.2 (c.f. [25]). Let $\rho > \theta > 0$ be two constants. Suppose $(y - x)^2 + t^2 \geq \rho^2, t > 0$ and $\alpha > N$. Then, when $0 < \beta < N$, it holds that

$$\int_{\mathbb{R}^N} \frac{1}{(t + |z|)^\alpha |y - z - x|^\beta} \leq C \left( \frac{1}{(1 + |y - x|)^\beta t^{\alpha-N}} + \frac{1}{(1 + |y - x|)^{\alpha+\beta-N}} \right), \quad (A.1)$$

and

$$\int_{\mathbb{R}^N \setminus B_\delta(0)} \frac{1}{(t + |z|)^\alpha |y - z - x|^\beta} \leq \frac{C}{(1 + |y - x|)^\beta t^{\alpha-N}};$$

when $N < \beta$, we have that

$$\int_{\mathbb{R}^N \setminus B_\delta(y-x)} \frac{1}{(t + |z|)^\alpha |y - z - x|^\beta} \leq C \left( \frac{1}{(1 + |y - x|)^\beta t^{\alpha-N}} + \frac{1}{(1 + |y - x|)^{\alpha+\beta-N}} \right),$$

where $C > 0$ is a constant independent of $\theta$.

Remark A.3. For some $\epsilon \to 0$, in the case of $\beta < N$, (A.1) implies directly that

$$\int_{\mathbb{R}^N} \frac{1}{(t + |z|)^\alpha (1 + \frac{|y - z - x|}{\epsilon})^\beta} \leq C \epsilon^\beta \int_{\mathbb{R}^N} \frac{1}{(t + |z|)^\alpha |y - z - x|^\beta} \leq C \epsilon^\beta \left( \frac{1}{(1 + |y - x|)^\beta t^{\alpha-N}} + \frac{1}{(1 + |y - x|)^{\alpha+\beta-N}} \right).$$

While in the case of $\beta > N$, following the proof in [25], we can get indeed that,

$$\int_{\mathbb{R}^N} \frac{1}{(t + |z|)^\alpha (1 + \frac{|y - z - x|}{\epsilon})^\beta} \leq C \epsilon^\beta \left( \frac{1}{(1 + |y - x|)^\beta t^{\alpha-N}} + \frac{\epsilon^N}{(1 + |y - x|)^\alpha} \right) \leq C \epsilon^N \left( \frac{1}{(1 + |y - x|)^\beta t^{\alpha-N}} + \frac{1}{(1 + |y - x|)^\alpha} \right).$$

Now let $\Gamma \in C^2$ be a closed hypersurface in $\mathbb{R}^N$. For $y \in \Gamma$, let $\nu(y)$ and $T(y)$ denote respectively the outward unit normal to $\Gamma$ at $y$ and the tangent hyperplane to $\Gamma$ at $y$. The curvatures of $\Gamma$ at a fixed point $y_0 \in \Gamma$ are determined as follows. By a rotation of coordinates, we can assume that $y_0 = 0$ and $\nu(0)$ is the $x_N$-direction, and $x_j$-direction is the $j$-th principal direction. In some neighborhood $\mathcal{N} = \mathcal{N}(0)$ of 0, we have

$$\Gamma = \{ x : x_N = \varphi(x') \},$$

where $x' = (x_1, \cdots, x_{N-1})$,

$$\varphi(x') = \frac{1}{2} \sum_{j=1}^{N-1} \kappa_j x_j^2 + O(|x'|^3),$$

where $\kappa_j$ is the $j$-th principal curvature of $\Gamma$ at 0. The Hessian matrix $[D^2 \varphi(0)]$ is given by

$$[D^2 \varphi(0)] = \text{diag}[\kappa_1, \cdots, \kappa_{N-1}].$$

Suppose that $W$ is a smooth function, such that $W(x) = a$ for all $x \in \Gamma$.

Lemma A.4 (c.f. [30]). We have

$$\frac{\partial W(x)}{\partial x_l} \Big|_{x=0} = 0, \quad l = 1, \cdots, N-1,$$
\[
\frac{\partial^2 W(x)}{\partial x_m \partial x_l} \bigg|_{x=0} = -\frac{\partial W(x)}{\partial x_N} \bigg|_{x=0} \kappa_i \delta_{ml}, \text{ for } m, l = 1, \ldots, N - 1,
\]
where \(\kappa_1, \ldots, \kappa_{N-1}\) are the principal curvatures of \(\Gamma\) at 0.

**B. Linearization**

**Lemma B.1.** Let \(\xi = (\xi_1, \xi_2, \ldots, \xi_k)\) be the solution of the following problem

\[
(-\Delta)^s \xi_i + (1 - pU^{p-1})\xi_i = -\frac{p-1}{ka_s} U(x) \sum_{j=1}^{k} \int_{\mathbb{R}^N} U^p(x) \xi_j(x), \quad i = 1, \ldots, k. \tag{A.1}
\]

Then, there holds that

\[
\xi_i(x) = \sum_{j=0}^{N} \gamma_{i,j} \Psi_j,
\]

where \(\gamma_{i,j}\) are constants, and

\[
\Psi_0 = U + \frac{p-1}{2s} x \cdot \nabla U, \quad \Psi_j = \frac{\partial U}{\partial x_j}, \quad j = 1, \ldots, N. \tag{A.3}
\]

Moreover, \(\gamma_{i,0} = \gamma_{l,0}\) for \(i, l = 1, \ldots, k\).

**Proof.** Setting

\[\bar{L}(u) = (-\Delta)^s u + (1 - pU^{p-1})u\text{ and }\Psi^j = (\Psi_j, \ldots, \Psi_j),\]

then it is easy to find that \(\bar{L}(\Psi^j) = 0\) and \(\Psi^j\) solve (A.1) for \(j = 1, \ldots, N\). On the other hand, we denote

\[
\Psi = (U + \frac{p-1}{2s} x \cdot \nabla U, \ldots, U + \frac{p-1}{2s} x \cdot \nabla U),
\]

and then by checking the extension equation associated with (A.1), we could obtain that \(\Psi\) solves (A.1) too. Noting that \(\Psi, \Psi^j\) are linearly independent, we obtain (A.2).

Moreover, putting (A.2) into (A.1), we obtain that \(\gamma_{i,0} = \gamma_{l,0}\) for any \(i, l = 1, \ldots, k\).

\[\square\]

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