THE EDGE STATES OF THE BF SYSTEM 
AND 
THE LONDON EQUATIONS

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Abstract

It is known that the 3d Chern-Simons interaction describes the scaling limit of a quantum Hall system and predicts edge currents in a sample with boundary, the currents generating a chiral $U(1)$ Kac-Moody algebra. It is no doubt also recognized that in a somewhat similar way, the 4d $BF$ interaction (with $B$ a two form, $dB$ the dual $^*j$ of the electromagnetic current, and $F$ the electromagnetic field form) describes the scaling limit of a superconductor. We show in this paper that there are edge excitations in this model as well for manifolds with boundaries. They are the modes of a scalar field with invariance under the group of diffeomorphisms (diffeos) of the bounding spatial two-manifold. Not all of this group seem implementable by operators in quantum theory, the implementable group being a subgroup of volume preserving diffeos. The $BF$ system in this manner can lead to the $w_{1+\infty}$ algebra and its variants. Lagrangians for fields on the bounding manifold which account for the edge observables on quantization are also presented. They are the analogues of the $1+1$ dimentional massless scalar field Lagrangian describing the edge modes of an abelian Chern-Simons theory with a disk as the spatial manifold. We argue that the addition of “Maxwell” terms constructed from $F\wedge^* F$ and $dB\wedge^* dB$ do not affect the edge states, and that the augmented Lagrangian has an infinite number of conserved charges- the aforementioned scalar field modes- localized at the edges. This Lagrangian is known to describe London equations and a massive vector field. A $(3+1)$ dimensional generalization of the Hall effect involving vortices coupled to $B$ is also proposed.
1. INTRODUCTION

When a physical system undergoes spontaneous symmetry breakdown, its behavior at low energy and momentum is well approximated by the dynamics of a Nambu-Goldstone mode. The latter is a field valued in a homogeneous space $G/H$, $G$ being the Lagrangian symmetry group and $H$ the unbroken one.

The group $G$ is a gauged symmetry group for numerous physical systems. That is the case in electroweak theory which involves the spontaneous reduction of $SU(2) \times U(1)$ to the $U(1)$ group of electromagnetism. It is the case in superconductivity where the electromagnetic $U(1)$ breaks down to the discrete group $Z_2$.

Many years ago, it was pointed out by London that the essential phenomenology of superconductivity is captured by the constituent equation

$$\partial_\mu J_\nu - \partial_\nu J_\mu = \lambda F_{\mu\nu}, \quad \lambda = \text{constant}$$

relating the current $J_\mu$ to the electromagnetic field $F_{\mu\nu}$. The approach to superconductivity based on the Nambu-Goldstone mode incorporates this fundamental equation. Thus for superconductivity, the mode is a complex field $e^{i\varphi}$ of unit modulus and charge $2e$ and responds to the gauge transformation $A_\mu \to A_\mu + \frac{1}{e} \partial_\mu \Lambda$ according to $e^{i\varphi} \to e^{2i\Lambda} e^{i\varphi}$. If $\langle H \rangle$ is the (real) vacuum value of the Higgs or order parameter field $H$, then the current

$$J_\mu = 4ie\langle H \rangle^2 e^{-i\varphi} D_\mu e^{i\varphi},$$

$$D_\mu = \partial_\mu - 2ieA_\mu$$

of the Landau-Ginsburg Lagrangian in the London limit is gauge invariant. Furthermore, the London ansatz is an identity with $\lambda = 8e^2 \langle H \rangle^2$.

There is an alternative approach to the London ansatz which leads to the $BF$ system and which is of particular interest in this paper. It begins with the remark that $J_\mu$ fulfills
the continuity equation

$$\partial^\mu J_\mu = 0 \quad (1.3)$$

in addition to (1.1). (Our metric has signature $-,+,+,+,$.) Its expression in (1.2) based on the Nambu-Goldstone field can be thought of as solving (1.1) as an identity and obtaining (1.3) as a field equation from the Lagrangian

$$\int d^3x \left\{ -\langle H \rangle^2 \left( D^\mu e^{i\varphi} \right)^* \left( D_\mu e^{i\varphi} \right) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right\}. \quad (1.4)$$

In the alternative approach, we solve the continuity equation instead as an identity by setting

$$J_\mu = -\epsilon_{\mu\nu\lambda\rho} \partial^\nu B^{\lambda\rho} \quad (1.5)$$

where the convention $\epsilon^{0123} = +1$ is adopted for the Levi-Civita symbol $\epsilon^{\mu\nu\lambda\rho}$. The constituent equation is then obtained as a field equation from the Lagrangian

$$L = \int d^3x \mathcal{L},$$

$$\mathcal{L} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} B_{\mu\nu} F_{\lambda\rho} - \frac{1}{3 \lambda} H^{\mu\nu\lambda} H_{\mu\nu\lambda} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu},$$

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}. \quad (1.6)$$

Lagrangians of this sort have come into prominence in modern times in connection with topological field theories[3, 4] and exotic statistics[5, 6], as a method to generate mass for gauge fields distinct from the Higgs mechanism[7] and in connection with quantum hair for black holes[8]. For spatial manifolds devoid of boundaries, the classical and quantum aspects of $L$ have been developed to an advanced level particularly by Allen et al.[7]. In this paper, we propose to investigate (1.6) when the underlying manifold $\Sigma$ has a boundary. It can for instance be a three dimensional ball, a solid torus, or a solid cylinder. For reasons of simplicity, we will exclusively consider these examples and assume also that $\Sigma$ is a submanifold of $\mathbb{R}^3$ (with the metric induced from the Euclidean metric of $\mathbb{R}^3$ and with local Cartesian coordinates $x^i$). We will show that there are edge
states localized at the boundary of $\Sigma$ which are similar to the edge states in the quantum Hall system[8-11]. There is for example a diffeomorphism (diffeo) group (or a central extension thereof) acting on them just as the Virasoro group[13] acts on the Hall edge states[3, 9, 12].

Our central attention in this paper is focussed on these edge states. Their basic features are not sensitive to the dynamics in the interior $\Sigma^0$ of $\Sigma$. It is therefore possible, although not necessary, to assume the minimum energy configuration in $\Sigma^0$ with no essential loss of content. With this assumption, the edge states are described by the $BF$ system [4, 5, 3] with the Lagrangian

$$L^* = \int d^3x L^*,$$
$$L^* = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} B_{\mu\nu} F_{\lambda\rho}.$$ (1.7)

In this way, we discover that the edge states are well accounted for by a topological field theory.

In Section 2, we briefly describe the canonical formalism for (1.6) and its edge observables. The important result that these observables form an infinite number of constants of motion, with a relation to gauge transformations similar to that of charges in gauge theories, is also established. The existence of such an infinite number of constants of motion (or edge observables) depends only on gauge invariance and the presence of the boundary $\partial \Sigma$, and not on the specific Lagrangian (1.6). We also display the canonical generators of the aforementioned diffeo group and establish their Sugawara form[13]. Next, in Section 3, we pull back or restrict the canonical formalism to the part of the phase space with zero energy in $\Sigma^0$. It results in the $BF$ Lagragian (1.7). All these activities are as yet classical.

Section 4 quantizes the basic edge observables of Section 2 for $\Sigma$ a ball or a solid torus. (The solid cylinder has a special interest, having an association with the $w_{1+\infty}$ algebra[14]. It is hence separately discussed in Section 5, although much of what is did
here applies there as well.) They are described by an infinite number of independent harmonic oscillators which can be associated with the modes of a scalar field. There is also a Lagrangian defined on $\partial \Sigma$ which leads to these excitations, namely

$$L^2 \int_{\partial \Sigma} \partial_t \Psi dC, \quad L = \text{constant}$$

(1.8)

where $\Psi$ is a scalar field and $C$ a one form. It is the precise analogue of the Lagrangian

$$\pm L^2 \int_{S^1} \partial_t \varphi \partial_\theta \varphi d\theta$$

(1.9)

which describe the edge observables for the Chern-Simons field on a disc $D$ with the circle $S^1$ as its boundary $\partial D$.

Now there is much to be said about (1.8). It has for instance to be quantised. There is also a second order Lagrangian which leads to (1.8) just as the Lagrangian

$$L_i^3 \int \left[ (\partial_t \varphi)^2 - \left( \frac{1}{L^2} \partial_\theta \varphi \right)^2 \right] d\theta, \quad L_i = \text{constants}$$

(1.10)

leads to (1.9) on chiral decomposition. We will discuss these matters elsewhere[15]. In this paper, we will instead discuss the diffeo group acting on the edge states in a little detail. We have alluded to this group before, while its existence is also suggested by the invariance of (1.8) under the diffeos of $\partial \Sigma$.

The following point is worthy of note in this context. The association of edge observables with a quantum scalar field is not unique. This is because the definition of the latter involves the choice of a dispersion relation connecting frequency and mode labels. It is with its help that we normally define creation and annihilation operators and a Fock space.

[ A more general method of quantisation involving the choice of a complex structure and a metric on the phase space of fields[16] is also possible. For our present purposes, it is enough to consider the method based on dispersion relation.] This dispersion relation is not given by (1.6), (1.7) or (1.8) and must be supplied externally. [But the second order Lagrangian analogous to (1.10) does provide adequate data for quantisation, as we shall...]

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discuss in [17]. An analogous condition prevails in the Chern-Simons problem[9, 10], where too the quantum edge field can be shown to be indeterminate without the dispersion relation as a new input. The latter can be constructed (although still not uniquely) by the insistence that the Virasoro algebra acts properly on the Fock space of the scalar field.

But a similar approach does not quite work here, for (1.6), (1.7) or (1.8). This is because, in so far as we can tell, it is impossible to implement the Lie algebra of the entire \( \partial \Sigma \) diffeo group (or a possible central extension thereof) on a Fock space. (We will not be too careful in the Introduction to distinguish a diffeo group from its central extension.) We argue that the implementable diffeos are those preserving a volume form \( \mu \) on \( \partial \Sigma \) (at least for simple choices \( \partial \Sigma \)). This group of diffeos is often denoted by \( SDiff(\partial \Sigma) \) (and its identity component by \( SDiff_0(\partial \Sigma) \)) in the literature[18]. It preserves the Lagrangian

\[
L = \frac{1}{2} \int_{\partial \Sigma} \mu \left( (\partial_t \varphi)^2 - \omega_0^2 \varphi^2 \right)
\]  

(1.11)

of the field \( \varphi \), \( \omega_0 \) being a positive constant. The canonical treatment of (1.11) leads to essentially all our edge states and implementable infinitesimal diffeos. Thus, (1.11) also may provide a satisfactory account of our edge states. The constant \( \omega_0 \) in (1.11) can not be determined by any reasonable consideration based on (1.6) or (1.7), in the same way that the speed of the field in (1.10) can not be inferred from the Chern-Simons action.

The Lagrangian (1.11) is the two dimensional analogue of the Einstein model for specific heat in one dimension[19].

Section 5 deals with the solid cylinder for which \( \partial \Sigma = S^1 \times \mathbb{R}^1 \). In this case, with \( \mu \) a rotationally and translationally invariant volume form, \( SDiff_0(\partial \Sigma) \) is the group with the algebra \( w_{1+\infty} \) of conformal field theories[14]. The group which occurs in quantum theory is perhaps not this one, but a central extension thereof, but we have not done the necessary calculations to verify this possibility.

The fields \( A \) and \( B \) naturally admit two kinds of sources, namely point charges and
vortices or strings. Elsewhere, the $BF$ Lagrangian with such sources has been studied in investigations of exotic statistics\cite{5,6}. In a paper under completion\cite{15}, we will demonstrate that like the Chern-Simons sources, these sources also get ‘framed’, or rather acquire spin degrees of freedom, on regularization. Vertex operators for the creation of these sources, similar to the Fubini-Veneziano vertex operator \cite{13}, will be constructed and a spin-statistics theorem for vortices connecting a certain exchange and a $2\pi$ rotation of the spin variables alluded to above will be established. It must be remarked that neither this exchange nor this rotation are those appropriate for the conventional spin-statistics theorem so that the result shown here is genuinely novel. As for the conventional theorem, that too can be proved, in fact easily and without recourse to relativistic quantum fields, as will be seen in that paper.

The quantum Hall edge states have a simple physical interpretation\cite{12}. There is an analogue of the Hall effect for vortices coupled to $B$ and we shall explain it in Section 6. It is our expectation that the edge states of (1.6) or (1.7) will find a similar interpretation in the context of this phenomenon. Such an interpretation will be helpful for the observation of these states and merits attention.

A basic physical issue not addressed in this paper and \cite{15} concerns the reproducibility of their results in the Higgs field description. Its answer seems affirmative. We plan to report on this matter, and on the nonabelian version of the preceding edge states and sources, in future publications.

In references \cite{9,10}, the edge states of the Chern-Simons Lagrangian were considered in the absence of the “Maxwell” or “kinetic energy” term for the $U(1)$ Chern-Simons potential $A$ [the analogue of the term in $L$ from the last term in $\mathcal{L}$ of (1.6)] and of the similar term for its nonabelian counterpart. It is possible however to generalize that work by including these terms and proceeding along the lines of this paper. This task has been carried out by the authors of \cite{9,10} in unpublished work.
2. THE CANONICAL FORMALISM

We follow Dirac[20] for the canonical treatment of (1.6).

Let $\pi_\mu$ and $P_{\mu\nu}$ be the momenta conjugate to $A_\mu$ and $B_{\mu\nu}$. The Legendre map of (1.6) gives the primary constraints

$$\pi_0 \approx 0, \quad P_{0i} \approx 0, \quad \text{(2.1)}$$

and the expressions

$$\pi_i = F_{oi} + \epsilon_{ijk} B_{jk},$$

$$P_{ij} = \frac{4}{\lambda} H_{0ij} \quad \text{(2.2)}$$

where

$$\epsilon_{ijk} = \epsilon^{0ijk}, \quad 1 \leq i, j \leq 3,$$

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu} \quad \text{(2.3)}$$

and $\approx$ stands for weak equality.

The Hamiltonian is

$$H = \int d^3x \left[ \frac{1}{2} [\pi_i - \epsilon_{ijk} B_{jk}]^2 + \frac{\lambda}{16} P_{ij}^2 + \frac{1}{4} F_{ij}^2 + \frac{1}{3\lambda} H_{ijk}^2 \right.$$

$$- A_0 \partial_i \pi_i - B_{0i}(\partial_j P_{ji} + 2\epsilon_{ijk} \partial_j A_k)$$

$$\left. + \psi^0 \pi_0 + \psi^i P_{0i} \right], \quad \text{(2.4)}$$

$\psi^0$ and $\psi^i$ being Lagrange multiplier fields. It leads to the secondary constraints

$$\partial_i \pi_i \approx 0, \quad \text{(2.5)}$$

$$\partial_j P_{ji} + 2\epsilon_{ijk} \partial_j A_k \approx 0. \quad \text{(2.6)}$$

There are no tertiary constraints.
The constraints are all first class. Of these, (2.1) eliminate $A_0$ and $B_{0i}$ as observables. They will henceforth be ignored along with $\pi_0$ and $P_{0i}$, as is permissible.

The Gauss law constraints (2.5) and (2.6) require delicate treatment on manifolds with boundary for reasons outlined in [9]. Following that work, we first rewrite them in the form

$$G_0(\lambda(0)) = \int_\Sigma d^3x \lambda(0) \partial_i \pi \approx 0, \quad (2.7)$$

$$G_1(\lambda(1)) = \int_\Sigma d^3x \lambda_i^{(1)} [\partial_j P_{ji} + 2\epsilon_{ijk} \partial_j A_k] \approx 0. \quad (2.8)$$

The allowed class of “test functions” $\lambda(0)$ and $\lambda_i^{(1)}$ are then fixed by requiring that (2.7) and (2.8) are differentiable in the fields $\pi_i$, $P_{lk}$ and $A_m$. Thus, consider the variations

$$\delta G_0(\lambda(0)) = \int_\Sigma d^3x \lambda(0) \partial_i \delta \pi_i = \int_{\partial\Sigma} d^2x \lambda(0) n^i \delta \pi_i - \int_\Sigma d^3x \partial_i \lambda(0) \delta \pi_i, \quad (2.9)$$

$$\delta G_1(\lambda(1)) = \int_\Sigma d^3x \lambda_i^{(1)} [\partial_j \delta P_{ji} + 2\epsilon_{ijk} \partial_j \delta A_k] = \int_{\partial\Sigma} d^2x \lambda_i^{(1)} n^j [\delta P_{ji} + 2\epsilon_{ijk} \delta A_k] - \int_\Sigma d^3x \partial_j \lambda_i^{(1)} [\delta P_{ji} + 2\epsilon_{ijk} \delta A_k], \quad (2.10)$$

$n^j$ defining the outward drawn normal to $\partial\Sigma$. The functionals $G_j$ are differentiable only if the boundary terms (first terms) in (2.9) and (2.10) are absent. In this way, we are led to the conditions

$$\lambda(0) |_{\partial\Sigma} = 0, \quad \vec{n} \times \vec{\lambda}^{(1)} |_{\partial\Sigma} = 0. \quad (2.11)$$

It is better to write (2.7) and (2.8) entirely in terms of forms. Thus let

$$B \equiv B_{ij} dx^i dx^j = \frac{1}{2} \epsilon_{ijk} \pi_k dx^i dx^j,$$

$$A \equiv A_i dx^i = \left[ A_i - \frac{1}{4} \epsilon_{ijk} P_{jk} \right] dx^i,$$

$$\lambda^{(1)} = \lambda_i^{(1)} dx^i. \quad (2.12)$$
(The wedge symbols between differential forms are being omitted.) Then

\[ G_0(\lambda^{(0)}) = \int_{\Sigma} \lambda^{(0)} dB \approx 0, \]
\[ G_1(\lambda^{(1)}) = 2 \int_{\Sigma} \lambda^{(1)} dA \approx 0 \quad (2.13) \]

if \( \lambda^{(0)}, \lambda^{(1)} \) belong to the test function space \( I^{(0)}, I^{(1)} \), defined by the conditions

\[ \lambda^{(0)} \mid_{\partial \Sigma} = 0, \quad \lambda^{(1)} \mid_{\partial \Sigma} = 0. \quad (2.14) \]

Here by the notation \( \lambda^{(N)} \mid_{\partial \Sigma} = 0 \) for an \( N \) form \( \lambda^{(N)} \), we mean the pull back of \( \lambda^{(N)} \) to \( \partial \Sigma \).

Next consider

\[ q(w^{(1)}) = \int_{\Sigma} w^{(1)} B, \]
\[ p(w^{(2)}) = -\int_{\Sigma} w^{(2)} A \quad (2.15) \]

where \( w^{(j)} \) are closed \( j \) forms:

\[ dw^{(j)} = 0. \quad (2.16) \]

(2.13) and (2.16) are differentiable in \( \pi_k, P_{jk} \) and \( A_t \) even if

\[ w^{(j)} \mid_{\partial \Sigma} \neq 0. \quad (2.17) \]

Their Poisson brackets (PB’s) with the constraints vanish. (All PB’s are at equal times.) For example, on using

\[ \{ B_{ij}(x), A_k(y) \} = -\frac{1}{2} \epsilon_{ijk} \delta^3(x - y), \quad (2.18) \]

we get

\[ \{ p(w^{(2)}), G_0(\lambda^{(0)}) \} = \int_{\Sigma} w^{(2)} d\lambda^{(0)} \]
\[ = \int_{\partial \Sigma} w^{(2)} \lambda^{(0)} = 0 \quad (2.19) \]

by (2.14). Thus (2.17) are observables.
They are furthermore constants of motion for the Hamiltonian \( (2.4) \). This can be shown using the PB’s

\[
\begin{align*}
\{q(w^{(1)}), A_i\} &= -w_i^{(1)}, \\
\{q(w^{(1)}), \pi_i\} &= 0, \\
\{q(w^{(1)}), B_{ij}\} &= 0, \\
\{q(w^{(1)}), P_{ij}\} &= 0, \\
\{p(w^{(2)}), A_i\} &= 0, \\
\{p(w^{(2)}), \pi_i\} &= -\epsilon_{ijk}w^{(2)}_{jk}, \\
\{p(w^{(2)}), B_{ij}\} &= -w_{ij}^{(2)}, \\
\{p(w^{(2)}), P_{ij}\} &= 0 \\
\end{align*}
\]

(2.20)

where \( w^{(1)} = w^{(1)}_i dx^i \) and \( w^{(2)} = w^{(2)}_{jk} dx^j dx^k \) with an antisymmetric \( w^{(2)}_{jk} \). Thus our dynamical system has an infinite number of constants of motion. They are the exact analogues of charges in conventional gauge theories, with a relation to Gauss laws similar to those of charges \([9]\).

Note next that since \( q(w^{(1)} + d\lambda^{(0)}) \approx q(w^{(1)}), p(w^{(2)} + d\lambda^{(1)}) \approx p(w^{(2)}) \) in view of (2.13) and (2.14), test functions differing by \( d\lambda^{(j-1)} \) define equivalent (or weakly equal) observables.

Actually, if a \( \lambda^{(j-1)} \) modulo a closed form belongs to \( I^{(j-1)} \), then \( w^{(j)} \) and \( w^{(j)} + d\lambda^{(j-1)} \) define equivalent observables. This is because a closed form drops out of \( d\lambda^{(j-1)} \).

When \( \Sigma \) is a ball \( B_3 \), a closed \( w^{(j)} \) is also exact. In this case, \( w^{(j)} = d\xi^{(j-1)} \) with \( \xi^{(j-1)} \) and \( \xi^{(j-1)} + \lambda^{(j-1)} \) giving equivalent observables. An observable is consequently sensitive only to \( \xi^{(j-1)}|_{\partial \Sigma} \) and can be regarded as localized at the edge. For suitable choices of \( \xi^{(j-1)} \), they are also localized on an arbitrary contractible open set at the edge.
The following PB’s give the fundamental classical algebra of these edge excitations:

\[
\{ q(d\xi(0)), q(d\bar{\xi}(0)) \} = \{ p(d\xi(1)), p(d\bar{\xi}(1)) \} = 0,
\]

\[
\{ q(d\xi(0)), p(d\xi(1)) \} = \int_{\Sigma} d\xi(0) d\xi(1)
= \int_{\partial \Sigma} \xi(0) d\xi(1).
\] (2.21)

Next consider the solid torus \( T_3 \). For \( T_3 \), all \( w^{(2)} \) are exact, \( w^{(2)} = d\xi^{(1)} \). As for \( w^{(1)} \), consider \( \partial T_3 = \) the two-torus \( T^2 \). It has coordinates \( \theta^1, \theta^2 \) with \( \theta^i \) and \( \theta^i + 2\pi \) being identified. Let the cycle obtained by increasing \( \theta^2 \) by \( 2\pi \) (with fixed \( \theta^1 \)) be the one contractible to a point by shrinking it within \( T_3 \). Then \( d\theta^1 \) can be extended from \( \partial T_3 \) to \( T_3 \) as a globally defined closed but inexact one form on \( T_3 \). Any closed one form on \( T_3 \) is a linear combination of such an extension and an exact one form \( d\xi^{(0)} \). Just as for the ball \( B_3 \), the observables \( q(d\xi^{(0)}) \) and \( p(d\xi^{(1)}) \) are localized on \( \partial \Sigma \) and fulfill (2.21). For suitable choices of \( \xi^{(j-1)} \), they are also localized on an arbitrary contractible open set on \( \partial \Sigma \). As for \( q(d\theta^1) \), it too gives an observable living only on \( \partial \Sigma \) since \( q(d\theta^1) \approx q(d\theta^1 + d\lambda^{(0)}) \). But \( q(\theta^1) \), or \( q(d\theta^1 + d\xi^{(0)}) \) for any choice of \( \xi^{(0)} \), are not localized on a contractible open set on \( \partial \Sigma \), \( d\theta^1 + d\xi^{(0)} \) being cohomologically nontrivial. \( q(d\theta^1) \) has zero PB with \( q(d\xi^{(1)}) \) while

\[
\{ q(d\theta^1), p(d\xi^{(1)}) \} = \int_{\partial \Sigma} \xi^{(1)} d\theta^1.
\] (2.22)

The generators of diffeos of \( \partial \Sigma \) will now be examined. Let \( \eta = \eta^i \partial_i \) be a vector field on \( \Sigma \) tangent to \( \partial \Sigma \) at \( \partial \Sigma \). It generates a diffeo \( \Sigma \rightarrow \Sigma \) and acts on fields by the Lie derivative \( L_\eta \). For example,

\[
L_\eta(\xi^{(0)}) = \eta^i \partial_i \xi^{(0)},
\]

\[
L_\eta(\xi^{(1)}) = \left[ \partial_i(\xi_j^{(1)} \eta^j) + \eta^j (\partial_j \xi_i^{(1)} - \partial_i \xi_j^{(1)}) \right] dx^i,
\]

\[
L_\eta \bar{\eta}^i \partial_j = \left[ \eta^i \partial_i, \bar{\eta}^j \partial_j \right].
\] (2.23)

We want to construct an observable \( l(\eta) \) depending on \( \eta \) with the following properties:

1) Its canonical action via PB’s is that of an infinitesimal diffeo on the edge observables:
\[ \begin{align*}
\{ l(\eta), q(w^{(1)}) \} &= q \left( \mathcal{L}_\eta w^{(1)} \right), \\
\{ l(\eta), p(w^{(2)}) \} &= p \left( \mathcal{L}_\eta w^{(2)} \right). 
\end{align*} \] (2.24)

2) It becomes a constraint if the restriction \( \eta|_{\partial \Sigma} \) of \( \eta \) to \( \partial \Sigma \) is zero. With this requirement, it becomes also an edge observable, which is satisfactory as we are after a theory of edge observables insensitive to \( \Sigma^0 \).

It is remarkable that such an \( l \) exists. It is

\[ l(\eta) = \int (\mathcal{L}_\eta A)B. \] (2.25)

For showing that \( l(\eta) \) has the correct properties, the following identities are useful: Let \( i_\eta \) be the contraction on the vector field \( \eta \) so that for example \( i_\eta A_j dx^j = \eta^j A_j \). Then we have the identity [21]

\[ \mathcal{L}_\eta = di_\eta + i_\eta d \] (2.26)

Therefore, if \( w = w_{ijk} dx^i dx^j dx^k \) is a three form, \( w_{ijk} \) being totally antisymmetric,

\[ \int \mathcal{L}_\eta w = \int (di_\eta + i_\eta d)w = \int_{\partial \Sigma} i_\eta w \]

by Stokes theorem as \( dw = 0 \). Now \( i_\eta w \) in the last integral is \( \eta^i w_{ijk} dx^j dx^k \), the differentials being tangent to \( \partial \Sigma \). Since \( \eta|_{\partial \Sigma} \) is also similarly tangent, we have \( i_\eta w|_{\partial \Sigma} = 0 \). The basic result

\[ \int \Sigma \mathcal{L}_\eta w = 0 \] (2.27)

and its consequence

\[ \int \Sigma (\mathcal{L}_\eta \alpha) \beta = - \int \alpha \mathcal{L}_\eta \beta, \quad \alpha = \alpha_i dx^i, \beta = \beta_{ij} dx^i dx^j \] (2.28)

thus follow.
(2.28) helps us to show the differentiability of $l(\eta)$ in all the dynamical fields. Thus if $A$ for instance is varied,

$$\delta l(\eta) = \int_{\Sigma} (\mathcal{L}_{\eta} \delta A) B = -\int_{\Sigma} \delta A \mathcal{L}_{\eta} B,$$

which shows its differentiability in $A$.

We must now check that $l(\eta)$ is an observable having properties 1) and 2). It is an observable if it is weakly invariant under the gauge transformations generated by $G_{i}$. The best way to verify this may be as follows. A transformation due to $G_{0}(\lambda^{(0)})$ for instance converts $A$ to $A + d\lambda^{(0)}$. Using $\mathcal{L}_{\eta}d = d\mathcal{L}_{\eta}[21]$, we therefore have

$$\{G_{0}(\lambda^{(0)}), l(\eta)\} = \int_{\Sigma} d(\mathcal{L}_{\eta}\lambda^{(0)}) B$$
$$= -\int_{\Sigma} \mathcal{L}_{\eta}\lambda^{(0)} dB \approx 0$$

(2.30)

as $\mathcal{L}_{\eta}\lambda^{(0)}|_{\partial \Sigma} = 0$. In a similar way, we can show its weak invariance under $B \rightarrow B + d\lambda^{(1)}$, $\lambda^{(1)} \in \mathcal{I}^{(1)}$.

As for 1), it follows easily from (2.18). Only 2) now remains. Let $\bar{\eta}$ be a vector field vanishing at $\partial \Sigma$. Then

$$l(\bar{\eta}) = \int_{\Sigma} [(di_{\bar{\eta}} + i_{\bar{\eta}}d)A] B$$
$$= -\int_{\Sigma} (i_{\bar{\eta}} A) dB + \int_{\Sigma} B i_{\bar{\eta}} dA$$
$$\approx 0,$$

(2.31)

the two terms being constraints (of types $G_{0}$ and $G_{1}$).

We have now established that $l(\eta)$ is an edge observable creating edge diffeos.

The Fourier analysis of edge observables is important for quantization and useful for establishing that $l(\eta)$ has the generalized Sugawara form. For performing this analysis, we must first fix a volume form $\mu$ on $\partial \Sigma$. For $\partial \Sigma = S^{2}$, it can for instance be constant $\times d(cos \theta)d\varphi$, with $\theta$ and $\varphi$ being polar and azimuthal angles. For $\partial \Sigma = T^{2}$, it
can be constant \( \times d\theta^1 d\theta^2 \), \( \theta^i \) (mod \( 2\pi \)) being angular coordinates. The choice of \( \mu \) is not of course unique. It defines a Hilbert space \( \mathcal{H} = L^2(\mu, \partial\Sigma) \) of square integrable functions with respect to \( \mu \). Thus, if \( \psi, \chi \in \mathcal{H} \), then

\[
(\psi, \chi) \equiv \int_{\Sigma} \mu \psi^* \chi < \infty. \tag{2.32}
\]

Let \( e_n \) be an orthonormal basis of smooth functions for \( \mathcal{H} \) with \( e_0 \) the constant function:

\[
(e_n, e_m) = \delta_{nm}, \ e_0 = \text{constant}. \tag{2.33}
\]

For \( S^2 \), for

\[
\mu = \frac{A}{4\pi^2} d(\cos \theta) d\varphi \tag{2.34}
\]

[with \( \theta \) and \( \varphi \) being the usual polar and azimuthal angles], we can assume the correspondences

\[
n \to J_m,
\]

\[
e_n \to \left(\frac{4\pi}{A}\right)^{1/2} Y_{jm}, \ Y_{jm} = \text{Spherical harmonics}, \tag{2.35}
\]

\( e_0 \) becoming \( \left(\frac{4\pi}{A}\right)^{1/2} \). For \( T^2 \), for

\[
\mu = \frac{A}{4\pi^2} d\theta^1 d\theta^2, \tag{2.36}
\]

we can assume the correspondences

\[
n \to \vec{N} \equiv (N_1, N_2),
\]

\[
e_n(\theta^1, \theta^2) \to e_{\vec{N}} = \frac{1}{\sqrt{A}} e^{i\vec{N} \cdot \vec{\theta}},
\]

\[
\vec{N} \cdot \vec{\theta} \equiv N_1 \theta^1 + N_2 \theta^2, \ N_i \in \mathbb{Z}, \tag{2.37}
\]

\( e_0 \) becoming \( \frac{1}{\sqrt{A}} \). In both these cases, \( A \) is the area of \( \partial \Sigma \):

\[
\int_{\partial\Sigma} \mu = A. \tag{2.38}
\]
With this setup, the observables can be Fourier analyzed. Consider first $q(d\xi^{(0)})$. All $d\xi^{(0)}$ with equal boundary value $\xi^{(0)}|_{\partial \Sigma}$ give equivalent observables. Let $\langle q(d\xi^{(0)}) \rangle$ denote this equivalence class. We will hereafter call it as a single ($q$ type) observable. It depends only on $\xi^{(0)}|_{\partial \Sigma}$. A basis of such $q$ type observables is thus obtained by choosing $\xi^{(0)}|_{\partial \Sigma}$ to be $e_n$:

$$q_n = \langle q(d\xi^{(0)}_n) \rangle, \quad \xi^{(0)}_n|_{\partial \Sigma} = e_n, \ n \neq 0. \quad (2.39)$$

$q(d\xi^{(0)}_0)$ here is the null observable. This is because $\xi^{(0)}_0|_{\partial \Sigma}$, being a constant, can be extended as a constant function to all of $\Sigma$, and in that case $d\xi^{(0)}_0 = 0$. It is for this reason that we have excluded $n = 0$ from (2.39).

In addition to $q_n$, there is also one more $q$ type observable $Q$. It is the equivalence class of observables weakly equal to $1/\sqrt{A}q(d\theta^1)$.

$$Q = \langle q(d\xi^{(0)}) \rangle. \quad (2.40)$$

Note that $Q \neq \langle q(d\xi^{(0)}_0) \rangle$, the equivalence class of observables weakly equal to $q(d\xi^{(0)}_0)$.

As for $p(d\xi^{(1)})$, let $\langle p(d\xi^{(1)}) \rangle$ denote the equivalence of weakly equal observables, all with the same $d\xi^{(1)}|_{\partial \Sigma}$. We will hereafter call it a single ($p$ type) observable. We can choose $e^*_n \mu$ ($n \neq 0$) for $d\xi^{(1)}|_{\partial \Sigma}$ to obtain a class of $p$ type observables, $e^*_n$ being the complex conjugate of $e_n$. This is because $e^*_n \mu$ integrates to zero on $\partial \Sigma$ if $n \neq 0$, and for $S^2$ or $T^2$, this means that it is exact, $e^*_n \mu = d\chi^{(1)}_n$. But $\chi^{(1)}_n$ can always be extended (in fact in many ways) from $\partial \Sigma$ to a form $\xi^{(1)}_n$ on $\Sigma$. Hence the choice $\xi^{(1)}_n$ for $\xi^{(1)}$ gives us $e^*_n \mu$ for $d\xi^{(1)}_n|_{\partial \Sigma}$. We can thus take

$$p_n = \langle p(d\xi^{(1)}_n) \rangle, \quad n \neq 0 \quad (2.41)$$

as a class of $p$ type observables.

The observables $(2.41)$ form a basis for $p$ type observables for a ball. For a solid torus, to obtain a basis for $p$ type observables, we must also consider, for example,

$$p(d\psi^{(1)})$$
\begin{equation}
\psi^{(1)} = \text{Any one form fulfilling } \psi^{(1)}|_{\partial \Sigma} = -\frac{\sqrt{A}}{4\pi^2} d\theta^2.
\end{equation}

Let $P$ denote the equivalence class of observables of the type (2.42), all with the same $\psi^{(1)}|_{\partial \Sigma}$. Then $P$ and $p_n$ form a basis for $p$ type observables for $T_3$. The observable $P$ is missing from the set $\{p_n\}$ in (2.41) because $d\psi^{(1)}|_{\partial \Sigma} = 0$. [Note that $P \neq \langle p(d\xi^{(1)}_0)\rangle$, the equivalence class of observables weakly equal to $p(d\xi^{(1)}_0)$.] The PB’s of these observables are determined by (2.21) and (2.22). The nonzero PB’s which involve them are given by

\begin{align*}
\left\{ q(d\xi^{(0)}_m), p(d\xi^{(1)}_m) \right\} &= \int_{\partial \Sigma} \xi^{(0)}_m d\xi^{(1)}_n = \delta_{nm}, \\
\left\{ \frac{1}{2\pi} q(d\theta^1), p(d\psi^{(1)}) \right\} &= \frac{1}{4\pi^2} \int_{\partial \Sigma} d\theta^1 d\theta^2 = 1
\end{align*}

and read

\begin{equation}
\left\{ q_m, p_n \right\} = \delta_{mn},
\end{equation}

\begin{equation}
\left\{ Q, P \right\} = 1.
\end{equation}

There is an interpretation of the observables $Q$ and $P$ which will be briefly alluded to here, a fuller discussion being reserved for ref. [15]. The observable $Q$ is associated with the operator which creates magnetic flux loops which loops are homologous to the cycle on $\partial \Sigma$ obtained by varying $\theta^2$ from 0 to $2\pi$ with fixed $\theta^1$. The value of the observable $P$ is a measure of the flux on these loops.

Now for the Fourier components of the equivalence class $\langle l(\eta) \rangle$ of observables, all with the same $\eta|_{\partial \Sigma}$, we adopt the choices (2.34)-(2.38). We may then set

\begin{equation}
\eta = \eta_{J_m,\alpha}, \tag{2.44}
\end{equation}

\begin{equation}
\eta_{J_m,\alpha}|_{\partial \Sigma} = Y_{J_m} L_{\alpha} \tag{2.45}
\end{equation}

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for $S^2$, $L_\alpha$ being the angular momentum generators:

$$L_{\alpha} \mu = 0, \quad [L_\alpha, L_\beta] = \epsilon_{\alpha\beta\gamma} L_\gamma.$$ (2.46)

(2.43) is permissible as its right hand side can be extended to all of $\Sigma$. The corresponding $\langle l(\eta_{J,m,\alpha}) \rangle$’s are

$$l_{J,m,\alpha} = \langle l(\eta_{J,m,\alpha}) \rangle.$$ (2.47)

For $\partial \Sigma = T^2$, we can analogously set

$$\eta = \eta_{\vec{N},j}$$

$$\eta_{\vec{N},j} \big|_{\partial \Sigma} = \epsilon^{i\vec{N} \cdot \vec{\theta}} \frac{\partial}{\partial \theta^j},$$

$$\mathcal{L} \frac{\partial}{\partial \theta^j} \mu = 0,$$

$$l_{\vec{N},j} = \langle l(\eta_{\vec{N},j}) \rangle.$$ (2.48)

We next give the PB’s involving $l_{\vec{N},j}$. Those involving (2.47) are also straightforward to derive, but involve Clebsch-Gordan coefficients and are omitted for simplicity. We have,

$$\{l_{\vec{N},j}, q_{\vec{M}}\} = i M_j q_{\vec{M} + \vec{N}},$$

$$\{l_{\vec{N},j}, Q\} = \delta_{j1} q_{\vec{N}},$$

$$\{l_{\vec{N},j}, p_{\vec{M}}\} = -i (M_j - N_j) p_{\vec{M} - \vec{N}} - \delta_{j1} \delta_{\vec{N},\vec{M}} P,$$

$$\{l_{\vec{N},j}, P\} = 0,$$

$$\{l_{\vec{N},j}, l_{\vec{M},k}\} = i M_j l_{\vec{N} + \vec{M},k} - i N_k l_{\vec{N} + \vec{M},j}.$$ (2.49)

where $\delta_{\vec{N},\vec{M}} = \delta_{N_1,M_1} \delta_{N_2,M_2}$. The derivations of only the second, the last term in the third, and the fourth equations merit comment. The second equation follows easily from the first equation of (2.24) [with $w^{(1)} = \frac{1}{\sqrt{A}}d\theta^1$] and (2.26). As for the last term in the third equation, we have, $\{l(\eta_{\vec{N},j}), p(d\xi_{\vec{N}}^{(1)})\} = p\{d\{i_{\eta_{\vec{N},j}} \xi_{\vec{N}}^{(1)}\}\}$ after using (2.26). Now
$i \eta_{\mathbf{N},j} \xi_{\mathbf{N}}^{(1)}|_{\partial \Sigma} = \frac{\sqrt{A}}{4\pi^2} [\delta_{j1}d\theta^2 - \delta_{j1}d\theta^1]$. The second term here can be extended to all of $\Sigma$ as a closed form and thus contributes nothing. The first term gives the term in question. As for the fourth equation, one has, $\{l(\eta_{\mathbf{N},j}), p(d\psi^{(1)})\} = p \left( d\mathcal{L}_{\eta_{\mathbf{N},j}} \psi^{(1)} \right)$. Now $\left( \mathcal{L}_{\eta_{\mathbf{N},j}} \psi^{(1)} \right)|_{\partial \Sigma} = \mathcal{L}_{\eta_{\mathbf{N},j}}|_{\partial \Sigma} \left( -\frac{\sqrt{A}}{4\pi^2} d\theta^2 \right)$ which on using (2.26) becomes $d \left( -\frac{\sqrt{A}}{4\pi^2} e^{i \tilde{\mathbf{N}} \cdot \tilde{\mathbf{\theta}}} \delta_{j2} \right)$. This can be extended as an exact form $\chi^{(1)}$ to all of $\Sigma$ so that $p(d\chi^{(1)}) = 0$ (in a trivial way). The fourth equation above follows readily.

We next note that the observables

$$\hat{l}_{\mathbf{N},j} = -i \sum_{\mathbf{M}} M_j q_{\mathbf{M}+\mathbf{S}p_{\mathbf{M}} - \delta_{j1}q_{\mathbf{N}}} \quad (2.50)$$

have the same PB’s as $l_{\mathbf{N},j}$. It must therefore the case that

$$l_{\mathbf{N},j} = \hat{l}_{\mathbf{N},j} \quad (2.51)$$

which is the generalized classical Sugawara formula.

We can show (2.51) explicitly in the following way. Choose a Euclidean metric on (the interior of) $T_3$. Relative to this metric, let $r$ be the radial distance from the central thread of the solid torus $T_3$ with $\partial \Sigma$ having $r = 1$. Then $(r, \theta^1, \theta^2)$ are coordinates for $T_3$. Let $\Lambda$ be a function of $r$ alone with

$$\Lambda(r) = 0, \quad r < \epsilon < 1, \quad \Lambda(1) = 1, \quad d\Lambda(1) = 0, \quad (2.52)$$

$\epsilon$ being a small positive number. We may then set

$$\eta_{\mathbf{N},j}(r, \theta^1, \theta^2) = \Lambda(r) e^{i \tilde{\mathbf{N}} \cdot \tilde{\mathbf{\theta}}} \frac{\partial}{\partial \theta^j}. \quad (2.53)$$

For this choice,

$$l_{\mathbf{N},j} = \int_{\partial \Sigma} e^{i \tilde{\mathbf{N}} \cdot \tilde{\mathbf{\theta}}} A_j(1, \tilde{\theta}) B(1, \tilde{\theta}) - \int_{\Sigma} \left( \Lambda(r) e^{i \tilde{\mathbf{N}} \cdot \tilde{\mathbf{\theta}}} A_j(r, \tilde{\theta}) \right) dB + \int_{\Sigma} \Lambda(r) e^{i \tilde{\mathbf{N}} \cdot \tilde{\mathbf{\theta}}} B(\tilde{y}) \mathcal{F}_{ja}(\tilde{y}) dy^a \quad (2.54)$$
where $F_j = \partial_j A - \partial A_j$.[There is a slight (and inconsequential) abuse of notation in (2.54) which is really $l(\eta_{N,j})$ and not $l_{N,j}$. There are similar unimportant inaccuracies in what follows.]

In (2.54) and in the equations up to (2.67), the components of forms and vector fields are with reference to the coordinate system $(r, \theta^1, \theta^2) \equiv (y^1, y^2, y^3) \equiv \vec{y}$. [Elsewhere, we have instead used Cartesian coordinates.] The Levi-Civita symbols $\epsilon^{\alpha\beta\gamma}[1 \leq \alpha, \beta, \gamma \leq 3]$ and $\epsilon_{jk}[2 \leq j, k \leq 3]$ for these equations are so defined that $\epsilon^{123} = 1 = \epsilon_{23}$. Also $B_{jk}(1, \vec{\theta})$ denotes $B_{jk}(1, \vec{\theta})d\theta^j d\theta^k$.

Let
\[
\Lambda e_{\vec{M}},
\]
\[
\frac{A}{4\pi^2} \Lambda \frac{e_{\vec{M}}}{-iM_j} \epsilon_{jk} d\theta^k \quad \text{(no } j \text{ summation)} \tag{2.55}
\]
be the extensions of $\xi_{\vec{M}}^{(0)}|_{\partial \Sigma} = e_{\vec{M}}$ and $\xi_{\vec{M}}^{(1)}|_{\partial \Sigma}$ to all of $\Sigma$. We then have
\[
q_{\vec{M}} = \int_{\partial \Sigma} e_{\vec{M}} B - \int_{\Sigma} \Lambda e_{\vec{M}} dB,
\]
\[
- iM_j p_{\vec{M}} = \int_{\partial \Sigma} A_j e_{\vec{M}}^* \mu - \frac{A}{4\pi^2} \int_{\Sigma} \Lambda e_{\vec{M}}^* \epsilon_{jk} d\theta^k dA, \quad M_j \neq 0. \tag{2.56}
\]

We next examine
\[
- i \sum_{\vec{M} \text{ with } M_j \neq 0} M_j q_{\vec{M} + \vec{N}} p_{\vec{M}}
\]
\[
- \frac{A}{4\pi^2} \sum_{\vec{M} \text{ with } M_j = 0} \int_{\Sigma} \Lambda(r') e_{\vec{M} + \vec{N}}(\vec{\theta}') B(r', \vec{\theta}') \int_{\Sigma} \Lambda(r) e_{\vec{M}}^* (\vec{\theta}) e_{jk} d\theta^k A(r, \vec{\theta})
\]
\[
= - i \sum_{\vec{M}} \left[ \int_{\partial \Sigma} e_{\vec{M} + \vec{N}} B - \int_{\Sigma} \Lambda e_{\vec{M} + \vec{N}} dB \right] \left[ \int_{\partial \Sigma} A_j e_{\vec{M}}^* \mu - \frac{A}{4\pi^2} \int_{\Sigma} \Lambda e_{\vec{M}}^* \epsilon_{jk} d\theta^k dA \right]. \tag{2.57}
\]

Consider the second term on the left hand side of (2.57). We will now show that it is the same as the last term on the right hand side of (2.50). First consider $j = 2$. For this $j$, the second integral in this term is
\[
- \frac{1}{\sqrt{A}} \int_{\Sigma} d \left[ \Lambda(r) e^{-iM_1 \theta^1} d\theta^1 \right] A(r, \vec{\theta}). \tag{2.58}
\]
\( e^{-iM_1 \theta^1} d\theta^1 \) being a globally defined closed one form on \( T_3 \), this is equal to

\[
- \frac{1}{\sqrt{A}} \int_{\Sigma} d \left[ (\Lambda(r) - 1) e^{-iM_1 \theta^1} d\theta^1 \right] A(r, \bar{\theta}) .
\]  \tag{2.59}

As

\[
(\Lambda(r) - 1) e^{-iM_1 \theta^1} d\theta^1 \bigg|_{\partial \Sigma} = 0 ,
\]  \tag{2.60}

this expression, and hence the term in question, are weakly zero. This is what we want, the last term in (2.50) being zero for \( j = 2 \).

For \( j = 1 \), the analogue of (2.58) is

\[
\frac{1}{\sqrt{A}} \int_{\Sigma} d \left[ \Lambda(r) e^{-iM_2 \theta^2} d\theta^2 \right] A(r, \bar{\theta})
\]  \tag{2.61}

which for \( M_2 \neq 0 \) is

\[
\frac{1}{\sqrt{A}} \int_{\Sigma} d \left[ \Lambda(r) e^{-iM_2 \theta^2} d\theta^2 - d \left( \frac{\Lambda(r) e^{-iM_2 \theta^2}}{-iM_2} \right) \right] A(r, \bar{\theta}) .
\]  \tag{2.62}

Since

\[
\left. \left[ \Lambda(r) e^{-iM_2 \theta^2} d\theta^2 - d \left( \frac{\Lambda(r) e^{-iM_2 \theta^2}}{-iM_2} \right) \right] \right|_{\partial \Sigma} = 0
\]  \tag{2.63}

in view of (2.52), this term is weakly zero. When both \( M_1 \) and \( M_2 \) are zero, the second term on the left hand side of (2.57) gives just \(-q_S P\). Thus the left hand side of (2.57) is the same as the right hand side of (2.50), that is \( \hat{l}_{\vec{N},j} \).

The right hand side of (2.57) gives

\[
\hat{l}_{\vec{N},j} = \int_{\partial \Sigma} e^{i\vec{N} \cdot \bar{\theta}} A_j(1, \bar{\theta}) B(1, \bar{\theta}) + \int_{\Sigma} \Lambda(r) e^{i\vec{N} \cdot \bar{\theta}} B(1, \bar{\theta}) \mathcal{F}_{\alpha}(\bar{y}) dy^\alpha
\]

\[
- \int_{\Sigma} \Lambda(r) e^{i\vec{N} \cdot \bar{\theta}} A_j(1, \bar{\theta}) dB(r, \bar{\theta})
\]

\[
+ \int_{\Sigma} \Lambda(r, \bar{\theta}) e^{i\vec{N} \cdot \bar{\theta}} \left[ \int_0^\infty dr' \Lambda(r') \partial_\lambda B_{\rho\sigma}(r', \bar{\theta}) e^{\lambda \rho \sigma} \right] \epsilon_{j k} d\theta^k dA(r, \bar{\theta})
\]  \tag{2.64}

where the completeness relation

\[
\sum_{\tilde{N}} e^{i\tilde{N} \cdot (\bar{\theta} - \bar{\theta})} = 4\pi^2 \delta(\theta_1 - \theta'_1) \delta(\theta_2 - \theta'_2)
\]  \tag{2.65}
has been used.

The test function for the Gauss law generator $d\mathcal{A}$ in the last term involves the Gauss law generator $d\mathcal{B}$. We therefore interpret it as weakly zero and get

\[
\hat{l}_{N,j} = \int_{\partial\Sigma} e^{i\vec{N} \cdot \vec{\theta}} A_j(1, \vec{\theta}) B(1, \vec{\theta}) + \int_{\Sigma} \Lambda(r) e^{i\vec{N} \cdot \vec{\theta}} B_j \mathcal{F}_{j\alpha}(\vec{\gamma}) dy^\alpha \\
- \int_{\Sigma} \Lambda(r) e^{i\vec{N} \cdot \vec{\theta}} A_j(1, \vec{\theta}) dB(r, \vec{\theta}). \tag{2.66}
\]

Now we can write $BF_{j\alpha} dy^\alpha$ as $-2B_{j\alpha} dy^\alpha dB$. Also, in view of (2.52), and the fact that $j$ refers to components tangent to $\partial \Sigma$ when $r = 1$, it is clear that

\[
\left[ \Lambda(r) e^{i\vec{N} \cdot \vec{\theta}} A_j(r, \vec{\theta}) - \Lambda(r) e^{i\vec{N} \cdot \vec{\theta}} A_j(1, \vec{\theta}) \right] |_{\partial \Sigma} = 0,
\]

\[
\left[ \Lambda(r) e^{i\vec{N} \cdot \vec{\theta}} B_{j\alpha}(\vec{\gamma}) dy^\alpha - \Lambda(r) e^{i\vec{N} \cdot \vec{\theta}} B_{j\alpha}(1, \vec{\theta}) d\theta^\alpha \right] |_{\partial \Sigma} = 0. \tag{2.67}
\]

We thus find (2.51).

For $\partial \Sigma = S^2$, we can find the analogues $\hat{l}_{Jm,\alpha}$ of (2.50) which have the same PB’s as $l_{Jm,\alpha}$. This leads to the generalized classical Sugawara formula

\[
\hat{l}_{Jm,\alpha} = l_{Jm,\alpha}. \tag{2.68}
\]

[We do not reproduce the formula for $\hat{l}_{Jm,\alpha}$ as it is complicated and not very illuminating.]

It should be possible to verify (2.68) explicitly as for the torus, although we have not done so.
3. THE TOPOLOGICAL BF THEORY

In this Section, we show that when the system of Section 2 is in its ground state, it is described by the topological $BF$ field theory. The latter also predicts edge states with properties similar to those of the last Section.

For the Hamiltonian (2.4), the energy density in $\Sigma^0$ is zero if

$$\pi_i - \epsilon_{ijk} B_{jk} = 0, \quad P_{ij} = 0,$$

$$F_{ij} = 0, \quad H_{ijk} = 0. \quad (3.1)$$

The symplectic one form

$$\theta = \int \left( \pi_i \delta A_i + \frac{1}{2} P_{ij} \delta B_{ij} \right) d^3 x$$

appropriate to Section 2 restricted (pulled back) to the surface (3.1) becomes

$$\theta^* = \int \epsilon_{ijk} B_{jk} \delta A_i d^3 x, \quad (3.2)$$

while the constraints $G_i$ pulled back to (3.1) become

$$G^*_0 = \int d^3 x \lambda^{(0)} dB \approx 0,$$

$$G^*_1 = 2 \int d^3 x \lambda^{(1)} dA \approx 0. \quad (3.3)$$

We can also pull back the observables $q(w^{(1)}), p(w^{(2)})$ and $l(\eta)$. They become

$$q^*(w^{(1)}) = \int w^{(1)} B,$$

$$p^*(w^{(2)}) = - \int w^{(2)} A,$$

$$l^*(\eta) = \int (L_\eta A) B. \quad (3.4)$$

The PB’s of $A$ and $B$ follows from (3.2):

$$\{A_i(x), A_j(y)\} = \{B_{ij}(x), B_{kl}(y)\} = 0,$$
\[ \{B_{ij}(x), A_k(y)\} = -\frac{1}{2} \epsilon_{ijk} \delta^3(x - y). \] (3.5)

All fields here are evaluated at equal times.

It is to be observed that (3.3)-(3.5) can be obtained from Section 2 by substituting \( A, B \) for \( \mathcal{A}, \mathcal{B} \). The entire previous description of constraints and edge excitations can therefore be transferred intact to the surface (3.1).

It is also to be observed that (3.2)-(3.5) are consequences of the topological \( BF \) action

\[ S^* = \int dt L^* \] (3.6)

of the Lagrangian (1.7). Thus when a system described by London equations is in its ground state, its edge excitations are described by a topological field theory.

Henceforth, we will work with the Lagrangian (1.6), but it will be obvious that we could equally well have worked with the topological field theory.
4. QUANTIZATION AND THE GROUP $SDIFF(\partial \Sigma)$

We here consider $\Sigma = B_3$ or $T_3$. The quantum operators for $q., p., l.$ will be denoted by the corresponding capital letters. The quantum operators for $Q$ and $P$ will be denoted by $Q$ and $P$. We will also adopt the choice (2.32)-(2.38) for describing the Fourier components.

Now since
\[ Y_{Jm}^* = (-1)^m Y_{J,-m}, \quad (e^{i\vec{N} \cdot \vec{\theta}})^* = e^{-i\vec{N} \cdot \vec{\theta}}, \] (4.1)
we have
\[ Q_{Jm}^\dagger = (-1)^m Q_{J,-m}, \quad P_{Jm}^\dagger = (-1)^m P_{J,-m}, \]
\[ Q_{\vec{N}}^\dagger = Q_{-\vec{N}}, \quad P_{\vec{N}}^\dagger = P_{-\vec{N}}. \] (4.2)

Let \( \omega : n \to \omega(n) (> 0) \) be a frequency function invariant under the substitution
\[ n = Jm \to n^* \equiv J, -m \]
or
\[ n = \vec{N} \to n^* \equiv -\vec{N}. \] (4.3)

The dispersion relation is otherwise left arbitrary for the moment.

For the moment, let us set aside the modes $Q$ and $P$ which exist for $T_3$.

We now form the annihilation and creation operators
\[ a_n = \frac{1}{\sqrt{2}} [\omega(n) Q_n + iP_n^\dagger], \]
\[ a_n^\dagger = \frac{1}{\sqrt{2}} [\omega(n) Q_n^\dagger - iP_n]. \] (4.4)

Their only nonzero commutator is
\[ [a_n, a_m^\dagger] = \omega(n) \delta_{nm}. \] (4.5)
The Fock space quantization of (4.5) is standard. Let \( |0\rangle \) denote the Fock space vacuum:

\[
a_n |0\rangle = 0.
\] (4.6)

The quantum version \( L_n \) of diffeo generators are obtained from their classical expressions after normal ordering. We now argue the following: a) Not all \( L_n \) can be implemented on the preceding Fock space regardless of the dispersion relation. b) Let \( SDiff_0(\partial \Sigma) \) denote the group of diffeos leaving \( \mu \) invariant with classical generators \( s_n \). Their quantum versions \( S_n \) can be implemented on the preceding Fock space if \( \omega(n) \) is independent of \( n \).

In this way, by demanding the implementability of \( S_n \), we gain some control over the dispersion relation just as in the Chern-Simons case. As remarked in the Introduction, this requirement will also suggest interesting field theories for describing the edge excitations.

As for a), consider for example the squared norm

\[
\mathcal{N}^2 = \langle 0 | L_{N,j}^{(k)} L_{N,j}^{(k)} | 0 \rangle
\] (4.7)
of the state \( L_{N,j} |0\rangle \) for \( \partial \Sigma = T^2 \). Here

\[
L_{N,j} = -\frac{1}{2} \sum_{\hat{M}} \frac{M_j}{\omega(M + N)} (a_{\hat{M}+N} + a_{-\hat{M}-N}^\dagger)(a_{-\hat{M}} - a_{\hat{M}}^\dagger) :
\] (4.8)

where we have ignored a term containing \( \mathcal{P} \).

The computation of (4.7) requires regularisation. We interpret it as

\[
\lim_{k \to \infty} \mathcal{N}^2_k ,
\]

\[
\mathcal{N}^2_k = \langle 0 | L_{N,j}^{(k)} L_{N,j}^{(k)} | 0 \rangle
\] (4.9)

where

\[
L_{N,j}^{(k)} = -\frac{1}{2} \sum_{|M_i| < k \text{ for } i=1,2} \frac{M_j}{\omega(M + N)} : (a_{\hat{M}+N} + a_{-\hat{M}-N}^\dagger)(a_{-\hat{M}} - a_{\hat{M}}^\dagger) :
\] (4.10)
Substitution of (4.10) in (4.9) shows that

$$N_k^2 = 1 \frac{1}{4} \sum_{M_i \text{ for } i=1,2} M_j^2 \frac{\omega(M)}{\omega(M + N)} - \frac{1}{4} \sum_{M_i \text{ and } |M_i| < k} (M_j^2 + N_j M_j) \prod_{l=1,2} \delta_{\bar{M}_l, -M_l - N_l}$$

If both $N_i$ for example are positive, this becomes

$$N_k^2 = 1 \frac{1}{4} \sum_{|M_i| \leq k} M_j^2 \frac{\omega(M)}{\omega(M + N)} - \frac{1}{4} \sum_{M_1 = -k}^{k-N_1} \sum_{M_2 = -k}^{k-N_2} (M_j^2 + N_j M_j). \quad (4.11)$$

We can find no function $\omega$ for which this expression is finite as $k \to \infty$ and therefore (tentatively) conclude that there is no choice of $\omega$ for which $L_{\bar{N},j}$ is well defined on our Fock space.

Now there is a local scalar field Lagrangian invariant under $SDiff(\partial \Sigma)$ and with a prescribed dispersion relation, namely (1.11). It turns out that the Lie algebra of the group $SDiff_0(\partial \Sigma)$ is implementable by operators in the quantum theory of this Lagrangian. We shall now study this Lagrangian, argue that it can describe the edge states and finally show the implementability of the algebra of the group $SDiff_0(\partial \Sigma)$ in the quantum theory of (1.11).

The field $\varphi$ described by (1.11) is characterized by the frequency $\omega(n) = \omega_0$ independent of $n$. It becomes a quantum field $\Phi$ if we set

$$\Phi = \frac{1}{\sqrt{2\omega_0}} \sum_n \left( a_n e_n + a_n^\dagger e_n^* \right)$$

$$\dot{\Phi} = -i \sqrt{\frac{\omega_0}{2}} \sum_n \left( a_n e_n - a_n^\dagger e_n^* \right) \quad (4.12)$$

where

$$[a_n, a_m^\dagger] = \delta_{nm} \quad (4.13)$$

and define the vacuum $|0 \rangle$ by

$$a_n |0 \rangle = 0 \quad (4.14)$$

Its Hamiltonian has the expression $\sum \omega_0 a_n^\dagger a_n$. 

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In comparison with the previous $a_n$, the field $\Phi$ also has the modes $a_0, a_0^\dagger$. They can be eliminated by considering $L_\alpha \Phi$ (for $S^2$) or $\frac{\partial}{\partial \theta} \Phi$ (for $T^2$). Thus the modes we find are those of these gradient fields. In the same way, the Chern-Simons theory on a disc $D$ describes a suitable derivative of a scalar field on the boundary $\partial \Sigma$. As in that theory, these zero modes become relevant when $\partial \Sigma$ has more than one connected component or there are sources, when $a_0$ and $a_0^\dagger$ acquire an interpretation in terms of electric charge excitations, $\frac{1}{\sqrt{2i}}(a_0 - a_0^\dagger)$ being the charge operator. We will show these results in a second paper \[15\]. But there is a problem with the Lagrangian (1.11) when the zero modes are important. The zero mode part $\omega_0 a_0^\dagger a_0$ of its Hamiltonian is not diagonal when charge is diagonal. For this reason, (1.11) must be used with caution when zero modes are significant. We emphasize that (1.11) has been used here only to motivate our conclusion that $SDiff_0(\partial \Sigma)$ is implementable for the edge states of (1.6) and (1.7). None of our substantive results depend crucially on the use of (1.11).

We have yet to attend to the modes $Q$ and $P$. As we will argue in \[15\], $P$ measures the magnetic flux tangent to $\partial \Sigma$ (such as that in vortices winding around $\partial \Sigma$) and $Q$ is associated with the creation of such flux. We will regard $P$ as a superselected operator with eigenvalue $\Theta$ for state $|0\rangle$ and $Q$ as associated with vertex operators for the creation of such flux. This assumption is similar to the one adopted in Chern-Simons dynamics or conformal field theories in the treatment of charge or ‘momentum’ \[9, 10, 13\]. Thus $|0\rangle$ is a unit norm state with $P |0\rangle = \Theta |0\rangle$. Note that the inclusion of the term $-\frac{\delta}{\sqrt{2\omega(\mathcal{N})}}(a_\mathcal{N}^\dagger - a_{-\mathcal{N}})P$, ignored in (4.8), does not affect our conclusion regarding the divergence of $\mathcal{N}^2$.

The following vector fields $\eta|_{\partial \Sigma}$ preserve $\mu$:

\begin{align*}
S^2 : \quad \eta|_{\partial \Sigma} &= v_{lm}, \\
v_{lm} &= \frac{\partial Y_{lm}}{\partial (\cos \theta)} \frac{\partial}{\partial \phi} - \frac{\partial Y_{lm}}{\partial \phi} \frac{\partial}{\partial (\cos \theta)}
\end{align*}

\begin{align*}
T^2 : \quad \eta|_{\partial \Sigma} &= v_{\mathcal{N}}
\end{align*}

\[4.15\]
or \( \eta |_{\partial \Sigma} = t_i \),

\[
v_{\tilde{N}} = \frac{\partial e^*_N}{\partial \theta^1} \frac{\partial}{\partial \theta^2} - \frac{\partial e^*_N}{\partial \theta^2} \frac{\partial}{\partial \theta^1} = -ie^*_N \left( N_1 \frac{\partial}{\partial \theta^2} - N_2 \frac{\partial}{\partial \theta^1} \right),
\]

\[t_i = \frac{\partial}{\partial \theta^i}. \quad (4.16)\]

For the field \( \varphi \), the classical generators of transformations due to these vector fields are

\[
T^2 : \ s^\varphi_0 = \int \mu \varphi \mathcal{L}_{\eta \dot{\varphi}} \quad (4.17)
\]

and their quantum versions for the torus are

\[
S^\varphi_{\tilde{N}} = \sum_{\tilde{M}} \frac{i}{2} (\tilde{N} \times \tilde{M}) : \left( a_{\tilde{N}+\tilde{M}} + a^\dagger_{-\tilde{M}-\tilde{N}} \right) \left( a_{-\tilde{M}} - a^\dagger_{\tilde{M}} \right) : \text{ for } \eta |_{\partial \Sigma} = v_{\tilde{N}} , \quad (4.18)
\]

\[
S^\varphi_i = -\frac{1}{2} \sum_{\tilde{M}} M_i : \left( a_{\tilde{M}} + a^\dagger_{-\tilde{M}} \right) \left( a_{-\tilde{M}} - a^\dagger_{\tilde{M}} \right) : \text{ for } \eta |_{\partial \Sigma} = t_i . \quad (4.19)
\]

where \( \tilde{N} \times \tilde{M} = (N_1 M_2 - N_2 M_1) \). [There are similar expresions for \( S^2 \), but they are long and will not be displayed here.] These are also the generators one obtains from \( I(\eta) \) with the choices \((4.15)-(4.16)\) for \( \eta \) if \( Q \) and \( P \) are ignored.

It should not be a matter for surprise to note that \( SDiff f_0(\partial \Sigma) \) does not mix the zero modes \( a_0 \) and \( a^\dagger_0 \) with the rest, and indeed that they are entirely absent, in \((4.18)\) and \((4.19)\). For if \( D \) is in this group with the action \( p \to Dp \) on points of \( \partial \Sigma \), its action \( D^* : f \to D^* f \) on functions \( f \) is by pull back: \( (D^* f)(p) = f(D(p)) \). Constant functions being invariant under this action, \( S^\varphi_{\tilde{N}} \) and \( S^\varphi_i \) must commute with \( a_0 \) and \( a^\dagger_0 \). It is for this reason that \( a_0 \) and \( a^\dagger_0 \) do not occur in \( S^\varphi_{\tilde{N},i} \).

We have now established the connection of our edge states to the modes of the field \( \Phi \), and at the same time motivated the dispersion relation \( \omega(n) = \omega_0 \). But point b) is not yet fully covered, as we have not said anything about the implementability of \((4.18)\)
and (4.19) on our Fock space. For this purpose, let us first consider the expression (4.18) for $S_{\Phi}\vec{N}$. In (4.18), the term with two creation and two annihilation operators commute with any $a_{\vec{L}}$ and $a_{\vec{L}}^\dagger$ for fixed $\vec{L}$ if $k$ is large enough, suggesting that we can discard them. We can get zero for these term also by noting that they (formally) change sign if we do the substitution $\vec{M} = -\vec{M}' - \vec{N}$. For these reasons, we will discard the terms with two creation and two annihilation operators in (4.18) and define $S_{\Phi}\vec{N}$ to be

$$S_{\Phi}\vec{N} = \sum_{\vec{M}} \frac{i}{2} (\vec{N} \times \vec{M}) \left[ a_{-\vec{M}-\vec{N}}^\dagger a_{-\vec{M}} - a_{-\vec{M}+\vec{N}} a_{\vec{M}}^\dagger \right].$$

As for $S_{\Phi}i$, if we regulate the sum as before by first summing over $|M_i| \leq k$ and then letting $k \to \infty$, then the terms with two creation and two annihilation operators vanish. We thus set

$$S_{\Phi}i = \sum_{\vec{M}} M_i a_{\vec{M}}^\dagger a_{\vec{M}}.$$  \hfill (4.21)

Now we note that that the operators $S_{\Phi}\vec{N}$ and $S_{\Phi}i$ are well defined on the vacuum as they just annihilate the latter. It is furthermore easy to see that they are well defined on any vector in the Fock space.

We must also check the commutators and make sure that we do not get divergent central terms. The calculation using (4.20) and (4.21) is straightforward. There are no central terms, divergent or otherwise. We find

$$[S_{\Phi}\vec{M}, S_{\Phi}\vec{N}] = i(\vec{M} \times \vec{N}) S_{\Phi}\vec{M} + \vec{N},$$

$$[S_{\Phi}\vec{M}, S_{\Phi}i] = M_i S_{\Phi}\vec{M},$$

$$[S_{\Phi}i, S_{\Phi}j] = 0.$$  \hfill (4.22)

The Lagrangian (1.11) is not of course the only one with $SDiff_0(\partial \Sigma)$ invariance, as we can for instance replace $\omega_0\varphi^2$ by any potential $V(\varphi)$. We can in particular change $\varphi^2$ to $(\varphi - \varphi_0)^2$ for a constant $\varphi_0$. This model is of particular interest if the value of $A$ is macroscopic since the shift $\varphi \to \varphi + \varphi_0$ is not allowed as $A \to \infty$ as in Lagrangians which
spontaneously break symmetries. In any case, for $A$ finite, (1.11) seems the simplest choice which accounts for the properties of the edge states.

For the Lagrangian (1.6), $SDiff_0(\partial \Sigma)$ generators are $S^\Phi_N - i N_2 Q_N P$ and $S^\Phi_i$. It is clear that this group continues to be implementable even after this modification of $S^\Phi_N$.

It may be remarked that the $SDiff_0(\partial \Sigma)$ generators from (1.6) and (1.7) can be regularized if $\omega(n)$ approaches $\omega_0$ fast enough as $J_i |m|$ or $|N_i|$ become large. It is not necessary that $\omega(n)$ is exactly a constant for all $n$. Similar uncertainties exist for the dispersion relation of Chern-Simons edge excitations. They have to be resolved using appropriate physical inputs.
5. THE ALGEBRA $w_{1+\infty}$

In this Section, we briefly consider the case $\Sigma = \text{solid cylinder}$. It has a cylinder $S^1 \times \mathbb{R}^1$ as its boundary $\partial \Sigma$. Let $\theta \,(\text{mod } 2\pi)$ and $z \,(-\infty < z < \infty)$ be coordinates on this $\partial \Sigma$ and let

$$\mu = d\theta dz$$

(5.1)

It is then known\[14\] that the Lie algebra of $SDiff_0(\partial \Sigma)$ is the algebra $w_{1+\infty}$.

There is an easy way to see this result. Following earlier work \[14\], we can regard $\mu$ as a symplectic form with the associated PB\[14\]

$$\{f, g\} = \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial \theta}$$

(5.2)

(This PB is different from the PB for (1.6), (1.7) or (1.11).) The group $SDiff(\partial \Sigma)$ is then the group of canonical transformations for the form $\mu$, the elements of its Lie algebra being functions $f$ with Lie brackets given by (5.2). A basis for this Lie algebra is the set of functions

$$U^l_m = -i e^{im\theta} z^{l+1}$$

(5.3)

with the PB’s

$$\{U^l_m, U^k_n\} = [m(k+1) - n(l+1)] U^{l+k}_{m+n}$$

(5.4)

which are exactly the defining relations of $w_{1+\infty}$.

We must now pass to quantum theory. For this purpose, we can either work with the $\varphi$ Lagrangian (1.11) or equivalently use one of the two Lagrangians (1.6) or (1.7) for $A$ and $B$. Let us work with $\varphi$. Quantization requires a choice of basis $e_m$ for the Hilbert space $\mathcal{H} = L^2(\mu, \partial \Sigma)$ of functions on $\partial \Sigma$ with the scalar product

$$(\alpha, \beta) = \int_{\partial \Sigma} d\theta dz \alpha^* \beta(\theta, z)$$

(5.5)
But identification of \( e_m \) with the functions \( U^l_m \) is not correct. They are not normalizable if \( l \geq -1/2 \), nor do they have an interpretation along the lines of plane waves in quantum mechanics. In this way, we are led to treat \( w_{1+\infty} \) in a basis different from \((5.3)\). A simple basis adopted to \( \mathcal{H} \) is given by the correspondence

\[
e_n \rightarrow e_{n,N} = \frac{1}{\sqrt{2^N N! \pi^{1/2}}} \exp(i n \theta) \exp\left(-\frac{z^2}{2}\right) H_N(z) \quad n \in \mathbb{Z}, \quad N \in \mathbb{Z}^+,
\]

\( H_N \) being Hermite polynomials, and \( \mathbb{Z} \) and \( \mathbb{Z}^+ \) the set of integers and of nonnegative integers respectively. \( e_{n,N} \) has the symmetry

\[
e^*_{n,N} = e_{-n,N}.
\]

We next expand the quantum versions \( \Phi \) and \( \dot{\Phi} \) of \( \varphi \) and \( \dot{\varphi} \):

\[
\Phi = \frac{1}{\sqrt{2\omega_0}} \sum_{n,N} \left( a_{n,N} e_{n,N} + a^\dagger_{n,N} e^*_{n,N} \right),
\]

\[
\dot{\Phi} = -i \sqrt{\frac{\omega_0}{2}} \sum_{n,N} \left( a_{n,N} e_{n,N} - a^\dagger_{n,N} e^*_{n,N} \right).
\]

Here the only nonzero commutator involving \( a \)'s and \( a^\dagger \)'s is

\[
\left[ a_{n,N}, a^\dagger_{m,M} \right] = \delta_{nm} \delta_{NM}.
\]

The fields \( \Phi, \dot{\Phi} \) are next realized by introducing a Fock space. Let \( \left| 0 \right> \) denote its vacuum.

The classical generators of \( S D f f_0 (\partial \Sigma) \) can be chosen to be

\[
s_{n,N} = \int d^3 x \dot{\varphi} \left\{ e_{n,N}, \varphi \right\}_\mu
\]

\[
= \int d^3 x \dot{\varphi} \left( \frac{\partial e_{n,N}}{\partial \theta} \frac{\partial}{\partial z} - \frac{\partial e_{n,N}}{\partial z} \frac{\partial}{\partial \theta} \right) \varphi.
\]

On quantization they become

\[
S_{n,N} =: \int d^3 x \dot{\Phi} \left( \frac{\partial e_{n,N}}{\partial \theta} \frac{\partial}{\partial z} - \frac{\partial e_{n,N}}{\partial z} \frac{\partial}{\partial \theta} \right) \Phi:
\]

Although it is of interest to compute the commutators of \( S_{n,N} \), we will not attempt that task here.
6. WHAT IS 3d HALL EFFECT

In Section 1, the relevance of discovering a three dimensional analogue of Hall effect in the context of our problem was motivated. In this last Section, we make a proposal for the same.

Let us briefly recall the Hall effect. Here there is a particle of charge $e$ in the $1-2$ plane in the presence of fields $F_{12}$ and $F_{a0}$ ($a = 0, 1$) which we take to be time independent constants for simplicity. The particle is subject to the force

$$e \left( F_{a0} + \epsilon_{ab} \dot{x}^b F_{12} \right), \quad \epsilon_{ab} = \epsilon^{0ab3} \quad (6.1)$$

where $x^a$ are its (Cartesian) coordinates and $\dot{x}^a$ the components of its velocity. The force vanishes if

$$\dot{x}^a = \epsilon^{ab} \frac{F_{b0}}{F_{12}}, \quad \epsilon^{ab} \equiv \epsilon_{ab}, \quad (6.2)$$

and this equation embodies the Hall effect.

In three dimensions, what corresponds to $F_{12}$ is $H_{123}$. As it is the vortex or the string which has coupling to this field, we look for the analogue of (6.2) when vortices interact with $H$, or rather with $B$. This interaction is \[22\]

$$\mathcal{L}_{INT} = \frac{\lambda}{2} \epsilon^{ab} B_{\mu\nu}(y) \partial_{a} y^{\mu}(\sigma) \partial_{b} y^{\nu}(\sigma), \quad \partial_{a} \equiv \frac{\partial}{\partial \sigma^a} \quad (6.3)$$

where $\sigma = (\sigma^0, \sigma^1)$, $\sigma^0$ is the evolution parameter, $y : \sigma \rightarrow y(\sigma)$ describes the spacetime history of the vortex, $\lambda$ is a constant and the Levi-Civita symbol is defined by $\epsilon^{01} = -\epsilon^{10} = 1$. [It is thus different from the $\epsilon$'s in (6.1) and (6.2).]

The “force” term from (6.3) and what corresponds to (6.1) is

$$\frac{\partial \mathcal{L}_{INT}}{\partial y^{\mu}} - \partial_{a} \frac{\partial \mathcal{L}_{INT}}{\partial (\partial_{a} y^{\mu})} = \frac{\lambda}{2} H_{\mu\nu\lambda}(y) \epsilon^{ab} \partial_{a} y^{\nu} \partial_{b} y^{\lambda} \quad (6.4)$$

which, with the choice $y^0(\sigma) = \sigma^0$ becomes, for $\mu = i = 1, 2, 3$,

$$\lambda \left[ H_{123} \epsilon_{ijk} \partial_{0} y^{j} \partial_{1} y^{k} - H_{ij0} \partial_{1} y^{j} \right], \quad \epsilon_{ijk} \equiv \epsilon^{0ijk} \quad (6.5)$$
\( H_{123}, H_{ij0} \) are here regarded as time independent constants as in (6.1).

(6.5) vanishes for

\[
(\partial_0 - c\partial_1) y^j = -\frac{1}{2} \epsilon^{jkl} \frac{H_{k\ell 0}}{H_{123}}, \quad \epsilon^{jkl} = \epsilon^{0jkl},
\]

where \( c \) is any real constant. It is this equation which describes the vortex Hall effect. It implies the wave equation

\[
(\partial_0 + c\partial_1) (\partial_0 - c\partial_1) y^j = 0
\]

and can thus provide solutions of the field equation for the Lagrangian

\[
\int d^2\sigma \left( \mathcal{L}_0 + \mathcal{L}_{INT} \right),
\]

\[
\mathcal{L}_0 = \frac{1}{2} \left[ \frac{1}{c^2} (\partial_0 y^j)^2 - (\partial_1 y^j)^2 \right].
\]

It seems reasonable to propose (6.6) as describing the three dimensional analogue of the Hall effect. It displays the response of a vortex to charge density \( J^0 = -2H_{123} \) and current density \( J^j = \epsilon^{jkl} H_{k\ell 0} \). Depending on the sign of \( c \), it claims that the left or right moving vortex mode has velocity parallel to current density.

It is easy to solve (6.6). The general solution is

\[
y^j(\sigma^0, \sigma^1) = -\frac{1}{2} \epsilon^{jkl} \frac{H_{k\ell 0}}{H_{123}}(\sigma^0 - \frac{1}{c} \sigma^1) + z^j(\sigma^0 + \frac{1}{c} \sigma^1)
\]

where the functions \( z^j \) are not determined by (6.6) alone.

Now a closed vortex is periodic in \( \sigma^1 \). In this case, then,

\[
z^j(\sigma^0 + \frac{1}{c} \sigma^1) = -\frac{1}{2} \epsilon^{jkl} \frac{H_{k\ell 0}}{H_{123}}(\sigma^0 + \frac{1}{c} \sigma^1) + \tilde{z}^j(\sigma^0 + \frac{1}{c} \sigma^1),
\]

\( \tilde{z}^j \) being a periodic function, and

\[
y^j(\sigma^0, \sigma^1) = -\epsilon^{jkl} \frac{H_{k\ell 0}}{H_{123}} \sigma^0 + \tilde{z}^j(\sigma^0 + \frac{1}{c} \sigma^1).
\]

But if the vortex is not closed, and its ends terminates on \( \partial \Sigma \), we can not conclude (6.10) and (6.11).
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References

[1] T.P. Chang and L.F. Li, *Gauge Theory of Elementary Particle Physics* (Oxford University Press, New York, 1984).

[2] P.G. de Gennes, *Superconductivity of Metals and Alloys*, (W.A. Benjamin, Inc., New York, 1966)

[3] E. Witten, Commun. Math. Phy. 121 (1989) 351-399; I. Kogan, Phys.Lett. B231 (1989) 377; D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Phys. Rept. 209 (1991), 129-340 and references therein.

[4] G.T. Horowitz, Commun. Math. Phys. 125 (1989) 417; V. Husain, Phys. Rev. D43 (1991) 1803.

[5] C. Aneziris, A.P. Balachandran, L. Kauffman and A.M. Srivastava, Int. J. Mod. Phys. A6 (1991), 2519-2559. For earlier work, see A.P. Balachandran, V.P. Nair, B.-S. Skagerstam and A. Stern, Phys. Rev. D26 (1982) 1443 and references therein.

[6] A. Harvey and J. Liu, Phys. Lett. B240 (1990), 369-374.

[7] T.J. Allen, M. Bowick and A. Lahiri, Mod. Phys. Lett. A6 (1991) 559-572; J.A. Minahan and R.C. Warner, Florida preprint UFIFT-HEP-89-15 (1989).

[8] M.J. Bowick, S.B. Giddings, J.A. Harvey, G.T. Horowitz and A. Strominger, Phys. Rev. Lett. 61 (1988) 2823; T.J. Allen, M. Bowick and A. Lahiri, Phys. Lett. B237 (1989) 47; T.J. Allen, Wisconsin preprint MAD/TH-91-14 (1991); S. Coleman, J. Preskill and F. Wilczek, Institute for Advanced Study preprint IASSNS-HEP-91/64 (1991); A. Lahiri, Los Alamos preprint LA-UR-92-471 (1992).
[9] A.P. Balachandran, G. Bimonte, K.S. Gupta and A. Stern, Alabama, Naples and Syracuse preprint UAHEP 9113, INFN-NA-IV-91/92 and SU-4228-487 (1991) and Int. J. Mod. Phys. A (in press).

[10] A.P. Balachandran, G. Bimonte, K.S. Gupta and A. Stern, Syracuse, Naples and Alabama preprint SU-4228-481, INFN-NA-IV-91/13 and UAHEP 9114 (1992) and Int. J. Mod. Phys. A (in press).

[11] R. Iengo and K. Lercher, Nucl. Phys. B364 (1991) 551 and references therein.

[12] For a review, see A.P. Balachandran and A.M. Srivastava, Minnesota and Syracuse preprint TPI-MINN-91-38-T and SU-4228-492 (1991) [to be published in a volume in honour of B. Vijayaraghavan]. It also contain references.

[13] For a review, see P. Goddard and D. Olive, Int. J. Mod. Phys. A1 (1986) 303.

[14] For a review, see C.N. Pope, L.J. Romans and X. Shen, Texas A&M University preprint CTP-TAMU-89-90 (1990).

[15] A.P. Balachandran and P. Teotonio-Sobrinho (in preparation).

[16] M.J. Bowick and S.G. Rajeev, Nucl. Phys. B296 (1988) 1007.

[17] A.P. Balachandran, G. Bimonte and P. Teotonio-Sobrinho (in preparation).

[18] V.I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, 1978), App. 2.

[19] A. Einstein, Ann. Physik 22 (1906) 800; 34 (1911) 170.

[20] P.A.M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science Monographs Series No. 2 (Yeshiva University, New York, 1964); A.P. Balachandran,
G. Marmo, B.-S. Skagerstam and A. Stern, *Classical Topology and Quantum States* (World Scientific, 1991) and references therein.

[21] Y. Choquet-Bruhat, C. DeWitt-Morette and M. Dillard-Bleick, *Analysis, Manifolds and Physics* (North-Holland, 1989).

[22] M. Kalb and P. Ramond, Phys. Rev. D9 (1974) 2273.