Decoherence in Josephson qubits

G. Falci\(^{(1)}\), E. Paladino\(^{(1)}\), and R. Fazio\(^{(2)}\)

\(^{(1)}\) Dipartimento di Metodologie Fisiche e Chimiche (DMFCI), Università di Catania, viale A. Doria 6, 95125 Catania, Italy & Istituto Nazionale per la Fisica della Materia, UdR Catania.

\(^{(2)}\) Scuola Normale Superiore, 56126 Pisa, Italy & NEST-INFM, Pisa.

Abstract

A high degree of quantum coherence is a crucial requirement for the implementation of quantum logic devices. Solid state nanodevices seem particularly promising from the point of view of integrability and flexibility in the design. However decoherence is a serious limitation, due to the presence of many types low energy excitations in the “internal” environment and of “external” sources due to the control circuitry. Here we study both kind of dephasing in a special implementation, the charge Josephson qubit, however many of our results are applicable to a large class of solid state qubits. This is the case of \(1/f\) noise for which we introduce and study a model of an environment of bistable fluctuators. External sources of noise are analyzed in terms of a suitable harmonic oscillator environment and the explicit mapping on the spin boson model is presented. We perform a detailed investigation of various computation procedures (single shot measurements, repeated measurements) and discuss the problem of the information needed to characterize the effect of the environment. For a fluctuator environment with \(1/f\) spectrum memory effects turn out to be important. Although in general information beyond the power spectrum is needed, in many situations this results in the knowledge of only one more microscopic parameter of the environment. This allows to determine which degrees of freedom of the environment are effective sources of decoherence in each different physical situation considered.
I. INTRODUCTION

Solid state coherent system are at the forefront of present day research because of the perception that large scale integration may be combined with new physical properties to yield new paradigms for nanoelectronics. A concrete example is quantum computation [1–3] with solid state devices, with several proposals [4–8] and few recent experiments [9–11]. More generally the ability of controlling the dynamics of a complex quantum system would open a wide scenario for both fundamental and applied physics.

Controlled dynamics may be achieved if first it is possible to prepare (write) and measure (read) a set of $N$ observables $\{Q_i\}$ [2]. This set defines the basis of the so called computational states $|\{q_i\}\rangle$. If moreover it is possible to tune the Hamiltonian of the system, then the dynamics of a generic state

$$|\psi, t\rangle = \sum_{q_1,\ldots,q_N} c_{q_1\ldots q_N}(t) |q_1,\ldots,q_N\rangle$$  \hspace{1cm} (1)$$

may be controlled. In general $|\psi, t\rangle$ is a superposition of computational states and moreover if $c_{q_i}(t)$ cannot be factorized in individual functions $c_{q_i}(t)$, then $|\psi, t\rangle$ is entangled [1,2]. Quantum algorithms use superpositions to produce constructive interference towards the correct answer and entanglement for the speed up of information processing. Thus certain calculations which are practically impossible on a classical computer could be performed if coherence could be preserved. Coherence usually denotes situations when a well defined relation between the components $c_{q_i}(t)$ of the state Eq.(1) exists. In our case this simply means that the nanodevice can be described by the pure state Eq.(1), the relation between the $c_{q_i}(t)$ being the solution of the Schrödinger equation. Loss of coherence is due to the fact that the Hilbert space of the device is much larger than the computational space [12,13]. The system (defined by the set $\{Q_i\}$) interacts with the environment (defined by all the other observables needed to complete the set) [14]. Even a weakly coupled environment may cause decoherence [12], i.e. it may destroy the phase relation between $c_{q_i}(t)$. The system should be described by a density matrix $\rho(t)$ rather than a pure state as Eq.(1).

Considering a larger Hilbert space is needed because the nanodevice is a many-body object and $\{Q_i\}$ is only a small set of collective variables. Decoherence from these “internal” sources represents a serious limitation due to the presence of many low energy excitations in the solid state environment. Clever protocols and technological progress may reduce these effects, but even in an idealized situation there is an “external” environment of apparata (for preparation, measurement, tuning of the Hamiltonian during time evolution) which are themselves quantum systems enlarging the overall Hilbert space. To appreciate the importance of the external environment, notice that of course we would be happy with a system we can manipulate at will, but to obtain easy tunability we have to open a port to the external world and this determines decoherence.
In this work we consider the charge-Josephson qubit [7]. It is a superconducting island, where the charge $Q$ can be stored, connected to a circuit via a Josephson junction and a capacitance $C_2$ (Fig.1a). Adding a charge polarizes the surroundings and costs energy, provided by the voltage source $V_x$. Then $V_x$ may be used to fix $Q$ to the desired value. On the other hand Josephson tunneling mixes the charge states. Under suitable conditions (charging energy $E_C = e^2/2(C_1 + C_2)$ much larger than the Josephson coupling $E_J$ and temperatures $k_BT \ll E_J$) only two charge states are important and the system realizes a qubit with Hamiltonian

$$\mathcal{H}_Q = \frac{\varepsilon}{2} \sigma_z - \frac{E_J}{2} \sigma_x \quad ; \quad \varepsilon(V_x) = 4E_C(1 - C_2V_x/e)$$

where the eigenstates $\{|n\rangle : n = 0, 1\}$ of $\sigma_z$ represent a well defined extra number $n$ of Cooper pairs in the island and span the computational space. Superconducting qubits [5,6,8] are important because they are the only solid state implementations where coherence in a single qubit has been observed in the time domain [9,10] and promising experiments exist for two-qubit systems [11]. Problems on decoherence can thus be posed in a realistic perspective.

We will study models of environments which describe decoherence due to fluctuations of the external circuit and to $1/f$ noise produced by charges which may be trapped close to the device (Fig.1b). These have been recognized to be major sources of errors in charge-Josephson qubits, because the computational states $\{|n\rangle\}$ are coupled to charges moving in the environment\(^1\). The system plus environment Hamiltonian reads

$$\mathcal{H} = \mathcal{H}_Q - \frac{1}{2} \hat{E} \sigma_z + \mathcal{H}_E$$

where $\mathcal{H}_E$ describes the environment, coupled to the system via the operator $\hat{E}$, which acts as an extra contribution to the polarization $\varepsilon$.

\(^1\)We will not discuss here other applications of our work, but we stress that charge noise is ubiquitous in the solid state and it is important also for solid state qubits based on the electron spin, where proposed implementations of two-qubits gates use the Coulomb interaction. Moreover the models we consider can be directly applied to flux noise in flux-Josephson qubits [28,29].
Since we are interested on measurements on the qubit, the standard road-map is to calculate the reduced density matrix $\rho(t) = \text{Tr}_E\{W(t)\}$ where $W(t)$ is the system plus environment density matrix. Equations for $\rho(t)$ can be written in terms of statistical information on the environment, i.e. correlation functions like $\langle \hat{E}(t_1)\hat{E}(t_2) \ldots \rangle_E$ and system-environment correlations, which makes the problem formidable. In practice we hope that in order to make predictions we do not really need such a detailed knowledge of the microscopic parameters which define $\mathcal{H}_E$. This may happen in two remarkable cases, namely if the environment is weakly coupled and fast, and if the environment is modeled by harmonic oscillators. Then all the information needed on the environment is encoded in the power spectrum of the coupling operator $\hat{E}$

$$S(\omega) = \int_{-\infty}^{\infty} dt \frac{1}{2} \langle \hat{E}(t)\hat{E}(0) + \hat{E}(0)\hat{E}(t) \rangle e^{i\omega t}$$

(4)

Roughly speaking if the environment is weakly coupled and fast on the time scales typical of decoherence, then the system is unable to probe it in great detail. If the environment is made of harmonic oscillators all its equilibrium statistical properties can be derived from the power spectrum\(^2\). Unfortunately this is not enough to describe all the effects of a solid state environment, since low-energy excitations may determine memory effects and in general specific gates are sensitive to different details of the environment.

To conclude this introduction we notice that although decoherence comes from the entanglement of the system with its environment, the reduction of the amplitude of the coherent signal in specific experiments may often be studied in less fundamental terms. For instance one may think to the environment as producing a classical stochastic field which couples to the qubit. In this way spontaneous emission is missed but, apart from that, this may be a useful point of view for practical estimates of the effect of solid state environments.

**II. SIMPLE ESTIMATES OF DECOHERENCE**

**A. Master Equation**

For a weakly coupled environment the trace on the environmental degrees of freedom can be approximately calculated in several ways, in second order in the interaction [16]. A standard approach leads to the Markovian Master equation in the basis of the eigenstates of $\mathcal{H}_Q$ [17]

$$\dot{\rho}_{ij} = -i\omega_{ij} \rho_{ij} + \sum_{mn} R_{ijmn} \rho_{mn}$$

(5)

where $\omega_{ij}$ is the difference of the energy eigenvalues. The relaxation tensor $R_{ijmn}$ depends on combinations of quantities of the kind $\int_0^\infty dt C_{ijkl}^z(t)$ and may be calculated if the Green’s function of the environment $i G^z(t) = \langle \hat{E}(t)\hat{E}(0) \rangle_E$, or its Fourier transform $2 S(\omega)/(1 + $\ldots$

\(^2The Caldeira Leggett model has been proposed in this spirit to describe the environment in Macroscopic Quantum Tunneling and Coherence in Josephson circuits [15]
e^{-\beta \omega} ), is known \(^3\). In deriving this equation it has been assumed that the environment is always at equilibrium and it is fast, i.e. \( C^>(t) \) should decay on a time scale \( \tau_c \) which is the smallest scale in the problem. Often the sum in Eq.(5) can be restricted to the secular terms (terms such that \( \omega_{ij} = \omega_{mn} \)) and the result for the relaxation rate \( \Gamma_R \), governing the exponential decay of the populations \( \rho_{ii} \) towards equilibrium, and for the decoherence rate \( \Gamma_\phi \) which describes the vanishing of the coherences \( \rho_{ij} \), is readily found

\[
\Gamma_R = \frac{1}{2} \sin^2 \theta S(\Omega) \quad ; \quad \Gamma_\phi = \Gamma_0 + \frac{1}{2} \Gamma_R = \frac{1}{2} \cos^2 \theta S(0) + \frac{1}{2} \Gamma_R
\]

where \( \Omega = \sqrt{\epsilon^2 + E_J^2} \) is the bare level splitting and \( \tan \theta = -E_J/\epsilon \) is the angle characterizing \( \mathcal{H}_Q \) in the Bloch sphere representation. As we will see the major problems in solid state come from the so called adiabatic term \( \Gamma_\phi^0 \), which depends on low frequencies of the environment. The result Eqs.(6) suggests that an optimal operation point is \( \theta = \pi/2 \) and that dephasing only depends on the power spectrum of the environment at the operating frequency \( \Omega = E_J \). A typical protocol is depicted in Fig.2a.

FIG. 2. Bloch sphere representation of the typical protocol used to generate a phase shift in a qubit, by operating with a slightly noisy gate (a) at the optimal operation point, \( \theta = \pi/2 \), and (b) at \( \theta = 0 \), when “pure” dephasing occurs. Notice that for charge-Josephson qubit charge states can be easily set and measured, so preparation and redout occur along the \( \hat{z} \) axis.

B. Pure dephasing

For \( \theta = 0 \) the system is sensitive to low frequencies of the environment. This case, usually referred to as “pure dephasing”, is special in that the Hamiltonian (3) commutes with \( \sigma_z \). The charge in the island is conserved and no relaxation occurs. However if we prepare the qubit in a superposition of charge states the system will dephase and consequently the coherences will decay (a typical protocol is depicted in Fig.2b).

If the initial density matrix is factorized, \( W(0) = w_E(0) \otimes \rho(0) \), it is possible to write an exact expression for the coherences only in terms of the environment \([13,18]\).

\(^3\)The correlators \( C_{ijkl}^>(t) \) read \( C_{ijkl}^>(t) = Tr_E \left\{ w_E(0) \langle i | e^{iH_E t} H_{int} e^{-iH_E t} | j \rangle \langle l | H_{int} | k \rangle \right\} \) and \( C_{ijkl}^<(t) = Tr_E \left\{ w_E(0) \langle i | H_{int} | j \rangle \langle l | e^{iH_E t} H_{int} e^{-iH_E t} | k \rangle \right\} \), for a factorized initial density matrix \( W(0) = w_E(0) \otimes \rho(0) \). The system-environment interaction term for the Hamiltonian Eq.(3) reads \( H_{int} = -\hat{E} \sigma_z/2 \).
\[
\rho_{01}(t) = \rho_{01}(0) e^{-\Gamma(t) - i\delta E(t)} \quad ; \quad \Gamma(t) = -\ln|\text{Tr}_E\{w_E(0) e^{i\mathcal{H}_0 t} e^{-i\mathcal{H}_1 t}\}| \tag{7}
\]
where \(\mathcal{H}_\eta = \mathcal{H}_E - (\eta/2) \hat{E}\).

C. Model for circuit fluctuations

In order to apply the above results we now specify a model for the environment which describes fluctuations of the external circuit, modeled by an external impedance \(Z(\omega)\) (Fig.1b). On the classical level the effect is accounted for by substituting \(V_x \rightarrow V_x + X(t)\) in Eq.(2), where \(X(t)\) are the classical voltage fluctuations \(X(\omega)\) at the impedance. The phenomenological quantum model is obtained by applying the following rules. First we substitute \(V_x \rightarrow V_x + X\) in Eq.(2). Then we introduce a set of harmonic oscillators to describe the impedance and to weight quantum fluctuations of \(X\)

\[
\mathcal{H}_E = \sum_\alpha \left\{ \frac{\hat{p}_\alpha^2}{2m_\alpha} + \frac{m_\alpha\omega_\alpha^2}{2} \hat{x}_\alpha^2 \right\} \quad ; \quad X = \sum_\alpha c_\alpha x_\alpha
\]

which specifies the Hamiltonian (3). Finally the power spectrum of \(X\) is identified by the voltage fluctuations at \(Z(\omega)\)

\[
S_X(\omega) = |\omega| \text{Re} \frac{Z(\omega)}{1 + i\omega Z(\omega) C_{\text{eff}}} \text{ctgh} \frac{\beta |\omega|}{2} = J(\omega) \text{ctgh} \frac{\beta |\omega|}{2}
\]

where \(C_{\text{eff}} = C_1 C_2/(C_1 + C_2)\) and to make contact with the standard notation \[14\] we introduced the spectral density of the environment \(J(\omega)\). This allows to identify \(S(\omega) = (4E_C C_2/e)^2 S_X(\omega)\) and to estimate the rates using Eq.(6). At the optimal point the quality factor \(E_J/\Gamma_\phi = (C_1/C_{\text{eff}})^2 2R_Q/(\pi R) \sim 10^6\) for typical values of the parameters (we took \(Z(\omega) = R\) and \(R_Q = \hbar/(4e^2)\) is the superconducting quantum of resistence), which would allow a single qubit gate to work perfectly. Adiabatic dephasing would lead to an even larger \(E_J/\Gamma_\phi\), by a factor \(E_J/K_B T\).

D. 1/f noise

Numerous experiments have shown that the performance of single electron tunneling (SET) devices strongly suffer from fluctuations of background charges (BC) located in the vicinity of the junctions \[19,20\]. Their behavior can be visualized as a random extra polarization \(E(t)\) which produces voltage fluctuations at the device. Voltage noise can be measured in SET transistors, and it has been observed to have the \(1/f\) form up to \(1\ GHz\) \[19\]. Using experimental data we may identify \(S(\omega) = 16\pi A E_C^2 / \omega\) where \(A\) is about \(10^{-6}\) \[19\] and Eq.(6) at the optimal point leads to a quality factor \(E_J/\Gamma_\phi \sim 10^3\). This procedure underlies two questionable assumptions: first, one should extrapolate observed \(1/f\) noise up to frequencies \(E_J \sim 10\ GHz\), second, one should make the ad hoc assumption that there are
nonequilibrium impurity charges at such frequencies, much larger than the temperature\(^4\). Apart from this Eqs.(6) badly fails in estimating \(\Gamma_0\) which is proportional to the inverse of the small (and unmeasurable) low-frequency cut-off \(\gamma_m\) of 1/f noise. In order to have a more reliable estimate it has been proposed that one may describe the environment as a set of harmonic oscillators [26]. In this case the exact expression Eq.(7) may be evaluated [13], yielding

\[
\Gamma_{osc}(t) = \int_0^\infty \frac{d\omega}{\pi} S(\omega) \frac{1 - \cos \omega t}{\omega^2}.
\]

(8)

For times \(t \ll 1/\gamma_M\), where \(\gamma_M\) is the high-frequency cut-off of the 1/f noise, \(\Gamma_{osc}(t)\) is approximated by \(\Gamma_{osc}(t) \approx 8AE_C^2 \ln \left(\frac{\gamma_m}{\gamma_M}\right) t^2\). The result is still dependent on \(\gamma_m\) in a relevant way: each decade of noise, including noise produced by very slow fluctuators, gives exactly the same contribution to dephasing. It has been proposed the recipe to use instead an ad hoc effective low-frequency cut-off which gives resonable results, but in order to have a clear picture of what is going on one should take more seriously the discrete character of charge noise. In particular the main difficulty in treating the 1/f environment by the above standard methods is that the large majority of degrees of freedom are much slower than all time scales of the evolution of the system, so they give rise to important memory effects. As a consequence different gates are sensitive to different details of the environment.

### III. MODEL FOR 1/F NOISE

In this section we introduce a simple model of an environment which yields 1/f noise, which is a suitable set of bistable fluctuators [18]. We first consider the fluctuators to be sources of a classical stochastic process, i.e. each fluctuator switches between two configurations with total rate \(\gamma_i\) and determines an extra polarization of the qubit given by \(\frac{v_i}{2}p_i(t)\), where \(p_i(t) = \pm 1\). The total extra polarization, \(E(t) = \sum_i \frac{v_i}{2}p_i(t)\), has power spectrum \(S(\omega) = \sum_i (v_i/2)^2 \int_{-\infty}^{\infty} dt (p_i(t)p_i(0) - \delta p^2) e^{i\omega t} = \sum_i v_i^2/2(1 - \bar{p}^2) \gamma_i/(\gamma_i^2 + \omega^2)\), where the overline means average on the stochastic processes. The standard assumption [21] of a distribution of switching rates \(P(\gamma) \propto 1/\gamma\) for \(\gamma \in [\gamma_m, \gamma_M]\) and zero elsewhere leads to 1/f noise, \(S(\omega) = \{\pi(1 - \bar{p}^2)n_d v^2/(4 \ln 10)\} \omega^{-1}\) for frequencies \(\omega \in [\gamma_m, \gamma_M]\) (\(n_d\) is the number of fluctuators per noise decade). Already at this level it is clear that the environment presents a large number of slow fluctuators, so memory effects are important.

We introduce a quantum model [18,23] by describing each fluctuator as a localized impurity level connected to a band [24]. The system plus environment Hamiltonian reads

\[
H = H_Q - \frac{1}{2} \sigma_z \sum_i v_i b_i^\dagger b_i + \sum_i \mathcal{H}_i
\]

(9)

\(^4\)Measurements on the accuracy of single-electron traps showed indirect effects of 1/f noise at high frequencies [22]. They were interpreted within a similar “one photon” approach which requires the ad hoc assumption that there are nonequilibrium impurity charges at such frequencies. This analysis yields the better value \(A \sim 10^{-8}\) but is certainly non conclusive.
\[ \mathcal{H}_i = \varepsilon_i b_i^\dagger b_i + \sum_k \left[ T_{ki} c_{ki}^\dagger b_i + \text{h.c.} \right] + \sum_k \varepsilon_{ki} c_{ki}^\dagger c_{ki}. \]

Here \( \mathcal{H}_i \) describes an isolated BC: the operators \( b_i \) (\( b_i^\dagger \)) destroy (create) an electron in the localized level \( \varepsilon_{ci} \). This electron may tunnel, with amplitude \( T_{ki} \) to a band described by the operators \( c_{ki} \), \( c_{ki}^\dagger \) and the energies \( \varepsilon_{ki} \). For simplicity we assume that each localized level is connected to a distinct band. An important scale is the total switching rate

\[ \gamma_i = 2\pi N(\varepsilon_{ci})|T_i|^2 \]

(\( N \) is the density of states of the electronic band, \( |T_{ki}|^2 \approx |T_i|^2 \)), which characterizes the classical relaxation regime of each BC. Finally the coupling with the qubit is such that each BC produces a bistable extra bias \( v_i \).

It may be argued that the impurity model introduced here describes impurities in metals, rather than in insulating substrates. However our aim is to discuss consequences of the discrete nature of the BCs and this is the simplest model embedding this feature. Moreover this model has been introduced and used in Ref. [23] to successfully explain experiments on charge trapping in systems of small tunnel junctions very similar to the charge-qubit.

Our aim is to investigate the effect of the BC environment on the dynamics of the qubit. The picture which emerges from our analysis is that the contribution of the single BC in dephasing the qubit depends on the ratio \( g_i \equiv v_i/\gamma_i \). As a consequence, we distinguish between two different kinds of BC: the ones with \( v/\gamma \ll 1 \), which we call weakly coupled and the ones in the other regime, which we call strongly coupled. Concerning these latter notice that we are interested to a physical situation where the \( v_i \) are so small that the energy scale associated to the total extra bias produced by the set of BCs is much smaller than \( \Omega \), so strongly coupled charges means small \( \gamma_i \).

Strongly coupled BCs give rise to memory effects and moreover differences of their statistical properties from those of an oscillator environment (cumulants higher than the second are nonvanishing) may be relevant [21]. As a consequence the results Eqs.(6) may be inapplicable. One possibility is to calculate higher orders in the Master equation, but even if one assumes that the environment is still modelled by harmonic oscillators, the corrections for \( 1/f \) spectrum are of the same order of the leading terms [25]. Here we follow a different strategy, namely we enlarge the “system” which allows to treat more accurately the dynamics of the BCs, at all orders in the couplings \( v_j \). In the modified roadmap we consider only the continuos electron bands in Eq.(9) as the environment and investigate the reduced dynamics of the system composed by the qubit and the BCs. We find that weakly coupled charges behaves as a source of gaussian noise, whose effect is fully characterized by the power spectrum \( S(\omega) \). On the other hand, from our quantum mechanical treatment it emerges that the decoherence due to strongly coupled charges shows pronounced features of their discrete character. In the following we will specialize our analysis to the two cases \( \theta = 0, \pi/2 \).

**IV. QUBIT AT THE OPTIMAL POINT**

If the environment is made of a single BC we can implement the new roadmap by solving the Master equation Eq.(5) for a system composed by the qubit and the BC. This method is unpractical for a large number \( N \) of BCs since the number of equations becomes \( 2^{2N} - 1 \). So we study the general problem by using the Heisenberg equations of motion. For the average values of the qubit observables \( \langle \sigma_\alpha \rangle, \alpha = x, y, z \) we obtain \( (\hbar = 1) \)
\[
\langle \dot{\sigma}_x \rangle = \sum_i^N v_i \langle \hat{b}_i \hat{b}_i \sigma_y \rangle \quad ; \quad \langle \dot{\sigma}_y \rangle = E_J \langle \sigma_z \rangle - \sum_i^N v_i \langle \hat{b}_i \hat{b}_i \sigma_x \rangle \quad ; \quad \langle \dot{\sigma}_z \rangle = -E_J \langle \sigma_y \rangle
\]

In the rhs averages of new operators which involve also the localized levels and the bands are generated and we can write new equations for them. This procedure could be iterated and we would obtain an infinite chain of equations, but instead, at this stage, we factorize high order averages, so we are left with a set of \(3(N + 1)\) equations. In practice, we ignore the cumulants \(\langle \hat{b}_i \hat{b}_i \hat{b}_j \hat{b}_j \rangle_c, \langle \hat{b}_i \hat{b}_i \hat{b}_j \hat{b}_j \sigma_\alpha \rangle_c\) for \(i \neq j\) and insert the relaxation dynamics for the BCs in the approximated terms. This method gives accurate results for general values of \(g_i\) even if \(v_i/E_J\) is not very small, as we checked by comparing with numerical evaluation of the reduced density matrix of the qubit with one and two BCs. Results are presented in Fig. 3 where the time Fourier transform of \(\langle \sigma_z(t) \rangle\), proportional to the average charge on the island, is shown. We assumed factorized initial condition for the qubit and the BCs. We first consider a set of weakly coupled BCs in the range \([10^{-2}, 10] E_J\) which determine \(1/f\) noise in a frequency interval around the operating frequency. The coupling strengths \(v_i\) have been generated uniformly with approximately zero average and with magnitudes chosen in order to yield the amplitude of typical measured spectra [19,20,22] (extrapolated at GHz frequencies).

**FIG. 3.** (a) The Fourier transform \(\sigma_z(\omega)\) for a set of weakly coupled BCs plus a single strongly coupled BC (solid line). The separate effect of the coupled slow BC alone \((g_0 = 8.3, \text{dashed line})\) and of the set of weakly coupled BCs (dotted line), is shown for comparison. In the inset the corresponding power spectra: notice that at \(\omega = E_J\) the power spectrum of the extra charge alone (dashed line) is very small. In all cases the noise level at \(E_J\) is fixed to the value \(S(E_J)/E_J \approx 3.18 \times 10^{-4}\). (b) The Fourier transform \(\sigma_z(\omega)\) for a set of weakly coupled BCs plus a strongly coupled BC \((v_0/\gamma_0 = 61.25)\) prepared in the ground (dash-dotted line) or in the excited state (solid line).

These BCs are weakly coupled, and determine a dephasing rate which reproduces the prediction of Eq.(6) (Fig.3 dotted line \(\Gamma_\phi/E_J \approx 1.65 \times 10^{-4}\)). Now we add a slower (and strongly coupled \(g_0 = v_0/\gamma_0 = 8.3\)) BC, in order to extend the \(1/f\) spectrum to lower frequencies. The added BC gives negligible contribution to \(S(E_J)\) so according to Eq.(6) it should not modify \(\Gamma_\phi\). Instead, as shown in Fig.3a, we find that the strongly coupled BC alone determines a dephasing rate comparable with that of the weakly coupled BCs. The overall \(\Gamma_\phi\) is more than twice the prediction of Eq.(6).

The above result shows that \(\Gamma_\phi\) does not depend only on \(S(E_J)\). Moreover we checked that information beyond the full \(S(\omega)\) is needed: we considered sets of charges with different
and $v_i$ which realize the same power spectrum $S(\omega)$ and we find they yield different values of $\Gamma_{\phi}$. The decoherence is larger if BCs with $g \gtrsim 1$ are present in the set. In summary these results show that Eq.(6) underestimates the effect of strongly coupled BCs and in particular that a finite $\Gamma_{\phi}$ at the optimal point can be obtained even if the $1/f$ spectrum does not extend up to frequencies $\sim E_J$.

If we further slow down the added BC we find that $\Gamma_{\phi}$ increases toward values $\sim \gamma_0$, the switching rate of the BC. This indicates that the effect of strongly coupled BCs on decoherence tends to saturate. We discuss later similar results for the case of pure dephasing, where this conclusion can be made sharp, but we stress here that this is a consequence of the nonlinearity inherent to the discrete nature of the BC environment. In this regime we observe also effects related to the initial preparation of the strongly coupled BC (see Fig.3b). As we will discuss later we may describe different experimental procedures using different preparations of the environment. Thus we have here a first example of the fact that different measurement protocols should be analyzed separately as far as dephasing due to BCs is concerned.

V. PURE DEPHASING

In the absence of the tunneling term Eq.(9) is a model for pure dephasing. The analysis can be carried out using the exact result Eq.(7). We notice that for our model Eq.(9) $\mathcal{H}_{\eta}$ can be decomposed in a sum of commuting terms each referring to a BC, so if we assume factorized $w_E(0)$ Eq.(7) also factorizes $\rho_{01}(t) = \rho_{01}(0) \prod_{j=1}^{N} \exp\{-i(\varepsilon - v_j/2)t\} f_j(t)$ in averages referring to a single BC. Using a real-time path-integral technique, the general form of $f_j(t)$ in Laplace space is obtained [18]

$$f_j(\lambda) = \frac{\lambda + K_{1,j}(\lambda) - i v_j/2 \delta p^0_j}{\lambda^2 + (v_j/2)^2 + \lambda K_{1,j}(\lambda) + v_j/2 K_{2,j}(\lambda)},$$  

where $\delta p^0_j = 1 - 2 \langle b_j^\dagger b_j \rangle_{t=0}$ specify the initial conditions for the charges. The kernels $K_{1,j}(\lambda)$ and $K_{2,j}(\lambda)$ are expressed by formal series expressions in the tunneling amplitudes $T_j$.

An interesting explicit form of the kernels is obtained in the Non Interacting Blip Approximation (NIBA) [14], which amounts to approximate the kernels $K_{1,j}(\lambda)$ and $K_{2,j}(\lambda)$ at lowest order in the couplings $v_j$. The result is

$$K_{1,j}(\lambda) = \gamma_j; \quad K_{2,j}(\lambda) = -\frac{\gamma_j^2}{\pi} [\psi(1/2 + \beta/2\pi (\lambda - i\epsilon_{c_j})) - \psi(1/2 + \beta/2\pi (\lambda + i\epsilon_{c_j}))].$$

In order to appreciate the physical meaning of the NIBA result, we notice that it is also obtained using the Heisenberg equations of motion, which yield a closed set of equations provided that the electronic band is assumed in thermal equilibrium. Thus the NIBA result, besides describing the dynamics of the qubit plus BC at all orders in the couplings $v_j$, is valid even if the couplings $T$ with the bands are not small. What is possibly missed are details of the dynamics of the bands, which is in total agreement with the philosophy of the modified roadmap outlined in Sec.III.
A. Nearly incoherent BC dynamics

We now want to apply these results to a set of BCs which produce $1/f$ noise. In order to produce a clear physical picture, we will consider only the physically relevant limit where the BCs have an incoherent dynamics. This limit can be formally obtained from the NIBA result by approximating the kernel $K_{2,j}(\lambda) \sim K_{2,j}(0)$, and can be expressed in analytic form

$$\rho_{01}(t) = \frac{\rho_{01}(0)}{e^{i\varepsilon t} N \prod_{j=1}^{N} e^{iv_j t/2} \left\{ A_j e^{-\frac{\gamma_j}{2}(1-\alpha_j)t} + (1-A_j) e^{-\frac{\gamma_j}{2}(1+\alpha_j)t} \right\}}$$

where

$$\alpha_j = \sqrt{1 - g_j^2 - 2i \varepsilon_{cj} \tgh (\beta \varepsilon_{cj}/2)} ; \quad A_j = \frac{1}{2\alpha_j} \left( 1 + \alpha_j - i \delta p_0^0 g_j \right).$$

In the framework of the NIBA this result is valid if $\varepsilon_{ci}, v_i, \gamma_i \ll K_B T$, but again its validity is broader. Indeed Eq.(12) it may be obtained in different ways, for instance by analyzing the system of qubit and BC by a master equation and taking the limit where $\varepsilon_{ci}$ is the largest scale, or even as the exact result of a model where the coupling operator $\sum_i v_i b_i^\dagger b_i$ is substituted by a classical stochastic process $E(t)$ which is the sum of random telegraph processes (see App. B).

The form of Eqs.(12,13) elucidates the different role of weakly and strongly coupled BCs in the decoherence process. We focus on the long time limit $\gamma t \gg 1$. Dephasing due to each BC comes from the sum of two exponential terms. For extremely weakly coupled charges, $g_j \ll 1$, we have $\alpha_j \approx 1$ and only the first term is important, which describes decay of the coherences with a rate $\approx 1/4 \cosh^2(\beta \varepsilon_{cj}/2) v_j^2/\gamma_j$. This is precisely the result of Eq.(6) for the adiabatic rate $\Gamma_0$. For strongly coupled charges, $g_j \gg 1$, each of the two exponentials in Eq.(12) expresses roughly the same decay rate $\propto \gamma_j$, the switching rate of the individual BC, and moreover they come with the same weight. The individual $\alpha_j$s have large imaginary parts so the main effect of strong coupling with the qubit is not decay but rather an energy shift. The short time limit, which is more important in actual experiments, will be discussed in the next subsection.

The long time limit allows to draw a simple picture of the effect of a BC: the qubit is practically insensitive to very fast BCs ($g_j \gg 1$), which are averaged completely. The dephasing effect increases as $v_j^2/\gamma$ as the BC gets slower but eventually saturates. Indeed a very slow BC ($g_j \gg 1$) will dephase only when it switches (effect $\propto \gamma$) so for most of the time it provides a static extra polarization for the qubit. Based on this picture one could conclude that slow charges could be neglected, however it is not a priori clear if this is a valid choice for the $1/f$ spectrum, where the number of slow charges is large due to the $P(\gamma) \propto \gamma$ distribution. Eq.(12) allows to answer to this question, but we defer this point to section VI and we first discuss in detail other aspects of the result for a single BC.

B. Single BC

Results for a single BC are concentrated in Fig. 4a. We consider Eq.(12) for $N = 1$ and look at the quantity

$$\Gamma(t) = -\ln \left| \frac{\rho_{01}(t)}{\rho_{01}(0)} \right|$$
In Fig.4 we compare this quantity with the exact result for an oscillator environment, Eq.(8), which is reported as the thick dashed line. We show the effect of a single BC for various initial conditions and different values of $g$. Weakly coupled charges ($g = 0.1$) give a contribution close to $\Gamma_{osc}(t)$, whereas deviations are observed in the other cases. In particular BCs such that $g > 1$ show slower dephasing compared to an oscillator environment with the same $S(\omega)$, as a manifestation of saturation effects. Recurrences at times comparable with $1/v$ are visible in $\Gamma(t)$.

Fig. 4a allows to discuss initial conditions of the BC. For each value of $g$ three cases are shown. The thick lines correspond to $\delta p_0 = 1$ (the dotted lines to $\delta p_0 = -1$), i.e. the BC is initially in the lower (higher) energy classical configuration. These are typical initial situations to be considered in a single shot process. In this case $\Gamma(t)$ describes dephasing 	extit{during} time evolution. Differences in the behavior of $\Gamma(t)$ reflect memory effects. In particular for $\gamma t \ll 1$ the approximate behavior is $\Gamma(t) \approx v^2 t^2 (1 - \delta p_0^2)/8 - \gamma v^2 t^3 (1 + 2\delta p_0 \overline{\delta p} - 3\delta p_0^2)/24$, whereas $\Gamma_{osc}(t) \approx v^2 t^2 (1 - \overline{\delta p}^2)/8$. Thus if $\delta p_0 = \pm 1$, $\Gamma(t) \propto t^3$ which accounts for the fact that a two level system is stiffer than a set of oscillators. On the other hand if we choose $\delta p_0 = \overline{\delta p}$, the equilibrium value, $\Gamma(t)$ decays more rapidly, $\Gamma(t) \approx \Gamma_{osc}(t)$. These latter choice of initial conditions corresponds physically to repeated measurements in which we do not control the preparation of the BC so $\Gamma(t)$ describes both dephasing 	extit{during} time evolution and the blurring of the total signal of several realizations of the time evolution with slightly different characteristic frequency, a sort of inhomogeneous broadening. Notice finally that if we use the above equilibrium initial conditions $\Gamma(t)$ follows $\Gamma_{osc}(t)$ for short times, the two curves becoming indistinguishable only if moreover $g \ll 1$. This observation is important for the subsequent analysis of inhomogeneous broadening (Sec. VIII).

The above analysis of decoherence due to a single BC has clearly evidenced the different qualitative influence on the qubit dynamics of weakly and strongly coupled BCs. A single weakly coupled, $v/\gamma \ll 1$, BC behaves as a source of gaussian noise. Thus decoherence only depends on the power spectrum of the fluctuator, and does not show any dependence on the initial condition of the BC. On the other side decoherence due a single strongly coupled BC, $v/\gamma \gg 1$, displays saturation effects and dependence on the initial condition.

FIG. 4. (a) Reduced $\Gamma(t)$ due to a BC with different preparations (thin lines) for the indicated values of $g$. Inset: longer time behaviour for stable state preparation. The thick dashed line represents the oscillator approximation; (b) Saturation effect of slow BCs for a $1/f$ spectrum. Relevant parameters ($\overline{\nu} = 9.2 \times 10^7$ Hz, $n_d = 1000$) give typical experimental measured noise levels and reproduce the observed decay of the echo signal [20] in charge Josephson qubits. Couplings $\nu_i$ are distributed with dispersion $\Delta \nu/\overline{\nu} = 0.2$. $\Gamma(t)$ is almost unaffected by strongly coupled charges (the label is the number of decades included).
VI. 1/F NOISE IN SINGLE SHOT MEASUREMENTS

We now want to check how the intuitive picture for an environment made of a single fluctuator may apply to a set of charges producing 1/f noise. The distribution of BCs involves both weakly and strongly coupled fluctuators, so that no typical time scale is present. Moreover since a very large number of slow fluctuators is present, it is not a priori clear how saturation effects will manifest.

In Fig.4b we show the results for a sample with a number of BCs per decade $n_d = 1000$ and with $v_i$ distributed with small dispersion around the value $|v| = 9.2 \times 10^7$ Hz. We choose initial conditions $\delta p_j^0 = \pm 1$ randomly distributed on the set of $N$ charges to reproduce equilibrium conditions in the ensemble, $(1/N) \sum_j \delta p_j^0 \approx \bar{\delta p}$. We checked that for every realization of the initial conditions we obtained roughly the same overall polarization and the same dephasing. What we are going to calculate is then the result of single shot measurements, i.e. the average signal of several experiments on the time evolution of the qubit where the total initial polarization of the environment is recalibrated before each experiment. This quantity is essentially dephasing during time evolution and the corresponding protocol is such to minimize the effects of the environment. To make reference to a concrete situation, the results we present were calculated with parameters close to the experiments [20].

In order to illustrate the different role played by the BCs with $g_j \ll 1$ and $g_j \gg 1$, we now perform a spectral analysis of the effects of the environment. We consider sets of BCs with the same $\gamma_M = 10^{12}$ Hz$^5$ and decreasing $\gamma_m$, all producing 1/f noise with the same amplitude $\mathcal{A}$ in the corresponding frequency range. Solid lines in Fig.4b represent $\Gamma(t)$ calculated with Eq.(12). In this example the dephasing is given by BCs with $\gamma_j > 10^7$ Hz $\approx |v|/10$. The main contribution comes from three decades at frequencies around $|v|$. The overall effect of the strongly coupled BCs ($\gamma_j < |v|/10$) is minimal, despite of their large number, showing the saturation effect of low-frequency noise. Dashed lines represent $\Gamma_{osc}(t)$, the result of the approximation of the environment with a set of harmonic oscillators, Eq.(8). In this case low-frequency noise does not saturate, each decade of noise equally contributes to dephasing and $\Gamma_{osc}(t)$ depends on the low-frequency cut-off of 1/f noise. We notice finally that for this single shot protocol $\Gamma(t)$ is roughly given by $\Gamma_{osc}(t)$ provided we use $\omega \sim |v|$ as a low frequency cut-off.

This observation also explains the fact that our results are not very sensitive to the value of $n_d$ we choose. Indeed in order to reproduce a given amplitude $\mathcal{A}$ we must keep constant for each decade $\sum_i v_i^2 \approx n_d v^2$, meaning that the effective low-frequency cut-off $\sim |v|$ varies as $n_d^{-1/2}$. For $n_d \to \infty$ the low-frequency cut-off goes to zero, as in the result Eq.(8) for the oscillator environment.

Finally we point out that even if we performed our calculation by assigning all the microscopic parameters of the environment, a reliable estimate of dephasing for single shot measurements turns out to depend on a single additional parameter besides the power spectrum $S(\omega)$, namely the average coupling $|v|$ or equivalently the number of charges producing a decade of noise, $n_d$.

---

5 This value of $\gamma_M$ is surely too large, but in this section we are interested to the way dephasing changes by adding low-frequency noise decade after decade.
VII. COMPARISON WITH THE OSCILLATOR ENVIRONMENT

In this section we make a more quantitative comparison of our results with the results for an equivalent oscillator environment. It is useful to consider our result from the point of view of the classical stochastic approach of App. B which expresses the off diagonal element of the reduced density matrix of the qubit as an average over classical stochastic processes generated by a set of random telegraph fluctuators. The result for quantum oscillators Eq.(8) is the second cumulant expansion of Eq.(12). Alternatively one may verify that

$$\Gamma_{osc}(t) = \frac{1}{2} \sum_i g_i^2 \left[ \frac{\partial^2 \Gamma(t)}{\partial g_i^2} \right]_{g_i=0}$$

where $\delta p_i^0 = \delta \bar{p}$ has been posed in $\Gamma(t)$. This result is intuitive in that in the limit $v_i \to 0$ the noise is produced by $n_d \to \infty$ fluctuators per decade, and its discrete nature is lost. This agrees with the fact that in this limit all BCs are weakly coupled.

To give an idea of the numbers involved we checked this conclusion by calculating $\Gamma(t)$ from Eq.(12), using different sets of BCs. The results are shown in Fig.5a. The power spectrum $S(\omega)$ is identical for all the curves shown, which differ in the choice of $n_d$. The gaussian behavior is recovered in the long time limit $\gamma_m t \gg 1$ for large enough $n_d$ (all the BCs are weakly coupled). If in addition we take $\delta p_j^0 = \delta \bar{p}$, $\Gamma(t)$ approaches $\Gamma_2(t)$ also at short times.

VIII. REPEATED MEASUREMENTS AND INHOMOGENEOUS BROADENING

Measurements of the state of a charge-Josephson qubit involve the measurement of a single extra Cooper pair in the superconducting island. Different strategies of measurements have been proposed and experimental prototypes have been produced, but in practice measurements are not single-shot. In the simple scheme used in Ref. [9,20] the time evolution procedure is repeated $\sim 10^5$ times and what is measured is the total current due to the possible presence of the extra Cooper pair in all the repetitions. The signal is then the sum over different possible time evolutions of the BCs with initial conditions which are also randomly fluctuating. The average effective precession frequency of the qubit is then different in different realizations and the overall signal decays faster than a single shot measurement would yield. In this respect this additional decay of the signal is analogous to inhomogeneous broadening in NMR, where the signal is collected by many noninteracting “qubits” each with its specific environment which determines different average precession frequencies. One difference may possibly be that strong correlations may exist between the different repetitions, due to the strong correlations in time of $1/f$ noise.
FIG. 5. (a) $\Gamma(t)/\Gamma_2(t)$ for a $1/f$ spectrum between $\gamma_m = 2 \times 10^7$ and $\gamma_M = 2 \times 10^9$ with different numbers of BCs per decade: (a) $n_d = 10^2$, (b) $n_d = 4 \times 10^3$, (c) $n_d = 8 \times 10^3$, (d) and (dI) $n_d = 4 \times 10^4$, (e) and (eI) $n_d = 4 \times 10^5$. Full lines correspond to $\delta p_j^0 = \pm 1$, dashed lines to equilibrium initial conditions for the BCs. (b) Different averages over $\delta p_j^0$ for $1/f$ spectrum reproducing the measured noise level of Ref. [20]: $\overline{|v|} = 9.2 \times 10^6 H z$, $n_d = 10^5$, $\gamma_m = 1 H z$, $\gamma_M = 10^9 H z$. Dashed lines correspond to the oscillator approximation with a lower cut-off at $\omega = \min\{|v|, 1/t_m\}$.

In order to study the overall signal we have to sum Eq.(12) over different sets of $\{\delta p_j^0\}$ representing initial conditions for each repetition starting at $t = t_\alpha$. Equivalently we may average Eq.(12) over a suitable distribution of $\{\delta p_j^0\}$ which depends on time. If BCs are assumed to be independent this distribution factorizes and since $\{\delta p_j^0\}$ enters linearly the coefficient $A$ we have only to evaluate Eq.(12) with $\delta p_j^0 = \delta p_j^0(t_m)$, where $\delta p_j^0(t_m)$ is the average of the values of $\delta p_j$ sampled at regular times $t_\alpha$ for the overall time $t_m$. As a rough estimate we may let $\delta p_j^0(t_m) = \delta p_j^0(0) = \pm 1$ if $\gamma_j t_m < 1$ and $\delta p_j^0(t_m) = \overline{\delta p}$ for $\gamma_j t_m > 1$, i.e. slow charges do not change roughly initial condition (but still they may dephase during time evolution) whereas fast charges completely average during $t_m$. In this way the additional scale $t_m$ enters the problem.

We consider only the case of long overall measurement time $|v| t_m \gg 1$ which is pertinent to the present day experimental conditions. From the results of Sec. V B and Sec. VII we may infer that BCs with $\gamma < 1/t_m < |v|$, being strongly coupled and with $\delta p_j^0 = \pm 1$, are ineffective whereas for the other BCs, being averaged, we may take $\Gamma(t) \approx \Gamma_{osc}(t)$ for small enough times and, as a result, $\Gamma(t) \approx \int_0^\infty d\omega S(\omega)(1 - \cos \omega t)/(\pi \omega^2)$. This would proof the recipe proposed in Ref. [26]. It is interesting to notice that this result and possible extensions to higher order effects [25] in the qubit-environment can be obtained with no reference to the quantum nature of the environment, and depend on the classical statistical properties of an equivalent random process.

In Fig.5b, (dotted lines) we show that indeed dephasing calculated as outlined above, is roughly given at short times by the oscillator environment approximation with a lower cut-off taken at $\omega \approx \min\{|v|, 1/t_m\}$ (dashed line). We also show results with a different averaging procedure $\delta p_j^0(t_m) = 1/t_m \int_0^{t_m} dt \overline{\delta p(t)}$ which takes into account the strongly correlated dynamics of $1/f$ noise (solid lines in Fig.5b.). These correlation do not affect the results except possibly for $t_m \approx |v|$.
FIG. 6. (a) Bloch sphere description of the echo protocol. (b) Decay of the echo amplitude for a $1/f$ spectrum, with noise level as in Fig.(4b) with $\gamma_m = 1Hz$ and different $\gamma_M$. The dashed lines correspond to the oscillator environment approximation.

IX. CHARGE ECHO

Echo-type techniques have been suggested [9,26] and experimentally tested [20] as a tool to reduce inhomogeneous broadening due to the low-frequency fluctuators of the $1/f$ spectrum. In the experiment of Ref. [20] the echo protocol consists of a $\pi/2$ preparation pulse, a $\pi$ swap pulse and a $\pi/2$ measurement pulse. Each pulse is separated by the delay time $t$ (see Fig. 6). Since the duration of each pulse is negligible with respect to $t$ we can estimate the decay of the coherence only looking at the evolution during the delay times. Within the semiclassical approach we obtain [30]

$$
\frac{\rho_{01}(t)}{\rho_{01}(0)} = \prod_j \left\{ A_1^j e^{-(1-Re\alpha_j)\gamma t} + A_2^j e^{-(1+Re\alpha_j)\gamma t} + A_3^j e^{-(1-im\alpha_j)\gamma t} + A_4^j e^{-(1+im\alpha_j)\gamma t} \right\} 
$$

(14)

where $\alpha_j$ is given by Eq.(13) and information on $\delta p^j_0$ is contained in $A_k^j$. We are interested to $\Gamma^{(2)}(t) = -\ln |\rho_{01}(t)/\rho_{01}(0)|$. Again the expansion of Eq.(14) to the second cumulant gives the result for an equivalent environment of quantum oscillators

$$
\Gamma^{(2)}_{osc}(t) = \frac{2}{\pi} \int_0^\infty d\omega S(\omega) \frac{(1 - \cos \omega t)^2}{\omega^2}
$$

Results are shown in Fig. 6 for the same set of BCs of Fig. 4 whose parameters are close to the experimental situation of Ref. [20]. First of all we checked that the dependence of $\Gamma^{(2)}(t)$ on initial conditions $\delta p^j_0$ is extremely weak even if we start from out of equilibrium configurations of the set of BCs. Moreover $\Gamma^{(2)}(t) \approx \Gamma^{(2)}_{osc}(t)$. This means that the echo procedure actually cancels the effect of strongly coupled charges.\(^7\) Only weakly coupled charges

---

\(^6\)As it is clear from the results of Sec. VI, for a given measured $A$ we had to infer $n_d$. This we made by comparing our results of this section with the echo experiment. This allows only a rough estimate of $n_d$ because, as in the other cases, the dependence on $n_d$ is weak.

\(^7\)This conclusion is valid as long as the delay time is short, $|tv| \ll 1$.
contribute to $\Gamma^{(2)}(t)$ and the environment behaves as a suitable set of harmonic oscillators. We remark that in the regime of parameters we consider, for given noise amplitude the echo signal is strongly dependent on the high frequency cut-off, as studied in Fig. 6.

X. CONCLUSIONS

In conclusion we have studied dephasing due to charge fluctuations in solid state qubits. We analyzed and compared two models for the environment, namely an environment of harmonic oscillators and an environment made of fluctuators with applications to $1/f$ noise which is probably the most serious limitation for these devices. Even if decoherence comes in general from the entanglement of the system with its environment, for solid state devices measurements of system-environment correlations are extremely hard and the reduction of the amplitude of the coherent signal in specific experiments may often be studied in less fundamental terms. We discussed the problem of the information needed to characterize the effect of the environment. For environments which are weakly coupled and fast or environments made of harmonic oscillators the information needed is entirely contained in the power spectrum of the operator which couples the environment to the system. For a fluctuator environment with $1/f$ spectrum memory effects and higher order moments are important so additional information is needed. We discussed in detail the remarkable case of pure dephasing where the reduction of the amplitude of the coherent signal is given by correlation functions of the environment alone (see Eq. (7)) and provided exact results for the fluctuator environment. In this case the additional information of the environment needed depends on the protocol but often reduces to a single parameter. A new energy scale emerges, the average coupling $\langle |v| \rangle$ of the qubit with the BCs, which is the additional information needed to discuss single shot experiments (alternatively one should know the order of magnitude of $n_d$, the number of BCs per decade of noise). For repeated experiments the relevant scale is instead $\min\{\langle |v| \rangle, 1/t_m\}$ where $t_m$ is the overall measurement time. Finally echo measurements are sensitive to the high-frequency cut-off $\gamma_M$ of the $1/f$ spectrum.

Our results are directly applicable to other implementations of solid state qubits. We only mention Josephson flux qubits [27,28] which suffer from similar $1/f$ noise, originated from trapped vortices. In that case the eigenstates of $\sigma_z$ are the flux states of the device. Also the parametric effect of $1/f$ noise on the coupling energy of a Josephson junction [29] can be analyzed within our model, as long as individual fluctuators do not determine large variations of $E_J$. In this case it may be possible that the same sources generate both charge noise and fluctuations of $E_J$. This can be accounted for in our model by choosing the “noise axis” as the $\hat{z}$ axis.

In this work we have also briefly discussed the possibility of finding optimal operating points for the qubit. This idea has been successfully implemented in the experiment of Ref. [10] and consists in operating with external parameters tuned at the points where the energy splittings of the system are less sensitive to fluctuations. However a slight deviation from the optimal point determines a strong degradation of the performances. An equivalent point of view is that the system should be tuned at the point where adiabatic dephasing cancels in lowest order, since the effect of a low frequency environment (in particular the $1/f$ environment) is minimized. An explicit example is the recent proposal of implementing a communication protocol with Josephson junctions [31]. Low-frequency noise can also be
minimized by echo techniques, but the flexibility in the implementation of gates is greatly reduced. A possibility is to implement quantum computation using Berry phases [8], where the design of gates includes an echo procedure, but a detailed analysis of the effect of 1/f noise is still missing.

Finally we mention that the sensitivity of coherent devices may be used to investigate high frequency noise [32]. In particular an accurate matching between measured inhomogeneous broadening, echo signal and relaxation may give reliable information on the actual existence of BCs at GHz and on the high-frequency cut-off of the 1/f spectrum.

APPENDIX A: LINEAR QUANTUM NOISE

Fluctuations of the electromagnetic circuit can be modeled by coupling the system to an environment of harmonic oscillators [14,33] which mimicks the external impedences (see Fig.1). In this Appendix we present a model for the electromagnetic environment and we derive an effective Hamiltonian $H_{\text{eff}}$ using no phenomenological argument [34]. This has two motivations. First, since in principle bare circuit parameters are well defined and tunable, we want to know precisely how this reflects on $H$. Second, in general coupling to the environment produces decoherence and energy shifts, which may in principle be large. In dissipative quantum mechanics shifts are usually treated either by introducing counterterms [33] or by writing $H_{\text{eff}}$ in terms of renormalized quantities [33]. The role of induced shifts, which is minor in the devices of Refs. [9,10], may be crucial in various situations (e.g. geometric quantum computation [8], dynamics of registers and error correction devices).

We consider the Cooper pair box [5] of Fig. 1. The external impedance is modeled by a suitable LC transmission line and the Lagrangian of the system is

$$L = \sum_{i=1,2} C_i \dot{\phi}_i^2 - V_J(\phi_1) + \sum_{\alpha} \left[ \frac{C_\alpha \dot{\phi}_\alpha^2}{2} - \frac{\phi_\alpha^2}{2L_\alpha} \right]$$

where $\dot{\phi}$ are voltage drops and the Josephson energy is $V_J(\phi_1) = -E_J \cos(2e\phi_1/\hbar)$. The environment is fully specified by the elements $C_\alpha$ and $L_\alpha$. The circuit is introduced by the constraint $\phi_1 + \phi_2 + \sum_{\alpha} \phi_\alpha = V_x$, which allows to eliminate one variable and to write

$$L = \frac{C_{\Sigma} \dot{\eta}^2}{2} - V_J(2e(\kappa_2 \Phi - \eta)\hbar) + L_b$$

$$L_b = \frac{1}{2} C_e \dot{\Phi}^2 + \sum_{\alpha} \left[ \frac{C_\alpha \dot{\phi}_\alpha^2}{2} - \frac{1}{2L_\alpha} \phi_\alpha^2 \right]$$

where $\eta = \kappa_2 \phi_2 - \kappa_1 \phi_1$, $C_{\Sigma} = \sum_i C_i$, $C_e = C_1 C_2 / C_{\Sigma}$, $\kappa_{1,2} = C_{1,2} / C_{\Sigma}$, and $\dot{\Phi} = V_x - \sum_{\alpha} \dot{\phi}_\alpha$. We next diagonalize $L_b$ and obtain the form

$$L_b = \sum_{\alpha} \left[ \frac{m_\alpha \dot{x}_\alpha^2}{2} - \frac{m_\alpha \omega_\alpha^2}{2} x_\alpha^2 \right] - C_e V_x \sum_{\alpha} d_\alpha \dot{x}_\alpha$$

In order to determine the parameters, notice that $L_b$ describes a series $C_e - Z$ circuit, with $\sum_\beta d_\beta \dot{x}_\beta = \sum_\alpha \dot{\phi}_\alpha = V_Z$. By comparing the linear response with the known classical
dynamics of $V_Z$ we determine the spectral density ($\omega > 0$)

$$J'(\omega) = \sum_{\alpha} \frac{\pi d_0^2 \delta(\omega - \omega_\alpha)}{2m_\alpha\omega_\alpha} = \text{Re}\left[\frac{Z(\omega)/\omega}{1 + i\omega Z(\omega)C_e}\right]$$

Notice that since $L_b$ is quadratic this procedure is an exact way to perform the diagonalization, due to circuit theory, and is valid after quantization due to the Ehrenfest theorem. We then perform a (canonical) transformation $\chi = \eta - \kappa_2 \Phi$ and obtain the total Lagrangian $L = L_a + L_b$ with

$$L_a = \frac{C_\Sigma}{2}(\dot{\chi} + \kappa_2 V_x - \kappa_2 \sum_\alpha d_\alpha \dot{x}_\alpha)^2 + V_J(2e\chi/\hbar)$$

One can verify that the variable canonically conjugated to $\chi$ is the charge $Q$ in the island. The Hamiltonian corresponding to $L$ reads

$$H = \frac{Q^2}{2C_1} + \frac{Q}{\kappa_2} \sum_\alpha \frac{d_\alpha}{m_\alpha} p_\alpha - E_J \cos\left(\frac{2e}{\hbar} \chi\right)$$

$$+ \sum_\alpha \left[ \frac{p_\alpha^2}{2m_\alpha} + \frac{m_\alpha \omega_\alpha^2}{2} x_\alpha^2 \right] + C_e V_x \sum_\alpha d_\alpha p_\alpha$$

where $p_\alpha$ are conjugated to $x_\alpha$ and we used the relation $\sum_\alpha d_\alpha^2/m_\alpha = 1/C_e$. A further canonical transformation of the environment ($\tilde{x}_\alpha = p_\alpha/(m_\alpha \omega_\alpha), \tilde{p}_\alpha = -m_\alpha \omega_\alpha x_\alpha$) yields

$$H_{eff} = \frac{Q^2}{2C_1} + V_J(\frac{2e}{\hbar} \chi) + \kappa_2 Q \sum_\alpha c_\alpha \tilde{x}_\alpha$$

$$+ \sum_\alpha \left[ \frac{\tilde{p}_\alpha^2}{2m_\alpha} + \frac{m_\alpha \omega_\alpha^2}{2} \tilde{x}_\alpha^2 \right] + C_e V_x \sum_\alpha c_\alpha \tilde{x}_\alpha$$

where $c_\alpha = d_\alpha \omega_\alpha$ and we introduce $J(\omega) = \pi \sum_\alpha \delta(\omega - \omega_\alpha)c_\alpha^2/(2m_\alpha \omega_\alpha) = \omega^2 J'(\omega)$, the spectral density.

If we isolate the dc bias $V_x(t) = V_x + \delta V_x(t)$ and redefine $\tilde{x}_\alpha + c_\alpha C_e V_x/(m_\alpha \omega_\alpha^2) \to x_\alpha$ we finally obtain

$$H_{eff} = \frac{Q^2}{2C_1} - \kappa_2 V_x Q + V_J(\frac{2e}{\hbar} \chi) + \kappa_2 \sum_\alpha c_\alpha x_\alpha$$

$$+ \sum_\alpha \left[ \frac{p_\alpha^2}{2m_\alpha} + \frac{m_\alpha \omega_\alpha^2}{2} x_\alpha^2 \right] + C_e \delta V_x(t) \sum_\alpha c_\alpha x_\alpha$$

Notice that the capacitance $C_1$ (and not $C_\Sigma$) enters the $Q^2$ term. and if we put $c_\alpha = 0$ we do not obtain the Cooper pair box hamiltonian. This is correct because the environment represents global fluctuations of the circuit, not only of $Z$. Notice that a static $Q$ shifts the equilibrium points of the oscillators and also produces a $Q$-dependent shift of the zero of their energies. This latter can be reabsorbed in the charging energy if we write the oscillator hamiltonian using the shifted values $x_\alpha + c_\alpha \kappa_2 Q/(m_\alpha \omega_\alpha^2)$, which produces the extra term $-\kappa_2^2 Q^2 \sum_\alpha c_\alpha^2/(2m_\alpha \omega_\alpha^2) = -Q^2/(2C_e)$ and
\[ H_{\text{eff}} = \frac{Q^2}{2C} - \frac{\kappa_2 V_x Q}{2} + V_J \left( \frac{2e\chi}{\hbar} \right) + C_e \delta V_x \sum_{\alpha} c_{\alpha} x_{\alpha} \]
\[ + \sum_{\alpha} \left[ \frac{p_{\alpha}^2}{2m_\alpha} + \frac{m_\alpha \omega_\alpha^2}{2} \left( x_{\alpha} + \frac{c_{\alpha} \kappa_2 Q}{m_\alpha \omega_\alpha^2} \right)^2 \right] \]

(A1)

which reduces to the non dissipative form by letting \( c_{\alpha} = 0 \). This is a convenient starting point for a weak coupling analysis also because a static shift in the equilibrium points of the oscillators has no effect even if it is large.

**APPENDIX B: DEPHASING DUE TO CLASSICAL STOCHASTIC FLUCTUATIONS**

In the semiclassical approach the coupling operator \( \sum_i v_i b_i^\dagger b_i \) is substituted by a classical stochastic process \( E(t) \) which is the sum of random telegraph processes. The system is initially in a given superposition of eigenstates of \( \mathcal{H}_Q \) Eq.(2), say \( \mid \psi, 0 \rangle = \alpha \mid a \rangle + \beta \mid b \rangle \). We make now a first assumption, namely we neglect relaxation and we write the time evolution using the adiabatic theorem \( \mid \psi, t \rangle = \alpha \mid a, t \rangle + \beta e^{-i \int_0^t dt' \Omega(t')} \mid b, t \rangle \), where now we use the instantaneous basis of \( \mathcal{H}_{ad} = \mathcal{H}_Q - \frac{1}{2} E(t) \sigma_z \) and the instantaneous level splitting \( \Omega(t) = \sqrt{[\varepsilon + E(t)]^2 + E_J^2} \). The error introduced by the random field \( E(t) \) is mostly due to the phase factor and we make a second assumption, namely we neglect the difference between the instantaneous basis and the eigenbasis of \( \mathcal{H}_Q \). In this case the diagonal elements of the density matrix \( \mid \psi, t \rangle \langle \psi, t \mid \) are constant whereas the off diagonal matrix element is

\[ \rho_{ba}(t) = \rho_{ba}(0) e^{-i \int_0^t dt' \Omega(t')} \].

(B1)

At this stage the standard assumption of the semiclassical approach is that averages over the environment correspond to averages over the stochastic process \( E(t) \). Thus decoherence during time evolution is estimated by

\[ \Gamma(t) = -\ln \left| \overline{e^{-i \int_0^t dt' \Omega(t')}} \right| \],

(B2)

where the overline denotes an average over different stochastic processes with given initial conditions. The effect of inhomogeneous broadening is included by a further averages on initial conditions.

Let us discuss the validity of this result. If \( E_J = 0 \) the two assumption leading to Eq.(B1) are rigorously verified and the result is exact. By working out the average we shall arrive to the result Eq.(12). In the general case the first assumption is valid if \( E(t) \) is slow on the time scale set by \( \Omega \) and the second if \( E(t) \ll \Omega \). The condition of slow environment is equivalently expressed by requiring that \( S(\omega) \) is small at frequencies \( \omega \sim \Omega \). In this case relaxation in leading order (see Eq.(6)) is negligible. Then Eq.(B2) describes the effect of a large part of the degrees of freedom composing the \( 1/f \) environment, if not all of them.

We may apply Eq.(B2) to repeated measurements as described in Sec.VIII. If one assumes that for small times we can approximate the environment to a suitable set of harmonic oscillators then we may approximate Eq.(B2) to the second cumulant. Moreover, under the
assumption of small $E(t)$ it is easy to derive an expression for the decay of the off diagonal matrix element

$$\Gamma(t) = \cos^2 \theta \Gamma_{osc}(t) + \frac{\sin^4 \theta}{8\Omega} \int_0^t dt' dt'' [E^2(t')E^2(t'') - \overline{E}^2]$$

This expression has been also derived in Ref. [25] where the oscillator environment has been studied in detail and we will not pursue this analysis. We only stress that the correction to the term containing $\Gamma_{osc}(t)$ depends on the so called second spectrum [21] of the environment which is known to differ for gaussian and non gaussian stochastic processes. So if the correction is important, i.e. if the system turns out to be sensitive to the second spectrum, then the approximation of Eq.(B2) to the second cumulant is inaccurate.

We turn now to the derivation of Eq.(12) for an environment of classical fluctuators. Eq.(B2) has to be averaged over the possible realizations of the stochastic process $E(t)$ with given initial conditions $\delta p_0$. As already discussed we need to perform the calculation only for a single Random Telegraph process switching between the two values $E_a(t) = -v/2$ and $E_b(t) = v/2$ with rates $\gamma_+$ and $\gamma_-$. (see Fig.(7a)).

FIG. 7. (a) Random Telegraph fluctuator switching between valor $\pm v/2$ with rates $\gamma_{\pm}$; (b) Each fluctuator can follow four kinds of paths. If intial and final states are equal, the path has an even number of transitions, an odd number of flips occurs if initial and final states are different.

The average $\overline{E(t)}$ is related to the average population difference between the two state $a$, $b$ of the BC, $\overline{\delta p(t)} = (\overline{\delta p^0} - \overline{\delta p}) e^{-\gamma t} + \overline{\delta p}$, by $\overline{E(t)} = -v/2\overline{\delta p(t)}$, where $\gamma = \gamma_+ + \gamma_-$ is the total switching rate, $\overline{\delta p^0}$ and $\overline{\delta p} = (\gamma_- - \gamma_+)/\gamma$ are respectively the initial and equilibrium population difference. For a stationary process $\overline{\delta p^0} = \overline{\delta p}$ the Fourier transform of the second cumulant $\overline{[E(t)E(t+\tau)]_c}$ gives the power spectrum of the polarization fluctuations $v^2/2(1-p^2) \gamma/(\gamma^2 + \omega^2)$.

In order to calculate the generating functional $Z(t) \equiv \exp\{-i \int_0^t dt' \overline{E(t')}\}$, we have to take into account all the possible realizations of the stochastic processes $E(t)$ with fixed initial conditions for the fluctuator, $\delta p^0$. To this end we have to consider that the fluctuator can follow one of the four kinds of paths illustrated in Fig.7b.

Then we have to perform (i) a sum over all possible number of switches, (ii) an integration over the corresponding transition times. For each process in Fig.7b we can define a probability density $\pi_{\alpha,\beta}(t; t_m-t_1|0), (\alpha, \beta \text{ can be either } a \text{ or } b)$ of finding the fluctuator in the state $\beta$ at time $t$ once it has been prepared in the state $\alpha$ at time $t = 0$ after $m$ transitions.
at times $t_i, i = 1, \ldots, m$. The probability densities of the elementary processes of having at most a single transition in a fixed time interval are easily found from the statistics of the populations of the two states $a$ and $b$, $|\pi_{\alpha}(t), \alpha = a, b$. The probability densities to stay in the states $a$ and $b$ are $\pi_{aa}(t_2|t_1) = \exp[-\gamma_+(t_2 - t_1)]$, and $\pi_{bb}(t_2|t_1) = \exp[-\gamma_-(t_2 - t_1)]$. The probability density of a single flip at time $t_1$ in the interval $t - t_0$ for instance from $a$ to $b$ reads $\pi_{ab}(t; t_1|t_0) = \exp[-\gamma_-(t - t_1)]\gamma_+ \exp[-\gamma_+(t_1 - t_0)]$. With this, the probability density of the complex processes $\pi_{\alpha,\beta}(t; t_m - t_1|0)$ are given by:

$$
\pi_{aa}(t; t_{2N} - t_1|0) = (\gamma_+\gamma_-)^N e^{-\gamma_+T_{2N} - \gamma_-T_{2}} = (\gamma_+\gamma_-)^N e^{-\gamma_+t - \gamma_-T_{2N}} \\
\pi_{ab}(t; t_{2N+1} - t_1|0) = (\gamma_+\gamma_-)^N \gamma_+ e^{-\gamma_+T_{2N+1} - \gamma_-T_{2}} = (\gamma_+\gamma_-)^N \gamma_+ e^{-\gamma_+t - \gamma_-T_{2N+1}}
$$

(B3)

The probability densities $\pi_{bb}(t; t_m - t_1|0)$, $\pi_{ba}(t; t_m - t_1|0)$ are found from the expressions $\pi_{aa}(t; t_m - t_1|0)$ and $\pi_{ba}(t; t_m - t_1|0)$ with the replacements $\gamma_+ \rightarrow \gamma_-$ and $\delta p \rightarrow -\delta p$. Each $\pi_{\alpha,\beta}(t; t_m - t_1|0)$ only depends on the number $m = 2N, (2N + 1)$ of switches and on the total times $T_\alpha$ during the random fluctuator has assumed value $E_\alpha$. Thus we have:

$$
Z(t) = \sum_{\alpha,\beta} p_0^\alpha z_{\alpha,\beta}(t) \\
z_{\alpha,\beta}(t) = \sum_{m=0}^{\infty} \int_0^t dt_m \ldots \int_0^{t_2} dt_1 \pi_{\alpha,\beta}(t; t_m - t_1|0) e^{-i(E_\alpha T_\beta + E_\beta T_\alpha)}
$$

(B4)

where $p_0^\alpha$ is the probability to find the fluctuator in the state $\alpha$ at time $t = 0$, and $m = 2N$ if $\alpha = \beta$ (the term $m = 0$ being 1), $m = 2N + 1$ if $\alpha \neq \beta$. The total times spent by the fluctuators in the state $a$ or $b$, $T_a$ and $T_b$, are given respectively by

$$
T_b = \sum_{n=1}^{2N} (-1)^n t_n \equiv T_{2N} \quad \quad \quad T_a = t - T_{2N} \quad \quad \quad \alpha = \beta \\
T_b = \sum_{n=1}^{2N} (-1)^n t_n + t - T_{2N+1} \equiv t + T_{2N+1} \quad \quad \quad T_a = -T_{2N+1} \quad \quad \quad \alpha \neq \beta
$$

(B5)

From eqs. (B3), (B5) $Z(t)$ can be written in the simple form

$$
Z(t) = Ae^{-\frac{\gamma}{\tau}(1-\alpha)t} + (1 - A)e^{\frac{\gamma}{\tau}(1+\alpha)t}
$$

(B6)

where $A$ and $\alpha$ are defined in Eq. (13). An interesting approximation of Eq.(B6) is obtained by taking its second cumulant expansion. This amounts to consider a Gaussian process $E(t)$ having the same power spectrum $S(\omega)$. Estimating the average given in Eq.(B6) by its second cumulant and taking $\delta p^0 = \delta p$ we get Eq.(8).

The above exact results of the semiclassical analysis represent an important limiting case of decoherence due to a quantum discrete environment. Eq. (B6) reproduces in fact the results presented in Section VA for the pure dephasing decay due to an environment of quantum bistable fluctuators, in the regime in which each fluctuator performs a relaxation dynamics.

We thank L. Faoro, A. D’Arrigo and F. Taddei who collaborated to part of the work presented here. We warmly thank G. Giaquinta who initiated the work on mesoscopic physics.
at the University of Catania. We thank M. Palma, A. Shnirman, G. Schön and C. Urbina for discussions which greatly sharpened the point of view presented in this work. Very useful discussion with L. Amico, J. Clarke, D. Esteve, F. Hekking, P. Lafarge, J. Friedman, M. Grifoni, D. van Harlingen, J.E. Mooji, Y. Nakamura, Y. Nazarov, F. Plastina, J. M. Raimond, V. Tognetti, U. Weiss, D. Vion and A. Zorin are finally acknowledged.
REFERENCES

[1] A. Ekert and A. Jozsa, Rev. Mod. Phys., 68, 733 (1996); Quantum Computation and Quantum Information Theory, edited by C. Macchiavello, G.M. Palma, A. Zeilinger, World Scientific (2000).
[2] M. Nielsen and I. Chuang Quantum Computation and Quantum Communication, Cambridge University Press, (2000).
[3] Experimental implementation of quantum computation, edited by R.G. Clark, Rinton Press, Princeton (2001).
[4] D. Loss and P. Di Vincenzo Phys. Rev. A, 57, 120 (1998).
[5] Y. Makhlin, G. Schön and A. Shnirman, Rev. Mod. Phys., 73, 357 (2001).
[6] D.A. Averin Sol. State Comm., 105, 659 (1998); L.B. Ioffe et al. Nature, 398, 679 (1999); J.E. Mooij et al. Science, 285, 1036 (1999).
[7] Y. Makhlin, G. Schön and A. Shnirman Nature 398, 305 (1999); A. Shnirman, G. Schön and Z. Hermon Phys. Rev. Lett. 79, 2371 (1997).
[8] G. Falci et al. Nature, 407, 355 (2000).
[9] Y. Nakamura, Yu.A. Pashkin, J.S. Tsai Nature, 398, 786 (1999).
[10] D. Vion et al. Science 296, 886 (2002); Y. Yu et al. Science, 296, 889 (2002); J. Martinis et al. Phys. Rev. Lett., 89, 117901 (2002); J. Friedman et al. Nature, 406, 43 (2000); I. Chiorescu et al. private communication.
[11] Yu. A. Pashkin et al. cond-mat/0212314.
[12] W. Zurek Physics Today, 44, 36 (1991).
[13] G.M. Palma, K.-A. Suominen and A.K. Ekert Proc. Roy. Soc. London A, 452, 567 (1996).
[14] U. Weiss Quantum Dissipative Systems 2nd Ed (World Scientific, Singapore 1999).
[15] A. Leggett et al Rev. Mod. Phys., 59, 1 (1987).
[16] A. G. Redfield IBM J. Research Develop, 1, 19 (1957); M. Lax Phys. Rev., 145, 110 (1966).
[17] C. Cohen-Tannoudji, J. Dupont-Roc and G. Grynberg Atom-Photon Interactions, Wiley-Interscience (1993)
[18] E. Paladino et al. Phys. Rev. Lett., 88, 228304 (2002).
[19] A.B. Zorin et al. Phys. Rev. B, 53, 13682 (1996).
[20] Y. Nakamura et al. Phys. Rev. Lett., 88, 047901 (2002).
[21] M.B. Weissman Rev. Mod. Phys., 60, 537 (1988).
[22] M. Covington et al. Phys. Rev. Lett., 84, 5192 (2000).
[23] R. Bauernschmitt and Y.V. Nazarov Phys. Rev. B, 47, 9997 (1993).
[24] G.D. Mahan Many-Particle Physics Kluwer Academic, New York, (2000)
[25] A. Shnirman, Y. Makhlin, G. Schön, Physica Scripta, T102, 147, (2002).
[26] A. Cottet et al. in Macoscopic Quantum Coherence and Quantum Computing edited by D.V. Averin, B. Ruggiero and P. Silvestrini, (Kluwer Pub., 2001),pg.111.
[27] L. Tian et al. in Proceedings of the NATO-ASI on Quantum Mesoscopic Phenomena and Mesoscopic Devices in Microelectronics, edited by I.O. Kulik and R. Elliatoglu (Kluwer Pub. 2000), pg. 429.
[28] J.E. Mooij et al., Science, 285, 1036 (1999).
[29] D. J. Van Harlingen et al. in Quantum Computing and Quantum Bits in Mesoscopic Systems Proceedings of the International Workshop on “Macroscopic Quantum Coherence and Computing”, Napoli 3-7 Giugno 2002 (Kluwer Academic Plenum Press) (2003).
[30] E. Paladino et al. in Quantum Computing and Quantum Bits in Mesoscopic Systems, Proceedings of the International Workshop on “Macroscopic Quantum Coherence and Computing”, Napoli 3-7 Giugno 2002 (Kluwer Academic Plenum Press) (2003).
[31] F. Plastina and G. Falci, Phys. Rev. B, 67, 224514 (2003)
[32] R. Aguado and L. P. Kouwenhowen Phys. Rev. Lett. 84, 1986 (2000)
[33] A. O. Caldeira and A. J. Leggett Ann. Phys. 149, 374 (1983).
[34] E. Paladino et. al Physica E 18, 39 (2003)