The non-Abelian Chern-Simons path integral on $M = \Sigma \times S^1$ in the torus gauge: a review

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Abstract

In the present paper we review the main results of a series of recent papers on the non-Abelian Chern-Simons path integral on $M = \Sigma \times S^1$ in the so-called “torus gauge”. More precisely, we study the torus gauge fixed version of the Chern-Simons path integral expressions $Z(\Sigma \times S^1, L)$ associated to $G$ and $k \in \mathbb{N}$ where $\Sigma$ is a compact, connected, oriented surface, $L$ is a framed, colored link in $\Sigma \times S^1$, and $G$ is a simple, simply-connected, compact Lie group.

We demonstrate that the torus gauge approach allows a rather quick explicit evaluation of $Z(\Sigma \times S^1, L)$. Moreover, we verify in several special cases that the explicit values obtained for $Z(\Sigma \times S^1, L)$ agree with the values of the corresponding Reshetikhin-Turaev invariant. Finally, we sketch three different approaches for obtaining a rigorous realization of the torus gauge fixed CS path integral.

It remains to be seen whether also for general $L$ the explicit values obtained for $Z(\Sigma \times S^1, L)$ agree with those of the corresponding Reshetikhin-Turaev invariant. If this is indeed the case then this could lead to progress towards the solution of several open questions in Quantum Topology.

1 Introduction

Let $M$ be a compact, connected, oriented 3-manifold, let $k \in \mathbb{N}$, and let $G$ be a simple, simply-connected, compact Lie group. At a heuristic level, the map which associates to every (colored) link $L$ in $M$ the (informal) Chern-Simons path integral $Z(M, L)$ corresponding to $(G, k)$ is then a link invariant. In a celebrated paper, cf. [79], Witten succeeded in evaluating $Z(M, L)$ and $Z(M) := Z(M, \emptyset)$ explicitly using arguments from Conformal Field Theory. Later, Reshetikhin and Turaev found rigorous versions $RT(M)$ and $RT(M, L)$ of Witten’s invariants $Z(M)$, $Z(M, L)$ using the representation theory of quantum groups, cf. [67, 68] and [78].

At present it is not clear if/how one can derive the algebraic expressions $RT(M, L)$ directly from the (informal) path integral expressions $Z(M, L)$ and whether it is possible to find a rigorous

\footnote{This was first observed in [73] where it was also suggested that $Z(M, L)$ might be related to the Jones polynomial.}
path integral realization $Z_{\text{rig}}(M, L)$ of $Z(M, L)$ (or, alternatively, of a gauge fixed version of $Z(M, L)$). These two open problems, “Problem (P1)” and “Problem (P2)”, are important by themselves, cf. [51] [72]. Moreover, if Problems (P1) and (P2) can be solved this could lead to progress towards the solution of several other open problems in the field of 3-manifold quantum topology, cf. Sec. 3.2 below.

In the present paper we will restrict our attention to the special base manifolds $M$ of the form $M = \Sigma \times S^1$ for which “torus gauge fixing” is available. This gauge fixing was first applied to the Chern-Simons path integral by Blau and Thompson in [17] where it was shown (cf. Remark 3.3 in Sec. 3.1 below) that this allows a remarkably quick and simple (informal) evaluation of $Z(M)$ and of $Z(M, L)$ in the special case where $L$ is a “fiber link” in $M = \Sigma \times S^1$, i.e. a link consisting only of loops which are “parallel to the $S^1$-component” (or, more precisely, a link consisting only of loops each of which is contained completely in some $S^1$-fiber of $M = \Sigma \times S^1$). In more recent work Blau and Thompson generalized torus gauge fixing first to non-trivial $S^1$-bundles $M$ (cf. [19]) and then to Seifert fibered spaces $M$ (cf. [20]) and used this to evaluate $Z(M)$ and $Z(M, L)$ for “fiber links” $L$ in $M$. By doing so they recovered the explicit expressions obtained earlier in [15] [16] where non-Abelian localization was applied to the CS path integral (cf. Remark 3.34 in Sec. 3.3 below).

Instead of trying to generalize the torus gauge fixing approach to the CS path integral to more general manifolds like in [19] [20] one can also try to generalize this approach to general (colored) links $L$ in the original (trivial) $S^1$-bundles $M = \Sigma \times S^1$. This question is studied in the series of papers [35] [36] [37] [25] [38] [39] [41] [40] where apart from the explicit evaluation of $Z(\Sigma \times S^1, L)$ for general $L$ we also consider the issue of finding a rigorous realization of $Z(\Sigma \times S^1, L)$. The short term goal of the program initiated in [35] [36] [37] [25] [38] [39] [41] [40] is to obtain a complete solution of the aforementioned problems (P1) and (P2) for manifolds $M$ of the form $M = \Sigma \times S^1$ (cf. Sec. 3.5.2 and Sec. 4 below). The medium term goal is to solve problems (P1) and (P2) for general links in all those manifolds $M$ considered in [19] [20] (by combining the ideas/methods in the present paper with those in [19] [20], cf. Sec. 5.1 below). The long term goal is to exploit this for making progress regarding some of the open problems in Quantum Topology hinted at above (and described in more detail in Sec. 5.2 below).

The present paper reviews and extends the results of [35] [36] [37] [25] [38] [39] [41] [40]. The emphasis is on the explicit evaluation of $Z(\Sigma \times S^1, L)$. The rigorous realization of (the torus gauge fixed version of) $Z(\Sigma \times S^1, L)$ is only outlined, cf. Sec. 4 below.

The present paper is organized as follows:

In Sec. 2 we give a self-contained rederivation of the formula, found in [38], for the general (informal) Chern-Simons path integral on $M = \Sigma \times S^1$ in the torus gauge, see Eq. (2.30) below. (Eq. (2.30) below is later rewritten in a suitable way, cf. Eq. (2.47) and cf. also Eq. (3.39b) in Sec. 3.2 below).

In Sec. 3 we evaluate $Z(\Sigma \times S^1, L)$ explicitly in several situations. First we give a complete evaluation of $Z(\Sigma \times S^1, L)$ in three interesting special cases (cf. Secs 3.1, 3.3, and 3.4) and then we sketch the evaluation of $Z(\Sigma \times S^1, L)$ in the case of general $L$ (cf. Sec. 3.5 and cf. also Appendix B.2).

**Footnotes:**

1. We have included some new material, cf. Sec. 3.2.3, Sec. 4.3, Sec. 5.2, Appendix B.2, Appendix B.6, and Appendix D. Moreover, we have streamlined the presentation in [35] [36] [37] [25] [38] [39] [41] [40], see, in particular, Sec. 2.2, Sec. 2.3, Sec. 5.2, and Appendices B.3, B.5.

2. i.e. for general, colored links $L$ in $M = \Sigma \times S^1$.

3. From the knot theoretic point of view the most interesting of these special cases is the case considered in Sec. 3.3 where $L$ belongs to a large class of colored torus (ribbon) knots in $S^3 \times S^1$. The explicit formula for $Z(\Sigma \times S^1, L)$ in this special case (cf. Eq. (3.38) below) can be generalized in a straightforward way (cf. Eq. (3.89) and the rewritten version Eq. (3.91) below). By combining Eq. (3.91) with a special case of Witten’s surgery formula (cf. Eq. (3.93)) we then arrive (for all $G$) at the so-called “Rosso-Jones formula” for colored torus knots in $S^3$ (cf. Eq. (3.97) below).
Appendix D. We refer to the beginning of Sec. 3 for a more detailed summary of the content of Sec. 3.

In Sec. 4 we summarize and sketch the various approaches studied in [35, 36, 37, 25, 38, 39, 41, 40] for obtaining a rigorous realization of $Z(\Sigma \times S^1, L)$ and of the computations in Sec. 3.

In Sec. 5 we conclude the main part of the present paper with a short outlook explaining in more detail the medium and long term goals mentioned above.

The present paper has four appendices.

In Appendix A we list the Lie theoretic and quantum algebraic notation used in the paper.

In Appendix B we fill in some technical details omitted in Sec. 2.

In Appendix C we recall the definition of Turaev’s shadow invariant in the special case relevant for us and discuss its relation with the Reshetikhin-Turaev invariant.

Appendix D is a supplement to Sec. 3.5.2.

2 The Chern-Simons path integral in the torus gauge

2.1 The original Chern-Simons path integral

Let $M$ be a compact, connected, oriented 3-manifold and let $G$ be a simple, simply-connected, compact Lie group. We denote the Lie algebra of $G$ by $\mathfrak{g}$ and we set

$$\mathcal{A} := \Omega^1(M, \mathfrak{g}),$$

$$\mathcal{G} := C^\infty(M, G)$$

(2.1) (2.2)

Let $k \in \mathbb{N}$ and let $\langle \cdot, \cdot \rangle$ be the unique $\text{Ad}$-invariant scalar product on $\mathfrak{g}$ normalized such that $\langle \check{\alpha}, \check{\alpha} \rangle = 2$ for every short real coroot $\check{\alpha}$ (w.r.t. to any fixed Cartan subalgebra of $\mathfrak{g}$).

The “Chern-Simons action function” associated to $M$, $G$, and the “level” $k$ is the function $S_{CS} : \mathcal{A} \rightarrow \mathbb{R}$ given by

$$S_{CS}(A) = -k\pi \int_M \langle A \wedge dA \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle \quad \forall A \in \mathcal{A}$$

(2.3)

where $[\cdot \wedge \cdot]$ denotes the wedge product associated to the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ and $\langle \cdot \wedge \cdot \rangle$ the wedge product associated to the scalar product $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$. The normalization of the scalar product $\langle \cdot, \cdot \rangle$ chosen above ensures that

$$\mathcal{A} \ni A \mapsto \exp(iS_{CS}(A)) \in \mathbb{C}$$

is “gauge invariant”, i.e. invariant under the standard right-action of the group $\mathcal{G}$ on $\mathcal{A}$ (cf. Eq. (2.9) below), see Sec. 1 in [79] for the case $G = SU(N)$ and, e.g., [69] for the case of general $G$.

A (smooth) knot in $M$ is a smooth embedding $K : S^1 \rightarrow M$. Note that, using the surjection $i : [0, 1] \ni s \mapsto e^{2\pi is} \in \{z \in \mathbb{C} \mid |z| = 1\} \cong S^1$, we can consider each knot $K$ in $M$ as a (smooth) loop $l : [0, 1] \rightarrow M$, $l(0) = l(1)$, in the obvious way.

In the following let us fix an (ordered) “link” in $M$, i.e. a finite tuple $L = (l_1, l_2, \ldots, l_m)$, $m \in \mathbb{N}$, of pairwise non-intersecting knots $l_i$. We equip each $l_i$ with a “color”, i.e. an irreducible, finite-dimensional, complex representation $\rho_i$ of $G$. By doing so we obtain a “colored link” $((l_1, l_2, \ldots, l_m), (\rho_1, \rho_2, \ldots, \rho_m))$, which will also be denoted by “$L$” in the following.

Here and in the following $\Omega^p(N, V)$ denotes, for every finite-dimensional real vector space $V$, the space of $V$-valued $p$-forms on a smooth manifold $N$. 

3
The “Chern-Simons path integral associated to \((M, G, k)\) and \(L\)” is the informal integral expression given by (cf. Sec. 1 in [79])

\[
Z(M, L) := \int_A \left( \prod_{i=1}^m \text{Tr}_{\rho_i}(\text{Hol}_{l_i}(A)) \right) \exp(iS_{CS}(A)) \, DA
\]

(2.4)

where \(DA\) is the (ill-defined) “Lebesgue measure” on the infinite-dimensional space \(A\) and where \(\text{Hol}_{l_i}(A) \in G\) is the holonomy of \(A \in A\) around the knot \(l = l_i, i \leq m\) (considered as a smooth loop), cf. Eqs (2.10) below. It will sometimes be convenient to work with the “normalization” \(\langle L \rangle\) of \(Z(M, L)\) given by

\[
\langle L \rangle := \frac{Z(M, L)}{Z(M)}
\]

(2.5)

where \(Z(M) := \int_A \exp(iS_{CS}(A)) \, DA\).

(2.6)

### 2.1.1 Restriction to the case of matrix Lie groups

Since every compact Lie group is isomorphic to a matrix Lie group we can (and will) assume, without loss of generality, that \(G \subset \text{GL}(N, \mathbb{R})\) (and hence \(g \subset \text{Mat}(N, \mathbb{R})\)) for some fixed \(N \in \mathbb{N}\). (This will be very convenient in Sec. 2.3 and in parts B.3–B.5 of the appendix below.) We can then rewrite Eq. (2.3) as

\[
S_{CS}(A) = k\pi \int_M \text{Tr}(A \wedge dA + \tfrac{3}{2} A \wedge A \wedge A)
\]

(2.7)

where \(\wedge\) is the wedge product for \(\text{Mat}(N, \mathbb{R})\)-valued forms and where \(\text{Tr} : \text{Mat}(N, \mathbb{R}) \to \mathbb{R}\) is the trace functional normalized such that

\[
\text{Tr}(CD) = -\langle C, D \rangle \quad \text{for all } C, D \in g \subset \text{Mat}(N, \mathbb{R}).
\]

(2.8)

(This is always possible because, by assumption, \(G\) is simple.)

For two \(\text{Mat}(N, \mathbb{R})\)-valued forms \(\alpha\) and \(\beta\) we will simply write \(\alpha\beta\) instead of \(\alpha \wedge \beta\) if \(\alpha\) or \(\beta\) is a 0-form. The standard right-action of \(G\) on \(A\) mentioned above can then be written as

\[
A \cdot \Omega = \Omega^{-1} A \Omega + \Omega^{-1} d\Omega \quad \forall A \in A, \Omega \in G
\]

(2.9)

Moreover, for every \(A \in A\) and every smooth loop \(l : [0, 1] \to M\) we then have

\[
\text{Hol}_l(A) = P_1(A)
\]

(2.10a)

where \(P(A) = (P_s(A))_{s \in [0, 1]}\) is the unique smooth map \([0, 1] \to \text{Mat}(N, \mathbb{R})\) such that

\[
\forall s \in [0, 1] : \frac{d}{ds}P_s(A) = P_s(A) \cdot A(l'(s)), \quad P_0(A) = 1
\]

(2.10b)

where “\(\cdot\)” is the multiplication of \(\text{Mat}(N, \mathbb{R})\).

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6In the physics literature the notation \(P \exp(\int_l A)\) is often used instead of \(\text{Hol}_l(A)\).
2.2 Torus gauge fixing

As mentioned in Sec. 1 “torus gauge fixing” was introduced in [17] and used for the (informal)
evaluation of \( Z(\Sigma \times S^1) \) and \( Z(\Sigma \times S^1, L) \) for “fiber links” \( L \) in \( M = \Sigma \times S^1 \). In [36, 37, 38] the
formula (7.9) in [17] was generalized\(^7\) to arbitrary links \( L \) in \( M = \Sigma \times S^1 \), cf. Eq. (2.23), Eq. (2.36),
and Eq. (2.47) below. In Sec. 2.2, Sec. 2.3, and Appendix B we give a shortened (but
self-contained) rederivation of these three equations.

In order to motivate the derivation of Eq. (2.23) in Sec. 2.2 below we will first derive an
analogous formula for the manifold \( S^1 \).

2.2.1 Motivation: Torus gauge fixing for the manifold \( S^1 \)

We will now fix, for the rest of this paper, a maximal torus \( T \) of \( G \) and denote by \( t \) the Lie
algebra of \( T \) and by \( k \) the \( \langle \cdot, \cdot \rangle \)-orthogonal complement of \( t \) in \( g \).

By \( \frac{\partial}{dt} \) we will denote the vector field on \( S^1 \) which is induced by the map \( i_{S^1} : [0, 1] \ni s \mapsto \exp(2\pi is) \in \{ z \in \mathbb{C} | |z| = 1 \} \cong S^1 \) and by \( dt \) we denote the 1-form on \( S^1 \) which is dual to \( \frac{\partial}{dt} \).

Moreover, we set

\[
A_{S^1} := \Omega^1(S^1, g), \quad \mathcal{G}_{S^1} := C^\infty(S^1, G). \tag{2.11a}
\]

The group \( \mathcal{G}_{S^1} \) acts on \( A_{S^1} \) from the right by the obvious analogue of Eq. (2.9) in Sec. 2.1
above.

In the following we want to show, informally, that for every continuous\(^8\) \( \mathcal{G}_{S^1} \)-invariant function \( \chi : A_{S^1} \to \mathbb{C} \) we have

\[
\int_{A_{S^1}} \chi(A) DA \sim \int_t \chi(b \ dt) \det(1_k - \exp(\text{ad}(b))|_t) db \tag{2.12}
\]

where \( \sim \) denotes equality up to a multiplicative constant independent of \( \chi \) and where \( 1_k \) is the
identity on \( k \). Moreover, \( DA \) is the (informal) Lebesgue measure on \( A_{S^1} \) and \( db \) denotes the
normalized Lebesgue measure on \( t = (t, \langle \cdot, \cdot \rangle) \). (Here \( \int_t \cdots \ db \) and \( \int \cdots DA \) are sloppy notations
for the improper integrals \( \int_t \cdots \ db \) and \( \int \cdots DA \) appearing in Proposition 2.1 and Remark 2.2
below.)

Eq. (2.12) can be derived using a standard Faddeev-Popov determinant computation. In the
present section we will give an alternative derivation of Eq. (2.12) which is based on a corollary
of the Weyl integral formula (cf. Proposition 2.1 and Appendix B.1 below) and which has at
least the following two advantages: Firstly, it is probably more accessible for mathematicians
and secondly, and more importantly, it can be extended successfully to the situation in Sec.
2.2.2 below. [By contrast, the argument using the Faddeev-Popov determinant computations
leads to certain difficulties when applied to \( M = \Sigma \times S^1 \) if the surface \( \Sigma \) is compact, cf. the
second paragraph after Eq. (2.19) below.]

\(^7\)cf. part (ii) of Remark 2.4 in Sec. 2.2.2 below and Remark 5.1 in Sec. 5.1 for more comments regarding the
relation between Eq. (7.9) in [17] and Eq. (6.8) in [18] on the one hand and our Eq. (2.23), Eq. (2.36),
and Eq. (2.47) on the other hand.

\(^8\)Here we assume that \( A_{S^1} \) is equipped with a suitable topology, which we do not specify since the derivation
of Eq. (2.12) is informal anyway.
**Proposition 2.1** For every continuous conjugation invariant function \( f : G \to \mathbb{C} \) we have

\[
\int_G f(g) dg \sim \int_1^\infty f(\exp(b)) \det(1_t - \exp(\text{ad}(b))_t) db
\]  

(2.13)

where \( dg \) is the normalized Haar measure on \( G \) and where \( \int_1^\infty \phi(b) db \) is a suitably defined improper integral which extracts the “mean value” of a periodic function \( \phi \) on \( t \).

Using Proposition 2.1 we can now derive Eq. (2.12) above as follows:

Let \( \hat{G}_{S^1} := \{ \Omega \in G_{S^1} \mid \Omega(1) = 1 \} \). It is not difficult to see that\(^{11}\) \( \psi : A_{S^1}/\hat{G}_{S^1} \ni [A] \mapsto \mathrm{Hol}_{i_{S^1}}(A) \in G \) is a well-defined bijection. From the bijectivity of \( \psi \) it follows that for every \( \hat{G}_{S^1} \)-invariant \( \chi : A_{S^1} \to \mathbb{C} \) there is a \( \tilde{\chi} : G \to \mathbb{C} \) such that

\[ \chi = \tilde{\chi} \circ p \]

where

\[ p : A_{S^1} \ni A \mapsto \mathrm{Hol}_{i_{S^1}}(A) \in G. \]

Accordingly, we obtain, informally, for every \( \hat{G}_{S^1} \)-invariant function \( \chi : A_{S^1} \to \mathbb{C} \)

\[
\int_{A_{S^1}} \chi(A) DA = \int_{A_{S^1}} \tilde{\chi}(p(A)) DA \sim \int_G \tilde{\chi}(g) dg
\]  

(2.14a)

where step (*) is justified in Remark 2.2 below.

Next observe that as \( \chi \) was not only \( \hat{G}_{S^1} \)-invariant but even \( G_{S^1} \)-invariant the function \( \tilde{\chi} \) will be conjugation invariant. Accordingly, we now obtain from Proposition 2.1 above and the relation \( \exp(b) = \mathrm{Hol}_{i_{S^1}}(b)dt = p(b)dt \), \( b \in t \),

\[
\int_G \tilde{\chi}(g) dg \sim \int_1^\infty \tilde{\chi}(\exp(b)) \det(1_t - \exp(\text{ad}(b))_t)db
\]

\[
\sim \int_1^\infty \chi(b) dt \det(1_t - \exp(\text{ad}(b))_t) db
\]  

(2.14b)

By combining Eq. (2.14a) with Eq. (2.14b) we arrive at Eq. (2.12) above.

**Remark 2.2** In order to justify step (*) in Eq. (2.14a) above we will now (re)interpret the informal integral \( \int \chi(A) DA \) appearing above as the (informal) improper integral \( \int_1^\infty \chi(A) DA := \lim_{\epsilon \to 0} \int \chi(A) d\mu_\epsilon(A) \) where \( d\mu_\epsilon \) is the informal Gaussian measure on \( A_{S^1} \) given by \( d\mu_\epsilon(A) := e^{-\|A\|^2} DA / \int e^{-\|A\|^2} DA \) where \( \| \cdot \|_2 \) is the \( L^2 \)-norm on \( A_{S^1} \) associated to any fixed Riemannian metric on \( S^1 \). Step (*) in Eq. (2.14a) above then follows at an informal level provided that we can argue that \( p_\epsilon(d\mu_\epsilon) \to dg \) weakly as \( \epsilon \to 0 \) where \( p_\epsilon(d\mu_\epsilon) \) is the pushforward of \( d\mu_\epsilon \) under \( p \). By using standard techniques in probability theory one can obtain rigorous versions of this informal result. (One such rigorous version will be included in an additional part of the appendix in the next version of the present paper.)

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\(^9\)For example, we can use the definition \( \int_1^\infty \phi(b) db := \lim_{R \to -\infty} \lim_{Q \to R^{-1}} \int_{R \cap Q} \phi(b) db \) where \( Q \) is, e.g., a unit hyper-cube centered around 0 in \( t \) or the unit ball around 0 or we can make the ansatz \( \int_1^\infty \phi(b) db := \lim_{R \to 0} \int \phi(b) d\mu_\epsilon(b) \) where \( d\mu_\epsilon(b) := e^{-\|b\|^2} db / \int e^{-\|b\|^2} db \).

\(^{10}\)Note that \( \det(1_t - \exp(\text{ad}(b))_t) = \det(1_t - \mathrm{Ad}(\exp(b))_t) \) so \( \phi(b) = f(\exp(b)) \det(1_t - \exp(\text{ad}(b))_t) \) is periodic.\(^{11}\)The surjectivity of \( \psi \) follows from the fact that, since \( G \) was assumed to be compact and connected, the exponential map \( \exp : \mathfrak{g} \to G \) is surjective. The injectivity follows from a short explicit calculation.
2.2.2 Torus gauge fixing for \( M = \Sigma \times S^1 \)

Let \( M \) be a smooth 3-manifold of the form \( M = \Sigma \times S^1 \) where \( \Sigma \) is a connected, orientable surface. As in Sec. 2.1 we use the notation \( \mathcal{A} = \Omega^1(\Sigma \times S^1, \mathfrak{g}) \) and \( \mathcal{G} = C^\infty(\Sigma \times S^1, G) \).

Recall that at the beginning of Sec. 2.2.1 we introduced a vector field \( \frac{\partial}{\partial t} \) and a 1-form \( dt \) on \( S^1 \). The obvious “lift”/pullback of \( \frac{\partial}{\partial t} \) and \( dt \) to \( M = \Sigma \times S^1 \) will also be denoted by \( \frac{\partial}{\partial t} \) and \( dt \) in the following. Observe that we have

\[
\mathcal{A} = \mathcal{A}^\perp \oplus \mathcal{A}^\parallel
\]

where we have set

\[
\mathcal{A}^\perp := \{ A \in \mathcal{A} \mid A(\frac{\partial}{\partial t}) = 0 \},
\]

\[
\mathcal{A}^\parallel := \{ A_0 dt \mid A_0 \in C^\infty(\Sigma \times S^1, \mathfrak{g}) \}.
\]

In the following we set

\[
B := C^\infty(\Sigma, t)
\]

and we will make the identification

\[
B \cong \{ A_0 \in C^\infty(\Sigma \times S^1, t) \mid \forall \sigma \in \Sigma : A_0(\sigma, \cdot) \text{ is constant} \}
\]

Now let \( \chi : \mathcal{A} \to \mathbb{C} \) be a \( \mathcal{G} \)-invariant function, which we assume to be continuous w.r.t. a suitable topology on \( \mathcal{A} \) (cf. Footnote 8 above). By applying (in a naive way) a standard Faddeev-Popov determinant argument to the situation at hand one arrives, informally, at the following analogue of Eq. (2.12) above

\[
\int_{\mathcal{A}} \chi(A) \mathcal{D}A \sim \int_{\mathcal{B}} \left[ \int_{\mathcal{A}^\perp} \chi(A^\perp) \mathcal{D}A^\perp \right] \det(1_t - \exp(\text{ad}(B))) \mathcal{D}B
\]

where \( \sim \) denotes equality up to a multiplicative constant independent of \( \chi \). Moreover, \( \mathcal{D}B \) denotes the (informal) “Lebesgue measure” on \( \mathcal{B} \), \( 1_t - \exp(\text{ad}(B)) \) is the linear operator on \( C^\infty(\Sigma, \mathfrak{t}) \) given by \( (1_t - \exp(\text{ad}(B))) \cdot f(\sigma) = (1_t - \exp(\text{ad}(B(\sigma))) \cdot f(\sigma) \) for all \( \sigma \in \Sigma, f \in C^\infty(\Sigma, \mathfrak{t}) \) and \( \det(1_t - \exp(\text{ad}(B))) \) is its (informal) determinant. (See Sec. 2.3.2 below for a rigorous realization of \( \det(1_t - \exp(\text{ad}(B))) \).)

However, a more careful analysis of “torus gauge fixing” and its properties shows that, as a result of certain topological obstructions, Eq. (2.19) is not correct if \( \Sigma \) is compact, cf. Sec. 6 in \[18\] and Sec. 2.2 in \[38\]. (It will not be necessary to repeat the analysis in \[18\] or \[38\] here. The origin of the aforementioned topological obstructions will get obvious during our derivation of Eq. (2.23) below in Appendix B.2.)

Before we state the corrected version of Eq. (2.19) we need some notation. Let

\[
\mathcal{G}_\Sigma := C^\infty(\Sigma, G).
\]

The group \( \mathcal{G}_\Sigma \) acts on \( C^\infty(\Sigma, G/T) \) from the left by

\[
\Omega \cdot \tilde{g} = \Omega \tilde{g} \quad \text{for all } \tilde{g} \in C^\infty(\Sigma, G/T) \text{ and } \Omega \in \mathcal{G}_\Sigma
\]

More precisely, in Sec. 2.3.2 below we will make rigorous sense of the informal expression which we obtain after combining \( \det(1_t - \exp(\text{ad}(B))) \) with another factor, cf. Eq. (2.41b) and Eq. (2.46) in Sec. 2.3.1.
where $\Omega \bar{g} \in C^\infty(\Sigma, G/T)$ is given by $(\Omega \bar{g})(\sigma) = \Omega(\sigma)\bar{g}(\sigma)$ for all $\sigma \in \Sigma$. The corresponding orbit space will be denoted by \[ C^\infty(\Sigma, G/T)/G_{\Sigma}. \] (2.21)

Moreover, we set \[ B_{\text{reg}} := C^\infty(\Sigma, t_{\text{reg}}) \] (2.22)

where $t_{\text{reg}} \subset t$ is the union of the Weyl alcoves of $t$, cf. Appendix A.1 below.

Finally, we set $\bar{g}b\bar{g}^{-1} := gb\bar{g}^{-1} \in \mathfrak{g}$ for each $\bar{g} \in G/T$ and $b \in t$ where $g$ is an arbitrary element of $G$ fulfilling $gT = \bar{g}$.

In Appendix B.2 below we will derive (at an informal level) the following corrected version of Eq. (2.19)

\[ \int_{A} \chi(A) \mathcal{D}A \sim \sum_{h \in C^\infty(\Sigma, G/T)/G_{\Sigma}} \int_{B_{\text{reg}}} \left[ \int_{A^1} \chi(A^1 + \bar{g}_hB\bar{g}_h^{-1} dt) \mathcal{D}A^1 \right] \times \det(1 - \exp(\text{ad}(B))|t) \mathcal{D}B \] (2.23)

where $(\bar{g}_h)_{h \in C^\infty(\Sigma, G/T)/G_{\Sigma}}$ is any fixed system of representatives of $C^\infty(\Sigma, G/T)/G_{\Sigma}$ and where $\bar{g}_hB\bar{g}_h^{-1} \in C^\infty(\Sigma, \mathfrak{g})$, for $\bar{g}_h \in C^\infty(\Sigma, G/T)$ and $B \in B = C^\infty(\Sigma, t)$, is given by $(\bar{g}_hB\bar{g}_h^{-1})(\sigma) := \bar{g}_h(\sigma)B(\sigma)\bar{g}_h(\sigma)^{-1}$ for all $\sigma \in \Sigma$.

See Remark 2.1 in Appendix B.2 for a quick explanation regarding the origin of the differences between the RHS of Eq. (2.23) and the RHS of Eq. (2.19), and see also part (ii) of Remark 2.4 below for a comparison between our Eq. (2.23) and formula (6.8) in [15].

**Remark 2.3**

(i) From the assumption that $\dim(\Sigma) = 2$ and that $G$ is simply-connected it follows that two maps $\bar{g}_1, \bar{g}_2 \in C^\infty(\Sigma, G/T)$ are in the same $G_{\Sigma}$-orbit iff they are homotopic (cf. Proposition 3.2 in [10]). Accordingly, we can identify $C^\infty(\Sigma, G/T)/G_{\Sigma}$ with the set $[\Sigma, G/T]$ of homotopy classes of (smooth or continuous) maps $\Sigma \to G/T$.

On the other hand, every\( \text{\textsuperscript{13}} \) non-compact (connected, orientable) surface is homotopy equivalent to a 1-dimensional CW-complex. Since $G/T$ is simply-connected (cf. Prop. 7.6 in Chap. V in [10]) this means that every continuous map $\bar{g} : \Sigma \to G/T$ is null-homotopic. This implies that for non-compact $\Sigma$ we have $[\Sigma, G/T] = \{[1_T]\}$ where $1_T : \Sigma \to G/T$ is the constant map taking only the value $T \in G/T$. Accordingly, Eq. (2.23) then reduces to Eq. (2.19) (with $B$ replaced by $B_{\text{reg}}$).

(ii) In part (i) of the present remark we observed that if $\Sigma$ is non-compact then every continuous map $\bar{g} : \Sigma \to G/T$ is null-homotopic. Since $\pi_{G/T} : G \to G/T$ is a fiber bundle and therefore possesses the homotopy lifting property (cf., e.g., [14]) we conclude that if $\Sigma$ is non-compact then every continuous map $\bar{g} : \Sigma \to G/T$ can be lifted to a continuous map $\Omega : \Sigma \to G$. Moreover, if $\bar{g}$ is smooth then $\Omega$ can be chosen to be smooth as well. This observation will play an important role in Sec. 2.3 below, cf. the paragraph after Eq. (2.23) below.

\(\text{\textsuperscript{13}}\)The standard notation for this orbit space would be $G_{\Sigma}\backslash C^\infty(\Sigma, G/T)$ but in order to be consistent with the notation in [20] where we worked with a right-action of $G_{\Sigma}$ on $C^\infty(\Sigma, G/T)$ we will use the notation $C^\infty(\Sigma, G/T)/G_{\Sigma}$ in the following.

\(\text{\textsuperscript{14}}\)That this indeed holds for every non-compact (connected, orientable) surface $\Sigma'$ is a rather deep result in low-dimensional topology. It follows, e.g., from the result by Behnke and Stein (1948) that every non-compact, connected Riemann surface is a Stein manifold. I emphasize that in the main part of the present paper we need this result only in the special case $\Sigma' = \Sigma \backslash \{\sigma_0\}$ where $\Sigma$ is a compact, connected, orientable surface and $\sigma_0 \in \Sigma$ a fixed point. Using the classification theorem for compact, connected, orientable surfaces it is not difficult to show directly that $\Sigma \backslash \{\sigma_0\}$ is indeed homotopy equivalent to a 1-dimensional CW-complex.
Remark 2.4 (i) In Sec. 2.3 below we will apply Eq. (2.23) to the situation relevant for the CS path integral, i.e. we take $\chi = \chi_L$ where $\chi_L$ is given by Eq. (2.24) below. Using a suitable change of variable argument we then arrive at the very simple equation Eq. (2.30) below (which is later rewritten as Eq. (2.40) and Eq. (2.47)).

(ii) In the special case where $L$ is either the “empty link” or a “fiber link”, which is the only case treated in [17] (cf. Eq. (7.9) in [17] and Remark 3.1 in Sec. 3 below), Eq. (2.47) mentioned in part (i) simplifies considerably leading to Eq. (3.1) below. In order to compare Eq. (3.1) below with Eq. (7.9) in [17] observe first that according to Remark 2.3 there is a natural 1-1-correspondence between the elements of the orbit space $C^\infty(\Sigma, G/T)/G_{\Sigma}$ and the elements of $[\Sigma, G/T]$. On the other hand, since $G$ is path-connected and simply-connected the $T$-bundle $G \to G/T$ is “2-universal” (cf. Sec. 16 in [21]) so there is a 1-1-correspondence between the elements of $[\Sigma, G/T]$ and the set of equivalence classes of $T$-bundles over $\Sigma$ (cf. Theorem 16.1 in [21]). Accordingly, it is clear that Eq. (3.1) below and Eq. (7.9) in [17] are very closely related. (Note that the derivation of Eq. (7.9) in [17] involves a sum over equivalence classes of $T$-bundles over $\Sigma$ and so does Eq. (6.8) in [18].) And indeed, the explicit evaluation of Eq. (7.9) in [17] gives rise to the same concrete values as the explicit evaluation of Eq. (3.1) below, so from a computational point of view both equations are equivalent. On the other hand, from a conceptual point of view Eq. (7.9) in [17] does not seem to be equivalent to Eq. (3.1) below, cf. Remark 3.1 in Sec. 3 below.

Remark 2.5 (i) In Eq. (2.23) we can replace the space $B_{reg}$ by $C^\infty(\Sigma, P)$ (for any fixed Weyl alcove $P$) or by each of the two spaces $B_{reg}^{ess}$ or $\{B \in B_{reg}^{ess} | B(\sigma_0) \in P\}$ introduced in Sec. 3.2.3 and Appendix [21.6] below. In the present section we chose to work with $B_{reg}$ for stylistic reasons. In Sec. 3.1 it will be convenient to work with the space $C^\infty(\Sigma, P)$ (cf. the last paragraph before Sec. 3.1.1). From Sec. 3.2.6 on we will work with the space $B_{reg}^{ess}$, for reasons explained in Remark 3.8 in Sec. 3.2.3 below.

(ii) Observe that, similarly to the situation in Eq. (2.12) in Sec. 2.2.1 above, the integral $\int \cdots DB$ on the RHS of Eq. (2.23) should be interpreted as a suitable improper integral $\int \cdots DB$, cf. Remark 2.2 above. (The same applies if we use the space $B_{reg}^{ess}$ instead of $B_{reg}$.) However, since the integral on the RHS of Eq. (2.23) is informal anyway we do not use a notation like $\int \cdots DB$.

2.3 Torus gauge fixing applied to the Chern-Simons path integral

Let us now go back to Eq. (2.4) in Sec. 2.1. We will consider the special case of Eq. (2.4) where $M$ is of the form $M = \Sigma \times S^1$ where $\Sigma$ is a compact, connected, oriented surface and where $L = ((l_1, l_2, \ldots, l_m), (\rho_1, \rho_2, \ldots, \rho_m))$ is a colored link in $M = \Sigma \times S^1$. Obviously, the RHS of Eq. (2.4) then agrees with the LHS of Eq. (2.23) if we choose $\chi$ to be the $G$-invariant function $\chi_L : A \to \mathbb{C}$ given by

$$\chi_L(A) = \left(\prod_{i=1}^m \operatorname{Tr}_{\rho_i}(\operatorname{Hol}_{l_i}(A))\right) \exp(iS_{CS}(A)) \quad \forall A \in A \quad (2.24)$$

From Eq. (2.4) and Eq. (2.23) we therefore obtain

$$Z(\Sigma \times S^1, L) \sim \sum_{h \in C^\infty(\Sigma, G/T)/G_{\Sigma}} \int_{B_{reg}} \left[ \int_{A^\perp} \left(\prod_{i=1}^m \operatorname{Tr}_{\rho_i}(\operatorname{Hol}_{l_i}(A^+ + \tilde{g}_h B_{g^{-1}} dt))\right) \times \exp(iS_{CS}(A^+ + \tilde{g}_h B_{g^{-1}} dt)) DA^\perp \right] \det(1_t - \exp(\operatorname{ad}(B))|_t) DB \quad (2.25)$$
Recall from Sec. 2.1.1 above that we have been assuming (without loss of generality) that $h$ is a matrix Lie group, i.e. $G$, and where $\mathcal{G}$ where $\mathcal{C}$ where $\mathcal{H}$

For each fixed $B \in \mathcal{B}$ and $h \in C^\infty(\Sigma, G/T)/\mathcal{G}_\Sigma$ we will now simplify the integral

$$
\int_{A^+} \left( \prod_{i=1}^{m} \Tr_{\rho_i}(\Hol_{\xi_i}(A^+ + \bar{g}_h B\bar{g}_h^{-1}dt)) \right) \exp(iS_{CS}(A^+ + \bar{g}_h B\bar{g}_h^{-1}dt)) DA^+
$$

by applying a suitable change of variable (and by exploiting the special properties of the functions $\Tr_{\rho_i}(\Hol_{\xi_i}(\cdot))$ and $S_{CS}$).

As a preparation let us set $l_i^\Sigma := \pi_S \circ l_i$ for each loop $l_i$ appearing in the link $L$ where $\pi_S : \Sigma \times S^1 \to \Sigma$ is the canonical projection and let us fix $\sigma_0 \in \Sigma$ such that

$$
\sigma_0 \notin \bigcup_{i=1}^{m} \Image(l_i^\Sigma)
$$

(2.26)

Since the (connected, oriented) surface $\Sigma \setminus \{\sigma_0\}$ is non-compact, according to Remark 2.2 in Sec. 2.2 above the map $(\bar{g}_h)_{\Sigma \setminus \{\sigma_0\}} \in C^\infty(\Sigma \setminus \{\sigma_0\}, G/T)$ is null-homotopic and can therefore be lifted (in the fiber bundle $\pi : G \to G/T$) to a map $\Omega_h \in \mathcal{G}_{\Sigma \setminus \{\sigma_0\}} := C^\infty(\Sigma \setminus \{\sigma_0\}, G)$. We will keep $\Omega_h$ fixed in the following.

Since $\Tr_{\rho_i}(\Hol_{\xi_i}(\cdot))$ is $\mathcal{G}$-invariant we have, in particular,

$$
\Tr_{\rho_i}(\Hol_{\xi_i}(A^+ + \Omega B\Omega^{-1}dt)) = \Tr_{\rho_i}(\Hol_{\xi_i}(A^+ \cdot \Omega + Bdt))
$$

(2.27)

for all $\Omega \in \mathcal{G}_{\Sigma} \subset \mathcal{G}$. Now observe that as $\sigma_0$ was chosen to that condition (2.26) above is fulfilled, Eq. (2.27) also holds for all $\Omega \in \mathcal{G}_{\Sigma \setminus \{\sigma_0\}} = C^\infty(\Sigma \setminus \{\sigma_0\}, G)$ if the expressions $\Hol_{\xi_i}(A^+ \cdot \Omega + Bdt)$ and $\Hol_{\xi_i}(A^+ + \Omega B\Omega^{-1}dt)$ are defined in the obvious way. Applying this to the special case $\Omega = \Omega_h$ we then obtain

$$
\Tr_{\rho_i}(\Hol_{\xi_i}(A^+ + \bar{g}_h B\bar{g}_h^{-1}dt)) = \Tr_{\rho_i}(\Hol_{\xi_i}(A^+ \cdot \Omega_h + Bdt)) = \Tr_{\rho_i}(\Hol_{\xi_i}(A^+ \cdot \Omega_h + Bdt))
$$

(2.28)

Let us now derive a similar formula for the factor $\exp(iS_{CS}(A^+ + \bar{g}_h B\bar{g}_h^{-1}dt))$ in Eq. (2.25). Recall from Sec. 2.1.1 above that we have been assuming (without loss of generality) that $G$ is a matrix Lie group, i.e. $G \subset \text{GL}(N, \mathbb{R}) \subset \text{Mat}(N, \mathbb{R})$ for some $N \in \mathbb{N}$. From Eq. (2.7) in Sec. 2.1.1 above we then obtain

$$
S_{CS}(A^+ + Bdt)
$$

$$
= k \pi \int_M \left[ \Tr(A^+ \wedge dA^+) + 2 \Tr(A^+ \wedge Bdt \wedge A^+) + 2 \Tr(A^+ \wedge dB \wedge dt) \right]
$$

$$
= -k \pi \int_{S^1} \int_{\Sigma} \left[ \Tr(A^+(t) \wedge (\partial/\partial t + \text{ad}(B)) \cdot A^+(t)) - 2 \Tr(A^+(t) \wedge dB) \right] dt
$$

(2.29)

where in the last expression we used the identification (cf. Sec. 2.3.1 in [35])

$$
A^+ \cong C^\infty(S^1, A_\Sigma)
$$

(2.30)

where

$$
A_\Sigma := \Omega^1(\Sigma, g)
$$

(2.31)

and where $C^\infty(S^1, A_\Sigma)$ is the space of maps $f : S^1 \to A_\Sigma$ which are “smooth” in the sense that $\Sigma \times S^1 \ni (\sigma, t) \mapsto (f(t))(X_\sigma) \in g$ is smooth for every smooth vector field $X$ on $\Sigma$. The operator $\partial/\partial t : C^\infty(S^1, A_\Sigma) \to C^\infty(S^1, A_\Sigma)$ is defined in the obvious way.
For reasons which become clear later we will rewrite Eq. (2.29) using a suitable improper integral, as

\[
S_{CS}(A^\perp + Bdt) = -k\pi \int_{S^1} \lim_{\epsilon \to 0} \left[ \int_{\Sigma \setminus B_\epsilon(\sigma_0)} \Tr(A^\perp(t) \wedge (\partial/\partial t + \text{ad}(B)) \cdot A^\perp(t)) - 2 \Tr(A^\perp(t) \wedge dB) \right] dt
\] (2.32)

where \( B_\epsilon(\sigma_0) \), \( \epsilon > 0 \), denotes the closed \( \epsilon \)-ball around \( \sigma_0 \) w.r.t. an arbitrary fixed Riemannian metric \( g_\Sigma \) on \( \Sigma \). In Appendix B.3 below we will therefore give a careful justification for the change of variable \( n \)

\[ n(h) := \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(\sigma_0)} \pi_t(\Omega_h^{-1} d\Omega_h) \in t \] (2.34)

where \( \pi_t \) is the \( \langle \cdot, \cdot \rangle \)-orthogonal projection \( g \to t \) and where the orientation of \( \partial B_\epsilon(\sigma_0) \) is opposite to the orientation induced by \( B_\epsilon(\sigma_0) \) (cf. Footnote 100 in Appendix B.3).

The following proposition will be proven in Appendix B.4 below.

**Proposition 2.6**

(i) The limit in Eq. (2.34) exists.

(ii) \( n(h) \) is independent of the special choice of \( \bar{g}_h \) and \( \Omega_h \) above.\(^{15}\)

(iii) The map \( C^\infty(\Sigma, G/T)/\mathcal{G}_\Sigma \ni h \mapsto n(h) \) is injective and its image is \( I := \ker(\exp|_t) \subset t \).

From Eq. (2.28) and Eq. (2.33) we obtain

\[
\int_{A^\perp} \left( \prod_{i=1}^m \Tr_{\rho_i}(\text{Hol}_{\rho_i}(A^\perp + \bar{g}_h B\bar{g}_h^{-1} dt)) \right) \exp(iS_{CS}(A^\perp + \bar{g}_h B\bar{g}_h^{-1} dt)) \exp(-2\pi i k \langle n(h), B(\sigma_0) \rangle) \]

\[
\times \int_{A^\perp} \left( \prod_{i=1}^m \Tr_{\rho_i}(\text{Hol}_{\rho_i}(A^\perp \cdot \Omega_h + Bdt)) \right) \exp(iS_{CS}(A^\perp \cdot \Omega_h + Bdt)) \exp(-2\pi i k \langle n(h), B(\sigma_0) \rangle) \]

\[
\times \int_{A^\perp} \left( \prod_{i=1}^m \Tr_{\rho_i}(\text{Hol}_{\rho_i}(A^\perp + Bdt)) \right) \exp(iS_{CS}(A^\perp + Bdt)) \exp(-2\pi i k \langle n(h), B(\sigma_0) \rangle) \]

\[
\times \int_{A^\perp} \left( \prod_{i=1}^m \Tr_{\rho_i}(\text{Hol}_{\rho_i}(A^\perp + Bdt)) \right) \exp(iS_{CS}(A^\perp + Bdt)) \exp(-2\pi i k \langle n(h), B(\sigma_0) \rangle) \]

where in step (*) we have applied the change of variable \( A^\perp \rightarrow A^\perp \cdot \Omega_h^{-1} = \Omega_h A^\perp \Omega_h^{-1} - d\Omega_h \Omega_h^{-1} \) (and used a similar argument as in step (*) in Eq. (2.37) in Appendix B.2 below.)

**Remark 2.7** Note that because of the discontinuity of \( \Omega_h \) and the singularity of \( d\Omega_h \) in the point \( \sigma_0 \) the change of variable \( A^\perp \rightarrow A^\perp \cdot \Omega_h^{-1} \) we used above is not really a transformation of the space \( A^\perp \) and we can not be sure whether this change of variable will lead to the correct results. In Appendix B.7 below we will therefore give a careful justification for the change of variable \( A^\perp \rightarrow A^\perp \cdot \Omega_h^{-1} \) in Eq. (2.33) above. That such a justification is necessary also becomes clear

\(^{15}\)Part (ii) of Proposition 2.6 is used in the proof of part (iii) in Appendix B.4 below. Note that, not surprisingly, \( n(h) \) is also independent of the special choice of \( g_\Sigma \) and \( \sigma_0 \) above but this plays no role in the proof of part (iii).
from the following observation: If we rewrite some of the formulas above using the expression $S_{CS}(A^\perp \cdot \Omega_\alpha + Bdt)$ introduced in Eq. (B.13) in Appendix B.3 below and then perform the change of variable $A^\perp \rightarrow A^\perp \cdot \Omega_\alpha^{-1}$ in a naive way we would arrive at an incorrect result, cf. Remark B.8 in Appendix B.5 below.

By applying Eq. (2.35) for each fixed $B \in \mathcal{B}$ and $h \in C^\infty(\Sigma, G/T)/G\Sigma$ and by taking into account part (iii) of Proposition 2.6 above we finally obtain from Eq. (2.25)

$$Z(\Sigma \times S^1, L) \sim \sum_{y \in I} \int_{\mathcal{B}_{reg}} \left[ \int_{A^\perp} \prod_i \text{Tr}_{\rho_i} (\text{Hol}_{l_i}(A^\perp + Bdt)) \exp(iS_{CS}(A^\perp + Bdt)) DA^\perp \right]$$

$$\times \exp\left(-2\pi i k \langle y, B(\sigma_0) \rangle \right) \det(1_t - \exp(\text{ad}(B)_{|t}) DB$$

(2.36)

### 2.3.1 Rewriting Eq. (2.36)

It will be convenient to rewrite the RHS of Eq. (2.36) as an iterated “Gauss-type” integral (cf. Remark 2.8 below). In order to do so let us set

$$A_{\Sigma, t} := \Omega^1(\Sigma, t),$$  (2.37a)

$$A_{\Sigma, t} := \Omega^1(\Sigma, t)$$  (2.37b)

$$\tilde{A}^\perp := \{ A^\perp \in A^\perp | \int A^\perp(t) dt \in A_{\Sigma, t} \}$$  (2.37c)

$$A^\perp_c := \{ A^\perp \in A^\perp | A^\perp \text{ is constant and } A_{\Sigma, t}-\text{valued} \} \cong A_{\Sigma, t}$$  (2.37d)

(Recall that we made the identification $A^\perp \cong C^\infty(S^1, A_{\Sigma})$ where $C^\infty(S^1, A_{\Sigma})$ is as in the paragraph before Eq. (2.32) above). Observe that

$$\tilde{A}^\perp = \tilde{A}^\perp \oplus A^\perp_c$$  (2.38)

From Eq. (2.29) it follows easily that

$$S_{CS}(\tilde{A}^\perp + A^\perp_c + Bdt) = S_{CS}(\tilde{A}^\perp + Bdt) + S_{CS}(A^\perp_c + Bdt)$$  (2.39)

for $\tilde{A}^\perp \in \tilde{A}^\perp$, $A^\perp_c \in A^\perp_c$, and $B \in \mathcal{B}$. Taking this into account we can rewrite Eq. (2.36) as

$$Z(\Sigma \times S^1, L) \sim \sum_{y \in I} \int_{\tilde{A}^\perp \times A^\perp_c \times B} \left\{ \text{Det}_{FP}(B) 1_{\mathcal{B}_{reg}}(B) \right\}$$

$$\times \left[ \int_{\tilde{A}^\perp} \prod_i \text{Tr}_{\rho_i} (\text{Hol}_{l_i}(\tilde{A}^\perp + A^\perp_c, B)) \exp(iS_{CS}(\tilde{A}^\perp, B)) DA^\perp \right]$$

$$\times \exp\left(-2\pi i k \langle y, B(\sigma_0) \rangle \right) \exp(iS_{CS}(A^\perp_c, B))(DA^\perp_c \otimes DB)$$  (2.40)

where $D\tilde{A}^\perp$, $DA^\perp_c$, and $DB$ are the informal “Lebesgue measures” on $\tilde{A}^\perp$, $A^\perp_c$, and $\mathcal{B}$, where we have introduced the short notation

$$S_{CS}(A^\perp, B) := S_{CS}(A^\perp + Bdt)$$  (2.41a)

$$\text{Det}_{FP}(B) := \det(1_t - \exp(\text{ad}(B))_{|t})$$  (2.41b)

$$\text{Hol}_{l_i}(A^\perp, B) := \text{Hol}_{l_i}(A^\perp + Bdt)$$  (2.41c)

and where $1_{\mathcal{B}_{reg}}$ is the indicator function of the subset $\mathcal{B}_{reg}$ of $\mathcal{B}$.

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16 On the other hand, as explained in Remark 2.8 in Appendix B.5 below, also when working with $S_{CS}(A^\perp \cdot \Omega_\alpha + Bdt)$ we arrive at the correct result if we replace the argument based on the aforementioned naive change of variable by a more careful argument.
Remark 2.8 For any (fixed) Riemannian metric $g_\Sigma$ on $\Sigma$ we have

$$S_{CS}(\dot{A}) = \pi k \ll \dot{A}^\perp, \star \left(\frac{\partial}{\partial t} + \text{ad}(B)\right)\dot{A}^\perp \gg \dot{A}^\perp$$

(2.42)

$$S_{CS}(\dot{A}) = -2\pi k \ll \dot{A}^\perp, \star dB \gg \dot{A}^\perp$$

(2.43)

for $B \in B$, $\dot{A}^\perp \in \dot{A}^\perp$, and $\dot{A}^\perp \in A^\perp$ where $\star$ is the Hodge star operator\(^{17}\) associated to $g_\Sigma$ and $\ll \cdot, \cdot \gg_{A^\perp}$ and $\ll \cdot, \cdot \gg_{A^\perp}$ are the scalar products on $A^\perp$ and $A^\perp \cong C^\infty(S^1, A^\perp)$ which are induced by $g_\Sigma$.

In view of Eq. (2.42) and Eq. (2.43) it is clear that both measures on the RHS of Eq. (2.40) are (complex) “Gauss-type” measures. This greatly simplifies the tasks of finding a rigorous realization of the RHS of Eq. (2.40), cf. Sec. 4 below. It also simplifies the explicit evaluation of the RHS of Eq. (2.40) which will be the topic in Sec. 3 below. We would like to point out, however, that in an informal evaluation of the RHS of Eq. (2.40) it is useful to rewrite the outer “Gauss-type” integral $\int \cdots \exp(iS_{CS}(A^\perp, B))(DA^\perp \otimes DB)$ as an iterated integral $\int [\cdots \exp(iS_{CS}(A^\perp, B))DA^\perp] DB$, cf. Sec. 3.2.1, Sec. 3.3.3, and Sec. 3.4.7 below.

Eq. (2.40) is not yet our final formula for $Z(\Sigma \times S^1, L)$. There are three more things to do before we obtain our final formula, i.e. Eq. (3.39b) in Sec. 3.2.6 below:

1. Recall that our goal is to find a rigorous realization of Witten’s CS path integral expressions which reproduces the Reshetikhin-Turaev invariants (in the special situation described in the Introduction). Since the Reshetikhin-Turaev invariants are defined for ribbon links (or, equivalently\(^{18}\) for framed links) we will later introduce a ribbon analogue of Eq. (2.40), cf. Sec. 3.2.1 below.

2. For reasons explained in Sec. 3.2.3 below we will replace the subspace $B_{\text{reg}}$ of $B$ appearing in Eq. (2.40) by the slightly larger subspace $B_{\text{reg}}^{\text{ess}}$ of $B$, cf. Eq. (3.28) below.

3. We will set, for each $B \in B_{\text{reg}}$ (and later for $B \in B_{\text{reg}}^{\text{ess}}$),

$$\bar{Z}(B) := \int \exp(iS_{CS}(\dot{A}^\perp, B))(DA^\perp),$$

(2.44)

$$d\mu_{B} := \frac{1}{Z(B)} \exp(\text{i}S_{CS}(\dot{A}^\perp, B))DA^\perp$$

(2.45)

and

$$\text{Det}(B) := \text{Det}_{FP}(B)\bar{Z}(B)$$

(2.46)

and we will then find a rigorous realization $\text{Det}_{\text{rig}}(B)$ of $\text{Det}(B)$.

Since neither ribbons nor the space $B_{\text{reg}}^{\text{ess}}$ are necessary for the simple situation of “fiber links” treated in Sec. 3.2.1 below we will, for now, only incorporate point 3 above. Doing so we obtain from Eq. (2.40)

$$Z(\Sigma \times S^1, L) \sim \sum_{y \in I} \int_{A^\perp \times B} \left\{ 1_{B_{\text{reg}}}(B) \text{Det}_{\text{rig}}(B) \right\}$$

$$\times \left[ \int_{A^\perp} \left( \prod_{i} \text{Tr}_{P_i}(\text{Hol}_{p_i}(\dot{A}^\perp + A^\perp, B)) \right) d\mu_{B}(\dot{A}^\perp) \right]$$

$$\times \exp\left(-2\pi i k(y, B(\sigma_0))\right) \exp(iS_{CS}(A^\perp, B))(DA^\perp \otimes DB)$$

(2.47)

\(^{17}\)More precisely, $\star$ denotes both the Hodge star operator $\star : A^\perp \to A^\perp$ and the linear automorphism $\star : C^\infty(S^1, A^\perp) \to C^\infty(S^1, A^\perp)$ given by $(\star A^\perp)(t) = \star(A^\perp(t))$ for all $A^\perp \in A^\perp$ and $t \in S^1$.

\(^{18}\)From the knot theory point of view the framed link picture and the ribbon link picture are equivalent. However, the ribbon picture seems to be better suited for the study of the Chern-Simons path integral in the torus gauge.
with $\text{Det}_{\text{rig}}(B)$ as defined in Sec. 2.3.2 below.

### 2.3.2 Rigorous realization $\text{Det}_{\text{rig}}(B)$ of $\text{Det}(B)$

We will now introduce a rigorous realization $\text{Det}_{\text{rig}}(B)$ of $\text{Det}(B)$. In order to do so we will use a standard $\zeta$-function regularization argument and a variant of the heat kernel regularization method used in Sec. 6 in [17], cf. Remark 3.2 in Sec. 3.1.1 below.

As above let us fix an auxiliary Riemannian metric $g_{\Sigma}$ on $\Sigma$ and let $\ast$ and $\ll \cdot , \cdot \gg_{A^\perp}$ be as in Remark 2.3 above. Then we obtain, informally, for $B \in \mathcal{B}_{\text{reg}} = C^\infty(\Sigma, t_{\text{reg}})$

\[
\tilde{Z}(B) = \int \exp(iS_{\text{CS}}(A^\perp, B))D\tilde{A}^\perp
= \int \exp(i\pi k \ast (\frac{\partial}{\partial t} + \text{ad}(B))\tilde{A}^\perp \gg_{A^\perp})D\tilde{A}^\perp
\sim \text{det}(\frac{\partial}{\partial t} + \text{ad}(B))^{-1/2}
\]

(2.48)

where "\sim" denotes equality up to a multiplicative constant independent of $B$.

In order to make sense of the RHS of Eq. (2.48) we first consider the analogous problem of making sense of the determinant of the linear operator $\frac{\partial}{\partial t} + \text{ad}(b) : C^\infty(S^1, t) \to C^\infty(S^1, t)$ with $b \in t_{\text{reg}}$. This problem can be solved using a standard $\zeta$-function regularization argument. Using this one arrives at

\[
\text{det}(\frac{\partial}{\partial t} + \text{ad}(b)) \sim \text{det}((1_t - \exp(\text{ad}(b))|_t)) \quad \forall b \in t_{\text{reg}}
\]

Now let $(1_t - \exp(\text{ad}(b))|_t)^{(p)}$, for $p \in \{0, 1, 2\}$, denote the linear operator on $\Omega^p(\Sigma, t)$ given by

\[
\left( (1_t - \exp(\text{ad}(B))|_t)^{(p)} \cdot \alpha \right)(X_\sigma) = (1_t - \exp(\text{ad}(B(\sigma))|_t) \cdot \alpha(X_\sigma)
\]

for all $\alpha \in \Omega^p(\Sigma, t)$, $\sigma \in \Sigma$, $X_\sigma \in \wedge^p T_\sigma \Sigma$.

In view of Eq. (2.49) we then have, informally, for $B \in \mathcal{B}_{\text{reg}}$

\[
\text{det}(\frac{\partial}{\partial t} + \text{ad}(B)) \sim \text{det}((1_t - \exp(\text{ad}(B))|_t)^{(1)})
\]

(2.51)

and combining this with Eq. (2.48) and the (informal) definition of $\text{Det}(B)$ (cf. Eq. (2.46) and Eq. (2.41b) above) we therefore obtain for $B \in \mathcal{B}_{\text{reg}}$

\[
\text{Det}(B) = \text{det}\left((1_t - \exp(\text{ad}(B))|_t)^{(0)}\right) \text{det}\left((1_t - \exp(\text{ad}(B))|_t)^{(1)}\right)^{-1/2}
\]

(2.52)

It will be convenient to rewrite Eq. (2.52) in a suitable way. In order to do so let us denote by $M_{(p)}^f$, for $p = 0, 1, 2$ and $K \in \{\mathbb{R}, \mathbb{C}\}$, the multiplication operator $\Omega^p(\Sigma, \mathbb{K}) \to \Omega^p(\Sigma, \mathbb{K})$ obtained by multiplication with a smooth function $f : \Sigma \to \mathbb{K}$. For every $B \in \mathcal{B}_{\text{reg}}$ and $p = 0, 1, 2$ we then obtain, informally,

\[
\text{det}\left((1_t - \exp(\text{ad}(B))|_t)^{(p)}\right) = \text{det}_{\mathbb{C}}\left((1_t - \exp(\text{ad}(B))|_t)^{(p)} \otimes_{\mathbb{R}} \mathbb{C}\right) \overset{\text{(a)}}{=} \prod_{\alpha \in \mathbb{R}_+} \text{det}_{\mathbb{C}}\left(M_{(1-\epsilon^2\sin(B(\cdot)))}^f\right) \prod_{\alpha \in \mathbb{R}_+} \text{det}_{\mathbb{C}}\left(M_{(2\pi\alpha(B(\cdot)))}^f\right)^2
\]

(2.53)

\[\textbf{Note that under the identification } C^\infty(\Sigma, t) \cong \Omega^0(\Sigma, \mathbb{K}) \text{ the operator } (1_t - \exp(\text{ad}(B))|_t)^{(0)} \text{ coincides with what above we call } 1_t - \exp(\text{ad}(B))|_t.\]
where we use the notation of Appendix A.1 and where in step (*) we applied a suitable diagonalization argument. (Note that since $B \in \mathcal{B}_{\text{reg}}$ the function $|\sin(\pi \alpha(B(\cdot)))| : \Sigma \ni \sigma \mapsto |\sin(\pi \alpha(B(\sigma)))| \in \mathbb{R}$ is smooth.) Setting
\[ O_\alpha^{(p)}(B) := M_2^{(p)}|\sin(\pi \alpha(B(\cdot)))| \]
we can therefore rewrite Eq. (2.52) as\(^{20}\)
\[ \det(B) = \prod_{\alpha \in \mathcal{R}_+} \det(O_\alpha^{(0)}(B))^{2} \det(O_\alpha^{(1)}(B))^{-1} \] (2.55)

We will now use Eq. (2.55) as the starting point for obtaining a rigorous realization $\det_{\text{rig}}(B)$ of $\det(B)$ for $B \in \mathcal{B}_{\text{reg}}$ (by means of a suitable “heat kernel regularization” argument).

Recall that above we have fixed an auxiliary Riemannian metric $g_\Sigma$ on $\Sigma$. Let us now equip the two spaces $\Omega^i(\Sigma, \mathbb{R})$, $i = 0, 1$, with the scalar product which is induced by $g_\Sigma$. By $\overline{\Omega^i(\Sigma, \mathbb{R})}$ we will denote the completion of the pre-Hilbert space $\Omega^i(\Sigma, \mathbb{R})$, $i = 0, 1$, and by $\Delta_i$ the (closure of the) Hodge Laplacian on $\overline{\Omega^i(\Sigma, \mathbb{R})}$.

**Definition 2.9** In view of Eq. (2.55) above we now define
\[ \det_{\text{rig}}(B) := \prod_{\alpha \in \mathcal{R}_+} \det_{\text{rig}, \alpha}(B) \] (2.56a)
with
\[ \det_{\text{rig}, \alpha}(B) := \lim_{\epsilon \to 0^+} \left[ \det_\epsilon(O_\alpha^{(0)}(B))^{2} \det_\epsilon(O_\alpha^{(1)}(B))^{-1} \right] \] (2.56b)
where for $i = 0, 1$ we have set\(^{21}\)
\[ \det_\epsilon(O_\alpha^{(i)}(B)) := \exp(\text{Tr}(e^{-\epsilon \Delta_i} \log(O_\alpha^{(i)}(B)))) \] (2.56c)

Note that each of the operators $O_\alpha^{(i)}(B)$, $i = 0, 1$, is a symmetric, bounded, positive operator whose spectrum is bounded away from zero. Hence also $\log(O_\alpha^{(i)}(B))$ is a well-defined symmetric, bounded operator. Moreover, $e^{-\epsilon \Delta_i}$ is trace-class so the product $e^{-\epsilon \Delta_i} \log(O_\alpha^{(i)}(B))$ is also trace-class and the expression $\text{Tr}(e^{-\epsilon \Delta_i} \log(O_\alpha^{(i)}(B)))$ in Eq. (2.56c) is well-defined. In Sec. 3.2.5 we will show that the $\epsilon \to 0$ limit on the RHS of Eq. (2.56c) exists for all $B \in \mathcal{B}_{\text{reg}}$ and $\det_{\text{rig}, \alpha}(B)$ and $\det_{\text{rig}}(B)$ are therefore well-defined.

### 2.4 A remark on the relation between $Z(M, L)$ and the Reshetikhin-Turaev invariant $RT(M, L)$

Let $\mathfrak{g}_\mathbb{C}$ be a simple complex Lie algebra, and $q \in U(1)$ a root of unity (of sufficiently high order). Moreover, let $M$ be a compact, connected, oriented 3-manifold and $L$ a framed, colored link in $M$. The Reshetikhin-Turaev invariant $RT(M, L)$ associated to the quantum group $U_q(\mathfrak{g}_\mathbb{C})$ is believed to be equivalent to Witten’s informal path integral expression $Z(M, L)$ based on the Chern-Simons action function associated to $(G, k)$ where $G$ is the simply connected, compact

---

\(^{20}\)The reader may wonder we do not rewrite Eq. (2.55) in terms of the (informal) determinants of the multiplication operators $M_2^{(p)}|\sin(\pi \alpha(B(\cdot)))|$. The advantage of using Eq. (2.55) will become clear in Sec. 3.2.5 below where we will generalize Eq. (2.55) to the case where $B \in \mathcal{B}_{\text{ess}}^{\text{reg}}$.

\(^{21}\)This ansatz is, of course, motivated by the rigorous formula $\det(A) = \exp(\text{Tr}(\log(A)))$ which holds for every strictly positive (self-adjoint) operator $A$ on a finite-dimensional Hilbert-space.
Lie group corresponding to the compact real form $\mathfrak{g}$ of $\mathfrak{g}_C$ and $k \in \mathbb{N}$ is chosen suitably. It it often assumed that this relationship between $q$ and $k \in \mathbb{N}$ is given by

$$ q = e^{2\pi i/(k+c_g)} $$

(2.57)

where $c_g$ is the dual Coxeter number of $\mathfrak{g}$. The appearance of $k + c_g$ instead of $k$ is the famous “shift of the level” $k$. However, several authors (cf., e.g., [32]) have argued that the occurrence (and magnitude) of such a shift in the level depends on the regularization procedure and renormalization prescription which is used for making sense of the informal path integral. Accordingly, it should not be surprising that there are several papers (cf. the references in [32]) where the shift $k \rightarrow k + c_g$ is not observed and one is therefore led to the following relationship between $q$ and $k \in \mathbb{N}$ with \(22 \leq k > c_g\)

$$ q = e^{2\pi i/k} $$

(2.58)

This is also the case in [38, 39, 40, 41] and the present paper. By contrast, in [17, 19, 20] (and also in [25]) it is assumed that $q$ and $k$ are related by (2.57), cf. Remark 3.2 below.

**Remark 2.10** In Sec. 3.3 below we will compare $Z(\Sigma \times S^1, L)$ directly with the Reshetikhin-Turaev invariant $RT(\Sigma \times S^1, L)$. By contrast, in Secs 3.4 and 3.5 below we will compare $Z(\Sigma \times S^1, L)$ with the reformulation of $RT(\Sigma \times S^1, L)$ in terms of Turaev’s shadow invariant $|L|$, cf. Eq. (3.98) at the beginning of Sec. 3.4 and Eq. (C.7) in Appendix C below.

## 3 Explicit evaluation of $Z(\Sigma \times S^1, L)$

We will now evaluate $Z(\Sigma \times S^1, L)$ explicitly (at an informal level), first in several special cases and then, in Sec. 3.5 below, we will consider the case of general (“admissible”) $L$.

We begin in Sec. 3.1 with the special case of “fiber links” $L$, which is the only class of links considered in [17], cf. Remark 3.1 below. Even though from a knot theoretic point of view fiber links are trivial the study of such links is still very interesting, due to the relationship to the Verlinde formula for the WZW model, cf. Remark 3.3 below.

Sec. 3.2 is a preparation for Secs 3.3–3.5. In Sec. 3.2 we will derive our final formula for $Z(\Sigma \times S^1, L)$, Eq. (3.39b) below. (We do this by incorporating into Eq. (2.40) above the first two points of the list appearing in Sec. 2.3.1, cf. the beginning of Sec. 3.2 for more details.)

In Sec. 3.3 we will then study an interesting class of non-trivial knots in $S^2 \times S^1$, namely the class of all torus knots in $S^2 \times S^1$ of “standard type” (cf. Definition 3.18 and Definition 3.20 below). We will first derive an $S^2 \times S^1$-analogue of the so-called Rosso-Jones formula (cf. Footnote 4 in Sec. 1) and we will then show how a simple argument based on Witten’s surgery formula allows us to derive for arbitrary (simple, simply-connected, compact) $G$ the original version of the Rosso-Jones formula, which is concerned with torus knots in $S^3$.

In Sec. 3.4 we then study links in $\Sigma \times S^1$ without “double points”. Even though such links are not very interesting from a knot theoretic point of view (although they are not trivial) they are interesting in so far as they allow us to see how major building blocks of the shadow invariant $|L|$ arise.

Finally, in Sec. 3.5 we consider the case of general (“strictly admissible”) $L$ and sketch the strategy for evaluating $Z(\Sigma \times S^1, L)$. I want to emphasize that with some extra work we can expect to obtain an explicit formula for $Z(\Sigma \times S^1, L)$ also for general (strictly admissible) $L$, cf. the paragraph before Remark 3.46 in Sec. 3.5.2 below. The difference with respect to the treatment of the three special cases mentioned above is that in the present paper we do not
verify that the explicit expressions obtained for $Z(\Sigma \times S^1, L)$ for general $L$ agree with those in $RT(\Sigma \times S^1, L)$ (even though we do give some plausibility arguments later, cf. Appendix D below.)

Note: In Sec. 3.1 we essentially give a rederivation of the main result of [17]. Sec. 3.2 is based on [11] (with the exception of Sec. 3.2.3 which is new). In Sec. 3.3 we have rewritten the rigorous, “simplicial” treatment in [10] using the continuum setting introduced in Sec. 3.2 below. Similarly, in Sec. 3.4 we have rewritten the informal computations in [37, 25] within the continuum setting of Sec. 3.2. Finally, in Sec. 3.5 we have modified and generalized the treatment in Sec. 5.3 in [37].

3.1 Special case I. “Fiber links” in $M = \Sigma \times S^1$

Let us now consider the special case where $L = (l_1, l_2, \ldots, l_m)$ consists only of “fiber loops”, i.e. loops which are “parallel” to the $S^1$-component of $\Sigma \times S^1$. (Note that we could also work with ribbons, cf. Sec. 3.2.1 below, but in the present section it is sufficient to work with loops.)

More precisely, we assume that each $l_i$, $i \leq m$, is given by $l_i(s) = (\sigma_i, i S_1(s))$ for all $s \in [0, 1]$ for some fixed point $\sigma_i$ in $\Sigma$. (This special case was already treated in [17], cf. Remark 3.1 below for a comparison). Observe that in this special case we have $\mathrm{Hol}_{l_i}(A^1 + B dt) = \exp(B(\sigma_i))$ and therefore

$$\int_{A^1} \left( \prod_{i} \mathrm{Tr}_{\rho_i}(\mathrm{Hol}_{l_i}(\hat{A}^1 + A^1_{\cdot}, B)) \right) d\mu_B(\hat{A}^1) = \prod_{i=1}^{m} \mathrm{Tr}_{\rho_i}(\exp(B(\sigma_i)))$$

so Eq. (2.47) reduces to

$$Z(\Sigma \times S^1, (\sigma_i)_i, (\rho_i)_i) \sim \sum_{y \in I} \int_{B} \left\{ \int_{A^1} 1_{B_{\rho_0}}(B) \mathrm{Det}_{\text{rig}}(B) \left( \prod_{i=1}^{m} \mathrm{Tr}_{\rho_i}(\exp(B(\sigma_i))) \right) \right. \times \exp\left(-2\pi ik \langle y, B(\sigma_0) \rangle \right) \right\} \exp(iS_{CS}(A^1_{\cdot}, B)) DA^1_{\cdot} DB \quad (3.1)$$

where we have written $Z(\Sigma \times S^1, (\sigma_i)_i, (\rho_i)_i)$ instead of $Z(\Sigma \times S^1, L)$ and where $\sim$ denotes equality up to a multiplicative constant which is independent of $(\sigma_i)_i$ and $(\rho_i)_i$.

**Remark 3.1** In view of the relation $S_{CS}(A^1_{\cdot}, B) = 2\pi k \int_{\Sigma} \mathrm{Tr}(B \cdot dA^1_{\cdot})$ it is clear that Eq. (3.1) is closely related to the formula (7.9) in [17], or rather, the obvious generalization/modification of (7.9) in [17] which one obtains after including the analogue of the factor $\prod_{i=1}^{m} \mathrm{Tr}_{\rho_i}(\exp(B(\sigma_i)))$ appearing above (cf. Sec. 7.6 in [17]), replacing “$k + h$” by $k$ and replacing the group $SU(n)$ by a general simple, simply-connected, compact Lie group $G$. Both formulas turn out to be “computationally” equivalent in the sense that their explicit evaluation of their RHS leads to the same explicit expressions, cf. Eq. (3.12) below.

On the other hand, from a conceptual point of view these two formulas do not seem to be equivalent. Observe that in Eq. (3.1) we have a sum $\sum_{y \in I}$, a factor $\exp(-2\pi ik \langle y, B(\sigma_0) \rangle)$), and the integration $\int \cdots DA^1_{\cdot}$. By contrast in (the generalization of) formula (7.9) in [17] we have a sum $\sum_{\lambda \in \Lambda}$ over the weight lattice $\Lambda$, a factor “$\exp(-i \int_{\Sigma} \mathrm{tr}(\lambda F))$”, and the integration $\int \cdots DF$ where $F$ runs over the space of all 2-forms on $\Sigma$. As a result of the appearance of arbitrary 2-forms, formula (7.9) in [17] does not seem to have a natural generalization to the situation of general links $L$ (while Eq. (3.1) above obviously has such a generalization, namely Eq. (2.47) above).

23 There are two differences, though: we begin our computations with Eq. (2.47) above as the starting point rather than with Eq. (7.9) in [17], cf. Remark 3.1 below. The second difference is described in Remark 3.2 below.

24 Here $h$ is the notation in [17] for the dual Coxeter number of $g$ (which we denote by $c_g$). We refer to Sec. 2.4 above and Remark 3.2 below for a comment regarding the replacement $k + h \rightarrow k$. 
Instead of working with the original version of Eq. (3.1) let us now switch, for simplicity, to its $1_{C^\infty(\Sigma, \rho)}$-analogue (cf. Remark 2.5 in Sec. 2.2.2 above), i.e. to

$$Z(\Sigma \times S^1, (\sigma_i), (\rho_i)) \sim \sum_{y \in I} \int_B \int_{A_\perp^1} \left\{ 1_{C^\infty(\Sigma, \rho)}(B) \operatorname{Det}_{rig}(B) \left( \prod_{i=1}^m \operatorname{Tr}_{\rho_i}(\exp(B(\sigma_i))) \right) \right. \times \exp \left(-2\pi ik \langle y, B(\sigma_0) \rangle \right) \exp \left(iS_{CS}(A_\perp^1, B) \right) \left. \right\} DB \quad (3.2)$$

### 3.1.1 Explicit evaluation of the RHS of Eq. (3.2)

In order to evaluate the RHS of Eq. (3.2) we first integrate out the variable $A_\perp^1$. Since the only term in Eq. (3.2) depending on $A_\perp^1$ is the factor

$$\exp(iS_{CS}(A_\perp^1, B)) = \exp\left(-2\pi ik \ll A_\perp^1, *dB \gg A_\perp^1 \right)$$

(cf. Remark 2.8 above) we obtain, informally, a delta function expression $\delta(dB)$. In view of this delta-function the $\int \cdots DB$-integral can be replaced by an integral over the subspace $B_c := \{ B \in B \mid B \text{ is constant} \} \cong \mathbb{R}$. From Eq. (3.2) we therefore obtain

$$Z(\Sigma \times S^1, (\sigma_i), (\rho_i)) \sim \sum_{y \in I} \int_t \left\{ 1_{P}(b) \operatorname{Det}_{rig}(b) \times \left( \prod_{i=1}^m \operatorname{Tr}_{\rho_i}(\exp(b)) \right) \exp\left(-2\pi ik \langle y, b \rangle \right) \right\} db \quad (3.3)$$

In order to evaluate the expression $\operatorname{Det}_{rig}(b)$, $b \in t_{reg}$, given by Eqs (2.56a)–(2.56c) above observe first that

$$\lim_{\epsilon \to 0} (2 \text{Tr}(e^{-\epsilon \Delta_0}) - \text{Tr}(e^{-\epsilon \Delta_1})) \overset{(s)}{=} 2 \dim(ker(\Delta_0)) - \dim(ker(\Delta_1)) \overset{(**)}{=} 2 \dim(H^0(\Sigma, \mathbb{R})) - \dim(H^1(\Sigma, \mathbb{R})) = \chi(\Sigma) \quad (3.4)$$

where $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$. Here step $(s)$ in Eq. (3.4) follows from a well-known argument by McKean & Singer (cf. 61) and step (**) in Eq. (3.4) follows because according to the Hodge theorem we have $\ker(\Delta_1) \cong H^1(\Sigma, \mathbb{R})$. From Eq. (2.56b), Eq. (2.56c), and Eq. (3.4) we obtain

$$\operatorname{Det}_{rig, \alpha}(b) = (2 |\sin(\pi \alpha(b))|)^{\chi(\Sigma)} = (2 \sin(\pi \alpha(b)))^{\chi(\Sigma)} \quad (3.5)$$

Combining this with Eq. (2.56a) and Eq. (A.4) in Appendix A we arrive at

$$\operatorname{Det}_{rig}(b) = \det(1_t - \exp(ad(b)))^{\chi(\Sigma)/2} \quad (3.6)$$

(In particular, the value of $\operatorname{Det}_{rig}(b)$ is independent of the auxiliary Riemannian metric $g_{\Sigma}$.)

From Eq. (3.3), Eq. (3.6), and the Poisson summation formula (at an informal level) we therefore obtain

$$Z(\Sigma \times S^1, (\sigma_i), (\rho_i)) \sim \sum_{\lambda \in \Lambda} 1_P(\lambda/k) \left\{ \det(1_t - \exp(ad(\lambda/k)))^{\chi(\Sigma)/2} \left( \prod_{i=1}^m \operatorname{Tr}_{\rho_i}(\exp(\lambda/k)) \right) \right\} \quad (3.7)$$

where $\Lambda$ is the lattice dual to $I$, i.e. the (real) weight lattice of $(g, t)$, cf. Appendix A.1 below.
Remark 3.2  As mentioned in Sec. 2.3.2 above, the approach for obtaining a rigorous realization Det\textsubscript{rig}(B) of Det(B) used in the present paper is a variant of the approach in Sec. 6 in [17]. In the present paper we use the exponentials $e^{-\epsilon \Delta_i}$ of the original (="plain") Hodge Laplacians $\Delta_i$ for defining Det\textsubscript{rig}(B) and we later use the classical Gauss-Bonnet theorem for proving the well-definedness and for the explicit evaluation of Det\textsubscript{rig}(B), cf. Sec. 3.2.5 below). By contrast in Sec. 6 in [17] "covariant Hodge Laplacians" are used (and instead of the Gauss-Bonnet theorem the index theorem for the Dolbeault operator is applied). This leads to an additional term containing the dual Coxeter number $c_g$ of $g$. The overall effect in the present situation where $L$ is a "fiber link" is precisely the "shift" $k \rightarrow k + c_g$ mentioned in Sec. 2.4. Since in the present paper we are using the plain Hodge Laplacian we do not obtain such a shift, but according to Sec. 2.4 this is not a problem. Anyway, it would be interesting to study whether also in the case of general links $L$ the use of the covariant Hodge Laplacian can produce the shift $k \rightarrow k + c_g$ in all places where this is necessary. (Note that such a shift must also appear in the expressions $T_{\text{cl}}^c(A_c^\perp, B)$, $cl \in Cl_2(L, D)$ appearing in Sec. 3.5 below.)

3.1.2 Rewriting Eq. (3.7) using quantum algebraic notation

First we apply the change of variable $\lambda \rightarrow \lambda - \rho$ where $\rho \in \Lambda$ is the half sum of positive roots of $(g, t)$. We then obtain

$$Z(\Sigma \times S^1, (\sigma_i)_i, (\rho_i)_i) \sim \sum_{\lambda \in \Lambda \cap (kP - \rho)} \det(1 - \exp(ad((\lambda + \rho)/k))|_{\mathfrak{t}})^{(\Sigma)/2} (\prod_{i=1}^m \text{Tr}_{\rho_i}(\exp((\lambda + \rho)/k)))$$ \hspace{1cm} (3.8)

Without loss of generality we can assume that $P$ is the fundamental Weyl alcove, cf. Appendix A.1. Then, using the notation of Appendix A.2 below we have

$$\Lambda \cap (kP - \rho) = \Lambda^k$$

Let $\mu_i \in \Lambda_+^k$ be the highest weight of the representation $\rho_i$. In the following we will restrict our attention to the special case where $\mu_i \in \Lambda_+^k$. According to Eq. (A.12) in Appendix A we then have for all $\lambda \in \Lambda_+^k$

$$\text{Tr}_{\rho_i}(\exp((\lambda + \rho)/k)) = \frac{S_{\lambda\mu_i}}{S_{\lambda\lambda_0}}$$ \hspace{1cm} (3.9)

where $(S_{\mu\nu})_{\mu, \nu \in \Lambda_+^k}$ is the "S-matrix" defined by Eq. (A.8) in Appendix A (cf. Remark A.4 in Appendix A.2). Moreover, we have for every $\lambda \in \Lambda_+^k$ (cf. Eq. (A.11))

$$\det(1 - \exp(ad((\lambda + \rho)/k))|_{\mathfrak{t}}) \sim (S_{\lambda\lambda_0})^2$$ \hspace{1cm} (3.10)

Using this we can rewrite Eq. (3.8) above as

$$Z(\Sigma \times S^1, (\sigma_i)_i, (\rho_i)_i) \sim \sum_{\lambda \in \Lambda_+^k} \left(\prod_{i=1}^m \frac{S_{\lambda\mu_i}}{S_{\lambda\lambda_0}}\right) (S_{\lambda\lambda_0})^{\chi(\Sigma)}$$ \hspace{1cm} (3.11)

In other words, we have

$$Z(\Sigma \times S^1, (\sigma_i)_i, (\rho_i)_i) = C(\Sigma, G, k) \sum_{\lambda \in \Lambda_+^k} \left(\prod_{i=1}^m \frac{S_{\lambda\mu_i}}{S_{\lambda\lambda_0}}\right) (S_{\lambda\lambda_0})^{\chi(\Sigma)}$$ \hspace{1cm} (3.12)

where $C(\Sigma, G, k)$ is a constant depending only on (the homeomorphism class of) $\Sigma$, $G$, and $k$. 

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Remark 3.3 (i) Let us now first consider the special case $\Sigma \cong S^2$. Let $N_{\mu_1\mu_2\mu_3}$ be the dimension of the vector space $V_{k,(\rho_1,\rho_2,\rho_3)} := V_{S^2,G,k,(\rho_1,\rho_2,\rho_3)}$ of conformal blocks of the WZW model with group $G$ at level $k \in \mathbb{N}$ on the punctured surface $\Sigma = S^2$ with three punctures at $(\sigma_1, \sigma_2, \sigma_3)$ with colors $(\rho_1, \rho_2, \rho_3)$. According to [79] we have

$$N_{\mu_1\mu_2\mu_3} = Z(S^2 \times S^1, (\sigma_1, \sigma_2, \sigma_3), (\rho_1, \rho_2, \rho_3))$$  \hspace{1cm} (3.13)

if $Z(S^2 \times S^1, (\sigma_1, \sigma_2, \sigma_3), (\rho_1, \rho_2, \rho_3))$ is evaluated explicitly using the method in [79] for $k = \bar{k}$.

Now recall from Sec. 2.4 above that we expect that, for general $L$, the explicit values for $Z(\Sigma \times S^1, L)$ which we obtain by using the method in the present paper coincide with the values for $Z(\Sigma \times S^1, L)$ obtained in [79] up to a “shift” $k \rightarrow k + c_0$. In particular, in the present special case where $L$ is the fiber link described above we expect to obtain Eq. (3.13) provided that we choose $k = \bar{k} + c_0$. By taking into account that $N_{000} = 1$ or, equivalently, $Z(S^2 \times S^1) = 1$ (cf. Sec. 4.4 in [79]) we obtain from Eq. (3.13) and Eq. (3.12) above (applied to the special case $m = 3$)

$$N_{\mu_1\mu_2\mu_3} = \sum_{\lambda \in \Lambda_k^+} \frac{S_{\lambda m_1} S_{\lambda m_2} S_{\lambda m_3}}{S_{\lambda 0}}$$  \hspace{1cm} (3.14)

where $\Lambda_k^+$ and $(S_{\mu})_{\mu \nu}$ are defined as above with $k = \bar{k} + c_0$ (cf. Remark A.4 in Appendix A.2).

Eq. (3.14) is called the “fusion rules” in [79].

(ii) The second special case we consider is the case $m = 0$ (for general $\Sigma$). According to [79] we have for $k = \bar{k}$

$$\dim(V_{\Sigma,\bar{k}}) = Z(\Sigma \times S^1)$$  \hspace{1cm} (3.15)

where $V_{\Sigma,\bar{k}} := V_{\Sigma,G,\bar{k}}$ is the vector space of conformal blocks of the WZW model on $\Sigma$ with group $G$ at level $\bar{k}$. From what we said in part (i) of the present remark (cf. again Sec. 2.4) we expect Eq. (3.15) to hold (for our value of $Z(\Sigma \times S^1)$) if we choose $k = \bar{k} + c_0$. Combining Eq. (3.15) with Eq. (3.12) above we obtain

$$\dim(V_{\Sigma,\bar{k}}) = C(\Sigma, G, k) \sum_{\lambda \in \Lambda_k^+} (S_{\lambda 0})^{2-2g}$$  \hspace{1cm} (3.16)

where $g$ is the genus of $\Sigma$. The correct value of $C(\Sigma, G, k)$ turns out to be 1. Accordingly, we arrive at the “Verlinde formula”

$$\dim(V_{\Sigma,\bar{k}}) = \sum_{\lambda \in \Lambda_k^+} (S_{\lambda 0})^{2-2g}$$  \hspace{1cm} (3.17)

Observe that in view of the Weyl denominator formula and Eq. (A.81) in Appendix A below (and the paragraph after Eq. (A.81)) Eq. (3.17) is indeed equivalent to Eq. (1.2) in [17].

3.2 Preparations for Secs 3.3–3.5

As mentioned in Sec. 2.3.1 above, Eq. (2.47) is not yet our final formula for $Z(\Sigma \times S^1, L)$. Before we arrive at the final formula (cf. Eq. (3.39b) below) we will need to incorporate the first two points appearing in the list after Remark 2.8 in Sec. 2.3.1 above, which is what we will do in the present section. In particular, we will introduce ribbon holonomies (and later

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27In view of Eq. (A.9) in Appendix A.2 and $C_{00} = \delta_{00}$ the relation $Z(S^2 \times S^1) = 1$ implies $C(S^2, G, k) = 1$.

28In Sec. 4.5 of this formula is called the “Verlinde formula” but in the present paper we restrict the use of the term “Verlinde formula” to Eq. (3.17) below.

29Of course, Eq. (3.16) does not contain any information as long as we know nothing about $C(\Sigma, G, k)$. Still it is interesting to see how the expression $\sum_{\lambda \in \Lambda_k^+} (S_{\lambda 0})^{2-2g}$ on the RHS of Eq. (3.17) below appears automatically in the torus gauge approach to the CS path integral. Moreover, as is sketched in Sec. 7.5 in [17] it seems to be possible in principle to obtain the full Eq. (3.17) by using suitable additional informal arguments.
regularized ribbon holonomies), we will give the definition of the space $B_{\text{reg}}^{\text{ess}}$ mentioned above, we will show the existence of $\det_{\text{reg}}(B)$ for each $B \in B_{\text{reg}}$ (and generalize its definition to all $B \in B_{\text{reg}}^{\text{ess}}$), and for a suitable choice of the auxiliary Riemannian metric $g_\Sigma$ on $\Sigma$ we will give an explicit evaluation of $\det_{\text{reg}}(B)$ for those $B$ which will be relevant in Secs 3.3–3.5.

### 3.2.1 Closed ribbons and ribbon holonomies

**Definition 3.4** (i) A closed ribbon in $M = \Sigma \times S^1$ is a smooth embedding $R : S^1 \times [0, 1] \rightarrow \Sigma \times S^1$.

(ii) A ribbon link in $M = \Sigma \times S^1$ is a finite tuple $L = (R_1, R_2, \ldots, R_m)$, $m \in \mathbb{N}$, of non-intersecting closed ribbons $R_i$ in $M = \Sigma \times S^1$.

(iii) A colored ribbon link in $M = \Sigma \times S^1$ is a pair $L = ((R_1, R_2, \ldots, R_m), (\rho_1, \rho_2, \ldots, \rho_m))$, $m \in \mathbb{N}$, where $(R_1, R_2, \ldots, R_m)$ is a ribbon link in $M = \Sigma \times S^1$ and each $\rho_i$, $i \leq m$, is an irreducible, finite-dimensional, complex representation of $G$.

**Definition 3.5** A closed ribbon $R$ in $M = \Sigma \times S^1$ is called “horizontal” iff every $t \in S^1$ has an open neighborhood $U$ such that the restriction of $R^i := \pi_\Sigma \circ R$ to $U \times [0, 1] \rightarrow \Sigma$ is a smooth embedding. A ribbon link $L = (R_1, R_2, \ldots, R_m)$, $m \in \mathbb{N}$, in $M = \Sigma \times S^1$ is called horizontal iff each $R_i$, $i \leq m$, is horizontal.

**Definition 3.6** Let $L = (R_1, R_2, \ldots, R_m)$, $m \in \mathbb{N}$, be a ribbon link in $M = \Sigma \times S^1$. Then we will denote by $L^0$ the proper link in $M = \Sigma \times S^1$ given by $L^0 = (l_1, l_2, \ldots, l_m)$ where $l_i = R_i(\cdot, 1/2)$ for each $i \leq m$. Note that each $R_i$ induces a framing of the loop $l_i$ in a natural way. If $R_i$ is horizontal then the framing on $l_i$ induced by $R_i$ will also be called “horizontal”.

In the following we will assume that the (proper) link $L = (l_1, l_2, \ldots, l_m)$ in $M = \Sigma \times S^1$ which we fixed in Sec. 2.3 above has the property that $L = L_{\text{ribb}}$ for some horizontal ribbon link $L_{\text{ribb}} = (R_1, R_2, \ldots, R_m)$ in $M = \Sigma \times S^1$. We will keep $L_{\text{ribb}}$ fixed in the following and we will usually write $L$ instead of $L_{\text{ribb}}$.

From now on we will assume that $\sigma_0 \in \Sigma$ was chosen such that

$$\sigma_0 \notin \bigcup_{i=1}^m \text{Image}(R^i_{\Sigma})$$

(3.18)

where we have set $R^i_{\Sigma} := (R_i) : \pi_\Sigma \circ R_i$. For every $R \in \{R_1, R_2, \ldots, R_m\}$ we define

$$\text{Hol}_R(A) := P_1(A)$$

(3.19a)

where $(P_s(A))_{s \in [0, 1]}$ is the unique solution of

$$\frac{d}{ds} P_s(A) = P_s(A) \cdot \left( \int_0^1 A(l_u(s)) du \right), \quad P_0(A) = 1$$

(3.19b)

where $l_u$, $u \in [0, 1]$, is the loop $[0, 1] \rightarrow \Sigma \times S^1$ associated to the knot $K_u := R(\cdot, u)$ in $\Sigma \times S^1$, cf. Sec. 2.1 (In other words: $l_u$ is given by $l_u(s) = K_u(i_{S^1}(s)) = R(i_{S^1}(s), u)$ for all $s \in [0, 1]$.)

---

30 Note that this will always be the case if $L$ is admissible in the sense of Definition 3.37 in Sec. 3.3 below. See also Remark 3.38.

31 More precisely, $P(A) = (P_s(A))_{s \in [0, 1]}$ is the unique smooth map $[0, 1] \rightarrow \text{Mat}(N, \mathbb{R})$ fulfilling Eq. (3.19b).
From Eq. (2.47) we now obtain, after replacing each $l_i, i \leq m$, by $R_i$,

\[
Z(\Sigma \times S^1, L) \sim \sum_{y \in I} \int_{A^\perp_c \times B} \left\{ 1_{B_{reg}}(B) \text{Det}_{\text{rig}}(B) \right. \\
\times \left[ \int_{A^\perp} \left( \prod_i \text{Tr}_{p_i}(\text{Hol}_{R_i}(A^\perp + A^\perp_c, B)) \right) d\mu^\perp_B(A^\perp) \right. \\
\left. \times \exp\left( -2\pi i k \langle y, B(\sigma_0) \rangle \right) \right\} \exp(iS_{CS}(A^\perp_c, B))(DA^\perp_c \otimes DB) \quad (3.20)
\]

Observe that when working with ribbon holonomies instead of the usual loop holonomies there are two “complications”:

- While the original holonomy $\text{Hol}_l(A)$ is invariant under a reparametrization of the loop $l$, the ribbon holonomy $\text{Hol}_R(A)$ is not invariant under a reparametrization of $R$. More precisely, if $R' = R \circ \phi$ for some diffeomorphism $\phi : S^1 \times [0,1] \rightarrow S^1 \times [0,1]$ then in general we will have $\text{Hol}_R(A) \neq \text{Hol}_{R'}(A)$.

- While the functions $A \ni A \mapsto \text{Tr}_p(\text{Hol}_l(A)) \in \mathbb{C}$ are $G$-invariant, the functions $A \ni A \mapsto \text{Tr}_p(\text{Hol}_R(A)) \in \mathbb{C}$ are not. This is one reason why we did not introduce a ribbon analogue of Eq. (2.4) above. Instead we postponed the introduction of (closed) ribbons until now, that is, after the gauge has been fixed.\(^{32}\)

We can “defuse”/bypass these two complications by “sending the ribbon widths to zero” in the following sense:

For $s \in (0,1)$ and $i \leq m$ let $R^{(s)}_i(t,u)$ be the (closed) ribbon obtained from $R_i$ by

\[
R^{(s)}_i(t,u) := R_i(t,s \cdot (u - 1/2) + 1/2) \quad \text{for all } t \in S^1 \text{ and } u \in [0,1]
\]

(Observe that each $R^{(s)}_i$ is a rescaling of the restriction of $R_i$ onto $S^1 \times [1/2 - s/2, 1/2 + s/2]$).

By “sending the ribbon widths to zero” we mean that in Eq. (3.20) above we replace each $R_i$ by $R^{(s)}_i$ and add $\lim_{s \to 0}$ in front of the RHS of Eq. (3.20). Since

\[
\forall A \in \mathcal{A} : \lim_{s \to 0} \text{Hol}_{R^{(s)}_i}(A) = \text{Hol}_{R_i}(A) \quad (3.21)
\]

the use of ribbon holonomies whose ribbon widths we then send to zero can be considered as a “point-splitting” regularization (in the sense of Sec. 2.1 in \[79\]) of the original loop holonomies $\text{Hol}_{R_i}(A)$.

For reasons explained in the paragraph before Eq. (3.24) below we will introduce in Sec. 3.2.2 an additional regularization.

### 3.2.2 Regularized ribbon holonomies

Above we introduced ribbon holonomies $\text{Hol}_R(A), A \in \mathcal{A}$. Let us now consider the special situation $A = A^\perp + B dt$ where $A^\perp \in \mathcal{A}^\perp$ and $B \in \mathcal{B}$ and introduce the notation

\[
\text{Hol}_R(A^\perp, B) := \text{Hol}_R(A^\perp + B dt) \quad (3.22)
\]

According to the definition we have

\[
\text{Hol}_R(A^\perp, B) = P_1(A^\perp, B) \quad (3.23a)
\]

\(^{32}\)One should note that this is totally analogous to what is done in the Lorenz gauge approach to the CS path integral mentioned in Sec. 5.2 below where one introduces a framing of the link components after the Lorenz gauge has been fixed.
where \((P_s(A^\perp, B))_{s \in [0, 1]}\) is the unique solution of

\[
\frac{d}{ds} P_s(A^\perp, B) = P_s(A^\perp, B) \cdot D_s(A^\perp, B), \quad P_0(A^\perp, B) = 1. \tag{3.23b}
\]

Here we have set

\[
D_s(A^\perp, B) := \int_0^1 A^\perp(l_u'(s))du + \int_0^1 (Bdt)(l_u'(s))du, \tag{3.23c}
\]

where (as in Sec. 3.2.1 above) \(l_u, u \in [0, 1]\) is the loop \(l_u : [0, 1] \ni s \mapsto R(i_{s\Sigma}(s), u) \in \Sigma \times S^1\).

Let us fix \(s, u \in [0, 1]\) temporarily. Moreover, let us fix for the rest of Sec. 3 an arbitrary orthonormal basis \((T_a)_{a \leq \dim(g)}\) of \(g\).

During the informal calculation below it would be very convenient to be able to write \(A^\perp(l_u'(s))\) as a scalar product in a suitable way. Informally, we have

\[
A^\perp(l_u'(s)) = A^\perp((l_u')_\Sigma(s)) = \sum_a T_a \ll A^\perp, T_a(l_u')_\Sigma(s) \delta_{l_u(s)} \gg A^\perp \tag{3.24}
\]

where we have set \((l_u)_{\Sigma} := \pi_{\Sigma} \circ l_u\), where \(\ll \cdot, \gg \) is the scalar product on \(A^\perp\) induced by the Riemannian metric \(g := g_{\Sigma}\) fixed in Sec. 2.3 above, and where \(\delta_p\) for \(p = l_u(s)\) is the “Dirac delta function” in the point \(p\).

A well-defined version of the last term in Eq. (3.24) can be obtained if, instead of working, for every fixed \(p \in \Sigma \times S^1\), with the “Dirac delta function” \(\delta_p\) we work with a suitable “Dirac family”\(^{33}\) \((\delta_{\epsilon}^\ell)_{\epsilon < \epsilon_0}\), \(\epsilon_0 > 0\), w.r.t. the measure \(d\mu_g \otimes dt\), cf. the second paragraph after Eq. (3.26b) below. Here \(d\mu_g\) is the volume measure on \(\Sigma\) associated to \(g = g_{\Sigma}\).

After these preparations we will now replace for every \(\epsilon < \epsilon_0\) (with \(\epsilon_0 > 0\) as in the “bullet point” after Eq. 3.26b below) the expression \(A^\perp(l_u'(s))\) in Eq. (3.23c) by the “regularized” expression\(^{34}\)

\[
(A^\perp(l_u'(s)))^{(\epsilon)} := \sum_a T_a \ll A^\perp, T_a(l_u)_{\Sigma}(s) \delta_{l_u(s)} \gg A^\perp \tag{3.25}
\]

where for every \(v \in T_{\sigma_0\Sigma}, \sigma_0 \in \Sigma\), we denote by \(X_v\) the local vector field on \(\Sigma\) around \(\sigma_0\) which is obtained by parallel transport with the Levi-Civita connection, cf. the paragraph after Eq. 3.26c below. Next we replace the expression \(D_s(A^\perp, B)\) in Eq. 3.23b by

\[
D^*_s(A^\perp, B) := \int_0^1 (A^\perp(l_u'(s)))^{(\epsilon)}du + \int_0^1 (Bdt)(l_u'(s))du \tag{3.26a}
\]

and arrive at the regularized holonomy

\[
\text{Hol}^*_R(A^\perp, B) := P^*_1(A^\perp, B) \tag{3.26b}
\]

where \((P^*_s(A^\perp, B))_{s \in [0, 1]}\) is the unique solution of

\[
\frac{d}{ds} P^*_s(A^\perp, B) = P^*_s(A^\perp, B) \cdot D^*_s(A^\perp, B), \quad P^*_0(A^\perp, B) = 1 \tag{3.26c}
\]

- The vector field \(X_v\) where \(v \in T_{\sigma_0\Sigma}, \sigma_0 \in \Sigma\) mentioned above is obtained as follows:

  Let \(d\mu\) be the Riemannian distance function on \(\Sigma\) induced by \(g\). Since \(\Sigma\) is compact there is an \(\epsilon_0 > 0\) such that for all \(\sigma_0, \sigma_1 \in \Sigma\) with \(d\mu(\sigma_0, \sigma_1) < \epsilon_0\) there is a unique (geodesic) segment starting in \(\sigma_0\) and ending in \(\sigma_1\). Using parallel transport along this geodesic segment w.r.t. the Levi-Civita connection of \((\Sigma, g)\) we can transport every tangent vector \(v \in T_{\sigma_0\Sigma}\) to a tangent vector in \(T_{\sigma_0\Sigma}\). Thus every \(v \in T_{\sigma_0\Sigma}\) induces in a natural way a vector field \(X_v\) on the open ball \(B_{\epsilon_0}(\sigma_0) \subset \Sigma\).

---

\(^{33}\)i.e. \(\delta_{\epsilon}^\ell, \epsilon \in (0, \epsilon_0)\), is a non-negative and smooth function \(\Sigma \times S^1 \to \mathbb{R}\). Moreover, \(\int \delta_{\epsilon}^\ell d\mu_g \otimes dt = 1\), and we have \(\delta_{\epsilon}^\ell \to \delta_p\) weakly as \(\epsilon \to 0\) where \(\delta_p\) is the Dirac measure on \(\Sigma \times S^1\) in the point \(p\).

\(^{34}\)Here we interpret \(T_aX_{(l_u)_{\Sigma}(s)}\delta_{l_u(s)}\) as an element of \(A^\perp \cong C^\infty(S^1, A_{\Sigma})\) using the identification \(A_{\Sigma} \cong g \otimes VF(\Sigma)\) (induced by \(g_{\Sigma}\)) where \(VF(\Sigma)\) is the space of smooth vector fields on \(\Sigma\).
• The Dirac family \((\delta^i_p)_t < \epsilon_0\) on \(\Sigma \times S^1\), mentioned above, is obtained as follows: We first we choose, for each \(t \in S^1\), a Dirac family \((\delta^i_t)_t < \epsilon_0\) around \(t\) w.r.t. the measure \(dt\) on \(S^1\). Moreover, we choose for each \(\sigma \in \Sigma\), a Dirac family \((\delta^\sigma_p)_t < \epsilon_0\) around \(\sigma\) w.r.t. \(d\mu_\sigma\). For technical reasons we will assume that for each \(\epsilon \in \Sigma\) the support of \(\delta^\sigma_\epsilon\) is contained in the open ball \(B_\epsilon(\sigma)\).

For every \(p = (\sigma, t) \in \Sigma \times S^1\) and \(\epsilon < \epsilon_0\) we define \(\delta^\epsilon_p \in C^\infty(\Sigma \times S^1, \mathbb{R})\) by

\[
\delta^\epsilon_p(\sigma', t') := \delta^\sigma_\epsilon(\sigma') \delta^i_t(t') \quad \text{for all } \sigma' \in \Sigma \text{ and } t' \in S^1.
\]

Remark 3.7

(i) Instead of \(P^e_R(A^1, B)\) and \(D^\alpha_R(A^1, B)\) we later also use the notation \(P^e_{R,s}(A^1, B)\) and \(D^\alpha_{R,s}(A^1, B)\) i.e. in situations where more than one (closed) ribbon \(R\) is involved, cf. Sec. 3.7.

(ii) Above \(R\) was a closed ribbon in \(M = \Sigma \times S^1\), i.e. a smooth map \(R : S^1 \times [0,1] \to M\), which can be considered as a map \(R : [0,1] \times [0,1] \to M\) where \(R(0,u) = R(1,u)\) for all \(u \in [0,1]\). The definition of \(P^e_{R,s}(A^1, B)\) and \(D^\alpha_{R,s}(A^1, B)\) can be generalized in an obvious way to all “ribbons”, i.e. all smooth maps \(R : [0,1] \times [0,1] \to M\) (where the condition \(R(0,u) = R(1,u)\) for all \(u \in [0,1]\) need not be fulfilled, in which case we will call \(R\) “open”). (This will be useful in Sec. 3.5 below.) Instead of \(P^e_{R,1}(A^1, B)\) we will simply write \(P^e_R(A^1, B)\).

3.2.3 The space \(B_{reg}^{ess}\)

Observe that \(t_{reg} = t \setminus (\bigcup_{\alpha \in \mathcal{R}, k \in \mathbb{Z}} H_{a,k})\) where \(H_{a,k} : \alpha \in \mathcal{R}, k \in \mathbb{Z}\) is the hyperplane in \(t\) given by \(H_{a,k} := \alpha^{-1}(k)\), cf. Appendix A.1 below. Accordingly, the space \(B_{reg}\) defined in Sec. 2.2 above is the space of those \(B \in B\) whose image does not meet any of these hyperplanes \(H_{a,k}\), \(\alpha \in \mathcal{R}, k \in \mathbb{Z}\).

For reasons explained in Remark 3.8 below we will now replace in Eq. (3.20) above the space \(B_{reg}\) by the slightly larger space \(B_{reg}^{ess}\) of those \(B \in B\) which, whenever their image does meet a hyperplane \(H_{a,k}\), they intersect \(H_{a,k}\) “properly”. More precisely, we demand that for all \(\sigma \in \Sigma\) for which \(B(\sigma) \in H_{a,k}\) holds for some \(\alpha \in \mathcal{R}\) and \(k \in \mathbb{Z}\) the differential \(dB_\sigma(\sigma)\) of the map \(B_\sigma := \alpha \circ B : \Sigma \to \mathbb{R}\) in the point \(\sigma\) does not vanish. More briefly, \(B_{reg}^{ess}\) is given by

\[
B_{reg}^{ess} := \{ B \in B : \forall \sigma \in \Sigma : \alpha \in \mathcal{R} : |B_\sigma(\sigma) \in \mathbb{Z} \Rightarrow dB_\sigma(\sigma) \neq 0\}\)

Observe that if \(B \in B_{reg}^{ess}\) then the set \(\mathcal{N} := \{ \sigma \in \Sigma : B(\sigma) \notin \mathbb{Z}\}\) is a \(d\mu_\sigma\)-zero set of \(\Sigma\) and the singularities of the functions \(\log(\sin(\pi \alpha(\sigma))) : \Sigma \to \mathbb{C}, \alpha \in \mathcal{R}\), are so mild that the RHS of Eq. (3.30) in Sec. 3.2.5 below exists. (This will allow us to generalize the definition of \(\text{Det}_{reg}(B)\) to all \(B \in B_{reg}^{ess}\).)

Remark 3.8

(i) The replacement \(B_{reg} \to B_{reg}^{ess}\) will be justified in Appendix B.6 below (cf. also Remark 3.17 in Sec. 3.7.2 below).

(ii) In certain special cases one can actually continue to work with the space \(B_{reg}\) rather than having to work with \(B_{reg}^{ess}\). For example, this is the case when \(L\) is a vertical link, cf. Sec. 3.7. Moreover, if \(L\) is as is in Sec. 3.4 (and, possibly, also if \(L\) is as in Sec. 3.7) and each \(\rho_i\) is a fundamental representation of \(G\) and \(G = SU(N)\) (or, possibly, also for general \(G\)) then

---

35In order to see this note that in the special case where each \(\rho_i\) is a fundamental representation of \(G\) the step functions \(B\) appearing on the RHS of Eq. (3.105) in Sec. 3.4 below all have small “step sizes”. In the special case where \(G = SU(N)\) it is easy to show that this implies that those \(B\) for which \(1_{B_{reg}^{ess}}(B) = 0\) (i.e. those \(B\) whose image is not contained in a single Weyl alcove) will have at least one “step” whose (constant) value lies in one of the hyperplanes \(H_{a,k}\), \(\alpha \in \mathcal{R}, k \in \mathbb{Z}\). Accordingly, we then also have \(1_{B_{reg}^{ess}}(B) = 0\). On the other hand, if \(1_{B_{reg}^{ess}}(B) = 1\) then trivially also \(1_{B_{reg}^{ess}}(B) = 1\). Consequently, the value of the RHS of Eq. (3.105) below does not change if we replace \(1_{B_{reg}^{ess}}(B)\) by \(1_{B_{reg}}(B)\).
can also work with the space $\mathcal{B}_{\text{reg}}$ instead of $\mathcal{B}_{\text{reg}}^{\text{ess}}$. However, in the majority of cases we have to work with $\mathcal{B}_{\text{reg}}^{\text{ess}}$ if we want to have a chance of obtaining the correct result for the value of $Z(\Sigma \times S^1, L)$.

### 3.2.4 A convenient choice of the auxiliary Riemannian metric $g_{\Sigma}$ on $\Sigma$

Let $L = (R_1, R_2, \ldots, R_m)$ be the (horizontal) ribbon link in $M = \Sigma \times S^1$ fixed in Sec. 3.2.1 above. Set $R_i^2 := \pi_\Sigma \circ R_i$, $i \leq m$, where $\pi_\Sigma : \Sigma \times S^1 \to \Sigma$ is the canonical projection and let $S_i$ denote the interior of $\text{Image}(R_i^2)$ in $\Sigma$.

Recall that above we have fixed an auxiliary Riemannian metric $g_{\Sigma}$ on $\Sigma$. In order to simplify our life we will assume from now on that $g := g_{\Sigma}$ fulfills the following condition:

**Condition 1** The auxiliary Riemannian metric $g$ on $\Sigma$ was chosen such that for each $i \leq m$ the pullback of $g_{S_i}$ via $R_i^2 : S^1 \times (0, 1) \to \Sigma$ coincides with the Riemannian product metric $g_{S^1} \otimes g_{(0,1)}$ on $S^1 \times (0,1)$ where $g_{(0,1)}$ is the standard Riemannian metric on $(0,1)$ and $g_{S^1}$ is the translation-invariant Riemannian metric on $S^1$, which we assume to be normalized such that $\text{vol}(S^1) = 1$. (Note that $g$ is uniquely determined on each $S_i$.)

There are two reasons why we choose $g_{\Sigma}$ such that Condition 1 is fulfilled:

1. The evaluation of the inner integral in Eq. (3.40) in Sec. 3.3 and in Sec. 3.4 as well as the evaluation of the expressions $T_{A_{c}}^{\sigma}(s, B)$, $cl \in \text{Cl}_1(L, D)$, in Sec. 3.5 becomes much easier.

2. The explicit evaluation of $\text{Det}_{rig}(B)$ in Sec. 3.2.5 below leads to the correct formula. This may also be the case without Condition 1 (cf. Remark 3.13 below), however, this point is not yet clarified.

**Remark 3.9** Recall that the original (informal) path integral expression in Eq. (2.41) above is topologically invariant. In particular, it does not involve a Riemannian metric. However, for technical reasons, we work with the auxiliary Riemannian metric $g_{\Sigma}$ breaking topological invariance.

Clearly, whenever one introduces an auxiliary object $O$ in order to make sense of an informal expression in a natural way it would be good to have either (or a combination) of the following:

(i) The auxiliary object $O$ can be chosen arbitrarily and the final result does not depend on it.

(ii) There is a distinguished/canonical choice of $O$ and this is the choice which we use.

Condition 1 is a combination of these two cases. The restriction of $g = g_{\Sigma}$ to $S := \bigcup_{i=1}^{m} S_i$ is given canonically. On the other hand, the restriction of $g$ to $S^c := \Sigma \setminus S$ can essentially be chosen arbitrarily (as long as $g_{|S}$ and $g_{|S^c}$ “fit together” smoothly, i.e. induce a smooth Riemannian metric on all of $\Sigma$).

**Remark 3.10** For the concrete ribbon links $L$ which we will consider in Sec. 3.3 and Sec. 3.4 below Condition 1 can always be fulfilled and, according to Remark 3.11, is natural.

On the other hand, for general “admissible” ribbon links $L$ as defined in Sec. 3.2 below Condition 1 can in general only be fulfilled after replacing each $R_i$ by a suitable reparametrization.

So in the general case it may be more natural to use the following “infinitesimal version” of Condition 1 (which is suggested by Observation 3.12 below):

---

\[36\text{Observe, for example, that the two maps } R_i^2 \text{ and } R_j^2 \text{ will in general induce a different Riemannian metric on } U_{ij} := S_i \cap S_j \neq \emptyset \text{ if } i \neq j. \text{ On the other hand, if the ribbon link } L \text{ fixed above is admissible in the sense of Definition 3.20 in Sec. 3.5 below then for every } i, j \leq m \text{ it is always possible to reparametrize } R_i \text{ and } R_j \text{ such that the reparametrized versions of } R_i^2 \text{ and } R_j^2 \text{ induce the same Riemannian metric on } U_{ij}.\]
The auxiliary Riemannian metric $g$ on $\Sigma$ was chosen such that for each $i \leq m$ and every $p \in C_i := \text{Image}(l_i)$ with $l_i := R_i(\cdot, 1/2)$ the geodesic curvature $k^C_i(p)$ of $C_i$ in the point $p$ vanishes.

Observe that this “infinitesimal version” of Condition 4 can always be fulfilled but it involves considerably more work when evaluating $Z(\Sigma \times S^1, L)$.

Note also that, as mentioned in Remark 3.12 below, it may be possible that Condition 4 or its infinitesimal version can be dropped altogether. (Of course, this increases even further the amount of work we have to do for evaluating $Z(\Sigma \times S^1, L)$.)

3.2.5 Generalization of $\operatorname{Det}_{\text{rig}}(B)$ for $B \in B^\text{ess}_{\text{reg}}$

In the present section we will modify (slightly) the definition of $\operatorname{Det}_{\text{rig}}(B)$ for $B \in B_{\text{reg}}$, given in Sec. 2.3.2 above, and we will then generalize the new definition to the case of all $B \in B^\text{ess}_{\text{reg}}$. Moreover, for the class of $B$ relevant for us in Sec. 3.3–3.5 below we will give an explicit formula for $\operatorname{Det}_{\text{rig}}(B)$ by applying the Gauss-Bonnet formula for surfaces with boundary.

Recall that for $B \in B_{\text{reg}}$ we rewrote $\operatorname{Det}(B)$ informally as (cf. Eq. (2.55))

$$\operatorname{Det}(B) = \prod_{\alpha \in \mathcal{R}_+} \det(O^{(0)}_{\alpha}(B))^2 \det(O^{(1)}_{\alpha}(B))^{-1}$$

(3.29)

where for each fixed $\alpha \in \mathcal{R}_+$ the operators $O^{(i)}_{\alpha}(B) : \Omega^i(\Sigma, \mathbb{R}) \to \Omega^i(\Sigma, \mathbb{R})$, $i = 0, 1$, are the multiplication operators obtained by multiplication with the function $2|\sin(\pi \alpha(B(\cdot)))|$. Then we used this as the motivation to define $\operatorname{Det}_{\text{rig}}(B)$ by Eqs. (2.56a)–(2.56c) above.

Let us now (re)define $\operatorname{Det}_{\text{rig}}(B)$ for $B \in B_{\text{reg}}$ again by Eqs. (2.56a)–(2.56c) in Sec. 2.3.2 above where now we take $O^{(i)}_{\alpha}(B)$ to be the multiplication operators obtained by multiplication with the function $2\sin(\pi \alpha(B(\cdot)))$ (rather than $2|\sin(\pi \alpha(B(\cdot)))|$) and we take the “operator-logarithm” log appearing in Eq. (2.56a) above to come from the restriction to $\mathbb{R}\setminus\{0\}$ of the principal branch of the complex logarithm. (We will give the definition of $\operatorname{Det}_{\text{rig}}(B)$ for general $B \in B^\text{ess}_{\text{reg}}$ below.)

Remark 3.11 (i) Observe that when working with the new ansatz for $O^{(i)}_{\alpha}(B)$, the RHS of Eq. (3.29) depends explicitly (via $\mathcal{R}_+$) on the Weyl chamber $C$ fixed in Appendix A below. However, one can argue at an informal level that the value of $\operatorname{Det}(B)$ does not depend on $C$. (This implies also that if $B \in B_{\text{reg}}$ then the value of the expression Eq. (3.29) is independent of whether we use the original or the new ansatz for the operators $O^{(i)}_{\alpha}(B)$.) Moreover, for the special maps $B \in B^\text{ess}_{\text{reg}}$ relevant below one can see from Eq. (3.36) below (by taking into account that $\sum_i \chi(Y_i) = \chi(\Sigma)$ is even) that also the value of $\operatorname{Det}_{\text{rig}}(B)$ does not depend on the choice of the Weyl chamber $C$.

(ii) Taking $O^{(i)}_{\alpha}(B)$ to be the multiplication operator with the function $2\sin(\pi \alpha(B(\cdot)))$ looks natural and, as we will see below, leads to the correct value

$^37$of $Z(\Sigma \times S^2, L)$. Anyway, it would be desirable to obtain a direct justification for the new ansatz above (i.e. for taking $O^{(i)}_{\alpha}(B)$ to be the multiplication operator with the function $2\sin(\pi \alpha(B(\cdot)))$), for example by using an argument which involves the computation of the $\eta$-invariant of a suitable operator.

---

$^37$By contrast, if we choose again $O^{(i)}_{\alpha}(B)$ to be the multiplication operator with the function $2|\sin(\pi \alpha(B(\cdot)))|$ then we will only get the the correct values for $Z(\Sigma \times S^2, L)$ in the cases mentioned in Remark 3.13 in Sec. 3.2.3 above. But these are exactly those cases where we do not have to work with the space $B^\text{ess}_{\text{reg}}$ in the first place but can continue to work with the space $B_{\text{reg}}$. 

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If $B \in \mathcal{B}_{\text{reg}}$, then $\text{Tr}(e^{-\epsilon \Delta_1} \log(O^{(i)}_\alpha(B)))$ is well-defined (cf. the argument at the end of Sec. 2.3.2 above) and we can rewrite the RHS of Eq. (2.56c) as

$$\exp(\text{Tr}(e^{-\epsilon \Delta_1} \log(O^{(i)}_\alpha(B)))) = \exp(\int_\Sigma \text{Tr}(K^{(i)}_\epsilon(\sigma, \sigma)) \log(2\sin(\pi\alpha(B(\sigma)))) d\mu_g(\sigma))$$ (3.30)

where $K^{(0)}_\epsilon : \Sigma \times \Sigma \to \mathbb{R} \cong \text{End}(\mathbb{R})$ is the integral kernel of $e^{-\epsilon \Delta_0}$ and $K^{(1)}_\epsilon : \Sigma \times \Sigma \to \bigcup_{\sigma_1, \sigma_2 \in \Sigma} \text{Hom}(T_{\sigma_1} \Sigma, T_{\sigma_2} \Sigma)$ is the integral kernel of $e^{-\epsilon \Delta_1}$ (and where $\log : \mathbb{R} \setminus \{0\} \to \mathbb{C}$ is the restriction to $\mathbb{R} \setminus \{0\}$ of the principal branch of the complex logarithm). Observe that the RHS of Eq. (3.30) is well-defined not only when $B \in \mathcal{B}_{\text{reg}}$ but even when $B \in \mathcal{B}^{\text{ess}}_{\text{reg}}$. Accordingly, let us now set, for general $B \in \mathcal{B}^{\text{ess}}_{\text{reg}},$

$$\det_\epsilon(O^{(i)}_\alpha(B)) := \exp(\int_\Sigma \text{Tr}(K^{(i)}_\epsilon(\sigma, \sigma)) \log(2\sin(\pi\alpha(B(\sigma)))) d\mu_g(\sigma))$$ (3.31)

and define $\text{Det}_{\text{rig}, \alpha}(B)$ and $\text{Det}_{\text{rig}}(B)$ again by Eqs. (2.56a)–(2.56b) in Sec. 2.3.2 above.

According to a well-known result in [61] the negative powers of $\epsilon$ that appear in the asymptotic expansion of $K^{(i)}_\epsilon$, $i = 0, 1$ as $\epsilon \to 0$ cancel each other and we obtain

$$[2 \text{Tr}(K^{(0)}_\epsilon(\sigma, \sigma)) - \text{Tr}(K^{(1)}_\epsilon(\sigma, \sigma))] \to \frac{1}{4\pi} R_g(\sigma) \quad \text{uniformly in } \sigma \text{ as } \epsilon \to 0$$ (3.32)

where $R_g$ is the scalar curvature (= twice the Gaussian curvature) of $(\Sigma, g)$.

From Eqs. (3.31) and (3.32) it follows that $\text{Det}_{\text{rig}, \alpha}(B)$ is well-defined for all $B \in \mathcal{B}^{\text{ess}}_{\text{reg}}$ (i.e. that the $\epsilon \to 0$ limit in Eq. (2.56b) really exists) and that we have

$$\text{Det}_{\text{rig}, \alpha}(B) = \exp\left(\int_\Sigma \log(2\sin(\pi\alpha(B(\sigma)))) \frac{1}{4\pi} R_g(\sigma)d\mu_g(\sigma)\right)$$ (3.33)

Let us now evaluate $\text{Det}_{\text{rig}}(B)$ explicitly for all $B$ relevant for us.

Recall that in Sec. 3.1 above only the special case $\Sigma = \Sigma$ (with $\Sigma \in \mathcal{t}_{\text{reg}}$) was relevant. (Observe that we can rederive Eq. (3.5) and hence also Eq. (3.6) in Sec. 3.1 above directly from Eq. (3.33) above by applying the classical Gauss-Bonnet Theorem $4\pi\chi(\Sigma) = \int_\Sigma R_g d\mu_g$.)

Now in Secs 3.3–3.5 below a larger class of maps $B : \Sigma \to \Sigma$ will be relevant for the explicit evaluation of $Z(\Sigma \times S^1, L)$ (after the $\epsilon \to 0$-limit on the RHS of Eq. (3.40) below have been carried out), namely the class of all those maps $B$ which are constant on each connected component $Y_i$, $i \leq r$, of $\Sigma \setminus \bigcup_{i=1}^n \text{Image}(R_{\Sigma}^b)$.

In order to deal with this larger class of maps $B$ we will need the following, more general version\footnote{In fact, if we want to have a chance of dealing successfully with the case of general (strictly admissible) $L$ as in Sec. 3.5 below we will have to work with yet another generalization of Eq. (3.34) where the boundary $\partial Y$ is only a piecewise smooth (rather than a smooth) submanifold of $\Sigma$ (cf. Remark D.8 in Appendix D). In the generalized formula there will be an extra term on the RHS involving a sum over the finite number of those points $p$ of $\partial Y$ where $\partial Y$ is not smooth (and containing the corresponding “angle” of $\partial Y$ at $p$).} of the classical Gauss-Bonnet Theorem mentioned above:

Let $Y \subset \Sigma$ be such that the boundary $\partial Y$ is (either empty or) a smooth 1-dimensional submanifold of $\Sigma$. We equip $\partial Y$ with the Riemannian metric induced by $g = g_\Sigma$ and denote by $ds$ the corresponding “line element” on $\partial Y$. Then we have

$$4\pi\chi(Y) = \int_Y R_g d\mu_g + 2 \int_{\partial Y} k^{\partial Y}_g d\mu_{\partial Y}$$ (3.34)

where $k^{\partial Y}_g(p)$ for $p \in \partial Y$ is the geodesic curvature of the curve $\partial Y$ in the point $p$.\footnote{In fact, if we want to have a chance of dealing successfully with the case of general (strictly admissible) $L$ as in Sec. 3.5 below we will have to work with yet another generalization of Eq. (3.34) where the boundary $\partial Y$ is only a piecewise smooth (rather than a smooth) submanifold of $\Sigma$ (cf. Remark D.8 in Appendix D). In the generalized formula there will be an extra term on the RHS involving a sum over the finite number of those points $p$ of $\partial Y$ where $\partial Y$ is not smooth (and containing the corresponding “angle” of $\partial Y$ at $p$).}
**Observation 3.12** Condition \( \mathcal{I} \) implies that the scalar curvature \( R_g \) vanishes on \( \bigcup_{i=1}^{m} \text{Image}(R_{\Sigma}^i) \). Moreover, on each of the curves \( C^i_u = \text{Image}((R^i_u)_{\Sigma}), i \leq m, u \in [0, 1] \), the geodesic curvature \( k^C_{g^i} \) vanishes.

Now let \( B : \Sigma \to \mathfrak{t} \) be as above\(^{39}\), i.e. \( B \) is constant on each of \( Y_i, i \leq r \), where (\( Y_i \)\),\( \Sigma \)) is the family of connected components of \( \Sigma \setminus \bigcup_{i=1}^{m} \text{Image}(R_{\Sigma}^i) \). Let \( b_i \) denote the unique value of \( B \) on \( Y_i \). From Eq. (3.33) and Observation 3.12 we conclude that then

\[
\text{Det}_{\text{rig}, \alpha}(B) = \exp \left( \sum_i \int_{Y_i} \log(2\sin(\pi\alpha(b_i))) \frac{1}{4\pi} R_g(\sigma) d\mu_g(\sigma) \right)
= \prod_i \exp \left( \log(2\sin(\pi\alpha(b_i))) \left[ \int_{Y_i} \frac{1}{4\pi} R_g(\sigma) d\mu_g(\sigma) + \frac{1}{2\pi} \int_{\partial Y_i} k^\partial_{\Sigma} ds \right] \right)
= \prod_i \exp \left( \log(2\sin(\pi\alpha(b_i))) \left[ \chi(Y_i) \right] \right)
= \prod_i (2\sin(\pi\alpha(b_i)))^{\chi(Y_i)}
\]

(3.35)

From this we obtain

\[
\text{Det}_{\text{rig}}(B) = \prod_i \det^{1/2}(1_t - \exp(ad(b_i)))^{\chi(Y_i)}
\]

(3.36)

where \( \det^{1/2}(1_t - \exp(ad(\cdot))) : t \to \mathbb{R} \) is given by

\[
\det^{1/2}(1_t - \exp(ad(b))) = \prod_{\alpha \in \mathcal{R}_+} 2\sin(\pi\alpha(b)) \quad \forall b \in \mathfrak{t}
\]

**Remark 3.13** Recall from Remark 3.10 above that when working with general (ribbon) links \( L \) it may be better to work with the infinitesimal version of Condition \( \mathcal{I} \) mentioned in Remark 3.10 above. It fact, it may even be possible to avoid the use of Condition \( \mathcal{I} \) (or its infinitesimal version) completely. This may be surprising since for the derivation of Eq. (3.36) above it was crucial that the geodesic curvature terms \( k^\partial_{\Sigma} \) appearing in Eq. (3.35) vanish. However, it is possible that during the computations later on these geodesic curvature terms appear when evaluating the inner integral in Eq. (3.39a). (This is because, when Condition \( \mathcal{I} \) is not fulfilled, the “covariances” \( \ll \phi^\partial_i, C(B)\phi^\partial_j \) appearing Eq. (3.50b) will in general not vanish anymore.)

**Remark 3.14** One can rewrite Eq. (3.29) in a more symmetric way, which will be useful in Remark 3.15 below.

In order to do so observe first that for \( B \in \mathbb{B} \) and \( \alpha \in \mathcal{R}_+ \) we have \( O^0_{\alpha}(B) = \star^{-1} \circ O^2_{\alpha}(B) \circ \star \) where \( \star : \Omega^0(\Sigma, \mathbb{T}) \to \Omega^2(\Sigma, \mathbb{T}) \) is the Hodge star operator induced by any fixed Riemannian metric \( g_\Sigma \) on \( \Sigma \). Thus we obtain, informally,

\[
O^0_{\alpha}(B) = O^2_{\alpha}(B)
\]

(3.37)

and we can rewrite Eq. (3.29) as

\[
\text{Det}(B) = \prod_{\alpha \in \mathcal{R}_+} \det(O^0_{\alpha}(B)) \det(O^1_{\alpha}(B))^{-1} \det(O^2_{\alpha}(B))
\]

(3.38)

\(^{39}\) Observe that on \( \bigcup_{i=1}^{m} \text{Image}(R_{\Sigma}^i) \) \( B \) does not have to be smooth.
Remark 3.15 There is an alternative method for making rigorous sense of $\text{Det}(B)$ which is very natural when using the rigorous frameworks (F1) and (F3) described in Sec. 4.3 below for making sense of $Z(\Sigma \times S^1, L)$ This alternative method consists in introducing a “simplicial analogue” $\text{Det}^\text{disc}(B)$ of the RHS of Eq. (3.38), cf. Sec. 3.6 in [40]. (For the definition of $\text{Det}^\text{disc}(B)$ it is crucial to use the RHS of Eq. (3.38) here. If instead one uses the RHS of Eq. (3.29) as the starting point for a definition of $\text{Det}^\text{disc}(B)$ one obtains incorrect results.)

3.2.6 The (regularized) torus gauge fixed CS path integral $Z^{t.g.f}(\Sigma \times S^1, L)$

In order to arrive at our final formula for the Chern-Simons path integral $Z(\Sigma \times S^1, L)$ we will now incorporate the constructions in Secs 3.2.1–3.2.5. In particular, we will now replace the colored (proper) link $L = ((l_1, l_2, \ldots, l_m), (\rho_1, \rho_2, \ldots, \rho_m))$, $i \leq m$, in $M = \Sigma \times S^1$ which we fixed in Sec. 2.3 above by the colored ribbon link $L_{\text{ribb}} = ((R_1, R_2, \ldots, R_m), (\rho_1, \rho_2, \ldots, \rho_m))$ chosen in Sec. 3.2.1. (Recall that we assume that $L_{\text{ribb}}$ is horizontal and fulfills $L = L_{\text{ribb}}^0$, cf. Definition 3.5 and Definition 3.6). Instead of $L_{\text{ribb}}$ we will simply write $L$ in the following and we set

$$
Z^{t.g.f}(\Sigma \times S^1, L) := \lim_{s \to 0} \lim_{\epsilon \to 0} \sum_{y \in I} \int_{A_c^+ \times B} \left\{ 1_{G_{\text{CS}}^{\text{rig}}}(B) \text{Det}_{\text{rig}}(B) \right\} \times \left[ \int_{A_c^+} \left( \prod_i \text{Tr}_{\rho_i} \left( \text{Hol}_{R_i^{(0)}}^\epsilon (A_c^+ + A_c^+) \right) \right) d\mu_B(A_c^+) \right] \times \exp(-2\pi i k y, B(\sigma_0)) \exp(i S_{\text{CS}}(A_c^+, B)) (DA_c^+ \otimes DB) \quad (3.39a)
$$

According to what we said in Secs 3.2.1–3.2.3 above $Z^{t.g.f}(\Sigma \times S^1, L)$ can be considered as a regularized and gauge-fixed version of $Z(\Sigma \times S^1, L)$ and we should therefore have

$$
Z(\Sigma \times S^1, L) \sim Z^{t.g.f}(\Sigma \times S^1, L). \quad (3.39b)
$$

In the special situations in Sec. 3.3 and Sec. 3.4 below (and probably also in the situation of general strictly admissible $L$, cf. Sec. 3.5) the $s \to 0$-limit in Eq. (3.39a) will turn out to be trivial, i.e. Eq. (3.39b) will reduce to

$$
Z(\Sigma \times S^1, L) \sim \lim_{s \to 0} \sum_{y \in I} \int_{A_c^+ \times B} \left\{ 1_{G_{\text{CS}}^{\text{rig}}}(B) \text{Det}_{\text{rig}}(B) \right\} \times \left[ \int_{A_c^+} \left( \prod_i \text{Tr}_{\rho_i} \left( \text{Hol}_{R_i^{(0)}}^\epsilon (A_c^+ + A_c^+) \right) \right) d\mu_B(A_c^+) \right] \times \exp(-2\pi i k y, B(\sigma_0)) \exp(i S_{\text{CS}}(A_c^+, B)) (DA_c^+ \otimes DB) \quad (3.40)
$$

for any fixed $s_0 \in (0, 1)$. (We exclude the case $s_0 = 1$ for technical reasons, cf. the paragraph before Eq. (3.50a) below.)

Convention 3.16 In the following we will sometimes write $\tilde{R}_i$ instead of $R_i$. Moreover, we will often write $R_i$ instead of $R_i^{(s_0)}$. In other words, $R_i$ can refer both to $\tilde{R}_i$ and to $R_i^{(s_0)}$.  

\footnote{It can also be useful with the rigorous continuum approach (F2) if it is combined with a suitable continuum limit argument.}
Remark 3.17 One point which needs to be better understood is why, in the generalization of the definition of $\text{Det}_{\text{reg}}(B)$ for all $B \in B^{\text{ess}}_{\text{reg}}$ which we gave in Sec. 3.2.2 above, we need to define $O_B(i)(B)$ to be the multiplication operator with the function $2\sin(\pi \alpha(B(\cdot)))$ rather than $2|\sin(\pi \alpha(B(\cdot)))|$, cf. Remark 3.11 above.

Moreover, one should try to understand better why, as remarked in Remark 3.8 above, apart from some special cases we cannot work with the indicator function $1_{B_{\text{reg}}}(B)$ instead of $1_{B_{\text{ess}}}(B)$ if we want to obtain the correct values for $Z(\Sigma \times S^1, L)$, even though both Eq. (3.39b) and the modification of Eq. (3.39a) where in Eq. (3.39a) the factor $1_{B_{\text{ess}}}(B)$ is replaced by $1_{B_{\text{reg}}}(B)$ can be derived/justified at an informal level.

Finally, it would be desirable to check whether Condition 1 above can be dropped (cf. Remark 3.13 above), i.e. whether in Secs. 3.3–3.5 we arrive at the correct values for $Z(\Sigma \times S^1, L)$ for an arbitrary auxiliary Riemannian metric $g$ on $\Sigma$.

3.3 Special case II. Torus knots in $M = S^2 \times S^1$

We will now evaluate the RHS of Eq. (3.40) in the special case where $L$ belongs to a large class of (colored) “torus ribbon knots” in $M = S^2 \times S^1$ (cf. Definition 3.20 below). By doing so we obtain a $S^2 \times S^1$-analogue of the Rosso-Jones formula, cf. Eq. (3.38) below. In Sec. 3.3.5 we will combine the straightforward generalization Eq. (3.39) of Eq. (3.38) with a short surgery argument and obtain, for arbitrary (simple, simply-connected, compact Lie group) $G$, the original Rosso-Jones formula, which is concerned with arbitrary (colored) torus knots in $S^3$.

Recall that a torus knot in $S^3$ is a knot $\tilde{K} : S^1 \to S^3$ whose image is contained in an unknotted torus $\mathcal{T} \subset S^3$. (Note, for example, that two of the four simplest non-trivial knots in $S^3$ are torus knots, namely the trefoil knot and the cinquefoil knot.) We take this as the motivation for the following definition.

Definition 3.18 (i) A torus knot in $S^2 \times S^1$ of standard type is a knot $K : S^1 \to S^2 \times S^1$ whose image is contained in a torus $\mathcal{T}$ in $S^2 \times S^1$ fulfilling the following condition:

(C) $\mathcal{T}$ is of the form $\mathcal{T} = \psi(C_0 \times S^1)$ where $C_0$ is an embedded circle in $S^2$ and $\psi : S^2 \times S^1 \to S^2 \times S^1$ is a diffeomorphism.

In the special case where $\psi = \text{id}_{S^2 \times S^1}$, i.e. $\mathcal{T} = C_0 \times S^1$, we will call $K$ “canonical”.

(ii) For every canonical torus knot $K$ in $S^2 \times S^1$ of standard type we denote by $p(K)$ and $q(K)$ the two winding numbers of $K$ where we consider $K$ as a continuous map $S^1 \to C_0 \times S^1 \cong S^1 \times S^1$ in the obvious way. (As a side remark we mention that $p(K)$ and $q(K)$ will always be coprime.)

Remark 3.19 Note that every unknotted torus $\tilde{T}$ in $S^3$ can be obtained from a torus $T$ in $S^2 \times S^1$ fulfilling condition (C) by performing a suitable Dehn surgery on a separate knot in $S^2 \times S^1$. Consequently, every torus knot $\tilde{K}$ in $S^3$ can be obtained from a torus knot $K$ in $S^2 \times S^1$ of standard type by performing such a Dehn surgery. Moreover, every torus knot $\tilde{K}$ in $S^3$ can be obtained up to equivalence from some canonical torus knot in $S^2 \times S^1$ of standard type by performing a suitable Dehn surgery. We will makes use of this observation in Sec. 3.3.5 below where we will derive the aforementioned Rosso-Jones formula for torus knots in $S^3$.

Definition 3.20 (i) A canonical torus ribbon knot in $S^2 \times S^1$ of standard type is a closed ribbon $R : S^1 \times [0,1] \to S^2 \times S^1$ which is horizontal (cf. Definition 3.17 above) and which has the property that each of the knots $K_u := R(\cdot, u)$, $u \in [0,1]$, is a canonical torus knot in $S^2 \times S^1$ of standard type. We set $p(R) := p(K_u) \in \mathbb{Z}$ and $q(R) := q(K_u) \in \mathbb{Z}$ for any $u \in [0,1]$.

(ii) A canonical torus ribbon knot $R$ in $S^2 \times S^1$ of standard type is called “strictly canonical” iff $p := p(R) \neq 0$ and the following two conditions are fulfilled:
Remark 3.21

(i) Clearly, if $R$ is a strictly canonical torus ribbon knot in $S^2 \times S^1$ of standard type then the knot $K := R(\cdot, 1/2)$ will be a canonical torus knot in $S^2 \times S^1$ of standard type with $p(K) \neq 0$. Conversely, up to equivalence, every canonical torus knot $K$ in $S^2 \times S^1$ of standard type with $p(K) \neq 0$ can be obtained in this way. For the derivation of the Rosso-Jones formula in Sec. 3.3.8 below (cf. Remark 3.19 above) it will therefore be sufficient to consider only strictly canonical torus ribbon knots in $S^2 \times S^1$ of standard type.

(ii) The advantage of working with strictly canonical torus ribbon knots in $S^2 \times S^1$ of standard type is that for them Condition 1 above can always be fulfilled. This simplifies the computations considerably, cf. Remark 3.10 above. On the other hand, if one is prepared to do the extra work one can use the infinitesimal version of Condition 1 mentioned in Remark 3.10 above instead of the original Condition 1. By doing so it should be possible to generalize the computations in the present section to general canonical torus ribbon knots in $S^2 \times S^1$ of standard type.

In the following let $L := (R_1, \rho_1)$ where $R_1$ is a strictly canonical torus ribbon knot of standard type in $S^2 \times S^1$ with winding numbers $p = p(R_1) \in \mathbb{Z} \setminus \{0\}$ and $q = q(R_1) \in \mathbb{Z}$ and where $\rho_1$ is an irreducible, finite-dimensional complex representation of $G$ with highest weight $\lambda_1 \in \Lambda_+$. Without loss of generality we can assume that $p > 0$ and that Condition 1 above is fulfilled.

Convention 3.22

Even though in the present section, i.e. Sec. 3.3, we have $\Sigma = S^2$ we will often write $\Sigma$ instead of $S^2$. In particular, we will do this whenever $\Sigma = S^2$ appears as a subscript like, e.g., in $A_{\Sigma,4}$ or $(R_1)_\Sigma$ or $g_\Sigma$.

3.3.1 Evaluation of the inner integral in Eq. (3.40)

We will first evaluate, for fixed $A^\perp \in A^\perp, B \in \mathcal{B}^{ess \ reg}$, and $\epsilon \in (0, \epsilon_0)$ (with $\epsilon_0$ as in Sec. 3.2.2) the “inner integral” in Eq. (3.40), which in the present special case where $m = 1$ is the integral

$$\int_{\tilde{A}^\perp} \operatorname{Tr}_{\rho_1}(\operatorname{Hol}_{R_1}(\tilde{A}^\perp + A^\perp, B))d\mu^B_B(\tilde{A}^\perp), \quad (3.41)$$

cf. Convention 3.16 above. In order to do so we will exploit an important property of the (informal) complex measure $d\mu^B_B$ introduced in Eq. (2.45). Observe that according to Remark 2.8 above $d\mu^B_B$ can be written as

$$d\mu^B_B = \left(\frac{1}{Z(B)}\right) \exp\left(i\frac{1}{2} \ll \tilde{A}^\perp, S(B)\tilde{A}^\perp \gg \right)DA^\perp \quad (3.42a)$$

where $\ll, \gg$ is the restriction of the scalar product $\ll, \gg_{A^\perp}$ to $\tilde{A}^\perp$ and where we have set

$$S(B) := 2\pi k\left(\left\{ \frac{d}{d} + \operatorname{ad}(B) \right\} \right) \quad (3.42b)$$

Accordingly, $d\mu^B_B$ is an informal, normalizedootnote{In Sec. 4 below we will study rigorous analogues of $d\mu^B_B$, cf. frameworks (F1) and (F3). There it will be important to check that for those $B$ for which the regularized version of $1_{\mathcal{B}_{reg}}(B)$ or $1_{\mathcal{B}_{ess \ reg}}(B)$ does not vanish the rigorous version of $Z(B)$ is non-zero. On the other hand, this issue can be circumvented in the rigorous continuum framework (F2) where we do not try to make sense of $d\mu^B_B$ itself but only of the associated integral functional $\Phi^B = \int \cdots d\mu^B_B$.} (“Gauss-type”) complex measure on the pre-Hilbert space $(\tilde{A}^\perp, \ll, \gg)$.
Similarly, we have
\[ \int \Phi : \mathbb{R} \rightarrow \mathbb{R} \]
for every polynomial function \( \Phi : \mathbb{R} \rightarrow \mathbb{R} \). (We remark that if \( B \in \mathcal{B}_{\text{reg}} \) then \( C(B) \) has full domain and is bounded.)

Observe however, that \( C(B) \) is neither positive nor negative definite, i.e. there are non-zero elements \( j \in \text{dom}(C(B)) \subset \mathbb{R}^n \) such that \( \langle j, C(B)j \rangle = 0 \). This property of \( d\mu_{\frac{1}{2}} \) has important consequences, which we will exploit below. As a preparation let us first study the following two examples which deal with the analogous properties of suitable (“Gauss-type”) complex measures on finite-dimensional spaces.

**Example 3.23** Let \( d\mu \) be the (well-defined) normalized (“Gauss-type”) complex measure on \( \mathbb{R}^2 \) which is given by

\[ d\mu(x) = \frac{1}{2\pi} \exp(i\frac{1}{2}(x, Sx))dx \quad \text{where} \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

and where \( \langle \cdot, \cdot \rangle \) is the standard scalar product on \( \mathbb{R}^2 \). Clearly, \( S \) is symmetric and invertible but its inverse \( C := S^{-1} (= S) \) is obviously neither positive-definite nor negative-definite since there are non-zero vectors \( v \in \mathbb{R}^2 \) fulfilling \( \langle v, Cv \rangle = 0 \), for example \( v = (1,0) \) or \( v = (0,1) \). For such \( v \) we obtain (using a suitable analytic continuation argument and the well-known formulas for the moments of a genuine Gaussian measure)

\[ \int \sim \langle x, v \rangle^n d\mu(x) = 0 \quad \forall n \in \mathbb{N} \]

where we have introduced the regularized integral functional

\[ \int \sim \cdots d\mu := \lim_{\epsilon \to 0} \int \cdots e^{-\epsilon|x|^2} d\mu(x) \]

Similarly, we have

\[ \int \Phi(\langle x, v \rangle)d\mu(x) = \Phi(0) \]

for every polynomial function \( \Phi : \mathbb{R} \rightarrow \mathbb{R} \) and, more generally, for every entire analytic function \( \Phi : \mathbb{R} \rightarrow \mathbb{R} \) whose sequence of Taylor coefficients satisfies a suitable growth condition.

**Example 3.23** can easily be generalized to arbitrary dimension:

**Example 3.24** Let \( d\mu \) the (well-defined) normalized (“Gauss-type”) complex measure on \( \mathbb{R}^n \) given by

\[ d\mu(x) = \frac{1}{2\pi} \exp(i\frac{1}{2}(x, Sx))dx \]

where \( \langle \cdot, \cdot \rangle \) is the standard scalar product on \( \mathbb{R}^n \), where \( S \) is an arbitrary symmetric, invertible endomorphism on \( \mathbb{R}^n \) and where \( Z \subset \mathbb{C}\setminus\{0\} \) is chosen such that \( \int_{\sim} 1 \ d\mu = 1 \) where we have again introduced the regularized integral functional

\[ \int_{\sim} \cdots d\mu := \lim_{\epsilon \to 0} \int \cdots e^{-\epsilon|x|^2} d\mu(x) \]

This follows from the observation that for \( b \in \mathcal{B}_{\text{reg}} \), the linear operator \( \partial_t + \text{ad}(b) : C^\infty(S^1, \mathbb{C}) \rightarrow C^\infty(S^1, \mathbb{C}) \) is invertible and its inverse is given by \( ((\partial_t + \text{ad}(b))^{-1})f(t) = (e^{\text{ad}(b)}t - 1)^{-1} : \int_0^t e^{\gamma t + \text{ad}(b)}f(t + i\lambda s)ds \) for all \( a \in S^1 \) and \( f \in C^\infty(S^1, \mathbb{C}) \).

In the sense that \( \int_{\sim} 1d\mu = 1 \) with \( \int_{\sim} \cdots d\mu \) as below.

\( Z \) is given explicitly by \( Z = \frac{(2\pi)^{n/2}}{\text{det}^{1/2}(S)} \) with \( \text{det}^{1/2}(S) := \prod_{k=1}^n \sqrt{\lambda_k} \) where \( \lambda_k \) are the (real) eigenvalues of \( S \) and \( \sqrt{\cdot} : \mathbb{C}\setminus(-\infty, 0) \rightarrow \mathbb{C} \) is the standard square root.
Moreover, let \((v_i)_{i \leq m}, m \in \mathbb{N}\), be a sequence of vectors of \(\mathbb{R}^n\) such that\(^{45}\)

\[
\langle v_i, Cv_j \rangle = 0 \quad \forall i, j \leq m
\]  

(3.44)

where \(C := S^{-1}\). Then we have for every polynomial function \(\Phi : \mathbb{R}^n \to \mathbb{R}\) (and, more generally, for every entire analytic function \(\Phi : \mathbb{C} \to \mathbb{R}\) whose family of Taylor coefficients satisfies a suitable growth condition)

\[
\int_{\mathbb{R}}^{\infty} \Phi((\langle x, v_i \rangle)_{i \leq m})d\mu(x) = \Phi((0, v_i)_{i \leq m}) = \Phi(0)
\]  

(3.45)

**Remark 3.25** Above we have assumed that \(S\) is an invertible endomorphism. In fact, one can also define \(\int_{\cdots} \cdots d\mu\) if \(S\) is not invertible (in which case we call \(d\mu\) “degenerate”) by setting

\[
\int_{\cdots} \cdots d\mu := \lim_{\epsilon \to 0} (\epsilon/\pi)^{n/2} \int \cdots e^{-\epsilon|x|^2}d\mu(x)
\]

where \(n := \dim(\ker(S))\). This will be relevant in Sec. 4.1.1 and 4.3.1 below.

Let us now go back to our infinite dimensional (informal) integral (3.41) above. By applying a suitable limit argument we will show in Subsec. 3.3.2 below that the argument in Example 3.25 can be used to evaluate the integral (3.41) above and that by doing so we obtain for all fixed \(A_c^\perp \in A_c^\perp, B \in \mathcal{B}_{reg}^{ess}\), and \(\epsilon < \epsilon_0\)

\[
\int_{A_c^\perp} \text{Tr}_{\rho_1} (\text{Hol}_{R_1}^\epsilon (A_c^\perp + A_c^\perp, B))d\mu_B^1(A_c^\perp) = \text{Tr}_{\rho_1} (\text{Hol}_{R_1}^\epsilon (0 + A_c^\perp, B)) = \text{Tr}_{\rho_1} (\text{Hol}_{R_1}^\epsilon (A_c^\perp, B))
\]  

(3.46)

Of course, in order to arrive at Eq. (3.46) we need to verify a condition analogous to Eq. (3.44) and the analyticity & growth assumption in Example 3.25. We will do this in Sec. 3.3.2 below by using the assumptions on \(R_1 = R_1\) made at the beginning of Sec. 3.3 and Condition 1 for the auxiliary Riemannian metric \(g_\Sigma\), cf. Sec. 3.2.1 above.

**Remark 3.26** Similar arguments will play a crucial role in Sec. 3.4 below and for the explicit evaluation of the expressions “\(T\)”, \(cl \in \mathcal{C} \subseteq (L, D)\), introduced in Sec. 3.3 below.

### 3.3.2 Justification of Eq. (3.46)

As in Sec. 3.3.1 above let \(A_c^\perp \in A_c^\perp, B \in \mathcal{B}_{reg}^{ess}\), and \(\epsilon \in (0, \epsilon_0)\) be fixed.

Recall that \(\rho := \rho_1\) was a representation of \(G\) over a finite-dimensional complex vector space \(V\). Now observe that we have (with \(R_1 = R_1^{(s)}\), cf. Convention 3.16 above)

\[
\text{Tr}_{\rho}(\text{Hol}_{R_1}^\epsilon (A_c^\perp, B)) = \text{Tr}(\rho(\text{Hol}_{R_1}^\epsilon (A_c^\perp, B))) = \text{Tr}(P_1^\epsilon (\rho; A_c^\perp, B))
\]  

(3.47a)

where \((P_1^\epsilon (\rho; A_c^\perp, B))_{s \in [0,1]}\) is the unique solution of

\[
\frac{d}{ds}P_s^\epsilon (\rho; A_c^\perp, B) = P_s^\epsilon (\rho; A_c^\perp, B) \cdot \rho_\epsilon (D_s^\epsilon (A_c^\perp, B)), \quad P_0^\epsilon (\rho; A_c^\perp, B) = \text{id}_V
\]  

(3.47b)

where \(\cdot\) is the multiplication of \(\text{End}(V)\) and where \(\rho_\epsilon : \mathfrak{g} \to \mathfrak{gl}(V)\) is the derived representation of \(\rho\) and \(D_s^\epsilon (A_c^\perp, B)\) is defined by 3.26a in the situation \(R = R_1\).

\(^{45}\)Clearly, this condition is only interesting if \(C\) is neither positive-definite nor negative-definite since otherwise Eq. (3.44) can only be fulfilled if \(v_i = 0\) for all \(i \leq m\). In the latter case Eq. (3.45) is trivially fulfilled.
Let us now expand \( P^e_1(\rho; A^\perp, B) \) in a Picard-Lindelöf series

\[
P^e_1(\rho; A^\perp, B) = \text{id}_V + \int_0^1 \rho_*(D_s^e(A^\perp, B))ds_1 + \int_0^1 \int_0^{s_2} \rho_*(D^e_{s_1}(A^\perp, B))\rho_*(D^e_{s_2}(A^\perp, B))ds_1ds_2 \\
+ \int_0^1 \int_0^{s_2} \rho_*(D^e_{s_1}(A^\perp, B))\rho_*(D^e_{s_2}(A^\perp, B))\rho_*(D^e_{s_3}(A^\perp, B))ds_1ds_2ds_3 + \ldots
\]

where

\[
\Delta_n := \{ s \in [0,1]^n \mid s_1 \leq s_2 \leq \cdots \leq s_n \}
\]

In view of Eq. (3.47a), Eq. (3.48), and Eqs (3.26a) and (3.25) in Sec. 3.2.2 above we now see that, for each fixed \( A^\perp, B \) we can approximate

\[
\tilde{A} := A^\perp \mapsto \text{Tr}_\rho(\text{Hol}_{R_1}(\tilde{A}^\perp + A^\perp c, B)) \in \mathbb{C}
\]

by a sequence \((F_n)_{n \in \mathbb{N}}\) of functions \( F_n : \tilde{A} \mapsto \mathbb{C} \) (depending on \( A^\perp, B \)) of the form

\[
F_n(\tilde{A}) = \Phi_n((\ll \phi^e_i, \tilde{A} \gg)_i \leq d_n)
\]

where \( \ll \cdot, \cdot \gg := \ll \cdot, \cdot \gg_{A^\perp} \), where each \( \Phi_n : \mathbb{R}^{d_n} \to \mathbb{C}, d_n \in \mathbb{N} \), is a polynomial function (depending on \( A^\perp, B \)), and each \( \phi^e_i \in \tilde{A}^\perp, i \leq d_n \), is of the form

\[
\phi^e_i = \pi_{A^\perp}(T_{a_i}X_{(u_i)_\Sigma}(s_i)\delta_{\pi_{l_{u_i}}(s_i)})
\]

for some \( s_i, u_i \in [0,1] \) and \( a_i \leq \dim(t) \) \((i \leq d_n)\) where \( \pi_{A^\perp} : A^\perp \mapsto \tilde{A}^\perp \) is the \( \ll \cdot, \cdot \gg_{A^\perp} \)-orthogonal projection, where \( l_{u_i}, u_i \in [0,1] \), is the loop \( [0,1] \ni s \mapsto R_1(\pi_{s_1}(s), u) \in S^2 \times S^1 \), and where we have set \((u_i)_\Sigma := \pi_{\Sigma} \circ l_{u_i}, \) cf. Convention 3.37 above. (Note that if \( B \in \mathcal{B}_{\text{reg}} \) it is possible that \( \phi^e_i \notin \text{dom}(C(B)) \). Since \( \text{dom}(C(B)) \) is dense in \( (\tilde{A}^\perp, \ll \cdot, \cdot \gg) \) we can in this case simply replace \( \phi^e_i \) by an element \( \text{dom}(C(B)) \) which is sufficiently close to \( \phi^e_i \).)

Now recall that above we have assumed that \( R_1 = R_1 \) (cf. Convention 3.16 above) is a strictly canonical torus ribbon knot of standard type, that Condition 14 is fulfilled, and that the number \( s_0 \) fixed in Sec. 3.2.6 above is strictly smaller than 1. From these assumptions it follows that for all sufficiently small \( \epsilon > 0 \) we have

\[
\ll \phi^e_i, \phi^e_j \gg = 0 \quad \forall i, j \leq d_n
\]

and therefore

\[
\ll \phi^e_i, C(B)\phi^e_j \gg = 0 \quad \forall i, j \leq d_n
\]

i.e., the analogue of Eq. (3.44) in Example 3.22 above is fulfilled. So, for each \( n \in \mathbb{N} \), we obtain an analogue of Eq. (3.45). We now obtain Eq. (3.46) by sending \( n \to \infty \) and interchanging \( \lim_{n \to \infty} \int_{A^\perp} \ldots d\mu^\perp_\rho(\tilde{A}^\perp) \) appearing in Eq. (3.46):

\[
\int_{A^\perp} \text{Tr}_{\rho_1}(\text{Hol}_{R_1}(\tilde{A}^\perp + A^\perp c, B))d\mu^\perp_\rho(\tilde{A}^\perp) = \int_{A^\perp} \lim_{n \to \infty} \Phi_n((\ll \phi^e_i, \tilde{A} \gg)_i \leq d_n)d\mu^\perp_\rho(\tilde{A}^\perp) \\
= \lim_{n \to \infty} \int_{A^\perp} \Phi_n((\ll \phi^e_i, \tilde{A} \gg)_i \leq d_n)d\mu^\perp_\rho(\tilde{A}^\perp) = \text{Tr}_{\rho_1}(\text{Hol}_{R_1}(A^\perp + B))
\]

\[\text{(3.51)}\]
where in step (*) we have applied the aforementioned analogue\(^\text{17}\) of Eq. (3.45).

**Remark 3.27** In the rigorous continuum framework (F2) described in Sec. 4.2 below there is indeed a rigorous realization of the aforementioned limit argument (similar to the argument given in Proposition 6 in [34]).

In the simplicial framework (F1) described in Sec. 4.1 the limit argument can be avoided altogether and we can work with the original argument in Example 3.24.

### 3.3.3 Evaluation of the outer integral(s) in Eq. (3.40)

By combining Eq. (3.40) (in the special situation under consideration in the present section) with Eq. (3.44) we arrive at

\[
Z(S^2 \times S^1, L) \sim \lim_{\epsilon \to 0} \sum_{y \in I} \int_B \left[ \int_{A^c_1} \{1_{B_{rig}}(B) \ Det_{rig}(B) \exp(-2\pi i k \langle y, B(\sigma_0) \rangle) \right.
\]

\[\times \ Tr_{\rho_1}(\text{Hol}_{R_1}(A^c_1, B)) \} \exp(iS_{CS}(A^c_1, B)) DA^c_1 \] \[\left. \right] DB \tag{3.52} \]

where we have written the double integral as an iterated integral, cf. Remark \text{2.28} in Sec. 2.3.1 above.

For simplicity let us now interchange, informally, in Eq. (3.52) the \(\epsilon \to 0\)-limit with the sum \(\sum_y \cdots\) and the two integrals. (We could avoid this informal interchange but we would then have to work a bit harder, cf. Remark \text{3.29} below.) Since \(\lim_{\epsilon \to 0} \text{Hol}_{R_1}(A^c_1, B) = \text{Hol}_{R_1}(A^c_1, B)\) we then obtain

\[
Z(S^2 \times S^1, L) \sim \sum_{y \in I} \int_B \left[ \int_{A^c_1} \{1_{B_{rig}}(B) \ Det_{rig}(B) \exp(-2\pi i k \langle y, B(\sigma_0) \rangle) \right.
\]

\[\times \ Tr_{\rho_1}(\text{Hol}_{R_1}(A^c_1, B)) \} \exp(iS_{CS}(A^c_1, B)) DA^c_1 \] \[\left. \right] DB \tag{3.53} \]

Let \(D_s(A^c_1, B), s \in [0, 1], \) and \(l_u, u \in [0, 1], \) be as in Sec. \text{3.2.2} above for \(R = R_1.\) (In particular, we have \(l_u(s) = R_1(i_{S^1}(s), u)\) for all \(s, u \in [0, 1].\) \) Observe that since \(D_s(A^c_1, B) \in \mathfrak{t}\) we have

\[
\text{Hol}_{R_1}(A^c_1, B) = \exp\left(\int_0^1 D_s(A^c_1, B) ds\right)
\]

\[
= \exp\left(\int_0^1 \int_0^1 [A^c_1(l_u'(s))] ds du\right) \exp\left(\int_0^1 \int_0^1 [(B dt)(l_u'(s))] ds du\right) \tag{3.54} \]

Let \(u \in [0, 1]\) be fixed for a while. As in Sec. 3.2.2 set \((l_u)_\Sigma := \pi_\Sigma \circ l_u\) (cf. Convention 3.22 above). Recall that \(R_1\) is a strictly canonical torus ribbon knot in \(M = S^2 \times S^1\) of standard type (cf. Definition 3.20 above). This implies\(^\text{18}\) that the restriction of \((l_u)_\Sigma : [0, 1] \to S^2\) to the subinterval \([0, 1/p]\) is a Jordan loop in \(\Sigma = S^2\), which we will denote by \(l_u\). Another implication

\(^{17}\) In fact, if we include one more step where we use the pushforward measure \(\pi_\Sigma(d\mu^\Sigma_{\cdot \cdot \cdot})\) where \(\pi : \hat{A}^c \to V\) is the orthogonal projection onto the finite-dimensional subspace \(V\) of \(\hat{A}^c\) which is spanned by \((\phi^c_i)_{i \leq d}\), then we do not have to use an analogue of Eq. (3.45) but can use Eq. (3.35) itself.

\(^{18}\) In order to see this observe that if \(R := R_1\) then \(R_{S^2}(s, u)\) in the notation of Definition 3.20 coincides with what now is denoted by \((l_u)_\Sigma(s)\).
is that \((l_u)_\Sigma\) is the \(p\)-fold concatenation of the loop \(l_u\) with itself. Using this (for each \(u \in [0, 1]\)) and Stokes’ Theorem we now obtain for every \(A_c^+ \in A_c^+ \cong A_{\Sigma,1}\)

\[
\int_0^1 \int_0^1 [A_c^+(l_u'(s))] dsdu = \int_0^1 \left( \int_{l_u} A_c^+ \right) du = \int_0^1 \left( \int_{(l_u)_\Sigma} A_c^+ \right) du
\]

\[
= \int_0^1 \left( p \int_{l_u} A_c^+ \right) du = \int_0^1 \left( p \int_{\partial X_u} A_c^+ \right) du = \int_0^1 \left( p \int_{X_u} dA_c^+ \right) du = p \int_{S^2} dA_c^+ f_1 \tag{3.55}
\]

where \(X_u\) is the connected component of \(S^2 \setminus \text{Image}(l_u)\) chosen such that the orientation of \(\text{Image}(l_u) = \partial X_u\) induced by \(X_u\) is the same as the orientation induced by \(l_u\) and where \(f_1 = f_{R_1}\) is the function \(S^2 \to \mathbb{R}\) given by

\[
f_1(\sigma) = \int_0^1 1_{X_u}(\sigma) du \quad \text{for all } \sigma \in S^2, \tag{3.56}
\]

1\(X_u\) being the indicator function of \(X_u\). Next observe that

\[
\text{Tr}_{\rho_1} (\exp(b)) = \sum_{\alpha \in \Lambda} m_{\lambda_1}(\alpha) e^{2\pi i (\alpha, b)} \quad \forall b \in \mathfrak{t} \tag{3.57}
\]

where we use the notation of Appendix A below. In particular, \(m_{\lambda_1}(\alpha)\) is the multiplicity of \(\alpha \in \Lambda\) as a weight of \(\rho_1\). (Recall that \(\lambda_1 \in \Lambda_+^{1}\) is the highest weight of \(\rho_1\).)

From Eq. (3.54), Eq. (3.55), Eq. (3.57), Remark 2.8 in Sec. 2.3.2 above, and the relation 
\[
\alpha \left( \int_{S^2} dA_c^+ f_1 \right) = \alpha \left( \int_{S^2} A_c^+ \wedge df_1 \right) = \ll A_c^+, \alpha \star df_1 \gg A_+ \tag{3.58}
\]

we therefore obtain

\[
\text{Tr}_{\rho_1} (\text{Hol}_{R_1}(A_c^+, B)) \exp(iSCS(A_c^+, B)) = \sum_{\alpha \in \Lambda} m_{\lambda_1}(\alpha) \left[ \exp \left( 2\pi i \alpha \left( \int_0^1 \int_0^1 [(Bdt)(l_u'(s))] dsdu \right) \right) \right. \\
\times \exp \left( 2\pi i \ll A_c^+, -k \star dB + \alpha p(\star df_1) \gg A_+ \right) \tag{3.59}
\]

Informally, we have

\[
\int \exp \left( 2\pi i \ll A_c^+, -k \star dB + \alpha p(\star df_1) \gg A_+ \right) DA_c^+ \sim \delta(\star d(-k \star B + \alpha p f_1)) \sim \delta(d(B - \frac{1}{k} \alpha p f_1)) \tag{3.60}
\]

where \(\delta(\cdot)\) is the (informal) delta function on \(A_c^+\). From Eq. (3.58) and the last two equations\(^49\) we obtain

\[
Z(S^2 \times S^1, L) \sim \sum_{y \in I} \sum_{\alpha \in \Lambda} m_{\lambda_1}(\alpha) \int_0^1 db \left[ 1_{G^{rig}_{\alpha}}(B) \text{Det}_{rig}(B) \exp \left( -2\pi ik \langle y, B(\sigma_0) \rangle \right) \right] \\
\times \exp \left( 2\pi i \alpha \left( \int_0^1 \int_0^1 [(Bdt)(l_u'(s))] dsdu \right) \right) \bigg|_{B=b+\frac{1}{k}\alpha p f_1} \tag{3.60}
\]

\[\text{Observation 3.28}\ Let R_1^\Sigma := (R_1)_\Sigma := \pi_\Sigma \circ R_1 \text{ (cf. Convention 3.22) and let } X_1^+ \text{ and } X_1^- \text{ be the two connected components of } S^2 \setminus \text{Image}(R_1^\Sigma) = \Sigma \setminus \text{Image}(R_1). \text{ Here } X_1^+ \text{ is determined}
\]

\[\text{Recall that according to Remark 2.8 in Sec. 2.2 above the integral } \int \cdots DB \text{ on the RHS of Eq. 2.23 (and consequently also on the RHS of Eq. 3.60)) should be interpreted as a suitable improper integral } \int \cdots DB. \text{ cf. Remark 2.22 above. This is why in Eq. 3.60 we use the improper integral } \int db \text{ (cf. Proposition 2.1 in Sec. 2.21).} \]

\[\text{Note: } k \text{ is always a positive integer.}\]
by the condition $X^+_1 \subset X_u$ for all $u \in [0,1]$ where $X_u$ is as above. (By contrast, $X^-_1 \cap X_u = \emptyset$ for every $u \in [0,1]$.) Observe that we have $X^+_1 = X_u$ either for $u = 1$ (“Case I”) or for $u = 0$ (“Case II”). From the definitions it follows that $f_1$ is given explicitly by

$$f_1(\sigma) = 1 \quad \text{for } \sigma \in X^+_1, \quad f_1(\sigma) = 0 \quad \text{for } \sigma \in X^-_1,$$

(3.61a)

and

$$f_1(\sigma) = \begin{cases} 
  u_\sigma & \text{in Case I} \\
  1 - u_\sigma & \text{in Case II}
\end{cases} \text{ for } \sigma \in \text{Image}(R_{\Sigma}^2)$$

(3.61b)

where $u_\sigma$ is the unique $u \in [0,1]$ such that $\sigma \in \text{Image}(l_u) = \text{Image}((l_u)_{\Sigma})$.

Recall that above we interchanged the $\epsilon \to 0$-limit in Eq. (3.52) with the sum $\sum_y \cdots$ and the two integrals. This informal argument simplified the calculations above but it has an obvious disadvantage. The function $f_1$ is only piecewise smooth a 1-form of the form $\alpha \star df_1$ will in general not be an element of $\mathcal{A}_{\Sigma,t} \cong A^1_c$ and a map $B$ of the form $B = b + \frac{1}{k}\alpha p f_1$ will in general not be an element of $B$. So strictly speaking, the expressions $1_{B_{\text{essreg}}}(B)$ and $\text{Det}_{\text{essreg}}(B)$ for $B = b + \frac{1}{k}\alpha p f_1$ appearing on the RHS of Eq. (3.60) above are not defined unless $\alpha \neq 0$ even though there are obvious candidates for such definitions, namely

$$\text{Det}_{\text{rig}}(B) = \prod_{i=1}^{2} \det^{1/2}(1_t - \exp(\text{ad}(B(Y_i))))_t) Y_i$$

(3.62a)

and

$$1_{B_{\text{essreg}}}(B) = \prod_{i=1}^{2} 1_{4_{\text{essreg}}}(B(Y_i))$$

(3.62b)

where we write $Y_1$ instead of $X^+_1$ and $Y_2$ instead of $X^-_1$ and where $B(Y_i), i = 1,2$, is the (constant) value of $B = b + \frac{1}{k}\alpha p_1 f_1$ on $Y_i$.

In the following remark we will sketch a more careful (informal) derivation of Eq. (3.60) above which avoids the aforementioned informal interchange of limit procedures and where the “candidates” for $\text{Det}_{\text{reg}}(B)$ and $1_{B_{\text{essreg}}}(B)$, given by Eqs. (3.62) above appear automatically.

**Remark 3.29** Here is an alternative derivation of Eq. (3.60) above which is preferable to the derivation above from a conceptual point of view and which is relevant for the rigorous implementation of the informal calculations in Sec. 3.2.2 within the rigorous continuum framework (F2) introduced in Sec. 4.4 below.

Observe that since $D^\epsilon_s(A^1_c,B) = D^\epsilon_{R_{1,s}}(A^1_c,B) \in t$ (cf. Eq. (3.26a) in Sec. 3.2.2 above) we obtain (with $l_u, u \in [0,1]$, as above)

$$\text{Hol}^\epsilon_{R_{1}}(A^1_c,B) = \exp \left( \int_0^1 D^\epsilon_s(A^1_c,B) ds \right)$$

$$= \exp \left( \sum_{a=1}^{\dim(t)} T_a \ll A^1_c, T_a j_{R_{1}} \gg A^1_c \right) \exp \left( \int_0^1 \int_0^1 [(Bdt)(l_u'(s))] ds du \right)$$

(3.63)

where we have assumed, for convenience, that the ortho-normal basis $(T_a)_{a \leq \dim(g)}$ of $g$ fixed in Sec. 3.2.2 above was chosen such that $T_a \in t$ for $a \leq \dim(t)$ and where we have set (cf. Convention 3.22 above)

$$j_{R_{1}} := \int_0^1 \int_0^1 X_{(l_u)_{\Sigma}(s)} Y_{(l_u)_{\Sigma}(s)} ds du \in \mathcal{A}_{\Sigma,t} \cong A^1_c$$

(3.64)
From Eq. (3.63), Eq. (3.57), and Remark 2.8 in Sec. 2.3.2 above and \( \sum_{q=1}^{\dim(t)} (\alpha, T_a) T_a = 0 \) we therefore obtain

\[
\text{Tr}_{\rho_1}(\text{Hol}_{R_1}(A_c^\perp, B)) \exp(iS\text{CS}(A_c^\perp, B)) = \sum_{\alpha \in \Lambda} m_{\lambda_1}(\alpha) \left[ \exp\left( 2\pi i \alpha \left( \int_0^1 \int_0^1 [(B dt)(l_u'(s))] ds du \right) \right) \times \exp\left( 2\pi i A_c^\perp \cdot [-(k \ast dB) + \alpha j_{R_1}^\perp \gamma_{A_c^\perp}] \right) \right] \quad (3.65)
\]

Moreover, we have, informally

\[
\int \exp(2\pi i A_c^\perp \cdot [-(k \ast dB) + \alpha j_{R_1}^\perp \gamma_{A_c^\perp}]) DA_c^\perp \sim \delta(- \ast dB + \frac{1}{\rho} \alpha j_{R_1}^\perp) \quad (3.66)
\]

So far we have been working with a rather general choice of Dirac families \( \{G_\delta^\perp \}_{\delta < \epsilon_0} \mid \delta \in \Sigma \}, \) cf. Sec. 3.2.2 above. For the next argument it will be convenient to work with a restricted (but canonical) choice. More precisely, we will assume that the family (of Dirac families) \( \{G_\delta^\perp \}_{\delta < \epsilon_0} \mid \delta \in \Sigma \} \) is “translation invariant” on \( \text{Image}((R_1^{(s_0)})_{\Sigma}^{(S^1 \times (0,1)))}) \) when the latter set is embedded into \( S^1 \times S^1 \) in a suitable way. (This can always be achieved provided that \( \epsilon_0 > 0 \) in Sec. 3.2.2 was chosen sufficiently small.) One can show that then

\[
j_{R_1}^\perp \in \text{Image}(\ast d) \quad (3.67)
\]

and

\[
\lim_{\epsilon \to 0} ((\ast d)^{-1} j_{R_1}^\perp)(\sigma) = pf_1(\sigma) + C \quad \text{uniformly in } \sigma \quad (3.68)
\]

where \( C \in \mathbb{R} \) is a constant (depending only on \( R_1 \)) and \( \ast d \) is the restriction of \( B \supset B \mapsto \ast dB \in A_{\Sigma,1} \) onto the orthogonal complement \( B' \) of \( B_c \cong t \) (w.r.t. to the scalar product on \( B \) induced by the Riemannian metric \( g_{\Sigma,1} \)).

By combining Eq. (3.52) with Eq. (3.65), Eq. (3.66), and Eq. (3.68) we arrive at Eq. (3.60) above with \( \text{Det}_{\text{reg}}(B) \) and \( 1_{[0,1]}(B) \) given by Eqs. (3.62) above.

### 3.3.4 Some simplifications

Let \( B : S^2 \to t \) be of the form \( B = b + b_1 f_1 \) for some \( b, b_1 \in t \) with \( f_1 \) as in Sec. 3.3.3 above. From Eq. (3.61) above it follows that either \( f_1((l_u)_{\Sigma}(s)) = u \) or \( f_1((l_u)_{\Sigma}(s)) = 1 - u \). In particular, \( B((l_u)_{\Sigma}(s)) \) is independent of \( s \). Observe also that \( \int_0^1 dt(l_u'(s)) ds = q \) for every \( u \in [0,1] \). Taking this into account we obtain

\[
\int_0^1 \int_0^1 [(B dt)(l_u'(s))] ds du = \int_0^1 \int_0^1 [B((l_u)_{\Sigma}(s)) \cdot dt(l_u'(s))] ds du = q \int_0^1 B((l_u)_{\Sigma}(0)) du = q(b + b_1/2) = \frac{q(b + b_1)}{2} \quad (3.69)
\]

---

More precisely: in view of Condition 4 in Sec. 3.2.4 we have \( \text{Image}((R_1^{(s_0)})_{\Sigma}^{(S^1 \times (0,1)))} \cong S^1 \times (0,1) \cong S^1 \times S^1 \). Here we have equipped each space with the “obvious” Riemannian metric. In particular, \( S^1 \times S^1 \) is equipped with the product of \( g_{S^1} \) with itself (cf. Sec. 3.2.4). Since \( g_{S^1} \) is translation-invariant we also have a natural notion of translation invariance for families of Dirac families on \( S^1 \times S^1 \), which gives rise to a notion of translation invariance for families of Dirac families on the Riemannian submanifold \( \text{Image}((R_1^{(s_0)})_{\Sigma}^{(S^1 \times (0,1)))}) \) of \( S^1 \times S^1 \).

More precisely: we arrive at the modification of Eq. (3.60) above where \([\ldots]_{B=b+b_1 f_1}^{\ast d} \) is replaced by \([\ldots]_{B=b+b_1 f_1+\gamma_{\Sigma}} \). This constant \( C \) can be eliminated by performing, for each fixed \( \alpha \), the change of variable \( b \to b - \frac{1}{\rho} \alpha C \) and by taking into account that each of the four functions appearing in Eqs. (3.55) in Sec. 3.3.3 below is \( I \)-periodic.
where as in in Sec. 3.3.3 above we have set $Y_1 := X^+_1$ and $Y_2 := X^-_1$ and where we write $B(Y_i)$, $i = 1, 2$, for the unique value of $B|_{Y_i}$.

Let us now assume without loss of generality\(^{52}\) that

$$\sigma_0 \in Y_2$$

(3.70)

Then $f_1(\sigma_0) = 0$ and $B(\sigma_0) = b$. Combining Eq. (3.69) with Eq. (3.60) and Eqs (3.62) in Sec. 3.3.3 above we arrive at

$$Z(S^2 \times S^1, L) \sim \sum_{\alpha \in \Lambda} m_{\lambda_1}(\alpha) \sum_{y \in I} \int_{\mathfrak{t}} e^{-2\pi ik\langle y, b \rangle} F_\alpha(b) \, db$$

(3.71)

where for $b \in \mathfrak{t}$ and $\alpha \in \Lambda$ we have set

$$F_\alpha(b) := \left[ \left( \prod_{i=1}^{2} 1_{\imath_{\text{reg}}}(B(Y_i)) \right) \left( \exp(\pi i q(\alpha, B(Y_1) + B(Y_2))) \right) \right] \left( \prod_{i=1}^{2} \det^{1/2}(1_{\mathfrak{t}} - \exp(\text{ad}(B(Y_i)))_{\mathfrak{t}}) \right)^{X(Y_i)}$$

(3.72)

Now observe that the function $F_\alpha(b)$ (and the whole integrand in Eq. (3.71) above) is $I$-invariant. This follows from

$$e^{2\pi i \epsilon(\alpha, b + x)} = e^{2\pi i \epsilon(\alpha, b)} \quad \text{for all } \alpha \in \Lambda, \epsilon \in \mathbb{Z}$$

(3.73a)

$$\det^{1/2}(1_{\mathfrak{t}} - \exp(\text{ad}(b + x))_{\mathfrak{t}}) = \det^{1/2}(1_{\mathfrak{t}} - \exp(\text{ad}(b))_{\mathfrak{t}})$$

(3.73b)

$$1_{\imath_{\text{reg}}}(b + x) = 1_{\imath_{\text{reg}}}(b)$$

(3.73c)

$$\exp\left( -2\pi ik\langle y, b + x \rangle \right) = \exp\left( -2\pi ik\langle y, b \rangle \right),$$

(3.73d)

for all $b \in \mathfrak{t}$ and $x \in I$. The first equation follows because by definition $\Lambda$ is the lattice dual to $\Gamma = I$. The second equation follows\(^{53}\) from Eq. (A.4) in Appendix A below by taking into account that $(-1)\sum_{\alpha \in R^+_1} (\alpha, x) = (-1)^2(\rho, x) = 1$ for $x \in \Gamma = I$ because $\rho = \frac{1}{2} \sum_{\alpha \in R^+_1} \alpha$ is an element of the weight lattice $\Lambda$. The last two equations follow from Remark A.2 in Appendix A.

Since the integrand in Eq. (3.71) is $I$-periodic we obtain

$$Z(S^2 \times S^1, L) \sim \sum_{\alpha \in \Lambda} m_{\lambda_1}(\alpha) \sum_{y \in I} \int_{Q} e^{-2\pi ik\langle y, b \rangle} F_\alpha(b) \, db$$

(3.74)

where we have set

$$Q := \left\{ \sum_{i} x_i e_i \mid x_i \in (0, 1) \text{ for all } i \leq m \right\} \subset \mathfrak{t},$$

(3.75)

Here $(e_i)_{i \leq m}$ is an (arbitrary) fixed basis of $I$.

By applying the Poisson summation formula at an informal level\(^{54}\) to the RHS of Eq. (3.74) we obtain

$$Z(S^2 \times S^1, L) \sim \sum_{\alpha \in \Lambda} m_{\lambda_1}(\alpha) \sum_{b \in R^+_1} 1_{\Lambda} Q(b) F_\alpha(b)$$

(3.76)

\(^{52}\) It is not difficult to see that if $\sigma_0 \in Y_1$ we will get the same explicit expression for $Z(S^2 \times S^1, L)$ as if $\sigma_0 \in Y_2$.

\(^{53}\) Since $\det^{1/2}(1_{\mathfrak{t}} - \exp(\text{ad}(b))_{\mathfrak{t}})$ is a square root of $\det(1_{\mathfrak{t}} - \exp(\text{ad}(b))_{\mathfrak{t}})$ and $\text{exp}(\text{ad}(b)) = \text{Ad}(\text{exp}(b))$ it is immediately clear that $\det^{1/2}(1_{\mathfrak{t}} - \exp(\text{ad}(b + x))_{\mathfrak{t}}) = \pm \det^{1/2}(1_{\mathfrak{t}} - \exp(\text{ad}(b))_{\mathfrak{t}})$. The argument above is necessary in order to show that the sign appearing on the RHS of the last equation is “+” rather than “−”.

\(^{54}\) In a rigorous treatment of this argument one needs to regularize the indicator functions $1_{\imath_{\text{reg}}}$ appearing on the RHS of Eq. (3.72) above, cf. Secs 3.7 and 5.4 in [40].
where in step (*) we performed the change of variable \( b \rightarrow \alpha_0 := kb \) and wrote \( \alpha_1 \) instead of \( \alpha \). From Eq. (3.76) and Eq. (3.72) we obtain (by taking into account that \( \chi(Y_1) = \chi(Y_2) = 1 \))

\[
Z(S^2 \times S^1, L) \sim \sum_{\alpha_0, \alpha_1 \in \Lambda} 1_{kQ}(\alpha_0)m_{\lambda_1}(\alpha_1)
\times \left[ \left( \prod_{i=1}^{2} 1_{t_{reg}}(B(Y_i)) \right) \left( \prod_{i=1}^{2} \det^{1/2}(1_t - \exp(\text{ad}(B(Y_i)))_{|B}) \right) \right.
\times \exp(\pi i q(\alpha_1, B(Y_1) + B(Y_2))))
\left|_{B=\frac{1}{k}(\alpha_0 + \alpha_1 p f_1)} \right]
\tag{3.77}
\]

3.3.5 **Rewriting Eq. (3.77) in quantum algebraic notation**

As a preparation for Sec. 3.3.8 below we will now rewrite Eq. (3.77) using the quantum algebraic notation of Appendix A.2.

Recall that \( \lambda_1 \in \Lambda_+ \) is the highest weight of \( \rho_1 \). In the following we will assume, for simplicity, that \( \lambda_1 \in \Lambda^k \) where \( \Lambda^k \) is as in Appendix A.2. For each \( \alpha_0, \alpha_1 \in \Lambda \) we set

\[
B_{(\alpha_0, \alpha_1)} := \frac{1}{k}(\alpha_0 + \alpha_1 p f_1)
\]

where in step (*) we defined

\[
\eta_{(\alpha_0, \alpha_1)}(Y_i) := kB_{(\alpha_0, \alpha_1)}(Y_i) - \rho \quad i = 1, 2
\tag{3.78}
\]

Observe that for \( \eta = \eta_{(\alpha_0, \alpha_1)} \) and \( B = B_{(\alpha_0, \alpha_1)} \) we have, using the notation of Appendix A.2 below (and recalling Eq. (3.70) in Remark 3.29 above)

\[
\alpha_0 = kB(Y_1) = \eta(Y_1) + \rho,
\tag{3.79a}
\]

\[
\alpha_1 = \frac{1}{p}(kB(Y_1) - kB(Y_2)) = \frac{1}{p}(\eta(Y_1) - \eta(Y_2))
\tag{3.79b}
\]

\[
\exp(\pi i q(\alpha_1, B(Y_1) + B(Y_2))
\]

\[
= \exp(\pi i q(\frac{1}{p}(\eta(Y_1) - \eta(Y_2)), \frac{1}{k}(\eta(Y_1) + \eta(Y_2) + 2p))) = \frac{\theta^q_{\eta(Y_1)}}{\theta^q_{\eta(Y_2)}},
\tag{3.79c}
\]

\[
\det^{1/2}(1_t - \exp(\text{ad}(B(Y_i)))_{|B}) = \det^{1/2}(1_t - \exp(\text{ad}(\frac{1}{k}(\eta(Y_i) + \rho)))_{|B}) \sim d_{\eta_{(\alpha_0, \alpha_1)}},
\tag{3.79d}
\]

In view of the previous equations it is clear that we can rewrite Eq. (3.77) in the following form

\[
Z(S^2 \times S^1, L) \sim \sum_{\alpha_0, \alpha_1 \in \Lambda} m_{\lambda_1}(\frac{1}{p}(\eta(Y_1) - \eta(Y_2))) 1_{kQ}(\eta(Y_1) + \rho) \left( \prod_{i=1}^{2} 1_{t_{reg}}(\frac{1}{k}(\eta(Y_i) + \rho)) \right)
\times d_{\eta(Y_1)}d_{\eta(Y_2)} \theta^q_{\eta(Y_1)} \theta^q_{\eta(Y_2)}
\left|_{\eta = \eta_{(\alpha_0, \alpha_1)}} \right.
\]

\[
\sim \sum_{\eta_1, \eta_2 \in \Lambda} m_{\lambda_1}(\frac{1}{p}(\eta_1 - \eta_2)) \left( \frac{1}{k}(\eta_1 - \eta_2) \right) 1_{kQ \cap t_{reg}}(\eta_1 + \rho) \left( \frac{1}{k}(\eta_2 + \rho) \right) d_{\eta_1}d_{\eta_2} \theta^q_{\eta_1} \theta^q_{\eta_2}
\tag{3.80}
\]

where in step (*) we made the change of variable \( (\alpha_0, \alpha_1) \rightarrow (\eta_1, \eta_2) := (\eta_{(\alpha_0, \alpha_1)}(Y_1), \eta_{(\alpha_0, \alpha_1)}(Y_2)) \).

Let \( P \) be the fundamental Weyl alcove (w.r.t. to the Weyl chamber \( C \) fixed above), cf. Eq. (A.2) in Appendix A. Observe that the map

\[
\mathcal{W}_{aff} \times P \ni (\tau, b) \mapsto \tau \cdot b \in t_{reg}
\tag{3.81a}
\]
is a well-defined bijection, cf. part (ii) of Remark A.2 in Appendix A below. Moreover, there is a finite subset $W$ of $\mathcal{W}_{aff}$ such that

$$W \times P \ni (\tau, b) \mapsto \tau \cdot b \in Q \cap t_{reg}$$

(3.81b)

is a bijection, too. Clearly, these two bijections above induce two other bijections

$$\mathcal{W}_{aff} \times (kP - \rho) \ni (\tau, b) \mapsto \tau \ast b \in kt_{reg} - \rho$$

(3.82a)

$$W \times (kP - \rho) \ni (\tau, b) \mapsto \tau \ast b \in k(Q \cap t_{reg}) - \rho$$

(3.82b)

where $\ast : \mathcal{W}_{aff} \times t \to t$ is given by

$$\tau \ast b = k(\tau \cdot \frac{1}{k}(b + \rho)) - \rho, \quad \text{for all } \tau \in \mathcal{W}_{aff} \text{ and } b \in t$$

(3.83)

Observe that for all $\tau \in \mathcal{W}_{aff}$, $\eta, \eta_1, \eta_2 \in \Lambda$ we have (cf. Remark 3.30 below)

$$d_{\tau \ast \eta} = (-1)^{\tau} d_{\eta},$$

(3.84a)

$$m_{\lambda_1}(\frac{1}{P}(\tau \ast \eta_1 - \tau \ast \eta_2)) \frac{q}{\tau \ast \eta_1} \frac{q}{\tau \ast \eta_2} = m_{\lambda_1}(\frac{1}{P}(\eta_1 - \eta_2)) \frac{q}{\eta_1} \frac{q}{\eta_2}$$

(3.84b)

**Remark 3.30**

(i) In fact, we also have $\theta_{\tau \ast \eta} = \theta_{\eta}$ and in the special case where $\tau \in \mathcal{W}$ we even have $\frac{q}{\tau \ast \eta} = \frac{q}{\eta}$. However, if $p \neq \pm 1$ we cannot expect $\frac{q}{\tau \ast \eta} = \frac{q}{\eta}$ to hold for a general element $\tau$ of $\mathcal{W}_{aff}$.

(ii) According to Remark A.2 in Appendix A below the affine Weyl group $\mathcal{W}_{aff}$ is generated by $\mathcal{W}$ and the translations associated to the lattice $\Gamma = I$ so it is enough to check Eq. (3.84a) and Eq. (3.84b) for elements of $\mathcal{W}$ and the aforementioned translations. If $\tau \in \mathcal{W}$ then $\tau \ast \eta = \tau \cdot \eta + \tau \cdot \rho - \rho$. On the other hand if $\tau$ is the translation by $y \in \Gamma$ we have $\tau \ast \eta = \eta + ky$. Using this Eq. (3.84a) follows from Eq. (A.10b) and Eq. (A.85) and Eq. (3.84b) follows by taking into account Eq. (A.5) in Appendix A below.

Combining Eq. (3.80) with Eqs (3.84) we therefore obtain

$$Z(S^2 \times S^1, L) \sim \sum_{\eta_1, \eta_2 \in (kP - \rho) \cap \Lambda} \sum_{\tau_1 \in W, \tau_2 \in \mathcal{W}_{aff}} m_{\lambda_1}(\frac{1}{P}(\tau_1 \ast \eta_1 - \tau_2 \ast \eta_2)) d_{\tau_1 \ast \eta_1} d_{\tau_2 \ast \eta_2} \frac{q}{\tau_1 \ast \eta_1} \frac{q}{\tau_2 \ast \eta_2}$$

$$= \sum_{\eta_1, \eta_2 \in (kP - \rho) \cap \Lambda} \sum_{\tau_1 \in W, \tau_2 \in \mathcal{W}_{aff}} m_{\lambda_1}(\frac{1}{P}(\tau_1 \ast \eta_1 - \tau_2 \ast \eta_2)) (-1)^{\tau_1} (-1)^{\tau_2} d_{\eta_1} d_{\eta_2} \frac{q}{\tau_1 \ast \eta_1} \frac{q}{\tau_2 \ast \eta_2}$$

$$\overset{(\ast)}{=} \sum_{\eta_1, \eta_2 \in (kP - \rho) \cap \Lambda} \sum_{\tau_1 \in W, \tau \in \mathcal{W}_{aff}} m_{\lambda_1}(\frac{1}{P}(\eta_1 - \tau \ast \eta_2)) (-1)^{\tau} d_{\eta_1} d_{\eta_2} \frac{q}{\tau \ast \eta_1} \frac{q}{\tau \ast \eta_2}$$

$$\overset{(**)}{=} \sum_{\eta_1, \eta_2 \in \Lambda_{+} \cap \Lambda} \sum_{\tau \in \mathcal{W}_{aff}} m_{\lambda_1 P}(\tau) d_{\eta_1} d_{\eta_2} \frac{q}{\tau \ast \eta_1} \frac{q}{\tau \ast \eta_2}$$

(3.85)

where we have set for $\lambda \in \Lambda_{+}$, $\mu, \nu \in \Lambda$, $p \in \mathbb{Z} \setminus \{0\}$, and $\tau \in \mathcal{W}_{aff}$

$$m_{\lambda 1 P}(\tau) := (-1)^{\tau} m_{\lambda}(\frac{1}{P}(\mu - \tau \ast \nu)) \in \mathbb{Z}$$

(3.86)

Above in step $(\ast)$ we applied Eq. (3.84b) and made the change of variable $\tau_2 \rightarrow \tau := \tau_1^{-1} \tau_2$ and in step $(**) \text{we used Eq. (A.7)}$ of Appendix A

\[ \text{\footnote{This is relevant only in the special case where $\tau$ is a translation by $y \in \Gamma$. If $\tau \in \mathcal{W}$ then the validity of Eq. (3.84b) follows from the $\mathcal{W}$-invariance of $m_{\lambda_1}(\cdot)$ and the relations mentioned above.}} \]
Analogous (but considerably simpler) computations for the empty link \( L = \emptyset \) lead to
\[
Z(S^2 \times S^1) = Z(S^2 \times S^1, \emptyset) \sim \frac{1}{S_{00}^2}
\]
where the multiplicative (non-zero) constant represented by \( \sim \) is the same as that in Eq. (3.85) above. Combining Eq. (3.85) and Eq. (3.87) we conclude (cf. Eq. (2.5) in Sec. 2.1)
\[
\langle L \rangle = \frac{Z(S^2 \times S^1, L)}{Z(S^2 \times S^1)} = \sum_{\eta_1, \eta_2} m^{\eta_1 \eta_2}_{\Lambda_{1+B}}(\tau) d_{\eta_1} d_{\eta_2} \theta_{\eta_1}^{r} \theta_{\eta_2}^{-r} (3.88)
\]

### 3.3.6 A useful generalization of Eq. (3.88)

As a preparation for Sec. 3.3.8 below let us now generalize Eq. (3.88) above in a straightforward way. Let \( L = (R_1, R_2) \) be a 2-component link where \( R_1 \) is the torus ribbon knot above and where \( R_2 : S^1 \times [0, 1] \to S^2 \times S^1 \) is a “fiber ribbon”, i.e. each of the loops \( R_2(\cdot, u) : S^1 \to S^2 \times S^1, u \in [0, 1] \), is a fiber loop in the sense of Sec. 3.1 above.

Assume that \( L \) is colored with \((\rho_1, \rho_2)\) and that \( \lambda_1, \lambda_2 \in \Lambda^k_+ \) where \( \lambda_1, \lambda_2 \) are the highest weights of \( \rho_1 \) and \( \rho_2 \). By generalizing the computations above in a straightforward way (cf. Sec. 5.5 in [40] for analogous computations within the simplicial setting of [40]) we obtain
\[
\langle L \rangle = S_{00} \sum_{\eta_1, \eta_2} m^{\eta_1 \eta_2}_{\Lambda_{1+B}}(\tau) d_{\eta_1} d_{\eta_2} S_{2\lambda_1 \rho_1}^{\eta_1} S_{2\lambda_2 \rho_2}^{\eta_2} (3.89)
\]

**Remark 3.31** Instead of using the original version of Eq. (3.89) we can also work with the variant of Eq. (3.89) which is obtained by replacing in \( L \) the “fiber ribbon” \( R_2 \) by a fiber loop \( l_2 : [0, 1] \to S^2 \times S^1 \), \( l_2(s) = (\sigma_2, i_{S^1}(s)) \) for all \( s \in [0, 1] \) (and for fixed \( \sigma_2 \in S^2 \)). More precisely, we replace \( L = (R_1, R_2) \) by the “mixed” ribbon/loop link \( L = (R_1, l_2) \) and denote by \( Z(S^2 \times S^1, L) \) and \( \langle L \rangle \) the obvious path integrals. Clearly, due to the “mixing” of ribbons and proper loops, from a conceptual point of view this variant of Eq. (3.89) is less natural than its original version but it has the advantage of being somewhat easier to derive than Eq. (3.89).

Here is a sketch of this derivation:

Since \( \text{Tr}_{\rho_2}(H_{l_2}(A_+ + A^+_k, B)) = \text{Tr}_{\rho_2}(\exp(B(\sigma_2))) \) an extra factor \( \text{Tr}_{\rho_2}(\exp(B(\sigma_2))) \) will appear in (the obvious) modifications of the equations in Sec. 3.3.4, for example, in Eq. (3.52) and in Eq. (3.77). Let us now assume without loss of generality that the point \( \sigma_2 \) lies in the region \( Y_2 \) (cf. the beginning of Sec. 3.3.4 above). Then the extra factor \( \text{Tr}_{\rho_2}(\exp(B(\sigma_2))) \) in (the modification of) Eq. (3.77) gives rise to an extra factor \( \text{Tr}_{\rho_2}(\exp(\frac{1}{2} \eta_2 + \rho)) = \frac{S_{2\lambda_2}}{S_{2\rho_2}} \) in Eq. (3.80), cf. Eq. (A.12) in Appendix A. Since apart from Eq. (3.81) we also have \( S_{(\tau \eta_2)\lambda_2} = (-1)^\tau S_{\eta_2 \lambda_2} \) we then obtain from a computation completely analogous to the one in Eq. (3.85) above
\[
Z(S^2 \times S^1, L) \sim \sum_{\eta_1, \eta_2} m^{\eta_1 \eta_2}_{\Lambda_{1+B}}(\tau) d_{\eta_1} d_{\eta_2} \theta_{\eta_1}^{r} \theta_{\eta_2}^{-r} (3.90)
\]
Combining this with Eq. (3.87) above and taking into account that \( d_{\eta_2} = \frac{S_{\eta_2 \rho_0}}{S_{00}} \) (cf. Eq. (A.10H) in Appendix A) we arrive at Eq. (3.89).

### 3.3.7 Change of notation

As a preparation for Sec. 3.3.8 below we will now modify our notation. We will write

- \( T_{p,q} \) instead of \( R_1 \),
• $C$ instead of the fiber loop $l_2$ appearing in Remark 3.31
• $\lambda$ instead of $\lambda_1$ and $\rho_\lambda$ instead of $\rho_1$.
• $\alpha$ instead of $\lambda_2$ and $\rho_\alpha$ instead of $\rho_2$.
• $Z(S^2 \times S^1, (T_{p,q}, \rho_\lambda), (C, \rho_\alpha))$ instead of $Z(S^2 \times S^1, L)$,
• $\langle (T_{p,q}, \rho_\lambda), (C, \rho_\alpha) \rangle$ instead of $\langle L \rangle$.

Using the new notation we can rewrite Eq. (3.89) (or rather, the variant of Eq. (3.89) appearing in Remark 3.31 above) as

$$\langle (T_{p,q}, \rho_\lambda), (C, \rho_\alpha) \rangle = S_{00} \sum_{\eta_1, \eta_2 \in \Lambda^k_+} \sum_{\tau \in W_{aff}} m_{\alpha \eta_2}^{\eta_1} (\tau) d_{\eta_1} S_{\alpha \eta_2} \theta_\eta \theta_{\tau \gamma \eta_2}$$

(3.91)

3.3.8 Derivation of the general Rosso-Jones formula for torus knots in $S^3$

Let us now combine Eq. (3.91) above with Witten’s surgery formula and derive the original Rosso-Jones formula (for general $G$) which is concerned with (colored) torus knots in $S^3$. More precisely, we will use the following two informal arguments from [79]:

• $Z(S^2 \times S^1) = 1$, which implies

$$Z(S^2 \times S^1, (T_{p,q}, \rho_\lambda), (C, \rho_\alpha)) = \langle (T_{p,q}, \rho_\lambda), (C, \rho_\alpha) \rangle$$

(3.92)

• Witten’s surgery formula:

$$Z(S^3, (\tilde{T}_{p,q}, \rho_\lambda)) = \sum_{\alpha \in \Lambda^k_+} S_{00} Z(S^2 \times S^1, (T_{p,q}, \rho_\lambda), (C, \rho_\alpha))$$

(3.93)

where $\tilde{T}_{p,q}$ is the torus ribbon knot in $S^3$ which is obtained from $T_{p,q}$ by performing the surgery on $C$ which transforms $S^2 \times S^1$ into $S^3$ (cf. Fig. 16 on p. 389 in [79]).

Combining Eq. (3.91) with Eq. (3.92) and Eq. (3.93) (for every $\alpha \in \Lambda^k_+$) we obtain

$$Z(S^3, (\tilde{T}_{p,q}, \rho_\lambda)) = \sum_{\alpha \in \Lambda^k_+} S_{00} \left( \sum_{\eta_1, \eta_2 \in \Lambda^k_+} \sum_{\tau \in W_{aff}} m_{\alpha \eta_2}^{\eta_1} (\tau) d_{\eta_1} S_{\alpha \eta_2} \theta_\eta \theta_{\tau \gamma \eta_2} \right)$$

(3.94)

Here in Step (*) we used $S^2 = C$ and the fact that $S$ is a symmetric matrix and in Step (**) we used $C_{0\mu} = \delta_{0\mu} = \delta_{0\mu}$ (cf. Appendix A below).

For simplicity we will now assume that $k$ is “sufficiently large”. (It can be shown that this restriction on $k$ can be dropped, i.e. we can actually derive Eq. (3.97) for all $k > c_0$.) If $k$ is sufficiently large then the sum $\sum_{\tau \in W_{aff}} \cdot \cdot \cdot$ appearing in the last term in Eq. (3.94) can

---

Footnotes:

56 Here we use a notation which is very similar to Witten’s notation; note, however, that we write $(C, \rho_\alpha)$ where Witten writes $R_\alpha$.

57 Note that up to equivalence and a change of framing every torus ribbon knot in $S^3$ can be obtained in this way.
be replaced by $\sum_{\tau \in \mathcal{W}} \cdots$. Moreover, (for fixed $\lambda$, $p$, and $\tau \in \mathcal{W}$) the coefficients $m_{\lambda,p}^{\eta_1}(\tau)$ are non-zero only for a finite number of values of $\eta_1 \in \Lambda^k_\pm$. So if $k$ is large enough we can replace the index set $\Lambda^k_\pm$ in Eq. (3.94) by $\Lambda^+$. Making these two replacements, writing $\mu$ instead of $\eta_1$, and taking into account that for $\tau \in \mathcal{W}$ we have $\theta_{3\tau^0} = \theta_0 = 1$ (cf. Remark 3.30 above) we arrive at

$$Z(S^3, (\tilde{T}_{p,q}, \rho_{\lambda})) = S_{00} \sum_{\mu \in \Lambda^+} \left( \sum_{\tau \in \mathcal{W}} m_{\lambda,p}^{\mu_0}(\tau) \right) d_{\mu} \theta_{\mu}^p$$

(3.95)

According to Eq (3.86) and Eq. (3.83) above we have

$$\sum_{\tau \in \mathcal{W}} m_{\lambda,p}^{\mu}(\tau) = c_{\lambda,p}^\mu$$

(3.96)

where $c_{\lambda,p}^\mu$ is defined by Eq. (A.15) in Appendix A below. So we finally obtain

$$Z(S^3, (\tilde{T}_{p,q}, \rho_{\lambda})) = S_{00} \sum_{\mu \in \Lambda^+} c_{\lambda,p}^\mu d_{\mu} \theta_{\mu}^p,$$

(3.97)

which is a version\(^58\) of the Rosso-Jones formula, cf. \[68\] and Eq. (10) in \[28\].

Remark 3.32 In Appendix A below we define the coefficients $(c_{\lambda,p}^\mu)_{\mu, \lambda \in \Lambda^+}$ appearing in the Rosso-Jones formula by formula \(\text{(A.15)}\). In fact, these coefficients $(c_{\lambda,p}^\mu)_{\mu, \lambda \in \Lambda^+}$ are usually defined using a different formula, cf. Eq. (9) in \[28\] and Eq. (7) in Sec. 2 in \[29\]. So Eq. \(\text{(A.15)}\) of Appendix A mentioned above is then not a definition but an identity, which was discovered only recently in \[29\] (cf. Lemma 2.1 in \[29\]) and rediscovered in \[40\].

To my knowledge there are three other approaches for the informal evaluation of $Z(S^3, K)$ for colored torus knots $K$ in $S^3$, firstly the “knot operator” approach by \[57\] and then the two path integral approaches by \[15\], \[16\] and \[20\]. In the following two remarks we comment on these three approaches.

Remark 3.33 The “knot operator” approach by \[57\], mentioned above, was applied in the series\(^59\) of papers \[23\], \[42\], \[74\], \[57\], \[72\] to the explicit evaluation of $Z(S^3, K)$ for colored torus knots $K$ in $S^3$ and equivalence with the Rosso-Jones formula was shown. Note that the knot operator approach uses the Hamiltonian formulation of Chern-Simons theory. Accordingly, it involves only very few genuine path integral arguments (i.e. arguments which deal directly/explicitly with the CS path integral).

Remark 3.34 The two path integral approaches by \[16\] and \[20\] rely on the observation by Moser (cf. Ref. “[85]” in \[16\]) that a knot $K$ in $S^3$ is a torus knot iff it can be represented as a “fiber loop” in the sense of \[7\] above by considering $S^3$ as Seifert fibered space in a

\[^{58}\] By replacing $Z(S^3, (\tilde{T}_{p,q}, \rho_{\lambda}))$ with the Reshetikhin-Turaev invariant $RT(S^3, (\tilde{T}_{p,q}, \rho_{\lambda}))$ one obtains a rigorous version of Eq. (3.94) which – as is shown in Appendix A in \[10\] – is equivalent to the original version of the Rosso-Jones formula. (Observe that the original version of the Rosso-Jones formula deals with unframed torus knots while in the present paper we are working with ribbon torus knots).

\[^{59}\] The first five of these papers deal with the following special cases: 1. $G = SU(2)$ and $K$ is colored with the defining representation of $G = SU(2)$ (cf. \[53\]); 2. $G = SU(2)$ and arbitrary knot colors (cf. \[15\]); 3. $G = SU(N)$ and $K$ is colored with the defining representation of $G = SU(N)$ (cf. \[53\]); 4. $G = SO(N)$ and $K$ is colored with the defining representation of $G = SO(N)$ (cf. \[53\]); 5. $G = SU(N)$ and arbitrary knot colors (cf. \[53\]). The case of arbitrary simple, simply-connected, compact Lie group $G$ and arbitrary knot colors was then covered in \[74\].

\[^{60}\] “Seifert loop” in the terminology of \[16\].
suitable way. By combining this observation\footnote{This observation plays a crucial role both in \cite{16} and in \cite{20}. Accordingly, since general knots/links cannot be represented as fiber loops/links, the computation of $Z(M,L)$ for general $L$ is not considered in \cite{16} or \cite{20}.} with results from orbifold theory and the theory of moduli spaces, $Z(S^3,K)$ was evaluated explicitly in \cite{10} for colored torus knots $K$ in $S^3$ using non-Abelian localization. For the special case $G = SU(2)$ it was verified in \cite{10} that the explicit expression obtained for $Z(S^3,K)$ agrees with the corresponding expression in the Rosso-Jones formula. (To my knowledge this has not yet been verified for general $G$ even though it should not be too difficult to do so by using formula (A.15) of Appendix A below.)

Recall from the introduction that in \cite{16,20} torus gauge fixing was generalized first to the case where $M$ is a non-trivial $S^1$-bundle and later to the case where $M$ is a Seifert fibered space and this can be used to obtain an explicit evaluation of $Z(M)$ and $Z(M,L)$, where $L$ is a fiber link in $M$. As observed above, torus knots in $S^3$ can be represented as “fiber loops” when $S^3$ is considered as Seifert fibered space in a suitable way. By doing so the approach in \cite{20} allows the explicit evaluation of $Z(S^3,K)$ for colored torus knots $K$ in $S^3$ and it can be expected that this evaluation will lead to expressions which are equivalent to those in \cite{10} (which, as mentioned above, in the case $G = SU(2)$ were shown to be equivalent to those in the Rosso-Jones formula). In contrast to the treatment in \cite{16} the treatment in \cite{20} does not require any arguments involving moduli spaces but some technical results from orbifold theory are necessary. (In particular, a version of Hodge decomposition for orbifolds, due to Baily, the Gysin sequence for moduli spaces but some technical results from orbifold theory are necessary. (In particular, a version of Hodge decomposition for orbifolds, due to Baily, the Gysin sequence for moduli spaces but some technical results from orbifold theory are necessary. (In particular, a version of Hodge decomposition for orbifolds, due to Baily, the Gysin sequence for moduli spaces but some technical results from orbifold theory are necessary. (In particular, a version of Hodge decomposition for orbifolds, due to Baily, the Gysin sequence for moduli spaces but some technical results from orbifold theory are necessary. (In particular, a version of Hodge decomposition for orbifolds, due to Baily, the Gysin sequence for moduli spaces but some technical results from orbifold theory are necessary.)

The evaluation of $Z(S^3,K)$ (with $K = (\tilde{T}_{p,q},\rho_\lambda)$) which we have given above has the advantage of avoiding both the use of moduli space arguments and orbifold theory. On the other hand it has the disadvantage of having to rely on the use of Witten’s surgery formula (which was derived in \cite{79} using the Hamiltonian formulation of Chern-Simons theory). So in contrast to the evaluations in \cite{16} and \cite{20} our evaluation of $Z(S^3,K)$ is not a pure path integral evaluation. It would be desirable to find a way to eliminate Witten’s surgery argument and, indeed, by combining the ideas/methods in the present paper with those in \cite{19} this is probably possible, cf. Sec. 5.4 below.

3.4 Special case III. Links in $M = \Sigma \times S^1$ without “double points”

In Sec. 3.5 below we will consider the case of general (strictly admissible) colored ribbon links $L = ((R_1,R_2,\ldots,R_m),(\rho_1,\rho_2,\ldots,\rho_m))$, $m \in \mathbb{N}$, in $M = \Sigma \times S^1$, where $Z(\Sigma \times S^1,L)$ is expected to be given by\footnote{Recall from Sec. 2.3 above that we expect $Z(\Sigma \times S^1,L) \sim RT(\Sigma \times S^1,L)$ and according to Eq. (3.94) in Appendix C we have $RT(\Sigma \times S^1,L) \sim |L|$.}

$$Z(\Sigma \times S^1,L) \sim RT(\Sigma \times S^1,L) \sim |L| = \sum_{\eta \in \text{col}(L)} |L|_1^\eta |L|_2^\eta |L|_3^\eta |L|_4^\eta$$

(3.98)

with $|L|_i^\eta$, $i \leq 4$, $\eta \in \text{col}(L)$, as in Appendix C below and with $\text{col}(L)$ denoting the set of maps $Y(L) \to \Lambda^4_\Sigma$ where

$$Y(L) = \{Y_0,Y_1,\ldots,Y_{m'}\}, \ m' \in \mathbb{N},$$

is the set of connected components of $\Sigma \setminus \bigcup_{i=1}^m \text{Image}(R^i_\Sigma)$. (Recall from Sec. 3.2.4 that $R^i_\Sigma := (R_i)_{\Sigma} := \pi_{\Sigma} \circ R_i$.)

As a preparation for Sec. 3.5 (and in particular, in order to show how major building blocks of the “shadow invariant” $|L|$ appear naturally) we will now consider the following simplified situation where $L = ((R_1,R_2,\ldots,R_m),(\rho_1,\rho_2,\ldots,\rho_m))$, $m \in \mathbb{N}$, fulfills the following two conditions:
(C1) The maps \( R^1_\Sigma \) neither intersect each other nor themselves.\(^{63}\)

(C2) Each of the maps \( R^i_\Sigma \), \( i \leq m \) is null-homotopic.

In this special situation Eq. \( (3.98) \) reduces to (cf. Eqs. \((C.3)\) in Appendix \(C\) and Remark \(5.35\) below)

\[
Z(\Sigma \times S^1, L) \sim |L| = \sum_{\eta \in \text{col}(L)} |L_1^0| |L_2^0| |L_3^0| = \sum_{\eta \in \text{col}(L)} \left( \prod_{i=1}^m N_i \right) \frac{\eta(Y^+_i)}{\lambda_i \eta(Y^-_i)} \prod_{Y \in Y(L)} (d_{\eta(Y)} \chi(Y) \theta_{\eta(Y)} \text{gleam}(Y))
\]

(3.99)

where \( \lambda_i \in \Lambda^+ \) is the highest weight of \( \rho_i \) and where \( \text{gleam}(Y) \) is given by Eq. \( (3.101) \) below.

(For simplicity, we will assume in the following that \( \lambda_i \in \Lambda^c \).)

Let \( L^0 = (l_1, l_2, \ldots, l_m) \) be the proper link associated to \( L \), cf. Definition \(3.6\) above. Let us set \( l^1_\Sigma := (l_i) \Sigma := \pi \Sigma \circ l_i \) and \( l^1_{S1} := (l_i)_{S1} := \pi_{S1} \circ l_i \).

Remark 3.35 \( (i) \) In the special case where \( L \) fulfills conditions \((C1)\) and \((C2)\) above the set \( V(L^0) \) is empty and the set \( E_{\text{loop}}(L^0) \) coincides with \( E(L^0) \), so Eq. \((C.2)\) then indeed reduces to Eq. \(\ref{3.99}\) above.

\( (ii) \) Observe also that in the special case where \( L \) fulfills conditions \((C1)\) and \((C2)\) we have \( m' = m \) and there is \( Y \in Y(L) \cong Y(L^0) \) such that

\[
\chi(Y) = 2 - 2g - \# \{ j \leq m \mid \text{Image}(l^1_\Sigma) \subset \partial Y \} \tag{3.100a}
\]

where \( g \) is the genus of \( \Sigma \) while for all the other \( Y \in Y(L) \) we have

\[
\chi(Y) = 2 - \# \{ j \leq m \mid \text{Image}(l^1_i) \subset \partial Y \} \tag{3.100b}
\]

Moreover, for every \( Y \in Y(L) \cong Y(L^0) \) we have the explicit formula

\[
\text{gleam}(Y) = \sum_{i \text{ with } \text{Image}(l^1_i) \subset \partial Y} \text{wind}(l^1_{S1}) \cdot \text{sgn}(Y; l^1_\Sigma) \in \mathbb{Z} \tag{3.101}
\]

where \( \text{wind}(l^1_{S1}) \) is the winding number of the loop \( l^1_{S1} \) and where \( \text{sgn}(Y; l^1_\Sigma) \) is given by

\[
\text{sgn}(Y; l^1_\Sigma) := \begin{cases} 
1 & \text{if } Y \subset \bar{X}^+_i \\
-1 & \text{if } Y \subset \bar{X}^-_i \end{cases} \tag{3.102}
\]

where \( \bar{X}^+_i \) and \( \bar{X}^-_i \) are the two connected components of \( \Sigma \setminus \text{Image}(l^1_\Sigma) \) (cf. Condition \((C2)\) above). More precisely, \( \bar{X}^+_i \) is determined by the condition that the orientation of \( \text{Image}(l^1_\Sigma) = \partial \bar{X}^+_i \) induced by \( \bar{X}^+_i \) coincides with the orientation induced by \( l^1_\Sigma \).

In the rest of this section we will now derive Eq. \(\ref{3.99}\) by evaluating the RHS of Eq. \(\ref{3.40}\) explicitly for the special \( L \) described above.

\(^{63}\)e.g. these maps have pairwise disjoint images and each \( R^i_\Sigma : S^1 \times [0, 1] \to \Sigma \) is an embedding.

\(^{64}\)Observe that \( Y^+_e = Y^+_e \) with \( e = l^1_\Sigma \) where \( Y^+_e \) is as in Appendix \(C\).
3.4.1 Explicit evaluation of $Z(\Sigma \times S^1, L)$

Let us first evaluate the RHS of Eq. (3.40) explicitly for the special $L$ described at the beginning of Sec. 3.4 above. In Sec. 3.4.2 below we will then rewrite the explicit expression on the RHS of Eq. (3.107) in a suitable way and by doing so we will obtain Eq. (3.99).

We begin by evaluating the inner integral on the RHS of Eq. (3.40). In a completely analogous way as in Secs. 3.3.1–3.3.2 we obtain (cf. Remark 3.26 above) with $R_i = R_i^{(s_0)}$ (cf. Convention 3.16 in Sec. 3.2.6 above)

$$
\int_{A^1} (\prod_i \Tr_{\rho_i} (\Hol_{R_i} (A^1, B))) d\mu_B(A^1) = \prod_i \Tr_{\rho_i} (\Hol_{R_i} (A^1, B))
$$

so the RHS of Eq. (3.40) simplifies and we obtain

$$
Z(S^2 \times S^1, L) \sim \lim_{\epsilon \to 0} \sum_{y \in I} \int_B \left[ \int_{A^1} \left\{ 1_{B_{reg}^{(\epsilon)}}(B) \Det_{reg}(B) \exp(-2\pi i k(y, B(\sigma_0))) \times \left( \prod_i \Tr_{\rho_i} (\Hol_{R_i} (A^1, B)) \right) \exp(iS_{CS}(A^1, B)) \right\} dA_{\epsilon} \right] DB
$$

For each $i \leq m$ let $X_i^+$ and $X_i^-$ be the two connected components of $\Sigma \setminus \Image(R_i^{(\epsilon)})$ (cf. Condition (C2) above) where $X_i^+$ is given by $X_i^+ \subset \widetilde{X}_i^+$ with $\widetilde{X}_i^+$ as in Remark 3.35 above.

Moreover, let $f_i := f_{R_i}$ be the function $\Sigma \to [0, 1]$ defined in an analogous way as the function $f_1$ in Sec. 3.3.3 above. Instead of working with $f_i$ it will be more convenient to work with the functions $f_i : \Sigma \to [0, 1]$ given by $f_i = f_i + C_i$ where $C_i \in \mathbb{R}$ is chosen such that $f_i(\sigma_0) = 0$.

Finally, for each $i \leq m$ we denote by $Y_i^+$ (or $Y_i^-$, respectively) the unique $Y \in Y(L)$ which is contained in $X_i^+$ (or $X_i^-$, respectively) and has a common boundary with $\Image(R_i^{(\epsilon)})$.

By an obvious modification of the arguments in Sec. 3.3.3 above we then obtain

$$
Z(\Sigma \times S^1, L) \sim \sum_{y \in I} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m \in \Lambda} \left( \prod_j m_{\lambda_j}(\alpha_j) \right) \int_{A^1} \left[ 1_{B_{reg}^{(\epsilon)}}(B) \Det_{reg}(B) \exp(-2\pi i k(y, B(\sigma_0))) \times \left( \prod_j \exp(2\pi i \alpha_j \left( \int_0^1 \int_0^1 [(Bdt)(t'(s))] ds du \right)) \right] \right] \left| B = b + \frac{1}{2} \sum_i \alpha_i f_i \right| (3.105)
$$

where for each $j \leq m$ and $u \in [0, 1]$ we have set

$$
l_{\alpha} := (l_j)_{\alpha} := R_j(\cdot, u)
$$

Note that according to Footnote 51 in Sec. 3.3.3 above the value of the RHS of Eq. (3.105) above does not change if we replace each $f_i$ by $f_i$. We will do this in the following but will simply write $f_i$ instead of $f_i$.

By modifying the calculations in Sec. 3.3.3 in a straightforward way we obtain

$$
Z(\Sigma \times S^1, L) \sim \sum_{\alpha_0, \alpha_1, \ldots, \alpha_m \in \Lambda} 1_{kQ}(\alpha_0) \left( \prod_{j=1}^m m_{\lambda_j}(\alpha_j) \right) \times \left[ \left( \prod_{j=0}^m 1_{B_{reg}}(B(Y_j)) \right) \left( \prod_{j=0}^m \det^{1/2}(i_t - \exp(\text{ad}(B(Y_j)))) |_{t=0} \chi(Y_j) \right) \times \left( \prod_{j=1}^m \exp(\pi i \text{wind}(l_{\alpha_0})^j(\alpha_j, B(Y_j^+), B(Y_j^-))) \right) \right] |_{B = b + \frac{1}{2} (\alpha_0 + \sum_i \alpha_i f_i)}
$$

51Note that if $m = 1$ we have $Y(L) = \{ Y_0, Y_1 \}$ and $Y_1^+ = X_1^+$ and $Y_1^- = X_1^-$. This probably makes it easier to compare the notation here with the one in Sec. 3.33.
Using Eqs (3.109) we see that we can rewrite Eq. (3.107) as
\[ B_{(\alpha_i)_i} := \frac{1}{k} \left( \alpha_0 + \sum_{j=1}^{m} \alpha_j \cdot f_j \right) \]
and introduce the map \( \eta_{(\alpha_i)_i} : Y(L) = \{ Y_0, Y_1, \ldots, Y_m \} \rightarrow \Lambda \) given by
\[ \eta_{(\alpha_i)_i}(Y) := kB_{(\alpha_i)_i}(Y) - \rho = \alpha_0 + \sum_{i=1}^{m} \alpha_i \cdot f_i(Y) - \rho \quad \forall Y \in Y(L) \] (3.108)

Then with \( B = B_{(\alpha_i)_i} \) and \( \eta = \eta_{(\alpha_i)_i} \) we have
\[ \alpha_j = \eta(Y_j^+) - \eta(Y_j^-) \quad \text{for } 1 \leq j \leq m , \] (3.109a)
\[ \det^{1/2} \left( 1_{\mathbb{R}} - \exp(\text{ad}(B(Y_i))) \right) \sim d_{\eta(Y_i)} , \] (3.109b)
\[
\prod_{j} \exp(\pi i \ \text{wind}(l^j_{\Sigma_1})(\alpha_j, B(Y_j^+) + B(Y_j^-))) \\
= \prod_{j} \exp \left( \frac{\pi i}{k} \ \text{wind}(l^j_{\Sigma_1}) \left[ \langle \eta(Y_j^+), \eta(Y_j^+) + 2\rho \rangle - \langle \eta(Y_j^-), \eta(Y_j^-) + 2\rho \rangle \right] \right) \\
= \prod_{Y \in Y(L)} \exp \left( \frac{\pi i}{k} \left( \sum_j \text{Image}(l^j_{\Sigma_1}) \ \text{wind}(l^j_{\Sigma_1}) \ \text{sgn}(Y; l^j_{\Sigma}) \right) \langle \eta(Y), \eta(Y) + 2\rho \rangle \right) \\
= \left( \prod_{Y \in Y(L)} \left( \theta_{\eta(Y)} \text{gleam}(Y) \right) \right) \] (3.109c)

where in step (*) we used Eq. (3.104) above.

Let us now assume without loss of generality that \( \sigma_0 \in Y_0 \). Then we have
\[ \alpha_0 = \eta_{(\alpha_i)_i}(Y_0) + \rho \] (3.109d)

Using Eqs (3.109) we see that we can rewrite Eq. (3.107) as
\[
Z(\Sigma \times S^1, L) \sim \sum_{(\alpha_i)_i} \left[ \prod_{Y \in Y(L)} 1_{k_{t_{reg}} - \rho}(\eta(Y)) \right] \\
\times \left( \prod_{j=1}^{m} \text{m}_{\lambda_j}(\eta(Y_j^+) - \eta(Y_j^-)) \right) \\
\times \left( \prod_{Y \in Y(L)} (d_{\eta(Y)})^{\chi(Y)(\theta_{\eta(Y)})} \text{gleam}(Y) \right) \right] |_{\eta = \eta_{(\alpha_i)_i}} \] (3.110)

Recall that \( \text{col}(L) \) denotes the set of maps \( \eta : Y(L) = \{ Y_0, Y_1, \ldots, Y_m \} \rightarrow \Lambda^k \). In the following let \( \text{col}'(L) \) denote the set of maps \( \eta : \{ Y_0, Y_1, \ldots, Y_m \} \rightarrow \Lambda \cap (k_{t_{reg}} - \rho) \).
Observation 3.36 The map

\[ \Phi : \{ (\alpha_i)_{0 \leq i \leq m} \in \Lambda^{m+1} | \eta(\alpha_i) \in (k_{treg} - \rho) \text{ for all } t \in \{0, \ldots, m\} \} \rightarrow \text{col}'(L) \]
given by \( \Phi((\alpha_i)_{0 \leq i \leq m}) = \eta(\alpha_i) \) is a well-defined bijection. (That \( \Phi \) is well-defined and surjective is easy to see. That \( \Phi \) is also injective follows from Eq. (3.109a) and Eq. (3.109c) above.)

Combining Eq. (3.110) with Observation 3.36 we obtain

\[
Z(\Sigma \times S^1, L) \sim \sum_{\eta \in \text{col}'(L)} \left[ 1_{k(Q_{treg}-\rho)}(\eta(Y_0)) \left( \prod_{j=1}^m m_{\lambda_j}(\eta(Y_j^+ - \eta(Y_j^-)) \right) \right. \\
\times \left. \left( \prod_{Y \in Y(L)} (d_{\eta(Y)})(\chi(Y)(\theta_{\eta(Y)})^{\text{gleam}(Y)}) \right) \right] 
\] (3.111)

Let \((W_{\text{aff}})^{Y(L)}\) denote the space of maps \( \tau : \{Y_0, Y_1, \ldots, Y_m\} \rightarrow W_{\text{aff}} \). We can then rewrite Eq. (3.111) as

\[
Z(\Sigma \times S^1, L) \sim^{(*)} \sum_{\eta \in \text{col}(L)} \sum_{\tau \in (W_{\text{aff}})^{Y(L)}, \tau(Y_0) \in W} \left[ \left( \prod_{j=1}^m m_{\lambda_j}(\tau(Y_j^+ - \eta(Y_j^-)) \right) \right. \\
\times \left. \left( \prod_{Y \in Y(L)} (d_{\eta(Y)})(\chi(Y)(\theta_{\eta(Y)})^{\text{gleam}(Y)}) \right) \right] 
\]

\[
\sim^{(**)} \sum_{\eta \in \text{col}(L)} \sum_{\tau \in (W_{\text{aff}})^{Y(L)}, \tau(Y_0) \in W} \left[ \left( \prod_{j=1}^m m_{\lambda_j}(\tau(Y_j^+ - \eta(Y_j^-)) \right) \right. \\
\times \left. \left( \prod_{Y \in Y(L)} ((-1)^{\tau(Y)})(\chi(Y)(\theta_{\eta(Y)})^{\text{gleam}(Y)}) \right) \right] 
\]

\[
\sim^{(+)} \sum_{\eta \in \text{col}(L)} \sum_{\tilde{\tau} \in (W_{\text{aff}})^m, \tilde{\tau}_0 \in W} \left[ \left( \prod_{j=1}^m m_{\lambda_j}(\eta(Y_j^+ - \tilde{\tau}_j \eta(Y_j^-)) \right) \right. \\
\times \left. \left( \prod_{Y \in Y(L)} ((-1)^{\tilde{\tau}_j})(\chi(Y)(\theta_{\eta(Y)})^{\text{gleam}(Y)}) \right) \right] 
\] (3.112)

with \( W \subset W_{\text{aff}} \) as in Sec. 3.35, and where in step (*) we used the two bijections \( \Phi \), in step (**) we used Eqs (3.84a) and the first relation in part (i) of Remark 3.30 above, and in step (+) we used that

\[
m_{\lambda_j}(\tau(Y_j^+ - \eta(Y_j^-)) = m_{\lambda_j}(\eta(Y_j^+ - (\tau(Y_j^+ - \eta(Y_j^-)) \right)
\]

and then made the change of variable \( \tau = (\tau_0, \tau_1, \tau_2, \ldots, \tau_m) \rightarrow \tilde{\tau} = (\tilde{\tau}_0, \tilde{\tau}_1, \ldots, \tilde{\tau}_m) \) given by \( \tilde{\tau}_0 = \tau(Y_0) \) and \( \tilde{\tau}_j := \tau(Y_j^-) \cdot \tau(Y_j^-) \), for \( j \leq m \), and took into account that

\[
\prod_{Y \in Y(L)} ((-1)^{\tau(Y)})(\chi(Y)) \left( \prod_{j=1}^m m_{\lambda_j}(\eta(Y_j^+ - \tilde{\tau}_j \eta(Y_j^-)) \right)
\]

\[
= \prod_{j=1}^m ((-1)^{\tau(Y_j^+)}(-1)^{\tau(Y_j^-)} = \prod_{j=1}^m ((-1)^{\tilde{\tau}_j})
\]

Here step (++) follows from Eqs. (3.100) in Remark 3.35 above. By combining Eq. (3.112) with the relation

\[
\sum_{\tilde{\tau}_j \in W_{\text{aff}}} ((-1)^{\tilde{\tau}_j}) m_{\lambda_j}(\eta(Y_j^+ - \tilde{\tau}_j \eta(Y_j^-)) = m_{\lambda_j}(\eta(Y_j^+ - \eta(Y_j^-)) \quad \text{for all } j \leq m
\]

\[\text{Here we use } 1_{k(Q_{treg}-\rho)}(\eta(Y_0)) = 1_{k(Q_{treg}-\rho)}(\eta(Y_0)) = 1_{k(Q_{treg}-\rho)}(\eta(Y_0)). \]

\[\text{Observe that the map } \tau \rightarrow \tilde{\tau} \text{ introduced above is a (well-defined) bijection from } \{ \tau \in W_{\text{aff}}^{Y(L)} | \tau(Y_0) \in W \} \text{ onto } W \times (W_{\text{aff}})^m. \]
3.5 The case of “generic” links in $M = \Sigma \times S^1$

We will finally consider the case of “generic” links $L$ in $M = \Sigma \times S^1$. In contrast to Sec. 3.3 and Sec. 3.4 where for the special ribbon links considered there we gave a complete evaluation of $Z(\Sigma \times S^1, L)$ we will now only sketch the overall strategy for evaluating $Z(\Sigma \times S^1, L)$. It should be noted, though, that with some extra work one can indeed obtain an explicit combinatorial formula for $Z(\Sigma \times S^1, L)$ also for general “strictly admissible” $L$ (cf. Definition 3.45 below). The difference with respect to the treatment of the three special cases mentioned above is that we do not verify here that the explicit expressions obtained for $Z(\Sigma \times S^1, L)$ for general strictly admissible $L$ agrees with those in $RT(\Sigma \times S^1, L)$, cf. the paragraph after Eq. (3.128) in Sec. 3.5.2 below.

3.5.1 Admissible links and (strictly) admissible ribbon links in $M = \Sigma \times S^1$

As a preparation for Definition 3.42 and Definition 3.43 below we first introduce several definitions for proper links $L = (l_1, l_2, \ldots, l_m)$, $m \in \mathbb{N}$, in $M = \Sigma \times S^1$.

For given $L$ we call $p \in \Sigma$ a “double point” (resp. a “triple point”) of $L$ if the intersection of $\pi^{-1}_\Sigma\{\{p\}\}$ with the union of the images of $l_1, l_2, \ldots, l_m$ contains at least two (or at least three, respectively) elements. We set

$$V(L) := \{p \in \Sigma \mid p \text{ is a double point of } L\}$$  \hspace{1cm} (3.113)

Moreover, we will denote by $E(L)$ the set of curves in $\Sigma$ into which the loops $l^1_\Sigma, l^2_\Sigma, \ldots, l^m_\Sigma$ (defined again by $l^i_\Sigma := (l_i)_\Sigma := \pi_\Sigma \circ l_i$) are decomposed when being “cut” in the points of $V(L)$.

**Definition 3.37** Let $L = (l_1, l_2, \ldots, l_m)$, $m \in \mathbb{N}$, be a (smooth) link in $M = \Sigma \times S^1$. We call $L$ admissible iff

- There are no triple points of $L$,
- The projected loops $l^1_\Sigma, l^2_\Sigma, \ldots, l^m_\Sigma$ only have transversal intersections. (In particular, $V(L)$ is a finite set.)
- Each $l^i_\Sigma$, $i \leq m$, is an immersion (i.e. none of the tangent vectors $(l^i_\Sigma)'(t)$, $t \in [0, 1]$ vanishes).

**Remark 3.38** Observe that every (smooth) link $L$ in $M = \Sigma \times S^1$ is equivalent (i.e. isotopic) to an admissible one.

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68 Note that in view of Remark 3.38 and Remark 3.44 below, “admissible” ribbon links (in the sense of Definition 3.44 below) can be considered to be “generic”. Moreover, in the special case $\Sigma \cong S^2$ also “strictly admissible” ribbon links (in the sense of Definition 3.45 below) are “generic”.

69 More precisely: for each $p \in V(L)$ the two corresponding tangent vectors in $T_p\Sigma$ (which are given by $(l^i_\Sigma)'(\bar{t})$ and $(l^j_\Sigma)'(\bar{u})$ where $\bar{t}, \bar{u} \in [0, 1]$, $i, j \leq n$, such that $p = l^i_\Sigma(\bar{t}) = l^j_\Sigma(\bar{u})$) are not parallel to each other.
A decomposition of a link \( L = (l_1, l_2, \ldots, l_m) \), \( m \in \mathbb{N} \), in \( M = \Sigma \times S^1 \) is a tuple \( D = (D_i)_{i \leq m} \) where each \( D_i \) is a finite decomposition\(^7\) of \( \text{dom}(l_i) = [0,1] \) into subintervals. Using \( D_i \) the loop \( l_i \), \( i \leq m \), decomposes into finitely many (sub)curves. We will denote this set of curves by \( \mathcal{C}(l_i, D_i) \) and we set \( \mathcal{C}(L, D) := \bigcup_i \mathcal{C}(l_i, D_i). \) For each \( c \in \mathcal{C}(L, D) \) we set \( c_\Sigma := \pi_\Sigma \circ c. \) By identifying \( \text{dom}(c) \) with \([0,1]\) in the obvious way we can consider each \( c \in \mathcal{C}(L, D) \) as a map \([0,1] \rightarrow \Sigma \times S^1 \) and each \( c_\Sigma \) as a map \([0,1] \rightarrow \Sigma \).

**Definition 3.39** (i) A decomposition \( D \) of \( L \) is called a “cluster decomposition” of \( L \) iff the following two conditions are fulfilled

- For every \( c \in \mathcal{C}(L, D) \) the projected curve \( c_\Sigma \) is a smooth embedding and none of the two endpoints of \( c_\Sigma \) lies in \( V(L) \)
- For every \( c \in \mathcal{C}(L, D) \) we have either of the following:  
  Case 1: For every \( c' \in \mathcal{C}(L, D), c \neq c' \), the two curves \( c_\Sigma \) and \( c'_\Sigma \) may have an endpoint in common but they do not intersect “properly”\(^7\).
  Case 2: There is exactly one \( c' \in \mathcal{C}(L, D), c \neq c' \), such that the two curves \( c_\Sigma \) and \( c'_\Sigma \) intersect properly. Moreover, \( c_\Sigma \) and \( c'_\Sigma \) intersect in exactly one point \( p \in \Sigma \) and they intersect transversally (cf. Footnote \(^7\) above).

(ii) A “1-cluster” of \( L \) (induced by \( D \)) is a set of the form \( \{c\} \) where \( c \in \mathcal{C}(L, D) \) is as in Case 1 above.

(iii) A “2-cluster” of \( L \) (induced by \( D \)) is a set of the form \( \{c,c'\} \) where \( c,c' \in \mathcal{C}(L, D) \) are as in Case 2 above.

**Observation 3.40** A (smooth) link \( L \) in \( M = \Sigma \times S^1 \) is admissible iff it possesses a cluster decomposition.

Let us now go back to the case of ribbon links in \( M = \Sigma \times S^1 \). Let \( L = (R_1, R_2, \ldots, R_m), m \in \mathbb{N} \), be a horizontal ribbon link in \( M = \Sigma \times S^1 \).

We say that \( p \in \Sigma \) is a double point of \( L \) iff the intersection of \( \pi_\Sigma^{-1}(\{p\}) \) with \( \bigcup_{i=1}^m \text{Image}(R_i) \) contains at least two elements. A decomposition of \( L \) is a tuple \( D = (D_i)_{i \leq m} \) where each \( D_i \) is a finite decomposition of \([0,1]\) into subintervals \((I^*_i)_j\). Via \( D_i \) the (closed) ribbon \( R_i, i \leq m \), decomposes into finitely many “pieces”, i.e. maps of the form \( S : I^*_i \times [0,1] \rightarrow \Sigma \times S^1 \). We will denote this set of maps by \( S(R_i, D_i) \). Moreover, we set \( S(L, D) := \bigcup_i S(R_i, D_i) \) and we fix an (arbitrary) order relation on \( S(L, D) \) (which will be kept fixed in the following). Finally, for each \( S \in S(L, D) \) we set \( S_\Sigma := \pi_\Sigma \circ S. \) By identifying each \( I^*_i \) with \([0,1]\) in the obvious way we can consider each \( S \) as a map \([0,1] \times [0,1] \rightarrow \Sigma \times S^1 \) and each \( S_\Sigma \) as a map \([0,1] \times [0,1] \rightarrow \Sigma \).

**Definition 3.41** We say that two smooth maps \( S^1_\Sigma : [0,1] \times [0,1] \rightarrow \Sigma, i \in \{1,2\}, \) “intersect transversally” if – using suitable reparametrizations and a suitable chart of \( \Sigma \) – we can transform \( (S^1_\Sigma, S^2_\Sigma) \) into \((S^1_\text{can}, S^2_\text{can})\) where \( S^i_\text{can} : [0,1] \times [0,1] \rightarrow \mathbb{R}^2, i \in \{1,2\}, \) are given by \( S^i_\text{can}(s,u) = (2s - 1, u - 1/2) \) and \( S^2_\text{can}(s,u) = (u - 1/2, 2s - 1) \) for all \( s,u \in [0,1] \).

\(^7\)More precisely: each \( D_i \) is a finite sequence \((t_j)_{j \leq n}, n \in \mathbb{N}, \) of elements of \([0,1]\) fulfilling \( 0 = t_1 < t_2 < \cdots < t_n = 1. \) Clearly, \( D_i \) induces a sequence \((t_{j,i} + 1)_{j \leq n-1}\) of subintervals of \([0,1]\).

\(^7\)More precisely, \( c_\Sigma(s) = c'_\Sigma(s') \) can only occur for some \( s,s' \in [0,1] \) iff \( s = 0 \lor s' = 1 \) or \( s = 1 \land s' = 0. \)

\(^7\)More precisely: \( S^\delta_i \), \( i \in \{1,2\}, \) intersect transversally if it is possible to find smooth reparametrizations \( \psi_i : [0,1] \times [0,1] \rightarrow [0,1] \times [0,1] \) and an open chart \((U, \varphi)\) of \( \Sigma \) such that each \( \text{Image}(S^\delta_i), i \in \{1,2\}, \) is contained in \( U \) and such that \( S^\delta_i = \varphi \circ S^\delta_i \circ \psi_i \) for \( i \in \{1,2\}. \)
Definition 3.42  (i) A decomposition \( D \) of \( L \) is called a “cluster decomposition” of \( L \) iff the following two conditions are fulfilled

- For every \( S \in \mathcal{S}(L, D) \) the projected map \( S_\Sigma \) is injective and no endpoint of \( S_\Sigma \) (i.e. none of the points \( p \in \{ S_\Sigma(s, u) \mid s \in \{0, 1\}, u \in [0, 1]\} \)) is a double point of \( L \).
- For every \( S \in \mathcal{S}(L, D) \) we have either of the following:
  
  Case 1: For every \( S' \in \mathcal{S}(L, D) \), \( S \neq S' \), the two maps \( S_\Sigma \) and \( S'_\Sigma \) “do not intersect properly” \footnote{More precisely: \( S_\Sigma(s, u) = S'_\Sigma(s', u') \) for \( s, s', u, u' \in [0, 1] \) can only occur when \( s = 0 \land s' = 1 \) or \( s = 1 \land s' = 0 \).}
  
  Case 2: There is exactly one \( S' \in \mathcal{S}(L, D) \), \( S \neq S' \), such that the two maps \( S_\Sigma \) and \( S'_\Sigma \) intersect properly. Moreover, \( S_\Sigma \) and \( S'_\Sigma \) intersect transversally in the sense of Definition 3.41 above.

(ii) A 1-cluster of \( L \) induced by \( D \) is a set \( \{ S \} \) where \( S \in \mathcal{S}(L, D) \) is as in Case 1.

(iii) A 2-cluster of \( L \) induced by \( D \) is a set \( \{ S_1, S_2 \} \) where \( S_1, S_2 \in \mathcal{S}(L, D) \) are as in Case 2.

In view of the order relation on \( \mathcal{S}(L, D) \) fixed above we can consider a 2-cluster as an ordered pair \((S_1, S_2)\). (This will convenient below). We will denote the set of 1-clusters (and 2-clusters, respectively) of \( L \) induced by \( D \) by \( Cl_1(L, D) \) (and \( Cl_2(L, D) \), respectively). We set \( Cl(L, D) := Cl_1(L, D) \cup Cl_2(L, D) \). The order relation on \( \mathcal{S}(L, D) \) fixed above induces an order relation on \( Cl(L, D) \).

Observation \ref{obs:3.40} above motivates the following definition.

Definition 3.43 An admissible ribbon link in \( M = \Sigma \times S^1 \) is a horizontal ribbon link in \( M = \Sigma \times S^1 \) which possesses a cluster decomposition.

Remark 3.44 The set of admissible ribbon links is very large. Indeed, every admissible proper link when equipped with a horizontal framing leads to an admissible ribbon link. More precisely, we have the following:

Let \( L_{pr} \) be a (proper) link in \( M = \Sigma \times S^1 \) which is admissible in the sense of Definition 3.37 above. Let \( L \) be a horizontal ribbon link such that \( L^0 = L_{pr} \), cf. Definition 3.5 and Definition 3.6 in Sec. 3.2. Now let \( L(s) = (R_1(s), R_2(s), \ldots, R_m(s)) \), for \( s \in (0, 1] \), where each \( R_i(s) \) is given as in Sec. 3.2.4 above. Then if \( s \) is sufficiently small, \( L(s) \) will be an admissible ribbon link.

Definition 3.45 An admissible ribbon link \( L = (R_1, R_2, \ldots, R_m) \), \( m \in \mathbb{N} \), in \( M = \Sigma \times S^1 \) is called “strictly admissible” iff

- Each \( R_i \), \( i \leq m \), is null-homotopic, and
- It is possible to choose the auxiliary Riemannian metric \( g_\Sigma \) in Sec. 3.2 such that Condition 1 in Sec. 3.2.4 can be fulfilled for \( L \).

(Note that every admissible ribbon link \( L \) can be reparametrized so that Condition 1 in Sec. 3.2.4 can be fulfilled).
3.5.2 Evaluation of $Z(\Sigma \times S^1, L)$ for general (strictly admissible) $L$: sketch

Let $L = ((R_1, R_2, \ldots, R_m), (\rho_1, \rho_2, \ldots, \rho_m)), m \in \mathbb{N}$, be an arbitrary colored, strictly admissible ribbon link in $M = \Sigma \times S^1$. For the evaluation of $Z(\Sigma \times S^1, L)$ we will essentially follow the same steps as in Sec. 3.3 and Sec. 3.4 above Step 1.

**Step 1:** We evaluate (for each fixed $A_c^+ \in A_c^+$, $B \in \mathcal{B}$, and $\epsilon < \epsilon_0$) the inner integral

$$I^*_L(A_c^+, B) := \int_{A_c^+} \left( \prod_i \text{Tr}_{\rho_i} \left( \text{Hol}_{R_i}(A_c^+ + A_c^+, B) \right) \right) d\mu_B(A_c^+)$$

in Eq. (3.40) (with $R_i = R_i^{(s_0)}$, cf. Convention 3.16 in Sec. 3.2.6 above).

Observe that in contrast to the situation in the two special cases II and III in Sec. 3.3 and Sec. 3.4 above Step 1 is now no longer trivial. Still, using the following procedure it is possible to perform Step 1.

- Let us fix a cluster decomposition $D$ of $L$.

Recall that each color $\rho_i$, $i \leq m$, is an irreducible representation $\rho_i : G \to \text{Aut}(V_i)$ where $V_i$ is a finite-dimensional, complex vector space. For every $S \in S(L, D)$ we set $\rho_S := \rho_i$ and $V(S) := V_i$ where $i \leq m$ is the unique index such that $S \in S(R_i, D_i)$ (i.e. such that $S$ is a “piece of the closed ribbon $R_i$”). Moreover, if $cl = \{S\} \in Cl_1(L, D)$ we will write $V(cl)$ instead of $V(S)$ and if $cl = (S_1, S_2) \in Cl_2(L, D)$ we will write $V_i(cl)$ instead of $V(S_i)$, $i = 1, 2$.

For each $\epsilon > 0$, $A_c^+ \in A_c^+$, $B \in \mathcal{B}$, and $cl \in Cl(L, D)$ we define

$$\mathcal{P}^\epsilon_{cl}(A_c^+, B) \in \begin{cases} \text{End}(V(cl)) & \text{if } cl \in Cl_1(L, D) \\ \text{End}(V_1(cl)) \otimes \text{End}(V_2(cl)) & \text{if } cl \in Cl_2(L, D) \end{cases}$$

by

$$\mathcal{P}^\epsilon_{cl}(A_c^+, B) := \begin{cases} \rho_{S_1}(P_{S_1}^\epsilon(A_c^+, B)) & \text{if } cl = \{S_1\} \in Cl_1(L, D) \\ \rho_{S_1}(P_{S_1}^\epsilon(A_c^+, B)) \otimes \rho_{S_2}(P_{S_2}^\epsilon(A_c^+, B)) & \text{if } cl = (S_1, S_2) \in Cl_2(L, D) \end{cases} \quad (3.115)$$

where we use the notation of Remark 3.7 in Sec. 3.2.

It is not difficult to see that there is a linear form $\beta_{L,D}$ on

$$(\otimes_{cl \in Cl_1(L, D)} \text{End}(V(cl))) \otimes (\otimes_{cl \in Cl_2(L, D)} (\text{End}(V_1(cl)) \otimes \text{End}(V_2(cl))))$$

such that for all $\epsilon > 0$ we have

$$\prod_i \text{Tr}_{\rho_i} (\text{Hol}_{R_i}(A_c^+, B)) = \prod_i \text{Tr}_{\rho_i} (\mathcal{P}^\epsilon_{cl}(A_c^+, B)) = \beta_{L,D} (\otimes_{cl \in Cl(L, D)} \mathcal{P}^\epsilon_{cl}(A_c^+, B)) \quad (3.118)$$

(See also Eq. (D.9) in Appendix D below for an explicit formula for $\beta_{L,D}$.) Above we interpret “$\otimes_{cl \in Cl(L, D)} \mathcal{P}^\epsilon_{cl}(A_c^+, B)$” on the RHS of Eq. (3.118) as an element of the tensor product (3.117) in the obvious way.

- Let us introduce the short notation $\Phi_B^1 := \int_{A_c^+} \cdots d\mu_B(A_c^+)$ and let us denote by $\mathcal{P}^\epsilon_{cl}(\cdot + A_c^+, B)$ (for $A_c^+ \in A_c^+$, $B \in \mathcal{B}$) the map

$$\mathcal{A}_c^+ \ni \mathcal{A}_c^+ \mapsto \mathcal{P}^\epsilon_{cl}(\mathcal{A}_c^+ + A_c^+, B) \in \begin{cases} \text{End}(V(cl)) & \text{if } cl \in Cl_1(L, D) \\ \text{End}(V_1(cl)) \otimes \text{End}(V_2(cl)) & \text{if } cl \in Cl_2(L, D) \end{cases}$$

\footnote{We write $\Phi_B^1(f)$ both for functions $f : \mathcal{A}_c^+ \to \mathbb{C}$ and for vector valued maps $f : \mathcal{A}_c^+ \to V$ where $V$ is a real vector space.}
The key observation now is that the family of maps $(\mathcal{P}_d^\epsilon(\cdot + A_c^\perp, B))_{d \in \text{Cl}(L,D)}$ has the following "independence" property w.r.t. the functional $\Phi_B$\textsuperscript{73}:

$$\Phi_B\left((\otimes_{d \in \text{Cl}(L,D)} \mathcal{P}_d^\epsilon(\cdot + A_c^\perp, B))\right) = \otimes_{d \in \text{Cl}(L,D)} \Phi_B^\epsilon(\mathcal{P}_d^\epsilon(\cdot + A_c^\perp, B))$$

(3.119)

for all sufficiently small $\epsilon > 0$. From Eq. (3.118) and Eq. (3.119) and the linearity of $\beta_{L,D}$ we obtain

$$I_L(A_c^\perp, B) = \Phi_B^\epsilon\left(\beta_{L,D} \circ (\otimes_{d \in \text{Cl}(L,D)} \mathcal{P}_d^\epsilon(\cdot + A_c^\perp, B))\right) = \beta_{L,D} \left(\otimes_{d \in \text{Cl}(L,D)} T_d^\epsilon(A_c^\perp, B)\right)$$

(3.120)

for all sufficiently small $\epsilon > 0$ where we have set

$$T_d^\epsilon(A_c^\perp, B) := \Phi_B^\epsilon(\mathcal{P}_d^\epsilon(\cdot + A_c^\perp, B))$$

(3.121)

- We evaluate $T_d^\epsilon(A_c^\perp, B)$ for $cl \in \text{Cl}_1(L, D)$. This is easy. Using the same argument as in Sec. 3.3.1 and 3.3.2 (cf. Remark 3.26 in Sec. 3.3.1) we obtain for $cl = \{S\} \in \text{Cl}_1(L, D)$

$$T_d^\epsilon(A_c^\perp, B) = \int_{\tilde{A}_c^\perp} \mathcal{P}_d^\epsilon(\tilde{A}_c^\perp + A_c^\perp, B) d\mu_B^\perp(\tilde{A}_c^\perp) = \int_{\tilde{A}_c^\perp} \rho_S(\mathcal{P}_S^\epsilon(\tilde{A}_c^\perp + A_c^\perp, B)) d\mu_B^\perp(\tilde{A}_c^\perp)$$

$$= \rho_S(\mathcal{P}_S^\epsilon(0 + A_c^\perp, B)) = \rho_S(\mathcal{P}_S^\epsilon(A_c^\perp, B)) = \exp\left(\int_0^1 \rho_S^\epsilon(\det_{\mathbb{R}^d}(\tilde{A}_c^\perp, B)) ds\right)$$

(3.122)

where we use the notation of Remark 3.7 in Sec. 3.2.

- Next, we evaluate $T_d^\epsilon(A_c^\perp, B)$ for $cl \in \text{Cl}_2(L, D)$.

In contrast to the case $cl \in \text{Cl}_1(L, D)$, the evaluation of $T_d^\epsilon(A_c^\perp, B)$ for $cl \in \text{Cl}_2(L, D)$ is now non-trivial. However, using Eq. (3.48) and the well-known formula for the moments of Gaussian measures it is not too difficult to write down an explicit formula involving a (rather complicated) infinite series of powers of $1/k$, similarly to the derivation of Eq. (6.7) in \textsuperscript{34}. (It should be noted that, in contrast to Eq. (6.7) in \textsuperscript{34}, the coefficients in front of each power of $1/k$ will now also depend on $k$).

- Finally, by combining Eq. (3.10) with Eq. (3.120) we obtain

$$Z(\Sigma \times S^1, L) \sim \lim_{\epsilon \to 0} \sum_{y \in \mathbb{S}} \int_{A_c^\perp \times B} \left\{1_{B_{\epsilon y}(B)} \det_{\mathbb{R}^d}(B) \exp(-2\pi i k(y, B(\sigma_0)))\right\}$$

$$\times \beta_{L,D}(\otimes_{d \in \text{Cl}(L,D)} T_d^\epsilon(A_c^\perp, B)) \exp(iS_{CS}(A_c^\perp, B)(DA_c^\perp \otimes DB))$$

(3.123)

Let us now rewrite Eq. (3.123) in such a way that its RHS becomes more similar to the RHS of Eq. (C.2) in Appendix \textsuperscript{C} (with the explicit expressions (C.3) inserted into it). As in Appendix \textsuperscript{C} let $L^0$ be the proper link associated to $L$ (cf. Definition 3.6 in Sec. 3.2.1) and set

$$V(L) := V(L^0), \quad E(L) := E(L^0)$$

Observe that there is an obvious 1-1-correspondence between $\text{Cl}_2(L, D)$ and $V(L)$. Moreover, we can assume without loss of generality\textsuperscript{76} that the cluster decomposition $D$ was

\textsuperscript{73} Cf. Eq. (5.35) in \textsuperscript{37} and Eq. (6.3) in \textsuperscript{34} which are closely related (rigorous) results. In fact, the derivation of Eq. (3.119) is a slightly more complicated than the derivation of Eq. (5.35) in \textsuperscript{37}. We need to use Condition \textsuperscript{1} above and also the assumption that $s_0 < 1$. cf. Sec. 3.2.6.

\textsuperscript{76} This can, of course, always be achieved by "merging" every "chain" of consecutive 1-clusters to a single 1-cluster.
chosen to be “minimal” in the sense that every 1-cluster $cl \in Cl_1(L, D)$ “connects” two different 2-clusters. Then there is also a 1-1-correspondence between $Cl_1(L, D)$ and $E(L)$. For $x \in V(L)$ (and $e \in E(L)$, respectively) let $cl(x) \in Cl_2(L, D)$ (or $cl(e) \in Cl_1(L, D)$, respectively) denote the corresponding 2-cluster (1-cluster). Accordingly, we can rewrite Eq. (3.122) as

$$Z(\Sigma \times S^1, L) \sim \lim_{\epsilon \to 0} \sum_{y \in I} \int_{A_c^\perp \times B} \left( 1_{B_{rig}}(B) \det_{rig}(B) \exp(-2\pi i k \langle y, B(\sigma_0) \rangle) \right) \times \beta_{L,D}\left( (\otimes_{e \in E(L)} T^x_e(A_c^\perp, B)) \otimes (\otimes_{x \in V(L)} T^x_T(A_c^\perp, B)) \right) \times \exp(iS_{CS}(A_c^\perp, B))(DA_c^\perp \otimes DB)$$

(3.124)

where, for $x \in V(L)$ and $e \in E(L)$, we have set

$$T^x_e(A_c^\perp, B) := T^x_{cl}(A_c^\perp, B) \quad \text{and} \quad T^x_T(A_c^\perp, B) := T^x_{cl(x)}(A_c^\perp, B).$$

(3.125)

**Steps 2–4:** We perform in Eq. (3.124) above the integral, the sum $\sum_y$, and the $\epsilon \to 0$-limit, cf. Appendix D below (and the paragraph after Eq. (3.129) below).

**Step 5:** We rewrite the algebraic expression obtained after performing Steps 2–4 in quantum algebraic notation.

From Eq. (3.122) we easily obtain an explicit (and “closed”) expression for $T^x_e(A_c^\perp, B)$. By contrast, even though each $T^x_e(A_c^\perp, B), x \in V(L)$, can be written explicitly as suitable infinite sums (cf. the paragraph after Eq. (3.122) above) so far we do not have a closed expression for $T^x_T(A_c^\perp, B)$.

Since the integral on the RHS of Eq. (3.124) only involves the Abelian fields $A_c^\perp$ and $B$ and since the informal complex measure $\exp(iS_{CS}(A_c^\perp, B))(DA_c^\perp \otimes DB)$ is of Gauss-type one would expect that the evaluation of the integral is straightforward.

There is, however, one potential complication: the informal measure $\exp(iS_{CS}(A_c^\perp, B))(DA_c^\perp \otimes DB)$ is “degenerate”. More precisely, the kernel of (the symmetric bilinear form associated to) the quadratic form $S_{CS}(A_c^\perp, B)$ on $A_c^\perp \oplus B \cong A_{\Sigma,t} \oplus B$ is $A_{closed} \oplus B_c$ where $B_c := \{ B \in B \mid B \text{ is constant} \}$ and $A_{closed} := \{ \alpha \in A_{\Sigma,t} \mid d\alpha = 0 \}$. On the other hand, if it turns out that the function

$$F^L(A_c^\perp, B) := \beta_{L,D}\left( (\otimes_{e \in E(L)} T^x_e(A_c^\perp, B)) \otimes (\otimes_{x \in V(L)} T^x_T(A_c^\perp, B)) \right)$$

(3.126)

fulfills

$$F^L(A_{closed} + A_{coex}, B) = F^L(A_{coex}, B)$$

(3.127)

for all $B \in B, A_{closed} \in A_{closed}$, and $A_{coex} \in A_{coex} := \{ *df \mid f \in \Omega^0(\Sigma, t) \} \in A_{\Sigma,t}$ then the aforementioned complication does not have any serious consequences. And indeed, for the special links $L$ considered in Sec. 3.4 above, where $V(L) = \emptyset$ and each $R_{\Sigma}$ is null-homotopic, it is easy to check\(^7\) that (3.127) is indeed fulfilled provided that we use the “canonical” choice for the family (of Dirac families) $(\delta^e_{A \leq e_0})_{A \in A_{\Sigma}}$ as in Remark 3.29 above. This is the deeper reason why we did not get any problems in Sec. 3.4 above when evaluating $Z(\Sigma \times S^1, L)$. Recall that by doing so we arrived at

$$Z(\Sigma \times S^1, L) \sim |L| = \sum_{\eta \in col(L)} |L|_{\eta}^1 |L|_{\eta}^2 |L|_{\eta}^3$$

(3.128)

where the factors $|L|_{\eta}^i, i \leq 3$, are as in Sec. 3.4 above.

\(^7\)Since $V(L) = \emptyset$ in Sec. 3.4 this follows easily from the aforementioned explicit expression for $T^x_e(A_c^\perp, B)$, cf. Eq. (3.122) above.
The obvious two questions now are whether Eq. (3.127) also holds for general strictly admissible \( L \) and whether we obtain indeed

\[
Z(\Sigma \times S^1, L) \sim |L| = \sum_{\eta \in C(L)} |L|_1 |L|_2 |L|_3 |L|_4
\]

(3.129)
after carrying out Steps 2–5 above. (Here \( |L|_4 \) is given by \( |L|_4 = \text{cont}_{D(L)}(\otimes_{x \in V(L)} T(x, \eta)) \) with \( \text{cont}_{D(L)} \) and \( T(x, \eta) \) as in Appendix [C].)

It should be possible to verify each of these two questions “perturbatively” by using the aforementioned explicit infinite series for \( T_x^c(A_c^\perp, B) \), \( x \in V(L) \), cf. the “bullet point” after Eq. (3.122) above. Note that since at present we do not have a closed formula for

\[
T_x^c(A_c^\perp, B) := \lim_{\epsilon \to 0} T_x^c(A_c^\perp, B)
\]

we cannot yet give a completely explicit non-perturbative treatment of these two questions (in contrast to the situation in Sec. 3.3 and Sec. 3.4 where for the special class of links \( L \) considered there a non-perturbative treatment was possible). On the other hand, as we will show in Appendix D below, even without having a closed formula for \( T_x^c(A_c^\perp, B) \) we can still get quite far with a non-perturbative evaluation of \( Z(\Sigma \times S^1, L) \). In particular, we will make it plausible that also for general strictly admissible \( L \) we have a good chance of obtaining Eq. (3.129), cf. Remark 3.46 in Appendix [D].

Remark 3.46 In order to clarify if Eq. (3.129) holds for general strictly admissible \( L \) it may be useful to do consider first the special case \( G = SU(2) \) and to restrict one’s attention to those ribbon links \( L \) in \( \Sigma \times S^1 \) which stay inside \( \Sigma \times (S^1 \setminus \{1\}) \). In this special case we can use the formulas in [G] which allow us to rewrite the RHS of Eq. (3.129) using “R-matrices” instead of quantum 6j-symbols.

Remark 3.47 The strategy used in Step 1 is similar to the strategy used in Sec. 6 in [24] (see also [24, 52] and [6]) for the evaluation of the Chern-Simons path integral \( Z(\mathbb{R}^3, L) \) on \( \mathbb{R}^3 \) in the axial gauge where \( L \) is a colored framed link in \( \mathbb{R}^3 \). However, as was observed in [34, 24, 52] the Chern-Simons path integral \( Z(\mathbb{R}^3, L) \) in the axial gauge is problematic.

It turns out that the values of \( Z(\mathbb{R}^3, L) \) do not agree with those expected in the standard literature even for links \( L \) without double points.\(^{78}\) However, the explicit expressions obtained for \( Z(\mathbb{R}^3, L) \) are surprisingly close to the correct expressions, cf. Sec. 6 and Sec. 7 in [34]. For example, for a restricted class of the “loop smearing” regularization procedure which we use, the values of \( Z(\mathbb{R}^3, L) \) are invariant under Reidemeister I and Reidemeister II moves. Moreover, in the special case \( G = \text{Spin}(N) \) one “almost recovers” Kauffman’s state models for the HOMFLY polynomial at special values. (It is easy to imagine that by studying this special case one could have rediscovered Kauffman’s state models. I emphasize that we really use \( G = \text{Spin}(N) \) here even though the HOMFLY polynomial is usually associated to the groups \( G = SU(N), N \geq 2 \).)

Because of this I am optimistic that we will indeed obtain Eq. (3.129) for general strictly admissible ribbon links \( L \) when evaluating \( Z(\Sigma \times S^1, L) \) given by Eq. (3.120) or Eq. (3.395) (or by a suitable modification of Eq. (3.395), e.g. the modification sketched in Remark 3.48 below).

\(^{78}\) In [54] we left it open what the deeper reason for this is. One explanation could be that axial gauge fixing is so “singular” that we can in general not expect meaningful results when applying it. Alternatively, the problems with \( Z(\mathbb{R}^3, L) \) could indicate that something is “wrong” with the Chern-Simons path integral \( Z(M, L) \) when \( M \) is a non-compact 3-manifold.

\(^{79}\) When considering the Chern-Simons path integral \( Z(\mathbb{R}^3, L) \) on \( \mathbb{R}^3 \) in the axial gauge it is unclear why quantum groups (or rather, the corresponding R-matrices) should enter the computations. Note that a quantum group \( U_q(\mathfrak{g}) \), \( q \in \mathbb{C} \setminus \{-1, 0, 1\} \), is obtained from the classical enveloping algebra \( U(\mathfrak{g}) \) by a deformation process that involves a fixed Cartan subalgebra \( \mathfrak{t} \) of \( \mathfrak{g} \). But such a Cartan subalgebra does not play any role when considering the Chern-Simons path integral \( Z(\mathbb{R}^3, L) \) in the axial gauge. By contrast, when considering the Chern-Simons path integral \( Z(\Sigma \times S^1, L) \) in the torus gauge a Cartan subalgebra \( \mathfrak{t} \) plays an important role right from the beginning.

\(^{80}\) In Sec. 7 in [54] we actually considered the non-simply connected group \( G = \text{SO}(N) \). This is, however, equivalent to the situation where \( G = \text{Spin}(N) \) and where each link color \( \rho_i \) comes from a \( \text{SO}(N) \)-representation.
Remark 3.48 It may useful to perform what in \[39\] was called the “Transition to the BF-theoretic setting”. This amounts to applying a suitable linear change of variable to the CS path integral on \( M = \Sigma \times S^1 \) with group \( G \times G \) and parameters \((k, -k)\), cf. Sec. 7 in \[39\] and Appendix D in \[39\]. Note that as long as one works with proper loop holonomies the formulas which we obtain after performing the aforementioned linear change of variable will be equivalent to the original formulas. However, once we switch to ribbon holonomies this no longer needs to be the case and, accordingly, the aforementioned change of variable can be expected to lead to something new. In particular, the change of variable discussed in Appendix D in \[39\], which involves a complex structure on \( g \oplus g \), could be interesting, since it allows the complexification \( g_C \) of \( g \) to enter the picture. This reduces the gap between the CS path integral approach and the algebraic approach in \[67, 66, 78\] where complex Lie algebras play an essential role.

4 Rigorous realization \( Z_{\text{rig}}^{t\cdot g \cdot f}(\Sigma \times S^1, L) \) of \( Z^{t\cdot g \cdot f}(\Sigma \times S^1, L) \).

Sec. 3 was dedicated to what in Sec. 1 we called “Problem (P1)”. Now we will focus on Problem (P2), i.e. the problem of making rigorous sense of the torus gauge fixed CS path integral \( Z^{t\cdot g \cdot f}(\Sigma \times S^1, L) \) given by Eq. (3.39a) (or, alternatively, the RHS of Eq. (3.20) above). This is desirable for the following reasons:

(R1) In order to have a chance of making progress regarding the open problems in Quantum Topology mentioned in Sec. 5.2 below we need to solve both Problem (P1) and Problem (P2) of Sec. 1.

(R2) The study of the informal CS path integral gives rise to an interesting “paradox”: On the one hand the study of the informal CS path integral leads to very deep mathematics but, from the purist’s point of view, since it is only informal it does not contain any real mathematics at all. Obviously, once we have a rigorous realization \( Z_{\text{rig}}^{t\cdot g \cdot f}(\Sigma \times S^1, L) \) of \( Z^{t\cdot g \cdot f}(\Sigma \times S^1, L) \) this paradox is resolved for manifolds of the form \( M = \Sigma \times S^1 \) (cf. Remark 4.1 in Sec. 4.6 below for additional comments).

(R3) Torus gauge fixing is quite “singular”\(^{81}\) so it would be good to have a rigorous definition and evaluation of the torus gauge fixed CS path integral \( Z^{t\cdot g \cdot f}(\Sigma \times S^1, L) \).

In Sec. 4.1\--\4.3 we will sketch three different approaches/frameworks (F1), (F2), and (F3) for obtaining a rigorous realization \( Z_{\text{rig}}^{t\cdot g \cdot f}(\Sigma \times S^1, L) \) of \( Z^{t\cdot g \cdot f}(\Sigma \times S^1, L) \).

For each of these three approaches/frameworks we proceed in the following way:

- First we replace the spaces \( \hat{A}^\perp, A^\perp \), and \( B \) by suitable modifications \( \hat{A}_\text{mod}^\perp, (A^\perp)_\text{mod} \), and \( B_{\text{mod}} \) (e.g. finite-dimensional analogues like in framework (F1) and (F3) or suitably extended spaces like in framework (F2)).

- Then we find a rigorous realization of the informal integral functionals \( \mathcal{S}_{\text{CS}}(\hat{A}^\perp, B) \) and \( S_{\text{CS}}(A^\perp; B) \) for \( \hat{A}^\perp \in \hat{A}_\text{mod}^\perp, A^\perp \in (A^\perp)_\text{mod} \), and \( B \in B_{\text{mod}} \). (Recall that \( d\mu_B(\hat{A}^\perp) \) is the normalization of \( \exp(iS_{\text{CS}}(\hat{A}^\perp, B))D\hat{A}^\perp \), cf. Eq. (2.44) in Sec. 2.3.1 above.) By contrast, in framework (F2) we construct the two

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\(^{81}\)One manifestation of this “singularity” is the “instability” phenomenon described in Sec. 4.6 below.

\(^{82}\)When working with the original spaces of smooth 1-forms and maps \( \hat{A}^\perp, A^\perp \), and \( B \) the informal integral functionals given by Eqs (1.1) below cannot be defined in a satisfactory way. This is related to the well-known fact in measure theory that on most infinite-dimensional topological vector spaces \( E \) a (non-trivial) cylinder set measure does not extend to a true measure. An important exception is the case where \( E \) is the dual of a nuclear Frechet space, and it is precisely this exception which plays a crucial role for approach (F2).

\(^{83}\)In frameworks (F1) and (F3) this boils down to finding a rigorous realization of the expressions \( S_{\text{CS}}(\hat{A}^\perp, B) \) and \( S_{\text{CS}}(A^\perp; B) \) for \( \hat{A}^\perp \in \hat{A}_\text{mod}^\perp, A^\perp \in (A^\perp)_\text{mod} \), and \( B \in B_{\text{mod}} \). (Recall that \( d\mu_B(\hat{A}^\perp) \) is the normalization of \( \exp(iS_{\text{CS}}(\hat{A}^\perp, B))D\hat{A}^\perp \), cf. Eq. (2.44) in Sec. 2.3.1 above.) By contrast, in framework (F2) we construct the two
A concrete/full “implementation” of framework (F1) was given in [40] (which improves the original implementation in [38]).

Next, we find, for each $\tilde{A}^\perp \in \tilde{A}^\perp_{\text{mod}}$, $A^\perp_c \in (A^\perp_c)^{\text{mod}}$, and $B \in B_{\text{mod}}$ a rigorous analogue (or regularized version) of the expression (this is the choice for framework (F1))

$$\Phi^\perp_B := \int \cdots d\mu^\perp_B(\tilde{A}^\perp), \quad B \in B_{\text{mod}}$$

and

$$\Psi := \int \cdots \exp(iS_{CS}(A^\perp_c, B))(DA^\perp_c \otimes DB)$$

These rigorous realizations $\Phi^\perp_B$ and $\Psi$ must have reasonably large domains $\text{dom}(\Phi^\perp_B) \subset \text{Fun}(\tilde{A}^\perp_{\text{mod}}, \mathbb{C})$ and $\text{dom}(\Psi) \subset \text{Fun}((A^\perp_c)^{\text{mod}} \times B_{\text{mod}}, \mathbb{C})$. (Here $\text{Fun}(X, \mathbb{C}) = \mathbb{C}^X$.)

The final definition for $Z^t_{\text{rigf}}(\Sigma \times S^1, L)$ is now obtained by combining the first three points above in the obvious way (according to the RHS of Eq. (3.39a) above) and by adding suitable limits for the elimination of the regularization parameters which are involved in the definition of $F_{\text{rig}}$. In the case of framework (F3), we also have to perform a continuum limit.

4.1 The simplicial framework (F1)

A concrete/full “implementation” of framework (F1) was given in [40] (which improves the original implementation in [38]).

\footnote{In fact, for reasons explained in Sec. 4.5 below, in Framework (F2) it will not be the informal integral functional $\int_{\tilde{A}^\perp \times B} \cdots \exp(iS_{CS}(\tilde{A}^\perp, B))(DA^\perp \otimes DB)$ for which we will find a rigorous realization but another (closely related) informal integral functional.}
4.1.1 Outline

- The spaces $\hat{A}^\perp_{\text{mod}}$, $(A^\perp_c)_{\text{mod}}$, and $B_{\text{mod}}$ are chosen to be (finite-dimensional) simplicial analogues of the continuum spaces $\hat{A}^\perp$, $A^\perp_c$, and $B$.

- The two informal integral functionals (1.1) appearing above are defined as follows:

  First we introduce natural simplicial analogues $S_{CS}^{\text{disc}}(\hat{A}^\perp, B)$ and $S_{CS}^{\text{disc}}(A^\perp_c, B)$ of the functions $S_{CS}(\hat{A}^\perp, B)$ and $S_{CS}(A^\perp_c, B)$. This gives us two well-defined complex measures $\exp(i S_{CS}(\hat{A}^\perp, B)) D\hat{A}^\perp$ and $\exp(i S_{CS}(A^\perp_c, B)) DA^\perp_c \otimes DB$, where $D\hat{A}^\perp$, $DA^\perp_c$, and $DB$ are the (normalized) Lebesgue measures on the finite dimensional spaces $\hat{A}^\perp_{\text{mod}}$, $(A^\perp_c)_{\text{mod}}$, and $B_{\text{mod}}$ (equipped with a natural scalar product). By normalizing the complex measure $\exp(i S_{CS}(\hat{A}^\perp, B)) D\hat{A}^\perp$ we obtain a simplicial analogue of $d\mu_B^\perp$. The simplicial analogues of the two integral functionals (1.1) above can now be realized rigorously as regularized integrals in the same way as the functional $\int \cdots d\mu$ appearing in Sec. 3.3.1 above. (For the second integral functional we need to use the definition in Remark 3.25 in Sec. 3.3.1).

- The rigorous function $F_{\text{rig}}$ is obtained by constructing a natural “simplicial analogue” of the continuum expression Eq. (4.2a) above (cf. Sec. 4.5 below for more details and Sec. 3 in [40] for full details).

- Depending on the concrete implementation that we use a limit for the elimination of one regularization parameter may be necessary (cf. Sec. 4.5 below).

4.1.2 Comments

Plus points:

- Framework (F1) is very simple, even elementary. As a result rigorous proofs are easy to obtain, cf. Theorems 5.7 and 5.8 in [40] and Theorem 3.5 in [39] for a rigorous version and proof of the informal results in Sec. 3.3 and Sec. 3.4 above.

- So far (F1) is the only rigorous approach/framework which was carried out completely in the three special situations covered in Sec. 5.1 Sec. 5.3 and Sec. 5.4 above.

Drawbacks:

- If one wants to have a reasonable chance of obtaining a rigorous treatment for the case of general $L$ (i.e. of generalizing Theorem 3.5 in [39] and Theorems 5.7 and 5.8 in [40] to the case of general admissible ribbon links $L$, cf. Sec. 8.5) within (F1) it seems to be necessary to make the transition to the $BF$-theoretic setting, cf. Remark 8.38 in Sec. 8.5.2 above and Sec. 7 in [39].

- In view of the observations in Appendix D in [38] it seems very unlikely that a suitable “(approximative) unfixing the gauge”-procedure (cf. the last paragraph of Sec. 5.2 below) can be implemented within (F1).

85For every $p \in \{0, 1, 2\}$ and every real vector space $V$ the space of $V$-valued $p$-cochains $C^p(K,V)$ for some fixed finite smooth triangulation (or polyhedral cell decomposition) $K$ of $\Sigma$ is a simplicial analogue of the space of $V$-valued $p$-forms $\Omega^p(\Sigma,V)$. 
4.2 The continuum framework (F2)

This approach/framework was inspired by [6], which (to my knowledge) was the first paper to study, for non-Abelian $G$, the rigorous realization of $Z(M, L)$ for any manifold $M$, namely the non-compact manifold $M = \mathbb{R}^3$.

A concrete implementation of framework (F2) was given in [41] (using many ideas from [6]), cf. Sec. 4.5 below for more details.

4.2.1 Outline

- The spaces $\mathcal{A}_c^\perp$, $(\mathcal{A}_c^\perp)_\text{mod}$, and $\mathcal{B}_\text{mod}$ are chosen to be considerably larger than $\mathcal{A}_c^\perp$, $(\mathcal{A}_c^\perp)_\text{mod}$, and $\mathcal{B}$. They consist of distributional elements, for example, we take $\mathcal{A}_c^\perp := \mathcal{N}'$ (with the weak topology) where $\mathcal{N}$ is the nuclear Frechet space obtained from $\mathcal{A}_c^\perp$ by equipping that space with a suitable family of semi-norms, cf. Footnote 82 above.

- The framework of White Noise Analysis (WNA) is used for the rigorous construction of the integral functionals $\Psi$ and $\Phi^\perp_B$ (or, rather $\Phi^\perp_{B^r}$ where $B^r$ is a suitable regularization of $B \in \mathcal{B}_\text{mod}$). (We refer to [6, 7] for a summary of the constructions of WNA which are relevant here.) For example, $\Phi^\perp_{B^r}$ can be realized rigorously as a continuous linear functional $\Phi^\perp_{B^r} : (\mathcal{N}') \to \mathbb{C}$ where $(\mathcal{N}')$ is a suitable subspace of $L^2(\mathcal{N}', d\mu_{\text{can}})$, $d\mu_{\text{can}}$ being the canonical Gaussian Borel measure on $\mathcal{N}' = \mathcal{A}_c^\perp$. Regarding the implementation of $\Psi$ we refer the reader to Sec. 4.5 below (cf. also Footnote 84 above).

- We need to construct $F_{\text{rig}}$ such that for each fixed $A_c^\perp \in (\mathcal{A}_c^\perp)_\text{mod}$ and $B \in \mathcal{B}_\text{mod}$ the function $F_{\text{rig}}(\cdot, A_c^\perp, B)$ is an element of $(\mathcal{N})$. (This takes care of Eq. (4.3a) above.) Since we are working with regularized holonomies this is easy. What remains to be done is to prove Eq. (1.35) above, or, rather, the analogue of Eq. (1.35) obtained after introducing suitable regularizations of the two terms $1_{\mathcal{B}_\text{rig}^r}(B)$ and $\text{Det}_{\text{rig}}(B)$, cf. Sec. 4.5 below.

- Three limits for the elimination of the regularization parameters involved are necessary.

4.2.2 Comments

Plus points:

- Framework (F2) is closest to the informal treatment in Sec. 3. If the informal calculations sketched in Sec. 3.5 lead to the correct result for $Z(\Sigma \times S^1, L)$ for general admissible $L$ then it is almost certain that Framework (F2) will allow us to make these computations rigorous. (In particular, a “transition to the BF-theoretic setting” will not be necessary, unless it is necessary already for the informal treatment, cf. Remark 3.48 in Sec. 3.5.2 above).

- Framework (F2) provides the most direct way for resolving the “paradox” mentioned in (R2) above. (Also when using framework (F1) and (F3) this paradox can be resolved but the argument is then less direct, cf. Remark 4.1 in Sec. 4.6 below.)

Drawbacks:

86 Apart from the parameters $\epsilon$ and $s$ appearing in Eq. (4.2c) in the concrete implementation given in [41] there is a third parameter $n$ which is used for the regularization $B^r$ of $B$ mentioned above and the regularization of the two terms $1_{\mathcal{B}_\text{rig}^r}(B)$ and $\text{Det}_{\text{rig}}(B)$, cf. Sec. 4.5 below.

87 And in fact, in Sec. 3 we (indirectly) referred to Framework (F2) several times in order to justify some of the informal arguments appearing there, cf., e.g., Remark 3.27 in Sec. 3.3.2 and Eq. (3.119) in Sec. 3.5.2 above.)
• Framework (F2) is quite technical. Partly because of this, full proofs have not been given yet.

• It seems unlikely that “(approximative) unfixing the gauge”, cf. the last paragraph of Sec. 5.2 below, can be implemented within Framework (F2).

4.3 The “mixed” framework (F3)

Framework (F3) was briefly sketched in Sec. 3.10 in [40] but a concrete implementation has not been given yet.

The basic idea of Framework (F3) is to combine constructions from the simplicial and the continuum setting. By doing so we can combine some of the advantages of the simplicial setting (like finite dimensional realizations of the space $\mathcal{A}_{\text{mod}}, \mathcal{A}_{\text{c}}^{\perp}_{\text{mod}}$, and $\mathcal{B}_{\text{mod}}$) with some of the advantages of the continuum setting (like a very natural definition for the rigorous realization $F_{\text{rig}}(\mathcal{A}, \mathcal{A}_{\text{c}}^{\perp}, \mathcal{B})$ of the function (4.2b) above).

4.3.1 Outline

• The spaces $\mathcal{A}_{\text{mod}}^{\perp}, (\mathcal{A}_{\text{c}}^{\perp})_{\text{mod}}$, and $\mathcal{B}_{\text{mod}}$ are obtained by embedding suitable simplicial spaces into the spaces $\mathcal{A}_{\text{pw}}^{\perp}, (\mathcal{A}_{\text{c}}^{\perp})_{\text{pw}}$, and $\mathcal{B}_{\text{pw}}$ which are the extensions of $\mathcal{A}^{\perp}, \mathcal{A}_{\text{c}}^{\perp}$, and $\mathcal{B}$ containing piecewise smooth elements.

• The two informal integral functionals (4.1) are defined in an analogous way as in (F1). The advantage now is that the functions $S_{\text{CS}}(\mathcal{A}^{\perp}, \mathcal{B})$ and $S_{\text{CS}}(\mathcal{A}_{\text{c}}^{\perp}, \mathcal{B})$ for $\mathcal{A}^{\perp} \in \mathcal{A}_{\text{mod}}^{\perp}, \mathcal{A}_{\text{c}}^{\perp} \in (\mathcal{A}_{\text{c}}^{\perp})_{\text{mod}},$ and $\mathcal{B} \in \mathcal{B}_{\text{mod}}$ are given canonically.

• The definition of $F_{\text{rig}}(\mathcal{A}^{\perp}, \mathcal{A}_{\text{c}}^{\perp}, \mathcal{B})$ is as follows: the last two factors on the RHS of Eq. (4.2b) can be defined in a canonical way. For the factor $\text{Det}_{\text{rig}}(\mathcal{B})$ on the RHS of Eq. (4.2b) we have several options: we can use again the “continuum” definition given by Eqs (2.56) in Sec. 2.3.2 or we use the simplicial definition mentioned in Remark 3.15 above. The realization of $1_{\mathcal{B}_{\text{rig}}}$ requires a suitable regularization.

• The regularization parameters have to be eliminated using suitable limits. Moreover, we also have to perform a continuum limit.

4.3.2 Comments

Plus points:

• Framework (F3) is not as simple as the simplicial framework (F1) but still quite simple and not very technical.

• I consider the chances that framework (F3) will allow a successful treatment of general $L$ to be fairly good (but not as good as when working with framework (F2)). In particular, for framework (F3) a “transition to the BF-theoretic setting” will not be necessary, unless it is necessary already for the informal treatment, cf. Remark 3.48 in Sec. 3.5.2 above.

• In framework (F3) the chances for a successful implementation of a suitable “(approximative) unfixing the gauge” procedure (cf. the last paragraph of Sec. 5.2 below) seem to be fairly good.

Drawbacks:
The basic idea in (F3) of combining/mixing constructions from the simplicial and the continuum setting makes (F3) less natural/elegant than a purely simplicial framework or a pure continuum framework.

As mentioned above, a continuum limit is necessary in (F3).

### 4.4 Comparison of the three frameworks

| Framework | Simplicity | Continuum limit necessary? | Concrete implementation given? / Carried out for all 3 special cases? | Chances for a successful treatment of general $L$ | Chances for a successful implementation of “unfixing of the gauge” |
|-----------|------------|---------------------------|---------------------------------------------------------------------|-----------------------------------------------|---------------------------------------------------|
| (F1)      | very simple | no                        | yes / yes                                                           | negligible                                    | negligible                                        |
| (F2)      | quite technical | no                     | yes / no                                                           | very good                                    | low                                              |
| (F3)      | rather simple | yes                     | no / no                                                            | fair                                         | fair                                             |

**Comments:**

- The “three special cases” referred to in the fourth column are those considered in Sec. 3.1, Sec. 3.3, and Sec. 3.4 above.

- “Chances for a successful treatment of general $L$”: This refers to the chances of being able to find a rigorous derivation of Eq. (3.129) above for general (strictly admissible) $L$ within the original framework, i.e. without having to make a serious modification of the approach/framework (provided that Eq. (3.129) can be derived on an informal level). For framework (F1) these chances seem to be zero unless we make the transition to the BF-theoretic setting, cf. Sec. 4.1 above.

- For “unfixing of the gauge” see the last paragraph of Sec. 5.2 below.

### 4.5 Some remarks regarding the concrete implementation of (F1), (F2), and (F3)

In [40] and [41] we have given concrete implementations of the frameworks (F1) and (F2), respectively. Here are some brief remarks regarding these implementations (plus a remark regarding the implementation of (F3)):

(F1) We first fix a (sufficiently fine) finite (polyhedral) cell decomposition $K$ of $\Sigma$. For technical reasons we then construct from $K$ another cell decomposition $qK$ which is finer than $K$ but coarser than the barycentric subdivision $bK$ of $K$. Using $qK$ we then obtain simplicial analogues $\tilde{A}^\perp(qK)$, $A_+^\perp(qK)$, and $B(qK)$ of the three spaces $\tilde{A}^\perp$, $A_+^\perp$, and $B$. The spaces $\tilde{A}^\perp_{mod}$, $A_+^\perp_{mod}$, and $B_{mod}$ are then defined as suitable subspaces of $\tilde{A}^\perp(qK)$, $A_+^\perp(qK)$, and $B(qK)$ (for reasons explained in Sec. 3 in [40]). The definitions of the expressions $S_{CS}(\tilde{A}^\perp, B)$ and $S_{CS}^\text{disc}(A_+^\perp, B)$ mentioned in Sec. 4.1.1 above are very natural. The definition of the first three of the four factors appearing on the RHS of Eq. (4.2a) are also very natural.\(^{88}\) The simplicial analogues for the terms $\text{Hol}_{R_i}(A)$ appearing in the fourth

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\(^{88}\)The simplicial analogue for the first factor is obvious. As a simplicial analogue of $\text{Det}(B)$ we use $\text{Det}^\text{disc}(B)$, mentioned in Remark 3.15 in Sec. 3.2. The simplicial analogue of the factor $1_{\text{reg}}(B)$ is essentially straightforward. The only point worth mentioning is that, for technical reasons (cf. Secs 3.7 and 5.4 in [40]) the simplicial analogue of $1_{\text{reg}}(B)$ requires a suitable regularization.
factor in Eq. (4.2a) used in [40] are also natural but in contrast to the implementation of \( \text{Hol}_{\mathcal{R}}(A) \) in frameworks (F2) and (F3) definitely not canonical. In fact, among a number of possible definitions of \( \text{Hol}_{\mathcal{R}}(A) \) which would all be natural one has to choose the one definition that gives rise to the correct values for \( Z_{\text{rig}}^{l,g,f}(\Sigma \times S^1, L) \). (This is an illustration of the “instability” phenomenon which we discuss in Sec. 4.6 below.) Finally, for reasons explained in Sec. 3.11 in [40] we need to apply a suitable regularization (cf. (M2) in Sec. 3.11 in [40]).

(F2) In Sec. 4.2.1 we already explained roughly how the space \( \mathcal{A}^\bot_{\text{mod}} \) and the functional(s) \( \Phi^\bot_B \) are defined in framework (F2).

The definition/realization of the functional \( \Psi \) is somewhat trickier. For reasons explained now \( \Psi \) will not be constructed as a rigorous realization of the informal integral functional \( \int_{\mathcal{A}^\bot_{\text{mod}} \times \mathcal{B}} \cdots \exp(iS_{CS}(A^\bot_c, B))(DA^\bot_c \otimes DB) \) but of a closely related (informal) integral functional. Recall from Sec. 3.5.2 above that in the special case considered in Sec. 3.4 we have Eq. (3.127) with \( F^L_\gamma(A^\bot_c, B) \) as in Eq. (3.126) (provided that we use the “canonical” choice for the family of Dirac families \( (\delta^\gamma_c)_{\gamma < \gamma_0} \) as in Remark 3.29 above) and that we expect Eq. (3.127) to hold also for general \( L \). If this is indeed the case then the expression inside the \( \int_{\mathcal{A}^\bot_{\text{mod}} \times \mathcal{B}} \cdots \exp(iS_{CS}(A^\bot_c, B))(DA^\bot_c \otimes DB) \)-integral on the RHS of Eq. (3.39a) only depends on the \( A_{\text{coex}} \)-component of \( A^\bot_c = A_{\text{closed}} + A_{\text{coex}} \), which means that the integration over the \( A_{\text{closed}} \)-component is trivial and can be carried out right away. After doing so we are left with an (informal) integral functional of the form \( \int_{A_{\text{coex}} \times \mathcal{B}'} \cdots \exp(iS_{CS}(A_{\text{coex}}, B'))(DA_{\text{coex}} \otimes DB') \). For technical reasons we also make use of the decomposition \( \mathcal{B} = \mathcal{B}' \oplus \mathcal{B}_c \) with \( \mathcal{B}' \) and \( \mathcal{B}_c \), as in Remark 3.29 above. The rigorous functional \( \Psi \) mentioned above is constructed as a rigorous realization of the (informal) integral functional \( \int_{A_{\text{coex}} \times \mathcal{B}'} \cdots \exp(iS_{CS}(A_{\text{coex}}, B'))(DA_{\text{coex}} \otimes DB') \) using the framework of WNA.

Finally, note that for making sure that Eq. (4.35) holds we need to regularize the two factors \( \text{Det}_{\text{rig}}(B) \) (B) appearing in Eq. (4.2a) above in a suitable way. In Appendix B in [11] we give a concrete suggestion for such regularizations, which are, however, not very elegant.

(F3) A concrete implementation of (F3) has not been written up explicitly yet, but due to the observation above that almost all constructions in (F3) are “canonical” it should be straightforward to do so.

4.6 Some comments on the “instability” of \( Z^{l,g,f}(\Sigma \times S^1, L) \)

The informal torus gauge fixed CS path integral \( Z^{l,g,f}(\Sigma \times S^1, L) \) shows a certain degrees of “instability” in the sense that the explicit values of certain candidates for a rigorous realization \( Z_{\text{rig}}^{l,g,f}(\Sigma \times S^1, L) \) of \( Z^{l,g,f}(\Sigma \times S^1, L) \) depend in a delicate way on the details of this realization. This is most obvious when using the simplicial framework (F1) for obtaining \( Z_{\text{rig}}^{l,g,f}(\Sigma \times S^1, L) \). \( Z_{\text{rig}}^{l,g,f}(\Sigma \times S^1, L) \) can then be considered as a kind of “lattice regularization” of the informal expression \( Z^{l,g,f}(\Sigma \times S^1, L) \). Usually, when one works with a lattice regularization in Quantum Field Theory one has to perform a suitable continuum limit. As we saw above, in the simplicial framework (F1) such a continuum limit is actually not necessary, which is a great advantage of the simplicial approach. (Of course, if we want to apply a continuum limit anyway, this is still possible but the limit will turn out to be trivial.) On the other hand there is a price to pay: in

\[^{69}\text{They rely on choosing a cell decomposition of } \Sigma \text{ which is not in the spirit of a continuum approach.}\]
contrast to the standard situation in QFT where the continuum limit is usually independent of the lattice regularization the value of $Z^t,g,f(\Sigma \times S^1, L)$ will depend on the details of the lattice regularization and even if we choose to apply a continuum limit anyway this dependence on the details will not disappear. As a result, only for a distinguished subclass of lattice regularizations the expression $Z^t,g,f(\Sigma \times S^1, L)$ will have the correct value.

In the following remark we give one possible (but somewhat speculative) interpretation of the instability phenomenon mentioned above.

**Remark 4.1** Suppose that

1. Eq. (3.129) holds for general strictly admissible $L$, i.e. the informal evaluation of $Z^t,g,f(\Sigma \times S^1, L)$ leads to the correct result.

2. Using framework (F3) above (or a suitable variant of (F3)) it is possible to obtain a rigorous realization $Z_{rig}^t,g,f(\Sigma \times S^1, L)$ of $Z^t,g,f(\Sigma \times S^1, L)$.

3. The rigorous framework which we use allows “(approximative) unfixing of the gauge” (cf. the last paragraph of Sec. 5.2 below) and therefore allows us to relate the rigorous expression $Z_{rig}^t,g,f(\Sigma \times S^1, L)$ to (the informal expression) $Z(\Sigma \times S^1, L)$ in a suitable way.

If all this is the case then both $Z^t,g,f(\Sigma \times S^1, L)$ and $Z(\Sigma \times S^1, L)$ can be considered as idealized (informal) continuum limits of the rigorously defined expression $Z_{rig}^t,g,f(\Sigma \times S^1, L)$. In particular, it would then be natural to consider $Z_{rig}^t,g,f(\Sigma \times S^1, L)$ as “primary” and $Z^t,g,f(\Sigma \times S^1, L)$ as “secondary”. Now observe that it is quite possible that the definition of $Z_{rig}^t,g,f(\Sigma \times S^1, L)$ will involve certain technical features (like, e.g., a distinguished class of lattice “regularizations” which will no longer be visible after the (idealized) continuum limit has been carried out. The “instability phenomenon” mentioned above is created “artificially” by changing one’s mind about what is primary and what is secondary. It is only when one considers $Z^t,g,f(\Sigma \times S^1, L)$ (or $Z(\Sigma \times S^1, L)$) as “primary” and $Z_{rig}^t,g,f(\Sigma \times S^1, L)$ as “secondary” (namely as a kind of regularization) that one feels the need to explain why $Z_{rig}^t,g,f(\Sigma \times S^1, L)$ involves a distinguished subclass of lattice regularizations.

5 Outlook

5.1 Generalization to the case where $M$ is the total space of a non-trivial $S^1$-bundle (or a more general Seifert fibered space)

As mentioned in Sec. 1 the results in [17] on the CS path integral on manifolds of the form $M = \Sigma \times S^1$ were generalized in [19] to non-trivial $S^1$-bundle spaces $M$ and in [20] to Seifert

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90 Cf. the discussion regarding the possible definitions of $\text{Hol}_{Ri}(A)$ in the concrete implementation of framework (F1) sketched in Sec. 4.2 for a concrete example.

91 Observe that from what we said in Sec. 5.3 there are good chances that Eq. (3.129) in Sec. 5.3 above holds for general strictly admissible $L$. If so then the rigorous implementation within framework (F2) above is almost certainly possible, and there are fair chances that the rigorous implementation is also possible within Framework (F3). Finally, there seem to be fair chances that “(approximative) unfixing of the gauge” is possible within Framework (F3).

92 In some sense the non-gauge fixed CS theory would be an effective theory. In contrast to what is normally the case with effective theories this effective theory would have a rather special property: The expression $Z(\Sigma \times S^1, L)$ would not just be an approximation of the “primary”/“real” expression $Z_{rig}^t,g,f(\Sigma \times S^1, L)$ but the informal perturbative evaluation of $Z(\Sigma \times S^1, L)$ would lead exactly to the same values as the evaluation of $Z_{rig}^t,g,f(\Sigma \times S^1, L)$.

93 If we consider $Z_{rig}^t,g,f(\Sigma \times S^1, L)$ to be primary the word “regularization” is actually not really appropriate anymore.
fibered spaces $M$. It is natural to try to combine the ideas/methods in the present paper with those in \[19, 20\] and to try to extend the results in \[19, 20\] to general links $L$ in the aforementioned 3-manifolds $M$. (Recall from the introduction that the only links $L$ considered in \[19, 20\] are fiber links.)

For the applications to the open problems (OP1) and (OP3) mentioned in Sec. 5.2 below it is sufficient to consider the special case where $M = S^3$ (which can be considered as a non-trivial $S^3$-bundle via the Hopf fibration). As a first step one should therefore study whether by combining the ideas/methods in \[19\] with those in Secs 2–3 of the present paper one can “define” and evaluate a suitable torus gauge fixed CS path integral $Z^{l,g,f}(S^3, L)$ explicitly for general colored links $L$ in $S^3$ (such that the explicit value of $Z^{l,g,f}(S^3, L)$ coincides with $RT(S^3, L)$) and whether one can obtain a rigorous realization $Z^{l,g,f}_{\text{rig}}(S^3, L)$ of $Z^{l,g,f}(S^3, L)$ by adapting the frameworks (F1)–(F3) in Sec. 4 above in a suitable way (or by using another framework).

If this can be done successfully the next step would be to introduce/define $Z^{l,g,f}(M, L)$ and $Z^{l,g,f}_{\text{rig}}(M, L)$ for general colored links $L$ in all those Seifert fibered spaces $M$ considered in \[20\].

### 5.2 Potential applications to Quantum Topology

Several open conjectures in Quantum Topology can be “proven” on an informal level by assuming the equivalence between $RT(M, L)$ and $Z(M, L)$ and by applying informal path integral methods for the perturbative evaluation of $Z(M, L)$. This is, e.g., the case for the following open problems (OP1), (OP2), and (OP3):

- **(OP1)** As is shown in \[31, 12, 10, 11, 22, 8, 75\] the informal CS path integral $Z^{l,g,f}(S^3, L)$ in the Lorenz gauge can be evaluated on a perturbative level for general colored, framed links $L$ in $S^3$. More precisely, $(L)^{l,g,f} = Z^{l,g,f}(S^3, L)/Z^{l,g,f}(S^3)$ can be expanded as an asymptotic series of powers of $1/k$ and the coefficients in this series involve complicated analytic expressions called “configuration space integrals”. In view of the expected equivalence between $Z(S^3, L)$ and $RT(S^3, L)$ one therefore arrives at the conjecture that these configuration space integrals also appear (with the same coefficients as predicted by \[31, 12, 10, 11, 22, 8, 75\]) when expanding $RT_{\text{norm}}(S^3, L)$ as an asymptotic series of powers of $1/k$. To my knowledge, to date this conjecture has been proven only up to order 6, cf. Sec. 7 in \[58\] and Sec. 3.2 in \[14\].

- **(OP2)** By using the equivalence of $RT(M)$ and the original (= non-gauge fixed) CS path integral $Z(M)$ and by applying standard techniques from asymptotic analysis (including the stationary phase method) to the (informal) evaluation of $Z(M)$ as $k \to \infty$ one arrives at the so-called “Perturbative Expansion Conjecture”, which relates $RT(M)$ to geometric/topological concepts like moduli spaces, Reidemeister torsion, spectral flow, and the Chern-Simons invariant, cf. \[9\].

- **(OP3)** The so-called “Volume Conjecture”, which relates the colored Jones polynomial of a (hyperbolic) knot $K$ in $S^3$ to the hyperbolic volume $vol_{hyp}(S^3 \setminus K)$ of the knot complement $S^3 \setminus K$, is one of the most important open problems in Knot Theory. So far, it has only been verified for a small number of special knots $K$. There is, however, a very promising path integral approach (cf. \[33, 80\]) which goes a long way towards giving an informal “proof” of the volume conjecture for general $K$ by using arguments involving the CS path integral for $G = SU(2)$ and $M = S^3$ on the one hand, the CS path integral for $G = SL(2)$ and $M = S^3 \setminus K$ on the other hand, and suitable analytic continuation arguments relating the two types of CS path integrals.

\[\text{In the case of (OP3) the informal approach of \[33, 80\] does not quite give a full informal “proof” but goes a long way towards such a proof.}\]
If it were possible to obtain rigorous realizations $Z_{\text{rig}}^{L\cdot\text{g.f.}}(S^3, L)$, $Z_{\text{rig}}(M)$, and $Z_{\text{rig}}(S^3, L)$ of $Z^{L\cdot\text{g.f.}}(S^3, L)$, $Z(M)$, and $Z(S^3, L)$ then one could hope to be able to turn the informal path integral “proofs” mentioned above into rigorous proofs.

Now, as we have seen in Sec. 3 and Sec. 4 above, for the manifolds of the form $M = \Sigma \times S^1$ there are good chances that the informal evaluation of the torus gauge fixed CS path integral $Z^{t\cdot\text{g.f.}}(M, L)$ (given by the RHS of Eq. (3.20) or Eq. (3.39a) above) leads to the same values as $RT(M, L)$ and that one can obtain a rigorous realization $Z_{\text{rig}}^{t\cdot\text{g.f.}}(M, L)$ of $Z^{t\cdot\text{g.f.}}(M, L)$. As we explained in Sec. 5.1 above, it is probably possible to generalize our approach to the case where $M$ is a non-trivial $S^1$-bundle, which would lead, in particular, to a rigorous realization $Z_{\text{rig}}^{t\cdot\text{g.f.}}(S^3, L)$ of the suitably “defined” informal expression $Z^{t\cdot\text{g.f.}}(S^3, L)$. In a future version of the present paper I plan to include an additional part of the appendix where I sketch the possible implementation of a suitable “(approximative) unfixing of the gauge”-procedure and an “(approximative) changing the gauge”-procedure and how this could be exploited for relating $Z_{\text{rig}}^{t\cdot\text{g.f.}}(S^3, L)$ directly to the expressions appearing in the (informal) perturbative evaluation of $Z(S^3, L)$ and $Z^{L\cdot\text{g.f.}}(S^3, L)$.

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\section{Summary of Lie theoretic and quantum algebraic notation}

In the present part of the Appendix we list the Lie theoretic and quantum algebraic notation used in the paper and we also recall some useful identities.

\subsection{Lie theoretic notation}

Recall that in Sec. 2 we fixed a simple, simply-connected compact Lie group $G$ with Lie algebra $\mathfrak{g}$ and a maximal torus $T$ of $G$ with Lie algebra $\mathfrak{t}$. We set

- $G_{\text{reg}} := \{ g \in G \mid g \text{ is regular} \}$, cf. Remark A.1 below,
- $\mathfrak{g}_{\text{reg}} := \exp^{-1}(G_{\text{reg}})$,
- $T_{\text{reg}} := T \cap G_{\text{reg}}$,
- $\mathfrak{t}_{\text{reg}} := \exp^{-1}(T_{\text{reg}}) = \mathfrak{t} \cap \mathfrak{g}_{\text{reg}}$.

\begin{remark}
(i) An element $g$ of $G$ is called “regular” iff it is contained in exactly one maximal torus of $G$, cf. Sec. 3 in Chap. IV in [23].
(ii) The connected components of $\mathfrak{t}_{\text{reg}}$ are called “Weyl alcoves”.
\end{remark}
Using the scalar product \( \langle \cdot, \cdot \rangle \) on \( g \) which we fixed in Sec. 2 above we now make the obvious identification \( t \cong t^* \).

- \( R \subset t^* \) denotes the set of real roots associated to \((g, t)\).
- \( \tilde{R} \) denotes the set of real coroots, i.e. \( \tilde{R} := \{ \tilde{\alpha} \mid \alpha \in R \} \subset t \) where \( \tilde{\alpha} := \frac{2\alpha}{(\alpha, \alpha)} \).
- \( \Gamma \subset t \) denotes the lattice generated by the set of real coroots.
- \( \Lambda \subset t^* \) denotes the real weight lattice associated to \((g, t)\), i.e. the lattice in \( t \) which is dual to \( \Gamma \).
- \( W \) denotes the Weyl group associated to \((g, t)\). For \( \tau \in W \) we denote by \((-1)^\tau\) the determinant of \( \tau \in \text{Aut}(t) \).
- \( W_{\text{aff}} \) denotes the affine Weyl group associated to \((g, t)\), i.e. the group of isometries of \( t \cong t^* \) generated by the orthogonal reflections on the hyperplanes \( H_{\alpha, k}, \alpha \in R, k \in \mathbb{Z} \), where \( H_{\alpha, k} := \alpha - 1(k) \).
- \( I := \ker(\exp|_t) \).

Remark A.2

(i) From the assumption that \( G \) is simply-connected it follows that \( \Gamma = I \) (cf. Theorem 7.1 in Chap. V in [23]).

(ii) \( W_{\text{aff}} \) is generated by \( W \) and the translations \( T_x : t \ni b \mapsto b + x \in t \), \( x \in \Gamma = I \). In fact, \( W_{\text{aff}} \) is the semi-direct product of \( W \) and the group \( \{ T_x \mid x \in \Gamma \} \cong \Gamma \), so we can make the identification \( W_{\text{aff}} \cong W \times \Gamma \) (cf. Proposition 7.9 in Chap. V in [23]). For \( \tau = (\sigma, x) \in W_{\text{aff}} \cong W \times \Gamma \) we write \((-1)^\tau\) instead of \((-1)^{\tau_1}\).

(iii) We have \( t_{\text{reg}} = t \setminus \bigcup_{\alpha \in R, k \in \mathbb{Z}} H_{\alpha, k} \) where \( H_{\alpha, k}, \alpha \in R, k \in \mathbb{Z} \) is as above.

(iv) Every \( \tau \in W_{\text{aff}} \) leaves \( t_{\text{reg}} \) invariant and transfers each Weyl alcove into another Weyl alcove. Accordingly, \( W_{\text{aff}} \) acts on the set of Weyl alcoves. One can show that this action is free and transitive (cf. Proposition 7.10 in Chap. V in [23]).

(v) \( x, y \in \Gamma : \langle x, y \rangle \in \mathbb{Z} \), and \( \forall x, y \in \Gamma : \langle x, x \rangle \in 2\mathbb{Z} \). In order to see this it is enough to show that

\[
\langle \tilde{\alpha}, \tilde{\beta} \rangle \in \mathbb{Z} \quad \text{for all coroots } \tilde{\alpha}, \tilde{\beta} \tag{A.1}
\]

According to the general theory of semi-simple Lie algebras we have \( 2\frac{\langle \tilde{\alpha}, \tilde{\beta} \rangle}{(\alpha, \alpha)} \in \mathbb{Z} \). Moreover, there are at most two different (co)roots lengths and the quotient between the square lengths of the long and short coroots is either 1, 2, or 3. Since the normalization of \( \langle \cdot, \cdot \rangle \) was chosen such that \( \langle \tilde{\alpha}, \tilde{\alpha} \rangle = 2 \) holds if \( \tilde{\alpha} \) is a short coroot we therefore have \( \langle \tilde{\alpha}, \tilde{\alpha} \rangle/2 \in \{1, 2, 3\} \) and (A.1) follows.

Let us now also fix a Weyl chamber \( C \).

- \( R_+ \) denotes the set of positive (real) roots associated to \((g, t)\) and \( C \).
- \( \Lambda_+ \) denotes the set of dominant (real) weights associated to \((g, t)\) and \( C \).
- \( \rho \) denotes the half-sum of the positive (real) roots.
- \( \theta \) denotes the unique long (real) root in \( C \).
- We set \( c_g := 1 + (\theta, \rho) \). (Note that \( c_g \) is the dual Coxeter number of \( g \).)
- The fundamental Weyl alcove is the unique Weyl alcove $P$ which is contained in the Weyl chamber $C$ fixed above and which has $0 \in t$ on its boundary. $P$ is given explicitly by

$$P = \{b \in C \mid \langle b, \theta \rangle < 1\}.$$  \hspace{1cm} (A.2)

- We have

$$\det(1_t - \exp(\text{ad}(b))|_t) = \prod_{\alpha \in \mathcal{R}^+} 4 \sin^2(\pi \alpha(b)) \quad b \in t$$  \hspace{1cm} (A.3)

In Sec. 3.2 above we introduced the “square root” $\det^{1/2}(1_t - \exp(\text{ad}(\cdot))|_t) : t \to \mathbb{R}$ of $\det(1_t - \exp(\text{ad}(\cdot))|_t) : t \to \mathbb{R}$ by setting

$$\det^{1/2}(1_t - \exp(\text{ad}(b))|_t) := \prod_{\alpha \in \mathcal{R}^{+} \in C} 2 \sin(\pi \alpha(b)) \quad \forall b \in t$$  \hspace{1cm} (A.4)

- For $\lambda \in \Lambda_+$ let $\lambda^* \in \Lambda_+$ denote the weight conjugated to $\lambda$ and $\bar{\lambda} \in \Lambda_+$ the weight conjugated to $\lambda$ “after applying a shift by $\rho$”. More precisely, $\bar{\lambda}$ is given by $\bar{\lambda} + \rho = (\lambda + \rho)^*$.

For every $\lambda \in \Lambda_+$ we denote by $\rho_\lambda$ the (up to equivalence) unique irreducible, finite-dimensional, complex representation of $G$ with highest weight $\lambda$. For every $\mu \in \Lambda$ we will denote by $m_{\lambda}(\mu)$ the multiplicity of $\mu$ as a weight in $\rho_\lambda$. It will be convenient to introduce $\check{m}_{\lambda} : t \to \mathbb{Z}$ by

$$\check{m}_{\lambda}(b) = \begin{cases} m_{\lambda}(b) & \text{if } b \in \Lambda \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (A.5)

Instead of $\check{m}_{\lambda}$ we often write $m_{\lambda}$.

### A.2 Quantum algebraic notation

Recall that in Sec. 2 above we fixed $k \in \mathbb{N}$. We set

$$\Lambda^k_+ := \{\lambda \in \Lambda_+ \mid \langle \lambda + \rho, \theta \rangle < k\} = \{\lambda \in \Lambda_+ \mid \langle \lambda, \theta \rangle \leq k - c_\theta\}$$  \hspace{1cm} (A.6)

In Sec. 3 the following formula proved to be very useful

$$\Lambda^k_+ = \Lambda \cap (kP - \rho)$$  \hspace{1cm} (A.7)

where $P$ is the fundamental Weyl alcove, cf. Eq. (A.2) above.

**Remark A.3** In order to see that Eq. (A.7) holds observe first that according to Eq. (A.2) above we have

$$(kP - \rho) = \{kb - \rho \mid b \in C \text{ and } \langle b, \theta \rangle < 1\} = \{\bar{b} \in t \mid \bar{b} + \rho \in C \text{ and } \langle \bar{b} + \rho, \theta \rangle < k\}$$

and therefore

$$\Lambda \cap (kP - \rho) = \{\lambda \in \Lambda \mid \lambda + \rho \in C \text{ and } \langle \lambda + \rho, \theta \rangle < k\} \text{ (s) } = \{\lambda \in \Lambda \cap \overline{C} \mid \langle \lambda + \rho, \theta \rangle < k\} = \Lambda^k_+$$

Here in step (s) follows because for each $\lambda \in \Lambda$, $\lambda + \rho$ is in the open Weyl chamber $C$ iff $\lambda$ is in the closure $\overline{C}$ (cf. the last remark in Sec. V.4 in [19]).
Let $C$ and $S$ be the $\Lambda_+^k \times \Lambda_+^k$ matrices with complex entries given by

$$C_{\lambda\mu} := \delta_{\lambda\mu}, \quad (A.8a)$$

$$S_{\lambda\mu} := \frac{i^{\# R_+}}{k^{\dim(t)/2} |\Lambda/\Gamma|^{1/2}} \sum_{\tau \in W} (-1)^{\tau} e^{-\frac{2\pi}{k} \langle \lambda + \rho, \tau \cdot (\mu + \rho) \rangle} \quad (A.8b)$$

for all $\lambda, \mu \in \Lambda_+^k$ where $\# R_+$ is the number of elements of $R_+$ and $|\Lambda/\Gamma|$ is the index of $\Gamma$ in $\Lambda$. (Note that according to Proposition 7.16 in Chap. V in [23] $|\Lambda/\Gamma|$ coincides with the order of the center of $G$.) We have

$$S^2 = C \quad (A.9)$$

**Remark A.4.** The matrix $S$ introduced above coincides with the $S$-matrix of the modular category associated to $U_q(\mathfrak{g}_C)$ with $q := \exp(\frac{2\pi i}{k})$ (cf. Sec. 1.4 in Chap. II in [78]) or, equivalently, the $S$-matrix of the WZW model associated to $\Sigma$, $G$, and the level $k := k - e_\theta$. Moreover, $\Lambda_+^k$ is the set of dominant weights which are “integrable at level $k$”.

Let $\theta_\lambda$ and $d_\lambda$ for $\lambda \in \Lambda$ be given by

$$\theta_\lambda := e^{\frac{2\pi}{k} \langle \lambda, \lambda + 2\rho \rangle} \quad (A.10a)$$

$$d_\lambda := \frac{S_{\lambda 0}}{S_{00}} = \prod_{\alpha \in R_+} \frac{\sin(\frac{\pi}{k} \langle \lambda + \rho, \alpha \rangle)}{\sin(\frac{\pi}{k} \langle \rho, \alpha \rangle)} \quad (A.10b)$$

where in Eq. (A.10b) we have generalized the definition of $S_{\lambda 0}$ to the situation of general $\lambda, \mu \in \Lambda$ using again Eq. (A.8b) and where step $(\ast)$ follows from the Weyl denominator formula (cf., e.g., part (iii) in Theorem 1.7 in Chap. VI in [23]).

From Eq. (A.4) and Eq. (A.10b) we obtain for $\lambda \in \Lambda_+^k$

$$\det^{1/2}(1_t - \exp(ab((\lambda + \rho)/k))|_t) \sim S_{\lambda 0} \quad (A.11)$$

where $\sim$ denotes equality up to a multiplicative constant which is independent of $\lambda$. Moreover, according to Weyl’s character formula we have for all $\lambda, \mu \in \Lambda_+^k$:

$$\text{Tr}_{\rho_\lambda}(\exp((\mu + \rho)/k)) = \frac{S_{\mu \lambda}}{S_{\rho 0}} \quad (A.12)$$

For $\lambda, \mu, \nu \in \Lambda_+^k$ we set

$$N_{\lambda \mu \nu} := \sum_{\alpha \in \Lambda_+^k} S_{\alpha \lambda} S_{\alpha \mu} S_{\alpha \nu} \quad (A.13a)$$

$$N_{\mu \nu} := N_{\lambda \mu \nu} \quad (A.13b)$$

The numbers $N_{\lambda \mu \nu}$, $\lambda, \mu, \nu \in \Lambda_+^k$, are called “Verlinde numbers” and $N_{\lambda \mu}^\nu$, $\lambda, \mu, \nu \in \Lambda_+^k$, are the so-called “fusion coefficients”.

Observe that the following identity holds (called the “quantum Racah formula” in [71])

$$N_{\mu \nu}^\lambda = \sum_{\tau \in W_{\text{aff}}} (-1)^{\tau} m_\lambda(\mu - \tau \cdot \nu) \quad (A.14)$$

for all $\lambda \in \Lambda_+$, $\mu, \nu \in \Lambda$ where $\ast : W_{\text{aff}} \times t \to t$ is as in Eq. (3.83) above.

Finally, we set for all $\lambda \in \Lambda_+$, $\mu \in \Lambda$, and $p \in \mathbb{Z} \setminus \{0\}$ (cf. Remark 3.32 in Sec. 3.3.8)

$$c_{\lambda, p}^\mu := \sum_{\tau \in W_{\text{aff}}} (-1)^{\tau} m_\lambda(\frac{1}{p}(\mu - \tau \cdot p + \rho)) \quad (A.15)$$

95 For $r \in \mathbb{Q}$ we will write $\theta_\lambda^r$ instead of $e^{\frac{2\pi i}{k} \langle \lambda, \lambda + 2\rho \rangle}$. Note that this notation is somewhat dangerous since $\theta_{\lambda 1} = \theta_{\lambda 2}$ does, of course, in general not imply $\theta_{\lambda 1}^r = \theta_{\lambda 2}^r$.

96 The notation $N_{\lambda \mu \nu}$ is motivated by Eq. (3.14) in Sec. 3.3.
B Some technical details for Sec. 2

We will now fill in some of the technical details omitted in Sec. 2 above. Observe that Sec. B.1, Sec. B.3, and Sec. B.4 are rigorous while in Sec. B.2, Sec. B.5, and Sec. B.6 we use several informal arguments. (Secs B.2 and B.6 are new, Secs B.3–B.4 are based on [36] and Sec. B.5 is based both on [36] and on Appendix B of [38].)

B.1 Proof of Proposition 2.1

First proof: Let \( f, dg, t, \) and \( 1_t \) be as in Proposition 2.1. From Theorem 1.11 in Chap. IV in [23] we obtain

\[
\int_G f(g) dg \sim \int_T f(t) \det(1_t - \Ad(t^{-1})|_T) dt
\]

where \( dt \) is the normalized Haar measure on \( T \). Eq. (2.13) in Proposition 2.1 now follows from \( \Ad(\exp(b)^{-1}) = \Ad(\exp(-b)) = \exp(\ad(-b)), b \in t, \) and the relation \( \det(1_t - \exp(\ad(-b)))|_T = \det(1_t - \exp(\ad(b)))|_T, \) cf. Eq. (A.3) in Appendix A above.

Here is a second proof of Eq. (2.13), which is more complicated than the first one but which is a useful preparation for Appendix B.2 below.

Second proof: Recall the notation \( G_{reg} \) and \( t_{reg} \) introduced in Appendix A.1 above. Let \( P \subset t_{reg} \) be a fixed Weyl alcove. We will use the following three observations:

- The map \( q : P \times G/T \ni (b, \bar{g}) \mapsto \bar{g} \exp(b) \bar{g}^{-1} \in G_{reg} \) is a (well-defined) diffeomorphism. (Here we set, for each \( \bar{g} \in G/T \) and \( t \in T, \ bar{g}t \bar{g}^{-1} := gtg^{-1} \in G \) where \( g \) is an arbitrary element of \( G \) fulfilling \( gT = \bar{g}. \) In order to see this, observe first that, according to Prop. 7.11 in Sec. V.7 in [23] \( q \) is a (connected) smooth covering. Moreover, according to Lemma 7.3 in Sec. V.7 in [23] the assumption that \( G \) is simply-connected implies that also \( G_{reg} \) is simply-connected. These two observations imply that \( q \) is a trivial, connected, smooth covering, i.e. a diffeomorphism.

- From Proposition 1.8 in Chap. IV in [23] it follows\(^{70}\) that the pushforward measure \( (q^{-1})_*(dg) \) of \( dg \) under \( q^{-1} \) is given by

\[
(q^{-1})_*(dg) \sim \det(1_t - \exp(\ad(b))|_T) db \otimes d\bar{g}
\]

where \( d\bar{g} \) is the normalized left-invariant Borel measure on \( G/T \) and where “\( \otimes \)” denotes the product of two measures.

- \( G \setminus G_{reg} \) is a zero-set w.r.t. to the Haar measure \( dg \) on \( G \). This follows, e.g., from the observation in the proof of Lemma 7.3 in Sec. V.7 in [23] that \( G \setminus G_{reg} \) is contained in a submanifold \( N \) of \( G \) with codimension at least 3 (in the sense explained there).

From the three points above we now obtain for every conjugation-invariant continuous func-
tion \( f : G \rightarrow C \)

\[
\int_G f(g) dg = \int_{G_{\text{reg}}} f(g) dg
\]

\[
= \int (f \circ q) (q^{-1})_* (dg)
\]

\[
\sim \int_{P \times G/T} f(q(b, \tilde{g})) \det(1_t - \exp(\text{ad}(b))|_t) db \otimes d\tilde{g}
\]

\[
= \int_P \int_{G/T} f(\tilde{g} \exp(b) \tilde{g}^{-1}) \det(1_t - \exp(\text{ad}(b))|_t) d\tilde{g} db
\]

\[
= \int_P f(\exp(b)) \det(1_t - \exp(\text{ad}(b))|_t) db
\]  

(B.3)

Finally, observe that the multiplicative constant hidden in the symbol “\( \sim \)” is independent of the Weyl alcove \( P \) chosen above. Since \( t_{\text{reg}} \) is the disjoint union of all Weyl alcoves we obtain from Eq. (B.3) by a trivial “averaging” procedure

\[
\int_G f(g) dg \sim \int \sim 1_{t_{\text{reg}}}(b) f(\exp(b)) \det(1_t - \exp(\text{ad}(b))|_t) db
\]

where \( \int \sim \cdots db \) is as in Proposition 2.1 in Sec. 2.2.1 Since the complement of \( t_{\text{reg}} \) in \( t \) is a \( db \)-zero set we arrive (again) at Eq. (2.13).

### B.2 Derivation of Eq. (2.23)

Our approach for deriving Eq. (2.23) in Sec. 2.2.2 will be similar (but not totally analogous) to the derivation of Eq. (2.12) in Sec. 2.2.1 above.

**Remark B.1** Here are two aspects of the derivation of Eq. (2.23) which are not analogous to the derivation of Eq. (2.12) above and which are responsible for the differences between the RHS of Eq. (2.23) and the RHS of (the incorrect) Eq. (2.19) in Sec. 2.2.2:

1. While the standard left-action of \( G \) on \( G/T \) is transitive the left-action of \( G \Sigma = C^\infty(\Sigma,G) \) on \( C^\infty(\Sigma,G/T) \) defined in Sec. 2.2.2 above is not transitive if \( \Sigma \) is compact. In other words: The orbit space \( C^\infty(\Sigma,G/T)/G \Sigma \) is then non-trivial (cf. part (iii) of Proposition 2.6 in Sec. 2.3).

2. While the complement of \( t_{\text{reg}} \) in \( t \) is a zero set w.r.t. the measure \( db \) one cannot argue at an informal level that \( C^\infty(\Sigma,t) \setminus C^\infty(\Sigma,t_{\text{reg}}) \) is a zero-set w.r.t. the (informal) measure \( DB \). Accordingly, we cannot argue at an informal level that in Eq. (2.23) we can replace \( \int_{C^\infty(\Sigma,t_{\text{reg}})} \cdots DB \) by \( \int_{C^\infty(\Sigma,t)} \cdots DB \).

Let \( \chi : \mathcal{A} \rightarrow C \) be a (not necessarily \( G \)-invariant) function. Using the observation that \( \mathcal{A} = \mathcal{A}^\perp \oplus \mathcal{A}^\parallel \) (cf. Eq. (2.15) in Sec. 2.2.2) let us first rewrite the LHS of Eq. (2.23) as

\[
\int_{\mathcal{A}} \chi(A) DA = \int_{\mathcal{A}^\parallel} \chi_{\text{red}}(A^\parallel) DA^\parallel
\]  

(B.4)

where \( DA^\parallel \) is the (informal) Lebesgue measure on \( \mathcal{A}^\parallel \) and where we have introduced the function \( \chi_{\text{red}} : \mathcal{A}^\parallel \rightarrow C \) by

\[
\chi_{\text{red}}(A^\parallel) := \int_{\mathcal{A}^\perp} \chi(A^\perp + A^\parallel) DA^\perp \quad \forall A^\parallel \in \mathcal{A}^\parallel,
\]  

(B.5)

\(^98\) Observe that \( \text{codim}(t \setminus t_{\text{reg}}) = 1 \) so a generic smooth map \( \Sigma \rightarrow t \) will in general not remain inside \( t_{\text{reg}} \).
$DA^\perp$ being the informal Lebesgue measure on $A^\perp$. Now consider the $\mathcal{G}$-action on $A^\parallel$ given by
\[
(A^\parallel \cdot \Omega)(\sigma) = A^\parallel(\sigma) \cdot \Omega(\sigma) \quad \forall \sigma \in \Sigma
\]
where we have made the obvious identifications
\[
A^\parallel \cong C^\infty(\Sigma, A_{S^1}) \quad \text{(B.6a)}
\]
\[
\mathcal{G} \cong C^\infty(\Sigma, \mathcal{G}_{S^1}) \quad \text{(B.6b)}
\]
where the two spaces $C^\infty(\Sigma, A_{S^1})$ and $C^\infty(\Sigma, \mathcal{G}_{S^1})$ are defined in a totally analogous way as the space $C^\infty(S^1, \mathcal{A}_\Sigma)$ in Sec. 2.3 above.

**Observation B.2** If $\chi$ is $\mathcal{G}$-invariant then $\chi_{\text{red}}$ is $\mathcal{G}$-invariant as well.

"Proof": If $\chi : \mathcal{A} \to \mathbb{C}$ is $\mathcal{G}$-invariant we have, informally, for all $A^\parallel \in A^\parallel$ and $\Omega \in \mathcal{G}$
\[
\chi_{\text{red}}(A^\parallel \cdot \Omega) = \chi_{\text{red}}(\Omega^{-1}A^\parallel \Omega + (\Omega^{-1}d\Omega)^\parallel) = \int_{A^\perp} \chi(A^\parallel + \Omega^{-1}A^\parallel \Omega + (\Omega^{-1}d\Omega)^\parallel) DA^\perp
\]
\[
= \int_{A^\perp} \chi((A^\parallel + A^\parallel) \cdot \Omega) DA^\perp = \int_{A^\perp} \chi(A^\parallel + A^\parallel) DA^\perp = \chi_{\text{red}}(A^\parallel)
\]
\[
\text{(B.7)}
\]
where $(\Omega^{-1}d\Omega)^\perp$ and $(\Omega^{-1}d\Omega)^\parallel$ are the components of $\Omega^{-1}d\Omega \in A_{\Sigma} \subset A^\perp$ w.r.t. the decomposition (2.15) in Sec. 2.2 and where in step (*) we have used the change of variable\footnote{This change of variable argument was inspired by a similar argument used in [18], cf. Eqs. (6.5) and (6.6) in Sec. 6 of [18].} $A^\perp \to \Omega^{-1} \cdot A^\perp \cdot \Omega + (\Omega^{-1}d\Omega)^\perp$ and have taken into account that $DA^\perp$ is invariant under this affine transformation whose linear part has, informally, determinant 1. (Note that since $G$ is compact and connected we have $\det(\text{Ad}(g)) = 1$ for all $g \in G$.)

From now on we will assume that $\chi$ is $\mathcal{G}$-invariant. According to Observation B.2 above $\chi_{\text{red}}$ is then $\mathcal{G}$-invariant as well. In particular, $\chi_{\text{red}}$ is $\mathcal{G}$-invariant where we have set $\tilde{\mathcal{G}} := \{ \Omega \in \mathcal{G} \mid \forall \sigma \in \Sigma : \Omega(\sigma, 1) = 1 \}$. As in Sec. 2.2.1 it follows that there is a $\chi_{\text{red}} : C^\infty(\Sigma, G) \to \mathbb{C}$ such that
\[
\chi_{\text{red}} = \tilde{\chi}_{\text{red}} \circ \tilde{p}
\]
\[
\text{(B.8)}
\]
where $\tilde{p} : C^\infty(\Sigma, A_{S^1}) \to C^\infty(\Sigma, G)$ is given by $(\tilde{p}(A^\parallel))(\sigma) = p(A^\parallel(\sigma))$ for all $A^\parallel \in C^\infty(\Sigma, A_{S^1})$ and $\sigma \in \Sigma$. (Here $p$ is as in Sec. 2.2.1). Taking this into account we obtain the following analogue of Eq. (2.14a) in Sec. 2.2.1
\[
\int_{A^\parallel} \chi_{\text{red}}(A^\parallel) DA^\parallel = \int_{C^\infty(\Sigma, A_{S^1})} \chi_{\text{red}}(A^\parallel) DA^\parallel
\]
\[
= \int_{C^\infty(\Sigma, A_{S^1})} \tilde{\chi}_{\text{red}}(\tilde{p}(A^\parallel)) DA^\parallel \sim \int_{C^\infty(\Sigma, G)} \tilde{\chi}_{\text{red}}(\Omega) D\Omega \quad \text{(B.9)}
\]
where $D\Omega$ is the (informal) normalized Haar measure on $C^\infty(\Sigma, G) = \mathcal{G}_\Sigma$.

Since $\chi_{\text{red}}$ is not only $\tilde{\mathcal{G}}$-invariant but even $\mathcal{G}$-invariant we can conclude that the function $\tilde{\chi}_{\text{red}}$ is conjugation invariant, i.e. invariant under the $\mathcal{G}_\Sigma$-action on itself by conjugation. We will exploit this in a similar way as we exploited the conjugation invariance of the function $f$ appearing in Eq. (13.3) in Appendix B.1 above. Before we do this we need some preparations.
First observe that, by assumption, \( \dim(\Sigma) = 2 \), and since \( G \setminus G_{\text{reg}} \) is contained in a submanifold of \( G \) with codimension at least 3 (cf. Appendix B.1 above) a “generic” map \( \Omega : \Sigma \to G \) will remain inside \( G_{\text{reg}} \) so, informally, the set \( C^\infty(\Sigma,G) \setminus C^\infty(\Sigma,G_{\text{reg}}) \) can be considered as a zero-set w.r.t. \( D\Omega \).

Let \( q : P \times G/T \to G_{\text{reg}} \) be the diffeomorphism introduced in Appendix B.1 (where \( P \) is the Weyl alcove fixed there). Let \( \hat{q} : C^\infty(\Sigma,P) \times C^\infty(\Sigma,G/T) \to C^\infty(\Sigma,G_{\text{reg}}) \) be given by \( \hat{q}(B,\tilde{g})(\sigma) = q(B(\sigma),\tilde{g}(\sigma)) \) for all \( B \in C^\infty(\Sigma,P), \tilde{g} \in C^\infty(\Sigma,G/T) \), and \( \sigma \in \Sigma \). As \( q \) is a diffeomorphism we conclude that \( \hat{q} \) is a bijection. In analogy with Eq. (B.3) in Appendix B.1 above we now obtain, informally,

\[
\int_{C^\infty(\Sigma,G)} \bar{\chi}_{\text{red}}(\Omega) D\Omega = \int_{C^\infty(\Sigma,G_{\text{reg}})} \bar{\chi}_{\text{red}}(\Omega) D\Omega
\]

and therefore

\[
\int_{C^\infty(\Sigma,P) \times C^\infty(\Sigma,G/T)} (\bar{\chi}_{\text{red}} \circ \hat{q})(\hat{q}^{-1}) D\Omega
\]

\[
\sim \int_{C^\infty(\Sigma,P)} \int_{C^\infty(\Sigma,G/T)} \bar{\chi}_{\text{red}}(\hat{q}(B,\tilde{g})) \det(1_t - \exp(\text{ad}(B)_{|t})) D\tilde{g} DB
\]

where \( D\tilde{g} \) is the (informal) \( G_\Sigma \)-invariant measure on \( C^\infty(\Sigma,G/T) \) normalized such that every \( G_\Sigma \)-orbit has volume 1.

Similarly to the second part of Eq. (B.3) in Appendix B.1 above we obtain

\[
\int_{C^\infty(\Sigma,P)} \int_{C^\infty(\Sigma,G/T)} \bar{\chi}_{\text{red}}(\hat{q}(B,\tilde{g})) \det(1_t - \exp(\text{ad}(B)_{|t})) D\tilde{g} DB
\]

\[
= \int_{C^\infty(\Sigma,P)} \left[ \int_{C^\infty(\Sigma,G/T)} \bar{\chi}_{\text{red}}(\hat{\tilde{g}} \exp(B) \tilde{g}^{-1}) D\tilde{g} \right] \det(1_t - \exp(\text{ad}(B)_{|t})) DB
\]

\[
\overset{(*)}{=} \sum_{h \in C^\infty(\Sigma,G/T)/G_\Sigma} \int_{C^\infty(\Sigma,P)} \bar{\chi}_{\text{red}}(\hat{g}_h \exp(B) \tilde{g}_h^{-1}) \det(1_t - \exp(\text{ad}(B)_{|t})) DB
\]

where \( (\hat{g}_h)_{h \in C^\infty(\Sigma,G/T)/G_\Sigma}, \tilde{g}_h \in C^\infty(\Sigma,G/T), \) is an arbitrary system of representatives of \( C^\infty(\Sigma,G/T)/G_\Sigma \) and where, for \( t \in C^\infty(\Sigma,T) \) and \( \tilde{g} \in C^\infty(\Sigma,G/T) \), we have denoted by \( \tilde{g}t\tilde{g}^{-1} \) the element of \( C^\infty(\Sigma,G) \) given by \( (\tilde{g}t\tilde{g}^{-1})(\sigma) := \tilde{g}(\sigma) t(\sigma) \tilde{g}(\sigma)^{-1} \) for all \( \sigma \in \Sigma \).

Note that in step \((*)\) we used that, as a result of the conjugation invariance of \( \bar{\chi}_{\text{red}} \), for each fixed \( B \) the function \( C^\infty(\Sigma,G/T) \ni \tilde{g} \mapsto \bar{\chi}_{\text{red}}(\hat{g} \exp(B) \tilde{g}^{-1}) \in \mathbb{C} \) is constant on each \( G_\Sigma \)-orbit.

By combining Eq. (B.4), Eq. (B.9), Eq. (B.10a), Eq. (B.10b), and Eq. (B.10c) and by taking into account that

\[
\bar{\chi}_{\text{red}}(\hat{g}_h \exp(B) \tilde{g}_h^{-1}) = \bar{\chi}_{\text{red}}(\exp(\tilde{g}_h B \tilde{g}_h^{-1})) = \bar{\chi}_{\text{red}}(\hat{\tilde{g}}(\tilde{g}_h B \tilde{g}_h^{-1} dt)) = \chi_{\text{red}}(\tilde{g}_h B \tilde{g}_h^{-1} dt)
\]

we now obtain

\[
\int_A \chi(A) DA \sim \sum_{h \in C^\infty(\Sigma,G/T)/G_\Sigma} \int_{C^\infty(\Sigma,P)} \bar{\chi}_{\text{red}}(\hat{g}_h B \tilde{g}_h^{-1} dt) \det(1_t - \exp(\text{ad}(B)_{|t})) DB
\]
Observe that the multiplicative constant implicit in \( \sim \) is independent of the choice of the Weyl alcove \( P \). Since \( B_{reg} = C^\infty(\Sigma, t_{reg}) \) is the disjoint union of \( \{ C^\infty(\Sigma, P') \mid P' \text{ is a Weyl alcove of } t \} \), we now obtain from Eq. \( (B.11) \) by a trivial averaging procedure analogous to the one used in the last paragraph in Appendix B.1.

\[
\int_A \chi(A) DA \sim \sum_{h \in C^ \infty(\Sigma; G/T) / G_\Sigma} \int_{\tilde{g}_h} 1_{B_{reg}}(B) \chi_{red}(\tilde{g}_h B \tilde{g}_h^{-1}) dt \det \left( 1_t - \exp(\text{ad}(B)|t) \right) DB
\]

(B.12)

if \( \sim \cdots DB \) is a suitable (informal) “improper integral”, cf. Remark 2.7 and part (ii) of Remark 2.2 in Sec. 2.2 above. Combining Eq. \( (B.12) \) with Eq. \( (B.5) \) above we now arrive at Eq. \( (2.23) \).

### B.3 Derivation of Eq. \( (2.33) \)

Recall from Sec. 2.1.1 above that if \( \alpha \) or \( \beta \) is a 0-form we write \( \alpha \beta \) instead of \( \alpha \wedge \beta \).

Let \( \mathcal{A}_{\Sigma \setminus \{ \sigma_0 \}} := \Omega^1(\Sigma \setminus \{ \sigma_0 \}, g) \) and let \( C^\infty(S^1, \mathcal{A}_{\Sigma \setminus \{ \sigma_0 \}}) \) be defined in a completely analogous way as the space \( C^\infty(S^1, \mathcal{A}_\Sigma) \) in Sec. 2.3 above. For \( A^+ \in C^\infty(S^1, \mathcal{A}_{\Sigma \setminus \{ \sigma_0 \}}) \) and \( B \in C^\infty(\Sigma \setminus \{ \sigma_0 \}, t) \) let us set

\[
S'_{CS}(A^+ + B) := -k \pi \int_{S^1} \lim_{\epsilon \to 0} \left[ \int_{\Sigma \setminus B_\epsilon(\sigma_0)} \text{Tr}(A^+(t) \wedge (\partial/\partial t + \text{ad}(B)) \cdot A^+(t)) \right. \\
- 2 \text{Tr}(d(A^+(t))B) dt \quad (B.13)
\]

if the limit exists. (Above \( d \) is the differential for differential forms on \( \Sigma \).) Observe that if we consider \( A^+ \cong C^\infty(S^1, \mathcal{A}_\Sigma) \) as a subspace of \( C^\infty(S^1, \mathcal{A}_{\Sigma \setminus \{ \sigma_0 \}}) \) and \( B = C^\infty(\Sigma, t) \) as a subspace of \( C^\infty(\Sigma \setminus \{ \sigma_0 \}, t) \) in the obvious way then from Stokes’ Theorem it follows that\( S'_{CS}(A^+ + B) = S_{CS}(A^+ + B) \) it \( A^+ \in A^+ \) and \( B \in B \). Moreover, we also have

\[
S'_{CS}(A^+ \cdot \Omega_h B \Omega_h^{-1} dt) = S_{CS}(A^+ \cdot \Omega_h B \Omega_h^{-1} dt) \quad (B.14)
\]

where \( S_{CS}(A^+ \cdot \Omega_h B \Omega_h^{-1} dt) \) is given as in Sec. 2.3. On the other hand, in general we have

\[
S'_{CS}(A^+ \cdot \Omega_h + B) \neq S_{CS}(A^+ \cdot \Omega_h + B) \quad (B.15)
\]

where \( S_{CS}(A^+ \cdot \Omega_h + B) \) is given as in Sec. 2.3 and where “\( \cdot \)” refers to the obvious \( \mathcal{G}_{\Sigma \setminus \{ \sigma_0 \}} \) action on \( C^\infty(S^1, \mathcal{A}_{\Sigma \setminus \{ \sigma_0 \}}) \). In order to see this observe

\[
S'_{CS}(A^+ \cdot \Omega_h + B) = S_{CS}(A^+ \cdot \Omega_h + B) \\
= 2\pi k \lim_{\epsilon \to 0} \int_{\Sigma \setminus B_\epsilon(\sigma_0)} \left( \text{Tr}(d(A^+_c \cdot \Omega_h)B) - \text{Tr}(\left( A^+_c \cdot \Omega_h \right) \wedge dB) \right) \\
= 2\pi k \lim_{\epsilon \to 0} \int_{B_\epsilon(\sigma_0)} d\left( \text{Tr}(\left( A^+_c \cdot \Omega_h \right)B) \right) \quad (*) \\
= 2\pi k \left[ \lim_{\epsilon \to 0} \int_{\Sigma \setminus B_\epsilon(\sigma_0)} \text{Tr}(\Omega_h^{-1} A^+_c \Omega_h B) + \lim_{\epsilon \to 0} \int_{B_\epsilon(\sigma_0)} \text{Tr}(\Omega_h^{-1} dB) \right] \\
= 2\pi k \left[ 0 + \text{Tr}(\left( \lim_{\epsilon \to 0} \int_{\Sigma \setminus B_\epsilon(\sigma_0)} \Omega_h^{-1} dB \right) \right] = -2\pi k \left< n(h), B(\sigma_0) \right> \quad (B.15)
\]

where we have set \( A^+_c := \int \pi(A^+(t)) dt \in \mathcal{A}_{\Sigma \setminus \{ \sigma_0 \}} \) and where in step (*) in the last line we have used Stokes’ Theorem.\(^{100}\) (We also used Eq. \( (2.8) \) and Eq. \( (2.31) \).)
Above step (+) holds for all $B$ provided that the lift $\Omega_h$ of $\bar{\Omega}_{\Sigma}$ chosen in Sec. 2.3 does not oscillate vary too wildly around $\sigma_0$. (Such a choice of $\Omega_h$ is always possible.)

Eq. (2.33) now follows by combining Eq. (B.14) and Eq. (B.15) with (B.16)

$$S'_{CS}(A^\perp \cdot \Omega_h + B dt) = S'_{CS}(A^\perp + \Omega_h B \Omega_h^{-1} dt) \quad (\text{B.16})$$

**Proof of Eq. (B.16):** Let $B \in \mathcal{B} \subset C^\infty(\Sigma\setminus\{\sigma_0\}, t)$ and $A^\perp \in \mathcal{A} \cong C^\infty(S^1, \mathcal{A}_\Sigma) \subset C^\infty(S^1, \mathcal{A}_{\Sigma\setminus\{\sigma_0\}})$. Observe that

$$(A^\perp \cdot \Omega_h)(t) = A^\perp(t) \cdot \Omega_h \quad (\text{B.17})$$

where the ‘‘‘’ on the RHS refers to the obvious $\mathcal{G}_{\Sigma\setminus\{\sigma_0\}}$ action on $\mathcal{A}_{\Sigma\setminus\{\sigma_0\}}$. Taking this into account we obtain

$$S'_{CS}(A^\perp \cdot \Omega_h + B dt)$$

$$= -k\pi \lim_{\epsilon \to 0} \int_{S^1} \left[ \int_{\Sigma\setminus\mathcal{B}_\epsilon(\sigma_0)} \left[ \text{Tr}\left( (A^\perp(t) \cdot \Omega_h) \wedge (\partial/\partial t + \text{ad}(B)) \cdot (A^\perp(t) \cdot \Omega_h) \right) 
- 2 \text{Tr}(d(A^\perp(t) \cdot \Omega_h) B) \right] dt 
\right.

$$

$$\left. + \text{Tr}(d\Omega_h \Omega_h^{-1} \wedge \frac{\partial}{\partial t} A^\perp(t)) \right] dt$$

$$= S'_{CS}(A^\perp + \Omega_h B \Omega_h^{-1} dt) - k\pi \lim_{\epsilon \to 0} \int_{S^1} \frac{d}{dt} \left[ \int_{\Sigma\setminus\mathcal{B}_\epsilon(\sigma_0)} \text{Tr}(d\Omega_h \Omega_h^{-1} \wedge A^\perp(t)) \right] dt$$

$$= S'_{CS}(A^\perp + \Omega_h B \Omega_h^{-1} dt) - k\pi \lim_{\epsilon \to 0} [0] = S'_{CS}(A^\perp + \Omega_h B \Omega_h^{-1} dt)$$

Here step (*) follows – setting $\Omega := \Omega_h$ – from

$$\text{Tr}\left( (A^\perp(t) \cdot \Omega) \wedge (\partial/\partial t + \text{ad}(B)) \cdot (A^\perp(t) \cdot \Omega) \right)$$

$$= \text{Tr}(\Omega^{-1} A^\perp(t) \Omega \wedge (\partial/\partial t + \text{ad}(B)) \cdot \Omega^{-1} A^\perp(t) \Omega)$$

$$+ \{ \text{Tr}(\Omega^{-1} A^\perp(t) \Omega \wedge \text{ad}(B) \cdot \Omega^{-1} d\Omega) + \text{Tr}(\Omega^{-1} d\Omega \wedge \text{ad}(B) \cdot \Omega^{-1} \Omega^{-1} A^\perp(t) \Omega) \}$$

$$+ \{ \text{Tr}(\Omega^{-1} d\Omega \wedge \text{ad}(B) \cdot \Omega^{-1} d\Omega) 
\}$$

$$= \text{Tr}(A^\perp(t) \wedge (\partial/\partial t + \text{ad}(\Omega B \Omega^{-1})) \cdot A^\perp(t))$$

$$+ \{ 2 \text{Tr}(A^\perp(t) \wedge \text{ad}(\Omega B \Omega^{-1}) \cdot d\Omega \Omega^{-1}) + \text{Tr}(d\Omega \Omega^{-1} \wedge \frac{\partial}{\partial t} A^\perp(t)) \}$$

$$- 2 \text{Tr}(d\Omega \Omega^{-1} \wedge d\Omega B \Omega^{-1})$$

---

101 On the other hand step (+) holds even for wildly oscillating $\Omega_h$ if $B$ is locally constant around $\sigma_0$, which is the only case which will be relevant later, cf. Remark 2.7 in Appendix B.3 below.

102 As Eq. (B.16) is simpler than Eq. (2.33) the reader may wonder why in Sec. 2.3 we did not work with $S'_{CS}(\cdot)$ instead of $S_{CS}(\cdot)$. The reason for this is explained in Remark 2.7 in Sec. 2.3 above.
and

\[
\text{Tr}(d(A^\perp(t) \cdot \Omega)B) \\
= \text{Tr}(d(\Omega^{-1}A^\perp(t)\Omega)B) + \text{Tr}(d(\Omega^{-1}d\Omega))B) \\
= \text{Tr}((-\Omega^{-1}d\Omega^{-1}) \wedge A^\perp(t)\Omega B) + \text{Tr}(\Omega^{-1}d(A^\perp(t))\Omega B) - \text{Tr}(\Omega^{-1}A^\perp(t) \wedge d\Omega B) \\
+ \text{Tr}(d(\Omega^{-1}d\Omega))B) \\
= \text{Tr}(\Omega^{-1}d(A^\perp(t))\Omega B)) + \text{Tr}(A^\perp(t) \wedge (\text{ad}(\Omega B\Omega^{-1}) \cdot d\Omega \Omega^{-1})) \\
- \text{Tr}(\Omega^{-1}d\Omega \wedge \Omega^{-1}d\Omega)B) \\
= \text{Tr}(d(A^\perp(t))\Omega B\Omega^{-1}) + \text{Tr}(A^\perp(t) \wedge (\text{ad}(\Omega B\Omega^{-1}) \cdot d\Omega \Omega^{-1})) \\
- \text{Tr}(d\Omega \Omega^{-1} \wedge d\Omega B \Omega^{-1})
\]

\[\square\]

### B.4 Proof of Proposition 2.6

**Proof of part (i):** Let \( h, \bar{g}_h, \) and \( \Omega_h \) be as in Sec. 2.3 and set \( \bar{g} := \bar{g}_h \) and \( \Omega := \Omega_h. \) Let \( s \) be a (smooth) local section of the bundle \( \pi_G/T : G \to G/T \) such that \( \bar{g}(\sigma_0) \in \text{dom}(s) \) and let \( U \) be an open neighborhood of \( \sigma_0 \) fulfilling \( \text{Image}(\bar{g}_U) \subset \text{dom}(s). \) Then \( \Omega_0 := s \circ \bar{g}_U \in C^\infty(U, G) \) and there is a (unique) \( t \in C^\infty(U \setminus \{\sigma_0\}, T) \) such that \( \Omega|_{U \setminus \{\sigma_0\}} = (\Omega_0)|_{U \setminus \{\sigma_0\}} \cdot t. \) For sufficiently small \( \epsilon > 0 \) we have \( B_\epsilon(\sigma_0) \subset U \) and \( 103 \)

\[
\int_{\partial B_\epsilon(\sigma_0)} \pi_t(\Omega^{-1}d\Omega) = \int_{\partial B_\epsilon(\sigma_0)} \pi_t(\Omega_0^{-1}d\Omega) + \int_{\partial B_\epsilon(\sigma_0)} t^{-1}dt \quad (B.18)
\]

Since \( d\Omega_0 \) is bounded \( 104 \) on the (compact) set \( \overline{B_\epsilon(\sigma_0)} \) we have \( \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(\sigma_0)} \pi_t(\Omega_0^{-1}d\Omega) = 0. \) On the other hand, \( \int_{\partial B_\epsilon(\sigma_0)} t^{-1}dt \) is independent \( 105 \) of \( \epsilon. \) (This follows, e.g., from Stokes’ theorem and \( d(t^{-1}dt) = -t^{-1}dt \wedge t^{-1}dt = 0. \)) We conclude that \( \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(\sigma_0)} \pi_t(\Omega^{-1}d\Omega) \) exists. \[\square\]

**Proof of part (ii):** Let \( h \) and \( \Omega_h \) be as in Sec. 2.3. Moreover, let \( \Omega \) be an arbitrary element of \( G_\Sigma = C^\infty(\Sigma, G) \) and \( t \) an arbitrary element of \( C^\infty(\Sigma \setminus \{\sigma_0\}, T). \) Finally, let us set \( \Omega_h := \Omega_h t. \) The assertion then follows easily from

\[
\lim_{\epsilon \to 0} \int_{\partial B_\epsilon(\sigma_0)} \pi_t((\Omega_0^\perp)^{-1}d\Omega^\perp_0) \overset{(*)}{=} \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(\sigma_0)} \pi_t((\Omega_0^{-1}d(\Omega_0^\perp)) \overset{(**)}{=} \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(\sigma_0)} \pi_t(\Omega^{-1}d\Omega_h)
\]

Here step \((*)\) follows from a short computation taking into account that \( \Omega_0 \) is defined an all of \( \Sigma \) and \( d\Omega \) is therefore bounded and step \((**)\) follows from an argument analogous to the one in Eq. (11.18) above and by taking into account that \( \int_{\partial B_\epsilon(\sigma_0)} t^{-1}dt = -\int_{\Sigma \setminus B_\epsilon(\sigma_0)} t^{-1}dt \wedge t^{-1}dt = 0 \) by Stokes’ theorem.

\[\square\]

Let \( h \in C^\infty(\Sigma, G/T)/G_\Sigma. \) It is not difficult to see that \( h \) has a representative \( \bar{g}_h \in C^\infty(\Sigma, G/T) \) which is constant on a neighborhood \( U \) of \( \sigma_0 \) taking only the value \( T \in G/T \) there. Observe that every lift \( \Omega_h \in C^\infty(\Sigma \setminus \{\sigma_0\}, G) \) of \( \bar{g}_h|_{\Sigma \setminus \{\sigma_0\}} \) will then only take values in \( T \) on \( U \setminus \{\sigma_0\} \). We will call a \( \bar{g}_h \in C^\infty(\Sigma, G/T) \) and a \( \Omega_h \) with the properties just described “a standard representative of \( h \)” and “a standard lift associated to \( h \),” respectively.

---

103Recall that \( \pi_t : g \to t \) is the orthogonal projection w.r.t. the Ad-invariant scalar product \( \langle \cdot, \cdot \rangle \) on \( g. \) From this it follows that \( \pi_t(a) = a \) for all \( a \in g \) and \( t \in T. \)

104Here the notion “bounded” is defined in terms of the Riemannian metric \( g, \) cf. Footnote 106 below.

105Observe that in contrast to the situation in the proof of part (ii) below we cannot conclude \( \int_{\partial B_\epsilon(\sigma_0)} t^{-1}dt = 0 \) as \( t \) is only defined on \( U \setminus \{\sigma_0\} \) and not on all of \( \Sigma \setminus \{\sigma_0\}. \)
Observation B.3 Let $h \in C^\infty(\Sigma, G/T)/G_\Sigma$ and let $\Omega_h$ be a standard lift associated to $h$. Then for sufficiently small $\epsilon > 0$ we have $t := (\Omega_h)|_{\partial B_\epsilon(\sigma_0)} \in C^\infty(\partial B_\epsilon(\sigma_0), T)$ and

$$n(h) = \int_{\partial B_\epsilon(\sigma_0)} t^{-1} dt$$

Moreover, every $\Omega \in C^\infty(\Sigma \setminus \{\sigma_0\}, G)$ which on a neighborhood of $\sigma_0$ only takes values in $T$ is a standard lift associated to some $h \in C^\infty(\Sigma, G/T)/G_\Sigma$.

Observation B.4 Let $\psi : C^\infty(S^1, T) \to t \mapsto \int_S t^{-1} dt \in t$. Then we have:

- $\text{Image}(\psi) = I = \ker(\exp_t)$
- $\psi(t_1) = \psi(t_2)$ for $t_1, t_2 \in C^\infty(S^1, T)$ implies that $t_1$ and $t_2$ are homotopic.

Proof of part (iii): From Observation B.3 and Observation B.4 it follows immediately that $\text{Image}(n) \subset I$. From Observation B.3 and Observation B.4 it will also follow that $\text{Image}(n) \supset I$ provided that we can show for all sufficiently small $\epsilon > 0$ every smooth map $t : \partial B_\epsilon(\sigma_0) \to T$ can be extended smoothly to a map $\Omega : \Sigma \setminus \{\sigma_0\} \to G$ which on $B_\epsilon(\sigma_0) \setminus \{\sigma_0\}$ takes only values in $T$. But this follows easily from the assumption that $G$ is simply-connected.

It remains to be shown that $n$ is injective. Let $h_1, h_2 \in C^\infty(\Sigma, G/T)/G_\Sigma$ with $n(h_1) = n(h_2)$. For $i = 1, 2$ let $\tilde{g}_i$ be a standard representative of $h_i$ and $\Omega_i$ be lift of $(\tilde{g}_i)|_{\Sigma \setminus \{\sigma_0\}}$ (and hence a standard lift associated to $h_i$). For sufficiently small $\epsilon > 0$ we then have $t_i := (\Omega_i)|_{B_\epsilon(\sigma_0) \setminus \{\sigma_0\}} \in C^\infty(B_\epsilon(\sigma_0) \setminus \{\sigma_0\}, T)$ and $t := t_1 t_2^{-1} \in C^\infty(B_\epsilon(\sigma_0) \setminus \{\sigma_0\}, T)$. Observation B.3 then implies that

$$\int_{\partial B_\epsilon(\sigma_0)} t^{-1} dt = \int_{\partial B_\epsilon(\sigma_0)} t_1^{-1} dt_1 - \int_{\partial B_\epsilon(\sigma_0)} t_2^{-1} dt_2 = n(h_1) - n(h_2) = 0$$

According to Observation B.4 $t|_{\partial B_\epsilon(\sigma_0)}$ is null-homotopic, which implies that $t : B_\epsilon(\sigma_0) \setminus \{\sigma_0\} \to T$ can be extended smoothly to a map $\tilde{t} : \Sigma \setminus \{\sigma_0\} \to T$. Clearly, $\Omega := \Omega_1^{-1} \Omega_2^{-1} \in G_{\Sigma \setminus \{\sigma_0\}}$ is locally constant around $\sigma_0$ and can therefore be extended to an element $\Omega$ of $G_\Sigma$ in a trivial way. So we finally obtain $\tilde{g}_2 = \tilde{g}_1$ and therefore $h_1 = h_2$. \hfill $\square$

Remark B.5 In view of the relation $C^\infty(\Sigma, G/T)/G_\Sigma \cong [\Sigma, G/T]$ (cf. Remark 2.3 above) and the relation $n(h) = \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(\sigma_0)} \pi_t(\Omega_h^{-1} d\Omega_h) = \pm \int_{\Sigma \setminus \{\sigma_0\}} \pi_t(\Omega_h^{-1} d\Omega_h \wedge \Omega_h^{-1} d\Omega_h)$ where we have set $\int_{\Sigma \setminus \{\sigma_0\}} \pi_t(\Omega_h^{-1} d\Omega_h \wedge \Omega_h^{-1} d\Omega_h) \,:= \lim_{\epsilon \to 0} \int_{\Sigma \setminus \{\sigma_0\}} \pi_t(\Omega_h^{-1} d\Omega_h \wedge \Omega_h^{-1} d\Omega_h)$ it is clear that Proposition 2.4 is very closely related to the argument in Sec. 5 in [IT] (cf., in particular Eq. (5.5) in [IT]), which involves the Kirillov-Kostant symplectic forms on the regular coadjoint orbits of $G$ (each of which can be identified with $G/T$) and the winding numbers of their pullbacks on $\Sigma$ via smooth maps $\tilde{g} : \Sigma \to G/T$. By elaborating this argument in a suitable way it should not be difficult to obtain an alternative (but less direct) proof of Proposition 2.6.

B.5 Justification of the change of variable $A^\perp \to A^\perp \cdot \Omega_h^{-1}$ in Eq. (2.35)

Let $h \in C^\infty(\Sigma, G/T)/G_\Sigma$ and $B \in \mathcal{B}$ be fixed. In the following we will justify the change of variable $A^\perp \to A^\perp \cdot \Omega_h^{-1}$ or, equivalently, $A^\perp \cdot \Omega_h \to A^\perp$ appearing in step (1) in Eq. (2.35) in Sec. B.3 above. First observe that in view of $A^\perp \cdot \Omega_h = \Omega_h^{-1} A^\perp \Omega_h + \Omega_h^{-1} d\Omega_h$ the change of variable $A^\perp \cdot \Omega_h \to A^\perp$ in step (1) in Eq. (2.35) can be realized by performing the following two changes of variable one after the other:

$$(\text{CoV1}) \quad \Omega_h^{-1} A^\perp \Omega_h \to A^\perp,$$
(CoV2) \( A^\perp + \Omega_h^{-1} d\Omega_h \rightarrow A^\perp \).

It is enough to justify each of these two changes of variable separately.

**Justification of (CoV1):** It is a standard procedure in Constructive Quantum Field Theory (CQFT) for making rigorous sense of a given informal path integral expression to extend the "path space" in a suitable way, the implicit assumption being that this does not change the value of the corresponding path integral, cf. Remark B.6 below.

In order to justify (CoV1) we can do something similar. We replace the space \( A^\perp \cong C^\infty(S^1, A_\Sigma) \) appearing in the second line in Eq. (2.35) by the extended space
\[
\overline{A}^\perp := C^\infty(S^1, \overline{A}_\Sigma)
\]
where \( \overline{A}_\Sigma \) is a suitable extension of the space \( A_\Sigma \). For example, we can choose \( \overline{A}_\Sigma \) to be the space of \( g \)-valued 1-forms \( A_c \) on \( \Sigma \) which are smooth on \( \Sigma \setminus \{\sigma_0\} \) and bounded in a suitable sense.\(^{106}\) (Observe that the integrand in the second line in Eq. (2.35) makes sense for every \( A^\perp \in \overline{A}^\perp \).)

Now the justification of the change of variable (CoV1) is straightforward since \( A^\perp \rightarrow \Omega_h A^\perp \Omega_h^{-1} \) is a well-defined linear transformation of \( \overline{A}^\perp \) whose determinant equals 1, informally. (Recall that since \( G \) is compact and connected we have \( \det(\text{Ad}(g)) = 1 \) for all \( g \in G \).) After carrying out (CoV1) we replace \( \overline{A}^\perp \) again by \( A^\perp \) (assuming again that by doing so the value of the path integral does not change). By doing so we arrive at the expression
\[
\exp(-2\pi i \langle n(h), B(\sigma_0) \rangle) \times \int_{A^\perp} \left( \prod_{i=1}^m \text{Tr}_{\rho_i}(\text{Hol}_i(A^\perp + A_{sg}(h) + Bdt)) \right) \times \exp(iSCS(A^\perp + A_{sg}(h) + Bdt)) DA^\perp \quad \text{(B.19)}
\]
where we have set \( A_{sg}(h) := \Omega_h^{-1} d\Omega_h \).

**Remark B.6** The extended spaces used in CQFT are usually rather large, for example they often consist of spaces of distributions (or distributional forms) while the original spaces usually consist of smooth functions/forms. By contrast, the replacement \( A_\Sigma \rightarrow \overline{A}_\Sigma \) used for the change of variable (CoV1) above is quite modest.

Another difference with respect to the standard procedure in CQFT is that here we only extend the space \( A^\perp \) "temporarily", i.e. even though we initially make the replacement \( A^\perp \rightarrow \overline{A}^\perp \) we later go back\(^{107}\) from \( \overline{A}^\perp \) to \( A^\perp \), assuming again that this will not change the value of the corresponding path integral.

Note that the change of variable (CoV1) above, even though it is not a well-defined transformation of \( A^\perp \), does not involve any singularities but only points of discontinuity. This made it easy to use a “temporary extension of space”-argument (cf. Remark B.6 above) for the justification of (CoV1). By contrast, the 1-form \( A_{sg}(h) = \Omega_h^{-1} d\Omega_h \) appearing in the change of variable (CoV2) has a singularity in the point \( \sigma_0 \) and the map \( A^\perp \rightarrow A^\perp - A_{sg}(h) \) is therefore far from being a well-defined transformation of \( A^\perp \). Accordingly, we can not be sure that the naive change of variable (CoV2) will lead to the correct result. (Remark 2.7 in Sec. 2.3 above and Remark B.8 below illustrate how things could go wrong.) One could still find a way to justify

\(^{106}\) For example, \( \| \cdot \|_{\infty} \)-bounded where \( \| \cdot \|_{\infty} \) is the norm given by \( \| A \|_{\infty} := \sup_{\sigma \in \Sigma \setminus \{\sigma_0\}} \| A_\sigma \|_{\sigma, g} \) where \( \| \cdot \|_{\sigma, g} \) is the norm on \( \text{Hom}(T\Sigma, g) \) induced by the Riemannian metric \( g \) on \( \Sigma \) and \( \langle \cdot , \cdot \rangle \).

\(^{107}\) On the other hand, in the “continuum framework” (F2) for making rigorous sense of \( Z^{t, s, f}(\Sigma \times S^1, L) \) (cf. Sec. 1.2.1 above) we later replace the space \( A^\perp \) again by a modified space which then is a large extension of \( A^\perp \).
the change of variable (CoV2) with the help of a suitable “extension of space”-argument\textsuperscript{108} but it is safer to use the following more careful argument.

**Justification of (CoV2):** Let $U \subset \Sigma$ be an open neighborhood of $\sigma_0$ such that
\[ U \subset \Sigma \setminus \left( \bigcup_{j=1}^{m} \text{Image}(l_j^2) \right). \]

Moreover, choose a pair $(V, \varphi)$ where $V$ is an open neighborhood of $\sigma_0$ with $\nabla \subset U$ and where $\varphi$ is a smooth function $\Sigma \to [0, 1]$ fulfilling
\[ \varphi \equiv 1 \quad \text{on } V \quad \text{and} \quad \varphi \equiv 0 \quad \text{on } \Sigma \setminus U \]

From the assumptions above it follows that for all $A^\perp \in A^\perp$ we have
\[ \text{Tr}_{l_i}(\text{Hol}_i(A^\perp + A_{sg}(h) + Bdt)) = \text{Tr}_{l_i}(\text{Hol}_i(A^\perp + (1 - \varphi)A_{sg}(h) + Bdt)) \quad (B.20) \]

We can consider $(1 - \varphi)A_{sg}(h) \in A_{\Sigma \setminus \{\sigma_0\}}$ as an element of $A_{\Sigma} \subset A^\perp$ (by trivially extending $(1 - \varphi)A_{sg}(h)$ in the point $\sigma_0$). Accordingly, we obtain
\[
\begin{align*}
\int_{A^\perp} \left( \prod_{i=1}^{m} \text{Tr}_{l_i}(\text{Hol}_i(A^\perp + A_{sg}(h) + Bdt)) \right) \exp(iSCS(A^\perp + A_{sg}(h) + Bdt))DA^\perp \\
= \int_{A^\perp} \left( \prod_{i=1}^{m} \text{Tr}_{l_i}(\text{Hol}_i(A^\perp + (1 - \varphi)A_{sg}(h) + Bdt)) \right) \exp(iSCS(A^\perp + A_{sg}(h) + Bdt))DA^\perp \\
\overset{(*)}{=} \int_{A^\perp} \left( \prod_{i=1}^{m} \text{Tr}_{l_i}(A^\perp + Bdt) \right) \exp(iSCS(A^\perp + Bdt))DA^\perp \\
= \left[ \int_{A^\perp} \prod_{i=1}^{m} \text{Tr}_{l_i}(A^\perp + Bdt) \right] \times \\
\times \exp \left( i2\pi k \int_{\Sigma \setminus \{\sigma_0\}} \text{Tr}(\varphi A_{sg}(h) \wedge dB) \right) \\
\end{align*}
\]

where in step $(*)$ we applied the informal change of variable $A^\perp \to A^\perp - (1 - \varphi)A_{sg}(h)$ (which is now justified since $(1 - \varphi)A_{sg}(h)$ is an element of $A^\perp$, cf. the paragraph after Eq. (B.20) above).

Observe that Eq. (B.21) holds for all $U$, $V$, and $\varphi$ satisfying the assumption above. Moreover, the term $\int_{\Sigma \setminus \{\sigma_0\}} \text{Tr}(\varphi A_{sg}(h) \wedge dB) = \int_{\Sigma \setminus \{\sigma_0\}} \text{Tr}(A_{sg}(h) \wedge \varphi dB)$ can be made arbitrarily small by choosing $\text{supp}(U)$ small enough. From this we conclude that
\[
\begin{align*}
\int_{A^\perp} \left( \prod_{i=1}^{m} \text{Tr}_{l_i}(\text{Hol}_i(A^\perp + A_{sg}(h) + Bdt)) \right) \exp(iSCS(A^\perp + A_{sg}(h) + Bdt))DA^\perp \\
= \int_{A^\perp} \prod_{i=1}^{m} \text{Tr}_{l_i}(A^\perp + Bdt) \exp(iSCS(A^\perp + Bdt))DA^\perp \quad (B.22)
\end{align*}
\]

(Note that this is exactly the formula which one would obtain by performing the naive change of variable (CoV2).) Applying Eq. (B.22) to the expression (B.19) above we then obtain the last expression in Eq. (2.35).

\textsuperscript{108}In contrast to the 2-form $dA_{sg}(h)$, appearing in the change of variable (CoV2)’ in Footnote\textsuperscript{110} below, the 1-form $A_{sg}(h)$ appearing in (CoV2) is locally integrable.
Remark B.7 Observe that Eq. (B.21) and Eq. (B.22) taken together seem to imply that
\[ \exp\left(i2\pi k \int_{\Sigma\setminus\{\sigma_0\}} \text{Tr}(\varphi A_{sg}(h) \wedge dB)\right) = 1, \]
which would be a contradiction. However, a closer look shows that Eq. (B.21) and Eq. (B.22) together in fact only imply that Eq. (B.23) holds unless the integral on the RHS of Eq. (B.22) vanishes. In Appendix B.2 in [38] it is shown (on an informal level) that the RHS of Eq. (B.22) indeed vanishes unless \(B\) is constant on each connected component \(Y\) of \(\Sigma\setminus\left(\bigcup_{j=1}^m \text{Image}(l_j)\right)\). But in this case Eq. (B.23) does hold so there is no contradiction.

Remark B.8 In Sec. 2.3 instead of working with the expression \(S_{CS}(A^\perp + Bdt)\) given by Eq. (2.32) in Sec. 2.3, we could have decided to work with the expression \(S'_{CS}(A^\perp + Bdt)\) given by Eq. (B.13). Then instead of Eq. (2.33) in Sec. 2.3 we would use Eq. (B.16) above and, accordingly, the first equation in Eq. (2.35) in Sec. 2.3 would read

\[ \int_{A^\perp} \left(\prod_{i=1}^m \text{Tr}_{\rho_i}(\text{Hol}_{l_i}(A^\perp + \tilde{g}_h B \bar{g}_h^{-1}dt))\right) \exp\left(iS_{CS}(A^\perp + \tilde{g}_h B \bar{g}_h^{-1}dt)\right) \text{DA}^\perp = \int_{A^\perp} \left(\prod_{i=1}^m \text{Tr}_{\rho_i}(\text{Hol}_{l_i}(A^\perp \cdot \Omega_h + Bdt))\right) \exp\left(iS'_{CS}(A^\perp \cdot \Omega_h + Bdt)\right) \text{DA}^\perp \]

Now, using the change of variable \((CoV1)\) (which can again be justified in a similar way as above) we can rewrite the last expression in the previous equation as

\[ \int_{A^\perp} \left(\prod_{i=1}^m \text{Tr}_{\rho_i}(\text{Hol}_{l_i}(A^\perp \cdot \Omega_h + Bdt))\right) \exp\left(iS'_{CS}(A^\perp \cdot \Omega_h + Bdt)\right) \text{DA}^\perp = \int_{A^\perp} \left(\prod_{i=1}^m \text{Tr}_{\rho_i}(\text{Hol}_{l_i}(A^\perp + A_{sg}(h) + Bdt))\right) \exp\left(iS'_{CS}(A^\perp + A_{sg}(h) + Bdt)\right) \text{DA}^\perp \]

Observe that if in the last expression we performed the change of variable \(A^\perp + \Omega_h^{-1}d\Omega_h \rightarrow A^\perp\), which we will call \((CoV2)\)' (in order to distinguish it from the analogous change of variable \((CoV2)\) appearing above in the main text) in a naive way we would arrive at

\[ \int_{A^\perp} \left(\prod_{i=1}^m \text{Tr}_{\rho_i}(\text{Hol}_{l_i}(A^\perp + A_{sg}(h) + Bdt))\right) \exp\left(iS'_{CS}(A^\perp + A_{sg}(h) + Bdt)\right) \text{DA}^\perp = \int_{A^\perp} \left(\prod_{i=1}^m \text{Tr}_{\rho_i}(\text{Hol}_{l_i}(A^\perp + Bdt))\right) \exp\left(iS'_{CS}(A^\perp + Bdt)\right) \text{DA}^\perp \]

which is not correct. In fact, according to the careful (informal) argument in [38] Appendix B.3

\footnote{Observe that there is an obvious complication here: Those \(B\) which have this property will actually not be smooth (unless they are constant) and will therefore not be well-defined elements of \(B\). This complication can be eliminated if we advance the switch from loops to closed ribbons (and regularized ribbon holonomies) from Secs 3.2.1 and 3.2.2 to Sec. 2.3. More precisely, the introduction of (smeared) ribbon holonomies (which we consider as a regularization of the original loop holonomies) must take place after we have applied Eq. (2.22) since in Eq. (2.23) we use an argument based on \(G\)-invariance, which is only valid for loop holonomies (cf. the second bullet point after Eq. (3.20) in Sec. 3.2.1).}

\footnote{It is interesting to analyze what would happen if we tried to use an “extension of space”-argument in order to justify the (incorrect) change of variable \((CoV2)\)'\textsuperscript{2}. The replacement \(S'_{CS}(A^\perp + A_{sg}(h) + Bdt) \rightarrow S_{CS}(A^\perp + Bdt)\) involves a replacement \(dA^\perp + dA_{sg}(h) \rightarrow dA^\perp\). Now observe that the differential \(dA_{sg}(h)\) of \(A_{sg}(h)\) is not locally integrable around \(\sigma_0\) w.r.t. the measure \(d\Omega_h\), cf. Sec. 3.2.2. This rules out any reasonable “extension of space”-argument for justifying this change of variable \((CoV2)\)'\textsuperscript{2}. In particular, we cannot even use the large extended spaces containing distributional forms mentioned in Remark B.6 above.}

\footnote{Note that in [38] the notation \(A_{sg}(h)\) is used for the 1-form \(\pi_1(\Omega_h^{-1}d\Omega_h)\) rather than \(\Omega_h^{-1}d\Omega_h\) but it is obvious how to modify the argument there.}
in \cite{[38]} we have
\begin{equation}
\int_{A^\perp} \left( \prod_{i=1}^{m} \mathrm{Tr}_{\rho_i}(\text{Hol}_t(A^\perp + A_{sg}(h) + Bdt)) \right) \exp(iS'_{CS}(A^\perp + A_{sg}(h) + Bdt)) DA^\perp \\
= \exp(-2\pi i k(n(h), B(\sigma_0))) \int_{A^\perp} \left( \prod_{i=1}^{m} \mathrm{Tr}_{\rho_i}(\text{Hol}_t(A^\perp + Bdt)) \right) \exp(iS'_{CS}(A^\perp + Bdt)) DA^\perp
\end{equation}
(B.24c)

From Eqs. (B.24a), (B.24b), and (B.24c) and $S'_{CS}(A^\perp + Bdt) = S_{CS}(A^\perp + Bdt)$ we now obtain again Eq. (2.35).

### B.6 Justification of the replacement $B_{\text{reg}} \to B_{\text{reg}}^{\text{ess}}$ in Sec. 3.2.3

Let us now justify the replacement $B_{\text{reg}} \to B_{\text{reg}}^{\text{ess}}$ in Sec. 3.2.3. More precisely, we will show that in Eq. (2.23) in Sec. 2.2 we can replace $B_{\text{reg}}$ by $B_{\text{reg}}^{\text{ess}}$. (This then implies that also in Sec. 2.3 we can replace $B_{\text{reg}}$ by $B_{\text{reg}}^{\text{ess}}$ everywhere.) In order to do so we will modify/extend some of the arguments in Appendix B.2.

Recall from Appendix B.2 that the diffeomorphism $q : P \times G/T \ni (b, g) \mapsto \hat{g} \exp(b) \tilde{g}^{-1} \in G_{\text{reg}}$ induces a bijection $\hat{q} : C^\infty(\Sigma, P) \times C^\infty(\Sigma, G/T) \to C^\infty(\Sigma, G_{\text{reg}})$. Similarly, the smooth map $\hat{q}' : t \times G/T \ni (b, g) \mapsto \hat{g} \exp(b) \tilde{g}^{-1} \in G$ induces an injection

$$\hat{q}' : \{ B \in B_{\text{reg}}^{\text{ess}} \mid B(\sigma_0) \in P \} \times C^\infty(\Sigma, G/T) \to C^\infty(\Sigma, G)$$

By modifying Eqs (B.10) in Appendix B.2 above in a suitable way we obtain
\begin{equation}
\int_{C^\infty(\Sigma, G)} \bar{\chi_{\text{reg}}}(\Omega) D\Omega
\sim \int_{\{ B \in B_{\text{reg}}^{\text{ess}} \mid B(\sigma_0) \in P \}} \left[ \sum_{h \in C^\infty(\Sigma, G/T)/G_{\Sigma}} \bar{\chi_{\text{reg}}} (\hat{g}_h \exp(B) \tilde{g}_h^{-1}) \right] \det(1_t - \exp(\text{ad}(B)|_{t})) DB
\end{equation}
(B.25)

which then leads to
\begin{equation}
\int_{A} \chi(A) DA \sim \sum_{h \in C^\infty(\Sigma, G/T)/G_{\Sigma}} \int_{\{ B \in B_{\text{reg}}^{\text{ess}} \mid B(\sigma_0) \in P \}} \left[ \int_{A^\perp} \chi(A^\perp + (\hat{g}_h B \tilde{g}_h^{-1}) dt) DA^\perp \right] \times \det(1_t - \exp(\text{ad}(B)|_{t})) DB
\end{equation}
(B.26)

By applying Eq. (B.26) to all different choices of $P$ (and using that $t \setminus t_{\text{reg}}$ is a $db$-zero set) we arrive at the version of Eq. (2.23) in Sec. 2.2 where $B_{\text{reg}}$ is replaced by $B_{\text{reg}}^{\text{ess}}$.

### C The shadow invariant $|L|$

The algebraic approach to the quantum invariants by Reshetikhin and Turaev, which works with quantum group representations and surgery operations, can be reformulated leading to the so-called “shadow world” approach (cf. \cite{[48], [77], and part II of [78]}) which also works with quantum group representations but eliminates the use of surgery operations. Let us briefly recall the definition of the shadow invariant in the situation relevant for us, i.e. for the base manifold $M = \Sigma \times S^1$.

Let $L = (l_1, l_2, \ldots, l_m), m \in \mathbb{N}$, be an admissible link in $M = \Sigma \times S^1$ (cf. Definition 3.37) such that each $l^i_{\Sigma} : S^1 \to \Sigma, i \leq m$, is null-homotopic. We will assume that each $l_i, i \leq m$, is equipped with a horizontal framing, cf. Definition 3.6 in Sec. 3.2.1 above.
As in Sec. 3.5 we will denote by $V(L)$ the set of double points of $L$ (i.e. the set of those $p \in \Sigma$ where the loops $l_i^\pm$, $i \leq m$, cross themselves or each other). Moreover, we will denote by $E(L)$ the set of curves in $\Sigma$ into which the loops $l_1^\pm, l_2^\pm, \ldots, l_m^\pm$ are decomposed when being “cut” in the points of $V(L)$.

From the assumption that $L$ is admissible it follows that there are only finitely many connected components $Y_0, Y_1, Y_2, \ldots, Y_{m'}$, $m' \in \mathbb{N}$ (“faces”) of $\Sigma \setminus (\bigcup_i \text{Image}(l_i^\pm))$. We set

$$Y(L) := \{Y_0, Y_1, Y_2, \ldots, Y_{m'}\}. \quad (C.1)$$

By $\text{col}(L)$ we denote the set of all maps $\eta : Y(L) \to \Lambda^k_\pm$ (= “area colorings”).

As explained in [77] one can associate to each face $Y \in Y(L)$ in a natural way a half integer called the “gleam” of $Y$ (notation: gleam$(Y)$).

From now on we will assume that each loop $l_i$ in the link $L$ is equipped with a “color” $\rho_i$, i.e. an irreducible finite-dimensional complex representation of $G$. By $\gamma_i \in \Lambda_+$ we denote the highest weight of $\rho_i$ and we set $\gamma(e) := \gamma_i(e)$ for each $e \in E(L)$ where $i(e)$ denotes the unique index $i \leq m$ such that Image(e) $\subset$ Image(l_i). We can now define the “shadow invariant” $|L|$ associated to the pair $(g, k)$ and the colored, horizontally framed link $L$ by (cf. Remark C.1 below)

$$|L| := \sum_{\eta \in \text{col}(L)} |L|_1^0 |L|_2^0 |L|_3^0 |L|_4^0 \quad (C.2)$$

with

$$|L|_1^0 = \prod_{Y \in Y(L)} (d_{\eta(Y)})^{\chi(Y)} \quad (C.3a)$$
$$|L|_2^0 = \prod_{Y \in Y(L)} (\theta_{\eta(Y)})^{\text{gleam}(Y)} \quad (C.3b)$$
$$|L|_3^0 = \prod_{e \in E_{\text{loop}}(L)} N_+^{\eta(Y)^+}(Y^+) \quad (C.3c)$$
$$|L|_4^0 = \text{contr}_{D(L)}(\bigotimes_{x \in V(L)} T(x, \eta)) \quad (C.3d)$$

Here $d_\lambda$, $\theta_\lambda$, $N_{\mu\nu}^\lambda$ (for $\lambda, \mu, \nu \in \Lambda_+^k$) are as in Appendix A above and $E_{\text{loop}}(L)$ is the subset of those $e \in E(L)$ which are loops. Moreover, $Y^+_e$ (or $Y^-$, respectively) denotes the unique face $Y$ such that Image(e) $\subset$ $\partial Y$ and, additionally, the orientation on Image(e) induced by the standard orientation on Image(l_i(e)) coincides with (or is opposite to, respectively) the orientation induced on e $\subset$ $\partial Y$ by the orientation on $Y$.

Each factor $T(x, \eta)$ appearing in $|L|_4^0$ is an element of a certain finite-dimensional complex vector space $W(x, \eta)$. The definitions of both $W(x, \eta)$ and $T(x, \eta)$ involve six elements of $\Lambda_+^k$, firstly, the values $\eta(Y_i(x))$ for the four faces $Y_i(x) \in Y(L)$, $i \leq 4$, having $x$ on their boundary and, secondly, the highest weights $\gamma_1(x)$ and $\gamma_2(x)$ of the two colors $\rho_i$ and $\rho_j$ associated to the two loops $l_i$ and $l_j$ whose $\pi_2$-projections intersect in $x$. Moreover, the definition of $T(x, \eta)$ involves the so-called (normalized) “quantum $6j$-symbols” associated to the quantum group $U_q(\mathfrak{g})$ where $q := \exp(\frac{2\pi i}{k})$ (cf. Chap. VI and Chap. XI in [78]). Finally, $D(L)$ is the graph $(V(L), E_{\text{red}}(L))$ where $E_{\text{red}}(L) := E(L) \setminus E_{\text{loop}}(L)$ and $\text{contr}_{D(L)} : \bigotimes_{x \in V(L)} W(x, \eta) \to \mathbb{C}$ is a suitable linear functional which depends on the structure of $D(L)$. (See the last paragraph of Remark C.2 below for additional comments regarding $T(x, \eta)$ and $\text{contr}_{D(L)}$.)

**Remark C.1** The definition of $|L|$ above generalizes the definition of the “shadow invariant” in [77]. More precisely, the invariant defined in [77] is the special case of $|L|$ where $\mathfrak{g} = \mathfrak{su}(2)$. By combining the ideas in Sec. 5 in [77] with those in Chapters VI, VIII and XI in [78] it should not be too difficult to show that $|L|$ is indeed a topological invariant also if $\mathfrak{g}$ is the Lie algebra.
of a general simple, (simply-connected) compact Lie group \(G\). However, to my knowledge such a proof has not yet been written down anywhere\(^{112}\).

Instead of proving the topological invariance of \(|L|\) directly we will prove it indirectly by expressing \(|L|\) in terms of the topological invariant \(|M, \Gamma|\) defined in Sec. X.7.3 in [78]. More precisely, we have

\[
|\Sigma \times S^1, L^*| \sim |CY(\Sigma \times S^1, L^*)| \sim |L|
\]

where

\[
\bullet \ L^* \text{ denotes the framed, colored link in } \Sigma \times S^1 \text{ obtained from the framed, colored link } L = (l_1, l_2, \ldots, l_m), \text{ fixed above by replacing each color } \rho_i \text{ of } L \text{ with the dual color } \rho_i^*,
\]

\[
\bullet \ \text{the first } "\cdot" \text{ refers to the invariant } |M, \Gamma| \text{ defined by}\(^{113}\) the first equation in Sec. X.7.3 in [78],
\]

\[
\bullet \ \text{"CY}(\Sigma \times S^1, L^*)" \text{ is as explained in part (i) of the present remark},
\]

\[
\bullet \ \text{the second } "\cdot" \text{ is the map described in part (ii) of the present remark},
\]

\[
\bullet \ \text{"~" denotes equality up to a multiplicative constant independent of } \Sigma \text{ for the case } \Sigma = \Sigma_0.
\]

Note that, apart from showing the topological invariance of \(|L|\), Eq. (C.4) will have the additional benefit of allowing us to find the explicit relation between \(|L|\) and the Reshetikhin-Turaev invariant \(RT(\Sigma \times S^1, L)\), cf. Eq. (C.7) below.

We will now sketch the definition/construction of \(CY(\Sigma \times S^1, L^*)\) and the second map \(|\cdot|\) appearing in Eq. (C.4) above. In Remark C.2 below we will then sketch how one can verify directly (for an important special case) that the second "~" in Eq. (C.4) indeed holds. Note that the treatment here is definitely not self-contained but we hope that it will still be helpful for the reader, in particular because it will probably allow the reader to navigate more quickly through the sections of [78] which are relevant for us.

(i) The notation \(CY(\Sigma \times S^1, L^*)\) appearing above refers to a “shadowed 2-polyhedron” (in the sense of Sec. VIII.1.2 in [78]) which is constructed as follows:

\[
\bullet \ \text{First we choose a suitable “skeleton” of } M = \Sigma \times S^1, \text{ i.e. an orientable “simple 2-polyhedron”}\(^{115}\) \(X_0\) contained in \(M\) (and with } \partial X_0 = \emptyset) \text{ such that } M \setminus X_0 \text{ is the disjoint union of open 3-balls, cf. Sec. IX.2.1 in [78]. Note that we cannot choose } X_0 = \Sigma \times \{1\} \text{ but we can choose } X_0 \text{ such that } X_0 \supset \Sigma \times \{1\} \equiv \Sigma \text{ and such that } X_0 \text{ has no “vertices” and only “circle 1-strata” in the terminology of [78].}
\]

\(^{112}\)In the special case where the set \(E_{\text{loop}}(L)\) is empty the RHS of Eq. (C.2) above coincides with the RHS of Eq. (25) in [63] where the topological invariance of the RHS of their Eq. (25) is mentioned but not proven. In [63] it is also mentioned that their formula (25) follows from the general results in [78]. Eq. (C.4) below makes this explicit.

\(^{113}\)The first equation in Sec. X.7.3 in [78] reads (using the notation of [78]) \(|M, \Gamma| = D^{\dim(\Gamma)}|CY(M, \Gamma)| \in H(\Gamma)\). The two observations important for us are that in the special case where \(\Gamma\) is a link we have, firstly, \(H(\Gamma) = \mathbb{C}\) and, secondly, the factor \(\dim(\Gamma)\) is then trivial. In order to understand the first point note that the notation \(H(\Gamma)\) in Sec. X.7.3 is a short notation for \(H(\Gamma, \lambda)\), defined as in Sec. X.1.1 in [78], where \(\lambda\) is the coloring of \(\Gamma\) and note also that in the special case where \(\Gamma\) is a link it does not contain any vertices. The second point follows from Eq. (5.5b) in Sec. X.5.5 in [78] by taking into account that the boundary \(\partial X = CY(M, \Gamma)\) only has “circle 1-strata” in the terminology of Sec. VI.4.1 in [78] if \(\Gamma = L\), cf. the fourth bullet point in the list in point (i) below.

\(^{114}\)By contrast, this “constant” may depend on \(\Sigma, G,\) and \(k\).

\(^{115}\)Cf. Sec. VIII.1.1 in [78].
• Once we have $X_0$ we can construct from the framed link $L^*$ a “shadowed system of loops” $(l^i_{X_0})_{i \leq m}$ in $X_0$ in the sense of Sec. VIII.3.1 in [78]. Each loop $l^i_{X_0}$ is obtained from the corresponding loop $l_i$ of $L^*$ by a kind of “projection” into $X_0$. The details of this construction are explained in Sec. IX.3.2 in [78].

• The shadowed 2-polyhedron $CY(\Sigma \times S^1, L^*)$ appearing above (= “the cylinder over the shadowed system of loops” $(l^i_{X_0})_{i \leq m}$ in the terminology of [78]) is constructed from the shadowed system of loops $(l^i_{X_0})_{i \leq m}$ in $X_0$ by attaching an annulus $\bar{Z}_i \cong S^1 \times [0,1]$, $i \leq m$, along each of the loops $l^i_{X_0}$ in a suitable way. This construction is alluded to at the end of Sec. IX.3.3 in [78] but the details of this construction are actually given earlier, namely in Sec. VIII.3.2 in [78].

• The boundary $\partial X$ of $X = CY(\Sigma \times S^1, L^*)$ is the disjoint union of $m$ copies of $S^1$. (This is the result of the aforementioned attachment of $m$ annuli $\bar{Z}_i$.) In the terminology of Sec. VI.4.1 in [78] we say that $\partial X$ consists only of “circle 1-strata”. We equip each of the $m$ connected components of $\partial X \cong \bigsqcup_{i=1}^m S^1$ with the corresponding color $\rho^i$ coming from the colored link $L^*$. By doing so we obtain a “coloring” $\lambda$ of $\partial X$ in the sense of Sec. X.1.2 in [78].

(ii) The second map $|\cdot|$ appearing in Eq. (C.4) above is the map defined in Sec. X.1.2 in [78]. More precisely: What is denoted above by $|X|$ with $X = CY(\Sigma \times S^1, L^*)$ is actually a short notation for $[X, \lambda]$ defined by the last equation in Sec. X.1.2 in [78], which after dropping a factor independent of $L$ reads

$$|X, \lambda| \sim \sum_{\varphi \in \text{col}(X)} |X|_1 |X|_2 |X|_3 |X|_4 |X|_5$$  \hspace{1cm} (C.5)

where $\lambda$ is the coloring of $\partial X$ induced by $L^*$ (cf. the fourth bullet point of part (i) below), where $\text{col}(X)$ is the set of “colorings” of $X$ and where $|X|_i$ for $i \leq 5$ and $\varphi \in \text{col}(X)$ is defined as in Sec. X.1.2 in [78]. (We will give some more details regarding $\text{col}(X)$ and the terms $|X|_i$ in Remark C.2.2 below.)

Remark C.2 For the convenience of the reader we will now sketch (for the special case where $L \subset \Sigma \times (S^1 \setminus \{1\})$) a proof for the assertion above that we have indeed

$$|CY(\Sigma \times S^1, L^*)| \sim |L|.$$  \hspace{1cm} (C.6)

(The details for the general case will be postponed to a later version of the present paper.) We will use the notation introduced in Remark C.1 above.

(i) In the special case where $L \subset \Sigma \times (S^1 \setminus \{1\})$ we can choose the skeleton $X_0$ above such that the projected loops $l^i_{X_0}$ do not meet the (circle) 1-strata of $X_0$, which implies that there is a 1-1-correspondence between the vertices of $X = CY(\Sigma \times S^1, L^*)$ and our set $V(L)$.

(ii) By contrast, as $X_0$ cannot be chosen to be equal to $\Sigma \times \{1\}$ the “regions” of the shadowed system of loops $(l^i_{X_0})_{i \leq m}$ in $X_0$ (in the sense of Sec. VIII.3.1 in [78]) are not in 1-1-correspondence with the elements of our set $V(L)$. Moreover, as a result of the aforementioned attachment of $m$ annuli $\bar{Z}_i$, the set of “regions” $\text{Reg}(X)$ of $X = CY(\Sigma \times S^1, L^*)$ (in the sense of Sec. VIII.1.1

---

116 The adjective “shadowed” refers to the family of “gleams”, i.e. the half integers which are associated canonically to the “regions” of $(l^i_{X_0})_{i \leq m}$ in $X_0$, where the notion of “region” is defined as in Sec. VIII.3.1 in [78].

117 Or later, namely in Sec. IX.8.3 in [78] where the general definition of $CY(M, \Gamma)$ is given. In the special situation where $\Gamma$ is a link (and where $M = \Sigma \times S^1$) it is more convenient to follow Sec. VIII.3.2 in [78].

118 By contrast, for general $L$ in $\Sigma \times S^1$ this is not the case, which complicates matters considerably.
in [78] is even larger. There are \( m \) regions \( Z_i \cong S^1 \times (0, 1) \) of \( X \), coming from the annuli \( \tilde{Z}_i \), \( i \leq m \), which do not correspond to any of the elements of \( Y(L) \).

This means that, while the area colorings \( \eta \) appearing in the formula for \( |L| \) (cf. Eq. (C.2) above) are elements of \((\Lambda^k)^n \cong (\Lambda^k)^n_+\) where \( n := \# Y(L) \), the colorings \( \varphi \) appearing in Eq. (C.5) above for \( X = CY(\Sigma \times S^1, L^*) \) can be considered as elements of \((\Lambda^k)^{n+m} \) where \( n' > n \) is the number of regions of the shadowed system of loops \( \{l_{X_0}\}_{i \leq m} \) in \( X_0 \).

A second difference is that, while the sum in Eq. (C.2) above is over all \( \eta \), the sum in Eq. (C.5) above is only over those \( \varphi \) obeying the relevant boundary condition. So in the case of Eq. (C.2) above we have a sum over \((\Lambda^k)^n \) while in Eq. (C.5) we have a sum over \((\Lambda^k)^{n'} \).

Fortunately, it is quite easy to reduce the sum over \((\Lambda^k)^{n'} \) to a sum over \((\Lambda^k)^n \) by summing out \( n' - n \) suitably chosen components of \((\Lambda^k)^{n'} \) and by taking into account that \( \sum_{\alpha} N_{\alpha \beta \gamma} d_{\alpha} = d_{\beta} d_{\gamma} \) and \( \sum_{\alpha} S_{\alpha 0}^2 = 1 \). (Note that \( h^{\alpha \beta \gamma} \) and \( \dim(\alpha) \) in the notation of [78] coincide with our \( N_{\alpha \beta \gamma} \) and \( d_{\alpha} \).) The details of this argument will be given in later versions of the present paper.

After summing out the aforementioned \( n' - n \) components of \((\Lambda^k)^{n'} \) we then arrive at a sum over \((\Lambda^k)^n \), which can be identified with the sum over all those “reduced” colorings \( \varphi : Y(L) \cup \{Z_i \mid i \leq m\} \to \Lambda^k_+ \) which fulfill the boundary condition mentioned in Footnote 119 above. The important point is that the value of \( [X, \lambda] \) is again given by Eq. (C.5) above where now each \( |X|^{\varphi}_{\eta} \), \( i \leq 5 \), is reexpressed in terms of the “reduced” \( \varphi \) and where the sum over \( \varphi \) is reinterpreted as the sum over the set of “reduced” colorings \( \varphi \).

(iii) In the following let us fix \( \varphi : Y(L) \cup \{Z_i \mid i \leq m\} \to \Lambda^k_+ \) and let \( \eta \) denote the area coloring \( Y(L) \to \Lambda^k_+ \) corresponding to \( \varphi \); i.e. \( \eta = \varphi|_{Y(L)} \). We will now compare the five factors \( |X|^{\varphi}_{\eta} \), \( |X|^{\varphi}_{\eta} - |X|^{\varphi}_{\eta} \), \( |X|^{\varphi}_{\eta} \), and \( |X|^{\varphi}_{\eta} \) appearing in Sec. X.1.2 in [78] (reexpressed in terms of the “reduced” \( \varphi \)) with the four factors \( |L|^{\varphi}_{\lambda} \), \( |L|^{\varphi}_{\lambda} \), \( |L|^{\varphi}_{\lambda} \), and \( |L|^{\varphi}_{\lambda} \) appearing in our Eq. (C.5).

- \( |X|^{\varphi}_{\eta} \) is trivial for \( X = CY(\Sigma \times S^1, L^*) \) since in this case \( \partial X \) consists of circle 1-strata and therefore does not contain any vertices (cf. the last paragraph in Remark [C.4] above).

- In order to see that the term \( |X|^{\varphi}_{\eta} \) leads to our term \( |L|^{\varphi}_{\lambda} \) one uses the relation \( \chi(Z_i) = \chi(S^1 \times (0, 1)) = 0 \).

- In order to see that the term \( |X|^{\varphi}_{\eta} \) gives rise to our term \( |L|^{\varphi}_{\lambda} \) note that \( (v')^2 \) in the notation of [78] corresponds to our \( \theta_{\lambda} \) if \( \lambda = i \) and that the gleams of the \( m \) regions \( Z_i \cong S^1 \times (0, 1) \) equal the framing numbers of the \( m \) loops \( l_i \) in \( L \), cf. Sec. VIII.3.2 in [78]. In the special case where \( L \) is horizontally framed (which we have assumed above) all these framing numbers are equal to 0.

- In order to see that the term \( |X|^{\varphi}_{\eta} \) gives rise to our term \( |L|^{\varphi}_{\lambda} \) recall first that \( h^{ijk} \) in the notation of [78] coincides with our \( N_{ijk} \) and that we have \( N_{ijk}^0 = N_{ijk} \). Moreover, one should note that the asymmetric treatment of the two faces \( Y^+_{\varphi} \) and \( Y^-_{\varphi} \) in each factor \( N_{i,j}^{(Y^+_{\varphi})} \) on the RHS of Eq. (C.33) above is mirrored in the definition of the term \( |X|^{\varphi}_{\eta} \) in [78] even though at first look\(^{120}\) each factor \( h_{\varphi}(g) \) of \( |X|^{\varphi}_{\eta} \) seems to have a symmetric definition.

\(^{119}\) Note that the value of \( \varphi \) on each \( Z_i \) is fixed by the boundary condition \( \partial \varphi = \lambda \) appearing under the \( \Sigma \)-sign in Eq. (C.3) above.

\(^{120}\) The point to note is that when each face \( Y \in Y(L) \) is equipped with the orientation induced by the orientation on \( \Sigma \) (as we have assumed implicitly above) then \( Y^+_{\varphi} \) and \( Y^-_{\varphi} \) will induce different orientations on the bounding edge \( e \). By contrast, in each factor \( h_{\varphi}(g) \) (where \( g \) is our \( e \)) the orientation of the three relevant regions, i.e. \( Y^+_{\varphi} \), \( Y^-_{\varphi} \), and \( Z_{\lambda(e)} \) is supposed to induce the same orientation on \( e \). So under the correspondence between \( \varphi \) and \( \eta \) which we are using \( \varphi(Y^+_{\varphi}) \) (or \( \varphi(Y^-_{\varphi}) \), respectively) must be replaced by \( \varphi(Y^+_{\varphi}) = \varphi(Y^-_{\varphi}) \) (or \( \varphi(Y^-_{\varphi}) \), respectively) where \( Y^\pm_{\varphi} \) denotes the face \( Y^\pm_{\varphi} \) when equipped with the opposite orientation and where \( \lambda \) for \( \lambda = \varphi(Y^\pm_{\varphi}) \) is the dual weight.
• Finally note that the second factor of the tensor product appearing in the formula for $|X|_0^2$ is trivial since $\partial X$ only consists of circle 1-strata and therefore does not contain any vertices. So the expression for $|X|_0^2$ reduces to “$\text{cntr}(\otimes_x |x|_\varphi) = \text{cntr}(\otimes_x |\Gamma_T^e|)$” in the notation of Sec. 2.1.2 in [78]. Note that “$\text{cntr}$” in the notation of [78] coincides with what we denote by $\text{cntr}_D(L)$ and $|\Gamma_T^e|$, $x \in V(L)$, coincides with what we denote by $T(x, \eta)$ (where $\eta$ and $\varphi$ are related as described above).

Let us now go back to the case of ribbon links. Let $L$ be a general strictly admissible ribbon link in $M = \Sigma \times S^1$ and let $L^0$ be the proper (framed) link associated to $L$, cf. Definition 3.5 in Sec. 3.2.1. Observe that $L^0$ is then automatically admissible and horizontally framed. We will write $|L|$ instead of $|L^0|$ and $|L|^0_i$ instead of $|L^0|^0_i$. Moreover, we will identify the set $Y(L^0)$ (defined at the beginning of Appendix C) with the set $Y(L)$ defined as in Sec. 3.3 above. According to Theorem 7.3.1 in Sec. X.7.3 in [78] and Remark C.1 above we have

$$RT(\Sigma \times S^1, L) \sim |\Sigma \times S^1, L^*| \sim |L|.$$  
(C.7)

(Observe that we use the notation $RT(M, L)$ for what in [78] is denoted by $\tau(M, L)$.)

D Performing Steps 2–4 in Sec. 3.5.2

Note: The present part of the appendix is somewhat speculative and will not be included in the print version of the present paper.

We will now sketch the strategy for performing Steps 2–4 in Sec. 3.5.2. (At the end of Appendix D we will also make some comments regarding Step 5.) In contrast to the calculations in Sec. 3.3 above we do this not as a preparation for a rigorous treatment but simply in order to make it plausible that also for general strictly admissible $L$ we have a good chance of obtaining

$$Z(\Sigma \times S^1, L) \sim |L| = \sum_{\eta \in \text{col}(L)} |L|^0_1 |L|^0_2 |L|^0_3 |L|^0_4$$  
(D.1)

Accordingly, in the present part of the appendix we take the liberty to use several somewhat sloppy (informal) arguments (cf., e.g., Footnotes 121, 125 and 124 below).

As in Sec. 3.3.3 above let us (informally) interchange the $\epsilon \to 0$-limit with the integral and the $\sum_y$-sum. By doing so we arrive at the following variant of Eq. (3.124)

$$Z(\Sigma \times S^1, L) \sim \sum_{y \in I} \int_{A^*_c \times B} \left\{ 1_{B_{\text{rig}}}(B) \text{Det}_{\text{rig}}(B) \exp(-2\pi i k(y, B(\sigma_0))) \right. 
\times \left. \beta_{L,D} \left( (\otimes_{e \in E(L)} T_e(A^*_c, B)) \otimes (\otimes_{x \in V(L)} T_x(A^*_c, B)) \right) \right\} 
\times \exp(i S_{CS}(A^*_c, B))(DA^*_c \otimes DB)$$  
(D.2)

where we have set

$$T_e(A^*_c, B) := \lim_{\epsilon \to 0} T^*_e(A^*_c, B), \quad \text{and} \quad T_x(A^*_c, B) := \lim_{\epsilon \to 0} T^*_x(A^*_c, B).$$  
(D.3)

Let us now rewrite Eq. (D.2) as an iterated integral[121]

$$Z(\Sigma \times S^1, L) \sim \sum_{y \in I} \int_B \left\{ 1_{B_{\text{rig}}}(B) \text{Det}_{\text{rig}}(B) \exp(-2\pi i k(y, B(\sigma_0))) F_{L,D}(B) \right\} DB$$  
(D.4)

[121] In fact, instead of working with $\int \cdots DA_\Sigma$ and $\int \cdots DB$ it would have several conceptual advantages to work with suitable improper integrals $\int^\cdots DA_\Sigma$ and $\int^\cdots DB$ as in Remark 24a and Remark 24b above, cf. Footnote 124 and Footnote 127 below.
with
\[ F_{L,D}(B) := \int_{A_{\Sigma,1}} \beta_{L,D}(\langle \otimes_{e \in E(L)} T_e(A_\Sigma, B) \otimes (\otimes_{x \in V(L)} T_x(A_\Sigma, B) \rangle) \exp(iS_{CS}(A_\Sigma, B)) \]  

(D.5)

where \( DA_\Sigma \) is the informal Lebesgue measure on \( A_{\Sigma,1} \approx \mathbb{A}_c \).

Recall that the ribbon link \( L = (R_1, R_2, \ldots, R_m) \) fixed in Sec. \[ 3.5.2 \] above is colored, i.e. each \( R_j \) is equipped with an irreducible representation \( \rho_j : G \to \text{Aut}(V_j) \) where \( V_j \) is a finite-dimensional complex vector space.

**Convention D.1**

(i) For each \( e \in E(L) \) we denote by \( S(e) \) the unique ("open") ribbon \([0,1] \times [0,1] \to \Sigma \times S^1 \) given by \( \text{cl}(e) = \{S(e)\} \). Similarly, for each \( x \in V(L) \) we denote by \( S_1(x) \) and \( S_2(x) \) the two ("open") ribbons \([0,1] \times [0,1] \to \Sigma \times S^1 \) given by \( \text{cl}(x) = (S_1(x), S_2(x)) \).

(ii) For \( e \in E(L) \) and \( x \in V(L) \) we denote by \( j(e) \) (or \( j_1(x) \) or \( j_2(x) \), respectively) the unique index \( j \leq m \), such that \( S(e) \) (or \( S_1(x) \) or \( S_2(x) \), respectively) "is a piece" of the closed ribbon \( R_j \).

From Eq. [3.122] above we obtain the following explicit formula for \[ T_e(A_\Sigma, B) \in \text{Aut}(V_{j(e)}), \]

\[ e \in E(L): \]

\[ T_e(A_\Sigma, B) = \lim_{\epsilon \to 0} \exp \left( \int_0^1 (\rho_{j(e)})(s)(D_{S(e),s}(A_\Sigma, B))ds \right) \]

\[ = \exp \left( \int_0^1 \int_0^1 \left[ (\rho_{j(e)})(s)(A_\Sigma((c_{e,u})_\Sigma(s))) + (\rho_{j(e)})(s)((Bdt)(c'_{e,u}(s))) \right]dsdu \right) \]

\[ = \rho_{j(e)} \exp \left( \int_0^1 \int_0^1 A_\Sigma((c_{e,u})_\Sigma(s))dsdu + \int_0^1 \int_0^1 (Bdt)(c'_{e,u}(s))dsdu \right) \]  

(D.6)

where \( c_{e,u} : [0,1] \to \Sigma \times S^1, u \in [0,1], \) are the curves given by \( c_{e,u}(s) = S(e)(s,u) \) for all \( s \in [0,1] \).

It will be convenient to fix, for each \( j \leq m \), a basis \((v^j_a)_{a \leq \dim(V_j)}\) of \( V_j \) such that \( (\rho_j)|_{T} \) leaves each of the 1-dimensional subspaces \( \mathbb{C}v^j_a, a \leq \dim(V_j) \) invariant. The real weight corresponding to \( v^j_a \) will be denoted by \( \alpha^j_a \). In other words, \( \alpha^j_a \in \Lambda \) is given by

\[ \rho_j(\exp(b))v^j_a = e^{2\pi i \langle \alpha^j_a, b \rangle} v^j_a \quad \forall b \in \mathfrak{t} \]  

(D.7)

Observe that Eq. (D.6) and Eq. (D.7) imply

\[ (T_e(A_\Sigma, B))_{a \in \Lambda} = \delta_{a_a} e^{2\pi i \langle \alpha_{a_e}, t \rangle} \int_0^1 \int_0^1 A_\Sigma((c_{e,u})_\Sigma(s))dsdu \]  

(D.8)

where we have written \( \alpha_{a_e} \) instead of \( \alpha^j_a \).

**Convention D.2**

(i) For each \( j \leq m \) we set \( \mathcal{I}_j := \{1, 2, \ldots, \dim(V_j)\} \).

(ii) We set \( \mathcal{I}(I) := \times e \in E(L) \mathcal{I}_j(e) \).

(iii) For an element \((a(e))_{e \in E(L)}\) of \( \mathcal{I}(I) \) we will usually use the short notation \( (a) \).

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122 Recall from the paragraph before Remark [3.46] in Sec. 3.5.2 above that we can also obtain an explicit formula for \( T_x(A_\Sigma, B), x \in V(L) \), as an infinite sum of powers of \( 1/k \) (whose coefficients also depend on \( k \)) but we do not yet have a closed formula for this sum.

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decomposition property" iff for every (piecewise) smoothly embedded circle $\Sigma$ contractible in $T$ and from Eq. (D.8) we conclude that $S$ lie in the open ribbon $x$ while $e_2(x)$ and $e_4(x)$ lie in the open ribbon $S_2(x)$, cf. Convention [D.7] above.

From the characterization of the linear form $\beta_{L,D}$ which we introduced in Sec. 3.5.2 above and from Eq. (D.8) we conclude that

$$
\beta_{L,D}(\bigotimes_{e\in E(L)} T_e(A_e^\perp, B) \otimes (\bigotimes_{x\in V(L)} T_x(A_x^\perp, B))) = \sum_{\{a\in I\}} \left[ \prod_{x\in V(L)} (T_x(A_\Sigma, B))^{a(e_1(x))a(e_2(x))}_{a(e_3(x))a(e_4(x))} \right. \\
\left. \times \prod_{e\in E(L)} (T_e(A_\Sigma, B))_{a(e)a(e)} \right] (D.9)
$$

where $(T_x(A_\Sigma, B))_{a,a'}$ are the components of $T_x(A_\Sigma, B)$ w.r.t. to the basis $(v_j^a)_{a\leq \dim(V_j)}$ with $j = j(e)$ and where $(T_x(A_\Sigma, B))^{ab}_{a'b'}$ are the components of $T_x(A_\Sigma, B)$ w.r.t. to the basis $(v_j^a \otimes v_j^{a'})_{a,b}$ (with $j_1 = j_1(x)$, $j_2 = j_2(x)$), i.e. given by $T_x(A_\Sigma, B)\cdot (v_j^a \otimes v_j^{a'}) = \sum_{a,b} (T_x(A_\Sigma, B))^{ab}_{a'b'} (v_j^a \otimes v_j^{a'})$.

Clearly, from Eq. (D.5) and Eq. (D.9) we obtain

$$
F_{L,D}(B) = \sum_{\{a\in I\}} F_a(B) (D.10)
$$

where we have set, for each $(a) \in (I)$

$$
F_a(B) := \int_{A_\Sigma,1} \left[ \prod_{x\in V(L)} (T_x(A_\Sigma, B))^{a(e_1(x))a(e_2(x))}_{a(e_3(x))a(e_4(x))} \right. \\
\left. \times \prod_{e\in E(L)} (T_e(A_\Sigma, B))_{a(e)a(e)} \exp(iS_{CS}(A_\Sigma, B)) \right] DA_\Sigma (D.11)
$$

The following remark will play an important role later.

**Remark D.4** (i) We say that a functional $F : \Omega^p(\Sigma, V) \to \mathbb{C}$, $p \in \{0,1\}$, has the "loop decomposition property" iff for every (piecewise) smoothly embedded circle $C \subset \Sigma$ which is contractible in $\Sigma$ there are functionals $F_{K_j} : \Omega^p(K_j, V) \to \mathbb{C}$, $j = 1,2$, such that for all $\omega \in \Omega^p(\Sigma, V)$ we have

$$
F(\omega) = F_{K_1}(\omega|_{K_1}) \times F_{K_2}(\omega|_{K_2}) (D.12)
$$

where $K_1$ and $K_2$ are the two closed subsets of $\Sigma$ obtained as the closure of the two connected components of $\Sigma\setminus C$.

If $F$ has the "loop decomposition property" in the sense above we obtain, informally\(^\ddagger\)

$$
\int_{\Omega^p(\Sigma, V)} F(\omega) D\omega = \int_{\Omega^p(K_1, V)} F_{K_1}(\omega_1) \left[ \int_{\Omega^p(K_2, V)} F_{K_2}(\omega_2) \delta(\omega_2|_{C}=(\omega_1|_{C}) D\omega_2 \right] D\omega_1 (D.13)
$$

\(^\ddagger\)Note that we could rewrite Eq. (D.13) in a more symmetric way similarly to the last equation on page 633 in [69]. Even though Eq. (D.13) is less elegant than its symmetric version it is more convenient for the calculations below.
where $D\omega$, $D\omega_i$, $i = 1, 2$, are the obvious informal Lebesgue measures on $\Omega^p(\Sigma, V)$, $\Omega^p(K_i, V)$, $i \leq 2$, and where $\delta_{[|\omega_2| = (\omega_1)|C]} D\omega_2$ is a somewhat sloppy notation for the (informal) measure on $\{\omega_2 \in \Omega^p(K_2, V) \mid (\omega_2)|C = (\omega_1)|C\}$ obtained by “disintegration” or “conditioning”\(^{124}\).

(ii) The definition of the notion “loop decomposition property” and the (informal) Eq. (D.13) can be generalized in an obvious way to the situation where instead of one embedded circle $C_i$ we have a finite family $(C_i)_{i \leq n}$ of non-intersecting, (piecewise) smoothly embedded circles which are contractible in $\Sigma$.

For each $x \in V(L)$ we set $D_x := \text{Image}(\pi_\Sigma \circ S_1(x)) \cap \text{Image}(\pi_\Sigma \circ S_2(x)) \subset \Sigma$, cf. Convention [D.1] above. Moreover, by $\Sigma'$ we denote the closure of $\Sigma \setminus \left(\bigcup_{x \in V(L)} D_x\right)$ (in $\Sigma$).

From the definitions above (cf., in particular, Eq. [D.3] above and Eq. (3.125), Eq. (3.121), and Eq. (3.116) in Sec. 3.5.2) we see that for each fixed $A_\Sigma \in A_{\Sigma,t}$ each of the factors $(T_x(A_\Sigma, B))_{a(\varepsilon_1(x))a(\varepsilon_2(x))}^{a(e_1(x))a(e_2(x))}$ in Eq. (D.11) above only depends on $A_{D_x} := (A_\Sigma)|_{D_x}$. Similarly, each factor $(T_x(A_\Sigma, B))_{a(e)a(e)}$ only depends on $A_{D_x} := (A_\Sigma)|_{\Sigma}$. On the other hand, we have

$$\exp(iS_{CS}(A_\Sigma, B)) = \exp(iS_{CS}(A_\Sigma', B)) \prod_{x \in V(L)} \exp(iS_{CS}(A_{D_x}, B))$$

where we have set $S_{CS}(A_\Sigma, B) := 2\pi k \int_{\Sigma} \text{Tr}(dA_\Sigma \cdot B)$ and $S_{CS}(A_{D_x}, B) := 2\pi k \int_{D_x} \text{Tr}(dA_{D_x} \cdot B)$. Taking this into account (as well as the relation $\partial D_x \cong S_1$, $x \in V(L)$) we obtain from the generalization of Eq. (D.13) which is mentioned in part (ii) of Remark [D.4] above\(^{125}\)

$$F_{(a)}(B) = \left[\int_{A_{\Sigma,t}} \prod_e (T_x(A_\Sigma', B))_{a(e)a(e)} \exp(iS_{CS}(A_\Sigma', B)) D\Sigma'\right] \prod_{x \in V(L)} J_x(B) \tag{D.14}$$

where

$$J_{x,(a)}(B) := \int_{A_{Dx,t}} (T_x(A_{D_x}, B))_{a(\varepsilon_1(x))a(\varepsilon_2(x))}^{a(e_1(x))a(e_2(x))} \exp(iS_{CS}(A_{D_x}, B)) D\Sigma' \tag{D.15}$$

and where we have set $A_{\Sigma,t} := \Omega^1(\Sigma', t)$ and $A_{D_x,t} := \Omega^1(D_x, t)$.

We will call $(a) \in (I)$ admissible if there is a $f_{(a)} : \Sigma' \to t$ such that

- $f_{(a)}$ is constant on each\(^{124}\) $Y \subset Y(L)$,
- $f_{(a)}$ is continuous,
- For each $e \in E(L)$ the restriction of $f_{(a)}$ to $\text{Image}(\pi_\Sigma \circ S(e)) \cong [0,1] \times [0,1]$ is given by

$$f_{(a)}((s,u)) = \begin{cases} \alpha_{a(e)} \cdot u + \text{const} & \text{in Case I} \\ \alpha_{a(e)} \cdot (1-u) + \text{const} & \text{in Case II} \end{cases} \quad \text{for all } (s,u) \in [0,1] \times [0,1]$$

---

\(^{124}\)In measure theory the intuitive notion of restricting a measure $d\mu$ on a measurable space $X$ to a $d\mu$-zero-subset of $X$ can be implemented rigorously in many cases (and, in particular, in situations which are analogous to the situation here) with the help of the so-called disintegration theorem or — in the case where $d\mu$ is a probability measure (which is relevant also for the application in the present remark, provided that we reinterpret $\int \cdots D\omega_2$ as a suitable improper integral $\int \cdots D\omega_2$ as in Remark [D.16] in Sec. 2.2.1 above) — with the help of the rigorous notion of conditional expectations/measures.

\(^{125}\)This argument is a bit sloppy. The integral $\int \cdots D\Sigma'$ in Eq. (D.14) should actually be a “conditioned integral” as in Remark [D.4]. For example, in the special case where $V(L)$ consists of only one element $x$ we can take $K_1 = D_x$ and $K_2 = \Sigma'$ and apply (the original version of) Eq. (D.13), which would lead to an expression involving a $\int \cdots \delta_{|A_\Sigma|}|_{\Sigma'} = (A_{D_x})|_{\Sigma'} D\Sigma'$-integral. Strictly speaking we should therefore rewrite some of the $\int \cdots D\Sigma'$-integrals appearing below as conditioned integrals in a suitable way. Doing so would, however, not lead to anything new, i.e. we would again arrive at Eq. (D.19) below.

\(^{126}\)From the definition of $Y(L)$ and of $\Sigma'$ it follows each $Y \subset Y(L)$ is contained in $\Sigma'$. 
where “Case I” and “Case II” are defined in a similar way as the two cases in Observation 3.28 in Sec. 3.3.3 above.

If \( f(a) \) exists it is uniquely determined up to an additive constant, which we will fix by demanding that

\[
 f(a)(\sigma_0) = 0 \tag{D.16}
\]

**Example D.5** In the special situation on Sec. 3.4 where \( V(L) = \emptyset \) and \( E(L) = \{l_1, l_2, \ldots, l_m\} \) every \( (a) \in (I) \) is admissible and we have \( f(a) = \sum_{i=1}^{m} \alpha_i f_i \) where \( \alpha_i := \alpha_a(l_i) \) and \( f_i \) as in Sec. 3.4 above.

Observe that for \( (a) \in (I)_{adm} := \{(a) \in (I) \mid (a) \text{ is admissible}\} \) we have

\[
 \sum_{e \in E(L)} \left( \alpha_{a(e)}, \int_0^1 \int_{(c_e,u)_{\Sigma}(s)} A_{\Sigma'}((c_e,u)_{\Sigma}(s))dsdu \right) = \sum_{e \in E(L)} \left( \alpha_{a(e)}, \int_0^1 \int_{(c_e,u)_{\Sigma}} A_{\Sigma'}du \right) = \ll A_{\Sigma'}, df(a) \gg A_{\Sigma'}, \tag{D.17}
\]

for \( A_{\Sigma'} \in A_{\Sigma',t} \) where \( \ll \cdot, \cdot \gg A_{\Sigma',t} \) is the scalar product on \( A_{\Sigma',t} \) induced by \( g_{\Sigma} \). Using this as well as

\[
 S_{CS}(A_{\Sigma'}, B) = -2\pi k \ll A_{\Sigma'}, dB \gg A_{\Sigma',t}
\]

and Eq. (D.8) we obtain for \( (a) \in (I)_{adm} \)

\[
 \int_{A_{\Sigma',t}} \prod_e (T_e(A_{\Sigma'}, B))_{a(e)a(e)} \exp(ikS_{CS}(A_{\Sigma'}, B))DA_{\Sigma'} = \delta(B_{|\Sigma'} - \frac{1}{k}f(a)) \left( \prod_e e^{2\pi i(\alpha_a(e), f_0^1(Bdt)(c'_{e,u}(s))dsdu)} \right) \tag{D.18}
\]

On the other hand, for \( (a) \notin (I)_{adm} \) we have

\[
 \int_{A_{\Sigma',t}} \prod_e (T_e(A_{\Sigma'}, B))_{a(e)a(e)} \exp(ikS_{CS}(A_{\Sigma'}, B))DA_{\Sigma'} = 0
\]

Combining the last two equations with Eq. (D.14) we obtain

\[
 F(a)(B) = \delta(B_{|\Sigma'} - \frac{1}{k}f(a)) \left( \prod_e e^{2\pi i(\alpha_a(e), f_0^1(Bdt)(c'_{e,u}(s))dsdu)} \right) \prod_{x \in V(L)} J_{x,(a)}(B) \tag{D.19a}
\]

if \( (a) \in (I)_{adm} \) and

\[
 F(a)(B) = 0 \tag{D.19b}
\]

if \( (a) \notin (I)_{adm} \). Combining this with Eq. (D.4) and Eq. (D.10) we now obtain

\[
 Z(\Sigma \times S^1, L) \sim \sum_{y \in I} \sum_{(a) \in (I)_{adm}} \left( \int_{B \cap \Sigma^b} \left( \delta(B_{|\Sigma'} - \frac{1}{k}f(a)) \right) 1_{B_{\Sigma^b}m}(B) \text{Detrig}(B) \exp(-2\pi ik(y, B(\sigma_0))) \right) \times \left( \prod_e e^{2\pi i(\alpha_a(e), f_0^1(Bdt)(c'_{e,u}(s))dsdu)} \right) \prod_{x \in V(L)} J_{x,(a)}(B) \right) DB \tag{D.20}
\]

Now observe that each factor \( J_{x,(a)}(B) \) is actually a function of \( B_{|D_{xe}} \). On the other hand, the factors \( \exp(-2\pi ik(y, B(\sigma_0))) \), \( \int_0^1 f_0^1(Bdt)(c'_{e,u}(s))dsdu \), and \( \delta(B_{|\Sigma'} - \frac{1}{k}f(a)) \) only depend on \( B_{|\Sigma'} \). From Eq. 3.33 and Observation 3.12 in Sec. 3.2.5 above it follows that the same is true for
for $K(D.20)$ and Eq. (D.21) in Sec. 3.3.4 above (and, in particular, the Poisson summation formula), we obtain from Eq. (D.21) above the factor $\det_{rig}(B)$. (Actually, we will sometimes write $\det_{rig}(B|\Sigma')$ instead of $\det_{rig}(B)$ in the following.) Finally, observe that we have $1_{B_{reg}^{ss}}(B) = 1_{B_{reg}^{ss}(\Sigma')}(B_{\Sigma'}) \prod_{x \in V(L)} 1_{B_{reg}^{ss}(D_x)}(B_{\partial D_x})$ where we have set $B_{reg}^{ss}(K) := \{ B \in C^\infty(K, t) \mid \forall \sigma \in K : \alpha \in \mathcal{R} : [B_\alpha(\sigma) \in \mathbb{Z} \Rightarrow dB_\alpha(\sigma) \neq 0]\}$ for $K = \Sigma'$ or $K = \partial D_x, x \in V(L)$.

Consequently, for each fixed $(a) \in (T)_{adm}$ the integrand on the RHS of Eq. (D.20) has the “loop decomposition property” in the sense of Remark D.4 above and by applying (in step (*) part (ii)) of Remark D.4 we obtain \[ \int_B \left\{ \delta\left(d(B|\Sigma' - \frac{1}{\pi} f(a))\right)1_{B_{reg}^{ss}}(B) \det_{rig}(B) \exp\left(-2\pi i k(y, B(\sigma_0))\right) \times \left(\prod_e e^{2\pi i (\alpha_0(e) \cdot j_0^1(Bdt)(c'_{e,u}(s))dsdu)}\right) \prod_{x \in V(L)} J_{x,(a)}(B) \right\} DB \]

\[ = \int_{B_{\Sigma'}} \left\{ \delta\left(d(B|\Sigma' - \frac{1}{\pi} f(a))\right)1_{B_{reg}^{ss}(\Sigma')}(B|\Sigma') \det_{rig}(B|\Sigma') \times \exp\left(-2\pi i k(y, B|\Sigma'(\sigma_0))\right) \left(\prod_e e^{2\pi i (\alpha_0(e) \cdot j_0^1(B|\Sigma'dt)(c'_{e,u}(s))dsdu)}\right) \right\} DB_{\Sigma'} \]

\[ = \int_t db \; e^{-2\pi i k(y,b)} \left[ 1_{B_{reg}^{ss}(\Sigma')}(B|\Sigma') \det_{rig}(B|\Sigma') \times \left(\prod_e e^{2\pi i (\alpha_0(e) \cdot j_0^1(B|\Sigma'dt)(c'_{e,u}(s))dsdu)}\right) \right] \left(\prod_{x \in V(L)} J_{x,(a)}(B|\Sigma')\right)_{\Sigma'} \]

where we have set $B_{\Sigma'} := C^\infty(\Sigma', t)$ and $B_{\partial D_x} := C^\infty(D_x, t)$, where $DB_{\Sigma'}$ is the (informal) Lebesgue measure on $B_{\Sigma'}$ and $DB_{D_x}$ is the (informal) Lebesgue measure on $B_{D_x}$ and where we have set for $b_{\partial D_x} : \partial D_x \to t$

\[ J_{x,(a)}(b_{\partial D_x}) := \int_{B_{D_x}} J_{x,(a)}(B_{D_x})1_{B_{reg}^{ss}(D_x)}(B_{D_x}) \delta_{[(B_{D_x})|\partial D_x=b_{\partial D_x}, DB_{D_x}]} DB_{D_x} \]

The notation $\delta_{[(B_{D_x})|\partial D_x=b_{\partial D_x}, DB_{D_x}]} DB_{D_x}$ here and the notation $\delta_{[(B_{D_x})|\partial D_x=(B_{\Sigma'})|\partial D_x], DB_{D_x}}$ in Eq. (D.21) above is explained in Remark D.4 above.

In step (***) we used Eq. (D.10) above.

Let us set $F(a)(b) := F(B|\Sigma')_{\Sigma'}=b + \frac{1}{\pi} f(a)$ where $F(B|\Sigma')$ is the expression appearing inside $[\cdots]$ in the last two lines of Eq. (D.21) above. It is plausible to expect that $t \ni b \mapsto F(a)(b) \in \mathbb{C}$ is $I$-periodic for each $(a) \in (T)$. If this is the case then, by using analogous arguments as in Sec. 3.3.4 above (and, in particular, the Poisson summation formula), we obtain from Eq. (D.20) and Eq. (D.21)

\[ Z(\Sigma \times S^1, L) \sim \sum_{(a) \in (T)_{adm}} \sum_{a_0 \in A} \left| kQ(a_0)1_{B_{reg}^{ss}(\Sigma')}(B) \det_{rig}(B) \right| \prod_{e \in E(L)} e^{2\pi i (\alpha_0(e) \cdot j_0^1(Bdt)(c'_{e,u}(s))dsdu)} \prod_{x \in V(L)} J_{x,(a)}(B|\partial D_x) \right|_{b=B(a) \cdot a_0} \]

\[ \text{Note that in contrast to the situation in Eq. (D.21) in Remark D.3 above, where $F_{K_1}(\omega_1)$ is a proper function, we now have a delta-function expression $\delta(d(B_{\Sigma'} - \frac{1}{\pi} f(a)))$. It is therefore not totally clear if the application of the argument in Remark D.3 is really justified. This complication can be "defused" somewhat if (as mentioned in Footnote 21 above) we work with a suitable (informal) improper integral $\int \cdots D\mathcal{A}$ as in Remark 2.2 instead of working with the original informal integral $\int \cdots D\mathcal{A}$.) By doing so the informal expressions $\delta^{(c)}(d(B_{\Sigma'} - \frac{1}{\pi} f(a)))$, $\epsilon > 0$ appear instead of the delta-function $\delta(d(B_{\Sigma'} - \frac{1}{\pi} f(a)))$. Here $\delta^{(c)} : B_{\Sigma'} \to \mathbb{R}$ is given, informally, by $\delta^{(c)}(B) = \exp(-\frac{1}{\epsilon^2}\|B\|^2)/\int \exp(-\frac{1}{\epsilon^2}\|B\|^2) DB$.}
with $Q$ given by Eq. (3.75) in Sec. 3.3.3 above and where we have set $B_{(a),a_0} := \frac{1}{k}(a_0 + f_{(a)})$ for each $a_0 \in \Lambda \cap (kQ)$ and $(a) \in (I)$.

**Remark D.6** Recall from the paragraph before Remark 3.46 in Sec. 3.5.2 above that even though we can write $T_x(A_\Sigma, B)$, $x \in V(L)$, explicitly as an infinite series of powers of $1/k$ (whose coefficients also depend on $k$) we do not yet have a closed formula for $T_x(A_\Sigma, B)$ and therefore neither for $J_x(a)((B_{(a),a_0})|_{\partial D_x})$. This means that we have not yet carried out Steps 2–4 of Sec. 3.5.2 completely. Anyway, Eq. (D.23) above is explicit enough in order to allow us to make some comments regarding Step 5 and, in particular, to make it plausible that also for general strictly admissible $L$ we have a good chance of arriving at Eq. (D.1).

On the RHS of Eq. (D.23) we can now make the change of variable $B_{(a),a_0} \to \eta_{(a),a_0}$ where, for each $a_0 \in \Lambda \cap (kQ)$ and $(a) \in (I)$, we have set $\eta_{(a),a_0} := kB_{(a),a_0} - \rho = a_0 + f_{(a)} - \rho$. (In view of Example D.5 above this essentially generalizes the change of variable used in Sec. 3.4.2 above.) Now observe that each $\eta_{(a),a_0}$ takes values in $\Lambda$ and is constant on each $Y \in Y(L)$. Moreover, we can express $(a)$ and $a_0$ by the values of $(\eta(Y))_{Y \in Y(L)}$ with $\eta = \eta_{(a),a_0}$. Accordingly, we can rewrite the RHS of Eq. (D.23) as a sum over maps $\eta: Y(L) \to \Lambda$.

**Observation D.7** Let $J_x(\eta)$ be the expression by which the factor $J_x(a)((B_{(a),a_0})|_{\partial D_x})$, $x \in V(L)$, gets replaced when rewriting the RHS of Eq. (D.23) as a sum over maps $\eta: Y(L) \to \Lambda$ as explained above. Then $J_x(\eta)$ only depends\footnote{In order to see this recall Eq. (D.22) and Eq. (D.13) above and note that the RHS of Eq. (D.13) only depends on $(a) \in (I)$ via the four components $a(e_i(x))$, $i = 1, 2, 3, 4$.} on the values of $\eta(Y)$ for the four faces $Y \in Y(L)$ having $D_x$ on its boundary and on the representations $p_{j_1(x)}$ and $p_{j_2(x)}$, cf. Convention D.7 above, i.e., in other words, on the colors of the two loops $l_i$ and $l_j$ (contained in the proper link $L^0$ associated to $L$) whose $\pi_\Sigma$-projections intersect in $x$. (This observation will be useful below, cf. the last paragraph before Remark D.8.)

It is not difficult to see that those $\eta: Y(L) \to \Lambda$ which do not take values in $\Lambda \cap (kt_{reg} - \rho)$ do not contribute to the sum mentioned above. (This is a consequence of the presence of the factor $1_{t_{reg} \eta(S)}(\Sigma)(B)$ on the RHS of Eq. (D.23)). Accordingly, we can rewrite the RHS of Eq. (D.23) as a sum over maps $\eta: Y(L) \to \Lambda \cap (kt_{reg} - \rho)$, and, by using the bijections (3.82) in Sec. 3.3.3 above, we can rewrite the RHS of Eq. (D.23) as sum over all maps $Y(L) \to \Lambda^k_+$, i.e. as a sum of the form $\sum_{\eta \in col(L)} \cdots$. (Observe that apart from the latter sum we also have a sum $\sum_{\tau \in (W_{aff})_\Lambda Y(L)} \cdots$ as in Eq. (3.112) in Sec. 3.4.2 above.)

We have now seen how the sum $\sum_{\eta \in col(L)} \cdots$ on the RHS of Eq. (D.1) above arises when rewriting the RHS of Eq. (D.23) in the way explained above. Let us now make some comments regarding the four factors $\vert L_1 \vert_1^0$, $\vert L_2 \vert_2^0$, $\vert L_3 \vert_3^0$, and $\vert L_4 \vert_4^0$ on the RHS of Eq. (D.1). Recall that in Sec. 3.4.2 above we showed already in the special case $V(L) = \emptyset$ how the first three factors $\vert L_1 \vert_1^0$, $\vert L_2 \vert_2^0$, $\vert L_3 \vert_3^0$ appear in the heuristic evaluation of $Z(\Sigma \times S^1, L)$. It is easy to believe that this will also be the case in the situation $V(L) \neq \emptyset$, cf. Remark D.8 below.

By far the most interesting and complicated factor on the RHS of Eq. (D.1) is the factor $\vert L_4 \vert_4^0$. At the moment it is totally open whether we really obtain this factor after rewriting the RHS of Eq. (D.23) in the way explained above. However, the following observation gives reason for optimism.

From the definition of $\vert L_4 \vert_4^0$ in Eq. (C.3d) we see that $\vert L_4 \vert_4^0$ is a linear combination of products of matrix elements\footnote{W.r.t. to a suitable choice of basis of the vector space $W(x, \eta)$ mentioned in Appendix C above.} of $T(x, \eta)$. Note that this is analogous to the situation on the RHS of Eq. (D.23) where we have the products $\prod_{x \in V(L)} J_x(a)((B_{(a),a_0})|_{\partial D_x})$ which later lead to products
\[ \prod_{x \in V(L)} J_x(\eta) \] with \( J_x(\eta) \) as in Observation \[ \text{D.7} \] above. Moreover, as we observed in Observation \[ \text{D.7} \] the factors \( J_x(\eta) \) have several properties that are analogous\(^\text{130}\) to the properties of \( T(x,\eta) \) which are mentioned in the second paragraph after Eq. \( \text{(C.3d)} \) in Appendix \[ \text{C} \] above.

**Remark D.8** Above I mentioned that also in the situation \( V(L) \neq \emptyset \) it is plausible to expect that the three factors \( |L|^1, |L|^2, |L|^3 \) appear during the heuristic evaluation of \( Z(\Sigma \times S^1, L) \).

In fact, for the factor \( |L|^3 \) this is definitely the case. Regarding the factor \( |L|^2 \) note that the formula for \( \text{gleam}(Y) \) in Eq. \( \text{(3.101)} \) in Sec. \[ \text{3.4} \] is now not sufficient anymore. The general formula for \( \text{gleam}(Y) \) contains contributions coming form the vertices \( x \in V(L) \). I expect that these contributions will come from the factors \( J_x(\eta) \) mentioned in Observation \[ \text{D.7} \] above. The factors \( |L|^1 \) are somewhat problematic. Observe that for general strictly admissible links \( L \) Eq. \( \text{(3.34)} \) in Sec. \[ \text{3.2.3} \] is in general not applicable as the faces \( Y \in Y(L) \) in general will have “corners”, which are associated to the vertices \( x \in V(L) \). We will therefore have to use a generalization of Eq. \( \text{(3.34)} \) where on the RHS of the generalization of Eq. \( \text{(3.34)} \) extra terms involving the angles in the corners of \( Y \) appear, cf. Footnote \[ \text{38} \] in Sec. \[ \text{3.2.3} \]. It is conceivable that these extra terms also come from the \( J_x(\eta) \)-factors. In fact, if it turns out that Condition \[ \text{7} \] in Sec. \[ \text{3.2.3} \] on the auxiliary Riemannian metric \( g_\Sigma \) can be dropped (cf. Remark \[ \text{3.17} \] in Sec. \[ \text{3.2.6} \]) then there are good chances that this is indeed the case. If Condition \[ \text{7} \] cannot be dropped we still have the alternative of using a different approach for defining \( \det_{\text{rig}}(B) \), for example, the approach sketched in Remark \[ \text{3.15} \] in Sec. \[ \text{3.2.5} \] above where the Gauss-Bonnet formula is irrelevant.

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\[ ^{130} \text{Note that we cannot expect a complete analogy at this stage. According to Remark \[ \text{D.8} \] below we expect the factors } J_x(\eta) \text{ to contribute not only to the factor } |L|^1 \text{ but also to the factors } |L|^2 \text{ and } |L|^3 \text{ on the RHS of Eq. \[ \text{(D.1)} \]. Note also that after rewriting the RHS of Eq. \[ \text{(D.23)} \] in the way explained above we not only have a sum } \sum_{\eta \in \text{col}(L)} \cdots \text{ but also a sum } \sum_{x \in \{V_{\text{aff}}(Y(L)) \}} \cdots . \]

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