Asymptotic expansion of a variation with anticipative weights

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Summary Asymptotic expansion of a variation with anticipative weights is derived by the theory of asymptotic expansion for Skorohod integrals having a mixed normal limit. The expansion formula is expressed with the quasi-torsion, quasi-tangent and other random symbols. To specify these random symbols, it is necessary to classify the level of the effect of each term appearing in the stochastic expansion of the variable in question. To solve this problem, we consider a class \( \mathcal{L} \) of certain sequences \( (I_n)_{n \in \mathbb{N}} \) of Wiener functionals and we give a systematic way of estimation of the order of \( (I_n)_{n \in \mathbb{N}} \). Based on this method, we introduce a notion of exponent of the sequence \( (I_n)_{n \in \mathbb{N}}, \) and investigate the stability and contraction effect of the operators \( D_{u_n} \) and \( D \) on \( \mathcal{L} \), where \( u_n \) is the integrand of a Skorohod integral. After constructed these machineries, we derive asymptotic expansion of the variation having anticipative weights. An application to robust volatility estimation is mentioned.

Keywords and phrases Asymptotic expansion, variation, mixed normal distribution, Malliavin calculus, random symbol, exponent, Watanabe’s delta functional, compact group.

1 Introduction

Let \( \mathcal{W} = \{ \mathcal{W}(h) \}_{h \in H} \) be an isonormal Gaussian process on a real separable Hilbert space \( \mathcal{H} \) with inner product \( \langle \cdot, \cdot \rangle \). That is, \( \mathcal{W} \) is a system of centered Gaussian variables on a probability space

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\((\Omega, \mathcal{F}, P)\) with
\[
E[\mathcal{W}(h_1)\mathcal{W}(h_2)] = \langle h_1, h_2 \rangle
\]
for \(h_1, h_2 \in \mathcal{H}\). Then we have the Malliavin derivative \(D\) and the divergence operator \(\delta\), i.e., the adjoint operator \(D^*\) of \(D\). The Sobolev space with differentiability index \(s\) and integrability index \(p\) is denoted by \(\mathbb{D}^{s,p}\). The \(q\)-fold multiple Wiener integral is denoted by \(I_q(f) = \delta^k(f)\) for \(f \in \mathcal{H}^{\otimes q}\), the space of symmetric elements of \(\mathcal{H}^\otimes q\). For the Malliavin calculus, we refer the reader to Watanabe [73], Ikeda and Watanabe [26], and Nualart [46]. Nourdin and Pecati [45] is an excellent exposition of applications of the Malliavin calculus to central limit theorems by Stein’s method.

In this paper, we are specially interested in asymptotic expansion of a variation of a Wiener process with random weights. This problem is deeply related to the expansion of the power variation of a diffusion process time discretely observed; see Podolskij and Yoshida [55] and Podolskij et al. [54]. In what follows, we suppose that \(\mathcal{H} = L_2(\mathbb{R}_+)\), the \(L_2\)-space of \(\mathbb{R}\)-valued functions on \([0, \infty)\), and that
\[
\langle h_1, h_2 \rangle = \int_0^\infty h_1(t)h_2(t)dt
\]
for \(h_1, h_2 \in \mathcal{H}\). Let \(w_t = \mathcal{W}(1_{[0,t]}(t) \in \mathbb{R}_+\) is a one-dimensional standard Wiener process.

Let \(t_j = t^n_j = j/n\) \((j = 0, 1, ..., n)\) and let denote by \(I_{t_j}\) the indicator function \(1_{t_j}\), \(I_j = [t_{j-1}, t_j]\). Suppose that \(Q\) is a finite subset of \(\{2, 3, 4, ..., \}\). Let us consider the variation
\[
V_n = \sum_{q \in Q} n^{2^{-1}(q-1)/2} \sum_{j=1}^n a_j(q)I_q(1_j^{\otimes q})
\]
with random weights \(a_j(q) = a^n_j(q)\) depending on \(j \in J_n = \{1, ..., n\}\) and \(n \in \mathbb{N}\). The weight \(a_j(q)\) is not necessarily predictable with respect to the Brownian filtration.

For each \(q \in Q\), we consider an \(\mathcal{H}\)-valued functional
\[
u_n = \sum_{q \in Q} \nu_n(q)
\]
given by
\[
u_n(q) = n^{-1/2} \sum_{j=1}^n a_j(q)I_{q-1}(n^{1/2}1_j^{\otimes (q-1)})n^{1/2}1_j
\]
\[
= n^{2^{-1}(q-1)/2} \sum_{j=1}^n a_j(q)I_{q-1}(1_j^{\otimes (q-1)})1_j.
\]

For example, \(a_j(q) = a_q(w_{t_j}, w_{t_{j+1}})\) for a function \(a_q\) and a Wiener process \(w\), however we will treat more general functionals \(a_j(q)\). Suppose for the meantime that
\[
a_j(q) \in \mathbb{D}^{1,2^+} = \cup_{p>2} \mathbb{D}^{1,p} \quad (j \in J_n; q \in Q)
\]
In particular \( u_n(q) \in \text{Dom}(\delta) \) and

\[
M_n := \delta(u_n)
\]

is well defined and in \( L^2 \). Equivalently,

\[
M_n = \sum_{q \in Q} M_n(q) \tag{1.6}
\]

where

\[
M_n(q) = \delta(u_n(q)) = n^{2-1(q-1)} \sum_{j=1}^n a_j(q) I_q(1_j^{\otimes q}) - n^{2-1(q-1)} \sum_{j=1}^n (D_1 a_j(q)) I_{q-1}(1_j^{\otimes (q-1)}) \tag{1.7}
\]

Here \( D_1 a_j(q) = \langle Da_j(q), 1_j \rangle \).

Consider a sequence of random variables

\[
Z_n = M_n + n^{-1/2}N_n \quad (n \in \mathbb{N}) \tag{1.8}
\]

where \((N_n)_{n \in \mathbb{N}}\) is a sequence of random variables. Variable \( N_n \) is of order one and satisfying some regularity conditions, as specified more clearly later. Therefore \( Z_n \) is a perturbation of \( M_n \) by \( n^{-1/2}N_n \). In particular, \( V_n \) is an example of \( Z_n \) if one sets

\[
\sum_{q \in Q} n^{3/2} \sum_{j=1}^n (D_1 a_j(q)) I_{q-1}(1_j^{\otimes (q-1)}) \tag{1.9}
\]

to \( N_n \). The variable \( Z_n \) of the form (1.8) is standard in analysis of the variation of a process driven by \( w \) and statistics for volatility as well. The perturbation (1.8) appeared in [79], [80], [84], [83], [13], [55], [53], [48] etc.

Statistics appearing here is the so-called non-ergodic statistics, where the limit distribution of \( M_n \) is typically a mixed normal distribution. In the statistical context, the Fisher information becomes random even in the limit, and it causes a variance mixture of normal distributions. In Section 2, we will provide asymptotic expansion of the distribution of \((V_n, X_{\infty})\) for a reference variable \( X_{\infty} \). Section 4 treats the general functional \( Z_n \) of (1.8) without the specific expression (1.9) for \( N_n \), and derives asymptotic expansion of the joint distribution of \((Z_n, X_n)\) for a sequence of reference variables \( X_n \).

Asymptotic expansion for a sequence of random variables asymptotically having a mixed normal distribution was derived in Yoshida [84] for continuous martingales by introducing the notion of random symbols called \textit{adaptive random symbol} for tangent and \textit{anticipative random symbol} for torsion, and applied to the power variation by Podolskij and Yoshida [55], the quadratic variation under microstructure noise by Podolskij et al. [54], and to a precise error estimate for the Euler-Maruyama scheme by Podolskij et al. [53]. Even if one starts with the martingale framework, asymptotic expansion is not closed in it, and the Malliavin derivative appears in the anticipative random symbol. Nualart and Yoshida [48] introduced new random symbols called \textit{quasi-tangent} (\( q \)-tangent) and \textit{quasi-torsion} (\( q \)-torsion), and provided asymptotic
expansion of the Skorohod integral with these random symbols. They applied this scheme to some examples including a randomly weighted quadratic form of the increments of a fractional Brownian motion. Section 3.2 recalls the theory of asymptotic expansion for Skorohod integrals.

As summarized in Section 3.2, it is necessary to identify each limit of the $q$-tangent and the $q$-torsion for deriving the asymptotic expansion formula. When we apply the theory to $V_n$, it is not easy to distinguish the effect of each term in the limit since the expression of random symbols becomes fairly complicated, and really it is more difficult for $M_n$ of (1.5) than the quadratic form treated in Nualart and Yoshida [48]. To solve this problem, we consider a class $\mathcal{L}$ of sequences $(\mathcal{I}_n)_{n\in\mathbb{N}}$ of Wiener functionals; the definition of $\mathcal{L}$ is in Section 7.1. In Section 12, we present estimates to $(\mathcal{I}_n)_{n\in\mathbb{N}}$ in $\mathcal{L}$ in order to make the classifications easier in derivation of the expansion formula. The author hopes this section is of interest even apart from asymptotic expansion. Based on the estimate in Section 12, we introduce a notion of exponent of the sequence $(\mathcal{I}_n)_{n\in\mathbb{N}}$ in Section 7, and investigate the effect of the operators $D_{u_n(q)}$ and $D$ applied on $\mathcal{L}$. Section 8 gives a proof to the results in Section 2 by applying the theory of asymptotic expansion for Skorohod integrals recalled in Section 3.2, while Section 10 validates the results in Section 4. Our results are general. There are many applications though we only mention two examples in Sections 5 and 6. The latter section treats asymptotic expansion for an anticipative functional appearing in robust volatility estimation.

Concluding Introduction, we mention developments of the theory of asymptotic expansion (Edgeworth expansion) in the central limit cases, listing some basic literature in the early days (because giving a complete list in a paragraph is impossible) to guide the reader to the history of asymptotic expansion and to indicate where we are now. When the limit of a sequence of random variables is exactly Gaussian, as it is ordinarily true under a sufficiently fast mixing property, asymptotic expansion was proved in various scenarios. Basic references are Cramér [11, 12], Gnedenko and Kolmogorov [18], Bhattacharya [6], Petrov [51], Bhattacharya and Ranga Rao [8] and Bhattacharya and Ghosh [7] among huge historical literature for independent observations. Asymptotic expansion has been established as a basic tool supporting the modern statistical theory. Higher-order asymptotic theory cannot exist without asymptotic expansion: Akahira and Takeuchi [1], Pfanzagl [52] and Ghosh [17]. Historically, multivariate analysis is one of the fields where asymptotic expansion techniques have been well developed (Okamoto [49]). Informative textbooks for asymptotic expansion in the multivariate analysis are Anderson [3] and Fujikoshi, Ulyanov and Shimizu [16]. Theory of Bootstrap methods (Efron [15]) has a theoretical basis on asymptotic expansion (Hall [21]). The reader is referred to Barndorff-Nielsen [4], Jensen [28] and Pace and Salvan [50] for saddle-point approximations. Combining asymptotic expansion with the notion of the $\alpha$-connection, information geometry gave an interpretation of the higher-order efficiency of estimation in terms of the curvature of the fibre associated with the estimator (Amari [2]). Asymptotic expansion was also applied to construction of information criteria for model selection as well as prediction problems; e.g., Konishi and Kitagawa [30], Uchida and Yoshida [70, 72], Komaki [29]. Bhattacharya et al. [5] is an excellent textbook including a concise exposition of asymptotic expansion. Dependent cases were studied by Götze and Hipp [19, 20] for (approximate) Markov chains having a mixing property. It should be commented that in this short paragraph, we can not list a

\footnote{We remark that the correspondence between the two pairs of random symbols is not one to one. In particular, the quasi-tangent does not correspond to the tangent expressed by the adaptive random symbol.}
large number of studies behind the epoch-making paper by Götze and Hipp [19], but a few related papers to be mentioned are Nagaev [43] and Tikhomirov [66], and Lahiri [36] after [19]. Specialized treatments are possible for Gaussian processes (Taniguchi [65]). With stochastic analysis, asymptotic expansion was studied by Kusuoka and Yoshida [32] and Yoshida [82] for continuous-time mixing Markov processes, and by Mykland [41, 42] and Yoshida [79, 80] for martingales about moments and distributions, respectively. The martingale expansion with mixed normal limit distribution (Yoshida [84]) has been applied to power variations and the Euler-Maruyama approximation of a stochastic differential equation, as already mentioned. The Malliavin calculus played an essential role in the above studies [79, 80, 32, 82, 84]. Asymptotic expansion of the heat kernel was an important application of the Malliavin calculus (Watanabe [74], Ikeda and Watanabe [26]), though purely distributional treatment is not necessary, differently from the limit theorems we are now aiming at. In statistics and finance, asymptotic expansion has various applications even if limited to the objects in stochastic analysis: for example, Yoshida [76, 77, 78, 81] based on the Watanabe theory [74] in the Malliavin calculus, Dermoune and Kutoyants [14], Kutoyants [34, 33], Sakamoto and Yoshida [57, 58, 59, 60, 61, 62] together with an application (Sakamoto et al. [56]) to an admissibility problem (Stein’s phenomenon), Masuda and Yoshida [40], Kutoyants and Yoshida [35], Dalalyan and Yoshida [13], Li [37] in statistics for stochastic processes, and in finance Yoshida [75], Kunitomo and Takahashi [31], Takahashi and Yoshida [63, 64], Uchida and Yoshida [71], Li [38, 39] and Cai et al. [10] for option pricing among plenty of literature; a related paper is Hayashi and Ishikawa [23]. YUIMA has implemented asymptotic expansion techniques for automatic fast computation of the expectation of a generic functional defined by a system of stochastic differential equations ([9], [25]). We refer the reader to Yoshida [85] for a short exposition of asymptotic expansion and additional references therein. Today asymptotic expansion has been motivated once again in the surge of studies on central limit theorems for Wiener functionals after the forth moment theorem (Nualart and Peccati [47]). Tudor and Yoshida [68] provided asymptotic expansion for general Wiener functionals. Arbitrary order of asymptotic expansion for Wiener functionals was obtained by Tudor and Yoshida [67] in the ergodic case and applied to a stochastic wave equation. This scheme was also applied to a mixed fractional Brownian motion in Tudor and Yoshida [69]. Yoshida [84] and Nualart and Yoshida [48] derived asymptotic expansion having a mixed normal limit. The last paper is the starting point of this article.

2 Asymptotic expansion of $V_n$

We will present asymptotic expansion to the joint law $\mathcal{L}\{(V_n, X_\infty)\}$ for $V_n$ defined in (1.2) and a reference variable $X_\infty$. The random asymptotic variance of $V_n$ is an example of $X_\infty$ but a generic variable $X_\infty$ will be treated. In this section, we give priority to simplicity of presentation though we can generalize the results in this section by replacing $X_\infty$ by a sequence of variables $X_n (n \in \mathbb{N})$, as will do it for $Z_n$ of (1.8) with a generic $N_n$ together with $X_n$ in Section 4.
2.1 Asymptotic expansion of $E[f(V_n, X_\infty)]$ for a smooth function $f$

The following set of conditions is concerning certain regularity of $a_j(q)$ and limits of some related variables, in particular, it involves $a(t, q)$ for $(t, q) \in [0, 1] \times \mathcal{Q}$. We will write

$$G_\infty = \sum_{q \in \mathcal{Q}} q! \int_0^1 a(t, q)^2 dt. \quad (2.1)$$

To write the asymptotic expansion formula, we use measurable random fields $(a(t, q))_{t \in [0, 1]}$, $(\tilde{a}(t, s, q))_{(t, s) \in [0, 1]^2}$, $(\ddot{a}(t, s, q))_{(t, s) \in [0, 1]^2}$ $(q \in \mathcal{Q})$ and $(\dddot{a}(t))_{t \in [0, 1]}$ (if $2 \in \mathcal{Q}$).

[A] The following properties hold.

(i) $a_j(q) \in D_\infty$ for $j \in J_n$, $n \in \mathbb{N}$ and $q \in \mathcal{Q}$, and that

$$\sup_{n \in \mathbb{N}} \sup_{j \in J_n} \sup_{t \in [0, 1]} \|D_t \cdots D_{t_i} a_j(q)\|_p < \infty \quad (2.2)$$

for every $i \in \mathbb{Z}_+ = \{0, 1, \ldots\}$, $p > 1$ and $q \in \mathcal{Q}$. (Recall that $a_j(q)$ depends on $n$.)

(ii) For every $q \in \mathcal{Q}$ and every $\epsilon > 0$,

$$E\left[ \int_{[0, 1]} 1 \left\{ \left| \sum_{j=1}^n 1_{t_j(s) a_j(q) - a(s, q)} \right| > \epsilon \right\} ds \right] \to 0 \quad (2.3)$$

as $n \to \infty$.

(iii) For every $q \in \mathcal{Q}$ and every $\epsilon > 0$,

$$E\left[ \int_{[0, 1]^2} 1 \left\{ \left| \sum_{j,k=1}^n 1_{t_j(t) a_j(q) - \tilde{a}(t, s, q)} \right| > \epsilon \right\} ds dt \right] \to 0 \quad (2.4)$$

as $n \to \infty$.

(iv) For every $\epsilon > 0$,

$$E\left[ \int_{[0, 1]} 1 \left\{ \left| \sum_{k=1}^n 1_{t_k(t) a_k(2) - \ddot{a}(t, 2)} \right| > \epsilon \right\} dt \right] \to 0 \quad (2.5)$$

as $n \to \infty$ if $2 \in \mathcal{Q}$.

(v) For every $q \in \mathcal{Q}$ and every $\epsilon > 0$,

$$E\left[ \int_{[0, 1]^2} 1 \left\{ \left| \sum_{j,k=1}^n 1_{t_j(t) a_j(q) - \dddot{a}(t, s, q)} \right| > \epsilon \right\} ds dt \right] \to 0 \quad (2.6)$$

as $n \to \infty$.

(vi) If $2 \in \mathcal{Q}$, then

$$E\left[ \int_{[0, 1]^2} 1 \left\{ \left| \sum_{j=1}^n 1_{t_j(t) a_j(q) - \dddot{a}(t, 2)} \right| > \epsilon \right\} ds dt \right] \to 0 \quad (2.7)$$

as $n \to \infty$ for every $\epsilon > 0$. 

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(vii) $G_\infty \in D^\infty$ and
\[
\sup_{t_1, \ldots, t_i \in [0, 1]} \|D_{t_i} \cdots D_{t_1} G_\infty\|_p < \infty
\]
for every $i \in \mathbb{N}$ and $p > 1$. Moreover,
\[
n^{-1} \sum_j \sum_{q \in Q} q! a_j(q)^2 - G_\infty = o_p(n^{-1/2})
\]
(2.8)
as $n \to \infty$.

(viii) $X_\infty \in D^\infty$ and
\[
\sup_{t_1, \ldots, t_i \in [0, 1]} \|D_{t_i} \cdots D_{t_1} X_\infty\|_p < \infty
\]
for every $i \in \mathbb{N}$ and $p > 1$. Moreover, there exists a measurable process $\tilde{X}_\infty = (\tilde{X}_\infty(t))_{t \in [0, 1]}$ such that
\[
E \left[ \int_{[0,1]} 1 \left\{ \sum_{j=1}^n 1_{j_j}(t) n^2 D_{t_j} D_{t_{j-1}} X_\infty - \tilde{X}_\infty(t) \right\} dt \right] \to 0
\]
(2.9)
as $n \to \infty$.

Remark 2.1. $D_{t_i} \cdots D_{t_1} F$ denotes a density that represents the $i$-th Malliavin derivative $D^i F$ of a functional $F$. $D_{t_i} \cdots D_{t_1} F$ is also denoted by $D_{t_1, \ldots, t_i} F$. In the condition, the infinite order of smoothness of $a_j(q)$ is for $L^\infty$-estimate, as explained later. $X_\infty \in D^\infty$ is assumed for simplicity in presentation but the differentiability can be relaxed. Condition [A] (vi) is for handling the specific $N_n$ with expression (1.9) in the case of $V_n$.

In statistics of volatility, special interest is in even polynomials of the increments of the process, i.e., the case $Q \subset 2\mathbb{N}$. However, we will treat a more general case where
\[
Q \cap (Q+1) = \phi,
\]
(2.10)
in other words,
\[
\min\{|q_1 - q_2|; q_1, q_2 \in Q, q_1 \neq q_2\} \geq 2.
\]
This condition is satisfied when one considers a variation with an even power, e.g. The Hermite power variation with anticipative weights is a simple example of the case $\# Q = 1$, that satisfies (2.10). It is possible to remove Condition (2.10) but we do not pursue it here because the results and the presentation will be much more complicated.

Let
\[
a^{(3,0)}(t, s, q_1, q_2) = a_{\circ \circ}(t, s, q_1) a(s, q_2) a(t, 2) + a_{\circ}(t, s, q_1) a_{\circ}(t, s, q_2) a(t, 2)
\]
\[+ a_{\circ}(t, s, q_1) a(s, q_2) a(t, 2).
\]
(2.11)
We need the notion of random symbols (Yoshida [84]). In this paper, we will consider only polynomial random symbols. A polynomial random symbol is a random polynomial of \((z, x)\) the coefficients of which are integrable random variables. We introduce three random symbols. For \(\bar{q}(3) = q_1 + q_2 + q_3\), let

\[
\mathcal{G}^{(3,0)}(iz, ix) = \frac{1}{3} \sum_{q_1, q_2, q_3 \in \mathbb{Q}} 1_{\{\bar{q}(3) = \text{even}\}}(q_1 + q_2)(q_1 - 1) c_{0.5\bar{q}(3)-2}(q_1 - 2, q_2 - 1, q_3 - 1) \\
\times \int_0^1 a(q_1) a(q_2) a(q_3) dt \ (iz)^3
\]

\[
+ \sum_{q_1 \in \mathbb{Q}} 1_{\{q_1 \in \mathbb{Q}\}} q_1! \int_{[0,1]^2} a^{(3,0)}(t, s, q_1) ds dt \ (iz)^3
\]

\[
+ \sum_{q_1 \in \mathbb{Q}} 1_{\{q_1 \in \mathbb{Q}\}} q_1! \int_{t=0}^1 \left( 2^{-1} D_t G_\infty(iz)^5 + D_t X_\infty(iz)^3(ix) \right) \\
\times \int_0^1 \dot{a}(t, s, q_1)a(s, q_1) ds \] \(a(t, 2) dt, \quad (2.12)
\]

where the random field \(a^{(3,0)}(t, s, q_1, q_2)\) in (2.12) is given by (2.11) and \(c_\nu(q_1, q_2, q_3)\) is defined by (7.7). The integrals appearing in the expression (2.12) exist a.s. under \([A]\), since e.g. \(\dot{a} \in L^\infty(dPdt ds)\) by Fatou’s lemma. Let

\[
\mathcal{G}^{(1,1)}(iz, ix) = 1_{\{q_1 \in \mathbb{Q}\}} \left\{ \int_0^1 \bar{X}_\infty(t)a(t, 2) dt \right\}
\]

\[
\times \left( \int_0^1 \bar{D}_t \bar{X}_\infty(a(t, 2) dt ) \right) (ix) + D_t X_\infty(iz)^3 \right) \\
+ \int_0^1 (D_t X_\infty)^2 a(t, 2) dt (iz)^2 \right\} \\
(2.13)
\]

and let

\[
\mathcal{G}^{(1,0)}(iz) = 1_{\{q_1 \in \mathbb{Q}\}} \left\{ \int_0^1 \frac{1}{2} (D_t G_\infty) \dot{a}(t, 2) dt (iz)^3 \\
+ \int_0^1 (D_t X_\infty) \dot{a}(t, 2) dt (ix) + \int_0^1 \ddot{a}(t, 2) dt (iz) \right\}. \quad (2.14)
\]

Then the full random symbol for Theorem 2.2 is given by

\[
\mathcal{G}(iz, ix) = \mathcal{G}^{(3,0)}(iz, ix) + \mathcal{G}^{(1,1)}(iz, ix) + \mathcal{G}^{(1,0)}(iz, ix) \quad (2.15)
\]

where the random symbols \(\mathcal{G}^{(3,0)}(iz, ix)\), \(\mathcal{G}^{(1,1)}(iz, ix)\) and \(\mathcal{G}^{(1,0)}(iz, ix)\) are given by (2.12), (2.13) and (2.14), respectively.
The space of bounded functions on $\mathbb{R}^2$ having bounded derivatives up to order $K$ is denoted by $C^K_b(\mathbb{R}^2)$, equipped with the norm $\|f\| = \sum_{k=0}^{K} \sup_{(z,x)\in\mathbb{R}^2} |\partial^k_{x,z}f(z,x)|$. The following is the first result, the proof of which is given in Section 8.

**Theorem 2.2.** Suppose that Condition [A] and (2.10) are fulfilled. Then

$$\sup_{f \in \mathcal{B}} \left| E \left[ f(V_n, X_{\infty}) \right] - \left\{ E \left[ f(G_\infty^{1/2}\zeta, X_{\infty}) \right] + n^{-1/2} E \left[ \mathcal{G}(\partial_x, \partial_z) f(G_\infty^{1/2}\zeta, X_{\infty}) \right] \right\} \right| = o(n^{-1/2})$$

as $n \to \infty$ for any bounded set $\mathcal{B}$ in $C^8_b(\mathbb{R}^2)$, where $\mathcal{G}$ is given by (2.15) and $\zeta \sim N(0,1)$ independent of $\mathcal{F}$.

**Remark 2.3.** Theorem 2.2 gives stable convergence of $V_n$ to $G_\infty^{1/2}\zeta$, in particular. Of course, it is possible to reduce the conditions for the asymptotic expansion only to prove the stable convergence, but it is an exercise and so details are omitted.

### 2.2 Expansion of $E[f(V_n, X_{\infty})]$ for a measurable function $f$

If $(V_n, X_{\infty})$ is asymptotically non-degenerate in Malliavin’s sense, the differentiability of $f$ can be removed. Application of the Malliavin calculus to the non-degeneracy issue requires more smoothness of functionals. We say $U_n = O_D(b_n)$ as $n \to \infty$ for a sequence of variables $U_n$ and a sequence of positive numbers $b_n$ if $\|U_n\|_{s,p} = O(b_n)$ as $n \to \infty$ for all $(s,p) \in \mathbb{R} \times (1,\infty)$. For a multi-dimensional differentiable functional $F = (F^i)_{i=1,\ldots,i}$, we write $\Delta_F = \det(DF^{i_1} \ldots DF^{i_2})_{i_1, i_2 = 1, \ldots, i}$.

[\text{A}^2] (I) [A] (i), (ii), (iii), (iv), (v), (viii) and the following three conditions hold.

(iii) For every $t, s \in [0,1]$ and $q \in \mathcal{Q}$, $a(t,q) \in D^{5,\infty}$, $\hat{a}(t,s,q) \in D^{5,\infty}$, $\hat{a}(t,2) \in D^{5,\infty}$ (if $2 \in \mathcal{Q}$) and $\hat{a}^{\circ\circ}(t,s,q) \in D^{5,\infty}$, and

$$\max_{q \in \mathcal{Q}} \sup_{t \in [0,1]} \|a(t,q)\|_{5,p} + \max_{q \in \mathcal{Q}} \sup_{(t,s) \in [0,1]^2} \|\hat{a}(t,s,q)\|_{5,p}$$

$$+ \{2 \in \mathcal{Q}\} \sup_{t \in [0,1]} \|\hat{a}(t,2)\|_{5,p} + \max_{q \in \mathcal{Q}} \sup_{(t,s) \in [0,1]^2} \|\hat{a}^{\circ\circ}(t,s,q)\|_{5,p} < \infty \quad (2.16)$$

for every $p > 1$.

(vii) If $2 \in \mathcal{Q}$, then sup$t \in [0,1] \|\hat{a}(t,2)\|_{5,p} < \infty$ for every $p > 1$, and

$$E \left[ \int_{[0,1]^2} 1\{ \sum_{j=1}^{i_1,t(t^n)^2 D_{t,j} \hat{a}(t,2) - \hat{a}(t,2) > \epsilon} \} ds\, dt \right] \to 0 \quad (2.17)$$

as $n \to \infty$ for every $\epsilon > 0$.

(vii) $G_\infty \in D^{\infty}$ and

$$\sup_{t_1, \ldots, t_i \in [0,1]} \|D_{t_1} \cdots D_{t_i} G_\infty\|_p < \infty$$
for every $i \in \mathbb{N}$ and $p > 1$. Moreover, there exists a positive number $\kappa$ such that

$$
n^{-1} \sum_{j=1}^{n} \sum_{q \in \mathcal{Q}} q! a_j(q)^2 - G_\infty = O_{\mathbb{D}^\infty}(n^{-\frac{j}{2} + \kappa}) \tag{2.18}
$$
as $n \to \infty$.

\hspace{1cm} (II) $G_\infty^{-1} \Delta_{X_\infty}^{-1} \in L^{\infty}$.

Condition (2.18) is realistic. An example will be given in Section 8.6.

For a polynomial random symbol $\varsigma(iz,ix) = \sum_\alpha c_\alpha(iz,ix)^\alpha$, where each $c_\alpha$ is a random variable and the multi-index $\alpha$ is in $\mathbb{Z}^2_+$, the expected action of the adjoint $\varsigma(\partial_z,\partial_x)^\ast$ to $\phi(z;0,G_\infty)\delta_x(X_\infty)$ is defined by

$$
E \left[ \varsigma(\partial_z,\partial_x)^\ast \left\{ \phi(z;0,G_\infty)\delta_x(X_\infty) \right\} \right] = \sum_\alpha (-\partial_z, -\partial_x)^\alpha \left[ c_\alpha \phi(z;0,G_\infty)\delta_x(X_\infty) \right]
$$

$$
= \sum_\alpha (-\partial_z, -\partial_x)^\alpha \left\{ E[c_\alpha \phi(z;0,G_\infty)|X_\infty = x] p^{X_\infty}(x) \right\}
$$

(2.19)

Here $\delta_x(X_\infty)$ is Watanabe’s delta function, that is, the pull-back of the delta function $\delta_x$ by $X_\infty$, that is a generalized Wiener functional in $\mathbb{D}^{\infty} = \cup_{s \in \mathbb{R}} \cap_{p>1} \mathbb{D}^{s,p}$, suppose that $X_\infty$ is non-degenerate in Malliavin’s sense, as well as smoothness of functionals appearing. $p^{X_\infty}(x)$ is a good version of density function of the distribution $\mathcal{L}\{X_\infty\}$. It and its continuous derivatives exist so that the differentiations in (2.19) make the ordinary sense. See Section 3.1.

Define the (polynomial) random symbol $\mathcal{S}_n$ by

$$
\mathcal{S}_n = 1 + n^{-1/2} \mathcal{S},
$$

where the random symbol $\mathcal{S}$ is given in (2.15) for Theorem 2.4 forthcoming. The asymptotic expansion formula is expressed by the density

$$
p_n(z,x) = E \left[ \mathcal{S}_n(\partial_z, \partial_x)^\ast \left\{ \phi(z;0,G_\infty)\delta_x(X_\infty) \right\} \right]. \tag{2.20}
$$

The density function $p_n$ is well defined under Condition $[A^2]$.

Let $\mathcal{V}(M, \gamma)$ denote the set of measurable functions $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying $|f(z,x)| \leq M(1 + |z| + |x|)^\gamma$ for all $(z,x) \in \mathbb{R}^2$.

**Theorem 2.4.** Suppose that Condition $[A^3]$ is satisfied. Then

$$
\sup_{f \in \mathcal{V}(M, \gamma)} \left| E[f(V_n, X_\infty)] - \int_{\mathbb{R}^2} f(z,x)p_n(z,x)dzdx \right| = o(r_n)
$$
as $n \to \infty$ for every $(M, \gamma) \in (0, \infty)^2$. Here the density $p_n(z,x)$ is defined by (2.20) for the random symbol $\mathcal{S}$ given in (2.15).

Proof of Theorem 2.4 is in Section 9. In Theorems 2.2 and 2.4, $N_n$ and $X_n$ were specific. We will generalize them in Section 4 after clarifying what kind conditions should be posed by the general theory recalled in Section 3.2.
3 Basic results in asymptotic expansion

3.1 Donsker-Watanabe’s delta function

Denote by $S(\mathbb{R}^d)$ the space of rapidly decreasing smooth functions on $\mathbb{R}^d$. The dual of $S(\mathbb{R}^d)$ is the space of Schwartz distributions and denoted by $S'(\mathbb{R}^d)$. Let $A = 1 + |x|^2 - \frac{1}{2} \Delta$, where $\Delta$ is the Laplacian on $\mathbb{R}^d$. Define the norm $\|u\|_{2k} = \sup_{x \in \mathbb{R}^d} |A^k u(x)|$ for $u \in S(\mathbb{R}^d)$ and $k \in \mathbb{Z}$. For negative $k$, $A^k$ is well defined as an integral operator.

Denote $\hat{C}(\mathbb{R}^d)$ the space of continuous functions $g$ on $\mathbb{R}^d$ such that $\lim_{|x| \to \infty} g(x) = 0$. The spade $\hat{C}(\mathbb{R}^d)$ is equipped with the sup-norm $\| \cdot \|_\infty$. For $m \in \mathbb{Z}_+$, defined $\hat{C}^{-2m}(\mathbb{R}^d)$ by

$$\hat{C}^{-2m}(\mathbb{R}^d) = \left\{ g \in S'(\mathbb{R}^d); A^{-m} g \in \hat{C}(\mathbb{R}^d) \right\}.$$

For $g \in \hat{C}^{-2m}(\mathbb{R}^d)$, we defined $\|g\|_{-2m}$ by

$$\|g\|_{-2m} = \|A^{-m} g\|_\infty.$$

By the representation theorem of $S'(\mathbb{R}^d)$, we know

$$S'(\mathbb{R}^d) = \bigcup_{m \in \mathbb{Z}_+} \hat{C}^{-2m}(\mathbb{R}^d).$$

More precisely, every $g \in S'(\mathbb{R}^d)$ has a representation

$$g(x) = (1 + |x|^2)^{s/2} \partial^\alpha f(x)$$

for some $\alpha \in \mathbb{Z}_+^d$, $s \geq 0$ and some bounded measurable function $f$, where the partial derivatives is in the sense of the generalized derivatives. Then $\partial^\nu A^{-m} \partial^\lambda g \in \hat{C}(\mathbb{R}^d)$ whenever $2m > |\nu| + |\lambda| + |\alpha| + s$. Moreover,

$$\|\partial^\nu A^{-m} \partial^\lambda g\|_\infty \leq C\|f\|_\infty$$

for some constant $C$ depending on $m, |\nu|, |\lambda|, |\alpha|, s$ and $d$, but independent of $f$. In the case of the Dirac delta function $\delta_a$ on $\mathbb{R}^d$,

$$\partial^\nu_x \delta_a(x) \in \hat{C}^{-2m}(\mathbb{R}^d)$$

for $m \in \mathbb{Z}_+$ satisfying $m > (|\mathbf{n}| + d)/2$. Therefore, $\partial^\nu_a \delta_a(F)$ is well defined for $F \in D^{2m+1,\infty}(\mathbb{R}^d)$ and $m > (|\mathbf{n}| + d)/2$ if $\Delta_{F}^{-1} \in L^\infty$. The composite functional $\partial^\nu_a \delta_a(F)$ requires the same regularity. In this paper, we use $\delta_x(X_\infty)$ and its derivatives of limited order. For example, the differentiability requested to $X_\infty$ in Condition $[C]$ in Section 3.2 is sufficient for the use. We refer the reader to Watanbe [73] and Ikeda and Watanabe [26] for details of the notion of the generalized Wiener functionals.
3.2 Asymptotic expansion of Skorohod integrals

This section overviews the machinery given by Nualart and Yoshida [48] for asymptotic expansion of Skorohod integrals, and extends their results by a perturbation method. Since there are many other applications than this paper gives and the extension is general, in this section, we keep the multi-dimensional setting in [48]. In this section, we do not need the structure of the Hilbert space for a standard Brownian motion. So we will work on a probability space \((\Omega, \mathcal{F}, P)\) equipped with an isonormal Gaussian process \(\mathcal{W} = \{W(h)\}_{h \in \mathcal{H}}\) on a real separable Hilbert space \(\mathcal{H}\) having an inner product \(\langle \cdot, \cdot \rangle\) and the norm \(| \cdot |_\mathcal{H} = (\langle \cdot, \cdot \rangle)^{1/2}\).

We will consider a sequence of \(d\)-dimensional random variables \(Z_n\) having a decomposition

\[
Z_n = M_n + r_n N_n
\]

where \(M_n = \delta(u_n) = (\delta(u^i_n))_{i=1,\ldots,d}\) is the Skorohod integral for a generic \(u_n = (u^i_n)_{i=1,\ldots,d} \in \text{Dom}(\delta)^d\), \(N_n\) is a \(d\)-dimensional random variable and \((r_n)_{n \in \mathbb{N}}\) is a sequence of positive numbers such that \(\lim_{n \to \infty} r_n = 0\). Later, we will set \(r_n = n^{-1/2}\) as in (1.8), when using the asymptotic expansion for Skorohod integrals to the applications of this paper. A reference variable will be denoted by a \(d_1\)-dimensional random variable \(X_n\). Our interest is in deriving asymptotic expansion for the joint distribution \(\mathcal{L}\{\{Z_n, X_n\}\}\). To be considered here is the situation where \(M_n \to^d G_\infty \zeta\) as \(n \to \infty\), where \(G_\infty\) is a \(d \times d\) positive symmetric random matrix and \(\zeta\) is a \(d\)-dimensional standard normal random vector independent of \(G_\infty\) and possibly defined on an extension of \((\Omega, \mathcal{F}, P)\). See Nourdin, Nualart and Pecati [44] for the stable limit theorems on the Wiener space.

The \(r\) times tensor product of a vector \(v\) is denoted by \(v \otimes^r\). For a tensor \(T = (T_{i_1,\ldots,i_k})_{i_1,\ldots,i_k}\) and vectors \(v_1 = (v^i_1)_{i_1,\ldots,i_k}, \ldots, v_k = (v^i_k)_{i_1,\ldots,i_k}\), we write

\[
T[v_1, \ldots, v_k] = T[v_1 \otimes \cdots \otimes v_k] = \sum_{i_1,\ldots,i_k} T_{i_1,\ldots,i_k} v^i_1 \cdots v^i_k.
\]

The brackets \([\ ]\) stands for a multilinear mapping. For \(\mathcal{H}\)-valued tensors \(S = (S_i)\) and \(T = (T_j)\), the tensor with components \((\langle S_i, T_j \rangle)_{i,j}\) is denoted by \(\langle S, T \rangle\). This rule is also applied to tensor-valued tensors. The Malliavin derivative of a matrix-valued functional \(F = (F^{ij})\) is denoted by \(DF\).

Nualart and Yoshida [48] defined three random symbols analogous to the adaptive random symbol (tangent) and the anticipative random symbol (torsion) introduced in Yoshida [84], though these two sets of random symbols are not corresponding one-to-one. The **quasi-tangent** (q-tangent) is defined by

\[
\text{qTan}[iz_1, iz_2] = r_n^{-1}\left(\langle DM_n[iz_1], u_n[iz_2] \rangle - G_\infty[iz_1, iz_2]\right)
\]

for \((iz_1, iz_2) \in (i\mathbb{R}^d)^2\). The **quasi-torsion** (q-torsion) and **modified quasi-torsion** (modified q-torsion) are defined by

\[
\text{qTor}[iz_1, iz_2, iz_3] = r_n^{-1}\left(D\langle DM_n[iz_1], u_n[iz_2] \rangle_{\mathcal{H}}, u_n[iz_3]\right)
\]

for \((iz_1, iz_2, iz_3) \in (i\mathbb{R}^d)^3\).
and

\[ \text{mqTor}[iz_1, iz_2, iz_3] = r_n^{-1}(DG_\infty[iz_1, iz_2], u_n[iz_3]) \]

for \((iz_1, iz_2, iz_3) \in (i\mathbb{R}^d)^3\), respectively. These random symbols are well defined under suitable differentiability of variables. They are related by the formula

\[ \langle DqTan[(iz)^{\otimes 2}], u_n[iz] \rangle = \text{qTor}[(iz)^{\otimes 3}] - \text{mqTor}[(iz)^{\otimes 3}], \]

or symbolically,

\[ D_{un}qTan = \text{qTor} - \text{mqTor}. \]

We could replace \(D_{un}M_n = \langle DM_n, u_n \rangle\) by its symmetric version

\[ \frac{1}{2}(\langle DM_n, u_n \rangle + \langle u_n, DM_n \rangle) \]

when defining \(qTan\), but we go with the above definition.

When \((Z_n, X_n) \rightarrow^d (G^{1/2}_\infty, X_\infty)\) as \(n \rightarrow \infty\), we have

\[ E[f(Z_n, X_n)] - E[f(G^{1/2}_\infty, X_\infty)] = o(1) \]

as \(n \rightarrow \infty\) for any bounded continuous function \(f : \mathbb{R}^d \rightarrow \mathbb{R}, \ d = d + d_1\), by definition of the convergence in distribution. Asymptotic expansion gives more precise approximation to \(E[f(Z_n, X_n)]\) than (3.2) as

\[ E[f(Z_n, X_n)] - E[f(G^{1/2}_\infty, X_\infty)] - r_n\mathbb{L}(f) = o(r_n), \]

where \(\mathbb{L}(f)\) is a linear functional of \(f\). The approximation (3.3) holds for smooth function \(f\) in general. However, if the distribution \(\mathbb{L}\{(Z_n, X_n)\}\) is regular to some extent, then (3.3) is valid for non-differentiable functions \(f\), and moreover uniformly in \(f\) in a certain class of measurable functions. Then the notion of local density is necessary and a truncation technique will be required to carry out the validation. For this purpose, we introduce a random variable \(\psi_n : \Omega \rightarrow [0, 1]\) depending on \(n \in \mathbb{N}\).

Write \(X_n = r_n^{-1}(X_n - X_\infty)\),

\[ G_n^{(2)} = r_nqTan = D_{un}M_n - G_\infty \]

and

\[ G_n^{(3)} = r_nmqTor = D_{un}G_\infty. \]

We write

\[ D_{un}^k = D_{un} \cdots D_{un}. \]
Define the following random symbols:

\[
\begin{align*}
\mathcal{S}_n^{(3,0)}(iz) &= \mathcal{S}_n^{(3,0)}(iz, ix) = \frac{1}{3} r_n^{-1} D_{n[iz]}^2 M_n[iz] = \frac{1}{3} q \tan [(iz)^{\otimes 3}], \\
\mathcal{S}_0,n^{(2,0)}(iz) &= \mathcal{S}_0,n^{(2,0)}(iz, ix) = \frac{1}{2} r_n^{-1} C_n^{(2)}(iz) \\
&= \frac{1}{2} r_n^{-1} (D_{n[iz]} M_n[iz] - G_{\infty}[(iz)^{\otimes 2}]) = \frac{1}{2} q \tan [(iz)^{\otimes 2}], \\
\mathcal{S}_n^{(1,1)}(iz, ix) &= r_n^{-1} D_{n[iz]} X_{n[ix]}, \\
\mathcal{S}_n^{(1,0)}(iz) &= \mathcal{S}_n^{(1,0)}(iz, ix) = N_n[iz], \\
\mathcal{S}_n^{(0,1)}(iz) &= \mathcal{S}_n^{(0,1)}(iz, ix) = X_n[ix], \\
\mathcal{S}_1,n^{(2,0)}(iz) &= \mathcal{S}_1,n^{(2,0)}(iz, ix) = D_{n[iz]} N_n[iz], \\
\mathcal{S}_1,n^{(1,1)}(iz, ix) &= D_{n[iz]} X_n[ix]
\end{align*}
\]

for \( z \in \mathbb{R}^d \) and \( x \in \mathbb{R}^{d_1} \).

Let

\[
\Psi(z, x) = \exp \left( 2^{-1} G_{\infty}[(iz)^{\otimes 2}] + X_{\infty}[ix] \right)
\]

for \( z \in \mathbb{R}^d \) and \( x \in \mathbb{R}^{d_1} \). Suppose that there are some (polynomial) random symbols \( \mathcal{S}_n^{(3,0)}(iz, ix), \mathcal{S}_0,n^{(2,0)}(iz, ix), \mathcal{S}_n^{(1,1)}(iz, ix), \mathcal{S}_n^{(1,0)}(iz, ix), \mathcal{S}_n^{(0,1)}(iz, ix), \mathcal{S}_1,n^{(2,0)}(iz, ix) \) and \( \mathcal{S}_1,n^{(1,1)}(iz, ix) \). Condition \([B]\) in Section 3.2 of Nualart and Yoshida [48] is rephrased as follows.

\[ \begin{align*}
[B]\ (i) \quad & u_n \in \mathbb{D}^{4p}(S \otimes \mathbb{R}^d), G_{\infty} \in \mathbb{D}^{3p}(\mathbb{R}^d \otimes \mathbb{R}^d), N_n \in \mathbb{D}^{3p}(\mathbb{R}^d), X_n, X_{\infty} \in \mathbb{D}^{3p}(\mathbb{R}^{d_1}) \text{ and } \psi_n \in \mathbb{D}^{2p_1} \text{ for some } p \text{ and } p_1 \text{ satisfying } 5p^{-1} + p_1^{-1} \leq 1. \\
(ii) \quad & \text{As } n \to \infty, \\
\|u_n\|_{1,p} &= O(1) \quad (3.4) \\
\sum_{k=2,3} \|G_n^{(k)}\|_{p/2} &= O(r_n) \quad (3.5) \\
\|D_{u_n} G_n^{(2)}\|_{p/3} &= O(r_n) \quad (3.6) \\
\|D_{u_n} G_n^{(3)}\|_{p/3} &= o(r_n) \quad (3.7) \\
\sum_{k=2,3} \|D_{u_n}^{2} G_n^{(k)}\|_{p/4} &= o(r_n) \quad (3.8) \\
\|D_{u_n} X_{\infty}\|_{p} &= O(r_n) \quad (3.9)
\end{align*} \]
\[ \|D^2_{u_n} X_n\|_{p/3} = o(r_n) \]  (3.10)

\[ \|D^3_{u_n} X_n\|_{p/4} = o(r_n) \]  (3.11)

\[ \|N_n\|_{3,p} + \|\diamond X_n\|_{3,p} = O(1) \]  (3.12)

\[ \|D^2_{u_n} N_n\|_{p/3} + \|\diamond D^2_{u_n} X_n\|_{p/3} = o(1) \]  (3.13)

\[ \|D^3_{u_n} N_n\|_{p/4} + \|\diamond D^3_{u_n} X_n\|_{p/4} = o(1) \]  (3.14)

\[ \|1 - \psi_n\|_{2,p_1} = o(r_n). \]  (3.15)

(iii) For every \( z \in \mathbb{R} \) and \( x \in \mathbb{R} \),

\[ \lim_{n \to \infty} E\left[ \Psi(z, x) \overline{\Theta_n(i z, i x)} \psi_n \right] = E\left[ \Psi(z, x) \overline{\Theta(i z, i x)} \right] \]  (3.16)

for the pairs of random symbols \((\overline{\Theta_n}, \overline{\Theta})= (\overline{\Theta^{(3,0)}}, \overline{\Theta^{(3,0)}}), (\overline{\Theta^{(2,0)}}, \overline{\Theta^{(2,0)}}), (\overline{\Theta^{(1,1)}}, \overline{\Theta^{(1,1)}}), (\overline{\Theta^{(1,0)}}, \overline{\Theta^{(1,0)}}), (\overline{\Theta^{(0,1)}}, \overline{\Theta^{(0,1)}}), (\overline{\Theta^{(2,0)}}, \overline{\Theta^{(2,0)}}), \) and \((\overline{\Theta^{(1,1)}}, \overline{\Theta^{(1,1)}})\).

Here we are assuming that each \( \overline{\Theta} \) is a random polynomial of \((i z, i x)\) with integrable coefficients. The convergence in (iii) of Condition \([B]\) does not necessarily mean the \( L^1 \)-convergence of the random symbol \( \overline{\Theta} \) to \( \overline{\Theta} \). It only requires “weak” convergence in that the projection of \( \overline{\Theta}_n \) onto \( \Psi \) converges to that of \( \overline{\Theta} \) onto \( \Psi \). Indeed, the degree of the random polynomial \( \overline{\Theta} \) may differ from that of \( \overline{\Theta}_n \).

In some cases, \( N_n \) admits a decomposition \( N_n = N'_n + N''_n \) such that Conditions (3.13) and (3.14) can be verified easily for \( N'_n \). For \( N''_n \), we can try the perturbation method with the help of a limit theorem. The perturbation method was used in Yoshida [79, 80, 84] and Sakamoto and Yoshida [59].

The random symbol \( \mathcal{G}(i z, i x) \) is defined by

\[ \mathcal{G}(i z, i x) = \mathcal{G}^{(3,0)}(i z, i x) + \mathcal{G}^{(2,0)}_0(i z, i x) + \mathcal{G}^{(1,1)}_0(i z, i x) + \mathcal{G}^{(1,0)}(i z, i x) + \mathcal{G}^{(0,1)}(i z, i x) + \mathcal{G}^{(2,0)}_1(i z, i x) + \mathcal{G}^{(1,1)}_1(i z, i x). \]  (3.17)

Denote by \( \beta_0 \) the degree in \((z, x)\) of the random symbol \( \mathcal{G} \). Let \( \beta = \max\{7, \beta_0\} \). Now Theorem 3.3 of Nualart and Yoshida [48] is rephrased by the following theorem.

**Theorem 3.1.** Suppose that Condition \([B]\) is satisfied. Then

\[ \sup_{f \in \mathcal{B}} \left| E\left[ f(Z_n, X_n) \right] - \left\{ E\left[ f(G_{\infty}^{1/2} \zeta, X_\infty) \right] + r_n E\left[ \mathcal{G}(\partial_z, \partial_x) f(G_{\infty}^{1/2} \zeta, X_\infty) \right] \right\} \right| = o(r_n) \]

as \( n \to \infty \) for any bounded set \( \mathcal{B} \) in \( C_{b}^{\beta+1}(\mathbb{R}^d) \), where \( \zeta \sim N_d(0, I_d) \) independent of \( \mathcal{F} \).
The role of \( \psi_n \) is implicit in Theorem 3.1. Simplification by setting \( \psi_n = 1 \) is obvious if the other conditions are satisfied. However, the truncation functional \( \psi_n \) is indispensable when asymptotic expansion of \( E[f(Z_n, X_n)] \) for measurable functions \( f \), as discussed below.

Asymptotic expansion of \( E[f(Z_n, X_n)] \) is possible for measurable functions if the law \( \mathcal{L}\{(Z_n, X_n)\} \) is sufficiently regular. In the present context, it is natural to assess the regularity by the local non-degeneracy of the Malliavin covariance of the variables in question. Here we recall a set of conditions among others considered in Nualart and Yoshida [48]. It involves a parameter \( \ell = d + 8 \) concerning differentiability. We denote by \( \beta_x \) the maximum degree in \( x \) of \( \mathcal{G} \). Let \( d_2 = (\ell + \beta_x - 7) \vee \left( 2 \left( (d_1 + 2)/2 \right) + 2 \left( (\beta_x + 1)/2 \right) \right) \). Let \( \mathbb{D}^{d+\infty} = \bigcap_{p>1} \mathbb{D}^{d+p} \). For a multi-dimensional functional \( F \), we write \( \Delta_F = \det \sigma_F \), \( \sigma_F \) being the Malliavin covariance matrix of \( F \). The following condition rephrases Condition \([C^\delta]\) of Section 7 of Nualart and Yoshida [48].

\[ \begin{array}{l}
\text{[C]} \quad \text{(i)} \quad u_n \in \mathbb{D}^{\ell+1,\infty} (\mathcal{G} \otimes \mathbb{R}^d), \quad G_{\infty} \in \mathbb{D}^{(\ell+1)\vee d_2,\infty} (\mathbb{R}^d \otimes \mathbb{R}^d), \quad N_n \in \mathbb{D}^{\ell,\infty} (\mathbb{R}^d), \quad X_n \in \mathbb{D}^{\ell,\infty} (\mathbb{R}^{d_1}), \quad X_\infty \in \mathbb{D}^{\ell_0(d_1+1),\infty} (\mathbb{R}^{d_1}).
\end{array} \]

\( \text{(ii)} \) There exists some positive number \( \kappa \) such that the following estimates hold for every \( p > 1 \):

\[ \begin{align*}
\| u_n \|_{\ell, p} &= O(1) \\
\| G_n^{(2)} \|_{\ell-2, p} &= O(r_n) \\
\| G_n^{(3)} \|_{\ell-2, p} &= O(r_n) \\
\| D_{u_n} G_n^{(2)} \|_{\ell-1, p} &= O(r_n^{1+\kappa}) \\
\| D_{u_n} G_n^{(2)} \|_{\ell-3, p} &= O(r_n^{1+\kappa}) \\
\| D_{u_n} X_\infty \|_{\ell-3, p} &= O(r_n) \\
\| D_{u_n} X_\infty \|_{\ell-2, p} &= O(r_n^{1+\kappa}) \\
\| N_n \|_{\ell-1, p} + \| \tilde{X}_n \|_{\ell-1, p} &= O(1)
\end{align*} \]

\[ \begin{align*}
\| D_{u_n}^2 N_n \|_{\ell-2, p} + \| D_{u_n}^2 \tilde{X}_n \|_{\ell-2, p} &= O(r_n^\kappa).
\end{align*} \]

\( \text{(iii)} \) For each pair of random symbols \( (\mathcal{Z}_n, \mathcal{X}) = (\mathcal{G}^{(3,0)}_n, \mathcal{G}^{(3,0)}_0), (\mathcal{G}^{(2,0)}_n, \mathcal{G}^{(2,0)}_0), (\mathcal{G}^{(1,1)}_n, \mathcal{G}^{(1,1)}_0), (\mathcal{G}^{(1,0)}_n, \mathcal{G}^{(1,0)}_0), (\mathcal{G}^{(0,1)}_n, \mathcal{G}^{(0,1)}_0), (\mathcal{G}^{(2,0)}_1, \mathcal{G}^{(2,0)}_0), \text{ and } (\mathcal{G}^{(1,1)}_1, \mathcal{G}^{(1,1)}_0), \) the following conditions are fulfilled:
(a) \( \mathfrak{T} \) is a (polynomial) random symbol the coefficients of which are in \( \mathbb{D}^{d+\beta_x+1,1} = \bigcup_{p>1} \mathbb{D}^{d+\beta_x+1,p} \).

(b) For some \( p > 1 \), there exists a (polynomial) random symbol \( \mathfrak{T}_n \) that has \( L^p \) coefficients and the same degree as \( \mathfrak{T} \), such that
\[
E[\Psi(z,x)\mathfrak{T}_n(iz,i x)] = E[\Psi(z,x)\mathfrak{T}(iz,i x)]
\]
and \( \mathfrak{T}_n \to \mathfrak{T} \) in \( L^p \) as \( n \to \infty \).

(iv) (a) \( G_{\infty}^{-1} \in L^{\infty} \).

(b) There exists a positive number \( \kappa \) such that
\[
P[\Delta(M_n,X_{\infty}) < s_n] = O(r_n^{1+\kappa})
\]
for some positive random variables \( s_n \in \mathbb{D}^{\ell-2,\infty} \) satisfying
\[
\sup_{n \in \mathbb{N}} (\|s_n^{-1}\|_p + \|s_n\|_{\ell-2,p}) < \infty
\]
for every \( p > 1 \).

In the case \( d = d_1 = 1 \), the indices appearing in \( [C] \) are
\[
\ell = 10 \quad \text{and} \quad d_2 = (3 + \beta_x) \vee (2 + 2[(\beta_x + 1)/2] ).
\]
In Condition \( [C] \), the convergence \( \mathfrak{T}_n \to \mathfrak{T} \) in \( L^p \) means each coefficient of \( \mathfrak{T}_n \) converges to the corresponding coefficient of \( \mathfrak{T} \) in \( L^p \). The functional \( s_n \) in Condition \( [C] \) (iv) (b) makes a suitable truncation functional \( \psi_n \), though we just mention it here; see Nualart and Yoshida [48] for details of a construction of \( \psi_n \). The truncation method is essential because \( (M_n,X_{\infty}) \) is not necessarily uniformly non-degenerate, but locally asymptotically non-degenerate. In statistics, variables are not always defined as a smooth Wiener functional, and then there is no way the uniform non-degeneracy can be obtained. Condition \( [C] \) is much stronger than Condition \( [B] \) even with \( \psi_n = 1 \).

Define the (polynomial) random symbol \( \mathfrak{G}_n \) by
\[
\mathfrak{G}_n = 1 + r_n \mathfrak{G}, \quad (3.27)
\]
where \( \mathfrak{G} \) is the general random symbol defined by (3.17). In this general setting, the asymptotic expansion formula is given by the density
\[
p_n(z,x) = E \left[ \mathfrak{G}_n(\partial_z, \partial_x)^* \left\{ \phi(z;0,G_{\infty})\delta_x(X_{\infty}) \right\} \right]. \quad (3.28)
\]
The density function \( p_n \) is well defined under Condition \( [C] \).

The following theorem rephrases Theorem 7.6 of of Nualart and Yoshida [48]. In the present context, we denote by \( \mathcal{E}(M,\gamma) \) the set of measurable functions \( f: \mathbb{R}^d \to \mathbb{R} \) satisfying \( |f(z,s)| \leq M(1+|z|+|x|)^{\gamma} \) for all \( (z,x) \in \mathbb{R}^2 \).

**Theorem 3.2.** Suppose that Condition \( [C] \) is satisfied. Then
\[
\sup_{f \in \mathcal{E}(M,\gamma)} \left| E[f(Z_n,X_n)] - \int_{\mathbb{R}^d} f(z,x)p_n(z,x)dzdx \right| = o(r_n)
\]
as \( n \to \infty \) for every \( (M,\gamma) \in (0,\infty)^2 \).

Proofs of Theorems 3.2 and 3.2 are in Nualart and Yoshida [48].
3.3 Perturbation method

We recall the perturbation method used in Yoshida [79, 80, 84] and in Sakamoto and Yoshida [59]. Consider a $d$-dimensional random variable $S_T$ that has the decomposition

$$S_T = X_T + r_T Y_T$$

where $X_T$ and $Y_T$ are $d$-dimensional random variables for $T > 0$, and $r_T$ is a positive number such that $\lim_{T \to \infty} r_T = 0$. Let $m \in \mathbb{N}$. Suppose that positive numbers $M$, $\gamma$, $\ell_1$ and $\ell_2$, and a functional $\xi_T$ satisfy the following conditions:

(i) $m > \frac{1}{2} \gamma + 1$, $\ell_1 \geq \ell_2 - 1$, $\ell_2 \geq (2m + 1) \lor (d + 3)$.

(ii) $\sup_T \|X_T\|_{\ell_2,p} + \sup_T \|Y_T\|_{\ell_2,p} < \infty$ for every $p > 1$.

(iii) $(X_T, Y_T) \to^d (X_\infty, Y_\infty)$ as $T \to \infty$ for some random variables $X_\infty$ and $Y_\infty$.

(iv) $\sup_T \|\xi_T\|_{\ell_1,p} < \infty$ for every $p > 1$.

(v) $P[|\xi_T| > 1/2] = O(r_T^\alpha)$ for some $\alpha > 1$.

(vi) $\sup_T E[1_{(|\xi_T| < 1)} \Delta_T^{-p}] < \infty$ for every $p > 1$.

(vii) There exists a signed measure $\Psi_T$ on $\mathbb{R}^d$ such that

$$\sup_{f \in \mathcal{E}(M, \gamma)} \left| E[f(X_T)] - \int_{\mathbb{R}^d} f(z) \Psi_T(dz) \right| = o(r_T)$$

as $n \to \infty$, where $\mathcal{E}(M, \gamma)$ is the set of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $|f(z)| \leq M(1 + |x|)^\gamma$.

Under these conditions, Sakamoto and Yoshida [59] showed the following result.

**Theorem 3.3.** It holds that

$$\sup_{f \in \mathcal{E}(M, \gamma)} \left| E[f(S_T)] - \int_{\mathbb{R}^d} f(z) \left\{ \Psi_T(dz) + r_T g_\infty(z)dz \right\} \right| = o(r_T)$$

as $T \to \infty$, where

$$g_\infty(z) = -\partial_z \left\{ E[Y_\infty | X_\infty = z] p^{X_\infty}(z) \right\}.$$
4 Generalization

We will consider the variable \( Z_n \) having the expression (1.8). The functional \( M_n \) is defined by (1.5) with the kernel \( u_n \) given by (1.3) and (1.4). The principal part \( M_n \) of \( Z_n \) is the same as before, however, the variables \( N_n \) and \( X_n \) \((n \in \mathbb{N})\) are general in this section. We do not assume \( N_n \) is specified by (1.9). Such a generalization is motivated by asymptotic expansion of a parametric estimator for a stochastic differential equation, for example, where \( N_n \) involves derivatives (with respect to the statistical parameter) of the quasi-log likelihood function of the model and is utterly different from and more complicated than (1.9). A multi-dimensional extension is straightforward but we will only consider one-dimensional variables for notational simplicity.

A sufficient condition for asymptotic expansion of the joint distribution \( \mathcal{L}\{(Z_n, X_n)\} \) will be in part similar to the one presented in the general theory in Section 3.2 since we do not assume any specific form of \( N_n \) or \( X_n \). We keep \( G_\infty \) defined by (2.1) as well as measurable random fields \((a(t, q))_{t \in [0,1]}, (\hat{a}(t, 2))_{t \in [0,1]} \) \((q \in \mathcal{Q})\). Let \( X_\infty \) be a one-dimensional random variable in \( D_\infty \). According to the theory in Section 3.2, \( \Psi(z, x) \) is defined by

\[
\Psi(z, x) = \exp \left( 2^{-1}G_\infty(iz)^2 + X_\infty ix \right)
\]

for \( z \in \mathbb{R} \) and \( x \in \mathbb{R} \), in the present situation. Denote \( \overset{\circ}{X}_n = n^{1/2}(X_n - X_\infty) \). Following the theory, let

\[
\mathcal{G}_n^{(1,0)}(iz) = \mathcal{G}_n^{(1,0)}(iz, ix) = N_n iz,
\]

\[
\mathcal{G}_n^{(0,1)}(ix) = \mathcal{G}_n^{(0,1)}(iz, ix) = \overset{\circ}{X}_n ix,
\]

\[
\mathcal{G}_{1,n}^{(2,0)}(iz) = \mathcal{G}_{1,n}^{(2,0)}(iz, ix) = D_{un} N_n (iz)^2.
\]

and let

\[
\mathcal{G}_{1,n}^{(1,1)}(iz, ix) = D_{un} \overset{\circ}{X}_n (iz)(ix).
\]

We suppose that (polynomial) random symbols \( \mathcal{G}^{(1,0)}(iz, ix), \mathcal{G}^{(0,1)}(iz, ix), \mathcal{G}_{1}^{(2,0)}(iz, ix) \) and \( \mathcal{G}_{1}^{(1,1)}(iz, ix) \) are given. Condition (2.10) is assumed in this section.

4.1 Asymptotic expansion of \( E[f(Z_n, X_n)] \) for a differentiable function \( f \)

We derive asymptotic expansion of the expectation \( E[f(Z_n, X_n)] \) for differentiable functions \( f \).

[\( D \) (I)] Condition \( [A] \) is satisfied except for \( [A] \) (vi).

(II) \( N_n, X_n \in D_\infty \) and the properties (3.12), (3.13) and (3.14) hold for every \( p > 1 \).
(III) For every \( z \in \mathbb{R} \) and \( x \in \mathbb{R} \),
\[
\lim_{n \to \infty} E \left[ \Psi(z, x) \mathfrak{T}_n(iz, ix) \right] = E \left[ \Psi(z, x) \mathfrak{T}(iz, ix) \right]
\]
for the pairs of random symbols \((\mathfrak{T}_n, \mathfrak{T}) = (\mathfrak{S}_n^{(1,0)}, \mathfrak{S}^{(1,0)}), (\mathfrak{S}_n^{(0,1)}, \mathfrak{S}^{(0,1)}), (\mathfrak{S}_{1,n}^{(1,0)}, \mathfrak{S}_1^{(2,0)}), \) and \((\mathfrak{S}_{1,n}^{(1,1)}, \mathfrak{S}_1^{(1,1)})\).

The random symbol \( \mathfrak{S} \) is defined by
\[
\mathfrak{S}(iz, ix) = \mathfrak{S}^{(3,0)}(iz, ix) + \mathfrak{S}^{(1,1)}(iz, ix) + \mathfrak{S}^{(1,0)}(iz, ix) + \mathfrak{S}^{(0,1)}(iz, ix) + \mathfrak{S}_1^{(2,0)}(iz, ix) + \mathfrak{S}_1^{(1,1)}(iz, ix),
\]
where the components \( \mathfrak{S}^{(3,0)}(iz, ix) \) and \( \mathfrak{S}^{(1,1)}(iz, ix) \) are given by \((2.12)\) and \((2.13)\), respectively, and \( \mathfrak{S}^{(1,0)}(iz, ix), \mathfrak{S}^{(0,1)}(iz, ix), \mathfrak{S}_1^{(2,0)}(iz, ix) \) and \( \mathfrak{S}_1^{(1,1)}(iz, ix) \) are characterized by Condition \([D]\) (III). Remark that \( \mathfrak{S}_1^{(2,0)}(iz, ix) = 0 \) in the general expression of \( \mathfrak{S} \) in \((3.17)\). We define the random symbol \( \mathfrak{S}_n \) by
\[
\mathfrak{S}_n = 1 + n^{-1/2} \mathfrak{S},
\]
(4.2)

Let \( \beta_1 \) is the maximum degree of the (polynomial) random symbols \( \mathfrak{S}^{(1,0)}(iz, ix), \mathfrak{S}^{(0,1)}(iz, ix), \mathfrak{S}_1^{(2,0)}(iz, ix) \) and \( \mathfrak{S}_1^{(1,1)}(iz, ix) \). Remark that the possible maximum degree of \( \mathfrak{S}^{(3,0)}(iz, ix) \) and \( \mathfrak{S}^{(1,1)}(iz, ix) \) is 5. The proof of the following theorem is given in Section 10.1.

**Theorem 4.1.** Suppose that Condition \([D]\) is satisfied. Then
\[
\sup_{f \in \mathcal{B}} \left| E \left[ f(Z_n, X_n) \right] - E \left[ \mathfrak{S}_n(\partial_z, \partial_x) f(G_{1/2}^{1/2} \zeta, X_\infty) \right] \right| = o(n^{-1/2})
\]
as \( n \to \infty \) for any bounded set \( \mathcal{B} \) in \( C_b^{\beta+1}(\mathbb{R}^2) \), where \( \beta = \max\{7, \beta_1\} \) and \( \zeta \sim N(0, 1) \) independent of \( \mathcal{F} \).

## 4.2 Expansion of \( E[f(Z_n, X_n)] \) for a measurable function \( f \)

The index \( \beta_x \) is the maximum degree in \( x \) of \( \mathfrak{S} \) in Section 4.1. The condition we will work in this section is

\([D^1]\) \[(I)\] Satisfied are \([A] \) (i), (ii), (iii), (iv), (v), (viii), \([A^\ddagger] \) (vii\ddagger), \([A^\sharp] \) (II) and

\[(i')\] For every \( t, s \in [0, 1] \) and \( q \in \mathcal{Q}, a(t, q) \in \mathbb{D}^{3+\beta_x, \infty}, \overset{\circ}{a}(t, s, q) \in \mathbb{D}^{3+\beta_x, \infty}, \overset{\circ}{a}(t, 2) \in \mathbb{D}^{3+\beta_x, \infty} \) if \( 2 \in \mathcal{Q} \) and \( \overset{\circ}{a}(t, s, q) \in \mathbb{D}^{3+\beta_x, \infty}, \) and
\[
\max_{q \in \mathcal{Q}} \sup_{t \in [0, 1]} \left\| a(t, q) \right\|_{3+\beta_x, p} + \max_{q \in \mathcal{Q}} \sup_{(t,s) \in [0,1]^2} \left\| \overset{\circ}{a}(t, s, q) \right\|_{3+\beta_x, p} + 1_{\{2 \in \mathcal{Q}\}} \sup_{t \in [0, 1]} \left\| \overset{\circ}{a}(t, 2) \right\|_{3+\beta_x, p} + \max_{q \in \mathcal{Q}} \sup_{(t,s) \in [0,1]^2} \left\| \overset{\circ}{a}(t, s, q) \right\|_{3+\beta_x, p} < \infty
\]
(4.3)

for every \( p > 1 \).
Theorem 4.2. Suppose that Condition [D^2] is satisfied. Then

$$\sup_{f \in E(M, \gamma)} \left| E[f(Z_n, X_n)] - \int_{\mathbb{R}^2} f(z, x)p_n(z, x)dzdx \right| = o(r_n)$$

(4.4)

as $n \to \infty$ for every $(M, \gamma) \in (0, \infty)^2$.

Theorem 4.2 can provide an asymptotic expansion formula of $L\{Z_n\}$ without a reference variable $X_n$. Indeed, even when $X_n$ is not given, we can apply Theorem 4.2 to the joint distribution $L\{(Z_n, X_\infty)\}$ by adding an independent variable $X_n = X_\infty \sim N(0, 1)$ and extending the Malliavin calculus for $Z_n$ to incorporate $X_\infty$. Though this is a possible way, reconstruction of the proof without $X_n$ has a merit that it reduces the differentiability index. So we directly apply Theorem 7.7 of Nualart and Yoshida [48] to the case without the reference variable $X_n$ nor $X_\infty$. Let \( \hat{\Psi}(z) = \exp(2^{-1}G_\infty(iz)^2) \) for $z \in \mathbb{R}$.

[D^2] (I) Satisfied are [A] (i), (ii), (iii), (iv), (v), [A^2] (vi iv) and

(iii) For every $t, s \in [0, 1]$ and $q \in Q$, $a(t, q) \in D^{3, \infty}$, \( \overset{\circ}{a}(t, s, q) \in D^{3, \infty} \), \( \overset{\circ}{a}(t, 2) \in D^{3, \infty} \) (if $2 \in Q$) and \( \overset{\circ}{a}(t, s, q) \in D^{3, \infty} \), and

$$\max_{q \in Q} \sup_{t \in [0, 1]} \| a(t, q) \|_{3, p} + \max_{q \in Q} \sup_{(t, s) \in [0, 1]^2} \| \overset{\circ}{a}(t, s, q) \|_{3, p}$$

$$+1_{\{2 \in Q\}} \sup_{t \in [0, 1]} \| \overset{\circ}{a}(t, 2) \|_{3, p} + \max_{q \in Q} \sup_{(t, s) \in [0, 1]^2} \| \overset{\circ}{a}(t, s, q) \|_{3, p} < \infty$$

(4.5)

for every $p > 1$.

(I) $N_n \in D^{\infty}$, $\|N_n\|_{8, p} = O(1)$ and $\|D^2_{an}N_n\|_{7, p} = O(r_n^\kappa)$ ($p > 1$) as $n \to \infty$ for some positive number $\kappa$. 

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The random symbols $S^{(1,0)}$, and $S^{(2,0)}$ are a (polynomial) random symbol the coefficients of which are in $D_2^{1+}$. Moreover, for each pairs of random symbols $(\mathcal{T}_n, \mathcal{S}) = (S^{(1,0)}_n, S^{(1,0)}_1)$, and $(S^{(2,0)}_1, S^{(2,0)}_1)$, and for some $p > 1$, there exists a (polynomial) random symbol $\mathcal{T}_n$ that has $L^p$ coefficients and the same degree as $\mathcal{T}$ such that

$$E[\hat{\Psi}(z)\mathcal{T}_n(iz)] = E[\hat{\Psi}(z)\mathcal{T}(iz)]$$

and $\mathcal{T}_n \to \mathcal{T}$ in $L^p$ as $n \to \infty$ for every $z \in \mathbb{R}$.

(IV) $G_{-1}^{-1} \in L^\infty$.

In the present situation with no reference variable $X_n$, we define the random symbol $\mathcal{S}$ by

$$\mathcal{S}(iz) = S^{(3,0)}(iz) + S^{(2,0)}_0(iz) + S^{(1,0)}(iz) + S^{(2,0)}_1(iz).$$

Let

$$\hat{p}_n(z) = E[\hat{\mathcal{S}}_n(\partial_z)^\ast \phi(z; 0, G_\infty)]$$

for $\hat{\mathcal{S}}_n(iz) = 1 + n^{-1/2}\hat{\mathcal{S}}(iz)$. Denote by $\mathcal{E}_1(M, \gamma)$ the set of measurable functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $|f(z)| \leq M(1 + |z|)^\gamma$ for $M, \gamma > 0$.

**Theorem 4.3.** Suppose that Condition $[D]$ is satisfied. Let $(M, \gamma) \in (0, \infty)^2$. Then

$$\sup_{f \in \mathcal{E}_1(M, \gamma)} \left| E[f(Z_n)] - \int_{\mathbb{R}} f(z)\hat{p}_n(z)dz \right| = o(r_n) \quad (4.6)$$

as $n \to \infty$.

5 An example

We will consider an example in this section. For a Wiener process $w = (w_t)_{t \in [0,1]}$ realized by an isonormal Gaussian process over a real separable Hilbert space $\mathcal{H}$. We will consider a quadratic variation with anticipative weights

$$V_n = \sum_{j=1}^n a(w_{1-t_j})(\Delta_j w)^2$$

where $a \in C_p^\infty(\mathbb{R})$ and $t_j = j/n$. Let us simply denote $f_t = f(w_t)$ as before. It is easy to see

$$V_n \to_p V_\infty$$

as $n \to \infty$ for $V_\infty = \int_0^1 a_{1-t}dt$. Let $Z_n = \sqrt{n}(V_n - V_\infty)$. Then

$$Z_n := \sqrt{N}(V_n - V_\infty) = M_n + n^{-1/2}N_n$$

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where $M_n = \delta(u_n)$ for

$$u_n = n^{1/2} \sum_{j=1}^{n} a_{1-t_j} I_1(1_j)\frac{1}{n}$$

and

$$N_n = n \sum_{j=1}^{n} (D_1 a_{1-t_j}) I_1(1_j) + n \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (a_{1-t_j} - a_{1-t}) dt =: N_n^{(1)} + N_n^{(2)}. \quad (5.2)$$

This $Z_n$ is an example of that in (1.8) if one sets $Q = \{2\}$ and $a_j(2) = a_{1-t_j}$. For a reference variable, we shall consider $X_\infty = w_1$. Therefore the asymptotic expansion of the joint law $\mathcal{L}\{(Z_n, X_\infty)\}$ is given by Theorem 4.2.

We will verify Condition $[D^\#]$. Condition $[A]$ (i) is obvious. For $a(s, 2) = a_{1-s}$,

Condition $[A]$ (ii) holds. Since

$$D_{t_k} a_{1-t_j} = a'(w_1-t_j) n^{-1} 1_{\{t_k \leq 1-t_j\}},$$

Condition $[A]$ (iii) is satisfied with

$$\overset{\circ}{a}(t, s, 2) = a'(w_1-s) 1_{\{t \leq 1-s\}},$$

and Condition $[A]$ (iv) with

$$\overset{\cdot}{a}(t, 2) = a'(w_1-t) 1_{\{t \leq 1/2\}}.$$

We have

$$D_{t_k} D_{t_k} a(w_1-t_j) = a''(w_1-t_j) n^{-2} 1_{\{t_k \leq 1-t_j\}}$$

and hence Condition $[A]$ (v) is satisfied by

$$\overset{\circ\circ}{a}(t, s, 2) = a''(w_1-s) 1_{\{t \leq 1-s\}}.$$

Condition $[A]$ (viii) is met for $X_\infty = w_1$ and $\dot{X}_\infty(t) = 0$.

In the present situation,

$$G_\infty = 2 \int_0^1 a_{1-s}^2 ds = 2 \int_0^1 a_s^2 ds.$$

Then Condition $[A^\#]$ (vii) is satisfied. In fact, Section 8.6 shows (2.18). We assume that $G_\infty^{-1} \in L^\infty$. Then Condition $[A^\#]$ (II) is fulfilled. Condition $[D^\#]$ (i) is easy to check.

The variable $X_\infty = w_1$ satisfies the conditions in $[D^\#]$ (II) concerning it, so we verify it for $N_n$. According to Section 7.1, the exponent of the first component of (5.2) is $e(N_n^{(1)}) = 0$, which
means $N_n^{(1)} = O_D(1)$. Moreover, Proposition 7.1 applied to $N_n^{(1)}$ represented by (5.2) ensures $D_{un}N_n^{(1)} = O_D((n^{-1/2})$, and hence $D_{un}^2N_n^{(1)} = O_D((n^{-1/2})$. Thus $N_n^{(1)}$ satisfies the related part of Condition $[D^2]$ (II).

For $N_n^{(2)}$, we have

$$N_n^{(2)} = -n \sum_{j=1}^{n} \int_{t_j-1}^{t_j} (a_t - a_{t_{j-1}}) dt$$

Let

$$A_n(j; x) = -\int_0^1 a'(w_x + v(w_x - w_{t_{j-1}})) dv.$$ 

for $x \in \mathbb{T}$, the one-dimensional torus identified with $[0, 1]$, and let

$$\mathcal{H}_n^{(0)}(x) = n \sum_{j=1}^{n} A_n(j; x) I_1(1_{[t_{j-1}, x]}1_j(x))$$

Then

$$N_n^{(2)} = \int_\mathbb{T} \mathcal{H}_n^{(0)}(x) dx,$$

where $dx$ is the Haar measure on the compact group $\mathbb{T}$ compatible with the Lebesgue measure. The functional $\mathcal{H}_n(x)$ of (12.9) is

$$\mathcal{H}_n^{(0)}(x) = n \sum_{j=1}^{n} A_n(j; t_{j-1} + x) I_1(1_{[t_{j-1}, t_{j-1}+x]}1_j(t_{j-1} + x)).$$

for $\mathcal{H}_n^{(0)}(x)$. The exponent

$$e \left( \mathcal{H}_n^{(0)}(x) \right) = 1$$

and hence

$$\mathcal{H}_n^{(0)}(x) = O_D(n)$$

uniformly in $x \in \mathbb{T}$ because this estimate follows from Theorem 12.3. Therefore we obtain

$$N_n^{(2)} = O_{L}^{\infty}(1)$$

by Theorem 12.4 applied to the case where

$$e_{n,j}^{(1)}(x) = I_1(1_{[t_{j-1}, x]}1_j(x)) = I_1(1_{[t_{j-1}, x]}1_j(x - t_{j-1}))1_{[0,n^{-1}]}(x - t_{j-1}),$$

$$g_{n,j}^{(1)} = -t_{j-1},$$

$$\chi_n^{(1)} = [0, n^{-1}].$$
The $i$-th derivative $D^i N^{(2)}_n$ can be estimated in a similar manner. Indeed, from (12.10),

$$D^i N^{(2)}_n = \int_{\mathbb{T}} D^i H_n^{(0)}(x) 1_{[0,n-1]}(x) dx$$  \hspace{1cm} (5.3)

for $i \in \mathbb{Z}_+$. The (uniform-in-$x$ version of) stability of the operation $D^i$ proved by Proposition 7.2 shows

$$\| D^i N^{(2)}_n \|_p \leq \int_{\mathbb{T}} \| D^i H_n^{(0)}(x) \|_p 1_{[0,n-1]}(x) dx = O(1)$$

for every $p > 1$. We obtained the estimate $N^{(2)}_n = O_{D^\infty}(1)$ as $n \to \infty$.

Operating $D_{u_n}$ to (5.3) in $i = 0$, we have

$$D^k D_{u_n} N^{(2)}_n = \int_{\mathbb{T}} D^k D_{u_n} H_n^{(0)}(x) 1_{[0,n-1]}(x) dx$$  \hspace{1cm} (5.4)

for $k \in \mathbb{Z}_+$. Since $H_n^{(0)}(x)$ involved only the first chaos, Proposition 7.1 (ii) undertakes the estimate

$$e(D_{u_n} H_n^{(0)}(x)) \leq 0.5,$$

and the stability Proposition 7.2 gives

$$\| D^i D_{u_n} H_n^{(0)}(x) \|_p = O(n^{0.5})$$

for every $p > 1$. Combining this and the representation (5.4), we conclude $D_{u_n} N^{(2)}_n = O_{D^\infty}(n^{-0.5})$. We can also say $D^k_{u_n} N^{(2)}_n = O_{D^\infty}(n^{-0.5})$ for $k \geq 1$ since $u_n = O_{D^\infty}(1)$. Thus, the part concerning $N^{(2)}_n$ in [D$^2$] (II) has been verified. [D$^2$] (II) has been checked. In particular, $D_{u_n} N_n = O_{D^\infty}(n^{-0.5})$.

The random symbol $\mathcal{S}^{(3,0)}$ defined by (2.12) is in this case

$$\mathcal{S}^{(3,0)}(iz, ix) = \frac{4}{3} \int_0^1 a_{1-t}^3 dt (iz)^3 + 2 \int_{[0,1]^2} a^{(3,0)}(t,s,2,2) ds dt (iz)^3$$

$$+ \int_0^1 D_t G_{\infty} \left[ \int_0^{1-t} a'_{1-s} a_{1-s} ds \right] a_{1-t} dt (iz)^5$$

$$+ 2 \int_0^1 \left[ \int_0^{1-t} a'_{1-s} a_{1-s} ds \right] a_{1-t} dt (iz)^3 (ix)$$  \hspace{1cm} (5.5)

where $a^{(3,0)}(t,s,q_1,q_2)$ given by (2.11) is

$$a^{(3,0)}(t,s,2,2) = a''_{1-s} a_{1-s} a_{1-t} 1_{\{t \leq 1-s\}} + (a'_{1-s})^2 a_{1-t} 1_{\{t \leq 1-s\}}$$

$$+ a'_{1-s} a_{1-s} a'_{1-t} 1_{\{t \leq 1-s\}} 1_{\{t \leq 1/2\}}.$$
The random symbol $\mathcal{S}^{(1,1)}$ defined in (2.13) is now

$$
\mathcal{S}^{(1,1)}(iz, ix) = \int_0^\frac{1}{2} a'(w_{1-t})dt(iz)(ix) + \frac{1}{2} \int_0^1 (D_t G_\infty)a_{1-t}dt (iz)^3(ix) + \int_0^1 a_{1-t}dt(iz)(ix)^2.
$$

(5.6)

It is now obvious that $\mathcal{S}^{(0,1)} = 0$, $\mathcal{S}^{(2,0)} = 0$ and $\mathcal{S}^{(1,1)} = 0$. We have

$$
N_n^{(2)} = -n \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (a_t - a_{t_{j-1}})dt
$$

$$
= -n \sum_{j=1}^{n} a'_{t_{j-1}} I_1(g_j) - n \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} 2^{-1}a''_sdsdt + O_L(n^{-0.5})
$$

where $g_j$ is defined in (11.1), and we used the estimate

$$
-n \sum_{j=1}^{n} a''_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} dw_r dw_s dt = O_L(n^{-0.5}).
$$

Then, for $\mathcal{S}^{(1,0)}(iz) = N_n(iz)$, we have

$$
E\left[\Psi(z, x)\mathcal{S}^{(1,0)}(iz)\right] = E\left[\Psi(z, x)N_n\right]iz
$$

$$
= E\left[\Psi(z, x)n \sum_{j=1}^{n} (D_{1_j}a_{1-t_j})I_1(1_j)\right]iz
$$

$$
- E\left[\Psi(z, x) \sum_{j=1}^{n} a'_{t_{j-1}} I_1(g_j)\right]iz
$$

$$
- E\left[\Psi(z, x)n \sum_{j=1}^{n} \int_{t_{j-1}}^{t} \int_{t_{j-1}}^{t} 2^{-1}a''_sdsdt\right]iz + o(1)
$$

$$
= \sum_{j=1}^{n} E\left[D_{1_j}\left(\Psi(z, x)a'_{1-t_j}1(t_j \leq 1/2)\right)\right]iz
$$

$$
- \sum_{j=1}^{n} E\left[D_{g_j}\left(\Psi(z, x)a'_{t_{j-1}}\right)\right]iz
$$

$$
- \sum_{j=1}^{n} E\left[\Psi(z, x)n \int_{t_{j-1}}^{t} \int_{t_{j-1}}^{t} 2^{-1}a''_sdsdt\right]iz + o(1)
$$

In the present problem,

$$
\Psi(z, x) = \exp(2^{-1}G_\infty(iz)^2 + w_1ix).
$$
Therefore, we obtain

\[
\sum_{j=1}^{n} E \left[D_{ij} (\Psi(z,x)\alpha_{1-t_j}1_{\{t_j \leq 1/2\}}) \right] iz 
\]

\[
= \sum_{j=1}^{n} E \left[\Psi(z,x) \left\{ 2^{-1} D_{ij} G_{\infty}(iz)^2 + n^{-1} ix \right\} \alpha_{1-t_j}1_{\{t_j \leq 1/2\}} \right] iz 
\]

\[
+ \sum_{j=1}^{n} E \left[\Psi(z,x)\alpha''_{1-t_j}1_{\{t_j \leq 1/2\}} n^{-1} \right] iz 
\]

\[
\rightarrow E \left[\Psi(z,x) \left\{ \int_{0}^{1/2} 2^{-1} D_{i} G_{\infty} a'_{1-t} dt (iz)^3 + \int_{0}^{1/2} a'_{1-t} dt (iz)(ix) + \int_{0}^{1/2} a''_{1-t} dt (iz) \right\} \right], 
\]

\[- \sum_{j=1}^{n} E \left[ D_{gj} (\Psi(z,x)\alpha'_{t_j-1}) \right] iz 
\]

\[
= - \sum_{j=1}^{n} E \left[\Psi(z,x) \left\{ 2^{-1} D_{gj} G_{\infty}(iz)^2 + 2^{-1} n^{-1} ix \right\} \alpha'_{t_j-1} \right] iz 
\]

\[- \sum_{j=1}^{n} E \left[\Psi(z,x) D_{gj} \alpha'_{t_j-1} \right] iz 
\]

\[
\rightarrow E \left[\Psi(z,x) \left\{ - \int_{0}^{1} 4^{-1} D_{i} G_{\infty} a'_t dt (iz)^3 - \int_{0}^{1} 2^{-1} a'_t dt (iz)(ix) \right\} \right] 
\]

and

\[- \sum_{j=1}^{n} E \left[\Psi(z,x) n \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} 2^{-1} a''_s ds dt \right] iz \rightarrow E \left[\Psi(z,x) (-4^{-1}) \int_{0}^{1} a''_t dt iz \right] 
\]

Thus, we conclude

\[ E \left[\Psi(z,x)\mathcal{S}^{(1,0)}(iz) \right] \rightarrow E \left[\Psi(z,x)\mathcal{S}^{(1,0)}(iz) \right] 
\]

as \( n \rightarrow \infty \) for

\[
\mathcal{S}^{(1,0)}(iz) = \int_{0}^{1/2} 2^{-1} D_{i} G_{\infty} a'_{1-t} dt (iz)^3 - \int_{0}^{1} 4^{-1} D_{i} G_{\infty} a'_t dt (iz)^3 
\]

\[
+ \int_{0}^{1/2} a'_{1-t} dt (iz)(ix) - \int_{0}^{1} 2^{-1} a'_t dt (iz)(ix) 
\]

\[
+ \int_{0}^{1} a''_{1-t} dt (iz) - 4^{-1} \int_{0}^{1} a''_t dt (iz). 
\]

(5.7)

In the present case, the full random symbol

\[
\mathcal{S} = \mathcal{S}_{(3,0)}^{(3,0)} + \mathcal{S}_{(1,1)}^{(1,1)} + \mathcal{S}_{(1,0)}^{(1,0)} 
\]

is expressed more precisely with (5.5), (5.6) and (5.7). For the density function \( p_n(z,x) \) for \( \mathcal{S}_n \) defined by (4.2), the asymptotic expansion of \( \mathcal{L}' \{ (Z_n, w_1) \} \) can be obtained from Therome 4.2 under \( G_{\infty}^{-1} \in L^{\infty} \) for measurable functions.
Robust realized volatility

Robustness is an established notion in statistics (Huber [24], Hampel et al. [22]). According to the concept of robust statistics, the most effective part of the estimator should be removed or diminished to attain robustness because that part is most sensitive to contamination of the data. In volatility estimation, it is known that the realized volatility is weak against jumps, and multi-power variations and (predictable) thresholding methods for example have been studied to approach this issue. Recently, Inatsugu and Yoshida [27] proposed global jump filters to realize robust estimation for volatility of an Itô process. For correct rejection of contaminated observations, their filters use a selector depending on the whole data. The resulting estimator obviously destroys the martingale structure, however resists jumps well, especially consecutive jumps, for which traditional methods like by-power variation do not work. Related to such construction, we will consider another type of robust variation with a global filter and investigate asymptotic expansion as an example of our scheme.

Consider a unique strong solution $X = (X_t)_{t \in \mathbb{R}_+}$ to the stochastic differential equation
\[
\begin{align*}
&dX_t = \sigma(X_t)dw_t + b(X_t)dt, \quad t \in \mathbb{R}_+ \\
&X_0 = x_0.
\end{align*}
\]
Here $w = (w_t)$ is a Brownian motion, $x_0 \in \mathbb{R}$, $\sigma, b \in C^\infty$ and we suppose every positive order of derivative of $\sigma$ and $b$ is bounded. The uniform nondegeneracy $\inf_x \sigma(x) > 0$ is assumed.

The most popular volatility estimator in finance is the realized volatility $U_n$ defined by
\[
U_n = \sum_{j=1}^n (\Delta_j X)^2
\]
on the interval $[0, 1]$, where $\Delta_j X = X_{t_j} - X_{t_{j-1}}$ with $t_j = j/n$. It is well known that $U_n$ is sensitive to jumps. A robust variation of the realized volatility we will consider is
\[
\Psi_n = \sum_{j=1}^n \Phi(U_n, L_{n,j}) (\Delta_j X)^2
\]
(6.1)
where $\Phi \in C^\infty_p(\mathbb{R}^2)$, the space of smooth functions with derivatives of at most polynomial growth. The functional $L_{n,j}$ is defined by
\[
L_{n,j} = \eta_{n,j}^{-1} \sum_{k \in K_j} (\Delta_k X)^2
\]
with $\eta_{n,j} = \#K_j/n$, where
\[
K_j = K_{n,j} = \{ k \in \mathbb{N}; \underline{K_j} \leq k \leq \overline{K_j} \}
\]
and
\[
\underline{K_j} = \underline{K_{n,j}} = (j - \lfloor n\lambda \rfloor + 1) \lor 1, \quad \overline{K_j} = \overline{K_{n,j}} = (j + \lfloor n\lambda \rfloor - 1) \land n
\]
for the locality parameter $\lambda \in (0, 1)$. 

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An example of robust volatility measures is

$$V^*_n = \sum_{j=1}^{n} \varphi \left( U_n^{-1} L_{n,j} \right) (\Delta_j X)^2$$

where $\varphi \in C^\infty_b(\mathbb{R})$, the space of smooth bounded functions having bounded derivatives. The statistic $V^*_n$ selects increments $\Delta_j X$ according to the relative magnitude $L_{n,j}$ of the local volatility to the estimated average spot volatility. For example, $\varphi$ is a smooth function taking 1 in \( \{ x; |x| \leq C \} \) and 0 in \( \{ x; |x| \geq C \} \) for some positive constant $C$. Then, especially when $\lambda$ is small, an increment $\Delta_j X$ will be removed from the sum if an increment in either $[t_{j-1}, t_j]$ or intervals near around is contaminated with a jump. Let $\psi : \mathbb{R}_+ \to [0, 1]$ be a smooth function such that $\psi(x) = 1$ for $x \in [0, 1/c_0]$ and $\psi(x) = 0$ for $x \in [2/c_0, \infty)$ for $c_0 = \inf_{x \in \mathbb{R}} \sigma(x)^2/2$. Then it is easy to see the asymptotic expansion of $V^*_n$ coincides with that of

$$V^{**}_n = \sum_{j=1}^{n} \psi(U_n^{-1}) \varphi \left( U_n^{-1} L_{n,j} \right) (\Delta_j X)^2$$

The statistic $V^{**}_n$ is exactly an example of $V_n$ for $\Phi(x, y) = \psi(|x|^{-1}) \varphi(|x|^{-1} |y|)$.

There is no mathematical difficulty even if one adopts various local estimators but we consider $L_{n,j}$ for notational simplicity. Indeed, as for $U_n$, it was possible to choose a multipower variation or another global volatility estimator.

Simply denote $f_t = f(X_t)$ for a function $f$. Let

$$U_\infty = \int_0^1 \sigma^2_t \, dt, \quad L_{\infty, t} = \eta_{\infty, t}^{-1} \int_{(t-\lambda) \vee 0}^{(t+\lambda) \wedge 1} \sigma^2_s \, ds$$

for

$$\eta_{\infty, t} = \{ (t + \lambda) \wedge 1 \} - \{ (t - \lambda) \vee 0 \}.$$ 

We will consider estimation of the filtered integrated volatility

$$\nabla_n = \int_0^1 \Phi(U_n, L_{n,j}) \sigma^2_t \, dt,$$  \hspace{1cm} (6.2)

while estimation of the limit

$$\nabla_\infty = \int_0^1 \Phi(U_\infty, L_{\infty, t}) \sigma^2_t \, dt.$$ 

of $\nabla_n$ also makes sense though the description will be more involved.

### 6.1 Stochastic expansion of the error

We introduce some systematic notation. We write $b^{[1]} = \sigma$ and $b^{[2]} = b$. For a $C^2$ function $f$, denote $f^{[1]} = \sigma f', f^{[2]} = b^{[1]} f'', f^{[2]} = \frac{1}{2} \sigma^2 f'' + bf' = \frac{1}{2} (b^{[1]})^2 f'' + b^{[2]} f'$. Itô’s formula is then written by

$$f_t = f_s + \int_s^t f^{[1]}_r \, dw_r + \int_s^t f^{[2]}_r \, dr$$

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for \( t > s \geq 0 \). For \( b^{[1]} = \sigma \) and \( b^{[2]} = b \), define \( b^{[i_1, i_2, \ldots, i_m]} \) inductively by
\[
b^{[i_1, i_2, \ldots, i_m]} = (b^{[i_1, i_2, \ldots, i_{m-1}]} )^{[m]}
\]
for \( i_1, \ldots, i_m \in \{1, 2\}, m \geq 2 \).

Let
\[
\overline{\theta}_t = \Phi(U_\infty, L_\infty, t) \cdot
\]
To simplify notation, we will often use the symbol \( \beta = (b^{[1]})^2 \). Then
\[
\beta^{[1]} = 2(b^{[1]})^2(b^{[1]})' = 2b^{[1]}b^{[1,1]}
\]
and
\[
\beta^{[2]} = (b^{[1,1]})^2 + 2b^{[1]}b^{[1,2]}.
\]
Define \( S_j^{(6.3)} \) by
\[
S_j^{(6.3)} = b^{[1]}_{t_{j-1}} b^{[1,1]}_{t_{j-1}} I_3(1_j^{\otimes 3}) + 2hb^{[1]}_{t_{j-1}} b^{[1,1]}_{t_{j-1}} I_1(1_j) + h^2b^{[1]}_{t_{j-1}} b^{[1,2]}_{t_{j-1}} + h^2b^{[1]}_{t_{j-1}} b^{[2,1]}_{t_{j-1}}
\]
\[
+ h^2(b^{[2]}_{t_{j-1}})^2 + 2^{-1}h^2(b^{[1,1]}_{t_{j-1}})^2 - h\beta^{[1]}_{t_{j-1}} I_1(g_j) - 2^{-1}h^2\beta^{[2]}_{t_{j-1}}.
\] (6.3)

Let
\[
F_j^{(6.4)} = (b^{[1]}_{t_{j-1}})^2 I_2(1_j^{\otimes 2})
\] (6.4)
and let
\[
\Theta_j = \Phi(U_\infty, L_\infty, t_{j-1})
\] (6.5)

Let
\[
\Psi_{j,k}(x, y) = \Psi_{n,j,k}(x, y) = 1_{\{k \in J_n\}} \partial_1 \Phi(x, y) + 1_{\{k \in K_j\}} \partial_2 \Phi(x, y) \eta_{n,j}^{-1}.
\] (6.6)

Let
\[
\Xi_{j,k,\ell}(x, y) = \Xi_{n,j,k,\ell}(x, y)
\]
\[
= \frac{1}{2} 1_{\{k, \ell \in J_n\}} \partial_1^2 \Phi(x, y) + \partial_1 \partial_2 \Phi(x, y) 1_{\{k \in J_n\}} 1_{\{\ell \in K_j\}} \eta_{n,j}^{-1}
\]
\[
+ \frac{1}{2} \partial_2^2 \Phi(x, y) 1_{\{k \in K_j\}} \eta_{n,j}^{-2}
\] (6.7)

We introduce a process \((\omega_s)_{s \in [0,1]} \) defined by
\[
\omega_s = \Phi(U_\infty, L_\infty, s) \beta_s.
\] (6.8)
Let
\[
G_\infty = \int_0^1 2 \omega_t^2 \, dt.
\] (6.9)

Now the scaled error
\[
Z_n = n^{1/2}(\Psi_n - \overline{\Psi}_n)
\]
admits the following stochastic expansion.
Lemma 6.1.

\[ Z_n = M_n + n^{-1/2} N_n \]  
for \( M_n = \delta \left( u_n \right) \) for \( u_n \) defined in (11.44) to be mentioned subsequently, and

\[ N_n = \sum_{j=1}^{n} \left( D_{1j} \left( \Theta_j \left( b_{j-1}^{[1]} \right)^2 \right) \right) I_1 \left( 1_j \right) + n \sum_{j=1}^{n} \Theta_j \mathcal{S}_j^{(6.3)} + n \sum_{j,k=1}^{n} \Psi_{j,k} \left( U_{\infty}, L_{\infty,t_{j-1}} \right) \mathcal{F}_j^{(6.4)} \mathcal{F}_k^{(6.4)} + R_n^{(6.12)}, \]  

where \( R_n^{(6.12)} \in \mathbb{D}^\infty \) satisfying

\[ R_n^{(6.12)} = O_{\mathbb{D}^\infty} \left( n^{-0.5} \right). \]  

Proof of Lemma 6.1 is in Section 11.1.

6.2 Asymptotic expansion of \( Z_n \)

We can now apply the results in Section 4. However, the asymptotic expansion term \( N_n \) consists of two terms of different nature. Because of this, it is useful to treat them in different ways. Based on the stochastic expansion (6.10), we consider the following decomposition of \( Z_n \):

\[ Z_n = Z_n^0 + n^{-1/2} N_n^\times, \]  

with

\[ Z_n^0 = M_n + n^{-1/2} N_n^\circ \]  

where

\[ N_n^\circ = \sum_{j=1}^{n} \left( D_{1j} \left( \Theta_j \left( b_{j-1}^{[1]} \right)^2 \right) \right) I_1 \left( 1_j \right) + n \sum_{j=1}^{n} \Theta_j \mathcal{S}_j^{(6.3)} + R_n^{(6.12)} \]  

and

\[ N_n^\times = \sum_{j,k=1}^{n} \Psi_{j,k} \left( U_{\infty}, L_{\infty,t_{j-1}} \right) \mathcal{F}_j^{(6.4)} \mathcal{F}_k^{(6.4)} \]  

We will apply the results in Section 4 to \( Z_n^0 \) of (6.14) to obtain asymptotic expansion for the law \( \mathcal{L}\{ (Z_n^0, X_1) \} \). Then the variable \( Z_n \) is a perturbation of \( Z_n^0 \) by \( n^{-1/2} N_n^\times \), as in (6.13). To incorporate the effect of \( N_n^\times \), the perturbation method will be used.

Among several possible exposition, we only give a result on the asymptotic expansion for a measurable function \( f \) in the existence of the reference variable \( X_1 \) though the situation can be easily generalized. A non-degeneracy condition for \( G_\infty \) will be assumed because it can degenerate in general. On the other hand, the non-degeneracy of \( X_1 \) is quite standard. Proof of the following theorem is in Section 11.2.
Theorem 6.2. Suppose that $G^{-1}_\infty$ and $\Delta^{-1}_{x_1}$ are in $L^\infty$. Then there exists a (polynomial) random symbol $\mathfrak{A}(iz, ix)$ with coefficients in $\mathbb{D}^\infty$ such that

$$\sup_{f \in \mathcal{E}(M, \gamma)} \left| E[f(Z_n, X_1)] - \int_{\mathbb{R}^2} f(z, x) p_n(z, x) dzdx \right| = o(r_n)$$

as $n \to \infty$ for every $(M, \gamma) \in (0, \infty)^2$, where

$$p_n(z, x) = E\left[ \mathfrak{A}_n(\partial_z, \partial_x)^* \{ \phi(z; 0, G_\infty) \delta_z(X_1) \} \right].$$

and

$$\mathfrak{A}_n(iz, ix) = 1 + n^{-1/2} \mathfrak{A}(iz, ix).$$

An expression of $\mathfrak{A}$ is presented in Section 11.

7 Action of the operators $D_{u_n(q)}$ and $D^i$

7.1 Exponent of a functional

We will consider the functional $\mathcal{I}_n$ defined by

$$\mathcal{I}_n = n^\alpha \sum_{j_1, \ldots, j_m=1}^n A_n(j_1, \ldots, j_m) I_{q_1}(f^{(1)}_{j_1}) \cdots I_{q_n}(f^{(m)}_{j_m}) \quad (7.1)$$

where $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $m \in \mathbb{N}$ and $(q_1, \ldots, q_n) \in \mathbb{Z}^m$. Suppose that $f^{(i)}_{j_i} = f^{(i), n} \in \mathcal{F}^\infty_{q_i}$ such that $\text{supp}(f^{(i)}_{j_i}) \subset I_{j_i}^{q_i}$ and that

$$\max_{i=1, \ldots, m} \sup_{n \in \mathbb{N}, j_i=1, \ldots, n} \sup_{t_1, \ldots, t_{q_i} \in [0, 1]} |f^{(i)}_{j_i}(t_1, \ldots, t_{q_i})| \leq C_1 \quad (7.2)$$

for some finite constant $C_1$. The following condition will be assumed for the coefficients $A_n(j_1, \ldots, j_m)$ in this section. We are writing $J_n = \{1, \ldots, n\}$.

[S] $A_n(j_1, \ldots, j_m) \in \mathbb{D}^\infty$ for all $j_1, \ldots, j_m \in J_n$, $n \in \mathbb{N}$, and

$$C(i, p) := \sup_{n \in \mathbb{N}} \sup_{j_1, \ldots, j_m \in J_n} \sup_{t_1, \ldots, t_{q_i} \in [0, 1]} \| D^i_{t_1, \ldots, t_{q_i}} A_n(j_1, \ldots, j_m) \|_p < \infty$$

for every $i \in \mathbb{Z}_+$ and $p > 1$.

Let $\tilde{q}(m) = q_1 + \cdots + q_m$. For $(q_1, \ldots, q_m) \in \mathbb{Z}^m$, define $m_1(q_1, \ldots, q_m)$ and $m_0(q_1, \ldots, q_m)$ by

$m_1(q_1, \ldots, q_m) = \# \{ m' \in \{1, \ldots, m\}; q_{m'} > 0 \}$ and $m_0(q_1, \ldots, q_m) = \# \{ m' \in \{1, \ldots, m\}; q_{m'} = 0 \}$,
respective. We define the exponent \( e(I_n) \) of \( I_n \) admitting an expression (7.5) by
\[
e(I_n) = \alpha - \frac{1}{2} (q(m) - m_1(q_1, \ldots, q_m)) + m_0(q_1, \ldots, q_m) - \infty 1_{\{\min(q_1, \ldots, q_m) < 0\}}
\]
\[
= \alpha - \frac{1}{2} q(m) + m - \frac{1}{2} m_1(q_1, \ldots, q_m) - \infty 1_{\{\min(q_1, \ldots, q_m) < 0\}}.
\]

The exponent is defined for the sequence \( (I_n)_{n \in \mathbb{N}} \); in this sense, it is more rigorous to write \( e((I_n)_{n \in \mathbb{N}}) \) for \( e(I_n) \), but we adopt simpler notation. For example, \( e(I_n) = 0 \) for \( I_n \) admitting the representation (7.5) if \( \alpha = 2^{-1}(q(m) - m) \) and \( (q_1, \ldots, q_m) \in \mathbb{N}^m \). The exponent \( e(I_n) \) depends on an expression (7.1) of \( I_n \). Often we will not write explicitly which expression is considered when writing an exponent if there is no fear of confusion. Theorem 12.3 ensures
\[
I_n = O_{L_\infty}(n^{e(I_n)}) \tag{7.3}
\]
as \( n \to \infty \) under \( [S] \) if \( (q_1, \ldots, q_m) \in \mathbb{Z}^m_+ \). In fact, if \( q_1, \ldots, q_k > 0 = q_k+1 = \cdots = q_m \), for example, then
\[
\|I_n\|_p \leq n^\alpha \sum_{j_k+1, \ldots, j_m} \left\| \sum_{j_1, \ldots, j_k} A_n(j_1, \ldots, j_m) I_{q_1}(f_{j_1}^{(1)}) \cdots I_{q_m}(f_{j_m}^{(m)}) \right\|_p
\]
\[
\leq n^{\alpha + (m-k)-2^{-1}(q_1+\cdots+q_k-k)}
\]
\[
= O(n^{e(I_n)})
\]
for \( p > 1 \). If \( (q_1, \ldots, q_m) \) involves a negative component, then \( e(I_n) = -\infty \) and (7.3) is valid in the sense that
\[
0 = O_{L_\infty}(n^{-\infty}).
\]
Thus, (7.3) holds for all \( (q_1, \ldots, q_m) \in \mathbb{Z}^m \).

Let \( S \) be the set of sequences \( (I_n)_{n \in \mathbb{N}} \) such that each \( I_n \) admits a representation (7.1) for some \( \alpha \in \mathbb{R}, m \in \mathbb{N} \) and \( (q_1, \ldots, q_m) \in \mathbb{Z}^m \) (independent of \( n \)) and coefficients \( A_n(j_1, \ldots, j_m) \) satisfying \([S]\). Let \( \mathcal{L} \) be the linear space generated by all finite linear combination of elements of \( S \). For \( J_n \in \mathcal{L} \), we define the exponent \( e(J_n) \) of \( J_n \) by the maximum exponent of the summands in \( J_n \). We remark that each summand of \( J_n \) has a specific representation to determine its exponent, and hence \( e(J_n) \) depends on these representations. Furthermore, we define the exponent \( e(K_n) \) by
\[
e(K_n) = \max_{i=1, \ldots, \bar{i}} e(J_n^{(i)})
\]
for a sequence \( (K_n)_{n \in \mathbb{N}} \) of variables that satisfies
\[
|K_n| \leq \sum_{i=1}^{\bar{i}} |J_n^{(i)}|
\]
for some sequences \( (J_n^{(i)})_{n \in \mathbb{N}} \in \mathcal{L}, i \in \{1, \ldots, \bar{i}\} \). The exponent \( e(K_n) \) depends on the dominating sequences \( (J_n^{(i)})_{n \in \mathbb{N}, i \in \{1, \ldots, \bar{i}\}} \) and their representation. The set of such sequences \( (K_n)_{n \in \mathbb{N}} \) is denoted by \( \mathcal{M} \). By definition,
\[
K_n = O_{L_\infty}(n^{e(K_n)}) \tag{7.4}
\]
for \( (K_n)_{n \in \mathbb{N}} \in \mathcal{M} \).
7.2 Operator $D_{u_n(q)}$

We are interested in the effect of the action of the projector $D_{u_n}$ on the functional $I_n$. Though the following argument is easily applied to more general functions $f_j$ but we will specially pay attention to $I_n$ re-defined by

$$I_n = n^\alpha \sum_{j_1,\ldots,j_m=1}^n A_n(j_1,\ldots,j_m)I_{q_1}(1_{j_1}^{\otimes q_1})\cdots I_{q_m}(1_{j_m}^{\otimes q_m})$$  \hspace{1cm} (7.5)

where $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $m \in \mathbb{N}$ and $(q_1,\ldots,q_m) \in \mathbb{Z}^m$. By convention,

$$I_q(1_{j}^{\otimes q}) = \begin{cases} 1 & (q = 0) \\ 0 & (q < 0) \end{cases}$$

In the present problem, the product formula has the expression

$$I_{q_1}(1_{j_1}^{\otimes q_1})I_{q_2}(1_{j_2}^{\otimes q_2})\cdots I_{q_n}(1_{j_n}^{\otimes q_n}) = \sum_{\nu \in \mathbb{Z}_+} c_\nu(q_1,q_2,\ldots,q_\mu)I_{\sum_{i=1}^n q_i-2\nu}(1_{j}^{\otimes (\sum_{i=1}^\mu q_i-2\nu)})n^{-\nu}$$  \hspace{1cm} (7.6)

for some constants $c_\nu(q_1,q_2,\ldots,q_\mu)$ depending on $\nu \in \mathbb{Z}_+$ and $(q_1,q_2,\ldots,q_\mu) \in \mathbb{Z}_+^\mu$. The function $(q_1,q_2,\ldots,q_\mu) \mapsto c_\nu(q_1,q_2,\ldots,q_\mu)$ is symmetric, and may take 0; e.g., $c_\nu(q_1,q_2,\ldots,q_\mu) = 0$ whenever $\sum_{i=1}^\mu q_i - 2\nu < 0$. We have

$$c_\nu(q_1,q_2) = c_\nu(q_1,q_2,0) = 1_{(0 \leq \nu \leq (q_1 \wedge q_2))}\nu!\left(\begin{array}{c} q_1 \\ \nu \end{array}\right)\left(\begin{array}{c} q_2 \\ \nu \end{array}\right)$$

for $q_1,q_2 \in \mathbb{N}$, and

$$c_\nu(q_1,q_2,q_3) = \sum_{\nu_1 \in \mathbb{Z}_+} \left[1_{(0 \leq \nu_1 \leq (q_2 \wedge q_3))}1_{(0 \leq \nu \leq ((q_1+\nu_1) \wedge (q_2+q_3-\nu_1)))}\times\nu_1!\left(\begin{array}{c} q_2 \\ \nu_1 \end{array}\right)\left(\begin{array}{c} q_3 \\ \nu_1 \end{array}\right)(\nu-\nu_1)!\left(\begin{array}{c} q_1 \\ \nu-\nu_1 \end{array}\right)\left(\begin{array}{c} q_2 + q_3 - 2\nu_1 \\ \nu-\nu_1 \end{array}\right)\right]$$  \hspace{1cm} (7.7)

for $q_1,q_2,q_3 \in \mathbb{Z}_+$ (it is easy to verify that this three-factor product formula is valid for indices possibly being 0). Let $c_x = 0$ when $x \notin \mathbb{Z}$.

If $c_\nu(q_1,q_2,q_3) \neq 0$ for $\nu = \bar{q}(3)/2$, then $\bar{q}(3)$ is even and

$$\nu = \frac{q_1 + q_2 + q_3}{2} \leq \min\{(q_1+\nu_1),(q_2+q_3-\nu_1)\}$$  \hspace{1cm} (7.8)

for some $\nu_1$ such that

$$0 \leq \nu_1 \leq \min\{q_2,q_3\}.$$  \hspace{1cm} (7.9)
Remark that if \((a + b)/2 \leq \min\{a, b\}\) for two real numbers \(a\) and \(b\), then \(a = b = (a + b)/2\). Therefore,

\[ q_1 + \nu_1 = q_2 + q_3 - \nu_1 = \frac{q_1 + q_2 + q_3}{2}, \tag{7.10} \]

which implies

\[ \nu_1 = \frac{-q_1 + q_2 + q_3}{2}. \tag{7.11} \]

From (7.9) and (7.10), we obtain

\[ q_1 \leq q_2 + q_3, \quad q_2 \leq q_1 + q_3, \quad q_3 \leq q_1 + q_2. \tag{7.12} \]

Let \(\mathbb{T}\) be the set of \((q_1, q_2, q_3) \in \mathbb{Z}_+^3\) satisfying the “triangular condition” (7.12). For \(\nu = \bar{q}(3)\) and \(\nu_1\) given by (7.11), simple calculus gives

\[ c_{\bar{q}(3)/2}(q_1, q_2, q_3) = \frac{q_1!q_2!q_3!}{(\frac{q_1 + q_2 + q_3}{2})! (\frac{q_2 + q_3 + q_4}{2})! (\frac{q_1 + q_3 + q_4}{2})!} \tag{7.13} \]

from (7.7). In conclusion, \(c_{\bar{q}(3)/2}(q_1, q_2, q_3)\) is possibly non-zero only when \(\bar{q}(3)\) is even and \((q_1, q_2, q_3) \in \mathbb{T}\), and then \(c_{\bar{q}(3)/2}(q_1, q_2, q_3)\) is given by (7.13) (for \(\nu = \bar{q}(3)/2\) and \(\nu_1\) in (7.11)).

The exponent \(e(D_{u_n(q)}\mathcal{I}_n)\) is estimated as follows.

**Proposition 7.1.** Suppose that Conditions \([S]\) and (2.2) of \([A]\) are fulfilled. Suppose that \(u_n(q)\) is given by (1.4) with some \(q \in \{2, 3, \ldots\}\). Then, for \(\mathcal{I}_n\) with the representation (7.5),

(i) \((D_{u_n(q)}\mathcal{I}_n)_{n \in \mathbb{N}} \in \mathcal{L}\) and

\[ e(D_{u_n(q)}\mathcal{I}_n) \leq e(\mathcal{I}_n). \]

(ii) If there is no \(m' \in \{1, \ldots, m\}\) such that \(q_{m'} = q\), then

\[ e(D_{u_n(q)}\mathcal{I}_n) \leq e(\mathcal{I}_n) - \frac{1}{2}. \]

**Proof.** Recall \(u_n(q)\) defined in (1.4):

\[ u_n(q) = n^{2^{-1}(q-1)} \sum_{j=1}^n a_j(q)\mathcal{I}_q \mathcal{I}_{\bar{q}(3)}(1_j^{\otimes(q-1)})1_j \]

where \(q \in \{2, 3, \ldots\}\). Let \(\bar{q}(m) = \bar{q}(m) + q\). We have

\[
D_{u_n(q)}\mathcal{I}_n = n^{\alpha+0.5\bar{q}-1.5} \sum_{j_1, \ldots, j_m, j_{m+1}=1}^n (nD_{A_n(j_1, \ldots, j_m)}^{j_{m+1}}(q)\mathcal{I}_{1}^{(\otimes q_1)} \cdots \mathcal{I}_{m}^{(\otimes q_m)}\mathcal{I}_{q-1}^{(\otimes q_{m+1})} \\
+ \sum_{m'=1}^m n^{\alpha+0.5\bar{q}-1.5} \sum_{j_1, \ldots, j_m} A_n(j_1, \ldots, j_m)_{q_{m'}} \mathcal{I}_{1}^{(\otimes q_1)} \cdots \mathcal{I}_{m'}^{(\otimes q_{m'-1})} \\
\times \mathcal{I}_{q-1}^{(\otimes q_{m'+1})} \cdots \mathcal{I}_{m}^{(\otimes q_m)} \\
=: \mathcal{I}_n(1) + \sum_{m'=1}^m \mathcal{I}_n(2, m'). \tag{7.14} \]

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We notice that
\[ \|nD_{1,m+1}A_n(j_1, \ldots, j_m)\|_p = \left\| n \int_0^1 D_tA_n(j_1, \ldots, j_m)1_{j_{m+1}}(t)dt \right\|_p \]
\[ \leq n \int_0^1 \|D_tA_n(j_1, \ldots, j_m)\|_p 1_{j_{m+1}}(t)dt \]
\[ \leq \sup_{t \in [0,1]} \|D_tA_n(j_1, \ldots, j_m)\|_p \]
for every \( p > 1 \). Therefore \((I_n(1))_{n \in \mathbb{N}} \in \mathcal{S}\) and
\[ e(I_n(1)) = (\alpha + 0.5q - 1.5) - 0.5(q(m) + q - 1) + (m + 1) \]
\[ -0.5m_1(q_1, \ldots, q_m, q - 1) - \infty 1_{\{\min\{q_1, \ldots, q_m, q - 1\}<0\}} \]
\[ = \alpha - 0.5q(m) + m - 0.5m_1(q_1, \ldots, q_m) - \infty 1_{\{\min\{q_1, \ldots, q_m\}<0\}} - 0.5 \]
\[ = e(I_n) - 0.5. \quad (7.15) \]

If \( q_{m'} \leq 0 \), then \( e(I_n(2, m')) = -\infty \). If \( q_{m'} = 1 \), then for the expression
\[ I_n(2, m') = n^{\alpha + 0.5q - 1.5} \sum_{j_1, \ldots, j_m} A_n(j_1, \ldots, j_m)q_{m'}I_q(1^{\otimes q_1}) \cdots I_{q_{m'-1}}(1^{\otimes q_{m'-1}}) \]
\[ \times I_{q-1}(1^{\otimes (q-1)}I_{q_{m'+1}}(1^{\otimes q_{m'+1}}) \cdots I_{q_{m}}(1^{\otimes q_{m}}) \]
of \( I_n(2, m') \), we may say
\[ e(I_n(2, m')) = (\alpha + 0.5q - 1.5) - 0.5(q(m) - 1 + q - 1) + m \]
\[ -0.5m_1(q_1, \ldots, q_{m'-1}, q - 1, q_{m'+1}, \ldots, q_m) - \infty 1_{\{\min\{q_1, \ldots, q_{m'-1}, q - 1, q_{m'+1}, \ldots, q_m\}<0\}} \]
\[ = (\alpha + 0.5q - 1.5) - 0.5(q(m) - 1 + q - 1) + m \]
\[ -0.5m_1(q_1, \ldots, q_{m'-1}, q_{m'}, q_{m'+1}, \ldots, q_m) - \infty 1_{\{\min\{q_1, \ldots, q_{m'-1}, q_{m'}, q_{m'+1}, \ldots, q_m\}<0\}} \]
\[ = e(I_n) - 0.5. \quad (7.16) \]

If \( q_{m'} \geq 2 \), then we apply the product formula (7.6) on p.34 to obtain
\[ I_{q_{m'-1}}(1^{\otimes (q_{m'-1})})I_{q-1}(1^{\otimes (q-1)}) = \sum_{\nu \in \mathbb{Z}_+} c_{\nu}(q_{m'} - 1, q - 1)I_{q_{m'+q-2-2\nu}}(1^{\otimes (q_{m'} + q-2-2\nu)})n^{-\nu} \]

Therefore,
\[ I_n(2, m') = \sum_{\nu \in \mathbb{Z}_+} c_{\nu}(q_{m'} - 1, q - 1)I_n(2, m', \nu) \quad (7.17) \]
where
\[ I_n(2, m', \nu) = n^{\alpha + 0.5q - 1.5 - \nu} \sum_{j_1, \ldots, j_m} A_n(j_1, \ldots, j_m)a_{j_{m'}}(q)q_{m'}q_{m'}I_q(1^{\otimes q_1}) \cdots I_{q_{m'-1}}(1^{\otimes q_{m'-1}}) \]
\[ \times I_{q_{m'+q-2-2\nu}}(1^{\otimes (q_{m'} + q-2-2\nu)}) \]
\[ \times I_{q_{m'+1}}(1^{\otimes q_{m'+1}}) \cdots I_{q_{m}}(1^{\otimes q_{m}}) \quad (7.18) \]
If \( q_{m'} \geq 2 \) and \( q_{m'} = q \), then
\[
e(I_n(2, m', \nu)) = (\alpha + 0.5q - 1.5 - \nu) - 0.5(q(m) + q - 2 - 2\nu) + m
-0.5m_1(q_1, ..., q_{m'-1}, q_{m'} + q - 2 - 2\nu, q_{m'+1}, ..., q_m)
-\infty 1_{\{\min\{q_1, ..., q_{m'-1}, q_{m'} + q - 2 - 2\nu, q_{m'+1}, ..., q_m\} < 0\}}.
\]
In this case, for \( \nu = q - 1 \),
\[
e(I_n(2, m', 2^{-1}(q_{m'} + q) - 1)) = (\alpha - 0.5q - 0.5) - 0.5(q(m) - q) + m
-0.5(m_1(q_1, ..., q_{m'-1}, q_{m'}, q_{m'+1}, ..., q_m) - 1)
-\infty 1_{\{\min\{q_1, ..., q_{m'-1}, q_{m'}, q_{m'+1}, ..., q_m\} < 0\}}
= e(I_n) - 0.5.
\]
So, if a term satisfying \( q_{m'} \geq 2 \) and \( q_{m'} = q \) appears, then in general \( e(D_{an}I_n) \leq e(I_n) \), according to the other estimates (7.21) below.

If \( q_{m'} \geq 2 \) and \( q_{m'} \neq q \), then \( q_{m'} + q - 2 - 2\nu \) never attains 0. Indeed, the terms for \( \nu \leq \min\{q_{m'} - 1, q - 1\} \) can only remain in the product formula, and this implies \( \nu \) can be \( 2^{-1}(q_{m'} + q) - 1 \) (equivalently, \( q_{m'} + q - 2 - 2\nu = 0 \)) among such \( \nu \) vs only when \( q_{m'} = q \). We may consider \( \nu \) such that \( q_{m'} + q - 2 - 2\nu > 0 \) for the least favorable case; otherwise \( q_{m'} + q - 2 - 2\nu < 0 \) and then the exponent is \( -\infty \).

In this situation,
\[
I_n(2, m', \nu) = (\alpha + 0.5q - 1.5 - \nu) - 0.5(q(m) + q - 2 - 2\nu) + m
-0.5m_1(q_1, ..., q_{m'-1}, q_{m'} + q - 2 - 2\nu, q_{m'+1}, ..., q_m)
-\infty 1_{\{\min\{q_1, ..., q_{m'-1}, q_{m'} + q - 2 - 2\nu, q_{m'+1}, ..., q_m\} < 0\}}
= e(I_n) - 0.5.
\]

**7.3 Operator \( D^i \)**

Recall the sequence of functionals \( I_n \) admitting the representation (7.5):
\[
I_n = n^\alpha \sum_{j_1, ..., j_m=1}^n A_n(j_1, ..., j_m)I_{q_1}(1_{j_1}^{\otimes q_1}) \cdots I_{q_m}(1_{j_m}^{\otimes q_m})
\]
where \( n \in \mathbb{N}, \alpha \in \mathbb{R}, m \in \mathbb{N} \) and \( (q_1, ..., q_m) \in \mathbb{Z}^m \). In this section, we will assess the effect of the operator \( D^i \) applied to \( I_n \).

**Proposition 7.2.** Suppose that Conditions [S] and (2.2) of [A] are fulfilled. Then, for \( I_n \) with the representation (7.5) and \( i \in \mathbb{Z}_+ \),
\[
|D^iI_n|_{S^\otimes i} = O_{L^\infty}(n^{e(I_n)})
\]
as \( n \to \infty \).
Proof. We may assume that \( q_1, \ldots, q_m \geq 0 \) since otherwise the claim is true due to \( I_n = 0 \). We have

\[
|D I_n|_p^2 \leq J_n^{(1)} + J_n^{(2)}
\] (7.22)

where

\[
J_n^{(1)} = 2^n n^{2\alpha} \sum_{j_1, \ldots, j_m = 1}^{n} \langle DA_n(j_1, \ldots, j_m), DA_n(j'_1, \ldots, j'_m) \rangle
\]

\[
\times I_{q_1}(1^{\otimes q_1}) \cdots I_{q_m}(1^{\otimes q_m}) I_{q_1}(1^{\otimes q_1}) \cdots I_{q_m}(1^{\otimes q_m})
\]

and

\[
J_n^{(2)} = 2^n n^{2\alpha - 1} \sum_{m' = 1}^m q_{m'}^2 \sum_{j_1, \ldots, j_m, j'_1, \ldots, j'_m, j_{m'} = 1}^{n} A_n(j_1, \ldots, j_m) A_n(j'_1, \ldots, j'_m)
\]

\[
\times I_{q_1}(1^{\otimes q_1}) \cdots I_{q_{m'-1}}(1^{\otimes q_{m'-1}}) I_{q_{m'}}(1^{\otimes q_{m'}}) \cdots I_{q_m}(1^{\otimes q_m})
\]

\[
\times I_{q_1}(1^{\otimes q_1}) \cdots I_{q_{m'-1}}(1^{\otimes q_{m'-1}}) I_{q_{m'}}(1^{\otimes q_{m'}}) \cdots I_{q_m}(1^{\otimes q_m})
\]

\[
\times I_{q_m}(1^{\otimes(q_m-1)})^2
\]

Then

\[
e(J_n^{(1)}) = 2\alpha - \bar{q}(m) + 2m - m_1(q_1, \ldots, q_m) - \infty 1_{\{m_1(q_1, \ldots, q_m) < 0\}} = 2e(I_n).
\] (7.23)

For \( J_n^{(2)} \), we apply the product formula

\[
I_{q_m}(1^{\otimes(q_m-1)})^2 = \sum_{\nu} c_{\nu}(q_m - 1, q_m - 1) I_{2q_m - 2 - 2\nu}(1^{\otimes(2q_m - 2 - 2\nu)})n^{-\nu}
\]

to obtain

\[
e(J_n^{(2)}) \leq 2e(I_n)
\] (7.24)

since

\[
(2\alpha - 1 - \nu) - 0.5(2\bar{q}(m) - 2 - 2\nu) + (2m - 1)
\]

\[
-0.5 \times m_1(q_1, \ldots, q_{m'-1}, q_{m'+1}, \ldots, q_m, q_1, \ldots, q_{m-1}, q_{m'+1}, \ldots, q_m, 2q_m - 2 - 2\nu)
\]

\[
-\infty 1_{\{\min(q_1, \ldots, q_{m'-1}, q_{m'+1}, \ldots, q_m, 2q_m - 2 - 2\nu) < 0\}}
\]

\[
\leq 2\alpha - 2 \times 0.5\bar{q}(m) + 2m - 1
\]

\[
-0.5 \times m_1(q_1, \ldots, q_{m'-1}, q_{m'+1}, \ldots, q_m, q_1, \ldots, q_{m'-1}, q_{m'+1}, \ldots, q_m)
\]

\[
= 2\alpha - 2 \times 0.5\bar{q}(m) + 2m - 1 - 0.5 \times (2m_1(q_1, \ldots, q_m) - 21_{\{q_m > 0\}})
\]

\[
\leq 2e(I_n)
\]

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due to \( \min\{q_1, \ldots, q_m\} \geq 0 \) by assumption. From (7.22), (7.23) and (7.24), we obtain

\[
e(\|D I_n\|^2) \leq 2e(I_n)
\]

and hence

\[
|D I_n|_{\ell^p} = O(\eta^e(I_n)).
\]

The derivative \( D^s I_n \) for \( s \geq 2 \) can be estimated in a similar manner. We consider

\[
\mathcal{K}_n = \left| n^a \sum_{j_1, \ldots, j_m = 1}^{n} D^k A_n(j_1, \ldots, j_m) I_{p_1}(1_{j_1}^{\otimes p_1}) \cdots I_{p_m}(1_{j_m}^{\otimes p_m}) \otimes 1_{j_1}^{\otimes (q_1-p_1)} \otimes \cdots \otimes 1_{j_m}^{\otimes (q_m - p_m)} \right|^2_{\ell^p}
\]

for \( k, p_1, \ldots, p_m \in \mathbb{Z}_+ \) such that \( p_i \leq q_i \) (\( i = 1, \ldots, m \)) and \( k + \sum_{i=1}^{m} (q_i - p_i) = s \). In general, \( p_{\ell+1} = \cdots = p_m = 0 \), and in this situation, we may consider

\[
\mathcal{K}_n' = \left| n^a \sum_{j_1, \ldots, j_m = 1}^{n} D^k A_n(j_1, \ldots, j_m) I_{p_1}(1_{j_1}^{\otimes p_1}) \cdots I_{p_\ell}(1_{j_\ell}^{\otimes p_\ell}) I_{q_{\ell+1}}(1_{j_{\ell+1}}^{\otimes q_{\ell+1}}) \cdots I_{q_\kappa}(1_{j_\kappa}^{\otimes q_\kappa}) \otimes 1_{j_1}^{\otimes (q_1-p_1)} \otimes \cdots \otimes 1_{j_\ell}^{\otimes (q_\ell-p_\ell)} \right|^2_{\ell^p}
\]

where \( p_1, \ldots, p_\ell \geq 0 \). This expression causes, when \( q_1, \ldots, q_\ell, q_{\ell+1}, \ldots, q_\kappa > 0 \) and \( q_{\kappa+1} = \cdots = q_m = 0 \), from the \( q_i - p_i > 0 \) times derivative of \( I_{p_1}(1_{j_1}^{\otimes p_1}) \), \( \ldots \), \( I_{p_m}(1_{j_m}^{\otimes p_m}) \) and no derivative of \( I_{q_{\ell+1}}(1_{j_{\ell+1}}^{\otimes q_{\ell+1}}), \ldots, I_{q_\kappa}(1_{j_\kappa}^{\otimes q_\kappa}) \) with \( q_{\ell+1}, \ldots, q_\kappa > 0 \). Now

\[
\mathcal{K}_n' = n^{2a - \sum_{i=1}^{\ell} (q_i - p_i) + 2(m - \kappa)}
\]

\[
\times \sum_{j_1, \ldots, j_\ell = j_1, \ldots, j_\ell}^{j_1, \ldots, j_\ell, \ldots, j_m, j_{\ell+1}, \ldots, j_m} \left\{ n^{-2(m - \kappa)} \sum_{j_1', \ldots, j_\ell' = j_1', \ldots, j_\ell'}^{j_1', \ldots, j_\ell', \ldots, j_m, j_{\ell+1}', \ldots, j_m'} \langle D^k A_n(j_1, \ldots, j_m), D^k A_n(j_1', \ldots, j_m') \rangle_{\ell^p} \right\} \times I_{p_1}(1_{j_1}^{\otimes p_1})^2 \cdots I_{p_\ell}(1_{j_\ell}^{\otimes p_\ell})^2 I_{q_{\ell+1}}(1_{j_{\ell+1}}^{\otimes q_{\ell+1}}) \cdots I_{q_\kappa}(1_{j_\kappa}^{\otimes q_\kappa}) I_{q_{\ell+1}}(1_{j_{\ell+1}}^{\otimes q_{\ell+1}}) \cdots I_{q_\kappa}(1_{j_\kappa}^{\otimes q_\kappa}) \right.^{(7.25)}
\]

where \( j_1 = j_1', \ldots, j_\ell = j_\ell' \). The factor \( I_{p_1}(1_{j_1}^{\otimes p_1})^2 \cdots I_{p_\ell}(1_{j_\ell}^{\otimes p_\ell})^2 \) is a weighted sum of the terms

\[
I_{2p_1-2\nu_1}(1_{j_1}^{2p_1 - 2\nu_1}) \cdots I_{2p_\ell-2\nu_\ell}(1_{j_\ell}^{2p_\ell - 2\nu_\ell}) n^{-(\nu_1 + \cdots + \nu_\ell)}
\]

with \( \nu_i \in \{0, \ldots, p_i\} \) for \( i = 1, \ldots, \ell \). Therefore, for the representation (7.25) of \( \mathcal{K}_n' \) expanded
with (7.26), we have \(e(K'_n)\) as the maximum (in \(\nu_i\)'s) of
\[
2\alpha - \sum_{i=1}^{\ell}(q_i - p_i) + 2(m - \kappa) - \sum_{i=1}^{\ell}\nu_i
\]
\[
-\frac{1}{2}\left\{\sum_{i=1}^{\ell}(2p_i - 2\nu_i) + \sum_{i=\ell+1}^{\kappa}2q_i\right\} + (\ell + 2(\kappa - \ell))
\]
\[
-\frac{1}{2}m_1(2p_1 - 2\nu_1, ..., 2p_\ell - 2\nu_\ell, q_{\ell+1}, ..., q_\kappa, q_{\ell+1}, ..., q_\kappa)
\]
\[
\leq 2\alpha - \sum_{i=1}^{\kappa}q_i + 2m - \kappa
\]
\[
= 2\alpha - \sum_{i=1}^{m}q_i + 2m - m_1(q_1, ..., q_m)
\]
\[
= 2e(I_n)
\]
Consequently,
\[
|D^i I_n|^2_{\mathcal{B}^{\otimes i}} = O(n^{2e(I_n)})
\]
for \(i \in \mathbb{Z}_+\).

7.4 Stability of the class \([S]\)

Proposition 7.3. Suppose that \(I_n\) admits an expression (7.1) with \(A_n(j_1, ..., j_m)\) satisfying Condition \([S]\). Then a version of density \(D^i_1, ..., D^i_m I_n\) satisfies
\[
\sup_{n \in \mathbb{N}} \sup_{t_1, ..., t_i \in [0, 1]} \|D^i_1, ..., D^i_m I_n\|_p \leq K(i, p)n^{e(I_n)}
\]
for every \(i \in \mathbb{Z}\) and \(p > 1\), where \(K(i, p)\) is a constant determined by a finite number of \(C(i, p)\)'s in Condition \([S]\).

Proof. According to the discussion below, we may show the claim for \(i = 1\). We have
\[
D_s I_n = n^\alpha \sum_{j_1, ..., j_m = 1}^n D_s A_n(j_1, ..., j_m) I_{q_1}(f^{(1)}_{j_1}) \cdots I_{q_m}(f^{(m)}_{j_m})
\]
\[
+ n^\alpha \sum_{j_1, ..., j_m = 1}^n A_n(j_1, ..., j_m) q_1 I_{q_1}(f^{(1)}_{j_1(s)}) I_{q_2}(f^{(2)}_{j_2}) \cdots I_{q_m}(f^{(m)}_{j_m}) + \cdots
\]
\[
=: k_n^{(0)} + k_n^{(1)} + \cdots + k_n^{(m)}
\]
where \(j_1(s) = j_1\) such that \(s \in I_{j_1}\). Obviously, \(\sup_{t \in [0, 1]} \sup_{n \in \mathbb{N}} \|k_n^{(0)}\|_p = O(n^{e(I_n)})\) for every \(p > 1\) because \(e(I_n) = e(K_n^{(0)})\). For \(k_n^{(1)}\), the \(L^p\)-norm \(\|k_n^{(1)}\|_p\) is bounded by the product of
\[ q_1 \| I_{q_1-1}(f^{(1)}_{j_1(s)}) \|_{2p} = O(n^{-0.5(q_1-1)}) \] and
\[
\sup_{n \in \mathbb{N}} \sup_{j_1 \in J} \left\| n^\alpha \sum_{j_2, \ldots, j_m=1}^n A_n(j_1, \ldots, j_m) I_{q_2}(f^{(2)}_{j_2}) \cdots I_{q_m}(f^{(m)}_{j_m}) \right\|_{2p} = O(n^\xi)
\]
for
\[
\xi = \alpha - 0.5(q_2 + \cdots + q_m) + (m - 1) - 0.5m_1(q_2, \ldots, q_m).
\]
Therefore,
\[
\| \xi^{(1)}_n \|_p = O(n^\xi)
\]
for
\[
\xi' = \alpha - 0.5(q_2 + \cdots + q_m) + (m - 1) - 0.5m_1(q_2, \ldots, q_m) - 0.5(q_1 - 1)
\]
\[
\leq \alpha - 0.5(q_1 + \cdots + q_m) + m - 0.5m_1(q_1, \ldots, q_m)
\]
\[
\leq e(I_n)
\]
We obtain the result since similar estimates hold for other terms and it is easy to see each estimate does not depend on \( s \in [0, 1] \).

We can strengthen Proposition 7.1 (ii).

**Corollary 7.4.** Under the conditions in Proposition 7.1, if there is one \( m' \in \{1, \ldots, m\} \) such that \( q_{m'} \neq q \), then
\[
e(D_{u_n(q)} I_n) \leq e(I_n) - \frac{1}{2}.
e(D_{u_n(q)} I_n)
\]

**Proof.** We may suppose that \( q_1, \ldots, q_m \geq 0 \) and \( q_1 \neq q \). Then
\[
I_n = n^{\alpha-e} \sum_{j_1} A'_n(j_1) I_{q_1}(1^{\otimes q_1})
\]
for
\[
A'_n(j_1) = n^{e} \sum_{j_2, \ldots, j_m} A_n(j_1, j_2, \ldots, j_m) I_{q_2}(1^{\otimes q_2}) \cdots I_{q_m}(1^{\otimes q_m}),
\]
where
\[
e = e(A'_n(j_1)) = -0.5(q_2 + \cdots q_m) + (m - 1) - 0.5m_1(q_2, \ldots, q_m).
e(A'_n(j_1))
\]

By Proposition 7.3, we see \( A'_n(j_1) \) satisfies Condition [S]. Then, by Proposition 7.1 (ii), the representation (7.27) gives
\[
e(D_{u_n(q)} I_n) \leq (\alpha + e - 0.5q_1 + 1 - 0.5m_1(q_1)) - 0.5
\]
\[
e(D_{u_n(q)} I_n) \leq e(I_n) - 0.5.
e(D_{u_n(q)} I_n)
\]
8 Proof of Theorem 2.2

8.1 The first projection $D_{u_n} M_n$

For the first projection $D_{u_n(q_2)} M_n(q_1) = \langle DM_n(q_1), u_n(q_2) \rangle$, we have

$$D_{u_n(q_2)} M_n(q_1) = D_{u_n(q_2)} \left( n^{2-1(q_1-1)} \sum_{j=1}^{n} a_j(q_1) I_{q_1}(1_j^{\otimes q_1}) \right)$$

$$- D_{u_n(q_2)} \left( n^{2-1(q_1-1)} \sum_{j=1}^{n} (D_1 a_j(q_1)) I_{q_1-1}(1_j^{\otimes (q_1-1)}) \right)$$

$$= \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 + \mathbb{I}_4$$

where

$$\mathbb{I}_1 = n^{2-1(q_1+q_2)-1} \sum_{j,k} (q_1 a_j(q_1) a_k(q_2) (1_j, 1_k) I_{q_1-1}(1_j^{\otimes (q_1-1)}) I_{q_2-1}(1_k^{\otimes (q_2-1)}),$$

$$\mathbb{I}_2 = n^{2-1(q_1+q_2)-1} \sum_{j,k} (D_1 a_j(q_1)) a_k(q_2) I_{q_1}(1_j^{\otimes q_1}) I_{q_2-1}(1_k^{\otimes (q_2-1)}),$$

$$\mathbb{I}_3 = -n^{2-1(q_1+q_2)-1} \sum_{j,k} (q_1 - 1) (D_1 a_j(q_1)) a_k(q_2) (1_j, 1_k) I_{q_1-2}(1_j^{\otimes (q_1-2)}) I_{q_2-1}(1_k^{\otimes (q_2-1)})$$

and

$$\mathbb{I}_4 = -n^{2-1(q_1+q_2)-1} \sum_{j,k} (D_1 D_1 a_j(q_1)) a_k(q_2) I_{q_1-1}(1_j^{\otimes (q_1-1)}) I_{q_2-1}(1_k^{\otimes (q_2-1)}).$$

8.2 The second projection $D_{u_n}^2 M_n$ and the quasi-torsion

In this section, we will investigate the quasi-torsion. The second projection $D_{u_n}^2 M_n = D_{u_n} D_{u_n} M_n = \langle D(DM_n, u_n), u_n \rangle$ involves various patterns of terms. Let

$$\beta_{j,k} = \langle 1_j, 1_k \rangle = n^{-1} 1_{\{j=k\}},$$

the Kronecker’s delta, and simply denote

$$\bar{q}(3) = q_1 + q_2 + q_3$$

for $(q_1, q_2, q_3) \in \mathcal{Q}^3$. For random variables $V_n$ and positive numbers $b_n$, we write $V_n = O_{L^n}(b_n)$ if $\|V_n\|_p = O(b_n)$ as $n \to \infty$ for every $p > 1$. Let $z \in \mathbb{R}$ and $x \in \mathbb{R}$. 

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8.2.1 \( D_{u_n(q_3)} \|_4 \)

Recall the expression equivalent to (8.5):

\[
\|_4 = -n^{2(q_1 + q_2) - 3} \sum_{j,k} (n^2 D_{1_q} D_{j,1} a_j(q_1)) a_k(q_2) I_{q_1 - 1}(1_{j}^{\otimes(q_1 - 1)}) I_{q_2 - 1}(1_{k}^{\otimes(q_2 - 1)}).
\]

The exponent of \( \|_4 \) by this representation is

\[
e(\|_4) = (0.5(q_1 + q_2) - 3) - 0.5(q_1 + q_2 - 2) + 2 - 0.5 \times 2 = -1
\]

(8.7) since \( q_1, q_2 \geq 2 \). Then Proposition 7.1 gives

\[
e(D_{u_n} \|_4) \leq -1
\]

(8.8)

Therefore,

\[
D_{u_n(q_3)} \|_4 = O_{L^{\infty}}(n^{-1})
\]

(8.9) by (7.3). In particular,

\[
E[\Psi(z, x) D_{u_n(q_3)} \|_4] = O(n^{-1})
\]

(8.10) as \( n \to \infty \) for every \( z \in \mathbb{R} \) and \( x \in \mathbb{R} \).

8.2.2 \( D_{u_n(q_3)} \|_1 \)

Recall

\[
\|_1 = n^{2(q_1 + q_2) - 3} \sum_{j,k} q_1 a_j(q_1) a_k(q_2) \beta_{j,k} I_{q_1 - 1}(1_{j}^{\otimes(q_1 - 1)}) I_{q_2 - 1}(1_{k}^{\otimes(q_2 - 1)}).
\]

Then

\[
D_{u_n(q_3)} \|_1 = n^{0.5(q_3 - 3)} \sum_{j,k,\ell} q_1 (q_1 - 1) a_j(q_1) a_k(q_2) a_\ell(q_3) \beta_{j,k} \beta_{j,\ell} \\
\times I_{q_1 - 2}(1_{j}^{\otimes(q_1 - 2)}) I_{q_2 - 1}(1_{k}^{\otimes(q_2 - 1)}) I_{q_3 - 1}(1_{\ell}^{\otimes(q_3 - 1)})
\]

\[
+ n^{0.5(q_3 - 3)} \sum_{j,k,\ell} q_1 (q_2 - 1) a_j(q_1) a_k(q_2) a_\ell(q_3) \beta_{j,k} \beta_{j,\ell} \\
\times I_{q_1 - 1}(1_{j}^{\otimes(q_1 - 1)}) I_{q_2 - 2}(1_{k}^{\otimes(q_2 - 2)}) I_{q_3 - 1}(1_{\ell}^{\otimes(q_3 - 1)})
\]

\[
+ n^{0.5(q_3 - 3)} \sum_{j,k,\ell} q_1 (D_{1_q}(a_j(q_1) a_k(q_2))) a_\ell(q_3) \beta_{j,k} \\
\times I_{q_1 - 1}(1_{j}^{\otimes(q_1 - 1)}) I_{q_2 - 1}(1_{k}^{\otimes(q_2 - 1)}) I_{q_3 - 1}(1_{\ell}^{\otimes(q_3 - 1)})
\]

(8.11)

In the case (8.6),

\[
\|_{1,1} = n^{0.5q_3 - 3.5} \sum_{j} q_1 (q_1 - 1) a_j(q_1) a_j(q_2) a_j(q_3) \\
\times I_{q_1 - 2}(1_{j}^{\otimes(q_1 - 2)}) I_{q_2 - 1}(1_{j}^{\otimes(q_2 - 1)}) I_{q_3 - 1}(1_{j}^{\otimes(q_3 - 1)}).
\]

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By (7.6),
\[ I_{1,1} = I_{1,1,1} + I_{1,1,2}, \]
where
\[ I_{1,1,1} = n^{-1.5} \sum_j q_1(q_1 - 1)a_j(q_1)a_j(q_2)a_j(q_3)c_{0.5\tilde{q}(3)-2}(q_1 - 2, q_2 - 1, q_3 - 1) \]  
and
\[ I_{1,1,2} = \sum_{\nu, \tilde{q}(3)-4-2\nu > 0} \langle E, 1 \rangle \nu \tilde{q}(3)-3.5 - \nu n^{0.5\tilde{q}(3)-3.5 - \nu} \sum_j q_1(q_1 - 1)a_j(q_1)a_j(q_2)a_j(q_3)I_{\tilde{q}(3)-4-2\nu}(1^\otimes(\tilde{q}(3)-4-2\nu)). \]

We have
\[ e(I_{1,1,1}) = -1.5 - 0.5 \times 0 + 1 - 0.5 \times 0 = -0.5 \]  
and
\[ e(I_{1,1,2}) = (0.5\tilde{q}(3) - 3.5 - \nu) - 0.5(\tilde{q}(3) - 4 - 2\nu) + 1 - 0.5 = -1. \]

Thus, \( I_{1,1,1} \) determines the order of the symbol. More precisely,
\[ E[\Psi(z, x) I_{1,1,1}] = 1_{(q(3):even)} q_1(q_1 - 1)c_{0.5\tilde{q}(3)-2}(q_1 - 2, q_2 - 1, q_3 - 1) \]
\[ \times n^{-1.5} \sum_{j=1}^n E[\Psi(z, x)a_j(q_1)a_j(q_2)a_j(q_3)] + O(n^{-1}). \]  

Similarly,
\[ I_{1,1}' = I_{1,1,1}' + I_{1,1,2}' \]
where
\[ I_{1,1,1}' = n^{-1.5} \sum_j q_1(q_2 - 1)a_j(q_1)a_j(q_2)a_j(q_3)c_{0.5\tilde{q}(3)-2}(q_1 - 1, q_2 - 2, q_3 - 1) \]
and \( I_{1,1,2}' \in L \). Moreover,
\[ e(I_{1,1,1}') = -0.5 \]  
and
\[ e(I_{1,1,2}') = -1 \]

In particular, since \( e(I_{1,1,2}') = O_L \approx -n^{-1/2} \),
\[ E[\Psi(z, x) I_{1,1}'] = 1_{(q(3):even)} q_1(q_2 - 1)c_{0.5\tilde{q}(3)-2}(q_1 - 1, q_2 - 2, q_3 - 1) \]
\[ \times n^{-1.5} \sum_{j=1}^n E[\Psi(z, x)a_j(q_1)a_j(q_2)a_j(q_3)] + O(n^{-1}). \]
To estimate $E[\Psi(z,x)\mathbb{I}_{1,2}]$, we apply the product formula to $I_{q-1}(1_j^\otimes(q-1))^2$ first and obtain

$$\mathbb{I}_{1,2} = \mathbb{I}_{1,2,1} + \mathbb{I}_{1,2,2} \quad (8.19)$$

where

$$\mathbb{I}_{1,2,1} = n^{0.5q_3-1.5} \sum_{\ell} \left\{ n^{-1} \sum_{j} 1_{\{q_1=q_2\}} q_1!(nD_1(q_j)D_2(q_j))a_\ell(q_3) \right\} I_{q_3-1}(1_\ell^\otimes(q_3-1)) \quad (8.20)$$

and

$$\mathbb{I}_{1,2,2} = \sum_{\nu, q_1+q_2-2-2\nu > 0} c_\nu(q_1-1,q_2-1)n^{0.5\bar{q}(3)-3.5-\nu} \sum_{j,\ell} nq_1(D_1(q_j)a_j(q_3))a_\ell(q_3) \times I_{q_1+q_2-2-2\nu}(1_j^\otimes(q_1+q_2-2-2\nu))I_{q_3-1}(1_\ell^\otimes(q_3-1))$$

Then

$$e(\mathbb{I}_{1,2,1}) = -0.5 \quad (8.21)$$

and

$$e(\mathbb{I}_{1,2,2}) = \nu_{q_1+q_2-2-2\nu > 0} \max \left\{ 0.5\bar{q}(3) - 3.5 - \nu, 0.5\bar{q}(3) - 3 - 2\nu, 2 - 0.5 \times 2 \right\}$$

Thus, the IBP gives

$$E[\Psi(z,x)\mathbb{I}_{1,2}] = n^{0.5q_3-1.5} 1_{\{q_1=q_2\}} q_1! \times \sum_{j,\ell} E\left[ \Psi(z,x)(D_1(q_j)a_j(q_3))a_\ell(q_3)I_{q_3-1}(1_\ell^\otimes(q_3-1)) \right] + O(n^{-1})$$

$$= n^{0.5q_3-1.5} 1_{\{q_1=q_2\}} q_1! \times \sum_{j,\ell} E\left[ D_1(q_1)^{-1}(\Psi(z,x)(D_1(q_j)a_j(q_3))a_\ell(q_3)) \right] + O(n^{-1})$$

$$= 1_{\{q_1+q_2, \text{ even}\}} O(n^{-0.5(q_3-1)}) + O(n^{-1})$$

Only when $q_1 = q_2$ and $q_3 = 2$, this term has effect to the asymptotic expansion. Therefore

$$E[\Psi(z,x)\mathbb{I}_{1,2}] = 1_{\{q_1=q_2, q_3=2\}} n^{-0.5} q_1! \sum_{j,\ell} E\left[ D_1(q_1)^2 (\Psi(z,x)(D_1(q_j)a_j(q_3))a_\ell(2)) \right] + O(n^{-1}). \quad (8.22)$$
From (8.11), (8.15), (8.18) and (8.22), it follows that
\[
E[\Psi(z, x)D_{u_n(q_3)}]|_1 = 1_{\{q(3):even\}}q_1(q_1 - 1)c_{0.5q(3)-2}(q_1 - 2, q_2 - 1, q_3 - 1) \\
\times n^{-1.5} \sum_{j=1}^{n} E[\Psi(z, x)a_j(q_1)a_j(q_2)a_j(q_3)] \\
+ 1_{\{q(3):even\}}q_1(q_2 - 1)c_{0.5q(3)-2}(q_1 - 1, q_2 - 2, q_3 - 1) \\
\times n^{-1.5} \sum_{j=1}^{n} E[\Psi(z, x)a_j(q_1)a_j(q_2)a_j(q_3)] + O(n^{-1}) \\
+ 1_{\{q_1=q_2, q_3=2\}} n^{-0.5} q_1! \\
\times \sum_{j, \ell} E[D_{1, \ell}\{\Psi(z, x)(D_{1, \ell}(a_j(q_1)^2))a_\ell(2)\}] + O(n^{-1}) \tag{8.23}
\]

We remark that, on the right-hand side of (8.23), the contribution possibly occurs from the first term only when \((q_1 - 1, q_2 - 2, q_3 - 1) \in \mathbb{T}\), and from the second term only when \((q_1 - 1, q_2 - 2, q_3 - 1) \in \mathbb{T}\).

From the above discussion, we know
\[
D_{u_n(q_3)}|_1 = 1 + \hat{l}_1 \tag{8.24}
\]
where
\[
\hat{l}_1 = l_{1,1,1} + l'_{1,1,1} + l_{1,2,1} \tag{8.25}
\]
with
\[
e(\hat{l}_1) = -0.5
\]
and
\[
e(\hat{l}_1) = -1. \tag{8.26}
\]

### 8.2.3 \(D_{u_n(q_3)}|_2\)

Recall
\[
l_2 = n^{2-1(q_1+q_2)-1} \sum_{j,k} (D_{1,\ell}(a_j(q_1)))a_k(q_2)I_{q_1}(1_{j}^{\otimes q_1})I_{q_2-1}(1_{k}^{\otimes (q_2-1)}).
\]

Then the projection \(D_{u_n(q_3)}|_2\) admits the decomposition
\[
D_{u_n(q_3)}|_2 = n^{0.5q(3)-3} \sum_{j,k,\ell} (D_{1,\ell}(a_j(q_1)))a_k(q_2)q_1I_{q_1-1}(1_{j}^{\otimes (q_1-1)})\beta_{j,\ell}a_{\ell}(q_3)I_{q_2-1}(1_{k}^{\otimes (q_2-1)})I_{q_3-1}(1_{\ell}^{\otimes (q_3-1)}) \\
+ n^{0.5q(3)-3} \sum_{j,k,\ell} (D_{1,\ell}(a_j(q_1)))a_k(q_2)I_{q_1}(1_{j}^{\otimes q_1})(q_2 - 1)I_{q_2-2}(1_{k}^{\otimes (q_2-2)})\beta_{k,\ell} \\
\times a_{\ell}(q_3)I_{q_3-1}(1_{\ell}^{\otimes (q_3-1)}) \\
+ n^{0.5q(3)-3} \sum_{j,k,\ell} n^2D_{1,\ell}((D_{1,\ell}(a_j(q_1)))a_k(q_2)))a_{\ell}(q_3)I_{q_1}(1_{j}^{\otimes q_1})I_{q_2-1}(1_{k}^{\otimes (q_2-1)})I_{q_3-1}(1_{\ell}^{\otimes (q_3-1)}) \\
=: l_{2,1} + l_{2,2} + l_{2,3}. \tag{8.27}
\]
The sum \( l_{2,1} \) has the same pattern as \( l_{1,2} \). Thus,

\[
l_{2,1} = n^{0.5(\bar{q}(3) - 3) - 1} \sum_{j,k} (D_{1_k} a_j(q_1)) a_k(q_2) a_j(q_3) q_1 I_{q_1-1} (1_j^{\otimes(q_1-1)}) I_{q_3-1} (1_j^{\otimes(q_3-1)}) I_{q_2-1} (1_k^{\otimes(q_2-1)})
\]

\[
= n^{0.5(\bar{q}(3) - 3) - 1} \sum_{j,k} (D_{1_k} a_j(q_1)) a_k(q_2) a_j(q_3) q_1 \\
\times \sum_{\nu \in \mathbb{Z}_+} c_\nu (q_1 - 1, q_3 - 1) I_{q_1+q_3-2-2\nu} (1_j^{\otimes(q_1+q_3-2-2\nu)}) n^{\nu} I_{q_2-1} (1_k^{\otimes(q_2-1)})
\]

\[
= l_{2,1,1} + l_{2,1,2}
\]

(8.28)

where

\[
l_{2,1,1} = n^{0.5q_2-1.5} \sum_k \left\{ n^{-1} \sum_j (nD_{1_k} a_j(q_1)) a_k(q_2) a_j(q_1) \right\} \\
\times 1_{(q_1=q_3)} q_1 ! I_{q_2-1} (1_k^{\otimes(q_2-1)})
\]

and

\[
l_{2,1,2} = \sum_{\nu \in \mathbb{Z}_+, q_1+q_3-2-2\nu > 0} n^{0.5q(3) - 3.5 - \nu} \sum_{j,k} (nD_{1_k} a_j(q_1)) a_k(q_2) a_j(q_3) q_1 \\
\times c_\nu (q_1 - 1, q_3 - 1) I_{q_1+q_3-2-2\nu} (1_j^{\otimes(q_1+q_3-2-2\nu)}) I_{q_2-1} (1_k^{\otimes(q_2-1)}).
\]

Then

\[
e(l_{2,1,1}) = (0.5q_2 - 1.5) - 0.5(q_2 - 1) + 1 - 0.5 = -0.5,
\]

therefore this sum can remain and is treated below. On the other hand,

\[
e(l_{2,1,2}) = \max_{\nu; q_1+q_3-2-2\nu > 0} \left\{ (0.5\bar{q}(3) - 3.5 - \nu) - 0.5(\bar{q}(3) - 3 - 2\nu) + 2 - 1 \right\} = -1,
\]

therefore this term is negligible.

By these observations, we compute the projection of \( l_{2,1} \) to \( \Psi(z, x) \):

\[
E[\Psi(z, x) l_{2,1}] = q_1 ! 1_{(q_1=q_3)} n^{0.5q_2-1.5} \\
\times \sum_{j,k} E \left[ \Psi(z, x) (D_{1_k} a_j(q_1)) a_k(q_2) a_j(q_1) I_{q_2-1} (1_k^{\otimes(q_2-1)}) \right] + O(n^{-1})
\]

\[
= q_1 ! 1_{(q_1=q_3)} n^{0.5q_2-1.5} \\
\times \sum_{j,k} E \left[ D_{1_k}^{q_2-1} (\Psi(z, x) (D_{1_k} a_j(q_1)) a_k(q_2) a_j(q_1)) \right] + O(n^{-1})
\]

by the IBP. The order of the order of this term is therefore

\[
(0.5q_2 - 1.5) + 2 \left( \sum_{j,k} q_2 \right) = -0.5q_2 + 0.5,
\]

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and only the case $q_2 = 2$ remains. Consequently, we obtain

$$E[\Psi(z, x) | \mathbb{I}_{2,1}] = 1_{\{q_1 = q_3, q_2 = 2\}} n^{-0.5} q_1!$$

$$\times E \left[ \sum_{j,k} \{ D_{1k} (\Psi(z) (D_{1k} a_j(q_1)) a_k(2) a_j(q_1)) \} \right] + O(n^{-1}) \tag{8.29}$$

The sum $\mathbb{I}_{2,2}$ is evaluated as follows:

$$\mathbb{I}_{2,2} = n^{0.5(q(3) - 3) - 1} \sum_{j,k} (D_{1k} a_j(q_1)) a_k(q_2) a_k(q_3) I_{q_1} (1_j^{\otimes q_1}) (q_2 - 1) I_{q_2 - 2} (1_k^{\otimes (q_2 - 2)}) I_{q_3 - 1} (1_k^{\otimes (q_3 - 1)})$$

$$= n^{0.5(q(3) - 3) - 1} \sum_{j,k} (D_{1k} a_j(q_1)) a_k(q_2) a_k(q_3) (q_2 - 1) I_{q_1} (1_j^{\otimes q_1})$$

$$\times \sum_{\nu \in \mathbb{Z}^+} c_\nu (q_2 - 2, q_3 - 1) I_{q_2 + q_3 - 3 - 2\nu} (1_k^{\otimes (q_2 + q_3 - 3 - 2\nu)}) n^{-\nu}$$

$$= \mathbb{I}_{2,2,1} + \mathbb{I}_{2,2,2} \tag{8.30}$$

where

$$\mathbb{I}_{2,2,1} = n^{0.5q_1 - 1} \sum_j \left\{ n^{-1} \sum_k (nD_{1k} a_j(q_1)) a_k(q_2) a_k(q_2 - 1) (q_2 - 1)! \right\} I_{q_1} (1_j^{\otimes q_1})$$

$$\times 1_{\{q_2 = q_3 + 1\}} \tag{8.31}$$

and

$$\mathbb{I}_{2,2,2} = \sum_{\nu; q_2 + q_3 - 3 - 2\nu > 0} c_\nu (q_2 - 2, q_3 - 1) n^{0.5(q(3) - 3) - \nu}$$

$$\times \sum_{j,k} (nD_{1k} a_j(q_1)) a_k(q_2) a_k(q_3) (q_2 - 1) I_{q_1} (1_j^{\otimes q_1}) I_{q_2 + q_3 - 3 - 2\nu} (1_k^{\otimes (q_2 + q_3 - 3 - 2\nu)}). \tag{8.32}$$

From the representation (8.31),

$$e(\mathbb{I}_{2,2,1}) = (0.5 q_1 - 1) - 0.5 q_1 + 1 - 0.5 = -0.5 \tag{8.33}$$

and

$$e(\mathbb{I}_{2,2,2}) = \max_{\nu; q_2 + q_3 - 3 - 2\nu > 0} \left\{ (0.5q(3) - 3.5 - \nu) - 0.5(q(3) - 3 - 2\nu) + 2 - 0.5 \times 2 \right\}$$

$$= -1. \tag{8.34}$$

Then

$$E[\Psi(z, x) | \mathbb{I}_{2,2}] = 1_{\{q_2 = q_3 + 1\}} n^{0.5q_1 - 1} \sum_{j,k} (q_2 - 1)!$$

$$\times E \left[ \Psi(z, x) (D_{1k} a_j(q_1)) a_k(q_2) a_k(q_2 - 1) I_{q_1} (1_j^{\otimes q_1}) \right] + O(n^{-1})$$

$$= 1_{\{q_2 = q_3 + 1\}} n^{0.5q_1 - 1} (q_2 - 1)!$$

$$\times \sum_{j,k} E \left[ D_{1k}^{q_1} \{ \Psi(z, x) (D_{1k} a_j(q_1)) a_k(q_2) a_k(q_2 - 1) \} \right] + O(n^{-1})$$

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by IBP. The exponent to \( n \) of this expression is
\[
(0.5q_1 - 1) + 2\left(\sum_{j,k} q_1(D_{i,j}^{\otimes n}) - 1(D_{i,k})\right) = \ -0.5q_1 \leq -1 \quad (q_1 \geq 2)
\]
Thus, \( I_{2,2} \) asymptotically has no effect to the random symbol (even without Condition (2.10)):
\[
E[\Psi(z, x)|I_{2,2}] = O(n^{-1}).\tag{8.35}
\]
We remark that, under Condition (2.10), \( I_{2,2,1} = 0 \) since \( q_2 \neq q_3 + 1 \) for \( q_2, q_3 \in \mathcal{Q} \), hence
\[
e(\Psi) = -1, \quad \tag{8.36}
\]
and in particular, (8.35) follows directly.

Based on the representation of \( I_{2,3} \) in (8.27),
\[
e(\Psi) = \ (0.5\bar{q}(3) - 3.5) - 0.5(\bar{q}(3) - 2) + 3 - 0.5 \times 3 = -1
\]
and we conclude
\[
I_{2,3} = O_L(n^{-1}) \tag{8.37}
\]
and
\[
E[\Psi(z, x)|I_{2,3}] = O(n^{-1}).\tag{8.38}
\]
Consequently,
\[
E[\Psi(z, x)D_{u_n(q_3)}I_{2}] = 1_{(q_1 = q_3, q_2 = 2)}n^{-0.5}q_1!
\times E\left[\sum_{j,k} D_{i,k}(\Psi(z, x)(D_{i,j}^{(q_1)})a_j(2)a_j(q_1))\right] + O(n^{-1}) \tag{8.39}
\]
by (8.27), (8.29), (8.35) and (8.38).

### 8.2.4 \( D_{u_n(q_3)}I_{3} \)

In the present case of a Brownian motion,
\[
I_{3} = J_{3,1} + J_{3,2} \tag{8.40}
\]
where
\[
J_{3,1} = -n^{-1.5}\sum_{j} 1_{(q_1 = q_2 + 1)}(q_1 - 1)!(nD_{i,j}^{(q_1)}a_j(q_1))a_j(q_2)I_0(1^{\otimes 0}) \tag{8.41}
\]
and
\[
J_{3,2} = -\sum_{\{\nu: q_1 + q_2 - 3 - 2\nu > 0\}} c_{\nu}(q_1 - 2, q_2 - 1)(q_1 - 1)
\times n^{0.5(q_1 + q_2) - 3 - \nu}\sum_{j} (nD_{i,j}^{(q_1)}a_j(q_1))a_j(q_2)I_{q_1 + q_2 - 3 - 2\nu}(1^{\otimes (q_1 + q_2 - 3 - 2\nu)}) \tag{8.42}
\]

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Then
\[ e(J_{3,1}) = -0.5 \quad (8.43) \]
and
\[ e(J_{3,2}) = -1. \quad (8.44) \]
Applying Proposition 7.1 to \( J_{3,1} \) having the representation (8.41) and the exponent (8.43), we see \( Du_n(q_3)J_{3,1} \in \mathcal{L} \) and
\[ e(Du_n(q_3)J_{3,1}) \leq -1 \quad (8.45) \]
since \( 0 \neq q_3 \in \mathcal{Q} \). Simple application of Proposition 7.1 gives
\[ e(Du_n(q_3)J_{3,2}) \leq -1 \quad (8.46) \]
Thus, from (8.40), (8.45) and (8.46), we conclude \( Du_n(q_3)I_3 \in \mathcal{L} \) and
\[ e(Du_n(q_3)I_3) \leq -1. \quad (8.47) \]
In particular,
\[ Du_n(q_3)I_3 = O_{L^\infty}(n^{-1}) \quad (8.48) \]
has no effect asymptotically.

8.2.5 qTor

From (1.3) and (1.6), we have
\[ Du_n Du_n M_n = \sum_{q_1,q_2,q_3 \in \mathcal{Q}} Du_n(q_3)Du_n(q_2)M_n(q_1) \]
and, from (8.1),
\[ Du_n(q_3)Du_n(q_2)M_n(q_1) = Du_n(q_3)I_1 + Du_n(q_3)I_2 + Du_n(q_3)I_3 + Du_n(q_3)I_4. \]
Then, by the estimates (8.23), (8.39), (8.48) and (8.10), and by the symmetry of the functions $c_r(\cdots)$, we obtain
\[
E[\Psi(z, x)D_{u_n}D_{u_n}M_n] = \sum_{q_1, q_2, q_3 \in \mathbb{Q}} E[\Psi(z, x)D_{u_n(q_3)}D_{u_n(q_2)}M_n(q_1)]
\]
\[
= \sum_{q_1, q_2, q_3 \in \mathbb{Q}} 1_{\{q(3): \text{even}\}}(q_1 + q_2)(q_1 - 1)c_{0.5q(3) - 2}(q_1 - 2, q_2 - 1, q_3 - 1)
\times n^{-1.5} \sum_{j=1}^{n} E[\Psi(z, x)a_j(q_1)a_j(q_2)a_j(q_3)]
\]
\[
+ \sum_{q_1, q_2, q_3 \in \mathbb{Q}} 1_{\{q_1 = q_2, q_3 = 2\}} n^{-0.5} q_1!
\times \sum_{j,k=1}^{n} E[D_{1k}\{\Psi(z, x)(D_{1k}(a_j(q_1)^2)a_k(2))\}]
\]
\[
+ \sum_{q_1, q_2, q_3 \in \mathbb{Q}} 1_{\{q_1 = q_3, q_2 = 2\}} n^{-0.5} q_1!
\times \sum_{j,k=1}^{n} E[D_{1k}\{\Psi(z, x)(D_{1k}a_j(q_1)a_k(2)a_j(q_1))\}]
\]
\[+ O(n^{-1})
\]
or equivalently,
\[
n^{-0.5}E[\Psi(z, x)q_{\text{Tor}}(iz)^3]
\]
\[
= E[\Psi(z, x)D_{u_n}D_{u_n}M_n(iz)^3]
\]
\[
= \sum_{q_1, q_2, q_3 \in \mathbb{Q}} 1_{\{q(3): \text{even}\}}(q_1 + q_2)(q_1 - 1)c_{0.5q(3) - 2}(q_1 - 2, q_2 - 1, q_3 - 1)
\times n^{-1.5} \sum_{j=1}^{n} E[\Psi(z, x)a_j(q_1)a_j(q_2)a_j(q_3)](iz)^3
\]
\[
+ 3 \sum_{q_1 \in \mathbb{Q}} 1_{\{2 \in \mathbb{Q}\}} n^{-0.5} q_1! \sum_{j,k=1}^{n} E[\Psi(z, x)D_{1k}\{(D_{1k}a_j(q_1))a_j(q_1)a_k(2)\}](iz)^3
\]
\[
+ 3 \sum_{q_1 \in \mathbb{Q}} 1_{\{2 \in \mathbb{Q}\}} n^{-0.5} q_1! \sum_{j,k=1}^{n} E[\Psi(z, x)(2^{-1}D_{1k}G_{\infty}(iz)^5 + D_{1k}X_{\infty}(iz)^3(iz))^3]
\times (D_{1k}a_j(q_1))a_j(q_1)a_k(2)]
\]
\[+ O(n^{-1})
\]
(8.49)

### 8.2.6 Approximate projection of $q_{\text{Tor}}$

Define the random symbol $G_n^{(3,0)}(iz)$ by
\[
G_n^{(3,0)}(iz) = G_n^{(3,0)}(iz, ix) = \frac{1}{3} q_{\text{Tor}}(iz)^3.
\]
Then
$$E[\Psi(z, x) \mathcal{E}^{(3,0)}_n(iz)] = E[\Psi(z, x) \mathcal{E}^{(3,0)}_n(z, x)] + O(n^{-0.5})$$  \hfill (8.50)

for an approximate projection, that is a random symbol $\tilde{\mathcal{E}}^{(3,0)}(z, x)$ for $\mathcal{E}^{(3,0)}_n(z)$ defined by

$$\tilde{\mathcal{E}}^{(3,0)}(z, x) = \frac{1}{3} \sum_{q_1, q_2, q_3 \in Q} 1_{\{q(3) : \text{even}\}}(q_1 + q_2)(q_1 - 1)c_{0.5\bar{q}(3)-2}(q_1 - 2, q_2 - 1, q_3 - 1)$$

$$\times n^{-1} \sum_{j=1}^{n} a_j(q_1)a_j(q_2)a_j(q_3)(iz)^3$$

$$+ \sum_{q_1 \in Q} 1_{\{q_1 \in Q\}} q_1! \sum_{j, k=1}^{n} D_{1k} \{ (D_{1k}a_j(q_1))a_j(q_1)a_k(2) \} (iz)^3$$

$$+ \sum_{q_1 \in Q} 1_{\{q_1 \in Q\}} q_1! \sum_{j, k=1}^{n} \left[ (2^{-1}D_{1k}G_\infty(iz)^5 + D_{1k}X_\infty(iz)^3(ix)) \right.$$ 

$$\times (D_{1k}a_j(q_1))a_j(q_1)a_k(2) \left. \right].$$  \hfill (8.51)

Recall (2.11):
$$a^{(3,0)}(t, s, q_1, q_2) = \overset{a}{a}(t, s, q_1)a(s, q_2)a(t, 2) + \overset{a}{a}(t, s, q_1)\overset{a}{a}(t, s, q_2)a(t, 2)$$

$$+ \overset{a}{a}(t, s, q_1)a(s, q_2)\overset{a}{a}(t, 2).$$

The random symbol $\mathcal{E}^{(3,0)}(iz, ix)$ is defined by (2.12) on p.8.

Let $r > 0$.

[$A'$], Conditions (i),(vii) and (viii) of [$A$] are satisfied. Moreover, the following conditions are fulfilled (in place of (ii)-(vi) of [$A$]).

(ii) For every $q \in Q$,
$$\sum_{j=1}^{n} \int_0^1 1_{I_j}(s) |a_j(q) - a(s, q)|^r ds \to^p 0$$  \hfill (8.52)

as $n \to \infty$.

(iii) For every $q \in Q$,
$$\sum_{j, k=1}^{n} \int_0^1 \int_0^1 1_{I_k \times I_j}(t, s) |nD_{1k}a_j(q) - \overset{\hat{a}}{a}(t, s, q)|^r ds dt \to^p 0$$  \hfill (8.53)

as $n \to \infty$.

(iv) As $n \to \infty$,
$$\sum_{k=1}^{n} \int_0^1 1_{I_k}(t) |nD_{1k}a_k(2) - \overset{\hat{a}}{a}(t, 2)|^r dt \to^p 0$$  \hfill (8.54)

if $2 \in Q$. 

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(v) For every \( q \in \mathcal{Q} \),
\[
\sum_{j,k=1}^{n} \int_{[0,1]^2} 1_{I_k \times I_j}(t,s) \left| n^2 D_{1_k} D_{1_k} a_j(q_1) - \overset{\infty}{\overset{\circ}{a}}(t,s,q) \right|^r \, ds \, dt \to^p 0 \tag{8.55}
\]
as \( n \to \infty \).

(vi) An \( n \to \infty \),
\[
\sum_{j=1}^{n} \int_{0}^{1} 1_j(t) \left| n^2 D_{1_j} a_j(2) - \overset{\circ}{a}(2) \right|^r \, dt \to^p 0 \tag{8.56}
\]
if \( 2 \in \mathcal{Q} \).

Lemma 8.1. The following properties are equivalent:

(a) \([A]\) holds.

(b) \([A']_r\), holds for some \( r > 0 \).

(c) \([A']_r\), holds for all \( r > 0 \).

Proof. Let
\[
\varphi_n(t,s) = \sum_{j,k} 1_{I_k \times I_j}(t,s) n^2 D_{1_k} D_{1_k} a_j(q_1) - \overset{\infty}{\overset{\circ}{a}}(t,s,q_1).
\]
Under (2.2) and (2.6), it follows from Fatou’s lemma applied to the measure \( dPds\) that
\[
E \left[ \int_{[0,1]^2} \left| \overset{\infty}{\overset{\circ}{a}}(t,s,q_1) \right|^r \, ds \, dt \right] < \infty \tag{8.57}
\]
for every \( r > 1 \). We have
\[
E \left[ \sum_{j,k} \int_{[0,1]^2} 1_{I_k \times I_j}(t,s) \left| n^2 D_{1_k} D_{1_k} a_j(q_1) - \overset{\infty}{\overset{\circ}{a}}(t,s,q_1) \right|^r \, ds \, dt \right]
\]
\[
= E \left[ \int_{[0,1]^2} \left| \varphi_n(t,s) \right|^r \, ds \, dt \right]
\]
\[
\leq E \left[ \int_{[0,1]^2} \left| \varphi_n(t,s) \right|^r 1_{\{ \left| \varphi_n(t,s) \right| > A \}} \, ds \, dt \right] + E \left[ \int_{[0,1]^2} \left| \varphi_n(t,s) \right|^r 1_{\{ \left| \varphi_n(t,s) \right| \leq A \}} \, ds \, dt \right]
\]
\[
+ E \left[ \int_{[0,1]^2} \left| \varphi_n(t,s) \right|^r 1_{\{ \left| \varphi_n(t,s) \right| \leq \epsilon \}} \, ds \, dt \right]
\]
\[
\leq A^{-1} \int_{[0,1]^2} E \left[ \left| \varphi_n(t,s) \right|^{r+1} \right] \, ds \, dt + A \epsilon E \left[ \int_{[0,1]^2} 1_{\{ \left| \varphi_n(t,s) \right| > \epsilon \}} \, ds \, dt \right] + \epsilon^r
\]

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for constants $A > \epsilon > 0$. This shows (2.6) implies (8.55) for any $r > 0$ under (2.2) and (8.57). Other implications are verified in the same way. Therefore we proved $(a) \Rightarrow (c)$. It is easy to show $(c) \Rightarrow (b) \Rightarrow (a)$. 

We will show the convergence of $\sum_{j,k}(D_{1_k}D_{1_k}a_j(q_1))a_j(q_2)a_k(2)$ when $2 \in \mathcal{Q}$. We have

$$
\Phi_1 := \left| \sum_{j,k}(D_{1_k}D_{1_k}a_j(q_1))a_j(q_2)a_k(2) - \int_{[0,1]^2} \mathcal{O}(t, s, q_1)a(s, q_2)a(t, 2)dsdt \right|
$$

$$
\leq \Phi_{1,1} + \Phi_{1,2} + \Phi_{1,3}
$$

for

$$
\Phi_{1,1} = \sum_{j,k} \int_{[0,1]^2} 1_{I_k \times I_j}(t, s) \left\{ n^2D_{1_k}D_{1_k}a_j(q_1) - \mathcal{O}(t, s, q_1) \right\} a_j(q_2)a_k(2)dsdt,
$$

$$
\Phi_{1,2} = \sum_{j,k} \int_{[0,1]^2} 1_{I_k \times I_j}(t, s) \mathcal{O}(t, s, q_1) \{ a_j(q_2) - a(s, q_2) \} a_k(2)dsdt,
$$

$$
\Phi_{1,3} = \sum_{j,k} \int_{[0,1]^2} 1_{I_k \times I_j}(t, s) \mathcal{O}(t, s, q_1)a(s, q_2) \{ a_k(2) - a(t, 2) \} dsdt.
$$

Hölder’s inequality gives

$$
\Phi_{1,1} \leq \left[ \sum_{j,k} \int_{[0,1]^2} 1_{I_k \times I_j}(t, s) \left| n^2D_{1_k}D_{1_k}a_j(q_1) - \mathcal{O}(t, s, q_1) \right|^r dsdt \right]^{1/r}
$$

$$
\times \left[ \sum_{j,k} \int_{[0,1]^2} 1_{I_k \times I_j}(t, s) |a_j(q_2)a_k(2)|^{r'} dsdt \right]^{1/r'}
$$

where $r' = r/(r - 1)$ for any $r > 1$.

Then, by (8.55) and (2.2), we see $\Phi_{1,1} \to^p 0$ as $n \to \infty$. Similarly, the convergences $\Phi_{1,i} \to^p 0$ ($i = 2, 3$) follow from (8.52), (2.2) and (8.57). Therefore, $\Phi_1 \to^p 0$ as $n \to \infty$. Condition (2.2) ensures boundedness of the variables $\{ \sum_{j,k}(D_{1_k}D_{1_k}a_j(q_1))a_j(q_2)a_k(2) \}_{n \in \mathbb{N}}$ in $L^\infty$, and so we obtain

$$
\sum_{j,k}(D_{1_k}D_{1_k}a_j(q_1))a_j(q_2)a_k(2) \to \int_{[0,1]^2} \mathcal{O}(t, s, q_1)a(s, q_2)a(t, 2)dsdt \quad \text{in } L^\infty \quad (8.58)
$$

as $n \to \infty$; the limit is in $L^\infty$ since $E[\int_{[0,1]} |a(s, q_2)|^r ds] < \infty$ for every $r > 1$, which follows from Fatou’s lemma and the assumed convergence in measure. Similarly we obtain

$$
\sum_{j,k}(D_{1_k}a_j(q_1))(D_{1_k}a_j(q_2))a_k(2) \to \int_{[0,1]^2} \mathcal{O}(t, s, q_1) \mathcal{O}(t, s, q_2)a(t, 2)dsdt \quad \text{in } L^\infty \quad (8.59)
$$
from (8.52) and (8.53), and also
\[\sum_{j,k} (D_{1k} a_j(q_1)) a_j(q_2) D_{1k} a_k(2) \to \int_{[0,1]^2} \dot{a}(t, s, q_1) a(s, q_2) \dot{a}(t, 2) ds dt \quad \text{in } L^\infty \tag{8.60}\]
as \(n \to \infty\) from (8.52), (8.53) and (8.54). Since
\[D_{1k} \{(D_{1k} a_j(q_1)) a_j(q_2) a_k(2)\} = (D_{1k} D_{1k} a_j(q_1)) a_j(q_2) a_k(2) + (D_{1k} a_j(q_1))(D_{1k} a_j(q_2)) a_k(2) + (D_{1k} a_j(q_1)) a_j(q_2) D_{1k} a_k(2),\]
it follows from (8.58), (8.59), (8.60) and (2.11) that
\[\sum_{j,k=1}^n D_{1k} \{(D_{1k} a_j(q_1)) a_j(q_2) a_k(2)\} \to \int_{[0,1]^2} a^{(3,0)}(t, s, q_1, q_2) ds dt \quad \text{in } L^\infty \tag{8.61}\]
as \(n \to \infty\).
Similarly, it is possible to prove
\[n^{-1} \sum_j a_j(q_1) a_j(q_2) a_j(q_3) \to \int_0^1 a(q_1) a(q_2) a(q_3) ds \quad \text{in } L^\infty. \tag{8.62}\]
Since the mapping \(t \to D_t G_\infty(\omega)\) is in \(L^2([0,1], dt)\) for almost all \(\omega\), with approximation in \(L^2([0,1], dt)\) by a continuous function if necessary, we see
\[\int_{[0,1]} \left| \sum_{j=1}^n 1_{I_j}(t) nD_{1j} G_\infty - D_t G_\infty \right| dt \to 0 \quad \text{a.s.} \tag{8.63}\]
as \(n \to \infty\); in fact, for a continuous process \(g = (g(t))_{t \in [0,1]}\), one has the estimate
\[\int_{[0,1]} \left| \sum_j 1_{I_j}(t) (nD_{1j} G_\infty - g(t)) \right| dt \leq \int_{[0,1]} \left| D_s G_\infty - g(s) \right| ds + \sup_{s, t \in [0,1], |s-t| \leq 1/n} |g(s) - g(t)|.\]
From (8.63) in particular,
\[E \left[ \int_0^1 \left\{ \left| \sum_{j=1}^n 1_{I_j}(t) nD_{1j} G_\infty - D_t G_\infty \right| > \epsilon \right\} dt \right] \to 0 \tag{8.64}\]
as \(n \to \infty\) for every \(\epsilon > 0\). Therefore, in the present situation, we get
\[\sum_{j=1}^n \int_0^1 1_{I_j}(t) |nD_{1j} G_\infty - D_t G_\infty|^r dt \to^p 0 \tag{8.65}\]
as \(n \to \infty\) for any \(r > 0\). In the same way,
\[E \left[ \int_0^1 \left\{ \left| \sum_{j=1}^n 1_{I_j}(t) nD_{1j} X_\infty - D_t X_\infty \right| > \epsilon \right\} dt \right] \to 0 \tag{8.66}\]
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as \( n \to \infty \) for every \( \epsilon > 0 \), and in particular,
\[
\sum_{j=1}^{n} \int_{0}^{1} 1_{I_j}(t) |n D_{1_j} X_\infty - D_t X_\infty|^r dt \to^p 0 \tag{8.67}
\]
as \( n \to \infty \) for any \( r > 0 \). Thus, we obtain
\[
\sum_{j,k=1}^{n} \left[ \left( 2^{-1} D_{1_k} G_\infty(iz)^5 + D_{1_k} X_\infty(iz)^3(ix) \right) (D_{1_k} a_j(q_1))a_j(q_1)a_k(2) \right]
\to \int_{0}^{1} \left[ \left( 2^{-1} D_t G_\infty(iz)^5 + D_t X_\infty(iz)^3(ix) \right) \int_{0}^{1} \tilde{a}(t, s, q_1)a(s, q_2)ds \right] a(t, 2)dt. \text{ in } L^\infty \tag{8.68}
\]
as \( n \to \infty \). [Though likely \( D_t G_\infty = \int_{s=0}^{1} \sum_{q \in Q} q! \tilde{a}(t, s, q)a(s, q)ds \), we do not assume it.]

Therefore, from (8.50), (8.51), (2.12), (8.61), (8.62) and (8.68), we obtain
\[
E[\Psi(z, x)S^{(3,0)}_n(iz)] \to E[\Psi(z, x)S^{(3,0)}(iz, ix)] \tag{8.69}
\]
as \( n \to \infty \), where \( S^{(3,0)}(iz, ix) \) is defined by (2.12).

8.2.7 Example

Let \( Q = \{2\} \), \( X_\infty = 0 \) and let \( a \in C^\infty(\mathbb{R}) \) such that any order of derivative of \( a \) is at most polynomial growth. Then \( q_1 = q_2 = q_3 = 2 \) and \( \bar{q}(3) = 6 \) for \( q_1, q_2, q_3 \in Q \). Let \( a_t = a(B_t) \) and \( a_j(2) = a(w_{t_{j-1}}) \) for \( j \in \{1, \ldots, n\} \). This is the case of a quadratic form of the increments of a Brownian motion with predictable weights
\[
M_n = n^{1/2} \sum_{j=1}^{n} a(w_{t_{j-1}})I_2(1^\otimes 2).
\]

We remark that the second term of the expression (1.7) vanishes in this example because of the strong predictability of \( a_j(q) \), while such a predictability condition is not assumed in the general context in this paper.

Let
\[
G_\infty = \int_{0}^{1} 2a_t^2 dt.
\]

Then
\[
\Psi(z, x) = \exp \left( - \int_{0}^{1} a_t^2 dt \right) = \exp \left( \int_{0}^{1} a_t^2 dt(iz)^2 \right).
\]
In this special case,

\[
E[\Psi(z, x)\mathcal{S}_n^{(3,0)}(iz)] = \frac{1}{3} E[\Psi(z, x)q\text{Tor}(iz)^3]
\]

\[
= \frac{4}{3} n^{-1} \sum_{j=1}^{n} E[\Psi(z, x)a_j(2)a_j(2)a_j(2)](iz)^3
\]

\[
+ 2 \sum_{j,k=1}^{n} E[\Psi(z, x)D_{1k}\{(D_{1k}a_j(2))a_j(2)a_k(2)\}](iz)^3
\]

\[
+ \sum_{j,k=1}^{n} E\left[\Psi(z, x)(D_{1k}G_\infty(iz)^5)(D_{1k}a_j(2))a_j(2)a_k(2)\right]
\]

\[
+ O(n^{-0.5}) \quad (8.70)
\]

We can use an approximate projection

\[
\tilde{\mathcal{S}}^{(3,0)}_n(iz) = \frac{4}{3} n^{-1} \sum_{j=1}^{n} a_j(2)a_j(2)a_j(2)(iz)^3
\]

\[
+ 2 \sum_{j,k=1}^{n} D_{1k}\{(D_{1k}a_j(2))a_j(2)a_k(2)\}(iz)^3
\]

\[
+ \sum_{j,k=1}^{n} \left[(D_{1k}G_\infty(iz)^5)(D_{1k}a_j(2))a_j(2)a_k(2)\right]
\]

Then it is not difficult to show from (8.70) that

\[
E[\Psi(z, x)\mathcal{S}_n^{(3,0)}(iz)] = E[\Psi(z, x)\tilde{\mathcal{S}}^{(3,0)}_n(iz)] + O(n^{-0.5}) \rightarrow E[\Psi(z, x)\sigma(iz)] \quad (8.71)
\]

where

\[
\sigma(iz) = \frac{4}{3} \int_0^1 a_s^3 dt (iz)^3 + 2 \int_0^1 \int_t^1 (aa')_s a_t ds dt (iz)^3 + 4 \int_0^1 \left(\int_t^1 a_s a'_s ds\right)^2 a_t dt (iz)^5.
\]

Here \(a'_s = a'(w_s)\) and \((aa')'_s = (aa')'(w_s)\).

The random symbol \(\sigma(iz)\) coincides with the full random symbol \(\sigma\) (without \(ix\)) of Yoshida ([84], p.918) and also with \(\mathcal{S}^{3,0}\) of Nualart and Yoshida ([48], p.37). In this example, it is known that the full random symbol, which is the sum of the adaptive random symbol (tangent) and the anticipative random symbol (torsion), in the martingale expansion corresponds to only the quasi-torsion in the theory of asymptotic expansion for Skorohod integrals. In other words, the quasi-tangent disappears in this example; see the above papers or Section 8.3.

### 8.3 The quasi-tangent

In this section, we will consider the quasi-tangent. For the meantime, we work without Condition (2.10) to clarify the necessity of the assumption. Recall the decomposition (8.1) on p.42.
of $D_{n(q_2)}M_n(q_1)$. By the product formula (7.6) on p.34,

\[ \mathbb{I}_1 = n^{2-1(q_1+q_2)-1} \sum_{j,k} q_1 a_j(q_1) a_k(q_2) \langle 1_j, 1_k \rangle I_{q_1-1}(1_j^{\ominus(q_1-1)}) I_{q_2-1}(1_k^{\ominus(q_2-1)}) \]

\[ = J_{1,1} + J_{1,2}, \quad (8.72) \]

where

\[ J_{1,1} = n^{-1} \sum_j 1_{\{q_1=q_2\}} q_1! a_j(q_1)^2 \quad (8.73) \]

and

\[ J_{1,2} = \sum_{\nu:q(2)-2-2\nu>0} c_\nu(q_1-1, q_2-1)n^{0.5(q_1+q_2)-2-\nu} \sum_j q_1 a_j(q_1) a_j(q_2) I_{q(2)-2-2\nu}(1_j^{\ominus(q(2)-2-2\nu)}). \quad (8.74) \]

Since

\[ e(\mathbb{I}_1 - J_{1,1}) = e(J_{1,2}) = -0.5, \quad (8.75) \]

the effect of $J_{1,2}$ to the random symbol can remain. After IBP, the exponent of $n$ for $E[\Psi(z, x) J_{1,2}]$ becomes

\[ \max_{\nu:q(2)-2-2\nu>0, \nu \leq \min(q_1-1,q_2-1)} \left\{ (0.5q(2) - 2 - \nu) + 1\left( \sum_j - (q(2) - 2 - 2\nu) \right) \right\} \]

\[ = \max_{\nu:q(2)-2-2\nu>0, \nu \leq \min(q_1-1,q_2-1)} \left\{ -0.5q(2) + 1 + \nu \right\} \]

\[ = -0.5|q_1 - q_2| - 1_{\{q_1=q_2\}} \]

The term is negligible when $|q_1 - q_2| > 1$ or $q_1 = q_2$. More precisely, remaining is Case $[|q_1 - q_2| = 1]$ giving the terms of order $n^{-1/2}$. Thus,

\[ E[\Psi(z, x) \mathbb{I}_1] = 1_{\{q_1=q_2\}} q_1! n^{-1} \sum_j E[\Psi(z) a_j(q_1)^2] \]

\[ + n^{-0.5} 1_{\{q_1-q_2=1\}} \Xi_n(q_1, q_2) + O(n^{-1}) \]

with

\[ \Xi_n(q_1, q_2) = \sum_j E[D_{1_j} \left\{ \Psi(z) \omega_1(q_1, q_2) a_j(q_1) a_j(q_2) \right\}] \]

In particular, Condition (2.10) simplifies the formula as

\[ E[\Psi(z, x) \mathbb{I}_1] = 1_{\{q_1=q_2\}} q_1! n^{-1} \sum_j E[\Psi(z) a_j(q_1)^2] + O(n^{-1}). \]
Though the simple estimate of the exponent of $\|_2$ becomes
\[
\varepsilon(\|_2) = -0.5, \quad (8.76)
\]
this is not enough to show $\|_2$ has no effect to the random symbol. The sum $\|_2$ is expressed by
\[
\|_2 = \|_{2,1} + \|_{2,2}
\]
where
\[
\|_{2,1} = n^{2^{-1}(q_1 + q_2) - 2} \sum_{j,k} (nD_{1}a_j(q_1))a_k(q_2)I_{q_2}^{-1}(1_j^{\otimes q_1} \odot k^{\otimes q_2})
\]
and
\[
\|_{2,2} = \sum_{\nu > 0} c_{\nu}(q_1, q_2 - 1) \\
\times n^{2^{-1}(q_1 + q_2) - 2} \sum_{j,k} (nD_{1}a_j(q_1))a_k(q_2)I_{q_2}^{-1} - 2\nu (1_j^{\otimes q_1 - \nu} \odot k^{\otimes (q_2 - 1 - \nu)}) (1_j^{\otimes \nu}, k^{\otimes \nu})_{\nu \nu}.
\]
We have an elementary estimate
\[
\sup_{n \in \mathbb{N}} \sup\sup_{j,k \in J_n} \| nD_{1}a_j(q_1) \|_p = \sup_{n \in \mathbb{N}} \sup\sup_{j,k \in J_n} \left\| n \int_0^1 D_{t}a_j(q_1)1_k(t)dt \right\|_p \\
\leq \sup_{n \in \mathbb{N}} \sup\sup_{j,k \in J_n} n \int_0^1 \| D_{t}a_j(q_1) \|_p 1_k(t)dt \\
\leq \sup_{n \in \mathbb{N}} \sup\sup_{j,k \in J_n} \sup_{t \in [0,1]} \| D_{t}a_j(q_1) \|_p < \infty \quad (8.77)
\]
for every $p > 1$ under (2.2) of [A]. Integration-by-parts gives
\[
E[\Psi(z, x)\|_{2,1}] = O(n^{0.5q(2) - 2 + (q(2) - 1)}) = O(n^{-0.5q(2) + 1}) = O(n^{-1}).
\]
For $\|_{2,2}$, we obtain
\[
\varepsilon(\|_{2,2}) = -1.
\]
Indeed,
\[
n^{2^{-1}(q_1 + q_2) - 2} \sum_{j,k} (nD_{1}a_j(q_1))a_k(q_2)I_{q_2}^{-1} - 2\nu (1_j^{\otimes q_1 - \nu} \odot k^{\otimes (q_2 - 1 - \nu)}) (1_j^{\otimes \nu}, k^{\otimes \nu})_{\nu \nu} \\
= n^{2^{-1}(q_1 + q_2) - 2 - \nu} \sum_{j} (nD_{1}a_j(q_1))a_j(q_2)I_{q_2}^{-1} - 2\nu (1_j^{\otimes (q(2) - 1 - 2\nu)})
\]
for $\nu > 0$, and hence
\[
\varepsilon(\|_{2,2}) = 0.5q(2) - 2 - \nu - 0.5(q(2) - 1 - 2\nu) + 1 - 0.5m(\bar{q}(2) - 1 - 2\nu) \\
= -0.5 - 0.5m(\bar{q}(2) - 1 - 2\nu) \\
\leq -1
\]
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since Condition (2.10) bans \( q_1 = q_2 - 1 \), which entails \( \bar{q}(2) - 1 - 2\nu > 0 \); if \( q_1 \neq q_2 - 1 \), then \( (\bar{q}(2) - 1)/2 > \min\{q_1, q_2 - 1\} \geq \nu \). Therefore, \( \|k_{2,2} \) is negligible in \( L^p \), and \( \|k_{2,1} \) is negligible in the random symbol. Therefore, \( \|k_2 \) is negligible in the random symbol.

For \( \|k_3 \)

\[
\|k_3 \| = -n^{2-1(q_1+q_2)-2} \sum_j (q_1 - 1)(D_{1_j}a_j(q_1))a_k(q_2)I_{q_1-2}(1^\otimes(q_1-2))I_{q_2-1}(1^\otimes(q_2-1))
\]

\[
= -n^{2-1(q_1+q_2)-2-\nu} \sum_j (q_1 - 1)(D_{1_j}a_j(q_1))a_k(q_2)\sum_\nu c_\nu(q_1 - 2, q_2 - 1)I_{\bar{q}(2)-3-2\nu}(1^\otimes(q_2-3-2\nu))
\]

The exponent of \( \|k_3 \) is

\[
e(\|k_3 \|) = -1
\] (8.78)

since \( \bar{q}(2) - 3 - 2\nu \neq 0 \) by \( \{(q_1 - 2) + (q_2 - 1)\}/2 > \min\{q_1 - 2, q_2 - 1\} \geq \nu \) due to Condition (2.10). The sum \( \|k_3 \) is negligible.

As for the sum

\[
\|k_4 \| = -n^{2-1(q_1+q_2)-3} \sum_{j,k} (n^2D_{1_k}D_{1_j}a_j(q_1))a_k(q_2)I_{q_1-3}(1^\otimes(q_1-3))I_{q_2-1}(1^\otimes(q_2-1)),
\]

the exponent is

\[
e(\|k_4 \|) = (0.5\bar{q}(2) - 3) - 0.5(\bar{q}(2) - 2) + 2 - 0.5 \times 2 = -1
\] (8.79)

The sum \( \|k_4 \) is also negligible.

To investigate \( q\Tan \), we need the approximation

\[
\sum_{q_1,q_2 \in \mathcal{Q}} E[\Psi(z,x)D_{u_n}M_n] = \sum_{q \in \mathcal{Q}} q!n^{-1} \sum_j E[\Psi(z,x)a_j(q)^2]
\]

\[
+ \sum_{q_1,q_2 \in \mathcal{Q}} n^{-0.5}1_{||q_1-q_2||=1} \Xi_n(q_1,q_2) + O(n^{-1})
\]

\[
= \sum_{q \in \mathcal{Q}} q!n^{-1} \sum_j E[\Psi(z,x)a_j(q)^2] + O(n^{-1})
\]

under Condition (2.10). In other words, for

\[
\Theta_{0,n}^{(2,0)}(iz) = 1/2 q\Tan(iz)^2
\]

\[
= 1/2 n^{1/2}(D_{u_n}M_n - G_\infty)(iz)^2,
\]

we have

\[
E[\Psi(z,x)\Theta_{0,n}^{(2,0)}(iz)] = 1/2 n^{1/2} E\left[\Psi(z,x) \left(n^{-1} \sum_{j} q_ja_j(q_j)^2 - G_\infty\right)\right](iz)^2
\]

\[
+ O(n^{-1/2})
\] (8.80)
under (2.10). In particular, it follows from (8.80) that
\[ E[\Psi(z, x)\mathcal{G}_{0,n}^{(2,0)}(iz)] = o(1), \quad (8.81) \]
therefore,
\[ \mathcal{G}_{0}^{(2,0)}(iz) = 0 \quad (8.82) \]
under (2.10), if (2.8) is satisfied.

**Remark 8.2.** If Condition (2.10) was not assumed, then
\[
E[\Psi(z, x)\mathcal{G}_{0,n}^{(2,0)}(iz)] = \frac{1}{2} n^{1/2} E \left[ \Psi(z, x) \left( n^{-1} \sum_{q \in \mathcal{Q}} q! a_j(q)^2 - G_{\infty} \right) \right] (iz)^2
\]
\[ + \frac{1}{2} n^{-0.15} \sum_{q_1, q_2 \in \mathcal{Q}} 1_{(|q_1 - q_2| = 1)} \Xi_n(q_1, q_2)
\]
\[ + O(n^{-1/2}). \]
This suggests the random symbol \( \mathcal{G}_{0,n}^{(2,0)}(iz) \) does not vanish though it vanished in all the examples in Nualart and Yoshida [48]; as a matter of fact, those examples satisfied (2.10).

### 8.4 Other random symbols

#### 8.4.1 \( \mathcal{G}_{n}^{(0,1)} \) and \( \mathcal{G}_{1,n}^{(1,1)} \)

In the setting of Theorem 2.2,
\[ \mathcal{G}_{n}^{(0,1)} = 0 \quad \text{and} \quad \mathcal{G}_{1,n}^{(1,1)} = 0 \]
since \( \mathcal{G}_{n}^{(0,1)} = 0 \) and \( \mathcal{G}_{1,n}^{(1,1)} = 0 \).

#### 8.4.2 \( \mathcal{G}_{n}^{(1,1)}(iz, ix) \)

By definition
\[
\mathcal{G}_{n}^{(1,1)}(iz, ix) = n^{1/2} D_{u_n} X_{\infty}(iz)(ix)
\]
\[ = \sum_{q \in \mathcal{Q}} \sum_{j=1}^{n} n^{0.5q-1} (nD_{1_j} X_{\infty}) a_j(q) I_{q-1}(1_{j}^{(q-1)})(iz)(ix) \]

Then
\[
E[\Psi(z, x)\mathcal{G}_{n}^{(1,1)}(iz, ix)] = \sum_{q \in \mathcal{Q}} n^{0.5q} \sum_{j} E \left[ \Psi(z, x) (D_{1_j} X_{\infty}) a_j(q) I_{q-1}(1_{j}^{(q-1)}) \right] (iz)(ix)
\]
\[ = \sum_{q \in \mathcal{Q}} n^{0.5q} \sum_{j} E \left[ D_{1_{j}^{q-1}} \Psi(z, x) (D_{1_j} X_{\infty}) a_j(q) \right] (iz)(ix). \]
Then the exponent of \( n \) for this expectation for \( q \in Q \) is
\[
0.5q + 1(\sum_j) - (q - 1)(D_{1j}^{q-1}) - 1(D_{1j}) = -0.5q + 1.
\]

Since \( q \geq 2 \), the effect of this term to the asymptotic expansion exists only when \( q = 2 \). Therefore,
\[
E[\Psi(z,x)\mathcal{G}_n^{(1,1)}(iz,ix)] = n \sum_j E\left[D_{1j} \{ \Psi(z,x)(D_{1j}X_\infty) a_j(2) \} \right](iz)(ix) + O(n^{-0.5})
\]
\[
= n \sum_j E\left[\Psi(z,x)(D_{1j}D_{1j}X_\infty) a_j(2) \right](iz)(ix) + O(n^{-0.5})
\]
\[
+ n \sum_j E\left[\Psi(z,x)(D_{1j}X_\infty) D_{1j} a_j(2) \right](iz)(ix) + O(n^{-0.5})
\]
\[
+ n \sum_j E\left[\Psi(z,x)\{2^{-1}D_{1j}G_\infty(iz)^2 + D_{1j}X_\infty ix \} (D_{1j}X_\infty) a_j(2) \right] (iz)(ix)
\]+ O(n^{-0.5}) \tag{8.83}
\]

Like the discussion around (8.67), for any \( r \geq 1 \), by path-wise in-\( L^r([0,1],dt) \)-approximation to the function \( t \mapsto D_tG_\infty \) by a continuous function, we see
\[
E\left[\int_0^1 \sum_j 1_j(s) |nD_{1j}G_\infty - D_nG_\infty|^r ds \right] \to 0.
\]
Similarly,
\[
E\left[\int_0^1 \sum_j 1_j(s) |nD_{1j}X_\infty - D_nX_\infty|^r ds \right] \to 0.
\]
Moreover,
\[
E\left[\int_{[0,1]} \sum_j 1_j(t) |n^2D_{1j}D_{1j}X_\infty - \bar{X}(t)|^r dt \right] \to 0
\]
as \( n \to \infty \). Therefore, from (8.83),
\[
E[\Psi(z,x)\mathcal{G}_n^{(1,1)}(iz,ix)] \to E[\Psi(z,x)\mathcal{G}^{(1,1)}(iz,ix)]
\]
as \( n \to \infty \), where \( \mathcal{G}^{(1,1)}(iz,ix) \) is defined by (2.13) on p.8.

8.4.3 \( \mathcal{G}_n^{(1,0)}(iz,ix) \)

In Theorem 2.2, \( N_n \) is given by (1.9), and
\[
\mathcal{G}_n^{(1,0)}(iz,ix) = \mathcal{G}_n^{(1,0)}(iz) \equiv N_niz
\]
\[
= \sum_{q \in Q} \sum_{j=1}^n n^{q/2} (D_{1j}a_j(q)) I_{q-1}(1_j^{\otimes(q-1)})iz.
\]
Then
\[
E[\Psi(z, x) \mathcal{G}_n^{(1,0)}(iz)] = \sum_{q \in Q} n^{q/2} \sum_{j=1}^{n} E \left[ \Psi(z, x)(D_1, a_j(q))I_{q-1}(1_j^{(q-1)}) \right] iz
\]
\[
= \sum_{q \in Q} n^{q/2} \sum_{j=1}^{n} E \left[ D_1^{(q-1)}(1_j) \{ \Psi(z, x)D_1, a_j(q) \} \right] iz
\]
\[
= 1_{\{2 \in Q\}} n \sum_{j=1}^{n} E \left[ D_1, \{ \Psi(z, x)D_1, a_j(2) \} \right] iz + o(1)
\]
\[
= 1_{\{2 \in Q\}} E \left[ \Psi(z, x) n \sum_{j=1}^{n} \{ 2^{-1}D_1, G_{\infty}(iz)^2 + D_1, X_{\infty}(ix) \} D_1, a_j(2) \right] iz
\]
\[
+ 1_{\{2 \in Q\}} E \left[ \Psi(z, x) n \sum_{j=1}^{n} D_1, D_1, a_j(2) \right] iz + o(1)
\]
Therefore,
\[
E[\Psi(z, x) \mathcal{G}_n^{(1,0)}(iz)] \to E[\Psi(z, x) \mathcal{G}^{(1,0)}(iz)]
\]
as \(n \to \infty\), where \(\mathcal{G}^{(1,0)}(iz)\) is defined by (2.14).

8.4.4 \(\mathcal{G}^{(2,0)}_{1,n}(iz, ix)\)

In Theorem 2.2, \(N_n\) is given by (1.9), and the exponent of \(N_n\) is \(e(N_n) = 0\). Moreover, \(q - 1 \not\in Q\) under Condition (2.10), which gives
\[
e(D_u, N_n) \leq -0.5
\]
by Proposition 7.1. In particular,
\[
E[\Psi(z, x) \mathcal{G}_{1,n}^{(2,0)}] = o(1)
\]
as \(n \to \infty\). Therefore,
\[
\mathcal{G}_{1}^{(2,0)} = 0.
\]

8.5 Condition [B]

We will verify Condition [B] for \(\psi_n = 1\), under Condition [A].

8.5.1 [B] (i)

Condition [B] (i) is verified with (2.2) of [A] for \(\psi_n = 1\) and any pair \((p, p_1)\) of indices satisfying \(5p^{-1} + p_1^{-1} \leq 1\).
8.5.2 Condition (3.4) of [B] (ii)

For later use, we show a stronger estimate.

**Lemma 8.3.** Suppose that (2.2) of [A] is satisfied. Then

\[ \| D^i u_n \|_p = O(1) \]

as \( n \to \infty \) for every \( i \in \mathbb{Z}_+ \) and \( p > 1 \).

**Proof.** For \( q \in \mathbb{Q} \) and \( i \in \mathbb{Z}_+ \),

\[ D^i u_n(q) = D^i n^{2-1(q-1)} \sum_{j=1}^n a_j(q) (D_{q-1}(1_j^{\otimes(q-1)})1_j \]

\[ = \sum_{\alpha=0}^i C_{i,\alpha} \widetilde{\Pi}(i, \alpha, q)_n \]

where \( \widetilde{\Pi}(i, \alpha, q)_n \) is the symmetrization of

\[ \Pi(i, \alpha, q)_n = n^{2-1(q-1)} \sum_{j=1}^n I_{q-1-\alpha}(1_j^{\otimes(q-1-\alpha)})(D_{q-1}(1_j^{\otimes(q-1-\alpha)})1_j^{\otimes(q-1-\alpha)}) \]

where \( C_{i,\alpha} = (q-1)!/(q-1-\alpha)!1_{\{q-1-\alpha\geq0\}} \). Therefore,

\[ \|\Pi(i, \alpha, q)_n\|_{\mathcal{B}^{\otimes(\alpha+1)}}^2 = n^{q-\alpha-2} \sum_{j=1}^n \|D_{q-1-\alpha}(1_j^{\otimes(q-1-\alpha)})1_j^{\otimes(q-1-\alpha)}\|^2 \]

Since the hypercontractivity yields

\[ \sup_{j,n} \| I_{q-1-\alpha}(1_j^{\otimes(q-1-\alpha)}) \|_p = O(n^{-(q-1-\alpha)}) \]

for every \( p > 1 \), we obtain

\[ \sup_n \|\Pi(i, \alpha, q)_n\|_{\mathcal{B}^{\otimes(\alpha+1)}} < \infty \]

for every \( p > 1 \), which concludes the proof. [This is a direct proof. On the other hand, we could switch it on the way by using Proposition 7.2.]

8.5.3 Condition (3.5) of [B] (ii)

**Lemma 8.4.** Suppose that (2.2) and (vii) of [A] except for (2.8) hold and that

\[ \left\| n^{-1} \sum_{q \in \mathbb{Q}} \sum_{j=1}^n q! a_j(q)^2 - G_\infty \right\|_p = O(n^{-1/2}) \quad (8.84) \]

as \( n \to \infty \) for every \( p > 1 \); in particular, (2.8) is sufficient for (8.84). Then

\[ \| G_n^{(2)} \|_p = O(n^{-1/2}) \]

as \( n \to \infty \) for every \( p > 1 \).
Proof. From the decomposition (8.1) on p.42, we have
\[
G_n^{(2)} = D_{u_{n\cdot}M_n - G_\infty} = \sum_{q_1,q_2 \in \mathcal{Q}} (\mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 + \mathbb{I}_4) - G_\infty \tag{8.85}
\]
where \(\mathbb{I}_i (i = 1, ..., 4)\) are given by (8.2)-(8.5). The variable \(\mathbb{I}_1\) has a decomposition \(\mathbb{I}_1 = J_{1,1} + J_{1,2}\) as in (8.72) with \(J_{1,1}\) of (8.73) and \(J_{1,2}\) of (8.74). From these representations,
\[
e(J_{1,1}) = 0 \quad \text{and} \quad e(J_{1,2}) = -0.5. \tag{8.86}
\]
Then
\[
\mathbb{I}_1 - 1_{(q_1=q_2)} n^{-1} \sum_j q_1! a_j(q_1)^2 = O_{L_\infty}(n^{-1/2}). \tag{8.87}
\]
Equality (8.76) gives
\[
\mathbb{I}_2 = O_{L_\infty}(n^{-0.5}). \tag{8.88}
\]
In the same way by (8.78),
\[
\mathbb{I}_3 = O_{L_\infty}(n^{-1}). \tag{8.89}
\]
Here we used Condition (2.10).

The sum \(\mathbb{I}_4\) can be estimated from (8.7) as
\[
\mathbb{I}_4 = O_{L_\infty}(n^{-1}). \tag{8.90}
\]
Now we conclude the proof by combining (8.85), (8.87), (8.88), (8.89), (8.90) and (8.84).

**Lemma 8.5.** Suppose that (2.2) and (vii) of \([A]\) are satisfied Then
\[
\|G_n^{(3)}\|_p = O(n^{-1/2}) \tag{8.91}
\]
as \(n \to \infty\) for every \(p > 1\).

**Proof.** Recall
\[
G_n^{(3)} = D_{u_{n\cdot}G_\infty} = \sum_{q \in \mathcal{Q}} D_{u_{n\cdot}(q)} G_\infty. \tag{8.92}
\]
Routinely,
\[
D_{u_{n\cdot}(q)} G_\infty = n^{q-1} (q^{-1}-1) \sum_j (n D_1 G_\infty) a_j(q) I_{q-1}(1_j^{\otimes(q-1)}) \tag{8.93}
\]
for \(q \in \mathcal{Q}\). Then \(e(D_{u_{n\cdot}(q)} G_\infty) = -0.5\), therefore
\[
e(G_n^{(3)}) = -0.5. \tag{8.94}
\]
This gives the result.
8.5.4 Condition (3.6) of \([B]\) (ii)

**Lemma 8.6.** Suppose that (2.2) and (vii) of \([A]\) are satisfied. Then

\[
\|D_{u_n} G_n^{(2)}\|_p = O(n^{-1/2})
\]  \( (8.95) \)

as \( n \to \infty \) for every \( p > 1 \).

**Proof.** We have the decomposition (8.85) of \( G_n^{(2)} \) and (8.72), and the exponents (8.86), (8.88), (8.89) and (8.90). Therefore, \( G_n^{(2)} \in \mathcal{L} \) by expressing

\[
s_{j_1, j_2} \in \mathbb{Q} \quad \sum_{j_1, j_2} J_{1,1} - G_{\infty} = n^{-1} \sum_{j=1}^n (q! a_j(q)^2 - G_{\infty}) I_0(1_{j})
\]

which completes the proof.

8.5.5 Condition (3.7) of \([B]\) (ii)

We use (i) and (vii) of \([A]\). In view of (8.93) and (8.94), since \( q-1 \notin \mathbb{Q} \) under Condition (2.10), Proposition 7.1 concludes that \( D_{u_n} G_n^{(3)} \in \mathcal{L} \) and \( e(D_{u_n} G_n^{(3)}) \leq -1/2 \), which completes the proof.

8.5.6 Condition (3.8) of \([B]\) (ii)

We use (i) and (vii) of \([A]\). From (8.85) and (8.24), we have

\[
D_{u_n} G_n^{(2)} = D_{u_n} D_{u_n} M_n - D_{u_n} G_{\infty} = \sum_{j_1, j_2, j_3 \in \mathbb{Q}} \hat{I}_1 + \hat{I} - G_n^{(3)}
\]  \( (8.96) \)

where

\[
\hat{I} = \sum_{j_1, j_2, j_3 \in \mathbb{Q}} (\hat{I}_1 + D_{u_n(q_3)} \hat{I}_2 + D_{u_n(q_3)} \hat{I}_3 + D_{u_n(q_3)} \hat{I}_4).
\]

We apply Proposition 7.1 with the properties (8.26), (8.78) and (8.8) to obtain

\[
e(\hat{I}_1 + D_{u_n(q_3)} \hat{I}_2 + D_{u_n(q_3)} \hat{I}_3) \leq -1
\]

and inductively

\[
e(D_{u_n}(\hat{I}_1 + D_{u_n(q_3)} \hat{I}_2 + D_{u_n(q_3)} \hat{I}_3)) \leq -1.
\]  \( (8.97) \)
Moreover, since $e(l_2) = -0.5$ from (8.76), Corollary 7.4 applied under Condition (2.10) concludes

$$e(D_{u_n} l_2) \leq -1. \quad (8.98)$$

Therefore,

$$e(D_{u_n} \tilde{l}) \leq -1. \quad (8.99)$$

Since $G_n^{(3)}$ is expressed by (8.92) and (8.93), Proposition 7.1 yields

$$e(D_{u_n} G_n^{(3)}) \leq -1 \quad (8.100)$$

from (8.94) under Condition (2.10).

The sum $l_1$ consists of the three terms, as in (8.25). The component $l_{1,2,1}$ has the expression (8.20) and $e(l_{1,2,1}) = -0.5$ according to (8.21). Then, under Condition (2.10), it follows from Proposition 7.1 that

$$e(D_{u_n} l_{1,2,1}) \leq -1. \quad (8.101)$$

The component $l_{1,1,1}$ has the representation (8.13) (multiplied by $I_0(1_{j^0})$) and the exponent $-0.5$ by (8.14). Since $0 \not\in Q$, Proposition 7.1 tells

$$e(D_{u_n} l_{1,1,1}) \leq -1. \quad (8.102)$$

In the same fashion, we can say

$$e(D_{u_n} l'_{1,1,1}) \leq -1. \quad (8.103)$$

from (8.16) and (8.17). Thus, from (8.96), (8.99), (8.100), (8.25), (8.101), (8.102) and (8.103), we conclude that

$$e(D_{u_n} D_{u_n} G_n^{(2)}) = -1, \quad (8.104)$$

and in particular that

$$D_{u_n} D_{u_n} G_n^{(2)} = O_{L^\infty}(n^{-1}). \quad (8.105)$$

For $G_n^{(3)}$, we obtain

$$e(D_{u_n} D_{u_n} G_n^{(3)}) \leq -1. \quad (8.106)$$

from (8.100), and in particular,

$$D_{u_n} D_{u_n} G_n^{(3)} = O_{L^\infty}(n^{-1}). \quad (8.107)$$

So Condition (3.8) has been verified by (8.105) and (8.107).
8.5.7 Conditions (3.9), (3.10) and (3.11) of \([B]\) (ii)

We use (i) of \([A]\) and the first half of (viii) of \([A]\). In the same way of deriving (8.94), (8.100) and (8.106), we see

\[
e(D_{un(q)}X_{\infty}) = -0.5.
\]  

and

\[
e(D_{un(q)}^iX_{\infty}) = -1 \quad (i = 2, 3)
\]

Properties (3.9), (3.10) and (3.11) follow from (8.108) and (8.109).

8.5.8 Condition (3.12), (3.13) and (3.14) of \([B]\) (ii)

We use (i) of \([A]\). In Theorem 2.2, \(X_n = X_{\infty}\), therefore \(\hat{X}_n = 0\). The variable \(N_n\) is given by

\[
N_n = \sum_{q \in \mathcal{Q}} n^{\eta/2} \sum_{j=1}^n (D_{lj}a_j(q))I_{q-1}(1_{j}^{\otimes(q-1)})
\]  

Then the exponent of \(N_n\) based on the expression (8.110) is

\[
e(N_n) = 0
\]

by definition of the exponent. Moreover, Proposition 7.1 ensures

\[
e(D_{un}N_n) \leq -0.5
\]

under Condition (2.10). Inductively from (8.112),

\[
e(D_{un}^iN_n) \leq -0.5 \quad (i = 2, 3)
\]

Now (8.111) and (8.113) verify Properties (3.12), (3.13) and (3.14).

8.5.9 Condition (3.15) of \([B]\) (ii)

Condition (3.15) holds obviously because \(\psi_n = 1\) now.

8.5.10 Condition \([B]\) (iii) and the full random symbol \(\mathcal{S}(iz, ix)\)

We have already verified Condition \([B]\) (iii) in Sections 8.2, 8.3 and 8.4. Now Theorem 3.1 applied to the present case completes the proof of Theorem 2.2.
8.6 An example that satisfies (2.18)

Suppose that $A \in C_c^\infty(\mathbb{R})$, the space of smooth functions on $\mathbb{R}$ with derivatives of at most polynomial growth. Fix a $q \in \mathbb{Q}$. For a Wiener process $w = (w_t)_{t \in [0,1]}$, let $a_j(q)^2 = A(w_{1-t_j})$. Let $a(t,q)^2 = A(w_{1-t})$. Then

\[ U_n := n^{-1} \sum_{j=1}^n a_j(q)^2 - \int_0^1 a(t,q)^2 dt \]

\[ = n^{-1} \sum_{j=1}^n A(w_{1-t_j}) - \int_0^1 A(w_{1-t}) dt \]

\[ = n^{-1} \sum_{j=1}^n \int_{I_j} \{ A(w_{1-t_j}) - A(w_{1-t}) \} dt \]

Let

\[ f_{n-j+1}(s) = f_{n-j+1,n}(s) = -n \int_{I_j} 1_{[1-t_j,1-t]}(s) dt \]

for $j \in \{1, ..., n\}$. Obviously, $\sup_{n \in \mathbb{N}} \sup_{j=1, ..., n} \sup_{s \in [0,1]} |f_j(s)| \leq 1$ and we have

\[ \int_{I_j} (w_{1-t_j} - w_{1-\cdot}) dt = \int_{I_j} I_1(1_{[0,1-t_j]} - 1_{[0,1-t]}) dt = - \int_{I_j} I_1(1_{[1-t_j,1-t]}) dt \]

\[ = -I_1 \left( \int_{I_j} 1_{[1-t_j,1-t]} dt \right) = n^{-1} I_1(f_{n-j+1}). \]

With

\[ R_n = \sum_j \int_{I_j} \int_0^1 (1 - s) A''(w_{1-t} + s(w_{1-t_j} - w_{1-\cdot})) ds (w_{1-t_j} - w_{1-\cdot})^2 dt, \]

we write

\[ U_n = \sum_j A'(w_{1-t_j}) \int_{I_j} (w_{1-t_j} - w_{1-\cdot}) dt + R_n \]

\[ = \mathbb{L}_n + R_n \]

where

\[ \mathbb{L}_n = n^{-1} \sum_{j=1}^n A'(w_{1-t_j}) I_1(f_{n-j+1}) = n^{-1} \sum_{j=1}^n A'(w_{t_j-1}) I_1(f_j). \]

Then it is not difficult to show

\[ R_n = O_{D^\infty}(n^{-1}) \]

as $n \to \infty$. Since the exponent

\[ e(\mathbb{L}_n) = -1 - 0.5 \times 1 + 1 - 0.5 \times 1 = -1, \]

we obtain

\[ \mathbb{L}_n = O_{D^\infty}(n^{-1}). \]

Consequently, $U_n = O_{D^\infty}(n^{-1})$ as $n \to \infty$. This means (2.18) is satisfied in the present example.

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9 Proof of Theorem 2.4

We will verify Condition \([C]\) to apply Theorem 3.2. Condition \([A^p]\) is assumed now.

9.1 \([C] \text{ (i) and (iv) (a)}\)

\(u_n, N_n \in \mathbb{D}_\infty\) under \([A] \text{ (i)}, G_\infty \in \mathbb{D}_\infty\) under \([A] \text{ (vii)}, \) and \(X_\infty = X_n \in \mathbb{D}_\infty\) under \([A] \text{ (viii)}.\)

Condition \([C] \text{ (iv) (a)}\) is ensured by \([A^\#]\) (II).

9.2 \([C] \text{ (ii)}\)

Since \([A^\#]\) is stronger than \([A]\), we can use all results proved in Section 8. Lemma 8.3 verifies the estimate (3.18).

According to (8.85), (8.72), (8.73) and (8.74), \(G_n^{(2)}\) admits the following decomposition:

\[ G_n^{(2)} = \tilde{J}_1 + \tilde{J}_2 \]

where

\[ \tilde{J}_1 = \sum_{q_1,q_2 \in \mathcal{Q}} (J_{1,1} - G_\infty) = \sum_{q \in \mathcal{Q}} \left\{ \sum_j n^{-1} q! a_j(q)^2 - q! \int_0^1 a(t,q)^2 dt \right\} \]

and

\[ \tilde{J}_2 = \sum_{q_1,q_2 \in \mathcal{Q}} (J_{1,2} + I_2 + I_3 + I_4). \]

We already know

\[ e(\tilde{J}_2) = -0.5 \]

from (8.86) on p.65, (8.76) on p.59, (8.78) on p.60 and (8.7) on p.43, therefore,

\[ \tilde{J}_2 = O_{\mathbb{D}_\infty}(n^{-0.5}) \]

(9.1)

Condition (2.18) of \([A^p]\) (vii) on p.10 assumes

\[ \tilde{J}_2 = O_{\mathbb{D}_\infty}(n^{-0.5-\kappa}). \]

(9.2)

Therefore, Proposition 7.2 implies

\[ G_n^{(2)} = O_{\mathbb{D}_\infty}(n^{-0.5}), \]

and hence Condition (3.19) is verified. On the other hand, Condition (3.22) follows from the estimate (8.104).

It follows from (8.94) that \(G_n^{(3)} = O_{\mathbb{D}_\infty}(n^{-1/2}),\) therefore (3.20) is satisfied. Moreover, we see \(D_{\alpha_n} G_n^{(3)} = O_{\mathbb{D}_\infty}(n^{-1})\) from (8.100). Thus we verified (3.20) and (3.21). Properties (3.23) and (3.24) follow from (8.108) and (8.109). Properties (3.25) and (3.26) follow from (8.111) and (8.113), respectively, since \(X_n = 0\) in the present situation. Condition \([C] \text{ (ii)}\) has been checked.
9.3 \([C] \, (iii)\)

Now \(\beta_x = 2\), and \([C] \, (iii) \, (a)\) is verified by using \([A^?] \, (i^?)\). We already observed the convergences requested by \([C] \, (iii) \, (b)\).

9.4 \([C] \, (iv) \, (b)\)

For \(q \in \mathcal{Q}\), the expression (1.7) gives

\[
DM_n(q) = D\delta(u_n(q)) = \|n\|^{(1)}(q) + \|n\|^{(2)}(q) + \|n\|^{(3)}(q) + \|n\|^{(4)}(q) \tag{9.3}
\]

where

\[
\|n\|^{(1)}(q) = n^{2-1(q-1)} \sum_{j=1}^{n} (Da_j(q)) I_q(1_{j}^{\otimes q}), \tag{9.4}
\]

\[
\|n\|^{(2)}(q) = n^{2-1(q-1)} \sum_{j=1}^{n} a_j(q) q I_{q-1}(1_{j}^{\otimes(q-1)}) 1_j, \tag{9.5}
\]

\[
\|n\|^{(3)}(q) = -n^{2-1(q-1)} \sum_{j=1}^{n} (D(D_1 a_j(q))) I_{q-1}(1_{j}^{\otimes(q-1)}) \tag{9.6}
\]

and

\[
\|n\|^{(4)}(q) = -n^{2-1(q-1)} \sum_{j=1}^{n} (D(D_1 a_j(q))(q - 1) I_{q-2}(1_{j}^{\otimes(q-2)}) 1_j. \tag{9.7}
\]

For non-zero \(v \in \mathcal{F}\),

\[
\|n\|^{(3)}(q)[v] = -n^{0.5q-1.5} \sum_{j=1}^{n} (D_{|v|\mathcal{F}}(n D_1 a_j(q))) I_{q-1}(1_{j}^{\otimes(q-1)}) |v| \beta =: \|n\|^{(3)}(q,v)[v] \beta
\]

The exponents of \(\|n\|^{(3)}(q,v)[v]\) is

\[
e(\|n\|^{(3)}(q,v)) = (0.5q - 1.5) - 0.5(q - 1) + 1 - 0.5 = -0.5.
\]

Then Proposition 7.2 ensures

\[
\left| D^i \|n\|^{(3)}(q,v)[v] \right|_{\mathcal{H}^{\otimes i}} \leq O_{L^\infty}(n^{-0.5})
\]

and this estimate is uniform in \(v \in \mathcal{F}\) since Proposition 7.2 is based on Theorem 12.3. Therefore

\[
\left| D^i \|n\|^{(3)}(q,v)[v] \right|_{\mathcal{H}^{\otimes i}} \leq O_{L^\infty}(n^{-0.5}) |v| \beta \quad (v \in \mathcal{F}) \tag{9.8}
\]
where the we can take a factor for $O_L \sim (n^{-0.5})$ independently from $v \in \mathcal{H}$. The inequality (9.8) means

$$|D^{(i)q_n}(q)|_{\mathcal{H}^{\otimes (i+1)}} = O_L \sim (n^{-0.5})$$

(9.9)

for $i \in \mathbb{Z}_+$. In this connection, for $\mathcal{B}^{(1)}[v], \mathcal{B}^{(2)}[v]$ and $\mathcal{B}^{(4)}[v]$, the exponents are given by

$$e(\mathcal{B}^{(1)}[v]) = 0, \quad e(\mathcal{B}^{(2)}[v]) = 0, \quad \text{and} \quad e(\mathcal{B}^{(4)}[v]) = \frac{1}{2} 1_{q>2},$$

(9.10)

respectively, and those sums are not negligible. The exponents (9.10) are accounted by the following representations:

$$\mathcal{B}^{(1)}[v] = n^{0.5q-1} \sum_{j=1}^{n} (D_{\nu^{-1}_j} a_j(q)) I_{q(1_{j}^{\otimes q})}|v|_{\mathcal{H}},$$

and

$$\mathcal{B}^{(2)}[v] = n^{0.5q-1} \sum_{j=1}^{n} a_j(q) q \langle n^{1/2} 1_j, |v|^{-1}_{\mathcal{H}} \rangle I_{q-1} (1_{j}^{\otimes (q-1)}) |v|_{\mathcal{H}},$$

and

$$\mathcal{B}^{(4)}[v] = -n^{0.5q-2} \sum_{j=1}^{n} (nD_1 a_j(q)) (q-1) \langle n^{1/2} 1_j, |v|^{-1}_{\mathcal{H}} \rangle I_{q-2} (1_{j}^{\otimes (q-2)}) |v|_{\mathcal{H}},$$

respectively.

Let

$$\mathcal{B}^{(2)} = \sum_{q \in \mathbb{Q}} \mathcal{B}^{(2)}[q] \quad \text{and} \quad \mathcal{B}^{(2)*} = \sum_{q \in \mathbb{Q}} (\mathcal{B}^{(1)}[q_] + \mathcal{B}^{(3)}[q] + \mathcal{B}^{(4)}[q]).$$

Then

$$\Delta_{(M_n, X_\infty)} = \det \begin{pmatrix} \langle DM_n, DM_n \rangle & \langle DM_n, DX_\infty \rangle \\ \langle DM_n, DX_\infty \rangle & \langle DX_\infty, DX_\infty \rangle \end{pmatrix}$$

$$= \det \begin{pmatrix} \langle \mathcal{B}^{(2)*}, \mathcal{B}^{(2)**} \rangle & \langle \mathcal{B}^{(2)*}, DX_\infty \rangle \\ \langle \mathcal{B}^{(2)**}, DX_\infty \rangle & \langle DX_\infty, DX_\infty \rangle \end{pmatrix}$$

$$= \det \begin{pmatrix} \langle \mathcal{B}^{(2)**}, DX_\infty \rangle & \langle \mathcal{B}^{(2)**}, DX_\infty \rangle \\ \langle \mathcal{B}^{(2)**}, DX_\infty \rangle & \langle DX_\infty, DX_\infty \rangle \end{pmatrix} + \det \begin{pmatrix} \langle \mathcal{B}^{(2)*}, DX_\infty \rangle & \langle \mathcal{B}^{(2)*}, DX_\infty \rangle \\ \langle \mathcal{B}^{(2)*}, DX_\infty \rangle & \langle DX_\infty, DX_\infty \rangle \end{pmatrix} + \mathcal{R}_n$$

(9.11)

where

$$\mathcal{R}_n = \det \begin{pmatrix} 0 & \langle \mathcal{B}^{(2)*}, DX_\infty \rangle \\ \langle \mathcal{B}^{(2)*}, DX_\infty \rangle & \langle DX_\infty, DX_\infty \rangle \end{pmatrix} + \det \begin{pmatrix} \langle \mathcal{B}^{(2)*}, DX_\infty \rangle & \langle \mathcal{B}^{(2)*}, DX_\infty \rangle \\ \langle \mathcal{B}^{(2)*}, DX_\infty \rangle & \langle DX_\infty, DX_\infty \rangle \end{pmatrix}$$

(9.12)
We dared compute the determinant in general way to demonstrate this passage was dimension free though the variables are one-dimensional in the present problem.

It is easy to see the components appearing in the matrices on the right-hand side of (9.11) and (9.12) are of $O_{D_\infty}(1)$. Now

$$\langle \mathbb{I}_n, DX_\infty \rangle = O_{D_\infty}(n^{-0.5}) \quad (9.13)$$

since the exponent

$$e\left(\langle \mathbb{I}_n^{(2)}(q), DX_\infty \rangle \right) = e\left(n^{0.5q-1.5} \sum_{j=1}^{n} a_j(q)(nD_{1,j}X_\infty)qI_{q-1}(1^{\otimes(q-1)})\right)$$

$$= (0.5q - 1.5) - 0.5(q - 1) + 1 - 0.5 = -0.5.$$ 

Let $q_1, q_2 \in \mathbb{Q}$. Then

$$\langle \mathbb{I}_n^{(1)}(q_1), \mathbb{I}_n^{(2)}(q_2) \rangle = O_{D_\infty}(n^{-0.5}) \quad (9.14)$$

since

$$e\left(\langle \mathbb{I}_n^{(1)}(q_1), \mathbb{I}_n^{(2)}(q_2) \rangle \right) = e\left(n^{0.5(q_1+q_2)-2} \sum_{j,k=1}^{n} (nD_{1,k}a_j(q_1))a_k(q_2)q_1I_{q_1}(1^{\otimes(q_1)})I_{q_2-1}(1^{\otimes(q_2-1)})\right)$$

$$= (0.5(q_1 + q_2) - 2) - 0.5(q_1 + q_2 - 1) + 2 - 0.5 \times 2 = -0.5.$$ 

On the other hand,

$$\langle \mathbb{I}_n^{(4)}(q_1), \mathbb{I}_n^{(2)}(q_2) \rangle = O_{D_\infty}(n^{-0.5}) \quad (9.15)$$

from (9.10) and the estimate

$$e\left(\langle \mathbb{I}_n^{(4)}(q_1), \mathbb{I}_n^{(2)}(q_2) \rangle \right) = e\left(n^{0.5q-2} \sum_{j=1}^{n} (nD_{1,j}a_j(2))a_j(q)qI_{q-1}(1^{\otimes(q-1)})\right)$$

$$= (0.5q - 2) - 0.5(q - 1) + 1 - 0.5 = -1$$

for $q \in \mathbb{Q}$, if $2 \in \mathbb{Q}$. Thus,

$$\langle \mathbb{I}^*, \mathbb{I}^{**} \rangle = O_{D_\infty}(n^{-0.5}) \quad (9.16)$$

from (9.9), (9.10), (9.14) and (9.15). The first term on the right-hand side of (9.11) is nonnegative. So it follows from (9.11), (9.12), (9.13) and (9.16) that

$$\Delta_{(M_nX_\infty)} \geq \langle \mathbb{I}^*, \mathbb{I}^* \rangle \langle DX_\infty, DX_\infty \rangle + \mathfrak{R}_n^* \quad (9.17)$$

for some functional $\mathfrak{R}_n^* \in D_\infty$ satisfying

$$\mathfrak{R}_n^* = O_{D_\infty}(n^{-0.5}).$$
For \( q_1, q_2 \in \mathbb{Q} \),

\[
\langle \mathbb{I}_n^{(2)}(q_1), \mathbb{I}_n^{(2)}(q_2) \rangle = n^{0.5(q_1 + q_2) - 2} \sum_{j=1}^{n} a_j(q_1)q_1 a_j(q_2)q_2 I_{q_1 - 1}(1_j^{\otimes(q_1 - 1)}) I_{q_2 - 1}(1_j^{\otimes(q_2 - 1)})
\]

\[
= \sum_{\nu} c_\nu(q_1 - 1, q_2 - 1) \times n^{0.5(q_1 + q_2) - 2 - \nu} \sum_{j=1}^{n} a_j(q_1)q_1 a_j(q_2)q_2 I_{q_1 + q_2 - 2 - 2\nu}(1_j^{\otimes(q_1 + q_2 - 2 - 2\nu)})
\]

\[
= 1_{\{q_1 = q_2\}} n^{-1} \sum_{j=1}^{n} a_j(q_1)^2 q_1 q_1! + \mathcal{R}_n(q_1, q_2)
\]

where

\[
\mathcal{R}_n(q_1, q_2) = \sum_{\nu: q_1 + q_2 - 2 - 2\nu > 0} c_\nu(q_1 - 1, q_2 - 1) \times n^{0.5(q_1 + q_2) - 2 - \nu} \sum_{j=1}^{n} a_j(q_1)q_1 a_j(q_2)q_2 I_{q_1 + q_2 - 2 - 2\nu}(1_j^{\otimes(q_1 + q_2 - 2 - 2\nu)}).
\]

We see

\[
\mathcal{R}_n(q_1, q_2) = O_D(1) (n^{-0.5})
\]

because \( e(\mathcal{R}_n(q_1, q_2)) \) is

\[
\max \left\{ (0.5(q_1 + q_2) - 2 - \nu) - 0.5(q_1 + q_2 - 2 - 2\nu) + 1 - 0.5 ; \nu < 0.5(q_1 + q_2) - 1 \right\} = -0.5.
\]

Therefore we obtain the representation

\[
\langle \mathbb{I}^*, \mathbb{I}^* \rangle = \sum_{q \in \mathbb{Q}} n^{-1} \sum_{q=1}^{n} a_j(q)^2 q q! + \mathcal{R}_{n}^{**}
\]

with some functional

\[
\mathcal{R}_{n}^{**} = O_{D}(n^{-0.5}).
\]

Let

\[
\mathcal{E}_n = n^{-1} \sum_{j=1}^{n} \sum_{q \in \mathbb{Q}} q! a_j(q)^2.
\]

From (9.17) and (9.19), we obtain

\[
\Delta_{(M_n, X_\infty)} \geq 2 \mathcal{E}_n(DX_\infty, DX_\infty) + \mathcal{R}_{n}^{***}
\]
with some functional $\mathcal{R}^{**\ast}_n$ satisfying
\[ \mathcal{R}^{**\ast}_n = O_{D^\infty}(n^{-0.5}) \] (9.21)

Let
\[ s_\infty = s_n = G_\infty |DX_\infty|_2. \]
Then $s_\infty \in D^\infty$ and $s^{-1}_\infty \in L^\infty$ under $[A^5]$. By (9.20), we have
\[ P[\Delta(M_n,X_\infty) < s_\infty] \leq P\left(2G_0 - G_\infty|DX_\infty|_2 + |\mathcal{R}^{**\ast}_n| > G_\infty|DX_\infty|_2 \right) \leq E\left((G_\infty|DX_\infty|_2)^{-L}\left\{2G_0 - G_\infty|DX_\infty|_2 + |\mathcal{R}^{**\ast}_n| \right\}^L \right) \]
for any positive number $L$. Then, from (9.21), (2.18) and $s^{-1}_\infty \in L^\infty$ (or $[A^4]$ (II)), we obtain
\[ P[\Delta(M_n,X_\infty) < s_\infty] = O(n^{-L}) \]
as $n \to \infty$ for every $L > 0$. In this way, Condition $[C]$ (iv) (b) has been verified. This is the last step of the proof of Theorem 2.4.

## 10 Proof for Section 4

### 10.1 Proof of Theorem 4.1

It suffices to check Condition $[B]$ for $p_n = 1$ and $(Z_n, X_n)$ in Section 4, that is, $Z_n = M_n + n^{1/2}N_n$, where $N_n$ and $X_n$ are general. Under the conditions in $[A]$ except for $[A]$ (vi), the argument in Sections 8.1, 8.2, 8.3 and 8.4.2 (without consideration for the specific $N_n$) is valid. In particular, we have the convergence of $\mathcal{G}_n^{(3,0)}$ to $\mathcal{G}^{(3,0)}$, $\mathcal{G}_0^{(2,0)}$ to $\mathcal{G}_0^{(2,0)} = 0$, and $\mathcal{G}_n^{(1,1)}$ to $\mathcal{G}^{(1,1)}$ in the sense of (3.16) in $[B]$ (iii). The convergences $\mathcal{G}_n^{(1,0)}$ to $\mathcal{G}^{(1,0)}$, $\mathcal{G}_n^{(0,1)}$ to $\mathcal{G}^{(0,1)}$, $\mathcal{G}_1^{(2,0)}$ to $\mathcal{G}_1^{(2,0)}$, and $\mathcal{G}_1^{(1,1)}$ to $\mathcal{G}_1^{(1,1)}$ in the sense of (3.16) in $[B]$ (iii) are assumed in $[D]$. Therefore, Condition $[B]$ (iii) is satisfied. Condition $[B]$ (i) is obvious if we combine the observations in Section 8.5.1 and Condition $[D]$ (II). Sections 8.5.2-8.5.7 verify the estimates (3.4)-(3.11) of $[B]$ (ii), and the estimates (3.12)-(3.14) are assumed in $[D]$ (II). Therefore $[B]$ (ii) has been checked. Now we can apply Theorem 3.1 to prove Theorem 4.1.

### 10.2 Proof of Theorem 4.2

We should check the conditions in $[C]$. Condition $[C]$ (i) holds obviously under $[D^2]$. Condition $[A^6]$ without $[A^4]$ (i) (vi), which holds under $[D^2]$, is stronger than $[A]$ without $[A]$ (vi), and $[A]$ without $[A]$ (vi) is sufficient to follow the argument in Section 9.2 except for the last part concerning (3.25) and (3.26). However, the last two properties are assumed by $[D^2]$ (II). Therefore $[C]$ (ii) is satisfied under $[D^2]$. Condition $[C]$ (iii) (a) is satisfied with $[D^2]$ (I) (i$^2$) and $[D^2]$ (III). We already found $\mathcal{G}_n$ for $\mathcal{G}_n^{(3,0)}$, $\mathcal{G}_0^{(2,0)}$ and $\mathcal{G}_n^{(1,1)}$, and their convergence in the sense
of \([C]\) (iii) (b), from the argument in Sections 8.1, 8.2, 8.3 and 8.4.2. Since Condition \([D^2]\) (III) takes care of the convergence of \(\mathcal{S}_n^{(1,0)}, \mathcal{S}_n^{(0,1)}, \mathcal{S}_n^{(2,0)}\) and \(\mathcal{S}_n^{(1,1)}\), Condition \([C]\) (iii) is verified. Condition \([C]\) (iv) has been verified with \([A^1]\) (II); see Section 9.4. This completes the proof of Theorem 4.2.

\section{11 Proof of Theorem 6.2}

\subsection{11.1 Proof of Lemma 6.1}

\subsubsection{11.1.1 Decomposition of \((\Delta_jX)^2\)}

Define \(f_j\) and \(g_j\) in \(\mathcal{H}\) by

\[ f_j(t) = 1_j(t)n(t - t_{j-1}) \quad (t \in \mathbb{R}_+) \]

and

\[ g_j(t) = 1_j(t)n(t_j - t) \quad (t \in \mathbb{R}_+) \quad (11.1) \]

respectively, for \(j \in \mathbb{J}_n\).

The increment \(\Delta_jX\) has a decomposition

\[ \Delta_jX = b_{j_{j-1}}^{[1]}I_1(1_j) + b_{j_{j-1}}^{[2]}h + 2^{-1}b_{j_{j-1}}^{[1,1]}I_2(1_j) + 6^{-1}b_{j_{j-1}}^{[1,1,1]}I_3(1_j) + h\mathcal{B}_{j_{j-1}}^{[1,2]}I_1(f_j) + h\mathcal{B}_{j_{j-1}}^{[2,1]}I_1(g_j) + R_{j_{j-1}}^{[11,3]} \quad (11.2) \]

where

\[ R_{j_{j-1}}^{[11,3]} = \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} \int_{t_{j-1}}^{s} \int_{t_{j-1}}^{r} b_p^{[1,1,1,1]} dw_p dw_r dw_s dw_t + \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} \int_{t_{j-1}}^{s} \int_{t_{j-1}}^{r} b_p^{[1,1,1,2]} dp dw_r dw_s dw_t + \int_{t_{j-1}}^{t} \int_{t_{j-1}}^{r} \int_{t_{j-1}}^{s} b_p^{[1,1,2]} dr dw_s dw_t + \int_{t_{j-1}}^{t} \int_{t_{j-1}}^{t} \int_{t_{j-1}}^{s} b_p^{[1,2,1]} dw_r ds dw_t + \int_{t_{j-1}}^{t} \int_{t_{j-1}}^{t} \int_{t_{j-1}}^{s} b_p^{[1,2,2]} dr ds dw_t + \int_{t_{j-1}}^{t} \int_{t_{j-1}}^{t} \int_{t_{j-1}}^{s} b_p^{[2,1,1]} dw_r dw_s dt + \int_{t_{j-1}}^{t} \int_{t_{j-1}}^{t} \int_{t_{j-1}}^{s} b_p^{[2,1,2]} dr dw_s dt + \int_{t_{j-1}}^{t} \int_{t_{j-1}}^{t} b_p^{[2,2]} ds dt. \quad (11.3) \]

It is easy to see

\[ R_{j_{j-1}}^{[11,3]} = \mathcal{O}_{D^\infty}(n^{-2}), \quad (11.4) \]
that is,

$$\sup_{n \in \mathbb{N}} \sup_{j \in J_n} \| R_j^{(11.3)} \|_{s,p} = O(n^{-2})$$

as $n \to \infty$ for any $s \in \mathbb{R}$ and $p > 1$.

From (11.2),

$$(\Delta_j X)^2 = P_j^{(11.6)} + R_j^{(11.7)}$$

(11.5)

where

$$P_j^{(11.6)} = (b_{t_j-1}^{[1]} - b_{t_j-1})^2 I_1(1_j)^2 + 2h b_{t_j-1}^{[1]} b_{t_j-1}^{[2]} I_1(1_j) + b_{t_j-1}^{[1]} b_{t_j-1}^{[1,1]} I_1(1_j) I_2(1_j^{(2)})$$

$$+ 2h b_{t_j-1}^{[1]} b_{t_j-1}^{[1,2]} I_1(1_j) I_1(f_j) + 2h b_{t_j-1}^{[1]} b_{t_j-1}^{[2,1]} I_1(1_j) I_1(g_j)$$

$$+ h^2 (b_{t_j-1}^{[2]})^2 + 2^{-2} (b_{t_j-1}^{[1,1]})^2 I_2(1_j^{(2)})^2$$

(11.6)

and

$$R_j^{(11.7)} = 3^{-1} b_{t_j-1}^{[1]} b_{t_j-1}^{[1,1]} I_1(1_j) I_3(1_j^{(3)}) + 2b_{t_j-1}^{[1]} I_1(1_j) R_j^{(11.3)}$$

$$+ h b_{t_j-1}^{[1]} b_{t_j-1}^{[1,1]} I_2(1_j^{(2)}) + 3^{-1} h b_{t_j-1}^{[1]} b_{t_j-1}^{[1]} I_3(1_j^{(3)})$$

$$+ 2h^2 b_{t_j-1}^{[2]} b_{t_j-1}^{[1,2]} I_1(f_j) + 2h^2 b_{t_j-1}^{[2]} b_{t_j-1}^{[2,1]} I_1(g_j) + 2h^2 b_{t_j-1}^{[2]} R_j^{(11.3)}$$

$$+ 6^{-1} b_{t_j-1}^{[1]} b_{t_j-1}^{[1]} I_2(1_j^{(2)}) b_{t_j-1}^{[1,1]} I_3(1_j^{(3)}) + b_{t_j-1}^{[1]} b_{t_j-1}^{[1]} I_2(1_j^{(2)}) b_{t_j-1}^{[1,2]} I_1(f_j)$$

$$+ b_{t_j-1}^{[2]} I_2(1_j^{(2)}) h b_{t_j-1}^{[1]} I_1(f_j) + b_{t_j-1}^{[1]} I_2(1_j^{(2)}) R_j^{(11.3)}$$

$$+ \left\{ 6^{-1} b_{t_j-1}^{[1]} I_3(1_j^{(3)}) + h b_{t_j-1}^{[1,2]} I_1(f_j) + h b_{t_j-1}^{[2,1]} I_1(g_j) + R_j^{(11.3)} \right\}^2.$$ (11.7)

### 11.1.2 A weighted sum of squares

Suppose that the double sequence $\theta = (\theta_{n,j})_{j \in J_n, n \in \mathbb{N}}$ in $\mathbb{D}^\infty$ satisfies

$$\sup_{n \in \mathbb{N}} \sup_{j \in J_n} \sup_{s_1, \ldots, s_{i} \in [0,1+\eta]} \| D_{s_1} \cdots D_{s_i} \theta_{n,j} \|_p < \infty$$

(11.8)

for every $i \in \mathbb{Z}_+$ and $p > 1$. We will write $\theta_j = \theta_{n,j}$ simply.

We see that

$$e \left( \sum_j \theta_j 3^{-1} b_{t_j-1}^{[1]} b_{t_j-1}^{[1,1]} I_1(1_j) I_3(1_j^{(3)}) \right)$$

$$= \max_{\nu=0,1} \left\{ (0 - \nu) - 0.5(4 - 2\nu) + 1 - 0.5 \times 1_{\{4-2\nu>0\}} \right\} = -1.5$$

(11.9)

with the product formula, and also that

$$e \left( \sum_j \theta_j h b_{t_j-1}^{[2]} b_{t_j-1}^{[1,1]} I_2(1_j^{(2)}) \right) = -1 - 0.5 \times 2 + 1 - 0.5 = -1.5.$$ 

(11.10)
From (11.7), (11.4), (11.9), and (11.10), we obtain
\[
\sum_j \theta_j R_j^{(11.7)} = O_{\mathcal{D}}(n^{-1.5}). \tag{11.11}
\]

On the other hand,
\[
P_j^{(11.6)} = P_j^{(11.13)} + F_j^{(6.4)} + S_j^{(11.14)} + R_j^{(11.15)} \tag{11.12}
\]
where
\[
P_j^{(11.13)} = (b_{i,j-1}^{[1]})^2 h, \tag{11.13}
\]
\[
S_j^{(11.14)} = b_{i,j-1}^{[1]} b_{i,j-1}^{[1,1]} I_{5}(1_j^{\otimes 3}) + 2h b_{i,j-1}^{[1]} b_{i,j-1}^{[1,1]} I_1(1_j) + h^2(b_{i,j-1}^{[2]})^2 \tag{11.14}
\]
and
\[
R_j^{(11.15)} = 2h b_{i,j-1}^{[1]} b_{i,j-1}^{[1,2]} I_{2}(1_j \odot f_j) + 2h b_{i,j-1}^{[1]} b_{i,j-1}^{[2,1]} I_2(1_j \odot g_j) + 2^{-2}(b_{i,j-1}^{[1,1]})^2 I_4(1_j^{\otimes 4}) + h(b_{i,j-1}^{[1,1]})^2 I_2(1_j^{\otimes 2}). \tag{11.15}
\]

Simply
\[
F_j^{(6.4)} = \overline{O}_{\mathcal{D}}(n^{-1}) \tag{11.16}
\]
and
\[
S_j^{(11.14)} = \overline{O}_{\mathcal{D}}(n^{-1.5}) \tag{11.17}
\]

Since
\[
e\left(\sum_j \theta_j R_j^{(11.15)}\right) = \max \{-1.5, -1.5, -1.5, -1.5\} = -1.5,
\]
we obtain
\[
\sum_j \theta_j R_j^{(11.15)} = O_{\mathcal{D}}(n^{-1.5}). \tag{11.18}
\]

Related with this, we also see
\[
e\left(\sum_j \theta_j F_j^{(6.4)}\right) = -0.5 \tag{11.19}
\]
and
\[
e\left(\sum_j \theta_j S_j^{(11.14)}\right) = \max\{-1, -1, -1, -1, -1, -1\} = -1. \tag{11.20}
\]
We observe
\[
\sum_j \theta_j \int_{t_{j-1}}^{t_j} \beta_i dt - \sum_j \theta_j \beta_{t_{j-1}} h = \sum_j \theta_j \beta_{t_{j-1}}^1 h I_1(g_j) + \sum_j \theta_j 2^{-1} h^2 \beta_{t_{j-1}}^{2} + R_n^{(11.22)}(11.21)
\]
where
\[
R_n^{(11.22)} = \sum_j \theta_j \int_{t_{j-1}}^{t_j} (t_j - s)(\beta_s^1 - \beta_{t_{j-1}}^1) dw_s + \sum_j \theta_j \int_{t_{j-1}}^{t_j} (\beta_s^2 - \beta_{t_{j-1}}^2) ds dt.
\]
Indeed,
\[
\int_{t_{j-1}}^{t_j} (\beta_s - \beta_{t_{j-1}}) dt = \int_{t_{j-1}}^{t_j} \left\{ \int_{t_{j-1}}^{t} \beta_s^1 dw_s + \int_{t_{j-1}}^{t} \beta_s^2 ds \right\} dt
= \int_{t_{j-1}}^{t_j} (t_j - s) \beta_s^1 dw_s + \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} \beta_s^2 ds dt
= \beta_{t_{j-1}}^{1} \int_{t_{j-1}}^{t_j} (t_j - s) dw_s + \int_{t_{j-1}}^{t_j} (t_j - s)(\beta_s^1 - \beta_{t_{j-1}}^1) dw_s
+ \beta_{t_{j-1}}^{2} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} ds dt + \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} (\beta_s^2 - \beta_{t_{j-1}}^2) ds dt.
\]

The sequence
\[
\sum_j \theta_j \int_{t_{j-1}}^{t_j} (t_j - s)(\beta_s^1 - \beta_{t_{j-1}}^1) dw_s
= \sum_j \theta_j \beta_{t_{j-1}}^{11} \int_{t_{j-1}}^{t_j} (t_j - s) \int_{t_{j-1}}^{s} dw_s dw_s + \sum_j \theta_j \int_{t_{j-1}}^{t_j} (t_j - s) \int_{t_{j-1}}^{s} (\beta_r^{11} - \beta_{t_{j-1}}^{11}) dw_r dw_s
+ \sum_j \theta_j \int_{t_{j-1}}^{t_j} (t_j - s) \int_{t_{j-1}}^{s} \beta_r^{12} dr dw_s
= O_D(n^{-1.5})
\]
since the exponent of the first term on the right-hand side is −1.5. Therefore
\[
R_n^{(11.22)} = O_D(n^{-1.5}).
\]

Let
\[
\forall_n(\theta) = \sum_{j=1}^{n} \theta_j (\Delta_j X)^2
\]
for the sequence \( \theta \) satisfying (11.8). For a generic random variable \( \forall \), From (11.5), (11.21) and
(11.12), the difference $\mathcal{V}_n - \mathcal{V}$ admits the expansion

$$
\mathcal{V}_n(\theta) - \mathcal{V} = \left( \sum_{j=1}^{n} \theta_j (\Delta_j X)^2 - \sum_{j=1}^{n} \theta_j \int_{t_{j-1}}^{t_j} \beta_t dt \right) + \left( \sum_{j=1}^{n} \theta_j \int_{t_{j-1}}^{t_j} \beta_t dt - \mathcal{V} \right)
$$

$$
= \left( \sum_{j=1}^{n} \theta_j F_j^{(11.6)} - \sum_{j=1}^{n} \theta_j \beta_{t_{j-1}} h - \sum_{j=1}^{n} h \theta_j \beta_{t_{j-1}}^{[1]} I_1(g_j) - \sum_{j=1}^{n} \theta_j 2^{-1} h^2 \beta_{t_{j-1}}^{[2]} \right)
$$

$$
+ \left( \sum_{j=1}^{n} \theta_j \int_{t_{j-1}}^{t_j} \beta_t dt - \mathcal{V} \right) + \left( \sum_{j=1}^{n} \theta_j R_j^{(11.7)} - R_n^{(11.22)} \right)
$$

$$
= \sum_{j=1}^{n} \theta_j F_j^{(6.4)} + \sum_{j=1}^{n} \theta_j S_j^{(6.3)} + \left( \sum_{j=1}^{n} \theta_j \int_{t_{j-1}}^{t_j} \beta_t dt - \mathcal{V} \right) + R_n^{(11.25)} \quad (11.24)
$$

for

$$
S_j^{(6.3)} = S_j^{(11.14)} - h \beta_{t_{j-1}}^{[1]} I_1(g_j) - 2^{-1} h^2 \beta_{t_{j-1}}^{[2]}
$$

and

$$
R_n^{(11.25)} = \sum_{j=1}^{n} \theta_j R_j^{(11.7)} - R_n^{(11.22)} + \sum_{j=1}^{n} \theta_j R_j^{(11.15)}. \quad (11.25)
$$

We can say

$$
\sum_{j=1}^{n} \theta_j F_j^{(6.4)} = O_{\Delta \infty} (n^{-0.5}) \quad (11.26)
$$

and

$$
\sum_{j=1}^{n} \theta_j S_j^{(6.3)} = O_{\Delta \infty} (n^{-1}) \quad (11.27)
$$

by using (11.19) and (11.20), respectively, and also

$$
R_n^{(11.25)} = O_{\Delta \infty} (n^{-1.5}) \quad (11.28)
$$

from (11.11), (11.23) and (11.18).

### 11.1.3 Expansion of $L_{n,j}$ and $U_n$

For each $j \in J_n$, applying the formula (11.24) to

$$
\theta_k = \eta_{n,j}^{-1} \mathbb{1}_{\{k \in K_j\}}
$$

and $\mathcal{V} = L_{\infty,t_{j-1}}$, we obtain

$$
L_{n,j} - L_{\infty,t_{j-1}} = \sum_{k \in K_j} \eta_{n,j}^{-1} F_k^{(6.4)} + \sum_{k \in K_j} \eta_{n,j}^{-1} S_k^{(6.3)} + \mathcal{E}_j + R_n^{(11.30)} \quad (11.29)
$$
where
\[
E_j = \sum_{k \in K_j} \eta_{n,j}^{-1} \int_{t_{k-1}}^{t_k} \beta_t dt - L_{\infty,t_{j-1}}
\]
and \(R_{n,j}^{(11.30)} \in \mathbb{D}^\infty\) such that uniformly in \(j \in J_n\),
\[
R_{n,j}^{(11.30)} = \overline{O}_{\mathbb{D}^\infty}(n^{-1.5}). \tag{11.30}
\]

We remark that
\[
E_j = \sum_{k \in K_j} \eta_{n,j}^{-1} \int_{t_{k-1}}^{t_k} \beta_t dt - L_{\infty,t_{j-1}}
\]
\[
= \sum_{k \in K_j} \eta_{n,j}^{-1} \int_{t_{k-1}}^{t_k} (\beta_t - \beta_{t_{j-1}}) dt - \eta_{\infty,t_{j-1}}^{-1} \int_{(t_{j-1}-\lambda)^{\vee}0}^{(t_{j-1}+\lambda)^{\land}1} (\beta_t - \beta_{t_{j-1}}) dt
\]
\[
= \sum_{k \in K_j} (\eta_{n,j}^{-1} - \eta_{\infty,t_{j-1}}^{-1}) \int_{t_{k-1}}^{t_k} (\beta_t - \beta_{t_{j-1}}) dt
\]
\[
+ \eta_{\infty,t_{j-1}}^{-1} \left\{ - \int_{(t_{j-1}+\lfloor n\lambda \rfloor/n)^{\land}1}^{(t_{j-1}+\lambda)^{\land}1} (\beta_t - \beta_{t_{j-1}}) dt + \int_{(t_{j-1}-(\lfloor n\lambda \rfloor-1)/n)^{\vee}0}^{(t_{j-1}-\lambda)^{\vee}0} (\beta_t - \beta_{t_{j-1}}) dt \right\}
\]
\[
= \overline{O}_{\mathbb{D}^\infty}(n^{-1})
\]
since \(\liminf_{n \to \infty} \inf_j \eta_{n,j} > 0\) and \(\inf_t \eta_{\infty,t} > 0\). Thus, \(E_j\) is not negligible in \(L^p\) and in \(\mathbb{D}^\infty\).

Similarly, (11.24) applied to \(\theta_j = 1\) and \(U_\infty\) gives
\[
U_n - U_\infty = \sum_{k=1}^{n} F_k^{(6.4)} + \sum_{k=1}^{n} S_k^{(6.3)} + R_n^{(11.32)} \tag{11.31}
\]
with
\[
R_n^{(11.32)} = O_{\mathbb{D}^\infty}(n^{-1.5}) \tag{11.32}
\]
thanks to the estimate (11.28).

**11.1.4 Expansion of \(\Phi(U_n, L_{n,j})\)**

Let
\[
\theta_j = \theta_{n,j} = \Phi(U_n, L_{n,j}). \tag{11.33}
\]
Then Condition (11.8) is satisfied. Recall $\Psi_{j,k}(x,y)$ defined by (6.6) and $\Xi_{j,k,\ell}(x,y)$ defined by (6.7). We have

$$\begin{align*}
\theta_j - \overline{\theta}_{t_{j-1}} &= \partial\Phi(U_\infty, L_\infty, t_{j-1}) \left[ (U_n - U_\infty, L_{n,j} - L_\infty, t_{j-1}) \right] \\
&\quad + \frac{1}{2} \partial^2 \Phi(U_\infty, L_\infty, t_{j-1}) \left[ (U_n - U_\infty, L_{n,j} - L_\infty, t_{j-1}) \right]^2 + R_j^{(11.35)} \\
&= \sum_{k=1}^n \Psi_{j,k}(U_\infty, L_\infty, t_{j-1}) F_k^{(6.4)} + \sum_{k=1}^n \Psi_{j,k}(U_\infty, L_\infty, t_{j-1}) S_k^{(6.3)} \\
&\quad + \partial_2 \Phi(U_\infty, L_\infty, t_{j-1}) \mathcal{E}_j \\
&\quad + \sum_{k,\ell=1}^n \Xi_{j,k,\ell}(U_\infty, L_\infty, t_{j-1}) F_k^{(6.4)} F_\ell^{(6.4)} + R_j^{(11.36)},
\end{align*}$$

where

$$R_j^{(11.35)} = \mathcal{O}_\mathcal{D}(n^{-1.5})$$

and

$$R_j^{(11.36)} = \mathcal{O}_\mathcal{D}(n^{-1.5})$$

by (11.32), (11.30) and (11.35). In particular,

$$\begin{align*}
\theta_j - \overline{\theta}_{t_{j-1}} &= \mathcal{O}_\mathcal{D}(n^{-0.5})
\end{align*}$$

as $n \to \infty$.

### 11.1.5 Expansion of $\mathcal{V}_n$

Recall that the target of the estimator is

$$\mathcal{V}_n = \int_0^1 \Phi(U_n, L_{n,j}) \sigma_t^2 dt = \sum_{j \in J_n} \int_{t_{j-1}}^{t_j} \theta_j \beta_t dt$$

for $\theta = (\theta_j)$ of (11.33). We apply (11.24) to $\theta = (\theta_j)$ and $\mathcal{V} = \mathcal{V}_n$ to obtain

$$\begin{align*}
\mathcal{V}_n - \mathcal{V}_n &= \sum_{j=1}^n \theta_j F_j^{(6.4)} + \sum_{j=1}^n \theta_j S_j^{(6.3)} + R_n^{(11.25)} \\
&= \sum_{j=1}^n \overline{\theta}_{t_{j-1}} F_j^{(6.4)} + \sum_{j=1}^n \overline{\theta}_{t_{j-1}} S_j^{(6.3)} \\
&\quad + \sum_{j=1}^n \sum_{k=1}^n \Psi_{j,k}(U_\infty, L_\infty, t_{j-1}) F_j^{(6.4)} F_k^{(6.4)} + \sum_{j=1}^n \sum_{k=1}^n \Psi_{j,k}(U_\infty, L_\infty, t_{j-1}) F_j^{(6.4)} S_k^{(6.3)} \\
&\quad + \sum_{j=1}^n \sum_{k=1}^n \Xi_{j,k,\ell}(U_\infty, L_\infty, t_{j-1}) F_k^{(6.4)} F_\ell^{(6.4)} + \sum_{j=1}^n \sum_{k,\ell=1}^n \Xi_{j,k,\ell}(U_\infty, L_\infty, t_{j-1}) F_k^{(6.4)} F_\ell^{(6.4)} + R_n^{(11.38)}.
\end{align*}$$
where $R_n^{(11.38)} \in D^n$ satisfying

$$R_n^{(11.38)} = O_D(n^{-1.5}).$$

The estimate (11.38) used (11.28).

We have

$$e\left(\sum_{j=1}^{n} \sum_{k=1}^{n} \Psi_{j,k}(U_\infty, L_\infty, t_{j-1}) F_j^{(6.4)} S_k^{(6.3)}\right) = \max \left\{0 - 0.5 \times 5 + 2 - 0.5 \times 2, \right.$$ 
$$\left. -1 - 0.5 \times 3 + 2 - 0.5 \times 2, \right.$$ 
$$\left.(1 - 2) - 0.5 \times 2 + 1 - 0.5\right\} = -1.5$$

(11.39)

since this sum consists of the three types of sums involving $I_3, I_1$ and $h^2$ respectively. Therefore, this sum is $O_D(n^{-1.5})$ and negligible in the asymptotic expansion. Similarly,

$$e\left(\sum_{j=1}^{n} \sum_{k,\ell=1}^{n} \Xi_{j,k,\ell}(U_\infty, L_\infty, t_{j-1}) F_j^{(6.4)} F_k^{(6.4)} F_\ell^{(6.4)}\right) = 0 - 0.5 \times 6 + 3 - 0.5 \times 3$$
$$= -1.5$$

(11.40)

Therefore, this sum is also $O_D(n^{-1.5})$ and negligible in the asymptotic expansion. Moreover,

$$e\left(\sum_{j=1}^{n} F_j^{(6.4)} \partial_2 \Phi(U_\infty, L_\infty, t_{j-1}) \mathcal{E}_j\right) = -1.5$$

(11.41)

Let

$$\Theta_j = \bar{\theta}_{t_{j-1}}$$

equivalently (6.5). In this way, (11.37) can be simplified as

$$\nabla_n - \nabla_n = \sum_{j=1}^{n} \Theta_j F_j^{(6.4)} + \sum_{j=1}^{n} \Theta_j S_j^{(6.3)}$$
$$+ \sum_{j=1}^{n} \sum_{k=1}^{n} \Psi_{j,k}(U_\infty, L_\infty, t_{j-1}) F_j^{(6.4)} F_k^{(6.4)} + R_n^{(11.43)}$$

(11.42)

with the residual term

$$R_n^{(11.43)} = O_D(n^{-1.5}).$$

(11.43)

We consider $M_n = \delta(u_n)$ for $u_n$ defined by

$$u_n = n^{1/2} \sum_{j=1}^{n} \Theta_j (b_{t_{j-1}}^{[j]})^2 I_1(1_j) 1_j.$$  

(11.44)
Then
\[
M_n = \delta(u_n) = n^{1/2} \sum_{j=1}^{n} \Theta_j(b_{t_j-1}^{[1]})^2 I_1(1_j) - n^{1/2} \sum_{j=1}^{n} \langle D(\Theta_j(b_{t_j-1}^{[1]})^2 I_1(1_j)), 1_j \rangle
\]
\[
= n^{1/2} \sum_{j=1}^{n} \Theta_j F_j^{(6.4)} - n^{1/2} \sum_{j=1}^{n} \langle D_{1_j}(\Theta_j(b_{t_j-1}^{[1]}))^2 I_1(1_j) \rangle \tag{11.45}
\]
from (6.4). Therefore we obtained (6.10). The estimate (6.12) follows from (11.43).

11.2 Proof of Theorem 6.2

11.2.1 Decomposition of \( Z_n \)

For the reference variable \( X_\infty \) in the general theory, let us consider \( X_1 \), the value of the process \( X = (X_t)_{t \in \mathbb{R}_+} \) evaluated at \( t = 1 \). It is known that \( X_1 \in \mathbb{D}^\infty \) and \( \Delta X_1 \) is non-degenerate if the stochastic differential equation is uniformly elliptic. In this situation, \( Q = \{2\} \) and
\[
a_j(2) = \Theta_j(b_{t_j-1}^{[1]})^2 = \Theta_j \beta_{t_j-1}. \tag{11.46}
\]

The density of the Malliavin derivative \( DX_\tau \) is expressed explicitly by
\[
D_sX_\tau = Y_sY_s^{-1}\sigma_s 1_{\{s \leq \tau\}}
\]
for \( \tau \in \mathbb{R}_+ \), where \( Y = (Y_t)_{t \in \mathbb{R}_+} \) is the unique solution to the variational equation
\[
\begin{cases}
  dY_t = \sigma'(X_t)Y_t dw_t + b'(X_t)Y_t dt \\
  Y_0 = 1.
\end{cases}
\]

Similarly,
\[
D_tD_sX_\tau = Y_sY_s^{-1}\sigma_s\sigma_t 1_{\{s < t \leq \tau\}} + Y_{\tau}Y_{t}^{-1}\sigma_t\sigma_s' 1_{\{t < s \leq \tau\}}
\]
\[
= Y_sY_s^{-1}\sigma_s\sigma_t 1_{\{s,t \leq \tau\}}.
\]

So it is natural to write
\[
D_tD_sX_\tau = Y_sY_s^{-1}\sigma_t 1_{\{t \leq \tau\}}.
\]
The derivatives of \( U_\infty \) and \( L_{\infty,t} \) up to the second order are expressed with those of \( X_\tau \) for \( \tau \in \mathbb{R}_+ \). For example, \( D_tD_tU_\infty \) is well defined through the expression of \( D_tD_tX_\tau \). In particular, these derivatives are stable in Lebesgue measure in limiting operations.

From (11.46) and (6.5),
\[
a_j(2) = \bar{\partial}_{t_j-1} \beta_{t_j-1}. \tag{11.47}
\]
To specify the terms in the asymptotic expansion formula, it is necessary to investigate certain projections of the Malliavin derivatives of \( a_j(2) \). The first two projections concerning \( a_j(2) \) are
\[
D_{1_k}a_j(2) = \partial_1\Phi(U_\infty, L_{\infty,t_j-1})\beta_{t_j-1}D_{1_k}U_\infty + \partial_2\Phi(U_\infty, L_{\infty,t_j-1})\beta_{t_j-1}D_{1_k}L_{\infty,t_j-1}
\]
\[
+ \Phi(U_\infty, L_{\infty,t_j-1})\beta_{t_j-1}D_{1_k}X_{t_j-1}
\]
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and

\[
D_{1k} D_{1k} a_j(2) = \partial^2_t \Phi(U_\infty, L_{\infty,t_j-1}) \beta_{t_j-1} (D_{1k} U_\infty)^2 \\
+ 2 \partial_1 \partial_2 \Phi(U_\infty, L_{\infty,t_j-1}) \beta_{t_j-1} (D_{1k} U_\infty) (D_{1k} L_{\infty,t_j-1}) \\
+ 2 \partial_1 \Phi(U_\infty, L_{\infty,t_j-1}) \beta_{t_j-1}' (D_{1k} X_{t_j-1}) (D_{1k} U_\infty) \\
+ \partial_1^2 \Phi(U_\infty, L_{\infty,t_j-1}) \beta_{t_j-1}^2 (D_{1k} D_{1k} U_\infty) \\
+ 2 \partial_2 \Phi(U_\infty, L_{\infty,t_j-1}) \beta_{t_j-1}' (D_{1k} X_{t_j-1}) (D_{1k} L_{\infty,t_j-1}) \\
+ \partial_2^2 \Phi(U_\infty, L_{\infty,t_j-1}) \beta_{t_j-1}'' (D_{1k} X_{t_j-1})^2 \\
+ \Phi(U_\infty, L_{\infty,t_j-1}) \beta_{t_j-1}' D_{1k} D_{1k} X_{t_j-1}.
\]

For Condition [A] (ii), the process \(a(s, 2)\) is corresponding to \(a_j(2)\). Similarly, for Condition [A] (iii), the process

\[
\dot{a}(t, s, 2) = \partial_1 \Phi(U_\infty, L_{\infty,s}) \beta_s D_t U_\infty + \partial_2 \Phi(U_\infty, L_{\infty,s}) \beta_s D_t L_{\infty,s} \\
+ \Phi(U_\infty, L_{\infty,s}) \beta_s^2 D_t X_t
\]

appears. In Condition [A] (iv),

\[
\dot{a}(t, 2) = \partial_1 \Phi(U_\infty, L_{\infty,t}) \beta_s D_t U_\infty + \partial_2 \Phi(U_\infty, L_{\infty,t}) \beta_s^2 D_t L_{\infty,t} \\
+ \Phi(U_\infty, L_{\infty,t}) \beta_s^3 D_t X_t
\]

for \(a_j(2)\). For Condition [A] (v),

\[
\overset{\circ}{\dot{a}}(t, s, 2) = \partial^2_t \Phi(U_\infty, L_{\infty,s}) \beta_s (D_t U_\infty)^2 \\
+ 2 \partial_1 \partial_2 \Phi(U_\infty, L_{\infty,s}) \beta_s (D_t U_\infty) (D_t L_{\infty,s}) \\
+ 2 \partial_1 \Phi(U_\infty, L_{\infty,s}) \beta_s^2 (D_t X_t) (D_t U_\infty) \\
+ \partial_1^2 \Phi(U_\infty, L_{\infty,s}) \beta_s D_t D_t U_\infty \\
+ \partial_2^2 \Phi(U_\infty, L_{\infty,s}) \beta_s (D_t L_{\infty,s})^2 \\
+ 2 \partial_2 \Phi(U_\infty, L_{\infty,s}) \beta_s' (D_t X_{t_j-1}) (D_t L_{\infty,s}) \\
+ \partial_2 \Phi(U_\infty, L_{\infty,s}) \beta_s D_t D_t L_{\infty,s} \\
+ \Phi(U_\infty, L_{\infty,s}) \beta_s'' (D_t X_t)^2 \\
+ \Phi(U_\infty, L_{\infty,s}) \beta_s^3 D_t X_t.
\]

As already noted, these functionals appearing as the limits are well defined and have an explicit expression, respectively. We could go further with involved expressions though we change the way of description below.

To simplify the notation, we will use the process \((\alpha_s)_{s \in \mathbb{R}^+}\) defined by (6.8) on p.30. From (11.46) and (6.5), we have

\[
a_j(2) = \Theta_j \beta_{t_j-1} = \Phi(U_\infty, L_{\infty,t_j-1}) \beta_{t_j-1}.
\]

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Then
\[
\sup_{j \in J} \sup_{s \in I_j} \sup_{t_1, \ldots, t_i \in [0,1]} \left\| D_{t_i} \cdots D_{t_1} a_j(2) - D_{t_i} \cdots D_{t_1} \eta \right\|_p = O(n^{-0.5}) \tag{11.48}
\]
for every \( i \in \mathbb{Z}_+ \) and \( p > 1 \). Thanks to (11.48), we can investigate functionals related with the Malliavin derivatives of \( a_j(2) \).

### 11.2.2 Asymptotic expansion of \( Z_n^0 \)

Now we apply Theorem 4.2 on p.21 to \( Z_n^0 \) of (6.14) in place of \( Z_n \). Condition \([A]\) (i) is routine. Condition \([A]\) (ii) is satisfied for
\[
a(s, 2) = \varphi_s. \tag{11.49}
\]
Condition \([A]\) (iii) holds for
\[
\dot{a}(t, s, 2) = D_t \varphi_s. \tag{11.50}
\]
Condition \([A]\) (iv) holds for
\[
\dot{a}(t, 2) = D_t \varphi_t. \tag{11.51}
\]
Condition \([A]\) (v) holds for
\[
\ddot{a}(t, s, 2) = D_t D_t \varphi_s. \tag{11.52}
\]
Condition \([A]\) (viii) holds for
\[
\dddot{X}_{\infty}(t) = D_t D_t X_1. \tag{11.53}
\]
In the present problem, we have set \( G_{\infty} \) in (6.9):
\[
G_{\infty} = \int_0^1 2 \varphi_t^2 \, dt.
\]
Let
\[
\overline{\Theta}_s = \Phi(U_{\infty}, L_s). \tag{11.54}
\]
Then
\[
\Theta_j - \overline{\Theta}_s = O_{D}(n^{-1}) \tag{11.55}
\]
uniformly in \( s \in I_j \) and \( j \leq n \) since \( |\eta_n - \eta| \leq n^{-1} \) and
\[
L_{\infty, \epsilon_{k-1}} - L_{\infty, r} = O_{D}(n^{-1})
\]
uniformly in \( r \in I_k \) and \( k \leq n \).
We have
\[
\begin{align*}
&n^{-1} \sum_{j=1}^{n} 2a_j(2)^2 - G_{\infty} \\
&= n^{-1} \sum_{j=1}^{n} 2\Theta_j^2 \beta_{t_{j-1}}^2 - \int_{0}^{t} 2a_1^2 \ dt \\
&= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}} 2\Theta_j^2 \beta_{t_{j-1}}^2 \ dt + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}} 2\Theta_j^2 (\beta_{t_{j-1}}^2 - \beta_t^2) \ dt \\
&= O_D(n^{-1}) - \sum_{j=1}^{n} n^{-1} 2\Theta_j^2 (\beta_{t_{j-1}}^2 I_1(g_j)) \\
&- \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}} 2\Theta_j^2 \left( \int_{t_{j-1}}^{t} \left\{ (\beta_{s}^{2})_{1} - (\beta_{s}^{2})_{t_{j-1}}^1 \right\} dw_s + \int_{t_{j-1}}^{t} (\beta_{s}^{2})_{s}^2 \ ds \right) \ dt
\end{align*}
\]
for \( g_j \) defined in (11.1) on 76. We know the exponent of the second term on the right-hand side of the above equation is \(-1\). Therefore
\[
\begin{align*}
n^{-1} \sum_{j=1}^{n} 2a_j(2)^2 - G_{\infty} &= O_D(n^{-1}),
\end{align*}
\]
which shows \([A^\sharp] \ (vii^\sharp)\). \([A^\sharp] \ (II)\) is now assumed. [This condition is not always satisfied. Our setting admits \( \Phi(x,y) = x^{-1} \) (e.g. it is possible to consider when \( \sigma \) is uniformly non-degenerate).] \([D^\sharp] \ (i^\sharp)\) is obvious.

The functional \( N_n^a \) of (6.15) satisfies
\[
\begin{align*}
e(D_i^{(1,n)} N_n^a) &= -0.5 \quad (11.56)
\end{align*}
\]
for \( i = 1 \) due to Proposition 7.1 applied to the expressions (6.15) and (6.3), and it entails the estimate (11.56) for \( i = 2 \). Thus, Conditions (3.25) and (3.26) are fulfilled for \( N_n^a \) since \( \tilde{X}_n = 0 \). Condition (II) of \([D^\sharp] \) has been verified. Related with these estimates, we conclude
\[
\begin{align*}
\mathcal{G}^{(0,1)}(ix) &= 0, \\
\mathcal{G}^{(1,1)}(iz,ix) &= 0
\end{align*}
\]
for \( \tilde{X}_n = 0 \), and
\[
\begin{align*}
\mathcal{G}^{(2,0)}(iz) &= 0
\end{align*}
\]
for \( N_n^a \).
For $\mathcal{G}_n^{(1,0)}(iz) = N_n^o iz$, the integration-by-parts is used in $E[\Psi(z,x)N_n^o]$ written by

$$
E[\Psi(z,x)N_n^o iz] = E \left[ \Psi(z,x) n \sum_{j=1}^{n} (D_{1j}(\Theta_j(b_{j_{i-1}}^{[1]}))^2) I_1(1_j) + \Psi(z,x) n \sum_{j=1}^{n} \Theta_j S_j^{(6,3)} \right] iz
$$

$$
= E \left[ \Psi(z,x) n \sum_{j=1}^{n} (D_{1j}(\Theta_j(b_{j_{i-1}}^{[1]}))^2) I_1(1_j) + \Psi(z,x) n \sum_{j=1}^{n} \Theta_j \left\{ b_{j_{i-1}}^{[1]} b_{j_{i-1}}^{[1,1]} I_3(1_j^{\otimes 3}) + 2h b_{j_{i-1}}^{[1]} b_{j_{i-1}}^{[1,1]} I_1(1_j) 
+ 2h b_{j_{i-1}}^{[1]} b_{j_{i-1}}^{[2]} I_1(1_j) + h^2 b_{j_{i-1}}^{[1]} b_{j_{i-1}}^{[2]} b_{j_{i-1}}^{[1,1]} b_{j_{i-1}}^{[2,1]} 
+ h^2 (b_{j_{i-1}}^{[2,1]})^2 + 2^{-1} h^2 (b_{j_{i-1}}^{[1,1]})^2 - h b_{j_{i-1}}^{[1]} I_1(g_j) - 2^{-1} h^2 b_{j_{i-1}}^{[2,1]} \right\} \right] iz.
$$

(11.57)

Obviously, the term involving the multiple integrals of order three vanishes in the limit by the IBP. Define the following three random symbols:

$$
b_t^{(1)}(iz,ix) = D_t D_t o_t(iz) + 2^{-1}(D_t G_{\infty}) D_t o_t(iz)^3 + (D_t X_{\infty}) D_t o_t(iz)(ix),
$$

$$
b_t^{(2)}(iz,ix) = D_t \hat{E}_t^{(2)}(iz) + 2^{-1}(D_t G_{\infty}) \hat{E}_t^{(2)}(iz)^3 + (D_t X_{\infty}) \hat{E}_t^{(2)}(iz)(ix),
$$

where

$$
\hat{E}_t^{(2)} = \mathcal{G}_t \left\{ 2h b_t^{[1]} b_t^{[1,1]} + 2h b_t^{[1]} b_t^{[2]} - 2^{-1} h b_t^{[1]} \right\}
$$

and

$$
b_t^{(3)}(iz,ix) = \left\{ b_t^{[1]} b_t^{[1,2]} + b_t^{[1]} b_t^{[2,1]} + (b_t^{[2]})^2 + 2^{-1} (b_t^{[1,1]})^2 - 2^{-1} b_t^{[2]} \right\}(iz).
$$

Let

$$
\mathcal{G}^{(1,0)}(iz,ix) = \int_0^1 \sum_{i=1}^{3} b_t^{(i)}(iz,ix) dt.
$$

Then

$$
E[\Psi(z,x)N_n^o iz] \rightarrow E[\Psi(z,x)\mathcal{G}^{(1,0)}(iz,ix)]
$$

as $n \rightarrow \infty$ from (11.57). Finally this proved $[D^t]$ (III).

Applying Theorem 4.2 to $Z_n^o$, we obtain the asymptotic expansion for $(Z_n^o, X_1)$ as

$$
\sup_{f \in \mathcal{E}(M, \gamma)} \left| E[f(Z_n^o, X_1)] - \int_{\mathbb{R}^2} f(z,x) p_n(z,x) dz dx \right| = o(n^{-1/2})
$$

(11.60)

as $n \rightarrow \infty$ for every $(M, \gamma) \in (0, \infty)^2$. Here the density $p_n(z,x)$ is defined by

$$
p_n(z,x) = E \left[ \mathcal{G}_n(\partial_z, \partial_x)^* \left\{ \phi(z,0, G_{\infty}) \delta_x(X_1) \right\} \right]
$$

(11.61)
for

\[ S_n = 1 + n^{-1/2} \mathcal{G} \]

with the random symbol \( \mathcal{G} \) given by

\[ \mathcal{G}(iz, ix) = \mathcal{G}^{(3,0)}(iz, ix) + \mathcal{G}^{(1,1)}(iz, ix) + \mathcal{G}^{(1,0)}(iz, ix) \]

(11.62)

with the components \( \mathcal{G}^{(3,0)}(iz, ix) \) and \( \mathcal{G}^{(1,1)}(iz, ix) \) are generally given by (2.12) and (2.13), respectively. In the present situation they are described by

\[
\mathcal{G}^{(3,0)}(iz, ix) = \frac{4}{3} \int_0^1 \varphi_t^3 dt (iz)^3 + 2 \int_{[0,1]^2} \varphi^{(3,0)}(t, s) ds dt (iz)^3 \\
+ 2 \int_{t=0}^1 \left[ \left( 2^{-1} D_t G_{\infty}(iz)^5 + D_t X_{\infty}(iz)^3(iix) \right) \int_0^1 (D_t a_s) ds \right] a_t dt
\]

(11.63)

where the random field \( \varphi^{(3,0)}(t, s) \) comes from \( a^{(3,0)}(t, s, q_1, q_2) \) defined by (2.11) is at present given by

\[ \varphi^{(3,0)}(t, s) = (D_t D_t a_s) a_s a_t + (D_t a_s)^2 a_t + (D_t a_s) a_s D_t a_t, \]

(11.64)

and by

\[
\mathcal{G}^{(1,1)}(iz, ix) = \int_0^1 D_t D_t X_1 a_t dt (iz)(ix) \\
+ \int_0^1 (D_t X_1) D_t a_t dt (iz)(ix) \\
+ \frac{1}{2} \int_0^1 (D_t G_{\infty})(D_t X_1) a_t dt (iz)^3(iix) \\
+ \int_0^1 (D_t X_1)^2 a_t dt (iz)(ix)^2.
\]

(11.65)

11.2.3 Perturbation method and asymptotic expansion of \( Z_n \)

The variable aimed at is \( Z_n \) having the decomposition (6.13):

\[ Z_n = Z_n^0 + n^{-1/2} N_n^x, \]

We need convergence of the joint law \( (M_n, N_n^x) \) to apply the perturbation method.

Though the following limit theorem is easily generalized to more sophisticated cases, we only consider functionals

\[ X_n = n^{0.5(q_0-1)} \sum_{j=1}^n A_n(j) I_{\phi_0}(i_j^{\otimes \phi_0}) \]

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\[
Y_n = n^{0.5(q-m-|m|)} \sum_{j_1, \ldots, j_{1,m_1}=1}^{n} \cdots \sum_{j_{\nu}, \ldots, j_{\nu,m_{\nu}}=1}^{n} B_n(j_1, \ldots, j_{\nu,m_{\nu}}) \times I_{q_1(1)} \cdots I_{q_1(j_{1,m_1})} \cdots I_{q_{\nu}(1)} \cdots I_{q_{\nu}(j_{\nu,m_{\nu}})}
\]
where \( \nu \in \mathbb{N}, m_1, \ldots, m_{\nu} \geq 1, q_0 \geq 2, 2 \leq q_1 < q_3 < \cdots < q_m, q \cdot m = m_1 q_1 + \cdots + m_{\nu} q_{\nu} \) and \( |m| = m_1 + \cdots + m_{\nu} \).

\(E\) (i) The coefficients \( A_n(j) \) and \( B_n(j_1, \ldots, j_{1,m_1}, \ldots, j_{\nu,1}, \ldots, j_{\nu,m_{\nu}}) \) satisfy \([S]\) on p.32, respectively.

(ii) There exist some continuous random fields \([0, 1] \ni s \mapsto A(s) \in \mathbb{D}^\infty \) and \([0, 1]^{|m|} \ni (t_{1,1}, \ldots, t_{1,m_1}, \ldots, t_{\nu,1}, \ldots, t_{\nu,m_{\nu}}) \mapsto B(t_{1,1}, \ldots, t_{1,m_1}, \ldots, t_{\nu,1}, \ldots, t_{\nu,m_{\nu}}) \in \mathbb{D}^\infty \) such that
\[
\lim_{n \to \infty} \sup_{s \in I_j, j \in J_n} \|A_n(j) - A(s)\|_{k,p} = 0
\]
and
\[
\lim_{n \to \infty} \sup_{s \in I_j, j \in J_n} \|B_n(j_1, m_1, \ldots, j_{\nu,m_{\nu}}) - B(t_{1,1}, \ldots, t_{1,m_1}, \ldots, t_{\nu,1}, \ldots, t_{\nu,m_{\nu}})\|_{k,p} = 0,
\]
for every \( k \in \mathbb{R} \) and \( p > 1 \), where the supremum is taken over
\[
(t_{1,1}, \ldots, t_{1,m_1}, \ldots, t_{\nu,1}, \ldots, t_{\nu,m_{\nu}}) \in I_{j_{1,1}} \times \cdots \times I_{j_{1,m_1}} \times I_{j_{\nu,1}} \times \cdots \times I_{j_{\nu,m_{\nu}}}, \quad j_{1,1}, \ldots, j_{1,m_1}, \ldots, j_{\nu,1}, \ldots, j_{\nu,m_{\nu}} \in J_n.
\]

In the following lemma, \( d_s \) stands for the \( \mathcal{F} \)-stable convergence.

**Lemma 11.1.** Suppose that Condition \([E]\) is satisfied. Then there exist Wiener processes \( W^{(0)}, W^{(1)}, \ldots, W^{(\nu)} \) independent of \( \mathcal{F} \) such that
\[
(X_n, Y_n) \rightarrow_{d_s} (X_\infty, Y_\infty)
\]
as \( n \to \infty \), where
\[
X_\infty = \int_0^1 (q_0!)^{1/2} A(s) dW_s^{(0)}
\]
and
\[
Y_\infty = \int_{[0,1]^{|m|}} (q_1!)^{m_1/2} \cdots (q_{\nu}!)^{m_{\nu}/2} B(t_{1,1}, \ldots, t_{1,m_1}, \ldots, t_{\nu,1}, \ldots, t_{\nu,m_{\nu}}) \cdot dW_t^{(1)} \cdots dW_t^{(1)} \cdots dW_t^{(\nu)} \cdots dW_t^{(\nu)} \quad (11.66)
\]
Moreover, \( W^{(1)}, \ldots, W^{(\nu)} \) are independent. \( W^{(0)} = W^{(q_i)} \) if \( q_i = q_0 \) for some \( i \in \{1, \ldots, \nu\} \); otherwise, \( W^{(0)}, W^{(1)}, \ldots, W^{(\nu)} \) are independent. The integral \( Y_\infty \) includes the diagonal elements.
Proof. $L^p$-estimate of the functional of type $\mathcal{Y}_n$ is discussed in Section 12, and it is observed that $\mathcal{Y}_n$ is near to 0 in $D^\infty$ if the coefficients are uniformly close to 0 in $D^\infty$. This implies that for any $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$\limsup_{n \to \infty} \| \mathcal{Y}_n - \tilde{\mathcal{Y}}_n \|_2 < \epsilon$$

for

$$\tilde{\mathcal{Y}}_n = n^{0.5(qm-|m|)} \sum_{j_1,1,..,j_1,m_1=1}^{n} \cdots \sum_{j_\nu,1,..,j_\nu,m_\nu=1}^{n} B(S(j_1,1,..,j_\nu,m_\nu))$$

$$\times I_{q_1}(1^{\otimes q_1}) \cdots I_{q_\nu}(1^{\otimes q_\nu}) \cdots I_{q_1}(1^{\otimes q_1}) \cdots I_{q_\nu}(1^{\otimes q_\nu})$$

where $S(j_1,1,..,j_\nu,m_\nu)$ is the nearest lower grid point in $\{0/K, ..., (K-1)/K\}^{qm}$ from $(j_1,1,..,j_\nu,m_\nu)/n$. Therefore, by the Bernstein blocks, the problem is reduced to the stable central limit theorem in sub-cubes made with vertices in $\{0/K, ..., (K-1)/K\}^{qm}$. Then the limit of $\tilde{\mathcal{Y}}_n$ becomes an elementary integral of $B$ with respect to the suitably scaled Wiener processes appearing as the stable limit. And it is near to $\mathcal{Y}_\infty$. \qed

In view of (11.45), (11.54), (11.55) and (6.4), let $\mathcal{X}_n$ be the principal part of $M_n$, that is,

$$\mathcal{X}_n = n^{1/2} \sum_{j=1}^{n} \Psi_{t_j-1} \beta_{t_j-1} I_2(1^{\otimes 2})$$

satisfies

$$M_n - \mathcal{X}_n \to^p 0.$$

Recall (6.16):

$$N_n^\mathcal{X} = n \sum_{j,k=1}^{n} \Psi_{j,k}(U_\infty, L_{\infty,t_j-1}) I_j^{(6.4)} I_k^{(6.4)}.$$

Define $\Lambda_{s,t}$ by

$$\Lambda_{s,t} = \partial_1 \Phi(U_\infty, L_{\infty,s}) \beta_s \beta_t + 1_{(s-\lambda)\vee 0 \leq t \leq (s+\lambda)\wedge 1} \eta_{\infty,s}^{-1} \partial_2 \Phi(U_\infty, L_{\infty,s}) \beta_s \beta_t.$$

Then

$$\mathcal{Y}_n := n \sum_{j,k=1}^{n} \Lambda_{t_j-1,t_k-1} I_2(1^{\otimes 2}) I_2(1^{\otimes 2})$$

satisfies

$$N_n^\mathcal{X} - \mathcal{Y}_n \to^p 0$$

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By Lemma 11.1, we obtain
\[
(Z_n^o, \mathcal{Y}_n) = (X_n, \mathcal{Y}_n) + o_p(1)
\]
\[
\rightarrow_{d_s} (X_\infty, \mathcal{Y}_\infty)
\]
\[
= \left( \int_{[0,1]} \sqrt{\zeta} \tilde{\varphi}_s \, dW_s, \int_{[0,1]^2} 2\Lambda_{s,t} \cdot dW_s dW_t \right)
\]
for some standard Wiener process \(W = (W_s)_{s \in [0,1]}\) independent of \(\mathcal{F}\). Remark that
\[
\int_{[0,1]^2} 2\Lambda_{s,t} \cdot dW_s dW_t = \int_0^1 \int_0^t 4\Lambda_{s,t} \, dW_s dW_t + \int_0^1 2\Lambda_{s,t} \, ds
\]
(11.67)
where the first term on the right-hand side of (11.67) is equal to a double Wiener integral and \(\Lambda_{s,t}\) is the symmetrization of \(\Lambda_{s,t}\).

We have
\[
E[Y_\infty|(X_\infty, X_1) = (z, x)] = E \left[ E[Y_\infty|(X_\infty, X_1) = (z, x), \mathcal{F}] \bigg| (X_\infty, X_1) = (z, x) \right]
\]
\[
= E \left[ \mathcal{C}(\omega, z) \bigg| (X_\infty, X_1) = (z, x) \right]
\]
where
\[
\mathcal{C}(z) = \mathcal{C}(\omega, z)
\]
\[
= \int_0^1 \int_0^t 8 G_\infty^{-2} \tilde{\varphi}_s \Lambda_{s,t} \tilde{\varphi}_s d\sigma dt (z^2 - G_\infty) + \int_0^1 2\Lambda_{s,s} \, ds
\]
\[
=: \mathcal{C}_3(z^2 - G_\infty) + \mathcal{C}_1.
\]
See e.g. Yoshida [76] for a formula of the above conditional expectation. In Theorem 3.3, we take \((Z_n, X_1), (Z_n^o, X_1), (N_n^0, 0)\) and \(n^{-1/2}\) for \(S_T, \mathcal{X}_T, \mathcal{Y}_T\) and \(r_T\), respectively. Then the asymptotic expansion of \(Z_n\) is adjusted by the function
\[
g(z, x) = -\partial_{(z,x)} \cdot \left\{ E \left[ \begin{pmatrix} Y_\infty \\ X_1 \end{pmatrix} \bigg| (X_\infty, X_1) = (z, x) \right] p^{(X_\infty, X_1)}(z, x) \right\}
\]
\[
= -\partial_z \left\{ E \left[ \mathcal{C}(\omega, z) \bigg| (X_\infty, X_1) = (z, x) \right] p^{(X_\infty, X_1)}(z, x) \right\}
\]
\[
= -\partial_z \left\{ E \left[ \mathcal{C}(\omega, z) \delta_z(X_\infty) \delta_x(X_1) \right] \right\}
\]
\[
= -\partial_z \left\{ E \left[ \mathcal{C}(\omega, z) \phi(z; 0, G_\infty) \delta_x(X_1) \right] \right\}
\]
\[
= -\partial_z \left\{ E \left[ (\mathcal{C}_3(z^2 - G_\infty) + \mathcal{C}_1) \phi(z; 0, G_\infty) \delta_x(X_1) \right] \right\}
\]
\[
= (-\partial_z)^3 \left\{ E \left[ \mathcal{C}_3 G_\infty^2 \phi(z; 0, G_\infty) \delta_x(X_1) \right] \right\}
\]
\[
- \partial_z \left\{ E \left[ \mathcal{C}_1 \phi(z; 0, G_\infty) \delta_x(X_1) \right] \right\}.
\]
Here we used a suitable representation of random variables on a probability space. Define the random symbol $\mathcal{G}$ by

$$\mathcal{G}(iz) = \mathcal{G}(iz, ix) = C_3 G_\infty^2(iz)^3 + C_1(iz).$$

Consequently,

$$\mathfrak{g}(z, x) = E[\mathcal{G}(\partial_z, \partial_x)\{\phi(z; 0, G_\infty)\delta_x(X_1)\}].$$

and the asymptotic expansion of $\mathcal{L}\{(Z_n, X_1)\}$ is given by the formula

$$\varphi_n(z, x) = p_n(z, x) + n^{-1/2} \mathfrak{g}(z, x).$$

Equivalently, the asymptotic expansion of $\mathcal{L}\{(Z_n, X_1)\}$ is given by the random symbol

$$\mathfrak{A}_n = 1 + n^{-1/2} \mathfrak{A}$$

where $\mathfrak{A} = \mathcal{G} + \mathcal{G}$. This completes the proof of Theorem 6.2.

## 12 Estimate of a multilinear form of multiple Wiener integrals

### 12.1 Basic estimates

Let $\mathcal{W} = \{\mathcal{W}(h)\}_{h \in \mathcal{H}}$ be a Gaussian process on a real separable Hilbert space $\mathcal{H}$. Let $q = (q_1, ..., q_m) \in \mathbb{N}^m$, for $m \in \mathbb{N} = \{1, 2, ...\}$. We will consider a sum of multilinear forms of multiple Wiener integrals

$$\mathbb{I}_n = \sum_{j_1, ..., j_m=1}^n A_n(j_1, ..., j_m) I_{q_1}(f_{j_1}^{(1)}) \cdots I_{q_m}(f_{j_m}^{(m)})$$

(12.1)

for $f_{j_i}^{(i)} \in \mathcal{H}^{\otimes n}$, the $q_i$-th symmetric tensors, $i = 1, ..., m$. The random coefficients $A_n(j_1, ..., j_m)$ can be anticipative. We already met many such functionals in the previous sections. An example of $\mathbb{I}_n$ is

$$\mathbb{I}_n = \sum_{j_1, ..., j_m=1}^n \sum_{k=1}^n \sum_{l=1}^n \left( (D_s D_t a_{t_{j_{k-1}}} a_{t_{l_{k-1}}} l_k(s)) \right)$$

$$\times ((B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1})) (B_t - B_{t_{k-1}}) (B_s - B_{t_{l-1}})$$

for $s, t \in [0, T]$, a fractional Brownian motion $B = (B_t)$ dwelling in $\mathcal{W}$ and $(a_r) \subset \mathbb{D}^\infty$. The monomial $\mathbb{I}_n^{2k}$ is also an example of $\mathbb{I}_n$, $k \in \mathbb{N}$. The coefficient $A_n(j_1, ..., j_m)$ can have the factor $\Psi_\infty$ appearing in the context of the theory of asymptotic expansion. To introduce the notion of exponent, an $L^p$-estimate for $\mathbb{I}_n$ and the generated algebra played an essential role. This section will give an $L^p$-estimate of $\mathbb{I}_n$.

Suppose that Denote $\bar{q} = \bar{q}_1 = \sum_{i=1}^m q_i$ and $\bar{r} = \bar{r}_1 = r_1 + \cdots + r_{m-2} + r_{m-1}$. Suppose that

$$A_n(j_1, ..., j_m) \in \mathbb{D}^{q_2}.$$  

(12.2)

The symmetrized $r$-contraction of a tensor product is denoted by $\otimes_r$. for $(r_1, ..., r_{m-1}) \in \mathbb{Z}_+^{m-1}$. We have the following estimate for $\mathbb{I}_n$ defined by (12.1).
Proposition 12.1. Suppose that Condition (12.2) is satisfied. Then
\[
|E[\mathbb{I}_n]| \leq \sum_{j_1, \ldots, j_m=1}^{n} |E[\mathbb{I}_{j_1}]| \leq C(q_1, \ldots, q_m) \sum_{r_1, \ldots, r_{m-1}}^{n} \sum_{j_m=1}^{n} \left| E\left[ \langle D_{\bar{q}_1}^{2r_1} A_n(j_1, \ldots, j_m), f_{j_1}^{(1)} \tilde{\otimes} r_1 \cdots \tilde{\otimes} r_{m-1} f_{j_m}^{(m)} \rangle_{\mathcal{G}^{\otimes (\bar{q}_1-2r_1)}} \right] \right|
\]
for some constant \(C(q_1, \ldots, q_m)\) depending only on \(q_1, \ldots, q_m\).

Proof. Repeatedly apply the product formula to \(I_{q_1}(f_{j_1}^{(1)}) \cdots I_{q_m}(f_{j_m}^{(m)})\) to reach a linear combination of Skorohod integrals of order \(\bar{q}_1 - 2r_1\). The sum \(\sum_{r_1, \ldots, r_{m-1}}\) is finite. Next use duality for \(\delta_{\bar{q}_1 - 2r_1}\).

Let
\[
\mathbb{I}_n(j_1, \ldots, j_m) = A_n(j_1, \ldots, j_m) I_{q_1}(f_{j_1}^{(1)}) \cdots I_{q_m}(f_{j_m}^{(m)})
\]

Corollary 12.2. Suppose that Condition (12.2) is satisfied. Then
\[
|E[\mathbb{I}_n(j_1, \ldots, j_m)]| \leq C(q_1, \ldots, q_m) \times \sum_{r_1, \ldots, r_{m-1}} \left| E\left[ \langle D_{\bar{q}_1}^{2r_1} A_n(j_1, \ldots, j_m), f_{j_1}^{(1)} \tilde{\otimes} r_1 \cdots \tilde{\otimes} r_{m-1} f_{j_m}^{(m)} \rangle_{\mathcal{G}^{\otimes (\bar{q}_1-2r_1)}} \right] \right|
\]
for some constant \(C(q_1, \ldots, q_m)\) depending only on \(q_1, \ldots, q_m\).

12.2 \(L^p\)-estimate for a multilinear form of multiple Wiener integrals for discretization

Consider a double sequence of numbers \(t_j = t_j^n (i = 1, \ldots, n, n \in \mathbb{N})\) such that
\[
0 = t_0 < t_1 < \cdots < t_n, \quad \max_{j=0,1,\ldots,n} (t_j - t_{j-1}) \leq T/n \quad (12.3)
\]
for a constant \(T > 0\). For example, \((t_j)\) is a double sequence such that \(0 = t_0 < t_1 < \cdots < t_n \leq T_0\) and that \(\max_{j=1,\ldots,n} (t_j - t_{j-1}) \leq 2T_0/n\) for \(T = 2T_0\).

Let \(I_j = I_j^n = 1_{(t_{j-1}, t_j)}\). From now on we let \(\mathcal{F}_j = L^2([0, T], dt)\). We will consider a double sequence \(f_{j_1}^{(i)} = f_{j_1}^{(i),n} \in \mathcal{F}_j^{\otimes q_j}\) such that \(\text{supp}(f_{j_1}^{(i)}) \subset I_j^n\) and that
\[
\max_{i=1,\ldots,m} \sup_{n \in \mathbb{N}} \sup_{j_1,\ldots,j_m=1} \sup_{t_1,\ldots,t_q \in [0,T]} |f_{j_1}^{(i)}(t_1, \ldots, t_q)| \leq C_1 \quad (12.4)
\]
for a constant \(C_1\). Let \(k \in \mathbb{N}\). Suppose that \(A_n(j_1, \ldots, j_m) \in D^{2k\bar{q},4k}\) (the derivatives up to 2k-th order are in \(L^{4k}\) and
\[
\max_{i=0,1,\ldots,2k} \sup_{n \in \mathbb{N}} \sup_{j_1,\ldots,j_m=1} \sup_{t_1,\ldots,t_q \in [0,T]} \|D_{t_1,\ldots,t_q} A_n(j_1, \ldots, j_m)\|_{4k} \leq C_0 \quad (12.5)
\]
for some constant $C_0$.

In the following theorem, we apply the duality of the Skorohod integrals, as it was used in Proposition 12.1 and Corollary 12.2, to

$$\tilde{I}_n(j) = A_n(j_{1,1}, \ldots, j_{1,m})A_n(j_{2,1}, \ldots, j_{2,m}) \cdots A_n(j_{2k,1}, \ldots, j_{2k,m}) \times I_{q_1}(f_{j_{1,1}}^{(1)}) \cdots I_{q_m}(f_{j_{1,m}}^{(m)})I_{q_1}(f_{j_{2,m}}^{(1)}) \cdots I_{q_m}(f_{j_{2,k,1}}^{(1)}) \cdots I_{q_m}(f_{j_{2k,m}}^{(m)})$$

for $j = (j_{1,1}, \ldots, j_{1,m}, \ldots, j_{2k,1}, \ldots, j_{2k,m}) \in \{1, \ldots, n\}^{2km}$. The exponent introduced in Section 7.1 is based on the following estimate.

**Theorem 12.3.** There exists a constant $C_2 = C(q_1, \ldots, q_m, k, T, C_0, C_1)$ such that

$$\sum_{j \in \{1, \ldots, n\}^{2km}} \sup^* |E[\tilde{I}_n(j)]| \leq C_2^{2k} n^{-k(q-m)}$$

for all $n \in \mathbb{N}$, where $\sup^*$ is the supremum taken over all elements $f^*_*$ and $A_n(*)$ satisfying (12.4) and (12.4), respectively. In particular,

$$\|\tilde{I}_n\|_{2k} \leq C_2 n^{-(q-m)}$$

for all $n \in \mathbb{N}$. More precisely, one can take

$$C_2 = C(q_1, \ldots, q_m, k, T)C_0C_1^m$$

for some constant $C(q_1, \ldots, q_m, k, T)$.

**Proof.** Let $\tilde{Q}_1 = 2k\bar{q}$. We will estimate the sum

$$S(R) := \sum_j \sup^* |E_n(j, R)|$$

for

$$E_n(j, R) = E\left[\left\langle D^{\tilde{Q}_1-2R_1}B_n(j), \Phi(j, R)\right\rangle_{j^R(\tilde{Q}_1-2R_1)}\right],$$

where

$$B_n(j) = A_n(j_{1,1}, \ldots, j_{1,m})A_n(j_{2,1}, \ldots, j_{2,m}) \cdots A_n(j_{2k,1}, \ldots, j_{2k,m}),$$

$$R = (R_{1,1}, \ldots, R_{1,m}, R_{2,1}, \ldots, R_{2,m}, \ldots, R_{2k,1}, \ldots, R_{2k,m-1})$$

$$\Phi(j, R) = f_{j_{1,1}}^{(1)} \otimes R_{1,1} \cdots \otimes R_{1,m-1} f_{j_{1,m}}^{(m)} \otimes R_{1,m} f_{j_{2,1}}^{(1)} \otimes R_{2,1} \cdots \otimes R_{2,m-1} f_{j_{2,m}}^{(m)} \otimes R_{2,m} \cdots$$

$$\otimes R_{2k-1,m} f_{j_{2k-1,1}}^{(1)} \otimes R_{2k-1,1} \cdots \otimes R_{2k,m-1} f_{j_{2k,m}}^{(m)}$$

and

$$\tilde{R}_1 = R_{1,1} + \cdots + R_{1,m} + R_{2,1} + \cdots + R_{2,m} + \cdots + R_{2k,1} + \cdots + R_{2k,m-1}. $$

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Remark that each nonnegative integer $R_{a,b}$ is bounded by a suitable number so that each contraction makes sense. Each contraction has implicitly chosen a set of arguments in the factor in front of it and the all factors after it. Later this set will vary between all possible cases, but we fix an $R$ for the meantime.

We call the factor $f_{j_i}^{(i)}$ is single if all its arguments are free from any contraction specifically allocated according to $R$. A single factor $f_{j_i}^{(i)}$ has $q_i$ arguments $(t_1, \ldots, t_{q_i})$, and this factor contributions to $S$ by a factor of order $O(n^{-(q_i-1)})$ after summation $\sum_{j_i=1}^n |\mathbb{E}_n(j,R)|$, due to the specific support of $f_{j_i}^{(i)}$.

We consider connection between two $f_s^{(s)}$s by a contraction. In a connected component $C$ (having at least two $f_s^{(s)}$s), suppose that $r (\geq 1)$ contractions exist. Let $q$ be the sum of the numbers of arguments of $m$ functions, $f_{j_1}^{(i_1)}, \ldots, f_{j_m}^{(i_m)}$ say, consisting of $C$. We have $q = q_1 + \cdots + q_m$ since each $f_{j_i}^{(i)}$ has $q_i$ arguments. Connected two arguments of (different) $f_s^{(s)}$s in $C$ give a factor of order $O(n^{-1})$ (before summing up with respect to involving $j$‘s). So, $2r$ arguments give totally a factor of order $O(n^{-r})$. On the other hand, the rest of the arguments appearing in $C$ gives a factor of order $O(n^{-(q-2r)})$. The multiple sum with respect to $j$‘s in $C$ becomes a single sum by the orthogonality, because $C$ is connected. Thus, after summation with respect $j_1 = \cdots = j_m$, the component $C$ contributes to $S(R)$ with a factor of the order

$$n \times n^{-(q-2r)} \times n^{-r} = n^{1+r-m-\sum_{i=1}^{m}(q_i-1)} \geq n^{-\sum_{i=1}^{m}(q_i-1)}.$$ (12.6)

Here we used the inequality $1 + r \geq m$. Inequality (12.6) suggests we should consider the allocations of contractions that do not involve any single $f_s^{(s)}$, in order to find the dominating term.

We consider an allocation of contractions according to $R$ that has $c$ connected components with $c \geq 1$ and has no single $f_s^{(s)}$. Then

$$S(R) = O(n^{c+R_1-2km-(Q_1-2km)}) = O(n^{c+R_1-Q_1}).$$

Since the contraction takes place between different two functions $f_s^{(s)}$, each connected component has at least two functions $f_s^{(s)}$, therefore, the maximum $c$ is $km$. The maximum $R_1$ is $kq$. (Indeed, there is a pattern of contractions for which $c = km$ and $R_1 = kq$.) Consequently,

$$S(R) = O(n^{km+kq-Q_1}) = O(n^{-k(q-m)})$$

for any pattern of allocation of contractions. The existence of a constant $C(q_1, \ldots, q_m, k, T)$ is obvious if one observes the above argument precisely. This completes the proof.

12.3 $L^p$-estimate of an integrated functional on a compact group

Let $m \in \mathbb{N}$ and $q_1, \ldots, q_m \in \mathbb{N}$. For each $i \in \{1, \ldots, m\}$, let $G^{(i)}$ be a compact group with a Haar measure $\mu^{(i)}$. Let $G = G_1 \times \cdots G_m$ and $\mu = \mu_1 \times \cdots \times \mu_m$. Let $e^{(i)}_{n,j} : G^{(i)} \to \tilde{S}^{\tilde{q}_n}$ for $j \in J_n = \{1, \ldots, n\}$, $n \in \mathbb{N}$ and $i = 1, \ldots, m$. Suppose that $A_n(j_1, \ldots, j_m; x)$ is a random variable for each $(j_1, \ldots, j_m) \in J_n^m$ and $x = (x^{(1)}, \ldots, x^{(m)}) \in G$. We assume that the mappings
\( A_n(j_1, ..., j_m; \cdot) \) and \( I_{q_i}(e^{(i)}_{j_i}(\cdot)) \) are measurable random fields on \( G \) and \( G^{(i)} \), respectively, and

\[
\int_{G} \left| A_n(j_1, ..., j_m; x) \prod_{i=1}^{m} I_{q_i}(e^{(i)}_{j_i}(x^{(i)})) \right| \mu(dx) < \infty \text{ a.s.}
\]

Let

\[
\mathbb{h}_n(x) = \sum_{j_1, ..., j_m = 1}^{n} A_n(j_1, ..., j_m; x) I_{q_i}(e^{(i)}_{j_i}(x^{(i)})) \cdots I_{q_m}(e^{(m)}_{j_m}(x^{(m)})).
\]  \( \text{(12.7)} \)

We are interested in the \( L^p \)-estimate of the integral

\[
\mathbb{H}_n = \int_{G} \mathbb{h}_n(x) \mu(dx).
\]

Assume the localization condition that for every \( j_i \in J_n \),

\[
e^{(i)}_{n,j_i}(x^{(i)}) = e^{(i)}_{n,j_i}(x^{(i)}) 1_{\chi^{(i)}}(L^{(i)}_{g^{(i)}_{n,j_i}} x^{(i)}) \quad (x^{(i)} \in G^{(i)})
\]  \( \text{(12.8)} \)

for some \( g^{(i)}_{n,j_i} \in G^{(i)} \) and some measurable set \( \chi^{(i)}_n \) in \( G^{(i)} \), where \( L^{(i)}_{g^{(i)}} \) is the left transformation.

[For example, when \( G^{(i)} = \mathbb{R}/\mathbb{Z}, \chi^{(i)}_n = [0, c/n] \), a fixed interval of bigger than the maximum length of \( (t_{j_i-1}, t_{j_i}) \). \( e^{(i)}_{n,j_i}(x^{(i)}) \) can still include an indicator function.]

Then

\[
\mathbb{H}_n = \int_{G} \mathbb{h}_n(x) \mu(dx)
\]

\[
= \int_{G} \int_{G_1} \cdots \int_{G_m} \sum_{j_1, ..., j_m = 1}^{n} A_n(j_1, ..., j_m; x) \prod_{i=1}^{m} I_{q_i}(e^{(i)}_{n,j_i}(x^{(i)})) \mu^{(1)}(dx^{(1)}) \cdots \mu^{(m)}(dx^{(m)})
\]

\[
= \int_{G} \mathbb{h}_n(x) \prod_{i=1}^{m} 1_{\chi^{(i)}_n}(x^{(i)}) \mu(dx)
\]

where

\[
\mathbb{h}_n(x) = \sum_{j_1, ..., j_m = 1}^{n} A_n(j_1, ..., j_m; (L^{(i)}_{g^{(i)}_{n,j_i}-1} x^{(i)})^{m}) \prod_{i=1}^{m} I_{q_i}(e^{(i)}_{j_i}(L^{(i)}_{g^{(i)}_{n,j_i}-1} x^{(i)})).
\]  \( \text{(12.9)} \)

Therefore,

**Theorem 12.4.** Suppose that Condition \( (12.8) \) is satisfied. Then

\[
\mathbb{H}_n = \int_{G} \mathbb{h}_n(x) \prod_{i=1}^{m} 1_{\chi^{(i)}_n}(x^{(i)}) \mu(dx)
\]  \( \text{(12.10)} \)
and

$$
\|H_n\|_p \leq \sup_{x \in G} \|\hat{g}_n(x)\|_p \mu \left( \prod_{i=1}^{m} \chi^{(i)}_n \right)
$$

(12.11)

for $p \geq 1$.

Here is an example of applications of Theorem 12.3. Let $H = L^2([0,1], dt)$. A functional that appeared in Nualart and Yoshida [48] on p.32 is

$$
\mathcal{I}_1 = \int_0^1 \int_0^1 \sqrt{n} \sum_{j=1}^n (D_s D_t a_{t_j}^{(k)}) q_j
$$

$$
\times \sqrt{n} \sum_{k=1}^n a_{t_k-1} (B_t - B_{t_k-1}) 1_k(t) dt
$$

$$
\times \sqrt{n} \sum_{\ell=1}^n a_{t_{\ell}-1} (B_s - B_{t_{\ell}-1}) 1_{\ell}(s) ds,
$$

where $t_j = j/n$, $1_j = 1_{(t_j-t_{j-1})}$, $q_j = (B_{t_j} - B_{t_{j-1}})^2 - n^{-1}$, and $a_t = a(B_t)$ with $a \in C^\infty(\mathbb{R})$ is of moderate growth and $B = (B_t)$ is a Brownian motion.

Let $G = \{0\} \times \mathbb{T} \times \mathbb{T}$ for $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The element $t \in \mathbb{T}$ is naturally embedded into $[0,1]$, so $B_t$ makes sense for $t \in \mathbb{T}$. For (12.7), let

$$
\hat{H}_n(x^{(1)}, x^{(2)}, x^{(3)}) = n^{1.5} \sum_{j,k,\ell=1}^n (D^{(1)}_{x^{(1)}} D^{(2)}_{x^{(2)}} a_{t_j}^{(k)}) a_{t_k-1} e^{(1)}_{n,j}(x^{(1)}) e^{(2)}_{n,k}(x^{(2)}) e^{(3)}_{n,\ell}(x^{(3)})
$$

with

$$
e^{(1)}_{n,j}(x^{(1)}) = I_2(1_{(0)}^{(2)})(x^{(1)} \equiv 0),
$$

$$
e^{(2)}_{n,k}(x^{(2)}) = I_1(1_{(t_{j-1},x^{(2)})}^{(2)})
$$

$$
e^{(3)}_{n,\ell}(x^{(3)}) = I_1(1_{(t_{\ell-1},x^{(3)})}^{(3)}).
$$

Then

$$
\mathcal{I}_1 = \mathbb{H}_n = \int_G \hat{H}_n(x^{(1)}, x^{(2)}, x^{(3)}) d\mu(x^{(1)} dx^{(2)} dx^{(3)}).
$$

For $(\chi^{(1)}, g^{(1)}_{n,j}) = (\{0\}, 0)$, $(\chi^{(2)}, g^{(2)}_{n,k}) = ([0,1/n], -t_{k-1})$ and $(\chi^{(3)}, g^{(3)}_{n,\ell}) = ([0,1/n], -t_{\ell-1})$, Condition (12.8) is satisfied because

$$
e^{(1)}_{n,j}(t) = I_2(1_{(0)}^{(2)})(L_0 x^{(1)})(x^{(1)} \equiv 0),
$$

$$
e^{(2)}_{n,k}(x^{(2)}) = I_1(1_{(t_{j-1},x^{(2)})}^{(2)})(1_k(x^{(2)}))
$$

$$
= I_1(1_{(t_{j-1},x^{(2)})}^{(2)})(1_{[0,1/n]}(x^{(2)} - t_{k-1})
$$

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and

$$
\epsilon^{(3)}_{n,\ell}(x^{(3)}) = I_1(1\ell(x^{(3)})1_{(t_{\ell-1},x^{(3)})}(\cdot))1_{[0,1/n]}(x^{(3)} - t_{\ell-1}).
$$

Now

$$
\|h_n(x^{(1)}, x^{(2)}, x^{(3)})\| = n^{3/2} \sum_{j,k,\ell=1}^{n}(D_{t_{\ell-1}+x^{(3)}}D_{t_{k-1}+x^{(2)}}a_{t_{k-1}}a_{t_{\ell-1}}
\times I_2(1_{j}) I_1(1_{(t_{k-1},t_{k-1}+x^{(2)})}1_{(x^{(2)})}) I_1(1_{(t_{\ell-1},t_{\ell-1}+x^{(3)})}1_{(x^{(3)})})).
$$

Then we obtain $I_1 = O_{L^\infty}(n^{-1})$ as $n \to \infty$ from (12.11) since $\mu_1(\prod_{i=1}^{n}x_i^{(i)}) = O(n^{-2})$ and $\|h_n(x)\| = O_{L^\infty}(n)$ uniformly in $x \in G$ due to the exponent $e(\|h_n(x^{(1)}, x^{(2)}, x^{(3)})\|) = 1$ by Theorem 12.3. The order $O(n^{-1})$ is different from the order $E[\Psi(z,x)\|I_1] = O(n^{-1.5})$, however these bounds do not conflict with each other because the bound obtained here is an $L^p$-bound. As for $L^p$-estimate of the derivative of the quasi-torsion, $\|D_{u_n}D_{u_n}M_n\|_p = o(r_n)$ follows from this estimate in the martingale expansion approach.

### 12.4 A bound for a polynomial of multiple Wiener integrals

Let $p = (p_1, ..., p_m) \in \mathbb{N}^m$ and $q = (q_1, ..., q_m) \in \mathbb{N}^m$. We will assume (12.4) and Condition that

$$
\max_{i=0,1,...,s} \sup_{n \in \mathbb{N}} \sup_{n \in \mathbb{N}} \sup_{n} \|D_{t_{i_1},...,t_{i_s}}A_n(j_1, ..., j_m)\|_4 \leq C_2 \quad (12.12)
$$

for $s = 2k \sum_{i=1}^{m} p_i q_i$ and some constant $C_2$. In this section, we will consider the functional

$$
\kappa_n = \sum_{j_1, ..., j_m=1}^{n} A_n(j_1, ..., j_m) I_{q_1}(f_{j_1}^{(1)})^{p_1} \cdots I_{q_m}(f_{j_m}^{(m)})^{p_m}.
$$

**Theorem 12.5.** Suppose that Conditions (12.4) and (12.12) are fulfilled. Then there exists a constant $C$ depending on $(C_0, C_2, p, q)$ such that

$$
\|\kappa_n\|_{2k} \leq C n^{-\frac{1}{2} - \xi}
$$

for

$$
\xi = p \cdot q - m - \# \{ i \in \{1, ..., m\}; p_i \geq 2 \text{ and } p_i q_i \text{ is even} \}.
$$

**Proof.** Without loss of generality, we may assume with $p_1, ..., p_{m-\ell} \geq 2$ and $p_{m-\ell+1} = \cdots = p_m = 1$; in particular, all $P_1, ..., P_3 \geq 2$ when $\ell = 0$. By the product formula applied to each factor $I_{q_i}(f_{j_i}^{(i)})^{p_i}$, we obtain

$$
\kappa_n = \sum_{n_{1}, ..., n_{m} \in \mathbb{Z}^+} \left\{ \prod_{i=1}^{\ell} g_{p_i}^{v_i}[q_i, p_i] \right\}
\times \sum_{j_1, ..., j_m=1}^{n} n^{-\sum_{i=1}^{\ell} v_i} A_n(j_1, ..., j_m) \left( \prod_{i=1}^{\ell} I_{p_i q_i - 2 v_i} \left( n^{p_i} \tilde{f}_{j_i}^{(i)} \right) \right) \left( \prod_{i=\ell+1}^{m} I_{q_i}(f_{j_i}^{(i)}) \right).
$$

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The desired estimate follows from the product formula and the estimate (7.3) in Section 7.1 based on Theorem 12.3. Indeed, the exponent
\[
e(K_n) = \max_{(\nu_1, \ldots, \nu_m) \in \mathbb{Z}_+^m} \left\{ -\sum_{i=1}^\ell \nu_i - 0.5 \left( \sum_{i=1}^\ell (p_i q_i - 2\nu_i) + \sum_{i=\ell+1}^m p_i q_i \right) + m \right. \\
\left. - 0.5 \left( \# \{ i \in \{1, \ldots, \ell\}; p_i q_i - 2\nu_i > 0 \} + (m - \ell) \right) \right\} \\
= -0.5 \mathbf{p} \cdot \mathbf{q} + 0.5m + 0.5 \# \{ i \in \{1, \ldots, \ell\}; p_i q_i \text{ is even} \}.
\]

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