Elementary excitations and the phase transition in the bimodal Ising spin glass model

N Jinuntuya\(^1\) and J Poulter\(^2\)

\(^1\) Department of Physics, Faculty of Science, Mahidol University, Rama 6 Road, Bangkok 10400, Thailand
\(^2\) Department of Mathematics, Faculty of Science, Mahidol University, Rama 6 Road, Bangkok 10400, Thailand
E-mail: fscinpr@ku.ac.th (N Jinuntuya) and scjpt@mahidol.ac.th

Received 23 October 2011
Accepted 14 December 2011
Published 16 January 2012

Abstract. We show how the nature of the phase transition in the two-dimensional bimodal Ising spin glass model can be understood in terms of elementary excitations. Although the energy gap with the ground state is expected to be 4\(J\) in the ferromagnetic phase, a gap 2\(J\) is found if the finite lattice is wound around a cylinder of odd circumference \(L\). This 2\(J\) gap is really a finite size effect that should not occur in the thermodynamic limit of the ferromagnet. The spatial influence of the frustration must be limited and not wrap around the system if \(L\) is large enough. In essence, the absence of 2\(J\) excitations defines the ferromagnetic phase without recourse to calculating the magnetization or investigating the system response to domain wall defects. This study directly investigates the response to temperature. We also estimate the defect concentration where the phase transition to the spin glass state occurs. The value \(p_c = 0.1045(11)\) is in reasonable agreement with the literature.

Keywords: classical phase transitions (theory), spin glasses (theory), extreme value statistics

ArXiv ePrint: 1109.3548
1. Introduction

Spin glasses [1]–[4] have attracted much interest for quite a while. Due to the considerable complexity of real materials much computational effort has gone into studies of a simplified model [5] that is nevertheless thought to include the essential ingredients that lead to spin glass behaviour. Since even this model is not trivial, a considerable industry has developed over time devoted to particular models that, although probably unphysical, have provided subjects for the development of numerical techniques [6, 7].

Systems known as spin glasses are disordered magnetic systems characterized by a random mixture of ferromagnetic and antiferromagnetic exchange interactions leading to frustration [8]. Typically, at low temperatures below a critical temperature $T_c$, a system undergoes a phase transition from a ferromagnet to a spin glass at some critical concentration $p_c$ of antiferromagnetic interactions.

The model studied in this work is the bimodal, or $\pm J$, Ising spin glass in two dimensions. This system has quenched bond (short range, nearest-neighbour) interactions of fixed magnitude $J$ but random sign. The Hamiltonian is

$$H = -\sum_{\langle ij \rangle} J_{ij} s_i s_j,$$

where the sum is over all nearest-neighbour spins. The spins $s_i$ are Ising spins, $s_i = \pm 1$. The distribution of the random interaction $J_{ij}$ is

$$P(J_{ij}) = p\delta(J_{ij} + J) + (1 - p)\delta(J_{ij} - J).$$

The concentration $p$ of negative, or antiferromagnetic, bonds is varied from zero up the canonical spin glass at $p = 0.5$. It is believed that the spin glass can only exist at zero temperature [9] where $p > p_c$ with [10, 11] $p_c = 0.103$. This is clearly below the concentration $p_n = 0.109$ at the (finite temperature) Nishimori point, indicating a re-entrant phase transition as confirmed by Monte Carlo work [12].

The ground state is highly degenerate with an entropy per spin [13, 14] of $0.07k$. Consequently spin correlation functions are not guaranteed to take values $\pm 1.0$. If a
nearest-neighbour bond correlation function does have a value $\pm 1.0$ then we call that bond a rigid bond \cite{15}. This means that the spin alignment across the bond is the same in all ground state configurations. A recent study \cite{16} suggests that the rigid lattice does not percolate in the spin glass phase. This is consistent with the idea that the ferromagnetic phase is characterized by percolation of rigid bonds.

Droplet theory \cite{17–21} has enjoyed much success with regard to understanding the spin glass phase. The essential idea is that reversing all the spins in a compact cluster with respect to a ground state provides a low energy excitation. Typical droplet excitations dominate the thermodynamic behaviour. A closely related idea is that of the domain wall defect \cite{22,23}—essentially a droplet perimeter that extends to infinity. With a continuous distribution of disorder these related views seem to be equivalent \cite{24} according to the predictions of droplet theory.

For the bimodal model, domain wall defects have, for example, been applied \cite{11,25} to the determination of the value of the critical defect concentration $p_c$. Nevertheless, it still remains unclear whether droplet theory is appropriate \cite{26}. The ground state is not unique and a droplet may represent some different ground state—not an excitation.

For this study the $L \times L$ square lattice is wound around a cylinder, that is we use periodic boundary conditions in one dimension. In the second dimension the system is nested in an infinite unfrustrated environment. There are no open boundaries. If the circumference $L$ of the cylinder is even, then the energy gap is $4J$. Otherwise it is $2J$. In the spin glass phase the distribution of degeneracies of the first excited state is extreme, with a long tail representing large values \cite{27,28}. We have also looked at systems with open boundaries and have found extreme distributions of $2J$ excitations in some agreement with Wang \cite{29}.

The issue of the size of the energy gap of the bimodal Ising spin glass dates back to the proposal of Wang and Swendsen \cite{30} that it should be $2J$ in the thermodynamic limit. It now seems clear that in fact there is no energy gap at all and the low temperature specific heat varies as a power law $c_v \sim T^{-\alpha}$. The first indications of this appeared in \cite{31} and confirmation from the evaluation of very large Pfaffians has recently appeared \cite{32}.

The issue that remains unclear is the value of the critical exponent $\alpha$. For the case of continuous (Gaussian) disorder, direct calculations \cite{33,34} report that the specific heat is linear with $\alpha = -1.0$. For bimodal disorder, Monte Carlo work \cite{35} reports that $\alpha = -4.21$, while droplet theory \cite{32} suggests that $\alpha = -3.0$ although the temperature range used is extremely narrow. Other Monte Carlo results \cite{31,35} for the correlation length with the assumption of hyperscaling give $\alpha = -7.1$. Universality is hard to prove.

The exponent $\alpha$ is difficult to estimate. One feature that makes this so for the bimodal case is that the specific heat is not normally distributed. We have performed some calculations with open boundaries and find that the distribution of the specific heat has a tail for low temperature and small values of linear sample size $L$. The methods used were the direct evaluation of Pfaffians as well as summing the density of states \cite{36}. Although it is reasonable to believe that the specific heat will be normally distributed in the thermodynamic limit, it is not clear what value to use from calculations with finite $L$.

It is at least clear now that the low temperature specific heat contains contributions from excitations having a range of energies. This fits well with droplet theory \cite{32} where it is predicted that $\alpha = 1 - 2/\theta_S$ with the fractal dimension of domain walls given by $d_f = 2\theta_S$. If $\theta_S = 0.5$ as reported \cite{32,36,37}, then $\alpha = -3.0$. However, other works \cite{38–41}
Elementary excitations and the phase transition

Figure 1. An example of a $2J$ excitation. On the left is a ground state configuration with six frustrated plaquettes and five unsatisfied (jagged) bonds. On the right is a first excited state obtained by flipping all spins on one side of the vertical broken line. The excited state has six unsatisfied bonds. Periodic BCs are indicated by the top and bottom dashed vertical lines.

predict values $d_l > 1$ that imply $\alpha > -3.0$. It seems unlikely that droplet theory can predict a value in agreement with $\alpha = -4.21$ or $-7.1$.

To obtain a simple description of the ferromagnetic phase we can start with the case of low defect concentration $p$. The defect bonds are widely separated and the ground state is unique (aside from global inversion). So the degeneracy of the ground state is $M_0 = 1$. We can find first excited states by flipping a spin at either end of a defect bond. Thus the degeneracy of the first excited state is $M_1 = 4pN$ where the square lattice has $N$ sites and $2N$ bonds. The value of the density of states $M_1/M_0$ per spin is $4p$. We can think of clusters of disorder each composed of one negative bond and two frustrated plaquettes.

As the concentration $p < p_c$ increases, the clusters of disorder grow in size and influence. The distribution of the density of states becomes less normal and its peak moves above $4p$. Nevertheless, the rigid lattice still percolates and there remains some finite magnetization. With a lattice of finite size, wound in one direction, the $4J$ excitations occur in two classes. Some are derived locally and are not influenced by the boundary condition—just like for the simple case of low concentration. Others are formed by extending all the way around the system.

Since it is not easy to distinguish between these two classes, we employ the device of fixing an odd value of the circumference $L$ of the cylinder. In this case the $2J$ excitations are entirely nonlocal. Figure 1 shows an example. The excitation depends on the boundary condition and would not exist otherwise. All $2J$ excitations involve flipping all spins on one side of some closed path around the system. Other closed paths can give excitations with energies equal to an odd multiple of $2J$. A $4J$ excitation requires two paths.

Our main message here is that it is possible to essentially define the ferromagnetic phase by the absence of these $2J$ excitations. Alternative approaches [11] include the imposition of domain wall defects and the calculation of magnetization. These nevertheless lack clear systematics due to the large degeneracy of the ground state. Domain wall defects may not represent excitations at all since they can correspond to alternative ground states. Sampling of domain walls needs to be done in a controlled way in order to obtain typical representative domain walls—a goal that might well be achieved using an algorithm [42] that is able to compute Pfaffians exactly.

doi:10.1088/1742-5468/2012/01/P01010
Calculation of the magnetization is also problematic as a result of the ground state degeneracy. In [11], for example, the algorithm starts with a ground state and proceeds with a Monte Carlo simulation to determine a typical value of the magnetization.

In this work we propose a simple picture of the ferromagnetic phase that is evaluated from the response to temperature alone. The number of lowest energy excitations is counted exactly. In the thermodynamic limit these excitations can only exist in the spin glass phase. Details of our results are given in section 3 after a brief account of our method.

2. Formalism

We use the Pfaffian method and degenerate state perturbation theory to calculate the degeneracies of the excited states. The planar Ising model can be mapped onto a system of noninteracting fermions. Each bond is decorated with two fermions, one either side. A square plaquette then has four fermions inside and four others across the bonds, as shown in figure 2. For a system with $N$ lattice sites we have $4N$ fermions in total. The partition function can be written as [43, 44]

$$Z = 2^N \left( \prod_{(ij)} \cosh(J_{ij}/kT) \right) (\det D)^{1/2}. \quad (3)$$

The product is over all nearest-neighbour bonds $J_{ij}$ on an $N$-site lattice. The matrix $D$ is a $4N \times 4N$ skew-symmetric matrix that comprises constant diagonal blocks, and off-diagonal blocks that depend on temperature $T$ through matrix elements $\pm \tanh J_{ij}/kT$. The factor $(\det D)^{1/2}$ is precisely the Pfaffian [43, 44]. This formalism is applicable to any distribution of disorder.

At zero temperature there are defect eigenstates of $D$ with eigenvalues equal to zero. Each defect eigenstate can be expressed as a linear combination of the fermions localized in a frustrated plaquette. The number of these defect eigenstates is exactly equal to the number of frustrated plaquettes. At low temperature each defect eigenvalue approaches...
Elementary excitations and the phase transition

\[ \epsilon = \pm \frac{1}{2} X \exp \left( -\frac{2Jr}{kT} \right) \]  \hspace{1cm} (4)

where \( r \) is an integer and \( X \) is a real number. These quantities \( r \) and \( X \) can be obtained using degenerate state perturbation theory [45]. The ground state energy is written as

\[ U_0 = -2NJ + 2J \sum d r_d \]  \hspace{1cm} (5)

where the sum are over all defect eigenstate pairs. The ground state degeneracy is

\[ M_0 = \prod d X_d \]  \hspace{1cm} (6)

and the ground state entropy can then be written as \( S_0 = k \sum d \ln X_d \).

At arbitrary low temperature the internal energy can be expanded as [28]

\[ U = U_0 + \sum_{m=1}^{\infty} e^{-2Jm/kT} U_m \]  \hspace{1cm} (7)

where the coefficient \( U_m \) is expressed as

\[ U_m = -2^m J \ \text{Tr} \ R^m \]  \hspace{1cm} (8)

with

\[ R = D_1 g_{c1}(1 + D_1 G_1)(1 + D_2 G_2) \cdots (1 + D_{r_{\text{max}}} G_{r_{\text{max}}}). \]  \hspace{1cm} (9)

The 2 \times 2 block diagonal matrix \( D_1 \) is defined according to \( D = D_0 + \delta D_1 \) where \( D_0 \) is the matrix \( D \) when \( T = 0 \) and \( \delta = 1 - \tanh J/kT \). \( D_1 \) has non-zero matrix elements joining two fermions across bonds only. The 4 \times 4 block diagonal matrix \( g_{c1} \) is derived from the continuum Green’s function [45] and has matrix elements connecting the fermions within a plaquette. \( D_2 \) is given by \( D_2 = D_1 g_{c1} D_1 \) and, for \( r > 2, D_r = D_{r-1}(1 + G_{r-2} D_{r-2}) \cdots (1 + G_1 D_1) g_{c1} D_1 \). The Green’s function \( G_r \) is given by [45]

\[ G_r = -\sum_{i=1}^{N(r)} |r, i\rangle \left( \frac{1}{\epsilon_r^i} \right) \langle r, i|, \]  \hspace{1cm} (10)

where \( |r, i\rangle \) is the ground state defect eigenstate of \( D \) with eigenvalue \( \epsilon_r^i \). The integer \( r \) represents the order of perturbation theory at which the degeneracy is lifted. It is also the index \( r \) in (4). The total number of such eigenstates is \( N(r) \).

The coefficient \( U_m \) can also be expressed in terms of the degeneracies of the excited states. We denote the degeneracy of the \( i \)th excited state as \( M_i \). The partition function of the bimodal Ising model can be expressed in terms of the degeneracies as

\[ Z = 2 M_0 e^{-U_0/kT} \left( 1 + \frac{M_1}{M_0} e^{-2J/kT} + \frac{M_2}{M_0} e^{-4J/kT} + \cdots \right). \]  \hspace{1cm} (11)

Using some thermodynamic relations together with the expansion of \( \ln Z \) using the Taylor
Elementary excitations and the phase transition

Figure 3. The probability $P_1$ of finding $M_1/M_0 > 0$, plotted as a function of system size $L$ for various values of antiferromagnetic bond concentration $p$.

From these relations the ratios $M_i/M_0$ for all excited states can be obtained recursively. Note that $M_1$ is the number of $2J$ excitations. We also emphasize that both the degeneracies $M_i$ and the quantities $U_m$ are independent of temperature.

3. Results

We have calculated $M_1/M_0$ for system sizes up to $L = 129$ and concentrations $p$ ranging from 0.050 to 0.150. The number of disorder realizations ranges from 20000 for the smallest size to 2000 for the largest. We denote as $P_1$ the probability of finding $M_1/M_0 > 0$. In figure 3, $P_1$ is plotted as a function of system size $L$ for various defect concentrations. The error bars are evaluated using the bootstrap method [46]. The transition concentration $p_c$ is indicated where the $L$ dependence of $P_1$ changes from decreasing to increasing. We can see that $P_1$ is decreasing for $p < 0.102$. The system can be regarded as ferromagnetic below this concentration. Since $P_1$ is increasing with $L$ for $p > 0.106$, the system can be regarded as a spin glass. We conclude from these results that the value of $p_c$ lies between 0.102 and 0.106.

We have constructed a scaling plot using the relation [25],

$$P_1 L^\psi = f((p - p_c) L^\phi).$$

(13)

doi:10.1088/1742-5468/2012/01/P01010

7
Figure 4. The scaling plot of $P_1$ as a function of the antiferromagnetic bond concentration $p$ with $p_c = 0.1045(11)$ and $\phi = 0.532(72)$.

Figure 5. The variation of $S_{\text{min}}$ as a function of $p_c$.

It is reasonable to fix $\psi = 0$ since the value of $P_1$ is bounded to the range $[0, 1]$. In any case, with $\psi$ not fixed, the best scaling plots have $\psi < 0.001$. The parameters $p_c$ and $\phi$ are chosen to minimize the quality parameter $S$ [34, 47]. The best fits give $p_c = 0.1045(11)$ and $\phi = 0.532(72)$ with $S = 0.62$. The resulting scaling plot is shown in figure 4. The error bars of each parameter are obtained using the method described in [11]. We fix the corresponding parameter at various values and minimize $S$ with respect to the other parameter. The range of the fixed parameter that gives $S$ double the minimum value is regarded as the error bar. For example we show in figure 5 the variation of the partial minimized value of $S(p_c, \phi)$ as a function of $p_c$. The error bar of $\phi$ can be obtained in the same way.
The above value of \( p_c \) agrees, within error bars, with \( p_c = 0.103(1) \) proposed in [11]. Note that we also performed the analysis using data from systems with \( L \leq 65 \) and get \( p_c \gtrsim 0.105 \). This indicates that the effect of finite size is the overestimation of \( p_c \). It is expected that if we perform this analysis using data with \( L > 129 \), we will get a smaller value of \( p_c \).

We have also investigated the distributions of the \( 2J \) excitations in the ferromagnetic phase. We denote as \( C_1(x) \) the probability of finding \( M_1/M_0 \leq x \). In figure 6, \( C_1(x) \) with \( p = 0.090 \) is plotted for various values of \( L \). It is clear that the most likely value of \( M_1/M_0 \) is zero. The probability of getting \( M_1/M_0 > 0 \) is decreasing with \( L \). We may expect that in the ferromagnetic phase the \( 2J \) excitations will vanish in the thermodynamic limit.

Since there are no \( 2J \) excitations in the ferromagnetic phase in the thermodynamic limit, the first excited state has energy \( 4J \). We have investigated the behaviour of the \( 4J \) excitations by calculating the ratio \( M_2/M_0 \) for system sizes up to \( L = 97 \). We denote as \( H_2(x) \) the probability density function for getting \( (1/L^2)(M_2/M_0) = x \). We use the kernel density estimation algorithm [48] to obtain \( H_2(x) \). In figure 7, \( H_2(x) \) with \( p = 0.090 \) is plotted for various odd values of \( L \). A sharp peak develops with increasing \( L \). We expect to get a definite value of \( (1/L^2)(M_2/M_0) \) in the thermodynamic limit. It is interesting that this behaviour does not depend on whether \( L \) is odd or even. In figure 8, \( H_2(x) \) with \( p = 0.090 \) is plotted for various even values of \( L \). The distributions are much the same and provide the same conclusions. From these results we have that the energy gap in the ferromagnetic phase is \( 4J \).

We can expect this also from the behaviour of the specific heat at low temperature. When the temperature is low enough the behaviour is dominated by the first excited state and can be expressed as [35]

\[
c_v = \frac{16J^2}{kT^2} \left( \frac{1}{L^2} \frac{M_2}{M_0} \right) e^{-4J/kT}.
\] (14)
Elementary excitations and the phase transition

Figure 7. The probability density \( H_2(x) \) for getting \( (1/L^2)(M_2/M_0) = x \) for \( p = 0.090 \) with odd \( L \).

Figure 8. The probability density \( H_2(x) \) for getting \( (1/L^2)(M_2/M_0) = x \) for \( p = 0.090 \) with even \( L \).

Sharpness of the distribution of \( (1/L^2)(M_2/M_0) \) satisfies the requirement of \( c_v \) being a physical quantity. We can expect that in the ferromagnetic phase the specific heat will have a definite value in the thermodynamic limit. At low temperature \( c_v \) is proportional to \( \exp(-4J/kT) \) and the energy gap can be regarded as \( 4J \).

The distribution of the \( 2J \) excitations in the spin glass phase is quite different. In figure 9, \( C_1(x) \) with \( p = 0.110 \) is plotted for various values of \( L \). Although the most likely value of \( M_1/M_0 \) is still at zero, the probability of getting \( M_1/M_0 > 0 \) is increasing with \( L \).
Elementary excitations and the phase transition

Figure 9. The probability $C_1(x)$ of finding $M_1/M_0 \leq x$ for $p = 0.110$.

Figure 10. The probability density $H_2(x)$ for getting $(1/L^2)(M_2/M_0) = x$ for $p = 0.110$ with odd $L$.

The distributions of $M_1/M_0$ do not have a sharp peak but broaden when $L$ is increasing. We have that the $2J$ excitations persist as $L$ increases.

The distribution of the $4J$ excitations in the spin glass phase is also different from that in the ferromagnetic phase. In figure 10, $H_2(x)$ with $p = 0.110$ is plotted for various odd values of $L$. The most likely value of $H_2(x)$ increases with $L$ and the distributions broaden. This behaviour of $H_2(x)$ is similar to that of the canonical spin glass ($p = 0.5$) with even $L$ [28]. In particular, the height of the peak of the distribution collapses with increasing $L$. We have also checked the distributions of $H_2(x)$ with $p = 0.110$ and even $L$. The results are shown in figure 11. The characteristics are the same for odd and even $L$. 

doi:10.1088/1742-5468/2012/01/P01010
4. Conclusions

We have proposed a simple view that distinguishes between the ferromagnetic and spin glass phases. The ferromagnetic phase is characterized by the absence of lowest energy, that is $2J$, excitations. Our method counts the number of excitations exactly without bias. It is not necessary to work with some typical ground or excited state.

Distributions of the number of $2J$ excitations are shown to differ in character between the phases. In the ferromagnetic phase the number declines as the (odd) circumference $L$ of the cylindrical winding increases. A finite size scaling analysis produces a data collapse of excellent quality to support our conclusion that $2J$ excitations do not exist in the thermodynamic limit of the ferromagnetic phase. In the spin glass phase the situation is reversed, with the degeneracy of the first excited state increasing with $L$.

The energy gap in the ferromagnetic phase is $4J$. For even values of $L$ the first excitations have energy $4J$. We have also presented distributions of $4J$ excitations so as to indicate that there is no essential dependence on whether $L$ is even or odd. In the ferromagnetic phase the peak grows taller and narrower with increasing $L$ and will presumably lead to a unique value of the low temperature specific heat in the thermodynamic limit.

In the spin glass phase the behaviour of the distributions is quite different. Essentially they are extreme with long tails. As $L$ increases the tails become fatter and the peak collapses. We believe that this is consistent with a power law behaviour for the low temperature specific heat. The extreme distributions only indicate spin glass behaviour; a proper statistical mechanical description of the model requires a summation over the entire density of states. This seems to suggest that thermally active droplets can indeed take many different values of energy.

Figure 11. The probability density $H_2(x)$ for getting $(1/L^2)(M_2/M_0) = x$ for $p = 0.110$ with even $L$. 

Elementary excitations and the phase transition
Finally, we have not found any evidence that indicates a random antiphase state [15], although we cannot rule out a situation where percolation of rigid bonds coexists with zero magnetization.

Acknowledgments

NJ thanks the National Science and Technology Development Agency, Thailand, for a scholarship. Some of the computations were performed on the Tera Cluster at the Thai National Grid Centre and on the Rocks Cluster at the Department of Physics, Kasetsart University.

References

[1] Binder K and Young A P, 1986 Rev. Mod. Phys. 58 801
[2] M´ezard M, Parisi G and Virasoro M, 1987 Spin Glass Theory and Beyond (Singapore: World Scientific)
[3] Fischer K H and Hertz J A, 1991 Spin Glasses (Cambridge: Cambridge University Press)
[4] Kawashima N and Rieger H, 2004 Frustrated Spin Systems ed T H Diep (Singapore: World Scientific)
[5] Edwards S F and Anderson P W, 1975 J. Phys. F: Met. Phys. 5 965
[6] Hartmann A K and Rieger H, 2002 Optimization Algorithms in Physics (Berlin: Wiley–VCH)
[7] Hartmann A K, 2009 Practical Guide to Computer Simulations (Singapore: World Scientific)
[8] Toulouse G, 1977 Commun. Phys. 2 115
[9] Ohzeki M and Nishimori H, 2009 J. Phys. A: Math. Theor. 42 332001
[10] Wang C, Harrington J and Preskill J, 2003 Ann. Phys. 303 31
[11] Amoruso C and Hartmann A K, 2004 Phys. Rev. B 70 134425
[12] Toldin F P, Pelissetto A and Vicari E, 2009 J. Stat. Phys. 135 1039
[13] Blackman J A, Gonçalves J R and Poulter J, 1998 Phys. Rev. E 58 1502
[14] Lukic J, Galluccio A, Marinari E, Martin O C and Rinaldi G, 2004 Phys. Rev. Lett. 92 117202
[15] Barahona F, Maynard R, Rammal R and Uhrig J P, 1982 J. Phys. A: Math. Gen. 15 673
[16] Romá F, Risau-Gusman S, Ramírez-Pastor A J, Nieto F and Vogel E E, 2010 Phys. Rev. B 82 214401
[17] McMillan W L, 1984 J. Phys. C: Solid State Phys. 17 3179
[18] Bovier A and Fröhlich J, 1986 J. Stat. Phys. 44 347
[19] Fisher D S and Huse D A, 1987 Heidelberg Colloquium on Glassy Dynamics ed J L van Hemmen and I Morgenstern (Berlin: Springer) p 121
[20] Bray A J and Moore M A, 1987 Heidelberg Colloquium on Glassy Dynamics ed J L van Hemmen and I Morgenstern (Berlin: Springer) p 121
[21] Fisher D S and Huse D A, 1988 Phys. Rev. B 38 386
[22] McMillan W L, 1984 Phys. Rev. B 29 4026
[23] McMillan W L, 1984 Phys. Rev. B 30 476
[24] McMillan W L, 1985 Phys. Rev. B 31 340
[25] Bray A J and Moore M A, 1984 J. Phys. C: Solid State Phys. 17 L463
[26] Bray A J and Moore M A, 1985 Phys. Rev. B 31 631
[27] Bray A J and Moore M A, 1987 Phys. Rev. Lett. 58 57
[28] Hartmann A K and Young A P, 2002 Phys. Rev. B 66 094419
[29] Kawashima N and Rieger H, 1997 Europhys. Lett. 39 85
[30] Hartmann A K, 2008 Phys. Rev. B 77 144418
[31] Dayal P, Trebst S, Wessel S, Würtz D, Troyer M, Sabhapandit S and Coppersmith S N, 2004 Phys. Rev. Lett. 92 097201
[32] Atisattapong W and Poulter J, 2008 New J. Phys. 10 093012
[33] Wang J-S, 2005 Phys. Rev. E 72 036706
[34] Wang J-S and Swendsen R H, 1988 Phys. Rev. B 38 4840
[35] Jörg T, Lukic J, Marinari E and Martin O C, 2006 Phys. Rev. Lett. 96 237205
[36] Thomas C K, Huse D A and Middleton A A, 2011 Phys. Rev. Lett. 107 047203
[37] Cheung H-F and McMillan W L, 1983 J. Phys. C: Solid State Phys. 16 7033
[38] Houdayer J and Hartmann A K, 2004 Phys. Rev. B 70 014418
[39] Katzgraber H G, Lee L W and Campbell I A, 2007 Phys. Rev. B 75 014412
[40] Saul L and Kardar M, 1993 Phys. Rev. E 48 R3221
[41] Saul L and Kardar M, 1994 Nucl. Phys. B 432 641
[42] Lukic J, Marinari E, Martin O C and Sabatini S, 2006 J. Stat. Mech. L10001

doi:10.1088/1742-5468/2012/01/P01010
Elementary excitations and the phase transition

[38] Romá F, Risau-Gusman S, Ramírez-Pastor A J, Nieto F and Vogel E E, 2007 Phys. Rev. B 75 020402
[39] Melchert O and Hartmann A K, 2007 Phys. Rev. B 76 174411
[40] Weigel M and Johnston D, 2007 Phys. Rev. B 76 054408
[41] Aromsawa A and Poulter J, 2007 Phys. Rev. B 76 064427
[42] Thomas C K and Middleton A A, 2009 Phys. Rev. E 80 046708
[43] Green H S and Hurst C A, 1964 Order–Disorder Phenomena (London: Interscience)
[44] Blackman J A, 1982 Phys. Rev. B 26 4987
[45] Blackman J A and Poulter J, 1991 Phys. Rev. B 44 4374
[46] Efron B, 1982 The Jackknife, the Bootstrap and Other Resampling Plans (Philadelphia, PA: Society of Industrial and Applied Mathematics)
[47] Melchet O, autoScale.py—a program for automatic finite-size scaling analyses: a user’s guide, 2009 arXiv:0910.5403v1
[48] Botev Z I, Grotowski J F and Kroese D P, 2010 Ann. Stat. 38 2916

doi:10.1088/1742-5468/2012/01/P01010