Pivotal bootstrap for quantile-based modal regression

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Abstract

In this paper, we develop uniform inference methods for the conditional mode based on quantile regression. Specifically, we propose to estimate the conditional mode by minimizing the derivative of the estimated conditional quantile function defined by smoothing the linear quantile regression estimator, and develop a novel bootstrap method, which we call the pivotal bootstrap, for our conditional mode estimator. Building on high-dimensional Gaussian approximation techniques, we establish the validity of simultaneous confidence rectangles constructed from the pivotal

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bootstrap for the conditional mode. We also extend the preceding analysis to the case
where the dimension of the covariate vector is increasing with the sample size. Fi-
ally, we conduct simulation experiments and a real data analysis using U.S. wage
data to demonstrate the finite sample performance of our inference method.

*Keywords:* quantile regression, kernel smoothing, modal regression, high-dimensional CLT,
pivotal bootstrap
1 Introduction

1.1 Overview

Modal regression is a principal statistical methodology to estimate and make inference on
the conditional mode. Modes provide useful distributional information missed by the mean
when the (conditional) distribution is skewed (Chen et al., 2016) and are known to be robust
under measurement errors (Bound and Krueger 1991; Hu and Schennach 2008). The global
mode offers intuitive interpretability by being understood as “the most likely” or “the most
common” (Heckman et al., 2001; Hedges and Shah 2003). As such, modal regression has
wide applications in various areas including astronomy (Bamford et al., 2008), medical
research (Wang et al., 2017), econometrics (Kemp and Santos-Silva, 2012), etc. We refer
the reader to Chacón (2018) and Chen (2018) for recent reviews on modal regression; see
also a literature review below.

In this paper, we consider estimating the conditional mode by “inverting” a quantile
regression model, which builds on the observation that the derivative of the conditional
quantile function coincides with the reciprocal of the conditional density so that the con-
ditional mode can be obtained by minimizing the derivative of the conditional quantile
function. Specifically, we estimate the conditional mode by minimizing the derivative
of the kernel smoothed Koenker-Bassett estimator of the conditional quantile function
(Koenker and Bassett, 1978) with a sufficiently smooth kernel. We develop asymptotic
theory for the proposed estimator $\hat{m}(x)$ of the conditional mode $m(x)$. In particular,
we consider simultaneous confidence intervals for the conditional mode at multiple de-
design points, $m(x_1), \ldots, m(x_L)$, where $L$ is allowed to grow with the sample size $n$, i.e.,
$L = L_n \to \infty$. To this end, we first show that $\hat{m}(x) - m(x)$ can be approximated by
the linear term $(nh^{3/2})^{-1} \sum_{i=1}^{n} \psi_x(U_i, X_i)$ uniformly over a range of design points $x$, where
$h = h_n \to 0$ is a sequence of bandwidths, $\psi_x$ is the influence function (that depends
on $n$) at design point $x$, $X_1, \ldots, X_n$ are covariate vectors, and $U_1, \ldots, U_n$ are uniform
random variables on $(0,1)$ independent of the covariate vectors. Building on high di-
mensional Gaussian approximation techniques developed in Chernozhukov et al. (2014)
we show that $\sqrt{n \ell^3} (\hat{m}(x_\ell) - m(x_\ell))_{\ell=1}^L$ can be approximated by an $L$-dimensional Gaussian vector uniformly over the hyperrectangles in $\mathbb{R}^L$, i.e., all sets $A$ of the form: $A = \{w \in \mathbb{R}^L : a_j \leq w_j \leq b_j \text{ for all } j = 1, \cdots, L\}$ for some $-\infty \leq a_j \leq b_j \leq \infty$, $j = 1, \cdots, L$, even when $L \gg n$.

The leading stochastic term in the prescribed expansion is conditionally “pivotal” in the sense that conditionally on $X_1, \ldots, X_n$, the distribution of the process

$$x \mapsto \left(\frac{nh^{3/2}}{2}\right)^{-1} \sum_{i=1}^{n} \psi_x(U_i, X_i)$$

is completely known up to some nuisance parameters. This suggests a version of bootstrap for the proposed estimator by sampling uniform random variables $U_i$ independent of the data. In practice, the influence function $\psi_x$ depends on nuisance parameters and we replace them by consistent estimates. We call the resulting bootstrap “pivotal bootstrap” and prove that the pivotal bootstrap can consistently estimate the sampling distribution of $\sqrt{n \ell^3} (\hat{m}(x_\ell) - m(x_\ell))_{\ell=1}^L$ uniformly over the rectangles in $\mathbb{R}^L$ even when $L \gg n$. In fact, our inference framework is more general and covers simultaneous inference for linear combinations of the vector $(m(x_\ell))_{\ell=1}^L$, which can be used to construct simultaneous confidence intervals for partial effects and test significance of certain covariates on the conditional mode. We also extend the previous analysis to the case where the dimension of the covariate vector increases with the sample size.

We conduct simulation experiments on various modal inference problems and a real data analysis to demonstrate the finite sample performance of the pivotal bootstrap. Our simulation experiments show that the pivotal bootstrap yields accurate pointwise and simultaneous confidence intervals for the conditional mode. Additionally, we apply our inference method to analyze a real U.S. wage dataset. Analysis of wage data is important in econometric and social science [Autor et al. 2008, Western and Rosenfeld 2011, Buchinsky 1994]. Wage data are often positively skewed and “the most common wage” as a representative of the majority of the population is usually of more interest. Common questions in the analysis of wage data include: i) What is the most likely wage for given covariates? How to construct pointwise and simultaneous confidence intervals for the estimated wages? ii) Is there an effect of a specific covariate on the most likely wage given the same other
covariates? We address those empirical questions using the inference method developed in the present paper.

From a technical perspective, the asymptotic analysis in this paper is highly nontrivial. Our program of the technical analysis proceeds as 1) first establishing a uniform asymptotic representation and 2) high-dimensional Gaussian approximation to our estimate, and 3) then proving the validity of the pivotal bootstrap building on 1) and 2). Each of these steps relies on modern empirical process theory and high-dimensional Gaussian approximation techniques recently developed by Chernozhukov et al. (2014, 2017). In particular, the pivotal bootstrap differs from the nonparametric or multiplier bootstraps that have been analyzed in the literature in the high-dimensional setup (Belloni et al., 2019; Chernozhukov et al., 2016; Deng and Zhang, 2017; Chen and Kato, 2020), and proving the validity of the pivotal bootstrap requires a substantial work.

In summary, the present paper contributes to the literature on modal regression in twofold. First, we propose a new quantile-based conditional mode estimate that enjoys both desirable computational and statistical guarantees. Second, we propose a new resampling method (pivotal bootstrap) that builds on an insight into the specific structure of our estimate, and establish theoretical validity of the pivotal bootstrap for a broad spectrum of inference tasks in a unified way.

1.2 Literature review

Starting from the pioneering work of Sager and Thisted (1982), there is now a large literature on modal regression. There are two major approaches to estimating the conditional mode comparable to our method; one is linear modal regression where the conditional mode is assumed to be linear in covariates (Lee, 1989, 1993; Kemp and Santos-Silva, 2012; Yao and Li, 2014), and the other is nonparametric estimation (Yao et al., 2012; Chen et al., 2016); see also Lee and Kim (1998); Manski (1991); Einbeck and Tutz (2006); Sasaki et al. (2016); Ho et al. (2017); Khardani and Yao (2017); Krief (2017) for alternative methods including semiparametric and Bayesian estimation. Lee (1989, 1993) assume symmetry of the error distribution to derive limit theorems for their proposed estimators, but the symmetry
assumption implies that the conditional mean, median, and mode coincide, thereby significantly reducing the complexity of estimating the conditional mode. Kemp and Santos-Silva (2012) and Yao and Li (2014) consider an alternative estimator defined by minimizing a kernel-based loss function for linear modal regression and develop limit distribution theory for the estimator without assuming symmetry of the error distribution. However, the optimization problem of Kemp and Santos-Silva (2012) and Yao and Li (2014) is (multidimensional and) nonconvex, and while they propose EM-type algorithms to compute their estimators, “there is no guarantee that the algorithm will converge to the global optimal solution” (Yao and Li 2014, p. 659). Compared with the method of Kemp and Santos-Silva (2012) and Yao and Li (2014), all three methods (including ours) enjoy the same rate of convergence, while our method is computationally attractive since linear quantile regression can be formulated as a linear programming problem (Koenker 2005), and minimizing the estimated derivative of the conditional quantile function is a one-dimensional optimization problem both of which can be solved accurately and efficiently.

Yao et al. (2012) consider local linear estimation of the conditional mode but their Condition (A6) is essentially the symmetry assumption on the error distribution, which makes their problem statistically equivalent to conditional mean estimation. Chen et al. (2016) study nonparametric estimation of the conditional mode based on kernel density estimation (KDE), and develop multiplier bootstrap inference for their KDE-based estimate. The nonparametric estimation is able to avoid model misspecification. Chen et al. (2016) also allow for multiple local modes, while we assume the existence of the unique global mode at each design point of interest. Thus, the setup of Chen et al. (2016) is more general than ours. However, the convergence rate of the KDE-based estimate of Chen et al. (2016) is slow even when the dimension of the covariate vector is moderately large (“curse of dimensionality”). Specifically, the convergence rate of the Chen et al. (2016) estimate is at best $n^{-2/(p+7)}$ where $p$ is the number of continuous covariates under the assumption of four times differentiability of the conditional density, while our estimate can achieve the $n^{-2/7}$ rate (up to logarithmic factors when evaluated under the uniform norm) assuming three times differentiability of the conditional density (albeit assuming a linear quantile
The present paper builds on (but substantially differs from) the recent work of Ota et al. (2019), which proposes a different quantile-based estimate of the conditional mode and develops pointwise limit distribution theory for their estimator. Contrary to ours, Ota et al. (2019) directly use the linear quantile regression estimate and minimize its difference quotient (as the linear quantile regression estimate is not smooth in the quantile index), which makes a substantial difference between their asymptotic analysis and ours. Indeed, Ota et al. (2019) show that the rate of convergence of their estimate is at best $n^{-1/4}$ that is slower than our $n^{-2/7}$ rate, and find that the pointwise limit distribution is a scale transformation of nonstandard Chernoff’s distribution. The nonstandard limit distribution poses a substantial challenge in inference using their estimate and Ota et al. (2019) only consider pointwise inference using a general purpose subsampling method (Politis et al., 1999). We overcome this limitation by employing kernel smoothing, and further, develop a model-based bootstrap method (pivotal bootstrap) that enables us to deal with much broader inference tasks including simultaneous confidence intervals and significance testing.

This paper also builds on the quantile regression literature. Quantile regression provides a comparatively full picture of how the covariates impact the conditional distribution of a response variable and has wide applications (Koenker, 2017). In particular, the pivotal bootstrap of the present paper is related to Parzen et al. (1994); Chernozhukov et al. (2009); He (2017); Belloni et al. (2019) who study resampling-based inference methods that build on (conditionally) pivotal influence functions in the quantile regression setup. Their scopes and methods are, however, substantially different from ours. To the best of our knowledge, exploiting pivotal influence functions to make inference for modal regression is new.

1.3 Organization

The rest of the paper is organized as follows. In Section 2 we introduce the setup and define the proposed quantile-based modal estimator. In Section 3 we present the main theoretical results for the proposed estimator. We first derive a uniform asymptotic linear representation for the proposed estimator. Then we present our pivotal bootstrap inference
framework together with its theoretical guarantees. In Section 4 we present the simulation results and a real data example. In Section 5 we extend the preceding analysis to the increasing dimension case. Finally, we summarize the paper in Section 6. The proofs of main results and additional technical details are relegated to the Appendix.

2 Mode Estimation via smoothed quantile regression

We begin with the setup and define our estimator. We are interested in making inference on the conditional mode of a scalar response variable \( Y \in \mathbb{R} \) given a \( d \)-dimensional covariate vector \( X \in \mathbb{R}^d \). We will assume that the dimension \( d \) is fixed in Section 3 but consider the extension to the case with \( d = d_n \to \infty \) in Section 5. In what follows, we assume that there exists a conditional density of \( Y \) given \( X \), \( f(y \mid x) \), which is (at least) continuous in \( y \) for each design point \( x \). We are interested in making inference on the conditional mode over a compact subset \( X_0 \) of the support of \( X \). We will maintain the assumption that for each \( x \in X_0 \), there exists a unique global mode \( m(x) \), i.e., \( m(x) \) is the unique maximizer of the function \( y \mapsto f(y \mid x) \),

\[
m(x) = \arg \max_{y \in \mathbb{R}} f(y \mid x).
\]  

(1)

Our strategy to estimate the conditional mode is based on “inverting” a quantile regression model. For \( \tau \in (0, 1) \), let \( Q_x(\tau) \) denote the conditional \( \tau \)-quantile of \( Y \) given \( X \). Observe that the derivative of the conditional quantile function with respect to the quantile index \( \tau \) coincides with the reciprocal of the conditional density at \( Q_x(\tau) \), i.e.,

\[
s_x(\tau) := Q_x'(\tau) := \frac{\partial Q_x(\tau)}{\partial \tau} = \frac{1}{f(Q_x(\tau) \mid x)}.
\]  

(2)

This suggests that the conditional mode \( m(x) \) can be obtained by minimizing the “sparsity” function \( s_x(\tau) := Q_x'(\tau) \). Specifically, let \( \tau_x \) denote the minimizer of \( s_x(\cdot) \), i.e.,

\[
\tau_x = \arg \min_{\tau \in (0, 1)} s_x(\tau).
\]

Then, we arrive at the expression \( m(x) = Q_x(\tau_x) \). Hence, estimation of \( m(x) \) boils down to estimation of \( Q_x(\cdot) \) and \( \tau_x \).
To estimate the conditional quantile function, we assume a linear quantile model, i.e.,
\[ Q_x(\tau) = x^T \beta(\tau), \quad \tau \in (0, 1). \]

Suppose that we are given i.i.d. observations \((Y_1, X_1), \ldots, (Y_n, X_n)\) of \((Y, X)\). We estimate the slope vector \(\beta(\tau)\) by the standard quantile regression estimator (Koenker and Bassett, 1978),
\[ \hat{\beta}(\tau) = \arg \min_{\beta \in \mathbb{R}^d} \sum_{i=1}^n \rho_{\tau}(Y_i - X_i^T \beta), \]
where \(\rho_{\tau}(u) = u \{\tau - I(u \leq 0)\}\) is the check function. However, the plug-in estimator \(\hat{Q}_x(\tau) := x^T \hat{\beta}(\tau)\) for the conditional quantile function is not smooth in \(\tau\). To overcome this difficulty, we propose to smooth the naive estimator \(\hat{Q}_x(\tau)\) by a kernel function, and estimate \(\tau_x\) by minimizing the derivative of the smoothed quantile estimator. To this end, let \(K: \mathbb{R} \to \mathbb{R}\) be a kernel function (a function that integrates to 1) that is smooth and supported in \([-1, 1]\) (see Assumption 1 (vii) in the following for more details). For a given sequence of bandwidth parameters \(h = h_n \to 0\), we modify the naive estimator \(\hat{Q}_x(\tau)\) by
\[ \hat{Q}_x(\tau) := \int_{\tau-h}^{\tau+h} \hat{Q}_x(t) K_h(\tau - t) dt, \quad \tau \in [\epsilon, 1 - \epsilon], \]
where \(K_h(\cdot) := h^{-1} K(\cdot/h)\) and \(\epsilon \in (0, 1/2)\) is some small user-chosen parameter. The restriction of the range of \(\tau\) is to avoid the boundary problem. Since \(K\) is supported in \([-1, 1]\), the integral \(\int_{\tau-h}^{\tau+h}\) above can be formally replaced by \(\int_{\mathbb{R}}\) with the convention that \(\hat{Q}_x(t) = 0\) for \(t \notin (0, 1)\).

Then, we can estimate \(s_x(\tau)\) by differentiating \(\hat{Q}_x(\tau)\), \(\hat{s}_x(\tau) := \hat{Q}_x'(\tau)\), and estimate \(\tau_x\) by minimizing \(\hat{s}_x(\tau)\),
\[ \hat{\tau}_x := \arg \min_{\tau \in [\epsilon, 1-\epsilon]} \hat{s}_x(\tau). \]
By the smoothness of \(K(\cdot)\), the map \(\tau \mapsto \hat{s}_x(\tau)\) is smooth, so \(\hat{\tau}_x\) is guaranteed to exist by compactness of \([\epsilon, 1-\epsilon]\). Finally, we propose to estimate the conditional mode \(m(x)\) by a plug-in method:
\[ \hat{m}(x) := \hat{Q}_x(\hat{\tau}_x). \]

Some remarks on the proposed estimator are in order.
Remark 1 (Linear quantile regression). The linear quantile regression model is common in the quantile regression literature and can cover many data generating processes (see Remark 1 in Ota et al. 2019). Importantly, the linear quantile regression problem can be solved efficiently since the optimization problem (3) can be formulated as a (parametric) linear programming problem whose solution path can be computed efficiently even for large-scale datasets (Koenker 2005). Having said that, the linear specification of the conditional quantile function is not essential and the theoretical results developed in the following sections can be extended to nonlinear quantile regression models.

Remark 2 (Comparison with other estimators). Compared with linear modal regression, our setting allows for nonlinear conditional mode functions even though the conditional quantile function is assumed linear in $x$ (see Remark 1 in Ota et al. (2019)). In fact, under linear quantile assumption, $m(x) = x^T \beta(\tau_x)$ and $\beta_x$ is allowed to be a (possibly nonlinear) function of $x$. In addition, computation of linear modal regression involves non-convex optimization (Yao and Li 2014; Cheng 1995; Einbeck and Tutz 2006), while the proposed method only relies on linear quantile regression that can be formulated as a linear programming problem, and an one-dimensional optimization. Chen et al. (2016) show the convergence rate $O_P(h^2+n^{−1/2}h^{−(p+3)/2})$ for the KDE-based mode estimator, where $h$ is the KDE bandwidth parameter and $p$ is the number of continuous covariates. This implies slow convergence for even moderate dimensions. In contrast, we show that the convergence rate of our estimator is $O_P(h^2+n^{−1/2}h^{−3/2})$ for any fixed dimension $d$ and thus our estimator is free from the “curse of dimensionality”.

3 Main Results

3.1 Notation and Conditions

We use $U(0,1)$ and $N(\mu, \Sigma)$ to denote the uniform distribution on $(0,1)$ and the normal distribution with mean $\mu$ and covariance matrix $\Sigma$, respectively. We use $\| \cdot \|$, $\| \cdot \|_1$, $\| \cdot \|_\infty$ to denote the Euclidean, $\ell^1$, and $\ell^\infty$-norms, respectively. For a smooth function $f(x)$, we write $f^{(r)}(x) = \partial^r f(x)/\partial x^r$ for any integer $r \geq 0$ with $f^{(0)} = f$. For vectors
Let $\mathcal{X} \subset \mathbb{R}^d$ denote the support of $\mathbf{X}$ and let $\mathcal{X}_0 \subset \mathcal{X}$ be the set over which we make inference on the conditional mode. In this section the dimension $d$ of $\mathbf{X}$ is assumed to be fixed. Recall the model assumptions in the last section that we are given i.i.d. observations $(Y_1, \mathbf{X}_1), \ldots, (Y_n, \mathbf{X}_n)$ of $(Y, \mathbf{X})$ where the conditional distribution of $Y$ given $\mathbf{X}$ has a unique mode and satisfies the linear quantile regression model. We make the following additional assumption.

**Assumption 1.** In addition to the model assumptions above, we assume the following conditions.

(i) The set $\mathcal{X}_0$ is compact in $\mathbb{R}^d$.

(ii) For any $\mathbf{x} \in \mathcal{X}_0$, $\tau_x \in (\epsilon, 1 - \epsilon)$.

(iii) The covariate vector $\mathbf{X}$ has finite $q$-th moment, $\mathbb{E}[\|\mathbf{X}\|^q] < \infty$, for some $q \in [4, \infty)$, and the Gram matrix $\mathbb{E}[\mathbf{X} \mathbf{X}^T]$ is positive definite.

(iv) The conditional density $f(y \mid \mathbf{x})$ is three times continuously differentiable with respect to $y$ for each $\mathbf{x} \in \mathcal{X}$. Let $f^{(j)}(y \mid \mathbf{x}) = \partial^j f(y \mid \mathbf{x})/\partial y^j$ for $j = 0, 1, 2, 3$. There exists a constant $C_1$ such that $|f^{(j)}(y \mid \mathbf{x})| \leq C_1$ for all $j = 0, 1, 2, 3$ and $(y, \mathbf{x}) \in \mathbb{R} \times \mathcal{X}$.

(v) There exists a positive constant $c_1$ (that may depend on $\epsilon$) such that $f(y \mid \mathbf{x}) \geq c_1$ for all $y \in [Q_{\mathbf{x}}(\epsilon), Q_{\mathbf{x}}(1 - \epsilon)]$ and $\mathbf{x} \in \mathcal{X}$.

(vi) There exists a positive constant $c_2$ such that $-f^{(2)}(m(\mathbf{x}) \mid \mathbf{x}) \geq c_2$ for all $\mathbf{x} \in \mathcal{X}_0$.

(vii) The kernel function $K$ is three times differentiable, symmetric, and supported in $[-1, 1]$.

(viii) The bandwidth $h = h_n \to 0$ satisfies that $nh^5/\log n \to \infty$.

Condition (i) is innocuous (recall that $\mathcal{X}_0$ is not the support of $\mathbf{X}$). Condition (ii) excludes the extreme quantile case where $\tau_x \to 0$ or 1 for some sequence of $\mathbf{x}$. Condition (iii) is a moment condition on the covariate vector $\mathbf{X}$. Conditions (iv) and (v) are standard.
smoothness conditions on the conditional density \( f(\cdot \mid \mathbf{x}) \) in the quantile regression literature (Koenker 2005). Similar conditions appear in Chen et al. (2016) and Ota et al. (2019). Smoothness of \( f(\cdot \mid \mathbf{x}) \) implies smoothness of conditional quantile function \( Q_x(\tau) \). Indeed, under Conditions (iv) and (v), \( Q_x(\tau) \) is four-times continuously differentiable. Condition (vi) ensures that the conditional mode \( m(\mathbf{x}) \) as a solution to the optimization problem (1) is nondegenerate. Condition (vi) also ensures that the map \( \mathbf{x} \mapsto -s''_x(\tau_x) \) is bounded away from zero on \( \mathcal{X}_0 \), as

\[
\frac{3f^{(1)}(Q_x(\tau)) | \mathbf{x}) - f(Q_x(\tau)) | \mathbf{x}) f^{(2)}(Q_x(\tau) | \mathbf{x})}{f(Q_x(\tau) | \mathbf{x})^5}
\]

and \( f^{(1)}(Q_x(\tau_x) | \mathbf{x}) = f^{(1)}(m(\mathbf{x}) | \mathbf{x}) = 0 \). Conditions (vii) and (viii) are concerned with the kernel function \( K \) and the bandwidth \( h_n \). We will use the biweight kernel \( K(t) = \frac{15}{16}(1 - t^2)^2I(|t| < 1) \) in our numerical studies. Condition (viii) ensures \( \hat{Q}_x^{(3)}(\tau) \) to be (uniformly) consistent; see Lemma 6 in the Appendix.

3.2 Uniform asymptotic linear representation

In this section, we derive a uniform asymptotic linear representation for our estimator \( \hat{m}(\mathbf{x}) \), which will be a building block for the pivotal bootstrap. Define

\[
J(\tau) := \mathbb{E}[f(\mathbf{X}^T \beta(\tau) \mid \mathbf{X}) \mathbf{X} \mathbf{X}^T].
\]

By Assumption [1] (iii) and (v), the minimum eigenvalue of the matrix \( J(\tau) \) is bounded away from zero for \( \tau \in [\epsilon, 1 - \epsilon] \). Further, for \( (u, \mathbf{x}') \in (0, 1) \times \mathbb{R}^d \), define

\[
\psi_x(u, \mathbf{x}') := -\frac{s_x(\tau_x)}{s''_x(\tau_x)} K' \left( \frac{\tau_x - u}{h} \right) \mathbf{x}'^T J(\tau_x)^{-1} \mathbf{x}',
\]

which will serve as an influence function for our estimator \( \hat{m}(\mathbf{x}) \). Let \( \kappa = \int t^2 K(t) dt \).

**Proposition 1** (Uniform asymptotic linear representation). Under Assumption [1], the following asymptotic linear representation holds uniformly in \( \mathbf{x} \in \mathcal{X}_0 \):

\[
\hat{m}(\mathbf{x}) - m(\mathbf{x}) + \frac{s_x(\tau_x) s^{(3)}_x(\tau_x)}{2s''_x(\tau_x)} \kappa h^2 + o_P(h^2)
\]

\[
= \frac{1}{nh^{3/2}} \sum_{i=1}^{n} \psi_x(U_i, \mathbf{X}_i) + O_P(n^{-1/2}h^{-1} + n^{-1}h^{-4} \log n),
\]

12
where $U_1, \ldots, U_n \sim U(0,1)$ i.i.d. independent of $X_1, \ldots, X_n$. In addition, we have

$$
\sup_{x \in \mathcal{X}_0} \left| \frac{1}{nh^{3/2}} \sum_{i=1}^{n} \psi_x(U_i, X_i) \right| = O_P(n^{-1/2}h^{-3/2} \sqrt{\log n}).
$$

The influence function $\psi_x(U_i, X_i)$ has mean zero when $h \leq \min\{\tau_x, 1 - \tau_x\}$ which holds for sufficiently large $n$, since

$$
\int_0^1 K'(\frac{\tau_x - u}{h}) du = h \int_{(\tau_x - 1)/h}^{\tau_x/h} K'(u) du = h \int_{\mathbb{R}} K'(u) du = 0 \quad (4)
$$

and by independence between $U_i$ and $X_i$. Proposition 1 in particular implies pointwise asymptotic normality of the proposed estimator.

**Corollary 1** (Pointwise asymptotic normality). Suppose that Assumption 1 holds. Then, for any fixed $x \in \mathcal{X}_0$, we have

$$
\sqrt{nh^3} \left[ \hat{m}(x) - m(x) + \frac{s_x(\tau_x)s_x(3)(\tau_x)}{2s''(\tau_x)} \kappa h^2 + o_P(h^2) \right] \xrightarrow{d} N(0, V_x),
$$

where $V_x = s_x(\tau_x)^2 \mathbb{E}[(x^T J(\tau_x)^{-1} X)^2] \kappa_1/s_x(\tau_x)^2$ and $\kappa_1 = \int K'(t)^2 dt$.

Proposition 1 shows that the uniform convergence rate of the proposed estimator is

$$
O_P(n^{-1/2}h^{-3/2} \sqrt{\log n} + h^2),
$$

which is dimension-free (i.e., independent of $d$). If we choose $h \sim (n/ \log n)^{-1/7}$, which balances between $n^{-1/2}h^{-3/2} \sqrt{\log n}$ and $h^2$, then the rate reduces to $O_P((n/ \log n)^{-2/7})$.

### 3.3 Pivotal bootstrap

We consider simultaneous inference for the conditional mode at several design points $x_1, \ldots, x_L \in \mathcal{X}_0$, where $L$ is allowed to depend on $n$, i.e., $L = L_n \to \infty$. Indeed, we aim at developing a general inference framework to construct confidence sets for linear combinations of the vector $(m(x_\ell))_{\ell=1}^L$. Specifically, we consider making inference on $D(m(x_\ell))_{\ell=1}^L$ where $D$ is a deterministic $M \times L$ matrix and the number of rows $M$ is also allowed to increase with $n$, i.e., $M = M_n \to \infty$. The following are a few examples of the matrix $D$.

See also Examples 3 and 4 ahead for more details.
**Example 1** (Simultaneous confidence intervals). Suppose that we are interested in constructing simultaneous confidence intervals for the conditional mode at design points $\mathbf{x}_1, \ldots, \mathbf{x}_L$. Construction of such simultaneous confidence intervals requires to approximate the distribution of the vector $(\hat{m}(\mathbf{x}_\ell) - m(\mathbf{x}_\ell))_{\ell=1}^L$, and thus $D = I_L$ ($L \times L$ identity matrix).

Another application is constructing simultaneous confidence intervals for partial effects of certain covariates on the conditional mode, i.e., the change of the conditional mode due to the change of one particular covariate while the rest of the covariates are controlled. Inference on partial effects is an important topic in econometrics and social science (Williams, 2012). For example, suppose that we have covariate $\mathbf{X} = (X_1, X_{-1})$ where $X_{-1}$ contains covariates other than $X_1$. Consider to construct simultaneous confidence intervals for partial effects of $X_1$ at $M$ different design points $x_1^{(1)}, \ldots, x_1^{(M)}$: $m(x_1^{(k)} + \delta, x_{-1}) - m(x_1^{(k)}, x_{-1})$ ($1 \leq k \leq M$) for some small user-chosen $\delta$ and fixed $x_{-1}$. To this end, we need to approximate the distribution of $(\hat{m}(x_{1}^{(k)} + \delta, x_{-1}) - \hat{m}(x_{1}^{(k)}, x_{-1}))_{k=1}^M$. If we take $\mathbf{x}_{2k-1} = (x_1^{(k)} + \delta, x_{-1})$ and $\mathbf{x}_{2k} = (x_1^{(k)}, x_{-1})$ for $k = 1, \ldots M$, then the corresponding $D$ matrix is

$$D = \begin{pmatrix}
1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & -1
\end{pmatrix}_{M \times 2M}$$

with $L = 2M$.

**Example 2** (Testing significance of covariates). Suppose first that we are interested in testing whether the conditional mode is constant over designs points $\mathbf{x}_1, \ldots, \mathbf{x}_L$, i.e., $m(\mathbf{x}_1) = \cdots = m(\mathbf{x}_L)$, which is equivalent to test $m(\mathbf{x}_{\ell+1}) - m(\mathbf{x}_\ell) = 0$ simultaneously for all $1 \leq \ell \leq L - 1$ (this corresponds to testing lack of significance of all covariates). Calibrating critical values for such tests boils down to approximating the null distribution
of the vector \(\hat{m}(x_{t+1}) - \hat{m}(x_t)\) for \(t = 1, \ldots, L-1\), and thus the matrix \(D\) is
\[
D = \begin{pmatrix}
1 & -1 & 0 & \cdots & 0 & 0 \\
0 & 1 & -1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -1 \\
\end{pmatrix}_{(L-1) \times L}.
\]

We can also consider testing significance of certain covariates on the conditional mode. For instance, suppose that we have three covariates (including 1): \(X = (1, X_1, X_2)^T\) with binary \(X_2\) (i.e., \(X_2 \in \{0, 1\}\)), and we are interested in testing lack of significance of the covariate \(X_2\), i.e., \(m(X_1, 0) = m(X_1, 1)\) (the constant 1 is omitted from the expression of \(m(X)\)). This can be carried out by picking designs points \(x_1^{(1)}, \ldots, x_1^{(M)}\) from the support of \(X_1\), and testing the simultaneous hypothesis that \(m(x_1^{(k)}, 0) = m(x_1^{(k)}, 1)\) (or equivalently \(m(x_1^{(k)}, 0) - m(x_1^{(k)}, 1) = 0\)) for all \(k = 1, \ldots, M\). Calibrating critical values for such tests requires to approximate the distribution of \(\hat{m}(x_{1}^{(k)}, 0) - \hat{m}(x_{1}^{(k)}, 1)\) for all \(k = 1, \ldots, M\). If we define \(x_{2k-1}^{(k)} = (x_{1}^{(k)}, 0)\) and \(x_{2k}^{(k)} = (x_{1}^{(k)}, 1)\) for \(k = 1, \ldots, M\), then the corresponding \(D\) matrix is the same as (5).

To cover above applications in a unified way, we consider to approximate the distribution of \(D(\hat{m}(x_t) - m(x_t))_{t=1}^{L}\). We will first show that, under regularity conditions, \(\sqrt{n h^3} D(\hat{m}(x_t) - m(x_t))_{t=1}^{L}\) can be approximated by an \(L\)-dimensional Gaussian vector uniformly over the hyperrectangles in \(\mathbb{R}^L\), even when \(L\) and \(M\) are possibly much larger then \(n\). This approximating Gaussian distribution is infeasible in practice since its covariance matrix is unknown. To deal with this difficulty, we propose a novel bootstrap to further approximate the sampling distribution.

### 3.3.1 Gaussian approximation

Define \(\Psi_i := (\psi_{x_1}(U_i, X_i), \ldots, \psi_{x_L}(U_i, X_i))^T\) and \(\Sigma := \mathbb{E} [\Psi_i \Psi_i^T]\). For \(k = 1, \ldots, M\), let \(D_k^T\) denote the \(k\)-th row of the matrix \(D\). We may assume without loss of generality that each row \(D_k\) is nonzero. Further, we will assume that the matrix \(D\) is sparse in the sense that the number of nonzero elements of each row \(D_k\) is of constant order, which is satisfied in
all the examples discussed above. We are primarily interested in inference for the vector 
\((m(x_\ell))_{\ell=1}^L\), so we consider to normalize the coordinates of the vector by their approximate 
standard deviations (technically the normalization does not matter for the Gaussian ap-
proximation, but we will replace the approximate standard deviations by their estimates in 
the bootstrap, whose effect has to be taken care of). Let 
\(S_k := \{\ell \in \{1,\ldots,L\} : D_{k,\ell} \neq 0\}\) 
denote the support of \(D_k\). Define \(\sigma^2_k := D_k^T \Sigma D_k\) for \(k = 1,\ldots,M\), which corresponds to 
the variance of \(D_k^T \Psi_i\), and \(\Gamma := \text{diag}\{\sigma_1,\ldots,\sigma_M\}\). Set 
\(A = (A_1,\ldots,A_M)^T := \Gamma^{-1} D\).

Related to the matrix \(D\), we make the following assumption.

**Assumption 2.** We assume the following conditions.

(i) \(\max_{1\leq k\leq M} |S_k| = O(1)\) and \(\max_{1\leq k\leq M; 1\leq \ell \leq L} |D_{k,\ell}| = O(1)\).

(ii) There exists a fixed constant \(c_3 > 0\) such that \(\min_{1\leq k\leq M} \sigma_k \geq c_3\).

Condition (i) is a sparsity assumption on the matrix \(D\) discussed above. Condition (ii) 
excludes the situation where \(D_k^T \Psi_i\) has vanishing variance.

The following theorem derives a Gaussian approximation result.

**Theorem 1** (Gaussian approximation). Suppose that Assumptions [1] and [2] hold. In addition, 
assume that

\[
\frac{\log^7 (Mn)}{nh} \sqrt{ \frac{\log^3 (Mn)}{n^{1-2/q} h} } \sqrt{ \frac{(\log^2 n) \log M}{nh^5} } \to 0 \quad \text{and} \quad (nh^7 \vee h) \log M \to 0. \tag{7}
\]

Then, we have

\[
\sup_{b \in \mathbb{R}^M} \left| \mathbb{P} \left( A \sqrt{nh^3}(\hat{m}(x_\ell) - m(x_\ell))_{\ell=1}^L \leq b \right) - \mathbb{P} (AG \leq b) \right| \to 0,
\]

where \(G\) is an \(L\)-dimensional Gaussian random vector with mean 0 and covariance \(\Sigma\).

Condition [7] allows \(M\) to be much larger than \(n\), i.e., \(M \gg n\). The condition that 
\(nh^7 \log M \to 0\) is an “undersmoothing” condition that ensures that the deterministic bias 
is negligible relative to the stochastic error. This condition can be relaxed by assuming 
additional smoothness conditions on the conditional density and using higher order kernels.

We do not pursue this extension for brevity. Discussion on the bandwidth selection can be 
found in Section 4.1.1.
The proof of Theorem 1 can be found in the Appendix. The proof builds on the uniform asymptotic linear representation developed in Proposition 1 coupled with the high dimensional Gaussian approximation techniques developed in Chernozhukov et al. (2014, 2017). From Theorem 1, we see that the distribution of $A\sqrt{n h^3}(\hat{m}(x_\ell) - m(x_\ell))_{\ell=1}^L$ can be approximated by the distribution of $AG$ uniformly over the rectangles. Still, the distribution of $AG$ is unknown since the covariance matrix of $G$ is unknown. We will use a new bootstrap called the pivotal bootstrap to further estimate the distribution of $AG$.

Remark 3 (Limit distribution of maximum deviation). It is of interest to find a limit distribution of the maximum deviation, $\zeta_n := \max_{1 \leq \ell \leq L} \sqrt{n h^3}|\hat{m}(x_\ell) - m(x_\ell)|/\sigma_x$ with $\sigma_x^2 = \mathbb{E}[\psi_x(U, X)^2]$, when $L = L_n \to \infty$ after a suitable normalization. Such a limit distribution enables us to find analytical critical values for simultaneous confidence intervals. Indeed, combining Theorem 1 with extreme value theory (cf. Leadbetter et al., 1983), we can derive a limit distribution for the maximal deviation under additional regularity conditions.

Proposition 2 (Limit distribution of maximal deviation). Suppose that Assumption 1 and Condition 7 with $M = L$ hold. Let $\zeta_n := \max_{1 \leq \ell \leq L} \sqrt{n h^3}|\hat{m}(x_\ell) - m(x_\ell)|/\sigma_x$ with $\sigma_x^2 = \mathbb{E}[\psi_x(U, X)^2]$. Assume $L = L_n \to \infty$, and define

$$a_n = (2 \log L_n)^{1/2} \quad \text{and} \quad b_n = (2 \log L_n)^{1/2} - \frac{1}{2}(2 \log L_n)^{-1/2}(\log \log L_n + \log \pi).$$

If, in addition, $\tau_{x_1}, \ldots, \tau_{x_L}$ are all distinct and $\min_{k \neq \ell} |\tau_{x_k} - \tau_{x_\ell}| > 2h$ for sufficiently large $n$, then $a_n(\zeta_n - b_n)$ converges in distribution to the Gumbel distribution, i.e.,

$$\lim_{n \to \infty} \mathbb{P}(a_n(\zeta_n - b_n) \leq t) = e^{-e^{-t}}, \ t \in \mathbb{R}.$$ 

Proposition 2 suggests that we can use the Gumbel approximation to construct simultaneous confidence intervals. The proof shows that if $\min_{k \neq \ell} |\tau_{x_k} - \tau_{x_\ell}| > 2h$, then $\Sigma$ is diagonal so that $\zeta_n$ can be approximated by the maximum in absolute value of $L$ independent $N(0, 1)$ random variables, which can be further approximated (after normalization) by the Gumbel distribution by extreme value theory. Compared with the pivotal bootstrap to be discussed in the following section, the Gumbel approximation leads to analytical critical
values, so from a computational perspective, using the Gumbel limit seems more attractive. However, the justification of the Gumbel approximation relies on a nontrivial spacing assumption on \( \tau_{x_k} \)'s (which the pivotal bootstrap does not). More importantly, convergence of normal suprema is known to be extremely slow [Hall, 1991], so simultaneous confidence intervals constructed from the Gumbel approximation may not have desirable coverage accuracy.

### 3.3.2 Pivotal bootstrap

The proof of Theorem [1] shows that the distribution of \( G \) comes from approximating the distribution of the process

\[
\mathbf{x} \mapsto \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_{x}(U_i, X_i)
\]

at \( \mathbf{x} \in \{x_1, \ldots, x_L\} \). Importantly, the process \( \mathbf{x} \mapsto \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_{x}(U_i, X_i) \) is “pivotal” in the sense that its distribution is completely known up to some estimable nuisance parameters given \( X_1, \ldots, X_n \) since \( U_1, \ldots, U_n \) are independent \( U(0, 1) \) random variables. The baseline idea of the pivotal bootstrap is to simulate the pivotal process \( \mathbf{x} \mapsto \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_{x}(U_i, X_i) \) (given the data) to estimate the distribution of \( G \) by generating \( U(0, 1) \) random variables.

To implement the pivotal bootstrap, we first have to estimate the nuisance parameters. We consider to estimate the matrix \( J(\tau) = \mathbb{E}[f(\mathbf{X}^T \beta(\tau) \mid \mathbf{X}) \mathbf{X} \mathbf{X}^T] \) by Powell’s kernel method [Powell, 1986], i.e.,

\[
\hat{J}(\tau) := \frac{1}{n} \sum_{i=1}^{n} \hat{K}_{\hat{h}_n}(Y_i - \mathbf{X}_i^T \hat{\beta}(\tau)) \mathbf{X}_i \mathbf{X}_i^T,
\]

where \( \hat{K} : \mathbb{R} \to \mathbb{R} \) is a kernel function and \( \hat{h}_n \) is a bandwidth. For simplicity of exposition, we will use \( \hat{K} = K \) and \( \hat{h}_n = h \). Then, we shall estimate the influence function \( \psi_{x} \) by

\[
\hat{\psi}_x(u, \mathbf{x}') := -\frac{\hat{s}_x'_{\hat{\tau}_x}}{\hat{s}_x''(\hat{\tau}_x)\sqrt{\hat{h}}} K'(\frac{\hat{\tau}_x - u}{h}) \mathbf{x}'^T \hat{J}(\hat{\tau}_x)^{-1} \mathbf{x}',
\]

where \( \hat{s}_x''(\tau) \) is the second derivative of \( \hat{s}_x(\tau) \) with respect to \( \tau \).

The pivotal bootstrap reads as follows. Generate \( U_1, \ldots, U_n \sim U(0, 1) \) i.i.d. that are independent of the data \( D_n := (Y_i, X_i)_{i=1}^{n} \). We denote the conditional probability \( \mathbb{P}(\cdot \mid D_n) \)
and conditional expectation $\mathbb{E}[\cdot \mid D_n]$ by $\mathbb{P}_U(\cdot)$ and $\mathbb{E}_U[\cdot]$, respectively. Define

$$
\hat{\Psi}_i := \left( \hat{\psi}_{x_1}(U_i, X_i), \ldots, \hat{\psi}_{x_L}(U_i, X_i) \right)^T \quad \text{and} \quad \hat{\Sigma} := \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_U[\hat{\Psi}_i \hat{\Psi}_i^T].
$$

Then, we shall estimate the distribution of $\Delta G$ (or $n^{-1/2} \sum_{i=1}^{n} A\hat{\Psi}_i$) by the conditional distribution of $n^{-1/2} \sum_{i=1}^{n} \hat{A} \hat{\Psi}_i$ given the data $D_n$, where $\hat{A} = \hat{\Gamma}^{-1} D$ and

$$
\hat{\Gamma} = \text{diag}\{\hat{\sigma}_1, \ldots, \hat{\sigma}_M\} := \text{diag}\left\{ \sqrt{D_1^T \hat{\Sigma} D_1}, \ldots, \sqrt{D_M^T \hat{\Sigma} D_M} \right\}.
$$

The conditional distribution can be simulated with arbitrary precision. The following theorem establishes consistency of the pivotal bootstrap over the rectangles.

**Theorem 2** (Validity of pivotal bootstrap). Suppose that Assumptions 1 and 2 hold with $q > 4$ in Condition (v) in Assumption 1. In addition, assume that

$$
\frac{\log^2 (Mn)}{n^{1/2/q_h}} \sqrt{\frac{\log^3 (Mn)}{n^{1-4/q_h}}} \sqrt{\frac{(\log n) \log^4 M}{nh^5}} \to 0 \quad \text{and} \quad h \log^2 M \to 0.
$$

Then, we have

$$
\sup_{b \in \mathbb{R}^M} \left| \mathbb{P}_U \left( n^{-1/2} \sum_{i=1}^{n} \hat{A} \hat{\Psi}_i \leq b \right) - \mathbb{P} (\Delta G \leq b) \right| \to 0.
$$

The proof of Theorem 2 can be found in the Appendix. The proof of Theorem 2 is nontrivial and does not follow directly from existing results since the pivotal bootstrap differs from the nonparametric or multiplier bootstraps that have been analyzed in the literature in the high-dimensional setup. The proof consists of two steps. First, noting that $\hat{\Psi}_1, \ldots, \hat{\Psi}_n$ are independent with mean zero conditionally on the data $D_n$ (cf. equation (4)), we apply the high dimensional CLT conditionally on $D_n$ to approximate the conditional distribution of $n^{-1/2} \sum_{i=1}^{n} \hat{A} \hat{\Psi}_i$ by the conditional Gaussian distribution $N(0, \hat{A} \hat{\Sigma} \hat{A}^T)$. Second, we compare the $N(0, \hat{A} \hat{\Sigma} \hat{A}^T)$ distribution with $\Delta G \sim N(0, A \Sigma A^T)$ by a Gaussian comparison technique.

As a byproduct of the proof of Theorem 2 we can show that the conclusion of Theorem 1 continues to hold even if the matrix $A$ acting on $(\hat{m}(x_\ell) - m(x_\ell))_{\ell=1}^{L}$ is replaced by its estimate $\hat{A}$.

**Proposition 3.** Suppose that the conditions of Theorem 1 holds. In addition, assume that

$$
\frac{(\log n) \log^2 M}{nh^5} \to 0 \quad \text{and} \quad h \log M \to 0.
$$

19
Then, we have
\[
\sup_{b \in \mathbb{R}^M} \left| \mathbb{P}\left( \hat{\mathbf{A}} \sqrt{n h^3 (\hat{m}(\mathbf{x}_t) - m(\mathbf{x}_t))_{t=1}^L} \leq b \right) - \mathbb{P}(AG \leq b) \right| \to 0.
\]

In what follows, we discuss applications of the pivotal bootstrap to constructions of pointwise and simultaneous confidence intervals and testing.

**Example 3 (Simultaneous confidence intervals).** Consider to construct a simultaneous confidence interval for \(m(\mathbf{x}_1), \ldots, m(\mathbf{x}_L)\). In this case, \(D = I_L (M = L)\), \(A = \text{diag}\{1/\sigma_{x_1}, \ldots, 1/\sigma_{x_L}\}\), and \(\hat{\mathbf{A}} = \text{diag}\{1/\hat{\sigma}_{x_1}, \ldots, 1/\hat{\sigma}_{x_L}\}\), where \(\sigma^2_{x} = \mathbb{E}[\psi_{x}(U, \mathbf{X})^2]\) and \(\hat{\sigma}^2_{x} = n^{-1} \sum_{i=1}^{n} \mathbb{E}_U[\hat{\psi}_{x}(U_i, \mathbf{X}_i)^2]\).

Then, Proposition 3 and Theorem 2 imply that, for \(G = (g_1, \ldots, g_L)^T \sim N(0, \Sigma)\),
\[
\sup_{b \in \mathbb{R}} \mathbb{P}\left( \max_{1 \leq \ell \leq L} \left| \sqrt{n h^3 (\hat{m}(\mathbf{x}_\ell) - m(\mathbf{x}_\ell))_{\ell=1}^L} \right| \leq b \right) - \mathbb{P}\left( \max_{1 \leq \ell \leq L} \left| g_\ell / \sigma_{x_\ell} \right| \leq b \right) \to 0, \quad \text{(9)}
\]

Denoting by \(\hat{q}_{1-\alpha} = \text{conditional } (1 - \alpha)\)-quantile of \(\max_{1 \leq \ell \leq L} \left| n^{-1/2} \sum_{i=1}^{n} \hat{\psi}_{x_\ell}(U_i, \mathbf{X}_i) / \hat{\sigma}_{x_\ell} \right|\), we can show that the data-dependent rectangle (interval when \(L = 1\))
\[
\prod_{\ell=1}^{L} \left[ \hat{m}(\mathbf{x}_\ell) \pm \frac{\hat{\sigma}_{x_\ell}}{\sqrt{n h^3}} \hat{q}_{1-\alpha} \right]
\]
contains the vector \((m(\mathbf{x}_\ell))_{\ell=1}^L\) with probability approaching \(1 - \alpha\).

Formally, the coverage guarantee of the preceding confidence rectangle follows from
\[
\mathbb{P}\left( \max_{1 \leq \ell \leq L} \left| \sqrt{n h^3 (\hat{m}(\mathbf{x}_\ell) - m(\mathbf{x}_\ell))_{\ell=1}^L} / \hat{\sigma}_{x_\ell} \right| \leq \hat{q}_{1-\alpha} \right) \to 1 - \alpha. \quad \text{(10)}
\]

The latter (10) follows from the preceding convergence result (9) coupled with the following Lemma 1 (note: since in general \(\max_{1 \leq \ell \leq L} \left| g_\ell / \sigma_{x_\ell} \right|\) need not have a limit distribution, it is not immediate that the former (9) implies the latter (10); cf. Lemma 23.3 in van der Vaart (2000)). A similar analysis can be done for constructing simultaneous confidence intervals for partial effects of certain covariates.
Lemma 1. Let $Y_n, W_n, Z_n$ be sequences of random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that (i) $Y_n$ is measurable relative to a sub-$\sigma$-field $\mathcal{C}_n$ (that may depend on $n$); (ii) $\sup_{t \in \mathbb{R}}|\mathbb{P}(Y_n \leq t) - \mathbb{P}(Z_n \leq t)| \to 0$ and $\sup_{t \in \mathbb{R}}|\mathbb{P}(W_n \leq t | \mathcal{C}_n) - \mathbb{P}(Z_n \leq t)| \to 0$; (iii) the distribution function of $Z_n$ is continuous for each $n$ ($Z_n$ need not have a limit distribution). Let $\hat{q}_n(\alpha)$ denote the conditional $\alpha$-quantile of $W_n$ given $\mathcal{C}_n$. Then $\mathbb{P}(Y_n \leq \hat{q}_n(\alpha)) \to \alpha$.

Example 4 (Testing significance of covariates). Consider testing the hypothesis $H_0 : m(x_1) = \cdots = m(x_L)$ for some $x_1, \ldots, x_L \in X_0$. In this case, the matrix $D$ is given by (6) with $M = L - 1$, and $A(\hat{m}(x_\ell) - m(x_\ell))_{\ell=1}^{L-1} = ((\hat{m}(x_{\ell+1}) - \hat{m}(x_{\ell}))/\sigma_{x_{\ell+1},x_\ell})_{\ell=1}^{L-1}$ under $H_0$, where $\sigma^2_{x_{\ell+1},x_\ell} = \mathbb{E}[(\psi_{x_{\ell+1}} - \psi_{x_\ell})^2(U,X)]$. Let $\hat{\sigma}^2_{x_{\ell+1},x_\ell} = n^{-1} \sum_{i=1}^{n} E_U[(\hat{\psi}_{x_{\ell+1}} - \hat{\psi}_{x_\ell})^2(U_i,X_i)]$. We shall consider the test of the form

$$\max_{1 \leq \ell \leq L-1} \sqrt{n} h^3 \frac{|\hat{m}(x_{\ell+1}) - \hat{m}(x_\ell)|}{\hat{\sigma}_{x_{\ell+1},x_\ell}} > c \Rightarrow \text{reject } H_0$$

(11)

for some critical value $c$. To calibrate critical values, we may use the pivotal bootstrap. For a given level $\alpha \in (0,1)$, let

$$\hat{c}_{1-\alpha} = \text{conditional } (1 - \alpha)-\text{quantile of } \max_{1 \leq \ell \leq L-1} \left| \frac{n^{-1/2} \sum_{i=1}^{n} (\hat{\psi}_{x_{\ell+1}} - \hat{\psi}_{x_\ell})(U_i,X_i)}{\hat{\sigma}_{x_{\ell+1},x_\ell}} \right|.$$ 

Then, Proposition 3 and Theorem 2 guarantee that, under regularity conditions, the test (11) with $c = \hat{c}_{1-\alpha}$ has level approaching $\alpha$ if $H_0$ is true (cf. the discussion at the end of the preceding example). The case where the $D$ matrix is given by (5) is similar; we omit the details for brevity.

4 Numerical Examples

4.1 Simulation Results

In this section, we evaluate the numerical performance of the pivotal bootstrap using synthetic data. We start with implementation details, in particular bandwidth selection.
4.1.1 Implementation details

In our simulation study, we use the biweight kernel, \( K(t) = \frac{15}{16}(1 - t^2)^2 I(|t| < 1) \), and use \( \epsilon = 0.1 \) when computing our modal estimator. We estimate the matrix \( J(\tau) \) by \( \hat{J}(\tau) = (2n\hat{h})^{-1} \sum_{i=1}^{n} I(|Y_i - X_i^T \hat{\beta}(\tau)| \leq \hat{h}) X_i X_i^T \), where \( \hat{h} \) is set to be the default bandwidth in \texttt{quantreg} package in R (the theory does not require the kernel used to estimate \( J(\tau) \) to be smooth). We find that computing \( \hat{s}''(\tau_x) \) by differentiating \( \hat{Q}_x(\tau) \) three times tends to be unstable in the finite sample. Instead, we use the alternative expression \( s''(\tau_x) = -f^{(2)}(Q_x(\tau_x) \mid x)s_x(\tau_x)^3 \) and estimate the derivative \( f^{(2)}(\cdot \mid x) \) by a kernel method as in Remark 9 of Ota et al. (2019) (we plug in \( \hat{Q}_x(\hat{\tau}_x) \) and \( \hat{s}_x(\hat{\tau}_x) \) for \( Q_x(\tau_x) \) and \( s_x(\tau_x) \), respectively).

Finally, we discuss bandwidth selection. Corollary 1 implies that the approximate MSE of \( \hat{m}(x) \) is

\[
\left[ \frac{s_x(\tau_x)s_x^{(3)}(\tau_x)}{2s_x^{(2)}(\tau_x)} \kappa \hat{h}^2 \right]^2 + \frac{\kappa_1 s_x(\tau_x)^2 \mathbb{E}[(x^T J(\tau_x)^{-1} X)^2]}{n\hat{h}^3 s_x''(\tau_x)^2}.
\]

The optimal \( h \) that minimizes the above approximate MSE is given by

\[
h_{opt}(x) := \left[ \frac{3\kappa_1 x^T J(\tau_x)^{-1} \mathbb{E}[XX^T] J(\tau_x)^{-1} x}{\kappa^2 s_x^{(3)}(\tau_x)^2} \right]^{1/7} n^{-1/7}.
\]

Here we make some remarks on the optimal bandwidth. First, we note direct use of \( h_{opt} \) will result in an asymptotic bias and a bias-correction will be needed. However, the asymptotic bias contains high order derivatives of underlying conditional quantile function which are hard to be estimated. Hence, we recommend a smaller bandwidth to be used in the finite sample implementation. In our numerical analysis, we take the bandwidth of order \( n^{-1/6} \) instead of \( n^{-1/7} \) and multiply \( h_{opt} \) by 0.8 to correct for too large bandwidth when the sample size is small. Additionally, for the unknown quantities in \( h_{opt} \), we plug in corresponding sample version estimators. However, considering that estimation of the fourth derivative of the conditional quantile function is highly unstable, we adopt a “rule of thumb” method by using the fourth derivative of the standard normal distribution instead of corresponding derivative estimator. For the uniform inference on multiple design points, we take bandwidth to be the median of the pointwise bandwidths of each design point. Our empirical results show above bandwidth selection approach works reasonably well.
4.1.2 Pointwise confidence intervals

We will consider two different models which correspond to linear and nonlinear mode functions respectively. Suppose covariate $X = (1, X_1)$ and $Y$ are generated from either of the following models,

- (Linear modal function) $Y = 1 + 3X_1 + \sigma(X_1) \cdot \xi,$
- (Nonlinear modal function) $Y = U^3/3 - X_1 \cdot U^2 + 1.5X_1 \cdot U.$

In the linear modal function case, we take $\sigma(x) = 1 + 2x$. For the distribution of $\xi$, we consider two cases: $\xi \sim N(0, 1)$ (lmNormal model) and $\log(\xi) \sim N(1, 0.64)$ (lmLognormal model). These two cases are interesting since the mode coincides with the conditional mean in the first case while they are different in the second. In particular, $m(X) = 1 + 3X_1$ for lmNormal model and $m(X) = 1 + 3X_1 + (1 + 2X_1) \cdot e^{0.36}$ for lmLognormal model, both of which are linear in $X$. Similar model has been considered in the simulation analysis of Yao and Li (2014) and Ota et al. (2019). For the nonlinear modal function case (Nonlinear model), we take $U \sim U(0, 1)$ and thus we have $m(X) = -2X_1^3/3 + 1.5X_1^2$ which is nonlinear in $X$. We generate the covariate $X_1$ in both models from the uniform distribution over $(0, 1)$.

For each model, we construct 95% and 99% confidence intervals for conditional modes with design $x = (1, 0.3)$, $(1, 0.5)$ and $(1, 0.7)$. We consider different sample sizes ranging from 500 to 2000 and repeat computing the confidence intervals for different conditional modes under different sample sizes for 500 times. The resultant empirical coverage probabilities and interval length statistics for each model are reported in Table 1 to Table 3. In the simulation, we find that some of the computed confidence intervals are extremely large, especially when the sample size is comparatively small ($n = 500$) due to unstable estimation of high order derivatives of the conditional quantile function. Therefore, we report the median length of the confidence intervals to exclude the influence of those extreme results.

From Table 1 to Table 3, the bootstrap confidence intervals achieve satisfying coverage probabilities in all three scenarios. We point out, in each case, the coverage probabilities
Table 1: Simulation results for pointwise confidence intervals of \text{lmNormal} model.

| Design point | Sample size | Coverage probability | Median length |
|--------------|-------------|----------------------|---------------|
|              |             | 95% | 99% | 95% | 99% |
| $X_1 = 0.3$  | $n = 500$   | 99.8% | 100% | 0.99 | 1.28 |
|              | $n = 1000$  | 100% | 100% | 0.68 | 0.89 |
|              | $n = 2000$  | 100% | 100% | 0.52 | 0.68 |
| $X_1 = 0.5$  | $n = 500$   | 95.8% | 98.4% | 0.91 | 1.21 |
|              | $n = 1000$  | 97.6% | 99.6% | 0.62 | 0.82 |
|              | $n = 2000$  | 97.6% | 99.4% | 0.50 | 0.64 |
| $X_1 = 0.7$  | $n = 500$   | 82.4% | 87.6% | 1.46 | 1.87 |
|              | $n = 1000$  | 87.2% | 93.8% | 0.86 | 1.15 |
|              | $n = 2000$  | 93.4% | 98.8% | 0.59 | 0.77 |

Table 2: Simulation results for pointwise confidence intervals of \text{lmLognormal} model.

| Design point | Sample size | Coverage probability | Median length |
|--------------|-------------|----------------------|---------------|
|              |             | 95% | 99% | 95% | 99% |
| $X_1 = 0.3$  | $n = 500$   | 98% | 98% | 30.26 | 40.38 |
|              | $n = 1000$  | 96.6% | 96.6% | 21.34 | 27.47 |
|              | $n = 2000$  | 100% | 100% | 18.25 | 24.07 |
| $X_1 = 0.5$  | $n = 500$   | 96.8% | 97.2% | 23.00 | 29.23 |
|              | $n = 1000$  | 94.4% | 94.4% | 13.79 | 18.11 |
|              | $n = 2000$  | 100% | 100% | 12.09 | 16.04 |
| $X_1 = 0.7$  | $n = 500$   | 96% | 96.6% | 39.31 | 48.78 |
|              | $n = 1000$  | 91.6% | 92.8% | 10.79 | 14.27 |
|              | $n = 2000$  | 98.8% | 98.8% | 9.44 | 12.43 |

of $X_1 = 0.7$ are slightly lower than the other two design points under the same sample size. This is because large $X_1$ results in a large variance of $Y$ which makes the estimation
Table 3: Simulation results for pointwise confidence intervals of Nonlinear model.

| Design point | Sample size | Coverage probability | Median length |
|--------------|-------------|----------------------|---------------|
|              |             | 95%  | 99%  | 95% | 99%  |
| $X_1=0.3$    | $n=500$     | 98.8% | 100% | 0.93 | 1.31 |
|             | $n=1000$    | 100% | 100% | 0.68 | 0.94 |
|             | $n=2000$    | 100% | 100% | 0.54 | 0.74 |
| $X_1=0.5$    | $n=500$     | 99.8% | 100% | 0.71 | 0.94 |
|             | $n=1000$    | 100% | 100% | 0.56 | 0.74 |
|             | $n=2000$    | 100% | 100% | 0.42 | 0.55 |
| $X_1=0.7$    | $n=500$     | 98.2% | 99.8% | 1.03 | 1.40 |
|             | $n=1000$    | 97.4% | 99.2% | 0.77 | 1.05 |
|             | $n=2000$    | 99%  | 99.4% | 0.62 | 0.83 |

more difficult. We report the mean squared error of our modal estimator, $\hat{m}_{x}$, in Table A1 in Section D.1 of Appendix which verifies this point. However, our bootstrap method still achieves approximate nominal coverage probabilities in such situation when the sample size is sufficiently large. Besides, we note the length of the confidence intervals decreases with the growing sample size for each design point across all three scenarios, which agrees with our asymptotic theories in the previous section. We also note that the resultant confidence intervals tend to be “conservative” in some cases. This is due to those extremely large confidence intervals.

### 4.1.3 Approximate confidence band

In this section, we investigate the finite sample performance of the pivotal bootstrap in simultaneous inference problems. In particular, we construct approximate confidence bands for the three different models considered in Section 4.1.2. To build an approximate confidence band, we compute simultaneous confidence intervals for a grid of $X_1$ over $[0.4, 0.6]$ with a break 0.01.
We repeat the simulation 100 times for each model, and calculate the empirical coverage probabilities and the median lengths defined by taking the median of the median length of the simultaneous confidence intervals in one simulation. The median is used to reduce the influence of potential extreme results in the simulations. The resultant empirical coverage probabilities and median lengths of the approximate confidence bands for each model are presented in the Table 4.

Table 4: Simulation results for approximate confidence bands of lmNormal, lmLognormal and Nonlinear models with design points falling in interval [0.4,0.6].

| Models     | Sample size | Coverage probability | Median length |
|------------|-------------|----------------------|---------------|
|            |             | 95%  | 99%  | 95% | 99% |               |
| lmNormal   | n = 500     | 94%  | 97%  | 1.31| 1.67|               |
|            | n = 1000    | 96%  | 100% | 0.86| 1.12|               |
|            | n = 2000    | 100% | 100% | 0.66| 0.88|               |
| lmLognormal| n = 500     | 98%  | 98%  | 25.38| 29.01|               |
|            | n = 1000    | 98%  | 98%  | 19.14| 23.67|               |
|            | n = 2000    | 100% | 100% | 17.91| 22.25|               |
| Nonlinear  | n = 500     | 100% | 100% | 1.17 | 1.43 |               |
|            | n = 1000    | 98%  | 100% | 0.99 | 1.24 |               |
|            | n = 2000    | 100% | 100% | 0.83 | 1.03 |               |

From Table 4, the approximate confidence bands successfully capture the modes simultaneously with probability either close to or above the nominal probability. Additionally, similar to the pointwise confidence interval, the lengths of the confidence bands decrease while the sample size grows.

4.1.4 Testing lack of significance

In this section, we consider testing significance of a covariate on the conditional mode. Suppose covariate $X = (1, X_1, X_2)$ where $X_1$ is continuous and $X_2$ is binary (0 or 1). We
want to test the null hypothesis $H_0 : m(X_1, 0) = m(X_1, 1)$ versus the alternative hypothesis $H_1 : m(X_1, 0) \neq m(X_1, 1)$, where $m(x_1, x_2)$ is the conditional mode of $Y$ given $X_1 = x_1$ and $X_2 = x_2$. We will generate $X$ according to $X_1 \sim \text{Unif}(0, 1)$ and $X_2 \sim \text{Binomial}(0.5)$. For the outcome $Y$, two generation schemes are considered: (1) $Y = 1 + 3X_1 + \xi$ and (2) $Y = 1 + 3X_1 + 3X_2 + \xi$, where we take $\xi \sim N(0, 1)$ in both models. The corresponding mode functions are $m(X) = 1 + 3X_1$ and $m(X) = 1 + 3X_1 + 3X_2$, respectively. Therefore, the two generation schemes correspond to $H_0$ being true and false respectively which allows us to evaluate both power and size of our bootstrap testing procedure. In the current setup, the limiting Gaussian distribution given by Theorem 1 is one dimensional and the corresponding variance can be calculated explicitly based on the above setup. Therefore, an oracle test procedure can be constructed by using the quantiles of the corresponding limiting Gaussian distribution to define the test rejection region. We will compare the performance of our bootstrap testing with this benchmark oracle testing.

We conduct hypothesis testing of nominal level 0.05 and 0.01 for $X_1$ taking value at 0.3, 0.5 and 0.7. For each value of $X_1$, three different sample sizes from 500 to 2000 are considered. We report the empirical size and power of both bootstrap testing and oracle testing based on 500 simulations in Table 5. It can be seen from Table 5 that both bootstrap testing and oracle testing perform reasonably well. The Type I errors are well preserved for both tests at three design points while the bootstrap testing committed slightly fewer Type I errors. The power of both tests approaches 1 when the sample size increases. Besides, we remark that the good performance of the oracle testing justifies our Gaussian approximation in Section 3.3. To further explore the performance of the pivotal bootstrap test when the alternative hypothesis is “close” to the null hypothesis, we provide extra simulation results in Section D.2 in the Appendix.

### 4.2 US wage data

In this section, we apply the pivotal bootstrap inference framework on a real US wage data. The data is extracted from 1980 U.S. Census micro data used in Angrist et al. (2006). We keep all the data of the black people, randomly select the same amount of data of the white
Table 5: Size and power for bootstrap testing and oracle testing.

| Design Point | Sample size | Bootstrap testing | Oracle testing |
|--------------|-------------|-------------------|----------------|
|              |             | Size   | Power  | Size   | Power  | Size   | Power  |
|              |             | 0.05   | 0.01   | 0.05   | 0.01   | 0.05   | 0.01   |
| $X_1 = 0.3$  | $n = 500$    | 0.002  | 0      | 0.962  | 0.908  | 0.04   | 0.01   |
| $X_1 = 0.5$  | $n = 500$    | 0      | 0      | 0.988  | 0.956  | 0.03   | 0.004  |
| $X_1 = 0.7$  | $n = 500$    | 0.002  | 0      | 0.966  | 0.922  | 0.026  | 0.01   |
|              | $n = 1000$   | 0      | 0      | 1      | 0.996  | 0.018  | 0      |
|              | $n = 2000$   | 0.004  | 0      | 1      | 1      | 0.05   | 0.008  |
|              | $n = 1000$   | 0      | 0      | 1      | 0.998  | 0.02   | 0      |
|              | $n = 2000$   | 0      | 0      | 1      | 1      | 0.04   | 0.002  |
|              | $n = 2000$   | 0      | 0      | 1      | 1      | 0.026  | 0      |

people and combine both of them as our new dataset. In the new dataset, log-transformed weekly income (logwk) of 9944 different people, which will be the response $Y$, are collected together with corresponding covariates including education (educ), work experience (exper) and race.

We investigate the marginal effect of race on the conditional modes using pivotal bootstrap testing. Specifically, we investigate whether the most common wage is different in black and white people given the same education and work experience. This can be formulated as a testing of covariate significance problem considered in the simulation study. Specifically, we take the two other covariates, education and work experience, to be the full-sample median of each covariate and test the equality of the resultant conditional mode between two races. The estimation and testing results are presented in Table 6.

To provide an intuitive evaluation of the estimation, in Figure 1, we collect the people with median values of education and work experience from the two groups and plot KDE-based density estimates superimposed on histograms of their log weekly wage respectively.
Table 6: Testing results of marginal effect of race.

| Estimated mode of wage | Testing result |
|------------------------|----------------|
| Black                  | 0.05           |
| White                  | 0.01           |
| 6.30                   | Not reject     |
| 6.45                   | Not reject     |

The estimated modes and sample means are also highlighted in Figure 1.

![Histograms of log weekly wage for black and white people with median values of education and work experience.](image)

Figure 1: Histograms of log weekly wage for black and white people with median values of education and work experience.

From Figure 1, we have several observations: first, both conditional distributions, especially for the black people, are skewed which justifies the usage of modal regression; second, our modal estimator provides accurate estimations of conditional modes for both groups. Though the estimated modes are slightly different in two groups, our bootstrap testing procedure suggests not rejecting the null hypothesis under both nominal sizes, which indicates the difference of the conditional modes between the two groups is not statistically significant.

5 Extension to the increasing dimension case

In this section, we extend the theoretical analysis to the case where the dimension $d$ of the covariate vector is allowed to increase with the sample size $n$, i.e., $d = d_n \to \infty$. 
Such situation arises when we approximate conditional quantile function $Q_x(\tau)$ by a linear combination of series terms and the series approximation error is negligible (in fact, the theory of this section holds as long as the approximation error is at most of the order as the residue term in the Bahadur’s representation; see Lemma 13 in the Appendix). In this case, $X$ is generated as basis functions of a fixed dimensional genuine covariate $Z$, i.e., $X = W(Z)$, where vector $W(Z)$ includes transformations of $Z$ that have good approximation properties such as Fourier series, splines, and wavelets; cf. Belloni et al. (2015, 2019). It is then of interest to draw simultaneous confidence intervals for the conditional mode along with values of $Z$ which has fixed dimension though the dimension of $X$ increases with $n$.

We first modify Assumption 1 to accommodate the case where $d = d_n \to \infty$. In what follows, constants refer to nonrandom numbers independent of $n$.

**Assumption 3.** We assume the following conditions.

(i) There exists a constant $C_2 \geq 1$ such that $C_2^{-1} \sqrt{d} \leq \|x\| \leq C_2 \sqrt{d}$ for all $x \in \mathcal{X}_0$.

(ii) There exists $\epsilon_1 \in (\epsilon, 1)$ such that $\tau_x \in [\epsilon_1, 1 - \epsilon_1]$ for all $x \in \mathcal{X}_0$.

(iii) There exists a positive constant $C_3$ such that $P(\|X\| \leq C_3 \sqrt{d}) = 1$. The Gram matrix $E[XX^T]$ is positive definite with smallest eigenvalue $\lambda_{\text{min}} \geq c_{\text{min}} > 0$ and largest eigenvalue $\lambda_{\text{max}} \leq c_{\text{max}} < \infty$ for some constants $c_{\text{min}}$ and $c_{\text{max}}$.

(iv) Conditions (iv)–(vii) in Assumption 1 hold.

(v) For any $\delta > 0$, there exists a positive constant $c_4$ (that may depend on $\delta$) such that

$$\inf_{x \in \mathcal{X}_0} \inf_{\tau \in [\epsilon, 1-\epsilon]} |\tau - \tau_x| \geq \delta \{ s_x(\tau) - s_x(\tau_x) \} \geq c_4.$$ 

(vi) $d^4 = o(n^{1-c_5})$ for some $c_5 \in (0, 1)$.

Condition (i) requires the design points of interest to be of the same order $\sqrt{d}$. We assume Condition (i) to state the results in a concise way, but the $\sqrt{d}$ order can be relaxed as long as $\inf_{x \in \mathcal{X}_0} \|x\|$ and $\sup_{x \in \mathcal{X}_0} \|x\|$ are of the same order. The modified condition (ii) is assumed to avoid boundary problems of $\tau_x$ when the dimension increases. We also assume that $\|X\|$ is bounded by $C_3 \sqrt{d}$ to avoid some technicalities. In particular, under
series approximation framework, this assumption is satisfied when \( \mathbf{X} \) is generated from basis functions such as Fourier series, B-splines and wavelet series; cf. Belloni et al. (2015). The condition on the Gram matrix is satisfied under mild conditions on the distribution of the genuine covariate \( \mathbf{Z} \) and basis functions; cf. Belloni et al. (2019). Condition (v) is a global identification condition on \( \tau_x \) that is needed to verify the uniform consistency of \( \hat{\tau}_x \). If \( d \) is fixed, then Condition (v) follows automatically as \( x \mapsto \tau_x \) is continuous under Assumption 1 (see the proof of Lemma 8), but if \( d = d_n \to \infty \), then \( s_x \) and \( \tau_x \) depend on \( n \), so that we require Condition (v). Condition (vi) is used to guarantee the Bahadur representation of \( \hat{\beta}(\tau) \); cf. Theorem 2 in Belloni et al. (2019).

Redefine \( \Psi_i \) as \( \Psi_i := (\xi_x(U_i, \mathbf{X}_i), \ldots, \xi_{xL}(U_i, \mathbf{X}_i))^T \) with

\[
\xi_x(u, \mathbf{x}') := \frac{s_x(\tau_x)}{s_x''(\tau_x)\sqrt{d}h} \int \mathbf{x}'^T J(t) \text{J}^{-1}(t) \left\{ t - I(U \leq t) \right\} \mathbf{K}'' \left( \frac{\tau_x - t}{h} \right) dt.
\]

Further, redefine the matrices \( \Sigma, \Gamma, \) and \( A \) as in Section 3.3.1 corresponding to the new definition of \( \Psi_i \). The reason to work with \( \xi_x \) instead of \( \psi_x \) is to better control the residual term in the proof of high dimensional Gaussian approximation result. Normalization by \( \sqrt{d} \) ensures that the norm of \( x/\sqrt{d} \) is bounded on \( \mathcal{X}_0 \). The Gaussian approximation with \( d = d_n \to \infty \) reads as follows.

**Theorem 3** (Gaussian approximation when \( d = d_n \to \infty \)). Suppose that Assumptions 2 and 3 hold and we also assume that

\[
\frac{d \log^7 (Mn)}{nh} \sqrt{\frac{d^4 (\log^2 n) \log^2 M}{nh^2}} \sqrt{\frac{d^3 (\log^2 n) \log M}{nh^5}} \to 0 \quad \text{and} \quad \frac{nh^7 \log M}{d} \to 0. \quad (12)
\]

Then, we have

\[
\sup_{b \in \mathbb{R}^n} \left| \mathbb{P} \left( A \sqrt{nh^3 \mathbb{E}}(\hat{m}(\mathbf{x}_t) - m(\mathbf{x}_t))_{t=1}^L \leq b \right) - \mathbb{P}(AG \leq b) \right| \to 0, \quad \text{with} \ G \sim \mathcal{N}(0, \Sigma).
\]

Suppose that \( \log M = O(\log n) \); then Condition (12) reduces to

\[
\frac{d^4 \log^4 n}{nh^2} \sqrt{\frac{d^3 \log^3 n}{nh^5}} \to 0 \quad \text{and} \quad \frac{nh^7 \log n}{d} \to 0.
\]

If we take \( h = (n/d)^{-1/7}(\log n)^{-2} \), then the condition on \( d \) reduces to \( d^8 \cdot \text{polylog}(n) = o(n) \). As before, this condition can be relaxed by assuming additional smoothness conditions on
the conditional density and using higher order kernels. Similar conditions on \( d \) appear in
the analysis of resampling methods for quantile regression under increasing dimensions;
see, e.g., Theorem 5 in Belloni et al. (2019), where we need \( d = o(n^{1/10}) \).

We turn to the pivotal bootstrap. Redefine \( \hat{\Psi}_i = (\hat{\psi}_{x_1}(U_i, X_i), \ldots, \hat{\psi}_{x_L}(U_i, X_i))^T \) with

\[
\hat{\psi}_x(u, x') := -\frac{\hat{s}_x'(\hat{\tau}_x)}{\hat{s}_x''(\hat{\tau}_x)} \frac{\sqrt{dh}}{K'\left(\frac{\hat{\tau}_x - U_i}{h}\right)} x^T J(\hat{\tau}_x)^{-1} X_i
\]

Let \( \hat{A} \) be defined as in Section 3.3.2 corresponding to the new definition of \( \hat{\psi}_x \).

**Theorem 4** (Validity of pivotal bootstrap when \( d = d_n \to \infty \)). Suppose that Assumptions 2 and 3 hold and we also assume that

\[
\frac{d \log^7 (Mn)}{nh} \sqrt{\frac{d^2(d \vee h^{-2})(\log n) \log^4 M}{nh^3}} \to 0 \quad \text{and} \quad h \log^2 M \to 0.
\]

Then, we have

\[
\sup_{b \in \mathbb{R}^M} \left| \mathbb{P}_U \left( n^{-1/2} \sum_{i=1}^n \hat{A} \hat{\Psi}_i \leq b \right) - \mathbb{P}(AG \leq b) \right| \overset{P}{\to} 0.
\]

**Remark 4.** The pivotal bootstrap above is the same as the one under the fixed dimension case as the extra normalization by \( \sqrt{d} \) is canceled by the multiplication by \( \hat{A} \) (we introduced normalization by \( \sqrt{d} \) to facilitate the proof).

### 6 Summary

In this paper, we propose a novel pivotal bootstrap for uniform inference on conditional modes based on a kernel-smoothed Koenker-Bassett quantile estimator. Our pivotal bootstrap inference framework allows for simultaneous inference on multiple linear functions of different conditional modes which is general to deal with numerous practical inference problems. Building on recent high dimensional probabilistic tools, we prove a high dimensional Gaussian approximation result and bootstrap consistency theorem for the validity of our pivotal bootstrap inference under both fixed dimension and increasing dimension settings. The numerical results not only provide strong support of our theoretical results, but also demonstrate that the new bootstrap inference framework is a flexible and powerful tool for modal regression.
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Appendix A  Technical tools

In this section, we collect technical tools that will be used in the subsequent proofs. For a probability measure $Q$ on a measurable space $(S, \mathcal{S})$ and a class of measurable functions $\mathcal{F}$ on $S$ such that $\mathcal{F} \subset L^2(Q)$, let $N(\mathcal{F}, \| \cdot \|_{Q,2}, \delta)$ denote the $\delta$-covering number for $\mathcal{F}$ with respect to the $L^2(Q)$-seminorm $\| \cdot \|_{Q,2}$. The class $\mathcal{F}$ is said to be pointwise measurable if there exists a countable subclass $\mathcal{G} \subset \mathcal{F}$ such that for every $f \in \mathcal{F}$ there exists a sequence $g_m \in \mathcal{G}$ with $g_m \to f$ pointwise. A function $F : S \to [0, \infty]$ is said to be an envelope for $\mathcal{F}$ if $F(x) \geq \sup_{f \in \mathcal{F}} |f(x)|$ for all $x \in S$. See Section 2.1 in van der Vaart and Wellner (1996) for details. For a vector-valued function $g$ defined over a set $T$, we define $\|g\|_T := \sup_{x \in T} \|g(x)\|.$

**Lemma 2** (Local maximal inequality). Let $X, X_1, \ldots, X_n$ be i.i.d. random variables taking values in a measurable space $(S, \mathcal{S})$, and let $\mathcal{F}$ be a pointwise measurable class of (measurable) real-valued functions on $S$ with measurable envelope $F$. Suppose that $\mathcal{F}$ is VC type, i.e., there exist constants $A \geq e$ and $V \geq 1$ such that

$$\sup_Q N(\mathcal{F}, \| \cdot \|_{Q,2}, \epsilon\|F\|_{Q,2}) \leq (A/\epsilon)^V, \quad 0 < \epsilon \leq 1,$$

where $\sup_Q$ is taken over all finitely discrete distributions on $S$. Furthermore, suppose that $0 < \mathbb{E}[F^2(X)] < \infty$, and let $\sigma^2 > 0$ be any positive constant such that $\sup_{f \in \mathcal{F}} \mathbb{E}[f^2(X)] \leq \sigma^2 \leq \mathbb{E}[F^2(X)]$. Define $B = \sqrt{\mathbb{E}[\max_{1 \leq i \leq n} F^2(X_i)]}$. Then

$$\mathbb{E} \left[ \left\| \frac{1}{\sqrt{n}} \sum_{j=1}^n \{f(X_j) - \mathbb{E}[f(X)]\} \right\|_{\mathcal{F}} \right] \leq C \left[ V\sigma^2 \log \left( \frac{A\sqrt{\mathbb{E}[F^2(X)]}}{\sigma} \right) + \frac{V}{\sqrt{n}} \log \left( \frac{A\sqrt{\mathbb{E}[F^2(X)]}}{\sigma} \right) \right],$$

where $C > 0$ is a universal constant.

**Proof.** See Corollary 5.1 in Chernozhukov et al. (2014). \qed

The following anti-concentration inequality for Gaussian measures (called Nazarov’s inequality in Chernozhukov et al. (2017a)), together with the Gaussian comparison inequality, will play crucial roles in proving the validity of the pivotal bootstrap.
Lemma 3 (Nazarov’s inequality). Let $Y = (Y_1, \ldots, Y_d)^T$ be a centered Gaussian vector in $\mathbb{R}^d$ such that $\mathbb{E}|Y_j|^2 \geq \sigma^2$ for all $j = 1, \ldots, d$ and some constant $\sigma > 0$. Then for every $y \in \mathbb{R}^d$ and $\delta > 0$,

$$
\mathbb{P}(Y \leq y + \delta) - \mathbb{P}(Y \leq y) \leq \frac{\delta}{\sigma} (\sqrt{2 \log d} + 2).
$$

Proof. See Lemma A.1 in Chernozhukov et al. (2017a); see also Chernozhukov et al. (2017b). □

Lemma 4 (Gaussian comparison). Let $Y$ and $W$ be centered Gaussian random vectors in $\mathbb{R}^d$ with covariance matrices $\Sigma_Y = (\Sigma_{j,k})_{1 \leq j, k \leq d}$ and $\Sigma_W = (\Sigma_{j,k})_{1 \leq j, k \leq d}$, respectively, and let $\Delta = \|\Sigma_Y - \Sigma_W\|_{\infty} := \max_{1 \leq j, k \leq d} |\Sigma_{j,k}^Y - \Sigma_{j,k}^W|$. Suppose that $\min_{1 \leq j \leq d} \Sigma_{j,j}^Y \vee \min_{1 \leq j \leq d} \Sigma_{j,j}^W \geq \sigma^2$ for some constant $\sigma > 0$. Then

$$
\sup_{b \in \mathbb{R}^d} |\mathbb{P}(Y \leq b) - \mathbb{P}(W \leq b)| \leq C \Delta^{1/3} \log^{2/3} d,
$$

where $C$ is a constant that depends only on $\sigma$.

Proof. Implicit in the proof Theorem 4.1 in Chernozhukov et al. (2017a). □

Appendix B  Proofs for Section 3

B.1 Uniform Convergence Rates

We first establish uniform convergence rates of $\hat{Q}_x^{(r)}(\tau)$. The following Bahadur representation of the linear quantile regression estimator $\hat{\beta}(\tau)$ will be used in the subsequent proofs.

Lemma 5 (Bahadur representation of $\hat{\beta}(\tau)$). Under Assumption $[\ ]$ we have

$$
\hat{\beta}(\tau) - \beta(\tau) = J(\tau)^{-1} \left[ \frac{1}{n} \sum_{i=1}^{n} \{\tau - I(U_i \leq \tau)\} X_i \right] + o_P(n^{-3/4} \log n),
$$

uniformly in $\tau \in [\varepsilon/2, 1 - \varepsilon/2]$, where $U_1, \ldots, U_n \sim U(0,1)$ i.i.d. that are independent of $X_1, \ldots, X_n$. In addition, we have

$$
\sup_{\tau \in [\varepsilon/2, 1 - \varepsilon/2]} \left\| \frac{1}{n} \sum_{i=1}^{n} \{\tau - I(U_i \leq \tau)\} X_i \right\| = O_P(n^{-1/2}).
$$
Proof. See Lemma 3 in Ota et al. (2019). See also Ruppert and Carroll (1980); Guten-
brunner and Jurecková (1992); He et al. (1996).

We first prove the following technical lemma.

Lemma 6. If Assumption 7 holds, then for \( r = 1, 2, 3 \), we have

\[
\sup_{x \in X_0, \tau \in [\epsilon, 1 - \epsilon]} \frac{1}{nh^r} \sum_{i=1}^{n} x^T J(\tau)^{-1} X_i \left\{ K^{(r-1)} \left( \frac{\tau - U_i}{h} \right) - hI(r = 1) \right\} = O_P \left( n^{-1/2}h^{-r+1/2}\sqrt{\log n} \right).
\]

Proof. Since \( K \) is supported in \([-1, 1]\), for sufficiently large \( n \),

\[
\mathbb{E} \left[ K^{(r-1)} \left( \frac{\tau - U}{h} \right) \right] = h \int_{(1 - r)/h}^{\tau/h} K^{(r-1)}(t)dt = h \int_{\mathbb{R}} K^{(r-1)}(t)dt = hI(r = 1).
\]

Consider the function class \( F_h := \{(u, x') \mapsto K^{(r-1)}((\tau - u)/h)x^T J(\tau)^{-1}x' : x \in X_0, \tau \in [\epsilon, 1 - \epsilon]\} \) (which depends on \( n \) since \( h = h_n \) does). It suffices to show that

\[
\mathbb{E}[\|G_n\|_{F_h}] = O(\sqrt{h\log n}) \quad \text{with} \quad G_n f = n^{-1/2} \sum_{i=1}^{n} \{f(U_i, X_i) - \mathbb{E}[f(U, X)]\}.
\]

To this end, we will apply Lemma 2. The function class \( F_h \) is a subset of the pointwise product of the following two function classes (that are independent of \( n \)): \( F' = \{(u, x') \mapsto x^T J(\tau)^{-1}x' : x \in X_0, \tau \in [\epsilon, 1 - \epsilon]\} \) and \( F'' = \{(u, x') \mapsto K^{(r-1)}(au + b) : a, b \in \mathbb{R}\} \).

The former function class \( F' \) has envelope \( F_1(u, x') = C\|x'\| \) and the latter function class \( F'' \) has envelope \( F_2(u, x') = C' \) where \( C, C' \) are some constants independent of \( n \). The function class \( F' \) is a subset of a vector space of dimension \( d \), so that it is a VC subgraph class with VC index at most \( d + 2 \) (cf. Lemma 2.6.15 in van der Vaart and Wellner (1996)).

Next, since \( K^{(r-1)} \) is of bounded variation (i.e., it can be written as the difference of two bounded nondecreasing functions) and the function class \( \{u \mapsto au + b : a, b \in \mathbb{R}\} \) is a VC subgraph class (as it is a vector space of dimension 2), the function class \( F'' \) is VC type in view of Lemma 2.6.18 in van der Vaart and Wellner (1996). Conclude that, for \( F(u, x') = CC'\|x'\| \), there exist positive constants \( A, V \) independent of \( n \) such that

\[
\sup_{Q} N(F_h, \| \cdot \|_{Q, 2}, \eta \|F\|_{Q, 2}) \leq (A/\eta)^V, \quad 0 < \forall \eta \leq 1,
\]

where \( \sup_{Q} \) is taken over all finitely discrete distributions on \((0, 1) \times \mathbb{R}^d\).
It is not difficult to verify that, by independence between $U$ and $X$,
\[
\sup_{x \in X_{0}} \sup_{\tau \in [\epsilon, 1-\epsilon]} \mathbb{E}[\{K^{(r-1)}((\tau - U)/h)x^T J(\tau)^{-1} X\}^2] \leq O(1) \int_{0}^{1} K^{(r-1)}((\tau - u)/h)^2 du = O(h).
\]
In addition, $\mathbb{E}[\max_{1 \leq i \leq n} F^2(U, X_i)] \leq O(1) \mathbb{E}[\max_{1 \leq i \leq n} \|X_i\|^2] = O(n^{1/2})$ (as $\mathbb{E}[\|X\|^4] < \infty$). Conclude from Lemma 2 that
\[
\mathbb{E}[\|G_n\|_{F_n}] = O(\sqrt{h \log n + n^{-1/4} \log n}) = O(\sqrt{h \log n}).
\]
This completes the proof. $\square$

The following lemma derives uniform convergence rates of $\hat{Q}_x^{(r)}(\tau)$.

**Lemma 7** (Uniform convergence rates $\hat{Q}_x^{(r)}(\tau)$). Under Assumption 1, we have
\[
\sup_{x \in X_{0}} \sup_{\tau \in [\epsilon, 1-\epsilon]} |\hat{Q}_x^{(r)}(\tau) - Q_x^{(r)}(\tau)| = \begin{cases} O_P \left( n^{-1/2} + h^2 \right) & \text{if } r = 0 \\ O_P \left( n^{-1/2} h^{-r+1/2} \sqrt{\log n + h^2} \right) & \text{if } r = 1 \text{ or } 2 \\ O_P \left( n^{-1/2} h^{-5/2} \sqrt{\log n + h} \right) & \text{if } r = 3 \end{cases}
\]

**Proof.** Consider first the case where $r = 0$. By definition,
\[
\sup_{x \in X_{0}} \sup_{\tau \in [\epsilon, 1-\epsilon]} |\hat{Q}_x(\tau) - Q_x(\tau)| = \sup_{x \in X_{0}} \sup_{\tau \in [\epsilon, 1-\epsilon]} \left| \int \hat{Q}_x(t) K_h(\tau - t) dt - Q_x(\tau) \right|
\]
\[
\leq \sup_{x \in X_{0}} \sup_{\tau \in [\epsilon, 1-\epsilon]} \left| \int [\hat{Q}_x(t) - Q_x(t)] K_h(\tau - t) dt \right|
\]
\[
+ \sup_{x \in X_{0}} \sup_{\tau \in [\epsilon, 1-\epsilon]} \left| \int Q_x(t) K_h(\tau - t) dt - Q_x(\tau) \right|
\]
\[
=: I + II.
\]
We have $I = O_P(n^{-1/2})$ by Lemma 5 and $II = O(h^2)$ by Taylor expansion.

Next, consider $1 \leq r \leq 3$. We note that
\[
\sup_{x \in X_{0}} \sup_{\tau \in [\epsilon, 1-\epsilon]} |\hat{Q}_x^{(r)}(\tau) - Q_x^{(r)}(\tau)| \leq \sup_{x \in X_{0}} \sup_{\tau \in [\epsilon, 1-\epsilon]} \left| \int [\hat{Q}_x(t) - Q_x(t)] K_h^{(r)}(\tau - t) dt \right|
\]
\[
+ \sup_{x \in X_{0}} \sup_{\tau \in [\epsilon, 1-\epsilon]} \left| \int Q_x(t) K_h^{(r)}(\tau - t) dt - Q_x^{(r)}(\tau) \right|
\]
\[
=: III + IV.
\]
We have $IV = O(h^2)$ for $r = 1, 2$ and $= O(h)$ for $r = 3$ by Taylor expansion (recall that $Q_x(\tau)$ is four-times continuously differentiable). Observe that, by Lemma 5 and change of variables,

$$
III \leq \sup_{x \in X_0, \tau \in [\epsilon, 1 - \epsilon]} \left| \frac{1}{nh^r} \sum_{i=1}^n \int x^T J(\tau - th)^{-1} X_i \left\{ \tau - th - I \left( U_i \leq \tau - th \right) \right\} K^{(r)}(t) dt \right|
$$

$$
+ \overline{o_P(n^{-3/4}h^{-r} \log n)} = \overline{o_P(n^{-1/2}h^{-r+1/2} \sqrt{\log n}).
$$

Replacing $J(\tau - th)$ by $J(\tau)$ in the first term on the right hand side results in an error of order $O_P(n^{-1/2}h^{-r+1})$; this can be verified by a similar argument to the proof of the preceding lemma. Thus, it remains to bound

$$
\sup_{x \in X_0, \tau \in [\epsilon, 1 - \epsilon]} \left| \frac{1}{nh^r} \sum_{i=1}^n \int x^T J(\tau)^{-1} X_i \left\{ \tau - th - I \left( U_i \leq \tau - th \right) \right\} K^{(r)}(t) dt \right|
$$

$$
= \sup_{x \in X_0, \tau \in [\epsilon, 1 - \epsilon]} \left| \frac{1}{nh^r} \sum_{i=1}^n x^T J(\tau)^{-1} X_i \left\{ K^{(r-1)} \left( \frac{\tau - U_i}{h} \right) \right\} + h \int tK^{(r)}(t) dt \right|
$$

where we have used the fact that $K^{(r)}$ integrates to 0. Here, by integration by parts,

$$
\int tK^{(r)}(t) dt = - \int K^{(r-1)}(t) dt = -I(r = 1).
$$

Thus, from Lemma 6, we have $III = O(n^{-1/2}h^{-r+1/2} \sqrt{\log n})$. This completes the proof. 

**Remark 5** (Bias of $\hat{Q}_x(\tau)$ at $\tau = \tau_x$). The bias of $\hat{Q}_x(\tau)$ can be improved to $O(h^4)$ at $\tau = \tau_x$ by $Q''_x(\tau_x) = 0$ and symmetry of $K$.

**Remark 6** (Expansion of $\hat{Q''}_x(\tau)$). Inspection of the proof shows that

$$
\hat{Q''}_x(\tau) - Q''_x(\tau) - \frac{Q^{(4)}(\tau)}{2} nh^2 + o(h^2) = \frac{1}{nh^2} \sum_{i=1}^n K' \left( \frac{\tau - U_i}{h} \right) x^T J(\tau)^{-1} X_i
$$

$$
+ \overline{o_P(n^{-1/2}h^{-1}) + o_P(n^{-3/4}h^{-2} \log n)} = \overline{o_P(n^{-1/2}h^{-1})}
$$

uniformly in $(\tau, x) \in [\epsilon, 1 - \epsilon] \times X_0$. Recall that $\kappa = \int t^2 K(t) dt$. 

5
B.2 Proofs for Section 3.2

We first prove the uniform consistency of $\hat{\tau}_x$.

**Lemma 8** (Uniform consistency of $\hat{\tau}_x$). *Under Assumption 1 we have* $\sup_{x \in X_0} |\hat{\tau}_x - \tau_x| \xrightarrow{P} 0$.  

*Proof.* We divide the proof into two steps.

**Step 1.** We will verify that for any $\delta > 0$,

$$\eta_{\delta} := \inf_{x \in X_0} \inf_{\tau \in [\epsilon, 1-\epsilon]} \{s_x(\tau) - s_x(\tau_x)\} > 0.$$  

This follows from the following two claims: (i) $s_x(\tau) - s_x(\tau_x)$ is jointly continuous in $(\tau, x)$, (ii) $S_{\delta} := \{ (\tau, x) : x \in X_0, \tau \in [\epsilon, 1-\epsilon], |\tau - \tau_x| \geq \delta \}$ is compact in $(0, 1) \times \mathbb{R}^d$ and the observation that $\tau_x$ is the unique minimizer of $s_x(\tau)$, i.e., $\tau_x = \arg\min_{\tau \in [\epsilon, 1-\epsilon]} s_x(\tau)$. Since $s_x(\tau) = \partial Q_x(\tau)/\partial \tau$ is continuous in $\tau$ for any fixed $x$ under Assumption 1 and also linear (thus convex) in $x$ by the linear quantile assumption, Theorem 10.7 in Rockafellar (1970) implies that $s_x(\tau)$ is jointly continuous in $(\tau, x)$. Now, by Berge’s maximum theorem (cf. Theorem 17.31 in Aliprantis and Border (2006): see also their Lemma 17.6), we see that $\tau_x$ is continuous in $x$. The preceding discussion also implies that $s_x(\tau) - s_x(\tau_x)$ is jointly continuous in $(\tau, x)$. Combining the continuity of $\tau_x$ and the definition of $S_{\delta}$, we can verify $S_{\delta}$ is closed and bounded and therefore compact. Thus, we have verified claims (i) and (ii) and the conclusion of this step follows.

**Step 2.** We will prove the uniform consistency of $\hat{\tau}_x$. Consider the event $A_{\delta} := \{ \sup_{x \in X_0} \{s_x(\hat{\tau}_x) - s_x(\tau_x)\} \geq \eta_{\delta} \}$. Observe that

$$\sup_{x \in X_0} \{s_x(\hat{\tau}_x) - s_x(\tau_x)\} \leq \sup_{x \in X_0} \{s_x(\hat{\tau}_x) - \hat{s}_x(\tau_x)\} + \sup_{x \in X_0} \{\hat{s}_x(\hat{\tau}_x) - \hat{s}_x(\tau_x)\} + \sup_{x \in X_0} \{\hat{s}_x(\tau_x) - s_x(\tau_x)\}.$$  

The first and third terms on the right hand side are $o_P(1)$ by Lemma 7 while the second term is nonpositive by the definition of $\hat{\tau}_x$. This implies that $\mathbb{P}(A_{\delta}) \leq \mathbb{P}(\eta_{\delta} \leq o_P(1)) = o(1)$. The uniform consistency of $\hat{\tau}_x$ follows from the fact that the event $\{ \sup_{x \in X_0} |\hat{\tau}_x - \tau_x| \geq \delta \}$ is included in $A_{\delta}$.


The uniform consistency guarantees that the first order condition for $\hat{\tau}_x$ holds for all $x \in \mathcal{X}_0$ with probability approaching one, i.e.,

$$\mathbb{P}\left(\hat{s}_x'(\hat{\tau}_x) = 0, \forall x \in \mathcal{X}_0\right) \to 1. \tag{A2.14}$$

Recall that $s'_x(\tau) = \hat{Q}_x''(\tau)$. Now, we derive an asymptotic linear representation for $\hat{\tau}_x$.

**Lemma 9** (Asymptotic linear representation of $\hat{\tau}_x$). Under Assumption 1, the following expansion holds uniformly in $x \in \mathcal{X}_0$:

$$\hat{\tau}_x - \tau_x + \frac{s^{(3)}_x(\tau_x)}{2s''_x(\tau_x)} kh^2 + o_P(h^2) = 0 = \hat{Q}_x''(\hat{\tau}_x) = \hat{Q}_x''(\tau_x) + \hat{Q}_x^{(3)}(\hat{\tau}_x)(\hat{\tau}_x - \tau_x).$$

where $\hat{\tau}_x$ lies between $\hat{\tau}_x$ and $\tau_x$. This yields that

$$\hat{\tau}_x - \tau_x = -\hat{Q}_x^{(3)}(\hat{\tau}_x)^{-1} \cdot \hat{Q}_x''(\tau_x).$$

The rest of the proof is divided into two steps.

**Step 1.** We will show that $\sup_{x \in \mathcal{X}_0} |\hat{\tau}_x - \tau_x| = O_P(n^{-1/2}h^{-3/2}\sqrt{\log n + h^2})$. Observe that $\hat{Q}_x^{(3)}(\hat{\tau}_x) = Q_x^{(3)}(\hat{\tau}_x) + O_P(n^{-1/2}h^{-5/2}\sqrt{\log n + h}) = Q_x^{(3)}(\hat{\tau}_x) + o_P(1)$ uniformly in $x \in \mathcal{X}_0$ by Lemma 6, and the map $x \mapsto Q_x^{(3)}(\tau_x)$ is bounded away from zero on $\mathcal{X}_0$. Thus, we have

$$\sup_{x \in \mathcal{X}_0} |\hat{\tau}_x - \tau_x| = O_P\left(\sup_{x \in \mathcal{X}_0} |\hat{Q}_x''(\tau_x)|\right).$$

However, since $Q_x''(\tau_x) = 0$, the right hand side on the above equation is $O_P(n^{-1/2}h^{-3/2}\sqrt{\log n} + h^2)$ by Lemma 6.
Step 2. We wish to derive the conclusion of the lemma. From the preceding discussion, we see that
\[
\hat{Q}_x(3)\hat{r}_x(3) = Q_x(3)x(3) - Q_x(3)x(3) + O_P(n^{-1/2}h^{-5/2}\sqrt{\log n} + h)
\]
uniformly in \( x \in X_0 \), so that
\[
\hat{r}_x - r_x = -Q_x(3)(\hat{r}_x - r_x) + O_P(n^{-1/2}h^{-5/2}\sqrt{\log n} + h^3)
\]
uniformly in \( x \in X_0 \). The conclusion of the lemma follows from combining the expansion (A2.13).

We are now in position to prove Proposition 1.

Proof of Proposition 1. We note that, uniformly in \( x \in X_0 \),
\[
\hat{m}(x) - m(x) = \{\hat{Q}_x(\hat{r}_x) - Q_x(\hat{r}_x)\} + \{Q_x(\hat{r}_x) - Q_x(r_x)\}
\]
\[
= Q_x(\hat{r}_x) - Q_x(r_x) + O_P(n^{-1/2}h^{-3/2}\sqrt{\log n}) + o_P(h^2) \quad \text{by Lemma 6 and Remark 5}
\]
\[
= Q_x(\hat{r}_x)(\hat{r}_x - r_x) + O_P\left(\sup_{x' \in X_0} |\hat{r}_x - r_x|^3\right) + O_P(n^{-1/2}h^{-1} + o_P(h^2)) \quad \text{by Lemma 6 and Remark 5}
\]
\[
= \frac{1}{nh^{3/2}} \sum_{i=1}^{n} \psi_x(U_i, X_i) - \frac{s_x(\hat{r}_x)s_x(3)(\hat{r}_x)}{2s''(r_x)} \kappa h^2
\]
\[
+ O_P(n^{-1/2}h^{-1} + n^{-1}h^{-4}\sqrt{\log n} + h^3) \quad \text{by Lemma 9}
\]
This completes the proof.

Proof of Corollary 1. Proposition 1 implies that, for any fixed \( x \in X_0 \),
\[
\hat{m}(x) - m(x) = \frac{1}{nh^{3/2}} \sum_{i=1}^{n} \psi_x(U_i, X_i) + o_P(n^{-1/2}h^{-3/2}).
\]
Thus, it suffices to show that \( n^{-1/2} \sum_{i=1}^{n} \psi_x(U_i, X_i) \to N(0, V_x) \). Recall that \( \psi_x(U_i, X_i) \) has mean zero. The above result follows from verifying the Lyapunov condition, together with the fact that \( E[\psi_x(U, X)^2] = V_x \). We omit the details for brevity.

Proof of Proposition 2. Since \( K \) is supported in \([-1, 1]\), if \( |\tau_{x_k} - \tau_{x_\ell}| > 2h \), then
\[
E \left[ K'(\frac{\tau_{x_k} - U}{h}) K'(\frac{\tau_{x_\ell} - U}{h}) \right] = 0.
\]
Thus, $\Sigma = \text{diag}\{\sigma_{x_1}^2, \ldots, \sigma_{x_L}^2\}$, so that Theorem 1 implies that
\[
\sup_{b \in \mathbb{R}} \left| \mathbb{P}(\zeta_n \leq b) - \mathbb{P}\left( \max_{1 \leq \ell \leq L} |W_\ell| \leq b\right) \right| \to 0,
\]
where $W_1, \ldots, W_L \sim N(0,1)$ i.i.d. The rest of the proof follows from standard extreme value theory; cf. Theorem 1.5.3 in Leadbetter et al. (1983).

**B.3 Proofs for Section 3.3**

**Proof of Theorem 1**. We divide the proof into two steps.

**Step 1.** We will show that
\[
\sup_{b \in \mathbb{R}} \left| \mathbb{P}\left( n^{-1/2} \sum_{i=1}^n A \Psi_i \leq b \right) - \mathbb{P}(AG \leq b) \right| \to 0. \tag{A2.15}
\]
To this end, we verify Conditions (M.1), (M.2), and (E.2) in Proposition 2.1 of Chernozhukov et al. (2017a).

**Condition (M.1):** For $k = 1, \ldots, M$, by definition,
\[
\mathbb{E}[(A_k^T \Psi_i)^2] = A_k^T \mathbb{E}[\Psi_i \Psi_i^T] A_k = D_k^T \Sigma D_k / \sigma_k^2 = 1.
\]

**Condition (M.2):** For $k = 1, \ldots, M$,
\[
\mathbb{E}\left[ |A_k^T \Psi_i|^3 \right] \leq \max_{\ell \in S_k} |A_{k,\ell}|^3 \mathbb{E}\left[ \|\psi_{x_\ell}(U, X)\|_{L^1} \right] \\
\leq \max_{\ell \in S_k} |A_{k,\ell}| \cdot |S_k|^2 \sum_{\ell \in S_k} \mathbb{E}[|\psi_{x_\ell}(U, X)|^3] \leq \max_{\ell \in S_k} |A_{k,\ell}| \cdot |S_k|^3 \max_{1 \leq \ell \leq L} \mathbb{E}[|\psi_{x_\ell}(U, X)|^3]
\]
Under our assumption, $\max_{1 \leq k \leq M, \ell \in S_k} |A_{k,\ell}| = O(1)$ and $\max_{1 \leq k \leq M} |S_k| = O(1)$. In addition,
\[
\max_{1 \leq \ell \leq L} \mathbb{E}[|\psi_{x_\ell}(U, X)|^3] \\
\leq O(h^{-3/2}) \mathbb{E}[\|X\|^3] \max_{1 \leq \ell \leq L} \int_0^1 \left| K' \left( \frac{\tau_{x_\ell} - u}{h} \right) \right|^3 du = O(h^{-1/2}).
\]
Likewise, we have $\max_{1 \leq k \leq M} \mathbb{E}[|A_k^T \Psi_i|^4] = O(h^{-1})$.

**Condition (E.2):** Similarly to the previous case (but bounding $h^{-1/2} K'(\tau_{x_\ell} - U)/h$ by $h^{-1/2}\|K'\|_\infty$), we can show that
\[
\mathbb{E}\left[ \max_{1 \leq k \leq M} |A_k^T \Psi_i|^9 \right] \leq O(h^{-9/2}) \mathbb{E}[\|X\|^9] = O(h^{-9/2}).
\]
Thus, we can apply Proposition 2.1 in Chernozhukov et al. (2017a), and the conclusion of this step follows as soon as
\[
\frac{\log^7(Mn)}{nh} \sqrt{\frac{\log^3(Mn)}{n^{1-2/q}h}} \to 0,
\]
but this is satisfied under our assumption.

**Step 2.** Define \(\delta_n = h^{1/2} + n^{-1/2}h^{-5/2} \log n + n^{1/2}h^{7/2} (nhh^{-3/2} - (nhh^{-3/2})^{-1} \sum_{i=1}^n \Psi_i).\) By Proposition 1, we know that \(\sqrt{nhh^3} \| AR_n\|_{\infty} = O_P(\delta_n),\) so that
\[
\sqrt{nhh^3} \| AR_n\|_{\infty} \leq \max_{1 \leq k \leq M} \sum_{\ell \in S_k} |A_{k,\ell}| \| \sqrt{nhh^3} R_{n,\ell} \|_{\infty} \leq \max_{1 \leq k \leq M} \sum_{1 \leq \ell \leq L} |A_{k,\ell}| \max_{1 \leq k \leq M} \sum_{\ell \in S_k} |\sqrt{nhh^3} R_{n,\ell}| \\
\leq \max_{1 \leq k \leq M} |A_{k,\ell}| \max_{1 \leq k \leq M} |S_k| \| \sqrt{nhh^3} R_{n}\|_{\infty} = O_P(\delta_n).
\]
Thus, for any \(B_n \to \infty,\) we have \(\mathbb{P}(\sqrt{nhh^3} \| AR_n\|_{\infty} > B_n \delta_n) = o(1).\) Now, for any \(b \in \mathbb{R}^M,\)
\[
\mathbb{P}
\left(
A \sqrt{nhh^3}(\hat{m}(x_\ell) - m(x_\ell))_{\ell=1}^L \leq b
\right)
\leq \mathbb{P}
\left(
(\sqrt{nhh^3} - B_n \delta_n)\mathbb{E}^1 \leq b
\right) + O(1)
\leq \mathbb{P}(AG \leq b + B_n \delta_n) + o(1) \quad (\text{by Step 1})
\leq \mathbb{P}(AG \leq b) + O(B_n \delta_n \sqrt{\log M}) + o(1), \quad (\text{by Nazarov’s inequality (Lemma 3)})
\]
where the \(o\) and \(O\) terms are independent of \(b.\) Likewise, we have
\[
\mathbb{P}
\left(
A \sqrt{nhh^3}(\hat{m}(x_\ell) - m(x_\ell))_{\ell=1}^L \geq b
\right) \geq \mathbb{P}(AG \leq b) - O(B_n \delta_n \sqrt{\log M}) - o(1).
\]
Since \(B_n \delta_n \sqrt{\log M} \to 0\) for sufficiently slow \(B_n \to \infty\) under our assumption, we obtain the conclusion of the theorem.

**B.3.1 Proof of Theorem 2**

We start with proving some technical lemmas. We use \(\| \cdot \|_{op}\) to denote the operator norm of a matrix.

**Lemma 10.** Under Assumption 1, we have
\[
\sup_{x \in \mathcal{X}_0} \| \hat{J}(\tau_x) - J(\tau_x) \|_{op} = O_P(n^{-1/2}h^{-3/2} \sqrt{\log n + h^2}).
\]
Proof. It suffices to show that \( \sup_{x \in X_0} |\hat{J}_{j,k}(\hat{\tau}_x) - J_{j,k}(\tau_x)| = O_P(n^{-1/2} h^{-3/2} \sqrt{\log n + h^2}) \) for any \( 1 \leq j, k \leq d \) (as the dimension \( d \) is fixed). Observe that

\[
\sup_{x \in X_0} |\hat{J}_{j,k}(\hat{\tau}_x) - J_{j,k}(\tau_x)| \\
\leq \sup_{x \in X_0} \left| \frac{1}{n} \sum_{i=1}^{n} K_h(Y_i - X_i^T \hat{\beta}(\hat{\tau}_x))X_{ij}X_{ik} - \mathbb{E} \left[ K_h(Y - X^T \beta)X_jX_k \right] \right|_{\beta = \hat{\beta}(\hat{\tau}_x)} \\
+ \sup_{x \in X_0} \mathbb{E} \left[ K_h(Y - X^T \beta \mid X_i)X_{j}X_{k} \right]_{\beta = \hat{\beta}(\hat{\tau}_x)} - \mathbb{E} \left[ f(X^T \beta \mid X)X_jX_k \right]_{\beta = \hat{\beta}(\hat{\tau}_x)} \\
+ \sup_{x \in X_0} \mathbb{E} \left[ f(X^T \beta \mid X)X_{j}X_{k} \right]_{\beta = \hat{\beta}(\hat{\tau}_x)} - \mathbb{E} \left[ f(X^T \beta \mid X)X_jX_k \right]_{\beta = \beta(\tau_x)}.
\]

It is routine to show that the first and second terms on the right hand side are \( O_P(n^{-1/2} h^{-1}) \) and \( O(h^2) \), respectively; cf. the proof of Lemma\([\bar{7}]\). By Taylor expansion, the last term can be bounded by \( O_P(\|\hat{\beta}(\hat{\tau}_x) - \beta(\tau_x)\|_{X_0}) \). Observe that

\[
\|\hat{\beta}(\hat{\tau}_x) - \beta(\tau_x)\|_{X_0} \leq \|\hat{\beta} - \beta\|_{[\epsilon, 1 - \epsilon]} + \|\beta(\tau_x) - \beta(\tau_x)\|_{X_0} \\
\leq O_P(n^{-1/2} + \|\hat{\tau}_x - \tau_x\|_{X_0}) = O_P(n^{-1/2} h^{-3/2} \sqrt{\log n + h^2}).
\]

This completes the proof. \( \Box \)

Lemma 11. Under Assumption\([\bar{1}]\) we have

\[
\|\hat{\Sigma} - \Sigma\|_{\infty} = O_P(n^{-1/2} h^{-5/2} \sqrt{\log n + h}).
\]

Proof. For simplicity of notation, let \( \hat{J}_{x_k} = \hat{J}(\hat{\tau}_{x_k}) \) and \( J_{x_k} = J(\tau_{x_k}) \). The difference \( \hat{\Sigma}_{j,k} - \Sigma_{j,k} \) can be decomposed as

\[
\begin{align*}
\frac{\hat{s}_{x_k}(\hat{\tau}_{x_k}) \hat{s}_{x_k}(\hat{\tau}_{x_k})'}{\hat{s}_{x_k}(\hat{\tau}_{x_k})} & \mathbb{E}_{U} \left[ \frac{1}{h} K' \left( \frac{\hat{\tau}_{x_k} - U}{h} \right) K' \left( \frac{\hat{\tau}_{x_k} - U}{h} \right) \right] x_k^T \hat{J}_{x_k}^{-1} \left[ \frac{1}{n} \sum_{i=1}^{n} X_i X_i^T \right] \hat{J}_{x_k}^{-1} x_k' \\
- & \frac{\hat{s}_{x_k}(\hat{\tau}_{x_k}) \hat{s}_{x_k}(\tau_{x_k})}{\hat{s}_{x_k}(\tau_{x_k})} \mathbb{E} \left[ \frac{1}{h} K' \left( \frac{\tau_{x_k} - U}{h} \right) K' \left( \frac{\tau_{x_k} - U}{h} \right) \right] x_k^T J_{x_k}^{-1} \mathbb{E}[X X^T] J_{x_k}^{-1} x_k'.
\end{align*}
\]

Observe that

\[
\max_{1 \leq k, \ell \leq L} \left| \frac{\hat{s}_{x_k}(\hat{\tau}_{x_k}) \hat{s}_{x_k}(\hat{\tau}_{x_k})'}{\hat{s}_{x_k}(\hat{\tau}_{x_k})} - \frac{\hat{s}_{x_k}(\tau_{x_k}) \hat{s}_{x_k}(\tau_{x_k})'}{\hat{s}_{x_k}(\hat{\tau}_{x_k})} \right| \\
\leq O_P \left( \|s_x(\hat{\tau}_x) - s_x(\tau_x)\|_{X_0} \sqrt{\|\hat{s}_{x_k}(\hat{\tau}_{x_k}) - s_{x_k}(\tau_{x_k})\|_{X_0}} \right), \text{ and}
\]

\[
\|\hat{s}_{x_k}(\hat{\tau}_x) - s_{x_k}(\tau_x)\|_{X_0} \leq \|\hat{s}_{x_k}(\tau) - s_{x_k}(\tau)\|_{[\epsilon, 1 - \epsilon]} \times X_0 + \|s_{x_k}(\hat{\tau}_x) - s_{x_k}(\tau_x)\|_{X_0} \\
= O_P(n^{-1/2} h^{-5/2} \sqrt{\log n + h}) \text{ for } r = 0, 2,
\]

11.
where we have used Lemma 7 in the last line.

Next, we note that

\[
\max_{1 \leq k, \ell \leq L} \left| x_k^T \hat{J}_{x_k}^{-1} \left[ \frac{1}{n} \sum_{i=1}^n X_i X_i^T \right] \hat{J}_{x_\ell}^{-1} x_\ell - x_k^T J_{x_k}^{-1} E \left[ X X^T \right] J_{x_\ell}^{-1} x_\ell \right| \leq O_P \left( \max_{1 \leq k \leq L} \| \hat{J}_{x_k}^{-1} - J_{x_k}^{-1} \|_{op} \sqrt{\frac{1}{n} \sum_{i=1}^n X_i X_i^T - E \left[ XX^T \right]} \right)
\]

where we have used Lemma 10 in the last line.

Finally, observe that

\[
\left| \mathbb{E}_U \left[ \frac{1}{h} K' \left( \frac{\hat{\tau}_{x_k} - U}{h} \right) K' \left( \frac{\hat{\tau}_{x_\ell} - U}{h} \right) \right] - \mathbb{E}_U \left[ \frac{1}{h} K' \left( \frac{\tau_{x_k} - U}{h} \right) K' \left( \frac{\tau_{x_\ell} - U}{h} \right) \right] \right| \leq \| K'' \|_{\infty} h^{-1} |\hat{\tau}_{x_k} - \tau_{x_k}| \mathbb{E}_U \left[ \left| \frac{1}{h} K' \left( \frac{\hat{\tau}_{x_\ell} - U}{h} \right) \right| \right] = O_P(n^{-1/2} h^{-5/2} \sqrt{\log n + h})
\]

uniformly in 1 \leq k, \ell \leq L. Likewise, we have

\[
\left| \mathbb{E}_U \left[ \frac{1}{h} K' \left( \frac{\tau_{x_k} - U}{h} \right) K' \left( \frac{\tau_{x_\ell} - U}{h} \right) \right] - \mathbb{E}_U \left[ \frac{1}{h} K' \left( \frac{\tau_{x_k} - U}{h} \right) K' \left( \frac{\tau_{x_\ell} - U}{h} \right) \right] \right| = O_P(n^{-1/2} h^{-5/2} \sqrt{\log n + h})
\]

uniformly in 1 \leq k, \ell \leq L. Combining these estimates, we obtain the conclusion of the lemma.

\[\square\]

**Lemma 12.** Under Assumptions 1 and 2, we have

\[
\max_{1 \leq k, \ell \leq M} |D_k^T (\hat{\Sigma} - \Sigma) D_\ell| = O_P(n^{-1/2} h^{-5/2} \sqrt{\log n + h}).
\]

**Proof.** This follows from the observation that

\[
\max_{1 \leq k, \ell \leq M} |D_k^T (\hat{\Sigma} - \Sigma) D_\ell| = \max_{1 \leq k, \ell \leq M} \left| \sum_{k' \in S_k} \sum_{\ell' \in S_\ell} D_{k,k'} (\hat{\Sigma}_{k',\ell'} - \Sigma_{k',\ell'}) D_{\ell,\ell'} \right|
\]

\[
\leq \max_{1 \leq k \leq M} |S_k|^2 \| D \|_{\infty} \| \hat{\Sigma} - \Sigma \|_{\infty} = O_P(n^{-1/2} h^{-5/2} \sqrt{\log n + h}).
\]

\[\square\]

We are now in position to prove Theorem 2.
Proof of Theorem 3. Let \( \hat{G} \) be an \( L \)-dimensional random vector such that conditionally on \( \mathcal{D}_n, \hat{G} \sim N(0, \Sigma) \). We begin with noting that

\[
\sup_{b \in \mathbb{R}^M} \left| \mathbb{P}_{U} \left( n^{-1/2} \sum_{i=1}^{n} \hat{\Psi}_i \leq b \right) - \mathbb{P}(AG \leq b) \right| \leq \sup_{b \in \mathbb{R}^M} \left| \mathbb{P}_{U} \left( n^{-1/2} \sum_{i=1}^{n} \hat{\Psi}_i \leq b \right) - \mathbb{P} \left( \hat{A}G \leq b \right) \right| + \sup_{b \in \mathbb{R}^M} \left| \mathbb{P}_{U}(\hat{A}G \leq b) - \mathbb{P}(AG \leq b) \right|.
\]

We first analyze \( \text{II} \) and \( \text{III} \). In view of the Gaussian comparison inequality (cf. Lemma 4), to show that \( \text{II} \lor \text{III} = o_P(1) \), it suffices to verify that

\[
\| \hat{A} \hat{\Sigma} \hat{A}^T - A \hat{\Sigma} A^T \|_\infty \lor \| A \hat{\Sigma} A^T - A \Sigma A^T \|_\infty \log^2 M = o_P(1). \tag{A2.16}
\]

Indeed, by Lemma 12, we can deduce that the bracket on the left hand side is \( O_P(n^{-1/2}h^{-5/2} \sqrt{\log n + h}) \). Thus, \( \text{(A2.16)} \) holds under our assumption.

To show that \( I = o_P(1) \), we apply Proposition 2.1 in Chernozhukov et al. (2017a) conditionally on \( \mathcal{D}_n \) (recall that conditionally on \( \mathcal{D}_n \), the vectors \( \hat{\Psi}_1, \ldots, \hat{\Psi}_n \) are independent with mean zero). By construction, \( n^{-1} \sum_{i=1}^{n} \mathbb{E}_{U}[(\hat{A}_k^T \hat{\Psi}_i)^2] = \hat{A}_k^T \hat{\Sigma} \hat{A}_k = D_k^T \hat{\Sigma} D_k/\hat{\sigma}_k^2 = 1 \). Similarly to the proof of Theorem 1, we can verify that \( \max_{1 \leq k \leq M} n^{-1} \sum_{i=1}^{n} \mathbb{E}_{U}[(\hat{A}_k^T \hat{\Psi}_i)^{2+r}] = O_P(h^{-r/2}) \) for \( r = 1, 2 \). Finally,

\[
\max_{1 \leq i \leq n} \mathbb{E}_{U} \left[ \max_{1 \leq k \leq M} |\hat{A}_k^T \hat{\Psi}_i|^q \right] \leq O_P(h^{-q/2}) \max_{1 \leq i \leq n} \| X_i \|_q = O_P(nh^{-q/2}).
\]

Hence, applying Proposition 2.1 in Chernozhukov et al. (2017a), we see that \( I = o_P(1) \) as soon as

\[
\frac{\log^7(Mn)}{n^{1-2/q}h} \sqrt{\log^3(Mn) \frac{n}{n^{1-4/q}h}} \to 0,
\]

but this is satisfied under our assumption. This completes the proof.

\[ \square \]

Proof of Proposition 3. Theorem 1 implies that

\[
\sup_{b \in \mathbb{R}^M} \left| \mathbb{P} \left( D \sqrt{nh^3}(\hat{m}(x_{\ell}) - m(x_{\ell}))_{\ell=1}^{L} \leq b \right) - \mathbb{P}(DG \leq b) \right| \to 0.
\]

Since the variances of the coordinates of \( G \) are bounded, we see that \( \mathbb{E}[\| G \|_\infty] = O(\sqrt{\log M}) \) by Lemma 2.2.2 in van der Vaart and Wellner (1996). Hence, we have

\[
\left\| D \sqrt{nh^3}(\hat{m}(x_{\ell}) - m(x_{\ell}))_{\ell=1}^{L} \right\|_\infty = O_P(\sqrt{\log M}).
\]
Combining Lemma 12, we see that
\[
\left\| (\hat{\Gamma}^{-1} - \Gamma^{-1}) D \sqrt{n h^{3}} (\hat{m}(x_{\ell}) - m(x_{\ell}))_{\ell=1}^{L} \right\|_{\infty} = o_{P}(1/\sqrt{\log M})
\]
under our assumption. The rest of the proof is analogous to the last part of Theorem 1.

We omit the details for brevity.

Finally, we prove the auxiliary lemma that appeared in Example 3.

Proof of Lemma 7. Let \( q_n(\alpha) \) denote the \( \alpha \)-quantile of \( Z_n \). By assumption, we may choose a sequence \( \delta_n \to 0 \) such that
\[
\sup_{t \in \mathbb{R}} |\mathbb{P}(Y_n \leq t) - \mathbb{P}(Z_n \leq t)| \leq \delta_n \quad \text{and} \\
\mathbb{P}\left( \sup_{t \in \mathbb{R}} |\mathbb{P}(W_n \leq t \mid C_n) - \mathbb{P}(Z_n \leq t)| > \delta_n \right) \leq \delta_n.
\]
The latter follows from the fact that the Ky Fan metric metrizes convergence in probability.

Define the event \( E_n = \{ \sup_{t \in \mathbb{R}} |\mathbb{P}(W_n \leq t \mid C_n) - \mathbb{P}(Z_n \leq t)| \leq \delta_n \} \). On this event,
\[
\mathbb{P}(W_n \leq q_n(\alpha + \delta_n) \mid C_n) \geq \mathbb{P}(Z_n \leq q_n(\alpha + \delta_n)) - \delta_n = \alpha,
\]
so that \( \hat{q}_n(\alpha) \leq q_n(\alpha + \delta_n) \). Thus,
\[
\mathbb{P}(Y_n \leq \hat{q}_n(\alpha)) \leq \mathbb{P}(Y_n \leq q_n(\alpha + \delta_n)) + \delta_n \leq \mathbb{P}(Z_n \leq q_n(\alpha + \delta_n)) + 2\delta_n = \alpha + 3\delta_n.
\]
Likewise, on the event \( E_n \),
\[
\mathbb{P}(Z_n \leq t \mid t = \hat{q}_n(\alpha)) \geq \mathbb{P}(W_n \leq t \mid C_n)_{t=\hat{q}_n(\alpha)} - \delta_n \geq \alpha - \delta_n,
\]
so that \( \hat{q}_n(\alpha) \geq q_n(\alpha - \delta_n) \). Arguing as in the previous case, we see that \( \mathbb{P}(Y_n \leq \hat{q}_n(\alpha)) \geq \alpha - 3\delta_n \). This completes the proof.

\( \square \)

Appendix C  Proofs for Section 5

Let \( S^{d-1} \) denote the unit sphere in \( \mathbb{R}^d \), i.e., \( S^{d-1} := \{ x \in \mathbb{R}^d : \|x\| = 1 \} \).
C.1 Proof of Theorem 3

Overall, the proof is analogous to that of Theorem 1. The following Banadur representation is taken from Belloni et al. (2019).

Lemma 13. Under Assumption 3, we have

\[
\hat{\beta}(\tau) - \beta(\tau) = J(\tau)^{-1} \left[ \frac{1}{n} \sum_{i=1}^{n} \{ \tau - I(U_i \leq \tau) \} X_i \right] + \hat{R}_n(\tau)
\]

with \( \| \hat{R}_n \| [\epsilon, 1 - \epsilon] = O_P(n^{-3/4}d\sqrt{\log n}) \) and \( \| n^{-1} \sum_{i=1}^{n} \{ \tau - I(U_i \leq \tau) \} X_i \| [\epsilon, 1 - \epsilon] = O_P(\sqrt{d/n}) \).

Proof. See Theorems 1 and 2 in Belloni et al. (2019).

The rates of convergence of \( \hat{Q}_x^{(r)}(\tau_x) \) change as follows.

Lemma 14. Under the conditions of Theorem 3, we have

\[
\sup_{x \in X, \tau \in [\epsilon, 1 - \epsilon]} | \hat{Q}_x^{(r)}(\tau) - Q_x^{(r)}(\tau) | = \begin{cases} 
O_P \left( n^{-1/2}d + h^2 \right) & \text{if } r = 0 \\
O_P \left( n^{-1/2}h^{-r+1/2}d\sqrt{\log n} + h^2 \right) & \text{if } r = 1 \text{ or } 2 \\
O_P \left( n^{-1/2}h^{-5/2}d\sqrt{\log n} + h \right) & \text{if } r = 3
\end{cases}
\]

Proof. We divide the proof into two steps.

Step 1. We will show that

\[
\sup_{x \in X_0, \tau \in [\epsilon, 1 - \epsilon]} \left| \frac{1}{nh^r} \sum_{i=1}^{n} x^T J(\tau)^{-1} X_i \left\{ K^{(r-1)} \left( \frac{\tau - U_i}{h} \right) - hI(r = 1) \right\} \right| = O_P \left( n^{-1/2}h^{-r+1/2}d\sqrt{\log n} \right),
\]

for \( r = 1, 2, 3 \). The proof is analogous to that of Lemma 6, so we only point out required modifications. The envelope function \( F \) should be modified to \( F(u, x') = C\sqrt{d\|x'\|} \) for some constant \( C \), and note that the VC constant \( V \) is of order \( V = O(d) \). Observe that

\[
\sup_{x \in X_0, \tau \in [\epsilon, 1 - \epsilon]} \mathbb{E} \left[ \left\{ K^{(r-1)}((\tau - U)/h)x^T J(\tau)^{-1} X \right\}^2 \right] \leq O(d) \int_{0}^{1} K^{(r-1)}((\tau - u)/h)^2 du = O(hd),
\]

and \( \mathbb{E} \left[ \max_{1 \leq i \leq n} F^2(U_i, X_i) \right] = O(d^2) \) (as \( \| X \| \leq C_3\sqrt{d} \)). Applying Lemma 2 leads to the above rates.
Step 2. We will show the conclusion of the lemma. This part is analogous to the proof of Lemma 7, so we only outline required modifications. The \( r = 0 \) follows from Lemma 13 and Taylor expansion. For \( 1 \leq r \leq 3 \), combining Lemma 13, change of variables, and Taylor expansion, we can bound super the limit as \( \tau \rightarrow \infty \) by

\[
\sup_{x \in X_0, \tau \in [\epsilon, 1 - \epsilon]} \left| \frac{1}{nh^{r}} \sum_{i=1}^{n} x^T J(t - th)^{-1} X_i \{ \tau - th - I(U_i \leq \tau - th) \} K^{(r)}(t) dt \right|
\]

Replacing \( J(t - th) \) by \( J(t) \) in the first term on the right-hand side results in an error of order \( O_P(n^{-1/2}d^{3/2}h^{-2} \sqrt{\log n}) \). Given Step 1, the rest of the proof is completely analogous to the last part of the proof of Lemma 7.

Remark 7 (Expansion of \( \hat{Q}_x^r(\tau) \)). Inspection of the proof shows that

\[
\hat{Q}_x^r(\tau) - Q_x^r(\tau) = \frac{1}{nh^{r}} \sum_{i=1}^{n} x^T J(t) X_i \{ t - I(U_i \leq t) \} K^{(r)} \left( \frac{\tau - t}{h} \right) dt + O_P(n^{-3/4}h^{-2} d^{3/2} \sqrt{\log n}) + O(h^2)
\]

uniformly in \( (\tau, x) \in [\epsilon, 1 - \epsilon] \times X_0 \), and the uniform rate over \( (\tau, x) \in [\epsilon, 1 - \epsilon] \times X_0 \) of the first term on the right-hand side is \( O_P(n^{-1/2}h^{-3/2}d\sqrt{\log n}) \).

Recall the definition of \( \xi_x \). In view of the proof of Lemma 8, the following lemma follows relatively directly from Lemma 14.

Lemma 15. Under the conditions of Theorem 3, the following asymptotic linear representation holds uniformly in \( x \in X_0 \):

\[
\hat{m}(x) - m(x) = \sqrt{d} \sum_{i=1}^{n} \xi_x(U_i, X_i) + O_P(n^{-3/4}h^{-2} d^{3/2} \sqrt{\log n} + n^{-1}h^{-4}d^2 \log n + h^2)
\]

where \( U_1, \ldots, U_n \sim U(0, 1) \) i.i.d. independent of \( X_1, \ldots, X_n \). In addition, we have

\[
\sup_{x \in X_0} \left| \frac{1}{nh^{3/2}} \sum_{i=1}^{n} \psi_x(U_i, X_i) \right| = O_P(n^{-1/2}h^{-3/2}d\sqrt{\log n}).
\]

We are now in position to prove Theorem 3.
Proof of Theorem 3. As before, we split the proof into two parts.

Step 1. We will apply Proposition 2.1 in Chernozhukov et al. (2017a) to \( n^{-1/2} \sum_{i=1}^n A \Psi_i \).

To this end, we will check Conditions (M.1), (M.2), and (E.1) of Chernozhukov et al. (2017a). Condition (M.1) follows automatically, so we will verify Conditions (M.2) and (E.1).

Condition (M.2). Recall that \( \|x\|/\sqrt{d} \leq C_2 \) for all \( x \in X_0 \). Observe that

\[
\max_{1 \leq k \leq M} \mathbb{E} \left[ \left\| (\xi_x(U_i, X_i))_{t \in S_k} \right\|^3 \right] \\
\leq O(h^{-3/2}) \sup_{\alpha \in \mathbb{S}^{d-1}} \mathbb{E}[|\alpha^T X|^3] \max_{1 \leq \ell \leq L} \int K^n \left( \frac{\tau_x - t}{h} \right)^3 dt = O(h^{-1/2}d^{1/2}),
\]

where we used the fact that

\[
\sup_{\alpha \in \mathbb{S}^{d-1}} \mathbb{E}[|\alpha^T X|^3] \leq C_3 \sqrt{d} \sup_{\alpha \in \mathbb{S}^{d-1}} \mathbb{E}[|\alpha^T X|^2] = C_3 \sqrt{d} \|\mathbb{E}[XX^T]\|_{op} = O(\sqrt{d}).
\]

This implies that \( \max_{1 \leq k \leq M} \mathbb{E}[|A_k^T \Psi_i|^3] = O(d^{1/2}h^{-1/2}) \). Likewise, \( \max_{1 \leq \ell \leq M} \mathbb{E}[|A_k^T \Psi_i|^4] = O(dh^{-1}) \).

Condition (E.2). Since \( \|X\| \leq C_3 \sqrt{d} \), we have \( |A_k^T \Psi_i| \leq \text{const. } h^{-1/2}d^{1/2} \).

Thus, applying Proposition 2.1 in Chernozhukov et al. (2017a), we have

\[
\sup_{b \in \mathbb{R}^M} \left| \mathbb{P}\left( n^{-1/2} \sum_{i=1}^n A \Psi_i \leq b \right) - \mathbb{P}(AG \leq b) \right| \to 0,
\]

provided that

\[
\frac{d \log^7 (Mn)}{nh} \to 0,
\]

which is satisfied under our assumption.

Step 2. Observe that

\[
\left\| A \sqrt{nh^3d^{-1}((\hat{m}(x_\ell) - m(x_\ell))_{\ell=1}^L - n^{-1/2} \sum_{i=1}^n A \Psi_i) } \right\|_\infty \\
= O_P(n^{-1/4}h^{-1/2}d^{3/2} \log n + n^{-1/2}h^{-5/2}d^{3/2} \log n + n^{1/2}h^{7/2}d^{-1/2}).
\]

(A3.17)

In view of the proof of Step 2 in Theorem 1, the desired conclusion follows if the right hand side on (A3.17) is \( o_P(1/\sqrt{\log M}) \), which is satisfied under our assumption.
C.2 Proof of Theorem 4

Define \( \Psi_i := (\hat{\psi}_{x_1}(U_i, X_i), \ldots, \hat{\psi}_{x_L}(U_i, X_i))^T \) with

\[
\hat{\psi}_x(u, x') := \frac{s_x(\tau_x)}{s_x''(\tau_x)} K' \left( \frac{\tau_x - u}{h} \right) x' J(\tau_x)^{-1} x'.
\]

Further, define \( \Sigma = \mathbb{E} [\tilde{\Psi}_i \tilde{\Psi}_i^T] \), \( \Gamma := \text{diag}\{\sigma_1, \ldots, \sigma_M\} \) with \( \sigma_i^2 := D_i^T \Sigma D_i \), and \( \tilde{\Lambda} := \tilde{\Gamma}^{-1} D \).

The following operator norm bound is in parallel to Lemma 10 for the fixed dimensional case.

**Lemma 16.** Under the conditions of Theorem 4, we have

\[
\sup_{x \in \mathcal{X}_0} \| \hat{J}(\hat{\tau}_x) - J(\tau_x) \|_{\text{op}} = O_P(n^{-1/2}h^{-3/2}d^{3/2}/\sqrt{\log n + h^2}).
\]

**Proof.** Observe that the left hand side can be bounded by

\[
\left\| \frac{1}{n} \sum_{i=1}^n \left( K_h(Y_i - X_i^T \hat{\beta}(\hat{\tau}_x))(\alpha T X_i)^2 - \mathbb{E} \left[ K_h(Y_i - X_i^T \beta)(\alpha T X)^2 \right] \bigg| \beta = \hat{\beta}(\hat{\tau}_x) \right) \right\|_{\mathbb{S}^{d-1} \times \mathcal{X}_0} + \left\| \mathbb{E} \left[ K_h(Y_i - X_i^T \beta)(\alpha T X)^2 \right] \bigg| \beta = \hat{\beta}(\hat{\tau}_x) \right\|_{\mathbb{S}^{d-1} \times \mathcal{X}_0} + \left\| \mathbb{E} \left[ f(X^T \beta \mid X)(\alpha T X)^2 \right] \bigg| \beta = \hat{\beta}(\hat{\tau}_x) \right\|_{\mathbb{S}^{d-1} \times \mathcal{X}_0}
\]

where \( \| \cdot \|_{\mathbb{S}^{d-1} \times \mathcal{X}_0} = \sup_{(\alpha, \beta) \in \mathbb{S}^{d-1} \times \mathcal{X}_0} | \cdot | \). By Taylor expansion and \( \sup_{\alpha \in \mathbb{S}^{d-1}} \mathbb{E}[(\alpha^T X)^2] = \| \mathbb{E}[XX^T] \|_{\text{op}} = O(1) \), we see that II = O(h^2). Next, applying the local maximal inequality (Lemma 2) combined with the fact that \( \sup_{\alpha \in \mathbb{S}^{d-1}} \mathbb{E}[|\alpha^T X|^4] = O(d) \), we can show that I = \( O_P(\sqrt{n^{-1}h^{-1}d^2 \log n}) \). Finally, the term III is bounded by

\[
C_1 \| \hat{\beta}(\hat{\tau}_x) - \beta(\tau_x) \|_{\mathcal{X}_0} \sup_{\alpha \in \mathbb{S}^{d-1}} \mathbb{E}[|\alpha^T X|^3] \quad \text{and} \quad \underline{O(d^{1/2})}
\]

\[
\| \hat{\beta}(\hat{\tau}_x) - \beta(\tau_x) \|_{\mathcal{X}_0} = O_P \left( \| \hat{\beta} - \beta \|_{\ell_1, 1-\epsilon} \sqrt{\| \beta(\hat{\tau}_x) - \beta(\tau_x) \|_{\mathcal{X}_0}} \right)
\]

\[
= O_P \left( n^{-1/2}d^{1/2} \sqrt{n^{-1/2}h^{-3/2}d \sqrt{\log n}} \right) = O_P(n^{-1/2}h^{-3/2}d^{3/2}/\sqrt{\log n}),
\]

where we used the observation that \( Q'(\tau) = x^T \beta'(\tau) \) is bounded in \( (\tau, x) \in [\epsilon, 1-\epsilon] \times \mathcal{X} \). Conclude that III = \( O_P(n^{-1/2}h^{-3/2}d^{3/2}/\sqrt{\log n}) \). \( \square \)
Similarly, we have the following lemma in parallel to Lemma 11.

**Lemma 17.** Under Assumption 3, we have
\[
\|\hat{\Sigma} - \tilde{\Sigma}\|_\infty = O_P(n^{-1/2}h^{-3/2}d(d^{1/2} \vee h^{-1})\sqrt{\log n + h}).
\]

**Proof.** The proof is analogous to the proof of Lemma 11, given that \(x_\ell/\sqrt{d} \leq C_2\) and we added normalization by \(\sqrt{d}\) in the definition of \(\psi_x\). The only missing part is a bound on
\[
\left\| \frac{1}{n} \sum_{i=1}^n X_i X_i^T - E[XX^T] \right\|_{op},
\]
but Rudelson’s inequality yields that the above term is \(O_P(\sqrt{d(\log d)/n})\); cf. Rudelson (1999).

We are now in position to prove Theorem 4.

**Proof of Theorem 4.** Observe that
\[
\begin{align*}
\sup_{b \in \mathbb{R}^M} \left| \mathbb{P}_U \left( n^{-1/2} \sum_{i=1}^n \hat{A}_i \hat{\Psi}_i \leq b \right) - \mathbb{P}(AG \leq b) \right| &
\leq \sup_{b \in \mathbb{R}^M} \left| \mathbb{P}_U \left( n^{-1/2} \sum_{i=1}^n \hat{A}_i \hat{\Psi}_i \leq b \right) - \mathbb{P}(\hat{A} \hat{G} \leq b) \right| \\
&+ \sup_{b \in \mathbb{R}^M} \left| \mathbb{P}(\hat{A} \hat{G} \leq b) - \mathbb{P}(AG \leq b) \right|,
\end{align*}
\]
where \(\hat{G} \sim N(0, \hat{\Sigma})\). The first term on the right hand side of (A3.18) is bounded by
\[
\begin{align*}
\sup_{b \in \mathbb{R}^M} \left| \mathbb{P}_U \left( n^{-1/2} \sum_{i=1}^n \hat{A}_i \hat{\Psi}_i \leq b \right) - \mathbb{P}(\hat{A} \hat{G} \leq b) \right| \\
&+ \sup_{b \in \mathbb{R}^M} \left| \mathbb{P}(\hat{A} \hat{G} \leq b) - \mathbb{P}(AG \leq b) \right| \\
&+ \sup_{b \in \mathbb{R}^M} \left| \mathbb{P}(AG \leq b) - \mathbb{P}(\hat{A} \hat{G} \leq b) \right|,
\end{align*}
\]
where \(\hat{G} \sim N(0, \hat{\Sigma})\) conditionally on \(D_n\). For I, we can apply Proposition 2.1 in Chernozhukov et al. (2017a) conditionally on \(D_n\). Similarly to the last part of the proof of Theorem 2, we can show that \(I = o_P(1)\) if
\[
\frac{d \log^7 (Mn)}{nh} \to 0,
\]
which is satisfied under our assumption. We can analyze \( II \) and \( III \) as in the proof of Theorem 2 and show that 
\begin{equation}
II \lor III = o_P(1) \text{ if } n^{-1/2}h^{-3/2}d(d^{1/2}\lor h^{-1})(\sqrt{\log n})\log^2 M = o(1)
\end{equation}
and \( h \log^2 M = o(1) \), which is satisfied under our assumption.

Finally, in view of the Gaussian comparison inequality (Lemma 4), we see that the second term on the right hand side of (A3.18) is \( o(1) \) if 
\begin{equation}
\|\hat{A}\hat{\Sigma}\hat{A}^T - A\Sigma A^T\|_\infty \log^2 M = o(1).
\end{equation}
It is not difficult to see that 
\begin{equation}
\|\hat{A}\hat{\Sigma}\hat{A}^T - A\Sigma A^T\|_\infty = O\left( \sup_{x_1,x_2 \in X_0} |E[\xi_{x_1}\xi_{x_2} - \hat{\psi}_{x_1}\hat{\psi}_{x_2}]| \right) = O_P(h) = o_P(1/\log^2 M).
\end{equation}
This completes the proof. \( \square \)

### Appendix D Additional simulation results

#### D.1 Mean squared error table of Section 4.1.2

In the following, we present the table of the mean squared error of our modal estimator \( \hat{m}_x \), which is defined by
\begin{equation}
MSE(\hat{m}_x) := s^{-1}\sum_{i=1}^{s}(\hat{m}_x^{(i)} - m_x)^2,
\end{equation}
where \( m_x \) is the true conditional mode and \( \hat{m}_x^{(i)} \) is our modal estimator in \( i \)-th repetition.

We will consider \( lmNormal, lmLognormal \) and \( Nonlinear \) models from Section 4.1.2.

The same subsample sizes, \( n = 500, 1000 \) and \( 2000 \), and repetition number \( s = 500 \) are considered as Section 4.1.2.

#### D.2 Additional simulation results for pivotal bootstrap testing

In this section, we examine the performance of our pivotal bootstrap testing when the alternative hypothesis and the null hypothesis are “close” which makes the testing lack of significance harder. We will again use the setup in Section 4.1.4 but we will replace the generation scheme that is in favor of the alternative hypothesis by 
\begin{equation}
Y = 1 + 3X_1 + \alpha X_2 + \xi,
\end{equation}
where we take \( \alpha = 1 \) and 0.8. As \( \alpha \) becomes smaller, the null hypothesis and the alternative hypothesis get “closer”. We present the results in Table A2 to Table A3.
Table A1: Mean squared error of $\hat{m}_x$.

| Design point | Sample size | lmNormal | lmLognormal | Nonlinear |
|--------------|-------------|----------|-------------|-----------|
|              | $n = 500$   | 0.047    | 0.276       | 0.003     |
| $X_1 = 0.3$  | $n = 1000$  | 0.030    | 0.218       | 0.001     |
|              | $n = 2000$  | 0.021    | 0.153       | 0.001     |
|              | $n = 500$   | 0.092    | 0.415       | 0.004     |
| $X_1 = 0.5$  | $n = 1000$  | 0.034    | 0.285       | 0.003     |
|              | $n = 2000$  | 0.019    | 0.263       | 0.002     |
|              | $n = 500$   | 1.323    | 0.723       | 0.009     |
| $X_1 = 0.7$  | $n = 1000$  | 0.215    | 0.476       | 0.007     |
|              | $n = 2000$  | 0.035    | 0.359       | 0.004     |

Table A2: Power for bootstrap testing and oracle testing with $\alpha = 1$.

| Design point | Sample size | Bootstrap testing | Oracle testing |
|--------------|-------------|-------------------|----------------|
|              |             | 0.05              | 0.01           | 0.05          | 0.01          |
|              | $n = 500$   | 0.635             | 0.425          | 0.945         | 0.915         |
| $X_1 = 0.3$  | $n = 1000$  | 0.835             | 0.7            | 0.995         | 0.975         |
|              | $n = 2000$  | 0.970             | 0.895          | 1             | 0.99          |
|              | $n = 500$   | 0.705             | 0.455          | 0.95          | 0.92          |
| $X_1 = 0.5$  | $n = 1000$  | 0.865             | 0.735          | 0.975         | 0.955         |
|              | $n = 2000$  | 0.965             | 0.93           | 0.995         | 0.995         |
|              | $n = 500$   | 0.67              | 0.465          | 0.98          | 0.925         |
| $X_1 = 0.7$  | $n = 1000$  | 0.885             | 0.755          | 0.99          | 0.985         |
|              | $n = 2000$  | 0.95              | 0.875          | 1             | 0.98          |

From the tables, as we expect, we can see the decrease of power of both bootstrap testing and oracle testing under the same design point ($X_1$) and subsample size ($n$) as
| Design point | Sample size | Bootstrap testing | Oracle testing |
|--------------|-------------|-------------------|----------------|
|              |             | 0.05   | 0.01      | 0.05  | 0.01  |
|              | $n = 500$   | 0.485  | 0.24      | 0.89  | 0.8   |
| $X_1 = 0.3$  | $n = 1000$  | 0.755  | 0.56      | 0.965 | 0.94  |
|              | $n = 2000$  | 0.915  | 0.785     | 0.995 | 0.965 |
|              | $n = 500$   | 0.48   | 0.245     | 0.88  | 0.795 |
| $X_1 = 0.5$  | $n = 1000$  | 0.775  | 0.585     | 0.975 | 0.935 |
|              | $n = 2000$  | 0.94   | 0.81      | 0.995 | 0.975 |
|              | $n = 500$   | 0.395  | 0.195     | 0.905 | 0.77  |
| $X_1 = 0.7$  | $n = 1000$  | 0.745  | 0.56      | 0.95  | 0.905 |
|              | $n = 2000$  | 0.88   | 0.77      | 0.98  | 0.96  |

$\alpha$ gets smaller. For a fixed design point, the power of both tests approaches 1 with the increasing subsample size which supports our theory. In particular, the good performance of the oracle testing justifies our normal approximation theory. The performance of the proposed bootstrap testing is inferior to the oracle testing under the same design point and subsample size. This may due to several reasons including the bootstrap approximation error and the bias in the estimation of the nuisance parameters. However, the performance of bootstrap testing is reasonable when the sample size is sufficiently large which agrees with our asymptotic theory.

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