Leapfrog/Finite Element Method for Fractional Diffusion Equation

Zhengang Zhao¹ and Yunying Zheng²

¹ Department of Fundamental Courses, Shanghai Customs College, Shanghai 201204, China
² School of Mathematical Sciences, Huaibei Normal University, Huaibei 235000, China

Correspondence should be addressed to Zhengang Zhao; zgzhao888@163.com

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We analyze a fully discrete leapfrog/Galerkin finite element method for the numerical solution of the space fractional order (fractional for simplicity) diffusion equation. The generalized fractional derivative spaces are defined in a bounded interval. And some related properties are further discussed for the following finite element analysis. Then the fractional diffusion equation is discretized in space by the finite element method and in time by the explicit leapfrog scheme. For the resulting fully discrete, conditionally stable scheme, we prove an $L^2$-error bound of finite element accuracy and of second order in time. Numerical examples are included to confirm our theoretical analysis.

1. Introduction

Fractional calculus and fractional partial differential equations (FPDEs) have many applications in various aspects such as in viscoelastic mechanics, power-law phenomenon in fluid and complex network, allometric scaling laws in biology and ecology, colored noise, electrode-electrolyte polarization, dielectric polarization, boundary layer effects in ducts, electromagnetic waves, quantitative finance, quantum evolution of complex systems, and fractional kinetics [1]. And a lot of attention has recently been paid to the problem of the numerical approximation of FPDEs.

Generally speaking, the finite difference method and the finite element method are the two main means to solve FPDEs. Recently, some typical fractional difference methods have been utilized to solve FPDEs numerically [2–4]. On the other hand, the finite element method has also been used to find the variational solution of FPDEs [5–14]. But there are still some interesting schemes that can be constructed to enhance the convergence order by using the finite difference/finite element mixed method.

In this paper, we use the explicit leapfrog difference/Galerkin finite element mixed method to numerically solve the space fractional diffusion equation in order to get a higher convergence order.

The fractional diffusion equation as a typical kind of fractional partial differential equation [15] is a generalization of the classical diffusion equation, which can be used to better characterize anomalous diffusion phenomena. Besides, the spatial fractional diffusion equation usually describes the Lévy flights. The operator $\mathbb{R}_L^\Delta_\alpha \mathbb{D}_\alpha \mathbb{I}^\beta$ is commonly referred to the left (right) sided Lévy stable distribution, where the underlying stochastic process is Lévy $\alpha$-stable flights; see [16–18]. And a more general form $\mathbb{R}_L^\Delta_\alpha \mathbb{D}_\alpha \mathbb{I}^\beta$ is widely used for mathematical modelling and numerical computation.

Here, we mainly focus on constructing and analyzing a kind of efficient numerical schemes for approximately solving space fractional diffusion equation. The considered problem reads as follows: for $1/2 < \alpha < 1$,

$$\partial_t u (x, t) - \Delta_\alpha (\lambda \cdot u (x, t)) = f (x, t), \quad x \in \Omega, \ t \in [0, T],$$

$$u (x, t) = \varphi (t), \quad x \in \partial \Omega, \ t \in [0, T],$$

$$u (x, 0) = \psi (x), \quad x \in \Omega,$$

(1)
where \( \Omega = [a, b], \) time \( T > 0. \) Here the spatial fractional differential operator \( \Delta^n \) is denoted by \( \kappa_1 \cdot RL_{a, x}^{2\alpha} + \kappa_2 \cdot RL_{x, b}^{2\alpha}, \) where \( 0 \leq \kappa_1, \kappa_2 \leq 1, \) and \( \kappa_1 + \kappa_2 = 1. \) When \( \alpha = 1, \) the problem models a Brownian diffusion process. And \( f \) is a source term, \( \lambda \) is a positive constant.

The rest of this paper is constructed as follows. In Section 2, the preliminary knowledge of fractional derivative and the generalized fractional derivative spaces are defined. And some related properties are further discussed. The approximate system of the equation, existence and uniqueness of the weak solution, and the error estimates of the fully discrete scheme for (1) are studied in Section 3. In Section 4, numerical examples are presented to demonstrate the efficiency of the theoretical results derived in Section 3.

### 2. Generalized Fractional Derivative Spaces

In this section, we first give the definition of fractional derivatives. There are several definitions for the fractional derivatives, which is a reasonable generalization of the classical derivative [1, 19–22]. Then we define the generalized fractional derivative spaces by using Riemann-Liouville derivative, which is extended from the \( L^2 \) sense to the \( L^p \) sense.

**Definition 1.** The \( n \)th order left and right Riemann-Liouville integrals of function \( u(x) \) are defined in a finite interval \( (a, b) \) as follows:

\[
\begin{align*}
RL_{a, x}^{-\alpha} u(x) & = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{u(s)}{(x-s)^{1-\alpha}} ds, \\
RL_{x, b}^{-\alpha} u(x) & = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{u(s)}{(s-x)^{1-\alpha}} ds,
\end{align*}
\]

where \( \alpha > 0. \)

**Definition 2.** The \( n \)th order left and right Riemann-Liouville derivatives of function \( u(x) \) defined in a finite interval \( (a, b) \) are given as

\[
\begin{align*}
RL_{a, x}^{\alpha} u(x) & = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-\tau)^{n-\alpha-1} u(\tau) d\tau, \\
RL_{x, b}^{\alpha} u(x) & = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (\tau-x)^{n-\alpha-1} u(\tau) d\tau,
\end{align*}
\]

in which \( n-1 < \alpha < n \in \mathbb{Z}^+. \) Obviously, they are the integer derivatives of the left and right fractional integrals, respectively.

Now, we give some lemmas and corollaries which are necessary to define the generalized fractional derivative spaces.

**Lemma 3** (see [5]). Let \( \Omega = [a, b] \) be bounded and \( \alpha > 0. \) Then \( u \in L^p(\Omega) \) satisfies

\[
\begin{align*}
\| RL_{a, x}^{-\alpha} u(x) \|_{L^p(\Omega)} & \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \| u \|_{L^p(\Omega)}, \\
\| RL_{x, b}^{-\alpha} u(x) \|_{L^p(\Omega)} & \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \| u \|_{L^p(\Omega)}.
\end{align*}
\]

**Lemma 4** (fractional integration by parts, see [20]). The relation

\[
\int_a^b RL_{a, x}^{-\alpha} u(x) \cdot v(x) dx = \int_a^b u(x) \cdot RL_{x, b}^{-\alpha} v(x) dx
\]

is valid under the assumption that

\[
u(x) \in L^p(\Omega), \quad v(x) \in L^q(\Omega), \quad \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha, \quad p \geq 1, \quad q \geq 1,
\]

with \( p \neq 1, q \neq 1 \) in the case \( 1/p + 1/q = 1 + \alpha. \)

**Corollary 5** (see [20]). The formula

\[
\int_a^b RL_{a, x}^{\alpha} u(x) \cdot v(x) dx = \int_a^b u(x) \cdot RL_{x, b}^{\alpha} v(x) dx
\]

is valid under the assumption that \( u(x) \in RL_{a, x}^{\alpha}(L^p(\Omega)), \ v(x) \in RL_{x, b}^{\alpha}(L^q(\Omega)), \) \( 1/p + 1/q \leq 1 + \alpha, \) where the function space \( RL_{a, x}^{\alpha}(L^p(\Omega)) = \{ f(x) \mid f(x) = RL_{a, x}^{\alpha} \phi(x), \phi(x) \in L^p(\Omega) \}, \ RL_{x, b}^{\alpha}(L^q(\Omega)) = \{ f(x) \mid f(x) = RL_{x, b}^{\alpha} \psi(x), \psi(x) \in L^q(\Omega) \}. \)

**Corollary 6** (see [13]). One can further give the following corollary:

\[
\int_a^b RL_{a, x}^{2\alpha} u(x) \cdot v(x) dx = \int_a^b RL_{a, x}^{\alpha} u(x) \cdot RL_{x, b}^{\alpha} v(x) dx
\]

under the assumption that \( u(x) \in RL_{a, x}^{2\alpha}(L^p(\Omega)), \ v(x) \in RL_{a, x}^{\alpha}(L^q(\Omega)), \) \( 1/p + 1/q \leq 1 + \alpha. \)

Note that the above assumption \( u(x) \in RL_{a, x}^{2\alpha}(L^p(\Omega)) \) implies \( u(x) \in RL_{a, x}^{\alpha}(L^p(\Omega)) \) one can prove that by using Lemma 3.

**Corollary 7** (see [13]). Consider

\[
\int_a^b RL_{x, b}^{2\alpha} u(x) \cdot v(x) dx = \int_a^b RL_{x, b}^{\alpha} u(x) \cdot RL_{a, x}^{\alpha} v(x) dx
\]

under the assumption that \( u(x) \in RL_{x, b}^{2\alpha}(L^p(\Omega)), \ v(x) \in RL_{a, x}^{\alpha}(L^q(\Omega)), \) \( 1/p + 1/q \leq 1 + \alpha. \)

Note that, from the definition of the function space \( RL_{a, x}^{\alpha}(L^p(\Omega)) \), we can get that if \( u(x) \in RL_{a, x}^{\alpha}(L^p(\Omega)), \)
then \( u(x) = \int_{\alpha}^{b} D_{\alpha}^{\alpha} p(x), \) and \( D_{\alpha}^{\alpha} u(x) = \phi(x), \) where \( \phi(x) \in L^p(\Omega), \) such that \( u \in L^p(\Omega), \) which is obtained by Lemma 3. And \( D_{\alpha}^{\alpha} u(x) \in L^p(\Omega) \) naturally holds. So, by the above idea, we define the following fractional derivative spaces from the \( L^2 \) sense to the \( L^p \) sense, which will be proved to be equivalent with the fractional Sobolev spaces under some certain conditions.

**Definition 8.** Define the following norms of the left (with symbol \( W_{\alpha}^{\alpha,p} \)) fractional derivative space and the right (with symbol \( W_{\alpha}^{\alpha,p} \)) fractional derivative space in a bounded interval \( \Omega = [a, b] \) as follows correspondingly, where \( 1 < p < +\infty: \)

\[
W_{\alpha}^{\alpha,p}(\Omega) = \left\{ u \in L^p(\Omega) : D_{\alpha}^{\alpha} u(x) \in L^p(\Omega) \right\}
\]

equipped with seminorm

\[
|u|_{W_{\alpha}^{\alpha,p}(\Omega)} = \left\| D_{\alpha}^{\alpha} u(x) \right\|_{L^p(\Omega)}
\]

and norm

\[
\|u\|_{W_{\alpha}^{\alpha,p}(\Omega)} = \left( \sum_{k=0}^{[\alpha]} \left\| D^k u \right\|^p_{L^p(\Omega)} + |u|_{W_{\alpha}^{\alpha,p}(\Omega)}^p \right)^{1/p}
\]

**Definition 9.** Define the symmetric fractional derivative space (with symbol \( H_{\alpha}^{\alpha} \)) in a bounded interval \( \Omega = [a, b] \) in the \( L^2 \) sense

\[
H_{\alpha}^{\alpha}(\Omega) = \left\{ u \in L^2(\Omega) : \int_{a}^{b} D_{\alpha}^{\alpha} u(x) \cdot D_{\alpha}^{\alpha} u(x) \, dx \in L^2(\Omega) \right\}
\]

equipped with seminorm

\[
|u|_{H_{\alpha}^{\alpha}(\Omega)} = \left\| \int_{a}^{b} D_{\alpha}^{\alpha} u(x) \cdot D_{\alpha}^{\alpha} u(x) \, dx \right\|^{1/2}
\]

and norm

\[
\|u\|_{H_{\alpha}^{\alpha}(\Omega)} = \left( \sum_{k=0}^{[\alpha]} \left\| D^k u \right\|^2_{L^2(\Omega)} + |u|_{H_{\alpha}^{\alpha}(\Omega)}^2 \right)^{1/2}
\]

**Definition 10.** Define the spaces \( W_{\alpha}^{\alpha,p}(\Omega) \) as the closures of \( C_0^{\alpha}(\Omega) \) under their respective norms.

From [6], we can get the following lemma, which is true in the \( L^2 \) sense.

**Lemma 11.** The spaces \( W_{\alpha}^{\alpha,2}(\Omega), W_{\alpha}^{\alpha,2}(\Omega), H_{\alpha}^{\alpha}(\Omega), \) and \( H_{\alpha}^{\alpha}(\Omega) \) are equal to equivalent seminorms and norms, where \( H_{\alpha}^{\alpha}(\Omega) \) is the fractional Sobolev space in terms of the Fourier transform.

Therefore, in this paper we always use \( H_{\alpha}^{\alpha} \) when \( p = 2, \) to denote the fractional derivative space equipped with the norm \( \| \cdot \|_\alpha \) which can be any one of (12), (15), and (18), and \( H_{\alpha}^{\alpha}(\Omega) \) is denoted as the dual space of \( H_{\alpha}^{\alpha}(\Omega), \) with norm \( \| \cdot \|_\alpha. \)

Moreover, we can present some new properties about norms for the above left and right fractional derivative spaces in the \( L^p \) sense.

**Lemma 12.** Let \( \alpha > 0 \) and \( \Omega = [a, b] \subset \mathbb{R} \) be bounded. Then the following mapping properties hold:

1. \( D_{\alpha}^{\alpha} u(x) : L^p(\Omega) \to L^p(\Omega) \) is a bounded linear operator;
2. \( D_{\alpha}^{\alpha} u(x) : L^p(\Omega) \to L^p(\Omega) \) is a bounded linear operator;
3. \( D_{\alpha}^{\alpha} u(x) : W_{\alpha}^{\alpha,p}(\Omega) \to L^p(\Omega) \) is a bounded linear operator;
4. \( D_{\alpha}^{\alpha} u(x) : W_{\alpha}^{\alpha,p}(\Omega) \to L^p(\Omega) \) is a bounded linear operator;
5. \( D_{\alpha}^{\alpha} u(x) : L^p(\Omega) \to W_{\alpha}^{\alpha,p}(\Omega) \) is a bounded linear operator;
6. \( D_{\alpha}^{\alpha} u(x) : L^p(\Omega) \to W_{\alpha}^{\alpha,p}(\Omega) \) is a bounded linear operator.

**Proof.** Properties (1) and (2) follow directly from Lemma 3. Property (3) follows directly from the definition of \( W_{\alpha}^{\alpha,p}(\Omega) \) and \( W_{\alpha}^{\alpha,p}(\Omega) \) as

\[
\| D_{\alpha}^{\alpha} u(x) \|_{L^p(\Omega)} \leq \left( \sum_{k=0}^{[\alpha]} \| D^k u \|_{L^p(\Omega)}^p + \| D_{\alpha}^{\alpha} u(x) \|_{L^p(\Omega)}^p \right)^{1/p}
\]

Property (4) follows similarly.
Property (5) follows from the definition of \( W_{L}^{α,p}(Ω) \) and the semigroup property of fractional operator,
\[
\left\| RL D_{α,x}^α u(x) \right\|_{W_{L}^{α,p}(Ω)} = \left( \sum_{k=0}^{[α]} \left\| D^k \cdot RL D_{α,x}^α u(x) \right\|_{L^p(Ω)}^p \right)^{1/p} + \left\| RL D_{α,x}^α u(x) \right\|_{L^p(Ω)}^p \leq \left( \sum_{k=0}^{[α]} \left\| D^k \cdot RL D_{α,x}^α u(x) \right\|_{L^p(Ω)}^p \right)^{1/p}.
\]
Using Lemma 3, there exist constants \( c_k, k = 0, \ldots, [α] \) such that
\[
\left\| RL D_{α,x}^α u(x) \right\|_{L^p(Ω)}^p \leq C_k \left\| u \right\|_{L^p(Ω)}^p.
\]
Therefore, we obtain the bound
\[
\left\| RL D_{α,x}^α u(x) \right\|_{W_{L}^{α,p}(Ω)} \leq \left( 1 + \sum_{k=0}^{[α]} c_k \right) \left\| u \right\|_{L^p(Ω)}.
\]
Property (6) follows similarly.

\[\textbf{Corollary 13.} \text{ Consider } W_{L}^{α,p}(Ω) \rightarrow W_{L}^{α-[α],p}(Ω), \]
\[
W_{R}^{α,p}(Ω) \rightarrow W_{R}^{α-[α],p}(Ω)
\]
for \( 1 ≤ p < ∞ \). And if \( 1 ≤ p ≤ q < ∞ \), one has
\[
W_{L}^{α,p}(Ω) \rightarrow W_{L}^{α,q}(Ω), \quad W_{R}^{α,p}(Ω) \rightarrow L^q(Ω).
\]

It is obviously true by using the norms of fractional derivative spaces and imbedding theorems for \( L^p(Ω) \).

\[\textbf{Lemma 14.} \text{ Let } Ω = [a, b] \subset R \text{ be bounded. Then for } u \in W_{L}^{α,p}(Ω), \text{ one has } \left\| u \right\|_{L^p(Ω)} ≤ C \left\| u \right\|_{L^{α,p}(Ω)}, \text{ for } 0 < s < α, \text{ one has } \left\| u \right\|_{L^{α,p}(Ω)} ≤ C \left\| u \right\|_{L^{α,s,p}(Ω)}.\]

Proof. If \( u \in W_{L}^{α,p}(Ω) \) by using Lemmas 3 and 12, we have that
\[
\left\| u \right\|_{L^p(Ω)} = \left\| RL D_{α,x}^α RL D_{α,x}^α u(x) \right\|_{L^p(Ω)} ≤ \frac{(b - a)^s}{Γ(α + 1)} \left\| RL D_{α,x}^α u(x) \right\|_{L^p(Ω)}.
\]
Therefore, (25) is true. And the following inequality certainly holds:
\[
\left\| u \right\|_{L^p(Ω)} ≤ C \left\| RL D_{α,x}^α u(x) \right\|_{L^p(Ω)}.
\]
So, we get that
\[
\left\| RL D_{α,x}^α u(x) \right\|_{L^p(Ω)} ≤ \left\| RL D_{α,x}^α RL D_{α,x}^α u(x) \right\|_{L^p(Ω)} ≤ \left\| RL D_{α,x}^α u(x) \right\|_{L^p(Ω)}.
\]
Therefore, (26) holds.

\[\textbf{3. Error Estimates of the Leapfrog/Finite Element Scheme}\]

In this section, we firstly give a fully discrete scheme, where we use the leapfrog difference method in the temporal direction and the finite element method in the spatial direction and then analyze the error estimate. Let \( S_h \) denote a uniform partition on \( Ω \), with grid parameter \( h \). For \( k ∈ N \), let \( P_k(Ω) \) denote the space of polynomials on \( Ω \) with degree not greater than \( k \). Then we define \( X_h \) as the finite element space on \( S_h \) with the basis of the piecewise polynomials of order \( k \in Z^+ \); that is,
\[
X_h = \{ v ∈ X | v \in C(Ω) : \forall \gamma ∈ P_k(D), \forall D ∈ S_h \},
\]
in which \( D \) is the unit of \( S_h \).

The following property of finite element spaces is necessary for our subsequent analysis [23]: for \( u ∈ H^{k+1}(Ω), 0 ≤ μ ≤ k + 1 \), there exists \( v ∈ X_h \) such that
\[
\left\| u - v \right\|_μ ≤ C H^{k+1-μ} \left\| u \right\|_{k+1}.
\]
The Gronwall's lemma is also needed for the error analysis.

\[\textbf{Lemma 15 (discrete Gronwall’s lemma, see [24]).} \text{ Let } Δt, H \text{ and } a_n, b_n, c_n, γ_n \text{ (for integer } n ≥ 0 \text{) be nonnegative numbers such that } \]
\[
a_n + Δt \sum_{n=0}^{N} b_n ≤ Δt \sum_{n=0}^{N} γ_n a_n + Δt \sum_{n=0}^{N} c_n + H,
\]
for \( N ≥ 0 \). Suppose that \( Δt γ_n < 1 \) for all \( n \), and set \( γ_n = (1 - Δt γ_n)^{-1} \); then
\[
a_n + Δt \sum_{n=0}^{N} b_n ≤ \exp \left( Δt \sum_{n=0}^{N} a_n γ_n \right) \left( Δt \sum_{n=0}^{N} c_n + H \right).
\]
for \( N ≥ 0 \).

In the following, we give the fully discrete scheme of (1). Let \( Δt \) denote the step size for \( t \) so that \( t_n = n Δt, n = 1, 2, \ldots, N - 1 \). For notational convenience, we denote \( u^n := u(t_n, t_n) \) and
\[
d_t u^n := \frac{u^{n+1} - u^{n-1}}{2 Δt}.
\]
Let $u^n_h$ of (1) be the finite element solution at time $t = t_n$ of the following fully discrete scheme:

$$
(d_t u^n_h, v) - (\Delta^\alpha (\lambda \cdot u^n_h), v) = (f^n, v), \quad \forall v \in X_h;
$$
(35)

that is,

$$
(u^{n+1}_h - u^n_h, v) - 2\Delta t (\Delta^\alpha (\lambda \cdot u^n_h), v) = 2\Delta t (f^n, v), \quad \forall v \in X_h,
$$
(36)

where $(\cdot, \cdot)$ is denoted by an $L^2$ inner product and $(\Delta^\alpha (\lambda \cdot u^n_h), v) = \kappa_1 \cdot (\langle \text{RL}^D_{\alpha \cdot a_\cdot c} (\lambda \cdot u^n_h), \text{RL}^D_{\alpha \cdot b \cdot c} v \rangle) + \kappa_2 \cdot (\langle \text{RL}^D_{\alpha \cdot a_\cdot c} (\lambda \cdot u^n_h), \text{RL}^D_{\alpha \cdot b_\cdot c} v \rangle).$ For brevity, we always use $(\Delta^\alpha (\lambda \cdot u^n_h), v)$ instead of the right hand side of this equation.

**Lemma 16.** For a sufficient small step size $\Delta t > 0$, there exists a unique solution $u^n_h \in X_h$ satisfying (36).

**Proof.** Firstly, we prove that $(u^n_h, u^n_h)/2\Delta t - (\Delta^\alpha (\lambda \cdot u^n_h), u^n_h)$ is positive, which is one of the sufficient conditions for the existence and uniqueness of $u^n_h$.

For $\Delta t > 0$ chosen sufficiently small, we have that

$$
\frac{(u^n_h, u^n_h)}{2\Delta t} - (\Delta^\alpha (\lambda \cdot u^n_h), u^n_h) \geq C\|u^n_h\|_{\alpha, h}^2.
$$
(37)

Besides, by using the fractional Poincare-Friedrichs formula, we can easily get the continuity of $(u^n_h, u^n_h)/2\Delta t - (\Delta^\alpha (\lambda \cdot u^n_h), u^n_h)$. Hence, by using the Lax-Milgram theorem, we have that (36) is uniquely solvable for $u^n_h$.

Now, we carry out the error analysis for the fully discrete problem. The following norms are also used in the analysis:

$$
\|u\|_{\text{cock}} = \max_{1 \leq n \leq N} \|u^n\|_k,
$$
(38)

$$
\|u\|_{0, \alpha} = \left( \sum_{n=1}^{N} \|u^n\|_{\alpha, h}^2 dt \right)^{1/2}.
$$

**Theorem 17.** Assume that (1) has a solution $u$ satisfying $u \in L^2(0, T; H^\alpha \cap H^{k+1}(\Omega))$, $u_t \in L^2(0, T; H^{k+1}(\Omega))$, and $u_{tt} \in L^2(0, T; L^2(\Omega))$, with $u^0 \in H^{k+1}(\Omega)$. $u^n_h$ is the solution of (36), and $u^n_h$ is computed in such a way that

$$
\|u^n_h(x) - u(x, \Delta t)\| \leq C(\Delta t)^2.
$$
(39)

Then, there exists a constant $C_0$ independent of $h$ and $\Delta t$, such that if

$$
\Delta t \cdot h^{-2\alpha} \leq C_0,
$$
(40)

then the finite element approximation (36) is convergent to the solution of (1) on the interval $(0, T)$ as $\Delta t, h \to 0$.

And the approximation solution $u^n_h$ satisfies the following error estimates:

$$
\|u - u^n_h\|_{0, \alpha} \leq C(h^{k+1}\|u_t\|_{0, k+1} + (\Delta t)^2\|u_{tt}\|_{0, 0}) + h^{k+1-\alpha}\|u_{tt}\|_{0, k+1},
$$
(41)

$$
\|u - u^n_h\|_{\infty, 0} \leq C(h^{k+1}\|u_t\|_{0, k+1} + (\Delta t)^2\|u_{tt}\|_{0, 0} + h^{k+1-\alpha}\|u_{tt}\|_{0, k+1} + h^{k+1}\|u|_{\infty, k+1}).
$$
(42)

**Proof.** In order to estimate (41) and (42), we first discuss the error at $t = t_n$, $n = 1, 2, \ldots, N - 1$. Let $u^n = u(t_n, \cdot)$ represent the solution of (1), define $\varepsilon^n = u^n - u^n_h$, and for $U^n \in X_h$, define $A^n$ and $E^n$ as $A^n = u^n - U^n$, $E^n = U^n - u^n_h$. So, we have $\varepsilon^n = A^n + E^n$.

Obviously, the true solution of this problem (1) $u^n$ also satisfies

$$
(d_t u^n, v) - (\Delta^\alpha (\lambda \cdot u^n), v) = (f^n, v), \quad \forall v \in X_h.
$$
(43)

Therefore, subtracting (36) from (43) gives

$$
(d_t u^n - d_t u^n_h, v) = 2\Delta t (\delta u^n - d_t u^n_h, v), \quad \forall v \in X_h;
$$
(44)

that is,

$$
(\varepsilon^{n+1} - \varepsilon^{n-1}, v) - 2\Delta t (\Delta^\alpha (\lambda \cdot \varepsilon^n), v) = 2\Delta t (d_t u^n - d_t u^n_h, v), \quad \forall v \in X_h.
$$
(45)

Substituting $\varepsilon^{n+1} = A^{n+1} + E^{n+1}$, $\varepsilon^n = A^n + E^n$ into (45) leads to

$$
(E^{n+1} - E^{n-1} + E^{n+1}) - 2\Delta t (\Delta^\alpha (\lambda \cdot E^n), E^{n+1} + E^{n-1}) = - (A^{n+1} - A^{n-1} + E^{n+1} + E^{n-1}) + 2\Delta t (\Delta^\alpha (\lambda \cdot A^n), E^{n+1} + E^{n-1}) + 2\Delta t (d_t u^n - d_t u^n_h, E^{n+1} + E^{n-1}).
$$
(46)

After adding $\|E^n\|^2$ to both sides of (46), we obtain the identity

$$
\|E^{n+1}\|^2 = \|E^n\|^2 - 2\Delta t (\Delta^\alpha (\lambda \cdot E^n), E^{n+1}) + \|E^{n-1}\|^2 + 2\Delta t (\Delta^\alpha (\lambda \cdot E^n), E^{n-1}) - (A^{n+1} - A^{n-1} + E^{n+1} + E^{n-1}) + 2\Delta t (\Delta^\alpha (\lambda \cdot A^n), E^{n+1} + E^{n-1}) + 2\Delta t (d_t u^n - d_t u^n_h, E^{n+1} + E^{n-1}).
$$
(47)

Define now the quantity $A^{n+1}$, for $1 \leq n \leq N - 1$, by

$$
A^{n+1} = \|E^{n+1}\|^2 + \|E^{n-1}\|^2 - 2\Delta t (\Delta^\alpha (\lambda \cdot E^n), E^{n+1}) + \|E^{n+1}\|^2 - 2\Delta t (\Delta^\alpha (\lambda \cdot E^n), E^{n-1}) + 2\Delta t (\Delta^\alpha (\lambda \cdot A^n), E^{n+1} + E^{n-1}) + 2\Delta t (d_t u^n - d_t u^n_h, E^{n+1} + E^{n-1}).
$$
(48)

We can rewrite (47) as

$$
A^{n+1} = A^n - (A^{n+1} - A^{n-1} + E^{n+1} + E^{n-1}) + 2\Delta t (\Delta^\alpha (\lambda \cdot E^n), E^{n-1}) + 2\Delta t (\Delta^\alpha (\lambda \cdot A^n), E^{n+1} + E^{n-1}) + 2\Delta t (\Delta^\alpha (\lambda \cdot E^n), E^n) + 2\Delta t (d_t u^n - d_t u^n_h, E^{n+1} + E^{n-1}).
$$
(49)
Denoting

\[
F(E^{n-1}, E^n, E^{n+1}) = - (\Lambda^{n+1} - \Lambda^{n-1}, E^{n+1} + E^{n-1}) + 2\Delta t (\Delta^a (\lambda \cdot E^n), E^{n-1}) + 2\Delta t (\Delta^a (\lambda \cdot \Lambda^n), E^{n+1} + E^{n-1}) + 2\Delta t (d_t u^n - u^n_t, E^{n+1} + E^{n-1}),
\]

then (49) can be abbreviated as

\[
A^{n+1} = A^n + F(E^{n-1}, E^n, E^{n+1}).
\]

We now estimate each term in \(F(E^{n-1}, E^n, E^{n+1})\). For the second term of the right hand side, one has

\[
2\Delta t (\Delta^a (\lambda \cdot E^n), E^{n-1}) = 2\Delta t \left( \kappa_1 \left( \text{RLD}_{ax}^\alpha (\lambda \cdot E^n), \text{RLD}_{xb}^\alpha E^{n-1} \right) + \kappa_2 \left( \text{RLD}_{xb}^\alpha (\lambda \cdot E^n), \text{RLD}_{ax}^\alpha E^{n-1} \right) \right) \leq C_1 \Delta t \|E^n\|_\alpha^2 + C_2 \Delta t \|E^{n-1}\|_\alpha^2 (\Lambda^{n+1} - \Lambda^{n-1}, E^{n+1} + E^{n-1}) \leq \|\Lambda^{n+1} - \Lambda^{n-1}\| \cdot \|E^{n+1} + E^{n-1}\| = 2\Delta t \|d_t \Lambda^n\| \cdot \|E^{n+1} + E^{n-1}\| \leq \Delta t \|d_t \Lambda^n\|^2 + \Delta t (\|E^{n+1}\|^2 + \|E^{n-1}\|^2) \leq C_3 \Delta t \cdot h^{2k+2} \|u^n\|^2_{0,k+1} + \Delta t \left( \|E^{n+1}\|^2 + \|E^{n-1}\|^2 \right),
\]

where

\[
\sum_{n=0}^{N} \Delta t \|d_t \Lambda^n\|^2 = \sum_{n=1}^{N} \Delta t \left( \frac{1}{\Delta t} \left[ \int_{t_{n-1}}^{t_n} 1 \frac{\partial \lambda}{\partial t} dt \right]^2 \right) \leq \Delta t \sum_{n=1}^{N} \left( \frac{1}{\Delta t} \right)^2 \int_{t_{n-1}}^{t_n} \left( \int_{t_{n-1}}^{t_n} \frac{\partial \lambda}{\partial t} dt \right)^2 \right) dx \leq C_3 h^{2k+2} \|u^n\|^2_{0,k+1}.
\]

For the third term of the right hand side, one has

\[
2\Delta t (\Delta^a (\lambda \cdot \Lambda^n), E^{n+1} + E^{n-1}) = 2\Delta t \cdot \kappa_1 \left( \text{RLD}_{ax}^\alpha (\lambda \cdot \Lambda^n), \text{RLD}_{xb}^\alpha (E^{n+1} + E^{n-1}) \right) + 2\Delta t \cdot \kappa_2 \left( \text{RLD}_{xb}^\alpha (\lambda \cdot \Lambda^n), \text{RLD}_{ax}^\alpha (E^{n+1} + E^{n-1}) \right) \leq C_4 \Delta t \|\Lambda^n\|_\alpha \cdot \|E^{n+1} + E^{n-1}\|_\alpha \leq C_5 \Delta t \cdot h^{2(k+1-\alpha)} \|u^n\|^2_{k+1} + C_6 \Delta t \left( \|E^{n+1}\|^2_\alpha + \|E^{n-1}\|^2_\alpha \right),
\]

in which

\[
\|\lambda \cdot \Lambda^n\|_\alpha \leq \|\lambda\|_{\infty} \cdot \|\Lambda^n\|_\alpha \leq C_2 h^{2k+1-\alpha} \|u^n\|^2_{k+1}.
\]

For the fourth term of the right hand side, one has

\[
2\Delta t (d_t u^n - u^n_t, E^{n+1} + E^{n-1}) = (u^{n+1} - u^{n-1} - 2\Delta t \cdot u^n_t, E^{n+1} + E^{n-1}) \leq C_9 (\Delta t)^5 \|u^n\|^2_{ttt} + \Delta t \left( \|E^{n+1}\|^2 + \|E^{n-1}\|^2 \right),
\]

where by Taylor's theorem

\[
\|u^{n+1} - u^{n-1} - 2\Delta t \cdot u^n_t\| \leq C_9 (\Delta t)^3 \|u^n\|^3_{ttt}.
\]

Hence, summing from \(n = 1\) to \(N - 1\), one has

\[
A^N - A^{N-1} \leq \sum_{n=1}^{N-1} F(E^{n-1}, E^n, E^{n+1});
\]

that is,

\[
A^N \leq A^{N-1} + C_{10} \sum_{n=1}^{N-1} \Delta t \left( h^{2k+2} \|u^n\|^2_{k+1} + h^{2k+2-2\alpha} \|u^n\|^2_{k+1} + (\Delta t)^5 \|u^n\|^3_{ttt} \right) + C_{11} \sum_{n=1}^{N-1} \Delta t \left( \|E^{n+1}\|^2_\alpha + \|E^{n-1}\|^2_\alpha + \|E^n\|^2_\alpha \right) + \|E^{n-1}\|^2_\alpha + \|E^{n+1}\|^2_\alpha.
\]
We now show that, under our stability assumption (40), $A^{n+1}$ is positive and comparable to $\|E^n\|_2 + \|E^{n+1}\|_2$. To this end, we use the inverse inequality $\|v\|_\alpha \leq C_{12} h^{-\alpha} \|v\|$, $v \in \mathcal{X}_h$, and this yields
\[
2\Delta t \left( \Delta^x (\lambda_i E^n, E^{n+1}) \right) \\
\leq \Delta t \left( \|E^n\|_\alpha^2 + \|E^{n+1}\|_\alpha^2 \right) \\
\leq C_{13} \Delta t \cdot h^{-2\alpha} \left( \|E^n\|_2^2 + \|E^{n+1}\|_2^2 \right).
\]
Hence, if $\Delta t \cdot h^{-2\alpha}$ is sufficiently small such that $C_{13} \Delta t \cdot h^{-2\alpha} \leq C_{13} \leq 1$, we get
\[
(1 - C_{13}) \left( \|E^{n-1}\|_2^2 + \|E^{n}\|_2^2 \right) \\
\leq A^N \leq (1 + C_{13}) \left( \|E^{n-1}\|_2^2 + \|E^{n}\|_2^2 \right).
\]
So we have that
\[
\|E^{n-1}\|_2^2 + \|E^n\|_2^2 + C_{14} \sum_{n=1}^N \Delta t \|E^n\|_\alpha^2 \\
\leq \|E^n\|_2^2 + \|E^{n+1}\|_2^2 \\
+ C_{15} \sum_{n=1}^{N-1} \Delta t \left( h^{2k+2} \|u^n\|_{i,k+1} + h^{2k-2+2\alpha} \|u^n\|_{k+1}^2 \right) \\
+ (\Delta t)^4 \|u^n\|_{2,k+1}^2 \\
+ C_{16} \sum_{n=1}^{N-1} \Delta t \left( \|E^{n+1}\|_2^2 + \|E^{n-1}\|_2^2 \right).
\]
Therefore, we obtain
\[
\|E^{n-1}\|_2^2 + \|E^n\|_2^2 + C_{14} \sum_{n=1}^N \Delta t \|E^n\|_\alpha^2 \\
\leq \|E^n\|_2^2 + \|E^{n+1}\|_2^2 \\
+ C_{15} \left( h^{2k+2} \|u^n\|_{i,k+1} + h^{2k-2+2\alpha} \|u^n\|_{k+1}^2 \right) \\
+ (\Delta t)^4 \|u^n\|_{2,k+1}^2 \\
+ C_{16} \sum_{n=1}^{N-1} \Delta t \left( \|E^{n+1}\|_2^2 + \|E^{n-1}\|_2^2 \right).
\]
By using the discrete Gronwall’s Lemma 15, we have
\[
\|E^n\|_2^2 + C_{14} \sum_{n=1}^N \Delta t \|E^n\|_\alpha^2 \\
\leq C_{17} \|E^1\|_2^2 \\
+ C_{15} \left( h^{2k+2} \|u^n\|_{i,k+1} + h^{2k-2+2\alpha} \|u^n\|_{k+1}^2 \right) \\
+ (\Delta t)^4 \|u^n\|_{2,k+1}^2 \\
+ C_{16} \sum_{n=1}^{N-1} \Delta t \|E^{n+1}\|_2^2.
\]
Now denoting $G(\Delta t, h) = h^{2k+2} \|u^n\|_{i,k+1} + h^{2k-2+2\alpha} \|u^n\|_{k+1}^2 + (\Delta t)^4 \|u^n\|_{2,k+1}^2$ and using the condition (39), we get that
\[
\|E^n\|_2 = \sum_{n=1}^N \Delta t \|E^n\|_2^2 \leq C_{18} (T + 1) G(\Delta t, h).
\]
By using the interpolation property and the following result
\[
\|u - u^n\|_{0,\alpha} \leq \|E^n\|_2 + \|A\|_{0,\alpha},
\]
estimate (41) follows.

Using estimate (66) and approximation properties, we have
\[
\|u - u^n\|_{0,0} \leq \|E^n\|_{0,0} + \|A\|_{0,0} \leq G(\Delta t, h) + h^{2k+2} \|u^n\|_{2,0,k+1},
\]
which yields estimate (42).

4. Numerical Examples for Piecewise Linear Polynomials

Let $S_h$ denote a uniform partition on $\Omega = [a,b]$ and $X_h$ the space of continuous piecewise linear functions on $S_h$; that is, $k = 1$. Then we use the Galerkin finite element method for the spatial variables. After the spatial discretization, we get classical ODEs systems with variables $u^n_i, i = 1, 2, \ldots, T/\Delta t$. In order to satisfy the condition (39) in Theorem 17, we use the two-order Runge-Kutta method to compute the variable $u^n_i$.

In this section, we present numerical calculations which support the error estimates in Theorem 17. If we suppose $\Delta t = Ch^{-\alpha}$, then we have the convergence rate
\[
\|u(t_{m+1}) - u_h^{n+1}\|_{0,\alpha} \sim \mathcal{O}(h^{2-\alpha}),\]
\[
\|u(t_{m+1}) - u_h^{n+1}\|_{0,0} \sim \mathcal{O}(h^{2-\alpha}).\]

Example 1. (i) Let
\[
u(x,t) = t^2x (1-x)\]
The experiential error results and convergence rates of Example 1 (i).

| $h$   | $\|u_h - u_0\|_{0,0}$ | cv. rate | $\|u - u_h\|_{0,0}$ | cv. rate |
|-------|------------------------|----------|----------------------|----------|
| 1/4   | $1.0569 \cdot 10^{-2}$ | —        | $3.1077 \cdot 10^{-2}$ | —        |
| 1/8   | $4.0416 \cdot 10^{-3}$ | 1.3869   | $6.5469 \cdot 10^{-3}$ | 2.2470   |
| 1/16  | $3.8027 \cdot 10^{-4}$ | 3.4098   | $4.4162 \cdot 10^{-4}$ | 3.8899   |
| 1/32  | $1.0910 \cdot 10^{-4}$ | 1.5382   | $1.4097 \cdot 10^{-4}$ | 1.6474   |
| 1/64  | $8.5999 \cdot 10^{-5}$ | 1.5382   | $6.4241 \cdot 10^{-5}$ | 1.3353   |

The experiential error results and convergence rates of Example 1 (ii).

| $h$   | $\|u_h - u_0\|_{0,0}$ | cv. rate | $\|u - u_h\|_{0,0}$ | cv. rate |
|-------|------------------------|----------|----------------------|----------|
| 1/4   | $5.6283 \cdot 10^{-3}$ | —        | $9.3921 \cdot 10^{-3}$ | —        |
| 1/8   | $1.9379 \cdot 10^{-3}$ | 1.5382   | $3.2521 \cdot 10^{-3}$ | 1.5301   |
| 1/16  | $7.1701 \cdot 10^{-4}$ | 1.4344   | $1.2910 \cdot 10^{-3}$ | 1.4274   |
| 1/32  | $2.6932 \cdot 10^{-4}$ | 1.4128   | $4.7917 \cdot 10^{-4}$ | 1.3353   |
| 1/64  | $1.0362 \cdot 10^{-4}$ | 1.3780   | $1.9584 \cdot 10^{-4}$ | 1.2909   |

5. Conclusion

In this paper, we study the finite element method for fractional diffusion equation. We use the simple, second order accurate explicit scheme, leapfrog difference method in time, and the finite element method in space. Under the suitably accurate initial conditions and the stability requirement that $\Delta t \cdot h^{-2\alpha}$ be sufficiently small, the error analysis for the fully discrete scheme is discussed, which is an $L^2$-error bound of finite element accuracy and of second order in time. Numerical examples are given to demonstrate the efficiency of the theoretical results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, New York, NY, USA, 1993.
[2] F. Liu, V. Anh, I. Turner, and P. Zhuang, “Time fractional advection-dispersion equation,” Journal of Applied Mathematics and Computing, vol. 13, no. 1-2, pp. 233–245, 2003.
[3] M. M. Meerschaert and C. Tadjeran, “Finite difference approximations for two-sided space-fractional partial differential equations,” Applied Numerical Mathematics, vol. 56, no. 1, pp. 80–90, 2006.
[4] S. B. Yuste and L. Acedo, “An explicit finite difference method and a new von Neumann-type stability analysis for fractional...
diffusion equations,” *SIAM Journal on Numerical Analysis*, vol. 42, no. 5, pp. 1862–1874, 2005.

[5] G. J. Fix and J. P. Roof, “Least squares finite-element solution of a fractional order two-point boundary value problem,” *Computers and Mathematics with Applications*, vol. 48, no. 7-8, pp. 1017–1033, 2004.

[6] V. J. Ervin and J. P. Roop, “Variational formulation for the stationary fractional advection dispersion equation,” *Numerical Methods for Partial Differential Equations*, vol. 22, no. 3, pp. 558–576, 2006.

[7] V. J. Ervin, N. Heuer, and J. P. Roop, “Numerical approximation of a time dependent, nonlinear, space-fractional diffusion equation,” *SIAM Journal on Numerical Analysis*, vol. 45, no. 2, pp. 572–591, 2007.

[8] F. H. Zeng, C. P. Li, F. W. Liu, and I. W. Turner, “The use of finite difference/element approaches for solving the time-fractional subdiffusion equation,” *SIAM Journal on Scientific Computing*, vol. 35, pp. A2976–A3000, 2013.

[9] Y. Zheng, C. Li, and Z. Zhao, “A note on the finite element method for the space-fractional advection diffusion equation,” *Computers and Mathematics with Applications*, vol. 59, no. 5, pp. 1718–1726, 2010.

[10] Y. Zheng, C. Li, and Z. Zhao, “A fully discrete discontinuous galerkin method for nonlinear fractional fokker-planck equation,” *Mathematical Problems in Engineering*, vol. 2010, Article ID 279038, 26 pages, 2010.

[11] C. Li, Z. Zhao, and Y. Chen, “Numerical approximation of nonlinear fractional differential equations with subdiffusion and superdiffusion,” *Computers and Mathematics with Applications*, vol. 62, no. 3, pp. 855–875, 2011.

[12] Z. G. Zhao and C. P. Li, “Fractional difference/finite element approximations for the time-space fractional telegraph equation,” *Applied Mathematics and Computation*, vol. 219, pp. 2975–2988, 2012.

[13] Z. G. Zhao and C. P. Li, “A numerical approach to the generalized nonlinear fractional Fokker-Planck equation,” *Computers and Mathematics with Applications*, vol. 64, pp. 3075–3089, 2012.

[14] C. P. Li and F. H. Zeng, “Finite element methods for fractional differential equation,” in *Recent Advances in Applied Nonlinear Dynamics with Numerical Analysis*, C. P. Li, Y. J. Wu, and R. S. Ye, Eds., pp. 49–68, World Scientific, Singapore, 2013.

[15] W. Wyss, “The fractional diffusion equation,” *Journal of Mathematical Physics*, vol. 27, no. 11, pp. 2782–2785, 1986.

[16] M. F. Shlesinger, B. J. West, and J. Klafter, “Lévy dynamics of enhanced diffusion: application to turbulence,” *Physical Review Letters*, vol. 58, no. 11, pp. 1100–1103, 1987.

[17] T. H. Solomon, E. R. Weeks, and H. L. Swinney, “Observation of anomalous diffusion and Lévy flights in a 2-dimensional rotating flow,” *Physical Review Letters*, vol. 71, no. 24, pp. 3975–3978, 1993.

[18] D. A. Benson, S. W. Wheatcraft, and M. M. Meerschaert, “The fractional-order governing equation of Levy motion,” *Water Resources Research*, vol. 36, no. 6, pp. 1413–1423, 2000.

[19] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, 1999.

[20] S. C. Samko, A. A. Kilbas, and O. I. Maxitchev, *Integrals and Derivatives of the Fractional Order and Some of Their Applications*, Nauka i Tekhnika, Minsk, Belarus, 1987, (Russian).

[21] N. Heymans and I. Podlubny, “Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives,” *Rheologica Acta*, vol. 45, no. 5, pp. 765–771, 2006.