GROWTH CONDITIONS FOR CONFORMAL TRANSFORMATIONS PRESERVING RIEMANNIAN COMPLETENESS

A. DIRMEIER

Abstract. For a complete Riemannian metric, a pointwise conformal transformation may lead to a complete or incomplete transformed Riemannian metric, depending on the behavior of the conformal factor. We establish conditions on the growth of the conformal factor towards the infinity of the Riemannian metric, such that the conformally transformed Riemannian metric remains complete.

In 1961 Nomizu and Ozeki [4] established the result that every manifold $M$, satisfying the second axiom of countability, admits complete and incomplete Riemannian metrics. Moreover these are connected by conformal transformations. Hence for every incomplete Riemannian metric we can find a conformal factor to make it complete and vice versa. Subsequently it has been established [2] and [3] that the complete and the incomplete Riemannian metrics are dense in the space of Riemannian metrics over the manifold $M$. Based on [2] it possible to establish a partial order for Riemannian metrics $g, h$ on $M$ by $g \leq h \iff g(x)[v, v] \leq h(x)[v, v]$ for all $x \in M$ and $v \in T_x M$. Now if $g$ is complete and $g \leq h$ then $h$ is complete. As usual we will call a Riemannian metric $g$ on $M$ or the Riemannian manifold $(M, g)$ complete if $M$ is complete with respect to the distance function $d_g(\cdot, \cdot)$ associated to $g$. The aim of the present paper is to establish conditions on a conformal factor $A: M \to (0, \infty)$, which transforms a given complete Riemannian metric $g$ by $g' = A^2 g$, such that $g'$ is still complete. Following [4] a conformal factor, which makes $g'$ incomplete is easy to find. Take for example exponential growth of $A$ towards $g$-infinity, then $g'$ is incomplete (if $M$ is non-compact). But the exact upper bound on the growth of $A$, where $g'$ changes from complete to incomplete has hitherto not been established. This is the content of the Theorem below.

Remark. A function $f: M \to (0, \infty)$ on a complete Riemannian manifold $(M, g)$ is said to grow at most linearly towards infinity if for all fixed $x_0 \in M$ and all fixed compact sets $K$ with $x_0 \in K \subset M$, there are constants $c_1, c_2 > 0$ such that $f(x) \leq c_1 d_g(x, x_0) + c_2$ holds for all $x \in M \setminus K$. Subsequently we call the function $f$ to grow superlinearly towards infinity if there is a $x_0 \in M$, a compact set $K$ with $x_0 \in K \subset M$ and an $\epsilon > 0$ such that $f(x) \geq c_1 (d_g(x, x_0))^{1+\epsilon} + c_2$ for all $x \in M \setminus K$.

Please note that if a function $f$ grows at most linearly towards infinity, then the constant $c_1$ can be chosen independent of the choice of the fixed point $x_0$, whereas
c_2 can depend on the choice of x_0. This can be deduced directly from the triangle inequality for d_g. Let x_1 ∈ M be another fixed point, then
\[ f(x) \leq c_1 d_g(x_0, x) + c_2 \leq c_1 d_g(x_1, x) + c_1 d_g(x_0, x_1) + c_2 =: c_1 d_g(x_1, x) + c'_2. \]
The constant c_2 may also depend on the choice of the compact set K. We can set c_2 := \max_{x \in K} f(x). Then we have that c_2 ≥ f(x_0) for any choice of a compact set K containing x_0 and c_1 is even independent of the choice of K.

The following theorem was established in [1], where it was applied to Lorentzian geometry of stationary spacetimes. Nevertheless, it is an independent result in Riemannian geometry.

**Theorem 1.** Let (M, g) be a complete Riemannian manifold and A(x) a function A: M → (0, ∞). The conformal metric g' = \frac{\sqrt{A}}{\sqrt{g}} is complete if and only if A(x) grows at most linearly towards g-infinity.

We will need the following Lemma, which is a consequence of the theorems established in [3].

**Lemma 1.** Let (M, g) be a Riemannian manifold. Assume the Riemannian metric g to be bounded, i.e. there is an r > 0, such that d_g(x, y) < r for all x, y ∈ M. If g is additionally complete, then M is compact.

In the Theorem we can assume M to be non-compact, because otherwise any Riemannian metric on M would be complete. Thus any bounded metric on M must be incomplete.

**Proof of Theorem:** “⇒”: We assume the function A(x) grows superlinearly towards g-infinity, i.e. there is an \( \epsilon > 0 \) such that \( A(x) \geq c_1(d_g(x_0, x))^{1+\epsilon} + c_2 \) for x outside a compact set containing x_0. As g is complete we can join x_0 and any point x ∈ M by a minimizing geodesic arc of length \( d_g(x_0, x) \) and this geodesic arc is extendible to infinite values of the curve parameter. Let \( \alpha \subset S \) be such a geodesic with \( \alpha(0) = x_0 \), \( \alpha(d_g(x_0, x)) = x \) and with velocity normalized by \( g[\dot{\alpha}, \dot{\alpha}] = 1 \).

Hence we have
\[ d_g(x_0, \alpha(s)) = \int_0^s \sqrt{g[\dot{\alpha}, \dot{\alpha}]} \, ds = \int_0^s d\sigma = s. \]
For the metric g' this yields
\[ d_{g'}(x_0, x) \leq \lim_{s \to \infty} \int_0^s \frac{1}{A(\alpha(\sigma))} \, d\sigma \leq \int_0^\infty \frac{1}{c_1 s^{1+\epsilon} + c_2} \, ds = \left( \frac{1}{c_1 c'_2} \right)^{-\frac{1}{\epsilon}} \Gamma\left(\frac{\epsilon}{1+\epsilon}\right) \Gamma\left(1 + \frac{\epsilon}{1+\epsilon}\right) =: \frac{r}{2} < \infty, \]
which is independent of x ∈ M. Thus we have for all x, y ∈ M
\[ d_{g'}(x, y) \leq d_{g'}(x_0, x) + d_{g'}(x_0, y) \leq r. \]
Hence the metric g' is bounded and, therefore, it has to be incomplete by the Lemma. Thus we have that for complete g', the function A(x) must obey for all \( \epsilon > 0 \)
\[ A(x) < c_1(d_g(x_0, x))^{1+\epsilon} + c_2, \]
and hence \( A(x) \leq c_1 d_g(x_0, x) + c_2 \).

\( \Leftarrow \): Under the assumption that the linear growth condition holds, we now show that \( g' \) is complete. This follows from the fact, that any relatively compact ball with constant radius, say \( \frac{1}{2c_1} \), with respect to the metric \( g' \)

\[
B'(x_0, \frac{1}{2c_1}) = \{ x \in M \mid d_{g'}(x_0, x) < \frac{1}{2c_1} \},
\]

is contained in a relatively compact ball of the metric \( g \) with radius \( R \)

\[
B(x_0, R) = \{ x \in M \mid d_g(x_0, x) < R \},
\]

i.e. \( B'(x_0, \frac{1}{2c_1}) \subset B(x_0, R) \) for all fixed \( x_0 \in M \). We will show that this is the case if \( R \geq \frac{1}{c_1} \). Let \( \gamma: [0, 1] \rightarrow M \) with \( \gamma(0) = x_0 \) and \( \gamma(1) = x \) be a curve in \( M \). Assume further that \( d_g(x_0, x) \geq \frac{c_2}{c_1} \), then the length \( L \) of \( \gamma \) with respect to the metric \( g \) is given by

\[
L = \int_0^1 \sqrt{g(\gamma(s))[\gamma', \gamma']} \, ds \geq \frac{c_2}{c_1}.
\]

Now we evaluate the length \( L' \) of the same curve in the metric \( g' \)

\[
L' = \int_0^1 \sqrt{\frac{g(\gamma(s))[\gamma', \gamma']} {A(\gamma(s))}} \, ds.
\]

If now \( L' \geq \frac{1}{2c_1} \), then \( B'(x_0, \frac{1}{2c_1}) \subset B(x_0, \frac{c_2}{c_1}) \). By a mean value theorem there is a \( \sigma \in [0, 1] \) such that

\[
L' = \frac{L}{A(\gamma(\sigma))}.
\]

Now by the linear growth condition of \( A \) and because we have \( d_g(x_0, \gamma(\sigma)) \leq L \) we get

\[
L' \geq \frac{L}{c_1 d_g(x_0, \gamma(\sigma)) + c_2} \geq \frac{L}{c_1 L + c_2} \geq \frac{1}{2c_1}.
\]

\( \square \)

As a consequence of the Theorem we can now establish a condition for the completeness of a Riemannian metric \( g = h - s \), emerging from a complete Riemannian metric \( h \) and a non-negative, symmetric \((0, 2)\)-tensor field \( s \) (i.e. \( s(x)[v, v] \geq 0 \) for all \( x \in M \) and all \( v \in T_x M \)) on a manifold \( M \). Obviously \( g \) is a non-degenerate Riemannian metric if \( \frac{s(x)[v, v]} {h(x)[v, v]} < 1 \) for all \( x \in M \) and all \( v \in T_x M \setminus 0 \). We can now define an \( h \)-norm for \((0, 2)\)-tensor fields on \( M \). For the tensor field \( s \) this norm is given at some point \( x \in M \) by

\[
\| s \|^h_{x} = \sup_{v \in T_x M \setminus 0} \frac{\sqrt{s(x)[v, v]}} {\sqrt[2]{h(x)[v, v]}}.
\]

Then \( g \) is complete if \( \sup_{x \in M} \| s \|^h_x < 1 \), because in this case \( h \leq g \). But if \( \sup_{x \in M} \| s \|^h_x = 1 \), the metric \( g \) can be complete anyway if \( \| s \|^h_x \) obeys a certain growth condition, which can be deduced from the Theorem above.

**Corollary 1.** Let \( (M, h) \) be a complete Riemannian manifold and \( s \) a non-negative, symmetric \((0, 2)\)-tensor field \( M \). Let the symmetric \((0, 2)\)-tensor field \( g \), given by \( g = h - s \), be a Riemannian metric for all \( x \in M \). Assume \( \sup_{x \in M} \| s \|^h_x = 1 \), then
$g$ is complete if for all fixed $x_0 \in M$ and all fixed compact sets $K$ with $x_0 \in K \subset M$, there are constants $c_1, c_2 > 0$ such that,

$$\left(\|s\|_h^b\right)^2 \leq 1 - \frac{1}{(c_1d_h(x, x_0) + c_2)^2}$$

for all $x \in M \setminus K$.

**Proof:** For the Riemannian metric $g$ we compute

$$g = h(1 - \frac{s}{h}) \geq h(1 - (\|s\|_h^b)^2) = \frac{h}{1 - (\|s\|_h^b)^2} =: h'.$$

So $g$ is complete if $h'$ is complete and by the Theorem for complete $h$, the metric $h'$ is complete if

$$\sqrt{\frac{1}{1 - (\|s\|_h^b)^2}} \leq c_1d_h(x, x_0) + c_2.$$

And this is obviously the case if $(\|s\|_h^b)^2 \leq 1 - \frac{1}{(c_1d_h(x, x_0) + c_2)^2}$ holds. \(\square\)

**Remark.** A special case of the Corollary is on hand if the tensor field $s$ is given by $s = \beta \otimes \beta$, with $\beta$ being a one-form on the manifold $M$. In this case the tensor norm $\| \cdot \|$ can be replaced by the usual norm for one-forms given by

$$\|\beta\|_v^b = \sup_{v \in T_{M\setminus0}} \sqrt{\frac{\beta^2(x)[v]}{h(x)[v,v]}}.$$ 

**Example 1.** For an instructive example we can look at the flat metric

$$\delta = dr^2 + r^2d\Omega^2$$

on the pointed euclidian space $\mathbb{R}^3 \setminus \{0\}$, with $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ being the usual metric on the two-sphere $S^2$. This metric is incomplete because for any fixed angle $\Omega_0$ we can find a radial line segment $x(t) = (-t, \Omega_0)$ with $t \in [-1, 0)$, approaching the removed origin from the unit sphere. For the flat metric $\delta$ this is a geodesic arc, not extendible to $t = 0$. So obviously with $\dot{x} = (-1, 0)$ we have

$$d_\delta((1, \Omega_0), (0 + \epsilon, \Omega_0)) = \int_{-1}^{0-\epsilon} \sqrt{\delta[x, x]} dt = \int_{-1}^{0-\epsilon} dt = 1 - \epsilon < \infty.$$

If we now consider the conformally transformed metric

$$\tilde{\delta} = \frac{\delta}{r^2} = \frac{dr^2}{r^2} + d\Omega^2$$

we observe that this metric is complete on $\mathbb{R}^3 \setminus \{0\}$. This is obvious by choosing a new radial coordinate $\rho = \ln r$. For $r \in (0, \infty)$ and $r = 1$, we now have $\rho \in (-\infty, \infty)$ and $\rho = 0$. Thus the curve $y(t) = (-t, \Omega_0)$ with $t \in [0, \infty)$ and fixed angle $\Omega_0$ in the new coordinates is a geodesic arc for $\tilde{\delta}$. This geodesic approaches negative $\rho$-infinity—which corresponds to $r = 0$ in the old coordinates—for the curve parameter $t \to \infty$. Thus one could say that the conformal transformation moved the origin to infinite distance. Clearly we have for $\dot{y} = (-1, 0)$

$$d_\tilde{\delta}((0, \Omega_0), (\infty, \Omega_0)) = \int_{-4}^{\infty} \sqrt{\delta[y, y]} dt = \int_{-4}^{\infty} dt = \infty.$$
so that $\tilde{\delta}$ is complete. Moreover one observes from this, that $(\mathbb{R}^3 \setminus \{0\}, \delta)$ is conformally equivalent to $(\mathbb{R} \times S^2, \tilde{\delta})$. If we would now like to impose a conformal transformation

$$\tilde{\delta} = \frac{\tilde{\delta}}{A(\rho)}$$

depending on the radial coordinate $\rho$ (resp. $r$) only, we have for the radial distance with fixed angle $\Omega_0$

$$d_\delta(\rho_0, \rho) = |\rho - \rho_0|.$$ 

So setting $\rho_0 = 0$ (resp. $r = 1$) we get from the Theorem that

$$A(\rho) \leq c_1|\rho| + c_2$$

must hold in order to keep $\tilde{\delta}$ complete. In the coordinate $r$ this means that a conformal factor $A(r)$ imposed on $\tilde{\delta}$ may at most grow by

$$A(r) \leq c_1|\ln r| + c_2$$

towards the removed origin $r = 0$ in order to keep the metric

$$\tilde{\delta}' = \frac{d\rho^2 + d\Omega^2}{c_1|\ln r| + c_2}$$

complete. As an example for the Corollary we consider the metric

$$h = \tilde{\delta} - \beta^2 = \left(1 - \frac{\rho^2}{\rho^2 + 1}\right) d\rho^2 + d\Omega^2$$

on $\mathbb{R} \times S^2$, with $\beta = \frac{\rho}{\sqrt{\rho^2 + 1}} d\rho$. By choosing $c_1 = c_2 = 1$ in the Corollary we get

$$(\|\beta\|_\rho^2)^2 = \frac{\rho^2}{\rho^2 + 1} \leq 1 - \frac{1}{(\rho + 1)^2} = \frac{\rho^2 + 2|\rho|}{\rho^2 + 2|\rho| + 1}$$

which holds true because $d_\delta(0, \rho) = |\rho|$ and $0 \leq 2|\rho|$, thus $(\mathbb{R} \times S^2, h)$ is complete.

REFERENCES

[1] A. Dirrmeier, M. Plaue, M. Scherfner, Growth Conditions, Riemannian Completeness and Lorentzian Causality, J. Geom. Phys., 62(3) (2012), 604-612.

[2] H. D. Fegan, R. S. Millman, Quadrants of Riemannian Metrics, Michigan Math. J., 25 (1978), 3–7.

[3] J. A. Morrow, The denseness of complete Riemannian metrics, J. Diff. Geom., 4(2) (1970), 225–226.

[4] K. Nomizu, H. Ozeki, The existence of complete Riemannian metrics, Proc. Amer. Math. Soc., 12 (1961), 889–891.

Department of Mathematics, Technische Universität Berlin, Str. d. 17. Juni 136, 10623 Berlin, Germany