An example of anti-dynamo conformal Arnold metric

by

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Abstract

A 3D metric conformally related to Arnold cat fast dynamo metric: $ds_A^2 = e^{-\lambda z}dp^2 + e^{\lambda z}dq^2 + dz^2$ is shown to present a behaviour of non-dynamos where the magnetic field exponentially decay in time. The Riemann-Christoffel connection and Riemann curvature tensor for the Arnold and its conformal counterpart are computed. The curvature decay as z-coordinates increases without bounds. Some of the Riemann curvature components such as $R_{pzpz}$ also undergoes dissipation while component $R_{qzqz}$ increases without bounds. The remaining curvature component $R_{pqpq}$ is constant on the torus surface. The Riemann curvature invariant $K^2 = R_{ijkl}R^{ijkl}$ is found to be 0.155 for the $\lambda = 0.75$. A simple solution of Killing equations for Arnold metric yields a stretch Killing vector along one direction and compressed along other direction in order that the modulus of the Killing vector is not constant along the flow. The flow is shown to be untwisted. The stability of the two metrics are found by examining the sign of their curvature tensor components. PACS numbers: 02.40.Hw-Riemannian geometries
I Introduction

Geometrical tools have been used with success [1] in Einstein general relativity (GR) have been also used in other important areas of physics, such as plasma structures in tokamaks as been clear in the Mikhailovskii [2] book to investigate the tearing and other sort of instabilities in confined plasmas [2], where the Riemann metric tensor plays a dynamical role interacting with the magnetic field through the magnetohydrodynamical equations (MHD). Recently Garcia de Andrade [3] has also made use of Riemann metric to investigate magnetic flux tubes in superconducting plasmas. Thiffault and Boozer [4] following the same reasoning applied the methods of Riemann geometry in the context of chaotic flows and fast dynamos. In this paper we use the other tools of Riemannian geometry, also user in GR, such as Killing symmetries , and Ricci collineations , to obtain Killing symmetries in the cat dynamo metric [5]. We also use the Euler equations for incompressible flows in Arnold metric [6]. Antidynamos or non-dynamos are also important in the respect that it is important to recognize when a topology or geometry of a magnetic field does force the field to decay exponentially for example. As we know planar dynamos does not exist and Anti-dynamos theorems are important in this respect. Thus in the present paper we also obtain antidynamos metrics which are conformally related to the fast dynamo metric discovered by Arnold. Levi-Civita connections [7] are found together Riemann curvature from the MAPLE X GR tensor package. The paper is organized as follows: In section 2 the curvature and connection are found and the Euler equation is found. In section 3 the Killing symmetries are considered. In section 4 the conformal anti-dynamo metric is presented with the new feature that the magnetic field decays exponentially in time along the longitudinal flux tube flow. Conclusions are presented in section 5.

II Riemann dynamos and dissipative manifolds and Euler flows

Arnold metric can be used to compute the Levi-Civita-Christoffel connection

$$\Gamma^p_{pz} = \frac{\lambda}{2}$$

(II.1)
\[ \Gamma^q_{qz} = \frac{\lambda}{2} \] (II.2)
\[ \Gamma^z_{pp} = \frac{\lambda}{2} e^{-\lambda z} \] (II.3)
\[ \Gamma^z_{qq} = -\frac{\lambda}{2} e^{-\lambda z} \] (II.4)

From these connection components one obtains the Riemann tensor components

\[ R_{pppq} = -\frac{\lambda^2}{4} \] (II.5)

Note that since this component is negative from the Jacobi equation [7] that the flow is unstable. The other components are

\[ R_{pzpz} = -\frac{\lambda^2}{2} e^{-\lambda z} \] (II.6)
\[ R_{zqzq} = -\frac{\lambda^2}{2} e^{\lambda z} \] (II.7)

One may immediately notice that at large values of \( z \) the curvature component \((pzpz)\) is bounded and vanishes, or undergoes a dissipative effect, while component \((zqzq)\) of the curvature increases without bounds, component \((pqqq)\) remains constant. As in GR or general Riemannian manifolds, to investigate singular curvature behaviours we compute the so-called Kretschmann scalar \( K^2 \) defined in the abstract as

\[ K^2 = R_{ijkl} R^{ijkl} = [R_{pzpz} g^{qq} g^{zz}]^2 + [R_{pzpz} g^{pp} g^{zz}]^2 + [R_{zqzq} g^{qq} g^{zz}]^2 = \frac{3}{16} \lambda^4 \] (II.8)

with the value of 0.75 for \( \lambda \) one obtains \( K^2 = 0.155 \). Which would give a almost flat on singular manifold. In GR for example when this invariant is \( \infty \) the metric is singular. This would be a useful method to find singularities in dynamos. Let us now compute the forced Euler equation. The forced Euler equation in 3D manifold \( \mathcal{R}^3 \) is

\[ \langle \vec{v}, \nabla \rangle \vec{v} = \vec{F} \] (II.9)

where \( \vec{v} \) is the speed of the flow and \( \vec{F} \) is the external force to the flow. By expressing the flow velocity in 3D curvilinear coordinates basis \( \vec{e}_i \) \((i, j = p, q, z)\) we obtain

\[ (v^j < \vec{e}_i, \vec{e}_k > \partial_k)v^i \vec{e}_k = F^k \vec{e}_k \] (II.10)
Since the Krönecker delta is given by \( < \vec{e}_i, \vec{e}_j > = \delta_{ij} \) we may write the Euler equation in the form

\[ (v^k \partial_k) v^l \vec{e}_l = F^k \vec{e}_k \]  

(II.11)

Expanding the derivative on the LHS one obtains

\[ < \vec{v}, \nabla > \vec{v} = [v^k D_k v^l] \vec{e}_l = F^l \vec{e}_l \]  

(II.12)

where \( D \) is the covariant Riemannian derivative as defined in reference 1. By making use of the Gauss equation

\[ \partial_k \vec{e}_p = \Gamma_{klt} \vec{e}_t \]  

(II.13)

The covariant derivative can be expressed by

\[ D_k v^l = \partial_k v^l - \Gamma_{klt} v^l \]  

(II.14)

Thus the Euler force equation becomes

\[ v^k D_k v^l = F^l \]  

(II.15)

Computation of the \( p \)-component of the force leads to

\[ v^z \partial_z v^p = F^p \]  

(II.16)

In the next section we shall compute the Killing vector equation and yield a simple solution.

### III  Killing equations for fast dynamos

The Killing symmetries are defined by the Killing equations

\[ \mathcal{L}_\chi g = 0 \]  

(III.17)

where \( \chi \) represent the Killing vector and \( g \) represents the metric tensor. Explicitly this equation reads

\[ [\partial_q g_{ik}] \chi^l + g_{il} \partial_k \chi^i + g_{it} \partial_i \chi^l = 0 \]  

(III.18)

which explicitly reads

\[ -\lambda g_{pp} \chi^z + 2g_{pp} \partial_p \chi^p = 0 \]  

(III.19)
\[ \lambda g_{qq} \chi^z + 2 g_{qq} \partial_q \chi^q = 0 \quad (\text{III.20}) \]
\[ e^{-\lambda z} \partial_z \chi^p + \partial_p \chi^z = 0 \quad (\text{III.21}) \]
\[ e^{-\lambda z} \partial_q \chi^p + e^{\lambda z} \partial_p \chi^q = 0 \quad (\text{III.22}) \]

Note that a very simple solution for this system can be obtained if we put \( \chi^p = c_1 \), \( \chi^q = c_2 \), and \( \chi^z = 0 \), where \( c_1 \) and \( c_2 \) are constants. Since this Killing vector has to satisfy the modulus condition
\[ |\chi|^2 = g_{pp}[\chi^p]^2 + g_{qq}[\chi^q]^2 = [c_1]^2 e^{-\lambda z} + [c_2]^2 e^{\lambda z} \quad (\text{III.23}) \]
one immediately notices that the modulus of the Killing vector cannot be constant along the flow, and is stretch along the \( q \)-direction and compressed along the \( p \)-direction. In the next section we shall analyze a new solution of MHD dynamo equation which is conformally related to the Arnold fast dynamo metrics where stretch and compressible behaviors of the magnetic field appear as well.

## IV Conformal anti-dynamo metric

Conformal metric techniques have been widely used as a powerful tool obtain new solutions of the Einstein’s field equations of GR from known solutions. By analogy, here we are using this method to yield new solutions of MHD anti-dynamo solutions from the well-known fast dynamo Arnold solution. We shall demonstrate that distinct physical features from the Arnold solution maybe obtained. The conformal metric line element can be defined as
\[ ds^2 = \lambda^{-2z} ds_A^2 = dx_+^2 + \lambda^{-4z} dx_-^2 + \lambda^{-2z} dz^2 \quad (\text{IV.24}) \]
where we have used here the Childress and Gilbert [5] notation for the Arnold metric in \( \mathcal{R}^3 \) which reads now
\[ ds_A^2 = \lambda^{2z} dx_+^2 + \lambda^{-2z} dx_-^2 + dz^2 \quad (\text{IV.25}) \]
where the coordinates are defined by
\[ \bar{x} = x_+ e_+^z + x_- e_-^z + z e_z \quad (\text{IV.26}) \]
where a right handed orthogonal set of vectors in the metric is given by

\[ \vec{f}_+ = \vec{e}_+ \]  
(IV.27)

\[ \vec{f}_- = \lambda^{2z} \vec{e}_- \]  
(IV.28)

\[ \vec{f}_z = \lambda^z \vec{e}_z \]  
(IV.29)

A component of a vector in this basis, such as the magnetic vector \( \vec{B} \) is

\[ \vec{B} = B_+ \vec{f}_+ + B_- \vec{f}_- + B_z \vec{f}_z \]  
(IV.30)

The vector analysis formulas in this frame are

\[ \nabla = [\partial_+, \lambda^{2z} \partial_-, \lambda^z \partial_z] \]  
(IV.31)

\[ \nabla^2 \phi = [\partial_+^2 \phi, \lambda^{4z} \partial_- \phi, \lambda^{2z} \partial_z^2 \phi] \]  
(IV.32)

The MHD dynamo equations are

\[ \nabla \cdot \vec{B} = \partial_+ B_+ + \lambda^{2z} \partial_- B_- + \lambda^z \partial_z B_z = 0 \]  
(IV.33)

\[ \partial_t \vec{B} + (\vec{u} \cdot \nabla) \vec{B} - (\vec{B} \cdot \nabla) \vec{u} = \epsilon \nabla^2 \vec{B} \]  
(IV.34)

where \( \epsilon \) is the conductivity coefficient. Since here we are working on the limit \( \epsilon = 0 \), which is enough to understand the physical behavior of the fast dynamo, we do not need to worry to expand the RHS of equation (IV.34), and it reduces to

\[ (\vec{u} \cdot \nabla) \vec{B} = \partial_+ [B_+ \vec{e}_+ + B_- \epsilon^{2\mu} \vec{e}_- + B_z \epsilon^{\mu} \vec{e}_z] \]  
(IV.35)

where we have used that \( (\vec{B} \cdot \nabla) \vec{u} = B_z \mu \epsilon^{\mu} \vec{e}_z \) and that \( \mu = \log \lambda \). This is one of the main differences between Arnold metric and ours since in his fast dynamo, this relation vanishes since in Arnold metric \( \vec{u} = \vec{e}_z \) where \( \vec{e}_z \) is part of a constant basis. Separating the equation in terms of the coefficients of \( \vec{e}_+, \vec{e}_- \) and \( \vec{e}_z \) respectively one obtains the following scalar equations

\[ \partial_z B_+ + \partial_t B_+ = 0 \]  
(IV.36)

\[ \partial_t B_- + \partial_t B_+ 2\mu B_- = 0 \]  
(IV.37)

\[ \partial_t B_z + \partial_z B_+ = 0 \]  
(IV.38)
Solutions of these equations allows us to write down an expression for the magnetic vector field $\vec{B}$ as

$$\vec{B} = \left[B^0_z, \lambda^{-(t+z)}B^0_-, B^0_z\right](t - z, y, x + y) \quad (IV.39)$$

From this expression we can infer that the field is carried in the flow, stretched in the $\vec{f}_z$ direction and compressed in the $\vec{f}_-$ direction, while in Arnold’s cat fast dynamo is also compressed along the $\vec{f}_-$ direction but is stretched along $\vec{f}_+$ direction while here this direction is not affected. But the main point of this solution is the fact that the solution represents an anti-dynamo since as one can see from expression (IV.39) the magnetic field fastly decays exponentially in time as $e^{\mu(t+z)}$. Let us now compute the Riemann tensor components of the new conformal metric to check for the stability of the non-dynamo flow. To easily compute this curvature components we shall make use of Elie Cartan [8] calculus of differential forms, which allows us to express the conformal metric as

$$ds^2 = dp^2 + e^{4\lambda z} dq^2 + e^{\lambda z} dz^2 \quad (IV.40)$$

or in terms of the frame basis form $\omega^i$ is

$$ds^2 = (\omega^p)^2 + (\omega^q)^2 + (\omega^z)^2 \quad (IV.41)$$

where we are back to Arnold’s notation for convenience. The basis form are write as

$$\omega^p = dp \quad (IV.42)$$

$$\omega^q = e^{\lambda z} dq \quad (IV.43)$$

and

$$\omega^z = e^{\frac{\lambda}{2} z} dq \quad (IV.44)$$

By applying the exterior differentiation in this basis form one obtains

$$d\omega^p = 0 \quad (IV.45)$$

$$d\omega^z = 0 \quad (IV.46)$$

and

$$d\omega^q = \lambda e^{-\frac{\lambda}{2} z} \omega^z \wedge \omega^q \quad (IV.47)$$
Substitution of these expressions into the first Cartan structure equations one obtains

\[ T^p = 0 = \omega^p_q \land \omega^q + \omega^p_z \land \omega^z \] (IV.48)

\[ T^q = 0 = \lambda e^{\frac{\lambda}{2} z} \omega^z \land \omega^q + \omega^q_p \land \omega^p + \omega^q_z \land \omega^z \] (IV.49)

and

\[ T^z = 0 = \omega^z_p \land \omega^p + \omega^z_q \land \omega^q \] (IV.50)

where \( T^i \) are the Cartan torsion 2-form which vanishes identically on a Riemannian manifold. From these expressions one is able to compute the connection forms which yields

\[ \omega^p_q = -\alpha \omega^p \] (IV.51)

\[ \omega^q_z = \lambda e^{\frac{\lambda}{2} z} \omega^q \] (IV.52)

and

\[ \omega^z_p = \beta \omega^p \] (IV.53)

where \( \alpha \) and \( \beta \) are constants. Substitution of these connection form into the second Cartan equation

\[ R^i_j = R^i_{jkl} \omega^k \land \omega^l = d\omega^i_j + \omega^i_l \land \omega^l_j \] (IV.54)

where \( R^i_j \) is the Riemann curvature 2-form. After some algebra we obtain the following components of Riemann curvature for the conformal antidynamo

\[ R^p_{qpq} = \lambda e^{-\frac{\lambda}{2} z} \] (IV.55)

\[ R^q_{zqz} = \frac{1}{2} \lambda^2 e^{-\lambda z} \] (IV.56)

and finally

\[ R^p_{zpq} = -\alpha \lambda e^{-\frac{\lambda}{2} z} \] (IV.57)

We note that only component to which we can say is positive is \( R^p_{zqz} \) which turns the flow stable in this q-z surface. This component also dissipates away when \( z \) increases without bounds, the same happens with the other curvature components [8].
V Conclusions

In conclusion, we have used a well-known technique to find solutions of Einstein’s field equations of gravity namely the conformal related spacetime metrics to find a new anti-dynamo solution in MHD nonplanar flows. The stability of the flow is also analysed by using other tools from GR, namely that of Killing symmetries. Examination of the Riemann curvature components enable one to analyse the stretch and compression of the dynamo flow. The Killing symmetries can be used in near future to classify the dynamo metrics in the same way they were useful in classifying general relativistic solutions of Einstein’s gravitational equations in four-dimensional spacetime [1].

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