Group-Theoretical Classification of BPS and Possibly Protected States in D=4 Conformal Supersymmetry

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Abstract

We give explicitly the reduction of supersymmetries of the positive energy unitary irreducible representations of the N-extended D=4 conformal superalgebras su(2,2/N). Further we give the classification of BPS and possibly protected states.

1Permanent address.
1 Introduction

Recently, superconformal field theories in various dimensions are attracting more interest, especially in view of their applications in string theory. From these very important is the AdS/CFT correspondence, namely, the remarkable proposal of Maldacena [1], according to which the large $N$ limit of a conformally invariant theory in $d$ dimensions is governed by supergravity (and string theory) on $d+1$-dimensional $AdS$ space (often called $AdS_{d+1}$) times a compact manifold. Actually the possible relation of field theory on $AdS_{d+1}$ to field theory on $\mathcal{M}_d$ has been a subject of long interest, cf., e.g., [2–4], and also [5] for discussions motivated by recent developments. The proposal of [1] was elaborated in [6] and [7] where was proposed a precise correspondence between conformal field theory observables and those of supergravity. More recently, there were developments of integrability in the context of the AdS/CFT correspondence, in which superconformal field theories, especially in 4D, were also playing important role. For this we refer to the reviews [8,9], and for earlier relevant papers also in the general context of AdS/CFT and superconformal symmetry, we refer to [10–25] and references therein.

In all cases, it was known for a long time that the classification of the UIRs of the conformal superalgebras is of great importance. For some time such classification was known only for the $D = 4$ superconformal algebras $su(2,2/1)$ [26] and $su(2,2/N)$ for arbitrary $N$ [27], (see also [28,29]). Then, more progress was made with the classification for $D = 3$ (for even $N$), $D = 5$, and $D = 6$ (for $N = 1,2$) in [30] (some results being conjectural), then for the $D = 6$ case (for arbitrary $N$) was finalized in [31]. Finally, the cases $D = 9,10,11$ were treated by finding the UIRs of $osp(1/2n)$, [32].

After the list of UIRs is found the next problem to address is to find their characters since these give the spectrum which is important for the applications. This problem is solved in principle, though not all formulae are explicit, for the UIRs of $D = 4$ conformal superalgebras $su(2,2/N)$ in [33]. The mathematical point of view this question is clear only for representations with conformal dimension above the unitarity threshold viewed as irreps of the corresponding complex superalgebra $sl(4/N)$ [35–41]. But for $su(2,2/N)$ even the UIRs above the unitarity threshold are truncated for small values of spin and isospin. Furthermore, in the applications the

\footnote{For another, more practical though not so rigorous, approach, see [34].}
most important role is played by the representations with “quantized” conformal dimensions at the unitarity threshold and at discrete points below. In the quantum field or string theory framework some of these correspond to operators with “protected” scaling dimension and therefore imply “non-renormalization theorems” at the quantum level, cf., e.g., [42,43]. Especially important in this context are the so-called BPS states, cf., [43–51].

These investigations require deeper knowledge of the structure of the UIRs, in particular, more explicit results on the decompositions of long superfields as they descend to the unitarity threshold. Fortunately, most of the needed information is contained in [27–29,33,52], see also [53–58].

The paper is organized as follows. In Section 2 we give the preliminaries. In Section 3 we give explicitly the reduction of supersymmetries. In Section 4 we give the classification of BPS and possibly protected states.
2 Preliminaries

2.1 Representations of D=4 conformal supersymmetry

The conformal superalgebras in $D = 4$ are $\mathcal{G} = su(2,2/N)$. The even subalgebra of $\mathcal{G}$ is the algebra $\mathcal{G}_0 = su(2,2) \oplus u(1) \oplus su(N)$. We label their physically relevant representations of $\mathcal{G}$ by the signature:

$$\chi = [d; j_1, j_2; z; r_1, \ldots, r_{N-1}] (2.1)$$

where $d$ is the conformal weight, $j_1, j_2$ are non-negative (half-)integers which are Dynkin labels of the finite-dimensional irreps of the $D = 4$ Lorentz subalgebra $so(3,1)$ of dimension $(2j_1 + 1)(2j_2 + 1)$, $z$ represents the $u(1)$ subalgebra which is central for $\mathcal{G}_0$ (and is central for $\mathcal{G}$ itself when $N = 4$), and $r_1, \ldots, r_{N-1}$ are non-negative integers which are Dynkin labels of the finite-dimensional irreps of the internal (or $R$) symmetry algebra $su(N)$.

We recall the root system of the complexification $\mathcal{G}^c$ of $\mathcal{G}$ (as used in [29]). The positive root system $\Delta^+$ is comprised of $\alpha_{ij}$, $1 \leq i < j \leq 4 + N$. The even positive root system $\Delta^+_0$ is comprised of $\alpha_{ij}$, with $i, j \leq 4$ and $i, j \geq 5$; the odd positive root system $\Delta^+_1$ is comprised of $\alpha_{ij}$, with $i \leq 4, j \geq 5$. The generators corresponding to the latter (odd) roots will be denoted as $X^+_{i,4+k}$, where $i = 1, 2, 3, 4$, $k = 1, \ldots, N$. The simple roots are chosen as in (2.4) of [29]:

$$\gamma_1 = \alpha_{12}, \gamma_2 = \alpha_{34}, \gamma_3 = \alpha_{25}, \gamma_4 = \alpha_{4,4+N}, \gamma_k = \alpha_{k,k+1}, 5 \leq k \leq 3 + N. (2.2)$$

Thus, the Dynkin diagram is:

![Dynkin diagram](image)

(2.3)

This is a non-distinguished simple root system with two odd simple roots [60].

Sometimes we shall use another way of writing the signature related to the above enumeration of simple roots, cf. [29] and (1.16) of [33]:

$$\chi = (2j_1; (\Lambda, \gamma_3); r_1, \ldots, r_{N-1}; (\Lambda, \gamma_4); 2j_2), (2.4)$$
(where \(\Lambda, \gamma_3\), \(\Lambda, \gamma_4\) are definite linear combinations of all quantum numbers), or even giving only the Lorentz and \(SU(N)\) signatures:

\[
\chi_N = \{ 2j_1; r_1, \ldots, r_{N-1}; 2j_2 \} .
\] (2.5)

**Remark:** We recall that the group-theoretical approach to \(D = 4\) conformal supersymmetry developed in [27–29] involves two related constructions - on function spaces and as Verma modules. The first realization employs the explicit construction of induced representations of \(G\) (and of the corresponding supergroup \(G = SU(2, 2/N)\)) in spaces of functions (superfields) over superspace which are called elementary representations (ER). The UIRs of \(G\) are realized as irreducible components of ERs, and then they coincide with the usually used superfields in indexless notation. The Verma module realization is also very useful as it provides simpler and more intuitive picture for the relation between reducible ERs, for the construction of the irreps, in particular, of the UIRs. For the latter the main tool is an adaptation of the Shapovalov form [59] to the Verma modules [27,52]. Here we shall need only the second - Verma module - construction.

We use lowest weight Verma modules \(V^\Lambda\) over \(G^\varrho\), where the lowest weight \(\Lambda\) is characterized by its values on the Cartan subalgebra \(\mathcal{H}\) and is in 1-to-1 correspondence with the signature \(\chi\). If a Verma module \(V^\Lambda\) is irreducible then it gives the lowest weight irrep \(L_\Lambda\) with the same weight. If a Verma module \(V^\Lambda\) is reducible then it contains a maximal invariant submodule \(I^\Lambda\) and the lowest weight irrep \(L_\Lambda\) with the same weight is given by factorization: \(L_\Lambda = V^\Lambda / I^\Lambda\) [61].

There are submodules which are generated by the singular vectors related to the even simple roots \(\gamma_1, \gamma_2, \gamma_5, \ldots, \gamma_{N+3}\) [29]. These generate an even invariant submodule \(I^\Lambda_c\) present in all Verma modules that we consider and which must be factored out. Thus, instead of \(V^\Lambda\) we shall consider the factor-modules:

\[
\tilde{V}^\Lambda = V^\Lambda / I^\Lambda_c
\] (2.6)

The Verma module reducibility conditions for the \(4N\) odd positive roots of \(G^\varrho\) were derived in [28,29] adapting the results of Kac [61]:

\[
d = d_{Nk}^1 - z\delta_{N4}
\] (2.7a)

\[
d_{Nk}^1 \equiv 4 - 2k + 2j_2 + z + 2m_k - 2m/N
\]
\[ d = d^2_{Nk} - z\delta_{N4} \quad (2.7b) \]
\[ d^2_{Nk} \equiv 2 - 2k - 2j_2 + z + 2m_k - 2m/N \]
\[ d = d^3_{Nk} + z\delta_{N4} \quad (2.7c) \]
\[ d^3_{Nk} \equiv 2 + 2k - 2N + 2j_1 - z - 2m_k + 2m/N \]
\[ d = d^4_{Nk} + z\delta_{N4} \quad (2.7d) \]
\[ d^4_{Nk} \equiv 2k - 2N - 2j_1 - z - 2m_k + 2m/N \]

where in all four cases of (2.7) \( k = 1, \ldots, N \), \( m_N \equiv 0 \), and

\[ m_k \equiv \sum_{i=k}^{N-1} r_i \quad m \equiv \sum_{k=1}^{N-1} m_k = \sum_{k=1}^{N-1} kr_k \quad (2.8) \]

Note that we shall use also the quantity \( m^* \) which is conjugate to \( m \):

\[ m^* \equiv \sum_{k=1}^{N-1} kr_{N-k} = \sum_{k=1}^{N-1} (N - k)r_k \quad (2.9) \]
\[ m + m^* = Nm_1 \quad (2.10) \]

We need the result of [27] (cf. part (i) of the Theorem there) that the following is the complete list of lowest weight (positive energy) UIRs of \( su(2, 2/N) \):

\[ d \geq d_{max} = \max(d^1_{N1}, d^3_{NN}) \quad (2.11a) \]
\[ d = d^4_{NN} \geq d^1_{N1} \quad j_1 = 0 \quad (2.11b) \]
\[ d = d^2_{N1} \geq d^3_{NN} \quad j_2 = 0 \quad (2.11c) \]
\[ d = d^2_{N1} = d^2_{NN} \quad j_1 = j_2 = 0 \quad (2.11d) \]

where \( d_{max} \) is the threshold of the continuous unitary spectrum. Note that in case (d) we have \( d = m_1, z = 2m/N - m_1 \), and that it is trivial for \( N = 1 \).

Next we note that if \( d > d_{max} \) the factorized Verma modules are irreducible and coincide with the UIRs \( L_\Lambda \). These UIRs are called \textbf{long} in the modern literature, cf., e.g., [43,46,53–57]. Analogously, we shall use for the cases when \( d = d_{max} \), i.e., (2.11a), the terminology of \textbf{semi-short} UIRs, introduced in [43,53], while the cases (2.11b,c,d) are also called \textbf{short} UIRs, cf., e.g., [43,46,54–58].
Next consider in more detail the UIRs at the four distinguished reducibility points determining the UIRs list above: \( d_{N_1}^1, d_{N_1}^2, d_{NN}^3, d_{NN}^4 \). The above reducibilities occur for the following odd roots, resp.:
\[
\alpha_{3,4+N} = \gamma_2 + \gamma_4, \quad \alpha_{4,4+N} = \gamma_4, \quad \alpha_{15} = \gamma_1 + \gamma_3, \quad \alpha_{25} = \gamma_3.
\]
We note a partial ordering of these four points:
\[
d_{N_1}^1 > d_{N_1}^2, \quad d_{NN}^3 > d_{NN}^4.
\]
Due to this ordering at most two of these four points may coincide.

First we consider the situations in which no two of the distinguished four points coincide. There are four such situations:

a : \( d = d_{max} = d_{N_1}^1 = d^s \equiv 2 + 2j_2 + z + 2m_1 - 2m/N > d_{NN}^3 \)\( (2.14a) \)

b : \( d = d_{N_1}^2 = d^6 \equiv z - 2j_2 + 2m_1 - 2m/N > d_{NN}^3, \quad j_2 = 0 \)\( (2.14b) \)

c : \( d = d_{max} = d_{NN}^3 = d^e \equiv 2 + 2j_1 - z + 2m/N > d_{N_1}^1 \)\( (2.14c) \)

d : \( d = d_{NN}^4 = d^d \equiv 2m/N - 2j_1 - z > d_{N_1}^1, \quad j_1 = 0 \)\( (2.14d) \)

where for future use we have introduced notations \( d^s, d^6, d^e, d^d \), the definitions including also the corresponding inequality.

We shall call these cases single-reducibility-condition (SRC) Verma modules or UIRs, depending on the context. In addition, as already stated, we use for the cases when \( d = d_{max} \), i.e., \( (2.14a,c) \), the terminology of semi-short UIRs, while the cases \( (2.14b,d) \), are also called short UIRs.

The factorized Verma modules \( \tilde{V}^\Lambda \) with the unitary signatures from \( (2.14) \) have only one invariant odd submodule which has to be factorized in order to obtain the UIRs. These odd embeddings and factorizations are given as follows:
\[
\tilde{V}^\Lambda \rightarrow \tilde{V}^{\Lambda+\beta}, \quad L_\Lambda = \tilde{V}^{\Lambda}/I^\beta,
\]
where we use the convention \([28]\) that arrows point to the oddly embedded module, and we give only the cases for \( \beta \) that we shall use later:

\[
\beta = \alpha_{3,4+N}, \quad \text{for } (2.14a), \quad j_2 > 0, \quad \text{(2.16a)}
\]
\[
\beta = \alpha_{3,4+N} + \alpha_{4,4+N}, \quad \text{for } (2.14a), \quad j_2 = 0, \quad \text{(2.16b)}
\]
\[
\beta = \alpha_{15}, \quad \text{for } (2.14c), \quad j_1 > 0, \quad \text{(2.16c)}
\]
\[
\beta = \alpha_{15} + \alpha_{25}, \quad \text{for } (2.14c), \quad j_1 = 0 \quad \text{(2.16d)}
\]
We consider now the four situations in which two distinguished points coincide:

\[ \begin{align*}
\mathbf{ac} : & \quad d = d_{\text{max}} = d^{\text{ac}} = 2 + j_1 + j_2 + m_1 = d^3_{N1} = d^3_{NN} \quad (2.17a) \\
\mathbf{ad} : & \quad d = d^{\text{ad}} = 1 + j_2 + m_1 = d^1_{N1} = d^4_{NN}, \quad j_1 = 0 \quad (2.17b) \\
\mathbf{bc} : & \quad d = d^{\text{bc}} = 1 + j_1 + m_1 = d^2_{N1} = d^3_{NN}, \quad j_2 = 0 \quad (2.17c) \\
\mathbf{bd} : & \quad d = d^{\text{bd}} = m_1 = d^2_{N1} = d^4_{NN}, \quad j_1 = j_2 = 0 \quad (2.17d)
\end{align*} \]

We shall call these **double-reducibility-condition (DRC)** Verma modules or UIRs. The cases in (2.17a) are semi-short UIR, while the other cases are short.

The odd embedding diagrams and factorizations for the DRC modules are [28]:

\[ \begin{align*}
\tilde{V}^\Lambda + \beta' & \rightarrow \tilde{V}^\Lambda + \beta + \beta' \\
\uparrow & \uparrow \\
\tilde{V}^\Lambda & \rightarrow \tilde{V}^\Lambda + \beta
\end{align*} \]

\[ L_\Lambda = \tilde{V}^\Lambda / I^{\beta,\beta'}, \quad I^{\beta,\beta'} = I^\beta \cup I^{\beta'} \]

and we give only the cases for \( \beta, \beta' \) to be used later:

\[ \begin{align*}
(\beta, \beta') & = (\alpha_{15}, \alpha_{3,4+N}), \quad \text{for (2.17a),} \quad j_1 j_2 > 0 \quad (2.19a) \\
& = (\alpha_{15}, \alpha_{3,4+N} + \alpha_{3,4+N}), \quad \text{for (2.17b),} \quad j_1 > 0, \quad j_2 = 0 \quad (2.19b) \\
& = (\alpha_{15} + \alpha_{25}, \alpha_{3,4+N}), \quad \text{for (2.17c),} \quad j_1 = 0, \quad j_2 > 0 \quad (2.19c) \\
& = (\alpha_{15} + \alpha_{25}, \alpha_{3,4+N} + \alpha_{3,4+N}), \quad \text{for (2.17d),} \quad j_1 = j_2 = 0 \quad (2.19d)
\end{align*} \]

## 2.2 Decompositions of long superfields

First we present the results on decompositions of long irreps as they descend to the unitarity threshold [33].

In the SRC cases we have established that for \( d = d_{\text{max}} \) there hold the two-term decompositions:

\[ \left( L_{\text{long}} \right)_{|d=d_{\text{max}}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda + \beta} , \quad r_1 + r_{N-1} > 0 , \quad (2.20) \]
where $\Lambda$ is a semi-short SRC designated as type $a$ (then $r_1 > 0$) or $c$ (then $r_{N-1} > 0$) and there are four possibilities for $\beta$ depending on the values of $j_1, j_2$ as given in (2.16). In cases (2.16a,c) also the second UIR on the RHS of (2.20) is semi-short, while in cases (2.16b,d) the second UIR on the RHS of (2.20) is short of type $b$, $d$, resp.

In the DRC cases we have established that for $N > 1$ and $d = d_{\text{max}} = d^{ac}$ hold the four-term decompositions:

$$\left(L_{\text{long}}\right)_{d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta} \oplus \hat{L}_{\Lambda+\beta'} \oplus \hat{L}_{\Lambda+\beta+\beta'}, \quad r_1 r_{N-1} > 0,$$

(2.21)

where $\Lambda$ is the semi-short DRC designated as type $ac$ and there are four possibilities for $\beta, \beta'$ depending on the values of $j_1, j_2$ as given in (2.19a,b,c,d). Note that in case (2.19a) all UIRs in the RHS of (2.21) are semi-short. In the case (2.19b) the first two UIRs in the RHS of (2.21) are semi-short, the last two UIRs are short of type $bc$. In the case (2.19c) the first two UIRs in the RHS of (2.21) are semi-short, the last two UIRs are short of type $ad$. In the case (2.19d) the first UIR in the RHS of (2.21) is semi-short, the other three UIRs are short of types $bc, ad, bd$, resp.

Next we note that for $N = 1$ all SRC cases enter some decomposition, while no DRC cases enter any decomposition. For $N > 1$ the situation is more diverse and so we give the list of UIRs that do not enter decompositions together with the restrictions on the $R$-symmetry quantum numbers:

- **SRC cases:**
  - $a$  \quad $d = d^a$, \quad $r_1 = 0$.
  - $b$  \quad $d = d^b$, \quad $r_1 \leq 2$.
  - $c$  \quad $d = d^c$, \quad $r_{N-1} = 0$.
  - $d$  \quad $d = d^d$, \quad $r_{N-1} \leq 2$.

- **DRC cases:**
  - all non-trivial cases for $N = 1$, while for $N > 1$ the list is:
    - $ac$  \quad $d = d^{ac}$, \quad $r_1 r_{N-1} = 0$.
    - $ad$  \quad $d = d^{ad}$, \quad $r_{N-1} \leq 2$, \quad $r_1 = 0$ for $N > 2$.
    - $bc$  \quad $d = d^{bc}$, \quad $r_1 \leq 2$, \quad $r_{N-1} = 0$ for $N > 2$.
    - $bd$  \quad $d = d^{bd}$, \quad $r_1, r_{N-1} \leq 2$ for $N > 2$, \quad $1 \leq r_1 \leq 4$ for $N = 2$. 

3 Reduction of supersymmetry in short and semi-short UIRs

Our first task in this paper is to present explicitly the reduction of the supersymmetries in the irreducible UIRs. This means to give explicitly the number $\kappa$ of odd generators which are eliminated from the corresponding lowest weight module, (or equivalently, the number of super-derivatives that annihilate the corresponding superfield).

3.1 R-symmetry scalars

We start with the simpler cases of $R$-symmetry scalars when $r_i = 0$ for all $i$, which means also that $m_1 = m = m^* = 0$. These cases are valid also for $N = 1$. More explicitly:

- $a$ \[ d = d_{m=0}^a = 2 + 2j_2 + z > 2 + 2j_1 - z, \quad j_1 \text{ arbitrary}, \]
  \[ \kappa = N + (1 - N)\delta_{j_2,0}, \quad \text{or casewise:} \]
  \[ \kappa = N, \quad \text{if} \quad j_2 > 0, \]
  \[ \kappa = 1, \quad \text{if} \quad j_2 = 0 \]

Here, $\kappa$ is the number of anti-chiral generators $X_{3,5+N-k}^+, \quad k = 1, \ldots, \kappa$, that are eliminated. Thus, in the cases when $\kappa = N$ the semi-short UIRs may be called semi-chiral since they lack half of the anti-chiral generators.

- $b$ \[ d = d_{m=0}^b = z > 2 + 2j_1 - z, \quad j_1 \text{ arbitrary}, \quad j_2 = 0, \]
  \[ \kappa = 2N \]

These short UIRs may be called chiral since they lack all anti-chiral generators $X_{3,4+k}^+, \quad X_{4,4+k}^+, \quad k = 1, \ldots, N$.

- $c$ \[ d = d_{m=0}^c = 2 + 2j_1 - z > 2 + 2j_2 + z, \quad j_2 \text{ arbitrary}, \]
  \[ \kappa = N + (1 - N)\delta_{j_1,0}, \quad \text{or casewise:} \]
  \[ \kappa = N, \quad j_1 > 0, \]
  \[ \kappa = 1, \quad j_1 = 0 \]
Here, $\kappa$ is the number of chiral generators $X_{1,4+k}^+, k = 1, \ldots, \kappa$, that are eliminated. Thus, in the cases when $\kappa = N$ the semi-short UIRs may be called semi–anti-chiral since they lack half of the chiral generators.

- $d = d^d = -z > 2 + 2j_2 + z, \quad j_2$ arbitrary, $j_1 = 0, \quad \kappa = 2N \quad (3.4)$

These short UIRs may be called anti-chiral since they lack all chiral generators $X_{1,4+k}^+, X_{2,4+k}^+, k = 1, \ldots, N$.

- $d = d^{ac} = 2 + j_1 + j_2, \quad z = j_1 - j_2, \quad \kappa = 2N + (1 - N)(\delta_{j_1,0} + \delta_{j_2,0}), \quad \text{or casewise:} \quad (3.5)$

Here, $\kappa$ is the number of mixed elimination: chiral generators $X_{1,4+k}^+$, $X_{2,4+k}^+$, $(k = 1, \ldots, \kappa)$, and anti-chiral generators $X_{3,5+N-k}^+$, $(k = 1, \ldots, N + (1 - N)\delta_{j_2,0})$. Thus, in the cases when $\kappa = 2N$ the semi-short UIRs may be called semi–chi ral–anti-chiral since they lack half of the chiral and half of the anti-chiral generators. (They may be called Grassmann-analytic following [43].)

- $d = d^{ad} = 1 + j_2 = -z, \quad j_1 = 0, \quad \kappa = 3N + (1 - N)\delta_{j_2,0}, \quad \text{or casewise:} \quad (3.6)$

Here, $\kappa$ is the number of mixed elimination: both types chiral generators $X_{1,4+k}^+$, $X_{2,4+k}^+$, $(k = 1, \ldots, N)$, and anti-chiral generators $X_{3,5+N-k}^+$, $(k = 1, \ldots, N + (1 - N)\delta_{j_2,0})$. Thus, in the cases when $\kappa = 3N$ the semi-short UIRs may be called semi–chiral and anti-chiral since they lack all the chiral and half of the anti-chiral generators.
\[ d = d^a = 2 + 2j_2 + z + 2m_1 - 2m/N > 2 + 2j_1 - z + 2m/N , \]
\[ j_1, j_2 \] arbitrary,
\[ \kappa = 1 + i_0(1 - \delta_{j_2,0}) \leq N - 1 . \]

3.2 R-symmetry non-scalars

Here we need some additional notation. Let \( N > 1 \) and let \( i_0 \) be an integer such that \( 0 \leq i_0 \leq N - 1 \), \( r_i = 0 \) for \( i \leq i_0 \), and if \( i_0 < N - 1 \) then \( r_{i_0+1} > 0 \). Let now \( i'_0 \) be an integer such that \( 0 \leq i'_0 \leq N - 1 \), \( r_{N-i} = 0 \) for \( i \leq i'_0 \), and if \( i'_0 < N - 1 \) then \( r_{N-1-i'_0} > 0 \).

With this notation the cases of \( R \)-symmetry scalars occur when \( i_0 + i'_0 = N - 1 \), thus, from now on we have the restriction:
\[ 0 \leq i_0 + i'_0 \leq N - 2 \]

Now we can make a list for the values of \( \kappa \), with the same interpretation as in the previous subsection, only the last case is added here.

\[ \bullet \text{ ad} \quad d = d^a = 2 + 2j_2 + z + 2m_1 - 2m/N > 2 + 2j_1 - z + 2m/N , \]
\[ j_1, j_2 \] arbitrary,
\[ \kappa = 1 + i_0(1 - \delta_{j_2,0}) \leq N - 1 . \]
Here are eliminated the anti-chiral generators \( X_{3,5+N-k}^+ \), \( k \leq \kappa \).

\[ \bullet \quad \text{b} \quad d = d^b = z + 2m_1 - 2m/N > 2 + 2j_1 - z + 2m/N , \]
\[ j_2 = 0 , \quad j_1 \text{ arbitrary}, \]
\[ \kappa = 2 + 2i_0 \leq 2N - 2 . \]  
(3.10)

Here are eliminated the anti-chiral generators \( X_{3,5+N-k}^+ \), \( X_{4,5+N-k}^+ \), \( k \leq 1 + i_0 \).

\[ \bullet \quad \text{c} \quad d = d^c = 2 + 2j_1 - z + 2m/N > 2 + 2j_2 + z + 2m_1 - 2m/N , \]
\[ j_1, j_2 \text{ arbitrary}, \]
\[ \kappa = 1 + i'_0(1 - \delta_{j_1,0}) \leq N - 1 . \]  
(3.11)

Here are eliminated the chiral generators \( X_{1,4+k}^+ \), \( k \leq \kappa \).

\[ \bullet \quad \text{d} \quad d = d^d = 2m/N - z > 2 + 2j_2 + z + 2m_1 - 2m/N , \]
\[ j_1 = 0 , \quad j_2 \text{ arbitrary}, \]
\[ \kappa = 2 + 2i'_0 \leq 2N - 2 . \]  
(3.12)

Here are eliminated the chiral generators \( X_{1,4+k}^+ \), \( X_{2,4+k}^+ \), \( k \leq 1 + i'_0 \).

\[ \bullet \quad \text{ac} \quad d = d^{ac} , \quad z = j_1 - j_2 + 2m/N - m_1 , \quad j_1, j_2 \text{ arbitrary}, \]
\[ \kappa = 2 + i_0(1 - \delta_{j_2,0}) + i'_0(1 - \delta_{j_1,0}) \leq N . \]  
(3.13)

Here are eliminated chiral generators \( X_{1,4+k}^+ \), \( k \leq 1 + i'_0(1 - \delta_{j_1,0}) \), and anti-chiral generators \( X_{3,5+N-k}^+ \), \( k \leq 1 + i_0(1 - \delta_{j_2,0}) \).

\[ \bullet \quad \text{ad} \quad d = d^{ad} , \quad j_1 = 0 , \quad z = 2m/N - m_1 - 1 - j_2 , \quad j_2 \text{ arbitrary}, \]
\[ \kappa = 3 + i_0(1 - \delta_{j_2,0}) + 2i'_0 \leq 1 + N + i'_0 \leq 2N - 1 . \]  
(3.14)

Here are eliminated chiral generators \( X_{1,4+k}^+ \), \( X_{2,4+k}^+ \), \( k \leq 1 + i'_0 \), and anti-chiral generators \( X_{3,5+N-k}^+ \), \( k \leq 1 + i_0(1 - \delta_{j_2,0}) \).
• bc \[ d = d^{bc}, \quad j_2 = 0, \quad z = 2m/N - m_1 + 1 + j_1, \quad j_1 \text{ arbitrary}, \]
\[ \kappa = 3 + 2i_0 + i'_0(1 - \delta_{j_1,0}) \leq 1 + N + i_0 \leq 2N - 1. \]  
(3.15)

Here are eliminated chiral generators \( X_{1,4+k}^+, \ k \leq 1 + i'_0(1 - \delta_{j_1,0}), \) and anti-chiral generators \( X_{3,5+N-k}^+, \ X_{4,5+N-k}^+, \ k \leq 1 + i_0. \)

• bd \[ d = d^{bd} = m_1, \quad j_1 = j_2 = 0, \quad z = 2m/N - m_1, \]
\[ \kappa = 4 + 2i_0 + 2i'_0 \leq 2N. \]  
(3.16)

Here are eliminated chiral generators \( X_{1,4+k}^+, \ X_{2,4+k}^+, \ k \leq 1 + i'_0, \) and anti-chiral generators \( X_{3,5+N-k}^+, \ X_{4,5+N-k}^+, \ k \leq 1 + i_0. \)

Note that the case \( \kappa = 2N \) is possible exactly when \( i_0 + i'_0 = N - 2, \) i.e., when there is only one nonzero \( r_i; \) namely, \( r_{i_0+1} \neq 0, \ i_0 = 0, 1, \ldots, N - 2: \)

• bd \[ \kappa = 2N : \quad d = m_1 = r_{i_0+1}, \quad j_1 = j_2 = 0, \quad z = r_{i_0+1} \frac{2 + 2i_0 - N}{N}. \]  
(3.17)

When \( d = m_1 = 1 \) these \( \frac{1}{2} \)-eliminated UIRs form the `mixed’ series of massless representations \([27]\), see also \([29, 33]\).\footnote{This series is absent for \( N = 1. \)}

Remark:  In this paper we use the Verma (factor-)module realization of the UIRs. We give here a short remark on what happens with the ER realization of the UIRs. As we know, cf. \([29]\), the ERs are superfields depending on Minkowski space-time and on 4\(N\) Grassmann coordinates \( \theta^a_i, \quad \bar{\theta}^a_i, \quad a, b = 1, 2, \)
\( i, k = 1, \ldots, N. \) There is 1-to-1 correspondence in these dependencies and the odd null conditions. Namely, if the condition \( X_{a,4+k}^+ |\Lambda\rangle = 0, \ a = 1, 2, \) holds, then the superfields of the corresponding ER do not depend on the variable \( \theta^k_a, \) while if the condition \( X_{a,5+N-k}^+ |\Lambda\rangle = 0, \ a = 3, 4, \) holds, then the superfields of the corresponding ER do not depend on the variable \( \bar{\theta}^k_a. \) These statements were used in the proof of unitarity for the ERs picture, cf. \([52]\), but were not explicated. They were analyzed in detail in the papers \([43–45, 54]\), using the notions of ’harmonic superspace analyticity’ and Grassmann analyticity. \( \diamond \)

In the next Section we shall use the above classification to the so-called BPS states.
4 BPS and possibly protected states

4.1 PSU(2,2/4)

The most interesting case is when $N = 4$. This is related to super-Yang-Mills and contains the so-called BPS states, cf., [43–51]. They are characterized by the number $\kappa$ of odd generators which annihilate them - then the corresponding state is called $\frac{\kappa}{4N}$-BPS state. Group-theoretically the case $N = 4$ is special since the $u(1)$ subalgebra carrying the quantum number $z$ becomes central and one can invariantly set $z = 0$.

We give now the explicit list of these states:

• $d = d_{41}^1 = 2 + 2j_2 + 2m_1 - \frac{1}{2}m > d_{44}^3$. The last inequality leads to the restriction:

$$2j_2 + r_1 > 2j_1 + r_3 .$$

(4.1)

In the case of $R$-symmetry scalars, i.e., $m_1 = 0$, follows that $j_2 > j_1$, i.e., $j_2 > 0$, and then we have:

$$\kappa = 4, \quad m_1 = 0, \quad j_2 > 0 .$$

(4.2)

In the case of $R$-symmetry non-scalars, i.e., $m_1 \neq 0$, we have the range: $i_0 + i'_0 \leq 2$, and thus:

$$\kappa = 1 + i_0(1 - \delta_{j_2,0}) \leq 3 .$$

(4.3)

• $d = d_{41}^2 = \frac{1}{2}m^* > d_{44}^3$, $j_2 = 0$. The last inequality leads to the restriction:

$$r_1 > 2 + 2j_1 + r_3 .$$

(4.4)

The latter means that $r_1 > 2$, i.e., $m_1 \neq 0$, $i_0 = 0$, and thus:

$$\kappa = 2 .$$

(4.5)

The next two cases are conjugate to the previous two so we present them shortly:

• $d = d_{44}^3 = 2 + 2j_1 + \frac{1}{2}m > d_{41}^1 \implies$

$$2j_1 + r_3 > 2j_2 + r_1 ,$$

(4.6)
\[ m_1 = 0 \implies j_1 > j_2 \implies j_1 > 0 \implies \kappa = 4, \quad m_1 = 0, \quad j_1 > 0. \tag{4.7} \]

\[ m_1 \neq 0 \implies i_0 + i_0' \leq 2 \implies \kappa = 1 + i_0'(1 - \delta_{j_1,0}) \leq 3. \tag{4.8} \]

\[ \bullet \quad d = d_{44}^4 = \frac{1}{2}m > d_{41}^1, \quad j_1 = 0, \implies r_3 > 2 + 2j_2 + r_1, \tag{4.9} \]
\[ \implies r_3 > 2 \implies m_1 \neq 0, \quad i_0' = 0 \implies \kappa = 2. \tag{4.10} \]

\[ \bullet \quad d = d^{ac} = 2 + j_1 + j_2 + m_1. \quad \text{From } z = 0 \text{ follows:} \]
\[ 2j_2 + r_1 = 2j_1 + r_3. \tag{4.11} \]

In the case of \( R \)-symmetry scalars, i.e., \( m_1 = 0 \), follows that \( j_2 = j_1 = j \), and then we have:
\[ \kappa = 8 - 6\delta_{j,0}, \quad d = 2 + 2j. \tag{4.12} \]

In the case of \( R \)-symmetry non-scalars, i.e., \( m_1 \neq 0, \quad i_0 + i_0' \leq 2 \), and thus:
\[ \kappa = 2 + i_0(1 - \delta_{j_2,0}) + i_0'(1 - \delta_{j_1,0}) \leq 4. \tag{4.13} \]

\[ \bullet \quad d = d^{ad} = 1 + j_2 + m_1 = 3 + 3j_2 + 2r_1 + r_2, \tag{4.14} \]
\[ \chi_4 = \{ 0; r_1, r_2, 2 + 2j_2 + r_1; 2j_2 \} . \]

\[ \bullet \quad \chi_4 = \{ 2j_1; 2 + 2j_2 + r_3, r_2, r_3; 0 \} . \tag{4.15} \]
\( \bullet \textbf{bd} \) From \( z = 0 \) follows: \( m = m^* \Rightarrow r_1 = r_3 = r \), thus, \( i_0 = i_0' = 0,1 \) and then we have:

\[
\kappa = 4(1 + i_0) , \\
d = d^{bd} = m_1 = 2r + r_2 \neq 0 , \quad r, r_2 \in \mathbb{Z}_+ , \\
\chi_4 = \{ 0 ; r, r_2, r ; 0 \} .
\]

Some of these BPS-cases are extensively studied in the literature, mostly those listed here as cases \textbf{ac, bd}, cf. [43–51].

From the above BPS states we list now the most interesting ones in Tables 1-3:

Table 1

\( PSU(2,2/4), \ \frac{1}{2} \) BPS states, \( (\kappa = 8) \)

| case | \( d \) | \( j_1, j_2 \) | \( r_1, r_2, r_3 \) | protected |
|------|------|--------|-----------------|---------|
| \textbf{ac} | \( 2 + 2j \geq 3 \) | \( j = j_1 = j_2 \geq \frac{1}{2} \) | \( m_1 = 0 \) |       |
| \textbf{bd} | \( r_2 \geq 1 \) | \( j_1 = j_2 = 0 \) | \( m_1 = r_2 \) |       |
Table 2

$PSU(2, 2/4)$, $\frac{1}{4}$-BPS states, ($\kappa = 4$)

|   | $d$          | $j_1; j_2$                            | $r_1; r_2; r_3$ | protected        |
|---|--------------|---------------------------------------|-----------------|------------------|
| a | $2 + 2j_2 \geq 3$ | $j_2 \geq \frac{1}{2}$                | $m_1 = 0$       |                  |
| c | $2 + 2j_1 \geq 3$ | $j_1 \geq \frac{1}{2}$                | $m_1 = 0$       |                  |
| ac| $2 + j_1 + j_2 + r_{1+i_0} \geq 7/2$ | $j_1 - j_2 = \frac{1}{2}r_{1+i_0}(1 - i_0)$, $j_1 + j_2 \geq 1/2$ | $m_1 = r_{1+i_0} > 0$, $i_0 = 0, 1, 2$ |                  |
| ad| $\frac{m}{2} \geq \frac{9}{2}$ | $j_1 = 0, j_2 \geq \frac{1}{2}$       | $r_1 = 0, r_3 = 2 + 2j_2$ | No               |
| bc| $\frac{m^*}{2} \geq \frac{9}{2}$ | $j_1 \geq \frac{1}{2}, j_2 = 0$       | $r_1 = 2 + 2j_1$, $r_3 = 0$ | No               |
| bd| $m_1 \geq 2$ | $j_1 = j_2 = 0$                        | $r_1 = r_3 \geq 1$ | No, if $r_1 > 2$ |
Finally, we remark that some of the above states would violate the protectedness conditions that we gave in Subsection 2.2. As indicated in the last column of the above Tables these would be the $\frac{1}{8}$-BPS cases listed as cases \textbf{ad,bc}, and in case \textbf{bd} for $r_1 = r_3 > 2$, while for the $\frac{1}{8}$-BPS cases that would be the cases \textbf{b,d}, and in case \textbf{ac} for $r_1r_3 > 0$.

### 4.2 SU(2,2/N), $N = 1, 2$

We can set $z = 0$ also for $N \neq 4$ though this does not have the same group-theoretical meaning as for $N = 4$. In this Subsection we treat separately the cases $N = 1, 2$, which are more peculiar.

#### 4.2.1 SU(2,2/1)

For $N = 1$ setting $z = 0$ is possible only for three cases \textbf{a,c,ac}:
\begin{itemize}
  \item \( a \quad d = 2 + 2j_2, \quad j_2 > j_1 \geq 0, \quad \kappa = 1, \ \frac{1}{4}\text{-BPS}; \)
  \item \( c \quad d = 2 + 2j_1, \quad j_1 > j_2 \geq 0, \quad \kappa = 1, \ \frac{1}{4}\text{-BPS}; \)
  \item \( ac \quad d = 2 + 2j, \quad j_1 = j_2 = j, \quad \kappa = 2, \ \frac{1}{2}\text{-BPS}. \)
\end{itemize}

Note that according to the result of Subsection 2.2 the first two cases would not be protected.

### 4.2.2 SU(2,2/2)

For \( N = 2 \) holds \( i_0 = i'_0 = 0,1. \) Setting \( z = 0 \) is possible for four cases \( a,c,ac,bd \) when we have:

\begin{itemize}
  \item \( a \quad d = 2 + 2j_2 + r_1, \quad j_2 > j_1 \geq 0, \quad \kappa = 1 + i_0 \leq 2; \)
  \item \( c \quad d = 2 + 2j_1 + r_1, \quad j_1 > j_2 \geq 0, \quad \kappa = 1 + i'_0 \leq 2; \)
  \item \( ac \quad d = 2 + 2j + r_1, \quad j_1 = j_2 = j, \quad \kappa = 4 - 2\delta_{i_0j,0} \leq 4; \)
  \item \( bd \quad d = r_1 > 0, \quad j_1 = j_2 = 0, \) (here \( z = 0 \) holds in all cases), \( \kappa = 4, \ \frac{1}{2}\text{-BPS}. \)
\end{itemize}

Note that according to the result of Subsection 2.2 the first three cases would not be protected when \( r_1 > 0, \) i.e., when \( i_0 = i'_0 = 0. \) In contradistinction, when \( r_1 = 0, \) i.e., \( i_0 = i'_0 = 1, \) the first two are \( \frac{1}{4}\text{-BPS}, \) and the third, when \( j > 0, \) a \( \frac{1}{2}\text{-BPS}. \) The fourth case would not be protected if \( r_1 > 4. \)

### 4.3 SU(2,2/N), \( N \geq 3 \)

The cases \( N \geq 3 \) are somewhat similar in these considerations to \( N = 4, \) (though some results differ), so we present them only in Tables 4,5,6 and 7.

As we see the case of \( \frac{1}{2}\text{-BPS} \) states can be presented in a table for all \( N. \)

The case of \( \frac{1}{4}\text{-BPS} \) states for \( N = 3 \) may be seen also in the tables for 20
general $N$, but it makes sense to be presented separately in Table 5.

### Table 4

$SU(2,2/N)$, $\frac{1}{2}$-BPS states, $\kappa = 2N$, $N \geq 1$

| $d$ | $j_1, j_2$ | $r_1, \ldots, r_{N-1}$ | protected |
|-----|-------------|-------------------------|-----------|
| ac  | $2 + 2j \geq 3$ | $j = j_1 = j_2 \geq \frac{1}{2}$ | $m_1 = 0$ |
| bd  | $r_{\frac{N}{2}} \geq 1$ | $j_1 = j_2 = 0$ | $m_1 = r_{\frac{N}{2}}$ | No, if $r_1 > 4$ for $N = 2$ |

### Table 5

$SU(2,2/3)$, $\frac{1}{4}$-BPS states, $\kappa = N = 3$

| $d$ | $j_1, j_2$ | $r_1, r_2$ | protected |
|-----|-------------|-------------|-----------|
| a   | $2 + 2j_2 \geq 3$ | $j_2 \geq \frac{1}{2}$ | $m_1 = 0$ |
| c   | $2 + 2j_1 \geq 3$ | $j_1 \geq \frac{1}{2}$ | $m_1 = 0$ |
| ac  | $2 + j_1 + j_2 + r_{1+i_0} \geq 6$ | $j_1 - j_2 = \frac{1}{3}(r_1 - r_2) = \pm 1, \pm 2, \ldots$ | $m_1 = r_{1+i_0} = 3, 6, \ldots, i_0 = 0, 1$ |
| ad  | $\frac{2}{3}m = \frac{2}{3}(r_1 + 2r_2) \geq 4$ | $j_1 = 0, j_2i_0 = 0$ | $r_2 = 3 + r_1 + 3j_2$ | No |
| bc  | $\frac{2}{3}m^* = \frac{2}{3}(2r_1 + r_2) \geq 4$ | $j_2 = 0, j_1i_0' = 0$ | $r_1 = 3 + r_2 + 3j_1$ | No |
Table 6  
SU(2, 2/N), 1/4-BPS states, \( \kappa = N, N > 4 \)

|   | \( d \) | \( j_1, j_2 \) | \( r_1, \ldots, r_{N-1} \) | protected                |
|---|--------|----------------|-------------------------|-------------------------|
| a | 2 + 2\( j_2 \) | \( j_2 \geq \frac{1}{2} \) | \( m_1 = 0 \) |                        |
| c | 2 + 2\( j_1 \) | \( j_1 \geq \frac{1}{2} \) | \( m_1 = 0 \) |                        |
| ac | 2 + \( j_1 + j_2 + r_{1+i_0} \) | \( j_1 - j_2 = \) \( r_{1+i_0} (1 - \frac{2}{N} (1 + i_0)) \) | \( m_1 = r_{1+i_0} > 0, \) \( i_0 \leq N - 2 \) |                        |
| ad | \( 1 + m_1 \) | \( j_1 = j_2 = 0 \) | \( i'_0 = \frac{N-3}{2} \) | No, if \( r_1 > 0 \) |
| N odd | | | | |
| ad | \( \frac{2m}{N} \) | \( j_1 = 0, j_2 \geq \frac{1}{2} \) | \( i_0 + i'_0 \leq N - 3 \) | No, if \( r_1 > 0 \) or \( r_{N-1} > 2 \) |
| bc | \( 1 + m_1 \) | \( j_1 = j_2 = 0 \) | \( i_0 = \frac{N-3}{2} \) | No, if \( r_{N-1} > 0 \) |
| N odd | | | | |
| bc | \( \frac{2m^*}{N} \) | \( j_1 \geq \frac{1}{2}, j_2 = 0 \) | \( i_0 + i'_0 \leq N - 3 \) | No, if \( r_{N-1} > 0 \) or \( r_1 > 2 \) |
| bd | \( m_1 > 0 \) | \( j_1 = j_2 = 0 \) | \( i_0 + i'_0 = \frac{N}{2} - 2 \) | No, if \( r_1, r_{N-1} > 2 \) |
Table 7

$SU(2,2/N)$, $\frac{1}{8}$-BPS states, $\kappa = N/2$, $N$ even, $N > 4$

| case | $d$ | $j_1, j_2$ | $r_1, \ldots, r_{N-1}$ | protected |
|------|-----|-----------|-----------------|-----------|
| a    | $2 + 2j_2 + \frac{2}{N}m^*$ | $j_1 - j_2 < \frac{m^* - m}{N}$ | $i_0 = \frac{N}{2} - 1$ |          |
| b    | $\frac{2}{N}m^*$ | $j_1 + 1 < \frac{m^* - m}{N}$, $j_2 = 0$ | $i_0 = \frac{N}{4} - 1$ |          |
| c    | $2 + 2j_1 + \frac{2}{N}m$ | $j_2 - j_1 < \frac{m^* - m}{N}$ | $i_0' = \frac{N}{2} - 1$ |          |
| d    | $\frac{2}{N}m$ | $j_2 + 1 < \frac{m^* - m}{N}$, $j_1 = 0$ | $i_0' = \frac{N}{4} - 1$ |          |
| ac   | $2 + j_1 + j_2 + m_1$ | $j_1 - j_2 = \frac{m^* - m}{N}$, $j_1 + j_2 > 0$ | $i_0(1 - \delta_{j_2,0}) + i_0'(1 - \delta_{j_1,0}) = \frac{N}{2} - 2$ |          |
| ad   | $\frac{2}{N}m$ | $j_2 + 1 = \frac{m^* - m}{N}$, $j_1 = 0$, $j_2 > 0$ | $i_0 + 2i_0' = \frac{N}{2} - 3$ | No, if $r_1 > 0$ or $r_{N-1} > 2$ |
| ad   | $\frac{2}{N}m$ | $j_1 = j_2 = 0$ | $\frac{m^* - m}{N} = 1$, $i_0' = \frac{1}{2}(\frac{N}{2} - 3)$ | No, if $r_1 > 0$ or $r_{N-1} > 2$ |
| bc   | $\frac{2}{N}m^*$ | $j_1 + 1 = \frac{m^* - m}{N}$, $j_1 > 0$, $j_2 = 0$ | $2i_0 + i_0' = \frac{N}{2} - 3$ | No, if $r_1 > 2$ or $r_{N-1} > 0$ |
| bc   | $\frac{2}{N}m^*$ | $j_1 = j_2 = 0$ | $\frac{m^* - m}{N} = 1$, $i_0 = \frac{1}{2}(\frac{N}{2} - 3)$ | No, if $r_1 > 2$ or $r_{N-1} > 0$ |
| bd   | $m_1$ | $j_1 = j_2 = 0$ | $m = m^*$, $i_0 + i_0' = \frac{N}{4} - 2$ | No, if $r_1, r_{N-1} > 2$ |
5 Outlook

In the present paper, we gave explicitly the reduction of supersymmetries of the positive energy unitary irreducible representations of the N-extended D=4 conformal superalgebras su(2,2/N). Further we give the classification of BPS and possibly protected states. Our considerations are group-theoretic and model-independent. Thus, we could give only the necessary conditions for protectedness, or equivalently, the sufficient conditions for unprotectedness.

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