Einstein’s gravitational lensing and nonlinear electrodynamics

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Einstein (1936) predicted the phenomenon presently known as gravitational lensing (GL). A prime feature of GL is the magnification, because of the gravitational field, of the star visible surface as seen from a distant observer. We show here that nonlinear electrodynamics (NLED) modifies in a fundamental basis Einstein’s general relativistic (GR) original derivation. The effect becomes apparent by studying the light propagation from a strongly magnetic ($B$) pulsar (SMP). Unlike its GR counterpart, the photon dynamics in NLED leads to a new effective GL, which depends also on the $B$-field permeating the pulsar. The apparent radius of a SMP appears then unexpectedly diminished, by a large factor, as compared to the classical Einstein’s prediction. This may prove very crucial in determining physical properties of high $B$-field stars from their X-ray emission.

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1. INTRODUCTION

Einstein’s general theory of relativity (GTR) has proved to be one of the most successful physical theories ever formulated. As a theory of the gravitational interaction, it has a number of predictions definitely corroborated by experiments or astronomical observations, which makes it the correct gravity theory here-to-fore. After readdressing an earlier study, Einstein (1936) came up with the prediction of the gravitational lensing effect. The idea is that the gravitational field produced by a massive astrophysical object, the Sun for instance, can act as a convergent lens able to deviate light rays flying-by. Observations of far away quasars and galaxies and more familiar total solar eclipses confirm the reality of this phenomenon. More than half of the star’s surface may be seen by a distant observer.
This means that a photon emitted at a given colatitude on the star’s surface, reaching the observer at infinity, must be emitted at a smaller angle with respect to the normal at that point. Because of this effect, a distant compact star must then appear to any observer larger than it actually is. The relation between the apparent radius $R_1$ of a spherical (non-rotating) star, as seen by a distant observer, and its physical radius $R$ is

$$R_1 = \frac{R}{1 - \left(\frac{R}{R_S}\right)^{1/2}}.$$  

(1)

This relation is obtained from the photon trajectory in polar coordinates $(r, \theta)$ by using a Schwarzschild metric

$$ds^2 = \left(1 - \frac{R_S}{r}\right)dt^2 - \frac{R_S}{r}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

(2)

Here $R_S = 2GM/c^2$ is the Schwarzschild radius of the star of mass $M$. As is clear from this line-element no effects from physical fields other than the gravitational have been taken into account in prescribing the star apparent size. This paper shows that the phenomenon changes in a fundamental fashion if NLED is called into play. The analysis below, based on well-known nonlinear Lagrangeans, as the Heisenberg-Euler (H-E) and the exact Born & Infeld (1934) (B-I), proves that when NLED is included to describe the photon dynamics, the trajectory depends on the background $B$-field pervading the star. We stress from the very beginning that the results below are obtained upon an idealization of the $B$-field structure. In a real star, the $B$-field is neither purely radial nor constant, so that the apparent radius hardly goes to zero at high $B$.

We stress in passing that this theory is not a new theory of gravitation. In our formalism Einstein’s general relativity remains definitively unaltered. Only the photons propagating out of the ultramagnetized neutron stars “see” the effective metric that appears once one incorporates NLED into a self-consistent physical description of light propagation from those stars. That NLED is a theory already tested in a number of experiments is demonstrated by the laboratory experiment described by Burke, Field, Horton-Smith, et al. (1997) as quoted below. Our theory attempts to call to the attention of the astrophysics community the fundamental changes that should appear in determining physical properties as radius, mass, and more crucial: the equation of state, of neutron star pulsars endowed with extremely strong magnetic fields, since a property as the star radius (which can be estimated from the x-ray variability of very short period of the star emission (light curve), or by measuring the gravitational redshift of absorption or emission lines received from the star surface, or etc.) can say too much about the stellar structure and its equation of state. A fundamental property that encodes basic information of the physical constituents of the neutron star matter.

2. Apparent radius in NLED

According to quantum electrodynamics (QED: see Delphenich 2003 for a complete review on NLED and QED) a vacuum has nonlinear properties which affect the photon propagation. A noticeable advance in the realization of this theoretical prediction has
been provided by Burke, Field, Horton-Smith, et al. (1997) who demonstrated experimentally that the inelastic scattering of laser photons by gamma-rays is definitely a nonlinear phenomenon. The propagation of photons in NLED has been examined by several authors (Bialynicka-Birula & Bialynicka-Birula 1970; Salazar et al. 1989; Dietrich & Gies 1998; De Lorenci 2000; Denisov & Svertilov 2003; Alarcon 1981; Garcia & Plebanski 1989; Boillat 1970; Vazquez et al. 1969; Torrence 1984; Garcia 1984) in the geometric optics approximation, it was shown by Novello et al. (2000) and Novello & Salim (2001) that when the photon propagation is identified with the propagation of discontinuities of the EM field in a nonlinear regime, a remarkable feature affloats: the discontinuities propagate along null geodesics of an \textit{effective} geometry which depends on the EM field on the background. This means that the NLED interaction can be geometrized, in analogy to gravity in GTR. A key outcome of this formalism is introduced in this paper.

2.1. \textit{Euler-Heisenberg approach}

The H-E \cite{2} Lagrangean

\[ \mathcal{L}(F; G) = \frac{1}{4} F^2 + \frac{1}{4} F^2 + \frac{7}{4} G^2 \]  

where \( \frac{2}{45} \) is a quantum parameter, is a gauge invariance description of the photon propagation in NLED that uses two invariants

\[ F = F \; F \; F \; ; \; G = F \; F \; F \; F \; F \; ; \]  

constructed upon the Maxwell EM tensor and its dual \(^a\). The method of characteristic surfaces or shock waves followed here (introduced by Hadamard (1903)\cite{20}) can be applied to any field theory having hyperbolic field equations, including electrodynamics. Hence, the result in Eq. (50) (see appendix), implies two possible paths of propagation or polarization modes, according to the double solution \( \). Using Eq. (3) to compute the Lagrangean derivatives \( L_F \; L_F F \; L_G F \), one arrives to the couple of effective metrics (first order in \( )

\[ q^+ = g + 8 F F F ; \; q^- = g + 14 F F F ; \]  

In terms of electric \((E)\) and magnetic \((B)\) fields the tensor \( F \) can be written as

\[ F = E V E V + \; \; V B \; ; \]  

where \( V \) is the normalized \( \sqrt{V} = 1 \), Eq. (2) velocity of the reference frame where the fields are measured. Our main concern here is the behavior of photons, in the NLED context, emitted from a highly magnetized neutron star. Corotating charges in the pulsar magnetosphere or a rotating magnetic dipole lead to induced \( E \)-fields in the star surface. We consider here slowly rotating neutron stars in order that the \( E \)-field contribution could

\(^a\)The attentive reader must notice that this first order approximation is valid only for \( B \)-fields smaller than \( B_q = \frac{\pi^2}{3} \frac{e^2}{\hbar c^3} = 4 \times 10^{13} \) G (Schwinger's critical \( B \)-field).
be neglected. Since $E = 0$, the above expression simplifies to: $F = V B$. Its self-product reads

$$F \cdot F = B B B^2 (g V V) ;$$

(7)

where $B^2 = B B$. After discarding a nonphysical conformal factor the effective metrics become

$$g^e = g \left(1 - \frac{2 B^2}{1 - \frac{2 B^2}{B}} V V \right) ;$$

(8)

The inverse (covariant) metric of Eq.(8) is obtained from the relation $g \cdot g = 1$, which then reads

$$g^e = g + \frac{-2 B^2}{1 - \frac{2 B^2}{B}} V V ;$$

(9)

By $\sim$ we mean $\sim = 14$ and $\sim = 8$. In order to pursue our calculations we assume a radial magnetic field. This is clearly not realistic, but is an approximate description of the field geometry near the polar caps of a magnetized neutron star. Thus

$$B \sim j B j ; \quad l \sim p g_{rr} r ; \quad V \sim p g_{00} 0 ;$$

(10)

and consequently one arrives to the effective metric components

$$g^e_{00} = (1 - \sim B^2) g_{00} ; \quad g^e_{rr} = (1 - \sim B^2) g_{rr} ;$$

(11)

Notice the structural similarity of both metric components. From Eq.(11) is straightforward to rewrite the expression for the apparent radius of the spherical star in Eq.(1) in the putative background metric given by Eq.(2). After doing so, one can compute the ratio of the star area with H-E NLED effects included to the area without (Einstein’s derivation), as seen from the distant observer. This quantity is represented here by $N$.

2.1.1. Qualitative description of the visible (apparent) star surface

As stated above, "$N$" is intended to be a quantity describing the ratio between the area of the star as seen by an observer at infinity that receives the actual traveling luminous ray, and the actual physical area of the neutron star, both projected on the plane of the sky. This ratio of areas can be consistently figured out by the following qualitative reasoning.

In a general relativistic framework, the local neutron star luminosity is defined as

$$L = 4 R^2 \frac{B^4}{\Delta} ;$$

(12)

We stress here that the NLED effects depend on the effective geometry, which in turn depends upon the strength of the magnetic field. Although the magnetic field $B$ is a solution of the NLED field equations, to compute the ratio "$N$" is only needed to know the local (on the star surface) magnitude or modulus of the field. In the particular case of the computation of the null geodesics discussed below, is indeed necessary to know the analytic expression for the field as a function of the coordinates, in particular of the radial $r$-coordinate. Unfortunately, such a solution for the magnetic field structure (or dependence on the coordinates) is only known for the linear (Maxwell’s) case.
where $R_\gamma$ is the neutron star radius and $T_\gamma$ is its temperature. Both measured locally at the star surface. The luminosity measured by the observer at infinity, which carries the information on the spacetime metric around the neutron star, is defined
\begin{equation}
L_1 = L \left( \frac{R_\gamma}{R_1} \right)^1 ;
\end{equation}
where $1 \left( \frac{R_\gamma}{R_1} \right)^1$ is the $tt$-metric component of a Schwarzschild spacetime, i.e.,
\begin{equation}
g_{tt} = 1 \left( \frac{R_\gamma}{R_1} \right)^1 ;
\end{equation}
Notice that $g_{rr} = g_{tt}$. On the other hand, the luminosity at infinity can be written as
\begin{equation}
L_1 = 4 \left( \frac{R_1}{R_{11}} \right)^2 sB \left( T_\gamma^{1/4} \right)^4 ;
\end{equation}
Thus, the temperature measured by an observer at infinity relates to the actual star temperature via
\begin{equation}
T_\gamma^{1/4} = T_\gamma \left( \frac{R_\gamma}{R_{11}} \right)^{1=2} ;
\end{equation}
Therefore, after manipulating all these relations one arrives to the relation between both the star radii (local and at infinity)
\begin{equation}
R_1 = R_\gamma \left( g_{tt} \right)^{1=2} ;
\end{equation}
In an equivalent approach, once one realizes that the star luminosity is modified by the spacetime geometry induced by the matter distribution, one can readily verify that in the case of NLED its geometric effects on the photon propagation from the ultra magnetized neutron star leads to an expression that is similar to the one above. Hence, one can write,
taking into account the general relativistic effects superposed to the NLED effects, the local neutron star luminosity as given by

$$L = 4 R^2_{SB} T_7^4.$$  \hspace{1cm} (18)

The luminosity measured by the observer at infinity, which carries the information on the spacetime metric as modified by NLED, as in the Heisenberg-Euler (H-E) NLED, now reads

$$L_{1 eff} = L \left( 1 + \frac{R_s}{R_7} \right) \frac{h}{1} \sim 2 B^2 i.$$  \hspace{1cm} (19)

or equivalently,

$$L_{1 eff} = 4 \left( \frac{R_{1 eff}}{R_7} \right)^2_{SB} (T_7^4)^4;$$  \hspace{1cm} (20)

where the term $1 + \frac{R_s}{R_7} \sim 2 B^2$ now substitutes the tt-metric component of the pure Schwarzschild spacetime presented above. Notice that in this case the effective tt-metric component reads

$$g_{tt}^{eff} = \frac{h}{1} \sim 2 B^2.$$  \hspace{1cm} (21)

This implies that

$$g_{tr}^{eff} = (g_{tt}) \frac{h}{1} \sim 2 B^2 = \frac{h}{g_{tt}};$$  \hspace{1cm} (22)

Thus, the temperature measured by an observer at infinity now reads

$$T_7^1 = T_7 \left( 1 + \frac{R_s}{R_7} \right)^{i=2} \frac{h}{1} \sim 2 B^2 i_{i=2};$$  \hspace{1cm} (23)

Once again, after manipulating these relations one obtains

$$R_{1 eff} = R_7; \quad g_{tr}^{eff} = 1.$$  \hspace{1cm} (24)

From these relations one can write the area ratio "N" that estimates the magnitude of the change in the apparent visible area of the star as seen from infinity compared to the actual star area, as measured by a local observer:

$$N = \frac{A_{real local}}{A_{real NLED}} = \frac{4 \left( \frac{R_{1 eff}}{R_7} \right)^2}{4 \left( \frac{R_{1 eff}}{R_7} \right)^2};$$  \hspace{1cm} (25)

Hence, by substituting the corresponding terms in these last equations one arrives to

$$N = \frac{g_{tr}}{g_{tr}^{eff}} = \frac{1}{1} \sim 2 B^2.$$  \hspace{1cm} (26)

As one can notice, in this new expression for the area ratio there is no any functional of the r coordinate but rather only a function of the magnetic field strength at the surface of the neutron star. This is the reason of why we took B as a constant in the numerical calculation leading to the plots in Figures 1, and 2.
2.2. Born-Infeld NLED

The propagation of light from hypermagnetized neutron stars can also be viewed within the framework of the Born-Infeld Lagrangian

\[ L = b^2 \left( 1 + \frac{F}{b^2} \right) G^2 \]

where \( b^2 = \frac{e^4}{c^2} = e^4 = m \) \( \alpha \) \( c^2 \) \( ! \) \( b = 9 \cdot 10^{15} \) \( \epsilon \) \( \text{m} \) \( \text{s} \) \( \text{u} \) \( \). As is well known, this is an exceptional Lagrangean. One of its remarkable properties is that it does not exhibit birefringence\( ^{21} \); \( ^{18} \). In this case, the deduction we present in the appendix fails since the quantities \( i \), with \( i = 1; 2; 3 \); vanish identically. We need hence to go back to Eq.\( ^{46} \); \( ^{47} \); and substituting in these equations the form for \( L (F; G) \) of Eq.\( ^{27} \) to obtain a unique characteristic equation, which then leads to the effective metric

\[ q_\alpha = g + \frac{2}{b^2 + F} F' F' : \]

Using the self-product of the tensor \( F' \) given by \( ^{17} \), and noting that in our case \( F' = F' F' = 2b^2 \), the effective metric then reads (see reference\( ^{18} \) for details)

\[ q_\alpha = g + \frac{2b^2}{b^2} (V V \ l \ l) : \]

By computing the inverse metric via \( q_\alpha q^\alpha = \), the covariant form of this effective metric is obtained as

\[ g^\alpha = g + \frac{2b^2}{b^2 + b^2} V V + \frac{2b^2}{(2b^2 + 1)} l l : \]

After recalling the relations for \( l \) and \( b \) given in Eq.\( ^{11} \), one can verify that the covariant \( \tau \tau \) effective metric component is then written as

\[ g^\alpha_{\tau \tau} = g_{\tau \tau} \frac{2b^2}{(2b^2 + 1)} g_{\tau \tau} - \frac{1}{1 + 2b^2} = g_{\tau \tau} : \]

By following steps similar to the H-E case, the area ratio is presented in Fig.\( ^{2} \). Although our hypothesis about the field geometry maximizes the NLED effect on the observed properties of the star, they should properly be taken into account when analyzing the X-ray emission. (This point will be considered in a forthcoming paper). It should also be relevant in pondering the effects on supernova dynamics of photons radiated either by highly magnetized proto-neutron stars (Miralles et al. (2002)\( ^{25} \)) or stellar-mass black holes enshrouded by super strong \( B \) -fields (see van Putten et al. (2004)\( ^{26} \)).

3. Photon trajectories as seen from a distance

A self-consistent treatment of this problem does require the study of the light propagation in the neutron star spacetime geometry modified by NLED. In NLED the Lagrangean defined by Eq.\( ^{2} \) becomes
As the photons travel along null geodesics, the Euler-Lagrange equation for the $rr$-component of the metric can be obtained from the (first integral) relation

$$g_{rr}(u^r)^2 + g_{r\tau}(u^\tau)^2 + g \ (u \ )^2 + g^e \ (\nu \ )^2 = 0 ;$$  

(36)

In analogy with the planetary system one can chose a particular “orbital” plane at $= -2$, so that $u = 0$ and $\sin^2 = 1$, and thus $g^e = r^2$. By making use of the standard
change of variables \((u = 1 = r, 0 = d = dr, \text{ and defining } A = 2B^2 = b^2)\) one can write the photon propagation equation as

\[
E_0^2 \left( \frac{1}{A + 1} \right)^2 \hbar^2 (u^0)^2 (1 + R_s u) \frac{1}{A + 1} \hbar^2 u^2 = 0 ;
\]

which one can in turn write as (\(E_0, \hbar \) constants)

\[
(u^0)^2 + u^2 = \frac{E_0}{\hbar^2} + R_s u^3 + 2 \frac{E_0}{\hbar^2} u^2 R_s u^3 A + \frac{E_0}{\hbar^2} A^2 ;
\]

After performing the derivative of equation (38) with respect to the angular coordinate, and taking into account the fact that the solution \(u = \text{constant}\) can be discarded by an analogous (physical) reason as it is done in the case of a pure photon propagation in a Schwarzschild geometry, one arrives to

\[
u^0 + (1 + A)u = \frac{3}{2} R_s (1 + A) u^2 + 1 + R_s U \frac{2E_0}{\hbar} (1 + A) u^2 \frac{dA}{dr} ;
\]

This equation is highly nonlinear and quite hard to solve by analytical means. (Notice that Eq.(39) is quite similar to that one obtains in the case of the photon propagation in a Schwarzschild spacetime. Nonetheless, we stress that in order to find such a solution; an analogous procedure as the one usually pursued in general relativity (the perturbation approach) does not work in the present case, since for the extremely large magnetic field we are considering the induced NLED effects are of the same order of magnitude as those purely obtained from gravitation.) Besides, one needs to know the function \(B(r, \tau)\), which is a solution of the Born-Infeld equations for this problem. Perhaps a numerical solution can be found, but that would require extra work that we think would not add too much to the new results we are presenting in this version of the paper. (A full detailed solution will be given elsewhere).

Eqs.(38,39) shows very clearly that NLED does modify the standard propagation of photons as compared to that one in a pure Schwarzschild gravitational field as described in most textbooks of GTR. A first look at Eq.(38) suggests that very large \(B\)-fields do reduce the effective star’s visible area.

4. Discussion and conclusion

The nonlinear electrodynamics is a well established theory of the electromagnetic interaction, which overcomes the failure of Maxwell’s theory in describing the stability of atomic structures. Since its introduction in physics by Euler and Heisenberg it has gained the status of a theory with a strong experimental support (for a recent complete review see Delphenich (2003)). Despite being known since long, its very key features have become recognized only in the late years (see references above). It had been shown in the last few years (Novello et al2000; Novello & Salim 2001) that the force acting along the photons trajectory can be geometrized in such a way that in an effective metric \(g^{\text{eff}} = g + g^{\text{NLED}}\) the photons follow geodesic paths in that effective geometry (Novello et al2000; Novello & Salim 2001). In particular, the effects of nonlinear electrodynamics in the physics of
strongly magnetized neutron stars have been recently studied by Mosquera Cuesta & Salim (2003; 2004)\textsuperscript{23,24}. It was shown there that for extremely supercritical magnetic fields nonlinear electrodynamics effects force photons to propagate along accelerated curves, in such a way that the surface gravitational redshift of emission lines from hypermagnetized neutron stars, as measured by a distant observer, is significantly modified. Such a new effect turns out to be of the same order of magnitude of the one produced by the pure gravitational field.

In this paper, we present a new astrophysical application of the peculiar effects brought by NLED, which is intimately related to the one previously discussed by (Mosquera Cuesta & Salim 2003; 2004)\textsuperscript{23,24}. We show that the famous gravitational lensing effect, originally introduced by Einstein, is also modified when the electromagnetic (nonlinear) effects are taken into the depiction of the physics around strongly magnetized neutron stars. Such a dynamics is described by using both the Euler-Heisenberg and the Born-Infeld approaches to NLED. Incidentally, we show, by comparing Eqs. (11) and (30), that Eq. (11) clearly exhibits a divergence. Such an unphysical behavior, in turn, invalidates the results presented by Denisov et al. in Ref. \textsuperscript{17}, obtained within the H-E approach, since they extended to the case of magnetars (neutron stars having $B$ -fields $10^{15}$ G) their formula to estimate the light-ray bending angle beyond the QED $B$ -field limit. That is inconsistent. Besides, both approaches: the H-E and the B-I NLED induce critical changes in the $rr$ metric component that modifies the quantification of the lensing effect. More relevant, yet, the area of a putative star may appear to a distant observer nonphysically diminished in H-E NLED, whereas it is physically largely reduced, and may even “disappear”, for fields $B > 10^{17}$ G in the B-I approach. The impact of this on the star flux may be dramatic. Thence, the effect introduced by (Mosquera Cuesta & Salim 2003; 2004)\textsuperscript{23,24} and this new here crucially alters the dynamics of photons from very high $B$ -field stars, while entangles (makes it difficult) the inference of their physical properties.

To the very end, what kind of observations can be performed in a search for this peculiar effect? As light is the prime messenger from the stars, one can look for variability of very short period in the x-ray (or γ-ray) light curves from most hypermagnetized pulsars. A comparative analysis of the characteristics of those light curves with the ones from canonical pulsars may render the clues on the presence of NLED effects. A similar analysis can be pursued by using the measurements of the surface gravitational redshift of that kind of hypermagnetic stars.

5. APPENDIX: The method of effective geometry

Following Hadamard (1903)\textsuperscript{20}, the surface of discontinuity $c$ of the EM field is denoted by $\Gamma$. The field is continuous when crossing $\Gamma$, while its first derivative presents a finite discontinuity. These properties are specified as follows: $[\mathbf{F}] = 0$; $[\mathbf{F}]_j = \mathbf{f}_k$, where the symbol $[\mathbf{F}] = \lim_{\sigma \to 0} (\mathbf{J} \cdot \mathbf{J})$ represents the discontinuity of the

\textsuperscript{5}Of course, the entire discussion onwards could alternatively be rephrased using concepts more familiar to the astronomy community as that of light rays used for describing propagation of EM waves in geometric optics.
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arbitrary function \( J \) through the surface \( \mathcal{S} \). The tensor \( f \) is called the discontinuity of the field, \( k = \partial \mathcal{S} \) is the propagation vector, and the symbol “\( \partial \)” stands for partial derivative.

Hereafter we investigate the effects of nonlinearities of very strong B-fields in the evolution of EM waves; described onwards as the surface of discontinuity of the EM field (represented here-to-fore by \( F \)). Extremizing the Lagrangian with respect to the potentials \( \Phi \) yields the following field equation:

\[
\mathbf{r} \cdot \left( L_F F F + L_G F \right) = 0; \tag{40}
\]

Besides this, we have the cyclic identity:

\[
\mathbf{r} \cdot F = 0, \quad F_j + F_j + F_j = 0; \tag{41}
\]

The field equation can be written explicitly as:

\[
L_F \mathbf{r} \cdot F + 2N \cdot \mathbf{r} = 0; \tag{42}
\]

where the tensor \( N \) is defined as

\[
N = L_{FF} F F F F + L_{FG} F F F F + L_{GG} F F F F; \tag{43}
\]

Taking the discontinuities of the field equation we get:

\[
f_k k = \frac{2}{L_F} N \cdot f_k; \tag{44}
\]

The discontinuity of the Bianchi identity yields:

\[
\mathbf{f} \cdot f_k + f_k + f_k = 0; \tag{45}
\]

From these equations we obtain (see [10] for details)

\[
k^2 = \frac{4}{L_F} F F k k \left( L_{FF} + L_{GG} \right) k^2 \frac{G}{L_F} \left( L_{FG} + L_{GG} \right); \tag{46}
\]

\[
k^2 = \frac{4}{L_F} F F k k \left( L_{FG} + L_{GG} \right) \frac{G}{L_F} k^2 \left( L_{FG} + L_{GG} \right) + \frac{2}{L_F} F F k^2 \left( L_{FG} + L_{GG} \right); \tag{47}
\]

where we introduce the notation: \( = F F, \quad = F F, \quad k^2 = g \cdot k k \) As the H-E QED Lagrangean is not a functional of the product \( F \cdot G \), then one obtains

\[
k^2 = \frac{4}{L_F} F F k k \left( L_{FF} \right) \frac{G}{L_F} k^2 \left( L_{GG} \right); \tag{48}
\]
We seek thus for a master relation representing the propagation of field discontinuities, which should be independent of the quantities $f$, that is, independent of $F$ and $G$. There is a simple way to achieve such a goal. We firstly isolate the common term $F$ which appears in both Eqs. (48,49). Then, by assuming that $k^2 \neq 0$, the difference of these equations can be put in the form of an algebraic linear relation between $F$ given as:

$$1 + 2 + 3 = 0;$$

where we define:

$$1 = L_{F F} - L_{G G} F$$

and

$$2 = L_{G G} + L_{F F} F$$

and

$$3 = L_{F F}$$

Solving the quadratic equation for $F$ we obtain: $F = \frac{g_{eff}}{k^2}$, with:

$$g_{eff} = \sqrt{\frac{1}{4} + \frac{2}{3} - \frac{1}{2}}.$$ Using this solution in Eqs. (48) and (49), and assuming $k^2 \neq 0$, after some algebra one gets the dispersion relation

$$g_{eff} F = 0;$$

Thence, one concludes that the discontinuities will follow geodesics in this effective metric $g_{eff}$.

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References
1. A. Einstein, Sci. 84, 506 (1936)
2. W. Heisenberg, H. Euler, Zeit. Phys. 98, 714 (1936)
3. M. Born, L. Infeld, Pr. Roy. Soc. Lond. A 144, 425 (1934)
4. D. H. Delphenich, report hep-th/0309108 (2003)
5. J. Schwinger, Phys. Rev. 82, 664 (1951)
6. D. L. Burke, R. C. Field and G. Horton-Smith, et al., Phys. Rev. Lett. 79, 1626 (1997)
7. Z. Bialynicka-Birula, I. Bialynicka-Birula, Phys. Rev. D 2, 2341 (1970)
8. A. Garcia, J. Plebanski, Math. Phys. 30, 2689 (1989)
9. W. Dietrich and H. Gies, Phys. Rev. D 58, 025004 (1998)
10. V. A. De Lorenci, et al., Phys. Lett. B 482, 134 (2000)
11. S. Aharone, A. Dall'Agata, J. Plebanski, J. Math. Phys. 22, 2835 (1981)
12. H. Salazar, A. Garcia, J. Plebanski, J. Math. Phys. 30, 2689 (1989)
13. A. Garcia, H. Salazar, J. Plebanski, Nuov. Cim. 84B, 65 (1984)
14. G. Boillat, J. Math. Phys. 11, 941 (1970)
15. L. Vazquez, J. Math. Phys. 18, 1259 (1977)
16. R. Pellicer, R. J. Torrence, J. Math. Phys. 10, 1718 (1969)
17. V. I. Denisov, I. P. Denisova, S. I. Svertilov, Dokl. Phys. 46, 705 (2001a); V. I. Denisov, I. P. Denisova, S. I. Svertilov, Dokl. Akad. Nauk Serf. Fiz. 380, 435 (2001b); V. I. Denisov, S. I. Svertilov, A & A. 399, L39 (2003)
18. M. Novello, et al., Phys. Rev. D 61, 045001 (2000)
19. M. Novello, J. M. Salim, Phys. Rev. D 63, 083511 (2001)
20. Hadamard, J., Leçons sur la propagation des ondes et les équations de l’Hydrodynamique (Hermann, Paris, 1903)
21. J. Plebski. Lectures on nonlinear electrodynamics, Nordita, Copenhagen, (1970).
22. V. A. De Lorenci, et al., A & A 369, 690 (2001)
23. H. J. Mosquera Cuesta, J. M. Salim, Astrophys. J. 608, 925 (2004). See also reports: astro-ph/0307519 (2003); and astro-ph/0403045 (2004)
24. H. J. Mosquera Cuesta, J. M. Salim, Mon. Not. Roy. Astron. Soc. 354: L55-L59 (2004)
25. J. A. Miralles, J. A. Pons, V. A. Urpin, Astrophys. J. 574, 356 (2002)
26. van Putten, M., et al., Phys. Rev. D 69, 044007 (2004)