On $\mathbb{Z}_2$-twisted representation of vertex operator superalgebras and the Ising model SVOA

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Abstract

We investigate a general theory of the $\mathbb{Z}_2$-twisted representations of vertex operator superalgebras. Certain one-to-one correspondence theorems are established. We also give an explicit realization of the Ising model SVOA and its $\mathbb{Z}_2$-twisted modules. As an application, we obtain the Gerald Höhn’s Babymonster SVOA $VB^\natural$ and its $\mathbb{Z}_2$-twisted module $VB^\natural_{tw}$ from the moonshine VOA $V^\natural$ by cutting off the Ising models. It is also shown in this paper that $\text{Aut}(VB^\natural)$ is finite.

1 Introduction

In the theory of vertex operator algebras (VOAs), we sometimes notice that it makes the theory simpler to use some representations of vertex operator superalgebras (SVOAs for short) instead of VOAs. For example, as one can see in [M1]-[M3], in the representation theory of the Ising model VOA $L(\frac{1}{2}, 0)$ it seems more natural that we should treat the theory in the view point of the SVOA $L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$, which is the most interested object in this paper. For the theory of VOAs, we have many remarkable results on the fundamental representation theory, so-called Zhu’s theory (cf. [Z], [FZ], [Li2], [DLM1], [DLM2], [Y]) and some of them are extended to that for SVOAs (cf. [KW]). The Zhu algebra $A(V)$ is an associative algebra associated to every VOA $V$. It is well-known that there exists a one-to-one correspondence between the category of irreducible $V$-modules and that of irreducible $A(V)$-modules (cf. [Z], [DLM2]). Since every SVOA has a canonical involution, we can think of the $\mathbb{Z}_2$-twisted representations of SVOAs and the above theory should be naturally extended to $\mathbb{Z}_2$-twisted representations for SVOAs.

To start the investigation of SVOAs, the Ising model SVOA $L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$ will be a good example. It is one of the smallest SVOAs and a fundamental object in the...
theory of rational VOAs. In spite of its simplicity, there are many applications and deep
depth theories. The fusion algebra of the Ising model VOA \(L(\frac{1}{2},0)\) has been determined and
many related results are obtained (cf. [DM2], [DGH], [M1]-[M3], etc). In particular, in
\[M3\] the moonshine vertex operator algebra \(V^\natural\) is reconstructed by Miyamoto by using
the Ising models. In the Miyamoto’s theory, some central extensions of 2-groups play an
important role. It seems that the appearance of 2-groups reflects the \(Z_2\)-gradation of
the structure of SVOAs. Since there are two ways to represent the \(Z_2\)-gradation of SVOA,
namely, we can consider the \(Z_2\)-graded representation and the \(Z_2\)-twisted representation,
the latter shall be as important as the former. One can also find the importance of the
\(Z_2\)-twisted representation from another point of view. In Frenkel-Lepowsky-Muerman’s
\(Z_2\)-orbifold construction [FLM], a \(Z_2\)-twisted representation of the Leech lattice VOA is
used to construct the moonshine VOA. Through the two different constructions of the
moonshine VOA, we come to believe that the \(Z_2\)-twisted representation of SVOA will be
an essential concept in the theory of SVOAs. In this respect, we introduce the \(Z_2\)-twisted
Zhu algebras and investigate the Ising model SVOA. We will give an explicit realization
of the Ising model SVOA and its all \(Z_2\)-twisted representations. Namely, we will give
a realization of the all unitary representation of the Virasoro algebra of central charge
\(\frac{1}{2}\) in terms of local and \(Z_2\)-twisted local systems. It enable us to compute all type of
intertwining operators in fermionic literature, which is simpler than the known bosonic
literature. Based on our realization, we will determine the structure of \(Z_2\)-twisted Zhu
algebra for the Ising model SVOA. As an application, we propose a method to construct
some SVOAs and its \(Z_2\)-twisted representations using the Ising models. Applying our
method on the moonshine VOA \(V^\natural\), we obtain the Gerald Höhn’s Babymonster SVOA
\(VB^\natural\) and its irreducible \(Z_2\)-twisted representation \(VB^\natural_{tw}\).

We organize this paper in the following way. In Sec. 2 we generalize Kac-Wang’s
Zhu algebra associated to SVOA to the twisted case and then we establish a bijective
correspondence which is well-known in the theory of VOAs. In the proof of the associat-
vativity of the Zhu algebra, by making use of the commutativity and associativity of an
SVA, we can make the proof much simpler. We also present Frenkel-Zhu’s bimodules for
\(Z_2\)-twisted Zhu algebras. The fusion rules for \(Z_2\)-twisted modules are described in terms
of our bimodules for \(Z_2\)-twisted Zhu algebras.

In Sec. 3 we present some basic facts on SVOA. Invariant bilinear forms for SVOA
and \(Z_2\)-conjugacy of the modules are treated.

In Sec. 4 we present a construction of the Ising models. Using the ideas of the
local system and twisted local system in [Li1], we attain an SVOA structure in \(M = L(\frac{1}{2},0) \oplus L(\frac{1}{2}, \frac{1}{2})\) and its \(Z_2\)-twisted representation in \(L(\frac{1}{2}, \frac{1}{16})\) in Subsec 4.2. Although some
ingredients in Subsec 4.1-4.2 are already known in some papers (cf. [FRW1], [FRW2], [La], etc); we will pick up and repeat the necessary parts with the suitable modifications because fermionic construction has one merit such that it enable us to compute intertwining operators of all type for the Ising models including the twisted part. In Subsec. 4.3 we determine \( \mathbb{Z}_2 \)-twisted Zhu algebras for the Ising model SVOA and classify all irreducible \( \mathbb{Z}_2 \)-twisted representations based on our realization given previously. At the last of this section, we consider an application of the Ising models. We present a method to construct an SVOA from a VOA containing the Ising models. This method is already shown in [H]. However, it is worth to present our method since it works in simpler situation.

In Sec. 5 we present a construction of the Gerald Höhn’s Babymonster SVOA and its \( \mathbb{Z}_2 \)-twisted module from the moonshine VOA. It is a simple SVOA whose automorphism group contains Fischer’s Baby monster sporadic simple group. Using some methods from the Quantum Galois theory, we will show that the full automorphism group of the Babymonster SVOA is finite.

2 \( \mathbb{Z}_2 \)-twisted Zhu theory for SVOAs

2.1 \( \mathbb{Z}_2 \)-twisted representations

Let \( (V,Y,\mathbb{1},\omega) \) be an SVOA, where \( V \) has a \( \mathbb{Z}_2 \)-grading \( V = V^0 \oplus V^1 \). We assume that \( V \) has a half-integer grading: \( V^0 = \bigoplus_{n \in \mathbb{Z}} V_n \) and \( V^1 = \bigoplus_{n \in \mathbb{Z}} V_{n+1/2} \), where \( V_i = \{ v \in V | L_0 v = sv \}, i \in \mathbb{Z}_2, s \in \frac{1}{2} \mathbb{Z} \). We also assume that each \( L_0 \)-weight space \( V_s \) is of finite dimension. Let \( p \) and \( q \) be the parity functions defined by \( p(a,b) = 1 \) for \( a, b \in V^1 \) and \( p(a,b) = 0 \) for \( \mathbb{Z}_2 \)-homogeneous \( a, b \in V \) of the other cases, and \( q(a) = i \) for \( a \in V^i \), respectively. In this paper, we will treat two distinct representations of \( V \). Even though these definitions are already shown in [Li1], we repeat them for convenience.

**Definition 2.1.** A \( \mathbb{Z}_2 \)-graded \( V \)-module is a pair \( (M, Y_M) \) consisting of a \( \mathbb{Z}_2 \)-graded vector space \( M = M^0 \oplus M^1 \) which comes from a \( \frac{1}{2} \mathbb{N} \)-grading \( M^i = \bigoplus_{n \in \mathbb{N} + \frac{1}{2}} M^i(n) \) \( (i = 0, 1) \) and a linear map \( Y_M(\cdot, z) \) from \( V \) to \( \text{End}(M)[[z, z^{-1}]] \) satisfying the following conditions:

1° For any \( a \in V, v \in M, a_n v = 0 \) for \( n \) sufficiently large;

2° \( Y_M(\mathbb{1}, z) = id_M \);

3° \( a_n M(s) \subset M(s + \text{wt}(a) - n - 1) \);

4° For any \( \mathbb{Z}_2 \)-homogeneous \( a, b \in V \), the following Jacobi identity holds:
\[
\begin{align*}
\delta(z) = \left(\frac{z_1 - z_2}{z_0}\right) Y_M(a, z_1) Y_M(b, z_2) - (-1)^{p(a, b)} \delta(z_0) \left(\frac{-z_2 + z_1}{z_0}\right) Y_M(b, z_2) Y_M(a, z_1) \\
= \delta(z) Y_M(Y_V(a, z_0)b, z_2).
\end{align*}
\] (2.1)

**Definition 2.2.** A \(\mathbb{Z}_2\)-twisted \(V\)-module is a pair \((M, Y_M)\) consisting of an \(\mathbb{N}\)-graded vector space \(M = \bigoplus_{n \in \mathbb{N}} M(n)\) and a linear map \(Y_M(\cdot, z)\) from \(V\) to \(\text{End}(M)[[z^{\frac{1}{2}}, z^{-\frac{1}{2}}]]\) satisfying the following conditions:

1. For \(a \in V\), the module vertex operator has the shape \(Y_M(a, z) = \sum_{n \in \mathbb{N}} a_n z^{-n-1}\);
2. For any \(a \in V\), \(v \in M\), \(a_n v = 0\) for \(n \in \frac{1}{2}\mathbb{Z}\) sufficiently large;
3. \(Y_M(1, z) = \text{id}_M\);
4. \(a_n M(s) \subset M(s + \text{wt}(a) - n - 1)\);
5. For any \(\mathbb{Z}_2\)-homogeneous \(a, b \in V\), the following \(\mathbb{Z}_2\)-twisted Jacobi identity holds:

\[
\begin{align*}
\delta(z) Y_M(a, z_1) Y_M(b, z_2) - (-1)^{p(a, b)} \delta(z) Y_M(b, z_2) Y_M(a, z_1) \\
= \delta(z) Y_M(Y_V(a, z_0)b, z_2).
\end{align*}
\] (2.2)

### 2.2 \(\mathbb{Z}_2\)-twisted Zhu algebras

As in the case of VOAs, we can define the Zhu algebras for SVOAs. Since the representations of Zhu algebras correspond to those of original SVOAs, we can introduce two type of Zhu algebras for corresponding type of the representations. First, we consider Zhu algebras for the \(\mathbb{Z}_2\)-graded representations. The following definition is due to Kac-Wang [KW].

**Definition 2.3.** We define the bilinear maps \(* : V \otimes V \rightarrow V, \circ : V \otimes V \rightarrow V\) as follows.

\[
a * b := \begin{cases} 
\text{Res}_z Y(a, z) \left(\frac{1 + z}{z}\right)^{\text{wt}(a)} b & \text{if } a \in V^0, \\
0 & \text{if } a \in V^1,
\end{cases}
\]

\[
a \circ b := \begin{cases} 
\text{Res}_z Y(a, z) \left(\frac{1 + z}{z^2}\right)^{\text{wt}(a)} b & \text{for } a \in V^0, \\
\text{Res}_z Y(a, z) \left(\frac{1 + z}{z}\right)^{\text{wt}(a)-\frac{1}{2}} b & \text{for } a \in V^1.
\end{cases}
\]
Extend to $V \otimes V$ linearly, denote by $O(V) \subset V$ the linear span of elements of the form $a \circ b$, and by $A(V)$ the quotient space $V/O(V)$.

**Remark 2.4.** It follows from the definition that $a \circ 1 = a$ for $a \in V^1$ so that $V^1 \subset O(V)$. Notice that $O(V^0) \subset O(V)$ where $O(V^0)$ is the kernel of the Zhu algebra $A(V^0)$ for a VOA $V^0$. Therefore, $A(V)$ is a quotient algebra of $A(V^0)$.

In [KW] one can find the followings.

**Theorem 2.5.** *(Theorem 1.1, 1.2, 1.3 in [KW])*

1. $O(V)$ is a two-sided ideal of $V$ under the multiplication $\ast$. Moreover, the quotient algebra $(A(V), \ast)$ is associative.
2. $1 + O(V)$ is the unit element of $A(V)$ and $\omega + O(V)$ is in the center of $A(V)$.
3. Let $M = \oplus_{n \in \frac{1}{2} \mathbb{N}} M_n$ be a $\mathbb{Z}_2$-graded $V$-module. Then the top level $M_0$ is an $A(V)$-module via $a + O(V) \mapsto o(a) = a_{\text{wt}(a)-1}$.
4. Given an $A(V)$-module $(W, \pi)$, there exists a $\mathbb{Z}_2$-graded $V$-module $M = \oplus_{n \in \frac{1}{2} \mathbb{N}} M_n$ such that the $A(V)$-modules $M_0$ and $W$ are isomorphic. Moreover, this gives a bijective correspondence between the set of irreducible $A(V)$-modules and the set of irreducible $\mathbb{Z}_2$-graded $V$-modules.

We can define another Zhu algebra, the one for $\mathbb{Z}_2$-twisted representations as follow.

**Definition 2.6.** *(\$\mathbb{Z}_2$-twisted Zhu algebra)*

For $L_0$-homogeneous $a, b \in V$, define the bilinear maps $\ast_t : V \otimes V \to V$, $\circ_t : V \otimes V \to V$ as follows.

\[
\begin{align*}
a \ast_t b & := \text{Res}_z Y(a, z) \frac{(1 + z)^{\text{wt}(a)}}{z} b, \\
a \circ_t b & := \text{Res}_z Y(a, z) \frac{(1 + z)^{\text{wt}(a)}}{z^2} b.
\end{align*}
\]

Extend to $V \otimes V$ linearly, denote by $O_t(V) \subset V$ the linear span of elements of the form $a \circ_t b$, and by $A_t(V)$ the quotient space $V/O_t(V)$.

We will show

**Theorem 2.7.** *(1) The quotient space $(A_t(V), \ast_t)$ is a $\mathbb{Z}_2$-graded associative algebra with unit element $1 + O_t(V)$.\n
2. Let $M = \oplus_{n \in \mathbb{N}} M(n)$ be a $\mathbb{Z}_2$-twisted V-modules. Then the top level $M(0)$ of $M$ is an $A_t(V)$-module under the action $a + O_t(V) \mapsto o(a) = a_{\text{wt}(a)-1}$ for homogeneous $a \in V$.\n
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(3) Let $(W, \pi)$ be an irreducible $A_r(V)$-module. Then there exists an irreducible \( \mathbb{Z}_2 \)-twisted $V$-module $M = \oplus_{n \in \mathbb{N}} M(n)$ such that the top level $M(0)$ of $M$ is isomorphic to $W$ as $A_r(V)$-modules.

Proof: Since we can prove the above statements by the same argument used in [DLM1], we give a slight one here. To prove (1), using the associativity of a vertex algebra, we can make proof very simpler. Let $A$ be an associative algebra $\mathbb{C}[\frac{1}{2}, t^{-\frac{1}{2}}]$. Then $(A, Y_A, 1, \frac{d}{dt})$ is a vertex algebra on which the vertex operator is defined by $Y(t^s, z) = e^{\frac{2\pi}{\sqrt{-1}} t^s} (t + z)^s$. $A$ has a vertex algebra grading $A = \oplus_{s \in \mathbb{Q}_Z} A_s$ defined by $\text{wt}(t^s) := -s$. Let $\hat{V} := A \otimes_{\mathbb{C}} V$ be a tensor product of vertex (super)algebras. It is clear that $(\hat{V}, Y_A \otimes Y_V, 1 \otimes 1, \frac{d}{dt} \otimes 1 + 1 \otimes L_{-1})$ is a vertex superalgebra and $\hat{V}$ also carries a $\frac{1}{2} \mathbb{Z}$-grading $\hat{V} = \oplus_{n \in \frac{1}{2} \mathbb{Z}} \hat{V}(n)$, where $\hat{V}(n) = \oplus_{i+j=n} A_i \otimes V_j$. Define a linear isomorphism $\hat{\theta}$ of $\hat{V}$ by $\hat{\theta}(t^s \otimes a) := (-1)^{q(a)} e^{2\pi i s \cdot t^s} a$. Then one can easily check that $\hat{\theta}$ is an isomorphism of vertex superalgebra $\hat{V}$. Take $\theta$-invariants of $\hat{V}$ and denote it by $\hat{V}^\theta$. It is obvious that $\hat{V}^\theta$ admits a $\mathbb{Z}$-grading decomposition

$$
\hat{V}^\theta = \bigoplus_{n \in \mathbb{Z}} \hat{V}^\theta(n), \quad \hat{V}^\theta(n) = \bigoplus_{i+j=n} A_i \otimes V_j.
$$

A subspace $\hat{V}^\theta(0)$ is an algebra with unit element $1 \otimes 1$ under the multiplication $X \cdot Y := X \bullet_{-1} Y$ for $X, Y \in \hat{V}^\theta(0)$, where $\bullet_{n}$ denotes the $n$-th product in $\hat{V}$. As a linear space, $\hat{V}^\theta(0) = \oplus_{s \in \frac{1}{2} \mathbb{Z}_+} A_{-s} \otimes V_s$ is isomorphic to $V$ under the mapping $t^{\text{wt}(a)} \otimes a \mapsto a$, so we can identify them. Then $O_t(V)$ is isomorphic to $\hat{V}^\theta(0) \bullet_{-2} \hat{V}^\theta(-1)$, i.e.

$$
a \circ_t b = \text{Res}_z Y(a, z)b \frac{(1 + z)^{\text{wt}(a)}}{z^2} 
\sim \text{Res}_z Y(a, z)(t + z)^{\text{wt}(a)} \cdot \frac{(t + z)^{\text{wt}(b)+1}b}{z^2} = (t^{\text{wt}(a)} \otimes a) \bullet_{-2} (t^{\text{wt}(b)+1} \otimes b),
$$

under the identification. Therefore, $A_t(V)$ is linearly isomorphic to $\hat{V}^\theta(0)/K$, $K = \hat{V}^\theta(0) \bullet_{-2} \hat{V}^\theta(-1)$. Furthermore, under the identification, the product $a \ast_t b$ in $A_t(V)$ corresponds to $(t^{\text{wt}(a)} \otimes a) \bullet_{-1} (t^{\text{wt}(b)+1} \otimes b)$. So to prove (1), we should show that $(\hat{V}^\theta(0)/K, \bullet_{-1})$ is an associative algebra. To show this, we need the following simple lemma which is a direct consequence of the definition of the tensor product of SVAs.

Lemma 2.8. (Lemma 2.1.2 in [2]) For any $n \in \mathbb{Z}$, $i, j \in \mathbb{N}$, we have the followings.

$$
\hat{V}^\theta(-i) \bullet_{-2-i} \hat{V}^\theta(n) \subset \hat{V}^\theta(0) \bullet_{-2} \hat{V}^\theta(n),
$$

$$
\hat{V}^\theta(0) \bullet_{-2-j} \hat{V}^\theta(n - j) \subset \hat{V}^\theta(0) \bullet_{-2} \hat{V}^\theta(n).
$$

In particular, $\hat{V}^\theta(-i) \bullet_{-2-i-j} \hat{V}^\theta(-1 - j) \subset K$ holds for any $i, j \in \mathbb{N}$. 

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Let \( E, F, G \in \hat{V}^\theta(0) \) and \( H \in \hat{V}^\theta(-1) \). To prove that \( K \) is an two-sided ideal of \( \hat{V}^\theta(0) \), it is suffice to show that \( E_{-1}F_{-2}H \) and \( (F_{-2}H)_{-1}E \) belong to \( K \). Note that \( \hat{V}^\theta \) is a sub SVA of \( \hat{V} \). The commutativity and associativity of SVA and Lemma 2.8 lead
\[
E_{-1}F_{-2}H = (-1)^{p(E,F)}F_{-2}E_{-1}H + [E_{-1},F_{-2}]H = (-1)^{p(E,F)}F_{-2}E_{-1}H + \sum_{i=0}^{\infty} \binom{-1}{i} (E_iF)_{-3-i}H \in K,
\]
\[
(F_{-2}H)_{-1}E = \sum_{i=0}^{\infty} (-1)^i \binom{-2}{i} \{ F_{-2-i}H_{-1+i} - (-1)^{p(F,H)}H_{-3-i}F_i \} E \in K.
\]
Thus \( K \) is a two-sided ideal of \( \hat{V}^\theta(0) \). Similarly, we have
\[
E_{-1}F_{-1}G - (E_{-1}F)_{-1}G
= E_{-1}F_{-1}G - \sum_{i=0}^{\infty} (-1)^i \binom{-1}{i} \{ E_{-1-i}F_{-1+i} - (-1)^{p(E,F)}F_{-2-i}E_i \} G
= -\sum_{i=1}^{\infty} E_{-1-i}F_{-1+i}G + \sum_{i=0}^{\infty} (-1)^{p(E,F)-1}F_{-2-i}E_i G \in W.
\]
Therefore \( \hat{V}^\theta(0)/K \) is associative. Since \( O_t(V) \) has a natural \( \mathbb{Z}_2 \)-grading induced from \( V \), \( A_t(V) = V/O_t(V) \) also has a natural \( \mathbb{Z}_2 \)-grading induced from \( V \). This proves (1).

(2) is similar to Theorem 2.1.2 of [Z] so that we consider (3). Denote the derivation operator \( \frac{d}{dt} \otimes 1 + 1 \otimes L_{-1} \) on \( \hat{V} \) by \( D \). Then \( (\hat{V}^\theta/D\hat{V}^\theta, \cdot_0) \) becomes a Lie superalgebra (cf. [Bo]). Denote this Lie superalgebra by \( g(V) \). Defining a \( \mathbb{Z} \)-gradation on \( g(V) \) by \( \deg (t^n \otimes a + D\hat{V}^\theta) := \text{wt}(a) - n - 1 \), we obtain a \( \mathbb{Z} \)-gradation decomposition \( g(V) = \oplus_{n \in \mathbb{Z}} g(V)_n \) of Lie superalgebra and using the grading we have a triangular decomposition of a Lie superalgebra \( g(V) \) as follow.
\[
g(V) = g(V)^- \oplus g(V)^0 \oplus g(V)^+,
\]
where \( g(V)^\pm = \oplus_{n>0} g(V)^n \) and \( g(V)^0 = g(V)_0 \). Let \( U(g(V)) \) be the universal enveloping algebra for \( g(V) \). Replacing \( U(V[g]) \) in [DLM1] by \( U(g(V)) \), we can prove (3) by exactly the same argument as that in [DLM1].

Remark 2.9. One can show that the image of \( \omega \) in \( A_t(V) \) is in the center of \( A_t(V) \) by a direct calculation. But our “affinization argument” can’t cover this proof.

Remark 2.10. Since \( V \) has a canonical involution \( \sigma \) which is identical on \( V^0 \) and acts as \(-1\) on \( V^1 \), we can consider a \( \sigma \)-twisted \( V \)-module, which is exactly a \( \mathbb{Z}_2 \)-twisted \( V \)-module in our notation. Contrary to the terminology ‘twisted’, it seems that \( \mathbb{Z}_2 \)-twisted Zhu algebras for SVOAs correspond to non-twisted Zhu algebras for VOAs and non-twisted ones for SVOAs correspond to twisted ones for VOAs.
2.3 Frenkel-Zhu’s bimodules and intertwining operators

In this subsection we define a bimodule $A_t(U)$ of $A_t(V)$ for every $\mathbb{Z}_2$-graded $V$-module $U$ as a generalization of [FZ] and [KW]. Then we give a description of the fusion rules among $\mathbb{Z}_2$-twisted modules in terms of $A_t(U)$.

Let $M^1$, $M^2$, $M^3$ be irreducible $\mathbb{Z}_2$-graded $V$-modules with $\frac{1}{2}\mathbb{N}$-grading $M^i = \oplus_{n \in \mathbb{N}} M^i(n)$ and $\mathbb{Z}_2$-grading $(M^i)^r = \oplus_{n \in \mathbb{N}} M^i(n + \frac{r}{2})$ for $i = 1, 2, 3$ and $r = 0, 1$, respectively. It is known that all irreducible $V$-modules admit the $L_0$-weight space decomposition so that we can find some $h_i \in \mathbb{C}$ such that $M^i(n) = M^i_n+h_i$ for $i = 1, 2, 3$, respectively, where $X_s$ denotes the $L_0$-weight space of $X$ with weight $s \in \mathbb{C}$.

**Definition 2.11.** Under the above setting, a $\mathbb{Z}_2$-graded intertwining operator of type $(M^3, M^3)$ is a linear map

$$I(\cdot, z) : u \in M^1 \mapsto I(u, z) = \sum_{s \in \mathbb{C}} u_s z^{-s-1} \in \text{Hom}_\mathbb{C}(M^2, M^3)\{z\}$$

satisfying the following conditions:

1° For every $v \in M^1$, $I(u, z) \in \text{Hom}_\mathbb{C}(M^2, M^3)\{z\}$ are $\mathbb{Z}_2$-homogeneous;

2° For any $s \in \mathbb{C}$, $u \in M^1$ and $v \in M^2$, $u_{s+N}v = 0$ for sufficiently large $N \in \mathbb{N}$;

3° $L_{-1}$-derivation: $I(L_{-1}u, z) = \frac{d}{dz}I(u, z)$;

4° For $\mathbb{Z}_2$-homogeneous $a \in V$ and $u \in M^1$, the following Jacobi identity holds:

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_{M^3}(a, z_1)I(u, z_2) - \frac{1}{2} p(a, u) z_0^{-1} \delta \left( -\frac{z_2 + z_1}{z_0} \right) I(u, z_2) Y_{M^2}(a, z_2) = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) I(Y_{M^1}(a, z_0)u, z_2),$$

where the pairing $p(\cdot, \cdot) : V \times U \to \mathbb{Z}_2$ is understood appropriately.

Frenkel-Zhu’s bimodules for SVOAs are introduced by Kac-Wang in [KW].

**Definition 2.12.** For a $\mathbb{Z}_2$-graded $V$-module $U$, we define bilinear operations $a \circ u$, $a * u$ and $u * a$, for $a \in V$ homogeneous and $u \in U$, as follows

$$a \circ u := \text{Res}_2 Y(a, z) \left( \frac{1 + z}{z} \right)^{\text{wt}(a)-\frac{r}{2}} u, \quad \text{for } a \in V^r,$$

$$a * u := \text{Res}_2 Y(a, z) \left( \frac{1 + z}{z} \right)^{\text{wt}(a)} u, \quad \text{for } a \in V^0,$$

$$u * a := \text{Res}_2 Y(a, z) \left( \frac{1 + z}{z} \right)^{\text{wt}(a)-1} u, \quad \text{for } a \in V^0,$$

$$a * u = u * a = 0, \quad \text{for } a \in V^1$$
and extend linearly. We also define $O(U) \subset U$ to be the linear span of elements of the form $a \circ u$ and $A(U)$ to be the quotient space $U^0/(O(U) \cap U^0)$.

**Remark 2.13.** Our definition of $A(U)$ differs from Kac-Wang’s original one. Namely, we define $A(U)$ to be a quotient space of $U^0$. See [K] for the validity of this change.

By definition, we know that $z^{h_1+h_2-h_3} I(\cdot, z) \in \text{Hom}_\mathbb{C}(M^2, M^3)[[z, z^{-1}]]$. It is convenient to set $I(u, z) = \sum_{n \in \mathbb{Z}} u(n) z^{-n-1-h_1-h_2+h_3}$ and $\deg(u) := \text{wt}(u) - h_1$ for $u \in M^1$ to state the following theorems which are shown in [KW].

**Theorem 2.14.** (Theorem 1.4 in [KW]) $A(U)$ is an $A(V)$-bimodule under the action $\ast$.

**Theorem 2.15.** (1) (Theorem 1.5 in [KW]) For a $\mathbb{Z}_2$-graded intertwining operator $I(\cdot, z)$ of type $(M^2_{U, M^1})$, its zero-mode action $o^I(u) := u_{(\deg(u)-1)}$ gives a linear injection from $I(M^2_{U, M^1})$ to $\text{Hom}_{A(V)} (A(U) \otimes_{A(V)} M^1(0), M^2(0))$.

(2) (Theorem 2.11 in [Li2]) Suppose that every $\mathbb{Z}_2$-graded $V$-module is completely reducible. Then the linear map $I(\cdot, z) \mapsto o^I$ given in (1) defines a linear isomorphism of vector spaces $I(M^2_{U, M^1})$ and $\text{Hom}_{A(V)} (A(U) \otimes_{A(V)} M^1(0), M^2(0))$.

**Remark 2.16.** As pointed out in [Li2], the assumption on completely reducibility in the above statement (2) is necessary. For details, see [Li2].

We extend the above results to $\mathbb{Z}_2$-twisted case. We begin with introducing the notion of $\mathbb{Z}_2$-twisted intertwining operators. Let $U$ be an irreducible $\mathbb{Z}_2$-graded $V$-module and $W^1, W^2$ be irreducible $\mathbb{Z}_2$-twisted $V$-modules with the weight space decompositions $U = \bigoplus_{n \in \mathbb{Z}} U_{n+h_0}$ and $W^i = \bigoplus_{n \in \mathbb{Z}} W^i_{n+h_i}$ for $i = 1, 2$, respectively.

**Definition 2.17.** Under the above setting, a $\mathbb{Z}_2$-twisted intertwining operator $I(\cdot, z)$ of type $(W^2_{U, W^1})$ is a linear map $I(\cdot, z) : u \in U \mapsto I(u, z) = \sum_{s \in \mathbb{C}} u_s z^{-s-1} \in \text{Hom}_\mathbb{C}(W^1, W^2)\{z\}$ satisfying the following conditions:

1° For $u \in U^r$, $I(u, z) \in \text{Hom}_\mathbb{C}(W^1, W^2)[[z, z^{-1}]]z^{-\frac{2}{r}-h_0-h_1+h_2}$, where $r = 0, 1$;

2° For any $s \in \mathbb{C}$, $u \in U$ and $w \in W^1$, $u_{s+N} w = 0$ for sufficiently large $N \in \mathbb{N}$;

3° $L_{-1}$-derivation: $I(L_{-1}u, z) = \frac{d}{dz} I(u, z)$;
4° For $\mathbb{Z}_2$-homogeneous $a \in V$ and $u \in U$, the following $\mathbb{Z}_2$-twisted Jacobi identity holds:

$$
\frac{z_1 - z_2}{z_0} Y_{W^2}(a, z_1) I(u, z_2) - (-1)^{p(a,u)} \frac{z_1 + z_2}{z_0} I(u, z_2) Y_{W^1}(a, z_1)
$$

$$
= \frac{z_1 - z_0}{z_2} \left( \frac{z_1 - z_0}{z_2} \right)^{\frac{p(a)}{2}} I(Y_{U}(a, z_0)u, z_2).
$$

As we did previously, we set $I(u, z) = \sum_{n \in \mathbb{Z}^{+}} u_{(n)} z^{-n-1-h_0 - h_1 + h_2}$ and $\text{deg}(u) := \text{wt}(u) - h_0$ for $u \in U^r$.

**Remark 2.18.** For a $\mathbb{Z}_2$-twisted $V$-module $W$, the module vertex operator $Y_W(\cdot, z)$ is a $\mathbb{Z}_2$-twisted intertwining operator of type $(t^W_{t^V})$ by definition.

We define the following products.

**Definition 2.19.** For a $\mathbb{Z}_2$-graded $V$-module $U$, we define bilinear operations $a \circ_t u$, $a \ast_t u$ and $u \ast_t a$, for homogeneous $a \in V$ and $u \in U$, as follows

$$a \circ_t u := \text{Res}_z Y(a, z) \frac{(1 + z)^{\text{wt}(a)}}{z^2} u,$$

$$a \ast_t u := \text{Res}_z Y(a, z) \frac{(1 + z)^{\text{wt}(a)}}{z} u,$$

$$a \ast_t u := (-1)^{p(a,u)} \text{Res}_z Y(a, z) \frac{(1 + z)^{\text{wt}(a)-1}}{z} u,$$

and extend linearly. We also define $O_t(U) \subset U$ to be the linear span of elements of the form $a \circ_t u$ and $A_t(U)$ to be the quotient space $U/O_t(U)$.

As an analogy of Theorem 2.14 and 2.15, we have the followings.

**Theorem 2.20.** (1) $A_t(U)$ is an $A_t(V)$-bimodule under the action $\ast_t$.

(2) For a $\mathbb{Z}_2$-twisted intertwining operator $I(\cdot, z)$ of type $(t^W_{t^V})$, its zero-mode action $\sigma^i(u) := u_{(\text{deg}(u)-1)}$ gives a linear injection from $I(t^W_{t^V})$ to $\text{Hom}_{A_t(V)} (A_t(U) \otimes_{A_t(V)} W^1(0), W^2(0))$.

(3) Suppose that every $\mathbb{Z}_2$-twisted $V$-module is completely reducible. Then the linear map $I(\cdot, z) \mapsto \sigma^i$ given in (2) defines a linear isomorphism of linear spaces $I(t^W_{t^V})$ and $\text{Hom}_{A_t(V)} (A_t(U) \otimes_{A_t(V)} W^1(0), W^2(0))$.

**Proof:** One can find similar proof in [Li2] and [Y].

**Remark 2.21.** In order to prove the above assertions rigorously, we have to introduce some cocycles for Lie superalgebra associated to an affinized SVOA. However, such modifications can be cleared by a little attentions.
3 Some facts on SVOAs

In this section we give some notes on the basic property of SVOA and its representations.

3.1 Invariant bilinear form

Let $V$ be an SVOA and $M = \oplus_{n \in \frac{1}{2} \mathbb{N}} M(n)$ be its module. We can find a natural $V$-module structure in the dual space $M^* = \oplus_{n \in \frac{1}{2} \mathbb{N}} M(n)^*$.

**Definition 3.1.** For the restricted dual space $M^*$ of a $\mathbb{Z}_2$-graded $V$-module $M$, we define the adjoint vertex operators $Y^*(a, z)$ by means of the linear map

$$V \ni a \mapsto Y^*(a, z) = \sum_{n \in \mathbb{Z}} a_n^* z^{-n-1} \in \text{End}(M^*)[[z, z^{-1}]]$$

determined by the condition

$$\langle Y^*(a, z)f|v \rangle := \langle f|Y(e^{zL_1}(\lambda z^{-2})L_0 \lambda^{-2L_0^2}a, z^{-1})v \rangle \quad (3.1)$$

for $a \in V$, $f \in M^*$ and $v \in M$, where $\lambda = e^{\pm \pi i}$.

**Remark 3.2.** In the above definition, it seems that we have two definitions according to the choice of the square root of unity $\lambda$, but each choice determines the same adjoint vertex operators since we have assumed that $V$ has $\frac{1}{2} \mathbb{Z}$-grading. So we may choose each root of unity.

Similar to the case of VOAs, we have the following.

**Proposition 3.3.** (*Theorem 5.2.1 in [FHL]*) $(M^*, Y^*)$ is a $\mathbb{Z}_2$-graded $V$-module. Moreover, if each homogeneous space $M(n)$ of $M$ is finite dimensional, then $(M^*)^* \simeq M$.

The proof is the same as that in [FHL]. One can also show the following.

**Proposition 3.4.** (1) (*Proposition 5.3.6 in [FHL]*) Assume that $V$ as $V$-module is isomorphic to $V^*$. Then the natural pairing $V \times V \to \mathbb{C}$ is automatically symmetric.

(2) (*Theorem 3.1 in [Li3]*) The space of invariant forms on $V$ is linearly isomorphic to $	ext{Hom}_{\mathbb{C}}(V_0/L_1V_1, \mathbb{C})$. Therefore, the existence of invariant forms on $V$ is equivalent to the existence of invariant forms on sub VOA $V^0$ of $V$.

We can also introduce the dual module for every $\mathbb{Z}_2$-twisted $V$-module in similar way. However, we won’t give the details here.
3.2 \( \mathbb{Z}_2 \)-conjugacy

Every SVOA \( V = V^0 \oplus V^1 \) has a canonical involution \( \sigma \) which is identical on \( V^0 \) and acts as \(-1\) on \( V^1 \), so we can think of the \( \sigma \)-conjugation of \( V \)-modules. That is, for a \( V \)-module \((M,Y_M)\), define another vertex operator \( Y_M^\sigma \) by

\[
Y_M^\sigma(a,z) := Y_M(\sigma a,z).
\]

Then \((M,Y_M^\sigma)\) is also a \( V \)-module and we will denote it simply by \( M^\sigma \). However, \( \sigma \)-conjugation of a \( \mathbb{Z}_2 \)-graded \( V \)-module is a trivial concept because every \( \sigma \)-conjugate is isomorphic to original \( \mathbb{Z}_2 \)-graded \( V \)-module as one can easily see. But, as the following proposition insists, the \( \sigma \)-conjugation of \( \mathbb{Z}_2 \)-twisted \( V \)-modules is not a trivial concept.

**Proposition 3.5.** Let \( V = V^0 \oplus V^1 \) be a simple SVOA with \( V^1 \neq 0 \) and \( W \) be an irreducible \( \mathbb{Z}_2 \)-twisted \( V \)-module. Then one of the followings holds:

1. \( W \) and \( W^\sigma \) are non-isomorphic \( V \)-modules if \( W \) is irreducible as \( V^0 \)-module.

2. If \( W \) is not an irreducible \( V^0 \)-module, then \( W \) is completely reducible \( V^0 \)-module and it has two irreducible components. Write \( W = N^1 \oplus N^2 \), then we have \( V^1 \cdot N^1 = N^2 \) and \( V^1 \cdot N^2 = N^1 \). Therefore, \( W \) has a \( \mathbb{Z}_2 \)-grading under the action of \( V \) and the conjugate module \( W^\sigma \) is isomorphic to \( W \) as \( V \)-module.

**Proof:** Assume that \( W \) as \( V^0 \)-module is irreducible and \( W \) and \( W^\sigma \) are isomorphic as \( V \)-modules. Then we can find a \( V \)-isomorphism \( f : W \to W^\sigma \). Clearly, \( f \) is also a \( V^0 \)-isomorphism. By definition, we have a linear map \( \phi : W \to W^\sigma \) satisfying \( \phi Y(a,z)w = Y(\sigma a,z)\phi w \) for \( a \in V \) and \( w \in W \). Then the linear map \( \phi^{-1}f \) becomes a \( V^0 \)-isomorphism of \( W \) so that we can find some non-zero scalar \( \alpha \in \mathbb{C} \) such that \( \phi^{-1}f = \alpha \) by Schur’s lemma. Let \( a \in V^1 \) and \( w \in W \) be non-zero elements. It is well-known that \( Y(a,z)w \neq 0 \) for simple SVOAs so that we get \( 0 \neq f(Y(a,z)w) = \alpha \phi Y(a,z)w = -\alpha Y(a,z)\phi(w) = -Y(a,z)f(w) = -f(Y(a,z)w) \), a contradiction. Therefore \( W \) and \( W^\sigma \) are not isomorphic \( V \)-modules. This proves (1) and so we may assume that \( W \) is not an irreducible \( V^0 \)-module. Let \( N \) be a proper \( V^0 \)-submodule of \( W \). Then the associativity implies \( V^1 \cdot N \) is also \( V^0 \)-submodule of \( W \) and we have \( N \cap (V^1 \cdot N) = 0 \) since it is a proper \( V \)-submodule of \( W \). Therefore \( W \) contains \( N \oplus V^1 \cdot N \) and by the irreducibility we obtain \( W = N \oplus V^1 \cdot N \). This implies that any proper \( V^0 \)-submodule \( N \) of \( W \) is irreducible \( V^0 \)-module. Therefore \( W \) has a \( \mathbb{Z}_2 \)-grading under the action of \( V^0 \oplus V^1 \) and so the conjugate \( W^\sigma \) is isomorphic to \( W \) as \( V \)-module. \( \blacksquare \)
4 Ising model SVOA

In this section we will give an explicit construction of the Ising model SVOA $L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$ and its $\mathbb{Z}_2$-twisted modules $L(\frac{1}{2}, \frac{1}{10})^\pm$. By calculating Zhu algebras explicitly, we shall prove that $L(\frac{1}{2}, \frac{1}{10})^\pm$ are all irreducible $\mathbb{Z}_2$-twisted modules for this SVOA. This construction is well-known and the most of contents in Sec. 4.1-4.2 can be found in [FRW1] and [FRW2].

4.1 Realization of Ising models

Here we consider a certain realizations of unitary highest weight representations of Virasoro algebras of central charge $\frac{1}{2}$. There are exactly three unitary representations $L(\frac{1}{2}, 0)$, $L(\frac{1}{2}, \frac{1}{2})$ and $L(\frac{1}{2}, \frac{1}{16})$, which are often called the Ising models. First two ones are realized as follow.

Let $\mathcal{A}_\psi$ be the algebra generated by $\{\psi_k | k \in \mathbb{Z} + \frac{1}{2}\}$ subject to the defining relations

$$[\psi_m, \psi_n]_+ := \psi_m \psi_n + \psi_n \psi_m = \delta_{m+n,0}, \quad m, n \in \mathbb{Z} + \frac{1}{2},$$

and denote a subalgebra of $\mathcal{A}_\psi$ generated by $\{\psi_k | k \in \mathbb{Z} + \frac{1}{2}, k > 0\}$ by $\mathcal{A}_\psi^+$. Let $\mathbb{C} \mathbf{1}$ be a trivial $\mathcal{A}_\psi^+$-module. Define a canonical induced $\mathcal{A}_\psi$-module $M$ by

$$M := \text{Ind}_{\mathcal{A}_\psi^+}^{\mathcal{A}_\psi} \mathbb{C} \mathbf{1} = \mathcal{A}_\psi \otimes \mathbb{C} \mathbf{1}.$$ 

As well-known, we can find Virasoro module structure in $M$. Following [KR], set

$$L_n := \frac{1}{2} \sum_{k > -n/2} (n + 2k) \psi_{-k} \psi_{n+k}, \quad (n \in \mathbb{Z}).$$

Then $\{L_n | n \in \mathbb{Z}\}$ gives a representation of Virasoro algebra of central charge $\frac{1}{2}$ on $M$ and as Virasoro module $M$ is decomposed as follow

$$M = L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}),$$

where $L(c, h)$ denotes the irreducible highest weight module of central charge $c$ with highest weight $h$. The component $L(\frac{1}{2}, 0)$ is generated by $\mathbf{1}$, whereas $L(\frac{1}{2}, \frac{1}{2})$ is generated by $\psi_{-\frac{1}{2}} \mathbf{1}$ under the Virasoro and it is also clear that the above decomposition coincides with the standard $\mathbb{Z}_2$-grading decomposition, i.e.

$$L\left(\frac{1}{2}, \frac{1}{2}\right) = \mathbb{C} \langle \psi_{-n_1} \cdots \psi_{-n_k} \mathbf{1} | n_j > 0, \ n_1 + \cdots + n_k \in \mathbb{Z} + \frac{1}{2} \rangle.$$
for \( i = 0, 1 \). Furthermore, one can introduce a symmetric contravariant Hermitian form \( \langle \cdot | \cdot \rangle \) on \( M \) such that \( \langle 1 | 1 \rangle = 1 \) and \( \langle \psi_n a | b \rangle = \langle a | \psi_{-n} b \rangle \). We also note that the above basis forms an orthonormal basis for \( (M, \langle \cdot | \cdot \rangle) \).

Similarly, \( L(\frac{1}{2}, \frac{1}{16}) \) shall be realized as follow. Let \( A_\psi \) be the other algebra generated by \( \{ \phi_n | n \in \mathbb{Z} \} \) whose defining relations are

\[
[\phi_m, \phi_n]_+ = \delta_{m+n,0}, \quad m, n \in \mathbb{Z}.
\]

Let \( A_\phi^+ \) be a subalgebra of \( A_\psi \) generated by \( \{ \phi_n | n > 0 \} \) and denote a trivial 1-dimensional \( A_\phi^+ \)-module by \( Cv_0 \). Then set \( N = \text{Ind}_{A_\phi^+}^{A_\phi} Cv_0 \) as we did previously. Again we can find Virasoro representation in \( N \). Set

\[
L'_n := \frac{1}{16} \delta_{n,0} + \frac{1}{2} \sum_{k > -n/2} (n + 2k) \phi_{-k} \phi_{n+k}, \quad (n \in \mathbb{Z}).
\]

Then \( \{ L'_n | n \in \mathbb{Z} \} \) satisfies the Virasoro relation of central charge \( \frac{1}{2} \) on \( N \). There are two distinct highest weight vectors \( v_0 \) and \( \phi_0 v_0 \) with highest weight \( \frac{1}{2} \) in \( N \) and as Virasoro module \( N \) decomposes as follow (cf. [KR]):

\[
N = L(\frac{1}{2}, \frac{1}{16}) \oplus L(\frac{1}{2}, \frac{1}{16}),
\]

where one of \( L(\frac{1}{2}, \frac{1}{16}) \) is generated by \( v_0 \) and the other one is generated by \( \phi_0 v_0 \) under the Virasoro. One can also introduce a symmetric contravariant Hermitian form \( \langle \cdot | \cdot \rangle \) on \( N \) such that \( \langle v_0 | v_0 \rangle = 1 \), \( \langle v_0 | \phi_0 v_0 \rangle = \langle \phi_0 v_0 | v_0 \rangle = 0 \) and \( \langle \phi_n a | b \rangle = \langle a | \phi_{-n} b \rangle \).

### 4.2 SVOA structure on Ising models

We keep the same notation as previous. By its construction, \( M \) is generated by \( 1 \) over \( A_\psi \). Define the generating series

\[
\psi(z) := \sum_{n \in \mathbb{Z}} \psi_{n+\frac{1}{2}} z^{-n-1}.
\]

Since \([\psi(z), \psi(w)]_+ = z^{-1} \delta(\frac{w}{z})\), \( \psi(z) \) is local with itself and it follows from the defining relations of \( A_\psi \) that \( \psi(z) \) satisfies \( L_{-1} \)-derivation property \( [L_{-1}, \psi(z)] = \frac{d}{dz} \psi(z) \). Therefore we can consider a subalgebra of a local system on \( M \) generated by \( \psi(z) \) and \( I(z) = \text{id}_M \) (cf. [Li1]). By a direct calculation, one sees that

\[
\frac{1}{2} \psi(z) \circ_{-2} \psi(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},
\]

where \( \circ_n \) denotes the \( n \)-th normal product (cf. [Li1]). That is, the coefficients in the left hand side of the above equality coincide with \( \langle 1 | 1 \rangle \). Recall that \( \psi_{-n_1+\frac{1}{2}} \psi_{-n_2+\frac{1}{2}} \cdots \psi_{-n_k+\frac{1}{2}} 1 \),
$n_1 > n_2 > \cdots > n_k > 0$, $k \geq 0$ form a basis of $M$. We shall define a vertex operator of each base $\psi_{-n_1+\frac{1}{2}} \psi_{-n_2+\frac{1}{2}} \cdots \psi_{-n_k+\frac{1}{2}}$ on $M$. For $k = 0$ we set $Y(\mathbb{1}, z) := \text{id}_M$ and inductively we define $Y(\psi_{-n+\frac{1}{2}}, z) := \psi(z) \circ_n Y(a, z)$. Then by the theory of the local system, we have the following well-known statement.

**Theorem 4.1.** (Theorem 2 in [FRW3]) By the above definition, $(L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}), Y(\cdot, z), 1 \frac{1}{2} \psi_{-\frac{1}{2}}, \psi_{-\frac{1}{2}} \mathbb{1})$ has a simple (and unique) SVOA structure with $L(\frac{1}{2}, 0)$ even part and $L(\frac{1}{2}, \frac{1}{2})$ odd part.

Note that the invariant bilinear form defined by (3.1) coincides with the contravariant bilinear form on $M$ since $\psi_n^* = \psi_{-n}$ for all $n \in \mathbb{Z} + \frac{1}{2}$. Next, we consider $L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$-module structure in $L(\frac{1}{2}, \frac{1}{16})$. Recall that in $N$ there are two highest weight vectors $v_0$ and $\phi_0 v_0$ and each of them generates $L(\frac{1}{2}, \frac{1}{16})$ under the Virasoro algebra. Since they are not stable under $\phi_0$, we have to change the highest weight vectors to suitable ones. We define $v^\pm_{\frac{1}{16}} := \phi_0 \mathbb{1} \pm \frac{1}{\sqrt{2}} \mathbb{1}$ to have $\phi_0 \cdot v^\pm_{\frac{1}{16}} = \pm \frac{1}{\sqrt{2}} v^\pm_{\frac{1}{16}}$. Then both $v^+_{\frac{1}{16}}$ and $v^-_{\frac{1}{16}}$ are highest weight vectors and each of them generates $L(\frac{1}{2}, \frac{1}{16})$ under the Virasoro. Denote the Virasoro modules generated by $v^\pm_{\frac{1}{16}}$ by $L(\frac{1}{2}, \frac{1}{16})^\pm_\phi$, respectively. Then we have $N = L(\frac{1}{2}, \frac{1}{16})^+ \oplus L(\frac{1}{2}, \frac{1}{16})^-$. Note that $L(\frac{1}{2}, \frac{1}{16})^+$ and $L(\frac{1}{2}, \frac{1}{16})^-$ are isomorphic as Virasoro modules but they are not isomorphic to each other as $\mathcal{A}_\phi$-modules. Consider the generating series

$$\phi(z) := \sum_{n \in \mathbb{Z}} \phi_n z^{-n-\frac{1}{2}}.$$ 

By direct calculations one can show that $\phi(z)$ is local with itself and the derivation property $[L'_1, \phi(z)] = \frac{d}{dz} \phi(z)$ holds. We realize Virasoro operator (4.2) by using $\phi(z)$. Since the powers of $z$ in $\phi(z)$ lie in $\mathbb{Z} + \frac{1}{2}$, we have to use the twisted normal product. Define a generating series $L(\cdot)$ of operators on $N$ by

$$L(z) := \frac{1}{2} \text{Res}_{z_0} \text{Res}_{z_1} z_0^{-2} \left( \frac{z_1 - z_0}{z_2} \right)^{\frac{1}{2}} \times \left\{ z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) \phi(z_1) \phi(z_2) + z_0^{-1} \delta \left( \frac{-z_2 + z_1}{z_0} \right) \phi(z_2) \phi(z_1) \right\}.$$ 

The following is a consequence of a direct computation.

**Lemma 4.2.** $L(z) = \sum_{n \in \mathbb{Z}} L'_n z^{-n-2}$, where $L'_n$ is defined by (4.2).

Thanks to the above lemma, we can find a $\mathbb{Z}_2$-twisted $L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$-module structure in $L(\frac{1}{2}, \frac{1}{16})^\pm$. To show it, we associate a vertex operator on $N$ for every element of $L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$ and then we prove that our association gives a homomorphism of vertex
superalgebras. Set $Y_N(1, z) := \text{id}_N$ and define inductively a vertex operator of $\psi_{-n+\frac{1}{2}}a$ on $N$ by

$$Y_N(\psi_{-n+\frac{1}{2}}a, z) := \frac{1}{2} \text{Res}_{z_0} \text{Res}_{z_1} z_0^{-n} \left( \frac{z_1 - z_0}{z_2} \right)^{\frac{1}{2}}$$

$$\times \left\{ z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) \phi(z_1) Y_N(a, z_2) - (-1)^{q(a)} z_0^{-1} \delta \left( \frac{-z_2 + z_1}{z_0} \right) Y_N(a, z_2) \phi(z_1) \right\},$$

where $a = \psi_{-n_1+\frac{1}{2}} \cdots \psi_{-n_k+\frac{1}{2}} 1$, $n > n_1 > \cdots > n_k > 0$ and extend linearly on $L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$. Let $\mathfrak{A}$ be a $\mathbb{Z}_2$-twisted local system on $N$ in which $\phi(z)$ contained. It is shown in [Li1] that $\mathfrak{A}$ is a vertex superalgebra under the $\mathbb{Z}_2$-twisted normal product (Theorem 3.14 in [Li1]). See [Li1] for the detailed description of the twisted normal product.

**Lemma 4.3.** The linear mapping $a \in L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}) \mapsto Y_N(a, z) \in \mathfrak{A}$ defined above gives an vertex superalgebra homomorphism.

**Proof:** We should show that $Y_N(a_m b, z) = Y_N(a, z) \circ_m Y_N(b, z)$ for any $a, b \in L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$ and $m \in \mathbb{Z}$, where $\circ_m$ denotes the $\mathbb{Z}_2$-twisted $m$-th normal product in $\mathfrak{A}$. We may assume that $a = \psi_{-n_1+\frac{1}{2}} \cdots \psi_{-n_k+\frac{1}{2}} 1$, $n_1 > \cdots > n_k > 0$. We proceed by induction on $k$. The case $k = 0$ is trivial and the case $k = 1$ is just the definition. Assume that $k \geq 1$ and $Y_N(a_m b, z) = Y_N(a, z) \circ_m Y_N(b, z)$ holds for arbitrary $b \in L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$ and $m \in \mathbb{Z}$. Take any $n > n_1$. Then we have

$$Y_N(\left(\psi_{-n+\frac{1}{2}}a\right)_m b, z)$$

$$= \sum_{i=0}^{\infty} (-1)^i \binom{-n}{i} Y_N(\psi_{-n-i+\frac{1}{2}}a_{m+i} b - (-1)^{q(a)-n} a_{-n+m-i} \psi_{i+\frac{1}{2}}b, z)$$

$$= \sum_{i=0}^{\infty} (-1)^i \binom{-n}{i} \left\{ \phi(z) \circ_{-n-i} Y_N(a_{m+i} b, z) \right\}$$

$$= \sum_{i=0}^{\infty} (-1)^i \binom{-n}{i} \left\{ \phi(z) \circ_{-n-i} \left( Y_N(a, z) \circ_{m+i} Y_N(b, z) \right) \right\}$$

$$= \left( \phi(z) \circ_{-n} Y_N(a, z) \right) \circ_m Y_N(b, z) \quad \text{(associativity in $\mathfrak{A}$)}$$

$$= Y_N(\psi_{-n+\frac{1}{2}}a, z) \circ_m Y_N(b, z).$$

Therefore by induction the mapping $a \mapsto Y_N(a, z)$ defines a vertex superalgebra homomorphism. \qed
Ler $V$ be an arbitrary SVOA. By Proposition 3.17 in [Li1], giving a $\mathbb{Z}_2$-twisted $V$-module structure on $N$ is equivalent to giving a vertex superalgebra homomorphism from $V$ to some local system of $\mathbb{Z}_2$-twisted vertex operators on $N$. Since both $L(\frac{1}{2}, \frac{1}{16})^+$ and $L(\frac{1}{2}, \frac{1}{16})^-$ are stable under the action $Y_N(\cdot, z)$, we arrive at the following conclusion.

**Theorem 4.4.** *(Theorem 2 in [FRW2])* $A_\phi$-modules $L(\frac{1}{2}, \frac{1}{16})^+$ and $L(\frac{1}{2}, \frac{1}{16})^-$ are non-isomorphic irreducible $\mathbb{Z}_2$-twisted $L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$-modules.

Note that the canonical contravariant bilinear form on $N$ can be naturally extended to the invariant bilinear form for the module vertex operators defined by (3.1). Therefore, by the natural symmetries of the fusion rules, we can explicitly calculate all intertwining operations of any type for the Ising models.

**Remark 4.5.** The $\mathbb{Z}_2$-twisted module vertex operators $Y_N(\cdot, z)|_{L(\frac{1}{2}, \frac{1}{16})^\pm}$ defined above give intertwining operators of type $L(\frac{1}{2}, h) \times L(\frac{1}{2}, \frac{1}{16})^\pm \to L(\frac{1}{2}, \frac{1}{16})^\pm$ for $h = 0, \frac{1}{2}$, respectively. Therefore, our construction gives another proof of Proposition 4.1-4.2 in [M2].

**Remark 4.6.** The $\mathbb{Z}_2$-twisted $L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$-modules $L(\frac{1}{2}, \frac{1}{16})^\pm$ can be characterized as follows. On the top level of $L(\frac{1}{2}, \frac{1}{16})^+$ (= $\mathbb{C}v^+$), a highest weight vector $\psi_{-\frac{1}{2}}$ in $L(\frac{1}{2}, \frac{1}{2})$ acts as $\sqrt{2}$, whereas on the top level of $L(\frac{1}{2}, \frac{1}{16})^-$ it acts as $-\frac{1}{\sqrt{2}}$. This observation will be used to determine the $\mathbb{Z}_2$-twisted Zhu algebra associated to the Ising model SVOA in the next subsection.

### 4.3 Zhu algebras for Ising model SVOA

In the following context, we will use the highest weight to denote the Virasoro module itself to simplify the notation. For example, we shall denote $[0]$ for $L(\frac{1}{2}, 0)$, $[\frac{1}{2}]$ for $L(\frac{1}{2}, \frac{1}{2})$, and so on.

Recall the following theorem in [KW].

**Theorem 4.7.** *(Theorem 4.1 in [KW])* $A([0] \oplus [\frac{1}{2}]) \simeq \mathbb{C}$. Therefore, SVOA $[0] \oplus [\frac{1}{2}]$ has a unique irreducible $\mathbb{Z}_2$-graded representation, namely $[0] \oplus [\frac{1}{2}]$ itself.

It is obvious that the canonical involution $\sigma$ is the only non-trivial automorphism on $[0] \oplus [\frac{1}{2}]$. Therefore the $\mathbb{Z}_2$-twisted representation is the only non-trivial twisted representation of $[0] \oplus [\frac{1}{2}]$. Let’s determine the $\mathbb{Z}_2$-twisted Zhu algebra $A_t([0] \oplus [\frac{1}{2}])$. Denote by $\iota$ a canonical surjection from $[0] \oplus [\frac{1}{2}]$ onto $A_t([0] \oplus [\frac{1}{2}])$.

**Lemma 4.8.** $\iota([0]) = \mathbb{C}v(1)$ and $\iota([\frac{1}{2}]) = \mathbb{C}v(\psi_{-\frac{1}{2}})$.
Proof: Again we proceed by induction on the length $k$ of a canonical basis

$$\psi_{-n_1 + \frac{1}{2}} \cdots \psi_{-n_k + \frac{1}{2}}$$

with $n_1 > \cdots > n_k > 0$. The case $k = 0$ is obvious. For $k = 1$, by Lemma 2.8 we have

$$\text{Res}_z Y(a,z) b \frac{(1 + z)^{\text{wt}(a)} }{z^{2+m}} \in O_t \left( [0] \oplus \left[ \frac{1}{2} \right] \right)$$

for any $a, b \in [0] \oplus \left[ \frac{1}{2} \right]$ and arbitrary $m \geq 0$. Putting $a = \psi_{-\frac{1}{2}} 1$, $b = 1$ and by induction on $n$, we obtain $\psi_{-n_1 + \frac{1}{2}} 1 \in O_t \left( [0] \oplus \left[ \frac{1}{2} \right] \right)$ for $n \geq 2$. Thus the case $k = 1$ is correct. Assume that our assertion is correct for all $\psi_{-n_1 + \frac{1}{2}} \cdots \psi_{-n_s + \frac{1}{2}} 1$ with $s \leq k$. Take any $n > n_1$, $n > 1$. Then

$$O_t \left( [0] \oplus \left[ \frac{1}{2} \right] \right) \ni \text{Res}_z \psi(z) \frac{(1 + z)^{\frac{1}{2}} \psi_{-n_1 + \frac{1}{2}} \cdots \psi_{-n_k + \frac{1}{2}} 1}{z^{2+(n-2)}}$$

$$= \psi_{-n_1 + \frac{1}{2}} \psi_{-n_1 + \frac{1}{2}} \psi_{-n_k + \frac{1}{2}} 1 + \sum_{i=1}^{\infty} \left( \frac{\sqrt{2}}{i} \right) \psi_{-n+i+\frac{1}{2}} \psi_{-n_1 + \frac{1}{2}} \cdots \psi_{-n_k + \frac{1}{2}} 1.$$  \hspace{1cm} (4.3)

It follows from the inductive assumption that the second term in the lower hand side of (4.3) can be written as the desired forms. Hence the first term in the lower hand side also has the desired form since each term in the lower hand side of (4.3) shares the same parity. So our assertion holds for $k + 1$. Thus we have the desired result.

The following theorem together with Theorem 4.7 completes the classification of all irreducible representations of the Ising model SVOA.

**Theorem 4.9.** There exists exactly two non-isomorphic irreducible $\mathbb{Z}_2$-twisted $[0] \oplus \left[ \frac{1}{2} \right]$-modules, namely, $[\left[ \frac{1}{16} \right]]^+$ and $[\left[ \frac{1}{16} \right]]^-$.  

**Proof:** We have already shown that both $[\left[ \frac{1}{16} \right]]^+$ and $[\left[ \frac{1}{16} \right]]^-$ are irreducible $\mathbb{Z}_2$-twisted $[0] \oplus \left[ \frac{1}{2} \right]$-modules in Theorem 4.4. By Theorem 2.7, we should determine all irreducible $A_t([0] \oplus \left[ \frac{1}{2} \right])$-modules. By Lemma 1.8, we know that $A_t([0] \oplus \left[ \frac{1}{2} \right])$ is generated by $\iota(1)$ and $\iota(\psi_{-\frac{1}{2}} 1)$. After a short computation along with the definition, we get the followings.

$$(\psi_{-\frac{1}{2}} 1) \circ_t (\psi_{-\frac{1}{2}} 1) = 2 \left( \omega - \frac{1}{16} \right), \quad (\psi_{-\frac{1}{2}} 1) *_t (\psi_{-\frac{1}{2}} 1) = \frac{1}{2} 1.$$

Hence we can deduce that $\omega_1 = L_0$ acts as $\frac{1}{16}$ on the top level of every irreducible $\mathbb{Z}_2$-twisted $[0] \oplus \left[ \frac{1}{2} \right]$-module and $A_t \left( [0] \oplus \left[ \frac{1}{2} \right] \right)$ is a homomorphic image of a ring $\mathbb{C}[x]/(x^2 - \frac{1}{2})$. But by Remark 1.6, the top levels $\mathbb{C}^{V_{\mathbb{Z}_2}^+}$ of $[\left[ \frac{1}{16} \right]]^+$ are non-isomorphic irreducible $A_t([0] \oplus \left[ \frac{1}{2} \right])$-modules on which $\iota(\psi_{-\frac{1}{2}} 1)$ acts as $\pm \frac{1}{\sqrt{2}}$, respectively, so that $A_t([0] \oplus \left[ \frac{1}{2} \right])$ must be isomorphic to $\mathbb{C}[x]/(x^2 - \frac{1}{2})$. Therefore, $[\left[ \frac{1}{16} \right]]^+$ and $[\left[ \frac{1}{16} \right]]^-$ are only non-isomorphic irreducible $\mathbb{Z}_2$-twisted $[0] \oplus \left[ \frac{1}{2} \right]$-modules.

\hspace{1cm} \blacksquare
Remark 4.10. Since \([\frac{1}{16}]\) as \([0]\)-module is irreducible, we know that \((\frac{1}{16})^\pm\) is not isomorphic to \([\frac{1}{16}]^\pm\) as \([0] \oplus [\frac{1}{2}]\)-module, respectively, by Proposition 3.3. This implies that if we denote the module vertex operator of \([\frac{1}{2}]\) on \([\frac{1}{16}]^+\) by \(I(\cdot, z)\), then the module vertex operator of \([\frac{1}{2}]\) on \([\frac{1}{16}]^-\) is given by \(-I(\cdot, z)\). Since \(I(\cdot, z)\) and \(-I(\cdot, z)\) are the only scalar multiples of \(I(\cdot, z)\) which satisfy the \(\mathbb{Z}_2\)-twisted Jacobi identity, we can also conclude that \([\frac{1}{16}]^\pm\) are all non-isomorphic \(\mathbb{Z}_2\)-twisted \(V\)-modules.

4.4 Application

In this subsection we consider an applications of the Ising models. One of the merit of using the Ising model is that it defines an automorphism, so-called “Miyamoto involution” of VOA. Concerning to this involution, we will find some \(\mathbb{Z}_2\)-twisted representations of SVOAs.

In the following, we will assume that \(V = \oplus_{n=0}^\infty V_n\) is a simple VOA whose weight zero space \(V_0\) is spanned by \(1\), i.e. \(V_0 = \mathbb{C}1\). Following [MI], we will call an element \(e \in V_2\) whose vertex operator \(Y(e, z) = \sum_{n \in \mathbb{Z}} L_n^e z^{-n-2}\) generates a copy of the Virasoro algebra of central charge \(c_e\) a conformal vector with c.c. \(c_e\). One can find the following lemma in [MI].

Lemma 4.11. An element \(e \in V_2\) is a conformal vector with c.c. \(c_e\) if and only if it satisfies \(e_1 e = 2e, e_2 e = 0\) and \(e_3 e = \frac{1}{2} c_e 1\).

A conformal vector \(e\) is called rational if the sub VOA \(\langle e \rangle\) generated by \(e\) becomes a rational Virasoro VOA. In this paper, we are especially interested in a rational conformal vector \(e\) with c.c. \(\frac{1}{2}\). One way to determine whether a conformal vector \(e\) with c.c. \(\frac{1}{2}\) is rational or not is to check \(64(L_{-2}^e)^3 1 + 93(L_{-3}^e)^2 1 - 264L_{-4}^e L_{-2}^e 1 - 108L_{-6}^e 1 = 0\) or not (cf. [DMZ]). Assume that \(e\) is a rational conformal vector with c.c. \(\frac{1}{2}\) in \(V\). Seen \(V\) as \(\langle e \rangle \simeq L(\frac{1}{2}, 0)\)-module, we can decompose \(V\) as follow.

\[
V = V_e(0) \oplus V_e(\frac{1}{2}) \oplus V_e(\frac{1}{16}),
\]

where \(V_e(h)\) is the sum of all irreducible \(\langle e \rangle\)-modules which are isomorphic to \(L(\frac{1}{2}, h)\) for \(h = 0, \frac{1}{2}, \frac{1}{16}\) as Virasoro modules. For \(h = 0, \frac{1}{2}, \frac{1}{16}\), let \(T_e(h)\) be the space \(\{v \in V | L_0^e v = hv\}\) of the highest weight vectors. Then \(T_e(h)\) is also the space of the multiplicity of irreducible components in \(V\) and so we get a linear isomorphism \(V_e(h) \simeq L(\frac{1}{2}, h) \otimes \mathbb{C} T_e(h)\). Since \([e_1, \omega_1] = 0\), \(T_e(h)\) admits a \(L_0\)-weight space decomposition \(T_e(h) = \oplus_{n \in \mathbb{N}} T_e(h)_n\), where \(T_e(h)_n = V_n \cap T_e(h)\). By a little calculation, one can show that \(\omega_2 e = 0\). So by [FZ] we know that \(f := \omega - e\) is also a conformal vector with c.c. \(\text{rank}(V) - \frac{1}{2}\) whose vertex operators are commutative with that of \(e\). Therefore we have a decomposition \(\omega = e + f\).
of the Virasoro vector of $V$ into two mutually commutative conformal vectors. It is also shown in [FZ] that if we set the commutants $U^e := \ker_V L^e_{-1}$ and $U^f := \ker_V L^f_{-1}$, then $(U^e, Y_{[\mu^e]}, I_V, e)$ and $(U^f, Y_{[\mu^f]}, I_V, f)$ forms mutually commutative sub VOAs of $V$, where we set $L^e_{-1} = e_{n+1}$ and $L^f_{-1} = f_{n+1}$. In general, $\ker_V L^e_{-1} \subseteq \ker_V L^e_0$ and $\ker_V L^f_{-1} \subseteq \ker_V L^f_0$, but in our case the equality holds for both cases and we have $U^e = \langle e \rangle$ and $U^f = T_e(0)$. Furthermore, we can show the following.

**Proposition 4.12.** $(T_e(0), Y_{T_e(0)}, I_V, f = \omega - e)$ is the maximal simple sub VOA of $V$ whose vertex operators are commutative with that of $\langle e \rangle$ element-wisely in $V$. Moreover, $V_e(0) = L(\frac{1}{2}, 0) \otimes T_e(0)$ is a simple sub VOA of $V$ whose Virasoro vector is the same as that of $V$.

**Proof:** We should only show the simplicity of $T_e(0)$. On $V$, the linear map $\tau_e$ which is identical on $V_e(0) \oplus V_e(\frac{1}{2})$ and acts $-1$ on $V_e(\frac{1}{2})$ defines an involution of $V$, which is known as “Miyamoto automorphism” in [M1]. On the $\tau_e$-invariants $V_e(0) \oplus V_e(\frac{1}{2})$, the linear map $\sigma_e$ which is identical on $V_e(0)$ and acts $-1$ on $V_e(\frac{1}{2})$ also defines an involution [M1]. Therefore the $\sigma_e$-invariants $V_e(0) = L(\frac{1}{2}, 0) \otimes T_e(0)$ is a simple sub VOA of $V$ since $V$ is simple. Then the irreducibility of $L(\frac{1}{2}, 0)$ implies $T_e(0)$ is also simple.

By this proposition, we may view $V$ as a $[0] \otimes T_e(0)$-module. It is not difficult to show that $T_e(h)$, $h = 0, \frac{1}{2}, \frac{1}{16}$ are $T_e(0)$-modules. Since the $\tau_e$-invariants $[0] \otimes T_e(0) \oplus \frac{1}{2} \otimes T_e(\frac{1}{2})$ is a simple sub VOA of $V$, $T_e(\frac{1}{2})$ is an irreducible $T_e(0)$-module. On the other hand, $[0] \oplus \frac{1}{2}$ has an SVOA structure. So it is natural for us to expect that $T_e(0) \oplus T_e(\frac{1}{2})$ also has an SVOA structure. Namely, we expect that the decomposition $V^\tau_e = [0] \otimes T_e(0) \oplus \frac{1}{2} \otimes T_e(\frac{1}{2})$ describes not only the tensor product of vector spaces but also that of SVOAs. As a generalization of Proposition 4.9 in [M1], the following theorem holds.

**Theorem 4.13.** Assume that $V_e(\frac{1}{2}) \neq 0$. Then there exists a simple SVOA structure on $T_e(0) \oplus T_e(\frac{1}{2})$ such that the even part of a tensor product of SVOAs $[0] \oplus \frac{1}{2}$ and $T_e(0) \oplus T_e(\frac{1}{2})$ is isomorphic to $V_e(0) \oplus V_e(\frac{1}{2})$ as VOAs.

**Proof:** We shall define vertex operators on an abstract linear space $T_e(0) \oplus T_e(\frac{1}{2})$. First, we show the existence of the intertwining operators. Let $\{a^\gamma | \gamma \in \Gamma \} \subset T_e(0)$ and $\{u^\lambda | \lambda \in \Lambda \} \subset T_e(\frac{1}{2})$ be sets of basis elements of $T_e(0)$ and $T_e(\frac{1}{2})$ consisting of $L^e_0$-eigen vectors, respectively. Using these bases, we obtain decompositions of $V_e(0)$ and $V_e(\frac{1}{2})$ as $[0]$-modules as follows.

$$V_e(0) = \bigoplus_{\gamma \in \Gamma} [0] \otimes a^\gamma, \quad V_e(\frac{1}{2}) = \bigoplus_{\lambda \in \Lambda} \frac{1}{2} \otimes u^\lambda.$$
Let $\pi_\gamma$ be a projection map $V_\epsilon(0) \to [0] \otimes a^\gamma$. We define intertwining operators $I^\gamma_{\lambda\mu}(\cdot,z)$ of type $[\frac{1}{2}] \otimes u^\lambda \times [\frac{1}{2}] \otimes u^\mu \to [0] \otimes a^\gamma$ as follow.

$$I^\gamma_{\lambda\mu}(x,z)y := \pi_\gamma Y(x \otimes u^\lambda, z) y \otimes u^\mu z^{-|\gamma|+|\lambda|+|\mu|},$$

where $x, y \in [\frac{1}{2}]$ and $|\gamma|, |\lambda|, |\mu|$ denote the $L^f_0$-weights of $a^\gamma, u^\lambda, u^\mu$, respectively. One can show that $I^\gamma_{\lambda\mu}(\cdot,z)$ satisfy $L^f_{-1}$-derivation, the condition of finiteness of negative powers and Jacobi identity for intertwining operators so that $I^\gamma_{\lambda\mu}(\cdot,z)$ are intertwining operators of type $[\frac{1}{2}] \times [\frac{1}{2}] \to [0]$. Since the space of intertwining operators of such a type is one dimensional, there exist suitable scalars $c^\gamma_{\lambda\mu} \in \mathbb{C}$ such that $I^\gamma_{\lambda\mu}(\cdot,z)$ are a multiple of the vertex operator $Y_{[\frac{1}{2}] \times [\frac{1}{2}]}(\cdot,z)$ by $c^\gamma_{\lambda\mu}$, which was constructed in Sec. 4.2. Therefore, the vertex operator of $x \otimes u^\lambda$ on $[\frac{1}{2}] \otimes T_e(\frac{1}{2})$ can be written as follow.

$$Y_V(x \otimes u^\lambda, z)y \otimes u^\mu = Y_{[\frac{1}{2}] \times [\frac{1}{2}]}(x, z)y \otimes \sum_{\gamma \in \Gamma} c^\gamma_{\lambda\mu} a^\gamma z^{-|\gamma|-|\lambda|-|\mu|}.$$

Thus, by setting $J(u^\lambda, z)u^\mu := \sum_{\gamma \in \Gamma} c^\gamma_{\lambda\mu} a^\gamma z^{-|\gamma|-|\lambda|-|\mu|}$, we obtain a decomposition

$$Y_V(x \otimes u^\lambda, z)y \otimes u^\mu = Y_{[\frac{1}{2}] \times [\frac{1}{2}]}(x, z)y \otimes J(u^\lambda, z)u^\mu$$

for $x \otimes u^\lambda, y \otimes u^\mu \in [\frac{1}{2}] \otimes T_e(\frac{1}{2})$. We claim that $J(\cdot,z)$ is an intertwining operator of type $T_e(\frac{1}{2}) \times T_e(\frac{1}{2}) \to T_e(0)$. It is not difficult to show that $J(\cdot,z)$ satisfies the condition of finiteness of negative powers and $L^f_{-1}$-derivation. So we should show the commutativity and associativity of $J(\cdot,z)$. Let $a \in T_e(0), u, v \in T_e(\frac{1}{2})$ be arbitrary elements. Take a sufficiently large $N \in \mathbb{N}$. Then the commutativity of vertex operators on $V$ leads

$$(z_1 - z_2)^{NY_V(\mathbf{1} \otimes a, z_1)Y_V(\psi_{-\frac{1}{2}} \mathbf{1} \otimes u, z_2)\psi_{-\frac{1}{2}} \mathbf{1} \otimes v} = (z_1 - z_2)^{NY_V(\psi_{-\frac{1}{2}} \mathbf{1} \otimes u, z_1)Y_V(\mathbf{1} \otimes a, z_2)\psi_{-\frac{1}{2}} \mathbf{1} \otimes v}.$$ 

Rewriting the above equality yields

$$(z_1 - z_2)^{NY_{[\frac{1}{2}] \times [\frac{1}{2}]}(\psi_{-\frac{1}{2}} \mathbf{1}, z_2)\psi_{-\frac{1}{2}} \mathbf{1} \otimes Y_{T_e}(0)(a, z_1)J(u, z_2)v} = (z_1 - z_2)^{NY_{[\frac{1}{2}] \times [\frac{1}{2}]}(\psi_{-\frac{1}{2}} \mathbf{1}, z_2)\psi_{-\frac{1}{2}} \mathbf{1} \otimes J(u, z_2)Y_{T_e}(0)(a, z_1)v}.$$ 

By comparing the coefficients of $(\psi_{-\frac{1}{2}} \mathbf{1})_0 \psi_{-\frac{1}{2}} \mathbf{1} = \mathbf{1}$, we get the commutativity:

$$(z_1 - z_2)^{NY_{T_e}(0)(a, z_1)J(u, z_2)v} = (z_1 - z_2)^{N_J(u, z_2)Y_{T_e}(0)(a, z_1)v}.$$ 

Similarly, by considering some coefficients of $Y_V(\mathbf{1} \otimes a, z_0)\psi_{-\frac{1}{2}} \mathbf{1} \otimes u, z_2)\psi_{-\frac{1}{2}} \mathbf{1} \otimes v$ in $V$, we can obtain the associativity. Hence, $J(\cdot,z)$ is the intertwining operator of the desired type.
Using $J(\cdot, \cdot)$, we introduce the vertex operations on $T_v(0) \oplus T_e(\frac{1}{2})$. Since we already know the action of $T_v(0)$ on $T_v(0) \oplus T_e(\frac{1}{2})$, we should define the action of $T_e(\frac{1}{2})$ on $T_v(0) \oplus T_e(\frac{1}{2})$. For $a \in T_v(0)$, $u, v \in T_e(\frac{1}{2})$, we set

$$Y_{Te(\frac{1}{2})}(u, z) a := e^{zL_{-1}} Y_{T_v(0)}(a, -z) u, \quad Y_{Te(\frac{1}{2})}(u, z) v := J(u, z) v.$$  

Since the above vertex operators are intertwining operators of type $T_v(\frac{1}{2}) \times T_v(0) \rightarrow T_v(\frac{1}{2})$ and $T_e(\frac{1}{2}) \times T_e(\frac{1}{2}) \rightarrow T_e(0)$, respectively, we only need to show the mutually commutativity of vertex operators. It follows from the definition that the vertex operator $Y_{Tv}(a \otimes b, z)$ in $V^\tau$ can be written as $Y_{Tv(0) \otimes Te(\frac{1}{2})}(b, z)$ for $a \in [0] \oplus [\frac{1}{2}]$ and $b \in T_v(0) \oplus T_e(\frac{1}{2})$.

Hence, by comparing suitable coefficients as we did previously, we can deduce the mutually commutativity of the vertex operators on $T_v(0) \oplus T_e(\frac{1}{2})$ since the vertex operators on $V$ and that on $[0] \oplus [\frac{1}{2}]$ satisfy the mutually commutativity. Therefore, by our definition, $(T_v(0) \oplus T_e(\frac{1}{2}), Y(\cdot, z), 1, f)$ becomes a simple SVOA. The rest of assertion is now clear.

Since $\tau^2 = 1$ on $V$, the space $V_v(\frac{1}{16})$ is irreducible $V^\tau$-module. As $(V^\tau)^{\sigma_e} = [0] \otimes T_v(0)$-module, $V_v(\frac{1}{16})$ can be written as $[\frac{1}{16}] \otimes T_v(\frac{1}{16})$. However, it is not clear that $T_v(\frac{1}{16})$ is irreducible under $T_v(0)$ in general. But we can insure that $T_v(\frac{1}{16})$ is irreducible under $T_v(0) \oplus T_e(\frac{1}{2})$.

**Theorem 4.14.** Assume that $V_v(\frac{1}{16}) \neq 0$. Then $T_v(\frac{1}{16})$ has an irreducible $Z_2$-twisted $T_v(0) \oplus T_e(\frac{1}{2})$-module structure such that $V_v(\frac{1}{16})$ is isomorphic to a tensor product of irreducible $Z_2$-twisted $[0] \oplus [\frac{1}{2}]$-module $[\frac{1}{16}]^+$ and irreducible $T_v(0) \oplus T_e(\frac{1}{2})$-module $T_e(\frac{1}{16})$.

**Proof:** Since we know how $[0] \oplus [\frac{1}{2}]$ acts on $[\frac{1}{16}]^+$, we can use the same strategy of the proof of Theorem 4.13. Since $V_v(\frac{1}{16}) = [\frac{1}{16}] \otimes T_v(\frac{1}{16})$ is an irreducible $V^\tau = [0] \otimes T_v(0) \oplus [\frac{1}{2}] \otimes T_e(\frac{1}{2})$-module, it is not hard to see that the vertex operator of $a \otimes b \in [h] \otimes T_e(0)$ on $[\frac{1}{16}] \otimes T_v(\frac{1}{16})$ can be written as $Y_{[h] \times [\frac{1}{16}]^+}(a, z) \otimes Y_{Te(\frac{1}{2})} \otimes T_v(\frac{1}{16}) \otimes T_e(\frac{1}{2})$ is an intertwining operator of type $[h] \times [\frac{1}{16}]^+ \otimes T_v(\frac{1}{16}) \otimes T_e(\frac{1}{2})$ for $h = 0, \frac{1}{2}$, respectively. As mentioned in Remark 4.11, by managing salar multiplications we may take $Y_{[h] \times [\frac{1}{16}]^+}(\cdot, z)$ to be the $Z_2$-twisted module vertex operators of $[0] \oplus [\frac{1}{2}]$ on $[\frac{1}{16}]^+$ for $h = 0, \frac{1}{2}$. Then the Jacobi identity on $V$ and the $Z_2$-twisted Jacobi identity on $[\frac{1}{16}]^+$ implies that the intertwining operators $Y_{T_v(0) \otimes T_e(\frac{1}{2})}(\cdot, z)$ on $T_v(\frac{1}{16})$ also satisfy the $Z_2$-twisted Jacobi identity for $h = 0, \frac{1}{2}$. Thus, $T_v(\frac{1}{16})$ is an irreducible $Z_2$-twisted $T_v(0) \otimes T_e(\frac{1}{2})$-module under the module vertex operators $Y_{T_v(0) \otimes T_e(\frac{1}{2})}(\cdot, z)$ for $h = 0, \frac{1}{2}$.  

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5 The Babymonster SVOA $VB^\natural_R$

In this section, we apply our result to the moonshine vertex operator algebra, which is of course the most important example of holomorphic VOAs. In this paper we treat the moonshine vertex operator algebra $V^2_\mathbb{R}$ over the real number field constructed in [M3]. As well-known, the full automorphism group of $V^2_\mathbb{R}$ is the Monster sporadic simple group $\mathbb{M}$ (cf. [M3]). In $V^2_\mathbb{R}$, there are many rational conformal vectors with c.c. $\frac{1}{2}$. It is shown in [M1] that each rational conformal vector $e \in V^2_\mathbb{R}$ defines an element $\tau_e$ of $2A$ conjugacy class of $\mathbb{M}$. It is also shown in [H] that this correspondence is one-to-one. In $\mathbb{M}$, we can find many sporadic simple groups. Let $e \in V^2_\mathbb{R}$ be a rational conformal vector with c.c. $\frac{1}{2}$. Then the centralizer $C_\mathbb{M}(\tau_e)$ is isomorphic to an extension $\langle \tau_e \rangle \cdot \mathbb{B}$ of the Baby monster sporadic simple group $\mathbb{B}$. So $\mathbb{B}$ acts on the $\tau_e$-invariants of $V^2_\mathbb{R}$ as an automorphism group of a VOA. Motivated by this fact, Höhn studied the $\tau_e$-invariants of $V^2_\mathbb{R}$ and found the Babymonster SVOA $VB^\natural$ on which $\mathbb{B}$ acts as an automorphism group of an SVOA in [H].

Our approach is the same as that of [H] and there are some overlap with this article. But our method seems simpler since it is based on explicit calculations of the intertwining operators for the Ising models so that we don’t need some assumptions that are supposed in [H].

Let $e$ be a rational conformal vector with c.c. $\frac{1}{2}$ in $V^2_\mathbb{R}$. (The existence of such a vector is clear.) It is shown in [M3] that the Ising model VOA $L(\frac{1}{2},0)_\mathbb{R}$ over $\mathbb{R}$ is also rational so that we can apply Theorem 4.13 to $V^2_\mathbb{R}$. By Theorem 4.13, we can obtain a simple SVOA $T^\natural_0(0) \oplus T^\natural_0(\frac{1}{2})$ from $V^2_\mathbb{R}$. Following Höhn [H], we set $VB^\natural_\mathbb{R} = T^\natural_0(0) \oplus T^\natural_0(\frac{1}{2})$ and call it the Babymonster SVOA. In $V^2_\mathbb{R}$, for every conformal vector $e$, its Miyamoto-involution $\tau_e$ is not trivial so that $V^\natural_\mathbb{R}(\frac{1}{16})$ is not zero. Therefore we also obtain an irreducible $\mathbb{Z}_2$-twisted $VB^\natural_\mathbb{R}$-module $T^\natural_\mathbb{R}(\frac{1}{16})$. We set $(VB^\natural_\mathbb{R})_\mathbb{R} := T^\natural_\mathbb{R}(\frac{1}{16})$. Note that the algebraic structures of $VB^\natural_\mathbb{R}$ and $(VB^\natural_\mathbb{R})_\mathbb{R}$ are independent of the choice of a conformal vector $e$ since every conformal vector with c.c. $\frac{1}{2}$ in $V^2_\mathbb{R}$ is conjugate under the Monster $\mathbb{M}$ so that the structures of them are uniquely determined by that of $V^2_\mathbb{R}$.

**Theorem 5.1.** (1) ([H]) The SVOA $VB^\natural_\mathbb{R}$ obtained from $V^2_\mathbb{R}$ by cutting off the Ising models is a simple SVOA whose full automorphism group contains $2 \times \mathbb{B}$.

(2) The piece $(VB^\natural_\mathbb{R})_\mathbb{R}$ obtained from $V^2_\mathbb{R}$ is an irreducible $\mathbb{Z}_2$-twisted $VB^\natural_\mathbb{R}$-module.

**Proof:** The first half assertion of (1) and (2) follow from Theorem 4.13. So it remains to show that $2 \times \mathbb{B}$ acts on $VB^\natural_\mathbb{R}$ as an automorphism group of SVOA. For any conformal vector $e \in V^2_\mathbb{R}$, the corresponding involution $\tau_e$ is unique so that it follows from $g\tau_e g^{-1} = \tau_{ge}$ for all $g \in \mathbb{M}$ that every element in $C_\mathbb{M}(\tau_e)$ fixes $e$. Hence, $C_\mathbb{M}(\tau_e)$ acts on $T^\natural_0(0)$. Furthermore, $C_\mathbb{M}(\tau_e)$ leaves the space $\psi_{\frac{1}{2}} \mathbb{I} \otimes T^\natural_0(\frac{1}{2}) \subset V^2_\mathbb{R}$ invariant since each element of
$C_{\mathbb{M}}(\tau_e)$ and the action of $e$ commute on $V^R_0$. Therefore we may think $C_{\mathbb{M}}(\tau_e)$ also acts on $T^e_0(\tfrac{1}{2})$. Since $C_{\mathbb{M}}(\tau_e) = \langle \tau_e \rangle \cdot \mathbb{B}$ and $\tau_e$ acts on $T^e_0(h)$ trivially for $h = 0, \frac{1}{2}$, we have an injection from $\mathbb{B}$ into the full automorphism group of the SVOA $V^R_0 = T^e_0(0) \oplus T^e_0(\tfrac{1}{2})$.

One can expect that the full automorphism group of $V^R_0$ is the Baby monster $\mathbb{B}$. However, we could not determine it is true or not in this paper. But we can prove that $\text{Aut}(V^R_0)$ is finite. To prove this, we need some results from the quantum Galois theory for SVOAs.

Let $V$ be a simple SVOA over $\mathbb{C}$ and $G$ be a finite subgroup of $\text{Aut}(V)$. For $\chi \in \text{Irr}(G)$, we set $V^\chi$ to be the sum of all irreducible components of $V$ on which $G$ acts as $\chi$. Clearly, $V^\chi$ has a $\mathbb{Z}_2$-grading and we denote such a $\mathbb{Z}_2$-grade decomposition by $V^\chi = V^0_\chi \oplus V^1_\chi$. Let $M_\chi$ be the irreducible representation of $G$ on which $G$ acts as $\chi$. Then we have a decomposition $V^i_\chi = M_\chi \otimes \text{Hom}_G(M_\chi, V^i)$ as a $G$-module. Setting $V^i_\chi := \text{Hom}_G(M_\chi, V^i)$ for $i = 0, 1$ and $\chi \in \text{Irr}(G)$, we have the following decomposition of $V$ as $\mathbb{C}[G] \otimes V^G$-module.

$$V = \bigoplus_{\chi \in \text{Irr}(G)} M_\chi \otimes (V^0_\chi \oplus V^1_\chi).$$

Then, as an extension of $[DM]$ and $[HMT]$, we have

**Theorem 5.2.** Under the above setting, the followings hold.

1. $V^\chi \neq 0$ for all $\chi \in \text{Irr}(G)$.
2. If $V^G$ is a VOA, then $V^i_\chi$, $i = 0, 1$, $\chi \in \text{Irr}(G)$ except trivial modules are non-isomorphic irreducible $V^G$-modules.
3. If $V^G$ is an SVOA, then $V^0_\chi \oplus V^1_\chi$ are non-isomorphic irreducible $\mathbb{Z}_2$-graded $V^G$-modules. In particular, none of $V^i_\chi$, $i = 0, 1$, $\chi \in \text{Irr}(G)$ is zero and $\mathbb{C}[G] \otimes V^G$ forms a dual pair over $V$.

Since the proof is just rewriting of the original quantum Galois theory for VOAs, we omit it.

**Remark 5.3.** It is case-by-case whether $V^G$ is a VOA or SVOA. If $G$ contains a canonical involution for SVOA, then $V^G$ must be a VOA.

The following theorem is an analogy of Theorem 9.2 in $[M3]$.

**Theorem 5.4.** $\text{Aut}(V^R_0)$ is finite.

**Proof:** Since $V^R_0$ is a framed VOA, $V^R_0$ is also a framed SVOA by its construction. (See $[DGH]$ for the definition of the framed VOAs). It follows from Miyamoto’s construction of the moonshine module $[M3]$ and our definition of the Babymonster SVOA that

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\( \mathcal{V}_R^\mathcal{N} \) has an invariant bilinear form which is positive definite on the even part \((\mathcal{V}_R^\mathcal{N})^0\) and negative definite on the odd part \((\mathcal{V}_R^\mathcal{N})^1\). Suppose that there exists a subgroup \( G \) of Aut(\( \mathcal{V}_R^\mathcal{N} \)) of infinite order. Since \((\mathcal{V}_R^\mathcal{N})^1 = 0\), we can apply Theorem 9.1 in \([M1]\) and so there are finitely many conformal vectors with c.c. \( \frac{1}{2} \) in \( \mathcal{V}_R^\mathcal{N} \). So we may assume that \( G \) fixes all conformal vectors with c.c. \( \frac{1}{2} \) in \( \mathcal{V}_R^\mathcal{N} \). Let \( \omega \) be the Virasoro vector of \( \mathcal{V}_R^\mathcal{N} \). Then we can find a set of mutually orthogonal conformal vectors \( e_1, e_2, \ldots, e_{47} \) with c.c. \( \frac{1}{2} \) in \( \mathcal{V}_R^\mathcal{N} \) such that \( \omega = e_1 + \cdots + e_{47} \). Set \( P = \langle \tau_{e_i}, \sigma | i = 1, \ldots, 47 \rangle \), where \( \tau_{e_i} \) are Miyamoto’s involutions and \( \sigma \) is a canonical involution on \( \mathcal{V}_R^\mathcal{N} \). By the definition of \( \tau_{e_i} \), \( P \) is an elementary abelian 2-group. Let \( \mathcal{V}_R^\mathcal{N} = \bigoplus_{\chi \in \text{Irr}(P)} (\mathcal{V}_R^\mathcal{N})^\chi \) be the decomposition of \( \mathcal{V}_R^\mathcal{N} \) into the direct sum of eigenspaces of \( P \). Since \( G \) fixes all \( e_i \), \([G, P] = 1\) and hence \( G \) leaves all \((\mathcal{V}_R^\mathcal{N})^\chi\) invariant. In particular, \( G \) acts on \((\mathcal{V}_R^\mathcal{N})^P\). Since \( P \) contains a canonical involution \( \sigma \), \((\mathcal{V}_R^\mathcal{N})^P\) is a VOA and is isomorphic to a code VOA \( M_D = \bigoplus_{\alpha \in D} M_\alpha \) for some even linear code \( D \). See \([M2]\) for the description of the code VOA \( M_D \). Since \( G \) fixes all \( e_i \), \( G \) fixes all elements in \( L(\frac{1}{2}, 0)^{\otimes 47} \subset M_D \) so that \( g \in G \) acts on \( M_\alpha \) as a scalar \( \lambda_\alpha(g) \). Since \((\mathcal{V}_R^\mathcal{N})^0\) has a positive definite invariant form, we have \( \lambda_\alpha(g) = \pm 1 \) for all \( g \in G \). Therefore, by taking finite index, we may assume that \( G \) fixes all elements in \((\mathcal{V}_R^\mathcal{N})^P\). By considering the complexification if necessary, we see that \((\mathcal{V}_R^\mathcal{N})^P\) is an irreducible \((\mathcal{V}_R^\mathcal{N})^P\)-module by Theorem 5.2 so that \( g \in G \) acts on \((\mathcal{V}_R^\mathcal{N})^\chi\) as a scalar \( \mu_\chi(g) \). By the same arguments as above, we have a contradiction even though if we make a complexification. Hence, Aut(\( \mathcal{V}_R^\mathcal{N} \)) is finite.

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