Abstract
We consider the following variational problem: minimize the \((n+1)\)st polynomial conserved quantity of KdV over \(H^n(\mathbb{R})\) with the first \(n\) conserved quantities constrained. Maddocks and Sachs (Comm Pure Appl Math 46:867–901, 1993) used that \(n\)-solitons are local minimizers for this problem in order to prove that \(n\)-solitons are orbitally stable in \(H^n(\mathbb{R})\).

Given \(n\) constraints that are attainable by an \(n\)-soliton, we show that there is a unique set of \(n\) amplitude parameters so that the corresponding multisolitons satisfy the constraints. Moreover, we prove that these multisolitons are the unique global constrained minimizers. We then use this variational characterization to provide a new proof of the orbital stability result from Maddocks and Sachs (Comm Pure Appl Math 46:867–901, 1993) via concentration compactness.

In the case when the constraints can be attained by functions in \(H^n(\mathbb{R})\) but not by an \(n\)-soliton, we discover new behavior for minimizing sequences.

1 Introduction

The Korteweg–de Vries (KdV) equation

\[
\frac{d}{dt} u = -u''' + 6uu' \tag{1.1}
\]

(where \(u' = \partial_x u\) denotes spatial differentiation) was derived over a century ago as a model for surface waves in a shallow channel of water. Although the equation was first proposed by Boussinesq [8], it did not gain traction until Korteweg and de Vries [25] used the explicit solutions

\[
q(t, x) = -2\beta^2 \text{sech}^2 \left[ \beta(x - 4\beta^2 t - x_0) \right] \tag{1.2}
\]

for \(\beta > 0\) and \(x_0 \in \mathbb{R}\) to explain the empirical observation of solitary traveling waves.
The solutions (1.2) are now commonly referred to as solitons, due to their particle-like behavior during interactions. This name was coined by Kruskal and Zabusky [39] when they numerically observed that two colliding solitons emerge with unchanged profiles and speeds. The interaction is nevertheless nonlinear, and this is manifested in a spatial shift of both waves in comparison to their initial trajectories.

We now know that this particle-like interaction can be well-approximated by a single explicit solution which resembles two solitons with distinct amplitudes as \( t \to \pm \infty \). In fact, together with the single soliton solutions (1.2), these are the beginning of a family of solutions called multisolitons which describe the interaction of an arbitrary number of distinct soliton profiles:

**Definition 1.1** (Multisoliton solutions) Fix \( N \geq 1 \). Given \( \beta_1, \ldots, \beta_N > 0 \) distinct and \( c_1, \ldots, c_N \in \mathbb{R} \), the multisoliton of degree \( N \) (or \( N \)-soliton) with these parameters is

\[
Q_{\beta, c}(x) = -2 \frac{d^2}{dx^2} \log \det[A(x)],
\]

where \( A(x) \) is the \( N \times N \) matrix with entries

\[
A_{jk}(x) = \delta_{jk} + \frac{1}{\beta_j + \beta_k} e^{-\beta_j (x - c_j) - \beta_k (x - c_k)}.
\]

The unique solution to KdV (1.1) with initial data \( u(0, x) = Q_{\beta, c}(x) \) is

\[
u(t, x) = Q_{\beta, c(t)}(x), \quad c_j(t) = c_j + 4 \beta_j^2 t.
\]

We define the multisoliton of degree zero to be the zero function.

Formula (1.3) was first discovered in [21] as part of a study of potentials for one-dimensional Schrödinger operators with vanishing reflection coefficient. Multisolitons have since been thoroughly examined by means of inverse scattering theory; see for example [14, 15, 19, 36, 37, 41].

The variational study of solitons (1.2) dates back to Boussinesq [9]. In addition to noting the conservation of the momentum and energy functionals

\[
E_1(u) = \int_{-\infty}^{\infty} \frac{1}{2} u^2 \, dx \quad \text{and} \quad E_2(u) = \int_{-\infty}^{\infty} \left[ \frac{1}{2} (u')^2 + u^3 \right] \, dx,
\]

he also realized that solitons are critical points for \( E_2 \) with \( E_1 \) constrained. We now know that the functionals (1.4) are the beginning of an infinite sequence of conserved quantities [32], because KdV is a completely integrable system. Their densities are defined recursively by [40]:

\[
\sigma_1(x) = u(x), \quad \sigma_{m+1}(x) = -\sigma_m''(x) - \sum_{j=1}^{m-1} \sigma_j(x) \sigma_{m-j}(x).
\]

For even \( m \) the density \( \sigma_m \) is a complete derivative, but for odd \( m \) we obtain a nontrivial conserved quantity

\[
E_n(u) = (-1)^n \frac{1}{2} \int_{-\infty}^{\infty} \sigma_{2n+1}(x) \, dx
\]

whose density is a polynomial in \( u, u', \ldots, u^{(n-1)} \). The first two functionals of this sequence are (1.4), and the next one is given by

\[
E_3(u) = \int_{-\infty}^{\infty} \left[ \frac{1}{2} (u'')^2 + 5u(u')^2 + \frac{5}{2} u^4 \right] \, dx.
\]
Nearly a century after Boussinesq’s work, Lax [26] studied two-soliton solutions of KdV in an effort to explain the particle-like interaction of solitons. Although he does not explicitly state it (until his later work [27]), his ODE for the two-soliton is the Euler–Lagrange equation for a critical point of $E_3$ with $E_1$ and $E_2$ constrained. In general, the $n$-soliton is known to be a critical point for the following variational problem:

**Problem 1.2** Given an integer $n \geq 0$ and constraints $e_1, \ldots, e_n$, minimize $E_{n+1}(u)$ over the set

$C_e = \{ u \in H^n(\mathbb{R}) : E_1(u) = e_1, \ldots, E_n(u) = e_n \}.$

In the case $n = 0$ there are no constraints, and the minimizer of the $L^2$-norm over the space $L^2(\mathbb{R})$ is simply the zero-soliton $q(x) \equiv 0$. We include this trivial observation because it will provide a convenient base case for an induction argument.

This variational problem has been heavily studied within the context of multisoliton stability. The first result was produced by Benjamin [6], who proved that solitons are orbitally stable in $H^1(\mathbb{R})$: solutions that start close to a soliton profile in $H^1(\mathbb{R})$ remain close to a soliton profile for all time. This was the introduction of a widely-applicable variational argument (cf. [38]) based on the fact that solitons are local constrained minimizers of $E_2$. Benjamin’s original work did contain some mathematical gaps however, but these were later resolved by Bona [7].

Solitons are not merely local minimizers for this problem, but are *global* minimizers [1] (see also Theorem 1.3 below). In order to employ this to prove orbital stability though, we need to know that profiles that almost minimize $E_2$ are close to a minimizing soliton. In general, we cannot expect minimizing sequences to admit convergent subsequences, because the manifold of minimizing solitons is translation-invariant and hence non-compact. This issue was solved by Cazenave and Lions [10] for a variety of NLS-like equations by a concentration compactness principle: minimizing sequences are precompact modulo translations. This powerful method is now the dominant way of proving orbital stability, but it has not yet been successfully applied to this variational problem because it requires a global understanding.

Nevertheless, Maddocks and Sachs [28] discovered that $n$-solitons are orbitally stable in $H^n(\mathbb{R})$. They first showed that $n$-solitons are indeed local minimizers of $E_{n+1}$ with $E_1, \ldots, E_n$ constrained. Then their argument relied on a careful study of the Hessian of $E_{n+1}$ on the manifold of minimizing $n$-solitons in directions tangent and perpendicular to the constraints. A key ingredient in analyzing the tangent directions was the formula

$$E_n(Q_{\beta,c}) = (-1)^{n+1} \sum_{m=1}^{N} \beta_m^{2n+1}$$

for the value of $E_n$ at an $N$-soliton, which is independent of $c$. This local analysis then implied the global result by employing the commuting flows of $E_1, \ldots, E_n$.

However, the variational problem remains unsolved: are multisolitons *global* constrained minimizers of $E_{n+1}$? If so, are they unique? In particular, affirmative answers to this problem would be a significant step towards applying concentration compactness to prove the orbital stability of multisolitons.

More generally, we would like to understand all solutions to this natural problem because they are basic building blocks. Indeed, special solutions to completely integrable models elucidate an avenue to a low-complexity understanding of the dynamics. A critical point for this variational problem must satisfy the Euler–Lagrange equation

$$\nabla E_{n+1}(u) = \lambda_1 \nabla E_1(u) + \lambda_2 \nabla E_2(u) + \cdots + \lambda_n \nabla E_n(u).$$

(1.8)
Solutions to these equations are called algebro-geometric solutions, and they are fundamental objects for integrable systems [17, 18]; in fact, solitons were discovered via (1.8). Naturally, we would like to understand all critical points for this variational problem.

In order to state our main results, we will introduce some notation. Given $n \geq 1$, we define the set of feasible constraints

$$\mathcal{F}^n = \{(e_1, \ldots, e_n) \in \mathbb{R}^n : \mathcal{C}_e \neq \emptyset\}$$

which are attainable by some function in $H^n(\mathbb{R})$. We also define the set of constraints attainable by multisolitons of degree at most $N \geq 0$:

$$\mathcal{M}_N^n = \{(e_1, \ldots, e_n) \in \mathbb{R}^n : \exists M \leq N \text{ and } \beta, c \in \mathbb{R}^M \text{ with } Q_{\beta, c} \in \mathcal{C}_e\}.$$ 

First we prove that when the constraints are attainable by a multisoliton of degree at most $n$, that multisoliton is the unique global minimizer:

**Theorem 1.3 (Variational characterization)** Fix an integer $n \geq 1$. Given constraints $(e_1, \ldots, e_n) \in \mathcal{M}_n^n$, there exists a unique integer $N \leq n$ and parameters $\beta_1 > \cdots > \beta_N > 0$ so that the multisoliton $Q_{\beta, c}$ lies in $\mathcal{C}_e$ for some (and hence all) $c \in \mathbb{R}^N$. Moreover, we have

$$E_{n+1}(u) \geq E_{n+1}(Q_{\beta, c}) \text{ for all } u \in \mathcal{C}_e,$$

with equality if and only if $u = Q_{\beta, c}$ for some $c \in \mathbb{R}^N$.

Together with an appropriate concentration compactness principle (Theorem 5.2) to analyze minimizing sequences, we also prove the orbital stability result of [28] via concentration compactness:

**Theorem 1.4 (Orbital stability)** Fix an integer $n \geq 1$ and distinct positive parameters $\beta_1, \ldots, \beta_n$. Given $\epsilon > 0$ there exists $\delta > 0$ so that for every initial data $u(0) \in H^n(\mathbb{R})$ satisfying

$$\inf_{c \in \mathbb{R}^n} \|u(0) - Q_{\beta, c}\|_{H^n} < \delta,$$

the corresponding solution $u(t)$ of KdV (1.1) satisfies

$$\sup_{t \in \mathbb{R}} \inf_{c \in \mathbb{R}^n} \|u(t) - Q_{\beta, c}\|_{H^n} < \epsilon.$$

It is now well-known that KdV is well-posed for initial data $u(0) \in H^n(\mathbb{R})$ with $n \geq 1$ [22]. Nevertheless, the $n = 1$ case [6] of Theorem 1.4 came nearly two decades before the corresponding well-posedness result [22]. This is because well-posedness plays only a small role in the proof of Theorem 1.4. Indeed, the crux of the problem is to prove Theorem 1.4 for Schwartz solutions, and the result for $H^n(\mathbb{R})$ solutions then follows immediately once well-posedness in $H^n(\mathbb{R})$ is known.

Many refinements have been made since the discovery of Theorem 1.4. While we choose to work in the space $H^n(\mathbb{R})$ because it is amenable to the functionals $E_1, \ldots, E_{n+1}$, the regularity of Theorem 1.4 in the scope of $H^s(\mathbb{R})$ spaces has since been significantly lowered [5, 24, 30, 31]. Moreover, the time-evolution of the parameter $c$ has been shown to remain close to the usual evolution $c_j + 4\beta_j^2 t$ uniformly for $t > 0$ [3]. In addition to Lyapunov stability statements like Theorem 1.4, the asymptotic stability of multisolitons has also been studied [29–31].

Recently, the $n = 2$ case of both Theorems 1.3 and 1.4 was resolved by Albert and Nguyen [4]. They also showed that for $n = 1$ we have

$$\mathcal{M}_1^1 = \{e_1 : e_1 \geq 0\} = \mathcal{F}^1,$$
Multisolitons are the unique constrained minimizers...

Fig. 1 The sets (1.9) and (1.11) of constraints. The three shaded regions correspond to qualitatively different behavior for minimizing sequences

but for \( n = 2 \) we have

\[
\mathcal{M}_2^2 = \left\{ (e_1, e_2) : e_1 > 0, \ e_2 \in \left[ -\frac{32}{5} \left( \frac{3}{8} \right)^3 e_1^3, -2 - \frac{2}{3} \frac{32}{5} \left( \frac{3}{8} \right)^3 e_1^3 \right] \right\} \cup \{(0, 0)\},
\]

(1.9)

These sets are depicted in Fig. 1. Theorem 1.3 says that for each \((e_1, e_2) \in \mathcal{M}_2^2\) the constrained minimizers of \(E_3\) are multisolitons of degree at most 2. Moreover, by the \( n = 1 \) case of Theorem 1.3, we know that for the constraints \(e_1 > 0\) and \(e_2 = -\frac{32}{5} \left( \frac{3}{8} \right)^3 e_1^3\) on the boundary of \(\mathcal{M}_2^2\) the minimizer is a single soliton. Likewise, in the case of \( n = 3 \) constraints, the boundary \(\mathcal{M}_3^3 \setminus (\text{int} \ \mathcal{M}_3^3)\) in \(\mathbb{R}^3\) looks like the graph of a continuous function on \(\mathcal{M}_2^2\), and so on. In general, we will show that \(\mathcal{M}_n^n\) is homeomorphic to the half-open simplex of parameters \(\beta \in \mathbb{R}^n\) corresponding to multisolitons of degree at most \( n \) (cf. Lemma 3.4).

Albert and Nguyen’s analysis does not easily extend to the general case however, because it makes crucial use of the fact that for \( n = 2 \) all solutions of the Euler–Lagrange equation (1.8) are one- or two-solitons [2]. Much is known about the ODEs (1.8); they are completely integrable Hamiltonian systems and thus can be formally integrated. Nevertheless, for \( n \geq 3 \) it is open whether all solutions to the Euler–Lagrange equation (1.8) are multisolitons of degree at most \( n \). (Specifically, it is difficult to show that if the Lagrange multipliers \(\lambda_1, \ldots, \lambda_n\) do not correspond to an \( n \)-soliton then solutions of (1.8) must be singular; see [2, §6] for details.)

Another advantage of our method is that it enables us to study the variational problem and minimizing sequences even when the constraints \((e_1, \ldots, e_n) \notin \mathcal{M}_n^n\) are not attainable by a multisoliton of degree at most \( n \). In order to present our results in this direction, we first need to recall some scattering theory. Ever since the seminal work [14], scattering theory for the one-dimensional Schrödinger operator \(-\partial_x^2 + u\) on the weighted Lebesgue space

\[
L_1^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \text{ such that } \int_{-\infty}^{\infty} |f(x)|(1 + |x|^2) \, dx < \infty \right\},
\]

the operator \(-\partial_x^2 + u\) on \(L^2(\mathbb{R})\) has purely absolutely continuous spectrum \([0, \infty)\) and finitely many simple negative eigenvalues \(-\beta_1^2, \ldots, -\beta_N^2\). For such \( u \) we can define the transmission and reflection coefficients at frequencies \( k \in \mathbb{R} \), both of which are bounded by 1 in modulus.
Multisolitons are distinguished in this class by having identically vanishing reflection coefficients. Additionally, the transmission coefficient \( T(k; u) \) extends to a meromorphic function of \( k \) in the upper half-plane \( \mathbb{C}^+ \) that is continuous down to \( \mathbb{R} \) and whose only singularities are simple poles at the square roots \( i \beta_1, \ldots, i \beta_N \) of the eigenvalues. Analytically, it is more convenient to work with the reciprocal \( a(k; u) \) of the transmission coefficient; this function is holomorphic in \( \mathbb{C}^+ \) with simple zeros at \( i \beta_1, \ldots, i \beta_N \).

The Zakharov–Faddeev trace formulas [40] for the polynomial conserved quantities \( E_n(u) \) generalize the formula (1.7) for \( u = Q_{\beta, e} \). Explicitly, this sequence begins

\[
E_1(u) = \frac{4}{\pi} \int_{-\infty}^{\infty} k^2 \log |a(k; u)| \, dk + \frac{8}{3} \sum_{m=1}^{N} \beta_m^3,
\]

\[
E_2(u) = \frac{16}{\pi} \int_{-\infty}^{\infty} k^4 \log |a(k; u)| \, dk - \frac{32}{5} \sum_{m=1}^{N} \beta_m^5,
\]

\[
E_3(u) = \frac{64}{\pi} \int_{-\infty}^{\infty} k^6 \log |a(k; u)| \, dk + \frac{128}{7} \sum_{m=1}^{N} \beta_m^7.
\]

These are valid for \( u \) Schwartz, where we have both the energies \( E_n(u) \) and the scattering data \( a(k; u) \) and \( \beta_1, \ldots, \beta_N \) at our disposal. The first terms on the RHS are the even moments of the nonnegative measure \( \log |a(k; u)| \) on \( \mathbb{R} \). The second terms are the odd moments of the positive numbers \( \beta_1, \ldots, \beta_N \), alternating in sign. For multisolitons \( q \) we have \( |a(k; q)| \equiv 1 \) on \( \mathbb{R} \), and so the log \( |a(k; u)| \) moments vanish and we recover (1.7).

After obtaining the \( n = 2 \) case of Theorems 1.3 and 1.4, Albert and Nguyen [4] made the reasonable conjecture for the remaining case \( (e_1, e_2) \in \mathcal{F}^2 \setminus \mathcal{M}_2^1 \) that minimizing sequences resemble a collection of solitons with at least two \( \beta \) parameters equal. Using our methods, we prove that this is partially true provided that the constraints are attainable by some multisoliton:

**Theorem 1.5** Given constraints \( (e_1, \ldots, e_N) \in \mathcal{M}_N^a \setminus \mathcal{M}_N^{a-1} \) for some \( N \geq n + 1 \), the infimum of \( E_{n+1}(u) \) over \( u \in \mathcal{C}_e \) is finite but not attained.

Moreover, if \( \{q_k\}_{k \geq 1} \subset \mathcal{C}_e \) is a minimizing sequence:

\[
E_1(q_k) \to e_1, \quad \ldots, \quad E_n(q_k) \to e_n, \quad E_{n+1}(q_k) \to \inf_{u \in \mathcal{C}_e} E_{n+1}(u) \quad \text{as} \quad k \to \infty,
\]

then there exist parameters \( \beta^1, \ldots, \beta^j \) of total degree \( \sum_{j=1}^{j} \#\beta^j = N \) taking at most \( n \) distinct values so that along a subsequence we have

\[
\inf_{e^1, \ldots, e^j} \left\| q_k - \sum_{j=1}^{j} Q_{\beta^j, e^j} \right\|_{H^n} \to 0 \quad \text{as} \quad k \to \infty.
\]

We must have multiple multisolitons in the conclusion of Theorem 1.5, because two multisolitons with a common \( \beta \) parameter necessarily become infinitely separated as \( k \to \infty \). On the other hand, we cannot guarantee \( N \) single-soliton profiles as originally conjectured in [4], because two distinct values of the minimizing parameters \( \beta \) can correspond to a multisoliton that does not resemble well-separated solitons.

Theorem 1.5 still does not account for all of the remaining feasible constraints \( \mathcal{F}^n \). In general, we can compute the boundary of \( \mathcal{M}_N^a \) by finding the extrema of \( E_{n+1} \) for \( \beta_1, \ldots, \beta_N > 0 \) distinct. For \( n = 2 \), it is not difficult to show that
\[ \mathcal{M}_N^2 = \left\{ (e_1, e_2) : e_1 > 0, e_2 \in \left[ -\frac{32}{3} \left( \frac{3}{8} \right)^5 e_1^5, -N^{-\frac{2}{3}} \frac{32}{3} \left( \frac{3}{8} \right)^5 e_1^5 \right) \right\} \cup \{(0, 0)\} \]

for all \( N \geq 2 \). Indeed, this is closely related to the equality inequality

\[ N^{-\frac{2}{5}} \left( \sum_{m=1}^{N} \beta_m^3 \right)^{\frac{5}{3}} \leq \sum_{m=1}^{N} \beta_m^5 \leq \left( \sum_{m=1}^{N} \beta_m^3 \right)^{\frac{5}{3}} \]

which expresses the equivalence of the \( \ell^3 \)- and \( \ell^5 \)-norms on \( \mathbb{R}^N \). The sets \( \mathcal{M}_N^2 \) are depicted in the phase diagram of Fig. 1, and can also be understood in terms of the Wigner–von Neumann example of a multisoliton, Albert–Nguyen’s conjecture cannot be true and entirely different behavior is exhibited. To illustrate this point, we characterize Schwartz minimizing sequences in the case \( n = 2 \):

**Theorem 1.6** Given constraints \( (e_1, e_2) \in \mathcal{F}^2 \) with \( (e_1, e_2) \notin \mathcal{M}_N^2 \) for all \( N \), the infimum of \( E_3(u) \) over \( u \in \mathcal{C}_e \cap \mathcal{S}(\mathbb{R}) \) is finite but not attained.

Moreover, if \( \{q_j\}_{j \geq 1} \subset \mathcal{C}_e \cap \mathcal{S}(\mathbb{R}) \) is a minimizing sequence:

\[ E_1(q_j) \to e_1, \quad E_2(q_j) \to e_2, \quad E_3(q_j) \to \inf_{u \in \mathcal{C}_e} E_3(u) \text{ as } j \to \infty, \]

then \( \beta_{j,m} \to 0 \) as \( j \to \infty \) for all \( m \), and \( k \mapsto \log |a(k; q_j)| \) converges in the sense of distributions to the even extension of a unique Dirac delta distribution (i.e. \( c_0(d\delta_{k_0}(k) + d\delta_{-k_0}(k)) \) for unique constants \( c_0, k_0 \geq 0 \)).

What might such a minimizing sequence look like? We can exhibit a family of these sequences by means of an ansatz inspired by the Wigner–von Neumann example of a Schrödinger potential with a positive eigenvalue. Given parameters \( c > 0 \) and \( k \geq 0 \), it is straightforward to check that the sequence

\[ q_n(x) = \sqrt{\frac{2}{n}} e^{-\frac{x^2}{2n^2}} \cos(2kx) \]

obeys

\[ E_1(q_n) \to \frac{\sqrt{\pi}}{4} c, \quad E_2(q_n) \to \sqrt{\pi} c k^2, \quad E_3(q_n) \to 4\sqrt{\pi} c k^4 \]

as \( n \to \infty \). In the proof of Theorem 1.6 we will explicitly compute the constrained infimum of \( E_3 \), and it is given by \( e_2^2 e_1^{-1} \) (cf. Lemma 7.2). We see that the limit of \( E_3(q_n) \) above is exactly equal to this quantity, and so \( \{q_n\}_{n \geq 1} \) is a Schwartz minimizing sequence. By Theorem 1.6, we deduce that this sequence has vanishing \( \beta \) parameters and \( \log |a(k; q_n)| \) converging to the even extension of a delta distribution.

This paper is organized as follows. In Sect. 2 we recall some scattering theory and facts about the energy functionals \( E_n \), with the Zakharov–Faddeev trace formulas (2.7) lying at the center. In Sect. 3, we then further analyze the functionals \( E_1, \ldots, E_{n+1} \) on the manifold of multisolitons and use this to describe the set \( \mathcal{M}_n^a \) of constraints.
The proof of Theorem 1.3 is then presented in Sect. 4. A key step is realizing that in order to minimize $E_{n+1}$, a minimizer $q$ must satisfy $\log |a(k; q)| \equiv 0$ on $\mathbb{R}$ (cf. (2.4) and (2.7)). We know that $\log |a(k; u)|$ can be brought all the way down to zero since the constraints can be met solely by the moments of $\beta \in \mathbb{R}^n$; this is why we must assume that $(e_1, \ldots, e_n) \in \mathcal{M}_n^n$ in Theorem 1.3. Next, we prove that $|a(k; q)| \equiv 1$ on $\mathbb{R}$ if and only if $q$ is a multisoliton. The “if” statement is already known from scattering theory (cf. (2.9)). For the reverse implication, we use some classical complex analysis to characterize $k \mapsto a(k; q)$ on $\mathbb{C}^+$ and conclude that $q$ is a multisoliton (cf. Lemma 4.6). It then remains to show that on the manifold of multisolitons, there is a unique minimizing set of $\beta$ parameters. First, we can rule out the case of $N \geq n + 1$ parameters by a variational argument (cf. Lemma 3.1). For the remaining $N \leq n$ unknown parameters $\beta_1, \ldots, \beta_N$, the formulas (1.7) for the $n$ constraints provide a nonlinear system of equations, which we show has a unique solution in Lemma 3.2.

In Sect. 5 we apply a concentration compactness principle (Theorem 5.2) to minimizing sequences in order to prove Theorem 1.4.

In Sect. 6, we prove Theorem 1.5 by adapting the methods of Sects. 4 to 5 to allow for non-distinct $\beta$ parameters.

Finally, in Sect. 7 we prove Theorem 1.6. The proof is again based on the trace formulas (1.10). The condition $(e_1, e_2) \notin \mathcal{M}_2^n$ requires that the log $|a(k; q_j)|$ moments are nonvanishing as $j \to \infty$. Consequently, the sequence $\{q_j\}_{j \geq 1}$ is minimizing a constrained moment problem for measures, and such minimizers are finite linear combinations of point masses. This particular moment problem for $n = 2$ is easily solved using the Cauchy–Schwarz inequality, but for general $n$ it is a Stieltjes moment problem. (We recommend [33] for an introduction to this classical analysis result.)

2 Preliminaries

In this section, we recall some facts about the energy functionals $E_n$ and some results from scattering theory for future reference. In particular, this will enable us to formulate the Zakharov–Faddeev trace formulas (2.7) that lie at the heart of our analysis.

The functionals (1.4) and (1.6) are the beginning of an infinite sequence of polynomial conserved quantities (1.5). We will only need certain properties of these functionals however, rather than their exact formula.

Proposition 2.1 ([32]) Given an integer $n \geq 0$, there exists a functional of the form

$$E_{n+1}(u) = \int_{-\infty}^{\infty} \left[ \frac{1}{2} (u^{(n)})^2 + P_{n+1}(u) \right] dx$$

(2.1)

that is conserved for Schwartz solutions $u(t)$ to the KdV equation (1.1), where $P_{n+1}$ is a polynomial in $u, u', \ldots, u^{(n-1)}$. Moreover, each term of $P_{n+1}$ is of the form $c_{\alpha_1, \ldots, \alpha_d} u^{(\alpha_1)} \cdots u^{(\alpha_d)}$ with $d \geq 3$ and obeys

$$\sum_{j=1}^{d} \alpha_j = 2n + 4 - 2d \quad \text{and} \quad 0 \leq \alpha_j \leq n - 1.$$

(2.2)

Each term of $P_{n+1}$ is of cubic or higher order, and the condition (2.2) simply says that they share the same scaling symmetry as the quadratic term $\frac{1}{2} (u^{(n)})^2$. In particular, this requires that the degree of $P_{n+1}$ is at most $n + 2$.

When combined with Sobolev embedding, a classical argument (cf. [27, Th. 3.1] in the periodic case) yields estimates for the functionals (2.1):
Lemma 2.2  Given $n \geq 0$, we have

\[ E_{n+1}(u) \lesssim \|u\|_{H^n} \quad \text{and} \quad \|u\|_{H^n} \lesssim E_1(u), \ldots, E_{n+1}(u) \quad (2.3) \]

uniformly for $u \in S(\mathbb{R})$. Moreover, $E_{n+1} : H^n(\mathbb{R}) \to \mathbb{R}$ is continuous.

Here (and throughout this paper), we are using the familiar $L^2$-based Sobolev spaces $H^s(\mathbb{R})$ (and the $L^p$-based spaces $W^{j,p}(\mathbb{R})$) of real-valued functions on $\mathbb{R}$. In addition to these classes, the scattering theory that we need to state our trace formulas will require that we work in the weighted $L^1$-spaces

\[ L^1_j(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R} \text{ such that } \int_{-\infty}^{\infty} |f(x)|(1+|x|^j) \, dx < \infty \right\} \]

with $j \geq 1$. When we need a common ground, we will use the class $S(\mathbb{R})$ of Schwartz functions.

Given a potential $q \in L^1_j(\mathbb{R})$ and $k \in \mathbb{R} \setminus \{0\}$, the Jost functions $f_j(x; k)$ are the unique solutions to the corresponding Schrödinger equation

\[ -f'' + qf_j = k^2 f_j, \quad j = 1, 2 \]

with the asymptotics

\[ f_1(x; k) \sim e^{ikx} \quad \text{as } x \to +\infty, \quad f_2(x; k) \sim e^{-ikx} \quad \text{as } x \to -\infty. \]

The transmission and reflection coefficients $T_j(k)$ and $R_j(k)$ are then uniquely determined by

\[ T_1(k) f_2(x; k) = R_1(k) f_1(x; k) + f_1(x; -k), \]

\[ T_2(k) f_1(x; k) = R_2(k) f_2(x; k) + f_2(x; -k). \]

Forward scattering theory tells us that the transmission and reflection coefficients satisfy the following properties. Proofs of these facts can be found in many introductory texts on the subject; however, we recommend the paper [12, §2 Th. 1] of Deift and Trubowitz for a complete and self-contained proof.

Proposition 2.3  (Forward scattering theory) If $q \in L^1_j(\mathbb{R})$, then the scattering matrix

\[ S(k) := \begin{pmatrix} T_1(k) & R_2(k) \\ R_1(k) & T_2(k) \end{pmatrix} \]

extends to a continuous function of $k \in \mathbb{R}$ satisfying the following properties:

(i)  (Symmetry) For all $k \in \mathbb{R}$,

\[ T_1(k) = T_2(k) =: T(k). \]

(ii)  (Unitarity) The matrix $S(k)$ is unitary for all $k \in \mathbb{R}$:

\[ T(k)R_2(k) + R_1(k) \overline{T}(k) \equiv 0, \quad |T(k)|^2 + |R_j(k)|^2 \equiv 1 \quad \text{for } j = 1, 2. \]

(iii)  (Analyticity) $T(k)$ is meromorphic in the open upper half-plane $\mathbb{C}^+$ and is continuous down to $\mathbb{R}$. Moreover, $T(k)$ has a finite number of poles $i\beta_1, \ldots, i\beta_N$, all of which are simple and on the imaginary axis, and $-\beta_1^2, \ldots, -\beta_N^2$ are the bound states of the operator $-\partial_x^2 + q$. 

Springer
(iv) (Asymptotics) We have
\[ T(k) = 1 + O\left(\frac{1}{k}\right) \text{ as } |k| \to \infty \text{ uniformly for } \text{Im} \ k \geq 0, \]
\[ R_j(k) = O\left(\frac{1}{k}\right) \text{ as } |k| \to \infty, \ k \in \mathbb{R}. \]

(v) (Rate at \( k = 0 \)) \( T(k) \) is nonvanishing for \( k \in \mathbb{C}^+ \setminus \{0\} \), and either
(a) \( |T(k)| \geq c > 0 \text{ for all } k \in \mathbb{C}^+ \), or
(b) \( T(k) = T'(0)k + o(k) \text{ for } k \in \mathbb{C}^+ \) with \( T'(0) \neq 0 \) and \( R_j(k) = -1 + R_j'(0)k + o(k) \) for \( k \in \mathbb{R} \).

(vi) (Reality) For all \( k \in \mathbb{R} \),
\[ \overline{T(k)} = T(-k), \quad \overline{R_j(k)} = R_j(-k) \text{ for } j = 1, 2. \]

Our trace formulas are most conveniently stated in terms of the reciprocal of the transmission coefficient:
\[ a(k; q) := \frac{1}{T(k)}. \]

For \( q \in L^1_2(\mathbb{R}) \), Proposition 2.3 tells us that \( k \mapsto a(k; q) \) is a holomorphic function on the open upper-half plane \( \mathbb{C}^+ \) and is continuous down to \( \mathbb{R} \). It has finitely many zeros \( i\beta_1, \ldots, i\beta_N \in \mathbb{C}^+ \), all of which are simple and on the imaginary axis. Moreover, we have the boundary conditions
\[ |a(k; q)| \geq 1 \text{ for all } k \in \mathbb{R}, \]
\[ |a(k; q) - 1| = O\left(\frac{1}{|k|}\right) \text{ as } |k| \to \infty \text{ uniformly for } \text{Im} \ k \geq 0, \]
along with the reality condition
\[ a(k; q) = a(-k; q) \text{ for all } k \in \mathbb{C}^+. \]

For \( u \in S(\mathbb{R}) \), the Zakharov–Faddeev trace formulas [40] provide an alternative representation of the polynomial conserved quantities:
\[ E_n(u) = \frac{2^n}{\pi} \int_{-\infty}^{\infty} k^{2n} \log |a(k; u)| \, dk + (-1)^{n+1} \frac{2^{2n+1}}{2n+1} \sum_{m=1}^{N} \beta_m^{2n+1}. \]  

The measure \( \log |a(k; u)| \, dk \) on \( \mathbb{R} \) is nonnegative and even, and the first terms on the RHS are its even moments (starting with the second), which are finite for \( u \in S(\mathbb{R}) \) [40]. The second terms are the odd moments of the distinct positive numbers \( \beta_1, \ldots, \beta_N \) (starting with the third) and are alternating in sign.

Later, we will deduce that a constrained minimizer \( q \) of \( E_{n+1} \) must have certain scattering data due to the trace formulas (2.7). Consequently, it will be useful to know when we can reconstruct the potential \( q \) from the scattering data [12, §5 Th. 3]:

**Proposition 2.4** (Inverse scattering theory) A matrix
\[ S(k) := \begin{pmatrix} T_1(k) & R_2(k) \\ R_1(k) & T_2(k) \end{pmatrix}, \quad k \in \mathbb{R} \]
is the scattering matrix for some \( q \in L^1_2(\mathbb{R}) \) without bound states if and only if
(i) (Symmetry) For all \( k \in \mathbb{R} \),
\[ T_1(k) = T_2(k) =: T(k). \]
(ii) (Unitarity) The matrix $S(k)$ is unitary for all $k \in \mathbb{R}$:

$$
T(k)R_2(k) + R_1(k)\overline{T}(k) \equiv 0, \quad |T(k)|^2 + |R_j(k)|^2 \equiv 1 \quad \text{for } j = 1, 2.
$$

(iii) (Analyticity) $T(k)$ is analytic in the open upper half-plane $\mathbb{C}^+$ and is continuous down to $\mathbb{R}$.

(iv) (Asymptotics) We have

$$
T(k) = 1 + O\left(\frac{1}{k}\right) \quad \text{as } |k| \to \infty \text{ uniformly for } \Im k \geq 0,
$$

$$
R_j(k) = O\left(\frac{1}{k}\right) \quad \text{as } |k| \to \infty, \quad k \in \mathbb{R}.
$$

(v) (Rate at $k = 0$) $T(k)$ is nonvanishing for $k \in \mathbb{C}^+ \setminus \{0\}$, and either

(a) $|T(k)| \geq c > 0$ for all $k \in \mathbb{C}^+$, or

(b) $T(k) = T'(0)k + o(k)$ for $k \in \mathbb{C}^+$ with $T'(0) \neq 0$ and $R_j(k) = -1 + R_j'(0)k + o(k)$ for $k \in \mathbb{R}$.

(vi) (Reality) For all $k \in \mathbb{R}$,

$$
\overline{T}(k) = T(-k), \quad \overline{R_j(k)} = R_j(-k) \quad \text{for } j = 1, 2.
$$

(vii) (Fourier decay) The Fourier transforms $F_j := \hat{R}_j$, $j = 1, 2$ are absolutely continuous and

$$
\int_{-\infty}^{a} |F_1'(|\kappa|)|1 + \kappa^2\rangle\,d\kappa < \infty \quad \text{and} \quad \int_{a}^{\infty} |F_2'(|\kappa|)|1 + \kappa^2\rangle\,d\kappa < \infty
$$

for all $a \in \mathbb{R}$.

The characterization in Proposition 2.4 is most easily stated in terms of potentials $q$ without bound states. This does not pose a problem though, because there is an explicit formula for modifying $q$ in order to prescribe any number of bound states [12, §3 Th. 6]. Rather than the explicit formula for $q$, we will only need to keep track of the changes in the transmission coefficient:

**Proposition 2.5** (Adding bound states) Let $q(x) \in L^1_1(\mathbb{R})$ be a potential without bound states and $\beta_1, \ldots, \beta_N > 0$ distinct. Then there exists a potential $q(x; +N) \in L^1_1(\mathbb{R})$ with the $N$ bound states $-\beta_1^2, \ldots, -\beta_N^2$. Moreover, the transmission coefficient is related to that of $q(x)$ via

$$
T(k; +N) = T(k) \prod_{m=1}^{N} \frac{k + i\beta_m}{k - i\beta_m}.
$$

(2.8)

Within this narrative, multisolitons are characterized by having vanishing reflection coefficients. In view of the preceding, this identifies the formula for $a(k; q)$:

**Corollary 2.6** (Characterization of multisolitons) Given distinct $\beta_1, \ldots, \beta_N > 0$ and $q \in \mathcal{S}(\mathbb{R})$, we have $q = Q_{\beta, \epsilon}$ for some $\epsilon \in \mathbb{R}^N$ if and only if

$$
a(k; q) = \prod_{m=1}^{N} \frac{k - i\beta_m}{k + i\beta_m}.
$$

(2.9)

Notice that the formula (2.9) is a finite Blaschke product from $\mathbb{C}^+$ to the unit disk, with zeros that are distinct and lie only on the imaginary axis. In particular, we see that multisolitons $Q_{\beta, \epsilon}$ satisfy $|a(k; Q_{\beta, \epsilon})| \equiv 1$ on $\mathbb{R}$, and so the log $|a|$ moments in the trace formulas (2.7) vanish. Consequently, the functionals $E_n$ for a multisoliton $Q_{\beta, \epsilon}$ are independent of $\epsilon$. 

\[\text{Springer}\]
3 The polynomial conserved quantities

The purpose of this section is to analyze the functionals $E_1, \ldots, E_{n+1}$ restricted to the manifold of multisolitons, and to then use this to describe the set $\mathcal{M}_n^m$ of constraints.

First, we will prove that as long as we have at least $n+1$ distinct positive $\beta$ parameters at our disposal, we can decrease $E_{n+1}$ while preserving $E_1, \ldots, E_n$. This surprisingly powerful fact will turn out to be an integral part of our analysis.

**Lemma 3.1** Fix $n \geq 1$, and suppose $Q_{\beta, c}$ is a multisoliton of degree $N \geq n+1$. Then there exist $\bar{\beta}_1, \ldots, \bar{\beta}_N > 0$ distinct so that

$$E_m(Q_{\bar{\beta}, c}) = E_m(Q_{\beta, c}) \quad \text{for } m = 1, \ldots, n, \quad \text{but } E_{n+1}(Q_{\bar{\beta}, c}) < E_{n+1}(Q_{\beta, c}).$$

**Proof** We will apply the implicit function theorem to the first $n+1$ trace formulas (2.7) as functions of $\beta$. Reorder $\beta$ so that $\beta_1 > \cdots > \beta_N > 0$. Define the function $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ by

$$f(x_1, \ldots, x_{n+1}) = \begin{pmatrix} x_1^3 + x_2^3 + \cdots + x_{n+1}^3 \\ x_1^2 + x_2^2 + \cdots + x_{n+1}^2 \\ \vdots \\ x_1^{2n+1} + x_2^{2n+1} + \cdots + x_{n+1}^{2n+1} \end{pmatrix}.$$  

(3.1)

This function has derivative matrix

$$Df(\beta_1, \ldots, \beta_{n+1}) = \begin{pmatrix} 3\beta_1^2 & \cdots & 3\beta_n^2 \\ 5\beta_1^4 & \cdots & 5\beta_n^4 \\ \vdots & \cdots & \vdots \\ (2n+1)\beta_1^{2n} & \cdots & (2n+1)\beta_n^{2n} \end{pmatrix}. \quad (3.2)$$

The left $n \times n$ block matrix is a Vandermonde matrix after pulling out common factors from each row and column, and thus has determinant

$$3 \cdot 5 \cdots (2n+1) \beta_1^2 \cdots \beta_n^2 \prod_{j<k}(\beta_k^2 - \beta_j^2). \quad (3.3)$$

This is nonvanishing as $\beta_1, \ldots, \beta_n$ are positive and distinct. The implicit function theorem then implies that there exists $\epsilon > 0$ and $C^1$ functions $x_1(x_{n+1}), \ldots, x_n(x_{n+1})$ defined on $(\beta_{n+1} - \epsilon, \beta_{n+1} + \epsilon)$ so that

$$f(x_1(x_{n+1}), \ldots, x_n(x_{n+1}), x_{n+1}) = \begin{pmatrix} \beta_1^3 + \cdots + \beta_{n+1}^3 \\ \beta_1^5 + \cdots + \beta_{n+1}^5 \\ \vdots \\ \beta_1^{2n+1} + \cdots + \beta_{n+1}^{2n+1} \end{pmatrix}.$$  

(3.4)

for $x_{n+1} \in (\beta_{n+1} - \epsilon, \beta_{n+1} + \epsilon)$.

It remains to show that we can pick $x_{n+1}$ in a way that decreases the next odd moment. To this end, we will compute its derivative at $x_{n+1} = \beta_{n+1}$:

$$\left. \frac{d}{dx_{n+1}} \right|_{x_{n+1} = \beta_{n+1}} [x_1(x_{n+1})^{2n+3} + \cdots + x_n(x_{n+1})^{2n+3} + x_{n+1}^{2n+3}]$$

$$= (2n+3) \begin{pmatrix} \beta_1^{2n+2} \cdots \beta_n^{2n+2} \\ \beta_1^{2n+2} \cdots \beta_n^{2n+2} \end{pmatrix}.$$  

(3.5)
The derivative of $x_1(x_{n+1})$, ..., $x_n(x_{n+1})$ is determined by differentiating (3.4) at $x_{n+1} = \beta_{n+1}$. This yields
\[
\begin{pmatrix}
  x'_1(\beta_{n+1}) \\
  \vdots \\
  x'_n(\beta_{n+1})
\end{pmatrix}
= - \begin{pmatrix}
  3\beta_1^2 & \cdots & 3\beta_n^2 \\
  \vdots & \ddots & \vdots \\
  (2n+1)\beta_1^{2n} & \cdots & (2n+1)\beta_n^{2n}
\end{pmatrix}^{-1}
\begin{pmatrix}
  3\beta_1^2 \\
  \vdots \\
  (2n+1)\beta_n^{2n}
\end{pmatrix}.
\tag{3.6}
\]
Inserting this into (3.5), we obtain an expression solely in terms of $\beta_1, \ldots, \beta_{n+1}$. In order to compute this, we will leverage that it is a Schur complement for the derivative matrix (3.2) but with an appended row. Specifically, if we define the $(n+1) \times (n+1)$ block matrix
\[
\begin{pmatrix}
  A & b \\
  c & d
\end{pmatrix}
= \begin{pmatrix}
  3\beta_1^2 & \cdots & 3\beta_n^2 & 3\beta_{n+1}^2 \\
  5\beta_1^4 & \cdots & 5\beta_n^4 & 5\beta_{n+1}^4 \\
  \vdots & \cdots & \vdots & \vdots \\
  (2n+3)\beta_1^{2n+2} & \cdots & (2n+3)\beta_n^{2n+2} & (2n+3)\beta_{n+1}^{2n+2}
\end{pmatrix},
\]
then LHS(3.5) is now given by
\[
d - cA^{-1}b.
\]
On the other hand, applying one step of Gaussian elimination to our block matrix yields
\[
\begin{pmatrix}
  A & b \\
  c & d
\end{pmatrix}
= \begin{pmatrix}
  I & 0 \\
  cA^{-1} & 1
\end{pmatrix}
\begin{pmatrix}
  A & b \\
  0 & d - cA^{-1}b
\end{pmatrix}.
\]
Taking the determinant of both sides, we deduce that LHS(3.5) is equal to
\[
\det \begin{pmatrix}
  A & b \\
  c & d
\end{pmatrix} (\det A)^{-1}.
\]
Both terms above can be computed by the Vandermonde determinant formula; the determinant of $A$ is given by (3.3) for $n$ and the determinant of the block matrix is given by (3.3) for $n + 1$.

Altogether, we conclude
\[
\frac{d}{dx_{n+1}}\bigg|_{x_{n+1} = \beta_{n+1}} \left[ x_1(x_{n+1})^{2n+3} + \cdots + x_n(x_{n+1})^{2n+3} + x_{n+1}^{2n+3} \right]
= (2n+3)\beta_{n+1}^2 \prod_{j=1}^{n} (\beta_{n+1}^2 - \beta_j^2).
\]
The RHS is nonvanishing since $\beta_1, \ldots, \beta_{n+1}$ are positive and distinct. As $\beta_1, \ldots, \beta_N$ were distinct to begin with, we conclude that there exists $x_{n+1} \in (\beta_{n+1} - \epsilon, \beta_{n+1} + \epsilon)$ sufficiently close to $\beta_{n+1}$ so that the values $x_1(x_{n+1}), \ldots, x_n(x_{n+1}), x_{n+1}$, and $\beta_{n+2}$ remain distinct, and
\[
(-1)^n \left[ x_1(x_{n+1})^{2n+3} + \cdots + x_n(x_{n+1})^{2n+3} + x_{n+1}^{2n+3} \right]
\]
is strictly less than its value at $x_{n+1} = \beta_{n+1}$. This quantity is the value of $E_{n+1}$ for the multisoliton with $\beta$ parameters $x_1(x_{n+1}), \ldots, x_n(x_{n+1}), x_{n+1}$. Replacing $\beta_1, \ldots, \beta_{n+1}$ by $x_1(x_{n+1}), \ldots, x_n(x_{n+1}), x_{n+1}$ in $\beta$, we obtain new distinct parameters $\beta_1 > \cdots > \beta_N > 0$ (with $\beta_j = \beta_j$ for $j \geq n + 2$) so that the multisoliton $Q_{\beta, e}$ decreases $E_{n+1}$ while preserving $E_1, \ldots, E_n$.

Our next step is to find the unique set of distinct $\beta$ parameters with at most $n$ values that attain the constraints. As $(e_1, \ldots, e_n) \in \mathcal{M}_n^e$, it only remains to show that there is at most one solution:
Lemma 3.2 Fix $n \geq 1$. Given constraints $e_1, \ldots, e_n$, there is at most one choice of $N \leq n$ and $\beta_1 > \cdots > \beta_N > 0$ so that

$$E_m(Q_{\beta,e}) = e_m \text{ for } m = 1, \ldots, n \text{ and any } e \in \mathbb{R}^N.$$ 

We will follow the clever argument from [34], which we learned about from [13, §3]. In fact, the result in [34] is even more general: it is shown that any $n$ power sums of $n$ distinct positive real numbers has at most one solution (up to permutation), in addition to some generalizations. However, we will provide a complete and self-contained proof here for future reference (in Corollary 3.3).

**Proof** Suppose towards a contradiction that there exist $\beta_1 > \cdots > \beta_N > 0$ and $\tilde{\beta}_1 > \cdots > \tilde{\beta}_{\tilde{N}} > 0$ with $\tilde{N} \leq N \leq n$ such that $E_k(Q_{\beta,e}) = E_k(Q_{\tilde{\beta},e})$ for $k = 1, \ldots, n$. By the trace formulas (2.1), this requires that

$$\sum_{m=1}^{N} \beta_m^{2k+1} = \sum_{m=1}^{\tilde{N}} \tilde{\beta}_m^{2k+1} \quad \text{for } k = 1, \ldots, n. \quad (3.7)$$

Consider the function $f : \mathbb{R} \to \mathbb{R}^n$ given by $f(x) = (x^3, x^5, \ldots, x^{2n+1})$. After canceling common terms and moving everything to the LHS, we obtain

$$\sum_{j=1}^{M} \epsilon_j f(\beta_j) = 0$$

for some $\beta_1 > \cdots > \beta_M > 0$, $M \leq 2N$, and signs $\epsilon_j \in \{\pm 1\}$.

Next, we append $2N - M$ copies of $\beta_j := 0$ and $\epsilon_j := -1$ for $j = M + 1, \ldots, 2N$ so that $\sum_{j=1}^{2N} \epsilon_j = 0$. Using summation by parts, this allows us to write

$$0 = \sum_{j=1}^{M} \epsilon_j f(\beta_j) + \sum_{j=M+1}^{2N} (-1) f(0) = \sum_{j=1}^{2N} \epsilon_j f(\beta_j) = \sum_{j=1}^{2N-1} \alpha_j [f(\beta_j) - f(\beta_{j+1})],$$

where $\alpha_j = \sum_{k=1}^{j} \epsilon_k$. By the fundamental theorem of calculus, we obtain

$$0 = \int_0^{\beta_1} \phi(s) f'(s) \, ds \quad (3.8)$$

for the step function $\phi$ which takes value $\alpha_j$ on the interval $(\beta_{j+1}, \beta_j)$. Let $I_1, \ldots, I_m$ denote the disjoint intervals in $[0, \beta_1]$ (in consecutive order) on which $\phi$ is nonvanishing and has constant sign. Note that we can have at most $n$ such intervals; indeed, $j$ must increment by two in order for $\alpha_j$ to change sign, which together with the first and last indices $j$ account for all of the $2N \leq 2n$ parameters. The equality (3.8) then tells us that the rows of the $m \times n$ matrix $A$ with entries $a_{jk} = \int_{I_j} [\phi(s)](f'(s))_k \, ds$, \quad $j \in \{1, \ldots, m\}$, \quad $k \in \{1, \ldots, n\}$

are linearly dependent.

We claim that $A$ is a strictly totally positive matrix—i.e. all of the minors of $A$ are strictly positive—which will contradict the linear dependence among the rows. Given two subsets
of indices $J \subset \{1, \ldots, m\}$ and $K \subset \{1, \ldots, n\}$, we can write the corresponding minor of $A$ as

$$\text{minor}_{J,K} \left( \begin{array}{c}
\int_{I_1} | \phi(s) | (f'(s))_1 \, ds \\
\vdots \\
\int_{I_m} | \phi(s) | (f'(s))_n \, ds
\end{array} \right)$$

$$= \int_{I_{j_1}} \cdots \int_{I_{j_{\ell}}} \text{minor}_{J,K} \left( \begin{array}{ccc}
|\phi(s_1)| (f'(s_1))_1 & \cdots & (f'(s_1))_n \\
\vdots & \ddots & \vdots \\
|\phi(s_m)| (f'(s_m))_1 & \cdots & (f'(s_m))_n
\end{array} \right) \, ds_{j_1} \cdots ds_{j_\ell}.$$ (Indeed, expanding the determinant on the LHS into a sum over permutations of the matrix entries, each term is a product of $|J|$ integrals which we combine into one $|J|$-fold integral.)

The integrand on the RHS above can be factored as

$$\text{minor}_{J,K} \left( \begin{array}{c}
|\phi(s_1)| \\
\vdots \\
|\phi(s_m)|
\end{array} \right) \left( \begin{array}{c}
(f'(s_1))_1 \\
\vdots \\
(f'(s_m))_n
\end{array} \right).$$

The first matrix has positive determinant because $\phi$ is nonvanishing on each $I_j$ by construction. Therefore, it suffices to show that $f'(s)$ is a strictly totally positive kernel, i.e. for all $0 < s_1 < \cdots < s_{\ell}$ and $k_1 < \cdots < k_{\ell}$ the matrix $$(f'(s_i))_{1 \leq i \leq \ell, 1 \leq j \leq \ell} = \begin{pmatrix}
(2k_1 + 1)s_1^{2k_1} & \cdots & (2k_{\ell} + 1)s_1^{2k_{\ell}} \\
\vdots & \ddots & \vdots \\
(2k_1 + 1)s_{\ell}^{2k_1} & \cdots & (2k_{\ell} + 1)s_{\ell}^{2k_{\ell}}
\end{pmatrix}$$

has positive determinant. Note that when $k_1, \ldots, k_{\ell}$ is an arithmetic progression, this matrix is essentially a Vandermonde matrix with rows and columns multiplied by constants. We will follow the classical argument for Vandermonde matrices.

First, we claim that the determinant is nonzero. Suppose towards a contradiction that the determinant vanishes. Then the columns would be linearly dependent, and so there would exist $\lambda_1, \ldots, \lambda_{\ell} \in \mathbb{R}$ so that

$$\sum_{j=1}^{\ell} \lambda_j s_i^{2k_j} = 0 \text{ for } i = 1, \ldots, \ell.$$ (We absorbed the coefficients $2k_j + 1$ into $\lambda_j$.) This means that the polynomial

$$P(x) := \sum_{j=1}^{\ell} \lambda_j x^{2k_j}$$

has $\ell$ positive roots $s_1, \ldots, s_{\ell}$. To obtain a contradiction, we will prove that nontrivial polynomials of the form (3.10) can have at most $\ell - 1$ positive roots by induction on $\ell$. The base case $\ell = 1$ is immediate. For the inductive step, note that if $P(x)$ has $\ell$ positive roots, then $x^{-2k_1} P(x)$ is a polynomial with the same $\ell$ positive roots. By Rolle’s theorem, the polynomial $(x^{-2k_1} P(x))'$ therefore has $\ell - 1$ positive roots. On the other hand, $(x^{-2k_1} P(x))'$ is a polynomial of the form (3.10) for $\ell - 1$, and so this contradicts the inductive hypothesis.

Lastly, we show that the determinant is positive. Now that we know the determinant is nonzero, its sign must be independent of the choice of $0 < s_1 < \cdots < s_{\ell}$ and $k_1 < \cdots < k_{\ell}$.
(but may depend on $\ell$). Therefore, we may pick $k_j = j$ for $j = 1, \ldots, \ell$ so that we essentially have a Vandermonde matrix with determinant
\[
\det \begin{pmatrix}
3s_1^2 & 5s_1^4 & \ldots & (2\ell + 1)s_1^{2\ell} \\
3s_2^2 & 5s_2^4 & \ldots & (2\ell + 1)s_2^{2\ell} \\
\vdots & \vdots & \ddots & \vdots \\
3s_\ell^2 & 5s_\ell^4 & \ldots & (2\ell + 1)s_\ell^{2\ell}
\end{pmatrix} = 3 \cdot 5 \cdots (2\ell + 1) s_1^2 s_2^2 \cdots s_\ell^2 \prod_{j < k} (s_j^2 - s_j^2).
\]
This is positive for any $\ell$ since $0 < s_1 < \cdots < s_\ell$.

For future reference (cf. Lemma 5.7), we note that the proof of Lemma 3.2 can allow for repeated $\beta$ parameters, as long as the total number of values is still at most $n$.

**Corollary 3.3** Fix $n \geq 1$. Given constraints $e_1, \ldots, e_n$, there is at most one choice of $N \geq 1$ and $\beta_1 \geq \cdots \geq \beta_N > 0$ attaining at most $n$ distinct values that satisfies
\[
(-1)^{m+1} \frac{2m+1}{2m+1} \sum_{j=1}^{N} \beta_j^{2m+1} = e_m \quad \text{for } m = 1, \ldots, n.
\]

**Proof** We repeat the proof of Lemma 3.2. Construct the step function $\phi$ so that (3.8) holds. It only remains to show that there are still at most $n$ intervals $I_j$ on which $\phi$ is nonvanishing and has constant sign. If $\beta_m = \beta_{m'}$ for some $m$ and $m'$, then these terms can be canceled from (3.7) while retaining equality. Consequently, the only new possibility for $\phi$ is that there may be a run of $\beta_{n}$ parameters with the same value and the same sign $e_j$. This increases the size of the jumps of $\phi$ but does not affect the number of sign changes, and so the claim follows. \(\square\)

By Lemma 3.2, the map $(\beta_1, \ldots, \beta_n) \mapsto (E_1(Q_{\beta,c}), \ldots, E_n(Q_{\beta,c}))$ from the half-open simplex
\[(\beta_1, \ldots, \beta_n) \in \mathbb{R}^n : \exists N \text{ with } \beta_1 > \cdots > \beta_N > 0, \ \beta_{N+1} = \cdots = \beta_n = 0 \]
(3.11)
into $\mathcal{M}_n^n$ has a well-defined inverse. In fact, the inverse is also continuous:

**Lemma 3.4** The function $(\beta_1, \ldots, \beta_n) \mapsto (E_1(Q_{\beta,c}), \ldots, E_n(Q_{\beta,c}))$ is a homeomorphism from the simplex (3.11) onto $\mathcal{M}_n^n$.

**Proof** Let
\[
\Phi(\beta_1, \ldots, \beta_n) = \left(\frac{8}{3} \sum_{m=1}^{n} \beta_m^3, -\frac{32}{5} \sum_{m=1}^{n} \beta_m^5, \ldots, (-1)^{n-1} \frac{2^{2n+1}}{2n+1} \sum_{m=1}^{n} \beta_m^{2n+1}\right)
\]
denote this function, which maps into $\mathcal{M}_n^n$ by definition of $\mathcal{M}_n^n$. Each component of $\Phi$ is a polynomial, and so $\Phi$ is smooth. By Lemma 3.2, we also know that $\Phi$ is a bijection from the simplex (3.11) onto the set of constraints $\mathcal{M}_n^n$.

It remains to show that $\Phi^{-1}$ is continuous. Fix an open subset $V \subset \mathcal{M}_n^n$ and let $\bigcup_{m=1}^{\infty} K_m$ be a compact exhaustion of $\mathcal{M}_n^n$. Recall the elementary topology fact that if $f : X \to Y$ is a continuous bijection between topological spaces with $X$ compact and $Y$ Hausdorff, then $f$ is a homeomorphism. As the map $\Phi$ is also proper, then $\Phi$ is a homeomorphism from $\Phi^{-1}(K_m)$ to $K_m$ for all $m$. Therefore $\Phi^{-1}(V \cap K_m)$ is relatively open in $\Phi^{-1}(K_m)$ for all $m$, and hence $\Phi^{-1}(V)$ is open. \(\square\)
Given constraints \((e_1, \ldots, e_n) \in \mathcal{M}_n^m\), let \(\beta_1 > \cdots > \beta_n > 0\) be the unique set of parameters with \(N \leq n\) and \(Q_{\beta, e}\) satisfying the constraints. We define

\[
C(e_1, \ldots, e_n) := E_{n+1}(Q_{\beta, e})
\]

(3.12)
to be the value of the next functional for these parameters. In proving Theorem 1.3, we will ultimately show that \(C(e_1, \ldots, e_n)\) is the minimum of \(E_{n+1}\) subject to the constraints \(e_1, \ldots, e_n\).

In order to do this, we will first need some properties of \(C\):

**Lemma 3.5** The function \(C : \mathcal{M}_n^m \to \mathbb{R}\) defined in (3.12) is continuous and is decreasing in each variable. Moreover, on the interior of \(\mathcal{M}_n^m\), \(C(e_1, \ldots, e_n)\) is continuously differentiable and satisfies \(\frac{\partial C}{\partial e_j} < 0\) for \(j = 1, \ldots, n\).

**Proof** We write

\[
C(e_1, \ldots, e_n) = (-1)^n \sum_{m=1}^{n} \beta_m^{2n+1} \tag{3.13}
\]

where \((\beta_1, \ldots, \beta_n)\) is the unique solution to

\[
e_1 = \frac{8}{3} \sum_{m=1}^{n} \beta_m^3, \quad e_2 = -\frac{32}{5} \sum_{m=1}^{n} \beta_m^5, \quad \ldots, \quad e_n = (-1)^{n-1} \sum_{m=1}^{n} \beta_m^{2n+1} \tag{3.14}
\]

in the simplex (3.11), guaranteed by Lemma 3.2. Note that \(C\) is continuous as the composition of the inverse of the homeomorphism in Lemma 3.4 with a polynomial.

Consequently, it suffices to show that \(\frac{\partial C}{\partial e_j} < 0\) for \(j = 1, \ldots, n\) on the interior of \(\mathcal{M}_n^m\). By Lemma 3.4, the interior of \(\mathcal{M}_n^m\) corresponds to the set of \(n\) positive parameters \(\beta_1 > \cdots > \beta_n > 0\). In other words, the interior of \(\mathcal{M}_n^m\) is the set of constraints \((e_1, \ldots, e_n)\) which correspond to \(n\)-solitons.

We will now compute \(\frac{\partial C}{\partial e_j}\) assuming \(\beta_1 > \cdots > \beta_n > 0\). Differentiating the constraints (3.14) with respect to \(e_j\), we see that

\[
\begin{pmatrix}
8 \beta_1^2 \\
-32 \beta_1^4 \\
\vdots \\
(-1)^{n-1} 2^{2n+1} \beta_1^{2n}
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n
\end{pmatrix}
= (-1)^{n-1} 2^{2n+1} \beta_1^{2n}
\]

is equal to the \(j\)th coordinate vector \((0, \ldots, 0, 1, 0, \ldots, 0)\). This is a Vandermonde matrix after pulling out common factors from each row and column, and thus it has determinant

\[
8 \cdot (-32) \cdots (-1)^{n-1} 2^{2n+1} \beta_1^2 \cdots \beta_n^2 \prod_{j < k} (\beta_k^2 - \beta_j^2).
\]

This expression is nonvanishing since \(\beta_1 > \cdots > \beta_n > 0\), and so we conclude that the partial derivatives \(\frac{\partial C}{\partial e_j}\) exist and are uniquely determined by the matrix product (3.15). On the other hand, differentiating (3.13) with respect to \(e_j\) yields

\[
\frac{\partial C}{\partial e_j} = (-1)^n \sum_{m=1}^{n} \beta_m^{2n+3} \begin{pmatrix}
\beta_1^{2n+2} \\
\beta_2^{2n+2} \\
\vdots \\
\beta_n^{2n+2}
\end{pmatrix} \frac{\partial e_j}{\partial \beta_m}
\]

(3.15)
We can determine the column vector on the RHS via (3.15). Collecting these equations for 
\( j = 1, \ldots, n \), we obtain the matrix equation
\[
\begin{pmatrix}
\frac{\partial C}{\partial e_1} & \cdots & \frac{\partial C}{\partial e_n}
\end{pmatrix}
\begin{pmatrix}
8\beta_1^2 & \cdots & 8\beta_n^2 \\
\vdots & \ddots & \vdots \\
(-1)^{n-1}2^{n+1}\beta_1^{2n} & \cdots & (-1)^{n-1}2^{n+1}\beta_n^{2n}
\end{pmatrix}
= (-1)^n 2^{2n+3} (\beta_1^{2n+2} \cdots \beta_n^{2n+2}),
\]
which we moved the matrix to the LHS to avoid inverting it. We have already seen that this matrix is invertible, and so we conclude that the partial derivatives \( \frac{\partial C}{\partial e_j} \) exist and are uniquely determined by the above equality.

In order to compute the derivatives \( \frac{\partial C}{\partial e_j} \), we will harness the classical role of Vandermonde matrices in polynomial interpolation. Specifically, reading off the \( n \) components of the equality (3.16), we see that the derivatives \( \frac{\partial C}{\partial e_j} \) are the coefficients \( C_j \) of the polynomial
\[
P(x) := 8C_1 x - 32C_2 x^2 + \cdots + (-1)^{n-1}2^{n+1}C_n x^n - (-1)^n 2^{2n+3} x^{n+1}
\]
which satisfies
\[
P(\beta_m^2) = 0 \quad \text{for } m = 1, \ldots, n.
\]
As \( \beta_1 > \cdots > \beta_n > 0 \), there is only one such polynomial, namely,
\[
P(x) = (-1)^{n+1}2^{2n+3} \prod_{m=1}^{n} (x - \beta_m^2).
\]
Therefore, the coefficients \( C_j \) are given by Vieta’s formulas:
\[
8C_1 = (-1)^{n+1}2^{2n+3}(-1)^n \prod_{j=1}^{n} \beta_j^2,
\]
\[
-32C_2 = (-1)^{n+1}2^{2n+3}(-1)^{n-1} \sum_{k=1}^{n} \prod_{j \neq k} \beta_j^2,
\]
\[
\vdots
\]
\[
(-1)^{n-1}2^{n+1}C_n = (-1)^{n+1}2^{2n+3}(-1)^{n} \sum_{j=1}^{n} \beta_j^2,
\]
where the RHS for \( C_j \) involves the \((n - j + 1)\)st elementary symmetric polynomial in \( \beta_1^2, \ldots, \beta_n^2 \). In particular, we see that each \( \frac{\partial C}{\partial e_j} = C_j \) is given by \((-1)^{2n+1} = -1\) times a strictly positive quantity, and hence is strictly negative as desired.

In order to employ that \( C \) is decreasing, we will also need to know that the set \( \mathcal{M}_n^N \) is downward closed within \( \bigcup_{N \geq 0} \mathcal{M}_N^N \) in the following sense:

**Lemma 3.6** If the constrains \( \tilde{e}_1, \ldots, \tilde{e}_n \) are in \( \mathcal{M}_N^n \) for some \( N \) and
\[
\tilde{e}_1 \leq e_1, \quad \ldots, \quad \tilde{e}_n \leq e_n
\]
for some \((e_1, \ldots, e_n) \in \mathcal{M}_n^N\), then \((\tilde{e}_1, \ldots, \tilde{e}_n) \in \mathcal{M}_n^n\).
Proof Let $\tilde{\beta}_1, \ldots, \tilde{\beta}_N > 0$ be the $\beta$ parameters of the multisoliton which witnesses the constraints $\tilde{e}_1, \ldots, \tilde{e}_n$. The case $N \leq n$ is immediate, so assume that $N > n + 1$. Let

$$\alpha_j = \sum_{m=1}^{N} \tilde{\beta}_m^j, \quad j = 3, 5, \ldots, 2n + 1$$

denote the odd moments of $\tilde{\beta}_1, \ldots, \tilde{\beta}_N$. Define

$$\Gamma := \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N : x_1, \ldots, x_N \geq 0, \sum_{m=1}^{N} x_m^j = \alpha_j \text{ for } j = 3, 5, \ldots, 2n + 1 \right\}$$

to be the set of parameters in $\mathbb{R}^N$ satisfying the constraints, which is nonempty because it contains ($\tilde{\beta}_1, \ldots, \tilde{\beta}_N$).

It suffices to show that the intersection of $\Gamma$ with the $n$-dimensional boundary face $\{ (x_1, \ldots, x_N) : x_{n+1}, \ldots, x_N = 0 \}$ is nonempty, since a point $(x_1, \ldots, x_n, 0, \ldots, 0)$ provides the $n$-soliton parameters that we seek. The case $n = 1$ is immediate as $\Gamma$ is just an $\ell^3$-sphere, and so we may assume that $\Gamma$ is the intersection of $n \geq 2$ constraints.

To accomplish this, consider the set “between” the constraints

$$\Omega := \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N : x_1, \ldots, x_N \geq 0, \quad (-1)^k \sum_{m=1}^{N} x_m^{2k+1} \leq (-1)^k \alpha_{2k+1} \text{ for } k = 1, \ldots, n \right\}.$$ 

Unlike $\Gamma$, we already know that the intersection of $\Omega$ with the boundary face $\{ (x_1, \ldots, x_N) : x_{n+1}, \ldots, x_N = 0 \}$ is nonempty by premise. Indeed, as $(e_1, \ldots, e_n) \in M_n^0$, then there exist $x_1, \ldots, x_n$ so that

$$(-1)^{k+1} \frac{2^{2k+1}}{2k+1} \sum_{m=1}^{n} \tilde{\beta}_m^{2k+1} = e_k \geq \tilde{e}_k = (-1)^{k+1} \frac{2^{2k+1}}{2k+1} \alpha_{2k+1} \quad \text{for } k = 1, \ldots, n.$$ 

(This premise is in fact necessary, as the sets $\Omega$ and $\Gamma$ may not intersect the boundary face $\{ (x_1, \ldots, x_N) : x_{n+1}, \ldots, x_N = 0 \}$ in general; cf. Lemma 6.1.) Note that $\Omega$ is also bounded, because each coordinate $x_j$ is bounded by $a_5^{1/5}$ since $n \geq 2$. As $\Omega \cap \{ (x_1, \ldots, x_N) : x_{n+1}, \ldots, x_N = 0 \}$ is nonempty, closed, and bounded, there exists some point $(\beta_1, \ldots, \beta_n, 0, \ldots, 0)$ in this intersection that minimizes the $(n + 1)$st odd moment $(-1)^n \sum x_m^{2n+3}$. This point must actually lie in $\Gamma$, because Lemma 3.5 tells us that the value $(-1)^n \sum x_m^{2n+3}$ is an individually decreasing function of the moments on $\Omega \cap \{ (x_1, \ldots, x_N) : x_{n+1}, \ldots, x_N = 0 \}$. \hfill $\square$

4 Global minimizers

We will prove Theorem 1.3 over the course of this section by induction on $n$. We begin with the base case $n = 0$. The conclusion is immediate since

$$E_1(u) = \int_{-\infty}^{\infty} \frac{1}{2} u^2 \, dx \geq 0 \quad \text{for all } u \in L^2(\mathbb{R}),$$

with equality if and only if $u$ is equal to the zero-soliton $q(x) \equiv 0$. 

$\odot$ Springer
Next, we turn to the inductive step. Suppose that \( n \geq 1 \) and that Theorem 1.3 holds for \( 1, 2, \ldots, n - 1 \). This inductive hypothesis yields the following fact for \( \mathcal{M}_n^\beta \):

**Lemma 4.1** The set \( \mathcal{M}_n^\beta \) is a relatively open subset of the feasible constraints \( \mathcal{F}^n \) (with respect to the topology on \( \mathbb{R}^n \)).

**Proof** Given \((e_1, \ldots, e_n)\) in the interior of \( \mathcal{M}_n^\beta \), we know that the number of \( \beta \) parameters is \( n \) by Lemma 3.4. Therefore, for each \( k \leq n - 1 \) we can increase and decrease \( E_k \) while preserving \( E_1, \ldots, E_k \) by Lemma 3.1 since \( n \) is strictly larger than \( k \). Moreover, Lemma 3.4 implies that the set \( \mathcal{M}_n^\beta \setminus (\text{int} \mathcal{M}_n^\beta) \) corresponds to multisolutions of degree at most \( n - 1 \), and thus lie on the boundary of \( \mathcal{F}^n \) by the inductive hypothesis that Theorem 1.3 holds for each \( k = 1, \ldots, n - 1 \). \( \square \)

Next, we will prove the first half of the inductive step: that multisolutions of degree at most \( n \) are global constrained minimizers.

**Theorem 4.2** Given constraints \((e_1, \ldots, e_n)\) in \( \mathcal{M}_n^\beta \), there exists a unique integer \( N \leq n \) and parameters \( \beta_1 > \cdots > \beta_N > 0 \) so that the multisolution \( Q_{\beta,e} \) lies in \( C \) for some (and hence all) \( c \in \mathbb{R}^N \). Moreover, we have

\[
E_{n+1}(u) \geq E_{n+1}(Q_{\beta,e}) \quad \text{for all } u \in C.
\]

**Proof** We first prove the inequality for \( u \) Schwartz. In this case, the trace formula (2.7) allows us to write

\[
E_{n+1}(u) = \frac{2^{2n+2}}{\pi} \int_{-\infty}^{\infty} k^{2n+2} \log |a(k; u)| \, dk + (-1)^n \sum_{m=1}^{N} \beta_m^{2n+3}.
\]

The integrand is nonnegative by (2.4), so we can omit the integral to obtain the inequality

\[
E_{n+1}(u) \geq (-1)^n \sum_{m=1}^{N} \beta_m^{2n+3}.
\]

Note that the \( \beta \) parameters do not satisfy the constraints \( e_j \), but rather the smaller constraints \( e_j - \frac{2^j}{\pi} \int k^{2j} \log |a| \, dk \) because the moments of \( \log |a(k; u)| \) may not vanish. Nevertheless, as \( \mathcal{M}_n^\beta \) is downward closed by Lemma 3.6, we know that these constraints are still attainable by a multisolution of degree at most \( n \) and so Lemma 3.1 implies that

\[
E_{n+1}(u) \geq C \left( e_1 - \frac{4}{\pi} \int k^2 \log |a| \, dk, \ldots, e_n - \frac{2^n}{\pi} \int k^{2n} \log |a| \, dk \right).
\]

Finally, \( C \) is individually decreasing in each variable by Lemma 3.5, and so we conclude

\[
E_{n+1}(u) \geq C(e_1, \ldots, e_n)
\]
as desired.

For general \( u \in H^n(\mathbb{R}) \), we approximate by a sequence of Schwartz functions. The constraints \( e_1, \ldots, e_n \) and minimum value \( C \) for the approximate functions will converge by the continuity of \( E_1, \ldots, E_n : H^n(\mathbb{R}) \to \mathbb{R} \) and \( C : \mathcal{M}_n^\beta \to \mathbb{R} \), the latter of which we proved in Lemma 3.5. Moreover, the constraints \( e_1, \ldots, e_n \) for the approximate functions eventually lie in \( \mathcal{M}_n^\beta \) by Lemma 4.1. \( \square \)

To conclude the inductive step of Theorem 1.3, it remains to show that any other constrained minimizer must also be a multisolution with the right \( \beta \) parameters.
Theorem 4.3  Suppose we have constraints \((e_1, \ldots, e_n) \in \mathcal{M}_n^b\) and that \(q \in \mathcal{C}_e\) minimizes \(E_{n+1}(u)\) over \(\mathcal{C}_e\). Then \(q = Q_{\beta, c}\) for some \(c \in \mathbb{R}^N\), where \(\beta \in \mathbb{R}^N\) are the unique parameters satisfying the constraints guaranteed by Theorem 4.2.

We break the proof of Theorem 4.3 into steps, with the overarching assumption that \(q \in H^n(\mathbb{R})\) is a constrained minimizer of \(E_{n+1}\).

In order to analyze \(q\) using the trace formulas, we first need to know that \(q\) is sufficiently regular so that we may construct \(a(k; q)\):

Lemma 4.4  If \(q\) is a constrained minimizer in the sense of Theorem 4.3, then \(q\) is Schwartz.

We will use that \(q\) solves the Euler–Lagrange equation (cf. (4.1)) to show that \(q\) is both infinitely smooth and exponentially decaying. As we will see shortly, smoothness follows from classical ODE theory because \(q \in H^n(\mathbb{R})\) \textit{a priori}. On the other hand, exponential decay is more delicate: even though we know \(q(x) \to 0\) as \(x \to \pm \infty\) (since \(q \in H^1(\mathbb{R})\)), there do exist multipliers \(\lambda_1, \ldots, \lambda_n\) so that (4.1) admits algebraically decaying solutions as \(x \to \pm \infty\). For example, \(u(x) = 2x^{-2}\) is a (meromorphic) solution to (4.1) for all \(n \geq 1\) with multipliers \(\lambda_1 = \cdots = \lambda_n = 0\). This is the beginning of an infinite family of solutions called the \textit{algebro-geometric solutions} to the stationary KdV hierarchy (see [17, §1.3] for details). Morally, this does not pose an obstruction here because the multipliers must be negative for a minimizer (cf. (4.3)), as is the case for any multisoliton.

Proof  As a critical point of \(E_{n+1}\), \(q\) satisfies the Euler–Lagrange equation

\[
\nabla E_{n+1}(q) = \lambda_1 \nabla E_1(q) + \lambda_2 \nabla E_2(q) + \cdots + \lambda_n \nabla E_n(q)
\]

(4.1)

for some Lagrange multipliers \(\lambda_1, \ldots, \lambda_n \in \mathbb{R}\). This assumes that the gradients \(\nabla E_1(q), \ldots, \nabla E_n(q)\) are linearly independent; however, the other case is analogous, since a linear dependence can be written as an equation of the form (4.1) for some smaller \(n\).

First we show that \(q\) is infinitely smooth. As \(q \in H^n(\mathbb{R})\), \(q\) only solves (4.1) in the sense of distributions \textit{a priori}. The highest order term in (4.1) is \(q^{(2n)}\), and it only appears in \(\nabla E_{n+1}(q)\). Isolating this term, we obtain

\[
q^{(2n)} = P(q, q', \ldots, q^{(2n-2)})
\]

(4.2)

for a polynomial \(P\). Note that product terms \(q^{(\gamma_1)} \cdots q^{(\gamma_2)}\) satisfy \(\sum \gamma_j \leq 2n - 2\) by the scaling requirement (2.2). In particular, if \(q \in H^s\) with \(s \geq n\), then RHS(4.2) is in \(H^{-s+2}\). (For example, \(qq^{(2n-2)} \in H^{-s+2}\) because \(q^{(2n-2)} \in H^{s-(2n-2)} \subset H^{-s+2}\) and \(q \in H^s \subset H^{s-2}\).)

Beginning with \(q \in H^n\), the equation (4.2) tells us that \(q^{(2n)}\) is in \(H^{-n+2}\), and so we conclude that \(q \in H^{n+2}\). Now taking \(q \in H^{n+2}\) as input, the equation (4.2) then tells us that \(q^{(2n)}\) is in \(H^{-n+4}\), and so we conclude that \(q \in H^{n+4}\). Iterating, we conclude that \(q\) is in \(H^s\) for all \(s > 0\) and hence is smooth.

Next, we claim that \(q\) decays exponentially as \(x \to \pm \infty\). Our salvation here is that because \(q\) is a minimizer (and not merely a critical point), we have restrictions on the Lagrange multipliers. If the constraints are in \(\mathcal{M}_n^b \setminus (\text{int} \mathcal{M}_n^b)\), then \(q\) is a minimizer of \(E_m\) for some \(m \leq n\) and thus \(q\) is a multisiton of degree at most \(m\) by the inductive hypothesis that Theorem 1.3 holds for \(m - 1\). So assume that the constraints \((e_1, \ldots, e_n)\) are in the interior of \(\mathcal{M}_n^b\). Consequently, we know from Lemma 3.5 that the minimum value \(C(e_1, \ldots, e_n)\) is a \(C^1\) function in a neighborhood of \((e_1, \ldots, e_n)\) and

\[
\lambda_j = \frac{\partial C}{\partial e_j}(e_1, \ldots, e_n) < 0 \quad \text{for} \quad j = 1, \ldots, n.
\]

(4.3)
The equality above is a general fact about Lagrange multipliers called the *envelope theorem*, and has applications to economics. (Cf. [35, Th. 1.F.4] and the corollary in Ex. 2 for a proof. As we know that all of the derivatives exist, this purely algebraic proof for the finite dimensional case still applies.)

From the quadratic terms of the energies (2.1), we see that the linear part of the Euler–Lagrange equation (4.1) is
\[
Lu := (-1)^n u^{(2n)} + (-1)^n \lambda_n u^{(2n-2)} + \cdots + \lambda_2 u'' - \lambda_1 u.
\]
The constant coefficients of this operator are alternating in sign by (4.3), and consequently it has no purely imaginary eigenvalues. Indeed, if $\xi$ is purely imaginary then all the terms in the polynomial
\[
(-1)^n \xi^{2n} + (-1)^n \lambda_n \xi^{2n-2} + \cdots + \lambda_2 \xi^2 - \lambda_1
\]
are nonnegative, and so the polynomial is bounded below by $-\lambda_1 > 0$. As the Euler–Lagrange equation (4.1) is an ODE of order $2n$, we may view it as a first-order system of ODEs in the variables $(q, q', \ldots, q^{(2n-1)}) \in \mathbb{R}^{2n}$. We just showed that the origin in $\mathbb{R}^{2n}$ is a hyperbolic fixed point for this system, and so the stable manifold theorem [11, Ch. 13 Th. 4.1] tells us that there exists a corresponding stable manifold in a neighborhood of the origin. We already know that $q^{(j)}(x) \to 0$ as $x \to \pm \infty$ for all $j \geq 0$ (since $q \in H^{j+1}$), and so eventually $(q, q', \ldots, q^{(2n-1)})$ remains in a small neighborhood of the origin in $\mathbb{R}^{2n}$ for all $x$ sufficiently large. By [11, Ch. 13 Th. 4.1], this can only happen if $q(x)$ is on the stable manifold and hence decays exponentially as $x \to \pm \infty$. 

Now that we know $q \in S(\mathbb{R})$, we have the trace formulas (2.7) at our disposal. Next, we show that the constrained minimizer $q$ must satisfy $|a(k; q)| \equiv 1$ on $\mathbb{R}$:

**Lemma 4.5** Let $q \in S(\mathbb{R})$ such that $|a(k; q)| \neq 1$ for some $k \in \mathbb{R}$. Then there exists some $\tilde{q} \in S(\mathbb{R})$ with
\[
E_m(\tilde{q}) = E_m(q) \quad \text{for } m = 1, \ldots, n, \quad \text{but} \quad E_{n+1}(\tilde{q}) < E_{n+1}(q).
\]
In particular, a constrained minimizer $q$ in the sense of Theorem 4.3 must satisfy $|a(k; q)| = 1$ for all $k \in \mathbb{R}$.

**Proof** We will only modify the transmission coefficient of $q$ and leave the bound states $-\beta_1^2, \ldots, \beta_N^2$ unchanged. Specifically, we will wiggle $\log |a(k; q)|$ via the implicit function theorem in a way that decreases its $(n + 1)$st moment in the trace formulas (2.7) while keeping the first $n$ moments constant. Then we will reconstruct the new potential $\tilde{q}$ using inverse scattering theory.

Let $\psi_1, \ldots, \psi_{n+1} \in C^\infty(\mathbb{R})$ be even functions to be chosen later. Define the function $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ by
\[
f(x_1, \ldots, x_{n+1}) = \begin{pmatrix}
\int k^2 \left[ \log |a(k; q)| + x_1 \psi_1(k) + \cdots + x_{n+1} \psi_{n+1}(k) \right] dk \\
\int k^4 \left[ \log |a(k; q)| + x_1 \psi_1(k) + \cdots + x_{n+1} \psi_{n+1}(k) \right] dk \\
\vdots \\
\int k^{2n} \left[ \log |a(k; q)| + x_1 \psi_1(k) + \cdots + x_{n+1} \psi_{n+1}(k) \right] dk
\end{pmatrix}.
\]
This function has derivative matrix
\[
Df(0, \ldots, 0) = \begin{pmatrix}
\int k^2 \psi_1(k) dk & \ldots & \int k^2 \psi_{n+1}(k) dk & \int k^2 \psi_{n+1}(k) dk \\
\int k^4 \psi_1(k) dk & \ldots & \int k^4 \psi_{n+1}(k) dk & \int k^4 \psi_{n+1}(k) dk \\
\vdots & \ddots & \vdots & \vdots \\
\int k^{2n} \psi_1(k) dk & \ldots & \int k^{2n} \psi_{n+1}(k) dk & \int k^{2n} \psi_{n+1}(k) dk
\end{pmatrix}.
\]
at the origin. If we replace each $\psi_j$ by the even extension $\frac{1}{2}(d\delta_{-k_j} + d\delta_{k_j})$ of a Dirac delta mass at $k_j > 0$, then the left $n \times n$ block matrix becomes the Vandermonde matrix

$$
\begin{pmatrix}
  k_1^2 & k_2^2 & \cdots & k_n^2 \\
  k_1^4 & k_2^4 & \cdots & k_n^4 \\
  \vdots & \vdots & \ddots & \vdots \\
  k_1^{2n} & k_2^{2n} & \cdots & k_n^{2n}
\end{pmatrix}
$$

with determinant $k_1^2 \cdots k_n^2 \prod_{i<j}(k_i^2 - k_j^2)$.

This determinant is nonvanishing provided that we pick the $k_i$ positive and distinct.

As $a(k; q)$ is a continuous and even function of $k \in \mathbb{R}$ by the reality condition (2.6), we may pick $n + 1$ distinct points $k_1, \ldots, k_{n+1}$ in $\{k > 0 : |a(k; q)| \neq 1\}$. We will take $\psi_1, \ldots, \psi_n$ to be mollifications of $\frac{1}{2}(d\delta_{-k_j} + d\delta_{k_j})$ for $j = 1, \ldots, n$ by a smooth and even function. If we take the mollifier to have sufficiently small support, then $\psi_1, \ldots, \psi_n$ will have disjoint supports within $\{k \neq 0 : |a(k; q)| \neq 1\}$. Taking the support of the mollifier to be even smaller if necessary, the above computation shows that the left $n \times n$ block of $Df(0, \ldots, 0)$ is invertible. Now the implicit function theorem implies that there exists $\epsilon > 0$ and $C^1$ functions $x_1(x_{n+1}), \ldots, x_n(x_{n+1})$ defined on $(-\epsilon, \epsilon)$ so that

$$
f(x_1(x_{n+1}), \ldots, x_n(x_{n+1}), x_{n+1}) = f(0, \ldots, 0) = \begin{pmatrix}
  \int k^2 \log |a(k; q)| \, dk \\
  \int k^4 \log |a(k; q)| \, dk \\
  \vdots \\
  \int k^{2n} \log |a(k; q)| \, dk
\end{pmatrix}
$$

(4.4)

for $x_{n+1} \in (-\epsilon, \epsilon)$.

It remains to show that we can pick $x_{n+1}$ in a way that decreases the next log $|a|$ moment.

To this end, we compute the derivative

$$
\left. \frac{d}{dx_{n+1}} \right|_{x_{n+1}=0} \int k^{2n+2} \left[ \log |a| + x_1(x_{n+1})\psi_1 + \cdots + x_n(x_{n+1})\psi_n + x_{n+1}\psi_{n+1} \right] \, dk
$$

$$
= \int k^{2n+2} \left[ (\psi_1 \ldots \psi_n) \begin{pmatrix} x_1'(0) \\ \vdots \\ x_n'(0) \end{pmatrix} + \psi_{n+1} \right] \, dk.
$$

The derivative of $x_1(x_{n+1}), \ldots, x_n(x_{n+1})$ is determined by differentiating (4.4) at $x_{n+1} = 0$. This yields

$$
\begin{pmatrix} x_1'(0) \\ \vdots \\ x_n'(0) \end{pmatrix} = - \begin{pmatrix} \int k^2 \psi_1 \cdots \int k^2 \psi_n \\ \vdots \\ \int k^{2n} \psi_1 \cdots \int k^{2n} \psi_n \end{pmatrix}^{-1} \begin{pmatrix} \int k^2 \psi_{n+1} \\ \vdots \\ \int k^{2n} \psi_{n+1} \end{pmatrix}.
$$

Recall that the matrix above is invertible by our choice of $\psi_1, \ldots, \psi_n$. Inserting this into the derivative of the $(2n + 2)$nd moment, the resulting matrix product is supported on the union of the supports of $\psi_1, \ldots, \psi_n$ which is disjoint from the support of the other term $\psi_{n+1}$ in the integrand. Therefore, we may pick another smooth and even function $\psi_{n+1}$ supported in a sufficiently small neighborhood of $\pm k_{n+1}$ so that the whole integral is nonzero. It then follows that there exists $x_{n+1} \in (-\epsilon, \epsilon)$ sufficiently small so that

$$
\log |a(k; q)| + x_1(x_{n+1})\psi_1 + \cdots + x_n(x_{n+1})\psi_n + x_{n+1}\psi_{n+1}
$$

(4.5)

is nonnegative for $k \in \mathbb{R}$, decreases $E_{n+1}$, and preserves $E_1, \ldots, E_n$. 
It only remains to show that the density (4.5) corresponds to \( \log |a(k; \tilde{q})| \) for some \( \tilde{q} \in \mathcal{S}(\mathbb{R}) \). To accomplish this, we will reconstruct \( \tilde{q} \) from its scattering data by verifying properties (i)-(vii) of Proposition 2.4. First, we require that the transmission coefficient satisfies 
\[
\frac{1}{|T(k; \tilde{q})|} = |a(k; \tilde{q})| = \exp \left\{ \log |a(k; q)| + x_1(x_{n+1})\psi_1 + \cdots + x_n(x_{n+1})\psi_n + x_{n+1}\psi_{n+1} \right\}
\]
for \( k \in \mathbb{R} \). We have \( |T(k; \tilde{q})| \leq 1 \) because (4.5) is nonnegative. As \( a(k, q) \) extended to a bounded holomorphic to all of \( \mathbb{C}^+ \), it is in the Hardy space \( \mathcal{H}^\infty(\mathbb{C}^+) \). By [16, Ch II Th. 4.4], given \( g \in L^\infty(\mathbb{R}) \) nonnegative, there exists a holomorphic function \( h \in \mathcal{H}^\infty(\mathbb{C}^+) \) with \( |h(k)| = g(k) \) for almost every \( k \in \mathbb{R} \) if and only if
\[
\int_{-\infty}^{\infty} \frac{\log |g(k)|}{1 + k^2} \, dk > -\infty.
\]
In particular, this property was satisfied by \( g(k) = |a(k; q)| \). As we have smoothly modified \( \log |a(k; q)| \) on a compact subset, this condition is also satisfied for \( g(k) = |a(k; \tilde{q})| \) and so there must also exist a holomorphic extension \( a(k; \tilde{q}) := h(k) \) to \( \mathbb{C}^+ \). This ensures that \( T(k; \tilde{q}) = \frac{1}{a(k; \tilde{q})} \) satisfies the analyticity condition (iii).

Next, we set \( T_1(k; \tilde{q}) = T_2(k; \tilde{q}) = T(k; \tilde{q}) \) in accordance with the symmetry condition (i). We then require the modulus of the reflection coefficients satisfy \( |R_1(k; \tilde{q})| = |R_2(k; \tilde{q})| = \sqrt{1 - |T(k; \tilde{q})|^2} \) and the phases satisfy
\[
\frac{\arg R_1(k; \tilde{q}) + \arg R_2(k; \tilde{q})}{2} = \frac{\pi}{2} - \arg T(k; \tilde{q})
\]
for \( k > 0 \) to ensure that the unitary condition (ii) holds. We also need to construct \( R_1, R_2 \) so that condition (v) on the rate at \( k = 0 \) still holds. We are still free to specify the difference \( \arg R_1 - \arg R_2 \), which if \( T(0; q) = 0 \) we take to satisfy \( \arg R_1 \to \pi \) and \( \arg R_2 \to \pi \) as \( k \downarrow 0 \). As we have only modified \( T(k; q) \) on a compact subset of \( \mathbb{R} \setminus \{0\} \), altogether we conclude that the condition (v) is satisfied. We then extend \( R_1 \) and \( R_2 \) to \( k < 0 \) according to the reality condition (vi).

Note that the Fourier decay condition (vii) is automatically satisfied because we have perturbed \( R_1(k; q) \) and \( R_2(k; q) \) smoothly. Likewise, the asymptotics condition (iv) is also satisfied because the coefficients have only been modified on a compact subset of \( \mathbb{R} \setminus \{0\} \). The resulting potential \( \tilde{q} \) is also automatically Schwartz. Indeed, inverse scattering theory reconstructs \( \tilde{q} \) via an explicit integral [12, §4 Eq. (1)] in terms of \( R_1 \) and the Jost function \( f_1(x; k) \), which have only been smoothly modified on a compact subset of \( \mathbb{R} \setminus \{0\} \).

Lastly, we use Proposition 2.5 to add the bound states \( \beta_1, \ldots, \beta_N \) of \( q \) back to \( \tilde{q} \). The formula (2.8) for the new transmission coefficient shows that \( |a(k; \tilde{q})| \) is unchanged for \( k \in \mathbb{R} \). This, together with the construction (4.5) of \( \log |a(k; \tilde{q})| \), shows that \( \tilde{q} \) decreases \( E_{n+1} \) while preserving \( E_1, \ldots, E_n \) as desired. \( \square \)

Next, we will show that the requirement \( |a(k; q)| \equiv 1 \) on \( \mathbb{R} \) forces \( a(k; q) \) to be the finite Blaschke product (2.9) on \( \mathbb{C}^+ \):

**Lemma 4.6** If \( q \) is a constrained minimizer in the sense of Theorem 4.3, then \( q \) is a multisoliton.

**Proof** Let \( \mathbb{C}^+ = \{ z \in \mathbb{C} : \text{Im} \, z > 0 \} \) denote the upper half-plane and \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) denote the unit disk. By Lemma 4.5, we know that \( |a(k; q)| \equiv 1 \) on \( \mathbb{R} \). We also know by the asymptotics (2.5) that \( a(k; q) \) tends to 1 as \( k \to \infty \) within \( \mathbb{C}^+ \). Applying the maximum modulus principle to the half-disks \( \mathbb{C}^+ \cap \{ z : |z| \leq R \} \) and taking \( R \to \infty \), we conclude that \( k \mapsto a(k; q) \) maps \( \mathbb{C}^+ \) into \( \overline{\mathbb{D}} \).
In particular, \( a(\cdot,q) \) is in the Hardy space \( \mathcal{H}^\infty(\mathbb{C}^+) \). We may therefore apply inner-outer factorization \([16,\text{ Ch. II Cor. 5.7}]\) to obtain

\[
a(k; q) = e^{i\theta} B(k) S(k) F(k),
\]

where \( \theta \in \mathbb{R} \) is a constant, \( B(k) \) is a Blaschke product, \( S(k) \) is a singular function, and \( F(k) \) is an outer factor.

First, we claim that \( F(k) \) is constant. Note that on \( \mathbb{R} \) we have \( |B(k)| \equiv 1 \) everywhere and \( |S(k)| \equiv 1 \) almost everywhere. Therefore \( |F(k)| \equiv 1 \) almost everywhere on \( \mathbb{R} \). As \( F \) is an outer factor, \( \log |F| \) in \( \mathbb{C}^+ \) is given by the Poisson integral over its boundary values. As \( \log |F| \equiv 0 \) almost everywhere on \( \mathbb{R} \), we conclude that \( \log |F| \equiv 0 \) on \( \mathbb{C}^+ \) and hence \( F \) is constant.

Next, we claim that \( S(k) \) is also constant. Recall that if \( S(k) \) is a singular function and \( |S(k)| \) is continuous from \( \mathbb{C}^+ \) to any \( k \in \mathbb{R} \cup \{\infty\} \), then \( k \) is not in the support of the singular measure that defines \( S \). In our case we know that \( a(k; q) \) extends continuously to \( \mathbb{R} \) and \( \infty \), and so we conclude that the measure for \( S(k) \) vanishes identically.

Altogether we now have \( a(k; q) \equiv e^{i\theta} B(k) \) for some constant \( \phi \in \mathbb{R} \). Taking \( k \to +i\infty \) we have \( a(k; q) \to 1 \) by (2.5) and \( B(k) \to 1 \), and so we conclude that \( a(k; q) \equiv B(k) \). By Proposition 2.3 we know that the zeros of \( a(k; q) \) are purely imaginary and simple. Therefore \( a(k; q) \) takes the form (2.9), and so we conclude that \( q \) is a multisoliton.

Finally, to conclude the proof of Theorem 4.3, we note that the degree of the multisoliton \( q \) is at most \( n \). Otherwise, Lemma 3.1 would imply that we could replace \( q \) by another multisoliton that decreases \( E_{n+1} \) while preserving \( E_1, \ldots, E_n \), which would contradict that \( q \) is a minimizer.

### 5 Orbital stability

The goal of this section is to prove Theorem 1.4. It will follow easily from the following property of minimizing sequences:

**Theorem 5.1** Fix an integer \( n \geq 1 \). If \((e_1, \ldots, e_n) \in \mathcal{M}_n \) and \( \{q_k\}_{k \geq 1} \subset H^n(\mathbb{R}) \) is a minimizing sequence:

\[
E_1(q_k) \to e_1, \quad \ldots, \quad E_n(q_k) \to e_n, \quad E_{n+1}(q_k) \to C(e_1, \ldots, e_n)
\]

as \( k \to \infty \), then there exists a subsequence which converges in \( H^n(\mathbb{R}) \) to the manifold of minimizing solitons \( \{Q_{\beta,c} : c \in \mathbb{R}^N\} \).

We begin the proof of Theorem 5.1 by fixing a minimizing sequence \( \{q_k\}_{k \geq 1} \subset H^n(\mathbb{R}) \) satisfying (5.1). The estimates (2.3) that prove that \( E_1, \ldots, E_{n+1} \) are continuous functionals on \( H^n(\mathbb{R}) \) show that the sequence \( \{q_k\}_{k \geq 1} \) is bounded in \( H^n(\mathbb{R}) \). Now that we know \( \{q_k\}_{k \geq 1} \) is bounded in \( H^n(\mathbb{R}) \), we are able to apply a concentration compactness principle adapted to our variational problem. Specifically, we will use the following statement associated to the embedding \( H^n(\mathbb{R}) \hookrightarrow W^{n-1,3}(\mathbb{R}) \), whose formulation is inspired by [23]. The choice of concentration compactness principle is not unique (cf. [4, §3]), but we will see below that this choice turns out to be efficient (see, for example, the proof of Lemma 5.4).
Theorem 5.2  Fix an integer \( n \geq 1 \). If \( \{q_k\}_{k \geq 1} \) is a bounded sequence in \( H^n(\mathbb{R}) \), then there exist \( J^* \in \{0, 1, \ldots, \infty\} \), \( J^* \)-many profiles \( \{\phi^j\}_{j=1}^{J^*} \subset H^n(\mathbb{R}) \), and \( J^* \)-many sequences \( \{x_k^j\}_{j=1}^{J^*} \subset \mathbb{R} \) so that along a subsequence we have the decomposition

\[
q_k(x) = \sum_{j=1}^{J^*} \phi^j(x - x_k^j) + r_k^j(x) \quad \text{for all } J \in \{0, \ldots, J^*\} \text{ finite}
\]

that satisfies:

\[
\lim_{J \to J^*} \limsup_{k \to \infty} \left\|r_k^j\right\|_{W^{n-1,3}} = 0, \quad (5.3)
\]

\[
\lim_{k \to \infty} \left( \|q_k\|_{H^n}^2 - \left( \sum_{j=1}^{J^*} \|\phi^j\|_{H^n}^2 + \|r_k^j\|_{H^n}^2 \right) \right) = 0 \quad \text{for all } J \text{ finite,} \quad (5.4)
\]

\[
\lim_{J \to J^*} \limsup_{k \to \infty} \left\|q_k\right\|_{W^{n-1,3}}^3 - \sum_{j=1}^{J^*} \left\|\phi^j\right\|_{W^{n-1,3}}^3 = 0, \quad (5.5)
\]

\[
|x_k^j - x_k^\ell| \to \infty \quad \text{as } k \to \infty \quad \text{whenever } j \neq \ell. \quad (5.6)
\]

The \( n = 1 \) case of Theorem 5.2 is well-known [20, Prop. 3.1]; for a textbook presentation of such concentration compactness principles, we recommend [23]. While it does not appear that Theorem 5.2 for \( n \geq 2 \) has been recorded in the literature, it can be proved by exactly the same method (e.g. [23, Th. 4.7]) and we omit the details.

We apply this concentration compactness principle to the minimizing sequence \( \{q_k\}_{k \geq 1} \) in Theorem 5.1. After passing to a subsequence, Theorem 5.2 provides a number \( J^* \in \{0, 1, \ldots, \infty\} \), \( J^* \)-many profiles \( \{\phi^j\}_{j=1}^{J^*} \subset H^n(\mathbb{R}) \), and \( J^* \)-many sequences \( \{x_k^j\}_{j=1}^{J^*} \subset \mathbb{R} \) so that along a subsequence we have the decomposition (5.2) satisfying the properties (5.3)-(5.6). We will ultimately show that each profile \( \phi^j \) is a constrained minimizer of \( E_{n+1} \), and hence is a multisoliton.

First, we will treat the case \( J^* = 0 \):

Lemma 5.3  If \( J^* = 0 \), then \( e_1 = \cdots = e_n = 0 \) and \( q_k \to 0 \) in \( H^n(\mathbb{R}) \) as \( k \to \infty \).

Proof  The decomposition (5.2) reads \( q_k = r_k^0 \), and so

\[
E_2(q_k) = E_2(r_k^0) = \int \left\{ \frac{1}{2} \left( r_k^0 \right)^2 + (r_k^0)^3 \right\} dx.
\]

The second term in the integrand contributes \( \|r_k^0\|_{L^3}^3 \), which we know vanishes in the limit \( k \to \infty \) by (5.3). The remaining term is nonnegative, and so we obtain

\[
\lim_{k \to \infty} E_2(q_k) \geq 0.
\]

On the other hand, we know that this limit is attainable by a multisoliton since \( e_1, \ldots, e_n \in \mathcal{M}_n \), and so there exists \( N \leq n \) and \( \beta_1 > \cdots > \beta_N > 0 \) so that

\[
\lim_{k \to \infty} E_2(q_k) = -\frac{32}{5} \sum_{m=1}^{N} \beta_m^5 \leq 0.
\]

Together, we see that \( E_2(q_k) \to 0 \) and \( N = 0 \). The only multisoliton that can attain this value is the zero-soliton \( q(x) \equiv 0 \), and so we conclude that \( e_1 = \cdots = e_n = 0 \).
Now we have
\[ 0 = \lim_{k \to \infty} E_1(q_k) = \lim_{k \to \infty} \frac{1}{2} \|q_k\|_{L^2}^2 , \]
and so \( q_k \to 0 \) in \( L^2(\mathbb{R}) \). Using the estimates (2.3) that prove that \( E_1, \ldots, E_{n+1} \) are continuous functionals on \( H^n(\mathbb{R}) \), we obtain \( q_k \to 0 \) in \( H^n(\mathbb{R}) \) as desired. \( \square \)

Lemma 5.3 proves Theorem 5.1 in the case \( J^* = 0 \), and so for the remainder of the section we assume \( J^* \geq 1 \).

Next, we show that our decomposition accounts for the entirety of the limiting value of each \( E_m(q_k) \):

**Lemma 5.4** For each \( m = 1, \ldots, n + 1 \) we have
\[ \lim_{j \to J^*} \lim_{k \to \infty} \left| E_m(q_k) - \left[ \sum_{j=1}^{J} E_m(\phi^j) + E_m(r^j_k) \right] \right| = 0. \]

**Proof** Fix \( m \), and insert the decomposition (5.2) for \( q_k \) into the expression (2.1) for \( E_m \). By the \( H^n \)-norm property (5.4), we see the quadratic terms of each \( E_m \) cancel, leaving only cubic and higher order terms:

\[ \lim_{k \to \infty} \left| E_m(q_k) - \left[ \sum_{j=1}^{J} E_m(\phi^j) + E_m(r^j_k) \right] \right| = \lim_{k \to \infty} \left| \int \left\{ P_m \left( \sum_{j=1}^{J} \phi^j(x - x^j_k) + r^j_k \right) - \left[ \sum_{j=1}^{J} P_m(\phi^j) + P_m(r^j_k) \right] \right\} \, dx \right|. \]

Consider an arbitrary term \( c_{\alpha_1, \ldots, \alpha_d} u^{(\alpha_1)} \cdots u^{(\alpha_d)} \) of \( P_m(u) \) for \( u = q_k, \phi^j, \) or \( r^j_k \). Expanding all products in the case \( u = q_k = \sum \phi^j(x - x^j_k) + r^j_k \), we are left with a term of the form
\[ u^{(\alpha_1)} \cdots u^{(\alpha_d)} \]
where each \( u_\ell \) for \( \ell = 1, \ldots, d \) is one of the profiles \( \phi^j \), its translation \( \phi^j(x - x^j_k) \), or the remainder \( r^j_k \).

We claim that all of the terms with \( u_1, \ldots, u_d \neq r^j_k \) cancel; in other words,
\[ \lim_{k \to \infty} \left| \int \left\{ P_m \left( \sum_{j=1}^{J} \phi^j(x - x^j_k) \right) - \sum_{j=1}^{J} P_m(\phi^j) \right\} \, dx \right| = 0 \quad (5.7) \]
for all \( J \leq J^* \) finite. When all of the \( u_\ell \) are given by the same translated profile \( \phi^j(x - x^j_k) \), we can change variables \( y = x - x^j_k \) in the integral and recover the corresponding term where \( u_1, \ldots, u_d = \phi^j(x) \). When there are at least two different translated profiles, the integral vanishes in the limit \( k \to \infty \) by the well-separation condition (5.6) and approximating each \( \phi^j \) by compactly-supported functions.

It remains to show that the remaining terms (which contain at least one factor of \( r^j_k \)) vanish in \( L^1 \). Note that by the scaling requirement (2.2), each order \( \alpha_\ell \) is at most \( m - 2 \leq n - 1 \). We estimate the highest order factor of \( r^j_k \) in \( L^3 \), which is vanishing in the limit \( k \to \infty \) and \( J \to J^* \) by the small-remainder condition (5.3). We then estimate the two other highest order factors \( \phi^j \) or \( r^j_k \) in \( L^3 \), and the remaining terms are bounded in \( L^\infty \) since \( \phi^j \) and \( r^j_k \) are uniformly bounded in \( H^n \to W^{n-1, \infty} \). \( \square \)
Next, we show that the quadratic term of $E_{n+1}(r^J_k)$ dominates as $k \to \infty$:

**Lemma 5.5** For each $m = 1, \ldots, n+1$ we have

$$\lim_{J \to J^*, \lim k \to \infty} \sup_{J \to J^*} \left| E_m(r^J_k) - \frac{1}{2} \| (r^J_k)^{(m-1)} \|^2_{L^2} \right| = 0.$$  

**Proof** Fix $m$. Using the expression (2.1) for $E_m$, we write

$$E_m(r^J_k) - \frac{1}{2} \| (r^J_k)^{(m-1)} \|^2_{L^2} = \int P_m(r^J_k) \, dx.$$  

Consider the contribution of an arbitrary term $c_{\alpha_1, \ldots, \alpha_n} u^{(\alpha_1)} \cdots u^{(\alpha_n)}$ of $P_m$, where each order $\alpha_k$ is at most $n - 1$ by the scaling requirement (2.2). We estimate the three highest order factors of $r^J_k$ in $L^3$, which vanish in the limit $k \to \infty$ and $J \to J^*$ by the small-remainder condition (5.3). We then estimate the remaining terms in $L^\infty$, which are all bounded since the sequence $r^J_k$ is uniformly bounded in $H^n \hookrightarrow W^{n-1, \infty}$. Altogether, we conclude that every term vanishes in the limit $k \to \infty$ and $J \to J^*$.  

Next, we show that each profile $\phi^j$ is a constrained minimizer:

**Lemma 5.6** For each $1 \leq j \leq J^*$ finite, the profile $\phi^j$ minimizes $E_{n+1}(u)$ over all $u \in C(e)$ with the constraints $E_1(\phi^j), \ldots, E_n(\phi^j)$, and hence is a multisoliton $Q_{\beta^j, \epsilon^j}$ of degree at most $n$.

**Proof** Suppose towards a contradiction that there exists $j$ for which $\phi^j$ does not minimize $E_{n+1}$. Then we can replace $\phi^j$ by another profile $\tilde{\phi}^j \in H^n(\mathbb{R})$ with

$$E_m(\tilde{\phi}^j) = E_m(\phi^j) \quad \text{for } m = 1, \ldots, n, \quad \text{but } E_{n+1}(\tilde{\phi}^j) < E_{n+1}(\phi^j).$$  

Construct a new sequence $\{\tilde{q}_k\}_{k \geq 1}$ given by the decomposition (5.2), but with $\tilde{\phi}^j$ in place of $\phi^j$. This new sequence still satisfies the properties (5.3)–(5.6), and so by Lemma 5.4 we have

$$\lim_{k \to \infty} E_m(\tilde{q}_k) = e_m \quad \text{for } m = 1, \ldots, n, \quad \lim_{k \to \infty} E_{n+1}(\tilde{q}_k) < \lim_{k \to \infty} E_{n+1}(q_k).$$  

(5.8)

However, this contradicts that $\{q_k\}_{k \geq 1}$ was a minimizing sequence. Therefore, we conclude that each $\phi^j$ is a constrained minimizer of $E_{n+1}$.

Applying our variational characterization (Theorem 1.3), we conclude that $\phi^j$ is a multisoliton of degree at most $n$.  

We now know that our profiles $\{\phi^j\}_{j=1}^{J^*}$ are a (possibly infinite) collection of multisolitons $\{Q_{\beta^j, \epsilon^j}\}_{j=1}^{J^*}$. Concatenate all of the vectors $\beta^j$ to form one (possibly infinite) string $\bigcup_{j=1}^{J^*} \beta^j$ of positive numbers which may contain repeated values. Next, we show that all of the parameters together minimize $E_{n+1}$ subject to the constraints $e_1, \ldots, e_n$:

**Lemma 5.7** The concatenation $\bigcup_{j=1}^{J^*} \beta^j$ is equal to the unique set of parameters $\beta_1 > \cdots > \beta_N > 0$ satisfying the constraints $e_1, \ldots, e_n$. In particular, $J^*$ is finite.

**Proof** Consider the relaxed variational problem where we minimize $E_{n+1}(u)$ over the larger set

$$\{u \in H^n(\mathbb{R}) : E_1(u) \leq e_1, \ldots, E_n(u) \leq e_n\}.$$  

(5.9)

As the minimum value $C(e_1, \ldots, e_n)$ is strictly decreasing in each constraint by Lemma 3.5 and the set $\mathcal{M}_n$ of constraints is downward closed by Lemma 3.6, then this relaxed
minimization problem enjoys the same conclusions of Theorem 1.3. We will ultimately show that the profiles \( \{Q_{\tilde{\beta}^j_{\epsilon,i}}\}_{1 \leq j < J^*} \) together form a minimizer for this relaxed problem.

In the proof of Theorem 1.3 we only needed to treat the case of finitely many \( \beta \) parameters, but this is easily resolved as follows. Suppose towards a contradiction that \( J^* = \infty \). We know that the third moment \( \sum_1^\beta \beta^3 \) is finite by the trace formula \( (2.1) \) for \( E_1 \), and so we have \( \beta_m \to 0 \) as \( m \to \infty \). In particular, even though there may be repeated values in \( \beta \), there are at least \( n+1 \) distinct values of \( \beta_m > 0 \). By Lemma 3.1, we may replace the first \( n+1 \) distinct values of \( \beta_m \) to obtain new parameters \( \tilde{\beta}^j \) so that \( E_1, \ldots, E_n \) are preserved, \( E_{n+1} \) is decreased, and each \( \tilde{\beta}^j \) is still a set of multisoliton parameters. Constructing a new sequence \( \{\tilde{q}_k\}_{k \geq 1} \) given by the decomposition \( (5.2) \) for the profiles \( \tilde{\beta}^j = Q_{\tilde{\beta}^j_{\epsilon,i}} \), we obtain a strictly better choice of minimizing sequence in the sense of \( (5.8) \). This contradicts that \( \{q_k\}_{k \geq 1} \) was a minimizing sequence.

Now that we know \( J^* < \infty \), Lemma 5.4 implies that

\[
\sum_{j=1}^{J^*} E_m(Q_{\tilde{\beta}^j_{\epsilon,i}}) \leq \limsup_{k \to \infty} \left[ E_m(q_k) - E_m(r_k^{J^*}) \right] \tag{5.10}
\]

for each \( m = 1, \ldots, n+1 \). For \( m \leq n \), we have \( E_m(q_k) \to e_m \) by construction and

\[
\liminf_{k \to \infty} E_m(r_k^{J^*}) \geq 0
\]

by Lemma 5.5, and so RHS\((5.10)\) is at most \( e_m \). In other words, the parameters \( \beta \) satisfy the relaxed constraints

\[
\sum_{j=1}^{J^*} E_m(Q_{\tilde{\beta}^j_{\epsilon,i}}) \leq e_m \quad \text{for } m = 1, \ldots, n. \tag{5.11}
\]

For \( m = n+1 \), we know that \( E_m(q_k) \) converges to the minimum value \( C(e_1, \ldots, e_n) \), and so RHS\((5.10)\) is at most \( C(e_1, \ldots, e_n) \). This yields

\[
\sum_{j=1}^{J^*} E_{n+1}(Q_{\tilde{\beta}^j_{\epsilon,i}}) \leq C(e_1, \ldots, e_n). \tag{5.12}
\]

Strict inequality here should not be possible since \( C \) is the minimum value of \( E_{n+1} \) over the set \( \{5.9\} \). Indeed, by \( (5.4) \) and \( (5.7) \) we have

\[
\sum_{j=1}^{J^*} E_{n+1}(Q_{\tilde{\beta}^j_{\epsilon,i}}) = \liminf_{k \to \infty} E_{n+1}\left( \sum_{j=1}^{J^*} Q_{\tilde{\beta}^j_{\epsilon,i}}(x-x_k^j) \right) \geq C(e_1, \ldots, e_n).
\]

Therefore, we conclude that equality holds in \( (5.12) \).

Altogether, we see that the finite collection \( \beta \) of parameters is a minimizer for the relaxed variational problem \( \{5.9\} \). As the minimum value \( C(e_1, \ldots, e_n) \) is strictly decreasing in each constraint by Lemma 3.5, we must have equality in \( (5.11) \). There cannot be \( n+1 \) distinct values in \( \beta \), since otherwise we could use Lemma 3.1 to replace the first \( n+1 \) distinct values in a way that preserves \( E_1, \ldots, E_n \) and decreases \( E_{n+1} \) in order to obtain a strictly better minimizing sequence. Now that we know there are at most \( n \) distinct values of \( \beta_m \), Corollary 3.3 implies that \( \beta \) is equal to the unique set of parameters \( \beta_1 > \cdots > \beta_N > 0 \) with \( N \leq n \) that satisfies the constraints \( (e_1, \ldots, e_n) \in M_n^n \). \( \Box \)

It remains to show that the whole sequence \( q_k \) converges strongly to the manifold of minimizing multisolitons. To this end, we will need:
Lemma 5.8  The remainders $r^J_k \rightarrow 0$ in $H^n(\mathbb{R})$ as $k \rightarrow \infty$.

Proof  By Lemma 5.7, the profiles $\{ Q_{\beta^j, c^j} \}_{1 \leq j < J^*}$ satisfy the constraints:

$$\sum_{j=1}^{J^*} E_m(Q_{\beta^j, c^j}) = e_m \quad \text{for } m = 1, \ldots, n.$$  

Combining this with Lemmas 5.4 and 5.5, we deduce

$$0 = \lim_{k \rightarrow \infty} E_m(r^J_k) = \lim_{k \rightarrow \infty} \frac{1}{2} \| (r^J_k)^{(m-1)} \|^2_{L^2}$$

for $m = 1, \ldots, n + 1$. \hfill \Box

The last ingredient that we will need is the following “molecular decomposition” of multisolitons, which says that our superposition $\sum Q_{\beta^j, c^j}(x - x_j^k)$ of well-separated multisolitons is close to the manifold of multisolitons:

**Proposition 5.9**  Fix integers $n \geq 0$ and $J \geq 1$. Suppose $\beta^j$ and $c^j$ are multisoliton parameters for each $1 \leq j \leq J$, and that all of the components $\beta^j_m$ of each $\beta^j$ are distinct for all $j$ and $m$. Then for any collection of $J$-many sequences $\{ x_j^k \}_{j=1}^J \subset \mathbb{R}$ satisfying the well-separation condition (5.6), there exists a sequence $c_k$ so that

$$Q_{\beta, c_k}(x) - \sum_{j=1}^{J} Q_{\beta^j, c^j}(x - x_j^k) \rightarrow 0 \quad \text{in } H^n(\mathbb{R}) \text{ as } k \rightarrow \infty,$$

where $\beta$ is the concatenation $\bigsqcup_{j=1}^{J} \beta^j$.

Proof  The $n = 0$ case is proved in [24, Prop. 3.1]. Given $n \geq 1$, we pick $c_k$ from the $n = 0$ case so that the desired convergence occurs in $L^2(\mathbb{R})$. Note that

$$\left\| Q_{\beta, c_k}(x) - \sum_{j=1}^{J} Q_{\beta^j, c^j}(x - x_j^k) \right\|_{H^{n+1}} \lesssim 1$$

uniformly in $k$, by the estimates (2.3) that prove that $E_1, \ldots, E_{n+2}$ are continuous, the trace formulas (2.7), and the well-separation condition (5.6). Using the inequality

$$\| f \|_{H^n} \leq \| f \|_{L^2} \| f \|_{H^{n+1}}$$

(which follows from Hölder’s inequality in Fourier variables), we conclude that the sequence converges in $H^n(\mathbb{R})$. \hfill \Box

We are now prepared to finish the proof of Theorem 5.1:

**Proof of Theorem 5.1**  It remains to show that the sequence $\{ q_k \}_{k \geq 1}$ converges to the manifold $\{ Q_{\beta, c} : c \in \mathbb{R}^N \}$. So far, we have the decomposition

$$q_k(x) = \sum_{j=1}^{J^*} Q_{\beta^j, c^j}(x - x_j^k) + r^J_k(x)$$

$q_k(x)$ converges to $Q_{\beta, c}(x)$ as $k \rightarrow \infty$. \hfill \Box

\[ \text{Springer} \]
Multisolitons are the unique constrained minimizers... with $J^*$ finite. Let $Q_{\beta,c_k}$ be the sequence of multisolitons guaranteed by the $H^m(\mathbb{R})$ molecular decomposition (Proposition 5.9). We estimate
\[ \|q_k - Q_{\beta,c_k}\|_{H^n} \leq \left\| Q_{\beta,c_k}(x) - \sum_{j=1}^{J^*} Q_{\beta^j,c_j}(x - x_k^j) \right\|_{H^n} + \| r_k^{J^*} \|_{H^n}. \]
The first term on the RHS converges to zero as $k \to \infty$ by Proposition 5.9. The second term on the RHS converges to zero by Lemma 5.8. Together, we conclude that
\[ \inf_{c \in \mathbb{R}^n} \|q_k - Q_{\beta,c}\|_{H^n} \to 0 \quad \text{as} \quad k \to \infty \]
as desired. \hfill \Box

As a corollary, we obtain orbital stability:

**Proof of Theorem 1.4** Suppose towards a contradiction that orbital stability fails. Then there exists a constant $\epsilon_0 > 0$, a sequence of initial data $\{q_k(0)\}_{k \geq 1} \subset H^n(\mathbb{R})$, and a sequence of times $\{t_k\}_{k \geq 1} \subset \mathbb{R}$ such that
\[ \inf_{c \in \mathbb{R}^n} \|q_k(0) - Q_{\beta,c}\|_{H^n} \to 0 \quad \text{as} \quad k \to \infty, \]
but the corresponding solutions $q_k(t)$ to KdV obey
\[ \inf_{c \in \mathbb{R}^n} \|q_k(t_k) - Q_{\beta,c}\|_{H^n} \geq \epsilon_0 \quad \text{for all} \quad k. \quad (5.13) \]
As $E_1, \ldots, E_{n+1}$ are continuous on $H^n(\mathbb{R})$ and are conserved by the KdV flow, we have
\[ \lim_{k \to \infty} E_m(q_k(t_k)) = \lim_{k \to \infty} E_m(q_k(0)) = E_m(Q_{\beta,c}) \]
for each $m = 1, \ldots, n+1$. There are $n$-many $\beta$ parameters, and so these are exactly the conditions (5.1) that the sequence $\{q_k(t_k)\}_{k \geq 1}$ is a minimizing sequence for $E_{n+1}$ with constraints $E_1(Q_{\beta,c}), \ldots, E_n(Q_{\beta,c})$ that are in $\mathcal{M}_N^n$. By Theorem 5.1, there exists a subsequence of $\{q_k(t_k)\}_{k \geq 1}$ which converges to the manifold $\{Q_{\beta,c} : c \in \mathbb{R}^N\}$ in $H^n(\mathbb{R})$, which contradicts our assumption (5.13). \hfill \Box

6 Proof of Theorem 1.5

In this section, we will adapt the methods of Sects. 3 to 5 in order to prove Theorem 1.5. Fix $N \geq n + 1$, and consider constraints $(e_1, \ldots, e_n) \in \mathcal{M}_N^n$ that are attainable by multisolitons of degree at most $N$. We aim to show that minimizing sequences resemble a superposition of multisolitons with at most $n$ distinct amplitudes.

As the constraints are attainable by finitely many parameters, compactness still guarantees that there exists a minimizing set of $N$-soliton parameters $\beta_1 \geq \cdots \geq \beta_N \geq 0$, provided that we allow for repeated values:

**Lemma 6.1** Given constraints $(e_1, \ldots, e_n) \in \mathcal{M}_N^n$, there exist $\beta_1 \geq \cdots \geq \beta_N \geq 0$ which minimize
\[ E_{n+1}(Q_{\beta,c}) = (-1)^n \frac{2^{n+3}}{2n+3} \sum_{m=1}^{N} \beta_m^{2n+3} \]
over the set of multisolitons in the constraint set $\mathcal{C}_e$. 

\[ \text{Springer} \]
Lemma 6.2

If there exist new values \( \tilde{\beta} \) from Lemma 3.1 to reduce the number of distinct components in the minimizer number of parameters. Instead, we can employ the implicit function theorem argument from \( e \) constraints.

Proof

Consider the set \( \Gamma \) of parameters

\[
\{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_1, \ldots, x_N \geq 0, \sum_{m=1}^{N} x_m^{2j+1} = \alpha_j \text{ for } j = 1, \ldots, n\}
\]

that satisfy the constraints, where

\[
\alpha_j = (-1)^{j+1} \frac{2j+1}{2^{j+1}} \epsilon_j
\]

are the prescribed odd moments. Note that the set \( \Gamma \) is compact, and it is nonempty since \( (e_1, \ldots, e_n) \in \mathcal{M}_N^m \). Therefore there exists a minimizer \( (\beta_1, \ldots, \beta_N) \) of the next odd moment

\[
(-1)^n \frac{2^{2n+3}}{2n+3} \sum_{m=1}^{N} x_m^{2n+3}
\]

in \( \Gamma \). As the odd moments are symmetric in \( \beta \), we may reorder them so that \( \beta_1 \geq \cdots \geq \beta_N \geq 0 \).

Unlike in the proof of Lemma 3.6, the set \( \Gamma \) no longer reaches the boundary \( \{(x_1, \ldots, x_N) : x_k = 0\} \) when \( (e_1, \ldots, e_n) \notin \mathcal{M}_N^m \), and so we cannot necessarily reduce the number of parameters. Instead, we can employ the implicit function theorem argument from Lemma 3.1 to reduce the number of distinct components in the minimizer \( (\beta_1, \ldots, \beta_N) \):

Lemma 6.2

If \( \beta_1 \geq \cdots \beta_N \geq 0 \) is a minimizer (in the sense of Lemma 6.1), then there are at most \( n \) distinct values of \( \beta_m \).

Proof

It suffices to show that if there are at least \( n + 1 \) distinct values in \( \beta_1, \ldots, \beta_N \), then there exist new values \( \tilde{\beta}_1, \ldots, \tilde{\beta}_N \) which preserve \( E_1, \ldots, E_n \) but decrease \( E_{n+1} \). To prove this, we repeat the proof of Lemma 3.1. Rather than recapitulating the whole proof, let us focus on the few minor alterations that need to be made.

For example, consider the case where we have \( N \geq n + 2 \), \( \beta_j = \beta_{j+1} \) for some \( j \), and all other \( \beta_m \) are distinct. Replace the function (3.1) by

\[
f(x_1, \ldots, x_{n+1}) = \begin{pmatrix}
x_1^3 + \cdots + x_{j-1}^3 + 2x_j^3 + x_{j+1}^3 + \cdots + x_{n+1}^3 \\
x_1^5 + \cdots + x_{j-1}^5 + 2x_j^5 + x_{j+1}^5 + \cdots + x_{n+1}^5 \\
\vdots \\
x_1^{2n+1} + \cdots + x_{j-1}^{2n+1} + 2x_j^{2n+1} + x_{j+1}^{2n+1} + \cdots + x_{n+1}^{2n+1}
\end{pmatrix}.
\]

This simply multiplies the \( j \)th column of the derivative matrix (3.2) by 2. Consequently, the left \( n \times n \) submatrix still has nonzero determinant and thus we may apply the implicit function theorem. We can then proceed with the remainder of the proof of Lemma 3.1.

In the general case, each column of the derivative matrix (3.2) is simply multiplied by a constant. Therefore the left \( n \times n \) submatrix is still invertible, and the proof of Lemma 3.1 proceeds as before.

Now that we know that every minimizer must possess at most \( n \) distinct \( \beta \) values, Corollary 3.3 immediately implies that the minimizer is unique.

We are now prepared to define our candidate value for the infimum of \( E_{n+1} \) subject to the constraints \( e_1, \ldots, e_n \). We extend the definition of \( C \) to \( (e_1, \ldots, e_n) \in \mathcal{M}_N^m \) via

\[
C(e_1, \ldots, e_n) = (-1)^n \frac{2^{2n+3}}{2n+3} \sum_{j=1}^{N} \beta_j^{2n+3}.
\]
This quantity still satisfies the properties from Lemma 3.5:

**Lemma 6.3** The function $C : \mathcal{M}_N^a \to \mathbb{R}$ is continuous and is decreasing in each variable. Moreover, $C$ is defined piecewise on finitely many connected subsets of $\mathcal{M}_N^a$, and on the interior of each such subset $C(e_1, \ldots, e_n)$ is continuously differentiable and satisfies $\frac{\partial C}{\partial e_j} < 0$ for $j = 1, \ldots, n$.

**Proof** Given a minimizer $\beta_1 \geq \cdots \geq \beta_N \geq 0$, Lemma 6.2 implies that there exist multiplicities $m_1, \ldots, m_N$ and distinct values $\overline{\beta}_1 > \cdots > \overline{\beta}_N \geq 0$ so that $N \leq n$, $\sum m_j = N$, and the string $\beta_1, \ldots, \beta_N$ consists of $m_1$ copies of $\overline{\beta}_1$, $m_2$ copies of $\overline{\beta}_2$, and so on. This allows us to write

$$C(e_1, \ldots, e_n) = (-1)^n \frac{2^{2n+3}}{2n+3} (m_1 \overline{\beta}_1^{2n+3} + \cdots + m_N \overline{\beta}_N^{2n+3}).$$

We will see that for $N = n$ and each fixed choice of multiplicities $m_1, \ldots, m_n$, we have $\frac{\partial C}{\partial e_j} < 0$ for $j = 1, \ldots, n$ as long as $\overline{\beta}_1 > \cdots > \overline{\beta}_N > 0$. In this way $C(e_1, \ldots, e_n)$ is a piecewise-defined function, and there are finitely many pieces because the number of possible multiplicities $m_1, \ldots, m_N$ is finite.

Fix multiplicities $m_1, \ldots, m_N$, and repeat the computation from Lemma 3.5. In fact, in the case $N = n$ the same computation applies! Indeed, in Lemma 3.5 we computed $\frac{\partial C}{\partial e_j}$ from the equality (3.16). We have now multiplied the columns of the matrix on the LHS and each entry on the RHS by the multiplicities $m_1, \ldots, m_N$, but this does not alter the system of equations. In the case $N < n$, the system of equations (3.14) is overdetermined. However, if we only consider the first $\overline{N}$ constraints, then the computation proceeds with $\overline{N}$ in place of $n$, and we conclude that $C$ as a function of $e_1, \ldots, e_{\overline{N}}$ (where $m_1, \ldots, m_{\overline{N}}$ are fixed) is $C^1$ and satisfies $\frac{\partial C}{\partial e_j} < 0$ for $j = 1, \ldots, \overline{N}$.

We are also able to compute $\frac{\partial \overline{\beta}_j}{\partial e_j}$ as long as $\overline{\beta}_1 > \cdots > \overline{\beta}_n > 0$, which implies that if we wiggle $e_1, \ldots, e_n$ then we can also wiggle $\overline{\beta}_1, \ldots, \overline{\beta}_n$ in a way that still satisfies the constraints. By uniqueness (Corollary 3.3), the perturbed values of $\overline{\beta}_1, \ldots, \overline{\beta}_n$ still minimize $E_{n+1}$. This defines an injective map $\Phi$ from the simplex

$$\{(\overline{\beta}_1, \ldots, \overline{\beta}_n) \in \mathbb{R}^n : \overline{\beta}_1 > \cdots > \overline{\beta}_n > 0\}$$

(6.2) into $\mathcal{M}_n^a$, and it is smooth up to its boundary. The image of $\Phi$ is exactly the interior of one of the components on which $C(e_1, \ldots, e_n)$ is defined by a single formula. The boundary of this component corresponds to some subset of the boundary of the simplex (6.2), which means that two values of $\overline{\beta}_j$ are colliding or that $\overline{\beta}_n$ is vanishing. Repeating the proof of Lemma 3.4, we conclude that $\Phi$ is a homeomorphism onto this component (including any boundary points it may contain). It then follows that $C$ is continuous by the same argument as in Lemma 3.5. $\square$

Next, we show that the set of constraints $\mathcal{M}_N^a$ is still downward closed:

**Lemma 6.4** If the constraints $\overline{e}_1, \ldots, \overline{e}_n$ are in $\mathcal{M}_N^a$, for some $\overline{N}$ and

$$\overline{e}_1 \leq e_1, \ldots, \overline{e}_n \leq e_n$$

for some $(e_1, \ldots, e_n) \in \mathcal{M}_N^a$, then $(\overline{e}_1, \ldots, \overline{e}_n) \in \mathcal{M}_N^a$.

**Proof** Let $\overline{\beta}_1, \ldots, \overline{\beta}_N > 0$ denote the $\beta$ parameters of the multisoliton which witnesses the constraints $\overline{e}_1, \ldots, \overline{e}_n$, and assume that we are in the nontrivial case $\overline{N} \geq N + 1$. Repeating
the proof of Lemma 3.6, we conclude that the set $\Gamma$ of parameters in $\mathbb{R}^{N}$ that satisfy the constraints must intersect the boundary $\{x_{1}, \ldots, x_{N} : x_{N+1}, \ldots, x_{\tilde{N}} = 0\}$. Any point in the intersection provides the desired $N$-soliton parameters. \hfill $\square$

We are now prepared to prove that $C(e_{1}, \ldots, e_{n})$ is the infimum of $E_{n+1}$:

**Proposition 6.5** Given $(e_{1}, \ldots, e_{n}) \in \mathcal{M}^{n}_{N}$ for some $N \geq n + 1$, we have

$$\inf\{E_{n+1}(u) : u \in \mathcal{C}_{e}\} = C(e_{1}, \ldots, e_{n}).$$

Moreover, if $(e_{1}, \ldots, e_{n}) \notin \mathcal{M}^{n}_{N}$, then this infimum is not attained by any $u \in \mathcal{C}_{e}$.

**Proof** First, we claim that

$$E_{n+1}(u) \geq C(e_{1}, \ldots, e_{n}) \quad \text{for all } u \in \mathcal{C}_{e}.$$ 

We repeat the proof of Theorem 4.2. This proof only required Lemmas 3.1, 3.5 and 3.6 as input, and we have established their analogues Lemmas 6.2 to 6.4 in this new setting.

To prove (6.3), it remains to show that $E_{n+1}(u)$ can be arbitrarily close to $C(e_{1}, \ldots, e_{n})$ for some choice of $u \in \mathcal{C}_{e}$. Recall that $C(e_{1}, \ldots, e_{n})$ is defined in terms of a minimizer $\beta_{1} \geq \cdots \geq \beta_{N} \geq 0$ in the sense of Lemma 6.1. The claim follows by taking $u$ to be an $N$-soliton with parameters $\beta_{1} > \cdots > \beta_{N} > 0$ that converge to the minimizer $\beta_{1} \geq \cdots \geq \beta_{N} \geq 0$ within the set (6.1).

Lastly, suppose towards a contradiction that $E_{n+1}(q) = C(e_{1}, \ldots, e_{n})$ for some $q \in \mathcal{C}_{e}$ and $(e_{1}, \ldots, e_{n}) \notin \mathcal{M}^{n}_{N}$. First, we show that $q$ is Schwartz by repeating the proof of Lemma 4.4. In the case where the number $\tilde{N}$ of multiplicities is equal to $n$, we have $\frac{\partial C}{\partial e_{j}} < 0$ for $j = 1, \ldots, n$ and the proof of Lemma 4.4 carries out unaltered. In the case where $\tilde{N} < n$, we recall from the proof of Lemma 6.3 that we may regard $C$ as a function of $e_{1}, \ldots, e_{\tilde{N}}$ and we have $\frac{\partial C}{\partial e_{j}} < 0$ for $j = 1, \ldots, \tilde{N}$. In either case, the minimizer $q$ satisfies an Euler–Lagrange equation of the form (4.1) with $\lambda_{1} < 0$ and all other $\lambda_{j} \leq 0$, and this is sufficient to conclude that $q$ is Schwartz. Then, by directly applying Lemma 4.5, 4.6, and 3.1 (without alteration!), we see that $q$ is a multisoliton of degree at most $n$, which contradicts that $(e_{1}, \ldots, e_{n}) \notin \mathcal{M}^{n}_{N}$. \hfill $\square$

When combined with concentration compactness, we can prove that minimizing sequences resemble a superposition of multisolitons with at most $n$ distinct values of $\beta_{m}$:

**Proof of Theorem 1.5** It only remains to prove the minimizing sequence statement. Fix $(e_{1}, \ldots, e_{n}) \in \mathcal{M}_{N}^{n} \setminus \mathcal{M}_{N-1}^{n}$ for some $N \geq n + 1$, and suppose that $\{q_{k}\}_{k \geq 1} \subset H^{n}(\mathbb{R})$ satisfies

$$E_{1}(q_{k}) \to e_{1}, \quad \ldots, \quad E_{n}(q_{k}) \to e_{n}, \quad E_{n+1}(q_{k}) \to C(e_{1}, \ldots, e_{n})$$

as $k \to \infty$.

First, we apply our concentration compactness principle. After passing to a subsequence, Theorem 5.2 provides us with a number $J^{*} \in \{0, 1, \ldots, \infty\}$, $J^{*}$-many profiles $\{\phi^{j}\}_{j=1}^{J^{*}} \subset H^{n}(\mathbb{R})$, and $J^{*}$-many sequences $\{x_{k}^{j}\}_{j=1}^{J^{*}} \subset \mathbb{R}$ so that along a subsequence we have the decomposition (5.2) which satisfies the properties (5.3)-(5.6).

The proof of Theorem 5.1 up through Lemma 5.6 still applies (without alteration), and so we conclude that each profile $\phi^{j}$ is a multisoliton $Q_{\beta^{j}, c^{j}}$. Repeating the proof of Lemma 5.7, we see that the concatenation $\beta = \bigsqcup_{j=1}^{J^{*}} \beta^{j}$ minimizes the inequality for $E_{n+1}$ in Proposition 6.5. Therefore $\beta$ is a minimizer in the sense of Lemma 6.1, and so by Lemmas 6.2 and 6.4
Multisolitons are the unique constrained minimizers...

we see that $J^*$ is finite, the total degree $\sum \#\beta^j$ is equal to $N$, and the components of $\beta$ attain at most $n$ distinct values.

We now have

$$\left\| q_k - \sum_{j=1}^{J^*} Q_{\beta^j, e^j} \right\|_{H^n} = \left\| r_k^{J^*} \right\|_{H^n}.$$  

Repeating the proof of Lemma 5.8, we see that the RHS converges to zero as $k \to \infty$. This yields

$$\inf_{e^1, \ldots, e^{J^*}} \left\| q_k - \sum_{j=1}^{J^*} Q_{\beta^j, e^j} \right\|_{H^n} \to 0 \text{ as } k \to \infty$$

as desired. \hfill \Box

7 Proof of Theorem 1.6

The goal of this section is to prove Theorem 1.6. Suppose that $(e_1, e_2) \in F_2$ and $(e_1, e_2) \not\in \mathcal{M}_{N}^2$ for all $N$. We aim to show that Schwartz minimizing sequences for these constraints have vanishing $\beta$ parameters and log $|a|$ converging to the even extension of a Dirac delta distribution.

By the explicit description (1.9) and (1.11) of $F_2$ and $\bigcup_{N \geq 0} \mathcal{M}_{N}^2$, we note that our conditions on $(e_1, e_2)$ are equivalent to $e_1 > 0$ and $e_2 \geq 0$.

First, we find a lower bound for the log $|a|$ contribution to $E_3$:

**Lemma 7.1** We have

$$E_3(u) \geq \frac{64}{\pi^2} \frac{\gamma_0^2}{\gamma_0} + C(e_1 - \frac{4}{\pi} \gamma_0, e_2 - \frac{16}{\pi} \gamma_1) \quad \text{for all } u \in \mathcal{C}_e \cap S(\mathbb{R}),$$

where

$$\gamma_0 = \int_{-\infty}^{\infty} k^2 \log |a(k; u)| \, dk \quad \text{and} \quad \gamma_1 = \int_{-\infty}^{\infty} k^4 \log |a(k; u)| \, dk.$$

**Proof** Fix $u \in S(\mathbb{R})$. Substituting $x = k^2$ into the trace formulas (2.7) and recalling the reality condition (2.6), we obtain

$$e_1 = \frac{4}{\pi} \int_0^{\infty} x^{\frac{1}{2}} \log |a(x^{\frac{1}{2}}; u)| \, dx + \frac{8}{5} \sum_{m=1}^{N} \beta_m^3,$$

$$e_2 = \frac{16}{\pi} \int_0^{\infty} x^{\frac{3}{2}} \log |a(x^{\frac{1}{2}}; u)| \, dx - \frac{32}{5} \sum_{m=1}^{N} \beta_m^5,$$

$$E_3(u) = \frac{64}{\pi} \int_0^{\infty} x^{\frac{5}{2}} \log |a(x^{\frac{1}{2}}; u)| \, dx + \frac{128}{7} \sum_{m=1}^{N} \beta_m^7.$$

The first constraint says that the positive measure

$$d\mu := x^{\frac{1}{2}} \log |a(x^{\frac{1}{2}}; u)| \, dx$$

on $[0, \infty)$ has total mass

$$\gamma_0 := \int_0^{\infty} 1 \, d\mu(x) = \frac{\pi}{4} \left( e_1 - \frac{8}{5} \sum_{m=1}^{N} \beta_m^3 \right) \in \left[ 0, \frac{\pi}{4} e_1 \right].$$
The first constraint also restricts how large the first moment of \(d\mu\) can be. As \(p \mapsto \|\beta\|_{\ell^p}\) is decreasing, we have
\[
\left( \sum_{m=1}^{N} \beta_m^5 \right)^{\frac{1}{5}} \leq \left( \sum_{m=1}^{N} \beta_m^3 \right)^{\frac{1}{3}} \leq \left( \frac{3}{8} e_1 \right)^{\frac{1}{3}}.
\]
This requires that the first moment obeys
\[
\gamma_1 := \int_0^\infty x \, d\mu(x) = \frac{\pi}{16} \left( e_2 + \frac{32}{5} \sum_{m=1}^{N} \beta_m^5 \right) \in \left[ \frac{\pi}{16} e_2, \frac{\pi}{16} \left( e_2 + \frac{32}{5} \left( \frac{3}{8} e_1 \right)^{\frac{5}{3}} \right) \right]. \tag{7.4}
\]
In order to bound \(E_3(u)\) below, we seek a lower bound for the second moment
\[
\gamma_2 := \int_0^\infty x^2 \, d\mu(x).
\]
By Cauchy–Schwarz we have
\[
\gamma_1 = \int_0^\infty x \, d\mu(x) \leq \left( \int_0^\infty 1 \, d\mu(x) \right)^{\frac{1}{2}} \left( \int_0^\infty x^2 \, d\mu(x) \right)^{\frac{1}{2}} = \gamma_0^{\frac{1}{2}} \gamma_2^{\frac{1}{2}},
\]
and so
\[
\gamma_2 \geq \frac{\gamma_1^2}{\gamma_0}. \tag{7.5}
\]
For future reference (cf. (7.8)), we note that equality occurs above if and only if the functions \(x^\frac{5}{2} \log |a(x^\frac{1}{2}; u)|\) and \(x^5 \log |a(x^\frac{1}{2}; u)|\) are proportional. As \(k \mapsto \log |a(k; u)|\) is a continuous nonnegative function on \(\mathbb{R}\) for \(u\) Schwartz, this can only happen when \(\log |a(k; u)| \equiv 0\) for \(k \in \mathbb{R}\).

The estimate (7.5) provides a lower bound for the \(\log |a|\) moment of \(E_3(u)\). The \(\beta\) moment is then bounded below by the infimum \(C\) of \(E_3(u)\) subject to the smaller constraints where the \(\log |a|\) moments are removed:
\[
\sum_{m=1}^{N} \beta_m^7 = E_3(Q_{\beta, e}) \geq C \left( e_1 - \frac{4}{\pi} \gamma_0, e_2 - \frac{16}{\pi} \gamma_1 \right).
\]
Together, this yields the inequality (7.1). \(\Box\)

We will now minimize the lower bound in (7.1) over all possible \(\gamma_0\) and \(\gamma_1\). At first glance, the first term \(\gamma_1^2/\gamma_0\) is smallest when \(\gamma_0\) is large and \(\gamma_1\) is small, while the second term \(C(e_1 - \frac{4}{\pi} \gamma_0, e_2 - \frac{16}{\pi} \gamma_1)\) is smallest when both \(\gamma_0\) and \(\gamma_1\) are small. We will see below that the first term is dominant, which yields the following inequality:

**Lemma 7.2** Given constraints \(e_1 > 0\) and \(e_2 \geq 0\), we have
\[
\inf \{E_3(u) : u \in \mathcal{C}_e \cap \mathcal{S}(\mathbb{R})\} = \frac{e_2^2}{e_1}. \tag{7.6}
\]
Moreover, this infimum is not attained by any \(u \in \mathcal{C}_e \cap \mathcal{S}(\mathbb{R})\).

**Proof** The domain of \((\gamma_0, \gamma_1)\) in \(\mathbb{R}^2\) is contained in the rectangle given by the product of the intervals in (7.3) and (7.4). Let \((\gamma_0, \gamma_1)\) be the minimizer of
\[
\frac{64}{\pi} \gamma_0^2 + C \left( e_1 - \frac{4}{\pi} \gamma_0, e_2 - \frac{16}{\pi} \gamma_1 \right)
\]
\(\Box\) Springer
over this compact rectangle. Differentiating with respect to $\gamma_1$, we have
\[
\frac{\partial}{\partial \gamma_1} \left\{ \frac{64 \gamma_1^2}{\pi \gamma_0} + C(e_1 - \frac{4}{\pi} \gamma_0, e_2 - \frac{16}{\pi} \gamma_1) \right\} = \frac{128 \gamma_1}{\pi \gamma_0} - \frac{16}{\pi} \frac{\partial C}{\partial e_2} \left(e_1 - \frac{4}{\pi} \gamma_0, e_2 - \frac{16}{\pi} \gamma_1\right).
\]
The derivative $\frac{\partial C}{\partial e_2}$ is nonpositive by Lemma 3.5, and so this quantity is positive for all $\gamma_1$ in the open interval $(\frac{7 \pi}{16} e_2, \frac{10}{16} (e_2 + 32 (3 \delta_1^2)^2))$. Therefore the minimizer must have $\gamma_1 = \frac{7 \pi}{16} e_2$.

Similarly, we have
\[
\frac{\partial}{\partial \gamma_0} \left\{ \frac{64 \gamma_1^2}{\pi \gamma_0} + C(e_1 - \frac{4}{\pi} \gamma_0, e_2 - \frac{16}{\pi} \gamma_1) \right\} = -\frac{64 \gamma_1^2}{\pi \gamma_0^2} - \frac{4}{\pi} \frac{\partial C}{\partial \gamma_1} \left(e_1 - \frac{4}{\pi} \gamma_0, e_2 - \frac{16}{\pi} \gamma_1\right).
\]

The derivative $\frac{\partial C}{\partial \gamma_1}$ is $O(\beta_1^2, \beta_2^2)$ by the computation (3.17), and hence vanishes as $\beta_1, \beta_2 \to 0$. Therefore, taking $\gamma_1 \to \frac{7 \pi}{16} e_2$ we obtain
\[
\frac{\partial}{\partial \gamma_0} \left\{ \frac{64 \gamma_1^2}{\pi \gamma_0} + C(e_1 - \frac{4}{\pi} \gamma_0, e_2 - \frac{16}{\pi} \gamma_1) \right\} \to -\frac{\pi e_1^2}{4 \gamma_0^2}
\]
for all $\gamma_0$ in the open interval $(0, \frac{\pi}{7} e_1)$. Therefore the minimizer has $\gamma_0 = \frac{\pi}{7} e_1$.

Altogether, we conclude that the minimum occurs at $\gamma_0 = \frac{\pi}{7} e_1, \gamma_1 = \frac{7 \pi}{16} e_2$ with value
\[
\left[ \frac{64 \gamma_1^2}{\pi \gamma_0} + C(e_1 - \frac{4}{\pi} \gamma_0, e_2 - \frac{16}{\pi} \gamma_1) \right] \bigg|_{\gamma_0 = \frac{\pi}{7} e_1, \gamma_1 = \frac{7 \pi}{16} e_2} = \frac{e_1^2}{e_1}.
\]

To prove (7.6), it remains to show that we can make $E_3(u)$ arbitrarily close to this value. Fix $(\tilde{\gamma}_0, \tilde{\gamma}_1)$ in the interior of the rectangle given by the product of the intervals in (7.3) and (7.4) that is arbitrarily close to the minimizer $(\gamma_0, \gamma_1)$. Pick a smooth and even function $k \mapsto \log |a(k; \tilde{u})|$ with compact support in $\mathbb{R} \setminus \{0\}$ which attains the moments $(\tilde{\gamma}_0, \tilde{\gamma}_1)$ (in the sense of (7.2)). Arguing as in Lemma 4.5, we can then use Proposition 2.4 to construct a function $\tilde{u} \in S(\mathbb{R})$ (with no bound states) so that $k \mapsto \log |a(k; \tilde{u})|$ attains the prescribed moments $(\tilde{\gamma}_0, \tilde{\gamma}_1)$.

Lastly, suppose that
\[
E_3(q) = \frac{e_1^2}{e_1}
\]
for some $q \in C \cap S(\mathbb{R})$. Then we would have $\gamma_0 = \frac{\pi}{7} e_1, \gamma_1 = \frac{7 \pi}{16} e_2$ and hence the $\beta$ moments $\sum \beta_m^3$ and $\sum \beta_m^5$ must vanish. As $e_1 > 0$ then this implies $\log |a(k; u)| \neq 0$, and so we must have strict inequality in (7.5):
\[
E_3(q) = \frac{64}{\pi} \gamma_2 > \frac{64 \gamma_1^2}{\pi \gamma_0} = \frac{e_1^2}{e_1}.
\]

This contradicts the assumption (7.7), and so such a minimizer $q$ cannot exist. \hfill $\Box$

Now that we have found the infimum of $E_3$, we are prepared to analyze Schwartz minimizing sequences:

**Proof of Theorem 1.6** Fix a minimizing sequence $\{q_j\}_{j \geq 1} \subset S(\mathbb{R})$, so that
\[
E_1(q_j) \to e_1, \quad E_2(q_j) \to e_2, \quad E_3(q_j) \to \frac{e_1^2}{e_1} \quad \text{as} \quad j \to \infty. \tag{7.9}
\]
In the inequality of Lemma 7.2, we see that we have equality in the limit $j \to \infty$. Therefore the moments $\sum_{m \geq 1} \beta_{j,m}^7$ vanish as $j \to \infty$; otherwise, we could construct a strictly better...
minimizing sequence with no $\beta$ parameters, because the constraints can be met solely in terms of the log $|a|$ moments. This implies

$$\beta_{j,m} \leq \left( \sum_{\ell \geq 1} \beta_{j,\ell}^2 \right)^{\frac{1}{2}} \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty$$

for all $m$.

We pass to an arbitrary subsequence of $\{q_j\}_{j \geq 1}$. We claim that there is a further subsequence with $\log |a| \, dk$ converging to the even extension of a unique point mass, from which it will follow that the whole sequence $\log |a(k; q_j)| \, dk$ converges to the same limit. Consider the measures

$$d\mu_j := x^\frac{1}{2} \log |a(x^{\frac{1}{2}}; q_j)| \, dx$$

on $[0, \infty)$. Changing variables $x = k^2$, the convergence (7.9) together with the trace formulas (2.7) and the reality condition (2.6) tell us that moments of $d\mu_j$ obey

$$\gamma^j_0 := \int_0^\infty 1 \, d\mu_j(x) \rightarrow \frac{\pi}{2} e_1 =: \gamma_0,$$

$$\gamma^j_1 := \int_0^\infty x \, d\mu_j(x) \rightarrow \frac{\pi}{16} e_2 =: \gamma_1,$$

$$\gamma^j_2 := \int_0^\infty x^2 \, d\mu_j(x) \rightarrow \frac{\gamma^2_2}{\gamma_0}.$$

We claim that the renormalized measures $d\mu_j/\gamma_0^j$ on $[0, \infty)$ are tight. For $R > 0$ we estimate

$$\frac{1}{\gamma_0^j} \mu_j((R, \infty)) = \frac{1}{\gamma_0^j} \int_R^\infty d\mu_j(x) \leq \frac{1}{R \gamma_0^j} \int_R^\infty x \, d\mu_j(x).$$

Note that $1/\gamma_0^j$ is bounded uniformly for $j$ large since $\gamma_0^j \rightarrow \gamma_0 > 0$. Also, the integral on the RHS is bounded uniformly in $j$ since $\gamma_1^j \rightarrow \gamma_1$. Together, we conclude that the RHS tends to zero as $R \rightarrow \infty$ uniformly in $j$.

Therefore, by Prokhorov’s theorem we may pass to a subsequence along which the probability measures $d\mu_j/\gamma_0^j$ converge weakly to some probability measure $d\mu/\gamma_0$. As $\gamma_0^j \rightarrow \gamma_0 > 0$, then the measures $d\mu_j$ converge weakly to $d\mu$.

The sequence of second moments $\gamma_2^j$ converges, and hence is bounded. It then follows that the zeroth and first moments converge to those of $\mu$:

$$\int_0^\infty d\mu_j(x) = \lim_{j \rightarrow \infty} \int_0^\infty d\mu_j(x) = \gamma_0,$$

$$\int_0^\infty x \, d\mu_j(x) = \lim_{j \rightarrow \infty} \int_0^\infty x \, d\mu_j(x) = \gamma_1.$$

For the second moments, we use Fatou’s lemma (which holds for weakly converging measures) to obtain

$$\frac{\gamma_2^j}{\gamma_0^j} \leq \int_0^\infty x^2 \, d\mu(x) \leq \liminf_{j \rightarrow \infty} \int_0^\infty x^2 \, d\mu_j(x) = \frac{\gamma_2}{\gamma_0}.$$

Altogether, we conclude that $\mu$ minimizes the second moment lower bound (7.5) from the Cauchy–Schwarz inequality. Therefore the distributions $d\mu$ and $x^2 \, d\mu(x)$ on $[0, \infty)$ are proportional, and hence $\mu$ is a Dirac delta mass. The support and total mass of this distribution are uniquely determined by $\gamma_0$ and $\gamma_1$. In turn, the limiting distribution

$$\frac{1}{\nu} (d\mu(k^2) + d\mu(-k^2))$$
of $\log |a| \, dk$ on $\mathbb{R}$ is then uniquely determined by the reality condition (2.6). Lastly, we note that weak convergence of measures implies convergence when integrated against bounded continuous test functions by the Portmanteau theorem, and hence implies convergence in distribution.

\begin{flushright}
\hfill $\square$
\end{flushright}

Acknowledgements I was supported in part by NSF grants DMS-1856755 and DMS-1763074. I would also like to thank my advisors, Rowan Killip and Monica Vișan, for their guidance, and Jaume de Dios Pont for pointing out the proof method from [34] for Lemma 3.2. Finally, I would like to thank the referee for their thoughtful comments and corrections.

References

1. Albert, J.P.: Concentration compactness and the stability of solitary-wave solutions to nonlocal equations, Applied analysis (Baton Rouge, LA, 1996), pp. 1-29 (1999). MR1647189
2. Albert, J.P.: A uniqueness result for 2-soliton solutions of the Korteweg-de Vries equation. Discrete Contin. Dyn. Syst. 39(7), 3635–3670 (2019). (MR3960482)
3. Albert, J.P., Bona, J.L., Nguyen, N.V.: On the stability of KdV multi-solitons. Differ. Integr. Equ. 20(8), 841–878 (2007). (MR2339841)
4. Albert, J.P., Nguyen, N.V.: A variational characterization of 2-soliton profiles for the KdV equation (2021). Preprint arXiv:2101.10574
5. Alejo, M.A., Muñoz, C., Vega, L.: The Gardner equation and the $L^2$-stability of the Nsoliton solution of the Korteweg-de Vries equation. Trans. Am. Math. Soc. 365(1), 195–212 (2013). (MR2984057)
6. Benjamin, T.B.: The stability of solitary waves. Proc. R. Soc. London Ser. A 328, 153–183 (1972). (MR338584)
7. Bona, J.: On the stability theory of solitary waves. Proc. R. Soc. London Ser. A 344(1638), 363–374 (1975). (MR386438)
8. Boussinesq, J.: Théorie des ondes et des remous qui se propagent le long d’un canal rectangulaire horizontal, en communicant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. J. Math. Pures Appl. 17, 55–108 (1872). (MR3363411)
9. Boussinesq, J.: Sur la théorie des eaux courantes. Mémoires à l’Acad. des Sci. Inst. Nat. France 23–24, 1–680 (1877)
10. Cazenave, T., Lions, P.-L.: Orbital stability of standing waves for some nonlinear Schrödinger equations. Comm. Math. Phys. 85(4), 549–561 (1982). (MR779977)
11. Coddington, E.A., Levinson, N.: Theory of Ordinary Differential Equations. McGraw-Hill Book Company, Inc., New York-Toronto-London (1955). (MR0069338)
12. Deift, P., Trubowitz, E.: Inverse scattering on the line. Comm. Pure Appl. Math. 32(2), 121–251 (1979). (MR512420)
13. Drury, S.W., Marshall, B.P.: Fourier restriction theorems for degenerate curves. Math. Proc. Cambridge Philos. Soc. 101(3), 541–553 (1987). (MR878901)
14. Gardner, C.S., Greene, J.M., Kruskal, M.D., Miura, R.M.: Method for solving the Korteweg-deVries equation. Phys. Rev. Lett. 19(19), 1095–1097 (1967)
15. Gardner, C.S., Greene, J.M., Kruskal, M.D., Miura, R.M.: Korteweg-deVries equation and generalization. VI. Methods for exact solution. Comm. Pure Appl. Math. 27, 97–133 (1974). (MR363122)
16. Garnett, J.B.: Bounded Analytic Functions, Graduate Texts in Mathematics, vol. 236, 1st edn. Springer, New York (2007). (MR2261424)
17. Gesztesy, F., Holden, H.: Soliton equations and their algebro-geometric solutions. Vol. I, Cambridge Studies in Advanced Mathematics, vol. 79, Cambridge University Press, Cambridge (2003). (1 + 1)-dimensional continuous models. MR1992536
18. Gesztesy, F., Holden, H., Michor, J., Teschl, G.: Soliton equations and their algebrogeometric solutions. Vol. II, Cambridge Studies in Advanced Mathematics, vol. 114. Cambridge University Press, Cambridge (2008). (1 + 1)-dimensional discrete models. MR2446594
19. Hirota, R.: Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons. Phys. Rev. Lett. 27(18), 1192–1194 (1971)
20. Himidi, T., Keraani, S.: Blowup theory for the critical nonlinear Schrödinger equations revisited. Int. Math. Res. Not. 46, 2815–2828 (2005). (MR2180464)
21. Kay, I., Moses, H. E.: Reflectionless transmission through dielectrics and scattering potentials, Div. Electromag. Res., Inst. Math. Sci., New York Univ. (1956). Res. Rep. No. EM-91. MR0090367
22. Kenig, C.E., Ponce, G., Vega, L.: Well-posedness of the initial value problem for the Korteweg-de Vries equation. J. Am. Math. Soc. 4(2), 323–347 (1991). (MR1086966)
23. Killip, R., Vişan, M.: Nonlinear Schrödinger equations at critical regularity. Evolution equations, pp. 325–437 (2013). MR3098643
24. Killip, R., Vişan, M.: Orbital stability of KdV multisolitons in H-1. Comm. Math. Phys. 389(3), 1445–1473 (2022). (MR4381177)
25. Korteweg, D.J., de Vries, G.: On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. Philos. Mag. 39(240), 422–443 (1895). (MR3363408)
26. Lax, P.D.: Integrals of nonlinear equations of evolution and solitary waves. Comm. Pure Appl. Math. 21, 467–490 (1968). (MR235310)
27. Lax, P.D.: Periodic solutions of the KdV equation. Comm. Pure Appl. Math. 28, 141–188 (1975). (MR369963)
28. Maddocks, J.H., Sachs, R.L.: On the stability of KdV multi-solitons. Comm. Pure Appl. Math. 46(6), 867–901 (1993). (MR1220540)
29. Martel, Y., Merle, F.: Asymptotic stability of solitons of the subcritical gKdV equations revisited. Nonlinearity 18(1), 55–80 (2005). (MR2109467)
30. Martel, Y., Merle, F., Tsai, T.-P.: Stability and asymptotic stability in the energy space of the sum of N solitons for subcritical gKdV equations. Comm. Math. Phys. 231(2), 347–373 (2002). (MR1946336)
31. Merle, F., Vega, L.: L2 stability of solitons for KdV equation. Int. Math. Res. Not. 13, 735–753 (2003). (MR1949297)
32. Miura, R.M., Gardner, C.S., Kruskal, M.D.: Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion. J. Math. Phys. 9, 1204–1209 (1968). (MR252826)
33. Simon, B.: The classical moment problem as a self-adjoint finite difference operator. Adv. Math. 137(1), 82–203 (1998). (MR1627806)
34. Steinig, J.: On some rules of Laguerre’s, and systems of equal sums of like powers. Rend. Mat. 4(1971), 629–644 (1972). (MR309867)
35. Takayama, A.: Mathematical Economics, 2nd edn. Cambridge University Press, Cambridge (1985). (MR832684)
36. Tanaka, S.: On the N-tuple wave solutions of the Korteweg-de Vries equation. Publ. Res. Inst. Math. Sci. 8, 419–427 (1972). (MR0328386)
37. Wadati, M., Toda, M.: The exact N-soliton solution of the Korteweg-de Vries equation. J. Phys. Soc. Japan 32, 1403–1411 (1972)
38. Weinstein, M.I.: Lyapunov stability of ground states of nonlinear dispersive evolution equations. Comm. Pure Appl. Math. 39(1), 51–67 (1986). (MR820338)
39. Zabuzsky, N.J., Kruskal, M.D.: Interaction of “solitons” in a collisionless plasma and the recurrence of initial states. Phys. Rev. Lett. 15, 240–243 (1965)
40. Zhabarov, V.E., Faddeev, L.D.: The Korteweg-de Vries equation is a fully integrable Hamiltonian system. Funkcional. Anal. i Priložen. 5(4), 18–27 (1971). (MR0303132)
41. Zakharov, V.E.: Kinetic equation for soliton. Soviet Phys. JETP 33(3), 538–541 (1971)

Publisher’s Note Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.