On the Initial Data Constraints on the Light cone for the Einstein-Vlasov system

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Abstract
This article is concerned with the derivation of the Gauss-Codazzi's constraints equations on the initial light cone for geometric transport equations in general relativity. Temporal-gauge-dependent constraints are addressed too and gauge-preservation is established. The global resolution of the constraints is studied, a large class of initial data sets is deduced from appropriate free data and their behavior at the vertex of the cone is examined.

Keywords: Characteristic Cauchy problem, Geometric-Transport equations, General relativity, Kinetic theory, initial data constraints

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1 Introduction and various issues

It is well known that by considering the characteristic Cauchy problem in general relativity, the treatment of the initial data constraints problem is considerably simplified as the constraints reduce to propagations equations along null geodesics generating the initial manifold carrying the data provided free data are well-chosen [2,6,9,10,12,17,18,19,20,21,22]. This is a particular interesting feature of the characteristic Cauchy problem in general relativity opposed to the ordinary spacelike Cauchy problem where the constraints equations are of elliptic type. However, a full description of the constraints in the characteristic Cauchy problem setting requires a priori investigation of the geometry of the null initial hypersurfaces in consideration and derivation on it of explicit expressions of various null geometry quantities [6,9,10,12,17,18,19,20,21,22]. Furthermore, while in the case of the ordinary spacelike Cauchy problem in general relativity the constraints are standard, depend only of the geometric nature of the Einstein’s equations (i.e. the so called Hamiltonian and momentum constraints) [3,5], in the case of the characteristic initial value problem, one faces the difficult task of highlighting additional gauge-dependent constraints essential to the construction of the full set of initial data [6,9,10,12,17,18,19,20,21,22]. These latter are induced by: the evolution system deduced of the splitting of the Einstein equations by the choice of a gauge, the form of the stress-energy momentum tensor of the matter involved, and their hierarchy depends heavily on the prescribed free data. On the other hand recent developments in the direction of the study of smoothness of Scri [10] besides the challenge for the global existence theory in general relativity [1,2,13,14,8] indicate the importance to enlarge the approach of the treatments of the characteristic initial data constraints problem in general relativity. As instance, in [10,17].
it is revealed that harmonic gauge is not appropriate to tackle some difficulties related to the occurrence of log-terms in the constraints at infinity. In this paper, the new approach consists in investigating the temporal gauge in the characteristic Cauchy problem setting. This requires that the shift is null and the time is in wave gauge. The characteristic initial value problem for the Einstein-Vlasov system on the light cone splits in the characteristic Cauchy problem on the light cone for the evolution system (2.6)-(2.7) and the initial data constraints problem on the light cone for the Einstein-Vlasov system. This gauge helps in particular in the treatment of the initial data constraints’s problem on a light cone when kinetic matter is involved. Indeed, the presence of all the components of the metric in each component of the momentum tensor of matter due to the Vlasov’s field makes difficult the use of the Rendall’s scheme of resolution of the initial data constraints’s problem. Such difficulties are revealed in [4],[17],[22]. The interest for the characteristic Cauchy problem in general relativity is well known [21], and there is a growing interest for this since the work of D. Christodoulou on the formation of Black holes in general relativity [7]. The gauge mostly used in this context is the harmonic or wave gauge (its generalization is the ”generalized wave map gauge”) [4],[6],[9],[10],[15],[17],[21],[22], which fits well to some types of matter. Another gauge ie, the ”Double null foliation gauge” is now also used and principally in vacuum [2],[16],[13]. For all these gauges, the question of existence of global solutions for the constraints’s equations is of great interest [9],[10],[16]. The rest of the paper is structured as follow: first we recall the evolution system (2.6)-(2.7) induced by the choice of the temporal gauge, thereby we identify the Cauchy data to be attached to this system and the type of constraints on concerned, then we analyze the constraints in coordinates adapted to the null geometry of the cone, this yields a full description of the constraints followed by their resolution from appropriate free data. The behavior of the solutions of the constraints at the tip of the cone is analyzed. The preservation of the gauge is established. The question left open is the study of the class of free data on a cone which leads to a smooth solution of the Einstein-Vlasov system on a neighborhood of the vertex of the cone.

2 The setting and the evolution system

The Einstein equations of general relativity are geometric in nature and do not take a specific partial differential equations type, unless a well-chosen of gauge is introduced. They describe the gravitational potential $g$. The Vlasov equation in turn appears in kinetic theory, it governs the density $\rho$ of moving particles. We are thus interested here in the characteristic Cauchy problem in a domain $Y_O$ above the light cone of vertex O for the combination of these equations, this models a spacetime $(Y_O,g)$ with collisionless matter, with $Y_O$ a Lorentzian manifold. In a global set of coordinates $x^\alpha = (x^0,x^1,x^a)$, $\alpha = 0,1,...,n;\ a = 2,...,n$ of $\mathbb{R}^{n+1} = \mathbb{R}^2 \times \mathbb{R}^{n-1},(n \geq 3)$, these equations read:

$$H_g: \ G_{\mu\nu} \equiv R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = T_{\mu\nu}; \quad (2.1)$$

$$H_\rho: \ p^a \frac{\partial \rho}{\partial x^a} - \Gamma^i_{\mu\nu} p^\mu p^\nu \frac{\partial \rho}{\partial p^i} = 0. \quad (2.2)$$
One considers that the particles are of rest mass $m$, and move towards the future $(p^0 > 0)$ in their mass shell

$$P := \{(x^\delta, p^\mu) \in Y_O \times \mathbb{R}^{n+1} / g_{\mu\nu}p^\mu p^\nu = -m^2, \ p^0 > 0\}. \quad (2.3)$$

The terms $R_{\mu\nu}$, $R$ and $G_{\mu\nu}$ design respectively the components of the Ricci tensor, the scalar curvature, the components of the Einstein tensor $G$, relative to the searched metric $g$, while the $T_{\mu\nu}$ are the components of the stress-energy momentum tensor of matter. The $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols of $g$, the $p^\lambda$ stand as the components of the momentum of the particles w.r.t. the basis $(\frac{\partial}{\partial x^\alpha})$ of the fiber $P_x := \{(p^\alpha) \in \mathbb{R}^{n+1} / g_{\mu\nu}(x)p^\mu p^\nu = -m^2, \ p^0 > 0\}$ of $P$, and

$$T_{\alpha\beta} = -\int_{\{g(p,p)=-m^2\}} \frac{\rho(x^\nu, p^\mu)p_\alpha p_\beta \sqrt{|g|}}{p^0} dp^1...dp^n. \quad (2.4)$$

As already mentioned, we investigate the temporal gauge \cite{3,5} requiring that

$$g_{0i} = 0, \ \Gamma^0 \equiv g^{\lambda\delta}\Gamma^0_{\lambda\delta} = 0, \ i = 1, ..., n; \ \lambda, \delta = 0, ..., n. \quad (2.5)$$

Following Choquet Bruhat, the evolution system $(H_{\overline{7}}, H_\rho)$ attached to the Einstein-Vlasov system $(H_g, H_\rho)$ and induced by this gauge \cite{3,5} is

$$H_{\overline{7}}: \ \partial_0 R_{ij} - \nabla_i \Lambda_{j0} - \nabla_j \Lambda_{i0} = \partial_0 \Lambda_{ij} - \nabla_i \Lambda_{j0} - \nabla_j \Lambda_{i0}; \quad (2.6)$$

$$H_\rho: \ p^\alpha \frac{\partial \rho}{\partial x^\alpha} - \Gamma^i_{\mu\nu}p^\mu p^\nu \frac{\partial \rho}{\partial p^i} = 0; \quad (2.7)$$

with $\Lambda_{\mu\nu} = T_{\mu\nu} + \frac{\mu\lambda}{\nu}T_{\lambda\nu} g_{\mu\nu}$, and where the system $H_{\overline{7}}$ replaces the Einstein equations and its principal part is $\Box \partial_0 \overline{7}_{ij}$, $\nabla$ denotes the connection w.r.t. the induced metric $\overline{7}$ on $\Lambda_t : x^0 = t$. In this paper, attention is focused on the construction and resolution of the constraints satisfied by a large class of initial data $(\overline{7}_0, k_0, \rho_0)$ on $C \times \mathbb{R}^n$ with $C$ of equation

$$C : \ x^0 - r = 0, \ r := \sqrt{\sum_{i=1}^n (x^i)^2}, \quad (2.8)$$

s.t. for any solution $(\overline{7}, \rho)$ of the evolution system $(H_{\overline{7}}, H_\rho)$ on $P$ satisfying $\overline{7}_{\mid C} = \overline{7}_0, \ (\partial_0 \overline{7})_{\mid C} = k_0, \ \rho_{\mid C} = \rho_0$, $(g, \rho)$ is solution of the Einstein-Vlasov system $(H_g, H_\rho)$ in $P$, where $g$ is of the form

$$g = -\tau^2 (dx^0)^2 + \overline{g}_{ij} dx^i dx^j, \quad (2.9)$$

with $\tau^2 = (c(x^i))^2|\overline{7}|_i$ and $c$ is a positive scalar density on $\Lambda_t$ which is determined by the prescribed data such that $\Gamma^0 = 0$ in $Y_O$.

### 3 The characteristic initial data constraints

In what follows, one sets $q_s = -\frac{\partial^2}{\partial x^0 \partial x^0}, \ q^s = g^{\mu\nu}q_l, \ X^{\mu\nu} \equiv G^{\mu\nu} - T^{\mu\nu}$. The start point of the construction of the constraints is the following proposition.
Proposition 1  For any $C^\infty$ solution $(\mathcal{T}, \rho)$ in $\mathbb{P}$ of the evolution system (2.6)–(2.7) such that with respect to the metric $g$ of the form (2.9) $X^\mu_{/C} = 0$, $(\partial_0 X_{0i})(O) = 0; s = 1, ..., n, s \neq s_0, q^{\mu_0}(O) \neq 0$, $(g, \rho)$ is a solution in $\mathbb{P}$ of the Einstein-Vlasov system.

Proof 1  If $(\mathcal{T}, \rho)$ is a $C^\infty$ solution in $\mathbb{P}$ of the evolution system (2.6)–(2.7), then with respect to the metric $g$ of the form (2.9) tied to $\mathcal{T}$ and in virtue of Bianchi identities, the tensor $(X^\mu)$ satisfies the homogeneous linear Leray-hyperbolic system ("see" [2], pages [407-414]):

\[
\partial_0 X^{00} + L^{00}(X^{00}, \partial_0 X^{00}) = 0 \quad (3.1)
\]
\[
\partial_0 X^{ij} + L^{ij}(X^{00}, \partial_0 X^{k0}) = 0 \quad (3.2)
\]
\[
\Box_g X^{00} + L^{ij}(X^{00}, \partial_0 X^{k0}) = 0. \quad (3.3)
\]

Another homogeneous third order Leray hyperbolic system derived from this one is the system

\[
\Box_g \partial_0 X^{ij} + T^{ij}(X^{00}, D^\epsilon X^{00}) = 0, \quad |\epsilon| \leq 2. \quad (3.4)
\]

Now if one has $X^\mu_{/C} = 0$, then on $\mathcal{C}$, $[\partial_0 X^{00}]$, $[\partial_0 X^{ij}]$ express in terms of $[\partial_0 X^{00}]$, $i = 1, ..., n$ as a consequence of restriction to $\mathcal{C}$ of the equations (3.3). Substituting these expressions in the system (3.3) restricted to $\mathcal{C}$, this latter appears then in turn as a homogeneous linear differential system of propagation equations on $\mathcal{C}$ of unknowns $[\partial_0 X^{00}]$, $i = 1, ..., n$. Now, thanks once more to the Bianchi relations $\nabla_\alpha X^{\alpha_i} = 0$, $i = 1, ..., n$ and the evolution system $H_{\mathcal{T}}$ written at $O$, it suffices that $\partial_0 X_{0k}(O) = 0$, $k = 1, ..., n, k \neq k_0$, $q^{k0} \neq 0$ so that $(\partial_0 X^{\mu})(O) = 0$. Finally $X^{\mu}_{/C} = 0$ in $Y_0$ thanks to (3.4).

3.1 Null geometry on the cone

The above proposition[[1]] is an indication that the Einstein equations must hold on the initial hypersurface $\mathcal{C}$ in order that the preservation of the gauge is established. This induces Gauss and Codazzi’s constraints on $\mathcal{C}$, but these standard constraints must be supplemented by other gauge-dependent constraints. To carry out the analysis, we introduce null adapted coordinates w.r.t. the trace of the metric on the cone $\mathcal{C}$ ($y^0, y^1, y^4$) (11, 10, 9, 13) defined by

\[
y^0 = x^0 - r, \quad y^1 = r = \sqrt{\sum_{i=1}^{n} (x^i)^2}, \quad y^A, \quad A = 2, ..., n; \quad (3.5)
\]

where $(y^A)$ design local coordinates in the sphere $S^{n-1}$: $\sum_{i=1}^{n} (x^i)^2 = 1$, then

\[
x^0 = y^0 + y^1, \quad x^i = y^1 \theta^i(y^A), \quad \sum_{i}(\theta^i)^2 = 1; \quad (3.6)
\]

and the $\theta^i(y^A)$ are $C^\infty$ functions of $y^A, A = 2, ..., n$; and require the assumption:

\[(A)\quad \text{The vector fields } \frac{\partial}{\partial y^A} \text{ is tangent to the null geodesics generating } \mathcal{C}, \quad (3.7)\]

which is an aspect of the affine parametrization condition of [11, 13, 9, 13], (since no a priori condition is given on the non-affinity constant on $\mathcal{C}$), and is equivalent to
the requirement that on the cone the lapse $\tau^{-2}$ is an eigenvalue of the Riemannian metric $\overline{g} = (g^{ij})$, the corresponding eigenvector being $-q = (-q_i)$, $q_i = -\frac{\partial}{\partial x}$. The components of tensors in coordinates $(y^\mu)$ are equipped with a tilde "$\sim$". The assumption $(A)$ induces that the trace on the null cone $C : y^0 = 0$, of the searched metric in temporal gauge is of the form

$$g_{iC} = \overline{g}_{01}dy^0dy^0 + \overline{g}_{01}(dy^0dy^1 + dy^1dy^0) + \overline{g}_{AB}dy^Ady^B, \ A, B = 2, \ldots, n.$$  

(3.8)

The non-zero components of the trace on $C$ of the inverse $g^{-1}$ of $g$ satisfy:

$$\overline{g}^{01}\overline{g}_{01} = 1, \overline{g}^{11} + \overline{g}^{01} = 0, (\overline{g}^{AB}) = (\overline{g}_{AB})^{-1}, \ A, B = 2, \ldots, n.$$  

(3.9)

We remark that $\overline{g}_{01} < 0$ according to the signature of $g$. Using the expression (3.8) of the trace on $C$ of the metric and its inverse (3.9), one derives, some algebraic relations between the gravitational data on $C$, the expressions of the restrictions on $C$ of the Christoffel symbols of the metric. The details are in appendix $\mathbf{B}$. One can then derive the restrictions on $C$ of the components of $X = (X_{\mu\nu})$.

### 3.2 Gauss and Codazzi’s constraints on $C$

The vectorfield $l = (0, -\overline{g}_{01}, 0, \ldots, 0)$ is outgoing normal to $C$, the projection operator on $C$ is $\pi$, s.t. $\pi_{\mu} = \delta_{\mu} + l_{\mu}l^{\nu}$, then the Gauss and Codazzi’s constraints on the light cone $C$ read

$$\bar{X}_{\mu\nu}l^\nu = 0, \bar{X}_{\mu\nu}l^\mu\pi_\lambda = 0,$$  

(3.10)

and resume to

$$\bar{X}_{1\lambda} = 0, \lambda = 0, 1, \ldots, n.$$  

(3.11)

These constraints involve naturally only the Cauchy data for the evolution system $(H_\pi, H_\rho)$. The other Einstein equations

$$\bar{X}_{00} = 0, \bar{X}_{0A} = 0, \bar{X}_{AB} = 0, \ A, B = 2, \ldots, n;$$  

(3.12)

do not play the role of constraints as their expressions on $C$ contain second order outgoing derivatives of the metric which are not part of the initial data of the third order characteristic problem for the evolution system $(H_\pi, H_\rho)$. Indeed, one has from straightforward computations the following expressions which reveal the harmful terms that one faces:

$$\bar{X}_{00} = \frac{1}{2}(\overline{g}^{01}\overline{g}^{AB}\overline{\partial}_0\overline{g}_{AB} + H_1(\overline{g}, \overline{\partial}g, \rho),$$  

(3.13)

$$\bar{X}_{AB} = \frac{1}{2}(\overline{g}^{01})^2\overline{\partial}_0\overline{g}_{11}\overline{g}_{AB} + H_2(\overline{g}, \overline{\partial}g, \rho),$$  

(3.14)

$$\bar{X}_{0A} = -\frac{1}{2}\overline{g}^{01}\overline{\partial}_0\overline{g}_{A1} + H_3(\overline{g}, \overline{\partial}g, \rho).$$  

(3.15)

### 3.3 Temporal gauge-dependent constraints

In this subsection we are led to finding modifications or combinations of the Einstein equations (3.12) in order to construct the gauge-dependent constraints, and this is lengthy and somewhat subtle. The Einstein equations $\bar{X}_{AB} = 0$ read:

$$\bar{X}_{AB} \equiv \bar{R}_{AB} - \frac{1}{2}\overline{g}_{AB}(2\overline{g}^{01}\overline{R}_{01} + \overline{g}^{11}\overline{R}_{11} + \overline{g}^{CD}\overline{R_{CD}}) - \overline{T}_{AB} = 0.$$  

(3.16)
They comprise the term \( \bar{R}_{01} \) which according to the expressions of the components of the Ricci tensor on \( \mathcal{C} \) ("see" appendix C), has second order outgoing derivative of the metric, ie:

\[
\bar{R}_{01} = -\frac{1}{2}(\bar{g}^{01})^2 \partial_{00}\bar{g}_{11} + H_4(\bar{g}, \partial\bar{g}).
\]

The treatment of this term induces the equations

\[
\bar{X}_{AB} - \bar{g}^{CD} \bar{X}_{CD} = 0, \quad A, B, C, D = 2, ..., n,
\]

which are equivalent to

\[
\bar{R}_{AB} - \bar{T}_{AB} - \bar{g}^{CD}(\bar{R}_{CD} - \bar{T}_{CD}) \frac{n}{n-1} \bar{g}_{AB} = 0, \quad A, B, C, D = 2, ..., n.
\]

These latter equations do not contain second order outgoing derivatives of the metric. The Einstein equation \( \bar{X}_{00} = 0 \) in turn reads:

\[
\bar{X}_{00} \equiv \bar{R}_{00} - \frac{1}{2}\bar{g}_{00}(2\bar{g}^{01}\bar{R}_{01} + \bar{g}^{11}\bar{R}_{11} + \bar{g}^{CD}\bar{R}_{CD}) - \bar{T}_{00} = 0. \tag{3.17}
\]

Second order outgoing derivatives in this equation are due to the terms \( \bar{R}_{00} \) and \( \bar{R}_{01} \) since ("see" appendix C)

\[
\bar{R}_{00} = \frac{1}{2}(\bar{g}^{01})^2 \partial_{00}\bar{g}_{11} + \frac{1}{2}(\bar{g}^{AB})^2 \partial_{00}\bar{g}_{AB} + H_5(\bar{g}, \partial\bar{g}).
\]

Dealing with these harmful terms led us to using of the expression

\[
\frac{\partial}{\partial y^0}(\bar{\Gamma}^0 + \bar{\Gamma}^1) = \frac{1}{2}(\bar{g}^{01})^2 \partial_{00}\bar{g}_{11} + \frac{1}{2}(\bar{g}^{AB})^2 \partial_{00}\bar{g}_{AB} + H_5(\bar{g}, \partial\bar{g}), \tag{3.18}
\]

which reflects the "time in wave gauge" property. This analysis results to considering the equation

\[
\bar{X}_{00} - \bar{g}_{01} \frac{\bar{g}^{CD}\bar{X}_{CD}}{n-1} + \bar{g}_{01} \frac{\partial(\bar{\Gamma}^0 + \bar{\Gamma}^1)}{\partial y^0} = 0, \quad A, B, C, D = 2, ..., n,
\]

which does not contain second order outgoing derivatives of the metric and is equivalent to

\[
\bar{R}_{00} - \bar{T}_{00} - \bar{g}_{01} \frac{\bar{g}^{AB}(\bar{R}_{AB} - \bar{T}_{AB})}{n-1} + \bar{g}_{01} \frac{\partial(\bar{\Gamma}^0 + \bar{\Gamma}^1)}{\partial y^0} = 0.
\]

We remark that the Einstein equations \( \bar{X}_{0A} = 0 \) are not involved in the construction of the gauge-dependent constraints and their realization on the cone \( \mathcal{C} \) will be obtained in a more indirect way. The construction of constraints in this section thus ends up by the following theorem.

**Theorem 1** Let \( (\bar{\gamma}, \rho) \) be any \( C^\infty \) solution of the evolution system \( (H_{\bar{\gamma}}, H_{\rho}) \) in a neighborhood \( \mathcal{V} \) of \( \mathcal{C} \times \mathbb{R}^n \), and let \( g \) associated to \( \bar{\gamma} \) of the form (2.7), s.t. the temporal gauge condition is satisfied in \( Y_O = \{y^0 \geq 0\} \). One sets \( \bar{X}_{\mu\nu} \equiv \bar{G}_{\mu\nu} - \bar{T}_{\mu\nu} \), and one assumes that w.r.t. the metric \( g \), the hypothesis \( (A) \) (3.7) and the relations

\[
\bar{X}_{1\lambda} = 0, \lambda = 0, ..., n, \tag{3.19}
\]
\[
\tilde{X}_{AB} - \frac{\tilde{g}^{CD} \tilde{X}_{CD}}{n-1} \tilde{g}_{AB} = 0, \quad (3.20)
\]

\[
\tilde{X}_{00} - \frac{\tilde{g}_{01} \tilde{g}^{CD} \tilde{X}_{CD}}{n-1} + \tilde{g}_{01} \frac{\partial(\tilde{G}^0 + \tilde{G}^1)}{\partial y^0} = 0, \quad A, B, C, D = 2, ..., n; \quad (3.21)
\]

are satisfied on \(\mathcal{C};\) if furthermore one has \(X_{0k}(O) = 0, \quad k = 1, ..., n;\) \((\partial_0 X_{0s})(O) = 0; \quad s = 1, ..., n, s \neq s_0, g^{00}(O) \neq 0;\) then \((g, \rho)\) is solution of the Einstein-Vlasov system \((H_g, H_\rho)\) in \(Y_O.\)

To prove the theorem, we establish the following lemma. One sets \([X_{\mu\nu}] = X_{\mu\nu}|_\mathcal{C}.

**Lemma 1** For any \(C^\infty\) solution \((\tilde{g}, \rho)\) of the evolution system \((H_{\tilde{g}}, H_\rho)\) in a neighborhood \(V \times \mathbb{R}^{n+1}\) of a smooth hypersurface \(\mathcal{I} \times \mathbb{R}^{n+1}\) with \(I\) of equation \(\mathcal{I}: x^0 - \phi(x^1) = 0, \quad \text{and} \quad g\) associated to \(\tilde{g}\) of the form (2.9) s.t. the temporal gauge condition is satisfied in \(V.\) One sets \(q_i = -\frac{\partial \phi}{\partial x^i}.\) Then the tensor \(X = (X_{\mu\nu})\) restricted to \(\mathcal{I}\) satisfies the homogeneous linear system of partial differential equations

\[
q^i \partial_j [X_{k0}] + q^i \partial_k [X_{j0}] - q_k g^{im} \partial_l [X_{m0}] + g^{ij} \partial_k [X_{jk}] + A^{\mu\nu}_k [X_{\mu\nu}] = 0. \quad (3.22)
\]

**Proof 2** \((\tilde{g}, \rho)\) is a \(C^\infty\) solution of the evolution system \((H_{\tilde{g}}, H_\rho),\) and according to the divergence free properties of the Einstein tensor \((G_{\mu\nu})\) and the stress energy momentum tensor of matter \((T_{\mu\nu})\) of \(g\) (2.9), one has:

\[
\nabla^\alpha X_{\alpha\beta}|_{\mathcal{I}} = 0, \quad (\partial_0 (R_{ij} - \Lambda_{ij}) - \nabla_i X_{j0} - \nabla_j X_{i0})|_{\mathcal{I}} = 0. \quad (3.23)
\]

Since \(g_{0i} = 0 = g^{0i},\) the Bianchi identities induce the following relations:

\[
g^{0\alpha} \nabla_0 X_{\alpha\beta} + g^{ij} \nabla_i X_{j\beta} = 0, \quad \forall \beta.
\]

For \(\beta = k,\) one has successively:

\[
g^{00} (\partial_0 X_{0k} - \Gamma^0_{00} X_{\alpha k} - \Gamma^0_{0k} X_{0\alpha}) + g^{ij} (\partial_i X_{jk} - \Gamma^0_{ij} X_{\alpha k} - \Gamma^0_{ik} X_{j\alpha}) = 0
\]

\[
g^{00} \partial_0 X_{0k} + g^{ij} \partial_i X_{jk} - g^{00} \Gamma^0_{0k} X_{0\alpha} - g^{00} \Gamma^0_{0k} X_{0\alpha} - g^{ij} \Gamma^0_{ij} X_{\alpha k} - g^{ij} \Gamma^0_{ij} X_{j\alpha} = 0.
\]

\([C_k] = 0. \quad (3.24)\]

Now on \(\mathcal{I},\) one has

\[
(\partial_i X_{jk}) = \partial_i [X_{jk}] + q_i [\partial_0 X_{jk}]; \quad (3.25)
\]

and the system (3.24) restricted to \(\mathcal{I}\) implies:

\[
g^{00} \partial_0 X_{0k} + q^i [\partial_0 X_{jk}] + g^{ij} \partial_i [X_{jk}] + C_k = 0; \quad (3.26)
\]

where the \(C_k\) are given by the relations (3.24). Furthermore, since \(X_{\mu\nu} = G_{\lambda\mu} - T_{\lambda\mu} = R_{\lambda\mu} - \Lambda_{\lambda\mu} - \frac{1}{2} g_{\lambda\mu} (R - \Lambda),\) one has:

\[
\partial_0 X_{jk} = \partial_0 (R_{jk} - \Lambda_{jk}) - \frac{1}{2} (\partial_0 (R - \Lambda)) g_{jk} - \frac{1}{2} (R - \Lambda) \partial_0 g_{jk} \quad (3.27)
\]

On the other hand, since \(R - \Lambda = g^{00} (R_{00} - \Lambda_{00}) + g^{kl} (R_{kl} - \Lambda_{kl}),\) one has

\[
\partial_0 (R - \Lambda) = (\partial_0 g^{00}) (R_{00} - \Lambda_{00}) + g^{00} \partial_0 (R_{00} - \Lambda_{00}) + (\partial_0 g^{kl}) (R_{kl} - \Lambda_{kl}) + g^{kl} \partial_0 (R_{kl} - \Lambda_{kl}). \quad (3.28)
\]
Considering the Bianchi identities for $\beta = 0$, one has successively:

$$g^{00}\nabla_0 X_{00} + g^{ij}\nabla_i X_{j0} = 0,$$

$$g^{00}(\partial_0 X_{00} - 2\Gamma^0_0 X_{00}) + g^{ij}\nabla_i X_{j0} = 0,$$

$$g^{00}\partial_0 [R_{00} - \Lambda_{00}] - \frac{1}{2} g^{00}(R - \Lambda) - 2g^{00}\Gamma^0_0 X_{00} + g^{ij}\nabla_i X_{j0} = 0,$$

$$g^{00}\partial_0 \{R_{00} - \Lambda_{00}\} = \frac{1}{2} g^{00}(\partial_0 g^{00})(R - \Lambda) - \frac{1}{2}\partial_0 (R - \Lambda) - 2g^{00}\Gamma^0_0 X_{00} + g^{ij}\nabla_i X_{j0} = 0. \tag{3.29}$$

Combining the relations (3.28) and (3.29), one has:

$$\frac{1}{2}\partial_0 (R - \Lambda) = g^{kl}\partial_0 (R_{kl} - \Lambda_{kl}) - g^{ij}\nabla_i X_{j0} + A, \tag{3.30}$$

where

$$A = 2^{-1}g^{00}(\partial_0 g^{00})(R - \Lambda) + 2g^{00}\Gamma^0_0 X_{00} + (\partial_0 g^{00})(R_{00} - \Lambda_{00}) + \frac{1}{2}\partial_0 (R_{kl} - \Lambda_{kl}).$$

Using the relations $R_{00} - \Lambda_{00} = X_{00} + \frac{1}{2}g^{00}(R - \Lambda), \ R_{kl} - \Lambda_{kl} = X_{kl} + \frac{1}{2}g_{kl}(R - \Lambda), \ R - \Lambda = \frac{1}{4}X := \frac{1}{2}n g^{\lambda\delta} X_{\lambda\delta}, \ A \ reads:\n
$$A = 1 - \frac{1}{n} g_{kl}(\partial_0 g^{kl})g^{\lambda\delta} X_{\lambda\delta} + 2g^{00}\Gamma^0_0 X_{00} + (\partial_0 g^{00})X_{00} + (\partial_0 g^{kl})X_{kl}. \tag{3.31}$$

The system (3.27) now reads:

$$\partial_0 X_{jk} = \partial_0 (R_{jk} - \Lambda_{jk}) - g_{jk}\{g^{lm}\partial_0 (R_{lm} - \Lambda_{lm}) - g^{lm}\nabla_l X_{m0}\}$$

$$- Ag_{jk} - \frac{(\partial_0 g_{jk})g^{\lambda\delta} X_{\lambda\delta}}{1 - n}. \tag{3.32}$$

The system (3.26) can then be written:

$$g^{00}\partial_0 X_{0k} + q^j (\partial_0 (R_{jk} - \Lambda_{jk}) - g_{jk}g^{lm}\partial_0 (R_{lm} - \Lambda_{lm}) + g_{jk}g^{lm}\nabla_l X_{m0} - Ag_{jk}) +$$

$$g^{ij}\partial_i [X_{jk}] + C_k - \frac{g^{\lambda\delta} X_{\lambda\delta}}{1 - n}(\partial_0 g_{jk})q^j = 0. \tag{3.33}$$

From the hypotheses $(\gamma_{ij}, \rho)$ satisfies

$$(\partial_0 (R_{ij} - \Lambda_{ij}) - \nabla_i X_{j0} - \nabla_j X_{i0}) |_{x} = 0, \tag{3.34}$$

the system (3.33) becomes:

$$g^{00}\partial_0 X_{0k} + q^j \{\nabla_j X_{k0} + \nabla_k X_{j0} - g_{jk}g^{lm}(\nabla_l X_{m0} + \nabla_m X_{l0}) + g_{jk}g^{lm}\nabla_l X_{m0}\} -$$

$$Aq_k + g^{ij}\partial_i [X_{jk}] + C_k - \frac{g^{\lambda\delta} X_{\lambda\delta}}{1 - n}(\partial_0 g_{jk})q^j = 0, \tag{3.35}$$
now, given the expressions of $\nabla_i X_{m0}$ and $\nabla_i X_{m0}$, one has
\[ g^{00} \partial_0 X_{0k} + q^i (\partial_i [X_{0k}] + q_j [\partial_0 X_{0k}]) + q^j (\partial_k [X_{0j}] + q_k [\partial_0 X_{0j}]) - q_k g^{lm} (\partial_l [X_{m0}] + q_l [\partial_0 X_{m0}]) + g^{ij} \partial_i [X_{jk}] + A_k = 0, \]
with:
\[ A_k \equiv C_k - 2q^i \Gamma^k_{jk} X_{n0} + q_k g^{lm} \Gamma^l_{m0} X_{k0} - q_k g^{lm} (\Gamma^0_{0m} X_{m0} + \Gamma^0_{l0} X_{m0} + \Gamma^0_{0s} X_{m0}) - Aq_k - \frac{g^{\lambda\delta} X_{k\delta}}{1 - n} (\partial_0 g_{jk}) q^i. \]

The terms $A$ and $C_k$ in $A_k$ are given in (3.21) and (3.24). Now, the hypersurface $I$ being characteristic ($g^{00} + q^i q_j = 0$ on $I$), the system (3.36) resumes to
\[ q^i \partial_i [X_{0k}] + q^j \partial_k [X_{0j}] - q_k g^{lm} \partial_l [X_{m0}] + g^{ij} \partial_i [X_{jk}] + A_k = 0. \]

The $A_k$ are linear combinations of $[X_{\mu\nu}] = X_{\mu\nu}|_I$.

**Proof 3 (Proof of theorem 1)** If the hypotheses of the theorem are satisfied together with the relations (3.14), (3.21), (3.21) for $g$ of the form (2.9), then straightforward calculations imply that on $C$ one has:
\[ X_{00} = \tilde{g}^{CD} \tilde{X}_{CD}/n - 1 = -\tilde{g}_{01} q^s X_{0s}, \]
\[ X_{0k} = \tilde{g}^{CD} \tilde{X}_{CD}/n - 1 = q_k + \frac{\partial y^A}{\partial x^k} \tilde{X}_{0A}, \]
\[ X_{jk} = (q_j \frac{\partial y^A}{\partial x^k} + q_k \frac{\partial y^A}{\partial x^j}) \tilde{X}_{0A} + (2\tilde{g}_{01} q_j q_k + g_{jk}) \frac{\tilde{g}^{CD} \tilde{X}_{CD}}{n - 1}, \]
\[ X_{jk} = q_j X_{0k} + q_k X_{0j} - g_{jk} q^s X_{0s}, \]
where $A, B, C, D = 2, ..., n; i, j, k, s = 1, ..., n$. By combining the relations (3.39), (3.42) and the result (3.23) of lemma 7 one obtains on $C$ the homogeneous linear differential system
\[ \frac{\partial [X_{0k}]}{\partial y^1} + L^s_k ([X_{0s}]) = 0, \quad k = 1, ..., n. \]

One deduces that if the subset $X_{0k} = 0, k = 1, ..., n$, of the Einstein equations are satisfied at the vertex $O$ of the cone, then $X_{0k} = 0, k = 1, ..., n$, on $C$, and after that $X_{\mu\nu} = 0, \forall \mu, \nu$, on $C$, thanks to the relations (3.39) - (3.43). The proof ends by using the proposition 7.

**Remark 1** We emphasize that the tensor $(Z_{AB})$, $Z_{AB} = \tilde{X}_{AB} - \frac{n-1}{\tilde{g}^{CD} \tilde{X}_{CD}} \tilde{g}_{AB}$, $A, B, C, D = 2, ..., n$, is a traceless tensor, consequently one equation chosen suitably will not be considered while solving the constraints (3.21) and will be proved automatically satisfied by the traceless property.
4 Constraints and Cauchy data for \((H_\Sigma, H_\rho)\)

The resolution of the constraints’ equations requires an exhaustive description of the constraints in terms of the Cauchy data for the evolution system \((H_\Sigma, H_\rho)\), and thereby the choice of free data. One sets:

\[
\tilde{g}_{01|C} = \theta, \quad \tilde{g}_{AB|C} = \Theta_{AB}, \quad \Theta = (\Theta_{AB}), \quad \rho_{|C} = f, \quad \partial_{\mu} = \frac{\partial}{\partial y^{\mu}},
\]

\[
\psi_{\mu} = \frac{\partial \tilde{g}_{\mu\nu}}{\partial y^{0}} |_{C}, \quad \pi^{\alpha} = \frac{\partial y^{\alpha}}{\partial x^{3}} \rho, \quad d\pi = d\pi^{1} \cdots d\pi^{n}.
\]  

(4.1)

4.1 The Hamiltonian constraint on \(C\)

The Hamiltonian constraint \(\bar{X}_{11} = 0\) is the so called Raychaudhuri equation. The equation \(\bar{X}_{11} = 0\) reads \(\bar{R}_{11} - \bar{T}_{11} = 0\). It is therefore necessary to compute the terms \(\bar{R}_{11}\) and \(\bar{T}_{11}\) in order to describe this equation. From the definition of the Ricci tensor and according to the expressions of the trace of the metric on \(C\) and the corresponding Christoffel symbols (“see” appendix B), on has:

\[
\bar{R}_{11} = \frac{1}{4} g_{01}^{-1} g^{-CB} \partial_{1} \bar{g}_{CB} \partial_{1} \bar{g} + \frac{1}{2} \partial_{1} (g_{CB} \partial_{1} \bar{g}_{CB}) + \frac{1}{4} (g_{CB} \partial_{1} \bar{g}_{BD})(\bar{g}^{DE} \partial_{1} \bar{g}_{CE}) -
\]

\[
\frac{1}{2} g_{01}^{-1} \partial_{1} g_{01} \bar{g}^{CB} \partial_{1} \bar{g}_{CB}.
\]  

(4.2)

Relying on the definition of the stress momentum tensor of Vlasov matter, one has:

\[
\bar{T}_{11} = - \int_{\mathbb{R}^{n}} f \left( \frac{(g_{01})^{2}(\pi^{0})^{2}}{\pi^{0} + \pi^{1}} \right) d\pi.
\]  

(4.3)

The mass shell equation on \(C \times \mathbb{R}^{n}\) reduces to

\[
2 \bar{g}_{01} \pi^{0} \pi^{1} = - (g_{01}(\pi^{0})^{2} + \bar{g}_{AB} \pi^{A} \pi^{B} + m^{2}).
\]  

(4.4)

On the other hand, the positivity of \(g_{ij}(\lambda p^{i} + q^{i})(\lambda p^{j} + q^{j})\) for every \(\lambda \in \mathbb{R}^{*}\), and the fact that the cone is characteristic imply that

\[
(p^{0})^{2} - (q_{i}p^{i})^{2} \geq \tau^{-2} m^{2},
\]  

(4.5)

one deduces then that for a non-zero mass \(m\), one has \(|p^{0}| > |q_{i}p^{i}|\), and hence \(\pi^{0} > 0\). For a zero mass (\(m = 0\)), one has on \(C \times \mathbb{R}^{n}\)

\[
(p^{0})^{2} = - g^{00} g_{ij} p^{i} p^{j} = - g^{00} \left( - \bar{g}_{01} q_{i} q_{j} + \frac{\partial y^{A}}{\partial x^{3}} \frac{\partial y^{B}}{\partial x^{3}} \bar{g}_{AB} \right) p^{i} p^{j}
\]

\[
= (q_{i} p^{i})^{2} - g^{00} \frac{\partial y^{A}}{\partial x^{3}} \frac{\partial y^{B}}{\partial x^{3}} \bar{g}_{AB} p^{i} p^{j}
\]

and subsequently

\[
(p^{0})^{2} = (q_{i} p^{i})^{2} - g^{00} \bar{g}_{AB} \pi^{A} \pi^{B}.
\]  

(4.6)

In the case of zero mass (\(m = 0\)), one thus assumes that the support of the initial density of particles \(f\) is a subset of \(\{ p^{i} \in \mathbb{R}^{n+1}, \sum_{A}(\pi^{A})^{2} > c_{2} > 0 \}\), and as a consequence of the symmetric positive definite character of the matrix \((\bar{g}_{AB})\), one
has $\pi^0 > 0$. Having $\pi^0 > 0$ induces in virtue of the relation (4.4) that $\pi^1 > 0$ since $\tilde{g}_{01} < 0$. The expression of $\pi^0$ on $C \times \mathbb{R}^n$ is therefore

$$\pi^0 = -\pi^1 + \sqrt{(\pi^1)^2 - \tilde{g}^{01}(\tilde{g}_{AB}\pi^A\pi^B + m^2)}, \quad (4.7)$$

and we end up with the following expression for $\tilde{T}_{11}$ where $\tilde{\gamma} = (\tilde{g}_{AB})$:

$$\tilde{T}_{11} = -\int_{\mathbb{R}^n} \frac{|\tilde{g}_{01}^3|(-\pi^1 + \sqrt{(\pi^1)^2 - \tilde{g}^{01}(\tilde{g}_{AB}\pi^A\pi^B + m^2)})^2}{\sqrt{(\pi^1)^2 - \tilde{g}^{01}(\tilde{g}_{AB}\pi^A\pi^B + m^2)}} \sqrt{g} \; d\pi. \quad (4.8)$$

We now combine this expression of $\tilde{T}_{11}$ with the one of $\tilde{R}_{11}$ (4.2) and use the notations (4.1) to obtain the following proposition.

**Proposition 2** Let $(\tilde{g}, \rho)$ be any $C^\infty$ solution of the evolution system $(H_\tilde{\gamma}, H_\rho)$ in a neighborhood $V$ of $C \times \mathbb{R}^n$, and let $q$ associated to $\tilde{g}$ of the form (2.7) s.t. the temporal gauge condition is satisfied in $Y_0 = \{y^0 \geq 0\}$. The Hamiltonian constraint $\tilde{X}_{11} = 0$ is satisfied on $C$ if and only if the Cauchy data $\theta$, $\Theta = (\Theta_{AB})$, $f$, $\psi_1$ verify the partial differential relation

$$\theta^{-1}(\Theta^{AB}\partial_A \theta)\psi_{11} + 2\partial_A(\Theta^{AB}\partial_A \Theta_{AB}) + (\Theta^{CB}\partial_B \Theta_{BD})(\Theta^{DE}\partial_E \Theta_{CE}) - 2\theta^{-1}(\Theta^{AB}\partial_A \Theta_{AB})\partial_A \theta = -4 \int_{\mathbb{R}^n} f|\theta^3\left(\sqrt{(\pi^1)^2 - \theta^{-1}(m^2 + \Theta_{AB}\pi^A\pi^B)} - \pi^1\right)^2 \sqrt{\tilde{\gamma}} \; d\pi. \quad (4.9)$$

### 4.2 The momentum constraints $\tilde{X}_{1A} = 0$ on $C$

The Einstein equations $\tilde{X}_{1A} = 0$ on $C$ read $\tilde{R}_{1A} = \tilde{T}_{1A}$. We are thus interested here in the expression of $\tilde{R}_{1A}$ resumed in (C.12) of appendix C and the expression of $\tilde{T}_{1A}$, which in its non-explicit form is

$$\tilde{T}_{1A} = \int_{\mathbb{R}^n} \frac{f(\tilde{g}_{01})^2 \tilde{g}_{AB}\pi^0\pi^1 - \tilde{\gamma}}{\pi^0 + \pi^1} \sqrt{\tilde{\gamma}} \; d\pi.$$ 

Combining this latter expression with the one of $\tilde{R}_{1A}$ (C.12), using the expression of $\pi^0$ (4.7) and notations (4.1), the momentum constraints $\tilde{X}_{1A} = 0$, $A = 2, \ldots, n$ reduce on $C$ to partial differential relations in terms of the Cauchy data for $\theta$, $\Theta = (\Theta_{AB})$, $f$, $\psi_{1i}$, $i = 1, \ldots, n$ for $(H_\tilde{\gamma}, H_\rho)$.

**Proposition 3** The hypotheses are those of proposition 2. Then the momentum constraints $\tilde{X}_{1A} = 0$ on $C$ are satisfied if and only if the Cauchy data $\theta$, $\Theta = (\Theta_{AB})$, $f$, $\psi_{1i}$, $i = 1, \ldots, n$ for $(H_\tilde{\gamma}, H_\rho)$, agree with the partial differential relations

$$\partial_A \psi_{11} + (\theta^{-1}(\Theta^{CB}\partial_C \Theta_{CB} - \theta^{-1}\partial_A \theta)\psi_{1A} + \frac{1}{2}(\theta^{-1}\psi_{11} - \theta^{-1}\partial_1 \theta - \frac{1}{2}\Theta^{CB}\partial_C \Theta_{CB})\partial_A \theta - \frac{1}{2}\partial_A(\Theta^{CB}\partial_C \Theta_{AB}) + \theta \partial_A(\Theta^{CB}\partial_C \Theta_{CB} - \theta^{-1}\partial_A \psi_{11} - \frac{1}{4}(\Theta^{EF}\partial_E \Theta_{EF})(\Theta^{CB}\partial_C \Theta_{BA}) + \theta \Theta^{DE} \Theta^{CB}(\partial_A \Theta_{DB})(\partial_A \Theta_{EC} + \partial_C \Theta_{EA} - \partial_E \Theta_{AC}) = 2 \int_{\mathbb{R}^n} f(\theta^3)\Theta_{AD} \left(-\pi^1 + \sqrt{(\pi^1)^2 - \theta^{-1}(m^2 + \Theta_{CB}\pi^C\pi^B)}\right) \pi^D \sqrt{\tilde{\gamma}} \; d\pi. \quad (4.10)$$
4.3 The momentum constraint \( \vec{X}_{01} = 0 \)

The momentum constraint \( \vec{X}_{01} = 0 \) is equivalent to

\[
\vec{g}^{AB} \vec{R}_{AB} = \vec{g}^{01} (\vec{R}_{11} - 2 \vec{T}_{01}).
\]  

(4.11)

According to the expression of \( \vec{R}_{AB} \) in appendix C, one has:

\[
\vec{g}^{AB} \vec{R}_{AB} = \vec{g}^{01} \partial_1 (\vec{g}^{AB} \partial_1 \vec{g}_{AB}) + \frac{1}{2} \vec{g}^{01} (\vec{g}^{AB} \partial_1 \vec{g}_{AB} - \vec{g}^{01} \partial_0 \vec{g}_{11}) (\vec{g}^{01} \partial_0 \vec{g}_{AB}) + \\
\frac{1}{2} \vec{g}^{01} (\vec{g}^{01} \partial_0 \vec{g}_{11} - \frac{1}{2} \vec{g}^{CD} \partial_1 \vec{g}_{CD}) (\vec{g}^{AB} \partial_1 \vec{g}_{AB}) + \vec{g}^{01} \vec{g}^{AB} \vec{R}^{D}_{AB} \partial_0 \vec{g}_{1D} - \\
\vec{g}^{01} \vec{g}^{AB} \partial_0 \vec{g}_{1B} - \frac{1}{2} \vec{g}^{01} \vec{g}^{AB} \partial^2 \vec{g}_{11} \vec{g}_{AB} - \vec{g}^{AB} \partial_C (\vec{g}^{01} \partial_1 \vec{g}_{AB}) + \frac{1}{2} \vec{g}^{01} \vec{g}^{AB} \partial_B [\vec{g}^{C}_{1\delta} \vec{g}^{\gamma}_{1\sigma}] \hspace{1cm} (\gamma, \delta) \in \{(0,1),(1,0),(A,B)\} \\
\frac{1}{2} (\vec{g}^{01})^2 \vec{g}^{AB} (\partial_0 \vec{g}_{1B} - \partial_B \vec{g}_{01}) \partial_0 \vec{g}_{1A} + \frac{1}{2} \vec{g}^{CD} \vec{g}^{AB} \partial_1 \vec{g}_{CA} \partial_1 \vec{g}_{DB} + \\
\vec{g}^{AB} \vec{R}^{D}_{AC} \vec{g}^{C}_{DB} - \vec{g}^{AB} (\vec{g}^{01} \partial_C \vec{g}_{01} + \vec{g}^{D}_{DC}) \vec{g}^{C}_{AB}.
\]  

(4.12)

On the other hand \( \vec{T}_{01} \) is given by:

\[
\vec{T}_{10} = \int_{\mathbb{R}^n} \frac{1}{\sqrt{|f|}} \left| \frac{\vec{g}^{01}}{\pi^0 + \pi^1} \right| \sqrt{|f|} d\pi.
\]  

(4.13)

Concerning the term \( \vec{R}_{11} \) in the right-hand side of equation (4.11), its expression is given in (4.2). We remark however that in the scheme of resolution of the constraints, the heavy term \( \vec{R}_{11} \) is replaced by the expression of \( \vec{T}_{11} \) as soon as the first constraint is satisfied. Now setting \( \chi = \Theta^{AB} \psi_{AB} \), combining the relations (4.12), (4.13), using the expression of \( \pi^0 \), the Christoffel symbols

\[
\vec{g}^{C}_{AB} = \frac{1}{2} \Theta^{CD} (\partial_A \Theta_{BD} + \partial_B \Theta_{AD} - \partial_D \Theta_{AB}), \hspace{0.5cm} A, B, C, D = 2, \ldots, n,
\]  

(4.14)

notations of (4.11), the momentum constraint \( \vec{X}_{10} = 0 \) is a partial differential relation on \( C \) in terms of the Cauchy data for \( \theta \), \( \Theta = (\Theta_{AB}) \), \( f \), \( \psi_{1i} \), \( i = 1, \ldots, n \), \( \chi = \Theta^{AB} \psi_{AB} \) for \( (H_\gamma, H_\rho) \).

**Proposition 4** The hypotheses are those of proposition 3. Then the momentum constraint \( \vec{X}_{10} = 0 \) on \( C \) is satisfied if and only if the Cauchy data \( \theta \), \( \Theta = (\Theta_{AB}) \), \( f \), \( \psi_{1i} \), \( i = 1, \ldots, n \), \( \chi = \Theta^{AB} \psi_{AB} \) for \( (H_\gamma, H_\rho) \), satisfy the partial differential relation

\[
\partial_1 \chi + 2 \Theta^{AB} \partial_1 (\Theta_{AB} - \theta^{-1} \psi_{11}) \chi + \frac{1}{2} (\theta^{-1} \psi_{11} - \frac{1}{2} \Theta^{CD} \partial_1 \Theta_{CD}) (\Theta^{AB} \partial_1 \Theta_{AB}) + \\
\Theta^{AB} (\vec{g}^{01} \partial_1 \psi_{1D} - \partial_A \psi_{1B} - \frac{1}{2} \vec{g}^{01} \Theta_{AB} - \theta \partial_C (\vec{g}^{01} \partial_1 \psi_{1C}) + \theta \partial_B [\theta^{-1} \partial_1 \theta]) + \\
\frac{1}{2} \Theta^{AB} (\partial_1 \Theta^{CD} \partial_A \Theta_{CD}) + \theta^{-1} (\psi_{11} - \theta \partial_B \psi_{1A} + \theta \partial_1 \Theta^{CD} \partial_A \Theta_{CD}) + \\
\Theta^{AB} (\vec{g}^{01} \partial_{1B} \Theta_{CD} - (\theta^{-1} \partial_1 \Theta_{CD})) \vec{g}^{C}_{AB} = \vec{R}_{11} - \\
2 \int_{\mathbb{R}^n} \left( \frac{\theta \pi^3}{\theta^{-1} (\pi^2 + \Theta_{AB} \pi^A \pi^B)} \right) \sqrt{|f|} d\pi.
\]  

(4.15)
4.4 The constraints $\tilde{X}_{AB} - \tilde{g}^{CD} \tilde{X}_{CD} \tilde{g}_{AB} = 0$

For the constraints $\tilde{X}_{AB} - \tilde{g}^{CD} \tilde{X}_{CD} \tilde{g}_{AB} = 0$, they are equivalent to

$$\tilde{R}_{AB} - \tilde{T}_{AB} - \tilde{g}^{CD} (\tilde{R}_{CD} - \tilde{T}_{CD}) \tilde{g}_{AB} = 0. \quad (4.16)$$

One needs the expression of $\tilde{R}_{AB}$ in appendix C and the one of $\tilde{T}_{AB}$ given by

$$\tilde{T}_{AB} = - \int_{\mathbb{R}^n} \tilde{g}_{AB} \tilde{g}_{BD} \Gamma^C_{CD} \sqrt{\gamma} \, d\pi. \quad (4.17)$$

We signal that in the hierarchy of resolution of constraints, the quantity $\tilde{g}^{CD} \tilde{R}_{CD}$ in the right-hand side of (1.16) is substituted by $\tilde{g}^{ij}(\tilde{T}_{11} - \tilde{T}_{01})$ as soon as the previous constraints are satisfied. All that precedes implies that the constraints $\tilde{X}_{AB} - \tilde{g}^{CD} \tilde{X}_{CD} \tilde{g}_{AB} = 0$ are partial differential in terms of the Cauchy data $\theta, \Theta = (\Theta_{AB}), f, \psi_{1i}, i = 1, \ldots, n, \psi_{AB}, A, B = 2, \ldots, n$ for $(H_{\pi}, H_{\rho})$.

**Proposition 5** The hypotheses are those of proposition 3. Then the constraints $\tilde{X}_{AB} - \tilde{g}^{CD} \tilde{X}_{CD} \tilde{g}_{AB} = 0$ on $\mathcal{C}$ are satisfied if and only if the Cauchy data $\theta, \Theta = (\Theta_{AB}), f, \psi_{1i}, i = 1, \ldots, n, \psi_{AB}, A, B = 2, \ldots, n$ for $(H_{\pi}, H_{\rho})$, satisfy the partial differential relations

$$\partial_1 \psi_{AB} -$$

$$\frac{1}{2} (\theta^{-1} \psi_{11} - \Theta^{EF} \partial_1 \Theta_{EF} \delta^C_A \delta^D_B + \Theta^{ED} \partial_1 \Theta_{EB} \delta^C_A + \Theta^{ED} \partial_1 \Theta_{DA} \delta^C_B) \psi_{CD} +$$

$$\frac{1}{2} (\sqrt{\gamma} f_{CD} \delta_{CD} \psi_{CE} + (\theta^{-1} \psi_{11} - \frac{1}{2} \Theta^{CD} \partial_1 \Theta_{CD})) \partial_1 \Theta_{AB} +$$

$$\Gamma^D_{AB} \psi_{1D} - \partial_A \psi_{1B} - \partial_B \psi_{1A} - \frac{1}{2} \partial^2_1 \Theta_{AB} - \theta \partial_C (\Gamma^C_{AB}) + \theta \partial_B (\theta^{-1} \partial_A \theta) +$$

$$\frac{1}{2} (\theta \partial_B (\Theta^{CD} \partial_A \Theta_{CD}) + \theta^{-1} (\psi_{1B} - \partial_B \theta) \psi_{1A} + \Theta^{CD} \partial_1 \Theta_{CA} \partial_1 \Theta_{DB}) +$$

$$\Gamma^D_{AC} \Gamma^C_{DB} - (\theta^{-1} \partial_C \theta + \tilde{\Gamma}^D_{DC} \Gamma^C_{AB} = \frac{\Theta^{CD} (\tilde{R}_{CD} - \tilde{T}_{CB})}{n - 1} \Theta_{AB} -$$

$$\int_{\mathbb{R}^n} f \frac{|\theta| \Theta_{AC} \Theta_{BD} \psi^C \psi^D}{\sqrt{(\pi^2)^2 - \theta^{-1} (m^2 + \Theta_{CD} \psi^C \psi^D)}} \sqrt{|\gamma|} \, d\pi. \quad (4.18)$$

**Remark 2** The constraints of the type in proposition 3 appear in the case of the "Double null foliation gauge" [2, 16].

4.5 The constraint $\tilde{X}_{00} - \tilde{g}_{01} \tilde{g}^{CD} \tilde{X}_{CD} \tilde{g}_{01} = 0$

The last constraint in our scheme corresponds to

$$\tilde{X}_{00} - \tilde{g}_{01} \tilde{g}^{CD} \tilde{X}_{CD} \tilde{g}_{01} = 0, \quad A, B, C, D = 2, \ldots, n;$$
it is equivalent to

$$\tilde{R}_{00} - \tilde{T}_{00} - \bar{g}_{01} \tilde{g}^{AB} (\tilde{R}_{AB} - \tilde{T}_{AB}) + \bar{g}_{01} \frac{\partial(\tilde{\Gamma}^0 + \tilde{\Gamma}^1)}{\partial y^0} = 0.$$  \hspace{1cm} \text{(4.19)}$$

One needs at this level the expression of $\tilde{R}_{00}$ (C.26) in appendix C and the one of $\frac{\partial(\tilde{\Gamma}^0 + \tilde{\Gamma}^1)}{\partial y^0}$ (D.6) in appendix D and $\tilde{T}_{00}$ which resums to

$$\tilde{T}_{00} = \int_{\mathbb{R}^n} f|\bar{g}_{01}|^3 (\pi^0 + \pi^1) \sqrt{|\gamma|} \, d\pi.$$  \hspace{1cm} \text{(4.20)}$$

**Proposition 6** The hypotheses are those of proposition [2]. Then the constraint $\bar{X}_{00} - \bar{g}_{01} \tilde{g}^{CD} \bar{X}_{0D} + \bar{g}_{01} \frac{\partial(\tilde{\Gamma}^0 + \tilde{\Gamma}^1)}{\partial y^0} = 0$; $C, D = 2, ..., n$ on $C$ is satisfied if and only if the Cauchy data $\theta, \Theta = (\Theta_{AB}), f, \psi_\alpha, \alpha = 0, 1, ..., n, \psi_{AB}, A, B = 2, ..., n$ for $(H_\theta, H_\rho)$, satisfy the partial differential relation

$$\partial_1 \psi_1 + \frac{1}{4} \Theta^{AB} \partial_1 \Theta_{AB} - \theta^{-1} (\partial_1 \theta + \psi_{11}) \psi_1 - \frac{1}{2} \Theta^{AB} \partial_1 \psi_{1A} +$$

$$\frac{\theta^{-1}}{4} \psi_{11} + \frac{\theta}{4} \Theta^{DB} \Theta^{CE} \psi_{BC} \psi_{ED} + \frac{\theta^{-1}}{4} \psi_{11} \partial_1 \theta - \frac{1}{2} \Theta^{CB} \psi_{1C} \partial_1 \theta +$$

$$+ \frac{\theta^{-1}}{2} \psi_{01}^2 + \frac{\theta}{2} \partial_1 \Theta^{AC} \partial_2 \theta + \frac{\theta}{2} \Theta^{AB} \partial_2 \theta + \frac{1}{4} \partial_1 \theta \Theta^{CB} (\psi_{CB} - \partial_1 \Theta_{CB}) +$$

$$\frac{\theta}{2} \Theta^{DB} (\partial_2 \theta) \gamma_{CD} = \frac{\Theta^{AB} (\tilde{R}_{AB} - \tilde{T}_{AB})}{n - 1} +$$

$$\theta \int_{\mathbb{R}^n} f(\psi)^3 (\sqrt{\gamma^1})^2 \psi_{11} (\pi^0 + \pi^1) (m^2 + \Theta_{AB} \pi^A \pi^B) \sqrt{|\gamma|} \, d\pi.$$  \hspace{1cm} \text{(4.21)}$$

**5 On the resolution of the initial data constraints**

The cone $C$ admits the equation $x^0 - \sqrt{\sum_i (x^i)^2} = 0$ in coordinates $(x^\mu)$ and this is viewed as the requirement that the coordinates $(x^\mu)$ coincide on $C$ with some normal coordinates based at $O$ and attached to an appropriate basis of vectors at $O$. The Christoffel symbols of the metric thus vanish at $O$. This induces that a regular metric in a neighborhood of the cone and Minkowskian at $O$ must satisfy the expansion

$$g = g_{\mu\nu} \, dx^\mu \, dx^\nu = (\eta_{\mu\nu} + O(r^2)) \, dx^\mu \, dx^\nu.$$  \hspace{1cm} \text{(5.1)}$$

It's behavior at $O$ (see also [R] for some details) in coordinates $(y^\mu)$ is thus:

$$\bar{g}_{01} = -1 + O(r^2), \quad \bar{g}_{AB} = r^2 \sum_i \frac{\partial \theta^i}{\partial y^A} \frac{\partial \theta^i}{\partial y^B} + O(r^4),$$

$$\bar{g}^{AB} = \sum_s \frac{\partial y^A}{\partial x^s} \frac{\partial y^B}{\partial x^s} + O(cste),$$  \hspace{1cm} \text{(5.2)}$$

where the expression of $\frac{\partial y^A}{\partial x^s}$ is given by:

$$\frac{\partial y^A}{\partial x^s} = \frac{1}{y^1} \left( \frac{\partial y^A}{\partial \theta^s} - \frac{\partial y^A}{\partial \theta^k} \theta^s \right),$$  \hspace{1cm} \text{(5.3)}$$

(14)
moreover, one has the following properties:

\[ [\partial_0 \tilde{g}_{00}] = O(r), \quad [\partial_0 \tilde{g}_{01}] = O(r), \quad [\partial_0 \tilde{g}_{0A}] = 0, \quad [\partial_0 \tilde{g}_{1A}] = O(r^2), \]
\[ [\partial_0 \tilde{g}_{11}] = O(r), \quad [\partial_0 \tilde{g}_{AB}] = O(r^3). \]  \hfill (5.4)

5.1 The free data and their many ways

The constraints (4.9), (4.10), (4.15), (4.18), (4.21) do not belong to a specific type of differential equations which can be solved using classical methods unless free data are well-chosen. Indeed, there are some free data which guarantees that the constraints (4.9), (4.10), (4.15), (4.18), (4.21) can be solved (hierarchically) algebraically or/and as propagations equations along null generators of the cone \( C \) in terms of the constrained Cauchy data. Their specificity rests on how to tackle the first constraint. To use a terminology which appears in other contexts \([2]\) and thereby highlight the many ways of the free data, we introduce for the tangent vectors \( X, Y \in TC_{\mu\nu} \) the one form \( \xi \) and the second fundamentals forms \( \chi, \hat{\chi} \) of \( C_{\mu\nu} := \{ y^0 = \mu, \ y^1 = \nu \} \), defined by

\[ \xi(X) = \frac{1}{2} g(\nabla^\partial_{X}, \partial_0), \]  \hfill (5.5)
\[ \chi(X, Y) = g(\nabla^\partial_{X}, Y), \quad \hat{\chi}(X, Y) = g(\nabla^\partial_{X}, Y), \]  \hfill (5.6)

and correspondingly the shear tensors \( \hat{\chi}, \chi \):

\[ \hat{\chi} = \chi - \frac{tr\chi}{n-1} \gamma, \quad \chi = \chi - \frac{tr\chi}{n-1} \gamma, \]  \hfill (5.7)

where \( tr\chi \) and \( tr\chi \) denote the trace of \( \chi \) and \( \hat{\chi} \) respectively, with respect to the induced metric \( \gamma \) of \( g \) on \( C_{\mu\nu} \). On \( C = \cup C_{ao} \), these geometric objects are linked to the gravitational Cauchy data, i.e.:

\[ \xi_A = \frac{1}{4} (\partial_A \tilde{g}_{01} - \partial_0 \tilde{g}_{A1}), \quad \chi_{AB} = \frac{1}{2} \partial_1 \tilde{g}_{AB}, \quad \hat{\chi}_{AB} = \frac{1}{2} \partial_0 \tilde{g}_{AB}. \]  \hfill (5.8)

The constraints (4.9), (4.10), (4.15), (4.18), (4.21) expressed then in terms of these geometric objects and other quantities identified as combinations or not of the Christoffel symbols of the metric on \( C \), as instance, one has:

\[ 2g(\nabla^\partial_{\partial_0}, \partial_1) = \partial_0 \tilde{g}_{11}, \quad 2g(\nabla^\partial_{\partial_1}, \partial_0) = \partial_0 \tilde{g}_{01}, \quad 2g(\nabla^\partial_{\partial_0}, \partial_1) = 2\partial_0 \tilde{g}_{01} - \tilde{g}_{01} \partial_1 \tilde{g}_{01}. \]

Here we make discussions according to the affine parametrization condition.

5.1.1 No initial condition on the non-affinity constant \( \kappa := \tilde{\Gamma}^{11} \)

The assumption (A) (3.7) guarantees only that the vector fields \( \nabla^\partial_{\partial_0} \) is tangent to the null generators of the cone. According to the notations of (4.1), one has \( \nabla^\partial_{\partial_0} = \kappa \partial_1 = \frac{1}{4} (2\partial_1 \theta - \psi_{11}) \partial_1 \). It appears thus clearly that \( \kappa \) is directly linked to the Cauchy data \( \psi_{11} \). If no condition is given on \( \kappa \), then the first constraint equation is solvable algebraically in term of \( \psi_{11} \) provided all the components of the metric \( \theta, \Theta_{AB} \) are given, with the divergence scalar \( tr\chi = \frac{1}{2} \Theta^{AB} \partial_1 \Theta_{AB} \) of \( C \) which is nowhere vanishing. In this case, the solution is global.
Remark 3 Given a metric $\tilde{\gamma} = (\tilde{\gamma}_{AB})$ on $\mathcal{C}$, a conformal class of metric $(e^{\omega}\tilde{\gamma}_{AB})$ satisfying the "nowhere vanishing condition for the divergence scalar" realizes for $\omega$ given by
\[
\omega(y') = -\frac{1}{n-1} \int_0^{y_0} (\tilde{\gamma}^{AB}\partial_1 \tilde{\gamma}_{AB} - \omega_0)(\lambda, y^A)d\lambda, \tag{5.9}
\]
where $\omega_0 \equiv \omega_0(y^1, y^A)$ is any function s.t. $|\omega_0| > 0$, $y^1 \neq 0$.

5.1.2 $\kappa$ and $(\tilde{\gamma}_{AB})$ as free data

Given $\kappa$ as free data is equivalent to the given of the Cauchy data $\psi_{11}$ as soon as $\theta < 0$ is known, and indicates the specification of an affine parameter though its expression is not necessarily trivial. The first constraint equation is solvable in terms of a conformal factor $\omega$ s.t. $\Theta_{AB} = e^{\omega} \tilde{\gamma}_{AB}$ as in [21], where the $\tilde{\gamma}_{AB}$ are prescribed freely and make up a symmetric positive definite matrix. Indeed,
\[
\Theta^{CB}\partial_1 \Theta_{CB} = (n-1)\partial_1 \omega + \tilde{\gamma}^{BC}\partial_1 \tilde{\gamma}_{BC},
\]
\[
\partial_1 (\Theta^{CB}\partial_1 \Theta_{CB}) = (n-1)\partial_1 (\partial_1 \omega) + \partial_1 (\tilde{\gamma}^{BC}\partial_1 \tilde{\gamma}_{BC}),
\]
\[
(\Theta^{CB}\partial_1 \Theta_{BD})(\Theta^{DE}\partial_1 \Theta_{CE}) = (n-1)[\partial_1 \omega]^2 + 2[\tilde{\gamma}^{EC}\partial_1 \tilde{\gamma}_{EC}][\partial_1 \omega] + \tilde{\gamma}^{BC}\tilde{\gamma}^{DE}(\partial_1 \tilde{\gamma}_{CE})(\partial_1 \tilde{\gamma}_{BD}).
\]
The first constraint equation then reads:
\[
\partial_1^2 \omega + \left(-\kappa + \tilde{\gamma}^{EC}\partial_1 \tilde{\gamma}_{EC} \right) \partial_1 \omega + \frac{(\partial_1 \omega)^2}{2} + \frac{\partial_1 (\tilde{\gamma}^{EC}\partial_1 \tilde{\gamma}_{EC})}{n-1}
\]
\[
- \frac{\kappa}{n-1} \tilde{\gamma}^{BC}\partial_1 \tilde{\gamma}_{BC} + \tilde{\gamma}^{BC}\tilde{\gamma}^{DE}\partial_1 \tilde{\gamma}_{CE} \partial_1 \tilde{\gamma}_{BD}
\]
\[
= \frac{2}{1-n} \int_{R^n} f |\theta|^3 \left( \frac{\sqrt{(\pi^1)^2 - \theta^{-1} (m^2 + e^{\omega} \tilde{\gamma}_{AB}\pi^A\pi^B)} - \pi^1}{\sqrt{(\pi^1)^2 - \theta^{-1} (m^2 + e^{\omega} \tilde{\gamma}_{AB}\pi^A\pi^B)}} \right)^2 e^{-\frac{\pi^1}{\sqrt{|\gamma|}}} d\pi.
\]
In term of $\Omega := e^{\tilde{\pi}}$, one has:
\[
2\partial_1^2 \Omega + 2 \left(-\kappa + \tilde{\gamma}^{EC}\partial_1 \tilde{\gamma}_{EC} \right) \partial_1 \Omega + \Omega
\]
\[
= \frac{2}{1-n} \int_{R^n} f |\theta|^3 \left( \frac{\sqrt{(\pi^1)^2 - \theta^{-1} (m^2 + \Omega^2 \tilde{\gamma}_{AB}\pi^A\pi^B)} - \pi^1}{\sqrt{(\pi^1)^2 - \theta^{-1} (m^2 + \Omega^2 \tilde{\gamma}_{AB}\pi^A\pi^B)}} \right)^2 \sqrt{|\gamma|} d\pi. \tag{5.10}
\]
One can also use the following relation to simplify some expressions above:
\[
\tilde{\gamma}^{EC}\partial_1 \tilde{\gamma}_{EC} = 2\partial_1 \ln \sqrt{|\gamma|}.
\]
This second order differential equation has a unique solution given Minkowskian initial values at $O$.
5.1.3 \( \kappa \) and components \( \sigma^B_A \) of the shear w.r.t. dual bases as free data

If the components \( \sigma^B_A \) of the shear w.r.t. dual bases \( \{ \frac{\partial}{\partial y^A} \}, \{ dy^A \} \) are given as free data together with \( \theta < 0 \) and \( \kappa \), the relation \( \chi_{AB} = \frac{1}{2} \partial_A \Theta_{AB} \) and the first constraint equation induce a system of differential equations, of unknowns the \( \Theta_{AB} \) and the divergence scalar \( \text{tr}\chi \). Indeed, one sets: \( \hat{\chi}^B_A = \sigma^B_A \), and one can deduce:

\[
\partial_1 \Theta_{AB} - 2(\sigma^C_B + \frac{\text{tr}\chi}{n-1} \delta^C_B)\Theta_{AC} = 0,
\]

(5.11)

\[
\partial_1 \text{tr}\chi - \frac{\theta}{2}(\theta + \partial_1 \theta)\text{tr}\chi + \frac{(\text{tr}\chi)^2}{n-1} + \sigma^B_A \sigma^A_B - \hat{T}_{11}(\theta, \Theta_{CD}, m^2, f) = 0.
\]

(5.12)

This system has a unique solution given Minkowskian initial values at 0. Setting

\[
V = e^{\int \frac{\text{tr}\chi}{n-1} dy^1},
\]

the equation (5.12) reads:

\[
\partial_{11} V - \frac{\theta}{2}(\theta + \partial_1 \theta)\partial_1 V + \left( -\sigma^B_A \sigma^A_B + \hat{T}_{11}(\theta, \Theta_{CD}, m^2, f) \right) V = 0.
\]

(5.13)

**Remark 4 (Shear prescribed)** One can prescribe rather the components \( \sigma_{AB} \) of the shear with respect to the basis \( \{ dy^A \} \), however the system (5.11), (5.12) seems simpler than the one obtained in this case.

**Remark 5 (On global solution in vacuum for prescribed \( \kappa \))** From the analysis above, it appears that in vacuum and for prescribed \( \kappa \), one has a global solution. Indeed, in this case, the equation (5.10) is a homogeneous linear second order differential equation in \( \Omega \), and the same property holds for the equation (5.13) of unknown \( V \) which can be solved independently of the equation (5.11).

5.2 Constraints’s solutions: the general idea

In this paper, we concentrate on the first case where no condition is given on \( \kappa \), this corresponds thus to unconstrained initial metric and the solutions of the constraints yield the first fundamental form of the initial hypersurface. The prescribed free data comprise precisely:

(a) \( C^\infty \) functions \( \tilde{\gamma}_{AB} = \tilde{\gamma}_{AB}(y^i) \) that make up (for \( y^1 \neq 0 \)) a symmetric positive definite matrix, and satisfy the ”nowhere vanishing” property for the divergence scalar, i.e.:

\[
\left| \tilde{\gamma}_{AB} \frac{\partial \tilde{\gamma}_{AB}}{\partial y^1} \right| > 0, \quad y^1 \neq 0.
\]

(5.14)

The existence of a large class of such free data \( (\tilde{\gamma}_{AB}) \) is analyzed in remark \( 3 \). (b) a smooth function \( \theta = \theta(y^i) \) on \( C \), and \( f \equiv f(y^i, \pi^j) \) on \( C \times \mathbb{R}^n \) s.t. \( \theta \) is negative, \( f \) is non negative of compact support contained in \( \{ \pi^1 > c_1 > 0 \} \) for a mass \( m \neq 0 \); and for a zero mass the support of \( f \) is contained in \( \{ \pi^1 > c_1 > 0, \sum_{A=2}^n (\pi^A)^2 > c_2 > 0 \} \), besides that, \( \text{Supp}(f) \cap \{O \times \mathbb{R}^n \} = \emptyset \). These free data satisfy

\[
\theta = -1 + O(r^2), \quad \tilde{\gamma}_{AB} = r^2 \sum_i \frac{\partial \theta^i}{\partial y^A} \frac{\partial \theta^i}{\partial y^B} + O(r^4),
\]

(5.15)
The constraints \( \tilde{\gamma}^{AB} = \sum_s \frac{\partial y^A}{\partial x^s} \frac{\partial y^B}{\partial x^s} + O(\text{este}). \) (5.16)

**Theorem 2** Given the free data as described above by \((a)-(b)\). Then, there exists a unique global solution \((\theta, \Theta_{AB}, \psi_{1A}, \psi_{AB}, f)\) on \(C \times \mathbb{R}^n\) of the initial data constraints (3.19), (3.20), (3.21) for the Einstein-Vlasov system.

**Proof 4** Given the free data \((a)-(b)\), one solves the constraints described by (4.17), (4.18), (4.21) in a hierarchical scheme. Indeed, one sets: \(\Theta_{AB}(y^1, y^A) = \gamma_{AB}(y^1, y^A)\), then \(|\Theta^{AB}\partial_i \Theta_{AB}| > 0\), and \(\psi_{11}\) solves algebraically the Hamiltonian constraint \(\tilde{X}_{11} = 0\) as described by (4.9), with \(\psi_{11} = O(y^1)\). For the constraints \(\tilde{X}_{1A} = 0\) (4.10) and \(\tilde{X}_{01} = 0\) (4.15), they can be written in the forms

\[
\frac{d\psi_{1A}}{dy^1} + \frac{(n-1)}{y^1} \psi_{1A} + \psi_A(y^i, \psi_{1C}) = 0, \quad (5.17)
\]

\[
\frac{d\psi}{dy^1} + \frac{(n-1)}{y^1} \chi + \psi(y^i, \chi) = 0, \quad (5.18)
\]

where the functions \(\psi_A\) and \(\psi\) are linear w.r.t. \(\psi_{1C}\) and respectively \(\chi\), their solutions satisfy the integral systems

\[
\psi_{1A} = \frac{1}{y^1} \int_0^{y^1} [-\lambda \psi_A + (2 - n) \psi_{1A}] (\lambda, y^A) d\lambda, \quad (5.19)
\]

\[
\chi = \frac{1}{y^1} \int_0^{y^1} [-\lambda \psi + (2 - n) \chi] (\lambda, y^A) d\lambda. \quad (5.20)
\]

Since the functions under the integral’s sign are continuous w.r.t. \(y^1\) and Lipschitzian w.r.t. the corresponding unknowns, the solutions of the systems (5.17), (5.18) exist, are unique and global thanks to linearity, furthermore \(\psi_{1A} = O((y^1)^2)\), \(\chi = O(y^1)\) according to the behavior near \(O\) of \(\psi_A\) and \(\psi\). One first solves the system in \(\psi_{1A}\) and, after that, the equation regarding \(\chi\). Now, we consider the constraints \(Z_{AB} = \tilde{X}_{AB} - \frac{2 y^D \tilde{X}_{CD}}{n-1} y_{AB} = 0\) described in (4.18) for \((A, B) \neq (2, 2)\) since \((Z_{AB})\), \((A, B = 2, \ldots, n)\) is a traceless tensor, of unknowns \(\psi_{AB}\) for \((A, B) \neq (2, 2)\) provided \(\psi_{22}\) takes the value \(\psi_{22} = \frac{1}{32\pi} (\chi - \sum_{(A,B)\neq(2,2)} \Theta_{AB} \psi_{AB})\), and where \(\Theta_{AB} \tilde{R}_{AB} \equiv \tilde{R}^{(n-1)}\) equals \((\tilde{X}_{11} - 2\tilde{X}_{01})\) since \(\tilde{X}_{01} = 0\) is satisfied. This system is of the form

\[
\frac{d\psi_{AB}}{dy^1} - \frac{2}{y^1} \psi_{AB} + \psi_{AB}(y^i, \psi_{BC}) = 0, \quad (5.21)
\]

its solution is unique, global, satisfies the integral system

\[
\psi_{AB} = \frac{1}{y^1} \int_0^{y^1} [-\lambda \psi_{AB} + 3 \psi_{AB}] (\lambda, y^A) d\lambda,
\]

and by closer inspection of the expression of \(\psi'_{AB}\), one proves that \(\psi_{AB} = O((y^1)^2)\). That \(Z_{22} = 0\) is also satisfied with \(\psi_{22} = \frac{1}{32\pi} (\chi - \sum_{(A,B)\neq(2,2)} \Theta_{AB} \psi_{AB})\) follows from...
the traceless property of \((Z_{AB})\). One also has \(\psi_{22} = O((y^1)^3)\). The last constraint (4.27) has the form

\[
\frac{d\psi_{01}}{dy^1} + \frac{n-1}{2y^1} \psi_{01} + \psi_{01}'(y^1, \psi_{01}) = 0,
\]

(5.22)

its solution is unique, global, agrees with the integral equation

\[
\psi_{01} = \frac{1}{y^1} \int_0^{y^1} \left[ -\lambda \psi_{01}' + \frac{(3-n)}{2} \psi_{01} \right](\lambda, y^1) d\lambda.
\]

At least by studying the behavior near \(O\) of the expression of \(\psi'\) one deduces that \(\psi_{01} = O(y^1)\).

6 On the evolution system \((H_\bar{g}, H_\rho)\)

The treatment of the evolution system requires more analysis of the free data, namely a complete description of their behavior near \(O\), and obviously the behavior near the vertex \(O\) of the constraints’ solutions. In order to apply Dossa’s well posedness results [11], it would be interesting in a subsequent work, to prove that the characteristic initial data on concerned here can arise as restrictions to the cone of functions smooth at the neighborhood of the tip of the cone. We remark that for this purpose, the lapse is related to the metric \(\bar{g}\) by the relation \(\tau = \frac{|D_0 g_{11}|}{\sqrt{|\bar{g}|}} \sqrt{|g|}\). Another more technical issue would be to derive the above results under lower regularity assumptions.

Moreover, the construction of a large class of initial data sets offers also here the possibility to study without any symmetry assumptions the global future evolution of small data with appropriate fall-off behavior at infinity. The strong nonlinear features of the Einstein equations requires one to rely on a quite rigid analytic approach based on energy estimates and other many tools, for this, one should take advantage of results on global existence of D. Christodoulou and S. Klainerman [8], S. Klainerman and F. Nicolo [13], H. Linblad and I. Rodniansky [15], Y. Choquet-Bruhat [3], L. Bieri and N. Zipser [H], P. G. LeFloch and Y. Ma [14].

Appendix A Some algebraic relations including the gravitational data on \(\mathcal{C}\)

From the zero-shift condition and algebraic properties between the metric and its inverse, one shows that the gravitational data satisfy the following relations on \(\mathcal{C}\).

\[
\partial_0 \bar{g}_{00} = 0, \quad \partial_0 \bar{g}_{01} = -(\bar{g}^{01})^2 (\partial_0 \bar{g}_{01} - \partial_0 \bar{g}_{11}) = -\partial_0 \bar{g}_{11},
\]

\[
\partial_0 \bar{g}_{00} = -(\bar{g}^{01})^2 \partial_0 \bar{g}_{11}, \partial_0 \bar{g}_{0C} = -\bar{g}^{01} \bar{g}^{CD} \partial_0 \bar{g}_{1D}, \partial_0 \bar{g}_{AD} = -\bar{g}^{AB} \bar{g}^{CD} \partial_0 \bar{g}_{CB},
\]

\[
\partial_0 \bar{g}_{00} = \partial_0 \bar{g}_{01}, \quad \frac{\partial}{\partial y^\nu} (\bar{g}^{01} + \bar{g}^{11}) = 0, \quad \nu = 0, \ldots, n, \quad i = 1, \ldots, n.
\]

(A.1)
Appendix B  Christoffel symbols of $\tilde{g}$ on $C$

The trace of the metric on $C$ induces the following expressions for the Christoffel symbols of the metric on $C$.

\[ \Gamma^0_{00} = \frac{1}{2} \gamma^0(2\partial_0 \tilde{g}_{00} - \partial_1 \tilde{g}_{00}), \quad \Gamma^0_{01} = \frac{1}{2} \gamma^0(2\partial_1 \tilde{g}_{01} - \partial_0 \tilde{g}_{01}), \quad \Gamma^0_{11} = \frac{1}{2} \gamma^0(2\partial_1 \tilde{g}_{11} - \partial_0 \tilde{g}_{11}), \]

\[ \Gamma^0_{0C} = \frac{1}{2} \gamma^0(\partial_0 \tilde{g}_{C0} + \partial_C \tilde{g}_{00}), \quad \Gamma^0_{1C} = 0, \quad \Gamma^0_{CD} = -\frac{1}{2} \gamma^0(\partial_0 \tilde{g}_{CD}), \]

\[ \tilde{\Gamma}_0^{10} = \frac{1}{2} \gamma^0(\partial_1 \tilde{g}_{01} - \partial_0 \tilde{g}_{01}), \quad \tilde{\Gamma}_1^{10} = \frac{1}{2} \gamma^0(\partial_1 \tilde{g}_{01} - \partial_0 \tilde{g}_{01}), \quad \tilde{\Gamma}_1^1 = -\frac{1}{2} \gamma^0(\partial_0 \tilde{g}_{11} - \partial_1 \tilde{g}_{11}), \]

\[ \tilde{\Gamma}_1^{01} = \frac{1}{2} \gamma^0(\partial_1 \tilde{g}_{01} - \partial_0 \tilde{g}_{01}), \quad \tilde{\Gamma}_1^{10} = \frac{1}{2} \gamma^0(\partial_1 \tilde{g}_{01} - \partial_0 \tilde{g}_{01}), \]

\[ \tilde{\Gamma}^1_{0D} = \frac{1}{2} \gamma^0(\partial_1 \tilde{g}_{0D} - \partial_0 \tilde{g}_{0D}), \quad \tilde{\Gamma}^C_{AB} = \frac{1}{2} \gamma^0(\partial_A \tilde{g}_{BD} + \partial_B \tilde{g}_{AD} - \partial_D \tilde{g}_{AB}), \quad \tilde{\Gamma}^C_{1D} = \frac{1}{2} \gamma^0(\partial_1 \tilde{g}_{BD} - \partial_D \tilde{g}_{1B}), \quad \tilde{\Gamma}^1_{11} = 0. \]

Appendix C  Ricci tensor on $C$

Using the collected formula and the expressions of the Christoffel symbols of the metric above, one establishes the expressions of the components of the Ricci tensor on $C$, following is the sketch.

C.1  Computation of $\tilde{R}_{11}$

By definition

\[ \tilde{R}_{11} = \partial_1 \tilde{\Gamma}^1_{11} - \partial_1 \tilde{\Gamma}^1_{01} + \tilde{\Gamma}^1_{01} \tilde{\Gamma}^1_{11} - \tilde{\Gamma}^1_{01} \tilde{\Gamma}^1_{11}. \]

Since $\tilde{\Gamma}^1_{11} = 0$ according to the expressions of appendix [B], the term $\partial_1 \tilde{\Gamma}^1_{11}$ splits as:

\[ \partial_1 \tilde{\Gamma}^1_{11} = \partial_1 \tilde{\Gamma}^0_{01} - \partial_0 \tilde{\Gamma}^0_{01} + \partial_1 \tilde{\Gamma}^0_{11} - \partial_0 \tilde{\Gamma}^0_{11} + \partial_1 \tilde{\Gamma}^C_{11} - \partial_C \tilde{\Gamma}^C_{11}. \]

On the other hand, from the expression of $\tilde{\Gamma}^0_{01}$ it results that:

\[ \partial_1 \tilde{\Gamma}^0_{01} = \frac{1}{2} \partial_1[\gamma^0(\partial_0 \tilde{g}_{11})] = \frac{1}{2} (\partial_1 \tilde{g}^0)(\partial_0 \tilde{g}_{11}) + \frac{1}{2} \gamma^0(\partial_1 \tilde{g}_{11}). \]

The term $\partial_0 \tilde{\Gamma}^0_{11}$ requires more attention due to the normal character of the derivative $\partial_0$. One has:

\[ \partial_0 \tilde{\Gamma}^0_{11} = \partial_0[\frac{1}{2} \gamma^0(2\partial_0 \tilde{g}_{11} - \partial_1 \tilde{g}_{11})] = \frac{1}{2} \partial_0(\tilde{g}^0)(2\partial_1 \tilde{g}_{11} - \partial_0 \tilde{g}_{11}) + \frac{1}{2} \gamma^0(\partial_0^2 \tilde{g}_{11} - \partial_1^2 \tilde{g}_{11}) = \frac{1}{2} \partial_0(\tilde{g}^0)(2\partial_1 \tilde{g}_{01} - \partial_0 \tilde{g}_{01}) + \frac{1}{2} \partial_0(\tilde{g}^0)(2\partial_1 \tilde{g}_{11} - \partial_0 \tilde{g}_{11}) + \frac{1}{2} \partial_0(\tilde{g}^0)(2\partial_1 \tilde{g}_{11} - \partial_0 \tilde{g}_{11}) + \frac{1}{2} \gamma^0(\partial_0^2 \tilde{g}_{11} - \partial_0 \tilde{g}_{11}). \]

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and after more simplifications thanks to properties of the metric and its inverse (3.3), (3.9), one ends up with:

$$\partial_0 \tilde{\Gamma}^{0}_{11} = \frac{1}{2} \partial_0 (g^{00}) (2 \partial_1 \tilde{g}_{01} - \partial_0 \tilde{g}_{11}) + \frac{1}{2} g^{01} \partial_1 (\partial_0 \tilde{g}_{11}).$$  \hspace{1cm} (C.2)

The expression of $\tilde{\Gamma}^C_{11}$ (appendix B) implies:

$$\partial_1 \tilde{\Gamma}^C_{11} = \frac{1}{2} \partial_1 (g^{CB} \partial_1 \tilde{g}_{CB}).$$  \hspace{1cm} (C.3)

Now, one deals with the term $\tilde{E}_{11} = \tilde{\Gamma}^2_{11} \tilde{\Gamma}^7_{11} - \tilde{\Gamma}^5_{17} \tilde{\Gamma}^3_{11}$, since $\tilde{\Gamma}^0_{11} = 0 = \tilde{\Gamma}^C_{11} = \tilde{\Gamma}^D_{11}$ (appendix B), this terms simplifies in:

$$\tilde{E}_{11} = \tilde{\Gamma}^0_{10} \tilde{\Gamma}^0_{01} + \tilde{\Gamma}^D_{1D} \tilde{\Gamma}^D_{C1} - \tilde{\Gamma}^0_{01} \tilde{\Gamma}^1_{11} - \tilde{\Gamma}^C_{C1} \tilde{\Gamma}^1_{11},$$

and yields:

$$\tilde{E}_{11} = \frac{1}{4} (g^{01} \partial_0 \tilde{g}_{11})^2 + \frac{1}{4} (g^{CB} \partial_1 \tilde{g}_{BD}) (g^{DE} \partial_1 \tilde{g}_{CE}) - \frac{1}{2} (g^{01} \partial_0 \tilde{g}_{11} + g^{CB} \partial_1 \tilde{g}_{CB}) \tilde{\Gamma}^1_{11}. \hspace{1cm} (C.4)$$

Combining the relations (C.1)-(C.4) it results

$$\tilde{R}_{11} = \frac{1}{2} (\partial_0 g^{01}) \partial_0 \tilde{g}_{11} - \frac{1}{2} (\partial_0 g^{00}) (2 \partial_1 \tilde{g}_{01} - \partial_0 \tilde{g}_{11}) + \frac{1}{2} \partial_1 (g^{CB} \partial_1 \tilde{g}_{CB}) +$$

$$+ \frac{1}{4} (g^{01} \partial_0 \tilde{g}_{11})^2 + \frac{1}{4} (g^{CB} \partial_1 \tilde{g}_{BD}) (g^{DE} \partial_1 \tilde{g}_{CE}) - \frac{1}{2} (g^{01} \partial_0 \tilde{g}_{11} + g^{CB} \partial_1 \tilde{g}_{CB}) \tilde{\Gamma}^1_{11}. \hspace{1cm} (C.5)$$

Using the expression of $\partial_0 g^{00}$ (appendix A), the one of $\tilde{\Gamma}^1_{11}$ (appendix B) and that $\partial_1 (g^{01} \tilde{g}_{01}) = 0$ the expression of $\tilde{R}_{11}$ resumes in:

$$\tilde{R}_{11} = \frac{1}{4} g^{01} \tilde{g}^{CB} \partial_1 \tilde{g}_{CB} \partial_0 \tilde{g}_{11} + \frac{1}{2} \partial_1 (g^{CB} \partial_1 \tilde{g}_{CB}) +$$

$$\frac{1}{4} (g^{CB} \partial_1 \tilde{g}_{BD}) (g^{DE} \partial_1 \tilde{g}_{CE}) - \frac{1}{2} g^{01} \partial_0 \tilde{g}_{01} g^{CB} \partial_1 \tilde{g}_{CB}. \hspace{1cm} (C.6)$$

### C.2 Computation of $\tilde{R}_{1A}$

By definition

$$\tilde{R}_{1A} = \partial_A \tilde{\Gamma}^\gamma_{1A} = \partial_A \tilde{\Gamma}^\gamma_{1A} + \tilde{\Gamma}^T_{A\gamma T} \tilde{\Gamma}^T_{\delta 1} = \tilde{\Gamma}^T_{A\gamma T} \tilde{\Gamma}^T_{\delta 1} \tilde{\Gamma}^T_{1A}. \hspace{1cm} A \neq \gamma \neq T \neq 1,$$

Furthermore:

$$\partial_A \tilde{\Gamma}^\gamma_{1A} = \partial_A \tilde{\Gamma}^0_{1A} + \partial_A \tilde{\Gamma}^1_{1A} + \partial_A \tilde{\Gamma}^C_{1A}. \hspace{1cm} A \neq \gamma \neq 1 \neq C,$$

Using the properties of the metric and its inverse (3.3), (3.9), the term $\partial_0 \tilde{\Gamma}^0_{1A}$ simplifies in:

$$\partial_0 \tilde{\Gamma}^0_{1A} = \frac{1}{2} (\partial_0 g^{00}) (\partial_A \tilde{g}_{01} - \partial_0 \tilde{g}_{A1}) + \frac{1}{2} (\partial_0 g^{0C}) \partial_1 \tilde{g}_{CA} + \frac{1}{2} g^{01} \partial_0 \tilde{g}_{A1}. \hspace{1cm} (C.7)$$

The other terms resume in:

$$\partial_A \tilde{\Gamma}^C_{1A} = \frac{1}{2} \partial_A [g^{01} \partial_0 \tilde{g}_{11} + 2 \tilde{\Gamma}^1_{11} + g^{CB} \partial_1 \tilde{g}_{CB}], \hspace{1cm} (C.8)$$

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By definition:

\[ \partial_1 \tilde{\Gamma}^A_{1A} = \frac{1}{2} (\partial_1 \tilde{g}^{01})(\partial_1 \tilde{\gamma}^0_0 - \partial_0 \tilde{\gamma}^0_1 A) + \frac{1}{2} \tilde{g}^{01}(\partial_1^2 \tilde{\gamma}^0_0 - \partial_1^2 \tilde{\gamma}^0_1 A), \quad (C.9) \]

\[ \partial_C \tilde{F}^C_{1A} = \frac{1}{2} \partial_C [\tilde{g}^{CB} \partial_1 \tilde{g}_{AB}]. \quad (C.10) \]

About the term \( \tilde{E}_{1A} \equiv \tilde{\Gamma}^\delta_{A\gamma} \tilde{\gamma}^\gamma_\delta \tilde{\Gamma}^1_{1A} \), it simplifies in:

\[ \tilde{E}_{1A} = \frac{1}{4} (\tilde{g}^{01})^2 (\partial_0 \tilde{g}_{11}1) (\partial_0 \tilde{g}_{11A} + \partial_1 \tilde{g}_{10}) + \frac{1}{2} \tilde{g}^{01}(\partial_1 \tilde{g}_{10} - \partial_0 \tilde{g}_{1A}) \tilde{\Gamma}^1_{11} + \]

\[ \frac{1}{4} \tilde{g}^{01}(\partial_1 \tilde{g}_{10} - \partial_0 \tilde{g}_{1A}) \tilde{g}^{CB} \partial_1 \tilde{g}_{BA} - \frac{1}{4} \tilde{g}^{01}(\partial_0 \tilde{g}_{1A}) \tilde{g}^{CB} (\partial_0 \tilde{g}_{1B} - \partial_B \tilde{g}_{01}) + \]

\[ \frac{1}{4} \tilde{g}^{DE} \tilde{g}^{CB} (\partial_1 \tilde{g}_{DB})(\partial_1 \tilde{g}_{EC} + \partial_C \tilde{g}_{EA} - \partial_E \tilde{g}_{AC}) - \frac{1}{4} \tilde{g}^{01} \tilde{g}^{\gamma_\delta}(\partial_1 \tilde{g}_{1A} - \partial_0 \tilde{g}_{1A}) \]

\[ - \frac{1}{4} \tilde{g}^{\gamma_\delta}(\partial_C \tilde{g}_{\gamma_\delta})(\tilde{g}^{CB} \partial_1 \tilde{g}_{BA}). \quad (C.11) \]

Combining the expressions \( C.7 \) - \( C.11 \) one obtains:

\[ \tilde{R}_{1A} = \frac{1}{4} \tilde{g}^{01} (\partial_1 (\partial_0 \tilde{g}_{1A}) + \frac{1}{2} (\frac{1}{2} \tilde{g}^{CB} \partial_1 \tilde{g}_{CB} - \tilde{g}^{01} \partial_0 \tilde{g}_{10}) \partial_0 \tilde{g}_{11A} + \]

\[ \frac{1}{2} \tilde{g}^{01}(\partial_0 \tilde{g}_{11}1 - \tilde{g}^{01} \partial_0 \tilde{g}_{11} - \frac{1}{2} \tilde{g}^{CB} \partial_1 \tilde{g}_{CB} - \partial_A \tilde{g}_{10} - \frac{1}{2} \partial_C (\tilde{g}^{CB} \partial_1 \tilde{g}_{AB}) + \]

\[ \frac{1}{2} \partial_A (\tilde{g}^{CB} \partial_1 \tilde{g}_{CB}) - \frac{1}{2} \partial_A (\partial_0 \tilde{g}_{11}) - \frac{1}{4} \tilde{g}^{EF} \partial_C \tilde{g}_{EF})(\tilde{g}^{CB} \partial_1 \tilde{g}_{BA}) + \]

\[ \frac{1}{4} \tilde{g}^{DE} \tilde{g}^{CB} (\partial_1 \tilde{g}_{DB})(\partial_1 \tilde{g}_{EC} + \partial_C \tilde{g}_{EA} - \partial_E \tilde{g}_{AC}). \quad (C.12) \]

### C.3 Computation of \( \tilde{R}_{AB} \)

By definition:

\[ \tilde{R}_{AB} = \partial_B \tilde{\Gamma}^\gamma_A - \partial_A \tilde{\Gamma}^\gamma_B + \tilde{\Gamma}^\delta_A \tilde{\gamma}^\gamma_\delta - \tilde{\gamma}^\gamma_\delta \tilde{\Gamma}^\delta_A B. \]

The term \( \partial_0 \tilde{\Gamma}^\gamma_A \) splits as:

\[ \partial_0 \tilde{\Gamma}^\gamma_A = \partial_0 \tilde{\Gamma}^0_{A0} + \partial_1 \tilde{\Gamma}^1_{A1} + \partial_C \tilde{\Gamma}^C_{A}. \]

The properties of the trace of the metric and its inverse \( \tilde{g}^{00}, \tilde{g}^{01} \), induce that:

\[ \partial_0 \tilde{g}^{01} = \frac{1}{2} (\partial_0 \tilde{g}^{00})(- \partial_0 \tilde{g}_{AB}) + \frac{1}{2} (\tilde{g}^{BC})(\partial_B \tilde{g}_{CA} + \partial_A \tilde{g}_{CB} - \partial_C \tilde{g}_{BA}) - \]

\[ \frac{1}{2} (\partial_0 \tilde{g}^{01})(\partial_1 \tilde{g}_{1A}) + \frac{1}{2} \tilde{g}^{01}(\partial_0 \tilde{g}_{1A} + \partial_1 \tilde{g}_{AB} - \partial_B \tilde{g}_{01}), \quad (C.13) \]

\[ \partial_B \tilde{\Gamma}^\gamma_A = \frac{1}{2} \partial_B [\tilde{g}^{\gamma_\delta}(\partial_\gamma \tilde{\gamma}_{\delta A} + \partial_{\delta A} \tilde{\gamma}_{\gamma A} - \partial_\gamma \tilde{\gamma}_{\delta A})] = \frac{1}{2} \partial_B [\tilde{g}^{\gamma_\delta} \tilde{\partial}_A \tilde{\gamma}_{\gamma_\delta}], \quad (C.14) \]

\[ \partial_1 \tilde{\Gamma}^A_{1B} = - \frac{1}{2} (\partial_1 \tilde{g}^{01})(\partial_0 \tilde{g}_{1B}) - \frac{1}{2} \tilde{g}^{01}(\partial_1^2 \tilde{g}_{1B}) + \]

\[ \frac{1}{2} (\partial_1 \tilde{g}^{01})(\partial_1 \tilde{g}_{1B}) + \frac{1}{2} \tilde{g}^{01}(\partial_1^2 \tilde{g}_{1B}). \quad (C.15) \]
\[ \partial_C \tilde{\Gamma}^C_{BA} = \frac{1}{2} \partial_C [g^{CD} (\partial_B \tilde{g}_{DA} + \partial_A \tilde{g}_{DB} - \partial_D \tilde{g}_{AB})]; \]  
\[ (C.16) \]

Now, one is interested in \[ \tilde{E}_{AB} = \tilde{\Gamma}^0_A \tilde{\gamma}^0_B - \tilde{\Gamma}^\gamma_A \tilde{\gamma}^\gamma_B. \] Straightforward computations using the properties of the trace of the metric and its inverse \[(C.3), (C.4), \] and some of the expressions of the Christoffel symbols of the metric on \( C \) (appendix \( [B] \), lead to the following non-explicit expression:

\[ \tilde{E}_{AB} = \frac{1}{4} (g^{01})^2 (\partial_0 \tilde{g}_{1A} + \partial_A \tilde{g}_{01}) (\partial_0 \tilde{g}_{1B} - \partial_B \tilde{g}_{01}) - \frac{1}{4} (g^{CD} \partial_0 \tilde{g}_{DA}) (g^{01} \partial_0 \tilde{g}_{CB}) + \]

\[ \frac{1}{4} (g^{01})^2 (\partial_0 \tilde{g}_{01} - \partial_0 \tilde{g}_{11}) (\partial_0 \tilde{g}_{01} - \partial_0 \tilde{g}_{11}) - \frac{1}{4} (g^{01} g^{CD} (\partial_0 \tilde{g}_{DA}) (\partial_0 \tilde{g}_{CB} - \partial_0 \tilde{g}_{01} + \partial_0 \tilde{g}_{11})) + \]

\[ - \frac{1}{4} g^{01} g^{CD} (\partial_0 \tilde{g}_{AC}) (\partial_0 \tilde{g}_{BD}) - \frac{1}{4} g^{01} g^{CD} (\partial_0 \tilde{g}_{AC} - \partial_0 \tilde{g}_{01} (\partial_0 \tilde{g}_{BD}) + \]

\[ + \tilde{\Gamma}^D_{AC} \Gamma^C_{DB} + \frac{1}{4} (g^{01}) \partial_1 \tilde{g}_{AB} \left[ g^{01} (2 \partial_0 \tilde{g}_{01} - \partial_0 \tilde{g}_{11}) + g^{CE} \partial_0 \tilde{g}_{CE} + \right] + \]

\[ + \frac{1}{4} (g^{01}) \left[ g^{01} \partial_0 \tilde{g}_{11} + 2 \tilde{\Gamma}^1_{11} + g^{CB} (\partial_0 \tilde{g}_{BC}) \right] (\partial_0 \tilde{g}_{AB} - \partial_1 \tilde{g}_{AB}) - \left[ g^{01} (\partial_0 \tilde{g}_{01}) + \tilde{\Gamma}^D_{DC} \Gamma^C_{AB} \right]. \]
\[ (C.17) \]

Combining the relations \[(C.13)-(C.17), \] it follows:

\[ \tilde{R}_{AB} = \tilde{g}^{01} \partial_0 \tilde{g}_{BA} + \frac{1}{2} \tilde{g}^{01} (-g^{01} \tilde{g}_{01} B + \frac{1}{2} g^{CB} \partial_0 \tilde{g}_{CB} \partial_0 \tilde{g}_{AB} + \]

\[ \frac{1}{2} g^{01} \left[ \tilde{\Gamma}^0_{AC} \partial_0 \tilde{g}_{BD} + \tilde{\Gamma}^0_{BD} \partial_0 \tilde{g}_{AC} \right] + \frac{1}{2} \tilde{g}^{CE} \partial_0 \tilde{g}_{AB} - \tilde{g}_{01} \partial_0 \tilde{g}_{11} \tilde{\gamma}_{AB} + \]

\[ \frac{1}{4} (g^{01})^2 \partial_0 \tilde{g}_{01} \partial_0 \tilde{g}_{AB} + \frac{1}{2} \tilde{g}^{01} g^{CD} (\partial_0 \tilde{g}_{DA}) (\partial_0 \tilde{g}_{BC} - \partial_0 \tilde{g}_{CB}) - \]

\[ \frac{1}{2} \tilde{g}^{01} (\partial_0 \tilde{g}_{1A} + \partial_0 \tilde{g}_{0A} \partial_0 \tilde{g}_{AB}) - \frac{1}{2} \tilde{C} [g^{CD} (\partial_0 \tilde{g}_{DA} + \partial_0 \tilde{g}_{DB} - \partial_0 \tilde{g}_{AB}) + \]

\[ \frac{1}{2} \tilde{g}^{01} g^{CD} (\partial_0 \tilde{g}_{AC}) (\partial_0 \tilde{g}_{BD}) - \frac{1}{4} g^{01} g^{CE} (\partial_0 \tilde{g}_{AB} + \tilde{\Gamma}^D_{DC} \Gamma^C_{AB}) - \tilde{g}^{01} (\tilde{\gamma}_{01}) + \tilde{\Gamma}^D_{DC} \Gamma^C_{AB}. \]
\[ (C.18) \]

**C.4 Computation of \( \tilde{R}_{01} \)**

By definition:

\[ \tilde{R}_{01} = \partial_0 \tilde{\Gamma}^0_{0} \tilde{\gamma}_{0}^{0} + \partial_0 \tilde{\Gamma}^0_{0} \tilde{\gamma}_{1}^{0} + \tilde{\Gamma}^0_{0} \tilde{\gamma}_{0}^{0} - \tilde{\Gamma}^0_{0} \tilde{\gamma}_{0}^{0} + \tilde{\Gamma}^0_{0} \tilde{\gamma}_{0}^{0}. \]

Thanks to the same properties as previously mentioned, one has:

\[ \partial_0 \tilde{\Gamma}^0_{0} = \partial_0 [g^{01} \partial_0 \tilde{g}_{01}] + \frac{1}{2} \partial_0 [\tilde{g}^{01} \partial_0 \tilde{g}_{11}] + \frac{1}{2} \partial_0 [\tilde{g}^{AB} \partial_0 \tilde{g}_{AB}], \]
\[ (C.19) \]

\[ \partial_0 \tilde{\Gamma}^0_{1} = \frac{1}{2} (\partial_0 \tilde{g}^{00}) \partial_0 \tilde{g}_{00} + \frac{1}{2} (\partial_0 \tilde{g}^{10}) \partial_0 \tilde{g}_{01} + \frac{1}{2} (\partial_0 \tilde{g}^{11}) \partial_0 \tilde{g}_{11} + \]

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Exploiting the relations (C.19)-(C.21), it follows:

\[ \frac{1}{2}(\partial_t \tilde{g}^{01})\partial_0 \tilde{g}_{11} + \frac{1}{2}(\partial_t \tilde{g}^{0C})\partial_0 \tilde{g}_{1C} - \partial_C \tilde{g}_{10} \] + 
\[ \frac{1}{2}(\partial_D \tilde{g}^{DC})\partial_0 \tilde{g}_{1C} - \partial_C \tilde{g}_{10} \] + \frac{1}{2} \tilde{g}^{01} \partial_0 \tilde{g}_{11} + \frac{1}{2} \tilde{g}^{01} \partial_1 \tilde{g}_{01} + 
\[ \frac{1}{2} \tilde{g}^{11} \partial_1 \tilde{g}_{11} + \frac{1}{2} \tilde{g}^{CD} (\partial^2_{CD} \tilde{g}_{1D} - \partial^2_{CD} \tilde{g}_{10}). \] (C.20)

Now, setting \( \bar{E}_{01} \equiv \bar{\Gamma}^D_{17} \bar{\Gamma}^2_{00} - \bar{\Gamma}^7,_{80} \bar{\Gamma}^D_{01} \), one obtains after straightforward computations:

\[ \bar{E}_{01} = \frac{1}{4} \tilde{g}^{01} \tilde{g}^{CB} \tilde{A}_0 \tilde{g}_{1B} - 2 \tilde{A}_0 \tilde{g}_{1C} + \tilde{\nabla}_C \tilde{g}_{01} + \frac{1}{2} \tilde{g}^{11} (\partial_1 \tilde{g}_{00} - \partial_0 \tilde{g}_{11}) - 
\frac{1}{4} \tilde{g}^{01} \tilde{g}^{BC} \partial_0 \tilde{g}_{00} (\partial_0 \tilde{g}_{1C} + \tilde{\nabla}_C \tilde{g}_{01}) - \frac{1}{4} \tilde{g}^{01} \tilde{g}^{CB} \partial_0 \tilde{g}_{1C} (\partial_0 \tilde{g}_{11} - \partial_0 \tilde{g}_{01}) + 
\frac{1}{4} \tilde{g}^{01} \tilde{g}^{CE} (\partial_0 \tilde{g}_{BC}) (\partial_0 \tilde{g}_{DE}) - \frac{1}{4} \tilde{g}^{01} \tilde{g}^{CB} (\partial_0 \tilde{g}_{CB}) (\partial_0 \tilde{g}_{11}) - 
\frac{1}{4} (\partial_1 \tilde{g}_{00} - \partial_0 \tilde{g}_{11}) (\tilde{g}^{01} \partial_0 \tilde{g}_{11} + \tilde{g}^{CB} \partial_1 \tilde{g}_{CB} - 2 \tilde{\nabla}^1_{11}) - 
\frac{1}{2} \tilde{g}^{CB} \{ \tilde{g}^{01} (\partial_C \tilde{g}_{01}) + \frac{1}{2} \tilde{g}^{DE} (\partial_C \tilde{g}_{DE} + \partial_D \tilde{g}_{CE} - \partial_E \tilde{g}_{DC}) \} (\partial_0 \tilde{g}_{01} - \partial_0 \tilde{g}_{00}). \] (C.21)

Exploiting the relations (C.19)-(C.21), it follows:

\[ \bar{R}_{01} = \frac{1}{2} \tilde{g}^{01} \partial_0 \tilde{g}_{01} + \tilde{g}^{01} \partial_1 (\partial_0 \tilde{g}_{01}) - (\tilde{g}^{01})^2 (\partial_1 \tilde{g}_{00})(\partial_0 \tilde{g}_{01}) - 
\frac{7}{4} (\tilde{g}^{01})^2 (\partial_1 \tilde{g}_{00})(\partial_0 \tilde{g}_{11}) - \frac{1}{2} \tilde{g}^{01} \partial_1 (\partial_0 \tilde{g}_{11}) + \frac{1}{2} \tilde{g}^{AB} \partial_0 \tilde{g}_{AB} + 
(\tilde{g}^{01} \partial_0 \tilde{g}_{01})^2 - \frac{7}{4} (\tilde{g}^{01})^2 (\partial_0 \tilde{g}_{11})(\partial_0 \tilde{g}_{01}) + \frac{1}{4} (\tilde{g}^{01} \partial_0 \tilde{g}_{11})^2 - 
\frac{1}{4} \tilde{g}^{01} \tilde{g}^{CB} (\partial_0 \tilde{g}_{CB})(\partial_0 \tilde{g}_{01} - \partial_0 \tilde{g}_{11}) + \frac{1}{2} \tilde{g}^{01} \tilde{g}^{CB} \partial_0 \tilde{g}_{1C} \partial_0 \tilde{g}_{1B} - 
\frac{1}{2} (\partial_D \tilde{g}^{DC})(\partial_0 \tilde{g}_{1C} - \tilde{\nabla}_C \tilde{g}_{10}) - \frac{1}{2} \tilde{g}^{01} \partial_1 \tilde{g}_{11} + \frac{1}{2} \tilde{g}^{01} \partial_0 \tilde{g}_{11} + 
\frac{1}{2} \tilde{g}^{CD} \partial_0 \tilde{g}_{1D} - \frac{1}{2} \tilde{\nabla}_D \bar{\Gamma}_{01} - \frac{1}{2} \tilde{g}^{CD} \partial_0 \tilde{g}_{01} - \partial_0 \tilde{g}_{11} + 
\frac{1}{4} \tilde{g}^{DE} \partial_0 \tilde{g}_{DE} - \tilde{g}^{01} \tilde{g}^{CB} (\partial_0 \tilde{g}_{CB})(\partial_0 \tilde{g}_{11}) - 
\frac{1}{4} \tilde{g}^{CB} (\partial_0 \tilde{g}_{01} - \partial_0 \tilde{g}_{00}) \bar{\Gamma}^D_{DC} \bar{\Gamma}_{01}. \] (C.22)
C.5 Computation of $\tilde{R}_{00}$

By definition:

$$\tilde{R}_{00} = \partial_0 \tilde{\Gamma}_{\gamma 0} - \partial_{\gamma} \tilde{\Gamma}_{00} + \tilde{\Gamma}_{0\gamma}^{\delta} \tilde{\Gamma}_{\delta 0}^{\gamma} - \tilde{\Gamma}_{\gamma 0}^{\delta} \tilde{\Gamma}_{\delta 00}^{\gamma}.$$ 

After simplifications, one obtains:

$$\partial_0 \tilde{\Gamma}_{\gamma 0} = \frac{1}{2} (\partial_0 \tilde{g}^{\gamma 0}) \partial_0 \tilde{g}_{\gamma 0} + \tilde{g}^{\gamma 0} \partial_0^2 \tilde{g}_{00} + \frac{1}{2} \tilde{g}^{11} \partial_0^2 \tilde{g}_{11} + \frac{1}{2} \tilde{g}^{AB} \partial_0^2 \tilde{g}_{AB}. \tag{C.23}$$

On the other hand one has:

$$\partial_{\gamma} \tilde{\Gamma}_{00} = \frac{1}{2} (\partial_{\gamma} \tilde{g}^{\gamma 0}) (2 \partial_0 \tilde{g}_{00} - \partial_{\gamma} \tilde{g}_{00}) + \tilde{g}^{\gamma 0} \partial_0^2 \tilde{g}_{0\gamma} +$$

$$\tilde{g}^{11} \partial_0^2 \tilde{g}_{10} - \frac{1}{2} \tilde{g}^{11} \partial_0^2 \tilde{g}_{00} - \frac{1}{2} \tilde{g}^{AB} \partial_0^2 \tilde{g}_{AB}. \tag{C.24}$$

Concerning $\tilde{E}_{00} = \tilde{\Gamma}_{\gamma 0}^{\delta} \tilde{\Gamma}_{\delta 00}^{\gamma} - \tilde{\Gamma}_{\gamma 0}^{\delta} \tilde{\Gamma}_{\delta 00}^{\gamma}$, it resumes in:

$$\tilde{E}_{00} = \frac{1}{4} (\tilde{g}^{01})^2 \partial_0 \tilde{g}_{11} (\partial_1 \tilde{g}_{01} - \partial_0 \tilde{g}_{10}) + \frac{1}{4} (\tilde{g}^{01})^2 (\partial_1 \tilde{g}_{01} - \partial_0 \tilde{g}_{11})^2 -$$

$$\frac{1}{4} \tilde{g}^{01} \tilde{g}^{BC} \partial_B \tilde{g}_{00} (\partial_0 \tilde{g}_{1C} + \partial_C \tilde{g}_{01}) - \frac{1}{2} \tilde{g}^{01} \tilde{g}^{CB} (\partial_0 \tilde{g}_{1B} - \partial_B \tilde{g}_{01}) \partial_0 \tilde{g}_{1C}$$

$$- \frac{1}{4} \tilde{g}^{01} (2 \partial_0 \tilde{g}_{10} - \partial_{\gamma} \tilde{g}_{00}) \left[ \tilde{g}^{01} (\partial_1 \tilde{g}_{10} - \partial_0 \tilde{g}_{11}) + \tilde{g}^{CB} \partial_0 \tilde{g}_{BC} \right] +$$

$$\frac{1}{4} \tilde{g}^{DB} \tilde{g}^{CE} \partial_0 \tilde{g}_{BC} \partial_0 \tilde{g}_{ED} - \frac{1}{4} \tilde{g}^{01} (\partial_1 \tilde{g}_{01} - \partial_0 \tilde{g}_{10}) (2 \tilde{\Gamma}_{11} + \tilde{g}^{CB} \partial_0 \tilde{g}_{BC}) +$$

$$\frac{1}{4} \tilde{g}^{01} \tilde{g}^{CB} \partial_B \tilde{g}_{00} (\partial_0 \tilde{g}_{10} - \partial_0 \tilde{g}_{1C}) - \frac{1}{2} \tilde{g}^{DB} \partial_B \tilde{g}_{00} \tilde{\Gamma}_{CD}. \tag{C.25}$$

Combining the expressions above $\text{[C.23]}-\text{[C.24]}$ leads to:

$$\tilde{R}_{00} = -\frac{1}{2} \tilde{g}^{01} \partial_0^2 \tilde{g}_{11} + \frac{1}{2} \tilde{g}^{AB} \partial_0^2 \tilde{g}_{00} + \tilde{g}^{01} \partial_0^2 \tilde{g}_{01} - \frac{1}{2} \tilde{g}^{AB} (\partial_0^2 \tilde{g}_{0B} - \partial_0^2 \tilde{g}_{01}) -$$

$$\frac{1}{2} \tilde{g}^{01} \partial_0^2 \tilde{g}_{10} + \frac{1}{2} \tilde{g}^{01} \left( \tilde{g}^{11} \partial_0 \tilde{g}_{11} - 2 \tilde{g}^{01} \partial_1 \tilde{g}_{01} + \tilde{g}^{CB} \left( \frac{1}{2} \partial_1 \tilde{g}_{0C} - \partial_0 \tilde{g}_{0C} \right) \right) \partial_0 \tilde{g}_{01}$$

$$- \frac{1}{4} (\tilde{g}^{01})^2 (\partial_1 \tilde{g}_{10} - \partial_0 \tilde{g}_{11})^2 + \frac{1}{2} \tilde{g}^{01} \tilde{g}^{AC} \partial_0 \tilde{g}_{1A} \partial_0 \tilde{g}_{0C} - \frac{1}{4} \tilde{g}^{DA-CB} \partial_0 \tilde{g}_{BA} \partial_0 \tilde{g}_{CD} +$$

$$\frac{1}{4} (\tilde{g}^{01})^2 (\partial_1 \tilde{g}_{01}) (\partial_0 \tilde{g}_{11} + \frac{1}{2} (\tilde{g}^{01} \partial_1 \tilde{g}_{01})^2 + \frac{1}{2} (\tilde{g}^{01} \partial_0 \tilde{g}_{11} + \partial_A \tilde{g}_{AC}) \partial_0 \tilde{g}_{01}$$

$$- \frac{1}{4} \tilde{g}^{CA} (\partial_1 \tilde{g}_{BC} - \partial_0 \tilde{g}_{BC}) \partial_0 \tilde{g}_{01} + \frac{1}{2} \tilde{g}^{DB} (\partial_B \tilde{g}_{01}) \tilde{\Gamma}_{CD}. \tag{C.26}$$

We do not give the expression of $\tilde{R}_{0A}$ here since it is not used directly in the construction of the constraints equations.
Appendix D  Computation of $\frac{\partial}{\partial y^\nu}(\tilde{\Gamma}^0 + \tilde{\Gamma}^1)$ on $\mathcal{C}$

By definition one has:

$$\frac{\partial}{\partial y^0}(\tilde{\Gamma}^0 + \tilde{\Gamma}^1) = \frac{\partial}{\partial y^0}[\tilde{g}^{\mu\nu}(\tilde{\Gamma}^0_{\mu\nu} + \tilde{\Gamma}^1_{\mu\nu})]$$

$$= (\frac{\partial}{\partial y^\beta}\tilde{g}^{\mu\nu})(\tilde{\Gamma}^0_{\mu\nu} + \tilde{\Gamma}^1_{\mu\nu}) + \tilde{g}^{\mu\nu} \frac{\partial}{\partial y^0}(\tilde{\Gamma}^0_{\mu\nu} + \tilde{\Gamma}^1_{\mu\nu}),$$

according to the properties of the trace of the metric and its inverse on the cone $\mathcal{C}$, one has on $\mathcal{C}$

$$\frac{\partial}{\partial y^0}(\tilde{\Gamma}^0_{01} + \tilde{\Gamma}^1_{01}) = 2\tilde{g}^{01}\frac{\partial}{\partial y^0}(\tilde{\Gamma}^0_{01} + \tilde{\Gamma}^1_{01}) + \tilde{g}^{11}\frac{\partial}{\partial y^0}(\tilde{\Gamma}^0_{11} + \tilde{\Gamma}^1_{11}) +$$

$$\tilde{g}^{AB}\frac{\partial}{\partial y^0}(\tilde{\Gamma}^0_{AB} + \tilde{\Gamma}^1_{AB}) + (\frac{\partial}{\partial y^0})(\tilde{\Gamma}^0_{\mu\nu} + \tilde{\Gamma}^1_{\mu\nu}).$$

Furthermore,

$$\frac{\partial}{\partial y^0}(\tilde{\Gamma}^0_{01} + \tilde{\Gamma}^1_{01}) =$$

$$\frac{1}{2} \frac{\partial}{\partial y^0}[\tilde{g}^{\mu\nu}(\partial_0\tilde{g}_{\mu1} + \partial_1\tilde{g}_{0\mu} - \partial_\mu\tilde{g}_{01})] + \tilde{g}^{\mu\nu}(\partial_0\tilde{g}_{1\mu} + \partial_1\tilde{g}_{0\mu} - \partial_\mu\tilde{g}_{01}) +$$

$$\frac{1}{2}(\tilde{g}^{\mu\nu} + \tilde{g}^{1\mu})(\partial_0\tilde{g}_{1\mu} + \partial_1\tilde{g}_{0\mu} - \partial_\mu\tilde{g}_{01}),$$

using the expressions of the trace of the metric and its inverse [3.8], [3.9], the relations [A.1] of appendix [A] this expression simplifies:

$$\frac{\partial}{\partial y^0}(\tilde{\Gamma}^0_{01} + \tilde{\Gamma}^1_{01}) = -\frac{1}{2}(\tilde{g}^{01})^2\partial_0\tilde{g}_{01}\partial_1\tilde{g}_{01} + \frac{1}{2}\tilde{g}^{01}\partial_0^2\tilde{g}_{00}. \quad \text{(D.1)}$$

The term $\frac{\partial}{\partial y^\nu}(\tilde{\Gamma}^0_{11} + \tilde{\Gamma}^1_{11})$ in turn reads:

$$\frac{\partial}{\partial y^0}(\tilde{\Gamma}^0_{11} + \tilde{\Gamma}^1_{11}) = \frac{1}{2}(\tilde{g}^{0\nu} + \tilde{g}^{1\nu})(2\partial_1\tilde{g}_{1\mu} - \partial_\mu\tilde{g}_{11}) +$$

$$\frac{1}{2}(\tilde{g}^{0\nu} + \tilde{g}^{1\nu})(2\partial_0^2\tilde{g}_{1\mu} - \partial_\mu\tilde{g}_{11}).$$

After simplifications thanks to the same arguments as above, this expression results in:

$$\frac{\partial}{\partial y^0}(\tilde{\Gamma}^0_{11} + \tilde{\Gamma}^1_{11}) = -\frac{1}{2}(\tilde{g}^{01})^2\partial_0\tilde{g}_{01}(2\partial_1\tilde{g}_{01} - \partial_0\tilde{g}_{11}) + \frac{1}{2}\tilde{g}^{01}(2\partial_0^2\tilde{g}_{01} - \partial_0^2\tilde{g}_{11}). \quad \text{(D.2)}$$

Concerning the terms $\frac{\partial}{\partial y^\nu}(\tilde{\Gamma}^0_{AB} + \tilde{\Gamma}^1_{AB})$, computations and various simplifications result in:

$$\frac{\partial}{\partial y^0}(\tilde{\Gamma}^0_{AB} + \tilde{\Gamma}^1_{AB}) = \frac{1}{2}(\tilde{g}^{01})^2\partial_0\tilde{g}_{0AB}(2\partial_1\tilde{g}_{0AB} + \partial_1\tilde{g}_{AB}0) +$$

$$\frac{1}{2}\tilde{g}^{01}(\partial_0^2\tilde{g}_{0AB} + \partial_0^2\tilde{g}_{0AB}). \quad \text{(D.3)}$$
Due to the fact that on the cone one has: $\partial_0 \tilde{g}_{0A} = 0$, $\partial_0 \tilde{g}_{00} = \partial_0 \tilde{g}_{01}$, it follows also that $\partial^2_{A0} \tilde{g}_{0B} = 0 = \partial^2_{0B} \tilde{g}_{0A}$, $\partial^2_{01} \tilde{g}_{00} - \partial^2_{01} \tilde{g}_{01} = 0$, and therefore at this step one has:

$$ \frac{\partial}{\partial y^\mu} (\Gamma^0 + \Gamma^1) = \left\{ \begin{array}{l}
\frac{1}{2} \left( \tilde{g}^{01} \right)^2 \partial_{00} \tilde{g}_{11} - \frac{1}{2} \tilde{g}^{01} \tilde{g}^{AB} (\partial_{00} \tilde{g}_{AB}) - \frac{1}{2} \left( \tilde{g}^{01} \right)^2 \partial_0 \tilde{g}_{01} \partial_0 \tilde{g}_{11} + \\
\frac{1}{2} \left( \tilde{g}^{01} \right)^2 \tilde{g}^{AB} \partial_0 \tilde{g}_{01} \partial_0 \tilde{g}_{AB} + (\frac{\partial g_{\mu\nu}}{\partial y^\mu}) (\Gamma^0_{\mu\nu} + \Gamma^1_{\mu\nu}).
\end{array} \right. $$(D.4)

Now, we are interested of the term $X_1$:

$$ X_1 = (\frac{\partial \tilde{g}^{\mu\nu}}{\partial y^\mu}) (\tilde{\Gamma}^0_{\mu\nu} + \tilde{\Gamma}^1_{\mu\nu}). $$

Although this term does not contain second order outgoing derivatives, it is important to highlight in it the presence of the term $\partial_0 \tilde{g}_{01}$. In virtue of the expressions of the Christoffel symbols of the metric on $C$ of appendix[A] the following computations hold on $C$:

$$ \tilde{\Gamma}^0_{00} + \tilde{\Gamma}^1_{00} = \frac{1}{2} \tilde{g}^{10} \partial_0 \tilde{g}_{00}, \quad \tilde{\Gamma}^0_{01} + \tilde{\Gamma}^1_{01} = \frac{1}{2} \tilde{g}^{10} \partial_1 \tilde{g}_{00}, $$

$$ \tilde{\Gamma}^0_{11} + \tilde{\Gamma}^1_{11} = \frac{1}{2} \tilde{g}^{10} (2 \partial_1 \tilde{g}_{01} - \partial_0 \tilde{g}_{11}), \quad \tilde{\Gamma}^0_{0A} + \tilde{\Gamma}^1_{0A} = \frac{1}{2} \tilde{g}^{10} \partial_A \tilde{g}_{00}, $$

$$ \tilde{\Gamma}^0_{1A} + \tilde{\Gamma}^1_{1A} = \frac{1}{2} \tilde{g}^{10} (\partial_A \tilde{g}_{10} - \partial_0 \tilde{g}_{1A}), \quad \tilde{\Gamma}^0_{AB} + \tilde{\Gamma}^1_{AB} = -\frac{1}{2} \tilde{g}^{10} \partial_A \tilde{g}_{AB}. $$

Exploiting these expressions, the properties of the trace of the metric and its inverse (3.8), (3.9), some relations of (A.1) (appendix[A]), $X_1$ expresses as:

$$ X_1 = \left\{ \begin{array}{l}
-\frac{1}{4} \left( \tilde{g}^{01} \right)^3 \partial_0 \tilde{g}_{11} \partial_0 \tilde{g}_{00} + \frac{1}{2} \tilde{g}^{01} \partial_0 \tilde{g}_{01} \partial_0 \tilde{g}_{11} (2 \partial_1 \tilde{g}_{01} - \partial_0 \tilde{g}_{11}) + \\
\left( \tilde{g}^{01} \right)^2 \tilde{g}^{AC} \partial_0 \tilde{g}_{01} \partial_0 \tilde{g}_{1A} + \frac{1}{2} \tilde{g}^{01} \tilde{g}^{AC} \tilde{g}^{BD} \partial_0 \tilde{g}_{AB} \partial_0 \tilde{g}_{CD}.
\end{array} \right. $$

and finally simplifies to:

$$ X_1 = \left\{ \begin{array}{l}
-\frac{1}{4} \left( \tilde{g}^{01} \right)^3 \partial_0 \tilde{g}_{11} \partial_0 \tilde{g}_{00} + \frac{1}{2} \tilde{g}^{01} \partial_0 \tilde{g}_{01} \partial_0 \tilde{g}_{11}^2 + \\
\left( \tilde{g}^{01} \right)^2 \tilde{g}^{AC} \partial_0 \tilde{g}_{01} \partial_0 \tilde{g}_{1A} + \frac{1}{2} \tilde{g}^{01} \tilde{g}^{AC} \tilde{g}^{BD} \partial_0 \tilde{g}_{AB} \partial_0 \tilde{g}_{CD}.
\end{array} \right. $$(D.5)

Combining the expressions (D.4), (D.5), one ends up by:

$$ \frac{\partial}{\partial y^\mu} (\Gamma^0 + \Gamma^1) = \frac{1}{2} \left( \tilde{g}^{01} \right)^2 \partial_{00} \tilde{g}_{11} - \frac{1}{2} \tilde{g}^{01} \tilde{g}^{AB} (\partial_{00} \tilde{g}_{AB}) - \frac{3}{2} \left( \tilde{g}^{01} \right)^3 \partial_0 \tilde{g}_{01} \partial_0 \tilde{g}_{11} + \\
\frac{1}{2} \left( \tilde{g}^{01} \right)^3 (\partial_0 \tilde{g}_{11})^2 + \frac{1}{2} \left( \tilde{g}^{01} \right)^2 \partial_0 \tilde{g}_{01} \tilde{g}^{AB} \partial_0 \tilde{g}_{AB} - \\
\left( \tilde{g}^{01} \right)^2 \tilde{g}^{AC} \partial_0 \tilde{g}_{01} \partial_0 \tilde{g}_{1A} + \frac{1}{2} \tilde{g}^{01} \tilde{g}^{AC} \tilde{g}^{BD} \partial_0 \tilde{g}_{AB} \partial_0 \tilde{g}_{CD}. $$ (D.6)

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