Adiabatic Evolution of Three ‘Constants’ of Motion for Greatly Inclined Orbits in Kerr Spacetime

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General orbits of a particle of small mass $\mu$ around a Kerr black hole of mass $M$ are characterized by three parameters: the energy, the angular momentum and the Carter constant. The time-averaged rates of change of the energy and the angular momentum can be obtained by computing the corresponding fluxes of gravitational waves emitted by the particle. By contrast, the time-averaged rate of change of the Carter constant cannot be expressed as a flux of gravitational waves. Recently a method to compute this rate of change was proposed by Mino, and we refined it into a simplified form. In this paper we further extend our previous work to give a new formulation without the aid of expansion in terms of a small inclination angle.

§1. Introduction

Thanks to recent advances in technology, an era of gravitational wave astronomy has almost arrived. There are already several large-scale laser interferometric gravitational wave detectors in operation. Among them are TAMA\textsuperscript{300\textsuperscript{,1}} LIGO\textsuperscript{,2} GEO-600\textsuperscript{,3} and VIRGO\textsuperscript{.4} The primary targets of these ground-based detectors are inspiralling compact binaries, which are expected to be detected in the near future.

There are also projects for space-based interferometric detectors. LISA is now on its research and development stage\textsuperscript{,5} and there is a future plan called DE-CIGO/BBO\textsuperscript{,6,7} These space-based detectors can detect gravitational waves from solar-mass compact objects orbiting supermassive black holes. To extract physical information concerning such binary systems, it is essential to know the theoretical gravitational waveforms with sufficient accuracy. The black hole perturbation approach is most suited for this purpose. In this approach, one considers gravitational waves emitted by a point particle that represents a compact object orbiting a black hole, assuming the mass of the particle $\mu$ is much less than that of the black hole $M$, i.e. $\mu \ll M$.

To lowest order in the mass ratio $\mu/M$, the orbit of the particle is a geodesic on the background geometry of a black hole. Already at this lowest order, combined
with the assumption of energy and angular momentum balance between the emitted gravitational waves and the orbital motion, this approach has proved to be very powerful in evaluating general relativistic corrections to the gravitational waveforms, even for neutron star-neutron star binaries, for which the assumption of this approach is maximally violated.\(^8\)

However, the deviation from the geodesic cannot be completely specified by the rates of change of the energy and angular momentum. In order to describe general orbits, we need to know the evolution of the third “constant” of motion, i.e. the Carter constant \(Q\). For this purpose, we need to evaluate the gravitational self-force acting on the particle directly. Here, the gravitational self-force is the force due to the metric perturbation caused by the particle itself.

Roughly speaking, there are two levels in the computation of the self-force in the linear approximation.\(^9\) The advanced level is the direct computation of the time-dependent self-force, in which one computes the self-force without any further approximation. The main problem here is that the full (bare) metric perturbation diverges at the location of the particle, which is assumed to be point-like, and hence so does the self-force. About a decade ago, a formal expression for the gravitational self-force was found, which contains the expression for the Green function divided into two parts: the direct part and the tail part.\(^10,11\) Later, this formula was reformulated in a more sophisticated manner by Detweiler and Whiting.\(^12\) In this new formulation, the direct part is replaced with the S part and the tail part with the R part. The R part has the improved property that it is a solution of source-free linearized Einstein equations. Thus, what we have to do to obtain a meaningful self-force is to compute the R part of the metric perturbation. The “mode sum” scheme, a practical calculation method for the R part, has been developed.\(^13\)–\(^19\) There are several implementations of this scheme in a scalar toy model: Ref. 20) for a static particle, Refs. 22) and 21) for radial infall, and Refs. 23), 24), 25) and 26) for a circular orbit. Extensions to the gravitational case are made in Ref. 27) for radial infall, and in Ref. 28) for a circular orbit.

Despite recent progress in the direct computation of the self-force, there seem to remain obstacles in computing it for general orbits on the Kerr background. Furthermore, all results obtained from the computation of the self-force are not equally meaningful. A few time-averaged combinations composed of the self-force are important for the prediction of the gravitational waveform. The most important quantities are the time-averaged rates of change of the “constants” of motion. Here a constant of motion means the quantity, such as the energy or angular momentum, that stays constant for background geodesics. Hence, the other, easier level of computing the self-force is to compute these rates of change of the “constants” of motion. Many years ago, Gal’tsov\(^29\) advocated using the radiative part of the metric perturbation to calculate \(dE/dt\) and \(dL/dt\). The radiative field is defined as half the retarded field minus half the advanced field. The advantage of using the radiative field is that this force is relatively easy to compute and is free from divergence. Hence we do not have to worry about how to subtract out the divergent part. Gal’tsov showed that the calculation using the radiative field correctly reproduces the results obtained by using the balance argument for \(dE/dt\) and \(dL/dt\) when they are averaged over an
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Recently, Mino directly verified the validity of applying the same scheme to the computation of the time averaged rates of change of the constants of motion including the Carter constant.\(^{30}\)

From the equivalence of the two calculations for \(dE/dt\) and \(dL/dt\) shown by Gal’tsov, we see that, in order to obtained time-averaged rates of change of the energy and angular momentum, all we have to do is to compute the fluxes of the energy and angular momentum evaluated at infinity and on the black hole horizon by using the Teukolsky formalism. Therefore, it is expected that the same might be true for the Carter constant. Recently, we have shown that this is indeed the case, obtaining a new simplified formula of the time-averaged rate of change for \(dQ/dt\) written in terms of the asymptotic amplitude of gravitational waves.\(^{31}\) This formula requires some knowledge of the particle orbit. In this sense, the rate of change is not expressed as a flux determined by the asymptotic form of gravitational waves. In our previous paper, using this new formula, we gave analytic expressions for the time-averaged rates of change of the energy, the angular momentum and the Carter constant in the post-Newtonian expansion.\(^{32}\) There, we also made use of the expansion in terms of the orbital inclination angle for a technical reason. Here, we extend our previous results, eliminating the limitation to small inclination angles.

This paper is organized as follows. In §2 we review the basic formalism about how to analytically compute the rates of change of the constants of motion due to gravitational wave emission for a particle orbiting a Kerr black hole, which is developed in Ref. 32). The formulas for the rates of change are written in terms of the amplitude of each partial wave of the emitted gravitational waves. In §3 we explain our new method for the analytic evaluation of this amplitude for a general orbit with a large inclination angle. In §4 substituting the results obtained in the preceding section into the formulas described in §2 we compute the time-averaged rates of change, \(dE/dt, dL/dt,\) and \(dQ/dt\). We also compute the phase evolution of gravitational waves. In §5 we summarize this paper.

§2. Basic formulation for adiabatic radiation reaction

In this section we give a brief review on the Teukolsky formalism\(^ {33},34)\) as well as the basic formulas obtained in Ref. 31). We consider the background Kerr spacetime in the Boyer-Lindquist coordinates:

\[
ds^2 = - \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar \sin^2 \theta}{\Sigma} dtd\varphi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r}{\Sigma} \sin^2 \theta\right) \sin^2 \theta d\varphi^2, \tag{2.1}\]

where

\[
\Sigma \equiv r^2 + a^2 \cos^2 \theta, \quad \Delta \equiv r^2 - 2Mr + a^2. \tag{2.2}\]

Here, \(M\) and \(a\) are the mass and the angular momentum of the black hole, respectively. In the Teukolsky formalism, the gravitational perturbation of a Kerr black
hole is described by a master variable \( \psi \), which satisfies the master equation

\[
_sO \psi_s = 4\pi \Sigma_s T,
\]

where

\[
sO \equiv -\left[ \frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta \right] \frac{d}{dr} - \frac{4Mar}{\Delta} \partial_r \partial_\varphi - \left[ \frac{a^2}{\Delta} - \frac{1}{\sin^2 \theta} \right] \frac{d^2}{d\varphi^2}
\]

\[
+ \Delta^{-s} \partial_r (\Delta^{s+1} \partial_r) + \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + 2s \left[ \frac{a(r - M)}{\Delta} + \frac{i \cos \theta}{\sin^2 \theta} \right]
\]

\[
\partial_\varphi + 2s \left[ \frac{M(r^2 - a^2)}{\Delta} - r - ia \cos \theta \right] \partial_t - s(s \cot^2 \theta - 1),
\]

and \( sT \) is the source term. The master variable \( \psi \) is equal to \( \psi_0 \) for \( s = 2 \) and \( (r - ia \cos \theta)^4 \psi_4 \) for \( s = -2 \), where

\[
\psi_0 \equiv -C_{\alpha\beta\gamma\delta}^\ell m^\delta \ell^\ell m^\delta, \quad \psi_4 \equiv -C_{\alpha\beta\gamma\delta}^\ell n^\alpha n^\beta n^\gamma n^\delta
\]

are the so-called Newman-Penrose quantities. Here, \( C_{\alpha\beta\gamma\delta} \) is the Weyl tensor, and the null vectors \( \ell, n, m \) are defined by

\[
\ell^\mu \equiv \left( (r^2 + a^2), \Delta, 0, a \right) / \Delta, \quad n^\mu \equiv \left( (r^2 + a^2), -\Delta, 0, a \right) / (2\Sigma),
\]

\[
m^\mu \equiv (ia \sin \theta, 0, 1, i / \sin \theta) / (\sqrt{2}(r + ia \cos \theta)).
\]

The bar denotes complex conjugation.

The master equation (2.3) can be solved by decomposing the master variable \( \psi \) as

\[
\psi = \sum_A \int d\omega s_{RA}(r)s_{SA}(\theta)e^{im\varphi}e^{-i\omega t},
\]

where \( A \) represents a set of separation constants, \( \{ \ell, m, \omega \} \). The equations for the radial and angular parts can be separated, and we obtain

\[
\left[ \Delta^{-s} \frac{d}{dr} \left( \Delta^{s+1} \frac{d}{dr} \right) + \left( \frac{K^2 - 2is(r - M)K}{\Delta} + 4is \omega r - \lambda \right) \right] s_{RA}(r) = s_{TA},
\]

\[
\left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) - a^2 \omega^2 \sin^2 \theta
\right]
\]

\[
- \left( \frac{(m + s \cos^2 \theta)^2}{\sin^2 \theta} - 2a s \omega \cos \theta + s + 2am \omega + \lambda \right) s_{SA}(\theta) = 0,
\]

where

\[
K \equiv (r^2 + a^2) \omega - ma,
\]

and \( \lambda \) is the eigenvalue determined by the equation for \( s_{SA} \). The angular function \( s_{SA} \) is called the spin-weighted spheroidal harmonic, which is usually normalized as

\[
\int_0^\pi (s_{SA})^2 \sin \theta d\theta = 1.
\]
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We define two independent homogeneous solutions of the radial Teukolsky equation:

\[ sR_A^{in} \rightarrow \begin{cases} 
B_A^{\text{trans}} e^{-iKr^*}, & \text{for } r \rightarrow r_+, \\
\rho_{2s-1} B_A^{\text{inc}} e^{-iKr^*}, & \text{for } r \rightarrow +\infty, 
\end{cases} \]
\[ sR_A^{up} \rightarrow \begin{cases} 
C_A^{\text{up}} e^{iKr^*} + \Delta^{-s} C_A^{\text{ref}} e^{-iKr^*}, & \text{for } r \rightarrow r_+, \\
\rho_{2s-1} C_A^{\text{trans}} e^{iKr^*}, & \text{for } r \rightarrow +\infty. 
\end{cases} \]  

(2.10)

Here, \( k \equiv \omega - ma/2Mr_+ \), and \( r^* \) is the tortoise coordinate defined by

\[ r^* \equiv r + \frac{2Mr_+}{r_+ - r_-} \ln \frac{r - r_+}{2M} - \frac{2Mr_-}{r_+ - r_-} \ln \frac{r - r_-}{2M}, \]  

(2.11)

with \( r_\pm \equiv M \pm \sqrt{M^2 - a^2} \). A systematic analytic method to compute homogeneous solutions for \( sR_A^{s} \) and \( sS_A^{s} \) in the post-Newtonian expansion has been developed in Refs. 35 and 36).

The explicit form of the source term \( T_A \) is given by

\[ T_A = 4 \int d\Omega dt \rho^{-5/2} (B_2' + B_2'^*) e^{-im\phi + i\omega t + 2sA_{\Delta}} \frac{2S_{\Delta}}{\sqrt{2\pi}}, \]  

(2.12)

where

\[ B_2' = -\frac{1}{2} \rho^4 \rho L_{-1} \left[ \rho^{-4} L_0 (\rho^{-2} \rho T_{nn}) \right] - \frac{1}{2\sqrt{\pi}} \rho^4 \rho^2 \Delta L_{-1} \left[ \rho^{-4} \rho^2 J_+ (\rho^{-2} \rho^2 \Delta^{-1} T_{mm}) \right], \]

\[ B_2'^* = -\frac{1}{4} \rho^4 \rho \Delta J_+ \left[ \rho^{-4} J_+ (\rho^{-2} \rho T_{mm}) \right] - \frac{1}{2\sqrt{\pi}} \rho^4 \rho^2 \Delta J_+ \left[ \rho^{-4} \rho^2 \Delta^{-1} L_{-1} (\rho^{-2} \rho^2 T_{mm}) \right], \]  

(2.13)

with

\[ \rho = (r - ia \cos)^{-1}, \quad L_s = \partial_\theta + \frac{m}{\sin \theta} - a \omega \sin \theta + s \cot \theta, \quad J_+ = \partial_r + \frac{iK}{\Delta}. \]

In the above, \( T_{nn}, T_{mm} \) and \( T_{mm} \) are the tetrad components of the energy-momentum tensor. Here and hereafter, for simplicity, we restrict our attention to the \( s = -2 \) case, and we omit the subscript \( s \).

We solve the radial Teukolsky equation using the Green function method. A solution of the Teukolsky equation that is purely out-going at infinity and is purely in-going on the horizon is given by

\[ R_A = \frac{1}{W_A} \left\{ R_A^{up} \int_{r_+}^r dr' R_A^{in} T_A \Delta^{-2} + R_A^{in} \int_r^\infty dr' R_A^{up} T_A \Delta^{-2} \right\}, \]  

(2.14)

where the Wronskian \( W_A \) is given by

\[ W_A = 2i\omega C_A^{\text{trans}} B_A^{\text{inc}}. \]  

(2.15)
Then, the solution has the following asymptotic behavior near the horizon:

\[ R_A(r \to r_+) \to \frac{B^{trans}}{2i\omega C_A^{trans} P^{inc}_A} \int_{r_+}^{\infty} dr'R_A^{up} T_A \Delta^{-2} \equiv Z^H_A \Delta^2 e^{-ikr}. \]  

(2.16)

Here we defined \( Z^H_A \), the amplitude of each partial wave labelled by \( A \). In the \( r \to \infty \) limit, the solution takes the form

\[ R_A(r \to \infty) \to \frac{r^3 e^{i\omega r}}{2i\omega B^{inc}_A} \int_{r_+}^{\infty} dr'T_A R_A^{in} \Delta^2 \equiv Z^\infty_A r^3 e^{i\omega r}. \]  

(2.17)

Hereafter, we focus on the gravitational waves emitted to infinity, partly because the horizon in-going wave does not give the dominant contribution in the present calculation, and partly because the extension is rather straightforward.

We consider the motion of a point particle, whose coordinates are \( z^\alpha = (t_z(\tau), r_z(\tau), \theta_z(\tau), \varphi_z(\tau)) \). Here \( \tau \) is the proper time along the orbit. For geodesic motion, there are three constants of motion,

\[ E \equiv -u^\tau \xi^\tau = \left(1 - \frac{2Mr_z}{\Sigma} \right) u^\tau + \frac{2Mar_z \sin^2 \theta_z}{\Sigma} u^\varphi, \]

\[ L \equiv u^\alpha \xi^\alpha - \frac{2Mar_z \sin^2 \theta_z}{\Sigma} u^\varphi + \frac{(r_z^2 + a^2)^2 - \Delta a^2 \sin^2 \theta_z}{\Sigma} \sin^2 \theta_z u^\psi, \]

\[ Q \equiv K_{\alpha\beta} u^\alpha u^\beta = \frac{(L - aE \sin^2 \theta_z)^2}{\sin^2 \theta_z} + a^2 \cos^2 \theta_z + \Sigma^2 (u^\theta)^2, \]  

(2.18)

where \( u^\alpha \equiv dz^\alpha/d\tau \) is the four velocity, and \( K_{\alpha\beta} \) is the Killing tensor defined by

\[ K_{\alpha\beta} \equiv 2\Sigma \ell_{(\alpha} n_{\beta)} + r^2 g_{\alpha\beta}. \]  

(2.19)

We often use alternative notation for the Carter constant, \( C \equiv Q - (aE - L)^2 \). For orbits on the equatorial plane, \( C \) vanishes. Using these constants of motion and a new parameter along the trajectory \( \lambda \) defined by \( d\lambda = d\tau/\Sigma \), the geodesic equations become

\[ \left( \frac{dr_z}{d\lambda} \right)^2 = R(r_z), \quad \frac{dt_z}{d\lambda} = -a(aE \sin^2 \theta_z - L) + \frac{r_z^2 + a^2}{\Delta} P(r_z), \]

\[ \left( \frac{d\cos \theta_z}{d\lambda} \right)^2 = \Theta(\cos \theta_z), \quad \frac{d\varphi_z}{d\lambda} = -aE + \frac{L}{\sin^2 \theta_z} + \frac{a}{\Delta} P(r_z), \]  

(2.20)

where

\[ P(r) = E(r^2 + a^2) - aL, \quad R(r) = [P(r)]^2 - \Delta [r^2 + Q], \]

\[ \Theta(\cos \theta) = C - (C + a^2 (1 - E^2) + L^2) \cos^2 \theta + a^2 (1 - E^2) \cos^4 \theta. \]  

(2.21)

It should be noted that the equations for the \( r \)-component and the \( \theta \)-component are decoupled when they are written in terms of \( \lambda \). Both \( R \) and \( \Theta \) are quartic functions of their arguments. Hence, both solutions are given by elliptic functions. Taking the amplitude of the radial oscillation as a small parameter, we can systematically
expand the radial solution $r_z(\lambda)$ as a Fourier series. For the motion in the $\theta$-direction, perturbative solutions in Fourier series can be systematically obtained even for a large inclination angle by taking the coefficient of the quartic term $a^2(1 - \mathcal{E}^2)$ as a small parameter. As we explain below, this expansion is a part of the post-Newtonian expansion. We stress that it is not necessary to restrict the amplitude of oscillation, $|\theta - \pi/2|$, to small values in this expansion.

The other two equations in (2.20) are integrated as

\[
\begin{align*}
t_z(\lambda) &= t^{(r)}(\lambda) + t^{(\theta)}(\lambda) + \left\langle \frac{dt_z}{d\lambda} \right\rangle_{\lambda}, \\
\varphi_z(\lambda) &= \varphi^{(r)}(\lambda) + \varphi^{(\theta)}(\lambda) + \left\langle \frac{d\varphi_z}{d\lambda} \right\rangle_{\lambda}, \quad (2.22)
\end{align*}
\]

where $\langle \cdots \rangle_{\lambda} \equiv \lim_{\Delta \lambda \to \infty} (2\Delta \lambda)^{-1} \int_{-\Delta \lambda}^{\Delta \lambda} d\lambda \cdots$ represents the time average along the geodesic, and $t^{(r)}$ and $t^{(\theta)}$, which are defined by

\[
\begin{align*}
t^{(r)}(\lambda) &\equiv \int d\lambda \left\{ \frac{(r_z^2 + a^2)P(r_z)}{\Delta} - \left\langle \frac{(r_z^2 + a^2)P(r_z)}{\Delta} \right\rangle_{\lambda} \right\}, \\
t^{(\theta)}(\lambda) &\equiv -\int d\lambda \left\{ a^2 \mathcal{E} \sin^2 \theta_z - \left\langle a^2 \mathcal{E} \sin^2 \theta_z \right\rangle_{\lambda} \right\} \quad (2.23)
\end{align*}
\]

are periodic functions of periods $2\pi/\Omega_r$ and $2\pi/\Omega_\theta$, respectively. The functions $\varphi^{(r)}$ and $\varphi^{(\theta)}$ are defined in the same way,

\[
\begin{align*}
\varphi^{(r)}(\lambda) &\equiv \int d\lambda \left\{ \frac{aP(r_z)}{\Delta} - \left\langle \frac{aP(r_z)}{\Delta} \right\rangle_{\lambda} \right\}, \\
\varphi^{(\theta)}(\lambda) &\equiv \int d\lambda \left\{ \frac{\mathcal{L}}{\sin^2 \theta_z} - \left\langle \frac{\mathcal{L}}{\sin^2 \theta_z} \right\rangle_{\lambda} \right\}. \quad (2.24)
\end{align*}
\]

Using Eqs. (2.20), the tetrad components of the energy momentum tensor are expressed as

\[
\begin{align*}
T_{nn} &= \mu \frac{C_{nn}}{\sin \theta} \delta(r - r(t))\delta(\theta - \theta(t))\delta(\varphi - \varphi(t)), \\
T_{mn} &= \mu \frac{C_{mn}}{\sin \theta} \delta(r - r(t))\delta(\theta - \theta(t))\delta(\varphi - \varphi(t)), \\
T_{mm} &= \mu \frac{C_{mm}}{\sin \theta} \delta(r - r(t))\delta(\theta - \theta(t))\delta(\varphi - \varphi(t)), \quad (2.25)
\end{align*}
\]

where

\[
\begin{align*}
C_{nn} &\equiv \frac{\rho^2 \rho_{\theta}^2}{4t} \left[ \mathcal{E}(r^2 + a^2) - a\mathcal{L} + \frac{dr}{d\lambda} \right]^2, \\
C_{mn} &\equiv -\frac{\rho^2 \rho_{\theta}^2}{2\sqrt{2}t} \left[ \mathcal{E}(r^2 + a^2) - a\mathcal{L} + \frac{dr}{d\lambda} \right] \left[ i\sin \theta \left( a\mathcal{E} - \frac{\mathcal{L}}{\sin^2 \theta} \right) - \frac{1}{\sin \theta} \frac{d\cos \theta}{d\lambda} \right], \\
C_{mm} &\equiv \frac{\rho^2}{2t} \left[ i\sin \theta \left( a\mathcal{E} - \frac{\mathcal{L}}{\sin^2 \theta} \right) - \frac{1}{\sin \theta} \frac{d\cos \theta}{d\lambda} \right]^2. \quad (2.26)
\end{align*}
\]
and \( i \equiv dt/d\lambda \). After some calculation, Eq. (2.12) finally becomes

\[
T_A = \mu \int dt \, e^{i\omega t - im\varphi(t)} \Delta^2 \left[ (A_{nn0} + A_{\bar{m}n0} + A_{\bar{m}\bar{m}0}) \delta(r - \ell) + \{(A_{\bar{m}n1} + A_{\bar{m}\bar{m}1}) \delta(r - \ell)\}_r + \{(A_{\bar{m}\bar{m}2}) \delta(r - \ell)\}_{rr}\right], \tag{2.27}
\]

where

\[
\mathcal{L}_\sigma^1 \equiv \partial_\sigma - \frac{m}{\sin \theta} + a \omega \sin \theta + \sigma \cot \theta,
\]

and

\[
\begin{align*}
A_{nn0} &= \frac{-2}{\sqrt{2\pi} \Delta^2} C_{nn} \rho^{-2} \bar{\rho}^{-1} \mathcal{L}_1^\dagger \left\{ \rho^{-4} \mathcal{L}_2^\dagger (\rho^3 S_A) \right\}, \\
A_{\bar{m}n0} &= \frac{2}{\sqrt{2\pi} \Delta} C_{\bar{m}n} \rho^{-3} \left[ (\mathcal{L}_2^\dagger S_A) \left( \frac{iK}{\Delta} + \rho + \bar{\rho} \right) - a \sin \theta S_A \frac{K}{\Delta} \left( \rho - \bar{\rho} \right) \right], \\
A_{\bar{m}\bar{m}0} &= -\frac{1}{\sqrt{2\pi}} \rho^{-3} \bar{\rho} C_{\bar{m}\bar{m}} S_A \left[ -i \left( \frac{K}{\Delta} \right)_r - \frac{K^2}{\Delta^2} + 2i\rho \frac{K}{\Delta} \right], \\
A_{\bar{m}n1} &= \frac{2}{\sqrt{2\pi} \Delta} \rho^{-3} C_{\bar{m}n} \left[ \mathcal{L}_2^\dagger S_A + i a \sin \theta (\bar{\rho} - \rho) S_A \right], \\
A_{\bar{m}\bar{m}1} &= -\frac{2}{\sqrt{2\pi}} \rho^{-3} \bar{\rho} C_{\bar{m}\bar{m}} S_A \left( \frac{iK}{\Delta} + \rho \right), \\
A_{\bar{m}\bar{m}2} &= -\frac{1}{\sqrt{2\pi}} \rho^{-3} \bar{\rho} C_{\bar{m}\bar{m}} S_A. \tag{2.28}
\end{align*}
\]

Inserting Eq. (2.27) into Eq. (2.17), we obtain \( Z_A \) as

\[
Z_A = \frac{\mu}{2i\omega B_{\text{inc}}} \int_{-\infty}^{\infty} dt e^{i\omega t - im\varphi} \left[ P_A \left\{ A_{nn0} + A_{\bar{m}n0} + A_{\bar{m}\bar{m}0} \right\} \\
- \frac{dP_A}{dr} \left\{ A_{nn1} + A_{\bar{m}n1} \right\} + \frac{d^2P_A}{dr^2} A_{\bar{m}\bar{m}2} \right]_{r=r(t)} \tag{2.29}
\]

Furthermore, when we consider bound orbits, the frequency spectrum of \( T_A \) becomes discrete. Therefore \( Z_A \) takes the form

\[
Z_A = \sum_{n_r, n_\theta} 2\pi \delta(\omega - \omega_{m, n_\theta}^{n_r}) Z_A \sqrt{4\pi} \omega, \tag{2.30}
\]

with

\[
\omega_{m, n_\theta}^{n_r} \equiv \left( \frac{dt_z}{d\lambda} \right)^{-1}_\Lambda \left( m \left( \frac{d\varphi_z}{d\lambda} \right)_\Lambda + n_r \Omega_r + n_\theta \Omega_\theta \right). \tag{2.31}
\]

Here \( \Lambda \) denotes the set of parameters \( \{ \ell, m, n_r, n_\theta \} \). Then the time-averaged luminosity is given by

\[
\left\langle \frac{d\mathcal{E}}{dt} \right\rangle = \mu^{-1} \sum_{\Lambda} \left| Z_\Lambda \right|^2, \tag{2.32}
\]
and the time-averaged angular momentum flux is given by

$$\langle \frac{dC}{dt} \rangle = \mu^{-1} \sum_{\Lambda} \frac{m}{\omega_{\mu^\prime\nu}} |\tilde{Z}_{\Lambda}|^2.$$  

(2.33)

Here $\langle \cdots \rangle$ represents the time average with respect to the coordinate time $t$. We quote the formulas for calculating $\langle \frac{dC}{dt} \rangle$ from our previous paper

$$\langle \frac{dC}{dt} \rangle = \langle \frac{dQ}{dt} \rangle - 2 (aE - L) \left( \frac{\langle dE/dt \rangle}{\Lambda} - \langle \frac{dL}{dt} \rangle \right),$$

$$\langle \frac{dQ}{dt} \rangle = 2 \left( \frac{r^2 + a^2}{\Lambda} \right) \langle \frac{\langle dE/dt \rangle}{\Lambda} \rangle - 2 \langle \frac{aP}{\Lambda} \rangle \langle \frac{dL}{dt} \rangle + \mu^{-1} \sum_{\Lambda} \frac{2n_r \Omega_r}{\omega_{\mu^\prime\nu}} |\tilde{Z}_{\Lambda}|^2.$$  

(2.34)

§3. Amplitude of partial wave

As mentioned above, taking the amplitude of the radial oscillation and the coefficient of the quartic term of $\Theta(\cos \theta)$, which are denoted by $e$ and $\epsilon_0 = a^2(1 - E^2)/L^2$, respectively, as small parameters, we can systematically expand the solutions as Fourier series. Indeed, the solutions are found to take the forms

$$r_z(\lambda) = p[1 + e \cos \Omega_r \lambda] + O(e^2), \quad \cos \theta_z(\lambda) = \frac{y}{\sqrt{y + 1}} \sin \Omega_\theta \lambda + O(\epsilon_0),$$  

(3.1)

where $p$ and $y$ are defined in Eqs. (3.2) and (3.3). There is a difficulty when we actually calculate the amplitude of a partial wave $\tilde{Z}_{\Lambda}$ using Eq. (2.29). Here we want to perform the inverse Fourier transformation analytically. This is possible when $Z_{\Lambda}$ is expanded as a power series in sinusoidal functions. However, $A_{\tilde{m}n\tilde{m}}$ and $A_{\tilde{m}m\tilde{n}}$ have factors of $1/\sin \theta_z$ and $1/\sin^2 \theta_z$ through $C_{\tilde{m}n\tilde{m}}$ and $C_{\tilde{m}m\tilde{n}}$, respectively. To make matters worse, $e^{i\phi}$ is proportional to $1/\sin \theta$. This means that $\tilde{Z}_{\Lambda}$ has a term proportional to $\sqrt{(y + 1)/(1 + y \cos^2 \Omega_\theta \lambda)}$, apparently. It would seem that this would prevent us from performing the inverse Fourier transformation analytically, unless we expanded this factor with respect to $y$, assuming a small inclination angle. If such problematic terms remain, however, it follows that we cannot truncate the series expansion labelled by $n_\theta$ at a finite order for general orbits. This seems quite counter intuitive. We expect, therefore, that we can somehow overcome this difficulty. In this subsection, we rewrite the source term into a tractable form, which does not contain any inverse power of $\sin \theta$. We present the results up through $O(e^2, v^5)$ as a demonstration, where $v$ is a typical velocity of the particle defined more precisely in Eq. (3.5).

3.1. General geodesic orbits in Kerr spacetime

Generic Kerr geodesics can be specified by fixing three orbital elements, say, the semi-latus rectum $p$, the eccentricity $e$, and a dimensionless inclination parameter $y$. We define $p$ and $e$ such that the turning points of the radial motion, the apastron
and periastron, are

\[ r_a = \frac{p}{1 - e}, \quad r_p = \frac{p}{1 + e}, \]  

(3.2)

respectively. We also define \( y \) by

\[ y \equiv \frac{C}{L^2}, \]  

(3.3)

which is something like the squared tangent of the inclination angle. This parametrization of orbits is useful for obtaining an intuitive understanding, but it is, of course, equivalent to specifying the three “constants” of motion, \( E, L, C \). Solving the above defining equations, \( E \) and \( L \) can be expressed in terms of \( p, e, \) and \( y \) as

\[ E = 1 - \frac{1}{2} v^2 + \frac{3}{8} v^4 - q Y v^5 + e^2 \left\{ \frac{1}{2} v^2 - \frac{3}{4} v^4 + 2 q Y v^5 \right\}, \]

\[ L = v + 3 Y v^3 - 3 q Y^2 v^4 + \left( \frac{q^2 Y^3 + 27 Y^4}{8} \right) v^5 \]

\[ + e^2 \left\{ \frac{3}{2} v^3 - q Y^2 v^4 + \left( \frac{q^2 Y^3 + 9 Y^4}{4} \right) v^5 \right\}, \]  

(3.4)

where

\[ v \equiv \sqrt{M/p}, \]  

(3.5)

and

\[ Y \equiv \frac{1}{\sqrt{y + 1}} = \frac{L}{\sqrt{C + L^2}}. \]  

(3.6)

Then, expanding the periodic parts with the period of the radial oscillations in the geodesic equations in powers of \( v \) and \( e \), the solutions up through \( O(e^2, v^5) \) are found to be given by

\[ r(\lambda) = p \sum_{n_r=0}^{n_{\text{max}}} \alpha_{n_r} \cos n_r \Omega_r \lambda, \]

\[ t^{(r)}(\lambda) = \sum_{n_r=0}^{n_{\text{max}}} t_{n_r}^{(r)} \sin n_r \Omega_r \lambda, \]

\[ \varphi^{(r)}(\lambda) = \sum_{n_r=0}^{n_{\text{max}}} \varphi_{n_r}^{(r)} \sin n_r \Omega_r \lambda, \]  

(3.7)

where \( n_{\text{max}} = 2 \), which is the truncated order in \( e \), and

\[ \alpha_0 = 1 + e^2 \left\{ \frac{1}{2} - \frac{1}{2} v^2 + q Y v^3 - \left( 3 - \frac{(1 - 2 Y^2) q^2}{2} \right) v^4 + 10 q Y v^5 \right\}, \]

\[ \alpha_1 = e, \]

\[ \alpha_2 = e^2 \left\{ \frac{1}{2} + \frac{1}{2} v^2 - q Y v^3 + \left( 3 - \frac{(1 - 2 Y^2) q^2}{2} \right) v^4 - 10 q Y v^5 \right\}, \]

\[ t_0^{(r)} = 0, \]
\[ t_1^{(r)} = e \left\{ 2 + 4v^2 - 6qYv^3 + \left(17 - (1 - 4Y^2)q^2\right)v^4 - 54qYv^5 \right\}, \]
\[ t_2^{(r)} = e^2 \left\{ \frac{3}{4} + \frac{7}{4}v^2 - \frac{13qY}{4}v^3 + \left(\frac{81}{8} + \frac{(20Y^2 - 7)q^2}{8}\right)v^4 - \frac{135qY}{4}v^5 \right\}, \]
\[ \varphi_0^{(r)} = 0, \]
\[ \varphi_1^{(r)} = e \left\{ -2qv^3 + 2q^2Yv^4 - 10qYv^5 \right\}, \]
\[ \varphi_2^{(r)} = e^2 \left\{ -q^2Yv^4 + \frac{q}{2}v^5 \right\}. \tag{3.8} \]

The angular velocity \( \Omega_r \) is obtained simultaneously when we solve \( r(\lambda) \), but we give it below, together with the angular velocity of the \( \theta \)-oscillations.

The other parts of the geodesic equations which are periodic with the period of the \( \theta \)-oscillations are expanded in powers of \( \epsilon_0 \equiv a^2(1 - \mathcal{E}^2)/\mathcal{L}^2 = \mathcal{O}(v^4) \) and obtained as
\[ \cos \theta(\lambda) = \sqrt{1 - Y^2} \sum_{n_\theta=0}^{n_{\max}} \beta_{n_\theta} \sin n_\theta \Omega_\theta \lambda, \]
\[ t^{(\theta)}(\lambda) = -\frac{pq^2v^4E}{\mathcal{L}} \sum_{n_\theta=0}^{n_{\max}} t^{(\theta)}_{n_\theta} \sin n_\theta \Omega_\theta \lambda, \tag{3.9} \]
where \( n_{\max} = 2 \times 1 + 2 = 4 \), which is twice the truncated order in \( \epsilon_0 \) plus two,
\[ \beta_0 = 0, \quad \beta_1 = 1 + \epsilon_0 \frac{(Y^2 - 9Y^4)}{16}, \quad \beta_2 = 0, \quad \beta_3 = \epsilon_0 \frac{(Y^2 - Y^4)}{16}, \quad \beta_4 = 0, \]
\[ t_0^{(\theta)} = 0, \quad t_1^{(\theta)} = 0, \quad t_2^{(\theta)} = (Y - Y^3) + \epsilon_0 \frac{Y^3(1 - 7Y^2)(1 - Y^2)}{16}, \]
\[ t_3^{(\theta)} = 0, \quad t_4^{(\theta)} = \epsilon_0 \frac{Y^3(1 - Y^2)^2}{64}. \tag{3.10} \]

For \( \varphi^{(\theta)} \), we introduce the variable \( X = \sin \theta e^{i\varphi^{(\theta)}} \). Then, it can be expanded as
\[ \Re(X) = \sum_{n_\theta} X_{n_\theta}^R \cos n_\theta \Omega_\theta \lambda, \quad \Im(X) = \sum_{n_\theta} X_{n_\theta}^\Im \sin n_\theta \Omega_\theta \lambda, \tag{3.11} \]
where
\[ X_0^R = \frac{(Y^2 + 1)}{2} - \epsilon_0 \frac{Y^2(1 - 9Y)(1 - Y^2)}{4}, \quad X_1^R = 0, \]
\[ X_2^R = \frac{(1 - Y)^2}{2} - \epsilon_0 \frac{Y^3(1 - Y^2)}{4}, \quad X_3^R = 0, \quad X_4^R = \epsilon_0 \frac{Y^2(1 + Y)(1 - Y^2)^2}{32}, \]
\[ X_0^\Im = 0, \quad X_1^\Im = 0, \quad X_2^\Im = -\frac{(1 - Y)}{2} + \epsilon_0 \frac{Y^2(1 + 5Y)(1 - Y^2)}{16}, \]
\[ X_3^\Im = 0, \quad X_4^\Im = -\epsilon_0 \frac{Y^2(1 + Y)(1 - Y)^2}{32}. \tag{3.12} \]

Using Eq. (3.4), we re-expand the above solutions in powers of \( v \) and \( e \) to obtain
\[ \cos \theta(\lambda) = \sqrt{1 - Y^2} \sum_{n_\theta=0}^{n_{\max}} \beta_{n_\theta} \sin n_\theta \Omega_\theta \lambda, \quad t^{(\theta)}(\lambda) = \sum_{n_\theta=0}^{n_{\max}} t^{(\theta)}_{n_\theta} \sin n_\theta \Omega_\theta \lambda, \]
Finally, the frequencies of radial and $\theta$-oscillations and the non-oscillating parts of $dt/d\lambda$ and $d\varphi/d\lambda$ are given by

\[
\begin{align*}
\frac{\Omega_r}{p^2} &= \frac{v^3}{M} \left[ 1 - \frac{3}{2} v^2 + 3 q Y v^3 + \left( \frac{(1 - 4 Y^2)q^2}{2} - \frac{45}{8} \right) v^4 + \frac{33 q Y}{2} v^5 \right. \\
&\quad \left. + e^2 \left\{ \frac{1}{2} v^2 - q Y v^3 + \left( \frac{1 + 2 Y^2}{4} q^2 v^4 \right) + 2 q Y v^5 \right\} \right], \\
\frac{\Omega_\theta}{p^2} &= \frac{v^3}{M} \left[ 1 + \frac{3}{2} v^2 - 3 q Y v^3 + \left( \frac{(-7 Y^2 + 1)q^2}{4} + \frac{27}{8} \right) v^4 - \frac{15 q Y}{2} v^5 \right. \\
&\quad \left. + e^2 \left\{ \frac{1}{2} v^2 - q Y v^3 + \left( \frac{1 + 2 Y^2}{4} q^2 + \frac{9}{4} \right) v^4 - 7 q Y v^5 \right\} \right], \\
\frac{1}{p^2} \left\langle \frac{dt}{d\lambda} \right\rangle &= \frac{v^3}{M} \left[ 1 + \frac{3}{2} v^2 + \frac{(1 - 2 Y^2)q^2}{2} + \frac{27}{8} \right] v^4 + 3 q Y v^5 \\
&\quad + e^2 \left\{ \frac{3}{2} v^2 - \frac{1}{4} v^2 + 2 q Y v^3 + \left( \frac{1 + 2 Y^2}{4} q^2 - \frac{99}{16} \right) v^4 + \frac{43 q Y}{2} v^5 \right\}, \\
\frac{1}{p^2} \left\langle \frac{d\varphi}{d\lambda} \right\rangle &= \frac{v^3}{M} \left[ 1 + \frac{3}{2} v^2 + (2 - 3 Y) q v^3 \\
&\quad + \left( \frac{-(1 + 7 Y)(1 - Y)q^2}{4} + \frac{27}{8} \right) v^4 + \frac{3(2 - 5 Y)q}{2} v^5 \\
&\quad + e^2 \left\{ \frac{1}{2} v^2 - q Y v^3 + \left( \frac{1 + 2 Y^2}{4} q^2 + \frac{9}{4} \right) v^4 + (4 - 7 Y) q v^5 \right\} \right].
\end{align*}
\]
3.2. Reformulation of the source term for the Teukolsky equation

Let us now discuss the method for rewriting the source term of the Teukolsky equation. The source term contains inverse powers of \( \sin \theta \) in \( C_{\text{new}}, C_{\text{old}} \), and \( e^{-im\phi} \sim (\sin \theta)^{-m} \). These factors prevent us from performing the inverse Fourier transformation analytically. Therefore we want to remove all inverse powers of \( \sin \theta \). Fortunately, the spheroidal harmonics \( S_A, L^1_A S_A \) and \( L^1_A C^1_A S_A \) are proportional to \((\sin \theta)^{|m|-2}, (\sin \theta)^{|m|-1}\) and \((\sin \theta)^{|m|}\), respectively. Therefore some of the inverse powers of \( \sin \theta \) cancel out, but we immediately find that some of them remain after this cancellation. In the following we show how we can eliminate all these annoying factors.

For convenience, we introduce the new angular functions

\[
2\tilde{S} \equiv \frac{S_A}{(\sin \theta)^{|m|-2}}, \quad 1\tilde{S} \equiv \frac{L^1_A S_A}{(\sin \theta)^{|m|-1}}, \quad 0\tilde{S} \equiv \frac{L^1_A C^1_A S_A}{(\sin \theta)^{|m|}},
\]

and

\[
\sigma \Xi_m \equiv \sigma \tilde{S} X_m, \quad X_m = (\sin \theta)^{|m|} e^{-im\phi}\theta = \begin{cases} 
X^m, & (m > 0) \\
1, & (m = 0) \\
X^m, & (m < 0)
\end{cases}
\]

If they are truncated at a finite post-Newtonian order, these new functions \( \sigma \tilde{S} \), and hence \( \sigma \Xi_m \), have no inverse powers of \( \sin \theta \), being expressed as polynomials of \( \cos \theta \). There are a few special cases in which \( \sigma \tilde{S} \) contains an additional overall factor of \( \sin \theta \). We have \( 2\tilde{S} = \sin^2 \theta(1 \pm \cos \theta) \times (\text{polynomial of } \cos \theta) \) for \( m = \pm 1 \), while \( 1\tilde{S} = \sin^2 \theta \times (\text{polynomial of } \cos \theta) \), and \( 2\tilde{S} = \sin^4 \theta \times (\text{polynomial of } \cos \theta) \) for \( m = 0 \). In addition, \( X_m \) can be expressed as a Fourier series of \( \exp(i\Omega \lambda) \), because it contains a positive power of \( X \) or \( \tilde{X} \). Using \( \sigma \Xi_m \), the amplitude of the partial wave is rewritten as

\[
Z_A = \frac{\mu}{2i\omega B m^8} \int_{-\infty}^{\infty} \frac{d\lambda e^{i\omega t - i(m\phi - \phi \theta)}}{(2\sqrt{2\pi} \rho) \left( \frac{-1}{2\sqrt{2\pi} \rho} \right) \left[ \frac{D^2}{\Delta^2} (a \Xi_m - 2ia \rho_1 \Xi_m) R_A \right.}
\]

\[
- \frac{2D_r D \theta}{\Delta} (a \Xi_m (J_+ - (\rho + \bar{\rho})) + ia \Xi_m(\bar{\rho} - \rho) J_-) R_A 
\]

\[
+ D^2_\theta \Xi_m (J_+^2 - 2\rho J_-) R_A \Bigg|_{r=r(\lambda)}
\]

where we have defined

\[
J_+ \equiv \partial_r - iK/\Delta, \\
D_r \equiv \mathcal{E}(\rho^2 + a^2) - a \mathcal{L} + \frac{dr}{\mathcal{L}}, \\
D \theta \equiv i \left( a \mathcal{E} - \frac{\mathcal{L}}{\sin^2 \theta} \right) - \frac{1}{\sin^2 \theta} \frac{d\cos \theta}{d\lambda}.
\]

In this expression, it is clear that inverse powers of \( \sin \theta \) are contained only through \( D \theta \), which contains the factor \((\sin \theta)^{-2}\), and therefore the index of the inverse powers
of \sin \theta is at most four. We emphasize that this index is less for \(|m| = 0 \text{ or } 1\), since \(2 \Xi_m\) and/or \(1 \Xi_m\) contain additional positive powers of \(\sin \theta\). Thus, we have to treat three cases, \(m = 0, m = \pm 1\), and \(|m| \geq 2\), separately. Here we discuss only the case with \(|m| \geq 2\), and the other simpler cases with \(|m| = 0 \text{ or } \pm 1\) are deferred to Appendix A.

Equation (3.18) has inverse powers of \(\sin \theta\) in \(D_\theta 1 \Xi_m\), \(D_\theta 2 \Xi_m\), and \(D_\theta^2 2 \Xi_m\). First we give a prescription for the terms containing the factor \((\sin \theta)^{-2}\), i.e. \(D_\theta 1 \Xi_m\) and \(D_\theta 2 \Xi_m\). After some calculation, we obtain (for \(\sigma \geq 1\))

\[
\frac{d}{d\lambda} \xi \Xi_m = -\frac{d \cos \theta}{d\lambda} \xi \Xi_m + m \left[ i \left( \frac{\mathcal{L}}{\sin^2 \theta} - \frac{1}{\sin^2 \theta} \frac{d \cos \theta}{d\lambda} \right) \xi \Xi_m \right. \\
+ \left[ a \omega \frac{d \cos \theta}{d\lambda} + im \left( \frac{\mathcal{L}}{\sin^2 \theta} \right)_\lambda - ima \xi \Xi_m \right] \sigma \Xi_m, 
\]

where we have used

\[
\frac{\mathcal{L}}{\sin^2 \theta} = \left\langle \frac{\mathcal{L}}{\sin^2 \theta} \right\rangle_\lambda + \frac{d \varphi(\theta)}{d\lambda}. 
\]

Using this relation, we can derive

\[
D_\theta \xi \Xi_m = F_m(\sigma \Xi_m, \xi \Xi_m), \\
F_m(A, B) \equiv \frac{1}{m} \left[ \frac{d \cos \theta}{d\lambda} A + \left( \frac{d}{d\lambda} - a \omega \frac{d \cos \theta}{d\lambda} \right) - im \left\langle \frac{\mathcal{L}}{\sin^2 \theta} \right\rangle_\lambda + ima \xi \Xi_m \right] B. 
\]

Since there are no inverse powers of \(\sin \theta\) in \(F_m(\sigma \Xi_m, \xi \Xi_m)\), the inverse Fourier transformation of this term can be easily performed. Obviously, the above formula is not valid for the case with \(m = 0\) because \(F_m\) contains \(m^{-1}\). In \(\left\langle \frac{\mathcal{L}}{\sin^2 \theta} \right\rangle_\lambda\) it seems that \((\sin \theta)^{-2}\) is remaining, but it is just a constant,

\[
\left\langle \frac{\mathcal{L}}{\sin^2 \theta} \right\rangle_\lambda = \frac{\Omega_\theta}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sin^2 \theta} = \frac{2 \Omega_\theta}{\pi} \int_0^{(\cos \theta)_{\text{max}}} d \cos \theta \frac{1}{\sqrt{1 - \cos^2 \theta}} \frac{\mathcal{L}}{\sqrt{1 - \cos \theta}},
\]

which can be analytically integrated in the post-Newtonian expansion.

Next, we consider the remaining term, \(D_\theta^2 2 \Xi_m\). Applying Eq. (3.22) twice, we obtain

\[
D_\theta^2 2 \Xi_m = D_\theta F_m(1 \Xi_m, 2 \Xi_m) \\
= F_m(D_\theta 1 \Xi_m, D_\theta 2 \Xi_m) - \frac{1}{m} \frac{dD_\theta}{d\lambda} 2 \Xi_m \\
= F_m(F_m(0 \Xi_m, 1 \Xi_m), F_m(1 \Xi_m, 2 \Xi_m)) - \frac{1}{m} \frac{dD_\theta}{d\lambda} 2 \Xi_m. 
\]

Using the geodesic equation for \(\cos \theta\), one can show

\[
\frac{dD_\theta}{d\lambda} = \cos \theta (-D_\theta^2 + 2ia \xi D_\theta + a^2). 
\]
Substituting this relation into Eq. (3.24) and applying Eq. (3.22) again, we obtain

\[ D_\theta^2 \Xi_m = \left( 1 + \frac{\cos \theta}{m} \right) \left\{ F_m(F_m(0\Xi_m, 1\Xi_m), F_m(1\Xi_m, 2\Xi_m)) - \frac{2iaE \cos \theta}{m} F_m(1\Xi_m, 2\Xi_m) - \frac{a^2 \cos \theta}{m} 2\Xi_m \right\} + \frac{1}{m^2}(1 - \sin^2 \theta)D_\theta^2 2\Xi_m. \]

Here, \( D_\theta^2 2\Xi_m \) appears also in the last term on the right-hand side. After solving this equation for \( D_\theta^2 2\Xi_m/m^2 \), we finally arrive at

\[ D_\theta^2 2\Xi_m = \frac{m^2}{m^2 - 1} \left( 1 + \frac{\cos \theta}{m} \right) \left\{ F_m(F_m(0\Xi_m, 1\Xi_m), F_m(1\Xi_m, 2\Xi_m)) - \frac{2iaE \cos \theta}{m} F_m(1\Xi_m, 2\Xi_m) - \frac{a^2 \cos \theta}{m} 2\Xi_m \right\} - \frac{1}{m^2 - 1}(D_\theta \sin^2 \theta) F_m(1\Xi_m, 2\Xi_m), \] (3.26)

which is free from inverse powers of \( \sin \theta \). Note that this formula is not valid for \( m = 0 \) and \( m = \pm 1 \) because of the factors of \( m^{-1} \) and \( (m^2 - 1)^{-1} \).

Combining the above results, the partial wave amplitude for \(|m| \geq 2\) is expressed as

\[ Z_A = \frac{\mu}{2i\omega B m c} \int_{-\infty}^{\infty} d\lambda e^{i\omega t - im(-\frac{d\phi}{d\lambda} + \phi(r))} \left( \frac{-1}{2\sqrt{2\pi} \rho} \right) \left[ D_r^2 (0\Xi_m - 2ia\rho(1\Xi_m)) R_A - \frac{2D_r^2}{\Delta} \{ F_m(0\Xi_m, 1\Xi_m)( J_+ - \rho + \bar{\rho}) + iaF_m(1\Xi_m, 2\Xi_m)( \bar{\rho} - \rho) J_- \} R_A + \frac{1}{m^2 - 1} \left( m + \cos \theta \right) \left\{ m F_m(F_m(0\Xi_m, 1\Xi_m), F_m(1\Xi_m, 2\Xi_m)) - \frac{2iaE \cos \theta}{m} F_m(1\Xi_m, 2\Xi_m) - \frac{a^2 \cos \theta}{m} 2\Xi_m \right\} \right]- \left( \frac{1}{m^2 - 1}(D_\theta \sin^2 \theta) F_m(1\Xi_m, 2\Xi_m) \right) \left( J^2 - 2\rho J_- \right) R_A \right]_{r = r(\lambda)}. \] (3.27)

This formula has no \( (\sin \theta)^{-1} \) factor in the integrand. The formulas for \(|m| \leq 1\) are presented in Appendix A

\section*{§4. Results}

\subsection*{4.1. The evolution of orbital parameters}

Substituting Eq. (3.27) into Eqs. (2.32), (2.33) and (2.34), the time-averaged rates of change for the three constants of motion up through \( O(e^2, v^5) \) are given by

\[ \left\langle \frac{dE}{dt} \right\rangle = -\frac{32}{3} \left( \frac{\mu}{M^2} \right) v^10(1 - e^2)^{3/2} \left[ \left( 1 + \frac{73}{24} e^2 \right) - \left( \frac{1247}{336} + \frac{9181}{672} e^2 \right) \right] v^2 \]
\[
\langle \frac{dL}{dt} \rangle = -\frac{32}{5} \left( \frac{\mu}{M^2} \right) M v^7 (1 - e^2)^{3/2} \left[ \left( Y + \frac{7}{8} e^2 \right) - \left( \frac{1247 Y}{336} + \frac{425 Y}{336} e^2 \right) v^2 \right.
- \left. \left( 4 + 97 e^2 \right) \pi v^3 \right] q v^3
- \left[ 4471 Y \frac{12}{9072} + 44531 Y \frac{12}{336} e^2 \right] \pi v^5 + \left( 3749 Y \frac{12}{336} + 1759 Y \frac{12}{56} e^2 \right) q v^5 \right],
\]

\[
\langle \frac{dC}{dt} \rangle = -\frac{64}{5} \left( \frac{\mu}{M^2} \right) M^2 v^6 (1 - e^2)^{3/2} \left( 1 - Y^2 \right) \left[ \left( 1 + \frac{7}{8} e^2 \right) \right.
- \left. \left( \frac{743}{336} - \frac{23 8}{42} e^2 \right) v^2 - \left( \frac{85 Y}{8} + \frac{211 Y}{8} e^2 \right) q v^3 \right.
+ \left. \left( 4 + 97 e^2 \right) \pi v^3 - \left( \frac{129193}{18144} + \frac{4035}{1728} e^2 \right) v^4 \right]
- \left[ \frac{329}{96} - \frac{53 Y^2}{8} + \left( \frac{929}{96} - \frac{163 Y^2}{8} \right) e^2 \right] q^2 v^4
+ \left. \left( \frac{2553 Y}{224} - \frac{553 Y}{192} e^2 \right) q v^5 - \left( \frac{4159}{672} + \frac{21229}{1344 e^2} \right) \pi v^5 \right].
\]

(4.1)

Here, a factor of \((1 - e^2)^{3/2}\) is factored out in order to make it easier to compare these results with the well-known formulas derived by Peters and Mathews.\(^{37}\)

We can derive the evolution of the orbital parameters, \(i^i = \{v, e, Y\}\), from the rates of change of the integrals of motion, \(I^i = \{E, L, C\}\),

\[
\langle \frac{dv}{dt} \rangle = \frac{32}{5} \left( \frac{\mu}{M^2} \right) v^9 (1 - e^2)^{3/2} \left[ \left( 1 + \frac{7}{8} e^2 \right) - \left( \frac{743}{336} + \frac{553}{21} e^2 \right) v^2 \right.
- \left. \left( 4 + 97 e^2 \right) \pi v^3 \right] q v^3
- \left[ \frac{329}{96} - \frac{53 Y^2}{8} + \left( \frac{929}{96} - \frac{163 Y^2}{8} \right) e^2 \right] q^2 v^4
+ \left. \left( \frac{2553 Y}{224} - \frac{553 Y}{192} e^2 \right) q v^5 - \left( \frac{4159}{672} + \frac{21229}{1344 e^2} \right) \pi v^5 \right].
\]

Using the above relation with Eqs. (3.4) and (3.6), the evolutions of the parameters \(v, e\) and \(Y\) are obtained as
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\[- \left( \frac{133Y}{12} + \frac{379Y}{24}e^2 \right) qv^3 + \left( 4 + \frac{97}{8}e^2 \right) \pi v^3 \]
\[+ \left( \frac{34103}{18144} - \frac{526955}{12096}e^2 \right) v^4 \]
\[- \left( \frac{329}{96} - \frac{815Y}{96} + \left\{ \frac{929}{96} - \frac{477Y^2}{32} \right\} e^2 \right) q^2 v^4 \]
\[- \left( \frac{1451Y^2}{56} + \frac{1043Y}{96}e^2 \right) \pi v^3 - \left( \frac{4159}{672} + \frac{48809}{1344}e^2 \right) \pi v^5 \],
\[\langle \text{d}e/\text{d}t \rangle = -\frac{304}{15} \left( \frac{\mu}{M^2} \right) v^3 e \left( 1 - e^2 \right)^{3/2} \left[ \left( 1 + \frac{121}{304}e^2 \right) - \left( \frac{6849}{2128} + \frac{4509}{2128}e^2 \right) v^2 \right.
\[- \left( \frac{879Y}{76} + \frac{515Y}{76}e^2 \right) qv^3 + \left( \frac{985}{152} + \frac{5969}{608}e^2 \right) \pi v^3 \]
\[- \left( \frac{286397}{38304} + \frac{2064415}{51072}e^2 \right) v^4 \]
\[- \left( \frac{3179}{608} - \frac{5869Y^2}{608} + \left\{ \frac{8925}{1216} - \frac{10747Y^2}{1216} \right\} e^2 \right) q^2 v^4 \]
\[- \left( \frac{1903Y}{304} + \frac{22373Y}{8512}e^2 \right) \pi v^5 \left[ \left( \frac{87947}{4256} + \frac{4072433}{68096}e^2 \right) \pi v^5 \right]. \tag{4.3} \]

Substituting $Y = 1$, it is found that these results are consistent with the previous results\(^{38)-40)}\) except for the errors reported in our previous paper.\(^{32)}\)

4.2. Phase evolution of gravitational waves

The time dependence of the orbital parameters gives us information about the phase evolution of gravitational waves. There are three fundamental frequencies, $\omega_r = \Omega_r \langle \text{d}t/\text{d}\lambda \rangle_{\lambda}^{-1}$, $\omega_\theta = \Omega_\theta \langle \text{d}t/\text{d}\lambda \rangle_{\lambda}^{-1}$, and $\omega_\phi = \langle \text{d}\varphi/\text{d}\lambda \rangle \langle \text{d}t/\text{d}\lambda \rangle_{\lambda}^{-1}$. $\omega_r$ is the angular velocity of radial oscillations, and it is not important for small eccentric orbits. By contrast, $\omega_\theta$ and $\omega_\phi$ are equally fundamental for greatly inclined orbits. Stated briefly, gravitational waves cannot be expressed by a single phase, even if we pick up the dominant quadrupole contribution.

When the orbit is in the equatorial plane, $\omega_\phi$ corresponds to the angular velocity used in the standard post-Newtonian calculation.\(^{41)}\) For general orbits, the post-Newtonian angular velocity is given by $\omega_N \equiv (1 - Y)\omega_\theta + Y\omega_\phi$. This can be understood in the following way. First we consider the frame rotating at the angular velocity of the orbital precession due to the spin-orbit coupling, $\omega_\theta - \omega_\phi$. In this frame, the orbital plane of the geodesic motion is fixed. The orbital angular velocity observed in this frame is given by $\omega_\theta$. Then, the effect of the rotating frame can be understood as the sum of two rotations, that around the axis perpendicular to the orbital plane and that around an axis in the orbital plane. The first rotation, whose angular velocity is $Y(\omega_\theta - \omega_\phi)$, is understood as the relative motion between the rotating frame and the non-rotating one. Therefore, the angular velocity of the
orbit projected onto the momentary orbital plane is given by \( \omega_N = (1 - Y)\omega_\theta + Y\omega_\phi \). This is in some sense the angular velocity observed from the direction perpendicular to the orbital plane. The latter rotation causes the change of the orbital plane. As the orbital plane itself changes, this post-Newtonian angular velocity \( \omega_N \) does not appear in the waveform with a fixed direction of the observer. When we are concerned with a short period of time in which the precession of the orbital plane can be ignored, of course, there is no practical way to distinguish these three angular velocities, \( \omega_N, \omega_\phi \) and \( \omega_\theta \).

In this subsection, we consider the evolution of the frequency \( F_\phi \equiv \omega_\phi / \pi \) and the corresponding phase \( \Phi_\phi(F_\phi) \equiv 2\pi \int F_\phi dt \), assuming that the evolution of all orbital parameters are governed by the time averaged expressions given in Eqs. (4.3). Here, we briefly describe the method for obtaining analytic expressions for them. The details, including the derivation of the Fourier transformed waveform under the stationary phase approximation and the results for \( F_\theta \equiv \omega_\theta / \pi \), are reported in Appendix B.

Using Eq. (3.15), we can express \( F_\phi \) in terms of \( \iota \). Solving this relation inversely with respect to \( v \), we can express \( v \) as a function of \( F_\phi, e \) and \( Y \) in the post-Newtonian expansion, i.e. expansion in powers of \( F_\phi \). Then, taking the derivative of \( F_\Phi(\iota) \), we rewrite it using Eqs. (4.3). Substituting \( v(F_\phi, e, Y) \) into this expression, we obtain

\[
\frac{dF_\phi}{dt} = \frac{96}{5\pi M^2 (\pi MF_\phi)^{\frac{3}{2}}} \left[ \left( 1 + \frac{157}{24}e^2 \right) - \left( \frac{743}{336} + \frac{3683}{112}e^2 \right) (\pi MF_\phi)^{\frac{3}{2}} \right.
\]

\[
- \left( \frac{10}{3} + \frac{73Y}{12} + \left\{ \frac{1193}{36} + \frac{857Y}{24} \right\} e^2 \right) q(\pi MF_\phi)
\]

\[
+ \left( 4 + \frac{2335}{48}e^2 \right) \pi(\pi MF_\phi) + \left( \frac{34103}{18144} - \frac{89353}{1728}e^2 \right) (\pi MF_\phi)^{\frac{5}{2}}
\]

\[
+ \left( -\frac{233}{96} + 2Y + \frac{527}{96}Y^2 + \left\{ -\frac{2875}{96} + \frac{281}{12}Y + \frac{4165}{96}Y^2 \right\} e^2 \right) q^2(\pi MF_\phi)^{\frac{5}{2}}
\]

\[
- \left( \frac{4159}{672} + \frac{20135}{96}e^2 \right) \pi(\pi MF_\phi)^{\frac{5}{2}}
\]

\[
+ \left( \frac{743}{72} - \frac{13907Y}{336} + \left\{ \frac{16687}{84} - \frac{84365Y}{336} \right\} e^2 \right) q(\pi MF_\phi)^{\frac{7}{2}} \right]
\]

where we have introduced the chirp mass, \( M = M^{2/5}\mu^{3/5} \). The evolution equation for \( e \) and \( Y \) can also be rewritten by using \( F_\phi \) instead of \( v \). We have to integrate these three differential equations simultaneously. This may appear to be difficult, but we can perform this integration iteratively, assuming a small eccentricity and the post-Newtonian expansion. It is important to note that \( \delta Y \equiv Y - Y_I \) is also small in the post-Newtonian expansion, where \( Y_I \) is the value of \( Y \) at \( F_\phi = 0 \). For this reason, in the lowest order approximation, we can fix the value of \( Y \) to \( Y_I \). The effect of the variation \( \delta Y \) can be taken into account perturbatively. Eliminating \( dt \) from these differential equations, \( e^2 \) and \( Y \) are integrated as functions of \( F_\phi \) as

\[
e^2 = e_I^2(\pi MF_\phi)^{\frac{3}{2}} \left[ 1 + \frac{3215}{1008}(\pi MF_\phi)^{\frac{2}{3}} \right]
\]
δe = \text{denoted the leading-order correction to } e

\delta e = \tilde{\delta} e

\text{Here, } \delta Y = (1 - Y_I^2)(\pi M F_\varphi)^2 + \cdots,

\delta Y = -(1 - Y_I^2)q(\pi M F_\varphi) \left( \frac{61}{72} - \frac{163}{384} \tilde{\delta} t (\pi M F_\varphi)^{-\frac{19}{9}} + \cdots \right). \tag{4.5}

Here, \tilde{\epsilon}_I \text{ denotes the limiting value of } e \times (\pi M F_\varphi)^{19/18} \text{ in the limit } F_\varphi \rightarrow 0. \text{ We have denoted the leading-order correction to } \tilde{\epsilon}^2 \text{ due to the evolution of } Y \text{ separately by } \delta e^2(\delta Y), \text{ although it is relatively order 3PN.}

Substituting the relations in Eq. (4.5) into Eq. (4.4) and integrating \(F_\varphi(dF_\varphi/dt)^{-1}\) over \(F_\varphi\), the phase \(\Phi_\varphi(F_\varphi)\) is calculated as

\begin{align*}
\Phi_\varphi(F_\varphi) &= \phi_\varphi^c - \frac{1}{16}(\pi M F_\varphi)^{-\frac{3}{2}} \left[ 1 + \frac{3715}{1008}(\pi M F_\varphi)^{\frac{3}{2}} \ight. \\
&+ \left. \begin{cases} 
q &+ \left( \frac{25}{3} + \frac{365}{24} Y_I \right) - 10\pi \right] (\pi M F_\varphi) \\
&+ \begin{cases} 
\frac{15293365}{1016064} + q^2 &\left( \frac{1165}{96} - 10Y_I - \frac{2635}{96}Y_I^2 \right) \right] (\pi M F_\varphi)^{\frac{3}{2}} \\
&+ \begin{cases} 
\frac{38645}{672} - q &\left( \frac{3715}{168} + \frac{688405}{2016} \right) \right] (\pi M F_\varphi)^{\frac{3}{2}} \ln(\pi M F_\varphi) \\
&+ \delta \Phi^{(\delta Y)}_\varphi + \delta \Phi^{(\epsilon)}_\varphi \right]
\end{cases}
\end{align*}

\delta \Phi^{(\delta Y)}_\varphi = q^2(1 - Y_I^2) \left( \frac{22265}{864} + \frac{21072205}{73728} \right) (\pi M F_\varphi)^{\frac{3}{2}},

\delta \Phi^{(\epsilon)}_\varphi = \tilde{\epsilon}_I^2(\pi M F_\varphi)^{-\frac{19}{9}} \left[ -\frac{785}{272} - \frac{2045665}{225792} \right] (\pi M F_\varphi)^{\frac{3}{2}}

\begin{align*}
&+ \begin{cases} 
\frac{65561}{2880} - q \left( \frac{2057}{90} + \frac{2953}{540} Y_I \right) \right] (\pi M F_\varphi) \\
&+ \begin{cases} 
\frac{111064865}{10948608} + q^2 &\left( \frac{698695}{101376} + \frac{16895}{1056} + \frac{650665}{101376} Y_I \right) \right] q^2 \right] (\pi M F_\varphi)^{\frac{3}{2}} \\
&+ \begin{cases} 
\frac{3873451}{86184} - q \left( \frac{1247185}{24624} + \frac{1899015067}{8273664} Y_I \right) \right] (\pi M F_\varphi)^{\frac{3}{2}} \right], \tag{4.6}
\end{cases}
\end{align*}

where \(\phi_\varphi^c\) is a constant of integration, and \(\delta \Phi^{(\epsilon)}_\varphi\) and \(\delta \Phi^{(\delta Y)}_\varphi\) express the corrections of \(O(\tilde{\epsilon}_I^2)\) and the terms associated with the time variation of the inclination angle, respectively. We have included the cross terms between these two effects in \(\delta \Phi^{(\delta Y)}_\varphi\).

Here, several remarks are in order. The terms of \(O(\tilde{\epsilon}_I^2)\) seem to be large in the sense of the post-Newtonian order. Those terms have relatively large inverse
powers of $F_\phi$. However, we should recall that $e^2 \approx \tilde{e}_2^2(\pi MF_\theta)^{-19/9}$. Therefore, under the current assumption of a small eccentricity ($e \ll 1$), the terms associated with the factor $\tilde{e}_2^2(\pi MF_\theta)^{-19/9}$ are much smaller than the other terms. In the above calculation, we have basically kept the terms up through $O(v^5, e^2)$. However, for $\delta \Phi_{(\delta Y)}$, we have also kept the terms of higher post-Newtonian order, $O(v^6)$, since the leading-order correction due to $\delta Y$ starts with this order.

As mentioned above, more explicit formulas including the Fourier transform of the waveform under the stationary phase approximation are given in Appendix B. Here, we would like to stress that the analytic method for computing the phase evolution presented here can be extended systematically to higher order in $v$ and $e$.

§5. Summary

In this paper, we have improved an analytic method for calculating the adiabatic evolution of the orbital parameters of a point particle orbiting a Kerr black hole. We have removed the previous limitation to an orbit with a small inclination angle. To do this, we rewrote the source term of the Teukolsky equation, removing inverse powers of $\sin \theta_z$ by using integration by parts, where $\theta_z$ is the polar angle of the particle position. As a result, our new expression for the source term is much more complicated than the original one, but it is possible to carry out an analytic Fourier transform of this source term. Moreover, even in a numerical approach along the line of Refs. 42) and 43), our expression might be useful when we consider the polar orbit, since the singular behavior of the source term near the poles in the original formulation is completely removed.

We have also shown that it is possible to integrate the time evolution of the orbital parameters under the assumption that it is governed by the time averaged values derived in the adiabatic approximation. Furthermore, analytic calculation of the Fourier transform of the waveform is also demonstrated in Appendix B. The results presented here are restricted to $O(v^5, e^2)$, where $v$ and $e$ are, respectively, the velocity and the eccentricity of the particle. Our method, however, can be extended systematically to higher order. Such analytic expressions for the waveform should be useful for the fast generation of templates of gravitational waveforms.

Our analytic expressions for the averaged evolutions of orbital parameters are written as polynomial functions with respect to $Y$ defined by (3.6). This result could be anticipated from the expressions for orbits given in §3.1. The only non-polynomial expression is $\sqrt{1 - Y^2}$ in $\cos \theta(\lambda)$. However, changing the signature of $\cos \theta(\lambda)$ alone corresponds to simply flipping the direction of the $z$-axis. This change should not affect the rates of change of the energy, the angular momentum and the Carter constant, which means that only even powers of $\cos \theta(\lambda)$ contribute to these rates of change. Here we do not pursue a rigorous proof of this statement. If we can prove that the averaged evolutions of orbital parameters can be written as polynomials in $Y$, we will not have to perform computations for a large inclination angle, at least for analytic calculations, because the expressions obtained in the expansion of a small inclination angle will be sufficient to determine all the coefficients.
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that we want to know. Then, extrapolation to a large inclination angle is exact. To
do this, we also need to know the maximum power of $Y$ at each post-Newtonian
order. This issue will be studied in a future publication.

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Appendix A

Reformulation of the Teukolsky equation for $|m| \leq 1$

- $m = 0$. In this case, $\sigma \ddot{S} = \sin^2 \theta \times$ (finite polynomial of $\cos \theta$) in the PN
approximation. Thus, we can write Eq. (3.18) as

$$Z_A = \frac{\mu}{2\lambda B^0} \int_{-\infty}^{\infty} d\lambda e^{i\omega t - im(\frac{\lambda}{\lambda}) + \psi^{(r)}} \left( \frac{-1}{2\sqrt{2\pi} \rho} \right) \left[ \frac{D^2}{\Delta^2} (\alpha \Sigma_2 - 2ia \rho_1 \Sigma_1) R_A ight.$$

$$- 2D_r (D_{\theta} \sin^2 \theta) \left( \frac{1}{\sin^2 \theta} (J_{-} - (\rho + \rho)) + i a \frac{2 \Sigma_1}{\sin^2 \theta} (\rho - \rho) J_{-} \right) R_A$$

$$+ (D_{\theta} \sin^2 \theta)^2 \frac{2 \Sigma_1}{\sin^2 \theta} (J_{-}^2 - 2\rho J_{-}) R_A \right]_{r = r(\lambda)}.$$  \hspace{1cm} (A.1)

This form has no factors of $1/\sin \theta$.

- $m = \pm 1$. In this case, the $D_{\theta} \Sigma_1$ and $D_{\theta} \Sigma_2$ terms have $1/\sin^2 \theta$ factors. We can treat the $D_{\theta} \Sigma_1$ term here in a manner similar to that for $|m| \geq 2$. Therefore, we have to consider the $D_{\theta} \Sigma_2$ term. Only in the case $m = \pm 1$, we can find the equation

$$\frac{d}{d\lambda} X_{\pm 1} = \left[ D_{\theta} + \left( \frac{1}{1 \pm \cos \theta} \frac{d \cos \theta}{d\lambda} - ia \mathcal{E} + i \left( \frac{L}{\sin^2 \theta} \right) \right) \right] X_{\pm 1}. \quad (A.2)$$

Since $\ddot{S}$ is proportional to $\sin^2 \theta (1 \pm \cos \theta)$, we obtain the relation

$$D_{\theta} \Sigma_{\pm 1} = (D_{\theta} \sin^2 \theta) \frac{2 \ddot{S}}{\sin^2 \theta (1 \pm \cos \theta)} \left[ \frac{d \cos \theta}{d\lambda} \right.$$

$$+ (1 \pm \cos \theta) \left( \pm \frac{d}{d\lambda} + ia \mathcal{E} - i \left( \frac{L}{\sin^2 \theta} \right) \right) \right] X_{\pm 1}. \quad (A.3)$$
Therefore, the amplitude of the partial wave for $m = \pm 1$ is

$$Z_A = \frac{\mu}{2i\omega B\text{inc}} \int_{-\infty}^{\infty} d\lambda e^{i\omega t - im((\vec{\rho} + \phi(r)) - \frac{1}{2}\sqrt{2}\rho)} \left( \frac{D_r^2}{\Delta^2} (\rho \bar{Z}_m - 2i\alpha_1 \bar{Z}_m) R_A - \frac{2D_r}{\Delta} \left\{ F_m(0, \bar{Z}_m, \bar{Z}_m)(J_-(\rho + \bar{\rho})) + ia(D_\theta \sin^2 \theta) \frac{2\bar{Z}_m}{\sin^2 \theta}(\rho - \bar{\rho}) \right\} R_A \right.$$ 

$$+ \frac{(D_\theta \sin^2 \theta) 2\bar{S}}{\sin^2 \theta(1 + m \cos \theta)} \left\{ -\frac{d\cos \theta}{d\lambda} X_m \right\} + (1 + m \cos \theta) \left\{ \frac{m d}{d\lambda} + iaE_e - i \left\{ \frac{L}{\sin^2 \theta} \right\} X_m \right\} \times (J_2^2 - 2\rho J_-) R_A \right|_{r = r(\lambda)} \right) \times (J_2^2 - 2\rho J_-) R_A \left. \right|_{r = r(\lambda)} \right) . \quad (A.4)$$

**Appendix B**

**Formulas for phase evolution**

In this appendix we present all the detailed formulas for the orbital and phase evolutions to supplement §4.2. The expression for $dF_\theta/dt$ corresponding to (4.4) is

$$dF_\theta/dt = \frac{96}{5\pi M^2} \left( \pi MF_\theta \right) \frac{11}{3} \left[ \left( 1 + \frac{157}{24} e^2 \right) - \left( \frac{743}{336} + \frac{3683}{112} e^2 \right) (\pi MF_\theta)^{\frac{1}{3}} \right.$$ 

$$- \left( \frac{73Y}{12} + \frac{857}{48} e^2 \right) q(\pi MF_\theta) \right.$$ 

$$+ \left( \frac{4 + \frac{2335}{48} e^2}{96} \right) \pi(\pi MF_\theta) + \left( \frac{34103}{18144} - \frac{89353}{1728} e^2 \right) (\pi MF_\theta)^{\frac{4}{3}} \right.$$ 

$$+ \left( \frac{233}{96} + \frac{527}{96} Y^2 \right) + \left( \frac{2875}{96} + \frac{4165}{96} e^2 \right) q^2(\pi MF_\theta) \right.$$ 

$$- \left( \frac{4159}{672} + \frac{2013}{96} e^2 \right) \pi(\pi MF_\theta)^{\frac{1}{3}} \right.$$ 

$$- \left( \frac{13907Y}{336} + \frac{84365Y}{336} e^2 \right) q(\pi MF_\theta)^{\frac{1}{3}} \right] . \quad (B.1)$$

As mentioned in §4.2, the standard post-Newtonian calculation uses a frequency that is defined differently, namely $F_N = (1 - Y)F_\theta + YF_\phi$. For comparison, we also give an expression for $dF_N/dt$,

$$dF_N/dt = \frac{96}{5\pi M^2} \left( \pi MF_N \right) \frac{11}{3} \left[ \left( 1 - \frac{743}{336} (\pi MF_N)^{\frac{1}{3}} + \left( \frac{4\pi - \frac{113Y}{12} q}{18144} \right) (\pi MF_N) \right.$$ 

$$+ \left( \frac{34103}{18144} + \left( \frac{233}{96} + 2Y + \frac{527}{96} Y^2 \right) q^2 \right) \right] . \quad (B.2)$$

This agrees with the standard post-Newtonian result, up to 1.5PN.\textsuperscript{41}
The expression for the phase evolution of the $\theta$-oscillation corresponding to (4.3)
is

$$
\Phi_{\theta}(F_\theta) = \phi_0^\prime - \frac{1}{16} (\pi MF_\theta)^{3/2} \left[ 1 + \frac{3715}{1008} (\pi MF_\theta)^{3/2} + \left\{ \frac{365}{24} Y_I q - 10\pi \right\} (\pi MF_\theta)^{3/2} 
\right.
+ \left\{ \frac{15293365}{1016064} + q^2 \left( \frac{1165}{96} - \frac{2635}{96} Y_I^2 \right) \right\} (\pi MF_\theta)^{3/2} 
+ \left\{ \frac{38645}{672} - q \frac{688405}{2016} Y_I \right\} (\pi MF_\theta)^{3/2} \ln(\pi MF_\theta) + \delta \Phi_\theta^{(\delta Y)} + \delta \Phi_\theta^{(c)} \right],
$$

$$
\delta \Phi_\theta^{(\delta Y)} = q^2 (1 - Y_I^2) \left( \frac{22265}{33} + \frac{21072205}{73728} e_f (\pi MF_\theta)^{1/2} \right) (\pi MF_\theta)^2,
$$

$$
\delta \Phi_\theta^{(c)} = e_f^2 (\pi MF_\theta)^{1/2} \left[ - \frac{785}{272} - \frac{2045665}{225792} (\pi MF_\theta)^{3/2} 
\right.
+ \left\{ \frac{65561}{2880} - q \frac{2953}{540} Y_I \right\} (\pi MF_\theta)^{3/2} 
- \left\{ \frac{111064865}{10948608} + \left( \frac{698695}{101376} - \frac{650665}{101376} Y_I^2 \right) q^2 \right\} (\pi MF_\theta)^{3/2} 
+ \left\{ \frac{3873451}{86184} - q \frac{1899015067}{8273664} Y_I \right\} (\pi MF_\theta)^{3/2} \right],
$$

where $\phi_0^\prime$ is a constant of integration.

In order to compute the waveform, we also need to know $t(F) \equiv \int (dF/dt)^{-1} dF$. This computation can be carried out in a manner similar to that for the calculation of the phase $\Phi(F)$. The results are

$$
t_\varphi(F_\varphi) = t_\varphi^c - \frac{5}{256} \mathcal{M}(\pi MF_\varphi)^{-3/2} \left[ 1 + \frac{743}{252} (\pi MF_\varphi)^{3/2} 
\right.
+ \left\{ q \left( \frac{16}{5} + \frac{146}{15} Y_I \right) - \frac{32}{5} \right\} (\pi MF_\varphi)^{3/2} 
+ \left\{ \frac{3058673}{508032} + \left( \frac{233}{48} - 4Y_I - \frac{527}{48} Y_I^2 \right) q^2 \right\} (\pi MF_\varphi)^{3/2} 
+ \left\{ \frac{743}{63} + \frac{137681}{756} Y_I \right\} - \frac{7729}{252} \pi \right\} (\pi MF_\varphi)^{3/2} + \delta t^{(\delta Y)} + \delta t^{(c)} \right],
$$

$$
\delta t_\varphi^{(\delta Y)} = -q^2 (1 - Y_I^2) \left( \frac{4453}{216} - \frac{4214441}{14400} e_f^2 (\pi MF_\varphi)^{1/2} \right) (\pi MF_\varphi)^2,
$$

$$
\delta t_\varphi^{(c)} = e_f^2 (\pi MF_\varphi)^{1/2} \left[ - \frac{157}{43} - \frac{409133}{37296} (\pi MF_\varphi)^{3/2} 
\right.
+ \left\{ \frac{65561}{2448} \pi - \frac{242}{9} + \frac{2953}{459} Y_I \right\} (\pi MF_\varphi)^{3/2} 
+ \left\{ \frac{22212973}{1928448} + \left( \frac{139739}{17856} + \frac{109}{6} Y_I + \frac{130133}{17856} Y_I^2 \right) q^2 \right\} (\pi MF_\varphi)^{3/2} 
+ \left\{ \frac{3873451}{79380} \pi - \frac{249437}{4536} + \frac{1899015067}{7620480} Y_I \right\} (\pi MF_\varphi)^{3/2} \right] \right].
$$
\[ t_\theta(F_\theta) = t_\phi - \frac{5}{256} M(\pi MF_\theta)^{-\frac{8}{3}} \left[ 1 + \frac{743}{252} (\pi MF_\theta)^\frac{2}{3} + \left\{ \frac{146}{15} Y_\theta - \frac{32}{5} \pi \right\} (\pi MF_\theta) \right. \\
+ \left\{ \frac{3058673}{508032} + \left( \frac{233}{48} - \frac{527}{48} Y^2_\theta \right) \right\} (\pi MF_\theta)^\frac{4}{3} \\
+ \left\{ \frac{137681}{756} Y_\theta - \frac{7729}{252} \pi \right\} (\pi MF_\theta)^\frac{2}{3} + \left( \delta t_{\theta}^{(e)} + \delta t_{\theta}^{(c)} \right) \right], \\
\delta t_{\theta}^{(c)} = -q^2 (1 - Y^2_\theta) \left( \frac{1575}{216} - \frac{4241441}{14400} e^2 (\pi MF_\theta)^{-\frac{19}{9}} \right) (\pi MF_\theta)^2, \\
\delta t_{\theta}^{(e)} = e^2 (\pi MF_\theta)^{-\frac{19}{9}} \left[ \frac{65561}{2448} - \frac{2953}{459} qY_\theta \right] (\pi MF_\theta) \\
+ \left\{ \frac{22212973}{1928448} - \left( \frac{139739}{17856} + \frac{130133}{17856} Y^2_\theta \right) \right\} (\pi MF_\theta)^\frac{4}{3} \\
+ \left\{ \frac{3873451}{79380} - \frac{1899015067}{7620480} qY_\theta \right\} (\pi MF_\theta)^\frac{2}{3} \right], \quad (B.5) \]

where \( t_\xi^c \) and \( t_\phi^c \) are constants of integration.

Writing the Fourier components of the waveforms as \( \tilde{h}(f) = A e^{i\psi(f)} \), \( \psi(f) \) is calculated as \( \psi(f) = -\Phi(f) + 2\pi ft(f) \), using the stationary phase approximation.44) The final results are

\[ \psi_\phi(f) = 2\pi ft_\phi^c - \phi_\phi^c + \frac{3}{128} (\pi MF)^{\frac{8}{3}} \left[ 1 + \frac{3715}{756} (\pi MF)^\frac{2}{3} \right. \\
+ \left\{ \frac{73}{3} Y_\theta q - 16\pi \right\} (\pi MF) \\
+ \left\{ \frac{15293365}{508032} + q^2 \left( \frac{1165}{48} - \frac{2635}{48} Y^2_\theta \right) \right\} (\pi MF)^\frac{4}{3} \\
+ \left\{ \frac{38645}{252} - \frac{688405}{756} qY_\theta \right\} (\pi MF)^\frac{2}{3} \ln(\pi MF) + \delta \psi_\phi^{(e)} + \delta \psi_\phi^{(c)} \right], \]

\[ \delta \psi_\phi^{(dY)} = q^2 (1 - Y^2_\theta) \left( \frac{111325}{1296} - \frac{71645497}{138240} e^2 (\pi MF)^{-\frac{19}{9}} \right) (\pi MF)^2, \\
\delta \psi_\phi^{(e)} = e^2 (\pi MF)^{-\frac{19}{9}} \left[ \frac{65561}{4080} - \frac{2953}{765} Y_\theta \right] (\pi MF) \\
+ \left\{ \frac{111064865}{14141952} + q^2 \left( \frac{698695}{130944} + \frac{650665}{130944} Y^2_\theta \right) \right\} (\pi MF)^\frac{4}{3} \\
+ \left\{ \frac{3873451}{100548} - \frac{1899015067}{9652608} Y_\theta \right\} (\pi MF)^\frac{2}{3} \right], \quad (B.6) \]

and

\[ \psi_\theta(f) = \psi_\phi(f) + 2\pi f \delta t^c - \delta \phi^c \\
+ \frac{3}{128} (\pi MF)^{-\frac{8}{3}} \left[ \frac{40}{3} q(\pi MF) - 20Y_\theta q^2 (\pi MF)^\frac{4}{3} \right] \]
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\[- \frac{3715}{189} q(\pi Mf)^{\frac{5}{3}} \ln(\pi Mf) + \delta\psi(e) \]

\[\delta\psi(e) = e^2(\pi Mf)^{-\frac{19}{15}} \left[ - \frac{212}{15} q(\pi Mf) + \frac{545}{44} q^2 Y_1(\pi Mf)^{\frac{4}{3}} 
- \frac{1247185}{28728} q(\pi Mf)^{\frac{5}{3}} \right]. \] (B.7)

Here, we define \(\delta t^c \equiv t^c - t^c_0\), and we have absorbed a constant phase into \(\phi^c\) and \(\delta\phi^c\).

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