U(3) and Pseudo-U(3) Symmetry of the Relativistic Harmonic Oscillator

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We show that a Dirac Hamiltonian with equal scalar and vector harmonic oscillator potentials has not only a spin symmetry but an U(3) symmetry and that a Dirac Hamiltonian with scalar and vector harmonic oscillator potentials equal in magnitude but opposite in sign has not only a pseudospin symmetry but a pseudo-U(3) symmetry. We derive the generators of the symmetry for each case.

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As is well-known, the non-relativistic spherical harmonic oscillator has degeneracies in addition to those due to rotational invariance. The energy spectrum depends only on the total harmonic oscillator quantum number \( N = 2n + \ell \), where \( n \) is the radial quantum number and \( \ell \) is the orbital angular momentum. Hence the states with \( \ell = N, N - 2, \ldots, 0 \) or 1 have the same energy. These degeneracies are produced by an U(3) symmetry [1]. This U(3) symmetry has been influential in connecting the shell model with collective motion [2]. Also the energy does not depend on the orientation of the spin and hence the non-relativistic harmonic oscillator has a spin symmetry as well.

Since relativistic models of nuclei are now so prevalent [3], we can ask if U(3) symmetry resides in the relativistic harmonic oscillator. Indeed the Dirac Hamiltonian, \( H \), for which the scalar, \( V_S(\vec{r}) \), and vector, \( V_V(\vec{r}) \), potentials are equal and harmonic has been solved analytically and is invariant under a spin symmetry [4, 5]; that is \([\vec{S}, H] = 0\) where \( \vec{S} \) is given in Eq.(2). Just as for the non-relativistic harmonic oscillator, the spherically symmetric relativistic harmonic oscillator energy spectrum depends only on the total harmonic oscillator quantum number \( N \), although the energy spectrum for the relativistic harmonic oscillator spectrum in general does not have a linear dependence on \( N \) as does the non-relativistic harmonic oscillator. This suggests that the relativistic harmonic oscillator does have an U(3) symmetry. If this is the case, the question is: what are the relativistic generators? In this letter we shall show that there is indeed a U(3) symmetry and we shall derive the generators.

The Dirac Hamiltonian for a spherical harmonic oscillator with spin symmetry is

\[
H = \vec{\alpha} \cdot \vec{p} + \beta M + (1 + \beta)V(r),
\]

where \( \vec{\alpha}, \beta \) are the Dirac matrices, \( \vec{p} \) is the momentum, \( M \) is the mass, \( V(r) = \frac{M\omega^2}{2} r^2 \), \( \vec{r} \) is
the radial coordinate, \( r \) its magnitude, and the velocity of light is set equal to unity, \( c = 1 \).

The generators for the spin SU(2) algebra and the orbital angular momentum SU(2) algebra, \( \vec{S}, \vec{L} \), which commute with the Dirac Hamiltonian, \( [H, \vec{S}] = [H, \vec{L}] = 0 \), are given by

\[
\vec{S} = \begin{pmatrix}
\vec{s} & 0 \\
0 & U_p \vec{s} U_p
\end{pmatrix}, \\
\vec{L} = \begin{pmatrix}
\vec{\ell} & 0 \\
0 & U_p \vec{\ell} U_p
\end{pmatrix},
\]

where \( \vec{s} = \frac{\vec{\sigma}}{2} \) are the usual spin generators, \( \vec{\sigma} \) the Pauli matrices, \( \vec{\ell} = (\vec{r} \times \vec{p}) / \hbar \), and \( U_p = \frac{\vec{\sigma} \cdot \vec{p}}{p} \) is the helicity unitary operator introduced in [7].

The non-relativistic U(3) generators are the orbital angular momentum \( \vec{\ell} \), the quadrupole operator \( q_m = \frac{1}{\hbar M \omega} \sqrt{\frac{3}{2}} (M^2 \omega^2 [rr]_m^{(2)} + [pp]_m^{(2)}) \), where \([rr]_m^{(2)} \) means coupled to angular momentum rank 2 and projection \( m \), and \( \mathcal{N}_{NR} = \frac{1}{2\sqrt{2}\hbar M \omega} (M^2 \omega^2 r^2 + p^2) - \frac{3}{2} \). They form the closed U(3) algebra

\[
[\mathcal{N}_{NR}, \vec{\ell}] = [\mathcal{N}_{NR}, q] = 0,
\]

with \( \mathcal{N}_{NR} \) generating a U(1) algebra whose eigenvalues are the total number of quanta \( N \) and \( \vec{\ell}, q \) generating an SU(3) algebra. In the above we use the coupled commutation relation between two tenors, \( T_1^{(t_1)}, T_2^{(t_2)} \) of rank \( t_1, t_2 \) which is

\[
[T_1^{(t_1)}, T_2^{(t_2)}]^{(t)} = [T_1^{(t_1)} T_2^{(t_2)}]^{(t)} - (-1)^{t_1+t_2-t} [T_2^{(t_2)} T_1^{(t_1)}]^{(t)}
\]

The relativistic orbital angular momentum generators \( \vec{L} \) are given in Eq. [2]. We shall now determine the the quadrupole operator \( Q_m \) and monopole operator \( \mathcal{N} \) that commute with the Hamiltonian in Eq. (1). In order for the quadrupole generator

\[
Q_m = \begin{pmatrix}
(Q_m)_{11} & (Q_m)_{12} \vec{\sigma} \cdot \vec{p} \\
\vec{\sigma} \cdot \vec{p} (Q_m)_{21} & \vec{\sigma} \cdot \vec{p} (Q_m)_{22}
\end{pmatrix},
\]
to commute with the Hamiltonian, \([Q_m, H] = 0\), the matrix elements must satisfy the conditions,

\[
(Q_m)_{12} = (Q_m)_{21}, \quad (5a)
\]

\[
[(Q_m)_{11}, V] + [(Q_m)_{12}, p^2] = 0, \quad (5b)
\]

\[
[(Q_m)_{12}, V] + [(Q_m)_{22}, p^2] = 0, \quad (5c)
\]

\[
(Q_m)_{11} = (Q_m)_{12} \ (V + 2M) + (Q_m)_{22} \ p^2. \quad (5d)
\]

One solution is

\[
Q_m = \lambda_2 \left( \frac{M \omega^2}{2} \left( \frac{M \omega^2}{2} r^2 + 2M \right) [rr]_m^{(2)} + [pp]_m^{(2)} \frac{M \omega^2}{2} [rr]_m^{(2)} \vec{\sigma} \cdot \vec{p} \right), \quad (6)
\]

where \(\lambda_2\) is an overall constant undetermined by the commutation of \(Q_m\) with the Dirac Hamiltonian.

For this quadrupole operator to form a closed algebra, the commutation with itself must be the orbital angular momentum operator as in Eq. (3). This commutation relation gives

\[
[Q, Q]^{(t)} = \sqrt{10} \lambda_2^2 M \omega^2 \hbar^2 \begin{pmatrix}
\frac{M \omega^2}{2} r^2 + 2M & \vec{\ell} \cdot \vec{\sigma} \cdot \vec{p} \\
\vec{\sigma} \cdot \vec{p} \vec{\ell} & 0
\end{pmatrix} = \sqrt{10} \lambda_2^2 M \omega^2 \hbar^2 (H + M) \vec{L} \delta_{t,1},
\]

and we get the desired result if \(\lambda_2 = \sqrt{\frac{3}{M \omega^2 \hbar^2 (H + M)}}\). The quadrupole operator then becomes
\[ Q_m = \sqrt{\frac{3}{M\omega^2\hbar^2(H + M)}} \left( \frac{M\omega^2}{2} \frac{r^2 + 2M}{[rr]^{(2)}_m} + [pp]^{(2)}_m \frac{M\omega^2}{2} \frac{\vec{\sigma} \cdot \vec{p}}{[rr]^{(2)}_m} \right). \] (8)

In order for the monopole generator
\[ \mathcal{N} = \begin{pmatrix} (\mathcal{N})_{11} & (\mathcal{N})_{12} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} (\mathcal{N})_{21} & \vec{\sigma} \cdot \vec{p} (\mathcal{N})_{22} \end{pmatrix} + N_0, \] (9)
to commute with the Hamiltonian, \([\mathcal{N}, H] = 0\), the matrix elements must satisfy the conditions in Eq.(13) with \(Q_m\) replaced by \(\mathcal{N}\). \(N_0\) is a constant. A solution is
\[ \mathcal{N} = \lambda_0 \begin{pmatrix} \frac{M\omega^2}{2} \frac{r^2 + 2M}{[rr]_m} + \frac{M\omega^2}{2} \frac{\vec{\sigma} \cdot \vec{p}}{r^2} \\ \vec{\sigma} \cdot \vec{p} \frac{M\omega^2}{2} \frac{r^2}{p^2} \end{pmatrix} + N_0. \] (10)

Straightforward calculations show that \(\mathcal{N}\) commutes with the the other generators as well as the Dirac Hamiltonian and consequently is the U(1) generator. However, the constants \(\lambda_0, N_0\) are undetermined by these commutation relations. These constants are determined instead by requiring that the eigenvalue of \(\mathcal{N}\) is the the total harmonic oscillator number, \(N\); that is, \(\mathcal{N} \Psi_N = N \Psi_N\), where \(\Psi_N\) are the eigenfunctions of the Dirac Hamiltonian, \(H \Psi_N = E_N \Psi_N\). Using the facts that \([4, 5]\)

\[ \Psi_N = \begin{pmatrix} g \\ \frac{\vec{\sigma} \cdot \vec{p}}{E_N + M} \end{pmatrix} [p^2 + (E_N + M)V(r) - 2 \hbar \sqrt{(E_N + M)M\omega^2(N + \frac{3}{2})}]g = 0, \] (11)

we derive that
\[ \mathcal{N} \Psi_N = [2 \hbar \lambda_0 \sqrt{(E_N + M)M\omega^2(N + \frac{3}{2}) + N_0}] \Psi_N = N \Psi_N, \] (12)

which determines \(\lambda_0 = \frac{1}{2 \hbar \sqrt{(H + M)M\omega^2}}, N_0 = -\frac{3}{2}\).
In the non-relativistic limit, \( H \to M, M \to \infty \),

\[
Q_m \to \frac{1}{\hbar M \omega} \sqrt{\frac{3}{2}} \begin{pmatrix}
M^2 \omega^2 \ [rr]_m^{(2)} + [pp]_m^{(2)} & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
q_m & 0 \\
0 & 0
\end{pmatrix},
\]

\[
\mathcal{N} \to \frac{1}{2 \sqrt{2} \hbar M \omega} \begin{pmatrix}
M^2 \omega^2 r^2 + p^2 & 0 \\
0 & 0
\end{pmatrix} - \frac{3}{2} = \begin{pmatrix}
\mathcal{N}_{NR} & 0 \\
0 & -\frac{3}{2}
\end{pmatrix},
\]

which agrees with the non-relativistic generators.

The commutation relations are then those of the U(3) algebra,

\[
[N, \bar{L}] = [N, Q_m] = 0.
\]

\[
[\bar{L}, \bar{L}]^{(t)} = -\sqrt{2} \bar{L} \delta_{t,1}, \quad [\bar{L}, Q]^{(t)} = -\sqrt{6} Q \delta_{t,2}, \quad [Q, Q]^{(t)} = 3\sqrt{10} \bar{L} \delta_{t,1}.
\]

The spin generators in Eq. (\ref{eq:spin}), \( S \), commute with the U(3) generators as well as the Dirac Hamiltonian, and so the invariance group is \( U(3) \times SU(2) \), where the \( SU(2) \) is generated by the spin generators, \([S, S]^{(t)} = -\sqrt{2} S \delta_{t,1} \).

The Dirac Hamiltonian for a spherical harmonic oscillator with pseudospin symmetry is\footnote{\ref{ref:dirac}}

\[
\bar{H} = \vec{\alpha} \cdot \vec{p} + \beta M + (1 - \beta) V(r),
\]

which explains the pseudospin doublets observed in nuclei\footnote{\ref{ref:pseudospin}}. This pseudospin Hamiltonian can be obtained from the spin Hamiltonian with a transformation by \( \gamma_5 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \) and \( M \to -M \), but \( M \omega^2 \to M \omega^2 \). The generators for the pseudospin\footnote{\ref{ref:pseudospin_algebra}} and pseudo- \( U(3) \) algebra which commute with the Dirac Hamiltonian, \([\bar{H}, \bar{S}] = [\bar{H}, \bar{L}] = [\bar{H}, \bar{Q}_m] = [\bar{H}, \mathcal{N}] = 0\).
0, are then obtained by the same transformation and are given by

\[ \vec{S} = \begin{pmatrix} U_p \vec{s} U_p & 0 \\ 0 & \vec{s} \end{pmatrix} , \quad \vec{L} = \begin{pmatrix} U_p \vec{\ell} U_p & 0 \\ 0 & \vec{\ell} \end{pmatrix} , \]

(17a)

\[ \tilde{Q}_m = \sqrt{\frac{3}{M \omega^2 \hbar^2 (H - M)}} \begin{pmatrix} [pp]^{(2)}_m \\ \frac{M \omega^2}{2} [rr]^{(2)}_m \sigma \cdot \vec{p} \\ \frac{M \omega^2}{2} [rr]^{(2)}_m [rr]^{(2)}_m \sigma \cdot \vec{p} \frac{M \omega^2}{2} r^2 - 2M [rr]^{(2)}_m + [pp]^{(2)}_m \end{pmatrix} , \]

(17b)

\[ \tilde{N} = \frac{1}{2 \hbar \sqrt{M \omega^2 (H - M)}} \begin{pmatrix} p^2 \\ \frac{M \omega^2}{2} \sigma \cdot \vec{p} r^2 \\ \frac{M \omega^2}{2} (M \omega^2 r^2 - 2M r^2 + p^2) \end{pmatrix} - \frac{3}{2} . \]

(17c)

The relativistic non-spherical harmonic oscillator has also been solved analytically \[5\] as well. The relativistic axially symmetric deformed harmonic oscillator will have a \( U(2) \) symmetry as does the non-relativistic axially symmetric deformed harmonic oscillator. This will be discussed in a forthcoming paper.

In summary, we have shown that a Dirac Hamiltonian with equal scalar and vector harmonic oscillator potentials has an \( U(3) \times SU(2) \) symmetry and that a Dirac Hamiltonian with scalar and vector harmonic oscillator potentials equal in magnitude but opposite in sign has a pseudo-\( U(3) \times pseudo-SU(2) \) symmetry and we have derived the corresponding generators for each case. If speculation that an anti-nucleon can be bound inside a nucleus is valid \[11\], the anti-nucleon spectrum will have an approximate spin symmetry and, most likely an approximate \( U(3) \) symmetry, because the vector and scalar potentials are approximately equal and are very strong \[4\].

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