Propagation of Chaos of Forward–Backward Stochastic Differential Equations with Graphon Interactions

Erhan Bayraktar 1 · Ruoyu Wu 2 · Xin Zhang 3

Accepted: 30 March 2023 / Published online: 12 May 2023
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2023

Abstract
In this paper, we study graphon mean field games using a system of forward–backward stochastic differential equations. We establish the existence and uniqueness of solutions under two different assumptions and prove the stability with respect to the interacting graphons which are necessary to show propagation of chaos results. As an application of propagation of chaos, we prove the convergence of $n$-player game Nash equilibrium for a general model, which is new in the theory of graphon mean field games.

Keywords Large population games · Graphon mean field games · Propagation of chaos · FBSDE · Convergence of Nash equilibrium

Mathematics Subject Classification 49N80 · 60F25 · 91A06 · 91A15

1 Introduction

The theory of mean field games was pioneered independently by Lasry, Lions (see [1–3]) and Caines, Huang, Malhamé (see [4, 5]). It is the study of strategic decision making by small interacting agents in large populations, and we refer the readers to [6–10] for finite state mean field games, to [11–14] for uniqueness of mean field...
game solutions, and to [15, 16] for a nice survey. Since then, the convergence of $n$-player game Nash equilibrium to the solution of mean field game has attracted lots of attention. There are three different ways to tackle this problem: (1) establish the regularity of solutions of the so-called Master equations, see [17]; (2) use compactness arguments to show the existence of weak limits of $n$-player game Nash equilibrium, and prove any weak limit is a weak solution of mean field game solution, see [18, 19]; (3) prove it using propagation of chaos results for forward–backward stochastic differential equations or backward stochastic differential equations, see respectively [20, 21].

In this paper, we investigate an analogous $n$-player game convergence problem for graphon mean field games. The standard mean field game theory assumes that interaction of different agents is symmetric. Recently, asymmetric graph connections among agents have been considered; see e.g. [22–25]. The heterogeneous interaction of different agents is symmetric. Recently, asymmetric graph connections among agents have been considered; see e.g. [22–25]. The heterogeneous interaction of players is modeled by graphons, which is a natural notion for the limit of a sequence of dense graphs. It was first introduced by Lovász et al.; see e.g. [26–28], and have been used to characterize heterogeneously interacting particle systems; see e.g. [29, 30], and also has important applications in $k$-core theory; see e.g. [31, 32]. In graphon mean field games, given a graphon $G : [0, 1] \times [0, 1] \rightarrow \mathbb{R}_+$, each $\lambda \in [0, 1]$ stands for a type of large subpopulation, and the correlation between two subpopulations $\lambda$ and $\kappa$ is characterized by $G(\lambda, \kappa)$. As far as we know, the convergence problem for graphon mean field game has only been solved by [22] for a linear-quadratic model, where their argument heavily relied on the existence of explicit solutions. In this work, we study the limiting system which is a family of forward–backward stochastic differential equations (FBSDEs), and prove the $n$-player game convergence for a general model using propagation of chaos.

More specially, we investigate a coupled system of $n$ FBSDEs of the form

$$
\begin{align*}
\begin{cases}
\frac{dx_{i,n}^i}{dt} &= \frac{1}{n} \sum_{j=1}^{n} G_n(x_n^{i,j}, \frac{j}{n}) \tilde{B}(t, x_{i,n}^i, x_{i,n}^j, y_{i,n}^j) dt + \sigma \, dW_{i,n}^j, \\
\frac{dy_{i,n}^j}{dt} &= -\frac{1}{n} \sum_{j=1}^{n} G_n(x_n^{j,i}, \frac{i}{n}) \tilde{F}(t, x_{i,n}^i, x_{i,n}^j, y_{i,n}^j) dt + \sum_{j=1}^{n} z_{i,n}^{j,j} \, dW_{i,n}^j, \\
x_{i,n}^0 &= \xi_i, \\
y_{i,n}^j &= \frac{1}{n} \sum_{j=1}^{n} G_n(x_n^{j,i}, \frac{i}{n}) \hat{Q}(x_{i,n}^i, x_{i,n}^j), \\
&\quad i = 1, \ldots, n,
\end{cases}
\end{align*}
$$

where $\tilde{B}, \tilde{F}, \hat{Q}$ are drifts and terminal, $(\xi^{\lambda})_{\lambda \in [0,1]}, (W^{\lambda})_{\lambda \in [0,1]}$ are independent initial positions and Brownian motions respectively, and $G_n$ is a sequence of graphons characterizing interactions between $(X_{i,n}^i)_{i=1,\ldots,n}$. As $n \rightarrow \infty$ and $G_n \rightarrow G$, we show the propagation of chaos result that the above particle system converges to the following graphon interacting particle system

$$
\begin{align*}
\begin{cases}
\frac{dx_{\lambda}}{dt} &= \int_0^1 \int_{\mathbb{R}} G(\lambda, \kappa) \hat{B}(t, x_{\lambda}, x, y_{\lambda}) \, \mathcal{L}(x_{\lambda})(dx) \, d\kappa \, dt + \sigma \, dW_{\lambda}, \\
\frac{dy_{\lambda}}{dt} &= -\int_0^1 \int_{\mathbb{R}} G(\lambda, \kappa) \hat{F}(t, x_{\lambda}, x, y_{\lambda}) \, \mathcal{L}(x_{\lambda})(dx) \, d\kappa \, dt + Z_{\lambda} \, dW_{\lambda}, \\
x_{\lambda}^0 &= \xi^{\lambda}, \\
y_{\lambda}^j &= \int_0^1 \int_{\mathbb{R}} G(\lambda, \kappa) \hat{Q}(x_{\lambda}^j, x) \, \mathcal{L}(x_{\lambda}^j)(dx) \, d\kappa, \\
&\quad \lambda \in [0, 1],
\end{cases}
\end{align*}
$$

where $\hat{B}, \hat{F}, \hat{Q}$ are drifts and terminal, $(\xi^{\lambda})_{\lambda \in [0,1]}, (W^{\lambda})_{\lambda \in [0,1]}$ are independent initial positions and Brownian motions respectively, and $G_n$ is a sequence of graphons characterizing interactions between $(X_{i,n}^i)_{i=1,\ldots,n}$. As $n \rightarrow \infty$ and $G_n \rightarrow G$, we show the propagation of chaos result that the above particle system converges to the following graphon interacting particle system

$$
\begin{align*}
\begin{cases}
\frac{dx_{\lambda}}{dt} &= \int_0^1 \int_{\mathbb{R}} G(\lambda, \kappa) \hat{B}(t, x_{\lambda}, x, y_{\lambda}) \, \mathcal{L}(x_{\lambda})(dx) \, d\kappa \, dt + \sigma \, dW_{\lambda}, \\
\frac{dy_{\lambda}}{dt} &= -\int_0^1 \int_{\mathbb{R}} G(\lambda, \kappa) \hat{F}(t, x_{\lambda}, x, y_{\lambda}) \, \mathcal{L}(x_{\lambda})(dx) \, d\kappa \, dt + Z_{\lambda} \, dW_{\lambda}, \\
x_{\lambda}^0 &= \xi^{\lambda}, \\
y_{\lambda}^j &= \int_0^1 \int_{\mathbb{R}} G(\lambda, \kappa) \hat{Q}(x_{\lambda}^j, x) \, \mathcal{L}(x_{\lambda}^j)(dx) \, d\kappa, \\
&\quad \lambda \in [0, 1],
\end{cases}
\end{align*}
$$
where \( \mathcal{L}(X_t^\kappa) \) denotes the law of \( X_t^\kappa \). By the stochastic maximum principle, the solution of the graphon mean field game can be characterized using (1.2). One can conclude the convergence of \( n \) player game using the convergence \([1.1 \Rightarrow (1.2)]\).

To carry out our analysis, first we study the existence and uniqueness of the limiting system (1.2). To properly define the interaction term

\[
\int_0^1 \int_\mathbb{R} G(\lambda, \kappa) \tilde{B}(t, X_t^\lambda, x, Y_t^\lambda) \mathcal{L}(X_t^\kappa)(dx) \, d\kappa \, dt,
\]
we must show the measurability of \( \lambda \to \mathcal{L}(X_t^\lambda) \). Since we are only interested in the marginals \( \mathcal{L}(X_t^\lambda)_{\lambda \in [0,1]} \) instead of the joint law \( \mathcal{L}(X_t) \), one can study another FBSDE system where all the \( (X_t^\lambda, Y_t^\lambda) \) are driven by the same Brownian motion. The marginal laws of these two systems are the same which is a useful observation in establishing measurability. Then under two commonly used monotonicity assumptions as in [33], we show the existence and uniqueness of solutions to (1.2). Next we study the stability of solutions with respect to the interacting graphon \( G \). As a natural notion of distance between two different graphons, the cut norm is widely used and is weaker than the \( L^1 \)-norm; see e.g. [28]. To make an estimation involving the cut norm, we adopt the argument of [29, Theorem 2.1] where the boundedness of solutions in the \( L^p \) norm for \( p > 2 \) is necessary. This is the reason that we prove existence results in general \( (L^p)_{p \geq 2} \) spaces. Finally, we prove the propagation of chaos for FBSDEs, where the stability is essential. Assuming that the cut norm convergence of the interacting graphons \( \|G_n - G\|_\square \to 0 \), we show that \([1.1 \Rightarrow (1.2)]\). Under a stronger condition that \( G_n \) is the uniform block sampling of \( G \), we can obtain the convergence rate.

The rest of the paper is organized as follows. In Sect. 2, we prove the unique existence of solutions of (2.1) under two different assumptions. In Sect. 3, assuming that the interaction is linear in \( G \), we show the stability of solutions with respect to \( G \). In Sect. 4, we prove the Propagation of Chaos result, and in Sect. 5 we apply previous results to a simple model of graphon mean field game.

### 1.1 Notation

Let us take \( \mathcal{P}_p(\mathbb{R}) \) to be the Wasserstein space of \( p \)-integrable probability measures on \( \mathbb{R} \). Denote by \( \mathcal{M}([0, T]; \mathcal{P}_p(\mathbb{R})) \) and \( \mathcal{C}([0, T]; \mathcal{P}_p(\mathbb{R})) \) the space of measurable functions from \([0, T] \) to \( \mathcal{P}_p(\mathbb{R}) \) and the space of continuous functions from \([0, T] \) to \( \mathcal{P}_p(\mathbb{R}) \) respectively. Define the space of families of probability flows as

\[
P F^p := \{ \mu : [0, 1] \to \mathcal{C}([0, T]; \mathcal{P}_p(\mathbb{R})) : \lambda \mapsto \mu^\lambda \text{ is measurable} \}.
\]

For any \( \mu, \tilde{\mu} \in \mathcal{P}(\mathcal{C}([0, T]; \mathbb{R}^2)) \), let us take

\[
\mathcal{W}_{2, T}(\mu, \tilde{\mu}) := \inf \left\{ \mathbb{E} \left[ \sup_{t \in [0, T]} |X_t - \tilde{X}_t|^2 + \sup_{t \in [0, T]} |Y_t - \tilde{Y}_t|^2 \right] : (X, Y) \sim \mu, (\tilde{X}, \tilde{Y}) \sim \tilde{\mu} \right\}.
\]
For a family of random variables \( \{X_\lambda\}_{\lambda \in [0,1]} \), we denote by \( \mathcal{L}^m(X) \) the set of laws \( \{\mathcal{L}(X_\lambda)\}_{\lambda \in [0,1]} \).

We define spaces of processes and random variables

- \( L^{p,2}_F \) to be the set of all \( \{F_t\}_{t \geq 0} \) progressively measurable real-valued process \( (X_t)_{t \geq 0} \) such that
  \[ \mathbb{E} \left[ \left( \int_0^T |X_t|^2 \, dt \right)^{p/2} \right] < +\infty. \]

- \( L^{p,c}_F \) to be the set of all \( \{F_t\}_{t \geq 0} \) progressively measurable real-valued continuous process \( (X_t)_{t \geq 0} \) such that
  \[ \mathbb{E} \left[ \sup_{t \in [0,T]} |X_t|^p \right] < +\infty. \]

- \( L^p_F \) to be the set of all \( F_t \)-measurable \( p \)-th integrable random variables.

- \( \mathcal{M}L^{p,2}_F \) to be the set of all measurable functions \( X \) from \( [0,1] \) to \( L^{p,2}_F \) such that
  \[ \max_{\lambda \in [0,1]} \mathbb{E} \left[ \left( \int_0^T |X^\lambda_t|^2 \, dt \right)^{p/2} \right] < +\infty, \]
  and similarly define \( \mathcal{M}L^{p,c}_F \), \( \mathcal{M}L^p_{F_t} \).

For any \( x \in \mathcal{M}L^{p,c}_F \), we define norms

\[ \|x\|_{k}^{I-p} := \mathbb{E} \left[ \int_{[0,1]} \int_0^T e^{kt} (x_\lambda^t)^p \, dt \, d\lambda \right], \]
\[ \|x\|_p := \max_{\lambda \in [0,1]} \mathbb{E} \left[ \int_0^T e^{kt} (x_\lambda^t)^p \, dt \right], \]
\[ \|x\|_S := \max_{\lambda \in [0,1]} \sup_{t \in [0,T]} \mathbb{E} \left[ |x_\lambda^t|^p \right]. \]

In particular, when \( p = 2 \), we write

\[ \|x\|_{k}^{I} := \mathbb{E} \left[ \int_{[0,1]} \int_0^T e^{kt} (x_\lambda^t)^2 \, dt \, d\lambda \right], \quad x \in \mathcal{M}L^{2,c}_F. \]

For any \( x \in L^{p,c}_F \), define its norm

\[ \|x\|_S := \sup_{t \in [0,T]} \mathbb{E}[|x_t|^p]. \]

\section{2 Unique Existence of Solutions}

In this section, we show the existence and uniqueness of solutions to a general FBSDE system (2.1) under two assumptions.

\[
\begin{aligned}
    dX^\lambda_t &= B^\lambda_G (t, X^\lambda_t, \mathcal{L}^m(X_t), Y^\lambda_t) \, dt + \sigma \, dW^\lambda_t, \\
    dY^\lambda_t &= -F^\lambda_G (t, X^\lambda_t, \mathcal{L}^m(X_t), Y^\lambda_t) \, dt + Z^\lambda_t \, dW^\lambda_t, \\
    X^\lambda_0 &= \xi^\lambda, \\
    Y^\lambda_T &= Q^\lambda_G (X^\lambda_T, \mathcal{L}^m(X_T)), \quad \forall \lambda \in [0,1].
\end{aligned}
\]
The proof is very similar to that of McKean Vlasov FBSDE, except the verification of measurability of $\lambda \mapsto \mathcal{L}(X^\lambda)$. Our main observation is that different types of players $(X^\lambda)$ are only interacting through their marginal laws $\mathcal{L}^m(X)$, and therefore by the weak uniqueness of FBSDE solutions, we can treat all processes $(X^\lambda, Y^\lambda, Z^\lambda)_{\lambda \in [0,1]}$ on one stochastic basis. Then we prove a stronger statement that $\lambda \mapsto X^\lambda$ is measurable in the $L^2$ sense, which implies the measurability of $\lambda \mapsto \mathcal{L}(X^\lambda)$.

**Definition 2.1** A family of processes $(X^\lambda, Y^\lambda, Z^\lambda)_{\lambda \in [0,1]}$ is said to be a solution of (2.1) if $\lambda \mapsto \mathcal{L}(X^\lambda)$ is measurable and $(X^\lambda, Y^\lambda, Z^\lambda)$ satisfies the FBSDE system (2.1) for each $\lambda \in [0,1]$.

For simplicity of notation, we suppress $G$ when the graphon is clear from the context. We make the following two assumptions in this section.

**Assumption 2.1** (i) $B^\lambda$ is Lipschitz in $x$, and there exists a constant $K_1 \in \mathbb{R}$ such that for any $\lambda \in [0,1]$, $(t, x, x', y) \in [0, T] \times \mathbb{R}^3$, $\eta \in \mathcal{M}([0,1]; \mathcal{P}_p(\mathbb{R}))$

$$(x - x') \cdot (B^\lambda(t, x, \eta, y) - B^\lambda(t, x', \eta, y)) \leq -K_1(x - x')^2$$

(ii) $F^\lambda$ is Lipschitz in $y$, and there exists a constant $K_2 \in \mathbb{R}$ such that for any $\lambda \in [0,1]$, $(t, x, y, y') \in [0, T] \times \mathbb{R}^3$, $\eta \in \mathcal{M}([0,1]; \mathcal{P}_2(\mathbb{R}))$

$$(y - y') \cdot (F^\lambda(t, x, \eta, y) - F^\lambda(t, x, \eta, y')) \leq -K_2(y - y')^2$$

(iii) $B^\lambda$ is $L_1$-Lipschitz in $y$, $F^\lambda$ is $L_2$-Lipschitz in $x$, $Q^\lambda$ is $L_3$-Lipschitz in $x$, and it holds that

$$|B^\lambda(t, x, \eta, y) - B^\lambda(t, x, \tilde{\eta}, y)| \leq \frac{L_1}{2} \mathcal{W}_p(\eta^\lambda, \tilde{\eta}^\lambda) + \frac{L_1}{2} \int_{[0,1]} \mathcal{W}_p(\eta^\lambda, \tilde{\eta}^\lambda) d\lambda,$$

$$|F^\lambda(t, x, \eta, y) - F^\lambda(t, x, \tilde{\eta}, y)| \leq \frac{L_2}{2} \mathcal{W}_p(\eta^\lambda, \tilde{\eta}^\lambda) + \frac{L_2}{2} \int_{[0,1]} \mathcal{W}_p(\eta^\lambda, \tilde{\eta}^\lambda) d\lambda,$$

$$|Q^\lambda(x, \eta) - Q^\lambda(x, \tilde{\eta})| \leq \frac{L_3}{2} \mathcal{W}_p(\eta^\lambda, \tilde{\eta}^\lambda) + \frac{L_3}{2} \int_{[0,1]} \mathcal{W}_p(\eta^\lambda, \tilde{\eta}^\lambda) d\lambda.$$

(iv) It holds that $pK_1 + pK_2 > (2p - 1)L_1 + (2p - 2)L_2$ and there exists a constant $k \in ((2p - 2)L_2 - pK_2, pK_1 - (2p - 1)L_1)$ such that

$$(k + pK_2 - (2p - 2)L_2) > 2^pL_1^pL_3^p + \frac{2^{p-1}L_1^2L_3^p + 2L_1L_2}{-k + pK_1 - (2p - 1)L_1}. \quad (2.2)$$

(v) We have that $\lambda \mapsto \mathcal{L}(\xi^\lambda)$ is measurable, $\sup_{\lambda \in [0,1]} \mathbb{E}[|\xi^\lambda|^p] < +\infty$, and $(B(\cdot, 0), F(\cdot, 0), Q) \in \mathcal{ML}_p^{2p} \times \mathcal{ML}_p^{2p} \times \mathcal{ML}_p^{p}$. 

 Springer
Assumption 2.2 (i) \((B^\lambda, F^\lambda)\) are \(L\)-Lipschitz in \((x, y)\), \(Q^\lambda\) is \(L\)-Lipschitz in \(x\), and it holds that
\[
|B^\lambda(t, x, \eta, y) - B^\lambda(t, x, \eta, y)| \leq lW_2(\eta^\lambda, \eta^\lambda) + l \int_{[0,1]} W_2(\eta^\lambda, \eta^\lambda) d\lambda.
\]
\[
|F^\lambda(t, x, \eta, y) - F^\lambda(t, x, \eta, y)| \leq lW_2(\eta^\lambda, \eta^\lambda) + l \int_{[0,1]} W_2(\eta^\lambda, \eta^\lambda) d\lambda.
\]
\[
|Q^\lambda(x, \eta) - Q^\lambda(x, \eta)| \leq lW_2(\eta^\lambda, \eta^\lambda) + l \int_{[0,1]} W_2(\eta^\lambda, \eta^\lambda) d\lambda.
\]

(ii) There exist a positive constant \(k > 3l\) such that for all \(\lambda \in [0, 1]\),
\[
-\Delta x^\lambda \left(F^\lambda(t, \theta^\lambda) - F^\lambda(t, \tilde{\theta}^\lambda)\right) + \Delta y^\lambda \left(B^\lambda(t, \theta^\lambda) - B^\lambda(t, \tilde{\theta}^\lambda)\right)
\leq -k(\Delta x^\lambda)^2 - k(\Delta y^\lambda)^2,
\]
\[
\Delta x^\lambda \left(Q^\lambda(x^\lambda, \eta) - Q^\lambda(\tilde{x}^\lambda, \eta)\right)
\geq k(\Delta x^\lambda)^2,
\]
where \(\Delta x^\lambda := x^\lambda - \tilde{x}^\lambda\), \(\Delta y^\lambda := y^\lambda - \tilde{y}^\lambda\), \(\theta^\lambda = (x^\lambda, \eta, y^\lambda), \tilde{\theta}^\lambda = (\tilde{x}^\lambda, \eta, \tilde{y}^\lambda)\).

(iii) We have that \(\lambda \mapsto \mathcal{L}(\xi^\lambda)\) is measurable, \(\sup_{\lambda \in [0, 1]} \mathbb{E}[|\xi^\lambda|^p] < +\infty\), and \((B(\cdot, 0), F(\cdot, 0), Q) \in \mathcal{ML}^{p,2}_m \times \mathcal{ML}^{p,2}_m \times \mathcal{ML}^{p}_m\).

Lemma 2.1 Under Assumption 2.1 or 2.2, there is a one-to-one correspondence between solutions to (2.1) and

\[
\begin{aligned}
\frac{dX^\lambda_t}{dt} &= B^\lambda_G(t, X^\lambda_t, \mathcal{L}^m(X_t), Y^\lambda_t) + \sigma dW_t, \\
\frac{dY^\lambda_t}{dt} &= -F^\lambda_G(t, X^\lambda_t, \mathcal{L}^m(X_t), Y^\lambda_t) + Z^\lambda_t dW_t, \\
X^\lambda_0 &= \xi^\lambda, \\
Y^\lambda_T &= Q^\lambda_G(X^\lambda_T, \mathcal{L}^m(X_T)), \quad \forall \lambda \in [0, 1].
\end{aligned}
\]

Proof Given a solution \((X, Y, Z)\) to (2.1), we plug in the law \(\mu(t) = [\mathcal{L}(X_t^\lambda)]_{\lambda \in [0, 1]}\) into the FBSDE for each \(\lambda \in [0, 1]\)

\[
\begin{aligned}
\frac{d\tilde{X}^\lambda_t}{dt} &= B^\lambda(t, \tilde{X}^\lambda_t, \mu(t), \tilde{Y}^\lambda_t) + \sigma dW_t, \\
\frac{d\tilde{Y}^\lambda_t}{dt} &= -F^\lambda(t, \tilde{X}^\lambda_t, \mu(t), \tilde{Y}^\lambda_t) + \tilde{Z}^\lambda_t dW_t, \\
\tilde{X}^\lambda_0 &= \tilde{\xi}^\lambda, \\
\tilde{Y}^\lambda_T &= Q^\lambda(\tilde{X}^\lambda_T, \mu(T)).
\end{aligned}
\]

It is well-known that there exists a pathwise unique solution to FBSDE (2.4) for each \(\lambda \in [0, 1]\) under Assumptions 2.1 or 2.2 (see e.g. [34, 35]). Due to Lemma B.1, its law \(\mathcal{L}(\tilde{X}^\lambda_t)\) coincides with \(\mathcal{L}(X^\lambda_t)\), and hence \(\mathcal{L}(\tilde{X}_t) = \mu(t)\). Therefore, the triple \((\tilde{X}, \tilde{Y}, \tilde{Z})\) solves (2.3). The proof for the converse is the same.  \(\square\)
Note that in (2.3) there is only one driven Brownian motion $W$ for all $\lambda \in [0, 1]$, and thus one can work on one stochastic basis and prove measurability more conveniently. As a result of Lemma 2.1, it is equivalent to solve (2.1) and (2.3), and thus we study only (2.3) in the remaining of this section.

2.1 Contraction Mapping

We will prove there exists a unique solution to (2.3) using contraction mapping theorem under Assumption 2.1 with a constant $p \geq 2$.

Given any measurable $y \in \mathcal{M}_{P,E}^{p,c}$, we define $\Psi(y) := x$ as the unique solution to
\[
\begin{aligned}
&dx^\lambda_t = B^\lambda(t, x^\lambda_t, \mathcal{L}^m(x_t), y^\lambda_t)\,dt + \sigma\,dW_t, \\
x^\lambda_0 = \xi^\lambda, \quad \forall \lambda \in [0, 1].
\end{aligned}
\] (2.5)

For any $x \in \mathcal{M}_{P,E}^{p,c}$, define $\Phi(x) := y$ to be the unique solution to backward stochastic equations
\[
\begin{aligned}
&dy^\lambda_t = -F^\lambda(t, x^\lambda_t, \mathcal{L}^m(x_t), y^\lambda_t)\,dt + z^\lambda_t\,dW_t, \\
y^\lambda_T = Q^\lambda(x_T, \mathcal{L}^m(x_T)), \quad \forall \lambda \in [0, 1].
\end{aligned}
\] (2.6)

Lemma 2.2 For $y \in \mathcal{M}_{P,E}^{p,c}$, there exists a unique solution $x$ to (2.5) such that $\lambda \mapsto x^\lambda$ is measurable, and for $x \in \mathcal{M}_{P,E}^{p,c}$, the solution $y = \Phi(x)$ to (2.6) is measurable with respect to $\lambda$.

Proof The proof is based on Picard iterations.

Step 1: For any $\mu \in PF^p$, the following stochastic differential equations can be solved
\[
\begin{aligned}
&dx^\lambda_t = B^\lambda(t, x^\lambda_t, \mu_t, y^\lambda_t)\,dt + \sigma\,dW_t, \\
x^\lambda_0 = \xi^\lambda,
\end{aligned}
\] (2.7)

via Picard iteration. Take $\tilde{x} \in \mathcal{M}_{P,E}^{p,c}$, and define
\[
\tilde{x}^\lambda_t = \xi^\lambda + \int_0^t B^\lambda(s, \tilde{x}^\lambda_s, \mu_t, y^\lambda_t)\,ds + \sigma W_t.
\]

As a result of Lemma A.4, $\lambda \mapsto (\tilde{x}^\lambda - \sigma W)$ is measurable, and so is $\lambda \mapsto \tilde{x}^\lambda$, i.e., $\tilde{x} \in \mathcal{M}_{P,E}^{p,c}$. Also $\bar{x} \mapsto \bar{x}$ is a contraction under the norm $\|\cdot\|_0$ with some $-\alpha$ large enough. Thus its fixed point $x$ solves (2.7), and $x \in \mathcal{M}_{P,E}^{p,c}$.

Take $\tilde{x} \in \mathcal{M}_{P,E}^{p,c}$, plug in $\mu = \mathcal{L}^m(\tilde{x})$ into (2.7), and obtain its solution $x := \Gamma(\tilde{x})$. By a modification of [36, Theorem 1.7], it can be easily shown that $\Gamma^k(\tilde{x}) \in \mathcal{M}_{P,E}^{p,c}$ converges, and its limit solves (2.5) and belongs to $\mathcal{M}_{P,E}^{p,c}$. 

 Springer
Step 2: Take any $\tilde{y} \in \mathcal{M}_{\mathcal{F}_T}^{\lambda, c}$, denote $f^\lambda(t) := F^\lambda(t, x^\lambda_t, \mathcal{L}^m(x_t), \tilde{y}^\lambda_t)$, $\forall \lambda \in [0, 1]$. We define

$$
\tilde{y}^\lambda_t := \mathbb{E}\left[ Q^\lambda(x^\lambda_T, \mathcal{L}^m(x_T)) - \int_t^T f^\lambda(s) \, ds \mid \mathcal{F}_t \right]
$$

By a modification of [36, Theorem 2.2], it can be easily shown that $\tilde{y} \mapsto \tilde{y}$ is a contraction under the norm $\| \cdot \|_\alpha^p$ for some $\alpha$ large enough. Then the unique fixed point is actually the solution to (2.6). For the measurability of $\lambda \mapsto \tilde{y}^\lambda \in L_{\mathcal{F}_T}^\lambda$, due to Lemma A.2 it suffices to show that $\lambda \mapsto \tilde{y}^\lambda_t \in L_{\mathcal{F}_t}^\lambda$ is measurable for any $t$. By Jensen’s inequality, it is readily seen that

$$L_{\mathcal{F}_T}^\lambda \ni \xi \mapsto \mathbb{E}[\xi \mid \mathcal{F}_t] \in L_{\mathcal{F}_t}^\lambda$$

is a contraction and thus is continuous. Due to Lemma A.4, $\lambda \mapsto Q^\lambda(x^\lambda_T, \mathcal{L}^m(x_T)) - \int_t^T f^\lambda(s) \, ds \in L_{\mathcal{F}_T}^\lambda$ is measurable. Therefore its composition with (2.8), $\lambda \mapsto \tilde{y}^\lambda_t \in L_{\mathcal{F}_t}^\lambda$, is measurable.

Let us prove that $\Phi \circ \Psi$ is a contraction, and thus the unique fixed point is the unique solution to (2.3). The proof is the same as in [35].

**Theorem 2.1** Under Assumption 2.1, the composition $\Phi \circ \Psi$ is a contraction under the norm $\| \cdot \|_k^p$.

**Proof** Take $y$, $\tilde{y}$, $x = \Psi(y)$, $\tilde{x} = \Psi(\tilde{y})$, and $Y = \Phi(x)$, $\tilde{Y} = \Phi(\tilde{x})$. Denote $\Delta x^\lambda_t = y^\lambda_t - \tilde{y}^\lambda_t$, $\Delta x^\lambda_t = x^\lambda_t - \tilde{x}^\lambda_t$, $\Delta Y^\lambda_t = Y^\lambda_t - \tilde{Y}^\lambda_t$. By Itô’s formula, we obtain that

$$e^{kt} |\Delta x^\lambda_s|^p = k \int_0^t e^{ks} |\Delta x^\lambda_s|^p \, ds + p \int_0^t e^{ks} |\Delta x^\lambda_s|^{p-2} \Delta x^\lambda_s B^\lambda(t, x^\lambda_s, \mathcal{L}^m(x_s), y^\lambda_s) \Delta Y^\lambda + \int_0^t e^{ks} |\Delta x^\lambda_s|^{p-2} \Delta x^\lambda_s B^\lambda(t, \tilde{x}^\lambda_s, \mathcal{L}^m(\tilde{x}_s), \tilde{y}^\lambda_s) \, ds.$$

According to Assumption 2.1, Young’s inequality and property of Wasserstein metric, we get that

$$|\Delta x^\lambda_s|^{p-2} \Delta x^\lambda_s \left( B^\lambda(t, x^\lambda_s, \mathcal{L}^m(x_s), y^\lambda_s) - B^\lambda(t, \tilde{x}^\lambda_s, \mathcal{L}^m(\tilde{x}_s), \tilde{y}^\lambda_s) \right) \leq -K_1 |\Delta x^\lambda_s|^p + L_1 \left( \frac{|\Delta x^\lambda_s|^p}{q} + \frac{\mathbb{E}[|\Delta x^\lambda_s|^p]}{p} \right) + \frac{L_1}{2} \left( \frac{|\Delta x^\lambda_s|^p}{q} + \int_{[0,1]} \frac{\mathbb{E}[|\Delta x^\lambda_s|^p]}{p} \, d\kappa \right) + L_1 \left( \frac{|\Delta x^\lambda_s|^p}{q} + \frac{|\Delta y^\lambda_s|^p}{p} \right).$$
where \( q \) is the conjugate of \( p \), i.e., \( \frac{1}{q} + \frac{1}{p} = 1 \). Thus we have

\[
e^{kt} \mathbb{E}[|\Delta x^\lambda_t|^p] + (-k + pK_1 - (4p - 3)L_1/2) \int_0^t e^{ks} \mathbb{E}[|\Delta x^\lambda_s|^p] \, ds
- \frac{L_1}{2} \int_0^t e^{ks} \int_{[0,1]} \mathbb{E}[|\Delta x^\kappa_s|^p] \, d\kappa \, ds
\leq L_1 \int_0^t \mathbb{E}[|\Delta y^\lambda_s|^p] \, ds,
\]

and hence

\[
\max_{\lambda \in [0,1]} \int_0^t e^{ks} \mathbb{E}[|\Delta x^\lambda_s|^p] \, ds
\leq \frac{L_1}{-k + pK_1 - (2p - 1)L_1} \max_{\lambda \in [0,1]} \int_0^t e^{ks} \mathbb{E}[|\Delta y^\lambda_s|^p] \, ds.
\]

As a result of \( k < pK_1 - (2p - 1)L_1 \), one can deduce from (2.9) that

\[
(-k + pK_1 - (2p - 1)L_1) \int_0^t e^{ks} \int_{[0,1]} \mathbb{E}[|\Delta x^\kappa_s|^p] \, d\kappa \, ds
\leq L_1 \int_0^t e^{ks} \int_{[0,1]} \mathbb{E}[|\Delta y^\kappa_s|^p] \, d\kappa \, ds
\leq \max_{\kappa \in [0,1]} L_1 \int_0^t e^{ks} \mathbb{E}[|\Delta y^\kappa_s|^p] \, ds,
\]

and also

\[
e^{kt} \mathbb{E}[|\Delta x^\lambda_t|^p] \leq L_1 \int_0^t e^{ks} \mathbb{E}[|\Delta y^\lambda_s|^p] \, ds + L_1/2 \int_0^t e^{ks} \int_{[0,1]} \mathbb{E}[|\Delta x^\kappa_s|^p] \, d\kappa \, ds.
\]

Taking maximum over \( \lambda \) in (2.12) and using the inequality (2.11), we obtain that

\[
\max_{\lambda \in [0,1]} e^{kt} \mathbb{E}[|x^\lambda_t - \tilde{x}^\lambda_t|^2]
\leq \left( L_1 + \frac{L_1^2}{2(-k + pK_1 - (2p - 1)L_1)} \right) \max_{\lambda \in [0,1]} \int_0^t e^{ks} \mathbb{E}[|y^\lambda_s - \tilde{y}^\lambda_s|^2] \, ds.
\]

For BSDEs, it can be easily seen that

\[
e^{kt} |\Delta Y^\lambda_t|^p = e^{kT} |Q^\lambda(\hat{x}^\lambda_T, \tau^m(\hat{x}^\lambda_T)) - Q^\lambda(\hat{x}^\lambda_T, \tau^m(\hat{x}^\lambda_T))|^p
- k \int_t^T e^{ks} |\Delta Y^\lambda_s|^p \, ds - p(p - 1)/2 \int_t^T e^{ks} |\Delta Y^\lambda_s|^p - |\Delta Z^\lambda_s|^2 \, ds
+ p \int_t^T e^{ks} |\Delta Y^\lambda_s|^{p-2} \Delta Y^\lambda_s \left( F^\lambda(s, x^\lambda_s, \tau^m(x^\lambda_s), Y^\lambda_s) - F^\lambda(s, \hat{x}^\lambda_s, \tau^m(\hat{x}^\lambda_s), \hat{Y}^\lambda_s) \right) \, ds
- p \int_t^T e^{ks} |\Delta Y^\lambda_s|^{p-2} \Delta Y^\lambda_s \Delta Z^\lambda_s \, dW_s.
\]
Using Assumption 2.1, Young’s inequality and properties of Wasserstein metric, we get that

$$
\mathbb{E}[|\Delta Y^\lambda_s|^{p-2} \Delta Y^\lambda_s (F^\lambda(s, x^\lambda_s, \mathcal{L}^m(x_s), Y^\lambda_s) - F^\lambda(s, \tilde{x}^\lambda_s, \mathcal{L}^m(\tilde{x}_s), \tilde{Y}^\lambda_s))]
\leq -K_2 \mathbb{E}[|\Delta Y^\lambda_s|^{p}] + \frac{L_2}{2} \left( \mathbb{E}[|\Delta Y^\lambda_s|^{p}] + \mathbb{E}[|\Delta x^\lambda_s|^{p}] \right)
+ \frac{L_2}{2} \left( \mathbb{E}[|\Delta Y^\lambda_s|^{p}] + \int_{[0,1]} \mathbb{E}[|\Delta x^\lambda_s|^{p}] \, d\kappa \right)
+ L_2 \left( \mathbb{E}[|\Delta Y^\lambda_s|^{p}] + \mathbb{E}[|\Delta x^\lambda_s|^{p}] \right).
$$

Therefore, one can obtain

$$
e^{kt} \mathbb{E}[|\Delta Y^\lambda_t|^{p}] + (k + pK_2 - (2p - 2)L_2) \int_t^T e^{k\kappa} \mathbb{E}[|\Delta Y^\lambda_t|^{p}] \, d\kappa
\leq (2^{p-2} + 2^{p-1})L_3^p e^{kT} \mathbb{E}[|\Delta x^\lambda_t|^{p}] + 2^{p-2}L_3^p e^{kT} \int_{[0,1]} \mathbb{E}[|\Delta x^\lambda_t|^{p}] \, d\kappa
+ 3L_2/2 \int_t^T e^{k\kappa} \mathbb{E}[|\Delta x^\lambda_t|^{p}] \, d\kappa + L_2/2 \int_t^T e^{k\kappa} \int_{[0,1]} \mathbb{E}[|\Delta x^\lambda_t|^{p}] \, d\kappa \, d\kappa ds
+ L_2 \int_t^T e^{k\kappa} \mathbb{E}[|\Delta x^\lambda_t|^{p}] \, d\kappa ds
$$

Plugging in (2.10) and (2.13), it can be readily seen that

$$(k + pK_2 - (2p - 2)L_2) \|Y - \tilde{Y}\|_k^p
\leq \left(2^p L_1 L_3^p + \frac{2^{p-1}L_1^2L_3^p + 2L_1L_2}{-k + pK_1 - (2p - 1)L_1} \right) \|y - \tilde{y}\|_k^p,$$

and $\Phi \circ \Psi$ a contraction due to (2.2). \hfill \Box

### 2.2 Method of Continuation

We consider the following family of FBSDEs parametrized by $\zeta \in [0, 1]$,

$$
\begin{aligned}
&dX^\zeta_t = \left( \zeta B^\lambda(t, X^\zeta_t, \mathcal{L}^m(X^\lambda_t), Y^\lambda_t) - (1 - \zeta)Y^\zeta_t + B_0^\lambda(t) \right) \, dt + \sigma \, dW_t, \\
&dY^\zeta_t = - \left( \zeta Y^\lambda(t, X^\zeta_t, \mathcal{L}^m(X^\lambda_t), Y^\lambda_t) + (1 - \zeta)X^\zeta_t + F_0^\lambda(t) \right) \, dt + Z^\zeta_t \, dW_t, \\
&X^\zeta_0 = \xi, \\
&Y^\zeta_T = 0.
\end{aligned}
$$

(2.14)
where $B_0, F_0, Q_0 \in \mathcal{ML}_{\mathcal{F}}^{p,2} \times \mathcal{ML}_{\mathcal{F}}^{p,2} \times \mathcal{ML}_{\mathcal{F}}^p$. In the case that $\zeta = 1$, $B_0^\lambda = F_0^\lambda = Q_0^\lambda = 0$, (2.14) reduces to (2.1). In the case that $\zeta = 0$, (2.14) becomes

$$
\begin{cases}
\frac{dX_t^\lambda}{dt} = -Y_t^\lambda + B_0^\lambda(t) dt + \sigma dW_t, \\
\frac{dY_t^\lambda}{dt} = -(X_t^\lambda + F_0^\lambda(t)) dt + Z_t^{0,\lambda} dW_t, \\
X_0^\lambda = \xi^\lambda, \\
Y_T^\lambda = X_T^\lambda + Q_0^\lambda.
\end{cases}
$$

(2.15)

For any $\Theta = (X, Y, Z) \in \mathcal{MP}[0, T] := \mathcal{ML}_{\mathcal{F}}^{p,c} \times \mathcal{ML}_{\mathcal{F}}^{p,c} \times \mathcal{ML}_{\mathcal{F}}^{p,2}$, we define its norm

$$\|\Theta\|_{\mathcal{MP}[0, T]} := \max_{\lambda \in [0, 1]} \mathbb{E} \left[ \sup_{t \in [0, T]} |X_t^\lambda|^p + \sup_{t \in [0, T]} |Y_t^\lambda|^p + \left( \int_0^T |Z_t^\lambda|^2 dt \right)^{p/2} \right].$$

The following lemma provides a sufficient condition for the $L^p$ boundedness of the limit $\lim_{k \to \infty} \Theta^k$.

**Lemma 2.3** Suppose $p \geq 2$ and the sequence $(\Theta^k)_{k \geq 0} \in \mathcal{MP}[0, T]$ satisfies

$$\|\Theta^k\|_{\mathcal{MP}[0, T]} \leq K, \quad k \geq 0,$$

$$\lim_{k \to \infty} \|\Theta - \Theta^k\|_{\mathcal{M}^2[0, T]} = 0.$$

Then $\Theta \in \mathcal{MP}[0, T]$.

**Proof** See [37, Lemma 4.1].

The following lemma establishes the existence result of (2.15). After that, we will present the main proposition of this subsection.

**Lemma 2.4** The FBSDE system (2.15) has a unique solution $\Theta = (X, Y, Z)$ in $\mathcal{MP}[0, T]$ and $\lambda \mapsto \mathcal{L}(X_t^{0,\lambda})$ is measurable for any $t \in [0, T]$. Furthermore, we have the bound

$$\|\Theta\|_{\mathcal{MP}[0, T]} \leq K \max_{\lambda \in [0, 1]} \mathbb{E} \left[ |\xi^\lambda|^p + |Q_0^\lambda|^p + \left( \int_0^T |B_0^\lambda(t)|^2 dt \right)^{p/2} + \left( \int_0^T |F_0^\lambda(t)|^2 dt \right)^{p/2} \right].$$

(2.16)

**Proof** The proof of measurability is the same as Lemma 2.2. Let us prove that there exists a unique solution $\Theta \in \mathcal{MP}[0, T]$.

We can solve the system $\lambda$ by $\lambda$. For each $\lambda \in [0, 1]$, consider the BSDE

$$
\begin{cases}
dP_t^\lambda = -(-P_t^\lambda + B_0^\lambda(t) + F_0^\lambda(t)) dt + Z_t^\lambda dW_t, \\
P_T^\lambda = Q_0^\lambda.
\end{cases}
$$

(2.17)
It is linear BSDE, and by [38, Proposition 4.1.2] we know that
\[ P_0^\lambda = e^{-t} \mathbb{E} \left[ e^{-T} Q_0^\lambda + \int_t^T e^{-s} (B_0^\lambda(s) + F_0^\lambda(s)) \, ds \right]. \]

Therefore we get
\[
\mathbb{E} \left[ \int_0^T |P_t^\lambda|^p \, dt \right] \leq K \mathbb{E} \left[ \left| Q_0^\lambda \right|^p + \left( \int_0^T |B_0^\lambda(t)|^2 \, dt \right)^{p/2} + \left( \int_0^T |F_0^\lambda(t)|^2 \, dt \right)^{p/2} \right].
\]

Also due to [39, Proposition 3.26], as the $p/2$-th power of quadratic variation of the martingale
\[ t \mapsto \mathbb{E} \left[ Q_0^\lambda + \int_0^T (-P_s^\lambda + B_0^\lambda(s) + F_0^\lambda(s)) \, ds \right], \]
we obtain that
\[
\mathbb{E} \left[ \left( \int_0^T |Z^\lambda|^2 \, dt \right)^{p/2} \right] \leq K \mathbb{E} \left[ \left| Q_0^\lambda \right|^p + \left( \int_0^T |B_0^\lambda(t)|^2 \, dt \right)^{p/2} + \left( \int_0^T |F_0^\lambda(t)|^2 \, dt \right)^{p/2} \right].
\]

Applying BDG inequality and Grönwall’s inequality to (2.17), we can easily get that
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |P_t^\lambda|^p \right] \leq K \mathbb{E} \left[ \left| Q_0^\lambda \right|^p + \left( \int_0^T |B_0^\lambda(t)|^2 \, dt \right)^{p/2} + \left( \int_0^T |F_0^\lambda(t)|^2 \, dt \right)^{p/2} \right].
\]

Now consider the SDE
\[ X_t = \xi^\lambda + \int_0^t (-X_s^\lambda - P_s^\lambda + B_0^\lambda(s)) \, ds + \sigma W_t. \]

Then by BDG inequality and Grönwall’s inequality, one can easily see that
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |X_t^\lambda|^p \right] \leq K \mathbb{E} \left[ \left| \xi^\lambda \right|^p + \left| Q_0^\lambda \right|^p + \left( \int_0^T |B_0^\lambda(t)|^2 \, dt \right)^{p/2} + \left( \int_0^T |F_0^\lambda(t)|^2 \, dt \right)^{p/2} \right].
\]

Note that $\Theta = (X^\lambda, X^\lambda + P^\lambda, Z^\lambda + \sigma)$ solves (2.15), and satisfies (2.16).
Proposition 2.1 Suppose there exists a \( \zeta \in [0, 1] \) such that for any \( B_0, F_0, Q_0 \in \mathcal{ML}^{p,2}_F \times \mathcal{ML}^{p,2}_F \times \mathcal{ML}^{p}_{FT} \) there exists a unique solution \( \Theta \) to (2.14) satisfying

\[
\|\Theta\|^{p}_{\mathcal{M}^{p}[0,T]} \leq K \max_{\lambda \in [0, 1]} \mathbb{E} \left[ |\xi|^p + |Q_0^{\lambda}|^p + \left( \int_0^T \|B^{\lambda}(t, 0)\|^p + \|B_{0}^{\lambda}(t)\|^2 + \int_0^T |F^{\lambda}(t, 0)|^2 + |F_{0}^{\lambda}(t)|^2 \, dt \right)^{p/2} \right].
\]

Then under Assumption 2.2, there exists an \( \delta_0 > 0 \) independent of \( \zeta \) such that for any \( \delta \in [0, \delta_0) \), \( (B_0, F_0, Q_0) \in \mathcal{ML}^{p,2}_F \times \mathcal{ML}^{p,2}_F \times \mathcal{ML}^{p}_{FT} \), (2.14) has a unique solution \( \Theta^{\zeta+\delta} = (X^{\zeta+\delta}, Y^{\zeta+\delta}, Z^{\zeta+\delta}) \), and the following estimate holds:

\[
\|\Theta^{\zeta+\delta}\|^{p}_{\mathcal{M}^{p}[0,T]} \leq K \max_{\lambda \in [0, 1]} \mathbb{E} \left[ |\xi|^p + |Q_0^{\lambda}|^p + \left( \int_0^T \|B^{\lambda}(t, 0)\|^p + \|B_{0}^{\lambda}(t)\|^2 + \int_0^T |F^{\lambda}(t, 0)|^2 + |F_{0}^{\lambda}(t)|^2 \, dt \right)^{p/2} \right].
\]

Proof Denote

\[
B^{\xi,\lambda}(t, x, \eta, y) = \xi B^{\lambda}(t, x, \eta, y) - (1 - \xi)y,
\]

\[
F^{\xi,\lambda}(t, x, \eta, y) = \xi F^{\lambda}(t, x, \eta) + (1 - \xi)x,
\]

\[
Q^{\xi,\lambda}(x, \eta) = \xi Q^{\lambda}(x, \eta) + (1 - \xi)x.
\]

For any pair \((x, y) \in L^2\) such that \( x_0^\lambda = \xi^\lambda \), according to our hypothesis, there exists a unique solution \((X, Y, Z)\) to

\[
\begin{aligned}
dX_t^\lambda &= (B^{\xi,\lambda}(t, X_t^\lambda, \mathcal{L}^m(X_t), Y_t^\lambda) + \delta B(t, X_t^\lambda, \mathcal{L}^m(x_t), y_t^\lambda) + \delta x_t^\lambda + B_{0}^{\lambda}(t)) \, dt + \sigma \, dW_t, \\
dY_t^\lambda &= - (F^{\xi,\lambda}(t, X_t^\lambda, \mathcal{L}^m(X_t), Y_t^\lambda) + \delta F(t, X_t^\lambda, \mathcal{L}^m(x_t), y_t^\lambda) - \delta x_t^\lambda + F_{0}^{\lambda}(t)) \, dt + Z^\lambda \, dW_t, \\
X_0^\lambda &= \xi^\lambda, \\
Y_0^\lambda &= Q^{\xi,\lambda}(X_T, \mathcal{L}^m(X_T)) + \delta Q(x_T, \mathcal{L}^m(x_T)) - \delta x_T^\lambda + Q_0^\lambda.
\end{aligned}
\]

Thus we have obtained maps

\[
\Pi : (x, y) \mapsto (X, Y),
\]

\[
\hat{\Pi} : (x, y) \mapsto (X, Y, Z).
\]

(2.18)

For any \((x, y) \in \mathcal{ML}^{2,\mathcal{C}}_F \times \mathcal{ML}^{2,\mathcal{C}}_F\), define a norm

\[
\|(x, y)\|^2 := \max_{\lambda \in [0, 1]} \left( \mathbb{E}[|x_T^\lambda|^2] + \mathbb{E} \left[ \int_0^T |x_t^\lambda|^2 + |y_t^\lambda|^2 \, dt \right] \right).
\]
Applying Itô’s formula to $\Delta \Theta^1$, we get that
\[
\mathbb{E} \left[ \Delta x^\lambda_2 \Delta y^\lambda_2 \right] = \mathbb{E} \left[ \Delta x^\lambda_2 \left( Q^\xi,\lambda (x^\lambda_2, \mathcal{L}^m (x_T)) - Q^\xi,\lambda (\tilde{x}^\lambda_2, \mathcal{L}^m (\tilde{x}_T)) \right) \right] + \delta \mathbb{E} \left[ \Delta x^\lambda_2 \left( Q^\lambda (x^\lambda_2, \mathcal{L}^m (x_T)) - Q^\lambda (\tilde{x}^\lambda_2, \mathcal{L}^m (\tilde{x}_T)) \right) - \Delta x^\lambda_2 \Delta x^\lambda_2 \right].
\]

Applying Itô’s formula to $\Delta x^\lambda_2 \Delta y^\lambda_2$, we also have
\[
\begin{align*}
\mathbb{E} \left[ \Delta x^\lambda_2 \Delta y^\lambda_2 \right] &= \int_0^T \mathbb{E} \left[ \Delta y^\lambda_2 \left( B^\xi,\lambda (t, \Theta^1_t) - B^\xi,\lambda (t, \tilde{\Theta}^1_t) \right) \right] dt \\
&\quad - \int_0^T \mathbb{E} \left[ \Delta x^\lambda_2 \left( \int_0^T \mathcal{L}^m (x^\lambda_t) \right) dt \right] \\
&\quad + \int_0^T \delta \mathbb{E} \left[ \Delta y^\lambda_2 \left( B^\lambda (t, \theta^1_t) - B^\lambda (t, \tilde{\theta}^1_t) \right) + \Delta y^\lambda_2 \Delta y^\lambda_2 \right] dt \\
&\quad - \int_0^T \delta \mathbb{E} \left[ \Delta x^\lambda_2 \left( \int_0^T \mathcal{L}^m (x^\lambda_t) \right) dt \right] dt.
\end{align*}
\]

According to our Assumption 2.2, we can easily get that
\[
\begin{align*}
(k - 2l - \epsilon) & \left( \mathbb{E} \left[ |\Delta x^\lambda_2|^2 \right] + \mathbb{E} \left[ \int_0^T |\Delta x^\lambda_2|^2 + |\Delta y^\lambda_2|^2 dt \right] \right) \\
&\leq C \delta \left( \mathbb{E} \left[ |\Delta x^\lambda_2|^2 \right] + \mathbb{E} \left[ \int_0^T |\Delta x^\lambda_2|^2 + |\Delta y^\lambda_2|^2 dt \right] \right) \\
&\quad + C \delta \left( \int_{[0,1]} \mathbb{E} \left[ |\Delta x^\lambda_2|^2 \right] d\kappa + \int_0^T \int_{[0,1]} \mathbb{E} \left[ |\Delta x^\lambda_2|^2 \right] d\kappa dt \right) \\
&\quad + l \left( \int_{[0,1]} \mathbb{E} \left[ |\Delta x^\lambda_2|^2 \right] d\kappa + \int_0^T \int_{[0,1]} \mathbb{E} \left[ |\Delta x^\lambda_2|^2 \right] d\kappa dt \right),
\end{align*}
\]

where $C$ is a constant only depends on $\epsilon$ and Lipchitz constant $l$, $L$. Taking maximum of both sides, one can obtain that
\[
\begin{align*}
(k - 2l - \epsilon) \max_{\lambda \in [0,1]} & \left( \mathbb{E} \left[ |\Delta x^\lambda_2|^2 \right] + \mathbb{E} \left[ \int_0^T |\Delta x^\lambda_2|^2 + |\Delta y^\lambda_2|^2 dt \right] \right) \\
&\leq C \delta \max_{\lambda \in [0,1]} \left( \mathbb{E} \left[ |\Delta x^\lambda_2|^2 \right] + \mathbb{E} \left[ \int_0^T |\Delta x^\lambda_2|^2 + |\Delta y^\lambda_2|^2 dt \right] \right) \\
&\quad + l \max_{\lambda \in [0,1]} \left( \mathbb{E} \left[ |\Delta x^\lambda_2|^2 \right] + \int_0^T \mathbb{E} \left[ |\Delta x^\lambda_2|^2 \right] dt \right).
\end{align*}
\]
and hence

\[
(k - 3l - \varepsilon) \max_{\lambda \in [0,1]} \left( \mathbb{E} \left[ |\Delta X^k_\ell|^2 \right] + \mathbb{E} \left[ \int_0^T |\Delta X^k_t|^2 + |\Delta Y^k_t|^2 \, dt \right] \right) \\
\leq C\delta \max_{\lambda \in [0,1]} \left( \mathbb{E} \left[ |\Delta x^k_\ell|^2 \right] + \mathbb{E} \left[ \int_0^T |\Delta x^k_t|^2 + |\Delta \lambda^k_t|^2 \, dt \right] \right). \tag{2.19}
\]

First choosing \( \varepsilon \) such that \( k - 3l > \varepsilon \), and choosing \( \delta \) small enough that \( k - 3l - \varepsilon > C\delta \), we finished proving that \( \Pi \) is a contraction.

**Step 2:** Take \( X^0 = Y^0 = 0 \), and define recursively \( \Theta^{k+1} = (X^{k+1}, Y^{k+1}, Z^{k+1}) = \tilde{\Pi}(X^k, Y^k) \), and the limit \( \Theta = (X, Y, Z) \). It is clear from our hypothesis that \( \lambda \mapsto (X^{k,\lambda}, Y^{k,\lambda}) \) is measurable for any \( k \in \mathbb{N} \), and therefore the limit \( \lambda \mapsto (X^\lambda, Y^\lambda) \) is also measurable. Using \( \lim_{k \to \infty} \|(X^k - X, Y^k - Y)\| = 0 \) and some standard estimate, we obtain that

\[
\lim_{k \to \infty} \|\Theta^k - \Theta\|_{\mathcal{M}^2[0,T]} = 0.
\]

**Step 3:** Invoking Lemma 2.3, it remains to show that

\[
\|\Theta^{k+1}\|_{\mathcal{M}^p[0,T]} \leq K \max_{\lambda \in [0,1]} \mathbb{E} \left[ |\xi|^p + |Q_0^\lambda|^p + \left( \int_0^T |B^\lambda(t,0)|^2 + |B^\lambda_0(t)|^2 \right)^{p/2} \right] + \left( \int_0^T |F^\lambda(t,0)|^2 + |F_0^\lambda(t)|^2 \, dt \right)^{p/2}.
\]

As a result of our hypothesis and the Lipschitz property of \( B^\lambda, F^\lambda, Q^\lambda \), we obtain that

\[
\|\Theta^{k+1}\|_{\mathcal{M}^p[0,T]} \leq K \max_{\lambda \in [0,1]} \mathbb{E} \left[ |\xi|^p + |Q_0^\lambda|^p + \left( \int_0^T |B^\lambda(t,0)|^2 + |B^\lambda_0(t)|^2 \right)^{p/2} \right] + K\delta \|\Theta^k\|_{\mathcal{M}^p[0,T]}.
\]

Choosing \( \delta \) small enough such that \( K\delta^p < 1/2 \), it is then clear that for each \( k \geq 1 \)

\[
\|\Theta^k\|_{\mathcal{M}^p[0,T]} \leq 2K \max_{\lambda \in [0,1]} \mathbb{E} \left[ |\xi|^p + |Q_0^\lambda|^p + \left( \int_0^T |B^\lambda(t,0)|^2 + |B^\lambda_0(t)|^2 \right)^{p/2} \right] + \left( \int_0^T |F^\lambda(t,0)|^2 + |F_0^\lambda(t)|^2 \, dt \right)^{p/2}.
\]

Letting \( k \to \infty \), we obtain the same bound for \( \Theta \).

\[\square\]

**Theorem 2.2** Under Assumption 2.2, there exists a unique solution to (2.3).
Proof The existence can be deduced directly from Lemma 2.4 and Proposition 2.1. Let us only prove the uniqueness. Suppose there are two different solutions \((X, Y, Z)\) and \((\tilde{X}, \tilde{Y}, \tilde{Z})\) to (2.3), and denote \(\Delta X = X - \tilde{X}\), \(\Delta Y = Y - \tilde{Y}\). Applying Itô’s formula to \(\Delta X^\lambda T\), \(\Delta Y^\lambda T\) and using similar estimation as in Step 1 of Proposition 2.1, we conclude that

\[
(k - 2l)\mathbb{E}[(\Delta X^\lambda T)^2] + (k - 2l) \int_0^T \mathbb{E}[\Delta X^\lambda T + (\Delta Y^\lambda T)^2] dt \\
\leq l \int_{[0,1]} \mathbb{E}[(\Delta X^\kappa)^2] d\kappa + l \int_0^T \int_{[0,1]} \mathbb{E}[(\Delta X^\kappa)^2] d\kappa dt.
\]

Taking maximum over all \(\lambda \in [0, 1]\), it can be readily seen that

\[
(k - 2l) \max_{\lambda \in [0, 1]} \left( \mathbb{E}[(\Delta X^\lambda T)^2] + \int_0^T \mathbb{E}[(\Delta X^\lambda T + (\Delta Y^\lambda T)^2] dt \right) \\
\leq l \left( \int_{[0,1]} \mathbb{E}[(\Delta X^\kappa)^2] d\kappa + \int_0^T \int_{[0,1]} \mathbb{E}[(\Delta X^\kappa)^2] d\kappa dt \right),
\]

which violates our assumption \(k > 3l\).

Remark 2.1 The method of continuation for FBSDEs developed by [40] is more flexible and complicated. Here we only work under a specific assumption.

3 Stability

Denote the solution to (2.1) by \((x_G, y_G, z_G)\). As in [29, Theorem 3.1], we prove that as \(\|G - \tilde{G}\|_k \to 0\),

\[
\int_0^1 \mathcal{W}_{2,T} \left( \mathcal{L}(x_G^\lambda, y_G^\lambda), \mathcal{L}(x_{\tilde{G}}^\lambda, y_{\tilde{G}}^\lambda) \right) d\lambda \to 0.
\]

The operator \(\Gamma := \Phi \circ \Psi\) depends on \(G\), and we denote it by \(\Gamma_G\) (see (2.5), (2.6) for the definition of \(\Phi\), \(\Psi\)). The proof stability result will be divided into three steps.

(i) The operator \(\Gamma_G\) is a contraction under the norm \(\|\cdot\|_k^f\).

(ii) The operator \(\Gamma_G\) is continuous in \(G\), i.e., as \(\|G - \tilde{G}\|_k \to 0\),

\[
\|\Gamma_G(y) - \Gamma_{\tilde{G}}(y)\|_k^f \to 0.
\]

(iii) It holds that

\[
\int_0^1 \mathcal{W}_{2,T} \left( \mathcal{L}(x_G^\lambda, y_G^\lambda), \mathcal{L}(x_{\tilde{G}}^\lambda, y_{\tilde{G}}^\lambda) \right) d\lambda \to 0.
\]
3.1 Contraction Mapping

**Assumption 3.1**

(i) $B_G^\lambda(t, x, \eta, y) = B_0(t, x, \eta^\lambda, y) + \int_{[0,1]} G(\lambda, \kappa) d\kappa \int \hat{B}(t, x, w, y) \eta^\kappa dw$.

(ii) $F_G^\lambda(t, x, \eta, y) = F_0(t, x, \eta^\lambda, y) + \int_{[0,1]} G(\lambda, \kappa) d\kappa \int \hat{F}(t, x, w, y) \eta^\kappa dw$.

(iii) $Q_G^\lambda(x, \eta) = Q_0(x, \eta^\lambda) + \int_{[0,1]} G(\lambda, \kappa) d\kappa \int \hat{Q}(x, w) \eta^\kappa dw$.

**Theorem 3.1**

Suppose $(x, y, z)$ and $(\tilde{x}, \tilde{y}, \tilde{z})$ are solutions of (2.1) with graphons $G$ and $\tilde{G}$ respectively. Then under Assumption 2.1 with any $p > 2$ and Assumption 3.1, we have that as $\|G - \tilde{G}\|_k \to 0$ and $E \left[ \int_{[0,1]} |x_0^\lambda - \tilde{x}_0^\lambda|^2 d\lambda \right] \to 0$,

$$
E \left[ \int_0^1 \left( \sup_{u \in [0,T]} |x_u^\lambda - \tilde{x}_u^\lambda|^2 + \sup_{u \in [0,T]} |y_u^\lambda - \tilde{y}_u^\lambda|^2 + \int_0^T |z_s^\lambda - \tilde{z}_s^\lambda|^2 ds \right) d\lambda \right] \to 0,
$$

which implies

$$
\int_0^1 W_{2,T}(L(x^\lambda, y^\lambda), L(\tilde{x}^\lambda, \tilde{y}^\lambda)) d\lambda \to 0.
$$

**Proof**

**Step 1:** By the same argument of Theorem 2.1, one can easily prove that $\Gamma$ is a contraction with the norm $||.||_k^{1.2}$ under Assumption 2.1. For any $x \in \mathcal{ML}_k^2$, since $||x||_k^2 \geq ||x||_k^{1.2}$, the fixed point of under $||.||_k^2$ must be the fixed under $||.||_k^{1.2}$.

**Step 2:** Take $y$, and denote $x = \Psi_G(y)$, $\tilde{x} = \Psi_{\tilde{G}}(y)$, $Y = \Phi_G(x)$, $\tilde{Y} = \Phi_{\tilde{G}}(\tilde{x})$.

Let us calculate

$$
e^{kt}|x^\lambda_t - \tilde{x}^\lambda_t|^2 = |x_0^\lambda - \tilde{x}_0^\lambda|^2 + k \int_0^t e^{ks}|x_s^\lambda - \tilde{x}_s^\lambda|^2 ds
$$

$$
+ 2 \int_0^t e^{ks} (x_s^\lambda - \tilde{x}_s^\lambda) \cdot (B_G^\lambda(s, x_s^\lambda, L^m(x_s), y_s^\lambda) - B_{\tilde{G}}^\lambda(s, \tilde{x}_s^\lambda, L^m(\tilde{x}_s), y_s^\lambda)) ds
$$

$$
+ 2 \int_0^t e^{ks} (x_s^\lambda - \tilde{x}_s^\lambda) \cdot (B_G^\lambda(s, x_s^\lambda, L^m(x_s), y_s^\lambda) - B_{\tilde{G}}^\lambda(s, \tilde{x}_s^\lambda, L^m(\tilde{x}_s), y_s^\lambda)) ds
$$

$$
\leq (k - 2K_1 + 2L_1 + \epsilon) \int_0^t e^{ks}|x_s^\lambda - \tilde{x}_s^\lambda|^2 ds
$$

$$
+ \frac{1}{\epsilon} \int_0^t e^{ks} \left( B_G^\lambda(s, x_s^\lambda, L^m(x_s), y_s^\lambda) - B_{\tilde{G}}^\lambda(s, x_s^\lambda, L^m(x_s), y_s^\lambda) \right)^2 ds.
$$

Taking expectation and integration both sides over $\lambda$, we get that

$$
E \left[ \int_{[0,1]} e^{kT}|x_T^\lambda - \tilde{x}_T^\lambda|^2 d\lambda \right] + (2K_1 - k - 2L_1 - \epsilon) E \left[ \int_0^T \int_{[0,1]} e^{ks}|x_s^\lambda - \tilde{x}_s^\lambda|^2 d\lambda ds \right]
$$

$$
\leq E \left[ \int_{[0,1]} |x_0^\lambda - \tilde{x}_0^\lambda|^2 d\lambda \right]
$$

$$
+ C E \left[ \int_0^T e^{ks} ds \int \left( B_G^\lambda(s, x_s^\lambda, L^m(x_s), y_s^\lambda) - B_{\tilde{G}}^\lambda(s, x_s^\lambda, L^m(x_s), y_s^\lambda) \right)^2 d\lambda \right].
$$

(3.3)
For the integrand of the last line, we show that as $\|G - \tilde{G}\|_\square \to 0$

$$
E \left[ \int_0^T e^{ks} \int \left( B^\lambda_G(s, x^\lambda_s, \lambda^m(x_s), y^\lambda_s) - B^\lambda_G(s, x^\lambda_s, \lambda^m(x_s), y^\lambda_s) \right)^2 d\lambda \right] \to 0.
$$

(3.4)

Due to Assumption 3.1, we have that

$$
\left( B^\lambda_G(s, x^\lambda_s, \lambda^m(x_s), y^\lambda_s) - B^\lambda_G(s, x^\lambda_s, \lambda^m(x_s), y^\lambda_s) \right)^2
\leq \left| \int_{[0,1]} (G(\lambda, \kappa) - \tilde{G}(\lambda, \kappa)) \, d\kappa \int \hat{B}(s, x^\lambda_s, w, y^\lambda_s) \lambda^s(w)(d\mu) \right|^2
\leq C \left( 1 + |x^\lambda_s| + \int_{[0,1]} \sqrt{E[|x^\lambda_s|^2]} \, d\kappa \right)
\times \left| \int_{[0,1]} (G(\lambda, \kappa) - \tilde{G}(\lambda, \kappa)) \, d\kappa \int \hat{B}(s, x^\lambda_s, w, y^\lambda_s) \lambda^s(w)(d\mu) \right|.
$$

Taking expectation of both sides, using the boundedness of $\sup_{\lambda \in [0,1]} E[|x^\lambda_s|^2]$, and taking integral with respect to $\lambda$, we get that

$$
E \left[ \int \left( B^\lambda_G(s, x^\lambda_s, \lambda^m(x_s), y^\lambda_s) - B^\lambda_G(s, x^\lambda_s, \lambda^m(x_s), y^\lambda_s) \right)^2 d\lambda \right]
\leq C E \left[ \int \int_{[0,1]} (G(\lambda, \kappa) - \tilde{G}(\lambda, \kappa)) \, d\kappa \int \hat{B}(s, x^\lambda_s, w, y^\lambda_s) \lambda^s(w)(d\mu) \big| d\lambda \right].
$$

By the estimation of $J^{n,3}$ in the proof of [29, Theorem 2.1] and the boundedness of $E[|x^\lambda_s|^p] + E[|y^\lambda_s|^p]$, we obtain that as $\|G - \tilde{G}\|_\square \to 0$

$$
E \left[ \int \left( B^\lambda_G(s, x^\lambda_s, \lambda^m(x_s), y^\lambda_s) - B^\lambda_G(s, x^\lambda_s, \lambda^m(x_s), y^\lambda_s) \right)^2 d\lambda \right] \to 0.
$$

Then (3.4) follows from the fact that $t \mapsto \hat{B}(t, x, w, y)$ is Lipschitz uniformly for $(x, w, y)$.

Then let us estimate $\tilde{Y} - Y$. From the equation

$$
e^{kt} \left| Q^\lambda_G(x^\lambda_T, \lambda^m(x_T)) - Q^\lambda_G(\tilde{x}^\lambda_T, \lambda^m(\tilde{x}_T)) \right|^2
= e^{kt} |Y^\lambda_T - \tilde{Y}^\lambda_T|^2 + k \int_0^T e^{ks} |Y^\lambda_s - \tilde{Y}^\lambda_s|^2 \, ds + \int_0^T e^{ks} |Z^\lambda_s - \tilde{Z}^\lambda_s|^2 \, ds
- 2 \int_t^T e^{ks} (Y^\lambda_s - \tilde{Y}^\lambda_s) \cdot \left( F^\lambda_G(s, x^\lambda_s, \lambda^m(x_s), Y^\lambda_s) - F^\lambda_G(s, \tilde{x}^\lambda_s, \lambda^m(\tilde{x}_s), \tilde{Y}^\lambda_s) \right) \, ds
- 2 \int_t^T e^{ks} (Y^\lambda_s - \tilde{Y}^\lambda_s) \cdot \left( F^\lambda_G(s, x^\lambda_s, \lambda^m(x_s), Y^\lambda_s) - F^\lambda_G(s, \tilde{x}^\lambda_s, \lambda^m(\tilde{x}_s), Y^\lambda_s) \right) \, ds
+ \int_t^T e^{ks} (Y^\lambda_s - \tilde{Y}^\lambda_s) (Z^\lambda_s - \tilde{Z}^\lambda_s) \, dW^\lambda_s,
$$

Springer
it can be easily seen that

\[
\begin{align*}
e^{kt}|Y^\lambda - \tilde{Y}^\lambda|^2 &+ k \int_t^T e^{ks}|Y^\lambda_s - \tilde{Y}^\lambda_s|^2\,ds + \int_t^T e^{ks}|Z^\lambda_s - \tilde{Z}^\lambda_s|^2\,ds \\
&\leq e^{KT}\left|Q^\lambda_G(x^\lambda_T, L^m(x_T)) - Q^\lambda_G(\tilde{x}^\lambda_T, L^m(\tilde{x}_T))\right|^2 + (2L_2 - 2K_2 + \epsilon) \int_t^T e^{ks}|Y^\lambda_s - \tilde{Y}^\lambda_s|^2\,ds \\
&\quad + L_2 \int_t^T e^{ks}|x^\lambda_s - \tilde{x}^\lambda_s|^2\,ds + \frac{L_2}{2} \int_t^T e^{ks}\mathbb{E}||x^\lambda_s - \tilde{x}^\lambda_s||^2\,ds \\
&\quad + \frac{L_2}{2} \int_t^T e^{ks}\int_{[0,1]}\mathbb{E}[|x^\lambda_s - \tilde{x}^\lambda_s|^2]\,d\kappa\,ds - \int_t^T e^{ks}(Y^\lambda_s - \tilde{Y}^\lambda_s)(Z^\lambda_s - \tilde{Z}^\lambda_s)\,dW^\lambda_s \\
&\quad + \frac{1}{\epsilon} \int_t^T e^{ks}\left(F^\lambda_G(s, x^\lambda_s, L^m(x_s), Y^\lambda_s) - F^\lambda_G(s, x^\lambda_s, L^m(x_s), Y^\lambda_s)\right)^2\,ds.
\end{align*}
\]

Noting that

\[
\begin{align*}
\left|Q^\lambda_G(x^\lambda_T, L^m(x_T)) - Q^\lambda_G(\tilde{x}^\lambda_T, L^m(\tilde{x}_T))\right|^2 &\leq C\left(|x^\lambda_T - \tilde{x}^\lambda_T|^2 + \mathbb{E}[|x^\lambda_T - \tilde{x}^\lambda_T|^2] + \int_{[0,1]}\mathbb{E}[|x^\beta_T - \tilde{x}^\beta_T|^2]\,d\beta\right) \\
&\quad + C\left(Q^\lambda_G(x^\lambda_T, L^m(x_T)) - \tilde{Q}^\lambda_G(x^\lambda_T, L^m(x_T))\right)^2,
\end{align*}
\]

therefore one conclude that

\[
\begin{align*}
(k + 2K_2 - 2L_2 - \epsilon)\mathbb{E}\left[\int_0^T \int_{[0,1]} e^{ks}|Y^\lambda_s - \tilde{Y}^\lambda_s|^2\,d\lambda\,ds\right] + \int_0^1 e^{ks}|Z^\lambda_s - \tilde{Z}^\lambda_s|^2\,d\lambda\,ds \\
&\leq C\left(\mathbb{E}\left[\int_{[0,1]} e^{kt}|x^\lambda_T - \tilde{x}^\lambda_T|^2\,d\lambda\right] + \mathbb{E}\left[\int_0^T \int_{[0,1]} e^{ks}|x^\lambda_s - \tilde{x}^\lambda_s|^2\,d\lambda\,ds\right]\right) \\
&\quad + C\mathbb{E}\left[\int_{[0,1]}\left(Q^\lambda_G(x^\lambda_T, isplacL^m(x_T)) - Q^\lambda_G(\tilde{x}^\lambda_T, L^m(\tilde{x}_T))\right)^2\,d\lambda\right] \\
&\quad + C\mathbb{E}\left[\int_0^T e^{ks}\int_{[0,1]}\left(F^\lambda_G(s, x^\lambda_s, L^m(x_s), Y^\lambda_s) - F^\lambda_G(s, x^\lambda_s, L^m(x_s), Y^\lambda_s)\right)^2\,d\lambda\right]. \quad (3.5)
\end{align*}
\]

Using the argument of (3.4), it can be easily seen that the last two lines converge to 0 as \(\|\tilde{G} - \tilde{G}\|_\square \to 0\). In conjunction with (3.3), we finish proving that \(\|\tilde{Y} - Y\|_k^f \to 0\) as \(\|\tilde{G} - G\|_\square \to 0\).

**Step 3:** Denote by \(y_G\) and \(y_{\tilde{G}}\) the fix point of \(\Gamma_G\) and \(\Gamma_{\tilde{G}}\) respectively. Then it is readily seen that

\[
\|y_G - y_{\tilde{G}}\|_k^f = \|\Gamma_G(y_G) - \Gamma_{\tilde{G}}(y_{\tilde{G}})\|_k^f \leq \|\Gamma_G(y_G) - \Gamma_{\tilde{G}}(y_{\tilde{G}})\|_k^f + \|\Gamma_{\tilde{G}}(y_G) - \Gamma_{\tilde{G}}(y_{\tilde{G}})\|_k^f \leq \|\Gamma_G(y_G) - \Gamma_{\tilde{G}}(y_{\tilde{G}})\|_k^f + \theta\|y_G - y_{\tilde{G}}\|_k^f.
\]
for some \( \theta \in (0, 1) \) by Theorem 2.1, and hence as \( \|G - \tilde{G}\|_\square \to 0 \)

\[
\|y_G - y_{G\theta}\|_k \leq \frac{1}{1 - \theta} \|\Gamma_G(y_G) - \Gamma_{G\theta}(y_G)\|_k \to 0.
\] (3.6)

Denote by \( y = y_G, \tilde{y} = y_{G\theta}, x = \Psi_G(y), \tilde{x} = \Psi_{G\theta}(\tilde{y}) \). Similar to the derivation of (3.3), one easily obtain that

\[
\int_{[0,1]} \sup_{u \in [0,T]} \mathbb{E}[|x_u^\lambda - \tilde{x}_u^\lambda|^2] \, d\lambda 
\leq \mathbb{E} \left[ \int_{[0,1]} |x_0^\lambda - \tilde{x}_0^\lambda|^2 \, d\lambda \right] + L_1 \mathbb{E} \left[ \int_{[0,1]} \int_0^T e^{\lambda s} |y_s^\lambda - \tilde{y}_s^\lambda|^2 \, d\lambda \, ds \right] 
+ C \mathbb{E} \left[ \int_0^T e^{\lambda s} ds \int_{[0,1]} \left( B_G^\lambda(s, x_s^\lambda, L^m(x_s), y_s^\lambda) - B_G^\lambda(s, x_s^\lambda, L^m(x_s), y_s^\lambda) \right)^2 \, d\lambda \right].
\]

In combination with (3.4), (3.5) and (3.6), it is clear that as \( \|G - \tilde{G}\|_\square \to 0 \) and \( \mathbb{E} \left[ \int_{[0,1]} |x_0^\lambda - \tilde{x}_0^\lambda|^2 \, d\lambda \right] \to 0 \),

\[
\int_0^1 \left( \sup_{u \in [0,T]} \mathbb{E}[|x_u^\lambda - \tilde{x}_u^\lambda|^2] + \int_0^T \mathbb{E}[|y_u^\lambda - \tilde{y}_u^\lambda|^2] \, d\lambda + \int_0^T \mathbb{E}[z_s^\lambda - \tilde{z}_s^\lambda]^2 \, ds \right) \, d\lambda \to 0.
\] (3.7)

**Step 4:** According to standard estimates, we have that

\[
\sup_{u \in [0,T]} (x_u^\lambda - \tilde{x}_u^\lambda)^2 
\leq |x_0^\lambda - \tilde{x}_0^\lambda|^2 + C \int_0^T |y_s^\lambda - \tilde{y}_s^\lambda|^2 + |x_s^\lambda - \tilde{x}_s^\lambda|^2 + \mathbb{E}[|x_s^\lambda - \tilde{x}_s^\lambda|^2] \, ds 
+ C \int_0^T \left( B_G^\lambda(s, x_s^\lambda, L^m(x_s), y_s^\lambda) - B_G^\lambda(s, x_s^\lambda, L^m(x_s), y_s^\lambda) \right)^2 \, ds,
\]

and

\[
\sup_{u \in [0,T]} (y_u^\lambda - \tilde{y}_u^\lambda)^2 + \int_0^T (z_s^\lambda - \tilde{z}_s^\lambda)^2 \, ds 
\leq C \left( |x_T^\lambda - \tilde{x}_T^\lambda|^2 + \mathbb{E}[|x_T^\lambda - \tilde{x}_T^\lambda|^2] \right) + \sup_{u \in [0,T]} \int_0^T (y_s^\lambda - \tilde{y}_s^\lambda)(z_s^\lambda - \tilde{z}_s^\lambda) \, dW_s^\lambda 
+ C \int_0^T |y_s^\lambda - \tilde{y}_s^\lambda|^2 + |x_s^\lambda - \tilde{x}_s^\lambda|^2 + \mathbb{E}[|x_s^\lambda - \tilde{x}_s^\lambda|^2] \, ds 
+ C \int_0^T \left( F_G^\lambda(s, x_s^\lambda, L^m(x_s), y_s^\lambda) - F_G^\lambda(s, x_s^\lambda, L^m(x_s), y_s^\lambda) \right)^2 \, ds.
\]
Taking expectation, using BDG inequality and integrating over $\lambda$, we can conclude (3.1) from (3.7).

\[\square\]

The next proposition gives a more explicit estimate than the above stability result in terms of the $L^p$ distance. It will be used to obtain the convergence rate of propagation of chaos in the next section.

**Proposition 3.1** Suppose $(x, y, z)$ and $(\tilde{x}, \tilde{y}, \tilde{z})$ are solutions of (2.1) with graphons $G$ and $\tilde{G}$ respectively. Then under Assumption 2.1 with any $p \geq 2$ and Assumption 3.1, we have that

$$
\mathbb{E}\left[\int_0^T \left( \sup_{u \in [0,T]} |x_u^\lambda - \tilde{x}_u^\lambda|^2 + \sup_{u \in [0,T]} |y_u^\lambda - \tilde{y}_u^\lambda|^2 + \int_0^T |z_u^\lambda - \tilde{z}_u^\lambda|^2 \, ds \right) \, d\lambda \right] 
\leq C \|G - \tilde{G}\|_2^2 + C \int_0^1 \mathcal{W}_2^2 \left( \mathcal{L}(x_0^\lambda), \mathcal{L}(\tilde{x}_0^\lambda) \right) \, d\lambda, 
$$

which implies

$$
\int_0^1 \mathcal{W}_{2,T}^2 \left( \mathcal{L}(x^\lambda, y^\lambda), \mathcal{L}(\tilde{x}^\lambda, \tilde{y}^\lambda) \right) \, d\lambda \leq C \|G - \tilde{G}\|_2^2 + C \int_0^1 \mathcal{W}_2^2 \left( \mathcal{L}(x_0^\lambda), \mathcal{L}(\tilde{x}_0^\lambda) \right) \, d\lambda. 
$$

**Proof** The arguments are very similar to those in the proof of Theorem 3.1, except that we have explicit estimates in terms of $\|G - \tilde{G}\|_2$. So here we only highlight the differences. In particular, in step 2, we have

$$
\left( B_G^\lambda(s, x_s^\lambda, \mathcal{L}^m(x_s), y_s^\lambda) - B_G^\lambda(s, x_s^\lambda, \mathcal{L}^m(x_s), y_s^\lambda) \right)^2 
\leq \left| \int_{[0,1]} |G(\lambda, \kappa) - \tilde{G}(\lambda, \kappa)| \, d\kappa \int B(s, x_s^\lambda, w, y_s^\lambda) \mathcal{L}(x_s^\kappa)(d\kappa) \right|^2 
\leq C \left( 1 + \int_{[0,1]} \mathbb{E}[|x_s^\kappa|^2] \, d\kappa \right) \int_{[0,1]} |G(\lambda, \kappa) - \tilde{G}(\lambda, \kappa)|^2 \, d\kappa.
$$

Therefore the estimate (3.4) can be replaced by

$$
\mathbb{E}\left[\int_0^T e^{ks} \, ds \int \left( B_G^\lambda(s, x_s^\lambda, \mathcal{L}^m(x_s), y_s^\lambda) - B_G^\lambda(s, x_s^\lambda, \mathcal{L}^m(x_s), y_s^\lambda) \right)^2 \, d\lambda \right] 
\leq C \|G - \tilde{G}\|_2^2.
$$

Similarly, the last two terms in (3.5) can be estimated by

$$
\mathbb{E}\left[\int_{[0,1]} \left( Q_G^\lambda(x_T^\lambda, \mathcal{L}^m(X_T)) - Q_G^\lambda(x_T^\lambda, \mathcal{L}^m(X_T)) \right)^2 \, d\lambda \right] 
+ \mathbb{E}\left[\int_0^T e^{ks} \, ds \int \left( F_G^\lambda(s, x_s^\lambda, \mathcal{L}^m(x_s), Y_s^\lambda) - F_G^\lambda(s, x_s^\lambda, \mathcal{L}^m(x_s), Y_s^\lambda) \right)^2 \, d\lambda \right] 
\leq C \|G - \tilde{G}\|_2^2.
$$
and hence \( \| \tilde{Y} - Y \|_k^l \leq C \| G - \tilde{G} \|_2^2 \). In step 3, we can replace (3.6) by \( \| y_G - y_{\tilde{G}} \|_k^l \leq C \| G - \tilde{G} \|_2^2 \) and hence replace (3.7) by

\[
\int_0^1 \left( \sup_{u \in [0,T]} \mathbb{E}[|x_u^\lambda - \tilde{x}_u^\lambda|^2] + \int_0^T \mathbb{E}[|y_u^\lambda - \tilde{y}_u^\lambda|^2] d\lambda + \int_0^T \mathbb{E}[z_s^\lambda - \tilde{z}_s^\lambda|^2] ds \right) d\lambda \\
\leq C \| G - \tilde{G} \|_2^2 + C \int_0^1 \mathcal{W}_2^2 \left( \mathcal{L}(x_0^\lambda), \mathcal{L}(\tilde{x}_0^\lambda) \right) d\lambda.
\]

The same argument in step 4 gives (3.8) and (3.9).

3.2 Method of Continuation

**Theorem 3.2** Suppose \((x, y, z)\) and \((\tilde{x}, \tilde{y}, \tilde{z})\) are solutions of (2.1) with graphons \(G\) and \(\tilde{G}\) respectively. Then under Assumption 2.2 with any \(p \geq 2\) and Assumption 3.1, we have the convergence (3.1) and (3.2) as \(\| G - \tilde{G} \| \to 0\) and

\[
\mathbb{E} \left[ \int_{[0,1]} |y_0^\lambda - \tilde{y}_0^\lambda|^2 d\lambda \right] \to 0.
\]

**Proof** The proof is a mix of Proposition 2.1 and Theorem 3.1. For any graphon \(G\), denote by \(\nu_G^\zeta(B_0, F_0, Q_0)\) the law of solution \(X^\zeta\) to (2.14). Let us recall the map \(\Pi\) defined in (2.18), and denote it by \(\Pi_G^\zeta\) to indicate the dependence on the parameter \(\zeta\) and coefficients \((B_G, F_G, Q_G)\).

**Step 1:** For any \((x, y) \in \mathcal{ML}_F^{2,c} \times \mathcal{ML}_F^{2,c}\), define a new norm

\[
\|(x, y)\|_{l,2} := \int_{[0,1]} \mathbb{E}[|x_T^\lambda|^2] + \int_0^T \mathbb{E}[|x_t^\lambda|^2 + |y_t^\lambda|^2] dt d\lambda.
\]

Under Assumption 2.2, it can be shown that \(\Pi_G\) is a contraction under this norm.

**Step 2:** Let us study \(\Pi_G^\zeta - \Pi_{\tilde{G}}^\zeta\). Take any \((x, y) \in \mathcal{ML}_F^{2,c} \times \mathcal{ML}_F^{2,c}\). Denote \((X, Y) = \Pi_G^\zeta(x, y), (\tilde{X}, \tilde{Y}) = \Pi_{\tilde{G}}^\zeta(x, y), \Delta X = X - \tilde{X}, \Delta Y = Y - \tilde{Y}\), and \(\theta_t^\lambda = (x_t^\lambda, L^m(x_t), y_t^\lambda), \Theta_t^\lambda = (X_t^\lambda, L^m(X_t), Y_t^\lambda), \tilde{\Theta}_t^\lambda = (\tilde{X}_t^\lambda, L^m(\tilde{X}_t), \tilde{Y}_t^\lambda)\).

Let us compute

\[
\begin{align*}
\mathbb{E} [\Delta X_T^\lambda \Delta Y_T^\lambda] &\geq (k - 2l - \epsilon) \mathbb{E} [(\Delta X_T^\lambda)^2] - l \int_{[0,1]} \mathbb{E} [(\Delta X_\kappa^\lambda)^2] d\kappa \\
&- C \mathbb{E} \left[ \left| Q_G^\lambda(X_T^\lambda, L^m(X_T)) - Q_{\tilde{G}}^\lambda(X_T^\lambda, L^m(X_T)) \right|^2 \right] \\
&+ \left| Q_G^\lambda(x_T^\lambda, L^m(x_T)) - Q_{\tilde{G}}^\lambda(x_T^\lambda, L^m(x_T)) \right|^2.
\end{align*}
\] (3.10)
Using Itô’s formula, we also obtain that

\[
\mathbb{E} \left[ \Delta X^\lambda_T \Delta Y^\lambda_T \right] - \mathbb{E} \left[ \Delta X^\lambda_0 \Delta Y^\lambda_0 \right] \\
\leq - (k - 2) \cdot (e) \int_0^T \mathbb{E} \left[ (\Delta X^\lambda_t)^2 + (\Delta Y^\lambda_t)^2 \right] dt + l \int_0^T \int_{[0,1]} \mathbb{E} [(\Delta X^\lambda)^2] d\kappa dt \\
+ C \int_0^T \mathbb{E} \left[ \left| B^\lambda_G(t, \Theta^\lambda_t) - B^\lambda_G(t, \Theta^\lambda_t) \right|^2 + \left| F^\lambda_G(t, \Theta^\lambda_t) - F^\lambda_G(t, \Theta^\lambda_t) \right|^2 \right] dt \\
+ C \int_0^T \mathbb{E} \left[ \left| B^\lambda_G(t, \theta^\lambda_t) - B^\lambda_G(t, \theta^\lambda_t) \right|^2 + \left| F^\lambda_G(t, \theta^\lambda_t) - F^\lambda_G(t, \theta^\lambda_t) \right|^2 \right] dt. \quad (3.11)
\]

Combining the above two inequalities (3.10), (3.11) and integrating over \( \lambda \in [0, 1] \), we get that

\[
(k - 3) \cdot (e) \int_0^T \mathbb{E} \left[ (\Delta X^\lambda_t)^2 + (\Delta Y^\lambda_t)^2 \right] d\kappa dt + \int_0^T \int_{[0,1]} \mathbb{E} [(\Delta X^\lambda)^2] d\kappa dt \\
\leq C \mathbb{E} \left[ \left| Q^\lambda_G(x^\lambda_T, \mathcal{L}^m(x_T)) - Q^\lambda_G(x^\lambda_T, \mathcal{L}^m(x_T)) \right|^2 \right] \\
+ C \mathbb{E} \left[ \left| Q^\lambda_G(x^\lambda_T, \mathcal{L}^m(x_T)) - Q^\lambda_G(x^\lambda_T, \mathcal{L}^m(x_T)) \right|^2 \right] \\
+ C \int_0^T \mathbb{E} \left[ \left| B^\lambda_G(t, \Theta^\lambda_t) - B^\lambda_G(t, \Theta^\lambda_t) \right|^2 + \left| F^\lambda_G(t, \Theta^\lambda_t) - F^\lambda_G(t, \Theta^\lambda_t) \right|^2 \right] dt \\
+ C \int_0^T \mathbb{E} \left[ \left| B^\lambda_G(t, \theta^\lambda_t) - B^\lambda_G(t, \theta^\lambda_t) \right|^2 + \left| F^\lambda_G(t, \theta^\lambda_t) - F^\lambda_G(t, \theta^\lambda_t) \right|^2 \right] dt \\
+ \int_{[0,1]} \mathbb{E} \left[ \Delta X^\lambda_0 \Delta Y^\lambda_0 \right] d\kappa. \quad (3.12)
\]

Using the same argument as in (3.4), we can show that the right hand side of the above inequality converges to 0 as \( \|G - \tilde{G}\|_\square \to 0 \) and \( \mathbb{E} \left[ \int_{[0,1]} |X^\lambda_0 - \tilde{X}^\lambda_0|^2 d\kappa \right] \to 0 \).

**Step 3:** Choose \( \delta \) as in Proposition 2.1, \( \zeta = 1 - \delta \), and \((X, Y), (\tilde{X}, \tilde{Y})\) to be the unique fixed point of \( \Pi^\zeta_G, \Pi^\zeta_G \) respectively. Then it is clear that

\[
\|(X, Y) - (\tilde{X}, \tilde{Y})\|_{1,2} = \|\Pi^\zeta_G(X, Y) - \Pi^\zeta_G(\tilde{X}, \tilde{Y})\|_{1,2} \\
\leq \|\Pi^\zeta_G(X, Y) - \Pi^\zeta_G(X, Y)\|_{1,2} + \|\Pi^\zeta_G(X, Y) - \Pi^\zeta_G(\tilde{X}, \tilde{Y})\|_{1,2}.
\]

Since \( \Pi^\zeta_G \) is \( \theta \)-Lipschitz with some \( \theta < 1 \) for all graphon \( \tilde{G} \), we have that

\[
\|(X, Y) - (\tilde{X}, \tilde{Y})\|_{1,2} \leq \frac{1}{1 - \theta} \|\Pi^\zeta_G(X, Y) - \Pi^\zeta_G(X, Y)\|_{1,2}.
\]
Due to Step 2, we know that

\[
\| \Pi_{G}^{\lambda}(X, Y) - \Pi_{\tilde{G}}^{\lambda}(X, Y) \|_{l^2} \to 0
\]  

(3.13)
as \| G - \tilde{G} \| \to 0 and \( \mathbb{E} \left[ \int_{[0,1]} |X_0^{\lambda} - \tilde{X}_0^{\lambda}|^2 d\lambda \right] \to 0.

Step 4: Recall the map (2.18) and note that the evolution of \((X, Y), (\tilde{X}, \tilde{Y})\) is given by (2.1) with graphon \(G\) and \(\tilde{G}\) respectively. By Itô’s formula, we have

\[
\begin{align*}
&\left| Q_{G}^{\lambda}(X_{T}^{\lambda}, L^{m}(X_{T}^{\lambda})) - Q_{\tilde{G}}^{\lambda}(\tilde{X}_{T}^{\lambda}, L^{m}(\tilde{X}_{T}^{\lambda})) \right|^2 \\
&= |Y_{t}^{\lambda} - \tilde{Y}_{t}^{\lambda}|^2 + \int_{t}^{T} |Z_{s}^{\lambda} - \tilde{Z}_{s}^{\lambda}|^2 ds \\
&\quad - 2 \int_{t}^{T} (Y_{s}^{\lambda} - \tilde{Y}_{s}^{\lambda}) \cdot \left( F_{G}^{\lambda}(s, X_{s}^{\lambda}, L^{m}(X_{s}^{\lambda}), Y_{s}^{\lambda}) - F_{\tilde{G}}^{\lambda}(s, \tilde{X}_{s}^{\lambda}, L^{m}(\tilde{X}_{s}^{\lambda}), \tilde{Y}_{s}^{\lambda}) \right) ds \\
&\quad + \int_{t}^{T} (Y_{s}^{\lambda} - \tilde{Y}_{s}^{\lambda})(Z_{s}^{\lambda} - \tilde{Z}_{s}^{\lambda}) dW_{s}^{\lambda}.
\end{align*}
\]

Taking expectations, integrating over \(\lambda\), and using (3.13), we get

\[
\int_{0}^{1} \int_{0}^{T} \mathbb{E}|Z_{s}^{\lambda} - \tilde{Z}_{s}^{\lambda}|^2 ds d\lambda \to 0.
\]

Lastly, using the argument of Step 4 in Theorem 3.1, we can easily conclude (3.1). \(\square\)

The next proposition gives a more explicit estimate than the above stability result in terms of the \(L^p\) distance. It will be used to obtain the convergence rate of propagation of chaos in the next section.

**Proposition 3.2** Suppose \((x, y, z)\) and \((\tilde{x}, \tilde{y}, \tilde{z})\) are solutions of (2.1) with graphons \(G\) and \(\tilde{G}\) respectively. Then under Assumption 2.2 with any \(p \geq 2\) and Assumption 3.1, we have the estimates (3.8) and (3.9).

**Proof** The arguments are very similar to those in the proof of Theorem 3.2, except that we have explicit estimates in terms of \(\| G - \tilde{G} \|_{2}\). So here we only highlight the differences. In particular, in step 2, from (3.12) we have

\[
(k - 3l - \epsilon) \left( \int_{[0,1]} [\mathbb{E}((\Delta X_{T}^{\lambda})^2)] d\lambda + \int_{[0,1]} \int_{[0,1]} [\mathbb{E}((\Delta X_{t}^{\lambda})^2 + (\Delta Y_{t}^{\lambda})^2)] d\lambda dt \right)
\]

\[
\leq C \| G - \tilde{G} \|_{2} + \int_{[0,1]} \mathbb{E} [\Delta X_{0}^{\lambda} \Delta Y_{0}^{\lambda}] d\lambda.
\]
In step 3, we have
\[
\| (X, Y) - (\tilde{X}, \tilde{Y}) \|^2 \leq \frac{1}{1 - \theta} \| \Pi_G^\varepsilon (X, Y) - \Pi_G^\varepsilon (X, Y) \|^2 \\
\leq C \| G - \tilde{G} \|^2 + C \int_0^1 \mathcal{W}_2^2 (\mathcal{L}(\lambda) - \mathcal{L}(\lambda')) d\lambda.
\]

Using the argument of Step 4 in Theorem 3.2, we have
\[
\int_0^1 \int_0^T \mathbb{E} \left| Z^\lambda_x - \tilde{Z}^\lambda_x \right|^2 ds d\lambda \leq C \| G - \tilde{G} \|^2 + C \int_0^1 \mathcal{W}_2^2 \left( \mathcal{L}(\lambda), \mathcal{L}(\lambda') \right) d\lambda.
\]
Using the argument of Step 4 in Theorem 3.1, we have the estimates (3.8) and (3.9).

Lastly, the following proposition shows the continuity of \( \lambda \mapsto \mathcal{L}(X^\lambda, Y^\lambda) \). The proof follows from standard coupling arguments similar to [29, Theorem 2.1].

**Proposition 3.3** Suppose Assumption 3.1 holds. Suppose either Assumption 2.1 or 2.2 holds with some \( p \geq 2 \). If \( G \) is (Lipschitz) continuous and \( \lambda \mapsto \mathcal{L}(X^\lambda, Y^\lambda) \) is (Lipschitz) continuous with respect to \( \mathcal{W}_2 \), then \( \lambda \mapsto \mathcal{L}(X^\lambda, Y^\lambda) \) is (Lipschitz) continuous with respect to \( \mathcal{W}_2 \).

### 4 Propagation of Chaos

Consider step graphon \( G_n \) such that \( \| G_n - G \|_\square \to 0 \) as \( n \to \infty \), and the following coupled systems of FBSDEs

\[
\begin{aligned}
&dX_{i,n}^j = B_0(t, X_{i,n}^j, Y_{i,n}^j) dt + \frac{1}{n} \sum_{j=1}^n \mathbb{E} G_n \left( \frac{i}{n}, \frac{j}{n} \right) \hat{B} \left( t, X_{i,n}^j, X_{i,n}^j, Y_{i,n}^j \right) dt + \sigma dW_{i,n}, \\
&dY_{i,n}^j = -F_0(t, X_{i,n}^j, Y_{i,n}^j) dt + \frac{1}{n} \sum_{j=1}^n \mathbb{E} G_n \left( \frac{i}{n}, \frac{j}{n} \right) \hat{F} \left( t, X_{i,n}^j, X_{i,n}^j, Y_{i,n}^j \right) dt + \sum_{j=1}^n \xi_{i,j} \sigma dW_{i,n}, \\
&X_{0,n}^j = \xi_{i,n}, \\
&Y_{0,n}^j = Q_0(X_{T,n}^j) + \frac{1}{n} \sum_{j=1}^n \mathbb{E} G_n \left( \frac{i}{n}, \frac{j}{n} \right) \hat{Q} \left( X_{T,n}^j, X_{T,n}^j \right), \; j = 1, \ldots, n,
\end{aligned}
\]

and the following limiting system

\[
\begin{aligned}
&dX_\lambda^j = B_0(t, X_\lambda^j, Y_\lambda^j) dt + \int_\mathbb{R} G(\lambda, \kappa) \hat{B} \left( t, X_\lambda^j, x, Y_\lambda^j \right) \mathcal{L}(X_\lambda^\kappa) (dx) d\kappa dt + \sigma dW_\lambda^j, \\
&dY_\lambda^j = -F_0(t, X_\lambda^j, Y_\lambda^j) dt + \int_\mathbb{R} G(\lambda, \kappa) \hat{F} \left( t, X_\lambda^j, x, Y_\lambda^j \right) \mathcal{L}(X_\lambda^\kappa) (dx) d\kappa dt + Z_\lambda^j dW_\lambda^j, \\
&X_0^j = \xi^j, \\
&Y_0^j = Q_0(X_T^j) + \int_\mathbb{R} G(\lambda, \kappa) \hat{Q} \left( X_T^\lambda, x \right) \mathcal{L}(X_T^\kappa) (dx) d\kappa, \; \lambda \in [0, 1].
\end{aligned}
\]

We will prove that solutions of (4.1) converge to that of (4.2).
4.1 Contraction Mapping

The following assumption summarizes Assumptions 2.1 (with $p = 2$) and 3.1.

**Assumption 4.1**

(i) $B_0$ is Lipschitz in $x$, and there exists a constant $K_1 \in \mathbb{R}$ such that for any $(t, x, x', y) \in [0, T] \times \mathbb{R}^3$

\[
(x - x') \cdot (B_0(t, x, y) - B_0(t, x', y)) \leq -K_1(x - x')^2.
\]

\(\hat{B}\) is $L_1$-Lipschitz in $x, x'$.

(ii) $F_0$ is Lipschitz in $y$, and there exists a constant $K_2 \in \mathbb{R}$ such that for any $(t, x, y, y') \in [0, T] \times \mathbb{R}^3$

\[
(y - y') \cdot (F_0(t, x, y) - F_0(t, x, y')) \leq -K_2(y - y')^2
\]

\(\hat{F}\) is $L_2$-Lipschitz in $x, x', y$.

(iii) $Q_0, \hat{Q}$ are Lipschitz.

(iv) $B_0, B$ are bounded in $y$.

(v) It holds that $2K_1 + 2K_2 > 3L_1 + 2L_2$ and there exists a constant $k \in (2L_2 - 2K_2, 2K_1 - 3L_1)$ such that

\[
(k + 2K_2 - 2L_2) > 4L_1 L_3^2 + \frac{2L_1^2 L_3^2 + 2L_1 L_2}{k + 2K_1 - 3L_1}. \]

(vi) It holds that $\sup_{x \in [0, 1]} \mathbb{E}[|\xi^k|^2] < +\infty$.

(vii) $\lambda \mapsto L(\xi^k)$ is continuous with respect to $\mathcal{W}_2$.

We introduce the following notation that will be used in this section. Given any measurable $y = (y^{i,n})_{i=1}^n \in \mathcal{ML}_{\mathcal{F}_t}^{p,c}$, we define $\hat{\Psi}_{G_n}(y) := x = (x^{i,n})_{i=1}^n$ as the unique solution to

\[
\begin{align*}
\left\{ \begin{array}{ll}
dx^{i,n}_t = B_0(t, x^{i,n}_t, y^{i,n}_t) dt + \frac{1}{n} \sum_{j=1}^n G_n(\frac{i}{n}, \frac{j}{n}) \hat{B}(t, x^{i,n}_t, x^{j,n}_t, y^{j,n}_t) dt + \sigma dW^{i,n}_t, \\
x^{i,n}_0 = \xi^{i,n}, & i = 1, \ldots, n,
\end{array} \right.
\end{align*}
\]

and define $\tilde{\Psi}_{G_n}(y) := x = (x^{i,n})_{i=1}^n$ as the unique solution to

\[
\begin{align*}
\left\{ \begin{array}{ll}
dx^{i,n}_t = B_0(t, x^{i,n}_t, y^{i,n}_t) dt + \frac{1}{n} \sum_{j=1}^n G_n(\frac{i}{n}, \frac{j}{n}) \hat{B}(t, x^{i,n}_t, x^{j,n}_t, y^{j,n}_t) \mathcal{L}(y^{j,n}_t)(dx) dt + \sigma dW^{i,n}_t, \\
x^{i,n}_0 = \xi^{i,n}, & i = 1, \ldots, n.
\end{array} \right.
\end{align*}
\]

For any $x = (x^{i,n})_{i=1}^n \in \mathcal{ML}_{\mathcal{F}_t}^{p,c}$, define $\hat{\Phi}_{G_n}(x) := y = (y^{i,n})_{i=1}^n$ to be the unique solution to backward stochastic equations

\[
\begin{align*}
\left\{ \begin{array}{ll}
dy^{i,n}_t = -F_0(t, x^{i,n}_t, y^{i,n}_t) dt - \frac{1}{n} \sum_{j=1}^n G_n(\frac{i}{n}, \frac{j}{n}) \hat{F}(t, x^{i,n}_t, x^{j,n}_t, y^{j,n}_t) dt + \sum_{j=1}^n Z^{j,n}_t dW^{j,n}_t, \\
y_T = Q_0(x^n_T) + \frac{1}{n} \sum_{j=1}^n G_n(\frac{i}{n}, \frac{j}{n}) \hat{Q}(x^{j,n}_T, x^{i,n}_T), & i = 1, \ldots, n,
\end{array} \right.
\end{align*}
\]
and define \( \tilde{\Phi}_G_n(x) := y = (y^{i,n})_{i=1}^n \) to be the unique solution to backward stochastic equations

\[
\begin{align*}
    dy^{i,n}_t &= -F_t(x^{i,n}_t, y^{i,n}_t) dt - \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_{\mathbb{R}} G_n(\frac{i}{n}, \frac{j}{n}) \tilde{f}(t, x^{i,n}_t, x, y^{i,n}_t) \mathcal{L}(x^{i,n}_t) dx dt + Z^{i,n}_t dW^i_t, \\
    y^{i,n}_T &= Q_0(x^{i,n}_T) + \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_{\mathbb{R}} G_n(\frac{i}{n}, \frac{j}{n}) \tilde{Q}(x^{i,n}_T, x) \mathcal{L}(x^{i,n}_T) dx, \quad i = 1, \ldots, n.
\end{align*}
\]

We note that \( \tilde{\Psi}_G_n \) and \( \tilde{\Phi}_G_n \) are simply the maps \( \Psi \) and \( \Phi \) with blockwise constant graphon \( G_n \) and associated piecewise constant initial states \( (\xi^{[n\lambda]}_{\lambda} \equiv 0 \leq \lambda \leq 1) \).

Let \( (\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n) \) be the unique solution of the limiting system with graphon \( G_n \) and initial states \( (\xi^{[n\lambda]}_{\lambda} \equiv 0 \leq \lambda \leq 1) \). Abusing notations, we write \( \tilde{X}^n = (\tilde{X}^{i,n}_{\lambda})_{i=1}^n = (\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n) \) and \( (\tilde{Y}^n, \tilde{Z}^n) \) respectively. The following result shows that \( \tilde{G}_G_n \) is a contraction map. The proof is similar to that of Theorem 2.1 and hence omitted.

**Lemma 4.1** Suppose Assumption 4.1 holds. Then there exists some \( \theta \in (0, 1) \) such that for each \( n \in \mathbb{N} \), the map \( \tilde{G}_G_n \) is a contraction under the norm \( \| \cdot \|_k \) with contraction constant \( \theta \).

**Theorem 4.1** Suppose Assumption 4.1 holds and \( \| G_n - G \|_0 \to 0 \). Then

\[
\begin{align*}
    \mathbb{E} \left[ \int_0^1 \left( \sup_{t \in [0,T]} |X_t^{[n\lambda],n} - X_t^{[\lambda],n}|^2 + \sup_{t \in [0,T]} |Y_t^{[n\lambda],n} - Y_t^{[\lambda],n}|^2 dt + \int_0^T |Z_t^{[n\lambda],n} - Z_t^{[\lambda],n}|^2 d\lambda \right) \right] \to 0. 
\end{align*}
\]

If in addition \( G \) is continuous, then

\[
\begin{align*}
    \frac{1}{n} \sum_{i=1}^n \mathbb{E} \sup_{0 \leq t \leq T} \left[ |X_t^{i,n} - X_t^{i/n}|^2 + |Y_t^{i,n} - Y_t^{i/n}|^2 \right] dt \to 0, \\
    \sup_{t \in [0,T]} \mathbb{E} \left[ |V_t|^2(v^i_t, v_t) \right] dt \to 0,
\end{align*}
\]

where \( v_t^n = \frac{1}{n} \sum_{i=1}^n \delta_{(X_t^{i,n}, Y_t^{i,n})} \) and \( v_t = \int_0^1 \mathcal{L}(X_t^\lambda, Y_t^\lambda) d\lambda \).

**Proof** Step 1: Take \( y = (y^k : k = 1, \ldots, n) \equiv (y^{k,n} : k = 1, \ldots, n) \) with independent coordinates \( y^{k,n} \), and denote \( x^n = \tilde{\Psi}_G_n(y) \), \( \tilde{x}^n = \tilde{\Psi}_G_n(y) \), \( y^n = \tilde{\Phi}_G_n(x^n) \), \( \tilde{y}^n = \tilde{\Phi}_G_n(\tilde{x}^n) \). Note that

\[
(y^{k,n}, \tilde{x}^{k,n}, \tilde{y}^{k,n}) \text{ are independent across } k = 1, \ldots, n.
\]
By Itô’s formula,

\[ e^{kt}|x_{t}^{i,n} - \tilde{x}_{t}^{i,n}|^2 = k \int_{0}^{t} e^{ks}|x_{s}^{i,n} - \tilde{x}_{s}^{i,n}|^2 \, ds \]

\[ + 2 \int_{0}^{t} e^{ks} (x_{s}^{i,n} - \tilde{x}_{s}^{i,n}) \cdot (B_0(s, x_{s}^{i,n}, y_{s}^{i,n}) - B_0(s, \tilde{x}_{s}^{i,n}, y_{s}^{i,n})) \, ds \]

\[ + 2 \int_{0}^{t} e^{ks} (x_{s}^{i,n} - \tilde{x}_{s}^{i,n}) \cdot \frac{1}{n} \sum_{j=1}^{n} G_n \left( \frac{i}{n}, \frac{j}{n} \right) \left( \hat{B}(s, x_{s}^{i,n}, x_{s}^{j,n}, y_{s}^{i,n}) - \hat{B}(s, \tilde{x}_{s}^{i,n}, \tilde{x}_{s}^{j,n}, y_{s}^{i,n}) \right) \, ds \]

\[ + 2 \int_{0}^{t} e^{ks} (x_{s}^{i,n} - \tilde{x}_{s}^{i,n}) \cdot \frac{1}{n} \sum_{j=1}^{n} G_n \left( \frac{i}{n}, \frac{j}{n} \right) \times \left( \hat{B}(s, \tilde{x}_{s}^{i,n}, \tilde{x}_{s}^{j,n}, y_{s}^{i,n}) - \int_{\mathbb{R}} \hat{B} \left( s, \tilde{x}_{s}^{i,n}, x, y_{s}^{j,n} \right) \mathcal{L}(\tilde{x}_{s}^{j,n})(dx) \right) \, ds \]

\[ \leq (k - 2K_1 + 3L_1 + \epsilon) \int_{0}^{t} e^{ks}|x_{s}^{i,n} - \tilde{x}_{s}^{i,n}|^2 \, ds + L_1 \frac{1}{n} \sum_{j=1}^{n} \int_{0}^{t} e^{ks}|x_{s}^{j,n} - \tilde{x}_{s}^{j,n}|^2 \, ds \]

\[ + \frac{1}{\epsilon} \int_{0}^{t} e^{ks} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \left[ \hat{B}(s, \tilde{x}_{s}^{i,n}, \tilde{x}_{s}^{j,n}, y_{s}^{i,n}) - \int_{\mathbb{R}} \hat{B} \left( s, \tilde{x}_{s}^{i,n}, x, y_{s}^{j,n} \right) \mathcal{L}(\tilde{x}_{s}^{j,n})(dx) \right]^2 \, ds. \]

Taking expectations and the average over \( i \), we have

\[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ e^{kt}|x_{t}^{i,n} - \tilde{x}_{t}^{i,n}|^2 \right] \]

\[ \leq (k - 2K_1 + 4L_1 + \epsilon) \int_{0}^{t} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ e^{ks}|x_{s}^{i,n} - \tilde{x}_{s}^{i,n}|^2 \right] \, ds \]

\[ + \frac{1}{\epsilon} \int_{0}^{t} e^{ks} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \left[ \hat{B}(s, \tilde{x}_{s}^{i,n}, \tilde{x}_{s}^{j,n}, y_{s}^{i,n}) - \int_{\mathbb{R}} \hat{B} \left( s, \tilde{x}_{s}^{i,n}, x, y_{s}^{j,n} \right) \mathcal{L}(\tilde{x}_{s}^{j,n})(dx) \right]^2 \, ds. \]

For the last line, we have the estimation

\[ \frac{1}{\epsilon} \int_{0}^{t} e^{ks} \mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^{n} G_n \left( \frac{i}{n}, \frac{j}{n} \right) \left( \hat{B}(s, \tilde{x}_{s}^{i,n}, \tilde{x}_{s}^{j,n}, y_{s}^{i,n}) - \int_{\mathbb{R}} \hat{B} \left( s, \tilde{x}_{s}^{i,n}, x, y_{s}^{j,n} \right) \mathcal{L}(\tilde{x}_{s}^{j,n})(dx) \right)^2 \right] \, ds \]

\[ = \frac{1}{\epsilon} \int_{0}^{t} e^{ks} \frac{1}{n^2} \sum_{j=1}^{n^2} G_n^2 \left( \frac{i}{n}, \frac{j}{n} \right) \mathbb{E} \left[ \hat{B}(s, \tilde{x}_{s}^{i,n}, \tilde{x}_{s}^{j,n}, y_{s}^{i,n}) - \int_{\mathbb{R}} \hat{B} \left( s, \tilde{x}_{s}^{i,n}, x, y_{s}^{j,n} \right) \mathcal{L}(\tilde{x}_{s}^{j,n})(dx) \right]^2 \, ds \]

\[ \leq \frac{C}{n}, \]

due to the independence (4.6), the boundedness of \( \mathbb{E}[\sup_{s \in [0, T]} |\tilde{x}_{s}^{i,n}|^2] \) and Lipschitz property of \( \hat{B} \). Therefore

\[ \frac{1}{n} \sum_{i=1}^{n} \sup_{t \in [0, T]} e^{kt}|x_{t}^{i,n} - \tilde{x}_{t}^{i,n}|^2 \leq \frac{C}{n}. \]  

(4.7)
Step 2: Then let us estimate $y^n - \bar{y}^n$. From the equation

$$e^{kT}E[y_T^n - \bar{y}_T^n]^2$$

$$= e^{kT}E|y_T^n - \bar{y}_T^n|^2 + kE \int_t^T e^{ks} |y_s^n - \bar{y}_s^n|^2 ds + \sum_{j=1}^n E \int_t^T e^{ks} |Z_s^{i,j,n} - \delta_{ij} \bar{Z}_s^n|^2 ds$$

$$- 2E \int_t^T e^{ks} (\varphi_s^n - \bar{\varphi}_s^n) \cdot \left( F_0(s, x_s^{i,n}, y_s^{i,n}) - F_0(s, \bar{x}_s^n, \bar{y}_s^n) \right) ds$$

$$- 2E \int_t^T e^{ks} (\varphi_s^n - \bar{\varphi}_s^n) \cdot \frac{1}{n} \sum_{j=1}^n G_n(i, j) \left( \hat{F}(s, x_s^{i,n}, x_s^{j,n}, y_s^{j,n}) - \hat{F}(s, \bar{x}_s^n, \bar{x}_s^n, \bar{y}_s^n) \right) ds$$

$$- 2E \int_t^T e^{ks} (\varphi_s^n - \bar{\varphi}_s^n) \cdot \frac{1}{n} \sum_{j=1}^n G_n(i, j) \left( \hat{F}(s, \bar{x}_s^n, \bar{x}_s^n, \bar{y}_s^n) - \int_{\mathbb{R}} \hat{F}(s, \bar{x}_s^n, x, \bar{y}_s^n) \mathcal{L}(\bar{x}_s^n)(dx) \right) ds,$$

it can be easily seen that

$$e^{kT}E[y_T^n - \bar{y}_T^n]^2 + kE \int_t^T e^{ks} |y_s^n - \bar{y}_s^n|^2 ds + \sum_{j=1}^n E \int_t^T e^{ks} |Z_s^{i,j,n} - \delta_{ij} \bar{Z}_s^n|^2 ds$$

$$\leq e^{kT}E|y_T^n - \bar{y}_T^n|^2 + (4L_2 - 2K_2 + \epsilon)E \int_t^T e^{ks} |y_s^n - \bar{y}_s^n|^2 ds$$

$$+ L_2E \int_t^T e^{ks} |x_s^n - \bar{x}_s^n|^2 ds + L_2 \sum_{j=1}^n \frac{1}{n} \sum_{j=1}^n E \int_t^T e^{ks} |x_s^{j,n} - \bar{x}_s^{j,n}|^2 ds + \frac{1}{\epsilon} \int_t^T e^{ks}$$

$$\times \left( \frac{1}{n} \sum_{j=1}^n G_n(i, j) \left( \hat{F}(s, \bar{x}_s^n, \bar{x}_s^n, \bar{y}_s^n) - \int_{\mathbb{R}} \hat{F}(s, \bar{x}_s^n, x, \bar{y}_s^n) \mathcal{L}(\bar{x}_s^n)(dx) \right) \right)^2 ds.$$

(4.8)

Note that

$$\frac{1}{\epsilon} \int_t^T e^{ks} \left( \frac{1}{n} \sum_{j=1}^n G_n(i, j) \left( \hat{F}(s, \bar{x}_s^n, \bar{x}_s^n, \bar{y}_s^n) - \int_{\mathbb{R}} \hat{F}(s, \bar{x}_s^n, x, \bar{y}_s^n) \mathcal{L}(\bar{x}_s^n)(dx) \right) \right)^2 ds = \frac{1}{\epsilon} \int_t^T e^{ks} \frac{1}{n^2} \sum_{j=1}^n G_n^2(i, j) \left( \hat{F}(s, \bar{x}_s^n, \bar{x}_s^n, \bar{y}_s^n) - \int_{\mathbb{R}} \hat{F}(s, \bar{x}_s^n, x, \bar{y}_s^n) \mathcal{L}(\bar{x}_s^n)(dx) \right)^2 ds$$

$$\leq \frac{C}{n},$$

due to the independence (4.6), the boundedness of $E[\sup_{s \in [0,T]} |\bar{x}_s^n|^2]$ and Lipschitz property of $\hat{F}$. Also,

$$e^{kT}E|y_T^n - \bar{y}_T^n|^2 \leq C E \left( |x_T^n - \bar{x}_T^n|^2 + \frac{1}{n} \sum_{j=1}^n |x_T^{i,j,n} - \bar{x}_T^{i,j,n}|^2 + \frac{1}{n} \right).$$

Therefore we conclude from (4.7) that

$$(k + 2K_2 - 4L_2 - \epsilon)E \left[ \int_0^T \frac{1}{n} \sum_{i=1}^n e^{ks} |y_s^{i,n} - \bar{y}_s^{i,n}|^2 ds \right] \leq \frac{C}{n}.$$  

(4.9)
Step 3: Recall the processes $Y^n$, $\tilde{Y}^n$ and $\tilde{Z}^n$. Note that $\tilde{Y}^n$ has independent coordinates. Using Lemma 4.1, we have

$$s \| Y^n - \tilde{Y}^n \|_{k} = \| \tilde{G}_n(Y^n) - \tilde{G}_n(\tilde{Y}^n) \|_{k}$$

$$\leq \| \tilde{G}_n(Y^n) - \tilde{G}_n(\tilde{Y}^n) \|_{k} + \| \tilde{G}_n(\tilde{Y}^n) - \tilde{G}_n(\tilde{Y}^n) \|_{k}$$

$$\leq \theta \| Y^n - \tilde{Y}^n \|_{k} + \| \tilde{G}_n(\tilde{Y}^n) - \tilde{G}_n(\tilde{Y}^n) \|_{k}.$$ 

It then follows from (4.9) that, as $n \to \infty$,

$$\| Y^n - \tilde{Y}^n \|_{k} \leq \frac{1}{1 - \theta} \| \tilde{G}_n(\tilde{Y}^n) - \tilde{G}_n(\tilde{Y}^n) \|_{k} \leq \frac{C}{n} \to 0.$$ 

Similar to the derivation of (4.7), one can easily obtain that

$$\frac{1}{n} \sum_{i=1}^{n} \sup_{t \in [0,T]} \mathbb{E} \left[ e^{kt} | X_{i,t}^{i,n} - \tilde{X}_{i,t}^{i,n} |^2 \right] \leq C \| Y^n - \tilde{Y}^n \|_{k} + \frac{C}{n} \to 0.$$ 

Combining these with (4.8) we have

$$\frac{1}{n} \sum_{i=1}^{n} \left( \sup_{t \in [0,T]} \mathbb{E} | X_{i,t}^{i,n} - \tilde{X}_{i,t}^{i,n} |^2 + \int_{0}^{T} \mathbb{E} | Y_{i,t}^{i,n} - \tilde{Y}_{i,t}^{i,n} |^2 \, dt + \sum_{j=1}^{n} \int_{0}^{T} \mathbb{E} \left[ Z_{i,j,t,n} - \delta_{ij} \tilde{Z}_{i,j,t,n} |^2 \, dt \right] \right)$$

$$\leq \frac{C}{n} \to 0.$$ 

Step 4: Similar to the arguments of Step 4 in Theorem 3.1, we have

$$\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \left( \sup_{t \in [0,T]} | X_{i,t}^{i,n} - \tilde{X}_{i,t}^{i,n} |^2 + \sup_{t \in [0,T]} | Y_{i,t}^{i,n} - \tilde{Y}_{i,t}^{i,n} |^2 + \sum_{j=1}^{n} \int_{0}^{T} | Z_{i,j,t,n} - \delta_{ij} \tilde{Z}_{i,j,t,n} |^2 \, dt \right) \right]$$

$$\leq \frac{C}{n} \to 0.$$ 

(4.10)

Using Theorem 3.1 and the assumptions that $\| G_n - G \|_{\square} \to 0$ and $\lambda \mapsto \mathcal{L}(\tilde{\xi}^\lambda)$ is continuous with respect to $\mathcal{W}_2$, we have

$$\mathbb{E} \left[ \int_{0}^{1} \left( \sup_{u \in [0,T]} | X_{u}^{\lambda} - \tilde{X}_{u}^{\lambda,n} |^2 + \sup_{u \in [0,T]} | Y_{u}^{\lambda} - \tilde{Y}_{u}^{\lambda,n} |^2 + \int_{0}^{T} | Z_{s}^{\lambda} - \tilde{Z}_{s}^{n,\lambda} |^2 \, ds \right) \, d\lambda \right] \to 0.$$ 

Combining the last two displays gives (4.3).

Finally we will show (4.4) and (4.5) under the assumption that $G$ is continuous. Using Proposition 3.3, we have the following convergence for the limiting system

$$\mathbb{E} \left[ \int_{0}^{1} \left( \sup_{t \in [0,T]} | X_{t}^{[n,\lambda]} - X_{t}^{\lambda} |^2 + \sup_{t \in [0,T]} | Y_{t}^{[n,\lambda]} - Y_{t}^{\lambda} |^2 \right) \, d\lambda \right] \to 0.$$
Combining this with (4.3) gives (4.4). By (4.4) we have
\[ \sup_{0 \leq t \leq T} \mathbb{E} \left[ W_2^2 (\tilde{\nu}^n_t, \tilde{\nu}'_t) \right] \to 0, \]
where \( \tilde{\nu}^n_t = \frac{1}{n} \sum_{i=1}^n \delta_{(X_i^{\lambda,n}, Y_i^{\lambda,n})} \). Using the independence and moment bound on \((X^{\lambda}_t, Y^{\lambda}_t)\) (see e.g. [30, Lemma A.1]), we have
\[ \sup_{0 \leq t \leq T} \mathbb{E} \left[ W_2^2 (\bar{\nu}^n_t, \mathbb{E} \tilde{\nu}^n_t) \right] \to 0. \]
From Proposition 3.3 we have
\[ \sup_{0 \leq t \leq T} \mathbb{E} \left[ W_2^2 (\mathbb{E} \bar{\nu}^n_t, \tilde{\nu}_t) \right] \to 0. \]
Combining these three displays gives (4.5). \qed

Under certain assumptions we can obtain the rate of convergence.

**Proposition 4.1** Suppose Assumption 4.1 holds. Then
\[
\mathbb{E} \left[ \int_0^1 \left( \sup_{t \in [0,T]} |X_{\lambda,n}^{\lambda} - X_\lambda|^2 + \sup_{t \in [0,T]} |Y_{\lambda,n}^{\lambda} - Y_\lambda|^2 \right) dt + \int_0^T |Z_{\lambda,n}^{\lambda} - Z_{\lambda}^{\lambda}|^2 dt \right] d\lambda \\
\leq C \frac{1}{n} + C \|G_n - G\|^2_2 + C \int_0^1 W_2^2 \left( \mathcal{L}(X_\lambda), \mathcal{L}(X_{\lambda,n}^{\lambda}) \right) d\lambda.
\]

**Proof** From Proposition 3.1 we have
\[
\mathbb{E} \left[ \int_0^1 \left( \sup_{u \in [0,T]} |X_u - \tilde{X}_u^{\lambda,n}|^2 + \sup_{u \in [0,T]} |Y_u - \tilde{Y}_u^{\lambda,n}|^2 \right) ds + \int_0^T |Z_s - \tilde{Z}^{\lambda,n}_s|^2 dt \right] d\lambda \\
\leq C \|G_n - G\|^2_2 + C \int_0^1 W_2^2 \left( \mathcal{L}(X_\lambda), \mathcal{L}(\tilde{X}_{\lambda,n}^{\lambda}) \right) d\lambda.
\]
The result then follows by combining this with (4.10) and the observation that \( \mathcal{L}(\tilde{X}_{\lambda,n}^{\lambda}) = \mathcal{L}(X_{\lambda,n}^{\lambda,n}) \). \qed

**Remark 4.1** Proposition 4.1 provides a rate of convergence in terms of the \( L^2 \) convergence \( \|G_n - G\|_2 \to 0 \). Such a convergence holds, for example, if \( \|G_n - G\|_{\square} \to 0 \) and \( G \in \{0, 1\} \) (see e.g. [41, Proposition 8.24]). It also holds (by dominated convergence theorem) if \( G \) is continuous and \( G_n \) is sampled from \( G \), namely \( G_n (\frac{i}{n}, \frac{j}{n}) := G(\frac{i}{n}, \frac{j}{n}) \).

The following is a more precise rate of convergence under Lipschitz conditions.
Corollary 4.1 Suppose Assumption 4.1 holds. Suppose $G$ is Lipschitz continuous, $\lambda \mapsto \mathcal{L}(\xi^{\lambda})$ is Lipschitz continuous with respect to $\mathcal{W}_2$, and $G_n$ is sampled from $G$, namely $G_n(i_n, j_n) = G(i_n, j_n)$. Then $\|G_n - G\|_2 \leq C/n$ and hence

$$
\mathbb{E}\left[ \int_0^1 \left( \sup_{t \in [0, T]} |X_t^{[n\lambda], n} - X_t^{\lambda}|^2 + \sup_{t \in [0, T]} |Y_t^{[n\lambda], n} - Y_t^{\lambda}|^2 \right) ds \right] \leq C/n,
$$

$$
\frac{1}{n} \sum_{i=1}^n \mathbb{E} \sup_{0 \leq t \leq T} \left[ |X_t^{i,n} - X_t^{i/n}|^2 + |Y_t^{i,n} - Y_t^{i/n}|^2 \right] dt \leq C/n,
$$

$$
\sup_{t \in [0, T]} \mathbb{E} \left[ \mathcal{W}_2^2(\nu_{t}^{n}, \nu_{t}^{\lambda}) \right] \leq C(n^{-1/2} + n^{-(p-2)/p}).
$$

Proof The estimate $\|G_n - G\|_2 \leq C/n$ follows from the Lipschitz continuity of $G$. Applying this to Proposition 4.1 gives (4.11). Combining (4.11) and Proposition 3.3, we have (4.12). From (4.12) we have

$$
\sup_{0 \leq t \leq T} \mathbb{E} \left[ \mathcal{W}_2^2(\nu_{t}^{n}, \nu_{t}^{\lambda}) \right] \leq \frac{C}{n}.
$$

Using the independence and moment bound on $(X_t^{\lambda}, Y_t^{\lambda})$ (see e.g. [30, Lemma A.1]), we have

$$
\sup_{0 \leq t \leq T} \mathbb{E} \left[ \mathcal{W}_2^2(\nu_{t}^{n}, \bar{\nu}_{t}^{n}) \right] \leq C(n^{-1/2} + n^{-(p-2)/p}).
$$

From Proposition 3.3 we have

$$
\sup_{0 \leq t \leq T} \mathbb{E} \left[ \mathcal{W}_2^2(\bar{\nu}_{t}^{n}, \bar{\nu}_{t}^{\lambda}) \right] \leq \frac{C}{n^2}.
$$

Combining these three displays gives (4.13).

Remark 4.2 Although we work for $\mathbb{R}$-valued stochastic process $X^{\lambda}$, similar arguments can be used to show that Theorem 4.1, Proposition 4.1 and Corollary 4.1 also hold for $\mathbb{R}^d$-valued setup. In that case, the rate in (4.13) will be slightly different (see e.g. [30, Lemma A.1])

4.2 Method of Continuation

Assumption 4.2 (i) $B_0, \hat{B}, F_0, \hat{F}, Q_0, \hat{Q}$ are $l$-Lipschitz.
(ii) There exist a positive constant \( k > 3l \) such that

\[
-\Delta x \left( F_0(t, \theta) - F_0(t, \tilde{\theta}) \right) + \Delta y \left( B_0(t, \theta) - B_0(t, \tilde{\theta}) \right) \leq -k(\Delta x)^2 - k(\Delta y)^2,
\]

\[
\Delta x (Q_0(x) - Q_0(\tilde{x})) \geq k(\Delta x)^2,
\]

where \( \Delta x := x - \tilde{x}, \Delta y := y - \tilde{y}, \theta = (x, y), \tilde{\theta} = (\tilde{x}, \tilde{y}). \)

(iii) It holds that \( \sup_{\xi \in [0, 1]} E[|\xi|^p] < +\infty, \) and \( (B(\cdot, 0), F(\cdot, 0), Q) \in \mathcal{ML}^{p, 2}_P \times \mathcal{ML}^{p, 2}_F \times \mathcal{ML}^{p, 2}_T. \)

(iv) \( \lambda \mapsto \mathcal{L}(E^\lambda) \) is continuous with respect to \( \mathcal{W}_2. \)

**Theorem 4.2** Suppose Assumption 4.2 holds and \( \|G_n - G\|_{\square} \to 0. \) Then (4.3) holds. If in addition \( G \) is continuous, then (4.4) and (4.5) hold.

**Proof** We will use the same notation as above Theorem 4.1. That is, let \( (\tilde{X}_n, \tilde{Y}_n, \tilde{Z}_n) \) be the unique solution of the limiting system with graphon \( G_n \) and initial states \( (\xi^{[n\lambda]}_\lambda)_{\lambda \in [0, 1]}, \) Abusing notations, we write \( \tilde{X}_n = (\tilde{X}_i)_i = (\tilde{X}^{n, \lambda})_{\lambda \in [0, 1]}, \) \( \tilde{Y}_n = (\tilde{Y}_i)_i = (\tilde{Y}^{n, \lambda})_{\lambda \in [0, 1]} \) and \( \tilde{Z}_n = (\tilde{Z}_i)_i = (\tilde{Z}^{n, \lambda})_{\lambda \in [0, 1]}. \) Let \( \Delta X_t^{i,n} = X_t^{i,n} - \tilde{X}_t^{i,n} \) and \( \Delta Y_t^{i,n} = Y_t^{i,n} - \tilde{Y}_t^{i,n}. \)

Let us compute

\[
\mathbb{E}[\Delta X_T^{i,n} \Delta Y_T^{i,n}] \geq (k - \frac{3l}{2} - \varepsilon) \mathbb{E}[\Delta X_T^{i,n}]^2 - \frac{l}{2} \sum_{j=1}^n \mathbb{E}[\Delta X_T^{j,n}]^2 \]

\[
- \frac{1}{4\varepsilon} \mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^n G_n \left( \frac{i}{n}, \frac{j}{n} \right) \left( \hat{\nu}(\tilde{X}_T^{i,n}, \tilde{X}_T^{j,n}) - \int_R \hat{\nu}(\tilde{X}_T^{i,n}, x) \mathcal{L}(\tilde{X}_T^{j,n})(dx) \right)^2 \right] \quad (4.14)
\]

Using Itô’s formula, we also obtain that

\[
\mathbb{E} \left[ \Delta X_T^{i,n} \Delta Y_T^{i,n} \right] \leq -(k - \frac{5l}{2} - \varepsilon) \int_0^T \mathbb{E}[\Delta X_t^{i,n}]^2 + \mathbb{E}[\Delta Y_t^{i,n}]^2 \, dt + \frac{l}{n} \sum_{j=1}^n \int_0^T \mathbb{E}[\Delta X_t^{j,n}]^2 \, dt \]

\[
+ \frac{1}{4\varepsilon} \int_0^T \mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^n G_n \left( \frac{i}{n}, \frac{j}{n} \right) \left( \hat{B}(t, \tilde{X}_t^{i,n}, \tilde{X}_t^{j,n}, \tilde{Y}_t^{i,n}) - \hat{B}(t, \tilde{X}_t^{i,n}, x, \tilde{Y}_t^{i,n}) \mathcal{L}(\tilde{X}_t^{j,n})(dx) \right) \right]^2 \, dt \]

\[
+ \frac{1}{4\varepsilon} \int_0^T \mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^n G_n \left( \frac{i}{n}, \frac{j}{n} \right) \left( \hat{F}(t, \tilde{X}_t^{i,n}, \tilde{X}_t^{j,n}, \tilde{Y}_t^{i,n}) - \hat{F}(t, \tilde{X}_t^{i,n}, x, \tilde{Y}_t^{i,n}) \mathcal{L}(\tilde{X}_t^{j,n})(dx) \right) \right]^2 \, dt. \quad (4.15)
\]

Combining the above two inequalities (4.14), (4.15) and averaging over \( i \in \{1, \ldots, n\}, \) we get that

\[ \square \]
\[
\left(k - \frac{7l}{2} - \epsilon\right) \left(\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[|\Delta X_i^{i,n}|^2] + \int_0^T \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[|\Delta X_i^{i,n}|^2 + |\Delta Y_i^{i,n}|^2] \, dt \right) \leq \frac{C}{n\epsilon} \to 0
\]

as \( n \to \infty \). Combining this with (4.14) and (4.15) gives

\[
\mathbb{E}[|\Delta X_i^{i,n}|^2] + \int_0^T \mathbb{E}[|\Delta X_i^{i,n}|^2 + |\Delta Y_i^{i,n}|^2] \, dt \leq \frac{C}{n} \to 0
\]

as \( n \to \infty \).

Using the argument of Step 4 in Theorem 3.2, we have

\[
\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^T \mathbb{E}|Z_t^{i,j,n} - \delta_{ij} Z_t^{i,n}|^2 \, dt \leq \frac{C}{n}.
\]

By the argument of Step 4 in Theorem 4.1, we have the estimate (4.10) and hence the desired results. \( \square \)

Under certain assumptions we can obtain the rate of convergence.

**Proposition 4.2** Suppose Assumption 4.2 holds. Then

\[
\mathbb{E} \left[ \int_0^1 \left( \sup_{t \in [0,T]} \|X_t^{(n,k),n} - X_t^k\|^2 + \sup_{t \in [0,T]} \|Y_t^{(n,k),n} - Y_t^k\|^2 \, ds + \int_0^T \|Z_t^{(n,k),n} - Z_t^k\|^2 \, dt \right) \right] \leq \frac{C}{n} + C\|G_n - G\|_2^2 + C \int_0^1 \mathcal{W}_2^2 \left( \mathcal{L}(X_0^k), \mathcal{L}(X_0^{(n,k),n}) \right) \, d\lambda.
\]

**Proof** From Proposition 3.2 we have

\[
\mathbb{E} \left[ \int_0^1 \left( \sup_{u \in [0,T]} \|X_u^{\lambda} - \tilde{X}_u^{n,\lambda}\|^2 + \sup_{u \in [0,T]} \|Y_u^{\lambda} - \tilde{Y}_u^{n,\lambda}\|^2 + \int_0^T \|Z_s^{\lambda} - \tilde{Z}_s^{n,\lambda}\|^2 \, ds \right) \right] \leq C\|G_n - G\|_2^2 + C \int_0^1 \mathcal{W}_2^2 \left( \mathcal{L}(X_0^\lambda), \mathcal{L}(\tilde{X}_0^{n,\lambda}) \right) \, d\lambda.
\]

The result then follows by combining this with (4.10) and the observation that \( \mathcal{L}(\tilde{X}_0^{n,\lambda}) = \mathcal{L}(X_0^{(n,k),n}) \). \( \square \)

Proposition 4.2 provides a rate of convergence in terms of the \( L^2 \) convergence \( \|G_n - G\|_2 \to 0 \). Such a convergence holds for examples mentioned in Remark 4.1. The following is a more precise rate of convergence under Lipschitz conditions. As mentioned in Remark 4.2, the rate in (4.13) will be slightly different if the process \( X^\lambda \) is \( \mathbb{R}^d \)-valued.

**Corollary 4.2** Suppose Assumption 4.2 holds. Suppose \( G \) is Lipschitz continuous, \( \lambda \mapsto \mathcal{L}(\xi^\lambda) \) is Lipschitz continuous with respect to \( \mathcal{W}_2 \), and \( G_n \) is sampled from \( G \), namely \( G_n(X_n^{(n,k),1/n}) = G(L_n^{(n,k),1/n}) \). Then \( \|G_n - G\|_2 \leq \frac{C}{n} \) and hence (4.11)–(4.13) hold.
**Proof** The proof is similar to that of Corollary 4.1, except that the use of Proposition 4.1 is replaced by Proposition 4.2, and hence omitted. \(\square\)

## 5 Graphon Mean Field Game and Convergence of \(n\)-Player Game

Let \(G : [0, 1] \times [0, 1] \to \mathbb{R}_+\) be a bounded graphon, and without loss of generality assume that \(|G(\lambda, \kappa)| \leq 1, \forall (\lambda, \kappa) \in [0, 1]^2\). Each \(\lambda \in [0, 1]\) represents a type of population, which consists of continuum many players. Let \(\eta^\lambda \in \mathcal{P}_2(\mathbb{R})\) denote the distribution of population of type \(\lambda\), and \(\eta\) denote the collection \(\{\eta^\lambda : \lambda \in [0, 1]\}\), i.e., \(\eta \in \mathcal{M}([0, 1]; \mathcal{P}_2(\mathbb{R}))\). Let \(A \subset \mathbb{R}^n\) be a convex control space. Take functions

\[
b_1, f_1, f_2 : [0, T] \times \mathbb{R} \to \mathbb{R}, \quad b_2, b_3 : [0, T] \to \mathbb{R},
\]

\[
q_1, q_2 : \mathbb{R} \to \mathbb{R}.
\]

For any \((\lambda, x, \eta, a) \in [0, 1] \times \mathbb{R} \times \mathcal{M}([0, 1]; \mathcal{P}_2(\mathbb{R})) \times A\), we define

\[
b^\lambda_G(t, x, \eta, a) := \int_{[0,1]} G(\lambda, \kappa) d\kappa \int_{\mathbb{R}} b_1(t, z) \eta^\kappa(dz) + b_2(t)x + b_3(t)a,
\]

\[
f^\lambda_G(t, x, \eta, a) := f_1(t, x) + \int_{[0,1]} G(\lambda, \kappa) d\kappa \int_{\mathbb{R}} f_2(t, z) \eta^\kappa(dz) + \frac{1}{2}a^2,
\]

\[
q^\lambda_G(x, \eta) := q_1(x) + \int_{[0,1]} G(\lambda, \kappa) d\kappa \int_{\mathbb{R}} q_2(z) \eta^\kappa(dz).
\]

Let \(W^\lambda\) be a family of independent standard Brownian motion, and \(\sigma > 0\) be a constant volatility. Denote by \(\mu^\lambda_G(t)\) the distribution of players of type \(\lambda\) at time \(t\), and \(\mu_G(t) := \{\mu^\lambda_G(t) : \lambda \in [0, 1]\}\), \(\mu_G := \{\mu_G(t) : t \in [0, T]\}\). Choosing \(\alpha^\lambda_G(t) \in A\), a representative player of type \(\lambda\) controls the dynamic

\[
\begin{cases}
    dX^\lambda_t = b^\lambda_G(t, X^\lambda_t, \mu_G(t), \alpha^\lambda(t)) \, dt + \sigma \, dW^\lambda_t, \\
    X^\lambda_0 = \xi^\lambda,
\end{cases}
\]

where \(\xi^\lambda\) is a square integrable random variable. The cost for the representative player \(\lambda\) is given by

\[
J(\alpha^\lambda, \mu_G) = \mathbb{E} \left[ \int_0^T f^\lambda_G(t, X^\lambda_t, \mu_G(t), \alpha^\lambda(t)) \, dt + q^\lambda_G(X^\lambda_T, \mu_G(T)) \right], \quad (5.1)
\]

and each representative player chooses control \(\alpha^\lambda(t)\) to minimize \(J(\alpha^\lambda, \mu_G)\).

For each \(\lambda \in [0, 1]\), define the Hamiltonian

\[
H^\lambda_G(t, x, \mu_G(t), y, a) := b^\lambda_G(t, x, \mu_G(t), a) \cdot y + f^\lambda_G(t, x, a, \mu_G(t)),
\]
and the minimizer
\[ \hat{\alpha}_{G}^{\lambda}(t, x, \mu_{G}(t), y) := \arg\min_{a \in A} H_{G}^{\lambda}(t, x, \mu_{G}(t), y, a) = -b_{3}(t)y. \]

Given \( \{\mu_{G}(t) : 0 \leq t \leq T\} \), by Pontryagin’s maximum principle, we obtain a family of BSDE
\[
\begin{cases}
    dY_{t}^{\lambda} = -\partial_{x} H_{G}^{\lambda}(t, X_{t}^{\lambda}, \mu_{G}(t), Y_{t}^{\lambda}, -b_{3}(t)Y_{t}^{\lambda}) dt + Z_{t}^{\lambda} dW_{t}^{\lambda}, \\
    Y_{T}^{\lambda} = \partial_{x} q_{G}^{\lambda}(X_{T}^{\lambda}, \mu_{G}(T)).
\end{cases}
\]
(5.2)

Since the law of \( X_{t}^{\lambda} \) should coincide with \( \mu_{G}^{\lambda}(t) \), after simplification we obtain the FBSDE of the graphon mean field game
\[
\begin{cases}
    dX_{t}^{\lambda} = \left( b_{2}(t)X_{t}^{\lambda} - |b_{3}(t)|^{2}Y_{t}^{\lambda} + \int_{0}^{1} \int_{\mathbb{R}} G(\lambda, \kappa) \mathbb{E}[b_{1}(t, X_{t}^{\kappa})] d\kappa \right) dt + \sigma dW_{t}^{\lambda}, \\
    dY_{t}^{\lambda} = -\left( b_{2}(t)Y_{t}^{\lambda} + \partial_{x} f_{1}(t, X_{t}^{\lambda}) \right) dt + Z_{t}^{\lambda} dW_{t}^{\lambda}, \\
    X_{0}^{\lambda} = \xi^{\lambda}, \\
    Y_{T}^{\lambda} = \partial_{x} q_{1}(X_{T}^{\lambda}), \quad \forall \lambda \in [0, 1].
\end{cases}
\]
(5.3)

In order for coefficients of (5.3) to satisfy Assumptions 2.1, 3.1, 4.1 or 2.2, 3.1, 4.2, we propose the following conditions.

**Assumption 5.1**

(i) \( b_{1} \) grows at most linearly in \( x \). \( b_{2}(t), b_{3}(t) \) are uniformly bounded.

(ii) \( f_{1}, f_{2}, q_{1} \) are differentiable, and of at most quadratic growth in \( x \).

(iii) \( b_{1}, \partial_{x} f_{1}, \partial_{x} f_{2}, \partial_{x} q_{1} \) are \( L \)-Lipschitz in \( x \), and \( \max_{t \in [0, T]} |b_{3}(t)| \leq L \) for some \( L > 1 \).

(iv) It holds that
\[
\max_{t \in [0, T]} b_{2}(t) < -100L^{4}.
\]
(5.4)

(v) We have \( \sup_{\lambda \in [0, 1]} \mathbb{E}[|\xi_{\lambda}|^{p}] < +\infty \) and \( \lambda \rightarrow \mathcal{L}(\xi^{\lambda}) \) is continuous with respect to \( \mathcal{W}_{2} \).

Due to the explicit structure of (5.3), we can easily check that its coefficients satisfy Assumptions 2.1, 3.1 and 4.1. Therefore we obtain the following result.

**Corollary 5.1** *Under Assumption 5.1, there exists a unique solution to (5.3), the solution is stable in the sense of Theorem 3.1, Proposition 3.1, and the propagation of chaos results hold as in Theorem 4.1, Proposition 4.1, Corollary 4.1.*

**Assumption 5.2**

(i) \( b_{1} \) grows at most linearly in \( x \). \( b_{2}(t), b_{3}(t) \) are uniformly bounded.

(ii) \( f_{1}, f_{2}, q_{1} \) are differentiable, and of at most quadratic growth in \( x \).

\[ \mathcal{L}(f_{2}) \]
(ii) There exist positive $\iota$ such that
\[ q_1(x') - q_1(x) - (x' - x)\partial_x q_1(x) \geq \iota(x' - x)^2, \]
and $f_1, f_2$ are convex in $x$.

(iii) $b_1$ and $\partial_x f_1$ are $L$-Lipschitz in $x$ for some $L \geq 1$.

(iv) It holds that
\[ \min \left\{ \inf_{t \in [0,T]} |b_3(t)|^2, \iota \right\} \geq 100L^2. \quad (5.5) \]

(v) We have $\sup_{\lambda \in [0,1]} \mathbb{E}[|\xi^\lambda|^p] < +\infty$ and $\lambda \rightarrow \mathcal{L}(\xi^\lambda)$ is continuous with respect to $\mathcal{W}_2$.

**Corollary 5.2** Under Assumption 5.2, there exists a unique solution to (5.3), the solution is stable in the sense of Theorem 3.2, Proposition 3.2, and the propagation of chaos results holds as in Theorem 4.2, Proposition 4.2, Corollary 4.2.

Now let us turn to the convergence of finite player game. Fix $n \in \mathbb{N}$ and $G_n, F_n$.

For any $(i, \underline{x}, a^i) \in \{1, \ldots, n\} \times \mathbb{R}^n \times A^n$, we define
\[ b^{i,n}(t, \underline{x}, a^i) := \frac{1}{n} \sum_{j=1}^{n} G_n(i, j) b_1(t, x^j) + b_2(t)x^i + b_3(t)a^i, \]
\[ f^{i,n}(t, \underline{x}, a^i) := f_1(t, x^i) + \frac{1}{n} \sum_{j=1}^{n} G_n(i, j) f_2(t, x^j) + \frac{1}{2}(a^i)^2, \]
\[ q^{i,n}(\underline{x}) := q_1(x^i) + \frac{1}{n} \sum_{j=1}^{n} G_n(i, j)q_2(x^j). \]

Let us compute the FBSDE system of this $n$-player game. Each player has the Hamiltonian
\[ H^{i,n}(t, \underline{x}, \underline{y}^i, a) = b^n(t, \underline{x}, a) \cdot \underline{y}^i + f^{i,n}(t, \underline{x}, a^i), \]
where $\underline{x}, \underline{y}^i \in \mathbb{R}^n, a \in A^n$. By our construction, it is clear that for any $t \geq 0, \underline{x} \in \mathbb{R}^n$ and $\underline{y} \in \mathbb{R}^{n \times n}$, functions $\hat{\alpha}^{i,n}(t, \underline{x}, \underline{y}) := -b_3(t)y^{i,i}$ satisfy that
\[ H^{i,n}(t, \underline{x}, \underline{y}^i, \hat{\alpha}^{n}(t, \underline{x}, \underline{y})) \leq H^{i,n}(t, \underline{x}, \underline{y}^i, (a^i, \hat{\alpha}^{n}(t, \underline{x}, \underline{y})^{-i})), \]
for all $a^i \in A$. 
Note that when $i = j$, we have

$$
\partial_{x_i} H^{i,n}(t, x, y, \hat{\alpha}^n(t, x, y)) = b_2(t) y^{i,i} + \frac{1}{n} \sum_{k=1}^{n} G_n(k, i) \partial_x b_1(t, x^i) y^{i,k} + \partial_x f_1(t, x^i)
$$

$$
+ \frac{1}{n} G_n(i, i) \partial_x f_2(t, x^i),
$$

$$
\partial_{x_i} q^{i,n}(x) = \partial_x q_1(x^i) + \frac{1}{n} G_n(i, i) \partial_x q_2(x^i),
$$

and when $i \neq j$,

$$
\partial_{x_j} H^{i,n}(t, x, y, \hat{\alpha}^n(t, x, y)) = b_2(t) y^{i,j} + \frac{1}{n} \sum_{k=1}^{n} G_n(k, j) \partial_x b_1(t, x^j) y^{j,k}
$$

$$
+ \frac{1}{n} G_n(i, j) \partial_x f_2(t, x^j),
$$

$$
\partial_{x_j} q^{i,n}(x) = \frac{1}{n} G_n(i, j) \partial_x q_2(x^j).
$$

(5.6)

We obtain the FBSDE system for the $n$-player game

$$
\begin{aligned}
&dX^{i,n}_t = b^{i,n}(t, X^{i,n}_t, \hat{\alpha}^{i,n}(t, X^{i,n}_t, Y^{i,n}_t)) dt + \sigma dW^{j,n}_t, \\
&dY^{i,j,n}_t = -\partial_{x_j} H^{i,n}(t, X^{i,n}_t, Y^{i,n}_t, \hat{\alpha}^{i,n}(t, X^{i,n}_t, Y^{i,n}_t)) dt + \sum_{k=1}^{n} Z^{i,j,k,n}_t dW^{k,n}_t, \\
&X^{i,n}_0 = \xi^{i,n}, \\
&Y^{i,j,n}_T = \partial_{x_j} q^{i,n}(X^{i,n}_T), \quad i, j = 1, \ldots, n.
\end{aligned}
$$

(5.7)

We briefly show the convergence result

$$
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \sup_{t \in [0,T]} |X^{i,n}_t - X^{i/n}_t|^2 + \sup_{t \in [0,T]} |\hat{\alpha}^{i,n}_t - \hat{\alpha}^{i/n}_t|^2 \right] \rightarrow 0 \text{ as } n \rightarrow \infty,
$$

which in our model is equivalent to

$$
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \sup_{t \in [0,T]} |X^{i,n}_t - X^{i/n}_t|^2 + \sup_{t \in [0,T]} |Y^{i,i,n}_t - Y^{i,j/n}_t|^2 \right] \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (5.8)
$$

The argument is divided into two steps.

**Theorem 5.1** Under Assumption 5.1, if $\|G_n - G\|_\square \rightarrow 0$ and $G$ is continuous, then the Nash equilibrium of $n$-player game converges to the corresponding graphon field game, i.e., (5.8) holds.
Proof  Step 1: Consider an auxiliary FBSDE system
\[
\begin{aligned}
\frac{d\tilde{X}_i^{i,n}}{n} &= b_2(t)\tilde{Y}_i^{i,n} + \frac{1}{n} \sum_{k=1}^{n} G_n(i,k) b_1(t, \tilde{X}_k^{i,n}) dt + \sigma dW_i^{i/n}, \\
\frac{d\tilde{Y}_i^{i,n}}{n} &= -\left(b_2(t)\tilde{Y}_i^{i,n} + \partial_x f_1(t, \tilde{X}_i^{i,n})\right) dt + \tilde{Z}_i^{i,j,n} dW_i^{i/n}, \\
\tilde{X}_0^{i,n} &= \xi_i, \\
\tilde{Y}_T^{i,n} &= \partial_x q_1(\tilde{X}_T^{i,n}),
\end{aligned}
\]
(5.9)

Invoking Theorem 4.1, we obtain that

\[
\frac{1}{n} \mathbb{E} \left[ \sup_{t \in [0,T]} |\tilde{X}_t^{i,n} - X_t^{i/n}|^2 + \sup_{t \in [0,T]} |\tilde{Y}_t^{i,n} - Y_t^{i/n}|^2 \right] \to 0 \text{ as } n \to \infty.
\]

Step 2: We will show that

\[
\frac{1}{n} \mathbb{E} \left[ \sup_{t \in [0,T]} |\tilde{X}_t^{i,n} - X_t^{i,n}|^2 + \sup_{t \in [0,T]} |\tilde{Y}_t^{i,n} - Y_t^{i,n}|^2 \right] \to 0 \text{ as } n \to \infty. \tag{5.10}
\]

Using the same computation as in Theorem 2.1, we can prove that

\[
\frac{1}{n} \mathbb{E} \left[ \sum_{i=1}^{n} \int_0^T |X_t^{i,n}|^2 + \sum_{j=1}^{n} |Y_t^{i,j,n}|^2 dt \right] \leq C,
\]
where C is some constant uniformly for any \( n \in \mathbb{N} \). Using this bound, the same computation shows that for any \( i \in \{1, \ldots, n\} \)

\[
\mathbb{E} \left[ \int_0^T |X_t^{i,n}|^2 + \sum_{j=1}^{n} |Y_t^{i,j,n}|^2 dt \right] \leq C,
\]
and also

\[
\mathbb{E} \left[ |X_t^{i,n}|^2 + \sum_{j=1}^{n} |Y_t^{i,j,n}|^2 \right] \leq C. \tag{5.11}
\]

According to (5.6), for any \( i \neq j \) we have the terminal \( Y_T^{i,j,n} = \frac{1}{n} G_n(i,j) \partial_x q_2(X_T^{i,n}) \) and its drift \( -\partial_x H^{i,n} \leq C(|y^{i,j,n}| + \frac{1}{n} \sum_{k \neq i} |y^{i,k,n}|) + O(1/n). \) Therefore, one can obtain

\[
\mathbb{E} \left[ |Y_t^{i,j,n}|^2 \right] \leq \frac{C}{n} \text{ for any } n \in \mathbb{N} \text{ and some } C > 0 \tag{5.12}
\]
as in [20, Lemma 22].

Using monotonicity conditions, and computing as in Theorem 2.1, it can be seen that
\[
\frac{1}{n} \int_0^T \mathbb{E} \left[ |\tilde{X}^{i,n}_t - X^{i,n}_t|^2 + |\tilde{Y}^{i,n}_t - Y^{i,i,n}_t|^2 \right] dt \\
\leq C \sum_{i=1}^n \left( \frac{1}{n} G_n(i, i) \mathbb{E} \left[ |\partial_x q_2(X^{i,n}_t)|^2 + \int_0^T |\partial_x f_2(t, X^{i,n}_t)|^2 dt \right] \right) \\
+ C \sum_{i,k=1}^n G_n(k, i) \mathbb{E} \left[ \int_0^T |\partial_x b_1(t, X^{i,n}_t) Y^{i,k,n}_t|^2 dt \right] \\
\leq C/n,
\]

where we use (5.11) and (5.12). Then by a similar argument as in Step 4 of Theorem 3.1, one can easily conclude (5.10). \(\square\)

**Remark 5.1** We want to point out that we are only able to prove the convergence under Assumption 5.1. It is just because it is difficult to show the uniform boundedness of solutions of (5.7) under Assumption 5.2. The rest of the proof actually works for both assumptions.

**Funding** E. Bayraktar is partially supported by the National Science Foundation under Grant DMS2106556 and by the Susan M. Smith chair.

**Declarations**

**Conflict of interest** The authors have not disclosed any competing interests.

**Appendix A: Measurability**

**Lemma A.1** \(\mu : [0, 1] \to C([0, T]; \mathcal{P}_p(\mathbb{R}))\) is measurable if and only if for any \(t \in [0, T]\),

\[
\lambda \mapsto \mu^\lambda(t) \in \mathcal{P}_p(\mathbb{R}) \text{ is measurable}. \quad (A.1)
\]

**Proof** The proof of ‘only if’ is trivial. For the proof of ‘if’ part, we note that with sup norm, \(C([0, T]; \mathcal{P}_p(\mathbb{R}))\) is a topological subspace of \(C^0([0, T]; \mathcal{P}_p(\mathbb{R}))\). For any \(n \in \mathbb{N}\), due to (A.1) we know that

\[
\lambda \mapsto (\mu^\lambda(T/n), \ldots, \mu^\lambda(T)) \text{ is measurable}.
\]

We construct \(\mu_n \in \mathcal{M}([0, T]; \mathcal{P}_p(\mathbb{R}))\)

\[
\mu_n^\lambda(t) := \mu^\lambda \left( \frac{i}{n} \right), \quad t \in \left( \frac{(i-1)T}{n}, \frac{iT}{n} \right], \quad i = 1, \ldots, n.
\]

Then it can be easily verified that \(\lambda \to \mu_n^\lambda\) is measurable. By the continuity of \(\mu^\lambda(\cdot)\), \(\lim_{n \to \infty} \mu_n^\lambda = \mu^\lambda\) is measurable in \(\lambda\). \(\square\)
Lemma A.2  A function \( x : \lambda \mapsto x^\lambda \in L^p_{\mathcal{F}} \) belongs to \( \mathcal{ML}^p_{\mathcal{F}} \) if and only if \( \lambda \mapsto x^\lambda_t \in L^p_{\mathcal{F}_t} \) is measurable for any \( t \in [0, T] \).

**Proof**  Note that \( L^p_{\mathcal{F}} \ni \eta \mapsto \eta_t \in L^p_{\mathcal{F}_t} \) is continuous. Therefore it can be readily seen that the measurability of \( \lambda \mapsto x^\lambda \) implies the measurability of \( \lambda \mapsto x^\lambda_t \) for any \( t \in [0, T] \).

Conversely, define \( x^N : \lambda \mapsto x^N_\lambda \in M_{L^p_{\mathcal{F}}} \) for \( N \in \mathbb{N} \) as follows,

\[
x^N_\lambda := x^{\lambda_{nT/N}}, \quad \forall t \in [nT/N, (n+1)T/N], \quad n = 0, \ldots, N-2, \quad \lambda \in [0, 1],
\]

\[
x^N_\lambda := x^{\lambda_{(N-1)T/N}}, \quad \forall t \in [(N-1)T/N, T], \quad \lambda \in [0, 1].
\]

According to our hypothesis, it can be easily seen that \( \lambda \mapsto x^N_\lambda \in \left( L^p_{\mathcal{F}}, \| \cdot \|_S \right) \) is measurable, and also the limit

\[
\lambda \mapsto x^\lambda \in \left( L^p_{\mathcal{F}}, \| \cdot \|_S \right).
\]

\( \square \)

Lemma A.3  Take a polish space \( \Omega \) and a Borel probability measure \( (\mathcal{F}, P) \) over \( \Omega \). Take another measure space \( (E, \Sigma, m) \). Suppose \( \rho : E \times \mathbb{R} \to \mathbb{R} \) is a real-valued function such that \( x \mapsto \rho(e, x) \) is continuous for any \( e \in E \), \( e \mapsto \rho(e, x) \) is measurable for any \( x \in \mathbb{R} \), and \( |\rho(e, x)| \leq C(1 + |x|) \), \( \forall (e, x) \in E \times \mathbb{R} \) for some positive constant \( C \). Then given any measurable mapping \( e \mapsto X(e) \in L^p(\Omega, \mathcal{F}, P) \), the Banach-valued function

\[
e \mapsto \rho(e, X(e)) \in L^p(\Omega, \mathcal{F}, P)
\]

is also measurable.

**Proof**  According to [42, Proposition 3.4.5], the Banach space \( L^p(\Omega, \mathcal{F}, P) \) is separable. Therefore as a result of Pettis measurability theorem, any measurable function \( X : E \to L^p(\Omega, \mathcal{F}, P) \) is also strongly measurable, i.e., \( X \) can be written as a pointwise limit of simple functions

\[
X^n = \sum_{i=1}^{m_n} 1_{S_{m_n}} x_{m_n},
\]

where \( m_n \in \mathbb{N}, S_1, \ldots, S_{m_n} \) is a finite collection of disjoint subsets of \( E \), and \( x_1, \ldots, x_{m_n} \in L^p(\Omega, \mathcal{F}, P) \). It is then readily seen that

\[
e \mapsto \rho(e, X^n(e))
\]

is measurable, and thus \( \rho(\cdot, X(\cdot)) = \lim_n \rho(\cdot, X^n(\cdot)) \) is measurable. \( \square \)
Lemma A.4 Suppose \( \psi : [0, 1] \times [0, T] \times \mathbb{R} \to \mathbb{R} \) is a measurable function such that \( x \mapsto \psi(\lambda, t, x) \) is continuous and grows at most linearly uniformly for \( (\lambda, t) \in [0, 1] \times [0, T] \). Given any measurable \( \lambda \mapsto X^\lambda \in L^{p, c}_x \), we have that
\[
\lambda \mapsto \int_0^t \psi(\lambda, s, X^\lambda(s)) \, ds \in L^{p, c}_x
\] (5.1)
is measurable.

Proof By our assumption, it is clear that \( (\lambda, s) \mapsto X^\lambda(s) \) is measurable. Applying Lemma A.3 with \( E = [0, 1] \times [0, T] \), it is readily seen that
\[
(\lambda, s) \mapsto \psi(\lambda, s, X^\lambda(s))
\] (5.2)
is measurable. The function (5.2) is also Bochner integrable due to our linear growth assumption in \( x \). Thus by the Fubini theorem of Bochner theorem,
\[
\lambda \mapsto \int_0^t \psi(\lambda, s, X^\lambda(s)) \, ds
\]
is measurable for any \( t \in [0, T] \). Now the measurability of (5.1) follows from Lemma A.2. \( \square \)

Remark A.1 Using approximation of simple functions, one can easily verify that Bochner integral coincides with Lebesgue integral.

Appendix B: Weak Uniqueness of FBSDE

The notion of weak existence and uniqueness for FBSDEs are almost the same to the ones considered for classical SDEs, see e.g. [43–45].

Definition B.1 A five-tuple \( (\Omega, \mathcal{F}, \mathbb{F}, P, W) \) is said to be a standard set-up if \( W \) is a Brownian motion over the probability space \( (\Omega, \mathcal{F}, \mathbb{F}, P) \) and \( \mathbb{F} := \{ \mathcal{F}_t \}_{t \geq 0} \) is complete and right continuous.

Consider an FBSDE
\[
\begin{cases}
X_t = x + \int_0^t B(s, X_s, Y_s) \, ds + \sigma \, W_t, \\
Y_t = Q(X_T) + \int_t^T F(s, X_s, Y_s) \, ds - \int_t^T Z_s \, dW_s,
\end{cases}
\] (B.1)
where \( B, F, Q \) are progressively measurable functions.

Definition B.2 A triple of processes \( (X, Y, Z) \) is said to be a weak solution of (B.1) if there exists a standard set-up \( (\Omega, \mathcal{F}, \mathbb{F}, P, W) \) such that \( (X, Y, Z) \) are adapted to the filtration \( \mathbb{F} \) and satisfy (B.1) a.s. If \( (X, Y, Z) \) and \( (\tilde{X}, \tilde{Y}, \tilde{Z}) \) are two weak solutions of (B.1) on the same set-up, we say that pathwise uniqueness holds if
\[
\mathbb{P}[(X_t, Y_t) = (\tilde{X}_t, \tilde{Y}_t), \forall t \in [0, T)] = 1.
\]
By Yamada-Watanabe Theorem for SDEs, pathwise uniqueness implies uniqueness in law. We have the same result for FBSDEs.

**Lemma B.1** Suppose the pathwise uniqueness property holds for FBSDE (B.1). Then for any two weak solutions \((X, Y, Z)\) on \((\Omega, \mathcal{F}, \mathbb{F}, P, W)\) and \((\tilde{X}, \tilde{Y}, \tilde{Z})\) on \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P}, \tilde{W})\), their distributions coincide.

**Proof** See [43, Theorem 5.1].

**References**

1. Lasry, J.-M., Lions, P.-L.: Jeux à champ moyen. I. Le cas stationnaire. C. R. Math. Acad. Sci. Paris 343, 619–625 (2006)
2. Lasry, J.-M., Lions, P.-L.: Jeux à champ moyen. II: horizon fini et contrôle optimal. C. R. Math. 343, 679–684 (2006)
3. Lasry, J.-M., Lions, P.-L.: Mean field games. Jpn. J. Math. 2, 229–260 (2007)
4. Huang, M., Caines, P.E., Malhame, R.P.: Large-population cost-coupled lqg problems with nonuniform agents: Individual-mass behavior and decentralized ε-nash equilibria. IEEE Trans. Autom. Control 52, 1560–1571 (2007)
5. Huang, M., Malhamé, R.P., Caines, P.E.: Large population stochastic dynamic games: closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle. Commun. Inf. Syst. 6, 221–251 (2006)
6. Bayraktar, E., Cecchin, A., Cohen, A., Delarue, F.: Finite state mean field games with wright-fisher common noise. J. Mat. Pures Appl. 147, 98–162 (2021)
7. Bayraktar, E., Cecchin, A., Cohen, A., Delarue, F.: Finite state mean field games with Wright-Fisher common noise as limits of \(N\)-player weighted games. Math. Oper. Res. 47, 2840–2890 (2022)
8. Bayraktar, E., Cohen, A.: Analysis of a finite state many player game using its master equation. SIAM J. Control Optim. 56, 3538–3568 (2018)
9. Cecchin, A., Fischer, M.: Probabilistic approach to finite state mean field games. Appl. Math. Optim. 81, 253–300 (2020)
10. Cecchin, A., Pelino, G.: Convergence, fluctuations and large deviations for finite state mean field games via the master equation. Stoch. Process. Appl. 129, 4510–4555 (2019)
11. Bayraktar, E., Zhang, X.: On non-uniqueness in mean field games. Proc. Am. Math. Soc. 148, 4091–4106 (2020)
12. Cardaliaguet, P., Rainer, C.: An example of multiple mean field limits in ergodic differential games. NoDEA Nonlinear Diff. Equ. Appl. 27, 25 (2020)
13. Cecchin, A., Pra, P.D., Fischer, M., Pelino, G.: On the convergence problem in mean field games: a two state model without uniqueness. SIAM J. Control Optim. 57, 2443–2466 (2019)
14. Delarue, F., FoguenTchuendom, R.: Selection of equilibria in a linear quadratic mean-field game. Stoch. Process. Appl. 130, 1000–1040 (2020)
15. Carmona, R., Delarue, F.: Probabilistic theory of mean field games with applications. I. In: Probability Theory and Stochastic Modelling, Mean Field FBSDEs, Control, and Games, vol. 83. Springer, Cham, (2018)
16. Carmona, R., Delarue, F.: Probabilistic theory of mean field games with applications. II. In: Probability Theory and Stochastic Modelling, Mean Field Games with Common Noise and Master Equations, vol. 84. Springer, Cham (2018)
17. Cardaliaguet, P., Delarue, F., Lasry, J.-M., Lions, P.-L.: The Master Equation and the Convergence Problem in Mean Field Games, vol. 201. Princeton University Press, Princeton (2019)
18. Lacker, D.: A general characterization of the mean field limit for stochastic differential games. Probab. Theory Relat. Fields 165, 581–648 (2016)
19. Lacker, D.: On the convergence of closed-loop Nash equilibria to the mean field game limit. Ann. Appl. Probab. 30, 1693–1761 (2020)
20. Laurière, M., Tangpi, L.: Convergence of large population games to mean field games with interaction through the controls. SIAM J. Math. Anal. 54, 3535–3574 (2022)

 Springer
21. Possamaï, D., Tangpi, L.: Non-asymptotic convergence rates for mean-field games: weak formulation and McKean–Vlasov BSDEs. arXiv:2105.00484 (2021)
22. Aurell, A., Carmona, R., Laurière, M.: Stochastic graphon games: II. The linear-quadratic case. Appl. Math. Optim. 85, 26–33 (2022)
23. Caines, P.E., Ho, D., Huang, M., Jian, J., Song, Q.: On the graphon mean field game equations: individual agent affine dynamics and mean field dependent performance functions. arXiv:2009.12144 (2020)
24. Caines, P.E., Huang, M.: Graphon mean field games and their equations. SIAM J. Control Optim. 59, 4373–4399 (2021)
25. Carmona, R., Cooney, D., Graves, C., Laurière, M.: Stochastic graphon games: I. The static case. Math. Oper. Res. (2022)
26. Borgs, C., Chayes, J.T., Lovász, L., Sós, V.T., Vesztergombi, K.: Convergent sequences of dense graphs. I. Subgraph frequencies, metric properties and testing. Adv. Math. 219, 1801–1851 (2008)
27. Borgs, C., Chayes, J.T., Lovász, L., Sós, V.T., Vesztergombi, K.: Convergent sequences of dense graphs. II. Multiway cuts and statistical physics. Ann. Math. 1, 151–219 (2012)
28. Lovász, L., Szegedy, B.: Limits of dense graph sequences. J. Comb. Theory B 96, 933–957 (2006)
29. Bayraktar, E., Chakraborty, S., Wu, R.: Graphon mean field systems. Ann. Appl. Probab. arXiv:2003.13180 (2020)
30. Bayraktar, E., Wu, R.: Stationarity and uniform in time convergence for the graphon particle system. Stoch. Process. Appl. 150, 532–568 (2022)
31. Bayraktar, E., Chakraborty, S., Zhang, X.: k-Core in percolated dense graph sequences. arXiv:2012.09730. (2020)
32. Riordan, O.: The k-core and branching processes. Combin. Probab. Comput. 17, 111–136 (2008)
33. Bayraktar, E., Zhang, X.: Solvability of infinite horizon McKean–Vlasov FBSDEs in mean field control problems and games. Appl. Math. Optim. 87, 13–26 (2023)
34. Hu, Y., Peng, S.: Solution of forward–backward stochastic differential equations. Probab. Theory Relat. Fields 103, 273–283 (1995)
35. Pardoux, E., Tang, S.: Forward–backward stochastic differential equations and quasilinear parabolic PDEs. Probab. Theory Relat. Fields 114, 123–150 (1999)
36. Carmona, R.: Lectures on BSDEs, Stochastic Control, and Stochastic Differential Games with Financial Applications, vol. 1. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (2016)
37. Yong, J.: Forward–backward stochastic differential equations with mixed initial-terminal conditions. Trans. Am. Math. Soc. 362, 1047–1096 (2010)
38. Zhang, J.: Backward stochastic differential equations. In: Probability Theory and Stochastic Modelling, From Linear to Fully Nonlinear Theory, vol. 86. Springer, New York (2017)
39. Karatzas, I., Shreve, S.E.: Brownian Motion and Stochastic Calculus. Graduate Texts in Mathematics, vol. 113, 2nd edn. Springer-Verlag, New York (1991)
40. Yong, J.: Finding adapted solutions of forward–backward stochastic differential equations: method of continuation. Probab. Theory Relat. Fields 107, 537–572 (1997)
41. Lovász, L.: Large Networks and Graph Limits, vol. 60. American Mathematical Society, London (2012)
42. Cohn, D.L.: Measure theory, Birkhäuser Advanced Texts: Basler Lehrbücher [Birkhäuser Advanced Texts: Basel Textbooks], 2nd edn. Birkhäuser/Springer, New York (2013)
43. Antonelli, F., Ma, J.: Weak solutions of forward–backward sde’s. Stoch. Process. Appl. 21, 493–514 (2003)
44. Delarue, F., Guatteri, G.: Weak existence and uniqueness for forward–backward sdes. Stoch. Process. Appl. 116, 1712–1742 (2006)
45. Ma, J., Zhang, J., Zheng, Z.: Weak solutions for forward–backward SDEs: a martingale problem approach. Ann. Probab. 36, 2092–2125 (2008)