UNIFORM $L^1$ STABILITY OF THE INELASTIC BOLTZMANN EQUATION WITH LARGE EXTERNAL FORCE FOR HARD POTENTIALS

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ABSTRACT. In this paper, we will study the uniform $L^1$ stability of the inelastic Boltzmann equation. More precisely, according to the existence result on the inelastic Boltzmann equation with external force near vacuum, we obtain the uniform $L^1$ stability estimates of mild solution for the hard potentials under the assumptions on the characteristic generated by force term which can be arbitrarily large. The proof is based on the exponentially decay estimate and Lu’s trick in [10].

1. Introduction. The spatially inhomogeneous Boltzmann equation describes the phase space evolution of a distribution function $f(t,x,v)$ of moderately dilute gas particles at time $t \geq 0$ and position $x \in \mathbb{R}^3$ with velocity $v \in \mathbb{R}^3$. In the presence of external forces $F(t,x)$, it reads

$$
\begin{align*}
\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F(t,x) \cdot \nabla_v f &= Q(f,f), \\
f(0,x,v) &= f_0(x,v),
\end{align*}
$$

where $Q(f,f)$ is the inelastic collision operator:

$$Q(f,f) = \int_{\mathbb{R}^3 x S^2} B(|u|, \theta) \frac{1}{e^{\gamma(J)}} f(v)f(v_*) - f(v)f(v_*) d\omega dv_*$$

with the notations $f'(v) = f(t,x',v)$, $f'(v_*) = f(t,x',v_*)$, $f(v_*) = f(t,x,v_*)$, $f(v) = f(t,x,v)$ for simplicity. Here $\omega \in S^2$ and the variable $u = v - v_*$ is the relative velocity between particles. $v'$, $v_*$ denote the velocities of two particles before collision, $v, v_*$ represent the velocities after collision, and

$$v = v' - \frac{1}{2} e^t (u \cdot \omega) \omega, \quad v_* = v_*' + \frac{1}{2} e^t (u \cdot \omega) \omega.$$
The parameter $e(0 < e < 1)$ is the restitution coefficient which is a function of $|u \cdot \omega|$. We denote $e' = e(|u \cdot \omega|)$ with $u = v - v_e$. The limit regimes $e = 1$ corresponds to elastic. $'J = J(|u \cdot \omega|)$ is the Jacobian of (2), and 
\[
J = \left| \frac{\partial (v, v_s)}{\partial (v', v_s')} \right| = \theta_z(|u \cdot \omega|),
\]
where $\theta(z) = z e(z)$, and $\theta_z(z)$ denote the derivative of $\theta$ with respect to $z$. $B(|u|, \theta) = b(\theta)|u| \gamma (\gamma > 0)$ is called the collision kernel. $-2 < \gamma < 0$ and $0 < \gamma < 1$ correspond to soft and hard potentials respectively. $\gamma = 1$ is the standard hard sphere model and $\gamma = 0$ is the so-called the Maxwellian molecules.

In this paper, we consider the case of hard potentials $0 \leq \gamma \leq 1$ (including Maxwellian molecules model and hard sphere model), and we assume that $b(\theta)$ satisfy Grad’s angular cut-off condition, i.e.:
\[
0 < \int_{S^2} b(\theta) dw = b_0 < \infty.
\]

Under the case of space inhomogeneous, the global existence of a mild solution with the inelastic Boltzmann equation is obtained by Duan et al. [5] obtained the $L^1$ stability of the Boltzmann equation near vacuum and the local Maxwellian. Moreover, for the Boltzmann equation with small external forces, Duan et al. [5] obtained the $L^1$ and BV-type stability by some Lyapunov functionals. For the Boltzmann equation with some large external forces, Cheng [3] established the $L^1$ stability in the case of soft potentials by direct method. But for the non-integrability of $|v - v_e|^p f(t, x, v)$, both above methods can not obtain the $L^1$ stability of the Boltzmann equation with some large external for hard potentials.

So, the aim of this paper is to study the $L^1$ stability of mild solutions obtained by Wei and Zhang [11] to the Cauchy problem for the inelastic Boltzmann equation with large external force for hard potentials (including hard sphere model). The main ideal come from Lu’s trick in [10].

The rest of this paper is organized as follows: In Section 2, we give the main result and the assumptions about external force and restitution coefficient. In Section 3, we present some basic estimates and establish the uniform $L^1$ stability estimate for hard potentials.

2. Assumptions and main results. The dynamic behavior of the particle trajectory is crucial in the existence and stability analysis of solutions, So for a given point $(t, v)$ in $\mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3$, we denote by $[X(s; t, x, v), V(s; t, x, v)]$ to be the particle trajectory which are the unique solutions of ODE system:
\[
\frac{dX(s; t, x, v)}{ds} = V(s; t, x, v), \quad \frac{dV(s; t, x, v)}{ds} = E(s, X(s; t, x, v)),
\]
We set
\[ [X(s; t, x, v), V(s; t, x, v)] := [X(s), V(s)] \]
along the bicharacteristic, we integrate (1.1) to get a mild form
\[ f(t, x, v) = f_0(X(0), V(0)) + \int_0^t Q(f, f)^2(s; t, x, v)ds. \]

The definition of mild solutions can be stated as follows.

**Definition 2.1.** A non-negative function \( f(t, x, v) \in C([0, T]), L^1(\mathbb{R}^3 \times \mathbb{R}^3) \) is a mild solution to (1.1) with a non-negative initial data \( f_0(x, v) \) if and only if \( f \) satisfies the integral equation (7) for all \( t \in [0, T] \) and a.e \( (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \).

In the following, we set the standard bounding functions decaying exponentially: For \( \alpha, \beta > 0 \)
\[ h(x) = \exp(-\alpha|x|^2), m(v) = \exp(-\beta|v|^2), \]
and we define a set \( S(\alpha, \beta, \delta) \) as
\[ S(\alpha, \beta, \delta) = \{ f \in C(\mathbb{R}^3 \times \mathbb{R}^3) : \sup_{t, x, v} \frac{|f(t, x, v)|}{h(X(0))m(V(0))} < \delta \} \]
Obviously, \( S(\alpha, \beta, \delta) \) is a Banach space with the norms:
\[ ||f|| = \sup_{t, x, v} \left\{ \frac{|f(t, x, v)|}{h(X(0))m(V(0))} \right\}. \]

Throughout this paper, the assumptions on the external force \( F(t, x) \) can be summarized as follows, which come from reference [6].

(A1) The external force \( F(t, x) \) ensures the existence of global-in-time smooth solution to the system (2.1) and (2.2). Furthermore, there are two functions \( \eta_1(s; t, x, v), \eta_2(s; t, x, v) \) satisfy following conditions:
\[ \begin{cases} X(0; s, X(s), V(s) - \xi) = X(0) + \eta_1(s; t, x, v)\xi, \\ V(0; s, X(s), V(s) - \xi) = V(0) - \eta_2(s; t, x, v)\xi, \end{cases} \]
for any \( \xi \in \mathbb{R}^3 \) and
\[ \begin{align*} 
\eta_1(s; t, x, v) &> 0, \eta_2(s; t, x, v) \geq \eta_0, \\
\eta(s; t, x, v) &\equiv \alpha_1'(s; t, x, v)\eta_2(s; t, x, v) - \eta_1(s; t, x, v)\alpha_2'(s; t, x, v) > 0, \\
(\eta_2(s; t, x, v))^\gamma &+ 1 \eta(s; t, x, v) \geq \bar{\eta}_0, 
\end{align*} \]
where \( \eta_i'(s; t, x, v) (i = 1, 2) \) is the derivative with respect to \( s \), and \( \eta_0, \bar{\eta}_0 \) are positive constants independent of \( s, t, x, v \).

**Remark 2.1.** For understanding the hypotheses of the external force, Duan et al. [6] list two examples which are obviously suitable for our explanation of the corresponding constructive conditions. If \( F(t, x) = F(t) \), then
\[ \eta_1(s; t, x, v) = s, \eta_2(s; t, x, v) = 1; \]
If \( F(t, x) = a^2x + F(t) \), then
\[ \eta_1(s; t, x, v) = \frac{e^{as} - e^{-as}}{2a}, \eta_2(s; t, x, v) = \frac{e^{as} + e^{-as}}{2}, \]
where $a$ is a positive constant.

**Remark 2.2.** Let $s = t$, (9) becomes

\[
\begin{align*}
X(0; t, x, v - \xi) &= X(0) + \eta_1(t; t, x, v)\xi, \\
V(0; t, x, v - \xi) &= V(0) - \eta_2(t; t, x, v)\xi.
\end{align*}
\]

(11)

For the coefficient of normal restitution we shall adopt the following assumptions, see [1]:

(A2) $e(r)$ is absolutely continuous and non-increasing from $[0, +\infty)$ to $(0, 1]$, $\theta(r) = re(r)$ is strictly increasing. And for any $c > 0, \gamma > -2$, it holds that

\[
\psi(r) = \exp\left(-\frac{c(1-e^2)}{2}r^2\right) \in L^\infty([0, +\infty)).
\]

Under above assumptions, the global existence of solutions was obtained. For details, refer [11]. Precisely, one has the following proposition.

**Proposition 2.1.** Let $-2 < \gamma \leq 1$, $0 \leq f_0(x, v) \in S(\alpha, \beta, \delta)$ and (A1), (A2), (1.7) hold. Then there exists a unique global in time mild solution $f(t, x, v)$ to the Cauchy problem (1.1) satisfying $f \in S(\alpha, \beta, \delta)$.

The main objective of this paper is to consider the uniform $L^1$ stability of mild solutions of Proposition 2.1 for hard potentials. Hence, we need an extra assumption on $\eta_1(s, t, x, v), \eta_2(s, t, x, v)$ appeared in (A1).

(A3) Define

\[
\psi(s) = \min(\sup_{s, x, v} \frac{1}{\eta_1(s, s, x, v)^{\gamma+3}}, \sup_{s, x, v} \frac{1}{\eta_2(s, s, x, v)^{\gamma+3}}), \quad 0 \leq \gamma \leq 1.
\]

And assume that

\[
\int_0^\infty \psi(s)ds < \infty.
\]

**Remark 2.3.** Obviously, the examples in Remark 2.2 satisfy assumption (A3).

Our main result about the stability is as follows.

**Theorem 2.1.** Let the main assumptions (A1)-(A3) hold, $0 \leq \gamma \leq 1$ and $f(t, x, v), g(t, x, v)$ be two mild solutions of Proposition 2.1 corresponding to initial data $f_0(x, v), g_0(x, v)$. Then we have

\[
\|f(t) - g(t)\|_1 \leq K\|f_0 - g_0\|_1^\theta.
\]

(12)

Here, $\theta$ and $K$ are positive constants depending on $\alpha, \beta, b_0, \delta$ and $\int_0^\infty \psi(s)ds$.

3. **Proof of Theorem 2.1.** For $L^1$ stability estimate of mild solutions, we employ the Gronwall type inequality given in [10].

**Lemma 3.1.** Let $\lambda(t)$ be a nonnegative function in $L^1((0, T))$, $T \in (0, +\infty)$. Suppose that a nonnegative function $\mu(t) \in L^\infty((0, T))$ satisfies

\[
\mu(t) \leq \mu_0 + R\int_0^t \lambda(s)\mu(s)ds + Ce^{-R}, \quad t \in [0, T)
\]

for any $R > 0$, where $\mu_0$ and $C$ are positive constants. Then we have

\[
\mu(t) \leq \mu_0 + C(T)\mu_0^\theta(T).
\]
Here
\[ \theta(T) = \frac{1}{2} \exp \left( - \int_0^T \lambda(s) ds \right), \quad C(T) = (e^2 C)^{1 - \theta(T)} \left[ 1 + \int_0^T \lambda(s) ds \right]. \]

Next, we present several useful estimates.

**Lemma 3.2.** For any \( 0 \leq \gamma \leq 1 \), it holds that
\[ \int_{\mathbb{R}^3} e^{\alpha |u|^\gamma} h(X(0) + u) du \leq C(\alpha) e^{\gamma |X(0)|}. \]

**Proof.** Using the following inequality for any \( 0 \leq \gamma \leq 1 \)
\[ |u|^\gamma \leq 1 + |u| \leq 1 + |u + X(0)| + |X(0)|, \]
we have
\[ \int_{\mathbb{R}^3} e^{\alpha |u|^\gamma} h(X(0) + u) du \leq e^\alpha \int_{\mathbb{R}^3} e^{\alpha |u+X(0)|} e^{\alpha |X(0)|} e^{-\alpha |u+X(0)|^2} du \]
\[ = e^\alpha e^{\gamma |X(0)|} \int_{\mathbb{R}^3} e^{\alpha |u|} e^{-\alpha |u|^2} du \]
\[ \leq C(\alpha) e^{\gamma |X(0)|}. \]

Then the proof of Lemma 3.2 is completed. \( \square \)

**Remark 3.1.** Similarly, for any \( 0 \leq \gamma \leq 1 \), it holds that
\[ \int_{\mathbb{R}^3} e^{\beta |u|^\gamma} m(V(0) - u) du \leq C(\beta) e^{\gamma |V(0)|}. \] (13)

In order to obtain the \( L^1 \) norm estimates of the collision term \( Q(f, g) \), we must consider the following integral:
\[ I_1 = \int_{\mathbb{R}^3} |v - v_*|^\gamma |f|(t, x, v_*) dv_*, \] (14)
where \( f \in S \).

**Lemma 3.3.** For \( I_1 \), we have following estimate:
\[ I_1 \leq \|f\| \left( RC(\alpha, \beta) \psi(t) + e^{-R} C(\alpha, \beta) e^{\alpha |X(0)| + \beta |V(0)|} \psi(t) \right) \]
for any \( R > 0 \).

**Proof.** By the definition of \( S(\alpha, \beta, \delta) \), it holds that
\[ |f|(t, x, v_*) \leq \|f\| \|h(X(0; t, x, v - u)) m(V(0; t, x, v - u)) \]
and we take \( s = t \) and \( \xi = u \) in (9) or Remark 2.2 to obtain that
\[ \begin{cases} 
X(0; t, x, v - u) = X(0; t, X(t), V(t) - u) = X(0) + \eta_1(t; t, x, v) u, \\
V(0; t, x, v - u) = V(0; t, X(t), V(t) - u) = V(0) - \eta_2(t; t, x, v) u.
\end{cases} \] (15)

Then,
\[ I_1 \leq \|f\| \int_{\mathbb{R}^3} |v - v_*|^\gamma h(X(0; t, x, v_*) m(V(0; t, x, v_*)) dv_* \]
\[ = \|f\| \int_{\mathbb{R}^3} |u|^\gamma h(X(0; t, x, v - u)) m(V(0; t, x, v - u)) du \]
\[ = \|f\| \int_{\mathbb{R}^3} |u|^\gamma h(X(0) + \eta_1(t; t, x, v) u) m(V(0) - \eta_2(t; t, x, v) u) du. \]
We now give two different estimates to above integrate based on decay properties in \((x,v)\) space.

Case one. Let \(\bar{u} = \eta_1(t; t, x, v)u\), we have

\[
I_1 \leq \frac{1}{\eta_1(t; t, x, v)^{\gamma+3}} \|f\| \int_{\mathbb{R}^3} |\bar{u}|^{\gamma} h(X(0) + \bar{u}) d\bar{u} \\
= \frac{1}{\eta_1(t; t, x, v)^{\gamma+3}} \|f\| \int_{\mathbb{R}^3} \alpha |\bar{u}|^{\gamma} h(X(0) + \bar{u}) d\bar{u}.
\]

Using inequality \(|\bar{u}|^\gamma \leq R + e^{[|\bar{u}|^\gamma] - R}\) for any \(R > 0\), we obtain that

\[
I_1 \leq \frac{1}{\eta_1(t; t, x, v)^{\gamma+3}} \|f\| \int_{\mathbb{R}^3} Rh(X(0) + \bar{u}) d\bar{u} \\
+ \frac{1}{\eta_1(t; t, x, v)^{\gamma+3}} \|f\| \int_{\mathbb{R}^3} e^{\alpha |\bar{u}|^\gamma} - Rh(X(0) + \bar{u}) d\bar{u} \\
\leq \|f\| \left( \frac{R}{\eta_1(s; s, x, v)^{\gamma+3}} + e^{-R} \frac{C(\alpha)}{\eta_1(s; s, x, v)^{\gamma+3}} e^{\alpha |X(0)|} \right),
\]

where we used Lemma 3.2.

Case two. Let \(\bar{u} = \eta_2(s; s, x, v)u\), we have

\[
I_1 \leq \|f\| \int_{\mathbb{R}^3} |u|^\gamma m(V(0)) - \eta_2(s; s, x, v)u) du \\
= \frac{1}{\eta_2(s; s, x, v)^{\gamma+3}} \|f\| \int_{\mathbb{R}^3} \beta |\bar{u}|^{\gamma} m(V(0) - \bar{u}) d\bar{u} \\
\leq \frac{1}{\eta_2(t; t, x, v)^{\gamma+3}} \|f\| \int_{\mathbb{R}^3} Rm(V(0) - \bar{u}) d\bar{u} \\
+ \frac{1}{\eta_2(t; t, x, v)^{\gamma+3}} \|f\| \int_{\mathbb{R}^3} e^{\beta |\bar{u}|^\gamma} - Rm(V(0) - \bar{u}) d\bar{u} \\
\leq \|f\| \left( \frac{C(\beta)}{\eta_2(s; s, x, v)^{\gamma+3}} + e^{-R} \frac{C(\beta)}{\eta_2(s; s, x, v)^{\gamma+3}} e^{\beta |V(0)|} \right).
\]

Hence,

\[
I_1 \leq \|f\| \left( \frac{C(\alpha, \beta)}{\eta_2(s; s, x, v)^{\gamma+3}} + e^{-R} \frac{C(\alpha, \beta)}{\eta_2(s; s, x, v)^{\gamma+3}} e^{\alpha |X(0)| + \beta |V(0)|} \right).
\]

This completes the proof of Lemma 3.3.

Remark 3.2. Let

\[
I_2 = \int_{\mathbb{R}^3} |v - v_\ast|^\gamma |g|(t, x, v_\ast) dv_\ast, g \in S,
\]

we have the same estimate:

\[
I_2 \leq \|g\| \left( RC(\alpha, \beta) \psi(t) + e^{-R} C(\alpha, \beta) e^{\alpha |X(0)| + \beta |V(0)|} \psi(t) \right)
\]

for any \(R > 0\).

Next, we'll give the the stability estimate. Let \(f, g\) be mild solutions of (1.1) with the initial data \(f_0, g_0\) respectively. Then we have

\[
f(t, x, v) = f_0(X(0), V(0)) + \int_0^t Q(f, f)(s, X(s), V(s)) ds,
\]
Then we have

\[ g(t, x, v) = g_0(X(0), V(0)) + \int_0^t Q(g, g)(s, X(s), V(s)) ds. \]

**Proof.** Taking difference of the above two equations and integrating it over \( \mathbb{R}^3 \times \mathbb{R}^3 \) to get

\[
\| (f(t) - g(t)) \|_1 \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} |(f_0 - g_0)(X(0), V(0))| dx dv + \\
\int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} |Q(f - g, f)(s, X(s), V(s))| dx dv ds + \\
\int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} |Q(g, f - g)(s, X(s), V(s))| dx dv ds \\
= \| f_0 - g_0 \|_1 + \int_0^t \| Q(f - g, f)(s) \|_1 ds + \int_0^t \| Q(g, f - g)(s) \|_1 ds.
\]

Define

\[
Q_+(f, g)(t, x, v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} b(\theta) \frac{|u| \gamma}{e^{\gamma(t')}} f(t, x', v') g(t, x, v) dv dx, \quad (19)
\]

\[
Q_-(f, g)(t, x, v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} b(\theta) |u| \gamma f(t, x, v) g(t, x, v) dv dx. \quad (20)
\]

Then we have

\[
\| Q_+(g, f)(s) \|_1 = \| Q_+(f, g)(s) \|_1 = \| Q_-(f, g)(s) \|_1 = \| Q_-(g, f)(s) \|_1,
\]

where we use \( u \cdot \omega = -'e('u \cdot \omega') \).

By the definition of \( Q(f - g, f) \) and \( Q(g, f - g) \):

\[
Q(f - g, f) = Q_+(f - g, f) + Q_-(f - g, f),
\]

\[
Q(g, f - g) = Q_+(g, f - g) + Q_-(g, f - g),
\]

we get

\[
\| (f(t) - g(t)) \|_1 \\
\leq \| f_0 - g_0 \|_1 + 2 \int_0^t \| Q_-(f - g, f)(s) \|_1 ds + 2 \int_0^t \| Q_-(g - f, g)(s) \|_1 ds.
\]

So, we only need to estimate

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} b(\theta) |u| \gamma |f - g|(s, x, v) f(s, x, v) dv dx dv ds dx \equiv J_1 \quad (21)
\]

and

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} b(\theta) |u| \gamma |f - g|(s, x, v) g(s, x, v) dv dx dv ds dx \equiv J_2. \quad (22)
\]

In fact, by the definition of \( J_1, J_2 \) and Lemma 3.3, we have

\[
J_1 = \int_{\mathbb{S}^2} b(\theta) dv \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} |u| \gamma |f - g|(s, x, v) g(s, x, v) dv dx dv ds dx \\
\leq b_0 \int_{\mathbb{R}^3 \times \mathbb{R}^3} |f - g|(s, x, v) \left( \int_{\mathbb{R}^3} |v - v_*| \gamma |g(s, x, v)| dv_* dx \\
= b_0 \int_{\mathbb{R}^3 \times \mathbb{R}^3} |f - g|(s, x, v) I_1 dx dv dx.
\]
Obviously, \(0 < \theta < 1/2\) and
\[
\|f(t) - g(t)\|_1 \leq (\|f_0 - g_0\|_1^{1-\theta} + C)\|f_0 - g_0\|_1^\theta
\]
\[
\leq (\|f_0\|_1 + \|g_0\|_1)^{1-\theta} + C)\|f_0 - g_0\|_1^\theta
\]
\[
\leq K\|f_0 - g_0\|_1^\theta,
\]
where \(K\) depend on \(\alpha, \beta, b_0, \delta\) and \(\int_0^\infty \psi(s)ds\). Then the proof of Theorem 2.1 is completed.
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