Thermodynamics of Bose-Einstein condensation of relativistic gas

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Abstract

We show that the free relativistic wave equation which describes the particle without or with rest mass has more than one part of energy spectrum. One part of energy spectrum is beginning with rest energy and it is not limited by above. This part of spectrum is called by us as normal. Another part of energy spectrum is beginning with the zero energy. This part of spectrum is called by us as anomalous since the zero energy corresponds to the infinite group velocity. The presence of the zero in the energy spectrum permits to consider the Bose-Einstein condensation. We show that the heat capacity has the finite discontinuity at the condensation temperature. The last means that we have the phase transition at the condensation point.

I. I. INTRODUCTION

It is well known that the construction of the thermodynamics requires the knowledge of particle energy spectrum because the principal thermodynamic characteristic, namely, equilibrium temperature distribution function depends on particle energy (frequency). We denote the last by $f_r$,

$$f_r = (\exp(h\omega - \mu/k_B T) + r)^{-1}, r = 0, \pm 1.$$ 

where the relativistic energy $h\omega$ is equal to

$$h\omega = hc\sqrt{k^2 + k_0^2},$$ (1)

Here $k^2$ is a square of 3-D momentum and $k_0$ is the inverse Compton wave length. We have superhigh energy at $k >> k_0$, and in this case energy is equal to $h\omega \approx hck$ (ultrarelativistic case). In the opposite case $k << k_0$ the energy spectrum of Eq(1) transforms into non relativistic energy spectrum. Energy spectrum of Eq(1) corresponds to eigenfunctions (plane waves) of Klein-Gordon -Fock (KGF) equation

$$\Psi(+) = \exp(i(-\omega t + k_2 z)) \exp(i(k_1 x_1 + k_2 x_2)).$$ (2)

Later we will explain the representation of the solution of Eq.(2) as a product of two exponential functions. The plane waves of Eq.(2) are the basis for Green functions (GF)
of KFG equation in the time-like Minkowski space $M^+$. It is known from long ago (see Stueckelberg, Feynman, Fierz [1-3]) that the construction of the causal relativistic propagator (CRP) requires the knowledge of GF in space-like Minkowski space $M^-$ as well. Considering relativistic quantum mechanics in so-called characteristic representation (CR) we can easily prove the last assertion [4]. Let us discuss the main principles of relativistic dynamics of CR in a brief form. In CR we have to single out one of the spatial coordinates in a wave operator (for the sake of definiteness we assume it to be $z$ and let us write the relativistic wave equation in the form of 1-D telegraphic equation with the operator parameter $\hat{a}^2$, 

$$\left( \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \hat{a}^2 \right) \Psi(x_1, x_2, z, t) = 0. \quad (3)$$

where 

$$\hat{a}^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - k_0^2 = \Delta_\perp - k_0^2$$

is elliptic operator.

Dealing with CR it is convenient to classify states by the eigenfunctions of the elliptic operator $\Delta_\perp$. This operator has two kinds of solutions. The first solution is a plane wave. The last gives the contribution to the GF in the time-like space $M^+$. The second solution is the Macdonald function

$$K_0(\hat{a}^2 x_\perp) = k_\perp^2 K_0(k_\perp, x_\perp). \quad (4)$$

where $\omega$ is equal now to

$$\omega = c \sqrt{k_\perp^2 - k_0^2}. \quad (5)$$

We have shown in [5] that the both solutions of Eqs.(2) and (4) are the basis for two GF. One is defined inside the light cone and another is out of the light cone. It is especially simply to obtain two GF for photons when rest mass is equal to zero. Taking into account that the 1-D telegraphic equation (3) has the fundamental solution in the terms of Bessel function (BF) of the first kind of order zero we obtain [4]

$$G_{ph}^{(+)} = J_0 \left( \sqrt{c^2 t^2 - z^2}(-\Delta_\perp) \right) |0\rangle = \frac{1}{2\pi} \frac{\delta(|t| - r/c)}{cr} = \frac{\delta(r^2 - c^2 t^2)}{2\pi},$$

where $\delta(z)$ is Dirac delta function, and here it’s taken into account that $\delta(|t| + r/c) = 0$ and also

$$G_{ph}^{(-)} = J_0 \left( \sqrt{z^2 - c^2 t^2}(\Delta_\perp) \right) |\bar{0}\rangle = \frac{1}{2\pi} \frac{1}{(r^2 - c^2 t^2)}.$$


Here
\[ |0\rangle = \frac{1}{4\pi^2} \int \exp i(k_1 x_1 + k_2 x_2) = \delta(x_1)\delta(x_2) \]
and
\[ |\tilde{0}\rangle = \frac{1}{2\pi} \int_0^\infty K_0(k_\perp x_\perp) k_\perp dk_\perp = \frac{1}{2\pi} \frac{1}{x_1^2 + x_2^2} \] (6)
are the initial values of GF which are given along the characteristics \( |t| = |z/c| \). The sum of these GF is CRP \( D_c \) [2]
\[ D_c = \frac{1}{2} G^{(+)}_{ph} + \frac{i}{\pi} G^{(-)}_{ph} . \]

The decomposition of the BF into the plane waves is presented in [5]. If rest mass is not equal to zero then the operator argument in the BF is changed: \(-\triangle_\perp \rightarrow k_\perp^2 - \triangle_\perp\). Inside the light cone we have the function \( G^{(+)} = \Delta_s \), where \( \Delta_s \) is the even solution of KFG equation [6] and outside the light cone we have
\[ G^{(-)} = J_0 \left( \sqrt{(z^2 - c^2t^2)(\Delta_\perp - k_\perp^2)} \right) |\tilde{0}\rangle = \frac{k_0 K_1(k_0 \sqrt{r^2 - c^2t^2})}{\sqrt{r^2 - c^2t^2}} \]
where \( |\tilde{0}\rangle \) is the initial value of propagator \( G^{(-)} \) along characteristics \( |t| = |z/c| \) (compare with (6))
\[ |\tilde{0}\rangle = \frac{1}{2\pi} \int_0^\infty K_0(k_\perp x_\perp) k_\perp dk_\perp = \frac{k_0}{2\pi} \frac{K_1(k_0 x_\perp)}{x_\perp} \] (7)

Let us write the group velocities along the determinate direction for energy spectra of Eqs. (1) and (5). For Eq.(1)
\[ v_{z}^{(+)} = \frac{\partial \omega}{\partial k_z} = \frac{ck_z}{\sqrt{k_z^2 + q_\perp^2 + k_0^2}} < |c| , \]
for Eq.(5)
\[ v_{z}^{(-)} = \frac{\partial \omega}{\partial k_z} = \frac{ck_z}{\sqrt{k_z^2 - k_\perp^2 + k_0^2}} \geq |c| . \]

What differs the spectrum of Eq.(5) from the spectrum of Eq.(1)? The main difference lies in the presence of minimal zeroes energy for which the velocity \( v_{z}^{(-)} \) is maximal and goes to infinity. The zero energy (frequency) is realized if the transverse momentum \( k_\perp \neq 0 \) and is equal to
\[ k_\perp = \sqrt{k_z^2 + k_0^2} \geq k_0 , \text{ at } \omega = 0 . \]
Due to such extraordinary property that the infinitely-moving particle has zero energy, the spectrum of Eq. (5) is called by us as anomalous, although the definition of this spectrum as anomalous is not good. The state describing this spectrum is needed as well as normal. As it has been mentioned above, the CRP $D_c$ consists of these two states (see Eqs. (2) and (4)).

Energy spectrum of Eqs. (1) and (5) (dispersion curves) are mapped on the plane $(\omega/c, k_z)$ by half-hyperbola which belong to different quadrants of the plane $(\omega/c, k_z)$ divided by asymptotes $|\omega/c| = |k_z|$.

In this paper we examine the special features of thermodynamic characteristics concerning statistical integral, internal and free energies, entropy etc. referring to relativistic gas in the anomalous state.

II. THERMODYNAMIC CHARACTERISTICS OF THE NORMAL STATES OF RELATIVISTIC GAS

For comparison we present in a brief form the known results of relativistic thermodynamics (RT) based on the spectrum of Eq. (1) \[7\]. First we consider the Maxwell-Boltzmann (M-B) statistics, that is $r = 0$. We denote the particle statistic integral in M-B statistics by $Z$

$$Z = \int d^3k \exp \left(-\frac{\hbar \omega}{k_B T}\right) = 4\pi k_0^3 \int_1^{\infty} \varepsilon \varepsilon^2 - 1 \exp \left(\frac{-\Theta \varepsilon}{T}\right) d\varepsilon,$$

where $\Theta = mc^2/k_B$ is typical parameter of RT having the temperature dimensionality and $k_B$ is Boltzmann constant. If $m$ is electron mass then $\Theta \approx 10^{10^9}\deg K$.

Let us denote the factor before the exponential function in integral of Eq. (8) by $g(\varepsilon)$

$$g(\varepsilon) = 4\pi k_0^3 \varepsilon \sqrt{\varepsilon^2 - 1},$$

or in dimensional units ($\varepsilon = \hbar \omega/mc^2$)

$$g(\omega/c) = 4\pi k_0^3 \frac{\hbar \omega}{mc^2} \sqrt{\frac{(\hbar \omega/mc^2)^2}{\Theta} - 1}.$$  \hspace{1cm} (9)

Eq. (9) gives the state density in the energy scale which is determined up to the factor $\frac{\Omega}{\Omega_{rev}}$, where $\Omega$ is the volume of relativistic gas. This density is equal to zero if $\hbar \omega \leq mc^2$ or in temperature units $T \leq \Theta$. Due to such temperature restriction it is said that RT has academic interest and it can be applicable in astrophysical phenomena where there are superhigh temperatures and densities. The integral of Eq. (8) is easily calculated by substitution $\varepsilon = \cosh \tau$ and with the help of the table integral $\int_0^{\infty} dz \exp(-q \cosh z) = K_0(q)$. It is respectively equal to

$$Z = 4\pi k_0^3 \frac{K_2(\frac{\Theta}{T})}{(\frac{\Theta}{T})},$$

where $K_2$ is the Macdonald function of the second kind.

Let us calculate the asymptotics of Eq. (10) at low temperature when $\frac{\Theta}{T} >> 1$.
\[ Z \approx (2\pi)^{3/2} k_0^3 \left( \frac{\Theta}{T} \right)^2 \exp \left( -\frac{\Theta}{T} \right) \]

Statistic integral of Eq.(10) does not depend on mass at high temperature when \( (\Theta/T) << 1 \)

\[ Z \approx 8\pi k_0^3 \left( \frac{T}{\Theta} \right)^3 = 8\pi \left( \frac{k_BT}{\hbar c} \right)^3. \]

Let us compute the internal energy density

\[ U = \langle \hbar \omega \rangle = \int d^3k \hbar \omega \exp \left( -\frac{\hbar \omega}{k_BT} \right) = k_BT^2 \frac{\partial Z}{\partial T}. \]

It is convenient to find the internal energy in per one particle so-called middle energy

\[ \overline{U} = \frac{\langle \hbar \omega \rangle}{Z} = k_BT^2 \frac{\partial}{\partial T} \ln Z = \mu c^2 \left( \frac{T}{\Theta} + \frac{K_1 + K_3}{2K_2} \right). \]

At low temperature the middle energy and heat capacity (HC) are equal to

\[ \overline{U} \approx \mu c^2 + 3/2k_BT, \quad c_v = \frac{3}{2}k_B. \]

The middle energy and HC at high temperature \( (\Theta/T) << 1 \) are equal to

\[ U \approx 3k_BT, \quad c_v = 3k_B. \]

If the relativistic gas obeys to Bose-Einstein (B-S) statistics that is \( r = -1 \) then all thermodynamic characteristics can be found from the distribution

\[ N = 4\pi k_0^3 \sum_{n=1}^{\infty} \exp(\mu n/k_BT)K_2(n\Theta/T)/(n\Theta/T). \]

III. THERMODYNAMICS OF ANOMALOUS STATES

Let us find the state density in an energy scale for the energy spectrum of Eq.(5). It is convenient to introduce dimensionless variables, namely, dimensionless energy (frequency) and dimensionless momentum variables

\[ \epsilon = \omega/k_0c, \quad q_z = k_z/k_0, \quad q_\perp = k_\perp/k_0. \]

We write dispersion equation (5) in dimensionless variables

\[ \epsilon = \sqrt{q_z^2 - q_\perp^2} + 1. \]

In fact, it is necessary to make the transition from the momentum representation to energy representation with the help of transformation formulae which depend on the connection between full energy \( \langle \hbar \omega \rangle \) and rest energy \( (mc^2) \). It means that in dimensionless variables
the value $\varepsilon$ can be more than one or less than one. Let $\varepsilon > 1$ then transformation formulae have the form

$$q_z = \sqrt{\varepsilon^2 - 1} \cosh v, \ q_1 = \sqrt{\varepsilon^2 - 1} \cos u \sinh v, \ q_2 = \sin u \sinh v.$$ 

The infinitely-small volume element $d^3 k$ ($d^3 q$ in dimensionless variables) in the new variables has the form

$$d^3 q = |I| d\varepsilon du dv,$$  \hspace{1cm} (11)

where $I$ is Jacobian of transformation. It is equal to $I = \varepsilon \sqrt{\varepsilon^2 - 1} \sinh v$.

To obtain the state density in energy scale it is needed to integrate Eq.(11) over two angles $u$ and $v$. The integration over the angle $u$ gives the factor $2\pi$. To integrate Eq.(11) over hyperbolic angle $v$ it is necessary to know its limits of integration. They are found from formula for the domain of the definition of transverse momentum $k_\perp$ ($q_\perp$ in dimensionless variables)

$$q_\perp = \sqrt{q_1^2 + q_2^2} = \sqrt{\varepsilon^2 - 1} \sinh v.$$  \hspace{1cm} (12)

If the upper limit of the value $q_\perp$ is equal to infinity then the upper limit of hyperbolic angle $v$ is equal to infinity too. In this case Eq.(11) becomes infinite after integration over hyperbolic angle $v$. Therefore we cut the momentum $k_\perp$ by the value $k_\perp^{\text{max}}$. It does not influence on the state with zero energy because

$$\omega = 0 \text{ at } k_\perp^{\text{max}} = \sqrt{(k^{\text{max}}_\varepsilon)^2 + k_0^2} > k_0 \text{ or } q_\perp^{\text{max}} > 1.$$ 

We can find the upper integral limit from Eq.(12)

$$\sinh v_{\text{max}} = \frac{q_\perp^{\text{max}}}{\sqrt{\varepsilon^2 - 1}}, \ \cosh v_{\text{max}} = \frac{\sqrt{(q_\perp^{\text{max}})^2 + \varepsilon^2 - 1}}{\sqrt{\varepsilon^2 - 1}}.$$ 

Integrating Eq.(11) over the angles $u$ and $v$ we obtain the state density in the energy scale

$$g(\varepsilon)d\varepsilon = 2\pi k_0^3 \left(\varepsilon \sqrt{\varepsilon^2 + (q_\perp^{\text{max}})^2 - 1} - \varepsilon^2\right) d\varepsilon$$  \hspace{1cm} (13) 

Further we consider the case $\varepsilon < 1$. It means that in the dimensional variables energy is less than rest energy $mc^2$ (or $\varepsilon < 1$). In this case the transformation formulae have the form

$$q_z = \sqrt{1 - \varepsilon^2} \sinh v, \ q_1 = \sqrt{1 - \varepsilon^2} \cos u \cosh v,$$ 

$$q_2 = \sqrt{1 - \varepsilon^2} \sin u \cosh v,$$ 

and the volume element in dimensionless variables has the form
\[ d^3q = |I| d\varepsilon d\varphi = \varepsilon \sqrt{1 - \varepsilon^2} d\varepsilon d\varphi (\sinh \varphi) \quad (14) \]

where \( I = \varepsilon \sqrt{1 - \varepsilon^2} \) \( \cosh \varphi \) is Jacobian of transformation.

We can find the limits of the angle \( \varphi \) from the expression

\[ q_\perp = \sqrt{q_1^2 + q_2^2} = \sqrt{1 - \varepsilon^2} \cosh \varphi. \]

Repeating preceding reasonings we find

\[ \cosh \varphi_{\text{min}} = \frac{1}{\sqrt{1 - \varepsilon^2}}, \sinh \varphi_{\text{min}} = \frac{\varepsilon}{\sqrt{1 - \varepsilon^2}}. \]

and

\[ \cosh \varphi_{\text{max}} = \frac{q_{\perp}^{\text{max}}}{\sqrt{1 - \varepsilon^2}}, \sinh \varphi_{\text{max}} = \frac{\sqrt{(q_{\perp}^{\text{max}})^2 + \varepsilon^2 - 1}}{\sqrt{1 - \varepsilon^2}}. \]

Performing in Eq.(14) the integration over angle variables we find the state density coinciding with Eq.(13).

Let us rewrite Eq.(13) in dimensional variables

\[ \frac{g(\omega/c)}{k_0} = 2\pi \left( \frac{\omega}{c} \sqrt{\frac{\omega^2}{c^2} + \left(\frac{k_{\perp}^{\text{max}}}{m} \right)^2 - k_0^2 - \frac{\omega^2}{c^2}} \right) \quad (15) \]

If the value \( k_0 \) in Eq.(15) goes to zero then the form of this equation is not changed. From Eq.(15) it follows

\[ \lim g(\omega/c) = \pi k_0 \left( (k_{\perp}^{\text{max}})^2 - k_0^2 \right) \text{ at } \omega \to \infty. \]

Introducing the notion of effective mass \( m^* = m \sqrt{(q_{\perp}^{\text{max}})^2 - 1} \), we rewrite the state density of Eq.(13) in a more compact form

\[ g(\omega/c) = 2\pi k_0^3 \left( \varepsilon \sqrt{\frac{m^*}{m}} - \varepsilon^2 \right) \quad (16) \]

Now we can consider the statistic integral of anomalous state in M-B statistics in energy representation

\[ Z = 2\pi k_0^3 \int_0^\infty \left( \varepsilon \sqrt{\frac{m^*}{m}} - \varepsilon^2 \right) \exp \left( -\frac{\Theta \varepsilon}{T} \right) d\varepsilon \quad (17) \]

One integral in Eq.(17) is calculated in an elementary manner

\[ \int_0^\infty \varepsilon^2 \exp \left( -\frac{\Theta \varepsilon}{T} \right) d\varepsilon = 2 \left( \frac{T}{\Theta} \right)^3 \]
The second integral of Eq.(17) is calculated with the help of substitution \( \varepsilon = \frac{m^*}{m} \sinh \theta \) and of table integral \( \int_0^\infty d\theta \exp(-q \sinh \theta) = \frac{\pi}{q} A_0(q) \), where \( A_0 = H_0 - N_0 \) and \( H_0 \) is Struve function and \( N_0 \) is the Bessel function of the second kind of the order zero. From what has been said above it follows that

\[
Z = \pi^2 \left( \frac{m^* c}{\hbar} \right) B \left( \frac{\Theta^*}{T} \right) \tag{18}
\]

where \( \Theta^* = \frac{m^* c^2}{k_B} = \frac{mc^2}{k_B} \sqrt{(q_{\perp}^{\text{max}})^2 - 1} \) is effective temperature and

\[
B(q) = 2A_1/q^2 - A_0/q - 4/\pi q^3, \quad A_i = H_i - N_i, \quad i = 1, 2 \tag{19}
\]

The spectrum of Eq.(5) unlike the spectrum of Eq.(1) permits to consider B-E condensation of relativistic gas. In this case the cutting parameter \( k_{\perp}^{\text{max}} \) (or effective mass \( m^* \)) can be expressed in the terms of condensation temperature \( T_c \) and gas particle number \( N_a \) per unit volume. The condensation temperature \( T_c \) is defined from the condition of vanishing of chemical potential \( \mu \) in the gas particle density \( N_a \). From the point of view of B-E statistics the \( N_a \) is equal to

\[
N_a = \int_0^\infty \frac{g(\varepsilon) d\varepsilon}{\exp \left( \frac{\Theta}{T} \right) - 1} = \pi^2 (k_0^*)^3 \sum_{n=1}^\infty B \left( \frac{n\Theta^*}{T_c} \right) \tag{20}
\]

where \( k_0^* = \frac{m^* c}{\hbar} \) and the value \( B(q) \) is defined from Eq.(19). At low temperature the condition \( \frac{n\Theta^*}{T_c} >> 1 \) is always performed and we can use the asymptotics of the function \( B(q) \approx 2/\pi q^2 \) at \( q >> 1 \),

\[
N_a \approx 2\pi \zeta(2) (k_0^*)^3 \left( \frac{T_c}{\Theta^*} \right)^2 = 2\pi \zeta(2) k_0^* \left( \frac{k_B T_c}{\hbar c} \right)^2 \tag{21}
\]

where \( \zeta(s) \) is zeta-function of Riemann.

From Eq.(21) it follows that the effective mass \( m^* \) satisfies the condition

\[
m^* T_c^2 = \text{const.}
\]

The gas particle density \( N_0 \) with the zero energy (condensation particles) and the density of overcondensation particles \( N_1 \) of relativistic gas are equal to

\[
N_0 = N_a \left( 1 - \left( \frac{T}{T_c} \right)^2 \right), \quad N_1 = N_a \left( \frac{T}{T_c} \right)^2.
\]

The internal energy density \( U \) of relativistic gas in temperature interval \( 0 \leq T \leq T_c \) is defined by integral

\[
U = mc^2 \int_0^\infty \frac{\varepsilon g(\varepsilon) d\varepsilon}{\exp \left( \frac{\Theta}{T} \right) - 1} = \pi^2 k_0^* m^* c^2 \sum_{n=1}^\infty D \left( \frac{n\Theta^*}{T} \right) \tag{22}
\]

where \( D(q) = 2/\pi q - 12/\pi q^4 - A_1/q - 3A_0/q^2 + 6A_1/q^3. \)
Taking into account that at low temperature \((q >> 1)\) the asymptotics of \(D(q)\) function is \(\sim 4/\pi q^3\) we have from Eq.(22)

\[
U \approx 4\pi\zeta(3)k_0^*m^*c^2\left(\frac{T}{\Theta^*}\right)^3 \approx 1,5N_ak_BT\left(\frac{T}{T_c}\right)^2
\]

(23)

In Eq. (23) the approximate equality has been used \(2\zeta(3)/\zeta(2) \approx 1.5\).

The HC \(C_v\) in the temperature interval \(0 \leq T \leq T_c\) is equal respectively to

\[
C_v \approx 4,5N_ak_BT\left(\frac{T}{T_c}\right)^2.
\]

Let us investigate the relativistic thermodynamics in the temperature interval \(T - T_c << T_c\). We take into account that \(N_0\) is equal to zero and \(|\mu|/k_BT << 1\). Then the particle density in relativistic gas has the form

\[
N_a = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp\left(\frac{\Theta(\varepsilon+\mu/mc^2)}{T}\right) - 1} \approx 2\pi(k_0^*)^3\zeta(2)\left(\frac{T}{\Theta^*}\right)^2 \exp\left(-\frac{\mu}{k_BT}\right)
\]

(24)

In Eq.(24) we have used the asymptotics of \(B(q)\) and the approximate summation with the help of Euler formula \(\sum_{n=1}^\infty \frac{\exp(-nq)}{n^2} \approx \zeta(2)\exp(-q)\) which is valid at \(q << 1\).

Substituting Eq.(21) in the left side of Eq.(24) we obtain the equation for the chemical potential \(\mu\),

\[
\left(\frac{T}{T_c}\right)^2 = \exp(\mu/k_BT) \rightarrow \mu = 2k_BT\ln\left(\frac{T}{T_c}\right) \text{ at } T \geq T_c.
\]

All the derivatives of chemical potential \(\mu\) are discontinuous at the condensation temperature \(T = T_c\), in contrary to non relativistic thermodynamics where the first derivative of chemical potential \(\mu\) is continuous at the condensation temperature \(T = T_c\),

\[
\mu'(T_c - 0) = 0 \text{ , } \mu'(T_c + 0) = 2k_B,
\]

and etc.

Let us calculate the internal energy density \(U\) in the temperature interval \(T \geq T_c\),

\[
U = mc^2\int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp\left(\frac{\Theta(\varepsilon+\mu/mc^2)}{T}\right) - 1} = 4\pi k_0^*m^*c^2\zeta(3)\left(\frac{T}{\Theta^*}\right)^2 \exp\left(-\frac{\mu}{k_BT}\right) \approx 1,5N_ak_BT,
\]

and HC is equal to \(C_v = 1,5N_ak_BT\). Thus, in the temperature interval \(T \geq T_c\) the internal energy and HC of relativistic gas in the anomalous state coincide with energy and HC of ideal non relativistic gas.

Let us write the expression for full internal energy using the Heaviside step-function \(\theta(t)\),

\[
U(T) = 1,5N_ak_BT\left\{\left(\frac{T}{T_c}\right)^2 \theta(T_c - T) + \theta(T - T_c)\right\},
\]

\[
U(T_c) = 1,5N_ak_BT_c.
\]
At the condensation temperature \( T = T_c \) we have the heat capacity jump

\[
\frac{C_v(T_c - 0) - C_v(T_c + 0)}{C_v(T_c + 0)} = 2
\]  

(25)

Eq. (25) does not contain the arbitrary parameters and therefore it admits the experimental verification.

Entropy variation can be found from the expression

\[ S(T) = \frac{9}{4} k_B N_a \left\{ \left( \frac{T}{T_c} \right)^2 \theta(T_c - T) + \left( \ln \left( \frac{T}{T_c} \right)^\frac{2}{3} + 1 \right) \theta(T - T_c) \right\} \]

and \( S(T_c) = \frac{9}{4} N_a k_B \).

We can find the expressions for free energy \( F = U - TS \), for an enthalpy \( H = U + PV = TS + \mu N_a \theta(T - T_c) \), the equation of state of relativistic gas etc. These main values have the fracture (or angle point) at the condensation temperature \( T_c \). It means that the derivative of these values with respect to temperature \( T \) has the discontinuity of the first kind at \( T = T_c \). Therefore these thermodynamic characteristics cannot be expanded into Taylor series in the vicinity of the critical point \( T = T_c \).

For comparison let us write the particle density \( N_{n,r} \) of non relativistic gas at the condensation temperature \( T_c \),

\[ N_{n,r} = (2\pi)^{3/2} \frac{s(3/2)}{\Theta} k_0^3 \left( \frac{T_c}{\Theta} \right)^{3/2} \]

The ratio of two densities (see Eq. (21))

\[ \frac{N_a}{N_{n,r}} = \frac{s(2)}{\sqrt{2\pi} s(3/2)} \frac{m^*}{m} \left( \frac{T_c}{\Theta} \right)^{1/2} \]

shows that at low temperature \( T_c \) the numbers \( N_a \ll N_{n,r} \). This inequality strengthens if \( m^* \ll m \).

IV. CONCLUSION

Our reasonings are at least in harmony with the papers of Bilaniuk, Terletskiy, Feinberg (see ref.[8] and refs. therein) in which it is proved that the relativistic quantum mechanics is incomplete if we are restricted by energy spectrum of Eq.(1). We know that the RCP \( D_c \) consists of the solutions of different nature (oscillatory and exponential damping). The last corresponds to energy spectrum of Eq.(5) which we call anomalous. We could call this spectrum as tachyon although tachyon term is referred to the solutions of KGF equation with imaginary mass[9]. We believe that the energy spectrum of Eq.(5) can be applied to the description of thermodynamics of models with the observed phase transition of the second kind.

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