Spread and asymmetry of typical quantum coherence and their inhibition in response to glassy disorder

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Keywords: quantum coherence, glassy disorder, spread and asymmetry of typical quantum coherence

Abstract
We consider the average quantum coherences of typical redits and qudits—vectors of real and complex Hilbert spaces—with the analytical forms stemming from the symmetry of Haar-uniformly distributed random pure states. We subsequently study the response to disorder in spread of the typical quantum coherence in response to glassy disorder. The disorder is inserted in the state parameters. Even in the absence of disorder, the quantum coherence distributions of redits and qudits are not uniform over the range of quantum coherence, and the spreads are relatively lower for higher dimensions. On insertion of disorder, the spreads decrease. This decrease in the spread of quantum coherence distribution in response to disorder is seen to be a generic feature of typical pure states: we observe the feature for different strengths of disorder and for various types of disorder distributions, viz. Gaussian, uniform, and Cauchy–Lorentz. We also find that the quantum coherence distributions become less asymmetric with increase in dimension and with infusion of glassy disorder.

1. Introduction

The phenomenon of quantum coherence emerges due to the ‘waveness’ of quantum systems. Entanglement [1–4], another quintessential quantum phenomenon, appears in shared states of at least two quantum systems, whereas quantum coherence appears due to interference of at least two quantum waves associated with a single system. While the concept of quantum coherence was known since the beginnings of quantum theory, a formal apparatus of the resource theory of quantum coherence is rather recent [5, 6]. There has been a large number of studies on different aspects of the resource theory, including references [7–23]. Along with its fundamental importance in the structure of quantum theory of physical systems, quantum coherence has also been connected to a variety of other physical phenomena, see e.g. [10, 24–40].

The importance of considering the response of disorder in a physical phenomenon can hardly be over-emphasized. Disorder appears naturally and almost universally in physical systems. The system under study, when realized, is expected to be affected by several such processes. Their individual effects can be of varied forms, and they may arise due to an array of physical effects, e.g. stray fields or imperfect system elements like a randomly-appearing defect in a lattice [41]. In our study, we have modeled these effects by incorporating a disordered state parameter (or a few of them) in the system state. The disorder itself may appear in different hues and arrangements in a substrate realizing a phenomenon. We will be interested in a category of disorder that has been called ‘glassy’ or ‘quenched’ in the literature [42–47], and refers to a disorder in which a system parameter picks up values randomly from a certain distribution, decided by its preparation procedure and physical nature, and the typical equilibration time of the disorder is far greater than the typical system manipulation times that we are interested in. We therefore assume that the physical effects that lead to the disordered parameter are ‘slow’ with respect to the timescales over which we observe our system. After we have chosen the relative equilibration timescale of the disorder, we still need to decide...
on the distribution of the disorder, and this will depend on the actual physical realization of the physical system. In the hope that the results obtained are somewhat generic, we consider three probability distributions for the disorder, one of which does not have a well-defined mean.

In this paper, we focus on the distribution of quantum coherence of Haar-random pure states, and the response of the same when the state parameters are inflicted by glassy disorder. Random pure states are the quantum analog of random numbers in classical information theory, and this is the reason that we chose to look at the effect of disorder in random pure states, as it would then provide us with a ‘typical’ response to the disorder. In the following section (section 2), we present an introductory discussion on the Haar uniformity of random pure states and the distribution functions used during the introduction of disorder in state parameters. In section 3, we present the analytical forms of the average quantum coherences (as quantified by the $l_1$-norm of quantum coherence [5, 6]) of Haar uniform random pure states, for both ‘rebits’ and ‘qudits’. In section 4, we discuss the variation of quantum coherence distributions obtained for Haar-uniformly generated pure states for Hilbert spaces of different dimensions. In section 5, we analyze the effect of disorder in the state parameters on the distributions of Haar-uniformly generated pure states. A conclusion is presented in section 6.

2. Haar uniformity and probability distribution functions

In this paper, we study the quantum coherence of typical real bits and quantum bits, and their $d$-dimensional versions, for Hilbert spaces of different dimensions. The quantum coherences are always considered in the computational basis. A $d$-dimensional pure quantum state can be written as

$$|\psi\rangle = \sum_{j=1}^{d} (c_j + ic_{\bar{j}})|j\rangle,$$

where $|j\rangle$ represents the $j$th orthonormal basis vector in the computational basis of the $d$-dimensional complex Hilbert space, $\mathbb{C}^d$, and $c_j$, $c_{\bar{j}}$ are real numbers, constrained by the normalization condition, $\langle \psi | \psi \rangle = 1$. It is called a quantum bit or qubit for $d = 2$, qutrit for $d = 3$, and qudit in general. If the imaginary parts of coefficients of the basis vectors in equation (1) are vanishing (i.e. if $c_{\bar{j}} = 0$ $\forall j$), then the corresponding vector can be called a real bit or rebit for $d = 2$, a retrit for $d = 3$, and a redit in general [48, 49]. A real bit has, in general, off-diagonal terms in the computation basis, and so is different from a probabilistic mixture of being in two orthogonal states, although both can be parametrized by a single real number. ‘Quantum coherence’ is typically defined for states of or defined on a complex Hilbert space. We however will use the same definition and terminology, also for states of or defined on real Hilbert spaces.

Haar uniformity is attained by choosing the $c_j$ independently from Gaussian distributions with vanishing mean and finite variance [50–54]. The obtained state will have to be normalized to unity. The probability density function for a Gaussian distribution is given by

$$f_G(x) = \frac{1}{\sigma_G \sqrt{2\pi}} e^{- \frac{(x - \mu_G)^2}{2\sigma_G^2}},$$

where $\mu_G$ is the mean and $\sigma_G$ is the standard deviation of the distribution. Before normalization, the states generated are distributed in a hyperspace of dimension $d$ when the real numbers are selected from the Gaussian distribution with zero mean and finite variance. After normalization, they are distributed uniformly over a hypersphere of unit radius in the same hyperspace. Note that the joint probability distribution of independent variables is spherically symmetric if the individual distributions are Gaussian.

Random generation of quantum states involves choosing the coefficients of the states from independent Gaussian distributions. We have used the Gaussian distribution to generate Haar uniform random pure states. We also used the Gaussian distribution in the process of insertion of disorder in those states. Beside Gaussian disorder we introduce disorder from uniform and Cauchy-Lorentz distributions as well. The uniform distribution may be represented by the probability density function

$$f_U(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \leq x \leq b, \\ 0, & \text{otherwise}. \end{cases}$$

The mean, standard deviation, and semi-interquartile range for the uniform distribution in terms of $a$ and $b$, that marks the terminal points of the non-trivial part of the uniform distribution, are given respectively by

$$\mu_U = \frac{b+a}{2}, \quad \sigma_U = \frac{b-a}{2\sqrt{3}}, \quad \text{and} \quad \gamma_U = \frac{b-a}{4}. \quad (4)$$
The probability density function of the uniform distribution can be rewritten in terms of its mean and standard deviation as

\[ f_U(x) = \begin{cases} \frac{1}{2\sqrt{3}\sigma_U}, & \text{if } \mu_U - \sqrt{3}\sigma_U \leq x \leq \mu_U + \sqrt{3}\sigma_U, \\ 0, & \text{otherwise}. \end{cases} \] (5)

The Cauchy-Lorentz probability density function is given by

\[ f_{C-L}(x|x_0, \gamma_{C-L}) = \frac{\gamma_{C-L}}{\pi \left[ \gamma_{C-L}^2 + (x-x_0)^2 \right]}, \] (6)

where \( x_0 \) is the median of the distribution and \( \gamma_{C-L} \) is the semi-interquartile range of the distribution. The Cauchy-Lorentz distribution has ‘long tails’, as a result of which the mean and variance do not exist. The Cauchy principal value of the mean is well-defined, and is equal to the median of the distribution. One may use the median as a measure of the central tendency of this distribution. In the absence of the standard deviation, the semi-interquartile range may be used as a measure of dispersion of the distribution. A useful function is the cumulative distribution function of the Cauchy-Lorentz distribution, given by

\[ F_{C-L}(x|x_0, \gamma_{C-L}) = \int_{-\infty}^{x} f_{C-L}(x'|x_0, \gamma_{C-L})dx' = \frac{1}{\pi} \tan^{-1} \left( \frac{x-x_0}{\gamma_{C-L}} \right) + \frac{1}{2}. \] (7)

The quantile function or the inverse cumulative distribution function is given by

\[ x = x_0 + \gamma_{C-L} \tan \left[ \pi \left( F_{C-L} - \frac{1}{2} \right) \right]. \] (8)

A random number from the Cauchy-Lorentz distribution is obtained using this quantile function when \( F_{C-L} \) is randomly chosen from a uniform distribution in the range 0 to 1.

3. Average quantum coherence of typical pure states

The phenomenon of quantum coherence emerges due to the ‘wave property’ of quantum entities. Quantum coherence describes the ability of a quantum system to maintain the same phase relationship between its components over time [55–57]. This can be compared with entanglement, that manifests itself in shared states of two or more quantum systems. When systems become entangled, their individual properties, such as spin or polarization, become intertwined so that the state of one particle cannot be described independently of the other particles [8, 58, 59]. A measure of entanglement for states of a quantum system is dependent on the way the whole system is partitioned into, and in parallel, a measure of quantum coherence is basis-dependent and related to the off-diagonal elements of the density matrix in the selected (chosen) basis. These off-diagonal elements capture the phase relationships between different components of the quantum system (in the chosen basis). By examining the off-diagonal elements, we can obtain information about the extent of coherence in the system [57, 60, 61].

The \( l_1 \)-norm of quantum coherence is a computable quantum coherence measure, and is directly related to the off-diagonal elements of the density matrix, expressed in the basis in which the quantum coherence is being measured. It is the sum of the absolute values of the off-diagonal elements of the density matrix \( \rho \) [8, 58, 62–65]:

\[ C_1(\rho) = \sum_{i\neq j} |\rho_{ij}| \] (9)

where \( \rho_{ij} \) is the (i,j)th matrix element of the state \( \rho \) in the computational basis. In our study, we use the \( l_1 \)-norm of quantum coherence to measure quantum coherence in the computational basis.

We analytically calculate the average \( l_1 \)-norm of typical qubits and qudits exploiting the spherical symmetry of Haar uniform random states. A Haar-random qubit in computational basis can be written as

\[ |q_2\rangle = (c_{11} + ic_{21}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (c_{12} + ic_{22}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \] (10)

The random coefficients \( c_{11}, c_{21}, c_{12}, c_{22} \) are chosen independently from a Gaussian distribution of mean zero and finite variance, and the state must then be normalized. Thus the qubits are distributed Haar uniformly
over the surface of a 4-dimensional unit hypersphere with $c_{11}, c_{21}, c_{12}, c_{22}$ being orthogonal axes. The corresponding $l_1$-norm of the qubit is

$$C_{l_1}(|q_2\rangle) = 2\sqrt{(c_{11}c_{12} + c_{21}c_{22})^2 + (c_{21}c_{12} - c_{11}c_{22})^2}.$$ 

We transform the equation to unit polar coordinates using the following transformations:

$$c_{11} = \cos \theta_1$$
$$c_{21} = \sin \theta_1 \cos \theta_2$$
$$c_{12} = \sin \theta_1 \sin \theta_2 \cos \theta_3$$
$$c_{22} = \sin \theta_1 \sin \theta_2 \sin \theta_3. \quad (11)$$

The surface element in this coordinate system is $ds_4 = \sin^2 \theta_1 \sin \theta_2 d\theta_1 d\theta_2 d\theta_3 d\theta_4$. Rewriting the $l_1$-norm in the unit polar coordinates and simplifying we get

$$C_{l_1}(|q_2\rangle) = 2\sin \theta_1 \sin \theta_2 \sqrt{1 - \sin^2 \theta_1 \sin^2 \theta_2}. \quad (12)$$

Therefore the average $l_1$-norm of the Haar uniformly distributed qubits is

$$\overline{C_{l_1}}(|q_2\rangle) = \frac{\int_{\theta_1=0}^{\pi/2} \int_{\theta_2=0}^{\pi/2} \int_{\theta_3=0}^{\pi/2} \{C_{l_1}(|q_2\rangle)\} ds_4}{\int_{\theta_1=0}^{\pi/2} \int_{\theta_2=0}^{\pi/2} \int_{\theta_3=0}^{\pi/2} ds_4} = \frac{\pi}{4}. \quad (13)$$

We choose the integration limits of $\theta_1, \theta_2$, and $\theta_3$ from 0 to $\pi/2$ to avoid keeping track of the signs in the sines and cosines in a larger interval. As the distribution of the qubits is uniform over the surface of the 4-dimensional unit hypersphere, the averaging over a part of the hypersphere where sines and cosines are positive and the same over the whole hypersphere are equivalent. We have also performed the calculations for higher-dimensional qudits. Note that it has been analytically shown in reference [63], by following an elegant mathematical procedure, that $\overline{C_{l_1}}(|q_d\rangle) = (d-1)\frac{\pi}{d}$. Our method involves simpler mathematics and is useful in the case of redits also, which will be discussed shortly afterwards. The average $l_1$-norm of quantum coherence for higher-dimensional qudits can be calculated in closed form using the same procedure as discussed above. We need $(2d-1)$ angles in the unit polar transformations, and it is a $(2d-1)$-variable integration problem. For any arbitrary dimension, $d$, the average $l_1$-norm of qudits is given by

$$\overline{C_{l_1}}(|q_d\rangle) = \frac{\int_{\theta_1=0}^{\pi/2} \int_{\theta_2=0}^{\pi/2} \cdots \int_{\theta_{2d-1}=0}^{\pi/2} \{C_{l_1}(|r_d\rangle)\} ds_{2d}}{\int_{\theta_1=0}^{\pi/2} \int_{\theta_2=0}^{\pi/2} \cdots \int_{\theta_{2d-1}=0}^{\pi/2} ds_{2d}}. \quad (14)$$

The above integration has been done analytically up to dimension $d = 3$. The result for average $l_1$-norm can be checked in even higher dimensions using numerically simulated Haar-uniform qudits, which we have checked until $d = 10$, and we propose that for a qudit, $\overline{C_{l_1}}(|q_d\rangle) = (d-1)\frac{\pi}{d}$. 

Real bits or rebits are the basic ingredients of the two-dimensional real vector space quantum mechanics [66]. It has been shown that quantum computation can be performed using rebits and gates that respectively belong to and act on real vector spaces [67–69]. The entanglement properties of rebits have also been investigated [48, 70, 71]. Recently, the entanglement of two rebits has been investigated in laboratories [72, 73]. Redits are qudits with real coefficients. A real vector space is a subset of the corresponding complex vector space, so that redits are a subset of qudits. We follow the same procedure to calculate the average typical $l_1$-norm for redits. A redit is represented by

$$|r_d\rangle = \sum_{j=1}^{d} c_j|j\rangle, \quad (15)$$

where $c_j$ are real numbers and $|j\rangle$ represents the $j$th orthonormal basis vector in the computational basis of the $d$-dimensional real Hilbert space. A Haar-random rebit in the computational basis may be written as

$$|r_2\rangle = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (16)$$
where \( c_1 \) and \( c_2 \) are random real numbers. The random coefficients \( c_1 \) and \( c_2 \) are chosen independently from a Gaussian distribution of mean zero, and then the state is normalized. The Haar-uniformly chosen rebits are uniformly distributed over the circumference of the unit circle. The \( l_1 \)-norm of quantum coherence of the rebit is given by \( C_{l_1}(|r_2\rangle) = 2|c_1c_2|\). In the polar coordinate system, with \( c_1 = \cos \theta \) and \( c_2 = \sin \theta \), the \( l_1 \)-norm becomes

\[
C_{l_1}(|r_2\rangle) = 2|\cos \theta \sin \theta| = |\sin 2\theta|.
\] (17)

The average of the \( l_1 \)-norms of quantum coherence of these Haar uniformly distributed rebits is given by

\[
\overline{C_{l_1}}(|r_2\rangle) = \frac{\int_0^{\pi/2} \sin 2\theta d\theta}{\int_0^{\pi/2} d\theta} = \frac{2}{\pi}.
\] (18)

Following a similar procedure, we find \( C_{l_1}(|r_3\rangle) = \frac{4}{\pi} \) and \( C_{l_1}(|r_4\rangle) = \frac{5}{\pi} \). For arbitrary dimension ‘\( d \)’, similar to the previous cases, the average \( l_1 \)-norm of quantum coherence of Haar uniformly distributed \( d \)-dimensional redits is given by

\[
\overline{C_{l_1}}(|r_d\rangle) = \frac{\int_0^{\pi/2} \ldots \int_0^{\pi/2} \{C_{l_1}(|r_d\rangle)\} ds_d}{\int_0^{\pi/2} \ldots \int_0^{\pi/2} ds_d}.
\] (19)

The above integration has been done numerically up to dimension \( d = 6 \). The result for average \( l_1 \)-norm can be checked in even higher dimensions using numerically simulated Haar-uniform states, and this we have checked until \( d = 10 \), and we propose that for a redit, \( \overline{C_{l_1}}(|r_d\rangle) = (d - 1) \frac{2}{\pi} \). The Haar-uniform simulation is discussed in section 2, and the numerical integration is performed using Gaussian quadratures.

4. Distribution of quantum coherence of typical pure states

In the preceding section, we have calculated the average \( l_1 \)-norms of quantum coherence for typical redits and qudits. The maximum values of these \( l_1 \)-norms (for redits and qudits) depend on the dimension \( d \) of the vector space, and is given by \( d - 1 \). In this section, we present a comparative study of average quantum coherences of typical pure states for different dimensions. For the comparative analysis, we modify the definition of the \( l_1 \)-norm so that for every dimension, it lies between zero and unity. The modified \( l_1 \)-norm of quantum coherence of a density matrix \( \rho_d \), acting on a \( d \)-dimensional Hilbert space, is therefore

\[
C_{l_1}^{\text{m}}(\rho_d) = \frac{1}{d - 1} \sum_{i \neq j} |\rho_{dij}| = \frac{C_{l_1}(\rho_d)}{d - 1},
\] (20)

where \( \rho_{dij} \) is the \((i,j)\)th matrix element of \( \rho_d \) in the computational basis.

After this normalization of the \( l_1 \)-norm of quantum coherence the average of the modified \( l_1 \)-norm \( \langle C_{l_1}^{\text{m}}(\rho_d) \rangle \) of typical qudits is \( \frac{4}{\pi} \approx 0.785 \) and of typical redits is \( \frac{5}{\pi} \approx 0.637 \) for any dimension. The numerically calculated mean values of typical qudits and typical redits as written in the insets of figures 1 and 3 are in agreement.

An arbitrary qudit pure state is given by equation (1). It can be Haar-uniformly generated by randomly choosing the real parameters in the coefficients independently from a Gaussian distribution of mean \( \mu_G = 0 \) and a finite standard deviation, followed by a normalization \( [52–54, 65, 74–77] \). Here we draw the random numbers independently from the Gaussian distribution with mean \( \mu_G = 0 \) and standard deviation \( \sigma_G = 1 \).

10⁶ such random pure states are generated, and the modified \( l_1 \)-norm (equation (20)) of each state is calculated. Thus, a quantum coherence distribution for random qudit pure states is obtained, ranging from 0 to 1. The relative frequency percentages of the distribution are plotted in figure 1.

The average value of the modified \( l_1 \)-norm, as obtained in our numerics, remains fixed at \( \approx 0.785 \) for every dimension, which is nearly equal to the analytical result \( \frac{4}{\pi} \), given in section 3. The distribution however hides further information. We quantify the spread of typical quantum coherences for different dimensions by the corresponding standard deviations and skewnesses.

Skewness is a statistical measure that indicates the degree of asymmetry or lack of symmetry in a probability distribution. In other words, it quantifies the extent to which a dataset is skewed or distorted.
from a symmetric distribution. If a distribution is symmetric, it means that the data values are evenly distributed around the mean. However, if a distribution is skewed, it means that the data values are more concentrated on one side of the mean than the other. Skewness can take on positive or negative values, or it can be zero. A positive skewness value indicates that the data is skewed to the right, meaning that the tail of the distribution extends further to the right than the left. A negative skewness value indicates that the data is skewed to the left, meaning that the tail of the distribution extends further to the left than the right. A skewness value of zero indicates that the data is symmetric. For a data set with data points, \( \{y_1, y_2, \ldots, y_N\} \), the skewness \( s \) is defined as

\[
s = \frac{\sum_{i=1}^{N} (y_i - \mu)^3}{N\sigma^3},
\]

where \( \mu \) and \( \sigma \) are the mean and the standard deviation of the data set [78].

We find that on increasing the dimension of the Hilbert space, the standard deviations and left skewnesses of the relative frequency percentage plots of figure 1, i.e. of the spread of typical quantum coherence as measured by the modified \( l_1 \)-norm of quantum coherence, decrease exponentially, as shown in figure 2. Note that the plot of skewness against \( l_1 \)-norm is identical with the plot of skewness against the modified \( l_1 \)-norm. Also note that the minimum value of skewness is zero, which corresponds to a symmetric distribution, and the positive (negative) sign corresponds to right (left) skewness. In the bottom panel of figure 2, the numeric value of skewness is negative and exponentially increasing with the dimension of the system, i.e. left skewness is decreasing.

Relative frequency percentages of redits for different dimensions are plotted in figure 3. These curves are similar to the corresponding curves for qudits. The typical average value of the modified \( l_1 \)-norm remains fixed at \( \approx 0.637 \) for every dimension, which is nearly equal to the value \( \frac{\pi}{4} \), found in section 3. Similar to the case of qudits, here, in the case of redits also, the standard deviation and left skewness decrease exponentially with increasing dimension of the Hilbert space, as depicted in figure 4. The equations of the exponential curves, for both qudits and redits, are given in the corresponding figure captions.

We therefore find that the distribution of typical normalized quantum coherence of qudits as well as redits become more and more concentrated around their average values as we go towards higher dimensional systems. The distributions also become more and more symmetric with increasing dimension, from being left-skewed in low dimensional systems. These observations are in agreement with those reported in [63], wherein it has been shown that the standard deviation of the distribution of the typical modified or normalized \( l_1 \)-norm concentrates around its mean value and goes to zero for infinite dimension. We find that the standard deviation of the \( l_1 \)-norm quantum coherence distributions of qudits as well as redits increases as a function of \( \sqrt{d} \) while the mean of the same distributions increases as a function of \( d \) (refer to figure 5) implying that the standard deviation of the typical normalized \( l_1 \)-norm distributions indeed concentrate around its mean and the standard deviation goes to zero as \( d \) goes to infinity for qudits as well as redits.

For the normalized \( l_1 \)-norm, the mean values are constant at \( \frac{\pi}{4} \approx 0.785 \) and \( \frac{\pi}{3} \approx 0.637 \) for qudits and redits respectively irrespective of dimension as shown in the inset of figure 1 and 3 respectively, and standard
deviation values decrease exponentially with increase in dimension as shown in figures 2(a) and 4(a) respectively for qudits and redits. The normalized or the modified $l_1$-norm is essential to compare the quantum coherence distributions of different dimensional qudits and redits.

5. Effect of disorder on the spread of quantum coherence of typical pure states

We now investigate the response of the distribution of quantum coherence for typical states on introduction of disorder in the parameters of the states. We introduce disorder in real parts of the coefficients $c_{1j}$, using Gaussian, uniform, or Cauchy-Lorentz distributions. We begin with the case of Gaussian disorder.

In the first step, we generate a million Haar-random pure qudits. In the second step, one hundred random pure states are generated for every state generated in the first step. It is in the second step that disorder is inserted. Each state $\sum_{j=1}^{d} (\tilde{c}_{1j} + i\tilde{c}_{2j})|j\rangle$ in the second step is created by randomly choosing numbers, $\tilde{c}_{1j}$, which are Gaussian distributed with mean $\mu_{G} = c_{1j}$, where $c_{1j}$ is the random number generated in the first
Figure 4. Variation with dimension of spread and asymmetry of modified $l_1$-norm distributions of typical redits. We plot the spread, as quantified by standard deviation ($\sigma_d$), and asymmetry, as quantified by skewness ($s_d$), of the different typical modified $l_1$-norm distributions for different dimensions, against the dimension ($d$). The curves are again fitted with exponentials, $f(d : \alpha, \beta, \gamma) = \alpha e^{-\beta d} + \gamma$, although the parameters are different, and so in (a), here, the fitting function is $\sigma_d = f(d : \alpha, \beta, \gamma)$, with $\alpha = 0.586, \beta = 0.591, \gamma = 0.127$. The corresponding goodness of fit is $8.38 \times 10^{-6}$. And in (b), the fitting function is $s_d = f(d : \alpha, \beta, \gamma)$, with $\alpha = -0.923, \beta = 0.697, \gamma = -0.267$. The goodness of fit is $2.02 \times 10^{-5}$. The inset plots are in the log-log scale. All quantities used are dimensionless.

Figure 5. Means (a) and standard deviations (b) of the relative frequency percentage plots using un-normalized $l_1$-norm are plotted against the dimension for qudits and redits. We fit a curve which is proportional to the square root of the dimension: $f(d : a, b) = a \sqrt{d} + b$, through the standard deviations of the relative frequency percentages plots using un-normalized $l_1$-norm for qudits and redits, although the parameters are different for qudits and redits. For qudits, $a = 0.282, b = -0.170$, and for redits, $a = 0.397, b = -0.247$.

step, and semi-interquartile-range $\gamma_{G} = 1/2$. It may be noted that $\gamma_{G} = 1/2$ corresponds to a standard deviation of $\approx 0.741$. The Gaussian distribution needed here to insert disorder should not be confused with the Gaussian distribution required in the context of Haar-uniform generation of states. The standard deviation and semi-interquartile range of the Gaussian distribution used here quantifies the strength of the error or disorder inserted. The $c_{ij}$ in the second step are the same as those selected in the first step of Haar uniform pure state generation after the normalization in the first step. The random pure states are normalized and their average quantum coherence calculated. We end up with another set of one million disorder averaged quantum coherence values. We plot the corresponding relative frequency percentages for the $c_{ij}$ for qubits, as red dots in figure 6. For a comparative study between different distributions of disorder,
we fix the semi-interquartile ranges of Gaussian, uniform, and Cauchy-Lorentz distributions at $\gamma_G = \gamma_U = \gamma_{C-L} = \frac{1}{2}$, while inserting disorders. The type of disorder we consider here is often referred to as ‘quenched disorder’ in the literature (see e.g. [42–47]). The averaging needs to be performed after all physical quantities for given realizations of the disorder have been calculated. Such an averaging has often been referred to as ‘quenched averaging’ in the literature (see e.g. [79–81]).

We see from figure 6 that the relative frequency percentage plot changes significantly with the introduction of disorder. The mean $C_{l_1}$ does not change much except when the disorder distribution is Cauchy-Lorentz. The spread of the plot is reduced significantly—the standard deviation decreases to about a third of its ordered case value—for every kind of disorder considered. The left skewnesses of the plots are also reduced, so that the disorder-averaged plots are more symmetric than the ordered one. The reduction in standard deviation is minimum for disorder from uniform distribution and maximum for the same from Cauchy-Lorentz distribution. The reduction in left skewness is minimum in the case of disorder from Cauchy-Lorentz distribution and maximum in the case of the same from Gaussian distribution.

Figure 7 shows the effect of Gaussian disorder on the relative frequency percentages of quantum coherence of qubits when we introduce the disorder in the imaginary parts of the coefficients. The red dots in figure 7 represent the relative frequency percentages of states when we introduce Gaussian disorder ($\gamma_G = \frac{1}{2}$) in the imaginary parts of the coefficients only. Note that the red dots of figure 6 completely overlap with the red dots of this figure and has therefore been omitted here. The relative frequency percentage distribution is identical, as intuitively expected, to the case when we introduce Gaussian disorder ($\gamma_G = \frac{1}{2}$) in the real parts of the coefficients only, vide the red dots in figure 6. The relative frequency percentage distribution represented by the blue dots in figure 7, where we insert Gaussian disorder ($\gamma_G = \frac{1}{2}$) in both the real and imaginary parts of the coefficients, is also qualitatively similar to the plots where we introduce Gaussian disorder in the real or the imaginary parts only i.e. the standard deviation of the distribution is drastically reduced. In the rest of the paper, we analyze the effect of disorder by inserting them in the real parts of the coefficients only.

Figure 8 shows the effect of variation of the strength of Gaussian disorder on the relative frequency percentages of quantum coherence of qubits. We find that the spreads of the distributions reduce with the increase of the strength of the disorder. Figure 9 depicts the plot of standard deviation against the strength of Gaussian disorder data from figure 8. A rational polynomial has been fitted to the data which is known to aid extrapolation. We find that $\sigma = 0.01647$ as $\gamma \to \infty$. It appears that although the average quantum coherence of typical Haar random states decreases marginally with increasing disorder, the quantum coherence of the individual random states huddle very close to the average value.

We conclude that the spread of quantum coherence of typical quantum states is inhibited, irrespective of the type and strength of the disorder distributions considered in this study. However, the amount of inhibition depends on the type and strength of disorder. To understand the reason for the inhibition of

![Figure 6. Inhibition of spread of quantum coherence of typical qubits in response to disorder. We plot here the relative frequency percentages of Haar-uniformly chosen pure qubits, with and without disorder. The disorder, whenever present, is in the real parts of all the coefficients of the states in the computational basis. This relative frequency is plotted against the $l_1$-norm values, with the latter lying between 0 and 1. The black asterisks correspond to percentages of random pure qubits generated Haar uniformly, and without any disorder inflicted. The other three curves depict quantum coherence distributions for random pure qubits with the disorder chosen from Gaussian (G, red dots), uniform (U, green crosses), and Cauchy-Lorentz (C−L, blue pluses) distributions. The disorder-averaged values plotted in the figure are for 100 disorder configurations for every ordered one. The disorder-averaged curves do not change up to the precision used for averaging over 50 configurations. The initial Haar-uniform generation used for the figure utilizes $10^9$ states. However, the same plot does not alter, up to the precision used, for $10^7$ states. The precision is checked to three significant figures. The skewnesses for the ordered case and the disordered cases from uniform, Gaussian, and Cauchy-Lorentz distributions are $s = -1.15, s_G = -0.177, s_C = 0.0161$, and $s_{C-L} = -0.225$ respectively. The means ($\mu_s$) and standard deviations ($\sigma_s$) for the curves are given in the legend. All quantities used are dimensionless.](image)
Figure 7. Inhibition of spread of quantum coherence of typical qubits when the disorders are inserted in the imaginary parts of the coefficients. The considerations are the same as in figure 6, except we consider Gaussian disorder in the imaginary parts only of all the coefficients in the red curve and Gaussian disorder in both real and imaginary parts of all the coefficients in the blue curve. The subscripts ‘im’ and ‘b’ are used in the inset to denote the disorder in imaginary parts and both parts, respectively. Note that the red dots of figure 6 completely overlap with the red dots of this figure and has therefore been omitted here.

Figure 8. Effect of variation of the strength of Gaussian disorder. The considerations are the same as in figure 6, except that only the Gaussian disorder is considered. Different semi-interquartile ranges are used with different colored symbols for the plots. Black asterisks are used for the case when there is no disorder, and is also present in figure 6. The curve with red dots is for $\gamma_G = 1/2$ and again was also present in figure 6. The deep-blue, blue, green, orange, magenta, brown, and lime dots are for $\gamma_G = 0.1, 0.2, 0.3, 0.4, 1.0, 1.5, 2.0$ respectively. Their skewnesses are $s_{0.1} = -0.000, s_{0.2} = -0.787, s_{0.3} = -0.462, s_{0.4} = -0.158, s_{1.0} = -0.0655, s_{1.5} = -0.202, and s_{2.0} = -0.219$ respectively. All quantities are dimensionless.

Figure 9. Rational polynomial function fit through the standard deviation versus semi-interquartile range data of figure 8. The fitting function is $\sigma = p_1 \gamma + p_2 \gamma^2 + q_1$, with $p_1 = 0.01647, p_2 = 0.0417$, and $q_1 = 1191$. The corresponding goodness of fit is $6.11 \times 10^{-5}$. Note that $\sigma = p_1 = 0.01647$ as $\gamma \to \infty$.

spread, we investigate further and introduce disorder in the same coefficients as before, but in qubit states with specific values of quantum coherence, $M_{in}$, and look at the disorder-averaged quantum coherence, $M_f$. For the analysis, we will need a window of quantum coherence for the input states, and we arbitrarily choose it to be $(M_{in} - .01, M_{in} + .01)$. It can be seen from figure 10 that the ‘final’ (i.e. disorder-averaged) quantum coherence $M_f$ is greater than the ‘initial’ value of quantum coherence $M_{in}$ when $M_{in} < \frac{\pi}{4}$, whereas the order is inverted when $M_{in} > \frac{\pi}{4}$. Note that the average quantum coherence of typical qubit states, without any disorder, is $\pi/4$. Therefore, it appears that as we perturb a quantum state which has a certain value of $C_{li}$, it has a greater probability to transform into a state with higher or lower $C_{li}$, depending on whether the parent
Response to disorder inserted in states of fixed quantum coherence. We analyze here the dependence of the disorder-averaged quantum coherence $M_f$ in response to Gaussian disorder, in the coefficients, on the initial quantum coherence $M_{in}$. We generate $x = 10^7$ Haar-uniform pure qubit states and choose all that have a quantum coherence (in computational basis) within the window $(M_{in} - 0.01, M_{in} + 0.01)$. For every such state, we generate $y = 100$ disordered states by randomly choosing the real parts of the coefficients in the computational basis from a Gaussian distribution with mean equal to the input coefficients and $\gamma_G = 1/2$, and calculate the quantum coherence in the computational basis of the states. We then plot the relative frequency percentages of the states against the quantum coherences, with a window of 0.025. The red asterisks, orange dots, green crosses, blue rhombuses, and black squares respectively represent the cases for $M_{in} = 0.95, 0.85, 0.75, 0.65, 0.55$. The values written in the upper left corner are correct up to two significant figures. All quantities are dimensionless. To check convergence of the percentages obtained, we re-do the calculations with $x = 10^8$, and do not find any appreciable change. Similarly, we re-do the calculations for $y = 50$, and do not find a significant difference of the numbers within the precision considered.

Inhibition of spread of quantum coherence of typical qutrits in response to disorder. We plot here the relative frequency percentages of Haar-uniformly chosen pure qutrits, with and without disorder. The disorder, whenever present, is in the real parts of all the coefficients of the states in the computational basis. This relative frequency is plotted against the $l_1$-norm values, with the latter lying between 0 and 2. The black asterisks correspond to percentages of random pure qubits generated Haar uniformly, and without any disorder inflicted. The other three curves depict quantum coherence distributions for random pure qubits with the disorder chosen from Gaussian ($G$, red dots), uniform ($U$, green crosses), and Cauchy-Lorentz ($C-L$, blue plus) distributions. The disorder-averaged values plotted in the figure are for 100 disorder configurations for every ordered one. The disorder-averaged curves does not change up to the precision used for averaging over 50 configurations. The initial Haar-uniform generation used for the figure utilizes $10^6$ states. However, the same plot does not alter, up to the precision used, for $10^7$ states. The precision is checked to three significant figures. The skewnesses for the ordered case and the disordered cases from uniform, Gaussian, and Cauchy-Lorentz distributions are $s = -0.865, s_U = 0.0479, s_G = -0.0139, s_{C-L} = -0.0437$ respectively.

state had a value of $C_{l_1}$ that was lower or higher than the average $C_{l_1}$ of Haar uniformly distributed states in the ordered case. Note that the shift of average quantum coherence, $|M_f - M_{in}|$, increases with $|M_{in} - \frac{\pi}{4}|$.

The effect of disorder on the quantum coherence spread of typical quantum states of higher dimensions is similar to its impact on qubits. Figures 11 and 12 show the effect of disorder on the quantum coherence spread of typical qutrits and typical four-dimensional quantum states.

For completeness, we have also studied the effect of disorder on quantum coherence of typical redits. We again notice that disorder inhibits its spread in any dimension. Figures 13 and 14 show the inhibition of spread of quantum coherence distributions of typical rebits and retrits for disorders introduced from Gaussian, uniform, and Cauchy-Lorentz distributions with semi-interquartile ranges $\gamma_{G} = 17\mu = \gamma_{C-L} = \frac{1}{7}$ respectively. The qudits and redits are vectors in different vector spaces, and therefore, the behavior of quantum coherences in the two spaces is expected to differ. They are so, but there is significant qualitative similarity. The rates of reduction of the spreads of the distributions of typical $l_1$-norms of quantum coherence for qudits and redits have similar behavior but are not the same.
Figure 12. Inhibition of spread of quantum coherence of typical four-dimensional quantum states in response to disorder. We plot here the relative frequency percentages of Haar-uniformly chosen pure four-dimensional qudits, with and without disorder. The disorder, whenever present, is in the real parts of all the coefficients of the states in the computational basis. This relative frequency is plotted against the $l_1$-norm values, with the latter lying between 0 and 3. The black asterisks correspond to percentages of random pure qubits generated Haar uniformly, and without any disorder inflicted. The other three curves depict quantum coherence distributions for random pure qudits with the disorder chosen from Gaussian ($G$, red dots), uniform ($U$, green crosses), and Cauchy-Lorentz ($C-L$, blue pluses) distributions. The disorder-averaged values plotted in the figure are for 100 disorder configurations for every ordered one. The disorder-averaged curves do not change up to the precision used for averaging over 50 configurations. The initial Haar-uniform generation used for the figure utilizes $10^6$ states. However, the same plot does not alter, up to the precision used, for $10^7$ states. The precision is checked to three significant figures. The skewnesses for the ordered case and the disordered cases from uniform, Gaussian, and Cauchy-Lorentz distributions are $s = -0.737, s_U = -0.0316, s_G = -0.0589, s_{C-L} = 0.0227$ respectively.

Figure 13. Inhibition of spread of quantum coherence of typical rebits in response to disorder. We plot here the relative frequency percentages of Haar-uniformly chosen pure rebits, with and without disorder. This relative frequency is plotted against the $l_1$-norm values, with the latter lying between 0 and 1. The black asterisks correspond to percentages of random pure qudits generated Haar uniformly, and without any disorder inflicted. The other three curves depict quantum coherence distributions for random pure qudits with the disorder chosen from Gaussian ($G$, red dots), uniform ($U$, green crosses), and Cauchy-Lorentz ($C-L$, blue pluses) distributions. The disorder-averaged values plotted in the figure are for 100 disorder configurations for every ordered one. The disorder-averaged curves do not change up to the precision used for averaging over 50 configurations. The initial Haar-uniform generation used for the figure utilizes $10^6$ states. However, the same plot does not alter, up to the precision used, for $10^7$ states. The precision is checked to three significant figures. The skewnesses for the ordered case and the disordered cases from uniform, Gaussian, and Cauchy-Lorentz distributions are $s = -0.498, s_U = -0.0564, s_G = -0.0338, s_{C-L} = -0.0337$ respectively.

Figure 14. Inhibition of spread of quantum coherence of typical retrits in response to disorder. We plot here the relative frequency percentages of Haar-uniformly chosen pure retrits, with and without disorder. This relative frequency is plotted against the $l_1$-norm values, with the latter lying between 0 and 2. The black asterisks correspond to percentages of random pure qudits generated Haar uniformly, and without any disorder inflicted. The other three curves depict quantum coherence distributions for random pure qudits with the disorder chosen from Gaussian ($G$, red dots), uniform ($U$, green crosses), and Cauchy-Lorentz ($C-L$, blue pluses) distributions. The disorder-averaged values plotted in the figure are for 100 disorder configurations for every ordered one. The disorder-averaged curves do not change up to the precision used for averaging over 50 configurations. The initial Haar-uniform generation used for the figure utilizes $10^6$ states. However, the same plot does not alter, up to the precision used, for $10^7$ states. The precision is checked to three significant figures. The skewnesses for the ordered case and the disordered cases from uniform, Gaussian, and Cauchy-Lorentz distributions are $s = -0.375, s_U = -0.0292, s_G = -0.0666, s_{C-L} = -0.0487$ respectively.
6. Conclusion

We analyze the average quantum coherence (as quantified by the $l_1$-norm of quantum coherence) of typical vectors of real- and complex-field Hilbert spaces (‘redits’ and ‘qudits’), and its response to disorder infusion. Along with the average quantum coherence of typical states, we have also studied the relative frequency distribution of quantum coherence for qudits and redits. We found that with increasing dimension, the distributions have less relative spread and become more symmetric. More precisely, the standard deviations and left skewnesses of the relative frequency distributions of the modified $l_1$-norm of qudits and redits were found to decrease exponentially with increase in dimension. Moreover, we found that introduction of disorder in the state parameters inhibits the spread of the relative frequency distributions of quantum coherence. One would intuitively expect that disorder would increase the spread of the quantum coherence distribution. But we find that disorder actually reduces the spread of the quantum coherence distributions. The more substantial the perturbation, stronger the reduction of the standard deviation. We observed that when perturbed, there is a significant probability for a state with a quantum coherence lower (higher) than the average quantum coherence (of the ordered case) to jump to one with more (less) quantum coherence than in the parent state. Here, the average quantum coherence is the mean $l_1$-norm of Haar uniformly distributed states in the entire relevant Hilbert space, which is $(d-1)\pi/4$ for qudits and $(d-1)2\pi$ for redits. This phenomenon results in a clustering effect which forces the disorder inflicted system to have a low standard deviation in the quantum coherence distribution. We claim that this feature of inhibition of the spread of typical quantum coherence is generic, irrespective of the strength and type of disorder and dimension of the vector space. We wish to prove that the results obtained in the manuscript are actual for arbitrary types of disorder distributions. Unfortunately, however, we were unable to verify that statement. Therefore, we used the Gaussian and uniform distributions as examples, as these are widely used in the community to consider disorder. We added the Cauchy-Lorentz distribution to our ‘list’, as this is in certain ways very different from Gaussian and uniform distributions. In particular, it does not have a well-defined mean (although it does have a Cauchy mean) and standard deviation. The results support our claim that inhibition in response to disorder is a generic feature for typical pure quantum states. Moreover, introduction of disorder results in the quantum coherence distribution to become more symmetric. We note that interaction of the systems with their environments could be modeled differently to what is done here, and in future works, to study the response of quantum coherence distributions to environmental disturbances, we intend to draw more definitive conclusions regarding the effect of environmental perturbations on the systems considered by considering more generic models.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgment

S S acknowledges support from the Council of Scientific and Industrial Research through the grant (Award Number: 09/1356/(11432)/2021-EMR-I). U S acknowledges partial support from the Department of Science and Technology, Government of India through the QuEST grant (Grant Number DST/ICPS/QUST/Theme-3/2019/120).

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