Error estimates for harmonic and biharmonic interpolation splines with annular geometry

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Abstract

The main result in this paper is an error estimate for interpolation biharmonic polysplines in an annulus \( A(r_1, r_N) \), with respect to a partition by concentric annular domains \( A(r_1, r_2), \ldots, A(r_{N-1}, r_N) \), for radii \( 0 < r_1 < \ldots < r_N \). The biharmonic polysplines interpolate a smooth function on the spheres \( |x| = r_j \) for \( j = 1, \ldots, N \) and satisfy natural boundary conditions for \( |x| = r_1 \) and \( |x| = r_N \). By analogy with a technique in one-dimensional spline theory established by C. de Boor, we base our proof on error estimates for harmonic interpolation splines with respect to the partition by the annuli \( A(r_{j-1}, r_j) \). For these estimates it is important to determine the smallest constant \( c_d(\Omega) \), where \( \Omega = A(r_{j-1}, r_j) \), among all constants \( c \) satisfying

\[
\sup_{x \in \Omega} |f(x)| \leq c \sup_{x \in \Omega} |\Delta f(x)|
\]

for all \( f \in C^2(\Omega) \cap C(\overline{\Omega}) \) vanishing on the boundary of the bounded domain \( \Omega \). In this paper we describe \( c_d(\Omega) \) for an annulus \( \Omega = A(r, R) \) and we will give the estimate

\[
\min \left\{ \frac{1}{2d}, \frac{1}{8} \right\} (R - r)^2 \leq c_d(A(r, R)) \leq \max \left\{ \frac{1}{2d}, \frac{1}{8} \right\} (R - r)^2
\]

where \( d \) is the dimension of the underlying space.

Keywords: multidimensional splines; harmonic splines; biharmonic splines; error estimates for interpolation splines; spherical harmonics

1 Introduction

Let \( C(\Omega) \) be the set of all continuous complex-valued functions defined on a subset \( \Omega \) in \( \mathbb{R}^d \). For an open subset \( \Omega \), let \( C^m(\Omega) \) be the set of all functions which have continuous partial derivatives of order \( \leq m \) on \( \Omega \), and for the closure of \( \Omega \), we denote by \( C^m(\overline{\Omega}) \) the set of all functions which have continuous partial derivatives of order \( \leq m \) on \( \overline{\Omega} \).

Recall that a spline of degree \( p \) defined on an interval \([t_1, t_N]\) with nodes \( t_1 < \ldots < t_N \) is a function \( S \in C^{p-1}[t_1, t_N] \) which on each subinterval \((t_j, t_{j+1})\) is identical with some polynomial of degree \( \leq p \) for \( j = 1, \ldots, N - 1 \). If \( p = 3 \) we call \( S \) a cubic spline. For \( p = 1 \) we obtain the definition of a linear spline. Note that \( S \) is a linear spline if and only if \( S \) a continuous function on \([t_1, t_N]\) such that on each subinterval \((t_j, t_{j+1})\) the function is linear, so a solution of the differential operator

\[
\frac{d^2}{dt^2} S(t) = 0.
\]
The aim of the present paper is to provide error estimates for interpolation by special types of multivariate splines, namely harmonic and biharmonic splines. Harmonic splines occur in a natural fashion in mathematical problems, see e.g., the discussion in [27]. Harmonic splines for block partitions have been discussed by various authors in the literature, see [5], [6], [7], [29]. Here a set \( \Omega \) of the form \([a_1, b_1] \times \cdots \times [a_d, b_d]\) is called a block in \( \mathbb{R}^d \), and a harmonic spline is a continuous function on \( \Omega \) which is harmonic on open and disjoint subdomains (called subblocks) \( \Omega_j \) of \( \Omega \) such that the closure of \( \Omega_1 \cup \cdots \cup \Omega_N \) is equal to \( \Omega \). Recall that a function \( f : \Omega \to \mathbb{C} \) is called harmonic if \( f \in C^2(\Omega) \) and
\[
\Delta f(x) := \sum_{j=1}^d \frac{\partial^2 f(x)}{\partial x_j^2} = 0 \quad \text{for all } x \in \Omega, \tag{2}
\]
and \( \Delta \) is the Laplace operator. It is convenient to introduce the following natural generalization: we say that a function \( S \) is a harmonic spline with respect to the partition \( \Omega_1, \ldots, \Omega_N \) if \( S \in C(\Omega) \) for \( \Omega = \Omega_1 \cup \cdots \cup \Omega_N \) and \( S \) is a harmonic function on each subdomain \( \Omega_j \) for \( j = 1, \ldots, N \). Thus a harmonic spline for a partition \( \Omega_1, \ldots, \Omega_N \) is a multivariate generalization of a linear spline.

For the partition \( \Omega_1, \ldots, \Omega_N \), we define harmonic spline interpolation in the following way: Given a function \( F \in C(\Omega) \), we say \( I_2(F) : \Omega \to \mathbb{R} \) is a harmonic spline interpolating the data function \( F \) if \( I_2(F) \) is a harmonic spline for \( \Omega_1, \ldots, \Omega_N \) and it satisfies the interpolation condition
\[
I_2(F)(\xi) = F(\xi) \quad \text{for all } \xi \in \partial \Omega = \partial \Omega_1 \cup \cdots \cup \partial \Omega_N. \tag{3}
\]
Note that we require in (3) an interpolation condition for infinitely many points, and in the literature this is often called transfinite interpolation. Transfinite interpolation was already considered by Gordon and Hall in [25] (cf. [22]) and this concept has found many applications in mesh generation, geometric modelling, finite element methods and spline analysis, see [10], [11], [21], [48], [50].

The existence of a harmonic spline interpolant is easy to establish when we assume that the Dirichlet problem is solvable\(^1\) for each domain \( \Omega_j \) for \( j = 1, \ldots, N \). The uniqueness of the harmonic interpolant is a simple consequence of the maximum principle. The main results in [7], [5], [6], [29] are optimal estimates for the error
\[
E(F) := F - I_2(F)
\]
for a given twice differentiable function \( F \in C^2(\Omega) \) in the case of block partitions with respect to the supremum norm defined by
\[
\|f\|_\Omega = \sup_{x \in \Omega} |f(x)|
\]
for \( f \in C(\Omega) \). In the present paper we shall present an explicit error estimate for harmonic splines for a partition given by annular subdomains of the form
\[
\Omega_j := A(r_j, r_{j+1}) := \{x \in \mathbb{R}^d : r_j < |x| < r_{j+1}\}
\]
\(^1\)The Dirichlet problem is solvable for a domain \( \Omega \) if for each continuous function \( f \) defined on the boundary \( \partial \Omega \) of \( \Omega \) there exists a continuous function \( h \) defined on the closure \( \overline{\Omega} \) of \( \Omega \) which is harmonic in \( \Omega \) and interpolates \( f \) on the boundary, i.e. \( h(\xi) = f(\xi) \) for all \( \xi \in \partial \Omega \).
for given positive radii \( r_1 < ... < r_N \). Indeed, we shall proof the following:

**Theorem 1** Let \( r_1 < ... < r_N \) be positive numbers, and let \( F \in C^2(\overline{\Omega}(r_1, r_N)) \). Assume that \( I_2(F) \) is the harmonic spline interpolating \( F \) for all \( x \) with \( |x| = r_j \) and \( j = 1, ... n \). Then

\[
\| F - I_2(F) \|_{A(r_1, r_N)} \leq C_d \max_{j=1,...,N-1} (r_{j+1} - r_j)^2 \| \Delta^2 F \|_{A(r_1, r_N)} .
\]

where \( d \) is the dimension of the space and \( C_d = \max \{ \frac{1}{t^2}, \frac{1}{r} \} \).

Note that this result is very similar to the error estimate for linear splines (see e.g. [33, p. 31]): Assume that \( S_1(F) \) is the (unique) linear spline interpolating a twice differentiable function \( F : [t_1, t_N] \to \mathbb{R} \) at the points \( t_1, ..., t_N \). Then

\[
\sup_{t \in [t_1, t_N]} |F(t) - S_1(F)(t)| \leq \frac{1}{8} \max_{j=1,...,N-1} |t_{j+1} - t_j|^2 \cdot \sup_{t \in [t_1, t_N]} |F''(t)| .
\]

The most difficult part in Theorem 1 is to establish the explicit nature of the constant \( C_d \) in the estimate (4). Let us also emphasize that Theorem 1 is an important ingredient to establish an error \( L^2 \)-estimate for interpolation with biharmonic splines – the next topic we want to discuss.

The definition of a harmonic spline can be traced back in old sources (see e.g. [27]), and the concept is very intuitive. For the definition of a biharmonic spline we use the approach given in [31] where the explicit description as a piecewise polyharmonic function is used and which emphasizes the analogy to the case of univariate cubic splines. For our purposes it is convenient to use the following definition:

**Definition 2** Let \( \Omega_1, ..., \Omega_N \) be open disjoint sets in \( \mathbb{R}^d \) and define \( \Omega = \Omega_1 \cup ... \cup \Omega_N \). A function \( f : \overline{\Omega} \to \mathbb{C} \) is a biharmonic spline for the partition \( \Omega_1, ..., \Omega_N \) if \( f \in C^2(\overline{\Omega}) \) and the restriction of \( f \) to each \( \Omega_j \) is biharmonic, i.e. \( f \in C^2(\overline{\Omega_j}) \) and

\[
\Delta^2 f(x) := \Delta \circ \Delta f(x) = 0
\]

for all \( x \in \Omega_j \) and for \( j = 1, ..., N \).

In the definition of a biharmonic spline the matching of the boundary behaviour of the biharmonic functions defined on \( \Omega_j \) is simply expressed by the requirement that \( f \) is a \( C^2 \)-function on \( \overline{\Omega} \). This corresponds to the definition of a cubic spline on the interval \([t_1, t_N]\) with nodes \( t_1 < ... < t_N \): it is a function \( S \in C^2([t_1, t_N]) \) which on each subinterval \((t_j, t_{j+1})\) is identical with a solution of the differential equation \( \frac{d^2}{dt^2} S = 0 \) for \( j = 1, ..., N - 1 \).

The existence of an interpolating biharmonic spline requires additional assumptions on the smoothness of the domains \( \Omega_j \) and higher regularity of the data function on the boundary \( \partial \Omega \) of \( \Omega \) which might be expressed in terms of Hölder or Sobolev spaces. On the other hand, we are interested only in error estimates, so at this place we do not need to dwell in the more difficult question of the existence of an interpolation biharmonic spline for a partition \( \Omega_1, ..., \Omega_n \) (see also Section 4 for some comments in the case of annuli, and for the details on the existence of interpolation polysplines consult the monograph [31], chapter 22).
Our main result is the following error estimate where we recall that the $L^p$-norm of a measurable function $f : \Omega \to \mathbb{C}$ is defined by

$$\|f\|_{L^p(\Omega)} = \left( \int_{\Omega} |f(x)|^p \, dx \right)^{1/p}.$$  

**Theorem 3** Let $0 < r_1 < \ldots < r_N$ and let $F \in C^4(\bar{A}(r_1, r_N))$. Assume that $I_4(F)$ is a biharmonic spline for the partition $A(r_1, r_2), \ldots, A(r_N, r_{N-1})$ which satisfies the (transfinite) interpolation conditions

$$I_4(x) = F(x) \text{ for all } x \text{ with } |x| = r_j \text{ and } j = 1, \ldots, N, \quad (7)$$

and the boundary conditions for the normal derivative \( \frac{\partial}{\partial n} \)

$$\frac{\partial I_4(F)}{\partial n}(x) = \frac{\partial F(x)}{\partial n} \text{ for all } |x| = r_1 \text{ and for all } |x| = r_N. \quad (8)$$

Then, with $C_d = \max \{ \frac{1}{24}, \frac{1}{8} \}$ as above, the following estimate holds:

$$\|F - I_4(F)\|_{L^2(\bar{A}(r_1, r_N))} \leq C_d^2 \gamma \max_{j = 1, \ldots, N-1} |r_{j+1} - r_j|^4 \|\Delta^2 F\|_{L^2(\bar{A}(r_1, r_N))}. \quad (9)$$

The above result should be compared with the $L^2$-error estimate of a one-dimensional cubic spline: Assume that $F \in C^4[t_1, t_N]$ and let $S_3(F)$ be a cubic spline interpolating $F$ at the points $t_1, \ldots, t_N$ and satisfying the additional boundary condition

$$\frac{d}{dt} F(t_1) = \frac{d}{dt} S_3(F)(t_1) \text{ and } \frac{d}{dt} F(t_N) = \frac{d}{dt} S_3(F)(t_N).$$

Then

$$\|F - S_3(F)\|_{L^2(t_1, t_N)} \leq \frac{4}{\pi^4} \max_{j = 1, \ldots, N} |t_{j+1} - t_j|^4 \left( \frac{d^4 F}{dt^4} \right)_{t = t_j},$$

see e.g. [4]. Let us also mention that for the supremum norm the following error estimate

$$\max_{t \in [t_1, t_N]} |F(t) - S_3(F)(t)| \leq \frac{1}{16} \max_{j = 1, \ldots, N} |t_{j+1} - t_j|^4 \max_{x \in [t_1, t_N]} \left| \frac{d^4 F}{dt^4}(x) \right| \quad (9)$$

holds, see [13] p. 55. We leave the question open whether in Theorem 3 one may replace the $L^2$-norm by the supremum norm. In passing we mention that in [39] the inequality (9) has been generalized to $L$-splines where $L$ is a differential operator with constant coefficients of order 4.

Let us briefly describe the structure of the paper: in Section 2 we shall discuss error estimate for harmonic interpolation splines with respect to a general partition $\Omega_1, \ldots, \Omega_N$. This problem is closely related to the problem of finding the smallest constant $c_d(\Omega)$ among all constants $c$ which satisfy the inequality

$$\|f\|_{\Omega} \leq c \|\Delta f\|_{\Omega} \quad (10)$$

for all $f \in C(\overline{\Omega}) \cap C(\Omega)$ vanishing on the boundary $\partial \Omega$. In Section 2 we will characterize the constant $c_d(\Omega)$.
Theorem 4 Let $\Omega$ be a bounded regular domain and let $h$ be the solution of the Dirichlet problem for the data function $|x|^2$. Then

$$c_d(\Omega) = \sup_{x \in \Omega} T_0(x) \quad \text{and} \quad T_0(x) = \frac{1}{2d} \left( h(x) - |x|^2 \right). \quad (11)$$

The function $T_0$ is the unique function which vanishes on the boundary of $\Omega$ and satisfies $\Delta T_0 = -1$.

The function $T_0$ plays an eminent role in various areas of mathematics and is called the torsion function, see e.g. the fundamental work of G. Pólya, G. Szegő about isoperimetric inequalities in [14], or the monography [51]. There is a vast literature on this subject with many ramifications and it would take too much space to survey the results, so we only mention a very incomplete list of new references [9], [15], [16], [17], [18]. In Section 2 we shall provide a self-contained proof of Theorem 4 which is based on a Green function approach.

In Section 3 we provide the proof of Theorem 3. It is remarkable that the $L^2$-error estimate in the biharmonic case can be performed by an iterative argument where the $L^2$-error estimate for harmonic splines is used twice.

In Section 4 we present a proof of an orthogonality relation for biharmonic interpolation splines which is used in Section 3.

In Section 5 we provide a computation of the best constant $c_d(\Omega)$ for the annular domain $A(r, R) = \{x \in \mathbb{R}^d : r < |x| < R\}$ and we prove the following inequalities:

$$\min \left\{ \frac{1}{2d}, \frac{1}{8} \right\} (R - r)^2 \leq c_d(A(r, R)) \leq \max \left\{ \frac{1}{2d}, \frac{1}{8} \right\} (R - r)^2.$$

Finally, let us mention that some of the presented concepts can be generalized. Recall that a function $f : \Omega \to \mathbb{C}$ is called polyharmonic of order $p$ if $f \in C^{2p}(\Omega)$ and

$$\Delta^p f(x) = 0 \quad \text{for all} \quad x \in \Omega$$

where $\Delta^p$ is the $p$-th iterate of $\Delta$, see [2], [3], [24]. Polyharmonic functions are often used in applied mathematics, see e.g. [8], [23], [53], [30], [37], [32], [47], [49].

Slightly more general than in [31] we define a function $f : \Omega \to \mathbb{C}$ to be polyspline of order $p$ for a partition $\Omega_1, ..., \Omega_N$ if $f \in C^{2p-2} (\Omega)$ for $\Omega = \Omega_1 \cup \ldots \cup \Omega_N$ and $\Delta^p f(x) = 0$ for all $x \in \Omega_j$ and for $j = 1, ..., N$. Cardinal polysplines of order $p$ on strips or annuli have been discussed by the first two authors in a series of papers [32], [33], [34].

In this paper we have dealt with transfinite interpolation and it might be of interest to compare our results with the thin plate splines of order $p > d/2$ (in $\mathbb{R}^d$) introduced by J. Duchon in [19] for the interpolation at a finite number of scattered points $x_1, ..., x_N \in \mathbb{R}^d$. Thin plate splines are polyharmonic functions of order $p$ on the set $\mathbb{R}^d \setminus \{x_1, ..., x_N\}$ since they are a finite linear combination of translates of the fundamental solution of $\Delta^p$ in $\mathbb{R}^d$. In contrast to a polyspline a thin plate spline is only a function in $C^{2p-d-1}(\mathbb{R}^d)$ which is the reason for the requirement $p > d/2$. By definition, an interpolating thin plate spline is defined as the unique minimizer of the integral functional

$$\langle f, f \rangle_{p, \mathbb{R}^d} := \int_{\mathbb{R}^d} \sum_{|\alpha| = p} \frac{p!}{\alpha!} D^\alpha f(x) \cdot D^\alpha f(x) dx$$
among all functions \( f : \mathbb{R}^d \to \mathbb{R} \) having all partial derivative \( D^\alpha f \) of total order \( |\alpha| = p \) in \( L^2 (\mathbb{R}^d) \) and interpolating the data. Here we used multi-index notation \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d \) with \( |\alpha| = \alpha_1 + \cdots + \alpha_d \) and \( \alpha! = \alpha_1! \cdots \alpha_d! \) and \( D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} \). In [20] one can find the error estimates which served as model example for error estimates for interpolation with radial basis functions, see e.g. [11], [33], [53], [54]. Unlike our results, in these references the constants for the error estimates are not explicit, and only Sobolev norms are used; only in two dimensions some explicit constants are found, cf. [45], [46]. It is our expectation that the error estimates for polysplines can be used to improve the error estimates for thin plates for data which is structured along curves, a subject we hope to address in a future paper.

2 Harmonic interpolation splines

We say that an open set \( \Omega \) is regular in \( \mathbb{R}^d \) if each boundary point is regular, see [1, p. 179] for definition, and [1, Theorem 6.5.5] for a characterization. It is known that for a regular bounded domain \( \Omega \) the Dirichlet problem is solvable, see [1, Theorem 6.5.5 and 6.5.4].

At first we discuss the error estimate for harmonic interpolation splines:

**Theorem 5** Assume that \( \Omega_1, \ldots, \Omega_N \) are pairwise disjoint bounded regular domains and define \( \Omega = \bigcup_{j=1}^N \Omega_j \). If \( F \in C^2(\overline{\Omega}) \) and \( I_2(F) \) is the harmonic spline interpolating \( F \) on \( \partial \Omega \) then

\[
\| F - I_2(F) \|_\Omega \leq \max_{j=1, \ldots, N} c_d(\Omega_j) \cdot \sup_{y \in \Omega} |\Delta F(y)|
\]

where \( c_d(\Omega_j) \) is the smallest constant \( c \) such that \( (10) \) holds for all function \( f \in C(\overline{\Omega_j}) \cap C^2(\Omega_j) \) vanishing on the boundary \( \partial \Omega_j \) for \( j = 1, \ldots, N \).

**Proof.** We consider \( f(x) = F(x) - I_2(F)(x) \). Then \( f(x) = 0 \) for all \( x \in \partial \Omega_j \subset \partial \Omega \). Further \( f \in C(\overline{\Omega_j}) \cap C^2(\Omega_j) \) since \( I_2(F) \) is harmonic on \( \Omega_j \) and continuous on \( \overline{\Omega_j} \). Hence for \( x \in \Omega_j \)

\[
|F(x) - I_2(F)(x)| \leq c_d(\Omega_j) \cdot \sup_{y \in \Omega_j} |\Delta F(y)|
\]

(12)

The statement is now obvious.

A fundamental theorem in potential theory states that for any open set \( U \) in \( \mathbb{R}^d \) with \( d \geq 3 \), or for any bounded open set \( U \) in \( \mathbb{R}^2 \), the Green function \( G_\Omega(x,y) \) exists, see [1, p. 90]. Further we denote by \( \omega_d \) the volume of the unit ball and define

\[
a_d = \left\{ \begin{array}{ll}
\frac{2}{\omega_{d-2} d \omega_d} & \text{for } d = 2 \\
\frac{\omega_{d-2} d \omega_d}{d} & \text{for } d \geq 3.
\end{array} \right.
\]

**Theorem 6** If \( \Omega \) is a bounded domain in \( \mathbb{R}^d \) then

\[
c_d(\Omega) \leq a_d \sup_{x \in \Omega} \int_\Omega G_\Omega(x,y) \, dy.
\]

Equality holds when \( \Omega \) is a regular domain.
Proof. Assume that \( f \in C(\Omega) \cap C^2(\Omega) \) vanishes on the boundary \( \partial\Omega \). If \( \Delta f \) is unbounded on \( \Omega \) the inequality (10) is trivial. If \( \Delta f \) is bounded it follows that

\[
\int_\Omega |\Delta f(y)| \, dy < \infty \quad \text{and} \quad \int_\Omega |\Delta f(y)|^p \, dy < \infty.
\]

for some \( p > d/2 \). Then it can be shown that the following representation formula holds

\[
f(x) = a_d \int_\Omega G_\Omega(x, y) \Delta f(y) \, dy
\]

for all \( x \in \Omega \), and all \( f \in C^2(\Omega) \cap C(\Omega) \) which vanish on the boundary \( \partial\Omega \). Clearly (15) implies that

\[
|f(x)| \leq \sup_{y \in \Omega} |\Delta f(y)| \cdot a_d \int_\Omega G_\Omega(x, y) \, dy
\]

for any \( x \in \Omega \). The first result follows since \( c_d(\Omega) \) is the smallest number satisfying (10).

Now assume that \( \Omega \) is regular, so the Dirichlet problem is solvable for \( \Omega \). Then there exists a harmonic function \( h \in C(\Omega) \) such that \( h(\xi) = |\xi|^2 \) for all \( \xi \in \partial\Omega \). Clearly the function

\[
T_0(x) = \frac{1}{2d} \left( h(x) - |x|^2 \right)
\]

is in \( C(\Omega) \cap C^2(\Omega) \) and vanishes on \( \partial\Omega \). Further \( \Delta T_0(x) = -1 \). Since \( c_d(\Omega) \) is the smallest constant satisfying (10) we infer that

\[
|T_0(x)| \leq c_d(\Omega) \sup_{\tau \in \Omega} |\Delta T_0(\tau)| = c_d(\Omega).
\]

The representation formula (15) shows that

\[
T_0(x) = -a_d \int_\Omega G_\Omega(x, y) \Delta T_0(x) \, dx = a_d \int_\Omega G_\Omega(x, y) \, dx.
\]

It follows that from (13), formulae (19) and (18) that

\[
c_d(\Omega) \leq a_d \sup_{x \in \Omega} \int_\Omega G_\Omega(x, y) \, dx = \sup_{x \in \Omega} T_0(x) \leq c_d(\Omega).
\]

The proof is complete. \( \blacksquare \)

Next we turn to \( L^p \)-estimates of harmonic splines:

**Theorem 7** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \). Assume that \( f \in C(\Omega) \cap C^2(\Omega) \) vanishes on the boundary \( \partial\Omega \) and that \( \Delta f \) is bounded. Then for any \( p > 1 \) and its conjugate exponent \( q \) defined by \( \frac{1}{q} + \frac{1}{p} = 1 \) we have

\[
\|f\|_{L^p(\Omega)} \leq \left( \max_{x \in \Omega} a_d G_\Omega(x, y) \int_\Omega \right)^{\frac{1}{p}} \cdot \|\Delta f\|_{L^p(\Omega)}.
\]
Proof. The representation formula (15) and the Hölder inequality show that
\[
|f(x)| \leq \int_{\Omega} (a_d G_{\Omega}(x, y))^\frac{1}{p} \cdot (a_d G_{\Omega}(x, y))^\frac{1}{p} |\Delta f(y)| \, dy
\leq \left( \int_{\Omega} a_d G_{\Omega}(x, y) \, dy \right)^\frac{1}{p} \left( \int_{\Omega} a_d G_{\Omega}(x, y) |\Delta f(y)|^p \, dy \right)^\frac{1}{p}.
\]
Write \( S(x) = a_d \int_{\Omega} G_{\Omega}(x, y) \, dy \), take the \( p \)-th power on both sides and integrate with respect to \( x \), then
\[
\int_{\Omega} |f(x)|^p \, dx \leq \int_{\Omega} (S(x))^\frac{1}{p} \int_{\Omega} a_d G_{\Omega}(x, y) |\Delta f(y)|^p \, dy \, dx.
\]
Then Fubini’s theorem shows that
\[
\int_{\Omega} |f(x)|^p \, dx \leq \int_{\Omega} \left( \int_{\Omega} a_d G_{\Omega}(x, y) (S(x))^{\frac{1}{p}} \, dy \right) |\Delta f(y)|^p \, dy. \tag{21}
\]
Further we see that
\[
\int_{\Omega} a_d G_{\Omega}(x, y) (S(x))^{\frac{1}{p}} \, dy \leq \max_{x \in \Omega} (S(x))^{\frac{1}{p}} \int_{\Omega} a_d G_{\Omega}(x, y) \, dx \tag{22}
\leq \max_{x \in \Omega} (S(x))^{\frac{1}{p} + 1} = \max_{x \in \Omega} (S(x))^{\frac{1}{p}}. \tag{23}
\]
Now take the \( p \)-th square root in (21) and we arrive at
\[
\|f\|_{L^p(\Omega)} \leq \max_{x \in \Omega} (S(x))^{\frac{1}{p}} \|\Delta f\|_{L^p(\Omega)}.
\]

We apply now the results to the case of annular domains.

Theorem 8 Let \( r_1 < \ldots < r_N \) be real numbers, and let \( F \in C^2(\overline{A(r_1, r_N)}) \). Assume that \( I_2(F) \) is a harmonic spline interpolating \( F \) for all \( x \) with \( |x| = r_j \) and \( j = 1, \ldots, N \), then for the supremum norm the estimate
\[
\|F - I_2(F)\|_{A(r_1, r_N)} \leq C_d \max_{j=1, \ldots, N-1} (r_{j+1} - r_j)^2 \|\Delta F\|_{A(r_1, r_N)}
\]
holds, and for the \( L^2 \)-norm
\[
\|F - I_2(F)\|_{L^2(A(r_1, r_N))} \leq C_d \max_{j=1, \ldots, N-1} (r_{j+1} - r_j)^2 \|\Delta F\|_{L^2(A(r_1, r_N))}
\]
where \( C_d = \max \left\{ \frac{1}{d_{a_1}}, \frac{1}{8} \right\} \).

Proof. The best constant \( c_d(\Omega) \) for annular domains is characterized and estimated in the last Section, see Theorem 13. Now Theorem 4 yields the first statement. For the second statement we note that
\[
\|F - I_2(F)\|_{L^2(A(r_1, r_N))}^2 = \sum_{j=1}^{N-1} \|F - I_2(F)\|_{L^2(A(r_j, r_{j+1}))}^2.
\]
Now apply Theorem 7 for \( p = q = 2 \) to the domain \( A(r_j, r_{j+1}) \) and we have
\[
\|F - I_2(F)\|_{L^2(A(r_j, r_{j+1}))}^2 \leq C_d^2 \cdot \|\Delta F\|_{L^2(A(r_j, r_{j+1}))}^2.
\]
By summing up over \( j = 1, \ldots, N \) and taking the square root we arrive at the second statement. \( \blacksquare \)
3 Biharmonic interpolation splines

In this section we want to provide the error estimate for biharmonic interpolation splines on annular domains, see Theorem 3 in the introduction. Note that a simple consequence of the error estimate is the uniqueness of the biharmonic interpolation spline for a given data function $F$.

We emphasize that the assumption for the data function, namely $F \in C^4(A(r_1, r_N))$, in Theorem 3 is weaker than the usual assumption for proving the existence of a biharmonic interpolation spline. Indeed, the first author has proved the existence of a biharmonic spline interpolating a function $F$ in [31, p. 446–453] with Dirichlet boundary conditions using a priori estimates for elliptic boundary values problems. For dimension 2 it is required that the data functions $F$ are from the fractional Sobolev space $H^{7/2}(T) \subset H^3(T)$. For dimension 2, A. Bejancu has shown in [11] the existence of a biharmonic spline interpolating a function $F$ with Beppo-Levi boundary conditions and a data function $F$ in the weighted Wiener algebra $W^2(T)$ (which is less restrictive) using Fourier series techniques, see also [10].

Our approach to the proof for the error estimate of biharmonic interpolation splines is inspired by the exposition of Carl de Boor in [13] for the error estimate of cubic interpolation splines which is deduced via error estimates with piecewise linear functions on $t_1 < \ldots < t_N$ in an iterative way. The main observation in the intermediate step of the proof in [13] is that the second derivative of interpolation cubic spline is the best $L^2$ approximation to the second derivative of the interpolated function. This fact depends on an orthogonality relation between the error and linear splines. We shall use a similar argument in our context of biharmonic and harmonic splines and for this we use the following well known fact from best approximation in Hilbert spaces. For convenience of the reader we include the short proof:

**Proposition 9** Let $U$ be a subspace of a Hilbert space $H$, and $f \in H$. Assume that $\varphi_0 \in U$ has the property that

$$\langle f - \varphi_0, \varphi \rangle = 0 \text{ for all } \varphi \in U. \quad (24)$$

Then $\|f - \varphi_0\| \leq \|f - \varphi\|$ for all $\varphi \in U$.

**Proof.** Put $g = f - \varphi_0$. Due to the orthogonality condition (24) we have

$$\|g - \varphi\|^2 = \|g\|^2 - 2 \langle g, \varphi \rangle + \|\varphi\|^2 = \|g\|^2 + \|\varphi\|^2 \geq \|g\|^2.$$

It follows that $\|g\| = \|f - \varphi_0\| \leq \|f - \varphi_0 - \varphi\|$. Since $U$ is a subspace we can replace $\varphi \in U$ by $\varphi_0 + \varphi \in U$, and the proof is finished. 

Let us recall the main result of the paper stated in the introduction:

**Theorem 10** Let $r_1 < \ldots < r_N$ and let $F \in C^4(A(r_1, r_N))$. Assume that $I_4(F)$ is a biharmonic spline for the partition $A(r_1, r_2), \ldots, A(r_{N-1}, r_N)$ which satisfies the transfinite interpolation conditions

$$I_4(x) = F(x) \text{ for all } x \text{ with } |x| = r_j \text{ and } j = 1, \ldots, N,$$

and

$$\frac{\partial I_4(F)}{\partial n}(x) = \frac{\partial F(x)}{\partial n} \text{ for all } |x| = r_1 \text{ and for all } |x| = r_N. \quad (25)$$
Then the following error estimate holds:
\[
\| F - I_4(F) \|_{L^2(A(r_1, r_N))} \leq (C_d)^2 \max_{j=1, \ldots, N-1} |r_{j+1} - r_j|^4 \| \Delta^2 F \|_{L^2(A(r_1, r_N))},
\]
where \( C_d = \max \{ \frac{1}{\pi^2}, \frac{1}{\pi^4} \} \).

**Proof.** At first we note that
\[
\| F - I_4(F) \|_{L^2(A(r_1, r_N))}^2 = \sum_{j=1}^{N-1} \| F - I_4(F) \|_{L^2(A(r_j, r_{j+1}))}^2.
\]
We apply (20) to the function \( f = F - I_4(F) \) on the annular domain \( A(r_j, r_{j+1}) \): note that \( f \) vanishes for any \( x \) with \( |x| = r_j \) for \( j = 1, \ldots, N \), hence
\[
\| F - I_4(F) \|_{L^2(A(r_j, r_{j+1}))}^2 \leq C_d^2 (r_{j+1} - r_j)^4 \| \Delta F - \Delta (I_4(F)) \|_{L^2(A(r_j, r_{j+1}))}^2.
\]
By summing up and taking the square root we see that
\[
\| F - I_4(F) \|_{L^2(A(r_1, r_N))} \leq C_d \max_{j=1, \ldots, N-1} (r_{j+1} - r_j)^2 \| \Delta F - \Delta (I_4(F)) \|_{L^2(A(r_1, r_N))}.
\]
Now we want to estimate the right hand side: let us put \( f_0 = \Delta F \) and \( \varphi_0 = \Delta (I_4(F)) \). Note that \( \varphi_0 \) is a harmonic spline – but unfortunately it does not interpolate the function \( f_0 \), so we cannot repeat the error estimate for interpolating harmonic splines. In Theorem [11] below we prove that the equality
\[
\langle f_0 - \varphi_0, \varphi \rangle_{L^2(A(r_1, r_N))} = 0
\]
holds for all harmonic splines \( \varphi \) for the partition \( A(r_1, r_2), \ldots, A(r_{N-1}, r_N) \). Then Proposition [8] implies that
\[
\| \Delta F - \Delta (I_4(F)) \|_{L^2(A(r_1, r_N))} = \| f_0 - \varphi_0 \|_{L^2(A(r_1, r_N))} \leq \| f_0 - \varphi \|_{L^2(A(r_1, r_N))}
\]
holds for all harmonic splines \( \varphi \) for the partition \( A(r_1, r_2), \ldots, A(r_{N-1}, r_N) \). Let us take the harmonic spline \( \varphi := I_2(\Delta F) \). Hence,
\[
\| f_0 - \varphi \|_{L^2(A(r_1, r_N))} = \| \Delta F - I_2(\Delta F) \|_{L^2(A(r_1, r_N))}
\]
is the error for the harmonic spline \( \varphi \) interpolating \( \Delta F \), and by Theorem [8] we obtain
\[
\| \Delta F - I_2(\Delta F) \|_{L^2(A(r_1, r_N))} \leq C_d \max_{j=1, \ldots, N-1} (r_{j+1} - r_j)^2 \| \Delta^2 F \|_{L^2(A(r_1, r_N))}.
\]
It follows that
\[
\| F - I_4(F) \|_{L^2(A(r_1, r_N))} \leq C_d^2 \max_{j=1, \ldots, N-1} (r_{j+1} - r_j)^4 \| \Delta^2 F \|_{L^2(A(r_1, r_N))}.
\]
This ends the proof. \( \blacksquare \)
4 Proof of the orthogonality relation

In this section we want to prove the following result:

**Theorem 11** Let \( r_1 < \ldots < r_N \) and let \( F \in C^4(\overline{\Omega}) \) and \( I_4(F) \) as in Theorem 3. Then for all harmonic splines \( \varphi \) for the partition \( A(r_j, r_{j+1}) \) for \( j = 1, \ldots, N \)

\[
\int_{A(r_1, r_n)} (\Delta F(x) - \Delta I_4(F)(x)) \cdot \varphi(x) \, dx = 0. \tag{26}
\]

Let us write \( f(x) = F(x) - I_4(F)(x) \). Then \( f \in C^4(\overline{A(r_1, r_N)}) \) and \( \varphi \in C^2(\overline{A(r_1, r_N)}) \cap C(\overline{A(r_1, r_N)}) \). Clearly we have

\[
\int_{A(r_1, r_n)} \Delta f(x) \cdot \varphi(x) \, dx = \sum_{j=1}^{n-1} \int_{A(r_j, r_{j+1})} \Delta f(x) \cdot \varphi(x) \, dx.
\]

A natural approach is to prove (26) by applying Green’s formula (see [1, p. 307]) to each summand on the right hand side. In order to apply Green’s formula, let us take \( \rho_j < \rho_{j+1} \) in the interval \((r_j, r_{j+1})\). Then \( \varphi \) and \( f \) are twice differentiable in a neighborhood of \( \overline{A}(\rho_j, \rho_{j+1}) \subset A(r_j, r_{j+1}) \) and we apply Green’s formula. Since \( \varphi \) is harmonic we obtain that

\[
\int_{A(\rho_j, \rho_{j+1})} \Delta f(x) \cdot \varphi(x) \, dx = R(\rho_{j+1}) - R(\rho_j) \tag{27}
\]

where \( R(\rho) \) is defined by

\[
R(\rho) = \int_{S_\rho} \frac{\partial f}{\partial n}(y) \varphi(y) \, d\sigma_\rho(y) - \int_{S_\rho} f(y) \frac{\partial \varphi}{\partial n}(y) \, d\sigma_\rho(y),
\]

and \( \partial/\partial n \) denotes the exterior normal derivative, and \( \sigma_\rho \) is the surface measure on the sphere \( S_\rho := \{ x \in \mathbb{R}^d : |x| = \rho \} \) for \( j = 1, 2 \). Now we want to take limits \( \rho_j \to r_j \) and \( \rho_{j+1} \to r_{j+1} \) in (27). The limit for the left hand side clearly exists, so

\[
\int_{A(r_1, r_N)} \Delta f(x) \cdot \varphi(x) \, dx = \sum_{j=1}^{N-1} \left( \lim_{\rho_j \to r_j, \rho_{j+1} \to r_{j+1}} R(\rho_{j+1}) - \lim_{\rho_j \to r_j, \rho_{j+1} \to r_{j+1}} R(\rho_j) \right).
\]

We know that \( f(y) \) vanishes for \(|y| = \rho_j \) but it seems to be unclear whether the expression

\[
\int_{S_\rho} f(y) \frac{\partial \varphi}{\partial n}(y) \, d\sigma_\rho(y)
\]

for \( \rho \to r_j \) has a limit, and whether this limit is 0 (as we would expect) since the harmonic spline \( \varphi(y) \) may have an unbounded gradient for \( y > r_j \). Thus this approach unfortunately does not provide a proof of the statement, and we will pursue a proof of Theorem 11 using facts from the theory of spherical harmonics.

We shall write \( x \in \mathbb{R}^d \) in spherical coordinates \( x = r\theta \) with \( \theta \in \mathbb{S}^{d-1} = \{ x \in \mathbb{R}^d : |x| = 1 \} \). Let \( d\theta \) be the surface measure of \( \mathbb{S}^{d-1} \) and define the inner product

\[
(f, g)_{\mathbb{S}^{d-1}} := \int_{\mathbb{S}^{d-1}} f(\theta) g(\theta) d\theta. \tag{28}
\]
Let $\mathcal{H}_k (\mathbb{R}^d)$ be the set of all harmonic homogeneous complex-valued polynomials of degree $k$. Then $f \in \mathcal{H}_k (\mathbb{R}^d)$ is called a solid harmonic and the restriction of $f$ to $\mathbb{S}^{d-1}$ a spherical harmonic of degree $k$ and we set $a_k := \dim \mathcal{H}_k (\mathbb{R}^d)$, see [52], [31] for details.

Assume that $Y_{k,\ell} : \mathbb{R}^d \to \mathbb{R}, \ell = 1, \ldots, a_k$, is an orthonormal basis of $\mathcal{H}_k (\mathbb{R}^d)$ with respect to (28). Since $Y_{k,\ell}$ is homogeneous of degree $k$ we have $Y_{k,\ell} (x) = r^k Y_{k,\ell} (\theta)$ for $x = r \theta$. For a continuous function $f : A (r_1, r_N) \to \mathbb{C}$ we define the Fourier-Laplace coefficient $f_{k,\ell} (r)$ for $r \in [r_1, r_N]$ by

$$f_{k,\ell} (r) = \int_{\mathbb{S}^{d-1}} f (r \theta) Y_{k,\ell} (\theta) \, d\theta.$$  \hfill (29)

The Fourier-Laplace series of $f : A (a, b) \to \mathbb{C}$ is defined by the formal expansion

$$f (r \theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} f_{k,\ell} (r) Y_{k,\ell} (\theta).$$ \hfill (30)

If $\theta \mapsto f (r \theta)$ is a continuous, it is a function in $L^2 (\mathbb{S}^{d-1})$ and since $(Y_{k,\ell} (\theta))_{k \in \mathbb{N}_0, \ell = 1, \ldots, a_k}$ is complete orthonormal basis one has

$$\int_{\mathbb{S}^{d-1}} |f (r \theta)|^2 \, d\theta = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} |f_{k,\ell} (r)|^2.$$

If $f, g$ are continuous on $A (r_1, r_N)$ with Fourier-Laplace coefficients $f_{k,\ell} (r)$ and $g_{k,\ell} (r)$ respectively we obtain

$$\int_{\mathbb{S}^{d-1}} f (r \theta) g (r \theta) \, d\theta = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} f_{k,\ell} (r) g_{k,\ell} (r)$$

and the series on the right hand side converges absolutely. Multiply this equation with $r^{d-1}$ and integrate with respect to $dr$, hence we have

$$\int_{A (r_1, r_N)} f (x) g (x) \, dx = \int_{r_1}^{r_N} \int_{\mathbb{S}^{d-1}} f (r \theta) g (r \theta) r^{d-1} \, d\theta \, dr = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \int_{r_1}^{r_N} f_{k,\ell} (r) g_{k,\ell} (r) r^{d-1} \, dr. \hfill (31)$$

Let us define the univariate differential operators

$$L_k (f) = \frac{\partial^2}{\partial r^2} f + \frac{d-1}{r} \frac{\partial}{\partial r} f - \frac{k(k+d-2)}{r^2} f.$$

The following result is now crucial for our arguments. Since we could not find a reference for this result we include a proof (in [31] it is proved under the stronger assumption that $\Delta f$ has a absolutely convergent Fourier-Laplace series).

**Theorem 12** Let $a < b$ be real numbers and assume that $f : A (a, b) \to \mathbb{C}$ is continuously partially differentiable of order 2. Then the Fourier-Laplace coefficient $f_{k,\ell}$ of $f$ is twice differentiable on $(a, b)$ and

$$\int_{\mathbb{S}^{d-1}} (\Delta f) (r \theta) \cdot Y_{k,\ell} (\theta) \, d\theta = L_k (f_{k,\ell}) (r).$$

Thus the $(k, \ell)$-th Fourier Laplace coefficient of $\Delta f$ is equal to $L_k (f_{k,\ell})$. 

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Theorem 12 applied to $f$. Let us now take $F$ and let $g$ have $k, \ell$ be the ($k, \ell$)-th Fourier-Laplace coefficient of $f$. By formula (31) applied to $\Delta f$ on the domain $A(r, r+h)$, we have

$$
R_r := \int_{S^{d-1}} \frac{\partial}{\partial r} f(r \theta) \cdot r^k Y_{k, \ell}(\theta) \, d\sigma_r(y) - \int_{S^{d-1}} f(r \theta) \frac{\partial}{\partial r} (r^k Y_{k, \ell}(\theta)) \, d\sigma_r(y)
$$

$$
= r^{k+1-1} \frac{\partial}{\partial r} \int_{S^{d-1}} f(r \theta) Y_{k, \ell}(\theta) \, d\sigma - r^d-1 \int_{S^{d-1}} f(r \theta) \frac{\partial}{\partial r} (r^k Y_{k, \ell}(\theta)) \, d\sigma
$$

$$
= r^{k+1-1} \frac{\partial}{\partial r} f_{k, \ell}(r) - kr^{k+d-2} f_{k, \ell}(r) =: F(r).
$$

Let us now take $r_1 = r$ and $r_2 = r + h$, then by Green’s formula (see above) we have

$$
\frac{1}{h} \int_{A(r, r+h)} \Delta f(x) \cdot Y_{k, \ell}(x) \, dx = \frac{F(r + h) - F(r)}{h}.
$$

We can take the limit on the right hand side and we see that

$$
F'(r) = r^{k+1-1} \frac{\partial}{\partial r} f_{k, \ell}(r) + (d - 1) r^{k+d-2} \frac{\partial}{\partial r} f_{k, \ell}(r) - k (k + d - 2) r^{k+d-3} f_{k, \ell}(r).
$$

On the other hand,

$$
\frac{1}{h} \int_{A(r, r+h)} \Delta f(x) \cdot Y_{k, \ell}(x) \, dx = \frac{1}{h} \int_r^{r+h} \left( \int_{S^{d-1}} \Delta f(r \theta) Y_{k, \ell}(\theta) \, d\sigma \right) r^{k+d-1} \, dr
$$

converges to

$$
\left( \int_{S^{d-1}} \Delta f(r \theta) Y_{k, \ell}(\theta) \, d\sigma \right) r^{k+d-1}.
$$

The proof is complete.

Proof of Theorem 11

Proof. Let $f := F - I_4 F \in C^2(A(r, r_N))$ and $\varphi$ a harmonic spline. Let $f_{k, \ell}(r)$ be the $(k, \ell)$-th Fourier-Laplace coefficient of $f$, so

$$
f_{k, \ell}(r) = \int_{S^{d-1}} f(r \theta) Y_{k, \ell}(\theta) \, d\theta.
$$

Then $f_{k, \ell}$ is twice differentiable and $f_{k, \ell}(r_j) = 0$ for $j = 1, \ldots, N$. Define $g := \Delta f$, and let $g_{k, \ell}$ and $\varphi_{k, \ell}$ be the Fourier-Laplace coefficients of $g$ and $\varphi$ respectively. By formula (31) applied to $\Delta f$ and $\varphi$ it suffices to show that

$$
I_{k, \ell} := \int_{r_1}^{r_N} g_{k, \ell}(r) \varphi_{k, \ell}(r) r^{d-1} \, dr = 0.
$$

Theorem 12 applied to $f$ on the domain $A(r_j, r_{j+1})$ shows that $g_{k, \ell}(r) = L_k f_{k, \ell}(r)$ for $r \in (r_j, r_{j+1})$. Thus we see that

$$
I_{k, \ell} = \int_{r_j}^{r_{j+1}} L_k f_{k, \ell}(r) \varphi_{k, \ell}(r) r^{d-1} \, dr.
$$

Partial integration shows that with $h_{k, \ell}(r) = \varphi_{k, \ell}(r) r^{d-1}$ shows that

$$
\int_{r_j}^{r_{j+1}} \frac{\partial^2 f_{k, \ell}(r)}{\partial r^2} h_{k, \ell}(r) \, dr = \frac{\partial f_{k, \ell}(r)}{\partial r} h_{k, \ell}(r) |_{r_j}^{r_{j+1}} - f_{k, \ell}(r) \frac{\partial}{\partial r} h_{k, \ell}(r) |_{r_j}^{r_{j+1}} + \int_{r_j}^{r_{j+1}} f_{k, \ell}(r) \frac{\partial^2 h_{k, \ell}(r)}{\partial r^2} \, dr.
$$

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Since \( f_{k,\ell}(r) = 0 \) and \( h_{k,\ell}(r) \) is analytic function for \( r > 0 \), the middle term vanishes. Next we see that
\[
\int_{r_j}^{r_{j+1}} \frac{d-1}{r} \frac{\partial}{\partial r} f_{k,\ell}(r) \cdot h_{k,\ell}(r) \, dr
= f_{k,\ell}(r) \frac{d-1}{r} h_{k,\ell}(r) |_{r_j}^{r_{j+1}} - (d-1) \int_{r_j}^{r_{j+1}} f_{k,\ell}(r) \frac{\partial}{\partial r} h_{k,\ell}(r) \, dr.
\]
It follows that
\[
\int_{r_j}^{r_{j+1}} L_k f_{k,\ell}(r) h_{k,\ell}(r) \, dr = \frac{\partial f_{k,\ell}(r)}{\partial r} h_{k,\ell}(r) |_{r_j}^{r_{j+1}} + \int_{r_j}^{r_{j+1}} f_{k,\ell}(r) M(h_{k,\ell})(r) \, dr
\]
and
\[
M(h_{k,\ell})(r) = \frac{\partial^2 h_{k,\ell}}{\partial r^2} - \frac{\partial}{\partial r} \left( \frac{d-1}{r} h_{k,\ell}(r) \right) - \frac{k(k+d-2)}{r^2} h_{k,\ell}(r)
= r^{d-1} \frac{\partial^2}{\partial r^2} \varphi_{k,\ell}(r) + 2(d-1) r^{d-2} \frac{\partial}{\partial r} \varphi_{k,\ell}(r) + \varphi_{k,\ell}(r) (d-1) (d-2) r^{d-3}
- (d-1) (d-2) r^{d-3} \varphi_{k,\ell}(r) - (d-1) r^{d-2} \frac{\partial}{\partial r} \varphi_{k,\ell}(r) - \frac{k(k+d-2)}{r^2} \varphi_{k,\ell}(r) r^{d-1}
\]
and we see that
\[
M(h_{k,\ell})(r) = r^{d-1} \left( \frac{\partial^2}{\partial r^2} \varphi_{k,\ell}(r) + (d-1) \frac{\partial}{\partial r} \varphi_{k,\ell}(r) - \frac{k(k+d-2)}{r^2} \varphi_{k,\ell}(r) \right).
\]
Theorem \([12]\) applied to the harmonic function \( \varphi \) on the domain \( A(r_j, r_{j+1}) \) shows that \( L_k \varphi_{k,\ell}(r) = 0 \) for \( r \in (r_j, r_{j+1}) \) and therefore \( M(h_{k,\ell})(r) = 0 \) for \( r \in (r_j, r_{j+1}) \). It follows that
\[
\sum_{j=1}^{N} \int_{r_j}^{r_{j+1}} L_k f_{k,\ell}(r) h_{k,\ell}(r) \, dr = \sum_{j=1}^{N} \frac{\partial f_{k,\ell}(r)}{\partial r} h_{k,\ell}(r) |_{r_j}^{r_{j+1}}
= \frac{\partial f_{k,\ell}(r_N)}{\partial r} h_{k,\ell}(r_N) - \frac{\partial f_{k,\ell}(r_1)}{\partial r} h_{k,\ell}(r_1).
\]
Further for \( r^* = r_N \) or \( r^* = r_1 \) we have
\[
\frac{\partial f_{k,\ell}(r)}{\partial r} h_{k,\ell}(r) = \lim_{r \to r^*} \frac{\partial}{\partial r} \int_{S_{d-1}} f(r, \theta) Y_{k,\ell} \, d\theta \cdot h_{k,\ell}(r)
= \lim_{r \to r^*} \int_{S_{d-1}} \frac{\partial}{\partial r} f(r, \theta) Y_{k,\ell} \, d\theta \cdot h_{k,\ell}(r)
= \int_{S_{d-1}} \frac{\partial}{\partial r} f(r, \theta) Y_{k,\ell} \, d\theta \cdot h_{k,\ell}(r)
= 0
\]
since \( f \) has normal derivative 0 at \( r^* \).

5 The constant \( c_d(\Omega) \) for the annulus

In this section we want to determine the constant \( c_d(A(r, R)) \) for the annulus. The Green function for the annulus is known, for a nice exposition see [26]. On
the other hand, Theorem 4 describes a simpler way to solve the problem for \( d \geq 2 \). We define for \( d \geq 3 \) the harmonic function \( h_d(x) = \left( \frac{|x|}{R} \right)^2 - 1 \) and \( h_2(x) = \log \left( \frac{|x|}{R} \right) \) for \( d = 2 \). Then

\[
T_0(x) = \frac{R^2}{2d} \left( 1 - \frac{|x|^2}{R^2} - \frac{1 - \frac{R^2}{h_d(r)} h_d(x)}{h_2(r)} \right).
\]

has the property that it vanishes for \( |x| = R \) and \( |x| = r \) and satisfies \( \Delta f = -1 \).

**Theorem 13** Let \( A(r, R) \) be the annulus in \( \mathbb{R}^d \) for \( d \geq 2 \) and set \( D = (d - 2)/2 \). Then

\[
c_d(A(r, R)) = \frac{(R - r)^2}{2d} H_d \left( \frac{r}{R} \right).
\]

where \( H_d \) is defined for \( d \geq 3 \) by

\[
H_d(\rho) = \frac{1}{(1 - \rho)^2} \left( 1 + \rho^{2D} \frac{1 - \rho^2}{1 - \rho^{2D}} - \frac{D + 1}{D} \left( D^2 \rho^2 - 1 \right) \frac{2}{1 - \rho^{2D}} \right).
\]

and for \( d = 2 \)

\[
H_2(\rho) = \frac{1}{(1 - \rho)^2} \left( 1 + \frac{1 - \rho^2}{2 \log \rho} - \frac{11}{2} \log \rho \log \left( \frac{1 - \rho^2}{2 \log \rho} \right) \right).
\]

**Proof.** At first we assume that \( d > 2 \). Then \( h_d(x) = \left( \frac{|x|}{R} \right)^2 - 1 \). Put \( u = |x|^2/R^2 \) and \( \rho = r/R < 1 \). Then

\[
T_0(x) = \frac{R^2}{2d} \left[ 1 - u - \frac{1 - \rho^2}{\rho^{2D} - 1} \left( u^{-D} - 1 \right) \right].
\]

It follows that

\[
c_d(A(r, R)) = \sup_{x \in A(r, R)} T_0(x) = \frac{R^2}{2d} \sup_{\frac{r^2}{R^2} \leq u \leq 1} f_d(u)
\]

where we define

\[
f_d(u) = 1 - u - B_d(\rho) (u^{-D} - 1) = 1 - u - B_d(\rho) u^{-D} + B_d(\rho)
\]

and

\[
B_d(\rho) = \frac{1 - \rho^2}{\rho^{2D} - 1} = \rho^{2D} \frac{1 - \rho^2}{1 - \rho^{2D}}.
\]

Obviously \( f_d(1) = 0 \) and \( f_d \left( \frac{r^2}{R^2} \right) = f_d \left( \rho^2 \right) = 0 \). Hence the maximum of \( f_d(u) \) on the interval \( \left[ \frac{r^2}{R^2}, 1 \right] \) is attained in the interior. Further

\[
\frac{d}{du} f_d(u) = -1 + DB_d(\rho) u^{-D-1}.
\]
Hence \( f_0'(u_0) = 0 \) implies that \( u_0^{D+1} = DB_d(\rho) \). It follows that with \( B = B_d(\rho) \) and inserting \( u_0 = (DB)^{D+1} \)

\[
\sup_{\rho = \frac{\rho}{2} \leq u \leq 1} f_d(u) = 1 - (DB)^{\frac{D+1}{D}} - B \left( (DB)^{\frac{D+1}{D}} - 1 \right) = 1 + B - \frac{D+1}{D}(DB)^{\frac{D+1}{D}}.
\]

(38) The identity (34) follows from (36) and (38) by writing

\[
\frac{R^2}{2d} = \frac{(R - r)^2}{2d} \frac{1}{(1 - \rho)^2}.
\]

For \( d = 2 \) we have \( h_2(x) = \log \left( \frac{|x|}{R} \right) = \frac{1}{2} \log \left( \frac{|x|}{R} \right)^2 \) and with \( u = |x|^2 / R^2 \)

\[
T_0(x) = \frac{R^2}{2d} \left[ 1 - u - \frac{1}{2} \frac{1 - \rho^2}{\log \rho} \log u \right].
\]

It follows that

\[
c_d(A(r, R)) = \sup_{x \in A(r, R)} T_0(x) = \frac{R^2}{2d} \sup_{\frac{r}{2} \leq u \leq 1} f_2(u)
\]

where we define \( f_2(u) = 1 - u - \frac{1}{2} \frac{1 - \rho^2}{\log \rho} \log u \). Then \( f_2(\rho^2) = 0 \) and \( f_1(1) = 0 \) and \( f_2'(u) = -1 - \frac{1}{2} \frac{1 - \rho^2}{\log \rho} \). Then \( u = \frac{1 - \rho^2}{2 \log \rho} \) is the critical point and

\[
\sup_{\rho = \frac{\rho}{2} \leq u \leq 1} f_2(u) = 1 + \frac{1 - \rho^2}{2 \log \rho} - \frac{1}{2} \frac{1 - \rho^2}{\log \rho} \log \left( \frac{1 - \rho^2}{-2 \log \rho} \right).
\]

Proposition 14 For dimension \( d = 3 \) the function \( H_3(\rho) \) in Theorem 13 is strictly decreasing on \([0, 1]\) and

\[
\frac{(R - r)^2}{8} \leq c_d(A(r, R)) \leq \frac{(R - r)^2}{6}.
\]

Proof. For \( d = 3 \) we have \( D = \frac{1}{2} \) and

\[
H_3(\rho) = \frac{1 + \rho \frac{1 - \rho^2}{1 - \rho} - 3 \left( \frac{1}{2} \rho \frac{1 - \rho^2}{1 - \rho} \right)^{\frac{1}{2}}}{(1 - \rho)^2}
\]

\[
= \frac{1 + \rho (1 + \rho) - 3 \left( \frac{1}{2} \rho (1 + \rho) \right)^{\frac{1}{2}}}{(1 - \rho)^2}.
\]

A computation shows that

\[
H_3'(\rho) = \frac{(-1) \sqrt{2}(\rho (\rho+1))^{\frac{1}{2}}}{\rho (1 - \rho)^{\frac{1}{2}} (\rho+1)} w(\rho)
\]
where \( w(\rho) = \rho^2 + 4\rho + 1 - 3(1 + \rho) \sqrt[3]{\frac{1}{2}\rho(\rho + 1)}. \) Then \( w(\rho) \) is non-negative since
\[
(\rho^2 + 4\rho + 1)^3 - 3^3(1 + \rho)^3 \frac{1}{2}\rho(\rho + 1) = \frac{1}{2}(\rho - 1)^4(2\rho^2 + 5\rho + 2) \geq 0.
\]
It follows that \( H_3'(\rho) < 0 \) for \( \rho \in (0, 1) \) and \( H_3 \) is strictly decreasing. Hence
\[
\frac{3}{4} = \lim_{\rho \to 1} \frac{1 + \rho(1 + \rho) - 3 \left( \frac{1}{2}\rho(1 + \rho) \right)^{\frac{3}{2}}}{(1 - \rho)^2} \leq H_3(\rho)
\]
\[
\leq H_3(0) = 1.
\]
The proof is complete. \( \blacksquare \)

**Proposition 15** For dimension \( d = 2 \) the function \( H_2(\rho) \) in Theorem 13 is strictly decreasing on \([0, 1]\) and
\[
\frac{(R - r)^2}{8} \leq c_d(A(r, R)) \leq \frac{(R - r)^2}{4}.
\]

**Proof.** Similarly one can see that the function \( H_2(\rho) \) is decreasing and
\[
\lim_{\rho \to 0} H_2(\rho) = \lim_{\rho \to 0} \frac{1 + \frac{1 - \rho^2}{2 \log \rho} - \frac{1}{2} \frac{1 - \rho^2}{\log \rho} \log \left( \frac{1 - \rho^2}{2 \log \rho} \right)}{(1 - \rho)^2} = 1
\]
and \( \lim_{\rho \to 1} H_2(\rho) = \frac{1}{2} \). The proof is complete. \( \blacksquare \)

The discussion for \( d \geq 4 \) is more technical and we need the following result:

**Lemma 16** Let \( d > 2 \), i.e. that \( D = (d-2)/2 > 0 \). Then the function
\[
B_d(\rho) = \rho^{2D} \frac{1 - \rho^2}{1 - \rho^{2D}} \text{ for } \rho \in (0, 1)
\]
is increasing and positive on \((0, 1)\) and the following limits exist:
\[
\lim_{\rho \to 1} B_d(\rho) = \frac{1}{D} \text{ and } \lim_{\rho \to 1} B_d'(\rho) = \frac{D + 1}{D} \text{ and } \lim_{\rho \to 1} B_d''(\rho) = 0.
\]
Further \( B_d'' \) is positive on \((0, 1)\) for \( D > 1 \), and negative for \( D \in \left(\frac{1}{2}, 1\right) \). Further
\[
\frac{B_d'(\rho)}{B_d(\rho)} = \frac{2(D - \rho^2 + \rho^{2D+2} - \rho^2 D)}{\rho(1 - \rho^{2D})} = \frac{D + 1}{D}, \quad (39)
\]

**Proof.** For \( D > 0 \) we see use the rule of l’Hospital \( \lim_{\rho \to 1} B_d(\rho) = \lim_{\rho \to 1} \rho^{2D} \frac{1 - \rho^2}{1 - \rho^{2D}} = \frac{1}{D}. \) For \( \rho \in (0, 1) \) we have
\[
B_d'(\rho) = \frac{2\rho^{2D-1}}{(1 - \rho^{2D})^2} (D - \rho^2 + \rho^{2D+2} - \rho^2 D) \to \frac{D + 1}{D}
\]
for \( \rho \to 1 \). Further we obtain equation \( (39) \) from the last formula. Consider the function \( g(x) = D - x + x^{D+1} - xD \), then
\[
g'(x) = (D + 1)x^D - (D + 1) = (D + 1)(x^D - 1) < 0
\]
for $0 < x < 1$ for $D > 0$. Hence $g$ is strictly decreasing on $[0, 1]$ and $g(1) = 0$, so $g(x) \geq g(1) = 0$ for all $x \in (0, 1)$. Thus $g$ is positive in $(0, 1)$ and it follows that $B_d(\rho)$ is strictly increasing for any $D > 0$. Further

$$\frac{d^3}{d\rho^3} \left( \rho^{2D} \frac{1 - \rho^2}{1 - \rho^{2D}} \right) = \frac{4D\rho^{2D-3}}{(1 - \rho^{2D})^4} f(\rho^2)$$

where

$$f(x) = (2D - 1)(D - 1) - x(D + 1)(2D + 1) + 2x^D(4D^2 - 1) - 2x^{D+1}(4D^2 - 1) + x^{2D}(2D^2 + 3D + 1) - x^{2D+1}(2D^2 - 3D + 1).$$

One can verify that $f(x)$ has a zero of order 5 at $x = 1$. Since $f$ is in the linear space generated by the basis functions $1, x, x^D, x^{D+1}, x^{2D}, x^{2D+1}$ the function $f$ has at exactly 5 zeros at $x = 1$ and no more positive zeros. It follows that $f$ is either positive or negative on $(0, 1)$, hence $f$ has the same sign as $f(0) = (2D - 1)(D - 1)$. Since $f$ has a zero of order 5 at $x = 1$ we see that

$$\lim_{\rho \to 1} B_d'''(\rho) = 0.$$  

The proof is complete.

**Theorem 17** For dimension $d = 4$ the function $H_4(\rho)$ in Theorem 13 is constant and

$$c_d(A(\rho, R)) = \frac{(R - r)^2}{8}.$$  

For dimension $d > 4$ the function $H_d$ in Theorem 13 is strictly increasing on $[0, 1]$ and

$$\frac{(R - r)^2}{2d} \leq c_d(A(\rho, R)) \leq \frac{(R - r)^2}{8}. \tag{40}$$

**Proof.** 1. For $d = 4$ we have $D = (d - 2)/2 = 1$ and

$$H_4(\rho) = \frac{1 + \rho^2 - \rho^2}{(1 - \rho)^2} - 2 \left( \frac{\rho^2 - 1}{1 - \rho^2} \right) = \frac{\rho^2 + 1 - 2\rho}{(1 - \rho)^2} = 1.$$

2. Assume now $d > 4$, so $D > 1$. We analyse the function

$$H_d(\rho) = \frac{G_d(\rho)}{(1 - \rho)^2}$$

with $G_d(\rho) := 1 + B_d(\rho) - \frac{D + 1}{D} (DB_d(\rho))^{D-1}$. For $\rho \in (0, 1)$ we compute

$$H_d'(\rho) = \frac{G_d'(\rho)(1 - \rho) + 2G_d(\rho)}{1 - \rho}.$$  

Consider the numerator of $H_d'(\rho)$ defined by

$$N_d(\rho) := G_d'(\rho)(1 - \rho) + 2G_d(\rho). \tag{41}$$
For $D > 1$ we will show that $N_d(\rho)$ is positive on $(0, 1)$. This clearly implies that $H_d$ is increasing for $D > 1$. The estimate \[^1\] is then a simple consequence of the monotonicity and

$$\lim_{\rho \to 1} H_d(\rho) = \lim_{\rho \to 1} \frac{B_D(\rho) + 1 - \frac{D+1}{D} (DB_D(\rho))^{\frac{1}{D+1}}}{(1 - \rho)^2} = \lim_{\rho \to 1} B'_D(\rho) \lim_{\rho \to 1} \frac{1 - (DB_D(\rho))^{\frac{1}{D+1}} - 1}{-2(1 - \rho)} = \frac{1}{2}(D + 1).$$

where we used the rule of L’Hospital for $\rho \to 1$ twice, and that $\frac{1}{232} (D + 1) = \frac{1}{8}$.

3. It remains to show that $N_d(\rho)$ is positive on $(0, 1)$. A short computation shows that for $\rho \in (0, 1)$

$$G'_d = B'_d - (DB_d)^{\frac{1}{D+1}} B'_D = B'_d \left( 1 - (DB_d)^{\frac{1}{D+1}} - 1 \right). \quad (42)$$

Lemma \[^1\] shows that $B'_d(\rho)$ converges for $\rho \to 1$ and $\lim_{\rho \to 1} DB_d(\rho) = 1$, hence $G'_d(\rho)$ → 0 for $\rho \to 1$. For $\rho \in (0, 1)$ we have

$$N'_d(\rho) = G''_d(\rho)(1 - \rho) + G'_d(\rho)$$

$$N''_d(\rho) = G''_d(\rho)(1 - \rho) .$$

If we can show that $G''_d(\rho) > 0$ for all $\rho \in (0, 1)$ we see that $N''_d(\rho) > 0$ for all $\rho \in (0, 1)$, hence $N'_d(\rho)$ is increasing on $(0, 1)$. Further

$$\lim_{\rho \to 1} N'_d(\rho) = \lim_{\rho \to 1} G''_d(\rho)(1 - \rho) + \lim_{\rho \to 1} G'_d(\rho) = 0$$

where we use the fact that $\lim_{\rho \to 1} G''_d(\rho)$ exists (see below for justification). It follows that $N'_d(\rho) \leq 0$ for all $\rho \in (0, 1)$, so $N'_d(\rho)$ is decreasing. It follows that $N_d(\rho) \geq N_d(1) = 0$.

4. It remains to show that for $D > 1$

$$G'''_d$$

is positive on $(0, 1)$ and $\lim_{\rho \to 1} G'''_d(\rho)$ exists.

We differentiate \[^1\] and using that $\frac{1}{D+1} - 1 = -\frac{D}{D+1}$ we obtain

$$G''_d = B''_d - B''_d (DB_d)^{\frac{1}{D+1}} - 1 + \frac{D^2}{D+1} B''_d (DB_d)^{\frac{1}{D+1}} - 2 .$$

Lemma \[^1\] shows that $\lim_{\rho \to 1} B^{(j)}_d(\rho)$ exists for $j = 0, 1, 2$, hence $\lim_{\rho \to 1} G''_d(\rho)$ exists. Further

$$G'''_d = B'''_d - B'''_d (DB_d)^{\frac{1}{D+1}} - 3 B''_d B'_d D^2 (DB_d)^{\frac{1}{D+1}} - 2 + \frac{D^3}{D+1} B''_d \left( \frac{1}{D+1} - 2 \right) (DB_d)^{\frac{1}{D+1}} - 3 .$$

We multiply $G'''_d$ with $(DB_d)^{3-\frac{1}{D+1}}$ and we obtain

$$(DB_d)^{3-\frac{1}{D+1}} G'''_d = (DB_d)^{3-\frac{1}{D+1}} B'''_d + \tilde{A}_d(\rho)$$
\[ \tilde{A}_d(\rho) = -D^2B''_d B^2_d + 3B''_d B'_d B_d \frac{D^3}{D+1} - D^3 \frac{2D+1}{(D+1)^2} B^3_d. \]

For \( D > 1 \) Lemma 16 shows that \( B''_d \geq 0 \) on \((0,1)\). Hence it suffices to show that \( \tilde{A}_d(\rho) \) is positive—which is a much easier function than \( G''_d \).

5. By dividing \( A_d(\rho) \) by \( D^2 \) and multiplying by \((D+1)\) it remains to show that
\[
A_d := -(D+1)B''_d B^2_d + 3DB''_d B'_d B_d - DB^3_d \frac{2D+1}{D+1}
\]
is positive. According to Lemma 16 we have
\[
\frac{B'_d(\rho)}{B_d(\rho)} = \frac{2(D - \rho^2 + \rho^2D + \rho^2D)}{\rho (1 - \rho^2D) (1 - \rho^2)}.
\]
Let \( p \) be the numerator and \( q \) the denominator. Then
\[
B'_d(\rho) = B_d(\rho) \frac{p}{q} \quad \text{and} \quad B''_d(\rho) = B_d(\rho) \frac{p}{q} + B_d(\rho) \frac{d}{d\rho} \left( \frac{p}{q} \right).
\]
Replace these expressions in the definition of \( A_d(\rho) \) and factor out \( B^3_d \). Then we have to show that
\[
\tilde{A}_d(\rho) := -(D+1) \left( \frac{p^3}{q^3} + \frac{3p}{q} \frac{d}{d\rho} \frac{p}{q} + \frac{d^2}{d\rho^2} \frac{p}{q} \right) + 3D \left( \frac{p^2}{q^2} + \frac{d}{d\rho} \frac{p}{q} \right) \frac{p}{q} - \frac{Dp^32D+1}{q^3(D+1)}
\]
is positive on \((0,1)\). Simplification gives
\[
\tilde{A}_d(\rho) = -(D+1) \left( \frac{d^2}{d\rho^2} \frac{p}{q} \right) - 3\frac{p}{q} \frac{d}{d\rho} \frac{p}{q} - \frac{1}{D+1} \frac{p^3}{q^3} \geq 0
\]
A calculation (e.g. with Maple) gives that
\[
\tilde{A}_d(\rho) = \frac{\text{Do}_d(\rho^2)}{\rho^3(D+1)(\rho^{2D} - 1)^3(\rho^2 - 1)^3}
\]
where \( \text{Do}_d(x) \) is equal to
\[
a_d(x) = 4(D - 1) + 12x(D + 1)
\]
\[- 4x^D(2D + 1)(D + 1)(2D + D^2 - 2) + 12x^{D+1}(-4D + 5D^2 + 7D^3 + 2D^4 - 2)
\]
\[- 12x^{D+2}(D + 1)(2D + 5D^2 + 2D^3 + 1) + 4x^{D+3}(2D + 1)(D + 1)^3
\]
\[- 4x^{2D}(2D + 1)(D + 1)^3 + 12x^{2D+1}(D + 1)(2D + 5D^2 + 2D^3 + 1)
\]
\[- 12x^{2D+2}(-4D + 5D^2 + 7D^3 + 2D^4 - 2) + 4x^{2D+3}(2D + 1)(D + 1)(2D + 2D^2 - 2)
\]
\[- 12x^{3D+2}(D + 1) - 4x^{3D+3}(D - 1). \]
Note that $a_d(x)$ is a linear combination of 12 power functions. It can be shown that the function $a_d(x)$ has a zero of order 7 at $x = 1$. Further

$$\lim_{x \to 1} \frac{d^7}{dx^7} a_d(x) = -168D^3(D-1)(2D+1)(D+2)(D+1)^3 < 0$$

Note that $a_d(0) > 0$. If $a_d(x_0) < 0$ for some $x_0 \in (0, 1)$ we see that $a_d$ has an a zero in $(0, x_0)$. Since $a_d(x) > 0$ for $x < 1$ close enough to 1 we see that $a_d$ has also a zero in $(x_0, 1)$. Due to the symmetry of the function

$$-x^{3D+3} a_d \left( \frac{1}{x} \right) = a_d(x)$$

we infer that the function $a_d$ has at least 11 positive zeros. This is impossible since the coefficients of the function $a_d$ have only 9 changes of sign.

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