LMI-based robust stability and stabilization analysis of fractional-order interval systems with time-varying delay

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ABSTRACT
This paper investigates the robust stability and stabilization analysis of interval fractional-order systems with time-varying delay. The stability problem of such systems is solved first, and then using the proposed results, a stabilization theorem is also included, where sufficient conditions are obtained for designing a stabilizing controller with a predetermined order, which can be chosen to be as low as possible. Utilizing efficient lemmas, the stability and stabilization theorems are proposed in LMI form, which is more suitable to check due to various existing efficient convex optimization parsers and solvers. Finally, some numerical examples have shown the effectiveness of our results.

1. Introduction

In the last decades, utilizing fractional-order calculus has opened new horizons in modeling real-world systems, since it can more concisely describe the behavior of systems having a response with long memory transients and other intrinsic features which are more suitable with fractional-order equations (Mondal et al. 2020; Mohsenipour and Jegarkandi 2019; Parvizian, Khandani, and Majd 2020). Moreover, it has been proven that fractional-order controllers have significant advantages over integer-order ones (Badri and Tavazoei 2016, 2015; Badri, Sojoodi, and Zavvari 2021; Huang and Xiang 2016; Zou and Xiang 2018). Hence, a lot of studies have been focused on fractional-order control systems (Muñoz-Vázquez et al. 2020; Ghorbani 2021).

Real systems in various areas, such as engineering, biology and economics, are sometimes confronted with time delays, which can lead to instability and oscillations in such systems (Phat and Ratchagit 2011; Mohsenipour and Jegarkandi 2019; He et al. 2004; Badri and Tavazoei 2019). Thus, in the past few years, the stability and stabilization problem of time-delay systems has attracted particular attention (Xu and Lam 2008), including fractional-order systems (Mohsenipour and Jegarkandi 2019; Moornani and Haeri 2010; Li et al. 2018; Liu et al. 2019; Zhang, Liu, and Cui 2019; Sadalla et al. 2020; Soukkou, Boukabou, and Goutas 2018; Phat, Khongtham, and Ratchagit 2012; Rajchakit 2012). In Gao (2017), the robust stabilization problem of interval fractional-order systems with
one time delay using fractional-order controllers by the Minkowski sum of value sets was investigated. Moreover, in Mohsenipour and Jegarkandi (2019), the robust stability of fractional-order interval systems with multiple time delays was discussed. In Liu et al. (2019), the robust stability of a fractional-order time-delay system is investigated in the frequency domain based on finite spectrum assignment. This algorithm is an extension of the traditional pole assignment method, which can change the undesirable system characteristic equation into a desirable one. Furthermore, in Zhang, Liu, and Cui (2019), a robust fractional-order PID controller is designed for fractional-order delay systems based on positive stability region (PSR) analysis. The finite-time stability of linear delay fractional-order systems is also investigated in Naifar et al. (2019) based on the generalized Gronwall inequality and the Caputo fractional derivative.

Moreover, modeling real-world processes usually lead to uncertain models due to neglected dynamics, uncertain physical parameters, parametric variations in time and so on. Therefore, robust stability and stabilization became an important problem for all control systems including fractional-order ones (Gao 2017; Lu and Chen 2009; Ma, Lu, and Chen 2014; Badri and Sojoodi 2019a; Rajchakit, Rojsiraphisal, and Rajchakit 2012). In Ma, Lu, and Chen (2014), the problems of the robust stability and stabilization of fractional-order linear systems with positive real uncertainty are proposed. As it has been declared that interval uncertainty is more convenient for the control system design problems (Alagoz 2016) and robust stability analysis (Henrion, Sebek, and Kucera 2001), stability and stabilization problems of fractional-order interval system are investigated in Lu and Chen (2009). A new sufficient condition in terms of LMI for the global asymptotic stability of a class of interval fractional-order nonlinear systems with time-varying delay was proposed in Li et al. (2018), where the state matrix of the linear part of the system is supposed to be diagonal.

In the majority of available controller design methods, high-order controllers are obtained suffering from costly implementation, high fragility, unfavorable reliability, maintenance difficulties and potential numerical errors. Designing a controller with a low and fixed order would be helpful because the desired closed-loop performance is not necessarily assured by available plant or controller order reduction procedures (Badri, Amini, and Sojoodi 2016; Badri, Zavary, and Sojoodi 2019). Therefore, in our previous works, fixed-order controllers have been designed for fractional-order systems (Badri and Sojoodi 2018, 2019b, 2019a).

Motivated by the aforementioned observations, our paper aims at solving the problem of stability and stabilization of interval fractional-order systems with time-varying delay in terms of linear matrix inequalities LMIs, which is suitable to be used in practice due to various efficient convex optimization parsers and solvers that can be applied to determine the feasibility of the LMI constraints and consequently calculate design parameters.

The main contributions of this paper can be summarized as follows:

- LMI conditions are obtained for stability check of fractional-order interval systems with time-varying delay.
- Robust stabilizing problem of such systems is investigated using proposed LMI stability conditions.
Compared to the commonly used state feedback control protocols for stability in FOMAS, the implementation in this article is based on the dynamic output feedback protocol with its well-known benefits.

Fixed-order dynamic output feedback controller order is designed whose order can be determined before design.

As far as we know, there is no result on the robust stability of uncertain FO-LTI systems, with time-varying delays in the literature. Moreover, the analytical design of a stabilizing dynamic output feedback controller for interval fractional-order systems with time-varying delay is investigated for the first time. It is worth noting that LMI stability conditions for uncertain FO systems with time delay, which are more comfortable to check, are obtained for the first time in this paper.

The fractional-order system with time-varying delay defined in this paper is affected by interval uncertainty. The main purpose of the paper is to derive some LMI constraints that can be easily checked by existing solvers and parsers to obtain the stabilizing controller parameters. It does not seem straightforward to derive linear stabilizing constraints for a linear system which is affected by time-varying delay. However using helpful lemmas which were introduced in Preliminaries and problem formulation section, and also utilizing LMI techniques which were used in the proof procedures of the theorems, interval uncertainty parameters were involved in the achieved LMI constraints. Then, by solving the mentioned LMIs, the unknown parameters of the controller can be achieved to establish the stability of the uncertain fractional-order system with time-varying delay. Therefore, stabilizing control systems plays a crucial role in control engineering which is an undeniable fact. Moreover, uncertainty and time delay are in real systems' nature. As a result, a dynamic robust controller covers a wide range of systems, and its advantages overtake static controller ones. Besides that, designing a dynamic robust controller leads to more unknown parameters in comparison with a static controller and makes controller design procedure more difficult due to more complex constraints which must be solved by the solvers. In this paper, using proper lemmas and theorems, LMI techniques and suitable solvers and parsers, the difficulty of designing such controllers has been overcome.

The rest of this paper is organized as follows: In Section 2, some preliminaries about interval uncertainty and fractional-order calculus together with the problem formulation are presented. LMI-based robust stability and stabilizing conditions using a dynamic output feedback controller are derived in Section 3. Some numerical examples are given in Section 4 to illustrate the effectiveness of the proposed theoretical results. Finally, the conclusion is drawn in Section 5.

Notations: In this paper, by $M^T$ we denote the transpose of matrix $M$, and $\text{Sym}(M)$ denote $M + M^T$. The notation $\ast$ is the symmetric component symbol in matrix, and $\dagger$ is the symbol of the pseudo inverse. Moreover, the notation $0$ denotes the zero matrix with appropriate dimensions.

2. Preliminaries and problem formulation

In this section, some basic concepts and lemmas of fractional-order calculus and interval uncertainty are presented.
Consider the following uncertain FO-LTI system for $0 < \alpha < 1$:

$$
\begin{cases}
D^\alpha x(t) = Ax(t) + Bu(t - d(t)), & t > 0, \\
x(t) = \phi(t), & t \in [-\tau, 0]
\end{cases}
$$

in which $x(t) \in \mathbb{R}^n$ denotes the pseudo-state vector, $u \in \mathbb{R}^l$ is the control input and $y \in \mathbb{R}^m$ is the output vector. Furthermore, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times l}$ are interval uncertain matrices as follows:

$$A \in A_I = [\bar{A}, \tilde{A}] = \{(a_{ij}) : a_{ij} \leq \bar{a}_{ij}, 1 \leq i, j \leq n\},$$ (2)

$$B \in B_I = [\bar{B}, \tilde{B}] = \{(b_{ij}) : b_{ij} \leq \bar{b}_{ij}, 1 \leq i \leq n, 1 \leq j \leq l\},$$ (3)

where $A = [a_{ij}]_{n \times n}$ and $\tilde{A} = [\bar{a}_{ij}]_{n \times n}$ satisfy $a_{ij} \leq \bar{a}_{ij}$ for all $1 \leq i, j \leq n$, and $B = [b_{ij}]_{n \times l}$ and $\tilde{B} = [\bar{b}_{ij}]_{n \times l}$ satisfy $b_{ij} \leq \bar{b}_{ij}$ for all $1 \leq i \leq n, 1 \leq j \leq l$. The time delay $d(t)$ is a time-varying continuous function that satisfies

$$0 \leq d(t) \leq \tau,$$ (4)

and

$$\dot{d}(t) \leq \mu < 1$$ (5)

where $\tau$ and $\mu$ are constants, and the initial condition $\phi(t)$ represents a continuous vector-valued initial function of $t \in [-\tau, 0]$.

In this article, the following Caputo definition for fractional derivatives of order $\alpha$ of function $f(t)$ is utilized (Podlubny 1998):

$$C_a^D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t - \tau)^{m-\alpha-1} \left( \frac{d}{d\tau} \right)^m f(\tau) d\tau$$

where $\Gamma(\cdot)$ is the Gamma function defined by $\Gamma(\epsilon) = \int_0^\infty e^{-t} t^{\epsilon-1} dt$, and $m$ is the smallest integer that is equal to or greater than $\alpha$.

The following notations are needed for dealing with interval uncertainties:

$$A_0 = 1/2(\bar{A} + \tilde{A}), \quad \Delta A = 1/2(\tilde{A} - \bar{A}) = \{\gamma_{ij}\}_{n \times n},$$ (6)

$$B_0 = 1/2(\bar{B} + \tilde{B}), \quad \Delta B = 1/2(\tilde{B} - \bar{B}) = \{\beta_{ij}\}_{n \times l},$$ (7)

It is evident that all elements of $\Delta A$ and $\Delta B$ are nonnegative; therefore, the following matrices are defined as follows:

$$M_A = \begin{bmatrix} \sqrt{\gamma_{11}^n e_1^1} & \cdots & \sqrt{\gamma_{n1}^n e_n^1} & \cdots & \sqrt{\gamma_{nn}^n e_n^n} \end{bmatrix}_{n \times n},$$ (8)

$$R_A = \begin{bmatrix} \sqrt{\gamma_{11}^m e_1^1} & \cdots & \sqrt{\gamma_{n1}^m e_n^1} & \cdots & \sqrt{\gamma_{nn}^m e_n^n} \end{bmatrix}_{n \times n}^T,$$ (9)

$$M_B = \begin{bmatrix} \sqrt{\beta_{11}^n e_1^1} & \cdots & \sqrt{\beta_{n1}^n e_n^1} & \cdots & \sqrt{\beta_{nn}^n e_n^n} \end{bmatrix}_{n \times n l},$$ (10)

$$R_B = \begin{bmatrix} \sqrt{\beta_{11}^l e_1^1} & \cdots & \sqrt{\beta_{n1}^l e_n^1} & \cdots & \sqrt{\beta_{nn}^l e_n^n} \end{bmatrix}_{n \times l}^T,$$ (11)

where $e_k^n \in \mathbb{R}^n$, $e_k^l \in \mathbb{R}^l$ and $e_k^m \in \mathbb{R}^m$ are column vectors with the $k$th element being 1 and all the others being 0. In addition, we have

$$H_A = \{\text{diag}(\delta_{11}, \ldots, \delta_{1n}, \ldots, \delta_{n1}, \ldots, \delta_{nn}) \in \mathbb{R}^{n^2 \times n^2}, |\delta_{ij}| \leq 1, i,j, \ldots, n\},$$ (12)
\[ H_B = \{ \text{diag}(\eta_{11}, \ldots, \eta_{1l}, \ldots, \eta_{n1}, \ldots, \eta_{nl}) \in \mathbb{R}^{(nl) \times (nl)}, \\
|\eta_{ij}| \leq 1, i = 1, \ldots, n, j = 1, \ldots, l \} \]  

The following lemmas are required to study the stability of interval fractional-order systems.

**Lemma 2.1 (Lu and Chen 2009):** Let

\[ A_I = \{ A = A_0 + M_AR_A | F_A \in H_A \}, B_I = \{ B = B_0 + M_BR_B | F_B \in H_B \}, \]  

then \( A_I = A_I \) and \( B_I = B_I \).

**Lemma 2.2 (Lu and Chen 2009):** For any matrices \( X \) and \( Y \) with appropriate dimensions, we have

\[ X^T Y + Y^T X \leq \eta X^T X + (1/\eta) Y^T Y \text{ for any } \eta > 0. \]  

**Lemma 2.3 (He et al. 2004):** For given scalars \( \tau > 0 \) and \( \mu < 1 \), the following certain integer-order system

\[ \begin{cases} 
\dot{x}(t) = Ax(t) + Bx(t - d(t)), & t > 0 \\
x(t) = \phi(t), & t \in [-\tau, 0] 
\end{cases} \]  

with fixed matrices \( A \) and \( B \) and a time-varying state delay \( d(t) \) satisfying (4) and (5) is asymptotically stable if there exist \( P = P^T > 0, Q = Q^T \geq 0, Z = Z^T > 0 \) and appropriately dimensioned matrices \( N_i \) and \( T_i \) \((i = 1, 2, 3)\) such that the following LMI holds:

\[ \Gamma = \begin{bmatrix} 
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \tau N_1 \\
\Gamma_{12}^T & \Gamma_{22} & \Gamma_{23} & \tau N_2 \\
\Gamma_{13}^T & \Gamma_{23}^T & \Gamma_{33} & \tau N_3 \\
\tau N_1^T & \tau N_2^T & \tau N_3^T & -\tau Z 
\end{bmatrix} < 0. \]  

where

\[ \begin{align*}
\Gamma_{11} &= Q + N_1 + N_1^T - A^T T_1^T - T_1 A, \\
\Gamma_{12} &= N_2^T - N_1 - A^T T_2^T - T_1 B, \\
\Gamma_{13} &= P + N_3^T + T_1 - A^T T_3^T, \\
\Gamma_{22} &= -(1 - \mu)Q - N_2 - N_2^T - T_2 B - B^T T_2^T, \\
\Gamma_{23} &= -N_3^T + T_2 - B^T T_3^T, \\
\Gamma_{33} &= \tau Z + T_3 + T_3^T. 
\end{align*} \]  

The proof of this lemma is presented in He et al. (2004), using the following Lyapunov–Krasovskii functional:

\[ V(x(t)) = x^T(t)Px(t) + \int_{t-d(t)}^{t} x^T(s)Qx(s) + \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)Z\dot{x}(s)ds d\theta. \]
Lemma 2.4 (Badri and Tavazoei 2019): Without loss of generality, suppose that $x = 0$ is the equilibrium point of integer and fractional-order time-delay systems

$$\dot{x} = f(x(t), x(t - \tau)), \tau \in [0, \infty). \quad (20)$$

$$C_t^0D^\alpha_t = f(x(t), x(t - \tau)), \tau \in [0, \infty), \alpha \in (0, 1). \quad (21)$$

If there exists a Lyapunov–Krasovskii functional in the form

$$V_I(t) = V(x(t)) + \int_{t-\tau}^{t} g(x(s))ds \quad (22)$$

for the system (20) such that $\dot{V}_I(t)$ is the negative definite and $V(x(t))$ is a convex function with respect to vector $x$, then the equilibrium point $x = 0$ of the system (21) is asymptotically stable.

Remark 2.1: For given scalars $\tau > 0$ and $\mu < 1$, the following certain fractional-order system

$$\begin{cases} D^\alpha x(t) = Ax(t) + Bx(t - d(t)), & t > 0 \\ x(t) = \phi(t), & t \in [-\tau, 0] \end{cases} \quad (23)$$

with fixed matrices $A$ and $B$ and a time-varying state delay $d(t)$ satisfying (4) and (5) is asymptotically stable for any $0 < \alpha < 1$ if there exist $P = P^T > 0$, $Q = Q^T \geq 0$, $Z = Z^T > 0$ and appropriately dimensioned matrices $N_i$ and $T_i$ ($i = 1, 2, 3$) such that the LMI constraint (17) holds.

Proof: As Lyapunov–Krasovskii functional (19) is in the form of (22) of Lemma 2.4., LMI constraint (17) of Lemma 2.3. can also stabilize the fractional-order system (23). $\blacksquare$

3. Main results

In this section first, a new robust stability condition is derived for interval delay system (1) using which an LMI approach is proposed for designing a dynamic output feedback control law to robustly stabilize it.

3.1. Robust stability

In this subsection, a robust stability sufficient condition is established for the asymptotic stability of the system (1) with $u(t) \equiv 0$.

Theorem 1: For given scalars $\tau > 0$ and $\mu < 1$, fractional-order interval system (1), with $0 < \alpha < 1$, $A \in A_I$, and $B \in B_I$ together with a time-varying state delay $d(t)$ satisfying (4) and (5) is asymptotically stable if there exist $P = P^T > 0$, $Q = Q^T \geq 0$, $Z = Z^T > 0$ and appropriately dimensioned matrices $N_i$ and $T_i$ ($i = 1, 2, 3$) such that the following LMI holds:

$$\begin{bmatrix} \phi & M^T \\ M & -\eta I \end{bmatrix} < 0, \quad (24)$$
in which

\[
\phi = \begin{bmatrix}
Q + N_1 + N_1^T & N_2^T - N_1 & P + N_3^T + T_1 & \tau N_1 \\
* & -(1 - \mu)Q - N_2 - N_2^T & -N_3^T + T_2 & \tau N_2 \\
* & * & \tau Z + T_3 + T_3^T & \tau N_3 \\
* & * & * & -\tau Z
\end{bmatrix}
\]

\[
+ \text{sym} \left\{ \begin{bmatrix}
-T_1 A_0 & -T_1 B_0 & 0 & 0 \\
-T_2 A_0 & -T_2 B_0 & 0 & 0 \\
-T_3 A_0 & -T_3 B_0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \right\}
\]

\[
+ \eta \begin{bmatrix}
R_A & 0 & 0 & 0 \\
0 & R_B & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}^T
\]

\[
M = \begin{bmatrix}
-T_1 M_A & -T_1 M_B & 0 & 0 \\
-T_2 M_A & -T_2 M_B & 0 & 0 \\
-T_3 M_A & -T_3 M_B & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

**Proof:** According to Remark 1, system (1) is asymptotically stable if

\[
\Gamma = \begin{bmatrix}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \tau N_1 \\
\Gamma_{12}^T & \Gamma_{22} & \Gamma_{23} & \tau N_2 \\
\Gamma_{13}^T & \Gamma_{23}^T & \Gamma_{33} & \tau N_3 \\
\tau N_1^T & \tau N_2^T & \tau N_3^T & -\tau Z
\end{bmatrix} < 0,
\]

in which

\[
\Gamma_{11} = Q + N_1 + N_1^T - (A_0 + M_A F_A R_A)^T T_1^T - T_1^T (A_0 + M_A F_A R_A),
\]

\[
\Gamma_{12} = N_2^T - N_1 - (A_0 + M_A F_A R_A)^T T_2^T - T_1 (B_0 + M_B F_B R_B),
\]

\[
\Gamma_{13} = P + N_3^T + T_1 - (A_0 + M_A F_A R_A)^T T_3^T,
\]

\[
\Gamma_{22} = -(1 - \mu)Q - N_2 - N_2^T - T_2 (B_0 + M_B F_B R_B) - (B_0 + M_B F_B R_B)^T T_2^T,
\]

\[
\Gamma_{23} = -N_3^T + T_2 - (B_0 + M_B F_B R_B)^T T_3^T,
\]

\[
\Gamma_{33} = \tau Z + T_3 + T_3^T.
\]

Inequality (26) can be rewritten as follows:

\[
\Gamma = \begin{bmatrix}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \tau N_1 \\
\Gamma_{12} & \Gamma_{22} & \Gamma_{23} & \tau N_2 \\
\Gamma_{13} & \Gamma_{23} & \Gamma_{33} & \tau N_3 \\
\tau N_1^T & \tau N_2^T & \tau N_3^T & -\tau Z
\end{bmatrix}
\]
\[
\begin{bmatrix}
Q + N_1 + N_1^T & N_2^T - N_1 & P + N_2^T + T_1 & \tau N_1 \\
* & -(1 - \mu)Q - N_2 - N_2^T & -N_3^T + T_2 & \tau N_2 \\
* & * & \tau Z + T_3 + T_3^T & \tau N_3 \\
* & * & * & -\tau Z
\end{bmatrix}
\]

\[
+ \text{sym} \begin{bmatrix}
-T_1 A_0 & -T_1 B_0 & 0 & 0 \\
-T_2 A_0 & -T_2 B_0 & 0 & 0 \\
-T_3 A_0 & -T_3 B_0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
+ \text{sym} \begin{bmatrix}
-T_1 M_A & -T_1 M_B & 0 & 0 \\
-T_2 M_A & -T_2 M_B & 0 & 0 \\
-T_3 M_A & -T_3 M_B & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
F_A & 0 & 0 & 0 \\
0 & F_B & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
R_A & 0 & 0 & 0 \\
0 & R_B & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
R_A & 0 & 0 & 0 \\
0 & R_B & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}^T < 0
\]

Applying Lemma 2.2 to the third part of the right-hand side of inequality (28), the following inequality can be obtained for a scalar \( \eta > 0 \):

\[
\Gamma = \begin{bmatrix}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \tau N_1 \\
\Gamma_{12}^T & \Gamma_{22} & \Gamma_{23} & \tau N_2 \\
\Gamma_{13}^T & \Gamma_{23}^T & \Gamma_{33} & \tau N_3 \\
\tau N_1^T & \tau N_2^T & \tau N_3^T & -\tau Z
\end{bmatrix}
\]

\[
\begin{bmatrix}
Q + N_1 + N_1^T & N_2^T - N_1 & P + N_2^T + T_1 & \tau N_1 \\
* & -(1 - \mu)Q - N_2 - N_2^T & -N_3^T + T_2 & \tau N_2 \\
* & * & \tau Z + T_3 + T_3^T & \tau N_3 \\
* & * & * & -\tau Z
\end{bmatrix}
\]

\[
+ \text{sym} \begin{bmatrix}
-T_1 A_0 & -T_1 B_0 & 0 & 0 \\
-T_2 A_0 & -T_2 B_0 & 0 & 0 \\
-T_3 A_0 & -T_3 B_0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
+ \eta^{-1} \begin{bmatrix}
-T_1 M_A & -T_1 M_B & 0 & 0 \\
-T_2 M_A & -T_2 M_B & 0 & 0 \\
-T_3 M_A & -T_3 M_B & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
-T_1 M_A & -T_1 M_B & 0 & 0 \\
-T_2 M_A & -T_2 M_B & 0 & 0 \\
-T_3 M_A & -T_3 M_B & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}^T
\]

\[
+ \eta \begin{bmatrix}
R_A & 0 & 0 & 0 \\
0 & R_B & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
R_A & 0 & 0 & 0 \\
0 & R_B & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}^T < 0
\]

Inequality (29) is nonlinear because of several multiplications of variables. Therefore, by applying Schur complement on the second part of the right side of the latter inequality,
one has
\[
\begin{bmatrix}
\phi & M^T \\
M & -\eta I
\end{bmatrix} < 0
\]
\[
\phi = \begin{bmatrix}
Q + N_1 + N_1^T & N_2^T - N_1 & P + N_2^T + T_1 & \tau N_1 \\
* & -(1 - \mu)Q - N_2 - N_2^T & -N_3^T + T_2 & \tau N_2 \\
* & * & \tau Z + T_3 + T_3^T & \tau N_3 \\
+ \text{sym} & & & -\tau Z
\end{bmatrix}
\]
\[
M = \begin{bmatrix}
-T_1 A_0 & -T_1 B_0 & 0 & 0 \\
-T_2 A_0 & -T_2 B_0 & 0 & 0 \\
-T_3 A_0 & -T_3 B_0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
R_A & 0 & 0 & 0 \\
0 & R_B & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
R_A & 0 & 0 & 0 \\
0 & R_B & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}^T
\]

which is equivalent to LMI in (24), and it completes the proof.

3.2. Robust stabilization

The main purpose of the authors in this subsection is to design a robust dynamic output feedback controller that asymptotically stabilizes the interval FO-LTI system (1) in terms of LMIs. Hence, the following dynamic output feedback controller is presented:

\[
\begin{align*}
D^\alpha x_C(t) &= A_C x_C(t) + B_C y(t), \quad 0 < \alpha < 1 \\
u(t) &= C_C x_C(t) + D_C y(t),
\end{align*}
\]

with \( x_C \in \mathcal{R}^{n_c} \), in which \( n_c \) is the arbitrary order of the controller and \( A_C, B_C, C_C, \) and \( D_C \) are corresponding matrices to be designed.

The resulting closed-loop augmented FO-LTI system using (1) and (31) is as follows:

\[
D^\alpha x_{Cl}(t) = A_{Cl} x_{Cl}(t) + A_{dCl} x_{Cl}(t - d(t)), \quad 0 < \alpha < 1
\]

with
\[
\begin{align*}
x_{Cl}(t) &= \begin{bmatrix} x(t) \\ x_C(t) \end{bmatrix}, \\
A_{Cl} &= \begin{bmatrix} A & 0 \\ B_C & A_C \end{bmatrix}, \\
A_{dCl} &= \begin{bmatrix} B D_C C & B C_C \\ 0 & 0 \end{bmatrix}.
\end{align*}
\]

Next, a robust stabilization theorem is established.
Theorem 2: For given scalars $\tau > 0$ and $\mu < 1$, closed-loop system (32), with $0 < \alpha < 1$, $A \in A_I$, $B \in B_I$ and output matrix $C$ together with and a time-varying state delay $d(t)$ satisfying (4) and (5), if there exist $P = P^T > 0$, $Q = Q^T \geq 0$, $Z = Z^T > 0$ and appropriately dimensioned matrices $N_i$ ($i = 1, 2, 3$) and $W_j$ ($j = 1, 2, 3, 4$) and matrix $T = T^T$ in the form of

$$T = \text{diag}(T_S, T_C), T_S \in R^{n \times n}, T_C \in R^{n_C \times n_C},$$

(34)

such that the following LMI constrain becomes feasible:

$$\phi + \eta \Sigma_1^T \Sigma_1 + \eta^{-1} \Sigma_2^T \Sigma_2 < 0,$$

(35)

in which

$$\phi = \text{sym} \begin{bmatrix} N_1 + Q & -N_1 & P + T & \tau N_1 \\ N_2 & -N_2 - (1 - \mu) Q & T & \tau N_2 \\ N_3 & -N_3 & T + \tau Z & \tau N_3 \\ 0 & 0 & 0 & -\tau Z \end{bmatrix},$$

(36)

$$\Sigma_1 = \begin{bmatrix} -M_A & 0 & -M_B & -M_B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -M_A & 0 & -M_B & -M_B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -M_A & 0 & -M_B & -M_B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Sigma_2 = \begin{bmatrix} R_A T_S & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_B D_C C T_S & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_B C_C T_C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

(37)

then, the dynamic output feedback controller parameters of

$$A_C = T_C^{-1} W_1, B_C = T_C^{-1} W_2, C_C = B^\dagger T_S^{-1} W_3, D_C = B^\dagger T_S^{-1} W_4,$$

make the closed-loop system (32) asymptotically stable.
**Proof:** The closed-loop system (32) can be considered as follows:

\[ D^\alpha x_{Cl}(t) = A_{Cl}x_{Cl}(t) + A_{dCl}x_{Cl}(t - d(t)), 0 < \alpha < 1, \]  

(38)

with

\[
x_{Cl}(t) = \begin{bmatrix} x(t) \\ x_C(t) \end{bmatrix}, A_{Cl} = A_{0Cl} + A_{\Delta Cl}, A_{dCl} = A_{0dCl} + A_{\Delta dCl}, A_{0Cl} = \begin{bmatrix} A_0 & 0 \\ B_{C}C & A_C \end{bmatrix},
\]

\[
A_{\Delta Cl} = \begin{bmatrix} M_{A}F_{A}R_A & 0 \\ 0 & 0 \end{bmatrix}, A_{0dCl} = \begin{bmatrix} B_{D}C & B_{C}C \\ 0 & 0 \end{bmatrix},
\]

\[
A_{\Delta dCl} = \begin{bmatrix} M_{B}F_{B}R_BD_{C}C & M_{B}F_{B}R_{B}C \\ 0 & 0 \end{bmatrix}.
\]

According to Remark 1, the closed-loop uncertain system (32) is asymptotically stable if

\[
\Gamma = \begin{bmatrix} \Gamma_{11}' & \Gamma_{12}' & \Gamma_{13}' & \tau N_1' \\ \Gamma_{12}' & \Gamma_{22}' & \Gamma_{23}' & \tau N_2' \\ \Gamma_{13}' & \Gamma_{23}' & \Gamma_{33}' & \tau N_3' \\ \tau N_1'^T & \tau N_2'^T & \tau N_3'^T & -\tau Z' \end{bmatrix} < 0
\]

(40)

in which we have

\[
\Gamma_{11}' = Q' + N_1' + N_1'^T - \text{sym}\{T_1(A_{0Cl} + A_{\Delta Cl})\},
\]

\[
\Gamma_{12} = N_2' - N_1 - (A_{0Cl} + A_{\Delta Cl})^T T_2^T - T_1(A_{0dCl} + A_{\Delta dCl}),
\]

\[
\Gamma_{13} = P + N_3' + T_1 - (A_{0Cl} + A_{\Delta Cl})^T T_3^T,
\]

\[
\Gamma_{22} = -(1 - \mu)Q - N_2 - N_2'^T - \text{sym}\{T_2(A_{0dCl} + A_{\Delta dCl})\},
\]

\[
\Gamma_{23} = -N_3^T + T_2 - (A_{0dCl} + A_{\Delta dCl})^T T_3^T,
\]

\[
\Gamma_{33} = \tau Z + T_3 + T_3^T.
\]

(41)

and \(Q' = Q'^T \geq 0, Z' = Z'^T > 0, P' = P'^T > 0, T_i' \) and \(N_i' (i = 1, 2, 3) \) are matrices with appropriate dimensions. By assuming \(T_i' = T', i = 1, 2, 3 \) in the form of (34), i.e. \(T' = T'^T = \text{diag}(T_S, T_C), T_S \in \mathbb{R}^n, T_C \in \mathbb{R}^{nc} \), and pre- and post-multiplying inequality (40) by \(\text{diag}(T'^{-1}, T'^{-1}, T'^{-1}, T'^{-1}) \) one has

\[
\begin{bmatrix} T'^{-1} & 0 & 0 & 0 \\ 0 & T'^{-1} & 0 & 0 \\ 0 & 0 & T'^{-1} & 0 \\ 0 & 0 & 0 & T'^{-1} \end{bmatrix} \times \begin{bmatrix} \Gamma_{11}' & \Gamma_{12}' & \Gamma_{13}' & \tau N_1' \\ \Gamma_{12}' & \Gamma_{22}' & \Gamma_{23}' & \tau N_2' \\ \Gamma_{13}' & \Gamma_{23}' & \Gamma_{33}' & \tau N_3' \\ \tau N_1'^T & \tau N_2'^T & \tau N_3'^T & -\tau Z' \end{bmatrix} < 0 \Rightarrow \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \tau N_1 \\ \Gamma_{12} & \Gamma_{22} & \Gamma_{23} & \tau N_2 \\ \Gamma_{13} & \Gamma_{23} & \Gamma_{33} & \tau N_3 \\ \tau N_1^T & \tau N_2^T & \tau N_3^T & -\tau Z \end{bmatrix} < 0
\]

(42)
\[
\Gamma'_{11} = T'^{-1}Q'T'^{-1} + T'^{-1}N_1'T'^{-1} + T'^{-1}N_1'T^{-1} - \text{sym}\{(A_0C_I + A_{\Delta C_I})T'^{-1}\}
\]
\[
\Gamma'_{12} = T'^{-1}N_2'T'^{-1} - T'^{-1}N_1'T'^{-1} - T'^{-1}(A_0C_I + A_{\Delta C_I})T - (A_0dC_I + A_{\Delta dC_I})T'^{-1}
\]
\[
\Gamma'_{13} = T'^{-1}P'T'^{-1} + T'^{-1}N_3'T'^{-1} + T'^{-1} - T'^{-1}(A_0C_I + A_{\Delta C_I})T
\]
\[
\Gamma'_{22} = -(1 - \mu)T'^{-1}Q'T'^{-1} - T'^{-1}N_2'T'^{-1} - T'^{-1}N_2'T^{-1}
\]
\[
- \text{sym}\{(A_0dC_I + A_{\Delta dC_I})T'^{-1}\}
\]
\[
\Gamma'_{23} = -T'^{-1}N_3'T'^{-1} + T'^{-1} - T'^{-1}(A_0dC_I + A_{\Delta dC_I})T,
\]
\[
\Gamma'_{33} = \tau T'^{-1}Z'T'^{-1} + 2T'^{-1},
\]

(43)

According to the symmetry of the matrix \(T'\), the following matrices can be defined:

\[
T = T'^{-1} = T^T, Q = T'^{-1}Q'T'^{-1} = Q^T \geq 0, N_1 = T'^{-1}N_1'T'^{-1}, N_2 = T'^{-1}N_2'T'^{-1},
\]
\[
N_3 = T'^{-1}N_3'T'^{-1}, P = T'^{-1}P'T'^{-1} = P^T > 0, Z = T'^{-1}Z'T'^{-1} = Z^T > 0.
\]

(44)

Therefore, inequality (42) can be rewritten as follows:

\[
\begin{bmatrix}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \tau N_1 \\
\Gamma_{12}^T & \Gamma_{22} & \Gamma_{23} & \tau N_2 \\
\Gamma_{13}^T & \Gamma_{23}^T & \Gamma_{33} & \tau N_3 \\
\tau N_1^T & \tau N_2^T & \tau N_3^T & -\tau Z
\end{bmatrix}
= \text{sym}
\begin{bmatrix}
N_1 + Q & -N_1 & P + T & \tau N_1 \\
N_2 & -N_2 - (1 - \mu)Q & T & \tau N_2 \\
N_3 & -N_3 & T + \tau Z & \tau N_3 \\
0 & 0 & 0 & -\tau Z
\end{bmatrix}
\]

\[
+ \text{sym}
\begin{bmatrix}
-A_0T_S & 0 & -B_0C_CT_S & -B_0C_CT_C & 0 & 0 & 0 & 0 \\
-B_0C_CT_S & -A_CT_C & 0 & 0 & 0 & 0 & 0 & 0 \\
-A_0T_S & 0 & -B_0C_CT_S & -B_0C_CT_C & 0 & 0 & 0 & 0 \\
-B_0C_CT_S & -A_CT_C & 0 & 0 & 0 & 0 & 0 & 0 \\
-A_0T_S & 0 & -B_0C_CT_S & -B_0C_CT_C & 0 & 0 & 0 & 0 \\
-B_0C_CT_S & -A_CT_C & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
+ \text{sym}
\begin{bmatrix}
-M_A & 0 & -M_B & -M_B & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-M_A & 0 & -M_B & -M_B & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-M_A & 0 & -M_B & -M_B & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
applying Lemma 2.2. to the third part of the right-hand side of the latter inequality, the following inequality can be obtained for a scalar $\eta > 0$:

$$\begin{bmatrix}
-F_A & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & F_B & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & F_B & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\times
\begin{bmatrix}
R_A T_S & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\times
\begin{bmatrix}
R_A T_S & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\leq \eta \Sigma_1 \Sigma_1 + \eta^{-1} \Sigma_2 \Sigma_2^T, \quad (46)$$

in which we have

$$\Sigma_1 = \begin{bmatrix}
-M_A & 0 & -M_B & -M_B & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-M_A & 0 & -M_B & -M_B & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-M_A & 0 & -M_B & -M_B & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-M_A & 0 & -M_B & -M_B & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}^T,$$
\[
\Sigma_2 = \begin{bmatrix}
R_A T_S & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & R_B D_C C T_S & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & R_B C_C T_C & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]  
(47)

Substituting (45) in (44) yields into

\[
\begin{bmatrix}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \tau N_1 \\
\Gamma_{12}^T & \Gamma_{22} & \Gamma_{23} & \tau N_2 \\
\Gamma_{13}^T & \Gamma_{23}^T & \Gamma_{33} & \tau N_3 \\
\tau N_1^T & \tau N_2^T & \tau N_3^T & -\tau Z
\end{bmatrix}
= sym
\begin{bmatrix}
N_1 + Q & -N_1 & P + T & \tau N_1 \\
N_2 & -N_2 - (1 - \mu)Q & T & \tau N_2 \\
N_3 & -N_3 & T + \tau Z & \tau N_3 \\
0 & 0 & 0 & -\tau Z
\end{bmatrix}
+ sym
\begin{bmatrix}
-A_0 T_S & 0 & -B_0 D_C C T_S & -B_0 C_C T_C & 0 & 0 & 0 \\
-B_C C T_S & -A_C T_C & 0 & 0 & 0 & 0 & 0 \\
-A_0 T_S & 0 & -B_0 D_C C T_S & -C_C T_C & 0 & 0 & 0 \\
-B_C C T_S & -A_C T_C & 0 & 0 & 0 & 0 & 0 \\
-A_0 T_S & 0 & -B_0 D_C C T_S & -C_C T_C & 0 & 0 & 0 \\
-B_C C T_S & -A_C T_C & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\]

(48)

Inequality (47) is nonlinear due to various multiplications of variables. Hence, by applying Schur complement on \(\eta^{-1} \Sigma_2^T \Sigma_2\), and changing variables as follows:

\[
W_1 = T_C A_C, \quad W_2 = T_C B_C, \quad W_3 = T_S B_C C, \quad W_4 = T_S B D_C
\]

one can obtain linear matrix inequality (35) with parameters in (36).

**Corollary 1:** Although Theorems 1 and 2 are, respectively, allocated to robust stability and stabilization of uncertain FO-LTI systems of form (1), the proposed method can be easily used for the case of certain systems by solving the LMI constraints \(\phi < 0\) in these theorems.

**Proof:** The proof is straightforward by assuming \(A_{\Delta_G I} = 0\) in the proof procedure of Theorems 1 and 2. \(\blacksquare\)

**Remark 2:** In terms of the feasibility of the solution for control design, it is shown in Diwekar (2020) that the domain of feasibility increases by having additional parameters.
in solving an inequality. In another word, when we have more parameters in solving a linear matrix inequality, our degree of freedom increases. Accordingly, the proposed dynamic output feedback controller guarantees the stability of the uncertain delayed system more effectively, and solver has more degree of freedom in solving the inequality in comparison to a static feedback controller.

**Remark 3:** In the state feedback control scheme, all individual states of the system are needed to be measured and used in the feedback line. However, in some practical situations, measuring all states is impossible or may sound difficult due to economic issues or physical limitations (Badri, Sojoodi, and Zavvari 2021; Du, Sun, and Wang 2014). In these cases, using output feedback control could be effective, since there is no need to measure all individual states of the system; only by measuring outputs of the system, the control action is done.

**4. Numerical examples**

In this section, some numerical examples are given to demonstrate the applicability of the proposed method. In this paper, we use YALMIP parser (Löfberg 2004) and SeDuMi (Sturm 1999) solver in Matlab tool (Higham and Higham 2016) in order to assess the feasibility of the proposed constraints to obtain the controller parameters.

**4.1. Example 1**

In Mohsenipour and Jegarkandi (2019), robust stability of the fractional-order interval system

\[
G(s) = \frac{[1.3, 1.7]s^{0.3} + [1.4, 1.6]}{[1.5, 2.5]s^{0.6} + [2.5, 3.5]s^{0.3} + [1.5, 2.5]} e^{-0.1s},
\]

with fractional-order PI controller \( C(s) = 2 + 0.5s^{-0.3} \), proposed in Gao (2017), is checked. The aim of this subsection is to check the robust stability of the closed-loop delayed system using proposed LMI constraints in Theorem 1. The pseudo-state space representation of form (1) for the given system is as follows:

\[
A = \begin{bmatrix} -2.3333 & 1 \\ -1.6667 & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -1 \\ -0.6000 \end{bmatrix}, \quad B = \begin{bmatrix} 0.52 \\ 0.56 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1.1333 \\ 1.0667 \end{bmatrix}, \quad \alpha = 0.3 \quad (51)
\]

Moreover, the pseudo-state space representation of the given controller is

\[
A_C = 0, B_C = 1, C_C = 0.5, D_C = 2 \quad (52)
\]

therefore, the closed-loop system of form (32) can be represented by the following parameters:

\[
A_{cl} = \begin{bmatrix} -2.3333 & 1 & 0 \\ -1.6667 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \bar{A}_{cl} = \begin{bmatrix} -1 \\ -0.6000 \\ 1 \end{bmatrix}, \quad B_{cl} = \begin{bmatrix} 0.52 \\ 0.56 \\ 1.1333 \end{bmatrix}, \quad \bar{B}_{cl} = \begin{bmatrix} 1.0667 \end{bmatrix}, \quad \alpha = 0.3 \quad (53)
\]

\[
A_{dcl} = \begin{bmatrix} 1.0400 & 0 & 0.2600 \\ 1.1200 & 0 & 0.2800 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{A}_{dcl} = \begin{bmatrix} 2.2666 & 0 & 0.5666 \\ 2.1334 & 0 & 0.5333 \end{bmatrix}, \quad d(t) = 0.1.
\]
Using Theorem 1, the following parameters can be obtained, which illustrate the robust stability of the uncertain system (49), which has been concluded in Mohsenipour and Jegarkandi (2019), by calculating a bound on the poles of fractional-order interval systems and extending the concept of the value set and zero exclusion principle to these systems.

\[
\eta = 0.0014 > 0, \quad P = \begin{bmatrix}
14.0163 & -16.3965 & -1.7118 \\
-16.3965 & 20.9606 & 0.6398 \\
-1.7118 & 0.6398 & 1.2590
\end{bmatrix} > 0,
\]

\[
Q = \begin{bmatrix}
13.4193 & -13.0917 & -3.4991 \\
-13.0917 & 18.3259 & -0.5569 \\
-3.4991 & -0.5569 & 3.8213
\end{bmatrix} \geq 0,
\]

\[
Z = \begin{bmatrix}
27.2212 & -24.7145 & -8.2742 \\
-24.7145 & 39.1922 & -4.9566 \\
-8.2742 & -4.9566 & 12.7109
\end{bmatrix} > 0,
\]

\[
T_1 = \begin{bmatrix}
11.6175 & -24.5352 & -2.3622 \\
24.5352 & -24.5352 & -2.3622
\end{bmatrix}
\quad T_2 = \begin{bmatrix}
0.6292 & 7.3876 & 4.6803 \\
7.3876 & -7.3876 & -4.6803
\end{bmatrix},
\]

\[
T_3 = \begin{bmatrix}
5.5947 & -12.9149 & -1.2126 \\
12.9149 & -12.9149 & -1.2126
\end{bmatrix}
\quad N_1 = \begin{bmatrix}
-6.1919 & -0.9171 & 2.4553 \\
-0.9171 & -6.2318 & 2.7583
\end{bmatrix},
\]

\[
N_2 = \begin{bmatrix}
1.1762 & 1.4902 & 1.4902 \\
1.4902 & 1.4902 & 1.4902
\end{bmatrix}
\quad N_3 = \begin{bmatrix}
9.6722 & -4.5325 & -2.8009 \\
-4.5325 & 2.0974 & 3.3851
\end{bmatrix}.
\]

\section*{4.2. Example 2 (stabilization)}

The dynamic output feedback stabilization problem of the certain fractional-order system in the form of (1) is considered with \( A \) and \( B \) presented in (54), and different scenarios for the time-varying delay \( d(t) \) are considered as presented in (55).

\[
A = \begin{bmatrix}
2 & 1 \\
1 & -3
\end{bmatrix}, \quad B = \begin{bmatrix}
-1 & 0 \\
-0.8 & -1
\end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix},
\]

I) \( d(t) = 0.3 \)

II) \( d(t) = \{d|\tau = 0.25, \mu = 0.15\} \)

III) \( d(t) = 0.15 \ast (\sin(t) + 1)(1 - e^{-t}) \Rightarrow \tau = 0.25, \mu = 0.15 \)

The location of eigenvalues of the certain open-loop system (54) with stability boundaries are depicted in Figure 1, which implies that the open-loop system (54) is unstable because some of the eigenvalues of the matrix \( A \) are located on the right side of the stability boundaries.

According to Corollary 1, it can be concluded that this fractional-order system is asymptotically stabilizable utilizing the obtained dynamic output feedback controllers in the form of (31), with controller order \( n_c = 2 \), tabulated in Table 1.

The time response of the closed-loop FO-LTI system of form (32), consisting of the unstable system in (54) and the obtained controller with \( n_c = 2 \), is illustrated in Figures 2, 3 and 4 for delay cases I, II and III, respectively, where all the states asymptotically converge to zero.
**Figure 1.** Location of eigenvalues of the certain open-loop system (54) in Example 2 with stability boundaries for $\alpha = 0.65$ (dotted), $\alpha = 0.8$ (dashed) and $\alpha = 0.95$ (solid).

**Table 1.** Obtained controller parameters for Example 2 using Corollary 1.

| $d(t)$ | I       | II      | III     |
|--------|---------|---------|---------|
| $A_C$  | $\begin{bmatrix} -0.7827 & 0.0006 \\ 0.0010 & -0.7819 \end{bmatrix}$ | $\begin{bmatrix} -0.8343 & 0.0023 \\ 0.0024 & -0.8347 \end{bmatrix}$ | $\begin{bmatrix} -0.8135 & -0.0080 \\ -0.0029 & -0.8622 \end{bmatrix}$ |
| $B_C$  | $\begin{bmatrix} 0.0017 \\ 0.0013 \end{bmatrix}$ | $\begin{bmatrix} 0.2433 \\ -0.4559 \times 10^{-3} \end{bmatrix}$ | $\begin{bmatrix} 0.0146 \\ 0.0009 \end{bmatrix}$ |
| $C_C$  | $\begin{bmatrix} -0.0047 & 0.0082 \\ 0.0100 & 0.0013 \end{bmatrix}$ | $\begin{bmatrix} -0.0045 & -0.0104 \\ 0.0145 & 0.0164 \end{bmatrix}$ | $\begin{bmatrix} 0.0588 & 0.0171 \\ 0.0246 & 0.0547 \end{bmatrix}$ |
| $D_C$  | $\begin{bmatrix} 2.8269 \\ -2.2677 \end{bmatrix}$ | $\begin{bmatrix} 2.5914 \\ -1.8437 \end{bmatrix}$ | $\begin{bmatrix} 2.9262 \\ -2.2767 \end{bmatrix}$ |

Besides, for the time delay of case II, a random curve of $d(t)$ and subsequently its obtained derivative $\dot{d}(t)$ are plotted in Figure 5, where it is clear that $\tau = 0.2$ and $\mu = 0.5$ and the time delay can take any value within these constraints.

As it is obvious from Figures 2, 3 and 4, in all cases of time delay in the present example, despite the change of the derivative order between 0 and 1, the proposed controller is still able to stabilize the fractional-order system.

### 4.3. Example 3 (robust stabilization)

The dynamic output feedback stabilization problem of the uncertain fractional-order system in the form of (1) is considered with $A \in A_I = [A, \bar{A}]$ and $B \in B_I = [B, \bar{B}]$ presented in (56), and time-varying delay $d(t)$ is considered as (57):

$$A = \begin{bmatrix} 1.89 & 0.95 \\ 0.95 & -3.12 \end{bmatrix}, \bar{A} = \begin{bmatrix} 2.11 & 1.05 \\ 1.05 & -2.88 \end{bmatrix},$$
Figure 2. Pseudo-state trajectory of the closed-loop FO-LTI system of form (32) in Example 2 with time delay case (I), via obtained controller with $n_C = 2$.

$$B = \begin{bmatrix} -1.06 & -0.02 \\ -0.82 & -1.10 \end{bmatrix}, \tilde{B} = \begin{bmatrix} -0.94 & 0.02 \\ -0.78 & -0.9 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$ \hspace{1cm} (57)

$$d(t) = 0.15(\sin(t) + 1)(1 - e^{-t}) \Rightarrow \tau = 0.25, \mu = 0.15$$ \hspace{1cm} (58)

The eigenvalues of some random matrices $A \in A_I = [A, \tilde{A}]$ with stability boundaries $\pm \alpha \pi / 2$, for different values of $\alpha$, related to the system (56) are drawn in Figure 6, which implies that the open-loop system (56) is unstable because some eigenvalues of $A$ are located on the right side of the stability boundaries.
Figure 3. Pseudo-state trajectory of the closed-loop FO-LTI system of form (32) in Example 2 with time delay case (II), via obtained controller with $n_C = 2$.

Table 2. Obtained controller parameters for Example 3 using Theorem 2.

| $n_c$ | $A_C$ | $B_C$ | $C_C$ | $D_C$ |
|-------|-------|-------|-------|-------|
| 2     | $\begin{bmatrix} -22.58 & 0.02 \\ 0.03 & -22.57 \end{bmatrix}$ | $\begin{bmatrix} 0.114 & -0.337 \\ 0.097 & -0.288 \end{bmatrix}$ | $\begin{bmatrix} 0.0135 & 0.0015 \\ -0.1692 & -0.0885 \end{bmatrix}$ | $\begin{bmatrix} 3.00 & -2.14 \\ -1.10 & -0.62 \end{bmatrix}$ |

However, using Theorem 2, it can be concluded that this uncertain fractional-order system with time-varying input delay can be asymptotically stabilized using the output feedback controller presented in Table 2.

The time response trajectories of the closed-loop FO-LTI system of form (32), consisting of the system in (56) and the obtained controller with $n_C = 2$, are illustrated in Figure 7, where all the states asymptotically converge to zero. As it is clear from Figure 7, despite the change of the derivative order between 0 and 1, the proposed controller is still able to stabilize the fractional-order system.
4.4. Example 4 (robust stabilization)

The dynamic output feedback stabilization problem of the uncertain fractional-order system of Example 1 in the form of (1) is considered with $A \in A_I = [\bar{A}, \tilde{A}]$ and $B \in B_I = [\bar{B}, \tilde{B}]$ presented in (50), and the time-varying delay $d(t)$ is considered as follows:

$$d(t) = 0.15(\sin(t) + 1)(1 - e^{-t}) \Rightarrow \tau = 0.25, \mu = 0.15$$

(59)

According to Theorem 2, it can be concluded that this uncertain fractional-order system is asymptotically stabilizable utilizing the obtained dynamic output feedback controllers in the form of (31), with controller orders $n_c = 0, 1, 2$, tabulated in Table 3.

The time response of the uncertain closed-loop FO-LTI system of form (32), consisting of a random system in the interval (50) and the obtained controller with $n_C = 0$ (static controller), is illustrated in Figure 8, where all the states asymptotically converge to zero.
Figure 5. Random curve of $d(t)$ and its obtained derivative $\dot{d}(t)$ for time delay case (II) in Example 2 with $\tau = 0.2$ and $\mu = 0.5$.

Table 3. Obtained controller parameters for Example 4 using Theorem 2.

| $n_c$ | $A_C$ | $B_C$ | $C_C$ | $D_C$ |
|-------|-------|-------|-------|-------|
| 0     | 0     | 0     | 0     | −1.4215 |
| 1     | −24.0413 | −0.2088 | −0.0061 | −1.6597 |
| 2     | $\begin{bmatrix} -21.4575 & -0.1151 \\ -0.1114 & -21.4721 \end{bmatrix}$ | $\begin{bmatrix} -0.1930 \\ -0.2219 \end{bmatrix}$ | $\begin{bmatrix} -0.0461 \\ -0.0330 \end{bmatrix}$ | −1.5227 |

The eigenvalues of $A_{CI}$, for some random systems in the above interval, and stability boundaries $±\alpha \pi/2$ are demonstrated in Figure 9, where all of the eigenvalues of $A_{CI}$ are located in the stability region. It is obvious from Figures 8 and 9 that stabilizing the interval FO-LTI system with time-varying delay is possible even with the proposed static controller with $n_c = 0$. 
Figure 6. Location of eigenvalues of the uncertain open-loop system (56) in Example 3 with stability boundaries for $\alpha = 0.65$ (dotted), $\alpha = 0.8$ (dashed) and $\alpha = 0.95$ (solid).

Figure 7. Pseudo-state trajectory of the closed-loop FO-LTI system of form (32) in Example 3 via obtained controller with $n_C = 2$. 
Figure 8. Pseudo-state trajectory of the closed-loop FO-LTI system of form (32) in Example 4, via obtained controller with $n_C = 0$.

Figure 9. Location of eigenvalues of the uncertain closed-loop system via obtained output feedback controller in Example 4 with $n_C = 0$.

5. Conclusion

This paper has solved the problem of stability and stabilization of interval fractional-order systems with time-varying delay, where the elements of the systems pseudo-state space matrices are uncertain parameters that each adopts a value in a real interval. The time-varying delay also offers more generality compared with time-constant one which has been adopted in previous works. Utilizing various lemmas, the stability and stabilization theorems are proposed in the form of LMIs, which are more suitable to check due to various existing efficient convex optimization parsers and solvers. Eventually, some numerical examples have shown the effectiveness of proposed robust stability and stabilization theorems.

Disclosure statement

No potential conflict of interest was reported by the author(s).
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