PRIMORDIAL MAGNETIC FIELDS AND THEIR DEVELOPMENT
(APPLIED FIELD THEORY)

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MOTIVATION

In this talk I discuss the non-linear development of magnetic fields in the early universe. Since this is based on a classical field theory, which turns out to have a rather complex structure, I thought it could be of interest for this meeting, under the heading of "applied field theory". I very briefly mention a number of particle physics mechanisms for generating magnetic fields in the early universe, but the emphasis is on the field theoretic aspects of the developments of these fields, mainly on the occurrence of inverse cascades, i.e. generation of order from disorder. In this connection it is also discussed how the Silk effect (photon diffusion) is counteracted by the inverse cascade, which moves energy from smaller to larger scales.

Many galaxies (including our own) are observed to have magnetic fields. One way to observe such fields is to study the polarization of light passing the galaxy. Due to the interaction with the field and the plasma there is a Faraday rotation of the polarization vector, proportional to the field and to the square of the wave length of the light. In this way fields are found to have the order of magnitude $10^{-6} \sim 10^{-8}$ Gauss on a scale of 100 kpc. If you have forgotten what a G(auss) is: the mean field on the sun is approximately one G.

Usually the galactic magnetic field is explained by the dynamo effect: turbulence (e.g. differential rotation) in the galaxy enhances the magnetic field exponentially up to some saturation value, corresponding to equipartition between kinetic and magnetic energy. The dynamics which governs these phenomena is called magnetohydrodynamics,

*A parsec is an astronomical unit, which has the physical value 1 pc $\approx$ 3.26 light year.
abbreviated as MHD, which is essentially the Maxwell plus Navier-Stokes equations. The dynamo can produce an enhancement factor of several orders of magnitude. An important feature is that the dynamo needs a seed field. It appears reasonable to assume that this field is of primordial origin, i.e. it has existed already in the early universe. Astrophysicists often say as a joke that a primordial magnetic field is a field which has existed for so long that everybody has forgotten how it was created. However, in particle physics we must be more serious since we have knowledge of the early universe, and hence we should explain the origin of these fields.

PRIMORDIAL MAGNETIC FIELDS IN THE EARLY UNIVERSE

In natural units magnetic fields have dimension \((\text{mass})^2\). At the electroweak scale, assuming the Higgs mass to be of order \(m_W\), there is essentially only one mass, \(m_W\), and we may therefore expect something like

\[
B_{\text{EW}} \lesssim m_W^2 \approx 10^{24} \, \text{G}
\]  

(1)

on a scale \(\sim 1/m_w\). This is a huge field, far larger than anything one has ever seen or produced on this earth. How does this compare with the rather weak fields found in galaxies?

In the standard cosmological model all distances are blown up by the scale factor \(R(t)\). It is useful for estimates that the scale factor is proportional to the inverse temperature. Thus, \(R_{\text{now}}/R_{\text{EW}} = T_{\text{EW}}/T_{\text{now}} \approx 10^{15}\). Hence, an initial correlation length of order \(\sim 1/m_W\) is of order 1 cm today, which has no astrophysical interest. We need fields on a scale of order 100 kpc \(\approx 3\times10^{23}\) cm.

If we assume that \(B\) is essentially random, we can estimate the field at any distance from a simple random walk. We have the field at the initial correlation length, but we want it at \(\approx 10^{23}\) times this length. Thus, in \(d\) dimensions we have

\[
<B_{\text{EW}} >_{\text{scale \, } 10^{23}/m_W} \approx 10^{24} \, \text{G}/(10^{23})^{d/2}.
\]  

(2)

So for \(d = 3\) we get \(<B_{\text{EW}} >\approx 10^{-10}\)G, whereas for \(d = 2\) and \(d = 1\) we have \(<B_{\text{EW}} >\) approximately equal 10 G and 10^{12} G, respectively, on the scale of \(10^{23}/m_W\).

In order to see if these fields are reasonable, we need to know the cosmological developments of \(<B >\). From MHD (with viscosity ignored) one has the result that the flux through a surface bounded by a curve following the fluid of charged particles is conserved. Since such a surface increases like \(R^2\), it follows that \(<B >\) decreases like \(1/R^2\) when the universe expands with the scale factor \(R\). It therefore follows that if today we need e.g. a primordial field of order \(10^{-15}\)G on a scale of 100 kpc, then on the corresponding scale \(10^{23}/m_W\) at the electroweak phase transition, we need \(<B >\approx 10^{15}\) G. Thus, from the random walk estimates above we see that only the case \(d = 1\) comes near this value, although a factor \(10^3\) is missing. Actually one could argue that the case \(d = 1\) is relevant, because in observing the magnetic field by Faraday rotation, a one dimensional average is made along the line of sight. However, this argument is not really convincing, since the dynamo effect is three dimensional, and hence the field relevant for this effect is the very small 3d average.

\(\text{The metric is}

\[
d\tau^2 = dt^2 - R(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right].
\]  

(3)

where \(k = +1, 0, -1\) for a closed, flat or open universe, respectively.
The conclusion is thus that if fields of the order of $m_W^2$ can be generated at the EW-scale, then there could still be missing a factor of order $10^x$, where $x$ is of order 3. However, it should be emphasized that a field of order $m_W^2$ is very large, and is not obtained in most mechanisms for creation of primordial fields. Hence, in most cases $x$ is larger than 3.

**MAGNETIC FIELDS FROM PARTICLE PHYSICS MECHANISMS**

In this section we very briefly discuss a number of proposed mechanisms for the generation of primordial fields. The list is by no means exhaustive.

**Fields From Inflation**

An inflationary creation of primordial magnetic fields has the advantage that the coherence scale is larger than in other mechanisms. As an example, we mention the work by Gasperini, Giovannini and Veneziano, which is based on a pre-big-bang cosmology inspired by superstrings, which is an alternative to the usual slow roll inflation. The dilaton field $\phi$ in the Lagrangian

$$\mathcal{L} = -\sqrt{g} e^{-\phi}(R + \frac{1}{2} \partial_{\mu}\phi \partial^{\mu}\phi + \frac{1}{4} F_{\mu\nu}F^{\mu\nu})$$

amplifies the quantum fluctuations of $F_{\mu\nu}$. The magnetic energy spectrum behaves like $\sim k^{0.8}$. The resulting magnetic fields are of the right order of magnitude on a 100 kpc scale. Recently, however, Turner and Weinberg have argued that this scenario requires fine tuning of the initial conditions in order to get enough inflation to solve the flatness and horizon problems.

**Bubble Formation at the EW Phase Transition and Magnetic Fields**

In a first order EW phase transition bubbles of new vacuum are formed. This was used by Baym, Bödecker and McLerran to obtain the generation of a magnetic field. The main point is that the bubbles, although overall neutral, have a dipole charge layer on the surface, so rotating bubbles generate a field. Although the field from each bubble is very small, there is a large number of bubbles, so depending on the subsequent development of $<B>$, in the end a reasonable magnitude can be produced. A different mechanism was considered by Kibble and Vilenkin: when the bubbles collide, the phase of the Higgs field varies, giving rise to currents and a magnetic field. Again, in this case one can get a reasonable magnitude provided the subsequent development of $<B>$ is favourable.

**Superconducting Cosmic Strings and Other Mechanisms**

There exists a number of other mechanisms for the generation of magnetic fields. Vachaspati pointed out that if the gradients of Higgs fields fluctuate, they can induce a magnetic field at the electroweak scale. The statistical averaging involved in this scenario was discussed in details by Enqvist and me. As mentioned already in the first section, the conclusion is that if line averaging is relevant, one obtains nearly the right order of magnitude, since by this mechanism the field at genesis is of order $m_W^2$ on a scale of $1/m_W$. However, this scenario operates with physically motivated fluctuations in gauge dependent quantities like gradients of Higgs fields. Such a procedure is not very clear to me.
Recently there has been discussions of generation of primordial magnetic fields from a network of Witten’s superconducting cosmic strings. These strings are current carrying, and hence produce magnetic fields. It turns out that if the strings are created at the GUT phase transition, where the current is very large, they can produce a field which is large enough over a sufficient scale, assuming that MHD does not give rise to any trouble. On the other hand, superconducting strings created at the EW phase transition cannot generate sufficient fields.

It has also been proposed that a Savvidy-type vacuum, where the energy is lowered relative to the trivial vacuum by having a magnetic field, can generate enough field. For SU($N$) the field produced at a temperature $T$ is of order

$$B \sim T^2 \exp \left( \frac{-48\pi^2}{11Ng^2} \right).$$

(5)

At the EW transition ($N = 3$), this field is far too small. At the GUT transition, however, it produces a large enough field for $N = 5$, due to the strong sensitivity of the exponent with respect to $N$. Whether this field is acceptable depends on the subsequent development according to MHD.

It was proposed long ago by Harrison that magnetic fields could be generated from vorticity present in eddies of plasma in the early universe. This idea was criticized, and the eddies were replaced by irrotational density fluctuations by Rees. A more modern version of this scenario is due to Vachaspati and Vilenkin, where the magnetic field is generated by vorticity arising in the wakes of ordinary (i.e. not superconducting) cosmic strings.

Finally we mention that recently Joyce and Shaposhnikov have presented a scenario which has the potential of leading to quite large fields. The standard model has charges with abelian anomaly only (e.g. right-handed electron number) which are essentially conserved in the very early universe, until a short time before the EW transition. A state with finite chemical potential of such a charge is unstable to the generation of hypercharge $U(1)$ fields. Such fields can turn into large magnetic fields, depending on their subsequent development.

It is clear that the physical validity of most, if not all, of these scenarios, depends on the subsequent non-linear development of the primordial field, due to MHD. This will be discussed in the next section. In the end of this talk we shall also discuss Silk diffusion, which is a mechanism for destroying magnetic fields by turning it into heat. We shall show that this linear diffusion is in fact counteracted by the non-linear terms of MHD.

**INVERSE CASCADE FROM MAGNETOHYDRODYNAMICS**

We shall now investigate what happens subsequently to a primordial magnetic field generated in the early universe. To simplify things, here we give the details only for the non-relativistic case, and mention without giving the arguments, what happens in the general relativistic case.

**The Non-Relativistic MHD Equations**

In the rest frame of a plasma consisting of charged particles with current $\mathbf{j}$, we have Ohm’s law,

$$\mathbf{j}_{\text{rest}} = \sigma \mathbf{E}_{\text{rest}},$$

(6)
where $\sigma$ is the conductivity. The universe is a good conductor, so $\sigma$ is very large. Thus it follows that

$$E_{\text{rest}} \approx 0.$$  
(7)

In a frame moving with bulk velocity $v$ one therefore has

$$E \approx -v \times B.$$  
(8)

The induction equation $\partial B/\partial t = -\nabla \times E$ therefore gives

$$\partial B/\partial t \approx \nabla \times (v \times B). \quad \text{(MHD I)}$$  
(9)

This is one (out of two) of the fundamental MHD equations. It tells us that the magnetic field is influenced by the velocity, and also, if you start from $B = 0$ no magnetic field can be generated. Therefore a seed field is needed in the dynamo mechanism.

The second fundamental MHD equation is the Navier-Stokes equation with the Lorentz force $j \times B$ on the right hand side. Here $j$ can be estimated from the Maxwell equation $j + \partial E/\partial t = \nabla \times B$. The time derivative can be estimated to be small in the non-relativistic case, so $j \approx \nabla \times B$, so the Navier-Stokes equation with the Lorentz force is given by

$$\partial \rho v/\partial t + (v \nabla) \rho v \approx -\nabla (p + \frac{1}{2} B^2) + (B \nabla) B. \quad \text{(MHD II)}$$  
(10)

Here for simplicity we have ignored the viscosity. For $\sigma$ large, this can be generalized to the relativistic case at the expense of a considerable increase in the complexity of the equations.

**Why and When does MHD have an Inverse Cascade: A Simple Scaling Argument**

Now let us suppose that by some particle physics mechanism a primordial magnetic field is generated. At the genesis the field has some correlation length, and the crucial question is then what happens as time passes. If the correlation length grows smaller, corresponding to a cascade, the situation is quite bad, even for the inflationary scenario. Such a cascade would appear if the system develops into a more chaotic direction. If, on the other hand, we have an inverse cascade, the correlation length grows and the system develops towards more order. In an inverse cascade, energy is thus transferred from smaller to larger scales.

It turns out that the situation depends on the initial spectrum. Roughly speaking, if the spectrum is concentrated at short (large) distances, it will develop into large (short) distances. To see this, one can make use of the fact that the MHD equations are invariant under the “self-similarity” equations,

$$x \rightarrow lx, \ t \rightarrow l^{1-h} t, \ v \rightarrow l^{h} v, \ B \rightarrow l^{h} B,$$  
(11)

where $l$ is some typical length. The velocity is given essentially by the Newtonian expression $v/\tau \approx eE/E$, where $E$ is the relativistic energy, and $\tau$ is the average time between collisions. Thus, $\tau \approx 1/n\sigma_x$, where $\sigma_x$ is a typical relativistic cross section. Thus, $j \approx ne(\tau eE/E)$, so $\sigma \approx ne^2\tau / E \approx e^2 / E\sigma_x$. A relativistic cross section goes like $e^4 / T^2$, since the temperature $T$ is a typical momentum transfer. Also, $e \sim T$. Thus $\sigma \approx T/e^2$, which is very large in the early universe, because the temperature is very high. At later stages the universe is still a good conductor, for different reasons.

†We have $E \sim vB$, so $\partial E/\partial t \sim (v/l)vB \sim (B/l)v^2$. But $|\nabla \times B| \sim B/l$, where $l$ is some typical length, so the time derivative of the electric field can be ignored relative to the curl of $B$. 

‡In the relativistic era this can be seen from the following estimate: The current is defined by $j = nev$. The velocity is given essentially by the Newtonian expression $v/\tau \approx eE/E$, where $E$ is the relativistic energy, and $\tau$ is the average time between collisions. Thus, $\tau \approx 1/n\sigma_x$, where $\sigma_x$ is a typical relativistic cross section. Thus, $j \approx ne(\tau eE/E)$, so $\sigma \approx ne^2\tau / E \approx e^2 / E\sigma_x$. A relativistic cross section goes like $e^4 / T^2$, since the temperature $T$ is a typical momentum transfer. Also, $e \sim T$. Thus $\sigma \approx T/e^2$, which is very large in the early universe, because the temperature is very high. At later stages the universe is still a good conductor, for different reasons.
and if the viscosity $\nu$ and Ohmic resistance are included, we further have

$$\nu \to \nu^{1+h}, \sigma \to \sigma^{-1-h}. \quad (12)$$

From this it is very easy to show that the magnetic and kinetic energy densities ($E$), given in 3+1 dimensions by expressions like

$$E(k, t) = \frac{2\pi k^2}{(2\pi)^3} \int d^3x \int d^3y \, e^{ik(x-y)} < B(x, t)B(y, t) >, \quad (13)$$

where

$$V \int dk E(k, t) = \frac{1}{2} \int d^3x < B^2 > = \text{total magnetic energy}, \quad (14)$$

must satisfy the scaling relation

$$E(k/l, l^{-1+2h}t) = l^{1+2h} E(k, t). \quad (15)$$

This is valid in the inertial range, where viscosity and Ohmic resistance can be ignored. The general solution of this equation is

$$E(k, t) = k^p \psi(k^{(3+p)/2}t), \quad (16)$$

with $p = -1 - 2h$ and $\psi$ some function of the single argument $k^{(3+p)/2}t$. The interpretation of this equation is that, if at the initial time $t = 0$ the spectrum is $k^p$ (from some particle physics mechanism), then at later times it will be governed by the function $\psi$. Hence the wave vector scales like

$$k \sim t^{-2/(3+p)} \quad (17)$$

Thus, if $p > -3$ there is an inverse cascade, because $k$ moves towards smaller values, whereas for $p < -3$, there is a cascade. Thus, if initially we have a random system corresponding to $p \geq 0$, then later the system becomes more ordered, as already announced. For $p = -3$ it follows from eq. (15) that the $k$ and $t$ dependence of the energy density become uncorrelated.

In the case when general relativity is included one obtains for a flat, expanding universe

$$R(t)^4 \, E(k, t) = k^p \psi(k^{(3+p)/2} \tilde{t}), \quad (18)$$

where $t$ is the Hubble time, $\tilde{t} = \int dt/R(t)$ is the conformal time, $\tilde{t} \propto \sqrt{t}$, and where $k$ is the comoving wave vector, so that the physical wave vector is $k_{\text{phys}} = k/R(t)$. Therefore the physical wave vector scales like

$$k_{\text{phys}} \sim \tilde{t}^{-2/(3+p)}/R(t) \propto t^{-(5+p)/2(3+p)}. \quad (19)$$

Thus, if the spectrum starts out with $p = 2$, corresponding to a Gaussian random initial field, we have a scaling of the physical wave vector by $t^{-0.7}$, instead of $t^{-0.5}$.

\footnote{If the initial field is given by

$$< B_i(x, 0)B_k(y, 0) > = \lambda \left( \delta_{ik} - \frac{\partial_i \partial_k}{\partial^2} \right) \delta^3(x-y), \quad (20)$$

then the initial energy is given by

$$E(k, 0) = \lambda (\delta_{ii} - k_i k_i/k^2)k^2 = 2\lambda k^2. \quad (21)$$

Thus the general relativistic scaling goes as $k_{\text{phys}} \sim t^{-3/5}$.
from pure expansion. If \( p \) decreases, the effective expansion increases. Thus, if the initial spectrum is characterized by \( p = 1 \), we get a scaling \( k_{\text{phys}} \sim t^{-0.75} \), and for \( p = 0 \) we have \( k_{\text{phys}} \sim t^{-0.8} \). These examples imply physically that the typical size of an eddy increases like \( t^{0.2} \), \( t^{0.25} \) and \( t^{0.3} \), respectively, on top of the expansion factor. Finally we mention that the “scale invariant” initial spectrum with \( p = -1 \) (\( dk/k = \text{scale inv.} \)) has an increase of the typical eddy size by \( t \), which means that the eddies follow the horizon.

**Numerical Simulations in 2+1 Dimensions**

From the general scaling arguments presented above one cannot deduce the value of the scaling function \( \psi \). Here numerical investigations are needed, since realistic analytic solutions of MHD are not known. However, a problem arises, since the Reynolds number \[ ^1 \] is very large in the early universe. For example, in the paper by Brandenburg, Enqvist and me\(^{14} \) the magnetic Reynolds number was estimated to be of order \( 10^{17} \). In numerical simulations one cannot reach this value, no matter how much computer time is used. We therefore did numerical simulations in 2+1 dimensions with a unrealistically low Reynolds number\(^{14} \), taken to be 10. So the non-linear terms are approximately ten times as important as the diffusion terms. These terms are, however, needed to achieve numerical stability (they act as a short distance cutoff).

We used the general relativistic MHD equations, which are considerably more complicated than the MHD equations discussed in a previous section. We took the initial conditions that the velocity vanishes and \( B \) is Gaussian random, so the magnetic energy spectrum goes like \( k \). Also, we took the energy density to be \( \rho = \text{const.}/R^4 \), and the pressure \( p = \rho/3 \).

In fig. 1 the numerical results\(^{14} \) are displayed. At the initial time there is a rather chaotic state, where the magnetic flux lines are either long random walk curves, or small closed loops. We used periodic boundary conditions, and satisfied \( \text{div}B = 0 \). We see that in a short time the typical length scale increases considerably. In the end of the simulation there are quite large eddies. Therefore we clearly see an inverse cascade, where order is produced from chaos, in contrast to the usual paradigm.

Also, the initial velocity \( v = 0 \) acquires a spectrum which shows an inverse cascade. The velocity is initially induced by the Navier-Stokes equation through the Lorentz force. The velocity generated this way then influences the magnetic field through the induction equation, etc. etc.

**Numerical Simulations in 3+1 Dimensions: The Shell Model**

As already mentioned, simulations of MHD with large Reynolds numbers is not possible with present day computers\(^{11} \). The situation gets worse when we go from 2+1 to 3+1 dimensions. Therefore one needs to make a model which has as many features of the real Navier-Stokes as is compatible with practical tractability. In recent years the so-called GOY (Gledzer, Ohkitani and Yamada) model has become increasingly popular. Another name for this model is the “shell model”. It gives results in good agreement with experiments, especially as far as the subtle intermittency effects are concerned. The model captures a basic feature of turbulence, namely the coupling of many different length scales. It is not known whether the model has relation to the real

\[ ^{11} \text{This number can be understood as the ratio between “typical” non-linear terms and the linear viscosity term. Thus, if } Re \text{ is large, turbulence is important.} \]

\[ ^{**} \text{This also applies to hydrodynamics, and is perhaps the reason why weather forecasts are pretty bad, at least in Denmark.} \]
Navier-Stokes and MHD. But it nicely illustrates the behaviour of a system in which numerical simulations are made difficult by the effect of a huge number of couplings between the different length scales. Also, real world conservation laws (energy, helicity) are buildt into the model.

To motivate the model, let us mention that in the Navier-Stokes equations and MHD one has terms like

\[(\mathbf{v} \nabla) \mathbf{v}, \quad \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (\mathbf{B} \nabla) \mathbf{B},\]

etc. In Fourier space they e.g. have the form

\[(\mathbf{v} \nabla) \mathbf{v} \rightarrow \int d^3 p \, v_i(p)(p_i - k_i)v_j(p - k)\].

Experience with numerical simulations show that the largest contributions come from triangles in \(k\)-space with similar side lengths. This is taken as a “phenomenological” input in the shell model.
At this stage, for numerical purposes, one would discretize \( k \)-space. In the shell model one of the basic ingredients is a hierarchical structure, where \( |k| \)-space is divided into shells

\[
k_n = \lambda^n k_0, \quad n = 1, 2, ..., N.
\]

Here \( \lambda \) is often taken to be 2. There furthermore exists a complex “velocity mode” \( v_n = v(k_n) \), which can be considered as the Fourier transform of the velocity difference \( \mathbf{v}(|x| + 2\pi/\lambda) - \mathbf{v}(|x|) \). Since \( k_n \) increases exponentially, it covers a wide range of corresponding length scales. The model then assumes couplings between neighbours and next nearest neighbours,

\[
(v \nabla) \mathbf{v} \rightarrow \sum_{i,j=\pm 2} c_{ij} v_{n+i} k_n v_{n+j},
\]

where the sum is over neighbours and/or next nearest neighbours to \( n \). The couplings \( C_{ij} \) in this sum should be made such that energy is conserved in the absence of diffusion. Thus, energy conservation

\[
\int (v^2 + B^2) d^3x = \text{const}
\]

now corresponds to

\[
\sum_{n=1}^{n=N} (|v_n|^2 + |B_n|^2) = \text{const}.
\]

Thus, we need to satisfy

\[
\sum_{n=1}^{n=N} \left( v_n \frac{dv_n^*}{dt} + B_n \frac{dB_n^*}{dt} + \text{complex conj.} \right) = 0,
\]

where, as before, \( \tilde{t} \) is the conformal time, \( \tilde{t} = \int dt/R(t) \). In this approach the vectorial character is thus lost, but the conservation of energy is kept as an essential feature.

We should now find equations for the time derivatives respecting the conservation of energy. Taking into account some factors from general relativity in an expanding universe (the expansion factor as well as the energy density and pressure) we get\(^\text{14}\)

\[
\frac{8}{3} \frac{dv_n}{d\tilde{t}} = ik_n(A + C)(v_{n+1} v_{n+2}^* - B_{n+1}^* B_{n+2}) + ik_n(B - \frac{1}{2} C)(v_{n+1}^* v_{n+2} - B_{n+1}^* B_{n+2}^*) + ik_n(\frac{1}{2} B + \frac{1}{4} A)(v_{n-2}^* v_{n-1} - B_{n-2}^* B_{n-1}) \tag{29}
\]

\[
\frac{dB_n}{d\tilde{t}} = ik_n(A - C)(v_{n+1} B_{n+2}^* - B_{n+1}^* v_{n+2}) + ik_n(B + \frac{1}{2} C)(v_{n+1} B_{n+2} - B_{n+1} v_{n+2}) + ik_n(\frac{1}{2} B - \frac{1}{4} A)(v_{n-2}^* B_{n-1} - B_{n-2}^* v_{n-1}) \tag{30}
\]

where with \( A, B, C \) arbitrary constants energy is conserved. In 3+1 dimensions, magnetic helicity is also conserved. In the continuum helicity is given by

\[
H = \int d^3x \, \mathbf{A} \cdot \mathbf{B},
\]

where \( \mathbf{A} \) is the vector potential. This conservation is trivial in 2+1 dimensions, since there \( H = 0 \). To mimic conservation of \( H \) in the shell model we require that the quantity

\[
H_{\text{shell}} = \sum_{n=1}^{n=N} (-1)^n k_n^{-1} B_n^* B_n \tag{32}
\]
Figure 2. Spectra of the magnetic energy at different times. The straight dotted-dashed line gives the initial condition \( t_0 = 1 \), the solid line gives the final time \( t = 3 \times 10^4 \), and the dotted curves are for intermediate times (in uniform intervals of \( \Delta \log\!(t-t_0) = 0.6 \)). \( A = 1, \ B = -1/2, \ C = 0 \). This figure is from ref. 14.

is conserved. The reason is that \( k^{-1}B_n \) is like the vector potential. The factor \((-1)^n\) is a more “phenomenological” factor. The corresponding conservation in hydrodynamics \( \int \! v(\nabla \! \times \! v) d^3x = \text{const} \) has been studied, and it was found that the integrand oscillates in sign. This is then taken into account in the shell model by the oscillating factor.

The requirement that helicity is conserved thus corresponds to taking into account 3+1 dimensions, and it leads to the following values for the otherwise arbitrary constants \( A, B, C, \)

\[
A = 1, \quad B = -1/4, \quad C = 0.
\]  

(33)

Using these values, we have \( 2N \) coupled set of equations. In our calculations we took \( N = 30 \), corresponding to solving 60 coupled equations. The resulting spectra at different times are shown in fig. 2. Again we see a nice inverse cascade, because as functions of the comoving wave vector \( k \) the spectra clearly move towards \( k = 0 \).

To give a more precise picture of the change of the spectrum towards large distances, we also computed a correlation length defined by averaging over the magnetic
energy density,
\[ l_0 \equiv \int \frac{2\pi}{k} \mathcal{E}(k, t) \left[ \int dk \mathcal{E}(k, t) \right]^{-1}. \] (34)

In turbulence theory this quantity is called the “integral scale”. It is a measure of the characteristic size of the largest eddies of turbulence.

The result is shown in fig. 3. We see that initially the system moves extremely rapidly towards larger scales. Clearly MHD (in the shell version) does not like the initial Gaussian random state for the magnetic field! The scaling arguments in eq. (18) predicts an increase in the eddy size like \( t^{0.2} \). This cannot directly be compared to the integral scale \( l_0 \), since the integrations in eq. (33) are limited by an ultraviolet cutoff, which also becomes scaled. However, a fit in ref. 14 gives \( l_0 \sim t^{0.25} \), if the steep initial increase in fig. 3 is ignored. Taking into account some uncertainty in the fitting, this is in good agreement with the scaling in eq. (18).

EFFECTS OF DIFFUSION: SILK DAMPING

The effect of diffusion has been ignored in the above discussion, except as a short distance cutoff in the numerical calculations. However, this is not realistic, as was pointed out by Siegl, Olinto, and Jedamzik\(^{15}\). This is connected to Silk damping, which occurs in the charged plasma because radiation can penetrate the plasma and carry away momentum by scattering off the charged particles. Around the time of recombination photon diffusion became very important and corresponded to a very large photon mean free path. The diffusion coefficient is proportional to the photon mean free path, and hence photon diffusion at that time cannot be ignored\(^ {15}\). In a linear approximation of MHD it was clearly demonstrated that the magnetic field must be destroyed, the magnetic energy beeing turned into heat\(^ {15}\). Silk diffusion would therefore remove the hope of understanding primordial magnetic fields from most points of view!

All hope is not lost, however, since the non-linear inverse cascade, discussed in the previous section, counteracts Silk diffusion. While the latter is busy removing magnetic energy at shorter scales, the former is active in removing the energy from these scales to large scales. As we have seen in fig. 3, this happens very quickly. Therefore, without doing any calculations it is clear that these two mechanisms compete against one another.

To be more precise, numerical simulations are needed. This was done by Brandenburg, Enqvist, and me\(^ {16}\), and the result is that even if Silk diffusion is included, the
inverse cascade is strong enough to make the magnetic field survive, at least until close to recombination. This should be enough for the dynamo effect to start to operate. We refer to the original paper\textsuperscript{16} for a full discussion of this.

**CONCLUSIONS**

In conclusion we mention that there are several particle physics models which can produce primordial magnetic fields. Of course, they are based on assumptions which may not turn out to be ultimately true. For example, there may not be a first order EW phase transition, superconducting or ordinary cosmic strings may not exist, etc. etc. So when the dust settles, there may not be so many mechanisms which survive. Also, it should be remembered that without the inverse cascade, there is no hope to produce large enough background fields (this does, of course, not apply to the inflationary mechanism), and it may be that for some or all of these models, the inverse cascade is not large enough.

Thus, in estimating the effect of various models one should take into account the combined effect of the inverse cascade and Silk diffusion. This will perhaps require rather complicated numerical calculations, although some results might conceivably be obtained or guessed from simple scaling arguments, as discussed in ref. 13.

Finally, we mention that there is a very interesting proposal for direct observation of a primordial background field\textsuperscript{17}. The idea is that gamma rays arising from strong sources can scatter in a background field, making pair production and delayed photons. The spectrum of these photons could then be observed, provided the field is of order $10^{-24}$ G or larger\textsuperscript{17}. If this is technically feasible, important information on the spectrum would be obtained, which could then be compared with different models.

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