Fluctuation universality for a class of directed solid-on-solid models

Benjamin J. Fleming and Peter J. Forrester

Department of Mathematics and Statistics, The University of Melbourne, Victoria 3010, Australia
email: bfleming@ms.unimelb.edu.au  p.forrester@ms.unimelb.edu.au

Abstract

Our interest is in a class of directed solid-on-solid models, which may be regarded as continuum versions of boxed plane partitions. In the case that the heights are chosen from a uniform distribution, the joint PDF of the heights is the same as that for the positions in a finitized bead process recently introduced by the authors and Nordenstam. We use knowledge of the correlation functions for the latter to show that upon a certain scaling the fluctuations of the heights along the back row of the solid-on-solid model are given by the Airy process from random matrix theory, as is the case for boxed plane partitions. Moreover, we show that this limiting distribution remains true if instead of the uniform distribution, the heights are sampled from a general absolutely continuous distribution.

1 Introduction

For more than a decade now, it has been demonstrated by numerous examples that fluctuation formulas relating to the largest eigenvalues of certain random matrix ensembles are also characteristic of fluctuation formulas for certain random growth models (for a recent review see [7]). Because the precise functional form of the random matrix quantities are known (see e.g. [9]), there are thus precise theoretical predictions for the functional form of the corresponding quantities in the growth models. Very recently, the validity of these theoretical predictions [16] has been demonstrated by a high precision experiment involving curved interface fluctuations of a growing droplet [19].

Crucial to the theoretical predictions is the hypothesis of universality of the fluctuation formulas: they depend on some global symmetry properties of the interface (in particular, is it curved or flat?), but not the details of the microscopic interactions. This hypothesis is generally unproven, as the results for many of the random growth models rely on the integrability of the particular microscopic choice of parameters. One recent advance has been to show that the random matrix fluctuation formulas predicted on the basis of exact calculation have been shown to hold true for the continuum KPZ equation with narrow wedge initial conditions [1,17]. But mostly universality results are out of reach.
An outstanding problem of this type relates to last passage percolation. Let \( \{a_{ij}\} \) be independent random variables drawn from a probability distribution with unit mean and finite variance. The random variable

\[
 l_N = \max_{(1,1)} \sum_{u/r(N,N)} a_{ij}
\]

finishing at \((N,N)\) and forming an up/right directed path when plotted on the integer grid, is referred to as the last passage time. In the case that each \( a_{ij} \) is drawn from an exponential distribution (or more generally a geometric distribution) the scaled fluctuation of \( l_N \) about its mean \( 4N \) is known to be identical to the fluctuation of the scaled largest eigenvalue of a random complex Hermitian matrix \([13]\). But for the \( a_{ij} \) with distributions different from these solvable cases, the corresponding theorem is not known.

In this paper we will demonstrate universality for fluctuations of a certain two-dimensional directed solid-on-solid model, defined in terms of a grid of height variables \( \{x_{ij}\}_{i=1,...,N} \). Each height variable is chosen independently from the same non-negative, absolutely continuous distribution \( D \), subject to a constraint on its value relative to its neighbours. The solid-on-solid model is defined in Section 2. In Section 3, in the particular case \( D = U_{[0,1]} \) (the uniform distribution on \([0,1]\)), use is made of a correspondence between this instance of the solid-on-solid model and the recently introduced finitized bead process \([8]\), to obtain a Fredholm determinant formula for the distribution of the largest heights along the back row of the former. In Section 4 the explicit form of the distribution function between scaled heights on the back row \( O(N^{2/3}) \) away from each other is computed. The functional form found is precisely that for the Airy process \([15, 11, 16]\), which in turn is identical to the correlation functions for the GUE minor process at the soft edge \([10]\). We show in Section 5 that, upon an appropriate choice of scaled variables, this same correlation function persists for the \( x_{ij} \) chosen from a general absolutely continuous distribution \( D \), thereby establishing the claimed universality.

2 Definition of the model

Consider the \( N \times N \) integer grid \( \{(i,j) : 1 \leq i,j \leq N\} \). At each site \((i,j)\), associate a height variable \( x_{i,j} \overset{d}{=} D \), where \( D \) is an absolutely continuous distribution with density \( h(x) \). Furthermore, require that for fixed \( i \)

\[
 x_{i,1} < x_{i,2} < \cdots < x_{i,N}
\]

(heights increase along rows) and that for fixed \( j \)

\[
 x_{1,j} < x_{2,j} < \cdots < x_{N,j}
\]

(heights increase up columns). Thus a directed up/right lattice path (recall text below (1.1)) must encounter successively larger heights.

Rotate the square grid by 45° anti-clockwise and mark lines parallel to the \( y \)-axis through the lattice points. For an \( N \times N \) grid there are \( 2N - 1 \) lines, and on the \( l \)-th line there are \( l \) grid points for \( l = 1, \ldots , N \), and \( 2N - l \) grid point for \( l = N + 1, \ldots , 2N - 1 \). Let \( y_i^{(l)} \) denote the height of the \( i \)-th lattice point counting from the top on line \( j \). We have that \( 1 \leq i \leq N(l) \),
where \( N(l) := l, (i = 1, \ldots, N), \) and \( N(l) := 2N - l \ (l = N + 1, \ldots, 2N) \) in keeping with the total number of particles on line \( j \), the relationship

\[
x_{jk} = \begin{cases} 
y^{(N+1-j+k)}_{(N+1-j+k)+1-k}, & j \geq k \\
y^{(N+1-j+k)}_{(N+1-j+k)+1-j}, & j < k,
\end{cases}
\]

and most importantly the interlacings

\[
y^{(l)}_{j+1} < y^{(l-1)}_{j} < y^{(l)}_{j}, \quad 1 \leq j \leq (l-1) \leq N \tag{2.1}
\]

and

\[
y^{(l)}_{j+1} < y^{(l+1)}_{j} < y^{(l)}_{j}, \quad 1 \leq j \leq 2N - (l+1) \leq (N-1). \tag{2.2}
\]

Let us denote the interlacings (2.1) and (2.2) by \( Y(\{y\}) \). The joint PDF for a configuration specified by the height coordinates \( \bigcup_{l=1}^{2N} \{y^{(l)}_{j}\}_{j=1,...,N(l)} \) will then be give in terms of the PDF for the heights \( h(x) \) by

\[
\frac{1}{C} \prod_{l=1}^{2N} \prod_{j=1}^{N(l)} h(y^{(l)}_{j}) \chi_{Y(\{y\})}, \tag{2.3}
\]

where \( \chi_A = 1 \) for \( A \) true and \( \chi_A = 0 \) otherwise.

As our first exact result in relation to the joint PDF (2.3), let us compute the normalization \( C \).

**Proposition 2.1** The normalization \( C \) in (2.3) is independent of the PDF \( h \), and is given by

\[
C = \prod_{j=0}^{N} \frac{\Gamma(1+j)}{\Gamma(N+j+1)}. \tag{2.4}
\]

Proof. The PDF \( h(x) \) maps \( \mathbb{R}_{\geq 0} \) to \( \mathbb{R}_{\geq 0} \). Its antiderivative

\[
H(x) = \int_0^x h(t) \, dt
\]

is therefore a non-decreasing function from \( \mathbb{R}_{\geq 0} \) to \([0,1]\). Thus if we change variables

\[
u^{(l)}_{j} = H(y^{(l)}_{j}) \tag{2.5}
\]

we see that the interlacings (2.1), (2.2) remain true in the variables \( \{\nu^{(l)}_{j}\} \). Furthermore

\[
\frac{du^{(l)}_{j}}{dy^{(l)}_{j}} = h(y^{(l)}_{j})
\]

and so

\[
\prod_{l=1}^{2N} \prod_{j=1}^{N(l)} h(y^{(l)}_{j}) \chi_{Y(\{y\})} \, d\vec{y} = \prod_{l=1}^{2N} \prod_{j=1}^{N(l)} U_{[0,1]}(\nu^{(l)}_{j}) \chi_{Y(\{\nu\})} \, d\vec{\nu}. \tag{2.6}
\]
The measure on the RHS of \((2.6)\) is an example (the case \(p = q = N\)) of the finitization of the bead process introduced in \([8]\). But note that the interpretations of the variables are different: in \([8]\) \(u_j^{(l)}\) is the position of the particle \(j\) on line \(l\) while here \(u_j^{(l)}\) is a (transformed) height at a lattice point as specified by \((2.5)\).

Also given in \([8]\) is a discrete version of \((2.6)\), and furthermore summation methods are presented to compute marginal PDFs for a single line. In \([9, \text{Prop. 10.2.3}]\) this working is adopted to the present continuous setting. In particular, it is shown \([9, \text{Eq. (10.66)}]\) that integrating over lines \(l = 1, 2, \ldots, \) with configuration \(\{y_{(N+1)}^{j}\}_{j=1, \ldots, N+1}\) gives

\[
\prod_{s=1}^{N} \frac{1}{s!} \prod_{1 \leq j < k \leq N+1} (y_{j}^{(N+1)} - y_{k}^{(N+1)}).
\]  

(2.7)

By symmetry, this same expression results by integrating over lines \(l = 2N, 2N-1, \ldots, N+1\) (in that order) with configuration \(\{y_{(N+1)}^{j}\}_{j=1, \ldots, N+1}\) on line \(N+1\). Hence

\[
C = \left(\prod_{s=1}^{N} \frac{1}{s!}\right)^2 \int_{\tilde{Y}} \prod_{1 \leq j < k \leq N+1} (y_{j}^{(N+1)} - y_{k}^{(N+1)})^2 \, d\tilde{y}^{(N+1)}
\]

(2.8)

where \(\tilde{Y}\) denotes the region

\[1 > y_{1}^{(N+1)} > y_{2}^{(N+1)} > \cdots > y_{N+1}^{(N+1)} > 0.\]

The integrand in \((2.8)\) is symmetric, so the region \(\tilde{Y}\) can be replaced by \([0, 1]^{N+1}\) provided we divide by \((N + 1)!\). Furthermore,

\[
\int_{[0,1]^{N+1}} \prod_{1 \leq j < k \leq N+1} (y_{j}^{(N+1)} - y_{k}^{(N+1)})^2 \, d\tilde{y}^{(N+1)} = \prod_{j=0}^{N} \frac{\Gamma(1 + j)\Gamma(2 + j)}{\Gamma(N + j + 1)}
\]

(this is a special case of the Selberg integral (see e.g. \([9, \text{Eq. (4.3)}]\)), or alternatively can be derived using knowledge of the normalization of single variable Jacobi polynomials \([9, \text{Eq. (5.74)}]\)) and \((2.8)\) results.

\[\square\]

### 3 Uniformly distributed height variables

Suppose the height variables are sampled from the uniform distribution on \([0, 1]\). It was remarked during the proof of Proposition \(2.1\) that another interpretation of the joint PDF \((2.3)\) for the heights in the directed solid-on-solid model is then as a joint PDF for interlaced particles on \(2N - 1\) parallel line segments \([0, 1]\), with \(N(l)\) particles on line \(l\). It was furthermore remarked that this latter model is the so called finitized bead process, studied in \([8]\), or more precisely the special case

\[p = N, \quad q = N\]

(3.1)

of the finitized bead process (general \(p\) and \(q\) would correspond to the height model of Section 2 being defined in a rectangle rather than square).
3.1 Sampling using random matrices

What does a typical configuration of the directed solid-on-solid model look like? Section 4 of [8] details how (a generalization) of the PDF (2.3) in the case that the heights are sampled from the uniform distribution $U_{[0,1]}$ can be obtained as the joint eigenvalue PDF of a sequence of random matrices. For a given matrix $A$, the construction is based on random corank 1 projections, $M = \Pi A \Pi$. Here $\Pi = I - \bar{x} \bar{x}^\dagger$, where $x$ is a normalized complex Gaussian vector of the same number of rows as $A$. The effect of the random corank 1 projection is to reduce by one the multiplicity of any degenerate eigenvalues in $A$, and also to create a new zero eigenvalue.

The first step is to form $M_1 = \Pi A_1 \Pi$, where $A_1 = \text{diag}(0, 1)$ (the notation $(k)^N$ is used to denote $k$ repeated $N$ times). The eigenvalue of $M_1$ then corresponds to the height $h^{(1)}_1$.

Next, for $r = 2, \ldots, N$, inductively generate $\{h^{(r)}_i\}_{i=1,\ldots,r}$ as the eigenvalues different from 0 and 1 of the matrix $M_r := \Pi A_r \Pi$, where

$$A_r = \text{diag}\left(0^{N-r+1}, h^{(r-1)}_1, \ldots, h^{(r-1)}_{r-1}, 1^{N-r+1}\right).$$

And after this, for $r = 1, \ldots, N-1$, inductively generate $\{h^{(p+r)}_i\}_{i=1,\ldots,N-r}$ ($r = 1, \ldots, N-1$) as the eigenvalues different from 0 of $M_{N+r} = \Pi A_{N+r} \Pi$ where

$$A_{N+r} = \text{diag}\left(h^{(N+r-1)}_1, \ldots, h^{(N+r-1)}_N\right).$$

Crucial to the practical implementation of this construction is the fact that if

$$A = \text{diag}\left((a_1)^{s_1}, (a_2)^{s_2}, \ldots, (a_n)^{s_n}\right),$$

the $n - 1$ eigenvalues different from $\{a_i\}_{i=1,\ldots,n}$ and 0 occur at the zeros of the random rational function

$$\sum_{i=1}^n \frac{q_i}{x - a_i},$$

where $(q_1, \ldots, q_n)$ has the Dirichlet distribution $D[s_1, \ldots, s_n] [12]$.

We have made use of the above theory to generate a typical configuration of the directed solid-on-solid model in the case $p = q = 10$, which is displayed graphically in Figure 3.1. We remark that if instead of sampling the individual heights from the continuous uniform distribution, they were sampled instead from the discrete uniform distribution on $[0, c]$ say ($c \in \mathbb{Z}^+$), then our directed solid-on-solid model would be equivalent to boxed plane partitions, of box size $N \times N \times c$. There is no longer a random matrix approach to the sampling of a typical configuration, but we point out that one recently introduced method [4] (see also [2] and [3]) has the interpretation as defining a dynamics on the underlying interlaced particle configurations.
3.2 Correlations and distribution functions

Let \((l, x)\) refer to the position \(x\) on line \(l\) of the bead process. The correlation function \(\rho(k)(X)\) for the configuration \(X = \bigcup_{i=1}^{s} \{l_i, x^{(l_i)}_1, \ldots, x^{(l_i)}_{r_i}\}\), where each \(x^{(l_i)}\) is on line \(l_i\) and \(k = \sum_{i=1}^{s} r_i\), defined as the average

\[
\rho(k)(X) = \left\langle \prod_{i=1}^{s} \sum_{m_1 \neq \cdots \neq m_{r_i}} \prod_{q=1}^{r_i} \delta(x^{(l_i)}_q - y^{(l_i)}_{m_q}) \right\rangle \tag{3.2}
\]

and thus requiring that \(k\) of the particles be fixed at positions \(X\), has been computed in [8, Prop. 5.1]. The correlations are given in terms of the rescaled Jacobi polynomials

\[
\tilde{P}^{(a,b)}_n(x) := P^{(a,b)}_n(1 - 2x),
\]

which satisfy the orthogonality

\[
\int_0^1 x^a(1 - x)^b \tilde{P}^{(a,b)}_j(x) \tilde{P}^{(a,b)}_k(x) \, dx = \mathcal{N}^{(a,b)}_j \delta_{j,k} \tag{3.4}
\]

where

\[
\mathcal{N}^{(a,b)}_n = \frac{1}{2n + a + b + 1} \frac{(n + a)!(n + b)!}{n!(n + a + b)!}. \tag{3.5}
\]
We also require the quantities

\[ a_l(x) = \begin{cases} (-x)^{N-l}(1-x)^{N-l}, & 1 \leq l \leq N \\ 1, & N + 1 \leq l \leq 2N - 1 \end{cases} \] (3.6)

\[ b_l(x) = \begin{cases} (-1)^{N-l}, & 1 \leq l \leq N \\ x^{l-N}(1-x)^{l-N}, & N + 1 \leq l \leq 2N - 1 \end{cases} \] (3.7)

\[ C_j(l) = \frac{(l-j)!}{(N-j)!}, \quad 1 \leq l \leq 2N - 1 \] (3.8)

\[ Q_j^{(l)}(x) = \begin{cases} P_{l-j}^{(N-l,N-l)}(x), & 1 \leq l \leq N \\ P_{2N+1-l-j}^{(l-N,l-N)}(x), & N + 1 \leq l \leq 2N - 1 \end{cases} \] (3.9)

\[ N_j^{(l)}(x) = \begin{cases} N_{l-j}^{(N-l,N-l)}, & 1 \leq l \leq N \\ N_{2N-l-j}^{(l-N,l-N)}, & N + 1 \leq l \leq 2N - 1 \end{cases} \] (3.10)

\[ \alpha(l, l') = \min\{2N + 1 - l, N\}. \] (3.11)

In terms of this notation Proposition 5.1 of [8] gives

\[ \rho_{(k)}(X) = \det[K(l_i, x_i; l_j, x_j)]_{i,j=1,...,k} \] (3.12)

where

\[ K(s, u; t, v) = \begin{cases} a_s(u)b_t(v) \sum_{l=1}^{\alpha(s,t)} \frac{C_l^{(s)} Q_l^{(s)}(u)Q_l^{(t)}(v)}{C_l^{(t)} N_l^{(t)}}, & s \geq t \\ -a_s(u)b_t(v) \sum_{l=-\infty}^{0} \frac{C_l^{(s)} Q_l^{(s)}(u)Q_l^{(t)}(v)}{C_l^{(t)} N_l^{(t)}}, & s < t \end{cases} \] (3.13)

To make use of knowledge of this bead process correlation function in the context of the solid-on-solid model, we must use it to specify the distribution of the position of specific particles. In fact it is sufficient to specify \( E_0(\{l_i; (u_i, 1)\}_{i=1,...,n}) \) — the probability there are no particles in the interval \((u_i, 1)\) of line \(l_i\) \((i = 1, \ldots, n)\). Thus the PDF \( p_{\text{max}}(\{l_i, u_i\}_{i=1,...,n}) \) for the position of the particles with maximum displacements on lines \(l_i, i = 1, \ldots, n\), then follows by partial differentiation according to

\[ p_{\text{max}}(\{l_i, u_i\}_{i=1,...,n}) = \frac{\partial^n}{\partial u_1 \cdots \partial u_n} E_0(\{l_i; (u_i, 1)\}_{i=1,...,n}). \] (3.14)

The crucial point from the viewpoint of the corresponding height model is that this PDF is identical to the PDF for the maximum heights on the same lines \(i = 1, \ldots, n\).

We are thus faced with the task of expressing \( E_0(\{l_i; (u_i, 1)\}_{i=1,...,n}) \) in terms of correlation functions, which is in fact a standard exercise [\S8.1]. First note that according to the definition

\[ E_0(\{l_i; (u_i, 1)\}_{i=1,...,n}) = \left\langle \prod_{i=1}^{n} \prod_{l=1}^{N(l_i)} (1 - \chi_{y_l}^{(l_i)}(u_i, 1)) \right\rangle. \]
Expanding the double product and recalling the definition (3.2) of the correlation function then shows

\[
E_0(\{l_i, (u_i, 1)\}_{i=1,\ldots,n}) = \sum_{m_1=0}^{N(l_1)} \cdots \sum_{m_n=0}^{N(l_n)} \frac{(-1)^{m_1+\cdots+m_n}}{m_1! \cdots m_n!} \times \int_{u_1}^1 dy_1^{(l_1)} \cdots \int_{u_1}^1 dy_n^{(l_n)} \rho(\sum_{i=1}^n m_i)\left( \bigcup_{i=1}^n \bigcup_{j=1}^{m_i} \{(l_i, y_j^{(l_i)})\} \right),
\]

where the term \(m_1 = \cdots = m_n = 0\) is taken to equal unity.

This is a general formula valid for any two-dimensional particle system confined to \(2N - 1\) parallel lines, with \(N(l)\) particles on line \(l (l = 1, \ldots, 2N - 1)\). It is furthermore the case that when the \(k\)-point \((k = \sum_{i=1}^n m_i)\) correlation function \(\rho_{(k)}(Y)\) has a determinantal form (3.12), the multiple sum (3.15) can be summed [9, §9.1]. This can be done by defining the \(n \times n\) matrix Fredholm integral operator \(K(\{(l_i; (u_i, 1))\}_{i=1,\ldots,n})\) with kernel

\[
\tilde{K}(x, y; \{l_i; (u_i, 1)\}_{i=1,\ldots,n}) = \left[ \chi_{x \in (u_i, 1)} K(x, l_i; y, l_j) \chi_{y \in (u_j, 1)} \right]_{i,j=1,\ldots,n}.
\]

We then have that

\[
E_0(\{l_i; (u_i, 1)\}_{i=1,\ldots,n}) = \det \left( 1 - K(\{l_i; (u_i, 1)\}_{i=1,\ldots,n}) \right),
\]

where the meaning of the determinant can be taken as the product over the eigenvalues of the operator.

### 4 Scaled limits for uniformly distributed heights

#### 4.1 The limiting shape

In the large \(N\) limit the global density on line of the finitized bead process has been computed in [8]. With the parameters as in [3.1] and the label \(j\) of each line scaled \(j \mapsto j/N =: S\) and thus \(0 < S < 2\), the support of the density was found to be the interval

\[
\left[ \frac{1}{2} - \frac{1}{2} \sqrt{S(2-S)}, \frac{1}{2} + \frac{1}{2} \sqrt{S(2-S)} \right] =: [c_S, d_S].
\]

And after dividing by \(N\), the explicit functional form of the density was shown to equal

\[
\rho_{(1)}(y, S) = \frac{1}{\pi} \sqrt{\frac{(d_S - y)(y - c_S)}{y(1-y)}}.
\]

To interpret these results in terms of the directed solid-on-solid model with heights sampled from \(U_{[0,1]}\), we first agree to scale the \(N \times N\) integer grid by \(1/N\) so that it is an \(N \times N\) grid within the unit square \([0,1]^2\). It then follows immediately from (4.1) that the limiting height
profiles, \( h(x, y) \) say, along \( y = 1, x = 1, y = 0 \) and \( x = 0 \) respectively are

\[
\begin{align*}
 h(x, 1) &= \frac{1}{2} \left( 1 + \sqrt{x(2-x)} \right) \\
 h(1, y) &= \frac{1}{2} \left( 1 + \sqrt{y(2-y)} \right) \\
 h(x, 0) &= \frac{1}{2} \left( 1 - \sqrt{1 - x^2} \right) \\
 h(0, y) &= \frac{1}{2} \left( 1 - \sqrt{1 - y^2} \right)
\end{align*}
\] (4.3)

We remark that (4.3) exhibits the general symmetry \( h(x, y) = h(y, x) \), which in turn follows from the symmetry with respect to the \( x \) and \( y \) directions of the rule for the interlacing of heights in the definition of the model.

To specify the heights at other positions of the unit square we need to work in the coordinates which relate directly to those used for the scaled bead process. The variable \( S \) used in (4.1), for \( 0 \leq S \leq 1 \), then identifies the line segment in \([0, 1]^2\) starting at \((0, 1 - S)\) and finishing at \((S, 1)\). Let \( t, 0 < t < 1 \) denote the scaled position along this line segment so that

\[
(x, y) = (tS, 1 + S(t - 1)).
\] (4.4)

We seek the height \( h(S, t) \) at position \( t \) along line \( S \). A little thought (cf. \[6\] eq. (3.5))] shows that \( h(S, t) \) is characterized by the equation

\[
tS = \int_{cS}^{h(S,t)} \rho(1)(u, s) \, du.
\] (4.5)

Thus in the bead model picture, the RHS of (4.5) gives the expected number of particles from the start of the line segment, to position \( h(S, t) \) along the segment. The LHS says this number of particles is equal to \( tS \). Hence particle number \( tS \) along this segment is expected to be at position \( h(S, t) \) on the line. In the solid-on-solid picture, the expected position of a particular numbered particle is the expected height at the point in the square corresponding to the numbering of the particle.

We note from (4.2) and (4.1) that \( \rho(1)(y, S) \) is symmetrical about \( y = 1/2 \). It follows that

\[
\int_{cS}^{1/2} \rho(1)(u, s) \, du = \frac{1}{2} \int_{cS}^{dS} \rho(1)(u, s) \, du = \frac{S}{2}, \quad 0 \leq S \leq 1
\]

and consequently

\[
h(S, \frac{1}{2}) = \frac{1}{2}, \quad 0 \leq S \leq 1
\]
or equivalently in terms of \( xy \)-coordinates,

\[
h(x, 1 - x) = \frac{1}{2}, \quad 0 \leq x \leq 1.
\] (4.6)

This supplements the exact profiles (4.3).
Along other line segments of the unit square, we have to make do with (4.5), although the integral can be evaluated explicitly. Thus recalling (4.1) and (4.2), use of computer algebra allows us to conclude

$$tS = \frac{1}{\pi} \left( \arcsin v + \pi/2 + \sqrt{1 - \tilde{d}^2} \left( \arctan \left( \frac{\sqrt{1 - \tilde{d}^2}}{\sqrt{1 - v^2}} \right) + \pi/2 \right) \right) \tag{4.7}$$

where

$$v := \frac{(2h(S,t) - 1)}{\tilde{d}}, \quad \tilde{d} = \sqrt{S(2 - S)}. \tag{4.8}$$

For practical determination of the height profile, we thus choose \((x,y)\) within the unit square, then determine \(S\) and \(t\) according to

$$S = 1 + x - y, \quad t = x/(1 + x - y) \tag{4.9}$$

as implied by (4.4). Substituting these values in (4.7) we obtain an equation for \(h\), which is solved using a root finding routine. A graph of the resulting shape is plotted in Figure 4.1.

### 4.2 Scaled correlation of heights along \(y = 1\)

The expected value of the heights along \(y = 1\) in the limiting solid-on-solid model is the first entry in (4.3). We seek a scaled limit of the joint distribution function (3.14) for \(k\) heights along this line, which according to (3.17) requires computing the large \(N\) form of (3.16). In the
limiting procedure $x$ and $y$ must first undergo a linear change of scale. Thus after introducing the scaled line number $S_j$, for $x_j$ corresponding to this line we must write

$$
x_j = x_0(S_j) + X_j \sigma N^{-2/3}
$$

(4.10)

where $x_0(S) := \frac{1}{2}(1 + \sqrt{S(2-S)})$, $\sigma$ is independent of $N$ and can be chosen at our convenience, and the factor $N^{-2/3}$ is chosen so that in the variable $X$ the spacing between the large heights on line $S$ is $O(1)$. We must also choose the spacing between scaled lines to scale with $N$. Thus we write

$$
S_j = S^* + t_j \tau N^{-1/3}
$$

(4.11)

where $S^*$ is the limiting line on the scale of a division by $N$, and $\tau_{S^*}$ is independent of $N$ to be chosen for convenience. The motivation for the choice (4.11) is that the scaled correlations are $O(1)$ in the variables $\{t_j\}$.

Since from §3.2 the correlations are given in terms of Jacobi polynomials (3.3), we require a uniform asymptotic expansion appropriate for the scaling (4.10). The relevant such expansion has recently been given by Johnstone [14]: it applies when the Jacobi polynomial parameters $a, b$ increase with $n$, and when $x$ is centred about the largest (or by replacing the roles of $a, b$, the smallest) zero, with a further scaling chosen so that the spacing between zeros is of order unity. For our purposes, due to the relation (3.3), we formulate the results of [14] about the smallest zero of $P_n^{(a,b)}(x)$, and thus the largest zero of $\tilde{P}_n^{(a,b)}(x)$.

**Proposition 4.1** Define variables $\kappa_n$, $\psi$ and $\gamma$ by

$$
\kappa_n = 2n + a + b + 1, \quad \cos \psi = \frac{a - b}{\kappa_n}, \quad \cos \gamma = \frac{a + b}{\kappa_n},
$$

(4.12)

and using these variables define

$$
M_n = \cos(\psi + \gamma), \quad \sigma_n^3 = \frac{2 \sin^4(\psi + \gamma)}{\kappa_n^2 \sin \psi \sin \gamma}.
$$

(4.13)

Then, for $x = M_n - \sigma_n X$, one has the uniform asymptotic expansion

$$
P_n^{(a,b)}(x) = \sqrt{\frac{\kappa_n \sigma_n^{2a+b+1} N_n^{(a,b)}}{(1-x)^a(1+x)^b(1-x^2)}} \left( \text{Ai}(X) + O(N^{-2/3}) \right) \left\{ \begin{array}{ll}
e^{-X/2}, & X > 0 \\
1, & X < 0 
\end{array} \right. .
$$

(4.14)

Equivalently, in terms of the rescaled Jacobi polynomials (3.3), for $x = (1 - M_n)/2 + \sigma_n X/2$,

$$
\tilde{P}_n^{(a,b)}(x) = \sqrt{\frac{\kappa_n \sigma_n N_n^{(a,b)}}{2x^a+1(1-x)^b+1}} \left( \text{Ai}(X) + O(N^{-2/3}) \right) \left\{ \begin{array}{ll}
e^{-X/2}, & X > 0 \\
1, & X < 0 
\end{array} \right. .
$$

(4.15)

According to (3.9), with $l$ scaling with $N$ such that $\bar{l} := l/N$ is fixed, we have $a = b = N - l$, and thus $\cos \psi = 0$, $\lim_{l \to \infty} \cos \gamma = 1 - \bar{l}$, which together imply

$$
\frac{1}{2} - \frac{1}{2} \lim_{l \to \infty} M_l = \frac{1}{2} + \frac{1}{2} \sqrt{1 - (1 - \bar{l})^2}.
$$

(4.16)
With \( \tilde{I} \) identified as \( S \), this is precisely \( x_0(S) \) in (4.10). We want to use Proposition 4.1 to show that with an appropriate choice of scaled variables as implied by (4.10) and (4.11), the summations (3.13) defining the correlation kernel \( K \) are slowing varying in the summation index \( l \) and as \( N \to \infty \) are given by explicit Riemann integrals. The latter we will recognise as specifying the well known Airy process from random matrix theory [13, 11]. In addition, for purposes of application to the scaled limit of the probability \( E_0 \) we need to demonstrate that the error term is integrable to the right of the soft edge.

**Proposition 4.2** Let

\[
\begin{align*}
x &= x_0(S_x) + X\sigma N^{-2/3}, & y &= x_0(S_y) + Y\sigma N^{-2/3}, \\
S_x &= S + s\tau N^{-1/3}, & S_y &= S + t\tau N^{-1/3},
\end{align*}
\]

with \( S \neq 1, x_0(S) \) as specified below (4.10) and \( \sigma, \tau \) such that

\[
\sigma^3 \frac{(1-S)^4}{16\sqrt{S(2-S)}} = \frac{(1-S)^2\sqrt{S(2-S)}}{2\sigma}.
\]

With \( K \) as in (3.13), define \( \tilde{K} \) by

\[
\tilde{K}(X, s; Y, t) = \lim_{N \to \infty} \sigma N^{-2/3} K(x, NS_x; y, NS_y).
\]

We then have

\[
\tilde{K}(X, s; Y, t) = \begin{cases} 
\frac{F(X, s)}{F(Y, t)} \int_{-\infty}^{\infty} e^{(t-s)u} \text{Ai}(u + X) \text{Ai}(u + Y) \, du, & s \geq t \\
-\frac{F(X, s)}{F(Y, t)} \int_{-\infty}^{0} e^{(t-s)u} \text{Ai}(u + X) \text{Ai}(u + Y) \, du, & s < t
\end{cases}
\]

and consequently

\[
\tilde{\rho}(k)(\{ (X_j, s_j) \}_{j=1,\ldots,k}) := \lim_{N \to \infty} (\sigma N^{-2/3})^k \rho(k)(\{ (x_j, NS_x) \}_{j=1,\ldots,k}) = \det[K^{\text{AiryP}}(X_j, s_j; X_l, s_l)]_{j,l=1,\ldots,k}.
\]

Moreover,

\[
\lim_{N \to \infty} (\sigma N^{-2/3})^k \int_{y_1}^{1} dx_1 \cdots \int_{y_k}^{1} dx_k \rho(k)(\{ (x_j, NS_x) \}_{j=1,\ldots,k}) = \int_{Y_1}^{\infty} dX_1 \cdots \int_{Y_k}^{\infty} dX_k \tilde{\rho}(k)(\{ (X_j, s_j) \}_{j=1,\ldots,k}).
\]

Proof. Let us introduce \( l_x \) according to \( S_x = l_x/N \). Substituting \( x \) as given in (4.17) into the uniform asymptotic expansion for the Jacobi polynomial (4.15), with \( n = l_x - j \), \( a = b = N - l_x \), gives

\[
P_{l_x-j,N-l_x}(x) = \sqrt{\frac{\kappa_{l_x-j}\sigma_{l_x-j}N_{l_x-j}^{N-N-l_x}}{2x^{N-l_x+1}(1-x)^{N-l_x+1}}} \left( \text{Ai}(X^*) + O(N^{-2/3}) \right) e^{-X^*/2} \begin{cases} 
eq 0, & X^* > 0 \\
1, & X^* < 0
\end{cases}
\]

(4.23)
where, in the notation of (4.13), $X^*$ is given by

$$X^* = \frac{2}{\sigma_{l_x-j}} \left( \frac{M_{l_x-j} - 1}{2} + x \right).$$  \hfill (4.24)

Recalling the definition (3.13) and using the expansion (4.23), $K(x, l_x; y, l_y)$ is given by a summation over $j$ of

$$\frac{(-1)^{l_x-l_y}}{2} \sqrt{\frac{x(1-x)}{(y-1-y)^{N-l_y+1}}} \left( \frac{N(N-l_x,N-l_x)}{N_{l_x-j}} \frac{N(N-l_y,N-l_y)}{N_{l_y-j}} \right) \left( \frac{(S_x-1)^2}{N S_x (2-S_x)} + \max \left( O(N^{-1}), O(j^2/N^2) \right) \right)$$ \hfill (4.25)

$$\times \left( \text{Ai}(X^*) \text{Ai}(Y^*) + O(N^{-2/3}) \left( \begin{array}{c} e^{-X^*/2}, \quad X^* > 0 \\ 1, \quad X^* < 0 \end{array} \right) \left( \begin{array}{c} e^{-Y^*/2}, \quad Y^* > 0 \\ 1, \quad Y^* < 0 \end{array} \right) \right).$$

In fact the summand (4.25) is a slowly varying function of $j$. To see this we first note from (4.13) that

$$M_{l_x-j} = -\sqrt{S_x (2-S_x)} - \frac{j}{N S_x (2-S_x)} + \max \left( O(N^{-1}), O(j^2/N^2) \right)$$ \hfill (4.26)

and $\sigma_{l_x-j} = 2\sigma N^{-2/3} + O(N^{-1})$. This substituted in (4.24) suggests we introduce a continuous summation label $w$ by $j = wN^{1/3}$ to obtain the large $N$ form

$$X^* = -w \frac{(S-1)^2}{2\sigma \sqrt{S(2-S)}} + X + O(N^{-1/3}).$$ \hfill (4.27)

Furthermore, in terms of $w$ use of Stirling’s formula shows that the first line of (4.25) (the factors before the Airy functions) reads

$$\frac{F(X,s)}{F(Y,t)} \left( \frac{8N^{1/3}S}{(S-1)^2} \right) \exp \left( \frac{(s-t)\tau w}{S(2-S)} \right) \left( 1 + O(N^{-1/3}) \right)$$ \hfill (4.28)

with the error term uniform in $w$ and where

$$F(X,s) = (-1)^{l_x} [x(1-x)]^{N-l_x-1/2} N^{s \tau N^{2/3}} (S(2-S))^{s \tau N^{2/3}/2} e^{-\frac{s^2}{2} \frac{1-S}{S(2-S)}} N^{-1/3} e^{\frac{3}{2} \frac{(2-S+2S^2)}{3S^2(2-S)^2}}$$ \hfill (4.29)

Thus, to leading order the summand depends only on the continuous summation label $w$, and thus can be converted to a Reimann integral to give

$$\sigma N^{-2/3} K(x, NS_x; y, NS_y) = \frac{F(X,s)}{F(Y,t)} \left( \frac{8\sigma^2}{(S-1)^2} \right) \int_0^\infty \exp \left( \frac{(s-t)\tau w}{S(2-S)} \right)$$

$$\times \text{Ai} \left( -w \frac{(S-1)^2}{2\sigma \sqrt{S(2-S)}} + X \right) \text{Ai} \left( -w \frac{(S-1)^2}{2\sigma \sqrt{S(2-S)}} + Y \right) \, dw$$

$$+ O(N^{-1/3}) O(e^{-(X+Y)/2}).$$ \hfill (4.30)

A change of variables

$$u = -w \frac{(S-1)^2}{2\sigma \sqrt{S(2-S)}}$$ \hfill (4.31)
turns this into
\[
\lim_{N \to \infty} \sigma N^{-2/3} K(x, NS_x; y, NS_y) = \frac{F(X, s)}{F(Y, t)} \int_0^\infty \exp \left( \frac{(t-s)\tau^2 \sigma u}{\sqrt{S(2-S)(1-S)^2}} \right) \\
\times \text{Ai} \left( u + X \right) \text{Ai} \left( u + Y \right) \, du
\]
(4.32)
and the choice of \( \tau \) as in (4.18) completes the proof in the case \( s \geq t \). Furthermore, essentially the same working holds true for the remaining case \( s < t \).

With the form of \( \tilde{K} \) now established, the formula (4.21) for the correlation function follows by noting that upon substitution in (3.12) the factor \( F(X_j, s_j)/F(X_i, s_i) \) cancels from the determinant. Finally, the validity of (4.22) follows from (3.12) combined with the error term in (4.30).

We can make use of Proposition 4.2 to compute the scaled limit of the probability \( E_0 \). Thus we know from [18], [3] that the convergence of the integrals (4.22) implies that the scaled limit can be applied term-by-term in (3.15). Since the limiting correlations are determinantal, the resulting sum can again be summed by appealing to Fredholm integral operator theory (see e.g. [9, §9.1]).

**Corollary 4.3** Let \( v_j \) and \( V_j \) be related as implied by the first line in (4.17), and \( s_j \) and \( S_j \) be related as implied by the second line. We have
\[
\lim_{N \to \infty} E_0(\{NS_i; (v_i, 1)\}_{i=1,...,n}) = \det \left( 1 - K^{\text{AiryP}}(\{s_i, (V_i, \infty)\}_{i=1,...,n}) \right)
\]
(4.33)
where \( K^{\text{AiryP}}(\{s_i, (V_i, \infty)\}_{i=1,...,n}) \) is the \( n \times n \) matrix Fredholm integral operator with kernel
\[
\tilde{K}^{\text{AiryP}}(X, Y; \{s_i; (V_i, \infty)\}_{i=1,...,n}) = \left[ \chi_{X \in (V_i, \infty)} K^{\text{AiryP}}(X, s_i; Y, s_j) \chi_{Y \in (V_j, \infty)} \right]_{i,j=1,...,n}.
\]
(4.34)

The expression (4.33) is precisely that for the cumulative distribution function of the scaled largest eigenvalue in the Dyson Brownian motion model of complex Hermitian matrices [16] (see [9, Ch. 11] for an account of the model). We remark that the statistical system defined by the scaling of the eigenvalues in the Dyson Brownian motion model of complex Hermitian matrices about the largest eigenvalue is referred to as the Airy process [16]. And as noted in the Introduction, the Airy process also occurs in random matrix theory as the correlation functions for the GUE minor process at the soft edge [10].

**5 Universality**

We know turn our attention to the case of a general absolutely continuous distribution \( \mathcal{D} \) for the height variables, characterized by the corresponding PDF \( h(x) \). We have already seen that the change of variables (2.5) maps the joint PDF for configurations of interlaced heights sampled from \( \mathcal{D} \) to the joint PDF for configurations of interlaced heights sampled from the particular case \( \mathcal{D} = U[0,1] \) — the uniform distribution on \([0,1]\). And in the previous section we have identified as the Airy process a particular scaling limit of the joint distribution of the heights along \( y = 1 \) in the case \( \mathcal{D} = U[0,1] \). Our aim in this subsection is to show that the change of
variables \((2.5)\) implies that there is a choice of scaled variables which also gives the Airy process as the scaling limit of the joint distribution of the heights along \(y = 1\).

Let \(N\) be large, and scale the vertical lines in the corresponding bead process so they are labelled by a continuous parameter \(S\) as specified at the beginning of \(\S 4.1\). Again in the bead process picture, \((4.1)\) gives the support of the density in the case \(D = U_{[0,1]}\). It follows immediately from the change of variables \((2.5)\) that the along line \(S\) the interval of support for a general absolutely continuous distribution with PDF \(h\) is specified by the equations

\[
\frac{1}{2} - \frac{1}{2}\sqrt{S(2-S)} = \int_{0}^{d_{S}(h)} h(t) \, dt, \quad \frac{1}{2} + \frac{1}{2}\sqrt{S(2-S)} = \int_{0}^{d_{S}(h)} h(t) \, dt.
\] (5.1)

We note that \(d_{S}(h)\) must be finite for \(S \neq 1\). Recalling \((4.10)\) in the case \(D = U_{[0,1]}\) for a scaling of positions in the bead model/heights in the directed solid-on-solid model about the upper edge of support, we introduce the renormalized parameter \(\tilde{\sigma}\) such that

\[
d_{S}(1) + X_{j}\sigma N^{-2/3} = \int_{0}^{d_{S}(h)+X_{j}\tilde{\sigma}N^{-2/3}} h(t) \, dt.
\] (5.2)

For \(h(t)\) continuous we can expand the RHS to leading order \(N^{-2/3}\) and so deduce that to leading order in \(N\),

\[
\sigma = \tilde{\sigma} h(d_{S}(h)).
\] (5.3)

Hence we conclude that for \(h(t)\) continuous and non-zero at \(t = d_{S}(h)\), the scaling of positions about the upper edge of the support

\[
x = d_{S_{x}}(h) + X_{j}/(h(d_{S_{x}}(h))N^{2/3})
\] (5.4)

is consistent with the change of variables \((2.5)\) required for the equality of joint probabilities \((2.6)\). Thus with the change of variables \((5.4)\) the scaling limit of the joint distribution of heights in the general case must be the same as in the particular case \(D = U_{[0,1]}\), so giving our sought universality result.

**Corollary 5.1** Let \(v_{j}\) and \(V_{j}\) be related as for \(x_{j}\) and \(X_{j}\) in \((5.4)\). Let \(s_{j}\) and \(S_{j}\) be related as given in the second line of \((4.17)\), where again it is required \(S \neq 1\). For the directed solid-on-solid model with heights sampled from a general absolutely continuous distribution \(D\) with corresponding PDF \(h(x)\), \(h(x)\) itself continuous and assumed non-zero at the boundary reference point \(x = d_{S}(h)\), the limit formula \((4.33)\) remains valid.

The requirement that \(S \neq 1\) is crucial (\(S\) is the reference line in the bead process picture about which, on separations of order \(N^{2/3}\), the correlations between heights along \(y = 1\) are being measured). To see this, let us consider the fluctuation of the height \(x_{N,N}\) in the top right corner of our directed solid-on-solid model. In the bead process picture, corresponds to the particle with the largest coordinate on the line \(S = 1\). Now according to the definition of the model, \(x_{N,N}\) is the largest of the heights. Hence, for \(x_{N,N}\) to be less than \(X\) we must have that all the \(N(N+1)/2\) height variables must be less that \(X\), and so

\[
Pr(x_{N,N} < X) = \left( \int_{0}^{X} h(t) \, dt \right)^{N(N+1)/2}
\] (5.5)
(here the interlacing requirement plays no role).

In the case $D = U_{[0,1]}$ we see from this that

$$
\lim_{N \to \infty} \Pr(x_{N,N} < 1 - X/N^2) = e^{-X/2}.
$$

(5.6)

On the other hand, for $D$ the exponential distribution it follows from (5.5) that

$$
\lim_{N \to \infty} \Pr(x_{N,N} < X + 2 \log N) = e^{-e^{-X}}.
$$

(5.7)

Thus for the fluctuations of the maximum height, both the scale required to get a well defined limiting distribution, and the limiting distribution itself, are dependent on $D$, in contrast to the situation exhibited in Corollary 5.1.

**Acknowledgements**

The work of the authors was supported by a Melbourne Postgraduate Research Award and the Australian Research Council respectively. PJF thanks Eric Nordenstam for a discussion on this topic at the MSRI random matrix semester, Fall 2010.

**References**

[1] G. Amir, I. Corwin, and J. Quastel, *Probability distribution of the free energy of the continuum directed random polymer in 1+1 dimensions*, Comm. Pure Appl. Math. 64 (2011), 466–537.

[2] A. Borodin and P. Ferrari, *Anisotropic growth of random surfaces in 2 + 1 dimensions*, arXiv:0804.3035, 2008.

[3] A. Borodin and P.J Forrester, *Increasing subsequences and the hard-to-soft transition in matrix ensembles*, J. Phys. A 36 (2003), 2963–2981.

[4] A. Borodin and V. Gorin, *Shuffling algorithm for boxed plane partitions*, Advance in Math. 220 (2009), 1739–1770.

[5] A. Borodin, V. Gorin, and E.M. Rains, *q-distributions on boxed plane partitions*, Sel. Math. New Ser. 16 (2010), 731–789.

[6] P.L. Ferrari and H. Spohn, *Step fluctuations for a faceted crystal*, J. Stat. Phys. 113 (2003), 1–46.

[7] P.L. Ferrari and H. Spohn, *Random growth models*, arXiv:1003.0881.

[8] B.J. Fleming, P.J. Forrester, and E. Nordenstam, *A finitization of the bead process*, Prob. Theory Relat. Fields (2011).

[9] P.J. Forrester, *Log-gases and random matrices*, Princeton University Press, Princeton, NJ, 2010.
[10] P.J. Forrester and T. Nagao, *Determinantal correlations for classical projection processes*, J. Stat. Mech. **2011** (2011) P08011

[11] P.J. Forrester, T. Nagao, and G. Honner, *Correlations for the orthogonal-unitary and symplectic-unitary transitions at the hard and soft edges*, Nucl. Phys. B **553** (1999), 601–643.

[12] P.J. Forrester and E.M. Rains, *Interpretations of some parameter dependent generalizations of classical matrix ensembles*, Prob. Theory Related Fields **131** (2005), 1–61.

[13] K. Johansson, *Shape fluctuations and random matrices*, Commun. Math. Phys. **209** (2000), 437–476.

[14] I.M. Johansson, *Multivariate analysis and Jacobi ensembles: Largest eigenvalue, Tracy-Widom limits and rates of convergence*, Ann. Stat. **36** (2008), 2683–2716.

[15] A.M.S. Macêdo, *Universal parametric correlations at the soft edge of the spectrum of random matrices*, Europhys. Lett. **26** (1994), 641–646.

[16] M. Prähofer and H. Spohn, *Scale invariance of the PNG droplet and Airy process*, J. Stat. Phys. **108** (2001), 1071–1106.

[17] T. Sasamoto and H. Spohn, *Exact height distributions for the KPZ equation with narrow wedge initial condition*, Nucl. Phys. B **834** (2010), 523–542.

[18] A. Soshnikov, *A note on the universality of the distribution of the largest eigenvalues in certain sample covariance matrices*, J. Stat. Phys. **108** (2002), 1033–1056

[19] K. Takeuchi and M. Sano, *Growing interfaces of liquid crystal turbulence: universal scaling and fluctuations*, Phys. Rev. Lett. **56** (2010), 889–892.