SINGLE-POINT GRADIENT BLOW-UP ON THE BOUNDARY FOR DIFFUSIVE HAMILTON-JACOBI EQUATION IN DOMAINS WITH NON-CONSTANT CURVATURE.

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Abstract. We consider the diffusive Hamilton-Jacobi equation $u_t - \Delta u = |\nabla u|^p$ in a bounded planar domain with zero Dirichlet boundary condition. It is known that, for $p > 2$, the solutions to this problem can exhibit gradient blow-up (GBU) at the boundary. In this paper we study the possibility of the GBU set being reduced to a single point. In a previous work [Y.-X. Li, Ph. Souplet, 2009], it was shown that single point GBU solutions can be constructed in very particular domains, i.e. locally flat domains and disks. Here, we prove the existence of single point GBU solutions in a large class of domains, for which the curvature of the boundary may be nonconstant near the GBU point.

Our strategy is to use a boundary-fitted curvilinear coordinate system, combined with suitable auxiliary functions and appropriate monotonicity properties of the solution. The derivation and analysis of the parabolic equations satisfied by the auxiliary functions necessitate long and technical calculations involving boundary-fitted coordinates.

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1. Introduction and first results

We consider the initial-boundary value problem for the diffusive Hamilton-Jacobi equation

\[
\begin{aligned}
    u_t - \Delta u &= |\nabla u|^p, & x \in \Omega, & t > 0, \\
    u &= 0, & x \in \partial \Omega, & t > 0, \\
    u(x, 0) &= u_0(x), & x \in \Omega,
\end{aligned}
\]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^2 \), \( p > 2 \) and \( u_0 \in X_+ := \{ v \in C^1(\Omega); v \geq 0, v|_{\partial \Omega} = 0 \} \).

Equation (1.1) is a typical model-case in the theory of nonlinear parabolic equations, being the simplest example of a parabolic equation with a nonlinearity depending on the gradient of the solution. It has been extensively studied in the past twenty years and it is well known that if \( p \leq 2 \) or if \( \Omega = \mathbb{R}^n \), then all solutions exist globally in the classical sense, see [3], [7], [8], [9], [14], [15], [21], [29], [31]. On the contrary, for the case of superquadratic growth of the nonlinearity, i.e. \( p > 2 \), with \( \Omega \neq \mathbb{R}^n \), solutions exhibit singularities for large enough initial data. The nature of this singularity is of gradient blow-up type, and occurs on some subset of the boundary of the domain, see [1], [2], [4], [6], [10], [11], [16], [18], [22], [29], [30], [32], [33].

In addition, equation (1.1) arises in stochastic control theory [23], and is involved in certain physical models, for example of ballistic deposition processes, where the solution describes the growth of an interface, see [17], [19], [20].

It follows from classical theory, see for example [12, Theorem 10, p. 206], that problem (1.1) admits a unique maximal, nonnegative classical solution \( u \in C^{2,1}(\Omega \times (0, T)) \cap C^{1,0}(\overline{\Omega} \times [0, T)) \), where \( T = T(u_0) \) is the maximal existence time. By the maximum principle, for problem (1.1) we have

\[
\|u(t)\|_{\infty} \leq \|u_0\|_{\infty}, \quad 0 < t < T.
\]

Since (1.1) is well posed in \( X_+ \), it follows that, if \( T < \infty \), then

\[
\lim_{t \to T} \|\nabla u(t)\|_{\infty} = \infty.
\]

This phenomenon of \( \nabla u \) blowing up with \( u \) remaining uniformly bounded is known as gradient blow-up. The gradient blow-up set of \( u \) is defined by

\[
\text{GBUS}(u_0) = \{ x_0 \in \partial \Omega; \limsup_{t \to T, x \to x_0} |\nabla u(x, t)| = \infty \}.
\]

We call gradient blow-up point (GBU point for short) any point in \( \text{GBUS}(u_0) \). The space profile at \( t = T \) is investigated in [10], [1], [33], [10], [24]. For results on the GBU rate, we refer to [10], [19], [34], [26]. Also, the existence and properties of a weak continuation of the solution after GBU are studied in [11], [6], [27], [29], [28].

From [33, Theorem 3.2], it follows that gradient blow-up can only occur at the boundary (see also [2], [4]). More precisely, the following estimate is given:

\[
|\nabla u| \leq C_1 \delta^{-\frac{1}{p-1}}(x, y) + C_2 \quad \text{in} \; \Omega \times [0, T),
\]

where \( C_1 = C_1(n, p) > 0 \) and \( C_2 = C_2(p, \Omega, \|u_0\|_{C^1}) > 0 \). Here, \( \delta(x, y) \) is the distance function to the boundary.
In this paper we are interested in the possibility of having isolated gradient blow-up points at the boundary. Up to now, the only available results of this kind, ensuring single-point GBU for suitable initial data, are those from [22], and they are restricted to very particular domains, namely disks and locally flat domains with some symmetry assumptions (see also [5] for a related problem with nonlinear diffusion in locally flat domains).

As it turns out, a key feature in the proofs in [22], [5] is the fact that the curvature of the boundary is constant near the GBU point. In this paper we are able to show that this can be considerably relaxed and we cover large classes of domains.

In order to give a good illustration of our main results without entering into too much technicality, let us right away formulate a single point gradient blow-up result for two typical classes of domains. More general results will be given in Section 2. We first treat the case of ellipses.

**Theorem 1.1.** Let \( p > 2 \) and \( \Omega \subset \mathbb{R}^2 \) be an ellipse. Then, there exist initial data \( u_0 \in X_+ \) such that \( T(u_0) < \infty \) and \( \text{GBUS}(u_0) \) contains only a boundary point of minimal curvature.

For our second class of domains, the main feature is that the GBU point has its center of curvature lying outside \( \Omega \) and is a local minimum of the curvature, along with suitable geometric conditions. Namely, we assume:

1. \( \Omega \) is symmetric with respect to the line \( x = 0 \) and convex in the \( x \)-direction,
2. \( \partial \Omega \) is tangent to the line \( y = 0 \) at the origin and \( \Omega \subset \{ y > 0 \} \),
3. The radius of curvature \( R(x) \) of \( \partial \Omega \) is a nonincreasing function for \( x > 0 \) small and \( \Omega \subset \{ y < R(0) \} \),

   For all \( X_0 \in \partial \Omega \cap \{ x > 0 \} \) close to the origin, the symmetric of \( \Omega_{X_0} \) with respect to \( \Lambda_{X_0} \), where \( \Lambda_{X_0} \) is the normal line to \( \partial \Omega \) at \( X_0 \), and \( \Omega_{X_0} \) is the part of \( \Omega \) to the right of \( \Lambda_{X_0} \).

See Figure 1 for an example of a domain satisfying these hypotheses. We point out that the function \( R(x) \) in (1.5) is valued in \( (0, \infty) \).

**Theorem 1.2.** Let \( p > 2 \) and suppose \( \Omega \subset \mathbb{R}^2 \) is a domain satisfying (1.3) – (1.6). Then, there exist initial data \( u_0 \in X_+ \) such that \( T(u_0) < \infty \) and \( \text{GBUS}(u_0) \) contains only the origin.

**Remark 1.3.**

(i) Observe that in the case of the locally flat domains studied in [22], condition (1.6) is a consequence of (1.3). In this case, for any \( X_0 \in \partial \Omega \cap \{ x > 0 \} \) near the origin, \( \Lambda_{X_0} \) will be parallel to the line \( x = 0 \). Also hypothesis (1.5) is trivially satisfied by locally flat domains.

(ii) Although it is possible to construct initial data for which the GBU set is arbitrarily concentrated close to any given boundary point (see Proposition 4.2), it is presently a (probably difficult) open question whether single point GBU may occur on points other than local minima of the curvature.

In the next section we give single point GBU results more general than Theorems 1.1 and 1.2 at the expense of more technical statements (see Theorems 2.3 and 2.5). The technical complexity of the statements comes from the fact that, in order to describe the hypotheses involved, we need to introduce a coordinate system adapted to the boundary.
near the gradient blow-up point (and actually this coordinate system is crucially used in the proof of our results).

2. General results

We introduce a class of symmetric domains with respect to the line $x = 0$, containing those described in the previous theorems, and for which we can construct single-point GBU solutions. A first step of our strategy is to prove that the solution $u$ is monotone in the parallel direction to the boundary in a neighborhood of the GBU point. It is therefore natural to introduce a curvilinear coordinate system adapted to the domain, allowing us to study the sign of the derivative of the solution in the parallel direction to the boundary. This coordinate system is sometimes called “boundary-fitted” coordinate system or “flow coordinates”. We point out that the use of these coordinates brings some technical difficulties, and that long computations and quite delicate arguments are required in order to control the terms related to the non-constant curvature (under appropriate assumptions on the domain). However, our attempts to prove such results, on single-point GBU in domains with nonconstant curvature, by merely using cartesian coordinates or local charts have turned out to be unsuccessful.

Next, we set the notation used throughout the rest of the paper and introduce the curvilinear coordinate system mentioned above. See Figure 2 for an illustration of this notation.

**Notation 2.1.**

- **$\Omega$ is a smoothly bounded domain of $\mathbb{R}^2$ and $\nu = (\nu_x, \nu_y)$ denotes the unit normal outward vector to $\partial \Omega$.**
- **$\Gamma \subset \partial \Omega$ is a connected boundary piece, with $(0,0) \in \Gamma$, and we assume that**

  \begin{equation}
  \Omega \text{ and } \Gamma \text{ are symmetric with respect to the line } x = 0.
  \end{equation}

- **For given $s_0 > 0$, the map**

  \[ \gamma(s) = (\alpha(s), \beta(s)), \quad s \in [-s_0, s_0], \]

  is an arclength parametrization of $\Gamma$ (i.e. $\alpha'(s)^2 + \beta'(s)^2 = 1$), with $\gamma(0) = (0,0)$.

- **We denote**

  \[ T(s) = (\alpha'(s), \beta'(s)), \quad N(s) = T^\perp(s) = (-\beta'(s), \alpha'(s)), \quad \text{for all } s \in [-s_0, s_0]. \]
We see that $T(s)$ is a unit tangent vector to $\partial \Omega$ at the point $\gamma(s)$ and, without loss of generality (replacing $s$ by $-s$ if necessary), we can assume that

$$N(s) \text{ is the inward normal vector to } \partial \Omega \text{ at the point } \gamma(s)$$

and that

$$\gamma(0) = (0, 0), \quad T(0) = (1, 0), \quad N(0) = (0, 1).$$

- We denote the curvature of the boundary by
  $$K(s) := \det(\gamma', \gamma'') = \alpha'' - \beta'' \alpha', \quad \text{for all } s \in [-s_0, s_0].$$
  By the regularity of $\partial \Omega$, this function is bounded and smooth.

- We introduce the map $M := \gamma + rN$, i.e.
  $$M : [0, \infty) \times [-s_0, s_0] \rightarrow \mathbb{R}^2, \quad (r, s) \mapsto M(r, s) = \gamma(s) + rN(s).$$

For a given domain $\Omega$ and a boundary piece $\Gamma$ as in Notation 2.1, our goal will be to prove the existence of initial data for which the GBU set is reduced to the origin. Using the coordinates given by the map $M$, we will use auxiliary functions to estimate the derivative of $u$ with respect to $s$. Then, an integration over the coordinate curves parallel to the boundary will give an upper estimate on $u$ which is sufficient to apply a nondegeneracy result (see Lemma 4.1 below) for each $s > 0$, proving that gradient blow-up can only take place at the origin.

In order to apply our methods, we need to make some extra geometric assumptions on the domain. Namely, we need to assume that $\Omega$ is locally convex near the origin and that the origin is a local minimum for the curvature of the boundary, i.e.

$$K(0) \geq 0 \quad \text{and} \quad K'(s) \geq 0 \quad \text{for all } s \in [0, s_0],$$

along with

$$\alpha'(s), \beta'(s) > 0, \quad \text{for all } s \in (0, s_0).$$

We note that (2.4) implies $K(s) \geq 0$ for $s \in (0, s_0]$. We point out that condition (2.5) excludes domains which are flat near the origin, but this case is comparatively easier and was treated in [22]. Hypotheses (2.4) and (2.5) are necessary for two reasons. On the one hand, they are needed to define a region where the parameterization $M$ is well defined. On the other hand, when deriving the parabolic inequalities satisfied by the auxiliary functions, they are needed to control some terms coming from the non-constant curvature.

Under the above assumptions, let us denote

$$R(s) = 1/K(s) \in (0, \infty], \quad s \in [0, s_0],$$

the radius of curvature of $\partial \Omega$ at $\gamma(s)$, and define the natural regions

$$Q_\Gamma = \{(r, s) \in \mathbb{R}^2; \ 0 \leq r < R(s), \ 0 \leq s \leq s_0\} \quad \text{and} \quad D_\Gamma = M(Q_\Gamma).$$

We observe that $D_\Gamma$ is the region bordered by the four curves: $\Gamma$, the $y$-axis, the normal line at $\gamma(s_0)$ and, from above, the evolute of $\Gamma$, i.e. the locus of the curvature centers

$$C(s) = \gamma(s) + R(s)N(s).$$

The following proposition shows that the region $D_\Gamma$ is well parametrized by $M$ and, consequently, that one can define there the derivative $u_s$, in the parallel direction to the
boundary. Although this fact is more or less standard, we give a proof in Section 3 for convenience.

**Proposition 2.2.** Let \( \Omega, \Gamma, \gamma, M \) be as in Notation 2.1 and assume (2.4), (2.5).

(i) Then, the map \( M \) is a diffeomorphism from \( Q_\Gamma \) to \( D_\Gamma \).

(ii) As a consequence, for any solution \( u \) of (1.1), the derivative

\[
\frac{\partial}{\partial s} [u(M(r,s),t)]
\]

is well defined in \( (\Omega \cap D_\Gamma) \times [0,T(u_0)) \).

The following result ensures that single-point GBU occurs for symmetric solutions satisfying a monotonicity condition near the origin.

**Theorem 2.3.** Let \( p > 2 \), let \( \Omega, \Gamma, \gamma, M \) be as in Notation 2.1 and assume (2.4), (2.5). Let \( u_0 \in X_+ \) be a symmetric function with respect to the line \( x = 0 \), such that \( T = T(u_0) < \infty \). Suppose that

\[
(2.9) \quad GBUS(u_0) \subset \gamma\left(-\frac{\alpha}{2}, \frac{\alpha}{2}\right)
\]

and that, for some \( t_0 \in (0,T), r_0 \in (0,R(s_0)) \), we have

\[
(2.10) \quad u_x, u_s < 0 \quad \text{in} \quad \omega_0 \times (t_0,T), \quad \text{with} \quad \omega_0 := \Omega \cap M((0,r_0) \times (0,s_0)).
\]

Then, \( GBUS(u_0) \) contains only the origin.

Hypothesis (2.9) is not difficult to guarantee. It is in fact satisfied whenever \( u_0 \) is sufficiently concentrated near the origin (cf. [22] and Proposition 3.2 below). On the contrary, the hypothesis \( u_x < 0 \) in (2.10) is in general more difficult to verify, and requires assumptions of more global nature.

The assumption \( u_x < 0 \) in (2.10) is required by the fact that the Laplace operator does not commute with the derivative in the \( s \)-direction. Therefore, we need to control a
term involving \( u_r \). This can be done by writing \( u_r \) as a linear combination of \( u_x \) and \( u_s \), see formula (3.10). The term \( u_x \) is obviously more tractable since the \( x \)-derivative does commute with the Laplace operator. This requires the use of two auxiliary functions \( J \) and \( \bar{J} \) in the proof of this Theorem (section 5), the first to control \( u_s \) and the second to control \( u_x \). The derivation and analysis of the parabolic equations satisfied by \( J \) and \( \bar{J} \) necessitate long and technical calculations involving boundary-fitted coordinates.

We next introduce the geometric hypotheses on the domain \( \Omega \) under which we are able to construct initial data satisfying condition (2.10). To this end we set the following further notation, which is motivated by moving plane arguments that we rely on.

**Notation 2.4.** For each \( s \in [0, s_0] \), we denote

- \( \Lambda_s \) the line \( \gamma(s) + \mathbb{R}N(s) \)
- \( T_s(\cdot) \) the symmetry with respect to \( \Lambda_s \)
- \( H_s \) the half-plane at the right of the line \( \Lambda_s \), i.e.:
  \[ H_s = \{ P \in \mathbb{R}^2; T(s) \cdot (P - \gamma(s)) > 0 \} \]
- \( \Omega_s = \Omega \cap H_s \).

Using Notations 2.1 and 2.4 the hypotheses that we shall assume are the following:

1. \( \omega_0 \subset D_\Gamma \), where \( \omega_0 := \Omega \cap D_\Gamma \cap \{ y < y_0 \} \), for some \( y_0 \in (0, \infty) \),
2. \( \nu_x \geq 0 \) on \( \partial \Omega \cap \{ x > 0 \} \),
3. \( \nu_y \geq 0 \) on \( \partial \Omega \cap \partial \omega_0 \cap \{ r > 0 \} \),
4. \( T_{s_0}(\Omega_{s_0}) \subset \Omega \),
5. \( T_{s}(\Omega^{+}) \subset \Omega \), where \( \Omega^{+} := \Omega \cap \{ y > y_0 \} \),
6. and \( T_s(\cdot) \) is the symmetry with respect to the line \( y = y_0 \).

See Figure 1 in section 1 and Figures 3 and 4 in section 6 for examples of domains satisfying these hypotheses. In view of Proposition 2.2 assumption (2.11) ensures that \( u_s \) is well defined in \( \omega_0 \). Our result reads as follows.

**Theorem 2.5.** Let \( p > 2 \) and let \( \Omega, \gamma, s_0, T_s, \Omega_s \) be as in Notations 2.1 and 2.4. Let \( D_\Gamma \) be defined by (2.7) and assume (2.4), (2.5), (2.11) – (2.15).

(i) There exist initial data \( u_0 \in X_+ \) such that \( T(u_0) < \infty \) and

1. \( u_0 \) is symmetric with respect to the line \( x = 0 \),
2. \( u_{0,x} \leq 0 \) in \( \Omega \cap \{ x > 0 \} \) and \( u_{0,s} \leq 0 \) in \( \omega_0 \),
3. \( u_0(P) \leq u_0(T_{s_0}(P)) \) for all \( P \in \Omega_{s_0} \),
4. \( u_0(P) \leq u_0(T_s(P)) \) for all \( P \in \Omega \cap \{ y > y_0 \} \).
5. \( GBUS(u_0) \subset \gamma(-\frac{s_0}{2}, \frac{s_0}{2}) \).

(ii) For any such \( u_0 \), \( GBUS(u_0) \) contains only the origin.
Remark 2.6. 

(i) If the domain $\Omega$ is sufficiently thin in the $y$-direction, then the center of curvature of the boundary lies outside $\Omega$ for all $s \in [0, s_0]$. In that case we can consider $y_0 = +\infty$ in (2.11) and conditions (2.13) and (2.14) disappear. When this is not the case, we can restrict $\omega$ to $\{y < y_0\}$, for some $y_0 > 0$, in order to be able to define the boundary-fitted coordinates. However, we then have to pay the price of assuming the reflection assumption (2.15), which allows us to prove $u_y \leq 0$ on $\Omega \cap \{y = y_0\}$ by a moving planes argument.

(ii) Hypothesis (2.12) implies that the domain is convex in the $x$ direction, and this, together with (2.1), allows one to construct solutions such that $u_x \leq 0$ in $\Omega \cap \{x > 0\}$.

(iii) On the other hand, hypotheses (2.13) and (2.14) are useful to construct solutions such that $u_s < 0$ in $\omega$. In particular, hypothesis (2.13) implies that on the upper piece of $\partial \omega$ which coincides with $\partial \Omega$, $u_s$ represents the derivative in a direction pointing outside $\Omega$, and therefore $u_s \leq 0$. Then, we prove that $u_s \leq 0$ on $\Lambda \cap \partial \omega$ by a moving planes argument, which can be applied only under hypothesis (2.14).

(iv) On $\partial \omega \cap \{y = y_0\}$, we prove $u_s \leq 0$ by expressing it as a linear combination of $u_x$ and $u_y$, that we can prove to be negative, see (i) and (ii) in this remark.

Observe that in figure 1 the domain is sufficiently thin so that we can consider $y_0 = +\infty$. Ellipses with non-zero eccentricity, i.e. ellipses which are not disks, are also examples of domains where it is possible to apply this result. In that case, we choose $y_0$ such that the line $\{y = y_0\}$ coincides with the major axis of the ellipse. The case of a disk is excluded since, in order to satisfy hypothesis (2.15), we must consider an $y_0$ bigger or equal than the radius of curvature of the disk, but then, hypothesis (2.11) cannot hold. However, the case of the disk can be treated using polar coordinates (see [22]).

Remark 2.7. Let $p > 2$ and $\Omega$ be as in Theorem 2.5. Denote $B^+_\rho := B_\rho(0,0) \cap \{x > 0\}$ and let $\rho > 0$ be such that

$$\Omega \cap B^+_\rho \subset \omega, \quad \partial \Omega \cap B^+_\rho \subset \gamma(0, s_0/2).$$

It follows from Theorem 2.5 and Proposition 4.3 below that $T(u_0) < \infty$ and $\text{GBUS}(u_0) = \{(0,0)\}$ whenever $u_0 \in \chi_+$, for instance satisfies (2.16), (2.17) and

$$\begin{align*}
\text{supp}(u_0) &\subset \bar{\Omega} \cap \bar{B}_{\rho/2}, \\
\|u_0\|_\infty &\leq C_2, \\
\inf_{B_{\tilde{\rho}}} u_0 &\geq C_1 \varepsilon^k \quad \text{with} \quad \tilde{\rho} = B_{\varepsilon/2}(0, \varepsilon), \quad \text{for some} \ \varepsilon \in (0, \rho/2),
\end{align*}$$

where $C_1(\rho) > 0$ and $C_2(p, \Omega, \rho) > 0$. Moreover, initial data satisfying these assumptions can be easily constructed. See the proof of Theorem 2.5(i) for details.

The outline of the rest of the paper is as follows. In section 3 we give some basic computations and notation on the “boundary-fitted” curvilinear coordinate system and we give the proof of Proposition 2.2. In section 4 we give some useful preliminary results, concerning nondegeneracy and localization of GBU as well as a Serrin type corner lemma. Theorems 2.3 and 2.5 are respectively proved in sections 5 and 7. Finally in section 7 we deduce Theorems 1.1 and 1.2 from Theorem 2.5.
3. Preliminary results I: basic computations in boundary-fitted curvilinear coordinates

In this section we give some basic computations in the coordinate system given by the map $M$ in (2.3). Here $\Omega$ and $\Gamma$ are as in Notation 2.1 and we assume conditions (2.4) and (2.5). By Proposition 2.2, that we will prove at the end of this section, $M$ is a diffeomorphism from $Q_\Gamma$ to $D_\Gamma$, where $Q_\Gamma$ and $D_\Gamma$ are defined in (2.7). To facilitate the change of coordinates throughout the paper, we adopt the following notation and conventions.

**Notation 3.1.** For any function $\psi(x,y)$ defined on (a part of) $D_\Gamma$, we express $\psi$ in terms of the variables $(r,s)$ by setting

$$\tilde{\psi} := \psi \circ M,$$

i.e. $\tilde{\psi}(r,s) = \psi(M(r,s))$ for $(r,s) \in Q_\Gamma$. The derivatives with respect to the variables $(r,s)$ of a function $\psi = \psi(x,y) \in C^1(D_\Gamma)$ are then defined by

$$\psi_r := \tilde{\psi}_r, \quad \psi_s := \tilde{\psi}_s.$$  

(3.1)

Similarly, for any function $\varphi(r,s)$ defined on (a part of) $Q_\Gamma$, we denote

$$\hat{\varphi} = \varphi \circ M^{-1}.$$  

In the rest of the paper, for any functions $\psi = \psi(x,y)$ and $\varphi = \varphi(r,s)$, when no risk of confusion arises, we will drop the tilde and the hat and will just write $\psi(r,s)$ in place of $\tilde{\psi}(r,s)$ and $\varphi(x,y)$ in place of $\hat{\varphi}(x,y)$.

Also, the gradient and the Laplacian operators will always be understood as

$$\nabla \psi = (\psi_x, \psi_y)$$

and

$$\Delta \psi = \text{div}(\nabla \psi) = \psi_{xx} + \psi_{yy},$$

either as functions of $(x,y)$, or as functions of $(r,s)$ (i.e., implicitly considering $(\nabla \psi) \circ M$ and $(\Delta \psi) \circ M$).

According to the chain rule, we have

$$\psi_r = \nabla \psi(M(r,s)) \cdot N(s) \quad \text{and} \quad \psi_s = \nabla \psi(M(r,s)) \cdot (\gamma'(s) + rN'(s)).$$

Using (3.7) and (3.8), we obtain

$$N'(s) = -K(s)(\alpha'(s), \beta'(s)) = -K(s)T(s),$$

and then, we can rewrite (3.2) as

$$\psi_r = \nabla \psi \cdot N(s) \quad \text{and} \quad \psi_s = (1 - rK(s))\nabla \psi \cdot T(s).$$

(3.3)

Note that

$$1 - rK(s) > 0 \quad \text{in } D_\Gamma,$$

owing to (2.6), (2.7). Since the vectors $N(s)$ and $T(s)$ are orthonormal, we then have

$$\nabla \psi(r,s) \equiv (\nabla \psi) \circ M = \psi_r N(s) + \frac{\psi_s}{1 - rK(s)} T(s),$$

(3.5)

as well as

$$\nabla \psi \cdot \nabla \varphi = \psi_r \varphi_r + \frac{\psi_s \varphi_s}{(1 - rK(s))^2}.$$  

(3.6)
We next recall two alternative expressions for the function curvature of the boundary $K(s)$. Since $\gamma(s) = (\alpha(s), \beta(s))$ is an arclength parametrization, we have
\[
\alpha'(s)\alpha''(s) + \beta'(s)\beta''(s) = \frac{(\alpha'(s)^2 + \beta'(s)^2)'}{2} = 0,
\]
and then we have $\alpha'(s)\alpha''(s) = -\beta'(s)\beta''(s)$. Using this identity, we can obtain
\[
(3.7) \quad K(s) = \alpha'(s)\beta''(s) - \beta'(s)\alpha''(s) = \alpha'(s)\beta''(s) + \frac{\beta'(s)\beta''(s)}{\alpha'(s)} = \frac{\beta''(s)}{\alpha'(s)},
\]
and in a similar way, recalling (2.5), we obtain
\[
(3.8) \quad K(s) = -\frac{\alpha''(s)}{\beta'(s)}, \quad s \neq 0.
\]
Now, we give some further identities relating the derivatives in boundary-fitted coordinates with the derivatives in cartesian coordinates. As we will see in our proofs, we have particular interest in expressing, when possible, $\psi_r$ as a linear combination of $\psi_x$ and $\psi_s$. In the following computations, and without risk of confusion, we omit the dependence on $s$ of the functions $K, \alpha', \beta'$. In view of (3.3), we have
\[
\psi_r = -\beta'\psi_x + \alpha'\psi_y,
\]
(3.9)
\[
\frac{\psi_s}{1 - rK} = \alpha'\psi_x + \beta'\psi_y.
\]
Then, recalling (2.5), we obtain the identity
\[
(3.10) \quad \psi_r = -\frac{1}{\beta'}\psi_x + \frac{\alpha'}{\beta'}\frac{\psi_s}{1 - rK}, \quad s \neq 0.
\]
We note that it is possible to write $\psi_r$ as a linear combination of $\psi_x$ and $\psi_s$ only when $\beta'(s) \neq 0$ (i.e., $s \neq 0$). This makes sense since, if $\beta'(s) = 0$, then $\psi_x = \psi_s$ and $\psi_r$ is the derivative in the $y$ direction, which is then orthogonal to the $x$ and $s$ directions.

The next result is a very useful expression of the Laplacian in flow coordinates.

**Proposition 3.2.** (i) Let $\psi = \psi(x,y) \in C^2(D_T)$. We have
\[
(3.11) \quad \Delta \psi \equiv (\Delta \psi) \circ M = \psi_{rr} - \frac{K}{1 - rK}\psi_r + \frac{1}{(1 - rK)^2}\psi_{ss} + \frac{rK'}{(1 - rK)^3}\psi_s, \quad \text{for } (r,s) \in Q_T.
\]
(ii) If $\varphi = \varphi(r,s) \in C^2(Q_T)$, then $\Delta \varphi \equiv [\Delta(\varphi \circ M^{-1})] \circ M$ is also given by (3.11) with $\psi$ replaced by $\varphi$.

**Proof.** (i) For $\varphi = \varphi(r,s)$ recall the notation $\hat{\varphi} = \varphi(x,y) := \varphi \circ M^{-1}$. For any $\psi = \psi(x,y) \in C^2(D_T)$, using (3.3), we obtain
\[
\nabla \psi = \hat{\psi}_r \hat{N} + \frac{\hat{\psi}_s}{1 - rK} \hat{T} \quad \text{in } D_T.
\]
It follows that
\[
(3.12) \quad \Delta \psi = \text{div}(\nabla \psi) = \nabla(\hat{\psi}_r) \cdot \hat{N} + \hat{\psi}_r \text{ div } \hat{N} + \frac{1}{1 - rK} \nabla(\hat{\psi}_s) \cdot \hat{T}
\]
\[
+ \hat{\psi}_s \nabla \left( \frac{1}{1 - rK} \right) \cdot \hat{T} + \frac{1}{1 - rK} \hat{\psi}_s \text{ div } \hat{T}.
\]
By (3.5), we have

\[ [(\nabla \varphi) \circ M] \cdot N = \varphi_r \quad \text{and} \quad (1 - rK) [(\nabla \varphi) \circ M] \cdot T = \varphi_s, \]

hence

\[ \nabla \varphi \cdot \hat{N} = \hat{\varphi}_r \equiv (\varphi \circ M)_r \circ M^{-1} \quad \text{and} \quad (1 - \hat{r}K)(\nabla \varphi) \cdot \hat{T} = \hat{\varphi}_s \equiv (\varphi \circ M)_s \circ M^{-1}. \]

Using this with \( \varphi = \hat{\psi}_r \), we can thus identify

\[ (3.13) \quad \nabla(\hat{\psi}_r) \cdot \hat{N} \equiv \nabla((\varphi \circ M)_r \circ M^{-1}) \cdot \hat{N} = (\varphi \circ M)_{rr} \circ M^{-1} \equiv \psi_{rr} \circ M^{-1}, \]

\[ (3.14) \quad (1 - \hat{r}K)\nabla(\hat{\psi}_s) \cdot \hat{T} \equiv (1 - \hat{r}K)\nabla((\varphi \circ M)_s \circ M^{-1}) \cdot \hat{T} = (\varphi \circ M)_{ss} \circ M^{-1} \equiv \psi_{ss} \circ M^{-1} \]

and

\[ (3.15) \quad \nabla \left( \frac{1}{1 - \hat{r}K} \right) \cdot \hat{T} = \frac{1}{1 - \hat{r}K} \frac{1}{1 - rK} \circ M^{-1} = \frac{rK'}{(1 - rK)^3} \circ M^{-1}. \]

On the other hand, since \( N(s) = -\beta'(s)(1,0) + \alpha'(s)(0,1) \), we have

\[ \text{div}(\hat{N}) = -\nabla \hat{\beta}' \cdot (1,0) + \nabla \alpha' \cdot (0,1). \]

Applying (3.3) with \( \psi = \hat{\beta}' \) and \( \psi = \hat{\alpha}' \), we obtain

\[ [\text{div}(\hat{N})] \circ M = -\frac{\beta''}{1 - rK} T(s) \cdot (1,0) + \frac{\alpha''}{1 - rK} T(s) \cdot (0,1) \]

\[ = -\frac{\beta'' \alpha'}{1 - rK} + \frac{\alpha'' \beta'}{1 - rK} = -\frac{K}{1 - rK}. \]

Similarly, since \( T(s) = \alpha'(s)(1,0) + \beta'(s)(0,1) \), hence

\[ \text{div}(\hat{T}) = \nabla \hat{\alpha}' \cdot (1,0) + \nabla \hat{\beta}' \cdot (0,1), \]

we have

\[ [\text{div}(\hat{T})] \circ M = \frac{\alpha''}{1 - rK} T(s) \cdot (1,0) + \frac{\beta''}{1 - rK} T(s) \cdot (0,1) \]

\[ = \frac{\alpha'' \alpha'}{1 - rK} + \frac{\beta'' \beta'}{1 - rK} = \frac{(\alpha'^2 + \beta'^2)}{2(1 - rK)} = 0. \]

Finally, plugging (3.13)–(3.17) in (3.12), we obtain (3.11).

(ii) It suffices to apply assertion (i) to \( \psi := \varphi \circ M^{-1} \), using (3.1) and the fact that \( \hat{\psi} \equiv \psi \circ M = \varphi \). \qed

We end this section with the proof of Proposition 2.2.

Proof of Proposition 2.2. It suffices to show assertion (i). We first establish the injectivity of \( M \) on \( Q_\Gamma \). Let \( C(s) = \gamma(s) + R(s)N(s) \) be the center of curvature. We note that \( D_\Gamma \) can be written as the union of half-open segments:

\[ D_\Gamma = \bigcup_{s \in [0,s_0]} \Sigma(s), \quad \text{where} \quad \Sigma(s) = [\gamma(s), C(s)]. \]

To show the injectivity, it suffices to verify that for any \( 0 \leq s_1 < s_2 \leq s_0 \), the segments \( \Sigma(s_1) \) and \( \Sigma(s_2) \) do not intersect. This amounts to showing that \( \Sigma(s_2) \) lies entirely in the
open half-plane to the right of the line $\Lambda_1$, defined as in Notation \ref{not:Lambda1}, which is the line containing the segment $\Sigma(s_1)$. This half-plane is defined by the inequality
\[ T(s_1) \cdot (x - \gamma(s_1)) > 0, \quad \text{with} \quad x \in \mathbb{R}^2. \]
Considering the extremes of the segment $\Sigma(s_2)$, this is thus equivalent to
\[ T(s_1) \cdot (\gamma(s_2) - \gamma(s_1)) > 0 \quad \text{and} \quad T(s_1) \cdot (C(s_2) - \gamma(s_1)) \geq 0. \]
To show (3.18), using $\gamma'(s) = T(s)$ and (2.5), we first compute
\[ \frac{d}{ds} \left( T(s_1) \cdot (\gamma(s) - \gamma(s_1)) \right) = T(s_1) \cdot T(s) > 0, \quad s_1 < s \leq s_0, \]
hence the first inequality in (3.18) follows. On the other hand, using $N'(s) = -K(s)T(s)$, (2.4) and (2.5), we get
\[ \frac{d}{ds} \left( T(s_1) \cdot N(s) \right) = -K(s)T(s_1) \cdot T(s) \leq 0, \]
Since $T(s_1) \cdot N(s_1) = 0$, we deduce that
\[ T(s_1) \cdot N(s) \leq 0, \quad s_1 < s \leq s_0. \]
Also, using $\gamma'(s) = T(s)$ and $N'(s) = -K(s)T(s)$, we have
\[ C'(s) = (1 - K(s)R(s))T(s) + R'(s)N(s) = R'(s)N(s). \]
Since $R'(s) \leq 0$ due to (2.4), it follows from (3.19) that
\[ \frac{d}{ds} \left( T(s_1) \cdot (C(s) - C(s_1)) \right) = R'(s)T(s_1) \cdot N(s) \geq 0, \quad s_1 < s \leq s_0, \]
hence, it follows from $\gamma(s_1) = C(s_1) - R(s_1)N(s_1)$ that
\[ T(s_1) \cdot (C(s_2) - \gamma(s_1)) = T(s_1) \cdot (C(s_2) - C(s_1)) \geq 0, \]
which guarantees the second inequality in (3.18). This completes the proof of the injectivity.

To prove that $M$ is a diffeomorphism from $Q_\Gamma$ to $D_\Gamma = M(Q_\Gamma)$, it thus suffices to show that the Jacobian of $M$ does not vanish in $Q_\Gamma$. For all $(r, s) \in Q_\Gamma$, using $\gamma' = T$ and $N' = -KT$ again, we compute
\[ \text{Jac}_M(r, s) = \text{det} \left( \frac{\partial M}{\partial r}, \frac{\partial M}{\partial s} \right) = \text{det} \left( N, (1 - Kr)T \right) = K(s)r - 1 < 0, \]
since $r < R(s) = 1/K(s)$, and the conclusion follows. \hfill \Box

4. Preliminary results II: Nondegeneracy and localization of GBU and corner lemma

In this section we give three preliminary results that we use in the proofs of Theorems \ref{thm:main} and \ref{thm:cornerlemma}. We start with the following nondegeneracy lemma, proved in \ref{lem:nondegeneracy}, which implies that, at any gradient blow-up point, the estimate \ref{eq:corner} is essentially optimal in the normal direction to the boundary.

**Lemma 4.1.** Let $\Omega \subseteq \mathbb{R}^2$ be a smoothly bounded domain and $x_0 \in \partial \Omega$. There exists $c_0 = c_0(p)$ such that, if
\[ u \leq c_0 \delta^{(p-2)/(p-1)}(x, y) \quad \text{in} \quad (B_\rho(x_0) \cap \Omega) \times [0, T), \]
for some $\rho > 0$, then $x_0$ is not a gradient blow-up point.
Consider the following problem:

\[ C \text{ such that } \frac{\partial C}{\partial \nu} > 0 \text{ on } \partial \Omega \]

By \cite[Thm 4.2]{29}, there exists a solution \( u_0 \) such that the solution blows up, with GBU set concentrated near an arbitrary given point. The idea of proof is based on that of \cite[Theorem 1.1]{22}, where a more particular example of initial data was given.

**Proposition 4.2.** Let \( p > 2, \Omega \subset \mathbb{R}^2 \) be a smoothly bounded domain and let \( x_0 \in \partial \Omega \) and \( \rho > 0 \). There exist constants \( C_1(p) > 0 \) and \( C_2(\rho, \Omega, p) > 0 \) with the following property:

If for some \( \varepsilon > 0 \) such that \( \tilde{B}_\varepsilon := B(x_0 + \varepsilon \nu(x_0), \varepsilon) \subset \Omega \), \( u_0 \in X_+ \) satisfies

\[
\begin{align*}
&\supp(u_0) \subset \overline{\Omega} \cap \overline{B}(x_0, \rho/2), \\
&\|u_0\|_\infty \leq C_2, \\
&\inf_{\tilde{B}_{\varepsilon/2}} u_0 \geq C_1 \varepsilon^k, \\
&\text{where } k = (p - 2)/(p - 1),
\end{align*}
\]

then \( T(u_0) < \infty \) and \( \text{GBUS}(u_0) \subset B_\rho(x_0) \cap \partial \Omega \).

**Proof.** We divide the proof into two steps.

**Step 1:** \( \nabla u \) blows up in finite time. The idea here is to use the auxiliary function introduced in \cite{22} as subsolution. Let \( \varphi \in C^\infty([0, \infty)) \) be a function satisfying

\[ \varphi' \leq 0, \quad \varphi(r) = 1, \text{ for } r \leq 1/4, \quad \varphi(r) = 0, \text{ for } r \geq 1/2. \]

Consider the following problem:

\[
\begin{align*}
&v_t - \Delta v = |\nabla v|^p, \quad x \in B_1(0), \ t > 0, \\
&v(x, t) = 0, \quad x \in \partial B_1(0), \ t > 0, \\
&v(x, 0) = \phi(x) := C_1 \varphi(|x|), \quad x \in B_1(0).
\end{align*}
\]

By \cite[Thm 4.2]{29} (see also \cite[Prop. 7.1]{33}), there exists \( C_0 = C_0(p) \) such that, if \( \|\phi\|_1 \geq C_0 \), then \( T(\phi) < \infty \). Therefore, we have \( T(\phi) < \infty \) whenever \( C_1 \) is bigger than some constant depending on \( p \). We now use the scale invariance of the equation. Namely we consider the rescaled function

\[ v_\varepsilon(x, t) = \varepsilon^k \varphi\left( \varepsilon^{-1}|x - \tilde{x}_0|, \varepsilon^{-2}t \right), \]

where \( \tilde{x}_0 = x_0 + \varepsilon \nu(x_0) \). Then \( v_\varepsilon \) solves \( 4.4 \) in \( \tilde{B}_\varepsilon \subset \Omega \).

Since we have

\[ v_\varepsilon(x, 0) = \varepsilon^k C_1 \varphi(\varepsilon^{-1}|x - \tilde{x}_0|) \leq \varepsilon^k C_1 \text{ in } \tilde{B}_{\varepsilon/2}, \]

and \( v_\varepsilon(x, 0) = 0 \) in \( \tilde{B}_\varepsilon \setminus \tilde{B}_{\varepsilon/2} \), we can use \( 4.3 \), together with the comparison principle to get

\[ u \geq v_\varepsilon \text{ in } \tilde{B}_\varepsilon \times (0, \tilde{T}), \quad \text{where } \tilde{T} = \min(T(u_0), T_\varepsilon) \text{ and } T_\varepsilon = \varepsilon^2 T(\phi). \]
Now we observe that $\tilde{B}_\varepsilon$ is tangent to $\partial \Omega$ at $x_0$, so we deduce
\[
-\frac{\partial u}{\partial \nu}(x_0, t) \geq -\frac{\partial v}{\partial \nu}(x_0, t), \quad 0 < t < \tilde{T}.
\]
On the other hand, as a consequence of the maximum principle applied to $\nabla v$ (see e.g. Prop. 40.3), we know that
\[
\max_{t \in [0, \tau]} \|\nabla v(\cdot, t)\|_\infty = \max \left( \|\nabla v(\cdot, 0)\|_\infty, \max_{\partial B(0) \times [0, \tau]} \left( \frac{\partial v}{\partial \nu} \right) \right), \quad 0 < \tau < T(\phi).
\]
Since $v$ is radially symmetric, it follow that
\[
\limsup_{t \to T_\varepsilon} \frac{\partial v}{\partial \nu}(x_0, t) = \infty,
\]
hence $T(u_0) \leq T_\varepsilon < \infty$.

**Step 2:** No GBU on $\partial \Omega \setminus B_\rho(x_0)$. For $\rho > 0$, consider a cut-off function $h \in C^\infty((0, \infty))$ satisfying
\[
h' \leq 0, \quad h(r) = 1, \quad \text{for } r \leq \rho/2, \quad h(r) = 0, \quad \text{for } r \geq 3\rho/4.
\]
Now, let $h_{x_0}$ be the function in $\Omega$ defined by
\[
h_{x_0}(x) := h(|x - x_0|).
\]
Let $\psi = \psi_{x_0}$ be the unique classical solution of the linear elliptic problem
\[
\begin{cases}
-\Delta \psi(x) = 1, & x \in \Omega, \\
\psi(x) = h_{x_0}(x), & x \in \partial \Omega.
\end{cases}
\]
(4.5)
We claim that there exists $c_1 > 0$, independent of $x_0$, satisfying
\[
\psi(x) \geq c_1, \quad \text{for all } x \in \Omega \cap B(x_0, \rho/2).
\]
We can prove this claim by using a contradiction and compactness argument. Suppose there exists a sequence $\{x_i\} \subset \partial \Omega$ such that
\[
\min_{\Omega \cap B(x_i, \rho/2)} \psi_{x_i}(x) \to 0, \quad \text{as } i \to \infty,
\]
(4.6)
where $\psi_{x_i}$ is the solution of (4.5) with boundary data $h_{x_i}$. Since $\partial \Omega$ is compact, we can suppose, by extracting a subsequence, that $x_i$ converges to some $x_\infty \in \partial \Omega$.

Now fix some $\alpha \in (0, 1)$ and observe that, by the construction of $h_{x_0}$ above, there exists $C > 0$, independent of $i$, such that $\|h_{x_i}\|_{C^{\alpha+\alpha}(\Omega)} \leq C$, and therefore $\|\psi_{x_i}\|_{C^{\alpha+\alpha}(\Omega)} \leq C'(C, \Omega)$ by interior-boundary elliptic Schauder estimates (see Theorem 47.2 (ii) in [29]). Hence, as $h_{x_i}$ converges to $h_{x_\infty}$ in $C^{2+\alpha}(\Omega)$, by compact embeddings and uniqueness for problem (4.5), we can deduce that $\psi_{x_i}$ converges to $\psi_{x_\infty}$ in $C^2(\Omega)$. It then follows from (4.6) that $\psi_{x_\infty}$ vanishes somewhere in $\Omega \cap B(x_\infty, \rho/2)$.

Since $h_{x_\infty}(x) = 1$ in $B(x_\infty, \rho/2)$, and then $\psi_{x_\infty}(x) = 1$ in $\partial \Omega \cap B(x_\infty, \rho/2)$, we deduce that $\psi_{x_\infty}$ vanishes somewhere in the interior of $\Omega$, contradicting the strong maximum principle. The claim is then proved.

On the other hand, applying elliptic estimates again, there exists $\tilde{C} = \tilde{C}(\rho, \Omega) > 0$ such that $\|\nabla \psi\|_\infty \leq \tilde{C}$. Choosing $c_2 = \tilde{C}^{-p/(p-1)}$, we then have $\|\nabla \psi\|^{-p/(p-1)} \geq c_2$, hence
\[
-\Delta(c_2 \psi) = c_2 \geq |\nabla (c_2 \psi)|^p, \quad \text{in } \Omega.
\]
And by (4.2) with $C_2 = c_1c_2$, we have
\[ c_2 \psi \geq c_2c_1 \geq u_0, \quad \text{in } B(x_0, \rho/2), \]
hence, using (4.1), we get $c_2 \psi \geq u_0$ in $\Omega$. By the comparison principle, it follows that $u \leq c_2 \psi$ in $\Omega \times (0, T(u_0))$. Therefore, since $\psi = 0$ on $\partial \Omega \setminus B_{3\rho/4}(x_0)$, we have
\[ 0 \leq -\frac{\partial u}{\partial \nu} \leq -c_2 \frac{\partial \psi}{\partial \nu} \leq C, \quad \text{on } (\partial \Omega \setminus B_{3\rho/4}(x_0)) \times (0, T(u_0)). \]
The conclusion then follows from Lemma 4.1.

We conclude this section with a parabolic version of “Serrin’s corner Lemma”, adapted to our parabolic problem and domain.

**Lemma 4.3.** Let $p > 2$ and $u_0 \in X_+$, let $\Omega, \Gamma, \gamma, M$ be as in Notation 2.1 and assume (2.4) and (2.5). Suppose that there exist $t_0 \in (0, T)$, $s_1 \in (0, s_0)$, $r_0 > 0$ and $c_1 > 0$ such that
\[
\omega_0 := M((0, r_0) \times (0, s_0)) \subset \Omega \cap D_\Gamma,
\]
(4.7) 
\[ u_x < 0 \quad \text{in } \omega_0 \times (t_0, T) \]
and
(4.8) 
\[ u_x \leq -c_1 r \quad \text{on } (0, r_0) \times \{s_1\} \times (t_0, T). \]
Then, for any fixed $r_1 \in (0, r_0)$ and $t_1 \in (t_0, T)$, there exists $\tilde{c}_1 > 0$ such that
\[ u_x(r, s, t_1) \leq -\tilde{c}_1 rs \quad \text{in } \omega_1, \]
where $\omega_1 := M((0, r_1) \times (0, s_1))$.

**Proof.** We fix a nontrivial smooth function $\phi \geq 0$ on $[0, r_0]$, with $\text{supp}(\phi) \subset \subset (0, r_0)$ and another smooth function $\psi$ on $[0, s_1]$ such that
\[ \psi = 0 \quad \text{on } [0, \frac{s_1}{2}], \quad \psi(s_1) = 1, \quad \psi', \psi'' \geq 0. \]
Fix a constant $M > 0$ such that
\[
M \geq \frac{K}{1 - rK} + p|\nabla u|^{p-1}, \quad M \geq \frac{rK'}{(1-rK)^3} + \frac{p|\nabla u|^{p-1}}{1-rK}, \quad \text{in } \omega_0 \times (t_0, t_1).
\]
Next, fix $t_2 \in (t_0, t_1)$ and let $v, V$ be the respective global solutions of
\[
v_t - v_{rr} = -M|v_r|, \quad r \in (0, r_0), \quad t > t_2,
\]
\[ v(0, t) = v(r_0, t) = 0, \quad t > t_2, \]
\[ v(r, t_2) = \phi(r), \quad r \in [0, r_0], \]
and
\[
V_t - V_{ss} = -MV_s, \quad s \in (0, s_1), \quad t > t_2,
\]
\[ V(0, t) = 0, \quad V(s_1, t) = 1, \quad t > t_2, \]
\[ V(s, t_2) = \psi(s), \quad s \in [0, s_1]. \]
By the maximum principle we have $v \geq 0$, $0 \leq V \leq 1$, and $V_s \geq 0$. Also, by (4.10), we deduce that $V_{ss}(s, t) \geq 0$, for $s \in \{0, s_1\}$ and $t > t_2$. Since $\psi'' \geq 0$, it follows from the maximum principle that $V_{ss} \geq 0$, for $s \in (0, s_1)$, $t > t_2$. Moreover, by Hopf’s lemma, for some $c_0 > 0$, we have
\[
v(r, t_1) \geq c_0 r \quad \text{in } (0, r_1), \quad V(s, t_1) \geq c_0 s \quad \text{in } (0, s_1).
\]
Let then \( z(r, s, t) = v(r, t)V(s, t) \). We compute
\[
\frac{z_t - z_{rr}}{(1 - rK)^2 z_{ss}} = V(v_t - v_{rr}) + v \left( V_t - \frac{1}{(1 - rK)^2} V_{ss} \right) \\
\leq -M |z_t| - M |z_s|.
\]
Hence, using (5.3), Proposition 3.2 and the choice of \( M \) in (4.9), we obtain
\[
z_t - \Delta z = z_t - z_{rr} + \frac{K}{1 - rK} z_r - \frac{1}{(1 - rK)^2} z_{ss} - \frac{rK'}{(1 - rK)^3} z_s
\]
\[
\leq - \left( M - \frac{K}{1 - rK} \right) |z_t| - \left( M - \frac{rK'}{(1 - rK)^3} \right) |z_s|
\]
\[
\leq p|\nabla u|^{p-2} \nabla u \cdot \nabla z.
\]
On the other hand, \( W := -u_x \) satisfies
\[
W_t - \Delta W = p|\nabla u|^{p-2} \nabla u \cdot \nabla W.
\]

For \( \mu \in (0, 1) \) small enough, due to (4.12), together with \( \text{supp}(\phi) \subset (0, r_0) \) and \( \psi \equiv 0 \) in \([0, s_1/2]\), we have
\[-u_x(r, s, t_2) \geq \mu \phi(r) \psi(s) = \mu z(r, s, t_2) \quad \text{in } \omega_1.
\]
Moreover, for possibly smaller \( \mu > 0 \), using (4.8), we see that
\[-u_x(r, s_1, t) \geq c_1 r \geq \mu v(r, t) = \mu z(r, s_1, t), \quad r \in (0, r_0), \ t \in [t_2, t_1].
\]
Since \( z = 0 \) on the rest of the lateral boundary of \( \omega_1 \times [t_2, t_1] \) (i.e. for \( r \in \{r_0, 1\} \) or \( s = 0 \)), it follows from (4.12), (4.13), the comparison principle and (4.11) that
\[-u_x(r, s, t_1) \geq \mu v(r, t_1) V(s, t_1) \geq c_1 rs \quad \text{in } \omega_1,
\]
with \( \hat{c}_1 = \mu c_3^2 \). \qed

5. Proof of Theorem 2.3

5.1. Auxiliary parabolic inequalities. Theorem 2.3 will be proved by using the techniques introduced in [22], that we here have to modify in a nontrivial way in order to adapt the method to the boundary with non constant curvature. These techniques are based on a Friedman-McLeod-type argument [13], which is very useful for solutions which are monotone in some sense. In our case, this monotonicity follows from the hypothesis (2.10).

Recall Notation 3.1 and 3.4. Let \( \sigma \in (0, \frac{1}{2(p-1)}) \) be fixed. For given \( \eta \in (0, s_0/2) \), we consider the auxiliary functions
\[
J = \frac{u_x}{1 - rK(s)} + c(s) d(r) F(u)
\]
and
\[
\bar{J} = u_x + \bar{c}(s) d(r) F(u),
\]
defined in \((D_{\Gamma} \cap \Omega) \times (0, T)\), where \( D_{\Gamma} \) is given in (2.7) and
\[
F(u) = u^q, \quad 1 < q < 2,
\]
\[
d(r) = r^{-\gamma}, \quad \gamma = (1 - 2\sigma)(q - 1),
\]
\[
c(s) = k(s - \eta), \quad k \in (0, 1),
\]
\[
\bar{c}(s) = ks,
\]
where $k, \gamma$ will be taken small (i.e., $q$ close to 1).

We start with a Lemma giving the equation satisfied by the first part of $J$.

**Lemma 5.1.** Let $\Omega, \Gamma, \gamma, M$ be as in Notation 2.1 and assume (2.4), (2.5). Then, the function $w = \frac{u_s}{1 - rK}$ satisfies

$$w_t - \Delta w = a_0 w + b_0 \cdot \nabla w + \frac{K'}{(1 - rK)^2} \frac{1}{\beta'} u_x$$

in $(D_T \cap \{s > 0\} \cap \Omega) \times (0, T)$, where

$$a_0 = \frac{K^2}{(1 - rK)^2} - \frac{pK}{1 - rK} |\nabla u|^{p-2} u_x - \frac{K'}{(1 - rK)^3} \frac{\alpha'}{\beta'},$$

$$b_0 = p |\nabla u|^{p-2} \nabla u - \frac{2K}{1 - rK} N(s).$$

The following lemma contains the key inequalities that enable one to apply the maximum principle to the auxiliary functions $J$ and $\bar{J}$.

**Lemma 5.2.** Let $\Omega, \Gamma, \gamma, M$ be as in Notation 2.1 and assume (2.4), (2.5). Let $J, \bar{J}$ be the functions defined in (5.5), (5.6) and define the parabolic operators

$$\mathcal{P}J := J_t - \Delta J - a J - b \cdot \nabla J$$

and

$$\mathcal{P}\bar{J} := \bar{J}_t - \Delta \bar{J} - \bar{a} \bar{J} - \bar{b} \cdot \nabla \bar{J},$$

with

$$a = \frac{pK}{1 - rK} |\nabla u|^{p-2} u_x - \frac{p}{1 - rK} c'dF |\nabla u|^{p-2} + \frac{K^2}{(1 - rK)^2}$$

$$- \frac{K'}{(1 - rK)^3} \frac{\alpha'}{\beta'} - \frac{2}{1 - rK} c'dF',$$

$$b = p |\nabla u|^{p-2} \nabla u - \frac{2K}{1 - rK} N(s),$$

$$\bar{a} = -\alpha' \frac{p}{1 - rK} c'dF |\nabla u|^{p-2} - \frac{2\alpha'}{1 - rK} c'dF',$$

$$\bar{b} = p |\nabla u|^{p-2} \nabla u.$$

Then we have,

$$\frac{\mathcal{P}J}{c'dF} \leq \Theta(A) + \frac{K'}{(1 - rK)^3} \frac{1}{\beta'c'dF} \bar{J},$$

in $(D_T \cap \{s > \eta\} \cap \Omega) \times (0, T)$,

and

$$\frac{\mathcal{P}\bar{J}}{c'dF} \leq \Theta(\bar{A}),$$

in $(D_T \cap \{s > 0\} \cap \Omega) \times (0, T)$,

with

$$\Theta(A) = -(p - 1)q \frac{|\nabla u|^p}{u} + \frac{pk}{1 - rK} \frac{u^q |\nabla u|^{p-2}}{r^\gamma} + p |\nabla u|^{p-1} A$$

$$- q(q - 1) \frac{|\nabla u|^2}{u^2} + \frac{2q}{r} \frac{|\nabla u|}{u} A + \frac{2qk}{1 - rK} \frac{u^{q-1}}{r^\gamma} + \frac{\gamma(\gamma + 1)}{r^2},$$

and $A = A(r, s) = \gamma + \frac{rK}{1 - rK}$, $\bar{A} = \bar{A}(r, s) = \gamma + \frac{\tau r}{1 - rK}$, for some $\tau = \tau(\Omega) > 0$. 

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In addition, there exists a constant \( L = L(p, \Omega, \|u_0\|_{C^1}) > 0 \) such that, for all real numbers \( X > 0 \), we have

\[
\Theta(X) \leq \left[ kB \left( pL^{q+p-2} + 2qL^{q-1} \right) + \frac{q}{q-1} X^2 + \frac{\sigma}{2} X - \gamma(\gamma + 1) \right] \frac{1}{r^2} + \left( \frac{p^2}{2\sigma} L^{p-1} - (p-1)q \right) \frac{|\nabla u|^p}{u},
\]

(5.10)

where \( B = B(r, s) = \frac{r(q-1)(2\sigma - \frac{q}{p}) + 2}{1 - rK(s)} \).

Since the proofs of these two Lemmas require long computations, we postpone them after the proof of Theorem 2.3.

5.2. Proof of Theorem 2.3

Step 1: Preparations. Fix any \( \eta \in (0, s_0/2) \) and recall the definition of the auxiliary function \( J \) given in (5.11)

\[
J = \frac{u_s}{1 - rK} + c(s)d(r)F(u) = \frac{u_s}{1 - rK} + k(s - \eta)r^{-\gamma}u^q \quad \text{in } \omega_0 \times (t_0, T),
\]

with \( 1 < q < 2, \gamma = (1 - 2\sigma)(q - 1) > 0 \), where \( \sigma \in \left( 0, \frac{1}{2(p-1)} \right) \) is fixed, and \( k \in (0, 1) \) and \( \gamma \) will be taken small (i.e., \( q \) close to 1). Without loss of generality, by taking \( r_0 > 0 \) possibly smaller, we may assume that

\[
\omega_0 = M((0, r_0) \times (0, s_0)) \subset \Omega,
\]

where \( M \) is the coordinate map defined in (2.3).

Observe that, for each \( t_0 < T' < T \), we have

\[
u \leq C \quad \text{in } \omega_0 \times [t_0, T'],
\]

(5.11)

for some \( C = C(T') > 0 \). Since \( \gamma < q \), we have in particular

\[
J \in C([\omega_0] \times [0, T)) \cap C^{2,1}(\omega_0 \times (0, T)).
\]

Fix \( t_1 = \frac{4s_1}{27}, s_1 = \frac{3}{4}s_0 \) and set \( K_1 = \max K(s) \). Our aim is to use the maximum principle to prove that

\[
J \leq 0 \quad \text{in } \omega_{1, \eta} \times (t_1, T),
\]

(5.12)

where \( \omega_{1, \eta} := M(((0, r_1) \times (\eta, s_1)) \)

for \( r_1 \in (0, \min(\frac{r_0}{4}, \frac{1}{2K_1})) \) to be chosen below.

Note that since \( 1 - rK \geq 1/2 \) in \( \omega_{1, \eta} \), inequality (5.13) implies

\[
u_s \leq -(1 - rK)cF \leq -\frac{k}{2}(s - \eta)r^{-1}u^q.
\]

Hence, if (5.13) is proved, then integrating (5.13) over the curve

\[
\{\gamma(\theta) + rN(\theta); \theta \in [\eta, s]\}
\]

for \( \eta < s < s_1, 0 < r < r_1 \) and \( t_1 < t < T \), we get

\[
u \leq C(s - \eta)^{-\frac{1}{q-1}}r^{1-2\sigma} \leq C(s - \eta)^{-\frac{1}{q-1}}\delta^{1-2\sigma}(x, y) \quad \text{in } \omega_{1, \eta} \times (t_1, T),
\]

for all real numbers \( X > 0 \).
for some constant $C = C(\eta) > 0$. Then, since $1 - 2\sigma > (p - 2)/(p - 1)$, it will follow from Lemma 4.1 and symmetry that $GBUS(u_0) \subset \gamma([-\eta, \eta])$. Since $\eta$ is arbitrarily small, we will conclude that $GBUS(u_0) = \{(0, 0)\}$. 

Step 2: Parabolic inequality for $J$.

It follows from (5.7) and 5.10 in Lemma 5.2 that, for the parabolic operator $P$ defined in (5.5), we have

$$
\frac{\partial J}{\partial t} \leq \left[ k B(p \bar{L}^q + p - 2 + 2q \bar{L}^{q - 1}) + \frac{q}{q - 1} A^2 + \frac{\sigma}{2} A - \gamma(\gamma + 1) \right] \frac{1}{r^2} 
$$

$$
+ \left( \frac{p^2 A}{2\sigma} \bar{L}^{p - 1} - (p - 1)q \right) \frac{\left| \nabla u \right|^p}{u} + \frac{K'}{(1 - rK)^{\beta/2}} \frac{1}{\beta} \bar{J}, \text{ in } \omega_1, t \times (t_0, T),
$$

with $L = L(p, \Omega, \|u_0\|_{C^1}) > 0$. At this point we fix $\tau$ and $r_1$ satisfying

$$
0 < \gamma < \sigma \min \left( \frac{1}{4}, \frac{1}{p^2 L^{p - 1}} \right) < 1
$$

and

$$
0 < r_1 < \min \left[ r_0, 1, \frac{\gamma^2}{2K_1}, \frac{\gamma^2}{2r}, \frac{3\gamma^2}{2(p \bar{L}^q + p - 2 + 2q \bar{L}^{q - 1})} \right],
$$

where $\tau = \tau(\Omega) > 0$ is given by Lemma 5.2 (some of the conditions in (5.16), (5.17) will be used only in Step 3), and we set

$$
\omega_1 := M((0, r_1) \times (0, s_1)).
$$

It follows, from $r_1 < \frac{1}{2K_1}$, that

$$
1 - rK \geq 1/2 \quad \text{in } \omega_1,
$$

hence

$$
A = \gamma + \frac{rK}{1 - rK} \leq \gamma(\gamma + 1) \quad \text{in } \omega_1,
$$

$$
B = \frac{r(q - 1)(2\sigma - \frac{1}{p} - 1) + 2}{1 - rK} \leq 2r_1 \quad \text{in } \omega_1,
$$

where we used $(q - 1) \left( \frac{2\sigma}{p} - \frac{1}{p} \right) + 2 \geq 1$, which follows from $1 < q < p$. As a consequence of (5.16) and (5.19), using $p > 2$ and $q > 1$, we first get

$$
\frac{p^2 A}{2\sigma} \bar{L}^{p - 1} - (p - 1)q \leq \frac{p^2 A}{2\sigma} \bar{L}^{p - 1} - 1 \leq 0 \quad \text{in } \omega_1.
$$

Next, since $\gamma = (1 - 2\sigma)(q - 1)$, we deduce from (5.16) and (5.19) that

$$
\frac{q}{q - 1} A^2 + \frac{\sigma}{2} A - \gamma(\gamma + 1) \leq \gamma(\gamma + 1) \left( (1 - 2\sigma + \gamma)(\gamma + 1) + \frac{\sigma}{2} - 1 \right)
$$

$$
= \gamma(\gamma + 1) \left( \gamma + 2(1 - \sigma) \right) - \frac{3\sigma}{2}
$$

$$
\leq 3\gamma(\gamma + 1) \left( \gamma - \frac{\sigma}{2} \right) \leq -3\gamma^2 \quad \text{in } \omega_1.
$$

In view of (5.17), (5.20), and recalling $k \in (0, 1)$, we obtain

$$
k B(p \bar{L}^q + p - 2 + 2q \bar{L}^{q - 1}) + \frac{q}{q - 1} A^2 + \frac{\sigma}{2} A - \gamma(\gamma + 1)
$$

$$
\leq 2r_1 (p \bar{L}^q + p - 2 + 2q \bar{L}^{q - 1}) - 3\gamma^2 \leq 0 \quad \text{in } \omega_1.
$$
It follows from (5.15), (5.21), (5.22) that, for all \( k \in (0, 1) \),
\[
(5.23) \quad \mathcal{P}J \leq \frac{K'}{\beta'(1-rK)^2} \tilde{J} \quad \text{in } \omega_1, \bar{\gamma} \times (t_0, T).
\]

Moreover, in view of (2.3), (2.5), (5.11) and (5.18), the coefficient \( a \) in \( \mathcal{P} \) satisfies
\[
(5.24) \quad \sup_{\omega_1, \bar{\gamma} \times (t_0, T')} |a| < \infty, \quad \text{for any } T' < T.
\]

**Step 3: Control of \( \tilde{J} \).**

We claim that under assumptions (5.16), (5.17), there exists \( \hat{k} \in (0, 1) \) such that, for all \( k \in (0, \hat{k}] \),
\[
(5.25) \quad \tilde{J} = u_x + \tilde{c}dF = u_x + ksr^{\bar{\gamma}}u^q \leq 0 \quad \text{in } \omega_1 \times (t_1, T),
\]

hence
\[
(5.26) \quad \mathcal{P}J \leq 0 \quad \text{in } \omega_1, \bar{\gamma} \times (t_1, T).
\]

By (5.8) and (5.10) in Lemma 5.2, we have the following inequality for the parabolic operator \( \mathcal{P} \) defined in (5.6):
\[
\frac{\mathcal{P}J}{cdF} \leq \left[ kB \left( pL^q + p^{-2} + 2qL^{q-1} \right) + \frac{q}{q-1} \bar{A}^2 + \frac{\bar{A}}{2} - \bar{A} \right] \frac{1}{r^2}
+ \left( \frac{p^2 \bar{A}}{2\sigma} L^{p-1} - (p-1)q \right) \frac{|\nabla u|^p}{u} \quad \text{in } \omega_1 \times (t_0, T),
\]

where
\[
\bar{A} = \gamma + \frac{\sigma(\Omega) r}{1-rK} \quad \text{and} \quad B = \frac{r^{(q-1)(2\sigma - p/2) + 2}}{1-rK}.
\]

Moreover, under assumptions (5.16), (5.17) (which in particular guarantee \( \bar{A} \leq \bar{\gamma}(\gamma + 1) \) in \( \omega_1 \)), the argument leading to (5.21), (5.22) yields:
\[
\frac{p^2 \bar{A}}{2\sigma} L^{p-1} - (p-1)q \leq 0 \quad \text{in } \omega_1,
\]

and
\[
kB(pL^q + p^{-2} + 2qL^{q-1}) + \frac{q}{q-1} \bar{A}^2 + \frac{\bar{A}}{2} - \bar{A} \leq 0 \quad \text{in } \omega_1.
\]

For any \( k \in (0, 1) \), we thus obtain
\[
(5.27) \quad \mathcal{P}J \leq 0, \quad \text{in } \omega_1 \times (t_0, T).
\]

By (2.9), there exists a constant \( C > 0 \) such that
\[
|\nabla u| \leq C \quad \text{in } \omega_0 \setminus M \left( (0, r_1/2) \times (0, \theta_1) \right) \times (t_0, T),
\]

for \( \theta_1 \in \left( \frac{\varphi}{2}, s_1 \right) \). Consequently, by parabolic estimates, \( u \) can be extended to a function such that
\[
(5.28) \quad u, \nabla u \in C^{2,1}(\bar{Q}) \quad \text{where } \bar{Q} = \left( \omega_0 \setminus M \left( (0, \frac{3r_1}{4}) \times (0, \theta_2) \right) \right) \times (t_0, T],
\]

with \( \theta_2 \in (\theta_1, s_1) \). Fix any \( t_2 \in (t_0, t_1) \) and \( r_2 \in (r_1, \min(r_0, \frac{1}{2r})] \). Since \( w = u_x \) satisfies
\[
(5.29) \quad w_t - \Delta w = p|\nabla u|^{p-2}\nabla u \cdot \nabla w \quad \text{in } \omega_0 \times (t_0, T),
\]

and
by Hopf’s Lemma, (5.28) and (2.10), there exist \( c_1, c_2 > 0 \) such that
\[
(5.29) \quad u_x \leq -c_1r \quad \text{on} \quad (0, r_2) \times \{s_1\} \times (t_2, T), \\
(5.30) \quad u_x \leq -c_1s \quad \text{on} \quad \{r_1\} \times (0, s_1) \times (t_2, T), \\
(5.31) \quad u \leq c_2r \quad \text{on} \quad (0, r_1) \times \{s_1\} \times (t_2, T).
\]
Moreover, in view of (2.10) and (5.28), we may thus apply the strong maximum principle and Lemma 4.3 to deduce the existence of \( \tilde{c}_1 > 0 \) such that
\[
(5.32) \quad u_x(r, s, t_1) \leq -\tilde{c}_1rs \quad \text{in} \quad (0, r_1) \times (0, s_1).
\]
Now, on the lateral boundary of \( \omega_1 \times (t_1, T) \), we have
\[
(5.33) \quad \bar{J}(0, s, t) = 0 \quad \text{on} \quad \{0\} \times (0, s_1) \times (t_1, T), \\
(5.34) \quad \bar{J}(r, 0, t) \leq 0 \quad \text{on} \quad (0, r_1) \times \{0\} \times (t_1, T), \\
(5.35) \quad \bar{J}(r_1, s, t) \leq -c_1s + ksr^{-\gamma}_0 \|u_0\|_{L_\infty}^q \leq 0 \quad \text{on} \quad \{r_1\} \times (0, s_1) \times (t_1, T), \\
(5.36) \quad \bar{J}(r, s_1, t) \leq -c_1r + ksc_2r^{q - \gamma} \leq 0 \quad \text{on} \quad (0, r_1) \times \{s_1\} \times (t_1, T),
\]
for any \( 0 < k \leq \tilde{k} \) with \( \tilde{k} > 0 \) sufficiently small, where we used \( q > \gamma + 1 \). And at the initial time \( t = t_1 \), for any \( 0 < k \leq \tilde{k} \) with possibly smaller \( \tilde{k} > 0 \), inequality (5.32) guarantees
\[
(5.37) \quad \bar{J}(r, s, t_1) \leq -\tilde{c}_1rs + ksc_2r^{q - \gamma} \leq 0 \quad \text{on} \quad (0, r_1) \times (0, s_1).
\]
Moreover, owing to (5.11) and (5.15), we have
\[
(5.38) \quad \sup_{\omega_1 \times (t_0, T')} \hat{a} < \infty, \quad \text{for any} \quad T' < T.
\]
Then, for any \( 0 < k \leq \tilde{k} \), claim (5.25) follows from (5.27), (5.33)–(5.38) and the maximum principle applied to \( \bar{J} \) in \( \omega_1 \times (t_1, T) \) (see Proposition 52.4 in [29]). Note that the use of the maximum principle is justified in view of the regularity property (5.12), which obviously also applies for \( \bar{J} \). Finally, (5.26) follows from (2.4), (2.5), (5.23) and (5.25).

**Step 4: Initial and boundary conditions for \( J \).**

Let \( w = \frac{u_s}{1 - rK} \). In view of Lemma 5.1, 2.4, 2.5 and (2.10), it follows that
\[
w_t - \Delta w - a_w w - b_w \cdot \nabla w = \frac{K'}{(1 - rK)^3} \beta' u_x \leq 0, \quad \text{in} \quad \omega_0 \times (t_0, T),
\]
with
\[
a_w = \frac{K^2}{(1 - rK)^2} - \frac{pK}{1 - rK} |\nabla u|^{p-2} u_x - \frac{K'}{(1 - rK)^3} \alpha', \\
b_w = p|\nabla u|^{p-2} u_x - \frac{2K}{1 - rK} N(s).
\]
Note in particular that \( \beta'(s) \) and \( 1 - rK \) are uniformly positive for \( s \in [\eta, s_1] \) by (2.5) and (5.18). In view of (2.4) and (5.28), we may thus apply the strong maximum principle and Hopf’s Lemma to deduce the existence of \( c_3, c_4, c_5 \) > 0 (possibly depending on \( \eta \)) such that
\[
(5.39) \quad u_s \leq -c_3r \quad \text{on} \quad (0, r_1) \times \{s_1\} \times (t_1, T), \\
(5.40) \quad u_s \leq -c_4 \quad \text{on} \quad \{r_1\} \times (\eta, s_1) \times (t_1, T), \\
(5.41) \quad u_s \leq -c_3r \quad \text{in} \quad \omega_{1, \eta} \times \{t_1\},
\]
as well as
\[
u \leq c_5r, \quad \text{on} \quad (0, r_1) \times \{s_1\} \times (t_1, T).
\]
Consequently, we may choose \( \tilde{k} > 0 \) small enough (possibly depending on \( \eta \)) such that, for any \( 0 < k \leq \tilde{k} \), on the lateral boundary of \( \omega_{1, \eta} \times (t_1, T) \), we have

\[
(5.42) \quad J(0, s, t) = 0 \quad \text{on } \{0\} \times (\eta, s_1) \times (t_1, T),
\]

\[
(5.43) \quad J(r, \eta, t) \leq 0 \quad \text{on } (0, r_1) \times \{\eta\} \times (t_1, T),
\]

\[
(5.44) \quad J(r_1, s, t) \leq -c_1 + k s r_1^{-1} \|u_0\|_{L^\infty}^q \leq 0 \quad \text{on } \{r_1\} \times (\eta, s_1) \times (t_1, T),
\]

\[
(5.45) \quad J(r, s_1, t) \leq -c_2 r + k s_1 c_q r^{q-\gamma} \leq 0 \quad \text{on } (0, r_1) \times \{s_1\} \times (t_1, T),
\]

where we used \( q > \gamma + 1 \), and at the initial time \( t = t_1 \),

\[
(5.46) \quad J(r, s, t_1) \leq -c_3 r + k s_1 c_q r^{q-\gamma} \leq 0, \quad \text{in } \omega_{1, \eta}.
\]

Then (5.13) follows from (5.24), (5.28), (5.42)–(5.46) and the maximum principle applied to \( J \) in \( \omega_{1, \eta} \times (t_1, T) \) (see Proposition 52.4 in [26]). Note that the use of the maximum principle is justified in view of (5.12).

In view of Step 1, this concludes the proof of the Theorem. \( \square \)

5.3. Proof of auxiliary parabolic inequalities (Lemmas 5.1 and 5.2).

**Proof of Lemma 5.1.** Let \( w = \frac{u_s}{1 - r K} \), and compute, in \( (D_\Gamma \cap \{s > 0\} \cap \Omega) \times (0, T) \).

\[
\begin{align*}

w_r &= \frac{u_{rs}}{1 - r K} + \frac{K}{(1 - r K)^2} u_s, \\

w_{rr} &= \frac{u_{rrs}}{1 - r K} + 2 \frac{K u_{rs}}{(1 - r K)^2} + 2 \frac{K^2}{(1 - r K)^3} u_s, \\

w_s &= \frac{u_{ss}}{1 - r K} + \frac{r K'}{(1 - r K)^2} u_s, \\

w_{ss} &= \frac{u_{sss}}{1 - r K} + 2 \frac{r K'}{(1 - r K)^2} u_{rs} + 2 \frac{r^2 K'^2}{(1 - r K)^3} u_s + \frac{r K''}{(1 - r K)^2} u_s.
\end{align*}
\]

Then, using Proposition 5.2 we get

\[
\begin{align*}

\Delta w &= w_{rr} - \frac{K}{1 - r K} w_r + \frac{1}{(1 - r K)^2} w_{ss} + \frac{r K'}{(1 - r K)^3} w_s \\

&= \frac{1}{1 - r K} u_{rrs} + \frac{K}{(1 - r K)^2} u_{rs} + \frac{1}{(1 - r K)^3} u_{sss} + \frac{K^2}{(1 - r K)^4} u_s \\

&\quad + \frac{r K'}{(1 - r K)^2} u_{rs} + \frac{r K''}{(1 - r K)^3} u_s + \frac{r^2 K'^2}{(1 - r K)^3} u_s
\end{align*}
\]

and also

\[
\begin{align*}

\frac{(\Delta u)_s}{1 - r K} &= \frac{1}{1 - r K} u_{rrs} - \frac{K}{(1 - r K)^2} u_{rs} + \frac{1}{(1 - r K)^3} u_{sss} + \frac{3}{(1 - r K)^4} u_s \\

&\quad + \frac{r K'}{(1 - r K)^2} u_{rs} + \frac{r^2 K'^2}{(1 - r K)^3} u_s - \frac{r K'}{(1 - r K)^2} u_r \\

&= \Delta w - \frac{1}{(1 - r K)^2} \left( \frac{K'}{1 - r K} u_r + 2 K u_{rs} + \frac{K^2}{1 - r K} u_s \right),
\end{align*}
\]
and replacing \( u_{rs} = (1 - rK)w_r - \frac{K}{1 - rK}u_s \) and using identity (3.10), we obtain

\[
\Delta w = \frac{(\Delta u)_s}{1 - rK} + \frac{2K}{1 - rK}w_r - \left( \frac{K^2}{(1 - rK)^2} - \frac{K'}{(1 - rK)^3} \beta' \right) w - \frac{1}{(1 - rK)^3 \beta' u_x}.
\]  
(5.48)

Note that the use of (3.10) is justified since \( s > 0 \), and then \( \beta' > 0 \). Then we get

\[
w_t - \Delta w = \frac{(\lvert \nabla u \rvert^p)_s}{1 - rK} + \frac{2K}{1 - rK}w_r + \left( \frac{K^2}{(1 - rK)^2} - \frac{K'}{(1 - rK)^3} \beta' \right) w + \frac{1}{(1 - rK)^3 \beta' u_x}.
\]  
(5.49)

Now, we write

\[
(\lvert \nabla u \rvert^p)_s = p \lvert \nabla u \rvert^{p-2} \nabla u \cdot (\nabla u)_s,
\]
and using (3.6), we obtain

\[
\nabla u \cdot (\nabla u)_s = \left( u_r N(s) + \frac{u_s}{1 - rK} T(s) \right) \cdot \left( u_{rs} N(s) + \frac{u_{ss}}{1 - rK} T(s) + u_r (N(s))_s \right) + \frac{rK'}{(1 - rK)^{3/2}} u_s T(s) + \frac{u_s}{1 - rK} (T(s))_s,
\]

where \( T(s) \) and \( N(s) \) are defined in Notation 2.1.

We observe that

\[
N(s) \cdot (N(s))_s = T(s) \cdot (T(s))_s = 0,
\]
\[
N(s) \cdot (T(s))_s + T(s) \cdot (N(s))_s = (N(s) \cdot T(s))_s = 0,
\]
so we have

\[
\nabla u \cdot (\nabla u)_s = \nabla u \cdot \nabla(u_s) + \frac{rK'}{(1 - rK)^3} u_s^2
\]
\[
= (1 - rK) \nabla u \cdot \nabla w - w \nabla u \cdot \nabla(rK) + w \frac{rK'}{(1 - rK)^2} u_s
\]
\[
= (1 - rK) \nabla u \cdot \nabla w - Ku_r w.
\]

Plugging this in (5.50), we obtain

\[
(\lvert \nabla u \rvert^p)_s = p \lvert \nabla u \rvert^{p-2} \nabla u \cdot \nabla w - \frac{pK}{1 - rK} \lvert \nabla u \rvert^{p-2} u_r w,
\]  
(5.51)

and combining this with (5.49), we obtain (5.4). □
Proof of Lemma 5.2.

Proof of inequality (5.49): Using Proposition 5.2.2 and (5.48), we compute, in \((D_T \cap \{s > \eta\} \cap \Omega) \times (0, T),\)

\[
J_t = \frac{u_{ts}}{1 - r K} + c d F' u_t,
\]

\[
\Delta J = \Delta w + c d F' \Delta u + c d F'' |\nabla u|^2 + \frac{2}{1 - r K} c' d F' w
\]

\[
= \frac{\Delta u}{1 - r K} + \frac{K^2}{(1 - r K)^2} - \frac{K'}{(1 - r K)^3} \beta' - \frac{2}{1 - r K} c' d F' w
\]

\[
= \frac{\Delta u}{1 - r K} + \frac{K^2}{(1 - r K)^2} - \frac{K'}{(1 - r K)^3} \beta' - \frac{2}{1 - r K} c' d F' w
\]

Then, it follows that

\[
J_t - \Delta J = \left( \frac{\Delta u}{1 - r K} + \frac{K^2}{(1 - r K)^2} - \frac{K'}{(1 - r K)^3} \beta' - \frac{2}{1 - r K} c' d F' w \right)
\]

\[
- \frac{2K}{1 - r K} w_r + c d F' |\nabla u|^p - c d F'' |\nabla u|^2 + \frac{K}{1 - r K} c d F'
\]

\[
= \frac{\Delta u}{1 - r K} + \frac{K^2}{(1 - r K)^2} - \frac{K'}{(1 - r K)^3} \beta' - \frac{2}{1 - r K} c' d F' w
\]

and plugging (5.51) here, we get

\[
J_t - \Delta J = \left( -\frac{pK}{1 - r K} |\nabla u|^{p-2} u_r + \frac{K^2}{(1 - r K)^2} - \frac{K'}{(1 - r K)^3} \beta' - \frac{2}{1 - r K} c' d F' w \right)
\]

\[
+ p |\nabla u|^{p-2} w_r + c d F' |\nabla u|^p - c d F'' |\nabla u|^2 - 2 c d F' u_r
\]

\[
+ \frac{K}{1 - r K} c d F' - c d F' - \frac{rK}{(1 - r K)^3} \beta' - \frac{2}{1 - r K} c' d F' w
\]

Now, we use the following identities:

\[
w = J - c d F,
\]

\[
w_r = N(s) \cdot \nabla J - c d F' u_r - c d F,
\]

\[
\nabla u \cdot \nabla w = \nabla u \cdot \nabla J - \nabla u \cdot \nabla (c d F),
\]

and

\[
\nabla u \cdot (c d F) = u_r (c d F + c d F' u_r) + \frac{u_{ts}}{(1 - r K)^2} (c' d F + c d F' u_r)
\]

(5.52)

\[
= c d F' |\nabla u|^2 + c d F u_r + c d F' u_r
\]

\[
\frac{u_{ts}}{(1 - r K)^2} (c' d F + c d F' u_r)
\]

hence

\[
\nabla u \cdot \nabla w = \nabla u \cdot \nabla J - \frac{1}{1 - r K} c' d F J - c d F' |\nabla u|^2 - c d F u_r + \frac{1}{1 - r K} c c' d^2 F^2,
\]
to obtain

\[ J_t - \Delta J = \left( -\frac{pK}{1-rK} |\nabla u|^{p-2}u_r + \frac{K'}{1-rK} \frac{1}{(1-rK)^2} \right) \frac{\alpha'}{\beta'} - \frac{2}{(1-rK)^{\beta'}} \frac{c'dF}{\beta'} \]

\[ \left( \frac{p}{1-rK} c'dF |\nabla u|^{p-2} J + \left( p|\nabla u|^{p-2} \nabla u - \frac{2K}{1-rK} N(s) \right) \cdot \nabla J \right) \]

\[ -(p-1)cdF |\nabla u|^p + \frac{p}{1-rK} cc'd^2 F^2 |\nabla u|^{p-2} \]

\[ -pcd' F |\nabla u|^{p-2} u_r + \frac{pK}{1-rK} cdF |\nabla u|^{p-2} u_r - cdF' |\nabla u|^2 \]

\[ + \frac{2K}{1-rK} cdF' u_r - 2cd' F' u_r + \frac{3K}{1-rK} cdF - cd'' F \]

\[ + \frac{2}{1-rK} cc'd^2 F' - \frac{rK'}{(1-rK)^3} c'dF - \frac{K^2}{(1-rK)^2} cdF \]

\[ + \frac{K'}{(1-rK)^3} \frac{1}{\beta'} (u_x + \alpha' cdF). \]

Let \( \mathcal{P} J := J_t - \Delta J - aJ - b \cdot \nabla J \), where \( a, b \) are defined in the statement of the Lemma. Using \( K, K' \geq 0 \) and the definitions of \( c, d, F \), along with \( \beta' > 0, 0 < \alpha' \leq 1 \) and \( 0 < c \leq \bar{c} \), we then have, in \( (D_T \cap \{ s > \eta \} \cap \Omega) \times (0, T) \):

\[ \frac{\mathcal{P} J}{cdF} = -(p-1) \frac{F'}{F} |\nabla u|^p + \frac{p}{1-rK} c'dF |\nabla u|^{p-2} - \frac{d'}{d} \frac{d'}{d} |\nabla u|^{p-2} u_r \]

\[ + \frac{pK}{1-rK} |\nabla u|^{p-2} u_r - \frac{F''}{F} |\nabla u|^2 + \frac{2K}{1-rK} \frac{F'}{F} u_r - \frac{d'd'}{d} u_r \]

\[ + \frac{3K}{1-rK} \frac{d'}{d} - \frac{d''}{d} \frac{d'}{d} + \frac{2}{1-rK} cc'd^2 F' - \frac{rK'}{(1-rK)^3} c'dF - \frac{K^2}{(1-rK)^2} \]

\[ + \frac{K'}{(1-rK)^3} \frac{1}{\beta'} (u_x + \alpha' cdF) \]

\[ \leq -(p-1) \frac{|\nabla u|^p}{u} + \frac{pk}{1-rK} u^{q-1} + \frac{|\nabla u|^{p-1}}{u} \left( \frac{rK}{1-rK} \right) \]

\[ -q(q-1) \frac{|\nabla u|^2}{u^2} + 2q \frac{|\nabla u|}{u} \left( \frac{rK}{1-rK} \right) + \frac{2kq}{1-rK} \left( \frac{u^{q-1}}{u^2} - \frac{\gamma}{r^2} \right) \]

\[ + \frac{K'}{(1-rK)^3} \frac{1}{\beta'} (u_x + \bar{c}dF) \]

that is, (5.7).
Proof of inequality (5.5):

In a similar but simpler way as in the computation for $J$ and using (3.9), we compute, in $(D_T \cap \{s > 0\} \cap \Omega) \times (0, T)$,

$$\tilde{J}_t = u_{tx} + \tilde{c}dF' u_t,$$

$$\Delta \tilde{J} = (\Delta u)_x + \tilde{c}dF' |\nabla u|^2 + \tilde{c}F \left( d'' - \frac{K}{1 - rK}d' \right) + \frac{rK'}{(1 - rK)^2} \tilde{c}' dF' + 2\tilde{c}dF' u_x + \tilde{c}dF' \Delta u$$

Then we obtain

$$\tilde{J}_t - \Delta \tilde{J} = \left( \left| \nabla u \right|^p \right)_x + \tilde{c}dF' |\nabla u|^p - \frac{2}{1 - rK} \tilde{c}dF' u_x - \tilde{c}dF' |\nabla u|^2$$

$$-2\tilde{c}dF' \left( u_r + \beta' - \frac{u_y}{1 - rK} \right) + \frac{rK'}{(1 - rK)^3} \tilde{c}' dF - \tilde{c}F \left( d'' - \frac{K}{1 - rK}d' \right).$$

In view of $u_x = \tilde{J} - \tilde{c}dF$, and using (3.9) and (5.52), we compute

$$\left( \left| \nabla u \right|^p \right)_x = p|\nabla u|^{p-2} \nabla u \cdot \nabla u_x$$

$$= p|\nabla u|^{p-2} \nabla u \cdot \nabla \tilde{J} - p|\nabla u|^{p-2} \nabla u \cdot \nabla (\tilde{c}dF')$$

$$= p|\nabla u|^{p-2} \nabla u \cdot \nabla \tilde{J} - p\tilde{c}dF'|\nabla u|^p - \alpha' \frac{p}{1 - rK} \tilde{c}' dF'|\nabla u|^{p-2}u_x$$

$$-p\tilde{c}dF'|\nabla u|^{p-2} \left( u_r + \beta' - \frac{u_y}{1 - rK} \right).$$

It then follows that

$$\tilde{J}_t - \Delta \tilde{J} = \tilde{a}\tilde{J} + \tilde{b} \cdot \nabla \tilde{J} + (p - 1)\tilde{c}dF'|\nabla u|^p + \alpha' \frac{p}{1 - rK} \tilde{c}' d^2 F'|\nabla u|^{p-2}$$

$$-p\tilde{c}dF'|\nabla u|^{p-2} \left( u_r + \beta' - \frac{u_y}{1 - rK} \right) - \tilde{c}dF'|\nabla u|^2$$

$$-2\tilde{c}dF' \left( u_r + \beta' - \frac{u_y}{1 - rK} \right) + \frac{2\alpha'}{1 - rK} \tilde{c}' d^2 F'$$

$$-\frac{rK'}{(1 - rK)^3} \tilde{c}' dF - \tilde{c}F \left( d'' - \frac{K}{1 - rK}d' \right),$$

(5.53)

where

$$\tilde{a} = -\alpha' \frac{p}{1 - rK} \tilde{c}' d^2 F'|\nabla u|^{p-2} - \frac{2\alpha'}{1 - rK} \tilde{c}' dF', \quad \tilde{b} = p|\nabla u|^{p-2} \nabla u.$$

In view of the symmetry of $\Omega$ and $\Gamma$ (assumption (2.3)), we have $\beta'(0) = 0$. By the regularity of $\partial \Omega$, it follows that there exists $\tau = \tau(\Omega) > 0$ such that

$$\beta'(s) \leq \tau s, \quad \forall s \in [0, s_0].$$

Let $\mathcal{P} \tilde{J} := \tilde{J}_t - \Delta \tilde{J} - \tilde{a}\tilde{J} - \tilde{b} \cdot \nabla \tilde{J}$, where $\tilde{a}, \tilde{b}$ are defined in the statement of the Lemma. Plugging the definitions of $\tilde{c}, d, F$ in the expression (5.53), and using the above inequality,
\( K' \geq 0 \) and \( \alpha' \leq 1 \), we obtain
\[
\frac{\ddot{p} \dot{J}}{\bar{c} \dot{d} F} = -(p-1) \frac{F'}{F} |\nabla u|^p + \alpha' \frac{p}{1-rK} \tilde{c}' \dot{d} F |\nabla u|^{p-2} \]
\[
- \frac{d'}{d} |\nabla u|^{p-2} \left( u_r + \beta' \frac{u_w}{1-rK} \tilde{c}' \dot{d} F \right) - \frac{F''}{F} |\nabla u|^2 \]
\[
- \frac{rK'}{(1-rK)^3} \tilde{c}' \dot{d} F - \frac{d''}{d} + \frac{K}{1-rK} \frac{d'}{d} \]
\[
\leq - (p-1) q \frac{|\nabla u|^p}{u^2} + \frac{p k}{1-rK} \frac{|\nabla u|^{p-2}}{r \gamma} + \frac{p |\nabla u|^{p-1}}{r} \left( \gamma \frac{1}{1-rK} \right) \]
\[
- q(q-1) \frac{|\nabla u|^2}{u^2} + \frac{2q}{r} \frac{|\nabla u|}{u} \left( \gamma \frac{1}{1-rK} \right) \frac{2qk}{1-rK} \frac{u^{q-1}}{r^2} - \gamma(\gamma + 1) \frac{1}{r^2} . \]

**Proof of inequality (5.10):**

Using Young's inequality we obtain, for any \( X > 0 \),
\[
\frac{2q}{r} \frac{|\nabla u|}{u} X \leq q(q-1) \frac{|\nabla u|^2}{u^2} + \frac{q}{q-1} \frac{X^2}{r^2} , \]
and
\[
p \frac{|\nabla u|^{p-1}}{r} X \leq \frac{\sigma}{2r^2} X + \frac{\rho^2}{2\sigma} X |\nabla u|^{2p-2} , \]
hence,
\[
(5.54) \quad \frac{2q}{r} \frac{|\nabla u|}{u} X - q(q-1) \frac{|\nabla u|^2}{u^2} - \frac{\gamma(\gamma + 1)}{r^2} \leq \left( \frac{q}{q-1} X^2 - \gamma(\gamma + 1) \right) \frac{1}{r^2} , \]
and
\[
(5.55) \quad -(p-1) q \frac{|\nabla u|^p}{u^2} + p \frac{|\nabla u|^{p-1}}{r} X \]
\[
\leq \left( \frac{\rho^2}{2\sigma} |\nabla u|^{p-2} - (p-1) q \right) \frac{|\nabla u|^p}{u} + \frac{\sigma}{2r^2} X . \]

Using (1.2), we obtain the following estimates
\[
(5.56) \quad \frac{u^q |\nabla u|^{p-2}}{r \gamma} \leq L^{q+p-2,(q-1) \frac{p-2}{r} - \gamma} = L^{q+p-2,(q-1) \frac{2\sigma - \frac{1}{r}}{r} - \gamma} , \]
\[
(5.57) \quad \frac{u^{q-1}}{r \gamma} \leq L^{q-1,(q-1) \frac{p-2}{r} \gamma} = L^{q-1,r(q-1) \frac{2\sigma - \frac{1}{r}}{r} } , \]
\[
(5.58) \quad u |\nabla u|^{p-2} \leq L^{p-1} , \]
where \( L = L(p, \Omega, \|u_0\|_{C^1}) > 0 \). Combining (5.54)-(5.58), we obtain
\[
\Theta(X) \leq \left[ k \left( pL^{q+p-2} + 2qL^q \right) \frac{r(q-1)(2\sigma - \frac{1}{r})+2}{1-rK} + \frac{q}{q-1} X^2 + \frac{\sigma}{2} X - \gamma(\gamma + 1) \right] \frac{1}{r^2} \]
\[
+ \left( \frac{\rho^2}{2\sigma} L^{p-1} - (p-1) q \right) \frac{|\nabla u|^p}{u} , \]
hence (5.10). \( \Box \)
6. Proof of Theorem 2.5

Proof. (i) We shall produce suitable initial data by means of Proposition 4.2. Fix \( \phi \in C^\infty([0,1]) \) such that \( \phi = 1 \) on \([0,1]\), \( \phi = 0 \) on \([3/2, \infty)\) and \( \phi' \leq 0 \). Take \( \rho > 0 \) so small that

\[
B_\rho(0,0) \cap \partial \Omega \subset \gamma \left( -\frac{s_0}{2}, \frac{s_0}{2} \right).
\]

Let \( C_1, C_2 \) be given by Proposition 4.2 pick any \( \varepsilon \in (0, \rho/4) \) such that \( C_1 \varepsilon^k < C_2 \) and set

\[
u_0(x,y) = C_2 \phi \left( \frac{\sqrt{x^2 + (y-\varepsilon)^2} - \sqrt{x^2 + (y+\varepsilon)^2}}{\varepsilon/2} \right).
\]

Then we immediately have (4.2) and \( \text{supp}(\nu_0) \subset B_\rho(0,\varepsilon) \subset B_{\rho/2}(0,0) \). Also, by taking \( \varepsilon > 0 \) possibly smaller, we get \( B_\rho(0,\varepsilon) \subset \Omega \), hence (6.1) and (6.2). It thus follows from Proposition 4.2 that \( T(\nu_0) < \infty \) and, in view of (6.1), that condition (2.20) is satisfied.

On the other hand, (2.10), and therefore \( u_{0,s} \leq 0 \) in (2.10), are clearly satisfied. Moreover, by considering \( \varepsilon > 0 \) possibly smaller, the reflection properties (2.13) and (2.19) hold trivially. In order to prove \( u_{0,s} < 0 \) in (2.10), we can use formula (3.9) to obtain

\[
\frac{u_{0,s}}{1-rK} = \frac{2C_2}{\varepsilon} \phi' \left( \frac{\sqrt{x^2 + (y-\varepsilon)^2} - \sqrt{x^2 + (y+\varepsilon)^2}}{\varepsilon/2} \right) \frac{\alpha'(y-\varepsilon)}{\sqrt{x^2 + (y-\varepsilon)^2}}.
\]

Then, in view of \( \phi' < 0 \) and the definition of the change of coordinates map \( (x,y) = (rN, s) = \gamma(s) + rN(s) \), it suffices to check that \( (\gamma', \gamma + rN - \varepsilon e_2) \geq 0 \) for all sufficiently small \( \varepsilon, s > 0 \). To do this, let us write the Taylor expansions

\[
\gamma'(s) = e_1 + sR_1(s), \quad \gamma(s) = se_1 + s^2R_2(s), \quad s > 0 \text{ is small,}
\]

where \( |R_1|, |R_2| \leq C_3 \) for some constant \( C_3 > 0 \). Using also \( N \perp \gamma' \), it follows that

\[
(\gamma', \gamma + rN - \varepsilon e_2) = (e_1 + sR_1(s), s(e_1 + sR_2(s)) - \varepsilon e_2) - \varepsilon e_2 \geq s(1 - C_3\varepsilon - 2C_3s - C_3^2s^2) \geq 0
\]

for all sufficiently small \( \varepsilon, s > 0 \).

(ii) The assertion will be derived as a consequence of Theorem 2.5. For this it suffices to establish the monotonicity properties (2.10). The proof is done in two steps.

**Step 1: Parabolic Inequality.** Consider the auxiliary function

\[
w = \frac{u_s}{1-rK} \quad \text{in } Q_T := \omega_0 \times [0,T).
\]

In view of (6.1), \( w \) satisfies

\[
w_t - \Delta w = a_w w + b_w \cdot \nabla w + \frac{K'}{(1-rK)^3} \frac{1}{\beta'(s)} u_x,
\]

with

\[
a_w = \frac{K^2}{(1-rK)^2} - \frac{pK}{1-rK} \frac{1}{|\nabla u|^{p-2} u_r} - \frac{K'}{(1-rK)^3} \frac{\alpha'(s)}{\beta'(s)},
\]

\[
b_w = p|\nabla u|^{p-2} \nabla u - \frac{2K}{1-rK} N(s).
\]

For any \( T' \in (0,T) \), we have \( \sup_{Q_{T'}} |\nabla u| < \infty \). Also, by hypothesis (2.11), \( 1-rK \) is bounded away from 0 in \( \omega_0 \). This, together with \( K' \geq 0, \alpha', \beta' > 0 \), implies

\[
\sup_{Q_{T'}} a_w < \infty.
\]
Since \( u_0 \geq 0 \) in \( \Omega \), by the strong maximum principle we have \( u > 0 \) in \( \Omega \times (0, T) \). Therefore, by Hopf’s lemma we get

(6.4) \[ u_\nu < 0 \quad \text{on } \partial \Omega \times (0, T), \]

where \( u_\nu \) is the derivative of \( u \) in the outward normal direction to the boundary. As consequence, by (2.12), we have

\[ u_x = \nu_x u_\nu \leq 0 \quad \text{on } (\partial \Omega \cap \{ x > 0 \}) \times (0, T). \]

By the symmetry of \( u_0 \) and \( \Omega \), we also have

\[ u_x = 0 \quad \text{on } [\Omega \cap \{ x = 0 \}] \times (0, T). \]

Now, we see that \( v = u_x \) satisfies

\[ v_t - \Delta v = p|\nabla u|^{p-2} \nabla u \cdot \nabla v \quad \text{in } [\Omega \cap \{ x > 0 \}] \times (0, T). \]

Then, after hypothesis (2.17) and the strong maximum principle, we have

(6.5) \[ u_x < 0 \quad \text{in } [\Omega \cap \{ x > 0 \}] \times (0, T). \]

Since \( \omega_0 \subset \Omega \cap \{ x > 0 \} \), it follows from (6.2), (6.5) and \( K' \geq 0, \beta' > 0 \) that

(6.6) \[ w_t - \Delta w - a_w w - b_w \cdot \nabla w \leq 0 \quad \text{in } Q_T. \]

**Step 2: Boundary conditions and conclusion.** We split the boundary of \( \omega_0 \) in five parts:

\[
\begin{align*}
\Gamma_1 &= \{(\alpha(s), \beta(s)); \quad 0 < s < s_0\}, \\
\Gamma_2 &= \Omega \cap \{ x = 0 \}, \\
\Gamma_3 &= \partial \Omega \cap \partial \omega_0 \cap \{ r > 0 \}, \\
\Gamma_4 &= \Omega \cap \Lambda_{s_0}, \\
\Gamma_5 &= \Omega \cap D_{\Gamma} \cap \{ y = y_0 \}.
\end{align*}
\]

See Figures 3 and 4 for illustrations of such partitions. Note that \( \Gamma_3 \) and/or \( \Gamma_5 \) may be empty. In that case, we need not to care about them.

\[ \Gamma_1 \]
\[ \Gamma_2 \]
\[ \Gamma_3 \]
\[ \Gamma_4 \]
\[ \Gamma_5 \]

\{y = y_0\}

\[ \omega_0 \]

\Lambda_{s_0}

\text{Figure 3. Illustration of the partition of } \partial \omega_0. \text{ In this case } \Gamma_3 = 0. \]

Since \( u = 0 \) on \( \partial \Omega \), we have

(6.7) \[ u_s = 0 \quad \text{on } \Gamma_1 \times [0, T). \]
By the symmetry of the domain, and using (3.9), we have $u_s = (1 - r K) u_x$ on $\Gamma_2$, and by (2.10), we deduce

$$u_s = 0 \quad \text{on } \Gamma_2 \times [0, T).$$

On the other hand, as consequence of (6.4), (2.12) and (2.13), we have

$$u_x = \nu_x u_\nu \leq 0, \quad u_y = \nu_y u_\nu \leq 0 \quad \text{on } \Gamma_3 \times [0, T).$$

Now, we recall from (3.9)

$$u_s = (1 - r K)(\alpha'(s) u_x + \beta'(s) u_y)$$

Then it follows from (2.5) that

$$u_s \leq 0 \quad \text{on } \Gamma_3 \times [0, T).$$

Next, we shall prove by a moving planes argument that

$$u_s \leq 0 \quad \text{on } \Gamma_4 \times [0, T).$$

We define in $\Omega_{s_0} \times (0, T)$ the functions

$$u_1(x, y, t) = u(x, y, t), \quad u_2(x, y, t) = u(T_{s_0}(x, y), t),$$

where $\Omega_{s_0}$ and $T_{s_0}$ are defined in Notation (2.4). We note that $u_2$ is well defined since by condition (2.14), $T_{s_0}(x, y) \in \Omega$, for all $(x, y) \in \Omega_{s_0}$. Both functions $u_1, u_2$ satisfy the equation

$$u_{i,t} - \Delta u_i = |\nabla u_i|^p \quad \text{in } \Omega_{s_0} \times (0, T),$$

for $i = 1, 2$. By condition (2.18), we have

$$u_1(x, y, 0) \leq u_2(x, y, 0) \quad \text{in } \Omega_{s_0}.$$

The boundary of $\Omega_{s_0}$ is composed of two parts:

$$\Gamma_{s_0}^1 := \partial \Omega \cap \partial \Omega_{s_0}, \quad \Gamma_{s_0}^2 := \Lambda_{s_0} \cap \Omega.$$

On $\Gamma_1^{s_0}$ we have $u_1(x, y, t) = 0$ and $u_2(x, y, t) \geq 0$ since $u \geq 0$ in $\Omega \times (0, T)$. On $\Gamma_2^{s_0}$ we have $u_1(x, y, t) = u_2(x, y, t)$, since $T_{s_0}(x, y) = (x, y)$, for all $(x, y) \in \Lambda_{s_0}$. So we conclude that $u_1(x, y, t) \leq u_2(x, y, t)$ on $\partial \Omega_{s_0} \times [0, T)$. As a consequence of the comparison principle, we get $u_1 \leq u_2$ in $\Omega_{s_0} \times [0, T)$. Letting $(x, y)$ go to $\Lambda_{s_0}$ in the normal direction to $\Lambda_{s_0}$, we deduce (6.11).
In order to show that
\begin{equation}
\tag{6.12}
\frac{\partial u}{\partial s} \leq 0 \quad \text{on } \Gamma_5 \times [0,T),
\end{equation}
we observe that, as a consequence of (2.15), (2.19) and of a similar moving planes argument as in the case of \(\Gamma_4\), we have
\begin{equation}
\tag{6.13}
\frac{\partial u}{\partial y} \leq 0 \quad \text{in } (\Omega \cap \{y = y_0\}) \times [0,T).
\end{equation}
Property (6.12) then follows from (6.5), (6.9), (6.13) and (2.5).

Then, since \(\partial \omega_0 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5\), it follows from (6.7), (6.8), (6.10)-(6.12) that
\begin{equation}
\tag{6.14}
\frac{\partial u}{\partial s} \leq 0 \quad \text{on } \partial \omega_0 \times (0,T).
\end{equation}
In view of (6.3), (6.6), (6.14), (2.17), it follows from the strong maximum principle that
\begin{equation}
\tag{6.15}
\frac{\partial u}{\partial s} < 0 \quad \text{in } \omega_0 \times (0,T).
\end{equation}
This, together with (6.5) and (2.20), allows us to apply Theorem 2.3, and the conclusion follows. \(\square\)

7. **Proof of Theorems 1.1 and 1.2**

Here we give the proofs of Theorems 1.1 and 1.2 as consequence of Theorem 2.5. We shall verify that the hypotheses of Theorem 2.5 hold for ellipses and for the domains satisfying the assumptions of Theorem 1.2.

**Proof of Theorem 1.1.** We only give the proof for ellipses with positive eccentricity. For disks, see [22]. Without loss of generality, we assume that the minor axis of the ellipse is on the half-line \(\{x = 0; \ y \geq 0\}\) and that the lower co-vertex is at the origin. Then, assumption (2.12) holds. If we consider \(\Gamma\) a connected boundary piece containing the origin and symmetric with respect to \(x = 0\), we can use Notation 2.1.

Now, take \(y_0 > 0\) such that the major axis of the ellipse is on the line \(y = y_0\). Hypothesis (2.15) is then satisfied. Moreover, in view of the position of the ellipse, it is well known that the center of curvature at any point of \(\partial \Omega \cap \{y < y_0\}\) lies in the half-plane \(\{y > y_0\}\). Considering \(s_0 > 0\) small enough so that \(\Gamma \subset \{y < y_0\}\), it follows that conditions (2.4), (2.5), (2.11) and (2.13) are satisfied.

Now let us verify that (2.14) also holds for this choice of \(\Gamma\). Here, we recall the definitions of \(H_{s_0}\) and \(\Lambda_{s_0}\) in Notation 2.4. We shall prove that the symmetric of \(\partial \Omega \cap H_{s_0}\) with respect to \(\Lambda_{s_0}\) lies in \(\Omega\), which guarantees (2.14) by convexity.

Let \(\partial \Omega\) be the original ellipse and \(T_{s_0}(\partial \Omega)\) its symmetric with respect to the line \(\Lambda_{s_0}\). We observe that the two ellipses intersect in at least two points, which are the two intersection points of \(\partial \Omega\) with \(\Lambda_{s_0}\). We also know that any two ellipses intersect in at most four points, counting the multiplicity. Since \(\Lambda_{s_0}\) is normal to \(\partial \Omega\) at \(\gamma(s_0)\), the two ellipses \(\partial \Omega\) and \(T_{s_0}(\partial \Omega)\) are tangent to each other at that point, which is then an intersection point of multiplicity at least 2.

Therefore, there can be at most one other intersection point between the two ellipses. By convexity, it cannot be on the segment \(\Lambda_{s_0} \cap \Omega\), and by symmetry with respect to the line \(\Lambda_{s_0}\), if there is an intersection point on one side of \(\Lambda_{s_0}\), there must be another one on the other side. Hence, the two ellipses only intersect in two points.
Finally, since the curvature of \( \partial \Omega \) increases from the origin up to the right vertex, near \( \gamma(s_0) \), the symmetric of \( \partial \Omega \cap H_{s_0} \) lies in \( \Omega \). As we have seen, it does not intersect again the boundary of \( \Omega \) until the other intersection of \( \Lambda_{s_0} \) with \( \partial \Omega \). Therefore, we conclude that \( T_{s_0}(\partial \Omega \cap H_{s_0}) \subset \Omega \). Hence, \( \Omega \) satisfies all the hypothesis of Theorem 2.5 and the conclusion follows.

**Proof of Theorem 1.2.** We give the proof for the case when \( \Omega \) is not locally flat at the origin. That is, in assumption (1.4) \( \partial \Omega \) only touches \( y = 0 \) at the origin. For locally flat domains, see [22].

As in the proof of Theorem 1.1 we shall verify that all the hypotheses of Theorem 2.5 hold. In view of assumptions (1.3) and (1.4), and considering a suitable boundary piece, we can use Notation 2.1 and hypothesis (2.12) is satisfied. By taking a smaller \( \Gamma \) if necessary, hypotheses (2.4), (2.5) and (2.14) are guaranteed by assumptions (1.5) and (1.6).

The assumption \( \Omega \subset \{ y < R(0) \} \) in (1.5), implies that the center of curvature of the boundary at the origin is at positive distance of \( \Omega \) (possibly at infinity). Since the curvature is a continuous function due to the regularity of the boundary, considering a smaller \( \Gamma \) if necessary, the evolute of \( \Gamma \) is also at positive distance of \( \Omega \). Therefore, hypothesis (2.11) is satisfied with \( y_0 = +\infty \), and then (2.15) is trivial.

As for hypothesis (2.13), we note that in view of (1.3) and since \( \Omega \) is smooth and connected, \( \Omega \cap \{ x = \eta \} \) is a segment for all \( \eta > 0 \) small. Therefore, (2.13) holds by considering a possibly smaller \( \Gamma \).

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**References**

[1] N. Alaa, Weak solutions of quasilinear parabolic equations with measures as initial data. *Annales Mathématiques Blaise Pascal* 3, no. 2 (1996): 1-15.

[2] N.D. Alikakos, P.W. Bates, C.P. Grant, Blow up for a diffusion-advection equation. *Proceeding of the Royal Society of Edinburgh*, Section A 113, no. 3-4 (1989): 181-90.

[3] L. Amour, M. Ben-Artzi, Global existence and decay for viscous Hamilton-Jacobi equations. *Nonlinear Analysis* 31 (1998): 621-28.

[4] J.M. Arrieta, A. Rodríguez-Bernal, Ph. Souplet, Boundedness of global solutions for nonlinear parabolic equations involving gradient blow-up phenomena. *Annali della Scuola Normale Superiore di Pisa*, Classe di Scienze (5)3, no. 1 (2004): 1-15.

[5] A. Attouchi, Ph. Souplet, Single point gradient blow-up on the boundary for a Hamilton-Jacobi equation with \( p \)-Laplacian diffusion, *Trans. Amer. Math. Soc.* 369 (2017), 935-974.

[6] G. Barles, F. Da Lio, On the generalized Dirichlet problem for viscous Hamilton-Jacobi equations. *Journal de Mathématiques Pures et Appliquées* 83 (2004): 53-75.

[7] S. Benachour, G. Karch, Ph. Laurençot, Asymptotic profiles of solutions to viscous Hamilton-Jacobi equations. *Journal de Mathématiques Pures et Appliquées* 83 (2004): 1275-308.

[8] S. Benachour, Ph. Laurençot, Global solutions to viscous Hamilton-Jacobi equations with irregular data. *Communications in Partial Differential Equations* 24 (1999): 1999-2021.

[9] M. Ben-Artzi, Ph. Souplet, F.B. Weissler, The local theory for viscous Hamilton-Jacobi equations in Lebesgue spaces. *Journal de Mathématiques Pures et Appliquées* (9) 81, no. 4 (2002): 343-78.

[10] G.R. Conner, C.P. Grant, Asymptotics of blowup for a convection-diffusion equation with conservation, *Differential Integral Equations* 9 (1996), 719-728.

[11] M. Filas, G.M. Lieberman, Derivative blow-up and beyond for quasilinear parabolic equations. *Differential and Integral Equations* 7, no. 3-4 (1994): 811-21.

[12] A. Friedman, Partial Differential equations of parabolic type, Englewood cliffs, NJ: Prentice-Hall, 1964.
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[13] A. Friedman, B. McLeod, Blow-up of positive solutions of semilinear heat equations, Indiana Univ. Math. J. 34 (1985), 425-447.

[14] B.H. Gilding, The Cauchy problem for $u_t = \Delta u + |\nabla u|^q$, large-time behaviour. Journal de Mathématiques Pures et Appliquées (9) 84, no. 6 (2005): 753-85.

[15] B.H. Gilding, M. Guedda, R. Kersner, The Cauchy problem for $u_t = \Delta u + |\nabla u|^q$, Journal of Mathematical Analysis and Applications 284 no. 2 (2003): 733-55.

[16] J.-S. Guo, B. Hu, Blowup rate estimate for the heat equation with a nonlinear gradient source term, Discrete Contin. Dyn. Syst. 20 (2008), 927-937.

[17] T. Halpin-Healy, Y.C. Zhang, Kinetic roughening phenomena, stochastic growth, directed polymers and all that, Aspects of multidisciplinary statistical mechanics. Phys. Rev. 254, 215-414 (1995).

[18] M. Hesaaraki, A. Moameni, Blow-up positive solutions for a family of nonlinear parabolic equations in general domain in $\mathbb{R}^N$. Michigan Mathematical Journal 52, no. 2 (2004): 375-89.

[19] M. Kardar, G. Parisi, Y.C. Zhang, Dynamic scaling of growing interfaces. Phys. Rev. Lett. 56(9), 889-892(1986).

[20] J. Krug, H. Spohn, Universality classes for deterministic surface growth. Phys. Rev. A 38, 4271-4283 (1988).

[21] Ph. Laurençot, Ph. Souplet, On the growth of mass for a viscous Hamilton-Jacobi equation. Journal d’Analyse Mathématique 89 (2003): 367-83.

[22] Y.-X. Li, Ph. Souplet, Single-Point Gradient Blow-up on the Boundary for Diffusive Hamilton-Jacobi Equations in Planar Domains, Comm. Math. Phys. 293 (2009), 499-517.

[23] P.-L. Lions, Generalized solutions of Hamilton-Jacobi equations. Research Notes in Mathematics, 69. Boston, MA: Pitman (Advanced Publishing Program), 1982.

[24] A. Porretta, Ph. Souplet, The profile of boundary gradient blow-up for the diffusive Hamilton-Jacobi equation, International Math. Research Notices 17 (2017), 5260-5301.

[25] A. Porretta, Ph. Souplet, Analysis of the loss of boundary conditions for the diffusive Hamilton-Jacobi equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 34 (2017), 1913-1923.

[26] A. Porretta, Ph. Souplet, Blow-up and regularization rates, loss and recovery of boundary conditions for the superquadratic viscous Hamilton-Jacobi equation, J. Math. Pures Appl., to appear (Preprint arXiv 1811.01612).

[27] A. Porretta, E. Zuazua, Null controllability of viscous Hamilton-Jacobi equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 29 (2012), 301-333.

[28] A. Quaas, A. Rodríguez, Loss of boundary conditions for fully nonlinear parabolic equations with superquadratic gradient terms, J. Differential Equations 264 (2018), 2897-2935.

[29] P. Quittner, Ph. Souplet, Superlinear parabolic problems. Blow-up, global existence and steady states. Birkhäuser Advanced Texts, 2007.

[30] Ph. Souplet, Gradient blow-up for multidimensional nonlinear parabolic equations with general boundary conditions. Differential and integral equations 15, no. 2 (2002): 237-56.

[31] Ph. Souplet, A remark on the large-time behavior of solutions of viscous Hamilton-Jacobi equations. Acta Mathematica Universitatis Comenianae (N.S.) 76 (2007): 11-13.

[32] Ph. Souplet, J.L. Vázquez, Stabilization towards a singular steady state with gradient blow-up for a diffusion-convection problem. Discrete and Continuous Dynamical Systems 14, no. 1 (2006): 221-34.

[33] Ph. Souplet, Q.S. Zhang, Global solutions of inhomogeneous Hamilton-Jacobi equations, J. Anal. Math. 99 (2006), 355-396.

[34] Z.-C. Zhang, Z. Li, A note on gradient blowup rate of the inhomogeneous Hamilton-Jacobi equations, Acta Math. Sci. Ser. B (Engl. Ed.) 33 (2013), 678-686.

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