RANK OF JACOBI OPERATOR AND EXISTENCE OF QUADRATIC PARALLEL DIFFERENTIAL FORM, WITH APPLICATIONS TO GEOMETRY OF ALMOST PARA-CONTACT METRIC MANIFOLDS

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Abstract. It is established that non-isotropic vector field with Jacobi operator of maximal rank is an obstacle for the existence of non-trivial second-order symmetric parallel tensor field. It follows that such manifold as pseudo-Riemannian manifold is locally non-reducible. In particular result is applied to widely studied classes of almost para-contact metric manifolds - para-contact metric, para-cosymplectic or para-Kenmotsu manifolds satisfying nullity and generalized nullity conditions. As corollary we have the following theorem: almost para-contact metric manifold with maximal rank Jacobi operator of characteristic vector field is locally non-isometric to Riemann product.

1. Introduction

Many authors has recently studied the problem of the existence of non-trivial parallel quadratic form on almost para-contact metric manifold which satisfies generalized nullity conditions. Let us recall it is said that almost para-contact metric manifold satisfies generalized nullity condition if

\[ R_{XY}\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)Y) + \nu(\eta(Y)h'X - \eta(X)h'Y), \]

\[ R \] is the operator of the Riemann curvature, and \( \kappa, \mu, \nu \) are functions, \( \xi \) is the characteristic vector field. The more elaborate terminology is that characteristic vector field belongs to generalized nullity distribution. For the studied classes of manifolds the results are that in generic case there is no parallel quadratic form, different from metric tensor up to non-zero multiplier. The proofs of these results are based on properties of Jacobi operator \( X \mapsto J_\xi = R_X\xi \), particularly its algebraic form. However more careful analysis shows that, for above mentioned generic cases, \( J_\xi \) has maximal rank. Going into this direction we have

**Proposition 1.** There is no non-trivial parallel quadratic form if Jacobi operator of characteristic vector field has maximal rank. Manifold as Riemannian manifold is locally irreducible.

The latter sentence comes from the fact that Riemann product always admit non-trivial parallel differential quadratic form.

Let’s denote by \( \chi(\xi, x) \) the characteristic polynomial of the Jacobi operator

\[ \chi(\xi, x) = x^n - \omega_1(\xi)x^{n-1} + \ldots (\pm 1)^{n-1}\omega_{n-1}(\xi)x, \]

we will show the following
Proposition 2. Assuming $\xi$ is non-isotropic, $J_\xi$ has maximal rank if and only if the coefficient $\omega_{n-1}$ at the lowest power term is non-zero $\omega_{n-1} \neq 0$.

Of course this assumption is always satisfied for characteristic vector field on almost para-contact metric manifold. Yet our results are remain valid in wider framework of pseudo-Riemannian manifolds. Then fact that $\xi$ is non-isotropic is essential.

Author would like to express his gratitude to Professor Quanxiang Pan for our fruitfull discussion.

2. Preliminaries

Let $(M, g)$ be pseudo-Riemannian $n$-dimensional manifold, $\nabla$ denote the Levi-Civita connection, and $R_{XY}Z$ its curvature
\begin{equation}
R_{XY}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z,
\end{equation}
it is assumed that letters $U, V, X, ...$ are used to denote vector fields, if it is not stated otherwise. Our convention for the Riemann covariant curvature tensor is
\begin{equation}
R(X, Y, Z, W) = g(R_{XY}Z, W).
\end{equation}
For vector field $\xi$ on $M$, $(1, 1)$-tensor field $X \mapsto J_\xi X = R_{X\xi} \xi$, is called Jacobi operator. From the properties of the curvature $J_\xi$ is $g$-self-adjoint and upper limit of its rank is
\[ g(J_\xi X, Y) = g(J_\xi Y, X), \quad r < \dim M. \]
If $g$ is definite, $J_\xi$ is semi-simple: eigenvalues are real and tangent space splits into orthogonal direct sum of corresponding eigen-spaces.

Given Riemannian manifold $M$, A. Gray [13], considered so-called $k$-nullity distribution $N_k$, $k = const.$, distribution where the curvature has algebraic form as curvature of constant sectional curvature manifold
\begin{equation}
R_{XY}Z = k(g(Y, Z)X - g(X, Z)Y),
\end{equation}
where $Z$ is section of $N_k$.

2.1. Almost para-contact metric manifolds. ([1], [16])

Let $M$ be $(2n + 1)$-dimensional manifold. Almost para-contact metric structure $(\varphi, \xi, \eta, g)$ is quadruple of tensor fields: $(1, 1)$-tensor field (affinor) $\varphi$, characteristic (or Reeb) vector field $\xi$, characteristic 1-form $\eta$, and pseudo-Riemannian metric $g$, such that, $\varepsilon = \pm 1$,
\begin{align*}
\varphi^2 &= \varepsilon(Id - \eta \otimes \xi), \quad \eta(\xi) = 1, \\
g(\varphi X, \varphi Y) &= \varepsilon(g(X, Y) - \eta(X)\eta(Y)).
\end{align*}

For $\varepsilon = -1$, structure is customary called almost contact metric. Eigenvalues of $\varphi$ are imaginary, spectrum contains $\{0, -i, i\}$. It is assumed that the metric is strictly positive. The latter condition implies that eigenvalues $-i, i$ have the same multiplicity $n$. The triple $(\varphi, \xi, \eta)$ is called almost contact structure.

For $\varepsilon = +1$, structure is called almost para-contact metric. Eigenvalues of $\varphi$ are real, the spectrum is $\{-1, 0, +1\}$. Tangent space decomposes into direct sum of one-dimensional kernel, and $n$-dimensional eigen-spaces $\mathcal{V}(\pm 1)$. From definition it
follows that restriction of the metric to any of eigen-space is null tensor, and \( \mathcal{V}(\pm 1) \) are maximal in dimension isotropic subspaces. Signature of \( g \) is
\[
\begin{bmatrix}
-1, \ldots, -1, +1, \ldots, +1
\end{bmatrix}_{n+1}
\]
Operators \( P_\pm = \phi \pm \text{Id} \), are orthogonal projectors \( P_\pm^2 = P_\pm \), \( P_+ P_- = P_- P_+ = 0 \), onto eigen-spaces \( \mathcal{V}(\pm 1) \). We have \( \text{Im} P_\pm = \mathcal{V}(\mp 1) \). In particular \( g(P_\pm X, P_\mp Y) = 0 \), \( g(P_\pm X, P_\mp Y) = g(X, Y) \), for \( \eta(X) = \eta(Y) = 0 \). Per analogy the triple \( (\phi, \xi, \eta) \), is called almost para-contact structure.

From now on we adopt in this paper the convention where both almost contact metric and almost para-contact metric structures are all together called almost para-contact metric manifolds. We just follow the line where some authors use term pseudo-Riemannian manifold in wider sense: manifold equipped with non-degenerate quadratic differential form. So the reader should be aware of this.

For almost para-contact metric structure tensor field \( \Phi(X, Y) = g(X, \phi Y) \), is skew-symmetric form called fundamental form. It satisfies
\[
\eta \wedge \Phi \neq 0,
\]
at every point, so \( \Phi \) has maximal rank everywhere, its kernel is spanned by characteristic vector field \( \xi \).

Manifold equipped with almost para-contact metric structure is called almost para-contact metric manifold. Such manifold is always orientable.

An important notion is normality. Almost para-contact metric manifold is called normal if
\[
[\phi, \phi](X, Y) - 2\varepsilon \, d\eta \otimes \xi = 0,
\]
where \([\phi, \phi] \) denotes Nijenhuis torsion of \( \phi \)
\[
[\phi X, \phi Y] = \phi^2[X, Y] + [\phi X, \phi Y] - \phi([\phi X, Y] + [X, \phi Y]).
\]

Non-degenerate hypersurface of almost para-Hermitian manifold can be equipped with almost para-contact metric structure. Thus such hypersurfaces are one of the fundamental examples of almost para-contact metric manifolds.

We just mention some classes of almost para-contact metric manifolds.

**Definition 1.** \((1)\), \((4)\), \((9)\). Almost para-contact metric manifold \((M, \phi, \xi, \eta, g)\) is called

1. **para-contact metric**
   \[d\eta = \Phi,\]
2. **almost para-cosymplectic (or almost para-coKahler)**
   \[d\eta = 0, \quad d\Phi = 0,\]
3. **almost para-Kenmotsu**
   \[d\eta = 0, \quad d\Phi = 2\eta \wedge \Phi.\]

Assuming additionally normality we obtain following classes of manifolds: para-Sasakian\(^1\), para-cosymplectic (or para-coKahler) and para-Kenmotsu.

Let define
\[
h = \frac{1}{2} \mathcal{L}_\xi \phi, \quad h' = h \circ \phi,
\]

\(^1\)Contact metric and normal, etc
$\mathcal{L}_\xi$ denotes the Lie derivative along $\xi$. Applying $\mathcal{L}_\xi$ to identity $\varphi_\xi = 0$, we obtain $h_\xi = 0$ ($h'_\xi = 0$ is evident). For given appropriate functions $\kappa, \mu, \nu$ - the choice depends on the structure in question - almost para-contact metric manifold, such that

$$R_{XY} \xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) + \nu(\eta(Y)h'X - \eta(X)h'Y),$$

is called $(\kappa, \mu, \nu)$-space or $(\kappa, \mu, \nu)$-almost para-contact metric manifold, etc. here authors adopt different naming conventions. The Jacobi operator of characteristic vector field on almost contact metric $(\kappa, \mu, \nu)$-space is

$$J_\xi : X \mapsto R_{X\xi} \xi = -\kappa \eta(X)\xi + (\kappa \text{Id} + \mu h + \nu h')X,$$

cf. [2], [3], [6], [9]. Note that $J_\xi$ has maximal rank if and only if

$$J_\xi|_{\eta = 0} = \kappa \text{Id} + \mu h + \nu h',$$

is invertible on $\eta = 0$.

3. Symmetric parallel tensors of pseudo-Riemannian manifolds

The goal of this section is to prove the following result.

**Proposition 3.** Let $(M, g)$ be $(n + 1)$-dimensional pseudo-Riemannian manifold. Let assume there is non-isotropic vector field $\xi$, $g(\xi, \xi) \neq 0$, with Jacobi operator of maximal rank. If $\alpha \neq 0$ is parallel differential quadratic form, then it is proportional to metric tensor $\alpha = cg$.

**Proof.** We may assume $g(\xi, \xi) = \epsilon = \pm 1$. So there is $(0, 2)$-tensor $\alpha(X, Y) = \alpha(Y, X)$ symmetric and parallel $\nabla \alpha = 0$, such that our quadratic differential form is just $X \mapsto \alpha(X, X)$. Moreover by assumption $r = \dim(\text{Im} J_\xi) = n$. We set

$$i_\xi \alpha(X) = \alpha(\xi, X), \quad i_\xi g(X) = g(\xi, X).$$

Let consider the case $i_\xi \alpha \neq 0$. For $\alpha$ and $g$ are both parallel

$$i_\xi \alpha(J_\xi \cdot) = 0, \quad i_\xi g(J_\xi \cdot) = 0,$$

equation{13}

to the both sides of the identity

$$\alpha(J_\xi Y, X) = \alpha(\xi, \xi)g(\xi, X),$$

equation{15}

we have to take into account that by assumption $\alpha$ is covariant constant. Subtracting resulting equations, having in mind that $R_{Y \xi} \xi = \nabla^2_{Y, \xi} - \nabla^2_{\xi, Y}$, we obtain

$$\alpha(J_\xi Y, X) = c \alpha(\xi, \xi)g(J_\xi Y, X),$$

equation{16}

which by maximality follows

$$\alpha = cg, \quad c = c \alpha(\xi, \xi) \neq 0,$$

equation{17}

and by $0 = \nabla \alpha = dc \otimes g$, we have $c = \text{const}$. 

In the case \( i_{\xi} \alpha = 0 \), in the similar way as above we find \( \alpha(X, Y) = 0 \), \( g(\xi, Y) = 0 \). For \( \xi \) is non-isotropic and \( i_{\xi} \alpha = 0 \), \( \alpha \) must vanish which contradicts our assumption \( \alpha \neq 0 \). Hence the case \( i_{\xi} \alpha = 0 \) is not possible. \( \square \)

3.1. Trace form. Let \( M \) be \((n+1)\)-dimensional pseudo-Riemannian manifold. Let recall formula for characteristic polynomial

\[
\chi(\xi, x) = \det(xI - J_\xi) = x^{n+1} - \omega_1(\xi)x^n + \ldots + (-1)^n\omega_n(\xi)x.
\]

Note that \( \omega_i \)’s as a functions

\[
: \xi \mapsto \omega_i(\xi)
\]

are all smooth differential forms. Indeed, let \( J_\xi^i \) denotes the \( i \)-th exterior power of \( J_\xi \). It is \((i, i)\)-tensor field - interpreted as endomorphism acting on \( i \)-th degree polivectors on \( M \). By definition on simple polivector \( W = V_1 \wedge \ldots \wedge V_i \)

\[
J_\xi^i(V_1 \wedge \ldots \wedge V_i) = J_\xi(V_1) \wedge \ldots \wedge J_\xi(V_i),
\]

then \( \omega_i = (-1)^ntr(J_\xi^i) \). From the definition of Ricci tensor we have \( Ric(\xi, \xi) = \omega_1(\xi) \). So we may think of \( \omega_i, i > 1 \), as a kind of Ricci tensors of higher degrees.

Let us recall the metric tensor gives rise to canonical symmetric bilinear form on polivectors: we denote it by \( g^{\wedge k} \)

\[
g^{\wedge k}(X_1 \wedge \ldots \wedge X_k) = \det[g(X_i, X_j)].
\]

The metric \( g^{\wedge k} \) and \( J_\xi^k \) are compatible in the sense that the latter is \( g^{\wedge k}\)-self-adjoint.

**Proposition 4.** Let assume \( J_\xi^n \neq 0 \). Then \( J_\xi^n \) has rank one. In particular there is \( n \)-form \( \tau_\xi \) and \( n \)-multivector \( W_\xi \), such that

\[
J_\xi^n = \tau_\xi \otimes W_\xi, \quad \omega_n(\xi) = \tau_\xi(W_\xi),
\]

and \( \omega_n(\xi) = 0 \) if and only if \( \xi \) is isotropic.

**Proof.** Let extend \( \xi \) to local frame \((\xi, X_1, \ldots, X_n)\), then

\[
W_i = \xi \wedge X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge X_n, \quad 1 \leq i \leq n,
\]

span the kernel of \( J_\xi^n \), we see that dimension of kernel is \( n \), so if \( J_\xi^n \neq 0 \) its image is one-dimensional subspace, so there is polivector \( W \) and \( n \)-form \( \tau \), such that

\[
J_\xi^n = \tau \otimes W, \quad \omega_n = tr(J_\xi^n) = \tau(W),
\]

\( \tau \) and \( W \) are determined up to re-scaling

\[
\tau \mapsto f\tau, \quad W \mapsto f^{-1}W,
\]

where \( f \) is arbitrary non-vanishing function.

Condition \( tr(J_\xi^n) = \tau(W) = 0 \) means that \( W \) itself belongs to the kernel of \( J_\xi^n \). Every element of the kernel is linear combination of polivectors as in (21), therefore \( W \) has decomposition

\[
W = \xi \wedge W_0,
\]

For some simple polivector \( X_1 \wedge \ldots \wedge X_n \) we have

\[
J_\xi^n(X_1 \wedge \ldots \wedge X_n) = W,
\]

Then

\[
\xi \wedge W_0 = W = J_\xi^n(X_1 \wedge \ldots \wedge X_n) = (R_{X_1\xi}\xi) \wedge \ldots (R_{X_n\xi}\xi),
\]
which follows that up to reoder
\begin{equation}
R_{X_{1}}\xi = c_0\xi + c_1X_1 + \ldots c_nX_n, \quad c_0 \neq 0,
\end{equation}
from other hand
\begin{equation}
0 = g(R_{X_{1}}\xi, \xi) = c_0 g(\xi, \xi) + \sum_{i=1}^{n} c_i g(X_i, \xi),
\end{equation}
If $g(\xi, \xi) \neq 0$, we would take all $X_1, \ldots X_n$, such that $g(X_i, \xi) = 0$ then from the above equation we will have $c_0 = 0$ - contradiction. Hence $g(\xi, \xi) = 0$. \hfill \Box

Note Jacobi operator has maximal rank if and only if its exterior power $J^n_\xi \neq 0$.

**Corollary 1.** Jacobi operator of non-isotropic vector field has maximal rank if and only if coefficient at lowest term of its characteristic polynomial is non-zero.

Note there are non-trivial parallel quadratic differential forms on Riemannian products.

**Corollary 2.** Let $(M, g)$ be pseudo-Riemannian manifold. If every point admits locally defined vector field with maximal rank Jacobi operator, then $M$ is locally irreducible.

4. **Applications to Almost Para-Contact Metric Manifolds**

Applications are direct. Assume $(M, \varphi, \eta, g)$ is almost para-contact metric manifold.

**Proposition 5.** If Jacobi operator of characteristic vector field has maximal rank, then non-zero parallel second order differential form is proportional to pseudo-length form. In particular manifold is locally irreducible.

In case of $\kappa$-nullity spaces, $\kappa \neq 0$, we just rephrase above result.

**Corollary 3.** Non-zero parallel second order differential form on $\kappa$-nullity almost para-contact metric manifold is proportional to pseudo-length form provided $\kappa \neq 0$.

The case of $(\kappa, \mu)$-spaces requires study of singular values of operator $\kappa Id + \mu h$. Let denote by $\omega(x) = \sum_{i=0}^{\infty} c_i x^{n-i}$, $c_0 = 1$, the characteristic polynomial of $h$. Those values are solutions $(\kappa, \mu)$ of polynomial equation
\begin{equation}
\text{det}(\kappa Id + \mu h) = \sum_{i=0}^{\infty} c_i \kappa^{n-i} \mu^i = 0,
\end{equation}
are singular values.

The case of almost contact metric manifold is simpler due to fact that $h$ is diagonalizable. Denoting by $\{\lambda_1, \ldots \lambda_k\}$ spectrum of $h$, we see that $(\kappa, \mu)$ are singular if they satisfy one of the equations
\begin{equation}
\kappa + \lambda_i\mu = 0, \quad i = 1, \ldots k.
\end{equation}

Let have a look at some particular classes of manifolds where we posses more detailed information concerning operator $h$.

**Example 1.** Almost Kenmotsu $(\kappa, \mu)$-nullity manifolds. There is strong result which asserts that $\kappa = -1$, and $\mu = 0$. So for such class of manifolds Jacobi operator of $\xi$ is of maximal rank. If we take instead $h' = h \circ \varphi$ (generalized $(\kappa, \mu)'$-nullity spaces), then $\kappa \leq -1$, for $\kappa = -1$, $h' = 0$ and for $\kappa < -1$, $\mu = -2$, eigenvalues of
$h'$ are 0, $\pm \sqrt{-k-1}$, from these conditions there is one-point singularity $(-2,-2)$. Beyond that pair of values vector field $\xi$ has maximal Jacobi operator.

**Example 2.** Contact metric $(\kappa,\mu)$-nullity spaces. For such manifold $\kappa \leq 1$, if $\kappa = 1$, then $h = 0$. For $\kappa < 1$, eigenvalues of $h$ are $0, \pm \sqrt{1-k}$ and singular values are pairs $(\kappa,\mu)$ which satisfy $\kappa^2 - (1-k)\mu^2 = 0$. So, beyond these points Jacobi operator has maximal rank. 

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2Minimal polynomial of $h$ is $x(x^2 - (1 - k))$