An update on Haiman’s conjectures

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Abstract. We revisit Haiman’s conjecture on the relations between characters of Kazhdan-Lusztig basis elements of the Hecke algebra over $S_n$. The conjecture asserts that, for purposes of character evaluation, any Kazhdan-Lusztig basis element is reducible to a sum of the simplest possible ones (those associated to so-called codominant permutations). When the basis element is associated to a smooth permutation, we are able to give a geometric proof of this conjecture. On the other hand, if the permutation is singular, we provide a counterexample.

1. Introduction

The group algebra $\mathbb{C}[S_n]$ admits a $q$-deformation called the Hecke algebra $H_n$, constructed as follows. Since every $w \in S_n$ can be written as a product of simple transpositions ($i, i + 1$), the group algebra $\mathbb{C}[S_n]$ can be described as the $\mathbb{C}$-algebra generated by $\{T_s\}$, where $s$ runs through all simple transpositions, with the relations

$$T_s^2 = 1$$  
for every simple transposition $s$,

$$T_s T_{s'} = T_{s'} T_s$$  
for every $s = (i, i + 1)$ and $s' = (j, j + 1)$ such that $|i - j| > 1$,

$$T_s T_{s'} T_s = T_{s'} T_s T_{s'}$$  
for every $s = (i, i + 1)$ and $s' = (j, j + 1)$ such that $|i - j| = 1$.

The algebra $H_n$ has the same generators as $\mathbb{C}[S_n]$ but with slightly different relations, although we abuse the notation and still write $T_s$ for these generators. Namely, $H_n$ is the $\mathbb{C}(q^\frac{1}{2})$-algebra\(^1\) generated by $\{T_s\}$, with the relations

$$T_s^2 = (q - 1)T_s + q$$  
for every simple transposition $s$,

$$T_s T_{s'} = T_{s'} T_s$$  
for every $s = (i, i + 1)$ and $s' = (j, j + 1)$ such that $|i - j| > 1$,

$$T_s T_{s'} T_s = T_{s'} T_s T_{s'}$$  
for every $s = (i, i + 1)$ and $s' = (j, j + 1)$ such that $|i - j| = 1$.

When $q = 1$, we recover the group algebra $\mathbb{C}[S_n]$. Since each $w \in S_n$ has a (non-unique) reduced expression $w = s_1 s_2 \ldots s_{\ell(w)}$ in terms of simple transpositions, the product

$$T_w := T_{s_1} T_{s_2} \ldots T_{s_{\ell(w)}}$$

is well defined, independent of the choice of reduced expression for $w$. Then as a $\mathbb{C}(q^\frac{1}{2})$-vector space, $(T_w)_{w \in S_n}$ is a basis of $H_n$.

To introduce the Kazhdan-Lusztig basis, we first define the Bruhat order of $S_n$: The length $\ell(w)$ of $w$ is the number of inversions of $w$, and given $z, w \in S_n$, we say that $z \leq w$ if for some (equivalently, for every) reduced expression $w = s_1 s_2 \ldots s_{\ell(w)}$ there exist $1 \leq i_1 < i_2 < \ldots < i_k \leq \ell(w)$ such that $z = s_{i_1} \ldots s_{i_k}$. Then letting $\iota$ denote the involution of $H_n$ given by

$$\iota: H_n \to H_n$$

$$q^\frac{k}{2} \mapsto q^{-\frac{k}{2}}$$

$$T_w \mapsto T_{\iota(w)}^{-1},$$

\(^1\)

Usually, the definition is over $\mathbb{Z}[q^\frac{1}{2}, q^{-\frac{1}{2}}]$.\)
the Kazhdan-Lusztig basis \( \{C'_w\}_{w \in S_n} \) of \( H_n \) is defined by the following properties:

\[
\begin{align*}
   t(C'_w) &= C'_w, \\
   q^{\ell(w)}C'_w &= \sum_{z \leq w} P_{z,w}(q)T_z,
\end{align*}
\]

(1a)

where \( P_{z,w}(q) \in \mathbb{Z}[q] \), \( P_{w,w}(q) = 1 \) and \( \deg(P_{z,w}) < \frac{(\ell(w) - \ell(z))}{2} \) for every \( z \neq w \). The existence of such a basis is proved in \([\text{KL79}]\) and the polynomials \( P_{z,w}(q) \) are called Kazhdan-Lusztig polynomials.

The Kazhdan-Lusztig elements and polynomials are closely related to the geometry of Schubert varieties in the flag variety. The flag variety \( \mathcal{B} \) is the projective variety parametrizing flags of vector subspaces of \( \mathbb{C}^n \), that is

\[
\mathcal{B} = \{ V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{C}^n; \dim_C(V_i) = i \}.
\]

We often abbreviate and write \( V_\bullet \) to denote \( V_1 \subset \ldots \subset V_n \). For each permutation \( w \), the relative Schubert variety \( \Omega_w \) and its open cell \( \Omega^o_w \) are defined as

\[
\begin{align*}
   \Omega_w := \{ (F_\bullet, V_\bullet); \dim V_i \cap F_j \geq r_{i,j}(w) \text{ for } i, j = 1, \ldots, n \} \subset \mathcal{B} \times \mathcal{B}, \\
   \Omega^o_w := \{ (F_\bullet, V_\bullet); \dim V_i \cap F_j = r_{i,j}(w) \text{ for } i, j = 1, \ldots, n \} \subset \mathcal{B} \times \mathcal{B},
\end{align*}
\]

where

\[
r_{i,j}(w) := |\{ k; k \leq i, w(k) \leq j \}|.
\]

Then \( \Omega_w = \bigcup_{z \leq w} \Omega^o_z \), where the disjoint union is taken over all permutations smaller than \( w \) in the Bruhat order of \( S_n \).

The Kazhdan-Lusztig polynomial \( P_{z,w}(q) \) measures the singularity of \( \Omega_w \) at \( \Omega^o_w \), in the sense that

\[
P_{z,w}(q) = \sum_i \dim H^i((IC_{\Omega_w})_p)q^i,
\]

where \( IC_{\Omega_w} \) is the intersection homology complex of \( \Omega_w \) and \( p \) is a point in \( \Omega^o_w \).

Note that not all conditions in Equation (1b) defining \( \Omega_w \) are necessary: The coessential set \( \text{Coess}(w) \) of \( w \) is the smallest set of pairs \( (i, j) \) such that

\[
\Omega_w = \{ (F_\bullet, V_\bullet); \dim V_i \cap F_j \geq r_{i,j}(w) \}.
\]

Equivalently, we have

\[
\text{Coess}(w) := \{ (i, j); w(i) \leq j < w(i+1), \; w^{-1}(j) \leq i < w^{-1}(j+1) \}.
\]

If a permutation \( w \) satisfies \( r_{i,j}(w) = \min(i,j) \) for every \( (i, j) \in \text{Coess}(w) \), we say that \( \Omega_w \) is defined by inclusions. Indeed, the condition \( \dim V_i \cap F_j = r_{i,j}(w) \) is equivalent to either \( V_i \subset F_j \) or \( F_j \subset V_i \).

If \( \Omega_w \) is defined by inclusions and for every \( (i_0, j_0), (i_1, j_1) \in \text{Coess}(w) \) with \( i_0 \leq j_0 \) and \( j_1 \leq i_1 \) we have that either \( j_0 \leq j_1 \) or \( i_1 \leq i_0 \), then we say that \( \Omega_w \) is defined by non-crossing inclusions.

Given \( w \in S_n \), it is well-known that the following conditions are equivalent:

(1) \( P_{w,w}(q) = 1 \),
(2) \( \Omega_w \) is smooth,
(3) \( \Omega_w \) is defined by non-crossing inclusions,
(4) \( w \) avoids the patterns 3412 and 4231.

**Definition 1.1.** A permutation satisfying any of the conditions above is called smooth, otherwise it is called singular.

If the inclusions defining \( \Omega_w \) are all of the form \( V_i \subset F_j \), that is, if \( i \leq j \) for every \( (i, j) \in \text{Coess}(w) \), we say that \( w \) is codominant. Codominant permutations are precisely the 312-avoiding permutations, and there is a natural bijection between codominant permutations and Hessenberg functions (or Dyck paths), that is, non-decreasing functions \( m: [n] \to [n] \) satisfying \( m(i) \geq i \) for \( i = 1, \ldots, n \). The codominant permutation \( w_m \) associated to \( m \) is the lexicographically greatest permutation satisfying \( w_m(i) \leq m(i) \) for all \( i \in [n] \) (see Figure 1A).
For codominant permutations \( w_{\mathbf{m}} \), the Schubert varieties are characterized by
\[
\Omega_{w_{\mathbf{m}}} = \{(V_\bullet, F_\bullet); V_i \subset F_{w_{\mathbf{m}}(i)}\}.
\]

The bijection between codominant permutations and Hessenberg functions can be extended to map from the set of smooth permutations to the set of Hessenberg functions. Indeed, for every smooth permutation \( w \), we can define a Hessenberg function \( \mathbf{m}_w \) as follows. Let \( I \subset [n] \) be the subset of indices \( i \) such that there exists \( j \geq i \) with either \( (i, j) \in \text{Coess}(w) \) or \( (j, i) \in \text{Coess}(w) \). We define \( \mathbf{m}_w \) by the conditions \( \mathbf{m}_w(i) = \mathbf{m}_w(i + 1) \) if \( i \notin I \) and \( \mathbf{m}_w(i) = j \) if \( i \in I \) and \( j \) is such that either \( (i, j) \) or \( (j, i) \) is in \( \text{Coess}(w) \). The non-crossing condition implies that \( \mathbf{m}_w \) is indeed an Hessenberg function and, if we enrich the set of Hessenberg functions with some extra datum (the datum where the inclusions change from \( V_i \subset F_j \) to \( F_i \subset V_j \)) we can achieve a bijection, see [GL20].

We now turn our attention to characters of the Hecke algebra. Each irreducible \( \mathbb{C} \)-representation of \( S_n \) lifts to an irreducible \( \mathbb{C}(q^{\frac{1}{2}}) \)-representation of \( H_n \) (see [GP00, Theorem 8.1.7]). Hence, if \( \chi^\lambda \) is the irreducible character of \( S_n \) associated to the partition \( \lambda \vdash n \) and, abusing notation, \( \chi^\lambda \) is the corresponding character of \( H_n \), we can define the (dual) Frobenius character of an element \( a \in H_n \) by
\[
\text{ch}(a) := \sum_{\lambda \vdash n} \chi^\lambda(a)s_\lambda(x) \in \mathbb{C}(q^{\frac{1}{2}}) \otimes \Lambda,
\]
where \( \Lambda \) is the algebra of symmetric functions in the variables \( x = (x_1, \ldots, x_m, \ldots) \) and \( s_\lambda(x) \) is the Schur symmetric function associated to the partition \( \lambda \). For a graded \( S_n \)-module \( L \) we also write \( \text{ch}(L) \) for its (graded) Frobenius character.

In [Hai93, Lemma 1.1] Haiman proved that \( \chi^\lambda(q^{\frac{1}{2}}C_w') \) is a symmetric unimodal polynomial in \( q \) with non-negative integer coefficients. We note that [Hai93, Lemma 1.1] implies that \( \text{ch}(q^{\frac{1}{2}}C_w') \) is Schur-positive, in the sense that its coefficients in the Schur-basis are polynomials in \( q \) with non-negative integer coefficients.

Haiman also made some conjectures regarding positivity of the characters \( \text{ch}(q^{\frac{1}{2}}C_w') \) and relations between them. A symmetric function in \( \mathbb{C}(q^{\frac{1}{2}}) \otimes \Lambda \) is called \( h \)-positive if its coefficients in the complete homogeneous basis \( \{h_\lambda\} \) are polynomials in \( q \) with non-negative coefficients.

**Conjecture 1.2** (Haiman). For any \( w \in S_n \) the (dual Frobenius) character \( \text{ch}(q^{\frac{1}{2}}C_w') \) of the Kazhdan-Lusztig element \( C_w' \) is \( h \)-positive.

If \( \mathbf{m} \) is a Hessenberg function and \( G_{\mathbf{m}} \) the associated indifference graph, we have by [CHSS16] (see also Corollary 3.6 below) that the character \( \text{ch}(q^{\frac{1}{2}}C_{w_{\mathbf{m}}}') \) is the omega-dual of the chromatic quasisymmetric function of \( G_{\mathbf{m}} \). In particular, Conjecture 1.2 implies the Stanley-Stembridge conjecture on \( e \)-positivity of the chromatic symmetric function of indifference graphs of \( 3+1 \) free posets.
(via results of Guay-Paquet, [GP13]) and the Shareshian-Wachs generalization of the Stanley-Stembridge conjecture on e-positivity of the chromatic quasisymmetric function of indifference graphs.

Haiman also made a conjecture about the relations between the characters $\text{ch}(C'_w)$, namely, he that every character $\text{ch}(C'_w)$ is a sum of characters of Kazhdan-Lusztig elements of codominant permutations.

**Conjecture 1.3** ([Hai93, Conjecture 3.1]). For any $w \in S_n$ there exist codominant permutations $w_1, \ldots, w_k$ such that

$$\text{ch}(C'_w) = \text{ch}(C'_{w_1}) + \text{ch}(C'_{w_2}) + \cdots + \text{ch}(C'_{w_k})$$

and

$$P_{e,w}(q) = \sum_{1 \leq i \leq k} q^{\ell(w) - \ell(w_i)}.$$

Conjecture 1.3 restricts to the following statement when $w$ is smooth.

**Conjecture 1.4.** If $w$ is a smooth permutation, there exists a single codominant permutation $w'$ such that

$$\text{ch}(C'_w) = \text{ch}(C'_{w'}).$$

Haiman pointed out in [Hai93] that Conjectures 1.4 and 1.3 should “reflect aspects of the geometry of the flag variety that cannot yet be understood using available geometric machinery”. Conjecture 1.4 was first proved combinatorially by Clearman-Hyatt-Shelton-Skan dera in [CHSS16].

The purpose of this article is to provide a geometric proof of the same result, as well as a counterexample to Conjecture 1.3.

1.1. Results. Let $X$ be an $n \times n$ matrix and $w$ be a permutation. The Lusztig variety associated to $X$ and $w$ is the subvariety of the flag variety defined by

$$\mathcal{Y}_w(X) := \{ V_i : X V_i \cap V_j \geq r_{i,j}(w) \text{ for } i, j = 1, \ldots, n \}.$$  

When $X$ is regular semisimple (has distinct eigenvalues), the intersection homology $IH^*(\mathcal{Y}_w(X))$ a natural $S_n$-module structure induced by the monodromy action of $\pi_1(\text{GL}_n^+, X)$ on $IH^*(\mathcal{Y}_w(X))$.

For $w$ a smooth permutation, so that $\mathcal{Y}_w(X)$ is smooth, this action can be explicitly characterized by a dot action $H^*(\mathcal{Y}_w(X))$ (as in [Tym08]). We have the following result due to Lusztig [Lus86], (see also [AN22]).

**Theorem 1.5** (Lusztig). For any $w \in S_n$, we have $\text{ch}(q^{\ell(w)/2}C'_w) = \text{ch}(IH^*(\mathcal{Y}_w(X)))$.

In Section 2 we will prove the following:

**Theorem 1.6.** Let $X \in SL_n(\mathbb{C})$ be regular semi-simple and $w \in S_n$ smooth. Then there exists a codominant permutation $w'$ such that $H^*(\mathcal{Y}_w(X))$ and $H^*(\mathcal{Y}_{w'}(X))$ are isomorphic as $S_n$-modules. In particular, $\text{ch}(C'_w) = \text{ch}(C'_{w'})$.

The main idea is to see that both $\mathcal{Y}_w(X)$ and $\mathcal{Y}_{w'}(X)$ are smooth GKM spaces, and hence their cohomologies are described by their moment graphs. Since the moment graph of $\mathcal{Y}_w(X)$ only depends on the transpositions which are smaller than $w$ in the Bruhat order, it suffices to see that there exists a codominant permutation whose set of smaller transpositions is equal to that of $w$. In fact, these transpositions are precisely the transpositions $(i,j)$ such that $i < j \leq m_w(i)$ (see, for example, [GL20]).

If $w$ and $w'$ are Coxeter elements, a stronger result holds, and we actually have that $\mathcal{Y}_w(X)$ is isomorphic to $\mathcal{Y}_{w'}(X)$ whenever $X$ is regular semisimple (see [AN22, Example 1.23]). Although for Coxeter elements, Conjecture 1.4 is a consequence of [Hai93, Proposition 4.2]. We note that our proof of Theorem 1.6 only proves the isomorphisms of cohomology groups and not of varieties (see Conjecture 3.9).

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2The condition on the Kazhdan-Lusztig polynomials is a consequence of the character equality.
Concerning singular permutations, we have the following theorems.

**Theorem 1.7.** Let \( w \in S_n \) be a singular permutation and \( s \) a simple transposition such that \( ws \) is smooth and \( sws < w \). Then
\[
\text{ch}(C'_w) = (q^{-1/2} + q^{1/2}) \text{ch}(C'_{ws}).
\]
The analogous equality holds if \( sw \) is smooth. Geometrically, if \( w \) and \( s \) satisfy the above conditions and \( X \) is regular semisimple, then \( \mathcal{Y}_{ws}(X) \) and \( \mathcal{Y}_w(X) \) fit into the following diagram
\[
\mathcal{Y}_{ws}(X) \quad \mathcal{Y}_w(X) \quad \begin{array}{c} f \\ \downarrow g \end{array} \quad \mathcal{Z}
\]
where \( f \) is a \( \mathbb{P}^1 \)-bundle and \( g \) is small.

Theorem 1.7 is a direct consequence of Corollary 3.2, Lemma 3.3 and Proposition 3.4. These results also apply when \( w \) is smooth, in which case we recover the so-called modular law for the chromatic quasisymmetric function of indifference graphs (see [AN21a]) and provide a geometric interpretation of it in Example 3.5 (See also [DCLP88] and [PS22]). The modular law also appears in other symmetric functions associated to indifference graphs, such as the LLT-polynomials ([Lee20]) and the symmetric function of increasing forests ([AN21b]).

**Theorem 1.8** (Counter-example to Conjecture 1.3). Let \( w = 62754381 \in S_8 \). Then \( P_{e,w}(q) = 1 + q \) and there do not exist codominant permutations \( w_0, w_2 \) such that
\[
\text{ch}(C'_w) = \text{ch}(C'_{w_0}) + \text{ch}(C'_{w_2}).
\]

**Proof.** Set \( s = (1, 2) \). Then \( sws = 16754382 < w \). Moreover, \( ws = 26754381 = w_{m_1} \), where \( m_1 = (2, 6, 7, 7, 7, 8) \) is a Hessenberg function. In particular, \( ws \) is codominant, hence smooth, so that \( P_{e,w}(q) = 1 + q \). Assume that there exist codominant permutations \( w_0 \) and \( w_2 \) such that
\[
(q^{-1/2} + q^{1/2}) \text{ch}(C'_{ws}) = \text{ch}(C'_{w_0}) + \text{ch}(C'_{w_2}).
\]
Then there exist Hessenberg functions \( m_0 \) and \( m_2 \) such that (recalling \( w_{m_1} = ws \))
\[
(1 + q) \text{csf}_q(G_{m_1}) = \text{csf}_q(G_{m_2}) + q \text{csf}_q(G_{m_0}).
\]
But by exhaustive search using the Algorithm in [AN21a], there do not exist \( m_0 \) and \( m_2 \) satisfying this condition, which finishes the proof.

![Figure 1B](image-url)

**Figure 1B.** The graphical representation of the Dyck path associated to the Hessenberg function \( m_1 = (2, 4, 5, 6, 6, 6) \) and of the permutation \( w = 62754381 \).

In view of Theorems 1.7 and 1.8, we propose a weaker version of Conjecture 1.3:

**Conjecture 1.9.** For each permutation \( w \in S_n \) there exists codominant permutations \( w_1, \ldots, w_k \in S_n \) such that \( \text{ch}(q^{\frac{m_1}{t(w)}} C'_w) \) is a combination of \( \text{ch}(q^{\frac{m_i}{t(w)}} C'_w) \) with coefficients in \( \mathbb{N}[q] \).
2. Proof of Theorem 1.6

We begin by recalling some properties of GKM-spaces (see [GKM98]). A GKM-space, is a smooth projective variety \( \mathcal{X} \) with an action of a torus \( T \) such that the number of fixed points and the number of 1-dimensional orbits are finite. The equivariant cohomology \( H^*_T(\mathcal{X}) \) is then encoded in a combinatorial object called the moment graph of \( \mathcal{X} \). The vertices of the moment graph are the fixed points, while the edges are the 1-dimensional orbits, each of which has exactly two fixed points on its closure.

If \( X \) is an \( n \times n \) diagonal regular semisimple matrix, the torus \( T \cong (\mathbb{C}^*)^n \) of diagonal matrices acts on the variety \( \mathcal{Y}_w(X) \). When \( w \) is smooth, this variety is a GKM-space because the action is a restriction of that of \( T \) on the whole flag variety, where the number of fixed points and 1-dimensional orbits are indeed finite. Moreover, the moment graph also encodes the action of \( S_n \) on the equivariant cohomology group \( H^*_T(\mathcal{Y}_w(X)) \) (induced by the monodromy action of \( \pi_1(GL^*_n, X) \)), usually called the dot action (see [Tym08]). In particular, if \( w \) and \( w' \) are smooth permutations and \( \mathcal{Y}_w(X) \) and \( \mathcal{Y}_{w'}(X) \) have the same moment graph, then \( \mathrm{ch}(H^*_T(\mathcal{Y}_w(X))) = \mathrm{ch}(H^*_T(\mathcal{Y}_{w'}(X))) \).

Since \( \mathcal{Y}_w(X) \) is a \( T \)-invariant subvariety of \( \mathcal{B} \), we have that the moment graph of \( \mathcal{Y}_w(X) \) is a subgraph of the moment graph of the flag variety \( \mathcal{B} \). We briefly recall the moment graph of \( \mathcal{B} \) (see [Car94] and [Tym08, Proposition 2.1]). The fixed points in \( \mathcal{B} \) are indexed by permutations \( w \in S_n \) (in fact, they are equal to \( \mathcal{Y}_w(X) \) for \( X \) a regular semisimple diagonal matrix). To see this, it is enough to see that a flag \( V \) is fixed by \( T \) if and only if each \( V_i \) is generated by eigenvectors of \( T \). However the eigenvectors of \( T \) are precisely the canonical basis vectors \( e_1, \ldots, e_n \), so there exists \( w \in S_n \) such that \( V_i = \langle e_{w(i)}(1), \ldots, e_{w(i)}(n) \rangle \).

The 1-dimensional orbits are associated to tuples \( (w_1, w_2, t) \), where \( w_1, w_2 \in S_n \) (corresponding to fixed points) with \( \ell(w_1) < \ell(w_2) \) and \( t \) is a transposition satisfying \( w_1 = w_2t \). Then the orbit can be described as follows: Write \( t = (ij) \) with \( i < j \) and define \( v_i = e_{w_2(i)} + ce_{w_2(j)} \) for \( c \in \mathbb{C}^* \). Then \( v_i \) generates the 1-dimensional orbit given by \( (w_1, w_2, t) \). The moment graph of \( V \) is the flag induced by \( t \), while when \( t \) goes to infinity, the limit of \( V \) is the flag \( V \). So the 1-dimensional orbit associated to \( (w_1, w_2, t) \) connects the fixed points corresponding to \( w_1 \) and \( w_2 \).

To describe the moment graph of \( \mathcal{Y}_w(X) \), we use the moment graph of \( \mathcal{B} \) is it is enough to see which fixed points and 1-dimensional orbits are contained in \( \mathcal{Y}_w(X) \). Since \( \mathcal{Y}_w(X) \subset \mathcal{Y}_w(X) \), we have that all fixed points of \( \mathcal{B} \) belong in \( \mathcal{Y}_w(X) \). We claim the following.

Lemma 2.1. The 1-dimensional orbit associated to \( (w_1, w_2, t) \) is contained in \( \mathcal{Y}_w(X) \) if and only if the transposition \( t \) is smaller than \( w \) in the Bruhat order of \( S_n \).

Proof. Consider the flag \( V \subset V_c \) in the 1-dimensional orbit \( (w_1, w_2, t) \). An easy computation shows that \( XV \cap V_c = r_{t,k}(t) \). In particular, \( V_c \subset (X_c)^{\circ} \). Since \( \mathcal{Y}_w(X) = \bigsqcup_{1 \leq w} (X_c)^{\circ} \) we have that \( V_c \subset \mathcal{Y}_w(X) \) if and only if \( t \leq w \). □

Lemma 2.2. Let \( w \) be a smooth permutations and \( m \) its associated Hessenberg function. A transposition \( t = (ij) \) with \( i < j \) is smaller than \( w \) in the Bruhat order of \( S_n \) if and only if \( j \leq m(i) \).

Proof. This is contained in [GL20, Theorem 5.1]. One can see this geometrically from the characterization of smooth Schubert varieties. Consider the pair \((V, F)\) where \( V \) is induced by the matrix \((e_1, \ldots, e_{i-1}, e_j, e_{i+1}, \ldots, e_{j-1}, e_i, e_{j+1}, \ldots, e_n)\) and \( F \) is induced by the identity matrix \((e_1, \ldots, e_n)\). Then we have \( V_i \subset F_j \) and \( F_i \subset V_j \), but \( V_i \not\subset F_{j-1} \) and \( F_i \not\subset V_{j-1} \). In particular, we have that \((V, F) \in \Omega_m \) if and only if \( j \leq m(i) \). Since \((V, F) \in \Omega_m \), the result holds. □

Proof of Theorem 1.6. Let \( w' \) be the codominant permutation associated to the Hessenberg function \( m \) associated to \( w \). By Lemmas 2.1 and 2.2, the moment graphs of \( \mathcal{Y}_w(X) \) and \( \mathcal{Y}_{w'}(X) \) are equal,
and since the dot action only depends on the moment graph, \( \text{ch}(H^*(\mathcal{Y}_w(X))) = \text{ch}(H^*(\mathcal{Y}_{w'}(X))) \). By Theorem 1.5 we have the result. \( \square \)

### 3. Proof of Theorem 1.7

To prove Theorem 1.7 we need a few algebraic results about Hecke algebras and singular permutations. Let \( w \in S_n \) be a permutation and \( s \) a simple transposition. Assume that \( sw < w < ws \).

Then by the multiplication rule of Kazhdan-Lusztig elements of the Hecke algebra (see [Hai93, Equation 8.8]) we have

\[
C_{w'}C_w' = C_{ws} + \sum_{z \leq w} \mu(z, w)C_z',
\]

\[
C_{w'}C_w' = (q^{-\frac{1}{2}} + q^{\frac{1}{2}})C_w',
\]

where \( \mu(z, w) \) is the coefficient of \( q^{\ell(z) - \ell(w)} \) in the Kazhdan-Lusztig polynomial \( P_{z,w}(q) \). Since \( \chi^\lambda(C_{w'}C_w') = \chi^\lambda(C_{w'}C_{w'}') \) for every partition \( \lambda \vdash n \), we have that

\[
\text{ch}(C_{w'}C_w') = \text{ch}(C_{w's}) + \sum_{z \leq w} \mu(z, w) \text{ch}(C_z').
\]

If \( w \) is smooth, then \( \mu(z, w) = 0 \) except for the permutations \( z \) such that \( z \leq w \) and \( \ell(z) = \ell(w) - 1 \), and in this case \( \mu(z, w) = 1 \). To simplify notation, we will write \( z < w \) to mean that \( z \leq w \) and \( \ell(z) = \ell(w) - 1 \). We will see below that if \( w \) is smooth and satisfies \( sw < w < ws \) for some simple reflection \( s \), then there exists at most one permutation \( z \) satisfying \( z < w \) and \( zs < z \).

**Proposition 3.1.** Let \( w \in S_n \) be a smooth permutation and \( s \) a simple reflection such that \( sw < w < ws \). Then one of the following holds:

1. The permutation \( ws \) is smooth and there exists precisely one \( z < w \) such that \( zs < z \). Moreover, \( z \) is smooth.
2. The permutation \( ws \) is singular and there does not exist any \( z < w \) such that \( zs < z \).

**Proof.** We first prove that there exists at most one \( z < w \) such that \( zs < z \). Write \( s = (l, l + 1) \) and assume that \( z \in S_n \) is a permutation satisfying \( z < w \) and \( zs < s \). Since \( z < w \) (which means that \( \ell(z) = \ell(w) - 1 \)), we have that there exist \( i_1, i_2 \) such that

- \( 1 \leq i_1 < i_2 \leq n \),
- \( z(j) = w(j) \) for every \( j \in [n] \setminus \{i_1, i_2\} \),
- \( z(i_k) = w(i_{k-1}) \),
- \( w(i_1) > w(i_2) \),
- for every \( i_1 < j < i_2 \) we have that either \( w(j) < w(i_2) \) or \( w(j) > w(i_1) \).

Since \( ws > w \) and \( zs < z \), we have \( w(l) < w(l + 1) \) and \( z(l) > z(l + 1) \). Hence either \( i_1 = l + 1 \) or \( i_2 = l \).

If \( i_1 = l + 1 \), we have

\[
w(j) < w(i_2) \text{ or } w(j) > w(i_1) = w(l + 1) \text{ for every } i_1 < j < i_2,
\]

\[
w(l + 1) = w(i_1) > w(l) > w(i_2).
\]

On the other hand, if \( i_2 = l \), we have

\[
w(j) < w(i_2) = w(l) \text{ or } w(j) > w(i_1) \text{ for every } i_1 < j < i_2,
\]

\[
w(i_1) > w(l + 1) > w(i_2) = w(l).
\]

See figures 3A and 3B below for a depiction of these conditions.
Assume that there exist two distinct permutations \( z, z' \) satisfying the conditions above, and let \( i_1, i_2 \) and \( i'_1, i'_2 \) be as above for \( z \) and \( z' \), respectively. We now compare the relative position of \( i_1, i_2, i'_1, i'_2 \).

- **Case 1.** Assume that \( i_2 = i'_2 = \ell \) and \( i_1 < i'_1 \) (the case \( i_1 < i'_1 \) being analogous). By Equation (3c), we have that \( w(i_1) > w(l + 1) > w(l), w(i'_1) > w(l + 1) > w(l) \). Since \( i_1 < i'_1 < i_2 \) and \( w(i'_1) > w(l) \), we have \( w(i'_1) > w(i_1) \) (again, by Equation (3c)). Hence \( w(i'_1) > w(i_1) > w(\ell + 1) > w(\ell) \) and this is a 3412 pattern on \( w \), which is a contradiction with the smoothness of \( w \). See Figure 3C.

- **Case 2.** Assume that \( i_1 = i'_1 = \ell + 1 \). This case is analogous to the previous one (just replace Equation (3c) with Equation (3b)).

- **Case 3.** Assume that \( i_2 = \ell \) and \( i'_1 = \ell + 1 \). In this case, we have that \( i_1 < i_2 = l < i'_1 = l + 1 < i'_2 \). By Equations (3c) and (3b), \( w(l + 1) > w(l) > w(i'_2) \) and \( w(i_1) > w(l + 1) > w(l) \), so \( w(i_1) > w(l + 1) > w(l) > w(i'_2) \), which is a 4231 pattern on \( w \), contradicting the smoothness of \( w \). See Figure 3D.
Similar considerations also prove that if $z$ exists, it must be smooth.

We now prove that if $ws$ is singular, there exists no $z < w$ with $zs < z$. Since $ws$ is singular, there exist $j_1 < j_2 < j_3 < j_4$ forming a 4231 or 3412 pattern in $ws$. Since $w$ is smooth, $\{l, l + 1\} \subset \{j_1, j_2, j_3, j_4\}$. Since $w(l) < w(l + 1)$ we have three cases.

- **Case 1.** Assume that we have a 4231 pattern in $ws$ with $j_1 = l$, $j_2 = l + 1$. Then $j_1, j_2, j_3, j_4$ induces a 2431 pattern on $w$ with $j_1 = l, j_2 = l + 1$. Let us assume that there exists $i_1 < i_2 := l = j_1$ satisfying Equation (3c). Then $w(i_1) > w(l + 1)$ and $i_1, j_1, \ldots, j_4$ induces a 52431 pattern on $w$, which contains a 4231 pattern, and this is a contradiction. Let us assume that there exists $l + 1 = j_2 := i_1 < i_2$ satisfying Equation (3b). Then $w(i_2) < w(l)$ and for every $l + 1 < k < i_2$ we have either $w(k) > w(l + 1)$ or $w(k) < w(i_2)$. Then $i_2 < j_3$ since $w(i_2) < w(l) < w(j_3) < w(l + 1)$. This means that $w$ contains either a 35241 or a 35412 pattern, but the first has a 4231 pattern, while the second has a 3412 pattern, which again contradicts the smoothness of $w$.

- **Case 2.** Assume that we have a 4231 pattern in $ws$ with $j_3 = l$, $j_4 = l + 1$. Then we have a 4213 pattern on $w$, and the argument is similar as above.

- **Case 3.** Assume that we have a 3412 pattern in $ws$ with $j_2 = l$, $j_3 = l + 1$, so that $j_1, j_2, j_3, j_4$ induces a 3142 pattern on $w$ with $j_2 = l, j_3 = l + 1$. Let us assume that there exists $i_1 < i_2 := l = j_2$ satisfying Equation (3c). Then $w(i_1) > w(l + 1)$, and for every $i_1 < k < l$ we have either $w(k) > w(i_1)$ or $w(k) < w(l)$. Then $i_1 > j_1$ and we have a 35142 pattern on $w$, a contradiction. Let us assume that there exist $l + 1 = j_3 := i_1 < i_2$ satisfying Equation (3b). Then $w(i_2) < w(l)$ and for every $l + 1 < k < i_2$ we have either $w(k) < w(i_2)$ or $w(k) > w(l + 1)$, so that $i_2 < j_4$ and we have a 42513 pattern on $w$, also a contradiction.

Finally, we will prove that if there is no $z < w$ with $zs < z$, then $ws$ is singular. First, assume that there exists $i < l$ such that $w(i) > w(l + 1)$ and consider the greatest possible such $i$. If $z = w \cdot (i, l)$, then $zs < z$ and $z < w$. This means that $z << w$, and that is equivalent to the existence of $i < j < l$ with $w(i) > w(j) > w(l)$. Since $i$ is the greatest $i < l$ with $w(i) > w(l + 1)$, we have that $w(i) > w(l + 1) > w(j) > w(l)$, which implies that $i, j, l, l + 1$ induces a 4213 pattern on $w$ and hence a 4231 pattern on $ws$. If there exists $i > l + 1$ with $w(i) < w(l)$, the argument is the same.

Therefore, let us assume that $w(i) < w(l + 1)$ for every $i < l$ and $w(i) > w(l)$ for every $i > l + 1$. In particular, we have that $w^{-1}(j) < l$ for every $j < w(l)$. Let $k$ be the maximum of $\{w(i)\}_{i \leq l}$, and note that $w(l) \leq k < w(l + 1)$. Assume that there exists $j < k$ with $w^{-1}(j) > l + 1$. By the argument above, we have that $j > w(l)$ (and hence $k > w(l)$), so $w^{-1}(k) < l < l + 1 < w^{-1}(j)$ and $w(l + 1) > k > j > w(l)$, which implies that $w^{-1}(k), l, l + 1, w^{-1}(j)$ induces a 3142 pattern on $w$, and hence a 4231 pattern on $ws$. On the other hand, if $w^{-1}(j) \leq l$ for every $j \leq k$, then

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{3D.png}
\caption{The relative position of $w(i_1)$, $w(i_2)$, $w(i_1')$, and $w(i_2')$. Note that we have a 4231 pattern on $w$.}
\end{figure}
\{w(1), \ldots, w(l)\} = \{1, \ldots, k\}, and in particular \(k = l\). But then \((l, l + 1)w > w\), a contradiction since \(sw < w\) by hypothesis. This finishes the proof. \hfill \square

We have the following direct corollary.

**Corollary 3.2.** Let \(w\) be a smooth permutation and \(s\) a simple transposition such that \(ws > w > sw\).

1. If \(ws\) is smooth and \(z\) is the only permutation \(z < w\) with \(zs < z\), then \((q^{-\frac{1}{2}} + q^{\frac{1}{2}}) \text{ch}(C'_w) = \text{dim}(\mathcal{G}^{ws}) + \text{ch}(C'_z)\).
2. If \(ws\) is singular, then \((q^{-\frac{1}{2}} + q^{\frac{1}{2}}) \text{ch}(C'_w) = \text{ch}(C'_w)\).

**Proof.** Follows directly from Equation (3a) and Proposition 3.1. \hfill \square

Corollary 3.2 has a geometric interpretation. Let \(w\) and \(s\) be as in Corollary 3.2, and let \(\mathcal{P}_s\) be the partial flag variety associated to \(s\), that is, if \(s = (l, l + 1)\) then

\[ \mathcal{P}_s = \{V_1 \subset V_2 \subset \ldots \subset V_{l-1} \subset V_{l+1} \subset \ldots \subset V_n = \mathbb{C}^n; \dim(V_i) = i\} \]

Using the algebraic group notation, we write \(G = GL_n\) and \(B\) for the Borel subgroup of \(G\) of uppertriangular matrices. For each permutation \(w \in S_n\) let \(\dot{w}\) denote the associated permutation matrix \(\dot{w} \in G\). We write \(P_s\) for the parabolic subgroup associated to \(s\), that is, \(P_s = B \cup B\dot{s}B\), so that \(\mathcal{P}_s = G/P_s\). In this notation, the Lusztig varieties are given by \(\mathcal{Y}_w(X)^{\circ} = \{gB; g^{-1}Xg \in B\dot{w}B\}\).

**Lemma 3.3.** Let \(w \in S_n\) be a permutation, \(s\) a simple transposition, and \(X\) a regular semisimple \(n \times n\) matrix. Then

1. If \(sw < w\) and \(ws < w\), then the forgetful map \(\mathcal{Y}_w(X) \rightarrow \mathcal{P}_s\) is a \(\mathbb{P}^1\)-bundle over its image.
2. If \(ws \neq sw\), then the forgetful map \(\mathcal{Y}_w(X)^{\circ} \rightarrow \mathcal{P}_s\) is injective.

**Proof.** We begin with item (1). For \(s = (l, l + 1)\) the hypothesis is equivalent to \(w(l) > w(l + 1)\) and \(w^{-1}(l) > w^{-1}(l + 1)\), and in particular, the coessential set of \(w\)

\[ \text{Coess}(w) := \{(a,b); w(a) \leq b < w(a + 1), w^{-1}(b) \leq a < w^{-1}(b + 1)\} \]

does not contain any pair \((a, b)\) with either \(a = l\) or \(b = l\). This means that the conditions involving \(\dim(XV_l \cap V_b)\) and \(\dim(XV_a \cap V_l)\) are redundant in \(\mathcal{Y}_w(X)\), hence \(V_i\) can be chosen arbitrarily.

Let us prove item (2). Since \(\mathcal{Y}_w(X) = \{gB; g^{-1}Xg \in B\dot{w}B\}\), to prove that the map \(\mathcal{Y}_w(X) \rightarrow \mathcal{P}_s\) is injective it suffices to prove that there do not exist \(g_1B\) and \(g_2B\) distinct such that \(g_1^{-1}Xg_1 \in B\dot{w}B\), \(g_2^{-1}Xg_2 \in B\dot{w}B\), and \(g_1 \in g_2P_s\). Assume by way of contradiction that such a pair \(g_1, g_2\) exists. Since \(P_s = B \cup B\dot{s}B\) and \(g_1B \neq g_2B\), we have that \(g_1 \in g_2B\dot{s}B\), in particular \(g_1 = g_2b_1\dot{s}b_2\) for some \(b_1, b_2 \in B\). Therefore

\[ g_2^{-1}Xg_2 \in B\dot{w}B, \]
\[ b_2^{-1}\dot{s}b_1^{-1}g_2^{-1}Xg_2b_1\dot{s}b_2 \in B\dot{w}B. \]

Since \(b_1, b_2 \in B\), we have

\[ b_1^{-1}g_2^{-1}Xg_2b_1 \in B\dot{w}B, \]
\[ b_1^{-1}g_2^{-1}Xg_2b_1 \in \dot{s}B\dot{w}B. \]

This means that \(B\dot{w}B \cap \dot{s}B\dot{w}B \neq \emptyset\). Let us assume, without loss of generality, that \(sw < w < ws\). Then by [MT11, proof of Lemma 11.14]

\[ B\dot{w}B \subset B\dot{w}B \cdot B\dot{s}B = B\dot{s}B, \]

and by [MT11, Lemma 11.14]

\[ \dot{s}B\dot{w}B \subset B\dot{s}B \cdot B\dot{s}B \subset B\dot{w}B \cup B\dot{w}\dot{s}B. \]
Since \(wsw \neq w\) (otherwise, \(wsw = ws\)), we have

\[
B\bar{w}B \cap (B\bar{s}wB \cup B\bar{w}swB) \neq \emptyset,
\]

which is a contradiction of the Bruhat decomposition of \(G\). \(\square\)

Let \(X\) be a regular matrix, \(w \in S_n\) an irreducible permutation, that is, a permutation that is not contained in any proper Young subgroup, and \(s\) a simple transposition satisfying the conditions in Corollary 3.2. Consider the forgetful map \(Y_{ws}(X) \to P_s\) and let \(Z\) be the image. By [AN22, Corollary 8.6], \(Y_{ws}(X)\) and \(Y_w(X)\) are irreducible, and so \(Z\) is as well. By Lemma 3.3, the map \(Y_{ws}(X) \to Z\) is a \(\mathbb{P}^1\)-bundle, while the map \(Y_w(X) \to P_s\) is injective. Since \(Y_w(X) \subset Y_{ws}(X)\) \((w < ws)\), the image of \(Y_w(X)\) is contained in \(Z\). Since \(Y_w(X) \to Z\) is injective and the dimensions agree, \(Y_w(X) \to Z\) is birational. Let \(z \in S_n\) be the permutation such that \(z < w\) and \(zs < z\). Then we have:

**Proposition 3.4.** The map \(Y_w(X) \to Z\) is semismall and the preimage of the relevant locus is precisely \(Y_z(X)\) (if \(z\) exists).

**Proof.** The fact that \(Y_w(X) \to Z\) is semismall follows from the fact that the map is birational and its fibers have dimension at most one (since they are contained in those of \(Y_{ws}(X) \to Z\)). We have that \(Y_w(X) = Y_w^o(X) \cup \bigcup_{s < w} Y_w(X)\), where \(Y_w(X)\) has codimension one in \(Y_w(X)\). We claim that the images of \(Y_w^o(X)\) and \(Y_w(X)\) are disjoint. Assume for contradiction that there exist \(g_1B\) and \(g_2B\) such that \(g_1^{-1}Xg_1 \in B\bar{w}B\), \(g_2^{-1}Xg_2 \in B\bar{w}swB\) and \(g_1P_s = g_2P_s\). Arguing as in the proof of Lemma 3.3, we have

\[
B\bar{w}swB \cap (B\bar{w}B \cup B\bar{w}swB) \neq \emptyset.
\]

However, \(\ell(ws) = \ell(w) + 1\), \(\ell(wsw) = \ell(w)\), and \(\ell(z) = \ell(w) - 1\), and \(B\bar{w}swB = \bigcup_{z' \leq z} B\bar{w}swB\). By the Bruhat decomposition, Equation (3d) is a contradiction.

Moreover, since the fibers have dimension at most one, the preimage of the relevant locus has codimension one in \(Y_w(X)\). By the discussion above, this preimage must be a union of \(Y_z(X)\) for some \(z' \prec w\). By the lifting property [Bre92, Proposition 2.2.7], either \(sz' < z'\) or \(z' = sw\). If \(z' = sw\), then \(z' = sw < wsw = zs'\) and \(zs' = w = sz'\), so by Lemma 3.3 \(Y_{zs'}(X) \to Z\) is injective, and hence \(Y_{zs'}(X)\) is not contained in the preimage of the relevant locus. If \(sz' < z'\) and \(sz' < z's\), so by Lemma 3.3 \(Y_{zs'}(X) \to Z\) is injective, and hence \(Y_{zs'}(X)\) is not contained in the preimage of the relevant locus. Finally, if \(sz' < z'\) and \(z's < z'\), then \(z' = z\), so by Lemma 3.3 \(Y_{z'}(X) \to Z\) is \(\mathbb{P}^1\)-bundle over its image, and hence \(Y_{z'}(X)\) is contained in the preimage of the relevant locus. Since the preimage of the relevant locus has codimension one, it is precisely \(Y_z(X)\). \(\square\)

By the decomposition theorem (we set \(Z_1\) as the image of \(Y_z(C)\) if \(z\) exists), \(IH^*(Y_{ws}(X)) = IH^*(z) \otimes (\mathbb{C} \oplus \mathbb{C}[−2])\), \(H^*(Y_w(X)) = IH^*(Z) \otimes IH^*(Z_1)[−2]\) and \(IH^*(Y_z(X)) = IH^*(Z_\infty) \otimes (\mathbb{C} \oplus \mathbb{C}[−2])\). Then

\[
\begin{align*}
\text{ch}(IH^*(Y_{ws}(X))) &= (1 + q)\text{ch}(IH^*(Z)), \\
\text{ch}(H^*(Y_w(X))) &= \text{ch}(IH^*(Z)) + q\text{ch}(IH^*(Z_1)), \\
\text{ch}(IH^*(Y_z(X))) &= (1 + q)\text{ch}(IH^*(Z_\infty)),
\end{align*}
\]

which implies

\[
(1 + q)\text{ch}(H^*(Y_w(X))) = \text{ch}(IH^*(Y_{ws}(X))) + q\text{ch}(IH^*(Y_z(X))).
\]

This, in turn, is equivalent by Theorem 1.5 to

\[
(1 + q)\text{ch}(q^{\ell(w)}C_w) = \text{ch}(q^{\ell(ws)+1}C_{ws}) + q\text{ch}(q^{\ell(w)-1}C'_w).
\]
When \( w \) is codominant and \( ws \) is smooth, then both \( ws \) and \( z \) are codominant as well. Below we give an example of what happens for Hessenberg varieties.

**Example 3.5** (Geometric interpretation of the modular law for indifference graphs). Let \( m_0, m_1, m_2 \) be Hessenberg functions and \( i \in [n] \) an integer such that \( m_0(j) = m_1(j) = m_2(j) \) for every \( j \neq i \), \( m_0(i) = m_1(i) - 1 = m_2(i) - 2 \) and \( m_1(m_1(i) + 1) = m_1(m_1(i)) \). Set \( l = m_1(1) \) and let \( s = (l, l + 1) \) be a simple transposition.

We claim that \( w_{m_1}s < w_{m_1} < w_{m_2} = sw_{m_1}, w_{m_0} < w_{m_1} \) and \( sw_{m_0} < w_{m_0} \), so we are in the hypothesis of Corollary 3.2. Indeed, since \( m_1(i) = l \) and \( m_1(i - 1) < l \), we have that \( w_{m_1}(i) = l \), while \( w_{m_1}^{-1}(l+1) > i \). So \( w_{m_1}s < w_{m_1} < sw_{m_1} \). Since \( m_2(i) = l+1 \) and \( m_2 \) agrees with \( m_1 \) everywhere else, \( w_{m_2} = sw_{m_1} \). Finally, \( w_{m_0} < w_{m_1} \), and since \( m_0(i) < l \) and \( m_0(i+1) > l \), we have \( sw_{m_0} < w_{m_0} \).

Let \( X \) be a regular semi-simple matrix, then the Hessenberg varieties are

\[
\mathcal{Y}_{m_0} = \{ V_*; XV_i \subset V_{i-1}; XV_j \subset V_{m_1(j)} \text{ for } j \in [n] \setminus \{i\} \},
\]

\[
\mathcal{Y}_{m_1} = \{ V_*; XV_i \subset V_{i-1}; XV_j \subset V_{m_1(j)} \text{ for } j \in [n] \setminus \{i\} \},
\]

\[
\mathcal{Y}_{m_2} = \{ V_*; XV_i \subset V_{i-1}; XV_j \subset V_{m_1(j)} \text{ for } j \in [n] \setminus \{i\} \}.
\]

Since \( m_1(l+1) = m_1(l) \), the conditions \( XV_i \subset V_{m_1(l)} \) and \( XV_{l+1} \subset V_{m_1(l+1)} = V_{m_1(l)} \) are redundant. In particular, there exists no condition involving \( V_k \) in \( \mathcal{Y}_{m_0}(X) \) and \( \mathcal{Y}_{m_2}(X) \). Then the forgetful maps

\[
\mathcal{Y}_{m_0}(X) \to \mathcal{P}_s
\]

\[
\mathcal{Y}_{m_2}(X) \to \mathcal{P}_s
\]

are \( \mathbb{P}^1 \)-bundles over their images, which are, respectively,

\[
\mathcal{Z}_0 = \{ V_*; XV_i \subset V_{l-1}; XV_j \subset V_{m_1(j)} \text{ for } j \in [n] \setminus \{i,l\} \},
\]

\[
\mathcal{Z}_2 = \{ V_*; XV_i \subset V_{l-1}; XV_j \subset V_{m_1(j)} \text{ for } j \in [n] \setminus \{i,l\} \},
\]

where we write \( V_* \) for a partial flag \( V_1 \subset \ldots \subset V_{l-1} \subset V_{l+1} \subset \ldots \subset V_n \) in \( \mathcal{P}_s \). The fibers of the map \( f: \mathcal{Y}_{m_1}(X) \to \mathcal{Z}_2 \) can be described as

\[
f^{-1}(V_*) = \{ V_*; V_j = V_j \text{ for } j \in [n] \setminus \{l\}, V_{l-1} + XV_i \subset V_{l+1} \}
\]

So \( f^{-1}(V_*) \) is isomorphic to \( \mathbb{P}^1 \) if \( XV_i \subset V_{l-1} \), as in this case \( \mathcal{V}_{l-1} + XV_i = \mathcal{V}_{l-1} \), or is a single point \( V_* \) with \( V_i = V_{k-1} + XV_j \). Note that \( \dim \mathcal{V}_{l-1} + XV_i \leq l \), as \( XV_{l-1} \subset V_{m_1(i-1)} \subset V_{l-1} \). In fact, \( \mathcal{Y}_{m_1}(X) \) is the blowup of \( \mathcal{Z}_2 \) along \( \mathcal{Z}_0 \).

\[
\begin{array}{ccc}
\mathcal{Y}_{m_0}(X) & \xrightarrow{p_1\text{-bundle}} & \mathcal{Y}_{m_2}(X) \\
\text{Isomorphism outside} & & \\
\mathcal{Y}_{m_0}(X) & \xrightarrow{p_1\text{-bundle}} & \mathcal{Z}_2
\end{array}
\]

This means that

\[
\text{ch}(H^*(\mathcal{Y}_{m_0}(X))) = (1 + q) \text{ch}(H^*(\mathcal{Z}_0))
\]

\[
\text{ch}(H^*(\mathcal{Y}_{m_1}(X))) = \text{ch}(H^*(\mathcal{P}_s\mathcal{Z}_2)) = \text{ch}(H^*(\mathcal{Z}_2)) + q \text{ch}(H^*(\mathcal{Z}_0))
\]

\[
\text{ch}(H^*(\mathcal{Y}_{m_2}(X))) = (1 + q) \text{ch}(H^*(\mathcal{Z}_2))
\]

and hence we get

\[
(1 + q) \text{csf}_q(m_1) = \text{csf}_q(m_2) + q \text{csf}_q(m_1).
\]
We refer to [AN22, Example 1.24] for an example where \( ws \) is singular.

A direct consequence of Example 3.5 is that characters of Kazhdan-Lusztig elements of codominant permutations are omega-dual to chromatic quasisymmetric functions of indifference graphs, first proved in [CHSS16].

**Corollary 3.6.** If \( \mathbf{m} : [n] \rightarrow [n] \) is a Hessenberg function, then
\[
\text{ch}(q^{\ell_{\mathbf{m}}}) C_{w_{\mathbf{m}}} = \omega(\text{csf}_q(G_{\mathbf{m}})).
\]

**Proof.** If \( \mathbf{m}_0, \mathbf{m}_1, \) and \( \mathbf{m}_2 \) are Hessenberg functions as in Example 3.5, then applying Corollary 3.2 to \( w_{\mathbf{m}_1} \), we see that \( w_{\mathbf{m}_1} s = w_{\mathbf{m}_2} \) and \( z = w_{\mathbf{m}_0} \). This means that the relation in item (1) is precisely the modular law (see [GP13] and [OS14]). By [AN21a, Theorem 1.1], the modular law is sufficient to characterize the values \( \text{ch}(C_w') \) for \( w \) codominant from the values \( \text{ch}(q^{\ell_{\mathbf{m}}}) C_{w_{\mathbf{m}}} \).

Since \( \text{ch}(q^{\ell_{\mathbf{m}}}) C_{w_{\mathbf{m}}} = \lambda_i h_\lambda = \omega(G_{\mathbf{m}}) \) the result follows. \( \square \)

**Remark 3.7.** We set \( H_n^{\text{cod}} \) to be the \( \mathbb{C}(q^{\frac{1}{2}}) \)-linear subspace of \( H_n \) generated by \( C_w' \), for \( w \) codominant. From [AN21a], the kernel of the linear map
\[
\text{ch} : H_n^{\text{cod}} \rightarrow \mathbb{C}(q^{\frac{1}{2}}) \otimes \Lambda,
\]
is generated by the relations in Corollary 3.2 item (1) for \( w \) codominant.

**Question 3.8.** Is the kernel of the linear map \( \text{ch} : H_n \rightarrow \mathbb{C}(q^{\frac{1}{2}}) \otimes \Lambda \) generated by the relations in Equation (3a)?

### 3.1. The geometry of \( \mathcal{Y}_w(X) \) when \( w \) is smooth.

In the proof of Theorem 1.6 in Section 2, we saw that for each smooth permutation \( w \in S_n \) there exists a codominant permutation \( w' \) such that the moment graphs of \( \mathcal{Y}_w(X) \) and \( \mathcal{Y}_{w'}(X) \) are the same and, in particular, they have isomorphic equivariant cohomology. We also saw that all the varieties \( \mathcal{Y}_w(X) \) associated to Coxeter elements \( w \) are isomorphic. We make the following conjecture which is a strengthening of Theorem 1.6.

**Conjecture 3.9.** Let \( X \in SL_n(\mathbb{C}) \) be regular semisimple and \( w \in S_n \) smooth. Then there exists a codominant permutation \( w' \) such that \( \mathcal{Y}_w(X) \) and \( \mathcal{Y}_{w'}(X) \) are homeomorphic.

We remark that the corresponding statement for Schubert varieties is false, for instance \( \Omega_{3142,F} \), is not homeomorphic to \( \Omega_{2341,F} \) (and this is the only Schubert variety associated with a codominant permutation with the same Poincaré polynomial as of \( \Omega_{3142,F} \)). On the other hand, both 3142 and 2341 are cocometer elements so that \( \mathcal{Y}_{3142}(X) \) is isomorphic to \( \mathcal{Y}_{2341}(X) \) if \( X \) is regular semisimple.

### References

[AN21a] Alex Abreu and Antonio Nigro, *Chromatic symmetric functions from the modular law*, J. Combin. Theory Ser. A **180** (2021), 105407, 30. MR 4199388 (cit. on pp. 5 and 13)

[AN21b] , *A symmetric function of increasing forests*, Forum Math. Sigma **9** (2021), Paper No. e35, 21. MR 4252214 (cit. on p. 5)

[AN22] , *A geometric approach to characters of hecke algebras*, 2022, arXiv:2205.14835, (cit. on pp. 4, 11, and 13)

[Bre92] Francesco Brenti, *Expansions of chromatic polynomials and log-concavity*, Trans. Amer. Math. Soc. **332** (1992), no. 2, 729–756. MR 1069745 (cit. on p. 11)

[Car94] James B. Carrell, *The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties*, Algebraic groups and their generalizations: classical methods (University Park, PA, 1991), Proc. Sympos. Pure Math., vol. 56, Amer. Math. Soc., Providence, RI, 1994, pp. 53–61. MR 1278709 (cit. on p. 6)
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[CHSS16] Samuel Clearman, Matthew Hyatt, Brittany Shelton, and Mark Skandera, Evaluations of Hecke algebra traces at Kazhdan-Lusztig basis elements, Electron. J. Combin. 23 (2016), no. 2, Paper 2.7, 56. MR 3512629 (cit. on pp. 3, 4, and 13)

[DCLP88] C. De Concini, G. Lusztig, and C. Procesi, Homology of the zero-set of a nilpotent vector field on a flag manifold, J. Amer. Math. Soc. 1 (1988), no. 1, 15–34. MR 924700 (cit. on p. 5)

[GKM98] Mark Goresky, Robert Kottwitz, and Robert MacPherson, Equivariant cohomology, Koszul duality, and the localization theorem, Invent. Math. 131 (1998), no. 1, 25–83. MR 1489894 (cit. on p. 6)

[GL20] Shoni Gilboa and Erez Lapid, Some combinatorial results on smooth permutations, Sém. Lothar. Combin. 84B (2020), Art. 81, 10. MR 4138708 (cit. on pp. 3, 4, and 6)

[GP00] Meinolf Geck and Görtz Pfeiffer, Characters of finite Coxeter groups and Iwahori-Hecke algebras, London Mathematical Society Monographs. New Series, vol. 21, The Clarendon Press, Oxford University Press, New York, 2000. MR 1778802 (cit. on p. 3)

[GP13] Mathieu Guay-Paquet, A modular relation for the chromatic symmetric functions of $(3+1)$-free posets, 2013. arXiv:1306.2400. (cit. on pp. 4 and 13)

[Hai93] Mark Haiman, Hecke algebra characters and immanant conjectures, J. Amer. Math. Soc. 6 (1993), no. 3, 509–595. MR 1186961 (cit. on pp. 3, 4, and 7)

[KL79] David Kazhdan and George Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (1979), no. 2, 165–184. MR 560412 (cit. on p. 2)

[Lee20] Seung Jin Lee, Linear relations on LLT polynomials and their $k$-Schar positivity for $k = 2$, Journal of Algebraic Combinatorics (2020), 1–18. (cit. on p. 5)

[Lus86] George Lusztig, Character sheaves. V, Adv. in Math. 61 (1986), no. 2, 103–155. MR 849848 (cit. on p. 4)

[MT11] Gunter Malle and Donna Testerman, Linear algebraic groups and finite groups of Lie type, Cambridge Studies in Advanced Mathematics, vol. 133, Cambridge University Press, Cambridge, 2011. MR 2850737 (cit. on p. 10)

[OS14] Rosa Orellana and Geoffrey Scott, Graphs with equal chromatic symmetric functions, Discrete Math. 320 (2014), 1–14. MR 3147202 (cit. on p. 13)

[PS22] Martha Precup and Eric Sommers, Perverse sheaves, nilpotent hessenberg varieties, and the modular law, 2022. (cit. on p. 5)

[Tym08] Julianna S. Tymoczko, Permutation actions on equivariant cohomology of flag varieties, Toric topology, Contemp. Math., vol. 460, Amer. Math. Soc., Providence, RI, 2008, pp. 365–384. MR 2428368 (cit. on pp. 4 and 6)