THE PILLARS OF RELATIVE QUILLEN–SUSLIN THEORY

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Abstract: We deduce the relative version of the equivalences relating the relative Local Global Principle and the Normality of the relative Elementary subgroups of the traditional classical groups; viz. general linear, symplectic and orthogonal groups. This generalizes our previous result for the absolute case; cf. [6].

1. Introduction

The main pillars of the Horrocks–Quillen–Suslin theory were developed in the papers [4], [20], [28]. In [4] the Monic Inversion Principle, in [20] the Local-Global Principle, and in [28] the Normality of the Elementary subgroup $E_n(R)$, were established. In [28] the $K_1$ analogues of both the Monic Inversion Principle and the Local-Global Principle were developed. In addition, Suslin established the Normality of the Elementary Linear subgroup $E_n(R)$ in the general linear group $GL_n(R)$ over a module finite ring $A$, when $n \geq 3$. This was appeared in [26].

In [6] the authors had established, for classical linear groups, viz. the linear, symplectic and orthogonal groups, that the Quillen–Suslin’s Local-Global Principle for the pair $(GL_n(R[X]), E_n(R[X]))$ and Suslin’s Normality Principle were equivalent in the sense that if one holds then so does the other. Recently, in [22] a further unification of these three principles was achieved.

In this article, we develop the equivalence of a relative Quillen’s Local-Global Principle and a normality of the relative elementary subgroup; cf. Theorem 3.1 for the precise equivalent statements.

We refer the reader to the Introduction of [6] where recent developments of the Quillen–Suslin theory are discussed in detail. The study of the relative Local-Global Principle with respect to an extended ideal began in [1]; and was developed in [2] for the Chevalley groups.

The proofs of the equivalent statements in this paper are done in an analogous manner to that done in [6]. This was possible due to a recent argument, which is detailed in [3], and which first appeared in the thesis of Anjan Gupta [10]. This argument works with the Noetherian excision ring $R \oplus I$ rather than the use of the (non-Noetherian) Excision ring $\mathbb{Z} \oplus I$, and the Excision theorem of W. van der Kallen in [15], as is commonly used. We refer [11] to see other interesting applications of the Noetherian Excision rings.
For the sake of being self-contained we have detailed the arguments of the various equivalences. However, we note that we could have alternatively deduced the implications from the corresponding implications done in [6] via this Noetherian Excision ring argument.

2. Definitions and Notations

Let $R$ be a commutative ring with 1, and $I \subset R$ an ideal. We refer [6] for the standard definitions and facts of the general linear, symplectic and orthogonal groups, and their elementary subgroups. Let $\sigma$ denote the permutation of the natural numbers given by $\sigma(2i) = 2i - 1$ and $\sigma(2i - 1) = 2i$. With respect to this permutation we define following classical groups.

For an integer $m > 0$, the symplectic group of size $2m \times 2m$ is defined with respect to the alternating matrix $\psi_m$ corresponding to the standard symplectic form

$$\psi_m = \sum_{i=1}^{m} e_{2i-1,2i} - \sum_{i=1}^{m} e_{2i,2i-1}.$$ 

For the orthogonal group we have considered symmetric matrix $\tilde{\psi}_m$ corresponding to the standard hyperbolic form

$$\tilde{\psi}_m = \sum_{i=1}^{m} e_{2i-1,2i} + \sum_{i=1}^{m} e_{2i,2i-1}.$$ 

**Definition 2.1. Symplectic Group** $\text{Sp}_{2m}(R)$: The group of all non-singular $2m \times 2m$ matrices $\{\alpha \in \text{GL}_{2m}(R) \mid \alpha^t \psi_m \alpha = \psi_m\}$.

**Definition 2.2. Orthogonal Group** $\text{O}_{2m}(R)$: The group of all non-singular $2m \times 2m$ matrices $\{\alpha \in \text{GL}_{2m}(R) \mid \alpha^t \tilde{\psi}_m \alpha = \tilde{\psi}_m\}$.

**Definition 2.3. Elementary Symplectic Group** $\text{ESp}_{2m}(R)$: For $1 \leq i \neq j \leq 2m$ we define,

$$se_{ij}(z) = I_{2m} + ze_{ij} \text{ if } i = \sigma(j)$$

$$= I_{2m} + ze_{ij} - (-1)^{i+j}ze_{\sigma(j)\sigma(i)} \text{ if } i \neq \sigma(j) \text{ and } i < j.$$ 

It is clear that when $z \in R$ all these matrices belong to $\text{Sp}_{2m}(R)$. We call them the elementary symplectic matrices over $R$ and the group generated by them is called elementary symplectic group.

**Definition 2.4. Elementary Orthogonal Group** $\text{ESp}_{2m}(R)$: For $1 \leq i \neq j \leq 2m$ we define,

$$oe_{ij}(z) = I_{2n} + ze_{ij} \text{ if } i \neq \sigma(j) \text{ and } i < j.$$ 

It is clear that when $z \in R$ all these matrices belong to $\text{O}_{2m}(R)$. We call them the elementary orthogonal matrices over $R$ and the group generated by them is called elementary orthogonal group.

**Notation 2.5.** In the sequel $M(n, R)$ will denote the set of all $n \times n$ matrices, $G(n, R)$ will denote either the linear group $\text{GL}_n(R)$, the symplectic group...
Sp\(_{2m}(R)\), or the orthogonal group \(O_{2m}(R)\), where \(2m = n\). \(S(n, R)\) will denote either the special linear group \(SL_n(R)\), the symplectic group \(Sp_{2m}(R)\), or the special orthogonal group \(SO_{2m}(R)\), when \(R\) is a commutative ring. Similarly, \(E(n, R)\) will denote the corresponding elementary subgroups \(E_n(R)\), \(ESp_{2m}(R)\), \(EO_{2m}(R)\) respectively. To denote the generators of \(E(n, R)\) we shall use the symbol \(ge_{ij}(x)\), \(x \in R\).

**Definition 2.6.** The elementary subgroup \(E(n, I)\) with respect to the ideal \(I\) is the subgroup of \(E(n, R)\) generated as a group by the elements \(ge_{ij}(x)\), for \(x \in I\). The **relative elementary group** \(E(n, R, I)\) is the normal closure of \(E(n, I)\) in \(E(n, R)\).

**Notation 2.7.** The relative subgroups of \(G(n, R)\) and \(S(n, R)\) will be denoted by \(G(n, R, I)\) and \(S(n, R, I)\) respectively. i.e.

\[
G(n, R, I) = \{ \alpha \in G(n, R) \mid \alpha \equiv I_n \text{ modulo } I \},
\]

\[
S(n, R, I) = \{ \alpha \in S(n, R) \mid \alpha \equiv I_n \text{ modulo } I \}.
\]

For an ideal \(I\) in \(R\), its extension in the ring \(R[X]\), i.e. \(I \otimes_R R[X]\) will be denoted by \(I[X]\).

Similarly, \(\hat{U}_m(R, I)\) will denote the set of all unimodular rows of length \(n\) which are congruent to \(e_1 = (1, 0, \ldots, 0)\) modulo \(I\).

We will mostly use localizations with respect to two types of multiplicatively closed subsets of \(R\). viz. \(S = \{1, s, s^2, \ldots\}\), where \(s \in R\) is a non-nilpotent, non-zero divisor, and \(S = R \setminus \text{Max}(R)\). By \(I_s[X]\) and \(I_m[X]\) we shall mean the extension of \(I[X]\) in \(R_s[X]\) and \(R_m[X]\) respectively.

**Blanket Assumption:** We assume that \(n \geq 3\), when dealing with the linear case and \(n = 2m\), with \(m \geq 2\), when considering the symplectic and orthogonal cases. While dealing with the orthogonal groups we shall consider only isotropic vectors; i.e. all such non-zero vectors which are orthogonal to themselves with respect to the given non-degenerate bilinear form. Throughout the article we shall assume 2 is invertible in the ring \(R\).

**Notation 2.8.** For any column vector \(v \in R^n\) we denote by \(\tilde{v} = v^t \psi_n\) in the symplectic case and \(\bar{v} = v^t \bar{\psi}_n\) in the orthogonal case.

**Definition 2.9.** We define the map \(M : R^n \times R^n \to M(n, R)\) and the inner product \(\langle \ , \ \rangle\) as follows: Let \(v, w\) be column vectors in \(R^n\). Then,

\[
M(v, w) = v.w^t, \quad \text{when dealing with the case } G(n, R) = GL_n(R).
\]

\[
= v.\tilde{w} + w.\bar{v}, \quad \text{when } G(n, R) = Sp_{2m}(R).
\]

\[
= v.\tilde{w} - w.\bar{v}, \quad \text{when } G(n, R) = O_{2m}(R).
\]

\[
\langle v, w \rangle = v^t.w, \quad \text{when } G(n, R) = GL_n(R).
\]

\[
= \tilde{v}.w, \quad \text{when } G(n, R) = Sp_{2m}(R) \text{ or } O_{2m}(R).
\]

**Notation 2.10.** For any \(\alpha \in G(n, R)\), as usual \(\alpha \perp I\) denotes its embedding in \(G(n + r, R)\), where \(r\) is even for non-linear cases.
To deduce the relative case from the absolute case we consider the “Noetherian Excision ring”.

**Definition 2.11. (The ring $R \oplus I$):** Let $I$ be an ideal in the ring $R$. We construct the new ring $R \oplus I$ by defining addition and co-ordinate wise multiplication as follows:

$$(r \oplus j)(s \oplus i) = rs \oplus (sj + ri + ij) \text{ for } r, s \in R \text{ and } i, j \in I.$$  

There is a natural homomorphism $\phi : R \oplus I \to R$ given by $(r \oplus i) \mapsto r + i \in R$.

**Notation 2.12.** Let $E(n, I) = \{\alpha \in S(n, R) \mid \alpha \equiv I_n \text{ modulo } I\}$. In general, $E(n, I)$ is not normal in $G(n, R)$. By $E(n, R, I)$ we mean the the normalization of $E(n, I)$ in $G(n, R)$, i.e. the relative elementary group generated by elements of the type $ge_{ij}(f)ge_{ji}(h)(ge_{ij}(f))^{-1}$, where $f \in R$ and $h \in I$. While working on the polynomial ring $R[X]$, by writing $\alpha(X) \in E(n, R[X], I[X])$ we mean $\alpha(X)$ is $I_n$ modulo $I$, and of the form $ge_{ij}(f(X))ge_{ji}(h(X))(ge_{ij}(f(X)))^{-1}$, where $f(X) \in R[X]$ and $h(X) \in I[X]$, as $E(n, R[X], I[X])$ is the normalization of $E(n, I[X])$ in $G(n, R[X])$.

**Lemma 2.13.** If $\epsilon \in E(n, R, I)$, then there exists $\bar{\epsilon} \in E(n, R \oplus I)$ such that $\phi(\bar{\epsilon}) = \epsilon$. (In fact, the converse is also true).

**Proof.** Let $\epsilon = (\epsilon_{ij})$ be a generator of the type $ge_{ij}(a)ge_{ji}(x)ge_{ij}(-a)$, where $a \in R$ and $x \in I$. We then have $1$'s, $1 + ax$ and $1 - ax$ on the diagonal of $\epsilon$ and zeros, $-a^2x$ and $x$ as the non-diagonal elements. We get a new matrix $\bar{\epsilon}$ by taking the corresponding diagonal elements as $(1,0)$, $(1,ax)$ and $(1,-ax)$ and the corresponding non-diagonal elements as $(0,0)$, $(0,-a^2x)$ and $(0,x)$ which are elements of the ring $R \oplus I$. Using the definition of multiplication in the ring $R \oplus I$, we can see that

$$
\bar{\epsilon} = ge_{ij}((a,0))ge_{ji}((0,x))ge_{ij}(-(a,0)) \in E(n, R \oplus I),
$$

and applying the homomorphism $\phi$ to it we obtain $\phi(\bar{\epsilon}) = \epsilon$. \hfill $\square$

**Lemma 2.14.** Let $\alpha \in G(n, R, I)$. Then there exists $\bar{\alpha} \in G(n, R \oplus I)$ such that $\phi(\bar{\alpha}) = \alpha$.

**Proof.** Let $\alpha = (\alpha_{ij}) \in G(n, R, I)$. Then $\alpha_{ii} = 1 + a_{ii}$ and $\alpha_{ij} = a_{ij}$ for $i \neq j$ where $a_{ij} \in I$ for all $i, j$. We get a new matrix $\bar{\alpha} = \bar{\alpha}_{ij}$, where $\bar{\alpha}_{ii} = (u_i, a_{ii})$ and $\bar{\alpha}_{ij} = (0, a_{ij})$ for $i \neq j$. The entries in $\bar{\alpha}$ are in the ring $R \oplus I$. Using the definition of multiplication in the ring $R \oplus I$, we can see that $\bar{\alpha} \in G(n, R \oplus I)$ and applying the homomorphism $\phi$ we obtain $\phi(\bar{\alpha}) = \alpha$. \hfill $\square$

Now we state the main theorem of this article. For the absolute case; i.e. for $I = R$ we refer to [6].

3. Equivalence: Relative L-G Principle and Normality

**Theorem 3.1.** Let $R$ be a commutative ring with identity, and $I \subseteq R$ an ideal of the ring $R$. Let $v, w$ be column vectors in $R^n$ with $w \in P^n$. Then the followings are equivalent:
Lemma 3.4. (Splitting Property): follows from the proofs of the absolute cases.

For a uniform proof cf. Lemma 3.3.

Remark 3.2. Since (6) will be established in Lemma 3.9, it follows that all the above statement (1)-(7) of Theorem 3.1 hold for commutative symplectic groups and ([29], §).

Let

\[ E(n, R, I) \]

Before proving the theorem we first collect a few lemmas.

Lemma 3.3. The group \( E(n, R, I) \) satisfies the property:

\[ [E(n, R, I), E(n, R)] = E(n, R, I). \]

Proof. cf. [5] for the general linear groups, ([16], Theorem 1.1) for the symplectic groups and ([29], §2) for the orthogonal groups.

Below we state few useful well-known lemmas. For the proofs cf. [5] for the linear groups, [16] for the symplectic groups, [29] for the orthogonal groups. For a uniform proof cf. [9], [5]. The analogue results for the relative cases follows from the proofs of the absolute cases.

Lemma 3.4. (Splitting Property): \( ge_{ij}(x+y) = ge_{ij}(x)ge_{ij}(y) \), \( \forall x, y \in R \).

Lemma 3.5. Let \( G \) be a group, and \( a_i, b_i \in G \), for \( i = 1, \ldots, n \). Then

\[ \prod_{i=1}^{n} a_i b_i = \prod_{i=1}^{n} r_i b_i r_i^{-1} \prod_{i=1}^{n} a_i, \text{ where } r_i = \prod_{j=1}^{i} a_j. \]

Lemma 3.6. The group \( G(n, R[X], (X)) \cap E(n, R[X], I[X]) \) is generated by the elements of the type \( ege_{ij}(Xh(X))e^{-1} \), where \( e \in E(n, R[X]), h(X) \in I[X] \).
Lemma 3.7. For $m > 0$, and $h(Y) \in I[Y]$, there are $h_t(X, Y, Z) \in I[X, Y, Z]$ such that
\[ ge_{pq}(Z)ge_{ij}(X^{2m}h(Y))ge_{pq}(-Z) = \prod_{t=1}^{k} ge_{pq}(X^{m}h_t(X, Y, Z)). \]

Corollary 3.8. If $\varepsilon = \varepsilon_1\varepsilon_2 \cdots \varepsilon_r$, where each $\varepsilon_j$ is an elementary generator, and $h(Y) \in I[Y]$, then there are $h_t(X, Y) \in I[X, Y]$ such that
\[ \varepsilon ge_{pq}(X^{2^m}h(Y))\varepsilon^{-1} = \prod_{t=1}^{k} ge_{pq}(X^{m}h_t(X, Y)). \]

Proof. Follows by induction on $r$ and using Lemma 3.7. □

We show that statement (6) of Theorem 3.1 is true over an arbitrary associative ring $R$ with 1.

Lemma 3.9. Let $R$ be a ring and $v \in E(n, R, I)e_1$. Let $w \in I^n$ be a column vector such that $\langle v, w \rangle = 0$. Then $I_n + M(v, w) \in E(n, R, I)$.

Proof. Let $v = \varepsilon e_1$, where $\varepsilon = (\varepsilon_{ij}) \in E(n, R, I)$. Hence $\varepsilon_{ii} = 1 + a_{ii}$ and $\varepsilon_{ij} = a_{ij}$ for $i \neq j$, where $a_{ij} \in I$ for all $i, j$. Let $\varepsilon = \varepsilon_{ij}$, where $\varepsilon_{ii} = (1, a_{ii})$, and $\tilde{\varepsilon}_{ij} = (0, a_{ij})$ for $i \neq j$. Let $\tilde{e}_1 = ((1, 0), (0, 0), \ldots, (0, 0))$, and
\[ \tilde{v} = ((1, v_1), (0, v_2), \ldots, (0, v_n)) \in (R \oplus I)^n, \]
\[ \tilde{w} = ((0, w_1), (0, w_2), \ldots, (0, w_n)) \in (0 \oplus I)^n. \]

Then it follows that
\[ I_n + M(\tilde{v}, \tilde{w}) = \tilde{\varepsilon}(I_n + M(\tilde{e}_1, \tilde{w}_1))\varepsilon^{-1}, \]
and $\tilde{w}_1 = \begin{cases} \tilde{\varepsilon} \tilde{w} & \text{for linear case} \\ \varepsilon^{-1} \tilde{w} & \text{otherwise.} \end{cases}$

Since $\langle (\tilde{e}_1, \tilde{w}_1) \rangle = \langle \tilde{v}, \tilde{w} \rangle = 0$, we get
\[ \tilde{w}_1^t = \begin{cases} ((0, 0), (0, w_{12}), (0, w_{13}), \ldots, (0, w_{1n})) & \text{for linear case} \\ ((0, w_{11}), (0, 0), (0, w_{13}), \ldots, (0, w_{1n})) & \text{otherwise.} \end{cases} \]

Therefore,
\[ I_n + M(\tilde{v}, \tilde{w}) = \begin{cases} \prod_{j=2}^{n} \tilde{\varepsilon} ge_{1j}(0, w_{1j})\tilde{\varepsilon}^{-1} & \text{for linear case} \\ \prod_{j=1}^{n} \tilde{\varepsilon} ge_{1j}(0, w_{1j})\tilde{\varepsilon}^{-1} & \text{otherwise.} \end{cases} \]

Hence $I_n + M(\tilde{v}, \tilde{w}) \in E(n, R \oplus I, 0 \oplus I)$. Now applying the homomorphism $\phi$ it follows that $I_n + M(v, w) \in E(n, R, I)$; as desired. □

Note that the above implication is true for any associative ring with identity.

Remark 3.10. It is well-known that every ring is a direct limit of Noetherian rings. Hence we may consider $R$ to be Noetherian.

We shall use following lemma frequently and sometime in a subtle way; e.g. for the implication (4) $\Rightarrow$ (3).
Lemma 3.11. (([12], Lemma 5.1)) Let $R$ be a Noetherian ring and $s \in R$. Then there exists a natural number $k$ such that the canonical homomorphism $G(n, s^k R) \rightarrow G(n, R_s)$ (induced by localization homomorphism $R \rightarrow R_s$) is injective. Moreover, it follows that the map $E(n, R, s^k R) \rightarrow E(n, R_s)$ for $k \in \mathbb{N}$ is injective.

Proof of Theorem 3.1: We shall assume the result for the absolute case; i.e. when $I = R$. The implication (7) $\Rightarrow$ (6): Obvious. We prove, (6) $\Rightarrow$ (5):

Note that we have assumed that (6) holds for any commutative ring, in particular for the ring $R[X]$, and the matrix $I_n + XM(v, w)$. Replacing $R$ by $R[X]$ in (6) we get that $I_n + XM(v, w) \in E(n, R[X], I[X])$. Let $v = v_1$, where $v \in E(n, R, I)$. As before, let $\tilde{v} = ((1, v_1), (0, v_2), \ldots, (0, v_n)) \in (R \oplus I)^n$, and $\tilde{w} = ((0, w_1), (0, w_2), \ldots, (0, w_n)) \in (0 \oplus I)^n$. Hence as in the proof of Lemma 3.9 we can write

$$I_n + XM(\tilde{v}, \tilde{w}) = \begin{cases} \prod_{j=1}^n \tilde{e} ge_{1j}((0, Xw_{1j})) \tilde{e}^{-1} & \text{for linear case} \\ \prod_{j=1}^n \tilde{e} ge_{1j}((0, Xw_{1j})) \tilde{e}^{-1} & \text{otherwise } \end{cases} \quad (\star)$$

Now we split the proof into following two cases.

Case I: $v$ is an elementary generator of the type $ge_{pq}(x), x \in R$. First applying the homomorphism $X \mapsto X^2$ and then applying Lemma 3.7 over $R[X]$ we get

$$I_n + X^2 M(\tilde{v}, \tilde{w}) = \prod_{j} \left( k \prod_{t=1}^k ge_{pq(t)q_j(t)}(Xh_{j(t)}(X)) \right),$$

where $h_{j(t)}(X) \in ((0 \oplus I)[X])$. Again, as before applying the homomorphism $\phi$ it follows that

$$I_n + X^2 M(v, w) = \prod_{j} \left( k \prod_{t=1}^k ge_{pq(t)q_j(t)}(Xh_{j(t)}(X)) \right),$$

where $h_{j(t)}(X) \in I[X]$; as desired. Hence the result also follows for $d > 0$.

Case II: $v$ is a product of elementary generators of the type $ge_{pq}(x)$. Let $\mu(v) = r$. Arguing as before, the result follows by applying the homomorphism $X \mapsto X^{2r}$ using the Corollary 3.8.

(5) $\Rightarrow$ (4): Given that $\alpha_s(X) \in E(n, R_s[X], I_s[X])$, where $s$ is non-nilpotent element in the ring $R$, and $\alpha(0) = I_n$. By Lemma 2.13 there exists $\tilde{\alpha}_{(s, 0)}(X) \in E(n, R_s[X] \oplus I_s[X])$, where the element $(s, 0)$ will remain non-nilpotent in the ring $R \oplus I$, and $\phi(\tilde{\alpha}_{(s, 0)}(X)) = \alpha_s(X)$.

Also, by Lemma 3.6 $\alpha_s(X)$ can be written as a product of the matrices of the form $\varepsilon_s ge_{ij}(Xh(X)) \varepsilon_s^{-1}$, with $h(X) \in I_s[X]$ and $\varepsilon_s \in E(n, R_s)$. Hence using the proof of Lemma 2.13 it follows that $\tilde{\alpha}_{(s, 0)}(X)$ can be written as a product of the matrices of the form $\tilde{\varepsilon}_{(s, 0)} ge_{ij}((0, Xh(X))) \tilde{\varepsilon}_{(s, 0)}^{-1}$, where $\phi(\tilde{\varepsilon}_{(s, 0)}) = \varepsilon_s$ and $((0, Xh(X)) \in ((R \oplus I)_{(s, 0)}[X])$. 

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Applying the homomorphism $X \mapsto XT^d$, where $d \gg 0$, from the polynomial ring $R[X]$ to the polynomial ring $R[X, T]$, we consider $\overline{\alpha}_{(s, 0)}(XT^d)$. Note that $R_s[X, T] \cong (R_s[X])[T]$. Now, using the Equation $(\star)$ as in the proof of (6) $\Rightarrow$ (5), we can rewrite $\overline{\alpha}_{(s, 0)}(XT^d)$ as the form $I_n + XT^d M(v, w)$; for some suitable $v, w$ over the ring $(R_s[X] \oplus I_s[X])[T]$. Hence by (5) we can write $\overline{\alpha}_{(s, 0)}(XT^d)$ as a product of elementary generators of general linear (symplectic/orthogonal resp.) group such that each of those elementary generator is congruent to identity modulo the ideal $(T)$ over the ring $((R_s \oplus I_s)[X])[T]$. Let $l$ be the maximum of the powers occurring in the denominators of those elementary generators. Again, as $R$ assumed to be Noetherian, by applying the homomorphism $T \mapsto (s, 0)^mT$, for $m \geq l$, it follows from Lemma 3.11 that by (uniquely) identifying it’s lift over the ring $(R \oplus I)[X, T]$ we can write $\overline{\alpha}_{(s, 0)}((s, 0)^mXT^d)$ as a product of elementary generators of the general linear (symplectic/orthogonal resp.) group such that each of those elementary generator is congruent to identity modulo $(T)$. i.e. there exists some $\overline{\beta}(X, T) \in E(n, (R \oplus I)[X, T])$ such that $\overline{\beta}(0, 0) = I_n$ and $\overline{\beta}_{(s, 0)}(X, T) = \overline{\alpha}_{(s, 0)}((s, 0)XT^d)$ for some $(b, 0) \in (s, 0)^m(R \oplus I)$. Finally, by substituting $T = (1, 0)$ and using Lemma 3.11 we get $\overline{\alpha}((b, 0)X) \in E(n, (R \oplus I)[X])$. Hence the result follows applying $\phi$ as before.

(4) $\Rightarrow$ (3): Since $\alpha_m(X) \in E(n, R_m[X], \overline{I}_m[X])$, for all $m \in \text{Max}(R)$, for each $m$ there exists $s \in R \setminus m$ such that $\alpha_s(X) \in E(n, R_s[X], I_s[X])$. Observe that

$$R_s[X] \oplus I_s[X] \cong (R_s \oplus I_s)[X] \cong (R \oplus I)_s[X].$$

Hence by Lemma 2.13 applied to the base ring $R_s[X]$, there exists $\overline{\alpha}_{(s, 0)}(X) \in E(n, (R \oplus I)_{(s, 0)}[X])$ such that $\phi_s(\overline{\alpha}_{(s, 0)}) = \alpha_s$. Let

$$\overline{\theta}(X, T) = \overline{\alpha}_{(s, 0)}(X + T)\overline{\alpha}_{(s, 0)}(T)^{-1}.$$ 

Then $\overline{\theta}(X, T) \in E(n, (R \oplus I)_{(s, 0)}[X, T])$ and $\overline{\theta}(0, T) = I_n$. By the condition (4) of the Theorem, applied to the base ring $(R \oplus I)[T]$, there exists $\overline{\beta}(X) \in E(n, (R \oplus I)[X, T])$ such that

$$\overline{\beta}_{(s, 0)}(X) = \overline{\theta}((b, 0)X, T).$$

with $(b, 0) \in (s, 0)^l(R \oplus I)$ for some $l \gg 0$.

Now, using the Noetherian property of $R \oplus I$, as mentioned in the Remark 3.10, we may consider a finite cover of $R \oplus I$, say $(s_1, 0) + \cdots + (s_r, 0) = (1, 0)$. Since for $l \gg 0$, the ideal $\langle (s_1, 0)^l, \ldots, (s_r, 0)^l \rangle = R \oplus I$, we choose $(b_1, 0), \ldots, (b_r, 0) \in R \oplus I$, with $(b_i, 0) \in (s_i, 0)^l(R \oplus I)$, $l \gg 0$ such that (I) holds and $(b_1, 0) + \cdots + (b_r, 0) = (1, 0)$. Hence for each $i = 1, \ldots, r$, there exists $\overline{\beta}^i(X) \in E(n, (R \oplus I)[X, T])$ such that $\overline{\beta}^i_{(s_i, 0)}(X) = \overline{\theta}((b_i, 0)X, T)$. Now,

$$\prod_{i=1}^r \overline{\beta}^i(X) \in E(n, (R \oplus I)[X, T]).$$

But,

$$\overline{\alpha}_{s'_1 \cdots s'_r}(X) = \left( \prod_{i=1}^{r-1} \theta_{s'_i \cdots s'_r} \overline{\beta}^i_{s'_i \cdots s'_r}((b_i, 0)X, T) \right)_{T=b_{i+1}X+\cdots+b_rX} \overline{\theta}_{s'_1 \cdots s'_{r-1}}((b_r, 0)X, 0),$$
where \( s'_i = (s_i, 0) \) and \( b'_i = (b_i, 0) \) for each \( i = 1, \ldots, r \). Now \( \alpha(0) = I_n \). Also, as a consequence of the Lemma \ref{Lem3.11} it follows that the map 

\[
E(n, R, (s, 0)^k (R \oplus I)[X]) \to E(n, (R \oplus I)_{(s, 0)}[X])
\]

for \( k \in \mathbb{N} \) is injective for each \( s = s_i \). Hence by (uniquely) identifying \( \tilde{\alpha}(s'_1 \cdots s'_r)(X) \) with its lift, we conclude \( \tilde{\alpha}(X) \in E(n, R[X] \oplus I[X]) \). Finally, applying the map \( \phi \) we get \( \alpha(X) \in E(n, R[X], I[X]) \); as desired.

(3) \( \Rightarrow \) (2): This is the implication where we use the commutative property of the base ring \( R \). Let \( \alpha(X) = I_n + XM(v, w) \), where \( v \in Um_n(R, I) \) and \( \langle v, w \rangle = 0 \) and \( w \in I^n \). Then \( \alpha(0) = I_n \). Let \( v = (1 + v_1, v_2, \ldots, v_n) \) and \( w = (w_1, w_2, \ldots, w_n) \in I^n \), with \( v_i, w_i \in I \) for \( i = 1, \ldots, n \). Then by Lemma \ref{Lem3.10}, \( \alpha_m(X) \) is elementary for every maximal ideal \( m \) in \( R \). Hence \( \alpha(X) \) is elementary by (3).

(2) \( \Rightarrow \) (1): Let \( \epsilon \in E(n, R, I) \) and \( \gamma = (\gamma_{ij}) \in G(n, R) \). There exist \( \tilde{\epsilon} \in E(n, R \oplus I) \) and \( \tilde{\gamma} = ((\gamma_{ij}, 0)) \in G(n, R \oplus I) \) respectively such that \( \phi(\tilde{\epsilon}) = \epsilon \) and \( \phi(\tilde{\gamma}) = \gamma \). Using (2) \( \Rightarrow \) (1) of the absolute case we get \( \tilde{\gamma} \tilde{\epsilon} \tilde{\gamma}^{-1} \in E(n, R \oplus I) \) as \( E(n, R \oplus I) \leq G(n, R \oplus I) \) and applying the homomorphism \( \phi \) it follows \( \gamma \gamma^{-1} \in E(n, R, I) \); as required.

(1) \( \Rightarrow \) (7): Let \( v = \gamma e_1 \) where \( \gamma \in G(n, R) \). Then there exists \( \tilde{\gamma} \in G(n, R \oplus I) \) such that \( \phi(\tilde{\gamma}) = \gamma \). Let \( \tilde{v} = \gamma e_1 \) and \( \tilde{w} = ((0, w_1), (0, w_2), \ldots, (0, w_n)) \in (0 \oplus I)^n \). We have \( \langle \tilde{v}, \tilde{w} \rangle = 0 \). Hence, using (1) \( \Rightarrow \) (7) of the absolute case it follows that \( I_n + M(\tilde{v}, \tilde{w}) \in E_n(R \oplus I) \). Now applying the homomorphism \( \phi \) we get \( I_n + M(v, w) \in E(n, R, I) \); as required.

The above implications prove the equivalence of the statements. \( \square \)

Remark 3.12. Assuming the result for the absolute case treated in \( [6] \) one can give simpler proofs of the steps (5) \( \Rightarrow \) (4), and (4) \( \Rightarrow \) (3). But, there is a gap in the proof of the absolute case in \( [6] \), as mentioned in \( [9] \). The gap was filled in \( [9] \) by proving results for Bak’s unitary groups, which cover linear, symplectic and orthogonal groups, and some more classical type groups. To make this note self-contained, we have given the detailed proofs of those steps.

4. RELATIVE L-G PRINCIPLE FOR TRANSVECTION SUBGROUPS

In this section we shall state auxiliary results without detailed proofs. For definitions of symplectic and orthogonal modules and their transvection subgroups we refer to \( [7] \).

In \( [7] \), the first and third authors together with Anthony Bak generalized Quillen-Suslin’s local-global principle for the transvection subgroups of the projective, symplectic and orthogonal modules. As before, all three cases were treated uniformly. We observe below how to obtain relative versions of that local-global principle. To state the results we need to recall a few notations.

Notation 4.1. In the sequel \( P \) will denote either a finitely generated projective \( R \)-module of rank \( n \), a symplectic \( R \)-module or an orthogonal \( R \)-module.
of even rank \( n = 2m \) with a fixed form \((,\)\). And \( Q \) will denote \( P \oplus R \) in the linear case, and \( P \perp R^2 \), otherwise. We will use the notation \( Q[X] \) to denote \((P \oplus R)[X] \) in the linear case and \((P \perp R^2)[X] \), otherwise. We assume that the rank of the projective module is \( n \geq 2 \), when dealing with the linear case, and \( n \geq 6 \), when considering the symplectic and the orthogonal cases. For a finitely generated projective \( R \)-module \( M \) we use the notation \( \text{G}(M) \) to denote \( \text{Aut}(M) \), \( \text{Sp}(M,\langle,\rangle) \) and \( \text{O}(M,\langle,\rangle) \) respectively; denote \( \text{SL}(M) \), \( \text{Sp}(M,\langle,\rangle) \) and \( \text{Trans}(M) \), \( \text{Trans}_{\text{Sp}}(M) \) and \( \text{Trans}_{\text{O}}(M) \) respectively; and \( \text{ET}(M) \) to denote \( \text{ET}_{\text{Sp}}(M) \), \( \text{ET}_{\text{Sp}}(M) \) and \( \text{ET}_{\text{O}}(M) \) respectively.

We shall also assume the following hypotheses:

(H1) for every maximal ideal \( m \) of \( R \), the symplectic (orthogonal) module \( Q_m \) is isomorphic to \( R_m^{2n+2} \) for the standard bilinear form \( \mathbb{H}(R_m^{n+1}) \).

(H2) for every non-nilpotent \( s \in R \), if the projective module \( Q_s \) is free \( R_s \)-module, then the symplectic (orthogonal) module \( Q_s \) is isomorphic to \( R_s^{2n+2} \) for the standard bilinear form \( \mathbb{H}(R_s^{n+1}) \).

We recall the following fact just to remind the reader that in the free case the transvection subgroups coincides with the elementary subgroups. Here the maps \( \varphi, \varphi_p, \sigma \) and \( \tau \) are as defined in [7].

**Lemma 4.2.** If the projective module \( P \) is free of finite rank \( n \) (in the symplectic and the orthogonal cases we assume that the projective module is free for the standard bilinear form), then \( \text{Trans}(P) = E_n(R) \), \( \text{Trans}_{\text{Sp}}(P) = E\text{Sp}_n(R) \) and \( \text{Trans}_{\text{O}}(P) = E\text{O}_n(R) \) for \( n \geq 3 \) in the linear case and for \( n \geq 6 \) otherwise.

**Proof.** In the linear case, for \( p \in P \) and \( \varphi \in P^* \) if \( P = R^n \) then \( \varphi_p : R^n \to R \to R^n \). Hence \( 1 + \varphi_p = I_n + v.w^t \) for some column vectors \( v \) and \( w \) in \( R^n \). Since \( \varphi(p) = 0 \), it follows that \( (v,w) = 0 \). Since either \( v \) or \( w \) is unimodular, it follows that \( 1 + \varphi_p = I_n + v.w^t \in E_n(R) \). Similarly, in the non-linear cases we have \( \sigma(u,v)(p) = I_n + v.\tilde{w} + w.\tilde{v} \), and \( \tau(u,v)(p) = I_n + v.\tilde{w} - w.\tilde{v} \), where either \( v \) or \( w \) is unimodular and \( (v,w) = 0 \). (Here \( \sigma(u,v) \) and \( \tau(u,v) \) are as in the definition of symplectic and orthogonal transvections.) Classically, these are known to be elementary matrices - for details see [24] for the linear case, [16] for the symplectic case, and [29] for the orthogonal case.

**Remark 4.3.** Lemma 4.2 holds for \( n = 4 \) in the symplectic case. This will follow from Remark 4.4.

**Remark 4.4.** \( \text{ESp}_4(A) \) is a normal subgroup of \( \text{Sp}_4(A) \) by ([16], Corollary 1.11). Also \( \text{ESp}_4(A[X]) \) satisfies the Dilation Principle and the Local-Global Principle by ([16], Theorem 3.6). Since we were intent on a uniform proof, these cases have not been covered above by us.

**Proposition 4.5.** (Relative Dilation Principle) Let \( R \) be a commutative ring with identity, and \( I \subseteq R \) an ideal in \( R \). Let \( P \) and \( Q \) be as in [4.1]. Assume that (H2) holds. Let \( s \) be a non-nilpotent in \( R \) such that \( P_s \) is free, and let
σ(X) ∈ G(Q[X], I[X]) with σ(0) = Id. Suppose

\[
σ_s(X) ∈ \begin{cases} E(n + 1, R_s[X], I_s[X]) & \text{in the linear case,} \\ E(n + 2, R_s[X], I_s[X]) & \text{otherwise.} \end{cases}
\]

Then there exists \(\tilde{σ}(X) \in ET(Q[X], I[X])\) and \(l > 0\) such that \(\tilde{σ}(X)\) localizes to \(σ(bX)\) for some \(b ∈ (s^l)\) and \(\tilde{σ}(0) = Id\).

Proof. Follows by imitating the technique explained in [7], and following steps mentioned in Section 2.

Theorem 4.6. (Relative Local-Global Principle) Let \(R\) be a commutative ring with identity, and \(I ⊆ R\) an ideal in \(R\). Let \(P\) and \(Q\) be as in 4.1. Assume that (H1) holds. Suppose \(σ(X) ∈ G(Q[X], I[X])\) with \(σ(0) = Id\). If

\[
σ_p(X) ∈ \begin{cases} E(n + 1, R_p[X], I_p[X]) & \text{in the linear case,} \\ E(n + 2, R_p[X], I_p[X]) & \text{otherwise} \end{cases}
\]

for all \(p ∈ \text{Spec}(R)\), then \(σ(X) ∈ ET(Q[X], I[X])\).

Proof. Follows by using similar technique as in (4) ⇒ (3) in Theorem 3.1 of [6], and arguing as in Section 2.

Remark 4.7. The authors believe that the above method using the “Noetherian Excision ring”, makes it possible to deduce the relative versions of almost all the results mentioned in [6], [7], [9], and the results mentioned in [8] between pgs 35–40.

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