QUASI-ANTICHAIN CHERMAK-DELGADO LATTICES OF
FINITE GROUPS

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Dedicated to Otto H. Kegel for the occasion of his eightieth birthday.

Abstract. The Chermak-Delgado lattice of a finite group is a dual, modular sublattice of the subgroup lattice of the group. This paper considers groups with a quasi-antichain interval in the Chermak-Delgado lattice, ultimately proving that if there is a quasi-antichain interval between subgroups $L$ and $H$ with $L \leq H$ then there exists a prime $p$ such that $H/L$ is an elementary abelian $p$-group and the number of atoms in the quasi-antichain is one more than a power of $p$. In the case where the Chermak-Delgado lattice of the entire group is a quasi-antichain, the relationship between the number of abelian atoms and the prime $p$ is examined; additionally, several examples of groups with a quasi-antichain Chermak-Delgado lattice are constructed.

This paper pursues the nature of the Chermak-Delgado lattice of a finite group. The Chermak-Delgado lattice was introduced by Chermak and Delgado [4]. Isaacs [6] re-introduced the lattice, sparking further study that resulted in [3] and [2]. In this article, we provide three primary contributions to the study of Chermak-Delgado lattices: a description of the structure of groups with a quasi-antichain (defined below) as an interval in the Chermak-Delgado lattice, results that narrow the possible structure of a quasi-antichain realized as a Chermak-Delgado lattice, and examples to illustrate the breadth of possibilities. Among these contributions is a proof that if a Chermak-Delgado lattice has an interval which is a quasi-antichain then the width must be a power of a prime plus 1.

Throughout the article, let $G$ be a finite group and $p$ be a prime. The Chermak-Delgado lattice of a finite group $G$ consists of subgroups $H \leq G$ such that $|H||C_G(H)|$ is maximal among all subgroups of $G$. For any subgroup $H$ of $G$, the product $|H||C_G(H)|$ is called the Chermak-Delgado measure of $H$ (in $G$) and is denoted by $m_G(H)$; if the group $G$ is clear from context then we write simply $m(H)$. To denote the maximum possible Chermak-Delgado measure of a subgroup in $G$ we write $m^*(G)$ and we refer to the set of all subgroups with measure attaining that maximum as the Chermak-Delgado lattice of $G$, or $CD(G)$.

It is known that the Chermak-Delgado lattice is a modular self-dual lattice and if $H, K \in CD(G)$ then $HK = KH = \langle H, K \rangle$. The duality of the Chermak-Delgado lattice is a result of the fact that if $H \in CD(G)$ then $C_G(H) \in CD(G)$ and $H = C_G(C_G(H))$. Moreover, if $M$ is the maximum subgroup in the Chermak-Delgado lattice of a group $G$ then the Chermak-Delgado lattices of $G$ and $M$ coincide. It is additionally known that the co-atoms in the Chermak-Delgado lattice are normal in $M$ and consequently the atoms, as centralizers of normal subgroups, are also normal in $M$. In [2], groups whose Chermak-Delgado lattice is a chain were studied; a chain of length $n$, where $n$ is a positive integer, is a totally ordered lattice with
n + 1 subgroups. We call a lattice consisting of a maximum, a minimum, and the atoms of the lattice a quasi-antichain and the width of the quasi-antichain is the number of atoms. A quasi-antichain of width 1 is also a chain of length 2.

Let \( L \leq H \leq G \); we use \([L, H]\) to denote the interval from \( L \) to \( H \) in a sublattice of the lattice of subgroups of \( G \). If \([L, H]\) is an interval in \( \text{CD}(G) \) then the duality of the Chermak-Delgado lattice tells us that \([C_{G}(H), C_{G}(L)]\) is an interval in \( \text{CD}(G) \).

Of course, these intervals may overlap or even coincide exactly. In Section 1 we make no assumption about the intersection of \([L, H]\) and \([C_{G}(H), C_{G}(L)]\); in the final two sections we study the situation where these two intervals not only are equal, but are the entirety of \( \text{CD}(G) \).

1. Quasi-antichain Intervals in Chermak-Delgado Lattices

Let \( G \) be a group with \( L < H \leq G \) such that \([L, H]\) is an interval in \( \text{CD}(G) \). The main theorem of this section establishes that if \([L, H]\) is a quasi-antichain of width \( w \geq 3 \) then there exists a prime \( p \) and positive integers \( a, b \) with \( b \leq a \) such that \( H/L \) is an elementary abelian \( p \)-group with order \( p^{2a} \) and \( w = p^b + 1 \). To make the role of the duality more transparent in the proofs, set \( H^* = C_{G}(L) \) and \( L^* = C_{G}(H) \). Observe that \( C_{G}(H^*) = L \) and \( C_{G}(L^*) = H \).

We start with a general statement about subgroups that are in the interval \([L, H]\) in \( \text{CD}(G) \). Let \( p \) be a prime dividing the index of \( L \) in \( H \). For this result, we remind the reader that the notation \( \Omega_{k}(M) \), where \( k \) is a positive integer and \( M \) is any group, denotes the subgroup of \( M \) generated by the elements whose order divides \( p^k \).

The hypothesis of Proposition 1 may initially sound restrictive: We require that \( G \) be a group with \([L, H]\) in \( \text{CD}(G) \) such that \([H H^*, H H^*] \leq L \cap L^* \). Note that \( L \leq H \) and \( L^* \leq H^* \), so the quotient groups described in Proposition 1 are well-defined. Moreover, notice that the hypotheses of the proposition occur when \( G \) is a \( p \)-group of nilpotence class 2 and \( H = G \in \text{CD}(G) \).

Proposition 1. Let \( G \) be a group with an interval \([L, H]\) in \( \text{CD}(G) \) such that \([H H^*, H H^*] \leq L \cap L^* \). Suppose that \( p \) is a prime dividing \( |H/L| \). The subgroups \( A_k(H), B_k(H) \leq H \) where \( A_k(H)/L = \Omega_{k}(H/L) \) and \( B_k(H) = \langle x^{p^k} \mid x \in H \rangle \) are members of \( \text{CD}(G) \) for all positive values of \( k \), as are the similarly defined subgroups \( A_k(H^*), B_k(H^*) \) of \( H^* \).

Proof. Let \( k \) be a positive integer. Without loss of generality, assume that \( |A_k(H)/L| \geq |A_k(H^*)/L^*| \). We first show that \( C_{G}(A_k(H)) = B_k(H^*) \) and that \( A_k(H) \in \text{CD}(G) \).

Observe that if \( x \in H \) and \( y \in H^* \) then \( [x, y] \in [H, H^*] \leq L \cap L^* = C_{G}(H^*) \cap C_{G}(H) \). Therefore \( [x^p, y] = [x, y^p] \) whenever \( x \in H \) and \( y \in H^* \). Moreover, if \( x \in A_k(H) \) then \( x^{p^k} \in L \), so \( [x^{p^k}, y] = 1 \) for all \( y \in C_{G}(L) = H^* \). Thus if \( x \in A_k(H) \) and \( y \) is a generator of \( B_k(H^*) \) then \( [x, y] = 1 \), therefore \( B_k(H^*) \leq C_{G}(A_k(H)) \).

Since the quotient \( H/L \) is abelian, \( |A_k(H)/L| = |H/B_k(H)| \) or equivalently \( |A_k(H)/L||B_k(H)| = |H| \). The same is true regarding \( |H^*| \) and the subgroups \( A_k(H^*), B_k(H^*) \). Thus the measure of \( A_k(H) \) in \( G \) can be calculated as follows:

\[
m(A_k(H)) = |A_k(H)||C_{G}(A_k(H))| \geq |A_k(H)||B_k(H^*)| = |A_k(H)/L||L||B_k(H^*)| \geq |A_k(H^*)/L^*||L||B_k(H^*)| = \frac{|H^*|L}{|L|} = \frac{|C_{G}(L)||L|}{L} = m^*(G).
\]
Therefore each inequality above is actually an equality, with \( C_G(A_k(H)) = B_k(H^*) \) and \( |A_k(H)/L| = |A_k(H^*)/L^*| \). Additionally \( A_k(H), B_k(H^*), A_k(H^*), B_k(H) \in CD(G) \).

For the rest of the paper, we study intervals that are quasi-antichains. Ultimately we will use Proposition 1 to show that \( A \) is a quasi-antichain of width \( w \geq 3 \). The next two propositions establish important facts about the atoms of a quasi-antichain interval in \( CD(G) \), as well as show that such an interval satisfies the hypothesis of Proposition 1.

Let \( [L, H] \) be a quasi-antichain of width \( w \) throughout the remainder of the article. Let the \( w \) atoms of the quasi-antichain be denoted by \( K_1, K_2, \ldots, K_w \). The interval \( [L^*, H^*] \) is also a quasi-antichain in \( CD(G) \), with atoms \( C_G(K_i) \) where \( 1 \leq i \leq w \). For each \( i \), let \( K_i^* = C_G(K_i) \) so that \( C_G(K_i^*) = K_i \).

**Proposition 2.** If \( K_1, K_2 \) are distinct atoms of the quasi-antichain then \( K_i \subseteq H \) for \( i = 1, 2 \), \( L \leq H \), and \( [K_1, K_2] \leq L \) and analogously for \( H^*, K_1^*, K_2^*, \) \( L^* \).

Moreover, \( [K_1 : L] = [K_2^* : L^*] \).

If \( w \geq 3 \) then \( K_i/L \cong K_j/L \) and \( K_i^*/L^* \cong K_j^*/L^* \) for all \( i, j \) with \( 1 \leq i, j \leq w \). Furthermore:

\[ |H/L| = |H/K_1|^2 = |H^*/K_1^*|^2 = |H^*/L^*|. \]

**Proof.** Let \( K_1, K_2 \in CD(G) \) with \( L < K_i < H \) for \( i = 1, 2 \). Because the interval \( [L, H] \) is a quasi-antichain, \( H = K_1 K_2 \) and \( K_1 \cap K_2 = L \). From this structure and because \( H \) cannot equal \( K_1 K_2^* \) for \( h \in H \), it follows that \( K_1 \leq H \) (similarly for \( K_2 \)). Therefore \( L \leq H \) and \( [K_1, K_2] \leq L \). The equality \( m^*(G) = m(H) = m(K_2) \) implies

\[ \frac{|K_2|}{|L|} = |C_G(K_2)| = |C_G(H)| = |K_2|/|C_G(K_2)|, \]

and consequently \( |K_1 : L| = |C_G(K_2) : C_G(H)| = |K_2^* : L^*| \).

In the situation that \( w \geq 3 \), then \( H = K_1 K_2 = K_3 K_2 \) where \( K_1 \cap K_2 = K_2 \cap K_3 = L \) and thus \( |K_i| = |K_j| \) for all \( i, j \) with \( 1 \leq i, j \leq w \). This additionally yields \( K_i/L \cong K_j/L \) for all \( i, j \). From the Isomorphism Theorems it follows that \( |H/K_1| = |K_1/L|^2 \).

The same arguments applied to the quasi-antichain \( [L^*, H^*] \) yield the remaining assertions.

**Proposition 3.** If \( w \geq 3 \) then \( [H, H^*] \leq L \) and \( H^*/L^* \) are isomorphic elementary abelian \( p \)-groups. In particular, if \( G = H \) and \( G \in CD(G) \) then \( G/Z(G) \) and \( [G, G] \) are elementary abelian \( p \)-groups.

**Proof.** Since \( w \geq 3 \), there exist at least three distinct atoms \( K_1, K_2, \) and \( K_3 \) in the interval \( [L, H] \) in \( CD(G) \). By Proposition 2

\[ [K_1, K_2 K_3] \leq ([K_1, K_3][K_1, K_2]) \leq L. \]

Therefore \( K_1/L \) centralizes \( K_2 K_3/L = H/L \). By symmetry, the same holds for \( K_2/L \) consequently \( H = K_1 K_2 \) centralizes \( H/L \) and \( H/L \) is abelian. Similarly \( H^* \) centralizes \( H^*/L^* \) and the latter is abelian.

Since \( K_1 \) normalizes every \( K_j \), it also normalizes every \( K_j^* \). Therefore \([K_i, H^*] = [K_i, K_i^* K_j^*] \leq K_j^* \) and \([K_i, H^*] = [K_i, K_i^* K_j^*] \leq K_j^* \), so that \([K_i, H^*] \leq L^* \). Similarly \([K_2, H^*] \leq L^* \) and thus \([H, H^*] \leq L^* \). In the same way, \([H, H^*] \leq L \). By the
Since $\text{CD of } K$ because $4 \leq i \leq w$ where $K_i = A$. At minimum, $K_i/L$ is an elementary abelian $p$-group but, as $K_i/L \cong K_j/L$ for all $i, j$, we have $H/L$ is an elementary abelian $p$-group. With similar reasoning, $H^*/L^*$ is an elementary abelian $p$-group and, since $[H/L] = [H^*/L^*]$, these quotients are isomorphic elementary abelian $p$-groups.

If $H = G$ and $Z(G) = L$ then $G/Z(G)$ is an elementary abelian $p$-group. Thus, for $x, y \in G$, we have $[x, y]^p = [x^p, y] = 1$; therefore $[G, G]$ is elementary abelian. □

**Theorem 4.** Let $G$ be a group such that $L, H \in \text{CD}(G)$ with the interval $[L, H]$ in $\text{CD}(G)$ a quasi-antichain of width $w \geq 3$. There exists a prime $p$ and positive integers $a, b$ with $b \leq 3$ such that $H/L$ and $G_\alpha(L)/G_\alpha(H)$ are elementary abelian $p$-groups of order $p^{2a}$ and $w = p^b + 1$.

**Proof.** The existence of the prime $p$ and the fact that $H/L$ and $H^*/L^*$ are elementary abelian $p$-groups were established in Proposition 3. From Proposition 2, we know $H/L = K_1/L \times K_2/L$. Let $i \geq 3$; the subgroup $K_i/L$ projects onto each coordinate under the natural projection maps and intersects each of $K_1/L$ and $K_2/L$ trivially. Thus $K_1/L$ is a subdirect product and there exists an isomorphism $\beta_i: K_1/L \rightarrow K_2/L$ such that $K_i/L = \{(kL)\beta_i(kL) \mid k \in K_1\}$. Choose $\beta_i(k) \in K_2$ with $\beta_i(kL) = \beta_i(k)L$; then $K_i/L = \{k\beta_i(k)L \mid k \in K_1\}$. Similarly, there exists an isomorphism $\overline{\beta_i}: K_i^*/L^* \rightarrow K_2^*/L^*$ where $\overline{\beta_i}(mL^*) = \alpha_i(m)L^*$ for each $m \in K_i^*$ and $K_i^*/L^* = \{m\alpha_i(m)L^* \mid m \in K_i^*\}$.

For $i, j$ such that $3 \leq i, j \leq w$, let $\Delta_{i,j} = \{k\beta_i(k)\beta_j(k) \mid k \in K_1\}$ and $\Delta_{i,j}^* = \{m\alpha_i(m)\alpha_j(m) \mid m \in K_i^*\}$; additionally define $K_{i,j} = \Delta_{i,j}L$ and $K_{i,j}^* = \Delta_{i,j}^*L^*$. Since $|K_1, K_2| \leq L$ and the functions $\beta_i, \beta_j$, are homomorphisms, it follows that $K_{i,j} \leq H$. Also observe that if $k\beta_i(k)\beta_j(k)L = k'\beta_i(k')\beta_j(k')L$ then $kL = k'L$, because $K_1 \cap K_2 = L$. Therefore $[K_{i,j}/L] = |K_1/L|$. Corresponding facts are true regarding $K_{i,j}^*$.

Our goal is to show that $K_{i,j}$ is one of the atoms in $[[L, H]]$, so we calculate $m(K_{i,j})$. From the definitions, clearly $[k_1, m_1] = [k_2, m_2] = 1$ when $k_1 \in K_i$ and $m_i \in K_i^*$ for $i = 1, 2$. By this information and the fact that $[H, H^*]$ is centralized by $H$ and $H^*$, if $k \in K_1$ and $m \in K_1^*$ then we obtain

$$1 = [k\beta_i(k), m\alpha_i(m)] = [k, \alpha_i(m)][\beta_i(k), m]$$

for all $i$ such that $3 \leq i \leq w$. Given $k \in K_1$ and $m \in K_1^*$, if $3 \leq i, j \leq w$ then

$$[k\beta_i(k)\beta_j(k), m\alpha_i(m)\alpha_j(m)]$$

$$= [k, \alpha_i(m)][\beta_i(k), \alpha_j(m)][\beta_j(k), \alpha_i(m)][\beta_j(k), m]$$

$$= [k, \alpha_i(m)][\beta_j(k), m]$$

$$= 1.$$ 

By $[H, L^*] = [H^*, L] = 1$ and the above calculation, $K_{i,j}^* \leq C_G(K_{i,j})$. Because $[K_{i,j}] = |K_1|$ and $|K_{i,j}^*| = |K_1|$, therefore $m(K_{i,j}) = m(K_1)$ and $K_{i,j} \in \text{CD}(G)$. Thus for each $i, j$ with $3 \leq i, j \leq w$, either $K_{i,j} = K_h$ for some $h$ such that $3 \leq h \leq w$ or $K_{i,j} = K_1$. Setting $\beta_2(k) = 1$ for all $k \in K_1$, it follows that $\{k\beta_2(k)\beta_j(k)L \mid k \in K_1\} = \{k\beta_2(k)L \mid k \in K_1\}$ for some $h$ with $2 \leq h \leq w$. Notice that if $k\beta_2(k)\beta_j(k)L = k'\beta_2(k')L$ then $kL = k'L$ and $\beta_i(k)\beta_j(k)L = \beta_h(k)L$, because $K_1 \cap K_2 = L$. 

Fix a $k_1 \in K_1 \setminus L$. Let $\Lambda = \{\beta_2(k_1), \beta_3(k_1), \ldots, \beta_w(k_1)\}$ and $R = \Lambda \cdot L$. By what we have shown in the preceding paragraphs, $R \leq H$ and, as $K_i \cap K_j = L$ for $i \neq j$, the set $\Lambda$ is a transversal for $L$ in $R$. Hence $|R/L| = |\Lambda| = w - 1$. Since $R \leq K_2$, it follows that $w - 1$ divides $p^a$.

2. Quasi-antichain Chermak-Delgado Lattices

We study here the groups $G$ such that $\CD(G)$ is a quasi-antichain and $G \in \CD(G)$, meaning that $G = H = H^*$ and $Z(G) = L = L^*$ in the notation of the first section. Additionally, the subgroups $K_i$ are now atoms of $[L, H]$; notice $K_i^* = K_i$ if and only if $K_i$ is abelian.

When studying groups of this type, the added condition that $[G, G]$ be cyclic imposes very strong restrictions on the structure of the group, as seen in the next proposition.

**Proposition 5.** Let $G \in \CD(G)$ and $[G, G]$ be cyclic. Then $\CD(G)$ is a quasi-antichain of width $w \geq 3$ with $G \in \CD(G)$ if and only if there exists a prime $p$ such that $|[G, G]| = p$ and $G/Z(G) \cong C_p \times C_p$. In this case $w = p + 1$.

**Proof.** Let $[G, G]$ is cyclic and $G \in \CD(G)$. Suppose first that $\CD(G)$ is a quasi-antichain of width $w \geq 3$. We know that there exists a prime $p$ such that $G/Z(G)$ and $[G, G]$ are elementary abelian $p$-groups by Proposition 3. Therefore $[G, G]$ has order $p$, but, more importantly, all $U \leq G$ such that $Z(G) \leq U$ are centralizers by [2, Satz]. In particular, a maximal subgroup $M < G$ is a centralizer so there exists $U > Z(P)$ with $M = C_G(U)$ and $m(M) = \frac{|G|}{|Z(G)|} \geq |G/Z(G)|$. Yet $G \in \CD(G)$, so $M, U \in \CD(G)$. The Chermak-Delgado lattice of $G$ is a quasi-antichain of width at least $3$ so by Proposition 2 $[M] = [U]$ and thus $|G/Z(P)| = p^2$.

Now suppose there exists a prime $p$ such that $|[G, G]| = p$ and $G/Z(G) \cong C_p \times C_p$. In this case, all $p + 1$ subgroups $U$ such that $Z(G) < U < G$ are abelian, have order $p|Z(G)|$, and have measure $p^3|Z(G)|^2$, which also equals the measure of $G$. Therefore $\CD(G) = \{U \leq G \mid Z(G) \leq U\}$ is a quasi-antichain of width $p + 1$ with $G \in \CD(G)$.

The next theorem justifies our attention on $p$-groups while studying groups with a quasi-antichain Chermak-Delgado lattice. The proof of Theorem 6 requires the observation: Let $M$ and $N$ be any pair of finite groups. The modularity of the Chermak-Delgado lattice implies that every maximal chain in the lattice has the same length. For example, all maximal chains in $\CD(G)$ have length $2$ because $\CD(G)$ is a quasi-antichain. That $\CD(M \times N) \cong \CD(M) \times \CD(N)$ as lattices gives that the length of a maximal chain in $M \times N$ is the sum of the lengths of maximal chains in $M$ and $N$.

**Theorem 6.** If $G$ is a group with $\CD(G)$ a quasi-antichain of width $w \geq 3$ and $G \in \CD(G)$ then $G$ is nilpotent of class $2$; in fact, there exists a prime $p$, a nonabelian Sylow $p$-subgroup $P$ with nilpotence class $2$, and an abelian Hall $p'$-subgroup $Q$ such that $G = P \times Q$, $P \in \CD(P)$, and $\CD(G) \cong \CD(P)$ as lattices. Moreover there exist positive integers $a, b$ with $b \leq a$ such that $|G/Z(G)| = |P/Z(P)| = p^a$ and $w = p^b + 1$.

**Proof.** Note that $G$ is nilpotent, by Proposition 3 but nonabelian and the length of a maximal chain in $\CD(G)$ is $2$. If $G = Q_1 \times Q_2$ where $Q_1$ and $Q_2$ are Hall $\pi$, $\pi'$-subgroups of $G$, respectively, then $\CD(G) \cong \CD(Q_1 \times Q_2)$; as a consequence of the
additivity of chain length over a direct product, if both $Q_1$ and $Q_2$ are nonabelian
then $\mathcal{CD}(Q_i) = \{Q_i, Z(Q_i)\}$ for $i = 1, 2$. However, this implies $\mathcal{CD}(Q_1 \times Q_2)$ is a
quasi-antichain of width 2. Consequently, exactly one of $Q_1$ or $Q_2$ is abelian and
the Chermak-Delgado lattice of the nonabelian factor is isomorphic (as lattices)
to $\mathcal{CD}(G)$. Therefore there exists a unique prime $p$ such that $G = P \times Q$ where
$P$ is a nonabelian Sylow $p$-subgroup of $G$ and $Q$ is an abelian Hall $p'$-subgroup
of $G$, with $\mathcal{CD}(G) \cong \mathcal{CD}(P)$ as lattices. The rest follows from Proposition \[2\] and
Theorem \[4\].

We investigated the number of abelian atoms that is permitted in a quasi-
antichain Chermak-Delgado lattice. The final theorem of this section records our
contributions in this direction.

**Theorem 7.** Let $G$ be a $p$-group with $\mathcal{CD}(G)$ a quasi-antichain of width $w \geq 3$
and suppose $|G/Z(G)| = p^{2a}$ for a positive integer $a$. Let $t$ be the number of abelian
atoms in $\mathcal{CD}(G)$ and $u$ be the number of pairs of nonabelian atoms.

1. If $t = 0$ then $p$ is odd.
2. If $t = 1$ then $p = 2$.
3. If $t \geq 2$ then there exists a positive integer $c \leq a$ such that $t = p^c + 1$. In
   particular, $p - 1$ divides $t - 2$; if $p$ is odd then $p^c$ divides $u$ and if $p = 2$
   then $t \geq 3$ and $2^{c-1}$ divides $u$.
4. If $t \geq 2$ and $u \geq 1$ then $3 \leq t \leq 2u + 1$ when $p = 2$ and $2 \leq t \leq u + 1$ when
   $p$ is odd.
5. If $t \geq 3$ then $t \geq p + 1$.

**Proof.** Theorem \[4\] tells us that $w = p^b + 1$ for some positive integer $b \leq a$, but also
$w = t + 2u$ as set up by the notation. If $t = 0$ then $w = 2u = p^b + 1$, necessarily
forcing $p$ to be odd. If $t = 1$ then $2u = p^b$; clearly $p$ must equal 2 in this case.

Suppose that $t \geq 2$; we continue here with the same notation and set up as in
the proof of Theorem \[4\] except we add the condition that $K_1$ and $K_2$ are abelian
atoms. Recall the fixed $K_1 \in K \setminus L$ and that $\beta_2(k) = 1$ for all $k \in K_1$. Set
$\Gamma = \{\beta_i(k_1) | 2 \leq i \leq w \text{ and } K_1 = K_1^*\}$, a subset of $\Lambda$.

Let $i, j \geq 3$ and assume that $K_i$ and $K_j$ are abelian atoms; we show that $K_{i,j}$ is
also abelian. We use the functions $\alpha_i$ defined in the proof of Theorem \[4\]. Observe
that $\alpha_i(k)$ now differs from $\beta_i(k)$ only by an element in $Z(G)$, for all $k \in K_1$. The
calculation below follows:

$$[k; \beta_i(k)\beta_j(k), k\beta_i(k)\beta_j(k')] = [k; \beta_i(k)\beta_j(k), k\alpha_i(k')\alpha_j(k')] = 1$$

for all $k, k' \in K_1$. Therefore $K_{i,j}$ is also an abelian atom in $\mathcal{CD}(G)$. Since
$K_{i,j} \neq K_2$, it follows that $\beta_i(k_1)\beta_j(k_1)Z(G) = \beta_h(k_1)Z(G)$ for some $\beta_h(k_1) \in \Gamma$.
Consequently $\Gamma Z(G) \leq K_2$ and since $\Gamma \subset \Lambda$, the elements of $\Gamma$ are distinct.
Therefore $|\Gamma Z(G)/Z(G)| = |\Gamma| = t - 1$ divides $p^a$ and there exists a positive integer $c$
such that $t - 1 = p^c$.

Since $t = p^c + 1$, clearly if $p = 2$ and $t \geq 2$ then $t = 2^c + 1 \geq 3$ but, for all primes
$p$, it is true that $p - 1$ divides $p^c - 1 = t - 2$. Observe, for part (5), that if $t \geq 3$
then $t \geq p + 1$ is necessary for $t - 2$ to be a multiple of $p - 1$. To complete the
proof of part (3), notice that $2u = w - t = p^c(p^{b-c} - 1)$, so that $u = \frac{2u}{p^c}\frac{p^{b-c} - 1}{p^c}$.
If $p = 2$ then $2^{c-1}$ divides $u$, and if $p$ is odd then $u$ must be divisible by $p^c$. This
completes the assertions in part (3).
Continuing with part (4), suppose that $t \geq 2$ and $u \geq 1$. Then $b > c$, so $p^{b-c} - 1 > p - 1$. If $p$ is odd, this implies $\frac{1}{2}(p^{b-c} - 1) \geq 1$ and consequently

$$t = p^c + 1 \leq p^c \left( \frac{p^{b-c} - 1}{2} \right) + 1 = u + 1.$$ 

If $p = 2$ then $2^{b-c} - 1 \geq 1$, so that

$$t = 2^c + 1 \leq 2^c(2^{b-c} - 1) + 1 = 2u + 1.$$ 

Thus when $t \geq 2$ and $u \geq 1$, the inequalities asserted in part (4) of the theorem are true. \qed

**Corollary 8.** While there exist finite groups with Chermak-Delgado lattice a quasi-antichain of width 6, there does not exist such a group with exactly 4 abelian atoms in its Chermak-Delgado lattice.

**Proof.** An extraspecial group of order $5^3$ has a Chermak-Delgado lattice that is a quasi-antichain of width 6 by Proposition 5. If we assume that $G$ is a finite group with $CD(G)$ a quasi-antichain of width 6 having exactly 4 abelian atoms then we know that $G$ has a Sylow 5-subgroup $P$ with $CD(G) \cong CD(P)$ as lattices by Proposition 6 and Theorem 4. Theorem 7 forces $4 = t \leq u + 1 = 2$; thus $G$ cannot exist. \qed

3. Examples

In this section we construct several examples of $p$-groups having a quasi-antichain for their Chermak-Delgado lattice. The first two examples show that every possible quasi-antichain of width 2 can be realized as the Chermak-Delgado lattice of a $p$-group.

1. **A group $G$ where $CD(G)$ is a quasi-antichain of width 2 with no abelian atoms:** Let $H$ be any group with $CD(H) = \{H, Z(H)\}$. A family of $p$-groups, each member of which having such a Chermak-Delgado lattice, was described in [2].

Define $G = H \times H$. In [3] it was established that $CD(G)$ is a quasi-antichain of width 2 with atoms $Z(H) \times H$ and $H \times Z(H)$. Clearly $H \times Z(H) = C_G(Z(H) \times H)$.

2. **A group $P$ such that $CD(P)$ is a quasi-antichain of width 2 with both atoms abelian:** Let $P = \langle m_1, m_2, n_1, n_2 \rangle$ where each element has order $p$ and

$$[m_1, m_2] = [n_1, n_2] = 1, \quad [m_i, n_j] = z_{ij} \in Z(P) \text{ for } i, j \in \{1, 2\},$$

and $Z(P) = \langle z_{i,j} \mid i, j \in \{1, 2\} \rangle$ is elementary abelian of order $p^4$. Clearly $P$ is nilpotent of class 2 with order $p^8$ and Chermak-Delgado measure $p^{12}$. Let $M = \langle m_1, m_2 \rangle Z(P)$ and $N = \langle n_1, n_2 \rangle Z(P)$. It’s a straightforward calculation to show that $C_P(m) = M$ whenever $m \in M \setminus Z(P)$ and $C_P(n) = N$ whenever $n \in N \setminus Z(P)$, whereas $C_P(x) = (x)Z(P)$ for all $x \in P \setminus (M \cup N)$. Thus of all subgroups containing $Z(P)$, only $M$ and $N$ have the largest measure, which is $p^{12}$. Since $m_P(M) = m_P(N) = m_P(Z(P)) = p^{12}$ and no other subgroups have this measure, $CD(P)$ is a quasi-antichain of width 2 (containing $P$) such that both atoms are abelian.
We now show that for every prime $p$ and every positive integer $n$, there exists a $p$-group whose Chermak-Delgado lattice is a quasi-antichain of width $p^n + 1$ with all atoms abelian.

**Proposition 9.** Let $p$ be a prime and $n$ a positive integer. Let $P$ be the group of all $3 \times 3$ lower triangular matrices over $\mathbf{GF}(p^n)$ with 1s along the diagonal. The Chermak-Delgado lattice of $P$ is a quasi-antichain of width $p^n + 1$ and all subgroups in the middle antichain are abelian.

**Proof.** By Exercise 39 in [3] III.16, $P$ has exactly $p^n + 1$ abelian subgroups of maximal order equal to $p^{2n}$; these subgroups have measure $p^{3n} = m(P)$. If $x \in P \setminus Z(P)$, it is easy to check that $|C_P(x)| = p^{2n}$. Therefore if $U \subseteq CD(P)$ with $Z(P) < U < P$ then $|C_P(U)| \leq p^{2n}$ and $|U| = |C_P(U)| \leq p^{2n}$. It follows that $m^*(P) = p^{3n}$ and $|U| = |C_P(U)| = p^{2n}$. If $U \neq C_P(U)$ then $U \cap C_P(U) = Z(P)$ since $U \cap C_P(U) \subseteq CD(P)$. But then for $x \in U \setminus Z(P)$ we have $C_P(x) = C_P(U)$ by order considerations and therefore $x \in U \cap C_P(U) = Z(P)$, a contradiction. Thus $U = C_P(U)$ is one of the abelian subgroups of maximal order and the assertion follows.

Extraspecial groups of order $p^3$ are examples where each of the $p + 1$ atoms in the quasi-antichain is abelian (Proposition 5); the next two propositions construct $p$-groups where the Chermak-Delgado lattice is a quasi-antichain of width $p^n + 1$ and, depending on the value of $p$ modulo 4, the number of abelian atoms is either 0, 1, or 2.

**Proposition 10.** Given any prime $p$ there exists a group $P$ of order $p^9$ such that $CD(P)$ is a quasi-antichain of width $p + 1$. In this example: if $p = 2$ then exactly one of the three atoms of $CD(P)$ is abelian, when $p \equiv 1 \mod 4$ then exactly two of the $p + 1$ atoms of $CD(P)$ are abelian, and if $p \equiv 3 \mod 4$ then none of the atoms in $CD(P)$ are abelian.

**Proof.** Let $P$ be generated by $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ with defining relationships $x_i^p = y_i^p = 1$ and $[x_i, y_j] = 1$ for all $i, j$ such that $1 \leq i, j \leq 3$, $Z(P) = \langle z_{1,2}, z_{1,3}, z_{2,3} \rangle$ is elementary abelian with order $p^3$, and $[x_i, x_j] = [y_i, y_j] = z_{ij}$ for every $i, j$ with $1 \leq i < j \leq 3$. Let $M_0 = \langle x_1, x_2, x_3 \rangle Z(P)$ and $M_p = \langle y_1, y_2, y_3 \rangle Z(P)$. For $1 \leq i \leq p - 1$, let $M_i = \langle x_1 y_i, x_2 y_i, x_3 y_i \rangle Z(P)$. We show that $CD(P) = \{P, Z(P), M_i \mid 0 \leq i \leq p\}$.

Observe that $P$ is the central product of $M_0$ with $M_p$ and $Z(P) = M_0 \cap M_p$. Additionally $C_P(M_0) = M_p$ and vice versa, yielding $m_P(P) = m_P(M_0) = m_P(M_p) = p^{12}$. It is easy to show that if $x \in M_0 \setminus Z(P)$ then $C_{M_0}(x) = \langle x \rangle Z(P)$ and $C_p(x) = \langle x \rangle M_p$. It follows that $C_P(\langle x, y \rangle) = \langle x, y \rangle Z(P)$ whenever $x \in M_0 \setminus Z(P)$ and $y \in M_p \setminus Z(P)$.

Let there exist $a_1, b_1 \in \mathbb{Z}/p\mathbb{Z}$ such that $x = x_1^{a_1} x_2^{a_2} x_3^{a_3} \in M_0 \setminus Z(P)$ and $y = y_1^{b_1} y_2^{b_2} y_3^{b_3} \in M_p \setminus Z(P)$. Assume that at least one of $a_1, a_2, a_3$ and one of $b_1, b_2, b_3$ are non-zero. For $x' y'$ with similar structure, $x' y' \in C_P(xy)$ if and only if $[x, x'] = [y', y]$. Further decomposing the commutators reveals

$$[x, x'] = \prod_{1 \leq i < j \leq 3} z_{ij}^{a_i a_j - a_j a_i}$$

and

$$[y', y] = \prod_{1 \leq i < j \leq 3} z_{ij}^{b_i b_j - b_j b_i}.$$ 

Thus $[xy, x'y'] = 1$ if and only if each of the three equations $a_i a_j - a_j a_i = b_i b_j - b_j b_i = 0$ hold where $1 \leq i < j \leq 3$. If $(a_1, a_2, a_3)$ and $(b_1, b_2, b_3)$ are not scalar multiples then $\langle x, y, m_1^{a_1} m_2^{a_2} m_3^{a_3}, n_1^{b_1} n_2^{b_2} n_3^{b_3} \rangle Z(P) = C_P(xy)$. On the other hand, if
there exists $k$ such that $(b_1, b_2, b_3) = k(a_1, a_2, a_3)$ then $C_{P}(xy) = \langle m, m_i n_i^{-k^{-1}} \mid 1 \leq i \leq 3 \rangle Z(P) = \langle m \rangle M_{-k^{-1}}$.

Therefore $C_{P}(M_k) = M_{-k^{-1}}$ for $1 \leq k \leq p - 1$ and $m_p(M_k) = p^{12}$.

It follows then that $m(U) < m(M_k)$ whenever $Z(P) < U < M_k$. Additionally, if $U < P$ and there exist $u_1, u_2 \in U$ where $u_1 \in M_k$ and $u_2 \in M_k'$ with $k \neq k'$ then $C_{P}(U) \leq Z(P)$. Hence $m^*(P) = p^{12}$ and $CD(P) = \{ P, Z(P), M_k \mid 0 \leq k \leq p \}$.

Since $C_{P}(M_k) = M_{-k^{-1}}$ for $1 \leq k \leq p - 1$, there exists an abelian atom of $CD(P)$ if and only if $p = 2$ or $p \equiv 1$ modulo 4. When $p = 2$ then $M_1$ is the unique abelian atom and if $p \equiv 1$ modulo 4 then $M_1$ and $M_{p-1}$ are both abelian, but no other atom in $CD(P)$ is abelian. When $p \equiv 3$ modulo 4 then there do not exist any abelian atoms in $CD(P)$. □

**Proposition 11.** Let $p$ be a prime. There exists a group $Y$ of order $p^n$ such that $CD(Y)$ is a quasi-antichain of width $p + 1$. In this example, if $p = 2$ then exactly one of the three atoms of $CD(Y)$ is abelian and if $p$ is odd then exactly two of the $p + 1$ atoms are abelian.

**Proof.** Define $P$ as in Proposition 10 except designate that $[n_i, n_j] = z_i^{-1}$. The same arguments made earlier will now show that $C_{P}(M_i) = M_{i-1}$ for $i = 1, 2, \ldots, p - 1$. This forces $M_1$ and $M_{p-1}$ to be abelian, yet the remaining facts still stand. □

In lattice theory [1 Chapter 1, Section 2], a lattice $L$ has a duality $\theta : L \to L$ if $\theta$ is a bijection and if $A, B \in L$ with $A \leq B$ implies $\theta(B) \leq \theta(A)$. Such a duality need not have order 2 as a function; in the case of the Chermak-Delgado lattice the duality does have order 2. In particular, the examples in this section show that all possible types of quasi-antichains of width 4 with duality of order 2 occur as Chermak-Delgado lattices of 3-groups and those of width 3 with duality of order 2 occur as Chermak-Delgado lattices of 2-groups.

This leaves several questions open for investigation, including: Which values of $t$ (in the notation of Theorem 1) are possible in quasi-antichain Chermak-Delgado lattices of width $w = p^n + 1$ when $n > 1$? That is, which dualities can be realized by the centralizer map? The first open case is when $w = 5$ and $t = 3$. And, are there examples of groups $G$ with $G \in CD(G)$ and $CD(G)$ a quasiantichain where $t = 0$ and either $p = 2$ or $p \equiv 1$ modulo 4?

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