Efficient Behavior of Small-World Networks

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We introduce the concept of \textit{efficiency} of a network, measuring how efficiently it exchanges information. By using this simple measure small-world networks are seen as systems that are both globally and locally efficient. This allows to give a clear physical meaning to the concept of small-world, and also to perform a precise quantitative analysis of both weighted and unweighted networks. We study neural networks and man-made communication and transportation systems and we show that the underlying general principle of their construction is in fact a small-world principle of high efficiency.

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We live in a world of networks. In fact any complex system in nature can be modeled as a network, where vertices are the elements of the system and edges represent the interactions between them. Coupled biological and chemical systems, neural networks, social interacting species, computer networks or the Internet are only few of such examples. Characterizing the structural properties of the networks is then of fundamental importance to understand the complex dynamics of these systems. A recent paper\textsuperscript{2} has shown that the connection topology of some biological and social networks is neither completely regular nor completely random. These networks, there named \textit{small-worlds}, in analogy with the concept of small-world phenomenon developed 30 years ago in social psychology\textsuperscript{3}, are in fact highly clustered like regular lattices, yet having small characteristics path lengths like random graphs. The original paper has triggered a large interest in the study of the properties of small-worlds (see ref.\textsuperscript{1} for a recent review). Researchers have focused their attention on different aspects: study of the inset mechanism\textsuperscript{4,5}, dynamics\textsuperscript{6} and spreading of diseases on small-worlds\textsuperscript{7}; applications to social networks\textsuperscript{8,9} and to the Internet\textsuperscript{10,11}. In this letter we introduce the concept of \textit{efficiency} of a network, measuring how efficiently information is exchanged over the network. By using efficiency small-world networks results as systems that are both globally and locally efficient. This formalization gives a clear physical meaning to the concept of small-world, and also allows a precise quantitative analysis of unweighted and weighted networks. We study several systems, like brains, communication and transportation networks, and show that the underlying general principle of their construction is in fact a small-world principle, provided attention is taken not to ignore an important observational property (closure).

We start by reexamining the original formulation proposed in ref.\textsuperscript{2}. There, a generic graph $G$ with $N$ vertices and $K$ edges is considered. $G$ is assumed to be \textit{unweighted}, i.e. edges are all equal, \textit{sparse} ($K \ll N(N-1)/2$), and \textit{connected}. i.e. there exists at least one path connecting any couple of vertices with a finite number of steps. $G$ is therefore represented by simply giving the adjacency (or connection) matrix, i.e. the $N \times N$ matrix whose entry $a_{ij}$ is 1 if there is an edge joining vertex $i$ to vertex $j$, and 0 otherwise. An important quantity of $G$ is the degree of vertex $i$, i.e. the number $k_i$ of edges incident with vertex $i$ (the number of neighbours of $i$). The average value of $k_i$ is $k = 2K/N$. Once $\{a_{ij}\}$ is given it can be used to calculate the matrix of the shortest path lengths $d_{ij}$ between two generic vertices $i$ and $j$. The fact that $G$ is assumed to be connected implies that $d_{ij}$ is positive and finite $\forall i \neq j$. In order to quantify the structural properties of $G$,\textsuperscript{2} proposes to evaluate two different quantities: the characteristic path length $L$ and the clustering coefficient $C$. $L$ is the average distance between two generic vertices $L = \frac{1}{N(N-1)} \sum_{i \neq j} d_{ij}$, and $C$ is a local property defined as $C = \frac{1}{K} \sum_{i} C_i$. Here $C_i$ is the number of edges existing in $G_i$, the subgraph of the neighbors of $i$, divided by the maximum possible number $k_i(k_i-1)/2$. In\textsuperscript{2} a simple method is considered to produce a class of graphs with increasing randomness. The initial graph $G$ is taken to be a one-dimensional lattice with each vertex connected to its $k$ neighbours and with periodic boundary conditions. Rewiring each edge at random with probability $p$, $G$ can be tuned in a continuous way from a regular lattice ($p = 0$) into a random graph ($p = 1$). For the regular lattice we expect $L \sim N/2k$ and a high clustering coefficient $C = 3/4(k-2)/(k-1)$, while for a random graph $L \sim \ln N/k_{av}(k-1)$ and $C \sim k/N$. Although in the two limit cases a large $C$ is associated to a large $L$ and vice versa a small $C$ to a small $L$, the numerical experiment reveals an intermediate regime at small $p$ where the system is highly clustered like regular lattices, yet having small characteristics path lengths like random graphs. This behavior is there called small-world and it is found to be a property of some social and
Now we propose a more general set-up to investigate real networks. We will show that:
- the definition of small-world behavior can be given in terms of a single variable with a physical meaning, the efficiency $E$ of the network.
- $1/L$ and $C$ can be seen as first approximations of $E$ evaluated resp. on a global and on a local scale.
- we can drop all the restrictions on the system, like unweightedness, connectedness and sparseness.

We represent a real network as a generic weighted (and possibly even non sparse and non connected) graph $G$. Such a graph needs two matrices to be described: the adjacency matrix $\{a_{ij}\}$ defined as for the unweighted graph, and the matrix $\{\ell_{ij}\}$ of physical distances. The number $\ell_{ij}$ can be the space distance between the two vertices or the strength of their possible interaction: we suppose $\ell_{ij}$ to be known even if in the graph there is no edge between $i$ and $j$. To make some examples, $\ell_{ij}$ can be the geographical distance between stations in transportation systems (in such a case $\ell_{ij}$ respects the triangle equality, though this is not a necessary assumption), the time taken to exchange a packet of information between routers in the Internet, or the inverse velocity of chemical reactions along a direct connection in a biological system. Of course, in the particular case of an unweighted graph $\ell_{ij} = 1 \forall i \neq j$.

The shortest path length $d_{ij}$ between two generic points $i$ and $j$ is the smallest sum of the physical distances throughout all the possible paths in the graph from $i$ to $j$. The matrix $\{d_{ij}\}$ is therefore calculated by using the information contained both in matrix $\{a_{ij}\}$ and in matrix $\{\ell_{ij}\}$. We have $d_{ij} \leq \ell_{ij} \forall i, j$, the equality being valid when there is an edge between $i$ and $j$. Let us now suppose that the system is parallel, i.e. every vertex sends information concurrently along the network, through its edges. The efficiency $\epsilon_{ij}$ in the communication between vertex $i$ and $j$ can be then defined to be inversely proportional to the shortest distance: $\epsilon_{ij} = 1/d_{ij} \forall i, j$. When there is no path in the graph between $i$ and $j$, $d_{ij} = +\infty$ and consistently $\epsilon_{ij} = 0$. The average efficiency of $G$ can be defined as:

$$E(G) = \frac{\sum_{i \neq j \in G} \ell_{ij}}{N(N-1)} = \frac{1}{N(N-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}} \tag{1}$$

To normalize $E$ we consider the ideal case $G_{id}$ in which the graph $G$ has all the $N(N-1)/2$ possible edges. In such a case the information is propagated in the most efficient way since $d_{ij} = \ell_{ij} \forall i, j$, and $E$ assumes its maximum value $E(G_{id}) = 1$. The efficiency $E(G)$ considered in the following of the paper is always divided by $E(G_{id})$ and therefore $0 \leq E(G) \leq 1$. Though the equality $E = 1$ is valid when there is an edge between each couple of vertices, real networks can reach a high value of $E$.

In our formalism, we can define the small-world behaviour by using the single measure $E$ to analyze both the local and global behaviour, rather than two different variables $L$ and $C$. The quantity in eq. (1) is the global efficiency of $G$ and we therefore name it $E_{glob}$. Since $E$ is defined also for a disconnected graph we can characterize the local properties of $G$ by evaluating for each vertex $i$ the efficiency of $G_i$, the subgraph of the neighbors of $i$. We define the local efficiency as the average efficiency of the local subgraphs, $E_{loc} = 1/N \sum_{i \in G} E(G_i)$. This quantity plays a role similar to the clustering coefficient $C$. Since $i \notin G_i$, the local efficiency $E_{loc}$ tells how much the system is fault tolerant, thus how efficient is the communication between the first neighbours of $i$ when $i$ is removed. The definition of small-world can now be rephrased and generalized in terms of the information flow: small-world networks have high $E_{glob}$ and $E_{loc}$, i.e. are very efficient in global and local communication. This definition is valid both for unweighted and weighted graphs, and can also be applied to disconnected and/or non sparse graphs.

It is interesting to see the correspondence between our measure and the quantities $L$ and $C$ of [2] (or, correspondingly, $1/L$ and $C$). The fundamental difference is that $E_{glob}$ is the efficiency of a parallel systems, where all the nodes in the network concurrently exchange packets of information (such are all the systems in [2], for example), while $1/L$ measures the efficiency of a sequential system (i.e. only one packet of information goes along the network). $1/L$ is a reasonable approximation of $E_{glob}$ when there are not huge differences among the distances in the graph, and this can explain why $L$ works reasonably well in the unweighted examples of [2]. But, in general $1/L$ can significantly depart from $E_{glob}$. For instance, in the Internet, having few computers with an extremely slow connection does not mean that the whole Internet diminishes by far its efficiency: in practice, the presence of such very slow computers goes unnoticed, because the other thousands of computers are exchanging packets among them in a very efficient way. Here $1/L$ would give a number very close to zero (strictly 0 in the particular case when a computer is disconnected from the others and $L = +\infty$), while $E_{glob}$ gives the correct efficiency measure of the Internet. We turn now our attention to the local properties of a network. $C$ is only one among the many possible intuitive measures of how well connected a cluster is. It can be shown that when in a graph most of its local subgraphs $G_i$ are not sparse, then $C$ is a good approximation of $E_{loc}$. Summing up there are not two different kinds of analysis to be done for the global and local scales, but just one with a very precise physical meaning: the efficiency in transporting information.

We now illustrate the onset of the small-world in an unweighted graph by means of the same example used in [2]. A regular lattice with $N = 1000$ and $k = 20$ is rewired...
with functions $p$ and $E_{\text{glob}}$ and $E_{\text{loc}}$ are reported in fig.1 as functions of $p$. For $p = 0$ we expect the system to be inefficient on a global scale ($E_{\text{glob}} \sim k/N \log(N/K)$) but locally efficient. The situation is inverted for the random graph. In fact at $p = 1$ $E_{\text{glob}}$ assumes a maximum value of 0.4, meaning 40% the efficiency of the ideal graph with an edge between each couple of vertices. This at the expenses of the fault tolerance ($E_{\text{loc}} \sim 0$).

The small-world behaviour appears for intermediate values of $p$. It results from the fast increase of $E_{\text{glob}}$ (for small $p$ we find a linear increase of $E_{\text{glob}}$ in the logarithmic horizontal scale) caused by the introduction of only a few rewired edges (short cuts), which on the other side do not affect $E_{\text{loc}}$. At $p \sim 0.1$, $E_{\text{glob}}$ has almost reached the value of the random graph, though $E_{\text{loc}}$ has only diminished by very little from the value of 0.82 of the regular lattice. Small worlds have high $E_{\text{glob}}$ and $E_{\text{loc}}$.

![FIG. 1. Global and local efficiency for the graph example considered in [4]. A regular lattice with $N = 1000$ and $k = 20$ is rewired with probability $p$. The small-world behaviour results from the increase of $E_{\text{glob}}$ caused by the introduction of only a few rewired edges (short cuts), which on the other side do not affect $E_{\text{loc}}$. At $p \sim 0.1$, $E_{\text{glob}}$ has almost reached the value of the random graph, though $E_{\text{loc}}$ has only diminished by very little from the value of 0.82 of the regular lattice. Small worlds have high $E_{\text{glob}}$ and $E_{\text{loc}}$.](image)

2) Communication Networks. We have considered two of the most important large-scale communication networks present nowadays: the World Wide Web and the Internet. Tab.2 shows that they have relatively high values of $E_{\text{glob}}$ (slightly smaller than the best possible values obtained for random graphs) and high $E_{\text{loc}}$. Despite the WWW is a virtual network and the Internet is a physical network, at a global scale they transport information essentially in the same way (as their $E_{\text{glob}}$'s are almost equal). At a local scale, the bigger $E_{\text{loc}}$ in the WWW case can be explained both by the tendency in the WWW to create Web communities (where pages talking about the same subject tend to link to each other), and by the fact that many pages within the same site are often quickly connected to each other by some root or menu page.

3) Transport Networks. Differently from previous databases the Boston subway transportation system (MBTA) can be better described by a weighted graph, the matrix $\{\ell_{ij}\}$ being given by the geographical distances between stations. If we consider the MBTA as an unweighted graph we obtain that it is apparently neither locally nor globally efficient (see Tab.3). On the other hand, when we take into account the geographical distances, we have $E_{\text{glob}} = 0.63$: this shows the MBTA is a very efficient transportation system on a global scale, only 37% less efficient than the ideal subway with a direct tunnel from each station to the others. Even in the weighted case $E_{\text{loc}}$ stays low (0.03), indicating a poor local behaviour: differently from a neural network
MBTA is not fault tolerant and a damage in a station will dramatically affect the connection between the previous and the next station. The difference with respect to neural networks comes from different needs and priorities in the construction and evolution mechanism: when we build a subway system, the priority is given to the achievement of global efficiency, and not to fault tolerance. In fact a temporary problem in a station can be solved by other means: for example, walking, or taking a bus from the previous to the next station. That is to say, the MBTA is not a closed system: it can be considered, after all, as a subgraph of a wider transportation network. This property is of fundamental importance when we analyze a system: while global efficiency is without doubt the major characteristic, it is closure that somehow leads a system to have high local efficiency (without alternatives, there should be high fault-tolerance). The MBTA is not a closed system, and thus this explains why, unlike in the case of the brain, fault tolerance is not a critical issue. Indeed, if we increase the precision of the analysis and change the MBTA subway network by taking into account, for example, the Boston Bus System, this extended transportation system comes back to be a small-world network ($E_{\text{glob}} = 0.72$, $E_{\text{loc}} = 0.46$). Qualitatively similar results, confirming the similarity of construction principles, have been obtained for other undergrounds and for a wider transportation system consisting of all the main airplane and highway connections throughout the world [21]. Considering all the transportation alternatives available at that scale makes again the system closed (there are no other reasonable routing alternatives), and so fault-tolerance comes back as a leading construction principle.

Summing up, the introduction of the efficiency measure allows to give a definition of small-world with a clear physical meaning, and provides important hints on why the original formulas of [2] work reasonably well in some cases, and where they fail. The efficiency measure allows a precise quantitative analysis of the information flow, and works both in the unweighted abstraction, and in the more realistic assumption of weighted networks. Finally, analysis of real data indicates that various existing (neural, communication and transport) networks exhibit the small-world behaviour (even, in some cases, when their unweighted abstractions do not), substantiating the idea that the diffusion of small-world networks can be interpreted as the need to create networks that are both globally and locally efficient.

| TABLE I. | Macaque and cat cortico-cortical connections |
|----------|---------------------------------------------|
|          | $E_{\text{glob}}$ | $E_{\text{loc}}$ |
| Macaque  | 0.52            | 0.70            |
| Cat      | 0.69            | 0.83            |
| C. elegans | 0.46         | 0.47            |

[1] Y. Bar-Yam, *Dynamics of Complex Systems* (Addison-Wesley, Reading Mass, 1997).
[2] D.J. Watts and S.H. Strogatz, *Nature* 393, 440 (1998).
[3] S. Milgram, *Physiol. Today* 2, 60 (1967).
[4] M.E.J. Newman, cond-mat/0011118.
[5] A. Barrat, M. Weigt, *Europ. Phys. J. B* 13, 547 (2000).
[6] M. Marchiori and V. Latora, *Physica A285*, 539 (2000).
[7] M. Barthelemy, L. Amaral, *Phys. Rev. Lett.* 82, 3180 (1999).
[8] L. F. Lago-Fernandez et al, *Phys. Rev. Lett.* 84, 2758 (2000).
[9] C. Moore and M.E.J. Newman, *Phys. Rev.* E61, 5678 (2000).
[10] M.E.J. Newman, cond-mat/0011144.
[11] L. A. N. Amaral, A. Scala, M. Barthélyeny, and H. E. Stanley, *Proc. Natl. Acad. Sci.* 97, 11149 (2000).
[12] R. Albert, H. Jeong, and A.-L. Barabási, *Nature* 406, 130 (1999).
[13] A.-L. Barabási and R. Albert, *Science* 286, 509 (1999).
[14] B. Bollobás, *Random Graphs* (Academic, London, 1985).
[15] Our concept of fault tolerance is different from the one adopted in R. Albert, H. Jeong, and A.-L. Barabási, *Nature* 406, 378 (2000); R. Cohen et al. *Phys. Rev. Lett.* 85, 2758 (2000), where the authors consider the response of the entire network to the removal of i.
[16] Here and in the following the matrix $\{d_{ij}\}_{i,j\in G}$ has been computed by using two different methods: the Floyd-Warshall ($(O(N^3))$) and the Dijkstra algorithm ($O(N^2 \log N)$).
[17] G. Gallo and S. Pallottino, *Ann. Oper. Res.* 13, 3 (1988).
[18] O. Sporns, G. Tononi, G.M. Edelman, *Celebiral Cortex 10*, 127 (2000).
[19] J.W. Scannell, *Nature* 386, 452 (1997).
[20] M.P. Young, *Phil.Trans.R.Soc* B252, 13 (1993).
[21] J.W. Scannell, M.P. Young and C. Blakemore, *J. Neurosci.* 15, 1463 (1995).
[22] E. Sivan, H. Parnas and D. Dolev, *Biol. Cybern.* 81, 11-23 (1999).
[23] J.G. White et al., *Phil. Trans. R. Soc. London* B314, 1 (1986).
[24] T.B. Achacoso and W.S. Yamamoto, 401 (1999).
[25] M. Marchiori and V. Latora, in preparation.

|          | $E_{\text{glob}}$ | $E_{\text{loc}}$ |
|----------|------------------|------------------|
| Macaque  | 0.52             | 0.70             |
| Cat      | 0.69             | 0.83             |
| C. elegans | 0.46            | 0.47             |
TABLE II. Communication networks. Data on the World Wide Web from [http://www.nd.edu/~networks](http://www.nd.edu/~networks) contains $N = 325729$ documents and $K = 1090108$ links [12], while the Internet database is taken from [http://moat.nlanr.net](http://moat.nlanr.net) and has $N = 6474$ nodes and $K = 12572$ links.

|        | $E_{\text{glob}}$ | $E_{\text{loc}}$ |
|--------|-------------------|-------------------|
| WWW    | 0.28              | 0.36              |
| Internet | 0.29                | 0.26              |

TABLE III. The Boston underground transportation system (MBTA) consists of $N = 124$ stations and $K = 124$ tunnels. The matrix $\{\ell_{ij}\}$ of the spatial distances between stations, used for the weighted case, has been calculated using databases from [http://www.mbta.com](http://www.mbta.com) and the U.S. National Mapping Division.

|        | $E_{\text{glob}}$ | $E_{\text{loc}}$ |
|--------|-------------------|-------------------|
| MBTA (unweighted) | 0.10              | 0.006             |
| MBTA (weighted)    | 0.63              | 0.03              |