Topical Review

Yang–Baxter deformations and generalized supergravity—a short summary

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Abstract

Integrable deformations of type IIB superstring theory on $\text{AdS}_5 \times S^5$ have played an important role over the last years. The Yang–Baxter deformation is a systematic way of generating such integrable deformations. Since its introduction, this topic has seen important conceptual progress and has among others led to the intriguing discovery generalized supergravity, a new low-energy effective theory. This review endeavors to not only introduce the historical development of the Yang–Baxter deformation, but also its relation to generalized supergravity, non-geometric backgrounds, non-abelian $T$-duality and preserved Killing spinors. We supplement the general treatment with a wealth of explicit examples.

Keywords: integrable deformations, generalized supergravity, Yang–Baxter deformations, non-geometric backgrounds

1. Introduction

The integrability of type IIB superstring on the $\text{AdS}_5 \times S^5$ background \cite{1} has led to many important advances as it allows the application of highly developed techniques from integrable...
systems to a variety of string-theoretic problems (see [2] for a comprehensive review of this subject). Finding deformations of $\text{AdS}_5 \times S^5$ which retain the property of integrability allows us to further expand the reach of such techniques to more and more string backgrounds. While at first, integrable deformations have been found on a case-by-case basis, finding systematic ways of generating integrable backgrounds has become more and more important.

A nice and systematic way of performing integrable deformations was originally developed for two-dimensional principal chiral models by Klimcik [3, 4] based on the modified classical Yang–Baxter equation (mCYBE). In this method, the deformations are labeled by classical $r$-matrices satisfying the mCYBE and hence it is often referred to as the Yang–Baxter deformation. In 2013, this technique was generalized to symmetric coset sigma models by Delduc et al [5] and found an immediate application to the $\text{AdS}_5 \times S^5$ superstring case [6]. Another possible generalization consists in considering the homogeneous classical Yang–Baxter equation (hCYBE) which extends the applicability of YB deformations to two-dimensional principal chiral models, symmetric coset sigma models [7], as well as the $\text{AdS}_5 \times S^5$ superstring [8]. The development of the mCYBE and the hCYBE has proceeded in parallel but independently and while they are technically similar, the resulting physical interpretation of the YB deformation is different. In addition, while for the mCYBE case the research has concentrated on the famous example of classical $r$-matrix of Drinfeld–Jimbo type, for the hCYBE case many classical $r$-matrices have been discussed in various contexts. For the benefit of the non-expert reader, we shall give a short summary of the development of the YB deformations for both the mCYBE and hCYBE cases.

Starting with the mCYBE case, Arutyunov, Borsato and Frolov realized a coset construction for the metric and Neveu–Schwarz (NS)–Neveu–Schwarz (NS) two-form [9]. This background is often called the $\eta$-deformation of $\text{AdS}_5 \times S^5$ or simply $\eta$-deformed $\text{AdS}_5 \times S^5$. It has proved to be technically difficult to obtain the corresponding supercoset construction by including the space–time fermions. This was done after two years, in 2015, when a full background including the Ramond (R)–Ramond (R) fluxes and dilaton was obtained [10]. A significant discovery [10] was that this background is not a solution to the standard type IIB supergravity. This lead Arutyunov, Frolov, Hoare, Roiban and Tseytlin to modify the usual definition supergravity to support the $\eta$-deformed $\text{AdS}_5 \times S^5$ as a solution [12]. This modified versions is called generalized supergravity. While this seemed initially artificial, it was later realized that the generalization has a fundamental origin in the context of string theory. In 2016, Tseytlin and Wulff succeeded in deriving the same rules by solving the $\kappa$-symmetry constraints in type IIB superstring on an arbitrary background [13], which also led to the construction of the equations of motion (EOM) of the dilatino and gravitino in generalized supergravity. This is in fact a possible solution to a long-standing problem regarding the solution to the $\kappa$-symmetry constraints. It is no exaggeration to say that this fundamental progress was made possible by the study of the YB deformation.

One of the advantages of the hCYBE in comparison to the mCYBE is that it allows partial deformations of $\text{AdS}_5 \times S^5$. That is, one can deform the $\text{AdS}_5$ and the $S^5$ part separately. In this way one can describe many integrable deformations of $\text{AdS}_5 \times S^5$ in a unified manner, including well-known backgrounds such as the Lunin–Maldacena background [14], the Maldacena–Russo background (a gravity dual of noncommutative gauge theory) [15] and the Schrödinger background [16]. The supercoset construction has been realized for arbitrary
classical $r$-matrices [17] and the associated deformed string backgrounds can be obtained systematically via a supercoset construction. A remarkable property in the hCYBE construction is the so called unimodularity condition found by Borsato and Wulff [18]. This condition for classical $r$-matrix ensures that the deformed spacetime satisfies the on-shell equations of the standard type IIB supergravity. Otherwise, the resulting background is a solution to the generalized supergravity. Another important observation is the fact that homogeneous YB deformations can be regarded as duality transformations [14–16, 19–32]. In fact, the above well-known backgrounds, such as the Lunin–Maldacena background, can also be reproduced via TsT-transformations [14–16, 19] which are sequences of $T$-dualities and a coordinate shift. The connection to duality transformations is further extended to the non-abelian $r$-matrix case. In [20], it was shown that homogeneous YB-deformations associated with a specific class of non-abelian $r$-matrices could also be realized as a generalization of the TsT-transformation. In this spirit, Hoare and Tseytlin [21] had conjectured that the homogeneous YB deformations are equivalent to (a certain class of) non-abelian $T$-dualities [33–37]. This conjecture was proven by Borsato and Wulff [22, 23]. Furthermore, some YB deformed backgrounds in generalized supergravity have been identified with non-geometric backgrounds like $T$-folds [38].

This review endeavors to put YB deformations into the context of these recent insights, highlighting their connections to generalized supergravity as well as understanding them as string duality transformations and relating certain subclasses to non-geometric fluxes. At the same time, we give a comprehensive list of explicit examples of the various types of YB-deformed backgrounds which have over the years appeared in the literature.

A review such as this one is necessarily incomplete. We will not report on a number of important developments that lie beyond its scope and instead refer the reader to the original literature. Due to the limitation of pages and possibly our knowledge, we could not include some significant issues such as the connection to Poisson–Lie $T$-duality [39–42] and the construction of the E-model [43, 44], the analysis of Hamiltonian integrability of deformed models [5, 45] and the relation [46–50] to 4D Chern–Simons theory [51–53]. In this review, we will concentrate on classical integrability only and not consider the question of quantum integrability.

We start out in section 2 with the very basics by introducing the AdS$_5 \times$ S$^5$ superstring [54] and its most important properties such as e.g. its classical integrability [1]. After this, we develop the YB deformations of the AdS$_5 \times$ S$^5$ superstring [6, 8] in section 3. After giving the action on the YB-deformed AdS$_5 \times$ S$^5$ superstring (section 3.1), we show its classical integrability (section 3.2) and $\kappa$-symmetry (section 3.3) [6, 8] and discuss YB-deformed backgrounds from the Green–Schwarz (GS) action [9, 10, 17] (section 3.4). Next, we introduce an important recent development, namely generalized supergravity [12, 13] (section 3.5). Whether a given $r$-matrix encoding a YB deformation gives rise to a supergravity solution or a solution of generalized supergravity depends on whether it satisfies the unimodularity condition [18], discussed in section 3.6. In section 3.7, we classify $r$-matrices according to whether they are abelian or non-abelian and their rank. In section 3.9, we show that YB deformations can be beautifully reinterpreted in the framework of string duality transformations [24, 25]. To do so, we include a brief review of double field theory (DFT) [55–61]. After having developed the general theory, we introduce a number of examples of homogeneous YB-deformed AdS$_5 \times$ S$^5$ backgrounds in section 4. Section 5 focuses on YB deformations of Minkowski [38, 62–65] and AdS$_5 \times$ S$^5$ backgrounds. We show that the deformed backgrounds we considered here are $T$-folds [38], a particular class of non-geometric backgrounds, again providing a number of explicit examples. In section 6, we finally turn to the interplay between integrability and preserved supersymmetries of a deformed background and present a formula for determining the
pre-served Killing spinors \[66, 67\]. In appendix A, we present our conventions and in appendix B we collect useful formulas on the \(\text{psu}(2,2|4)\) algebra.

2. The AdS\(_5\) × S\(_5\) superstring

In this section, we will briefly review some basic facts on the AdS\(_5\) × S\(_5\) superstring. For a comprehensive review, see [68].

2.1. Metsaev–Tseytlin action

The dynamics of the superstring on the AdS\(_5\) × S\(_5\) background is described by the supercoset

\[
\frac{\text{PSU}(2,2|4)}{\text{SO}(1,4) \times \text{SO}(5)}.
\]

(2.1)

The corresponding action in the GS formulation has been written down by Metsaev and Tseytlin [54] and has the form

\[
S = - \frac{T}{2} \int d\tau \ d\sigma \ P^{\alpha\beta} \text{Str}[A_\alpha d_-(A_\beta)],
\]

(2.2)

where \(T \equiv R^2 / 2\pi \alpha'\) is the effective string tension and \(R\) is the radius of AdS\(_5\) and S\(_5\). The \(P^{\alpha\beta}_\pm\) are linear combinations of the Weyl invariant metric \(\gamma^{ij}\) on the world-sheet and the antisymmetric tensor \(\varepsilon^{ij}\),

\[
P^{\alpha\beta}_\pm \equiv \frac{\gamma^{ij} \varepsilon^{ij}}{2}.
\]

(2.3)

We work in conformal gauge \(\gamma^{ij} = \text{diag}(-1, 1)\) and normalize the epsilon tensor as \(\varepsilon^{\tau\sigma} = 1\). \(A\) is the left-invariant one-form for an element \(g\) of \(\text{SU}(2,2|4)\) defined by

\[
A = g^{-1} \ dg, \quad g \in \text{SU}(2,2|4).
\]

(2.4)

It satisfies the Maurer–Cartan equation

\[
dA + A \wedge A = 0.
\]

(2.5)

Given the projection operators \(P^{0}(i=0,1,2,3)\) on each \(\mathbb{Z}_4\)-graded subspaces of \(g \equiv \text{su}(2,2|4)\), the projection operators \(d_\pm\) are defined as the linear combination of the \(P^{0}\)

\[
d_\pm \equiv \mp P^{(1)} + 2 P^{(2)} \pm P^{(3)},
\]

(2.6)

satisfying the relation

\[
\text{Str}[X d_\pm(Y)] = \text{Str}[d_\mp(X) Y].
\]

(2.7)

If we expand the left-invariant one-form \(A\) as

\[
A = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)}, \quad \text{where} \ A^{(i)} = P^{(i)}(A),
\]

(2.8)

the action (2.2) can be rewritten as

\[
S = \frac{T}{2} \int \text{Str}(A^{(2)} \wedge * \gamma A^{(2)} - A^{(1)} \wedge A^{(3)}),
\]

(2.9)
which is the sum of the kinetic and the Wess–Zumino term. The ratio of the coefficients of the two terms is determined by the requirement that the action be $\kappa$-symmetric.

### 2.2. Classical integrability

From a practical point of view, the property of integrability is of paramount importance, as it allows the application of powerful computational techniques. In the following, we will consider the kinematical integrability in the sense of the existence of Lax pair. This notion is weaker than the complete integrability in the sense of the construction of action-angle variables.

We will see that the EOM of the action (2.2) and the flatness condition (2.5) can be combined into a flatness condition for the Lax pair with a parameter $u$. From the Lax pair, we can construct infinitely many conserved charges via the monodromy matrix. In this sense, the AdS$_5 \times$ S$^5$ superstring is classically integrable as shown by Bena et al [1]. In this subsection, we show the classical integrability of the AdS$_5 \times$ S$^5$ superstring by constructing the Lax pair explicitly.

EOM. Let us start with the EOM of the action (2.2). They are given by

$$\mathcal{E} = \partial_\alpha d_-(A^\alpha_-) + \partial_\alpha d_+(A^\alpha_+) + [A_{(\alpha+)} - d_-(A^\alpha_-)] + [A_{(\alpha-)}, d_+(A^\alpha_+)] = 0, \quad (2.10)$$

where worldsheet vectors marked with ($\pm$) are have been acted on with the projection operator $P_{\pm}^\alpha$.

$$A^\alpha_{(\pm)} = P_{\pm}^\alpha A_\beta. \quad (2.11)$$

The flatness condition (2.5) of $A$ can be rewritten as

$$\mathcal{F} = \frac{1}{2} \epsilon^{\alpha\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta])$$

$$= \partial_\alpha A^\alpha_+ - \partial_\alpha A^\alpha_- + [A_{(\alpha-)}, A^\alpha_+] = 0. \quad (2.12)$$

For later convenience, we will decompose the EOM (2.10) and the flatness condition (2.12) on each of the $\mathbb{Z}_4$-graded components. The bosonic parts are

$$B_1 := \mathcal{F}^{(0)} = \partial_\alpha A^{(0)}_+ - \partial_\alpha A^{(0)}_- + [A^{(0)}_+, A^{(0)}_-] + [A^{(1)}_{(-)}, A^{(1)}_+] + [A^{(3)}_{(-)}, A^{(3)}_+] = 0, \quad (2.13)$$

$$B_2 := \frac{1}{4}(\mathcal{F}^{(2)} + 2 \mathcal{F}^{(2)}) = \partial_\alpha A^{(2)}_+ + [A^{(2)}_-, A^{(2)}_+] + [A^{(3)}_{(-)}, A^{(3)}_+] = 0, \quad (2.14)$$

$$B_3 := \frac{1}{4}(\mathcal{F}^{(2)} - 2 \mathcal{F}^{(2)}) = \partial_\alpha A^{(2)}_- - [A^{(2)}_-, A^{(2)}_+] - [A^{(1)}_{(-)}, A^{(1)}_+] = 0, \quad (2.15)$$

and the fermionic parts are

$$F_1 := \frac{1}{4}(3 \mathcal{F}^{(1)} - \mathcal{F}^{(1)})$$

$$= \partial_\alpha A^{(1)}_+ - \partial_\alpha A^{(1)}_- + [A^{(0)}_-, A^{(1)}_+] + [A^{(1)}_{(-)}, A^{(1)}_+] + [A^{(2)}_{(-)}, A^{(2)}_+] = 0, \quad (2.16)$$

$$F_2 := \frac{1}{4}(3 \mathcal{F}^{(3)} + \mathcal{F}^{(3)})$$

$$= \partial_\alpha A^{(3)}_+ - \partial_\alpha A^{(3)}_- + [A^{(0)}_-, A^{(3)}_+] + [A^{(1)}_{(-)}, A^{(1)}_+] + [A^{(3)}_{(-)}, A^{(3)}_+] = 0, \quad (2.17)$$


\[ F_3 := \frac{1}{4}(\mathcal{F}^{(1)} + \mathcal{F}^{(1)}) = [A^{(3)}_{-\lambda}, A^{(2)}_{(\lambda)}] = 0, \]  
\[ F_4 := \frac{1}{4}(-\mathcal{F}^{(3)} + 2\mathcal{F}^{(3)}) = [A^{(2)}_{-\lambda}, A^{(1)}_{(\lambda)}] = 0. \]  

Construction of the Lax pair. The Lax pair of the AdS$_5 \times$ S$^5$ superstring is given by (see [1])

\[ \mathcal{L}_\alpha \equiv M_{(-\lambda)} + L_{(+\lambda)}, \]

where $M^\alpha_{(-\lambda)}$ and $L^\alpha_{(\lambda)}$ are

\[ M^\alpha_{(-\lambda)} = A^{(0)}_{(-\lambda)} + u A^{(1)}_{(-\lambda)} + u^2 A^{(2)}_{(-\lambda)} + u^{-1} A^{(3)}_{(-\lambda)}, \]
\[ L^\alpha_{(\lambda)} = A^{(0)}_{(\lambda)} + u A^{(1)}_{(\lambda)} + u^{-2} A^{(2)}_{(\lambda)} + u^{-1} A^{(3)}_{(\lambda)}. \]

Here $u$ is the spectral parameter. The flatness condition of the Lax pair (2.20)

\[ \frac{1}{2} e^{\alpha \beta} (\partial_\alpha \mathcal{L}_\beta - \partial_\beta \mathcal{L}_\alpha + [\mathcal{L}_\alpha, \mathcal{L}_\beta]) = 0, \]  

is equivalent to the EOM (2.10) and the flatness condition (2.12). To see this observe that the condition can be rewritten as a sum over the EOM

\[ \frac{1}{2} e^{\alpha \beta} (\partial_\alpha \mathcal{L}_\beta - \partial_\beta \mathcal{L}_\alpha + [\mathcal{L}_\alpha, \mathcal{L}_\beta]) = u^0 B_1 + u^{-2} B_2 + u^{-3} B_3 + u^3 F_4 = 0. \]

Therefore, the Lax pair (2.20) is on-shell a flat current. We can then define the monodromy matrix

\[ T(u) = \mathcal{P} \exp \left( \int_0^{2\pi} d\sigma \mathcal{L}_\sigma(u) \right), \]

where $\mathcal{P}$ denotes the equal-time path ordering in terms of $\sigma$, and we assume all dynamical variables are periodic along the $\sigma$-direction. By using the flatness condition (2.23), we can obtain

\[ \frac{\partial}{\partial \tau} T(u) = \int_0^{2\pi} d\sigma \mathcal{P} \exp \left( \int_0^{2\pi} d\sigma' \mathcal{L}_\sigma(u) \right) \left( \partial_\tau \mathcal{L}_\sigma(u) \right) \exp \left( \int_0^\sigma d\sigma' \mathcal{L}_\sigma(u) \right) \]
\[ = \int_0^{2\pi} d\sigma \mathcal{P} \exp \left( \int_0^{2\pi} d\sigma' \mathcal{L}_\sigma(u) \right) \left( \partial_\tau \mathcal{L}_\sigma(u) - [\mathcal{L}_\tau, \mathcal{L}_\sigma] \right) \exp \left( \int_0^\sigma d\sigma' \mathcal{L}_\sigma(u) \right) \]
\[ = \int_0^{2\pi} d\sigma \left( \mathcal{P} \exp \left( \int_0^{2\pi} d\sigma' \mathcal{L}_\sigma(u) \right) \mathcal{L}_\tau \right) \exp \left( \int_0^\sigma d\sigma' \mathcal{L}_\sigma(u) \right) \]
\[ = \left[ \mathcal{L}_\tau \sigma = 0, T(u) \right], \]

where in the last equality, we have used the periodic boundary condition of the dynamical fields. Equation (2.26) indicates that $\text{Tr}(T(u)^n)$ ($n \in \mathbb{Z}_{\geq 0}$) does not depend on $\tau$. Therefore we obtain infinitely many conserved charges as the coefficients of an expansion in $u$ of the monodromy matrix [69].
2.3. \( \kappa \)-symmetry

The action (2.2) of the AdS\( _5 \times S^3 \) superstring is invariant under \( \kappa \)-symmetry [54], as we will briefly show in the following. In this subsection, we will work without taking the conformal gauge in order to discuss kappa symmetry.

The action of \( \kappa \)-symmetry is realized as a combination of the variation of the group element \( g \) and the Weyl invariant world-sheet metric \( \gamma^{\alpha\beta} \),

\[
g^{-1} \delta g = P^\alpha_\beta \left\{ Q^1_{\kappa_1 \alpha}, A^{(2)}_\beta \right\} + P^\alpha_\beta \left\{ Q^2_{\kappa_2 \alpha}, A^{(2)}_\beta \right\}, \tag{2.27}
\]

\[
\delta_g \gamma^{\alpha\beta} = \frac{1}{4} \text{STr} \left[ \kappa \left( \left[ Q^1_{\kappa_1}, A^{(1)}_{\gamma^{13}}, A^{(1)}_{\gamma^{23}} \right] + \left[ Q^2_{\kappa_2}, A^{(2)}_{\gamma^{13}}, A^{(2)}_{\gamma^{23}} \right] \right) \right], \tag{2.28}
\]

where \( \kappa_{\alpha}(I = 1, 2) \) are local fermionic parameters and we have defined \( \kappa = \text{diag} (1, -1, 1) \).

We first decompose the variation of the action into two parts,

\[
\delta g S = \delta_g S + \delta g S. \tag{2.29}
\]

The variation \( \delta g S \) coming from the group element \( g \) is given by

\[
\delta_g S = T \frac{2}{2} \int d^2 \sigma \text{STr} (g^{-1} \delta g \mathcal{E}), \tag{2.30}
\]

where \( \mathcal{E} \) is the EOM (2.10). If we take \( g^{-1} \delta g = \varepsilon = \varepsilon^{(1)} + \varepsilon^{(3)} \) with

\[
\varepsilon^{(1)} = P^\alpha_\beta \left\{ Q^1_{\kappa_1 \alpha}, A^{(2)}_\beta \right\}, \quad \varepsilon^{(3)} = P^\alpha_\beta \left\{ Q^2_{\kappa_2 \alpha}, A^{(2)}_\beta \right\}, \tag{2.31}
\]

\( \delta_g S \) can be rewritten as

\[
\delta_g S = T \frac{2}{2} \int d^2 \sigma \text{STr} \left[ \varepsilon^{(1)}(\varepsilon^{(3)} - 2\mathcal{E}^{(3)}) + \varepsilon^{(3)}(\varepsilon^{(1)} + 2\mathcal{E}^{(1)}) \right]
= -2T \int d^2 \sigma \text{STr} \left[ \varepsilon^{(1)}[A^{(2)}_{\gamma^{13}}, A^{(3)}_{\gamma^{13}}] + \varepsilon^{(3)}[A^{(2)}_{\gamma^{13}}, A^{(3)}_{\gamma^{13}}] \right], \tag{2.32}
\]

where we have used

\[
\mathcal{E}^{(1)} + \mathcal{E}^{(1)} = -4[A^{(2)}_{\gamma^{13}}, A^{(3)}_{\gamma^{13}}], \quad \mathcal{E}^{(3)} - \mathcal{E}^{(3)} = -4[A^{(2)}_{\gamma^{13}}, A^{(3)}_{\gamma^{13}}]. \tag{2.33}
\]

By using the expression (2.31), we can rewrite each of the terms in (2.32) as

\[
\text{STr} \left[ \varepsilon^{(1)}[A^{(2)}_{\gamma^{13}}, A^{(3)}_{\gamma^{13}}] \right] = \text{STr} \left[ A^{(2)}_{\gamma^{13}} A^{(3)}_{\gamma^{13}} A^{(1)}_{\gamma^{13}} Q^1_{\kappa_1} \right], \tag{2.34}
\]

\[
\text{STr} \left[ \varepsilon^{(3)}[A^{(2)}_{\gamma^{13}}, A^{(3)}_{\gamma^{13}}] \right] = \text{STr} \left[ A^{(2)}_{\gamma^{13}} A^{(3)}_{\gamma^{13}} A^{(1)}_{\gamma^{13}} Q^2_{\kappa_2} \right].
\]

An arbitrary grade-2 traceless element \( A^{(2)}_{\gamma^{13}} \) of \( su(2, 2|4) \) fulfills the relation

\[
A^{(2)}_{\gamma^{13}} A^{(2)}_{\gamma^{13}} = \frac{1}{8} \mathcal{Y} \text{STr}(A^{(2)}_{\gamma^{13}} A^{(2)}_{\gamma^{13}}) + c_{\alpha\beta} Z, \tag{2.35}
\]

where \( Z \) is the central charge of \( su(2, 2|4) \) and \( c_{\alpha\beta} \) is a symmetric function in \( \alpha \) and \( \beta \). By using the expressions (2.34), (2.35), \( \delta_g S \) becomes
\[ \delta g S = \frac{T}{4} \int d^2 \sigma \left[ \text{STr} \left( A_{\alpha}^{(2)} A_{\beta+}^{(2)} \right) \text{STr} \left( \{Q^1 \kappa_{+1}^\beta, A_{\alpha}^{(1)}\} \right) \right. \\
+ \left. \text{STr} \left( A_{\alpha+}^{(2)} A_{\beta+}^{(2)} \right) \text{STr} \left( \{Q^2 \kappa_{-2}^\beta, A_{\alpha}^{(3)}\} \right) \right]. \]  

(2.36)

Next we study the variation \( \delta \gamma S \) coming from the Weyl invariant world-sheet metric \( \gamma_{\alpha \beta} \). From (2.28), \( \delta \gamma S \) is

\[ \delta \gamma S = - \frac{T}{4} \int d^2 \sigma \text{STr} \left( A_{\alpha}^{(2)} A_{\beta-}^{(2)} \right) \text{STr} \left[ \left[ \{Q^1 \kappa_{+1}^\beta, A_{\alpha}^{(1)}\} + \{Q^2 \kappa_{-2}^\beta, A_{\alpha}^{(3)}\} \right] \right] \\
+ \text{STr} \left( A_{\alpha+}^{(2)} A_{\beta+}^{(2)} \right) \text{STr} \left( \{Q^2 \kappa_{-2}^\beta, A_{\alpha}^{(3)}\} \right). \]

(2.37)

Here, we used the relation

\[ A_{\alpha \pm} B_{\beta} = A_{\alpha \pm} B_{\beta}, \]  

(2.38)

where \( A_{\alpha}, B_{\alpha} \) are arbitrary vectors. This variation manifestly cancels out \( \delta g S \),

\[ \delta \gamma S = \delta g S + \delta \kappa S = 0. \]  

(2.39)

We see that the action (2.2) is invariant under the \( \kappa \)-symmetry transformations (2.27), (2.28).

2.4. The AdS\(_5\) × S\(^5\) background from the GS action

Next, let us explain how to read off the AdS\(_5\) × S\(^5\) background from the action (2.2).

2.4.1. The canonical form of the GS action. To read off the target space background from the action of the AdS\(_5\) × S\(^5\) superstring, we need to first introduce the canonical form of the GS action.

The canonical form of the type II GS superstring action at second order in \( \theta \) [70] is given by

\[ S = - T \int d^2 \sigma \left[ P^{\alpha \beta} (g_{mn} + B_{mn}) \partial_\alpha X^n \partial_\beta X^n \\
+ i \left( P^{\alpha \beta} \partial_\alpha X^n \Theta_1 \Gamma_m D_{\pm \beta} \Theta_1 + P^{\alpha \beta} \partial_\alpha X^n \Theta_2 \Gamma_m D_{\pm \beta} \Theta_2 \right) \\
- \frac{i}{8} P^{\alpha \beta} \Theta_1 \Gamma_m \tilde{F}_\alpha \Gamma_m \Theta_2 \partial_\alpha X^n \partial_\beta X^n + \mathcal{O} (\theta^4) \right], \]

(2.40)

where \( \Gamma_\alpha \) (\( \Gamma_m = e_m^a \Gamma_a \)) are the 32 × 32 gamma matrices. The differential operators \( D_{\pm \alpha} \) are defined by

\[ D_{\pm \alpha} \equiv \partial_\alpha + \frac{1}{4} \partial_\alpha X^m \omega_{\pm m}^{ab} \Gamma_{ab}, \]  

(2.41)

\[ \omega_{\pm m}^{ab} \equiv \omega_{m}^{ab} \pm \frac{1}{2} e_m^c H_{cab}, \]

(2.42)

where \( \omega^{ab} = \omega^{ab}_{m} dX^m \) is the spin connection on the target space, \( \Theta_1 \) and \( \Theta_2 \) are the 32-component Majorana spinor and its conjugate (see appendix B for details).
The metric $g_{mn}$ and the $B$-field $B_{mn}$ of the target space can be read off from the first line in (2.40). We can read off the dilaton $\Phi$ and the R–R field strengths $\hat{F}_{a_1 \ldots a_p}$ from the R–R bispinor $\hat{\mathcal{F}}$ which is defined by

$$\hat{\mathcal{F}} = \sum_p \frac{1}{p!} e^\Phi \hat{F}_{a_1 \ldots a_p} \Gamma^{a_1 \ldots a_p}.$$ (2.43)

If the Lagrangian (2.40) describes type IIB superstring, the summation runs over $p = 1, 3, 5, 7, 9$. Each R–R field strength satisfies

$$\hat{F}_p = (-1)^{\frac{p(p-1)}{2}} \ast \hat{F}_{10-p},$$ (2.44)

where the Hodge star $\ast$ is defined in appendix A. By comparing the action (2.2) expanded in terms of $\theta$ to the canonical form (2.40), we can obtain the explicit expression for the AdS$_5 \times$ S$^5$ superstring.

2.4.2. Group parametrization. To derive the AdS$_5 \times$ S$^5$ background from the GS action, we introduce a coordinate system via a parametrization of the group element $g$. We first decompose the group element into bosonic and fermionic parts,

$$g = g_b \cdot g_f \in SU(2, 2|4).$$ (2.45)

We parametrize the bosonic part $g_b$ as

$$g_b = g_{AdS_5} \cdot g_{S^5},$$

$$g_{AdS_5} \equiv \exp(x^\mu P_\mu) \cdot \exp(ln(z) D),$$

$$g_{S^5} \equiv \exp(\phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3) \cdot \exp(\xi J_56) \cdot \exp(\rho P_5).$$ (2.46)

Here, $P_\mu (\mu = 0, \ldots, 3)$ and $D$ are the translation and dilatation generators of the conformal algebra $so(2,4)$. We define the Cartan generators of the $so(6)$ algebra $h_i, (i = 1, 2, 3)$ by

$$h_1 \equiv J_{57}, \quad h_2 \equiv J_{68}, \quad h_3 \equiv P_9.$$ (2.47)

We parameterize the fermionic part $g_f$ as

$$g_f = \exp(Q^I \theta_I), \quad Q^I \theta_I = (Q^I)^{\hat{\alpha}\dot{\alpha}} \theta_{i\dot{\alpha}},$$ (2.48)

where the supercharges $(Q^I)^{\hat{\alpha}\dot{\alpha}} (I = 1, 2)$ are labeled by two indices $(\hat{\alpha}, \dot{\alpha} = 1, \ldots, 4)$ and $\theta_{i\dot{\alpha}} (I = 1, 2)$ are 16-component Majorana–Weyl fermions. A matrix representation of the above generators of $su(2,2|4)$ is given in appendix B.

2.4.3. Expansion of the left-invariant current. Next, we will expand the left-invariant current $A$ to second order in the spacetime fermions $\theta$,

$$A = A_0 + A_1 + A_2 + \mathcal{O}(\theta^3).$$ (2.49)

Now, since we chose the parametrization (2.45), (2.46), (2.48) for $g$, the left-invariant current $A$ can be expanded as

$$A = g_t^{-1} A_0 g_t + Q^I d\theta_I$$

$$= A_0 + [A_0, Q^I \theta_I] + \frac{1}{2} [[A_0, Q^I \theta_I], Q^J \theta_J] + Q^I d\theta_I + \mathcal{O}(\theta^3),$$ (2.50)
where $A_{(p)}$ is defined as $\partial^p(\theta^0)$ of the left-invariant current $A$, and $A_{(0)}$ is given by

$$A_{(0)} \equiv g_b^{-1} d g_b = \left( e_{\alpha}^a P_a - \frac{1}{2} \omega_{ab}^c J_{ab} \right) dX^m.$$  

(2.51)

The vielbein $e^a = e_{\alpha}^a dX^\alpha$ has the form

$$e^a = \left( \frac{dx^0}{z}, \frac{dx^1}{z}, \frac{dx^2}{z}, \frac{dx^3}{z}, dr, \sin r \, d \phi_1, \sin r \, \cos \phi_1, dr, \sin r \, d \phi_2, \cos r \, d \phi_3 \right),$$  

(2.52)

and $\omega_{ab}^c = \omega_{ab}^c dX^m$ is the associated spin connection.

Moreover, by using the commutation relations of $su(2,2|4)$ (see appendix A for our conventions), each commutator in (2.50) can be evaluated as

$$[A_{(0)}, \mathbf{Q}^j \theta_j] = \mathbf{Q}^j \left( \frac{1}{4} \delta^{ij} \omega^{ab} \gamma_{ab} + \frac{i}{2} \varepsilon^{ij} e^a \gamma_a \right) \theta_j,$$

(2.53)

$$[A_{(0)}, \mathbf{Q}^j \bar{\theta}_j], \mathbf{Q}^j \theta_j] = i \bar{\theta}_j \gamma_a \left( \frac{1}{4} \delta^{ij} \omega^{cd} \gamma_{cd} + \frac{i}{2} \varepsilon^{ij} e^b \gamma_b \right) \theta_j P_a \theta_i + \frac{1}{4} \varepsilon^{ikl} \bar{\theta}_j \gamma^{cd} \left( \frac{1}{4} \delta^{ij} \omega^{ab} \gamma_{ab} + \frac{i}{2} \varepsilon^{ij} e^c \gamma_c \right) \theta_j \varDelta X^m R_{cd} e^f J_{ef}$$

+ (irrelevant terms proportional to the central charge $Z$).  

(2.54)

Here, $R_{abcd}$ is the Riemann tensor in the tangent space of the $AdS_5 \times S^5$ background. For the derivation of (2.54), we have used $\delta^{ij} \bar{\theta}_j \gamma_a d \theta_j = 0$ and $\varepsilon^{ij} \bar{\theta}_j \gamma_{ab} d \theta_j = 0$.

From the above calculations, the left-invariant current $A$ up to second order in $\theta$ becomes

$$A = \left( e^a + \frac{i}{2} \bar{\theta}_i \gamma_i D^i \theta_j \right) P_a - \frac{1}{2} \left( \omega_{ab}^c + \frac{1}{4} \varepsilon^{ikl} \gamma_{cd} R_{cd} e^f J_{ef} \right) J_{ab}$$

$$+ \mathbf{Q}^j D^j \theta_j + \mathbf{O}(\theta^3),$$

(2.55)

where we defined the differential operator

$$D^i \equiv \delta^i \left( d + \frac{1}{4} \omega^{ab} \gamma_{ab} \right) + \frac{i}{2} \varepsilon^i e^a \gamma_a.$$  

(2.56)

In particular, $A_{(1)}$, $A_{(2)}$ are given by

$$A_{(1)} = \mathbf{Q}^j D^j \theta_j,$$

(2.57)

$$A_{(2)} = \frac{i}{2} \bar{\theta}_i \gamma_i D^i \theta_j P_a + \frac{1}{8} \varepsilon^{ikl} \bar{\theta}_j \gamma_{cd} R_{cd} e^f J_{ef} J_{ab}.$$  

(2.58)

2.4.4. Evaluation of the bi-linear current part. Using the expansion (2.55) we obtain

$$\frac{1}{2} \text{Str} \left[ A_{(2)} d(A_{(2)}) \right] = \eta_{ab} e^a e^b + i \left[ e_{\alpha}^a \left( \bar{\theta}_1 \gamma_a \partial_{\alpha} \theta_2 \right) + e_{\alpha}^a \left( \bar{\theta}_2 \gamma_a \partial_2 \theta_1 \right) \right]$$

$$+ \frac{i}{4} \left[ e_{\alpha}^a e_{\beta}^b \omega_{ab}^c \left( \bar{\theta}_1 \gamma_b \gamma_{cd} \theta_1 \right) e_{\alpha}^a e_{\beta}^b \omega_{ab}^c \left( \bar{\theta}_2 \gamma_a \gamma_{cd} \theta_2 \right) \right]$$

$$- e_{\alpha}^a e_{\beta}^b \bar{\theta}_1 \gamma_a \gamma_b \theta_2 + \mathbf{O}(\theta^3),$$

(2.59)
where \( e_{\alpha}^\mu \equiv e_m^\nu \partial_\mu X^m \). Further using (B.36), (B.40), and (B.41), we obtain
\[
\frac{1}{2} \text{Str}[A, d_-(A)] = g_{mn} \partial_\alpha X^m \partial_\beta X^n + i \left[ e_{\beta}^\alpha \theta_1 \Gamma_\alpha \partial_\beta \theta_1 + e_{\alpha}^\mu \theta_2 \Gamma_\mu \partial_\beta \theta_2 \right] + \frac{i}{4} \left[ e_{\beta}^\mu e_{\alpha}^\nu \omega_a^{cd} \theta_1 \Gamma_\beta \Gamma_\nu \theta_1 + e_{\alpha}^\nu e_{\beta}^\mu \omega_b^{cd} \theta_2 \Gamma_\alpha \Gamma_\beta \theta_2 \right] - \frac{i}{8} e_{\alpha}^\nu e_{\beta}^\mu \theta_1 \Gamma_\mu \theta_2 + \mathcal{O}(\theta^3),
\] (2.60)
where \( \mathcal{F}_5 \) is a bispinor
\[
\mathcal{F}_5 \equiv \frac{1}{5!} e^\phi \tilde{F}_{a_1 \cdots a_5} \Gamma^{a_1 \cdots a_5} = 4 \left( \Gamma^{01234} + \Gamma^{56789} \right).
\] (2.61)
This describes the R–R five-form field strength in the tangent space of the \( \text{AdS}_5 \times S^5 \) background.

The expression (2.60) implies that the action (2.2) takes the canonical form (2.40) of the GS action. Therefore, the target space of the action (2.2) is the familiar \( \text{AdS}_5 \times S^5 \) background with the RR five-form
\[
\begin{align*}
&dx^2 = dx_{\text{AdS}_5}^2 + dx_{S^5}^2, \\
e^\Phi F_5 = 4 \left( \omega_{\text{AdS}_5} + \omega_{S^5} \right).
\end{align*}
\] (2.62) (2.63)
Under the parametrization (2.46) of \( g_b \), the metrics of \( \text{AdS}_5 \) and \( S^5 \) are
\[
\begin{align*}
dx_{\text{AdS}_5}^2 &= -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + dx^5^2, \\
dx_{S^5}^2 &= dx^2 + \sin^2 r \, d\xi^2 + \cos^2 \xi \sin^2 r \, d\phi_1^2 + \sin^2 \xi \, d\phi_2^2 + \cos^2 \xi \, d\phi_3^2,
\end{align*}
\] (2.64) (2.65)
and the volume forms \( \omega_{\text{AdS}_5}, \omega_{S^5} \) of \( \text{AdS}_5 \) and \( S^5 \) are given by
\[
\begin{align*}
\omega_{\text{AdS}_5} &= -\frac{dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^5}{z^5}, \\
\omega_{S^5} &= \sin^3 r \, \cos r \, \sin \xi \, \cos \xi \, dr \wedge d\xi \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 \quad (\omega_{\text{AdS}_5} = *_{10} \omega_{S^5}).
\end{align*}
\] (2.66) (2.67)
In the following discussion, we set the dilaton to zero, \( \Phi = 0 \).

2.5. Killing vectors

For later use, we calculate the Killing vectors \( \tilde{T}_i \equiv \tilde{T}_i^m \partial_m \) associated to the bosonic symmetries \( T_i \) of the \( \text{AdS}_5 \) background. The Killing vectors on this background can be expressed as (see appendix C in [25] for more details)
\[
\tilde{T}_i = \tilde{T}_i^m \partial_m = \left[ \text{Ad}_{g_{b}^{-1}} \right]_i^a e_a^m \partial_m = \text{Str} \left( g_{b}^{-1} T_i g_b \mathbf{P}_a \right) e^a_m \partial_m,
\] (2.68)
where we introduced the notation \( g T_i g^{-1} \equiv [\text{Ad}_g]_i^a T_a \). Using our parametrization (2.46), the Killing vectors on the \( \text{AdS}_5 \) background are given by
\[
\tilde{P}_\mu \equiv \text{Str} \left( g_{b}^{-1} P_\mu g_b \mathbf{P}_a \right) e^a_m \partial_m = \partial_\mu,
\]
\[ K_\mu \equiv \text{STr} \left( g^{-1} K_\mu g \right) e^{a_m} \partial_m = (x^\nu x_\nu + z^2) \partial_\mu - 2 (x^\nu \partial_\nu + z \partial_z), \]
\[ \hat{M}_{\mu\nu} \equiv \text{STr} \left( g^{-1} M_{\mu\nu} g \right) e^{a_m} \partial_m = x_\nu \partial_\mu - x_\mu \partial_\nu, \]
\[ \hat{D} \equiv \text{STr} \left( g^{-1} D g \right) e^{a_m} \partial_m = x^\mu \partial_\mu + z \partial_z. \quad (2.69) \]

The Lie brackets of these vector fields satisfy the same commutation relations (B.25) as the conformal algebra so(2,4) (with negative sign, \([\hat{T}_i, \hat{T}_j] = -f_{ij}^k \hat{T}_k\)):

\[ [\hat{P}_\mu, \hat{K}_\nu] = -2 \left( \eta_{\mu\nu} \hat{D} - \hat{M}_{\mu\nu} \right), \]
\[ [\hat{D}, \hat{P}_\mu] = -\hat{P}_\mu, \]
\[ [\hat{D}, \hat{K}_\mu] = \hat{K}_\mu, \]
\[ [\hat{M}_{\mu\nu}, \hat{P}_\rho] = -\eta_{\mu\rho} \hat{P}_\nu + \eta_{\nu\rho} \hat{P}_\mu, \]
\[ [\hat{M}_{\mu\nu}, \hat{K}_\rho] = -\eta_{\mu\rho} \hat{K}_\nu + \eta_{\nu\rho} \hat{K}_\mu, \]
\[ [\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}] = -\eta_{\mu\rho} \hat{M}_{\nu\sigma} + \eta_{\nu\rho} \hat{M}_{\mu\sigma} + \eta_{\mu\sigma} \hat{M}_{\nu\rho} - \eta_{\nu\sigma} \hat{M}_{\mu\rho}. \quad (2.70) \]

3. Yang–Baxter deformations of the AdS5 × S5 superstring

Enlarging the reach of integrability techniques to other models brings many calculational advantages. A way of doing this is finding deformations of integrable models which retain the property of integrability. Being able to do so systematically instead of working case-by-case is a great advantage. The YB deformation provides a systematic way of generating integrable deformations of the AdS5 × S5 superstring. This section will give a comprehensive introduction to this topic.

3.1. The action of the YB-deformed AdS5 × S5 superstring

The action of the YB-deformed AdS5 × S5 superstring is given by [6, 8, 71]

\[ S_{\text{YB}} = \frac{T(1 - c^2 \eta^2)}{2} \int d^2 \sigma \, P^{03} \text{STr} \left[ A_\alpha \tilde{d}_- \circ \Theta^{-1}_- (A_\beta) \right], \quad (3.1) \]

where \( c^2 \) is a real parameter, \( \eta \in \mathbb{R} \) is a deformation parameter and \( \tilde{d}_\pm \) are the modified projection operators

\[ \tilde{d}_\pm \equiv \mp P^{(1)} \mp 2 \tilde{\eta} \mp P^{(2)} \pm P^{(3)}, \quad \tilde{\eta} = \sqrt{1 + c^2 \eta^2}. \quad (3.2) \]

The linear operators \( \Theta_\pm \) are defined by

\[ \Theta_\pm \equiv 1 \pm \eta R_8 \circ \tilde{d}_\pm. \quad (3.3) \]
For \( \eta = 0 \) the deformed action (3.1) reduces to the undeformed \( \text{AdS}_5 \times S^5 \) superstring sigma model action (2.2).

- **R-operator and classical r-matrix.** A key ingredient of the YB deformation is the \( R \)-operator which is a skew-symmetric linear operator \( R : g \rightarrow g \) and solves the classical Yang–Baxter equation (CYBE),

\[
\text{CYBE}(X, Y) \equiv [R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = -c^2 [X, Y], \quad X, Y \in g.
\]

(3.4)

The dressed \( R \)-operator \( R_g \) is defined by

\[
R_g(X) := g^{-1} R(gX g^{-1}) g = \text{Ad}_{g^{-1}} \circ R \circ \text{Ad}_{g}(X), \quad g \in SU(2, 2|4).
\]

(3.5)

The operator \( R_g \) is also a solution of the CYBE (3.4),

\[
\text{CYBE}_g(X, Y) \equiv [R_g(X), R_g(Y)] - R_g([R_g(X), Y] + [X, R_g(Y)]) = -c^2 [X, Y],
\]

(3.6)

if the linear operator \( R \) satisfies the CYBE. This is easily seen from the relation

\[
\text{CYBE}_g(X, Y) = \text{Ad}_g^{-1} \text{CYBE}(\text{Ad}_g(X), \text{Ad}_g(Y)).
\]

It is useful to rewrite the \( R \)-operator in tensorial notation. Then, the \( R \)-operator can be expressed via a skew-symmetric classical \( r \)-matrix \( r \in g \otimes g \) which is expressed as

\[
r = \frac{1}{2} r^{ij} T_i \wedge T_j, \quad r^{ij} = -r^{ji}(-1)^{i+j}, \quad T_i \in g.
\]

(3.7)

For \( T_i \) a bosonic generator, \( |i| = 0 \) and for \( T_j \) a fermionic generator, \( |j| = 1 \). Then, the action of the \( R \)-operator can be defined as

\[
R(X) = r^{ij} T_i \text{Str}(T_j X), \quad X \in g.
\]

(3.8)

This allows us to encode YB deformations by classical \( r \)-matrices.

- **Classification of the CYBE.** The CYBE can be of three types:

- (a) \( c^2 < 0 \),
- (b) \( c^2 = 0 \),
- (c) \( c^2 > 0 \).

The CYBE with \( c^2 \neq 0 \) is called the mCYBE. YB deformations of principal chiral models with \( c^2 < 0 \) were originally developed by Klimcik [3], and the integrability of the deformed models was shown in [4]. These deformations were generalized to symmetric coset sigma models [5] and the \( \text{AdS}_5 \times S^5 \) superstring [6, 71]. YB deformations based on the mCYBE with \( c^2 < 0 \) are called \( q \)-deformations.\(^9\) A typical solution of the mCYBE is the Drinfeld–Jimbo type \( r \)-matrix [72, 73],

\[
r_{DJ} = c \sum_{1 \leq i < j \leq 8} E_{ij} \wedge E_{ji} (-1)^{|i|},
\]

(3.9)

\(^9\) The \( q \)-deformed \( \text{AdS}_5 \times S^5 \) background is also called the \( \eta \)-deformed \( \text{AdS}_5 \times S^5 \) or the ABF background [9, 10].
where $E_{ij}(i, j = 1, \ldots, 8)$ are the $gl(4|4)$ generators and the super skew-symmetric symbol is defined by

$$E_{ij} \wedge E_{kl} \equiv E_{ij} \otimes E_{kl} - E_{kl} \otimes E_{ij}(-1)^{[i][k][j][l]}.$$  \hfill (3.10)

Here we determined the parity of the indices as $\bar{i} = 0$ for $i = 1, \ldots, 4$ and $\bar{i} = 1$ for $i = 5, \ldots, 8$. The associated $R$-operator acts on

$$R(E_{ij}) = \begin{cases} 
+c E_{ij} & \text{if } i < j \\ 0 & \text{if } i = j \\ -c E_{ij} & \text{if } i > j 
\end{cases} \hfill (3.11)$$

The $r$-matrix (3.9) with $c^2 < 0$ was used for a $q$-deformation of the AdS$_5 \times S^5$ superstring [6, 71]. We often normalize the complex parameter $c$ as $c = i$. The full explicit expression (3.68) of the $q$-deformed AdS$_5 \times S^5$ background is given in [10]. Remarkably, this deformed background does not solve the standard supergravity equations but the generalized supergravity equations (GSE) [12]. Interestingly, other $q$-deformed AdS$_2 \times S^2 \times T^6$ and AdS$_5 \times S^5$ backgrounds can constructed and shown to be solutions of standard supergravity using Drinfeld–Jimbo type $r$-matrices with different fermionic structures, as illustrated in [11]. For the case $c^2 > 0$, the associated YB deformations of the sigma model for the conformal algebra $su(2, 2)$ have been studied in [74].

The second class, $c = 0$, is frequently called the hCYBE. In terms of the $r$-matrix, the hCYBE (3.4) can be rewritten as

$$f_{1i_1j_1} r^{j_1}_{i_1} r^{k_2}_{l_2} + f_{1i_1j_1} r^{k_1}_{i_1} r^{j_2}_{i_2} + f_{1i_1j_1} k^{j_1}_{i_1} r^{k_2}_{i_2} = 0, \hfill (3.12)$$

where $f_{ij}^k$ are the structure constants $[T_i, T_j] = f_{ij}^k T_k$ of $g$. The homogeneous YB deformations of principal sigma models and symmetric coset sigma models had been developed in [7]. Moreover, it had been generalized to the AdS$_5 \times S^5$ superstring case in [8].

A remarkable feature of this class is that we can consider partial deformations of a given background. This is due to the fact that the right-hand side of (3.12) has no term proportional to $c^2$. Thanks to this, we can find many nontrivial solutions of this equation. In particular, as we will see later on, the associated deformations give deformed AdS$_5 \times S^5$ backgrounds which solve not only the standard supergravity equations but also the GSE.

### 3.2. Classical integrability

In this subsection, we will show that also the deformed action (3.1) admits a Lax pair. Therefore, YB deformations are integrable deformations of the AdS$_5 \times S^5$ superstring. To show this, we will explicitly give the Lax pair of the deformed system.

**EOM and the flatness condition.** To demonstrate the classical integrability of (3.1), we give the EOM of the deformed action (3.1). For this purpose, it is useful to introduce the deformed and the projected currents,

$$J_\alpha = \bar{O}^{-1} A_\alpha, \quad \bar{J}_\alpha = \bar{O}^{-1} A_\alpha,$$

$$J_\alpha^{(\pm)} = p^{\alpha\beta} J_\beta, \quad \bar{J}_\alpha^{(\pm)} = p^{\alpha\beta} \bar{J}_\beta. \hfill (3.14)$$

$^{10}$The action of the homogeneous YB-deformed AdS$_5 \times S^5$ superstring was constructed using the pure spinor formalism in [75].
Then, the EOM of the deformed action (3.1) are given by
\[ \delta E = \partial_a \bar{d} \cdot (J^a_{(-)}) + \partial_a \bar{d} \cdot (\tilde{J}^a_{(+)}) + [\bar{J}^a_{(+)}, \bar{d} \cdot (J^a_{(-)})] + [J^a_{(-)}, \bar{d} \cdot (\tilde{J}^a_{(+)})] = 0. \] (3.15)

The flatness condition for the left-invariant current is
\[ \bar{\delta} \epsilon = \frac{1}{2} \varepsilon^{\alpha \beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]) \]
\[ = \partial_\alpha \bar{J}^0_{(+) - \alpha} - \partial_\alpha \bar{J}^0_{(-)} + [J^0_{(-)}, \bar{J}^0_{(+) - \alpha}] \]
\[ - c^2 \eta^2 [\bar{d} \cdot (J^0_{(-)}), \bar{d} \cdot (\tilde{J}^0_{(+) - \alpha})] + \eta R_\alpha (\delta E) = 0. \] (3.16)

As in the undeformed case, we decompose the EOM (3.15) and the flatness condition (3.16) into the \( \mathbb{Z}_2 \)-graded components. The bosonic parts are
\[ B_1 := \bar{\delta} \epsilon^{(0)} = \partial_\alpha \bar{J}^{(0)}_{(+) - \alpha} - \partial_\alpha \bar{J}^{(0)}_{(-)} + [J^{(0)}_{(-)}, \bar{J}^{(0)}_{(+) - \alpha}] + \left(1 + c^2 \eta^2\right) \left[J^{(2)}_{(-)}, \bar{J}^{(2)}_{(+) - \alpha}\right] \]
\[ + (1 - c^2 \eta^2) \left[J^{(4)}_{(-)}, \bar{J}^{(2)}_{(+) - \alpha}\right] = 0. \] (3.17)
\[ B_2 := \frac{1}{4} (\bar{\delta} \epsilon^{(2)} + 2 \bar{\delta} \epsilon^{(2)}) = \partial_\alpha \bar{J}^{(2)}_{(+) - \alpha} - \partial_\alpha \bar{J}^{(2)}_{(-)} + [J^{(0)}_{(-)}, \bar{J}^{(0)}_{(+) - \alpha}] + [J^{(2)}_{(-)}, \bar{J}^{(2)}_{(+) - \alpha}] = 0, \] (3.18)
\[ B_3 := \frac{1}{4} (\bar{\delta} \epsilon^{(3)} - 2 \bar{\delta} \epsilon^{(2)}) = \partial_\alpha \bar{J}^{(0)}_{(+) - \alpha} - [J^{(2)}_{(-)}, \bar{J}^{(0)}_{(+) - \alpha}] - (1 + c^2 \eta^2) [J^{(0)}_{(-)}, \bar{J}^{(0)}_{(+) - \alpha}] = 0, \] (3.19)
and the fermionic parts are given by
\[ F_1 := \frac{1}{4} (3 \bar{\delta} \epsilon^{(1)} - \bar{\delta} \epsilon^{(1)}) = \partial_\alpha \bar{J}^{(1)}_{(+) - \alpha} - \partial_\alpha \bar{J}^{(1)}_{(-)} + [J^{(0)}_{(-)}, \bar{J}^{(1)}_{(+) - \alpha}] + [J^{(2)}_{(-)}, \bar{J}^{(0)}_{(+) - \alpha}] \]
\[ + \frac{1 - c^2 \eta^2}{1 + c^2 \eta^2} [J^{(1)}_{(-)}, \bar{J}^{(2)}_{(+) - \alpha}] = 0. \] (3.20)
\[ F_2 := \frac{1}{4} (3 \bar{\delta} \epsilon^{(3)} + \bar{\delta} \epsilon^{(3)}) = \partial_\alpha \bar{J}^{(2)}_{(+) - \alpha} - \partial_\alpha \bar{J}^{(2)}_{(-)} + [J^{(0)}_{(-)}, \bar{J}^{(2)}_{(+) - \alpha}] + [J^{(1)}_{(-)}, \bar{J}^{(1)}_{(+) - \alpha}] \]
\[ + \frac{1 - c^2 \eta^2}{1 + c^2 \eta^2} [J^{(1)}_{(-)}, \bar{J}^{(2)}_{(+) - \alpha}] = 0. \] (3.21)
\[ F_3 := \frac{1}{4} (\bar{\delta} \epsilon^{(1)} + 2 \bar{\delta} \epsilon^{(1)}) = [J^{(2)}_{(-)}, \bar{J}^{(2)}_{(+) - \alpha}] = 0, \] (3.22)
\[ \bar{F}_4 := \frac{1}{4} (\bar{\delta} \epsilon^{(3)} + 2 \bar{\delta} \epsilon^{(3)}) = [J^{(0)}_{(-)}, \bar{J}^{(1)}_{(+) - \alpha}] = 0. \] (3.23)

Construction of the Lax pair. Now, let us present the Lax pair of the YB-deformed AdS\( _5 \times S^5 \) superstring. It is given by [6, 8]
\[ \mathcal{L}_\alpha = L_{(+\alpha)} + M_{(-\alpha)}, \] (3.24)
where \( L_{(+\alpha)} \) and \( M_{(-\alpha)} \) are
\[ L_{(+\alpha)} = \bar{J}^{(0)}_{(+) - \alpha} + u \sqrt{1 - c^2 \eta^2} \bar{J}^{(2)}_{(+) - \alpha} + u^{-2} \sqrt{1 - c^2 \eta^2} \bar{J}^{(2)}_{(+) - \alpha} + u^{-1} \sqrt{1 - c^2 \eta^2} \bar{J}^{(2)}_{(+) - \alpha} \]
\[ M^0_{\alpha\beta} = J^{(0)}_{\alpha\beta} + u \sqrt{1 - c^2 \eta^2} J^{(1)}_{\alpha\beta} + u^2 \frac{1 - c^2 \eta^2}{1 + c^2 \eta^2} J^{(2)}_{\alpha\beta} + u^{-1} \sqrt{1 - c^2 \eta^2} J^{(3)}_{\alpha\beta}. \]  

(3.25)

The expression has a similar form as in the undeformed case (2.20). We now evaluate the flatness condition of the Lax pair,

\[ \varepsilon^{\alpha\beta} (\partial_\alpha \mathcal{L}_\beta - \partial_\beta \mathcal{L}_\alpha + [\mathcal{L}_\alpha, \mathcal{L}_\beta]) = 0. \]  

(3.26)

The left-hand side of this equation can be rewritten as

\[
\text{LHS of (3.26)} = u^0 B_1 + u^{-2} \frac{1 - c^2 \eta^2}{1 + c^2 \eta^2} B_2 - u^3 \frac{1 - c^2 \eta^2}{1 + c^2 \eta^2} B_3 \\
+ \sqrt{1 - c^2 \eta^2} \left( u F_1 + u^{-1} F_2 + u^{-3} \frac{1 - c^2 \eta^2}{1 + c^2 \eta^2} F_3 + u^3 \frac{1 - c^2 \eta^2}{1 + c^2 \eta^2} F_4 \right).
\]  

(3.27)

Therefore, the flatness condition (3.26) is equivalent to the EOM (3.15) and the flatness condition (3.16) on-shell. We see that indeed, also the deformed system (3.1) is classically integrable.

3.3. The $\kappa$-symmetry of the YB-deformed action

The deformed action (3.1) is also invariant under the $\kappa$-symmetry transformation [6, 8], as we will show in this subsection. As in the undeformed case, we will work without taking the conformal gauge.

The $\kappa$-symmetry transformation is given by [6, 8]

\[ G^{-1}_- g^{-1}_- \delta g = P^0_{-} \{ Q_1 \kappa_{1\alpha}, J^{(2)}_{\alpha\beta} \} + P^0_{+} \{ Q_2 \kappa_{2\alpha}, J^{(2)}_{\alpha\beta} \}, \]  

(3.28)

\[
\delta \kappa^\alpha_\beta = \frac{1 + c^2 \eta^2}{4} \text{Str} \left[ T \left( [Q_1 \kappa_{1(+)\alpha}, J^{(2)}_{\alpha\beta}^{(+)}] + [Q_2 \kappa_{2(+)\alpha}, J^{(2)}_{\alpha\beta}^{(+)}] \right) \right. \\
\left. + (\alpha \leftrightarrow \beta) \right].
\]  

(3.29)

It is easy to see that when we take $\eta = 0$, this expression reduces to the undeformed transformation (2.27), (2.28). As in the undeformed case, we decompose the variation of the deformed action (3.1) under the $\kappa$-symmetry transformation as

\[ \delta \kappa_S Y_B \equiv \delta \kappa_S Y_B + \delta \kappa_S Y_B, \]  

(3.30)

where $\delta \kappa_S Y_B$ and $\delta \kappa_S Y_B$ are the variations with respect to the group element and the world-sheet metric, respectively. Let us first consider $\delta \kappa_S Y_B$. By using (3.28), it is given by

\[
\delta \kappa_S Y_B = \frac{T(1 - c^2 \eta^2)}{2} \int d^2 \sigma \text{Str} \\
\times [\varepsilon^{(1)} P_2 \circ (1 + \eta R_2)(\bar{g}) + \varepsilon^{(3)} P_4 \circ (1 - \eta R_2)(\bar{g})] \\
= -2 T(1 - c^2 \eta^2) \int d^2 \sigma \text{Str} \left( \varepsilon^{(1)} [J^{(2)}_{\alpha\beta}, \bar{J}_{\alpha\beta}^{(1)}] + \varepsilon^{(3)} [J^{(2)}_{\alpha\beta}, \bar{J}_{\alpha\beta}^{(3)}] \right),
\]  

(3.31)
where $\varepsilon^{(1)}$ and $\varepsilon^{(3)}$ are

$$\varepsilon^{(1)} = (1 + \eta R^3_\gamma) P_+^{\alpha\beta} \{ Q^1 \kappa_{\alpha}, J^{(2)}_{\beta+} \}, \quad \varepsilon^{(3)} = (1 - \eta R^3_\gamma) P_+^{\alpha\beta} \{ Q^3 \kappa_{3\alpha}, J^{(2)}_{\beta+} \}. \quad (3.32)$$

In the second equation of (3.31), we have used

$$P_1 \circ (1 - \eta R^3_\gamma) (\partial) = -4[\bar{J}^{(2)}_{\alpha+}, J^{(3)}_{\beta+}], \quad (3.33)$$

$$P_3 \circ (1 + \eta R^3_\gamma) (\partial) = -4[\bar{J}^{(2)}_{\alpha+}, J^{(1)}_{\beta+}] + \bar{\mathcal{F}}^{(3)} \quad (3.34)$$

and ignored the flatness condition $\mathcal{F}$. Each of the terms in (3.31) can be rewritten as

$$\text{STr} \left( \varepsilon^{(1)} [\bar{J}^{(2)}_{\alpha+}, J^{(1)}_{\beta+}] \right) = \text{STr} \left( J^{(2)}_{\alpha+} [\bar{J}^{(1)}_{\beta+}, Q^1 \kappa_{\alpha}] \right), \quad (3.35)$$

$$\text{STr} \left( \varepsilon^{(3)} [\bar{J}^{(2)}_{\alpha+}, J^{(3)}_{\beta+}] \right) = \text{STr} \left( J^{(2)}_{\alpha+} [\bar{J}^{(3)}_{\beta+}, Q^3 \kappa_{3\alpha}] \right). \quad (3.36)$$

By using equation (2.35), the variation $\delta_{\gamma} S_{YB}$ becomes

$$\delta_{\gamma} S_{YB} = \frac{T(1 - c^2 \eta^2)}{4} \int d^2 \sigma \text{STr} \left( J^{(2)}_{\alpha+} J^{(2)}_{\beta+} \right) \text{STr} \left( [Q^1 \kappa_{\alpha}] J^{(1)}_{\beta+} \right)$$

$$+ \text{STr} \left( J^{(2)}_{\alpha+} J^{(2)}_{\beta+} \right) \text{STr} \left( [Q^3 \kappa_{3\alpha}] J^{(3)}_{\beta+} \right). \quad (3.37)$$

Next, let us consider the variation $\delta_{\gamma} S_{YB}$. For this purpose, it is convenient to rewrite the deformed action (3.1) in terms of the deformed current (3.13) as

$$S_{YB} = -\frac{T}{2} \left( 1 - c^2 \eta^2 \right) \int d^2 \sigma \gamma^{\alpha\beta} \text{STr} \left( J^{(2)}_{\alpha+} J^{(2)}_{\beta+} \right)$$

$$- \frac{T(1 - c^2 \eta^2)}{2} \int d^2 \sigma \varepsilon^{\alpha\beta} \text{STr} \left( J^{(1)}_{\alpha+} J^{(3)}_{\beta+} \right)$$

$$+ \eta \frac{T(1 - c^2 \eta^2)}{4} \int d^2 \sigma \varepsilon^{\alpha\beta} \text{STr} \left( \bar{d} - (J_\beta) R^3_\gamma \circ \bar{d} - (J_\beta) \right). \quad (3.38)$$

Using relation (2.38), $\delta_{\gamma} S_{YB}$ is given by

$$\delta_{\gamma} S_{YB} = -\frac{T(1 - c^2 \eta^2)}{4} \int d^2 \sigma \text{STr} \left( J^{(2)}_{\alpha+} J^{(2)}_{\beta+} \right) \text{STr} \left( [Q^1 \kappa_{\alpha}] J^{(1)}_{\beta+} + [Q^3 \kappa_{3\alpha}] J^{(3)}_{\beta+} \right)$$

$$= -\frac{T(1 - c^2 \eta^2)}{4} \int d^2 \sigma \text{STr} \left( J^{(2)}_{\alpha+} J^{(2)}_{\beta+} \right) \text{STr} \left( [Q^1 \kappa_{\alpha}] J^{(1)}_{\beta+} \right)$$

$$+ \text{STr} \left( J^{(2)}_{\alpha+} J^{(2)}_{\beta+} \right) \text{STr} \left( [Q^3 \kappa_{3\alpha}] J^{(3)}_{\beta+} \right). \quad (3.39)$$

This obviously cancels out the variation $\delta_{\gamma} S_{YB}$.

$$\delta_{\gamma} S_{YB} = (\delta_{\gamma} + \delta_{\kappa}) S_{YB} = 0. \quad (3.39)$$

As a result, the deformed action (3.1) is $\kappa$-symmetric.

Finally, let us comment on the implications of $\kappa$-invariance of the deformed GS action. As shown in [13], $\kappa$-invariance ensures that all deformed backgrounds are solutions either to the standard supergravity equations or to the generalized supergravity equations [12, 13]. In sections 4 and 5, we will present YB-deformed backgrounds which satisfy the GSE.
3.4. YB-deformed backgrounds from the GS action

In the following, we will rewrite the YB-deformed action in the form of the conventional GS action. In order to determine the deformed background, it is sufficient to expand the action up to quadratic order in the fermions,

\[ S_{YB} = S_{(0)} + S_{(2)} + \mathcal{O}(\theta^4). \] (3.40)

The explicit expression of the \( q \)-deformed AdS\(_5 \times S^5 \) background was given in the pioneering work [10]. It was subsequently generalized to the case of the homogeneous YB deformations in [17].

In this subsection, we provide the general formula for homogeneous YB deformed backgrounds. For simplicity, we limit our analysis to the cases where the \( r \)-matrices are composed only of the bosonic generators of \( su(2,2|4) \),

\[ r = \frac{1}{2} \gamma^i T_i \wedge T_j, \quad T_i \in \mathfrak{so}(2,4) \times \mathfrak{so}(6), \] (3.41)

which is a solution of the hCYBE provided that \( \gamma^i \) satisfies (3.12).11

3.4.1. Preliminaries.

To be able to expand the action (3.1) of the YB sigma model, we first need to introduce some notation. Since the \( r \)-matrix consists of bosonic generators only, the dressed \( R \)-operator acts on the generators as

\[ R_{gh}(P_a) = \lambda_a^b P_b + \frac{1}{2} \lambda_a^{bc} J_{bc}, \]

\[ R_{gh}(J_{ab}) = \lambda_{ab}^c P_c + \frac{1}{2} \lambda_{ab}^{cd} J_{cd}, \] (3.42)

\[ R_{gh}(Q_I) = 0. \]

The (dressed) \( R \)-operator is skew-symmetric,

\[ \text{STr} \left[ R_{gh}(X) Y \right] = -\text{STr} \left[ X R_{gh}(Y) \right]. \] (3.43)

If we take \( X \) and \( Y \) to be \( P_a \) or \( J_{ab} \), we obtain the relations

\[ \lambda_{ab} = \lambda_a^b \eta_{bc} = -\lambda_{ba}, \quad \lambda_{ab}^c = -\frac{1}{2} \eta^c_{def} R_{abcdefgh} \lambda_d^{ef}, \]

\[ \lambda_{ab}^{ef} R_{abcdef} = -\lambda_{cd}^{ef} R_{efab}, \] (3.44)

where \( R_{abcd} \) is the Riemann tensor in the tangent space of the AdS\(_5 \times S^5 \) background.

Using the action (3.42) for each generator of \( R_{gh} \) and the definitions (3.3) of \( \mathcal{O}_\pm \), the deformed currents \( \mathcal{O}_\pm A \) can be expanded as

\[ \mathcal{O}_\pm^\dagger A = \mathcal{O}_\pm^{0}(A_{(0)}) + \mathcal{O}_\pm^{(1)}(A_{(1)}) + \mathcal{O}_\pm^{(2)}(A_{(2)}) + \mathcal{O}(\theta^2) \]

\[ = e^\pm_a P_a - \frac{1}{2} W_{ab}^\pm J_{ab} + Q^I D^{IJ}_\pm \partial J^j + \mathcal{O}(\theta^2), \] (3.45)

11The YB sigma model action rewritten in the standard GS form based on \( \kappa \)-symmetry was given in [18] to all orders in the fermionic variables. There, also the deformed background associated to a general \( r \)-matrix was determined.
where we defined
\[ e^a_\pm \equiv e^b k_{a \pm}^b, \quad k_{a \pm}^b \equiv \left[ 1 \pm 2 \eta \lambda \right]_a^b, \quad W^{ab}_\pm \equiv \omega^{ab}_\pm \pm 2 \eta e^c_\pm \lambda_{ca}^b, \quad (3.46) \]
\[ D^U_\pm \equiv \delta^U_\pm D^\pm + \frac{i}{2} e^a_\pm D^\pm \gamma_a^U, \quad D^\pm \equiv \partial^\pm \pm \frac{1}{4} W^{ab}_\pm \gamma_{ab}, \quad (3.47) \]

Here, \( e^a_\pm \) and \( W^{ab}_\pm \) are the two vielbeins on the deformed background and the torsionful spin connections, respectively. In fact, \( e^a_\pm \) satisfy
\[ g^\prime_{mn} = \eta_{abe} e^a_m e^b_n, \quad (3.48) \]
and describe the deformed metric \( g^\prime_{mn} \). \( W^{ab}_\pm \) are given by
\[ W^{ab}_\pm = \omega^{(\pm)}_{ab} \pm \frac{1}{2} e^{\alpha}_\pm H^\prime_{(ab)}, \quad (3.49) \]
where \( \omega^{(\pm)}_{ab} \) are the spin connections (A.4) associated to the vielbeins \( e^a_\pm \) and \( H^\prime_3 \) is the \( H \)-flux on the deformed background.

3.4.2. The NS–NS sector. Metric and B-field. Let us first consider the metric and \( B \)-field of the YB-deformed action
\[ S(0) = -\frac{T}{2} \int d^2\sigma P^{ab} \text{Str}[A_{\lambda(0)} d_{\cdot \cdot} \circ \mathcal{O}^{-1}(A_{\lambda(0)})]. \quad (3.50) \]
By using the leading term of the expansions (2.55), (3.45) of \( A \) and \( J_- \) in fermions, the above action can be rewritten as
\[ S(0) = -T \int d^2\sigma P^{ab} \eta^{\alpha\beta} e^a_\alpha e^b_\beta k^\pm_{\alpha\beta}. \quad (3.51) \]
By comparing it with the canonical form (2.40) of the GS action, we can write down the expressions of the deformed metric and the \( B \)-field as
\[ g^\prime_{mn} = e^a_{(m} e^b_{n)} k^\pm_{\alpha\beta}, \quad B^\prime_{mn} = e^a_{(m} e^b_{n)} k^\pm_{\alpha\beta}, \quad (3.52) \]
where we used \( k_{\pm\alpha\beta} = k_{\alpha\beta} \pm \eta_{\alpha\beta} = k_{\alpha\beta} \).

Dilaton. Next we consider the YB-deformed dilaton \( \Phi^\prime \). The formula of the YB-deformed dilaton \( \Phi^\prime \) had been proposed in [17, 18]:
\[ e^{\Phi^\prime} = (\det k^\pm)^\frac{1}{4} = (\det k^-)^\frac{1}{4}. \quad (3.53) \]
This expression is consistent with the EOM of supergravity in the string frame and reproduces those of some well-known backgrounds (e.g. Lunin–Maldacena–Frolov [76, 77] and Maldacena–Russo backgrounds [78, 79]).

3.4.3. The R–R sector. Next, we determine the R–R fields from the quadratic part \( S(2) \) of the YB-deformed action.

As noted in [10, 17], the deformed action naively does not have the canonical form of the GS action (2.40), so we need to choose the diagonal gauge and perform a suitable redefinition of the bosonic fields \( X^m \). Since the analysis is quite complicated, we only give an outline here (see [25] for the details of the computation).
The quadratic part of the deformed action $S^{(2)}$ can be decomposed into two parts,

$$ S^{(2)} = S^{c(2)} + \delta S^{(2)}. \tag{3.54} $$

First, we focus only on the first part $S^{c(2)}$ since the second part $\delta S^{(2)}$ can be completely canceled by field redefinitions. The explicit expression of $S^{c(2)}$ in terms of the $32 \times 32$ gamma matrices is given by

$$ S^{c(2)} = -i T \int d^2 \sigma \left[ P^{\alpha \beta} \Theta_1 e_{-\alpha}^a \Gamma_a D_{+\beta} \Theta_1 + P^{\alpha \beta} \Theta_2 e_{+\alpha}^a \Gamma_a D_{-\beta} \Theta_2 \right. $$

$$ - \frac{1}{8} P^{\alpha \beta} \Theta_1 e_{-\alpha}^a \Gamma_a \tilde{F}_5 e_{+\beta}^b \Gamma_b \Theta_2 \right]. \tag{3.55} $$

where $D_{\alpha a} \Theta_I \equiv \left( \partial_\alpha + \frac{1}{2} W_{\alpha a}^{ab} \Gamma_{ab} \right) \Theta_I$ and $\tilde{F}_5$ is the undeformed R–R five-form field strength (2.61). We see that the quadratic action (3.55) is slightly different from the canonical form of the GS action.

In order to rewrite the action (3.55) in the canonical form of the GS action, we need to eliminate the vielbein $e_{+m}^a$ by using the relations

$$ e_{+m}^a = (\Lambda^{-1})^a_b e_{-m}^b = \Lambda_b^a e_{-m}^b, \quad \Lambda_b^a \equiv (k^{-1})_a^c k^c_b. \tag{3.56} $$

As discussed in [25], this procedure can be identified with the diagonal gauge fixing introduced in [80, 81]. Since the two vielbeins $e_{\pm m}^a$ describe the same metric, the above relations (3.56) can be regarded as a local Lorentz transformation. Indeed, it is easy to show that the matrix $\Lambda$ is an element of the ten-dimensional Lorentz group $SO(1,9)$. Furthermore, the matrix $\Lambda$ satisfies the identity

$$ \Omega^{-1} \Gamma_a \Omega = \Lambda_a^b \Gamma_b, \quad \Omega = (\det k)^{\frac{1}{2}} \mathcal{E} \left( -\eta \lambda^{ab} \Gamma_{ab} \right), \tag{3.57} $$

where $\Omega$ is a spinor representation of the local Lorentz transformation (3.56), and $\mathcal{E}$ is an exponential-like function with the gamma matrices totally antisymmetrized [82],

$$ \mathcal{E} \left( -\eta \lambda^{ab} \Gamma_{ab} \right) = \sum_{p=0}^{s} \frac{1}{2^p p!} (-2\eta \lambda_{a1a2}) \cdots (-2\eta \lambda_{a2p-1a2p}) \Gamma^{a1 \cdots a2p}. \tag{3.58} $$

By performing the local Lorentz transformation (3.56) and using the identity (3.57), the action (3.55) becomes

$$ S^{c(2)} = -i T \int d^2 \sigma \left[ P^{\alpha \beta} \Theta_1 e_{a}^a \Gamma_a D_{+\beta} \Theta_1 + P^{\alpha \beta} \Theta_2 e_{+\alpha}^a \Gamma_a D_{-\beta} \Theta_2 \right. $$

$$ - \frac{1}{8} P^{\alpha \beta} \Theta_1 e_{a}^a \Gamma_a \tilde{F}_5 e_{+\beta}^b \Gamma_b \Theta_2 \right]. \tag{3.59} $$

where we redefined the deformed vielbein $e_{-\alpha}^a$ as $e_{-\alpha}^a$. Next, we perform a redefinition of the fermionic variables $\Theta_I$,

$$ \Theta_I' \equiv \Theta_I, \quad \Theta_I' \equiv \Omega \Theta_2. \tag{3.60} $$

As the result of the redefinition, we obtain

$$ S^{c(2)} = -i T \int d^2 \sigma \left[ P^{\alpha \beta} \Theta_1 e_{a}^a \Gamma_a D_{+\beta} \Theta_1 + P^{\alpha \beta} \Theta_2 e_{+\alpha}^a \Gamma_a D_{-\beta} \Theta_2 \right. $$

$$ - \frac{1}{8} P^{\alpha \beta} \Theta_1 e_{a}^a \Gamma_a \tilde{F}_5 e_{+\beta}^b \Gamma_b \Theta_2 \right]. \tag{3.59} $$

where we redefined the deformed vielbein $e_{-\alpha}^a$ as $e_{-\alpha}^a$. Next, we perform a redefinition of the fermionic variables $\Theta_I$,
\[
S_c^{(2)} = -T \int d^2 \sigma \left[ P_{\alpha \beta}^{\gamma} i \hat{\Theta}_1 c_\alpha^a \Gamma_a D'_{\gamma \beta} \Theta'_1 + P_{\alpha \beta}^{\gamma} i \Theta_2 c_\alpha^a \Gamma_a D'_{\gamma \beta} \Theta'_2 \right.
\]
\[
- \frac{1}{8} P_{\alpha \beta}^{\gamma} i \hat{\Theta}_1 c_\alpha^a \Gamma_a \tilde{F}_5 \Omega^{-1} c_\beta^b \Gamma_b \Theta'_2 \bigg],
\]
(3.61)

where the derivatives \(D'_a\) are defined as
\[
D'_+ \equiv D_+ = d + \frac{1}{4} W^{ab} \Gamma_{ab},
\]
\[
D'_- \equiv \Omega \circ D_- \circ \Omega^{-1} = d + \frac{1}{4} W^{ab} \Gamma_{ab} \Omega^{-1} + \Omega \Gamma \Omega^{-1}
\]
\[
= d + \frac{1}{4} \left[ \Lambda^a \Lambda^b \kappa_{cd} W^{cd} + (\Lambda d\Lambda^{-1})_{ab} \right] \Gamma_{ab}.
\]
(3.62)

The spin connection \(\omega^{ab}\) associated with the deformed vielbein \(e'^a\) and the deformed H-flux \(H_{abc}\) satisfies
\[
\omega^{ab} + \frac{1}{2} c_\alpha^c H^{cab} = W^{ab},
\]
\[
\omega^{ab} - \frac{1}{2} c_\alpha^c H^{cab} = \Lambda^a \Lambda^b \kappa_{cd} W^{cd} + (\Lambda d\Lambda^{-1})_{ab}.
\]
(3.63)

Therefore, \(D'_a\) can be expressed as
\[
D'_\pm = d + \frac{1}{4} \left( \omega^{ab} \pm \frac{1}{2} c_\alpha^c H^{cab} \right) \Gamma_{ab}.
\]
(3.64)

In this way, the deformed action (3.61) becomes the conventional GS action at order \(O(\theta^2)\) by identifying the deformed R–R field strengths as
\[
\tilde{F}' = \tilde{F}_5 \Omega^{-1},
\]
\[
\tilde{F}' = \sum_{p=1,3,5,7,9} \frac{1}{p!} e^{\Phi'} \tilde{F}_{a_1...a_p} \Gamma^{a_1...a_p}.
\]
(3.65)

Here the deformed dilaton \(\Phi'\) is given by (3.53). The transformation rule (3.65) was originally given in [18]. A different derivation based on the \(\kappa\)-symmetry variation is given in appendix I of [25].

Finally, let us consider the remaining part \(\delta S^{(2)}\). It can be completely canceled by redefining the bosonic fields \(X^m\) [10, 17],
\[
X^m \rightarrow X^m + \frac{4}{7} \sigma_1^{\mu \nu} e^{cm} \lambda_\nu \gamma_1 \gamma_2 + O(\theta^4),
\]
(3.66)
as long as the \(r\)-matrix satisfies the hCYBE. Indeed, this redefinition results in a shift \(S^{(0)} \rightarrow S^{(0)} + \delta S^{(0)}\) and the sum of \(\delta S^{(0)}\) and \(\delta S^{(2)}\) is the quite simple expression
\[
\delta S^{(0)} + \delta S^{(2)} = -\frac{\eta^2 T}{2} \int d^2 \sigma \ P_{a}^{\beta} \sigma_1^{\alpha \beta} \left[ \text{CYBE}^{(0)}(X, Y) \right] \gamma_1 \gamma_2 + O(\theta^4),
\]
(3.67)

where CYBE\(^{(0)}\)(\(X, Y\)) represents the grade-0 component of CYBE\(_g\)(\(X, Y\)) defined in (3.6). This shows that \(\delta S^{(2)}\) is completely canceled out by \(\delta S^{(0)}\) when the \(r\)-matrix satisfies the hCYBE.
3.5. Generalized supergravity

In general, YB deformations correspond to solutions both of the usual supergravity equations (GSE). The GSE were originally proposed to support a $q$-deformed $\text{AdS}_5 \times S^5$ background as a solution [12]. It was shown subsequently that some homogeneous YB deformed $\text{AdS}_5 \times S^5$ backgrounds are also solutions to the GSE [20]. In this subsection, we will give the explicit expression of the $q$-deformed $\text{AdS}_5 \times S^5$ background and then introduce the GSE.

3.5.1 The $q$-deformed $\text{AdS}_5 \times S^5$ background. As explained in subsection 3.1, the $q$-deformed background can be realized as a YB deformation of the $\text{AdS}_5 \times S^5$ superstring [6] with a classical $r$-matrix of Drinfeld–Jimbo type satisfying the mCYBE (with $c = i$). The metric and $B$-field were originally derived in [9] and the full background including R–R fluxes and dilaton has been obtained by performing the supercoset construction in [10]. It is given by

$$\text{d}s^2 = \sqrt{1 + \kappa^2} \left[ -\frac{1 + \rho^2}{1 - \kappa \rho^2} \text{d}t^2 + \frac{\text{d}r^2}{1 - \kappa^2 \rho^2 (1 + \rho^2)} + \frac{\rho^2 (\text{d}\zeta^2 + \cos \zeta \text{d}\phi_1^2)}{1 + \kappa^2 \rho^4 \sin^2 \zeta} + \frac{\rho^4 \sin^2 \zeta \text{d}\phi_2^2}{1 + \kappa^2 \rho^4 \sin^2 \zeta} \right] + \frac{1 - \rho^2}{1 + \kappa^2 \rho^2} \text{d}\phi^2 + \frac{1}{1 + \kappa^2 \rho^2} \frac{\text{d}r^2}{(1 + \kappa^2 \rho^2)(1 - \rho^2)} + \frac{\rho^2 (\text{d}\zeta^2 + \cos \zeta \text{d}\phi_1^2)}{1 + \kappa^2 \rho^4 \sin^2 \zeta} + \frac{\rho^4 \sin^2 \zeta \text{d}\phi_2^2}{1 + \kappa^2 \rho^4 \sin^2 \zeta},
$$

$$B_2 = \sqrt{1 + \kappa^2} \left[ \frac{\kappa \rho}{1 - \kappa \rho^2} \text{d}t \wedge \text{d}r \wedge \text{d}\rho - \frac{\kappa \rho^2 \sin \zeta \cos \zeta}{1 + \kappa^2 \rho^2 \sin^2 \zeta} \text{d}\zeta \wedge \text{d}\psi_1 \right. + \left. \frac{\kappa \rho}{1 + \kappa \rho^2} \text{d}\rho \wedge \text{d}r + \frac{\kappa \rho^4 \sin \zeta \cos \zeta}{1 + \kappa^2 \rho^4 \sin^2 \zeta} \text{d}\zeta \wedge \text{d}\phi_1 \right],
$$

$$F_1 = 4\kappa^2 \sqrt{1 + \kappa^2} (\rho^4 \sin^2 \zeta \text{d}\psi_2 - \rho^4 \sin^2 \zeta \text{d}\phi_2),
$$

$$\tilde{F}_3 = \frac{\rho}{1 - \kappa \rho^2} \left[ \text{d}t \wedge \text{d}r \wedge \text{d}\rho \wedge (\kappa \rho^2 \sin^2 \zeta \text{d}\psi_2 + \rho^2 \sin^2 \zeta \text{d}\phi_2) \right. + \left. \frac{1 - \rho^2}{1 + \kappa \rho^4} \text{d}\phi \wedge \text{d}r \wedge (\kappa \rho^4 \sin^2 \zeta \text{d}\psi_2 + \rho^2 \sin^2 \zeta \text{d}\phi_2) \right. + \left. \frac{\rho^4}{1 - \kappa \rho^2} \text{d}r \wedge \text{d}\rho \wedge (\kappa \rho^2 \sin^2 \zeta \text{d}\psi_2 + \rho^2 \sin^2 \zeta \text{d}\phi_2) \right. + \left. \frac{\rho^2}{1 + \kappa \rho^4} \text{d}\rho \wedge \text{d}r \wedge (\kappa \rho^2 \sin^2 \zeta \text{d}\psi_2 + \rho^2 \sin^2 \zeta \text{d}\phi_2) \right] + \left. \frac{\rho^3}{1 - \kappa \rho^2} \text{d}r \wedge \text{d}t \wedge \text{d}\rho \wedge (\kappa \rho^2 \sin^2 \zeta \text{d}\psi_2 + \rho^2 \sin^2 \zeta \text{d}\phi_2) \right],
$$

$$\tilde{F}_5 = 4(1 + \kappa^2)^{1/2} \left[ \frac{\rho^3 \sin \zeta \cos \zeta}{(1 - \kappa \rho^2)(1 + \kappa \rho^2 \sin^2 \zeta)} \text{d}t \wedge \text{d}r \wedge \text{d}\zeta \wedge \text{d}\psi_1 \wedge (\text{d}\psi_2 + \kappa \rho^4 \rho^4 \sin^2 \zeta \text{d}\phi_2) \right. + \left. \frac{\rho^3 \sin \zeta \cos \zeta}{(1 + \kappa \rho^2)(1 + \kappa \rho^2 \sin^2 \zeta)} \text{d}\phi \wedge \text{d}r \wedge \text{d}\zeta \wedge \text{d}\phi_1 \wedge (\kappa^3 \rho^4 \rho^4 \sin^2 \zeta \text{d}\psi_2 - \rho^4 \sin^2 \zeta \text{d}\phi_2) \right. + \left. \frac{\rho^3 \rho^4}{(1 - \kappa \rho^2)(1 + \kappa \rho^2 \sin^2 \zeta)} \text{d}r \wedge \text{d}t \wedge \text{d}\rho \wedge (\rho^2 \sin^2 \zeta \text{d}\psi_2 + \rho^2 \sin^2 \zeta \text{d}\phi_2) \right. + \left. \frac{\rho^3 \rho^4}{(1 + \kappa \rho^2)(1 + \kappa \rho^2 \sin^2 \zeta)} \text{d}r \wedge \text{d}t \wedge \text{d}\rho \wedge (\rho^2 \sin^2 \zeta \text{d}\psi_2 + \rho^2 \sin^2 \zeta \text{d}\phi_2) \right. + \left. \frac{\rho^3 \rho^4}{(1 - \kappa \rho^2)(1 + \kappa \rho^2 \sin^2 \zeta)} \text{d}\phi \wedge \text{d}r \wedge \text{d}\zeta \wedge \text{d}\phi_1 \wedge (\rho^2 \sin^2 \zeta \text{d}\psi_2 + \rho^2 \sin^2 \zeta \text{d}\phi_2) \right. + \left. \frac{\rho^3 \rho^4}{(1 + \kappa \rho^2)(1 + \kappa \rho^2 \sin^2 \zeta)} \text{d}\phi \wedge \text{d}r \wedge \text{d}\zeta \wedge \text{d}\phi_1 \wedge (\rho^2 \sin^2 \zeta \text{d}\psi_2 + \rho^2 \sin^2 \zeta \text{d}\phi_2) \right. + \left. \frac{\rho^3 \rho^4 \rho^4}{(1 - \kappa \rho^2)(1 + \kappa \rho^2 \sin^2 \zeta)} \text{d}\phi \wedge \text{d}r \wedge \text{d}\zeta \wedge \text{d}\phi_1 \wedge (\rho^2 \sin^2 \zeta \text{d}\psi_2 + \rho^2 \sin^2 \zeta \text{d}\phi_2) \right. + \left. \frac{\rho^3 \rho^4 \rho^4}{(1 + \kappa \rho^2)(1 + \kappa \rho^2 \sin^2 \zeta)} \text{d}\phi \wedge \text{d}r \wedge \text{d}\zeta \wedge \text{d}\phi_1 \wedge (\rho^2 \sin^2 \zeta \text{d}\psi_2 + \rho^2 \sin^2 \zeta \text{d}\phi_2) \right].$$
Φ = \frac{1}{2} \log \left[ \frac{1}{(1 - \kappa^2 \rho^2)(1 + \kappa^2 \rho^2 \sin^2 \zeta)(1 + \kappa^2 \rho^4 \sin^2 \xi)} \right], \quad (3.68)

where we defined \( \kappa = \frac{2\eta_1}{\eta_2} \). Note here that we have kept total derivative terms of the \( B \)-field that are obtained after performing the supercoset construction (see for example footnote 19 of [10]).\(^{12}\) Remarkably, the deformed background is not a solution of type IIB supergravity [10] but of the GSE [12] when we introduce the extra Killing vector

\[ I = \frac{1}{\sqrt{1 + \kappa^2}} \left( -4\kappa \partial_t + 2\kappa \partial_{\phi_1} + 4\kappa \partial_{\phi_2} - 2\kappa \partial_{\phi_3} \right). \quad (3.69)\]

We will next give the explicit expression for the generalized type IIB supergravity equations.

3.5.2. Generalized supergravity equations. Our conventions for the type II GSE [12, 13, 83–85] are as follows:

\[ R_{mn} - \frac{1}{4} H_{mpq} H^{pq}_{\ \ n} + 2 D_m \partial_n \Phi + D_n U_m + D_m U_n = T_{mn}, \]

\[-\frac{1}{2} D^k H_{kmn} + \partial_k \Phi H^k_{\ \ mn} + U^k H_{kmn} + D_m I_n - D_n I_m = \mathcal{K}_{mn}, \]

\[ R - \frac{1}{2} |H|^2 + 4 D^n \partial_m \Phi - 4 |\partial \Phi|^2 - 4 (I^m I_m + U^m U_m + 2 U^m \partial_m \Phi - D_m U^n) = 0, \]

\[ d \ast \hat{F}_n = H_3 \wedge \ast \hat{F}_n - \frac{1}{2} \partial \mathcal{F}_n = 0, \]

\[ R = \frac{1}{2} |H|^2 + 4 D^n \partial_m \Phi - 4 |\partial \Phi|^2 + U^m \partial_m \Phi - D_m U^n = 0. \quad (3.70)\]

where we have defined \(|\alpha_p|^2 \equiv \frac{1}{p!} \alpha_{m_1 \cdots m_p} \alpha^{m_1 \cdots m_p} \). \( D_m \) is the usual covariant derivative associated to \( g_{mn} \), and \( T_{mn}, \mathcal{K}_{mn} \) are defined by

\[ T_{mn} \equiv \frac{1}{4} e^{2\Phi} \sum_p \left[ \frac{1}{(p - 1)!} \hat{F}_{(m_{k_1 \cdots k_{p-1}}} \hat{F}_{n_{k_1 \cdots k_{p-1}}} - \frac{1}{2} g_{mn} |\hat{F}|^2 \right], \quad (3.71)\]

\[ \mathcal{K}_{mn} \equiv \frac{1}{4} e^{2\Phi} \sum_p \frac{1}{(p - 2)!} \hat{F}_{k_1 \cdots k_{p-2}} \hat{F}_{mn}^{k_1 \cdots k_{p-2}}. \]

The relation between the R–R field strengths and potentials is given by (see [85] for details)

\[ \hat{F}_n = d \hat{C}_{n-1} + H_3 \wedge \hat{C}_{n-3} = \iota_1 B_2 \wedge \hat{C}_{p-1} - \iota_1 \hat{C}_{n+1}. \quad (3.72)\]

The Killing vector \( I = I^m \partial_m \) is defined to satisfy

\[ \mathcal{L}_I g_{mn} = 0, \quad \mathcal{L}_I B_2 + d (U - \iota_I B_2) = 0, \quad \mathcal{L}_I \Phi = 0, \quad I^m U_m = 0. \quad (3.73)\]

Here we have ignored the spacetime fermions. The full explicit expression of the type IIB GSE are given in [13].

\(^{12}\) Another expression for the dilaton \( \Phi \), which is different from the one in [12], is obtained due to the existence of the total derivative terms of the \( B \)-field.
We usually choose the particular gauge \( U_m = I^m B_{mn} \) (see [12, 85] for the details) in which the GSE (3.70) become

\[
R_{mn} - \frac{1}{4} H_{mpq} H_{n}^{pq} + D_m Z_n + D_n Z_m = T_{mn},
\]

\[
-\frac{1}{2} D^k H_{kmn} + Z_k H_{mn} + D_m I_n - D_n I_m = \mathcal{X}_{mn},
\]

\[
R - \frac{1}{2} |H_3|^2 + 4 \left( D^m Z_m - I^m I_m - Z_m Z_m \right) = 0,
\]

\[
d * \tilde{F}_n - H_3 \wedge * \tilde{F}_{n+2} - t_4 B_2 \wedge * \tilde{F}_n - t_4 * \tilde{F}_{n-2} = 0,
\]

where we defined

\[
Z_m \equiv \partial_m \Phi + I^m B_{mn},
\]

(3.75)

In this gauge, we can show that the \( q \)-deformed AdS_5 × S^5 background (3.68) with the Killing vector (3.69) solve the GSE. When \( I = 0 \), the GSE reduce to the usual supergravity EOM. Therefore, this deformation is characterized only by the Killing vector \( I^m \). Note that due to the presence of this Killing vector, the solutions of the GSE are effectively nine dimensional.

A remarkable feature of this theory is that the GSE can be reproduced from the requirement of \( \kappa \)-symmetry in the GS formalism. It has been known for a long time that the on-shell constraints of type II supergravity ensure \( \kappa \)-symmetry of the associated GS type string sigma model [86, 87]. At the time it had been conjectured that the \( \kappa \)-symmetry requires the type II supergravity equations. However, after about thirty years, Tseytlin and Wulff [13] solved this long-standing problem, showing that a general solution of the \( \kappa \)-symmetry constraint leads to solutions to the EOM of generalized supergravity.

3.5.3. Weyl invariance of string theory on generalized supergravity backgrounds.

Tseytlin–Wulff’s result implies that, at the classical level, string theory is consistently defined on generalized supergravity backgrounds. However, the quantum consistency of string theories defined on such backgrounds is not clear. Indeed, the GSE (3.70) were originally introduced as a scale-invariance condition for string theory. The Weyl invariance of string theory on such backgrounds has been studied in [85, 88, 89]. We will briefly comment on the current status of this subject.

For simplicity, we will consider the conventional (bosonic) string sigma model on a general background,

\[
S = -\frac{1}{4 \pi \alpha'} \int d^2 \sigma \left( g_{mn} \sqrt{-h} h^{\alpha \beta} - B_{mn} \varepsilon^{\alpha \beta} \right) \partial_\alpha X_m \partial_\beta X_n,
\]

(3.76)

where \( h^{\alpha \beta} \) is the world-sheet metric. The Weyl anomaly of this system takes the form

\[
2 \alpha' \left( T^\alpha \right) = \left( \beta_g^{\alpha \beta} h^{\alpha \beta} - \beta_B^{\alpha \beta} \varepsilon^{\alpha \beta} \sqrt{-h} \right) \partial_\alpha X_m \partial_\beta X_n.
\]

(3.77)

The \( \beta \)-functions at the one-loop level have been computed (for example in [90]) and have the form

\[
\beta_g^{\alpha \beta} = \alpha' \left( R_{mn} - \frac{1}{4} H_{mpq} H_n^{pq} \right), \quad \beta_B^{\alpha \beta} = \alpha' \left( -\frac{1}{2} D^k H_{kmn} \right).
\]

(3.78)

If the trace of the energy–momentum tensor (3.77) vanishes, the system is Weyl invariant.
Quantum scale invariance is satisfied by requiring \[ \beta^g_{mn} = -2 \alpha' D(mZ_n), \quad \beta^B_{mn} = -\alpha' \left( Z^k H_{kmn} + 2 D_{(mn)}I_n \right), \] (3.79)

where $I_m$ and $Z_m$ are certain vector fields in the target space. When the $\beta$-functions have the form (3.79), the Weyl anomaly (3.77) becomes

\[
\langle T^\alpha_{\alpha} \rangle = - \mathcal{D}_\alpha \left( Z_m h_{\alpha\beta} - I_m \frac{e^{\alpha\beta}}{\sqrt{-h}} \partial_{\beta} X^m \right) + Z^m \frac{2\pi\alpha'}{\sqrt{-h}} \frac{\delta S}{\delta X^m},
\]

(3.80)

where $\mathcal{D}_\alpha$ is the covariant derivative associated with $h_{\alpha\beta}$. Note that this scale invariance condition takes the same form as the NS sector of the GSE (3.70). The full set of the GSE is obtained by generalizing the scale invariance condition to include the R–R fields [12].

To obtain a consistent string theory at the quantum level, we need to cancel the Weyl anomaly (3.77). It is well known that when $Z_m = \partial_m \Phi$ and $I^m = 0$, we can cancel this by adding a counterterm, the so-called Fradkin–Tseytlin term \[ \frac{1}{4\pi} \int d^2 \sigma \sqrt{-h} R^{(2)} \Phi, \] (3.81)

to the original action (3.76). Compared to the sigma model action, the counterterm (3.81) is of higher order in $\alpha'$ and should be regarded as a quantum correction.

In the more general GSE case, the situation is more subtle as the counterterm (3.81) cannot cancel out the anomaly (3.80). However, it is important to note that $I$ and $Z$ are arbitrary vectors in the scale invariance conditions (3.79), while in the GSE case, $I$ is a Killing vector for $g_{mn}$ and $Z$ is given by (3.75). In [85, 88], by using this Killing property, a possible local and covariant counterterm for the bosonic string on generalized supergravity backgrounds was constructed as a generalization of the Fradkin–Tseytlin term (3.81) (see [85, 88] for details). The Weyl invariance of the type I superstring in generalized supergravity backgrounds was discussed in [89]. A detailed study of the conformal field theory (CFT) picture is however still outstanding and in this sense, the full consistency of string theory in generalized supergravity backgrounds is an open problem.

3.6. The unimodularity condition

As explained in the previous section, YB deformed backgrounds can be solutions not only of the usual supergravity equations but also of the GSE. Therefore, to obtain the usual supergravity solutions from YB deformations, we need to impose further constraints on the classical $r$-matrices. This unimodularity condition for the classical $r$-matrices is due to Borsato and Wulff [18]. We will call an $r$-matrix unimodular if it satisfies the condition

\[ r^{ij}[T_i, T_j] = 0, \quad T_i \in \text{su}(2, 2|4). \] (3.82)

Note that when we consider, for example, the YB deformations of the $\text{AdS}_5 \times S^5$ background, the unimodularity condition is a necessary and sufficient condition. In general, however, under certain conditions a YB deformation associated with a non-unimodular $r$-matrix can provide a solution to the standard supergravity equations (see for example [25, 92]).
Let us briefly explain the origin of the name of the condition (3.82). For simplicity, we will consider the bosonic subalgebra $\mathfrak{so}(2,4) \oplus \mathfrak{so}(6)$ of $\mathfrak{su}(2,2|4)$. A constant solution of the hCYBE (3.12) for a Lie algebra $\mathfrak{g}$ corresponds one-to-one to a subalgebra $\mathfrak{f} \subset \mathfrak{g}$ [94, 95]. Here, a constant solution means that $r^{ij}$ is a constant skew-symmetric matrix. Furthermore, restricting the range of the indices of $r^{ij}$ to $i, j = 1, \ldots, \dim \mathfrak{f}$, the matrices $r^{ij}$ are always invertible. This implies that $\mathfrak{f}$ is always even dimensional. Introduce the bi-linear map $\omega : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ defined by
\[
\omega(T_i, T_j) := (r^{-1})_{ij}.
\]
(3.83)
The hCYBE implies that $\omega$ is a two-cocycle
\[
\omega(x, y) = -\omega(y, x),
\]
(3.84)
where $x, y, z \in \mathfrak{f}$. This means that $\mathfrak{f}$ is a quasi-Frobenius Lie algebra. The two-cocycle condition can be rewritten as
\[
(r^{-1})_{ij} f_{ikl} = 0.
\]
(3.85)
By taking a contraction $r^{kl}$ with (3.85), we obtain
\[
0 = 3(r^{-1})_{ij} f_{ikl} r^{kl} = (r^{-1})_{ij} f_{ikl} r^{kl} + 2 f_{ij}.
\]
(3.86)
If the $r$-matrix satisfies the unimodularity condition, the equation becomes
\[
f_{ij} = -\frac{1}{2}(r^{-1})_{ij} f_{ikl} r^{kl} = 0,
\]
(3.87)
then $\mathfrak{f}$ is also an unimodular Lie algebra.

3.7 Classification of $r$-matrices

An $r$-matrix
\[
r = \frac{1}{2} r^{ij} T_i \wedge T_j,
\]
(3.88)
is called abelian if it consists of a set of generators which commute with each other, $[T_i, T_j] = 0$, otherwise it is called non-abelian. Most homogeneous YB deformations studied in the literature are based on abelian $r$-matrices, which are obviously unimodular. Moreover, when $\mathfrak{g}$ is a compact Lie algebra (for example $\mathfrak{su}(N)$, $\mathfrak{so}(N)$), all quasi-Frobenius Lie subalgebras $\mathfrak{f}$ are abelian [96]. Therefore, non-abelian unimodular $r$-matrices only exist for non-compact Lie algebras $\mathfrak{g}$.

Non-abelian unimodular $r$-matrices. We now discuss the classification of non-abelian unimodular $r$-matrices. We define the rank of an $r$-matrix as
\[
\text{Rank } r^{ij} := \dim \mathfrak{f}.
\]
(3.89)
Rank-2 unimodular $r$-matrices are by construction abelian. Non-abelian unimodular $r$-matrices with rank four have been fully classified.

\[\text{13Recently [93], homogeneous YB deformations associated with unimodular } r \text{-matrices including fermionic generators were considered, and the associated deformed backgrounds were constructed explicitly.}\]
In the following, we will for simplicity consider only $r$-matrices consisting of the generators of the conformal algebra $so(2, 4)$ for $r$-matrices with rank-4 and higher.

**Rank 4.** The rank-4 unimodular $r$-matrices for the bosonic isometries of AdS$_5$ have been classified in [18]. If we take a rank-4 $r$-matrix

$$r = T_1 \wedge T_2 + T_3 \wedge T_4, \quad T_{1,\ldots,4} \in so(2, 4),$$

we obtain the following four classes:

(i) $\mathfrak{h}_3 \oplus \mathbb{R}$ \quad $[T_1, T_3] = T_4$ (3.91)

(ii) $\mathfrak{t}_{3,-1} \oplus \mathbb{R}$ \quad $[T_1, T_3] = T_3$,  \quad $[T_1, T_4] = -T_4$ (3.92)

(iii) $\mathfrak{t}_{3,0} \oplus \mathbb{R}$ \quad $[T_1, T_3] = -T_4$, \quad $[T_1, T_4] = T_3$ (3.93)

(iv) $\mathfrak{n}_4$ \quad $[T_1, T_3] = -T_2$, \quad $[T_2, T_3] = T_4$. (3.94)

Classes (i)–(iii) are called almost abelian $r$-matrices [97] and cover most of the rank-4 examples studied in [18]. As argued in [18, 97], this class of YB deformations can be realized as a sequence of non-commuting $T_s T$-transformations (see [18] for the explicit form for the rank-4 examples). They are obtained as combination of the usual $T_s T$-transformations and appropriate diffeomorphisms. The last class (iv) cannot be generated by performing non-commuting $T_s T$-transformations.

**Rank 6.** There is to date no complete classification for rank 6 unimodular $r$-matrices. Some examples were given in [18].

**Rank 8.** No rank-8 non-abelian unimodular $r$-matrices exist.

**Non-unimodular $r$-matrices.** Finally, let us briefly comment on non-unimodular $r$-matrices. The simplest example is the rank-2 $r$-matrix

$$r = \frac{1}{2} T_1 \wedge T_2, \quad [T_1, T_2] = T_2.$$ (3.95)

It is easy to see that the $r$-matrix does not satisfy the unimodularity condition (3.82). The $r$-matrix is also called a Jordanian $r$-matrix. A generalization of the rank-2 Jordanian $r$-matrix was given in [98]. YB deformed backgrounds associated to non-unimodular $r$-matrices are solutions to the GSE [18, 20].

### 3.8. Homogeneous YB deformations as non-abelian T-dualities

What is so interesting about generalized supergravity? In the long history of the study of string theory, a number of so-called ‘pathological backgrounds’, that are not solutions to the standard supergravity equations, have been discovered. For example, it is well known that non-abelian $T$-duality produces such a pathological background only when the dualized algebra is non-unimodular. It wold be interesting to find out whether these backgrounds are solutions to the GSE (for a discussion see e.g. [32], where it is shown that the superstring on backgrounds obtained by non-abelian $T$-duality possess the kappa-symmetry of the GS formulation).

Indeed, Hoare and Tseytlin conjectured that homogeneous YB deformations are equivalent to (a class of) non-abelian $T$-dualities [21]. This conjecture was proved by Borsato and Wulff [22]. If a given YB deformed background is a solution to the GSE, the associated non-abelian $T$-dualized background is a solution to the GSE as well. Furthermore, in subsection 5.3, we will give examples of solutions to the GSE that cannot be obtained by performing a YB deformation but are generated by non-abelian $T$-dualities.
3.9. YB deformations as $O(d, d)$ T-duality transformations

In subsection 3.4, we derived the general formulas (3.52), (3.53), (3.65) for the homogeneous YB deformed $\text{AdS}_5 \times S^5$ backgrounds with an $r$-matrix composed of bosonic generators only. An important observation of [24] is that the homogeneous YB deformation of the $\text{AdS}_5 \times S^5$ background is nothing but the $\beta$-transformation which in turn is an $O(d, d)$ T-duality transformation.\(^{14}\) This observation has been further studied in [25, 29–31, 38, 99]. In this section, we will briefly discuss this interpretation.

3.9.1. A brief review of DFT

For this purpose, it is useful to utilize the manifestly T-duality covariant formulation of supergravity called DFT [55–61].

\textbf{DFT fields and their parametrization.} In DFT, we consider a doubled spacetime with coordinates $(x^M) = (x^m, \tilde{x}_m) (M = 1, \ldots, 2d; m = 1, \ldots, d)$. Here $x^m$ are the standard $d$-dimensional coordinates and $\tilde{x}_m$ are the dual coordinates. The bosonic fields in DFT are the generalized metric $H_{MN}$, the DFT dilaton $d(x)$, and an $O(d, d)$ spinor of R–R fields $\hat{F}$ (see [25] for our conventions).

The generalized metric $H_{MN}$ can be parameterized as

$$H \equiv (H_{MN}) = \left( \begin{array}{cc} (g - B g^{-1} B)_{mn} & B_{mn} \delta^m_n \\ -g_{kn} B_{kn} & g_{mn} \end{array} \right),$$

in terms of the metric $g_{mn}$ and the Kalb–Ramond $B$-field $B_{mn}$. The $2d$-dimensional indices $M, N, \ldots$ are raised and lowered with the $O(d, d)$ metric

$$(\eta_{MN}) = \left( \begin{array}{cc} 0 & \delta^m_n \\ \delta^m_n & 0 \end{array} \right), \quad (\eta^{MN}) = \left( \begin{array}{cc} 0 & \delta^n_m \\ \delta^n_m & 0 \end{array} \right).$$

The familiar properties of the generalized metric,

$$H^{TT} = H, \quad H^T \eta H = \eta,$$

follow from the above parametrization, and imply that $H$ is an element of $O(d, d)$. As $H \in O(d, d)$, the non-linear transformations of the $T$-duality group are covariantly realized as

$$H \rightarrow h^T H h, \quad h \in O(d, d).$$

Note that a single $T$-duality transformation along the $x^i$-direction corresponds to taking the matrix

$$(h^M_N) = \left( \begin{array}{cc} I_d - e_z & e_z \\ e_z & I_d - e_z \end{array} \right), \quad e_z \equiv \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0),$$

where $I_d$ is the $d \times d$ identity matrix. The DFT dilaton $d(x)$ is related to the conventional dilaton $\Phi$ by

$$e^{-2d(x)} = \sqrt{|g|} e^{-2\Phi},$$

and is invariant under $O(d, d)$ duality transformations.

\(^{14}\)The deformed background can be reproduced from the requirement of invariance of non-zero Page forms and associated Page charges [29, 31, 99].
For later discussions, it is convenient to introduce another parametrization of the generalized metric and the DFT dilaton,

\[
\mathcal{H} = (\mathcal{H}^M_N) = \begin{pmatrix} G_{mn} & G_{mn}\beta_{kn} \\ -\beta_{mk} G_{ln} & (G^{-1} - \beta G)^{mn} \end{pmatrix},
\]

\[
e^{-2d} = \sqrt{|G|} e^{-2\tilde{\phi}}.
\]

in terms of the dual fields \((G^{mn}, \beta_{mn}, \tilde{\phi})\) [100–103]. This parametrization (3.102) is referred to as the non-geometric parametrization of the generalized metric and the DFT dilaton. If the matrix \(E_{mn} = g_{mn} + B_{mn}\) is invertible, the relation between the conventional fields \((g_{mn}, B_{mn}, \Phi)\) and the dual fields \((G^{mn}, \beta_{mn}, \tilde{\phi})\) is given by

\[
E_{mn} = (E^{-1})^{mn} = G^{mn} - \beta_{mn},
\]

\[
e^{-2d} = \sqrt{|g|} e^{-2\Phi} = \sqrt{|G|} e^{-2\tilde{\phi}}.
\]

The dual metric \(G_{mn}\) coincides with the open-string metric [100, 104], to be distinguished from the initial closed-string metric \(g_{mn}\). As we will discuss in section 5, the non-geometric parametrization of the generalized metric is useful to discuss the non-geometric aspects of a given background.

**Section condition.** For the consistency of DFT, we require that arbitrary fields or gauge parameters \(A(x)\) and \(B(x)\) satisfy the so-called section condition [55, 58, 60],

\[
\eta^{MN} \partial_M A(x) \partial_N B(x) = 0, \tag{3.105}
\]

\[
\eta^{MN} \partial_M \partial_N A(x) = 0, \tag{3.106}
\]

where \(\partial_M = (\partial_m, \tilde{\partial}^m) \equiv \left( \frac{\partial}{\partial x^m}, \frac{\partial}{\partial \tilde{x}^m} \right)\). In general, under this condition, supergravity fields can depend on at most \(d\) physical coordinates out of the \(2d\) doubled coordinates \(x^M\).

In the canonical solution, all fields and gauge parameters are independent of the dual coordinates, \(\tilde{\partial}^m = 0\) and DFT reduces to conventional supergravity. If instead all fields depend on \((d - 1)\) coordinates \(x^i\) and only the DFT dilaton \(d(x)\) has an additional linear dependence on a dual coordinate \(\tilde{z}\), DFT reduces to generalized supergravity [83, 85].

**\(\beta\)-transformation.** A (local) \(\beta\)-transformation is a specific \(O(d, d)\) transformation generated by

\[
e^\beta = (e^{\beta M}_N) \equiv (h^M_N) = \begin{pmatrix} \delta^m_n & -\tilde{r}_{mn}^0(x) \\ 0 & \delta^m_n \end{pmatrix}, \quad (\rho_{mn} = -\tilde{r}_{mn}^0), \tag{3.107}
\]

which induces a shift of the \(\beta\)-field or \(E_{mn}\):

\[
\beta_{0mn}(x) \to \beta^{mn}(x) = \beta_{0mn}(x) - \tilde{r}_{mn}^0(x), \tag{3.108}
\]

where \(\beta_0\) is the \(\beta\)-field on the original background. The usual supergravity fields \((g_{mn}, B_{mn}, \Phi, \hat{F}, \hat{C})\) transform as

\[
\mathcal{H}' = e^{\beta} \mathcal{H} e^{\beta}, \quad d' = d, \tag{3.109}
\]

\[
\hat{F}' = e^{-\beta} e^{-\beta} e^{-\beta} e^{-B_{mn}} \hat{F}, \quad \hat{C}' = e^{-\beta} e^{-\beta} e^{-B_{mn}} \hat{C},
\]
where $\hat{F}$, $\hat{C}$ are polyforms which are formal summations of all the original R–R field strengths and potentials,

$$
\hat{F} \equiv \sum_{p=1,3,5,7,9} \hat{F}_p, \quad \hat{F}_p \equiv \frac{1}{p!} \hat{F}_{m_1 \ldots m_p} dx^{m_1} \wedge \ldots \wedge dx^{m_p},
$$

$$
\hat{C} \equiv \sum_{p=0,2,4,6,8} \hat{C}_p, \quad \hat{C}_p \equiv \frac{1}{p!} \hat{C}_{m_1 \ldots m_p} dx^{m_1} \wedge \ldots \wedge dx^{m_p},
$$

(3.110)

The operator $\beta \vee$ which acts on an arbitrary $p$-form $A_p$ is defined as

$$
\beta \vee A_p = \frac{1}{2} \beta^{mn} \epsilon_m \epsilon_n A_p,
$$

(3.111)

where $\epsilon_m$ is the inner product along the $x^m$-direction. Here it is also convenient to define the R–R fields $(F, A)$ and $(\hat{F}, \hat{C})$ as

$$
F \equiv \epsilon^{B_5} \hat{F}, \quad A \equiv \epsilon^{B_5} \hat{A}, \quad \tilde{F} \equiv \epsilon^{\beta \vee} F, \quad \tilde{C} \equiv \epsilon^{\beta \vee} C.
$$

(3.112)

In order to distinguish the three definitions of R–R fields, we call $(\hat{F}, \hat{C})$ B-untwisted R–R fields while we call $(\hat{F}, \hat{C})$ $\beta$-untwisted R–R fields. The B-untwisted fields are invariant under B-field gauge transformations while the $\beta$-untwisted fields are invariant under local $\beta$-transformations. $(F, A)$ will play an important role when we study the monodromy of $T$-folds in section 5.

Finally, we should stress that unlike the B-field gauge transformations, the local $\beta$-transformation is not a gauge transformation. This fact implies that in general, the $\beta$-transformed background may not satisfy the (generalized) supergravity equations (3.70) even if the original background is a solution of the supergravity (or DFT).

3.9.2. YB deformations and local $\beta$-transformations. Now, let us explain the relation between local $\beta$-deformations and homogeneous YB deformations. Since the original AdS$_5 \times S^5$ background does not have a B-field, the $\beta$-transformed background can be expressed as

$$
g'_{mn} + B'_{mn} = \left[ (G^{-1} - \beta) \right]^{-1}, \quad d' = d,
$$

$$
\tilde{F}' = e^{-B_5} e^{-\beta \vee} \tilde{F}_5, \quad \tilde{C}' = e^{-B_5} e^{-\beta \vee} \tilde{C}_4,
$$

(3.113)

where $G_{mn}$ is the metric of the original AdS$_5 \times S^5$ and $\tilde{F}_5$ is the undeformed R–R five-form field strength (2.63). An important observation made in [24] is that a YB deformed background associated with the $r$-matrix (3.41) can also be generated by a local $\beta$-transformation with the $\beta$-field

$$
\beta^{mn}(x) = -r^{mn}(x) = 2 \eta^{r/2} \tilde{T}^r_i(x) \tilde{T}_j^r(x),
$$

(3.114)

where $\tilde{T}^r_i(x)$ are Killing vector fields associated to the generators $T_i$ appearing in the $r$-matrix (3.41). This implies that the $\beta$-untwisted R–R fields $(\tilde{F}, \tilde{C})$ are invariant under homogeneous YB deformations.

In this review, we will only give a proof for the above relation in the NS sector (see [25] for the R–R sector). Since the original AdS$_5 \times S^5$ background does not include a B-field, $E_{mn}$ is simply

$$
E_{mn} = g_{mn} = e_a^m e_a^b \eta_{ab}, \quad E^{mn} = \eta^{ab} e_a^m e_b^n.
$$

(3.115)
On the other hand, by using (3.52), the inverse of $E_{mn}$ is deformed as
\[ (g' + B')^{-1} mn = (k_+^{-1}) ab e_a^m e_b^n = (\eta^m e_d^m e_b^n + 2 \eta \chi_{ab}) e_a^m e_b^n. \] (3.116)

Therefore, the deformation can be summarized as
\[ E_{mn} \rightarrow E'_{mn} = E_{mn} + 2 \eta \chi_{ab} e_d^m e_b^n. \] (3.117)

By comparing this to the $\beta$-transformation rule (3.107) or (3.108), the YB deformation can be regarded as a local $\beta$-transformation with the parameter
\[ r_{mn} = 2 \eta \chi_{ab} e_d^m e_b^n. \] (3.118)

Let us moreover rewrite $r_{mn}$ in (3.118) using the $\hat{r}$-matrix instead of $\chi_{ab}$. By using the $\hat{r}$-matrix
\[ r_{mn} = \frac{1}{2} r_{ij} T_i \wedge T_j, \]
and (3.118) becomes
\[ r_{mn} = -2 \eta r_{ij} \left[ \text{Ad}_{g_{-1}} \right]_i^a \left[ \text{Ad}_{g_{-1}} \right]_j^b e_a^m e_b^n. \] (3.120)

By using the Killing vectors (2.68), we obtain the very simple expression
\[ r_{mn} = -2 \eta r_{ij} \hat{T}_m \hat{T}_n. \] (3.121)

This implies that the $\beta$-field (3.121) is the bi-vector representation of the $\hat{r}$-matrix characterizing a YB deformation. If we compute the dual fields $G_{mn}, \beta_{mn}$ in the deformed background from the relation (3.103), we obtain
\[ G_{mn} = \eta_{ab} e_a^m e_b^n, \quad \beta_{mn} = 2 \eta r_{ij} \hat{T}_m \hat{T}_n. \] (3.122)

The dual metric is invariant under the deformation $G_{0, mn} \rightarrow G_{0, mn} = G_{mn}$, while the $\beta$-field, which is absent in the undeformed background, is shifted as $\beta_{mn} = 0 \rightarrow \beta_{mn} = -r_{mn}$.

Next, let us compare (3.53) with the $\beta$-transformation law (3.109) of the dilation. The invariance of $e^{-2d} = e^{-2\Phi} \sqrt{-g}$ under $\beta$-deformations shows
\[ e^{2\Phi} = \frac{\sqrt{-g}}{\sqrt{-g}} e^{-2\Phi} \frac{\det(e_{\pm}^m e^n)}{\det(e_{\pm}^m e^n)} e^{2\Phi} = (\det k_{\pm}) e^{2\Phi}. \] (3.123)

Recalling $\Phi = 0$ in the undeformed background, the transformation rule (3.53) can be understood as a $\beta$-transformation. Therefore, the homogeneous YB deformed NS–NS fields are precisely the $\beta$-transformed ones.

Finally, let us comment on the usefulness of the $\beta$-transformation rule (3.109). If the original backgrounds are not described by symmetric cosets or are supported by a non-trivial $H$-flux, it is not straightforward to define YB sigma models in general. However, the $\beta$-transformation rule (3.109) can be easily applied to almost any background. More concretely, if a given background has a non-trivial isometry $G$, we can look for a skew-symmetric $r$-matrix that satisfies the hCYBE for the Lie algebra $g$ of $G$. If such an $r$-matrix can be found, we can apply the associated $\beta$-transformation to a given background by using the $\beta$-field expressed in terms...
Indeed, we can consider deformations of Minkowski spacetime \([38, 62–65]\) and \(\text{AdS}_3 \times S^3 \times T^4\) supported by \(H\)-flux \([25–27]\), and show that the deformed backgrounds solve the (generalized) supergravity equations. This shows that \(\beta\)-transformations are a useful new tool to generate solutions to the (generalized) supergravity equations.

**R-flux.** When a \(\beta\)-field exists on a given background, we can consider the associated tri-vector \(R\) known as the *non-geometric R-flux*. This flux is defined as

\[
R \equiv [\beta, \beta]_S,
\]

where \([\cdot, \cdot]_S\) denotes the Schouten bracket, which is defined for a \(p\)-vector and a \(q\)-vector as

\[
[a_1 \wedge \cdots \wedge a_p, b_1 \wedge \cdots \wedge b_q]_S \\
\equiv \sum_{i,j} (-1)^{i+j} [a_i, b_j] \wedge a_1 \wedge \cdots \hat{a}_i \cdots \wedge a_p \wedge b_1 \wedge \cdots \hat{b}_j \cdots \wedge b_q,
\]

where the czech \(\hat{a}_i\) denotes the omission of \(a_i\).

The \(\beta\)-field on YB deformed backgrounds takes the form

\[
\beta^{mn} = -r^{mn} = 2 \eta_1 r_l^i \hat{T}_i^m \hat{T}_j^n, \\
\beta = \frac{1}{2} \beta^{mn} \partial_m \wedge \partial_n = 2 \eta_1 \left( \frac{1}{2} r_l^i \hat{T}_i \wedge \hat{T}_j \right).
\]

By using the Lie bracket for the Killing vector fields, \([\hat{T}_i, \hat{T}_j] = -f_{ij}^k \hat{T}_k\), we obtain

\[
R^{mpn} = 3 \beta^{[mq} \partial_n \beta^{np]} \\
= -8 \eta_1^2 \left( f_{ij}^k r_l^i \hat{\rho}^{jk} \rho_l^k + f_{ij}^k r_l^i \hat{\rho}^{jk} \rho_l^k + f_{ij}^k r_k^i \rho_l^j \hat{\rho}^{lk} \right) \hat{T}_m^i \hat{T}_n^j \hat{T}_k^p \\
= 0,
\]

upon using the hCYBE \((3.12)\) \([24]\). This shows the absence of the \(R\)-flux in homogeneous YB-deformed backgrounds.

### 3.9.3. The divergence formula

The Killing vector \(I\) in the GSE does not appear in the classical action of the string sigma model. Therefore, we have to check whether a given background has a Killing vector that allows it to be a solution of the GSE. Indeed, as discussed in \([85, 88]\), \(I\) appears in the counterterm which is introduced to cancel the Weyl anomaly of the string sigma model defined on generalized supergravity backgrounds.

However, when we consider YB deformations, we have a convenient formula to obtain the Killing vector \(I^m\) for YB deformed backgrounds. As discovered in \([105]\), the formula has a very simple form\(^{15}\)

\[
I^m = D_a r^{am},
\]

where \(D_a\) is the covariant derivative associated to the original metric, and \(r^{am}\) is given by \((3.121)\). If the \(r\)-matrix gives a non-zero \(I\), this implies the violation of the unimodularity condition \((3.82)\). To see this, we shall consider non-unimodular \(r\)-matrices satisfying

\[
r^{ij} [T_i, T_j] = r^{ij} f_{ij}^k T_k \neq 0.
\]

\(^{15}\)A general formula for \(I\) on the YB deformed \(\text{AdS}_3 \times S^5\) backgrounds was originally obtained in \([18]\).
By using the concrete expression (3.121) for $r^{mn}$, the divergence formula (3.128) can be rewritten as

$$I^{m} = -\eta r^{ij} [\hat{T}_{i}^{m}, \hat{T}_{j}^{m}] = \eta r^{ij} f^{k}_{ij} \hat{T}_{k}^{m},$$

(3.130)

where the Killing vectors $\hat{T}_{i}$ satisfy

$$[\hat{T}_{i}, \hat{T}_{j}]^{m} = \varepsilon_{i j}^{k} \hat{T}_{k}^{m} = -f^{k}_{ij} \hat{T}_{k}^{m}.$$ (3.131)

The Killing vector $I^{m}$ represents the amount of the violation of the unimodularity condition. In the next section, we will see that this formula works well for various non-unimodular $r$-matrices.

4. Examples of homogeneous YB-deformed $\text{AdS}_5 \times S^5$ backgrounds

In this section, we will present a number of examples of homogeneous YB deformed $\text{AdS}_5 \times S^5$ backgrounds.

4.1. Abelian $r$-matrices

First, let us consider homogeneous YB deformations of the $\text{AdS}_5 \times S^5$ background associated to abelian $r$-matrices. It is quite difficult to specify the gauge-theory duals for YB-deformed backgrounds in general. The abelian case is exceptional and the gauge-theory duals are well recognized as in [14–16]. As for the non-commutative geometric interpretation of the YB-deformed $\text{AdS}_5$, see [15, 97, 99, 105, 106].

4.1.1. The Maldacena–Russo background. To demonstrate how to use formula (3.113), let us consider the YB-deformed $\text{AdS}_5 \times S^5$ background associated to the classical $r$-matrix [15]

$$r = \frac{1}{2} P_1 \wedge P_2.$$ (4.1)

This $r$-matrix is abelian and satisfies the hCYBE (3.4). The associated YB deformed background was derived in [15, 17].

The classical $r$-matrix (4.1) leads to the associated $\beta$-field

$$\beta = \eta \hat{P}_1 \wedge \hat{P}_2 = \eta \partial_1 \wedge \partial_2.$$ (4.2)

Then, the $\text{AdS}_5$ part of the $10 \times 10$ matrix $(G^{-1} - \beta)$ is

$$(G^{-1} - \beta)^{mn} = \begin{pmatrix}
  z^2 & 0 & 0 & 0 & 0 \\
  0 & -z^2 & 0 & 0 & 0 \\
  0 & 0 & z^2 & -\eta & 0 \\
  0 & 0 & \eta & z^2 & 0 \\
  0 & 0 & 0 & 0 & z^2
\end{pmatrix},$$

(4.3)

where we have ordered the coordinates as $(z, x^0, x^1, x^2, x^3)$. By using the inverse of the matrix (4.3) and formula (3.113), we obtain the NS–NS fields of the YB-deformed background,

$$\begin{align*}
  ds^2 &= \frac{dz^2 - (dx^0)^2 + (dx^3)^2}{z^2} + \frac{z^2 [(dx^1)^2 + (dx^2)^2]}{z^4 + \eta^2} + dx_S^2, \\
  B_2 &= \frac{\eta}{z^4 + \eta^2} dx^1 \wedge dx^2, \\
  \Phi &= \frac{1}{2} \log \left[ \frac{z^4}{z^4 + \eta^2} \right].
\end{align*}$$

(4.4)
The next task is to derive the R–R fields of the deformed background. Using the undeformed R–R five-form field strength (2.63) of the AdS5 × S5 background, we find that the deformed field \( F = e^{-\beta \mathcal{F}} \) is given by

\[
F = e^{-\beta \mathcal{F}} = 4 (\omega_{\text{AdS5}} + \omega_S) - 4 \beta \mathcal{F} = 4 (\omega_{\text{AdS5}} + \omega_S) - 4 \beta \mathcal{F} = 4 (\omega_{\text{AdS5}} + \omega_S) - 4 \eta \frac{dz \wedge dx^0 \wedge dx^3}{z^5}.
\] (4.5)

which is a linear combination of the deformed R–R field strengths with different rank. Hence we can readily read off the following expressions:

\[
F_3 = -4 \eta \frac{dz \wedge dx^0 \wedge dx^3}{z^5}, \quad F_5 = 4 (\omega_{\text{AdS5}} + \omega_S).
\] (4.6)

The deformed R–R fields \( \mathcal{F}' \) can be computed as

\[
\mathcal{F}' = e^{-B_2 \mathcal{F}}
\]

\[
\mathcal{F}' = -4 \eta \frac{dz \wedge dx^0 \wedge dx^3}{z^5} + 4 \left( \frac{z^4}{z^4 + \eta^2} \omega_{\text{AdS5}} + \omega_S \right) - 4 B_2 \wedge \omega_S.
\] (4.7)

to obtain

\[
\mathcal{F}'_1 = 0, \quad \mathcal{F}'_3 = -4 \eta \frac{dz \wedge dx^0 \wedge dx^3}{z^5},
\]

\[
\mathcal{F}'_5 = 4 \left( \frac{z^4}{z^4 + \eta^2} \omega_{\text{AdS5}} + \omega_S \right),
\]

\[
\mathcal{F}'_7 = -4 B_2 \wedge \omega_S.
\] (4.8)

The full deformed background, given by (4.4) and (4.8), is a solution of standard type IIB supergravity. This is the gravity dual of non-commutative gauge theory [78, 79].

This example shows how, instead of the cumbersome supercoset construction, we can derive the full expression of the YB-deformed backgrounds using formula (3.113), which only requires the knowledge of the classical \( r \)-matrix.

4.1.2. Lunin–Maldacena–Frolov background. Next, we will consider \( r \)-matrix

\[
r = \frac{1}{2} (\mu_3 h_1 \wedge h_2 + \mu_1 h_2 \wedge h_3 + \mu_2 h_3 \wedge h_1),
\] (4.9)

which is composed of the Cartan generators \( h_1, h_2 \) and \( h_3 \) of su(4). Here, the \( \mu_i \) (\( i = 1, 2, 3 \)) are deformation parameters. The metric and B-field were computed in [14] and the full background was reproduced in [17] by performing the supercoset construction.

The associated \( \beta \)-field is

\[
\beta = 2 \eta (\mu_3 \partial_{\phi_1} \wedge \partial_{\phi_2} + \mu_1 \partial_{\phi_2} \wedge \partial_{\phi_3} + \mu_2 \partial_{\phi_3} \wedge \partial_{\phi_1}).
\] (4.10)

By using the formula (3.113), we obtain the deformed background
\( ds^2 = ds_{\text{AdS}_5}^2 + \sum_{i=1}^{3} (dp_i^2 + G(\hat{\gamma}_i)p_i^2d\phi_i^2) + G(\hat{\gamma}_i)p_1^2p_2^2p_3^2 \left( \sum_{i=1}^{3} \hat{\gamma}_id\phi_i \right)^2, \)
\( B_2 = G(\hat{\gamma}_i) \left( \hat{\gamma}_3 p_1^2p_2^2d\phi_1 \wedge d\phi_2 + \hat{\gamma}_1 p_2^2p_3^2d\phi_2 \wedge d\phi_3 + \hat{\gamma}_2 p_1^2p_3^2d\phi_3 \wedge d\phi_1, \right) \)
\( \Phi = \frac{1}{2} \log G(\hat{\gamma}_i), \)
\( \hat{F}_3 = -4 \sin^3 \alpha \cos \alpha \sin \theta \cos \theta \left( \sum_{i=1}^{3} \hat{\gamma}_i d\phi_i \right) \wedge d\alpha \wedge d\theta, \)
\( \hat{F}_5 = 4 \left( \omega_{\text{AdS}_5} + G(\hat{\gamma}_i)\omega_{S^5} \right), \)

where we defined new coordinates \( \rho_i (i = 1, 2, 3) \) as
\( \rho_1 = \sin r \cos \zeta, \quad \rho_2 = \sin r \sin \zeta, \quad \rho_3 = \cos r. \)

The deformation parameters \( \hat{\gamma}_i \) are defined by
\( \hat{\gamma}_i = 8\eta \mu_i, \)
and the scalar function \( G(\hat{\gamma}_i) \) is given by
\( G^{-1}(\hat{\gamma}_i) \equiv 1 + \sin^2 r(\hat{\gamma}_1^2 \cos^2 r \sin^2 \zeta + \hat{\gamma}_2^2 \cos^2 r \cos^2 \zeta + \hat{\gamma}_3^2 \sin^2 r \sin^2 \zeta \cos^2 \zeta). \)

The background (4.11) has originally been derived in [77]. Supersymmetry is completely broken.

This changes when all deformation parameters \( \hat{\gamma}_i \) are set equal,
\( \hat{\gamma}_1 = \hat{\gamma}_2 = \hat{\gamma}_3 \equiv \hat{\gamma}. \)

The background becomes
\( ds^2 = \sum_{i=1}^{3} (dp_i^2 + G\rho_i^2d\phi_i^2) + G^2 \rho_1^2\rho_2^2\rho_3^2 \left( \sum_{i=1}^{3} d\phi_i \right)^2, \)
\( B_2 = G \hat{\gamma}(\rho_1^2\rho_2^2d\phi_1 \wedge d\phi_2 + \rho_2^2\rho_3^2d\phi_2 \wedge d\phi_3 + \rho_3^2\rho_1^2d\phi_3 \wedge d\phi_1, \)
\( \Phi = \frac{1}{2} \log G, \)
\( \hat{F}_3 = -4\hat{\gamma} \sin^3 \alpha \cos \alpha \sin \theta \cos \theta \left( \sum_{i=1}^{3} d\phi_i \right) \wedge d\alpha \wedge d\theta, \)
\( \hat{F}_5 = 4 \left( \omega_{\text{AdS}_5} + G(\hat{\gamma})\omega_{S^5} \right), \)

where the scalar function \( G \) is
\( G^{-1} \equiv 1 + \hat{\gamma}^2 \left( \sin^2 2r + \sin^2 r \sin^2 2\zeta \right). \)
In this special case the background preserves 8 supercharges. The gauge dual of the background (4.16) is known as \( \beta \)-deformed \( \mathcal{N} = 4 \) super Yang–Mills (SYM) [76] which is an exactly marginal deformation of \( \mathcal{N} = 4 \) SYM [107] preserving \( \mathcal{N} = 1 \) supersymmetry.

### 4.1.3. Schrödinger spacetime.

Finally, we consider the abelian \( r \)-matrix [16]

\[
 r = \frac{1}{2} P_- \wedge (h_1 + h_2 + h_3),
\]

where \( P_- = (P_0 - P_3)/\sqrt{2} \) is a light-cone transformation generator in \( \mathfrak{so}(2,4) \). Using the coordinate system given in (4.21), (4.22), the \( \beta \)-field can be expressed as

\[
 \beta = -\eta \partial_- \wedge \partial_\chi. 
\]

Via the formula (3.113), the resulting deformed background is given by

\[
 ds^2 = -\frac{2d\chi^+ dx^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} - \eta^2 \frac{(dx^+)^2}{z^4} + dz_5^2, 
\]

\[
 B_2 = \frac{\eta}{z^2} dx^+ \wedge (d\chi + \omega), \quad \Phi = 0, \quad \tilde{F}_5 = 4 \left( \omega_{\text{AdS}_5} + \omega_{S^5} \right). 
\]

Here the metric of \( S^5 \) is described as an \( S^1 \)-fibration over \( \mathbb{C}P^2 \) and its explicit form is

\[
 ds^2_{S^5} = (d\chi + \omega)^2 + ds^2_{\mathbb{C}P^2}, \quad ds^2_{\mathbb{C}P^2} = d\mu^2 + \sin^2 \mu \left( \Sigma_1^2 + \Sigma_2^2 + \cos^2 \mu \Sigma_3^2 \right),
\]

where \( \chi \) is the fiber coordinate and \( \omega \) is a one-form potential of the Kähler form on \( \mathbb{C}P^2 \). \( \Sigma_i (i = 1, 2, 3) \) and \( \omega \) are defined by

\[
 \Sigma_1 \equiv \frac{1}{2}(\cos \psi \, d\theta + \sin \psi \, \sin \theta \, d\phi), \\
 \Sigma_2 \equiv \frac{1}{2}(\sin \psi \, d\theta - \cos \psi \, \sin \theta \, d\phi), \\
 \Sigma_3 \equiv \frac{1}{2}(d\psi + \cos \theta \, d\phi), \quad \omega \equiv \sin^2 \mu \Sigma_3.
\]

Note that the \( S^5 \) part of the metric, the R–R five-form field strength and the dilaton remain undeformed.

This deformed background is called Schrödinger spacetime and was first introduced in [108]. It is the gravity dual of dipole CFTs [109–111]. Its classical integrability was discussed from the perspective of \( T \)-duality [112] in [113]. The spectral problem was recently studied in [114] using integrability methods.

### 4.2. Non-unimodular classical r-matrices

In this section, we consider YB deformations associated to non-unimodular \( r \)-matrices. As explained in the previous sections, these deformed backgrounds are solutions to the GSE. We also show that some of them reduce to the original \( \text{AdS}_5 \times S^5 \) background after performing a generalized TdS transformation.

1. \( r = P_1 \wedge D \). As a first example, let us consider the non-abelian classical \( r \)-matrix

\[
 r = \frac{1}{2} P_1 \wedge D. 
\]
It is a solution of the hCYBE which was already used to study a YB deformation of four-dimensional Minkowski spacetime [62].

The corresponding $\beta$-field is

$$\beta = \eta \partial_1 \wedge (t \partial_1 + z \partial_3),$$

(4.24)

where we have rewritten the four-dimensional Cartesian coordinates as

$$x^0 = t \sinh \phi, \quad x^2 = t \cosh \phi \cos \theta, \quad x^3 = t \cosh \phi \sin \theta.$$  

(4.25)

Then, the deformed background is found to be

$$ds^2 = \frac{\varepsilon^2 [dt^2 + (dx^1)^2 + (dz + \eta dx^1)^2]}{z^4 + \eta^2 (z^2 + t^2)}$$

$$+ \frac{r^2 [-d\phi^2 + \cosh^2 \phi d\theta^2]}{z^2} + ds^2_{\mathrm{AdS}_5},$$

$$B_2 = -\eta t d\tau \wedge dz \wedge dx^1,$$

$$\Phi = \frac{1}{2} \log \left[ \frac{z^4}{z^4 + \eta^2 (r^2 + z^2)} \right],$$

(4.26)

$$F_3 = -\frac{4 \eta r^2 \cosh \phi}{z^4} \left[ dt \wedge d\theta \wedge d\phi - \frac{r}{z} d\theta \wedge d\phi \wedge dz \right],$$

$$F_5 = 4 \left[ \frac{z^4}{z^4 + \eta^2 (r^2 + z^2)} \omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}_5} \right],$$

$I = -\eta \partial_1$.

Note here that the $\phi$ direction has time-like signature. These fields do not satisfy the EOM of type IIB supergravity, but solve the equations of generalized type IIB supergravity. The GSE has the Killing vector $I = \eta \partial_1$ which satisfies the divergence formula

$$I^1 = \eta = -D_m \beta^{1m}.$$  

(4.27)

Let us now perform $T$-dualities on the deformed background (4.26). Following [12], the extra fields are traded for a linear term in the dual dilaton. $T$-dualizing along the $x^1$ and $\phi_3$ directions, we find:

$$\begin{align*}
\bar{ds}^2 &= \varepsilon^2 (dx^1)^2 + \frac{1}{z^4} \left[ (dt + \eta dx^1)^2 + (dz + \eta dx^1)^2 - r^2 d\phi^2 \right] \\
&\quad + \frac{r^2 \cosh^2 \phi d\theta^2}{z^2} + dr^2 + \sin^2 r d\xi^2 + \cos^2 \xi \sin^2 \phi d\phi_2^2 \\
&\quad + \sin^2 r \sin^2 \xi d\phi_2^2 + \frac{d\phi^2}{\cos^2 r},
\end{align*}$$

$$\begin{align*}
e^\Phi \bar{F}_5 &= \frac{4 r^2 \cosh \phi}{z^4} \left[ dt + \eta dx^1 \right] \wedge (dz + \eta dx^1) \wedge d\theta \wedge d\phi \wedge d\phi_3 \\
&\quad + 2z \sin^3 r \sin 2\xi dx^1 \wedge dr \wedge d\xi \wedge d\phi_1 \wedge d\phi_2, \\
\Phi &= \eta x^1 + \log \left[ \frac{z}{\cos r} \right].
\end{align*}$$

(4.28)

Remarkably, this is a solution of the usual type IIB supergravity equations rather than the generalized ones. Note that the dilaton has a linear dependence on $x^1$. This same strategy

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16 The metric and NS–NS two-form were computed in [106].
was used in [115] to show that the Hoare–Tseytlin solution is ‘$T$-dual’ to the $\eta$-deformed background.

The ‘$T$-dualized’ background in (4.28) is a solution to the standard type IIB equations and has a remarkable property: it is locally equivalent to undeformed $\text{AdS}_5 \times S^5$. Let us first perform the following change of coordinates:

$$t = \tilde{t}(1 - \eta \tilde{x}^1), \quad z = \tilde{z}(1 - \eta \tilde{x}^1), \quad x^1 = -\frac{1}{\eta} \log(1 - \eta \tilde{x}^1). \quad (4.29)$$

Note that the new coordinate system does not cover all of spacetime: the new coordinate $\tilde{x}^1$ has to be restricted to the region $\tilde{x}^1 < \eta - 1$. The signature of $\eta$ was fixed when we chose the deformation. This change of coordinates achieves the following points:

- It diagonalizes the metric;
- It absorbs the $x^1$-dependence of the dilaton into the $\tilde{z}$ variable, such that $\partial_1$ is now a symmetry of the full background.

Explicitly, we find

$$d\tilde{x}^2 = \tilde{z}^2 (d\tilde{x}^1)^2 + \frac{1}{\tilde{z}^2} \left[ d\rho^2 + d\tilde{z}^2 - \rho^2 d\phi^2 + \rho^2 \cosh^2 \phi d\theta^2 \right] + dr^2 + \sin^2 r d\xi^2$$

$$e^{\Phi} F_5 = \frac{4 \rho^2}{\tilde{z}^4} \frac{\cos \phi}{\cos r} d\rho \wedge d\tilde{z} \wedge d\theta \wedge d\phi \wedge d\phi_3$$

$$+ 2 \tilde{z} \sin^2 r \sin 2\tilde{x} \wedge dr \wedge d\xi \wedge d\phi_1 \wedge d\phi_2,$$

$$\Phi = \log \left[ \frac{\tilde{z}}{\cos r} \right]. \quad (4.30)$$

Now we can perform again the two standard $T$-dualities along $\tilde{x}^1$ and $\phi_3$ to find, as mentioned above, the undeformed $\text{AdS}_5 \times S^5$ background.\(^{17}\)

Let us summarize what we have done. We have started with a YB deformation of $\text{AdS}_5$ described by the non-abelian $r$-matrix (4.23). Using the formula (3.113), we have found the corresponding deformed background (4.26) which is a solution to the generalized equations described in section 3.5. Then we have ‘$T$-dualized’ this background using the rules of [12] to find a new background (4.28) which solves the standard supergravity equations, but whose dilaton depends linearly on one of the $T$-dual variables. Finally, we have observed that after a change of variables, this last background is locally equivalent to the $T$-dual of the undeformed $\text{AdS}_5 \times S^5$. This result implies that the YB deformation with the classical $r$-matrix in (4.23) can be interpreted as an integrable twist, just like in the case of abelian classical $r$-matrices (see for example [42, 77, 116, 117]).

2. $r = (P_0 - P_3) \wedge (D + M_{03})$. Our next example is the classical $r$-matrix

$$r = \frac{1}{2\sqrt{2}}(P_0 - P_3) \wedge (D + M_{03}), \quad (4.31)$$

\(^{17}\) The usual Poincaré coordinates are found using the same change of coordinates as in (4.25).
where $M_{(0)}$ is the generator of the Lorentz rotation in the plane $(x^0, x^3)$. Then the $\beta$-field is

$$\beta = \eta \partial_\perp \wedge (\rho \, d\rho + z \, dz), \quad (4.32)$$

where the Cartesian coordinates of four-dimensional Minkowski spacetime $x^\mu$ are

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3), \quad x^1 = \rho \cos \theta, \quad x^2 = \rho \sin \theta. \quad (4.33)$$

The divergence of the $\beta$-field is given by

$$I^- = -2\eta = D_m \beta^{-m}. \quad (4.34)$$

Performing the supercoset construction [17], we obtain the corresponding background:

$$d\mathbf{s}^2 = \frac{-2dx^+dx^- + d\rho^2 + \rho^2 d\theta^2 + dz^2}{\rho^2} - \eta^2 \left[ \frac{\rho^2}{\rho^2 + 1} \right] (dx^+)^2 + dz_5^2,
\quad B_2 = -\eta \frac{dx^+ \wedge (\rho d\rho + z \, dz)}{\rho^2},
\quad \hat{F}_3 = 4\eta \left[ \frac{\rho^2}{\rho^2 + 1} (dx^+ \wedge d\theta \wedge dz + \rho \, dx^+ \wedge d\rho \wedge d\theta) \right],
\quad \hat{F}_5 = 4(\omega_{\text{AdS}_5} + \omega_{S^5})$$

$$\Phi = \Phi_0 \text{(constant)}, \quad I = -2\eta \partial_\perp \cdot$$

This background is a solution of the GSE characterized by the Killing vectors $I = -2\eta \partial_\perp$.

Let us perform four ‘$T$-dualities’ along the $x^+, x^-, \phi_1$ and $\phi_2$ directions. \(^{18}\) The resulting background is given by

$$d\mathbf{s}^2 = -2z^2 dx^+ dx^- + \frac{(d\rho + \eta \rho dx^-)^2 + \rho^2 d\theta^2 + (dz + \eta z dx^-)^2}{\rho^2} + d\tau^2 + \sin^2 r \, d\xi^2 + \frac{d\phi_1^2}{\cos^2 \xi \sin^2 r} + \frac{d\phi_2^2}{\sin^2 r \sin^2 \xi} + \cos^2 r \, d\phi_3^2,$n\epsilon^\Phi \hat{F}_5 = \frac{4i\rho}{\sin^2 \xi} \frac{(d\rho + \eta \rho dx^-) \wedge d\theta \wedge (dz + \eta z dx^-) \wedge d\phi_1 \wedge d\phi_2}{\sin^2 r \sin \xi \cos \xi} + 4i\sin r \cos r \, d\rho^2 \wedge dx^- \wedge dr \wedge d\xi \wedge d\phi_3,
\quad \Phi = 2\eta x^- + \log \frac{\rho^2}{\sin^2 r \sin \xi \cos \xi},$$

where all other components are zero.

The ‘$T$-dualized’ background in (4.36) is a solution to the standard type IIB equations and is again $locally$ equivalent to undeformed $\text{AdS}_5 \times S^5$. Let us first change the coordinates as follows:

$$x^- = -\frac{1}{2\eta} \log(1 - 2\eta x^-), \quad \rho = \tilde{\rho} \sqrt{1 - 2\eta x^-}, \quad z = \tilde{z} \sqrt{1 - 2\eta x^-}. \quad (4.37)$$

\(^{18}\)To perform the $T$-dualities in the two light-like directions one can equivalently pass to Cartesian coordinates $(x^0, x^1)$, $T$-dualize in these and finally introduce light-like combinations for the $T$-dual variables.
Explicitly, we find

\[ ds^2 = -2\tilde{z}^2 dx^+ d\tilde{x}^- + \frac{d\tilde{\rho}^2 + \rho^2 d\theta^2 + dz^2}{\tilde{z}^2} + dr^2 + \sin^2 r d\xi^2 + \frac{d\phi_1^2}{\cos^2 \xi} + \frac{d\phi_2^2}{\sin^2 \xi \cos^2 \xi} + \cos^2 r d\phi_3^2, \]

\[ e^\Phi \hat{F}_5 = \frac{4i\tilde{\rho}}{\tilde{z}^3} \sin \xi \cos \xi \sin^2 r \frac{d\tilde{\rho} \wedge d\theta \wedge d\xi \wedge d\phi_1 \wedge d\phi_2}{\sin^2 \xi \cos \xi} \]

\[ + 4\tilde{z}^2 \sin r \cos r dx^+ \wedge d\tilde{x}^- \wedge dr \wedge d\xi \wedge d\phi_3, \]

\[ \Phi = \log \left[ \frac{\tilde{z}^2}{\sin^2 \xi \cos \xi} \right]. \]  

(4.38)

Now, rewriting the light-like coordinates in terms of the Cartesian coordinates as

\[ x^+ = \frac{1}{\sqrt{2}}(\tilde{x}^0 + \tilde{x}^3), \quad \tilde{x}^- = \frac{1}{\sqrt{2}}(\tilde{x}^0 - \tilde{x}^3), \]

(4.39)

and performing four $T$-dualities along $\tilde{x}^0$, $\tilde{x}^3$, $\phi_1$ and $\phi_2$, we reproduce the undeformed AdS$_5 \times$ S$^5$ background.

**Mixing of abelian and non-abelian classical $r$-matrices.** This example admits a generalization, obtained by mixing abelian and non-abelian classical $r$-matrices:

\[ r = \frac{1}{2\sqrt{2}}[P_0 - P_3] \wedge [a_1(D + M_{03}) + a_2M_{12}]. \]  

(4.40)

When $a_2 = 0$, the classical $r$-matrix reduces to the one described above; when $a_1 = 0$, the $r$-matrix becomes abelian and the associated background is the Hubeny–Rangamani–Ross solution of [118], as shown in [17].

In [17] it was shown that with a supercoset construction, one finds the following ten-dimensional background:

\[ ds^2 = -2dx^+ dx^- + \frac{d\rho^2 + \rho^2 d\theta^2 + dz^2}{z^2} - \eta^2 \left[ (a_1^2 + a_2^2) \frac{dz^2}{z^2} + \frac{a_1^2}{z^4} \right] (dx^+)^2 + ds^2, \]

\[ B_2 = -\frac{\eta}{z} dx^+ \wedge [a_1(\rho d\rho + z dz) - a_2 \rho^2 d\theta], \]

\[ \tilde{F}_3 = \frac{4i\rho}{z^3} dx^+ \wedge [a_1(\rho d\rho + z dz) \wedge d\theta + a_2 d\rho \wedge dz], \]

\[ \tilde{F}_5 = 4(\omega_{AdS_5} + \omega_{S^5}), \]

\[ \Phi = \Phi_0 \text{ (constant)}, \quad I = -2\eta a_1 \partial_-. \]  

(4.41)

This background is still a solution of the GSE with the Killing vector $I = -2\eta a_1 \partial_-$. This background can be reproduced by using the formula (3.113) with

\[ \beta = \eta \partial_- \wedge [a_1(\rho d\rho + z dz) - a_2 \partial_\theta]. \]  

(4.42)
The Killing vector \( I \) also satisfies
\[
I^\alpha = -2\eta a_1 = D^\mu \beta^{-\mu}.
\]
(4.43)

In the special case \( a_1 = 0 \), the above background reduces to a solution of standard type IIB supergravity.

Let us next perform four ‘\( T \)-dualities’ along the \( x^+, x^-, \phi_1 \) and \( \phi_2 \) directions. Then we can obtain a solution of the usual type IIB supergravity:

\[
dx^2 = -2\rho^2 \tr dr^2 + \frac{(d\rho + \eta a_1 \rho dx^-)^2 + \rho^2 (d\theta - \eta a_2 dx^-)^2 + (dz + \eta a_1 dz dx^-)^2}{\zeta^2} + dr^2 + \sin^2 \theta d\xi^2 + \frac{d\phi_1^2}{\cos^2 \xi \sin^2 r} + \frac{d\phi_2^2}{\sin^2 r \sin^2 \xi} + \cos^2 r d\phi_3^2,
\]

\[
e^\Phi \tilde{F}_5 = \frac{4i\rho}{\zeta^2 \sin \xi \cos \xi \sin^2 r} (d\rho + \eta a_1 \rho dx^-) \wedge (d\theta - \eta a_2 dx^-) \wedge (dz + \eta a_1 dz dx^-) \wedge d\phi_1 \wedge d\phi_2 + 4i\rho \sin r \cos r dx^+ \wedge dx^- \wedge dr \wedge d\xi \wedge d\phi_3,
\]

\[
\Phi = 2\eta a_1 x^- + \log \left\{ \frac{\zeta^2}{\sin^2 r \sin \xi \cos \xi} \right\},
\]

where all other components are zero. It is easy to see that this is just a twist of the previous solution (in (4.38)) and in fact there is a change of variables

\[
\rho = \tilde{\rho} \exp^{-\eta a_1 x^-}, \quad \zeta = \tilde{\zeta} \exp^{-\eta a_1 x^-}, \quad \theta = \tilde{\theta} - \eta a_2 x^-, \quad x^- = -\frac{1}{2\eta a_1} \log(1 - 2\eta a_1 \tilde{x}^-),
\]

(4.45)

that maps this background to the same local form:

\[
dx^2 = -2\rho^2 \tr dr^2 + \frac{d\tilde{\eta}^2 + \tilde{\rho}^2 d\tilde{\theta}^2 + d\tilde{z}^2}{\tilde{\zeta}^2} + dr^2 + \sin^2 \theta d\xi^2 + \frac{d\phi_1^2}{\cos^2 \xi \sin^2 r} + \frac{d\phi_2^2}{\sin^2 r \sin^2 \xi} + \cos^2 r d\phi_3^2,
\]

\[
e^{\Phi} \tilde{F}_5 = \frac{4i\tilde{\rho}}{\tilde{\zeta}^2 \sin \xi \cos \xi \sin^2 r} d\tilde{\rho} \wedge d\tilde{\theta} \wedge d\tilde{z} \wedge d\phi_1 \wedge d\phi_2 + 4i\tilde{\rho} \sin r \cos r dx^+ \wedge dx^- \wedge dr \wedge d\xi \wedge d\phi_3,
\]

\[
\Phi = \log \left\{ \frac{\zeta^2}{\sin^2 r \sin \xi \cos \xi} \right\},
\]

(4.46)

which is a \( T \)-dual of the undeformed \( S^5 \times S^5 \) background.

3. \( r = (P_0 - P_3) \wedge D \). Our last example is the classical \( r \)-matrix

\[
r = \frac{1}{2\sqrt{2}}(P_0 - P_3) \wedge D,
\]

(4.47)
which is another solution of the hCYBE. The associated $\beta$-field is

$$\beta = \eta \partial_- \wedge (\rho \, d\rho + z \, dz + x^+ \, \partial_+).$$  (4.48)

Here the following new coordinates have been introduced:

$$x^0 = \frac{x^+ + x^-}{\sqrt{2}}, \quad x^3 = \frac{x^+ - x^-}{\sqrt{2}}, \quad x^1 = \rho \cos \theta, \quad x^2 = \rho \sin \theta.$$  (4.49)

Using the formula (3.113), the associated background is found to be\footnote{The metric and NS–NS two-form were computed in [106].}

$$ds^2 = \frac{1}{\varepsilon^4 - \eta^2 (x^+)^2} \left[ \varepsilon^2 (-2 dx^+ \, dx^- + dz^2) + 2 \eta^2 z^{-2} x^+ \rho dx^+ d\rho \\
- \eta^2 z^{-2} \rho^2 (dx^+)^2 - \eta^2 (dx^+ - x^+ z^{-1} dz)^2 \right] + \frac{d\rho^2 + \rho^2 d\theta^2}{z^2} + ds^2_{S^5},$$

$$B_2 = - \eta \frac{dx^+ \wedge (z dz + \rho d\rho - x^+ dx^-)}{\varepsilon^4 - \eta^2 (x^+)^2},$$

$$\hat{F}_3 = 4 \eta \frac{\rho}{\varepsilon^4} \left[ \frac{\rho}{z} dx^+ \wedge d\theta \wedge dz + dx^+ \wedge d\rho \wedge d\theta + \frac{x^+}{z} d\rho \wedge d\theta \wedge dz \right],$$

$$\hat{F}_5 = 4 \left[ \frac{\varepsilon^4}{\varepsilon^4 - \eta^2 (x^+)^2} \omega_{AdS_5} + \omega_{S^5} \right],$$

$$\Phi = \frac{1}{2} \log \left[ \frac{\varepsilon^4}{\varepsilon^4 - \eta^2 (x^+)^2} \right], \quad I = - \eta \partial_-.$$  (4.50)

This background satisfies the GSE with the Killing vector $I = - \eta \partial_-$. In particular, the divergence formula (3.128) works well.

As of now, an appropriate $T$-dual frame in which this background is a solution to the standard type IIB equations with a linear dilaton has not been found. However, the deformed background can be reproduced by a generalized diffeomorphism which is a gauge symmetry of DFT (see (18) in [24]).

5. T-folds from YB deformations

In this chapter, we will concentrate on YB deformations of Minkowski and AdS$_5 \times S^5$ backgrounds, and show that the deformed backgrounds we consider here belong to a specific class of non-geometric backgrounds, called T-folds [119]. It is worth noting that these examples have the intriguing feature that also the R–R fields are twisted by the $T$-duality monodromy, as opposed to the well-known T-folds which include no R–R fields.

5.1. A brief review of T-folds

In this subsection, we briefly explain the notion of the T-fold. A T-fold is a generalization of the usual notion of manifold. It locally looks like a Riemannian manifold, but its patches are glued together not just by diffeomorphisms but also by $T$-duality. T-folds play a significant role in the study of non-geometric fluxes beyond the effective supergravity description. As illustrative
examples, we revisit two well-known cases in the literature corresponding to a chain of duality transformations [120, 121] and to a codimension-1 $S^1$-brane solution [122–124].

Different string theories are related by discrete dualities. It is possible that via such a duality transformations, a flux configuration turns into a non-geometric flux, meaning it cannot be realized in terms of the usual fields in 10/11-dimensional supergravity. This suggests that we need to go beyond the usual geometric isometries to fully understand flux compactifications.

For the case of $T$-duality, one proposal to address this problem is the so-called doubled formalism. This construction is based on the generalization of a manifold in which all the local patches are geometric. However, the transition functions that are needed to glue these patches include not only the usual diffeomorphisms and gauge transformations, but also $T$-duality transformations.

$T$-fold backgrounds are formulated in an enlarged space with a $T^n \times \tilde{T}^n$ fibration. The tangent space is the doubled torus $T^n \times \tilde{T}^n$ and is described by a set of coordinates $y^M = (y^m, \bar{y}_m)$ which transform in the fundamental representation of $O(n, n)$. The physical internal space arises as a particular choice of a subspace of the double torus, $T^n_{\text{phys}} \subset T^n \times \tilde{T}^n$ (this is called a polarization). Then $T$-duality transformations $O(n, n; \mathbb{Z})$ act by changing the physical subspace $T^n_{\text{phys}}$ to a different subspace of the enlarged $T^n \times \tilde{T}^n$. For a geometric background, we have a space-time which is a geometric bundle, $T^n_{\text{phys}} = T^n$.\footnote{We can also have $T^n_{\text{phys}} = \tilde{T}^n$, which corresponds to a dual geometric description.} More general non-geometric backgrounds do not fit together to form a conventional manifold: despite being locally well-defined, they do not have a valid global geometric description. Instead, they are globally well-defined as $T$-folds.

This formulation is manifestly invariant under the $T$-duality group $O(n, n; \mathbb{Z})$, which is broken by the choice of polarization. $T$-duality transformations allow to identify the backgrounds that belong to the same physical configuration or duality orbit and just differ by a choice of polarization.\footnote{These orbits have been determined in terms of a classification of gauged supergravities in [125].}

Let us now review some examples of $T$-folds that have been studied in the literature.

5.1.1. A toy example. We start by reviewing a toy model that involves several duality transformations of a given background. This example has been discussed in [120, 121]. Before introducing the $T$-fold, we will discuss geometric cases such as the twisted torus and the torus with $H$-flux as a warm-up.

**Twisted torus.** Let us consider the metric of a twisted torus,

$$ds^2 = dx^2 + dy^2 + (dz - mx dy)^2 , (m \in \mathbb{Z}).$$

Note that this is not a supergravity solution for $m \neq 0$, but can serve to exemplify the non-geometric global property. As this background has isometries along the $y$ and $z$ directions, these directions can be compactified with certain boundary conditions. For example, let us take

$$(x, y, z) \sim (x, y + 1, z), \quad (x, y, z) \sim (x, y, z + 1).$$

There is no isometry along the $x$ direction, but there is a Killing vector that can be thought of as a deformation with parameter $m$:

$$k = \partial_x + my \partial_z.$$
Also this isometry direction can be compactified imposing
\[ (x, y, z) \sim e^t(x, y, z) = (x + 1, y, z + m y). \] (5.4)

Under this identification, both the one-form \( e_z \equiv dz - mx dy \) and the metric (5.1) are globally well-defined [120].

This background can be regarded as a two-torus \( T^2_z \) fibered over a base \( S^1_x \). The metric of the two-torus is
\[ (g_{mn}) = \begin{pmatrix} 1 & -mx \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -mx & 1 \end{pmatrix}. \] (5.5)

Then, as one moves around the base \( S^1_x \), the metric is transformed by a \( \text{GL}(2) \) rotation. That is to say, for \( x \rightarrow x + 1 \), the metric is given by
\[ g_{mn}(x + 1) = [\Omega^T g(x) \Omega]_{mn}, \quad \Omega^m_n \equiv \begin{pmatrix} 1 & 0 \\ -m & 1 \end{pmatrix}. \] (5.6)

This monodromy twist can be compensated by a coordinate transformation
\[ y = y', \quad z = z' + my'. \] (5.7)

In other words, the metric is single-valued up to a diffeomorphism. In this sense this background is geometric.

**Torus with H-flux.** When a \( T \)-duality is performed on the twisted torus (5.1) along the \( y \) direction, we obtain the background
\[ ds^2 = dx^2 + dy^2 + dz^2, \quad B_2 = -mx dy \wedge dz, \] (5.8)
equipped with the \( H \)-flux
\[ H_3 = dB_2 = -m dx \wedge dy \wedge dz. \] (5.9)

If we consider the generalized metric (3.96) on the doubled torus \( (y, z, \tilde{y}, \tilde{z}) \) associated to this background, then we can easily identify the induced monodromy when \( x \rightarrow x + 1 \). The monodromy matrix is given by
\[ \mathcal{H}_{MN}(x + 1) = [\Omega^T \mathcal{H}(x) \Omega]_{MN}, \quad \Omega^M_N \equiv \begin{pmatrix} \delta^m_n & 0 \\ 2m \delta^m_n \delta^m_n & \delta^m_n \end{pmatrix} \in O(2, 2; \mathbb{Z}). \] (5.10)

The induced monodromy can be compensated by a constant shift in the \( B \)-field,
\[ B_{\tilde{y}z} \rightarrow B_{\tilde{y}z} - m. \] (5.11)

This shift transformation, which makes the background single-valued, is a gauge transformation of supergravity. In this sense, this background is also geometric.

**T-fold.** Finally, let us perform another \( T \)-duality transformation along the \( y \)-direction on the twisted torus (5.1). We obtain the background [120]
\[ ds^2 = dx^2 + \frac{dy^2 + dz^2}{1 + m^2 x^2}, \quad B_2 = \frac{mx}{1 + m^2 x^2} dy \wedge dz. \] (5.12)
In this case, neither general coordinate transformations nor B-field gauge transformations are enough to remove the multi-valuedness of the background. This can also be seen by calculating the monodromy matrix. The associated generalized metric is given by

\[
H(x) = \begin{pmatrix}
\delta^p_m & 0 \\
-2mx \delta^m_q \delta^p_l & \delta^m_p
\end{pmatrix}
\begin{pmatrix}
\delta^q_m & 0 \\
0 & \delta^q_p
\end{pmatrix}
\begin{pmatrix}
\delta^q_l & 2mx \delta^l_q \delta^q_z \\
0 & \delta^q_q
\end{pmatrix}.
\] (5.13)

We find that, upon the transformation \(x \rightarrow x + 1\), the induced monodromy is

\[
H_{MN}(x+1) = \left[\Omega^T H(x) \Omega \right]_{MN}, \quad \Omega^M_N \equiv \begin{pmatrix}
\delta^m_l & 2m \delta^m_q \delta^q_l \\
0 & \delta^m_m
\end{pmatrix} \in \text{O}(2,2; \mathbb{Z}).
\] (5.14)

The present O(2,2; \mathbb{Z}) monodromy matrix \(\Omega\) takes an upper-triangular form i.e. it corresponds to a \(\beta\)-transformation which is not part of the gauge group of supergravity. Hence, to keep the background globally well defined, the transition functions that glue the local patches should be extended to the full set of O(2,2; \mathbb{Z}) transformations beyond general coordinate transformations and B-field gauge transformations. This is what happens in the T-fold case.

In summary, we conclude that a non-geometric background with a non-trivial O(n,n; \mathbb{Z}) monodromy transformation, such as a \(\beta\)-transformation, is a T-fold. The background (5.12) is a simple example.

From the viewpoint of DFT, \(\beta\)-transformations can be realized as gauge symmetries by choosing a suitable solution of the section condition. Indeed, the above O(2,2; \mathbb{Z}) monodromy matrix \(\Omega\) can be canceled by a generalized coordinate transformation on the double torus coordinates \((y,z,\tilde{y},\tilde{z})\):

\[
y = y' + m \tilde{z}, \quad z = z', \quad \tilde{y} = \tilde{y}', \quad \tilde{z} = \tilde{z}'.
\] (5.15)

In this sense, the twisted doubled torus is globally well-defined in DFT.

It is also possible to make the single-valuedness manifest by using the dual fields \((G_{mn}, \beta_{mn}, \phi)\) defined by (3.102) or (3.103), (3.104). In the non-geometric parametrization (3.102), the background (5.13) becomes

\[
\text{d}x^2_{\text{dual}} = G_{mn} \text{d}x^m \text{d}x^n = \text{d}x^2 + \text{d}y^2 + \text{d}z^2, \quad \beta^z = mx,
\] (5.16)

and the O(2,2; \mathbb{Z}) monodromy matrix (5.14) corresponds to a constant shift in the \(\beta\)-field, \(\beta^z \rightarrow \beta^z + m\). We see that up to a constant \(\beta\)-shift, which is a gauge symmetry (5.15) of DFT, the background becomes single-valued.

We now define the non-geometric \(Q\)-flux \([126]\),

\[
Q_{p}^{mn} \equiv \partial_p \beta^{mn}.
\] (5.17)

After the transformation \(x \rightarrow x + 1\), the induced monodromy on the \(\beta\)-field can be measured by an integral of the \(Q\)-flux,

\[
\beta^{mn}(x+1) - \beta^{mn}(x) = \int_x^{x+1} \text{d}x' \partial_p \beta^{mn}(x') = \int_x^{x+1} \text{d}x' Q_{p}^{mn}(x').
\] (5.18)

This expression plays a central role in our argument.

After this illustrative example we conclude that \(Q\)-flux backgrounds are globally well-defined as T-folds. In the next subsection, we give a codimension-1 example of the exotic \(S^2\)-brane using the above \(Q\)-flux.
5.1.2. The codimension-1 522-brane background. Our second example is a supergravity solution studied in [122]. It is obtained by smearing the codimension-2 exotic 522-brane solution [123, 124], which is related to the NS5-brane solution by two T-duality transformations. It is also referred to as a Q-brane, as it is a source of Q-flux as we will see in the following. The codimension-1 version of this solution is given by

\[ ds^2 = m x (dx^2 + dy^2) + \frac{x (dz^2 + dw^2)}{m (x^2 + z^2)} + dz^2, \]

\[ B_2 = \frac{x}{m (x^2 + z^2)} dz \wedge dw, \]

\[ \Phi = \frac{1}{2} \log \left( \frac{x}{m (x^2 + z^2)} \right). \]

With the non-geometric parametrization (3.102), this solution simplifies:

\[ ds_{\text{dual}}^2 = m x (dx^2 + dy^2) + dz^2 + dw^2, \]

\[ \beta^{\text{ew}} = m y, \quad \tilde{\phi} = \frac{1}{2} \log \left( \frac{1}{m x} \right). \]

Assuming that the y direction is compactified with y \sim y + 1, the monodromy under y \rightarrow y + 1 corresponds to a constant shift of the field \beta:

\[ \beta^{\text{ew}} \rightarrow \beta^{\text{ew}} + m. \]  

As the background is twisted by a \beta-shift, this is an example of a T-fold with constant Q-flux

\[ Q_y^{\text{ew}} = m. \]

Finally, the monodromy matrix is given by

\[ \mathcal{H}^{MN}(y + 1) = [\Omega^T \mathcal{H}(y) \Omega]_{MN}, \quad \Omega_{\hat{M}N} \equiv \begin{pmatrix} \delta^m_n & 2m \delta^m_n \delta^{|n|}_w \\ 0 & \delta^{|m|}_n \delta^{m}_w \end{pmatrix} \in \text{O}(10, 10; \mathbb{Z}). \]

Employing the knowledge of T-folds introduced in this section, we will next elaborate on the non-geometric aspects of YB-deformed backgrounds when seen as T-folds.

5.2. Non-geometric aspects of YB deformations

Here we will show that various YB-deformed backgrounds can be regarded as T-folds.

5.2.1. T-duality monodromy of YB-deformed backgrounds. As we explained in section 3.9.2, the homogeneous YB-deformed background described by (\mathcal{H}', d', F') always has the structure

\[ \mathcal{H}' = e^{\beta} \mathcal{H} e^{\beta}, \quad d' = d, \quad F' = e^{-\beta} F, \]

\[ e^{\beta} = \begin{pmatrix} \delta_n^m & \beta^m_n \\ 0 & \delta^m_n \end{pmatrix}, \quad \beta^{mn} = 2 \eta^{ij} \tilde{T}_m^{i} \tilde{T}_n^{j}, \]

where (\mathcal{H}, d, F) represent the undeformed background and F is defined in (3.112). In the following examples, the B-field vanishes in the undeformed background. At this stage, we know only the local properties of the YB-deformed background.

In the examples considered in this section, the bi-vector \beta^{mn} always has a linear coordinate dependence. Pick a frame in which \beta^{mn} depends linearly on a coordinate \gamma.
\[ \beta^m = r^m y + \phi^m \quad (r^m : \text{constant}, \quad \phi^m : \text{independent of } y), \] (5.25)

and the \( \beta \)-untwisted fields are independent of \( y \). Then, from the abelian property

\[ e^{\beta_1 + \beta_2} = e^{\beta_1} e^{\beta_2} = e^{\beta_2} e^{\beta_1}, \quad e^{-\beta_1 + \beta_2} = e^{-\beta_1} e^{-\beta_2} = e^{-\beta_2} e^{-\beta_1}, \] (5.26)

we obtain

\[ \mathcal{H}_{MN}(y + a) = \left[ \Omega_{y}^T \mathcal{H}(y) \Omega_{a} \right]_{MN}, \quad d(y + a) = d(y), \]

\[ F(y + a) = e^{-\omega_{a}/y} F(y), \quad (\Omega_{a})^{M}_{N} \equiv \left( \begin{array}{c} \delta_{m}^{n} \\
0 \end{array} \right), \quad (\omega_{a})^{m}_{n} \equiv a r^{m}. \] (5.27)

This is in general an element of \( O(10, 10; \mathbb{R}) \). If there is a value \( a_0 \) such that the matrix \( \Omega_{a_0} \in O(10, 10; \mathbb{Z}) \), the background is consistent with an identification in the \( y \) direction of the type \( y \sim y + a_0 \). This is because \( O(10, 10; \mathbb{Z}) \) is a gauge symmetry of string theory and the background can be identified up to a gauge transformation. In this example of a \( T \)-fold, the monodromy matrices for the generalized metric and \( R \)–\( R \) fields are \( \Omega_{a_0} \) and \( e^{-\omega_{a_0}/y} \), respectively, while the dilaton \( d \) is single-valued. Note that the \( R \)–\( R \) potential \( A \) has the same monodromy as \( F \).

### 5.2.2. YB-deformed Minkowski backgrounds

In this subsection, we study YB-deformations of Minkowski spacetime \([62, 63]\). We begin with a simple example of an abelian YB deformation. Next we present two purely NS–NS solutions of the GSE and show that they are \( T \)-folds. These backgrounds have vanishing \( R \)–\( R \) fields and are the first examples of purely NS–NS solutions of the GSE.

**Abelian example.** Let us consider the simple abelian \( r \)-matrix \([62]\)

\[ r = -\frac{1}{2} P_1 \wedge M_{23}. \] (5.28)

The corresponding YB-deformed background is given by

\[ ds^2 = -(dx^0)^2 + \frac{(dx^1)^2 + \left[ 1 + (\eta x^2)^2 \right] (dx^3)^2 + \left[ 1 + (\eta x^3)^2 \right] (dx^3)^2 + 2 \eta^2 x^2 x^3 dx^2 dx^3}{1 + \eta^2 (x^2)^2 + (x^3)^2} \]
\[ + \sum_{i=4}^{9} (dx^i)^2, \]

\[ B_2 = \frac{\eta dx^1 \wedge (x^2 dx^3 - x^3 dx^2)}{1 + \eta^2 (x^2)^2 + (x^3)^2}, \quad \Phi = \frac{1}{2} \log \left[ \frac{1}{1 + \eta^2 (x^2)^2 + (x^3)^2} \right]. \] (5.29)

These expressions appear complicated, but after moving to an appropriate polar coordinate system (see section 3.1 of \([62]\)), the background (5.29) is found to be the dual to the well-known Melvin background \([127-132]\). In \([62]\), it was reproduced as a YB deformation with the classical \( r \)-matrix (5.28). For later convenience, we will keep using the expression in (5.29).

The dual parametrization of this background is given by

\[ ds^2_{\text{dual}} = -(dx^0)^2 + \sum_{i=1}^{9} (dx^i)^2, \quad \beta = \eta \left( x^2 \partial_1 \wedge \partial_3 - x^3 \partial_1 \wedge \partial_2 \right), \quad \tilde{\phi} = 0. \] (5.30)

Hence, under a shift \( x^2 \rightarrow x^2 + \eta^{-1} \), the background receives the \( \beta \)-transformation

\[ \beta \rightarrow \beta + \partial_1 \wedge \eta. \] (5.31)
Therefore, if the $x^2$ direction is compactified with the period $\eta^{-1}$, then the monodromy matrix becomes
\[
\mathcal{H}_{MN}(x^2 + \eta^{-1}) = \left[ \Omega^T \mathcal{H}(x^2) \Omega \right]_{MN},
\]
\[
\Omega^M_N \equiv \begin{pmatrix}
\delta^m_n & 2 \delta^m_1 \delta^1_n \\
0 & \delta^m_m
\end{pmatrix} \in O(10, 10; \mathbb{Z}).
\] (5.32)

Thus this background turns out to be a $T$-fold.

When the $x^3$ direction is also identified with the period $\eta^{-1}$, the corresponding monodromy matrix becomes
\[
\mathcal{H}_{MN}(x^3 + \eta^{-1}) = \left[ \Omega^T \mathcal{H}(x^3) \Omega \right]_{MN},
\]
\[
\Omega^M_N \equiv \begin{pmatrix}
\delta^m_n & -2 \delta^m_1 \delta^1_n \\
0 & \delta^m_m
\end{pmatrix} \in O(10, 10; \mathbb{Z}).
\] (5.33)

In terms of non-geometric fluxes, this background has a constant $Q$-flux. In the examples of $T$-folds given in section 5.1, a background with a constant $Q$-flux, $Q_{\mu \nu}$, is mapped to another background with a constant $H$-flux, $H_{\mu \nu \rho}$, under a double $T$-duality along the $x^\mu$ and $x^\nu$ directions. This is not possible in this case because $\partial_2$ and $\partial_3$ are not isometries and there is no $T$-dual frame in which the $H$-flux is constant.

**Non-unimodular example 1:** $r = \frac{1}{2}(P_0 - P_1) \wedge M_{01}$. Let us consider the non-unimodular classical $r$-matrix
\[
r = \frac{1}{2}(P_0 - P_1) \wedge M_{01}.
\] (5.34)

The corresponding YB-deformed background is given by
\[
d s^2 = \frac{- (dx^0)^2 + (dx^1)^2}{1 - \eta^2 (x^0 + x^1)^2} + \sum_{i=2}^{9} (dx^i)^2,
\]
\[
B_2 = -\frac{\eta (x^0 + x^1)}{1 - \eta^2 (x^0 + x^1)^2} dx^0 \wedge dx^1, \quad \Phi = \frac{1}{2} \log \left[ \frac{1}{1 - \eta^2 (x^0 + x^1)^2} \right].
\] (5.35)

This background has a coordinate singularity at $x^0 + x^1 = \pm 1/\eta$ which is removed in the dual parametrization (3.102) where the dual fields are given by
\[
d s^2_{\text{dual}} = - (dx^0)^2 + \sum_{i=1}^{9} (dx^i)^2, \quad \beta = \eta (x^0 + x^1) \partial_0 \wedge \partial_1, \quad \tilde{\phi} = 0,
\] (5.36)

and they are regular everywhere.\(^{22}\)

By introducing the Killing vector $I$ using the divergence formula (3.128),
\[
I = \tilde{D}_n \beta^m \partial_m = \partial_n \beta^m \partial_m = \eta (\partial_0 - \partial_1),
\] (5.37)

the background (5.35) with this $I$ solves the GSE. Here $\tilde{D}_n$ is the covariant derivative associated to the original Minkowski spacetime.

\(^{22}\) A similar resolution of singularities in the dual parametrization has been used in [133, 134] in the context of the exceptional field theory.
Since the $\beta$-field depends linearly on $x^1$, the background is twisted by the $\beta$-transformation as one moves along the $x^1$ direction. In particular, when the $x^1$ direction is identified with period $1/\eta$, this background becomes a $T$-fold with an $O(10,10;\mathbb{Z})$ monodromy,
\[
\mathcal{H}_{MN}(x^1 + \eta^{-1}) = [\Omega^T \Omega(x^1)]_{MN}, \quad \Omega^M_N \equiv \begin{pmatrix} \delta^n_m & 2 \delta_0^m \delta^n_1 \\ 0 & \delta^n_m \end{pmatrix}.
\]
Note that an arbitrary solution of the GSE can be regarded as a solution of DFT [85]. Indeed, by introducing light-cone coordinates and a rescaled deformation parameter,
\[
x^\pm \equiv x^0 \pm x^1 \sqrt{2}, \quad \bar{\eta} = \sqrt{2} \eta,
\]
the present YB-deformed background can be regarded as the DFT-solution
\[
\mathcal{H} = \begin{pmatrix} 0 & -1 & -\bar{\eta} x^+ & 0 \\ -1 & 0 & 0 & \bar{\eta} x^+ \\ -\bar{\eta} x^+ & 0 & 0 & (\bar{\eta} x^+)^2 - 1 \\ 0 & \bar{\eta} x^+ & (\bar{\eta} x^+)^2 - 1 & 0 \end{pmatrix}, \quad d = \bar{\eta} \tilde{x}^-,
\]
where only the $(x^+, x^-, \tilde{x}^+, \tilde{x}^-)$-components of $\mathcal{H}_{MN}$ are displayed. Note here that the dilaton has an explicit dual-coordinate dependence because we are now considering a non-standard solution of the section condition which makes this background a solution of the GSE rather than the usual supergravity.

Before performing the YB deformation (i.e. for $\bar{\eta} = 0$), there is a Killing vector $\chi \equiv \partial_+$, but the associated isometry is broken for non-zero $\bar{\eta}$. However, even after deforming the geometry, there exists a generalized Killing vector
\[
\hat{\chi} \equiv e^{\bar{\eta} \tilde{x}^-} \partial_+,
\]
which turns into the original Killing vector in the undeformed limit, $\bar{\eta} \to 0$. Indeed, we can show that the generalized metric and the DFT dilaton are invariant under the generalized Lie derivative $\hat{\xi}_\chi$ [58, 60] associated to $\chi$,
\[
\hat{\xi}_\chi \mathcal{H}_{MN} = 0, \quad \hat{\xi}_\chi e^{-2d} = 0.
\]
Here, the generalized Lie derivative acts on $\mathcal{H}_{MN}(x)$ and $d(x)$ as
\[
\hat{\xi}_\chi \mathcal{H}_{MN} = \chi^K \partial_K \mathcal{H}_{MN} + (\partial_M V^K - \partial^K V_M) \mathcal{H}_{KN} + (\partial_N V^K - \partial^K V_N) \mathcal{H}_{MK},
\]
\[
\hat{\xi}_\chi e^{-2d} = \partial_M (e^{-2d} \chi^M).
\]
In order to make the generalized isometry manifest, let us consider a generalized coordinate transformation,
\[
x'^+ = e^{-\eta \tilde{x}^-} x^+, \quad \tilde{x}'^- = -\bar{\eta}^{-1} e^{-\eta \tilde{x}^-}, \quad x'^M = x^M \text{ (others)}.
\]
By employing Hohm and Zwiebach’s finite transformation matrix [135],
\[
\mathcal{F}^N_M = \frac{1}{2} \left( \frac{\partial x^K}{\partial x^M} \frac{\partial x^K}{\partial x^N} + \frac{\partial x^M}{\partial x^N} \frac{\partial x^K}{\partial x^K} \right),
\]
the generalized Killing vector in the primed coordinates becomes constant, $\chi = \partial'_+$. We can also check that the generalized metric in the primed coordinate system is precisely the unde-
formed background. At least locally, the YB deformation can be undone by the generalized coordinate transformation. This fact is consistent with the fact that YB deformations can be realized as generalized diffeomorphisms [24].

Non-Riemannian background. Since the above background has a linear coordinate dependence on \( \tilde{x} \), let us rotate the solution to the canonical section (i.e. the section in which all of the fields are independent of the dual coordinates). By performing a \( T \)-duality along the \( x^- \) direction using the matrix in equation (3.100), we obtain

\[
\mathcal{H} = \begin{pmatrix}
0 & 0 & -\eta \tilde{x}^- & -1 \\
0 & 0 & (\bar{\eta} \tilde{x}^+)^2 - 1 & \bar{\eta} \tilde{x}^+ \\
-\eta \tilde{x}^+ & (\bar{\eta} \tilde{x}^+)^2 - 1 & 0 & 0 \\
-1 & \bar{\eta} \tilde{x}^+ & 0 & 0
\end{pmatrix}, \quad d = \bar{\eta} \tilde{x}^-. \tag{5.47}
\]

The resulting background is indeed a solution of DFT defined on the canonical section. However, this solution cannot be parameterized in terms of \((g_{mn}, B_{mn})\) and is called a non-Riemannian background in the terminology of [136]. This background does not even allow the dual parametrization (3.102) in terms of \((G_{mn}, \beta_{mn})\).

Non-unimodular example 2: \( r = \frac{1}{2\sqrt{2}} \sum_{\mu=0}^{4} (M_{0\mu} - M_{1\mu}) \wedge P^\mu \). Our next example is the classical \( r \)-matrix [63]

\[
r = \frac{1}{2\sqrt{2}} \sum_{\mu=0}^{4} (M_{0\mu} - M_{1\mu}) \wedge P^\mu. \tag{5.48}
\]

This classical \( r \)-matrix is a higher-dimensional generalization of the light-cone \( \kappa \)-Poincaré \( r \)-matrix in the four dimensional case. Using light-cone coordinates,

\[
x^\pm \equiv \frac{x^0 \pm x^1}{\sqrt{2}}, \tag{5.49}
\]

the corresponding YB-deformed background becomes

\[
ds^2 = -2 dx^+ dx^- - \eta^2 dx^+ \left[ \sum_{i=2}^{4} (x^i)^2 dx^+ - 2 x^+ \sum_{i=2}^{4} x^i dx^i - \sum_{i=2}^{9} (dx^i)^2 \right] + \sum_{i=2}^{9} (dx^i)^2, \tag{5.50}
\]

\[
B_2 = \eta dx^+ \wedge \left( x^+ dx^- - \sum_{i=2}^{4} x^i dx^i \right), \quad \Phi = \frac{1}{2} \log \left[ \frac{1}{1 - (\eta x^+)^2} \right].
\]

In terms of the dual parametrization, this background becomes

\[
d_{\text{dual}}^2 = -2 dx^+ dx^- + \sum_{i=2}^{9} (dx^i)^2, \quad \tilde{\phi} = 0, \tag{5.51}
\]

\[
\tilde{\beta} = \eta \sum_{\mu=0}^{4} M_{-\mu} \wedge \hat{P}^\mu = \eta \partial_- \wedge \left( x^+ \partial_+ + \sum_{i=2}^{4} x^i \partial_i \right).
\]

\[23\] In the study of YB deformations of \( \text{AdS}_5 \), the similar phenomenon has already been observed in [20].

\[24\] For another example of non-Riemannian backgrounds, see [136]. A classification of non-Riemannian backgrounds in DFT has been made in [137]. In the context of the exceptional field theory, non-Riemannian backgrounds have been found in [134] even before [136]. There, the type IV generalized metrics do not allow either the conventional nor the dual parametrization similar to our solution (5.47).
Again, by introducing the Killing vector \( I \) using the divergence formula (3.128),

\[
I = 4 \eta \partial_-, 
\]

the background (5.50) with this \( I \) solves the GSE. This background can also be regarded as the following solution of DFT:

\[
\mathcal{H} = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & -\eta x^2 & 0 & -\eta x^2 & -\eta x^3 & -\eta x^4 \\
-1 & 0 & 0 & 0 & 0 & \eta x^2 & 0 & \eta x^2 & \eta x^3 & \eta x^4 \\
0 & 0 & 1 & 0 & 0 & 0 & -\eta x^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -\eta x^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -\eta x^2 & 0 & 0 \\
-\eta x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\eta x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\eta x^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\eta x^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\eta x^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\( d = 4 \eta \tilde{x}_- \),

where only the \((x^+, x_-, x^2, x^3, x^4, \tilde{x}_+, \tilde{x}_-, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)\)-components of \( \mathcal{H}_{MN} \) are displayed.

When one of the \((x^2, x^3, x^4)\)-coordinates, say \( x^2 \), is compactified with the period \( x^2 \sim x^2 + \eta^{-1} \), the monodromy matrix is given by

\[
\mathcal{H}_{MN}(x^2 + \eta^{-1}) = [\Omega^T \mathcal{H}(x^2) \Omega]_{MN},
\]

\[
\Omega^M_N \equiv \begin{pmatrix} \delta^m_n & 2 \delta^m_0 \delta^n_2 \\ 0 & \delta^0_n \end{pmatrix} \in O(10, 10; \mathbb{Z}),
\]

and in this sense the compactified background is a T-fold. In terms of the non-geometric \( Q \)-flux, this background has the following components:

\[
Q_{++} = Q_{--} = Q_{-3} = Q_{4-4} = \eta.
\]

### 5.3. A non-geometric background from non-abelian T-duality

Before considering YB-deformations of AdS5 × S5, let us consider another example with a pure NS–NS background, which was found in [138] via a non-abelian T-duality. It takes the form

\[
\begin{align*}
\text{d}s^2 &= -\text{d}t^2 + \frac{(t^4 + y^2) \text{d}x^2 - 2xy \text{d}x \text{d}y + (t^4 + x^2) \text{d}y^2 + t^4 \text{d}z^2}{t^2 (t^4 + x^2 + y^2)} + \text{d}s^2_{\text{fl}}, \\
B_2 &= \frac{(x \text{d}x + y \text{d}y)}{t^4 + x^2 + y^2}, \\
\Phi &= \frac{1}{2} \log \left[ \frac{1}{t^2 (t^4 + x^2 + y^2)} \right],
\end{align*}
\]

where \( \text{d}s^2_{\text{fl}} \) is the flat metric of the six-torus. In terms of the dual parametrization, this background takes a Friedmann–Robertson–Walker-type form,

\[
\begin{align*}
\text{d}s^2_{\text{dual}} &= -\text{d}t^2 + t^{-2} \left( \text{d}x^2 + \text{d}y^2 + \text{d}z^2 \right) + \text{d}s^2_{\text{fl}}, \\
\beta &= (x \partial_x + y \partial_y) \wedge \partial_t, \\
\hat{\phi} &= -\log t.
\end{align*}
\]
Note that this background cannot be represented by a coset or a Lie group itself. Given the re-interpretation of YB deformations in the context of non-abelian $T$-duality and DFT, one may suspect that the background (5.56) can also be obtained by performing a YB deformation. However, the dual background (5.57) is different from the original one (see (2) and (37) with $a(t) = t$ in [138]). Furthermore, the background (5.56) is obtained by performing a non-abelian $T$-duality along the isometries associated with the three-dimensional Lie algebra of Bianchi V type, while the homogeneous YB transformations are realized as non-abelian $T$-dualities along even-dimensional isometries. For this reason the background (5.56) cannot be realized as a YB deformation and is not included in the discussion of [21, 22, 28].

The associated $Q$-flux is constant,

$$Q_{x^5} = Q_{x^3} = -1.$$  \hspace{1cm} (5.58)

If the $x$-direction is compactified as $x \sim x + 1$, the background fields are twisted by an O(10, 10; $\mathbb{Z}$) transformation:

$$H_{MN}(x+1) = [\Omega^T \mathcal{H}(x) \Omega]_{MN}, \quad \Omega^M_N \equiv \left( \begin{array}{cc} \delta^m_0 & 2 \delta^m_{[\eta]} \\ 0 & \delta^m_{\eta} \end{array} \right), \quad d(x+1) = d(x).$$  \hspace{1cm} (5.59)

Thus the background can be interpreted as a $T$-fold. If the $z$-direction is also compactified as $z \sim z + 1$, another twist is realized as

$$H_{MN}(y+1) = [\Omega^T \mathcal{H}(y) \Omega]_{MN}, \quad \Omega^M_N \equiv \left( \begin{array}{cc} \delta^m_0 & 2 \delta^m_{[\eta]} \\ 0 & \delta^m_{\eta} \end{array} \right), \quad d(y+1) = d(y).$$  \hspace{1cm} (5.60)

As stated in [138], this background is not a solution of the usual supergravity. However, using again the divergence formula $I_m = \tilde{D}_m \beta^m$ and introducing the vector field

$$I = -2 \partial_z,$$  \hspace{1cm} (5.61)

we can see that the background (5.56) together with the vector field $I$ satisfies the GSE. Thus, also this background can be regarded as a $T$-fold solution of DFT.

### 5.4. YB-deformed $AdS_5 \times S^5$ backgrounds

In this section, we will show that various YB deformations of the $AdS_5 \times S^5$ background are $T$-folds. We consider here examples associated to the following five classical $r$-matrices:

(a) $r = \frac{1}{2\eta} \left[ \eta_1 (D + M_{+\pm}) \land P_+ + \eta_2 M_{+2} \land P_3 \right],$

(b) $r = \frac{1}{2} P_0 \land D,$

(c) $r = \frac{1}{2} \left[ P_0 \land D + P^\mu \land (M_{0\mu} + M_{1\mu}) \right],$

(d) $r = \frac{1}{2} P_{-} \land (\eta_1 D - \eta_2 M_{-\pm}),$

(e) $r = \frac{1}{2} M_{-\mu} \land P^\mu.$

All the above $r$-matrices except the first are non-unimodular. Note here that the $S^5$ part remains undeformed and only the $AdS_5$ part is deformed. As shown in appendix A in [38] the second and third examples are reduced to the two examples discussed in the previous subsection taking a (modified) Penrose limit.

#### 5.4.1. Non-abelian unimodular $r$-matrix

Let us consider the non-abelian unimodular $r$-matrix (see $R_5$ in table 1 of [18]),

$$r = \frac{1}{2\eta} \left[ \eta_1 (D + M_{+\pm}) \land P_+ + \eta_2 M_{+2} \land P_3 \right].$$  \hspace{1cm} (5.62)
In light-cone coordinates,\(^{25}\)

\[ x^\pm \equiv x^0 \pm x^1 / \sqrt{2}, \quad (5.63) \]

the corresponding YB-deformed background is given by

\[
\begin{align*}
\text{ds}^2 &= -2z^2 \frac{dx^+ \, dx^- + (dx^3)^2 + (dx^2)^2 + dz^2}{z^4} \\
&+ \frac{\frac{\eta_1^2}{z^4} \, x^+ \, dz \, dx^- + \frac{\frac{\eta_2^2}{z^4} \, x^+ \, dz \, dx^-}{\eta_1^2 + \eta_2^2} + \frac{\frac{\eta_1 \eta_2}{z^4} \, x^+ \, dz \, dx^-}{\eta_1^2 + \eta_2^2}}{\eta_1 \eta_2}, \\
B_2 &= -\frac{\eta_1 ^2 \, [2x^2 \, x^+ \, dz \, dx^- + \eta_2 (2x^- \, x^+ \, dz \, dx^-) \, dx^-]}{\frac{1}{z^6} \, [\eta_1 \eta_2 \, x^+ \, dz \, dx^- + \eta_2 (2x^- \, x^+ \, dz \, dx^-) \, dz \, dx^-] + \eta_1 \eta_2 \, x^+ \, dz \, dx^-}, \\
\Phi &= \frac{1}{2} \log \left( \frac{z^8}{\eta_1 \eta_2 \, x^+ \, dz \, dx^- + \eta_2 (2x^- \, x^+ \, dz \, dx^-)} \right), \\
\hat{F}_1 &= \frac{4 \eta_1 \eta_2 \, x^+ \, dz \, dx^-}{\frac{1}{z^5}}, \\
\hat{F}_3 &= -B_2 \wedge F_1 + \frac{4 \eta_1}{z^4} \left( 2x^- \, dz \, dx^- \right) \wedge dx^2 \wedge dx^3 + \frac{4}{z} \, dz \wedge dx^- \left[ \eta_1 (x^3 \, dx^- - x^2 \, dx^3) + \eta_2 (x^- \, dx^+ - x^2 \, dx^2) \right], \\
\hat{F}_5 &= 4 \left( \frac{z^8}{\eta_1 \eta_2 \, x^+ \, dz \, dx^- + \eta_2 (2x^- \, x^+ \, dz \, dx^-)} \omega_{\text{AdS}_5 + \omega_{S^5}} \right), \\
\hat{F}_7 &= -B_2 \wedge F_5, \quad \hat{F}_9 = -\frac{1}{2} B_2 \wedge F_7. \\
\end{align*}
\]

(5.64)

The expressions are greatly simplified in terms of dual fields:

\[
\text{ds}^2_{\text{dual}} = -2 \frac{dx^+ \, dx^- + (dx^3)^2 + (dx^2)^2 + dz^2}{z^4} + ds^2_{S^5}, \quad \phi = 0, \quad (5.65)
\]

\[
\beta = \eta_1 \left( 2x^- \partial_+ + x^2 \partial_2 + x^3 \partial_3 + z \partial_4 \right) \wedge \partial_+ + \eta_2 \left( x^2 \partial_+ + x^- \partial_2 \right) \wedge \partial_3.
\]

(5.65)

The R–R field strengths are given by

\[
\tilde{F} = e^{-\beta} F, \quad F = e^{-\beta} \tilde{F}, \quad \tilde{F} = 4 \left( \omega_{\text{AdS}_5 + \omega_{S^5}} \right).
\]

(5.66)

\(^{25}\) In the following, we use the light-cone convention \(\xi^+ - 2\sqrt{\xi_0 \xi_5} = + \sqrt{|\xi|}.\)
We see that the $\beta$-untwisted $R$–$R$ fields $\tilde{F}$ are invariant under the YB deformation. The non-vanishing $Q$-flux components are

$$Q_{z^+} = \eta_1, \quad Q_{z^-} = 2\eta_1, \quad Q_{z^3} = \eta_1, \quad Q_{z^{23}} = \eta_2.$$

This means that we can understand this background as a $T$-fold if we compactify for example the $x^3$ direction with period $x^3 \sim x^3 + \eta_1^{-1}$. The corresponding monodromy is

$$\mathcal{H}_{MN}(x^3 + \eta_1^{-1}) = [\Omega^T \mathcal{H}(x) \Omega]_{MN}, \Omega^M_N \equiv \left( \begin{array}{c|c} \delta_m^m & 2\delta_m^m \delta^n_+ \\ \hline 0 & \delta^m_m \end{array} \right) \in O(10,10;\mathbb{Z}).$$

The $R$–$R$ fields $F$ are also twisted by the same monodromy,

$$F(x^3 + \eta_1^{-1}) = e^{-\omega^\vee} F(x^3), \quad \omega^m_n = 2\delta^m_m \delta^n_+.$$

5.4.2. $r = \frac{1}{2} P_0 \wedge D$. Let us next consider the classical $r$-matrix [20,106]

$$r = \frac{1}{2} P_0 \wedge D.$$ (5.70)

Since $[P_0, D] \neq 0$, this classical $r$-matrix does not satisfy the unimodularity condition. By introducing polar coordinates

$$x^1 = \rho \sin \theta \cos \phi, \quad x^2 = \rho \sin \theta \sin \phi, \quad x^3 = \rho \cos \theta,$$ (5.71)

the deformed background can be rewritten as [20,106]

$$dx^2 = \frac{z^2}{z^4 - \eta_1^2 (z^2 + \rho^2)} \left[ -(dx^0)^2 + d\rho^2 + dz^2 - \eta_2^2 (d\rho - \rho z^{-1} dz) \frac{d\theta^2 + \sin^2 \theta d\phi^2}{z^4} \right] + dz_5^2,$$

$$B_2 = -\eta \frac{dx^0 \wedge (\rho d\rho + z dz)}{z^4 - \eta_1^2 (z^2 + \rho^2)}, \quad \Phi = \frac{1}{2} \log \frac{z^4}{z^4 - \eta_1^2 (z^2 + \rho^2)}, \quad I = -\eta \partial_0,$$

$$\tilde{F}_1 = 0, \quad \tilde{F}_3 = 4\eta \rho^2 \sin \theta (z d\rho - \rho dz) \wedge d\theta \wedge d\phi,$$

$$\tilde{F}_5 = 4 \left[ \frac{z^4}{z^4 - \eta^2 (z^2 + \rho^2)} \omega_{\text{AdS}_5} + \omega_{\text{S}_5} \right],$$

$$\tilde{F}_7 = 4 \eta dx^0 \wedge (\rho d\rho + z dz) \wedge \omega_{\text{S}_5}, \quad \tilde{F}_9 = 0.$$ (5.72)

This background is not a solution of the usual type IIB supergravity, but of the GSE [12]. By setting $\eta = 0$, this background reduces to the original AdS$_5 \times$ S$^5$.

26 Only the metric and NS–NS two-form were computed in [106].
In the dual parametrization, the dual metric, the $\beta$-field and the dual dilaton are given by

$$d_{\text{dual}}^2 = dz^2 - (dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + dz^2 + d_{S5}^2,$$

$$\beta = \eta \hat{P}_0 \wedge \hat{D} = \eta \hat{\partial}_0 \wedge (x^1 \hat{\partial}_1 + x^2 \hat{\partial}_2 + x^3 \hat{\partial}_3 + z \hat{\partial}_z)$$

$$= \eta \hat{\partial}_0 \wedge (\rho \hat{\partial}_\rho + z \hat{\partial}_z).$$

The Killing vector $I^m$ satisfies the divergence formula,

$$I^0 = -\eta = D_m \beta^0_m.$$  (5.73)

The $Q$-flux has the following non-vanishing components:

$$Q_z^0 = Q_1^0 = Q_2^0 = Q_3^0 = \eta.$$  (5.74)

Thus, when at least one of the $(x^1, x^2, x^3)$ directions is compactified, the background can be interpreted as a $T$-fold. When for example the $x^1$ direction is compactified, the monodromy is given by

$$\mathcal{H}_{MN}(x^1 + \eta^{-1}) = [\Omega^T \mathcal{H}(x^1) \Omega]_{MN}, \quad \Omega^M_N \equiv \begin{pmatrix} \delta^m_n & 2 \delta^m_0 \delta^m_n \end{pmatrix}. \quad (5.75)$$

From (5.72), the R–R potentials are found to be

$$\hat{C}_0 = 0, \quad \hat{C}_2 = \frac{\eta \rho^3 \sin \theta}{z^4} d\theta \wedge d\phi,$$

$$\hat{C}_4 = \frac{\rho^2 \sin \theta}{z^4} dx^0 \wedge d\rho \wedge d\theta \wedge d\phi + \omega_4 - B_2 \wedge \hat{C}_2,$$

$$\hat{C}_6 = -B_2 \wedge \omega_4, \quad \hat{C}_8 = 0,$$

where the four-form $\omega_4$ satisfies $\omega_{S^5} = d\omega_4$. Via the $B$-twist, we obtain

$$F_1 = 0, \quad F_3 = \frac{4 \eta \rho^2 \sin \theta}{z^3} (\rho \, dz - z \, d\rho) \wedge d\theta \wedge d\phi,$$

$$F_5 = 4 \left( \omega_{AdS5} + \omega_{S^5} \right), \quad F_7 = 0, \quad F_9 = 0,$$

$$A_0 = 0, \quad A_2 = \frac{\eta \rho^3 \sin \theta}{z^4} d\theta \wedge d\phi,$$

$$A_4 = \frac{\rho^2 \sin \theta}{z^4} dx^0 \wedge d\rho \wedge d\theta \wedge d\phi + \omega_4, \quad A_6 = 0, \quad A_8 = 0.$$  (5.77)

We can further compute the $\beta$-untwisted fields,

$$\hat{F}_1 = 0, \quad \hat{F}_3 = 0, \quad \hat{F}_5 = 4 \left( \omega_{AdS5} + \omega_{S^5} \right), \quad \hat{F}_7 = 0, \quad \hat{F}_9 = 0,$$

$$\hat{C}_0 = 0, \quad \hat{C}_2 = 0, \quad \hat{C}_4 = \frac{\rho^2 \sin \theta}{z^4} dx^0 \wedge d\rho \wedge d\theta \wedge d\phi + \omega_4, \quad \hat{C}_6 = 0, \quad \hat{C}_8 = 0.$$  (5.78)
Let us now consider the classical

\[ A(x^1 + \eta^{-1}) = e^{-\omega_F} A(x^1), \quad F(x^1 + \eta^{-1}) = e^{-\omega_F} F(x^1), \quad \omega^{mn} = 2 \delta^{[m}_{[n} \delta^{n]}_{m]}, \quad (5.79) \]

5.4.3. A scaling limit of the Drinfeld–Jimbo r-matrix. Let us now consider the classical r-matrix \[20, 139\]

\[ r = \frac{1}{2} \left[ P_0 \wedge D + P^i \wedge (M_0 + M_i) \right], \quad (5.80) \]

which can be obtained as a scaling limit of the classical r-matrix of Drinfeld–Jimbo type \[72, 73\]. Using polar coordinates \((\rho, \theta)\),

\[ (dx^2)^2 + (dx^3)^2 = d\rho^2 + \rho^2 d\theta^2, \quad (5.81) \]

the YB-deformed background, which satisfies the GSE, is given by \[20, 139\]

\[
\begin{align*}
\mathrm{d}s^2 &= \frac{-(dx^0)^2 + dz^2}{z^2 - \eta^2} + \frac{2[(dx^1)^2 + d\rho^2]}{z^4 + \eta^2 \rho^2} + \frac{\rho^2 d\theta^2}{z^2} + d\eta^2_S, \\
B_2 &= \eta \left( -\frac{dx^0 \wedge dz}{z(z^2 - \eta^2)} - \frac{\rho dx^1 \wedge d\rho}{z^4 + \eta^2 \rho^2} \right), \\
\Phi &= \frac{1}{2} \log \left( \frac{\eta^2}{(z^2 - \eta^2)(z^4 + \eta^2 \rho^2)} \right), \quad I = -\eta(4\partial_0 + 2\partial_1), \\
\hat{F}_1 &= -\frac{4\eta^2 \rho}{z^4} d\theta, \\
\hat{F}_3 &= 4\eta \rho \left( -\frac{\rho dx^0 \wedge dz}{z(z^4 - \eta^2 z^2)} + \frac{dx^1 \wedge d\rho}{z^4 + \eta^2 \rho^2} \right) \wedge d\theta, \\
\hat{F}_5 &= 4 \left[ \frac{\eta^2}{(z^2 - \eta^2)(z^4 + \eta^2 \rho^2)} \omega_{AdS_5} + \omega_{S^5} \right], \\
\hat{F}_7 &= 4\eta \left( \frac{dx^0 \wedge dz}{z(z^2 - \eta^2)} + \frac{\rho dx^1 \wedge d\rho}{z^4 + \eta^2 \rho^2} \right) \wedge \omega_{S^5}, \\
\hat{F}_9 &= \frac{4\eta^2 \rho}{z(z^2 - \eta^2)(z^4 + \eta^2 \rho^2)} dx^0 \wedge dx^1 \wedge d\rho \wedge dz \wedge \omega_{S^5}.
\end{align*}
\]

The R–R potentials are given by

\[
\begin{align*}
\hat{C}_0 &= 0, \quad \hat{C}_2 = -\frac{\eta^2 \rho}{z^4} dx^0 \wedge d\theta, \quad \hat{C}_4 = \frac{\rho}{z^4 + \eta^2 \rho^2} dx^0 \wedge dx^1 \wedge d\rho \wedge d\theta + \omega_4, \\
\hat{C}_6 &= -B_2 \wedge \omega_4, \quad \hat{C}_8 = \frac{\eta^2 \rho}{z(z^2 - \eta^2)(z^4 + \eta^2 \rho^2)} dx^0 \wedge dx^1 \wedge d\rho \wedge dz \wedge \omega_4.
\end{align*}
\]

\[(5.83)\]
The corresponding dual fields in the NS–NS sector are given by
\[ ds_{\text{dual}}^2 = -\left(dx^0\right)^2 + \left(dx^1\right)^2 + d\rho^2 + \rho^2 d\theta^2 + d\varphi^2 + d\chi^2, \quad \tilde{\theta} = 0, \]
\[ \beta = \eta \left[ \tilde{P}_0 \wedge \tilde{D} + \tilde{P}^i \wedge \left(M_{0i} + M_{1i}\right) \right] \]
\[ = \eta \left(-x^2 \partial_1 \wedge \partial_2 - x^3 \partial_1 \wedge \partial_3 + z \partial_0 \wedge \partial_2 \right) \]
\[ = \eta \left(-\rho \partial_1 \wedge \partial_\rho + z \partial_0 \wedge \partial_\theta \right), \]
and the Killing vector \( I^m \) again satisfies the divergence formula,
\[ I^0 = -4\eta = \mathbb{D}_m \beta^{0m}, \quad I^1 = -2\eta = \mathbb{D}_m \beta^{1m}. \quad (5.84) \]

Providing the B-twist to the R–R field strengths, we obtain
\[ F_1 = -\frac{4\eta \rho^2}{\varepsilon^2} d\theta, \quad F_3 = \frac{4\eta \rho}{\varepsilon^3} \left( \rho dx^0 \wedge dx^1 \wedge dx^2 \right) \wedge d\theta, \quad (5.85) \]
\[ F_5 = 4 \left( \omega_{\text{AdS}_5} + \omega_{\text{S}^5} \right), \quad F_7 = 0, \quad F_9 = 0, \]
\[ A_0 = 0, \quad A_2 = -\frac{\eta \rho^2}{\varepsilon^2} dx^0 \wedge d\theta, \quad \]
\[ A_4 = \frac{\rho}{\varepsilon^2} dx^0 \wedge dx^1 \wedge d\rho \wedge d\theta + \omega_4, \quad A_6 = 0, \quad A_8 = 0. \]

Furthermore, the \( \beta \)-untwist leads to the following expressions:
\[ F_1 = 0, \quad F_3 = 0, \quad F_5 = 4 \left( \omega_{\text{AdS}_5} + \omega_{\text{S}^5} \right), \quad F_7 = 0, \quad F_9 = 0, \quad (5.86) \]
\[ \mathcal{C}_0 = 0, \quad \mathcal{C}_2 = 0, \quad \mathcal{C}_4 = \frac{\rho}{\varepsilon^2} dx^0 \wedge dx^1 \wedge d\rho \wedge d\theta + \omega_4, \quad \mathcal{C}_6 = 0, \quad \mathcal{C}_8 = 0. \]

These are the same as the undeformed R–R potentials.

The non-zero components of \( Q \)-flux are given by
\[ Q_{20} = \eta, \quad Q_{21} = -\eta, \quad Q_{31} = -\eta. \quad (5.87) \]

When the \( x^2 \)-direction is compactified as \( x^2 \sim x^2 + \eta^{-1} \), this background becomes a \( T \)-fold with monodromy
\[ \mathcal{H}_{MN}(x^2 + \eta^{-1}) = \left[ \Omega^M \mathcal{H}(x^2) \Omega^N \right]_{\text{MN}}, \quad \Omega^M_N \equiv \left( \begin{array}{cc} g^m_n & -2 \delta^m_1 \delta^n_2 \\ 0 & \delta^m_m \delta^n_n \end{array} \right). \]
\[ F(x^2 + \eta^{-1}) = e^{-\omega} F(x^2), \quad \omega_{mn} = -2 \delta^1_1 \delta^0_2. \quad (5.88) \]

5.4.4. \( r = \frac{1}{2\eta} P_- \wedge (\eta_1 D - \eta_2 M_{+-}). \) Let us next consider the non-unimodular \( r \)-matrix,\(^{27}\)
\[ r = \frac{1}{2\eta} P_- \wedge (\eta_1 D - \eta_2 M_{+-}). \quad (5.89) \]

\(^{27}\)This \( r \)-matrix includes the known examples studied in section 4.3 (\( \eta_1 = -\eta_2 = -\eta \)) and 4.4 (\( \eta_1 = -\eta, \eta_2 = 0 \)) of [20] as special cases.
The R–R potentials are given by
\[
x^\pm \equiv \frac{x^0 \pm x^1}{\sqrt{2}}, \quad (dx^2)^2 + (dx^3)^2 = d\rho^2 + \rho^2 d\theta^2. \tag{5.90}
\]
The YB-deformed background is given by
\[
dx^2 = \frac{-2z^2 dx^+ dx^-}{z^4 - (\eta_1 + \eta_2)^2 (x^+)^2} + \frac{d\rho^2 + \rho^2 d\theta^2 + dz^2}{z^2} + \eta_1 dx^+ \frac{2 x^+ (\eta_1 + \eta_2) (z dz + \rho d\rho) - \eta_1 (z^2 + \rho^2) dx^+}{z^2 [z^4 - (\eta_1 + \eta_2)^2 (x^+)^2]} + d\Sigma_5^2,
\]
\[
B_2 = -\left[ \eta_1 (dx^+ \wedge (\rho d\rho + z dz) - x^+ dx^+ \wedge dx^-) - \eta_2 x^+ dx^+ \wedge dx^- \right],
\]
\[
\Phi = \frac{1}{2} \log \left[ \frac{z^4}{z^4 - (\eta_1 + \eta_2)^2 (x^+)^2} \right], \quad I = -(\eta_1 - \eta_2) \partial_+,
\]
\[
\tilde{F}_1 = 0,
\]
\[
\tilde{F}_3 = -4 \rho \left[ \eta_1 (dx^+ \wedge (z d\rho - \rho dz) - x^+ dz \wedge d\rho) - \eta_2 x^+ dz \wedge d\rho \right] \wedge d\theta, \tag{5.91}
\]
\[
\tilde{F}_5 = 4 \left[ \frac{z^4}{z^4 - (\eta_1 + \eta_2)^2 (x^+)^2} \omega_{\Sigma 5} + \omega_{\Sigma 5} \right].
\]
\[
\tilde{F}_7 = 4 \left[ \eta_1 (dx^+ \wedge (\rho d\rho + z dz) - x^+ dx^+ \wedge dx^-) - \eta_2 x^+ dx^+ \wedge dx^- \right] \wedge \omega_{\Sigma 5},
\]
\[
\tilde{F}_9 = 0.
\]

The R–R potentials are given by
\[
\tilde{C}_0 = 0, \quad \tilde{C}_2 = \rho \left[ \eta_1 \rho dx^+ - (\eta_1 + \eta_2) x^+ d\rho \right] \wedge d\theta,
\]
\[
\tilde{C}_4 = \frac{\rho dx^+ \wedge [z^3 dx^- - \eta_1 (\eta_1 + \eta_2) x^+ dz] \wedge d\rho \wedge d\theta}{z^2 [z^4 - (\eta_1 + \eta_2)^2 (x^+)^2]} + \omega_4, \tag{5.92}
\]
\[
\tilde{C}_6 = -B_2 \wedge \omega_4, \quad \tilde{C}_8 = 0.
\]

The dual fields take the form
\[
dx^2_{\text{dual}} = -\frac{2 dx^+ dx^- + d\rho^2 + \rho^2 d\theta^2 + dz^2}{z^2} + d\Sigma_5^2, \quad \tilde{\phi} = 0,
\]
\[
\beta = P_\perp \wedge (\eta_1 \hat{D} + \eta_2 M_{++}) + \eta_1 \partial_+ \wedge (x^+ \partial_+ + \rho \partial_\rho + z \partial_z) + \eta_2 x^+ \partial_- \wedge \partial_+, \tag{5.93}
\]
\[
= \eta_1 \partial_+ \wedge (x^+ \partial_+ + x^2 \partial_2 + x^3 \partial_3 + z \partial_z) + \eta_2 x^+ \partial_- \wedge \partial_+,
\]
and the $Q$-flux has the following non-vanishing components:
\[
Q_z^{-+} = Q_+^{-+} = Q_2^{-+} = Q_3^{-+} = \eta_1, \quad Q_+^{-+} = \eta_2. \tag{5.94}
\]
In a similar manner as in the previous examples, by compactifying one of the \(x^1, x^2, \) and \(x^3\) directions with a certain period, this background can also be regarded as a \(T\)-fold. If we make for example the identification \(x^3 \sim x^3 + \eta_1 \), the associated monodromy becomes

\[
\mathcal{H}_{MN}(x^3 + \eta_1^{-1}) = \left[ \Omega^I \mathcal{H}^I(x^3) \Omega \right]_{MN}, \quad \Omega^M_N \equiv \begin{pmatrix} \delta^m_n & 2 \delta^m_0 \delta^n_1 \\ 0 & \delta^0_m \delta^n_0 \end{pmatrix},
\]

(5.95)

A solution of the generalized type IIA supergravity equations. In the background (5.82), by performing a \(T\)-duality along the \(x^1\)-direction (see [85] for the duality transformation rule), we obtain the following solution of the generalized type IIA EOM:

\[
ds^2 = - \frac{(dx^0)^2 + dz^2}{z^2 - \eta^2} + z^2 (dx^1)^2 + \left( \frac{d\rho + \eta \mu dx^1}{z^2} \right)^2 + d\tilde{s}^2,
\]

\[
B_2 = - \frac{\eta dx^0 \wedge dz}{z (z^2 - \eta^2)} \Phi = - 2 \eta x^1 - \frac{1}{2} \log \left( \frac{z^2 - \eta^2}{z^2} \right), \quad I = - 4 \eta \partial_0,
\]

\[
\tilde{F}_2 = \frac{4 \eta e^{2\eta x^1} \rho (d\rho + \eta \mu dx^1) \wedge d\theta}{z^2}, \quad \tilde{F}_4 = \frac{4 \eta e^{2\eta x^1} \rho (d\rho + \eta \mu dx^1) \wedge d\theta \wedge dz}{z^2 (z^2 - \eta^2)},
\]

\[
\tilde{F}_6 = - 4 \eta e^{2\eta x^1} dx^1 \wedge \omega_{S5}, \quad \tilde{F}_8 = \frac{4 \eta e^{2\eta x^1} dx^0 \wedge dx^1 \wedge dz \wedge \omega_{S5}}{z (z^2 - \eta^2)}.
\]

(5.96)

Here the R–R potentials are given by

\[
\hat{C}_1 = 0, \quad \hat{C}_3 = e^{2\eta x^1} \frac{\rho dx^0 \wedge (d\rho + \eta \mu dx^1) \wedge d\theta}{z^2},
\]

\[
\hat{C}_5 = e^{2\eta x^1} dx^1 \wedge \omega_4, \quad \hat{C}_7 = - e^{2\eta x^1} \frac{\eta dz \wedge dx^0 \wedge dx^1 \wedge \omega_4}{z (z^2 - \eta^2)}.
\]

(5.97)

This background cannot be regarded as a \(T\)-fold, but it is the first example of a solution for the generalized type IIA supergravity equations.

5.4.5. \( r = \frac{1}{2} M_{\nu \mu} \wedge P^\nu \). Our final example is associated to the \(r\)-matrix [20]

\[
r = \frac{1}{2} M_{\nu \mu} \wedge P^\nu.
\]

(5.98)

This \(r\)-matrix is called light-like \(\kappa\)-Poincaré. Again, by introducing the coordinates

\[
x^\pm \equiv \frac{x^0 \pm x^1}{\sqrt{2}}, \quad (dx^2)^2 + (dx^3)^2 = d\rho^2 + \rho^2 d\theta^2,
\]

(5.99)
the YB-deformed background is given by (see section 4.5 of [20])

\[
\begin{align*}
\text{ds}^2 &= \frac{z^2(-2 x^+ dx^- + dz^2)}{z^4 - (\eta x^+)^2} - \eta^2 \frac{\rho^2 (dx^+)^2 - 2 x^+ \rho \, dx^+ \, d\rho + (x^+)^2 \, dz^2}{z^2(z^4 - (\eta x^+)^2)} \\
&\quad + \frac{d\rho^2 + \rho^2 \, d\theta^2}{z^2}, \\
B_2 &= \frac{\eta \, dx^+ \wedge (x^+ \, dx^- - \rho \, d\rho)}{z^4 - (\eta x^+)^2}, \quad \Phi = \frac{1}{2} \log \left[ \frac{z^4}{z^4 - (\eta x^+)^2} \right], \quad I^- = 3 \eta,
\end{align*}
\]

\[
\begin{align*}
\hat{F}_1 &= 0, \quad \hat{F}_3 = \frac{4 \eta \rho}{z^4} (\rho \, dx^+ - x^+ \, d\rho) \wedge d\theta \wedge dz, \\
\hat{F}_5 &= 4 \left[ \frac{z^4}{z^4 - (\eta x^+)^2} \omega_{\text{AdS}_5} + \omega_{S^5} \right], \\
\hat{F}_7 &= -\frac{4 \eta}{\rho} \, dx^+ \wedge (x^+ \, dx^- - \rho \, d\rho) \wedge \omega_{S^5}, \quad \hat{F}_9 = 0.
\end{align*}
\]

(5.100)

The R–R potentials are found to be

\[
\begin{align*}
\hat{C}_0 &= 0, \quad \hat{C}_2 = \frac{\eta \rho}{z^4} (\rho \, dx^+ - x^+ \, d\rho) \wedge d\theta, \\
\hat{C}_4 &= \frac{\rho}{z^4 - (\eta x^+)^2} \, dx^+ \wedge dx^- \wedge d\rho \wedge d\theta + \omega_4, \quad \hat{C}_6 = -B_2 \wedge \omega_4, \quad \hat{C}_8 = 0.
\end{align*}
\]

(5.101)

The corresponding dual fields are given by

\[
\begin{align*}
\text{ds}_{\text{dual}}^2 &= -\frac{2 \, dx^+ \, dx^- + d\rho^2 + \rho^2 \, d\theta^2 + dz^2}{z^4} + dz^2, \quad \tilde{\phi} = 0, \\
\beta &= \eta \, M_{-\mu} \wedge \tilde{\rho}^m = \eta \, \partial_- \wedge (x^+ \, \partial_+ + \rho \, \partial_\rho) \\
&= \eta \, \partial_- \wedge (x^+ \, \partial_+ + x^2 \, \partial_2 + x^3 \, \partial_3),
\end{align*}
\]

and it is easy to check that the divergence formula is satisfied,

\[
I^- = 3 \eta = D_m \beta^{-m}.
\]

(5.102)

We can calculate the other types of R–R fields:

\[
\begin{align*}
F_1 &= 0, \quad F_3 = -\frac{4 \eta \rho}{z^4} (\rho \, dx^+ - x^+ \, d\rho) \wedge d\theta \wedge dz, \\
F_5 &= 4 \left( \omega_{\text{AdS}_5} + \omega_{S^5} \right), \quad F_7 = 0, \quad F_9 = 0, \\
A_0 &= 0, \quad A_2 = \frac{\eta \rho}{z^4} (\rho \, dx^+ - x^+ \, d\rho) \wedge d\theta, \\
A_4 &= \frac{\rho}{z^4} \, dx^+ \wedge dx^- \wedge d\rho \wedge d\theta + \omega_4, \quad A_6 = 0, \quad A_8 = 0.
\end{align*}
\]

(5.103)
\[ F_1 = 0, \quad F_3 = 0, \quad F_5 = 4 (\omega_{\text{AdS}_5} + \omega_{S^5}), \quad F_7 = 0, \quad F_9 = 0, \]
\[ \check{C}_0 = 0, \quad \check{C}_2 = 0, \quad \check{C}_4 = \frac{\rho}{2} \left( d\rho \wedge d\theta \wedge + \omega_{4} \right), \quad \check{C}_6 = 0, \quad \check{C}_8 = 0. \] (5.104)

The $\beta$-twisted fields are again invariant under the YB deformation. The non-geometric $Q$-flux has the non-vanishing components
\[ Q_{-^+} = Q_{2^{-2}} = Q_{3^{-3}} = \eta, \] (5.105)
and again by compactifying one of the $x^1, x^2,$ and $x^3$ directions, this background becomes a $T$-fold. If we compactify the $x^3$-direction as $x^3 \sim x^3 + \eta^{-1}$, the associated monodromy becomes
\[ H_{MN}(x^3 + \eta^{-1}) = \left[ \Omega_{\text{YB}}(x^3) \Omega \right]_{MN}, \quad \Omega^M_N \equiv \begin{pmatrix} \delta^m_n & 2 \delta^{|m|} n \omega^{|n|} \\ 0 & \delta^{|n|} m \end{pmatrix}, \] (5.106)

6. Killing spinors of the YB deformation

We have discussed in the previous sections how to associate an integrable system to $\text{AdS}_5 \times S^5$ and how the deformations of the integrable system translate into deformations of the ten-dimensional background. The starting system is remarkable for another reason, though: it has $\mathcal{N} = 4$ supersymmetry (in four dimensions). It is natural to wonder how this supersymmetric structure is affected by the integrable deformations discussed so far. One first, crucial, observation is that, just like in the case of isometries, if $T_i$ is a generator of the superalgebra $\mathfrak{g}$, it is preserved by the $R$-matrix of equation (3.8) if $R$ is equivariant with respect to the (adjoint) action of $T_i$ on $\mathfrak{g}$, i.e.
\[ [T,R(X)] = R([T,X]), \quad \forall \ x \in \mathfrak{g}. \] (6.1)

In the spirit of a geometrical interpretation, we want to describe the preserved supersymmetries in terms of Killing spinors of the deformed backgrounds. Solving the Killing spinor equation is however in general a difficult task. It is therefore much more convenient to have a frame-independent formalism allowing us to write an explicit formula for the spinors preserved by a given deformation. In [66, 67], such a formula was written in terms of the so-called non-commutative parameter, denoted by $\Theta^{mn}$. This is the same object defined via the Seiberg–Witten (SW) map in (3.103) [100, 104] and it coincides, up to a conventional sign, with the $\beta$ field $\beta^{mn}$.
\[ \Theta^{mn} \equiv -\beta^{mn}. \] (6.2)

In our context it encodes all the information about the integrable deformation. For consistency of our notation throughout the review, we will use $\beta$.

Such an explicit formalism is useful in a number of contexts. The bilinear formalism of Killing spinors is for example useful for the classification of supergravity solutions. It is necessary to solve the Killing spinor equations for supersymmetric localization calculations [140]. One needs to solve the Killing spinor equations in the construction of supersymmetric gauge
theories realized on the D-branes embedded in deformed supergravity backgrounds (see e.g. [141–143]).

Let us take the example of the $\Omega$-deformation of flat spacetime constructed in [130]. From the perspective of integrable deformations, it corresponds to a TsT transformations, where one of the $U(1)$ isometries acts freely. Placing a probe brane in different configurations, one can construct either a non-Lorentz-invariant gauge theory or a massive gauge theory that can be studied from the viewpoint of string theory. Having found these $\Omega$-deformed gauge theories as probe branes in a TsT-deformed spacetime, one may wonder if the near-horizon limit can be understood in terms of a similar deformation of the $AdS_5 \times S^5$ background.29

6.1. Killing spinors and T-dualities

We have seen that YB deformations are closely related to $T$-duality. As a first step we will see how $T$-duality transforms the Killing spinors.

Given an isometry generated by a Killing vector $\hat{T}_i$, a $T$-duality in this direction preserves only the Killing spinors that are covariantly constant with respect to the Kosmann–Lie (KL) derivative [36, 145, 146]

$$L_{\hat{T}_i} \epsilon \equiv \hat{T}^m \nabla_m \epsilon + \frac{1}{4} (\nabla \hat{T}_i)_{mn} \Gamma^m \epsilon.$$  (6.3)

This result can be extended to generalized $T$-duality using the same argument of [147] for non-abelian $T$-duality. There always exists a frame where the isometry $\hat{T}_i$ is represented as $\partial_z$. Then the metric is parametrized by

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + e^{2\phi}(dz + A_1(x))^2,$$  (6.4)

where $A_1$ is a one-form. In this metric, the KL derivative along $\partial_z$ is

$$L_{\partial_z} \epsilon = \partial_z \epsilon.$$  (6.5)

Consider the generalized type IIA background:

$$ds_{IIA}^2 = g_{\mu
u}(x)dx^\mu dx^\nu + e^{2\phi}(dz + A_1(x))^2,$$

$$B_2 = B + B_1 \wedge dz,$$

$$\tilde{f}_0 = m,$$

$$\tilde{f}_2 = g_2 + g_1 \wedge (dz + A_1(x)),$$

$$\tilde{f}_4 = g_4 + g_3 \wedge (dz + A_1(x)),$$

$$\Phi = -az + \varphi + \frac{1}{2} C,$$

$$I = \hat{a} \partial_z,$$

where $\hat{a}$ is a constant, and all the background fields $A_1, B_1, B, g_2, \varphi, C$ are functions of $x^\mu$ only.

The dilaton has a linear dependence in $z$ and we cannot use the standard Buscher rule along $\partial_z$, but we can still perform a generalized $T$-duality leading to the following generalized type

29 See for example [144] for related work, the Killing spinor analysis is also included.
IIB background:\footnote{30}
\[
d s_{\text{IIB}}^2 = g_{\mu \nu} dx^\mu dx^\nu + e^{-2C(x)}(dz + B_1(x))^2,
\]
\[
\tilde{B}_2 = B + A_1 \wedge (dz + B_1),
\]
\[
f_1 = m (dz + B_1(x)) - g_1,
\]
\[
f_3 = g_3 \wedge (dz + B_1(x)) - g_3,
\]
\[
f_5 = (1 + \star_{10})(g_4 \wedge (dz + B_1(x))),
\]
\[
\tilde{\Phi} = -\hat{a}z + \phi - \frac{1}{2}C,
\]
\[
\tilde{I} = a \partial_z.
\]
(6.7)

The background fluxes $f_5$ are usually rewritten in terms of generalized fluxes as
\[
\hat{F}_k = \begin{cases} e^{+C(x)/2} f_k & \text{type IIA}, \\ e^{-C(x)/2} f_k & \text{type IIB}. \end{cases}
\]
(6.8)

Now we need to compare the supersymmetry variations in the two backgrounds above. The supersymmetry variations of dilatini and gravitini for type IIA generalized supergravity are written as\cite{13}
\[
\delta \lambda_{\text{IIA}} = \frac{1}{2} \partial_\mu \Phi \Gamma^\mu \varepsilon + \frac{1}{2} (I^m B_{mn} + I_\mu \sigma_3) \Gamma^\mu \varepsilon - \frac{1}{8} H_3 \sigma_3 \varepsilon
\]
\[
+ \frac{1}{8} \left[ 5 \hat{F}_0 \sigma_1 + 3 \hat{F}_2 (i \sigma_2) + \hat{F}_4 \sigma_1 \right] \varepsilon,
\]
(6.9)
\[
\delta \Psi_{\text{IIA}} = \nabla_\mu \varepsilon - \frac{1}{8} H_{3mn} \Gamma^{mp} \sigma_3 \varepsilon + \frac{1}{8} \left[ \hat{F}_0 \sigma_1 + \hat{F}_2 (i \sigma_2) + \hat{F}_4 \sigma_1 \right] \Gamma_\mu \varepsilon,
\]
whereas for type IIB generalized supergravity
\[
\delta \lambda_{\text{IIB}} = \frac{1}{2} \partial_\mu \tilde{\Phi} \Gamma^\mu \tilde{\varepsilon} + \frac{1}{2} (I^m \tilde{B}_{mn} + \tilde{I}_\mu \sigma_3) \Gamma^\mu \tilde{\varepsilon} - \frac{1}{8} \tilde{H}_3 \sigma_3 \tilde{\varepsilon}
\]
\[
+ \frac{1}{8} \left[ \hat{F}_1 (i \sigma_2) + \frac{1}{2} \hat{F}_3 \sigma_1 \right] \tilde{\varepsilon},
\]
(6.10)
\[
\delta \Psi_{\text{IIB}} = \nabla_\mu \tilde{\varepsilon} - \frac{1}{8} H_{3mn} \Gamma^{mp} \sigma_3 \tilde{\varepsilon} - \frac{1}{8} \left[ \hat{F}_1 (i \sigma_2) + \hat{F}_3 (i \sigma_2) + \frac{1}{2} \hat{F}_5 \sigma_1 \right] \Gamma_\mu \tilde{\varepsilon},
\]
where the background fluxes without explicit indices are contracted with curved Gamma matrices and divided by the symmetry factors as in (2.43). Here we define the Killing spinors as the doublet
\[
\varepsilon \equiv \begin{pmatrix} \varepsilon^+ \\ \varepsilon^- \end{pmatrix}.
\]
(6.11)

Inserting the background data (6.6) and (6.7) into the variations leads to the relations
\[
\delta \Psi_{\text{IIB} \mu^+} = -\Gamma_\varepsilon \delta \Psi_{\text{IIA} \mu^+}, \quad \delta \Psi_{\text{IIB} \mu^-} = \delta \Psi_{\text{IIA} \mu^-},
\]
(6.12)
provided that
\[ \partial_z \varepsilon_\pm = 0 \quad (6.13) \]
and that
\[ \tilde{\varepsilon}_+ = -\Gamma \varepsilon_+, \quad \tilde{\varepsilon}_- = \varepsilon_- \quad (6.14) \]
where \( \Gamma \) and \( \Gamma^i \) are flat Gamma matrices in the initial background.

The conclusion is that a type IIA Killing spinor is mapped to a Killing spinor \( \tilde{\varepsilon} \) in type IIB if and only if the Killing spinor vanishes under the KL derivative along the isometry direction \( z \). In the case of a TsT transformation the Killing spinor has to be independent of both isometry directions in order to be preserved (and mapped to the final background).

6.2. Exponential factor from TsT transformation

We start with a concrete example of a TsT transformation. The simplest non-trivial configuration is obtained as a deformation on the ten-dimensional flat spacetime of Lunin–Maldacena type. The associated classical \( r \)-matrix is similar to the one in equation (4.9). First we show how to construct the projector matrix using the Kosmann Lie derivative. For flat space in polar coordinates,
\[ ds^2 = -(dx^0)^2 + (dx^1)^2 + \sum_{i=1}^3 (d\rho_i^2 + \rho_i^2 d\phi_i^2) + (dx^8)^2 + (dx^9)^2. \quad (6.15) \]
The Killing spinor equation is
\[ \nabla_m \varepsilon = 0, \quad (6.16) \]
which admits the solution
\[ \varepsilon = \prod_{i=1}^3 \exp \left[ \frac{\phi_i}{2} \Gamma_{i\phi_i} \right] (\eta_0 + i\chi_0), \quad (6.17) \]
where \( \eta_0, \chi_0 \) are constant Majorana–Weyl spinors and all the Gamma matrices are flat. To perform the deformation as in [76], we introduce the following adapted angles:
\[ \phi_1 = \psi - \varphi_1, \quad \phi_2 = \psi + \varphi_1 + \varphi_2, \quad \text{and} \quad \phi_3 = \psi - \varphi_2. \quad (6.18) \]
We now consider the TsT transformation in the angles \((\varphi_1, \varphi_2)\). First, note that the Killing spinor is rewritten as
\[ \varepsilon = e^{\frac{\varphi_1}{2}(\Gamma_{\rho_1\varphi_1} + \Gamma_{\rho_2\varphi_2} + \Gamma_{\rho_3\varphi_3})} e^{\frac{\varphi_2}{2}(\Gamma_{\rho_2\varphi_2} - \Gamma_{\rho_3\varphi_3})} e^{\frac{\varphi_3}{2}(\Gamma_{\rho_3\varphi_3} - \Gamma_{\rho_1\varphi_1})} \eta_0. \quad (6.19) \]
When performing a \( T \)-duality along \( \partial_{\varphi_1} \), in order to preserve supersymmetry we demand that the KL derivative along \( \partial_{\varphi_1} \) vanishes:
\[ \mathcal{L}_{\partial_{\varphi_1}} \varepsilon = \partial_{\varphi_1} \varepsilon = 0. \quad (6.20) \]
This is equivalent to acting with the following projection on the constant spinors \( \eta_0, \chi_0 \):
\[ \Pi_{\varphi_1} = \frac{1}{2} (1 - \Gamma_{\rho_1\varphi_1\rho_2\varphi_2}). \quad (6.21) \]
which removes the \( \varphi_1 \)-dependence from the Killing spinor. Now, we shift the angle \( \varphi_2 \) by \( +\eta \tilde{\varphi}_1 \), where \( \eta \) denotes the deformation parameter and \( \tilde{\varphi}_1 \) is the \( T \)-dual of \( \varphi_1 \), and we \( T \)-dualize on \( \tilde{\varphi}_1 \). Once more we demand the KL derivative in the direction \( \partial_{\tilde{\varphi}_1} \) to vanish. This is equivalent to asking for the derivative in the direction \( \partial_{\varphi_2} \) to vanish and in turn this is the same as inserting the projector

\[
\Pi^{\varphi_2} = \frac{1}{2}(1 - \Gamma_{\varphi_2} \partial_{\varphi_2} \psi_0).
\]

As remarked earlier, the KL derivative along \( \partial_{\tilde{\varphi}_1} \) acts equivalently to that along \( \partial_{\varphi_2} \). In total, the two projectors (6.21), (6.22) preserve one quarter of the supersymmetry.

Now we can determine the explicit form of the Killing spinors in the deformed background. The TsT transformation of interest leads to

\[
ds^2 = -(dx^0)^2 + (dx^1)^2 + \sum_{i=1}^3 (dx_i)^2 + (dx^9)^2 + \frac{1}{\Delta^2} \left[ \rho_1^2 (d\psi - d\varphi_1)^2 + \rho_2^2 (d\psi + d\varphi_1 + d\varphi_2)^2 + \rho_3^2 (d\psi - d\varphi_2)^2 + 9\lambda^2 \rho_1^2 \rho_2^2 \rho_3^2 d\psi^2 \right],
\]

\[
e^\Phi = \Delta^2,
\]

\[
B_2 = \frac{\Delta^2 - \eta \Delta}{\eta \Delta} d\varphi_1 \wedge d\varphi_2 - \frac{\Delta^2 - 1}{\eta \Delta} (d\varphi_1 - d\varphi_2) \wedge d\psi + \frac{3\eta}{\Delta} (\rho_1^2 \rho_2 d\varphi_1 - \rho_2^2 \rho_3 d\varphi_2) \wedge d\psi,
\]

\[
\Delta^2 = 1 + \eta^2 (\rho_1^2 \rho_2^2 + \rho_2^2 \rho_3^2 + \rho_3^2 \rho_1^2),
\]

and using the general formula in equation (6.29) we obtain the following Killing spinors:

\[
\bar{\varepsilon}_+ = \frac{1 + \eta (\rho_1 \rho_2 \Gamma_{\varphi_1 \varphi_2} + \rho_2 \rho_3 \Gamma_{\varphi_2 \varphi_3} + \rho_3 \rho_1 \Gamma_{\varphi_3 \varphi_1})}{1 + \eta^2 (\rho_1^2 \rho_2^2 + \rho_3^2 \rho_1^2 + \rho_2^2 \rho_3^2)} \varepsilon_+,
\]

\[
\bar{\varepsilon}_- = \varepsilon_-,
\]

where

\[
\varepsilon_+ + i\varepsilon_- = e^{i \Gamma_{\rho_1 \rho_2} + \Gamma_{\rho_2 \rho_3} + \Gamma_{\rho_3 \rho_1}} \Pi^{\varphi_2} \Pi^{\varphi_2} (\eta_+ + i\eta_-).
\]

In order to generalize this result to any YB-deformed background it is convenient to recast it in a form that depends explicitly on the parameters of the deformation encoded by the bivector \( \beta^{mn} \). Applying the SW map (3.103) to the background we find

\[
\beta = -\eta (\partial_{\varphi_1} \wedge \partial_{\varphi_2} + \partial_{\varphi_2} \wedge \partial_{\varphi_3} + \partial_{\varphi_3} \wedge \partial_{\varphi_1}).
\]

The deformed Killing spinor \( \bar{\varepsilon}_+ \) in (6.24) can be rewritten using this bi-vector as

\[
\bar{\varepsilon}_+ = \frac{1 - \frac{1}{2} \beta^{mn} \Gamma_{mn}}{1 + \frac{1}{2} \beta^{nm} \beta_{nm}} \varepsilon_+ = e^{\frac{1}{2} \beta^{nm} \Gamma_{nm} \varepsilon_+},
\]
where the Gamma matrices are curved in terms of the undeformed metric while the normalization factor \( \omega(\beta) \) satisfies

\[
\tan \left( \omega(\beta) \sqrt{\frac{1}{2} \beta_{mn} \beta_{mn}} \right) = \sqrt{\frac{1}{2} \beta_{mn} \beta_{mn}}. \quad (6.28)
\]

This expression is reminiscent of a quantum R-matrix in the spinor representation. This connection has not yet been explored in the literature. In [66] other concrete examples of TsT deformations of flat space and \( \text{AdS}_5 \times S^5 \) background were studied and it was found that the form of the preserved Killing spinors in equation (6.27) is generic.

To sum up, we conjecture the following structure of YB deformed Killing spinors expressed by the \( \beta \)-field only:

\[
\tilde{\varepsilon}_+ = e^{-\frac{1}{2} \omega(\beta) \Gamma_{mn} \Pi^{\text{TsT}} \varepsilon_+}, \quad \tilde{\varepsilon}_- = \Pi^{\text{TsT}} \varepsilon_-, \quad (6.29)
\]

where \( \varepsilon_\pm \) are undeformed Killing spinors, and \( \Pi^{\text{TsT}} \) is the projector derived from the KL derivatives. What remains to be done in order to find a complete formula that only depends on \( \beta_{mn} \) alone is to relate the projector \( \Pi^{\text{TsT}} \) to the bi-vector.

### 6.3. Supersymmetry projector formula

We have seen that the projector matrix is needed for preserving the Killing spinors under TsT transformations. In this section, we show how to find such a projector for non-TsT YB deformations, using only the \( \beta \)-field.

Let us assume that the initial undeformed background preserves some supersymmetry. This means that

\[
\delta \Psi_\mu = \nabla_\mu \varepsilon + \frac{1}{8} \mathcal{F} \Gamma_\mu \varepsilon = 0, \quad (6.30)
\]

where \( \mathcal{F} \) is the analog of the Ramond–Ramond flux bispinor in the initial background,

\[
\mathcal{F} = - \mathcal{F}_1 \otimes (i\sigma_2) - \mathcal{F}_3 \otimes \sigma_1 - \frac{1}{2} \mathcal{F}_5 \otimes (i\sigma_2). \quad (6.31)
\]

As discussed before, for TsT transformations, the preserved Killing spinors have to be independent of both isometry directions. So we ask the KL derivatives of the Killing spinor to vanish along all the directions \( T_i \) that are included in the classical \( r \)-matrix:

\[
\mathcal{L}_{T_i} \varepsilon = \hat{T}_i^m \nabla_m \varepsilon + \frac{1}{4} (\nabla \hat{T}_i)_{mn} \Gamma^{mn} \varepsilon = 0. \quad (6.32)
\]

Multiplying (6.30) by \( \hat{T}_i^m \), we get

\[
\hat{T}_i^m \delta \Psi_m = \hat{T}_i^m \nabla_m \varepsilon + \frac{1}{8} \hat{T}_i^m \mathcal{F} \Gamma_m \varepsilon = 0. \quad (6.33)
\]

Combining equations (6.32) and (6.33), we obtain

\[
\left[ \frac{1}{8} \hat{T}_i^m \mathcal{F} \Gamma_m - \frac{1}{4} (\nabla \hat{T}_i)_{mn} \Gamma^{mn} \right] \varepsilon = 0. \quad (6.34)
\]
and multiplying by $r^i T^j_l \Gamma_i$, one finds
\[
\left[ \frac{1}{8} r^i T^j_l \Gamma_i T^m_n \partial_m \Gamma_n - \frac{1}{4} r^i T^j_l (\nabla T)_m \Gamma^m \right] \varepsilon = 0. \tag{6.35}
\]

In this step, there is one caveat. If $\hat{T}_j$ is a light-like Killing vector, the matrix $r^i T^j_l \Gamma_i$ might not be invertible. In this case, equation (6.35) gives a necessary but not sufficient condition for the preservation of supersymmetry.

The above equation (6.35) can be further rewritten with respect to the bi-vector of the bi-Killing form. Recall that the $\beta$-field can be written in the basis of Killing vectors as
\[
\beta_{mn} = \eta r^i \hat{T}^m_i \hat{T}^n_j, \tag{6.36}
\]
and its covariant derivative
\[
\nabla_m \beta_{nl} = 2 \eta r^i \hat{T}^m_i [\nabla_m \hat{T}^n_j] \tag{6.37}
\]
in the form
\[
[\beta_{mn} \partial_m + \nabla_m \beta_{nl} \partial^m - 4 \nabla_m \beta_{n} \partial^m] \varepsilon = 0. \tag{6.38}
\]

This result implies that given any classical $r$-matrix (or $\beta$-field), we can determine a projection matrix $\Pi^{\text{TST}}$ to preserve the Killing spinor via
\[
[\beta_{mn} \partial_m + \nabla_m \beta_{nl} \partial^m - 4 \nabla_m \beta_{n} \partial^m] \Pi^{\text{TST}} = 0. \tag{6.39}
\]

In the case of the Lunin–Maldacena-like deformation of flat space in the previous section, due to the unimodularity of the classical $r$-matrix, only the second term in (6.38) contributes to the left-hand side. It is evaluated as
\[
[r^i \Gamma_{\rho_1 \phi_1} (\Gamma_{\rho_2 \phi_2} - \Gamma_{\rho_3 \phi_3}) - r^j \Gamma_{\rho_2 \phi_2} (\Gamma_{\rho_1 \phi_1} - \Gamma_{\rho_3 \phi_3}) + r^k \Gamma_{\rho_3 \phi_3} (\Gamma_{\rho_1 \phi_1} - \Gamma_{\rho_2 \phi_2})] \varepsilon = 0. \tag{6.40}
\]
From this we deduce two independent conditions,
\[
(\Gamma_{\rho_1 \phi_1} - \Gamma_{\rho_2 \phi_2}) \varepsilon = (\Gamma_{\rho_2 \phi_2} - \Gamma_{\rho_3 \phi_3}) \varepsilon = 0, \tag{6.41}
\]
which lead to the same projectors as (6.21) and (6.22):
\[
\Pi_1 = \frac{1}{2} (1 - \Gamma_{\rho_1 \phi_1 \rho_2 \phi_2}), \quad \Pi_2 = \frac{1}{2} (1 - \Gamma_{\rho_2 \phi_2 \rho_3 \phi_3}). \tag{6.42}
\]

6.4. Examples

In this section we corroborate the conjecture of the validity the formula for the Killing spinor in equation (6.29) and with the projector in equation (6.38) also for generic YB deformations not obtained as TsT transformations by reviewing two of the examples treated in [67]. One is supersymmetric and the other non-supersymmetric.

Since all the deformations here act on the AdS$_5$ part, we focus only on the Killing spinors derived from AdS$_5$ spacetime. Given the Poincaré coordinate system for AdS$_5$ space (2.64), the Killing spinor of complex form can be readily solved:
\[
\varepsilon_{\text{AdS}_5} = \left\{ \sqrt{z} + \frac{1}{\sqrt{z}} x^\mu \Gamma_\mu z \right\} \left[ \frac{1 - i \gamma \Gamma_3}{2} + \frac{1 + i \gamma \Gamma_3}{2} \right] (\psi_0 + i \chi_0). \tag{6.43}
\]
where $\varepsilon_0, \chi_0$ are constant Majorana–Weyl spinors. The matrix $\gamma$ is defined as $\gamma \equiv \Gamma_{56789}$, being the product of the flat $\Gamma$-matrices on $S^5$.

### 6.4.1. $r = \frac{1}{2} \{ P_1 \wedge P_3 + (P_0 + P_1) \wedge (M_{03} + M_{12}) \}$

In this case the corresponding $\beta$-field is also divergence-free, but gives rise to a deformed background that cannot be obtained via TsT transformation. The full background is presented in [18]. The $\beta$-field is approximated as

$$
\beta = 2\eta \partial_1 \wedge \partial_3 - 2\eta (x^0 - x^1) (\partial_0 + \partial_1) \wedge \partial_3 + \mathcal{O}(\eta^2).
$$

(6.44)

The projector formula (6.38) gives

$$
\left[ \Gamma_1 (1 + \gamma_0 \Gamma_z \otimes (i \sigma_2)) - (x^0 - x^1)(1 + \gamma_0 \Gamma_0 \otimes (i \sigma_2)) (\Gamma_0 + \Gamma_1) \right] \varepsilon = 0.
$$

(6.45)

It is not hard to see that non-zero solutions are constrained by two projectors

$$
\Pi_1 = \frac{1}{2} (1 + i \gamma_1 \Gamma_z), \quad \Pi_2 = \frac{1}{2} (1 + \Gamma_0),
$$

(6.46)

which implies that eight supercharges are preserved after the deformation. Using the fact that

$$
\frac{1}{2} \beta_{mn} \Gamma^{mn} = - \frac{2\eta}{z^2} (x^0 - x^1) (\Gamma_0 + \Gamma_1 - 1),
$$

(6.47)

we can write the Killing spinors at leading order:

$$
\tilde{\varepsilon}_+ = \left[ 1 + \frac{2\eta}{z^2} \{ (x^0 - x^1)(\Gamma_0 + \Gamma_1) - 1 \} \right] \Gamma_3 \varepsilon_+,
$$

$$
\tilde{\varepsilon}_- = \varepsilon_-,
$$

(6.48)

with

$$
\varepsilon_+ + i \varepsilon_- = \frac{1}{\sqrt{2}} \Pi_1 \Pi_2 \varepsilon_0, \quad \varepsilon_0 : \text{const.}
$$

(6.49)

We can verify this expression computing explicitly the supersymmetry variations. It is enough to consider the first order in the deformation where the $H$ and $F_3$ fluxes appear.

First let us look at the gravitino variations. Since the projector $\Pi_2$ acts on the $(x^0, x^1)$-plane in Poincaré coordinates of the AdS$_5$ space, we focus on the $x^0$ component. At linear order we find

$$
\nabla_0 = \partial_0 - \frac{1}{2z} \Gamma_{0z},
$$

$$
\frac{1}{8} H_{0mn} \Gamma^{mn} = - \frac{2\eta}{z^2} (x^0 - x^1) \gamma_1 \Gamma_{012},
$$

$$
\frac{1}{8} \mathcal{F}_0 = - \frac{\eta}{z^2} \Gamma_{0z} \left[ (x^0 - x^1)(1 + \Gamma_0) - 1 \right] \otimes \sigma_1 - \frac{1}{2z} \gamma_1 \Gamma_0 \otimes (i \sigma_2).
$$

(6.50)

The gravitino variation becomes

$$
\delta \Psi_{0+} = \frac{\eta}{z^2} \left[ 2(x^0 - x^1) \gamma_0 \Gamma_{012} \tilde{\varepsilon}_+ - \Gamma_{0z} \{ (x^0 - x^1)(1 + \Gamma_0) + 1 \} \tilde{\varepsilon}_- \right] = 0.
$$

(6.51)
The dilatino variation is more involved. The background fluxes contribute to the variation as

$$H_3 = \frac{8}{z^2} \Gamma_3 \left[ (x^0 - x^1)(\Gamma_0 + \Gamma_1) - \Gamma_1 \right] = \frac{1}{2} \nabla_m \beta^{mnp} \Gamma_{np},$$

$$\Gamma_m \mathcal{A}^m = -\frac{32}{z^2} \Gamma_2 \left[ (x^0 - x^1)(\Gamma_0 + \Gamma_1) - \Gamma_0 \right] \otimes \sigma_1 \quad (6.52)$$

$$= -2 \beta^{mnp} \Gamma_m \mathcal{A}_n \otimes \sigma_3,$$

where $\mathcal{A}_n$ is the bi-spinor analog evaluated on the undeformed AdS$_5 \times S^5$ background. In total, we obtain

$$\delta \lambda = -\frac{1}{8} \left[ \beta^{mnp} \Gamma_m \mathcal{A}_n + \nabla_m \beta^{mnp} \Gamma_{np} \right] \otimes \sigma_3 \varepsilon = 0. \quad (6.53)$$

Remarkably, we end up with the projector formula (6.38).

6.4.2. $r = \frac{1}{2} \mathcal{P}_1 \wedge D$. Finally let us comment on non-unimodular classical $r$-matrices which lead to a solution for the generalized supergravity EOM. Consider the background in (4.26). The corresponding $\beta$-field was computed in (4.24). Using the projector formula, we find

$$\left[ -\frac{1}{z} \Gamma_1 - \frac{2}{z} \Gamma_{023} \otimes (i\sigma_2) + \frac{2}{z^2} \left( x^1 \Gamma_0 + x^2 \Gamma_2 + x^3 \Gamma_3 \right) \Gamma_1 \varepsilon (1 - \Gamma_{0123} \otimes (i\sigma_2)) \right] \varepsilon = 0. \quad (6.54)$$

Since the whole matrix acting on the spinor has a non-vanishing determinant, only $\varepsilon = 0$ solves the above equation. All supersymmetries are broken in this deformed background.

6.5. Comments

It would be interesting to derive the exponential factor (6.27) as well as the projector formula (6.38) in an alternative supergravity framework, such as the $\beta$-supergravity [148]. The corresponding supersymmetry variations are given in [149] in absence of Ramond–Ramond fluxes.

We have restricted our attention to Killing spinors in type IIB supergravity. In [66], the analysis was extended to the so-called M-theory T$\delta$T transformations using the M-theory $T$-duality [150–152] on the AdS$_7 \times S^4$ background. It might be interesting to look for a general formula for T$\delta$T deformed Killing spinors in terms of an antisymmetric tri-vector from the viewpoint of non-commutativity in M2-brane. To this end, the notion of the generalized theta parameter in [153] might be useful. For the tri-vector deformation of M-theory backgrounds, see [154].

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Appendix A. Conventions

In this appendix, we collect our conventions.

The antisymmetrization is defined as

\[
A_{[m_1 \cdots m_n]} \equiv \frac{1}{n!} (A_{m_1 \cdots m_n} \pm \text{permutations}).
\] \hfill (A.1)

For conventions of differential forms, we use

\[
\varepsilon_{01} = \frac{1}{\sqrt{-g}}, \quad \varepsilon_{01} = -\frac{1}{\sqrt{-g}}, \quad d^2 \sigma = d\tau \wedge d\sigma,
\]

\[
(*_\gamma (\partial \sigma^\alpha_1 \wedge \cdots \wedge \partial \sigma^\alpha_q) = \frac{1}{(p+1-q)!} \varepsilon^{\alpha_1 \cdots \alpha_q \beta_1 \cdots \beta_{p+1-q}} \partial \sigma^{\beta_1} \wedge \cdots \wedge \partial \sigma^{\beta_{p+1-q}},
\]

\[
(*_\gamma (d\sigma^\alpha_1 \wedge \cdots \wedge d\sigma^\alpha_p) = \frac{1}{q!} \varepsilon_{\alpha_1 \cdots \alpha_q} \partial \sigma^{\alpha_1} \wedge \cdots \wedge \partial \sigma^{\alpha_q}.
\]

\[
(*_\gamma (d\sigma^\alpha_1 \wedge \cdots \wedge d\sigma^\alpha_q) = \frac{1}{(p+1-q)!} \varepsilon^{\alpha_1 \cdots \alpha_q \beta_1 \cdots \beta_{p+1-q}} \partial \sigma^{\beta_1} \wedge \cdots \wedge \partial \sigma^{\beta_{p+1-q}},
\]

\[
(*_\gamma (d\sigma^\alpha_1 \wedge \cdots \wedge d\sigma^\alpha_p) = \frac{1}{q!} \varepsilon_{\alpha_1 \cdots \alpha_q} \partial \sigma^{\alpha_1} \wedge \cdots \wedge \partial \sigma^{\alpha_q}.
\]

on the string worldsheet. In spacetime, we define

\[
\varepsilon^{1 \cdots D} = -\frac{1}{\sqrt{-g}}, \quad \varepsilon_{1 \cdots D} = -\sqrt{-g}, \quad d^1 \tau = d\tau \wedge d\sigma,
\]

\[
(*_\gamma (d\sigma^a_1 \wedge \cdots \wedge d\sigma^a_q) = \frac{1}{q!} \varepsilon^{a_1 \cdots a_q b_1 \cdots b_{p+1-q}} \partial \sigma^{b_1} \wedge \cdots \wedge \partial \sigma^{b_{p+1-q}},
\]

\[
(*_\gamma (d\sigma^a_1 \wedge \cdots \wedge d\sigma^a_p) = \frac{1}{q!} \varepsilon^{a_1 \cdots a_q b_1 \cdots b_{p+1-q}} \partial \sigma^{b_1} \wedge \cdots \wedge \partial \sigma^{b_{p+1-q}}.
\]

\[
(*_\gamma (d\sigma^a_1 \wedge \cdots \wedge d\sigma^a_p) = \frac{1}{q!} \varepsilon^{a_1 \cdots a_q b_1 \cdots b_{p+1-q}} \partial \sigma^{b_1} \wedge \cdots \wedge \partial \sigma^{b_{p+1-q}}.
\]

\[
(*_\gamma (d\sigma^a_1 \wedge \cdots \wedge d\sigma^a_p) = \frac{1}{q!} \varepsilon^{a_1 \cdots a_q b_1 \cdots b_{p+1-q}} \partial \sigma^{b_1} \wedge \cdots \wedge \partial \sigma^{b_{p+1-q}}.
\]

The spin connection is defined as

\[
\omega^a_{a'b} \equiv 2 \epsilon^{a'} a [\partial (e_{a'n})] b' - \epsilon^{a'} a b [\partial (e'_{b'n})] c' e_{mc},
\] \hfill (A.4)

which satisfies

\[
de^a + \omega^a_{a'b} \wedge e^b = 0,
\] \hfill (A.5)

where \(e^a \equiv e_m^a \, dx^m\) and \(\omega^a_{b'} \equiv \omega^a_{mb'} dx^m\). The Riemann curvature tensor is defined as

\[
R^a_{b'} \equiv \frac{1}{2} R^a_{bcd} e^c \wedge e^d \equiv \partial e^a_{b'} + \omega^a_{c'b'} \wedge \omega^c_{b'} \quad R^a_{bcd} = \epsilon_{mc} e^m_a e^p_b e^a_d R^m_{npq}.
\] \hfill (A.6)

Appendix B. \(psu(2,2|4)\) algebra

In this appendix, we collect our conventions and useful formulas on the \(psu(2,2|4)\) algebra (see for example [68] for more details).
B.1. Matrix realization

8 × 8 supermatrix representation. The super Lie algebra \( su(2, 2|4) \) can be realized by using 8 × 8 supermatrices \( \mathcal{M} \) satisfying \( \text{STr} \mathcal{M} = 0 \) and the reality condition

\[
\mathcal{M}^t H + H \mathcal{M} = 0, \quad \mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{B.1}
\]

where \( \text{STr} \mathcal{M} \equiv \text{Tr} A - \text{Tr} D \) and the Hermitian matrix \( H \) is defined as

\[
H \equiv \begin{pmatrix} \Sigma & 0_4 \\ 0_4 & 1_4 \end{pmatrix}, \quad \Sigma \equiv \begin{pmatrix} 0_2 & -i\sigma_3 \\ i\sigma_3 & 0_2 \end{pmatrix} = \sigma_2 \otimes \sigma_3. \tag{B.2}
\]

A trivial element satisfying the above requirement is the \( u(1) \) generator

\[
Z = i \begin{pmatrix} 1_4 & 0_4 \\ 0_4 & 1_4 \end{pmatrix}, \tag{B.3}
\]

and the \( psu(2, 2|4) \) is defined as the quotient \( su(2, 2|4)/u(1) \).

The \( psu(2, 2|4) \) has an automorphism \( \Omega \) defined as

\[
\Omega(\mathcal{M}) = -\mathcal{K} \mathcal{M}^a \mathcal{K}^{-1}, \quad \mathcal{K} = \begin{pmatrix} K & 0_4 \\ 0_4 & K \end{pmatrix}, \tag{B.4}
\]

where \( K \) is a 4 × 4 matrix

\[
K \equiv \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad K^{-1} = -K, \tag{B.5}
\]

and \( \mathcal{M}^a \) represents the supertranspose of \( \mathcal{M} \) defined as

\[
\mathcal{M}^a = \begin{pmatrix} A & -C \\ B & D \end{pmatrix}. \tag{B.6}
\]

By using the automorphism \( \Omega \) (of order four), we decompose \( g = psu(2, 2|4) \) as

\[
g = g^{(0)} \oplus g^{(1)} \oplus g^{(2)} \oplus g^{(3)}, \tag{B.7}
\]

where \( \Omega(g^{(k)}) = i^k g^{(k)} \) (\( k = 0, 1, 2, 3 \)) and the projector to each vector space \( g^{(k)} \) can be expressed as

\[
P^{(k)}(\mathcal{M}) \equiv \frac{1}{4} \left[ \mathcal{M} + i^{2k} \Omega(\mathcal{M}) + i^{2k} \Omega^2(\mathcal{M}) + i^k \Omega^3(\mathcal{M}) \right]. \tag{B.8}
\]

Bosonic generators. The bosonic generators of \( psu(2, 2|4) \) algebra, \( P_d \) and \( J_{ab} \), can be represented by the following 8 × 8 supermatrices:
\[ \{P_a\} \equiv \{P_{\hat{a}}, P_a\}, \quad \{J_{ab}\} \equiv \{J_{\hat{a}\hat{b}}, J_{a\hat{b}}\}, \]

\[
P_{\hat{a}} = \left( \begin{array}{cc} 1 & \gamma_{\hat{a}} \vspace{1mm} \\gamma_{\hat{a}} & 0 \end{array} \right), \quad J_{\hat{a}\hat{b}} = \left( \begin{array}{cc} -\frac{1}{2} \gamma_{\hat{a}\hat{b}} & 0 \vspace{1mm} \\0 & 0 \end{array} \right) \quad (\hat{a}, \hat{b} = 0, \ldots, 4), \]

\[
P_a = \left( \begin{array}{cc} 0 & 0 \vspace{1mm} \\gamma_a & 0 \end{array} \right), \quad J_{a\hat{b}} = \left( \begin{array}{cc} 0 & 0 \vspace{1mm} \\frac{1}{2} \gamma_{a\hat{b}} & 0 \end{array} \right) \quad (\hat{a}, \hat{b} = 5, \ldots, 9), \]

where we defined \(4 \times 4\) matrices \(\gamma_{\hat{a}} \equiv (\gamma_{\hat{a}j}) (\hat{i}, \hat{j} = 1, \ldots, 4)\) and \(\gamma_a \equiv (\gamma_{ij}) (i, j = 1, \ldots, 4)\).

The conformal basis, \(\{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_5\}\), and \(\{\gamma_{\hat{a}}, \gamma_{a}\}\) satisfy

\[
\{\gamma_{\mu}, \gamma_{\nu}\} = 2\eta_{\mu\nu}, \quad (\eta_{\mu\nu}) \equiv \text{diag}(-1, 1, 1, 1), \quad (\gamma_\mu) = K \gamma_\mu K^{-1}. \quad (B.12)
\]

The conformal basis, \(\{P_\mu, M_{\mu\nu}, D, K_\mu\}\), of a bosonic subalgebra \(\mathfrak{su}(2,2) \equiv \mathfrak{so}(2,4)\) that corresponds to the AdS isometries, can be constructed from \(P_a\) and \(J_{ab}\) as

\[
P_\mu \equiv P_\mu + J_{\mu\delta}, \quad K_\mu \equiv P_\mu - J_{\mu\delta}, \quad M_{\mu\nu} \equiv J_{\mu\nu}, \quad D \equiv P_4, \quad (B.13)
\]

where \(P_\mu, M_{\mu\nu}, D, K_\mu\) represent the translation generators, the Lorentz generators, the dilatation generator, and the special conformal generators, respectively. On the other hand, a bosonic subalgebra \(\mathfrak{su}(4) \equiv \mathfrak{so}(6)\) that corresponds to the isometries of \(S^5\) are generated by \(P_a\) and \(J_{ab}\). We choose the Cartan generators of \(\mathfrak{su}(4)\) as follows

\[
h_1 \equiv J_{57}, \quad h_2 \equiv J_{68}, \quad h_3 \equiv P_8. \quad (B.14)
\]

For later convenience, let us also define \(16 \times 16\) matrices \(\gamma_\mu, \gamma_{\hat{a}}, \gamma_a, \gamma_{ab}\) as

\[
(\gamma_\mu) \equiv (\gamma_{\hat{a}}, \gamma_a) = (\gamma_{\hat{a}} \otimes I_4, I_4 \otimes \gamma_a), \quad (\gamma_{\hat{a}}) \equiv (\gamma_{\hat{a}} \otimes I_4, I_4 \otimes i \gamma_a), \quad (\gamma_a) \equiv (\gamma_{a\hat{b}}, \gamma_{\hat{b}a}) = (\gamma_{a\hat{b}} \otimes I_4, I_4 \otimes \gamma_{\hat{b}a}). \quad (B.15)
\]
which satisfy
\[
\begin{align*}
(\gamma_0^a) &= \gamma_0^a \gamma_0^a, \\
(\gamma_i^a) &= \gamma_i^a \gamma_i^a, \\
(\tilde{\gamma}_0^a) &= -\gamma_0^a \gamma_0^a, \\
(\tilde{\gamma}_i^a) &= (K \otimes K)^{-1} \gamma_i^a (K \otimes K), \\
\{\gamma_a, \gamma_b\} &= 2\eta_{ab}, \\
\{\tilde{\gamma}_a, \tilde{\gamma}_b\} &= 2\eta_{ab}, \\
\{\gamma_a, \tilde{\gamma}_b\} &= -2\delta_{ab}.
\end{align*}
\]

We can easily see \(\gamma_{ab} = \gamma_a \gamma_b\) and \(\gamma_{ab} = \gamma_a \gamma_b\). If we also define \(\tilde{\gamma}_{ab} \equiv \tilde{\gamma}_a \tilde{\gamma}_b\) and \(\tilde{\gamma}_{ab} \equiv \tilde{\gamma}_a \tilde{\gamma}_b\), they satisfy
\[
\tilde{\gamma}_{ab} = -\frac{1}{2} R_{ab}^{cd} \gamma_{cd}.
\]

where \(R_{ab}^{cd}\) are the tangent components of the Riemann tensor in \(\text{AdS}_5 \times S^5\), whose non-vanishing components are
\[
R_{ab}^{\tilde{\gamma}d} = -2\delta_{[a}^\delta \delta_{b]}^\gamma, \quad R_{ab}^{\gamma d} = 2\delta_{[a}^\gamma \delta_{b]}^\delta.
\]

**Fermionic generators.** The fermionic generators \((Q^\gamma)^{\dot{\alpha}}_{\alpha}\) \((\dot{\alpha}, \alpha = 1, \ldots, 4)\) are given by
\[
(Q^\gamma)^{\dot{\alpha}}_{\alpha} = \begin{pmatrix} 0_4 & i \delta^\dot{\alpha}_i K^\dot{\alpha}_i \\ -\delta_\alpha^i K^\alpha_i & 0_4 \end{pmatrix}, \quad (Q^\gamma)^{\dot{\alpha}}_{\alpha} = \begin{pmatrix} 0_4 & -\delta^\dot{\alpha}_i K^\dot{\alpha}_i \\ i \delta^\alpha_i K^\alpha_i & 0_4 \end{pmatrix}.
\]

As discussed in [10], these matrices do not satisfy the reality condition \((B.1)\) but rather their redefinitions \(\mathcal{D}^\alpha \) do. The choice, \(Q^\dot{\gamma}\) or \(\mathcal{D}^\dot{\gamma}\), is a matter of convention, and we here employ \(Q^\dot{\gamma}\) by following [10]. We also introduce Grassmann-odd coordinates \(\theta_I \equiv (\theta_{\dot{\alpha} \alpha})_I\) which are 16-component Majorana–Weyl spinors satisfying
\[
(Q^\dot{\gamma} \theta_I)^\dagger H + H (Q^\dot{\gamma} \theta_I) = 0.
\]

Since the matrices \(Q^\dot{\gamma}\) satisfy
\[
(Q^\dot{\gamma})^\dagger_{\dot{\alpha} \alpha} = -i K^{-1}_\beta \delta_{\beta}^\dot{\alpha} (Q^\dot{\gamma})_{\dot{\alpha} \beta}^\alpha K^{-1}_\alpha, \\
H (Q^\dot{\gamma})_{\dot{\alpha} \alpha} H^{-1} = i (\gamma^0)_{\dot{\alpha} \dot{\beta}} (Q^\dot{\gamma})_{\dot{\alpha} \dot{\beta}},
\]

the condition \((B.20)\) is equivalent to the Majorana condition
\[
\tilde{\theta}_I \equiv \theta_I^\dagger \gamma^0 = \theta_I (K \otimes K),
\]

or more explicitly,
\[
\tilde{\theta}_{\dot{I}} \equiv \theta_{\dot{I}}^\dagger K^{\dot{\alpha}} \gamma^0 K^{\dot{\alpha}}.
\]

\[73\]}
Commutation relations. The generators of \( su(2, 2|4) \) algebra, \( P_a, J_{ab}, Q^I, \) and \( Z \) satisfy the following commutation relations:

\[
[P_a, P_b] = \frac{1}{2} R_{ab}^{\quad cd} J_{cd}, \quad [J_{ab}, P_c] = \eta_{ca} P_b - \eta_{cb} P_a,
\]

\[
[J_{ab}, J_{cd}] = \eta_{ac} J_{bd} - \eta_{ad} J_{bc} - \eta_{bd} J_{ac} + \eta_{cd} J_{ac},
\]

\[
[Q^I \theta_I, P_a] = \frac{i}{2} \varepsilon^{IJ} Q^J \tilde{\gamma}_a \theta_I, \quad [Q^I \theta_I, J_{ab}] = \frac{1}{2} \delta^{IJ} Q^J \gamma_{ab} \theta_I,
\]

\[
[Q^I \theta_I, Q^J \psi_J] = -i \delta^{IJ} \bar{\theta}_J \tilde{\gamma}_a \psi_J P_a - \frac{1}{4} \varepsilon^{IJ} \bar{\theta}_I \gamma_{ab} \psi_J R_{ab}^{\quad cd} J_{cd} - \frac{1}{2} \delta^{IJ} \tilde{\theta}_I \psi_J Z,
\]

(B.24)

and the \( psu(2, 2|4) \) algebra is obtained by dropping the last term proportional to \( Z \).

On the other hand, the bosonic generators \( \{ P_\mu, M_{\mu\nu}, D, K_\mu \} \) satisfy the \( so(2, 4) \) algebra,

\[
[P_\mu, K_\nu] = 2 \left( \eta_{\mu\nu} D - M_{\mu\nu} \right), \quad [D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu,
\]

\[
[M_{\mu\nu}, P_\rho] = \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu, \quad [M_{\mu\nu}, K_\rho] = \eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu,
\]

\[
[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\sigma} M_{\mu\rho}.
\]

Supertrace and projections. For generators of the \( psu(2, 2|4) \) algebra, the supertrace becomes

\[
\text{STr}(P_a P_b) = \eta_{ab}, \quad \text{STr}(J_{ab} J_{cd}) = R_{abcd},
\]

\[
\text{STr}(Q^I \theta_I Q^J \psi_J) = -2 \varepsilon^{IJ} \tilde{\theta}_I \lambda_J,
\]

(B.26)

where \( R_{abcd} \equiv R_{ab}^{\quad ef} \eta_{ef} \eta_{cd} \) and

\[
\eta_{ab} \equiv \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta_{ab} \end{pmatrix}, \quad \eta_{ab} \equiv \text{diag}(-1, 1, 1, 1, 1), \quad \eta_{ab} \equiv \text{diag}(1, 1, 1, 1, 1).
\]

(B.27)

Each \( Z_4 \)-component \( g^{(i)} \) is spanned by the following generators:

\[
g^{(0)} = \text{span}_R \{ J_{ab} \}, \quad g^{(1)} = \text{span}_R \{ Q^1 \},
\]

\[
g^{(2)} = \text{span}_R \{ P_a \}, \quad g^{(3)} = \text{span}_R \{ Q^2 \}.
\]

(B.28)

Then, from the definition of \( d_\pm \) (2.6),

\[
d_\pm \equiv \mp P^{(1)} + 2 P^{(2)} \pm P^{(3)}.
\]

(B.29)

we obtain

\[
d_\pm(P_a) = 2 P_a, \quad d_\pm(J_{ab}) = 0, \quad d_\pm(Q^I) = \mp \sigma_I^{IJ} Q^J.
\]

(B.30)
B.2. Connection to ten-dimensional quantities

By using the $16 \times 16$ matrices $\gamma_a$ defined in (B.15), the $32 \times 32$ gamma matrices $(\Gamma_a)_{\alpha \beta}$ are realized as

$$(\Gamma_a) \equiv (\Gamma_0, \Gamma_5) \equiv (\sigma_1 \otimes \gamma_0, \sigma_2 \otimes \gamma_0).$$

(B.31)

We can also realize the charge conjugation matrix as

$$C = i\sigma_2 \otimes K \otimes K.$$  

(B.32)

The 32-component Majorana–Weyl fermions $\Theta_I$ expressed as

$$\Theta_I = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \theta_I,$$  

(B.33)

which satisfies the chiral conditions

$$\Gamma^{11} \Theta_I = \Theta_I.$$  

(B.34)

The Majorana condition is given by

$$\bar{\Theta}_I \gamma_a \gamma_{a b} \theta_J = \bar{\Theta}_I \Gamma_a \Theta_J,$$  

(B.36)

and the following relations:

$$\bar{\Theta}_I \gamma_a \gamma_{a b} \gamma_{b c} \theta_J = \bar{\Theta}_I \Gamma_a \Gamma_{b c} \Theta_J,$$  

(B.41)

$$\bar{\Theta}_I \gamma_{a b} \gamma_{a b c d} \theta_J = -i \bar{\Theta}_I \Gamma_{a b c d} \Theta_J,$$  

(B.43)

Indeed, we obtain

$$\bar{\Theta}_I \gamma_a \gamma_{a b} \theta_J = \frac{i}{8} \bar{\Theta}_I \Gamma_a \bar{\mathcal{F}}_5 \Gamma_{b} \Theta_J.$$  

(B.40)

We can also show the following relations: \(^{31}\)

$$\bar{\Theta}_I \gamma_a \gamma_{a b c} \theta_J = \bar{\Theta}_I \Gamma_a \Gamma_{b c} \Theta_J,$$  

(B.41)

$$\bar{\Theta}_I \gamma_{a b} \theta_J = -i \bar{\Theta}_I \Gamma_{a b} \Theta_J = -i \bar{\Theta}_I \Gamma_{56789} \Gamma_{a b} \Theta_J,$$  

(B.42)

$$\bar{\Theta}_I \gamma_{a b c d} \theta_J = -i \bar{\Theta}_I \Gamma_{a b c d} \Gamma_{a b c d} \Theta_J = -i \bar{\Theta}_I \Gamma_{56789} \Gamma_{a b c d} \Gamma_{a b c d} \Theta_J.$$  

(B.43)

\(^{31}\)Recall that $\gamma_{a b}$ has only the components $(\gamma_{a b}) = (\gamma_{a b}, \gamma_{a b}).$
References

[1] Bena I, Polchinski J and Roiban R 2004 Phys. Rev. D 69 046002
[2] Beisert N et al 2012 Lett. Math. Phys. 99 3
[3] Klimčík C 2002 J. High Energy Phys. JHEP12(2002)051
[4] Klimčík C 2009 J. Math. Phys. 50 043508
[5] Delduc F, Magro M and Vicedo B 2013 J. High Energy Phys. JHEP11(2013)192
[6] Delduc F, Magro M and Vicedo B 2014 Phys. Rev. Lett. 112 051601
[7] Matsumoto T and Yoshida K 2015 Nucl. Phys. B 893 287–304
[8] Kawaguchi I, Matsumoto T and Yoshida K 2014 J. High Energy Phys. JHEP04(2014)153
[9] Arutyunov G, Gorsato R and Frolov S 2014 J. High Energy Phys. JHEP04(2014)002
[10] Arutyunov G, Gorsato R and Frolov S 2015 J. High Energy Phys. JHEP12(2015)049
[11] Hoare B and Seibold F K 2019 J. High Energy Phys. JHEP01(2019)125
[12] Arutyunov G, Frolov S, Hoare B, Roiban R and Tseytlin A A 2016 Nucl. Phys. B 903 262–303
[13] Tseytlin A A and Wulff L 2016 J. High Energy Phys. JHEP06(2016)174
[14] Matsumoto T and Yoshida K 2014 J. High Energy Phys. JHEP06(2014)135
[15] Matsumoto T and Yoshida K 2014 J. High Energy Phys. JHEP06(2014)163
[16] Matsumoto T and Yoshida K 2015 J. High Energy Phys. JHEP04(2015)180
[17] Kyono H and Yoshida K 2016 Prog. Theor. Exp. Phys. 2016 083B03
[18] Borsato R and Wulff L 2016 J. High Energy Phys. JHEP10(2016)045
[19] Osten D and van Tongeren S J 2016 Nucl. Phys. B 915 184
[20] Orlando D, Reffert S, Sakamoto J-i and Yoshida K 2016 J. Phys. A: Math. Theor. 49 445403
[21] Hoare B and Tseytlin A A 2016 J. Phys. A: Math. Theor. 49 494001
[22] Borsato R and Wulff L 2016 Phys. Rev. Lett. 117 251602
[23] Borsato R and Wulff L 2017 J. High Energy Phys. JHEP10(2017)024
[24] Sakamoto J, Sakatani Y and Yoshida K 2017 J. Phys. A: Math. Theor. 50 415401
[25] Sakamoto J and Sakatani Y 2018 J. High Energy Phys. JHEP06(2018)147
[26] Araujo T, Colgáin E Ó, Sheikh-Jabbari M M and Yavartanoo H 2019 J. High Energy Phys. JHEP03(2019)168
[27] Borsato R and Wulff L 2019 J. Phys. A: Math. Theor. 52 225401
[28] Hoare B and Thompson D C 2017 J. High Energy Phys. JHEP02(2017)059
[29] Bakhmatov I, Colgáin E O, Sheikh-Jabbari M and Yavartanoo H 2018 J. High Energy Phys. JHEP06(2018)161
[30] Lüst D and Osten D 2018 J. High Energy Phys. JHEP05(2018)165
[31] Araujo T, Colgáin E O and Yavartanoo H 2018 Eur. Phys. J. C 78 854
[32] Borsato R and Wulff L 2018 J. High Energy Phys. JHEP08(2018)027
[33] de la Ossa X C and Quevedo F 1993 Nucl. Phys. B 403 377−94
[34] Giveon A and Rocek M 1994 Nucl. Phys. B 421 173−90
[35] Álvarez E, Álvarez-Gaumé L and Lozano Y 1994 Nucl. Phys. B 424 155−83
[36] Sfetsos K and Thompson D C 2011 Nucl. Phys. B 846 21−42
[37] Lozano Y, Colgáin E Ó, Sfetsos K and Thompson D C 2011 J. High Energy Phys. JHEP06(2011)106
[38] Fernandez-Melgarejo J J, Sakamoto J, Sakatani Y and Yoshida K 2017 J. High Energy Phys. JHEP12(2017)108
[39] Klimčík C 2015 Nucl. Phys. B 900 259−72
[40] Klimčík C 2016 Phys. Lett. B 760 345−9
[41] Hoare B and Tseytlin A 2015 Nucl. Phys. B 897 448−78
[42] Vicedo B 2015 J. Phys. A: Math. Theor. 48 355203
[150] Sen A 1996 *Mod. Phys. Lett.* A **11** 827–34
[151] Ganor O J, Ramgoolam S and Taylor W 1997 *Nucl. Phys.* B **492** 191–204
[152] Aharony O 1996 *Nucl. Phys.* B **476** 470–83
[153] Berman D S, Cederwall M, Gran U, Larsson H, Nielsen M, Nilsson B E W and Sundell P 2002 *J. High Energy Phys.* JHEP02(2002)012
[154] Bakhmatov I, Deger N S, Musaev E T, Colgán E Ó and Sheikh-Jabbari M M 2019 *J. High Energy Phys.* JHEP08(2019)126