Corrections to finite–size scaling in the $\varphi^4$ model on square lattices

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Abstract

Corrections to scaling in the two–dimensional scalar $\varphi^4$ model are studied based on non–perturbative analytical arguments and Monte Carlo (MC) simulation data for different lattice sizes $L$ ($4 \leq L \leq 1536$) and different values of the $\varphi^4$ coupling constant $\lambda$, i. e., $\lambda = 0.1, 1, 10$. According to our analysis, amplitudes of the nontrivial correction terms with the correction–to–scaling exponents $\omega_\ell < 1$ become small when approaching the Ising limit ($\lambda \to \infty$), but such corrections generally exist in the 2D $\varphi^4$ model. Analytical arguments show the existence of corrections with the exponent $3/4$. The numerical analysis, supported by arguments of the conformal field theory, suggests that there exists also a correction with the exponent $1/2$, which is detectable at $\lambda = 0.1$. We have tested the consistency of susceptibility data with corrections, represented by an expansion in powers of $L^{-1/4}$. We conclude that a correction with exponent $\omega = 1/4$, probably, also exists.

Keywords: $\varphi^4$ model, corrections to scaling, Monte Carlo simulation

1 Introduction

The $\varphi^4$ model is one of the most extensively used tools in analytical studies of critical phenomena – see, e. g., [1,2]. These studies have risen also a significant interest in numerical testing of the theoretical results for this model. Recently, some challenging non–perturbative analytical results for the corrections to scaling in the $\varphi^4$ model have been obtained [3], which could be relatively easily verified numerically in the two–dimensional case. Therefore, we will further focus just on this case. Although the analytical studies are based on the continuous $\varphi^4$ model, its lattice version is more convenient for Monte Carlo (MC) simulations. Earlier MC studies of the 2D lattice model go back to the work by Milchev, Heermann and Binder [4]. The continuous version has been simulated, e. g., in [10]. In [8], effective critical exponents $\nu \approx 0.8$ for correlation length and $\gamma \approx 1.25$
for susceptibility have been obtained, based on the simulation data for lattices sizes up to \( L = 20 \). The considered there a scalar 2D \( \varphi^4 \) model should belong to the 2D Ising universality class with the exponents \( \nu = 1 \) and \( \gamma = 7/4 \), so that these effective exponents point to the presence of remarkable corrections to scaling. A later MC study \([11]\) of larger lattices, up to \( L = 128 \), has supported the idea that this model belongs to the 2D Ising universality class, stating that the asymptotic scaling is achieved for \( L \gtrsim 32 \). Apparently, numerical studies cause no doubts that the leading scaling exponents for the two–dimensional scalar \( \varphi^4 \) model and the 2D Ising model are the same. However, it is still important to refine further corrections to scaling. Indeed, the 2D \( \varphi^4 \) model can contain nontrivial correction terms, which do not show up or cancel in the 2D Ising model. We will focus on this issue in the following sections.

2 Analytical arguments

Here we will review the important non–perturbative analytical arguments of \([8]\) in the context of some other known results. These arguments are based on the continuous \( \varphi^4 \) model in the thermodynamic limit with the Hamiltonian \( \mathcal{H} \) given by

\[
\frac{\mathcal{H}}{k_B T} = \int \left( r_0 \varphi^2(x) + c(\nabla \varphi(x))^2 + w \varphi^4(x) \right) dx, \tag{1}
\]

where the order parameter \( \varphi(x) \) is an \( n \)-component vector with components \( \varphi_i(x) \), depending on the coordinate \( x \), \( T \) is the temperature, and \( k_B \) is the Boltzmann constant. It is assumed that the order-parameter field \( \varphi_i(x) \) does not contain the Fourier components \( \varphi_i(k) \) with \( k > \Lambda \), i. e., there exists the upper cut-off parameter \( \Lambda \). Here \( r_0 \) is the only \( T \)-dependent parameter in the right hand side of \((1)\), the dependence being linear. Assuming the power–law singularity of the correlation length, \( \xi \sim t^{-\nu} \) at small reduced temperature \( t \to 0 \), the leading singular part of specific heat \( C_V^{\text{sing}} \) can be expressed as \([8]\)

\[
C_V^{\text{sing}} \propto \xi^{1/\nu} \left( \int_{k<\Lambda'} \left( G(k) - G^*(k) \right) dk \right)^{\text{sing}}, \tag{2}
\]

where \( G(k) \) is the Fourier–transformed two–point correlation function, and \( G^*(k) \) is its value at the critical point. The superscript “\( \text{sing} \)” implies the leading singular contribution, \( \xi \) is the correlation length, and \( \Lambda' \) is a constant. This expression is valid for any positive \( \Lambda' < \Lambda \), since the leading singularity is provided by small wave vectors with the magnitude \( k = | k | \to 0 \), so that the contribution of finite wave vectors \( \Lambda' \leq k \leq \Lambda \) is irrelevant. In other words, \( C_V^{\text{sing}} \) is independent of the constant \( \Lambda' \).

The leading singularity of specific heat in the form of \( C_V^{\text{sing}} \propto (\ln \xi) \lambda \xi^{\alpha/\nu} \) and the two–point correlation function in the asymptotic form of \( G(k) = \sum_{\ell \geq 0} \xi^{(\gamma - \theta_i)/\nu} g_\ell(k \xi) \) have been considered in \([8]\). The latter two expressions represent the conventional scaling hypothesis, where \( \theta_0 = 0 \), \( \theta_\ell \) with \( \ell \geq 1 \) are the correction–to–scaling exponents, and \( g_\ell(k \xi) \) are the scaling functions. One has to note that, according to the self–consistent scaling theory of logarithmic correction exponents in \([12]\), logarithmic corrections can generally appear also in \( \xi \) as function of \( t \), as well as in \( G(k) \). Nevertheless, the consideration in \([8]\) covers the usual case of \( \lambda = 0 \), where no logarithmic corrections are present, as well as the important particular case of \( \alpha = 0 \) and \( \lambda = 1 \), where the logarithmic correction appears only in specific heat, as in the 2D Ising model \([12]\). The considered
in \( \phi^4 \) scaling forms appear to be general enough for our analysis of the \( \phi^4 \) model below the upper critical dimension \( d < 4 \), where \( \xi \) and \( G(k) \) have no logarithmic corrections according to the known results, except only for the case of the Kosterlitz–Thouless phase transition at \( n = 2 \) and \( d = 2 \).

A theorem has been proven in [8], stating that the two–point correlation function of the considered here \( \phi^4 \) model necessarily contains a correction with the exponent \( \theta_\ell = \gamma + 1 - \alpha - d\nu \), if \( C_{\text{sing}}^{(k)} \) can be calculated from (2), applying the considered here scaling forms, if the result is \( \Lambda' \)–independent, and if the condition \( \gamma + 1 - \alpha - d\nu > 0 \) is satisfied for the critical exponents. According to the known hyperscaling hypothesis \( \alpha + d\nu = 2 \), it yields \( \theta_\ell = \gamma - 1 \) for \( \gamma > 1 \). Apparently, all the listed here conditions of the theorem are satisfied for \( d = 3 \) and \( n \geq 1 \), as well as for \( d = 2 \) and \( n = 1 \). Indeed, the condition of \( \Lambda' \)–independence is generally meaningful, since the critical singularities are provided by long–wave fluctuations, whereas the assumed here scaling forms and the relations \( \alpha + d\nu = 2 \), \( \gamma > 1 \) and, consequently, \( \gamma + 1 - \alpha - d\nu > 0 \) hold in these cases, according to the current knowledge about the critical phenomena.

The consequences of the theorem have been discussed in detail in [8]. Here we only note that the existence of a correction with exponent \( \theta_\ell = 3/4 \) in the scalar \( n = 1 \) 2D \( \phi^4 \) model follows from this theorem, if \( \gamma = 7/4 \) and \( \nu = 1 \) hold here, as in the 2D Ising model. It corresponds to a correction exponent \( \omega_\ell = \theta_\ell/\nu = 3/4 \) in the critical two–point correlation function, as well as in the finite–size scaling. Since this exponent not necessarily describes the leading correction term, the prediction is \( \omega \leq 3/4 \) for the leading correction–to–scaling exponent \( \omega \). An evidence for a nontrivial correction with non–integer exponent (which might be, e. g., \( 1/4 \)) in the finite–size scaling of the critical real–space two–point correlation function of the 2D Ising model has been provided in [13], based on an exact enumeration by a transfer matrix algorithm. This correction, however, has a very small amplitude and is hardly detectable. Moreover, such a correction has not been detected in susceptibility. Usually, the scaling in the 2D Ising model is representable by trivial, i. e., integer, correction–to–scaling exponents when analytical background terms (e. g., a constant contribution to susceptibility) are separated – see, e. g., [14, 16] and references therein. The discussions have been focused on the existence of irrelevant variables [17, 18]. In particular, the high–precision calculations in [18] have shown that the conjecture by Aharony and Fisher about the absence of such variables [19, 20] fails.

The above mentioned theorem predicts the existence of nontrivial correction–to–scaling exponents in the 2D \( \phi^4 \) model. It can be expected that the nontrivial correction terms of the \( \phi^4 \) model usually do not show up or cancel in the 2D Ising model. This idea is not new. Based on the standard field–theoretical treatments of the \( \phi^4 \) model, \( \omega = 4/3 \) has been conjectured for the leading nontrivial scaling corrections at \( n = 1 \) and \( d = 2 \) in [2, 21]. However, it contradicts our theorem, which yields \( \omega \leq 3/4 \). This discrepancy is interpreted as a failure of the standard perturbative methods — see [7] and the discussions in [8]. One has to note that the alternative perturbative approach of [6], predicting \( \omega_\ell = \ell\eta \) (where \( \ell \geq 1 \) is an integer) with \( \eta = 2 - \gamma/\nu = 1/4 \) for \( n = 1 \) and \( d = 2 \), is consistent with this theorem.
3 Monte Carlo simulation of the lattice $\varphi^4$ model

We have performed MC simulations of the scalar 2D $\varphi^4$ model on square lattice with periodic boundary conditions. The Hamiltonian $\mathcal{H}$ is given by

$$\frac{\mathcal{H}}{k_B T} = -\beta \sum_{\langle ij \rangle} \varphi_i \varphi_j + \sum_i \left( \varphi_i^2 + \lambda \left( \varphi_i^2 - 1 \right)^2 \right),$$

where $-\infty < \varphi_i < \infty$ is a continuous scalar order parameter at the $i$-th lattice site, and $\langle ij \rangle$ denotes the set of all nearest neighbors. This notation is related to the one of $^22$ via $\beta = 2\kappa$ and $\varphi = \phi$. We have denoted the coupling constant at $\varphi_i \varphi_j$ by $\beta$ to outline the similarity with the Ising model.

Swendsen-Wang and Wolff cluster algorithms are known to be very efficient for MC simulations of the Ising model in vicinity of the critical point $^22$. However, these algorithms update only the spin orientation, and therefore are not ergodic for the $\varphi^4$ model. The problem is solved using the hybrid algorithm, where a cluster algorithm is combined with Metropolis sweeps. This method has been applied to the 3D $\varphi^4$ model in $^22$. In our simulations, we have applied one Metropolis sweep after each $N_W$ Wolff single cluster algorithm steps. Following $^22$, the order parameter is updated as $\varphi'_i = \varphi_i + s(r - 1/2)$ in one Metropolis step, where $s$ is a constant and $r$ is a random number from a set of uniformly distributed random numbers within $[0, 1]$. Here $N_W$ and $s$ are considered as optimization parameters, allowing to reach the smallest statistical error in a given simulation time. We have chosen $N_W$ such that $N_W \langle c \rangle / L^2$ is about $2/3$ or $0.6$, where $\langle c \rangle$ is the mean cluster size. The optimal choice of $s$ depends on the Hamiltonian parameters. Our simulations have been performed at $\lambda = 0.1$, $\lambda = 1$, $\lambda = 10$ and at such values of $\beta$, which correspond to $U = \langle m^4 \rangle / \langle m^2 \rangle^2 = 1.17$ and $U = 2$, $m$ being the magnetization per spin. At $U = 1.17$, we have chosen $s = 4$ for $\lambda = 0.1$, $s = 4$ for $\lambda = 1$ and $s = 3$ for $\lambda = 10$. At $U = 2$, the corresponding values are $s = 3.5$, $s = 3$ and $s = 2$. For comparison, $s = 3$ has been used in $^22$.

We have used the iterative method of $^24$ to find $\beta$, corresponding to certain value of $U$, as well as a set of statistical averaged quantities at this $\beta$, called the pseudo-critical coupling $\tilde{\beta}_c(L)$. We have performed high statistics simulations for evaluation of the derivative $\partial U / \partial \beta$ and the susceptibility $\chi = N \langle m^2 \rangle$, where $N = L^2$ is the total number of spins. For each lattice size $L$, these quantities have been estimated from 100 iterations (simulation bins) in vicinity of $\beta = \tilde{\beta}_c(L)$, collected from one or several simulation runs, discarding first 10 iterations of each run for equilibration. One iteration included $10^6$ steps of the hybrid algorithm, each consisting of one Metropolis sweep and $N_W$ Wolff algorithm steps, as explained before. To test the accuracy of our iterative method, we have performed some simulations (for $U = 2$ and $\lambda = 0.1$) with $2.5 \times 10^5$ hybrid algorithm steps in one iteration, and have verified that the results well agree with those for $10^6$ steps. Moreover, we have used two different pseudo-random number generators, the same ones as in $^25$, to verify that the results agree within the statistical error bars.

Note that the quantity $U$ is related to the Binder cumulant $B = 1 - U / 3$ $^29$. In the thermodynamic limit, we have $B = 0$ ($U = 3$) above the critical point, i.e., at $T > T_c$ or $\beta < \beta_c$, and $B = 2/3$ ($U = 1$) at $T < T_c$ or $\beta > \beta_c$. Thus, the pseudo-critical coupling $\tilde{\beta}_c(L)$, corresponding to a given $U$ in the range of $1 < U < 3$, tends to the true critical coupling $\beta_c$ at $L \to \infty$. We have chosen one $U$ value, $U = 2$ in the middle of the interval and the other one, $U = 1.17$, close to the critical value $U^*$ at $\beta = \beta_c$ and $L \to \infty$. 
Table 1: The values of $\tilde{\beta}_c$, as well as $\chi/L^{7/4}$, and $-(\partial U/\partial \beta)/L$ at $\beta = \tilde{\beta}_c$ for $\lambda = 0.1$ and $U = 2$ depending on the lattice size $L$.

| L   | $\tilde{\beta}_c$   | $\chi/L^{7/4}$ | $-(\partial U/\partial \beta)/L$ |
|-----|---------------------|----------------|-------------------------------|
| 4   | 0.549398(42)        | 0.60791(23)    | 2.4344(16)                    |
| 6   | 0.562326(28)        | 0.50107(21)    | 2.5045(17)                    |
| 8   | 0.570550(19)        | 0.44694(19)    | 2.5492(21)                    |
| 12  | 0.580455(14)        | 0.39460(21)    | 2.5900(22)                    |
| 16  | 0.5861408(94)       | 0.36991(17)    | 2.6112(26)                    |
| 24  | 0.5924039(62)       | 0.34936(16)    | 2.6459(26)                    |
| 32  | 0.5957584(45)       | 0.34116(15)    | 2.6663(30)                    |
| 48  | 0.5992406(34)       | 0.33538(14)    | 2.6881(27)                    |
| 64  | 0.6010332(23)       | 0.33412(13)    | 2.7129(31)                    |
| 96  | 0.6028383(15)       | 0.33359(14)    | 2.7273(36)                    |
| 128 | 0.6037470(11)       | 0.33396(14)    | 2.7351(40)                    |
| 192 | 0.60465804(69)      | 0.33486(14)    | 2.7592(37)                    |
| 256 | 0.6051333(60)       | 0.33537(12)    | 2.7614(38)                    |
| 384 | 0.60556996(40)      | 0.33648(12)    | 2.7745(38)                    |
| 512 | 0.60579773(41)      | 0.33701(11)    | 2.7780(40)                    |
| 768 | 0.60602518(20)      | 0.337618(99)   | 2.7836(37)                    |
| 1024| 0.60613849(13)      | 0.33780(13)    | 2.7827(44)                    |
| 1536| 0.606252278(88)     | 0.33825(11)    | 2.7890(40)                    |

In most of the cases, MC simulations have been performed for lattice sizes $4 \leq L \leq 128$. At $\lambda = 0.1$ and $U = 2$, the simulations have been extended up to $L = 1536$ for a refined analysis. A parallel algorithm, similar to that one used in [24], helped us to speed up the simulations for the largest lattice size $L = 1536$. The Wolff algorithm has been parallelized in this way, whereas the usual ideas of splitting the lattice in slices [23] have been applied to parallelize the Metropolis algorithm. In the current application, the parallel code showed a quite good scalability (for Wolff, as well as Metropolis, algorithms) up to 8 processors available on one node of the cluster. The simulation results are collected in Tabs. 1 to 6.

4 Monte Carlo analysis

4.1 Critical parameters

According to the finite–size scaling theory, $U$ behaves asymptotically as $U = F((\beta - \beta_c)L^{1/\nu})$ (see, e. g., the references in [22]) for large lattice sizes in vicinity of the critical point, where $F(z)$ is a smooth function of $z$. Hence, the pseudo-critical coupling $\tilde{\beta}_c$ behaves as

$$\tilde{\beta}_c = \beta_c + aL^{-1/\nu}$$

at large $L$, where the coefficient $a$ depends on $U$ and $\lambda$. Since $\nu = 1$ holds in this model, it is meaningful to plot $\tilde{\beta}_c$ vs $1/L$ as it is done in Fig. 1.

At the critical $U$ value, $U = U^*$, the coefficient $a$ vanishes and the asymptotic convergence of $\tilde{\beta}_c$ to the critical coupling $\beta_c$ is faster than $\sim 1/L$. As one can judge from Fig. 1, it
Table 2: The same quantities as in Tab. 1 for $\lambda = 0.1$ and $U = 1.17$.

| $L$ | $\tilde{\beta}_c$ | $\chi/L^{7/4}$ | $-(\partial U/\partial \beta)/L$ |
|-----|------------------|-----------------|---------------------------------|
| 4   | 0.656921(33)     | 2.33502(44)     | 0.81724(54)                      |
| 8   | 0.620328(19)     | 1.69631(38)     | 0.93296(66)                      |
| 16  | 0.609627(11)     | 1.38710(31)     | 1.04487(80)                      |
| 32  | 0.6070815(45)    | 1.25571(21)     | 1.12318(88)                      |
| 64  | 0.6065724(20)    | 1.20542(20)     | 1.16541(95)                      |
| 128 | 0.6064868(11)    | 1.18770(18)     | 1.1846(12)                       |

Table 3: The same quantities as in Tab. 1 for $\lambda = 1$ and $U = 2$.

| $L$ | $\tilde{\beta}_c$ | $\chi/L^{7/4}$ | $-(\partial U/\partial \beta)/L$ |
|-----|------------------|-----------------|---------------------------------|
| 4   | 0.512944(44)     | 0.315623(68)    | 1.27395(46)                     |
| 8   | 0.590002(23)     | 0.270748(70)    | 1.26492(60)                     |
| 16  | 0.633498(12)     | 0.251359(59)    | 1.25682(77)                     |
| 32  | 0.6567123(57)    | 0.244758(59)    | 1.25660(94)                     |
| 64  | 0.6686081(32)    | 0.243132(52)    | 1.26232(87)                     |
| 128 | 0.6745993(15)    | 0.242899(53)    | 1.2624(10)                      |

Table 4: The same quantities as in Tab. 1 for $\lambda = 1$ and $U = 1.17$.

| $L$ | $\tilde{\beta}_c$ | $\chi/L^{7/4}$ | $-(\partial U/\partial \beta)/L$ |
|-----|------------------|-----------------|---------------------------------|
| 4   | 0.721485(47)     | 1.03765(11)     | 0.43568(14)                     |
| 8   | 0.689059(25)     | 0.92831(10)     | 0.49442(21)                     |
| 16  | 0.682119(12)     | 0.878352(92)    | 0.52170(23)                     |
| 32  | 0.6808084(54)    | 0.858362(86)    | 0.53468(29)                     |
| 64  | 0.6805951(29)    | 0.850860(92)    | 0.53955(35)                     |
| 128 | 0.6805810(14)    | 0.848395(83)    | 0.54185(35)                     |

Table 5: The same quantities as in Tab. 1 for $\lambda = 10$ and $U = 2$.

| $L$ | $\tilde{\beta}_c$ | $\chi/L^{7/4}$ | $-(\partial U/\partial \beta)/L$ |
|-----|------------------|-----------------|---------------------------------|
| 4   | 0.287517(24)     | 0.367807(43)    | 1.53677(43)                     |
| 8   | 0.374876(16)     | 0.332360(53)    | 1.35076(57)                     |
| 16  | 0.4217519(94)    | 0.314668(54)    | 1.27009(55)                     |
| 32  | 0.4461113(40)    | 0.305725(47)    | 1.23283(56)                     |
| 64  | 0.4585487(24)    | 0.301387(50)    | 1.21446(69)                     |
| 128 | 0.4648281(11)    | 0.299070(47)    | 1.20451(70)                     |
Table 6: The same quantities as in Tab. [1] for $\lambda = 10$ and $U = 1.17$.

| L  | $\tilde{\beta}_c$       | $\chi L^{7/4}$       | $-(\partial U / \partial \beta) / L$ |
|----|-------------------------|----------------------|--------------------------------------|
| 4  | 0.464345(24)            | 1.002728(36)         | 0.538993(70)                         |
| 8  | 0.469597(15)            | 1.021750(44)         | 0.52244(11)                          |
| 16 | 0.470713(3)            | 1.028566(61)         | 0.51525(16)                          |
| 32 | 0.4710035(44)          | 1.031059(73)         | 0.51305(20)                          |
| 64 | 0.4710872(17)          | 1.031578(60)         | 0.51236(19)                          |
| 128| 0.4711247(12)          | 1.031885(70)         | 0.51233(25)                          |

Figure 1: The pseudo-critical coupling $\tilde{\beta}_c$ vs $1/L$ for $\lambda = 0.1$ (left), $\lambda = 1$ (middle) and $\lambda = 10$ (right). The upper plots (squares) and the lower plots (circles) refer to the cases $U = 1.17$ and $U = 2$, respectively. Statistical errors are much smaller than the symbol size.
occurs at $U$ about $1.17$ for all $\lambda$. In this sense $U^*$ is universal. Our techniques allow us to recalculate the data for slightly different $U$, such as $U = 1.16$ and $U = 1.18$, by using the Taylor series expansion. In this way, we have verified that $1.17$ is likely to be the correct rounded value of $U^*$. It agrees with the known MC estimate of the critical Binder cumulant $B^* = 1 - U^*/3 \approx 0.61$ referred in $[9]$. The fact that $U^* \approx 1.17$ is the correct rounded value is confirmed by an accurate estimation in $[17]$, yielding $U^* = 1.1679229 \pm 0.0000047$ for the 2D Ising model, corresponding to $\lambda \to \infty$.

Comparing different fits of $\beta_c$, we have estimated the critical coupling as $\beta_c = 0.606479 \pm 0.000004$ at $\lambda = 0.1$, $\beta_c = 0.68059 \pm 0.00003$ at $\lambda = 1$ and $\beta_c = 0.47116 \pm 0.00006$ at $\lambda = 10$. According to $[11]$, the fluctuations of $\varphi_i^2$ are suppressed at $\lambda \to \infty$ in such a way that $\varphi_i^2 \to 1$ holds for relevant spin configurations with finite values of $H/(k_B T)$ per spin. It means that the actual $\varphi^4$ model becomes equivalent to the Ising model, where $\varphi_i = \pm 1$, in the limit $\lambda \to \infty$, further called the Ising limit. Thus, it is not surprising that $\beta_c$ approaches the known exact value $\frac{1}{4} \ln (1 + \sqrt{2}) = 0.44068679 \ldots$ of the 2D Ising model $[20]$ when $\lambda$ becomes large.

It is somewhat unexpected that $\beta_c$ appears to be a non-monotonous function of $\lambda$. It can be explained by two competing effects. On the one hand, fluctuations increase with decreasing of $\lambda$, and therefore $\beta_c$ tends to increase. Indeed, $\beta_c$ at $\lambda = 1$ is remarkably larger than that at $\lambda = 10$. On the other hand, an effective interaction between spins becomes stronger for small $\lambda$ because $\langle | \varphi_i | \rangle$ and therefore also $\langle \varphi_i \varphi_j \rangle$ for neighboring spins increases in this case. It can explain the fact that $\beta_c$ at $\lambda = 0.1$ is slightly smaller than that at $\lambda = 1$.

### 4.2 Analysis of corrections to scaling

According to the idea that the actual $\varphi^4$ model is described by the same critical exponents $\gamma = 7/4$ and $\nu = 1$ as the 2D Ising model, it is expected that $\chi/L^{7/4}$ and $(\partial U/\partial \beta)/L$ at $\beta = \tilde{\beta}_c(L)$ tend to some nonzero constants at $L \to \infty$. The data in Tabs. $[1]$ to $[3]$ are consistent with this idea. The $L$–dependence of $\chi/L^{7/4}$ and $(\partial U/\partial \beta)/L$ is caused by corrections to scaling. Thus, we have

\begin{equation}
\frac{\chi}{L^{7/4}} = a_0 + \sum_{k \geq 1} a_k L^{-\omega_k},
\end{equation}

\begin{equation}
\frac{1}{L} \frac{\partial U}{\partial \beta} = b_0 + \sum_{k \geq 1} b_k L^{-\omega_k}
\end{equation}

for large $L$ at $\beta = \tilde{\beta}_c(L)$, where $a_k$ and $b_k$ are expansion coefficients and $\omega_k$ are correction–to-scaling exponents. The existence of trivial corrections to scaling with integer $\omega_k$ is expected, since such corrections appear in the 2D Ising model. Hence, if nontrivial corrections with $\omega_k < 1$ do not exist, then the convergence of $\chi/L^{7/4}$ and $(\partial U/\partial \beta)/L$ to the asymptotic values $a_0$ and $b_0$ is expected to be linear in $1/L$. The $\chi/L^{7/4}$ vs $1/L$ and $(\partial U/\partial \beta)/L$ vs $1/L$ plots are shown in Figs. $[2]$ and $[3]$ respectively.

The plots in Figs. $[2]$ and $[3]$ are rather linear at $\lambda = 10$ and become more nonlinear when $\lambda$ is decreased. The nonlinearity is most pronounced at $\lambda = 0.1$ and $U = 2$. It is interesting to note that the $\chi/L^{7/4}$ vs $1/L$ plots in Fig. $[2]$ show a very good linearity for small $L$, as indicated in Fig. $[2]$ by linear fits over $L \leq 12$. At $\lambda = 1$ and, particularly, $\lambda = 0.1$ these plots become more nonlinear for larger lattice sizes. In fact, an opposite behavior would be normally expected if these plots are asymptotically linear at $L \to \infty$. 
Figure 2: The $\chi/L^{7/4}$ vs $1/L$ plots for $U = 2$ (left) and $U = 1.17$ (right) at $\lambda = 0.1$ (circles), $\lambda = 1$ (diamonds) and $\lambda = 10$ (squares). Statistical errors are much smaller than the symbol size. The straight lines are the linear fits over $4 \leq L \leq 12$, except the case $\lambda = 10$ and $U = 2$, where all data points are included in the fit.

Figure 3: The $-(\partial U/\partial \beta)/L$ vs $1/L$ plots for $U = 2$ (left) and $U = 1.17$ (right) at $\lambda = 0.1$ (circles), $\lambda = 1$ (diamonds) and $\lambda = 10$ (squares). The plots at $\lambda = 0.1$ are shifted by $-0.8$ and $-0.25$ at $U = 2$ and $U = 1.17$, respectively. Statistical errors are within the symbol size.
We have performed a refined analysis in the case of $\lambda = 0.1$ and $U = 2$, where the strongest nonlinearity has been observed, in order to check the possible nontrivial corrections to scaling. We have found that the $(\partial U/\partial \beta)/L$ vs $L^{-1/2}$ plot is approximately linear within the whole range of sizes $4 \leq L \leq 1536$, as it can be seen in Fig. 4. The fit to

$$\frac{1}{L} \frac{\partial U}{\partial \beta} = b_0 + b_1 L^{-\omega}$$

with fixed $\omega = 1/2$ is fairly good within $16 \leq L \leq 1536$. The $\chi^2$ of the fit per degree of freedom, i.e., $\chi^2$/d.o.f. is 1.22 in this case. The fit is shown in Fig. 4 by straight line. Considering $\omega$ as a fit parameter, we obtain $\omega = 0.474(26)$ with $\chi^2$/d.o.f. = 1.22. A reasonable explanation of these results is such that (7) contains a term with the exponent 1/2, which is the leading term within $16 \leq L \leq 1536$, at least, for the actual parameters $\lambda = 0.1$ and $U = 2$. From the numerical analysis alone one can judge that this exponent is about 1/2. However, it is likely to be true that its value is exactly 1/2, owing to the arguments of the conformal field theory [27], predicting rational values of the critical exponents for 2D models and quite simple ones for the 2D Ising model.

According to the analytical arguments in Sec. 2 a correction term with exponent 3/4 exists in the two-point correlation function. Thus, it is expected in (5) – (6), as well. As explained in Sec. 2 extra correction terms with smaller exponents are also possible. The current analysis provides an evidence for such a correction with exponent 1/2. According to the predictions of (6), a correction term with exponent 1/4 is also expected. Our analysis of the $(\partial U/\partial \beta)/L$ data does not provide any evidence for such a correction. However, there is no contradiction with this conception, if we assume that the amplitude of the latter correction term is relatively small. In this case the behavior in Fig. 4 should be changed for large enough lattice sizes to the $\sim L^{-1/4}$ asymptotic convergence.

We have analyzed the $\chi/L^{1/4}$ data in Tab. 1 to clarify whether there could exist also a nontrivial correction with the exponent 1/4 in addition to those with correction–to–scaling

Figure 4: The $(\partial U/\partial \beta)/L$ vs $L^{-1/2}$ plot for $\lambda = 0.1$ and $U = 2$. The straight line represents the fit to (7) within $16 \leq L \leq 1536$. 

Table 7: The fit parameters \( a_k \) \((k = 0, 1, 2)\) in \([5]\) and the \(\chi^2/\text{d.o.f.}\) of the fit depending on the range of sizes \(L \in [L_{\text{min}}, L_{\text{max}}]\).

| \(L_{\text{min}}\) | \(L_{\text{max}}\) | \(a_0\)     | \(a_1\)     | \(a_2\)     | \(\chi^2/\text{d.o.f.}\) |
|-----------------|-----------------|-------------|-------------|-------------|-----------------|
| 192             | 1536            | 0.3383(18)  | 0.019(17)   | -0.120(39)  | 0.93            |
| 128             | 1024            | 0.3425(19)  | -0.020(16)  | -0.029(34)  | 1.70            |
| 96              | 768             | 0.3477(18)  | -0.064(14)  | 0.062(28)   | 1.36            |
| 64              | 512             | 0.3592(18)  | -0.154(13)  | 0.234(24)   | 2.59            |
| 48              | 384             | 0.3683(20)  | -0.219(13)  | 0.346(22)   | 2.76            |

exponents \(3/4\) and \(1/2\). For this purpose, first we have fit these data to the ansatz

\[
\chi/L^{7/4} = a_0 + a_1 L^{-1/4} + a_2 L^{-1/2}
\]

within certain interval \(L \in [L_{\text{min}}, L_{\text{max}}]\) with different values of \(L_{\text{min}}\) at \(L_{\text{max}}/L_{\text{min}} = 8\). The fit results for the coefficients \(a_k\) together with the values of \(\chi^2/\text{d.o.f.}\) are collected in Tab.7. The variations of \(a_1\) and \(a_2\) are dependent on the fit interval and indicate that the true asymptotic value of the expansion coefficient \(a_2\), very likely, is negative, whereas that of \(a_1\) might be positive. Note that the \(\chi^2/\text{d.o.f.}\) is about unity for moderately good fits and smaller for better fits. The remarkable variations of \(a_1\) and \(a_2\), as well as the values of \(\chi^2/\text{d.o.f.}\) in Tab.7 show that corrections of higher order than those included in \([8]\) are relevant. If the next expansion term is \(\propto L^{-3/4}\), then the \(O\left(L_{\text{min}}^{-1/2}\right)\) variation in \(a_1\) and the \(O\left(L_{\text{min}}^{-1/4}\right)\) variation in \(a_2\) are expected at \(L_{\text{min}} \to \infty\) at a fixed \(L_{\text{max}}/L_{\text{min}}\). It follows from the fact that the corresponding variations in \(a_1 L^{-1/4}\) and \(a_2 L^{-1/2}\) approximately compensate the neglected remainder term of the order \(O\left(L^{-3/4}\right)\) within \(L \in [L_{\text{min}}, L_{\text{max}}]\) to minimize the \(\chi^2\) of the fit.

If the remainder term is, indeed, of the order \(O\left(L^{-3/4}\right)\), then the variations in \(a_1\) and \(a_2\) can be remarkably reduced by adding a term \(a_3 L^{-3/4}\) to \([5]\). Following this idea, we have performed fits to

\[
\chi/L^{7/4} = a_0 + a_1 L^{-1/4} + a_2 L^{-1/2} + a_3 L^{-3/4}
\]

with different fixed values of \(a_3\). The latter coefficient has been fixed, since a consideration of \(a_3\) as an extra fit parameter results in too large statistical errors. We have found that the coefficients \(a_k\) with \(k = 0, 1, 2\) are stabilized and the quality of fits is greatly improved at certain values of \(a_3\) about 1.76. The results for \(a_3 = 1.76\) are shown in Tab.8.

The values of \(a_k\) in Tab.8 almost do not change within the considered range of sizes, and all fits are good. From this we conclude: (i) the actual \(\chi/L^{7/4}\) data are well consistent with the expansion \([9]\) in powers of \(L^{-1/4}\); (ii) if this is the correct expansion, then the true asymptotic expansion coefficients \(a_0, a_1\) and \(a_2\) are expected to be quite similar to those in Tab.8. It provides an evidence for the existence of a \(\propto L^{-1/4}\) correction term in addition to \(\propto L^{-1/2}\) and \(\propto L^{-3/4}\) corrections, supported by our previous arguments.

The overall behavior of the \((\partial U/\partial \beta)/L\) and \(\chi/L^{7/4}\) data can be interpreted in such a way that nontrivial corrections in the form of the expansion in powers of \(L^{-1/4}\) generally exist, although corrections with \(\omega_k < 1\) in \([5]\) and \([6]\) can be well detectable only for
Table 8: The fit parameters $a_k$ ($k = 0, 1, 2$) in (9) and the $\chi^2$/d.o.f. of the fit at a fixed coefficient $a_3 = 1.76$ depending on the range of sizes $L \in [L_{\text{min}}, L_{\text{max}}]$.

| $L_{\text{min}}$ | $L_{\text{max}}$ | $a_0$          | $a_1$          | $a_2$          | $\chi^2$/d.o.f. |
|------------------|------------------|----------------|----------------|----------------|----------------|
| 192              | 1536             | 0.3223(18)     | 0.253(17)      | -1.241(39)     | 0.72           |
| 128              | 1024             | 0.3203(19)     | 0.271(16)      | -1.278(34)     | 0.46           |
| 96               | 768              | 0.3209(18)     | 0.266(14)      | -1.269(28)     | 0.35           |
| 64               | 512              | 0.3223(18)     | 0.255(13)      | -1.248(24)     | 0.39           |
| 48               | 384              | 0.3225(20)     | 0.253(13)      | -1.246(22)     | 0.38           |

small values of $\lambda$, such as $\lambda = 0.1$, since the amplitudes of these correction terms decrease with increasing of $\lambda$ and approaching the Ising limit $\lambda \to \infty$. At large $\lambda$, the expansion coefficients at $1/L$ are relatively large, which explains the fact that the $(\partial U/\partial \beta)/L$ vs $1/L$ and $\chi/L^{7/4}$ vs $1/L$ plots look almost linear at $\lambda = 10$.

5 Summary and conclusions

Corrections to scaling in the scalar 2D $\varphi^4$ model have been studied based on analytical arguments (Sec. 2) and Monte Carlo analysis (Sec. 4.2). Our analysis supports the finite-size corrections near criticality, representable by an expansion of a correction factor in powers of $L^{-1/4}$. The analytical arguments show the existence of such an expansion term, which is proportional to $L^{-3/4}$, whereas the MC analysis of the $(\partial U/\partial \beta)/L$ data for lattice sizes $4 \leq L \leq 1536$ provides an evidence for a $\propto L^{-1/2}$ correction, taking into account also the arguments of the conformal field theory, suggesting that the exponents should be simple rational numbers in the two-dimensional case. Moreover, the analysis of the $\chi/L^{7/4}$ data shows that a non-vanishing $\propto L^{-1/4}$ term, most probably, also exists if the correct expansion is, indeed, representable by an expansion in powers of $L^{-1/4}$.

However, these nontrivial corrections can be well detectable only at small $\varphi^4$ coupling constants $\lambda$, such as $\lambda = 0.1$. They become relatively small and the trivial $\propto L^{-1}$ correction dominates at large $\lambda$ values, or approaching the Ising limit $\lambda \to \infty$.

Apart from corrections to scaling, we have estimated the critical parameters ($\beta_c$ depending on $\lambda$, as well as $U^*$) in Sec. 4.1 and have discussed an interesting phenomenon that the critical temperature $(1/\beta_c)$ appears to be a non-monotonous function of $\lambda$.

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References

[1] D. J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena*, World Scientific, Singapore, 1984.
[2] S. K. Ma, *Modern Theory of Critical Phenomena*, W. A. Benjamin, Inc., New York, 1976.

[3] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Clarendon Press, Oxford, 1996.

[4] H. Kleinert, V. Schulte-Frohlinde, *Critical Properties of $\phi^4$ Theories*, World Scientific, Singapore, 2001.

[5] A. Pelissetto, E. Vicari, Phys. Rep. 368 (2002) 549–727.

[6] J. Kaupužs, Ann. Phys. (Berlin) 10 (2001) 299–331.

[7] J. Kaupužs, Int. J. Mod. Phys. A 27, 1250114 (2012)

[8] J. Kaupužs, Canadian J. Phys. 9, 373 (2012)

[9] A. Milchev, D. W. Heermann, K. Binder, J. Stat. Phys. 44, 749 (1986)

[10] R. Toral, A. Chakrabarti, Phys. Rev. B 42, 2445 (1990)

[11] B. Mehling, B. M. Forrest, Z. Phys. B 89, 89 (1992)

[12] R. Kenna, D. A. Johnston, W. Janke, Phys. Rev. Lett. 97, 155702 (2006); Erratum – ibid 97, 169901 (2006)

[13] J. Kaupužs, Int. J. Mod. Phys. C 17, 1095 (2006)

[14] H. Au-Yang, J. H. H. Perk, Int. J. Mod. Phys. B 16, 2089 (2002)

[15] Y. Chan, A. J. Guttman, B. G. Nickel, J. H. H. Perk, J. Stat. Phys. 145, 549 (2011)

[16] M. Caselle, M. Hasenbusch, A. Pelissetto, E. Vicari, J. Phys. A 35, 4861 (2002)

[17] J. Salas, A. D. Sokal, J. Stat. Phys. 98, 551 (2000)

[18] W.P. Orrick, B. Nickel, A.J. Guttmann and J.H.H. Perk, J. Stat. Phys. 102, 795 (2001)

[19] A. Aharony, M. E. Fisher, Phys. Rev. Lett. 45, 679 (1980)

[20] A. Aharony, M. E. Fisher, Phys. Rev. B 27, 4394 (1983)

[21] M. Barma, M. Fisher, Phys. Rev. Lett. 53, 1935 (1984)

[22] M. Hasenbusch, J. Phys. A: Math. Gen. 32, 4851 (1999)

[23] M. E. J. Newman, G. T. Barkema, Monte Carlo Methods in Statistical Physics, Clarendon Press, Oxford, 1999

[24] J. Kaupužs, J. Rimšāns, R. V. N. Melnik, Phys. Rev. E 81, 026701 (2010).

[25] J. Kaupužs, J. Rimšāns, R. V. N. Melnik, Ukr. J. Phys. 56, 845 (2011)

[26] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London, 1989.

[27] P.D. Francesco, P. Mathieu, D. Sénéchal, *Conformal Field Theory*, Springer, New York, 1997.