Putting the Prisoner’s Dilemma in Context

L. A. Khodarinova† and J. N. Webb

†Magnetic Resonance Centre, School of Physics and Astronomy, University of Nottingham, Nottingham, England NG7 2RD, e-mail: LarisaKhodarinova@hotmail.com

Abstract. The standard iterated prisoner’s dilemma is an unrealistic model of social behaviour because it forces individuals to participate in the interaction. We analyse a model in which players have the option of ending their association. If the payoff for living alone is neither too high nor too low then the potential for cooperative behaviour is enhanced. For some parameter values it is also possible for a polymorphic population of defectors and conditional cooperators to be stable.

INTRODUCTION

The iterated, or repeated, prisoner’s dilemma is the most popular model of social interactions [1]. Since its inception the basic model has been modified in many ways (see Dugatkin [2] for a review). However, in all these versions it is assumed that the players must engage in the interaction and have no opportunity to end it. This unrealistic feature is just one facet of the more general assumption that one particular social interaction may be considered in isolation from all others that an individual may face.

In this paper we use the framework of stochastic games [3] to consider a version of the iterated prisoner’s dilemma in which the players may choose to discontinue their association. We assume that once the partnership has been dissolved by one or more of the players, then each receives the same, fixed, per-period payoff. This is probably the simplest way that an interaction can be considered as being dependent on other situations in which individuals find themselves during a complex and, at least partly, social life. We will use the standard replicator dynamics [4] to investigate the effect that the existence of this outside option has on the evolution of cooperative behaviour in a population of players.

THE MODEL

A general stochastic game has three major components: the set of states, the games played in each of these states and the (possibly behaviour-dependent) probabilities for transition between the states. In our model the states represent the different contexts in which players may interact, so we will refer to them as context games.

We consider an interaction described by the following multi-state, stochastic game. There are three possible context-games (states) \(G_0, G_1\) and \(G_2\). The interaction starts with context-game \(G_0\). In this game the players make the decision about whether or not they wish to initiate or
continue an association. The first player and the second player choose between two possible actions: \( A = \) “associate” or \( B = \) “break up”. There are no payoffs directly associated with this decision. Context-game \( G_1 \) represents some specific activity in which the individuals can participate together. It is modelled by the prisoner’s dilemma and the players choose between the possible actions: \( C = \) “cooperation” or \( D = \) “defection”. Context-game \( G_2 \) can be considered as a background state representing the situation when there is no interaction or association between the players. There is only one possible action: \( L = \) “be alone”.

The actions chosen define both immediate payoffs to the individuals and future transition probabilities. The immediate payoffs collected by the players are given in table 1. The first entry in each payoff pair contains the payoff to the player \( P_1 \), who selects the row action, the second is for the player \( P_2 \), who selects the column action. In this paper we are considering an extension of the standard iterated prisoner’s dilemma, for which the following inequalities hold in \( G_1 \).

\[
t > r > p > s \geq 0.
\]

Transition probabilities, which are determined by the choice of actions are presented in table 1 as a set of three numbers. This set of numbers appears in square brackets in each cell of the matrices. Here the first, second or third number is, respectively, the probability that context-game \( G_0 \), \( G_1 \) or \( G_2 \) is played at the next round. The probabilities are defined by the following rules. If context-game \( G_0 \) is played and action \( A = \) “associate” is chosen by both players, at the next round context-game \( G_1 \) is played; if action \( B = \) “break up” is chosen by at least one player, context-game \( G_2 \) is played at the next round with probability 1. Whatever actions are chosen when context-game \( G_1 \) is played,
context-game $G_0$ is played at the next round with probability 1. If context-game $G_2$ is played, at the next round context-game $G_2$ is played again with probability 1.

We assume that after playing context game $G_0$, players survive to play game $G_1$ or $G_2$ (as appropriate) with probability 1. After playing context games $G_1$ or $G_2$ players survive to the next round with probability $\beta$ ($0 \leq \beta < 1$). In principle, these survival probabilities could be different but, for simplicity, we will assume they are equal.

As with the iterated prisoner’s dilemma there is an infinite number of pure strategies that could be considered. We will initially restrict our attention to the following three strategies.

- **Conditional cooperation** (which we denote $\sigma_C$). A player following this strategy will initially “Associate” in $G_0$ then “Cooperate” in $G_1$; if this behaviour is reciprocated then the player will continue to associate and cooperate; otherwise it will choose “Break up” in $G_0$.
- **Defection** (which we denote $\sigma_D$). A player following this rather pathological strategy will “Associate” in $G_0$ and then “Defect” in $G_1$.
- **An unsociable strategy** (which we denote $\sigma_B$). A player following this strategy will “Break up” in $G_0$. Strictly speaking this is a set of strategies since any behaviour is allowed in $G_1$. However, since we do not consider the possibility that players make errors, the behaviour in $G_1$ does not affect payoffs. Consequently we ignore this technicality.

The consequences of introducing a fourth strategy of unconditional cooperation will be considered later.

**Evolutionary Dynamics**

We set up the evolutionary dynamics by considering an infinitely large population of individuals who adopt one of the three pure strategies. The payoffs in the repeated game, $\pi(\sigma, \sigma')$ for adopting strategy $\sigma$ against an opponent who adopts strategy $\sigma'$ are

$$A = \begin{bmatrix}
\pi(\sigma_C, \sigma_C) & \pi(\sigma_C, \sigma_D) & \pi(\sigma_C, \sigma_B) \\
\pi(\sigma_D, \sigma_C) & \pi(\sigma_D, \sigma_D) & \pi(\sigma_D, \sigma_B) \\
\pi(\sigma_B, \sigma_C) & \pi(\sigma_B, \sigma_D) & \pi(\sigma_B, \sigma_B)
\end{bmatrix} = \frac{1}{1 - \beta} \begin{bmatrix}
r & s(1 - \beta) + \beta z & z \\
t(1 - \beta) + \beta z & p & z \\
z & z & z
\end{bmatrix}. \quad (1)
$$

Let $x_1$ and $x_2$ be the proportions of individuals who adopt $\sigma_C$ and $\sigma_D$ respectively. The proportion of individuals using $\sigma_B$ is then $1 - x_1 - x_2$. The standard replicator dynamics is then two equations describing the evolution of a point $x = (x_1, x_2)$ in the domain

$$\Delta = \{(x_1, x_2) : (x_1 \geq 0) \cap (x_2 \geq 0) \cap (x_1 + x_2 \leq 1)\}. \quad (2)$$

Denote

$$a = (z - r) - \gamma (z - t); \quad b = \gamma (z - s) - (z - p);$$
$$c = (z - r); \quad f = \gamma (z - s) \quad \text{where} \quad \gamma = 1 - \beta.$$
Then the Replicator Dynamics can be written as the following system of equations.

\[
\dot{x}_1 = x_1 \left( c x_2^2 + (f + c - a) x_1 x_2 + (f - b) x_2^2 - c x_1 - f x_2 \right)
\]

\[
\dot{x}_2 = x_2 \left( c x_1^2 + (f + c - a) x_1 x_2 + (f - b) x_2^2 + (a - c) x_1 + (b - f) x_2 \right)
\]

Although this system is integrable for arbitrary choices of parameter values \([?]\), the general solution given in appendix A is not easy to work with. We will now introduce a commonly used set of values for the prisoner’s dilemma context game \(G_1\) to reduce the number of parameters, and we will use the standard linearization approach to study how the solution depends on the value of the outside option, \(z\), and the survival probability, \(\beta\). Accordingly we set

\[ r = 3, \quad s = 0, \quad t = 5, \quad p = 1. \]

The payoff matrix then becomes

\[
A = \frac{1}{1 - \beta} \begin{bmatrix}
3 & \beta z & z \\
5 (1 - \beta) + \beta z & 1 & z \\
\beta z & z & z
\end{bmatrix}.
\]

and the Replicator Dynamics is as follows.

\[
\dot{x}_1 = \frac{x_1}{1 - \beta} \left( (z - 3) x_1^2 + (1 - \beta) (2z - 5) x_1 x_2 + (z - 1) x_2^2 + (3 - z) x_1 + (\beta z - z) x_2 \right)
\]

\[
\dot{x}_2 = \frac{x_2}{1 - \beta} \left( (z - 3) x_1^2 + (1 - \beta) (2z - 5) x_1 x_2 + (z - 1) x_2^2 + (1 - \beta) (5 - z) x_1 + (1 - z) x_2 \right)
\]

There are four fixed points for this Dynamics and a standard linearization analysis produces the results shown in table 2.

| Point                      | eigenvectors | eigenvalues          |
|----------------------------|--------------|----------------------|
| \(\{0, 0\}\)              | \(e_1 = (1, 0)\) | \(\lambda_1 = 0\)   |
|                            | \(e_2 = (0, 1)\) | \(\lambda_2 = 0\)   |
| \(\{1, 0\}\)              | \(e_1 = (1, 0)\) | \(\lambda_1 = \frac{z-3}{1-\beta}\) |
|                            | \(e_2 = (-1, 1)\) | \(\lambda_2 = \frac{z-1}{1-\beta}\) |
| \(\{0, 1\}\)              | \(e_1 = (1, -1)\) | \(\lambda_1 = \frac{\beta z - 1}{1-\beta}\) |
|                            | \(e_2 = (0, 1)\) | \(\lambda_2 = \frac{1}{1-\beta}\) |
| \(\left\{\frac{\beta z - 1}{1+2\beta z - 5\beta}, \frac{\beta z + 2 - 5\beta}{1+2\beta z - 5\beta}\right\}\) | \(e_1 = (-1, 1)\) | \(\lambda_1 = \frac{(1-\beta)(\beta z - 5\beta + 2)}{(1-\beta)(2\beta z - 5\beta + 1)}\) |
|                            | \(e_2 = \left(1, \frac{\beta z - 5\beta + 2}{\beta z - 1}\right)\) | \(\lambda_2 = \frac{\beta(2-\beta)(z-5)+9+z}{\beta(2-\beta)(z-5)+9+z}\) |

Table 2. Eigenvalues and eigenvectors for the fixed points in the replicator dynamics system given by equations (3).

Depending on the values of the parameters \(z\) and \(\beta\) we obtain different solutions for the dynamics \(\textit{3}\). The \(\beta - z\) parameter space can be divided into 10 regions (see figure 1) which have
qualitatively different pictures of the dynamics (see figure 2). The main features of this overall picture can be summarized as follows. If \( z > 3 \) then the population evolves towards a monomorphic state in which every player uses the unsociable strategy \( \sigma_B \). If \( z < 1 \) then the picture resembles the iterated prisoner’s dilemma: if \( \beta \) is small then defection is stable, but if \( \beta \) is large populations using either defection or conditional cooperation are asymptotically stable and the population which arises depends on the initial conditions. The most interesting dynamics occur when \( \beta \) is large and \( 1 < z < 3 \) (labelled as regions VI to IX in figure 1). If \( \beta \) is large and \( z < \frac{5\beta - 2}{\beta} \) then conditional cooperation is the only asymptotically stable behaviour, and in region VII this is the endpoint of all trajectories which start in the interior of the simplex. In region VI a polymorphic population is stable: a proportion of players, \( x \), use the conditional cooperative strategy and a proportion, \( 1 - x \), defect where \( x = \frac{\beta z - 1}{1 + 2\beta - 3} \). In this population the proportion of individuals that would be observed in cooperative partnerships is \( x^2 \); the proportion of individuals involved in partnerships for which mutual defection was the norm would be \( (1 - x)^2 \); and a proportion \( 2x(1 - x) \) of individuals would be living alone.

**Introducing unconditional cooperators**

In region VII of the \( \beta - z \) parameter space we have found that conditional cooperation is asymptotically stable. It is pertinent to ask whether this property would be destroyed if we allowed individuals to use the “sucker” strategy of unconditional cooperation (which we denote \( \sigma_S \)). Recall that in the iterated prisoner’s dilemma, tit-for-tat is not asymptotically stable due to the presence of unconditional cooperators. Similarly, it is conceivable that the polymorphic population which is stable in region VI could be destabilized by the introduction of a strategy of unconditional cooperation.

We introduce a proportion \( x_3 \) of players who use the strategy of unconditional cooperation, \( \sigma_S \). These players associate in \( G_0 \) and cooperate in \( G_1 \) whatever their opponent does. (The proportion of individuals using \( \sigma_B \) is then \( 1 - x_1 - x_2 - x_3 \)) The new payoff matrix is given by

\[
A = \begin{bmatrix}
\pi(\sigma_C, \sigma_C) & \pi(\sigma_C, \sigma_D) & \pi(\sigma_C, \sigma_S) & \pi(\sigma_C, \sigma_B) \\
\pi(\sigma_D, \sigma_C) & \pi(\sigma_D, \sigma_D) & \pi(\sigma_D, \sigma_S) & \pi(\sigma_D, \sigma_B) \\
\pi(\sigma_S, \sigma_C) & \pi(\sigma_S, \sigma_D) & \pi(\sigma_S, \sigma_S) & \pi(\sigma_S, \sigma_B) \\
\pi(\sigma_B, \sigma_C) & \pi(\sigma_B, \sigma_D) & \pi(\sigma_B, \sigma_S) & \pi(\sigma_B, \sigma_B)
\end{bmatrix} = \frac{1}{1 - \beta} \begin{bmatrix}
3 & \beta z & 3 & z \\
5(1 - \beta) + \beta z & 1 & 5 & z \\
3 & 0 & 3 & z \\
z & z & z & z
\end{bmatrix}
\]

An analysis of the corresponding Replicator dynamics leads to the dynamics shown in figures 3 and 4 for regions VI and VII respectively (see appendix B for details). These figures show the dynamics for particular values of \( z \) and \( \beta \) but the pictures are qualitatively similar for any values of these parameters in the appropriate range. From these figures we can see that the polymorphic population remains asymptotically stable in region VI. In region VII, the population of conditional cooperators is no longer asymptotically stable. However, populations which consist of mixtures of
conditional and unconditional cooperators are the only end points of all solution trajectories which start in the interior of the simplex.

**Discussion**

A minimal version of the iterated prisoner’s dilemma deals with a population consisting of unconditional cooperators, unconditional defectors and conditional cooperators (such as tit-for-tat). In that model there is a threshold problem: cooperative behaviour only evolves if the initial proportion of conditional cooperators exceeds some value \([5, 2]\). Although it is sometimes suggested that the always defect strategy is an ESS or that the corresponding population is asymptotically
Figure 2. Qualitative pictures of the dynamics in the different regions of the $\beta - z$ parameter space. The regions are labelled according to figure 1. Point $a$ corresponds to a population which consists of 100% of players using strategy $\sigma_B$. Point $b$ corresponds to a population which consists of 100% of players using strategy $\sigma_C$. Point $c$ corresponds to a population which consists of 100% of players using strategy $\sigma_D$. Point $d$ corresponds to a polymorphic population with players using either $\sigma_C$ or $\sigma_D$. Asymptotically stable points are shown as solid circles, the other fixed points are shown as open circles.

stable, this is not the case. If sufficiently many varied strategies are introduced then the barrier can be removed [6].
Region VI: \( z = \frac{25}{10} \) and \( \beta = \frac{75}{100} \).

Region VII: \( z = \frac{25}{10} \) and \( \beta = \frac{9}{10} \).

**Figure 3.** Qualitative picture of the dynamics for the replicator system with payoff matrix given by equation (4). Vertex \( S \) corresponds to a population which consists of 100% of players using strategy \( \sigma_S \). Vertex \( C \) corresponds to a population which consists of 100% of players using strategy \( \sigma_C \). Vertex \( D \) corresponds to a population which consists of 100% of players using strategy \( \sigma_D \). The unlabelled vertex corresponds to a population in which all players live alone.

We have introduced an outside option into the iterated prisoner’s dilemma, which allows individuals to avoid being condemned to maintain an unprofitable interaction of permanent mutual defection. This provides another way of removing the barrier to the evolution of cooperative behaviour. The requirement is that the payoff from the outside option should be neither so poor that it is irrelevant nor so high that everyone opts for a solitary existence. The existence of the outside option also admits a range of parameter values for which a polymorphic population involving defectors and conditional cooperators is asymptotically stable, even in the presence of unconditional cooperators.

Some of the results we have obtained are similar to those obtained for optional public good games, which are multi-player generalizations of the prisoner’s dilemma \[7\]. In these games, as in ours, making participation voluntary enhances the possibilities for cooperation. One difference between the two models is that in the optional public good game rock-scissors-paper style cycles may occur. In our model, such cyclic behaviour does not arise. However, in both models the fixed point representing non-participatory behaviour may be non-hyperbolic. This leads to periods of cooperative behaviour, but eventually the population returns to a state in which everyone lives alone.
The iterated prisoner’s dilemma is an unrealistic model of social interactions because it treats one type of interaction between individuals in isolation from all others. We have shown, by means of a relatively simple example, that the methods of stochastic game theory can be employed to overcome this restriction. The prisoners dilemma has also been criticized as being an unrealistic model of social interactions on other grounds [5]. Our approach is not specific to the prisoner’s dilemma. That context game may be replaced by any other game or, indeed, a game which is randomly selected with a known probability from a set of games [9]. This allows quite complex social behaviour to be analyzed.

APPENDIX A

To integrate the Replicator Dynamics system (3) we make the following coordinate substitutions.

\[ k = \frac{x_2}{x_1} \quad \text{and} \quad l = \frac{1 - x_1 - x_2}{x_1} \]

or

\[ x_1 = \frac{1}{1 + k + l} \quad \text{and} \quad x_2 = \frac{k}{1 + k + l}. \]

Then we have

\[ \dot{k} = k \frac{a + bk}{1 + l + k} \]
\[ \dot{l} = l \frac{c + fk}{1 + l + k}. \]

Solution trajectories can be found by integrating

\[ \frac{dl}{dk} = \frac{l (c + fk)}{k (a + bk)} \]
\[ \frac{dl}{l} = \frac{c}{a} \frac{dk}{k} + \frac{af - bc}{a} \frac{dk}{(a + bk)} \]

This can be done analytically to obtain

\[ bc \ln|k| + (af - bc) \ln|a + bk| = ab \ln|l| + C \]
\[ |k|^bc |a + bk|^(af - bc) = C |l|^{ab} \]

where \( C \) is a constant that depends on the initial conditions. Finally, substituting the expressions for \( k \) and \( l \) into the above formula, we find that the solution trajectories are described by the expression.

\[ \left( \frac{x_2}{x_1} \right)^{bc} \left( \frac{a + b}{x_1} \right)^{(af - bc)} = C \left( \frac{1 - x_1 - x_2}{x_1} \right)^{ab}. \]
The fixed points together with their associated eigenvectors and eigenvalues are given in table 3.

| Population | Point | Eigenvectors | Eigenvalues |
|------------|-------|--------------|-------------|
| 100% of $\sigma_B$ | $\{0,0,0\}$ | $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_2=(0,0,1)$ | $\lambda_1=0$, $\lambda_2=0$, $\lambda_3=0$ |
| 100% of $\sigma_D$ | $\{0,1,0\}$ | $e_1=(-1,1,0)$, $e_2=(0,1,0)$, $e_2=(0,1,-1)$ | $\lambda_1=0$, $\lambda_2=\frac{z-1}{1-\beta}$, $\lambda_3=-\frac{1}{1-\beta}$ |
| 100% of $\sigma_C$ | $\{1,0,0\}$ | $e_1=(1,0,-1)$, $e_2=(-1,1,0)$, $e_3=(1,0,0)$ | $\lambda_1=0$, $\lambda_2=\frac{2-5\beta+\beta z}{1-\beta}$, $\lambda_3=\frac{z-3}{1-\beta}$ |
| 100% of $\sigma_S$ | $\{0,0,1\}$ | $e_1=(1,0,-1)$, $e_2=(0,1,-1)$, $e_3=(0,0,1)$ | $\lambda_1=0$, $\lambda_2=\frac{2}{1-\beta}$, $\lambda_3=\frac{z-3}{1-\beta}$ |
| $\alpha\%$ of $\sigma_C$ and $(1-\alpha)\%$ of $\sigma_S$ | $\{\alpha,0,1-\alpha\}$ | $e_1=(1,0,-1)$, $e_2=\left(\frac{\alpha z-2\alpha z+5\alpha z-2}{2-5\alpha z+\alpha z}, \frac{(\alpha-1)(2\alpha z-5\alpha z+2)}{2-5\alpha z+\alpha z}\right)$, $e_3=(1,0,1-\alpha)$ | $\lambda_1=0$, $\lambda_2=\frac{2-5\beta+\beta z}{1-\beta}$, $\lambda_3=\frac{z-3}{1-\beta}$ |
| $\frac{\beta z-1}{2\beta z-5\beta+1}\%$ of $\sigma_C$ and $\frac{2-5\beta+\beta z}{2\beta z-5\beta+1}\%$ of $\sigma_S$ | $\left\{\frac{\beta z-1}{2\beta z-5\beta+1}, \frac{2-5\beta+\beta z}{2\beta z-5\beta+1}, 0\right\}$ | $e_1=(-1,1,0)$, $e_2=\left(1,2-5\beta+\beta z-1, 0\right)$, $e_3=\left(1,\beta z-5\beta z+2, \frac{2\beta z-5\beta+1}{\beta z-1}\right)$ | $\lambda_1=\frac{(1-\beta)(\beta z-5\beta+2)}{(1-\beta)(\beta z-5\beta+1)}$, $\lambda_2\approx\frac{z\beta(2-\beta)(z-5)+4\beta}{(1-\beta)(2\beta z-5\beta+1)}$, $\lambda_3=\frac{2\beta z(2-\beta)+\beta z}{(1-\beta)(2\beta z-5\beta+1)}$ |

Table 3. Eigenvalues and eigenvectors for the fixed points of the replicator system with payoff matrix given by equation (4).
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