Finite Temperature Renormalization of the $(\phi^3)_6$- and $(\phi^4)_4$-Models at Zero Momentum

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Abstract

A self-consistent renormalization scheme at finite temperature and zero momentum is used together with the finite temperature renormalization group to study the temperature dependence of the mass and the coupling to one-loop order in the $(\phi^3)_6$- and $(\phi^4)_4$-models. It is found that the critical temperature is shifted relative to the naive one-loop result and the coupling constants at the critical temperature get large corrections. In the high temperature limit of the $(\phi^4)_4$-model the coupling decreases.
1 Introduction

Relativistic finite temperature field theory is an important tool when studying phase transitions in the early universe. A detailed understanding of the electroweak phase transition is needed to determine limits of the Higgs mass and possibly rule out the simplest version of the standard model as a theory of electroweak baryogenesis [1]. In this case, as in many others, infrared (IR) problems plague the perturbation theory since factors of $T/M$ ($T$ is the temperature and $M$ a typical mass) enter with higher powers in higher loops. Large $T$ singularities are thus similar to small $M$ singularities in this respect. One way to make the divergence weaker is to sum over an infinite set of diagrams and thereby give a thermal contribution to the mass. In the $(\phi^4)_4$-theory, where the one-loop correction to the mass is momentum independent, such a resummation gives only a $T$ dependent shift of the mass. At large $T$ the mass, corrected in this way, is

\[ M^2(T) = m^2 + \Sigma_\beta(m^2, \lambda, T) \simeq m^2 + \frac{\lambda T^2}{24}, \quad (1) \]

where $\Sigma_\beta$ is the finite temperature part of the self-energy and $m$ is the mass renormalized at zero temperature. When this summation is used iteratively on all internal lines a gap equation is obtained containing all *superdaisy* diagrams in the language of Ref.[2]

\[ M^2(T) = m^2 + \Sigma_\beta(M^2(T), \lambda, T). \quad (2) \]

This type of resummation, or a first iteration of it, was recently used for the standard electroweak theory [3]. A condition for being able to perform the resummation is that the correction is momentum independent, otherwise one gets complicated integral equations (Schwinger-Dyson equations). Such a simple resummation is not possible to carry out for the coupling constant though it may also get large corrections in the IR limit. Its one-loop correction has the following high $T$ behaviour

\[ \lambda(T) \simeq \lambda - \frac{3\lambda^2 T}{16\pi M}. \quad (3) \]

In the IR limit the corrections to mass and coupling are large and the perturbation theory in terms of the zero temperature parameters breaks down. This problem can be circumvented if the renormalization is performed at the temperature in which one interested, so that the mass and coupling take the physical values. Such a renormalization condition absorbs a dynamical change of the parameters into a constant at a given renormalization point. In next section this idea is further supplemented by the use of the temperature renormalization group equation, as derived by Matsumoto et al. [4], to get the $T$ dependence of the renormalized parameters.

The IR singularities occur not only in the high temperature limit but also at a
second order phase transition where the mass becomes zero at finite $T$. The phase transitions in $(\phi^3)_6$- and $(\phi^4)_4$-models are studied in section 3 and 4.

2 Renormalization at finite temperature

It is often emphasized that finite temperature does not introduce new UV divergences and that it is, therefore, enough to renormalize at zero temperature. The physical quantities should anyway be independent of the renormalization scheme. However, to finite order in perturbation theory the physical quantities do indeed depend on finite changes of the renormalization prescription and it is necessary to choose the prescription carefully. QCD is an example where this is important and the running of the coupling constant with temperature depends crucially on the vertex chosen for the renormalization [5, 6]. One strategy for renormalization is that as much as possible of the dynamics should be put into the expansion parameters. The same philosophy was discussed in Ref. [7], though their treatment of the problem was slightly different from the one in this paper.

In the usual renormalization group approach the mass and the coupling in the Feynman rules are chosen to coincide with the measured values at some relevant scale $\mu$. Perturbation theory is then used to compute the mass and the coupling at another scale $\mu'$. If the difference between $\mu$ and $\mu'$ is small one can hope that perturbation theory is good. Finally, the renormalization group is obtained as a differential equation in the limit $\mu' \rightarrow \mu$. The aim of this paper is to use this idea at finite temperature to the $(\phi^3)_6$- and $(\phi^4)_4$-models below the critical temperature.

The temperature renormalization group equations ((T)RGE) were introduced by Matsumoto et al. [4]. It has been pointed out that a naive analogue of the usual RGE is not valid since physical quantities do depend on the temperature. But that is not the point of the TRGE derived in [4]. There, it is only used that the physical quantities should be independent of the temperature at which one chooses to renormalize. This will then determine the $T$ dependence of the physical quantities.

An intuitive understanding of the TRGE can be gained in the following way. Suppose we renormalize the Lagrangian at a temperature $T$ and then compute the mass at $T + \Delta T$. We then get

$$M^2(T + \Delta T) = M^2(T) + \Sigma(M^2(T), T + \Delta T) - \Sigma(M^2(T), T),$$

(4)

where $\Sigma$ is the self-energy. If the difference between $\Sigma(T + \Delta T)$ and $\Sigma(T)$ is large we cannot trust perturbation theory but in the limit $\Delta T \rightarrow 0$ it is expected to be reliable. Therefore we instead derive

$$\frac{dM^2}{dT} = \frac{\partial \Sigma(M^2, T)}{\partial T},$$

(5)
and solve this differential equation. In each step of the integration of Eq. (5) the correction is small and we are more likely to stay within the range of validity of perturbation theory.

A derivation of the TRGE for an unbroken theory can be found in Ref. [4]. When the theory is spontaneously broken the renormalization condition (RC) has to be changed in order for the mass in the propagator to be the physical mass. Let us start from a bare Lagrangian

\[ L = L(\phi_0, m_0^2, g_0) \]

where the mass \( m_0^2 \) and the \( k \)-point coupling \( g_0 \) are fixed constants. They are temperature independent but they have to be infinite to make physical quantities finite in renormalizable theories. We choose to use dimensional regularization so that \( m_0^2 \) and \( g_0 \) depend on \( 1/\epsilon \). Next we rescale the field \( \phi_0 = \phi Z_{1/2} \) so the expectation value of \( \phi \) is finite. This can be considered as a renormalization of \( \bar{h} \).

Then we divide \( L \) into a finite part with which we define the perturbation theory and the rest is considered as counterterms. Typically we have

\[ \frac{m_0^2}{2} \phi_0^2 = \frac{m_0^2 Z}{2} \phi^2 = \frac{m^2}{2} \phi^2 + \frac{1}{2} (m_0^2 Z - m^2) \phi^2 , \]

and similarly for \( g_0 \to g \). This does not change \( L \) and everything computed from \( L \) is formally independent of \( m^2 \) and \( g \). Now we determine \( m_0^2, g_0 \) and \( Z \) from the measured values of the mass and coupling at a given temperature \( T_R \). The physical mass and coupling \( (M_R^2 \text{ and } g_R) \) are related to the excitations around the actual minimum (which may be different from \( \phi = 0 \)) so we shift the field \( \phi \to \phi + \sigma \) and use the following RC (called the \( T_R \)-scheme with the terminology of Ref. [4])

\[ \text{Re}\Gamma^{(1)}(T_R, \sigma_R, M_R^2, g_R) = 0 \]

\[ \text{Re}\Gamma^{(2)}(p(Q_R), T_R, \sigma_R, M_R^2, g_R) = p^2(Q_R) - M_R^2(T_R) \]

\[ \frac{\partial}{\partial p^2} \text{Re}\Gamma^{(2)}(p(Q_R), T_R, \sigma_R, M_R^2, g_R) = 1 \]

\[ \text{Re}\Gamma^{(k)}(p_i(Q_R), T_R, \sigma_R, M_R^2, g_R) = -g_R(T_R) , \]

where \( \Gamma^{(N)} \) are the \( N \)-point functions in the effective action computed from the shifted Lagrangian using \( M_R^2 \) as mass and \( g_R \) as coupling in the perturbation theory. The momentum is chosen in some suitable way (see the discussion in section 3). At finite temperature relaxation processes introduce imaginary parts in \( \Gamma^{(N)} \) so we take the real part in the RC. The first condition ensures that the effective

\footnote{Note that the gap equation (Eq. 2)) gives \( \frac{dM^2}{dT} = \frac{d\Sigma(M^2(T),T)}{dT} \), with total derivatives.}
action is expanded around a stationary point (a minimum should be chosen). These conditions determine \( m_0^2, g_0, Z \) and \( \sigma_R \) in terms of \( M_R^2, g_R, T_R \) and \( Q_R \) (and the scale of dimensional regularization \( \mu \) which is not spelled out). We can then write down the Lagrangian with unshifted fields in terms of the renormalized quantities and counterterms which depend on \( 1/\epsilon \). From this Lagrangian we compute finite \( N \)-point functions and denote them by \( \Gamma^{(N)}(p, T, \sigma; \sigma_R, M_R^2, g_R, T_R) \).

A change in renormalization temperature to \( tT_R \) means that \( \sigma_R, M_R^2 \) and \( g_R \) are different, but the final \( N \)-point function must be the same up to a rescaling of the field:

\[
\Gamma^{(N)}(p, T, \sigma; T_R, \sigma_R(T_R), M_R^2(T_R), g_R(T_R)) \]

\[= \rho(tT_R)^{-N/2} \Gamma^{(N)}(p, T, \sigma; tT_R, \sigma_R(tT_R), M_R^2(tT_R), g_R(tT_R)) , \tag{11}\]

where \( \rho \) is the finite wavefunction renormalization. This invariance imply

\[
\left( T_R \frac{\partial}{\partial T_R} + \eta \frac{\partial}{\partial \sigma_R} + \theta M_R \frac{\partial}{\partial M_R} + \beta \frac{\partial}{\partial g_R} - N\gamma \right) \Gamma^{(N)}(p, T, \sigma; T_R, \sigma_R, M_R^2, g_R) = 0 , \tag{12}\]

where

\[
\eta = T_R \frac{d\sigma_R}{dT_R} , \quad \theta = T_R \frac{dM_R}{dT_R} , \quad \beta = T_R \frac{dg_R}{dT_R} , \quad \gamma = \frac{T_R}{2\rho} \frac{d\rho}{dT_R} . \tag{13}\]

Let us abbreviate the notation and write

\[
\Gamma^{(N)}(p, T, \sigma) = \Gamma^{(N)}(p, T, \sigma; T_R, \sigma_R(T_R), M_R^2(T_R), g_R(T_R)) . \tag{14}\]

Using Eq.(7)-Eq.(10) and Eq.(11) one can derive the TRGE (see Ref.[4])

\[
0 = \left( T \frac{\partial}{\partial T} + \eta \frac{\partial}{\partial \sigma} \right) \Re \Gamma^{(1)}(T, \sigma)|_{T_R,\sigma_R} ,
\]

\[
\theta(T_R) = \gamma \frac{M_R^2 - p^2(Q_R)}{M_R^2} - \frac{1}{2M_R^2} \left( T \frac{\partial}{\partial T} + \eta \frac{\partial}{\partial \sigma} \right) \Re \Gamma^{(2)}(p, T, \sigma)|_{p(Q_R), T_R, \sigma_R} ,
\]

\[
\gamma(T_R) = - \frac{1}{2} \left( T \frac{\partial}{\partial T} + \eta \frac{\partial}{\partial \sigma} \right) \frac{\partial}{\partial p^2} \Re \Gamma^{(2)}(p, T, \sigma)|_{p(Q_R), T_R, \sigma_R} ,
\]

\[
\beta(T_R) = k\gamma - \left( T \frac{\partial}{\partial T} + \eta \frac{\partial}{\partial \sigma} \right) \Re \Gamma^{(k)}(p_i, T, \sigma)|_{p_i(Q_R), T_R, \sigma_R} . \tag{15}\]

Notice that the \( T \) and \( \sigma \) derivatives only act upon the explicit \( T \) and \( \sigma \) dependence. The right hand side can be computed perturbatively in the coupling and then one is
left with a set of coupled differential equations which defines a mass and a coupling running with the temperature.

The equations above are written in a Lorentz invariant way but the actual vertices do not respect the Lorentz symmetry because of the thermal heat-bath. The wavefunction renormalization condition has to be modified in some way and I use \( \frac{\partial}{\partial p^0} \) instead of \( \frac{\partial}{\partial p^\mu} \). The momentum point can be chosen to be \( p(Q_R) = (Q_R, 0, 0, 0) \).

The effective potential can be calculated from the 1-point function of the shifted theory using

\[
\frac{dV(T, \sigma)}{d\sigma} = -\Gamma^{(1)}(T, \sigma). \tag{16}
\]

When \( \Gamma^{(1)} \) is expressed in terms of the renormalized quantities \( \sigma_R, M^2_R \) and \( g_R \) all the infinities (\( \propto 1/\epsilon \)) and the \( \mu \) dependence disappear.

## 3 The \((\phi^3)_6\)-model

The scalar \((\phi^3)_6\)-model, being renormalizable and asymptotically free, has been investigated as a toy model of QCD. The potential is unbounded from below but for small coupling and positive mass it has a local minimum. At finite temperature it is expected that the decay-rate increases and that the system becomes completely unstable above a critical temperature. This was verified in Ref.\[8\] where the critical temperature was found to be

\[
T_{cr} = \left( \frac{180}{\pi} \right)^{1/4} \frac{M}{\sqrt{g}}. \tag{17}
\]

Here \( M \) and \( g \) are the zero temperature mass and coupling constant.

The expression for the effective mass derived in \[8\] can be obtained using the naive TRGE in section 2. The minimum of the effective potential is at \( \langle \phi \rangle = -\frac{g^2 \pi T^4}{360 M^2} \) in the high temperature limit (but below the critical point) and there the effective \( T \) dependent mass is

\[
M^2(T) = M^2 + g \langle \phi \rangle = M^2 - \frac{g^2 \pi T^4}{360 M^2}. \tag{18}
\]

This is a one-loop correction which is a polynomial in \( g \). If we instead solve the differential equation in Eq.\( \[3\] \) we get

\[
M^2(T) = M^2 \sqrt{1 - \frac{g^2 \pi T^4}{180 M^4}}, \tag{19}
\]

which corresponds to the sum of an infinite set of diagrams with leading power in temperature (see Ref.\[8\]) and it is no longer a polynomial in \( g \).
Let us now use the \((\phi^3)_6\)-theory as a simple model to see how the renormalization procedure described in section 8 works out in practise.

The Lagrangian in \(D = 6 - 2\epsilon\) dimensions is

\[
L = \frac{1}{2} (\partial_\mu \phi_0)^2 - h_0 \phi_0 - \frac{m_0^2}{2} \phi_0^2 - \frac{g_0}{6} \phi_0^3
\]

\[
= \frac{1}{2} (\partial_\mu \phi)^2 - \frac{M^2}{2} \phi^2 - \frac{g_\mu^\epsilon}{6} \phi^3 + \frac{Z - 1}{2} (\partial_\mu \phi)^2 - h_0 Z^{1/2} \phi - (m_0^2 Z + \frac{g_0 Z^{3/2} \sigma \mu^{-\epsilon}}{2}) \sigma \mu^{-\epsilon} \phi
\]

\[
- \frac{1}{2} (m_0^2 Z + g_0 Z^{3/2} \sigma \mu^{-\epsilon} - M^2) \phi^2 - \frac{g_0 Z^{3/2}}{6} - \frac{g_\mu^\epsilon}{6} \phi^3 .
\]  

(20)

where \(Z, \sigma, M^2 = m^2 + g\sigma\) and \(g\) are arbitrary parameters and a change of them is a change of the renormalization prescription. To finite order \((in\ g)\) the result depends on how we have chosen them. The Lagrangian above is obtained by scaling the field \((\phi_0 = Z\phi')\), separating the parameters into finite and infinite parts \((m_0^2 Z = m^2 + (m_0^2 Z - m^2) \text{ etc.})\) and finally shifting the field \((\phi' = \phi + \sigma \mu^{-\epsilon})\). The mass scale \(\mu\) is introduced to give \(g\) and \(\sigma\) the same dimensions as in \(D = 6\).

For simplicity I compute wavefunction renormalization for non-zero external momentum but let then the momentum go to zero. All other \(N\)-point functions are computed by taking derivatives of the 1-point function with respect to \(\sigma\), i.e. at zero momentum. It is, after all, the IR limit \((p \to 0)\) that is of most concern and that is presumably taken care of with this prescription. One could, on the other hand, also argue that \(p\) should be of order \(T\) since the particles in the gas typically have that momentum \([14]\). At the critical point, where the temperature is finite but the mass goes to zero, a finite \(p\) could act as a regulator.

In connection with this I would also like to remind the reader that the limit of taking the external momentum to zero is non-trivial at finite temperature. As was shown by Fujimoto et al. \([9]\) the limit depends on whether the momentum is space-like \((p^2 < 0)\) or time-like \((p^2 > 0)\). The calculation in Ref.\([8]\) was performed in the real-time formalism and coincidence with the imaginary time formalism was only found in the space-like limit. It does however not tell which limit is the correct one. I stick to the space-like limit for simplicity except for the wavefunction renormalization.

The easiest way to compute the effective potential, from which we can derive the \(N\)-point vertices at zero external momenta, is to calculate the tadpole diagram for the shifted Lagrangian in Eq.(20). The tadpole is equal to the derivative of the effective potential with respect to the field up to a factor \(i\). The result is the
following
\[ \Gamma^{(1)}(T, \sigma, M, g) = -h_0 Z^{1/2} - (m_0^2 Z + \frac{g_0 Z^{3/2} \sigma \mu^{-\epsilon}}{2}) \sigma \mu^{-\epsilon} \]
\[ - \frac{g \mu^{-\epsilon} M^4}{256 \pi^3} \left( \frac{1}{\epsilon} + \frac{3}{2} - \gamma - \ln \left( \frac{M^2}{4\pi \mu^2} \right) \right) - \frac{g}{24 \pi^3} F_1^4(T, M^2), \quad (21) \]
\[ \Gamma^{(2)}(p = 0, T, \sigma, M, g) = -(m_0^2 Z + g_0 Z^{3/2} \sigma \mu^{-\epsilon}) \]
\[ - \frac{g^2 M^2}{128 \pi^3} \left( \frac{1}{\epsilon} + 1 - \gamma - \ln \left( \frac{M^2}{4\pi \mu^2} \right) \right) + \frac{g^2}{16 \pi^3} F_1^2(T, M^2), \quad (22) \]
\[ \frac{\partial}{\partial p_0^2} \Gamma^{(2)}(p = 0, T, \sigma, m, g) \]
\[ = Z + \frac{g^2}{768 \pi^3} \left( \frac{1}{\epsilon} - \gamma - \ln \left( \frac{M^2}{4\pi \mu^2} \right) \right) + \frac{g^2}{192 \pi^3} F_5^1(T, M^2), \quad (23) \]
\[ \Gamma^{(3)}(p_i = 0, T, \sigma, m, g) \]
\[ = -g_0 Z^{3/2} - \frac{g^3 \mu^\epsilon}{128 \pi^3} \left( \frac{1}{\epsilon} - \gamma - \ln \left( \frac{M^2}{4\pi \mu^2} \right) \right) - \frac{g^3}{32 \pi^3} F_1^0(T, M^2). \quad (24) \]

We have defined some useful functions
\[ F_n^m(T, M^2) = \int_0^\infty dk \frac{k^m f_B(\omega)}{\omega^n}, \quad f_B(\omega) = \frac{1}{e^{\omega/T} - 1}, \quad \omega = \sqrt{k^2 + M^2}. \quad (25) \]

The RC in Eq.(7)-Eq.(10) are then used at \( T_R \) to determine the infinite constants \( h_0, m_0^2, g_0 \) and \( Z \) (\( h_0 \) and \( \sigma \) are not independent but we fix \( h_0 \) at zero temperature where \( \sigma = 0 \)). After that we compute \( \Gamma^{(N)}(T, \sigma) \) and express it terms of finite quantities. When we first fix \( M_R^2 \) at a shifted field \( \sigma_R \), shift the field back again to get the original Lagrangian and finally shift with an arbitrary \( \sigma \) to compute the effective potential from the tadpole, it effectively equals fixing \( m_R^2 = M_R^2 - g_R \sigma_R \) instead of \( M_R^2 \). So in the expression for \( \Gamma^{(1)}(T, \sigma) \) there is a \( \sigma \) dependence in \( M^2 = m_R^2 + g_R \sigma \).
The effective potential can be integrated explicitly from $\Gamma^{(1)}(T, \sigma)$ and we find

$$V(T, \sigma) = -\sigma - \sigma_R \left( \frac{g_R M^2_R}{512 \pi^3} + \frac{g_R^4}{24 \pi^3} F_1^4(T, M^2_R) \right)$$

$$+ \frac{(\sigma - \sigma_R)^2}{2} \left( M^2_R + \frac{g_R^2}{16 \pi^3} F_2^2(T, M^2_R) \right)$$

$$+ \frac{(\sigma - \sigma_R)^3}{6} \left( g_R + \frac{g_R^3}{128 \pi^3} - \frac{g_R^3}{32 \pi^3} F_0^6(T, M^2_R) \right)$$

$$+ \frac{M^6}{768 \pi^3} \left( \frac{5}{6} - \ln \left( \frac{M^2}{M^2_R} \right) \right) - \frac{1}{60 \pi^3} F_1^6(T, M^2) + \text{const.} ,$$

(26)

where we have used

$$\frac{\partial F_n^m}{\partial M^2} = -\frac{m - 1}{2} F_{n-2}^m, \ m \geq 2 .$$

(27)

If we choose the renormalization temperature to be equal to the actual temperature we get an effective action which has the values of the mass and the coupling at the minimum coinciding with the values of the running mass and coupling from the TRGE. Also, at $T = T_R$ the minimum is at $\sigma = \sigma_R$.

The form of the effective potential as a function of $\sigma$ is essentially the same as the zero temperature renormalization scheme but now the parameters have a non-trivial dependence of the temperature $T_R$. Therefore, we still have a minimum which becomes shallower as the temperature increases and eventually disappears. The value of the critical temperature is however different and depends on the zero temperature mass and coupling in a non-trivial way.

The $T_R$ dependence of $m^2_R$ and $g_R$ should also be calculated. By taking derivatives of $V(T, \sigma)$ with respect to $\sigma$ we get finite expressions for the other $\Gamma^{(N)}$'s at zero momentum, and they are used to get the one-loop TRGE for the $(\phi^3)_6$-theory. Dropping the subscript $R$ we get:

$$0 = \frac{g}{24 \pi^3} \frac{\partial F_1^4}{\partial T} + \frac{d\sigma}{dT} M^2$$

(28)

$$\frac{\gamma}{T} = -\frac{g^2}{384 \pi^3} \left( \frac{\partial F_5^4}{\partial T} - g \frac{d\sigma}{dT} \cdot \frac{1}{4M^2} - \frac{\partial F_5^4}{\partial M^2} \right)$$

(29)

$$\frac{dM}{dT} = M \frac{\gamma}{T} + \frac{g}{2M} \frac{d\sigma}{dT} - \frac{g^2}{32 \pi^3 M} \frac{\partial F_5^2}{\partial T}$$

(30)
These equations are solved by computer. The result for some different values of zero temperature mass and coupling are shown in Fig. 1 and 2. The effect of solving the TRGE instead of taking only the naive one-loop result is more important for large couplings.

It has sometimes been argued that an asymptotically free theory (in the sense that the usual $\beta$-function starts out negative for small couplings) should behave as a free theory also at high temperature. This conclusion is correct if the momentum is scaled at the same rate as $T$. Such a scheme was studied in Ref. [14]. In this paper the momentum is put to zero and we are thus studying the properties of a low lying excitation embedded in a hot heat-bath which is rather an IR limit. In this case the coupling increases with temperature.

### 4 The $(\phi^4)_4$-model

Spontaneously broken theories are very important in particle physics and the prototype is the $\phi^4$-model that we shall study in 4 dimensions. At low temperatures the field gets an expectation value but above a critical temperature it is expected that the symmetry is restored. Usual perturbation theory is bad close to the critical point where the mass goes to zero and IR divergences occur. A resummation of an infinite set of diagram to cure the IR problems was performed for the massless $(\phi^4)_4$-model in Ref. [10]. In contrast to the $(\phi^3)_6$-model the $(\phi^4)_4$-model is stable at all temperatures since the potential is bounded from below.

The Lagrangian in $D = 4 - 2\epsilon$ is given by

$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_0)^2 + \frac{m_0^2}{2} \phi_0^2 - \frac{\lambda_0}{4!} \phi_0^4$

$= \frac{1}{2} (\partial_\mu \phi)^2 - \frac{M^2}{2} \phi^2 - \frac{\lambda \sigma \mu^\epsilon}{3!} \phi^3 - \frac{\lambda \mu^{2\epsilon}}{4!} \phi^4 + \frac{Z - 1}{2} (\partial_\mu \phi)^2$

$-\sigma \mu^{-\epsilon} \left( \frac{\lambda_0 Z^2 \sigma^2 \mu^{-2\epsilon}}{6} - \frac{m_0^2 Z}{2} \right) \phi - \frac{1}{2} \left( \frac{(\lambda_0 Z^2 - \lambda \mu^{2\epsilon}) \sigma \mu^{-2\epsilon}}{2} - (m_0^2 Z - m^2) \right) \phi^2$

$- \frac{\lambda_0 Z^2 - \lambda \mu^{2\epsilon}}{3!} \sigma \mu^{-\epsilon} \phi^3 - \frac{(\lambda_0 Z^2 - \lambda \mu^{2\epsilon}) \sigma \mu^{-2\epsilon}}{4!} \phi^4$

where $M^2 = \frac{\lambda_0^2}{2} - m^2$. The calculation of the $N$-point function is similar to the $\phi^3$ case and we use the same convention regarding the momentum subtraction point.
We then find
\[ \Gamma^{(1)} = -\sigma \mu^{-\epsilon} \left( \frac{\lambda_0 Z^2 \sigma^2 \mu^{-2\epsilon}}{6} - m_0^2 Z \right) + \frac{\lambda \sigma M^2}{32\pi^2} \left( \frac{1}{\epsilon} + 1 - \gamma - \ln(\frac{M^2}{4\pi\mu^2}) \right) - \frac{\lambda \sigma}{4\pi^2} F_1^2(T, M^2). \]

Higher \(N\)-point functions at zero external momenta can be computed by taking successive derivatives with respect to \(\sigma\) (remembering the \(\sigma\) dependence in \(M^2\)). Since we also need the 2-point function at non-zero external momentum we compute it for \(p_\mu = (Q, 0, 0, 0)\) and let \(Q\) go to zero at the end. The \(N\)-point functions can also be calculated from the diagrammatic expansion in the shifted theory with the same result. It may seem unnecessary to compute the diagrams which are of higher order in \(\lambda\) but they are, on the other hand, more IR divergent. Using the tree level relation \(\lambda \sigma^2 = 3M^2\) at the local minimum in the broken phase, we find that they are all of the same leading order in \(\lambda T/M\). The loop expansion is an expansion in \(\bar{h}\) and not in \(\lambda\) so we can consistently keep all order terms in \(\lambda\). In the unbroken phase the \(\lambda \sigma^2\) terms do not contribute.

The other functions needed for renormalization are
\[ \Gamma^{(2)}(T, \sigma, M, \lambda) = -\left( \frac{\lambda_0 Z^2 \sigma^2}{2} - m_0^2 Z \right) + \frac{\lambda(M^2 + \lambda \sigma^2)}{32\pi^2} \left( \frac{1}{\epsilon} - \gamma - \ln(\frac{M^2}{4\pi\mu^2}) \right) \]
\[ + \frac{\lambda M^2}{32\pi^2} - \frac{\lambda}{4\pi^2} F_1^2(T, M^2) - \frac{\lambda^2 \sigma^2}{4\pi^2} \frac{\partial F_1^2}{\partial M^2}, \] (34)
\[ \frac{\partial}{\partial p_0^2} \Gamma^{(2)}(p = 0, T, \sigma, M, \lambda) = Z + \frac{\lambda^2 \sigma^2}{32\pi^2} \left( \frac{1}{6M^2} + F_5^2(T, M^2) \right), \] (35)
\[ \Gamma^{(4)}(p_i = 0, T, \sigma, M, \lambda) = -\lambda_0 Z^2 + \frac{3\lambda^2}{32\pi^2} \left( \frac{1}{\epsilon} - \gamma - \ln(\frac{M^2}{4\pi\mu^2}) \right) \]
\[ - \frac{3\lambda^3 \sigma^2}{16\pi^2 M^2} + \frac{\lambda^4 \sigma^4}{32\pi^2 M^4} - \frac{\lambda^2}{4\pi^2} \left( 3 \frac{\partial F_1^2}{\partial M^2} + 6\lambda \sigma^2 \frac{\partial^2 F_1^2}{\partial M^4} + (\lambda \sigma^2)^2 \frac{\partial^3 F_1^2}{\partial M^6} \right). \] (36)

A finite expression for the effective potential is obtained from the tadpole using the RC at \(\sigma_R\)
\[ V(\sigma) = \]
\[-\frac{\sigma^2}{2} \left( m_R^2 + \frac{R}{4\pi^2} (F_1^2 + \lambda_R \sigma^2 \frac{\partial F_1^2}{\partial M^2}) \right) - \]

\[\frac{\lambda_R \sigma_R^2}{8\pi^2} \left[ \frac{3\lambda_R \sigma_R^2}{4M_R^2} - \frac{(\lambda_R \sigma_R^2)^2}{8M_R^4} + 3 \frac{\partial F_1^2}{\partial M^2} + 6\lambda_R \sigma_R^2 \frac{\partial^2 F_1^2}{\partial M^4} + (\lambda_R \sigma_R^2)^2 \frac{\partial^3 F_1^2}{\partial M^6} \right] \]

\[+ \frac{\sigma^4}{24} \left( \lambda_R - \frac{\lambda_R^2}{4\pi^2} \left[ \frac{3\lambda_R \sigma_R^2}{4M_R^2} - \frac{(\lambda_R \sigma_R^2)^2}{8M_R^4} + 3 \frac{\partial F_1^2}{\partial M^2} + 6\lambda_R \sigma_R^2 \frac{\partial^2 F_1^2}{\partial M^4} + (\lambda_R \sigma_R^2)^2 \frac{\partial^3 F_1^2}{\partial M^6} \right] \right) \]

\[+ \frac{M^4}{64\pi^2} \left( \ln \left( \frac{M^2}{\lambda R} \right) - \frac{1}{2} \right) - \frac{1}{6\pi^2} F_1^4(T, M^2) - \frac{\lambda_R \sigma_R^2}{128\pi^2} (2\sigma^2 - \sigma_R^2) \quad (37)\]

where all functions $F_{m}^{n}$ without explicit arguments should be evaluated at $(T_R, M_R^2)$. But $T_R$ can, of course, be chosen to be equal to $T$ when the TRGE are solved. When deriving the expression for $V(T, \sigma)$ the RC in Eq.(7) was not used to eliminate the infinities. Therefore, the minimum of $V(T, \sigma)$ is not automatically at $\sigma_R$ as it was in the $(\phi^3)_6$-theory. However, if $\sigma_R(T_R)$ satisfies the TRGE and $\Gamma(1)$ is TRG invariant (i.e. satisfies Eq.(12)) we have

\[d\frac{\Gamma(1)}{dT_R}(T_R, \sigma_R; \sigma_R, M^2_R, \lambda_R, T_R) = 0 , \quad (38)\]

which ensures that the minimum is at $\sigma_R$ for all $T$ if we choose $T = T_R$. The one-loop tadpole is not TRG invariant so we determine $\sigma_R$ through the minimization of $V(T, \sigma)$ instead of solving the TRGE in order to get $M^2_R$ as the mass at the minimum in our approximation.

The TRGE become considerably more complex in the broken phase of the $(\phi^4)_4$-theory, but it is straightforward to derive them and we get

\[0 = \frac{\lambda_R \sigma R}{4\pi^2} \frac{\partial F_1^2}{\partial T} + \frac{d\sigma}{dT} M^2 , \quad (39)\]

\[\gamma \frac{T}{\lambda} = -\frac{\lambda}{32\pi^2} \left( \frac{\lambda R \sigma R}{2} \frac{\partial F_5^2}{\partial T} + \lambda_R \frac{d\sigma}{dT} \left[ \frac{1}{6M^2} - \frac{\lambda R \sigma R}{12M^4} + \frac{F_5^2}{2} + \frac{\lambda R \sigma R}{2} \frac{\partial F_5^2}{\partial M^2} \right] \right) , \quad (40)\]

\[\frac{dM}{dT} = M \frac{\gamma}{T} + \frac{\lambda R \sigma R}{2M} \frac{d\sigma}{dT} - \frac{\lambda}{8\pi^2 M} \left( \frac{\partial F_1^2}{\partial T} - \lambda_R \sigma R \frac{\partial^2 F_1^2}{\partial T \partial M^2} \right) \]

\[+ \lambda_R \sigma R \frac{d\sigma}{dT} \left[ \frac{5\lambda R \sigma R}{8M^2} - \frac{\lambda R \sigma R}{8M^4} + 5\lambda R \sigma R \frac{\partial^2 F_1^2}{\partial M^4} + (\lambda R \sigma R)^2 \frac{\partial^3 F_1^2}{\partial M^6} \right] , \quad (41)\]

11
\[
\frac{d\lambda}{dT} = 4\lambda \frac{\gamma}{T} - \frac{\lambda^2}{4\pi^2} \left( -\left[ 3 \frac{\partial^2 F_1^2}{\partial T \partial M^2} + 6\lambda\sigma^2 \frac{\partial^3 F_1^2}{\partial T \partial M^4} + (\lambda\sigma^2)^2 \frac{\partial^4 F_1^2}{\partial T \partial M^6} \right]\right)
\]

\[
\quad \quad + \lambda\sigma \frac{d\sigma}{dT} \left[ -\frac{15}{8M^2} + \frac{5\lambda\sigma^2}{4M^4} - \frac{(\lambda\sigma^2)^2}{4M^6} - 15\lambda\sigma^2 \frac{\partial^2 F_1^2}{\partial M^4} - 10\lambda\sigma^2 \frac{\partial^3 F_1^2}{\partial M^6} - (\lambda\sigma^2)^2 \frac{\partial^4 F_1^2}{\partial M^8} \right] \right).
\]

(42)

The unbroken phase, \(\sigma = 0\), has been studied earlier in Ref.\[11\] and I give the result for completeness. The TRGE simplify to

\[
\frac{dM}{dT} = \frac{\lambda}{8\pi^2 M} \frac{\partial F_1^2}{\partial T} ,
\]

\[
\frac{d\lambda}{dT} = \frac{3\lambda^2}{16\pi M} \frac{\partial^2 F_1^2}{\partial T \partial M^2} .
\]

(43)

It is the ratio \(T/M\) that defines the IR limit and if we expand the right hand side of Eq.(43) in \(T/M\) we get

\[
\frac{dM}{dT} = \frac{\lambda T}{24M} ,
\]

\[
\frac{d\lambda}{dT} = -\frac{3\lambda^2}{16\pi M} .
\]

(44)

For a constant coupling we would get back the usual formula in Eq.(1) but we see from Eq.(44) that \(\lambda(T)\) decreases and thus changes the large \(T\) behaviour of \(M(T)\). The asymptotic solutions are

\[
M(T) \simeq \frac{4\pi T}{9 \ln(T/T_0)} , \quad \lambda(T) \simeq \frac{128\pi^2}{27 \ln^2(T/T_0)} .
\]

(45)

It can be obtained by guessing that \(M \ln T/T\) and \(\lambda \ln^2 T\) approaches non zero constants as \(T \to \infty\) and proving it using Eq.(44). The constant \(T_0\) is introduced for dimensional reasons and is determined by the initial conditions. It must, therefore, by of the order of \(M(0)\). This analysis gives another asymptotic \(T\) dependence of the effective 4-point coupling \(\lambda(T)\) than the one given in Ref.\[13\] where the ordinary renormalization group (in \(\mu\)) was used. Like the case of the \((\phi^3)^6\)-model a different behaviour is expected if \(p \simeq T\) instead of \(p = 0\) as in this paper.

A two-loop calculation for the \(O(N)\)-symmetric model in the limit \(N \to \infty\) was carried out in Ref.[12] where it was found that the two-loop effects are important.
5 Conclusions

Self-consistent renormalization conditions at zero momentum together with the temperature renormalization group equations (TRGE) of Ref.\[4\] has been used to improve the IR behaviour of the scalar \((\phi^3)_6\) and \((\phi^4)_4\)-models. The renormalization condition replaces the gap equation which resums an infinite set of diagrams and the TRGE determine the running of the mass and coupling with temperature.

The \(T_R\)-renormalization scheme in Eq.(7)-Eq.(10) is not the only possible scheme. One can use the gap equation (Eq.(2)) as a renormalization prescription (called the \(T\)-scheme in Ref.\[4\]). For an unbroken theory it takes the form

\[
\text{Re} \Gamma^{(2)}(p, T, M(T), g(T))|_{p(Q_R)} = p^2(Q_R) - M^2(T), \quad \text{etc.}
\]  

Since the exact vertex functions should be independent of the renormalization prescription (up to a finite wavefunction renormalization factor) the \(M(T)\) obtained from Eq.(16) must be the same as the \(M(T)\) computed from the TRGE. It is, however, not true to finite order in perturbation theory. A careless use of the gap equation can lead to temperature dependent infinities (as pointed out in Ref.\[13\]) but with the analysis in this paper no such divergences occur.

From the numerical solutions of the TRGE we see in the figures that for small zero temperature coupling constants the corrections to the unrenormalized calculations of the critical temperature \(T_{cr}\) and the mass \(M(T)\) are small. The fact that the corrections in the \((\phi^4)_4\)-model are larger than in the \((\phi^3)_6\)-model for \(g = \lambda\) is due to a difference in the numerical factors in the TRGE. For zero temperature values of \(g\) and \(\lambda\) equal to 1 and 10, respectively, \(T_{cr}\) and the behaviour of \(M(T)\) are altered, but it is \(g(T)\) and \(\lambda(T)\) that get the most drastic corrections. The increase in \(T_{cr}\) can be explained by the decrease in \(g\) for the \((\phi^3)_6\)-model, and vice versa for the \((\phi^4)_4\)-model.

Since it is the coupling constants that get the largest corrections it would be interesting to use the TRGE to study a model with a first order phase transition (e.g. the electroweak theory). The height of the barrier between the coexisting phases depends strongly on the coupling constant.

When we use the renormalization conditions to express the vertex functions in terms of finite quantities all the dependence of the arbitrary scale \(\mu\) disappears. Instead we get a logarithmic dependence of the \(\sigma\) through \(M^2(\sigma)\) (see Eqs.(26, 37)). The effective potential is not to be trusted when these logarithms are large. A renormalization group in \(\sigma\), similar to the one in \(T\), should be used to improve the potential for \(\sigma\) far away from the minimum. The value of \(\sigma_R\), at which we renormalize, is arbitrary, but by choosing the minimum we get \(M(T_{R})\) to be the mass of the lowest physical excitation. At the minimum the logarithmic terms are zero and the form of the potential reliable. We can, therefore, follow the temperature dependence of
\[ \sigma_R. \] In the \((\phi^4)_4\)-model we find that it goes to zero at the same temperature as the mass. Thus, the phase transition is of second order.

In the high temperature limit of the \((\phi^4)_4\)-model we find that the ratio \(T/M\), which enters as a possible IR divergence in the perturbation expansion, goes like \(9 \ln T/4\pi\) for large \(T\) because also the mass grows with \(T\). But, for a given \(N\)-point function the IR factor is actually \((\lambda T/M)^V\) (\(V\) is the number of vertices in a diagram), apart from an overall factor that depends only on \(N\) and not on \(V\). In UV divergent diagrams there is an extra factor \(T/M\). If one also takes into account the \(T\) dependence of \(\lambda\) we actually find that

\[ \frac{\lambda T}{M} \approx \frac{32\pi}{3 \ln(T/T_0)}, \]

and it thus decreases at high temperature. This indicates that a perturbation expansion could be possible in the large \(T\) limit if only the renormalized parameters are used, but the extra factors of \(T/M\) from UV divergent diagrams may destroy the convergence. See Ref.[12] for the effects of higher loops. Note that the coupling is evaluated at zero external momenta. If we put \(p/T = \text{const.}\), as suggested in Ref.[14], we expect to get back the result from the usual renormalization group analysis. A renormalization group in \(p_0\) and \(|\vec{p}|\) should be used to study the momentum dependence in more detail.

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Figure Captions

**Figures** Numerical calculation of the temperature dependence of $M(T)$, $g(T)$ and $\lambda(T)$ in the $(\phi^3)_6$- and $(\phi^4)_4$-models. The temperature, mass and couplings are rescaled to make it easier to compare with the unrenormalized one-loop result in which cases the curves are straight lines.

1. The mass in the $(\phi^3)_6$-model.
2. The coupling in the $(\phi^3)_6$-model.
3. The mass in the $(\phi^4)_4$-model.
4. The coupling in the $(\phi^4)_4$-model.