On the relationship between extended state observer and unknown input observer

Jinfeng Chen\textsuperscript{a}, Zhiqiang Gao\textsuperscript{a}, Yu Hu\textsuperscript{a}, Sally Shao\textsuperscript{b}

\textsuperscript{a}Center for Advanced Control Technologies, Cleveland State University, Cleveland, OH 44115, United States
\textsuperscript{b}Department of Mathematics and Statistics, Cleveland State University, Cleveland, OH 44115, United States

Abstract

This paper explores the differences and similarities between unknown input observer (UIO) and extended state observer (ESO) for discrete-time linear time-invariant systems with uncertainties, both internal and external. It is shown that, with the relative degree of the plant higher than one, a UIO with delay produces exactly the same disturbance estimation as that of an ESO with poles all placed at origin. Hence, ESO has the ability to estimate arbitrary unknown disturbances like UIO. However, UIO with delay is not appropriate for real-time feedback control, while ESO is a popular tool for feedback control in industry. Our proofs in these differences and similarities show that the design of ESO must follow a principle of no invariant zero between the total disturbance input and the measurement output. Finally, based on above insights, a new, generalized ESO is proposed to add zero dynamics into the observer design unlike conventional ESO.

Key words: Extended state observer; Unknown input observer; Disturbance observer; Uncertain linear systems.

1 Introduction

State estimation is a pillar to control theory and a fundamental area in control research. This paper explores a possible connection, intimate and important, between two classes of state estimation techniques, known as the extended state observer (ESO) \cite{17} and the unknown input observer (UIO) \cite{35}.

The classical state observer is of course traced back to the one proposed by Luenberger \cite{30}, which estimates the states of a linear time-invariant dynamical system from known inputs and output signals. Left largely untouched by this original work, and much of the ensuing research papers and textbooks, are the problems associated with uncertainties that are widely existed in all industrial systems, such as uncertainties in plant dynamics and unknown external disturbances. Over time, researchers recognized that, when modeling a dynamical system, the effects of plant uncertainty and unknown external disturbance can generally be presented as arbitrary unknown inputs or inaccessible disturbances, which are state-dependent, time-varying, and totally unknown. Gradually, the problem of designing a state observer, which can estimate both the states and unknown inputs, began to draw considerable attention in recent decades, as seen in \cite{28}, \cite{2}, \cite{9}, \cite{32}, and the reference therein.

UIO, since it was first proposed by Basile and Marro in 1969 \cite{4}, has drawn much attention and produced rich literature in various settings, such as continuous-time UIO (\cite{22}; \cite{11}; \cite{19}) and discrete-time UIO (\cite{36}; \cite{1}) for systems with deterministic dynamics, as well UIO for those with stochastic dynamics (\cite{27}; \cite{10}; \cite{20}; \cite{15}; \cite{39}). Specifically, the necessary and sufficient conditions for the existence of all above UIOs, also called strong\textsuperscript{*} detectability conditions \cite{18}, consist of 1) a matrix rank matching condition and 2) a minimum phase condition, which may impact the range of practicality of such observers, even with the significant amount of research devoted to the relaxation of these two conditions (\cite{21}; \cite{12}; \cite{37}; \cite{28}; \cite{6}; \cite{5}; \cite{23}).

In particular, if a full state UIO does not exist, the necessary and sufficient conditions of existence of an unknown input functional observer are proposed in \cite{21} and \cite{12}. The first condition can be relaxed by using the deriv-
tives of the measurements in continuous time or the future measurements, i.e., time-delay, in discrete time ([2]; [35]; [13]). However, they cannot be used for real-time feedback control. A novel sliding mode observer has been proposed for systems without satisfying the above two conditions even when the system has less outputs than unknown inputs [37]. However, the chattering phenomenon is hard to be reduced due to its variable structure in sliding mode observer. In [28], by adding an internal model, the method to design a UIO, which can instantaneously and asymptotically obtain state and unknown inputs estimation, has been proposed. Other approaches to relax two conditions mentioned above can be found in [6], [5] and [23].

Parallel to UIO, there is another field of research, known as disturbance rejection, that, by nature, must also address the problem of estimating unknown disturbances that are otherwise unmeasurable in practice. We would like to clarify that such the notion of disturbance is similar to the concept of unknown inputs in the UIO literature: they are both unknown and inaccessible; and they can both represent internal and external uncertainties. While UIO has been mainly a field of rigorous academic research with applications to, among others, fault detection and isolation, the field of disturbance rejection grew more or less out of engineering practice, in which disturbances are first estimated in real-time, and then actively compensated in control law for improved performance and robustness.

Critical to the task of disturbance rejection is, of course, the disturbance estimator (DET) for which many techniques have been proposed, including the disturbance observer (DOB) ([31]; [8]), the perturbation observer (PO) [29], the generalized proportional integral observer (GPIO) [34], the equivalent input disturbance estimator (EIDE) [33], the uncertainty and disturbance estimator (UDE) [40], the extended state observer (ESO) [17] and high gain observer (HGO) [14], to just name a few.

This paper explores the deep-seated connection between these two seemingly quite different fields of study. In particular, the UIO, whose error dynamics is fully decoupled from unknown inputs, is reformulated in the form of the ESO, whose error dynamics is related with unknown inputs, thereby allowing the rigorous theoretical foundation behind UIO to be used to justify the machinery of ESO commonly used in practice. On the other hand, it will be shown that the matrix rank match condition from the UIO can be relaxed in light of ESO. Furthermore, it can be shown that the total disturbance obtained by the ESO is identical to that of the delayed UIO in discrete time, thereby establishing the relation between them, if the poles of ESO are all placed at the origin. This explains why delayed UIO is so sensitive to measurement noises and how to fix it by using ESO with reducing the observer bandwidth. This also implies that perhaps the UIO like functionality can be obtained without using time-delay, thereby making it more friendly for control applications.

By connecting ESO to UIO in discrete time manner also allows us to extend the necessary and sufficient condition for convergence in continuous time, as shown in [3], to discrete time. In fact, the corresponding condition for the continuous time and discrete time ESO is that no invariant zero exists between the output measurement and the disturbance. Furthermore, when such condition is met, a particular form, Luenburger-like ESO, can be used to estimate the total disturbance in all cases, rather than the cascade integral form commonly seen in the literature. And it is this special form of ESO that provided the basis for all analysis and proofs in this paper.

Finally, the insight developed in the ESO-UIO connection also helps to explain the similarity and difference between the original disturbance observer used in disturbance accommodation control (DAC) ([25]; [26]) and ESO. In the original ground breaking work of DAC from 50 years ago, the disturbance is assumed external to the plant and is generated from a given mathematical model whose input is assumed to be sparse impulses. The discrete time ESO as shown in this paper requires no such presumptions, as long as there is no invariant zero between the disturbance input and the measurement output. If, however, some partial disturbance model information is available, it can be readily added to the ESO for better performance, as shown in a simple example later in this paper.

This paper is organized as follows. In Section 2, the observation problem for an uncertain discrete-time linear system is introduced. Section 3 gives the basic ideas of ESO and UIO and illustrates their similarities and differences via simulations. Next, several proofs are given to explain their relationship from theoretical point of view in Section 4. Finally, Section 5 gives a new generalization of ESO using the property proposed in this paper.

2 Problem reformulation

Consider the following single-input single-output linear time-invariant discrete-time systems with uncertainty in a normal form [24]

\[
\begin{align*}
    x(k+1) &= A_0x(k) + B_0u(k) + D_0(F_0z(k) + f(k)) \\
    z(k+1) &= S_0z(k) + G_0x(k) + f_z(k) \\
    y(k) &= C_0x(k)
\end{align*}
\]

where \( x \in \mathbb{R}^n \) and \( z \in \mathbb{R}^m \) are the state variables, \( u \in \mathbb{R} \) is the known input, \( f \in \mathbb{R} \) is the disturbances transformed to the input channel, \( y \in \mathbb{R} \) is the measured...
In ESO, the total disturbance $f$ and the states $x$ are lumped together and formed a total disturbance. In this paper, for simplicity of comparison of ESO and UIO, we assume that there is no zero dynamics in the system. Moreover, in the section V, we will give an example to show how to generalize conventional ESO to add the zero dynamics in ESO design.

Now equation (1) can be rewritten as the following discrete-time linear time-invariant system with a known input and an unknown input (called total disturbance in ESO) are considered as follows

$$
\begin{align*}
\{ x(k+1) &= A_0 x(k) + B_0 u(k) + D_0 f(k) \\
y(k) &= C_0 x(k)
\end{align*}
$$

In ESO, the total disturbance $f$ is added in the state vector as an extended state. So, the Luenberger state observer can be designed to estimate the states $x$ and total disturbance $f$ simultaneously. However, in UIO, the states $x$ and the total disturbance $f$ can be estimated without the need of extending state vector. To compare UIO and ESO, the total disturbance $f$ is treated as an extended state, and the new augmented system is as follows

$$
\begin{align*}
\begin{pmatrix}
x(k+1) \\
f(k+1)
\end{pmatrix} &= A \begin{pmatrix}
x(k) \\
f(k)
\end{pmatrix} + B u(k) + D \Delta f(k) \\
y(k) &= C \begin{pmatrix}
x(k) \\
f(k)
\end{pmatrix}
\end{align*}
$$

where $\Delta f(k) = f(k+1) - f(k)$, $A = \begin{bmatrix} A_0 & D_0 \\ 0_{1 \times n} & 1 \end{bmatrix}_{(n+1) \times (n+1)}$, $B = [B^T_0, 0]_{1 \times (n+1)}^T$, $C = [C_0, 0]_{1 \times (n+1)}$, and $D = [0, \cdots, 0, 1]_{1 \times (n+1)}^T$. Note that $\Delta f(k)$ is a new unknown input in this new augmented system.

The objective of UIO and ESO is to design a Luenberger state observer to estimate the state vector accurately without using the knowledge of the unknown input $\Delta f$.

Since UIO is designed to general system description (i.e., the system matrices $A$, $B$, $C$, $D$ are any known constant matrices of suitable dimensions), the detectability (reconstructability) of the augmented system of equation (3) with an unknown input $\Delta f$ should be examined. The necessary and sufficient conditions for the existence of a UIO for augmented system of equation (3) are:

1) $\text{rank}(CD) = \text{rank}(D)$; 2) $\text{rank} \begin{bmatrix} zI_{n+1} - A - D \\ C \end{bmatrix} = n + 1 + \text{rank}(D)$, $\forall z \in \mathbb{C}$, $|z| \geq 1$ [36]. It is easy to verify that condition 1) is not satisfied, but condition 2) is satisfied. So, there is no full state real time UIO existed. However, a delayed UIO can be constructed [35]. Therefore, the properties of delayed UIO and ESO are compared in this paper.

### 3 The basic ideas of ESO and UIO

In this section, the basic ideas of ESO and UIO are introduced. And the numerical simulation examples are given to illustrate their similarities and differences. The results obtained in this section are the foundation for analyzing the theoretical relationship between ESO and UIO in the next section.

#### 3.1 ESO

In the framework of ESO, the unknown input $\Delta f$ in the new augmented system in equation (3) is ignored when designing a Luenberger state observer as follows:

$$
\begin{align*}
\dot{x}(k+1) &= A\dot{x}(k) + Bu(k) + L \begin{pmatrix} y(k) - C \dot{f}(k) \end{pmatrix} \\
\dot{f}(k+1) &= B_0 u(k) + C_0 x(k)
\end{align*}
$$

where $\dot{x}$ is the estimation of the state $x$, $\dot{f}$ is the estimation of the total disturbance $f$ in equation (2), $L \in \mathbb{R}^{n+1}$ is a column vector called the observer gain. Since $(A, C)$ in equation (3) is observable, the eigenvalues of $A - LC$ can be placed inside the unit circle. The Luenberger state observer in equation (4) converges. For the sake of simplicity of tuning, all eigenvalues of $(A, C)$ are placed at $\omega_0$, which is called the observer bandwidth of ESO. $\dot{x}$ and $\dot{f}$ have been proved in [16] and [38] to converge to $x$ and $f$ exponentially with the increase of magnitude of $\omega_0$ in continuous time domain.

Define the estimation error by

$$
e(k) = \begin{pmatrix} x(k) \\ f(k) \end{pmatrix} - \begin{pmatrix} \dot{x}(k) \\ \dot{f}(k) \end{pmatrix}.
$$

Then, the estimation error of the observer is updated according to the following equation:

$$
e(k+1) = (A - LC)e(k) + D\Delta f(k)$$

3
where $D \Delta f(k)$ is caused by the ignorance of the unknown input $\Delta f$ in equation (4).

3.2 UIO

The objective of UIO is to design a state observer which can decouple the unknown input $\Delta f$ from the dynamics of estimation error, that is, there is no $D \Delta f(k)$ in equation (6). To get rid of the unknown input $\Delta f$, we need to derive the response of system in equation (3) over $n+2$ time units as shown in equation (7) at the top of next page.

From the matrices $A$, $B$, $C$, $D$ defined in equation (3), we obtain the following conditions

$$
CB = 0, CAB = 0, \cdots, CA^{n-1}B = b_0, CA^nB = -a_1b_0, \quad (8) 
$$

$$
CD = 0, CAD = 0, \cdots, CA^{n-1}D = 0, CA^nD = 1. \quad (9)
$$

Assume a full-order state observer is of the form

$$
\dot{X}(k+1) = E\dot{X}(k) + FY[k : k + n + 1] + GU[k : k + n] \quad (10)
$$

where $\dot{X}(k) \in \mathbb{R}^{n+1}$ is the state estimate of the state $X(k)$, and $E$, $F$, and $G$ are matrices with appropriate dimensions. We need to find matrices $E$, $F$, and $G$ such that the estimation error $e(k) = X(k) - \dot{X}(k)$ satisfies

$$
e(k + 1) = Ee(k) \quad (11)
$$

and $E$ is asymptotically stable-nilpotent.

The dynamics of this observer estimation error is

$$
e(k + 1) = Ee(k) + Bu(k) + D\Delta f(k) - FY[k : k + n + 1] + (A - E)X(k) - GU[k : k + n]. \quad (12)
$$

Substituting equation (7) into equation (12) yields

$$
e(k + 1) = Ee(k) + (A - E - F\Theta_{n+1})X(k) + Bu(k) - (FN_{n+1} + G)U[k : k + n] + D\Delta f(k) - FM_{n+1}\Delta F[k : k + n]. \quad (13)
$$

Comparing equation (13) with equation (11), the following conditions should be satisfied:

(a) $FM_{n+1} = [D, 0, \cdots, 0]^{\top}$, $n+1$ columns

(b) $E = A - F\Theta_{n+1}$ must be a stable matrix,

(c) $Bu(k) - (FN_{n+1} + G)U[k : k + n] = 0.$

From equation (9), matrix $M_{n+1}$ has a quite simple form

$$
M_{n+1} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0
\end{bmatrix}^{(n+2) \times (n+1)} \quad (14)
$$

Then, condition (a) implies

$$
F = \begin{bmatrix}
\otimes \otimes \cdots \otimes \\
\otimes \otimes \cdots \otimes \\
\vdots & \vdots & \ddots & \vdots \\
\otimes \otimes \cdots \otimes
\end{bmatrix}^{(n+1) \times (n+2)} \quad (15)
$$

where $\otimes$ can be any real number, (i.e., all elements in $F$ can be any real number except the last column). For simplicity, we choose $\otimes$ to be zero and consider conditions (a) and (c) first. Condition (b) is the stability condition of the observer which will be determined later.

From equation (8), we have

$$
N_{n+1} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-b_0 & b_0 & \cdots & 0
\end{bmatrix}^{(n+2) \times (n+1)} \quad (16)
$$

Since there is only one unknown matrix $G$ in condition (c), solving condition (c) yields

$$
G = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-b_0 & b_0 & \cdots & 0
\end{bmatrix}^{(n+1) \times (n+1)} \quad (17)
$$

Therefore, matrices $F$ and $G$ designed above can eliminate the unknown input $\Delta f(k)$ and the known input $u(k)$ in the dynamics of observer estimation error in equation (13). In order to stabilize the observer, a new measurement $y(k)$ needs to be added like a Luenberger observer:

$$
\dot{X}(k + 1) = (A + LC)\dot{X}(k) - Ly(k) \quad (18)
$$

where $L \in \mathbb{R}^{n+1}$ is a column vector called the observer gain. Combining equation (10) and equation (18), we
have a new state observer

\[
\dot{X}(k+1) = (A + E + LC)\dot{X}(k) - Ly(k) + FY[k : k + n + 1] + GU[k : k + n].
\]

Then matrix \( E \) needs to be determined by eliminating state \( X(k) \) in estimation error equation (13) and make the new state observer observable.

The dynamics of this observer estimation error is

\[
e(k+1) = (A + E + LC)e(k) - (E + F\Theta_{n+1})X(k) + Bu(k) - (FN_{n+1} + G)U[k : k + n] + D\Delta f(k) - FM_{n+1}\Delta F[k : k + n] = (A + E + LC)e(k) - (E + F\Theta_{n+1})X(k).
\]

Therefore, condition (b) is equivalent to that \( (A+E,C) \) is observable and \( E + F\Theta_{n+1} = 0 \), or equivalently

\[
E = -\begin{bmatrix}
0 \\
0 \\
\vdots \\
CA^{n+1}
\end{bmatrix}_{(n+1)\times(n+1)}.
\]

And it is easy to verify that \( (A+E,C) \) is observable.

Remark 1: The derivation of UIO above is straightforward and has a clear physical meaning, while the UIO in [35] is a general form suitable for any system.

3.3 Numerical simulations – examples

A position control of a series elastic actuator system [7] is considered in this paper to compare ESO and UIO. After transforming the plant dynamics of series elastic actuator into the canonical form of cascade integrators and discretizing it with sampling time 20 ms, the discretized system in state space form is given by the matrices

\[
A_0 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix},
B_0 = \begin{bmatrix}
0 \\
0 \\
0.02612
\end{bmatrix},
\]

\[
C_0 = [1, 0, 0, 0], D_0 = [0, 0, 0, 1]^{T}.
\]

Since five time-step delay is used to design UIO, the first variable in the state vector is \( y(k-5) \). Fig. 2 shows the actual \( y(k-5) \) and estimated results by UIO. The difference between them is very small. However, it is impossible for UIO to be used in feedback control because this time-delay information cannot stabilize the plant, which is a reason why UIO is rarely used in control systems. Fig. 3 shows that ESO has the ability to estimate the current state vector although there exists a small error, which is crucial in controller design.

Moreover, ESO can smooth the estimation of the total disturbance. In motion control, the sampling time is usually 1 ms. After discretizing the series elastic actuator with this sampling time, the discretized system in state space form is given by the matrices
4 Analysis of the relationship between ESO and UIO

From the discussions in the sections of introduction and simulation, although UIO and ESO can both provide the same time-delay total disturbance estimation, ESO is more suitable for feedback control because observer bandwidth can adjust the smoothness of the estimations and current state vector can be estimated. In this section, we analyze these similarities and differences from the theoretic point of view. Try to understand why they have these properties.
4.1 The ESO and UIO – sharing the same estimations

In this subsection, for a more generalized proof, we consider the following discrete-time linear time-invariant system with a disturbance

\[
\begin{aligned}
    x(k + 1) &= A_0 x(k) + D_0 f(k) \\
    y(k) &= C_0 x(k)
\end{aligned}
\]  

(21)

where state vector \( x \in \mathbb{R}^n \), the disturbance \( f \in \mathbb{R} \), the output \( y \in \mathbb{R} \), \( C_0 = [1, 0, \cdots, 0]_{1 \times n} \), \( D_0 = [0, 0, \cdots, 1]_{1 \times n} \), and

\[
A_0 = \begin{bmatrix}
    a_{11} & a_{12} & 0 & \cdots & 0 & 0 \\
    a_{21} & a_{22} & a_{23} & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_{(n-2)1} & a_{(n-2)2} & a_{(n-2)3} & \cdots & a_{(n-2)(n-1)} & 0 \\
    a_{(n-1)1} & a_{(n-1)2} & a_{(n-1)3} & \cdots & a_{(n-1)(n-1)} & a_{(n-1)n} \\
    a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n(n-1)} & a_{nn}
\end{bmatrix}
\]

Moreover, \( a_{12} \neq 0, a_{23} \neq 0, \cdots, a_{(n-1)n} \neq 0 \). If any one of them is equal to zero, \( C_0, C_0 A_0, \cdots, C_0 A_0^{n-1} \) cannot span the whole \( n \)-dimension space, and the system is not observable. Since the known input does not affect the generality of the proof, the known input is ignored. The ones in \( C_0 \) and \( D_0 \) are set without losing generality by suitably adjusting the amplitude of the disturbance and measurements.

**Theorem 1** If matrix \( C_0 = [1, 0, \cdots, 0]_{1 \times n} \), then the following statements are equivalent:

(i) \( D_0 = [0, 0, \cdots, 0]_{n \times n} \), \( a_{12} \neq 0, a_{23} \neq 0, \cdots, a_{(n-1)n} \neq 0, d_n \neq 0 \), and \( d_n \neq 0 \).

(ii) \( C_0 D_0 = 0, C_0 A_0 D_0 = 0, \cdots, C_0 A_0^{n-2} D_0 = 0, C_0 A_0^{n-1} D_0 \neq 0 \).

(iii) \( \text{rank} \begin{bmatrix} A_0 - z I_n & D_0 \\ C_0 & 0 \end{bmatrix} = n + 1 \) for all \( z \in \mathbb{C} \), (i.e., the system has no invariant zero between disturbance and output).

**PROOF.** We first prove the equivalence of the statements (i) and (ii). \( (i) \Rightarrow (ii) \): \( C_0 \) has a nonzero element at the first entry. The second entry of \( C_0 A_0 \) is nonzero.

The \((n - 1)\)th entry of \( C_0 A_0^{n-2} \) is nonzero. Therefore, multiplying them by \( D_0 \) yields zero because \( D_0 \) has only one nonzero at the last entry. The \( n \)th entry of \( C_0 A_0^{n-1} \) is nonzero. Then \( C_0 A_0^{n-1} D_0 \neq 0 \).

Conversely, \( (ii) \Rightarrow (i) \): we show this by contradiction. Suppose \( A_0 \) does not have this form. Assume the second row is moved to the first row. The third entry of \( C_0 A_0 \) is nonzero. The fourth entry of \( C_0 A_0^2 \) is nonzero. The \( n \)th entry of \( C_0 A_0^{n-2} \) is nonzero. Therefore, multiplying \( C_0 A_0^{n-2} \) by \( D_0 \) yields nonzero, which is contrary to \( C_0 A_0^{n-2} D_0 = 0 \). Therefore, due to the special structure of \( A_0 \), if its form is changed, condition (ii) can not be satisfied.

Then we prove the equivalence of the statements (i) and (iii). \( (i) \Rightarrow (iii) \): \n
\[
\begin{bmatrix} A_0 - z I_n & D_0 \\ C_0 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix}
    a_{11} - z & a_{12} & 0 & \cdots & 0 & 0 \\
    a_{21} & a_{22} - z & a_{23} & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_{(n-2)1} & a_{(n-2)2} & a_{(n-2)3} & \cdots & a_{(n-2)(n-1)} & 0 \\
    a_{(n-1)1} & a_{(n-1)2} & a_{(n-1)3} & \cdots & a_{(n-1)(n-1)} & a_{(n-1)n} \\
    a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n(n-1)} & a_{nn}
\end{bmatrix}
\]

Since \( a_{12} \neq 0, a_{23} \neq 0, \cdots, a_{(n-1)n} \neq 0, d_n \neq 0 \), they are in different columns, and the entries above of them
all are zeros, the columns consisting of them are linear independent no matter what the variable $z$ is. And the first column is linear independent to them. Therefore, all columns are linear independent for any $z \in \mathbb{C}$.

Conversely, $(iii) \Rightarrow (i)$:

$$\text{rank} \left( \begin{bmatrix} A_0 - zI_n & D_0 \\ C_0 & 0 \end{bmatrix} \right)$$

$$= \text{rank} \begin{bmatrix} a_{11} - z & a_{12} & a_{13} & \cdots & a_{1n} & d_1 \\ a_{21} & a_{22} - z & a_{23} & \cdots & a_{2n} & d_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & a_{(n-1)3} & \cdots & a_{(n-1)n} & d_{n-1} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} - z & d_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$= n + 1 \quad \text{for all } z \in \mathbb{C}. \quad (22)$$

Therefore, the determinant of last matrix in equation (22) should be a nonzero constant. All entries including variable $z$ are in the subdiagonal. By computing the determinant from Laplace Expansion, we show that the determinant of the last matrix in equation (22) is a nonzero constant for all $z \in \mathbb{C}$, i.e., there is no $z$ in the determinant, if and only if it has the form in (i). \qed

Remark 2: Statement $(ii)$ implies $(A, C)$ is observable due to the structure of $A_0$, $C_0$ and $D_0$ in statement $(i)$. So, there is no need to assume $(A, C)$ is observable if statement $(ii)$ is satisfied as in [3].

From Theorem 1, the general discrete-time linear time-invariant system in equation (21) has no invariant zero between disturbance and output and satisfies condition (ii), which is the structure condition proposed in [3] in continuous-time. Therefore, this special structure is crucial for the Luenberger-like state observer in equation (4) to be able to estimate states $x$ and disturbance $f$ in the presence of the new unknown input $\Delta f$.

In the following, we are going to show why the disturbance $\hat{f}$ estimated by ESO and UIO are exactly the same mathematically which as we illustrated in Fig. 1. Since the error dynamics of UIO in equation (11) is asymptotically stable and the UIO uses the future measurements, the estimated disturbance $\hat{f}$ is equal to the actual disturbance $f$ with a suitable time delay. However, the error dynamics of ESO in equation (6) has an input term $\Delta f$.

The estimated disturbance $\hat{f}$ is still equal to the actual disturbance $f$ with time delay if the system dynamics has the form in equation (21) and all poles of Luenberger state observer in equation (4) are at the origin, which is shown in the following Theorem 2.

**Theorem 2** For the discrete-time linear time-invariant system of equation (21), the estimated disturbance $\hat{f}$ in equation (4) is equal to the actual disturbance $f$ with a delay of $n + 1$ steps if system (21) is augmented as in system of difference equation (3) and all poles of equation (4) are placed at the origin.

**Proof.** The dynamics of the estimation error of observer (4) for the augmented system of equation (21) is

$$e(k + 1) = (A - LC)e(k) + D\Delta f(k) \quad (23)$$

where

$$A = \begin{bmatrix} A_0 & D_0 \\ 0_{1 \times n} & 1 \end{bmatrix}(n+1) \times (n+1), \quad C = [C_0, 0]_{1 \times (n+1)}$$

$$D = [0, \ldots, 0, 1]_{1 \times (n+1)}, \quad L \in \mathbb{R}^{n+1}$$

and $A_0$, $C_0$, and $D_0$ are defined as in equation (21). Since $a_{12} \neq 0$, $a_{23} \neq 0$, \ldots, $a_{(n-1)n} \neq 0$, $(A, C_0)$ is observable. There exists an $(n + 1) \times (n + 1)$ invertible matrix $S_1 = \begin{bmatrix} S_0 & 0_{n \times 1} \\ 0_{1 \times n} & m \end{bmatrix}$ such that $S_1$ satisfies the following equation

$$S_1 A S_1^{-1} = A_1 \quad (24)$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ a_{11} & a_{12} & \cdots & a_{1n} & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad S_0 = \begin{bmatrix} C_0 \\ C_0A_0 \\ \vdots \\ C_0A_0^{n-1} \end{bmatrix}, \quad m = a_{12} \times a_{23} \times \cdots \times a_{(n-1)n}$$

which is the element in the row $n$ and column $n$ of $S_0$. and $[a_1, \ldots, a_n] \neq C_0A_0^n S_0^{-1}$. This invertible matrix $S_1$ can change the matrix $A$ into an augmented cascade form $A_1$.

Then, the dynamics of the estimation error in equation (23) can be written as

$$e_1(k + 1) = (A_1 - L_1C_1)e_1(k) + D_1\Delta f(k) \quad (25)$$

where $e_1(k) = S_1e(k), C_1 = CS_1^{-1} = [1, 0, \ldots, 0]_{1 \times (n+1)}, \quad D_1 = S_1D = [0, \ldots, 0, m]_{1 \times (n+1)}$, and $L_1 = S_1L$. $(A_1, C_1)$ can be changed into the observable canonical
form with invertible matrix $S_2$ with

$$S_2A_1S_2^{-1} = A_2, C_1S_2^{-1} = C_2$$

(26)

where $C_2 = [1, 0, \cdots, 0]_{1 \times (n+1)}$, $S_2 = \begin{bmatrix} C_1 \\ C_1A_1 \\ \vdots \\ C_1A_1^T \end{bmatrix}$, and

$$A_2 = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_1 & a_1 - a_2 & \cdots & a_{n-1} - a_n & a_n + 1 \end{bmatrix}.$$

Thus, the estimation error in equation (25) is transformed into

$$e_2(k+1) = (A_2 - L_2C_2)e_2(k) + D_2\Delta f(k)$$

(27)

where $e_2(k) = S_2e_1(k)$, $D_2 = S_2D_1 = [0, 0, \cdots, m]_T^{T \times (n+1)}$, and $L_2 = S_2L_1$.

Note that matrix $A_2$ can be transformed into the following observer companion form $A_3$ by using invertible matrix $Q_1$ so that we only need to change the entries in the first column of $A_3$ to place all poles, where

$$A_3 = \begin{bmatrix} a_n + 1 & 1 & \cdots & 0 & 0 \\ a_{n-1} - a_n & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 - a_2 & 0 & \cdots & 0 & 1 \\ -a_1 & 0 & \cdots & 0 & 0 \end{bmatrix}, Q_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 - a_n & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_3 - a_2 & a_4 - a_3 & \cdots & 1 & 0 \\ a_2 - a_1 & a_3 - a_2 & \cdots & -1 - a_n & 1 \end{bmatrix}.$$

By using the invertible matrix $Q_1$, estimation error dynamics in equation (27) can be written as

$$e_3(k+1) = (A_3 - L_3C_3)e_3(k) + D_3\Delta f(k)$$

(28)

where $e_3(k) = Q_1e_2(k)$, $A_3 = Q_1A_2Q_1^{-1}$, $L_3 = Q_1L_2$, $C_3 = C_2Q_1^{-1}$, $L_3 = [1, 0, \cdots, 0]_{1 \times (n+1)}$, and $D_3 = Q_1D_2 = [0, \cdots, 0, m]_T^{T \times (n+1)}$. Let $L_3 = [l_1, l_2, \cdots, l_{n+1}]^T$. Then

$$A_3 - L_3C_3 = \begin{bmatrix} a_n + 1 - l_1 & 1 & \cdots & 0 & 0 \\ a_{n-1} - a_n - l_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 - a_2 - l_n & 0 & \cdots & 0 & 1 \\ -a_1 - l_{n+1} & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Moreover, the characteristic polynomial of matrix

$$\begin{bmatrix} -\alpha_1 & 1 & 0 & \cdots & 0 \\ -\alpha_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\alpha_n & 0 & 0 & \cdots & 1 \\ -\alpha_{n+1} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

is $z^{n+1} + \alpha_1 z^n + \cdots + \alpha_n z + \alpha_{n+1}$. Therefore, if all poles are at the origin, the characteristic polynomial is $z^{n+1}$, which implies $\alpha_1 = \alpha_2 = \cdots = \alpha_{n+1} = 0$. Then, by equating coefficients,

$$A_3 = A_3 - L_3C_3 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

After pole placement, equation (28) can be written as

$$e_3(k+1) = \tilde{A}_3e_3(k) + D_3\Delta f(k).$$

(29)

Taking the $z$-transform of both sides of equation (29) without considering initial condition yields

$$\tilde{e}_3(z) = (zI - \tilde{A}_3)^{-1}D_3\Delta \hat{f}(z).$$

(30)

Since all poles are at the origin, we have

$$(zI - \tilde{A}_3)^{-1} = \begin{bmatrix} 1/z & 1/z^2 & \cdots & 1/z^{n+1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1/z^2 \\ 0 & \cdots & 1/z \end{bmatrix}.$$ 

(31)

Next, substituting $e_1(k) = S_1e(k)$, $e_2(k) = S_2e_1(k)$, $e_3(k) = Q_1e_2(k)$, $D_1 = S_1D$, $D_2 = S_2D_1$, and $D_3 = Q_1D_2$ into equation (30) yields

$$S_1\tilde{e}(z) = (Q_1S_2)^{-1}(zI - \tilde{A}_3)^{-1}(Q_1S_2)S_1D\Delta \hat{f}(z).$$

(32)

Due to $(Q_1S_2)S_1D = [0, \cdots, 0, m]_T^{T \times (n+1)}$, equation (32) can be written as

$$S_1\tilde{e}(z) = (Q_1S_2)^{-1}\begin{bmatrix} 1/z^{n+1} \\ 1/z^n \\ \vdots \\ 1/z \end{bmatrix}.$$ 

(33)
If the last row of \((Q_1S_2)^{-1}\) is \([1, 1, \cdots, 1]_{1 \times (n+1)}\), the last row of equation (33) is
\[
m \left( \tilde{f}(z) - \hat{f}(z) \right) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1/z^{n+1} & 1/z^n & \cdots & 1/z \\ \vdots & \vdots & \ddots & \vdots \\ 1/z & \cdots & \cdots & 1/z^{n+1} \end{bmatrix} m \Delta \hat{f}(z).
\]

Take the inverse \(z\)-transform of equation (34), we have
\[
f(k) - \hat{f}(k) = \Delta f(k-n-1) + \cdots + \Delta f(k-2) + \Delta f(k-1)
= f(k-n) - f(k-n-1) + \cdots + f(k-1)
- f(k-2) + f(k) - f(k-1)
= f(k) - f(k-n-1).
\]

Remark 3: From the derivation of UIO, the disturbance \(f(k)\) cannot be obtained until the measurement \(y(k+n+1)\) is fed into the observer, which, in other aspects, explains why the estimated disturbance \(\hat{f}(k)\) in ESO should have a delay of \(n+1\) steps.

Remark 4: The structure of the last row of matrix \(A\) is crucial for the estimation of disturbance \(f(k)\). Otherwise, the last row of \((Q_1S_2)^{-1}\) cannot be \([1, 1, \cdots, 1]_{1 \times (n+1)}\), and \(\hat{f}(k)\) cannot be equal to the actual disturbance with delay. Moreover, the structure of last row of matrix \(A\) also has a physical meaning in ESO design (i.e., we assume \(\Delta f(k) = 0\), which implies that the disturbance \(f(k)\) changes slow and this assumption is usually reasonable in practical situation).

4.2 A reason ESO and UIO have different properties

One of the most important features of ESO is the ability to adjust the smoothness of the estimations with the help of observer bandwidth according to different level of measurement noise. The following Theorem 3 shows that the estimation accuracy of ESO can be adjusted by observer bandwidth and why it can smooth the estimations under the influence of measurement noise.

**Theorem 3** For the discrete-time linear time-invariant system in equation (21), the estimation accuracy of the disturbance \(f\) in equation (4) can be adjusted by observer bandwidth \(\omega_o\) \((0 \leq \omega_o < 1)\), that is to say, the last row of estimation error vector \(e(k)\) decreases monotonically after \(n+1\) steps of the change of \(f\) if reducing the value of \(\omega_o\).

**Proof.** After a series of similarity transformations made in Theorem 2, the estimation error dynamics of observer in equation (4) for the augmented system of equation (21) can be written as
\[
e_3(k+1) = (A_3 - L_3C_3)e_3(k) + D_3\Delta f(k)
\]
where \(e_3(k) = (Q_1S_2S_1)e(k), A_3 = (Q_1S_2S_1)A(Q_1S_2S_1)^{-1}, L_3 = (Q_1S_2S_1)L, C_3 = C(Q_1S_2S_1)^{-1},\) and \(D_3 = (Q_1S_2S_1)D\). Since \(A_3\) is in observer companion form, the entries of the first column of \(A_3 - L_3C_3\) are determined by the coefficients of characteristic polynomial placed all poles at \(\omega_o\), which is
\[
A_3 - L_3C_3 = \begin{bmatrix} (-1)^2(n+1)\omega_o & 1 & \cdots & 0 \\ (-1)^3(n+1)\omega_o^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1}n\omega_o^{n+1} & 0 & \cdots & 0 \\ (-1)^{n+2}(n+1)\omega_o^{n+1} & 0 & \cdots & 0 \end{bmatrix}.
\]

Now we need to obtain the transfer function from \(\Delta f(k)\) to the last row of \(e(k)\), which is \(f(k) - \hat{f}(k)\). According to equation (38), it is not easy to compute \((zI - A_3 + L_3C_3)^{-1}\) directly. Similarity transformation of matrix \(Q_2\) is used to transform \(A_3 - L_3C_3\) to a controller companion form \(A_4\), where \(A_4 = \begin{bmatrix} (-1)^2(n+1)\omega_o & 1 & \cdots & 0 \\ (-1)^3(n+1)\omega_o^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1}n\omega_o^{n+1} & 0 & \cdots & 0 \\ (-1)^{n+2}(n+1)\omega_o^{n+1} & 0 & \cdots & 0 \end{bmatrix}.
\]
From equations (37), (39), (40) and (41), the
\[
\begin{bmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
\end{bmatrix}
\begin{bmatrix}
(-1)^{n+2(n+1)}\omega_0^{n+1} \\
(-1)^{n+1(n+1)}\omega_0^{n+2} \\
\vdots \\
(-1)^{n+2(n+1)}\omega_0^{n+1} \\
\end{bmatrix}
\]

\[
Q_2 = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
(-1)^{(n+1)}\omega_0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{(n-1)(n-1)}\omega_0^{n-1} & (-1)^{n-2} \omega_0^{n-2} & \cdots & 0 \\
(-1)^{(n+1)}\omega_0^{n} & (-1)^{n-1} \omega_0^{n-1} & \omega_0^{n-1} & 1 \\
\end{bmatrix}
\]

Then, equation (37) can be reformulated as
\[
e_4(k + 1) = A_4e_4(k) + D_4\Delta f(k) \tag{39}
\]
where \(e_4(k) = Q_2^{-1}e_3(k), A_4 = Q_2^{-1}(A_3 - L_3C_3)Q_2,\) and
\(D_4 = Q_2^{-1}D_3.\)

Taking the \(z\)-transform of both sides of equation (39) without considering initial condition yields
\[
\hat{e}_4(z) = (zI - A_4)^{-1}D_4\Delta \hat{f}(z). \tag{40}
\]

Since \(A_4\) is in a controller companion form and \(D_4 = [0, \cdots, 0, m]^T,\) the transfer function of equation (40) is
\[
(zI - A_4)^{-1}D_4 = \begin{bmatrix}
1/(z - \omega_o)^{n+1} \\
\vdots \\
z^{n-1}/(z - \omega_o)^{n+1} \\
z^n/(z - \omega_o)^{n+1} \\
\end{bmatrix} \tag{41}
\]

From equations (37), (39), (40) and (41), the \(z\)-transform of the original error estimation dynamics is
\[
S_1\hat{e}(z) = (Q_1S_2)^{-1}Q_2 \begin{bmatrix}
1/(z - \omega_o)^{n+1} \\
\vdots \\
z^{n-1}/(z - \omega_o)^{n+1} \\
z^n/(z - \omega_o)^{n+1} \\
\end{bmatrix} \Delta \hat{f}(z). \tag{42}
\]

We obtain the last row of equation (42)
\[
m \left( \hat{f}(z) - \hat{f}(z) \right) = [1, \cdots, 1]Q_2 \begin{bmatrix}
1/(z - \omega_o)^{n+1} \\
\vdots \\
z^{n-1}/(z - \omega_o)^{n+1} \\
z^n/(z - \omega_o)^{n+1} \\
\end{bmatrix} m\Delta \hat{f}(z). \tag{43}
\]

Since the original system is linear and satisfies additivity and homogeneity, the estimation error of the disturbance \(\hat{f}\) decreases with reducing the value of \(\omega_o\) for all \(k \geq 0\) if the inverse \(z\)-transform of the transfer function of equation (43) decreases with smaller \(\omega_o\) for all \(k \geq 0\). The transfer function of equation (43) is
\[
\hat{h}(z) = [1, \cdots, 1]Q_2 \begin{bmatrix}
1/(z - \omega_o)^{n+1} \\
\vdots \\
z^{n-1}/(z - \omega_o)^{n+1} \\
z^n/(z - \omega_o)^{n+1} \\
\end{bmatrix} = \sum_{i=1}^{n+1} \frac{(1 - \omega_o)^{i-1}(z - \omega_o)^{n+1-i}}{(z - \omega_o)^{n+1}}. \tag{44}
\]

Taking the inverse \(z\)-transform of equation (44) yields equation (45) (as shown at the top of next page).

The derivative of \(h(k)\) with respect to \(\omega_o\) is
\[
\frac{\partial h(k)}{\partial \omega_o} = \frac{1}{n!} (1 - \omega_o)^{n}(k - n - 1) \left( \prod_{j=0}^{n} (k + j) \right) \omega_o^{k-n-2}. \tag{46}
\]

Therefore, if \(0 < \omega_o < 1\), then \(\partial h/\partial \omega_o > 0\) for all \(k > n + 1\). And \(h(k)\) decreases with respect to \(\omega_o\) for all \(k > n + 1\). \(\square\)

The transfer function \(\hat{h}(z)\) in equation (44) has the form of the combination of low pass filters, which shows the reason why ESO has the ability to smooth the estimation when the measurements are polluted by noise. This may also give us a guide to change the structure of ESO to filter the noise more efficiently. Theorem 3 guarantees the performance of estimation of \(\hat{f}\) not only in steady state but also in transient state.

On the other hand, compared with UIO, ESO ignores the change of the total disturbance, i.e., \(\Delta f\), in observation equation, which means the assumption that the total disturbance is a constant disturbance is added into the

\[\text{For simplicity, let } 0! = 1, \prod_{j=0}^{1}(k + j) = 1.\]
whole system model. Moreover, we can borrow some well-known characteristics of Kalman filter to explain the efficacy of the added disturbance model. Kalman filter includes two parts: the time update equations and the measurements update equations. Time update equations are based on the reliability of the system model, while measurements update equations are based on the accuracy of measurement model. The Kalman gain is used to keep the best balance between time update equations and measurements update equations.

In ESO, observer gain (i.e., observer bandwidth) is like Kalman gain, which is used to balance the trust of system model and measurement model. If measurement noise is very small, the poles of ESO can be placed closer to the origin so that the measurement equations can be relied on. If measurement noise is large, the poles of ESO can be place further away from origin so that the system model is the only information it can be trusted. Due to the ignorance of the change of the total disturbance in system model, the smoothness of estimation can be adjusted by observer bandwidth.

5 A method to generalize ESO

As shown in Section 4, no invariant zero between disturbance and output is crucial in ESO design which explains the relative degree of the plant is the only information needed in conventional ESO [17]. The other plant information can be considered as an internal disturbance contained in the total disturbance, which is subtracted from the feedback control input so that the plant will act as a nominal model in controller design.

Moreover, one difference between ESO and UIO is that the dynamical model of uncertain disturbance acted on the system is added into the system model, which is like DAC proposed by Johnson in [25] and [26]. However, compared with DAC, there is only one mode of uncertain disturbance function in ESO, and the input of dynamical model of disturbance function in ESO can be any signal rather than a sequence of random impulses in DAC theory.

Theorem 2 and Theorem 3 in Section 4 may explain why DAC needs the assumption that the input of dynamical model of disturbance should be a vector sequence of random impulses with sparsely separation. Since there may be some invariant zeros between input of dynamical model of disturbance and output, the input signal can cause fluctuation in observer error dynamics. If the input signal is not sparsely separate impulses, it will lead continuous fluctuation and DAC controller will be unable to response properly. But we can borrow some ideas from DAC theory, the disturbance model in ESO doesn’t have to be limited to constant disturbance model as long as there is no invariant zero between input of disturbance dynamics and output.

Based on the above discussion, we can generalize ESO to accommodate more model information into the observer design in some cases. In the following, a simple example is given to illustrate. The series elastic actuator model used in Section 3 is still used. But the position of motor needs to be controlled and is the only measurable output. Assume the dynamics of series elastic actuator after discretization is

\[
\begin{cases}
(I_M z^2 + C_M z + K) \theta_M - K \theta_L = \tau_M \\
(I_L z^2 + C_L z + K) \theta_L - K \theta_M = \tau_L
\end{cases}
\]  

(47)

where \(I_M, I_L, C_M, C_L\) and \(K\) are parameters after discretization; \(\tau_M\) is the control input of the driving motor; \(\tau_L\) is the external torque applied on the load; \(\theta_M\) and \(\theta_L\) are positions of the driving motor and load.

For conventional ESO design, equation (47) needs to be converted to

\[
\left( z^2 + \frac{C_M}{I_M} z + \frac{K}{I_M} \right) \theta_M = \frac{1}{I_M} \tau_M + f
\]  

(48)

where \(f\) is the total disturbance which includes the external torque from \(\theta_L\) via the spring.

As shown in equation (48), the order of system is 2 (i.e., the relative degree of the system from input to output). After extending the state, the augmented system can be written in the form of equation (3), where

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
\frac{K}{I_M} - \frac{C_M}{I_M} & 1 \\
0 & 0 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
\frac{1}{I_M} \\
0
\end{bmatrix}, \quad D = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}, \quad C = [1, 0, 0],
\]

and \(x(k) = [\theta_M(k), \theta_M(k + 1)]^T\). However, the second equation in (47) is ignored because it is zero dynamics in the system.

In the new generalized ESO, the zero dynamics can be considered as a model of disturbance like that in DAC theory and the input of the disturbance model can be placed without adding any invariant zeros. The first

\[
h(k) = \begin{cases}
1 & 1 \leq k \leq n + 1 \\
\sum_{i=1}^{n+1} \frac{1}{(n+1)!} (1 - \omega_o)^{i-1} \left( \frac{1}{j-i+1} \right)^{k-j} \omega_o^{k-j} & k \geq n + 2
\end{cases}
\]

(45)
equation in (47) can be written as
\[
\theta_M(k + 2) = \frac{C_M}{I_M} \theta_M(k + 1) - \frac{K}{I_M} \theta_M(k) + \frac{1}{I_M} \tau_M(k) + \frac{K}{I_M} (\theta_L(k) + d_1(k)) \tag{49}
\]
where \( \frac{K}{I_M} (\theta_L(k) + d_1(k)) \) is the total disturbance on the motor side, \( d_1(k) \) is the disturbance rather than that caused by \( \theta_L(k) \). Set \( f_1(k) = \theta_L(k) + d_1(k) \). Substituting \( \theta_L(k) = f_1(k) - d_1(k) \) in the second equation of (47) yields
\[
f_1(k + 2) = -\frac{C_L}{I_L} f_1(k + 1) - \frac{K}{I_L} f_1(k) + \frac{K}{I_L} \theta_M(k) + 1 \frac{1}{I_L} \tau_L(k) + d_1(k) + \frac{C_L}{I_L} d_1(k + 1) + \frac{K}{I_L} d_1(k). \tag{50}
\]
Set the new total disturbance at the zero dynamics side
\[
\bar{f}(k) = \frac{1}{I_L} \tau_L(k) + d_1(k + 2) + \frac{C_L}{I_L} d_1(k + 1) + \frac{K}{I_L} d_1(k).
\]
Then, the new augmented system can be written in the form of equation (3), where \( C = [1, 0, 0, 0, 0], A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{K}{I_M} & \frac{C_M}{I_M} & \frac{K}{I_M} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{K}{I_L} & 0 & -\frac{K}{I_L} & -\frac{C_L}{I_L} & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, x(k) = [\theta_M(k), \theta_M(k + 1), f_1(k), f_1(k + 1)]^T, f(k) = \bar{f}(k), \) and \( \Delta \bar{f}(k) = \bar{f}(k + 1) - \bar{f}(k) \). By applying Theorem 1, there is no invariant zero between \( \bar{f}(k) \) and \( y(k) \).

Therefore, a Luenerberger state observer in equation (4) can be used to estimate states and disturbances like conventional ESO.

6 Conclusions

UIO and ESO are two well-known state observers capable of estimating states and unknown inputs. The observer error dynamics of UIO is decoupled from the unknown inputs while the error dynamics of ESO relates to the unknown inputs. UIO has a rigorous theoretical framework while ESO is a very popular tool among control practitioners. In this paper, a comparison of them for discrete-time linear time-invariant systems with unknown inputs is given.

We find that the disturbance estimated by UIO and ESO is exactly the same if the poles of ESO are placed at origin. Several proofs are given to explain their relationship. We also find that no invariant zero between disturbance and measurement output should be a design principle for ESO design. And a new generalized ESO is proposed at last. Since the total disturbance in ESO is fictitious disturbance in the system, it is very flexible to design ESO and use it in active disturbance rejection control. Moreover, with the help of this design principle and coordinate system transformation, the ESO design method proposed in this paper can be easily implemented in multi-input multi-output systems. We hope this paper can provide a solution to bridge the gap between control theory and practical engineering approaches.

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