Initial state design for suppressing undesirable effects of controller switches

Koichi Suyama* Noboru Sebe**

* Tokyo University of Marine Science and Technology
Etchujima, Koto-ku, Tokyo 135-8533, Japan
(e-mail: suyama@kaiyodai.ac.jp).
** Kyushu Institute of Technology
Kawazu, Iizuka, Fukuoka 820-8502, Japan
(e-mail: sebe@ai.kyutech.ac.jp).

Abstract: We propose a new initial state design procedure for a newly-activated controller at a controller switch. By minimizing the value of the state-dependent switching $L_2$ gain presented in this paper, we can obtain the optimal initial state for suppressing the difference between the actuality that a controller switch occurs and the virtual situation where it does not occur.

Keywords: Control systems; Controllers; Initial states; Linear systems; Switches.

1. INTRODUCTION

The switches from an operating controller to a more desirable one have been widely performed for letting a control system adjust to the changes in control objective, operating conditions, surrounding circumstances, and so on. Here, it is needless to say that we should reduce the undesirable effects of a controller switch. A practical technique for reducing the undesirable effect is to appropriately initialize a newly-activated controller.

Many studies have addressed the issue of suppressing the fluctuations in transient responses after a controller switch by designing the initial state of a newly-activated controller, e.g., Hanus et al. (1987), Kothare et al. (1994), Edwards and Postlethwaite (1998), Turner and Walker (2000), and Paxman and Vinnicombe (2000). Most of them aimed at making the output of a control system after a controller switch close to the virtual output in the case where the switch does not occur.

Asai (2003) designed a control system by reducing the value of the Hankel-type switching $L_2$ gain presented in Asai (2005) to suppress the fluctuations in transient responses. Suyama and Sebe (2019a) obtained the optimal switching matrix, which determines the initial state of a newly-activated controller, for suppressing the fluctuations in transient responses after a controller switch by minimizing the value of another switching $L_2$ gain. Moreover, Suyama and Sebe (2019b) presented a procedure for directly obtaining the optimal initial state by using the state-dependent switching $L_2$ gain. (Suyama and Sebe, 2018b).

For taking a desirable reference signal into consideration, Zaccarian and Teel (2005) and Hespanha et al. (2007) directly minimized the $L_2$ norm of the error between the plant output and reference signal to obtain the optimal initial state. Under the assumption that only the state of an integrator is assigned in a state-feedback control system, Saito et al. (1998) approximated its step response by assigning an initial value by using the observability Gramian. Also Nakano et al. (2018) directly evaluated the $L_2$ norm of the output signal by using the observability Gramian in an observer-based servo system to show that transient responses caused by state-feedback and observer gains switch can be suppressed by appropriately initializing a newly-activated observer.

In this paper, we first present a new state-dependent switching $L_2$ gain that focuses on the difference between the actuality that a controller switch occurs and the virtual situation where it does not occur. Then, for a given switching situation, by minimizing the value of the state-dependent switching $L_2$ gain, we obtain the optimal initial state for suppressing the difference between the actuality and virtual situation. We can make the output of a control system after a controller switch close to the virtual output in the case where the switch does not occur more effectively. The proposed initial state design procedure has the following advantages.

- The switching time does not affect its design result, and need not be known in advance.
- We need only solving a linear matrix inequality (LMI) problem. Complicated calculation is not necessary.
- We can always obtain the solution. It is not necessary to discuss the solvability.

They are important especially when the initial state should be determined immediately after a non-preplanned controller switch. Moreover, by the proposed initial state design, we can improve the safety of the operating-state transitions in the procedure of safe preventive maintenance of control systems presented in Suyama and Sebe (2017).

Notations. $F_t(G,K)$ denotes the lower linear fractional transformation (LFT) of $G$ and $K$, and $L_2(a,b)$: the Lebesgue space of all square-integrable and vector-valued functions defined on an interval $(a, b)$, i.e., $L_2(a,b) =$
\[
\|x(t)\|_2(a,b) < \infty, \text{ where } \|x(t)\|_2(a,b) \text{ denotes the } \mathcal{L}_2 \text{ norm defined by } \\
\|x(t)\|_2(a,b) = \left[ \int_a^b x^T(t)x(t)dt \right]^{\frac{1}{2}}.
\]

2. STATE-DEPENDENT SWITCHING \( \mathcal{L}_2 \) GAIN

2.1 System switch

Suppose that a linear time-invariant (LTI) system \( H_p \) switches to another LTI system \( H_f \) with a state transition at the switching time \( t = t_0 \).

Suppose that the pre-switch system is represented by

\[
H_p: \begin{cases}
\dot{x}_p(t) = A_p x_p(t) + B_p w(t), & t \leq t_0, \\
z(t) = C_p x_p(t) + D_p w(t),
\end{cases}
\]

where \( x_p(t) \in \mathbb{R}^{n_p} \) \((t \leq t_0)\) is the state-variable vector, \( w(t) \in \mathbb{R}^{n_w} \) is the input, and \( z(t) \in \mathbb{R}^{n_z} \) is the output. We assume the following.

**Assumption 2.1.**

(a) \( A_p \) is stable.

(b) \((A_p, B_p)\) is controllable and \((C_p, A_p)\) is observable.

(c) \( x_p(-\infty) = 0 \).

Suppose that the post-switch system is represented by

\[
H_f: \begin{cases}
\dot{x}_f(t) = A_f x_f(t) + B_f w(t), & t > t_0, \\
z(t) = C_f x_f(t) + D_f w(t),
\end{cases}
\]

where \( x_f(t) \in \mathbb{R}^{n_f} \) \((t > t_0)\) is the state-variable vector and \( w(t), z(t) \) are the same input and output as in the pre-switch system \( H_p \). We assume the following.

**Assumption 2.2.**

(a) \( A_f \) is stable.

(b) \((A_f, B_f)\) is controllable and \((C_f, A_f)\) is observable.

Note that \( n_f \) is not always equal to \( n_p \), because the system order can change. For example, in a controller switch, there can be the difference in orders between an operating controller and a newly-activated one.

Suppose that the following state transition occurs around the system switch:

\[
x_f(t_0+) = S x_p(t_0),
\]

where \( S \in \mathbb{R}^{n_f \times n_p} \) is a constant matrix.

2.2 Definition of state-dependent switching \( \mathcal{L}_2 \) gain

Under the condition that the switch to the post-switch system with the state transition does not occur at \( t = t_0 \) \(\) (strictly speaking, at any \( t \in [t_0, \infty) \)), let us extend the pre-switch system as follows:

\[
H_{p, \text{ext}}: \begin{cases}
\dot{x}_p(t) = A_p x_p(t) + B_p w(t), & t \leq t_0, \\
z(t) = C_p x_p(t) + D_p w(t), \\
z_{\text{vir}}(t) = C_p x_p(t) + D_p w(t), & t > t_0,
\end{cases}
\]

where \( z_{\text{vir}}(t) \in \mathbb{R}^{n_z} \) \((t > t_0)\) is the virtual output under the assumption that no system switch occurs.

The state-dependent switching \( \mathcal{L}_2 \) gain used in this paper is defined as follows.

**Definition 2.3.** For an \( x_0 \in \mathbb{R}^{n_p} \),

\[
\gamma_{\text{sd-dif}}(x_0) = \sup_{w(t) \in \mathcal{L}_2(-\infty, \infty), \{t\} \cap [0, \infty) = \emptyset} \frac{\|z(t) - z_{\text{vir}}(t)\|_2(t_0, \infty)}{\|w(t)\|_2(-\infty, \infty)}.
\]

Note that the switching time \( t_0 \) does not affect the gain value as shown in Theorem 2.4 later.

Suyama and Sebe (2016) presented the following switching \( \mathcal{L}_2 \) gain to analyze the magnitude of a system switch:

\[
\hat{\gamma}_{\text{dif}} = \sup_{w(t) \in \mathcal{L}_2(-\infty, \infty), \{t\} \cap [0, \infty) = \emptyset} \frac{\|z(t) - z_{\text{vir}}(t)\|_2(t_0, \infty)}{\|w(t)\|_2(-\infty, \infty)}.
\]

The relationship between \( \gamma_{\text{sd-dif}}(x_0) \) and \( \hat{\gamma}_{\text{dif}} \) is as follows:

\[
\hat{\gamma}_{\text{dif}} = \max_{x_0 \in \mathbb{R}^{n_p}} \gamma_{\text{sd-dif}}(x_0).
\]

2.3 Difference system

Consider the following difference system between the post-switch system \( H_f \) and the extended pre-switch system \( H_{p, \text{ext}} \) on the post-switch side:

\[
H_d: \begin{cases}
\dot{x}_d(t) = A_d x_d(t) + B_d w(t), \\
z_{\text{vir}}(t) = z(t) - z_{\text{vir}}(t) = C_d x_d(t) + D_d w(t), & t > t_0,
\end{cases}
\]

where

\[
x_d(t) = \begin{bmatrix} x_f(t) \\ x_p(t) \end{bmatrix}, \quad t > t_0
\]

and

\[
A_d = \begin{bmatrix} A_f & O \\ O & A_p \end{bmatrix}, \quad B_d = \begin{bmatrix} B_f \\ B_p \end{bmatrix}.
\]

\[
C_d = [C_f -C_p], \quad D_d = D_f -D_p.
\]

From Assumptions 2.1 (a) and 2.2 (a), \( H_d \) is stable. However, \( H_d \) is not always controllable and observable. If \( H_p \) and \( H_f \) have a common pole, there is the possibility that the pole is uncontrollable and/or unobservable. Thus, \( H_d \) is stabilizable and detectable in general.

We then consider the system switch from \( H_p \) to \( H_d \) with the state transition

\[
x_d(t_0+) = S_d x_p(t_0), \quad S_d = \begin{bmatrix} S \\ I \end{bmatrix}.
\]

It occurs at the switching situation \( x_p(t_0) = x_0 \) to discuss the state-dependent switching \( \mathcal{L}_2 \) gain \( \gamma_{\text{sd-dif}}(x_0) \).

2.4 Equation-based \( \mathcal{L}_2 \) gain condition

The following theorem presents an equation-based \( \mathcal{L}_2 \) gain condition. It shows that the switching time does not affect the value of the state-dependent switching \( \mathcal{L}_2 \) gain \( \gamma_{\text{sd-dif}}(x_0) \).

**Theorem 2.4.** Let \( \gamma > 0 \) and \( x_0(\neq 0) \in \mathbb{R}^{n_p} \). The state-dependent switching \( \mathcal{L}_2 \) gain \( \gamma_{\text{sd-dif}}(x_0) \) satisfies \( \gamma_{\text{sd-dif}}(x_0) < \gamma \) if and only if the following conditions are satisfied.

(a) \( \sigma(D_d) < \gamma \).

(b) There exists the stabilizing solution \( X_d \geq O \) to the Riccati equation

\[
\begin{align*}
X_d A_d + A_d^T X_d + C_d^T C_d + (X_d B_d + C_d^T D_d) \\
\times (\gamma^2 I - D_d^T D_d)^{-1}(X_d B_d + C_d^T D_d)^T = O.
\end{align*}
\]

(c) It holds that

\[
x_0^T (\gamma^2 X_p^{-1} - S_d^T X_d S_d) x_0 > 0,
\]

where \( X_p = \mathcal{L}_2(-\infty, \infty), \{t\} \cap [0, \infty) = \emptyset \).
where \( X_p \succ O \) is a unique solution to the Lyapunov equation
\[
A_p X_p + X_p A_p^T + B_p B_p^T = O. \tag{14}
\]

**Proof:** (i) Sufficiency: On the pre-switch side (i.e., \( t < t_0 \)), it follows from (14) that
\[
\begin{bmatrix}
X_p^{-1} x_p(t) \\
w(t)
\end{bmatrix}
= \begin{bmatrix}
A_p X_p + X_p A_p^T & B_p \\
B_p^T & -I
\end{bmatrix}
\begin{bmatrix}
X_p^{-1} x_p(t) \\
w(t)
\end{bmatrix}
\leq 0
\]
for any \( w(t) \in L_2(-\infty, t_0) \) and the corresponding \( x_p(t) \). Integrating (15) with respect to \( t \in (-\infty, t_0) \) and using Assumption 2.1 (c), we have
\[
\|w(t)\|_2^2 (-\infty, t_0) \geq x_0^T X_p^{-1} x_0. \tag{16}
\]
On the post-switch side (i.e., \( t > t_0 \)), using the stabilizing solution \( x_d \geq O \) to Riccati equation (12), we consider the following equation that holds for any \( w(t) \in L_2(t_0, \infty) \) and the corresponding \( x_d(t) \):
\[
\int_{t_0}^{\infty} \begin{bmatrix}
X_d A_d + A_d^T X_d & X_d B_d + W_{11} \\
B_d^T X_d & W_{22}
\end{bmatrix}
\begin{bmatrix}
x_d(t) \\
w(t)
\end{bmatrix}
dt \leq 0. \tag{17}
\]

where
\[
\begin{bmatrix}
X_d A_d + A_d^T X_d & X_d B_d + W_{11} \\
B_d^T X_d & W_{22}
\end{bmatrix}
= C_d C_d + C_d^T D_d,
\]
and
\[
W_{11} = C_d^T C_d,
\]
and
\[
W_{22} = -\gamma^2 I - D_d^T D_d.
\]
Here, it follows from the stability of \( H_d \) that \( x_d(\infty) = 0 \) for any \( w(t) \in L_2(t_0, \infty) \). We thus have
Left-hand side of (17)
\[
\|z_d(t)\|_2^2 (t_0, \infty) - \gamma^2 \|w(t)\|_2^2 (t_0, \infty) - S_d x_0 = 0.
\] (18)

Next, under Conditions (a), (b) and the detectability of \( H_d \), we have
\[
\int_{t_0}^{\infty} \begin{bmatrix}
X_d A_d + A_d^T X_d & X_d B_d + W_{11} \\
B_d^T X_d & W_{22}
\end{bmatrix}
\begin{bmatrix}
x_d(t) \\
w(t)
\end{bmatrix}
dt \leq 0.
\]
Thus, for any \( w(t) \in L_2(t_0, \infty) \) and the corresponding \( x_d(t) \),
Right-hand side of (17) \( \leq 0 \).
\[
\text{Therefore, from (18) and (19) we have}
\|z_d(t)\|_2^2 (t_0, \infty) - \gamma^2 \|w(t)\|_2^2 (t_0, \infty) \leq S_d x_0.
\] (20)

Multiplying (16) by \(-\gamma^2 \), adding it with (20), and using (11) and (13), we have
\[
\|z_d(t)\|_2^2 (t_0, \infty) - \gamma^2 \|w(t)\|_2^2 (-\infty, \infty) \leq -x_0 \gamma^2 X_p^{-1} - S_d x_0 x_0 < 0
\] (21)
for any \( w(t) \in L_2(-\infty, \infty) \). This implies that \( \tilde{\gamma}_{sd-diff}(x_0) < \gamma \).

(ii) Necessity: (ii-1) Conditions (a) and (b): Define the \( L_2 \) gain of \( H_d \) by
\[
\gamma_d = \sup_{w \in L_2(t_0, \infty)} \frac{\|z(t)\|_2(t_0, \infty)}{\|w(t)\|_2(t_0, \infty)},
\] (22)
where \( z_d(t) \in L_2(t_0, \infty) \) is the output of \( H_d \) against \( w(t) \in L_2(t_0, \infty) \) under the condition that \( x_d(t_0) = 0 \). Suppose that Condition (a) and/or (b) is not satisfied.
Then, it hold that \( \gamma_d \geq \gamma \). Define \( \epsilon_1 = \gamma_d - \gamma \geq 0 \). Let \( w_{\infty}(t) \) \((t > t_0)\) denote an input providing the value of \( \gamma_d \) under \( x_d(t_0) = 0 \). Let \( w_{dsc}(t) \) denote the output against \( w_{dsc}(t) \) under \( x_d(t_0) = 0 \). We consider the input \( w(t) \in L_2(-\infty, \infty) \) given by
\[
\tilde{w}(t) = \begin{bmatrix}
w_{p_0}(t), & t \leq t_0 \\
\eta \cdot w_{dsc}(t), & t > t_0
\end{bmatrix},
\] (23)
where \( \eta > 0 \), and \( w_{p_0}(t) \) \((t \leq t_0)\) realizes \( p(t) = x_0 \).
Then, the output against \( \tilde{w}(t) \) is given by
\[
\tilde{z}(t) = C_d e^{A_d(t-t_0)} S_d x_0 + \eta \cdot z_{dsc}(t), \quad t > t_0.
\] (24)

We thus have
\[
\tilde{\gamma}_{sd-diff}(x_0) = \gamma_d - \epsilon_2,
\] (25)
where \( \epsilon_2 > 0 \). By choosing \( \eta \) sufficiently large, we can achieve \( \epsilon_2 \leq \epsilon_1 \). This implies \( \tilde{\gamma}_{sd-diff}(x_0) \geq \gamma \). Thus, \( \tilde{\gamma}_{sd-diff}(x_0) < \gamma \) does not hold.

(ii-2) Condition (c): Suppose that Condition (c) is not satisfied. Consider the following input:
\[
\tilde{w}(t) = \begin{bmatrix}
B_d^T e^{A_d(t-t_0)} X_p^{-1} x_0, & t \leq t_0 \\
K_d e^{A_d(t-t_0)} S_d x_0, & t > t_0
\end{bmatrix},
\] (26)
where
\[
K_d = (\gamma^2 I - D_q^T D_d)^{-1} (X_d B_d + C_d^T D_d) .
\] (27)
Here, \(-A_p^T\) is anti-stable by Assumption 2.1 (a); \( A_d + B_d K_d \) is stable because \( X_p \) is the stabilizing solution to the Riccati equation (12). Thus, \( \tilde{w}(t) \) belongs to \( L_2(-\infty, \infty) \).

Since on the pre-switch side, the controllability Gramian
\[
X_p = \int_{t_0}^{\infty} e^{A_d^T \tau} B_p B_p^T e^{A_d \tau} d\tau \succ O
\]
is the solution to the Lyapunov equation (14), we have
\[
\|\tilde{w}(t)\|_2^2 (-\infty, t_0) = x_0^T X_p^{-1} x_0.
\] (28)

Furthermore, using Assumption 2.1 (c), we have \( x_p(t_0) = x_0 \).

On the post-switch side, using \( K_d \) given in (27) and Condition (b), we have
\[
\begin{bmatrix}
X_d A_d + A_d^T X_d & X_d B_d + W_{11} \\
B_d^T X_d & W_{22}
\end{bmatrix}
= \begin{bmatrix}
I & O \\
-K_d & I
\end{bmatrix}
\begin{bmatrix}
O & O \\
O & I
\end{bmatrix}
\begin{bmatrix}
I & O \\
-K_d & I
\end{bmatrix}.
\] (29)

Here, the input \( \tilde{w}(t) \) and corresponding \( \tilde{x}_d(t) \) with the initial condition \( x_d(t_0) = S_d x_0 \) satisfy \( \tilde{x}_d(t) = K_d \tilde{x}_d(t), \quad t > t_0 \). Then we have
\[
\begin{bmatrix}
I & O \\
-K_d & I
\end{bmatrix}
\tilde{w}(t) = \begin{bmatrix}
\tilde{x}_d(t) \\
0
\end{bmatrix}, \quad t > t_0.
\] (30)
By applying (29) and (30) to (19), we have that the inequality (19) has equality; thus, the inequality (20) also has equality as
\[ \|z(t)\|_{2,(t_0,\infty)}^2 - \gamma^2 \|\tilde{w}(t)\|_{2,(t_0,\infty)}^2 = (S_d x_0)^T X_d S_d x_0, \]
where \( z(t) \) is the output corresponding to \( \tilde{w}(t) \).

Multiplying (28) by \(-\gamma^2\) and adding it with (31), we have
\begin{align*}
\|z(t)\|_{2,(t_0,\infty)}^2 - \gamma^2 \|\tilde{w}(t)\|_{2,(t_0,\infty)}^2 &= -x_d^T (\gamma^2 X_p^{-1} - S_d^T X_d S_d) x_0.
\end{align*}
(32)
Since Condition (c) is not satisfied, \( \|z(t)\|_{2,(t_0,\infty)}^2 - \gamma^2 \|\tilde{w}(t)\|_{2,(t_0,\infty)}^2 \ngeq 0 \). Therefore, \( \gamma_{sd-dif}(x_0) < \gamma \) does not hold.

Let \( x_0 \neq 0 \). Suppose that Conditions (a) and (b) in Theorem 2.4 are satisfied, and \( x_d^T (\gamma^2 X_p^{-1} - S_d^T X_d S_d) x_0 = 0 \). Then, \( \gamma_{sd-dif}(x_0) = \gamma \). Furthermore, an input providing the value of \( \gamma_{sd-dif}(x_0) \) is given by (26).

Also, it follows from Conditions (a) and (b) in Theorem 2.4 that for any \( x_0 \neq 0 \) in \( \mathbb{R}^n \), the state-dependent switching \( \mathcal{L}_2 \) gain \( \gamma_{sd-dif}(x_0) \) satisfies
\begin{align*}
\gamma_{sd-dif}(x_0) &\geq \gamma_d.
\end{align*}
(33)

2.5 LMI-based \( \mathcal{L}_2 \) gain condition

The LMI conditions in the theorem presented below will play an essential role in the proposed initial state design.

Theorem 2.5. Let \( \gamma > 0 \) and \( x_0 \neq 0 \) in \( \mathbb{R}^n \). The state-dependent switching \( \mathcal{L}_2 \) gain \( \gamma_{sd-dif}(x_0) \) satisfies \( \gamma_{sd-dif}(x_0) < \gamma \) if and only if there exist \( \tilde{X}_p > O \) and \( \tilde{X}_d > O \) satisfying the following conditions:
\begin{align*}
\begin{bmatrix}
\tilde{X}_p A_p + A_p^T \tilde{X}_p & \tilde{X}_p B_p & -\gamma I \\
B_p^T \tilde{X}_p & -\gamma I & 0 \\
\end{bmatrix} &< O, \quad (34)
\begin{bmatrix}
\tilde{X}_d A_d^T + A_d^T \tilde{X}_d & B_d^T & C_d^T \\
B_d & -\gamma I & D_d^T \\
C_d & D_d & -\gamma I \\
\end{bmatrix} &< O, \quad (35)
\begin{bmatrix}
x_0^T S_d x_0 & \tilde{X}_d x_0 & \tilde{X}_d \tilde{X}_d x_0 & -\gamma I \\
S_d x_0 & \tilde{X}_d & \tilde{X}_d x_0 & -\gamma I \\
\end{bmatrix} &> O. \quad (36)
\end{align*}

Proof: The proof of the sufficiency is entirely analogous to that in Theorem 2.4. Thus, we only prove the necessity by obtaining \( \tilde{X}_p > O \) and \( \tilde{X}_d > O \) satisfying (34)–(36) from Conditions (a)–(c) in Theorem 2.4.

On the pre-switch side, it follows from Assumption 2.1 (a) that there exists \( X_p > O \) satisfying
\begin{align*}
A_p X_p + X_p A_p^T &< O, \quad (37)
\end{align*}

Multiplying (37) by \( \epsilon > 0 \) and adding it with (14), we have
\begin{align*}
A_p (X_p + \epsilon X_p') + (X_p + \epsilon X_p') A_p^T + B_p B_p^T &< O, \quad (38)
\end{align*}
where \( X_p + \epsilon X_p' > O \). Left- and right-multiplying (38) by \( (X_p + \epsilon X_p')^{-1} > O \) and using the Schur complement, we have
\begin{align*}
\begin{bmatrix}
(X_p + \epsilon X_p')^{-1} A_p & A_p^T (X_p + \epsilon X_p')^{-1} \\
B_p^T & -\gamma I \\
\end{bmatrix} &< O. \quad (39)
\end{align*}

Multiplying (39) by \( \gamma > 0 \) and defining \( \tilde{X}_p = \gamma (X_p + \epsilon X_p')^{-1} \), we can have (34).

By using \( \tilde{X}_p \) given in (40) with sufficiently small \( \epsilon > 0 \), we can have
\begin{align*}
x_0^T (\gamma \tilde{X}_p - S_d^T X_d S_d) x_0 > 0 \quad (41)
\end{align*}
from Condition (c) in Theorem 2.4. Consider the following Riccati matrix inequality related to (12):
\begin{align*}
X_d A_d + A_d^T X_d + C_d^T C_d + (X_d B_d + C_d^T D_d) \\
\times (\gamma I - D_d^T D_d)^{-1} (X_d B_d + C_d^T D_d)^T < O \quad (42)
\end{align*}
Since there exists the stabilizing solution \( X_d > O \) to (12), there exists a solution \( X_d' > O \) to (42). Furthermore, \( X_d > O \) is the minimum solution, i.e., \( X_d' \geq X_d \) (Zhou et al., 1996). Since \( \tilde{X}_p \) satisfying (34) and \( \tilde{X}_f \) also satisfy (41), there exists \( X_d' \) such that
\begin{align*}
x_0^T (\gamma \tilde{X}_p - S_d^T X_d S_d) x_0 > 0. \quad (43)
\end{align*}

We take \( \tilde{X}_d > O \) as
\begin{align*}
\tilde{X}_d = \frac{1}{\gamma} X_d'. \quad (44)
\end{align*}

Then, from (42) we can have
\begin{align*}
\begin{bmatrix}
\tilde{X}_d A_d + A_d^T \tilde{X}_d & \tilde{X}_d B_d & C_d' \\
B_d^T & -\gamma I & D_d^T \\
C_d & D_d & -\gamma I \\
\end{bmatrix} &< O. \quad (45)
\end{align*}

Furthermore, by using (44), from (43) we can have
\begin{align*}
x_0^T (\tilde{X}_p - S_d^T \tilde{X}_d S_d) x_0 > 0. \quad (46)
\end{align*}

Rewriting \( X_d^{-1} \) by \( \tilde{X}_d \) and using the Schur complement, we can have (35) and (36) from (45) and (46).

3. INITIAL STATE DESIGN

3.1 Problem statement

As shown in Fig. 1, we consider the pre-switch system \( H_p \) and post-switch system \( H_f \) described in the linear fractional transformation (LFT) framework. Here, \( w \) is the exogenous input, \( z \) is the evaluation output, \( u \) is the control input, and \( y \) is the measured output. The generalized plant \( G \) is not switched. Let \( x_d(t) \in \mathbb{R}^n \) denote its state-variable vector. On the other hand, the controller is switched from the operating \( K_p \) to a more desirable \( K_f \) at \( t = t_0 \) in some sense, such as performance and fault tolerance. Let \( x_{k_0}(t) \in \mathbb{R}^{n_{k_0}} \) and \( x_{k_1}(t) \in \mathbb{R}^{n_{k_1}} \) denote the state-variable vectors of \( K_p \) and \( K_f \), respectively. Then, by taking \( x(t) = [x_d(t) \ x_{k_0}(t)]^T \), the pre-switch system \( H_p \) with the output can be represented as follows:
\begin{align*}
H_p : \begin{bmatrix}
A_p \\
C_p \\
\end{bmatrix} = F_p(G, K_p). \quad (47)
\end{align*}

Furthermore, by taking \( x(t) = [x_d(t) \ x_{k_1}(t)]^T \), the post-switch system \( H_f \) can be represented as follows:
\begin{align*}
H_f : \begin{bmatrix}
A_f \\
C_f \\
\end{bmatrix} = F_f(G, K_f). \quad (48)
\end{align*}

We assume that \( H_p \) and \( H_f \) obtained above satisfy Assumptions 2.1 and 2.2, respectively.
Fig. 1. (a) Pre-switch system and (b) post-switch system in a controller switch. Since \( x_0(t_{0+}) = x_0(t_0) \), the switching matrix is of the following form:

\[
S = \begin{bmatrix} I & O \\
S_1 & S_2 \end{bmatrix},
\]

where \( S_1 \in \mathbb{R}^{n_k \times n_o} \) and \( S_2 \in \mathbb{R}^{n_k \times n_k} \). Then, we have

\[
x_f(t_{0+}) = \begin{bmatrix} x_0(t_0) \\
x_k(t_{0+}) \end{bmatrix} = \begin{bmatrix} x_0(t_0) \\
S_1 x_0(t_0) + S_2 x_k(t_0) \end{bmatrix}.
\]

The initial state design problem considered in this paper is as follows.

\section{Problem 3.1. (Initial state design problem).}
For given \( G, K_p, K_f \), and \( x_0 = x_0(t_0) \), find the optimal \( x_k(t_{0+}) \) that minimizes the state-dependent switching \( L_2 \) gain \( \tilde{\gamma}_{sd-dif}(x_0) \).

For a given switching situation \( x_0 \), this problem provides the optimal initial state of \( K_f \) for suppressing the difference between the actuality that this controller switch occurs and the virtual situation where it does not occur. Note that in this problem, the switching matrix \( S \) (i.e., \( S_1 \) and \( S_2 \)) is not determined.

On the other hand, we can consider the following problem for assigning the initial state of a newly-activated controller.

\section{Problem 3.2. (Switching matrix design problem).}
For given \( G, K_p, \) and \( K_f \), find the optimal \( S_1 \) and \( S_2 \) in (49) that minimizes the switching \( L_2 \) gain \( \tilde{\gamma}_{sd-dif}(x_0) \).

In this problem, the initial state of a newly-activated controller is assigned by \( x_k(t_{0+}) = S_1 x_0(t_0) + S_2 x_k(t_0) \). This problem focuses on the worst switching situation for the controller switch in the sense of the state-dependent switching \( L_2 \) gain \( \tilde{\gamma}_{sd-dif}(x_0) \). Thus, it is clear that by Problem 3.1 according to the switching situation, we can more effectively assign the initial state to suppress the difference between the actuality that this controller switch occurs and the virtual situation where it does not occur.

\subsection{3.2 Proposed initial state design procedure}
Consider Problem 3.1. Defining

\[
x_{01}' = \begin{bmatrix} 0 \\
x_k(t_{0+}) \end{bmatrix}, \quad x_{02}' = \begin{bmatrix} x_0(t_0) \\
0 \end{bmatrix},
\]

where \( x_k(t_{0+}) \) is the initial state of the newly-activated controller \( K_f \) to be obtained, then we have \( S_k x_0 = x_{01}' + x_{02}' \). Thus, the LMI (36) is equivalent to

\[
\begin{bmatrix} x_0^T \dot{X}_p x_0 & x_0^T \dot{x}_{02}^T + x_{02}^T X_d x_0 \\
x_0^T \dot{x}_{01} + x_{01}^T \dot{X}_d x_0 & x_{02}^T \end{bmatrix} > 0.
\]

We can then treat \( x_{01}' \) as an LMI variable. Thus, by solving the LMI problem under the constraint conditions (34), (35), and (52) to minimize \( \gamma \) by using the variables \( \dot{X}_p, \dot{X}_d \), and \( x_{01}' \), we can obtain the optimal \( x_k(t_{0+}) \) in \( x_{01}' \).

\section{Remark 3.3.}
We can solve Problem 3.2 in an entirely analogous fashion. Defining

\[
S_{d1} = \begin{bmatrix} O & O \\
O & O \\
O & O \end{bmatrix}, \quad S_{d2} = \begin{bmatrix} I & O \\
I & O \end{bmatrix},
\]

we can decompose \( S_d \) as \( S_d = S_{d1} + S_{d2} \). Then, by solving the LMI problem under the constraint conditions (34), (35), and

\[
\begin{bmatrix} \dot{X}_p + S_{d1}^T S_{d2} & S_{d1}^T X_d \\
S_{d1} X_d & X_d \end{bmatrix} > 0
\]

to minimize \( \gamma \) by using the variables \( \dot{X}_p, \dot{X}_d \), and \( S_{d1} \), we can obtain the optimal \( S_1 \) and \( S_2 \) in \( S_{d1} \).

\subsection{3.3 Features}
We can easily show that for any \( x_0 (\neq 0) \in \mathbb{R}^{n_o} \) and \( k (\neq 0) \in \mathbb{R} \),

\[
\tilde{\gamma}_{sd-dif}(x_0) = \tilde{\gamma}_{sd-dif}(k \cdot x_0).
\]

Thus, the value of the state-dependent switching \( L_2 \) gain \( \tilde{\gamma}_{sd-dif}(x_0) \) depends only on the direction in the state space \( x_p \in \mathbb{R}^{n_p} \).

Suppose that for a given switching situation \( x_0 (\neq 0) \in \mathbb{R}^{n_o} \), we obtain the optimal initial state \( x_k(t_{0+}) \in \mathbb{R}^{n_k} \) by solving Problem 3.1. Then, for a switching situation \( k \cdot x_0 (k \neq 0) \), the optimal initial state is \( k \cdot x_k \). This implies that the proposed initial state design procedure provides a mapping from a direction in the switching situation \( x_0 \in \mathbb{R}^{n_o} \) to a direction in \( x_k \in \mathbb{R}^{n_k} \).

For a given switching situation \( x_0 = x_0(t_0) \), the proposed initial state design provides the optimal initial state \( x_k(t_{0+}) \) for suppressing the difference between the actuality that this controller switch occurs and the virtual situation where it does not occur in the sense of the state-dependent switching \( L_2 \) gain \( \tilde{\gamma}_{sd-dif}(x_0) \). It has the following advantages.

(a) The switching time does not affect its design result, and need not be known in advance.

(b) It needs only solving an LMI problem. That is, complicated calculation is not necessary.

(c) It is not necessary to discuss the solvability, because we can always obtain the solution.

Conversely, the proposed initial state design cannot be applied to tracking control to a specified reference signal. For example, in a servo system for a stepped reference signal, by considering the fluctuations in the error against a controller switch in the steady state, the undesirable effects of the switch can be suppressed by the proposed design. Thus, by using it, we can improve the safety of the operating-state transitions in the procedure of safe preventive maintenance of control systems presented in Suyama and Sebe (2017). However, the virtual output is not always the reference signal to make the actual output track. The proposed design cannot provide an initial state for suppressing the fluctuations in tracking error for a specified reference signal.
Remark 3.4. We should use the initial state design and switching matrix design depending on the situation. For a pre-planned controller switch in the steady state, the proposed initial state design is more efficient for suppressing the undesirable effects of the switch. However, for a non-preplanned controller switch under a violent system fluctuations, the switching matrix design can provide a more desirable initial state.

3.4 Numerical example

The generalized plant $G$, operating controller $K_p$, and newly-activated controller $K_f$ are given as

$$
G : \begin{bmatrix}
-14 & -2 & 4 & -4 \\
5 & -9 & -2 & -1 \\
-4 & -12 & 0 & 0 \\
-1 & -13 & 0 & 0
\end{bmatrix}
$$

\( (56) \)

$$
K_p : \begin{bmatrix}
926.26 & -4190.34 & 326.88 \\
4.00 & -719.97 & 0 \\
-73.63 & 66562.80 & 73.54 \\
11257.06 & -726.36 & 2837.09
\end{bmatrix}
$$

\( (57) \)

$$
K_f : \begin{bmatrix}
86.54 & -16015.40 & 0 \\
-5.61 & 2925.70 & 5.39 \\
-3.62 & 0 & 0 \\
1.1077 & 0 & 0
\end{bmatrix}
$$

\( (58) \)

Consider the $L_2$ gain as the performance index. Table 1 shows the performance index values of $H_p$ and $H_f$; the values imply that this controller switch improves the control performance. Note that $\gamma_d = 0.5962$ is the lower bound of the switching $L_2$ gains $\gamma_{sd-dif}(x_0)$ and $\gamma_{dif}$.

|            | $L_2$ gain |
|------------|------------|
| Pre-switch system $H_p$ | 1.1077 |
| Post-switch system $H_f$ | 0.5635 |
| Difference system $H_d$ | 0.5962 |

Table 1. Performance index value.

Suppose that a controller switch from $K_p$ to $K_f$ occurs at $t = t_0$ and

$$
x_0 = x_p(t_0) = \begin{bmatrix} x_g(t_0) \\ x_{k_p}(t_0) \end{bmatrix} = \begin{bmatrix} 0.8614 \\ -0.0440 \\ 0.4980 \\ 0.0903 \end{bmatrix} \tag{59}
$$

In order to obtain the initial state $x_{k_f}(t_{0+})$ of the newly-activated controller $K_f$, we consider the following three design:

- Proposed initial state design (Problem 3.1)
- Switching matrix design (Problem 3.2)
- Zero initial state.

Table 2 shows the initial state design results. Note that in the switching design, we first obtain the following switching matrix:

$$
S_1 = \begin{bmatrix}
0.2920 & 1.1301 \\
-0.0074 & -0.0666
\end{bmatrix},
$$

\( S_2 = \begin{bmatrix}
0.4558 & -0.0028 \\
0.0216 & -0.0018
\end{bmatrix} \tag{60}
$$

The proposed initial state design directly provides the optimal initial state $x_{k_f}(t_{0+})$ of the newly-activated controller $K_f$ in the sense of the state-dependent switching $L_2$ gain $\gamma_{sd-dif}(x_0)$. That is, the proposed initial state design minimizes $\gamma_{sd-dif}(x_0)$ to obtain the value 0.5991. On the other hand, the switching matrix design minimizes $\gamma_{dif}(x_0)$ to obtain the value 0.7958.

|            | Proposed initial state design | Switching matrix design | Zero initial state |
|------------|-----------------------------|-------------------------|-------------------|
| $x_{k_f}(t_{0+})$ | 0.5025 | 0.4285 | 0 |
| $\gamma_{sd-dif}(x_0)$ | 0.5991 | 0.6129 | 1.1004 |
| $\gamma_{dif}$ | 0.7958 | 4.6482 |

Fig. 2. Responses in $z(t) - z_{vir}(t)$ against $\dot{w}(t)$ providing the value of $\gamma_{sd-dif}(x_0)$ normalized as $\|\dot{w}(t)\|_2(0,\infty) = 1$.

Fig. 3. Actual output $z(t)$ against $\dot{w}(t)$ providing the value of $\gamma_{sd-dif}(x_0)$ normalized as $\|\dot{w}(t)\|_2(0,\infty) = 1$.

We present the simulation results. We suppose without loss of generality that the controller switch occurs at $t = 0$, i.e., $t_0 = 0$. We can obtain an input $\dot{w}(t)$ given in (26) that provides the value of $\gamma_{sd-dif}(x_0)$ in each case, where it is normalized as $\|\dot{w}(t)\|_2(0,\infty) = 1$. Figures 2 and 3 show...
the responses in $z(t)−z_{vir}(t)$ and $dz(t)$, respectively, after the controller switch against the obtained input $\hat{w}(t)$ in the proposed initial state design case, switching matrix design case, and zero initial state case. As shown in Fig. 2, in the proposed initial state design case, even the worst fluctuation in $z(t)−z_{vir}(t)$ (i.e., the largest difference between the actuality that this controller switch occurs and the virtual situation where it does not occur) is well suppressed in comparison with the other cases.

This example implies that for given $G$, $K_p$, $K_f$, and $x_0$, the difference between the actuality and virtual situation can be well suppressed by appropriately designing the initial state $x_f(t_0+)$ of $K_f$ according to the situation at a switching time, i.e., $x_0$. The effectiveness of the initial state design by using the state-dependent switching $L_2$ gain $\tilde{\gamma}_{sd-diff}(x_0)$ shows its potential applicability as a design index.

4. CONCLUSIONS

We have proposed a new initial state design procedure for a newly-activated controller at a controller switch. By minimizing the value of the state-dependent switching $L_2$ gain presented in this paper, we can obtain the optimal initial state for suppressing the difference between the actuality that a controller switch occurs and the virtual situation where it does not occur. By the proposed initial state design, we can improve the safety of the operating-state transitions in the procedure of safe preventive maintenance of control systems presented in Suyama and Sebe (2017).

However, the virtual output is not always the reference signal to make the actual output track. The proposed design cannot provide an initial state for suppressing the fluctuations in tracking error for a specified reference signal. This is an important issue to be solved/improved in future.

Khargonekar et al. (1991) introduced the worst-case norm of the regulated output over all exogenous inputs and initial states as a performance measure. Balandin and Kogan (2010) obtained optimal time-invariant controllers by minimizing the measure by using LMIs. Another important future work is to clarify the relationship between the proposed initial state design and such $H_\infty$ controller design with transients.

REFERENCES

T. Asai. LMI-based synthesis for robust attenuation of disturbance responses due to switching. Proceedings of the 42nd IEEE Conference on Decision and Control, 5277–5282, 2003.

T. Asai. A general synthesis framework to attenuate disturbance responses due to switching. Transactions of SICE, 41: 846–855, 2005 (in Japanese).

D.V. Balandin and M.M. Kogan. LMI-based $H_\infty$-optimal control with transients. International Journal of Control, 83: 1664–1673, 2010.

C. Edwards and I. Postlethwaite. Anti-windup and bumpless-transfer schemes. Automatica, 34: 199–210, 1998.

R. Hanus, M. Kinnaert, and J.-L. Henrotte. Conditioning technique, a general anti-windup and bumpless transfer method. Automatica, 23: 729–739, 1987.

J.P. Hespanha, P. Sontosso, and G. Stewart. Optimal controller initialization for switching between stabilizing controllers. Proceedings of the 46th IEEE Conference on Decision and Control, 5634–5639, 2007.

P.P. Khargonekar, K.M. Nagpal, and K.R. Poolla. $H_\infty$ control with transients. SIAM Journal on Control and Optimization, 29: 1373–1393, 1991.

M.V. Kothare, P.J. Campo, M. Morari, and C.N. Nett. A unified framework for the study of anti-windup designs. Automatica, 30: 1869–1883, 1994.

K. Nakano, N. Sebe, S. Deguchi, and K. Suyama. State assignment method for improvement of transient response caused by controller switching. Proceedings of the 9th IFAC Symposium on Robust Control Design, 513–518, 2018.

J. Paxman and G. Vinnicombe. Optimal transfer schemes for switching controllers. Proceedings of the 39th IEEE Conference on Decision and Control, 1093–1098, 2000.

H. Saito, A. Kojima, and S. Ishijima. On the initial state of controllers for improving the transient. Transactions of SICE, 34: 1959–1961, 1998 (in Japanese).

K. Suyama and N. Sebe. Switching $L_2$ gain for analyzing the magnitude of a system switch. Proceedings of the 55th IEEE Conference on Decision and Control, 6495–6402, 2016.

K. Suyama and N. Sebe. Support technology for safe preventive maintenance of control systems. Proceedings of the 20th IFAC World Congress, 10858–10865, 2017.

K. Suyama and N. Sebe. State-dependent switching $L_2$ gain for analyzing the fluctuations in transient responses after a system switch. Proceedings of the 57th IEEE Conference on Decision and Control, 6263–6268, 2018.

K. Suyama and N. Sebe. Switching $L_2$ gain for evaluating the fluctuations in transient responses after an unpredictable system switch. International Journal of Control, 92: 1084–1093, 2019.

K. Suyama and N. Sebe. Initial state design for controller switches by using state-dependent switching $L_2$ gain. Proceedings of the 2019 European Control Conference, 1257–1262, 2019.

M.C. Turner and D.J. Walker. Linear quadratic bumpless transfer. Automatica, 36: 1089–1101, 2000.

L. Zaccarian and A.R. Teel. The $L_2$ ($l_2$) bumpless transfer problem for linear plants: Its definition and solution. Automatica, 41: 1273–1280, 2005.