Abstract

In a recent paper, Soner, Touzi and Zhang \cite{SonerTouziZhang} have introduced a notion of second order backward stochastic differential equations (2BSDEs for short), which are naturally linked to a class of fully non-linear PDEs. They proved existence and uniqueness for a generator which is uniformly Lipschitz in the variables $y$ and $z$. The aim of this paper is to extend these results to the case of a generator satisfying a monotonicity condition in $y$. More precisely, we prove existence and uniqueness for 2BSDEs with a generator which is Lipschitz in $z$ and uniformly continuous with linear growth in $y$. Moreover, we emphasize throughout the paper the major difficulties and differences due to the 2BSDE framework.

Key words: Second order backward stochastic differential equation, monotonicity condition, linear growth, singular probability measures

AMS 2000 subject classifications: 60H10, 60H30
1 Introduction

Backward stochastic differential equations (BSDEs for short) appeared in Bismut [11] in the linear case, and then have been widely studied since the seminal paper of Pardoux and Peng [14]. Their range of applications includes notably probabilistic numerical methods for partial differential equations, stochastic control, stochastic differential games, theoretical economics and financial mathematics.

On a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})\) generated by an \(\mathbb{R}^d\)-valued Brownian motion \(B\), a solution to a BSDE consists on finding a pair of progressively measurable processes \((Y, Z)\) such that

\[
Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad t \in [0, T], \quad \mathbb{P} - a.s.
\]

where \(f\) (also called the driver) is a progressively measurable function and \(\xi\) is an \(\mathcal{F}_T\)-measurable random variable.

Pardoux and Peng [14] proved existence and uniqueness of the above BSDE provided that the function \(f\) is uniformly Lipschitz in \(y\) and \(z\) and that \(\xi\) and \(f_s(0, 0)\) are square integrable. Then, in [15], they proved that if the randomness in \(f\) and \(\xi\) is induced by the current value of a state process defined by a forward stochastic differential equation, then the solution to the BSDE could be linked to the solution of a semilinear PDE by means of a generalized Feynman-Kac formula. Since their pioneering work, many efforts have been made to relax the assumptions on the driver \(f\). For instance, Lepeltier and San Martin [9] have proved the existence of a solution when \(f\) is only continuous in \((y, z)\) with linear growth.

More recently, motivated by applications in financial mathematics and probabilistic numerical methods for PDEs (see [6]), Cheredito, Soner, Touzi and Victoir [2] introduced the notion of second order BSDEs (2BSDEs), which are connected to the larger class of fully nonlinear PDEs. Then, Soner, Touzi and Zhang [20] provided a complete theory of existence and uniqueness for 2BSDEs under uniform Lipschitz conditions similar to those of Pardoux and Peng. Their key idea was to reinforce the condition that the 2BSDE must hold \(\mathbb{P} - a.s.\) for every probability measure \(\mathbb{P}\) in a non-dominated class of mutually singular measures (see Section 2 for precise definitions). Let us describe the intuition behind their formulation.

Suppose that we want to study the following fully non-linear PDE

\[
-\frac{\partial u}{\partial t} - h(t, x, u(t, x), Du(t, x), D^2u(t, x)) = 0, \quad u(T, x) = g(x).
\]

If the function \(\gamma \mapsto h(t, x, r, p, \gamma)\) is assumed to be convex, then it is equal to its double Fenchel-Legendre transform, and if we denote its Fenchel-Legendre transform by \(f\), we have

\[
h(t, r, p, \gamma) = \sup_{a \geq 0} \left\{ \frac{1}{2} a \gamma - f(t, x, r, p, a) \right\}
\]

Then, from (1.2), we expect, at least formally, that the solution \(u\) of (1.1) is going to verify

\[
u(t, x) = \sup_{a \geq 0} u^a(t, x),
\]
where \( u^a \) is defined as the solution of the following semi-linear PDE

\[
- \frac{\partial u^a}{\partial t} - \frac{1}{2} a D^2 u^a(t, x) + f(t, x, u^a(t, x), Du^a(t, x), a) = 0, \quad u^a(T, x) = g(x).
\]

Since \( u^a \) is linked to a classical BSDE, the 2BSDE associated to \( u \) should correspond (in some sense) to the supremum of the family of BSDEs indexed by \( a \). Furthermore, changing the process \( a \) can be achieved by changing the probability measure under which the BSDE is written. In these respects, the 2BSDE theory shares deep links with the theory of quasi-sure stochastic analysis of Denis and Martini [3] and the theory of G-expectation of Peng [17].

In addition to providing a probability representation for solutions of fully non-linear PDEs, the 2BSDE theory also has many applications in mathematical finance, especially within the context of markets with volatility uncertainty. Hence, Soner, Touzi and Zhang [19] solved the superhedging problem under volatility uncertainty within this quasi-sure framework (following the original approach of [3]). More recently, Matoussi, Possamaî and Zhou [12] proved that the solution of the utility maximization problem for an investor in an incomplete market with volatility uncertainty could be expressed in terms of a second order BSDE. The same authors also used this theory in [11] to give a superhedging price for American options under volatility uncertainty.

Our aim in this paper is to relax the Lipschitz-type hypotheses of [20] on the driver of the 2BSDE to prove an existence and uniqueness result. In Section 2, inspired by Pardoux [16], we study 2BSDEs with a driver which is Lipschitz in some sense in \( z \), uniformly continuous with linear growth in \( y \) and satisfies a monotonicity condition. We then prove existence and uniqueness and highlight one of the main difficulties when dealing with 2BSDEs. Indeed, the main tool in the proof of existence is to use monotonic approximations (as in [9]). However, since we are working under a family of non-dominated probability measures, the monotone or dominated convergence theorem may fail, which in turn raises subtle technical difficulties in the proofs.

## 2 Preliminaries

Let \( \Omega := \{ \omega \in C([0, T], \mathbb{R}^d) : \omega_0 = 0 \} \) be the canonical space equipped with the uniform norm \( \| \omega \|_\infty := \sup_{0 \leq t \leq T} |\omega_t| \), \( B \) the canonical process, \( \mathbb{P}_0 \) the Wiener measure, \( \mathbb{F} := \{ \mathcal{F}_t \}_{0 \leq t \leq T} \) the filtration generated by \( B \), and \( \mathbb{F}^+ := \{ \mathcal{F}_t^+ \}_{0 \leq t \leq T} \) the right limit of \( \mathbb{F} \). We first recall the notations introduced in [20].

### 2.1 The Local Martingale Measures

We will say that a probability measure \( \mathbb{P} \) is a local martingale measure if the canonical process \( B \) is a local martingale under \( \mathbb{P} \). By Karandikar [7], we know that we can give pathwise definitions of the quadratic variation \( \langle B \rangle_t \) and its density \( \hat{a}_t \).

Let \( \mathbb{P}_W \) denote the set of all local martingale measures \( \mathbb{P} \) such that

\[
\langle B \rangle_t \text{ is absolutely continuous in } t \text{ and } \hat{a}_t \text{ takes values in } S^d_{d+1}, \mathbb{P} - a.s.
\]
where $\mathbb{S}^d_+$ denotes the space of all $d \times d$ real valued positive definite matrices.

We recall from [20], the class $\mathcal{P}_S \subset \mathcal{P}_W$ consisting of all probability measures

$$\mathbb{P}^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1} \text{ where } X^\alpha_t := \int_0^t \alpha_s^{1/2} dB_s, \ t \in [0,1], \ \mathbb{P}_0 - \text{a.s.,} \quad (2.2)$$

and we concentrate on the subclass $\mathcal{P}_{S} : = \{ \mathbb{P}^\alpha \in \mathcal{P}_S, \ a \leq \alpha \leq \bar{a}, \ \mathbb{P}_0 - \text{a.s.} \}$, for fixed matrices $a$ and $\bar{a}$ in $\mathbb{S}^d_+$. We recall from [21] that every $\mathbb{P} \in \mathcal{P}_W$ (and thus in $\mathcal{P}_S$) satisfies the Blumenthal zero-one law and the martingale representation property.

We finish with a definition. Let $\mathcal{P}_0 \subset \mathcal{P}_W$.

**Definition 2.1.** We say that a property holds $\mathcal{P}_0$-quasi surely ($\mathcal{P}_0 - \text{q.s.}$ for short) if it holds $\mathbb{P} - \text{a.s.}$ for all $\mathbb{P} \in \mathcal{P}_0$.

### 2.2 The non-linear Generator

We consider a map $H_t(\omega, y, z, \gamma) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \rightarrow \mathbb{R}$, where $D_H \subset \mathbb{R}^{d \times d}$ is a given subset containing 0.

Following the PDE intuition outlined in the Introduction, we define the corresponding conjugate of $H$ w.r.t. $\gamma$ by

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2} \text{Tr}(a \gamma) - H_t(\omega, y, z, \gamma) \right\} \text{ for } a \in \mathbb{S}^d_+$$

$$\hat{F}_t(y, z) := F_t(y, z, \hat{a}_t) \text{ and } \hat{F}^0_t := \hat{F}_t(0, 0).$$

We denote by $D_{F_t(y, z)} := \{ a, \ F_t(\omega, y, z, a) < \infty \}$ the domain of $F$ in $a$ for a fixed $(t, \omega, y, z)$.

As in [20] we fix a constant $\kappa \in (1, 2]$ and restrict the probability measures in

**Definition 2.2.** $\mathcal{P}_H^\kappa$ consists of all $\mathbb{P} \in \mathcal{P}_S$ such that

$$\mathbb{E}^\mathbb{P} \left[ \left( \int_0^T |\hat{F}^0_t|^\kappa \, dt \right)^2 \right] < +\infty.$$ 

It is clear that $\mathcal{P}_H^\kappa$ is decreasing in $\kappa$, and $\hat{a}_t \in D_{F_t}$, $dt \times d\mathbb{P} - \text{a.s.}$ for all $\mathbb{P} \in \mathcal{P}_H^\kappa$. We will also denote $\overline{\mathcal{P}}_H^\kappa$ the closure for the weak topology of $\mathcal{P}_H^\kappa$.

**Remark 2.1.** Unlike in [20], we assume that the bounds on the density of the quadratic variation $\hat{a}$ are uniform with respect to the underlying probability measure. In particular, this ensures that the family $\mathcal{P}_H^\kappa$ is weakly relatively compact and that $\overline{\mathcal{P}}_H^\kappa$ is weakly compact.
We now state our main assumptions on the function $F$ which will be our main interest in the sequel

**Assumption 2.1.** (i) The domain $D_{F_1(y,z)} = D_{F_1}$ is independent of $(\omega, y, z)$.

(ii) For fixed $(y, z, a)$, $F$ is $\mathcal{F}$-progressively measurable.

(iii) We have the following uniform Lipschitz-type property

$$
\forall (y, z, z', a, t, \omega), \quad |F_t(\omega, y, z, a) - F_t(\omega, y, z', a)| \leq C \left| a^{1/2}(z - z') \right|.
$$

(iv) $F$ is uniformly continuous in $\omega$ for the $|| \cdot ||_\infty$ norm.

(v) $F$ is uniformly continuous in $y$, uniformly in $(z, t, \omega, a)$, and has the following growth property

$$
\exists C > 0 \ \text{s.t.} \ \forall (t, y, a, \omega), \quad |F_t(\omega, y, 0, a)| \leq |F_t(\omega, 0, 0, a)| + C(1 + |y|).
$$

(vi) We have the following monotonicity condition. There exists $\mu > 0$ such that

$$(y_1 - y_2)(F_t(\omega, y_1, z, a) - F_t(\omega, y_2, z, a)) \leq \mu |y_1 - y_2|^2, \text{ for all } (t, \omega, y_1, y_2, z, a)$$

(vii) $F$ is continuous in $a$.

**Remark 2.2.** Let us comment on the above assumptions. Assumptions 2.1 (i) and (iv) are taken from [20] and are needed to deal with the technicalities induced by the quasi-sure framework. Assumptions 2.1 (ii) and (iii) are quite standard in the classical BSDE literature. Then, Assumptions 2.1 (v) and (vi) were introduced by Pardoux in [16] in a more general setting (namely with a general growth condition in $y$, and only a continuity assumption on $y$) and are also quite common in the literature. Let us immediately point out that as explained in Remark 3.2 below, we must restrict ourselves to linear growth in $y$, because of the technical difficulties due to the 2BSDE framework. Moreover, we need to assume uniform continuity in $y$ to ensure that we have a strong convergence result for the approximation we will consider (see also Remark 3.2). Finally, Assumption 2.1 (vii) is needed in our framework to obtain technical results concerning monotone convergence in a quasi-sure setting.

### 2.3 The spaces and norms

We now recall from [20] the spaces and norms which will be needed for the formulation of the second order BSDEs. Notice that all subsequent notations extend to the case $\kappa = 1$.

For $p \geq 1$, $L^{p, \kappa}_H$ denotes the space of all $\mathcal{F}_T$-measurable scalar r.v. $\xi$ with

$$
||\xi||_{L^{p, \kappa}_H} := \sup_{\mathbb{P} \in \mathbb{P}_H^\kappa} \mathbb{E}^\mathbb{P} [||\xi||^p] < +\infty.
$$
\( \mathbb{H}_{H}^{p,\kappa} \) denotes the space of all \( \mathbb{F}^+ \)-progressively measurable \( \mathbb{R}^d \)-valued processes \( Z \) with

\[
\| Z \|_{\mathbb{H}_{H}^{p,\kappa}} := \sup_{\mathcal{P} \in \mathcal{P}_H} \mathbb{E}^\mathcal{P} \left[ \left( \int_0^T |a_t^{1/2} Z_t|^2 \, dt \right)^{\frac{p}{2}} \right] < +\infty.
\]

\( \mathbb{D}_H^{p,\kappa} \) denotes the space of all \( \mathbb{F}^+ \)-progressively measurable \( \mathbb{R} \)-valued processes \( Y \) with

\[
\mathcal{P}_H^\kappa - q.s. \text{ càdlàg paths, and } \| Y \|_{\mathbb{D}_H^{p,\kappa}} := \sup_{\mathcal{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathcal{P} \left[ \sup_{0 \leq t \leq T} |Y_t|^p \right] < +\infty.
\]

For each \( \xi \in L_{H}^{1,\kappa}, \mathcal{P} \in \mathcal{P}_H^\kappa \) and \( t \in [0, T] \) denote

\[
\mathbb{E}_{t}^{H,\mathcal{P}}[\xi] := \text{ess sup}_{\mathcal{P}' \in \mathcal{P}_H^\kappa(t^+,\mathcal{P})} \mathbb{E}_{t}^{\mathcal{P}'}[\xi] \text{ where } \mathcal{P}_H^\kappa(t^+,\mathcal{P}) := \left\{ \mathcal{P}' \in \mathcal{P}_H^\kappa : \mathcal{P}' = \mathcal{P} \text{ on } \mathcal{F}_t^+ \right\}.
\]

Here \( \mathbb{E}_{t}^{\mathcal{P}}[\xi] := \mathbb{E}^{\mathcal{P}}[\xi|\mathcal{F}_t] \). Then we define for each \( p \geq \kappa \),

\[
\mathbb{L}_{H}^{p,\kappa} := \left\{ \xi \in L_{H}^{1,\kappa} : \| \xi \|_{\mathbb{L}_H^{p,\kappa}} < +\infty \right\} \text{ where } \| \xi \|_{\mathbb{L}_H^{p,\kappa}} := \sup_{\mathcal{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathcal{P} \left[ \text{ess sup}_{0 \leq t \leq T} \left( \mathbb{E}_{t}^{H,\mathcal{P}}[|\xi|^\kappa] \right)^{\frac{p}{\kappa}} \right].
\]

Finally, we denote by \( \text{UC}_b(\Omega) \) the collection of all bounded and uniformly continuous maps \( \xi : \Omega \to \mathbb{R} \) with respect to the \( \| \cdot \|_{\infty} \)-norm, and we let

\[
\mathbb{L}_H^{p,\kappa} := \text{the closure of UC}_b(\Omega) \text{ under the norm } \| \cdot \|_{\mathbb{L}_H^{p,\kappa}}, \text{ for every } 1 \leq \kappa \leq p.
\]

### 2.4 Formulation

We shall consider the following second order BSDE (2BSDE for short), which was first defined in [20]

\[
Y_t = \xi + \int_t^T \tilde{F}_s(Y_s, Z_s) \, ds - \int_t^T Z_s \, dB_s + K_T - K_t, \quad 0 \leq t \leq T, \quad \mathcal{P}_H^\kappa - q.s. \tag{2.4}
\]

**Definition 2.3.** For \( \xi \in L_{H}^{2,\kappa} \), we say \( (Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa} \) is a solution to 2BSDE (2.4) if

- \( Y_T = \xi, \mathcal{P}_H^\kappa - q.s. \),
- \( \forall \mathcal{P} \in \mathcal{P}_H^\kappa, \text{ the process } K^\mathcal{P} \text{ defined below has non-decreasing paths } \mathcal{P} - a.s. \)

\[
K^\mathcal{P}_t := Y_0 - Y_t - \int_0^t \tilde{F}_s(Y_s, Z_s) \, ds + \int_0^t Z_s \, dB_s, \quad 0 \leq t \leq T, \quad \mathcal{P} - a.s. \tag{2.5}
\]

- The family \( \{ K^\mathcal{P}, \mathcal{P} \in \mathcal{P}_H^\kappa \} \) satisfies the minimum condition

\[
K^\mathcal{P}_t = \text{ess inf}_{\mathcal{P}' \in \mathcal{P}_H^\kappa(t^+,\mathcal{P})} \mathbb{E}_{t}^{\mathcal{P}'} \left[ K^\mathcal{P}'_T \right], \quad 0 \leq t \leq T, \quad \mathcal{P} - a.s., \quad \forall \mathcal{P} \in \mathcal{P}_H^\kappa. \tag{2.6}
\]

Moreover if the family \( \{ K^\mathcal{P}, \mathcal{P} \in \mathcal{P}_H^\kappa \} \) can be aggregated into a universal process \( K \), we call \( (Y, Z, K) \) a solution of 2BSDE (2.4).
Remark 2.3. Let us comment on this definition. As already explained, the PDE intuition leads us to think that the solution of a 2BSDE should be a supremum of solutions of standard BSDEs. Therefore, for each $\mathbb{P}$, the role of the non-decreasing process $K^\mathbb{P}$ is to "push" the process $Y$ so that it remains above the solution of the BSDE with terminal condition $\xi$ and generator $\tilde{F}$ under $\mathbb{P}$. In this regard, 2BSDEs share some similarities with reflected BSDEs.

Pursuing this analogy, the minimum condition (2.6) tells us that the processes $K^\mathbb{P}$ act in a "minimal" way (exactly as implied by the Skorohod condition for reflected BSDEs), and we will see in the next Section that it implies uniqueness of the solution. Besides, if the set $\mathcal{P}_H^\kappa$ was reduced to a singleton $\{\mathbb{P}\}$, then (2.6) would imply that $K^\mathbb{P}$ is a martingale and a non-decreasing process and is therefore null. Thus we recover the standard BSDE theory.

Finally, we would like to emphasize that in the language of G-expectation of Peng [17], (2.6) is equivalent, at least if the family can be aggregated into a process $K$, to saying that $-K$ is a G-martingale. This has been already observed in [19] where the authors proved the G-martingale representation property, which formally corresponds to a 2BSDE with a generator equal to 0.

Following [20], in addition to Assumption 2.1 we will always assume

Assumption 2.2. (i) $\mathcal{P}_H^\kappa$ is not empty.

(ii) The process $\hat{F}^0$ satisfies the following integrability condition

$$\phi_H^{2,\kappa} := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P} \left[ \sup_{0 \leq t \leq T} \mathbb{E}^{\mathbb{H},\mathbb{P}} \left[ \left( \int_0^T |\hat{F}_s^0|^\kappa \, ds \right)^{\frac{2}{\kappa}} \right] \right] < +\infty. \quad (2.7)$$

Before going on, let us recall one of the main results of [20]. For this, we first recall their assumptions on the generator $F$

Assumption 2.3. (i) The domain $D_{F_t(y,z)} = D_{F_t}$ is independent of $(\omega, y, z)$.

(ii) For fixed $(y, z, a)$, $F$ is $\mathbb{F}$-progressively measurable.

(iii) We have the following uniform Lipschitz-type property

$$\forall (y, y', z, z', a, t, \omega), \quad |F_t(\omega, y, z, a) - F_t(\omega, y', z', a)| \leq C \left( |y - y'| + |a^{1/2}(z - z')| \right).$$

(iv) $F$ is uniformly continuous in $\omega$ for the $|| \cdot ||_\infty$ norm.

Theorem 2.1 (Soner, Touzi, Zhang [20]). Let Assumptions 2.2 and 2.3 hold. Then, for any $\xi \in \mathbb{D}^{2,\kappa}_H$, the 2BSDE (2.4) has a unique solution $(Y, Z) \in \mathbb{D}^{2,\kappa}_H \times \mathbb{H}^{2,\kappa}_H$.

We now state the main result proved in this paper
Theorem 2.2. Suppose there exists $\epsilon > 0$ such that $\xi \in L^2_{H} \cap L^{2+\epsilon, \kappa}$ and

$$\sup_{P \in \mathcal{P}_H^\kappa} \mathbb{E}^P \left[ \int_0^T \left| \hat{F}_t^\kappa \right|^{2+\epsilon} dt \right] < +\infty. \quad (2.8)$$

Then, under Assumptions 2.1 and 2.2, there exists a unique solution $(Y, Z) \in \mathbb{D}^{2, \kappa}_H \times \mathbb{H}^{2, \kappa}_H$ of the 2BSDE (2.4).

Remark 2.4. Notice that the above Theorems both correspond to the one dimensional case (in the sense that the terminal condition belongs to $\mathbb{R}$). This is different from the standard BSDE literature (see [5] and [16]) where existence and uniqueness of a solution can be obtained in arbitrary dimensions under Lipschitz type assumptions on the generator. This difference is mainly due to the fact that the comparison theorem (only valid in dimension 1) is intensely used to obtain existence and uniqueness for 2BSDEs in [20]. This strategy of proof is (partly) due to the fact that the classical fixed-point argument for standard BSDEs no longer works in the 2BSDE framework. Indeed, as can be seen from the estimates of Theorem 4.5 in [20], trying to reproduce the classical fixed-point argument, would only lead to an application which is $1/2$-Hölder continuous, which is not sufficient to conclude. Extending these results to the multidimensional case is nonetheless an important and difficult problem.

2.5 Representation and uniqueness of the solution

We follow once more Soner, Touzi and Zhang [20]. For any $\mathbb{P} \in \mathcal{P}_H^\kappa$, $\mathcal{F}$-stopping time $\tau$, and $\mathcal{F}_\tau$-measurable random variable $\xi \in L^2(\mathbb{P})$, let $(y_\mathbb{P}, z_\mathbb{P}) := (y_\mathbb{P}(\tau, \xi), z_\mathbb{P}(\tau, \xi))$ denote the unique solution to the following standard BSDE (existence and uniqueness under our assumptions follow from Pardoux [10])

$$y_\mathbb{P}_t = \xi + \int_t^\tau \hat{F}_s(y_\mathbb{P}_s, z_\mathbb{P}_s)ds - \int_t^\tau z_\mathbb{P}_s dB_s, \quad 0 \leq t \leq \tau, \ P - a.s. \quad (2.9)$$

We then have similarly as in Theorem 4.4 of [20].

Theorem 2.3. Let Assumptions 2.1 and 2.2 hold. Assume $\xi \in L^2_{H} \kappa$ and that $(Y, Z) \in \mathbb{D}^{2, \kappa}_H \times \mathbb{H}^{2, \kappa}_H$ is a solution of the 2BSDE (2.4). Then, for any $\mathbb{P} \in \mathcal{P}_H^\kappa$ and $0 \leq t_1 < t_2 \leq T$,

$$Y_{t_1} = \text{ess sup}_{\mathbb{P}'} \int_{t_1}^{t_2} y_{t_1}^\mathbb{P}'(t_2, Y_{t_2}) dt, \ P - a.s. \quad (2.10)$$

Consequently, the 2BSDE (2.4) has at most one solution in $\mathbb{D}^{2, \kappa}_H \times \mathbb{H}^{2, \kappa}_H$.

Notice that the representation formula (2.10) corresponds exactly to the PDE intuition we described in the Introduction and Remark 2.3, and shows that the solution to a 2BSDE is indeed a supremum over a family of standard BSDEs.
and thus is unique. Then, since we have that $d(Y,B)_t = Z_t dB_t$, $P_H - q.s., Z$ is also unique. We shall now prove (2.10).

(i) Fix $0 \leq t_1 < t_2 \leq T$ and $P \in P_H$. For any $P' \in P_H(t_1^+, P)$, we have

$$Y_t = Y_{t_2} + \int_{t_2}^{t} \hat{F}_s(Y_s, Z_s) ds - \int_{t_2}^{t} Z_s dB_s + K_{t_2}^{P'} - K_{t_1}^{P'}$$

and that $K_{P'}$ is non-decreasing, $P' - a.s.$ Then, we can apply a generalized comparison theorem proved by Lepeltier, Matoussi and Xu (see Theorem 4.1 in [10]) under $P'$ to obtain $Y_{t_1} \geq Y_{t_2}(t_2, Y_{t_2}), P' - a.s.$ Since $P' = P$ on $\mathcal{F}_t^+$, we get $Y_{t_1} \geq y_{t_1}^{P'}(t_2, Y_{t_2}), P - a.s.$ and thus

$$Y_{t_1} \geq \text{ess sup}_{P' \in P_H(t_1^+, P)} y_{t_1}^{P'}(t_2, Y_{t_2}), P - a.s.$$

(ii) We now prove the reverse inequality. Fix $P \in P_H$. We will show in (iii) below that

$$C_{t_1} := \text{ess sup}_{P' \in P_H(t_1^+, P)} \mathbb{E}^{P'}_{t_1} \left[ \left( K_{t_2}^{P'} - K_{t_1}^{P'} \right)^{2} \right] < +\infty, P - a.s.$$

For every $P' \in P_H(t_1^+, P)$, denote

$$\delta Y := Y - y^{P'}(t_2, Y_{t_2}) \text{ and } \delta Z := Z - z^{P'}(t_2, Y_{t_2}).$$

By the Lipschitz Assumption (2.1)(iii) and the monotonicity Assumption (2.1)(vi), there exist a bounded process $\lambda$ and a process $\eta$ which is bounded from above such that

$$\delta Y_t = \int_{t}^{t_2} \left( \eta_s \delta Y_s + \lambda_s \tilde{a}_s^{1/2} \delta Z_s \right) ds - \int_{t}^{t_2} \delta Z_s dB_s + K_{t_2}^{P'} - K_{t_1}^{P'}, t \leq t_2, P' - a.s.$$

Define for $t_1 \leq t \leq t_2$

$$M_t := \exp \left( \int_{t_1}^{t} \left( \eta_s - \frac{1}{2} |\lambda_s|^{2} \right) ds - \int_{t_1}^{t} \lambda_s \tilde{a}_s^{-1/2} dB_s \right), P' - a.s.$$

By Itô's formula, we obtain, as in [20], that

$$\delta Y_{t_1} = \mathbb{E}^{P'}_{t_1} \left[ \int_{t_1}^{t_2} M_t dK_{t}^{P'} \right] \leq \mathbb{E}^{P'}_{t_1} \left[ \sup_{t_1 \leq t \leq t_2} (M_t)(K_{t_2}^{P'} - K_{t_1}^{P'}) \right],$$

since $K_{P'}$ is non-decreasing. Then, because $\lambda$ is bounded and $\eta$ is bounded from above, we have for every $p \geq 1$

$$\mathbb{E}^{P'}_{t_1} \left[ \sup_{t_1 \leq t \leq t_2} (M_t)^{p} \right] \leq C_p, P' - a.s.
Then it follows from the Hölder inequality that
\[\delta Y^t_1 \leq C(C^{p'}_{t_1})^{1/3} \left( \mathbb{E}^{P'}_t \left[ K^{p'}_{t_2} - K^{p'}_{t_1} \right] \right)^{1/3}, \, P' - a.s.\]

By the minimum condition (2.6) and since \( P' \in \mathcal{P}_H(t^+, \mathbb{P}) \) is arbitrary, this ends the proof.

(iii) It remains to show that the estimate for \( C^{p'}_{t_1} \) holds. By definition of the family \( \{ K^P, P \in \mathcal{P}_H \} \), the linear growth condition in \( y \) of the generator and the Lipschitz condition in \( z \), we have
\[
\sup_{P' \in \mathcal{P}_H(t_1^+, \mathbb{P})} \mathbb{E}^{P'}_t \left[ (K^{p'}_{t_2} - K^{p'}_{t_1})^2 \right] \leq C \left( 1 + \| Y \|^2_{H^{2,\kappa}} + \| Z \|^2_{H^{2,\kappa}} + \phi_{H}^2 \right) < +\infty.
\]

Then we proceed exactly as in the proof of Theorem 4.4 in [20].

\[\square\]

### 2.6 Non-dominated family of probability measures and monotone Convergence Theorem

Let \( \mathcal{P}_0 \subset \mathcal{P}_W \). In general in a non-dominated framework, the monotone convergence Theorem may not hold, that is to say that even if we have a sequence of random variables \( X_n \) which decreasingly converges \( \mathbb{P}_0 \)-a.s. to 0, then we may not have that
\[
\sup_{P \in \mathcal{P}_0} \mathbb{E}^P[X_n] \downarrow 0.
\]

Indeed, let us consider for instance the set
\[
\mathcal{P}_1 := \{ P^p := \mathbb{P}_0 \circ (\sqrt{p}B), \ p \in \mathbb{N}^* \}.
\]

Then, define \( Y_n := \frac{P^2}{n} \). It is clear that the sequence \( Y_n \) decreases \( \mathbb{P} \)-a.s. to 0 for all \( \mathbb{P} \in \mathcal{P}_1 \). But we have for all \( p \) and all \( n \)
\[
\mathbb{E}^{P^p}[Y_n] = \frac{p}{n},
\]
which implies that \( \sup_{P \in \mathcal{P}_1} \mathbb{E}^P[Y_n] = +\infty \).

It is therefore clear that it is necessary to add more assumptions in order to recover a monotone convergence theorem. Notice that those assumptions will concern both the family \( \mathcal{P}_0 \) and the random variables considered. For instance, in the above example, this is the fact that the set \( \mathcal{P}_1 \) considered is not weakly compact which implies that the monotone convergence theorem can fail.

Let us now provide some definitions
Definition 2.4. Let \((X^p)_{p \in \mathcal{P}_0}\) be a family of random variables. The family can be aggregated if there exists a random variable \(X\) such that 
\[X = X^p, \ P - a.s., \text{ for all } p \in \mathcal{P}_0.\]

\(X\) will be called an aggregator for this family.

Definition 2.5. (i) We say that a family of random variables \((X^p)_{p \in \mathcal{P}_0}\) is \(\mathcal{P}_0\)-uniformly integrable if
\[\lim_{C \to +\infty} \sup_{p \in \mathcal{P}_0} \mathbb{E}^p \left[|X^p| 1_{|X^p| > C}\right] = 0.\]

(ii) We say that a family of random variables \((X^p)_{p \in \mathcal{P}_0}\) is \(\mathcal{P}_0\)-quasi continuous if for all \(p \in \mathcal{P}_0\) and for all \(\epsilon > 0\), there exists an open set \(\mathcal{O}^c \subset \Omega\) such that 
\[\mathbb{P}(\mathcal{O}^c) \leq \epsilon \text{ and } X^p \text{ is continuous in } \omega \text{ outside } \mathcal{O}^c.\]

We will keep the same terminology if the family \((X^p)_{p \in \mathcal{P}_0}\) can be aggregated.

Remark 2.5. • A sufficient condition for a family of random variables \((X^p)_{p \in \mathcal{P}_0}\) to be \(\mathcal{P}_0\)-uniformly integrable is that 
\[\sup_{p \in \mathcal{P}_0} \mathbb{E}^p \left[|X^p|^{1+\epsilon}\right] < +\infty, \text{ for some } \epsilon > 0.\]

• The definition of \(\mathcal{P}_0\)-quasi continuity is inspired by the notion of quasi-continuity given in [4], but adapted to our context where we work without the theory of capacities and where the random variables are not necessarily aggregated.

We then have the following monotone convergence theorem proved in [4].

Theorem 2.4 (Denis, Hu, Peng). Let \(\mathcal{P}_0\) be a weakly compact family and let \(X_n\) be a sequence of \(\mathcal{P}_0\)-quasi continuous and \(\mathcal{P}_0\)-uniformly integrable random variables which verifies that 
\[X_n(\omega) \downarrow 0, \text{ for every } \omega \in \Omega \setminus \mathcal{N}, \text{ where } \sup_{p \in \mathcal{P}_0} \mathbb{P}(\mathcal{N}) = 0.\]

Then 
\[\sup_{p \in \mathcal{P}_0} \mathbb{E}^p [X_n] \downarrow 0.\]

It is obvious that this Theorem is particularly suited for a capacity framework, as considered in [3] or [4]. However in our case, we do not work with capacities, and all our properties hold only \(\mathbb{P}\)-a.s. for all probability measures in \(\mathcal{P}_0\), which generally makes this Theorem not general enough for our purpose. Nonetheless, we will see later that Theorem 2.4 will only be applied to quantities which are not only defined for every \(\omega\), but are also continuous in \(\omega\). In that case, this difference between capacities and probabilities is not important.
3 Proof of existence

3.1 Preliminary results

In order to prove existence, we need an approximation of continuous functions by Lipschitz functions proved by Lepeltier and San Martin in [9].

Define
\[ F^n_t(y, z, a) := \inf_{u \in \mathbb{Q}} \{ F_t(u, z, a) + n |y - u| \}, \]
where the minimum over rationals \( u \in \mathbb{Q} \) coincides with the minimum over real parameters by our continuity assumptions, and ensures the measurability of \( F^n \).

**Lemma 3.1.** Let \( C \) be the constant in Assumption 2.1. We have

(i) \( F^n \) is well defined for \( n \geq C \) and we have
\[ |F^n_t(y, z, a)| \leq |F_t(0, 0, a)| + C(1 + |y| + |a^{1/2}z|), \] for all \( y, z, a, n, t, \omega \).

(ii) \( |F^n_t(y_1, z_1, a) - F^n_t(y_2, z_2, a)| \leq C|a^{1/2}(z_1 - z_2)|, \) for all \( y_1, y_2, z_1, z_2, a, t, \omega \).

(iii) \( |F^n_t(y_1, z, a) - F^n_t(y_2, z, a)| \leq n |y_1 - y_2|, \) for all \( y_1, y_2, z, a, t, \omega \).

(iv) The sequence \( (F^n_t(y, z, a))_n \) is increasing, for all \( (t, y, z, a) \).

(v) If \( F \) is decreasing in \( y \), then so is \( F^n \).

**Proof.** The properties (i), (ii), (iii) and (iv) are either clear or proved in [9], thus we concentrate on (v).

We assume that \( F \) is decreasing in \( y \). In particular, \( F \) is differentiable in \( y \) for a.e. \( y \). Define for all \( y \), \( h_n(y, z, t, a)(u) := F_t(u, z, a) + n|y - u| \). For \( u \leq y \), \( h_n(y, z, t, a) \) is clearly decreasing in \( u \). Then, its minimum in \( u \) can only be attained at \( y \) or at a point strictly greater than \( y \).

Therefore we can write
\[ F^n_t(y, z, a) = \min \left\{ F_t(y, z, a), \inf_{u \in \mathbb{Q}, u > y} \{ F_t(u, z) + n(u - y) \} \right\} \]
\[ = \min \left\{ F_t(y, z, a), \inf_{u \in \mathbb{Q}, u > 0} \{ F_t(u + y, z, a) + nu \} \right\}, \]
and under this form it is clear that \( F^n \) is decreasing in \( y \). \( \square \)

We note that in above lemmas, and in all subsequent results, we shall denote by \( C \) a generic constant which may vary from line to line and depends only on the dimension \( d \), the maturity \( T \) and the constants in Assumptions 2.1 and 2.2. We shall also denote by \( C_\kappa \) a constant which may depend on \( \kappa \) as well.

Let us now note that we can always consider without loss of generality that the constant \( \mu \) in Assumption 2.1(vi) is equal to 0. Indeed, we have the following Lemma
Lemma 3.2. Let $\lambda > 0$, then $(Y_t, Z_t, \{K^P_t, \mathbb{P} \in \mathcal{P}_H^\kappa\})$ solve the 2BSDE (2.1) if and only if $(e^{\lambda t}Y_t, e^{\lambda t}Z_t, \{\int_0^t e^{\lambda s}dK^P_s, \mathbb{P} \in \mathcal{P}_H^\kappa\})$ solve the 2BSDE with terminal condition $\xi := e^{\lambda T} \xi$ and driver $\hat{F}^{(\lambda)}_t(y, z) := F^{(\lambda)}_t(y, z, \tilde{a}_t)$, where $F^{(\lambda)}_t(y, z, a) := e^{\lambda t} F_t(e^{-\lambda t} y, e^{-\lambda t} z, a) - \lambda y$.

Proof. The fact that the two solutions solve the corresponding equations is a simple consequence of Itô's formula. The only thing that we have to check is that the family $\{K^P_t, \mathbb{P} \in \mathcal{P}_H^\kappa\}$ satisfies the minimum condition (2.6) if and only if it is verified by the family $\{\int_0^t e^{\lambda s}dK^P_s, \mathbb{P} \in \mathcal{P}_H^\kappa\}$.

First of all, it is clear that

$$
\int_0^t e^{\lambda s}dK^P_s = \mathbb{E}^{\mathbb{P}}\left[\int_0^T e^{\lambda s}dK^P_s\right] \Leftrightarrow \mathbb{E}^{\mathbb{P}'}\left[\int_0^T e^{\lambda s}dK^P_s\right] = 0.
$$

Now for every $t \in [0, T]$, $\mathbb{P} \in \mathcal{P}_H^\kappa$ and $\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})$, the result follows from

$$
\mathbb{E}^{\mathbb{P}'}\left[\int_0^T e^{\lambda(s-T)}dK'^P_s\right] \leq \mathbb{E}^{\mathbb{P}'}\left[K'^P_T - K^P_T\right] \leq \mathbb{E}^{\mathbb{P}'}\left[\int_0^T e^{\lambda s}dK'^P_s\right], \mathbb{P} - a.s.
$$

Thus, if we choose $\lambda = \mu$ then $F^{(\mu)}$ satisfies

$$(y_1 - y_2)(F^{(\mu)}_t(\omega, y_1, z, a) - F^{(\mu)}_t(\omega, y_2, z, a)) \leq 0, \text{ for all } (t, \omega, y_1, y_2, z, a).$$

As a consequence of Lemma 3.2 we can assume without loss of generality that our driver is decreasing in $y$. Therefore, from now on this assumption will replace Assumption 2.1(i-vi).

As explained in Remark 3.2 we will actually need a strong convergence result for the sequence

$$
\hat{F}^n_t(y, z) := F^n_t(y, z, \tilde{a}_t).
$$

Let us define the following quantity

$$
\bar{F}^n_t := \sup_{(y, z, a) \in \mathbb{R}^{d+1} \times [\tilde{a}, \bar{a}]} \{F_t(y, z, a) - F^n_t(y, z, a)\}.
$$

We then have the following result

Lemma 3.3. Let Assumption 2.1 hold. Then the sequence $\hat{F}^n$ converges uniformly globally in $(y, z)$ and for all $0 \leq t \leq T$ and all $\epsilon > 0$

$$
\sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}\left[\bar{F}^n_t\right]^{2+\epsilon} = \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}\left[\bar{F}^n_t\right]^{2+\epsilon} \leq C,
$$

for some $C$ independent of $n$.
The proof is relegated to the Appendix.

**Remark 3.1.** Notice that in the above Lemma, we emphasize that we consider a supremum over \( \mathcal{P}_H^\kappa \) (even though it is equal to the supremum over \( \mathcal{P}_H^\kappa \)). This is important because we are going to apply the monotone convergence Theorem of [4] to the quantity \( |\tilde{F}_t^n| \) in the sequel, and this theorem can only be used under a weakly compact family of probability measures.

We are now in a position to prove the main result of this paper Theorem 2.2

### 3.2 Proof of the main result

For a fixed \( n \), consider the following 2BSDE

\[
Y_t^n = \xi + \int_t^T \tilde{F}_s^n(Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dB_s + K_T^n - K_t^n, \quad 0 \leq t \leq T, \quad \mathcal{P}_H^\kappa - q.s. \tag{3.1}
\]

By Lemma 3.1 and our Assumptions 2.1 and 2.2 we know that all the requirements of Theorem 4.7 of [20] are fulfilled. Thus, we know that for all \( n \) the above 2BSDE has a unique solution \((Y^n, Z^n)\) \( \in \mathbb{D}^{2,\kappa}_H \times \mathbb{H}^{2,\kappa}_H \). Moreover, if we introduce the following standard BSDEs for all \( \mathbb{P} \in \mathcal{P}_H^\kappa \)

\[
y_t^n = \xi + \int_t^T \tilde{F}_s^n(y_s^n, z_s^n)ds - \int_t^T z_s^n dB_s, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s., \tag{3.2}
\]

we have the already mentioned representation (see Theorem 4.4 in [20])

\[
Y_t^n = \operatorname{ess sup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})} y_t^{\mathbb{P}', n}, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s., \quad \forall \mathbb{P} \in \mathcal{P}_H^\kappa. \tag{3.3}
\]

The idea of the proof of existence is to prove that the limit in a certain sense of the sequence \((Y^n, Z^n)\) is a solution of the 2BSDE (2.4). We first provide a priori estimates which are uniform in \( n \) on the solutions of (3.1) and (3.2).

**Lemma 3.4.** There exists a constant \( C_\kappa > 0 \) such that for all \( n \) large enough

\[
\|Y^n\|_{\mathbb{D}^{2,\kappa}_H}^2 + \|Z^n\|_{\mathbb{H}^{2,\kappa}_H}^2 + \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P} \left[ \left| K_T^{\mathbb{P}, n} \right|^2 + \sup_{0 \leq t \leq T} |y_t^{\mathbb{P}, n}|^2 + \int_0^T |\tilde{a}_s^{1/2} z_s^{\mathbb{P}, n}|^2 ds \right] 
\leq C_\kappa \left( 1 + \|\xi\|_{L^{2,\kappa}_H}^2 + \phi_{2,\kappa} \right).
\]

**Proof.** Let us consider the following BSDE

\[
u_t^n = |\xi| + \int_t^T |\tilde{F}_s^n| + C \left( 1 + |u_s^n| + \|a_s^{1/2} v_s^n\| \right) ds - \int_t^T v_s^n dB_s, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s. \tag{3.4}
\]
Since its generator is clearly Lipschitz it has a unique solution. Moreover, we can apply the comparison theorem of El Karoui, Peng and Quenez\cite{Karoui89} to obtain that, due to our uniform growth assumption and (i), (ii), (iii) of Lemma 3.1

\[ \forall m \leq n \text{ large enough, } \forall \mathbb{P} \in \mathcal{P}_H^\kappa, y^{\mathbb{P},m} \leq y^{\mathbb{P},n} \leq u^\mathbb{P}, \mathbb{P} - a.s. \]

Now, following line-by-line the proof of Lemma 4.3 in\cite{Karoui89} we obtain that for all \( \mathbb{P} \in \mathcal{P}_H^\kappa \) and all \( 0 \leq t \leq T \)

\[ |u_t^\mathbb{P}| \leq C_\kappa \left( 1 + \mathbb{E}_t^\mathbb{P} \left[ |\xi|^\kappa + \int_t^T \| F_{s}^0 \|^\kappa ds \right]^{1/\kappa} \right). \]

Therefore by definition of the norms and the representation (3.3) we have

\[ \| Y^n \|_{L_{H}^{2,\kappa}}^2 + \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P} \left[ \sup_{0 \leq t \leq T} |Y_t^{n,\mathbb{P}}|^2 \right] \leq C_\kappa \left( 1 + \| \xi \|_{L_{H}^{2,\kappa}}^2 + \phi_{H}^{2,\kappa} \right). \]

We next apply Itô’s formula to \( (Y^n_t)^2 \) under each \( \mathbb{P} \in \mathcal{P}_H^\kappa \), we have \( \mathbb{P} - a.s. \)

\[ (Y^n_t)^2 + \int_t^T |\hat{a}_s^{1/2}Z_s^n|^2 ds \leq \xi^2 + 2 \int_t^T Y^n_s \hat{F}_s^n(Y^n_s, Z^n_s)ds - 2 \int_t^T Y^n_s Z^n_s dB_s + 2 \int_t^T Y^n_s dK^{\mathbb{P},n}_s. \]

Thus, since \( (Y^n, Z^n) \in \mathbb{D}_{H}^{2,\kappa} \times \mathbb{H}_{H}^{2,\kappa} \), by taking expectation we obtain that for all \( \mathbb{P} \in \mathcal{P}_H^\kappa \),

\[ \mathbb{E}^\mathbb{P} \left[ \int_t^T |\hat{a}_s^{1/2}Z_s^n|^2 ds \right] \leq \mathbb{E}^\mathbb{P} \left[ \int_t^T Y^n_s \hat{F}_s^n(Y^n_s, Z^n_s)ds + \sup_{0 \leq t \leq T} |Y^n_t|K^{\mathbb{P},n}_t \right]. \]

Now the uniform growth condition (i) of Lemma 3.1 and the elementary inequality \( 2ab \leq a^2/\varepsilon + \varepsilon b^2, \forall \varepsilon > 0 \) yield

\[ \mathbb{E}^\mathbb{P} \left[ \int_t^T |\hat{a}_s^{1/2}Z_s^n|^2 ds \right] \leq \| \xi \|_{L_{H}^{2,\kappa}}^2 + C \left( 1 + \phi_{H}^{2,\kappa} + \mathbb{E}^\mathbb{P} \left[ \int_t^T |Y^n_s|^2 ds \right] \right) + \frac{1}{3} \mathbb{E}^\mathbb{P} \left[ \int_t^T |\hat{a}_s^{1/2}Z_s^n|^2 ds \right] + 2 \mathbb{E}^\mathbb{P} \left[ \sup_{0 \leq t \leq T} |Y_t^n|K^{\mathbb{P},n}_t \right] \]

i.e. \[ \frac{2}{3} \mathbb{E}^\mathbb{P} \left[ \int_0^T |\hat{a}_s^{1/2}Z_s^n|^2 ds \right] \leq \| \xi \|_{L_{H}^{2,\kappa}}^2 + C \left( 1 + (1 + \varepsilon^{-1}) \| Y^n \|_{L_{H}^{2,\kappa}}^2 + \varepsilon \mathbb{E}^\mathbb{P} \left[ (K^{\mathbb{P},n}_t)^2 \right] \right). \]

But by definition of \( K_T^{\mathbb{P},n} \), it is clear that

\[ \mathbb{E}^\mathbb{P} \left[ (K_T^{\mathbb{P},n})^2 \right] \leq C_0 \left( 1 + \| \xi \|_{L_{H}^{2,\kappa}}^2 + \phi_{H}^{2,\kappa} + \| Y^n \|_{L_{H}^{2,\kappa}}^2 + \int_0^T |\hat{a}_s^{1/2}Z_s^n|^2 ds \right). \]

Choosing \( \varepsilon = \frac{1}{3C_0} \), reporting (3.5) in the previous inequality and taking supremum over \( \mathbb{P} \) yields

\[ \| Z^n \|_{L_{H}^{2,\kappa}}^2 \leq C_\kappa \left( 1 + \| \xi \|_{L_{H}^{2,\kappa}}^2 + \phi_{H}^{2,\kappa} \right), \]
from which we can then deduce the result for $K_{T}$.

Finally, we can show similarly, by applying Itô’s formula to $y_{t}^{P,n}$ instead, that

$$\sup_{P \in \mathcal{P}_{H}} \mathbb{E}^{P} \left[ \int_{0}^{T} |a_{s}^{1/2} z_{s}^{P,n}|^{2} \,ds \right] \leq C_{\kappa} \left( 1 + \|\xi\|_{L_{H}^{2}}^{2} + \phi_{H}^{2} \right),$$

Now from the comparison theorems for 2BSDEs (see Corollary 4.5 in [20]) and BSDEs and (iv) of Lemma 3.1, we recall that we have for all $0 \leq t \leq T$

$$y_{t}^{P,n} \leq y_{t}^{P,n+1} \mathbb{P} - a.s. \text{ for all } P \in \mathcal{P}_{H}, \text{ and } Y_{t}^{n} \leq Y_{t}^{n+1}, \mathbb{P}_{H} - q.s.$$

**Remark 3.2.** If we were in the classical framework, this $\mathbb{P}$-almost sure convergence of $y_{t}^{P,n}$ together with the estimates of Lemma 3.4 would be sufficient to prove the convergence in the usual $L^{2}$ space, thanks to the dominated convergence theorem. However, in our case, since the norms involve the supremum over a family of probability measures, this theorem can fail. This is exactly the major difficulty when considering the 2BSDE framework, since most of the techniques used in the standard BSDE literature to prove existence results involve approximations. In order to solve this problem, we need more regularity to be able apply the monotone convergence Theorem 31 of [4]. Indeed, restricting ourselves to linear growth in $y$ allows us to use the approximation by inf-convolution which has some very nice properties. If we had considered general growth in $y$, then it would have been extremely difficult to find reasonable conditions on the driver $\hat{F}$ in order to have uniform convergence of the approximation.

Next, we prove that

**Lemma 3.5.** Under the hypotheses of Theorem 2.2 we have

$$\sup_{0 \leq t \leq T} \sup_{P \in \mathcal{P}_{H}} \mathbb{E}^{P} \left[ |y_{t}^{P} - y_{t}^{P,n}|^{2+\epsilon'} \right] \xrightarrow{n \to +\infty} 0,$$

for any $0 < \epsilon' < \epsilon$, with the same $\epsilon$ as in (2.8).

**Proof.** Thanks to Lemma 3.3, we know that we can apply the same proof as that of Theorem 2.3 of [16] in order to get for each $P \in \mathcal{P}_{H}$

$$\mathbb{E}^{P} \left[ |y_{t}^{P} - y_{t}^{P,n}|^{2+\epsilon'} \right] \leq C \int_{0}^{T} \mathbb{E}^{P} \left[ |\hat{F}_{s}^{P}(y_{s}^{P}, z_{s}^{P}) - \hat{F}_{s}^{n}(y_{s}^{P}, z_{s}^{P})|^{2+\epsilon'} \right] ds \leq C \int_{0}^{T} \mathbb{E}^{P} \left[ |\hat{F}_{s}^{n}|^{2+\epsilon'} \right] ds.$$

By Lemma 3.3 we know that $F^{n}$ converges uniformly in $(y, z)$. Since $F^{n}$ and $F$ are also continuous in $a$ by Assumption 2.1(vii), the convergence is also uniform in $a \in [a, \bar{a}]$ by Dini’s lemma. Thus, $|\hat{F}_{s}^{n}|^{2+\epsilon'}$ decreases to 0 for every $\omega \in \Omega$. 

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Then, since \( F \) and \( F^n \) are uniformly continuous in \( \omega \) on the whole space \( \Omega \), then \( \left| \bar{F}^{n}_t \right|^{2+\varepsilon} \) is continuous in \( \omega \) on \( \Omega \), and therefore quasi-continuous in the sense of [1]. Moreover, we have again by Lemma 3.3 and the fact that \( \varepsilon' < \varepsilon \)

\[
\sup_{P \in \mathcal{P}_H} \mathbb{E}^P \left[ \left( \left| \bar{F}^{n}_t \right|^{2+\varepsilon'} \right)^{1+\varepsilon''} \right],
\]

for some \( \varepsilon'' > 0 \).

Hence, we have by classical arguments (namely Hölder and Markov inequalities) that

\[
\lim_{N \to +\infty} \sup_{P \in \mathcal{P}_H} \mathbb{E}^P \left[ \left| \bar{F}^{n}_t \right|^{2+\varepsilon'} \varepsilon_{N} \right] = 0.
\]

Therefore, we can apply the monotone convergence Theorem 2.4 under the family \( \mathcal{P}_H \) to obtain that

\[
\sup_{P \in \mathcal{P}_H} \mathbb{E}^P \left[ \left| \bar{F}^{n}_t \right|^{2+\varepsilon'} \right] = \sup_{P \in \mathcal{P}_H} \mathbb{E}^P \left[ \left| \bar{F}^{n}_t \right|^{2+\varepsilon'} \right] \to 0, \text{ for all } 0 \leq t \leq T.
\]

Finally, the required result follows from the standard dominated convergence Theorem for the integral with respect to the Lebesgue measure.

We continue with the following result

**Lemma 3.6.** Assume moreover that there exists an \( \varepsilon > 0 \) such that \( \xi \in L^{2+\varepsilon, \kappa} \) and

\[
\sup_{P \in \mathcal{P}_H} \mathbb{E}^P \left[ \int_0^T \left| \bar{F}^{0}_t \right|^{2+\varepsilon} \, dt \right] < +\infty.
\]

Then, we have

\[
\sup_{P \in \mathcal{P}_H} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} \left| Y^N_t - Y^P_t \right|^2 \right] \to 0, \text{ as } n, p \to +\infty.
\]

**Proof.** By the representation (3.3), we have for all \( n, p \) large enough

\[
\sup_{0 \leq t \leq T} \left| Y^N_t - Y^P_t \right| \leq \sup_{0 \leq t \leq T} \sup_{\mathcal{P}' \in \mathcal{P}_H(t+N, \mathcal{P})} \left| y^f_{t,n} - y^f_{t,p} \right|, \mathbb{P} - a.s. \text{ for all } \mathbb{P} \in \mathcal{P}_H.
\]

Then, we easily get

\[
\sup_{0 \leq t \leq T} \left| Y^N_t - Y^P_t \right|^2 \leq \sup_{0 \leq t \leq T} \left( \sup_{\mathcal{P}' \in \mathcal{P}_H(t+N, \mathcal{P})} \mathbb{E}^{\mathcal{P}'} \left[ \left( \sup_{0 \leq s \leq T} \left| y^f_{s,n} - y^f_{s,p} \right| \right)^2 \right] \right), \mathbb{P} - a.s. \text{ for all } \mathbb{P} \in \mathcal{P}_H.
\]

Taking expectations yields for all \( \mathbb{P} \in \mathcal{P}_H \)
\[
\mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |Y_t^n - Y'^n_t|^2 \right] \leq \sup_{P \in \mathcal{P}_H^+} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} \left( \text{ess sup}_{P' \in \mathcal{P}_H^+} (t^+) \left[ \sup_{0 \leq s \leq T} |y_{s,n}^P - y_{s,t}^P| \right] \right)^2 \right].
\]

We next use the generalization of Doob maximal inequality of Proposition A.1 in the Appendix (see also Lemma 6.2 in [19]), to obtain that for all \( \epsilon' < \epsilon \)

\[
\mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |Y_t^n - Y'^n_t|^2 \right] \leq C \sup_{P \in \mathcal{P}_H^+} \mathbb{E}^P \left[ \sup_{0 \leq s \leq T} |y_{s,n}^P - y_{s,t}^P|^{2+\epsilon'} \right].
\]

Thus it suffices to show that the right-hand side tends to 0 as \( n, p \to +\infty \). We start by stating some new a priori estimates with our new integrability assumptions for \( \xi \) and \( \hat{F}_0^0 \) (see the Appendix for the proof)

\[
\sup_n \left\{ \sup_{P \in \mathcal{P}_H^+} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |y_{t,n}^P|^2 \right] + \sup_{P \in \mathcal{P}_H^+} \mathbb{E}^P \left[ \left( \int_0^T \left| \tilde{a}_{t}^{1/2} z_{t,n}^P \right|^2 dt \right)^{2+\epsilon'} \right] \right\}
\leq C \left( 1 + \| \xi \|_{L_{2+\epsilon',H}^{2+\epsilon',H}}^2 + \sup_{P \in \mathcal{P}_H^+} \mathbb{E}^P \left[ \int_0^T |\hat{F}_s^P|^{2+\epsilon'} ds \right] \right). \tag{3.6}
\]

Let us denote \( \delta \hat{F}_t^{n,p} := \hat{F}_t^n(t, y_{t,n}^P, z_{t,n}^P) - \hat{F}_t^p(t, y_{t,p}^P, z_{t,p}^P) \). Applying Itô's formula to \( |y_{t,n}^P - y_{t,p}^P|^2 \) and taking conditional expectations yields, for all \( P \in \mathcal{P}_H^+ \)

\[
|y_{t,n}^P - y_{t,p}^P|^2 + \mathbb{E}^P \left[ \int_t^T \left( \tilde{a}_{s}^{1/2} (z_{s,n}^P - z_{s,p}^P) \right)^2 ds \right] \leq 2 \mathbb{E}^P \left[ \int_t^T |y_{s,n}^P - y_{s,p}^P| |\delta \hat{F}_s^{n,p}| ds \right]. \tag{3.7}
\]

Now since \( \epsilon' > 0 \), it follows from Doob maximal inequality (in the classical form under a single measure) and the Cauchy-Schwarz inequality that

\[
\mathbb{E}^P \left[ \sup_{0 \leq t \leq T} \left( \mathbb{E}^P \left[ \int_t^T \left( \tilde{a}_{s}^{1/2} (z_{s,n}^P - z_{s,p}^P) \right)^2 ds \right] \right)^{2+\epsilon'/2} \right] \leq C \mathbb{E}^P \left[ \left( \int_0^T |y_{s,n}^P - y_{s,p}^P| \int \left| \delta \hat{F}_s^{n,p} \right| ds \right)^{2+\epsilon'} \right] \leq C \left( \mathbb{E}^P \left[ \int_0^T |y_{s,n}^P - y_{s,p}^P|^{2+\epsilon'} ds \right] \right)^{1/2} \times \left( \mathbb{E}^P \left[ \int_0^T \left| \delta \hat{F}_s^{n,p} \right|^2 ds \right]^{2+\epsilon'/2} \right)^{1/2}. \tag{3.8}
\]

By the uniform growth property (i) of Lemma [3.1] we have for all \( t \in [0, T] \)

\[
|\delta \hat{F}_t^{n,p}|^2 \leq C \left( 1 + |\hat{F}_0^0|^2 + |y_{t,n}^P|^2 + |y_{t,p}^P|^2 + |\tilde{a}_{t}^{1/2} z_{t,n}^P|^2 + |\tilde{a}_{t}^{1/2} z_{t,p}^P|^2 \right), \ P - a.s., \ \forall P \in \mathcal{P}_H^+.
\]
Hence using the uniform a priori estimates of (3.16), we have for all \( P \in \mathcal{P}_H^\infty \)

\[
\mathbb{E}^P \left[ \left( \int_0^T |\hat{F}_t^{n,P}|^2 dt \right)^{2+\epsilon'} \right]^\frac{1}{2+\epsilon'} \leq C \left( 1 + \sup_{P \in \mathcal{P}_H^\infty} \mathbb{E}^P \left[ \left( \int_0^T |\hat{F}_s^{0}|^2 ds \right)^{2+\epsilon'} \right] \right) \\
+ C \left( \sup_{P \in \mathcal{P}_H^\infty} \mathbb{E}^P \left[ \left( \sup_{0 \leq t \leq T} |y_t^{P,n}|^{2+\epsilon'} \right) \right] \right) \\
+ C \left( \sup_{P \in \mathcal{P}_H^\infty} \mathbb{E}^P \left[ \left( \int_0^T |\hat{a}_s^{1/2} z_s^{P,n}|^2 ds \right)^\frac{2+\epsilon'}{2} \right] \right) \\
\leq C \left( 1 + \|\xi\|_{L^2+\epsilon,\infty}^{2+\epsilon} + \sup_{P \in \mathcal{P}_H^\infty} \mathbb{E}^P \left[ \left( \int_0^T |\hat{F}_s^{0}|^{2+\epsilon'} ds \right) \right] \right). \tag{3.9}
\]

We then have

\[
\sup_{P \in \mathcal{P}_H^\infty} \mathbb{E}^P \left[ \int_0^T |y_t^{P,n} - y_t^{P,p}|^{2+\epsilon'} dt \right] \leq \int_0^T \sup_{P \in \mathcal{P}_H^\infty} \mathbb{E}^P \left[ |y_t^{P,n} - y_t^{P,p}|^{2+\epsilon'} \right] dt \xrightarrow{n,p \to +\infty} 0, \tag{3.10}
\]

where we used Lemma 3.5 and the standard Lebesgue dominated convergence Theorem.

Therefore, plugging the estimate (3.9) in (3.8), using (3.10), and sending \( n, p \) to \( +\infty \), we see that

\[
\sup_{P \in \mathcal{P}_H^\infty} \mathbb{E}^P \left[ \left( \int_0^T |\hat{a}_s^{1/2} (z_s^{P,n} - z_s^{P,p})|^2 ds \right)^{2+\epsilon'} \right] \leq \sup_{P \in \mathcal{P}_H^\infty} \mathbb{E}^P \left[ \left( \mathbb{E}^P \left[ \left( \int_0^T |\hat{a}_s^{1/2} (z_s^{P,n} - z_s^{P,p})|^2 ds \right) \right] \right)^{2+\epsilon'} \right] \xrightarrow{n,p \to +\infty} 0. \tag{3.11}
\]

Then, by Itô’s formula we similarly obtain

\[
\sup_{P \in \mathcal{P}_H^\infty} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |y_t^{P,n} - y_t^{P,p}|^{2+\epsilon'} \right] \leq C \sup_{P \in \mathcal{P}_H^\infty} \mathbb{E}^P \left[ \left( \sup_{0 \leq t \leq T} \left( \int_0^T |(y_t^{P,n} - y_t^{P,p}) (z_t^{P,n} - z_t^{P,p}) d_B |^{2+\epsilon'} \right) \right)^{1/2} \right] \\
+ 2 \left( \sup_{P \in \mathcal{P}_H^\infty} \mathbb{E}^P \left[ \left( \int_0^T |y_t^{P,n} - y_t^{P,p}|^{2+\epsilon'} \right) \right] \right)^{1/2} \\
\times \left( \sup_{P \in \mathcal{P}_H^\infty} \mathbb{E}^P \left[ \left( \int_0^T |\hat{F}_s^{n,P}|^{2+\epsilon'} ds \right) \right] \right)^{1/2}. \tag{3.12}
\]

By previous calculations we know that the second term tends to 0 as \( n, p \to +\infty \). For the first one, we have by the Burkholder-Davis-Gundy inequality (where we recall that the
constants involved are universal and thus do not depend on $P$)

$$
\mathbb{E}^P \left[ \sup_{0 \leq t \leq T} \left| f_t^T (y^n_s - y^p_s) (z^n_s - z^p_s) dB_s \right| \right]^{2+\epsilon'} \leq C_0 \mathbb{E}^P \left[ \left( \sup_{0 \leq t \leq T} \left| y^n_t - y^p_t \right|^2 f_t^T \left( \frac{1}{2} (z^n_t - z^p_t) \right)^2 ds \right)^{2+\epsilon'} \right] \\
\leq C_0 \mathbb{E}^P \left[ \left( \sup_{0 \leq t \leq T} \left| y^n_t - y^p_t \right|^2 f_t^T \left( \frac{1}{2} (z^n_t - z^p_t) \right)^2 ds \right)^{2+\epsilon'} \right] \\
\leq \frac{1}{2C} \sup_{p \in \mathcal{P}^n_H} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} \left| y^n_t - y^p_t \right|^{2+\epsilon'} \right] \\
+ \frac{CC_0^2}{4} \sup_{p \in \mathcal{P}^n_H} \mathbb{E}^P \left[ \left( \int_0^T \left| \frac{1}{2} (z^n_t - z^p_t) \right|^2 ds \right)^{2+\epsilon'} \right],
$$

where $C$ is the constant in (3.12).

Reporting this in the above inequality (3.12) and letting $n, p$ go to $+\infty$ then finishes the proof.

\[ \square \]

**Remark 3.3.** In contrast with the classical case, we proved here the convergence of $Y^n$ in $\mathbb{D}_H^{2,\kappa}$ before proving any convergence for $Z^n$. Proceeding in this order is crucial because of the process $K^{P,n}$, which prevents us from using the usual techniques. Then, it is natural to use the representation formula for $Y^n$ to control the $\mathbb{D}_H^{2,\kappa}$ norm of $Y^n - Y^p$ by a certain norm of $y^{P,n} - y^{P,p}$. It turns out that we end up with a norm which is closely related to the $\mathbb{L}^{2,\kappa}$ norm. However, this norm for the process $y^{P,n}$ is not tractable for classical BSDEs, therefore, we have to use the generalized Doob inequality (which is currently conjectured to be the best possible, see Remark 2.9 in [20]) to return to the usual norm for $y^{P,n}$. This in turn forces us to assume stronger integrability assumptions on $\xi$ and $\hat{F}^0$.

We just have proved that the sequence $(Y^n)_n$ is Cauchy in the Banach $\mathbb{D}_H^{2,\kappa}$. Thus it converges to $Y$ in $\mathbb{D}_H^{2,\kappa}$. Let us now focus on $Z^n$ and $K^{P,n}$.

**Lemma 3.7.** There exist a process $Z \in \mathbb{H}_H^{2,\kappa}$ and a non-decreasing process $K^P \in \mathbb{D}^2(P)$ such that

$$
\|Z^n - Z\|^2_{\mathbb{H}_H^{2,\kappa}} + \sup_{p \in \mathcal{P}^n_H} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} \left| K^{P,n}_t - K^P_t \right|^2 \right] \to 0, \text{ as } n \to +\infty.
$$

**Proof.** Denote $\delta \hat{F}^{n,p}_t := \hat{F}^{n}_t (Y^n_t, Z^n_t) - \hat{F}^p_t (Y^p_t, Z^p_t)$. Applying Itô’s formula to $|Y^n_t - Y^p_t|^2$ and taking expectations yields, for all $P \in \mathcal{P}^n_H$

$$
\mathbb{E}^P \left[ |Y^n_0 - Y^p_0|^2 + \int_0^T |\hat{a}^{1/2}_t (Z^n_t - Z^p_t)|^2 dt \right] \leq 2\mathbb{E}^P \left[ \int_0^T (Y^n_t - Y^p_t) \delta \hat{F}^{n,p}_t dt \right] \\
+ 2\mathbb{E}^P \left[ \int_0^T (Y^n_t - Y^p_t) d(K^{P,n}_t - K^P_t) \right].
$$

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Then
\[
\mathbb{E}^p \left[ \int_0^T |a_t^{1/2}(Z_t^n - Z_t^p)|^2 dt \right] \leq 2 \left( \sup_{P \in \mathcal{P}_H^n} \mathbb{E}^P \left[ \int_0^T |Y_t^n - Y_t^p|^2 dt \right] \right)^{1/2} 
\times \left( \sup_{P \in \mathcal{P}_H^n} \mathbb{E}^P \left[ \int_0^T |\delta \hat{F}_t|_2^2 dt \right] \right)^{1/2} 
+ 2 \left( \sup_{P \in \mathcal{P}_H^n} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |Y_t^n - Y_t^p|^2 \right] \right)^{1/2} 
\times \left( \sup_{P \in \mathcal{P}_H^n} \mathbb{E}^P \left[ |K_T^n|_2 + |K_T^p|_2 \right] \right)^{1/2}.
\]

Notice that the right-hand side tends to 0 uniformly in \( P \) as \( n, p \to +\infty \) due to Lemmas 3.3 and 3.6 and (3.9). Thus \( (Z^n) \) is a Cauchy sequence in \( \mathbb{H}_H^{2,\kappa} \) and therefore converges to a process \( Z \in \mathbb{H}_H^{2,\kappa} \).

Now by (2.5), we have
\[
\sup_{P \in \mathcal{P}_H^n} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |K_t^{2,n} - K_t^{2,p}|^2 \right] \leq C \sup_{P \in \mathcal{P}_H^n} \mathbb{E}^P \left[ |Y_0^n - Y_0^p|^2 + \sup_{0 \leq t \leq T} |Y_t^n - Y_t^p|^2 \right] 
+ C \sup_{P \in \mathcal{P}_H^n} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} \left| \int_0^T (Z^n_s - Z^p_s) dB_s \right|^2 \right] 
+ C \sup_{P \in \mathcal{P}_H^n} \mathbb{E}^P \left[ \int_0^T |\delta \hat{F}_t|_2^2 dt \right].
\]

The first two terms on the right-hand side tend to 0 as \( n, p \to +\infty \) thanks to Lemma 3.6. For the last one, using BDG inequality and the result we just proved on the sequence \( (Z^n) \), we see that it also tends to 0. Thus, in order to finish the proof, we need to show that the term involving \( \delta \hat{F}^{n,p} \) converges to 0. This is deduced from the following facts

- \( Y^n \overset{\text{a.s.}}{\to} Y \) in \( \mathbb{D}_H^{2,\kappa} \) and \( dt \times \mathcal{P}_H^\kappa - q.s. \).

- By Lemma 3.3 \( \hat{F}^n \) converges uniformly to \( \hat{F} \) in \( (y, z) \). Moreover, we have from the Lipschitz property of \( F \) in \( z \) and its uniform continuity in \( y \)
\[
\left| \hat{F}_t(Y_t, Z_t) - \hat{F}_t^n(Y_t^n, Z_t^n) \right| \leq \rho(|Y_t^n - Y_t|) + \left| \hat{F}_t(Y_t^n, Z_t) - \hat{F}_t^n(Y_t^n, Z_t) \right| 
+ C|a_t^{1/2}(Z_t - Z_t^n)|,
\]
for some modulus of continuity \( \rho \).

When taking expectation under \( P \) and supremum over all \( P \in \mathcal{P}_H^\kappa \), the convergence to 0 for the term involving \( Z - Z^n \) is clear by our previous result. For the second one in the right-hand side above, we can use the same arguments as in the proof of Lemma
Finally, we recall that since the space $\Omega$ is convex, it is a classical result that we can choose the modulus of continuity $\rho$ to be concave, non-decreasing and sub-linear. We then have by Jensen inequality
\[
\sup_{P \in \mathcal{P}_H^\kappa} \mathbb{E}^P \left[ \rho(|Y^n_t - Y_t|) \right] \leq \rho \left( \sup_{P \in \mathcal{P}_H^\kappa} \mathbb{E}^P \left[ |Y^n_t - Y_t| \right] \right) \rightarrow 0,
\]
by Lemma 3.6.

Consequently, for all $P \in \mathcal{P}_H^\kappa$, there exists a non-decreasing and progressively measurable process $K^P$ such that
\[
\sup_{P \in \mathcal{P}_H^\kappa} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |K^P_{t,n} - K^P_t|^2 \right] \rightarrow 0, \quad n \to +\infty.
\]

Moreover, since the $K^P_{t,n}$ are càdlàg, so is $K^P_t$.

**Proof of Theorem 2.2.** Taking limits in the 2BSDE (3.1), we obtain that $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ is a solution of the 2BSDE (2.4). To conclude the proof of existence, it remains to check the minimum condition (2.6). But for all $P \in \mathcal{P}_H^\kappa$, we know that $K^P_{t,n}$ verifies (2.6). Then we can pass to the limit in the minimum condition verified by $K^P_{t,n}$. Indeed, we have
\[
K^P_t = \lim_{m \to +\infty} \mathbb{E}^{P_m} \left[ K^P_{T,m} \right], \quad P \text{- a.s.}
\]
Extracting a subsequence if necessary, the left-hand side converges to $K^P_t$, $\mathcal{P}_H^\kappa - q.s.$ Then as in the beginning of the proof of Lemma 3.6 we can write
\[
\lim_{m \to +\infty} \mathbb{E}^{P_m} \left[ K^P_{T,m} \right] = \lim_{m \to +\infty} \mathbb{E}^{P_m} \left[ K^P_{T,m} \right],
\]
where $(P_m)_{m \geq 0}$ is a sequence of probability measures belonging to $\mathcal{P}_H^\kappa(t^+, P)$.

Then by Lemma 3.7, we know that $\mathbb{E}^{P_m}[K^P_{T,m}]$ converges to $\mathbb{E}^{P}[K^P_T]$ uniformly in $P \in \mathcal{P}_H^\kappa$. Thus we can take the limit in $n$ in the above expression and switch the limits in $n$ and $m$. This shows that
\[
K^P_t = \lim_{m \to +\infty} \mathbb{E}^{P_m} \left[ K^P_{T,m} \right] \geq \mathbb{E}^{P_m} \left[ K^P_T \right].
\]
The converse inequality is trivial since the process $K^P$ is non-decreasing. Thus, the minimum condition is satisfied, and the proof of Theorem 2.2 is complete.

**Remark 3.4.** In comparison with the classical BSDE framework, we had to add some assumptions here to prove existence of a solution. The question is whether these assumptions can be weakened by using another construction for the solution. For instance, we may use the so called regular conditional probability distributions as in [20] and [18]. However, as mentioned in Remark 4.1 in [18], even though we could construct a candidate solution when
the terminal condition is in $UC_b(\Omega)$, when trying to check that the family $\{K^\mathbb{P}, \mathbb{P} \in \mathcal{P}_H^\kappa\}$ obtained verifies the minimum condition, our monotonicity assumption is not sufficient. Thus, regardless of the solution construction method, we have to add some assumptions to prove existence.

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A Appendix

**Proposition A.1.** Let $\xi$ be some $\mathcal{F}_T$ random variable, and let $n \geq 1$. Then, for all $p > n$

$$
\sup_{P \in \mathcal{P}_H^{\lambda_\infty}} \mathbb{E}_P \left[ \sup_{0 \leq t \leq T} \left( \text{ess sup}_{P' \in \mathcal{P}_H^{\lambda_\infty}(t+,P)} \mathbb{E}^P_{t} \left[ ||\xi||_t \right] \right)^n \right] \leq C \sup_{P \in \mathcal{P}_H^{\lambda_\infty}} \left( \mathbb{E}^P \left[ ||\xi||_P \right] \right)^{n/p}.
$$
Proof. The proof follows the ideas of the proof of Lemma 6.2 in [19], which closely follows the classical proof of the Doob maximal inequality (see also [23] for related results).

Fix some \( \mathbb{P} \), and let us note

\[
X^P_t := \text{ess sup}_{\mathbb{P'} \in \mathcal{P}^\kappa H(t^+, \mathbb{P})} \mathbb{E}^P_{t'}[|\xi|].
\]

Then, it can be shown that \( X^P_t \) is a \( \mathbb{P} \)-supermartingale, and thus admits a càdlàg version.

For all \( \lambda > 0 \), let us define the following \( \mathcal{F}^+ \) stopping times

\[
\tau^P_\lambda = \inf \left\{ t \geq 0, \ X^P_t \geq \lambda, \ \mathbb{P} - \text{a.s.} \right\}.
\]

Define \( X^{P, *} := \sup_{0 \leq t \leq T} X^P_t \). Then, we have

\[
\mathbb{P}(X^{P, *} \geq \lambda) = \mathbb{P}(\tau^P_\lambda \leq T) \leq \frac{1}{\lambda} \mathbb{E}^\mathbb{P}[X^P_{\tau^P_\lambda} 1_{\tau^P_\lambda \leq T}] .
\]

As previously in this chapter, we know that there exists a sequence \( (\mathbb{P}_n)_{n \geq 0} \subset \mathcal{P}^\kappa H(\tau^P_\lambda, \mathbb{P}) \) such that

\[
X^P_{\tau^P_\lambda} = \lim_{n \to +\infty} \uparrow \mathbb{E}^\mathbb{P}_n[|\xi|].
\]

Hence, using in this order the monotone convergence theorem and the fact that all the \( \mathbb{P}_n \) coincide with \( \mathbb{P} \) on \( \mathcal{F}_{\tau^P_\lambda} \), we have

\[
\mathbb{E}^\mathbb{P}[X^P_{\tau^P_\lambda} 1_{\tau^P_\lambda \leq T}] = \lim_{n \to +\infty} \uparrow \mathbb{E}^\mathbb{P}_n[|\xi|] 1_{\tau^P_\lambda \leq T}
\]

\[
= \lim_{n \to +\infty} \uparrow \mathbb{E}^\mathbb{P}_n[|\xi|] 1_{\tau^P_\lambda \leq T}
\]

\[
= \lim_{n \to +\infty} \uparrow \mathbb{E}^\mathbb{P}_n[|\xi|] 1_{\tau^P_\lambda \leq T}
\]

\[
\leq \lim_{n \to +\infty} \uparrow \left( \mathbb{E}^\mathbb{P}_n[|\xi|^p] \right)^{1/p} \mathbb{P}_n(\tau^P_\lambda \leq T)^{1 - \frac{1}{p}}
\]

\[
\leq \mathbb{P}(\tau^P_\lambda \leq T)^{1 - \frac{1}{p}} \lim_{n \to +\infty} \uparrow \left( \mathbb{E}^\mathbb{P}_n[|\xi|^p] \right)^{1/p}
\]

\[
\leq \mathbb{P}(\tau^P_\lambda \leq T)^{1 - \frac{1}{p}} \sup_{\mathbb{P} \in \mathcal{P}^\kappa H} \left( \mathbb{E}^\mathbb{P}[|\xi|^p] \right)^{1/p}.
\]

Using this estimate, we finally get

\[
\mathbb{P}(X^{P, *} \geq \lambda) \leq \frac{1}{\lambda^p} \sup_{\mathbb{P} \in \mathcal{P}^\kappa H} \mathbb{E}^\mathbb{P}[|\xi|^p].
\]
Now, for every \( \lambda_0 > 0 \), we have
\[
\mathbb{E}^P \left[ (X^{P,*})^n \right] = n \int_0^{\infty} \lambda^{-1} \mathbb{P}(X^{P,*} \geq \lambda) d\lambda \\
= n \int_0^{\lambda_0} \lambda^{-1} \mathbb{P}(X^{P,*} \geq \lambda) d\lambda + n \int_{\lambda_0}^{\infty} \lambda^{-1} \mathbb{P}(X^{P,*} \geq \lambda) d\lambda \\
\leq \lambda_0^n + n \sup_{P \in \mathcal{P}_H} \mathbb{E}^P[|\xi|^p] \int_{\lambda_0}^{\infty} \frac{d\lambda}{\lambda^{p-n+1}} \\
= \lambda_0^n + n \sup_{P \in \mathcal{P}_H} \mathbb{E}^P[|\xi|^p] \frac{\lambda_0^{n-p}}{p-n}.
\]

Choosing \( \lambda_0 = \sup_{P \in \mathcal{P}_H} (\mathbb{E}^P[|\xi|^p])^{1/p} \), this yields
\[
\mathbb{E}^P \left[ (X^{P,*})^n \right] \leq \left( 1 + \frac{n}{p-n} \right) \sup_{P \in \mathcal{P}_H} \left( \mathbb{E}^P[|\xi|^p] \right)^{n/p}.
\]

Hence the result. \(\square\)

**Proof of Lemma 3.3** The uniform convergence result is a simple consequence of a result already proved by Lasry and Lyons in \(\square\) (see the Theorem on page 4 and Remark (iv) on page 5). For the inequality, since \(\hat{F} \) is uniformly continuous in \( y \), there exists a modulus of continuity \( \rho \) with linear growth. Then, it follows that
\[
F_t - F_t^n = \sup_{u \in \mathcal{Q}, u \geq y} \{F_t(y, z, a) - F_t(u, z, a) - n |y - u|\} \\
\leq \sup_{u \in \mathcal{Q}, u \geq y} \{C \rho(|y - u|) - n |y - u|\} \\
= \sup_{u \geq 0} \{C \rho(u) - nu\}.
\]

Since \( \rho \) has linear growth in \( y \), the function on the right-hand side above is clearly decreasing in \( n \) and thus is dominated by a constant, which gives us the result.

Finally, the equality of the two suprema is clear because the quantity considered here is bounded and continuous in \( \omega \). \(\square\)

**Proposition A.2.** Assume that there exists an \( \epsilon > 0 \) such that \( \xi \in L_{H}^{2+\epsilon, \kappa} \) and
\[
\sup_{P \in \mathcal{P}_H} \mathbb{E}^P \left[ \int_0^T \left| \hat{F}_t^0 \right|^{2+\epsilon} dt \right] < +\infty.
\]
Then we have
\[
\sup_n \left\{ \sup_{F \in \mathcal{F}_n} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |y_t^{n} |^{2+\epsilon} \right] + \sup_{F \in \mathcal{F}_n} \mathbb{E}^P \left[ \left( \int_0^T \hat{a}_t^{1/2} z_t^{n} \, dt \right)^{2+\epsilon} \right] \right\} \\
\leq C \left( 1 + \|\xi\|_{L_H^{2+\epsilon}} + \sup_{F \in \mathcal{F}_n} \mathbb{E}^P \left[ \int_0^T \left| F_s^0 \right|^{2+\epsilon} \, ds \right] \right).
\]

**Proof.** Fix a \( \mathbb{P} \), and consider the BSDE [3.4]. As in the proof of Lemma 3.4, we have that for all \( n \), \( y^{n} \leq u^{P} \), \( \mathbb{P} \)-a.s. Now let \( \alpha \) be some positive constant which will be fixed later and let \( \eta \in (0,1) \). By Itô’s formula we have

\[
e^{\alpha t} \left| u_t^P \right|^2 + \int_t^T e^{\alpha s} \left| a_s^{1/2} u_s^P \right|^2 \, ds = e^{\alpha T} \left| \xi \right|^2 + 2 \int_t^T e^{\alpha s} u_s^P \left( \hat{F}_s^0 + C \right) \, ds \\
+ 2C \int_t^T u_s^P \left( e^{\alpha s} \left| u_s^P \right| + \left| a_s^{1/2} v_s^P \right| \right) \, ds - \alpha \int_t^T e^{\alpha s} \left| u_s^P \right|^2 \, ds \\
- 2 \int_t^T e^{\alpha s} u_s^P v_s^P dB_s \\
\leq e^{\alpha T} \left| \xi \right|^2 + 2 \int_t^T e^{\alpha s} \left| u_s^P \right| \left( \left| \hat{F}_s^0 \right| + C \right) \, ds \\
+ \left( 2C + \frac{C^2}{\eta} - \alpha \right) \int_t^T e^{\alpha s} \left| u_s^P \right|^2 \, ds + \eta \int_t^T e^{\alpha s} \left| a_s^{1/2} v_s^P \right|^2 \, ds \\
- 2 \int_t^T e^{\alpha s} u_s^P v_s^P dB_s.
\]

Now choose \( \alpha \) such that \( \nu := \alpha - 2C - \frac{C^2}{\eta} \geq 0 \). We obtain

\[
e^{\alpha t} \left| u_t^P \right|^2 + (1 - \eta) \int_t^T e^{\alpha s} \left| a_s^{1/2} v_s^P \right|^2 \, ds + \nu \int_t^T e^{\alpha s} \left| u_t^P \right|^2 \, ds \leq e^{\alpha T} \left| \xi \right|^2 + 2 \int_t^T e^{\alpha s} \left| u_s^P \hat{F}_s^0 \right| \, ds \\
+ 2C \int_t^T e^{\alpha s} \left| u_s^P \right| \, ds \\
- 2 \int_t^T e^{\alpha s} u_s^P v_s^P dB_s. \quad (A.1)
\]

Taking conditional expectation in (A.1) yields

\[
e^{\alpha t} \left| u_t^P \right|^2 \leq \mathbb{E}^P_{t} \left[ e^{\alpha T} \left| \xi \right|^2 + 2 \int_t^T e^{\alpha s} \left| u_s^P \right| \left( \left| \hat{F}_s^0 \right| + C \right) \, ds \right].
\]
By Doob's maximal inequality, we then get for all $\beta \in (0, 1)$, since $\epsilon > 0$

$$
\mathbb{E}^P \left[ \sup_{0 \leq t \leq T} e^{\alpha t \frac{2 + \epsilon}{2}} \left| u_t \right|^{2 + \epsilon} \right] \leq C \mathbb{E}^P \left[ e^{\alpha T \frac{2 + \epsilon}{2}} \left| \xi \right|^{2 + \epsilon} \right] 
$$

$$
+ C \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} \left( e^{\alpha t \frac{2 + \epsilon}{2}} \left| u_t \right|^{2 + \epsilon} \right) \left( \int_0^T C + \left| \tilde{F}_s \right| ds \right)^{\frac{2 + \epsilon}{2}} \right]
$$

$$
\leq C \left( 1 + \| \xi \|_{L^{2+\epsilon,\infty}^2} \right) + \beta \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} \left| u_t \right|^{2 + \epsilon} \right]
$$

$$
+ \beta \frac{C^2}{4} \mathbb{E}^P \left[ \int_0^T \left| \tilde{F}_s \right|^{2 + \epsilon} ds \right].
$$

Since $\alpha$ is positive and $y_{t,n}^P \leq u_t^P$, we get finally for all $n$

$$
\sup_{P \in \mathcal{P}_H} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} \left| y_t^{P,n} \right|^{2 + \epsilon} \right] \leq C \left( 1 + \left\| \xi \right\|_{L^{2+\epsilon,\infty}^2} + \mathbb{E}^P \left[ \int_0^T \left| \tilde{F}_s \right|^{2 + \epsilon} ds \right] \right). \quad (A.2)
$$

Now apply Itô's formula to $\left| y_{t,n}^P \right|^2$ and put each side to the power $\frac{2 + \epsilon}{2}$, we have easily

$$
\left( \int_0^T \left| a_t^{-1/2} z_{t,n}^P \right|^2 dt \right)^{\frac{2 + \epsilon}{2}} \leq C \left( \left| \xi \right|^{2+\epsilon} + \left( \int_0^T \left| y_t^{P,n} \tilde{F}_t \left( y_t^{P,n}, z_t^{P,n} \right) \right| dt \right)^{\frac{2 + \epsilon}{2}} \right)
$$

$$
+ C \left( \int_0^T \left| y_t^{P,n} z_t^{P,n} dB_t \right|^{2 + \epsilon} \right)^{\frac{2 + \epsilon}{2}}.
$$

Since $\tilde{F}^n$ satisfies a uniform linear growth property, we get

$$
\mathbb{E}^P \left[ \left( \int_0^T \left| a_t^{-1/2} z_{t,n}^P \right|^2 dt \right)^{\frac{2 + \epsilon}{2}} \right] \leq C \left\| \xi \right\|_{L^{2+\epsilon,\infty}^2} + C \left( 1 + \mathbb{E}^P \left[ \int_0^T \left| \tilde{F}_s \right|^{2 + \epsilon} ds \right] \right)
$$

$$
+ C \sup_{P \in \mathcal{P}_H} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} \left| y_t^{P,n} \right|^{2 + \epsilon} \right]
$$

$$
+ \frac{1}{3} \mathbb{E}^P \left[ \left( \int_0^T \left| a_t^{-1/2} z_{t,n}^P \right|^2 dt \right)^{\frac{2 + \epsilon}{2}} \right]
$$

$$
+ C \mathbb{E}^P \left[ \left( \int_0^T \left| y_t^{P,n} z_t^{P,n} dB_t \right|^{2 + \epsilon} \right)^{\frac{2 + \epsilon}{2}} \right].
$$

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Hence, we get for all $\gamma \in (0,1)$ by BDG inequality and (A.2)

\[
\mathbb{E}^P \left[ \left( \int_0^T \left| \alpha_t^{1/2} z_t^{\mathcal{P},n} \right|^2 dt \right)^{2+\epsilon} \right] \leq C \left( 1 + ||\xi||_{L^{2+\epsilon,H}}^{2+\epsilon} + \sup_{P \in \mathcal{P}_H^c} \mathbb{E}^P \left[ \left( \int_0^T \left| \hat{F}_t^0 \right|^2 dt \right)^{2+\epsilon} \right] \right)
\]

\[
+ C \mathbb{E}^P \left[ \left( \int_0^T \left| y_t^{\mathcal{P},n} \right|^2 \left| \alpha_t^{1/2} z_t^{\mathcal{P},n} \right|^2 dt \right)^{2+\epsilon} \right]
\]

\[
\leq C \left( 1 + ||\xi||_{L^{2+\epsilon,H}}^{2+\epsilon} + \sup_{P \in \mathcal{P}_H^c} \mathbb{E}^P \left[ \left( \int_0^T \left| \hat{F}_t^0 \right|^2 dt \right)^{2+\epsilon} \right] \right)
\]

\[
+ \gamma \mathbb{E}^P \left[ \left( \int_0^T \left| \alpha_t^{1/2} z_t^{\mathcal{P},n} \right|^2 dt \right)^{2+\epsilon} \right]
\]

\[
+ \frac{C^2}{4\gamma} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} \left| y_t^{\mathcal{P},n} \right|^{2+\epsilon} \right],
\]

which ends the proof. \(\square\)