ISOCLINISM IN LIE SUPERALGEBRAS

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Abstract. In this paper using the concept of isoclinism, we give the structure of all covers of Lie superalgebras when their Schur multipliers are finite dimensional. Further it has been shown that, each stem extension of a finite dimensional Lie superalgebra is a homomorphic image of a stem cover for it and as a corollary it is concluded that maximal stem extensions of Lie superalgebras are precisely same as the stem covers. Moreover, we have defined stem Lie superalgebra and show that a Lie superalgebra with finite dimensional derived subalgebra and finitely generated central factor is isoclinic to a finite dimensional Lie superalgebra.

1. Introduction

For a given group $G$, the notion of the Schur multiplier $\mathcal{M}(G)$ arose from the work of I. Schur on projective representation of groups as the second cohomology group with coefficients in $\mathbb{C}^*$ [1]. Let $0 \rightarrow R \rightarrow F \rightarrow G \rightarrow 0$ be a free presentation of the group $G$, then it can be shown by Hopf’s formula that $\mathcal{M}(G) \cong R \cap [F,F]/[R,F]$ [9]. The finite dimensional Lie algebra analogue to the Schur multiplier was developed in [3] and later it has been studied by several authors.

Let $A$ be a Lie algebra over a field $\mathbb{F}$ with a free presentation $0 \rightarrow R \rightarrow F \rightarrow A \rightarrow 0$, where $F$ is a free Lie algebra. Then the Schur multiplier of $A$, denoted by $\mathcal{M}(A)$, is defined to be the factor Lie algebra $(R \cap [F,F])/[R,F]$ [3]. It is easy to see that the Schur multiplier of a Lie algebra $A$ is abelian and independent of the choice of the free presentation of $A$. For more information about the Schur multiplier of Lie algebras one can refer [3, 4, 5] and the references therein. Batten in [3] showed that if $A$ is finite dimensional, then its Schur multiplier is isomorphic to $H^2(A,\mathbb{F})$, the second cohomology group of $A$. Similarly Schur multiplier for $n$-Lie algebras are defined and studied [7]. Schur multipliers are important because sometimes by just looking at the dimension of it, one can completely know the structure of corresponding Lie algebras [4, 11].

Here we recall some definitions of Lie superalgebra and fix some notations which we shall be using throughout. Write $\mathbb{Z}_2 = \{0, 1\}$ is a field. A $\mathbb{Z}_2$-graded vector space is simply a direct sum of vector spaces $V_0$ and $V_1$ such that $V = V_0 \oplus V_1$. A $\mathbb{Z}_2$-graded vector space (resp. $\mathbb{Z}_2$-graded algebra) is also referred to as a superspace (resp. superalgebra). We consider all vector superspaces and superalgebras over $\mathbb{F}$ (characteristic of $F \neq 2, 3$). Elements in $V_0$ (resp. $V_1$) are called even (rest. odd) elements. Non-zero elements of $V_0 \cup V_1$ are called homogeneous elements. For a homogeneous element $v \in V_\sigma$, with $\sigma \in \mathbb{Z}_2$ we set $|v| = \sigma$ is the degree of $v$. A vector subspace $U$ of $V$ is called $\mathbb{Z}_2$-graded vector subspace (or superspace) if $U = (V_0 \cap U) \oplus (V_1 \cap U)$. We adopt the convention that whenever the degree function appeared in a formula, the corresponding elements are supposed to be homogeneous. A Lie superalgebra
is a superspace $L = L_0 \oplus L_1$ with a bilinear mapping $[,] : L \times L \to L$ satisfying the following identities:

1. $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$, for $\alpha, \beta \in \mathbb{Z}_2$ (Z$_2$-grading),
2. $[x, y] = -(-1)^{|x||y|}[y, x]$ (graded skew-symmetry),
3. $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$ (graded Jacobi identity).

for all $x, y, z \in L$.

For a Lie superalgebra $L = L_0 \oplus L_1$, the even part $L_0$ is a Lie algebra and $L_1$ is a $L_0$-module. If $L_1 = 0$, then $L$ is just Lie algebra. But in general a Lie superalgebra is not a Lie algebra. Lie superalgebra without even part, i.e., $L_0 = 0$, is an abelian Lie superalgebra, as $[x, y] = 0$ for all $x, y \in L$. A subalgebra superalgebra of $L$ is a $\mathbb{Z}_2$ vector subspace which is closed under bracket operation. A $\mathbb{Z}_2$-graded subspace $I$ is a graded ideal of $L$ if $[I, L] \subseteq I$. The ideal $Z(L) = \{z \in L : [z, x] = 0 \text{ for all } x \in L\}$ is a graded ideal and it is called the center of $L$. Clearly, if $I$ and $J$ are graded ideals of $L$, then so is $[I, J]$. If $I$ is an ideal of $L$, the quotient Lie superalgebra $L/I$ inherits a canonical Lie superalgebra structure such that the natural projection map becomes a homomorphism.

By a homomorphism between superspaces $f : V \to W$ of degree $|f| \in \mathbb{Z}_2$, we mean a linear map satisfying $f(V_\alpha) \subseteq W_{\alpha+|f|}$ for $\alpha \in \mathbb{Z}_2$. In particular, if $|f| = 0$, then the homomorphism $f$ is called homogeneous linear map of even degree. A Lie superalgebra homomorphism $f : L \to M$ is a homogeneous linear map of even degree such that $f[x, y] = [f(x), f(y)]$ holds for all $x, y \in L$. The notions of epimorphisms, isomorphisms and automorphisms have the obvious meaning. Throughout for superdimension of Lie superalgebra $L$ we simply write $\text{dim}(L) = (m \mid n)$, where $\text{dim } L_0 = m$ and $\text{dim } L_1 = n$.

Now, we define the Schur multiplier of Lie superalgebra. Like in Lie algebra case, we define the multiplier of Lie superalgebra $L$ using its free presentation. The free Lie superalgebra on a $\mathbb{Z}_2$-graded set $X = X_0 \cup X_1$ is a Lie superalgebra $F(X)$ together with a degree zero map $i : X \to F(X)$ such that if $M$ is any Lie superalgebra and $j : X \to M$ is a degree zero map, then there is a unique Lie superalgebra homomorphism $h : F(X) \to M$ such that $j = h \circ i$. The existence of free Lie superalgebra is guaranteed by an analogue of Witt’s theorem. If $L$ is a Lie superalgebra generated by a $\mathbb{Z}_2$-graded set $X = X_0 \cup X_1$ and $\phi : X \to L$ is a degree zero map, then we have free Lie superalgebra $F$ and $\psi : F \to L$ extending $\phi$. Let $R = \ker(\psi)$. The extension

$$0 \to R \to F \to L \to 0$$

is called a free presentation of $L$ and is denoted by $(F, \psi)$.

**Definition 1.1.** Given a free presentation,

$$0 \to R \to F \to L \to 0$$

of $L$ with $F$ a free Lie superalgebra, we define multiplier of $L$ as

$$\mathcal{M}(L) = \frac{[F, F] \cap R}{[F, R]}.$$

It is easy to see that the Schur multiplier of a Lie superalgebra $L$ is abelian and independent of the choice of the free presentation of $L$.

Extensions of Lie superalgebras are studied by several authors [14, 15]. An extension of a Lie superalgebra $L$ is a short exact sequence

$$0 \to M \xrightarrow{e} K \xrightarrow{f} L \to 0. \tag{1.1}$$
Since $e : M \rightarrow e(M) = \ker(f)$ is an isomorphism we will usually identify $M$ and $e(M)$. An extension of $L$ is then same as an epimorphism $f : K \rightarrow L$. A homomorphism from an extension $f : K \rightarrow L$ to another extension $f' : K' \rightarrow L$ is a Lie superalgebra homomorphism $g : K \rightarrow K'$ satisfying $f = f' \circ g$; in other words, we have the following commutative diagram.

$$
\begin{array}{ccc}
K & \xrightarrow{g} & K' \\
\downarrow f & & \downarrow f' \\
L & & L
\end{array}
$$

A central extension of $L$ is an extension (1.1) such that $M = \ker f \subseteq Z(K)$. The central extension is said to be a stem extension of $L$ if $M \subseteq Z(K) \cap K'$. The stem extension is maximal if every epimorphism of any other stem extension of $L$ on to $0 \rightarrow M \rightarrow K \rightarrow L \rightarrow 0$ is necessarily an isomorphism. Finally, we call the stem extension a stem cover if $M \cong M(L)$ and in this case $K$ is said to be a cover of Lie superalgebra $L$.

In 1940, P. Hall introduced an equivalence relation on the class of all groups called isoclinism, which is weaker than isomorphism and plays an important role in classification of finite $p$-groups. In 1994, K. Moneyhun [2] gave a Lie algebra analogue of the concept of isoclinism. Now we introduce the isoclinism for Lie superalgebras as follows.

**Definition 1.2.** Let $L$ and $K$ be two Lie superalgebras, $\alpha : \frac{L}{Z(L)} \rightarrow \frac{K}{Z(K)}$ and $\beta : L' \rightarrow K'$ be Lie superalgebra homomorphisms such that the following diagram is commutative

$$
\begin{array}{ccc}
\frac{L}{Z(L)} \times \frac{L}{Z(L)} & \xrightarrow{\phi} & L' \\
\downarrow \alpha \times \alpha & & \downarrow \beta \\
\frac{K}{Z(K)} \times \frac{K}{Z(K)} & \xrightarrow{\psi} & K'
\end{array}
$$

where $\phi : (l, m) \rightarrow [l, m]$ for $l, m \in L$ and similarly for $\psi : (r, s) \rightarrow [r, s]$ for $r, s \in K$. Or, equivalently $\alpha$ and $\beta$ are defined in such a way that they are compatible, i.e., $\beta([l, m]) = [k, r]$, where $l, k, m, r \in L$ in which $k \in \alpha(l + Z(L))$ and $r \in \alpha(m + Z(L))$. Then the pair $(\alpha, \beta)$ is called homoclinism and if they are both isomorphisms, then $(\alpha, \beta)$ is called isoclinism.

If $(\alpha, \beta)$ is an isoclinism between $L$ and $K$, then $L$ and $K$ are said to be isoclinic, which is denoted by $L \sim K$. Obviously isoclinism is an equivalence relation, and hence it produces a partition on the class of all Lie superalgebras into equivalence classes called isoclinism classes. We show that each isoclinism class contains a special Lie superalgebra called a stem Lie superalgebra, such that its center is contained in its derived subalgebra. For the Lie algebra case one can see in [2].

Unlike the case of groups it has been shown that all covers of a finite dimensional Lie algebras are isomorphic [2]. Also in [12] it is shown that each perfect Lie algebras, up to isomorphism has a unique cover. But for arbitrary Lie algebra it is an open problem whether the above result holds or not. However Salmekar et al [6] have shown covers are at least isoclinic, for any Lie algebra with finite dimensional Schur multiplier. Here one of our main result is to show the same result holds for any arbitrary Lie superalgebra $L$, with $\dim(M(L)) = (m|n)$.

The organization of the paper is as follows. In Section 2 we give some consequences about isoclinism of Lie superalgebras. Section 3 is devoted to study structure of all covers of Lie
superalgebras with some condition. Finally in section we give a necessary and sufficient condition for Lie superalgebra to be stem Lie superalgebra and further show one more interesting result on isoclinism of Lie superalgebras.

2. Some properties of Isoclinism of Lie superalgebras

Here we show some elementary results on isoclinism of Lie superalgebras. Specifically, we give a criterion when two Lie superalgebras are isoclinic which we use in proving one of our main result.

Lemma 2.1. If $L$ is a Lie superalgebra and $A$ be an abelian Lie superalgebra, then $L \sim L \oplus A$.

Proof. $A$ is abelian, clearly we have $Z(L \oplus A) = Z(L) \oplus A$. Now define the map

$$\alpha : \frac{L}{Z(L)} \rightarrow \frac{L \oplus A}{Z(L) \oplus A}$$

by

$$x + Z(L) \mapsto x + (Z(L) \oplus A),$$

for all $x \in L$. It is easy to check that the map is well defined. Any homogenous element $x$ of the quotient Lie superalgebra $\frac{L}{Z(L)}$ is an element $x \in \frac{L + Z(L)}{Z(L)}$ for $\gamma \in \mathbb{Z}_2$. Also any homogeneous element $y$ of $\frac{L \oplus A}{Z(L) \oplus A}$ is $y \in \frac{L \oplus A + Z(L) \oplus A}{Z(L) \oplus A}$ for $\gamma \in \mathbb{Z}_2$. So, $\alpha$ is a homogeneous linear map of degree 0 and for $x, y \in L$ we have, $\alpha([x + Z(L), y + Z(L)]) = [\alpha(x + Z(L)), \alpha(y + Z(L))]$ holds, implies that $\alpha$ is a homomorphism. Further $\alpha$ is also a bijection, hence an isomorphism. Now consider the map $\beta : L' \rightarrow (L \oplus A)' = L'$ is the identity map. Finally, from the above construction the diagram

$$\begin{array}{ccc}
\frac{L}{Z(L)} \times \frac{L}{Z(L)} & \rightarrow & L' \\
\downarrow \alpha \times \alpha & & \downarrow \beta \\
\frac{L \oplus A}{Z(L) \oplus A} \times \frac{L \oplus A}{Z(L) \oplus A} & \psi \rightarrow & L'
\end{array}$$

is commutative as required. □

Lemma 2.2. Let $L$ be a Lie superalgebra and $H$ be a sub superalgebra of $L$. Then $H \sim H + Z(L)$. In particular, if $L = H + Z(L)$ then $L \sim H$. Conversely, if $L/Z(L)$ is of dimension $(m|n)$ and $L \sim H$ then $L = H + Z(L)$.

Proof. At first $Z(H + Z(L)) = Z(H) + Z(L)$. Define the map

$$\alpha : \frac{H}{Z(H)} \rightarrow \frac{H + Z(L)}{Z(H) + Z(L)}$$

by

$$x + Z(H) \mapsto x + (Z(H) + Z(L))$$

for all $x \in L$ and $\beta : L' \rightarrow H'$ is the identity map. One can easily see that $\alpha$ and $\beta$ are Lie superalgebra isomorphisms such that the diagram
\[
\begin{array}{c}
\frac{H}{Z(H)} \times \frac{H}{Z(H)} \\
\downarrow \alpha \times \alpha \\
\frac{H + Z(L)}{Z(H) + Z(L)} \times \frac{H + Z(L)}{Z(H) + Z(L)} \\
\downarrow \beta \\
H'
\end{array}
\]

is commutative. Hence \( H \sim H + Z(L) \) and if \( L = H + Z(L) \) then \( H \sim L \).

Conversely suppose \( K = H + Z(L) \). Here \( K \subseteq K \) is a sub superalgebra of \( L \) and let \( K \) be a proper sub superalgebra. Since \( L \sim H \) and also \( H \sim H + Z(L) = K \), \( L \sim K \).

Thus we have \( \frac{K}{Z(K)} \cong \frac{K}{Z(K)} \) and hence \( \frac{K}{Z(K)} \) is a quotient Lie superalgebra of dimension \((m|n)\).

Again \( Z(L) \subseteq Z(K) \) and suppose \( \dim(Z(L)) = (r|s) \). Then \( \dim\left(\frac{K}{Z(K)}\right) \leq \dim\left(\frac{K}{Z(L)}\right) \).

We have \((m|n) \leq \dim\left(\frac{K}{Z(L)}\right) \leq \dim\left(\frac{K}{Z(L)}\right) = (m|n) \) and this implies that \( \dim\left(\frac{K}{Z(L)}\right) = (m|n) \).

Therefore, \( \dim K = (m + r|n + s) = \dim L \) which is a contradiction to our assumption, and hence \( K = L \) as required.

**Lemma 2.3.** Let \( L \) be a Lie superalgebra and \( I \) be a graded ideal. Then \( L/I \sim L/(I \cap L') \). In particular, if \( I \cap L' = 0 \) then \( L \sim L/I \). Conversely, if \( L' \) is finite dimensional and \( L \sim L/I \) then \( I \cap L' = 0 \).

**Proof.** Let us denote \( \bar{L} := \frac{L}{I} \) and \( \bar{L} := \frac{L}{I \cap L'} \). Define map \( \alpha \) as follows,

\[
\alpha : \frac{L}{Z(L)} \to \frac{\bar{L}}{Z(\bar{L})}
\]

\[
\bar{l} + Z(\bar{L}) \mapsto \bar{l} + Z(\bar{L})
\]

where \( \bar{l} \in \bar{L}, \bar{I} \in \bar{L} \) with \( l \in L \). Our first claim is \( \bar{l} \in \bar{Z(\bar{L})} \) if and only if \( \bar{l} \in \bar{Z(\bar{L})} \). Consider, \( \bar{l} \in Z(\bar{L}) \) then we have \( [\bar{l}, \bar{L}] = 0 \) so \([l, L] \in I \) and also \([l, L] \in I' \) implies \([l, L] \in I \cap L' \). Now for \( x \in L \)

\[
[\bar{l}, \bar{L}] = [l + L' \cap I, x + I \cap L']
\]

\[
= [l, x] + I \cap L'
\]

\[
= 0
\]

which shows \( \bar{l} \in Z(\bar{L}) \). Similarly, we can show the converse part. Let us take \( \bar{l} + Z(\bar{L}) = \bar{m} + Z(\bar{L}) \) where \( \bar{l}, \bar{m} \in \bar{L} \). Then \((l - \bar{m}) \in \bar{Z(\bar{L})} \) and by previous argument, we have \((l - \bar{m}) \in \bar{Z(\bar{L})} \).

This implies that \( \bar{l} + Z(\bar{L}) = \bar{m} + Z(\bar{L}) \). Thus, the map \( \alpha \) is well defined and \( \alpha \) is also a bijection.

Further the way \( \alpha \) is defined, evidently it is a homogeneous map of degree zero. Consider \( \bar{l} \in \bar{L} \) and \( \bar{m} \in \bar{L} \),

\[
\alpha([\bar{l} + Z(\bar{L}), \bar{m} + Z(\bar{L})]) = \alpha([\bar{l}, \bar{m}] + Z(\bar{L}))
\]

\[
= [\bar{l}, \bar{m}] + Z(\bar{L})
\]

\[
= [\alpha(\bar{l} + Z(\bar{L})), \alpha(\bar{m} + Z(\bar{L}))]
\]

so \( \alpha \) is a Lie superalgebra homomorphism. Define the map \( \beta : \bar{L} \to \bar{L}' \) as \( \beta(\bar{l}) = \bar{l} \) where \( \bar{l} \in \bar{L}' \). We can easily check \( \beta \) is well defined map and is a bijection. For
In this section at first we show that stem cover do exist for every Lie superalgebras. Further, we show that for any given Lie superalgebra with finite dimensional Schur multiplier, all of its covers are isoclinic. Also it is shown that each stem extension of a finite dimensional Lie superalgebra is a homomorphic image of its stem cover. This is a result similar to work of Yamazaki \[13\] for group case and to work by Salmekar et al \[6\] for Lie algebra case.

**Lemma 3.1.** Stem covers exist for each Lie superalgebras.

**Proof.** Let \( L = L_0 \oplus L_1 \) be a Lie superalgebra and let \( 0 \longrightarrow R \longrightarrow F \overset{\pi}{\longrightarrow} L \longrightarrow 0 \) be a free presentation of \( L \). Since \( \frac{R}{[R,F]} \) is a central graded ideal of \( \frac{F}{[R,F]} \), so \( 0 \longrightarrow \frac{R}{[R,F]} \overset{\pi}{\longrightarrow} \frac{F}{[R,F]} \longrightarrow 0 \) is a central extension of \( L \). Let us consider \( \frac{S}{[R,F]} \) be a complementary \( \mathbb{Z}_2 \)-graded space of \( M(L) \) in \( \frac{R}{[R,F]} \) for some graded ideal \( S \) in \( F \). Set \( K := \frac{F}{S} \) and \( M := \frac{R}{S} \) and hence \( \frac{F}{S} \cong \frac{R}{S} \cong \frac{E}{S} \) implying \( K \cong M \). Now

\[
M = \frac{R}{S} \cong \frac{R/[R,F]}{S/[R,F]} = \frac{M(L) \oplus S/[R,F]}{S/[R,F]} \cong M(L).
\]

Again \( M = \frac{R}{S} \subseteq \frac{F\cap R+S}{S} \subseteq \frac{F+S}{S} = \left( \frac{F}{S} \right)' = K' \) and also \( M = \frac{R}{S} \subseteq Z\left( \frac{F}{S} \right) = Z(K) \), i.e. \( M \subseteq Z(K) \cap K' \). Hence, \( 0 \longrightarrow M \longrightarrow K \longrightarrow L \longrightarrow 0 \) is the required stem cover we are looking for. \( \square \)

The following lemma plays an essential role in our investigations.

**Lemma 3.2.** Let \( 0 \longrightarrow R \longrightarrow F \overset{\pi}{\longrightarrow} L \longrightarrow 0 \) be a free presentation of a Lie superalgebra \( L \) and let \( 0 \longrightarrow M \overset{\theta}{\longrightarrow} K \longrightarrow L \longrightarrow 0 \) be a central extension of another Lie
superalgebra $L$. Then for each homomorphism $\alpha : L \rightarrow L$, there exists a homomorphism $\beta : F/[R,F] \rightarrow K$ such that $\beta(R/[R,F]) \subseteq M$ and the following diagram is commutative:

$$
\begin{array}{c}
0 & \rightarrow & R/[R,F] & \rightarrow & F & \rightarrow & L & \rightarrow & 0 \\
\downarrow \beta_1 & & \downarrow \beta & & \downarrow \alpha & & & & \\
0 & \rightarrow & M & \rightarrow & K & \rightarrow & L & \rightarrow & 0.
\end{array}
$$

(3.1)

Proof. We have $0 \rightarrow R/[R,F] \rightarrow F/[R,F] \pi \rightarrow L \rightarrow 0$ is a central extension of $L$. As $F$ is free, there is an unique Lie superalgebra homomorphism $F \rightarrow \bar{L}$. But since we have $F \pi \rightarrow L \alpha \downarrow \downarrow K \theta \rightarrow \bar{L}$, so there must exists an unique Lie superalgebra homomorphism $\beta : F \rightarrow K$ such that the following commutes:

$$
\begin{array}{c}
F & \rightarrow & L \\
\downarrow \beta' & & \downarrow \alpha \\
K & \rightarrow & \bar{L},
\end{array}
$$

so there must exists an unique Lie superalgebra homomorphism $\beta' : F \rightarrow K$ such that the following commutes:

$$
\begin{array}{c}
F & \rightarrow & L \\
\downarrow \beta' & & \downarrow \alpha \\
K & \rightarrow & \bar{L},
\end{array}
$$

i.e., $\alpha \circ \pi = \theta \circ \beta'$. Our claim is $\beta'$ induces the homomorphism $\beta$. We have $R = \ker \pi$ is a graded ideal, now let $x \in \beta'(R_\gamma)$, so $x = \beta'(r)$ for $r \in R_\gamma$. Since $M = \ker \theta$ is an graded ideal of $K$, take $\theta(x) = \theta(\beta'(r)) = \alpha(\pi(r)) = 0$ which implies $x \in M_\gamma$, i.e., $\beta'(R_\gamma) \subseteq M_\gamma$. Consider $[x,y]$ for $y \in F_\delta$ and $x \in R_\gamma$, then $\beta'([x,y]) = [\beta'(x),\beta'(y)]$. But we have $\beta'(x) \in M_\gamma$ and $M_\gamma \subseteq Z(K)_\gamma$, implies $\beta'([x,y]) = 0$. We get $\beta'([R,F]) = 0$, hence $\beta'$ induces the homomorphism $\beta$ as required. Clearly $\beta(R/[R,F]) \subseteq M$ implies $\beta_1 = \beta|R/[R,F]$ is the restriction map, hence a homomorphism such that the diagram in (3.1) commutes.

Here we proof another crucial result which we will be using repeatedly, and the idea of proof is similar to that of Salmekar et al [6].

**Theorem 3.3.** Let $L$ be a Lie superalgebra such that its Schur multiplier is of dimension $(m|n)$ and $0 \rightarrow M \rightarrow R \rightarrow F \pi \rightarrow L \rightarrow 0$ be a free presentation of $L$. Then the extension $0 \rightarrow M \rightarrow K \psi \rightarrow L \rightarrow 0$ is a stem cover of $L$ if and only if there exists a graded ideal $S$ in $F$ such that:

(1) $K \cong F_S$ and $M \cong R_S$

(2) $R_{[R,F]} = \mathcal{M}(L) \oplus S_{[R,F]}.$

Proof. Suppose $0 \rightarrow M \rightarrow K \psi \rightarrow L \rightarrow 0$ is a stem cover of $L$. Then by Lemma [3.2] we have a homomorphism $\beta : F_{[R,F]} \rightarrow K$ such that $\psi \circ \beta = \bar{\pi}$ and $\beta \left( R_{[R,F]} \right) \subseteq M$. Let $x$
be an arbitrary element of $K$, then $\psi(x) \in L$. As $\pi$ is onto we have $\pi(y + [R, F]) = \psi(x)$ where $y \in F$. Now

$$\pi(y + [R, F]) = \psi \circ \beta(y + [R, F]) = \psi(x),$$

implies that $\psi(\beta(y + [R, F]) - x) = 0$ and $\beta(y + [R, F]) - x \in \text{Ker}\psi = M$, hence $K = \text{Im}\beta + M$. Here $\text{Im}\beta$ is a graded subalgebra and $M$ is a graded ideal of $K$. Now $M \subseteq K' = (\text{Im} \beta)' \subseteq \text{Im} \beta$, so $\beta$ is onto and $(R_{[R, F]} + M) = M$. Set $\ker \beta = \frac{S}{[R, F]} = \ker \beta_1$ for some graded ideal $S$ of $F$. Then $K \cong \frac{F}{S}$ and $M \cong \frac{R}{S}$. Let $x, y$ be an arbitrary element of $K$ then $[x, y] \in K'$. Since $\beta$ is onto, $x = \beta(r + [R, F])$ and $y = \beta(t + [R, F])$ for some $r, t \in R$. Using the fact that $\beta$ is a Lie superalgebra homomorphism, $[x, y] = \beta([r, t] + [R, F]) \in K'$. Further from definition, $\mathcal{M}(L) = \frac{F \cap R}{[R, F]}$ and also $M \subseteq K'$, combinely we have

$$\beta(\mathcal{M}(L)) \subseteq \beta \left( \frac{R}{[R, F]} \right) \cap \beta \left( \frac{F'}{[R, F]} \right) \subseteq M \cap K' = M.$$

Conversely, let $x = \beta(t + [R, F])$ for $t \in F'$ and $r \in R$. Then $(t - r) + [R, F] \in \frac{S}{[R, F]} \subseteq \frac{R}{[R, F]}$. This means $t \in R$ and $x \in \beta(\mathcal{M}(L))$, which implies $M \subseteq \beta(\mathcal{M}(L))$. Hence $\beta(\mathcal{M}(L)) = M$, i.e. $\beta|_{\mathcal{M}(L)}$ restricts to homomorphism onto $M$. Evidently $\mathcal{M}(L)$ is a proper sub superalgebra of $\frac{R}{[R, F]}$.

Now our claim is $\mathcal{M}(L) \cap \frac{S}{[R, F]} = 0$. Clearly $\beta \left( \mathcal{M}(L) \cap \frac{S}{[R, F]} \right) = 0$. So, either $\frac{S}{[R, F]} \subseteq \mathcal{M}(L)$ or $\mathcal{M}(L) \cap \frac{S}{[R, F]} = 0$. If first one holds $\ker \beta|_{\mathcal{M}(L)} = \frac{S}{[R, F]}$.

We already have $M \cong \frac{R}{S} \cong \frac{R}{[R, F]} / \frac{S}{[R, F]}$ and now also $M \cong \mathcal{M}(L) / \frac{S}{[R, F]}$ which implies $\frac{R}{[R, F]} \cong \mathcal{M}(L)$. We have $\dim(\mathcal{M}(L)) = (m|n)$, implies $\dim(\frac{R}{[R, F]}) = (m|n)$ which is a contradiction. Thus $\mathcal{M}(L) \cap \frac{S}{[R, F]} = 0$. Further we have the onto homomorphism $\beta : \frac{R}{[R, F]} \to M$ with $\ker \beta = \frac{S}{[R, F]}$, so $\frac{R}{[R, F]} = \mathcal{M}(L) \oplus \frac{S}{[R, F]}$. Converse is clear from Lemma 3.1.

Following are some important consequences of Theorem 3.3.

**Corollary 3.4.** Let $L$ be a Lie superalgebra such that its Schur multiplier is of dimension $(m|n)$. Then all covers are isoclinic.

**Proof.** Let $0 \to R \to F \to L \to 0$ be a free presentation of $L$. We show that all covers of $L$ are isoclinic to a factor Lie superalgebra $\frac{F}{[R, F]}$. Let $0 \to M \to K \to L \to 0$ be a stem cover of $L$, i.e. the Lie superalgebra $K$ is any cover of $L$. By Theorem 3.3, we have

$$\frac{R}{[R, F]} = \mathcal{M}(L) \oplus \ker \beta.$$  \hfill (3.2)

where $\beta : \frac{F}{[R, F]} \to K$ is an onto homomorphism. Clearly $\ker \beta \cap \left( \frac{F}{[R, F]} \right)' = 0$ and hence using Corollary 2.4, $K \sim \frac{F}{[R, F]}$. \hfill $\square$

**Corollary 3.5.** Any finite dimensional Lie superalgebra has at least one cover.

**Proof.** Let $\frac{F}{R} \cong L$ be a free presentation of Lie superalgebra $L$ and $\mathcal{M}(L) = \frac{F' \cap R}{[R, F]}$. For some graded ideal $S$ of $F$, consider $\frac{S}{[R, F]}$ is complement of $\mathcal{M}(L)$ in $\frac{R}{[R, F]}$. Then by Theorem 3.3 we get $0 \to \frac{R}{S} \to \frac{F}{S} \to L \to 0$ is a stem cover for $L$. Hence $\frac{F}{S}$ is the required cover of $L$. \hfill $\square$

Here is one another important result which leads to some important corollaries.
Theorem 3.6. Let $0 \to M \to K \to L \to 0$ be a stem extension of finite dimensional Lie superalgebra $L$. Then there is a cover say $L^*$ of $L$ such that $K$ is a homomorphic image of $L^*$.

Proof. Let $0 \to R \to F \to L \to 0$ be a free presentation of Lie superalgebra $L$. Then by Theorem 3.3, there is an onto homomorphism $\beta : \frac{F}{[R,F]} \to K$ such that the following diagram commutes,

$$
\begin{array}{cccccc}
0 & \to & \frac{R}{[R,F]} & \overset{\beta}{\to} & \frac{F}{[R,F]} & \overset{\#}{\to} & L & \to & 0 \\
& & \beta_1 \downarrow & & \beta \downarrow & & \theta_1 \downarrow & & \theta \downarrow & & 1_L \\
0 & \to & M & \overset{\beta}{\to} & K & \overset{\theta}{\to} & L & \to & 0
\end{array}
$$

(3.3)

where $\beta_1 = \beta|_{\frac{R}{[R,F]}}$. Set $\ker \beta = \frac{T}{[R,F]}$ for some grade ideal $T$ of $R$, clearly $\ker \beta_1 = \frac{T}{[R,F]} \cap \frac{R}{[R,F]} = \frac{R}{[R,F]}/[R,F]$. Now,

$$
\frac{R}{T} \cong \frac{R/([R,F])}{T/([R,F])} \cong M.
$$

Also in proof of Theorem 3.3 we have seen $\beta|_{\mathcal{M}(L)}$ is an onto homomorphism from $\mathcal{M}(L)$ onto $M$. Here $\ker \beta|_{\mathcal{M}(L)} = \left(\frac{(R \cap F')}{[R,F]}\right) \cap \frac{T}{[R,F]} = \frac{T \cap F'}{[R,F]}$ as $T$ is a graded ideal of $R$. So,

$$
M \cong \frac{(R \cap F')/([R,F])}{T \cap (R \cap F')} \cong \frac{R \cap F'}{T \cap (R \cap F')} \cong \frac{(R \cap F') + T}{T}.
$$

Since $M$ is of finite dimension, we have $R = (R \cap F') + T$. Suppose $\frac{S}{[R,F]}$ is $\mathbb{Z}_2$-graded complement subspace of $\frac{(R \cap F')}{[R,F]}$ in $\frac{T}{[R,F]}$, then $S \cap (R \cap F') = [R,F]$ and $R \cap F' + S = R$ which imply that $\frac{R}{[R,F]} = \mathcal{M}(L) \oplus \frac{S}{[R,F]}$. Thus by Theorem 3.3 $\frac{F}{S} : L^*$ is a cover of $L$. Moreover $K \cong \frac{F}{T} \cong \frac{L^*}{T/S}$, $K$ is a homomorphic image of $L^*$.

\[ \square \]

Corollary 3.7. The maximal stem extensions of Lie superalgebras are precisely the same as its stem covers.

Proof. Let $0 \to M \to K \to L \to 0$ be maximal stem extension of $L$. By Theorem 3.6 we have a stem cover $0 \to M^* \to L^* \to L \to 0$ of $L$ such that $\phi : L^* \to K$ is a homomorphism, hence necessarily an isomorphism.

\[ \square \]

Corollary 3.8. Let $0 \to M_i \to K_i \to L \to 0$ here $i = 1,2$, be two maximal stem extensions of a Lie superalgebra $L$ of finite dimension, then $\dim K_1 = \dim K_2 = (m|n)$ (say).

4. Structure of stem Lie superalgebra

Stem Lie algebras are first introduced and studied by Moneyhun [2]. Here we define stem Lie superalgebras.

Definition 4.1. A Lie superalgebra $L$ is called stem Lie superalgebra whenever $Z(L) \subseteq L'$.

Lemma 4.2. Suppose $\mathcal{C}$ is an isoclinic family of Lie superalgebras. Then

1. $\mathcal{C}$ contains a stem Lie superalgebra.
2. Each finite dimensional Lie superalgebra $T \in \mathcal{C}$ is stem if and only if $T$ has minimal dimension in $\mathcal{C}$.
Proof. Let $L = L_0 \oplus L_1$ be a Lie superalgebra and $L \in \mathcal{C}$. So $Z(L) \cap L' = L_0 \cap (Z(L) \cap L') \oplus L_1 \cap (Z(L) \cap L')$ is an graded ideal of $L$. Consider $S = L_0 \cap S \oplus L_1 \cap S$ be a $\mathbb{Z}_2$-graded vector space complement of $Z(L) \cap L'$ in $Z(L)$, i.e., $Z(L) = Z(L) \cap L \oplus S$. Clearly $S \cap L' = 0$. Now $S$ is $\mathbb{Z}_2$-graded vector subspace of $L$ and also $[S, L] \subseteq S$, making $S$ an graded ideal of $L$. Denote $\frac{L}{S} =: T$ and using Lemma 2.3, we have $T \sim L$. So $T \in \mathcal{C}$ and further

$$Z(T) = Z\left(\frac{L}{S}\right) = \frac{Z(L) + S}{S} = \left(\frac{Z(L) \cap L' + S}{S}\right) = \frac{S + Z(L) \cap L'}{S} \subseteq \frac{S + L'}{S} = \left(\frac{L}{S}\right)' = \left(\frac{L}{S}\right)^{'\prime}.$$ 

Hence $T$ is a stem Lie superalgebra which proves (1).

Consider $T \in \mathcal{C}$ and $T$ is stem Lie superalgebra with $\dim T = (m|n)$. Let $L \in \mathcal{C}$. Consider

$$\frac{L'}{L' \cap Z(L)} \cong \frac{L' + Z(L)}{Z(L)} = \left(\frac{L}{Z(L)}\right)' \cong \left(\frac{T}{Z(T)}\right)' = \frac{T' + Z(T)}{Z(T)} \cong \frac{T'}{T' \cap Z(T)} = \frac{T'}{Z(T)}$$

and also we have $L' \cong T'$. Hence $\dim Z(T) = \dim (Z(L) \cap L') = (r|s)$ (say) with $\dim Z(T)_0 = \dim (Z(L) \cap L')_0 = r$ and $\dim Z(T)_1 = \dim (Z(L) \cap L')_1$. Now $\dim Z(T) = \dim (Z(L) \cap L') = (r|s) \leq \dim Z(L) = (p|q)$ where $r \leq p$ and $s \leq q$. As $\frac{L}{Z(L)} \cong \frac{T}{Z(T)}$, so $\dim \frac{L}{Z(L)} = \dim \frac{T}{Z(T)} = (m - r|n - s)$. Hence,

$$\dim T = (m - r|n - s) + \dim Z(T) \leq (m - r|n - s) + \dim Z(L) = (m - r + p|n - s + q) = \dim L$$

i.e. $\dim T_0 \leq \dim L_0$ and $\dim T_1 \leq \dim L_1$, i.e. $T$ is of minimum dimension as required.

Conversely let $T \in \mathcal{C}$ and is of minimum dimension, our claim is $T$ is a stem Lie superalgebra. We can find a complementary $\mathbb{Z}_2$-graded vector subspace $R$ of $Z(T) \cap T'$ in $Z(T)$. Clearly $R \cap T' = 0$ and $R$ is an ideal in $T$. So $\frac{T}{R} \sim T$. But since $\dim T$ is minimum in $\mathcal{C}$, dim $R = 0$ implies $R = 0$. We have $Z(T) = Z(T) \cap T'$, i.e. $Z(T) \subseteq T'$ as required. □

A famous result of Schur is that if $G/Z(G)$ is finite than, so is $G'$. An analogue result of Schur for Lie algebra case is if $L/Z(L)$ is finite dimensional than so is $L'$, which is also well known. Further Niroomand [10] proved a converse to Schur’s theorem. We prove an analogous result of Niroomand for Lie superalgebra case here as converse of Schur’s theorem.
Theorem 4.3. Suppose $L = L_0 \oplus L_1$ is a Lie superalgebra. If $L/Z(L)$ is finitely generated and $L'$ is finite dimensional, then $L/Z(L)$ has dimension finite. More precisely, 

$$\dim (L/Z(L)) \leq (m + n) \dim L'$$

where $m + n$ is the minimum number of generators of $L/Z(L)$.

Proof. Let $\{x_1 + Z(L), \ldots, x_m + Z(L); x_{m+1} + Z(L), \ldots, x_{m+n} + Z(L)\}$ be a generating set for $L/Z(L)$ as a $\mathbb{Z}_2$-graded Lie algebra where $x_i \in L_0$ for $1 \leq i \leq m$ and $x_{m+j} \in L_1$ for $1 \leq j \leq n$. Let us define the map

$$f : L/Z(L) \to L' \oplus \cdots \oplus L'$$

$$y + Z(L) \mapsto ([x_1, y], \ldots, [x_m, y], [x_{m+1}, y], \ldots, [x_{m+n}, y]),$$

where $y \in L_0$, for $\gamma \in \mathbb{Z}_2$. One can easily check that the map is well defined. Next our claim is that $f$ is an injective linear transformation. Let us consider for any homogeneous elements $y, t$ of $L$, we have $f(y + Z(L)) = f(t + Z(L))$. That implies $[y, x_i] = [t, x_i]$ for $1 \leq i \leq m + n$. So, $[y - t, x_i] = 0,$ i.e., $x_i$’s are in the centralizer of $(y - t)$ in $L$. Again $L$ is generated by $x_i$ for $1 \leq i \leq m + n$ mod $Z(L)$. Combining these facts $y - t \in Z(L)$, i.e. $f$ is injective. Hence

$$\dim(L/Z(L)) \leq (m + n) \dim L'$$

as required. \qed

Finally using the previous two results we have the following theorem.

Theorem 4.4. Let $L$ be a Lie superalgebra such that $L/Z(L)$ is finitely generated. If $\dim L' = (r|s)$, then $L$ is isoclinic to a finite dimensional Lie superalgebra.

Proof. Let $C$ be an isoclinism class of Lie superalgebras with $L \in C$. Using Lemma 4.2, $L$ is isoclinic to a stem Lie superalgebra say $T$. We have $Z(T) \subseteq T'$ and $T' \cong L'$, so $\dim T' = (r|s)$. It follows $\dim Z(T) \leq r + s$. Further by Theorem 4.3, $L/Z(L)$ is finite dimensional. Suppose $\dim(L/Z(L)) = (m|n)$. This implies $\dim(T/Z(T)) = (m|n)$ as $T/Z(T) \cong L/Z(L)$. So, $T$ is finite dimensional, specifically $\dim T \leq m + r + n + s$ and we are done. \qed

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