QUOTIENT RINGS OF INTEGERS FROM A METRIC POINT OF VIEW

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Abstract. The theory of Gromov-Hausdorff convergence is applied to sequences of quotient rings of integers. It is shown the existence of limit rings (fields) as the Gromov-Hausdorff limits of sequences of metric quotient rings. The relation of these constructions with the field of the reals $\mathbb{R}$ is discussed, showing that they are dense in $\mathbb{R}$ but that they cannot be identified with the real field or with the rational field $\mathbb{Q}$, at least when $\mathbb{R}$ and $\mathbb{Q}$ are endowed with the usual metric structures. It is also shown that the limit rings can be endowed with an order relation.

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1. INTRODUCTION

In this article quotient rings of integers and their limits when the number of elements goes to infinite are investigated by means of methods of metric geometry. The questions considered are originally motivated in a context of certain theory of dynamical systems. According to the general theory of non-reversible dynamics developed in [2], time parameters are described by subsets of ordered fields. Subsects of the rational numbers $\mathbb{Q}$ or subsets of the real numbers $\mathbb{R}$ provide natural candidates for time parameters, but since the theory developed had a general scope, other discrete fields were also considered. Since the prime fields $\mathbb{Z}/p\mathbb{Z}$ with $p$ prime are endowed with a particular form of order relation, the prime fields are admissible to provide time parameter sets. Furthermore, we were interested in the case where the time parameter sets are equipped with a notion of infinitesimal time lapse, namely a time interval which is small compared with any macroscopic time interval for all practical purposes and that can be used in a definition of infinitesimal time difference. Therefore, in the context of the theory prime fields $\mathbb{Z}/p\mathbb{Z}$, it is natural to consider the limit $p \to +\infty$ for such purposes, since there is an infinitesimal time lapse given by the expression $1/(p - 1)$. 
The construction of the limits \( p \to +\infty \) of quotient rings \( \mathbb{Z}/p\mathbb{Z} \) or, in the case when \( p \) takes values on the set of prime numbers, prime quotient fields, is investigated in this paper. The main tool that we apply to show the existence of the limits is the theory of Gromov-Hausdorff convergence, as discussed in reference [1], Chapter 7. As a result, we will show the existence of certain rings obtained as limits in the sense of Gromov-Hausdorff of sequences composed by quotient rings endowed with metric structures. The limits provide examples of novel, discrete, non-finite, ordered rings.

Several questions arise on the nature and properties of the limit rings and fields that go beyond the metric point of view. In particular, the identification of the limit fields is an important open question. We show that for the case of successions of prime fields, the Gromov-Hausdorff limit fields, generically denoted by \( \mathbb{Z}/\infty\mathbb{Z} \), are fields dense in \( \mathbb{R} \), but that they cannot be identified with \( \mathbb{Q} \) or with \( \mathbb{R} \).

Besides the possible interest from the point of view of number theory, the Gromov-Hausdorff limit fields admit arbitrarily close approximations in the Gromov-Hausdorff distance topology, whose minimal time lapse is of the form \( \delta t_p = 1/(p - 1) \). For large \( p \), the time lapse \( \delta t_p \) can be arbitrarily small compared with the other scale that appear in the theory, namely the diameter of \( \mathbb{Z}/p\mathbb{Z} \), that for the metric used in the theory, is equal to 1 for every \( p \in \mathbb{N} \setminus \{1\} \). This construction provides a notion of infinitesimal time lapse, necessary for the formulation of differential equations in dynamical theories.

2. Preliminary considerations

Definition of the metric in \( \mathbb{Z}/p\mathbb{Z} \). The quotient ring \( \mathbb{Z}/p\mathbb{Z} \) is determined by the composition rules of the aggregate of equivalence classes \([k]\) module \( p \), \( [k] := \{n \in \mathbb{Z} \text{ s.t. } n - k = q p, q \in \mathbb{Z}\} \) with \( k \in \{0, 1, 2, ..., p - 1\} \). For each \( p \in \mathbb{N} \) there is a natural metric function \( d_p : \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \to \mathbb{R} \) defined by the expression

\[
(2.1) \quad d_p([n], [m]) := |n_0 - m_0|, \quad n_0 \in [n], \quad m_0 \in [m], \quad 0 \leq n_0, m_0 \geq p - 1.
\]

That the function \( d_p \) satisfies the axioms of a metric follows from the properties of the usual metric function on \( \mathbb{R} \) determined by the modulus function \(|\cdot| : \mathbb{R} \to \mathbb{R}^+ \cup \{0\} \). In this way, the distance associated to the metric function \( d_p \) between \([n]\) and \([m]\) in \( \mathbb{Z}/p\mathbb{Z} \) is \( d_p([n], [m]) \) and the norm of a class \([n]\) is just \( \|\cdot\|_p := d_p([0], [n]) \).

The following properties of the distance function \( d_p \) are easily proved:

1. The minimal distance \( d_{\min} := \min\{d_p([n], [m]) \mid [n], [m] \in \mathbb{Z}/p\mathbb{Z}\} \) is equal to 1 for any \( p \in \mathbb{N} \) greater or equal to 2.
(2) For any \( p \in \mathbb{N} \) greater or equal to 2 the diameter of \( \mathbb{Z}/p\mathbb{Z} \) is the maximum distance between elements in \( \mathbb{Z}/p\mathbb{Z} \),
\[
\text{diam}(\mathbb{Z}/p\mathbb{Z}) := \max\{d_p([n], [m]), [n], [m] \in \mathbb{Z}/p\mathbb{Z}\}.
\]
It follows that \( \text{diam}(\mathbb{Z}/p\mathbb{Z}) = p - 1 \).

**Normalized metric.** The normalized metric in \( \mathbb{Z}/p\mathbb{Z} \) is defined as
\[
(2.2) \quad \tilde{d}_p : \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \to \mathbb{R}, \quad ([n], [m]) \mapsto \frac{1}{\text{diam}(\mathbb{Z}/p\mathbb{Z})} d_p([n], [m]).
\]
For this metric function the following properties hold good:
- The quotient rings \( \mathbb{Z}/p\mathbb{Z} \) have finite diameters, \( \text{diam}_{\tilde{d}_p}(\mathbb{Z}/p\mathbb{Z}) = 1, \forall \ p \in \mathbb{N}\setminus\{1\} \). Therefore, the spaces \( \{\mathbb{Z}/p\mathbb{Z}, \tilde{d}_p\}, p \in \mathbb{N}\} \) cannot be points (according to the characterization 7.4.6 in [1]), since the diameter of each \( \mathbb{Z}/p\mathbb{Z} \) is non-zero.
- Since each metric space \( (\mathbb{Z}/p\mathbb{Z}, d_p) \) is compact, then \( (\mathbb{Z}/p\mathbb{Z}, \tilde{d}_p) \) cannot be identified with \( \mathbb{R} \) endowed with its usual metric structure.
- On each \( \mathbb{Z}/p\mathbb{Z} \) the minimal distance possible by using the metric \( \tilde{d}_p \) between elements of \( \mathbb{Z}/p\mathbb{Z} \) is given by \( 1/(p - 1) \). Therefore,
\[
\lim_{p \to +\infty} \text{d}_{\text{min}}(\mathbb{Z}/p\mathbb{Z}) = \lim_{p \to +\infty} \frac{1}{p - 1} = 0.
\]

2.1. **Order relation in \( \mathbb{Z}/p\mathbb{Z} \).** The ring \( \mathbb{Z}/p\mathbb{Z} \) can be endowed with an order relation in the following way.

**Definition 2.1.** If \([n], [m] \in \mathbb{Z}/p\mathbb{Z}\), we say that
\[
[n] \leq [m] \text{ if and only if } n_0 \leq m_0,
\]
with \( n_0 \in [n] \), \( m_0 \in [m] \), \( 0 \leq n_0, m_0 \leq p - 1 \).

It is direct that the relation \([n] \leq [m]\) as defined above determines a total order in \( \mathbb{Z}/p\mathbb{Z} \). Note that the total order relation determined by **Definition 2.1** is not unique. There are other possible total order relations defined in similar ways but that make use of different elements of each equivalence class in \( \mathbb{Z}/p\mathbb{Z} \).

Let us also remark that the order relations determined by the **Definition 2.1** are not compatible with the arithmetic operations of multiplication and addition of the ring \( \mathbb{Z}/p\mathbb{Z} \). However, the order relation defined above is compatible with the above definition of distance \( d_p \), because if \( d_p([0], [n]) < d_p([0], [m]) \), then \([n] < [m]\).
3. Gromov-Hausdorff distance between quotient rings

The main result of this paper comes from the application of Gromov-Hausdorff convergence theory to the class of compact metric spaces

\[ \mathcal{C} = \{(\mathbb{Z}/p\mathbb{Z}, \tilde{d}_p), p \in \mathbb{N} \setminus \{1\}\}. \]

The Gromov-Hausdorff distance between the metric spaces \((\mathbb{Z}/p_i\mathbb{Z}, d_{Z_{p_i}})\) and \((\mathbb{Z}/p_j\mathbb{Z}, d_{Z_{p_j}})\) is bounded by applying the corresponding definition given in \([1]\), section 7.3. Let us consider the isometries defining the metric functions \(d_p : \mathbb{Z}/p \to \mathbb{R}\), namely the maps

\[ v_p : \mathbb{Z}/p \to \mathbb{R}, [n] \mapsto n_0, \text{s.t. } 0 \leq n_0 \leq p - 1, n_0 \in [n] \subset \mathbb{R}. \]

With the standard metric structure on \(\mathbb{R}\) provided by the module function \(|\cdot| : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}\), \(v_p\) is an isometric embedding of \(\mathbb{Z}/p\mathbb{Z}\) in \(\mathbb{R}\). The Hausdorff distance between \(v(\mathbb{Z}/p_i\mathbb{Z}) \subset \mathbb{R}\) and \(v(\mathbb{Z}/p_j\mathbb{Z}) \subset \mathbb{R}\) can be evaluated for any pair \(p_i, p_j \in \mathbb{N}\) and is given by

\[ d_H(v_{p_i}(\mathbb{Z}/p_i\mathbb{Z}), v_{p_j}(\mathbb{Z}/p_j\mathbb{Z})) = |p_i - p_j|. \]

From the geometric representation of the above isometric embedding and from the notion of Gromov-Hausdorff distance, the Gromov-Hausdorff distance between the compact, metric spaces \((\mathbb{Z}/p_i\mathbb{Z}, d_{Z_{p_i}})\) and \((\mathbb{Z}/p_j\mathbb{Z}, d_{Z_{p_j}})\) is

\[ d_{GH}((\mathbb{Z}/p_i\mathbb{Z}, d_{Z_{p_i}}), (\mathbb{Z}/p_j\mathbb{Z}, d_{Z_{p_j}})) \leq |p_i - p_j|. \]

Let us consider the analogous construction in the case when the rings \(\mathbb{Z}/p\mathbb{Z}\) are endowed with metric structures associated to the normalized distance function \(\tilde{d}_p\) on each \((\mathbb{Z}/p\mathbb{Z}, d_{p}) \in \mathcal{C}\). In this case, isometric embeddings are given in the form \(\tilde{v}_p : \mathbb{Z}/p\mathbb{Z} \to \mathbb{R}, [n] \mapsto \frac{n_0}{p - 1}\), where \(n_0\) is the represent of \([n]\) in the interval \([0, p - 1]\). It is easy to observe that then the Hausdorff distance is given by the relation

\[ d_H(\tilde{v}_{p_i}(\mathbb{Z}/p_i\mathbb{Z}), \tilde{v}_{p_j}(\mathbb{Z}/p_j\mathbb{Z})) = \left| \frac{1}{p_i - 1} - \frac{1}{p_j - 1} \right|. \]

Furthermore, there is the bound

\[ d_H(\tilde{v}_{p_i}(\mathbb{Z}/p_i\mathbb{Z}), \tilde{v}_{p_j}(\mathbb{Z}/p_j\mathbb{Z})) \leq \frac{|p_i - p_j|}{\min\{p_i - 1, p_j - 1\}}. \]

The corresponding Gromov-Hausdorff distance associated to \(\tilde{d}_p\) is such that

\[ d_{GH}((\mathbb{Z}/p_i\mathbb{Z}, \tilde{d}_{p_i}), (\mathbb{Z}/p_j\mathbb{Z}, \tilde{d}_{p_j})) \leq \frac{|p_i - p_j|}{\min\{diam_{d_{p_i}}(\mathbb{Z}/p_i\mathbb{Z}), diam_{d_{p_j}}(\mathbb{Z}/p_j\mathbb{Z})\}}. \]
For the purposes of this paper, the convergence properties associated to the metrics $d_{GH}$ are more adequate than the metrics $d_{GH}$. Therefore, we will consider the convergence properties of $d_{GH}$ instead than the properties of $d_{GH}$.

Let us recall that a subset $\mathcal{N}$ of a metric space $X$ is an $\epsilon$-net if $d(x, \mathcal{N}) \leq \epsilon, \forall x \in X$. Then following Burago et al. (II, section 7.4) we consider the following definition,

**Definition 3.1.** A class $\Upsilon$ of compact metric spaces is uniformly totally bounded if the following two conditions hold:

1. There is a constant $\Delta > 0$ such that $\text{diam}(X) \leq \Delta$ for all $X \in \Upsilon$.
2. For every $\epsilon > 0$ there exists a natural number $N = N(\epsilon)$ such that every $X \in \Upsilon$ contains an $\epsilon$-net with no more than $N(\epsilon)$ points.

The following result is key for our considerations [I],

**Theorem 3.2.** Any uniformly totally bounded class $\Upsilon$ of compact metric spaces is pre-compact in the Gromov-Hausdorff topology. That is, any sequence of $\Upsilon$ contains a convergent sub-sequence.

The uniformly totally bounded property and the corresponding result on convergence of subsequences can be applied to sequences of finite rings in $\mathcal{C}$. In particular, let us consider the collection of metric spaces

$$\mathcal{C}_{\epsilon_0} := \{ (\mathbb{Z}/p\mathbb{Z}, \tilde{d}_p) \text{ s.t. } p > p_0 \text{ and } \frac{1}{p-1} < \epsilon_0 \}$$

For $\epsilon < \epsilon_0$ one has that $\mathcal{C}_\epsilon \subset \mathcal{C}_{\epsilon_0}$. Then we have the following result,

**Proposition 3.3.** For every $\epsilon_0 > 0$ the class $\mathcal{C}_{\epsilon_0}$ is uniformly totally bounded.

**Proof.** Condition (1) in **Definition 3.1** holds good, since $\text{diam}_{\tilde{d}_p}(\mathbb{Z}/p\mathbb{Z}) = 1$ for all $p \in \mathbb{N} \setminus \{1\}$.

In order to show that also the condition (2) is satisfied, for a given $\epsilon$ we distinguish two cases:

1. Let $p$ be a natural number such that

$$\tilde{d}_p([\alpha], \mathbb{Z}/p\mathbb{Z}) \leq \frac{1}{\text{diam}_{\tilde{d}_p}(\mathbb{Z}/p\mathbb{Z})} = \frac{1}{p-1} \leq \epsilon < \epsilon_0,$$

for each $[\alpha] \in \mathbb{Z}/p\mathbb{Z}$. Such a choice of $p$ satisfying $\frac{1}{p-1} \leq \epsilon$ implies that $\mathcal{N}(\epsilon) = \mathbb{Z}/p\mathbb{Z}$ is an $\epsilon$-net with no more than $p$ points.

2. If $p' > p$ we took $\mathcal{N}_{p'}$ to be the most equidistant distribution of $p$ points in $\mathbb{Z}/p'\mathbb{Z}$. In this case, we have

$$\tilde{d}_{p'}([\alpha]', \mathbb{Z}/p'\mathbb{Z}) \leq \frac{|p'|}{p'-1} = \frac{p' - r_0}{p' - 1} \frac{1}{p},$$
where \([p'/p]\) is the integer part of the quotient \(p'/p\) and \(r_0\) depends on \(p'\) and \(p\) and range in \(\{0, \ldots, p - 1\}\). It is direct that for a given \(\epsilon > 0\) we can choose \(p\) such that
\[
\tilde{d}_{p'}([a]', \mathbb{Z}/p'\mathbb{Z}) \leq 1/(p - 1) \leq \epsilon,
\]
from which follows that \(\mathcal{N}_{p'}\) is an \(\epsilon\)-net with \(p\) points, where \(p\) is a function of \(\epsilon_0\).

Therefore, for each \(p \in \mathbb{N}\) the \(\epsilon_0\)-net defined as discussed above contains a maximum of \(p\) points.

\[\square\]

Although strictly speaking we did not show that the property "2" of an \(\epsilon\)-net holds for any \(\epsilon\), we showed it holds for arbitrarily small \(\epsilon\), which is enough for our purposes. As a consequence of Proposition 3.3, Theorem 3.2 applies to the class \(\mathcal{C}_{\epsilon_0}\), since to find a convergent sub-sequence in \(\mathcal{C}_{\epsilon_0}\), the initial terms for \(p\) small are irrelevant. Therefore, we have the following result:

**Theorem 3.4.** For any \(\epsilon_0\), the class of compact spaces \(\mathcal{C}\) contains convergent sequences in the Gromov-Hausdorff topology of compact metric spaces.

By definition of the Gromov-Hausdorff convergence, if the subsequence \(\{S_n\} \subset \mathcal{C}\) is convergent, then there exist the limits
\[
\lim_{n \to +\infty} \tilde{d}_n := \tilde{d}_\infty, \quad \lim_{n \to +\infty} (\mathbb{Z}/n\mathbb{Z}) := \mathbb{Z}/\infty\mathbb{Z}
\]
and the limit metric quotient rings is
\[
(\mathbb{Z}/\infty\mathbb{Z}, \tilde{d}_\infty) := \lim_{n \to +\infty} (\mathbb{Z}/p_n\mathbb{Z}, \tilde{d}_{p_n}).
\]

Let us now consider the map
\[
(3.4) \quad \text{diam}_{\tilde{d}} : \mathcal{C} \to \mathbb{R}, \quad (\mathbb{Z}/p\mathbb{Z}, \tilde{d}_p) \mapsto \text{diam}_{\tilde{d}}(\mathbb{Z}/p\mathbb{Z}).
\]
Since the map \(\text{diam}_{\tilde{d}} : \mathcal{C} \to \mathbb{R}\) is constant, for a given convergent subsequence \(\{S_n\} \subset \mathcal{C}\) it holds that
\[
\text{diam}_{\tilde{d}_\infty} (\lim_{n \to +\infty} \mathbb{Z}/n\mathbb{Z}) = \lim_{n \to +\infty} \text{diam}_{\tilde{d}_n} (\mathbb{Z}/n\mathbb{Z}) = 1.
\]

Then we have that

**Lemma 3.5.** For any convergent subsequence \(\{S_n\} \subset \mathcal{C}\) convergent to the limit space \((\mathbb{Z}/\infty\mathbb{Z}, \tilde{d}_\infty)\), the relation
\[
(3.5) \quad \text{diam}_{\tilde{d}_\infty} (\mathbb{Z}/\infty\mathbb{Z}) = 1
\]
holds good.

Since the diameter \(\text{diam}_{\tilde{d}_\infty} (\mathbb{Z}/\infty\mathbb{Z})\) is finite but it is not zero, we have
Corollary 3.6. Given a convergent subsequence $\{S_n\} \subset C$, the limit space $\mathbb{Z}/\infty\mathbb{Z}$ is compact but it is not a point.

3.1. **Application to prime fields.** Let us consider the collection of prime fields as metric spaces,

$$\tilde{\mathcal{P}} := \{(\mathbb{Z}/p_i\mathbb{Z}, \tilde{d}_{\mathbb{Z}p_i}), \ p_i \in \Pi\},$$

where $\Pi \subset \mathbb{N}$ denotes the aggregate of the increasing prime numbers. Given the collection $\tilde{\mathcal{P}}$, there exist convergent subsequences in $\tilde{\mathcal{P}}$ in the Gromov-Hausdorff sense,

**Theorem 3.7.** The collection $\tilde{\mathcal{P}}$ contains a convergent subsequence of metric compact spaces.

The proof mimic the same than the one of Theorem 3.4 and it is not necessary to repeat the argument.

We would like to remark here two points. First, a direct construction of the set $\mathbb{Z}/\infty\mathbb{Z}$ as an aggregate of limits is not possible by the means considered until now, since the set $\mathbb{Z}/\infty\mathbb{Z}$ contains more elements than any $\mathbb{Z}/p\mathbb{Z}$ of the given sequence $S$. Second, one can consider the question of uniqueness of the limit $\mathbb{Z}/\infty\mathbb{Z}$ as independent of the convergent subsequence $\{S_n\} \subset \mathcal{P}$ the sense of metric spaces. In general, one should not expect uniqueness, but the specific example described below shows that there can be exceptions.

4. **Ring and field structures defined in the limits $\mathbb{Z}_\infty$**

A natural question that arises is related with the existence of algebraic structures in the limit space $\mathbb{Z}/\infty\mathbb{Z}$:

- Is $\mathbb{Z}/\infty\mathbb{Z} = \lim_{n \to +\infty} S_n$, with $S \subset \mathcal{C}$ a convergent subsequence, furnished with convenient operations, a ring?
- Is $\mathbb{Z}/\infty\mathbb{Z} = \lim_{n \to +\infty} S_n$ with $S \subset \tilde{\mathcal{P}}$ a convergent subsequence, furnished with the convenient operations, a field?

In order to address these questions we generalize the metrics structures considered above to the case of metric on product spaces.

**Lattices.** Let us consider the product $\mathcal{L} = \mathbb{Z}/p_1\mathbb{Z} \times \mathbb{Z}/p_2\mathbb{Z}$ endowed with a distance function $\tilde{d}$ defined by

$$\tilde{d}_{\mathcal{L}} : \mathcal{L} \times \mathcal{L} \to \mathbb{R}, \ (([\alpha_1], [\alpha_2]), ([\lambda_1], [\lambda_2])) \mapsto \sqrt{\tilde{d}^2_{\mathbb{Z}p_1}([\alpha_1], [\lambda_1]) + \tilde{d}^2_{\mathbb{Z}p_2}([\alpha_2], [\lambda_2])}.$$ 

The generalization of this metric structure to product spaces $\mathcal{L} := \prod_{i=1}^{n} \mathbb{Z}/p_i\mathbb{Z}$ is direct.
The following results answer positively the above questions.

**Theorem 4.1.** Let \((\mathbb{Z}/\infty \mathbb{Z}, \tilde{d}_\infty)\) be the limit in the Gromov-Hausdorff sense of a convergent subsequence in \(C \times C\). Then the limit space \(\mathbb{Z}/\infty \mathbb{Z}\) can be furnished with a ring structure by the limiting operations \(+_\infty\) and \(\cdot_\infty\) to be defined below. If the convergent subsequence \(S\) is in \(\tilde{P}\), then \((\mathbb{Z}/\infty \mathbb{Z}, +_\infty, \cdot_\infty)\) is endowed with operations making it a field.

**Proof.** Let us consider a convergent subsequence \(\{(\mathbb{Z}/p \mathbb{Z}, \tilde{d}_p)\}\) of \(C\) converging to \((\mathbb{Z}/\infty \mathbb{Z}, \tilde{d}_\infty)\). Then the algebraic addition operation \(+_p : \mathbb{Z}/p \mathbb{Z} \times \mathbb{Z}/p \mathbb{Z} \to \mathbb{Z}/p \mathbb{Z}\) is equivalent to the subset of \((\mathbb{Z}/p \mathbb{Z})^3\) of the form
\[
\tilde{+}_p := \{( [n]_p, [m]_p, [n + m]_p ) \in (\mathbb{Z}/p \mathbb{Z})^3, \ [n], [m] \in \mathbb{Z}/p \mathbb{Z}\}.
\]
Since \(S\) converges to \((\mathbb{Z}/\infty \mathbb{Z}, \tilde{d}_\infty)\) in the Gromov-Hausdorff sense, the sequence \(\{\tilde{+}_p, \tilde{d}_p\}\), with \(p\) in the same index set than \(\{S_p\}\) and with
\[
\tilde{d}_p := \left( \tilde{d}_p^2(1) + \tilde{d}_p^2(2) + \tilde{d}_p^2(3) \right)^{1/2}
\]
is convergent in the Gromov-Hausdorff sense towards the limit of the sequence \(\{\tilde{+}_\infty, \tilde{d}_\infty\}\), where
\[
\tilde{d}_\infty = \left( \tilde{d}_\infty^2(1) + \tilde{d}_\infty^2(2) + \tilde{d}_\infty^2(3) \right)^{1/2}
\]
and where \(\tilde{+}_\infty \subset (\mathbb{Z}/\infty \mathbb{Z})^3\). The limit \(\{\tilde{+}_\infty, \tilde{d}_\infty\}\) determines an addition operation \(+_\infty : \mathbb{Z}/\infty \mathbb{Z} \times \mathbb{Z}/\infty \mathbb{Z} \to \mathbb{Z}/\infty \mathbb{Z}\) by the action of the canonical projections
\[
pr_i : (\mathbb{Z}/\infty \mathbb{Z})^3 \to \mathbb{Z}/\infty \mathbb{Z}(i), \quad i = 1, 2, 3
\]
when restricted to the subset \(\tilde{+}_\infty \subset (\mathbb{Z}/\infty \mathbb{Z})^3\),
\[
+_\infty pr_1(\tilde{+}_\infty) \times pr_2(\tilde{+}_\infty) \to pr_3(\tilde{+}_\infty), \quad (pr_1(\alpha) \times pr_2(\alpha)) \mapsto pr_3(\alpha).
\]
Since \(\pi_i(\tilde{+}_\infty) \simeq \mathbb{Z}/\infty \mathbb{Z}\), the aggregate \(\tilde{+}_\infty\) and the projections \(pr_i\) determine an abelian group structure in \(\mathbb{Z}/\infty \mathbb{Z}\). One starts noting that since \([n]_p + [m]_p = [m]_p + [n]_p\) for all \(p \in \mathbb{N} \setminus \{1\}\), then the commutativity also hold for the limits; otherwise will be two limits for the same convergent subsequence. The rest of the group axioms can be proved by analogous arguments.

Similarly, the product operation \(\cdot_\infty : \mathbb{Z}/\infty \mathbb{Z} \times \mathbb{Z}/\infty \mathbb{Z} \to \mathbb{Z}/\infty \mathbb{Z}\) is constructed using the the limits of subsets of the product
\[
\tilde{\cdot}_p := \{([n]_p, [m]_p, [n \cdot m]_p) \in (\mathbb{Z}/p \mathbb{Z})^3, \ [n], [m] \in \mathbb{Z}/p \mathbb{Z}\}.
\]
By taking limits in a similar way as before, one obtains \(\tilde{\cdot}_\infty \subset (\mathbb{Z}/\infty \mathbb{Z})^3\). Then the projections \(pr_i\) and the corresponding limits, \(\tilde{\cdot}_\infty\), determines a product operation on \(\mathbb{Z}/\infty \mathbb{Z}\).
In the case of convergent sequences in \( \tilde{P} \), the limit inverse operation \( \frac{1}{\infty} \) is also well defined (except for the zero element), providing the limit \( \mathbb{Z}/\infty\mathbb{Z} \) with a field structure.

We introduce the notion of \( \epsilon \)-approximation, that will be useful to prove the existence of a natural order relation in \( \mathbb{Z}/\infty\mathbb{Z} \).

**Definition 4.2.** Let \( S \) be a subsequence of \( C \) convergent to \( (\mathbb{Z}/\infty\mathbb{Z}, \tilde{d}_\infty) \). Then \( (\mathbb{Z}/p\mathbb{Z}, \tilde{d}_p) \in S \) is an \( \epsilon \)-approximation to \( (\mathbb{Z}/\infty\mathbb{Z}, \tilde{d}_\infty) \) if and only if

\[
d_{GH}((\mathbb{Z}/p\mathbb{Z}, \tilde{d}_p), (\mathbb{Z}/\infty\mathbb{Z}, \tilde{d}_\infty)) < \epsilon.
\]

It is direct that if the sequence \( S \to (\mathbb{Z}/\infty\mathbb{Z}, \tilde{d}_\infty) \), then for every \( \epsilon > 0 \) there is a \( p \in \mathbb{N} \) such that \( (\mathbb{Z}/p\mathbb{Z}, \tilde{d}_p) \in S \) is an \( \epsilon \)-approximation.

Let us consider now the order relations

\[
[n]_{p_1} \leq [n]_{p_2} \leq [m]_{p_2} \leq [m]_{p_3} \leq [m]_{p_3}, \ldots
\]

where \( n, m < p_1 < p_2 < p_3 < \ldots \) are such that \( (\mathbb{Z}/p_i\mathbb{Z}, \tilde{d}_{p_i}) \in S \) and \( [n]_{p_i}, [m]_{p_i} \in \mathbb{Z}/p_i\mathbb{Z} \). For every \( \epsilon > 0 \) there is an \( \epsilon \)-approximation of \( (\mathbb{Z}/\infty\mathbb{Z}, \tilde{d}_\infty) \) such that the relation \( [n]_p \leq [m]_p \) is be valid for any \( p > p_0(\epsilon) \), for a large enough \( p_0 \). This condition is equivalent to define a function which is constant for \( p > p_0(\epsilon) \) and hence, continuous for large \( p \), that allows to define an order relation in the limit space \( (\mathbb{Z}/\infty\mathbb{Z}, \tilde{d}_\infty) \) as the value of the limits of all possible characteristic functions \( f(n, m)_p \) for \( p \to +\infty \) and all possible \( n, m \in \mathbb{N} \setminus \{1\} \), where the function \( f([m], [n])_p \) is defined by

\[
f([m], [n])_p := 1, \text{ if } [n]_p \geq [m]_p, [n]_p, [m]_p \in \mathbb{Z}/p\mathbb{Z},
\]

\[
0, \text{ if } [m]_p \geq [n]_p, [n]_p, [m]_p \in \mathbb{Z}/p\mathbb{Z}.
\]

The sequence of functions \( \{f_p\} \) is convergent in the limit \( p \to +\infty \). As a consequence of this construction, we have that \( f : \mathbb{Z}/\infty\mathbb{Z} \times \mathbb{Z}/\infty\mathbb{Z} \to \mathbb{R} \) determines a notion of total order in the limit space \( \mathbb{Z}/\infty\mathbb{Z} \): two classes \( [m], [n] \in \mathbb{Z}/\infty\mathbb{Z} \) are related by \( [m] \leq [n] \) if \( f([m], [n]) = 1 \), that is, if there is an \( \epsilon \)-approximation such that \( [m]_p \leq [n]_p \) for any \( p \) in the sub-sequence convergent \( \{S_p\} \) large enough.

The above arguments imply the following result,

**Theorem 4.3.** The limit number ring \( \mathbb{Z}/\infty\mathbb{Z} \) can be endowed with a total order, since neither the addition nor the multiplication operation are consistent with the ring operations.

Let us remark that although the limit field \( \mathbb{Z}/\infty\mathbb{Z} \) admits a total ordered relation, it is not an ordered field.
5. Relation of the limit fields with the real field \( \mathbb{R} \)

Each of the limits \( \mathbb{Z}/\infty \mathbb{Z} \) is dense in the reals \( \mathbb{R} \) in the following sense. Let us consider the injective map \( \varphi_p : \mathbb{Z}/p\mathbb{Z} \to \mathbb{S}^1, [n] \mapsto \exp(2\pi i n/p) \). Let \( \mathcal{S} \) be a convergent sequence of metric spaces \( (\mathbb{Z}/p\mathbb{Z}, \tilde{d}_p) \) in the Gromov-Hausdorff sense. The collection of maps \( \{ \varphi_p, \text{ s.t. } (\mathbb{Z}/p\mathbb{Z}, \tilde{d}_p) \in \mathcal{S} \} \) determines the limit map \( \varphi_\infty : \mathbb{Z}/\infty \mathbb{Z} \to \mathbb{S}^1 \), which is \( \epsilon \)-close to the maps \( \{ \varphi_p, \text{ s.t. } (\mathbb{Z}/p\mathbb{Z}, \tilde{d}_p) \in \mathcal{S} \} \) for \( p \) large enough. By considering the inverse of the projection map \( s^{-1} : \mathbb{R} \to \mathbb{S}^1 \setminus \{(0,1)\} \), then one has that \( s^{-1} \circ \varphi_\infty(\mathbb{Z}/\infty \mathbb{Z}) \) is dense in \( \mathbb{S}^1 \setminus \{\text{North}\} \). By the stereographic projection, however, a dense set in \( \mathbb{S}^1 \setminus \{\text{North}\} \) can be identified with a dense image in \( \mathbb{R} \).

However, the metric space \( (\mathbb{Z}/\infty \mathbb{Z}, \tilde{d}_\infty) \) cannot be identified with the reals as metric space \( (\mathbb{R}, ||) \) with the standard metric distance determined by the modulus function \( ||\mathbb{R} \to \mathbb{R} \), since the first one is compact while \( (\mathbb{R}, ||) \) is not compact. Therefore, we end up with the conclusion that in the case of the limits of prime fields, the limits \( (\mathbb{Z}/\infty \mathbb{Z}, \tilde{d}_\infty) \) constitute a new class of fields endowed with metric functions and order relations.

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