ON THE RIESZ BASIS PROPERTY OF ROOT VECTORS SYSTEM FOR 2 × 2 DIRAC TYPE OPERATORS

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ABSTRACT. The paper is concerned with the Riesz basis property of a boundary value problem associated in $L^2[0, 1] \otimes \mathbb{C}^2$ with the following $2 \times 2$ Dirac type equation

$$Ly = -iB^{-1}y' + Q(x)y = \lambda y, \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad y = \text{col}(y_1, y_2), \quad (0.1)$$

with a summable potential matrix $Q \in L^1[0, 1] \otimes \mathbb{C}^{2 \times 2}$ and $b_1 < 0 < b_2$. If $b_2 = -b_1 = 1$ this equation is equivalent to one dimensional Dirac equation. It is proved that the system of root functions of a linear boundary value problem constitutes a Riesz basis in $L^2[0, 1] \otimes \mathbb{C}^2$ provided that the boundary conditions are strictly regular.

By analogy with the case of ordinary differential equations, boundary conditions are called strictly regular if the eigenvalues of the corresponding unperturbed ($Q = 0$) operator are asymptotically simple and separated. As distinguished from the Dirac case there is no simple algebraic criterion of the strict regularity whenever $b_1 + b_2 \neq 0$. However under certain restrictions on coefficients of the boundary linear forms we present certain algebraic criteria of the strict regularity in the latter case. In particular, it is shown that regular separated boundary conditions are always strictly regular while periodic (antiperiodic) boundary conditions are strictly regular if and only if $b_1 + b_2 \neq 0$.

The proof of the main result is based on existence of triangular transformation operators for system (0.1). Their existence is also established here in the case of a summable $Q$. In the case of regular (but not strictly regular) boundary conditions we prove the Riesz basis property with parentheses. The main results are applied to establish the Riesz basis property of the dynamic generator of spatially non-homogenous damped Timoshenko beam model.

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1. Introduction

Spectral theory of non-selfadjoint boundary value problems (BVP) on a finite interval $I = (a,b)$ for $n$th order ordinary differential equations (ODE)

$$y^{(n)} + q_1 y^{(n-2)} + \ldots + q_{n-1} y = \lambda^n y, \quad x \in (a,b),$$  \hspace{1cm} (1.1)

with coefficients $q_j \in L^1[a,b]$ takes its origin in the classical papers by Birkhoff [4, 5] and Tamarkin [50, 51, 52]. They introduced the concept of regular boundary conditions for ODE and investigated the asymptotic behavior of eigenvalues and eigenfunctions of related BVP. Moreover, they proved that the system of root functions, i.e. eigenfunctions and associated functions, of the regular BVP is complete. Their results are also treated in the classical monographs (see [41, Section 2] and [14, Chapter 19]).

More subtle is the question of whether the system of root functions is a Riesz basis in $L^2[a,b]$. V.P. Mikhailov [38] and G.M. Keselman [21] independently proved that the system of root functions of a boundary value problem for equation (1.1) forms a Riesz basis provided that the boundary conditions are strictly regular. Similar results are also obtained in [14, Chapter 19.4]. Moreover, for boundary conditions which are only regular but not strictly regular, A.A. Shkalikov [45, 46] proved that in the case $q_j \in L^1(a,b), j \in \{1,\ldots,n-1\}$, the system of root functions forms a Riesz basis with parentheses. Recently A.M. Minkin [39] proved that the converse statement is almost true. Namely, he proved that if multiplicities of the eigenvalues are uniformly bounded, the Riesz basis property for the system of root functions of BVP implies the regularity (not necessarily strict regularity) of the boundary conditions.

Numerous papers are devoted to the completeness and Riesz basis property for the Sturm-Liouville operator (see the recent review [31] by A.S. Makin and the references cited therein). We especially mention the recent achievements for periodic (anti-periodic) Sturm-Liouville operator $-\frac{d^2}{dx^2} + q(x)$ on $[0,\pi]$. Namely, F. Gesztesy and V.A. Tkachenko [15, 16] for $q \in L^2[0,\pi]$ and P. Djakov and B.S. Mityagin [12] for $q \in W^{-1,2}[0,\pi]$ established by different methods a criterion for the system of root functions to contain a Riesz basis.

In this paper we consider a special case of the following first order system of ODE

$$Ly := L(Q)y := -iB^{-1}y' + Q(x)y = \lambda y, \quad y = \text{col}(y_1,\ldots,y_n),$$  \hspace{1cm} (1.2)

where $B$ is a nonsingular diagonal $n \times n$ matrix with complex entries,

$$B = \text{diag}(b_1,b_2,\ldots,b_n) \in \mathbb{C}^{n \times n},$$  \hspace{1cm} (1.3)

and $Q(\cdot) = (q_{jk}(\cdot))_{j,k=1}^n \in L^1([0,1];\mathbb{C}^{n \times n})$ is a potential matrix.

To obtain a BVP, we adjoin to equation (1.2) the following boundary conditions (BC)

$$Cy(0) + Dy(1) = 0, \quad C = (c_{jk}), \quad D = (d_{jk}) \in \mathbb{C}^{n \times n}.$$  \hspace{1cm} (1.4)

Moreover, in what follows we always impose the maximality condition $\text{rank}(C \; D) = n$.

Note that, systems (1.2) form a more general object than ordinary differential equations. Namely, the $n$th-order differential equation (1.1) can be reduced to the system (1.2) with $r = n$ and $b_j = \exp(2\pi i j/n)$ (see [33]). The systems (1.2) are of significant interest in some theoretical and practical problems. For instance, if $n = 2m, B = \text{diag}(I_m,-I_m)$ and $Q_{11} = Q_{22} = 0$, the system (1.2) is equivalent to the Dirac system [26, Section VII.1], [35]. Note also that...
equation (1.2) is used to integrate the problem of N waves arising in the nonlinear optics [42, Section III.4].

With the system (1.2) one associates, in a natural way, the maximal operator \( L_{\text{max}} = L_{\text{max}}(Q) \) acting in \( L^2([0,1]; \mathbb{C}^n) \) on the domain

\[
dom(L_{\text{max}}) = \{ y \in W^1_1([0,1]; \mathbb{C}^n) : L_{\text{max}}y \in L^2([0,1]; \mathbb{C}^n) \}. \tag{1.5}
\]

We denote by \( L_{C,D} := L_{C,D}(Q) \) the operator associated in \( L^2([0,1]; \mathbb{C}^n) \) with the BVP (1.2)-(1.4). It is defined as the restriction of \( L = L(Q) \) to the domain

\[
dom(L_{C,D}) = \{ y \in \dom(L_{\text{max}}) : Cy(0) + Dy(1) = 0 \}. \tag{1.6}
\]

Apparently, the spectral problem (1.2)-(1.4) has first been investigated by G. D. Birkhoff and R. E. Langer [6]. Namely, they have extended certain previous results of Birkhoff and Tamarkin on non-selfadjoint BVP for ODE to the case of BVP (1.2)-(1.4). More precisely, they introduced the concepts of regular and strictly regular boundary conditions (1.4) and investigated the asymptotic behavior of eigenvalues and eigenfunctions of the corresponding operator \( L_{C,D} \). Moreover, they proved a pointwise convergence result on spectral decompositions of the operator \( L_{C,D} \) corresponding to the BVP (1.2)-(1.4) with regular boundary conditions.

The completeness problem of the root vectors system of general BVP (1.2)-(1.4) has first been investigated in the recent paper [34] by one of the authors and L.L. Oridoroga. In this paper the concept of weakly regular boundary conditions (1.4) coincides with that of regular ones and reads as follows:

\[
det(CP_+ + DP_-) \neq 0 \quad \text{and} \quad det(CP_- + DP_+) \neq 0. \tag{1.7}
\]

Here \( P_+ \) and \( P_- \) are the spectral projections onto "positive" and "negative" parts of the spectrum of \( B = B^* \), respectively. In the recent papers [30, 29] the completeness of root vectors was established for certain classes of non-regular and even degenerated boundary conditions under certain algebraic assumptions on the boundary values \( Q(0), Q(1) \), of the matrix \( Q(\cdot) \).

Further, if Dirac type operator \( L_{C,D} \) is dissipative, then regularity of conditions (1.4) is equivalent to the first of conditions (1.7) only. It is proved in [27] that the resolvent \( (L_{C,D} - \lambda)^{-1} \) of any complete dissipative Dirac type operator \( L_{C,D} \) admits the spectral synthesis. In particular, the latter happens if the first of conditions (1.7) holds.

Finally, in [29, 30] it was established the Riesz basis property with parentheses for system (1.2) subject to various classes of boundary conditions with a potential \( Q \in L^\infty([0,1]; \mathbb{C}^{n \times n}) \). In [40] a stronger result was obtained for the Dirichlet BVP for \( 2m \times 2m \) Dirac equation \( n = 2m, \quad B = \text{diag}(I_m, -I_m) \) with a potential matrix \( Q \in L^2([0,1]; \mathbb{C}^{2m \times 2m}) \).

In this paper we investigate the Riesz basis property for \( 2 \times 2 \) Dirac type system

\[-iB^{-1}y' + Q(x)y = \lambda y, \quad y = \text{col}(y_1, y_2), \quad x \in [0,1], \tag{1.8}\]

subject to regular and strictly regular boundary conditions (1.4). Here

\[
B = \text{diag}(b_1, b_2), \quad b_1 < 0 < b_2, \quad \text{and} \quad Q = \begin{pmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{pmatrix} \in L^1([0,1]; \mathbb{C}^{2 \times 2}). \tag{1.9}
\]

First we note that in this case boundary conditions (1.4) are regular, i.e. conditions (1.7) are valid, if and only if they are equivalent to the following conditions

\[
\hat{U}_1(y) = y_1(0) + by_2(0) + ay_1(1) = 0, \quad \hat{U}_2(y) = dy_2(0) + cy_1(1) + y_2(1) = 0, \tag{1.10}
\]
with certain \( a, b, c, d \in \mathbb{C} \) satisfying \( ad - bc \neq 0 \). Clearly separated, periodic, and antiperiodic boundary conditions are regular.

Next we recall that regular BC (1.10) are called **strictly regular**, if the sequence \( \Lambda_0 = \{\Lambda_n^0\}_{n \in \mathbb{Z}} \) of the eigenvalues of the unperturbed \(( Q = 0)\) BVP (1.8)–(1.10), is asymptotically separated, i.e. there exist \( \delta > 0 \) and \( n_0 \in \mathbb{N} \) such that

\[
|\Lambda_j^0 - \Lambda_k^0| > 2\delta \quad \text{for} \quad j \neq k \quad \text{and} \quad |j|, |k| \geq n_0.
\]

In particular, the eigenvalues \( \{\Lambda_n^0\}_{|n| > n_0} \) are geometrically and algebraically simple.

For Dirac operator \(( B = \text{diag}(-1, 1))\) the strict regularity of BC reads as follows: \((a - d)^2 \neq -4bc\).

Going over to BVP (1.8)–(1.10) we note that a special case of \( 2 \times 2 \) Dirac operators \( L_{C,D}(Q) \), have been investigated much deeper. For instance, P. Djakov and B. Mityagin [10] imposing certain smoothness condition on \( Q \) proved equiconvergence of the spectral decompositions for \( 2 \times 2 \) Dirac equations subject to general regular boundary conditions.

Moreover, the Riesz basis property for \( 2 \times 2 \) Dirac operators \( L_{C,D}(Q) \) has been investigated in numerous papers (see [54, 55, 18, 8, 3, 9, 11, 12, 13] and references therein). The most complete result was obtained by P. Djakov and B. Mityagin in [9]. Namely, assuming that \( Q \in L^2[0, 1] \otimes \mathbb{C}^{2 \times 2} \) it is proved in [9] that the system of root vectors of the Dirac operator \( L(Q) \) with regular boundary conditions constitutes a Riesz basis with parentheses in \( L^2[0, 1] \otimes \mathbb{C}^2 \) and ordinary Riesz basis provided that BC are strictly regular. Note, that non-degenerate separated boundary conditions are always strictly regular, hence the root vectors of the corresponding BVP constitute a Riesz basis [9] (see Remark 6.10 in this connection).

The following theorem is the main result of the paper.

**Theorem 1.1.** Let \( L_{C,D}(Q) \) be the operator associated in \( L^2([0, 1]; \mathbb{C}^2) \) with the BVP (1.8)–(1.10) and let \( Q_{12}, Q_{21} \in L^1[0, 1] \). Assume that boundary conditions (1.10) are strictly regular. Then root vectors system of the operator \( L_{C,D}(Q) \) forms a Riesz basis in \( L^2[0, 1] \otimes \mathbb{C}^2 \).

While definition of the strict regularity in the case of \( b_1 \neq -b_2 \) is rather implicit, for certain classes of boundary conditions it can be expressed in purely algebraic terms. For instance, if \( bc = 0 \) and \( ad \neq 0 \), then BC (1.10) are strictly regular whenever \( b_1 \ln |d| + b_2 \ln |a| \neq 0 \). In particular, periodic \((a = d = -1)\) and antiperiodic \((a = d = 1)\) BC are strictly regular if and only if \( b_1 + b_2 \neq 0 \). Therefore Theorem 1.1 implies the following surprising result.

**Corollary 1.2.** Let \( Q_{12}, Q_{21} \in L^1[0, 1] \) and \( b_1 + b_2 \neq 0 \). Then the system of root vectors of the periodic (antiperiodic) operator \( L_{C,D}(Q) \) forms a Riesz basis in \( L^2[0, 1] \otimes \mathbb{C}^2 \).

This result demonstrates substantial difference between Dirac and Dirac type operators. Note in this connection that a criterion for the system of root functions of the periodic (necessarily non-strictly regular) BVP for \( 2 \times 2 \) Dirac equation to contain a Riesz basis (without parentheses) was obtained by P. Djakov and B. Mityagin in [12].

We also prove that the root vectors system of the operator \( L(Q) \) forms a Riesz basis with parentheses provided that BC are regular (see Proposition 6.9).

Emphasize that methods used in [9, 3] essentially use condition \( Q \in L^2[0, 1] \otimes \mathbb{C}^{2 \times 2} \) (i.e. the fact that the Fourier coefficients belong to \( l^2(\mathbb{Z}) \)) and most likely could not be applied even to Dirac operators with \( L^1 \)-potentials \( Q \). Note also that traditional methods of perturbations theory are also not applicable here since as opposed to the \( L^2 \)-case, the multiplication operator
by $Q(\in L^1[0, 1] \otimes \mathbb{C}^{2\times 2})$ is neither $B^{-1} \frac{d}{dx}$ – compact nor even subordinated to the unperturbed operator $B^{-1} \frac{d}{dx}$.

The Riesz basis property for abstract operators is investigated in numerous papers. Due to the lack of space we only mention [19, 36, 37, 1], the recent paper [47] and the references therein.

The main results of the paper including Theorem 1.1 were announced in [28] (partially with proofs). After appearance of [28] there appeared the paper by A.M. Savchuk and A.A. Shkalikov [44] where Theorem 1.1 was proved for the $2 \times 2$ Dirac operator. Note that approaches in [28] and [44] substantially differ. Moreover the case of Dirac type operators ($b_1 + b_2 \neq 0$) has interesting features (see e.g. Corollary 1.2) and turns out to be more complicated.

The paper is organized as follows. In Section 2 we prove the existence of triangular transformation operators for equation (1.8)–(1.9). In Section 3 we apply these operators to obtain asymptotic formulas for solutions to equation (1.8). In turn, these formulas are applied in Section 4 to obtain the following asymptotic formula

$$\lambda_n = \lambda_n^0 + o(1), \quad n \to \infty, \quad n \in \mathbb{Z}. \quad (1.11)$$

for the eigenvalues $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ of the operator $L_{C,D}(Q)$ with regular BC. In Section 5 we present certain necessary and sufficient algebraic conditions for equations (1.10) to determine strictly regular BC. In particular, we show in Proposition 5.5 that if $\alpha := -b_1/b_2 \notin \mathbb{Q}$ and $a = 0$, $bc, d \in \mathbb{R} \setminus \{0\}$, then BC (1.10) are strictly regular if and only if

$$d \neq -(\alpha + 1) \left( |bc|\alpha^{-\alpha} \right)^{\frac{1}{\alpha+1}}. \quad (1.12)$$

So, under the above restrictions condition (1.12) gives the algebraic criterion of the strict regularity of boundary conditions (1.10). In Section 6 we prove our main results on Riesz basis property of the root vectors system of the operator $L_{C,D}(Q)$ (Theorem 1.1 and Proposition 6.9).

Finally, in Section 7 we apply Theorem 1.1 to prove the Riesz basis property with parentheses for the dynamic generator of the Timoshenko beam model (see e.g. [53, 22, 49, 58, 57, 56]).

**Notation.** Let $T$ be a closed operator in a Hilbert space $\mathcal{H}$. Denote by $\rho(T)$ the set of regular points of $T$; $\sigma(T) = \mathbb{C} \setminus \rho(T)$ and $\sigma_p(T)$ denote the spectrum of $T$ and the point spectrum of $T$, respectively.

For the eigenvalue $\lambda_0 \in \sigma_p(T)$ denote by $m_a(\lambda_0)$ and $m_g(\lambda_0)$ the algebraic and geometric multiplicities of $\lambda_0$, respectively. Recall that $m_g(\lambda_0) = \dim(\ker(L - \lambda_0))$ and $m_a(\lambda_0)$ is a dimension of the root subspace corresponding to $\lambda_0$.

$\mathbb{D}_r(z) \subset \mathbb{C}$ denotes the disc of radius $r$ with a center $z$.

$\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{C}^n$; $\mathbb{C}^{n \times n}$ denotes the set of $n \times n$ matrices with complex entries; $I_n(\in \mathbb{C}^{n \times n})$ denotes the identity matrix.

2. **Triangular transformation operators**

2.1. **The Banach spaces $X_1$ and $X_\infty$.** Following [32] denote by $X_1 := X_1(\Omega)$ and $X_\infty := X_\infty(\Omega)$ the linear spaces composed of (equivalent classes of) measurable functions defined on $\Omega = \{(x, t) : 0 \leq t \leq x \leq 1\}$ satisfying

$$\|f\|_{X_1} := \operatorname{ess \ sup}_{t \in [0, 1]} \int_0^1 |f(x, t)| \, dx < \infty, \quad (2.1)$$

$$\|f\|_{X_\infty} := \operatorname{ess \ sup}_{x \in [0, 1]} \int_0^x |f(x, t)| \, dt < \infty, \quad (2.2)$$
respectively. It can easily be shown that the spaces $X_1$ and $X_\infty$ equipped with the norms (2.1) and (2.2) form Banach spaces that are not separable. Denote by $X_{1,0}$ and $X_{\infty,0}$ the subspaces of $X_1$ and $X_\infty$, respectively, obtained by taking the closure of continuous functions $f \in C(\Omega)$. Clearly, the set $C^1(\Omega)$ of smooth functions is also dense in both spaces $X_{1,0}$ and $X_{\infty,0}$.

To motivate appearance of the spaces $X_1$ and $X_\infty$ consider a Volterra type operator

$$N : f \to \int_0^x N(x,t)f(t)dt$$

(2.3)

with a measurable kernel $N(\cdot, \cdot)$ and denote by $\|N\|_p := \|N\|_{L^p[0,1] \to L^p[0,1]}$ the $L^p$-norm of the operator $N$ provided that it is bounded. The following simple lemma (cf. [32]) sheds light on appearance of the spaces $X_1$ and $X_\infty$.

Recall that a Volterra operator in a Banach space is a compact operator with zero spectrum.

**Lemma 2.1.** Let $N(\cdot, \cdot) \in X_1(\Omega) \cap X_{\infty}(\Omega)$ and generate the Volterra type operator (2.3). Then:

(i) The operator $N$ is bounded in $L^p[0,1]$ for each $p \in [1, \infty]$ and

$$\|N\|_p \leq \|N\|_{X_1(\Omega)}^{1/p} \cdot \|N\|_{X_{\infty}(\Omega)}^{1-1/p},$$

(2.4)

Moreover,

$$\|N\|_1 = \|N\|_{X_1(\Omega)}, \quad \|N\|_\infty = \|N\|_{X_{\infty}(\Omega)}.$$  

(2.5)

(ii) If $N(\cdot, \cdot) \in X_{1,0}(\Omega) \cap X_{\infty,0}(\Omega)$, then $N$ is a Volterra operator in $L^p[0,1]$ for each $p \in [1, \infty]$.

**Proof.** (i) The relations (2.5) are well known (see e.g. [33]) and can easily be proved. Combining the M. Riesz’s interpolation theorem with relations (2.5) yields

$$\|N\|_p \leq \|N\|_{X_1(\Omega)}^{1/p} \cdot \|N\|_{X_{\infty}(\Omega)}^{1-1/p} = \|N\|_{X_1(\Omega)}^{1/p} \cdot \|N\|_{X_{\infty}(\Omega)}^{1-1/p},$$

(2.6)

which proves estimate (2.4).

(ii) Since $N(\cdot, \cdot) \in X_{1,0}(\Omega) \cap X_{\infty,0}(\Omega)$, there exists a sequence $N_k(\cdot, \cdot) \in C^1(\Omega)$, $k \in \mathbb{N}$, such that $\lim_{k \to \infty}(\|N - N_k\|_{X_1(\Omega)} + \|N - N_k\|_{X_{\infty}(\Omega)}) = 0$. In accordance with (2.4)

$$\|N - N_k\|_p \leq \|N - N_k\|_{X_1(\Omega)}^{1/p} \cdot \|N - N_k\|_{X_{\infty}(\Omega)}^{1-1/p}, \quad p \in [1, \infty].$$

(2.7)

Since $N_k(\cdot, \cdot) \in C^1(\Omega)$, the operator $N_k$ of the form (2.3) is a Volterra operator. It follows from (2.7) that the operator $N$ of the form (2.3) is the uniform limit in $L^p[0,1]$ of the Volterra operators $N_k$ and is therefore itself a Volterra operator. \hfill \square

The following simple properties of the class $X_{\infty,0}(\Omega)$ will be useful in the sequel.

**Lemma 2.2.** For each $a \in [0,1]$ the trace mapping

$$i_a : C(\Omega) \to C[0,a], \quad i_a(N(x,t)) := N(a,t),$$

(2.8)

originally defined on $C(\Omega)$ admits a continuous extension (also denoted by $i_a$) as a mapping $X_{\infty,0}(\Omega) \to L^1[0,a]$ from $X_{\infty,0}(\Omega)$ onto $L^1[0,a]$.

**Proof.** Let $N(\cdot, \cdot) \in X_{\infty,0}(\Omega)$ and let $N_k(\cdot, \cdot) \in C(\Omega)$ be a sequence approaching $N$ in $X_{\infty}(\Omega)$. It follows from definition (2.2) of the norm in $X_{\infty}(\Omega)$ that

$$\int_0^a |N_k(a,t) - N_m(a,t)|dt \leq \|N_k - N_m\|_{X_\infty} \to 0 \quad \text{as} \quad n,m \to \infty,$$

(2.9)

i.e. the sequence $N_k(a, \cdot)$ is a Cauchy sequence in $L^1[0,a]$. Thus, there exists $f_a(\cdot) \in L^1[0,a]$ such the $\|f_a - N_k(a, \cdot)\|_{L^1[0,a]} \to 0$ as $k \to \infty$. We put $N(a, \cdot) := f_a(\cdot)$ and extend the mapping
\(i_a\) to the space \(X_{\infty,0}\) by setting \(i_a : N(\cdot, \cdot) \rightarrow f_a(\cdot) = N(a, \cdot)\). It is easily seen that this extension is well defined. Indeed, if \(\tilde{N}_k(\cdot, \cdot)\) is another sequence approaching \(N(\cdot, \cdot)\) in \(X_\infty\). Then \(\lim_{k \rightarrow \infty} \|N_k - \tilde{N}_k\|_{X_\infty} = 0\) and
\[
\lim_{k \rightarrow \infty} \|N_k(a, \cdot) - \tilde{N}_k(a, \cdot)\|_{L^1[0,a]} \leq \lim_{k \rightarrow \infty} \|N_k - \tilde{N}_k\|_{X_\infty} = 0. \tag{2.10}
\]
Hence \(\lim_{k \rightarrow \infty} \|N(a, \cdot) - \tilde{N}_k(a, \cdot)\|_{L^1[0,a]} = 0\) and the extension \(i_a\) is well defined. \(\square\)

Going over to the vector case we introduce the Banach spaces
\[
X^{2 \times 2}_1 := X^{2 \times 2}_1(\Omega) := X_1(\Omega) \otimes \mathbb{C}^{2 \times 2} \quad \text{and} \quad X^{2 \times 2}_\infty := X^{2 \times 2}_\infty(\Omega) := X_\infty(\Omega) \otimes \mathbb{C}^{2 \times 2} \tag{2.11}
\]
consisting of \(2 \times 2\) matrix functions \(f = (f_{jk})_{j,k=1}^2\) with entries from \(X_1\) and \(X_\infty\), respectively, and equipped with the norms
\[
\|f\|_{X_1} := \|f\|_{X_1 \otimes \mathbb{C}^{2 \times 2}} := \max\{\|f_{jk}\|_{X_1} : j,k \in \{1, 2\}\}, \tag{2.12}
\]
\[
\|f\|_{X_\infty} := \|f\|_{X_\infty \otimes \mathbb{C}^{2 \times 2}} := \max\{\|f_{jk}\|_{X_\infty} : j,k \in \{1, 2\}\}. \tag{2.13}
\]
We also put
\[
X^{2 \times 2}_{1,0} := X^{2 \times 2}_{1,0}(\Omega) := X_{1,0}(\Omega) \otimes \mathbb{C}^{2 \times 2} \quad \text{and} \quad X^{2 \times 2}_{\infty,0} := X^{2 \times 2}_{\infty,0}(\Omega) := X_{\infty,0}(\Omega) \otimes \mathbb{C}^{2 \times 2}.
\]

Further, equip the space \(L^p([0,1], \mathbb{C}^2) := L^p[0,1] \otimes \mathbb{C}^2\) of vector functions with the following norm
\[
\|f\|_p := \|\text{col}(f_1, f_2)\|_p := \|f_1\|_p + \|f_2\|_p, \quad p \in [1, \infty]. \tag{2.14}
\]

where \(\|f_j\|_p := \|f_j\|_{L^p[0,1]}, j \in \{1, 2\}\).

With each measurable kernel \((N_{jk}(\cdot, \cdot))_{j,k=1}^2\) one associates a Volterra type operator
\[
N : \left(\begin{array}{c} f_1 \\ f_2 \end{array}\right) \rightarrow \int_0^x \left(\begin{array}{c} f_1(t) \\ f_2(t) \end{array}\right) dt = \int_0^x \left(\begin{array}{cc} N_{11}(x,t) & N_{12}(x,t) \\ N_{21}(x,t) & N_{22}(x,t) \end{array}\right) \left(\begin{array}{c} f_1(t) \\ f_2(t) \end{array}\right) dt. \tag{2.15}
\]

Let us set \(\|N\|_p := \|N\|_{L^p[0,1] \otimes \mathbb{C}^{2 \times 2}} \rightarrow \|N\|_{L^p[0,1] \otimes \mathbb{C}^{2}}, p \in [1, \infty]\) provided that the norm is bounded.

**Lemma 2.3.** Let \(N(\cdot, \cdot) = \left(N_{jk}(\cdot, \cdot)\right)_{j,k=1}^2\), \(j,k \in X^{2 \times 2}_1(\Omega) \cap X^{2 \times 2}_\infty(\Omega)\) and generate the Volterra type operator by formula (2.15). Then:

(i) The Volterra type operator \(N\) is a bounded operator in \(L^p([0,1], \mathbb{C}^2)\) for each \(p \in [1, \infty]\) and
\[
\|N\|_p \leq \|N\|_{X^{2 \times 2}_1} \cdot \|N\|_{X^{2 \times 2}_\infty}. \tag{2.16}
\]

Moreover,
\[
\|N\|_1 = \|N\|_{X^{2 \times 2}_1} \quad \text{and} \quad \|N\|_{\infty} = \|N\|_{X^{2 \times 2}_\infty}. \tag{2.17}
\]

(ii) If \(N(\cdot, \cdot) \in X^{2 \times 2}_{1,0}(\Omega) \cap X^{2 \times 2}_{\infty,0}(\Omega),\) then \(N\) is a Volterra operator in \(L^p([0,1], \mathbb{C}^2)\) for each \(p \in [1, \infty]\).

The proof is similar to that of Lemma 2.1 and is omitted.

Next we demonstrate that the assumption \(N(\cdot, \cdot) \in X_{1,0}(\Omega) \cap X_{\infty,0}(\Omega)\) in Lemma 2.1(ii) is essential for the operator \(N\) to be a Volterra operator.
Proposition 2.4. Let $k(\cdot) \in L^1[0,1]$, $k(s)s^{-1} \in L^1[0,1]$, and let $\mathcal{M}_k(\alpha) := \int_0^1 k(s)s^{\alpha-1} \, ds$ be the Mellin transform of $k(\cdot)$, $\alpha \in \mathbb{C}_r := \{ z \in \mathbb{C} : \text{Re } z > 0 \}$. Then the Volterra type operator

$$
\mathcal{K} : f \rightarrow \frac{1}{x} \int_0^x k \left( \frac{t}{x} \right) f(t) \, dt.
$$

(2.18)

has the following properties:

(i) The operator $\mathcal{K}$ is bounded in $L^p[0,1]$ for each $p \in [1, \infty]$.

(ii) Its point spectrum is given by $\sigma_p(\mathcal{K}) = \text{range}(\mathcal{M}_k(\cdot)) = \{ \int_0^1 k(s)s^{\alpha-1} \, ds : \alpha \in \mathbb{C}_r \}$. In particular, $\mathcal{K}$ is not compact.

(iii) If $k(\cdot) \geq 0$, then the spectral radius of $\mathcal{K}$ is equal to its norm $\|\mathcal{K}\|_p$ in each $L^p[0,1]$.

Proof. (i) Let us check that $N(x,t) = \frac{1}{2}k(\frac{t}{x}) \in X_1(\Omega) \cap X_\infty(\Omega)$. Indeed setting $t/x = s$ one easily gets

$$
\int_t^1 |N(x,t)| \, dx = \int_t^1 \frac{1}{x} |k \left( \frac{t}{x} \right)| \, dx = \int_t^1 s^{-1}|k(s)| \, ds \leq \int_0^1 s^{-1}|k(s)| \, ds,
$$

(2.19)

and

$$
\int_0^x |N(x,t)| \, dt = \frac{1}{x} \int_0^x |k \left( \frac{t}{x} \right)| \, dt = \int_0^1 \frac{1}{x} |k(s)| x \, ds
$$

$$
= \int_0^1 |k(s)| \, ds \leq \int_0^1 s^{-1}|k(s)| \, ds.
$$

(2.20)

The boundedness of $\mathcal{K}$ in $L^p[0,1]$, $p \in [1, \infty]$, is now implied by Lemma 2.1.

(ii) Clearly, $f_\alpha = x^{\alpha-1} \in L^1[0,1]$ for $\alpha \in \mathbb{C}_r$ and

$$
(\mathcal{K}f_\alpha)(x) = \frac{1}{x} \int_0^x k \left( \frac{t}{x} \right) t^{\alpha-1} \, dt = \frac{1}{x} \int_0^1 k(s)(xs)^{\alpha-1} \, ds = M_k(\alpha)x^{\alpha-1}.
$$

(2.21)

Due to the assumption $k(\cdot)f_0(\cdot) \in L^1[0,1]$ one has

$$
|M_k(\alpha)| \leq \int_0^1 |k(s)s^{\alpha-1}| \, ds \leq \int_0^1 |k(s)s^{-1}| \, ds =: M_k(0).
$$

(2.22)

Since the function $M_k(\cdot)$ is holomorphic and bounded in $\mathbb{C}_r$, it might have at most countable (discrete in $\mathbb{C}_r(-1)$) set of zeros. For the rest of $\alpha$ $x^\alpha$ is the eigenvector of $R$ in $L^1[0,1]$ belonging to a non-zero eigenvalue $c(\alpha)$, i.e. $x^\alpha \in \ker(R - c(\alpha))$. Hence $\mathcal{K}$ is not a compact operator.

$\square$

2.2. Transformation operators.

Theorem 2.5. Let $Q_{12}, Q_{21} \in L^1[0,1]$. Assume that $e_\pm(\cdot; \lambda)$ are the solutions of the system (1.8) corresponding to the initial conditions $e_\pm(0; \lambda) = \left( \begin{smallmatrix} 1 \\ \pm 1 \end{smallmatrix} \right)$. Then $e_\pm(\cdot; \lambda)$ admits the following representation by means of the triangular transformation operator

$$
e_\pm(x; \lambda) = (I + K^\pm)e_\pm^0(x; \lambda) = e_\pm^0(x; \lambda) + \int_0^x K^\pm(x,t)e_\pm^0(t; \lambda) \, dt,
$$

(2.23)

where

$$
e_\pm^0(x; \lambda) = \left( \begin{array}{c} e^{ib_1\lambda x} \\ \pm e^{ib_2\lambda x} \end{array} \right), \quad K^\pm(x,t) = \left( K_{jk}^\pm(x,t) \right)_{j,k=1}^2.
$$

(2.24)
Proposition 2.8. Let \(K^\pm : f \to \int_0^x K^\pm(x,t)f(t)dt\), is a Volterra operator in each \(L^p[0,1]\), \(p \in [1, \infty]\), hence \(\sigma(K^\pm) = \{0\}\).

Our further considerations will substantially be relied on the following result which is a special case of [33, Theorem 1.2] where the general case of \(n \times n\) system (2.26)–(2.28) with the matrix \(B = B^* \in \mathbb{C}^{n \times n}\) and \(Q \in C[0,1] \otimes \mathbb{C}^{n \times n}\) was treated.

Proposition 2.6. [33] Let \(Q = \text{codiag}(Q_{12}, Q_{21}) \in C^1[0,1] \otimes \mathbb{C}^{2 \times 2}\). Then the boundary value problem

\[
\begin{align*}
B^{-1}D_x K^\pm(x,t) + D_t K^\pm(x,t)B^{-1} + iQ(x)K^\pm(x,t) &= 0, \\
K^\pm(x,x)B^{-1} - B^{-1}K^\pm(x,x) &= iQ(x), \quad x \in [0,1], \\
K^\pm(x,0)B\left(\frac{1}{\pm 1}\right) &= 0, \quad x \in [0,1].
\end{align*}
\]

has the unique solution \(K^\pm(\cdot, \cdot) = (K_{jk}^\pm(\cdot, \cdot))^2_{j,k=1} \in C^1(\Omega) \otimes \mathbb{C}^{2 \times 2}\). Moreover, \(K^\pm(\cdot, \cdot)\) is the matrix kernel of the transformation operator (2.23).

The proof of this result in [33] is divided in two steps. At first it is proved solvability (and uniqueness) of the certain auxiliary boundary value problem which in conformity to the 2 \(\times\) 2-case reads as follows

\[
\begin{align*}
B^{-1}D_x R(x,t) + D_t R(x,t)B^{-1} + iQ(x)R(x,t) &= 0, \\
R(x,x)B^{-1} - B^{-1}R(x,x) &= iQ(x), \quad x \in [0,1], \\
R_{11}(x,0) &= R_{22}(x,0) = 0, \quad x \in [0,1].
\end{align*}
\]

where \(R(x,t) = (R_{jk}(x,t))^2_{j,k=1}\). Let us recall the corresponding statement from [33].

Proposition 2.7. Let \(Q = \text{codiag}(Q_{12}, Q_{21}) \in C^1[0,1] \otimes \mathbb{C}^{2 \times 2}\). Then the auxiliary problem (2.28)–(2.30) has a solution \(R \in C^1(\Omega) \otimes \mathbb{C}^{2 \times 2}\). Moreover, it is unique in \(X^{2 \times 2}_\infty(\Omega)\).

Our first auxiliary result reads as follows.

Proposition 2.8. Assume that \(Q, \tilde{Q} \in C^1[0,1] \otimes \mathbb{C}^{2 \times 2}\) and \(\|Q\|_{L^1[0,1] \otimes \mathbb{C}^{2 \times 2}}, \|\tilde{Q}\|_{L^1[0,1]} \leq r\). Then there exists a constant \(C = C(r, b_1, b_2)\) such that

\[
\|R - \tilde{R}\|_{X^{2 \times 2}_\infty} \leq C\|Q - \tilde{Q}\|_{L^1[0,1] \otimes \mathbb{C}^{2 \times 2}}.
\]

Proof. We put

\[
a_j := b_j^{-1} \quad \text{and} \quad \kappa_{jk} := \frac{a_k}{b_j} = \frac{b_j}{a_k}, \quad j, k \in \{1, 2\},
\]

and

\[
\xi_{jk}(x,t) = \begin{cases} 
(a_k x - a_j t)(a_k - a_j)^{-1}, & j \neq k, \\
x - t, & j = k.
\end{cases}
\]

Let us rewrite boundary value problem (2.28)–(2.30) in the scalar form

\[
\begin{align*}
a_k (D_x R_{kk}(x,t) + D_t R_{kk}(x,t)) &= -iQ_{kj}(x)R_{jk}(x,t), \quad k \in \{1, 2\}, \\
a_j D_x R_{jk}(x,t) + a_k D_t R_{jk}(x,t) &= -iQ_{jk}(x)R_{kk}(x,t), \quad k \in \{1, 2\}, \quad j \neq k,
\end{align*}
\]

\[
R_{jk}(x,x) = \frac{iQ_{jk}(x)}{a_k - a_j}, \quad x \in [0,1],
\]

\[
R_{11}(x,0) = 0, \quad R_{22}(x,0) = 0, \quad x \in [0,1].
\]
The system (2.34) – (2.35) is hyperbolic. Integrating the Goursat problem (2.34) – (2.37) along characteristics we arrive at the following equivalent system of integral equations

\[ R_{kk}(x, t) = -\frac{i}{a_j} \int_{x-t}^{x} Q_{kj}(\xi) R_{jk}(\xi, \xi - x + t) d\xi, \quad k \in \{1, 2\}, \quad (2.38) \]

\[ R_{jk}(x, t) = \frac{i}{a_k - a_j} Q_{jk} \left( \frac{a_k x - a_j t}{a_k - a_j} \right) - \frac{i}{a_k} \int_{\xi_{jk}(x, t)}^{x} Q_{jk}(\xi) R_{kk}(\xi, \kappa_{jk} (\xi - x) + t) d\xi. \quad (2.39) \]

Here \( j, k \in \{1, 2\}, \ j \neq k \)

The functions \( \tilde{R}_{jk} \) satisfy the same system (2.38)–(2.39) with \( \tilde{Q}_{jk} \) instead of \( Q_{jk} \), \( j, k \in \{1, 2\} \). Next we put

\[ \tilde{R}_{jk} = \tilde{R}_{jk} - R_{jk}, \quad \tilde{Q}_{jk} = \tilde{Q}_{jk} - Q_{jk}, \quad j, k \in \{1, 2\}. \quad (2.40) \]

and

\[ \tilde{I}_{jk}(x) := \int_{0}^{x} |\tilde{R}_{jk}(x, t)| dt, \quad I_{jk}(x) := \int_{0}^{x} |R_{jk}(x, t)| dt, \quad j, k \in \{1, 2\}. \quad (2.41) \]

Making use the change of variables \( \xi = u, \xi - x + t = v \), we obtain from (2.38) that

\[ \tilde{I}_{11}(x) := \int_{0}^{x} |\tilde{R}_{11}(x, t)| dt \leq |b_1| \int_{0}^{x} dt \int_{x-t}^{x} |\tilde{Q}_{12}(\xi) R_{21}(\xi, \xi - x + t)| d\xi + |b_1| \int_{0}^{x} dt \int_{x-t}^{x} |\tilde{Q}_{12}(\xi) \tilde{R}_{21}(\xi, \xi - x + t)| d\xi \leq |b_1| \int_{0}^{x} |\tilde{Q}_{12}(u)| du \int_{0}^{u} |R_{21}(u, v)| dv + |b_1| \int_{0}^{x} |\tilde{Q}_{12}(u)| du \int_{0}^{u} |\tilde{R}_{21}(u, v)| dv \]

\[ = |b_1| \int_{0}^{x} |\tilde{Q}_{12}(u)| \tilde{I}_{21}(u) du + |b_1| \int_{0}^{x} |\tilde{Q}_{12}(u)| \tilde{I}_{21}(u) du. \quad (2.42) \]

Similarly, it follows from (2.39) and (2.41)

\[ \tilde{I}_{21}(x) = \int_{0}^{x} |\tilde{R}_{21}(x, t)| dt \leq \frac{1}{|a_1 - a_2|} \int_{0}^{x} \left| \tilde{Q}_{21} \left( \frac{a_1 x - a_2 t}{a_1 - a_2} \right) \right| dt + \frac{1}{a_2} \int_{0}^{x} dt \int_{\xi_{21}(x, t)}^{x} |\tilde{Q}_{21}(\xi) R_{21}(\xi, \kappa_{21} (\xi - x) + t)| d\xi + \frac{1}{a_2} \int_{0}^{x} dt \int_{\xi_{21}(x, t)}^{x} |\tilde{Q}_{21}(\xi) \tilde{R}_{21}(\xi, \kappa_{21} (\xi - x) + t)| d\xi. \quad (2.43) \]
Making use the change of variables $\xi = u, \frac{a_1}{a_2}(\xi - x) + t = v$, we obtain

$$\tilde{J}_{21}(x) = \frac{1}{a_2} \int_{\frac{a_1}{a_2}}^{x} |\tilde{Q}_{21}(u)|du + \frac{1}{a_2} \int_{\frac{a_1}{a_2}}^{x} |\tilde{Q}_{21}(u)|du \int_{\frac{a_1}{a_2}}^{u} |R_{11}(u, v)|dv$$

$$+ \frac{1}{a_2} \int_{\frac{a_1}{a_2}}^{x} |\tilde{Q}_{21}(u)|du \int_{\frac{a_1}{a_2}}^{u} |\tilde{R}_{11}(u, v)|dv$$

$$\leq \frac{1}{a_2} \int_{0}^{x} |\tilde{Q}_{21}(u)|du + \frac{1}{a_2} \int_{0}^{x} |\tilde{Q}_{21}(u)|du \int_{0}^{u} |R_{11}(u, v)|dv$$

$$+ \frac{1}{a_2} \int_{0}^{x} |\tilde{Q}_{21}(u)|du \int_{0}^{u} |\tilde{R}_{11}(u, v)|dv$$

$$= \frac{1}{a_2} \int_{0}^{x} |\tilde{Q}_{21}(t)|dt + \frac{1}{a_2} \int_{0}^{x} \tilde{J}_{11}(u)|\tilde{Q}_{21}(u)|du + \frac{1}{a_2} \int_{0}^{x} \tilde{J}_{11}(u)|\tilde{Q}_{21}(u)|du. \tag{2.44}$$

Let

$$C_j = \|J_{j1}\|_{L^\infty[0, 1]} = \|R_{j1}\|_{X^\infty(\Omega)}, \quad j \in \{1, 2\}. \tag{2.45}$$

Estimate (2.42) with account of (2.45) yields

$$\tilde{J}_{11}(x) \leq |b_1|C_2|\tilde{Q}_{12}||_{L^1} + |b_1| \int_{0}^{x} \tilde{J}_{21}(u)|\tilde{Q}_{12}(u)|du. \tag{2.46}$$

Combining this inequality with (2.44) implies

$$\tilde{J}_{21}(x) \leq \left(1 + \frac{C_1}{a_2}\right) \int_{0}^{x} |\tilde{Q}_{21}(u)|du + \frac{1}{a_2} \int_{0}^{x} \tilde{J}_{11}(u)|\tilde{Q}_{21}(u)|du$$

$$\leq C'_1\|\tilde{Q}_{21}\|_{L^1} + |b_1b_2|C_2\|\tilde{Q}_{12}\|_{L^1} \cdot \|\tilde{Q}_{21}\|_{L^1} + |b_1b_2| \int_{0}^{x} |\tilde{Q}_{21}(u)|du \int_{0}^{u} \tilde{J}_{21}(t)|\tilde{Q}_{12}(t)|dt$$

$$\leq C_3 \left(\|\tilde{Q}_{21}\|_{L^1} + \|\tilde{Q}_{12}\|_{L^1}\right) + |b_1b_2| \int_{0}^{x} \tilde{J}_{21}(t)|\tilde{Q}_{12}(t)|dt \int_{t}^{x} |\tilde{Q}_{21}(u)|du$$

$$\leq C_3 \left(\|\tilde{Q}_{21}\|_{L^1} + \|\tilde{Q}_{12}\|_{L^1}\right) + |b_1b_2| \cdot \|\tilde{Q}_{21}\|_{L^1} \int_{0}^{x} \tilde{J}_{21}(t)|\tilde{Q}_{12}(t)|dt, \tag{2.47}$$

where

$$C'_1 := \frac{1 + C_1}{a_2} \quad \text{and} \quad C_3 := \max\{C'_1, |b_1b_2|C_2\|\tilde{Q}_{21}\|_{L^1}\}.$$ 

Applying Cronwall's lemma to this inequality implies

$$\tilde{J}_{21}(x) \leq C_3 \left(\|\tilde{Q}_{21}\|_{L^1} + \|\tilde{Q}_{12}\|_{L^1}\right) \exp \left(|b_1b_2| \cdot \|\tilde{Q}_{21}\|_{L^1} \int_{0}^{x} |\tilde{Q}_{12}(t)|dt\right). \tag{2.48}$$

Inserting this inequality in (2.46) we arrive at the inequality

$$\tilde{J}_{11}(x) \leq |b_1| \left(\|\tilde{Q}_{12}\|_{L^1} + \|\tilde{Q}_{21}\|_{L^1}\right) \left(C_2 + C_3\|\tilde{Q}_{12}\|_{L^1} \cdot \exp \left(|b_1b_2| \cdot \|\tilde{Q}_{12}\|_{L^1} \cdot \|\tilde{Q}_{21}\|_{L^1}\right)\right) \tag{2.49}$$

Similar reasoning leads to similar estimates for $\tilde{J}_{12}$ and $\tilde{J}_{22}$. Combining these estimates with (2.48) and (2.49) we arrive at (2.31). \qed

**Proposition 2.9.** Let $Q = \text{codiag}(Q_{12}, Q_{21}) \in L^1[0, 1] \otimes \mathbb{C}^{2 \times 2}$. Then the system of integral equations (2.38) - (2.39) has a unique solution $R = (R_{jk})^2_{j,k=1}$ belonging to $X_{\infty, 0}^{2 \times 2}(\Omega)$. Moreover, $(R_{jk})^2_{j,k=1} \in X_{1, 0}^{2 \times 2}(\Omega) \cap X_{\infty, 0}^{2 \times 2}(\Omega)$. 


Further, let $Q_n = \text{diag}(Q_{12,n}, Q_{21,n}) \in C^1([0, 1] \otimes C^{2 \times 2}$ be any sequence approaching $Q$ in $L^1([0, 1])$-norm and let $R_n = (R_{jk,n})_{j,k=1}^{2} \in C^1(\Omega) \otimes C^{2 \times 2}$ be the corresponding system of solutions of the problem (2.28)–(2.30) with $Q_n$ instead of $Q$. Then there exists a constant $C = C(Q, b_1, b_2)$ not depending on $n$ and such that the following estimates hold
\[
\| R_{jk} - R_{jk,n} \|_{X_1} + \| R_{jk} - R_{jk,n} \|_{X_{\infty}} \leq C \| Q - Q_n \|_{L^1 \otimes C^{2 \times 2}}, \hspace{1cm} j, k \in \{1, 2\}, \hspace{1cm} n \in \mathbb{N}. \tag{2.50}
\]

Proof. (i) Choose sequences $\{Q_{12,n}\}_{n \in \mathbb{N}}, \{Q_{21,n}\}_{n \in \mathbb{N}} \subset C^1([0, 1])$ such that
\[
\| Q_{12} - Q_{12,n} \|_{L^1} + \| Q_{21} - Q_{21,n} \|_{L^1} \to 0 \hspace{1cm} \text{as} \hspace{1cm} n \to \infty. \tag{2.51}
\]
By Proposition 2.7, for each pair $Q_n = \{Q_{12,n}, Q_{21,n}\}$ there exists the unique matrix solution $R_n = (R_{jk,n})_{j,k=1}^{2} \in C^1(\Omega) \otimes C^{2 \times 2}$, $n \in \mathbb{N}$ of the system (2.38)–(2.39). It follows from (2.51) and (2.31) that there exists $R = (R_{jk})_{j,k=1}^{2} \in X^{2 \times 2}(\Omega)$ such that
\[
\lim_{n \to \infty} \| R_{jk,n} - R_{jk} \|_{X_{\infty}(\Omega)} = 0, \hspace{1cm} j, k \in \{1, 2\}. \tag{2.52}
\]
Let us show that $\{R_{jk}(\cdot, \cdot)\}_{j,k=1}^{2}$ satisfies the system (2.38)–(2.39). Let for instance equation $j \neq k$. Writing down equation (2.39) for $R_{jk,n}(\cdot, \cdot)$ and integrating it with respect to $t$ from 0 to $x$ one gets (cf. (2.44))
\[
\int_0^x R_{jk,n}(x, t) \, dt = -i \frac{i}{a_j} \int_{a_k-a_j}^a Q_{jk}(u) \, du - i \frac{i}{a_j} \int_{a_k-a_j}^a Q_{jk}(u) \, du \int_{a_j-u}^u R_{kk,n}(u, v) \, dv.
\]
It follows from estimate (2.48) that $\lim_{n \to \infty} \int_{v_1}^{v_2} R_{kk,n}(u, v) \, dv = \int_{v_1}^{v_2} R_{kk}(u, v) \, dv$ for any pair $v_1, v_2 \in [0, 1]$. Therefore and due to (2.52) the dominated convergence theorem applies as $n \to \infty$ in (2.53) and gives
\[
\int_0^x R_{jk}(x, t) \, dt = -i \frac{i}{a_j} \int_{a_k-a_j}^a Q_{jk}(u) \, du - i \frac{i}{a_j} \int_{a_k-a_j}^a Q_{jk}(u) \, du \int_{a_j-u}^u R_{kk}(u, v) \, dv
\]
\[
= \int_0^x \left[ \frac{i}{a_k-a_j} Q_{jk} \left( \frac{a_k-x-a_j}{a_k-a_j} \right) - \frac{i}{a_k} \int_{\xi_{jk}(x,t)}^x Q_{jk}(\xi) R_{kk}(\xi, \kappa_{jk}(\xi - x) + t) \, d\xi \right] \, dt. \tag{2.54}
\]
The latter is equivalent to (2.39). The equations for $R_{jj}(\cdot, \cdot), \ j \in \{1, 2\}$, is obtained similarly.

(ii) Since the sequence $Q_n(\cdot)$ approaches $Q(\cdot)$ in $L^1$-norm, it is bounded, $\|Q_n\|_{L^1 \otimes C^{2 \times 2}} \leq C_1 = C_1(Q, B), \ n \in \mathbb{N}$. Therefore Proposition 2.8 applies and gives
\[
\| R_{jk} - R_{jk,n} \|_{X_{\infty}} \leq C \| Q - Q_n \|_{L^1 \otimes C^{2 \times 2}}, \hspace{1cm} j, k \in \{1, 2\}. \tag{2.55}
\]
Next we prove similar estimate in $X_1(\Omega)$-norm. We let
\[
\tilde{R}_{jk,n} = R_{jk} - R_{jk,n}, \ \tilde{Q}_{jk,n} = Q_{jk} - Q_{jk,n}, \hspace{1cm} j, k \in \{1, 2\}, \hspace{1cm} n \in \mathbb{N}. \tag{2.56}
\]
First we prove estimate (2.50) for the case $j \neq k$. To this end we note that
\[
\int_t^1 \tilde{Q}_{jk,n} \left( \frac{a_k-x-a_j}{a_k-a_j} \right) \, dx = \int_t^1 \tilde{Q}_{jk,n} \left( \xi_{jk}(x,t) \right) \, dx = \frac{a_k-a_j}{a_k} \int_t^{\xi_{jk}(1,t)} \tilde{Q}_{jk,n}(u) \, du. \tag{2.57}
\]
where $\xi_{jk}(\cdot, \cdot)$ is given by (2.32).
Further, note that $R_{j,k,n}(\cdot, \cdot)$ satisfies equation (2.39) with $Q_{j,k,n}$ in place of $Q_{j,k}$, $j,k \in \{1,2\}$. Taking difference of this equation (2.39), then integrating the difference with respect to $x \in [t,1]$, and making use the change of variables $u = \xi$, $v = (\xi - x)\kappa_k + t = (\xi - x)\frac{a_k}{a_j} + t$, we obtain

$$\int_1^t |\tilde{R}_{j,k,n}(x,t)| dx = \int_1^t |R_{j,k}(x,t) - R_{j,k,n}(x,t)| dx$$

$$\leq \frac{1}{|a_k|} \int_t^1 |\tilde{Q}_{j,k,n}(u)| du + \frac{1}{|a_j|} \int_t^1 dx \int_{\xi_{j,k}(x,t)}^x |Q_{j,k}(\xi)\tilde{R}_{kk,n}(\xi, \kappa_{j,k}(\xi - x) + t)| d\xi$$

$$+ \frac{1}{|a_j|} \int_t^1 dx \int_{\xi_{j,k}(x,t)}^x |\tilde{Q}_{j,k,n}(\xi)\tilde{R}_{kk,n}(\xi, \kappa_{j,k}(\xi - x) + t)| d\xi$$

$$= \frac{1}{|a_k|} \int_t^1 |\tilde{Q}_{j,k,n}(u)| du + \frac{1}{|a_j|} \int_t^1 dx \int_0^{(v-t)\frac{a_j}{a_k} + 1} |Q_{j,k}(u)\tilde{R}_{kk,n}(u,v)| du$$

$$+ \frac{1}{|a_j|} \int_t^1 dx \int_0^{(v-t)\frac{a_j}{a_k} + 1} |\tilde{Q}_{j,k,n}(u)R_{kk,n}(u,v)| du$$

$$\leq \frac{1}{|a_k|} \int_t^1 |\tilde{Q}_{j,k,n}(u)| du + \frac{1}{|a_j|} \int_t^1 dv \int_0^1 |Q_{j,k}(u)\tilde{R}_{kk,n}(u,v)| du$$

$$+ \frac{1}{|a_j|} \int_t^1 dv \int_0^1 |\tilde{Q}_{j,k,n}(u)R_{kk,n}(u,v)| du$$

$$= \frac{1}{|a_k|} \int_t^1 |\tilde{Q}_{j,k,n}(u)| du + \frac{1}{|a_j|} \int_t^1 dv \int_0^1 |Q_{j,k}(u)| du \int_0^u |\tilde{R}_{kk,n}(u,v)| dv$$

$$+ \frac{1}{|a_j|} \int_t^1 dv \int_0^1 |\tilde{Q}_{j,k,n}(u)| du \int_0^1 |R_{kk,n}(u,v)| dv$$

$$\leq |b_k| \cdot \|\tilde{Q}_{j,k,n}\|_{L^1[0,1]} + |b_j| \cdot \|Q_{j,k}\|_{L^1[0,1]} \cdot \|\tilde{R}_{kk,n}\|_{X_\infty}$$

$$+ |b_j| \cdot \|Q_{j,k}\|_{L^1[0,1]} \cdot \|\tilde{Q}_{j,k,n}\|_{L^1} \cdot \|R_{kk,n}\|_{X_\infty}.$$  (2.58)

Here we use simple inequalities $\xi_{j,k}(1,t) \leq 1 \leq (v-t)\frac{a_j}{a_k} + 1 \leq 1$. The latter holds since $t \geq v$ and $a_j/a_k < 0$. It follows from (2.58) with account of definition (2.1)–(2.2) that

$$\|R_{j,k} - R_{j,k,n}\|_{X_1} \leq |b_k| \cdot \|\tilde{Q}_{j,k,n}\|_{L^1[0,1]}$$

$$+ |b_j| \left( \|Q_{j,k}\|_{L^1} \cdot \|\tilde{R}_{kk,n}\|_{X_\infty} + \|Q_{j,k}\|_{L^1} \cdot \|\tilde{Q}_{j,k,n}\|_{L^1} \cdot \|R_{kk,n}\|_{X_\infty} \right), \quad j \neq k.$$  (2.59)

On the other hand, estimate (2.55) implies $\lim_{n \to \infty} \|R_{j,k} - R_{j,k,n}\|_{X_\infty} = 0$. Therefore there exists $C_2 > 0$ such that $\max\{\|R_{j,k,n}\|_{X_\infty} : j,k \in \{1,2\}, n \in \mathbb{N}\} \leq C_2$. Combining this estimate with (2.55) yields the following estimate

$$\|R_{j,k} - R_{j,k,n}\|_{X_1} \leq C_3 \|Q - Q_n\|_{L^1 \otimes C^{2\times2}}, \quad j,k \in \{1,2\},$$  (2.60)

with a certain positive constant $C_3 > 0$ not depending on $n \in \mathbb{N}$. Combining this estimate with (2.55) implies (2.50) for $j \neq k$.

(iii) Going over to the case $j = k$ we start with equation (2.38) and similar equation for $R_{kk,n}(\cdot, \cdot)$ which holds with $Q_{j,k,n}$ in place of $Q_{j,k}$, $j,k \in \{1,2\}$. Taking difference of this equation and (2.38), then integrating the difference with respect to $x \in [t,1]$, and then making use the
change of variables $\xi = u, \xi - x + t = v$, obtain as

$$|a_j| \int_t^1 |\hat{R}_{jj,n}(x,t)|dx = |a_j| \int_t^1 |R_{jj}(x,t) - \hat{R}_{jj,n}(x,t)|dx,$$

$$\leq \int_t^1 dx \int_{x-t}^x |Q_{jk}(\xi)\hat{R}_{kj,n}(\xi,\xi - x + t)|d\xi + \int_t^1 dx \int_{x-t}^x |\hat{Q}_{jk,n}(\xi)R_{kj,n}(\xi,\xi - x + t)|d\xi$$

$$= \int_0^t dv \int_v^{v-t+1} |Q_{jk}(u)\hat{R}_{kj,n}(u,v)|du + \int_0^t dv \int_v^{v-t+1} |\hat{Q}_{jk,n}(u)R_{kj,n}(u,v)|du$$

$$\leq \int_0^t dv \int_v^1 |Q_{jk}(u)\hat{R}_{kj,n}(u,v)|du + \int_0^t dv \int_v^1 |\hat{Q}_{jk,n}(u)R_{kj,n}(u,v)|du$$

$$= \int_0^t |Q_{jk}(u)|du \int_0^u |\hat{R}_{kj,n}(u,v)|dv + \int_0^t |\hat{Q}_{jk,n}(u)|du \int_0^u |R_{kj,n}(u,v)|dv. \quad (2.61)$$

It follows with account of definition (2.1) – (2.2) that

$$\|R_{jj} - R_{jj,n}\|_{x_1} \leq |b_j| \left(\|Q_{jk}\|_{L^\infty} \cdot \|\hat{R}_{kj,n}\|_{x_\infty} + \|\hat{Q}_{jk,n}\|_{L^\infty} \cdot \|R_{kj,n}\|_{x_\infty}\right), \quad j \in \{1, 2\}. \quad (2.62)$$

Since $\|R_{kj,n}\|_{x_\infty} \leq C_2$ for $n \in \mathbb{N}$, this estimate together with (2.55) leads to the estimate (2.50) with $j = k$. \hfill $\square$

**Lemma 2.10.** Let $Q_{12}, Q_{21} \in L^1[0, 1]$ and let $R(\cdot, \cdot) = (R_{jk}(\cdot, \cdot))_{j,k=1}^2$ be a solution of the system of integral equations (2.38)–(2.39). Then $R(\cdot, \cdot) \in X_{2,0}^{2,2}(\Omega) \cap X_{\infty,0}^{2,2}(\Omega)$ and the operator

$$R : \left(\begin{array}{c} f_1 \\ f_2 \end{array}\right) \rightarrow \int_0^x R(x,t) \left(\begin{array}{c} f_1(t) \\ f_2(t) \end{array}\right) dt = \int_0^x \left(\begin{array}{cc} R_{11}(x,t) & R_{12}(x,t) \\ R_{21}(x,t) & R_{22}(x,t) \end{array}\right) \left(\begin{array}{c} f_1(t) \\ f_2(t) \end{array}\right) dt \quad (2.63)$$

is a Volterra operator in $L^p[0, 1] \otimes \mathbb{C}^2$ for each $p \in [1, \infty]$.

**Proof.** Let $Q_n = \text{c} \text{d} \text{i} \text{a} \text{g}(Q_{12,n}, Q_{21,n}) \in C^1[0, 1] \otimes \mathbb{C}^{2 \times 2}$ be any sequence approaching $Q$ in $L^1[0, 1]$-norm and let $R_n = (R_{jk,n})_{j,k=1}^2 \in C^{1}(\Omega) \otimes \mathbb{C}^{2 \times 2}$, $n \in \mathbb{N}$, be the corresponding system of solutions of the problem (2.28)–(2.30) with $Q_n$ instead of $Q$.

Since $R_n = (R_{jk,n})_{j,k=1}^2$ is a smooth kernel and $\lim_{n \to \infty} \|Q - Q_n\|_{L^1[0,1] \otimes \mathbb{C}^{2 \times 2}} = 0$, it follows from Proposition 2.9 (see estimates (2.50)) that $R(\cdot, \cdot) \in X_{2,0}^{2,2}(\Omega) \cap X_{\infty,0}^{2,2}(\Omega)$. Therefore by Lemma 2.3, $R$ is a Volterra operator in $L^p[0, 1] \otimes \mathbb{C}^2$, $p \in [1, \infty]$. This completes the proof. \hfill $\square$

**Proof of Theorem 1.** Let $P^\pm = \text{d} \text{i} \text{a} \text{g}(P_1^\pm, P_2^\pm)$ be a diagonal matrix function with entries $P_j^\pm \in L^1[0, 1]$, $j \in \{1, 2\}$. Define the convolution operator

$$P^\pm : f \rightarrow \int_0^x P^\pm(x-t)f(t)dt, \quad f = \text{c} \text{o} \text{l}(f_1, f_2) \in L^1[0, 1] \otimes \mathbb{C}^2. \quad (2.64)$$

Let $R(x,t) = (R_{jk}(x,t))_{j,k=1}^2$ be the solution of the system of integral equations (2.38)–(2.39). Starting from the operator $I + R$ and following the reasoning of [33, Theorem 1.2] we define the operator $K^\pm$ by the equality

$$I + K^\pm = (I + R)(I + P^\pm). \quad (2.65)$$

The latter means that the kernel $K^\pm(\cdot, \cdot)$ of $K^\pm$ is given by

$$K^\pm(x,t) = R(x,t) + P^\pm(x-t) + \int_t^x R(x,s)P^\pm(s-t)ds. \quad (2.66)$$
Let us show that $K^\pm(\cdot, \cdot)$ is the kernel of the transformation operator, i.e. representation (2.23)–(2.24) holds. First, we choose $P_1^\pm(\cdot)$ so that $K^\pm(\cdot, \cdot)$ will satisfy condition (2.27), i.e.

$$a_1K_{j_1}^\pm(x,0) + a_2K_{j_2}^\pm(x,0) = 0, \quad j \in \{1, 2\}.$$ 

Inserting representation (2.66) for $K^\pm(\cdot, \cdot)$ in these relations leads to the following system of Volterra integral equations

$$\begin{cases}
a_1 P_1^\pm(x) + \int_0^x [a_1 R_{11}(x,t) P_1^\pm(t) + a_2 R_{12}(x,t) P_2^\pm(t)] dt = \mp a_2 R_{12}(x,0) =: g_1^\pm(x), \\
a_2 P_2^\pm(x) + \int_0^x [a_1 R_{21}(x,t) P_1^\pm(t) + a_2 R_{22}(x,t) P_2^\pm(t)] dt = -a_1 R_{21}(x,0) =: g_2^\pm(x).
\end{cases} \quad (2.67)$$

Here the relations (2.37) have been taken into account. It follows from equation (2.39) that the functions $R_{jk}(x,0)$, $j, k \in \{1, 2\}$, are well defined. Moreover, the estimate (2.58) ensures that $g_j(\cdot) \in L^1[0, 1]$, $j \in \{1, 2\}$.

On the other hand, by Lemma 2.10 the operator $R$ of the form (2.63) is a Volterra operator in $L^1[0, 1]$. Therefore system (2.67) is the system of Volterra equations in $L^1[0, 1] \otimes C^2$ with respect to $\text{col}\{a_1 P_1^\pm(\cdot), a_2 P_2^\pm(\cdot)\}$, hence has the unique solution $\text{col}\{a_1 P_1^\pm(\cdot), a_2 P_2^\pm(\cdot)\} \in L^1[0, 1] \otimes C^2$.

Further, choose a sequence $Q_n = \text{cdig}(Q_{12,n}, Q_{21,n}) \in C^1[0, 1] \otimes C^{2 \times 2}$ approaching $Q$ in $L^1[0, 1]$-norm. Then according to Proposition 2.7 there exists the corresponding sequence of matrix solutions $R_n = (R_{jk,n})^2_{j,k=1} \in C^1(\Omega) \otimes C^{2 \times 2}$ of the problem (2.28)–(2.30) with $Q_n$ instead of $Q$. Moreover, by Proposition 2.9, the estimate (2.50) holds, hence $R_n$ approaches $R$ in $X_1$ and $X_\infty$ norms. Choose a sequence $P_n^\pm = \text{diag}(P_{1,n}^\pm, P_{2,n}^\pm)$ of diagonal matrix functions with entries $P_{j,n}^\pm(\cdot) \in C^1[0, 1]$, $j \in \{1, 2\}$, $n \in \mathbb{N}$, and assume that $P_n^\pm(\cdot)$ satisfies the following system of Volterra integral equations

$$\begin{cases}
a_1 P_{1,n}^\pm(x) + \int_0^x [a_1 R_{11,n}(x,t) P_{1,n}^\pm(t) + a_2 R_{12,n}(x,t) P_{2,n}^\pm(t)] dt = \mp a_2 R_{12,n}(x,0) =: g_{1,n}^\pm(x), \\
a_2 P_{2,n}^\pm(x) + \int_0^x [a_1 R_{21,n}(x,t) P_{1,n}^\pm(t) + a_2 R_{22,n}(x,t) P_{2,n}^\pm(t)] dt = -a_1 R_{21,n}(x,0) =: g_{2,n}^\pm(x).
\end{cases} \quad (2.68)$$

Next we define the kernels $K_n^\pm(\cdot, \cdot)$ by setting (cf. formula (2.66))

$$K_n^\pm(x, t) = R_n(x, t) + \Phi_n^\pm(x - t) + \int_t^x R_n(x, s) P_n^\pm(s - t) ds, \quad n \in \mathbb{N}. \quad (2.69)$$

Clearly, $K_n^\pm(\cdot, \cdot) \in C^1(\Omega)$ and in accordance with [33, Theorem 1.2], it is the unique solution of the boundary value problem (2.25)–(2.27). Note for instance, that condition (2.26) for the kernel $K_n^\pm(\cdot, \cdot)$ is satisfied since $R_n(\cdot, \cdot)$ satisfies this condition, $K_n^\pm(x, x) = R_n(x, x) + P_n^\pm(0)$, and the matrix $P_n^\pm(0)$ is diagonal.

Further, by Proposition 2.6, $K_n^\pm(\cdot, \cdot)$ is the kernel of transformation operator for equation (4.1)–(1.9) with $Q_n$ in place of $Q$, i.e. the solution $e_{\pm,n}(\cdot; \lambda)$ of this equation satisfying the initial condition $e_{\pm,n}(0; \lambda) = (\pm1)$ admits a representation

$$e_{\pm,n}(x; \lambda) = (I + K_n^\pm)e_0^\pm(x; \lambda) = e_0^\pm(x; \lambda) + \int_0^x K_n^\pm(x, t) e_0^\pm(t; \lambda) dt, \quad n \in \mathbb{N}. \quad (2.70)$$

Our aim is to pass to the limit in (2.69) and (2.70) as $n \to \infty$. It follows from (2.58) with $t = 0$ and the estimate (2.50) that

$$\lim_{n \to \infty} (\|g_{1,n}^\pm - g_1^\pm\|_{L^1[0,1]} + \|g_{2,n}^\pm - g_2^\pm\|_{L^1[0,1]}) = 0.$$
Combining this relation with Proposition 2.9 we obtain from (2.67) and (2.68) that

\[
\begin{pmatrix}
a_1 P_{1,n}^+(\cdot) \\
a_2 P_{2,n}^+(\cdot)
\end{pmatrix} = (I + R_n)^{-1} \begin{pmatrix} g_{1,n}^+ \\ g_{2,n}^+ \end{pmatrix} \rightarrow (I + R)^{-1} \begin{pmatrix} g_1^+ \\ g_2^+ \end{pmatrix} = \begin{pmatrix} a_1 P_1^+(\cdot) \\
a_2 P_2^+(\cdot)\end{pmatrix}.
\]  

(2.71)

Further, setting \( \hat{P}_n^\pm := P_n - P \) we derive from (2.66) and (2.69)

\[
\int_t^1 |\hat{K}_n^\pm(x,t)|dx = \int_t^1 |K_n^\pm(x,t) - K^\pm(x,t)|dx \leq \int_t^1 |\hat{R}_n(x,t)|dx + \int_t^1 |\hat{P}_n^\pm(x,t)|dx
\]

\[
+ \int_t^1 |P_n^\pm(s-t)|ds \int_s^1 |\hat{R}_n(x,s)|dx + \int_t^1 |\hat{P}_n^\pm(s-t)|ds \int_s^1 |R(x,s)|dx
\]

\[
\leq \|\hat{R}_n\|_{X_1(\Omega)} (1 + \|P_n^\pm\|_{L^1[0,1]}) + \|\hat{P}_n^\pm\|_{L^1[0,1]} (1 + \|R\|_{X_1(\Omega)}).
\]  

(2.72)

On the other hand, by Proposition 2.9, \( \lim_{n \rightarrow \infty} \|\hat{R}_n\|_{X_2^2(\Omega)} = \lim_{n \rightarrow \infty} \|R_n - R\|_{X_2^2(\Omega)} = 0 \), and due to (2.71) \( \lim_{n \rightarrow \infty} \|\hat{P}_n^\pm\|_{L^1[0,1]} = 0 \). Combining these relations with (2.72) yields

\[
\lim_{n \rightarrow \infty} \|\hat{K}_n^\pm\|_{X_2^2(\Omega)} = \lim_{n \rightarrow \infty} \|K^\pm - K_n^\pm\|_{X_2^2(\Omega)} = 0.
\]  

(2.73)

The latter means that \( K^\pm \in X_2^2(\Omega) \). In just the same way one proves the relation

\[
\lim_{n \rightarrow \infty} \|\hat{K}_n^\pm\|_{X_2^2} = \lim_{n \rightarrow \infty} \|K^\pm - K_n^\pm\|_{X_2^2} = 0.
\]  

(2.74)

Using relation (2.74) we can pass to the limit as \( n \rightarrow \infty \) in formula (2.70) and arrive at the required formula (2.23).

\( \square \)

**Remark 2.11.** (i) For Dirac 2 × 2 system \((B = \text{diag}(-1,1))\) with continuous \( Q \) the triangular transformation operators have been constructed in [26, Ch.10.3] and [35, Ch.1.2]. For \( Q \in L^1[0,1] \otimes \mathbb{C}^{2 \times 2} \) it is proved in [2] by an appropriate generalization of the Marchenko method.

(ii) Let \( J : f \rightarrow \int_0^x f(t)dt \) be a Volterra operator on \( L^p[0,1] \). Note that the similarity of Volterra operators given by (2.15) to the simplest Volterra operators of the form \( B \otimes J \) acting in the spaces \( L^p[0,1] \otimes \mathbb{C}^2 \) has been investigated in [33, 43].

### 3. Asymptotic Behavior of Solutions

Let \( K_{jk}(x,t) = (K_{jk}^\pm(x,t))^2 \) be the kernel of a triangular transformation operator constructed in Theorem 2.5 (see formulas (2.23)–(2.24)). To state the next result we put

\[
R_{jk}^\pm := 2^{-1}(K_{jk}^\pm + K_{jk}^-), \quad j, k \in \{1, 2\},
\]  

(3.1)

and let

\[
\Phi(\cdot, \lambda) = \begin{pmatrix} \varphi_{11}(\cdot, \lambda) & \varphi_{12}(\cdot, \lambda) \\ \varphi_{21}(\cdot, \lambda) & \varphi_{22}(\cdot, \lambda) \end{pmatrix} = (\Phi_1(\cdot, \lambda) \quad \Phi_2(\cdot, \lambda)), \quad \Phi(0, \lambda) = I_2,
\]  

(3.2)

be a fundamental matrix solution of the system (1.8). Here \( \Phi_k(\cdot, \lambda) \) is the \( k \)th column of \( \Phi(\cdot, \lambda) \).

Our investigating of the perturbation determinant relies on the following result.
Proposition 3.1. Let \( Q \in L^1[0,1] \otimes \mathbb{C}^{2 \times 2} \) and let \( \varphi_{jk}(\cdot, \lambda) \), \( j, k \in \{1, 2\} \), be the entries of the fundamental matrix solution (3.2). Then the functions \( \varphi_{jk}(\cdot, \lambda) \) admit the following representations

\[
\varphi_{11}(x, \lambda) = e^{ib_1 \lambda x} + \int_0^x R_{11}^+(x,t) e^{ib_1 \lambda t} dt + \int_0^x R_{12}^-(x,t) e^{ib_2 \lambda t} dt, \\
\varphi_{12}(x, \lambda) = \int_0^x R_{11}^+(x,t) e^{ib_1 \lambda t} dt + \int_0^x R_{12}^+(x,t) e^{ib_2 \lambda t} dt, \\
\varphi_{21}(x, \lambda) = \int_0^x R_{21}^+(x,t) e^{ib_1 \lambda t} dt + \int_0^x R_{22}^-(x,t) e^{ib_2 \lambda t} dt, \\
\varphi_{22}(x, \lambda) = e^{ib_2 \lambda x} + \int_0^x R_{21}^+(x,t) e^{ib_1 \lambda t} dt + \int_0^x R_{22}^+(x,t) e^{ib_2 \lambda t} dt,
\]

where \( R_{jk}^\pm \in X_{1,0}(\Omega) \cap X_{\infty,0}(\Omega), j, k \in \{1, 2\} \).

Proof. Comparing initial conditions and applying the Cauchy uniqueness theorem one easily gets \( \Phi_1(\cdot, \lambda) = \left( \varphi_{11}(\cdot, \lambda) \varphi_{21}(\cdot, \lambda) \right) = e_+(\cdot; \lambda) + e_-(\cdot; \lambda) \). Inserting in place of \( e_+(\cdot; \lambda) \) and \( e_-(\cdot; \lambda) \) their expressions from (2.23) one arrives at (3.3) and (3.5). Relations (3.4) and (3.6) are proved similarly.

Lemma 3.2. Let \( N(\cdot, \cdot) \in X_{\infty,0}(\Omega), b \in \mathbb{R} \setminus \{0\} \) and \( h > 0 \). Then the following asymptotic holds uniformly in \( x \in [0,1] \)

\[
\int_0^x N(x,t) e^{ib \lambda t} dt \to 0 \quad \text{as} \quad \lambda \to \infty, \quad |\text{Im} \lambda| \leq h.
\]

Proof. By the definition of the space \( X_{\infty,0}(\Omega) \), the inclusion \( N \in X_{\infty,0}(\Omega) \) ensures that for any \( \varepsilon > 0 \) there exists \( N_\varepsilon \in C^1(\Omega) \) such that

\[
\|N - N_\varepsilon\|_{\infty} = \sup_{x \in [0,1]} \int_0^x |N(x,t) - N_\varepsilon(x,t)| dt < \varepsilon.
\]

In particular, we get the following uniform estimate

\[
\left| \int_0^x (N(x,t) - N_\varepsilon(x,t)) e^{ib \lambda t} dt \right| \leq |\varepsilon| e^{|b| h}, \quad x \in [0,1], \quad |\text{Im} \lambda| \leq h.
\]

Since \( N_\varepsilon \in C^1(\Omega) \), integrating by parts the integral \( \int_0^x N_\varepsilon(x,t) e^{ib \lambda t} dt \) we obtain the following estimate uniformly in \( x \in [0,1] \) with some \( C > 0 \)

\[
\left| \int_0^x N_\varepsilon(x,t) e^{ib \lambda t} dt \right| < \frac{C}{|\lambda|}, \quad \lambda \neq 0, \quad |\text{Im} \lambda| \leq h.
\]

The desired formula (3.7) now directly follows from estimates (3.9) and (3.10).

Remark 3.3. We demonstrate that the assumption \( N(\cdot, \cdot) \in X_{\infty,0}(\Omega) \) is important for the validity of the statement of Lemma 3.2. More precisely, we show that for certain \( N(\cdot, \cdot) \in X_{\infty}(\Omega) \setminus X_{\infty,0}(\Omega) \) the pointwise convergence in (3.7) holds but is not uniform in \( x \in [0,1] \).

Let \( N(x,t) = \frac{1}{x} k(\frac{t}{x}) \) where \( k(\cdot) \) satisfies the conditions of Proposition 2.4. Then

\[
\int_0^x N(x,t) e^{i \lambda t} dt = \frac{1}{x} \int_0^x k \left( \frac{t}{x} \right) e^{i \lambda t} dt = \int_0^1 k(s) e^{i \lambda sx} ds \to 0 \quad \text{as} \quad \lambda \to \infty, \quad \lambda \in \Pi_h.
\]
for each \( x \in (0, 1) \) and \( \lambda \in \Pi_h \). However this convergence is not uniform in \( x \in [0, 1] \). Indeed, since \( k(\cdot) \not\equiv 0 \), its Fourier transform \( \hat{k}(\cdot) \) does not vanish identically, i.e. there exists \( a \in \mathbb{R} \) such that \( \hat{k}(a) \not= 0 \). Therefore for \( \lambda \in \mathbb{R} \) big enough and \( x = a/\lambda \in (0, 1) \) the right hand side of (3.11) is \( \hat{k}(a) \not= 0 \).

In the sequel we need the following result on the asymptotic behavior of solutions of the system (4.1) in the strip

\[ \Pi_h := \{ \lambda \in \mathbb{C} : |\text{Im} \lambda| \leq h \} . \]

**Proposition 3.4.** Let \( Q \in L^1[0, 1] \otimes \mathbb{C}^{2 \times 2} \). Then for any \( h > 0 \) the following asymptotic relations take place uniformly in \( x \in [0, 1] \)

\[ \varphi_{jk}(x, \lambda) = \delta_{jk}e^{ib\lambda x} + o(1) \quad \text{as} \quad \lambda \to \infty, \quad \lambda \in \Pi_h, \quad j, k \in \{1, 2\} . \tag{3.12} \]

**Proof.** The proof immediately follows by combining Proposition 3.1 with Lemma 3.2. \( \square \)

Applying the same approximation procedure as has just been used in the proof of Lemma 3.2 to the space \( L^1[0, 1] \) we obtain the following simple statement useful in the sequel.

**Lemma 3.5.** Let \( g \in L^1[0, 1] \) and \( c \in \mathbb{C} \setminus \{0\} \). Then for any \( \varepsilon > 0 \) there exists \( M = M_\varepsilon > 0 \) such that

\[ \left| \int_0^1 g(t)e^{c\lambda t}dt \right| < \varepsilon (e^{\text{Re}(c\lambda)} + 1), \quad |\lambda| > M . \tag{3.13} \]

Further, consider the adjoint system

\[ -iB^{-1}y' + Q^*(x)y = \lambda y, \quad x \in [0, 1] , \tag{3.14} \]

and introduce its fundamental matrix solution

\[ \Psi(\cdot, \lambda) = \begin{pmatrix} \psi_{11}(\cdot, \lambda) & \psi_{12}(\cdot, \lambda) \\ \psi_{21}(\cdot, \lambda) & \psi_{22}(\cdot, \lambda) \end{pmatrix} =: \begin{pmatrix} \Psi_1(\cdot, \lambda) & \Psi_2(\cdot, \lambda) \end{pmatrix}, \quad \Psi(0, \lambda) = I_2 . \tag{3.15} \]

Here \( \psi_k(\cdot, \lambda) \) is the \( k \)-th column of \( \Psi(\cdot, \lambda) \). Clearly, Proposition 3.4 holds for the matrix solution \( \Psi(\cdot, \lambda) \) as well. Hence (3.12) and similar relations for \( \psi_{jk}(\cdot, \lambda) \) imply the following result.

**Corollary 3.6.** Let \( h > 0 \). Then for \( \lambda \to \infty, \lambda \in \Pi_h \), the following asymptotic relations hold

\[ \left( \Phi_j(\cdot, \lambda), \Psi_k(\cdot, \lambda) \right) = \delta_{jk} + o(1), \quad j, k \in \{1, 2\} , \tag{3.16} \]

\[ \left( \Phi_1(\cdot, \lambda), \Phi_2(\cdot, \lambda) \right) = o(1) . \tag{3.17} \]

Moreover, there exist constants \( M > 0 \) and \( C_1, C_2 > 0 \), such that

\[ 0 < C_1 < \left| \left( \Phi_j(\cdot, \lambda), \Psi_j(\cdot, \lambda) \right) \right| < C_2, \quad \lambda \in \Pi_h, \quad |\lambda| > M, \quad j \in \{1, 2\} . \tag{3.18} \]

**Proof.** First let us evaluate \( \left( \Phi_1(\cdot, \lambda), \Phi_1(\cdot, \lambda) \right) \). Setting \( f(x) := \frac{e^{ia\lambda x} - 1}{x}, \ x \in \mathbb{R} \), and noting that \( e^{ia\lambda x}, \ a \in \mathbb{R} \), is bounded for \( (x, \lambda) \in [0, 1] \times \Pi_h \), one easily deduces from the uniform asymptotic relations (3.12)

\[ \left| \Phi_1(\cdot, \lambda) \right|^2 = (\Phi_1(\cdot, \lambda), \Phi_1(\cdot, \lambda)) = \int_0^1 \left( \varphi_{11}(x, \lambda)\overline{\varphi_{11}(x, \lambda)} + \varphi_{12}(x, \lambda)\overline{\varphi_{12}(x, \lambda)} \right) dx \]

\[ = \int_0^1 (e^{ib\lambda x} + o(1)) dx = \frac{e^{-2b_1\text{Im} \lambda} - 1}{-2b_1\text{Im} \lambda} + o(1) \]

\[ = f(-2b_1\text{Im} \lambda) + o(1) \quad \text{as} \quad \lambda \to \infty, \quad \lambda \in \Pi_h . \tag{3.19} \]
Clearly, there exists $C_1, C_2 > 0$ such that
\[ C_1 \leq |f(x)| \leq C_2, \quad |x| \leq 2|b_1|h. \] (3.20)
Combining (3.19) with (3.20) one proves (3.18) for $j = 1$. Relation (3.19) for $j = 2$ as well as relations (3.16), (3.17) are proved similarly.

4. Regular boundary conditions

Here we consider $2 \times 2$-Dirac type equation (4.1),
\[-iB^{-1}y' + Q(x)y = \lambda y, \quad y = \text{col}(y_1, y_2), \quad x \in [0, 1],\] (4.1)
subject to the following general boundary conditions
\[ U_j(y) := a_{j1}y_1(0) + a_{j2}y_2(0) + a_{j3}y_1(1) + a_{j4}y_2(1) = 0, \quad j \in \{1, 2\}.\] (4.2)
Denote by $L := L(Q, U_1, U_2)$ the operator associated in $L^2([0, 1]; \mathbb{C}^2)$ with the BVP (4.1)–(4.2). It is defined as the restriction of the maximal operator $L_{\max} = L_{\max}(Q) (1.5)$ to the domain
\[ \text{dom}(L) = \text{dom}(L(Q, U_1, U_2)) = \{y \in \text{dom}(L_{\max}) : U_1(y) = U_2(y) = 0\}.\] (4.3)

The eigenvalues of the problem (4.1)–(4.2) are the roots of the characteristic equation $\Delta(\lambda) := \det U(\lambda) = 0$, where
\[ U(\lambda) := \begin{pmatrix} U_1(\Phi_1(\cdot, \lambda)) & U_1(\Phi_2(\cdot, \lambda)) \\ U_2(\Phi_1(\cdot, \lambda)) & U_2(\Phi_2(\cdot, \lambda)) \end{pmatrix} = \begin{pmatrix} u_{11}(\lambda) & u_{12}(\lambda) \\ u_{21}(\lambda) & u_{22}(\lambda) \end{pmatrix}.\] (4.4)

Putting $A_{jk} = \begin{pmatrix} a_{1j} & a_{1k} \\ a_{2j} & a_{2k} \end{pmatrix}$, and $J_{jk} = \det(A_{jk}), \quad j, k \in \{1, \ldots, 4\}$, we obtain the following expression for the characteristic determinant
\[ \Delta(\lambda) = J_{12} + J_{34}e^{i(b_1+b_2)\lambda} + J_{32}\varphi_{11}(\lambda) + J_{13}\varphi_{12}(\lambda) + J_{42}\varphi_{21}(\lambda) + J_{14}\varphi_{22}(\lambda), \] (4.5)
where $\varphi_{jk}(\lambda) := \varphi_{jk}(1, \lambda)$. If $Q = 0$ then $\varphi_{12}(x, \lambda) = \varphi_{21}(x, \lambda) = 0$ and the characteristic determinant $\Delta_0(\cdot)$ becomes
\[ \Delta_0(\lambda) = J_{12} + J_{34}e^{i(b_1+b_2)\lambda} + J_{32}e^{ib_1\lambda} + J_{14}e^{ib_2\lambda}.\] (4.6)
In the case of Dirac system ($B = \text{diag}(-1, 1)$) this formula is simplified to
\[ \Delta_0(\lambda) = J_{12} + J_{34} + J_{32}e^{-i\lambda} + J_{14}e^{i\lambda}.\] (4.7)
Substituting formulas (3.3)–(3.6) at $x = 1$ to (4.5) and taking into account (4.6), we get the following expression for the characteristic determinant.

**Lemma 4.1.** The characteristic determinant $\Delta(\cdot)$ of the problem (4.1)–(4.2) is an entire function admitting the following representation
\[ \Delta(\lambda) = \Delta_0(\lambda) + \int_0^1 g_1(t)e^{ib_1\lambda}dt + \int_0^1 g_2(t)e^{ib_2\lambda}dt,\] (4.8)
with $g_1, g_2 \in L^1[0, 1]$. 


Proof. Consider representations (3.3)–(3.6) for $\varphi_{jk}(\cdot, \lambda)$, $j, k \in \{1, 2\}$. By Proposition 3.1, $R_{jk}^\pm(\cdot, \cdot) \in X_{1,0}(\Omega) \cap X_{\infty,0}(\Omega)$, $j, k \in \{1, 2\}$. Therefore by Lemma 2.2, the trace functions $R_{jk}^\pm(1, \cdot)$ are well defined and summable, $R_{jk}^\pm(1, \cdot) \in L^1[0, 1]$, $j, k \in \{1, 2\}$. Therefore one can substitute $x = 1$ in formulas (3.3)–(3.6) and obtain special representations for $\varphi_{jk}(\cdot, \lambda)$, $j, k \in \{1, 2\}$. For instance,

$$
\varphi_{jk}(\lambda) := \varphi_{jk}(1, \lambda) = \int_0^1 R_{jk}^+(1, t)e^{ib_k\lambda t} dt + \int_0^1 R_{jk}^-(1, t)e^{ib_j\lambda t} dt, \quad j \neq k. \quad (4.9)
$$

Inserting these expressions and similar expressions for $\varphi_0^0$ into account we arrive at (4.8) with $g_j(\cdot)$, $j \in \{1, 2\}$, being a linear combination of the functions $R_{jk}^\pm(1, \cdot)$, $j, k \in \{1, 2\}$.

In the sequel we need the following definitions (cf. [20]).

**Definition 4.2.**

(i) A sequence $\Lambda := \{\lambda_n\}_{n \in \mathbb{Z}}$ of complex numbers is said to be **separated** if for some positive $\delta > 0$,

$$
|\lambda_j - \lambda_k| > 2\delta \quad \text{whenever} \quad j \neq k. \quad (4.10)
$$

In particular, all entries of a separated sequence are distinct.

(ii) The sequence $\Lambda$ is said to be **asymptotically separated** if for some $n_0 \in \mathbb{N}$ the subsequence $\Lambda_{n_0} := \{\lambda_n\}_{|n| > n_0}$ is separated.

(iii) Let $\Lambda$ lie in the strip $\Pi_h$. It is called **incondensable** if for some $L > 0$ and $N \in \mathbb{N}$ every rectangle $[t - L, t + L] \times [-h, h] \subset \mathbb{C}$ contains at most $N$ entries of the sequence, i.e. for each $t \in \mathbb{R}$ the number of integers $\{n \in \mathbb{Z} : |\text{Re}\lambda_n - t| \leq L, |\text{Im}\lambda_n| \leq h\}$ does not exceed $N$.

We need the following simple property of incondensable sequences.

**Lemma 4.3.** Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ be an incondensable sequence lying in the strip $\Pi_h$. Then there exists $\varepsilon_0 > 0$ and $N_0 \in \mathbb{N}$ such that for any $\varepsilon \in (0, \varepsilon_0)$ every connected component of the union of discs $\cup_{n \in \mathbb{Z}} \mathbb{D}_\varepsilon(\lambda_n)$ has at most $N_0$ discs $\mathbb{D}_\varepsilon(\lambda_n)$.

**Proof.** Assume the contrary, i. e., for any $\varepsilon > 0$ and $K \in \mathbb{N}$ there exists connected component of $\Omega_\varepsilon := \cup_{n \in \mathbb{Z}} \mathbb{D}_\varepsilon(\lambda_n)$ that has at least $K$ discs $\mathbb{D}_\varepsilon(\lambda_n)$. By definition of incondensable sequence for some $L, N > 0$ every rectangle $[t - L, t + L] \times [-h, h]$, $t \in \mathbb{R}$, contains at most $N$ entries of the sequence $\Lambda$. Let $K > N$ be some positive integer and pick $\varepsilon$ to be such that $2K\varepsilon < L$. Consider some connected component of $\Omega_\varepsilon$ that has $M \geq K$ discs $\mathbb{D}_\varepsilon(\lambda_n)$, denote it by $C$. Let’s pick one of the discs $D_0 = \mathbb{D}_\varepsilon(\lambda_{n_0})$, $n_0 \in \mathbb{Z}$, in $C$, and let $t_0 = \text{Re}\lambda_{n_0}$. Due to above, rectangle $[t_0 - L, t_0 + L] \times [-h, h]$ contains at most $N$ entries of the sequence $\Lambda$. Consider the sequence $B$ of all discs $\mathbb{D}_\varepsilon(\lambda_n)$ in $C$ that have graph distance at most $K$ from $D_0$. Let’s shows that $B$ has more than $N$ discs. If no disc in $C$ has graph distance at least $K$ from $D_0$, then $B$ contains all discs from $C$ and thus cardinality of $B$ is $M \geq K > N$. Otherwise, $B$ has some disc $D$ with distance $K$ from $D_0$. All discs on the path from $D_0$ to $D$ belong to $B$ and hence $B$ has at least $K > N$ discs. For each disc $\mathbb{D}_\varepsilon(\lambda_n)$ in $B$ since graph distance from it to $D_0$ is at most $K$ and disc radii are $\varepsilon$ we have $|\lambda_n - \lambda_{n_0}| < 2K\varepsilon$. Thus, $\lambda_n \in [t_0 - L, t_0 + L] \times [-h, h]$ since $2K\varepsilon < L$. Thus centers of all discs in $B$ lie in $[t_0 - L, t_0 + L] \times [-h, h]$. Since there more than $N$ discs in $B$ it contradicts incondensability property of the sequence $\Lambda$. \[\Box\]

To get the asymptotic behavior of the eigenvalues of the problem (4.1)–(4.2) with regular boundary conditions we also need the following definition.
Definition 4.4. [24] An entire function $F(\cdot)$ of exponential type is said to be of sine-type if

(i) all zeros of $F(\cdot)$ lie in the strip $\Pi_h$ for some $h > 0$, and

(ii) there exists $C_1, C_2 > 0$ and $h_0 > h$ such that

$$0 < C_1 \leq |F(x + ih_0)| \leq C_2 < \infty, \quad x \in \mathbb{R}. \quad (4.11)$$

This definition is borrowed from [24] (see also [20]). It differs from that contained in [25]. Namely, it is assumed in [25] that the sequence of zeros of $F(\cdot)$ is separated and the indicator function $h_F(\cdot)$ of $F(\cdot)$,

$$h_F(\cdot) := \lim_{r \to +\infty} \frac{\ln |F(re^{i\varphi})|}{r}, \quad \varphi \in (-\pi, \pi], \quad (4.12)$$

satisfies the condition $h_F(\pi/2) = h_F(-\pi/2)$. The latter is imposed for convenience and can easily be achieved by multiplication of $F(\cdot)$ by a function $e^{iz}$ with an appropriate $\gamma \in \mathbb{R}$.

Recall also the definition of regular boundary conditions.

Definition 4.5. Boundary conditions (4.2) are called regular if

$$J_{14} J_{32} \neq 0. \quad (4.13)$$

In the case of regular boundary conditions the characteristic determinant $\Delta(\cdot)$ has certain important properties.

Proposition 4.6. Let the boundary conditions (4.2) be regular and let $\Delta(\cdot)$ be the characteristic determinant of the problem (4.1)–(4.2) given by (4.5). Then the following hold:

(i) The characteristic determinant $\Delta(\cdot)$ is a sine-type function with $h_\Delta(\pi/2) = -b_1$ and $h_\Delta(-\pi/2) = b_2$. In particular, $\Delta(\cdot)$ has infinitely many zeros

$$\Lambda := \{\lambda_n\}_{n \in \mathbb{Z}} \quad (4.14)$$

counting multiplicities and $\Lambda \subset \Pi_h$ for some $h > 0$.

(ii) The sequence $\Lambda$ is incondensable.

(iii) For any $\varepsilon > 0$ the determinant $\Delta(\cdot)$ admits the following estimate from below

$$|\Delta(\lambda)| \geq C_\varepsilon (e^{-b_1 \text{Im} \lambda} + e^{-b_2 \text{Im} \lambda}), \quad \lambda \in \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} \mathbb{D}_\varepsilon(\lambda_n), \quad (4.15)$$

with some $C_\varepsilon > 0$.

(iv) The sequence $\Lambda$ can be ordered in such a way that the following asymptotical formula holds

$$\lambda_n = \frac{2\pi n}{b_2 - b_1} (1 + o(1)) \quad \text{as} \quad n \to \infty. \quad (4.16)$$

Proof. (i). Let $\Delta_0(\cdot)$ be the characteristic determinant of the problem (4.1)–(4.2) with $Q = 0$. It easily follows from (4.6) that $\Delta_0(\cdot)$ admits a representation

$$\Delta_0(\lambda) = \int_{b_1}^{b_2} e^{i\lambda t} d\sigma_0(t), \quad \lambda \in \mathbb{C}, \quad (4.17)$$

with a piecewise constant function $\sigma_0(\cdot)$ having precisely four jump-points $\{0, b_1, b_1 + b_2, b_2\}$. In particular,

$$\sigma_0(b_1 + 0) - \sigma_0(b_1) = J_{32} \neq 0 \quad \text{and} \quad \sigma_0(b_2) - \sigma_0(b_2 - 0) = J_{14} \neq 0. \quad (4.18)$$
Let us set
\[ g(t) = \begin{cases} -\frac{1}{\pi} g_1 \left( \frac{t}{b_1} \right), & t \in [b_1, 0), \\ \frac{1}{b_2} g_2 \left( \frac{t}{b_2} \right), & t \in [0, b_2], \end{cases} \quad (4.19) \]
and
\[ \sigma(t) = \sigma_0(t) + \int_{b_1}^{t} g(s) ds. \quad (4.20) \]
Combining these notations with formulas (4.8) and (4.17) we arrive at the following representation for the characteristic determinant
\[ \Delta(\lambda) = \int_{b_1}^{b_2} e^{it\lambda} d\sigma(t), \quad \lambda \in \mathbb{C}, \quad (4.21) \]
It follows from (4.20) and (4.18) that
\[ \sigma(b_1 + 0) - \sigma(b_1) = J_{32} \neq 0 \quad \text{and} \quad \sigma(b_2) - \sigma(b_2 - 0) = J_{14} \neq 0. \quad (4.22) \]
Due to the property (4.22) representation (4.21) ensures that \( \Delta(\cdot) \) is a sine-type function with \( h_{\Delta_0}(\pi/2) = -b_1 \) and \( h_{\Delta_0}(-\pi/2) = b_2 \) (see [25]). Moreover, statement (i) is also implied by the representation (4.21) (see [23, Chapter 1.4.3]).

(ii) and (iii). These statements coincide with the corresponding statements of [20, Lemmas 3 and 4] for sine-type functions (see also [25, Lemma 22.1] in connection with part (iii)).

(iv) The determinant \( \Delta(\cdot) \) belongs to the class \( A \) since its zeros lie in the strip \( \Pi_h \) (this fact is also immediate from representation (4.21)). Therefore it follows from [23, Theorem 1.4.6] that for any \( \varepsilon \in (0, \pi/2) \)
\[ \lim_{t \to \infty} \frac{n_0^{(\varepsilon)}(t)}{t} = \frac{2\pi}{b_2 - b_1}. \quad (4.23) \]
Here \( n_0^{(\varepsilon)}(t) = \text{card} \{ n \in \mathbb{Z} : |\lambda_n| < t, |\arg \lambda_n| < \varepsilon \} \) is the number of zeros of \( \Delta(\cdot) \) in the domain \( \{ z : |\arg z| < \varepsilon, |z| < t \} \) counting multiplicity, and \( n_0^{(\varepsilon)}(t) = \text{card} \{ n \in \mathbb{Z} : |\lambda_n| < t, |\pi - \arg \lambda_n| < \varepsilon \} \). Since \( \Lambda \) lies in the strip \( \Pi_h \), asymptotic formula (4.16) directly follows from (4.23) (see e.g. [48, Proposition 13.1]). \( \square \)

Clearly, the conclusions of Proposition 4.6 are valid for the perturbation determinant \( \Delta_0(\cdot) \) given by \( (4.6) \). Let \( \Lambda_0 = \{ \lambda_n^0 \}_{n \in \mathbb{Z}} \) be the sequence of its zeros counting multiplicity. Let us order the sequence \( \Lambda_0 \) in a (possibly non-unique) way such that \( \text{Re} \lambda_n^0 \leq \text{Re} \lambda_{n+1}^0, \quad n \in \mathbb{Z} \).

**Proposition 4.7.** Let \( Q \in L^1[0, 1] \otimes \mathbb{C}^{2 \times 2} \), let boundary conditions (4.2) be regular, and let \( \Delta(\cdot) \) be the corresponding characteristic determinant. Then the sequence \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}} \) of its zeros can be ordered in such a way that the following asymptotic formula holds
\[ \lambda_n = \lambda_n^0 + o(1), \quad n \to \infty, \quad n \in \mathbb{Z}. \quad (4.24) \]

**Proof.** Let \( \varepsilon \in (0, 1) \). By Proposition 4.6(iii) there exists \( C_\varepsilon > 0 \) such that the estimate (4.15) holds. Combining Lemma 4.1 with Lemma 3.5 yields the following estimate
\[ |\Delta(\lambda) - \Delta_0(\lambda)| < 4^{-1} C_\varepsilon (e^{-b_1 \text{Im} \lambda} + e^{-b_2 \text{Im} \lambda} + 2\varepsilon) \leq 2^{-1} C_\varepsilon (e^{-b_1 \text{Im} \lambda} + e^{-b_2 \text{Im} \lambda}). \quad |\lambda| \geq M_\varepsilon, \quad (4.25) \]
with certain \( M_\varepsilon > 0 \). Here in the last inequality we have used that \( b_1 < 0 < b_2 \).

Due to estimates (4.15) and (4.25), the Rouche theorem implies that all zeros of \( \Delta(\cdot) \) lie in the domain
\[ \Omega_\varepsilon := \mathbb{D}_{M_\varepsilon}(0) \cup \Omega_\varepsilon, \quad \Omega_\varepsilon := \bigcup_{n \in \mathbb{Z}} \mathbb{D}_{\varepsilon}(\lambda_n^0), \quad (4.26) \]
and in each connected component of $\Omega_\varepsilon$ the functions $\Delta(\cdot)$ and $\Delta_0(\cdot)$ have the same number of zeros counting multiplicity. Since in accordance with Proposition 4.6(ii), the sequence of zeros \( \{\lambda_n^0\}_{n \in \mathbb{Z}} \) is incondensable, Lemma 4.3 implies that for $\varepsilon$ small enough each connected component of $\Omega_\varepsilon$ contains at most $N_0$ discs $\mathbb{D}_\varepsilon(\lambda_n^0)$ with $N_0$ not depending on $\varepsilon$. Hence the diameter of each connected component of $\Omega_\varepsilon$ does not exceed $2\varepsilon N_0$. Since $\varepsilon > 0$ is arbitrary small, the latter implies the desired asymptotic formula (4.24).

\[ \square \]

5. Strictly regular boundary conditions

Assuming boundary conditions (4.2) to be regular, let us rewrite them in a more convenient form. Since $J_{14} \neq 0$, the inverse matrix $A_{14}^{-1}$ exists. Therefore writing down boundary conditions (4.2) as the vector equation $(U_1(y)) = 0$ and multiplying it by the matrix $A_{14}^{-1}$ we transform them as follows

\[
\begin{align*}
\hat{U}_1(y) &= y_1(0) + by_2(0) + ay_1(1) = 0, \\
\hat{U}_2(y) &= dy_2(0) + cy_1(1) + y_2(1) = 0,
\end{align*}
\]

with some $a, b, c, d \in \mathbb{C}$. Now $J_{14} = 1$ and the boundary conditions (5.1) are regular if and only if $J_{32} = ad - bc \neq 0$. So, the characteristic determinants $\Delta_0(\cdot)$ and $\Delta(\cdot)$ take the form

\[
\begin{align*}
\Delta_0(\lambda) &= d + ae^{i(b_1 + b_2)\lambda} + (ad - bc)e^{ib_1\lambda} + e^{ib_2\lambda}, \\
\Delta(\lambda) &= d + ae^{i(b_1 + b_2)\lambda} + (ad - bc)\varphi_{11}(\lambda) + \varphi_{22}(\lambda) + c\varphi_{12}(\lambda) + b\varphi_{21}(\lambda).
\end{align*}
\]

Now we are ready to introduce a notion of strictly regular boundary conditions.

**Definition 5.1.** Boundary conditions (4.2) are called **strictly regular**, if they are regular, i.e. $J_{14}J_{32} \neq 0$, and the sequence of zeros $\Lambda = \{\lambda_n^0\}_{n \in \mathbb{Z}}$ of the characteristic determinant $\Delta_0(\cdot)$ is asymptotically separated. In particular, there exists $n_0$ such that zeros $\{\lambda_n^0\}_{|n| > n_0}$ are geometrically and algebraically simple.

It follows from Proposition 4.7 that the sequence $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ of zeros of $\Delta(\cdot)$ is asymptotically separated if the boundary conditions are strictly regular.

**Remark 5.2.** Let us list some types of strictly regular boundary conditions (5.1). In all of these cases the set of zeros of $\Delta_0$ is a union of finite number of arithmetic progressions.

**(i)** Separated boundary conditions $(a = d = 0, bc \neq 0)$ are always strictly regular.

**(ii)** Let $b_1/b_2 \in \mathbb{Q}$, i.e. $b_1 = -n_1b, b_2 = n_2b, n_1, n_2 \in \mathbb{N}, b > 0$. Since $ad \neq bc$, $\Delta_0(\cdot)$ is a polynomial at $e^{ib\lambda}$ of degree $n_1 + n_2$. Hence, boundary conditions (5.1) are strictly regular if and only if this polynomial does not have multiple roots. In particular, regular boundary conditions (5.1) for Dirac operator are strictly regular if and only if $(a - d)^2 \neq -4bc$. 

**Lemma 5.3.** Let $bc = 0$ and boundary conditions (5.1) are regular (i.e. $ad \neq 0$).

**(i)** Let

\[
b_1 \ln |d| + b_2 \ln |a| \neq 0. \tag{5.4}
\]

Then conditions (5.1) are strictly regular.

**(ii)** Let $b_1/b_2 \not\in \mathbb{Q}$. Then condition (5.4) is necessary for the strict regularity of boundary conditions (5.1).

**(iii)** Let $b_1/b_2 \in \mathbb{Q}$ and condition (5.4) is violated. Then boundary conditions (5.1) are strictly regular if and only if

\[
\frac{b_1 \arg(-d) + b_2 \arg(-a)}{2\pi \gcd(b_1, b_2)} \not\in \mathbb{Z}, \tag{5.5}
\]
where \(\gcd(b_1, b_2)\) is the greatest common divisor of real numbers \(b_1, b_2\), i.e. the largest number \(b > 0\) such that \(b_1/b\) and \(b_2/b\) are integers.

**Proof.** (i) Since \(bc = 0\), the characteristic determinant \(\Delta_0(\cdot)\) in (5.2) becomes

\[
\Delta_0(\lambda) = d + ae^{i(b_1+b_2)\lambda} + ade^{ib_1\lambda} + e^{ib_2\lambda} = (1 + ae^{ib_1\lambda})(d + e^{ib_2\lambda}).
\]

(5.6)

Let \(\Lambda_1 = \{\lambda_{1,n}\}_{n \in \mathbb{Z}}\) and \(\Lambda_2 = \{\lambda_{2,n}\}_{n \in \mathbb{Z}}\) be the sequences of zeros of the first and second factor, respectively. Clearly,

\[
\lambda_{1,n} = \frac{\arg(-a^{-1}) + 2\pi n}{b_1} + \frac{\ln |a|}{b_1}, \quad \lambda_{2,n} = \frac{\arg(-d) + 2\pi n}{b_2} - \frac{\ln |d|}{b_2}, \quad n \in \mathbb{Z}.
\]

(5.7)

Thus, \(\Lambda_1\) and \(\Lambda_2\) are algebraically simple and constitute two arithmetic progressions that lie on two lines parallel to the real axis. Condition (5.4) written in the form \(\frac{\ln |a|}{b_1} \neq \frac{\ln |d|}{b_2}\) implies that these horizontal lines are different. It follows that the sequence of zeros of \(\Delta_0(\cdot)\) is separated and hence boundary conditions (5.1) are strictly regular.

(ii) Now assume that \(\alpha := -b_1/b_2 \notin \mathbb{Q}\) and condition (5.4) is violated. In this case \(\text{Im } \lambda_{1,n} = \text{Im } \lambda_{2,m} = \frac{\ln |a|}{b_1} = -\frac{\ln |d|}{b_2}\) for \(n, m \in \mathbb{Z}\), i.e. the progressions \(\Lambda_1\) and \(\Lambda_2\) lie on the same line parallel to the real axis. Hence

\[
|\lambda_{1,n} - \lambda_{2,m}| = 2\pi b_1^{-1}|r + n + \alpha m|, \quad r = \frac{\arg(-a^{-1}) + \alpha \arg(-d)}{2\pi} \in \mathbb{R}.
\]

(5.8)

Since \(\alpha\) is irrational, the Kronecker theorem ensures that for any \(\varepsilon > 0\) and \(M > 0\) there exist \(n, m \in \mathbb{Z}\) such that \(|n|, |m| > M\) and \(|r + n + \alpha m| < \varepsilon\). This means that the zeros of \(\Delta_0(\cdot)\) are not asymptotically separated, which proves the result.

(iii) Finally, assume that condition (5.4) is violated while \(\alpha = -b_1/b_2 \in \mathbb{Q}\). Now as on the previous step, the progressions \(\Lambda_1\) and \(\Lambda_2\) lie on the same line parallel to the real axis and condition (5.8) holds. Since \(b_1/b_2\) is rational, the union of the arithmetic progressions \(\Lambda_1\) and \(\Lambda_2\) is asymptotically separated if and only if they have no common entries. Due to (5.8) this is equivalent to the fact that Diophantine equation \(n + \alpha m = -r\) does not have integer solutions \(n, m\). It is well-known that such equation has solutions if and only if \(r/\gcd(\alpha, 1) \in \mathbb{Z}\). Since \(\arg(-a^{-1}) = -\arg(-a)\), this is equivalent to the condition opposite to (5.5), which completes the proof. \(\square\)

**Remark 5.4.** Consider the case \(b = c = 0\). It includes periodic \((a = d = -1)\) and antiperiodic \((a = d = 1)\) boundary conditions. So, it follows from the statements (i) (see (5.4)) and (ii) that the periodic and antiperiodic BC are strictly regular if and only if \(b_1 + b_2 \neq 0\). This fact demonstrates a substantial difference between Dirac and Dirac type operators.

The following result demonstrates that in the case \(b_1/b_2 \notin \mathbb{Q}\) the problem of strict regularity of boundary conditions (5.1) is much more complicated than the one discussed in Remark 5.2 and Lemma 5.3.

**Proposition 5.5.** (i) Let \(\alpha := -b_1/b_2 \notin \mathbb{Q}\), \(a = 0\), \(bc, d \in \mathbb{R}\setminus\{0\}\), then boundary conditions (5.1) are strictly regular if and only if

\[
d \neq -((\alpha + 1)(|bc|\alpha^{-\alpha})^{1/\alpha+1}).
\]

(5.9)

(ii) Let \(a = 0\), \(bc \neq 0\), \(b_1 = -n_1 b\), \(b_2 = n_2 b\), \(n_1, n_2 \in \mathbb{N}\), \(b > 0\) and \(\gcd(n_1, n_2) = 1\). Then boundary conditions (5.1) are strictly regular if and only if

\[
n_1^{n_1} n_2^{n_2} (-d)^{n_1+n_2} \neq (n_1 + n_2)^{n_1+n_2} (-bc)^{n_2}.
\]

(5.10)
Proof. (i) It follows from (5.2) that
\[ \Delta_0(\lambda) = d - bc \cdot e^{ib_1\lambda} + e^{ib_2\lambda}. \] (5.11)
Let \( b_2\lambda = : \pi x + iy, x, y \in \mathbb{R} \). Then \( e^{ib_1\lambda} = e^{-i\alpha \pi x + \alpha y}, e^{ib_2\lambda} = e^{i\pi x - y} \). Since \( bc, d \in \mathbb{R} \), equation \( \Delta_0(\lambda) = 0 \) is equivalent to the system
\[ \begin{align*}
 e^{-y} \cos \pi x &= bc \cdot e^{\alpha y} \cos \alpha \pi x - d, \\
 e^{-y} \sin \pi x &= bc \cdot e^{\alpha y} \sin \alpha \pi x.
\end{align*} \] (5.12)
From the second equation in (5.12) we have
\[ y = \frac{-\ln \left( \frac{bc \cdot \sin \alpha \pi x}{\sin \pi x} \right)}{\alpha + 1}, \quad bc \cdot \sin \alpha \pi x < 0. \] (5.13)
Substituting it to the first equation we get
\[ \left( \frac{-bc \cdot \sin \alpha \pi x}{\sin \pi x} \right)^{\frac{1}{\alpha + 1}} \cos \pi x = bc \left( \frac{-bc \cdot \sin \alpha \pi x}{\sin \pi x} \right)^{\frac{\alpha}{\alpha + 1}} \cos \alpha \pi x - d, \] (5.14)
which is equivalent to
\[ \left( \frac{-bc \cdot \sin \alpha \pi x}{\sin \pi x} \right)^{\frac{1}{\alpha + 1}} \frac{\sin(\alpha + 1)\pi x}{\sin \alpha \pi x} = -d. \] (5.15)
For simplicity we assume that \( bc = -1 \) and \( 0 < \alpha < 1 \). Then summarizing all previous formulas we see that \( \lambda \) is zero of \( \Delta_0(\cdot) \) if and only if
\[ \begin{align*}
 b_2\lambda &= \pi x + iy, \quad x, y \in \mathbb{R}, \\
 \frac{\sin \alpha \pi x}{\sin \pi x} > 0, \\
 y &= -\ln \left( \frac{\sin \alpha \pi x}{\sin \pi x} \right), \\
 f(x) &= \left( \frac{\sin \alpha \pi x}{\sin \pi x} \right)^{\frac{1}{\alpha + 1}} \frac{\sin(\alpha + 1)\pi x}{\sin \alpha \pi x} = -d.
\end{align*} \] (5.16)
Note that the second relation in (5.16) is equivalent to
\[ x \in A := \left( \bigcup_{n \in \mathbb{Z}} (2n, 2n + 1) \right) \cap \left( \bigcup_{n \in \mathbb{Z}} \left( \frac{2n}{\alpha}, \frac{2n + 1}{\alpha} \right) \right) \]
\[ \cup \left( \bigcup_{n \in \mathbb{Z}} (2n - 1, 2n) \right) \cap \left( \bigcup_{n \in \mathbb{Z}} \left( \frac{2n - 1}{\alpha}, \frac{2n}{\alpha} \right) \right). \] (5.17)
Let us describe the set \( A \) in a more explicit way. Let \( n \in \mathbb{Z} \) be fixed and let’s find intersection \( A \cap (n, n + 1) \). Since \( \alpha < 1 \), then at most two intervals of the form \((m/\alpha, (m + 1)/\alpha)\), \( m \in \mathbb{Z} \), intersect with \((n, n + 1)\). There are two cases possible. First, for some \( m \in \mathbb{Z} \) interval \((m/\alpha, (m + 1)/\alpha)\) fully covers interval \((n, n + 1)\) (i.e. \( m/\alpha < n < n + 1 < (m + 1)/\alpha \)). In this case, if \( n \) and \( m \) are of the same parity then \((n, n + 1) \in A\), otherwise \( A \cap (n, n + 1) = \emptyset \). The second case, is when for some \( m \in \mathbb{Z} \) we have \( n < m/\alpha < n + 1 \). In this case it is clear that exactly one of the intervals \((n, m/\alpha)\) and \((m/\alpha, n + 1)\) belongs to \( A \) depending on parity of \( m - n \). Thus, the domain of the 4th equation in (5.16) is the union \( \cup_{n \in \mathbb{Z}} I_n \), where \( I_n \) is possibly empty subinterval of \((n, n + 1)\).
Put \( r(x) := \sin \alpha \pi x / \sin \pi x \). Straightforward calculation shows that for \( x \in I_n, n \in \mathbb{Z} \), we have

\[
f'(x) = \frac{1}{(\alpha + 1) \sin \alpha \pi x} \left( (\alpha + 1)^2 \cos(\alpha + 1) \pi x - (\cot \pi x + \alpha^2 \cot \alpha \pi x) \sin(\alpha + 1) \pi x \right) \]

\[
= \frac{-1}{(\alpha + 1) \sin \alpha \pi x} \left( (\alpha + 1)^2 \sin \pi x \sin \alpha \pi x + \left( \sqrt{r(x)} \cos \pi x + \frac{\alpha \cos \alpha \pi x}{\sqrt{r(x)}} \right)^2 \right). \tag{5.18}
\]

Since \( \sin \pi x \) and \( \sin \alpha \pi x \) are of the same sign on each interval \( I_n \) it is clear that \( f'(x) \) has fixed sign on \( I_n \). Hence, \( f(x) \) is strictly monotonic on \( I_n \). In particular, the equation \( f(x) = -d \), \( x \in I_n \), has at most one solution. Denote it by \( x_n \) if it exists and let \( y_n \) be the corresponding value of \( y \) from the third equation in (5.16). Clearly, all \( x_n \) are different and hence all zeros of \( \Delta_0(\cdot) \) are simple.

Now let (5.9) be satisfied and assume that boundary conditions are not strictly regular. Since zeros of \( \Delta_0(\cdot) \) are simple it means that there exists infinite set \( S \subset \mathbb{N} \) such that \( x_{n-1}, x_n \) exist for \( n \in S \) and

\[
x_n - x_{n-1} \to 0, \quad y_n - y_{n-1} \to 0 \quad \text{as} \quad n \to \infty, \quad n \in S. \quad \text{(5.19)}
\]

Since zeros of \( \Delta_0(\cdot) \) lie in the strip it follows that \( |y_n| \leq H, n \in S \), with some \( H > 0 \). Hence, the third relation in (5.16) implies that

\[
0 < C_1 < t_n := \frac{\sin \alpha \pi x_n}{\sin \pi x_n} < C_2, \quad n \in S, \quad \text{(5.20)}
\]

with some \( C_1, C_2 > 0 \). Since \( t_n = e^{-(\alpha + 1) y_n} \), it is clear that \( y_n - y_{n-1} \to 0 \) is equivalent to \( t_n - t_{n-1} \to 0 \).

Taking into account the form of the set \( A \) described after the formula (5.17) we see that there exists unique \( m = m_n \in \mathbb{Z} \) such that either \( x_{n-1} < m/\alpha < n < x_n \) or \( x_{n-1} < n < m/\alpha < x_n \). Moreover, \( m_n \) has the same parity as \( n \). Let \( S_1(S_2) \) be the set of those \( n \) in \( S \) for which the first (the second) inequality is satisfied. Since \( S \) is infinite and \( S = S_1 \cup S_2 \), either \( S_1 \) or \( S_2 \) is infinite. First consider the case when \( S_1 \) is infinite. For \( n \in S_1 \) we put

\[
\delta_{0n} := m/\alpha - x_{n-1}, \quad \varepsilon_n = n - m/\alpha, \quad \delta_{1n} := x_n - n. \tag{5.21}
\]

Since \( x_{n-1} < m/\alpha < n < x_n \), then \( \varepsilon_n, \delta_{0n}, \delta_{1n} > 0 \). Further, since \( x_n - x_{n-1} \to 0 \) as \( n \to \infty \), \( n \in S_1 \), then \( \varepsilon_n \to 0, \delta_{0n} \to 0, \delta_{1n} \to 0 \) as \( n \to \infty \), \( n \in S_1 \). Hence, for large \( n \in S_1 \) we have taking into account that \( m \) and \( n \) are of the same parity

\[
t_n = \frac{\sin \alpha \pi x_n}{\sin \pi x_n} = \frac{\sin \pi (m + \alpha (\delta_{1n} + \varepsilon_n))}{\sin \pi (n + \delta_{1n})} = \frac{\sin \pi \alpha (\delta_{1n} + \varepsilon_n)}{\sin \pi \delta_{1n}} > \alpha, \tag{5.22}
\]

\[
t_{n-1} = \frac{\sin \alpha \pi x_{n-1}}{\sin \pi x_{n-1}} = \frac{\sin \pi (m - \alpha \delta_{0n})}{\sin \pi (n - \delta_{0n} - \varepsilon_n)} = \frac{\sin \pi \alpha \delta_{0n}}{\sin \pi (\delta_{0n} + \varepsilon_n)} < \alpha. \tag{5.23}
\]

Here we used the inequality \( \sin \alpha u > \alpha \sin v \) for \( 0 < \alpha < 1 \) and \( 0 < v < u < \pi \).

Since \( t_n - t_{n-1} \to 0 \), it follows from (5.22) and (5.23) that \( t_n \to \alpha, t_{n-1} \to \alpha \) as \( n \to \infty \), \( n \in S_1 \). This implies that \( \varepsilon_n/\delta_{1n} \to 0 \) as \( n \to \infty \), \( n \in S_1 \). Indeed, since \( \varepsilon_n \to 0 \) and \( \delta_{1n} \to 0 \) as \( n \to \infty \), \( n \in S_1 \), then

\[
\alpha = \lim_{n \to \infty} \frac{t_n}{n} = \lim_{n \to \infty} \frac{\sin \pi \alpha (\delta_{1n} + \varepsilon_n)}{\sin \pi \delta_{1n}} = \lim_{n \to \infty} \frac{\alpha (\delta_{1n} + \varepsilon_n)}{\delta_{1n}} = \alpha + \alpha \lim_{n \to \infty} \frac{\varepsilon_n}{\delta_{1n}}. \tag{5.24}
\]
Further, note that
\[
\frac{\sin(\alpha + 1)\pi x_n}{\sin \alpha \pi x_n} = \frac{\sin \pi (n + \delta_{1n} + m + \alpha(\delta_{1n} + \varepsilon_n))}{\sin \pi (m + \alpha(\delta_{1n} + \varepsilon_n))} = \frac{\sin \pi ((\alpha + 1)\delta_{1n} + \alpha\varepsilon_n)}{\sin \pi (\alpha\delta_{1n} + \alpha\varepsilon_n)}.
\]
(5.25)

Finally, taking into account the last relation and the fact that \(\varepsilon_n/\delta_{1n} \to 0\) and \(t_n \to \alpha\) as \(n \to \infty\), \(n \in S_1\), we have
\[
-d = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} \frac{\sin(\alpha + 1)\pi x_n}{\sin \alpha \pi x_n} = \lim_{n \to \infty} \frac{\sin \pi ((\alpha + 1)\delta_{1n} + \alpha\varepsilon_n)}{\sin \pi (\alpha\delta_{1n} + \alpha\varepsilon_n)}
\]
\[
= \frac{\alpha^\alpha}{\alpha + 1 + \alpha\varepsilon_n/\delta_{1n}} = \frac{\alpha^\alpha}{\alpha + 1} = (\alpha + 1)^{-\frac{\alpha}{\alpha + 1}}.
\]
(5.26)

Since \(bc = -1\) this contradicts to (5.9). Therefore, zeros of \(\Delta_0(\cdot)\) are asymptotically separated. The case of infinite \(S_2\) is considered similarly.

Now let’s prove that opposite statement. As above, for simplicity we assume that \(bc = -1\) and \(\alpha < 1\). We need to prove that if \(d = -\alpha(\alpha + 1)\frac{\alpha}{\alpha + 1}\) then zeros of \(\Delta_0(\cdot)\) are not asymptotically separated. For \(n \in \mathbb{Z}\) we set \(m = m_n = \lfloor \alpha n \rfloor\). Let \(S \subset \mathbb{Z}\) be some infinite set such that \(m_n - n \to 0\) as \(n \to \infty\) and \(m_n\) is of the same parity as \(n\). Since \(\alpha \notin \mathbb{Q}\) it is clear that such set exists. Let us prove that for large enough \(n \in S\) the equation \(f(x) = -d\) has zeros \(x_{n-1}, x_n\) such that \(x_{n-1} < m/\alpha < n < x_n\) and \(x_n - x_{n-1} \to 0\) as \(n \to \infty\). Put
\[
g(u, v) := \left(\frac{\sin \alpha u}{\sin v}\right)^{\alpha + 1} \frac{\sin(\alpha u + v)}{\sin \alpha u}.
\]
(5.27)

Clearly
\[
f(x_{n-1}) = g(\pi \delta_{0m}, \pi(\delta_{0n} + \varepsilon_n)), \quad \text{and} \quad f(x_n) = g(\pi(\delta_{1n} + \varepsilon_n), \pi(\delta_{1n}))
\]
(5.28)

where \(\varepsilon_n, \delta_{0m}, \delta_{1n}\) are defined in (5.21). Due to the special form of \(d\), equation \(g(u, v) = -d\) is equivalent to
\[
f(u, v) := \left(\frac{\sin(\alpha u + v)}{\alpha + 1}\right)^{\alpha + 1} - \sin v \cdot \left(\frac{\sin \alpha u}{\alpha}\right)^\alpha = 0.
\]
(5.29)

It is easy to prove that
\[
f(u, 0) > 0, f(u, u) < 0, f(u, 2u) > 0, \quad 0 < u < 1/2.
\]
(5.30)

Hence for each \(u \in (0, 1/2)\) there exists \(v_+ \in (u, 2u)\) and \(v_- \in (0, u)\) such that \(f(u, v_\pm) = 0\). Clearly, \(v_\pm = v_\pm(u)\) is continuous at \(u\). Hence for sufficiently small \(\varepsilon > 0\) there exists \(u_\varepsilon^\pm, v_\varepsilon^\pm\) such that \(u_\varepsilon^\pm - v_\varepsilon^\pm = \pm \varepsilon\), \(g(u_\varepsilon^\pm, v_\varepsilon^\pm) = -d\) and \(u_\varepsilon^\pm, v_\varepsilon^\pm \to 0\) as \(\varepsilon \to 0\). Applying this fact for \(\varepsilon_n\) shows existence of needed \(x_{n-1}\) and \(x_n\). It is easy to prove from \(f(x_n) = f(x_{n-1}) = -d\) that \(t_n \to \alpha\) and \(t_{n-1} \to \alpha\) as \(n \to \infty\) where \(t_n, t_{n-1}\) are defined in (5.22)–(5.23). Which implies \(y_n - y_{n-1} \to \infty\) and shows that corresponding zeros of \(\Delta_0(\cdot)\) are not asymptotically separated.

(ii) Since \(b_1/b_2 \in \mathbb{Q}\) the set of zeros of \(\Delta_0(\cdot)\) is the union of finite number of arithmetic progressions. Hence zeros are asymptotically separated if and only if \(\Delta_0(\cdot)\) does not have multiple zeros, which is equivalent to the fact that \(\Delta_0(\cdot)\) and \(\Delta_0'(\cdot)\) have no common zeros. When \(a = 0\) we have \(\Delta_0'(\lambda) = ib_2 e^{ib_2 \lambda} - ib_1 \cdot bc \cdot e^{ib_1 \lambda}\). Hence zeros of \(\Delta_0'(\cdot)\) can be found explicitly. Substituting these values into \(\Delta_0(\lambda)\) and performing straightforward calculations we see that \(\Delta_0(\cdot)\) and \(\Delta_0'(\cdot)\) have no common zeros if and only if condition (5.10) is satisfied. \(\square\)
6. The Riesz basis property of root vectors system

6.1. Some auxiliary results. Recall the following definition.

**Definition 6.1.** (i) Let \( \mathfrak{H} \) be a separable Hilbert space. The vectors system \( \{f_k\}_{k \in \mathbb{Z}} \subset \mathfrak{H} \) is called Besselian in \( \mathfrak{H} \) if \( \{(f, f_k)\}_{k \in \mathbb{Z}} \subset l^2(\mathbb{Z}), f \in \mathfrak{H} \).

(ii) The vectors system \( \{f_k\} \) is called a Riesz basis in \( \mathfrak{H} \) if it constitutes a basis equivalent to an orthonormal one, i.e. there exists a linear homeomorphism \( T \) in \( \mathfrak{H} \) for which \( \{Tf_k\} \) is an orthonormal basis.

(iii) A sequence of subspaces \( \{\mathfrak{H}_k\}_{k=1}^\infty \) is called a Riesz basis of subspaces in \( \mathfrak{H} \) if there exists a complete sequence of mutually orthogonal subspaces \( \{\mathfrak{H}'_k\}_{k=1}^\infty \) and a bounded operator \( T \) in \( \mathfrak{H} \) with bounded inverse such that \( \mathfrak{H}_k = T\mathfrak{H}'_k, k \in \mathbb{N} \).

(iv) It is said that a sequence \( \{f_k\}_{k=1}^\infty \) of vectors in \( \mathfrak{H} \) forms a Riesz basis with parentheses if each its finite subsequence is linearly independent, and there exists an increasing sequence \( \{n_k\}_{k=0}^\infty \subset \mathbb{N} \) such that \( n_0 = 1 \) and the sequence \( \mathfrak{H}_k := \text{span}\{f_j\}_{j=n_k-1}^{n_k-1} \) constitutes a Riesz basis of subspaces in \( \mathfrak{H} \). Subspaces \( \mathfrak{H}_k \) are called blocks.

Our investigation of the Riesz basis property of the root vectors system of the operator \( L(Q) \) is heavily relied on the following well-known Bari criterion.

**Theorem 6.2.** [17, Theorem VI.2.1] Let \( \mathfrak{H} \) be a separable Hilbert space. The vectors system \( \{f_k\}_{k \in \mathbb{Z}} \subset \mathfrak{H} \) forms a Riesz basis in \( \mathfrak{H} \) if and only if it is complete, minimal and Besselian in \( \mathfrak{H} \), and the corresponding biorthogonal system \( \{g_k\}_{k \in \mathbb{Z}} \) is also complete and Besselian.

It is well-known that the root vectors system of the operator \( L(Q) \) after proper normalization is biorthogonal to the root vectors system of the adjoint operator \( L(Q)^* \). In this connection we give the explicit form of the operator \( L(Q)^* \) in the case of boundary conditions (5.1).

**Lemma 6.3.** Let \( L(Q) \) be an operator corresponding to the problem (4.1), (5.1). Then the adjoint operator \( L^* := L(Q, U_1, U_2)^* \) is \( L^* = L(Q^*, U_{s_1}, U_{s_2}) \), i.e. it is given by the differential expression (4.1) with \( Q^*(x) = \begin{pmatrix} 0 & Q_{21}(x) \\ Q_{12}(x) & 0 \end{pmatrix} \) instead of \( Q \) and the boundary conditions

\[
\begin{cases}
U_{s1}(y) = k\bar{y}1(0) + y_2(0) + \bar{y}2(1) & = 0, \\
U_{s2}(y) = \bar{y}1(0) + y_1(1) + k^{-1}\bar{y}2(1) & = 0,
\end{cases}
\]

where \( k := -b_2b_1^{-1} \). Moreover, boundary conditions (6.1) are regular (strictly regular) simultaneously with boundary conditions (5.1).

The following lemma plays the key role in the proof of Theorem 1.1.

**Lemma 6.4.** Let \( Q \in L^1[0,1] \otimes \mathbb{C}^{2 \times 2} \) and let \( \Phi(\cdot, \lambda) \) and \( \Psi(\cdot, \lambda) \) be the fundamental matrix solutions of the equations (4.1) and (3.14) satisfying \( \Phi(0, \lambda) = \Psi(0, \lambda) = I_2 \), given by formulas (3.2) and (3.15), respectively. Let also \( \Phi_j(\cdot, \lambda) \) and \( \Psi_j(\cdot, \lambda), j \in \{1,2\} \), be the columns of these matrices (cf. (3.2) and (3.15)). Then for any incondensible sequence \( \{\mu_n\}_{n \in \mathbb{Z}} \) the systems \( \{\Phi_j(\cdot, \mu_n)\}_{n \in \mathbb{Z}} \) and \( \{\Psi_j(\cdot, \mu_n)\}_{n \in \mathbb{Z}} \) are Besselian in \( L^2[0,1] \otimes \mathbb{C}^2, j \in \{1,2\} \).
Proof. Consider the case of the system \( \{ \Phi_1(\cdot, \mu_n) \}_{n \in \mathbb{Z}} \). Let \( f := \text{col}(f_1, f_2) \in L^2[0, 1] \otimes \mathbb{C}^2 \). Taking into account formulas (3.3), (3.5) we get

\[
(f, \Phi_1(\cdot, \mu_n))_{L^2[0, 1] \otimes \mathbb{C}^2} = \int_0^1 \left( f_1(x)\varphi_{11}(x, \mu_n) + f_2(x)\varphi_{21}(x, \mu_n) \right) \, dx
\]

\[
= \int_0^1 f_1(x)e^{ib_1\mu_n x} \, dx + \sum_{j,k=1}^2 \int_0^1 f_j(x) \left( \int_0^x N_{jk}(x,t)e^{ib_k\mu_n t} \, dt \right) \, dx,
\]

where \( N_{j1}(\cdot, \cdot) := R_{j1}^+(\cdot, \cdot), \ N_{j2}(\cdot, \cdot) := R_{j2}^-(\cdot, \cdot), \ j \in \{1, 2\} \). Further, note

\[
\int_0^1 f_j(x) \left( \int_0^x N_{jk}(x,t)e^{ib_k\mu_n t} \, dt \right) \, dx = \int_0^1 g_{jk}(t)e^{ib_k\mu_n t} \, dt
\]

where

\[
g_{jk}(t) := \int_t^1 N_{jk}(x,t)f_j(x) \, dx, \quad j, k \in \{1, 2\}.
\]

By Proposition 3.1, \( N_{jk}(\cdot, \cdot) \in X_1(\Omega) \cap X_\infty(\Omega) \). Therefore by Lemma 2.1 the Volterra type operators

\[
N_{jk} : f \rightarrow \int_0^x N_{jk}(x,t)f(t) \, dt \quad \text{and} \quad N_{jk}^* : f \rightarrow \int_1^x N_{jk}(x,t)f_j(x) \, dx
\]

are bounded in \( L^2[0, 1] \), hence \( g_{jk} \in L^2[0, 1], \ j, k \in \{1, 2\} \). Taking this inclusion into account and inserting expressions (6.3) into (6.2) one rewrites this equality as

\[
(f, \Phi_1(\cdot, \mu_n))_{L^2[0, 1] \otimes \mathbb{C}^2} = \left( f, e^{ib_1\mu_n x} \right)_{L^2[0, 1]} + \sum_{j,k=1}^2 \left( g_{jk}, e^{ib_k\mu_n t} \right)_{L^2[0, 1]}.
\]

Since the sequence \( \{ \mu_n \}_{n \in \mathbb{Z}} \) is incondensible, then by [20, Lemma 2] the sequence of exponents \( \{e^{ib_1\mu_n x}\}_{n \in \mathbb{Z}} \) is Besselian in \( L^2[0, 1] \). The latter implies

\[
\{(g_{jk}, e^{ib_k\mu_n t})_{L^2[0, 1]}\}_{n \in \mathbb{Z}} \in l^2(\mathbb{Z}) \quad \text{and} \quad \{(f, e^{ib_k\mu_n t})_{L^2[0, 1]}\}_{n \in \mathbb{Z}} \in l^2(\mathbb{Z}).
\]

Combining these inclusions with representation (6.5) shows that the system \( \{\Phi_1(\cdot, \mu_n)\}_{n \in \mathbb{Z}} \) is Besselian. The systems of functions \( \{\Phi_2(\cdot, \mu_n)\}_{n \in \mathbb{Z}} \) and \( \{\Psi_j(\cdot, \mu_n)\}_{n \in \mathbb{Z}} \) are treated similarly. \( \square \)

6.2. Proof of the main result. Now we are ready to prove the main result of the paper.

Proof of Theorem 1.1. According to Proposition 4.7 and Definition 5.1, the operator \( L(Q) \) has countably many eigenvalues \( \{\lambda_n\}_{n \in \mathbb{Z}} \). Moreover, they are of finite multiplicity; asymptotically simple and separated, and are located in the strip \( \Pi_h = \{ \lambda \in \mathbb{C} : |\text{Im} \lambda| \leq h \} \).

Since boundary conditions are regular, one can transform them to the form (5.1).

(i) In this step assume that \(|b| + |c| \neq 0\). Without loss of generality it suffices to consider the case \( b \neq 0 \). Let \( \mathfrak{F} = \{f_n\}_{n \in \mathbb{Z}} \) and \( \mathfrak{G} = \{g_n\}_{n \in \mathbb{Z}} \) be the system of root vectors of the operators \( L(Q) = L(Q, \hat{U}_1, \hat{U}_2) \) and \( L(Q)^* = L(Q^*, U_{11}, U_{12}) \), respectively. By [34, Theorem 1.2], each of the systems \( \mathfrak{F} \) and \( \mathfrak{G} \) is complete and minimal in \( L^2[0, 1] \otimes \mathbb{C}^2 \). Therefore these systems can be chosen to be biorthogonal to each other.

First we indicate the explicit form of the functions \( f_n \) and \( g_n \) for \( n \) large enough.
To this end one easily gets from (3.2) and (5.1) that
\[
\begin{align*}
\hat{U}_1(\Phi_1(\cdot, \lambda)) &= \varphi_{11}(0, \lambda) + b\varphi_{21}(0, \lambda) + a\varphi_{11}(1, \lambda) = 1 + a\varphi_{11}(\lambda), \\
\hat{U}_1(\Phi_2(\cdot, \lambda)) &= \varphi_{12}(0, \lambda) + b\varphi_{22}(0, \lambda) + a\varphi_{12}(1, \lambda) = b + a\varphi_{12}(\lambda).
\end{align*}
\] (6.6) (6.7)

Since \(\varphi_{12}(\lambda_n) = o(1)\) as \(n \to \infty\) (see (3.12)) and \(b \neq 0\), one gets that \(b + a\varphi_{12}(\lambda_n) \neq 0\) for \(|n| \geq n_1\) with some \(n_1 \in \mathbb{N}\). Therefore one derives from (6.6), (6.7), and (3.2) that the vector-function
\[
f_n(\cdot) := \hat{U}_1(\Phi_2(\cdot, \lambda_n))\Phi_1(\cdot, \lambda_n) - \hat{U}_1(\Phi_1(\cdot, \lambda_n))\Phi_2(\cdot, \lambda_n)
\]
\[
= (b + a\varphi_{12}(\lambda_n))\Phi_1(\cdot, \lambda_n) - (1 + a\varphi_{11}(\lambda_n))\Phi_2(\cdot, \lambda_n), \quad |n| \geq n_1,
\] (6.8)
is a non-trivial eigenfunction of the operator \(L(Q)\) corresponding to the eigenvalue \(\lambda_n\). Since boundary conditions (5.1) are strictly regular, it follows from Proposition 4.7 that the sequence \(\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}\) of the eigenvalues of \(L(Q)\), i.e. the zeros of \(\Delta(\cdot)\), is asymptotically separated. In particular, there exists \(n_0 \in \mathbb{N}\) such that the eigenvalues of \(L(Q)\) are geometrically and algebraically simple. Therefore \(f_n\) is the unique up to a multiplicative constant eigenfunction of the operator \(L(Q)\) corresponding to \(\lambda_n\) for \(|n| \geq \max\{n_0, n_1\}\).

Similarly, one easily gets from (6.1) that
\[
U_{\ast 1}(\Psi_1(\cdot, \lambda)) = k\overline{b} + \overline{d}\psi_{21}(\lambda), \quad U_{\ast 1}(\Psi_2(\cdot, \lambda)) = 1 + \overline{d}\psi_{22}(\lambda).
\] (6.9)

Moreover, in accordance with (3.12) there exists \(n_2 \in \mathbb{N}\) such that \(k\overline{b} + \overline{d}\psi_{21}(\lambda_n) \neq 0\) for \(|n| \geq n_2\). Therefore Lemma 6.3 ensures that the vector-function
\[
g_n(\cdot) := U_{\ast 1}(\Psi_2(\cdot, \overline{\lambda_n}))\Psi_1(\cdot, \overline{\lambda_n}) - U_{\ast 1}(\Psi_1(\cdot, \overline{\lambda_n}))\Psi_2(\cdot, \overline{\lambda_n})
\]
\[
= (1 + \overline{d}\psi_{22}(\overline{\lambda_n}))\Psi_1(\cdot, \overline{\lambda_n}) - (k\overline{b} + \overline{d}\psi_{21}(\overline{\lambda_n}))\Psi_2(\cdot, \overline{\lambda_n}), \quad |n| \geq n_2,
\] (6.10)
is a non-trivial eigenfunction of the operator \(L(Q)^*\) corresponding to the eigenvalue \(\overline{\lambda_n}\). Since the eigenvalues of \(L(Q)^*\) constitutes a sequence \(\{\overline{\lambda_n}\}_{n \in \mathbb{Z}}\), they are geometrically and algebraically simple simultaneously with \(\{\lambda_n\}_{n \in \mathbb{Z}}\), i.e. for \(|n| \geq n_0\). Therefore \(g_n\) is the unique up to a multiplicative constant eigenfunction of the operator \(L(Q)^*\) corresponding to \(\overline{\lambda_n}\) for \(|n| \geq \max\{n_0, n_2\}\).

Further, it follows from (3.12) and (3.17) that
\[
\|f_n\|^2 = |b + ao(1)|^2 \cdot \|\Phi_1(\cdot, \lambda_n)\|^2 + |1 + ae^{ib_1\lambda_n} + o(1)|^2 \cdot \|\Phi_2(\cdot, \lambda_n)\|^2
\]
\[
-2Re(b + a\overline{b}) + (1 + ae^{ib_1\lambda_n} + o(1))(\Phi_1(\cdot, \lambda_n), \Phi_2(\cdot, \lambda_n))
\]
\[
= |b|^2 \cdot \|\Phi_1(\cdot, \lambda_n)\|^2 + |1 + ae^{ib_1\lambda_n}|^2 \cdot \|\Phi_2(\cdot, \lambda_n)\|^2 + o(1) \quad \text{as} \quad n \to \infty.
\] (6.11)

Similar relation is valid for \(g_n\). Combining these relations with estimates (3.18) and noting that \(\lambda_n \in \Pi_h, n \in \mathbb{N}\), yields
\[
\|f_n\| \asymp 1, \quad \|g_n\| \asymp 1, \quad \text{for large} \quad n.
\] (6.12)

Here the symbol \(a_n \asymp 1\) for large \(n\) means that there exists \(n_0 \in \mathbb{N}\) and \(C_1, C_2 > 0\) such that \(C_1 < |a_n| < C_2, |n| > n_0\). In particular, vector-functions \(f_n, g_n\) are non-zero for large \(n\)
Similarly starting with (6.8), (6.10), and using relations (3.12), (3.16), and noting that \( k \in \mathbb{R} \), one arrives at the following important asymptotic relation

\[
(f_n, g_n) = (b + a \cdot o(1)) \left(1 + de^{-ib_2\lambda_n} + o(1)\right) (\Phi_1, \Psi_1)
+ \left(1 + ae^{ib_1\lambda_n} + o(1)\right) \left(kb + d \cdot o(1)\right) (\Phi_2, \Psi_2)
= b(de^{-ib_2\lambda_n} + kae^{ib_1\lambda_n} + k + 1) + o(1) \quad \text{as} \quad n \to \infty.
\]  

(6.13)

Further, using the formula (4.6) for the perturbation determinant \( \Delta_0(\cdot) \) one easily gets

\[
\frac{d}{d\lambda} \left( \Delta_0(\lambda) e^{-ib_1\lambda} \right) = \frac{d}{d\lambda} \left( de^{-ib_1\lambda} + ae^{ib_2\lambda} + ad - bc + e^{i(b_2-b_1)\lambda} \right)
= -ib_1 e^{i(b_2-b_1)\lambda} (de^{-ib_2\lambda} + kae^{ib_1\lambda} + k + 1).
\]  

(6.14)

Since boundary conditions (4.2) and (5.1) are equivalent, the latter are also strictly regular. Therefore in accordance with Definition 5.1 the sequence of zeros \( \Lambda := \{\lambda_n^0\}_{n \in \mathbb{Z}} \) of the determinant \( \Delta_0(\cdot) \) is asymptotically separated. In other words, there exist \( n_0 \in \mathbb{N} \) and \( \delta > 0 \) such that the separation condition (4.10) is satisfied for \( |j|, |k| \geq n_0 \).

Combining estimate (4.15) with the Minimum and Maximum Principles for analytic functions (cf. [25, Lemma 22.2]), we arrive at the following two-sided estimate

\[
C_1 < |\Delta_0'(\lambda)| < C_2, \quad \lambda \in \mathbb{D}_\delta(\lambda_n^0), \quad |n| > n_0.
\]  

(6.15)

with certain \( C_1 = C_1(\delta) > 0, C_2 = C_2(\delta) > 0 \). It follows from asymptotic formula (4.24) that \( \lambda_n = \lambda_n^0 + o(1) \in \mathbb{D}_\delta(\lambda_n^0) \) for \( n \) large enough. Combining this inclusion with estimates (6.15) yields \( \Delta_0'(\lambda_n) \approx 1 \) for large \( n \). Since \( |\text{Im} \lambda_n| \leq h \), relations (6.13), (6.14), and (6.15) imply

\[
(f_n, g_n) \approx 1 \quad \text{for large} \quad n.
\]  

(6.16)

Thus we can normalize the systems \( \{f_n\} \) and \( \{g_n\} \) by putting

\[
\tilde{f}_n := \frac{f_n}{\|f_n\|}, \quad \tilde{g}_n := \frac{g_n}{\|f_n\|}, \quad |n| \geq m := \max\{n_0, n_1, n_2\}.
\]  

(6.17)

Clearly, \( \|\tilde{f}_n\| = 1 \) and \( (\tilde{f}_n, \tilde{g}_n) = 1 \) for \( |n| > m \), i.e. the sequences

\[
\mathcal{F} := \left\{\tilde{f}_n\right\}_{|n| > m} \quad \text{and} \quad \mathcal{G} := \left\{\tilde{g}_n\right\}_{|n| > m}
\]  

(6.18)

are biorthogonal. It follows from (6.8), (6.10) and (6.17) that

\[
\tilde{f}_n(\cdot) = \frac{b + a\varphi_{12}(\lambda_n)}{\|f_n\|} \Phi_1(\cdot, \lambda_n) - \frac{1 + a\varphi_{11}(\lambda_n)}{\|f_n\|} \Phi_2(\cdot, \lambda_n),
\]  

(6.19)

\[
\tilde{g}_n(\cdot) = \frac{\|f_n\| (1 + \tilde{d}\psi_{22}(\lambda_n))}{(f_n, g_n)} \Psi_1(\cdot, \lambda_n) - \frac{\|f_n\| (k\tilde{b} + \tilde{d}\psi_{21}(\lambda_n))}{(f_n, g_n)} \Psi_2(\cdot, \lambda_n).
\]  

(6.20)

By Proposition 4.6, the sequence of eigenvalues \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \) is condensible. Therefore, by Lemma 6.4, the sequences of vector-functions \( \{\Phi_j(\cdot, \lambda_n)\}_{|n| > m} \) and \( \{\Psi_j(\cdot, \lambda_n)\}_{|n| > m}, j \in \{1, 2\}, \) are Besselian in \( L^2[0, 1] \otimes \mathbb{C}^2 \). It follows from (3.12), (6.12), and (6.16) with account of the inclusion \( \Lambda \subset \Pi_h \), that the coefficients at \( \Phi_j(\cdot, \lambda_n) \) and \( \Psi_j(\cdot, \lambda_n), j \in \{1, 2\}, \) in (6.19) and (6.20) are bounded in \( n \). Hence, the sequences \( \mathcal{F} \) and \( \mathcal{G} \) given by (6.18) are also Besselian in \( L^2[0, 1] \otimes \mathbb{C}^2 \).

Clearly, the systems \( \mathfrak{F} \) and \( \mathfrak{G} \) contain \( \mathcal{F} \) and \( \mathcal{G} \), respectively. Since \( \mathfrak{F} \) differs from \( \mathcal{F} \) by a finite number (in fact, \( 2m + 1 \)) entries and \( \mathcal{F} \) is Besselian in \( L^2[0, 1] \otimes \mathbb{C}^2 \), the system \( \mathfrak{F} \) is Besselian.
Similarly to (6.22) one derives composition \( \Lambda = \Lambda_1 \). Combining this representation with Proposition 3.4 and (6.21) yields the existence of \( \delta > 0 \). Since the boundary conditions are strictly regular, by the assumption, the arithmetic progressions \( \Lambda_1^0 \) and \( \Lambda_2^0 \) are separated, i.e., \(|\lambda_{1,n}^0 - \lambda_{2,m}^0| > 2\delta, \ m, n \in \mathbb{Z} \) for some \( \delta > 0 \). This implies the following asymptotic relations

\[ 1 + a e^{ib_1 \lambda_{1,n}^0} \approx 1, \quad d + e^{ib_2 \lambda_{2,n}^0} \approx 1 \quad \text{for large } |n|. \]

In accordance with Proposition 4.7 the sequence \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \) of zeros of \( \Delta(\cdot) \) admits a decomposition \( \Lambda = \Lambda_1 \cup \Lambda_2 \) where \( \Lambda_j = \{\lambda_{j,n}\}_{n \in \mathbb{Z}}, \ j \in \{1, 2\}, \) and \( \lambda_{j,n} = \lambda_{j,n}^0 + o(1) \) as \( n \to \infty \). Combining this representation with Proposition 3.4 and (6.21) yields

\[ 1 + a \varphi_{11}(\lambda_{1,n}) \approx 1, \quad d + \varphi_{22}(\lambda_{2,n}) \approx 1 \quad \text{for large } |n|. \]

Similarly to (6.22) one derives

\[ \overline{\sigma} + \psi_{11}(\overline{\lambda_{1,n}}) \approx 1, \quad (1 + d \psi_{22}(\overline{\lambda_{2,n}})) \approx 1 \quad \text{for large } |n|. \]

Combining this relation with (6.22) yields the existence of \( n'_1 \in \mathbb{N} \) such that

\[ (1 + a \varphi_{11}(\lambda_{1,n}))(\overline{\sigma} + \psi_{11}(\overline{\lambda_{1,n}})) \neq 0 \quad \text{for } |n| \geq n'_1. \]

Therefore it follows from (6.8) (with \( b = 0 \) and \( a \neq 0 \)) that for \( |n| \geq n'_1 \) the vector-function

\[ f_{1,n}(\cdot) := a \varphi_{12}(\lambda_{1,n}) \Phi_1(\cdot, \lambda_{1,n}) - (1 + a \varphi_{11}(\lambda_{1,n})) \Phi_2(\cdot, \lambda_{1,n}), \]

is the non-trivial eigenfunction of the operator \( L(Q) \) corresponding to the eigenvalue \( \lambda_{1,n} \). Moreover, it follows from the second equation in (6.1) and (6.24) that the vector-function

\[ g_{1,n}(\cdot) := U_{\sigma_2}(\Psi_2(\cdot, \overline{\lambda_{1,n}})) \Psi_1(\cdot, \overline{\lambda_{1,n}}) - U_{\sigma_2}(\Psi_1(\cdot, \overline{\lambda_{1,n}})) \Psi_2(\cdot, \overline{\lambda_{1,n}}) \]

\[ := \psi_{12}(\lambda_{1,n}) \Psi_1(\cdot, \lambda_{1,n}) - (\overline{\sigma} + \psi_{11}(\overline{\lambda_{1,n}})) \Psi_2(\cdot, \overline{\lambda_{1,n}}), \quad |n| \geq n'_1, \]

is a non-trivial eigenfunction of the operator \( L(Q)^* \) corresponding to the eigenvalue \( \overline{\lambda_{1,n}} \). Since boundary conditions (5.1) are strictly regular, Proposition 4.7 ensures existence of \( n'_0 \in \mathbb{N} \) such that the eigenvalues \( \{\lambda_{1,n}\} \) of \( L(Q) \) are geometrically and algebraically simple for \( |n| \geq n'_0 \). Therefore \( f_{1,n}(\cdot) (g_{1,n}(\cdot)) \) is the unique up to a multiplicative constant eigenfunction of the operator \( L(Q) \) \( (L(Q)^*) \) corresponding to the eigenvalue \( \lambda_{1,n} \) \( (\overline{\lambda_{1,n}}) \) for \( |n| \geq \max\{n'_0, n'_1\} \).

Further, combining (6.22) with relation (6.23) for large enough \( |n| \) and applying, Proposition 3.4 and Corollary 3.6, we arrive at the following asymptotic relations

\[ ||f_{1,n}|| \approx 1, \quad ||g_{1,n}|| \approx 1, \quad (f_{1,n}, g_{1,n}) \approx 1 \quad \text{for large } n, \]

(cf. (6.12) and (6.13)). Performing normalization of the sequences \( \{f_{2,n}\}_{n \in \mathbb{Z}} \) and \( \{g_{2,n}\}_{n \in \mathbb{Z}} \) in the same way as for the sequence (6.17) and repeating the same argument we get that their normalizations are besselian biorthogonal sequences.

Going over to the second branch \( \{\lambda_{2,n}\}_{n \in \mathbb{Z}} \) of eigenvalues we obtain from (3.2) and (5.1) with account of the assumption \( c = 0 \) that

\[ f_{2,n}(\cdot) := \tilde{U}_2(\Phi_2(\cdot, \lambda_{2,n}))) \Phi_1(\cdot, \lambda_{2,n}) - \tilde{U}_2(\Phi_1(\cdot, \lambda_{2,n}))) \Phi_2(\cdot, \lambda_{2,n}) \]

\[ = (d + \varphi_{22}(\lambda_{2,n})) \Phi_1(\cdot, \lambda_{2,n}) - \varphi_{21}(\lambda_{2,n}) \Phi_2(\cdot, \lambda_{2,n}) \]

(6.28)
It follows from the second relation in (6.22) that for \( n \) big enough \( f_{2,n} (\cdot) \) is a non-trivial eigenfunction of the operator \( L(Q) \) corresponding to the eigenvalue \( \lambda_{2,n} \). Similarly, it follows from the first equation in (6.1) (with \( b = 0 \)) and (6.23) that for \( n \) big enough the vector-function
\[
g_{2,n} (\cdot) := U_{*1} (\Psi_2 (\cdot, \lambda_{2,n})) \Psi_1 (\cdot, \lambda_{2,n}) - U_{*1} (\Psi_1 (\cdot, \lambda_{2,n})) \Psi_2 (\cdot, \lambda_{2,n})
\]
\[
= (1 + \overline{d\psi_{22} (\lambda_{2,n})}) \Psi_1 (\cdot, \lambda_{2,n}) - \overline{d\psi_{11} (\lambda_{2,n})} \Psi_2 (\cdot, \lambda_{2,n})
\] (6.29)
is a non-trivial eigenfunction of the operator \( L(Q)^* \) corresponding to the eigenvalue \( \lambda_{2,n} \). Starting with (6.28) and (6.29) one completes the proof in the case of strictly regular BC by repeating the above reasonings.

**6.3. The case of general potential matrix.** In applications systems (4.1)–(4.2) appear with potential matrices having non-trivial diagonal, i.e. of the form
\[
Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \in L^1[0,1] \otimes \mathbb{C}^{2 \times 2}.
\] (6.30)
First we apply gauge transformation to reduce system (4.1)–(4.2), with a potential matrix \( Q(\cdot) \) of the form (6.30) to similar system with off-diagonal potential matrix \( \tilde{Q} \). To this end we put
\[
w_j(x) := \exp \left( -ib_j \int_0^x Q_{jj}(t) dt \right), \quad x \in [0,1], \quad j \in \{1,2\},
\] (6.31)

**Lemma 6.5.** Let \( Q \) be a summable matrix given by (6.30). Then the operator \( L(Q) = L(Q, \tilde{U}_1, \tilde{U}_2) \) is similar to the operator \( L(\tilde{Q}) = L(Q, \tilde{U}_1, \tilde{U}_2) \) given by (4.1)–(4.2) with the same \( B \), a potential off-diagonal matrix \( \tilde{Q}(\cdot) \),
\[
\tilde{Q}(x) := \begin{pmatrix} 0 & k(x)Q_{12}(x) \\ k^{-1}(x)Q_{21}(x) & 0 \end{pmatrix}, \quad k(x) := w_{1}^{-1}(x)w_2(x),
\] (6.32)
instead of \( Q \) and the boundary conditions
\[
\tilde{U}_j(y) := a_{j1}y_1(0) + a_{j2}y_2(0) + w_1(1)a_{j3}y_1(1) + w_2(1)a_{j4}y_2(1) = 0, \quad j \in \{1,2\}.
\] (6.33)

**Proof.** See the first part of the proof of [30, Proposition 3.4].

**Corollary 6.6.** Let \( Q \) be a summable potential matrix given by (6.30) and let boundary conditions (4.2) be separated and regular, i.e.
\[
ay_1(0) + by_2(0) = cy_1(1) + dy_2(1) = 0 \quad \text{and} \quad abcd \neq 0.
\]
Then the eigenvalues of the corresponding operator \( L_{C,D}(Q) \) are asymptotically separated and there exists \( \beta \in \mathbb{C} \) such that the following asymptotic formula holds
\[
\lambda_n = \frac{2\pi n}{b_2 - b_1} + \beta + o(1) \quad \text{as} \quad |n| \to \infty.
\]
Moreover, the system of root vectors of \( L_{C,D}(Q) \) forms a Riesz basis in \( L^2[0,1] \otimes \mathbb{C}^2 \).

**Lemma 6.7.** Let boundary conditions (4.2) be regular. Then there exists \( w \neq 0 \) such that the boundary conditions
\[
\tilde{U}_j(y) := a_{j1}y_1(0) + a_{j2}y_2(0) + wa_{j3}y_1(1) + a_{j4}y_2(1) = 0, \quad j \in \{1,2\},
\] (6.34)
are strictly regular.
Proof. Since boundary conditions (4.2) are regular, \( J_{14}J_{32} \neq 0 \), we can assume without loss of generality that \( J_{14} = 1 \). Let \( \Delta_0(\cdot) \) be the characteristic determinant corresponding to the BVP (4.1), (6.34). It is easily seen that
\[
\Delta_0(\lambda) = J_{12} + wJ_{34}e^{ib_1+b_2}\lambda + wJ_{32}e^{ib_1}\lambda + e^{ib_2}\lambda.
\] (6.35)
Let us assume for simplicity that \( b_1/b_2 \in \mathbb{Q} \), i.e. \( b_1 = -n_1b, b_2 = n_2b, n_1, n_2 \in \mathbb{N}, b > 0 \). In this case we can rewrite \( \Delta_0(\cdot) \) in the following form
\[
\Delta_0(\lambda) = e^{ib_1}\lambda P_w(e^{ib_1}), \quad P_w(z) := z^{n_1+n_2} + J_{12}z^{n_1} + wJ_{34}z^{n_2} + wJ_{32}.
\] (6.36)
Since \( J_{32} \neq 0 \), one can choose \( w \neq 0 \) such that the zeros of the polynomial \( P_w(z) \) are simple. Clearly, for such \( w \) the zeros of \( \Delta_0(\cdot) \) are asymptotically separated and thus boundary conditions (6.34) are strictly regular.

To state the next result we recall that \( m_a(\lambda_0) \) and \( m_g(\lambda_0) \) denote the algebraic and geometric multiplicities of \( \lambda_0 \), respectively. Moreover, if \( \lambda_0 \) is an isolated eigenvalue, then \( m_a(\lambda_0) \) equals the dimension of the Riesz projection.

We need the following known abstract result (see e.g. [44]).

**Proposition 6.8.** Let \( L \) be an operator with compact resolvent in a separable Hilbert space \( \mathfrak{H} \) and let \( \{\lambda_n\}_{n \in \mathbb{Z}} \) be the sequence of its distinct eigenvalues. Assume that \( m_a(\lambda_n) < \infty \) for \( n \in \mathbb{N} \) and that \( A \) has finitely many associative vectors, i.e. there exists \( n_0 \in \mathbb{N} \) such that \( m_a(\lambda_n) = m_g(\lambda_n) \) for \( |n| > n_0 \). Further, assume that
\[
|\lambda_n| \geq C|n|, \quad |\text{Im} \lambda_n| \leq \tau, \quad n \in \mathbb{Z},
\] (6.37)
for some \( C, \tau > 0 \). Finally, let the system of root vectors of the operator \( L \) forms a Riesz basis in \( \mathfrak{H} \). Then for any bounded operator \( T \) in \( \mathfrak{H} \) the system of root vectors of the perturbed operator \( A = L + T \) forms a Riesz basis with parentheses in \( \mathfrak{H} \).

**Proof.** Since \( L \) has finitely many associated vectors, there exists a finite-dimensional operator \( K \) such that the operator \( L + K \) has no associative vectors, i.e. \( m_a(\lambda_n) = m_g(\lambda_n) \) for \( n \in \mathbb{N} \). Then the system of eigenvectors of \( L + K \) constitutes a Riesz basis in \( \mathfrak{H} \), i.e. the operator \( L + K \) is similar to a normal operator \( H \). The latter admits a representation \( H = H_R + iH_I \) where the operators \( H_R := (H + H^*)/2 = \) and \( H_I := (H - H^*)/2i \) are self-adjoint and commute (see [7, Theorem 6.6.1]). Since the spectrum of \( H \) lie in a strip \( \Pi_\tau \), its imaginary part \( H_I \) is bounded, \( \|H_I\| \leq \tau \). Clearly, inequality (6.38) remains valid for eigenvalues of \( H \) maybe with another constant \( C > 0 \). Therefore the operator \( A \) is similar to a bounded perturbation of the self-adjoint operator \( H \) whose eigenvalues satisfy (6.38). Hence, by [19, Theorem 3.1] (Katsnel’son-Markus-Matsaev theorem, see also [36, 37]) the system of root vectors of \( A \) constitutes a Riesz basis with parentheses.

\[ \square \]

**Proposition 6.9.** Let \( Q \) be a summable potential matrix given by (6.30) and let \( L_{C,D}(Q) \) be the operator associated in \( L^2[0,1] \otimes \mathbb{C}^2 \) with the BVP (4.1)–(4.2). Assume that boundary conditions (4.2) are regular. Then root vectors system of the operator \( L_{C,D}(Q) \) forms a Riesz basis with parentheses in \( L^2[0,1] \otimes \mathbb{C}^2 \).

**Proof.** It is clear that the regularity of boundary conditions is preserved under gauge transformation used in Lemma 6.5. Therefore one can assume that \( Q \) is off-diagonal. Now let us consider a perturbation of the operator \( L(Q) \) by a constant diagonal potential matrix \( Q_0 = \text{diag}(q_0, 0) \), \( q_0 \in \mathbb{C} \). Applying Lemma 6.5 again we see that the operator \( L_{C,D}(Q + Q_0) \) is similar to the
operator $L_{\alpha D}(\tilde{Q})$ with off-diagonal $\tilde{Q}$ and with boundary conditions $C y(0) + \tilde{D} y(1) = 0$, where 
$\tilde{D} = D \cdot \text{diag}(w, 1)$, $w = e^{-i b y_0}$. By Lemma 6.7, we can choose $y_0 \in \mathbb{C}$ such that the boundary 
conditions $C y(0) + \tilde{D} y(1) = 0$ are strictly regular. Let us verify that the operator $L(Q + Q_0)$ satisfies conditions of Proposition 6.8. Since eigenvalues of $L(Q + Q_0)$ are asymptotically separated, it follows that their algebraic multiplicities are finite and $L(Q + Q_0)$ has finitely many 
associated vectors.

According to Theorem 1.1 the root vectors system of the operator $L(Q + Q_0)$ forms a Riesz basis in $L^2[0, 1] \otimes \mathbb{C}^2$. On the other hand, inequalities (6.38) are implied by Proposition 4.6(iv). Thus, the operator $L(Q + Q_0)$ satisfies the conditions of Proposition 6.8, and hence the root 
vectors system of the original operator $L(Q) = L(Q + Q_0) - Q_0$ forms a Riesz basis with parentheses.

\begin{remark} \textbf{(i)} Note that inequalities (6.38) are valid for the roots of any entire function of 
exponential type $\sigma < \infty$ with infinitely many zeros. Namely, the following inequalities hold

$$|\lambda_n| \geq \frac{|n|}{\epsilon \sigma}, \quad |n| \geq N, \quad (6.38)$$

for all but finitely many numbers. In particular, they are valid for the roots of $\Delta(\cdot)$. However, 
Proposition 4.6(iv) gives sharp asymptotic.

\textbf{(ii)} The Riesz basis property for $2 \times 2$ operators $L_{\alpha D}(Q)$ with separated boundary conditions was established earlier than for the operators with general regular boundary conditions. Namely, this property was proved firstly in [55] and later on in [8] and [3] for $B = \text{diag}(-1, 1)$, $Q \in L^2[0, 1] \otimes \mathbb{C}^{2 \times 2}$, and in [18] for $B = \text{diag}(b_1, b_2)$, $Q \in C^1[0, 1] \otimes \mathbb{C}^{2 \times 2}$.

\textbf{(iii)} The Bari-Markus property of the Riesz projectors of unperturbed and perturbed BVPs for separated, periodic and antiperiodic boundary conditions was established in [8] and reproved by another method in [3]. In [9] similar results have been obtained for general regular boundary conditions. Finally, in the recent paper [40] the results of [8] regarding the Bari-Markus property in $L^2[0, 1] \otimes \mathbb{C}^{2m}$ were extended to the case of the Dirichlet BVP for $2m \times 2m$ Dirac equation with $Q \in L^2([0, 1]; \mathbb{C}^{2m \times 2m})$.

\section{Application to the Timoshenko beam model}

Consider the following linear system of two coupled hyperbolic equations for $t \geq 0$

$$I_{\rho}(x) \Phi_{tt} = K(x)(W_x - \Phi) + (EI(x)\Phi_x)x - p_1(x)\Phi_t, \quad x \in [0, \ell], \quad (7.1)$$
$$\rho(x)W_{tt} = (K(x)(W_x - \Phi))x - p_2(x)W_t, \quad x \in [0, \ell]. \quad (7.2)$$

The vibration of the Timoshenko beam of the length $\ell$ clamped at the left end is governed by the system (7.1)–(7.2) subject to the following boundary conditions for $t \geq 0$ [53]:

$$W(0, t) = \Phi(0, t) = 0, \quad (7.3)$$
$$EI(x)\Phi_x(x, t) + \alpha_1 \Phi_t(x, t) + \beta_1 W_t(x, t) \big|_{x=l} = 0, \quad (7.4)$$
$$K(x)(W_x(x, t) - \Phi(x, t)) + \alpha_2 W_t(x, t) + \beta_2 \Phi_t(x, t) \big|_{x=l} = 0. \quad (7.5)$$

Here $W(x, t)$ is the lateral displacement at a point $x$ and time $t$, $\Phi(x, t)$ is the bending angle at a point $x$ and time $t$, $\rho(x)$ is a mass density, $K(x)$ is the shear stiffness of a uniform cross-
section, $I_{\rho}(x)$ is the rotary inertia, $EI(x)$ is the flexural rigidity at a point $x$, $p_1(x)$ and $p_2(x)$ are locally distributed feedback functions, $\alpha_j, \beta_j \in \mathbb{C}$, $j \in \{1, 2\}$. Boundary conditions at the
right end contain as partial cases most of the known boundary conditions if \( \alpha_1, \alpha_2 \) are allowed to be infinity.

Regarding the coefficients we assume that they satisfy the following general conditions:

\[
\rho, I_\rho, K, EI \in C[0, \ell], \quad p_1, p_2 \in L^1[0, \ell], \quad 0 < C_1 \leq \rho(x), I_\rho(x), K(x), EI(x) \leq C_2, \quad x \in [0, \ell].
\] (7.6)

The energy space associated with the problem (7.1)–(7.5) is

\[
\mathcal{S} := \tilde{H}_0^1[0, \ell] \times L^2[0, \ell] \times \tilde{H}_0^1[0, \ell] \times L^2[0, \ell],
\] (7.8)

where \( \tilde{H}_0^1[0, \ell] := \{ f \in W^{1,2}[0, \ell] : f(0) = 0 \} \). The norm in the energy space is defined as follows:

\[
\|y\|^2 = \int_0^\ell (EI|y_1'|^2 + I_\rho|y_2|^2 + K|y_3' - y_1|^2 + \rho|y_4|^2) \, dx, \quad y = \text{col}(y_1, y_2, y_3, y_4).
\] (7.9)

The problem (7.1)–(7.5) can be rewritten as

\[
y_t = i\mathcal{L}y, \quad y(x, t)|_{t=0} = y_0(x),
\] (7.10)

where \( y \) and \( \mathcal{L} \) are given by

\[
y = \begin{pmatrix}
\Phi_t(x, t) \\
\Phi_t^*(x, t) \\
W_t(x, t) \\
W_t^*(x, t)
\end{pmatrix}, \quad \mathcal{L} \begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{pmatrix} = \frac{1}{i} \begin{pmatrix}
y_2 \\
y_4 \\
\frac{1}{\rho(x)} \left( (K(x)y_3' - y_1) - p_2(x)y_4 \right)
\end{pmatrix}
\] (7.11)
on the domain

\[
\text{dom}(\mathcal{L}) = \{ y = \text{col}(y_1, y_2, y_3, y_4) : y_1, y_2, y_3, y_4 \in \tilde{H}_0^1[0, \ell], \quad EI \cdot y_1' \in AC[0, \ell], \quad (EI \cdot y_1')' - p_1y_2 \in L^2[0, \ell], \quad K \cdot (y_3' - y_1) \in AC[0, \ell], \quad (K \cdot (y_3' - y_1))' - p_2y_4 \in L^2[0, \ell], \quad (EI \cdot y_1')(\ell) + \alpha_1y_2(\ell) + \beta_1y_4(\ell) = 0, \quad (K \cdot (y_3' - y_1))(\ell) + \alpha_2y_4(\ell) + \beta_2y_2(\ell) = 0 \}. \] (7.12)

Timoshenko beam model is investigated in numerous papers (see [53, 22, 49, 58, 57, 56] and the references therein). A number of stability, controllability, and optimization problems were studied. Note also that the general model (7.1)–(7.5) of spatially non-homogenous Timoshenko beam with both boundary and locally distributed damping covers the cases studied by many authors. Geometric properties of the system of root functions of the operator \( \mathcal{L} \) play an important role in investigation of different properties of the problem (7.1)–(7.5).

Below we establish the Riesz basis property with parentheses of the operator \( \mathcal{L} \), without analyzing its spectrum. For convenience we impose the following additional algebraic assumption on \( \mathcal{L} \):

\[
\nu(x) := \frac{EI(x)\rho(x)}{K(x)I_\rho(x)} = \text{const}, \quad x \in [0, \ell],
\] (7.13)

Clearly, (7.13) is satisfied whenever \( I_\rho(x) = R\rho(x) \), where \( R = \text{const} \) is a cross-sectional area of the beam, \( EI \) and \( K \) are constant functions, while \( \rho \in AC[0, \ell] \) and is arbitrary positive (cf. condition (7.19)). Our approach to the spectral properties of the operator \( \mathcal{L} \) is based on the
similarity reduction of \( \mathcal{L} \) to a special \( 4 \times 4 \) Dirac-type operator. To state the result we need some additional preparations.

Let \( \gamma(\cdot) \) be given by

\[
\sqrt{\frac{I_\rho(x)}{EI(x)}} = b_1 \gamma(x), \quad \text{where} \quad b_1 > 0 \quad \text{and} \quad \int_0^\ell \gamma(x) dx = 1. \tag{7.14}
\]

Conditions (7.6) and (7.7) imply together that \( \gamma \in C[0, \ell] \) and is positive. Further, in view of (7.13) we have

\[\sqrt{\frac{\rho(x)}{K(x)}} = b_2 \gamma(x), \quad \text{where} \quad b_2 > 0. \tag{7.15}\]

Let

\[
B := \text{diag}(-b_1, b_1, -b_2, b_2), \tag{7.16}
\]

\[
\Theta(x) := -2i \text{diag}(I_\rho(x), I_\rho(x), \rho(x), \rho(x)), \tag{7.17}
\]

\[
h_1(x) := \sqrt{EI(x)}I_\rho(x), \quad h_2(x) := \sqrt{K(x)}\rho(x). \tag{7.18}
\]

In the sequel we assume that

\[h_1, h_2 \in AC[0, \ell]. \tag{7.19}\]

Therefore, according to (7.6)-(7.7) the following matrix function is well-defined:

\[
\hat{Q}(x) := \Theta^{-1}(x) \begin{pmatrix}
p_1 + h'_1 & p_1 - h'_1 & h_2 & -h_2 \\
p_1 + h'_1 & p_1 - h'_1 & h_2 & -h_2 \\
-h_2 & -h_2 & p_2 + h'_2 & p_2 - h'_2 \\
h_2 & h_2 & p_2 + h'_2 & p_2 - h'_2
\end{pmatrix}. \tag{7.20}
\]

Next, we set

\[
t(x) = \int_0^x \gamma(s) ds, \quad x \in [0, \ell]. \tag{7.21}
\]

Since \( \gamma \in C[0, \ell] \) and is positive, the function \( t(\cdot) \) strictly increases on \( [0, \ell] \), \( t(\cdot) \in C^1[0, \ell] \), and due to (7.14) \( t(\ell) = 1 \). Hence, the inverse function \( x(\cdot) := t^{-1}(\cdot) \) is well defined, strictly increasing on \( [0, 1] \), and \( x(\cdot) \in C^1[0, 1] \). Next, we put

\[
Q(t) := \hat{Q}(x(t)) =: (q_{jk}(t))_{j,k=1}^4, \quad t \in [0, 1]. \tag{7.22}
\]

Finally, let

\[
C = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} \alpha_1 - h_1(\ell) & \alpha_1 + h_1(\ell) & 0 & 0 \\ 0 & 0 & \beta_1 & \beta_1 \\ \beta_2 & \beta_2 & \alpha_2 - h_2(\ell) & \alpha_2 + h_2(\ell) \end{pmatrix}. \tag{7.23}
\]

**Proposition 7.1.** [29, 30, Proposition 6.1] Let functions \( \rho, I_\rho, K, EI, p_1, p_2, h_1, h_2 \) satisfy conditions (7.6), (7.7), (7.13) and (7.19). Then the operator \( \mathcal{L} \) is similar to the \( 4 \times 4 \) Dirac-type operator \( L := L_{C,D}(Q) \) with the matrices \( B, C, D \), and \( Q(\cdot) \) given by (7.16), (7.23) and (7.22).

**Theorem 7.2.** Let conditions (7.6), (7.7), (7.13), (7.19) be satisfied and let also

\[
\beta_1 = \beta_2 = 0, \quad \alpha_1 \neq \pm h_1(\ell), \quad \alpha_2 \neq \pm h_2(\ell). \tag{7.24}
\]

Then the system of root functions of the operator \( \mathcal{L} \) forms a Riesz basis with parentheses in \( \mathfrak{H} \).
Hence Theorem 7.2.

Proof. Consider the operator $L_{C,D}(Q)$ defined in Proposition 7.1. Since $\beta_1 = \beta_2 = 0$ we can represent it as bounded perturbation of the direct sum of two $2 \times 2$ Dirac operators:

$$L_{C,D}(Q) = L(U_1, V_1, Q_1) \oplus L(U_2, V_2, Q_2) + \tilde{Q},$$

(7.25)

$$L(Q_j) = -i \begin{pmatrix} -b_j & 0 \\ 0 & b_j \end{pmatrix} y' + Q_1 y, \quad y = \text{col}(y_1, y_2),$$

(7.26)

$$U_j(y) := y_1(0) + y_2(0), \quad V_j(y) := (\alpha_j - h_j(\ell))y_1(1) + (\alpha_j + h_j(\ell))y_2(1), \quad j \in \{1, 2\},$$

(7.27)

$$Q_1(t) = \tilde{Q}_1(x(t)), \quad \tilde{Q}_1 = (-2iI)\rho^{-1} \begin{pmatrix} p_1 + h_1' & p_1 - h_1' \\ p_1 + h_1' & p_1 - h_1' \end{pmatrix},$$

(7.28)

$$Q_2(t) = \tilde{Q}_2(x(t)), \quad \tilde{Q}_2 = (-2i\rho)^{-1} \begin{pmatrix} p_2 + h_2' & p_2 - h_2' \\ p_2 + h_2' & p_2 - h_2' \end{pmatrix},$$

(7.29)

$$\tilde{Q}(t) = \tilde{Q}(x(t)), \quad \tilde{Q} = \Theta^{-1} \text{codiag} \begin{pmatrix} h_2 & -h_2 \\ h_2 & -h_2 \end{pmatrix}.$$

(7.30)

It follows from (7.6), (7.7), (7.19) that $Q_1, Q_2 \in L^1[0,1] \otimes \mathbb{C}^{2 \times 2}$ and $Q \in L^\infty[0,1] \times \mathbb{C}^{2 \times 2}$. Due to conditions (7.24), the operator $L(U_j, V_j, Q_j)$ is a $2 \times 2$ Dirac operator with separated regular boundary conditions. By Corollary 6.6, the system of its root vectors forms a Riesz basis in $L^2[0,1] \otimes \mathbb{C}^2$ and its eigenvalues have a proper asymptotic, in particular, inequality (6.38) is satisfied for them. It is also clear that the direct sum $L := L(U_1, V_1, Q_1) \oplus L(U_2, V_2, Q_2)$ has the same properties. Since $Q$ is bounded, the operator $L_{C,D}(Q)$ is a bounded perturbation of "spectral" operator $L$. Hence by Proposition 6.8, the system of root vectors of the operator $L_{C,D}(Q)$ forms a Riesz basis with parentheses in $L^2[0,1] \otimes \mathbb{C}^4$. Since, by Proposition 7.1, $L$ is similar to the operator $L_{C,D}(Q)$, the system of root functions of $L$ forms a Riesz basis with parentheses in $\mathfrak{H}$. □

Remark 7.3. In [29, 30] the same result was proved under additional smoothness assumptions

$$p_1, p_2 \in L^\infty[0, \ell], \quad h_1, h_2 \in \text{Lip}_1[0, \ell].$$

(7.31)

Hence Theorem 7.2 is considerable generalization to the most general conditions on coefficients.

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