WEIERSTRASS MODELS

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1. Introduction

This paper draws inspiration from lagrangian fibrations on irreducible symplectic manifolds in general, and more specifically, from the question whether all these fibrations have smooth base. Recall that lagrangian fibrations on irreducible symplectic manifolds are equidimensional, at least in the framework of projective varieties, see Matsushita [6]. And when such a base $S$ is also assumed to be smooth, $S$ is then a projective space by Hwang [5].

In this direction, we proved the following theorem 1 in [9]. The theorem 2 is a further effort in studying a certain class of these fibrations.

**Theorem 1.** — Let $S$ be a locally noetherian normal algebraic space of residue characteristic zero with regular formal fibers. Assume that there is a morphism $f : X \to S$, which is locally of finite type, equidimensional and admits étale local sections around every point of $S$, where $X$ is a regular algebraic space. Then $S$ is regular and $f$ is flat.

**Corollary.** — If a lagrangian fibration $f : X \to S$ on an irreducible symplectic manifold $X$ has local holomorphic sections around a point $s$ of $S$, and if $X$ and $S$ are both projective varieties, then $S$ is smooth over $\mathbb{C}$ at $s$.

For then $f$ admits formal sections around $s$, and one can apply Theorem 1.

**Definition.** — Keeping the hypotheses of Theorem 1, we say that $f$ is a Weierstrass model on $X$ if furthermore:

a) $f$ is projective.

b) Every geometric codimension $\leq 1$ fiber of $f$ is irreducible.

c) Every geometric maximal fiber of $f$ admits an abelian variety structure.

The definition is motivated by elliptic $K3$, whose study often benefits from their contraction to plane models, the Weierstrass models. It may happen that such a contraction is trivial, a phenomenon that
is generalized here. The following theorem says that Weierstrass models are, to a large extent, group theoretical, being compactification of torsors in a very tight form. We propose a possible application below.

Note that if \( f \) is a Weierstrass model on \( X \), then \( f \times_{S} S' \) is a Weierstrass model on \( X \times_{S} S' \) for every regular morphism \( S' \to S \).

**Theorem 2.** — Let \( f : X \to S \) be a Weierstrass model on \( X \). Then there exist:

a) An \( S \)-smooth group algebraic space \( A \) with connected fibers, which is Néron at every codimension 1 point of \( S \).

b) An \( A \)-torsor \( Q \) on \( S \) for the étale topology.

c) An open \( S \)-immersion \( j : Q \to X \).

Such \((A, Q, j)\) is unique up to unique isomorphisms and of formation compatible with every base change by a regular morphism \( S' \to S \).

The proof, unlike that of Theorem 1, does not need assume \( S \) has excellent local rings, and is insensitive to characteristics.

Keeping the hypotheses of Theorem 2, assume moreover that \( X \) is an irreducible symplectic manifold with its symplectic structure \( \Psi \) and hyperkahler metric \( \omega \) (cf. [2]). Then \( H^{0}(\mathcal{O}_{S}) = H^{0}(f_{*}\mathcal{O}_{X}) = H^{0}(\mathcal{O}_{X}) = \mathbb{C} \), \( S \) is proper smooth over \( \mathbb{C} \), and \( f \) is a lagrangian fibration, at least when \( S \) is projective over \( \mathbb{C} \) ([6] Theorem 1). Notice that all terms of the exact sequence

\[
0 \to f^{*}\Omega^{1}_{S}|Q \to \Omega^{1}_{Q} \to \Omega^{1}_{Q/S} \to 0
\]

have natural action by \( A \). So by descent one has the exact sequence

\[
0 \to \Omega^{1}_{S} \to \omega^{1}_{Q} \to \omega^{1}_{Q/S} \to 0
\]

with \( \omega^{1}_{Q} \) and \( \omega^{1}_{Q/S} \) consisting of the \( A \)-invariant forms of \( \Omega^{1}_{Q} \) and \( \Omega^{1}_{Q/S} \), respectively. Also, \( \Psi|Q \) by descent induces a symplectic form \( \psi \) on \( \omega^{1}_{Q} \), relative to which \( \Omega^{1}_{S} \) is lagrangian.

We now define a hermitian metric on \( S \): for each local holomorphic 1-form \( \alpha \) on \( S \), let it be considered as an \( A \)-invariant form on \( Q \) and then extended uniquely to \( X \) (\( \text{codim}(X - Q, X) \geq 2 \)), say \( \alpha' \); let \( |\alpha'|_{g} \) denote its norm relative to the hyperkahler metric \( \omega_{g} \) on \( X \). Clearly, the fiberwise integral

\[
\alpha \mapsto |\alpha|^{2} := \int_{X/S} |\alpha'|_{g}^{2} \omega_{g}^{n}
\]

where \( n \) is the relative dimension of \( X/S \), induces a hermitian metric on \( S \), which I speculate is Kähler-Einstein given the symmetry it enjoys.
2. Proof of Theorem 2

Recall that Néron models are of formation compatible with all base change by regular morphisms of spectra of discrete valuation rings [3] 7.2/1. The question is thus local for the étale topology of $S$. We may assume that $S$ is a scheme, quasi-compact, irreducible and with generic point $t$.

I) Case where $S = t$:

By assumptions of a Weierstrass model, $X \times_t \overline{t}$ admits a $\overline{t}$-abelian variety structure, for $\overline{t}$ a geometric point of $t = S$. The assertion then follows by [8] 7.2: $X = Q$ admits a unique torsor structure under its albanese $A$, dual abelian variety of $\text{Pic}^0_{X/t}$.

II) Case where $S$ is the spectrum of a discrete valuation ring:

Let $s$ be the closed point of $S$, and let $U$ consist of all those points of $X$ where $f$ is smooth. Since we assume that $f$ admits étale local sections around $s$, $U_s$ is non-empty and hence dense in the geometrically irreducible closed fiber $X_s$.

We have shown that the generic fiber $U_t = X_t$ is a torsor under its albanese $A_t$. It follows that $U$ is weak $S$-Néron [3] 3.5/1 by definition, and, as $U_s$ is irreducible, is also minimal, loc.cit. 4.3/2, and finally, by loc.cit. 6.5 Corollary 3, that $U$ is $S$-dense open in the $S$-Néron model $Q$ of $U_t = X_t$, and that $Q$ is a torsor under the Néron model $A$ of $A_t$.

The fiber $Q_s$, thus $A_s$ as well, is geometrically irreducible, since $U_s$ is geometrically irreducible and dense in $Q_s$.

To finish, it suffices to show that $U = Q$.

Let $L$ be an ample invertible module on $X$; such $L$ exists by the hypothesis that $X$ be projective over $S$. Notice that $Q - U$ (resp. $X - U$) is of codimension $\geq 2$ in $Q$ (resp. $X$). As $Q$ is regular, $Q$ is parafactorial along $Q - U$, so $L | U$ uniquely extends to an invertible module $N$ on $Q$. It remains only to see that $N$ is ample on $Q$, because

$$X = \text{Proj} \sum H^0(L \otimes^n) = \text{Proj} \sum H^0(L \otimes^n | U) = \text{Proj} \sum H^0(N \otimes^n).$$

That $N$ is ample follows immediately from Raynaud’s theory of cube [7] VI 2.1; it is enough to find an integer $n > 0$ and a section $\sigma \in H^0(N \otimes^n)$ such that

$$W := \{ y \in Q, \sigma(y) \neq 0 \}$$

is contained in $U$, affine, with $W_s \neq \emptyset$. Such a section $\sigma$ exists, since

$$H^0(N \otimes^n) = H^0(L \otimes^n | U) = H^0(L \otimes^n)$$

and $L$ is ample.
III) General case :

III.1) Uniqueness and existence of $A$ :

i) If $A$ exists, it is unique up to unique isomorphisms :

This follows by [4] Note (1) 3: The functor $A \mapsto A_t$ is fully faithful for those $A$ that is Néron at all codimension 1 points of $S$.

ii) In order that the $S$-smooth group algebraic space $A$ exists, it suffices that it exists over an $S$-algebraic space $S'$ that is smooth surjective over $S$ :

For, by i) and I), the asserted $S'$-smooth group algebraic space $A'$ will admit a descent datum relative to $S' \to S$. Such a descent is effective by Artin [1] 6.3.

iii) Let $S'$ be the open sub-scheme of $X$ consisting of all points where $f$ is smooth. Then $S' \to S$ is surjective :

Because $S$ is regular and we assume that $f$ admits étale local sections around every point of $S$, cf. [3] 3.1/3.

iv) For the existence of $A$, one can assume that $f$ admits an $S$-section, say $e : S \to X$ :

One takes the base change $S' \to S$, with $S'$ as in iii), and applies ii).

v) Let $e : S \to X$ be as in iv), let $U$ consist of all points of $X$ where $f$ is smooth, and let $V$ consist of all those points $x$ of $U$ such that $x$ and $e(f(x))$ lie in the same connected component of $U_{f(x)}$. Then $V$ is an open sub-scheme of $U$ and has geometrically irreducible fibers over $S$ :

That $V$ is open in $U$ follows from EGA IV 15.6.5. Its $S$-fibers are by definition connected, thus geometrically connected, since the $S$-section $e : S \to X$ factors through $V$, and finally geometrically irreducible, since $V \to S$ is smooth.

vi) Let $V$ be as in v). There exists on $V$ an $S$-birational group law with zero section $e : S \to V$. Such a structure is unique up to unique isomorphisms :

Notice that $V_t = X_t$, which is, in a unique way, an abelian variety $A_t$ with identity $e(t)$. And, we have shown above that, when localized at any codimension 1 point $s$ of $S$, $V = U$ coincides with the Néron model of $A_t$ over $\text{Spec}(\mathcal{O}_{S,s})$. The sought-after $S$-birational composition law $m$ on $V$ therefore, by the technique of “passage à la limite projective”, exists over an open sub-scheme $S'$ of $S$, with $S - S'$ of codimension $\geq 2$ in $S$. 
Now [7] IX 1.1 implies that the domain of definition of \( m \) is an open neighborhood of \((e \times_S e)(S)\), thus \( m \) exists throughout; it verifies the axioms of commutativity, associativity, etc, because it does over \( t \).

**viii)** There exists an open neighborhood \( V' \) of \( e(S) \) in \( V \) such that \((V', m|V', e)\) admits an \( S \)-dense open immersion into an \( S \)-group algebraic space \( A \), compatible with the group laws.

One applies SGA 3 XVIII 3.2+3.5+3.7 to the birational group law of **vii)**.

**ix)** The \( S \)-group algebraic space \( A \) of **viii)** is a scheme, separated, smooth, of finite type, and with connected fibers over \( S \):

That \( A \) is a scheme (resp. \( S \)-separated, resp. of finite type over \( S \)) follows by [3] 6.2 b), (resp. a), resp. b)), because \( V' \) is quasi-projective over \( S \). And, \( A \) is \( S \)-smooth with connected fibers, because \( V'' \) is \( S \)-smooth with geometrically connected fibers and the morphism \( V' \times_S V' \rightarrow A \), induced by the group law of \( A \), is faithfully flat, which one can verify fiber by fiber.

**x)** One has \( V = A \):

Let \( L \) be an \( S \)-ample invertible module on \( X \), which exists since \( f \) is by assumption projective. The restriction \( L|V' \) extends uniquely to an invertible module \( N \) on \( A \). Indeed, as \( A \), being smooth over \( S \), is regular and \( \text{codim}(A - V', A) \geq 2 \), one has that \( A \) is parafactorial along \( A - V' \). Observe also that \( X - V' \) is of codimension \( \geq 2 \) in \( X \). Now, \( N \) is \( S \)-ample on \( A \), which one proves by localizing at each point \( s \) of \( S \) and argues similarly as in II). This concludes the proof, because

\[
X = \text{Proj} \sum f_\ast L^\otimes n = \text{Proj} \sum (f|V')_\ast (L^\otimes n|V') = \text{Proj} \sum a_\ast N^\otimes n,
\]

where \( a : A \rightarrow S \) denotes the structural morphism of \( A \).

**III.2)** When \( f \) admits an \( S \)-section \( e \), the \((A, Q, j)\) exists as desired and remains so after every base change by a regular morphism:

One takes \( j : Q \rightarrow X \) to be the inclusion of \( A = V \) in \( X \).

**III.3)** The \((A, Q, j)\) in III.2 is independent of the choice of \( S \)-sections of \( f \), when such sections exist:

Let \( e_1, e_2 \) be two \( S \)-sections of \( f \), and let \((A_i, Q_i, j_i)\) denote the triple constructed relative to \( e_i \), \( i = 1,2 \). Identify \( Q_i \) with its image in \( X \), and write \( Q = Q_1 \cap Q_2 \).

Observe that \( Q_i - Q \) is of codimension \( \geq 2 \) in \( Q_i \). Indeed, \( Q_1 \) and \( Q_2 \) coincide over every codimension \( \leq 1 \) point of \( S \), by I)+II). It follows that, this intersection \( Q \), when viewed as a birational correspondence between \( Q_1 \) and \( Q_2 \), is, according to Weil [3] 4.4/1, in fact biregular. So the claim \( Q_1 = Q_2 \) is justified.
Write $+$ for the group law of $A = Q$ with origin $e_1$. There exists a unique element $\alpha_{12} \in A(S)$ satisfying
\[ \alpha_{12} + e_1 = e_2. \]

III.4) Completion of the proof:
Clearly, varying the sections of $f$ over all $S$-smooth schemes $S'$, we have produced, through the system of $\alpha_{12}$, a unique descent datum gluing the locally constructed $(A, Q, j)$. This finishes the proof.

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