EXISTENCE OF SOLUTIONS FOR SINGULARLY PERTURBED SCHRÖDINGER EQUATIONS WITH NONLOCAL PART

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Abstract. In the present paper we study the existence of solutions for a nonlocal Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = \left( \int_{\mathbb{R}^3} \frac{|u|^p}{|x-y|^\mu} dy \right) |u|^{p-2} u,$$

where $0 < \mu < 3$ and $\frac{6-m}{2} < p < 6 - \mu$. Under suitable assumptions on the potential $V(x)$, if the parameter $\varepsilon$ is small enough, we prove the existence of solutions by using Mountain-Pass Theorem.

1. Introduction. In this paper we study the existence of semiclassical solutions for the following nonlocal semilinear problem:

$$-\varepsilon^2 \Delta u + V(x)u = \alpha \left( \int_{\mathbb{R}^3} K(x-y)|u(y)|^p dy \right) |u|^{p-2} u. \tag{1}$$

Equation (1) is closely related to the following nonlocal Schrödinger equation

$$i\hbar \partial_t \Psi = -\hbar^2 \Delta \Psi + W(x) \Psi - \alpha \left( \int_{\mathbb{R}^3} K(x-y)|\Psi(y)|^p dy \right) |\Psi|^{p-2} \Psi, \quad x \in \mathbb{R}^3. \tag{2}$$

Here $m$ is the mass of the bosons, $\hbar$ is the planck constant, $W(x)$ is the external potential and $K(x)$ is the function which possesses information on the mutual interaction between the bosons. The scattering length $\alpha$, whose sign determines the type of interactions, negative for repulsive interaction and positive for attractive interaction. It is clear that $\Psi(x, t) = u(x)e^{-iEt}$ solves (2) if and only if $u(x)$ solves equation (1) with $V(x) = W(x) - E$ and $\varepsilon^2 = \frac{\hbar^2}{2m}$.

When the response function $K(x) = \delta(x)$, the nonlinear response is local, and the nonlinear Schrödinger equation (2) becomes a local one for which many more things are known. But nonlocality appears naturally in optical systems with a thermal [14], it is also known to influence the propagation of electromagnetic waves in plasmas [4] and plays an important role in the theory of Bose-Einstein condensation.

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where it accounts for the finite-range many-body interactions [6]. Nonlocal nonlinearities have attracted considerable interest as a means of eliminating collapse and stabilizing multidimensional solitary waves.

Here we will consider the attractive case, i.e. the scattering length $\alpha = +1$. In general, if the response function $K(x)$ is the delta function, then the equation (1) becomes a standard semilinear one

$$-\varepsilon^2 \Delta u + V(x)u = g(u), \ x \in \mathbb{R}^3. \quad (3)$$

The study of existence and concentration of the semiclassical states of Schrödinger equation (3) goes back to the pioneer work [9] by Floer and Weinstein. Assuming that $V(x)$ is a globally bounded potential having a nondegenerate critical point and $\inf V(x) > 0$, Floer and Weinstein [9] considered $N = 1$, $g(u) = u^3$ and studied firstly the existence of single and multiple spike solutions based on a Lyapunov-Schmidt reductions. This result was extended to higher dimension and for $g(u) = |u|^{p-2}u$ by Oh in [16, 17]. Since then, equation (3) has attracted the interest of many mathematicians under various assumptions on the potential $V(x)$. In [2] Ambrosetti et al. combined the Lyapuno-Schmit reduction method and variational arguments to study concentration phenomena of the solutions at isolated local minima and maxima of $V(x)$ with polynomial degeneracy. Without assumption of non-degeneracy on critical points of $V(x)$, the existence of (positive) solutions was handled in [20] by Rabinowitz purely via variational methods. In [20], still assuming that $\inf V(x) > 0$, Rabinowitz proved the existence of a positive ground state for any $\varepsilon > 0$ by further assuming that

$$0 < a \leq V(x) \leq \liminf_{|x| \to \infty} V(x), \ \text{for all} \ x \in \mathbb{R}^3 \ \text{and some} \ a > 0,$$

with strict inequality on a set of positive measure. Using a local variational approach, del Pino and Felmer [18, 19] constructed positive solutions concentrating around any topologically nontrivial critical point of the potential $V(x)$. A simple example is that of a local minimum. Assuming for instance $\Lambda \subset \mathbb{R}^3$ is a bounded open set such that

$$V_0 := \inf_{x \in \Lambda} V(x) < \inf_{x \in \partial \Lambda} V(x).$$

Then if $\inf V(x) > 0$ there also exists a positive semiclassical solution. We also refer authors to [5] for the case $\inf V(x) = 0$ and [8] for the case where the potential $V(x)$ was allowed to change sign.

If the response function $K(x)$ is a function of Coulomb type, for example $\frac{1}{|x|}$, then we arrive at the nonlocal Schrödinger equation, which is also called Choquard-Pekar equation,

$$-\varepsilon^2 \Delta u + V(x)u = (\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy)u. \quad (4)$$

Many efforts have been made to study the existence of nontrivial solutions for problem (4) with constant $\varepsilon = 1$. In [13], by using critical point theory, P.L. Lions obtained a solution $u \in H^1(\mathbb{R}^3)$, $u \neq 0$. Ackermann [1] proposed an approach to prove the existence of infinitely many geometrically distinct weak solutions to the problem with potential $V(x)$ periodic in $x_i$ and 0 is not in the spectrum of $-\Delta + V(x)$. In [15] Menzala also applied variational method to show the existence of a nontrivial weak solution of the nonlinear Schrödinger equation. For other nonlocal problems, we also want to mention the Schrödinger-Maxwell system which
has also been widely considered. For example, Ruiz [21] considered
\[
\begin{aligned}
-\Delta \psi + \psi + \mu \phi \psi &= |\psi|^{p-1} \psi \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= 4\pi \psi^2 \quad \text{in } \mathbb{R}^3.
\end{aligned}
\]
(5)

By reducing the system into a single nonlocal Schrödinger equation and work in
the radial functions subspace of \(H^1(\mathbb{R}^3)\), the author is able to obtain the existence
results of (5) depending on the parameter \(p\) and \(\mu > 0\). When \(\varepsilon \to 0\) i.e. the semi-
classical case is involved, the existence and concentration phenomena of solutions
for equation (5) have also been deeply studied, see [10, 11] and the references there-
in. The arguments there depend greatly on the Lyapuno-Schmit reduction method
and the non-degeneracy property of the ground state solutions of
\[
- \Delta u + u = u^p,
\]
(6)
existence and concentration phenomena of solutions for equation (5) have been
deeply studied. However there is few works about the existence of semiclassical
solutions for equation (4), since little is known about the ground states of the
corresponding autonomous limit problem
\[
- \Delta u + u = \left( \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \right) u.
\]
(7)

Recently Wei and Winter [23] proved the non-degeneracy property of the ground
state solution of (7), and then they studied the existence of multi-bump solutions
for equation (4) under the assumptions that \(\inf V(x) > 0\) and \(V(x) \in C^2(\mathbb{R}^3)\).

The aim of the present paper is to continue to study the existence of solutions
for the nonlocal Schrödinger equation with small parameter \(\varepsilon\)
\[
- \varepsilon^2 \Delta u + V(x)u = \left( \int_{\mathbb{R}^3} \frac{|u|^p}{|x-y|^\alpha} dy \right) |u|^{p-2} u,
\]
(8)
where \(0 < \mu < 3\) and \(\frac{6-\mu}{3} < p < 6 - \mu\). In order to obtain the existence of solutions,
we will apply variational methods. Under suitable assumptions on the potentials
we prove that for small \(\varepsilon\), there is at least one ground state solution \(u_\varepsilon\) for (8). To
establish the existence results, we assume that the potential \(V(x)\) satisfy
\(P_1\) \(V \in C(\mathbb{R}^3)\) and there is \(b > 0\) such that the set \(V^b := \{x \in \mathbb{R}^3 : V(x) < b\}\) has
finite Lebesgue measure.
\(P_2\) \(0 = V(0) \leq V(x)\).
\(P_3\) There exists \(0 < \tau < \frac{1}{2}\) such that
\[
\lim_{|x| \to 0} \frac{V(x)}{|x|^{1+\tau}} = 0.
\]

The main results of this paper are

**Theorem 1.1.** Let \((P_1) - (P_3)\) be satisfied. Then for any \(\sigma > 0\) there is \(\mathcal{E}_\sigma > 0\)
such that if \(\varepsilon \leq \mathcal{E}_\sigma\), equation (8) has at least one ground state solution \(u_\varepsilon\) satisfying
\[
\frac{p-1}{2p} \int_{\mathbb{R}^3} \left( \varepsilon^2 |\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2 \right) dx \leq \sigma \varepsilon^{\frac{p+\mu-4}{p-4}}.
\]

In general, the solutions of (1) might change sign. The following result shows
that, under further restrictions, the problem possesses solutions which change sign.
Recall that a map \(\theta : \mathbb{R}^3 \to \mathbb{R}^3\) is called an orthogonal involution if \(\theta \neq I\) and
\(\theta^2 = I\) where \(I\) denotes the identity map in \(\mathbb{R}^3\).
Theorem 1.2. Let \((P_1) - (P_3)\) be satisfied. Assume moreover

(S) there is an orthogonal involution \(\theta\) such that \(V(\theta x) = V(x)\) for all \(x \in \mathbb{R}^3\).

Then for any \(\sigma > 0\) there is \(E_\sigma > 0\) such that if \(\varepsilon \leq E_\sigma\), equation (8) has at least
one solution \(u_\varepsilon\) changing sign and satisfying
\[
\frac{p-1}{2p} \int_{\mathbb{R}^3} (\varepsilon^2 |\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2)dx \leq \sigma \varepsilon^{\frac{p+1}{p-1}}.
\]

Let \(E\) be a real Banach space and \(I : E \to \mathbb{R}\) a functional of class \(C^1\). We say
that \((u_n) \subset E\) is a Palais-Smale ((P.S) for short) sequence at \(c\) if \((u_n)\) satisfies
(i) \(I(u_n) \to c\) and (ii) \(I'(u_n) \to 0\), as \(n \to \infty\). \(I\) satisfies the (P.S) condition at \(c\), if
any (P.S) sequence at \(c\) possesses a convergent subsequence.

This paper is organized as follows. In Sect.2, we introduce the variational framework
and restate the problems in equivalent forms. In Sect.3, we analysis the
behaviors of the bounded (P.S) sequences. In Sect.4, we prove the existence of semi-classical solutions for the nonlocal Schrödinger equation (8) by using Mountain-Pass
Theorem.

2. Notations and variational framework. In this paper we use \(C, C_i\) to denote
positive constants and \(B_R\) the open ball centered at the origin with radius \(R > 0\).
\(C_\infty^0(\mathbb{R}^3)\) denotes functions infinitely differentiable with compact support in \(\mathbb{R}^3\).
\(H^1(\mathbb{R}^3)\) is the usual Sobolev spaces with norm
\[
\|u\|_{H^1} := \left( \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2)dx \right)^{1/2}
\]
and \(L^s(\mathbb{R}^3), 1 \leq s \leq \infty\), denotes the Lebesgue space with the norms
\[
|u|_s := \left( \int_{\mathbb{R}^3} |u|^s dx \right)^{1/s}.
\]
The best Sobolev constant \(S\) is defined by:
\[
S\|u\|_{2^*}^2 \leq \int_{\mathbb{R}^3} |
\nabla u|^2 dx \quad \text{for all } u \in H^1(\mathbb{R}^3).
\]
To prove the existence of semiclassical solutions of (8) for small \(\varepsilon\), we may rewrite
(8) in an equivalent form, let \(\lambda = \varepsilon^{-2}\), (8) reads then as
\[
-\Delta u + \lambda V(x)u = \lambda \int_{\mathbb{R}^3} \frac{|u|^p}{|x-y|^\mu}dy |u|^{p-2}u \quad (9)
\]
for \(\lambda \to \infty\). To find solutions of problem (9) we will apply variational methods. To
this end, we introduce the Hilbert spaces
\[
E := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < \infty \right\}
\]
with inner products
\[
(u, v) := \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv)dx
\]
and the associated norms
\[
\|u\|^2 = (u, u).
\]
Obviously, it follows from \((P_1)\) that \(E\) embeds continuously in \(H^1(\mathbb{R}^3)\) (see [7, 22]).
Note that the norm \(\|\cdot\|\) is equivalent to \(\|\cdot\|_\lambda\) deduced by the inner product
\[
(u, v)_\lambda := \int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda V(x)uv)dx
\]
Lemma 2.2

Proof. Orthogonal with respect to ($\eta$)

Correspondingly, one has $\eta$ for each $\lambda > 0$. It is thus clear that, for each $s \in [2, 2^*]$, there is $\nu_s > 0$ (independent of $\lambda$) such that if $\lambda \geq 1$

$$|u|_s \leq \nu_s \|u\| \leq \nu_s \|u\|_\lambda \quad \text{for all } u \in E. \quad (10)$$

In order to investigate the problems in suitable variational framework, we use $A_\lambda := -\Delta + \lambda V$ in $L^2(\mathbb{R}^3)$ to denote the selfadjoint operator related to the Schrödinger operator. By $\sigma(A_\lambda)$, $\sigma_c(A_\lambda)$ and $\sigma_d(A_\lambda)$ we denote the spectrum, the essential spectrum and the eigenvalues of $A_\lambda$ below $\lambda_c := \inf \sigma_c(A_\lambda)$, respectively. Note that each $\mu \in \sigma_d(A_\lambda)$ is of finite multiplicity. The following two Lemmas are proved in [8], we sketch the proofs here for the completeness of the paper.

Lemma 2.1 ([8]). Suppose that the assumption $(P_1)$ is satisfied, then there holds $\lambda_c \geq \lambda b$.

Proof. Set $W_\lambda(x) = \lambda(V(x) - b)$, $W_\lambda^\pm = \max\{\pm W_\lambda, 0\}$ and $D_\lambda = -\Delta + \lambda b + W_\lambda^-$. By $(P_1)$, the multiplicity operator $W_\lambda^-$ is compact relative to $D_\lambda$, hence $\sigma_c(D_\lambda) \subset \sigma_c(A_\lambda) \subset (\lambda b, \infty)$.

Fix in the following a number $b'$ close to $b$ with $0 < b' < b$ and $k_\lambda$ be the number of the eigenvalues of $A_\lambda$ which is smaller than $\lambda b'$. We write $\eta_{\lambda,j}$ and $h_{\lambda,j}$ ($1 \leq j \leq k_\lambda$), for the eigenvalues and eigenfunctions and set

$$L_\lambda^d = \text{Span}\{h_{\lambda,1}, \ldots, h_{\lambda,k_\lambda}\}.$$ 

We will also use the following orthogonal decomposition

$$L^2(\mathbb{R}^3) = L_\lambda^d \oplus L_\lambda^c, \quad u = u^d + u^c.$$ 

Correspondingly, one has

$$E = E_\lambda^d \oplus E_\lambda^c \quad \text{with} \quad E_\lambda^d = L_\lambda^d \cap E \quad \text{and} \quad E_\lambda^c = L_\lambda^c \cap E \quad (11)$$

orthogonal with respect to $(\cdot, \cdot)_{L^2}$ and $(\cdot, \cdot)_E$. From Lemma 2.1, we have

$$\lambda b'|u|^2 \leq \|u\|^2_{\lambda} \quad \text{for all } u \in E_\lambda^c. \quad (12)$$

Lemma 2.2 ([8]). For each $s \in [2, 6]$, there is $c_s > 0$ independent of $\lambda$ such that

$$c_s \lambda^{\frac{s-2}{2}} \|u\|^s \leq \|u\|^s_{\lambda} \quad \text{for all } u \in E_\lambda^c.$$

Proof. For $s \in (2, 6)$, by Sobolev inequality and (12),

$$|u|^s \leq \left(\int_{\mathbb{R}^3} |u|^2 dx \right)^{\frac{3}{s-2}} \left(\int_{\mathbb{R}^3} |u|^6 dx \right)^{\frac{s-2}{3}} \leq \left((b')^{-1} \|u\|^2_{\lambda} \right)^{\frac{s-2}{2}} \left(S^{-1} \|u\|^6_{\lambda} \right)^{\frac{s-2}{3}}$$

for all $u \in E_\lambda^c$, the conclusion then follows.

The following inequality will be frequently used to study the nonlocal problems.

Proposition 1 ([12], Hardy-Littlewood-Sobolev inequality). Let $p, r > 1$ and $0 < \mu < 3$ with $1/p + \mu/3 + 1/r = 2$. Let $f \in L^p(\mathbb{R}^3)$ and $h \in L^r(\mathbb{R}^3)$. There exists a sharp constant $C(p, \mu, r)$, independent of $f, h$, such that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(x)h(y)}{|x-y|^{\mu}} dy dx \leq C(p, \mu, r) \|f\|_p \|h\|_r.$$
Let us set
\[ I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda V(x)u^2) dx - \frac{\lambda}{2p} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^p |u(y)|^p}{|x-y|^\mu} dxdy, \]
from \(0 < \mu < 3\), \(\frac{6-n}{2} < p < 6 - \mu\) and Proposition 2.3, we know the energy functional \(I_\lambda(u)\) is well defined and belongs to \(C^1(E, \mathbb{R})\). Consequently, in order to obtain solutions of (9), we only need to look for critical points of the energy functional \(I_\lambda(u)\).

Now the existence results can be restated as

**Theorem 2.3.** Let \((P_1) - (P_3)\) be satisfied. Then for any \(\delta > 0\) there is \(\Lambda_\delta > 0\) such that if \(\lambda \geq \Lambda_\delta\), equation (9) has at least one ground state solution \(u_\lambda\) satisfying
\[ \frac{p-1}{2p} \int_{\mathbb{R}^3} (|\nabla u_\lambda|^2 + \lambda V(x)u_\lambda^2) dx \leq \delta \lambda^{\frac{4-p-\mu}{2(p-\mu)}}. \]
Then if \(4 < p + \mu < 6\) we have \(u_\lambda \to 0\) as \(\lambda \to \infty\).

**Theorem 2.4.** Let \((P_1) - (P_3)\) be satisfied. Assume moreover

\((S)\) there is an orthogonal involution \(\theta\) such that \(V(\theta x) = V(x)\) for all \(x \in \mathbb{R}^3\).

Then for any \(\delta > 0\) there is \(\Lambda_\delta > 0\) such that if \(\lambda \geq \Lambda_\delta\), equation (9) has at least one solution \(u_\lambda\) changing sign and satisfying
\[ \frac{p-1}{2p} \int_{\mathbb{R}^3} (|\nabla u_\lambda|^2 + \lambda V(x)u_\lambda^2) dx \leq \delta \lambda^{\frac{4-p-\mu}{2(p-\mu)}}. \]
Then if \(4 < p + \mu < 6\) we have \(u_\lambda \to 0\) as \(\lambda \to \infty\).

Let \(\theta : \mathbb{R}^3 \to \mathbb{R}^3\) be an orthogonal involution. Then \(\theta\) induces an involution on \(E\) which we denote again by \(\theta : E \to E\) as follows \((\theta u)(x) = -u(\theta x)\). If \((S)\) is satisfied, then \(I_\lambda\) is \(\theta\)-invariant: \(I_\lambda(\theta u) = I_\lambda(u)\), and \(I_\lambda\) is \(\theta\)-equivalent: \(I'(\theta u) = \theta I_\lambda'(u)\). In particular, if \(\theta u = u\) then \(I_\lambda'(u) = I_\lambda'(u)\). Let \(E^\theta := \{u \in E : \theta u = u\}\), it is known that critical points of the restriction of \(I_\lambda\) on \(E^\theta\) are solutions of (9) satisfying \(u(\theta x) = -u(x)\).

We will use the following standard Mountain Pass Theorem.

**Theorem 2.5** ([3]). Let \(E\) be a real Banach space and \(I : E \to \mathbb{R}\) a functional of class \(C^1\). Suppose that \(I(0) = 0\) and:

\(I_1\). There exist \(r, \rho > 0\) such that \(|I|_S \geq \rho > 0\) for all \(u \in S = \{u \in E : \|u\| = r\}\);
\(I_2\). there is \(e\) with \(\|e\| > r\) such that \(I(e) \leq 0\).

Then \(I\) possesses a \((P.S)_e\) sequence with \(c \geq \rho > 0\) given by
\[ c := \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)), \]
where
\[ \Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \quad I(\gamma(1)) < 0\}. \]

3. **Behavior of the \((P.S)\) sequences**. In this section we will analysis the behaviors of the \((P.S)\) sequences of the functional \(I_\lambda\).

**Lemma 3.1.** Suppose that the assumption \((P_3)\) holds. For fixed \(\lambda \geq 1\), let \((u_n)\) be a \((P.S)_e\)-sequence for \(I_\lambda\). Then \(c \geq 0\) and \((u_n)\) is bounded in \(E\).
Proof. The conclusion simply follows from the fact that
\[ c + o(1)\|u_n\|_\lambda \geq I_\lambda(u_n) - \frac{1}{2p}(I'_\lambda(u_n), u_n) \]
\[ = \left( \frac{1}{2} - \frac{1}{2p} \right) \int_{\mathbb{R}^3} (|\nabla u_n|^2 + \lambda V(x)u_n^2) \, dx \]
\[ = \left( \frac{1}{2} - \frac{1}{2p} \right) \|u_n\|_\lambda^2. \]
\[
\]
Hence, without loss of generality, we may assume \( u_n \to u \) in \( E \) and \( L^2(\mathbb{R}^3) \), \( u_n \to u \) in \( L^s(\mathbb{R}^3) \) for \( 1 \leq s < 2^* \), and \( u_n(x) \to u(x) \) a.e. for \( x \in \mathbb{R}^3 \). Clearly \( u \) is a critical point of \( I_\lambda \).

**Lemma 3.2.** One has along a subsequence:

1. \( I_\lambda(u_n - u) \to c - I_\lambda(u) \);
2. \( I'_\lambda(u_n - u) \to 0. \)

**Proof.** (1). Direct computation shows that
\[
I_\lambda(u_n - u) = \frac{1}{2}\|u_n - u\|^2 - \frac{\lambda}{2p} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x) - u(x)|^p|u_n(y) - u(y)|^p}{|x - y|^\mu} \, dy \, dx
\]
\[
= \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u\|^2 - \frac{\lambda}{2p} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x) - u(x)|^p|u_n(y) - u(y)|^p}{|x - y|^\mu} \, dy \, dx + o(1)
\]
\[
= I_\lambda(u_n) - I_\lambda(u) + \Gamma_n + o(1)
\]
where
\[
\Gamma_n = \frac{\lambda}{2p} \left\{ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x)|^p|u_n(y)|^p}{|x - y|^\mu} \, dy \, dx - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^p|u(y)|^p}{|x - y|^\mu} \, dy \, dx \right.
\]
\[
- \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x) - u(x)|^p|u_n(y) - u(y)|^p}{|x - y|^\mu} \, dy \, dx \}
\]
From the nonlocal Brezis-Lieb type Lemma 3.4 in [1], we know \( \Gamma_n \to 0 \).

(2). For any \( \varphi \) with \( \|\varphi\|_\lambda \leq 1 \), we have
\[
(I'_\lambda(u_n - u), \varphi) = \int_{\mathbb{R}^3} (\nabla(u_n - u) \nabla \varphi + \lambda V_1(x)(u_n - u) \varphi) \, dx
\]
\[
- \lambda \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(y) - u(y)|^p|u_n(x) - u(x)|^p|u_n(x) - u(x)|^p - 2(u_n(x) - u(x)) \varphi(x)}{|x - y|^\mu} \, dy \, dx
\]
\[
= (I_\lambda(u_n), \varphi) - (I'_\lambda(u), \varphi) + \tilde{\Gamma}_n + o(1)
\]
where
\[
\tilde{\Gamma}_n = \lambda \left\{ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(y)|^p|u_n(x)|^p - 2u_n(x) \varphi(x)}{|x - y|^\mu} \, dy \, dx
\]
\[
- \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(y)|^p|u(x)|^p - 2u(x) \varphi(x)}{|x - y|^\mu} \, dy \, dx
\]
\[
- \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(y) - u(y)|^p|u_n(x) - u(x)|^p - 2(u_n(x) - u(x)) \varphi(x)}{|x - y|^\mu} \, dy \, dx \}
\]
From Lemma 3.4 in [1], we also know \( \tilde{\Gamma}_n \to 0 \) uniformly in \( \varphi \) with \( \|\varphi\|_\lambda \leq 1 \).
In the following we will utilize the decomposition (11): $E = E^d_\lambda \oplus E^c_\lambda$. Recall that $\dim(E^d_\lambda) < \infty$. Write
$$w_n := u_n - u$$
and decompose $w_n$ by
$$w_n = w_n^d + w_n^c,$$
with $w_n^d \in E^d_\lambda$ and $w_n^c \in E^c_\lambda$.

From $w_n \to 0$ it is easy to see $w_n^d \to 0$ since $\dim(E^d_\lambda) < \infty$. And also $u_n \to u$ if and only if $w_n \to 0$.

**Lemma 3.3.** Suppose that the assumption $(F_1)$ holds. There is a constant $\alpha_0 > 0$ independent of $\lambda$ such that, for any $(P.S)_c$ sequence $(u_n)$ for $I_\lambda$ with $u_n \to u$, either $w_n \to 0$ along a subsequence or
$$c - I_\lambda(u) \geq \alpha_0 \lambda^{\frac{4-p-\mu}{2p-3}}.$$

**Proof.** Assume $(u_n)$ has no convergent subsequence, then $\liminf_{n \to \infty} \|w_n\|_\lambda > 0$. Lemma 3.2 implies that along a subsequence, one has
$$I_\lambda(w_n) \to c - I_\lambda(u) > 0$$ and $I'_\lambda(w_n) \to 0$.

It follows that
$$I_\lambda(w_n) - \frac{1}{2}(I'_\lambda(w_n), w_n) = \frac{\lambda(p-1)}{2p} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|w_n(x)|^p|w_n(y)|^p}{|x-y|^\mu} dydx,$$

i.e.
$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|w_n(x)|^p|w_n(y)|^p}{|x-y|^\mu} dx dy \leq \frac{2p}{p-1} \lambda \frac{c - I_\lambda(u) + o(1)}{\lambda^\frac{p}{p-1}}.$$

Since $I'_\lambda(w_n) \to 0$, using the the Hardy-Littlewood-Sobolev inequality again, we know
$$\begin{align*}
\|w_n\|_\lambda^2 &= \lambda \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|w_n(x)|^p|w_n(y)|^p}{|x-y|^\mu} dydx + o(1) \\
&\leq C_0 \lambda \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|w_n(x)|^p|w_n(y)|^p}{|x-y|^\mu} dydx \right)^\frac{1}{p} \left( \int_{\mathbb{R}^3} |w_n|^{\frac{6p}{6p-\mu}} dx \right)^\frac{6p-\mu}{3p} + o(1) \\
&\leq C_0 \left( \frac{2p}{p-1} \right)^\frac{1}{p} \lambda \left( c - I_\lambda(u) + o(1) \right)^\frac{1}{p} \left( \int_{\mathbb{R}^3} |w_n|^{\frac{6p}{6p-\mu}} dx \right)^\frac{6p-\mu}{3p} + o(1).
\end{align*}$$

Since $w_n^d \to 0$, it follows from Lemma 2.2 that
$$\begin{align*}
\|w_n^c\|_\lambda^2 + o(1) &\leq C_0 \left( \frac{2p}{p-1} \right)^\frac{1}{p} \lambda \left( c - I_\lambda(u) + o(1) \right)^\frac{1}{p} \left( \int_{\mathbb{R}^3} |w_n^c|^{\frac{6p}{6p-\mu}} dx \right)^\frac{6p-\mu}{3p} + o(1) \\
&\leq C_1 \lambda \left( c - I_\lambda(u) + o(1) \right)^\frac{1}{p} \|w_n^c\|_\lambda^2,
\end{align*}$$

consequently
$$1 + o(1) \leq C_1 \lambda \left( c - I_\lambda(u) + o(1) \right)^\frac{1}{p}.$$

We thus get
$$\alpha_0 \lambda^{\frac{4-p-\mu}{2p-3}} \leq c - I_\lambda(u)$$
with $\alpha_0 > 0$ independent of $\lambda$, proving the Lemma.

From Lemma 3.3, we have the following convergence criterion for the $(P.S)$ sequences.
Corollary 1. Suppose that the potential $V(x)$ satisfies the assumption $(P_1)$. Then $I_\lambda$ satisfies the $(P.S)$ condition for all $c < \alpha_0 \lambda^{\frac{4-p}{4}}$.

4. **Proof of the main results.** In the following we will prove that $I_\lambda$ satisfies the geometry conditions of the classical Mountain Pass Theorem. And then we can construct small minimax values for $I_\lambda$ at levels where the $(P.S)$ condition holds if the parameter $\lambda$ is large enough.

**Proposition 2.**

$$\inf \left\{ \int_{\mathbb{R}^3} |\nabla \varphi|^2 \, dx : \varphi \in C_0^\infty(\mathbb{R}^3), \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\varphi(x)|^p|\varphi(y)|^p}{|x-y|^\mu} \, dy \, dx = 1 \right\} = 0. \quad (14)$$

**Proof.** In fact, for all fixed $\varphi$ satisfying

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\varphi(x)|^p|\varphi(y)|^p}{|x-y|^\mu} \, dy \, dx = 1,$$

let us define

$$\varphi_t = t^\frac{6-\mu}{p} \varphi(tx), \quad t > 0,$$

then we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\varphi_t(x)|^p|\varphi_t(y)|^p}{|x-y|^\mu} \, dy \, dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\varphi(x)|^p|\varphi(y)|^p}{|x-y|^\mu} \, dy \, dx = 1$$

and

$$\int_{\mathbb{R}^3} |\nabla \varphi_t|^2 \, dx = t^\frac{6-\mu}{p} - 1 \int_{\mathbb{R}^3} |\nabla \varphi|^2 \, dx.$$  

Since $6 - \mu > p$, we know

$$\int_{\mathbb{R}^3} |\nabla \varphi_t|^2 \, dx \to 0$$

as $t \to 0$. Thus the proposition is proved. \hfill \square

**Lemma 4.1.** Assume that the potential $V(x)$ satisfies condition $(P_1) - (P_3)$, then the functional $I_\lambda$ satisfies the geometry of Mountain Pass:

1. $I_\lambda(0) = 0$ and there exist $\alpha_\lambda$, $\rho_\lambda > 0$ such that $I_\lambda(u) \geq \alpha_\lambda$ for all $\|u\|_\lambda = \rho_\lambda$;
2. For any $\sigma > 0$ there exists $\Lambda_\sigma > 0$, such that, for each $\lambda \geq \Lambda_\sigma$, there is $e_\lambda$ such that $I_\lambda(e_\lambda) < 0$.

**Proof.** (1) First note that, for each fixed $\lambda$, $I_\lambda(0) = 0$. By the Hardy-Littlewood-Sobolev inequality, we know

$$I_\lambda(u) = \frac{1}{2} \|u\|_\lambda^2 - \frac{\lambda}{2p} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^p|u(y)|^p}{|x-y|^\mu} \, dy \, dx$$

$$\geq \frac{1}{2} \|u\|_\lambda^2 - \frac{\lambda}{2p} \left( \int_{\mathbb{R}^3} |u|^{\frac{6p}{6-p}} \, dx \right)^{\frac{6-\mu}{6-p}}$$

$$\geq \frac{1}{2} \|u\|_\lambda^2 - C_0 \|u\|_\lambda^{2p}. \quad (15)$$

Since $p > 1$, the conclusion follows if $\|u\|_\lambda$ is small enough.

(2) From Proposition 2, we know

$$\inf \left\{ \int_{\mathbb{R}^3} |\nabla \varphi|^2 \, dx : \varphi \in C_0^\infty(\mathbb{R}^3), \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\varphi(x)|^p|\varphi(y)|^p}{|x-y|^\mu} \, dy \, dx = 1 \right\} = 0.$$
Thus for any $\delta > 0$ one can choose $\varphi_\delta \in C_0^\infty(\mathbb{R}^3)$ with $\text{Supp}\varphi_\delta \subset B_r(0)$ such that
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\varphi_\delta(x)|^p|\varphi_\delta(y)|^p}{|x-y|^\mu} \, dy \, dx = 1
\]
and $|\nabla \varphi_\delta|^2 < \delta$. For any $\alpha \geq \frac{3-\mu}{4(p-1)}$ and $\tau$ in assumption $(P_3)$, set
\[
e_\lambda(x) := \lambda^\alpha \varphi_\delta(\lambda^\tau x),
\]
then $\text{Supp} e_\lambda \subset B_{\lambda^{-\tau}r_\delta}(0)$.

It is easy to see that
\[
\int_{\mathbb{R}^3} |\nabla e_\lambda(x)|^2 \, dx = \lambda^{2\alpha - \tau} \int_{\mathbb{R}^3} |\nabla \varphi_\delta|^2 \, dx,
\]
\[
\int_{\mathbb{R}^3} V(x)e_\lambda^2(x) \, dx = \lambda^{2\alpha - 3\tau} \int_{\mathbb{R}^3} V(\lambda^{-\tau}x)\varphi^2_\delta(x) \, dx,
\]
and
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|e_\lambda(x)|^p|e_\lambda(y)|^p}{|x-y|^\mu} \, dy \, dx = \lambda^{2p\alpha + (\mu-6)\tau}.
\]
Note that $\text{Supp}\varphi_\delta \subset B_{r_\delta}(0)$ and
\[
\lim_{|x| \to 0} \frac{V(x)}{|x|^{1-2\tau}} = 0,
\]
there is $\Lambda_{\delta,0} > 0$ such that
\[
V(\lambda^{-\tau}x) \leq \frac{\delta}{\lambda^{1-2\tau}|\varphi_\delta|^2}
\]
uniformly for $x \in B_{r_\delta}(0)$. From the above equalities, we get
\[
I_\lambda(e_\lambda) = \frac{1}{2}\|e_\lambda\|^2_\lambda - \frac{\lambda}{2p} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|e_\lambda(x)|^p|e_\lambda(y)|^p}{|x-y|^\mu} \, dy \, dx
\]
\[
= \frac{\lambda^{2\alpha - \tau}}{2} \int_{\mathbb{R}^3} |\nabla \varphi_\delta|^2 \, dx + \frac{\lambda^{2\alpha - 3\tau + 1}}{2} \int_{\mathbb{R}^3} V(\lambda^{-\tau}x)\varphi^2_\delta \, dx
\]
\[
- \frac{\lambda^{2p\alpha + (\mu-6)\tau + 1}}{2p} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\varphi_\delta(x)|^p|\varphi_\delta(y)|^p}{|x-y|^\mu} \, dy \, dx
\]
\[
\leq \delta \lambda^{2\alpha - \tau} - \frac{\lambda^{2p\alpha + (\mu-6)\tau + 1}}{2p}.
\]
Since $\alpha \geq \frac{3-\mu}{4(p-1)}$ and $0 < \tau < \frac{1}{2}$, then
\[
2\alpha - \tau < 2p\alpha + (\mu - 6)\tau + 1,
\]
thus we know there exists $\Lambda_{\delta} > \Lambda_{\delta,0}$ such that, for any $\lambda > \Lambda_{\delta} > 1$ there is a $e_\lambda$
such that
\[
I_\lambda(e_\lambda) < 0.
\]

\[\square\]

**Remark 1.** In general, since $p > 1$, to prove (2) of lemma 4.1, we only need to notice that, for any $u \in E$,
\[
I_\lambda(tu) \to -\infty
\]
as $t \to \infty$. Here the purpose of choosing $e_\lambda$ is to construct small Minimax values below the threshold where $(P.S)$ condition holds.
As a consequence of Lemma 4.1, we have the following conclusion.

**Corollary 2.** Assume that (P1) – (P3) hold. For any $\sigma > 0$ there exists $\Lambda_\sigma > 0$, such that for each $\lambda \geq \Lambda_\sigma$ there is $c_\lambda > 0$ and a (P.S) sequence $(u_n)$ satisfying

$$I_\lambda(u_n) \to c_\lambda, \quad I'_\lambda(u_n) \to 0,$$

where $c_\lambda$ is given by

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I_\lambda(\gamma(t)),$$

and

$$\Gamma = \{ \gamma \in C([0,1], E) : \gamma(0) = 0, \quad I_\lambda(\gamma(1)) < 0 \}.$$

**Proof of Theorem 2.3.** Consider the functional $I_\lambda$. In virtue of Corollary 2, for any $0 < \delta < \alpha_0$, there exists $\Lambda_\delta > 0$, such that for each $\lambda \geq \Lambda_\delta$ there is $c_\lambda > 0$ and a (P.S) sequence $(u_n)$ satisfying

$$I_\lambda(u_n) \to c_\lambda, \quad I'_\lambda(u_n) \to 0, \quad n \to \infty.$$

Moreover, for such fixed $\lambda > \Lambda_\delta$, since $0 < \delta < 1$,

$$c_\lambda \leq \max_{t \in [0,1]} I_\lambda(te_\lambda) = \max_{t \in [0,1]} \left\{ \frac{t^2 \lambda_{2\alpha-\tau}}{2} \left( \int_{\mathbb{R}^3} |\nabla \varphi_\delta(x)|^2 dx + \int_{\mathbb{R}^3} \frac{t^2 \lambda_{2\alpha-3\tau+1}}{2} \int_{\mathbb{R}^3} V(\lambda^{-\tau}x)\varphi_\delta^3(x)dx \right) \right. \right.$$

$$- \frac{t^2 \lambda^{2\alpha+(\mu-6)\tau+1}}{2p} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\varphi_\delta(x)|^p |\varphi_\delta(y)|^\mu}{|x-y|^\mu} dydx \left. \right\} \leq \delta \lambda^{\frac{4-p-\mu}{2(p-\tau)}} \max_{t \in [0,1]} \left\{ t^2 \lambda^{2\alpha-\tau} \frac{4-p-\mu}{2(p-\tau)} - \frac{t^2 \lambda^{2\alpha+(\mu-6)\tau+1}}{2p} \right\}.$$

Direct computation shows

$$t^2 = 2\lambda \frac{-2(p-1)\alpha-(\mu-5)\tau-1}{p-1},$$

and then

$$\max_{t \in [0,1]} I_\lambda(te_\lambda) \leq \delta \lambda^{\frac{4-p-\mu}{2(p-\tau)}} \cdot \lambda^{\frac{(1-2\tau)\alpha+\mu-6}{p-1}} \leq \delta \lambda^{\frac{4-p-\mu}{2(p-\tau)}}.$$

since $p < 6 - \mu$, $\tau \leq \frac{1}{2}$ and $\lambda > 1$. Then Corollary 1 implies that $I_\lambda$ satisfies the (P.S) condition at $c_\lambda$, thus there is $u_\lambda \in E$ such that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = c_\lambda$. Additionally, if $\mathcal{M}$ denotes the Nehari manifold for $I_\lambda$

$$\mathcal{M} = \left\{ u \in E \setminus \{0\} : (I'_\lambda(u), u) = 0 \right\} = \left\{ u \in E \setminus \{0\} : \|u\|_2^2 = \lambda \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^p|u(y)|^\mu}{|x-y|^\mu} dydx \right\}.$$

Then the solution obtained above is a ground state solution in the following sense

$$I_\lambda(u_\lambda) = \inf_{w \in \mathcal{M}} I_\lambda(w).$$

And it is easy to see there holds

$$\frac{p-1}{2p} \lambda \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^p|u(y)|^\mu}{|x-y|^\mu} dydx \leq \delta \lambda^{\frac{4-p-\mu}{2(p-\tau)}}$$

and

$$\frac{p-1}{2p} \int_{\mathbb{R}^3} (|\nabla u_\lambda|^2 + \lambda V(x)u_\lambda^3)dx \leq \delta \lambda^{\frac{4-p-\mu}{2(p-\tau)}}.$$
Moreover, the solution $u_\lambda$ must be positive. In fact, since it is a minimizer on the Nehari manifold $\mathcal{M}$, then $|u_\lambda|$ is also a minimizer. Then we may assume that $u_\lambda$ is non-negative. The maximum principle then implies that $u_\lambda > 0$. \hfill \Box

**Remark 2.** Assume that (S) holds. Remark that if $u \in E$ then the function $\tilde{u} := (u + \theta u)/2$ satisfies $\theta \tilde{u} = \tilde{u}$, i.e., $\tilde{u} \in E^\theta$. It is clear that if $(\varphi_j) \subset C_0^\infty(\mathbb{R}^3)$, $\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\nabla \tilde{\varphi}_j|^2 dx dy = 1$ and $|\nabla \varphi_j|^2 \to 0$, then $\tilde{\varphi}_j = (\varphi_j + \tau \varphi_j)/2 \in E^\theta$ and $|\nabla \tilde{\varphi}_j|^2 \to 0$. Arguing as before, we see the conclusion: Assume that $(P_1) - (P_3)$ and (S) are satisfied. Then for any $\sigma > 0$ there exists $\Lambda_\sigma > 0$ such that for each $\lambda \geq \Lambda_\sigma$ there exists $0 \neq e_\lambda \in E^\theta$ such that $(I'_\lambda(e_\lambda), e_\lambda) = 0$ and

$$I_\lambda(e_\lambda) \leq \sigma \lambda^{\frac{4}{4-(\sigma+1)}}.$$ 

**Proof of Theorem 2.4.** If $u$ is a solution of (9) standard regularity theory shows that it is of class $C^2$ and $\theta$ induces bijection the connected components of $\{x \in \mathbb{R}^3 : u(x) > 0\}$ and those of $\{x \in \mathbb{R}^3 : u(x) < 0\}$. So $u$ changes sign an odd number of times.

Define the $\theta$-Nehari manifold

$$\mathcal{M}_\lambda^\theta := \{u \in E^\theta : u \neq 0, (I'_\lambda(u), u) = 0\}.$$ 

Then critical points of the restriction of $I_\lambda$ on $\mathcal{M}_\lambda^\theta$ are solutions of equation (9). Set

$$c_\lambda^\theta := \inf \{I_\lambda(u) : u \in \mathcal{M}_\lambda^\theta\}.$$ 

Then by Remark 1, for any $\sigma \in (0, \alpha_0)$, there is $\Lambda_\sigma > 0$ such that

$$0 < c_\lambda^\theta \leq \sigma \lambda^{\frac{4}{4-(\sigma+1)}}, \quad \text{if } \lambda \geq \Lambda_\sigma.$$ 

By Corollary 1, $I_\lambda$ satisfies the $(PS)_{c_\lambda^\theta}$ condition. Thus $c_\lambda^\theta$ is a critical value of $I_\lambda$. Let $u_\lambda \in E^\theta$ be the relative critical point, then it is a solution of (9) with $u\lambda(\theta x) = -u_\lambda(x)$. \hfill \Box

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