Abstract. Given a dense subset $A$ of the first $n$ positive integers, we provide a short proof showing that for $p = \omega(n^{-2/3})$ the so-called randomly perturbed set $A \cup [n]_p$ a.a.s. has the property that any 2-colouring of it has a monochromatic Schur triple, i.e. a triple of the form $(a, b, a+b)$. This result is optimal since there are dense sets $A$, for which $A \cup [n]_p$ does not possess this property for $p = o(n^{-2/3})$.

§1. Introduction

The model of randomly perturbed graphs was introduced by Bohman, Frieze and Martin in [5], where they considered for which $p = p(n)$ a Hamilton cycle appears a.a.s. if one studies the model $G_\alpha \cup G(n, p)$, where $G_\alpha$ is a graph on $n$ vertices with minimum degree $\delta(G) \geq \alpha n$ (for a fixed $\alpha > 0$) and $G(n, p)$ is the usual binomial random graph, i.e. each of the $\binom{n}{2}^2$ possible edges appears independently with probability $p$. Their result was a discovery of a phenomenon, that it suffices to take $p = C(\alpha)/n$, which is lower by a log-term needed for the appearance in the random graph alone, see e.g. [6]. Further properties were studied in [4, 17, 25] and, even more recently, various spanning structures in randomly perturbed graphs and hypergraphs were investigated in [1, 2, 3, 7, 8, 10, 13, 15, 16, 19].

The study of Ramsey properties in random graphs was initiated by Łuczak, Ruciński and Voigt [18] and the the so-called symmetric edge Ramsey problem was settled completely by Rödl and Ruciński in a series of papers [20, 21, 22]. Ramsey properties in randomly perturbed graphs were investigated first by Krivelevich, Sudakov, and Tetali [17], who proved that whenever one perturbs a sufficiently large $n$-vertex dense graph $G$ by adding to it $\omega(n^{2-2/(t-1)})$ edges chosen uniformly at random then the resulting graph almost surely possesses the property that any 2-colouring of its edges admits either a monochromatic triangle (in the first colour) or a monochromatic copy of $K_t$ (in the second colour), where here $t \geq 3$. Generally, one writes $G \rightarrow (K_3, K_t)$ to denote this fact (and shortens it to $G \rightarrow (K_t)_r$ in the symmetric case and $r$ colors). Moreover, the result of [17] is asymptotically best possible in terms of the number of random edges added.

In this short note we will study randomly perturbed dense sets of integers. More precisely, we denote by $[n]_p$ the model where each of the integers

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from \([n] := \{1, 2, \ldots, n\}\) is chosen independently with probability \(p = p(n)\). The model \([n]_p\) itself was thoroughly investigated with respect to extremal properties (e.g. Szemerédi’s theorem [9, 14, 23]) and Ramsey-type properties (partition regularity of equations [11, 12]).

Given a dense set \(A \subseteq [n]\), i.e. with at least \(\alpha n\) elements for some fixed positive \(\alpha\), we will study for which \(p = p(n)\), the set \(A \cup [n]_p\) a.a.s satisfies the property that no matter how one colours \(A \cup [n]_p\) with two colours, there will always be a monochromatic Schur triple, i.e. a triple of integers \(x, y, z\) with \(x + y = z\). An old result of Schur [24] states that if \(\mathbb{N}\) is finitely coloured, then there is always such a monochromatic triple. Moreover, the threshold \(\Theta(n^{-1/2})\) for the random set \([n]_p\) was determined by Graham, Rödl and Ruciński [12] (for two colours) and by Friedgut, Rödl and Schacht [11] (for any constant number of colours). Our main result on Schur triples in the model \(A \cup [n]_p\) and two colours is as follows.

**Theorem 1.** For every \(\alpha > 0\) and every \(p = \omega(n^{-2/3})\) it holds that whenever \(A \subseteq [n]\) has \(|A| \geq \alpha n\) then a.a.s. \(A \cup [n]_p\) has the property that in any 2-colouring of it a monochromatic Schur triple appears.

Let us say a few words about the nature of the threshold \(\omega(n^{-2/3})\). The point is that, above \(n^{-2/3}\), the set \(A \cup [n]_p\) contains a.a.s. some 10-element set, which, no matter how coloured by red and blue, always contains a monochromatic Schur triple. A similar situation occurs for \(G_\alpha \cup G(n, p) \rightarrow (K_3)_2\): it was shown in [4] that the threshold for the appearance of \(K_6\) is then \(p = \omega(1/n)\) and noted in [17] that, since \(K_6 \rightarrow (K_3)_2\), this is already the threshold.

The condition \(p = \omega(n^{-2/3})\) from Theorem 1 is asymptotically best possible. The expected number of Schur triples in the random set \([n]_p\) is \(O(p^3n^2)\); this quantity vanishes whenever \(p = o(n^{-2/3})\). A typical set at this density is a.a.s. sum-free then (i.e., has no Schur triples). This in turn implies that the perturbation of \([n/2 + 1, n]\), namely \([n/2 + 1, n] \cup [n]_p\), a.a.s. admits a 2-colouring with no monochromatic Schur triples. Indeed, one may colour the (dense) sum-free set \([n/2 + 1, n]\) with one colour and all elements (not already in \([n/2 + 1, n]\)) coming from \([n]_p\) with the complementary colour.

Moreover, for \(r > 2\) colours the colouring in which the set \(A := [n/2 + 1, 1]\) is coloured with one colour and \([n]_p\) with the remaining \(r - 1\) shows that, asymptotically, one does not gain on probability since the threshold for \([n]_p\) having the property that any \((r - 1)\)-colouring (for any constant \(r \geq 3\)) of it admits a monochromatic Schur triple is \(\Theta(n^{-1/2})\) [11, 12].

**§2. Proof of Theorem 1**

We refer to a set of integers as 2-Schur-Ramsey if any 2-colouring of it admits a monochromatic Schur triple. For \(x, y, d \in [n]\) satisfying \(y > x + d\) and \(x > d\) we define the 10-tuple \(\mathcal{L}(x, y, d) \in \mathbb{N}^{10}\) as follows

\[
(2) \quad \mathcal{L}(x, y, d) := (x - d, x, x + d, x + 2d, y - d, y, y + d, d, y - x - d, y - x).
\]
Its first four entries form a 4-term arithmetic progression (4AP, hereafter), the next three form a 3-term arithmetic progression (3AP, hereafter), and its last three points form a Schur triple denoted \(S(x, y, d) := (d, y-x-d, y-x)\).

**Lemma 3.** Let \(n \geq 1\) be an integer and let \(x, y, d \in [n]\) satisfy \(y > x + d, x > d\). Then \(L(x, y, d)\) is 2-Schur-Ramsey.

*Proof.* The proof proceeds via a simple case analysis. Fix a 2-colouring of \(L(x, y, d)\) and assume towards a contradiction that it admits no monochromatic Schur triple. We may assume that \(x, y, \) and \(d\) are not coloured in the same colour; for otherwise the assumption of no monochromatic Schur triple would force a contradiction in the shape of the Schur triple \((y-x-x-d, y-d)\) all coloured with the complimentary colour to the one used for \(x, y, \) and \(d\).

Nevertheless, two of \(x, y, \) and \(d\) are coloured using the same colour. We consider the three possible cases arising here. Suppose, firstly, that \(y\) and \(d\) are coloured, say, red, and that \(x\) is coloured blue. We may assume that \(y-d\) and \(y+d\) are both coloured blue in this setting. Then the Schur triple \((y-x-d, x, y-d)\) forces \(y-x-d\) to be red and consequently \(y-x\) is blue owing to \((y-x-d, d, y-x)\). The triple \((y-x, x-d, y-d)\) mandates that \(x-d\) is red and then one concludes that \(x+d\) is blue due to \((y-x-d, x+d, y)\).

At this point one notices that \((y-x, x+d, y+d)\) is a blue Schur triple, a contradiction.

Suppose, secondly, that \(x\) and \(d\) are coloured, say, red, and that \(y\) is coloured blue. We may assume that \(x-d\) and \(x+d\) are both coloured blue. The triple \((y-x-d, x+d, y)\) implies that \(y-x-d\) is red and then \(y-x\) is blue due to \((d, y-x-d, y-x)\). Consequently, \(y-d\) is red owing to \((y-x, x-d, y-d)\) and then one arrives at \((y-x-d, x, y-d)\) as a red Schur triple, a contradiction.

Suppose, thirdly (and finally), that \(x, y\) are coloured, say, blue and that \(d\) is coloured red. Due to \((y-x, x, y)\) we may assume that \(y-x\) is coloured red. Next, we have that \(y-x-d\) is blue due to \((d, y-x-d, y-x)\), that \(y-d\) is red due to \((y-x-d, x, y-d)\), that \(x+d\) is red due to \((y-x-d, x+d, y)\), that \(y+d\) is blue due to \((y-x, x+d, y+d)\), that \(x-d\) is blue due to \((y-x, x-d, y-d)\), and that \(x+2d\) is blue due to \((x+d, d, x+2d)\). All this ends with \((y-x-d, x+2d, y+d)\) being a blue Schur triple; contradiction.

Facilitating our proof is the following result due to Varnavides [26]: which roughly speaking asserts that dense sets in \([n]\) have \(\Omega(n^2)\) 4APs. Originally Varnavides’ result was phrased for 3APs, but a standard supersaturation argument generalizes to arithmetic progressions of any fixed length.

**Theorem 4.** (Varnavides [26]) For every \(\alpha > 0\) there exists an \(n_0 := n_0(\alpha)\) and a \(g(\alpha) > 0\) such that whenever \(n \geq n_0\) and \(A \subseteq [n]\) satisfy \(|A| \geq \alpha n\) then \(A\) contains at least \(g(\alpha)n^2\) 4APs.
Dense sets in \([n]\) have an abundance of 4APs then. This in turn implies that there are elements in \([n]\) that are repeatedly being used by 4APs found in the dense set as their common steps; in fact there are many such popular common steps. Making this precise, define for a set \(A \subseteq [n]\) and a parameter \(\varepsilon > 0\) the set

\[
D_\varepsilon(A) := \{d \in [n] : A \text{ has } \geq \varepsilon n \text{ 4APs with common step } d\}
\]
to be the set of common steps popular in the sense that each of these participants in at least \(\varepsilon n\) 4APs (in \(A\)).

**Lemma 5.** (Linearly many popular common steps) For every \(\alpha > 0\) there exists an \(\varepsilon := \varepsilon(\alpha) > 0\) such that for \(n\) sufficiently large \(|D_\varepsilon(A)| \geq \varepsilon n\) holds whenever \(A \subseteq [n]\) satisfies \(|A| \geq \alpha n\).

**Proof.** Given \(\alpha\), let \(g(\alpha)\) be as asserted by Theorem 4. Set \(\varepsilon := g(\alpha)/2\) and let \(\#4AP(A)\) denote the number of 4APs in \(A\). By Theorem 4, the set \(A\) has at least \(g(\alpha)n^2\) 4APs whenever \(n\) is sufficiently large. Then owing to

\[
g(\alpha)n^2 \leq \#4AP(A) < |D_\varepsilon(A)|n + n \cdot \varepsilon n
\]
the lemma follows.

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** By Lemma 3 it suffices to show that a.a.s. \(A \cup [n]_p\) contains \(L(x, y, d)\) for some \(x, y, d \in [n]\) (with \(y > x + d\) and \(x > d\)). By Theorem 4 the set \(A\) contains at least \(g(\alpha)n^2\) 4APs. Moreover, Lemma 5 asserts that there is an \(\varepsilon := \varepsilon(\alpha) > 0\) such that \(|D_\varepsilon(A)| \geq \varepsilon n\). Given \(d \in D_\varepsilon(A)\) and a pair of distinct 4APs both with common step \(d\), we may write them parametrically as \((x - d, x, x + d, x + 2d)\) and \((y' - d, y', y' + d, y' + 2d)\) with \(y' > x\). Setting \(y := y' + d\), we see that the Schur triple \(S(x, y, d) = (d, y - x - d, y - x)\) is well-defined.

Let \(T\) be the set of all such Schur triples \(S(x, y, d)\). Since there are at least \(\varepsilon n\) choices for \(d \in D_\varepsilon(A)\) and \(A\) contains at least \(\varepsilon n\) 4APs with common difference \(d\), there are, for a fixed \(d \in D_\varepsilon(A)\), at least \(\varepsilon n - 1\) different possible values for \(y - x\). Therefore, the number of such Schur triples is at least \(\varepsilon^2 n^2/4\), i.e., \(|T| \geq \varepsilon^2 n^2/4\).

It remains to argue that a.a.s. \([n]_p\) contains a member of \(T\). The expected number of members of \(T\) captured by \([n]_p\) is at least \(\varepsilon^2 p^3 n^2/4\). We write \(X_t\) for the indicator random variable indicating whether \(t = S(x, y, d) \in T\) is captured by \([n]_p\) and then put \(X := \sum_{t \in T} X_t\). Using Chebyshev’s inequality we arrive at:

\[
\Pr[X = 0] \leq \frac{\Var(X)}{(E(X))^2} \leq \frac{1}{E(X)} + \frac{\sum_{t \neq t' \in T} E[X_t X_{t'}]}{E(X)^2}.
\]

The number of pairs \(t, t' \in T, t \neq t'\), having a single entry in common is at most \(3n^3\), and each such pair satisfies \(E[X_t X_{t'}] \leq p^5\) so in total all such pairs contribute \(O(p^5 n^3)\) to the sum appearing on the r.h.s. above. Next,
the number of pairs $t, t' \in T$, $t \neq t'$, having two entries in common is at most $2n^2$, with the total contribution of $O(n^2 p^4)$. Using the lower bound on $E(X) \geq \varepsilon^2 p^3 n^2/2$ and $p = \omega(n^{-2/3})$ we have

$$
\Pr \left[ X = 0 \right] \leq \frac{1}{\Omega(p^3 n^2)} + \frac{O(p^5 n^3)}{\Omega(p^6 n^4)} + \frac{O(p^4 n^2)}{\Omega(p^6 n^4)} = o(1),
$$

finishing the proof. ■

§3. Concluding remarks

In this note we studied the first Ramsey-type question for randomly perturbed sets of integers, i.e. the model $A \cup [n]_p$. It would be interesting to determine the thresholds for more general partition regular systems of equations. For the threshold results in random sets see the work of Friedgut, Rödl and Schacht [11].

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