AN EDGE-OF-THE-WEDGE THEOREM FOR
HYPERSURFACE CR FUNCTIONS

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1. Introduction

The Lewy extension theorem asserts the holomorphic extendability of CR functions defined in a neighborhood of a point on a hypersurface in $\mathbb{C}^{n+1}$. The edge-of-the-wedge theorem asserts the extendability of holomorphic functions defined in wedges in $\mathbb{C}^{n+1}$ with edge a maximally real submanifold. In this article we prove under suitable hypotheses the holomorphic extendability to an open set in $\mathbb{C}^{n+1}$ of CR functions defined in the intersection of a hypersurface with a wedge whose edge is contained in the hypersurface. Unlike the situation for the classical edge-of-the-wedge theorem, for this hypersurface version extendability generally depends on the direction of the wedge.

Equip $\mathbb{R}^{n+1}$ with its standard inner product and let $\sigma \in \mathbb{R}^{n+1}$ be a unit vector. By the round cone in $\mathbb{R}^{n+1}$ of aperture $\delta > 0$, extent $\ell > 0$, and axis $\sigma$, we shall mean

$$\{ x \in \mathbb{R}^{n+1} : |x - \langle x, \sigma \rangle \sigma| < \delta \langle x, \sigma \rangle \text{ and } |x| < \ell \}.$$ 

Let $E \subset \mathbb{C}^{n+1}$ be a maximally real submanifold, i.e. $E$ is a totally real submanifold of maximal dimension $n+1$. Using $J$ to denote multiplication by $i$ and $TE$ to denote the tangent bundle to $E$, for $p \in E$ the subspace $J(T_pE)$ is transverse to $T_pE$. Now $J(T_pE)$ inherits an inner product from that on $\mathbb{C}^{n+1}$, hence may be identified with $\mathbb{R}^{n+1}$ so that round cones in $J(T_pE)$ are defined. We may view $J(T_pE)$ as a real affine subspace of $\mathbb{C}^{n+1}$. Shrinking $E$ if necessary, we assume that for some $r > 0$ the balls of radius $r$ in the subspaces $J(T_pE)$ sweep out a tubular neighborhood of $E$ in $\mathbb{C}^{n+1}$. By a one-sided wedge in $\mathbb{C}^{n+1}$ of aperture $\delta > 0$, extent $\ell \in (0, r)$, and edge $E$ we shall mean the union of the round cones of aperture $\delta$, extent $\ell$, and axes $\sigma(p)$ for some smooth family of unit vectors $\sigma(p) \in J(T_pE)$ for $p \in E$. By the opposite of such a one-sided wedge in $\mathbb{C}^{n+1}$ we shall mean the union of

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the round cones of aperture \( \delta \), extent \( \ell \), and axes \(-\sigma(p)\), and by a \textit{two-sided wedge} in \( \mathbb{C}^{n+1} \) we shall mean the union of such a one-sided wedge with its opposite. The classical edge-of-the-wedge theorem (e.g. [3]) asserts that a holomorphic function defined on a two-sided wedge in \( \mathbb{C}^{n+1} \) and continuous across its edge extends as a holomorphic function to a neighborhood of this edge.

Now suppose that \( M \) is a smooth hypersurface in \( \mathbb{C}^{n+1} \). The holomorphic tangent bundle \( H \equiv TM \cap J(TM) \) on \( M \) together with \( J|_H \) defines the CR structure of \( M \) intrinsically. Let \( E \subset M \) be a smooth submanifold which is maximally real as a submanifold of \( \mathbb{C}^{n+1} \). Define a subbundle \( N \subset TM|_E \) by \( N = J(TE) \cap TM \). Observe that \( N \subset H \) and for \( p \in E \), \( N_p \) is a complement to \( T_pE \) in \( T_pM \). By a one-sided wedge in \( M \) with edge \( E \) we shall mean the intersection \( W^+ \) of \( M \) with a one-sided wedge in \( \mathbb{C}^{n+1} \) with edge \( E \) and with axes \( \sigma(p) \) everywhere tangent to \( M \). We denote by \( W^- \) the intersection of \( M \) with the opposite one-sided wedge in \( \mathbb{C}^{n+1} \). By a two-sided wedge in \( M \) with edge \( E \) we shall mean the union \( W = W^+ \cup W^- \) of two such opposite one-sided wedges in \( M \). Observe that the axes of a wedge in \( M \) satisfy \( \sigma(p) \in N_p \). We shall say that a non-zero vector \( \tau \in N_p \) is a \textit{direction} of \( W^+ \) (resp. \( W^- \)) at \( p \) if \( \tau \) is in the intersection of \( T_pM \) with the round cone of infinite extent in \( J(T_pE) \) defining \( W^+ \) (resp. \( W^- \)). We say that \( \tau \) is a direction of \( W \) if it is a direction of either \( W^+ \) or \( W^- \).

Locally we may choose a smooth real defining function \( r \) for \( M \) so that \( M = \{ r = 0 \} \) and \( dr \neq 0 \) everywhere. The \textit{Levi form} associated to \( r \) is the complex-valued quadratic form on \( H \), Hermitian with respect to \( J \), given by \( L(\sigma, \tau) = \bar{\delta}\partial r(\sigma - iJ\sigma, \tau + iJ\tau) \) for \( \sigma, \tau \in H \). For \( p \in M \) we say that \( L_p \) is \textit{indefinite} if there are \( \sigma_+, \sigma_- \in H_p \) such that \( L(\sigma_+, \sigma_+) > 0 \) and \( L(\sigma_-, \sigma_-) < 0 \). We say that \( \sigma \in H_p \) is \textit{null} if \( L(\sigma, \sigma) = 0 \).

Our main theorem is as follows.

**Theorem 1.1.** Let \( M \) be a smooth hypersurface in \( \mathbb{C}^{n+1} \) and \( E \subset M \) a maximally real submanifold. Let \( W \) be a two-sided wedge in \( M \) with edge \( E \). Suppose that \( p_0 \in E \) is a point at which the Levi form of \( M \) is indefinite and that \( W \) has a direction \( \sigma \in N_{p_0} \) which is null. Then there is a neighborhood of \( p_0 \) in \( \mathbb{C}^{n+1} \) to which every CR function on \( W \), continuous across \( E \), extends as a holomorphic function.

Note that by replacing the round cones defining \( W^+ \) by smaller ones, we can suppose, without loss of generality, that the axis \( \sigma(p_0) \) is null. Note also that the Levi form \( L_p \) is indefinite for all \( p \in M \) near \( p_0 \). Therefore by the Lewy extension theorem applied to points \( p \neq p_0 \), any CR function on \( W \) is, close enough to \( p_0 \), the restriction of a holomorphic function, so in particular is a smooth function. The Lewy
extension theorem also allows us to rephrase the conclusion of Theorem 1.1 intrinsically by saying that there is a neighborhood of \( p_0 \) in \( M \) to which we obtain extension as a CR function.

As indicated earlier, Theorem 1.1 generalizes at the same time the (two-sided) Lewy extension theorem and the edge-of-the-wedge theorem. In the Lewy theorem, the CR function is required to be defined in a full neighborhood in \( M \) of \( p_0 \) in order to get extension; in the edge-of-the-wedge theorem, the holomorphic function is required to be defined in a full two-sided wedge in \( \mathbb{C}^{n+1} \).

Since Theorem 1.1 requires the Levi form to be indefinite, the hypotheses cannot be satisfied unless \( n \geq 2 \). The hypersurface edge-of-the-wedge phenomena of interest to us do not occur in \( \mathbb{C}^2 \) in any case since a maximally real submanifold \( E \) of a hypersurface \( M \) of \( \mathbb{C}^2 \) has codimension 1, so a two-sided wedge in \( M \) together with its edge \( E \) actually fills out a full neighborhood of \( E \). We consider two illustrative examples in \( \mathbb{C}^3 \). Let \((z_1, z_2, w) = (x_1 + iy_1, x_2 + iy_2, u + iv)\) denote the coordinate functions on \( \mathbb{C}^3 \).

**Example 1.2.** Let \( M = \{ v = y_1^2 - y_2^2 \} \) and \( E = \mathbb{R}^3 = \{ y = v = 0 \} \). Then \( H = \text{span}\{\partial_{x_1}, \partial_{x_2}, \partial_{y_1}, \partial_{y_2}\} \) and \( N = \text{span}\{\partial_{y_1}, \partial_{y_2}\} \). The Levi-null directions in \( N \) are multiples of \( \partial_{y_1} \pm \partial_{y_2} \), so Theorem 1.1 implies that if one of these is a direction of \( W \), then extension must hold. On the other hand, any non-null direction in \( N \) is a direction of a two-sided wedge \( W \) in \( M \) for which there do exist CR functions on \( W^\pm \) with equal boundary values on \( E \) for which extension fails. To see this, consider \( W = M \cap \{ v > 0 \} = W^+ \cup W^-, \) where \( W^\pm = M \cap \{ \pm y_1 > |y_2| \} \). Sufficiently near \( E \), \( W^+ \) is the intersection of \( M \) with the union over points of \( E \) of the round cones of aperture 1 and axes in the \( y_1 \)-direction. Consequently, the intersection of \( W \) with a small enough neighborhood of \( E \) is a two-sided wedge in \( M \) with edge \( E \), having all directions \( a\partial_{y_1} + b\partial_{y_2} \) with \( |a| > |b| \). Let \( f \) be a holomorphic function in the upper half plane in \( \mathbb{C} \) which extends smoothly to the closed upper half plane but which does not have a holomorphic extension across 0. Then \( f(w)|_W \) defines CR functions on \( W^\pm \) with the required properties. Clearly a similar construction can be carried out for \( M \cap \{ v < 0 \} \).

**Example 1.3.** Let \( M = \{ v = \Re(z_1 z_2) = y_1 x_2 - y_2 x_1 \} \), and again let \( E = \mathbb{R}^3 \). This \( M \) is the standard mixed signature hyperquadric in \( \mathbb{C}P_3 \) viewed in an affine chart. In this case \( L \) is indefinite and every direction \( \sigma \in N \) is null, so extension holds for any wedge. We obtain a result for CR functions completely analogous to the classical edge-of-the-wedge theorem for holomorphic functions.
For the case of two-sided wedges in $M$ which point in directions which are not null, we have the following one-sided extension theorem.

**Theorem 1.4.** Let $M$ be a smooth hypersurface in $\mathbb{C}^{n+1}$ with defining function $r$ and let $E \subset M$ be a maximally real submanifold. Let $W$ be a two-sided wedge in $M$ with edge $E$. Suppose that $p_0 \in E$ and that $W$ has a direction $\sigma \in N_{p_0}$ satisfying $L(\sigma, \sigma) > 0$. Then there is an open set $U \subset \{ r < 0 \}$ with the following properties:

(a). For each $p \in E$ sufficiently close to $p_0$ and each unit vector $\tau \in J(T_pE)$ such that $dr(\tau) < 0$, there is a round cone in $J(T_pE)$ with axis $\tau$ which is contained in $U$.
(b). $U$ contains a two-sided subwedge $\tilde{W}$ in $M$ of $W$ having $\sigma$ as a direction at $p_0$.
(c). To every continuous CR function $f$ on $W$, continuous across $E$, there is a holomorphic function on $U$, continuous in $U$, which restricts to $f$ on $\tilde{W}$. In particular, the extension agrees with $f$ on $E$.

We remark that Theorem 1.4 can be applied to a two-sided wedge in $M$ with direction $\sigma$ satisfying $L(\sigma, \sigma) < 0$ upon replacing $r$ by $-r$. We also remark that in (a) above, the aperture and extent of the cone with axis $\tau$ may be chosen to depend continuously on $p$ and $\tau$, but they will in general go to 0 as $dr(\tau) \to 0$.

Theorem 1.4 may be seen as a refinement of one-sided Lewy extension in which the CR function need only be defined in a wedge in $M$. Example 1.2 shows that extension to all of $\{ r < 0 \}$ need not hold in this case.

In this paper we prove Theorems 1.1 and 1.4 by analytic disc methods combined with an adaptation of the Baouendi-Treves approximation theorem to wedges. Such a proof of the $\mathbb{C}^{n+1}$-edge-of-the-wedge theorem for $C^1$ edges was given by Rosay [3]. The discs required in Theorem 1.4 are direct modifications of the usual discs used in Lewy extension. For Theorem 1.1 we first use Lewy extension away from the edge to obtain an open set in $\mathbb{C}^{n+1}$ containing $W$; this set has a parabolic “spike” approach transverse to $M$. We then use a version of the “folding screen lemma” to enlarge each side of the wedge to a full wedge in $\mathbb{C}^{n+1}$, whereupon we can apply the usual edge-of-the-wedge theorem. We remark that if in Theorem 1.1, $L|_{N_{p_0}}$ is indefinite in the sense that there are $\tau_+, \tau_- \in N_{p_0}$ with $L(\tau_+, \tau_+) > 0$ and $L(\tau_-, \tau_-) < 0$, then a proof can be given using Theorem 1.4. In fact, in this case any null $\sigma \in N_{p_0}$ can be perturbed to $\sigma_+, \sigma_- \in N_{p_0}$ satisfying $L(\sigma_+, \sigma_+) > 0$ and $L(\sigma_-, \sigma_-) < 0$, so Theorem 1.4 gives extension to one-sided wedges in $\mathbb{C}^{n+1}$ on either side of $M$. The usual edge-of-the-wedge theorem then
HYPERSURFACE EDGE-OF-WEDGE

yields holomorphic extension to a neighborhood of \( p_0 \). This argument can be used for Example 1.2. However, it seems that Example 1.3 requires a more sophisticated proof.

In [7], Tumanov considers extension of a CR function from a one-sided wedge. In our situation his results imply that there is a holomorphic extension to an ambient one-sided wedge but give no information on the directions or axis of this ambient wedge. His results apply equally to Example 1.3, for which the intrinsic edge-of-the-wedge theorem holds, and to Example 1.2, for which it generally fails. In order to apply the ambient classical edge-of-the-wedge theorem as we do in Theorem 1.1, it is crucial that we obtain holomorphic extension to a wedge with the same axis as the intrinsic wedge (as detailed in Remark 4.3).

In another paper [3] we will present a microlocal approach to results of this type formulated in terms of the hypo-analytic wave-front set of [1]. This in particular will contain analogues of Theorems 1.1 and 1.4 for higher codimension CR manifolds, and, in fact, an intrinsic version on general hypo-analytic manifolds.

Throughout this article, smooth will mean infinitely differentiable.

2. Folding Screens

In [4], Hörmander proved Bochner’s tube theorem by a geometrical arrangement dubbed a ‘folding screen’ by Komatsu [5]. We shall use this arrangement to study the polynomial hull of certain sets of wiggling spikes in \( \mathbb{C}^2 \). If \( S \subset \mathbb{C}^{n+1} \), we denote by \( \hat{S} \) the polynomial hull of \( S \):

\[
\hat{S} = \{ z : |p(z)| \leq \sup_{\zeta \in S} |p(\zeta)| \text{ for all holomorphic polynomials } p \}.
\]

Lemma 2.1. Set

\[
S = \left\{ (\zeta_1, \zeta_2) = (\xi_1 + i\eta_1, \xi_2 + i\eta_2) \in \mathbb{C}^2 : 0 < \eta_1 < 2 \text{ and } |\eta_2| < 2\eta_1^2 - |\xi|^2\eta_1/4 \right\}.
\]

Then \( \hat{S} \) contains

\[
T = \{ (\zeta_1, \zeta_2) = (i\eta_1, i\eta_2) : 0 < \eta_1 < 1 \text{ and } |\eta_2| < \eta_1/2 \}.
\]

Proof. We construct analytic discs with boundaries contained in \( S \). For each \( t \in (5, 6) \), we claim that the non-singular quadric

\[
\{(\zeta_1 + i)^2 + (\zeta_2 - 2i)^2 + t = 0 \}
\]

meets the square pillar

\[
\{(\zeta_1, \zeta_2) = (\xi_1 + i\eta_1, \xi_2 + i\eta_2) : 0 < \eta_1, \eta_2 < 1 \}
\]
in an analytic disc $D_t$ with piecewise smooth boundary in the folding screen

$$\{ \xi + i\eta : 0 < \eta_1 \leq 1, \eta_2 = 0 \} \cup \{ \xi + i\eta : \eta_1 = 1, 0 \leq \eta_2 < 1 \}.$$

In fact, the disc in question, with its boundary, may be parameterized by

$$\{ z = x + iy : t - 4 + y^2/16 \leq x \leq 4 - y^2/16 \} \subset \mathbb{C},$$

for it is easy to check that any such $z$ may be written uniquely both as

$$-(\zeta - 2i)^2$$

for some $\zeta = \xi + i\eta$ with $0 \leq \eta < 1$

and

$$(\zeta + i)^2 + t$$

for some $\zeta = \xi + i\eta$ with $0 < \eta \leq 1$.

Since

$$\Re \left[ (\zeta_1 + i)^2 + (\zeta_2 - 2i)^2 + t \right] = |\xi|^2 + t - [(\eta_1 + 1)^2 + (\eta_2 - 2)^2],$$

it follows that the disc $D_t$ is contained in the hypersurface

(2.1) \quad \{(\eta_1 + 1)^2 + (\eta_2 - 2)^2 = |\xi|^2 + t\}

which intersects the folding screen as

$$\{0 < \eta_1 = \sqrt{|\xi|^2 + t - 4} - 1 \leq 1, \eta_2 = 0\} \cup
\{\eta_1 = 1, 0 \leq \eta_2 = 2 - \sqrt{|\xi|^2 + t - 4} < 1\}.$$

This intersection is empty unless $|\xi|^2 \leq 3$ and, in this case, it is easily verified that

$$|\xi|^2/8 \leq \sqrt{|\xi|^2 + 1} - 1 < \sqrt{|\xi|^2 + t - 4} - 1$$

and

$$2 - |\xi|^2/4 > 2 - \sqrt{|\xi|^2 + 1} > 2 - \sqrt{|\xi|^2 + t - 4}.$$

Since $\partial D_t$ is the intersection of $D_t$ with the folding screen, from these inequalities it follows that $\partial D_t \subset S$ for all $t \in (5, 6)$.

The following diagram shows what is happening in the $(\eta_1, \eta_2)$-plane when $\xi = 0$.
Lemma 2.2. For any $\epsilon, K > 0$, there is a $\delta > 0$ such that if we set

$$S_{\epsilon, K} = \left\{ (\zeta_1, \zeta_2) = (\xi_1 + i\eta_1, \xi_2 + i\eta_2) \in \mathbb{C}^2 : \right.$$

$$0 < \eta_1 < \epsilon \text{ and } |\eta_2| < \epsilon\eta_1^2 - K|\xi|^2\eta_1 \left. \right\},$$

then $\overline{S_{\epsilon, K}}$ contains

$$T_\delta \equiv \{(\zeta_1, \zeta_2) = (i\eta_1, i\eta_2) : 0 < \eta_1 < \delta \text{ and } |\eta_2| < \delta\eta_1\}.$$

Proof. Consider the linear change of coördinates:

$$\phi(\zeta_1, \zeta_2) = (A\zeta_1, 2A^2\zeta_2/\epsilon) = (\hat{\zeta}_1, \hat{\zeta}_2),$$

where $A = \max\{2/\epsilon, \epsilon/2, 8K/\epsilon\}$. We claim that $\phi^{-1}(S) \subset S_{\epsilon, K}$. Were this to be shown, the result would follow immediately from Lemma 2.1.

Since $A \geq 2/\epsilon$,

$$0 < \hat{\eta}_1 < 2 \Rightarrow 0 < \eta_1 < \epsilon.$$

Since $A \geq \epsilon/2$,

$$|\hat{\xi}|^2 = (A\xi_1)^2 + (2A^2\xi_2/\epsilon)^2 \geq (A\xi_1)^2 + (A\xi_2)^2 = A^2|\xi|^2.$$

Thus,

$$|\hat{\eta}_2| < 2\hat{\eta}_1^2 - |\hat{\xi}|^2\hat{\eta}_1/4 \Rightarrow |2A^2\eta_2/\epsilon| < 2(A\eta_1)^2 - 2|\xi|^2(A\eta_1)/4$$

$$\Rightarrow |\eta_2| < \epsilon\eta_1^2 - K|\xi|^2\eta_1$$

since $A \geq 8K/\epsilon$. Altogether, $\phi^{-1}(S) \subset S_{\epsilon, K}$ as required.

We define the spike $Sp(\beta, \ell, 0, 0, m) \subset \mathbb{R}^2$ of sharpness $\beta > 0$, length $\ell > 0$, slope $m \in \mathbb{R}$, and with vertex at the origin, by

$$Sp(\beta, \ell, 0, 0, m) \equiv \left\{ (\eta_1, \eta_2) \in \mathbb{R}^2 : 0 < \eta_1, \right.$$  

$$\left. |(\eta_1, \eta_2)| < \ell, \text{ and } |\eta_2 - m\eta_1| < \beta\eta_1^2 \right\}.$$

Let $Sp(\beta, \ell, q_1, q_2, m)$ denote the translation of this spike with vertex at $(q_1, q_2) \in \mathbb{R}^2$.

Lemma 2.3. Suppose that $A, \beta, \ell, r$ are positive constants and suppose

$$S \supset \left\{ (\zeta_1, \zeta_2) = (\xi_1 + i\eta_1, \xi_2 + i\eta_2) : |\xi| < r \text{ and } \right.$$  

$$\left. (\eta_1, \eta_2) \in Sp(\beta, \ell, q_1(\xi), q_2(\xi), m(\xi)) \right\},$$

where $q_1(\xi), q_2(\xi), m(\xi)$ are smooth functions satisfying

$$|q_1(\xi)| \leq A|\xi|^2 \quad |q_2(\xi)| \leq A|\xi|^4 \quad |m(\xi)| \leq A|\xi|^2.$$

Then there exists $\delta > 0$ depending only on $A, \beta, \ell, r$ such that $T_\delta \subset \widehat{S}$. 

It is clear that every point in $T \cap \{\eta_2 > 0\}$, not already in $S$, lies in $D_t$ for some $t \in (5, 6)$. A similar family of discs sweeps out $T \cap \{\eta_2 < 0\}$. The result thus follows from the maximum modulus principle.
Lemma 3.1. Suppose that \( (\eta_1, \eta_2) \in S_{e,K} \). Then \( 0 < \eta_1 < \epsilon \) and \( \eta_1^2 - K|\xi|^2 \eta_1 > 0 \), so \( |\xi| < \epsilon/\sqrt{K} \). If we take \( \epsilon \leq 1 \) and \( K \geq 1/\epsilon^2 \), then \( |\xi| < r \) which is one of the conditions forcing \( \xi \) to be in \( S \). If we also take \( K \geq A \), then

\[
\eta_1 > K|\xi|^2/\epsilon \geq A|\xi|^2 \geq q_1(\xi),
\]

which is the first condition for \((\eta_1, \eta_2) \in Sp(\beta, \ell, q_1(\xi), q_2(\xi), m(\xi))\). Now \( |\eta_2| < \epsilon \eta_1^2 \). Therefore, \( |\eta| \leq 2\epsilon \). Thus,

\[
|\eta - (q_1(\xi), q_2(\xi))| \leq |\eta| + |q_1(\xi)| + |q_2(\xi)| \leq 2\epsilon + A|\xi|^2 + A|\xi|^4 \leq 2\epsilon + A(1/K + 1/K^2)
\]

which we can arrange to be less than \( \ell \) by taking \( \epsilon \) sufficiently small and \( K \) sufficiently large. This leaves one condition to be ensured. It is that

\[
\beta(\eta_1 - q_1(\xi))^2 - |\eta_2 - q_2(\xi) - m(\xi)(\eta_1 - q_1(\xi))| > 0
\]

when \( \zeta = \xi + i\eta \in S_{e,K} \). Now \( |\eta_2| < \epsilon \eta_1^2 - K|\xi|^2 \eta_1 \) on \( S_{e,K} \) so it suffices to show that

\[
\beta(\eta_1 - q_1(\xi))^2 - \epsilon \eta_1^2 + K|\xi|^2 \eta_1 - |q_2(\xi) + m(\xi)(\eta_1 - q_1(\xi))| > 0
\]

when \( \eta_1 > K|\xi|^2/\epsilon \). Choosing \( \epsilon < \beta/2 \), this expression is strictly bounded below by

\[
\beta \eta_1^2/2 + (K|\xi|^2 - 2\beta q_1(\xi) - |m(\xi)|)\eta_1 - |q_2(\xi)| - |m(\xi)q_1(\xi)|
\]

which, for \( \eta_1 > K|\xi|^2/\epsilon \), is bounded below by

\[
\beta K^2|\xi|^4/2\epsilon^2 + (K|\xi|^2 - 2\beta A|\xi|^2 - A|\xi|^2)\eta_1 - A|\xi|^4 - A^2|\xi|^4.
\]

Taking \( K > (2\beta + 1)A \), we may neglect the second term, leaving

\[
(\beta K^2/2\epsilon^2 - A - A^2)|\xi|^4,
\]

which is non-negative if we choose \( K^2/\epsilon^2 \geq 2(A + A^2)/\beta \). \( \square \)

3. A Normal Form

The following normal form was suggested to us by Vladimir Ezhov.

Lemma 3.1. Suppose that \( M \) is a smooth real hypersurface in \( \mathbb{C}^{n+1} \), \( E \subset M \) is a maximally real submanifold, and \( p \in E \). Then we may choose holomorphic coordinates near \( p \)

\[
(z_1, z_2, \ldots, z_n, w) = (x_1 + iy_1, x_2 + iy_2, \ldots, x_n + iy_n, u + iv)
\]

so that \( z = w = 0 \) at \( p \), \( M \) has a defining function with Taylor series

\[
r = -v + y^i\Lambda y + x^i\Omega y + \text{cubic and higher order terms in } x, y, \text{ and } u,
\]
and \( E \) osculates \( \mathbb{R}^{n+1} \) to order three at the origin, i.e. \( E \) may be written near the origin as the graph of a smooth function \( \mathbb{R}^{n+1} \to i\mathbb{R}^{n+1} \) whose Taylor series begins with terms of order at least four. Here, \( \Lambda \) is a symmetric \( n \times n \) matrix and \( \Omega \) is a skew \( n \times n \) matrix.

Proof. We first translate \( p \) to the origin, and then by a complex linear transformation, we may assume that

\[
T_0 M = \{ v = 0 \} \quad \text{and} \quad T_0 E = \{ y_1 = y_2 = \cdots = y_n = v = 0 \}.
\]

Then, near the origin,

\[
E = \{ y_1 = f_1(x, u), y_2 = f_2(x, u), \ldots, y_n = f_n(x, u), v = g(x, u) \}
\]

for smooth functions \( f(x, u) \) and \( g(x, u) \) whose Taylor series begin with terms of order at least two. Let \( F(x, u) \) and \( G(x, u) \) denote their third order Taylor polynomials. Then the complex polynomial change of coordinates

\[
z = \hat{z} + iF(\hat{z}, \hat{w}) \quad w = \hat{w} + iG(\hat{z}, \hat{w})
\]
is invertible near the origin and, in these new coordinates, \( E \) osculates \( \mathbb{R}^{n+1} \) to third order, as required. We shall now assume that this is done and drop the hats. The equation for \( M \) takes the form

\[
2v = z^t(\Lambda + i\Omega)z - \Re(z^t\Lambda z) + z^t\Gamma z + w^t\mu + \cdots
\]

where \( L = \Lambda + i\Omega \) is an \( n \times n \) Hermitian matrix (the Levi form), \( \Gamma \) is a real symmetric matrix, \( \mu \) is a real vector, and the ellipsis \( \cdots \) indicates cubic and higher order terms. The change of coordinates

\[
w \mapsto w + \frac{1}{2}(z^t\Gamma z + wz^t\mu)
\]
preserves \( \mathbb{R}^{n+1} \) and gives a new equation

\[
2v = z^t(\Lambda + i\Omega)z - \Re(z^t\Lambda z) + \cdots = 2y^t\Lambda y + 2x^t\Omega y + \cdots,
\]
as required. \( \square \)

Remark 3.2. If the discussion of §1 is applied to this normal form then, at the origin,

\[
H = \operatorname{span}\{ \partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n} \} \quad \text{and} \quad N = \operatorname{span}\{ \partial_{y_1}, \ldots, \partial_{y_n} \}.
\]

Upon identifying \( H \) with \( \mathbb{C}^n \) and \( N \) with \( \mathbb{R}^n \), the Levi form of \( r \) on \( H \) may be identified with the Hermitian matrix \( L = \Lambda + i\Omega \), and its restriction to \( N \) with the symmetric matrix \( \Lambda \).

Remark 3.3. Save for the initial complex linear change of coordinates, the change of coordinates constructed in the proof is completely determined. It follows easily that locally near a given point of \( E \), the new
coordinates, the new defining function for $M$, and the graphing function for $E$ can be chosen to depend smoothly on $p \in E$. Moreover, given $\tau \in J(T_p E)$ transverse to $M$, by suitably normalizing the initial linear change of coordinates one can guarantee that $\tau = \partial_v$ in the normal coordinates, with smooth dependence on $\tau$ as well.

4. Proof of Theorem 1.1

Let $W$ be a two-sided wedge in $M$ as in the statement of Theorem 1.1. We will show that $\hat{W}$ contains a neighborhood of $p_0$ in $\mathbb{C}^{n+1}$. Theorem 1.1 follows from this and from an extension of the Baouendi-Treves approximation theorem [2] to the setting of functions defined on a two-sided wedge in $M$ with a maximally real edge. This extension states that given a point on the edge of such a wedge and a direction of the wedge at that point, there is a two-sided subwedge having the given direction at the given point and on which any continuous CR function on the wedge, continuous across its edge, may be uniformly approximated by a sequence of polynomials. Since the subwedge also satisfies the hypotheses of Theorem 1.1, its hull contains a neighborhood of $p_0$, so the sequence of approximating polynomials also converges on this neighborhood and therefore defines the desired extension. The validity of the extended Baouendi-Treves theorem can be seen in either of two ways. Rosay [6] has observed that the proof of the usual Baouendi-Treves theorem applies to holomorphic functions on wedges in $\mathbb{C}^{n+1}$ continuous across the edge, and the same observation holds for CR functions, or, more generally, solutions of the involutive structure on a hypo-analytic manifold, defined on wedges and continuous across a maximally real edge. Alternatively, one may deduce the extended theorem by applying the usual Baouendi-Treves theorem on the blow-up of $M$ along $E$. This blow-up has a natural hypo-analytic structure induced from the embedded CR structure on $M$, and any continuous CR function on a two-sided wedge in $M$ lifts as a continuous solution of the underlying involutive structure on the blow-up on a full open set. (The hypo-analytic structure on the blow-up is discussed in [3].)

Without loss of generality, we may shrink $M$ to suppose that its Levi form is indefinite throughout $M$. Recall then that by the Lewy extension theorem, any CR function defined on $M$ uniquely extends as a holomorphic function to a neighborhood thereof in $\mathbb{C}^{n+1}$. Also, the usual proof as in [4] uses an analytic family of discs whose boundaries are contained in a small neighborhood of a point on $M$. From the existence of these discs one obtains an estimate on the polynomial hull of subsets of $M$. Let $B(z, r)$ denote the usual Euclidean ball of radius $r$ centered at $z \in \mathbb{C}^{n+1}$. Then from the Lewy discs it follows that for
any relatively compact open subset \( U \subset \overline{U} \subset M \) there are \( \alpha, r_0 > 0 \) such that if \( 0 < r < r_0 \) and \( z \in U \), then
\[
B(z, \alpha r^2) \subset (B(z, r) \cap M).
\]

As previously noted, we may assume that the axis \( \sigma(p_0) \) of the defining cone at \( p_0 \) for the wedge \( W \) in \( M \) is null. For \( z \in W \), let \( r(z) \) denote the distance from \( z \) to \( E \). By halving the aperture and shortening the extent of the cones used in defining \( W \), we obtain another two-sided wedge \( \tilde{W} \) in \( M \) with the same axes as \( W \) and with the property that there is a constant \( \kappa > 0 \) such that \( M \cap B(z, \kappa r(z)) \subset W^{\pm} \) for all \( z \in \tilde{W}^{\pm} \). Combining this with (4.1) and absorbing \( \kappa \) into \( \alpha \), it follows that for some \( \alpha > 0 \),
\[
\bigcup_{z \in \tilde{W}^{\pm}} B(z, \alpha r(z)^2) \subset \tilde{W}^{\pm}.
\]

(4.2)

If we could replace \( r(z)^2 \) by \( r(z) \) in (4.2), then we would have a wedge in \( \mathbb{C}^{n+1} \) and we could apply the classical edge-of-the-wedge theorem to finish the proof. Roughly speaking, the rest of the proof consists of using Lemma 2.3 to go the extra distance.

For \( p \in E \) near \( p_0 \), we can choose normal coordinates as in Lemma 3.1 and Remark 3.3 depending smoothly on \( p \), and we can certainly arrange that the null axis \( \sigma(p_0) \) of the cone defining \( W^+ \) at \( p_0 \) points in the positive \( y_1 \)-direction at the origin in the normal coordinates for \( p_0 \). In one of these normal coordinate systems, we have
\[
E = \{ y = f(x, u), v = g(x, u) \}
\]
and
\[
M = \{ v = y^t \Lambda y + x^t \Omega y + \phi(x, y, u) \},
\]
where
\[
|f(x, u)|, |g(x, u)| = O(|x|^4 + |u|^4)
\]
and
\[
|\phi(x, y, u)| = O(|x|^3 + |y|^3 + |u|^3).
\]

(4.5)

Since \( E \subset M \), it follows easily that also
\[
|\phi(x, 0, u)| = O(|x|^4 + |u|^4).
\]

(4.7)

For fixed small
\[
t = (t_2, t_3, \ldots, t_n) \in \mathbb{R}^{n-1}
\]
(say \( |t| \leq 1 \) at least), consider the linear embedding \( \Psi_t : \mathbb{C}^2 \hookrightarrow \mathbb{C}^{n+1} \) given by
\[
\Psi_t(\zeta_1, \zeta_2) = \zeta_1(1, t_2, \ldots, t_n, 0) + \zeta_2(0, 0, \ldots, 0, 1).
\]
Write \((\zeta_1, \zeta_2) = (\xi + i\eta_1, \xi + i\eta_2) = \xi + i\eta\) and \(\Pi_t = \Psi_t(\mathbb{C}^2) \subset \mathbb{C}^{n+1}\). We may compute how \(\Pi_t\) intersects \(M\). On \(\Pi_t\),

\[
v - y^t \Lambda y - x^t \Omega y = \eta_2 - Q(t)\eta_2
\]

where

\[
Q(t) = (1, t) \Lambda \left( \frac{1}{t} \right).
\]

Thus from (4.4) it follows that \(\Pi_t \cap M\) is given by the equation

\[
\eta_2 = \chi(t, \xi, \eta_1)
\]

where \(\chi(t, \xi, \eta_1) = Q(t)\eta_1^2 + \phi(\xi_1(1, t), \eta_1(1, t), \xi_2)\). Taylor expanding in \(\eta_1\), we can write

\[
\chi(t, \xi, \eta_1) = Q(t)\eta_1^2 + a(t, \xi) + b(t, \xi)\eta_1 + c(t, \xi)\eta_1^2 + d(t, \xi, \eta_1)\eta_3
\]

for smooth functions \(a, b, c, d\), and by (4.6) and (4.7) we have for some constant \(A > 0\) and for \(k = 0, 1, 2\)

\[
|a(t, \xi)| \leq A|\xi|^4, |b(t, \xi)| \leq A|\xi|^2, |c(t, \xi)| \leq A|\xi|, |\partial_{\eta_1}^k d(t, \xi, \eta_1)| \leq A.
\]

For given \((t, \xi)\), we denote by \((Y(t, \xi), V(t, \xi)) \in \mathbb{R}^n \times \mathbb{R}\) the imaginary part of the unique point of \(E\) whose real part is \(\Psi_t(\xi_1, \xi_2) = (\xi_1(1, t), \xi_2)\). From (4.3) we have \(Y(t, \xi) = f(\xi_1(1, t), \xi_2)\) and \(V(t, \xi) = g(\xi_1(1, t), \xi_2)\). From (4.3) it follows that there is a constant \(B > 0\) so that

\[
|Y(t, \xi)| \leq B|\xi|^4, |V(t, \xi)| \leq B|\xi|^4.
\]

We sometimes write \(Y = (Y_1, Y')\), where \(Y' = (Y_2, \ldots, Y_n)\).

The wedge \(\tilde{W}^+\) in (1.2) has axis \(\sigma(p_0)\) at \(p_0\). As we chose our coordinates to make this the positive \(y_1\)-axis in the normal coordinates centered at \(p_0\), it follows that there is a fixed round cone \(C\) in \(\mathbb{R}^{n+1}\) with axis in the positive \(y_1\)-direction so that for all \(p\) sufficiently close to \(p_0\), \(M \cap (E + iC) \subset \tilde{W}^+\) sufficiently near the origin in the normal coordinates at \(p\).

**Lemma 4.1.** There are constants \(\epsilon, \gamma, K > 0\) so that if \(|t| \leq \gamma\) and

\[
K|\xi|^4 < \eta_1 - Y_1(t, \xi) \leq \epsilon, \\
\eta_2 = \chi(t, \xi, \eta_1),
\]

then \(\Psi_t(\zeta_1, \zeta_2) \in \tilde{W}^+\).
Proof. First observe that (4.12) and (4.11) imply that
\begin{equation}
|\xi|^4 \leq \epsilon / K \quad \text{and} \quad |\eta_1| \leq \epsilon (1 + B / K),
\end{equation}
so it follows easily upon choosing \(\epsilon\) small enough that \((\zeta_1, \zeta_2)\) lies in a small neighborhood of the origin. The second line of (4.12) implies that \(\Psi_t(\zeta_1, \zeta_2) \in M\), so it suffices to show that \(\Psi_t(\zeta_1, \zeta_2) \in E + iC\). If \(\delta\) denotes the aperture of \(C\), this follows from the inequalities
\[ |\eta t - Y'(t, \eta)| < \frac{\delta}{2} (\eta_1 - Y_1(t, \xi)) \]
and
\[ |\eta_2 - V(t, \xi)| < \frac{\delta}{2} (\eta_1 - Y_1(t, \xi)) \].
Now
\[ |\eta t - Y'(t, \xi)| \leq |(\eta_1 - Y_1(t, \xi)) t| + |Y_1(t, \xi)| t| + |Y'(t, \xi)| \]
\[ \leq \gamma (\eta_1 - Y_1(t, \xi)) + 2B |\xi|^4 \]
\[ < (\gamma + 2B / K) (\eta_1 - Y_1(t, \xi)), \]
so the first inequality is satisfied if \(\gamma \leq \delta / 4\) and \(K \geq 8B / \delta\).

For the second inequality, we will need to estimate
\[ \chi(t, \xi, \eta_1) = a(t, \xi) + (b(t, \xi) + (Q(t) + c(t, \xi)) \eta_1 + d(t, \xi, \eta_1) \eta_1^2) \eta_1, \]
Observe first from (4.10) and (4.13) that
\[ |b(t, \xi) + (Q(t) + c(t, \xi)) \eta_1 + d(t, \xi, \eta_1) \eta_1^2| \leq \tilde{A} \sqrt{\epsilon}, \]
where \(\tilde{A}\) is a constant depending only on \(A, B\) and a bound for the matrix \(\Lambda\). Therefore we obtain
\[ |\eta_2 - V(t, \xi)| \leq |\chi(t, \xi, \eta_1)| + |V(t, \xi)| \]
\[ \leq |a(t, \xi)| + \tilde{A} \sqrt{\epsilon} |\eta_1| + B |\xi|^4 \]
\[ \leq (A + B) |\xi|^4 + \tilde{A} \sqrt{\epsilon} (\eta_1 - Y_1(t, \xi)) + \tilde{A} \sqrt{\epsilon} |Y_1(t, \xi)| \]
\[ \leq (A + B + \tilde{A} \sqrt{\epsilon}) |\xi|^4 + \tilde{A} \sqrt{\epsilon} (\eta_1 - Y_1(t, \xi)) \]
\[ < [\tilde{A} \sqrt{\epsilon} + (A + B + \tilde{A} B \sqrt{\epsilon}) / K] (\eta_1 - Y_1(t, \xi)). \]
If we choose \(\epsilon\) so that \(\tilde{A} \sqrt{\epsilon} \leq \delta / 4\) and \(K\) so that
\[ K \geq 4 (A + B + \tilde{A} B \sqrt{\epsilon}) / \delta, \]
then the second inequality holds as well. \(\square\)

We now insist that \(|t| \leq \gamma\) and we apply (4.12) to the points \(z = \Psi_t(\zeta_1, \zeta_2) \in \tilde{W}^+\) of Lemma 4.1. Observe that for such \(z\), \(r(z)\) is comparable to \(\eta_1 - Y_1(t, \xi)\). We deduce that there is \(\alpha > 0\) so that if we define
\[ S_t = \{ (\zeta_1, \zeta_2) : K |\xi|^4 < \eta_1 - Y_1(t, \xi) \leq \epsilon \quad \text{and} \]
\[ |\eta_2 - \chi(t, \xi, \eta_1)| \leq \alpha (\eta_1 - Y_1(t, \xi))^2 \} , \]
then $\Psi_t(S_t) \subset \mathcal{W}^\pm$. It follows that $\Psi_t(S_t) \subset \mathcal{W}^\pm$, where $S_t$ denotes the hull in $\mathbb{C}^2$. We intend to apply Lemma 2.3 to $S_t$. To this end, define

$$q_1(t, \xi) = Y_1(t, \xi) + K|\xi|^4,$$

$$q_2(t, \xi) = \chi(t, \xi, q_1(t, \xi)),$$

$$m(t, \xi) = (\partial_\eta \chi)(t, \xi, q_1(t, \xi)).$$

**Lemma 4.2.** There are positive constants $\beta, \ell, r, \hat{\alpha}$ so that if $|Q(t)| \leq \hat{\alpha}$, then

$$S_t \supset \left\{ (\zeta_1, \zeta_2) : |\varsigma| < r \quad \text{and} \quad (\eta_1, \eta_2) \in Sp(\beta, \ell, q_1(t, \xi), q_2(t, \xi), m(t, \xi)) \right\}.$$

**Proof.** If $|\varsigma| < r$ and $(\eta_1, \eta_2) \in Sp(\beta, \ell, q_1(t, \xi), q_2(t, \xi), m(t, \xi))$, then $0 < \eta_1 - q_1(t, \xi) < \ell$, which implies that

$$K|\varsigma|^4 < \eta_1 - Y_1(t, \xi) < \ell + K|\varsigma|^4 \leq \ell + Kr^4.$$

Therefore we can guarantee the first condition defining $S_t$ by choosing $\ell \leq \epsilon/2$ and $r$ so small that $Kr^4 \leq \epsilon/2$. For the second condition, we have

$$|\eta_2 - \chi(t, \xi, \eta_1)| \leq |\eta_2 - q_2(t, \xi) - m(t, \xi)(\eta_1 - q_1(t, \xi))|$$

$$+ |\chi(t, \xi, \eta_1) - [\chi(t, \xi, q_1(t, \xi)) + (\partial_\eta \chi)(t, \xi, q_1(t, \xi))(\eta_1 - q_1(t, \xi))]|.$$

By definition of $Sp(\beta, \ell, q_1, q_2, m)$, the first term is at most $\beta(\eta_1 - q_1(t, \xi))^2$. The second term is equal to

$$\left| \int_{\partial \chi(t, \xi)} \partial_\lambda^2 \chi(t, \xi, \lambda)(\eta_1 - \lambda)d\lambda \right|.$$

Now (4.9) and (4.11) give

$$|\partial_\lambda^2 \chi(t, \xi, \lambda)| = |2(Q(t) + c(t, \xi)) + \partial_\lambda^2(d(t, \xi, \lambda)\lambda^3)|$$

$$\leq 2|Q(t)| + 2A|\varsigma| + A'|\lambda|$$

for some constant $A'$, so the second term is at most

$$(|Q(t)| + A|\varsigma| + A'(|\eta_1| + |q_1(t, \xi)|))(\eta_1 - q_1(t, \xi))^2.$$

Collecting the terms, we deduce that

$$|\eta_2 - \chi(t, \xi, \eta_1)| \leq |\beta + |Q(t)| + A|\varsigma| + A'||\eta_1| + A'(B + K)|\varsigma|^4(\eta_1 - q_1(t, \xi))^2.$$

Since $|\varsigma|$ and $|\eta_1|$ can be made small by choosing $\ell$ and $r$ small, the result follows. \qed
Now $Q(0) = 0$ in the normal coordinates for $p_0$ since the $y_1$-axis is null in that case. By continuity it follows that for all $p$ sufficiently close to $p_0$ and sufficiently small $t$ we have $|Q(t)| \leq \hat{\alpha}$. From (4.11) we obtain $|q_1(t, \xi)| = O(|\xi|^4)$. Then (4.9) and (4.10) show that $|q_2(t, \xi)| = O(|\xi|^4)$ and $|m(t, \xi)| = O(|\xi|^2)$. Therefore we can apply Lemma 2.3 to deduce that there is a $\delta > 0$ so that $\Psi_t(\mathbb{T}_\delta) \subset \hat{W}^+$. As $t$ varies, the $\Psi_t(\mathbb{T}_\delta)$ sweep out a cone in $i\mathbb{R}^{n+1}$ with axis in the $y_1$-direction. As $p$ varies, these cones in the original variables sweep out a one-sided wedge in $\mathbb{C}^{n+1}$. The same argument applies to $W^-$, giving a two-sided wedge in $\mathbb{C}^{n+1}$ contained in $\hat{W}$. Finally, Rosay [6] has shown that the hull of such a two-sided wedge in $\mathbb{C}^{n+1}$ contains a neighborhood of a point on the edge, concluding the proof of Theorem 1.1.

**Remark 4.3.** Up to the last step, the construction of the analytic discs in the above argument proceeds separately on the two sides of the wedge. Therefore the argument establishes the following result about extension from a one-sided wedge. Let $E \subset M$ be a maximal real submanifold of a hypersurface $M \subset \mathbb{C}^{n+1}$, suppose that the Levi form of $M$ at $p_0 \in E$ is indefinite, and let $W$ be a one-sided wedge in $M$ with edge $E$ having a null direction $\sigma \in N_{p_0}$. Then there is a one-sided wedge $W'$ in $\mathbb{C}^{n+1}$ near $p_0$ with edge $E$ and axis $\sigma$ at $p_0$ with the property that every CR function on $W$, continuous in $W \cup E$, extends to a holomorphic function in $W'$, continuous in $W' \cup E$. From this result one can derive versions of the edge-of-the-wedge theorem in $M$ from those in $\mathbb{C}^{n+1}$ for extension from unions of one-sided wedges which are not opposite.

5. **Proof of Theorem 1.4**

Let $W$ be a two-sided wedge in $M$ with edge $E$ as in the statement of Theorem 1.4. We will show that $\hat{W} \cup E$ contains an open set $U \subset \{r < 0\}$ with properties (a) and (b). The result then follows upon application of the Baouendi-Treves approximation theorem as in the proof of Theorem 1.1.

For $p$ near $p_0$ and $\tau \in J(T_pE)$ such that $dr(\tau) < 0$, we may choose normal coordinates as in Lemma 3.1 and Remark 3.3. The defining function in the statement of Theorem 1.1 may be replaced by that in Lemma 3.1 since they are positive smooth multiples of one another. In the normal coordinates, $\sigma$ is represented by a vector in the $y$-plane satisfying $\sigma^t \Lambda \sigma > 0$. By scaling we may suppose that $\sigma^t \Lambda \sigma = 2$. We may write $E$ and $M$ in the forms (1.3), (1.4) such that (4.3) and (4.6) hold.
We now make the further coordinate change
\[ \hat{z} = z, \quad \hat{w} = w + \frac{1}{2} i \hat{z} \hat{\Lambda} \hat{z}. \]
To first order at the origin this is the identity but \( M \) is now defined by the vanishing of
\( \hat{r}(\hat{x}, \hat{y}, \hat{u}) = -\hat{v} + \frac{1}{2} \hat{z}'(\Lambda + i \Omega) \hat{z} + \hat{\phi}(\hat{x}, \hat{y}, \hat{u}) \)
and \( E \) by equations of the form
\( \hat{y} = \hat{f}(\hat{x}, \hat{u}), \quad \hat{v} = \frac{1}{2} \hat{x}' \hat{\Lambda} \hat{x} + \hat{g}(\hat{x}, \hat{u}), \)
where
\[ |\hat{\phi}(\hat{x}, \hat{y}, \hat{u})| = O(|\hat{x}|^3 + |\hat{y}|^3 + |\hat{u}|^3) \text{ and } |\hat{f}|, |\hat{g}| = O(|\hat{x}|^4 + |\hat{u}|^4). \]
The form (5.1) is that usually adopted in the proof of Lewy extension [4]. The usual Lewy discs are obtained for small \( \delta > 0 \) by intersecting \{ \hat{r} < 0 \} with the image of the holomorphic embedding \( \mathbb{C} \hookrightarrow \mathbb{C}^{n+1} \) given by
\[ (5.2) \quad \xi + i \eta = \zeta \mapsto (\hat{z}, \hat{w}) = (\zeta \sigma, i \delta^2). \]
This intersection corresponds to the subset \( \Delta_\delta \subset \mathbb{C} \) given by
\[ |\zeta|^2 + \hat{\phi}(\zeta \sigma, \eta \sigma, 0) < \delta^2, \]
whose connected component containing the origin is a star-shaped domain with smooth boundary.
Consider first the case \( \hat{f} = \hat{g} = 0 \), which can be arranged if \( E \) is real-analytic. Then the images of \( \zeta = \pm \delta \) lie on \( E \). Therefore \( \zeta = \pm \delta \in \partial \Delta_\delta \). Since the \( \hat{y} \)-components of the images of all other points in \( \partial \Delta_\delta \) are multiples of \( \sigma \), it follows that these boundary points lie in \( W \) for \( \delta \) sufficiently small. Note that the images as \( \delta \) varies of the two pure imaginary points of \( \partial \Delta_\delta \) form curves in \( W^\pm \) which are tangent to \( (\pm i \sigma, 0) \) at the origin. Since the boundary of the disc is completely contained in \( W \cup E \), the disc itself is contained in \( \hat{W} \cup \hat{E} \). The image of \( \zeta = 0 \) is the point \( (0, i \delta^2) \), so upon varying \( \delta \) we obtain a line segment in the \( v \)-direction contained in \( \hat{W} \cup \hat{E} \). In the original variables this corresponds to a curve emanating from \( p \) in the direction \( \tau \). Now let \( \tau, \sigma \), and \( \sigma \) vary. From the above properties it follows easily that the union of the resulting discs contains an open set \( U \) with the required properties.

The above discs must be modified in case \( \hat{f} \) or \( \hat{g} \) do not vanish, since in that case they might miss \( E \). In general, the points
\[ (\pm \delta \sigma + i \hat{f}(\pm \delta \sigma, 0), i(\delta^2 + \hat{g}(\pm \delta \sigma, 0)) \)
lie on $E$. Modify the embedding \((5.2)\) by sending $\zeta = \pm \delta$ to these two points and uniquely extend to be of the form $\zeta \mapsto \zeta A + B$. Then it can be shown that for sufficiently small $\delta$ the intersection of $\{\hat{r} < 0\}$ with the image of this embedding is an analytic disc with boundary in $W \cup E$ crossing between $W^+$ and $W^-$ at just these two points on $E$. The image of $\zeta = 0$ no longer lies on the $v$-axis, but upon varying $\delta$ we still obtain a curve tangent to it at the origin, so upon varying $\tau$, $p$, and $\sigma$ we still obtain a set $U$ as before.

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