A BETTER UPPER BOUND ON THE NUMBER OF TRIANGULATIONS OF A PLANAR POINT SET

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Abstract. We show that a point set of cardinality \( n \) in the plane cannot be the vertex set of more than \( 59^n O(n^{-6}) \) straight-edge triangulations of its convex hull. This improves the previous upper bound of \( 276.75^n + O(\log(n)) \).

Introduction

A triangulation of a finite point set \( A \subset \mathbb{R}^2 \) in the Euclidean plane is a geometric simplicial complex covering \( \text{conv}(A) \) whose vertex set is precisely \( A \). Equivalently, it is a maximal non-crossing straight-edge graph with vertex set \( A \). In this paper we prove that a point set of cardinality \( n \) cannot have more than \( 59^n O(n^{-6}) \) triangulations.

An upper bound of type \( 2^{O(n)} \) for this number is a consequence of the general results of \[1, 2\]. Upper bounds of \( 173000^n, 7.18752^n \) and \( 276.75^n + O(\log(n)) \) have been given, respectively, in \[10, 9\] and \[5\]. The precise statement of our new upper bound is:

**Theorem 1.** The number of triangulations of a planar point set is bounded above by

\[
\frac{59^v \cdot 7^b}{\binom{v+b+6}{6}}
\]

where \( v \) and \( b \) denote the numbers of interior and boundary points, meaning by this points lying in the interior and the boundary of the convex hull, respectively.

In Section 1 we prove this result and in Section 2 we briefly review what is known about the maximum and minimum number of triangulations of point sets of fixed cardinality. In particular, we mention that every point set in general position has at least \( \Omega(2.012^n) \), as proved in \[2\]. As a reference, compare these upper and lower bounds to the number of triangulations of \( n \) points in convex position, which is the Catalan number \( C_{n-2} = \frac{1}{n-1} \binom{2n-4}{n-2} = \Theta(4^n n^{-\frac{3}{2}}) \).

1. Proof of the upper bound

We will assume that our point set is in general position, i.e., that no three of its points are collinear. This is no loss of generality because if \( A \) is not in general position and we perturb it to a point set \( A' \) in general position without making boundary points go to the interior, then every triangulation of \( A \) is a triangulation of \( A' \) as well. In particular, the maximum number of triangulations of planar point
sets of given cardinality is achieved in general position. The same may not be true in higher dimension.

For the proof of Theorem 1 we will need the fact that the total number of vertices in a triangulation is bounded by a linear combination of the number of low-degree vertices:

**Lemma 2.** Let $T$ be a triangulation in the plane. For each integer $i \geq 3$, let $v_i$ denote the number of interior vertices of degree $i$ in $T$. For each integer $j \geq 2$, let $b_j$ denote the number of boundary vertices of degree $j$ in $T$. Then

$$\sum (6-i)v_i + \sum (4-j)b_j = 6,$$

and therefore

$$|A| + 6 \leq 4v_3 + 3v_4 + 2v_5 + v_6 + 3b_2 + 2b_3 + b_4.$$

**Proof.** Let $v$ and $b$ be the numbers of interior and boundary vertices in $T$, respectively. Let $e$ and $t$ be the numbers of edges and triangles in $T$. Counting the edges of $T$ according to their incidences to triangles shows that $3t = 2e - b$. Euler’s formula says that $t - e + b + v = 1$. These two equations give:

$$6v + 4b = 6 + 2e.$$

On the other hand, counting the edges of $T$ according to their incidences to vertices shows that

$$2e = \sum iv_i + \sum jb_j.$$

Substituting this into the previous equality and noting that $v = \sum_{i \geq 3} v_i$ and $b = \sum_{j \geq 2} b_j$ gives the first claimed equation. Adding $|A| = \sum_{i} i v_i + \sum_{j} j b_j$ on both sides of this equality and dropping the negative summands on the left hand side yields the claimed inequality.

Let $T$ be a triangulation of $A$ and $p \in A$ be one of its points. We say that a triangulation $T'$ of $A \setminus \{p\}$ is obtained by deleting $p$ from $T$ if all the edges of $T$ not incident to $p$ appear in $T'$. In the same situation we also say that $T$ is obtained by inserting $p$ into $T'$. Observe that neither the deletion nor the insertion of a point into a triangulation is a unique process: more than one triangulation of $A'$ can result from the deletion of $p$ from $T$ and more than one triangulation of $A'$ can result from the insertion of $p$ into $T'$. However, these numbers can be bounded in terms of Catalan numbers.

**Lemma 3.** Let $T$ be a triangulation of $A$ and let $p$ be a vertex in $A$ with degree $i$. The number of triangulations of $A'$ that can be obtained from $T$ by deleting $p$ is at least 1 and at most $C_i - 2$. In particular $h_3 = 1$, $h_4 \leq 3$, $h_5 \leq 9$ and $h_6 \leq 28$.

**Proof.** It suffices to note that the number in question is the number of ways in which the area formed by the intersection of $conv(A')$ and the triangles in $T$ incident to $p$ can be triangulated.

Let $T'$ be a triangulation of $A \setminus \{p\}$. For each $i \in \mathbb{N}$, let $h_i$ be the number of triangulations of $A$ in which $p$ has degree $i$ and which can be obtained by inserting $p$ into $T'$.

- **If $p$ is an interior point of $A$,** then $h_i \leq C_{i-1} - C_{i-2} = \frac{3}{2i-3} \binom{2i-3}{i-3}$. In particular $h_3 = 1$, $h_4 \leq 3$, $h_5 \leq 9$ and $h_6 \leq 28$. 


• If $p$ is a boundary point of $A$, then $h_i \leq C_{i-2} = \frac{1}{i-1} \binom{2i-4}{i-2}$. In particular $h_2 \leq 1$, $h_3 \leq 1$ and $h_4 \leq 2$.

Proof. Let us first assume that $p$ is interior. Let $\Delta$ be the triangle of $T'$ that contains $p$. After inserting $p$ with degree $i$ in $T''$, the union of the triangles incident to $p$ is a starshaped polygon $Q$, with $p$ in its kernel, with no other point of $A$ in its interior, and obtained as the union of $i-2$ triangles from $T'$. Conversely, any polygon with those properties provides a way of inserting $p$ with degree $i$.

Any triangle $t$ of $T'$ in such a polygon will be visible from $p$, meaning that $\text{conv}(t \cup \{p\})$ contains no vertex of $T'$ in its interior. Let $G'$ be the dual graph of $T'$, whose nodes are the triangles in $T'$ with two of them adjacent if the triangles share a common edge. Let $W$ be the subgraph in $G'$ induced by the triangles visible from $p$. This subgraph cannot contain a cycle (by non-degeneracy $p$ is not collinear with any two vertices of $T'$) and thus it is a forest. Let $V$ be the tree in $W$ that contains $\Delta$. The number $h_i$ coincides with the number of subtrees of $V$ with $i-2$ nodes that include $\Delta$.

We can view $V$ as a planted tree with root $\Delta$, which has degree at most 3, and all of whose subtrees are binary trees. The number of $(i-2)$-node subtrees of $V$ that contain the root is upperbounded by the number of such subtrees of the infinite tree $Z$ whose root has exactly three children each of which is root of an infinite binary tree. In turn, this number equals the number of binary trees with $(i-1)$ nodes whose right spine is not empty, because $Z$ can be also described as the tree obtained contracting the first edge of the right spine in the infinite binary tree. The number of those trees is clearly $C_{i-1} - C_{i-2}$.

Suppose now that $p$ is a boundary point. Let $e_1, \ldots, e_k$ be the edges for which $p$ is beyond, in order along the boundary of $T''$, and let $t_i = \text{conv}(e_i \cup \{p\})$. Enlarge the triangulation $T'$ by the triangles $t_1, \ldots, t_k$ to a triangulation $T''$ and proceed as in the first case but with $t_1$ playing the role of $\Delta$. The desired upper bound then turns out to be given by the number of $(i-1)$-node binary trees with exactly $k$ nodes on the right spine.

Proof of Theorem 1 Let $N(v, b)$ denote, for every pair of integers $v \geq 0$ and $b \geq 3$, the maximum number of triangulations among all point sets with $v + b$ points and with at most $v$ of them interior. We will prove by induction on $v + b$ that $N(v, b) \leq 59^v \cdot 7^b / \binom{v+b+6}{6}$. Induction starts with $b = 3$ and $v = 0$, which gives $59^0 \cdot 7^3 / \binom{3+3+6}{6} = 49/12 \geq 1 = N(3, 0)$.

Let $A$ be a point set with $b$ boundary points and $v$ interior points. For each $i \geq 3$, let $V_i$ denote the sum over all triangulations of $A$ of the numbers of interior vertices of degree $i$. For each $j \geq 2$, let $B_j$ denote the sum over all triangulations of $A$ of the numbers of boundary vertices of degree $j$. Let $N$ be the number of triangulations of $A$. Observe that deleting an interior point from $A$ gives a point configuration with $b$ boundary points and $v - 1$ interior points, while deleting a boundary point gives a point configuration with $v + b - 1$ points, at most $v$ of which are interior.

The number of triangulations of $T$ in which a certain vertex $p$ has degree $i$ is at most equal to the number of ways of inserting $p$ with degree $i$ in triangulations of $A \setminus \{p\}$. The inequality may by strict since insertions from different triangulations of $A \setminus \{p\}$ can lead to the same triangulation of $A$. Then, Lemma 4 implies that

\[ V_3 \leq vN(v-1, b), \quad V_4 \leq 3vN(v-1, b), \]
Theorem 1: the interior points as vertices. We can bound the number of triangulations in this
\( N \) vertices. This gives the following upper bound, where setting by adding the bounds of Theorem 1 for the different subset s of interior
\( T \)

The following table, taken from [1], gives
\( (1) \)

is necessary to assume general position since
\( T \)

Among all point sets in the plane in general position and of cardinality
\( n \)

Let \( T \) to the number of triangulations of the convex
\( n \)

Concerning the asymptotic behaviour of \( t(n) \) and \( T(n) \) we know that:
\( \Omega(2.0129^n) \leq t(n) \leq O(12^{n/2}) = O(3.46410^n) \),
\( \Omega(8^n n^{-7/2}) \leq T(n) \leq O(59^n n^{-6}) \)

Compare this with \( C_{n-2} = \Theta(4^n n^{-\frac{4}{3}}) \) for the convex \( n \)-gon. The lower bound for \( t(n) \) comes from [2]. The upper bound for \( T(n) \) is our Theorem [4]. The other two bounds come from the computation of the number of triangulations of the following point sets:

- **A double chain:** Let \( A \) consist of two convex chains of \( k = n/2 \) points each, facing one another and so that every pair of segments in different chains are visible from one another. See the center picture in Figure [1] for the case \( k = 9 \). The edges drawn in the figure are “unavoidable”, i.e., present in every triangulation. They divide \( A \) into two convex \( k \)-gons, with \( C_{k-2} \)
triangulations each, and a non-convex 2k-gon which is easily seen to have \( \binom{2k-2}{k-1} \) triangulations (see \( \text{(1)} \)). Hence, the number of triangulations of \( A \) is:

\[
\left( \frac{2k - 2}{k - 1} \right) C_{k - 2}^2 = \Theta(64^n k^{-\frac{3}{2}}) = \Theta(8^n n^{-\frac{3}{2}}).
\]

**A double circle:** Let \( A \) be a convex \( k \)-gon \((k = n/2)\) together with an interior point sufficiently close to each boundary edge. See the left part of Figure 1 for the case \( k = 9 \). Again, the edges drawn are unavoidable, and triangulating \( A \) is the same as triangulating the central non-convex \( n \)-gon. An inclusion-exclusion argument (see Proposition 1 in \( \text{(2)} \)) gives the exact number of triangulations of this polygon, which is

\[
\sum_{i=0}^{k} (-1)^i \binom{k}{i} 2^{k - i - 2} \leq 12^k = 12^\frac{n}{2}.
\]

It is interesting to observe that the double circle actually gives the minimum possible number of triangulations for \( n \leq 10 \). See \( \text{(2)} \), where this is conjectured to be true for all \( n \). (If \( n \) is odd, the double circle has to be modified with an extra interior point.)

On the other hand, the double chain has only 6, 80 and 1750 triangulations for \( n = 6, 8 \) and 10 respectively, which is less than \( T(n) \). There is a simple way to modify it and get more triangulations, as shown in the right picture of Figure 1. The big non-convex polygon is a double chain with \( n - 2 \) vertices. This modified double chain has exactly

\[
\left( \frac{2k - 4}{k - 2} \right) C_{k - 1}^2 = \frac{(2k - 3)(2k - 2)}{k^2} \left( \frac{2k - 2}{k - 1} \right) C_{k - 2}^2
\]

triangulations, where \( n = 2k \) as in the previous examples. That number is (asymptotically) four times the number of triangulations of the double chain. Still, the modified double chain has 8, 150 and 3920 triangulations for \( n = 6, 8, 10 \), which is not (always) the maximum. The numbers of triangulations of these configurations for \( n = 18 \) appear in Figure 1. The greatest number of triangulations for \( n = 18 \) known so far is 17 309 628 327 \( \text{(3)} \).

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