On the Structure of Weakly Acyclic Games

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Consider the best-response (or better-response) dynamics of a normal-form game:

**Pure NE = no outbound edges**
Convergence to pure Nash

- *Existence* of pure Nash is good, but will the game converge to one?
- *Some* kinds of games always converge to pure Nash:
  - congestion/potential games [Rosenthal’73; Monderer&Shapley ’96]
  - ordinal potential games (fully general for better-response)
  - dominance-solvable games [Moulin ’79]
- But what of games that don’t always converge?
Convergence to pure Nash

Is this divergence interesting?

|       | Play          | Stop          |
|-------|--------------|---------------|
|       | H            | T             |
| H     | -1,1,0       | 1,-1,0        |
| T     | 1,-1,0       | -1,1,0        |
| H     | 3,3,-1       | 3,3,-1        |
| T     | 3,3,-1       | 3,3,3         |

Random player ordering ⇒ stochastic convergence a.s. Other natural dynamics, like no-regret, also converge (Young, et al.)
Convergence to pure Nash

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- Weakly acyclic games: every state has a better-response path to a pure Nash (no non-singleton sinks) [Young’93; Milchtaich’96]
Convergence to pure Nash

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```

- Weakly acyclic games: every state has a better-response path to a pure Nash (no non-singleton sinks)
  - Weakly acyclic under best response
Convergence to pure Nash

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| **H**         | **H**         |
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| 1,-1,0        | 3,3,-1        |
| **T**         | **T**         |
| 1,-1,0        | 3,3,3         |
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The convergence map

Has pure NE

Weakly acyclic under Better Response

Weakly acyclic under Best Response

Strongly acyclic under Best Response

Strongly acyclic under Better Response

= Ordinal potential games
Characterizing weak acyclicity

Our contribution: combinatorial sufficient conditions that link subgame equilibria and weak acyclicity

- Subgame: each player gets a subset $S'_i \subseteq S_i$ of her strategies
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Start with *subgame stability*: each subgame has pure NE

- **Not** rare: necessary and not sufficient for ordinal potential
- Originally from networking (BGP routing): subgame stability $\Leftrightarrow$ stability under failures
The general result

2-player game [Yamamori&Takahashi’02]¹:

- Has pure NE
- Weakly acyclic under Better Response
- Weakly acyclic under Best Response
- Subgame stable
  - Strongly acyclic = Ordinal potential games

¹ Aaronson Conjecture: w.h.p., the cute combinatorial lemma in your TCS paper was already proven in the ’60s and published in Hungarian.
The general result

- 2-player game \([\text{Yamamori&Takahashi'02}]^2\):

  Has pure NE

  Weakly acyclic under Better Response

  Weakly acyclic under Best Response

  **Subgame stable**

  Strongly acyclic = Ordinal potential games

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\(^2\text{Our corollary: w.h.p., the cute combinatorial lemma in your AGT paper was already proven by economists, and published in Economese.}\)
The general result

- $n$-player game:
  - Has pure NE
  - Weakly acyclic under Better Response
  - Weakly acyclic under Best Response
  - Strongly acyclic = Ordinal potential games
  - Subgame stable
  - Unique subgame stable

- Unique Subgame Stability: each subgame has a unique pure NE
Two-player Subgame Stability [YT’02]:

- Not weakly acyclic $\Rightarrow$ BR dynamics has a \textit{sink equilibrium} [Goemans, et al.’05] of size $> 1$
Two-player Subgame Stability [YT’02]:

- Take the *span* of this component – subgame that includes all strategies used in the sink
Two-player Subgame Stability [YT’02]:

- This subgame has a pure NE, and the sink has a node in the same column
- The pure NE cannot be in the sink
Two-player Subgame Stability [YT’02]:

But where does the BR by row player go?
Thus, 2-player SS
⇒ Weak Acyclicity under Best Response
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For 2 players, there is a sink state within 1 player’s move from a pure NE

For \( n \) players, within \( \leq n - 1 \) players’ moves
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For $n$ players, within $\leq n - 1$ players’ moves
Idea: fix players’ strategies, one player at a time
$n$ players: not so easy

- For 2 players, there is a sink state within 1 player’s move from a pure NE
- For $n$ players, within $\leq n - 1$ players’ moves
- Idea: fix players’ strategies, one player at a time
- We’ll need *unique* subgame stability
Unique SS $\Rightarrow$ weak acyclicity (sketch)

Similar: take the span of a hypothetical big sink, find its pure NE, follow best response to have one player match his strategy in the NE
Introduction 2-player games $n$-player games More SS classes? Open problems

**Unique SS $\Rightarrow$ weak acyclicity (sketch)**

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Unique SS $\Rightarrow$ weak acyclicity (sketch)

- Remove all of that player’s strategies $\Rightarrow$ smaller subgame, also has a pure Nash
Unique SS $\Rightarrow$ weak acyclicity (sketch)

- Unique SS $\Rightarrow$ it’s the same Nash as before (non-trivial, but easy). Recurse to get closer.
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- Recursion builds up a path built of chunks of **BR paths** from different subgames: cheating? (no; see paper)
A finer subgame stability picture

- Has pure NE
- Subgame stable
  - ⇒ WA for 2 players

- n-player WA under Best Response
- n-player WA games
- Unique subgame stable
  - ⇒ WA for n players

Strongly acyclic = Ordinal potential games
A finer subgame stability picture

- Strict Subgame Stability: each subgame has a pure NE which each player strictly prefers
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- Has pure NE
  - $\Rightarrow$ WA for $2 \times M$ & $2 \times 2 \times 2$

- Subgame stable
  - $\Rightarrow$ WA for 2 players

- Strict subgame stable
  - $\Rightarrow$ WA for $2 \times 2 \times M$ & $2 \times 3 \times M$

- Unique subgame stable
  - $\Rightarrow$ WA for $n$ players

- Strongly acyclic $=$ Ordinal potential games

- The distinctions are tight w.r.t. game size
Open: Is there more structure to this space?

- Maybe there’s an interesting intermediate property between SSS and USS?
  - (Our proof doesn’t quite use full USS...)
- $\text{HasPNE} \supseteq \text{SS} \supseteq \text{SSS} \supseteq \text{USS}$ ...more to this hierarchy?
Why SS/USS/SSS?

*Our* interest started from BGP routing, where both SS and Unique SS have relevant incarnations.
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- First combinatorial sufficient condition for weak acyclicity in general games.
- Significantly lower complexity class:
  - [Mirrokni&Skopalik’09]: Weak acyclicity in several interesting succinct games is PSPACE-Complete.
  - For reasonable succinct games, all our conditions are low-ish in PH ($\Sigma_2^P$ and $\Sigma_3^P$).
Open problems

Are there interesting game classes which obey USS by design, or can be tractably checked for USS?

More broadly-applicable sufficient conditions of weak acyclicity?
Open problems

- Are there interesting game classes which obey USS by design, or can be tractably checked for USS?
- More broadly-applicable sufficient conditions of weak acyclicity?
- Weak acyclicity doesn’t have to be tied to myopic dynamics...
Open problem: the elephant in the room

- Weakly Acyclic games converge stochastically
- Bad *worst-case convergence time*, even in nice, strongly acyclic games. E.g., exponential in network congestion games [F,Papadimitriou,Talwar’04]
Open problem: the elephant in the room

- Weakly Acyclic games converge stochastically
- What about the expected time until convergence, assuming, e.g., u.a.r. player orderings?
- Random walk mixing time for particularly-shaped directed graphs — maybe need more basic tools?
- “Good” news: no worse than exponential, but when is it actually good?
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- “Good” news: no worse than exponential, but when is it actually good?
- Interesting: Without strictness, clean exponentially-bad examples [Ferraioli, over lunch]. But ties are fragile...
Thank you
| Introduction | 2-player games | \(n\)-player games | More SS classes? | Open problems |
|--------------|----------------|---------------------|------------------|--------------|
