On the applicability of the Hasselmann kinetic equation to the Phillips spectrum.

Alexander O. Korotkevich\textsuperscript{1,2} and Vladimir E. Zakharov\textsuperscript{3,4,5,2}

\textsuperscript{1}Department of Mathematics \& Statistics, The University of New Mexico, MSC01 1115, 1 University of New Mexico, Albuquerque, New Mexico, 87131-0001, USA  
\textsuperscript{2}L. D. Landau Institute for Theoretical Physics, 2 Kosygin Str., Moscow, 119334, Russian Federation  
\textsuperscript{3}University of Arizona, Department of Mathematics, Tucson, AZ 85721, USA  
\textsuperscript{4}P. N. Lebedev Physical Institute, 53 Leninskiy prospekt, Moscow, 119334, Russian Federation  
\textsuperscript{5}Laboratory of Nonlinear Wave Processes, Novosibirsk State University, Novosibirsk, Russian Federation

Abstract. We investigate applicability of the Hasselmann kinetic equation to the spectrum of surface gravity waves at different levels of nonlinearity in the system, which is measured as average steepness. It is shown that even in the case of relatively high average steepness, when Phillips spectrum is present in the system, the spectral lines are still very narrow, at least in the region of direct cascade spectrum. It allows us to state that even in the case of Phillips spectrum the kinetic equation can be applied to the description of the ensembles of ocean waves.

1 Introduction

Two really seminal papers were published in the area of Physical Oceanography about half a century ago. We mean the article by Phillips (1958) and the work of Hasselmann (1962). Both of them were concentrated on the same problem: what is going on with ocean waves growing under influence of wind?

O. Phillips suggested that this growth is arrested by wave-breaking, in other words by formation of “white caps” (or “white horses”). Wave breaking is the main mechanism of energy dissipation. In “white caps” mechanical energy of waves transforms to turbulence on small scales, then to heat. This is a strongly nonlinear phenomenon which cannot be studied by perturbative methods. An analytical theory of wave breaking is not developed yet. The Phillips’ assumption on predomination of wave-breaking effects made him possible to offer, on the base of dimensional considerations, the universal energy spectrum of wind driven waves, so called Phillips’ spectrum.

K. Hasselmann developed a completely different theory. He noticed that a typical ensemble of ocean waves hoards a small parameter – average steepness (slope) of the surface $\mu$. One can define it, for instance, as $\mu = \sqrt{\langle |\nabla \eta|^2 \rangle}$, where $\eta$ is the shape of the surface. This definition has the most clear geometrical meaning. Another definition estimates it as average amplitude $A$ of the waves multiplied by wave-vector of the spectral peak $k_p$. This definition is easy to use for experimental observations. Later we shall consider another definition, which involves spectral distribution function. All these definitions give very close values of the average steepness. Typically $\mu \approx 0.1$. This fact allows to assume that the wind-driven sea is a community of weakly interacting waves, which can be described statistically by the use of expansion in powers of average steepness $\mu$. This is a tedious procedure because one have to expand up to order $\mu^4$. However K. Hasselmann coped with this hard work and derived his famous Hasselmann kinetic equation for squared wave amplitudes.

Then it was found that Hasselmann equation has exact stationary solutions (Kolmogorov-Zakharov or KZ spectra) (Zakharov and Filonenko, 1967a,b; Zakharov and Zaslavskii, 1982; Zakharov et al., 1992) decaying at high frequency region slower than Phillips spectrum. A question arises which theory is more correct? The answer is given by experiment. Numerous measurements made in lakes and ocean showed that the real spectrum is a combination of weak-turbulent KZ spectra and the Phillips spectrum. If $\omega_p$ is spectrum peak frequency, in the energy containing range $\omega_p < \omega < 3\omega_p$ KZ spectra are realized, while the Phillips spectrum predominates in the high-frequency range. This fact has the following explanation. Wave turbulence in the wind driven sea is a mixture of weak and strong turbulence. This is a question of phase correlation. Weakly nonlinear interaction provides correlation of waves phases on large scales – hundreds of characteristic wavelengths. At the same time the wave breaking and white capping is the localized phenomenon. As a result looking at the wind driven sea one observes...
formation of short living localized wave breaking events embedded into homogeneous weakly nonlinear background.

Similar situation, coexistence of weak turbulence and localized coherent structures, is typical for wave turbulence. It takes place, for instance, in nonlinear optics [Dyachenko et al. 1992], where optical turbulence coexists with self-focusing or in isotropic plasma where weak Langmuir turbulence coexists with Langmuir collapses [Zakharov et al. 1992]. Similar phenomena were observed recently in numerical simulations for gravity waves [Zakharov et al. 2009]. The question is it still reasonable to use kinetic equation as an adequate model of waves sets) (Zakharov et al., 2009). The question is it still reasonable to use kinetic equation as an adequate model of waves interaction in the presence of coherent structures?

In this paper we give positive answer to this question. To do this we perform massive numerical simulation of primordials of the dynamical equations describing potential flow of Euler equations of ideal fluid with a free surface. In our numerical simulations the usual course of action is to augment corresponding kinetic equations of weak turbulence by introduction of an additional term, describing dissipation of energy in the coherent structures (corresponding to white capping onsets) [Zakharov et al., 2009]. The question is it still reasonable to use kinetic equation as an adequate model of waves interaction in the presence of coherent structures?

2 Basic model

We solve numerically weakly nonlinear Euler equations for dynamics of incompressible deep fluid with free surface in the presence of gravity by the pseudo-spectral code described in Korotkevich et al. (2012). The code was verified in our previous papers (Dyachenko et al., 2003a,b; 2004; Zakharov et al., 2005, 2007; Korotkevich et al., 2008; Korotkevich 2008, 2012). The equations are written for surface elevation \( \eta(x,y,t) \) and hydrodynamic velocity potential on the surface \( \psi(x,y,t) \). Equations are result of weakly nonlinear expansion of the Hamiltonian up to the fourth order terms in steepness [Zakharov et al. 1992].

\[
\begin{align*}
\dot{\psi} &= \frac{1}{2}k\frac{\partial}{\partial r} i k e^{-i k r} d^2 r. \\
\end{align*}
\]

Normal variables are introduced as follows

\[
\eta_k = \frac{1}{2\pi} \int \eta_r e^{-i k r} d^2 r.
\]

This definition gives a value of average steepness which differs from the geometrical definition \( \mu = \sqrt{\langle |\psi|^2 \rangle} \) in our experiments at max by value close to 0.005, which is just several percent of the characteristic average steepness.

For the spatial harmonic with wave vector \( k \) we defined the time Fourier transform as follows

\[ a(k,\omega) = \frac{1}{T} \int_0^T a(k,t) e^{-i \omega t} dt. \]

Here \( T = 2\pi/\omega_k \) – period of the chosen wave. Thereafter \( I(k,\omega) = |a(k,\omega)|^2 \). In equations (1) pumping had the form

\[ P_k = f_k e^{i\delta_k} \int_{t_0}^t F_0 (k_0,\omega_0 \rightarrow k,\omega) \frac{\langle \delta_k \rangle}{k_0^{\mu_1} k^{\mu_2}} \, dt, \]

where \( k_0 = 28, k_2 = 32 \) and \( F_0 = 1.5 \times 10^{-5} \); \( R_k(t) \) was a uniformly distributed random number in the interval \((0,2\pi)\) for each \( k \) and \( t \). The initial condition was a low amplitude noise in all harmonics. Time step was \( \Delta t = 6.7 \times 10^{-4} \). The dissipation function \( \gamma_k = \gamma_k^{(1)} + \gamma_k^{(2)} \).
Artificial viscosity $\gamma_k^{(1)}$ was the same in all experiments

$$\gamma_k^{(1)} = \begin{cases} \gamma_0(k-k_d)^2, & k > k_d, \\ 0, & \text{if } k \leq k_d; \end{cases} \tag{5}$$

where $k_d = 256$ and $\gamma_0 = 0.97 \times 10^2$. Due to presence of this dissipation the most part of our phase space, actually for $k > 256$, was "passive". But this was necessary in order to get rid of aliasing harmonics and to simulate dissipation of energy.

$\gamma_k^{(2)}$ was dissipation concentrated in small wave numbers. It was zero only in one of three experiments.

3 Description of experiments

We performed three series of experiments, choosing different functions $\gamma_k^{(2)}$, describing damping in the area of small wave-numbers.

3.1 Without condensate and inverse cascade

In the first series of experiments we assumed that it was linearly growing in small wave-numbers $k < 28$ and equal to

$$\gamma_k^{(2)} = \begin{cases} 0.2|k-28|, & k \leq 28, \\ 0, & \text{if } k > 28. \end{cases}$$

In this case the inverse cascade of wave action was completely suppressed. We observed formation of the direct cascade, reasonably described by the standard KZ-spectrum $|\tilde{a}_k|^2 \simeq \beta_1 k^{-4}$, with $\beta_1 \simeq 1.2 \times 10^{-4}$. This spectrum was observed in the range of scales $32 < k < 150$. The spectra in the linear and logarithmic scales are presented in Figures 1-2.

The steepness grows in this range approximately as shown in Figure 3. At $k_{max} \simeq 200$ steepness reaches its saturation level $\mu_{max} \simeq 0.104$. The shapes of spectral lines at $k$ equal to 50, 100, 150, 200, and 250 are represented in Figures 4-8.

One can see that the spectral lines are narrow. Only at area of large $k \simeq 250$ we observe bounded (slave) harmonics which do not obey linear dispersion relation.
4 Korotkevich, A. O., Zakharov, V. E.: On the applicability of the kinetic equation...

Fig. 4. Spectral line of the harmonic $k = (0,50)^T$. Both condensate and inverse cascade were suppressed.

Fig. 5. Spectral line of the harmonic $k = (0,100)^T$. Both condensate and inverse cascade were suppressed.

Fig. 6. Spectral line of the harmonic $k = (0,150)^T$. Both condensate and inverse cascade were suppressed.

Fig. 7. Spectral line of the harmonic $k = (0,200)^T$. Both condensate and inverse cascade were suppressed.

Fig. 8. Spectral line of the harmonic $k = (0,250)^T$. Both condensate and inverse cascade were suppressed.
In the second series of experiments the low frequency damping was presented only in small wave numbers $k < 10$, where the dissipation rate was constant $\gamma_k^{(2)} = 0.05$. In this experiment we observed formation of both direct and inverse cascades. The dynamic range for inverse cascade was too short for accurate evaluation of the power of the spectrum, the only statement which can be made is that it is relatively close to the predicted $|a_k|^2 \sim k^{-23/6}$ spectrum, with power exponent $\alpha$ in $k^{-\alpha}$ ranging from 3.1 to 4.0. In the area of direct cascade we observed slightly steeper spectrum than KZ-spectrum slope. In the area $32 < k < 150$ the spectrum can be approximated by $|a_k|^2 \sim k^{-4.2}$. This deviation from pure KZ-slope $k^{-4}$ can be explained by the influence of wave breaking and white capping effects (Korotkevich, 2012). The spectra are presented in Figures 9, 10. The steepness in this case reaches the level $\mu \simeq 0.130$ (see Figure 11). The spectral lines in this series of experiments were still narrow for $k = 50, 100, 150, 200$. In the area of significant damping $k = 250$ we observed intensive formation of slave (bond) harmonics (see Figures 12, 16).
Fig. 13. Spectral line of the harmonic $k = (0,100)^T$. Only condensate was suppressed.

Fig. 14. Spectral line of the harmonic $k = (0,150)^T$. Only condensate was suppressed.

Fig. 15. Spectral line of the harmonic $k = (0,200)^T$. Only condensate was suppressed.

Fig. 16. Spectral line of the harmonic $k = (0,250)^T$. Only condensate was suppressed.
3.3 With both condensate and inverse cascade

In the last series of experiments we completely eliminated low-frequency dissipation \( \gamma_k^{(2)} = 0 \). In this case we observed formation of intensive inverse cascade leading to creation of the condensate at \( k \approx 5 - 4 \). The spectrum in the area of inverse cascade was the same as in the previous series of experiments, while in the area of direct cascade we observed the Phillips spectrum \( \left| a_k \right|^2 \sim k^{-9/2} \) instead of a weak-turbulent KZ-spectrum (Korotkevich, 2008). The spectra are presented in Figures 17, 18. In order to check that spectrum slope is not changing any more, long time calculation was performed on a smaller grid, which showed motion of the condensate position by one wave-number grid step without any significant influence on the inverse cascade region. The average steepness was essentially higher than in the previous case, which looks quite small in absolute value (0.130 and 0.142), is really quite strong, because the dissipation rate depends strongly on the average steepness in this range of values (Zakharov et al., 2009) as well as probability of white capping grows pretty fast (Banner et al., 2000).

It is important to notice that the steepness of the condensate was quite moderate \( \mu \approx 0.06 \) (it is worth to note, that average steepness getting significant contribution from small scales, quite far from the spectral peak, which means that often used definition of the average steepness through the product of mean amplitude and wave-number of spectral peak can deviate significantly from the geometrical definition through the average slope of the surface). However modulation of the short waves by long waves caused intensive micro-breaking of waves which forms the Phillips spectrum (the process described qualitatively in Korotkevich (2008, 2012)). The most interesting and important question is about the shapes of spectral lines in the area of Phillips spectrum. They are presented in Figures 20-24. For this experiment we analysed the longest time series which resulted in higher frequency resolution. After obtaining the frequency spectrum we used moving averaging in order to get rid of noise. One can see that in the most interesting area \( 32 < k < 150 \) the spectral lines are still narrow while essentially “contaminated” by the slave harmonics. The maxima of the spectral peaks are shifted to the high-frequency area according to the theoretically predicted nonlinear frequency shift. In the area of shorter waves \( k > 200 \) the spectrum is a chaotic mixture of leading and slave harmonics.

4 Conclusions

We believe that our experiments make possible a “marriage of Heaven and Hell” in spirit of William Blake. Both outstanding scientists – O. Phillips and K. Hasselmann are right.
If the local steepness $\mu$ is small ($\mu \leq 0.1$) the Hasselmann kinetic equation is valid without any augmentation by any additional dissipation terms. For $\mu$ significantly higher than 0.1, in the area of very short waves the kinetic equation is not applicable in its "pure" form. It is impossible to separate "leading" and "slave" harmonics in this area. This part of the ocean spectrum cannot be described analytically by the use of perturbative methods. Nevertheless, the Phillips spectrum in this area is still applicable. The only theoretical reason for this statement is dimensional considerations supported by experimental data. At the same time we claim that there is a "grey area", which is especially interesting, because it is containing most of energy and generate most of steepness, where micro-breaking have equal foot with weakly nonlinear resonant waves interaction. Spectra in this area could be described by an "augmented" Hasselmann equation, including an additional term describing dissipation energy due to wave-breaking. Similar additional "dissipation terms" $S_{diss}$ are widely used in well developed operational wave forecasting models. But they are introduced "out of the blue" and not supported neither by theoretical consideration nor by experimental observations. Moreover, it is not clear a priori that one can use the Hasselmann kinetic equation in a situation where wave-breaking events are frequent enough. We hope that our experiments showed that this is possible. A necessary and sufficient condition of applicability of the Hasselmann equation is narrowness of the spectral line. In the present article we assert that in the "grey area" with a frequency range in one half of the decade the frequency spectra of harmonics are still narrow lines. It means that the Hasselmann equation is applicable there. Of course it must be augmented by a proper dissipative term. The existing and widely used dissipative terms hardly can be correct. They do not satisfy to the minimal requirement – the Hasselmann...
equation in the presence of this term has to have the Phillips spectrum as an exact solution. Hence, the urgent problem is a construction of a “really good” dissipative term. One possible and hopefully plausible variant was recently offered by Zakharov et al. (2012). The dependence of the dissipation term on the average steepness was recently measured directly from the numerical experiment, the preliminary results can be found in Zakharov et al. (2009). In our next paper we shall analyse what information about possible shape of the dissipative term can be extracted from massive numerical experiments.

Acknowledgements. The authors gratefully wish to acknowledge the following contributions: KAO was supported by the NSF grant 1131791, and during the summer visit by the grant NSH-6885.2010.2. ZVE was partially supported by the NSF grant 1130450 and by the Grant No. 11.G34.31.0035 of the Government of Russian Federation.

Also authors would like to thank developers of FFTW (Frigo and Johnson, 2005) and the whole GNU project (GNU, 1984-2012) for developing, and supporting this useful and free software.

References

Banner, M. L., Babanin, A. V., and Young, I. R.: Breaking probability for dominant waves on the sea surface, J. Phys. Oceanogr., 30, 3145–3160, 2000.

Dias, F., I. Dyachenko, A., and Zakharov, V. E.: Theory of weakly damped free-surface flows: A new formulation based on potential flow solutions, Phys. Lett. A, 372, 1297–1302, 2008.

Dyachenko, A. I., Newell, A. C., Pushkarev, A., and Zakharov, V. E.: Optical turbulence: weak turbulence, condensates and collapsing fragments in the nonlinear Schroedinger equation, Physica D, 57, 96–160, 1992.

Frigo, M. and Johnson, S. G.: The design and implementation of FFTW 3, Proc. IEEE, 93, 216–231, [http://fftw.org], 2005.

FFTW (Frigo and Johnson, 2005) and the whole GNU project (GNU, 1984-2012) for developing, and supporting this useful and free software.

Fig. 24. Spectral line of the harmonic \( k = (0, 250) \). Condensate was present.