A Note on the Relation between Recognisable Series and Regular Sequences, and Their Minimal Linear Representations

Clemens Heuberger, Daniel Krenn, Gabriel F. Lipnik

Abstract

In this note, we precisely elaborate the connection between recognisable series (in the sense of Berstel and Reutenauer) and \( q \)-regular sequences (in the sense of Allouche and Shallit) via their linear representations. In particular, we show that the minimisation algorithm for recognisable series can also be used to minimise linear representations of \( q \)-regular sequences.

1 Introduction

1.1 Overview

Every regular sequence can also be seen as a recognisable series—definitions of both notions are recalled below—and both can be described by a linear representation using a collection of square matrices and two vectors. So when the authors of this note implemented both concepts in SageMath \([6]\), this relation and property played fundamental roles. For recognisable series, there exists an algorithm to minimise the dimension of
their linear representations based on methods of Schützenberger [7, 8]; see Berstel and Reutenauer [3, Chapter 2]. So it seemed to be reasonable that this algorithm can also be used for regular sequences.

When implementing the results of [5], the authors of this note suddenly encountered a situation where the minimisation algorithm for recognisable series failed for a regular sequence. At first, we were quite puzzled. It soon turned out that the linear representation we used for our regular sequence did not fulfill a certain eigenvector property that should be fulfilled for regular sequences, but we were quite unsure whether fixing this completely solves the problem. The answer is yes, and the details are the topic of this note.

1.2 Recognisable Series and Regular Sequences

Let \( \mathbb{N}_0 \) denote the set of non-negative integers and \( K \) be an arbitrary field. Moreover, let \( q \geq 2 \) be an integer and set \( \mathcal{A}_q := \{0, \ldots, q - 1\} \).

We first recall the definition of a recognisable series; the book of Berstel and Reutenauer [3, Chapter 2] provides an introduction to these series.

**Definition 1.1.** Let \( \mathcal{A} \) be a finite set. A sequence \( x \in K^{\mathcal{A}^*} \) is said to be a recognisable series if there are a non-negative integer \( D \), a family \( M = (M(a))_{a \in \mathcal{A}} \) of \( D \times D \) matrices over \( K \) and vectors \( u \in K^{1 \times D}, w \in K^{D \times 1} \) such that for all \( b = b_0 \ldots b_{\ell - 1} \in \mathcal{A}^* \), we have

\[
x(b) = uM(b)w
\]

with

\[
M(b) := M(b_0) \cdots M(b_{\ell - 1}). \tag{1}
\]

We call \((u, M, w)\) a linear representation of \( x \) and \( D \) the dimension of the linear representation of \( x \).

Note that we will use the convention (1) throughout this note. Next, we recall the definition of a regular sequence; see Allouche and Shallit [1, 2] for characterisations, properties, and an abundance of examples. Asymptotic properties and further examples have been studied; cf. [4], [5], and the references therein.

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1We did not find any reference for that. If a reader is aware of such a reference, please contact the authors.

2See [https://github.com/sagemath/sage/issues/32921#issuecomment-1418154841](https://github.com/sagemath/sage/issues/32921#issuecomment-1418154841) A smaller example is presented in Example 1.3.

3In other words, we extend the map \( M: \mathcal{A} \to K^{D \times D} \to \) to a monoid homomorphism from \( \mathcal{A}^* \) to \( K^{D \times D} \).

By convention, if \( b \) is the empty word in \( \mathcal{A}^* \), then \( M(b) \) is the \( D \)-dimensional identity matrix. If the dimension \( D \) equals 0, then the product of an empty vector, \( M(b) \) (for any \( b \)) and another empty vector is an empty double sum, so equals 0.

4Strictly speaking, this is an algorithmic characterisation of a regular sequence which is equivalent to the definition given by Allouche and Shallit [1], who first introduced this concept: they define a sequence \( y \) to be \( q \)-regular if the kernel

\[
\{ y \circ (n \mapsto q^j n + r) \mid j, r \in \mathbb{N}_0 \text{ with } 0 \leq r < q^j \}
\]

is contained in a finite dimensional vector space.
**Definition 1.2.** A sequence $y \in K^{N_0}$ is said to be $q$-regular\(^3\) if there are a non-negative integer $D$, a family $M = (M(a))_{a \in A_q}$ of $D \times D$ matrices over $K$, a vector $u \in K^{1 \times D}$ and a vector-valued sequence $v \in (K^{D \times 1})^{N_0}$ such that for all $n \in N_0$, we have

$$y(n) = uv(n),$$

and such that for all $r \in A_q$ and all $n \in N_0$, we have

$$v(qn + r) = M(r)v(n).$$

We call $(u, M, v)$ a linear representation of $x$ and $D$ the dimension of the linear representation of $x$.

By induction using (2), it is easily seen that for all $n \in N_0$, we have

$$y(n) = uM(digits_q(n))w$$

where $digits_q(n) = n_0 \ldots n_{\ell - 1}$ is the standard $q$-ary expansion of $n$, i.e., $n = \sum_{0 \leq j < \ell} n_j q^j$ with $n_{\ell - 1} \neq 0$, and $w := v(0)$. In other words, given a $q$-regular sequence $y$ with linear representation $(u, M, w)$ and considering the recognisable series $x$ with linear representation $(u, M, w)$ over the alphabet $A_q$, we can write $y = x \circ digits_q$.

As mentioned in Section 1.1, this is how the authors of this note implemented regular sequences in SageMath: these were a special case of recognisable series—technically speaking, the class `RegularSequence` is a subclass of the class `RecognizableSeries`—where accessing the values for integer values is translated accordingly and additional properties (such as subsequences) are implemented. To construct a regular sequence in SageMath, the input is a family of square matrices $M$ and two vectors $u$ and $w$.

### 1.3 Minimisation Failing?

Definitions 1.1 and 1.2 are worded in such a way as to allow several different linear representations of the same recognisable series or regular sequence. When working with such objects algorithmically, inevitably, the question of minimality of the dimension of the linear representation arises.

As mentioned in Section 1.1, Berstel and Reutenauer\(^3\) describe an algorithm to determine a linear representation of minimal dimension for a recognisable series, given by some linear representation. The remarks at the end of Section 1.2 imply that this algorithm is also used in SageMath to find a linear representation of minimal dimension of a regular sequence. As also mentioned in Section 1.1, this led to a problem, and here is a simplified example illustrating it.

**Example 1.3.** Let $q = 2$,

$$u = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M(0) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad M(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad w = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

\(^3\)In the standard literature, the basis is frequently denoted by $k$ instead of our $q$ here.
and consider the sequence $y$ defined by (3) for these values of $u$, $M$, and $w$.

We have $y(0) = uw = 0$ and for all positive integers $n$, the standard binary expansion $\text{digits}_2(n)$ ends on a 1, so writing $\text{digits}_2(n) = b1$ for some $b \in \{0, 1\}^*$, we have

$$y(n) = uM(b)M(1)w = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$ 

Hence, we have shown $y(n) = 0$ for all $n \in \mathbb{N}_0$. For the zero sequence, the minimal linear representation is the representation of dimension 0, i.e., the left and right vectors as well as the matrices $M(a)$ are empty and all matrix products, vector-matrix products, and matrix-vector products are empty sums and therefore 0, as required.

We now compare our result here with the one in SageMath. The input

```python
S = RecognizableSeriesSpace(QQ, [0, 1])

u = vector([1, 0])
M_0 = matrix([[1, 1], [0, 0]])
M_1 = matrix([[1, 0], [0, 0]])
w = vector([0, 1])
x = S([M_0, M_1], u, w)
x.minimized().linear_representation()
```

yields the output

```
((1, 0), Finite family {0: [0 1] [0 1], 1: [1 0] [1 0]}, (0, 1))
```

We see that SageMath returns a linear representation of dimension 2 (not identical to the input linear representation) and claims it to be minimal. Note that we used a recognisable series here because the algorithm by Berstel and Reutenauer is formulated for recognisable series.

So we do have a problem: we easily saw that our regular sequence $y$ has a linear representation of dimension 0, but SageMath answered that all linear representations of the underlying recognisable series $x$ have dimension at least 2.

A few possibilities come to mind. Almost unthinkably, there could be an error in the algorithm of Berstel and Reutenauer, or, more probably, in our implementation of that algorithm in SageMath. Despite a clear peer-reviewing policy for contributions into SageMath, we might have overlooked something. We ask SageMath once more to get a partial answer.

**Example 1.4** (Continuation of Example 1.3). The input

```python
x
```

gives the first few terms of the recognisable series as
So it seems that the recognisable series does not vanish. The output suggests that the recognisable series $x$ with the linear representation defined in (4) is one exactly for those input words with trailing zeros. Indeed,

$$x(b_0) = uM(b_0)M(0)w = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} M(b_0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} M(b_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

by induction.

This means that $x(b) = 1$ if 0 is a suffix of $b$ and $x(b) = 0$ otherwise. It is therefore clear that there cannot be a linear representation of $x$ of dimension 0 (because that would lead to all zeros due to empty sums). It is not hard to see that $x$ cannot have a linear representation of dimension 1, either: then the matrices $M(0)$ and $M(1)$ of that linear representation would forcibly commute and $x(b)$ could only depend on the number of occurrences of the letters 0 and 1 in $b$, but not on their position. Thus, independently of the algorithm of Berstel and Reutenauer (and its implementation in SageMath), we conclude that any linear representation of $x$ must have dimension at least 2.

So with Examples 1.3 and 1.4, we did not construct a counterexample to the validity of the minimisation algorithm of Berstel and Reutenauer for recognisable series, however in this particular example, we see that we cannot apply the algorithm to find a minimal representation of the regular sequence. More generally, this means that choosing an arbitrary family of matrices $M$ and left and right vectors $u$ and $w$, respectively, and defining a regular sequence by (3) can lead to situations where the algorithm of Berstel and Reutenauer for recognisable series does not return a minimal linear representations for the regular sequence.

Is that the final word? Or can we somehow find at least some situations where using the minimisation algorithm for recognisable series is valid for the “corresponding” regular sequence?

### 1.4 Why this Note is Needed

The short answer is, we need this note to discuss the questions raised by the example above. So, let us briefly come back to Example 1.3. A key feature was that at first we only inserted words with trailing one (or the empty word) into the recognisable series because standard binary expansions of positive integers have exactly this property. And indeed, as the discussion in the examples shows, inserting any other binary expansion (with trailing zeros) instead of the standard binary expansion would lead to another result. This seems to be an important distinction between recognisable series (all words allowed) and regular sequences (only words without trailing zeros inserted into the corresponding recognisable series).

At first glance, the following observation seems to be a technical detail: If we insert $n = r = 0$ into (2), we obtain $v(0) = M(0)w(0)$. In other words, if not zero, then $v(0)$ is
an eigenvector of $M(0)$ associated with the eigenvalue 1. It seems to be a minor detail because once we replace the formulation (2) by (3), it does not seem to be relevant any more. However, we note that this condition is not fulfilled in Example 1.3 as $M(0)w = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \neq w$. So $(u, M, w)$ as given by (4) is not a linear representation of the 2-regular sequence $y$ considered in Example 1.3 (and we carefully never claimed it to be one).

This raises two questions. Suppose we have a regular sequence $y$, take a linear representation $(u, M, w)$ of that regular sequence (thus implying $M(0)w = w$), take it as a linear representation of a recognisable series $x$, and then run the minimisation algorithm by Berstel and Reutenauer on it. Will this approach yield a minimal linear representation of $y$? And will that linear representation still fulfil the essential eigenvector property?

The answer to both questions is yes. But according to the saying “fool me once, shame on you; fool me twice, shame on me”, we should make sure to have a proof. The nature of SageMath as an open source software system also means that this proof should not be a “well-known fact in the community” or some kind of an urban myth, but something which can be clearly referenced. In particular, such a reference is put in the documentation of SageMath. This note sets out to provide that proof and to clarify the relation between recognisable series and regular sequences, their linear representations, and their minimal linear representations.

1.5 Structure of this Note

In Section 2, we collect information on recognisable series and their minimal linear representations. Finally, in Section 3, we consider regular sequences, their connection to recognisable series, and prove the main result of this paper (Theorem 3.6) as outlined in the last two paragraphs of Section 1.4. We close by Example 3.7 providing another angle on what can go wrong with minimisation.

2 Recognisable Series

Definition 2.1. A linear representation of a recognisable series $x$ is said to be a minimal linear representation of $x$ if its dimension is minimal over all linear representations of $x$.

Berstel and Reutenauer present a characterisation for minimal linear representations [3, Proposition 2.1]. In this note we only need the direction of the following lemma and for reasons of self-containedness, we give an ad-hoc proof here.

Lemma 2.2 (Berstel–Reutenauer [3, Proposition 2.1]). Let $A$ be a finite set, $x \in K^A^*$ be a recognisable series and $(u, M, w)$ be a minimal linear representation of $x$ of dimension $D$. Then span\{\{uM(b) \mid b \in A^*\} = K^{1 \times D}$.

6 In order to construct a linear representation for a regular sequence out of a linear representation of a recognisable series as given by (4), one can follow the proof of [1, Lemma 4.1].

7 In [3], these linear representations are called reduced instead of minimal.
The basic idea of the proof is that if that span had lower dimension, then everything would take place in a proper subspace. Taking matrix representations with respect to a basis of the subspace would then give a lower dimensional linear representation.

We mention that by symmetry, we also have \( \text{span}\{M(b)w \mid b \in A^*\} = K^{D \times 1} \), but we will not need this property here.

**Proof of Lemma 2.2.** Let \( S := \{uM(b) \mid b \in A^*\} \). Toward a contradiction, assume that we have \( \text{span}(S) = W \) for some proper subspace \( W \) of \( K^{1 \times D} \) of dimension \( D' \) and let \( B \) be a basis of \( W \). Let \( \Phi_B : K^{1 \times D'} \to W \) be the coordinate map with respect to the basis \( B \).

As an implication of the definition of \( W \), we have \( u \in W \), and so there exists a \( u' \in K^{1 \times D'} \) such that \( \Phi_B(u') = u \). For all \( a \in A \), the map \( v \mapsto vM(a) \) is an endomorphism of \( W \) by construction of \( W \); for \( b \in A^* \), we have \( uM(b)M(a) = uM(ba) \in S \), which implies that the map under consideration maps \( S \) into itself and therefore maps \( W = \text{span}(S) \) into itself. We now construct a family \( M' = (M'(a))_{a \in A} \) of \( D' \times D' \) matrices over \( K \) as follows. For all \( a \in A \), let \( M'(a) \) be the matrix representation of the endomorphism \( v \mapsto vM(a) \) of \( W \) with respect to the basis \( B \), i.e., \( \Phi_B(v'M'(a)) = \Phi_B(vM(a)) \) holds for all \( v' \in K^{1 \times D'} \). Let \( w' \in K^{D' \times 1} \) be the matrix representation of the homomorphism \( v \mapsto vw \) from \( W \) to \( K \) with respect to the basis \( B \) of \( W \) and the standard basis of \( K \), i.e., \( \Phi_B(v')w = v'w' \) holds for all \( v' \in K^{1 \times D'} \).

For all \( b \in A^* \), this implies that

\[
x(b) = uM(b)w = \Phi_B(u')M(b)w = \Phi_B(u'M'(b))w = u'M'(b)w'.
\]

In other words, \((u', M', w')\) is a linear representation of \( x \), and its dimension is \( D' < D \), a contradiction to \((u, M, w)\) being a minimal linear representation of \( x \).

Let us consider a linear representation \((u, M, w)\) of a recognisable series \( x \in K^{A^*} \), \( A \) a finite set. If there is a \( z \in A \) with \( M(z)w = w \), then it is clear that \( x(bz) = x(b) \) holds for all \( b \in A^* \). It turns out that the converse is true if the linear representation is minimal. This is the assertion of the following proposition.

**Proposition 2.3.** Let \( A \) be a finite set, \( x \in K^{A^*} \) be a recognisable series and \((u, M, w)\) be a minimal linear representation of \( x \). Let \( z \in A \) be such that \( x(bz) = x(b) \) holds for all \( b \in A^* \). Then we have \( M(z)w = w \).

**Proof.** Let \( D \) be the dimension of the linear representation \((u, M, w)\), and set \( S := \{uM(b) \mid b \in A^*\} \). As

\[
uM(b)w = x(b) = x(bz) = uM(bz)w = uM(b)M(z)w
\]

holds for all \( b \in A^* \), the linear maps \( v \mapsto vw \) and \( v \mapsto vM(z)w \) from \( K^{1 \times D} \) to \( K \) coincide on \( S \). As \( S \) generates \( K^{1 \times D} \) by Lemma 2.2, these maps also coincide on \( K^{1 \times D} = \text{span}(S) \). Therefore, their matrix representations \( w \) and \( M(z)w \) coincide.

**Definition 2.4.** Let \( x \in K^{A_q^*} \) be a recognisable series such that \( x(b0) = x(b) \) holds for all \( b \in A_q^* \). Then \( x \) is said to be *compatible with regular sequences* (or simply *compatible*).
Remark 2.5. Let $x$ be a recognisable series with minimal linear representation $(u, M, w)$. Then by Proposition 2.3, $x$ being compatible is equivalent to the condition $M(0)w = w$.

The following example shows, however, that non-minimal linear representations $(u, M, w)$ of a compatible recognisable series do not necessarily satisfy the property $M(0)w = w$.

Example 2.6. Consider the constant recognisable series $x \in \mathbb{C}^{(0,1)^*}$ with $x(b) = 1$ for all $b \in \{0, 1\}^*$. It is clear that $x$ is compatible and a minimal linear representation $(u, M, w)$ is given by

$$ u = (1) \in \mathbb{C}^{1 \times 1}, \quad M(0) = M(1) = (1) \in \mathbb{C}^{1 \times 1} \quad \text{and} \quad w = (1) \in \mathbb{C}^{1 \times 1}. $$

So $M(0)w = w$ holds, as stated in Remark 2.5.

Moreover, $(u', M', w')$ with

$$ u' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad M'(0) = M'(1) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad w' = \begin{pmatrix} 1 \\ 1 \end{pmatrix} $$

is also a linear representation of $x$: for a word $b \in \{0, 1\}^*$ of length $\ell$, we have

$$ u'M'(b)w' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^\ell \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2^\ell \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2^\ell \end{pmatrix} = 1 = x(b); $$

the lower right entry in $M'(b)$ is annihilated by the zero in $u'$. However, $M'(0)w' = w'$ does not hold. This is no contradiction to Remark 2.5 because $(u', M', w')$ is not minimal.

### 3 Regular Sequences

**Definition 3.1.** A linear representation of a regular sequence $y$ is said to be a *minimal* linear representation of $y$ if its dimension is minimal over all linear representations of $y$.

The first statement of the following lemma corresponds to [1, Lemma 4.1] and has already been discussed in [3]; the second statement has been discussed towards the end of Section 1.4. Nevertheless, we restate it here for completeness and refer to the mentioned references for proofs.

**Lemma 3.2.** Let $y \in K^{\mathbb{N}_0}$ be a $q$-regular sequence with linear representation $(u, M, w)$ and let $n \in \mathbb{N}_0$. Then we have

$$ y(n) = uM(\text{digits}_q(n))w. \quad (5) $$

Furthermore, we have $M(0)w = w$.  

8
In the following lemma, for \( b = a_0 \ldots a_{\ell-1} \in A_q^\ast \), we set

\[
\text{value}(b) := \sum_{j=0}^{\ell-1} a_j q^j,
\]

with the usual convention that if \( b \) is the empty word in \( A_q^\ast \), then \( \text{value}(b) = 0 \).

**Lemma 3.3.** Let \( y \in K^{N_0} \) be a \( q \)-regular sequence and \((u, M, w)\) a linear representation of \( y \). Then

\[
y(\text{value}(b)) = uM(b)w
\]

holds for all \( b \in A_q^\ast \). In particular, the value of \( uM(b)w \) is independent of the particular choice of the linear representation \((u, M, w)\) of \( y \) and of trailing zeros of \( b \).

**Proof.** Let \( b \in A_q^\ast \) and write \( b = c0^\ell \) for some \( c \in A_q^\ast \) and some \( \ell \in N_0 \) such that \( \ell \) is maximal. Then \( \text{value}(b) = \text{value}(c) \), and by (3), we have

\[
y(\text{value}(c)) = uM(c)w.
\]

As \( M(0)w = w \) by Lemma 3.2 we also have

\[
uM(b)w = uM(c)M(0)^\ell w = uM(c)w,
\]

as required. \( \square \)

**Definition 3.4.** Let \( y \in K^{N_0} \) be a \( q \)-regular sequence with linear representation \((u, M, w)\). Then the recognisable series \( x \in KA^\ast \) with linear representation \((u, M, w)\) is called the recognisable series associated to \( y \).

**Remark 3.5.** From Lemma 3.3 we see that the recognisable series associated to a \( q \)-regular sequence is well defined. From Lemma 3.2 we see that a recognisable series associated to a \( q \)-regular sequence is compatible. Moreover, we see that every linear representation of a \( q \)-regular sequence is also a linear representation of its associated recognisable series.

**Theorem 3.6.** Let \( y \) be a \( q \)-regular sequence and \((u, M, w)\) be a minimal linear representation of the recognisable series associated to \( y \). Then \((u, M, w)\) is a linear representation of \( y \), and it is also minimal.

In other words, to find a minimal linear representation of a regular sequence, we can use the minimisation algorithm presented by Berstel and Reutenauer [3, Chapter 2] on the associated recognisable series, i.e., the recognisable series with the same linear representation as the regular sequence.

**Proof of Theorem 3.6.** Let \( x \) denote the recognisable series associated to \( y \) and \( D \) denote the dimension of \((u, M, w)\). By Remark 3.5 \( x \) is compatible, and therefore, by Proposition 2.3 (see Remark 2.5), we have \( M(0)w = w \).

We define a vector-valued sequence \( v \in (K^{D \times 1})^{N_0} \) by \( v(0) := w \) and (2) for all \( n \in N_0 \) and \( r \in A_q \). Note that \( M(0)w = w \) implies the validity of (2) for \( n = 0 \) and \( r = 0 \).
By the above definition of \( v \) and Lemma 3.3, \((u, M, w)\) is indeed a linear representation of \( y \).

Now, any linear representation of the \( q \)-regular sequence \( y \) of dimension \( D' \) is also a linear representation of the recognisable series \( x \) by Remark 3.5. Therefore, due to minimality of \((u, M, w)\), we have \( D' \geq D \). In particular, by choosing a minimal linear representation of \( y \), we see that \((u, M, w)\) is a minimal linear representation of \( y \) as well.

At last, we can relax the assumptions of Theorem 3.6 and ask: Given a \( q \)-regular sequence with minimal linear representation, can we find a recognisable series that gives the same values for each standard \( q \)-ary expansion of a non-negative integer, but whose minimal linear representation has a smaller dimension than that of the regular sequence? The following example provides an affirmative answer.

**Example 3.7.** Let us consider the recognisable series \( x \in C^{\{0,1\}^*} \) with \( x(b) = 2^t \) and \( t \) counting the letter 0 in \( b \in \{0,1\}^* \). Note that \( x(b0) = 2x(b) \neq x(b) \) for all \( b \in \{0,1\}^* \); in particular, \( x \) is not compatible. Moreover, a minimal linear representation \((u, M, w)\) of \( x \) is given by

\[
\begin{align*}
u &= (1) \in C^{1 \times 1}, & M(0) &= (2) \in C^{1 \times 1}, & M(1) &= (1) \in C^{1 \times 1} \quad \text{and} \quad w = (1) \in C^{1 \times 1}.
\end{align*}
\]

In contrast, let \( y \in C^{N_0} \) be the 2-regular sequence with \( y(n) = x(\text{digits}_2(n)) \) for all \( n \in N_0 \). Then \((u', M', w')\) with

\[
\begin{align*}
u' &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & M'(0) &= \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}, & M'(1) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad w' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{align*}
\]

is a minimal linear representation of \( y \).

Therefore, starting with the 2-regular sequence \( y \) whose minimal representation has dimension 2 can lead to a minimal representation of dimension 1 of a recognisable series when ignoring trailing zeros.

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