Estimating Differential Latent Variable Graphical Models with Applications to Brain Connectivity

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Abstract

Differential graphical models are designed to represent the difference between the conditional dependence structures of two groups, thus are of particular interest for scientific investigation. Motivated by modern applications, this manuscript considers an extended setting where each group is generated by a latent variable Gaussian graphical model. Due to the existence of latent factors, the differential network is decomposed into sparse and low-rank components, both of which are symmetric indefinite matrices. We estimate these two components simultaneously using a two-stage procedure: (i) an initialization stage, which computes a simple, consistent estimator, and (ii) a convergence stage, implemented using a projected alternating gradient descent algorithm applied to a nonconvex objective, initialized using the output of the first stage. We prove that given the initialization, the estimator converges linearly with a nontrivial, minimax optimal statistical error. Experiments on synthetic and real data illustrate that the proposed nonconvex procedure outperforms existing methods.

Keywords: Alternating projected gradient descent, differential network, functional connectivity, latent variable Gaussian graphical model.

1 Introduction

Gaussian graphical models (Lauritzen, 1996) are routinely used to capture complex relationships among observed variables in a variety of fields, ranging from computational biology (Friedman, 2004), genetics (Lauritzen and Sheehan, 2003), to neuroscience (Smith et al., 2011). Each node in a graphical model represents an observed variable and the (undirected) edge between two nodes is present if the nodes are conditionally dependent given all the other variables; thus (sparse) graphical models are highly interpretable and have been adopted for a wide variety of applications.

Of particular interest in this manuscript are applications to cognitive neuroscience, specifically functional connectivity; the study of functional interactions between brain regions, thought to be necessary for cognition (Bullmore and Sporns, 2009; van den Heuvel et al., 2009; Power et al., 2010). Importantly, functional connectivity is a promising biomarker for mental disorders (Castellanos et al., 2013), where the primary object of study is the differential network, that is the differences in connectivity between healthy individuals and patients. We point reader to Zuo et al. (2011),

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Monti et al. (2014), and Preti et al. (2017) for prior work on functional connectivity estimation and Bielza and Larrañaga (2014) for detailed reviews. Another relevant scientific discipline is genetics, where scientists are interested in understanding differences in gene networks between experimental conditions (that is, the case-control study), to elucidate potential mechanisms underlying genetic functions. In this case, the differential network between two groups provides important signals for detecting differences. The interested reader can find more details on estimating genetic network differences in Hudson et al. (2009), de la Fuente (2010), and Ideker and Krogan (2012).

In many applications, it is clear that relationships between the observed variables are confounded by the presence of unobserved, latent factors. For example, physiological and demographic factors may have confounding effects for both functional connectivity in neuroscience and genetics (Gaggiotti et al., 2009; Willi and Hoffmann, 2009; Durkee et al., 2012). The standard approach of estimating sparse Gaussian graphical models is of limited usage here as, due to the confounding, the marginal precision matrix is not sparse (Chandrasekaran et al., 2012; Meng et al., 2014). Thus, classical techniques for estimating sparse precision matrices fail (Drton and Maathuis, 2017). Instead, sparsity of the marginal graph, latent variable Gaussian graphical models exploit the observation that both the joint distribution of the observed and latent variables, and the graph of the observed variables conditioned on the latent variables are sparse – further, the marginal graph of the observed variables can be decomposed into a superposition of a sparse matrix and a low-rank matrix (Chandrasekaran et al., 2012; Meng et al., 2014).

This manuscript addresses the estimation of differential networks with latent factors. Suppose two groups of observed variables are drawn from latent variable Gaussian graphical models and one is interested in differences in the conditional dependence structure between the two groups, which can be reduced to estimating the difference of their respective precision matrices. To this end, we develop a novel estimation procedure that does not require separate estimation for each group, which allows for robust estimation even if each group contains hub nodes. We propose a two-stage algorithm to optimize the objective. In the first stage, we derive a simple, consistent estimator, which then serves as initialization for the next stage. In the second stage, we employ projected alternating gradient descent with a constant step size. The iterates are proven to linearly converge to a region around the ground truth, whose radius is characterized by the statistical error. Compared with existing convex approaches, our nonconvex approach enjoys lower computation costs and is thus more time efficient. Extensive experiments validate our conceptual and theoretical claims. Our code is available at https://github.com/senna1128/Differential-Network-Estimation-via-Nonconvex-Approach.

2 Background

2.1 Notations

Throughout the paper, we use $\mathbb{S}^{d \times d}$, $\mathbb{O}^{d \times d}$, $I_d$ to denote the set of $d \times d$ symmetric, orthogonal matrices, and matrices respectively. Given an integer $d$, we let $[d] = \{1, 2, \ldots, d\}$ be the index set. For any two scalars $a$ and $b$, we denote $a \leq b$ if $a \leq cb$ for some constant $c$. Similarly, $a \geq b$ if $a \leq cb$ for some constant $c$. We write $a \approx b$ if $a \leq b$ and $b \leq a$. We use $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$. For matrices $A, B \in \mathbb{S}^{d \times d}$, we write $A < B$ if $B - A$ is positive definite and $A \leq B$ if $B - A$ is positive semidefinite. We use $\langle A, B \rangle = \text{tr}(A^T B)$. For a matrix $A$, $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$ denote the minimum and maximum singular values, respectively. For a vector $a$, $\|a\|_p$ denotes its $\ell_p$ norm, $p \geq 1$, and $\|a\|_0 = |\text{supp}(a)|$ denotes the number of nonzero entries of $a$. For a matrix $A$, $\|A\|_p$
denotes the matrix induced p-norm, \( \|A\|_F \) denotes the Frobenius norm, \( \|A\|_* \) denotes the nuclear norm, and \( \|A\|_{p,q} = (\sum_{ij} |A_{ij}|^p)^{1/p} \). Given a set \( C \subseteq \mathbb{R}^{d \times r} \), the projection operator \( P_C(\cdot) \) is defined as \( P_C(U) = \arg \min_{V \in C} \|V - U\|_F \).

### 2.2 Preliminaries and related work

A Gaussian graphical model (Lauritzen, 1996) consists of a graph \( G = (V,E) \), where \( V = \{1, \ldots, d\} \) is the set of vertices and \( E \) is the set of edges, and a \( d \)-dimensional random vector \( X = (X_1, \ldots, X_d)^\top \sim N(\mu_X, \Sigma_X) \) that is Markov with respect to \( G \). The precision matrix of \( X \), \( \Omega_X = (\Sigma_X)^{-1} \), encodes the conditional independence relationships underlying \( X \) and the graph structure \( G \) where

\[
X_i \perp X_j \mid \{X_k : k \in V \setminus \{i, j\}\} \iff (i, j) \notin E \iff (\Omega_X^*)_{i,j} = 0.
\]

Learning the structure of a Gaussian graphical model is an important problem that has wide applications to fields ranging from genetics (Friedman, 2004; Wille et al., 2004; Werhli et al., 2006; Yu et al., 2016) and neuroscience (Jordan et al., 2001; Smith et al., 2011) to social science (Zhang et al., 2007; Dobra and Lenkoski, 2011). See Drton and Maathuis (2017) for a recent overview.

Latent variable Gaussian graphical models extend the applicability of Gaussian graphical models by assuming the existence of latent hidden factors, \( X_H \in \mathbb{R}^r \), that confound the observed conditional independence structure of the observed variables \( X_O \in \mathbb{R}^d \). In particular, the observed and hidden components are assumed to be jointly normally distributed as \( (X_O^\top, X_H^\top)^\top \sim N(\mu^*, \Sigma^*) \) with

\[
\mu^* = \begin{pmatrix} \mu^*_X \\ \mu^*_H \end{pmatrix}, \quad \Sigma^* = \begin{pmatrix} \Sigma^*_{OO} & \Sigma^*_{OH} \\ \Sigma^*_{HO} & \Sigma^*_{HH} \end{pmatrix}, \quad \Omega^* = (\Sigma^*)^{-1} = \begin{pmatrix} \Omega^*_{OO} & \Omega^*_{OH} \\ \Omega^*_{HO} & \Omega^*_{HH} \end{pmatrix}.
\]  

While the joint precision matrix \( \Omega \) is commonly assumed sparse, the marginal precision matrix of the observed component \( X_O \sim N(\mu^*_X, \Sigma^*_{OO}) \) is given as

\[
(\Sigma^*_{OO})^{-1} = \Omega^*_{OO} - \Omega^*_{OH}(\Omega^*_{HH})^{-1}\Omega^*_{HO}.
\]  

and in general is not sparse. In decomposition (2), the first term \( \Omega^*_{OO} = (\Sigma^*_{OO} - \Sigma^*_{OH}(\Sigma^*_{HH})^{-1}\Sigma^*_{HO})^{-1} \) is the precision matrix of the conditional distribution of \( X_O \) given \( X_H \), which is sparse and positive definite. The second term is a rank-\( r \) positive semidefinite matrix, which in general is not sparse. Therefore, the marginal precision matrix of observed variables \( X_O \) has the sparse plus low-rank structure. Chandrasekaran et al. (2012) and Meng et al. (2014) have developed methods for estimating precision matrices with such structure.

In this paper, we study the problem of estimating the differential network, which is characterized by the difference between two precision matrices, from two groups of samples distributed according to latent variable Gaussian graphical models. More specifically, suppose that we have independent observations of \( d \) variables from two groups of subjects: \( X_i = (X_{i1}, \ldots, X_{id})^\top \sim N(\mu_X, \Sigma_X) \) for \( i = 1, \ldots, n_X \) from one group and \( Y_i = (Y_{i1}, \ldots, Y_{id})^\top \sim N(\mu_Y, \Sigma_Y) \) for \( i = 1, \ldots, n_Y \) from the other. The differential network is defined as the difference between two precision matrices, denoted as \( \Delta^* = \Omega_X^* - \Omega_Y^* \), where \( \Omega_X^* = (\Sigma_X^*)^{-1} \) and \( \Omega_Y^* = (\Sigma_Y^*)^{-1} \). We assume that the differential network can be decomposed as

\[
\Delta^* = S^* + R^*,
\]  

where
where $S^*$ is sparse and $R^*$ is low-rank and they are both symmetric, but indefinite matrices. Such structure arises under the assumption that the group specific precision matrices have the sparse plus low rank structure as in (2). However, imposing the sparse plus low rank structure on the differential networks puts fewer restrictions on the data generating process.

Estimating the differential network $\Delta^*$ can be naively achieved by estimating group-specific precision matrices first and then taking their difference. However, such an approach requires imposing strong assumptions on the individual precision matrices and is less robust in practice. Fortunately, some of the most restrictive assumptions can be avoided by directly estimating the differential network. For example, hub nodes are commonly found in real world networks such as transcriptional networks (Barabási and Oltvai, 2004; Barabási et al., 2010). However, when hub nodes are present in a group-specific network, estimation of an individual precision matrix is challenging as sparsity assumption is violated, while the differential network may remain valid.

Chandrasekaran et al. (2012) estimated a precision matrix under a latent variable Gaussian graphical model by minimizing the penalized negative Gaussian log-likelihood

$$
(\hat{S}_X, \hat{R}_X) = \arg \min_{S,R} \quad \text{tr} \left[ (S + R) \hat{\Sigma}_X \right] - \log \det(S + R) + \lambda_n(\gamma \|S\|_1, 1 + \|R\|_*),
$$

subject to $S + R > 0, \quad -R \geq 0,$

where $\hat{\Sigma}_X$ is a sample covariance based on $n_X$ samples. Under suitable identifiability and regularity conditions, $\gamma^{-1}\|\hat{S}_X - S_X\|_{\infty, \infty} \lor \|\hat{R}_X - R_X\|_2 \lesssim (d/n_X)^{1/2}$ when $\lambda_n \asymp (d/n_X)^{1/2}$. Meng et al. (2014) developed an alternating direction method of multipliers for more efficient minimization of (4) and showed that $\|\hat{\Omega}_X - \Omega^*_X\|_F \lesssim (s \log d/n_X)^{1/2} + (r d/n_X)^{1/2}$. The main drawback of minimizing (4) arises from the fact that in each iteration of the algorithm, the matrix $R$ is updated without taking its low-dimensional structure into account. Xu et al. (2017) explicitly represented the low-rank matrix as $R = -UU^T$ for $U \in \mathbb{R}^{d \times r}$ and minimized the resulting nonconvex objective using the alternating gradient descent. Our alternating gradient descent procedure is closely related to this work, but more challenging in several aspects. First, the log-likelihood is not readily available for differential networks. We hence rely on a quasi-likelihood, which reaches its minimum at $\Delta^*$. Second, the low-rank matrix $R^*$ in our setup is indefinite, so we have to estimate the positive index of inertia for $R^*$ as well. Third, in order to establish theoretical properties of our estimator, we avoid relying on the concentration of $\|\hat{\Sigma}_X\|_1$ that requires $n_X \asymp d^2$. By a more careful analysis, we improve the sample complexity to $n_X \asymp d \log d$.

Several methods have been proposed to learn the group-specific precision matrices jointly. For example, Chiquet et al. (2011), Guo et al. (2011), Danaher et al. (2014), and Mohan et al. (2014) all maximized the penalized joint likelihood of samples from both groups with a penalty that encourages the estimated precision matrices to have the same support. Such methods work well when the individual precision matrices are sparse. Zhao et al. (2014) directly estimated the differential network $\Delta^*$ by minimizing $\|\Delta\|_{1,1}$ subject to the constraint $\|\hat{\Sigma}_X \Delta \hat{\Sigma}_Y - (\hat{\Sigma}_Y - \hat{\Sigma}_X)\|_{\infty, \infty} \leq \lambda$. Under suitable conditions and when the differential network is sparse, the truncated and symmetrized estimator satisfies $\|\hat{\Delta} - \Delta^*\|_F \lesssim \{(n_X \wedge n_Y)^{-1}\|\Delta^*\|_{0,1} \log d\}^{1/2}$. Yuan et al. (2017) instead minimized the $\ell_1$ penalized quadratic loss. Liu et al. (2014) and Kim et al. (2019) developed procedures for estimation and inference of differential networks under more general distributional assumptions, which allow for $X$ and $Y$ to follow an exponential family distribution. Our paper contributes to this literature by developing methodology to learn the differential network from latent variable Gaussian graphical models. In the presence of latent factors, the differential network is not guaranteed to be sparse,
therefore, the aforementioned methods are not applicable.

Finally, our work is related to a growing literature on robust estimation where parameter matrices have the sparse plus low-rank structure. For example, Candès et al. (2011), Chandrasekaran et al. (2011), Hsu et al. (2011), Chen and Wainwright (2015), Klopp et al. (2017), and Yi et al. (2016) studied robust PCA where the goal is to recover the underlying low-rank matrix from sparsely corrupted observations. Fazel et al. (2008), Waters et al. (2011), and Chen and Chi (2014) studied robust matrix sensing, where the goal is to recover both sparse component and low-rank component. Chen et al. (2011) and Gong et al. (2012) studied robust multi-task learning, where the task relationships are characterized by low-rank matrix while outliers are sparse. Chen and Huang (2012), She (2017), and Yu et al. (2017) studied estimation of parameter matrices that are simultaneously sparse and low-rank. Zhang et al. (2018) proposed a unified framework to establish the convergence of alternating gradient descent when applied on sparse plus low-rank recovery. However, our problem is more challenging and does not satisfy conditions required by their framework. In particular, we use noisy covariance matrices to recover the difference of their true inverses via a quadratic loss. The Hessian matrix in our problem is $(\hat{\Sigma}_Y \otimes \hat{\Sigma}_X + \hat{\Sigma}_X \otimes \hat{\Sigma}_Y)/2$ with $\otimes$ denoting the Kronecker product, which is different compared to examples in robust estimation where the expectation of the Hessian is identity. As a result, the Condition 4.4 in Zhang et al. (2018) fails to hold and hence we need a problem-oriented analysis.

3 Methodology

3.1 Empirical loss

We introduce the estimator of the differential network $\Delta^*$ based on observations from latent variable Gaussian graphical models described in §2.2. Since $\Delta^*$ satisfies $(\Sigma^*_X \Delta^* \Sigma^*_Y + \Sigma^*_Y \Delta^* \Sigma^*_X) = 0$, one can minimize the quadratic loss $L(\Delta) = \text{tr} \{(\Delta \Sigma^*_X \Delta \Sigma^*_Y/2 - \Delta (\Sigma^*_Y - \Sigma^*_X))\}$. This loss has been used in Xu and Gu (2016) and Yuan et al. (2017) to learn sparse differential networks. Using the decomposition of $\Delta^*$ in (3) and substituting the true covariance matrices with sample estimates, we arrive at the following empirical loss

$$L_n(S, R) = \text{tr} \left\{ (S + R) \hat{\Sigma}_X (S + R) \hat{\Sigma}_Y/2 - (S + R) (\hat{\Sigma}_Y - \hat{\Sigma}_X) \right\},$$

where $S \in \mathbb{S}^{d \times d}$ denotes the sparse component, $R \in \mathbb{S}^{d \times d}$ denotes the low-rank component with rank $r$, and $\hat{\Sigma}_X = n^{-1}_X \sum^n_{i=1} (X_i - \hat{\mu}_X)(X_i - \hat{\mu}_X)^T$ with $\hat{\mu}_X = n^{-1}_X \sum^n_{i=1} X_i$ and $\hat{\Sigma}_Y$ is similarly defined. The empirical loss $L_n(S, R)$ in (5) is convex with respect to the pair $(S, R)$ and strongly convex if either of the two components is fixed.

Directly minimizing $L_n(S, R)$ over a suitable constraint set would be computationally challenging as in each iteration $R$ would need to be updated in $\mathbb{R}^{d \times d}$, without utilizing its low-rank structure. To that end, we explicitly factorize $R$ as $R = U \Lambda U^T$, where columns of $U \in \mathbb{R}^{d \times r}$ are aligned with eigenvectors that correspond to nonzero eigenvalues, and $\Lambda \in \mathbb{R}^{r \times r}$ is the diagonal sign matrix with diagonal elements being the sign of each eigenvalue. Without loss of generality, we assume $\Lambda$ has +1 entries on the diagonal first, followed by −1 entries. This factorization implicitly imposes the constraints that $\text{rank}(R) = r$ and $R = R^T$. Different from estimating the single latent variable Gaussian graphical model in (2), where the low-rank component is positive semidefinite and can be factorized as $R = U \Lambda U^T$, $\Delta^*$ in our model (3) is only symmetric as it corresponds to the difference of
two low-rank positive semidefinite matrices. Thus, \( R^* = U^* \Lambda^* U^{*T} \) and we need estimate \( \Lambda^* \) as well. Plugging the factorization into (5), we aim to minimize the following empirical nonconvex objective

\[
\hat{\mathcal{L}}_n(S, U, \Lambda) = \mathcal{L}_n(S, UA U^T) = \text{tr} \left\{ (S + UA U^T) \hat{\Sigma}_X (S + UA U^T) \bar{\Sigma}_Y / 2 - (S + UA U^T)(\bar{\Sigma}_Y \Sigma_X) \right\},
\]

over a suitable constraint set that we discuss next.

We assume \( S^* \in \mathbb{S}^{d \times d} \) has at most \( s \) nonzero entries overall and each column (row) has at most a certain fraction of nonzero entries. In particular, we assume

\[
S^* \in \mathcal{S}(\alpha, s) = \{ S \in \mathbb{S}^{d \times d} : \| S \|_{0,1} \leq s, \| S \|_{0,\infty} \leq \alpha d \}
\]

for some integer \( s \) and fraction \( \alpha \in (0, 1) \). Furthermore, to make the low-rank component separable from the sum \( S^* + R^* \), we require \( R^* \) to be not too sparse. One way to ensure identifiability is to impose the incoherence condition (Candès and Romberg, 2007). Specifically, suppose \( R^* = L^* \Xi^* L^{*T} \) is the reduced eigenvalue decomposition, where \( L^* \in \mathbb{R}^{d \times r} \) satisfies \( L^{*T} L^* = I_r \) and \( \Xi^* = \text{diag}(\lambda_1^{R*}, \ldots, \lambda_r^{R*}) \). Then, we assume \( L^* \) satisfies \( \beta \)-incoherence condition, that is,

\[
L^* \in \mathcal{U}(\beta) = \mathcal{U}(\beta) = \left\{ L \in \mathbb{R}^{d \times r} \mid \| L^T \|_{2,\infty} \leq (\beta r/d)^{1/2} \right\}.
\]

Without loss of generality, the eigenvalues are ordered so that, for some integer \( r_1 \in \{0, \ldots, r\} \), \( \text{sign}(\lambda_i^{R*}) = 1 \) for \( 1 \leq i \leq r_1 \) and \( \text{sign}(\lambda_i^{R*}) = -1 \) for \( r_1 + 1 \leq i \leq r \). Here, \( r_1 \), so called the positive index of inertia of \( R^* \), is unique by Sylvester’s law of inertia (cf. Theorem 4.5.8 in Horn and Johnson, 2013), although eigenvalue decomposition is not.

### 3.2 Two-stage algorithm

We develop a two-stage algorithm to estimate the tuple \((S^*, U^*, \Lambda^*)\). We start from introducing the second stage. Given a suitably chosen initial point \((S^0, U^0, \Lambda^0)\), obtained by the first stage that we introduce later, we use the projected alternating gradient descent procedure to minimize the following nonconvex optimization problem

\[
\min_{S, U} \hat{\mathcal{L}}_n(S, U, \Lambda^0) + \frac{1}{2} \| U_1^T U_2 \|_F^2,
\]

subject to \( S \in \mathcal{S}(\gamma_1, \gamma_2 s), \quad U \in \mathcal{U}(\alpha \beta \| U \|_2^2), \)

where \( U = (U_1, U_2) \) with \( U_1 \in \mathbb{R}^{d \times \tilde{r}_1}, U_2 \in \mathbb{R}^{d \times (r - \tilde{r}_1)} \), \( \tilde{r}_1 \) is the number of +1 entries of \( \Lambda^0 \), used as an estimate of \( r_1 \), and \( \gamma_1, \gamma_2 \) are user-defined tuning parameters. The quadratic penalty in (7) biases the estimates \( U_1, U_2 \) of the matrix \( U \) to be orthogonal and can also be written as \( \| U^T U - \Lambda^0 U^T U \Lambda^0 \|_F^2 / 16 \).

Before we detail steps of the algorithm, we define two truncation operators that correspond to two different sparsity structures. For any integer \( s \) and \( A \in \mathbb{R}^{d \times d} \), the hard-truncation operator \( \mathcal{J}_s(\cdot) : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{d \times d} \) is defined as

\[
[\mathcal{J}_s(A)]_{i,j} = \begin{cases} A_{i,j} & \text{if } |A_{i,j}| \text{ is one of the largest } s \text{ elements of } A, \\ 0 & \text{otherwise}. \end{cases}
\]
Algorithm 1. Stage II: projected alternating gradient descent for solving (7).

Input: Sample covariance matrices $\hat{\Sigma}_X$, $\hat{\Sigma}_Y$; Initial point tuple $(S^0, U^0, \Lambda^0)$; Step sizes $\eta_1, \eta_2$;
Tuning parameters $\hat{\alpha} = \gamma_1 \alpha$, $\hat{\beta} = \gamma_2 s$, $\hat{\beta}$.

For $k = 0$ to $k = K - 1$
\begin{align*}
S^{k+1/2} &= S^k - \eta_1 \nabla_S \mathcal{L}_n(S^k, U^k, \Lambda^0);
S^{k+1} &= \mathcal{T}_\alpha \{ \mathcal{J}_s(S^{k+1/2}) \}; \\
C^k &= U(4\beta \|U^k\|_2^2);
U^{k+1/2} &= U^k - \eta_2 \nabla_U \mathcal{L}_n(S^k, U^k, \Lambda^0) - \frac{\eta_2}{2} U^k (U^{kT}U^k - \Lambda^0 U^{kT}U^k \Lambda^0);
U^{k+1} &= \mathcal{P}_{C^k}(U^{k+1/2});
\end{align*}
Output $S^K$, $U^K$.

For any $\alpha \in (0, 1)$, the dispersed-truncation operator $\mathcal{T}_\alpha(\cdot) : \mathbb{R}^{d \times d} \mapsto \mathbb{R}^{d \times d}$ is defined as
$$
[\mathcal{T}_\alpha(A)]_{i,j} = \begin{cases} A_{i,j} & \text{if } |A_{i,j}| \text{ is one of the largest } \alpha d \text{ elements for both } A_{i, \cdot} \text{ and } A_{\cdot,j}, \\ 0 & \text{otherwise.} \end{cases}
$$

In the above definitions, $\mathcal{J}_s(A)$ keeps the largest $s$ entries of $A$, while $\mathcal{T}_\alpha(A)$ keeps the largest $\alpha$ fraction of entries in each row and column. Therefore, the operator $\mathcal{J}_s(\cdot)$ projects to the constraint set $|S|_{0,1} \leq s$, while $\mathcal{T}_\alpha(\cdot)$ projects to the set $\|S\|_{0,\infty} \leq \alpha d$.

We summarize the projected alternating gradient descend procedure in Algorithm 1. Both the sparse and low-rank components are updated, with the other component being fixed, by the gradient descent step with a constant step size, followed by a projection step. Explicit formulas for $\nabla_S \mathcal{L}_n$ and $\nabla_U \mathcal{L}_n$ are provided in Appendix A. The sign matrix $\Lambda^0$ is not updated in the algorithm. We will show later that, under suitable conditions, the first stage estimate consistently recovers $\Lambda^*$, that is, $\Lambda^0 = \Lambda^*$. Computationally, the update of the low-rank matrix in each iteration requires only updating the factor $U$, which can be done efficiently.

Next, we describe how to get a good initial point, $(S^0, U^0, \Lambda^0)$, needed for Algorithm 1 in the first stage. The requirements on the initial point are presented in Theorem 1 in next section. Our initial point is obtained from a rough estimator of $\Delta^*$. Let $\hat{\Delta}^0 = (\hat{\Sigma}_X)^{-1} - (\hat{\Sigma}_Y)^{-1}$, where $\hat{\Sigma}_X = n_X/(n_X - d - 2) \hat{\Sigma}_X$ (similarly for $\hat{\Sigma}_Y$) is the scaled sample covariance matrix. The scaled covariance matrix, so called Kaufman-Hartlap correction (Paz and Sánchez, 2015), is used for the initialization step so to have $\mathbb{E}(\hat{\Sigma}_X^{-1}) = \Omega_X^*$. By rescaling the sample covariance, we are able to show that $\| (\hat{\Sigma}_X)^{-1} - \Omega_X^* \|_{\infty,\infty} \approx (\log d/n_X)^{1/2}$ with high probability, leading to a better sample size compared to $\| (\hat{\Sigma}_X)^{-1} - \Omega_X^* \|_{\infty,\infty} \approx d/n_X + (\log d/n_X)^{1/2}$. Given $\hat{\Delta}^0$, we calculate $S^0$ by directly truncating $\hat{\Delta}^0$. Then, we perform the eigenvalue decomposition on the residual matrix $R^0 = \hat{\Delta}^0 - S^0$ to extract $r$ eigenvectors that correspond to top $r$ eigenvalues in magnitude. $U^0$ and $\Lambda^0$ are further derived from the reduced matrix. Details are shown in Algorithm 2. In Theorem 2 we show that $\Lambda^0 = \Lambda^*$, that is the positive index of inertia is correctly recovered by the initial step, and $(S^0, U^0)$ lies in a sufficiently small neighborhood of $(\Sigma^*, U^*)$.

Throughout the two-stage algorithm, we only compute (reduced) eigenvalue decomposition once in the first stage. Therefore, it is computationally efficient compared to related convex approaches, mentioned in Appendix G, where in each iteration one needs to compute an eigenvalue decomposition to update $R$. 

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We establish the convergence rate of iterates generated by Algorithm 1 by first assuming that the constraint sets Assumption 1. The statistical rate appears due to the approximation of population loss by the empirical loss, and by the positive index of inertia $r$. From the reduced eigenvalue decomposition of $S$ be the reduced eigenvalue decomposition of the rank-$r$ matrix $R^*$. There exist $\alpha$, $\beta$ and $s$ such that $S^* \in S(\alpha, s)$ and $L^* \in U(\beta)$.

**Assumption 2.** There exist $0 < \sigma_d^X \leq \sigma_1^X < \infty$ and $0 < \sigma_d^Y \leq \sigma_1^Y < \infty$ such that $\sigma_d^X I_d \leq \Sigma_X \leq \sigma_1^X I_d$ and $\sigma_d^Y I_d \leq \Sigma_Y \leq \sigma_1^Y I_d$.

We start by defining the distance function that will be used to measure the convergence rate of the low-rank component. From the reduced eigenvalue decomposition of $R^*$, $R^* = L^* \Sigma^* L^* = U^* \Lambda^* U^*$ with $\Lambda^* = \text{sign}(\Sigma^*) = \text{diag}(I_r, -I_{r-r_1})$ and $U^* = L^*(\Sigma^* \Lambda^*)^{1/2}$. While $\Lambda^*$ is uniquely characterized by the positive index of inertia $r_1$, $U^*$ is not unique in the sense that it is possible to have $U^* \Lambda^* U^T = U \Lambda^* U^T$ but $U \neq U^*$. We deal with this non-uniqueness issue by using the following distance function.

**Definition 1** (Distance function). Given two matrices $U_1, U_2 \in \mathbb{R}^{d \times r}$ and an integer $r' \in \{0, \ldots, r\}$,
we define $d_r(U_1, U_2) = \inf_{Q \in \mathcal{Q}^{d \times r}} \| U_1 - U_2 Q \|_F$, where

$$Q^{d \times r}_r = \{ Q \in \mathcal{Q}^{d \times r} : \Lambda A_Q^T = \Lambda \text{ with } \Lambda = \text{diag}(I_r, -I_{r-r'}) \}
= \{ Q \in \mathcal{Q}^{d \times r} : Q = \text{diag}(Q_1, Q_2) \text{ with } Q_1 \in \mathcal{Q}^{r \times r}, Q_2 \in \mathcal{Q}^{(r-r') \times r} \}.$$

In the following, we will simply use $d(\cdot, \cdot)$ to represent $d_{r_1}(\cdot, \cdot)$ with $r_1$ being the positive inertia of $R^*$. Based on the following lemma, we see that $d(\cdot, \cdot)$ measures $\| U^* A^T - U^* A^* U^T \|_F$.

**Lemma 1** (Properties of $d(\cdot, \cdot)$). Suppose $U^* \in \mathbb{R}^{d \times r}$ has orthogonal columns and $\Lambda^* = \text{diag}(I_{r_1}, I_{r-r_1})$. Let $\sigma_1(\sigma_r)$ be the largest (smallest) singular value of $U^*$ and let $U \in \mathbb{R}^{d \times r}$.

(a) If $d(U, U^*) \leq \sigma_1$, then $\| U^* A^T - U^* A^* U^T \|_F \leq 3\sigma_1 d(U, U^*)$.

(b) If $\| U^* A^T - U^* A^* U^T \|_F \leq \sigma_r^2/2$, then $d(U, U^*) \leq ((\sqrt{2} - 1)/2)^{-1}\| U^* A^T - U^* A^* U^T \|_F$.

By Lemma 1, $U^* A^T = U^* A^* U^T \iff d^2(U, U^*) = 0$. Thus, once we can correctly recover $\Lambda^*$, that is, $\hat{\Lambda} = \Lambda^*$, the distance function in Definition 1 is a reasonable surrogate for $\| \hat{R} - R^* \|_F$, since $\| \hat{R} - R^* \|_F = \| U^* \hat{A}^T - U^* A^* U^T \|_F = \| U^* \hat{A}^T - U^* A^* U^T \|_F \approx d(\hat{U}, U^*)$.

Let $\sigma_1^{R^*} = \sigma_{\max}(R^*), \sigma_r^{R^*} = \sigma_{\min}(R^*)$ and define the condition numbers $\kappa_X = \sigma_1^X/\sigma_d^X, \kappa_Y = \sigma_1^Y/\sigma_d^Y$, and $\kappa_{R^*} = \sigma_1^{R^*}/\sigma_r^{R^*}$. We further define the following quantities that depend only on the covariance matrices

\begin{align*}
T_1 &= \left\{ \frac{\kappa_X^2 \kappa_Y^2 \left( (\Omega_1^X)^2_1 \Sigma_X^1 + (\Omega_1^Y)^2_1 \Sigma_Y^1 \right)}{\sigma_d^X \sigma_d^Y} \right\}^2, \\
T_2 &= \left( \frac{1}{\sigma_d^X} + \frac{1}{\sigma_d^Y} \right)^2, \\
T_3 &= \left( \frac{\Sigma_X^1}{\sigma_d^X} \right)^2 + \left( \frac{\Sigma_Y^1}{\sigma_d^Y} \right)^2, \\
T_4 &= \frac{\left( \sigma_d^X \sigma_d^Y \right)^2}{\kappa_X^4 \kappa_Y^4 \left( \sigma_d^X \Sigma_X^1 + \sigma_d^Y \Sigma_Y^1 \right)^2}, \\
T_5 &= \left\{ \| \Omega_1^X \|_1^2 + \| \Omega_1^Y \|_1^2 \right\}^2, \\
T_6 &= \left( \frac{\kappa_X^2 \sigma_d^X + \kappa_Y^2 \sigma_d^Y}{\sigma_d^X \sigma_d^Y} \right)^2.
\end{align*}

Finally, for $S \in \mathbb{S}^{d \times d}$ and $U \in \mathbb{R}^{d \times r}$, we define the total error distance to be

$$TD(S, U) = \| S - S^* \|^2_2/\sigma_1^{R^*} + d^2(U, U^*).$$

The error for the sparse component is scaled by $\sigma_r^{R^*}$ in order to have the two error terms on the same scale, based on the first part of Lemma 1. With this, we have the following result on the convergence of iterates obtained by Algorithm 1.

**Theorem 1** (Convergence of Algorithm 1). Suppose Assumptions 1 and 2 hold. Furthermore, suppose the following conditions hold: (a) sample size

$$\min\{n_X, n_Y\} \geq C_1 \left\{ \frac{d \log d}{\beta^2 \gamma^2} \sqrt{\frac{\kappa_X \kappa_Y}{\eta_1}} \left( \frac{1}{T_3} \cdot s \log d \cdot T_2 \cdot r_d \right) \right\},$$

and sparsity proportion $\alpha \leq C_4 T_4/((\beta^2 \gamma^2)$; (b) step sizes $\eta_1 \leq C_2 \left( \frac{\kappa_X}{\sigma_d^X} \kappa_X \kappa_Y \right), \eta_2 = C_3 \eta_1/\sigma_1^{R^*}$, and tuning parameters $\gamma_1 = \gamma_2 \geq C_5 \left( \frac{\kappa_X}{\sigma_d^X} \kappa_X \kappa_Y \right)^2$; (c) initialization point $\Lambda^0 = \Lambda^*, S^0 \in \mathbb{S}^{d \times d}, U^0 \in \mathcal{U}(9 \sigma_1^{R^*})$ with $TD(S^0, U^0) \leq \varepsilon_4 \sigma_r^{R^*}/(\kappa_X \kappa_Y)^2$; then the iterates $(S^k, U^k)$ of Algorithm 1 satisfy $S^k \in \mathbb{S}^{d \times d}$ and

$$TD(S^k, U^k) \leq \left( 1 - \frac{C_5}{\kappa_X^2 \kappa_Y^2 \kappa_X \kappa_Y} \right)^k \frac{T_2 \cdot s \log d + T_2 \cdot r_d}{n_X \cdot n_Y},$$

with probability at least $1 - C_4/d^2$ for some fixed constants $(C_i)_{i=1}^4$ sufficiently large and $(C_i)_{i=1}^5$ sufficiently small.
The two terms in (9) correspond to the algorithmic and statistical rate of convergence, respectively. The statistical error is of the order $O((s \log d + rd) / (n_X \wedge n_Y))$, which is minimax optimal rate (Chandrasekaran et al., 2012). In particular, the term $O((s \log d) / (n_X \wedge n_Y))$ corresponds to the statistical error of estimating $S^*$, while $O(rd / (n_X \wedge n_Y))$ corresponds to the statistical error of estimating $R^*$. The sample complexity requirement in (8) has an extra $d \log d / \beta$ term compared to typical results in robust estimations (see Corollary 4.11 and Corollary 4.13 in Zhang et al. (2018) for results in robust matrix sensing and robust PCA). This increased sample complexity is common in estimation of latent variable Gaussian graphical models. For example, Xu et al. (2017) requires the convergence of $\| \hat{\Sigma}_X \|_1$, for which they need $n_X \gtrsim d^2$. Theorem 1 improves the requirement on the sample size to $d \log d$. In the loss function (6), the low-rank component we need to control is $\hat{\Sigma}_X U^*$ (and $\hat{\Sigma}_Y U^*$). The extra term appearing in the sample complexity guarantees that the incoherence condition can transfer from $U^*$ to $\hat{\Sigma}_X U^*$. On the other hand, the covariance matrices $\hat{\Sigma}_X$ and $\hat{\Sigma}_Y$ work as design matrices in (6) and bring additional challenges compared to robust estimation problems. For concreteness, the design matrix in robust PCA is identity, while in robust matrix sensing its expectation is also identity. Thus, their loss functions all satisfy Condition 4.4 in Zhang et al. (2018), which is not the case for (6). Finally, we observe that the algorithmic error rate decreases exponentially and, after $O(\log (n_X \wedge n_Y / (s \log d + rd)))$ iterations, the statistical error is the dominant term.

Next, we show that the output $(S^0, U^0, \Lambda^0)$ of Algorithm 2 satisfies requirements on the initialization point of Algorithm 1 presented in condition (c) in Theorem 1. The requirement that $S^0 \in S^{d \times d}$ is easy to achieve. The following lemma suggests that $\Lambda^0 = \Lambda^*$ is implied by an upper bound on $\| R^0 - R^* \|_2$, which further connects to the upper bound on $TD(S^0, U^0)$ by Lemma 1.

**Lemma 2.** For any $R \in S^{d \times d}$, let $R = L \Xi L^T$ be the eigenvalue decomposition. Let $\Xi_r \in \mathbb{R}^{r \times r}$ be the diagonal matrix with $r$ largest entries of $\Xi$ in magnitude, and let $\hat{r}_1$ be the number of positive entries of $\Xi_r$. If $\| R - R^* \|_2 \leq \sigma_{r^*}^R / 3$, then $\hat{r}_1 = r_1$ and $\Lambda_r = \text{diag}(I_{\hat{r}_1}, -I_{r-r_1}) = \Lambda^*$.

Based on Lemma 2, if $\| R^0 - R^* \|_2 \leq \sigma_{r^*}^R / 3$, then $\Lambda^0 = \Lambda^*$. The next theorem shows the sample complexity under which the conditions on the initial point are satisfied and $\| R^0 - R^* \|_2 \leq \sigma_{r^*}^R / 3$ holds as well.

**Theorem 2** (Initialization). Suppose Assumptions 1 and 2 hold. If $\hat{\alpha} \geq \alpha, \hat{s} \geq s$, the sample sizes and dimension satisfy

$$n_X \wedge n_Y \geq C_1 \frac{(T_5 \hat{s} \log d + T_6 d)}{(\sigma_{r^*}^R)^2}, \quad d \geq C_2 \frac{\hat{s} \beta s^{1/2} r \kappa R^*}{},$$

then $S^0 \in S^{d \times d}$, $\| R^0 - R^* \|_2 \leq \sigma_{r^*}^R / 4$, $U^0 \in U(9 \beta \sigma_{1^*}^R)$, and

$$TD(S^0, U^0) \leq C_3 \left\{ \frac{r (T_5 \cdot \hat{s} \log d + T_6 \cdot d)}{\sigma_{r^*}^R (n_X \wedge n_Y)} + \frac{\hat{s} \beta^2 s^{3/2} \kappa R^* \cdot \sigma_{1^*}^R}{d^2} \right\}$$

with probability $1 - C_4 / d^2$ for some fixed constants $(C_1)_{i=1}^4$ sufficiently large. Furthermore, if $\hat{s} \asymp s$,

$$n_X \wedge n_Y \gtrsim \frac{r \kappa X \kappa Y}{(\sigma_{r^*}^R)^2} (T_5 \cdot s \log d + T_6 \cdot d), \quad d \gtrsim \beta s^{1/2} r^{3/2} \kappa R^* \kappa X \kappa Y,$$

then $TD(S^0, U^0) \leq \sigma_{r^*}^R / (\kappa X \kappa Y)^2$.
From the above theorem, the requirement for the initial point is satisfied under condition (10). Our sample complexity for initialization, $O((rs \log d + rd)/\sigma^2_r)$, is smaller than the sample complexity for convergence, $O(d \log d/\beta + (s \log d + rd)/\sigma^2_r)$. Combining Theorem 2 with Theorem 1, when $\alpha \leq T_4/(\beta \kappa_R \kappa_Y)$ and $d \geq \beta s^{1/2} \kappa_R \kappa_X \kappa_Y$, the iterates generated by our two-stage algorithm converge to the true value linearly with an unavoidable minimax optimal statistical error. In next section, we will validate the efficacy and efficiency of the proposed nonconvex method via extensive numerical experiments.

## 5 Simulations

### 5.1 Data generation and implementation details

We compare the performance of our estimator with two procedures that directly learn the differential network under the sparsity assumption, the $\ell_1$-minimization (Zhao et al., 2014) and $\ell_1$-penalized quadratic loss (Yuan et al., 2017), and two procedures that separately learn latent variable Gaussian graphical models, sparse plus low-rank penalized Gaussian likelihood (4) (Chandrasekaran et al., 2012) and constrained Gaussian likelihood (Xu et al., 2017). Table 1 summarizes the procedures.

| Abbr. | Reference | Type | Setup |
|-------|-----------|------|-------|
| M1    | Zhao et al. (2014) | joint, convex | differential network is sparse |
| M2    | Yuan et al. (2017) | joint, convex | differential network is sparse |
| M3    | Chandrasekaran et al. (2012) | separate, convex | single network is sparse + low-rank |
| M4    | Xu et al. (2017) | separate, nonconvex | single network is sparse + low-rank |
| M*    | present paper | joint, nonconvex | differential network is sparse + low-rank |

Data are generated from the latent variable Gaussian graphical model (1) described in §2.2. We set $\mu_X = \mu_Y = \mu = 0$. The blocks of $\Omega$ are generated separately. For $\Omega_{OO} \in \mathbb{R}^{d \times d}$, we set diagonal entries to be one and, following Xia et al. (2015), off-diagonal entries to be generated according to one of the following four models.

Model 1: $(\Omega_{OO})_{i,i+1} = (\Omega_{OO})_{i+1,i} = 0.6$, $(\Omega_{OO})_{i,i+2} = (\Omega_{OO})_{i+2,i} = 0.3$;

Model 2: $(\Omega_{OO})_{i,j} = 0.5$ for $i = 10k - 9, 10k - 6 \leq j \leq 10k, 1 \leq k \leq d/10$;

Model 3: $(\Omega_{OO})_{i,j} \sim 0.8 \cdot \text{Bernoulli}(0.1)$ for $i + 1 \leq j \leq i + 3$;

Model 4: $(\Omega_{OO})_{i,j} \sim 0.5 \cdot \text{Bernoulli}(0.5)$ for $i = 2k - 1, 2k \leq j \leq (2k + 2) \wedge d, 1 \leq k \leq d/2$.

The blocks $\Omega_{OH}, \Omega_{HO}$ are generated entrywise from the following mixture distribution

$$(\Omega_{OH})_{i,j} \sim 0.1 \cdot \delta_0 + 0.9 \cdot \text{Uniform}(0.5, 1), \quad i = 1, \ldots, d, \quad j = 1, \ldots, r,$$

and $\Omega_{HH} = I_r$. Combining the blocks, we get the following four models

$$\Omega^{(i)} = \begin{pmatrix} \Omega_{OO}^{(i)} & \Omega_{OH} \\ \Omega_{HO} & \Omega_{HH} \end{pmatrix}, \quad i = 1, 2, 3, 4.$$

Last, we let $\Sigma_i = [D^{1/2} \{\Omega^{(i)} + (\delta_i + 1)I_d\}]^{1/2}$, where $\delta_i = \min \\{\text{eig}(\Omega^{(i)})\}$ and $D \in \mathbb{R}^{(d+r) \times (d+r)}$ is a diagonal scaling matrix with $D_{d+i,i} \sim \text{Uniform}(0.5, 2.5)$. In our models, each latent variable is
connected to roughly 90% of observed covariates and hence the effect of latent variables is spread-out and incoherent. We generate $X$ using $\Sigma_i^*$ and denote it as the control group, while generate $Y$ using $\Sigma_i^*$, $i = 2, 3, 4$, and denote it as the test $i - 1$ group. Under this generation process, both $X_O$ and $Y_O$ have precision matrices with sparse plus low-rank structure.

Throughout simulations, we set the sample size equal for both groups, $n_X = n_Y = n$. For each combination of the tuple $(n, d, r)$, we generate a training and validation set with sample size $n$. For each method, we choose the corresponding tuning parameters that minimize the empirical loss $L_n(\hat{S}, \hat{R})$ on the validation set. We measure the performance by $\|\hat{S} - S^*\|_F$ and $\|\hat{\Delta} - \Delta^*\|_F/\sqrt{\sigma_{\text{max}}(R^*)}$, where the latter is used as a surrogate for the total error distance $TD(\hat{S}, \hat{U})$. Errors are computed on test sets with the same sample size based on 40 independent runs. For our method, the step sizes are set as $\eta_1 = 0.5$, $\eta_2 = \eta_1/\sigma_{\text{max}}(U^0)$, where $U^0$ is the output of the initialization step; the sparsity proportion $\hat{\alpha} = \gamma_1 \alpha (= \hat{\alpha})$ is chosen from $\{0.01, 0.03, 0.05, 0.1, 0.3, 0.5, 0.8\}$ and $\hat{s} = \gamma_2 s (= \hat{s})$ from $\{2d, 4d, 6d, 15d, 25d, 30d\}$; the rank used in Algorithm 1 and 2 is chosen from $\{0, 2, 4, 6\}$; and the incoherence parameter $\beta$ is chosen from $\{1, 3\}$. For methods of Zhao et al. (2014) and Yuan et al. (2017), we choose between 5 different $\lambda$ values, denoting tuning parameters in their papers, generated automatically by their packages. For the method of Chandrasekaran et al. (2012), we use the implementation in Ma et al. (2013), which greedily chooses the tuning parameters $\alpha \in \{0.01, 0.05, 0.1\}$ and $\beta \in \{0.15, 0.25, 0.35\}$ (see (2.1) in Ma et al. (2013)). For the method of Xu et al. (2017), we select the rank and sparsity in the same way as for our method, while the other parameters are chosen as in Xu et al. (2017).

5.2 Results

Simulation results are summarized in Table 2 and 3. From Table 3 we see that our method outperforms other methods when $r = 2$, corresponding to the case where rank($R^*$) = 4 as $R^*$ is the difference of two low-rank components. When $r = 1$, Chandrasekaran et al. (2012) is comparable with our method on the first two data generating models, while our method compares favourably in the third case. Table 2 reports results for the case where $r = 0$ and there are no latent variables. In this setting, our method is comparable to methods of Zhao et al. (2014) and Yuan et al. (2017) that are specifically designed for sparse differential network estimation without considering latent variables. In comparison, the approach of Chandrasekaran et al. (2012) misestimates the low-rank component. Overall, the proposed nonconvex method accurately estimates both the low-rank and sparse components at a low computational cost.

We further validate theoretical results established in previous section from different aspects. First, we illustrate the statistical rate of convergence by plotting $\|\hat{S} - S^*\|_F$ versus $(d \log d/n)^{1/2}$, since $s \approx d$ in the experiments, and $\|\hat{R} - R^*\|_F$ versus $(rd/n)^{1/2}$. Although the estimation errors for $S^*$ and $R^*$ are combined in Theorem 1, we expect a linear increasing trend in both figures since $d \log d/n \approx rd/n$. We set $d = 50$, $r = 0, 1, 2$, and vary $n$ only. Results are shown in Figure 1(a) and 1(b), which illustrate the linear trend established for the statistical error in Theorem 1. Next, for $d = 50$, $r = 1$ and varying $n$, we test whether the rank chosen by cross-validation is consistent with the true rank and how often we consistently estimate the positive index of inertia. From the results shown in Figure 1(c) we observe that $r$ and $r_1$ are consistently selected when $\sqrt{d \log d/n} \leq 0.25$. 


(a) Statistical rate of convergence of estimating $S^*$. From left to right, $r = 0, 1, 2$, respectively. We let $d = 50$ and vary $n$ only. The dot on the figure represents the average error over 40 independent replicates, and the distance between two bars represents the standard deviation.

(b) Statistical rate of convergence of estimating $R^*$. From left to right, $r = 1, 2$, respectively. Other setups are same as the above figure.

(c) Recovery of rank (left) and positive index of inertia (right) of $R^*$. $d = 50$ and $r = 1$. The left figure takes value either 0 or 1, where 1 represents the rank, chosen by cross-validation, is consistent with the true rank. The right figure is the proportion over 40 independent replicates that have correct estimate of the positive index of inertia of $R^*$. In both figures, the blue line is covered by the green line.

Figure 1: Statistical rate of convergence and rank, positive index of inertia recovery. The upper two panels show the trend of the convergence rate of estimating $S^*$ and $R^*$, respectively, with respect to the corresponding ratio. All of trends in figures increase linearly, which validates the results in Theorem 1. The bottom panel shows the recovery of rank and positive index of inertia of $R^*$. When sample size is large enough, both of them are correctly recovered.
Table 2: Simulation results for five algorithms when latent factors are present. The estimation errors of the differential network and its sparse component are averaged over 40 independent replications, with standard error given in parentheses. The control group is generated by covariance $\Sigma_1$ while the test $i$ group is generated by $\Sigma_{i+1}$ for $i = 1, 2, 3$. Throughout the table, the smallest error under the same setup is highlighted.

| Method | $\|\hat{S} - S^*\|_F$ | $\frac{1}{\sqrt{\sigma_{\text{max}}(R^*)}} \|\hat{\Delta} - \Delta^*\|_F$ | $\|\hat{S} - S^*\|_F$ | $\frac{1}{\sqrt{\sigma_{\text{max}}(R^*)}} \|\hat{\Delta} - \Delta^*\|_F$ | $\|\hat{S} - S^*\|_F$ | $\frac{1}{\sqrt{\sigma_{\text{max}}(R^*)}} \|\hat{\Delta} - \Delta^*\|_F$ |
|--------|------------------|-----------------|------------------|-----------------|------------------|-----------------|
| $n = 1000, d = 100, r = 1$ | | | | | | |
| Control - Test 1 | Control - Test 2 | Control - Test 3 |
| M* | 20.02(0.56) | 10.35(0.45) | 18.73(0.71) | 9.33(0.53) | 18.59(0.59) | 7.94(0.20) |
| M1 | 26.40(0.67) | 11.18(0.27) | 27.58(0.93) | 11.26(0.37) | 30.22(0.67) | 12.07(0.26) |
| M2 | 30.05(0.32) | 12.48(0.14) | 31.04(0.53) | 12.41(0.21) | 32.77(0.46) | 12.75(0.18) |
| M3 | 22.49(0.35) | 9.52(0.14) | 22.54(0.44) | 9.23(0.17) | 22.91(0.47) | 9.18(0.18) |
| M4 | 33.72(0.61) | 14.16(0.26) | 33.62(0.63) | 13.61(0.26) | 34.45(0.58) | 13.62(0.23) |
| $n = 10000, d = 100, r = 2$ | | | | | | |
| Control - Test 1 | Control - Test 2 | Control - Test 3 |
| M* | 12.55(0.35) | 4.87(0.13) | 11.10(0.38) | 4.52(0.14) | 10.61(0.38) | 4.37(0.15) |
| M1 | 39.50(0.87) | 14.91(0.33) | 50.09(0.36) | 19.49(0.14) | 37.64(0.56) | 14.816(0.22) |
| M2 | 27.86(0.25) | 10.41(0.09) | 32.99(0.22) | 12.77(0.09) | 29.58(0.22) | 11.56(0.09) |
| M3 | 30.54(0.17) | 11.51(0.06) | 34.11(0.20) | 13.27(0.08) | 31.80(0.14) | 12.47(0.056) |
| M4 | 18.88(0.19) | 6.58(0.07) | 17.44(0.21) | 6.44(0.09) | 14.63(0.30) | 5.60(0.11) |

$M^*$, the proposed method; M1, $\ell_1$-minimization in Zhao et al. (2014); M2, $\ell_1$-penalized quadratic loss in Yuan et al. (2017); M3, penalized Gaussian likelihood in Chandrasekaran et al. (2012); M4, constrained Gaussian likelihood in Xu et al. (2017); detailed descriptions of each method are in Table 1 and setups of tuning parameters are discussed in the paper.
Table 3: Simulation results for five algorithms when latent factors are absent. Description of this table is referred to Table 2.

\[
\begin{array}{cccccc}
\text{Method} & \|\hat{S} - S^*\|_F & \frac{1}{\sqrt{\sigma_{\text{max}}(R^*)}} \|\hat{\Delta} - \Delta^*\|_F & \|\hat{S} - S^*\|_F & \frac{1}{\sqrt{\sigma_{\text{max}}(R^*)}} \|\hat{\Delta} - \Delta^*\|_F & \|\hat{S} - S^*\|_F & \frac{1}{\sqrt{\sigma_{\text{max}}(R^*)}} \|\hat{\Delta} - \Delta^*\|_F \\
M^* & 11.40(0.41) & 11.40(0.41) & 11.73(0.24) & 11.73(0.24) & 9.86(0.44) & 9.86(0.44) \\
M1 & 10.88(0.33) & 10.88(0.33) & 11.92(0.27) & 11.92(0.27) & 10.64(0.25) & 10.64(0.25) \\
M2 & 11.37(0.54) & 11.37(0.54) & 10.81(0.40) & 10.81(0.40) & 10.37(0.38) & 10.37(0.38) \\
M3 & 11.04(0.16) & 10.85(0.17) & 11.23(0.18) & 10.86(0.17) & 10.48(0.20) & 10.37(0.21) \\
M4 & 27.27(1.13) & 27.27(1.13) & 14.79(0.65) & 14.79(0.65) & 12.71(0.67) & 12.71(0.67) \\
\end{array}
\]

\( n = 200, \ d = 50, \ r = 0 \)
6 Application to fMRI functional connectivity

We apply our method to the task of estimating differential brain functional connectivity from functional Magnetic Resonance Imaging (fMRI). In particular, we analyze the Center for Biomedical Research Excellence (COBRE) dataset, which is publicly available in the nilearn module in Python (Abraham et al., 2014). This dataset includes fMRI data from 146 subjects across two groups: 74 subjects are healthy controls and 72 subjects are diagnosed with schizophrenia. Each subject data includes resting-state fMRI time series with 150 samples. We remove time points with excessive motion as recommended by standard analyses, and apply Harvard-Oxford Atlas to automatically generate 48 regions of interests. This dataset has been carefully analyzed using the NeuroImaging Analysis Kit\(^1\). For more information about dataset and detailed preprocessing steps, we point to Giove et al. (2009), Bellec et al. (2010), Power et al. (2012), and Chai et al. (2012).

We first estimate the differential network between the schizophrenia and control groups. Following the approach suggested in Belilovsky et al. (2016), we collect all fMRI series in one group across all subjects, thus assuming individuals in the same group share the same brain functional connectivity. Equivalently, we simply stack all time series from the subjects together to obtain one dataset for each group. The proposed approach and all the baseline methods remain the same as described in §5. The sparse plus low-rank decomposition is reasonable for estimating the differential network as it considers potential confounds such as age and gender. The sparse component of the differential network is the parameter of scientific interest.

The estimated sparse component is reported in Figure 2. In Figure 2, each region corresponds to a vertex, and each edge corresponds to an entry of the precision matrix, the color varying from dark blue to dark red corresponds to the magnitude of entry varying from negative to positive. Since Harvard-Oxford Atlas is a 3D parcellation atlas with lateralized labels, we show the detected connectomes in the left hemisphere only. From Figure 2, we see Zhao et al. (2014) approach fails to recover a clear pattern; Chandrasekaran et al. (2012) recovers one negative edge in Central Opercular Cortex and one negative edge in Middle Frontal Gyrus; our method, together with methods in Yuan et al. (2017) and Xu et al. (2017), show that the sparse network has two obvious edges, one positive and one negative, in Central Opercular Cortex area, which is also consistent with some recent analysis that also discovered Central Opercular Cortex is one of regions differs the most for the schizophrenia (Sheffield et al. (2015); Geng et al. (2019)). Upon closer analysis, we find the network estimated by the proposed method is much sparser than other methods and has the smallest test loss. Specifically, we have 1060 out of 2304 zero entries while the second sparsest network, produced by Chandrasekaran et al. (2012), has 1906 zero entries. Our test loss is -5.15, which is also smaller than other methods that are all greater than 10. We attribute this phenomenon to the superior sparse-low-rank separation ability of the method.

To further quantitatively validate our claims, we consider an individual-level analysis. In particular, we select 10 subjects from each group and consider the 190 possible pairs among them – 100 out of 190 pairs are across-group while the remaining 90 pairs are within-group. We estimate the differential network for each pair and calculate \(\|S\|_F\). Based on the group differences, one expects the sparse differential network for within-group pairs to have smaller norms than across-group pairs. Applying an unpaired two-sample t test, the p-value for the proposed method is 0.09 while greater than 0.16 for M1 to M4. This further validates that our method also outperforms other methods at the individual level.

\(^1\)https://github.com/SIMEXP/niak
Figure 2: Glass brains for the estimated sparse component of the differential network.
7 Discussion

We study the estimation of differential networks in the setting where the effects of the latent variables are diffused across all the observed variables leading to a low-rank component, and where the low-rank component of the difference satisfies an incoherence condition. In this setting, we are able to estimate the subspace spanned by the unobserved variables. Extending our approach to identify the difference in the complete connectivity of the graph, which includes latent variables, is of additional interest. Vinyes and Obozinski (2018) studied the problem of identification and estimation of the complete connectivity of the graph in the presence of latent variables using a carefully designed convex penalty. Direct extension of their technique to a high-dimensional differential network estimation is not possible due to the indefiniteness of the low-rank component $R^\ast$. One possible approach to developing a nonconvex estimation procedure for high-dimensional differential network estimation could be based on a thresholding step for the low-rank component (Yu et al., 2018a).

Recent work on differential networks have focused on statistical inference, including developing statistical tests for the global null $H_0 : \Delta^\ast = 0$ (Xia et al., 2015; Cai et al., 2019) and development of confidence intervals for elements of the differential network (Kim et al., 2019). The regression approach of Ren et al. (2015) can be used to construct asymptotically normal estimators of the elements of the differential network in the presence of latent variables. Such an approach would require both the individual precision matrices to be sparse and the correlation between latent and observed variables to be weak. How to develop an inference procedure that requires only week conditions on the differential network remains an open problem.

In our simulation and real data application, we propose to choose the tuning parameters using cross-validation. Zhao et al. (2014) proposed to tune the parameters by optimizing approximate Akaike information criterion in the context of sparse differential network estimation, however, there are no theoretical guarantees associated with the chosen parameters. Extending ideas of Foygel and Drton (2010) in the context of sparse plus low-rank estimation and showing that Akaike or Bayesian information criterion can be used for consistent recovery is of both practical and theoretical interest, as it would allow for faster parameter tuning compared to cross-validation.

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A Main Lemmas

In this section, we state lemmas needed to prove Theorem 1 and 2. Their proofs are presented in Appendix B. We first introduce additional notations. Partial derivatives of the empirical loss functions in (5) and (6) are given as

\[
\begin{align*}
\nabla_S \tilde{L}_n(S,U,\Lambda) &= \frac{1}{2} \tilde{\Sigma}_X(S + U\Lambda U^T)\tilde{\Sigma}_Y + \frac{1}{2} \tilde{\Sigma}_Y(S + U\Lambda U^T)\tilde{\Sigma}_X - (\tilde{\Sigma}_Y - \tilde{\Sigma}_X), \\
\nabla_U \tilde{L}_n(S,U,\Lambda) &= \tilde{\Sigma}_X(S + U\Lambda U^T)\tilde{\Sigma}_Y U\Lambda + \tilde{\Sigma}_Y(S + U\Lambda U^T)\tilde{\Sigma}_X U\Lambda - 2(\tilde{\Sigma}_Y - \tilde{\Sigma}_X)U\Lambda, \\
\nabla_S \tilde{L}_n(S,R) &= \nabla_R \tilde{L}_n(S,R) = \frac{1}{2} \tilde{\Sigma}_X(S + R)\tilde{\Sigma}_Y + \frac{1}{2} \tilde{\Sigma}_Y(S + R)\tilde{\Sigma}_X - (\tilde{\Sigma}_Y - \tilde{\Sigma}_X).
\end{align*}
\]

(A.1)

Partial derivatives of the population loss are similarly obtained by replacing \(\tilde{\Sigma}_X, \tilde{\Sigma}_Y\) with \(\Sigma_X, \Sigma_Y\) in (A.1). We further define the following quantities

\[
\begin{align*}
\Upsilon_1 &= \sigma_1^X, \quad \Upsilon_2 = \sigma_2^X, \quad \Upsilon_3 = \sigma_3^X, \quad \Upsilon_4 = \sigma_4^X, \\
\psi^2 &= \frac{\sigma_r^R}{\kappa_X^2 d}, \quad C(\gamma_1, \gamma_2) = \left( 1 + \left( \frac{2}{\gamma_1 - 1} \right)^{1/2} \right)^2 \left( 1 + \frac{2}{(\gamma_2 - 1)^{1/2}} \right).
\end{align*}
\]

(A.2)

For ease of presentation, we generically use \(C_i\) to denote constants and their values may vary for each appearance.

The following two lemmas characterize the error based on one-step iteration of Algorithm 1.

**Lemma A.1** (One-step iteration for sparse component). Suppose Assumptions 1 and 2, \(n_X \gtrsim \kappa_X^2 d\), \(n_Y \gtrsim \kappa_Y^2 d\), and following conditions hold

\[
\Lambda^0 = \Lambda^*, \quad S^k \in S_0, \quad U^k \in U(9\beta\sigma_1^R) \cap \{ U \in \mathbb{R}^{d \times r} : d(U, U^*) \leq \sqrt{\sigma_1^R/2} \}.
\]

If \(\eta_1 \leq 8/(3 \Upsilon_2)\), then \(S^{k+1} \in S_0\) and

\[
\begin{align*}
\frac{\|S^{k+1} - S^*\|_F^2}{C(\gamma_1, \gamma_2)} &\leq \left( 1 - \frac{9 \Upsilon_3}{4 \Upsilon_2} \eta_1 \right) \|S^k - S^*\|_F^2 + 3(1 + \gamma_2)s \left( \frac{\Upsilon_2 \eta_1}{\Upsilon_3} + \eta_2^2 \right) \|\nabla_S \tilde{L}_n(S^*, R^*)\|_F^2, \\
&\quad + \frac{3 \Upsilon_2}{2 \Upsilon_3} (1 + \gamma_1) \alpha \sigma_1^R \eta_1 d(U^k, U^*) + \frac{9 \Upsilon_3}{4 \Upsilon_2} \|S^k - S^*\|_F^2, \\
&\quad + \frac{C_1 \Upsilon_2 \Upsilon_1}{\Upsilon_3} (1 + \gamma_1) \alpha \sigma_1^R \eta_1 d(U^k, U^*) + \frac{C_2 \Upsilon_3}{16 \Upsilon_2} \|S^k - S^*\|_F^2,
\end{align*}
\]

with probability \(1 - C_2/d^2\), where \((C_i)_{i=1}^2\) are fixed constants.

**Lemma A.2** (One-step iteration for low-rank component). Suppose conditions of Lemma A.1 hold. If \(\eta_2 \leq 1/(18 \Upsilon_2 \sigma_1^R)\), then

\[
\begin{align*}
d^2(U^{k+1}, U^*) &\leq \left\{ 1 - \frac{3 \sigma_r^R \Upsilon_3 \eta_2}{2 \Upsilon_2} + \frac{C_1 \Upsilon_2 \Upsilon_1}{\Upsilon_3} (1 + \gamma_1) \alpha \sigma_1^R \eta_2 \right\} d^2(U^k, U^*) + \frac{27 \Upsilon_3 \eta_2}{16 \Upsilon_2} \|R^k - R^*\|_F^2, \\
&\quad + \frac{5 \kappa_X \kappa_Y \eta_2 d(U^k, U^*)}{9 \Upsilon_3} + \eta_2 \left( \frac{9 \Upsilon_3}{4 \Upsilon_2} + C_2 \sigma_1^R \sigma_1^Y \eta_2 \right) \|S^k - S^*\|_F^2, \\
&\quad + \eta_2 \left( \frac{16 \Upsilon_2}{9 \Upsilon_3} + \frac{4}{\Upsilon_2} + 54 \sigma_1^R \eta_2 \right) \|\nabla_R \tilde{L}_n(S^*, R^*)\|_F^2,
\end{align*}
\]

with probability \(1 - C_3/d^2\), where \((C_i)_{i=1}^2\) are fixed constants.

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Combining the above two lemmas, we obtain the decrease of the total error in one iteration.

**Lemma A.3.** Suppose Assumptions 1 and 2 hold. Furthermore, suppose the following conditions hold: (a) sample sizes and sparsity proportion

\[(n_X \wedge n_Y) \geq \left[\frac{(\sigma_1^X \sigma_1^Y)^2}{\mathcal{T}_4} \frac{d \log d}{\beta} + \{(\kappa_X \vee \kappa_Y)^2 d\}\right], \quad \alpha \leq \frac{\mathcal{Y}_3^2}{C_1 \mathcal{T}_4 \mathcal{Y}_4 \gamma_2}, \quad \frac{1}{\beta r \kappa R^*},\]

(b) step sizes \(\eta_1 \leq 1/(C_2 \kappa_X \kappa_Y \mathcal{Y}_2), \quad \eta_2 = \eta_1/(36 \sigma_1^r R^*),\) tuning parameters \(\gamma_1 \geq 1 + 8 \mathcal{Y}_2^2/(\mathcal{Y}_3 \gamma_1^2), \quad \gamma_2 \geq (1 + \gamma_1)/2;\) (c) the \(k\)-th iterate satisfies

\[\Lambda^0 = \Lambda^*, \quad S^k \in \mathbb{S}^{d \times d}, \quad U^k \in \mathcal{U}(9 \beta \sigma_1^R) \cap \{U \in \mathbb{R}^{d \times r} : d^2(U, U^*) \leq \Psi^2/C_3\}.

Then, with probability at least \(1 - C_4/d^2\),

\[TD(S^{k+1}, U^{k+1}) \leq \left(1 - \frac{\sigma^r R^* \gamma_2 \eta_2}{2 \mathcal{Y}_2} \right) TD(S^k, U^k) + \frac{1}{\kappa_X \kappa_Y \gamma_3 \sigma_1^R \gamma_2} \left\{ \gamma_2 s \|\nabla_S \mathcal{L}_n(S^*, R^*)\|_{\infty, \infty}^2 \right. + \left. \left( \frac{\log d}{n_X \wedge n_Y} \right)^{1/2} \gamma_1 \right\},\]

where \((C_i)_{i=1}^4\) are fixed constants.

From Lemma A.3, we observe that the successive total error distance decreases with linear contraction rate \(\rho = 1 - (\sigma^r R^* \gamma_3 \eta_2)/(2 \mathcal{Y}_2) < 1\) up to a statistical error, which comes from the approximation of population loss \(\mathcal{L}(S, R)\). The statistical error bound is given in the next lemma.

**Lemma A.4** (Statistical error bound). The gradients of \(\mathcal{L}_n(S, R)\), defined in (A.1), satisfy

\[
\Pr \left\{ \left. \|\nabla_R \mathcal{L}_n(S^*, R^*)\|_2 \geq \right( \frac{\kappa_X \sigma_1^Y + \kappa_Y \sigma_1^X}{n_X \wedge n_Y} \right) \left( \frac{d \log d}{\beta \kappa \kappa R^*} \right)^{1/2} \right] \leq \frac{1}{d^2},
\]

\[
\Pr \left\{ \left. \|\nabla_S \mathcal{L}_n(S^*, R^*)\|_{\infty, \infty} \geq \right( \frac{\log d}{n_X \wedge n_Y} \right)^{1/2} \right\} \leq \frac{1}{d^2}.
\]

The proof of Theorem 1 combines Lemma A.1, A.2, A.3, A.4 and is given in Appendix C. The next lemma establishes the error bound for \(S^0\) in the initialization step.

**Lemma A.5** (Error bound for \(S^0\)). Suppose Assumptions 1 and 2 hold. If \(\widehat{\alpha} \geq \alpha, \widehat{s} \geq s, \quad d \leq c(n_X \wedge n_Y)\) for \(c \in (0, 1/2)\), then

\[
\|S^0 - S^*\|_F \leq 17 \widehat{s}^{1/2} \left\{ \left( \frac{\log d}{n_X \wedge n_Y} \right)^{1/2} \right\} \left( \frac{\log d}{n_X \wedge n_Y} \right)^{1/2} + \frac{\beta r \sigma_1^R}{d},
\]

with probability at least \(1 - 8/d^2\).
B Proofs of Main Lemmas

B.1 Proof of Lemma A.1

We study the \( k \)-th iteration for updating sparse component in Algorithm 1. Define

\[
\tilde{S}^{k+1/2} = J_{\gamma_2s}(S^{k+1/2}), \quad \Omega^k = \text{supp}(S^*) \cup \text{supp}(S^k), \quad \hat{\Omega}^k = \tilde{\Omega}^k \cup \text{supp}(\tilde{S}^{k+1/2}). \quad (B.1)
\]

Since \( \nabla_S \bar{\mathcal{L}}_n(S^k, U^k, \Lambda^0) \) and \( S^k \) are symmetric, so is \( S^{k+1/2} \). From Lemma F.1 and F.2, we have \( S^{k+1} \in \mathbb{S}^{d \times d} \) and, therefore, \( \Omega^k, \hat{\Omega}^k \subseteq V \times V \) are two symmetric index sets. With some abuse of the notation, we use \( \mathcal{P}_\Omega(\cdot) \) to denote the projection onto the \( \Omega \). For a matrix \( A \in \mathbb{R}^{d \times d} \), \( \mathcal{P}_\Omega(A) \in \mathbb{R}^{d \times d} \) with elements \( \mathcal{P}_\Omega(A)_{i,j} = A_{i,j} \cdot 1_{(i,j) \in \Omega} \), where \( 1_{(i)} \) is an indicator function. From the updating rule in Algorithm 1,

\[
\mathcal{P}_\Omega(S^{k+1/2}) = \mathcal{P}_\Omega(S^k) - \eta_1 \mathcal{P}_\Omega \left\{ \nabla_S \bar{\mathcal{L}}_n(S^k, U^k, \Lambda^0) \right\} = S^k - \eta_1 \mathcal{P}_\Omega \left\{ \nabla_S \bar{\mathcal{L}}_n(S^k, U^k, \Lambda^0) \right\}. \quad (B.2)
\]

Combining (B.1) and (B.2), and noting that \( \text{supp}(\tilde{S}^{k+1/2}) \subseteq \Omega^k \),

\[
\tilde{S}^{k+1/2} = J_{\gamma_2s}(S^{k+1/2}) = J_{\gamma_2s} \left\{ \mathcal{P}_\Omega(S^{k+1/2}) \right\} = J_{\gamma_2s} \left[ S^k - \eta_1 \mathcal{P}_\Omega \left\{ \nabla_S \bar{\mathcal{L}}_n(S^k, U^k, \Lambda^0) \right\} \right].
\]

Further, from Lemma F.1 and F.2,

\[
\|S^{k+1} - S^*\|^2_F = \|T_{\gamma_1}(\tilde{S}^{k+1/2}) - S^*\|^2_F \leq \left\{ 1 + \left( \frac{2}{\gamma_1 - 1} \right)^{1/2} \right\} \|\tilde{S}^{k+1/2} - S^*\|^2_F
\]

\[
= \left\{ 1 + \left( \frac{2}{\gamma_1 - 1} \right)^{1/2} \right\} \||S^k - \eta_1 \mathcal{P}_\Omega \left\{ \nabla_S \bar{\mathcal{L}}_n(S^k, U^k, \Lambda^0) \right\} - S^*\|^2_F
\]

\[
\leq \left\{ 1 + \left( \frac{2}{\gamma_2 - 1} \right)^{1/2} \right\} \||S^k - \eta_1 \mathcal{P}_\Omega \left\{ \nabla_S \bar{\mathcal{L}}_n(S^k, U^k, \Lambda^0) \right\} - S^*\|^2_F
\]

\[
= C(\gamma_1, \gamma_2) \left\{ \|S^k - S^*\|^2_F - 2\eta_1 \mathcal{I}_1 + \eta_1^2 \mathcal{I}_2 \right\}, \quad (B.3)
\]

where \( C(\gamma_1, \gamma_2) \) is defined in (A.2) and

\[
\mathcal{I}_1 = \langle S^k - S^*, \mathcal{P}_\Omega \left\{ \nabla_S \bar{\mathcal{L}}_n(S^k, U^k, \Lambda^0) \right\} \rangle, \quad \mathcal{I}_2 = \|\mathcal{P}_\Omega \left\{ \nabla_S \bar{\mathcal{L}}_n(S^k, U^k, \Lambda^0) \right\}\|^2_F.
\]

Using Lemma D.1 to lower bound \( \mathcal{I}_1 \) and Lemma D.2 to upper bound \( \mathcal{I}_2 \),

\[
\frac{\|S^{k+1} - S^*\|^2_F}{C(\gamma_1, \gamma_2)} \leq \left( 1 - \frac{9 \Gamma_3}{4 \Gamma_2} \eta_1 \right) \|S^k - S^*\|^2_F - \eta_1 \left( \frac{8}{\Gamma_2} - 3 \eta_1 \right) \|\nabla_S \bar{\mathcal{L}}_n(S^k, U^k, \Lambda^0) - \nabla_S \bar{\mathcal{L}}_n(S^*, U^k, \Lambda^0)\|^2_F
\]

\[
+ \frac{C_1^1 \Gamma_1 \eta_1}{\Gamma_2} (1 + \gamma_1) \alpha \sigma_1^R \eta_1 d^2(U^k, U^*) + \frac{27 (\sigma_1^X \sigma_1^Y)^2 \eta_1^2}{4} \|U^k \Lambda^0 U^{kT} - U^* \Lambda^* U^{*T}\|^2_F
\]

\[
+ 3(1 + 2 \gamma_2) s \left( \frac{4 \Gamma_3}{9 \Gamma_3} \eta_1 + \eta_1^2 \right) \|\nabla_S \mathcal{L}_n(S^*, R^*)\|^2_{F(2, \infty)},
\]
with probability 1 \(- C_2/d^2\) for some large enough constants \(C_1, C_2 > 0\). With \(\eta_1 \leq 8/(3\gamma_2)\),

\[
\frac{\|S^{k+1} - S^*\|_F^2}{C(\gamma_1, \gamma_2)} \leq \left(1 - \frac{9\gamma_3}{4\gamma_2^2}\eta_1\right) \left\|S^k - S^*\right\|_F^2 + 3(1 + 2\gamma_2) s \left(\frac{4\gamma_2}{9\gamma_3}\eta_1 + \eta_1^2\right) \left\|\nabla S\mathcal{L}_n(S^*, R^*)\right\|_{\infty, \infty}^2 + \frac{C_1\gamma_2 T_3}{\gamma_3} (1 + \gamma_1) a_\sigma^R \eta_1 d^2(U^k, U^*) + \frac{27(\sigma_1^X\sigma_1^Y)^2\eta_1^2}{4} \left\|U^k\Lambda^0 U^{kt} - U^*\Lambda^0 U^{*t}\right\|_F^2,
\]

which completes the proof.

### B.2 Proof of Lemma A.2

Suppose \(Q^k \in Q_r^{n \times r}\) satisfies \(d(U^k, U^*) = \left\|U^k - U^* Q^k\right\|_F\). Using the bound on \(\|U^k\|_2\) in (D.10), we know \(U^* \in \mathcal{U}(4\beta\|U^k\|_2) = \mathcal{C}^k\), so does \(U^* Q^k\). Let \(\mathcal{P}_{\mathcal{C}^k}(\cdot)\) be the projection operator onto \(\mathcal{C}^k\). Due to the non-expansion property of \(\mathcal{P}_{\mathcal{C}^k}(\cdot)\),

\[
d^2(U^{k+1}, U^*) \leq \left\|U^{k+1} - U^* Q^k\right\|_F^2 = \left\|\mathcal{P}_{\mathcal{C}^k}(U^{k+1/2}) - \mathcal{P}_{\mathcal{C}^k}(U^* Q^k)\right\|_F^2 \leq \left\|U^{k+1/2} - U^* Q^k\right\|_F^2,
\]

\[
= \left\|U^k - \eta_2 \nabla U \mathcal{L}_n(S^k, U^k, \Lambda^0) - \frac{\eta_2}{2} \left\|U^{kt} U^{kt} - \Lambda^0 U^{kt} U^{kt} \Lambda^0\right\|\right. - \left. \left\|U^* Q^k\right\|_F^2 \right.
\]

\[
\leq d^2(U^k, U^*) - 2\eta_2 \mathcal{I}_3 + 2\eta_2 \mathcal{I}_4 - 2\eta_2 \mathcal{I}_5 + \frac{\eta_2^2}{2} \mathcal{I}_6,
\]

where

\[
\mathcal{I}_3 = (U^k - U^* Q^k, \nabla U \mathcal{L}_n(S^k, U^k, \Lambda^0)), \quad \mathcal{I}_4 = \left\|\nabla U \mathcal{L}_n(S^k, U^k, \Lambda^0)\right\|_F^2,
\]

\[
\mathcal{I}_5 = (U^k - U^* Q^k, U^{kt} U^k - \Lambda^0 U^{kt} U^k \Lambda^0), \quad \mathcal{I}_6 = \left\|U^{kt} U^k - \Lambda^0 U^{kt} U^k \Lambda^0\right\|_F.
\]

Similar to (B.3), we will lower bound \(\mathcal{I}_3, \mathcal{I}_5\) and upper bound \(\mathcal{I}_4, \mathcal{I}_6\). Using Lemma D.3, we bound the term \(\mathcal{I}_3\). Using (A.1) and the triangle inequality,

\[
\mathcal{I}_4 = 4\left\|\nabla R \mathcal{L}_n(S^k, R^k) U^k \Lambda^0\right\|_F^2 \leq 12 \left\{\nabla R \mathcal{L}_n(S^k, R^k) - \nabla R \mathcal{L}_n(S^k, R^*)\right\} \left\|U^k\right\|_F^2 + 12 \left\{\nabla R \mathcal{L}_n(S^k, R^*) - \nabla R \mathcal{L}_n(S^*, R^*)\right\} \left\|U^k\right\|_F^2.
\]

Using Hölder’s inequality and the bound in (D.10), we further have

\[
\mathcal{I}_4 \leq 27\sigma_1^R \left\|\nabla R \mathcal{L}_n(S^k, R^k) - \nabla R \mathcal{L}_n(S^k, R^*)\right\|_F^2 + 12\sigma_1^R \left\|\nabla R \mathcal{L}_n(S^*, R^*)\right\|_F^2 + 27\sigma_1^R \left\|\nabla R \mathcal{L}_n(S^k, R^k) - \nabla R \mathcal{L}_n(S^*, R^k)\right\|_F^2 + 27\sigma_1^R \left\|\nabla R \mathcal{L}_n(S^*, R^*)\right\|_F^2.
\]

By Lemma F.10,

\[
\mathcal{I}_5 \geq \frac{1}{8} \left\|U^{kt} U^k - \Lambda^0 U^{kt} U^k \Lambda^0\right\|_F^2 - \frac{1}{2} d^4(U^k, U^*),
\]

(p. 22)
and by (D.10),

\[ \mathcal{I}_6 \leq \|U_k\|^2 \cdot \|U^{kT} U^k - \Lambda^0 U^{kT} U^k \Lambda^0\|^2_F \leq \frac{9\sigma_R^2}{4} \|U^{kT} U^k - \Lambda^0 U^{kT} U^k \Lambda^0\|^2_F. \] (B.8)

Combining pieces in (B.5), (B.6), (B.7), (B.8) and Lemma D.3, there exist constants \( C_1, C_2, C_3 > 0 \), such that

\[
d^2(U^{k+1}, U^*) \leq \left(1 + \frac{C_1(1 + \gamma_1)\alpha r \sigma_1 R^*}{C_{32,1}}\right) d^2(U^k, U^*) - \eta_2 \frac{9Y_3}{2Y_2} - (2r)^{1/2} C_{33,1}\right) \|R^k - R^*\|^2_F
\]

\[- \eta_2 \left\{ \frac{9\sigma_1 R^* \eta_2}{8} - 2 \eta_2 \left( \frac{9Y_3}{2Y_2} - \frac{1}{C_{31}} - 54\sigma_1^2 \eta_2 \right) \right\} \|S^k - S^*\|^2_F
\]

\[+ \eta_2 \left\{ \frac{C_{32,1} + \frac{9\sigma_1^2 \sigma_3^2 C_{32,2}}{4} + C_2 \sigma_1 R^* (\sigma_1^X \sigma_1^Y \eta_2)^2}{C_{33,1} C_{33,2}} \right\} \|S^k - S^*\|^2_F,
\]

with probability at least \( 1 - C_3/d^2 \) for any \( C_{31}, C_{32,1}, C_{32,2}, C_{33,1}, C_{33,2} > 0 \). We let

\[ C_{31} = \frac{Y_2}{2}, \quad C_{32,1} = \frac{9Y_3}{8Y_2}, \quad C_{32,2} = \frac{\sigma_3^2 \sigma_4^2}{2Y_2}, \quad C_{33,1} = \frac{9Y_3}{8 (2r)^{1/2} Y_2}, \quad C_{33,2} = \frac{Y_2}{2 (2r)^{1/2}}. \]

With \( \eta_2 \leq 1/(18Y_2 \sigma_1^R \sigma_1^R) \), there exists a constant \( C_4 > 0 \) such that

\[
d^2(U^{k+1}, U^*) \leq \left\{1 + \frac{C_4 Y_2 Y_3}{Y_3^2} (1 + \gamma_1)\alpha r \sigma_1 R^* \eta_2 \right\} d^2(U^k, U^*) - \frac{27Y_3 \eta_2}{8Y_2} \|R^k - R^*\|^2_F
\]

\[- \eta_2 \left\{ \frac{9\sigma_1 R^* \eta_2}{8} - \frac{16r Y_3}{9Y_3} + \frac{4r}{Y_2} + 54\sigma_1^2 \eta_2 \right\} \|\nabla_R \mathcal{L}_n(S^k, R^*)\|^2_2
\]

\[+ \eta_2 \left\{ \frac{9Y_3}{4Y_2} + C_2 \sigma_1 R^* (\sigma_1^X \sigma_1^Y \eta_2)^2 \right\} \|S^k - S^*\|^2_F. \] (B.9)

Focusing on the second and the third term in above inequality, we write

\[
\frac{27Y_3 \eta_2}{8Y_2} \|R^k - R^*\|^2_F + \left( \frac{\eta_2}{8} - \frac{9\sigma_1 R^* \eta_2}{8} \right) \|U^{kT} U^k - \Lambda^0 U^{kT} U^k \Lambda^0\|^2_F
\]

\[= \frac{27Y_3 \eta_2}{16Y_2} \|R^k - R^*\|^2_F + \eta_2 \left( \frac{27Y_3}{4Y_2} \frac{1}{4} \|R^k - R^*\|^2_F + \frac{1}{16} \|U^{kT} U^k - \Lambda^0 U^{kT} U^k \Lambda^0\|^2_F \right)
\]

\[+ \frac{\eta_2}{8} \left( \frac{1}{2} - 9\sigma_1 R^* \eta_2 \right) \|U^{kT} U^k - \Lambda^0 U^{kT} U^k \Lambda^0\|^2_F. \]

\[23 \]
Without loss of generality, $\Upsilon_2 > 1$ and $27\Upsilon_3/(4\Upsilon_2) < 1$. Then, by Lemma F.10,

$$\frac{27\Upsilon_3\eta_2}{8\Upsilon_2} \|R^k - R^*\|_F^2 + \left(\frac{\eta_2}{8} - \frac{9\sigma_1^2\eta_2}{8}\right) \|U^{kT}U^k - \Lambda^0 U^{kT}\Lambda^0\|_F^2 \geq \frac{27\Upsilon_3\eta_2}{16\Upsilon_2} \|R^k - R^*\|_F^2 + \frac{3\eta_2\sigma_1^2\Upsilon_3}{2\Upsilon_2} d^2(U^k, U^*).$$

Plugging into (B.9), we obtain the error recursion for one-step iteration for the low-rank component

$$d^2(U^{k+1}, U^*) \leq \left\{1 - \frac{3\eta_2\sigma_1^2\Upsilon_3}{2\Upsilon_2} + \frac{C_4\Upsilon_2\Upsilon_1}{\Upsilon_3}(1 + \gamma_1)\alpha r\sigma_1^2\eta_2\right\} d^2(U^k, U^*) - \frac{27\Upsilon_3\eta_2}{16\Upsilon_2} \|R^k - R^*\|_F^2$$

$$+ 5\eta_2\kappa X\kappa Y\Upsilon_2 d^4(U^k, U^*) + \eta_2\left\{\frac{9\Upsilon_3}{4\Upsilon_2} + C_2\sigma_1^2 r^2 (\sigma_1^X\sigma_1^Y)^2\eta_2\right\} \|S^k - S^*\|_F^2$$

$$+ \eta_2\left(\frac{16\Upsilon_2}{9\Upsilon_3} + \frac{4r}{\Upsilon_2} + 54\sigma_1^2\eta_2\right)\|\nabla_R\mathcal{L}_n(S^*, R^*)\|_2, \tag{B.10}$$

which completes the proof.

### B.3 Proof of Lemma A.3

Under the assumptions of the lemma, the conditions of Lemma A.1 are satisfied. By the definition of the total error distance, we combine (B.4) and (B.10) to get

$$TD(S^{k+1}, U^{k+1}) \leq M_1 \frac{\|S^k - S^*\|_F^2}{\sigma_1^R} + M_2 d^2(U^k, U^*) + M_3 \|R^k - R^*\|_F^2$$

$$+ M_4 \|\nabla_S\mathcal{L}_n(S^*, R^*)\|_{2,\infty}^2 + M_5 \|\nabla_R\mathcal{L}_n(S^*, R^*)\|_2^2, \tag{B.11}$$

with probability at least $1 - C_4/d^2$, where

$$M_1 = \left\{1 - \frac{9\Upsilon_3}{4\Upsilon_2}\eta_1\right\} C(\gamma_1, \gamma_2) + \sigma_1^R\eta_2\left\{\frac{9\Upsilon_3}{4\Upsilon_2} + C_1\sigma_1^R (\sigma_1^X\sigma_1^Y)^2\eta_2\right\},$$

$$M_2 = 1 - \frac{3\sigma_1^2\Upsilon_3}{2\Upsilon_2}\eta_2 + \frac{C_2\Upsilon_2\Upsilon_1}{\Upsilon_3}(1 + \gamma_1)\alpha r\sigma_1^2\eta_2 + \frac{C_2\Upsilon_2\Upsilon_1}{\Upsilon_3} C(\gamma_1, \gamma_2)(1 + \gamma_1)\alpha r\eta_1$$

$$+ 5\eta_2\kappa X\kappa Y\Upsilon_2 d^4(U^k, U^*),$$

$$M_3 = \frac{27(\sigma_1^X\sigma_1^Y)^2\eta_1^2}{4\sigma_1^R} C(\gamma_1, \gamma_2) - \frac{27\Upsilon_3\eta_2}{16\Upsilon_2},$$

$$M_4 = \frac{3(1 + 2\gamma_2)s}{\sigma_1^R}\left(\frac{4\Upsilon_2}{9\Upsilon_3}\eta_1 + \eta_2^2\right) C(\gamma_1, \gamma_2),$$

$$M_5 = \eta_2 r\left(\frac{16\Upsilon_2}{9\Upsilon_3} + \frac{4}{\Upsilon_2} + 54\sigma_1^2\eta_2\right),$$

for some constants $(C_i)_{i=1}^4$. We proceed to simplify $(M_i)_{i=1}^5$ under the assumptions. Under the conditions on $\gamma_1$ and $\gamma_2$,

$$\left(1 - \frac{9\Upsilon_3\eta_1}{4\Upsilon_2}\right) C(\gamma_1, \gamma_2) \leq 1 - \frac{\Upsilon_3\eta_1}{4\Upsilon_2}. \tag{B.12}$$
Furthermore, using the bounds on $\eta_1$ and $\eta_2$, we obtain

$$\sigma_1^{R^*} \eta_2 \left\{ \frac{9 \Upsilon_3}{4 \Upsilon_2} + C_1 \eta_2 \sigma_1^{R^*} (\sigma_1^X \sigma_1^Y)^2 \right\} = \frac{\Upsilon_3 \eta_1}{16 \Upsilon_2} + \frac{C_1 \eta_1^2}{36^2} (\sigma_1^X \sigma_1^Y)^2 \leq \frac{\Upsilon_3 \eta_1}{8 \Upsilon_2}.$$  

Combining the last two inequalities, $M_1 \leq 1 - \Upsilon_3 \eta_1 / (8 \Upsilon_2)$. Similarly,

$$M_2 \leq 1 - \frac{3 \sigma_r^{R^*} \Upsilon_3 \eta_2}{2 \Upsilon_2} + \frac{C_2 \Upsilon_2 \Upsilon_1}{\Upsilon_3} (1 + \gamma_1) \alpha \sigma_1^{R^*} \{1 + 36C(\gamma_1, \gamma_2)\} \eta_2 + 5 \kappa_X \kappa_Y \Upsilon_2 \eta_2 d^2(U^k, U^*).$$

Since $\eta_X \wedge \eta_Y \geq \beta^{-1} \left\{ \left( \Sigma_X \|1/\sigma_1^X \right)^2 + \left( \Sigma_Y \|1/\sigma_1^Y \right)^2 \right\}^{-1} d \log d$, we have $Y_1 \leq 2 Y_4 \beta$. Furthermore, if following conditions hold

$$\alpha \leq \frac{\Upsilon_3^2}{4 C_2 \Upsilon_2^2 \Upsilon_4 (1 + \gamma_1) \{1 + 36C(\gamma_1, \gamma_2)\}} \cdot \frac{1}{\beta \kappa \kappa^{R^*}},$$

$$d^2(U^k, U^*) \leq \frac{\Upsilon_3}{10 \kappa_X \kappa_Y \Upsilon_2^2} \cdot \sigma_r^{R^*},$$

we can get

$$M_2 \leq 1 - \frac{3 \sigma_r^{R^*} \Upsilon_3 \eta_2}{2 \Upsilon_2} + \frac{\sigma_r^{R^*} \Upsilon_3 \eta_2}{2 \Upsilon_2} + \frac{\sigma_r^{R^*} \Upsilon_3 \eta_2}{2 \Upsilon_2} = 1 - \frac{\sigma_r^{R^*} \Upsilon_3 \eta_2}{2 \Upsilon_2}.$$

The above conditions on $\alpha$ and $d^2(U^k, U^*)$ are implied by the ones in lemma, noting that $1 + \gamma_1 \leq 2 \gamma_2$, $C(\gamma_1, \gamma_2) \leq 2$ due to the setup of $\eta_1$ and (B.12), and $\Upsilon_3 \sigma_r^{R^*} / (10 \kappa_X \kappa_Y \Upsilon_2^2) = \sigma_r^{R^*} / (\kappa_X^2 \kappa_Y^2) \approx \Psi^2$. For the term $M_3$, we note that $M_3 \leq 0 \iff \eta_1 \leq (144C(\gamma_1, \gamma_2) \kappa_X \kappa_Y \Upsilon_2)^{-1}$. Since $C(\gamma_1, \gamma_2) \leq 2$, by choosing the constant in the learning rate $\eta_1$ big enough, the right hand side condition holds so that $M_3 \leq 0$. For the term $M_4$, we have

$$M_4 \leq 6(1 + 2 \gamma_2) \left\{ \frac{4 \Upsilon_2}{9 \Upsilon_3} \eta_1 + \eta_1^2 \right\} \frac{s}{\sigma_1^{R^*}} \leq \frac{\gamma_2 s}{\kappa_X \kappa_Y \Upsilon_3 \sigma_1^{R^*}},$$

and for $M_5$,

$$M_5 = \frac{\eta_1}{36} \left( \frac{16 \Upsilon_2 + 4}{9 \Upsilon_3} + \frac{3 \eta_1}{2} \right) \frac{r}{\sigma_1^{R^*}} \leq \frac{\Upsilon_2 \eta_1 r}{\Upsilon_3 \sigma_1^{R^*}} \leq \frac{r}{\kappa_X \kappa_Y \Upsilon_3 \sigma_1^{R^*}}.$$

Plugging all the bounds back into (B.11),

$$TD(S^{k+1}, U^{k+1}) \leq \left( 1 - \frac{\Upsilon_3 \eta_1}{8 \Upsilon_2} \right) \frac{\|S^k - S^*\|_F^2}{\sigma_1^{R^*}} + \left( 1 - \frac{\sigma_r^{R^*} \Upsilon_3 \eta_2}{2 \Upsilon_2} \right) d^2(U^k, U^*)$$

$$+ \frac{1}{\kappa_X \kappa_Y \Upsilon_3} \left\{ \frac{\gamma_2 s}{\sigma_1^{R^*}} \|\nabla_S \mathcal{L}_n(S^*, R^*)\|_{2, \infty}^2 + \frac{r}{\sigma_1^{R^*}} \|\nabla_R \mathcal{L}_n(S^*, R^*)\|_2^2 \right\}.$$  

The proof is completed by noting $(\Upsilon_3 \eta_1) / (8 \Upsilon_2) \geq (\sigma_r^{R^*} \Upsilon_3 \eta_2) / (2 \Upsilon_2)$. 

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B.4 Proof of Lemma A.4

From (A.1), we have

$$\nabla_R \mathcal{L}_n(S^*, R^*)$$

where we use the Hölder’s inequality:

$$C$$

and

$$\text{pr} \left( \| \hat{\Sigma}_X - \Sigma_X^* \|_{\infty, \infty} \leq \sqrt{\frac{\log d}{n_X}} \right) \geq 1 - \frac{C_2}{d^2}.$$ 

Therefore, the last term in (B.14) only contributes a high-order term and

$$\text{pr} \left( \| \nabla_S \mathcal{L}_n(S^*, R^*) \|_{\infty, \infty} \leq \left( \| \Omega_X^* \|_1 \| \Sigma_X^* \|_1 + \| \Omega_Y^* \|_1 \| \Sigma_Y^* \|_1 \right) \frac{\log d}{n_X \wedge n_Y} \right) \geq 1 - \frac{C_3}{d^2},$$

for a constant $C_3 > 0$. 

(B.13)
B.5 Proof of Lemma A.5

Let $\vec{S}^0 = \mathcal{J}_S(\hat{\Delta})$. Then $S^0 = \mathcal{T}_S(\vec{S}^0)$ Since $\text{supp}(S^* - S^0) \subseteq \text{supp}(S^*) \cup \text{supp}(S^0)$, we consider the following cases that depend on $(i, j)$ location.

Case 1. If $(i, j) \in \text{supp}(S^0)$, then

$$|S^*_i - S^0_i| = |\hat{\Delta}_{i,j} - \Delta_{i,j}| = |\Delta_{i,j} - \hat{\Delta}_{i,j} - R_{i,j}| \leq \|\Delta^* - \hat{\Delta}\|_{\infty, \infty} + \|R^*\|_{\infty, \infty}. \quad (B.15)$$

Case 2. If $(i, j) \in \{\text{supp}(S^*) \setminus \text{supp}(S^0)\} \cap \text{supp}(\vec{S}^0)$, then

$$|S^*_i - S^0_i| = |\Delta_{i,j}| \leq |\Delta^*| + |R_{i,j}| \leq \|\Delta^* - \hat{\Delta}\|_{\infty, \infty} + \|R^*\|_{\infty, \infty}. \quad (B.16)$$

We claim that

$$|\hat{\Delta}_{i,j}| = |\vec{S}^0_{i,j}| \leq \|\vec{S}^0 - S^*\|_{\infty, \infty}. \quad (B.17)$$

Consider otherwise. Since $S^*$ has an $\alpha$-fraction of nonzero entries per row and column, $\vec{S}^0 - S^*$ differs from $\vec{S}^0$ on at most $\alpha$-fraction positions per row and column. If $|\vec{S}^0_{i,j}| > \|\vec{S}^0 - S^*\|_{\infty, \infty}$, then $\vec{S}^0_{i,j}$ is one of the largest $\alpha d$ entries in the $i$th row and $j$th column of $\vec{S}^0$. Furthermore, it is one of the largest $\alpha d$ entries, since $\alpha > \alpha$. This contradicts the assumption that $(i, j) \notin \text{supp}(S^0)$. Therefore, from (B.16) and (B.17),

$$|S^*_i - S^0_i| \leq \|\vec{S}^0 - S^*\|_{\infty, \infty} + \|\Delta^* - \hat{\Delta}\|_{\infty, \infty} + \|R^*\|_{\infty, \infty}. \quad (B.18)$$

Next, we bound $\|\vec{S}^0 - S^*\|_{\infty, \infty}$. We have two subcases. For any $(k, l) \in \text{supp}(\vec{S}^0)$,

$$|\vec{S}^0_{k,l} - S^*_k| = |\hat{\Delta}_{k,l} - \Delta^*_{k,l} + R^*_{k,l}| \leq \|\hat{\Delta} - \Delta^*\|_{\infty, \infty} + \|R^*\|_{\infty, \infty}. \quad (B.19)$$

For any $(k, l) \in \text{supp}(S^*) \setminus \text{supp}(\vec{S}^0)$,

$$|\vec{S}^0_{k,l} - S^*_k| = |S^*_k| = |\Delta^*_{k,l} + \|\Delta^* - \hat{\Delta}\|_{\infty, \infty} + \|R^*\|_{\infty, \infty}. \quad (B.20)$$

In this case, we claim

$$|\Delta^*_{k,l}| \leq \|\hat{\Delta} - S^*\|_{\infty, \infty}. \quad (B.21)$$

Otherwise, assume $|\Delta^*_{k,l}| > \|\hat{\Delta} - S^*\|_{\infty, \infty}$. Then $\Delta^*_{k,l}$ is one of the largest $s$ entries of $\hat{\Delta}$, since $S^*$ only has $s$ nonzero entries overall and $\hat{\Delta} - S^*$ differs from $\hat{\Delta}$ on at most $s$ positions. Moreover, since $\hat{s} \geq s$, $(k, l) \in \text{supp}(\vec{S}^0)$, which contradicts the condition. By (B.20) and (B.21),

$$|\vec{S}^0_{k,l} - S^*_k| \leq \|\hat{\Delta} - S^*\|_{\infty, \infty} + \|\Delta^* - \hat{\Delta}\|_{\infty, \infty} + \|R^*\|_{\infty, \infty} \leq 2\|\Delta^* - \hat{\Delta}\|_{\infty, \infty} + 2\|R^*\|_{\infty, \infty}. \quad (B.22)$$

Combining (B.19) and (B.22),

$$\|\vec{S}^0 - S^*\|_{\infty, \infty} \leq 2\|\Delta^* - \hat{\Delta}\|_{\infty, \infty} + 2\|R^*\|_{\infty, \infty}. \quad (B.23)$$

Finally, plugging (B.23) back into (B.18),

$$|S^*_i - S^0_i| \leq 3\|\Delta^* - \hat{\Delta}\|_{\infty, \infty} + 3\|R^*\|_{\infty, \infty}. \quad (B.24)$$
Case 3. If \((i, j) \in \text{supp}(S^*) \setminus \{ \text{supp}(S^0) \cup \text{supp}(\tilde{S}^0) \}\), then (B.22) gives us
\[
|S_{i,j}^* - S_{i,j}^0| = |S_{i,j}^*| \leq 2 \| \Delta^* - \hat{\Delta} \|_{\infty,\infty} + 2 \| R^* \|_{\infty,\infty},
\] (B.25)
since \(\text{supp}(S^*) \setminus \{ \text{supp}(S^0) \cup \text{supp}(\tilde{S}^0) \} \subseteq \text{supp}(S^*) \setminus \text{supp}(\tilde{S}^0)\). Combining (B.15), (B.24), (B.25), we complete the proof.

From Lemma F.12, with probability at least \(1 - 8/d^2\),
\[
\| \Delta^* - \hat{\Delta} \|_{\infty,\infty} \leq \| \Omega^*_X - (\hat{\Sigma}_X)^{-1} \|_{\infty,\infty} + \| \Omega^*_Y - (\hat{\Sigma}_Y)^{-1} \|_{\infty,\infty} \leq 4 \left\{ \| (\Omega^*_X)^{1/2} \|_1^2 + \| (\Omega^*_Y)^{1/2} \|_1^2 \right\} \left( \frac{\log d}{n_X \wedge n_Y} \right)^{1/2}.
\]
Since \(R^* = L^* \Xi L^{*T}\) with \(L^* \in \mathcal{U}(\beta)\), \(\| R^* \|_{\infty,\infty} \leq \| \Xi^* \|_{\infty,\infty} \| L^{*T} \|_{2,\infty} \leq \beta r \sigma_1 R^* / d\), and further
\[
\| S^0 - S^* \|_{\infty,\infty} \leq 12 \left\{ \| (\Omega^*_X)^{1/2} \|_1^2 + \| (\Omega^*_Y)^{1/2} \|_1^2 \right\} \left\{ \frac{\log d}{n_X \wedge n_Y} \right\}^{1/2} + \frac{3 \beta r \sigma_1 R^*}{d}.
\]
With this and
\[
\| S^0 - S^* \|_2 \leq \| S^0 - S^* \|_F \leq (2 \bar{s})^{1/2} \| S^0 - S^* \|_{\infty,\infty},
\]
we complete the proof.

C Proofs of Main Theorems

C.1 Proof of Theorem 1

We show that, under assumptions of Theorem 1, we can apply Lemma A.3 by replacing \((Y_i)_i^4 = 2\) with its corresponding orders. First, we check the conditions on the step sizes. Since \(Y_2 = \sigma_Y^2 \Rightarrow \kappa_{X \Sigma_Y} \hat{Y}_2 = \kappa_{X \Sigma_Y} \Sigma_X^2 \hat{Y}_1\), we immediately see that the conditions on the step sizes in Lemma A.3 are satisfied. Furthermore, since \(1 + 8 \hat{Y}_2 / (\hat{Y}_2 \hat{Y}_3) = \hat{Y}_2 / (\hat{Y}_3 \hat{Y}_4) = (\kappa_{X \Sigma_Y})^2\), the conditions on \(\gamma_1\) and \(\gamma_2\) are also satisfied. For the sparsity proportion \(\alpha\), since \(\gamma_2 = \kappa_X^2 \kappa_Y^4\), we have \(\hat{Y}_2 / (\hat{Y}_2 \hat{Y}_4) \approx T_1\), which implies the condition on \(\alpha\) in Lemma A.3 is satisfied. Since \(\hat{Y}_4 / (\sigma_1^2 \sigma_1^1)^2 \approx T_3\), condition (a) in Theorem 1 implies the sample complexity in Lemma A.3. Finally, we verify that the condition (c) in Lemma A.3 holds for all iterations from 0 to \(k\). For \(k = 0\) the condition is satisfied, since, for any constant \(C_1 > 0\),
\[
TD(S^0, U^0) \leq \Psi^2 / C_1 \implies U^0 \in \{ U \in \mathbb{R}^{d \times r} : d^2(U, U^*) \leq \Psi^2 / C_1 \}.
\]
Therefore, we can apply Lemma A.3 and A.4 for \(k = 0\). Let \(\rho = 1 - \left( \sigma^2 \hat{Y}_2 \right) / (2 \hat{Y}_2)\). Since \(\gamma_2 (\kappa_{X \Sigma_Y} \hat{Y}_3)^{-1} (\| \Omega^*_X \|_1 \| \Sigma_X^1 \|_1 + \| \Omega^*_Y \|_1 \| \Sigma_Y^1 \|_1)^2 = T_1\) and \((\kappa_{X \Sigma_Y} \hat{Y}_3)^{-1} (\kappa_{X \Sigma_Y} \hat{Y}_1 + \kappa_{Y \Sigma_Y} \hat{Y}_1)^2 = T_2\),
\[
TD(S^1, U^1) \leq \rho TD(S^0, U^0) + C_2 \frac{T_1 \log d + T_2 r d}{\sigma_1^R (n_X \wedge n_Y)},
\]
with probability at least \(1 - C_3 / d^2\) for constants \(C_2, C_3\). Let \(\tau' = T_1 \log d + T_2 r d\). Since \(1 - \rho \approx 1 / \left( \kappa_X^2 \kappa_Y^2 \kappa_{R^*} \right)\) and \((n_X \wedge n_Y) \geq (\sigma_X^2)^{-2} \kappa_X^2 \kappa_Y^2 \tau'\), we can bound \(C_2 \tau' / (\sigma_1^R (n_X \wedge n_Y)) \leq \)
We further get \(TD(S^1, U^1) \leq \Psi^2/C_1\). Moreover, since \(U^1 \in \mathcal{U}(4\beta\|U^0\|_2)\) by the Algorithm 1 and \(\|U^0\|_2\) satisfies (D.10), \(U^1 \in \mathcal{U}(9\beta\sigma^R_1)\). Therefore the condition (c) in Lemma A.3 holds for \((S^1, U^1)\). Applying this reasoning iteratively and noting, for any iteration \(k\),

\[
TD(S^k, U^k) \leq \rho TD(S^{k-1}, U^{k-1}) + \frac{C_2\tau'}{\sigma^R_1(n_X \land n_Y)},
\]

we have

\[
TD(S^k, U^k) \leq \rho^k TD(S^0, U^0) + \frac{C_2\tau'}{(1 - \rho)\sigma^R_1(n_X \land n_Y)},
\]

with probability at least \(1 - C_3/d^2\). Plugging the order of \(\rho\) completes the proof.

### C.2 Proof of Theorem 2

The proof follows in few steps. First, we use Lemma A.5 to upper bound \(\|S^0 - S^*\|_F\). Next, we upper bound \(\|R^0 - R^*\|_2\) and show that under the sample size assumption the bound guarantees \(\|R^0 - R^*\|_2 \leq \sigma^R_1/4\). Finally, we upper bound \(d^2(U^0, U^*)\) and \(TD(S^0, U^0)\).

It directly follows from the algorithm that \(S^0\) is symmetric. By Lemma A.5, with probability at least \(1 - C_1/d^2\),

\[
\|S^0 - S^*\|_F \leq C_2 \xi^{1/2} \left\{ \|\Omega_X^+\|_F^2 + \|\Omega_Y^+\|_F^2 \right\} \left\{ \frac{\log d}{n_X \land n_Y} \right\}^{1/2} + \frac{\beta r\sigma^R_1}{d}.
\]

(C.1)

We bound \(\|R^0 - R^*\|_2\) as

\[
\|R^0 - R^*\|_2 = \|\Delta - S^0 - \Delta^* + S^*\|_2 \leq \|\bar{\Sigma}_X\|_2 + \|\bar{\Sigma}_Y\|_2.
\]

(C.2)

For the term \(\|\bar{\Sigma}_X\|_2\), we know

\[
\|\bar{\Sigma}_X\|_2 = \frac{n_X - d}{n_X} \|\Sigma_X^* - (\Sigma_X^* - (\bar{\Sigma}_X)\|_2
\]

\[
\leq \frac{n_X - d}{n_X} \|\Sigma_X^* - (\Sigma_X^* - (\bar{\Sigma}_X)\|_2
\]

\[
\leq \frac{n_X - d}{n_X} \|\Sigma_X - (\bar{\Sigma}_X)\|_2 + \frac{d + 2}{n_X - d} \|\bar{\Sigma}_X\|_2,
\]

where the first inequality follows from Hölder’s inequality and (D.1). From Lemma F.11, with probability \(1 - C_3/d^2\),

\[
\|\bar{\Sigma}_X\|_2 + \|\bar{\Sigma}_Y\|_2 \leq C_4 \left( \frac{\kappa_X}{\sigma_d} + \frac{\kappa_Y}{\sigma_d} \right) \left( \frac{d}{n_X \land n_Y} \right)^{1/2}.
\]

(C.3)

Combining (C.1), (C.2), and (C.3),

\[
\|R^0 - R^*\|_2 \leq C_5 \left( \tilde{\xi}^{1/2} \varphi_1 + \varphi_2 \right)
\]

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with probability at least $1 - C_6/d^2$, where

$$\varphi_1 = \left\{ \left\| (\Omega_X)^{1/2} \right\|^2 + \left\| (\Omega_Y)^{1/2} \right\|^2 \right\} \left( \frac{\log d}{n_X \wedge n_Y} \right)^{1/2} + \frac{\beta r \sigma^{R^*_r}}{d}, \quad \varphi_2 = \left( \frac{\kappa_X}{\sigma_d^{R^*_r}} + \frac{\kappa_Y}{\sigma_d} \right) \left( \frac{d}{n_X \wedge n_Y} \right)^{1/2}.$$ 

Under the assumptions of the theorem, $\| R^0 - R^* \|^2 \leq \sigma^{R^*_r} / 4$. From Lemma 2, $\Lambda^0 = \Lambda^*$. By Weyl’s inequality, $\| R^* \|^2_2 \leq \| \tilde{U}_0 \|^2_2 = | R^0 |^2_2 \leq 3 | R^* |^2_2 / 2$. Therefore, $U^0 \in \mathcal{U}( \beta | \tilde{U} |^2_2) \subseteq \mathcal{U}(9 \beta \sigma^{R^*_r})$ and $U^* \in \mathcal{U}( \beta | \tilde{U} |^2_2) = C$. Moreover,

$$\| \tilde{U}_0 \Lambda^0 \tilde{U}_0^\top - U^* \Lambda^0 U^* \|^2_2 = \| \tilde{U}_0 \Lambda^0 \tilde{U}_0^\top - U^* \Lambda^0 U^* \|^2_2 = \| L_{r_{\tilde{U}_0}}^0 \Lambda^0 L_{r_{\tilde{U}_0}}^0 - R^* \|^2_2$$

$$\leq \| L_{r_{\tilde{U}_0}}^0 \Lambda^0 L_{r_{\tilde{U}_0}}^0 - R^* \|^2_2 + \| R^0 - R^* \|^2_2 \leq 2 | R^0 - R^* |^2_2 \leq \sigma^{R^*_r} / 2,$$ \hspace{1cm} (C.4)

since $\| L_{r_{\tilde{U}_0}}^0 \Lambda^0 L_{r_{\tilde{U}_0}}^0 - R^* \|^2_2 \leq \sigma_{r+1}(R^0) = \sigma_{r+1}(R^0) - \sigma_{r+1}(R^*) \leq | R^0 - R^* |^2_2$ with $\sigma_{r+1}(R^0)$ denoting the $(r+1)$-th singular value of $R^0$. Suppose $d^2(\tilde{U}_0, U^*) = \| \tilde{U}_0 - U^* Q \|^2_F$, then

$$d^2(\tilde{U}_0, U^*) \leq \| U^0 - U^* Q \|^2_F = \| \mathcal{P}_C(\tilde{U}_0) - \mathcal{P}_C(U^* Q) \|^2_F \leq \| \tilde{U}_0 - U^* Q \|^2_F = d^2(\tilde{U}_0, U^*).$$

By Lemma 1 and (C.4),

$$d^2(\tilde{U}_0, U^*) \leq \frac{1}{(\sqrt{2} - 1)\sigma^{R^*_r}} \| \tilde{U}_0 \Lambda^0 \tilde{U}_0^\top - U^* \Lambda^0 U^* \|^2_F$$

$$\leq \frac{2r}{(\sqrt{2} - 1)\sigma^{R^*_r}} \| \tilde{U}_0 \Lambda^0 \tilde{U}_0^\top - U^* \Lambda^0 U^* \|^2_2 \leq \frac{8r}{(\sqrt{2} - 1)\sigma^{R^*_r}} | R^0 - R^* |^2_2.$$ \hspace{1cm} (C.5)

Combining Lemma A.5 with (C.5), with probability at least $1 - C_7/d^2$,

$$TD(S^0, U^0) = \frac{\| S^0 - S^* \|^2_F}{\sigma^{R^*_r}} + d^2(\tilde{U}_0, U^*) \leq C_8 \frac{r}{\sigma^{R^*_r}} \left( s^{1/2} \varphi_1 + \varphi_2 \right)^2.$$

Since

$$\frac{r}{\sigma^{R^*_r}} \left( s^{1/2} \varphi_1 + \varphi_2 \right)^2 = \frac{r s \varphi_1^2}{\sigma^{R^*_r}} + \frac{r \varphi_2^2}{\sigma^{R^*_r}} = \frac{r (T_5 s \log d + T_6 d)}{\sigma^{R^*_r} (n_X \wedge n_Y)} + \frac{s \beta^2 r^3 \kappa_{R^*_r} \sigma^{R^*_r}}{d^2},$$

the first part of the theorem follows. The second part follows, since $r \left( s^{1/2} \varphi_1 + \varphi_2 \right)^2 / \sigma^{R^*_r} \leq \Psi^2 = \sigma^{R^*_r} / (\kappa_X \kappa_Y)^2$ under the sample complexity in (10). This completes the proof.

### D Complementary Lemmas

**Lemma D.1.** Under the conditions of Lemma A.1, we have

$$\mathcal{L}_1 \geq \frac{9}{8} Y_3 \| S^k - S^* \|^2_F + \frac{4}{T_2} \| \nabla S \tilde{L}_n(S^k, U^k, \Lambda^0) - \nabla S \tilde{L}_n(S^*, U^k, \Lambda^0) \|^2_F$$

$$\quad - \frac{4}{9} Y_2 \left\{ C_1 \cdot (1 + \gamma_1) \sigma \sigma^{R^*_r} Y_1 d^2(U^k, U^*) + (1 + \gamma_2) s \| \nabla S \mathcal{L}_n(S^*, R^*) \|^2_{\infty, \infty} \right\}$$

with probability $1 - C_2/d^2$ for some fixed constants $C_1, C_2 > 0$ large enough.
Proof. We start by bounding the sample covariance matrices \( \hat{\Sigma}_X \) and \( \hat{\Sigma}_Y \). Let \( \delta = 1/2d^2 \) in Lemma F.11. Then, for some constant \( C_1 > 0 \),

\[
\text{pr} \left \{ \| \hat{\Sigma}_X - \Sigma_X \|_2 \leq C_1 \| \Sigma_X^* \|_2 \left( \frac{d}{n_X} \right)^{1/2} \text{ and } \| \hat{\Sigma}_Y - \Sigma_Y^* \|_2 \leq C_1 \| \Sigma_Y^* \|_2 \left( \frac{d}{n_Y} \right)^{1/2} \right \} \geq 1 - \frac{1}{d^2}.
\]

We will on the event

\[
\mathcal{E} = \left \{ \sigma_d^i/2 \leq \sigma_{\min}(\hat{\Sigma}_i) \leq \sigma_{\max}(\hat{\Sigma}_i) \leq 3\sigma_d^i/2, \text{ for } i \in \{X, Y\} \right \}, \tag{D.1}
\]

which occurs with probability at least \( 1 - 1/d^2 \) by Lemma F.11, since \( n_X \gtrsim \kappa_X^2d \) and \( n_Y \gtrsim \kappa_Y^2d \). By definition of \( \mathcal{I}_1 \) and (A.1),

\[
\mathcal{I}_1 = \langle S^k - S^*, \mathcal{P}_{\Omega^k} \left \{ \nabla_S \tilde{\mathcal{L}}_n(S^k, U^k, \Lambda^0) \right \} \rangle = \langle S^k - S^*, \nabla_S \tilde{\mathcal{L}}_n(S^k, U^k, \Lambda^0) \rangle
\]

\[
\quad = \langle S^k - S^*, \nabla_S \tilde{\mathcal{L}}_n(S^k, U^k, \Lambda^0) - \nabla_S \tilde{\mathcal{L}}_n(S^*, U^k, \Lambda^0) \rangle + \langle S^k - S^*, \nabla_S \tilde{\mathcal{L}}_n(S^*, U^*, \Lambda^*) \rangle
\]

\[
\quad = \langle S^k - S^*, \hat{\Sigma}_X(S^k - S^*) \hat{\Sigma}_Y \rangle + \langle S^k - S^*, \nabla_S \tilde{\mathcal{L}}_n(S^*, U^*, \Lambda^*) \rangle
\]

\[
\quad + \frac{1}{2} \langle S^k - S^*, \hat{\Sigma}_X(U^k \Lambda^0 U^k - U^* \Lambda^* U^*) \hat{\Sigma}_Y \rangle = \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13}. \tag{D.2}
\]

We bound \( \mathcal{I}_{11}, \mathcal{I}_{12}, \mathcal{I}_{13} \) from below separately. By Lemma F.8 and on the event \( \mathcal{E} \) in (D.1),

\[
\mathcal{I}_{11} \geq \frac{9 \gamma_3}{4Y_2} \| S^k - S^* \|_F \| \hat{\Sigma}_X(S^k - S^*) \hat{\Sigma}_Y + \hat{\Sigma}_Y(S^k - S^*) \hat{\Sigma}_X \|_F
\]

\[
\quad = \frac{9 \gamma_3}{4Y_2} \| S^k - S^* \|_F \| \nabla_S \tilde{\mathcal{L}}_n(S^k, U^k, \Lambda^0) - \nabla_S \tilde{\mathcal{L}}_n(S^*, U^k, \Lambda^0) \|_F. \tag{D.3}
\]

Using Hölder’s inequality, for any \( C_{12} > 0 \) to be determined later,

\[
\mathcal{I}_{12} \geq - \| S^k - S^* \|_{1,1} \| \nabla_S \tilde{\mathcal{L}}_n(S^*, U^*, \Lambda^*) \|_{\infty, \infty}
\]

\[
\quad \geq - \{(\gamma_2 + 1)s\}^{1/2} \| S^k - S^* \|_F \| \nabla_S \tilde{\mathcal{L}}_n(S^*, U^*, \Lambda^*) \|_{\infty, \infty}
\]

\[
\quad \geq - \frac{(\gamma_2 + 1)s}{2C_{12}} \| \nabla_S \tilde{\mathcal{L}}_n(S^*, R^*) \|_{2, \infty}^2 - \frac{C_{12} \{(\gamma_2 + 1)s\}^{1/2}}{2} \| S^k - S^* \|_F. \tag{D.4}
\]

where the second inequality is due to the fact that \( \text{supp}(S^k - S^*) \subseteq \bar{\Omega}^k \) and \( |\bar{\Omega}^k| \lesssim (\gamma_2 + 1)s \), and the third inequality uses the equation \( \nabla_S \tilde{\mathcal{L}}_n(S^*, U^*, \Lambda^*) = \nabla_S \tilde{\mathcal{L}}_n(S^*, R^*) \) with \( \bar{\Omega}^k \) defined in (B.1).

Similarly, for any \( C_{13} > 0 \),

\[
\mathcal{I}_{13} \geq - \frac{1}{2} \| S^k - S^* \|_F \| \mathcal{P}_{\Omega^k} \left \{ \hat{\Sigma}_X(U^k \Lambda^0 U^k - U^* \Lambda^* U^*) \hat{\Sigma}_Y \right \} \|_F
\]

\[
\quad \geq - \| S^k - S^* \|_F \| \mathcal{P}_{\Omega^k} \left \{ \hat{\Sigma}_X(U^k \Lambda^0 U^k - U^* \Lambda^* U^*) \hat{\Sigma}_Y \right \} \|_F
\]

\[
\quad \geq - \frac{C_{13}}{2} \| S^k - S^* \|_F^2 - \frac{1}{2C_{13}} \| \mathcal{P}_{\Omega^k} \left \{ \hat{\Sigma}_X(U^k \Lambda^0 U^k - U^* \Lambda^* U^*) \hat{\Sigma}_Y \right \} \|_F^2, \tag{D.5}
\]

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where the second inequality is due to the fact that $\tilde{\Omega}^k$ is symmetric and, hence, from Lemma F.7,

$$\mathcal{P}_{\tilde{\Omega}^k} \left\{ \tilde{\Sigma}_X(U^k \Lambda^0 U^{kT} - U^* \Lambda^* U^{*T})\tilde{\Sigma}_Y \right\} = \left[ \mathcal{P}_{\tilde{\Omega}^k} \left\{ \tilde{\Sigma}_X(U^k \Lambda^0 U^{kT} - U^* \Lambda^* U^{*T})\tilde{\Sigma}_Y \right\} \right]^T.
$$

We further upper bound the second term on the right hand side of (D.5). Let $Q^k \in \mathbb{Q}_{x^k}^{r \times r}$ be the optimal rotation of $U^*$ such that $d^2(U^k, U^*) = \|U^k - U^* Q^k\|_F^2$, and define

$$\tilde{A}^k = \tilde{\Sigma}_X U^k, \quad \tilde{A}^* = \tilde{\Sigma}_X U^* Q^k, \quad A^* = \Sigma_X U^* Q^k,$$$$

$$\tilde{B}^k = \tilde{\Sigma}_Y U^k, \quad \tilde{B}^* = \tilde{\Sigma}_Y U^* Q^k, \quad B^* = \Sigma_Y U^* Q^k.$$

Since $\Lambda^0 = \Lambda^*$,

$$\left\| \mathcal{P}_{\tilde{\Omega}^k} \left\{ \tilde{\Sigma}_X(U^k \Lambda^0 U^{kT} - U^* \Lambda^* U^{*T})\tilde{\Sigma}_Y \right\} \right\|_F^2 = \left\| \mathcal{P}_{\tilde{\Omega}^k} \left\{ \tilde{\Sigma}_X(U^k \Lambda^0 U^{kT} - U^* Q^k \Lambda^* Q^k U^{*T})\tilde{\Sigma}_Y \right\} \right\|_F^2 = \sum_{(i,j) \in \tilde{\Omega}^k} \left( \tilde{A}_{i,j}^k - \tilde{A}_{i,j}^* \right)^T \Lambda^* (\tilde{B}_{j,i}^k - \tilde{B}_{j,i}^*) + \left( \tilde{A}_{i,j}^* \right)^T \Lambda^* (\tilde{B}_{j,i}^k - \tilde{B}_{j,i}^*) + \left( \tilde{B}_{j,i}^* \right)^T \Lambda^* (\tilde{A}_{i,j}^k - \tilde{A}_{i,j}^*) \right)^2 \leq 3 \max_{i \in [d]} \left\| \tilde{A}_{i,j}^k - \tilde{A}_{i,j}^* \right\|_2 \sum_{j \in [d]} \sum_{i \in \tilde{\Omega}^k_{i,j}} \left\| \tilde{B}_{j,i}^k - \tilde{B}_{j,i}^* \right\|_2^2 + \frac{3}{2} \max_{j \in [d]} \left\| \tilde{B}_{j,i}^k - \tilde{B}_{j,i}^* \right\|_2 \sum_{i \in [d]} \sum_{j \in \tilde{\Omega}^k_{i,j}} \left\| \tilde{A}_{i,j}^k - \tilde{A}_{i,j}^* \right\|_2^2 + \frac{3}{2} \max_{j \in [d]} \left\| \tilde{B}_{j,i}^* \right\|_2 \sum_{i \in [d]} \sum_{j \in \tilde{\Omega}^k_{i,j}} \left\| \tilde{A}_{i,j}^k - \tilde{A}_{i,j}^* \right\|_2^2,$$

where $\tilde{\Omega}^k_{i,j} = \{ j \mid (i,j) \in \tilde{\Omega}^k \}$ and $\tilde{\Omega}^k_{i,j} = \{ i \mid (i,j) \in \tilde{\Omega}^k \}$. For any $i, j \in [d], \tilde{\Omega}^k_{i,j} \cup \tilde{\Omega}^k_{i,j} \leq (1 + \gamma_1)ad$ and, therefore,

$$\left\| \mathcal{P}_{\tilde{\Omega}^k} \left\{ \tilde{\Sigma}_X(U^k \Lambda^0 U^{kT} - U^* \Lambda^* U^{*T})\tilde{\Sigma}_Y \right\} \right\|_F^2 \leq \frac{3(1 + \gamma_1)ad}{2} \left\| (\tilde{A}^k - \tilde{A}^*)^T \right\|_{2,\infty} \left\| \tilde{B}^k - \tilde{B}^* \right\|_F^2 + \frac{3(1 + \gamma_1)ad}{2} \left\| (\tilde{B}^k - \tilde{B}^*)^T \right\|_{2,\infty} \left\| \tilde{A}^k - \tilde{A}^* \right\|_F^2 + 3(1 + \gamma_1)ad \left\| \tilde{B}^{k*} \right\|_{2,\infty} \left\| \tilde{A}^k - \tilde{A}^* \right\|_F^2 + 3(1 + \gamma_1)ad \left\| \tilde{B}^{k*} \right\|_{2,\infty} \left\| \tilde{A}^k - \tilde{A}^* \right\|_F^2 \right. \left. + 3(1 + \gamma_1)ad \left\| \tilde{A}^k - \tilde{A}^* \right\|_F^2 \left\{ \frac{1}{2} \left\| (\tilde{A}^k - \tilde{A}^*)^T \right\|_{2,\infty} + \left\| \tilde{A}^{k*} \right\|_{2,\infty} \right\} \right. \left. + 3(1 + \gamma_1)ad \left\| \tilde{B}^k - \tilde{B}^* \right\|_F^2 \left\{ \frac{1}{2} \left\| (\tilde{B}^k - \tilde{B}^*)^T \right\|_{2,\infty} + \left\| \tilde{B}^{k*} \right\|_{2,\infty} \right\} \right\}. \quad (D.6)$$

We bound each term involving $\tilde{A}$ above as

$$\left\| \tilde{A}^k - \tilde{A}^* \right\|_F^2 = \left\| \tilde{\Sigma}_X(U^k - U^* Q^k) \right\|_F^2 \leq \frac{9(q_{X})^2}{4} \left\| U^k - U^* Q^k \right\|_F^2 = \frac{9(q_{X})^2}{4} d^2(U^k, U^*) \quad (D.7)$$
and
\[
\|\hat{A}^* - A^*\|_{2,\infty} \leq \|\hat{A}^* - A^*\|^2_{2,\infty} + \|A^*\|_{2,\infty} = \left\| \left( \hat{\Sigma}_X - \Sigma_X^* \right) U^* Q^k \right\|_{2,\infty} + \left\| (\Sigma_X^* U^* Q^k)^T \right\|_{2,\infty}
\]
\[
\leq \left\| \left( \hat{\Sigma}_X - \Sigma_X^* \right) U^* Q^k \right\|_{2,\infty} + \|\Sigma_X^*\|_1 \left\| U^* T \right\|_{2,\infty} \leq \left\| \left( \hat{\Sigma}_X - \Sigma_X^* \right) U^* Q^k \right\|_{2,\infty} + \|\Sigma_X^*\|_1 \sqrt{\frac{\beta r \sigma_1^{R^*}}{d}},
\]
where the second inequality comes from Lemma F.9 and the last inequality is due to the incoherence condition in Assumption 1. From Lemma F.13, with probability at least \(1 - 2/d^2\),
\[
\left\| \left( \hat{\Sigma}_X - \Sigma_X^* \right) U^* Q^k \right\|_{2,\infty} \leq 11 \left\| U^* T \Sigma_X^* U^*\right\|_{2,\infty}^{1/2} \left( \frac{r \log d}{n_X} \right)^{1/2}.
\]
Since \(\|U^*\|^2_2 \leq \sigma_1^{R^*}\), with probability at least \(1 - 2/d^2\),
\[
\|\hat{A}^*\|_{2,\infty} \leq 11 \left( \frac{r \sigma_1^{R^*}}{\sigma_1^X} \right)^{1/2} \left( \frac{\log d}{n_X} \right)^{1/2} + \frac{\|\Sigma_X^*\|_1 \left( \beta \frac{d}{d} \right)^{1/2}}{d},
\]
(D.8)
We bound \(\|\hat{A}^* - A^*\|_{2,\infty}\) analogously. We have
\[
\left\| \left( \hat{\Sigma}_X - \Sigma_X^* \right) (U^k - U^* Q^k) \right\|_{2,\infty} = \left\| \left( \hat{\Sigma}_X - \Sigma_X^* \right) (U^k - U^* Q^k) \right\|_{2,\infty}^T
\]
\[
\leq \left\| \left( \hat{\Sigma}_X - \Sigma_X^* \right) (U^k - U^* Q^k) \right\|_{2,\infty}^T + \|\Sigma_X^*\|_1 \left\| U^k T \right\|_{2,\infty} + \|U^* T\|_{2,\infty}
\]
\[
\leq \left\| \left( \hat{\Sigma}_X - \Sigma_X^* \right) (U^k - U^* Q^k) \right\|_{2,\infty}^T + 4\|\Sigma_X^*\|_1 \left( \beta \frac{d}{d} \right)^{1/2}
\]
(D.9)
where the second inequality is due to Lemma F.9, the last inequality is due to the incoherence condition and assumption that \(U^k \in \mathcal{U}(9\sigma_1^{R^*})\). For the first term in (D.9), from Lemma F.13, with probability at least \(1 - 2/d^2\),
\[
\left\| \left( \hat{\Sigma}_X - \Sigma_X^* \right) (U^k - U^* Q^k) \right\|_{2,\infty} \leq 11 \left\| U^k T \Sigma_X^* (U^k - U^* Q^k)\right\|_{2\sigma_1^X}^{1/2} \left( \frac{r \log d}{n_X} \right)^{1/2}
\]
Since \(d(U^k, U^*) \leq \sigma_1^{R^*} / 2\) and \(d(U^k, U^*) \geq \|U^k - U^* Q^k\|_2\),
\[
\frac{\sqrt{\sigma_1^{R^*}}}{2} \leq \sqrt{\sigma_1^{R^*}} - d(U^k, U^*) \leq \|U^k\|_2 \leq \sigma_1^{R^*} + d(U^k, U^*) \leq 3\sqrt{\sigma_1^{R^*}} / 2.
\]
(D.10)
Thus,
\[
\left\| U^k - U^* Q^k \right\|_2 \Sigma_X^* (U^k - U^* Q^k) \leq 2.5^2 \sigma_1^{R^*} \sigma_1^X,
\]
and further
\[
\left\| \left( \hat{\Sigma}_X - \Sigma_X^* \right) (U^k - U^* Q^k) \right\|_{2,\infty} \leq 28\sigma_1^X \left( \frac{r \sigma_1^{R^*} \log d}{n_X} \right)^{1/2}.
\]
Combining with (D.9), with probability at least 1 \(- 2/d^2\),
\[
\|(\tilde{A}^k - \tilde{A}^*)^\top\|_{2,\infty} \leq 28 \left( r \sigma_1^{R^k} \right)^{1/2} \left\{ \sigma_X \left( \frac{\log d}{n_X} \right)^{1/2} + \|\Sigma_X^*\|_1 \left( \frac{\beta}{d} \right)^{1/2} \right\}. \tag{D.11}
\]
Similar to (D.7), (D.8), (D.11), we can bound \(\|\tilde{B}^k - \tilde{B}^*\|_F^2\), \(\|\tilde{B}^*\|_{2,\infty}\), and \(\|(\tilde{B}^k - \tilde{B}^*)^\top\|_{2,\infty}\). Putting everything together and combining with (D.6), we know for some constant \(C_2 > 0\), with probability at least 1 \(- 8/d^2\),
\[
\|\mathcal{P}_{\Omega^k} \left\{ \tilde{S}_X (U^k \Lambda^0 U^k - U^* \Lambda^* U^*) \tilde{\Sigma}_Y \right\} \|_F^2 \leq C_2 \left( 1 + \gamma_1 \right) \alpha r \sigma_1^{R^k} \gamma_1 d^2 (U^k, U^*), \tag{D.12}
\]
where \(\gamma_1\) is defined (A.2). Together with (D.5), we have
\[
\mathcal{I}_{13} \geq - \frac{C_{13}}{2} \|S^k - S^*\|_F^2 - \frac{C_2}{2C_{13}} \left( 1 + \gamma_1 \right) \alpha r \sigma_1^{R^k} \gamma_1 d^2 (U^k, U^*). \tag{D.13}
\]
Setting \(C_{12}\) and \(C_{13}\) in (D.4) and (D.13) as
\[
C_{12} = \frac{9}{8}, \quad C_{13} = \frac{9}{8},
\]
and combining with (D.2), (D.3), (D.4), (D.13), we have
\[
\mathcal{I}_1 \geq \frac{9}{8} \left( \frac{\gamma_3}{\gamma_2} \right) \|S^k - S^*\|_F^2 + \frac{4}{\gamma_2} \|\nabla_S \tilde{\Sigma}_n (S^k, U^k, \Lambda^0) - \nabla_S \tilde{\Sigma}_n (S^*, U^k, \Lambda^0)\|_F^2
\]
\[
- \frac{4}{\gamma_3} \left\{ C_2 \cdot (1 + \gamma_1) \alpha r \sigma_1^{R^k} \gamma_1 d^2 (U^k, U^*) + (1 + \gamma_2) s \|\nabla_S \tilde{\Sigma}_n (S^*, R^*)\|_{\infty,\infty}^2 \right\}
\]
with probability at least 1 \(- C_3/d^2\) for some constant \(C_2, C_3\) large enough. This completes the proof. \(\square\)

**Lemma D.2.** Under the conditions of Lemma A.1, we have
\[
\mathcal{I}_2 \leq 3 \|\nabla_S \tilde{\Sigma}_n (S^*, U^*, \Lambda^*) - \nabla_S \tilde{\Sigma}_n (S^*, R^*)\|_F^2 + 3(1 + 2\gamma_2) s \|\nabla_S \tilde{\Sigma}_n (S^*, R^*)\|_{\infty,\infty}^2 + \frac{27r^2 \sigma_1^{R^k}}{4} \|U^k \Lambda^0 U^k - U^* \Lambda^* U^*\|_F^2
\]
with probability at least 1 \(- 1/d^2\).

**Proof.** Using the fact that \(\nabla_S \tilde{\Sigma}_n (S^*, U^*, \Lambda^*) = \nabla_S \tilde{\Sigma}_n (S^*, R^*)\) and \(|\Omega^k| \leq (2\gamma_2 + 1) s\), by definition of \(\mathcal{I}_2\),
\[
\mathcal{I}_2 = \|\mathcal{P}_{\Omega^k} \left\{ \nabla_S \tilde{\Sigma}_n (S^k, U^k, \Lambda^0) \right\} \|_F^2
\]
\[
\leq 3 \|\mathcal{P}_{\Omega^k} \left\{ \nabla_S \tilde{\Sigma}_n (S^k, U^k, \Lambda^0) - \nabla_S \tilde{\Sigma}_n (S^*, U^k, \Lambda^0) \right\} \|_F^2 + 3 \|\mathcal{P}_{\Omega^k} \left\{ \nabla_S \tilde{\Sigma}_n (S^*, R^*) \right\} \|_F^2
\]
\[
\leq 3 \|\nabla_S \tilde{\Sigma}_n (S^k, U^k, \Lambda^0) - \nabla_S \tilde{\Sigma}_n (S^*, U^k, \Lambda^0)\|_F^2 + 3(1 + 2\gamma_2) s \|\nabla_S \tilde{\Sigma}_n (S^*, R^*)\|_{\infty,\infty}^2
\]
\[
+ \frac{3}{4} \|\mathcal{P}_{\Omega^k} \left\{ \tilde{\Sigma}_X (U^k \Lambda^0 U^k - U^* \Lambda^* U^*) \tilde{\Sigma}_Y + \tilde{\Sigma}_Y (U^k \Lambda^0 U^k - U^* \Lambda^* U^*) \tilde{\Sigma}_X \right\} \|_F^2
\]
\[
\leq 3 \|\nabla_S \tilde{\Sigma}_n (S^k, U^k, \Lambda^0) - \nabla_S \tilde{\Sigma}_n (S^*, U^k, \Lambda^0)\|_F^2 + 3(1 + 2\gamma_2) s \|\nabla_S \tilde{\Sigma}_n (S^*, R^*)\|_{\infty,\infty}^2
\]
\[
+ 3 \|\tilde{\Sigma}_X (U^k \Lambda^0 U^k - U^* \Lambda^* U^*) \tilde{\Sigma}_Y \|_F^2,
\]
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where the second inequality is due to the fact that $|\Omega^k| \leq (1 + 2\gamma_2)s$. For the last term above, using Hölder’s inequality and event $\mathcal{E}$ in (D.1),
\[
\|\left(\hat{\Sigma}_X(U^k\Lambda^0U^k) - U^*\Lambda^*U^*\right)\Sigma_Y\|^2_F \leq \frac{9(\sigma_1^X\sigma_1^Y)^2}{4} \|U^k\Lambda^0U^k - U^*\Lambda^*U^*\|^2_F,
\]
with probability at least $1 - 1/d^2$. The proof follows by combining the last two displays.

**Lemma D.3.** Under the conditions of Lemma A.2, for any $C_{31}, C_{32,1}, C_{32,2}, C_{33,1}, C_{33,2} > 0$
\[
\mathcal{I}_3 \geq \left\{ \frac{9\gamma_3}{4\gamma_2} - \frac{(2r)^{1/2} C_{33,1}}{2} \right\} \left[ \|R^k - R^*\|^2_F - \left( \frac{C_{32,1}}{2} + \frac{9\gamma_3}{4} \right) \left( \frac{C_{32,2}}{8} \right) \|S^k - S^*\|^2_F \right.
\]
\[
+ \left( 4 - \frac{1}{2C_{31}} \right) \|\nabla_R \mathcal{L}_n(S^k, R^k) - \nabla_R \mathcal{L}_n(S^k, R^k)\|^2_F
\]
\[
- \frac{C_1}{2C_{32,1}} \left[ (1 + \gamma_1) \alpha \sigma_1^R \gamma_1 \delta^2(U^k, U^*) - \left( \frac{C_{31}}{2} + \frac{(2r)^{1/2} (C_{33,1} + C_{33,2})}{2C_{33,1}C_{33,2}} \|\nabla_R \mathcal{L}_n(S^k, R^k)\|^2_F \right) \right]
\]
\[
\left. \left. d^2(U^k, U^*) \right) \right)\right),
\]
with probability at least $1 - C_2/d^2$ for some fixed constants $C_1, C_2 > 0$ large enough.

**Proof.** Let $R^k = U^k\Lambda^0U^k$ and $\Theta^k = U^k - U^*Q^k$. Using formulas in (A.1),
\[
\mathcal{I}_3 = \left( U^k - U^*Q^k, \nabla_R \mathcal{L}_n(S^k, U^k, \Lambda^0) \right) = \left( U^k - U^*Q^k, 2\nabla_R \mathcal{L}_n(S^k, R^k)U^k\Lambda^0 \right)
\]
\[
= 2\left( U^k - U^*Q^k \right)\Lambda^0U^k, \nabla_R \mathcal{L}_n(S^k, R^k) = \langle R^k - R^* + \Theta^k\Lambda^*\Theta^k, \nabla_R \mathcal{L}_n(S^k, R^k) \rangle
\]
\[
= \mathcal{I}_{31} + \mathcal{I}_{32} + \mathcal{I}_{33},
\]
(D.14)

where
\[
\mathcal{I}_{31} = \langle R^k - R^* + \Theta^k\Lambda^*\Theta^k, \nabla_R \mathcal{L}_n(S^k, R^k) - \nabla_R \mathcal{L}_n(S^k, R^*) \rangle,
\]
\[
\mathcal{I}_{32} = \langle R^k - R^* + \Theta^k\Lambda^*\Theta^k, \nabla_R \mathcal{L}_n(S^k, R^*) - \nabla_R \mathcal{L}_n(S^k, R^*) \rangle,
\]
\[
\mathcal{I}_{33} = \langle R^k - R^* + \Theta^k\Lambda^*\Theta^k, \nabla_R \mathcal{L}_n(S^k, R^*) \rangle.
\]

We bound the three terms separately. First, using (A.1)
\[
\mathcal{I}_{31} = \langle R^k - R^* + \Theta^k\Lambda^*\Theta^k, \frac{1}{2} \hat{\Sigma}_X(R^k - R^*)\hat{\Sigma}_Y + \frac{1}{2} \hat{\Sigma}_Y(R^k - R^*)\hat{\Sigma}_X \rangle
\]
\[
= \langle R^k - R^*, \hat{\Sigma}_X(R^k - R^*)\hat{\Sigma}_Y \rangle + \langle \Theta^k\Lambda^*\Theta^k, \frac{1}{2} \hat{\Sigma}_X(R^k - R^*)\hat{\Sigma}_Y + \frac{1}{2} \hat{\Sigma}_Y(R^k - R^*)\hat{\Sigma}_X \rangle.
\]

By Lemma F.8,
\[
\langle R^k - R^*, \hat{\Sigma}_X(R^k - R^*)\hat{\Sigma}_Y \rangle \geq \frac{9\gamma_3}{4\gamma_2} \|R^k - R^*\|^2_F + \frac{1}{4} \|\hat{\Sigma}_X(R^k - R^*)\hat{\Sigma}_Y + \hat{\Sigma}_Y(R^k - R^*)\hat{\Sigma}_X\|^2_F
\]
\[
= \frac{9\gamma_3}{4\gamma_2} \|R^k - R^*\|^2_F + \frac{4}{4} \|\nabla_R \mathcal{L}_n(S^k, R^k) - \nabla_R \mathcal{L}_n(S^k, R^k)\|^2_F.
\]

For any $C_{31} > 0$ to be determined later,
\[
\langle \Theta^k\Lambda^*\Theta^k, \frac{1}{2} \hat{\Sigma}_X(R^k - R^*)\hat{\Sigma}_Y + \frac{1}{2} \hat{\Sigma}_Y(R^k - R^*)\hat{\Sigma}_X \rangle
\]
\[
\geq -\|\Theta^k\Lambda^*\Theta^k\|^2_F \|\nabla_R \mathcal{L}_n(S^k, R^k) - \nabla_R \mathcal{L}_n(S^k, R^k)\|_F
\]
\[
\geq -\frac{C_{31}}{2} \|\Theta^k\|^4_F - \frac{1}{2C_{31}} \|\nabla_R \mathcal{L}_n(S^k, R^k) - \nabla_R \mathcal{L}_n(S^k, R^k)\|^2_F.
\]
Combining the last two inequalities,
\[
\mathcal{I}_{31} \geq \frac{9\gamma^3}{4\bar{Y}_2} \| R^k - R^* \|_2^2 - \frac{C_{31}}{2} d^4(U^k, U^*) + \left( \frac{4}{2} - \frac{1}{2C_{31}} \right) \| \nabla_R \mathcal{L}_n(S^k, R^k) - \nabla_R \mathcal{L}_n(S^*, R^*) \|_F^2. \tag{D.15}
\]

Next, we bound the term \( \mathcal{I}_{32} \) as (notation of \( \bar{\Omega}^k \) is in (B.1))
\[
\mathcal{I}_{32} = \langle R^k - R^* + \Theta^k \Lambda^* \Theta^k, 1 \bar{\Omega} \rangle X (S^k - S^*) \bar{\Omega} + \frac{1}{2} \bar{\Omega} Y (S^k - S^*) \bar{\Omega} X \rangle
\]
\[
\geq \frac{1}{2} \mathcal{P}_{\bar{\Omega}^k} \left\{ \bar{\Omega} X (R^k - R^* + \Theta^k \Lambda^* \Theta^k) \bar{\Omega} Y \right\} + \mathcal{P}_{\bar{\Omega}^k} \left\{ \bar{\Omega} Y (R^k - R^* + \Theta^k \Lambda^* \Theta^k) \bar{\Omega} X \right\}, S^k - S^*angle
\]
\[
\geq -\| S^k - S^* \|_F \mathcal{P}_{\bar{\Omega}^k} \left\{ \bar{\Omega} X (R^k - R^*) \bar{\Omega} Y \right\} \|_F - \| S^k - S^* \|_F \mathcal{P}_{\bar{\Omega}^k} \left\{ \bar{\Omega} X \Theta^k \Lambda^* \Theta^k \bar{\Omega} Y \right\} \|_F
\]
\[
\geq -\| S^k - S^* \|_F \mathcal{P}_{\bar{\Omega}^k} \left\{ \bar{\Omega} X (R^k - R^*) \bar{\Omega} Y \right\} \|_F - \frac{9\sigma_1^X \sigma_1^Y}{4} \| S^k - S^* \|_F \mathcal{P}_{\bar{\Omega}^k} \left\{ \bar{\Omega} X \Lambda^k \Theta^k \bar{\Omega} Y \right\} \|_F
\]
where the first inequality is due to Lemma F.7. Using the same derivation as in (D.5) to obtain (D.12), for any \( C_{32,1} > 0 \),
\[
-\| S^k - S^* \|_F \mathcal{P}_{\bar{\Omega}^k} \left\{ \bar{\Omega} X (R^k - R^*) \bar{\Omega} Y \right\} \|_F \geq -\frac{C_{32,1}}{2} \| S^k - S^* \|_F^2 - \frac{C_1}{2C_{32,1}} (1 + \gamma_1) \alpha r \sigma_1^R \bar{Y} d^2(U^k, U^*).
\]
with probability at least \( 1 - 8/d^2 \) for some constant \( C_1 \). Also, for any \( C_{32,2} > 0 \),
\[
-\frac{9\sigma_1^X \sigma_1^Y}{8} \| S^k - S^* \|_F \mathcal{P}_{\bar{\Omega}^k} \left\{ \bar{\Omega} X \Theta^k \Lambda^k \Theta^k \bar{\Omega} Y \right\} \|_F \geq -\frac{9\sigma_1^X \sigma_1^Y}{8} \| S^k - S^* \|_F^2 - \frac{9\sigma_1^X \sigma_1^Y}{8} d^4(U^k, U^*).
\]
Therefore, with probability at least \( 1 - 8/d^2 \),
\[
\mathcal{I}_{32} \geq -\left( \frac{C_{32,1}}{2} + \frac{9\sigma_1^X \sigma_1^Y C_{32,2}}{8} \right) \| S^k - S^* \|_F^2 - \frac{C_1}{2C_{32,1}} (1 + \gamma_1) \alpha r \sigma_1^R \bar{Y} d^2(U^k, U^*)
\]
\[
-\frac{9\sigma_1^X \sigma_1^Y}{8} d^4(U^k, U^*). \tag{D.16}
\]
For the term \( \mathcal{I}_{33} \), for any \( C_{33,1}, C_{33,2} > 0 \),
\[
\mathcal{I}_{33} = \langle R^k - R^* + \Theta^k \Lambda^* \Theta^k, \nabla_R \mathcal{L}_n(S^*, R^*) \rangle
\]
\[
\geq -\| \nabla_R \mathcal{L}_n(S^*, R^*) \|_2 (\| R^k - R^* \|_F + \| \Theta^k \Lambda^* \Theta^k \|_F)
\]
\[
\geq - (2r)^{1/2} \| \nabla_R \mathcal{L}_n(S^*, R^*) \|_2 (\| R^k - R^* \|_F + \| \Theta^k \Lambda^* \Theta^k \|_F)
\]
\[
\geq - (2r)^{1/2} \left( \frac{C_{33,1} + C_{33,2}}{2C_{33,1}C_{33,2}} \| \nabla_R \mathcal{L}_n(S^*, R^*) \|_2^2 + \frac{C_{33,1}}{2} \| R^k - R^* \|_F^2 + \frac{C_{33,2}}{2} d^4(U^k, U^*) \right). \tag{D.17}
\]
Combining (D.14), (D.15), (D.16), and (D.17), we complete the proof. □
E Proofs of Other Lemmas

E.1 Proof of Lemma 1

Since $U^*$ has orthogonal columns, $\sigma_r^2$ ($\sigma_r^2$) is the largest (smallest) singular value of $U^* \Lambda^* U^{\ast T}$. Then, for any $Q \in \mathbb{C}^{r \times r}$,

$$
\|U \Lambda^* U^{\ast T} - U^* \Lambda^* U^*\|_F
= \|U \Lambda^* U^{\ast T} - U^* Q \Lambda^* Q^T U^{\ast T}\|_F
= \|(U - U^* Q) \Lambda^* (U - U^* Q)^T + U^* Q \Lambda^* (U - U^* Q)^T\|_F
\leq 3 \|U - U^* Q\|_F^2 + 6 \sigma_1^2 \|U - U^* Q\|_F
\leq 3 \|U - U^* Q\|_F^2 + 6 \sigma_1^2 \|U - U^* Q\|_F^2.
$$

We minimize the right hand side over $Q$ to obtain

$$
\inf_{Q \in \mathbb{C}^{r \times r}} \{3 \|U - U^* Q\|_F^2 + 6 \sigma_1^2 \|U - U^* Q\|_F \}
= 3 \inf_{Q \in \mathbb{C}^{r \times r}} \|U - U^* Q\|_F^2 + 6 \sigma_1^2 \inf_{Q \in \mathbb{C}^{r \times r}} \|U - U^* Q\|_F
= 3 d^4(U, U^*) + 6 \sigma_1^2 d^2(U, U^*)
\leq 9 \sigma_1^2 d^2(U, U^*),
$$

which completes the proof of the first part. For the second part of the result, based on Lemma F.4,

$$
\inf_{Q \in \mathbb{C}^{r \times r}} \|U - U^* Q\|_F^2 + \|U \Lambda^* - U^* \Lambda^* Q\|_F^2
= \inf_{Q \in \mathbb{C}^{r \times r}} -2 \text{tr}(U^T U^* Q) - 2 \text{tr}(\Lambda^* U^T U^* \Lambda^* Q)
= -2 \sup_{Q \in \mathbb{C}^{r \times r}} \text{tr}(U^T U^* + \Lambda^* U^T U^* \Lambda^*) Q
\leq \frac{2}{(\sqrt{2} - 1) \sigma_1^2} \|U \Lambda^* U^{\ast T} - U^* \Lambda^* U^{\ast T}\|_F^2.
$$

Let $U = (U_1, U_2)$ with $U_1 \in \mathbb{R}^{d \times r_1}$ and $U_2 \in \mathbb{R}^{d \times (r - r_1)}$, and analogously $U^* = (U_1^*, U_2^*)$. Furthermore, suppose the following singular value decompositions $U_1^T U_1^* = A_1 \Sigma_1 B_1^T$, $U_2^T U_2^* = A_2 \Sigma_2 B_2^T$. Then,

$$
U^T U^* + \Lambda^* U^T U^* \Lambda^* = 2 \begin{pmatrix}
U_1^T U_1^* \\
U_2^T U_2^*
\end{pmatrix}
= 2 \begin{pmatrix}
A_1 & A_2 \\
\Sigma_1 & \Sigma_2
\end{pmatrix}
\begin{pmatrix}
B_1^T \\
B_2^T
\end{pmatrix}
.$$

The optimal $Q$ that achieves the supremum in (E.1) is

$$
Q = \begin{pmatrix}
B_1 A_1^T \\
B_2 A_2^T
\end{pmatrix}.
$$

Since $Q \in \mathbb{C}^{r \times r}$,

$$
\inf_{Q \in \mathbb{C}^{r \times r}} \|U - U^* Q\|_F^2 + \|U \Lambda^* - U^* \Lambda^* Q\|_F^2
= \inf_{Q \in \mathbb{C}^{r \times r}} \|U - U^* Q\|_F^2 + \|U \Lambda^* - U^* \Lambda^* Q\|_F^2
= \inf_{Q \in \mathbb{C}^{r \times r}} \|U - U^* Q\|_F^2 + \|U - U^* \Lambda^* Q\|_F^2
\leq 2 \inf_{Q \in \mathbb{C}^{r \times r}} \|U - U^* Q\|_F^2
= 2 d^2(U, U^*).
$$

Combining (E.2) with (E.1) completes the proof.
E.2 Proof of Lemma 2

By Sylvester’s law of inertia (cf. Theorem 4.5.8 Horn and Johnson, 2013), $R^*$ has $r_1$ positive eigenvalues and $r - r_1$ negative eigenvalues, denoted as $\lambda_i^* > \cdots > \lambda_{r_1}^* > 0 > \lambda_{r_1+1}^* \geq \cdots \lambda_r^*$. Therefore $\sigma_{R^*} = \min\{\lambda_i^*, |\lambda_{r_1+1}^*|\}$. Similarly, we denote the eigenvalue of $R$ as $\lambda_i \geq \cdots \geq \lambda_{\tilde{r}_1} \geq \epsilon_1 \geq \cdots \geq \epsilon_{d-r} \geq \lambda_{\tilde{r}_1+1} \geq \cdots \lambda_r$, where $\{\lambda_i\}_{i=1}^r$ denote the $r$ eigenvalues with the largest magnitude, $\tilde{r}_1$ of which are positive. In particular, $|\epsilon_i| \leq |\lambda_j|$, $i = 1, \ldots, d-r$, $j = 1, \ldots, r$.

Suppose $\tilde{r}_1 < r_1$. Then $\epsilon_1$ will correspond to $\lambda_j^*$ for some $j \in \{\tilde{r}_1 + 1, \ldots, r\}$. Based on Weyl’s inequality (cf. Theorem 4.3.1 Horn and Johnson, 2013),

$$\epsilon_1 \geq \lambda_j^* - |\epsilon_1 - \lambda_j^*| \geq \lambda_j^* - \|R - R^*\|_2 \geq \lambda_j^* - \|R - R^*\|_2 \geq 2\sigma_{R^*}/3.$$  \hspace{1cm} (E.3)

On the other hand, $\lambda_j^*$ for some $j \in \{\tilde{r}_1 + 1, \ldots, r\}$ corresponds to the zero eigenvalue of $R^*$. Then, by Weyl’s inequality,

$$|\lambda_j^*| \leq \|R - R^*\|_2 \leq \sigma_{R^*}/3.$$  \hspace{1cm} (E.4)

Combining (E.3) and (E.4), $\epsilon_1 > |\lambda_j^*|$, which is a contradiction. Using the same technique, assuming that $\tilde{r}_1 > r_1$ allows us to show that $\epsilon_{d-r}$ will be one of the $r$ largest eigenvalues of $R$, leading to a contradiction. Therefore $\tilde{r}_1 = r_1$.

F Auxiliary Results

Lemma F.1 (Lemma B.3 in Zhang et al. (2018)). Let $S^* \in S^{d \times d}$ satisfy $\|S^*\|_{0,c} \leq \alpha d$. Then for any $S \in S^{d \times d}$ and $\gamma > 1$, we have $\mathcal{T}_{\gamma \alpha}(S) \in S^{d \times d}$ and

$$\|\mathcal{T}_{\gamma \alpha}(S) - S^*\|_F^2 \leq \left\{1 + \left(\frac{2}{\gamma-1}\right)^{1/2}\right\}^2 \|S - S^*\|_F^2.$$  \hspace{1cm} (E.5)

Lemma F.2 (Lemma 3.3 in Li et al. (2016)). Let $S^* \in S^{d \times d}$ satisfy $\|S^*\|_{0,1} \leq s$. Then for any $S \in S^{d \times d}$ and $\gamma > 1$, we have $\mathcal{J}_{\gamma s}(S) \in S^{d \times d}$ and

$$\|\mathcal{J}_{\gamma s}(S) - S^*\|_F^2 \leq \left\{1 + \frac{2}{(\gamma - 1)^{1/2}}\right\} \|S - S^*\|_F^2.$$  \hspace{1cm} (E.6)

Lemma F.3 (Proposition 1.1 in Hsu et al. (2012)). Let $X \sim \mathcal{N}(0, \Sigma)$. Then, for any $\delta > 0$,

$$\Pr \left(\|X\|_2^2 \leq \|\Sigma\|_2 \left[d + 2 \{d \log (1/\delta)\}^{1/2} + 2 \log (1/\delta)\] \right] \geq 1 - \delta.$$  \hspace{1cm} (E.7)

Lemma F.4 (Lemma 5.14 in Tu et al. (2016)). Let $M_1, M_2 \in \mathbb{R}^{d_1 \times d_2}$ be rank-$r$ matrices with the reduced singular value decomposition $M_i = U_i \Sigma_i V_i^T$, $i = 1, 2$. Let $X_i = U_i \Sigma_i^{1/2}$ and $Y_i = V_i \Sigma_i^{1/2}$. If $\|M_2 - M_1\|_2 \leq \sigma_r(M_1)/2$, then

$$\inf_{Q \in \mathbb{Q}^{r \times r}} \|X_2 - X_1 Q\|_F^2 + \|Y_2 - Y_1 Q\|_F^2 \leq \frac{2}{\sqrt{2} - 1} \frac{\|M_2 - M_1\|_F^2}{\sigma_r(M_1)},$$  \hspace{1cm} (E.8)

where $\sigma_r(M_1)$ denotes the $r$-th singular value of $M_1$.  

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Lemma F.5 (Lemma C.1 in Wang et al. (2017)). Let $U \in \mathbb{R}^{d_1 \times r}$, $V \in \mathbb{R}^{d_2 \times r}$,

$$Z = \begin{pmatrix} U \\ V \end{pmatrix}, \quad \breve{Z} = \begin{pmatrix} U \\ -V \end{pmatrix}.$$ 

For any $Z^* \in \mathbb{R}^{(d_1 + d_2) \times r}$, we define $Q = \arg \inf_{Q \in \mathbb{Q}^{r \times r}} \|Z - Z^*Q\|_F$. Then

$$\langle \breve{Z} \breve{Z}^T Z, Z - Z^*Q \rangle \geq \frac{1}{4} \| \breve{Z} \breve{Z}^T Z \|_F^2 - \frac{1}{4} \|Z - Z^*Q\|_F^2.$$ 

Lemma F.6 (Lemma 5.4 in Tu et al. (2016)). Let $U, V \in \mathbb{R}^{d \times r}$ and let $\sigma_r$ be the $r$-th singular value of $V$. Then

$$\inf_{Q \in \mathbb{Q}^{r \times r}} \|U - VQ\|_F^2 \leq \frac{1}{2(\sqrt{2} - 1)\sigma_r^2} \|UU^T - VV^T\|_F^2.$$ 

Lemma F.7. Suppose $\Omega \subseteq [d] \times [d]$ is a symmetric index set. For any matrix $A \in \mathbb{R}^{d \times d}$, we have $(\mathcal{P}_\Omega(A))^T = \mathcal{P}_\Omega(A^T)$.

Proof. For any $(i, j) \in [d] \times [d]$, we have

$$[\mathcal{P}_\Omega(A^T)]_{i,j} = \begin{cases} [A^T]_{i,j} & \text{if } (i, j) \in \Omega \\ A_{j,i} & \text{if } (i, j) \notin \Omega \end{cases}$$ 

This completes the proof. \qed

Lemma F.8. Suppose $S_1, S_2 \in \mathbb{S}^{d \times d}$ are two symmetric matrices and $\Sigma_1, \Sigma_2 \in \mathbb{R}^{d \times d}$ satisfy $0 < \sigma_{i,d} I_d \leq \Sigma_i \leq \sigma_{i,1} I_d$ for $i = 1, 2$. Then

$$\langle \Sigma_1 (S_1 - S_2) \Sigma_2, S_1 - S_2 \rangle \geq \frac{1}{\sigma_{1,1} \sigma_{2,1} \sigma_{1,d} \sigma_{2,d}} \| S_1 - S_2 \|_F^2 + \frac{1}{4(\sigma_{1,1} \sigma_{2,1} + \sigma_{1,d} \sigma_{2,d})} \| \Sigma_1 (S_1 - S_2) \Sigma_2 + \Sigma_2 (S_1 - S_2) \Sigma_1 \|_F^2.$$ 

Proof. For a symmetric matrix $S \in \mathbb{S}^{d \times d}$, let $\mathcal{F}(S) = \frac{1}{2} \text{tr}(S \Sigma_1 S \Sigma_2)$. Then, for any $S_1, S_2 \in \mathbb{S}^{d \times d}$,

$$\mathcal{F}(S_1) - \mathcal{F}(S_2) - \langle \nabla \mathcal{F}(S_2), S_1 - S_2 \rangle = \frac{1}{2} \text{tr} \left[ (S_1 - S_2) \Sigma_1 (S_1 - S_2) \Sigma_2 \right].$$ 

Therefore,

$$\frac{\sigma_{1,d} \sigma_{2,d}}{2} \| S_1 - S_2 \|_F^2 \leq \mathcal{F}(S_1) - \mathcal{F}(S_2) - \langle \nabla \mathcal{F}(S_2), S_1 - S_2 \rangle \leq \frac{\sigma_{1,1} \sigma_{2,1}}{2} \| S_1 - S_2 \|_F^2,$$

implying that $\mathcal{F}(\cdot)$ is a $\sigma_{1,1} \sigma_{2,1}$-smooth and $\sigma_{1,d} \sigma_{2,d}$-strongly convex function. Furthermore,

$$\langle \Sigma_1 (S_1 - S_2) \Sigma_2, S_1 - S_2 \rangle = \langle \nabla \mathcal{F}(S_1) - \nabla \mathcal{F}(S_2), S_1 - S_2 \rangle \geq \frac{\sigma_{1,1} \sigma_{2,1} \sigma_{1,d} \sigma_{2,d}}{\sigma_{1,1} \sigma_{2,1} + \sigma_{1,d} \sigma_{2,d}} \| S_1 - S_2 \|_F^2 + \frac{4}{\sigma_{1,1} \sigma_{2,1} + \sigma_{1,d} \sigma_{2,d}} \| \Sigma_1 (S_1 - S_2) \Sigma_2 + \Sigma_2 (S_1 - S_2) \Sigma_1 \|_F^2,$$

where the inequality is due to Lemma 3.5 in Bubeck (2015). \qed

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Lemma F.9. Let $A \in \mathbb{R}^{d \times d}$, $U \in \mathbb{R}^{d \times r}$, and $Q \in \mathbb{Q}^{r \times r}$. Then $\left\| (AUQ)^T \right\|_{2,\infty} \leq \left\| A^T \right\|_1 \left\| U^T \right\|_{2,\infty}$.

Proof. Note that $\left\| (AUQ)^T \right\|_{2,\infty} = \left\| Q^T U^T A^T \right\|_{2,\infty} = \left\| U^T A^T \right\|_{2,\infty}$. Suppose $U^T = (U_1, \ldots, U_d)$, where $U_i \in \mathbb{R}^r$ is the $i$th row of $U$. Then

$$\left\| U^T A^T \right\|_{2,\infty} = \max_{j \in [d]} \left\| \sum_{i=1}^{d} A_{j,i} U_i \right\|_2 \leq \max_{j \in [d]} \sum_{i=1}^{d} |A_{j,i}| \max_{i \in [d]} \left\| U_i \right\|_2 \leq \left\| U^T \right\|_{2,\infty} \max_{j \in [d]} \left\| A_j \right\|_1 = \left\| U^T \right\|_{2,\infty} \left\| A^T \right\|_1 = \left\| U^T \right\|_{2,\infty} \left\| A^T \right\|_1.$$ 

\qed

Lemma F.10. Let $U^* \in \mathbb{R}^{d \times r}$ have orthogonal columns with the $r$-th singular value being $\sigma_r$. For any $U \in \mathbb{R}^{d \times r}$ and $r_1 \in \{0, \ldots, r\}$, let $Q = \arg \inf_{Q \in \mathbb{Q}^{r \times r}} \| U - U^* Q \|$ and $\Lambda = \text{diag}(I_{r_1}, -I_{r-r_1})$. Then the following two inequalities hold

$$\langle U - U^* Q, U(U^T U - \Lambda U^T U\Lambda) \rangle \geq \frac{1}{8} \| U^T U - \Lambda U^T U\Lambda \|_F^2 - \frac{1}{2} d^4(U, U^*) ,$$

$$\| U^T U - \Lambda U^T U\Lambda \|_F^2 \geq 8(\sqrt{2} - 1)\sigma_r^2 d^2(U, U^*) - 4\| U\Lambda U - U^* \Lambda U^* \|_F^2.$$ 

Furthermore, we have

$$\langle U - U^* Q, U(U^T U - \Lambda U^T U\Lambda) \rangle \geq (\sqrt{2} - 1)\sigma_r^2 d^2(U, U^*) - \frac{1}{2} \| U\Lambda U - U^* \Lambda U^* \|_F^2 - \frac{1}{2} d^4(U, U^*).$$

Proof. We only prove two inequalities. The last argument comes from two inequalities immediately. Let $V = U\Lambda$, $V^* = U^*\Lambda$, and 

$$Z = \begin{pmatrix} U \\ V \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} U \\ -V \end{pmatrix}, \quad Z^* = \begin{pmatrix} U^* \\ V^* \end{pmatrix}, \quad \tilde{Z}^* = \begin{pmatrix} U^* \\ -V^* \end{pmatrix}.$$ 

The $r$-th singular value of $Z^*$ is $\sqrt{2}\sigma_r$. From (E.1) and (E.2), we know $Q = \arg \inf_{Q \in \mathbb{Q}^{r \times r}} \| Z - Z^* Q \|_F$. By Lemma F.5,

$$\langle \tilde{Z} \tilde{Z}^T Z, Z - Z^* Q \rangle \geq \frac{1}{4} \| \tilde{Z} \tilde{Z}^T Z \|_F^2 - \frac{1}{4} \| Z - Z^* Q \|_F^2. \quad \text{(F.1)}$$

We can rewrite the left hand side of (F.1) as

$$\langle \tilde{Z} \tilde{Z}^T Z, Z - Z^* Q \rangle = \text{tr}\left( (U^T U - V^T V) \tilde{Z}^T (Z - Z^* Q) \right) = \text{tr}\left( (U^T U - V^T V) (U^T (U - U^* Q) - V^T (V - V^* Q)) \right) = \text{tr}\left( (U^T U - V^T V) U^T (U - U^* Q) - \text{tr}(U^T U - V^T V)V^T (V - V^* Q) \right).$$

Since $\Lambda Q \Lambda = Q$ and by the definition of $V$, the second term in above equation can be written as

$$-\text{tr}(U^T U - V^T V)V^T (V - V^* Q) = -\text{tr}(U^T U - V^T V) \Lambda U^T (U - U^* \Lambda Q \Lambda) \Lambda = -\text{tr}(U^T U - V^T V) \Lambda U^T (U - U^* Q) \Lambda = -\text{tr}(\Lambda (U^T U - V^T V) U^T (U - U^* Q)) = \text{tr}(U^T U - V^T V)U^T (U - U^* Q),$$

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which is the same as the first term. Using the definition of $V$ and plugging the above display into (F.2), we have
\[
\langle U - U^*Q, U(U^TU - \Lambda U^TU) \rangle = \frac{1}{2} \langle \tilde{Z} \tilde{Z}^T Z, Z - Z^*Q \rangle. \tag{F.3}
\]

For the right hand side of (F.1), by the definition of $Z$, $\tilde{Z}$, and (E.2), we have following relations
\[
\| \tilde{Z}^T Z \|_F^2 = \| U^TU - \Lambda U U^T \|_F^2, \quad \| Z - Z^*Q \|_F^2 = 2d^2(U, U^*). \tag{F.4}
\]

Combine (F.4) with (F.1), (F.3) and we prove the first inequality in the argument. Moreover, since $U^*$ has orthogonal columns, $\tilde{Z}^* Z^* = U^* U^* - V^* V^* = U^* U^* - \Lambda U^* U^* \Lambda = 0$. Therefore,
\[
\| \tilde{Z}^T Z \|_F^2 = \langle Z Z^T, \tilde{Z} \tilde{Z}^T \rangle = \langle Z Z^T, \tilde{Z} \tilde{Z}^T \rangle + \langle Z^* Z^*T, \tilde{Z} \tilde{Z}^T \rangle + \langle \tilde{Z} \tilde{Z}^T, Z^* Z^*T \rangle \\
\geq \langle Z Z^T - Z^* Z^*T, \tilde{Z} \tilde{Z}^T - \tilde{Z}^* \tilde{Z}^T \rangle \\
= \| U U^T - U^* U^* T \|_F^2 - 2\text{tr}(U V^T - U^* V^T) + \| V V^T - V^* V^T \|_F^2 \\
= 2\| U U^T - U^* U^* T \|_F^2 - 2\| U \Lambda U^T - U^* \Lambda U^* \|_F^2 \\
= \| Z Z^T - Z^* Z^*T \|_F^2 - 4\| U \Lambda U^T - U^* \Lambda U^* \|_F^2. \tag{F.5}
\]

By Lemma F.6 and (E.2),
\[
\| Z Z^T - Z^* Z^*T \|_F^2 \geq 2(\sqrt{2} - 1)(\sqrt{2}\sigma_r)^2 \inf_{Q \in \mathbb{Q}^{p \times r}} \| Z - Z^*Q \|_F^2 = 8(\sqrt{2} - 1)\sigma_r^2 d^2(U, U^*).
\]

Combining with relations in (F.4) and (F.5), we prove the second inequality. \qed

**Lemma F.11** (Concentration of sample covariance). Let $X_1, \ldots, X_n$ be independent realizations of $X$, which is $d$-dimensional random vector distributed as $\mathcal{N}(\mu, \Sigma)$. Let
\[
\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} (X_i - \hat{\mu})(X_i - \hat{\mu})^T, \quad \hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i, \tag{F.6}
\]

be the sample covariance and mean, respectively. Then there exists a constant $C_1 > 0$ such that
\[
\Pr \left( \| \hat{\Sigma} - \Sigma \|_2 \leq C_1 \| \Sigma \|_2 \left\{ \frac{d + \log(1/\delta)}{n} \right\}^{1/2} \sqrt{\frac{d + \log(1/\delta)}{n}} \right) \geq 1 - \delta.
\]

Moreover, suppose $\sigma_d \leq \sigma_{\text{min}}(\Sigma) \leq \sigma_{\text{max}}(\Sigma) \leq \sigma_1$, then if $n \geq \kappa^2 d$ with $\kappa = \sigma_1/\sigma_d$ being the condition number,
\[
\Pr \left( \frac{\sigma_d}{2} \leq \sigma_{\text{min}}(\hat{\Sigma}) \leq \sigma_{\text{max}}(\hat{\Sigma}) \leq \frac{3\sigma_1}{2} \right) \geq 1 - \frac{1}{d^2}.
\]

**Proof.** Since $\hat{\Sigma} = n^{-1} \sum_{i=1}^{n} (X_i - \mu)(X_i - \mu)^T - (\hat{\mu} - \mu)(\hat{\mu} - \mu)^T$, we have
\[
\| \hat{\Sigma} - \Sigma \|_2 \leq \left\| \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)(X_i - \mu)^T - \Sigma \right\|_2 + \| \hat{\mu} - \mu \|_2^2.
\]
Using equation (6.12) in Wainwright (2019), there exists a constant $C > 0$, so that
\[
\frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)(X_i - \mu)^T - \Sigma \leq C \|\Sigma\|_2 \left[ \left\{ \frac{d + \log(1/\delta)}{n} \right\}^{1/2} \sqrt{\frac{d + \log(1/\delta)}{n}} \right]
\]
with probability at least $1 - \delta$. For the second term, we have $\tilde{\mu} - \mu \sim \mathcal{N}(0, \Sigma/n)$. By Lemma F.3,
\[
\|\tilde{\mu} - \mu\|_2 \leq \frac{\|\Sigma\|_2}{n} \left[ d + 2 \left\{ d \log(1/\delta) \right\}^{1/2} + 2 \log(1/\delta) \right]
\]
with probability at least $1 - \delta$. The proof of the first part follows by combining the last two displays. Moreover, let $\delta = 1/d^2$. We see if $n \gtrsim \kappa^2 d$ then $\|\Sigma - \Sigma\|_2 \leq \sigma_d/2$ with probability at least $1 - 1/d^2$. By Weyl's inequality (cf. Theorem 4.3.1 Horn and Johnson, 2013), we can further get
\[
\sigma_{\min}(\tilde{\Sigma}) \geq \sigma_{\min}(\Sigma) - \|\tilde{\Sigma} - \Sigma\|_2 \geq \frac{\sigma_d}{2}.
\]
Similarly, the upper bound satisfies
\[
\sigma_{\max}(\tilde{\Sigma}) \leq \sigma_{\max}(\Sigma) + \|\tilde{\Sigma} - \Sigma\|_2 \leq \frac{3\sigma_1}{2}.
\]
This completes the second part of proof. \qed

**Lemma F.12.** Let $X_1, \ldots, X_n$ be independent copies of $X \sim \mathcal{N}(\mu, \Sigma) \in \mathbb{R}^d$ and let $\tilde{\Sigma} = n/(n - d - 2)\Sigma$ with $\tilde{\Sigma}$ defined in (F.6) be the scaled sample covariance. Suppose $d \leq cn$ for $c \in (0, 1/2)$,
\[
\Pr \left\{ \|\tilde{\Sigma}^{-1} - \Sigma^{-1}\|_{\infty, \infty} \geq \|\Sigma^{-1/2}\|_1 \left( \frac{8 \log d}{n} \right)^{1/2} \right\} \leq \frac{4}{d^2}.
\]

**Proof.** We only prove the result for $\mu = 0$. The same technique can be applied for a general $\mu$. Let $Y_i = \Sigma^{-1/2}X_i \sim \mathcal{N}(0, I_d)$ for $i = 1, \ldots, n$. Then
\[
\|\tilde{\Sigma}^{-1} - \Sigma^{-1}\|_{\infty, \infty} = \|\Sigma^{-1/2}(\tilde{\Sigma}^{-1/2} \Sigma^{-1/2} - I_d) \Sigma^{-1/2}\|_{\infty, \infty}
\]
\[
\leq \|\Sigma^{-1/2}\|_1 \left\| \left( \frac{1}{n - d - 2} \sum_{i=1}^{n} Y_i Y_i^T \right)^{-1} - I_d \right\|_{\infty, \infty}.
\]
Let $S_{n,d} = \frac{1}{n - d - 2} \sum_{i=1}^{n} Y_i Y_i^T$, $\Delta_{n,d} = S_{n,d} - I_d$, and $\omega_{j,k}^{(n,d)} = (S_{n,d}^{-1})_{j,k}$ for $j, k = 1, \ldots, d$. We first consider the case when $d = 2$. For a large enough sample size $n$, we have $\|\Delta_{n,2}\|_1 \leq 1/2$ and
\[
(I_2 + \Delta_{n,2})^{-1} = \sum_{k=0}^{\infty} (-1)^k \Delta_{n,2}^k = I_2 - \Delta_{n,2} + \Delta_{n,2}(I_2 + \Delta_{n,2})^{-1} \Delta_{n,2}.
\]
Then
\[
\|S_{n,d}^{-1} - I_2\|_{\infty, \infty} = \|(I_2 + \Delta_{n,2})^{-1} - I_2\|_{\infty, \infty} \leq \|\Delta_{n,2}\|_{\infty, \infty} + \|\Delta_{n,2}(I_2 + \Delta_{n,2})^{-1} \Delta_{n,2}\|_{\infty, \infty}.
\]
Since $\|\Delta_{n,2}\|_1 \leq 1/2$, we have
\[
\|\Delta_{n,2}(I_2 + \Delta_{n,2})^{-1} \Delta_{n,2}\|_{\infty, \infty} \leq \|\Delta_{n,2}\|_{\infty, \infty} \|\Delta_{n,2}\|_1 \sum_{k=0}^{\infty} \|\Delta_{n,2}\|_1^k \leq \frac{\|\Delta_{n,2}\|_{\infty, \infty} \|\Delta_{n,2}\|_1}{1 - \|\Delta_{n,2}\|_1} \leq \|\Delta_{n,2}\|_{\infty, \infty}.
\]
Combining with (F.8), we have \( \|S_{n,2}^{-1} - I_2\|_{\infty,\infty} \leq 2\|\Delta_{n,2}\|_{\infty,\infty} \). By Lemma 1 in Rothman et al. (2008), there exists a constant \( \kappa > 0 \), such that

\[
\text{pr}\left( \|S_{n,2}^{-1} - I_2\|_{\infty,\infty} > t \right) \leq \text{pr}\left( \|\Delta_{n,2}\|_{\infty,\infty} > \frac{t}{2} \right) \leq 4\exp(-nt^2), \quad |t| \leq \kappa. \tag{F.9}
\]

Next, we consider a general \( d \). We divide \( S_{n,d} \) into \( 2 \times 2 \) block matrix as

\[
S_{n,d} = \begin{pmatrix} S_{n,d,1,1} & S_{n,d,1,2} \\ S_{n,d,2,1} & S_{n,d,2,2} \end{pmatrix}
\]

where \( S_{n,d,1,1} \in \mathbb{R}^{2\times2} \) and \( S_{n,d,2,2} \in \mathbb{R}^{(d-2)\times(d-2)} \). Let \( S_{n,d,1,2} = S_{n,d,2,1}^{-1}S_{n,d,2,2}^{-1}S_{n,d,1,1}^{-1} \). Due to the structure of \( S_{n,d}^{-1} \), it suffices to show concentration of two representative entries: \( (1,1) \)-entry \( \omega_{1,1} \) and \( (1,2) \)-entry \( \omega_{1,2} \). By the block matrix inversion formula (cf. Theorem 2.1 in Lu and Shiu, 2002), \( \omega_{1,1} = [(S_{n,d}^{-1})^{-1}]_{1,1} \) and \( \omega_{1,2} = [(S_{n,d}^{-1})^{-1}]_{1,2} \). By Proposition 8.7 in Eaton (2007), we have that \( S_{n,d}^{-1} \) is equal in distribution to \( (n-d-1)^{-1}\sum_{i=1}^{n-d+2} Z_i Z_i^\top \) where \( Z_i, i = 1, \ldots, n-d+2, \) are independently drawn from \( \mathcal{N}(0, I_2) \). In particular, we have that \( S_{n,d}^{-1} \) is equal in distribution to \( (n-d+2)/(n-d-2) \cdot S_{n-d+2,2} \). Therefore,

\[
|\omega_{1,1} - 1| \vee |\omega_{1,2}| \leq \left\| \frac{n-d-2}{n-d+2} S_{n-d+2,2}^{-1} - I_2 \right\|_{\infty,\infty} \leq \left\| S_{n-d+2,2}^{-1} - I_2 \right\|_{\infty,\infty} + \frac{4}{n-d}.
\]

Further, we have

\[
\text{pr}\left( |\omega_{1,1} - 1| > t \right) \vee \text{pr}\left( |\omega_{1,2}| > t \right) \leq \text{pr}\left( \|S_{n-d+2,2}^{-1} - I_2\|_{\infty,\infty} > t - \frac{4}{n-d} \right) \leq 4\exp\left\{ -(n-d+2) \left( t - \frac{4}{n-d} \right)^2 \right\},
\]

for \( |t - 4/(n-d)| \leq \kappa \), using (F.9). Setting \( t = (8\log d/n)^{1/2} \), taking union bound over all entries, ignoring smaller order term \( 4/(n-d) \), and combining with (F.7), we finally complete the proof. \( \square \)

**Lemma F.13.** Let \( X_1, \ldots, X_n \) be independent copies of \( X \sim \mathcal{N}(\mu, \Sigma) \in \mathbb{R}^d \) and let \( \hat{\Sigma} \) be the sample covariance defined in (F.6). For any \( U \in \mathbb{R}^{d\times r} \) and \( Q \in \mathbb{Q}^{r\times r} \),

\[
\text{pr}\left( \|((\hat{\Sigma} - \Sigma)UQ)^\top\|_{2,\infty} > 11\sqrt{\|U\Sigma U\|_2\|\Sigma\|_2}\sqrt{\frac{r \log d}{n}} \right) \leq \frac{2}{d^2}.
\]

**Proof.** Let \( Y_i = X_i - \mu, i = 1, \ldots, n \). Then

\[
\|((\hat{\Sigma} - \Sigma)UQ)^\top\|_{2,\infty} = \|((\hat{\Sigma} - \Sigma)U)^\top\|_{2,\infty} \leq \|U^\top \left( \frac{1}{n} \sum_{i=1}^{n} Y_i Y_i^\top - \Sigma \right) \|_{2,\infty} \leq \|U^\top (\hat{\mu} - \mu) (\hat{\mu} - \mu)^\top \|_{2,\infty}.
\tag{F.10}
\]

For the first term in (F.10), we have

\[
\|U^\top \left( \frac{1}{n} \sum_{i=1}^{n} Y_i Y_i^\top - \Sigma \right) \|_{2,\infty} = \max_{j \in [d]} \sup_{v \in \mathbb{R}^r, \|v\|_2 \leq 1} \frac{1}{n} \sum_{i=1}^{n} v^\top U^\top Y_i Y_i^\top e_j - E (v^\top U^\top YY^\top e_j),
\]

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where $e_j \in \mathbb{R}^d$ denotes the $j$-th canonical basis of $\mathbb{R}^d$. Let $\mathcal{R}$ be a 1/2-net of $\{v \in \mathbb{R}^d : \|v\|_2 \leq 1\}$. Then $|\mathcal{R}| \leq 6^d$ and

$$
\max_{j \in [d]} \sup_{v \in \mathcal{R}} \frac{1}{n} \sum_{i=1}^n v^T U^T Y^t_i Y^t_i e_j - \mathbb{E} (v^T U^T Y Y^T e_j) \leq 2 \max_{j \in [d]} \sup_{v \in \mathcal{R}} \frac{1}{n} \sum_{i=1}^n v^T U^T Y^t_i Y^t_i e_j - \mathbb{E} (v^T U^T Y Y^T e_j)
$$

using (4.10) and Lemma 4.4.1 of Vershynin (2018). Note that $v^T U^T Y^t_i Y^t_i e_j$ is a sub-Exponential random variable with the Orlitz $\psi_1$-norm (c.f. Definition 2.7.5 Vershynin, 2018) bounded as $\|v^T U^T Y^t_i Y^t_i e_j\|_{\psi_1} \leq \|U^T \Sigma U\|_2^{1/2} \|\Sigma\|_2^{1/2}$. Using Bernstein’s inequality (cf. Theorem 2.8.1 in Vershynin, 2018) with $t = 5\|U^T \Sigma U\|_2^{1/2} \|\Sigma\|_2^{1/2} (r \log d/n)^{1/2}$ and the union bound over $j \in [d]$ and $v \in \mathcal{R}$, we have

$$
\Pr \left\{ \|U^T (\frac{1}{n} \sum_{i=1}^n Y^t_i Y^t_i - \Sigma)\|_{2,\infty} > 5\|U^T \Sigma U\|_2^{1/2} \|\Sigma\|_2^{1/2} (r \log d/n)^{1/2} \right\} \leq \frac{1}{d^2}. \tag{F.11}
$$

For the second term in (F.10), we proceed similarly. With $w = \hat{\mu} - \mu$, for any $t > 0$,

$$
\Pr \left( \|U^T w w^T\|_{2,\infty} > t \right) \leq d^6 \Pr \left( v^T U^T w w^T e_j > \frac{t}{2} \right) \leq d^6 \exp \left( -\frac{nt}{2\|U^T \Sigma U\|_2^{1/2} \|\Sigma\|_2^{1/2}} \right).
$$

Setting $t = 6\|U^T \Sigma U\|_2^{1/2} \|\Sigma\|_2^{1/2} r \log d/n$, we get

$$
\Pr \left\{ \|U^T (\hat{\mu} - \mu)(\hat{\mu} - \mu)^T\|_{2,\infty} > 6\|U^T \Sigma U\|_2^{1/2} \|\Sigma\|_2^{1/2} r \log d/n \right\} \leq \frac{1}{d^2}. \tag{F.12}
$$

Combining (F.11) and (F.12) with (F.10) completes the proof. \qed

### G Convex approaches

We propose two convex relaxation approaches for estimating $\Delta^*$ defined in (3). Given the empirical objective function in (5), one can relax sparse and low-rank constraints by adding $\ell_1$ and nuclear penalties and solve the following regularized convex problem over $S^{d \times d}$

$$
\min_{S, R} \mathcal{L}_n (S, R) + \lambda_1 \|S\|_{1,1} + \lambda_2 \|R\|_*.
$$

Here, $\lambda_1$ biases the overall sparsity of $S$ and $\lambda_2$ biases the rank of $R$. A related CLIME-type procedure is formulated as

$$
\min_{S, R} \|S\|_{1,1} + \lambda \|R\|_*, \tag{G.1}
$$

s.t. $\|\hat{\Sigma}_X (S + R) \hat{\Sigma}_Y - (\hat{\Sigma}_Y - \hat{\Sigma}_X)\|_{\infty, \infty} \leq \eta / \lambda$, 

$$
\|\hat{\Sigma}_X (S + R) \hat{\Sigma}_Y - (\hat{\Sigma}_Y - \hat{\Sigma}_X)\|_2 \leq \eta.
$$

The constraints in (G.1) are derived by plugging $\nabla_S \mathcal{L}_n (S, R)$ and $\nabla_R \mathcal{L}_n (S, R)$ into the dual function of the objective. Both of them are typically solved by alternating direction method of multipliers, which suffers from high computational cost since each iteration requires an eigenvalue decomposition of $R$ to compute the proximal update corresponding the nuclear norm penalty. Ma et al. (2013) and Wang et al. (2013) proposed accelerated algorithms targeting the above mentioned drawback. In comparison, nonconvex procedures are widely used to speed up estimation problems involving low-rank matrices (Tu et al., 2016; Park et al., 2018; Chi et al., 2018; Yu et al., 2018b).
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