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DYNAMICAL SYSTEMS APPROACH TO $G_2$ COSMOLOGY

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Abstract

In this paper we present a new approach for studying the dynamics of spatially inhomogeneous cosmological models with one spatial degree of freedom. By introducing suitable scale-invariant dependent variables we write the evolution equations of the Einstein field equations as a system of autonomous partial differential equations in first-order symmetric hyperbolic format, whose explicit form depends on the choice of gauge. As a first application, we show that the asymptotic behaviour near the cosmological initial singularity can be given a simple geometrical description in terms of the local past attractor on the boundary of the scale-invariant dynamical state space. The analysis suggests the name “asymptotic silence” to describe the evolution of the gravitational field near the cosmological initial singularity.

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1 Introduction

The simplest cosmological models are the Friedmann–Lemaître (FL) cosmologies, which describe an expanding Universe that is exactly spatially homogeneous and spatially isotropic. It is widely believed that on a sufficiently large spatial scale the Universe can be described by such a model, at least since the time of last scattering of primordial photons with unbound electrons.

There are, however, compelling reasons for studying cosmological models more general than the FL models. Firstly, the observable part of the Universe is not exactly spatially homogeneous and isotropic on any spatial scale and so, from a practical point of view, one is interested in models that are “close to FL” in some appropriate dynamical sense. The usual way to study deviations from an FL model is to apply linear perturbation theory. However, it is not known how reliable the linear theory is and, moreover, in using it one is a priori excluding the possibility of finding important non-linear effects. Secondly, it is necessary to consider more general models in order to investigate the constraints that observations impose on the geometry of spacetime. Thirdly, it is important to classify all possible asymptotic states near the cosmological initial singularity (i.e., near the Planck time) that are permitted by the Einstein field equations (EFE), with a view to explaining how the real Universe may have evolved.

For these and other reasons it is of interest to consider a spacetime symmetry-based hierarchy of cosmological models that are more general than FL. On the first level above the FL models are the spatially homogeneous (SH) models, i.e., models which admit a 3-parameter isometry group acting transitively on spacelike 3-surfaces, and expand anisotropically. This class has been studied extensively, and a detailed account of the results obtained up to 1997 is contained in the book edited by Wainwright and Ellis (WE) [52]. On the second level of the hierarchy are cosmological models with two commuting Killing vector fields (i.e., models which admit a 2-parameter Abelian isometry group acting transitively on spacelike 2-surfaces), which thus admit one degree of freedom as regards spatial inhomogeneity. This class of models, which are referred to briefly as $G_2$ cosmologies, are the focus of the present paper. In generalising from

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SH cosmologies to $G_2$ cosmologies one makes the transition from ordinary differential equations (ODE) to partial differential equations (PDE) in two independent variables as regards the evolution system of the EFE, with the inevitable increase in mathematical difficulty. For both classes of models one has available the four standard methods of systematic investigation:

(i) derivation and analysis of exact solutions,
(ii) approximation methods of a heuristic nature,
(iii) numerical simulations and experiments, and
(iv) rigorous qualitative analysis.

All four methods have been used to study $G_2$ cosmologies with varying degrees of success, subject to significant limitations.

We now give a brief history of $G_2$ cosmologies. To the best of our knowledge, the first development was the study by Gowdy of a class of solutions of the vacuum EFE with compact space sections and an Abelian $G_2$ isometry group, now called Gowdy spacetimes [24, 25]. Although vacuum, they can be regarded as idealised cosmological models because they have a preferred timelike congruence, start at a curvature singularity, and either expand indefinitely or recollapse. These solutions could represent the early stages of the Universe during which the energy–momentum–stress content is not dynamically significant. As regards $G_2$ cosmologies with a perfect fluid matter source, the earliest paper was by Liang [35], who used approximation methods to study the evolution of matter density fluctuations. A variety of exact perfect fluid $G_2$ cosmologies have been discovered, starting with Wainwright and Goode [53], and more recently by Senovilla and Vera [45, 47]. Most exact solutions have been derived by imposing a separability assumption on the metric components, so that the EFE decouple into two sets of ODE. As regards numerical simulations, work began in the 1970s (see, e.g., Centrella and Matzner [10]). Recent work (e.g., Berger and Moncrief [8] and Berger and Garfinkle [7]) has focussed on investigating the nature of the cosmological initial singularity in Gowdy vacuum spacetimes. Rigorous qualitative analysis has also focused on the past asymptotic behaviour of these spacetimes, starting with the paper of Isenberg and Moncrief [33] on the diagonal subcase. Recent work by Kichenassamy and Rendall [34] and by Anguige [2] has considered the Gowdy vacuum spacetimes with spatial topology $T^3$ and diagonal $G_2$ perfect fluid cosmologies, respectively. We also refer to Rein [40] for related results for a different matter model.

In summary, almost all of the research using methods (iii) and (iv) above has focused on the Gowdy vacuum spacetimes. The mathematical reasons for this choice are twofold: vacuum $G_2$ models are much more tractable than non-vacuum ones, and the assumption of compact space sections makes numerical simulations easier since it avoids the problem of boundary conditions at spatial infinity. These works are nevertheless of considerable physical interest in view of a conjecture by Belinski–Khalatnikov and Lifshitz (BKL) that “matter does not matter” close to the cosmological initial singularity, i.e., that matter is not dynamically significant in that epoch (see Lifshitz and Khalatnikov [36], p200, and Belinski–Khalatnikov et al [5], p532 and p538). We shall refer to this conjecture as BKL I.

In this paper we focus on $G_2$ cosmologies with perfect fluid matter content, incorporating vacuum models as an important special case. The overall goal is to provide a flexible framework for analysing the evolution of these models in a dynamical systems context. Our approach, which uses the orthonormal frame formalism,\(^1\) has three distinctive features:

(i) first-order autonomous equation systems,
(ii) scale-invariant dependent variables,
(iii) evolution equations that form a system of symmetric hyperbolic PDE.

Dynamical formulations employing (i) and (ii) have proved effective in studying SH cosmologies (see WE [52], Ch. 5, for motivation). We expect similar advantages to be gained in the study of $G_2$ cosmologies. In the SH case the scale-invariant dependent variables are defined by normalisation with the volume expansion rate of the $G_2$–orbits, i.e., the Hubble scalar $H$. In the present case, however, we define scale-invariant dependent variables by normalisation with the area expansion rate of the $G_2$–orbits,\(^2\) in order

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1See, e.g., MacCallum [37], and for an extended set of equations, van Elst and Uggla [19].
2We refer to Hewitt and Wainwright [29] for a dynamical formulation of perfect fluid $G_2$ cosmologies using Hubble-normalised dependent variables.
to obtain the evolution equations as a system of PDE in first-order symmetric hyperbolic (FOSH) format. In this way we make available an additional set of powerful analytical tools, that ensures local existence, uniqueness and stability of solutions to the Cauchy initial value problem for $G_2$ cosmologies and provides methods for estimating asymptotic decay rates.\footnote{For details on the theory of FOSH evolution systems see, e.g., Courant and Hilbert \cite{CourantHilbert} or Friedrich and Rendall \cite{FriedrichRendall}.} The FOSH format also provides a natural framework for formulating a concept of geometrical information propagation, by which we mean the propagation at finite speeds of jump discontinuities in the initial data set.\footnote{See, e.g., van Elst \textit{et al} \cite{vanElst}.}

We now digress briefly to describe some aspects of SH dynamics. The use of scale-invariant dependent variables to study SH models led to an important discovery, namely that self-similar solutions of the EFE play a key role in describing the dynamics of SH models, in that they can approximate the early, intermediate and late time behaviour of more general models. We refer to WE \cite{WainwrightEllis}, Ch. 5, for details and other references. A self-similar solution admits a homothetic vector field, which in physical terms means that as the cosmological model expands, its physical state differs only by an overall change in the length scale, i.e., the dynamical properties of the model are scale-invariant. The above discovery had been anticipated some years earlier by Eardley \cite{Eardley}, who observed that SH models of Bianchi Type–I, while not self-similar, are asymptotically self-similar. By this one means that in the asymptotic regimes, i.e., near the cosmological initial singularity and at late times, their evolution is approximated by self-similar models. In other words, these simple models have well-defined asymptotic regimes that are scale-invariant. In general, however, SH cosmologies are not asymptotically self-similar. For example, the well-known Mixmaster models (vacuum solutions of Bianchi Type–IX; see Ref. \cite{SN}) oscillate indefinitely as the cosmological initial singularity is approached into the past, and thus do not have a well-defined asymptotic state (see for example Ref. \cite{Dale} and WE \cite{WainwrightEllis}, Ch. 11). Nevertheless, as follows from the dynamical systems analysis, the Mixmaster models are successively approximated by an infinite sequence of self-similar models (Kasner vacuum solutions) as they evolve into the past towards the cosmological initial singularity. The mathematical reason for the above phenomena is that the self-similar solutions arise as equilibrium points (i.e., fixed points) of the evolution equations. These equilibrium points, in conjunction with the Bianchi classification of the $G_3$ isometry group, determine various invariant submanifolds of increasing generality that provide a hierarchical structure for the SH dynamical state space. In other words, the self-similar solutions play a key role as building blocks in determining the structure of the SH dynamical state space. We anticipate that self-similar models will play an analogous role in building the skeleton of the $G_2$ dynamical state space.

In studying $G_2$ cosmologies we expect to make use of insights into cosmological dynamics obtained from analysing SH cosmologies, for the following reasons. $G_2$ cosmologies can be regarded as spatially inhomogeneous generalisations of SH models of all Bianchi isometry group types except Type–VIII and Type–IX, since, apart from these two cases, the $G_3$ admits an Abelian $G_2$ as a subgroup. In the language of dynamical systems the dynamical state space of SH cosmologies with an Abelian $G_2$ subgroup is an invariant submanifold of the dynamical state space of $G_2$ cosmologies. It thus follows that orbits in the $G_2$ dynamical state space that are close to the SH submanifold will shadow orbits in that submanifold, thereby providing a link between $G_2$ dynamics and SH dynamics. A further link is provided by the famous conjecture of BKL to the effect that near a cosmological initial singularity the EFE effectively reduce to ODE, i.e., the spatial derivatives have a negligible effect on the dynamics (see Belinski\textit{ii et al} \cite{Belinski}, p656). In this asymptotic regime of near-Planckian order spacetime curvature it is plausible that SH dynamics will approximate $G_2$ dynamics locally, i.e., along individual timelines. We shall refer to this conjecture as BKL II.

As indicated above, we expect that SH dynamics will play a considerable role in determining the dynamics of $G_2$ cosmologies, and that analogies with the SH case will be helpful. In two respects, however, the $G_2$ problem differs considerably from the SH problem. Firstly, at any instant of time, the state of a $G_2$ cosmology is described by a finite-dimensional dynamical state vector of functions of the spatial coordinate $x$. In other words, the dynamical state space of $G_2$ cosmologies is a function space and, hence, is infinite-dimensional. The evolution of a $G_2$ cosmology is thus described by an orbit in this infinite-dimensional dynamical state space. Secondly, there is the so-called gauge problem, which we now describe. For SH cosmologies the $G_3$ isometry group determines a geometrically preferred timelike congruence, namely the normal congruence to the $G_3$–orbits, and hence there is a natural choice for the timelike vector field of the orthonormal frame. The remaining freedom in the choice of the orthonormal
frame is a time-dependent rotation of the spatial frame vector fields, which we refer to as the gauge freedom. On the other hand, in a $G_2$ cosmology there is a preferred timelike 2-space at each point that is orthogonal to the $G_2$-orbits. Thus there is an infinite family of geometrically preferred timelike congruences and the gauge freedom in the choice of the orthonormal frame is correspondingly more complicated. There is also gauge freedom associated with the choice of the local coordinates. One of the goals of the present paper is to discuss various gauge fixing options that arise for $G_2$ cosmologies.

The plan of the paper is as follows. In section 2 we derive the equation system that arises from the EFE and the matter equations. Working in the orthonormal frame formalism, we adapt the orthonormal frame to the $G_2$-orbits and then simply specialise the general orthonormal frame relations in Ref. [19] to get the desired equation system in dimensional form. We then introduce the scale-invariant dependent variables and transform the equation system to dimensionless form. In section 3 we address the gauge problem and introduce four specific gauge choices, showing that our approach has the flexibility to incorporate all previous work. In section 4 we discuss some features of the infinite-dimensional dynamical state space. In section 5 we give a simple geometrical representation of the past attractor as an invariant submanifold on the boundary of the infinite-dimensional dynamical state space. The nature of the past attractor illustrates the conjecture BKL II concerning cosmological initial singularities, and suggests the name “asymptotic silence” to describe the dynamical behaviour of the gravitational field as one follows a family of timelines into the past towards the singularity. We conclude in section 6 with a discussion of future research directions. Useful mathematical relations such as expressions for the scale-invariant components of the Weyl curvature tensor for $G_2$ cosmologies and the propagation laws for the constraint equations have been gathered in an appendix.

2 Framework and dynamical equation systems

2.1 Dimensional equation system

A cosmological model is a triple $(M, g, \mathbf{u})$, where $M$ is a 4-dimensional manifold, $g$ a Lorentzian 4-metric of signature $(-+++)$, and $\mathbf{u}$ is the matter 4-velocity field. We will assume that the EFE are satisfied with the matter content being a perfect fluid with a linear barotropic equation of state,

$$\tilde{p}(\tilde{\rho}) = (\gamma - 1) \tilde{\rho}, \quad 1 \leq \gamma \leq 2.$$  

(1)

The most important cases physically are radiation ($\gamma = \frac{4}{3}$) and dust ($\gamma = 1$). We will also include a non-zero cosmological constant $\Lambda$. Throughout our work we will employ geometrised units characterised by $c = 1 = 8\pi G$.  

We assume that an Abelian $G_2$ isometry group acts orthogonally transitively on spacelike 2-surfaces (cf. Ref. [9]), and introduce a group-invariant orthonormal frame $\{e_a\}$, with $e_2$ and $e_3$ tangent to the $G_2$-orbits. We regard the frame vector field $e_0$ as defining a timelike reference congruence. Since $e_0$ is orthogonal to the $G_2$-orbits, it is hypersurface orthogonal, and hence is orthogonal to a locally defined family of spacelike 3-surfaces $S: \{t = \text{const}\}$. We introduce a set of symmetry-adapted local coordinates $\{t, x, y, z\}$ that are tied to the frame vector fields $e_a$ in the sense that

$$e_0 = N^{-1} \partial_t, \quad e_1 = e_1^1 \partial_x, \quad e_A = e_A^B \partial_x^B, \quad A, B = 2, 3,$$  

(2)

where the coefficients are functions of the independent variables $t$ and $x$ only.\(^5\) The only non-zero frame variables are thus given by

$$N, \quad e_1^1, \quad e_A^B,$$  

(3)

which yield the following non-zero connection variables:

$$\alpha, \beta, a_1, n_+, \sigma_-, n_\times, \sigma_\times, n_-, \tilde{u}_1, \Omega_1;$$  

(4)

their interrelation is given in the appendix. Here we have followed Ref. [19] in doing a $(1+3)$-decomposition of the connection variables. The variables $\alpha, \beta, \sigma_-$ and $\sigma_\times$ are related to the Hubble volume expansion rate $H$ and the shear rate $\sigma_{\alpha\beta}$ of the timelike reference congruence $e_0$ according to

$$\alpha := (H - 2\sigma_+) \quad \beta := (H + \sigma_+),$$  

(5)

\(^5\)In the terminology of Arnowitt, Deser and Misner [4], $N$ is the lapse function, and we have chosen a zero shift vector field, $N^1 = 0$.  

where $\sigma_+$ is one of the components in the following decomposition of the symmetric tracefree shear rate tensor $\sigma_{\alpha\beta}$:

$$\sigma_+ := \frac{1}{2} (\sigma_{22} + \sigma_{33}) = -\frac{1}{2} \sigma_{11} , \quad \sigma_- := \frac{1}{2\sqrt{3}} (\sigma_{22} - \sigma_{33}) , \quad \sigma_\times := \frac{1}{\sqrt{3}} \sigma_{23} .$$

(6)

A consequence of this decomposition is that the shear rate scalar assumes the form $\sigma^2 := \frac{1}{2}(\sigma_{\alpha\beta}\sigma^{\alpha\beta}) = 3(\sigma_+^2 + \sigma_-^2 + \sigma_\times^2)$. We will use similar decompositions for the electric and magnetic Weyl curvature variables $E_{\alpha\beta}$ and $H_{\alpha\beta}$, as given in the appendix. The “non-null-null” variables $\alpha$ and $\beta$ (cf. Refs. [51, 15]) turn out to be more convenient to use than $H$ and $\sigma_+$, since they are naturally adapted to the characteristic structure of the evolution equations that arise from the Ricci identities when the latter are applied to the timelike reference congruence $\mathbf{e}_0$ (see Ref. [17]). The variables $a_1, n_+, n_\times$ and $n_-$ describe the non-zero components of the purely spatial commutation functions $a^\alpha$ and $n_{\alpha\beta}$, where

$$n_+ := \frac{1}{2} (n_{22} + n_{33}) , \quad n_- := \frac{1}{2\sqrt{3}} (n_{22} - n_{33}) , \quad n_\times := \frac{1}{\sqrt{3}} n_{23} ,$$

(7)

(see WE [52] for this type of decomposition of the spatial commutation functions). Finally, the variable $\dot{u}_1$ is the acceleration of the timelike reference congruence $\mathbf{e}_0$, while $\Omega_1$ represents the rotational freedom of the spatial frame $\{ \mathbf{e}_n \}$ in the $(\mathbf{e}_2, \mathbf{e}_3)$-plane. Setting $\Omega_1$ to zero corresponds to the choice of a Fermi-propagated orthonormal frame $\{ \mathbf{e}_n \}$. It should be pointed out that within the present framework the dependent variables

$$\{ N, \dot{u}_1, \Omega_1 \}$$

(8)

enter the evolution system as freely prescribable “gauge source functions” in the sense of Friedrich [21].

Since the $G_2$ isometry group acts orthogonally transitively, the 4-velocity vector field $\dot{u}$ of the perfect fluid is orthogonal to the $G_2$-orbits, and hence has the form

$$\dot{u} = \Gamma (\mathbf{e}_0 + v \mathbf{e}_1) ,$$

(9)

where the Lorentz factor is $\Gamma := (1 - v^2)^{-1/2}$. It turns out to be useful to replace the matter energy density $\bar{\mu}$ in the fluid rest frame with [19]

$$\mu = \frac{G_+}{(1 - v^2)} \bar{\mu} ,$$

(10)

where it is convenient to introduce the auxiliary quantities

$$G_\pm := 1 \pm (\gamma - 1) v^2 .$$

(11)

Thus $\mu$, which represents the matter energy density in the rest frame of $\mathbf{e}_0$, and $v$, which describes the fluid’s peculiar velocity relative to the same frame, describe, for a given value of the equation of state parameter $\gamma$, the fluid degrees of freedom.

The orthonormal frame version of the EFE and matter equations as given in Ref. [19], when specialised to the orthogonally transitive Abelian $G_2$ case with the dependent variables presented above, take the following form:

**Commutator equations**

Gauge fixing condition:

$$0 = (C_3)_{11} := N^{-1} \epsilon_1^1 \partial_x N - \dot{u}_1 .$$

(12)

Evolution equations:

$$N^{-1} \partial_t \epsilon_1^1 = - \alpha \epsilon_1^1$$

(13)

$$N^{-1} \partial_t \epsilon_2^A = - (\beta + \sqrt{3} \sigma_-) \epsilon_2^A - (\sqrt{3} \sigma_\times + \Omega_1) \epsilon_3^A$$

(14)

$$N^{-1} \partial_t \epsilon_3^A = - (\beta - \sqrt{3} \sigma_-) \epsilon_3^A - (\sqrt{3} \sigma_\times - \Omega_1) \epsilon_2^A .$$

(15)

Constraint equations:

$$0 = (C_{\text{com}})_{A12} := (\epsilon_1^1 \partial_x - a_1 - \sqrt{3} n_\times) \epsilon_2^A - (n_+ - \sqrt{3} n_-) \epsilon_3^A$$

(16)

$$0 = (C_{\text{com}})_{A31} := (\epsilon_1^1 \partial_x - a_1 + \sqrt{3} n_\times) \epsilon_3^A + (n_+ + \sqrt{3} n_-) \epsilon_2^A .$$

(17)
Einstein field equations and Jacobi identities

Evolution equations:

\[ N^{-1} \partial_t \alpha = -\alpha^2 + \beta^2 - 3(\sigma_x^2 - n_x^2 + \sigma_x - n_x) - a_1^2 - \frac{1}{2} G_+^{-1} \mu (1 - v^2) + (e_1^1 \partial_x + \dot{u}_1) a_1 \]
\[ N^{-1} \partial_t \beta = -\frac{3}{2} \beta^2 - \frac{3}{2}(\sigma_x^2 + n_x^2 + \sigma_x + n_x) - \frac{1}{2} (2u_1 - a_1) a_1 \]
\[ N^{-1} \partial_t a_1 = -\beta (\dot{u}_1 + a_1) - 3(n_x \sigma_+ - n_+ \sigma) - \frac{1}{2} G_+^{-1} \mu v \]
\[ N^{-1} \partial_t n_+ = -\alpha a_1 + 6(\sigma_- n_- + \sigma_+ n_+) - (e_1^1 \partial_x + \dot{u}_1) \Omega \]
\[ N^{-1} \partial_t \sigma_+ + e_1^1 \partial_x n_+ = -(\alpha + 2 \beta) \sigma_+ - 2n_+ n_- (\dot{u}_1 - 2a_1) n_+ - 2\Omega_1 \sigma_+ \]
\[ N^{-1} \partial_t \sigma_- + e_1^1 \partial_x n_- = -(\alpha + 2 \beta) \sigma_- - 2n_+ n_- (\dot{u}_1 - 2a_1) n_- + 2\Omega_1 \sigma_- \]
\[ N^{-1} \partial_t \sigma_x - e_1^1 \partial_x \sigma_x = -\alpha n_- + 2 \sigma_- n_+ + \dot{u}_1 \sigma_x - 2\Omega_1 n_x \]

Constraint equations:

\[ 0 = (C_{\text{Gauß}}) := 2(2e_1^1 \partial_x - 3a_1) a_1 - 6(n_x^2 + n^2) + 2(2\alpha + \beta) - 6(\sigma_- + \sigma_x) - 2\mu - 2\Lambda \]
\[ 0 = (C_{\text{Codazzi}})_1 := e_1^1 \partial_x \beta + a_1 (\alpha - \beta) - 3(n_x \sigma_- - n_- \sigma_x) - \frac{1}{2} G_+^{-1} \mu v \]

Source Bianchi identities (Relativistic Euler equations)

Evolution equations:

\[ \frac{f_1}{f_1} (N^{-1} \partial_t + \frac{\gamma}{G_+} v e_1^1 \partial_x) \mu + f_2 e_1^1 \partial_x v = -\frac{\gamma}{G_+} f_1 [\alpha (1 + v^2) + 2 \beta + 2(\dot{u}_1 - a_1) v] \]
\[ \frac{f_1}{f_1} (N^{-1} \partial_t - \frac{f_1}{G_+ G_-} v e_1^1 \partial_x) v + f_2 e_1^1 \partial_x \mu = -\frac{f_2}{f_1 G_-} \mu (1 - v^2) [2(\gamma - 1) \alpha v + 2(\gamma - 1) \beta v + G_- \dot{u}_1 + 2(\gamma - 1) a_1 v^2], \]

where

\[ f_1 := \frac{(\gamma - 1)}{\gamma G_-} (1 - v^2)^2, \quad f_2 := \frac{(\gamma - 1)}{G_+^2} (1 - v^2)^2, \quad f_3 := (3\gamma - 4) - (\gamma - 1)(4 - \gamma) v^2. \]

Dynamical features exhibited by the dimensional evolution system for orthogonally transitive \( G_2 \) cosmologies, that are independent of our later transformation to \( \beta \)-normalised scale-invariant dependent variables, are the following: Eq. (13) evolves the only dynamically important frame variable (being part of the metric), Eqs. (18) and (19) evolve the longitudinal components of the tensorial expansion rate of the timelike reference congruence \( \mathbf{e}_0 \), Eqs. (20) and (21) evolve a scalar and a non-tensorial spatial connection variable, respectively, Eqs. (22) – (25) provide the propagation laws for (transverse) gravitational waves, while, finally, the relativistic Euler equations (28) and (29) yield the propagation laws for (longitudinal) acoustic or pressure waves. Viewing the gauge source functions \( N, \dot{u}_1 \) and \( \Omega_1 \) as arbitrary prescribable real-valued functions, the evolution system is already in FOSH format. The characteristic propagation velocities \( \lambda \) relative to a family of observers comoving with the timelike reference congruence \( \mathbf{e}_0 \) are

\[ \lambda_1 = 0, \quad \lambda_{2,3} = \pm 1, \quad \lambda_{4,5} = \frac{(2 - \gamma)}{G_-} v \pm \frac{(1 - v^2)}{G_-} (\gamma - 1)^{1/2}. \]

The right-propagating and left-propagating characteristic eigenfields associated with the non-zero \( \lambda \)'s are (for \( 1 < \gamma \leq 2 \))

\[ \lambda_{2,3} : (\sigma_\pm n_x), \quad (\sigma_\mp n_-), \quad \lambda_{4,5} : \frac{\mu}{h_2(\gamma, v)} \left( h_2(\gamma, v) \mp h_2(\gamma, v) \pm \frac{\gamma G_-^2}{(\gamma - 1)^{3/2} v} \right), \]

\(^6\)Here we take the opportunity to correct for some sign errors in the expressions given in Refs. [17] and [18].
where \( h_1(\gamma, v) \) and \( h_2(\gamma, v) \) are complicated expressions of their arguments that for \( v = 0 \) have the limits \( h_1 = 0 \) and \( h_2 = 1 \), respectively. All of the \( \lambda \)’s are real-valued in the parameter range \( 1 \leq \gamma \leq 2 \) of Eq. (1), and so contains the dust case (\( \gamma = 1 \)) and also the stiff fluid case (\( \gamma = 2 \)). Note, however, that the former must be treated in terms of a modified version of the relativistic Euler equations, since for \( \gamma = 1 \) some of the coefficients in the principal part of the present version become zero. In summary, for our first-order dynamical formulation, the Cauchy initial value problem for the orthogonally transitive \( G_2 \) cosmologies is well-posed in the range \( 1 \leq \gamma \leq 2.7 \).

The values of \( \lambda_{4,5} \) reflect the anisotropic distortion of the sound characteristic 3-surfaces relative to the family of observers comoving with \( e_0 \). This distortion may be viewed as a manifestation of the Doppler effect. In the limit \( v \to 0 \), the magnitude \( |\lambda_{4,5}| \) reduces to the isentropic speed of sound, \( c_s = (\gamma - 1)^{1/2} \). In the extreme cases \( \gamma = 1 \) and \( \gamma = 2 \) we obtain \( \lambda_{4,5} = v \) and \( \lambda_{4,5} = \pm 1 \), respectively.

Note that in this non-fluid-comoving form [even with the linear equation of state (1)] the principal part of the relativistic Euler equations is highly non-linear. This feature could lead to the formation of shocks in the fluid dynamical sector of the evolution system and should be kept in mind in a numerical analysis of the given equation system. The effective semi-linearity of the principal part of the gravitational field sector, on the other hand, is less likely to lead to numerical problems of this kind. For the latter jump discontinuities can be specified in the initial data for \( \partial_v(\sigma \pm n_v) \) and \( \partial_v(\sigma_x \mp n_x) \).

The area density \( A \) of the \( G_2 \)-orbits plays a prominent rôle for \( G_2 \) cosmologies. It is defined (up to a constant factor) by

\[
A^2 := (\xi_a \xi^a)(\eta \eta^a) - (\xi_a \eta^a)^2 ,
\]

where \( \xi \) and \( \eta \) are two independent commuting spacelike Killing vector fields.\(^8\) Expressed in terms of the coordinate components of the frame vector fields \( e_A \) tangent to the \( G_2 \)-orbits this becomes

\[
A^{-1} = e_2^2 e_3^3 - e_2^3 e_3^2 .
\]

The key equations for \( A \), derivable from the commutator equations (14) – (17), are given by

\[
N^{-1} \partial_\alpha A = 2 \beta , \quad e_1^1 \partial_\xi A = -2 a_1 ,
\]

i.e., \( \beta \) is the area expansion rate of the \( G_2 \)-orbits. The frame variables \( e_A B \), which play a subsidiary rôle as regards the dynamics of \( G_2 \) cosmologies, are governed by Eqs. (14) – (17). Since these equations are decoupled from the remaining equations, we will not consider them further.

### 2.2 Scale-invariant reduced equation system

We will now introduce new dimensionless dependent variables that are invariant under arbitrary scale transformations. However, we will not normalise with the Hubble scalar \( H \) (i.e., the volume expansion rate of the \( G_3 \)-orbits), as is usually done for SH models. Instead we will use the area expansion rate \( \beta \) of the \( G_2 \)-orbits, since this leads to significant mathematical simplifications. We thus introduce \( \beta \)-normalised frame, connection and curvature variables as follows:

\[
(N^{-1}, E_1) := (N^{-1}, e_1)/\beta ,
\]

\[
(A, \dot{U}, (1 - 3\Sigma_+), \Sigma_, N_\xi, \Sigma_\xi, N_-, N_+, R) := (a_1, \dot{u}_1, a, \sigma_, n\xi, \sigma_\xi, n_, n_+, \Omega_1)/\beta ,
\]

\[
(\Omega, \Omega_\Lambda) := (\mu, \Lambda)/(3\beta^2) .
\]

Note that we maintain the same notation that was introduced in WE [52] for the \( H \)-normalised case. The two different normalisation procedures are linked through the relation

\[
H = (1 - \Sigma_+)/\beta .
\]

The above definition of \( \Sigma_+ \) in terms of \( \alpha \) is motivated by the relation \( \alpha = \beta - 3\sigma_+ \), which follows from Eqs. (5), and is equivalent to defining \( \Sigma_+ := \sigma_+ / \beta \). As a result of this definition, the \( \beta \)-normalised shear

\(^7\)Clearly well-posedness is lost for any value of \( \gamma \) in the range \( 0 \leq \gamma < 1 \) (cf. Ref. [22]).

\(^8\)In terms of symmetry-adapted local coordinates \( x^2 = y \) and \( x^3 = z \) such that \( \xi = \partial / \partial y \) and \( \eta = \partial / \partial z \), the area density is given by \( A = \sqrt{g_{AB} g_{AB}} \), where \( g_{AB} \), \( A, B = 2, 3 \), is the metric induced on the \( G_2 \)-orbits.
rate scalar $\Sigma^2 := (\sigma_0 \sigma_0^\beta)/(6\beta^2)$ has the form $\Sigma^2 = \Sigma_+^2 + \Sigma_-^2 + \Sigma_x^2$. Note that in the units we have chosen the matter variable is already dimensionless.

The dimensional equation system in subsection 2.1 leads to an equation system for the scale-invariant dependent variables (36) – (38). In order to make this change it is necessary to introduce the time and space rates of change of the normalisation factor $\beta$. In analogy with $H$-normalisation (see WE [52]), we define variables $q$ and $r$ by

\[
N^{-1} \partial_t \beta := - (q + 1) \beta \quad (40)
\]

and

\[
0 = (C_\beta) := (E_1 \partial_x + r) \beta \quad (41)
\]

Here $q$ plays the rôle of an “area deceleration parameter”, analogous to the usual “volume deceleration parameter”, while $r$ plays a rôle analogous to a “Hubble spatial gradient”. Using Eqs. (40) and (41) and the definitions (36) – (38), it is straightforward to transform the dimensional equation system to a $\beta$-normalised dimensionless form. A key step is to use the evolution equation (19) for $\beta$ and the Codacci constraint equation (27) to express $q$ and $r$, as defined above, in terms of the remaining scale-invariant dependent variables. The key result, which is essential for casting the scale-invariant evolution system in FOSH format, is that the expressions for $q$ and $r$ are purely algebraical. We refer to these equations as the defining equations for $q$ and $r$ [see Eqs. (54) and (55) below]. The relation

\[
N^{-1} \partial_r r - E_1 \partial_x q = (q + 3\Sigma_+) r - (r - \dot{U})(q + 1) + (q + 1) (C_U)
\]

arises as an integrability condition for the decoupled $\beta$-equations (40) and (41).

Scale-invariant equation system

Evolution system:

\[
N^{-1} \partial_t E_1 = (q + 3\Sigma_+^2) E_1 \\
3N^{-1} \partial_t \Sigma_+ = - (q + 3\Sigma_+) (1 - 3\Sigma_+) - 2q + 6 (\Sigma_+^2 + \Sigma_-^2) + \frac{3}{2} G_+^{-1} [(3\gamma - 2) + (2 - \gamma) v^2] \Omega - 3 \Omega \Lambda - (E_1 \partial_x - r + \dot{U} - 2A) \dot{U}
\]

\[
N^{-1} \partial_t A = (q + 3\Sigma_+) A + (r - \dot{U})
\]

\[
N^{-1} \partial_t N_+ = (q + 3\Sigma_+) N_+ + 6 (\Sigma_- N_- + \Sigma_x N_x) - (E_1 \partial_x - r + \dot{U}) R
\]

\[
N^{-1} \partial_t \Omega = 2 (q + 1) \Omega \Lambda
\]

\[
N^{-1} \partial_t \Sigma_+ + E_1 \partial_x N_+ = (q + 3\Sigma_+ - 2) \Sigma_- - 2 N_+ N_- + (r - \dot{U} + 2A) N_+ - 2R \Sigma_-
\]

\[
N^{-1} \partial_t N_+ + E_1 \partial_x \Sigma_+ = (q + 3\Sigma_+) N_+ + 2 \Sigma_x N_x + (r - \dot{U}) \Sigma_+ - 2A R
\]

\[
N^{-1} \partial_t \Sigma_+ + E_1 \partial_x N_+ = (q + 3\Sigma_+ - 2) \Sigma_- - 2 N_+ N_- + (r - \dot{U} + 2A) N_+ + 2 R \Sigma_
\]

\[
N^{-1} \partial_t N_+ - E_1 \partial_x \Sigma_+ = (q + 3\Sigma_+) N_+ + 2 \Sigma_x N_x - (r - \dot{U}) \Sigma_+ - 2A R
\]

\[
\frac{f_1}{\Omega} (N^{-1} \partial_t + \frac{\gamma}{G_+} v E_1 \partial_x) \Omega + f_2 E_1 \partial_x v = 2 \frac{\gamma}{G_+} f_1 \left[ \frac{G_+}{\gamma} (q + 1) - \frac{1}{2} (1 - 3\Sigma_+) (1 + v^2) \right] - 1 + (r - \dot{U} + A) v \]

\[
\frac{f_2}{f_1} \Omega (N^{-1} \partial_t - \frac{f_3}{G_+ G_-} v E_1 \partial_x) v + f_2 E_1 \partial_x \Omega = 2 \frac{f_2}{f_1 G_-} \Omega (1 - v^2) \left[ \frac{(\gamma - 1)}{\gamma} (1 - v^2) r - \frac{1}{2} (2 - \gamma) (1 - 3\Sigma_+ + v - \frac{1}{2} G_- \dot{U} \right],
\]

where $f_1, f_2, f_3$ and $G_\pm$ are defined by Eqs. (30) and (11), respectively.

Defining equations for $q$ and $r$:

\[
q := \frac{1}{2} + \frac{1}{2} (2 \dot{U} - A) A + \frac{3}{2} (\Sigma_+^2 + N_+^2 + \Sigma_x^2 + N_x^2) + \frac{3}{2} (\gamma - 1) + \frac{1}{2} (\gamma - 1) \Omega - \frac{1}{2} \Omega \Lambda
\]

\[
r := -3 A \Sigma_+ - 3 (N_+ \Sigma_+ - N_- \Sigma_-) - \frac{3}{2} \frac{\gamma}{G_+} \Omega v.
\]
3 GAUGE CHOICES

Constraint equations:

\[ 0 = (C_{\text{Gauß}}) = \Omega_k - 1 + 2\Sigma_+ + \Sigma_+^2 + \Sigma_\times^2 + \Omega + \Omega_A \tag{56} \]
\[ 0 = (C_A) = (E_1^1 \partial_x - 2r) \Omega_A , \tag{57} \]

where

\[ \Omega_k := -\frac{2}{3} (E_1^1 \partial_x - r) A + A^2 + N_\times^2 + N_\times^2 . \tag{58} \]

Gauge fixing condition:

\[ 0 = (C_U) := N_\times^{-1} E_1^1 \partial_x N + (r - \dot{U}) . \tag{59} \]

Supplementary equations

The quantity \((q + 3\Sigma_+)\) occurs frequently in the scale-invariant equation system. Combining the definition of \(q\) given in Eq. (54) with the Gauß constraint equation (56) solved for \(\Sigma_+\), one can express this quantity by

\[ (q + 3\Sigma_+) = 2 + (E_1^1 \partial_x - r) A + A^2 + N_\times^2 + N_\times^2 . \tag{60} \]

In terms of our scale-invariant dependent variables, the area density \(A\) of the \(G_2\)-orbits satisfies the relations

\[ A^{-1} N^{-1} \partial_t A = 2 , \quad A^{-1} E_1^1 \partial_x A = -2 A . \tag{61} \]

Combining the two, the magnitude of the spacetime gradient \(\nabla_a A\) is

\[ (\nabla_a A) (\nabla^a A) = -4\beta^2 (1 - A^2) A^2 , \tag{62} \]

which is timelike for \(A^2 < 1\).

3 Gauge choices

In this section we discuss the gauge problem.

3.1 Gauge freedom

The scale-invariant equation system in subsection 2.2 contains evolution equations for the dependent variables

\[ \{ E_1^1, \Sigma_+, A, N_+, \Omega_\times, \Sigma_-, N_\times, \Sigma_\times, N_\times, \Omega, v \} , \tag{63} \]

but not for the gauge source functions

\[ \{ N, \dot{U}, R \} , \tag{64} \]

and thus does not uniquely determine the evolution of the \(G_2\) cosmologies. The reason for this deficiency is that the orthonormal frame \(\{ e_a \}\) and the local coordinates \(\{ t, x \}\) were not specified uniquely in subsection 2.1. We now summarise the remaining freedom, which we refer to as the gauge freedom.

(i) Choice of timelike reference congruence \(e_0\).

The gauge freedom is a position-dependent boost

\[ \left( \begin{array}{c} \hat{e}_0 \\ \hat{e}_1 \end{array} \right) = \Gamma \left( \begin{array}{cc} 1 & w \\ w & 1 \end{array} \right) \left( \begin{array}{c} e_0 \\ e_1 \end{array} \right) , \quad \Gamma := \frac{1}{\sqrt{1 - w^2}} , \quad w = w(t, x) , \tag{65} \]

in the timelike 2-spaces orthogonal to the \(G_2\)-orbits.

(ii) Choice of local time and space coordinates \(t\) and \(x\).

The gauge freedom is the coordinate reparametrisation

\[ \dot{t} = \dot{t}(t) , \quad \dot{x} = \dot{x}(x) . \tag{66} \]
3 GAUGE CHOICES

(iii) Choice of spatial frame vector fields $e_2$ and $e_3$.

The gauge freedom is a position-dependent rotation

$$
\begin{pmatrix}
\dot{e}_2 \\
\dot{e}_3
\end{pmatrix} =
\begin{pmatrix}
\cos\varphi & \sin\varphi \\
-\sin\varphi & \cos\varphi
\end{pmatrix} \begin{pmatrix}
e_2 \\
e_3
\end{pmatrix}, \quad \varphi = \varphi(t, x),
$$

(67)

in the spacelike 2-spaces tangent to the $G_2$-orbits.

We say that (i) and (ii) constitute the temporal gauge freedom and (iii) represents the spatial gauge freedom.

Table 1 shows possible ways of fixing the spatial gauge by requiring one frame vector field or a combination of frame vector fields to be parallel to a Killing vector field. Each of these sets of conditions is preserved under evolution and under a boost. These choices are essentially all equivalent. We will routinely make the first choice, namely

$$
N_+ - \sqrt{3}N_- = 0 = R + \sqrt{3}\Sigma_x
$$

(68)

Table 1: Spatial gauge conditions for aligning a combination of the frame vector fields $e_A$ with a KVF.

| Spatial gauge condition | Frame vector field parallel to a KVF |
|------------------------|------------------------------------|
| $N_+ - \sqrt{3}N_- = 0 = R + \sqrt{3}\Sigma_x$ | $e_2$ |
| $N_+ + \sqrt{3}N_- = 0 = R - \sqrt{3}\Sigma_x$ | $e_3$ |
| $N_+ + \sqrt{3}N_\times = 0 = R + \sqrt{3}\Sigma_\times$ | $e_2 - e_3$ |
| $N_+ - \sqrt{3}N_\times = 0 = R - \sqrt{3}\Sigma_\times$ | $e_2 + e_3$ |

With this choice the evolution equation (46) becomes identical to Eq. (51), and thus can be omitted from the full scale-invariant equation system. Other interesting choices for fixing the spatial gauge do exist, however, such as a Fermi-propagated frame, for which $R = 0$.

3.2 Fixing the temporal gauge

Within the present scale-invariant formulation of the dynamics of orthogonally transitive $G_2$ cosmologies with perfect fluid matter source we will fix the temporal gauge by adapting the evolution of the gauge source function $\dot{U}$ to the following four geometrical features, listed in order of subsequent discussion.

(i) Adapt the evolution to a family of null characteristic 3-surfaces.

(ii) Adapt the evolution to the integral curves determined by the spacetime gradient of the area density of the $G_2$-orbits, $\nabla_{\nu}A$.

(iii) Adapt the evolution to the family of fluid sound characteristic 3-surfaces.

(iv) Adapt the evolution to zero-velocity characteristic 3-surfaces associated with a family of freely-falling observers.

The idea is to specialise $e_0$ in such a way that either $N^{-1} \partial_t \dot{U}$ or $\dot{U}$ itself is determined in terms of the other dependent variables. Then $N$ is determined from Eq. (59) up to an arbitrary dimensionless multiplicative function $f(t)$. We then use a reparametrisation of $t$ to choose

$$
f(t) = e^{Ct},
$$

(69)

where $C$ is an arbitrary constant. This coordinate choice leads to an autonomous differential equation for $N$ which we include in the evolution system, giving a fully determined autonomous scale-invariant equation system.

It should be pointed out that apart from the above four choices of temporal gauge other interesting possibilities such as, e.g., a constant area expansion rate gauge, where $r = 0 \Leftrightarrow \beta = \beta(t)$, do exist.
3 GAUGE CHOICES

3.2.1 Null cone gauge

The first choice of gauge, which we call the null cone gauge, is motivated by the identity
\[ N^{-1} \partial_t (r - \dot{U}) - E_1^1 \partial_x (q + 3\Sigma_+) = -N^{-1} E_1^1 \partial_t \partial_x [\ln(N E_1^1)] + N^{-1} \partial_t (C_U), \]  
(70)
which follows from combining Eqs. (59) and (43). It suggests that we impose the condition
\[ 0 = N^{-1} \partial_t (r - \dot{U}) - E_1^1 \partial_x (q + 3\Sigma_+) \]  
(71)
on \[ N^{-1} \partial_t \dot{U}. \]  
It follows immediately from these two relations and Eq. (42) that
\[ N = \frac{f(t)g(x)}{E_1^1} \]  
(72)
and
\[ \frac{1}{2} N^{-1} \partial_t \dot{U} + E_1^1 \partial_x \Sigma_+ = \frac{1}{2} (q + 3\Sigma_+) \dot{U} - \frac{1}{3} (r - \dot{U}) (1 - 3\Sigma_+) , \]  
(73)
provided that the gauge fixing condition (59) propagates along \( e_0 \) according to Eq. (188) in the appendix. It follows from Eq. (72) and (59) that
\[ \partial_x E_1^1 = (r - \dot{U}) + \frac{1}{g(x)} \frac{dg(x)}{dx} E_1^1 . \]  
(74)
We now use the reparametrisations (66) to set \( f(t) = e^{C_{uc} t} \) and \( g(x) = e^{D_{uc} x} \), where \( C_{uc} \) and \( D_{uc} \) are constants. Then Eqs. (72) and (74) form a set of two constraints,
\[ N E_1^1 = e^{C_{uc} t + D_{uc} x} \]  
(75)
\[ \partial_x E_1^1 = (r - \dot{U}) + D_{uc} E_1^1 . \]  
(76)
On differentiating Eq. (72) and using Eq. (43), we obtain an evolution equation for \( N \) that reads
\[ N^{-1} \partial_t N = - (q + 3\Sigma_+) N + C_{nc} . \]  
(77)
Note that in the null cone gauge Eqs. (44) and (73) form the \( (\Sigma_+, \dot{U}) \)–branch of an autonomous evolution system in FOSH format. The associated characteristic propagation velocities are \( \lambda = \pm 1 \).

Choosing the null cone gauge permits one to introduce the familiar conformal coordinates \( \{ t, x \} \) in the timelike 2-spaces orthogonal to the \( G_2 \)–orbits. Referring to Eq. (72), we use the coordinate reparametrisation (66) to set \( f(t) = 1 \) and \( g(x) = 1 \), so that \( N E_1^1 = 1 \). It follows from Eq. (36) that \( N e_1^1 = 1 \), which implies, using Eqs. (2), that the line element in the timelike 2-spaces orthogonal to the \( G_2 \)–orbits has the form
\[ (2)ds^2 = N^2 (-dt^2 + dx^2) . \]  
(78)
Conformal coordinates have been frequently used in the analytical study of vacuum \( G_2 \) cosmologies, and in the derivation of exact solutions, both for vacuum and for models with a perfect fluid matter source. Selected references from the literature are Gowdy [24, 25], Isenberg and Moncrief [33], Hübner [32], Kichenassamy and Rendall [34], Senovilla and Vera [46] and Anguige [2].

3.2.2 Area gauges

The separable area gauge is determined by imposing the condition
\[ 0 = (r - \dot{U}) , \]  
(79)
which determines \( \dot{U} \) algebraically through Eq. (55). There is thus no need to determine an evolution equation for \( \dot{U} \). It follows immediately from the gauge fixing condition (59) that \( N = f(t) \). We now use the \( t \)-reparametrisation (66) to set \( f(t) = N_0 \), a constant, i.e.,
\[ N = N_0 . \]  
(80)
3 GAUGE CHOICES

In this case the evolution equation for $N$ is trivial, i.e.,

$$N^{-1} \partial_t N = 0 .$$

(81)

It follows from Eqs. (61) and (80) that the area density has the form

$$A = \ell^2_0 e^{2N_0 t} m(x) ,$$

(82)

where here and throughout $\ell_0$ denotes the unit of the physical dimension [length], and $m(x)$ is a positive function of $x$. The gauge fixing condition (79) propagates along $e_0$ according to Eq. (188) in the appendix subject to an auxiliary equation for $N^{-1}_0 \partial_t \bar{U}$. Note that the separable area gauge does not in general yield an evolution system in FOSH format.

For the class of $G_2$ cosmologies in which the spacetime gradient $\nabla_a A$ is timelike, we can strengthen the separable area gauge condition (79) by requiring in addition that

$$A = \ell^2_0 e^{2N_0 t} .$$

(83)

which we achieve by choosing $e_0$ to be parallel to $\nabla_a A$. It follows from Eq. (61) that $\partial_x A = 0$, and Eq. (82) reduces to

$$A = \ell^2_0 e^{2N_0 t} .$$

(84)

This defines the so-called area time coordinate. Observe that condition (83) is invariant, by virtue of Eqs. (45) and (79). We shall refer to the gauge choices (79) and (83) as the timelike area gauge. We note that in this case, with $\Sigma_+$ and $\bar{U}$ algebraically determined in terms of the other scale-invariant dependent variables from, respectively, the Gauss constraint equation (56) and Eqs. (79) and (55), the evolution system becomes unconstrained and does assume FOSH format. We give the resultant equation system in subsection 4.4. Selected references from the literature using the timelike area gauge are Berger and Moncrief [8], Hern and Stewart [26] and Rendall and Weaver [42].

3.2.3 Fluid-comoving gauge

The fluid-comoving gauge is determined by choosing $e_0$ to be equal to the fluid 4-velocity field $\bar{u}$. By virtue of Eq. (9), this choice is equivalent to imposing the condition

$$0 = v .$$

(85)

The evolution equations (52) and (53) for $\Omega$ and $v$ now reduce to

$$N^{-1} \partial_t \Omega = 2 \left[ (q + 3\Sigma_+) - 2 + \frac{3}{2} (2 - \gamma) (1 - \Sigma_+) \right] \Omega ,$$

$$0 = (C_{FF}) := \left[ (\gamma - 1) (E_1^1 \partial_x - 2r) + \bar{U} \right] \Omega .$$

(86)

(87)

The evolution equation for $\bar{U}$ (which is now identified with the $\beta$-normalised fluid acceleration) results from demanding that the new constraint equation (87) propagates along $\bar{u}$. This leads to

$$\frac{1}{3} N^{-1} \partial_t \bar{U} + (\gamma - 1) E_1^1 \partial_x \Sigma_+ = \frac{1}{3} (q + 3\Sigma_+) \bar{U} - (\gamma - 1) (1 - \Sigma_+) (r - \bar{U}) .$$

(88)

It then follows that the gauge fixing condition (59) propagates along $\bar{u}$ according to Eq. (190) in the appendix. Furthermore, it follows from Eqs. (87) and (59) that

$$N = f(t) (\ell_0^2 \beta^2 \Omega)^{-(\gamma - 1)/\gamma} .$$

(89)

We now use the $t$-reparametrisation (66) to set $f(t) = e^{C_{co} t}$. On differentiating Eq. (89) and using Eqs. (40) and (86), we obtain an evolution equation for $\dot{N}$ that reads

$$N^{-1} \partial_t \dot{N} = - \left[ (q + 3\Sigma_+) - 2 + 3 (2 - \gamma) (1 - \Sigma_+) \right] \dot{N} + C_{co} .$$

(90)

9This equation is a scale-invariant form of the well-known dimensional relation $N = a(t) \mu^{-(\gamma - 1)/\gamma}$ for perfect fluid models with equation of state $p(\mu) = (\gamma - 1) \mu$ in fluid-comoving gauge; see, e.g., Ref. [49].
In order to obtain an evolution system in FOSH format, we need to multiply Eq. (44) by a factor of $(\gamma - 1)$, and write it in the form

$$
(\gamma - 1) \left( 3N^{-1} \partial_t \Sigma_+ + E_1^1 \partial_x \dot{U} \right) = - (\gamma - 1) \left[ (q + 3\Sigma_+) (1 - 3\Sigma_+) + 2q - 6 (\Sigma_+^2 + \Sigma_+^2) 
- (r - \dot{U} + 2A) \dot{U} - \frac{3}{2} (3\gamma - 2) \Omega + 3\Omega_A \right].
$$

(91)

When we adjoin Eqs. (88), (90) and (91) to the full scale-invariant equation system, with the fluid evolution equations (52) and (53) replaced by Eq. (86) and the new constraint equation (87), and Eq. (44) replaced by Eq. (91), we obtain a new evolution system that has FOSH format, with the fluid dynamical sector being shifted from the $(\Omega, \nu)$-branch to the $(\Sigma_+, \dot{U})$-branch, with characteristic velocities given by $\lambda_{4,5} = \pm (\gamma - 1)^{1/2}$. The family of sound characteristic 3-surfaces thus becomes symmetrically embedded inside the family of null characteristic 3-surfaces. In the case of a pressure-free fluid, $\gamma = 1 \Leftrightarrow \dot{U} = 0$, Eq. (91) does apply without the common factor $(\gamma - 1)$.

When doing numerical experiments in the present framework, the fluid-comoving gauge will have the advantage that, in view of the linear equation of state (1), the effective semi-linearity of the principal part of Eqs. (91) and (88) will prevent the development of shocks in the fluid dynamical sector of the evolution system. Hence, only the propagation of so-called contact discontinuities is possible in both the gravitational field and the fluid dynamical sectors. Another advantage of this gauge is that it makes direct physical interpretation possible in terms of kinematical fluid quantities.

Examples of references employing the fluid-comoving gauge are Eardley et al. [14], Liang [35], Wainwright and Goode [53] and Senovilla [44].

### 3.2.4 Synchronous gauge

The *synchronous gauge* is determined by choosing $e_0$ to be a timelike reference congruence that is geodesic, i.e., we impose the condition

$$
0 = \dot{U}.
$$

(92)

It follows from the gauge fixing condition (59) and Eq. (41) that

$$
\mathcal{N} = f(t) (e_0 \beta) .
$$

(93)

We now use the $t$-reparametrisation (66) to set $f(t) = e^{C_{\text{sync}} t}$. On differentiating Eq. (93) and using Eq. (40), we obtain an evolution equation for $\mathcal{N}$ that reads

$$
\mathcal{N}^{-1} \partial_t \mathcal{N} = -(q + 1) \mathcal{N} + C_{\text{sync}} .
$$

(94)

The gauge fixing condition (59) presently propagates along $e_0$ according to Eq. (191) in the appendix. When we adjoin Eq. (94) to the full scale-invariant evolution system, simplified using Eq. (92), we again obtain FOSH format. In a more general context, the synchronous gauge has been made prominent in particular by the work of BKL [5, 6].

## 4 Scale-invariant dynamical state space

The description of the dynamics of $G_2$ cosmologies is complicated by the fact that the scale-invariant dynamical state vector, and hence the structure of the scale-invariant dynamical state space, depends on the choice of temporal gauge. In this section we discuss this issue, and we explain which properties of the scale-invariant equation system and of the dynamical state space are independent of the choice of gauge. For simplicity, in the present discussion we set the cosmological constant to zero, $\Omega_\Lambda = 0$.

### 4.1 Overview

We assume that the spatial gauge has been fixed according to Eq. (68). Once we choose a specific temporal gauge, the equation system derived in subsection 2.2 gives an explicit set of evolution and constraint equations for a finite-dimensional dynamical state vector $X$. These equations can be written concisely in the following form, where the FOSH nature of the evolution part is indicated by the fact that
the coefficient matrices $A(X)$ and $B(X)$ are symmetric, with $A(X)$ being positive definite.

Evolution system:

$$A(X) \partial_0 X + B(X) \partial_1 X = F(X),$$  \hspace{1cm} (95)

and

Constraint equations:

$$0 = C(X, \partial_1 X),$$  \hspace{1cm} (96)

where $\partial_0 := \mathcal{N}^{-1} \partial_t$ and $\partial_1 := E_1 \partial_x$. The dynamical state vector $X$ depends on the choice of temporal gauge as follows:

$$X = X_g \oplus X_w,$$  \hspace{1cm} (97)

where $X_g$ is the temporal gauge-dependent part, and

$$X_w = (\Sigma_-, N_x, \Sigma_x, N_-)^T,$$  \hspace{1cm} (98)

the temporal gauge-independent part describing the dynamical degrees of freedom in the gravitational field.

| Temporal gauge | Gauge-dependent dynamical state vector | Constraint equations | No. of independent functions in $X_g$ |
|----------------|--------------------------------------|----------------------|--------------------------------------|
| Null cone      | $(\mathcal{N}, E_1^1, A, \Sigma_+, U, \Omega, v)^t$ | $(\mathcal{L}_Y), (\mathcal{C}_{\text{Gauß}})$, Eqs. (75), (76) | 3 |
| Timelike area  | $(E_1^1, \Omega, v)^t$ | none | 3 |
| Fluid-comoving | $(\mathcal{N}, E_1^1, A, \Sigma_+, U, \Omega)^t$ | $(\mathcal{L}_Y), (\mathcal{C}_{\text{Gauß}}), (\mathcal{C}_{\text{PF}})$ | 3 |
| Synchronous    | $(\mathcal{N}, E_1^1, A, \Sigma_+, \Omega, v)^t$ | $(\mathcal{L}_Y), (\mathcal{C}_{\text{Gauß}})$ | 4 |

Table 2: Form of $X_g$, relevant constraint equations, remaining number of independent functions in $X_g$.

In Tab. 2 we give the form of $X_g$, the number of constraint equations to be taken into account, and the remaining number of independent functions in $X_g$. In each case, except for the null cone gauge, one can use the reparametrisation (66) of $x$ to set $E_1^1 = 1$ initially, thereby reducing the number of independent functions by one. In the timelike area and fluid-comoving gauges $\mathbf{e}_0$ is uniquely fixed, whereas freedom remains in the null cone and synchronous gauges, which reduces the number of independent functions by one in these cases. (This freedom corresponds to a conformal coordinate freedom in the timelike 2-spaces orthogonal to the $G_2$-orbits.) Thus, when the gauge is uniquely fixed, $X_g$ will contain two independent functions, which, with Eq. (98), gives a total of six dynamical degrees of freedom.

### 4.2 Familiar solutions as invariant submanifolds

The $G_2$ cosmologies contain a rich variety of familiar classes of solutions as special cases. In this subsection we indicate how these classes of solutions arise as invariant subsets in the scale-invariant dynamical state space. In each case the related equation system can be obtained by specialising the general scale-invariant equation system and making an appropriate choice of gauge.

#### 4.2.1 Vacuum $G_2$ cosmologies

These solutions are described by the subset

$$0 = \Omega,$$  \hspace{1cm} (99)

which is invariant since $\Omega = 0$ implies $\partial_t \Omega = 0$ by Eq. (52). If the spacetime gradient $\nabla_n A$ is timelike, one can use the timelike area gauge. The resulting unconstrained evolution system, which has FOSH format, is given in subsection 4.4 [Eqs. (125) – (127)]. If the spatial topology is $T^3$, these equations describe the Gowdy vacuum spacetimes that can contain gravitational radiation with two polarisation states [24, 25].

It should be noted that the evolution equation (53) for $v$ is singular on the vacuum boundary $\Omega = 0$ due to the fact that if one solves for $\partial_x v$, one obtains the singular term $\partial_x \Omega / \Omega$. This fact means that care has to be taken in taking limits as $\Omega \to 0$, unless one is working in the fluid-comoving gauge, in which case this problem does not arise.
4 SCALE-INvariant Dynamical State Space

4.2.2 Diagonal $G_2$ cosmologies

The orthogonally transitive $G_2$ cosmologies have in general four dynamical degrees of freedom in the gravitational field, of two different polarisation states, that are associated with the null characteristic eigenfields ($\Sigma_- + N_\chi$) and ($\Sigma_\chi + N_- \mp \Sigma_\pm + N_- \pm \Sigma_- N_\chi$). With the spatial gauge choice (68), it follows that the conditions

$$0 = \Sigma_\chi = N_-$$

(100)

define an invariant submanifold, which corresponds to $G_2$ cosmologies with one possible polarisation state only. We shall refer to this class of solutions as diagonal $G_2$ cosmologies, because for them the line element can be written in diagonal form (since both Killing vector fields are hypersurface orthogonal; cf. WE [52]).

4.2.3 Plane symmetrical $G_2$ cosmologies

Specialising further, we can eliminate both polarisation states by considering the invariant submanifold

$$0 = \Sigma_\chi = N_- = \Sigma_\pm = N_\chi$$

(101)

which describes the class of plane symmetrical $G_2$ cosmologies (the isometry group here is a $G_3$ acting multiply-transitively on flat spacelike 2-surfaces). These solutions are the plane symmetrical analogues of the well-known spherically symmetrical Lemaître–Tolman–Bondi models, in general with non-zero fluid pressure (see Stewart and Ellis [48] and Eardley et al [14]). When $\gamma = 1$ the evolution system reduces to a set of ODE.

4.2.4 Self-similar $G_2$ cosmologies

It is of interest to consider the $G_2$ cosmologies that correspond to the equilibrium points (i.e., fixed points) of the evolution system (95), that are defined by the condition

$$0 = \partial_t X$$

(102)

This condition means that the dynamical state vector $X$ is constant on those timelike 3-surfaces whose spacelike normal congruence is $e_1$. It follows that these 3-surfaces are the orbits of a 3-parameter homothety group $H_3$, i.e., solutions of this kind are self-similar. It is important to note that the condition (102) should be imposed before specifying the temporal gauge, since it uniquely fixes the gauge by specifying the timelike frame vector field $e_0$. Indeed, Eq. (102) implies that both the separable area gauge condition (79) and the null cone gauge condition (71) are satisfied.

Under these conditions, the evolution system (95) reduces to a set of ODE that govern the spatial dependence of the models. In other words, the condition (102) defines a finite-dimensional submanifold of the infinite-dimensional dynamical state space. Specialising further, the condition $v = 0$ defines a smaller invariant set of solutions consisting of self-similar models whose fluid 4-velocity field $\tilde{u}$ is tangent to the $H_3$–orbits. These solutions, which we shall refer to as fluid-aligned self-similar $G_2$ cosmologies, have been analysed qualitatively in some detail by Hewitt et al [28, 31, 27]. They are of interest as potential future asymptotic states for more general $G_2$ cosmologies.

4.2.5 Spatially homogeneous $G_2$ cosmologies

The conditions

$$0 = \partial_t X, \quad 0 = \dot{U}$$

(103)

define a finite-dimensional invariant submanifold of the infinite-dimensional dynamical state space corresponding to $G_2$ SH models, which admit a $G_3$ isometry group acting transitively on the spacelike 3-surfaces orthogonal to $e_0$. As with Eq. (102), the condition (103) should be imposed before specifying the temporal gauge, since it likewise fixes the gauge by specifying the timelike frame vector field $e_0$. Indeed, Eq. (103) implies that both the separable area gauge condition (79) and the null cone gauge condition (71) are satisfied.

10 All perfect fluid SH cosmologies of Bianchi Type-I to Type-VII, apart from the exceptional Type-VI$_{1/2}$, admit an Abelian $G_2$ subgroup which acts orthogonally transitively, and are hence included.
Under these conditions, the evolution system (95) reduces to a set of ODE which determines the dynamical evolution of the models. Since $\mathbf{\theta}_F \mathbf{X} = 0$, the $E_1^1$-equation (43) decouples from the full system. Thus, if one is only interested in the evolution of $G_2$ SH models, all relevant information is given by the remaining equations, which are analogous to the equation systems studied in WE, but with $H$-normalisation replaced by $\beta$-normalisation. However, since we are interested in how the $G_2$ SH models are related to the $G_2$ cosmologies, it is necessary to retain the $E_1^1$-equation. Specialising further, the condition $v = 0$ defines a smaller invariant set of solutions, the so-called non-tilted SH models, in which the fluid 4-velocity field $\mathbf{u}$ is orthogonal to the $G_3$-orbits.

The equilibrium points of SH dynamics, i.e., cosmologies that admit an $H_4$ acting transitively on spacetime, play an important rôle in the $G_2$ dynamical state space. There are two main subclasses. Firstly, those equilibrium points that satisfy $(q + 3\Sigma_+) \neq 0$ must satisfy $E_1^1 = 0$, on account of Eq. (43). They are thus constrained to lie in the unphysical boundary $E_1^1 = 0$ (see subsection 4.3), and hence can potentially affect the $G_2$ dynamics near the cosmological initial singularity. The most important examples are the Kasner equilibrium set (see subsections 4.3 and 5.2) and the flat FL equilibrium point (see subsection 5.4). Secondly, those SH equilibrium points that satisfy $(q + 3\Sigma_+) = 0$ lie in the physical part of the dynamical state space and hence can potentially affect the evolution at late times. The most important of these are the so-called plane-wave equilibrium points (see WE [52], Ch. 9).

In the present formulation it is possible to solve globally for $\Sigma_+$, from the Gauß constraint equation (56). So, for example, in the vacuum case one thus automatically obtains a reduced dynamical system whose dimension is equal to the number of dynamical degrees of freedom.\(^{11}\)

It is worth noting that if one introduces the standard Fermi-propagated diagonal frame for SH models of class A, which is not the default frame choice in our formulation, the Kasner set is represented by a parabola given by $2\Sigma_+ + \Sigma_+^2 = 1$, while the Type-II vacuum solutions are straight lines.

### 4.3 Unphysical boundary

The evolution equation (43) for the scale-invariant frame variable $E_1^1$ shows that the set $E_1^1 = 0$ defines an invariant submanifold in an arbitrary gauge. This invariant submanifold divides the dynamical state space into two disjoint invariant submanifolds given by $E_1^1 > 0$ and $E_1^1 < 0$. The full scale-invariant equation system is, however, invariant under the discrete symmetry

$$\begin{align*}
(x, E_1^1) &\quad \mapsto \quad (-x, -E_1^1) .
\end{align*}$$

The two invariant submanifolds are thus physically equivalent, and without loss of generality we can restrict our considerations to the case

$$E_1^1 > 0 .$$

The set $E_1^1 = 0$ corresponds to unphysical states for which the area expansion rate $\beta$ diverges ($\beta \to +\infty$), typically leading to a spacetime singularity. We shall refer to the invariant submanifold $E_1^1 = 0$ as the unphysical boundary of the infinite-dimensional dynamical state space. It is significant that the evolution system is well-defined on the unphysical boundary $E_1^1 = 0$. Indeed Eq. (95) reduces to

$$\mathbf{A}(\mathbf{X}) \mathbf{\theta}_F \mathbf{X} = \mathbf{F}(\mathbf{X}) ,$$

i.e., a system of ODE, on the unphysical boundary. It is important to note that the solutions of Eq. (106), regarded as solutions of the full evolution system (95), have arbitrary $x$-dependence.\(^{12}\) These solutions thus do not, in general, correspond to solutions of the EFE, and in this sense they are unphysical. Nevertheless, they do play a significant rôle in the evolution of $G_2$ cosmologies. The key point is that if an orbit in the physical part of the dynamical state space with $E_1^1 > 0$ approaches the unphysical boundary as $t \to -\infty$, then it will shadow orbits in this boundary, i.e., the dynamics in the unphysical boundary will determine the asymptotic dynamics of $G_2$ cosmologies that are solutions to the EFE and are thus regarded physical. In the unphysical boundary there is a hierarchy of invariant submanifolds that influence the asymptotic dynamics of $G_2$ cosmologies. Firstly note that on the unphysical boundary the

\(^{11}\)This there exists no drawback with the orthonormal frame approach as recently indicated by Szydowski and Demaret [50] (of course, a $\beta$-normalisation can also be done in the Type-VIII and Type-IX cases).

\(^{12}\)An important example of such a solution is a Kasner metric whose Kasner exponents, instead of being constants, depend on the local coordinate $x$.\)
where the dependence of the gauge fixing condition (59) reduces to \((r - U) = 0\), which, in conjunction with the integrability condition (42), implies that

\[
\mathcal{N}^{-1} \partial_t \dot{U} = (q + 3\Sigma_+ \dot{U}) \ . 
\]

It follows that, in an arbitrary gauge, the condition \(\dot{U} = 0\) defines an invariant submanifold in the unphysical boundary. The evolution equations for the invariant submanifold \(0 = E_1^1 = \dot{U}\) are precisely the evolution equations for SH models, as follows from subsection 4.2.

Within the invariant submanifold \(0 = E_1^1 = \dot{U}\), the vacuum subset \(\Omega = 0\) is invariant, and within this set is the Kasner invariant set, defined by

\[
0 = A = N_\chi = N_- \ . 
\]

These conditions imply, on account of the Gauß constraint equation (56) and Eqs. (58) and (60), that

\[
(q + 3\Sigma_+) = 2 \ , 
\]

and

\[
2\Sigma_+ + \Sigma_-^2 + \Sigma_\chi^2 = 1 \ . 
\]

The remaining evolution equations are

\[
\begin{align*}
\mathcal{N}^{-1} \partial_t \Sigma_+ & = 0 \quad (111) \\
\mathcal{N}^{-1} \partial_t \Sigma_- & = 2\sqrt{3} \Sigma_-^2 \quad (112) \\
\mathcal{N}^{-1} \partial_t \Sigma_\chi & = -2\sqrt{3} \Sigma_- \Sigma_\chi \ . 
\end{align*} 
\]

where the dependence of \(\Sigma_+\), \(\Sigma_-\) and \(\Sigma_\chi\) on the local coordinate \(x\) is unrestricted. The Kasner equilibrium points are given by

\[
\Sigma_\chi = 0 \ , \quad 2\Sigma_+ + \Sigma_-^2 = 1 \ , 
\]

where \(\Sigma_-\) is an arbitrary function of \(x\). These conditions define a Kasner parabola \(K\). Intuitively speaking, the orbits in the Kasner invariant set, including the Kasner equilibrium points, describe a \(G_2\) cosmology with the evolution of a Kasner vacuum solution of the EFE, but with unrestricted \(x\)-dependence.

### 4.4 Timelike area gauge

In this subsection we give the evolution system in the timelike area gauge, which was introduced in subsection 3.2.2. For simplicity, we assume that the cosmological constant is zero, \(\Omega_{\Lambda} = 0\). This gauge has the advantage of leading to an unconstrained evolution system in FOSH format, as follows.

\[
\begin{align*}
\mathcal{N}_0^{-1} \partial_t E_1^1 & = (q + 3\Sigma_+) E_1^1 \\
\mathcal{N}_0^{-1} \partial_t \Sigma_- + E_1^1 \partial_x N_- & = (q + 3\Sigma_- - 2) \Sigma_- + 2\sqrt{3} \Sigma_-^2 - 2\sqrt{3} N_-^2 \\
\mathcal{N}_0^{-1} \partial_t N_\chi + E_1^1 \partial_x \Sigma_- & = (q + 3\Sigma_+) N_\chi \\
\mathcal{N}_0^{-1} \partial_t \Sigma_\chi - E_1^1 \partial_x N_- & = (q + 3\Sigma_+ - 2 - 2\sqrt{3} \Sigma_-) \Sigma_\chi - 2\sqrt{3} N_\chi N_- \\
\mathcal{N}_0^{-1} \partial_t N_- - E_1^1 \partial_x \Sigma_\chi & = (q + 3\Sigma_+ + 2\sqrt{3} \Sigma_- N_- + 2\sqrt{3} \Sigma_\chi N_\chi \\
\end{align*} 
\]

\[
\frac{f_1}{\Omega} (\mathcal{N}_0^{-1} \partial_t + \frac{\gamma}{G_+} v E_1^1 \partial_x) \Omega + f_2 E_1^1 \partial_x v = \frac{2\gamma}{G_+} f_1 \\
\times [ \frac{G_+}{\gamma} (q + 1) - \frac{1}{2} (1 - 3\Sigma_+) (1 + v^2) - 1 ] \\
\frac{f_2}{f_1} \Omega (\mathcal{N}_0^{-1} \partial_t - \frac{f_3}{G_+ G_-} v E_1^1 \partial_x) v + f_2 E_1^1 \partial_x \Omega & = -\frac{f_2}{f_1 G_-} \Omega (1 - v^2) [ \frac{2 - \gamma}{\gamma} G_+ \dot{U} \\
& + (2 - \gamma) (1 - 3\Sigma_+) v - 2 (\gamma - 1) v ] \quad . 
\]

where

\[
(q + 3\Sigma_+) = 2 - \frac{3}{2} \frac{(2 - \gamma)}{G_+} (1 - v^2) \Omega \ . 
\]
The auxiliary variables $\Sigma_+$ and $\dot{U}$ are obtained from the Gauß constraint equation (56) and Eqs. (79) and (55) as

$$\Sigma_+ = \frac{1}{2} (1 - \Sigma_-^2 - \Sigma_+^2 - N_-^2 - \Omega) \quad (123)$$

$$\dot{U} = r = -3 (N_+ \Sigma_+ - N_- \Sigma_-) - \frac{3}{2} \frac{\gamma}{G_+} \Omega v . \quad (124)$$

Note that in the present case we have from Eqs. (122) and (123) that $q \geq \frac{1}{2}$, which, on account of Eq. (40), guarantees that $\beta$ is a monotone function.

4.4.1 Gowdy vacuum spacetimes

The vacuum subcase of Eqs. (115) – (122) describes amongst others the Gowdy spacetimes with spatial topology $\mathbb{R}^3$ [24, 25]. In characteristic normal form these equations can be written as

$$N_0^{-1} \partial_t E_1^1 = 2E_1^1 \quad (125)$$

$$(N_0^{-1} \partial_t \pm E_1^1 \partial_x) (\Sigma_+ \pm N_+) = (\Sigma_+ \pm N_+) - (\Sigma_+ \pm N_+) + 2\sqrt{3} (\Sigma_+ \mp N_-) (\Sigma_+ \pm N_-) \quad (126)$$

$$(N_0^{-1} \partial_t \pm E_1^1 \partial_x) (\Sigma_+ \mp N_-) = (\Sigma_+ \mp N_-) - \left[1 + 2\sqrt{3} (\Sigma_\pm \pm N_\pm)\right] (\Sigma_\pm \mp N_\mp). \quad (127)$$

Note that in the present case we can use a reparametrisation (66) of $x$ to set $\partial_x E_1^1 = 0$. We use this representation to exemplify the following three aspects, which hold for the whole class of $G_2$ cosmologies and, in suitably generalised form, indeed for any general cosmological model.

(i) As the source terms on the RHS of the gravitational field equations (126) and (127) and $E_1^1$ must be continuous for the PDE system to be well-defined in the ordinary sense, so are the four characteristic first derivatives on the LHS. This implies that there are four unrestricted first derivatives given by $(N_0^{-1} \partial_t \pm E_1^1 \partial_x) (\Sigma_+ \pm N_+)$ and $(N_0^{-1} \partial_t \mp E_1^1 \partial_x) (\Sigma_\pm \mp N_-)$, which can be thus interpreted as the arbitrary information (four free real-valued functions) that gravitational radiation can propagate.

(ii) Spacetimes with $0 = (\Sigma_\pm - N_-) = (\Sigma_\pm + N_-)$ do form an invariant submanifold of the dynamical state space; as mentioned before they correspond to the diagonal subcase for which the dynamical degrees of freedom in the gravitational field are “polarised”.

(iii) Right-propagating and left-propagating characteristic eigenfields of the gravitational field do not form invariant submanifolds of the dynamical state space; they cannot in general be separated from each other. Referring to the Weyl curvature components listed in the appendix, this reflects the fact that a typical $G_2$ cosmology (and any general cosmological model) is of algebraic Petrov type I.

5 Past asymptotics and the past attractor

In this section we discuss the asymptotic evolution of the class of orthogonally transitive $G_2$ cosmologies near the cosmological initial singularity. We present evidence to support the claim that the past attractor lies on the unphysical boundary, and is a subset of the Kasner parabola $K$. This analysis leads to a discussion of the notion of asymptotic silence, and the related conjecture BKL II. Because of orthogonal transitivity of the $G_2$ isometry group, we do not expect to find a past attractor of oscillatory nature. We also linearise the evolution equations about the flat FL equilibrium point, and relate the results to the concept of an isotropic initial singularity.

5.1 Past attraction to the unphysical boundary

We use the timelike area gauge characterised by Eqs. (79), (80) and (83), and the associated evolution system (115) – (122). The evolution equation (115) for $E_1^1$, in conjunction with Eq. (122), reads

$$N_0^{-1} \partial_t E_1^1 = \left[2 - 3 \left(\frac{2 \gamma}{G_+} + (1 - v^2) \Omega\right) \right] E_1^1 . \quad (128)$$

$^{13}$On the characteristic normal form of a FOSH evolution system, see Ref. [12].
5 PAST ASYMMETRIC AND THE PAST ATTRAJECTOR

For vacuum models, i.e., $\Omega = 0$, we can solve this ODE, obtaining $E_1^1 = b(x) \exp(2N_\delta t)$, which implies

$$\lim_{t \to -\infty} E_1^1 = 0 .$$

(129)

This equation will also hold for perfect fluid models which satisfy the requirement \(^{14}\)

$$\lim_{t \to -\infty} \Omega = 0 .$$

(130)

If the orbit of a $G_2$ cosmology satisfies Eq. (129), the orbit will approach the unphysical boundary, and then we expect that it will shadow orbits in the boundary, which are described by the system of ODE obtained from Eq. (95) by setting $E_1^1 = 0$. On the unphysical boundary, $\bar{U}$ satisfies the evolution equation (107) which is of the same form as Eq. (128). It follows that along orbits in the unphysical boundary $\lim_{t \to -\infty} \bar{U} = 0$, and so we expect that typical orbits will approach the invariant submanifold $0 = E_1^1 = \bar{U}$. As mentioned in subsection 4.3, the evolution system in this invariant submanifold is precisely the evolution system for SH models. We thus expect that SH dynamics will approximate $G_2$ dynamics asymptotically as $t \to -\infty$. These heuristic considerations suggest that we should consider the Kasner parabola $K$ in order to localise a possible past attractor.

5.2 Linearisation about the Kasner equilibrium set

In this subsection we perform a linearisation of the evolution system about the Kasner equilibrium points that form the parabola $K$, given by Eqs. (108) and (114). We thus linearise Eqs. (115) -- (122) about the values

$$\Sigma_\_ = \phi \Sigma_\_ (x) , \quad 0 = E_1^1 = N_\times = \Sigma_\_ = N_\_ = \Omega = v ,$$

(131)

where $\phi \Sigma_\_ (x)$ is an arbitrary function of $x$. We have to treat the evolution equation (121) for $v$ in a special manner, due to the fact that the term $E_1^1 \partial_t \Omega \Sigma_\_ \Sigma_\_ \Sigma_\_ \Omega$ is singular on $K$. We first linearise Eq. (120) for $\Omega$ and then use the solution of this equation to show that the singular term in Eq. (121) can be neglected when linearising it. Rescaling the time variable so that $N_\delta = 1$, we then obtain the following system of linear ODE in $t$, with $x$-dependent coefficients:

$$\partial_t E_1^1 = 2 E_1^1$$

(132)

$$\partial_t \Sigma_\_ = - \frac{3}{2} (2 - \gamma) \phi \Sigma_\_ \Omega$$

(133)

$$\partial_t N_\times = 2 N_\times - (\partial_x \Sigma_\_) E_1^1$$

(134)

$$\partial_t \Sigma_\_ = - 2 \sqrt{3} \phi \Sigma_\_ \Sigma_\_$$

(135)

$$\partial_t N_\_ = 2 \left(1 + \sqrt{3} \phi \Sigma_\_ \right) N_\_$$

(136)

$$\partial_t \Omega = \frac{3}{2} (2 - \gamma) \left(1 + \phi \Sigma_\_ \right) \Omega$$

(137)

$$\partial_t v = \frac{1}{2} \left(3 \gamma - 2 - 3 (2 - \gamma) \phi \Sigma_\_ \right) v + 3 \left(\frac{2 - \gamma}{\gamma}\right) \phi \Sigma_\_ N_\_ .$$

(138)

The general solution of Eq. (132) is

$$E_1^1 = b(x) e^{2t} .$$

(139)

We can use a reparametrisation (66) of $x$ to set $b(x) = 1$. The resulting general solution of the linear ODE system is then given by

$$\Sigma_\_ = - \frac{1}{\sqrt{3}} \phi k(x) \left[1 - \frac{a_5(x)}{1 + \frac{1}{2} \phi k(x)^2} e^{\frac{3}{2}(2 - \gamma) |1 + \frac{1}{2} \phi k(x)^2| t}\right]$$

(140)

$$N_\times = \left[a_2(x) + \frac{1}{2} \phi k(x)^2 \partial_x k(x)\right] e^{2t}$$

(141)

$$\Sigma_\_ = a_3(x) e^{2k(x)t}$$

(142)

$$N_\_ = a_4(x) e^{2(1 - k(x))t}$$

(143)

$$\Omega = a_5(x) e^{\frac{3}{2}(2 - \gamma) |1 + \frac{1}{2} k(x)^2| t}$$

(144)

\(^{14}\)The physical interpretation of this requirement is that matter does not affect the dynamics near the cosmological initial singularity. According to conjecture BKL I this condition will be satisfied except for special classes of models.
\[ v = a_6(x) e^{\frac{1}{2}[3\gamma - 2 - (2-\gamma)k^2(x)]t} - \frac{2}{\sqrt{3}} \frac{k(x)}{\gamma [1 + \frac{1}{4}k^2(x)]} e^{2t} \]

\[ \times \left[ a_2(x) + \frac{1}{\sqrt{3}} \partial_x k(x) \left( t - \frac{2}{3(2-\gamma)[1 + \frac{1}{4}k^2(x)]} \right) \right]. \] \hfill (145)

For convenience, and to agree with the notation of Rendall and Weaver [42], we write

\[ \delta \Sigma_-(x) = -\frac{1}{\sqrt{3}} k(x) \] \hfill (146)

for the limiting value of \( \Sigma_- \) as \( t \to -\infty \). The solution of the linear ODE system suggests that an arc \( K_A \) of the Kasner equilibrium set \( K \) attracts neighbouring orbits (i.e., is a local attractor). This arc is defined by the requirement that in each exponential function in the solution (140) - (145) the independent variable \( t \) has a positive coefficient, so that the solution approaches the equilibrium point (131) as \( t \to -\infty \). The size of the arc depends on whether the model is polarised (i.e., diagonal) or not, and on the equation of state parameter \( \gamma \) of the fluid, as shown in Tab. 3. We stress that this linear analysis does not prove that

| Class of models | Attracting arc \( K_A \) |
|-----------------|--------------------------|
| Vacuum/polarised| all of \( K \) |
| Fluid/polarised | \(-\frac{(3\gamma - 2)^{1/2}}{\sqrt{3(2-\gamma)^{1/2}}} < \delta \Sigma_- < 0\) |
| Vacuum, or fluid with \( 1 \leq \gamma < 2 \)/unpolarised | \(-\frac{1}{\sqrt{3}} \leq \delta \Sigma_- < 0\) |

Table 3: Attracting arc \( K_A \) on the Kasner parabola \( K \)

\( K_A \) is a local attractor.

Over the past 11 years a number of rigorous analyses of the past asymptotic behaviour of \( G_2 \) cosmologies have been given which enable us to make precise statements about \( K_A \). Firstly, Isenberg and Moncrief [33] have proved that every polarised Gowdy vacuum solution with spatial topology \( T^3 \) is past asymptotic to a Kasner solution, showing that the Kasner equilibrium set \( K \) is the global past attractor for this class of models. Secondly, Kichenassamy and Rendall [34] used an analysis based on the Fuchsian algorithm to prove that a general family of unpolarised Gowdy vacuum solutions with spatial topology \( T^3 \) is past asymptotic to the arc \( K_A \), as given in Tab. 3. In a recent development, Rendall [41] has used the \( \beta \)-normalised scale-invariant FOSH evolution system for Gowdy vacuum spacetimes to argue that the arc \( K_A \) is in fact a local past attractor in the unpolarised case, for models which satisfy the so-called “low velocity” condition \( 0 \leq v_{Gowdy} < 1 \), where the Gowdy “velocity parameter” corresponds to

\[ v_{Gowdy} = \sqrt{3} (\Sigma^2 + \Sigma_x^2)^{1/2}, \] \hfill (147)

and thus quantifies the magnitude of the transverse shear rate of the timelike reference congruence \( \xi_0 \). It is known, however, that the arc \( K_A \) is not the global past attractor for Gowdy vacuum spacetimes since solutions which develop so-called spikes violate the inequality \( 0 \leq v_{Gowdy} < 1 \) at those points at which a spike occurs (see Rendall and Weaver [42], Berger and Moncrief [8] and Hern and Stewart [26]). Finally, Anguige [2] has proved that a general family of diagonal \( G_2 \) cosmologies with a perfect fluid matter source is past asymptotic to the arc \( K_A \) as given in Tab. 3.

It also follows from Refs. [33], [34] and [2] that the solution (139) - (145) to the linear equations gives the correct past asymptotic form of a general class of solutions in a neighbourhood of the local attractor \( K_A \) for polarised and unpolarised Gowdy vacuum spacetimes and for diagonal \( G_2 \) cosmologies with a perfect fluid matter source. We anticipate that it will also do so for orthogonally transitive perfect fluid \( G_2 \) cosmologies, but this remains to be proven.

Finally, we note that if the restriction \( \delta \Sigma_- > -\frac{(3\gamma - 2)^{1/2}}{\sqrt{3(2-\gamma)^{1/2}}} \), which arises in the polarised perfect fluid \( G_2 \) case in Tab. 3, does not hold, then the peculiar velocity \( v \) of the fluid will remain significant as \( t \to -\infty \), hinting at the existence of another local attractor distinct from \( K_A \). Experience with SH cosmologies (see, e.g., Ref. [30]) suggests that \( v \) will approach its extreme values, i.e., \( \lim_{t \to -\infty} v = \pm 1 \). This matter requires further investigation.

\(^{15}\)That is, a family whose initial data depends on four arbitrary real-valued functions.
5.3 Asymptotic silence

We now give a brief discussion of the conjecture BKL II, in the light of the previous two subsections. This conjecture is part of the folklore of mathematical cosmology and does not have a precise statement. We can best explain the essence of the conjecture by quoting from BKL [6], p656:

“...in the asymptotic vicinity of the singular point the Einstein equations are effectively reduced to a system of ordinary differential equations with respect to time: the spatial derivatives enter these equations ‘passively’ without influencing the character of the solution.”

Another way of expressing the idea heuristically is to say that the evolution at different spatial points decouples near the cosmological initial singularity. The FOSH format of the evolution system that we have given, namely

\[
A \frac{\partial}{\partial t} X + B \frac{\partial}{\partial x} X = F(X),
\]

(using the timelike area gauge with \(N_0 = 1\)), sheds light on this idea of spatial decoupling, since we have shown that \(\lim_{t \to -\infty} E_1^{\frac{1}{2}} = 0\). This result means that as one follows a timeline into the past towards the cosmological initial singularity, the local null cone (and hence the local fluid sound cone embedded therein) collapses onto the timeline, showing that geometrical information propagation between neighbouring timelines is asymptotically eliminated, as illustrated in Fig. 1. We shall refer to this phenomenon as “asymptotic silence” of the gravitational field dynamics as the cosmological initial singularity is approached.\(^{16}\) As regards the quotation from BKL [6], there are two ways of reducing the evolution system of PDE to a system of ODE. Firstly, one can set \(E_1^{\frac{1}{2}} = 0\) in Eq. (148), obtaining the system of ODE

\[
A \frac{\partial}{\partial t} X = F(X),
\]

(149)

Secondly, one can consider the system of linear ODE in the neighbourhood of the Kasner equilibrium set, Eqs. (132) – (138). We have seen that the system of linear ODE do produce the correct past asymptotic form of the solutions near the locally attracting Kasner arc \(\mathcal{K}_A\). One can also consider the relation between Eqs. (149) and (148). In view of the fact that \(\lim_{t \to -\infty} E_1^{\frac{1}{2}} = 0\), one might expect that the spatial derivative term \(B \frac{\partial}{\partial x} X\) in Eq. (148) would be negligible compared to the other two terms, as \(t \to -\infty\). Calculating each term in Eq. (148) using the asymptotic solution (139) – (145) shows that the spatial derivative terms are in fact negligible asymptotically in a neighbourhood of the Kasner arc \(\mathcal{K}_A\). This property does not hold at isolated points in solutions which develop spikes, since certain partial derivatives in \(\frac{\partial}{\partial x} X\) become unbounded, so that the term \(B \frac{\partial}{\partial x} X\) is not negligible there. Nevertheless, the spatial derivative terms appear to act “passively”, so that one still has silent, Kasner-like dynamics locally.

5.4 Isotropic initial singularities

Our discussion in subsection 5.2 concerns the past asymptotic behaviour of general classes of \(G_2\) cosmologies, which, according to conjecture BKL I, satisfy \(\lim_{t \to -\infty} \Omega = 0\). There are, however, special classes of models that violate this condition, the most important being models with an isotropic initial singularity (see Goode and Wainwright [23]), which satisfy

\[
\lim_{t \to -\infty} \Omega = 1, \quad \lim_{t \to -\infty} v = 0, \quad \lim_{t \to -\infty} \Sigma^2 = 0,
\]

(150)

whose evolution near the cosmological initial singularity is approximated by the flat FL model. We can understand the generality of the isotropic initial singularity by linearising the evolution equations (115) – (122) about the flat FL equilibrium point which lies on the unphysical boundary, and is given by

\[
\Omega = \frac{2}{3}, \quad 0 = E_1^{\frac{1}{2}}, \quad 0 = \Sigma_n = N_x = \Sigma_x = N_n = v.
\]

(151)

\(^{16}\)On the original idea of “silent cosmological models”, see Matarrese et al [38]; on their dynamical consequences, Ref. [20].

\(^{17}\)This is related to the so-called “velocity-dominated” system obtained by dropping the spatial derivatives from the evolution system but not from the constraint equations; see, e.g., Andersson and Rendall [1].
6 CONCLUDING REMARKS AND OUTLOOK

Setting $N_0 = 1$, we thus obtain the following system of ODE:

\[
\begin{align*}
\partial_t E_1^1 &= \frac{1}{2} (3\gamma - 2) E_1^1 \\
\partial_t \Sigma_{-} &= -\frac{3}{2} (2 - \gamma) \Sigma_{-} \\
\partial_t \Sigma_{+} &= -\frac{3}{2} (2 - \gamma) \Sigma_{+} \\
\partial_t \Omega &= \frac{3}{2} (2 - \gamma) (1 - \Omega)
\end{align*}
\]

Using a reparametrisation (66) of $x$, we obtain $E_1^1 = \exp(\frac{1}{2} (3\gamma - 2) t)$, and

\[
\begin{align*}
\Sigma_{-} &= a_1(x) e^{-\frac{3}{2} (2 - \gamma) t} \\
\Sigma_{+} &= a_3(x) e^{-\frac{3}{2} (2 - \gamma) t} \\
\Omega &= 1 + a_5(x) e^{-\frac{3}{2} (2 - \gamma) t} \\
N_{-} &= a_2(x) e^{\frac{1}{2} (3\gamma - 2) t} \\
N_{+} &= a_4(x) e^{\frac{1}{2} (3\gamma - 2) t} \\
v &= a_0(x) e^{\frac{1}{2} (3\gamma - 2) t}
\end{align*}
\]

Note that it follows from the present result that for a $G_2$ cosmology to have an isotropic initial singularity, the conditions

\[0 = a_1(x) = a_3(x) = a_5(x)\]

need to be satisfied. This amounts to setting precisely half the initial data compared to the full $G_2$ case. For rigorous results see Claudel and Newman [11] and Anguige and Tod [3].

6 Concluding remarks and outlook

In this paper we have shown how to formulate the EFE for orthogonally transitive $G_2$ cosmologies with a perfect fluid matter source as an autonomous system of PDE with evolution equations in FOSH format, using scale-invariant dependent variables. As stated in the introduction, one of our goals is to provide a flexible framework for analysing $G_2$ dynamics. A potential user of this paper, someone who wishes to apply the equation systems that we have derived to do numerical or rigorous analyses, need not be familiar with the orthonormal frame formalism. For such a reader the heart of the paper is the scale-invariant equation system in section 2.2, the discussion of the gauge choices in section 3, and the overview in section 4.1. The explicit equation system in section 4.4 for the timelike area gauge may also be of use.

On the other hand, for someone interested in the structure of the space of cosmological solutions of the EFE, the relevant sections are 4.2, 4.3 and 5.

An important aspect of our formulation is that it incorporates the SH models as a special case, thereby shedding light on how SH dynamics influences $G_2$ dynamics. The $H$-normalised scale-invariant dependent variables, when used in a dynamical systems setting, have proved effective in all four aspects of analysis of SH models, i.e., exact solutions, heuristic, numerical and rigorous. In particular, the dynamical systems framework suggested various heuristic ways of gaining insight into the SH dynamics, using local stability arguments and the notion of shadowing of orbits in the dynamical state space, in conjunction with the hierarchy of Bianchi invariant submanifolds. These methods have in turn led to proofs of various conjectures concerning SH dynamics (e.g., Ringström [43] and to the prediction of new dynamical phenomena (e.g., asymptotic self-similarity breaking and Weyl curvature dominance; see Wainwright et al [54]). Our initial success in describing some aspects of the $G_2$ dynamics at early times in terms of a local past attractor, as given in section 5, suggests that our new formulation of the evolution for perfect fluid $G_2$ cosmologies will prove equally effective. The three main problems concerning $G_2$ dynamics that we intend to investigate in the future are

(i) the asymptotic dynamics at early times when the peculiar velocity is dynamically significant,

(ii) the local stability of self-similar models and the asymptotic dynamics at late times, and

(iii) the dynamics of $G_2$ cosmologies that are close to FL in some epoch.

In conclusion, we believe that many of the ideas discussed in the present paper, when appropriately modified, will be of relevance in a much broader context in mathematical cosmology. For example, the notion of an infinite-dimensional scale-invariant dynamical state space with a hierarchical skeleton structure will be a useful guide for exploring more general cosmological spacetimes. We anticipate that
Non-zero 3-Ricci curvature variables

expressions

on the unphysical boundary of the dynamical state space, whose dynamics is described by a system of ODE, will play an important rôle in clarifying the dynamical content of the conjectures BKL I and BKL II concerning cosmological initial singularities. Likewise, these ingredients should be useful for studying almost-FL dynamical states near the cosmological initial singularity or at intermediate and late times, as well as other, more generic, aspects of these dynamical regimes.

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A Appendix

A.1 Connection components in terms of frame variables

The inverse area density of the $G_2$-orbits is given by

$$A^{-1} = e_2^2 e_3^3 - e_2^3 e_3^2.$$ (160)

Then it follows from the dimensional commutator equations (12) – (17) that

$$\alpha = -N^{-1} \frac{\partial e_1^1}{e_1^1}, \quad \dot{\alpha} = e_1^1 \frac{\partial \Omega}{\Omega}.$$ (161)

$$\beta = \frac{1}{2} N^{-1} \frac{\partial A}{A}, \quad \dot{\beta} = -\frac{1}{2} e_1^1 \frac{\partial A}{A}.$$ (162)

$$\sigma_+ = -\frac{1}{2 \sqrt{3}} A N^{-1} (e_3^3 \partial e_2^2 - e_2^2 \partial e_3^3 + e_3^3 \partial e_3^2 - e_2^3 \partial e_2^3).$$ (163)

$$\sigma_\times = \frac{1}{2 \sqrt{3}} A e_1^1 (e_3^3 \partial e_2^2 - e_2^2 \partial e_3^3 + e_3^3 \partial e_3^2 - e_2^3 \partial e_2^3).$$ (164)

$$\Omega_1 = \frac{1}{2} A N^{-1} (e_3^3 \partial e_2^2 - e_2^2 \partial e_3^3 + e_3^3 \partial e_3^2 - e_2^3 \partial e_2^3).$$ (165)

$$n_+ = -\frac{1}{2} A e_1^1 (e_3^3 \partial e_2^2 - e_2^2 \partial e_3^3 + e_3^3 \partial e_3^2 - e_2^3 \partial e_2^3).$$ (166)

$$n_\times = -\frac{1}{2} A e_1^1 (e_3^3 \partial e_2^2 - e_2^2 \partial e_3^3 + e_3^3 \partial e_3^2 - e_2^3 \partial e_2^3).$$ (167)

$$n_+ = -\frac{1}{2} A e_1^1 (e_3^3 \partial e_2^2 - e_2^2 \partial e_3^3 + e_3^3 \partial e_3^2 - e_2^3 \partial e_2^3).$$ (168)

A.2 Scale-invariant curvature variables

We define additional $\beta$-normalised curvature variables by

$$(\Omega_k, S_+, \mathcal{E}_-, \mathcal{H}_-) := (-\frac{1}{2} R^* S_+, E_+, H_-)/(3\beta^2).$$ (169)

Then we obtain for orthogonally transitive $G_2$ cosmologies with a perfect fluid matter source the following expressions

Non-zero 3-Ricci curvature variables:

$$\Omega_k = -\frac{2}{3} (E_1^1 \partial z - r - \frac{2}{3} A) A + (N_x^2 + N_x^2).$$ (170)

$$S_+ = -\frac{1}{6} (E_1^1 \partial z - r) A + \frac{2}{3} (N_x^2 + N_x^2).$$ (171)

$$S_- = \frac{1}{4} (E_1^1 \partial z - r) N_x + \frac{2}{3} N_x N_x.$$ (172)

$$S_\times = -\frac{1}{3} (E_1^1 \partial z - r) N_x + \frac{2}{3} N_x N_x.$$ (173)
\[ \mathcal{E}_+ = -\frac{1}{9} (E_1^e \partial_x - r^e) A + \frac{2}{3} (N^2_\Sigma + N^2_\Sigma) + \frac{1}{3} \Sigma^2_\Sigma - \frac{1}{3} (\Sigma^2_\Sigma + \Sigma^2_\Sigma) + \frac{1}{6} \gamma G^1_\Sigma \Omega v^2 \] (174)
\[ \mathcal{H}_+ = -N_\Sigma \Sigma - N_x \Sigma_x \] (175)
\[ (\mathcal{E}_- + \mathcal{H}_x) = \pm \frac{1}{3} (E_1^e \partial_x - r^e + 3 \Sigma^2_\Sigma - A) (\Sigma^2_\Sigma + N_x) \pm \frac{2}{3} N_x (\Sigma^2_\Sigma + N_x) + \frac{1}{2} (\Sigma^2_\Sigma - AN_x) \] (176)
\[ (\mathcal{E}_x + \mathcal{H}_-) = \pm \frac{1}{3} (E_1^e \partial_x - r^e + 3 \Sigma^2_\Sigma - A) (\Sigma^2_\Sigma + N_-) \pm \frac{2}{3} N_+ (\Sigma^2_\Sigma + N_-) + \frac{1}{2} (\Sigma^2_\Sigma + AN_-) \] (177)

### A.3 Line element and scale-invariant dependent variables for area gauges

Introducing in the separable area gauge a line element of the form
\[ ds^2 = \ell^2_0 \left[ -e^{2f(t,x)} dt^2 + e^{2g(t,x)} dx^2 + e^{2N_0} m(x) \left( e^{P(t,x)} (dy + Q(t,x) dz)^2 + e^{-P(t,x)} dz^2 \right) \right], \] (178)
the area expansion rate is given by
\[ \beta = \ell^{-1}_0 N_0 e^{-f}. \] (179)

Then we obtain the following expressions for the non-zero scale-invariant dependent variables:
\[ (N^{-1}, E_1^1) = (N^{-1}_0, N^{-1}_0 e^{f - g}) \] (180)
\[ (\Sigma^2_\Sigma, U) = \left( \frac{1}{2} (1 - N^{-1}_0 \partial_g), E_1^e \partial_x f \right) \] (181)
\[ (\Sigma_x, N_x) = \frac{1}{2\sqrt{3}} (N^{-1}_0 \partial_t P - E_1^e \partial_x P) \] (182)
\[ (\Sigma_x, N_-) = \frac{1}{2\sqrt{3}} e^P (N^{-1}_0 \partial_t Q, E_1^e \partial_x Q) \] (183)
\[ A = -\frac{1}{2} \frac{d}{dx} \ln m(x) e^{E_1^1}, \] (184)
and \( R = -\sqrt{3} \Sigma_x \) and \( N_+ = \sqrt{3} N_- \). Employing an \( x \)-reparametrisation (66) to set \( m(x) = \exp(-2D_{sa} x) \), with \( D_{sa} \) a constant, implies \( A = D_{sa} E_1^1 \). Apart from the sign of \( t \), these expressions reduce to the Gowdy vacuum line element of Rendall and Weaver [42] when \( N_0 = \frac{1}{2}, m(x) = 1, f(t,x) = \frac{1}{2} (\lambda(t,x) + 3t) \) and \( g(t,x) = \frac{1}{2} (\lambda(t,x) - t) \). Note that then \( \partial_x E_1^1 = 0 \).

### A.4 Propagation of constraint equations

**Propagation of dimensional constraint equations:**
\[ N^{-1} \partial_t (C_{\text{Gauß}}) = - (\alpha + \beta) (C_{\text{Gauß}}) - 4 \left( \dot{U}_1 + a_1 \right) (C_{\text{Codacci}})_1 \] (185)
\[ N^{-1} \partial_t (C_{\text{Codacci}})_1 = - (\alpha + 3 \beta) (C_{\text{Codacci}})_1 - \frac{1}{3} (\dot{U}_1 - a_1) (C_{\text{Gauß}}) \] (186)
\[ N^{-1} \partial_t (C_\beta) = - (1 - 3 \Sigma_+ ) (C_\beta). \] (187)

**Propagation of dimensionless gauge fixing conditions:**
\[ N^{-1} \partial_t (C_U)_\text{nc} = (g + 1) (C_U)_\text{nc} \] (188)
\[ N^{-1} \partial_t (C_U)_\text{sa} = (g + 3 \Sigma_+ + \dot{U} A - A^2 - 3\Omega \lambda ) (C_U)_\text{sa} - \frac{3}{2} (\dot{U} - A) (C_G) - \frac{3}{2} (C_\lambda) \] (189)
\[ N^{-1} \partial_t (C_U)_\text{co} = - 2 (2 - 3 \Sigma_+ ) (C_U)_\text{co} + 3 \gamma (1 - \Sigma_+ ) (C_U)_\text{co} + \frac{3}{2} (U + A - 1) (C_G) \] (190)
\[ - \frac{3}{2} (C_{PF}) + \frac{3}{2} (C_\lambda) \] (190)
\[ N^{-1} \partial_t (C_U)_\text{sync} = -(1 - 3 \Sigma_+ ) (C_U)_\text{sync} - \frac{3}{2} A (C_G) + \frac{3}{2} (C_\lambda). \] (191)

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\[ ^{18} \text{Again, we correct some sign errors in the expressions given in Refs. [17] and [18].} \]
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