The Proca field in loop quantum gravity

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Abstract

In this paper we investigate the Proca field in the framework of loop quantum gravity. It turns out that the methods developed there can be applied to the symplectically embedded Proca field, giving a rigorous, consistent, non-perturbative quantization of the theory. This can be achieved by introducing a scalar field, which has completely different properties than those used in spontaneous symmetry breaking. The analysis of the kernel of the Hamiltonian suggests that the mass term in the quantum theory has a different role than in the classical theory.

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1. Introduction

Although loop quantum gravity (LQG) is one of the most successful non-perturbative, covariant theories for quantum general relativity, there is still the question of whether its classical limit gives back general relativity ($\hbar \to 0$) or quantum field theory ($G \to 0$). For this reason we investigate a question that has been answered in quantum field theory, namely the case of the massive vector field. In quantum field theory mass is generated via spontaneous symmetry breaking due to the fact that the mass term in the Lagrangian of the massive vector field (the Proca field) is non-renormalizable. On the other hand, LQG is a non-perturbative theory, therefore the question arises of how to quantize this theory in the framework of LQG, and how the results are related to those of the quantum field theory. In this paper we will show that the Proca field can be quantized in a non-perturbative, diffeomorphism covariant way. We will see that just as in quantum field theory, a scalar field enters the formalism, but this scalar field will play a completely different role in LQG.

The paper is organized as follows. In section 2, we will perform the (3+1) decomposition of the Proca field in curved spacetime and in section 3 we will quantize the theory using the tools developed in LQG. In section 4 we will discuss the results which will be summarized at the end of this paper.
2. Classical theory

The action of the Proca field coupled to gravity has the form

\[ S = \int d^4x \, \mathcal{L} \]

\[ \mathcal{L} = \mathcal{L}_G + \sqrt{-g} g^{ac} g^{bd} \left[ -\frac{1}{4} F_{ab}^{\alpha} F_{\alpha}^{\beta} - \frac{1}{2} m^2 g_{ab} A_{\alpha}^4 A_{\beta}^4 \right], \]

where \( \mathcal{L}_G \) is the Lagrangian density of the gravitational field, \( g^{ab} \) is the metric tensor, \( g \) is its determinant and \( A_{\alpha}^4 \) is a \( U(1) \) connection with curvature \( F_{ab}^{\alpha} \). To apply the framework of loop quantum gravity to this system, first we have to do a (3+1) decomposition: introduce on the spacetime manifold \( M \) a smooth function \( t \) whose gradient is nowhere vanishing and a vector field \( t^a \) with affine parameter \( t \) satisfying \( t^a \nabla_a t = 1 \). This gives a foliation of spacetime, i.e. each \( t \) defines a three-dimensional hypersurface \( \Sigma_t \). Let us decompose \( t^a \) into its normal and tangential part

\[ t_a = N n_a + N_a, \]

where \( n_a \) is the unit normal of the hypersurface \( \Sigma_t \), \( N \) is the lapse function and \( N_a \) is the shift vector. Define the induced, positive-definite metric on \( \Sigma_t \) via

\[ q_{ab} = g_{ab} + n_a n_b. \]

Now we use (3) and express \( n_a \) with \( t_a \) and \( N \), exploit the fact that \( \sqrt{-g} = N \sqrt{q} \) and introduce the pull-backs of the quantities \( A_{\alpha}^4, D_{\alpha}^a \) to \( \Sigma_t \) respectively \( A_a = q_a^b A_b^\alpha, D_a = q_a^b D_b^\alpha \) and define \( A_0 = t^a A_a^4, A_0 = t^4 A_a^4 \). The canonical momenta are the following:

\[ \Pi_0 = \frac{\delta S}{\delta \dot{A}_0} = 0, \quad \Pi'_0 = \frac{\delta S}{\delta \dot{A}_0} = 0 \]

\[ \Pi_N = \frac{\delta S}{\delta \dot{N}} = 0, \quad \Pi_a = \frac{\delta S}{\delta \dot{N}_a} = 0 \]

\[ E_a^i = \frac{\delta S}{\delta A_a^i}, \quad E_0 = \frac{\delta S}{\delta A_0}, \]

so we have primary constraints \( \Pi_0, \Pi'_0, \Pi_N, \Pi_a \). Putting everything together (for details see [2, 6, 11]) we obtain for the Hamiltonian of the Proca field

\[ H = \int_{\Sigma} \left( N \mathcal{H} + N^a \mathcal{H}_a + A_0^i G_i + A_0 G + \sqrt{q} \frac{m^2}{2} \left( -\frac{A_0^2}{N^2} + \frac{2}{N^2} A_0 N^b A_b - \frac{1}{N^2} (N^b A_b)^2 \right) \right) \]

\[ \mathcal{H} = \frac{1}{\kappa \sqrt{q}} \text{tr}(2[K_a, K_b] - F_{ab}[E_a, E_b]) + \frac{q_{ab}}{2 \sqrt{q}} (F_a^b E_b + B_a^b B_b) + \frac{\sqrt{q} m^2}{2} q_{ab} A_a A_b \]

\[ \mathcal{H}_a = F_{ab} E_b^a + \epsilon_{abc} B^b B^c \]

\[ G = D_a E^a \]

\[ G_i = D_a E_i^a, \]

where \( F_{ab} \) is the curvature of the \( SU(2) \) connection \( A_a^4, K_a \) is the extrinsic curvature and \( B^a \) is the dual of the Maxwell connection \( A_a \). The phase space carries a symplectic structure where the (non-trivial) Poisson brackets are
\[ \{ A^a_i(x), E^b_j(y) \} = \delta^b_a \delta^{ij} \delta(x-y) \]  
(12)
\[ \{ A_a(x), E^b_j(y) \} = \delta^b_a \delta(x-y). \]  
(13)

Since \( \Pi_0 = 0 \) etc must hold, we get the following consistency conditions (secondary constraints):

\[ 0 = \{ \Pi_N, H \} = \mathcal{H} + \frac{1}{2} \sqrt{q} m^2 \tilde{A}^2 := \tilde{\mathcal{H}} \]  
(14)
\[ 0 = \{ \Pi_a, H \} = \mathcal{H}_a + \sqrt{q} m^2 \tilde{A} \Lambda_a := \tilde{\mathcal{H}}_a \]  
(15)
\[ 0 = \{ \Pi_0, H \} = \mathcal{G} - \sqrt{q} m^2 \tilde{A} := \tilde{\mathcal{G}} \]  
(16)
\[ 0 = \{ \Pi'_0, H \} = G_i, \]  
(17)

where we used the notation \( \tilde{A} = \frac{A^a - \Lambda^a}{N} \). One can verify that the above constraints have a second-class constraint algebra. To show this, first let us introduce the following linear combination of the primary constraints:

\[ \tilde{\Pi}_0 := \Pi_N + \tilde{A} \Pi_0 \]  
(18)
\[ \tilde{\Pi}_N := \Pi_N + N^a \Lambda_a + A_0 \Pi_0 \]  
(19)
\[ \tilde{\Pi}_a := \Pi_a + \Lambda_a \Pi_0. \]  
(20)

It is easy to show that the above linear combinations have weakly vanishing Poisson brackets with all constraints, so these are first class constraints. The constraints \( \mathcal{G}_i \) (gravitational Gauss constraint) and \( \tilde{\mathcal{H}}_a + \tilde{A} \tilde{\mathcal{G}} + \Lambda_i \mathcal{G}_i \) (diffeomorphism constraint) are also first class (as in the \( m = 0 \) case, the first generates infinitesimal \( SU(2) \) gauge transformations, while the latter generates infinitesimal spacial diffeomorphisms). There are only two second-class constraints: \( \tilde{\mathcal{G}} \) and \( \tilde{\mathcal{H}} \). Their Poisson bracket is

\[ \{ \tilde{\mathcal{H}}(x), \tilde{\mathcal{G}}(y) \} = m^2 \sqrt{q(x)} \sqrt{q(y)} \delta(x-y) \]
\[ -\sqrt{q(x)} m^2 \left( \tilde{A}^a(x) N^a(x) \right) \frac{D^{(y)}_a}{N(x)} \delta(x-y) \]
\[ := M(x, y), \]  
(21)

where \( D^{(y)}_a \) means that the derivative should be calculated in the \( y \) variable.

To deal with second-class systems, one needs to introduce the so-called Dirac brackets instead of the Poisson brackets. In the case of field theories, it is done in the following way (see [1] for details): first one calculates the matrix \( M_{ij}(x, y) := \{ C_i(x), C_j(y) \} \), where \( C_i(x) \) are the second-class constraints in the theory. After that one calculates the inverse of \( M_{ij}(x, y) \) in the following sense (since \( M_{ij}(x, y) \) is a distribution):

\[ \int d^3 z \, M_{jk}(x, z) (M^\dagger)^{ij} \delta(z, y) = \delta_{ij} \delta(x-y). \]  
(22)

After this the Dirac bracket is defined as

\[ \{ f, g \}_D := \{ f, g \} - \int d^3 x \, d^3 y \, \{ f, C_i(x) \} (M^\dagger)^{ij} \{ C_j(y), g \}. \]  
(23)

In our case, \( M_{ij}(x, y) \) is a \( 2 \times 2 \) matrix with components

\[ M_{11}(x, y) = M_{22}(x, y) = 0 \]
\[ M_{12}(x, y) = -M_{21}(y, x) = M(x, y). \]  
(24)
The inverse of this matrix has the same structure:

\[
(M^{-1})_{11}(x, y) = (M^{-1})_{22}(x, y) = 0
\]

\[
(M^{-1})_{12}(x, y) = -(M^{-1})_{21}(y, x) = \bar{M}(x, y),
\]

where \(\bar{M}(x, y)\) satisfies the following differential equation

\[
m^2 \frac{E^a(x)N^b(x)}{N(y)} \sqrt{q(x)} \bar{M}(x, y) - \sqrt{q(x)} m^2 \left( A^a(x) + \partial^a \phi \right) \left( A^b(y) + \partial^b \phi \right) = \delta(x - y),
\]

(26)

thus the Dirac-bracket has the form

\[
[f, g]_D := \{ f, g \} + \int d^3x d^3y \left( \{ f, \bar{H}(x) \bar{M}(y, x) \bar{G}(y), g \} - \{ f, \bar{G}(x) \bar{M}(x, y) \bar{H}(y), g \} \right).
\]

(27)

The above construction has two drawbacks: the first is that the above Lagrangian is not gauge invariant. The cause of this is the mass term, since if one replaces \(m = 0\), we will have a first-class constraint algebra. This affects only the \(U(1)\) gauge, \(SU(2)\) symmetries are still valid (since the mass term contains gravitational variables in the form of the metric tensor and its determinant). The second problem is that one is led to a system with second-class constraints. The latter problem is more troublesome because the canonical quantization becomes quite difficult due to the fact that it is far too non-trivial to implement the Dirac brackets in the quantum theory [3].

There is an elegant way of curing both problems [3–5], and that is to introduce an auxiliary scalar field and modify the Lagrangian to have the following form:

\[
\mathcal{L}_m = \mathcal{L}_G + \sqrt{-g} g^{abc} g^{bd} \left[ -\frac{1}{4} e_{ab}^2 e_{cd}^2 - \frac{1}{2} m^2 g_{ab} (A^a + \partial^a \phi) (A^b + \partial^b \phi) \right] = \mathcal{L}_G + \mathcal{L}_{YM} + \mathcal{L}_{M}.
\]

(28)

Note that the above Lagrangian is gauge invariant if the transformation rule for the fields under gauge transformation is

\[
\delta A^a_{\mu} = \partial^a_{\mu} \Lambda
\]

\[
\delta \phi = \Lambda
\]

and the original Lagrangian is obtained via the gauge fixing \(\partial^a_{\mu} \phi = 0\).

To see that the system is first class, we perform the (3+1) decomposition to the modified Lagrangian as well. Using the notations already introduced we obtain

\[
H_m = \int_\Sigma \left( N \mathcal{H} + N^a \mathcal{H}_a + A^a_0 G_i + A_0 G \right)
\]

(31)

\[
\mathcal{H} = \frac{1}{\kappa \sqrt{q}} \text{tr} \left[ 2 [K_a, K_b] - F_{ab}] [E_a, E_b] + \frac{q_{ab}}{2 \sqrt{q}} (E^a E^b + B^a B^b) \right.
\]

\[
+ \frac{\pi^2}{2 \sqrt{q} m^2} - \sqrt{q} m^2 \left( A^a + \partial^a \phi \right) \left( A^b + \partial^b \phi \right)
\]

\[
+ q_{ab} (A_a + \partial_a \phi) (A_b + \partial_b \phi)
\]

(32)

\[
\mathcal{H}_a = F_{ab}^j E_b^j + \epsilon_{abc} E_b^c + (A_a + \partial_a \phi) \pi
\]

(33)

\[
\mathcal{G} = D_a E^a - \pi
\]

(34)
where $\pi$ is the conjugate momenta for $\phi$:
\[
\pi = \frac{\delta S}{\delta \dot{\phi}} = \sqrt{qm^2}(A_0 - N^a A_a + \mathcal{L}_\phi - N^a \partial_a \phi).
\] (36)

We also have the primary constraints $\Pi_N = \Pi_a^i = \Pi_0 = 0$ which are the same as in the previous case. If we calculate $\dot{\Pi}_N$ etc, we find that $\mathcal{H}, \mathcal{H}_a, \mathcal{G}_b, \mathcal{G}$ are secondary constraints in the theory. These constraints are referred to as the Hamiltonian, the diffeomorphism (modulo gauge transformations), the gravitational Gauss and Maxwell Gauss constraints. The phase space carries a symplectic structure where the (non-trivial) Poisson brackets are
\[
\{A^a_i(x), E^b_j(y)\} = \delta^b_a \delta^i_j \delta(x - y)
\] (37)
\[
\{A^a_i(x), E^b_i(y)\} = \delta^b_a \delta(x - y)
\] (38)
\[
\{\pi(x), \phi(y)\} = \delta(x - y).
\] (39)

Before we turn to the quantization, it is worth observing some of the properties of the above system:

- It is easy to check that the above system is first class, i.e. the constraint algebra is closed. This is due to the fact that the canonical momenta of the scalar field appear in the Gauss constraint.
- Note that the mass only appears in the scalar constraint, which means that gauge and diffeomorphism symmetries are independent of $m$.
- The Hamiltonian is a linear combination of constraints, which is not true for the case where there is no scalar field, since there the Hamiltonian is quadratic in the Lagrange multipliers.
- The scalar field and the Yang–Mills field are only coupled to each other in the scalar constraint and only through a derivative term and no scalar mass term required, which means that if we quantize this system the scalar field will have a totally different role than the one introduced via symmetry breaking.
- The term $A^a_i + \partial_a \phi$ is gauge invariant with respect to the gauge transformations generated by (34), so this is different to the case when we couple a scalar field to a gauge field via covariant derivatives (actually its more like an affine field).
- One cannot replace $m = 0$ in the Hamiltonian formalism to obtain the usual Maxwell field. This is not unfamiliar in loop quantum gravity, since it resembles the case of the Immirzi parameter (though by this analogy we do not mean any deeper connection). There the connection and the electric field can be rescaled as $A^a_i \rightarrow A^a_i + \beta K^a_i, E^i_a \rightarrow E^i_a / \beta$. This is a canonical transformation since the Poisson brackets are invariant under this transformation. If we substitute the new quantities in the Hamiltonian, we find that the Gauss and diffeomorphism constraints are unchanged, but one part of the Hamiltonian constraint will have a term proportional with $\beta^2 + 1$. (see eg. [10, 11] and references therein). Now consider the following canonical transformation:
\[
\pi \rightarrow m \pi \quad \phi \rightarrow \frac{\phi}{m}
\]
\[
E^i_a \rightarrow m E^i_a \quad A^a_i \rightarrow \frac{A^a_i}{m}.
\]
This will remove the $m$ parameter from the mass term and furthermore this parameter will appear only in the Hamiltonian constraint, the other constraints will be independent of $m$. 

\[
G_i = D_a E^a_i,
\] (35)
3. Quantization

In this section we will summarize the tools necessary to quantize the Proca field in loop quantum gravity. The main advantage of the theory is that it gives a covariant, non-perturbative method to quantize any physical system with first-class constraint algebra. Since the symplectically embedded Proca field is of this type, we can directly apply the results achieved in loop quantum gravity. From these the most important for us will be the quantization of general gauge systems [7–12] and the scalar field [16, 17]. In the next sections we will give a brief summary on how one can apply these results to the present case, and we refer the reader to the above articles for details and proofs.

3.1. Quantization of a gauge field

Let us consider a Yang–Mills gauge field with a compact gauge group G. The Hilbert space can be constructed in the following way: let γ be an oriented graph in Σ with e₁, . . . , e_E edges and v₁, . . . , v_V vertices. Let h_i be the holonomy of the G-valued connection of the field evaluated along the e_i edge. Let us define a cylindrical function with respect to a γ graph in the following way:

\[ f_γ (A) := f (h_{e_1}, \ldots, h_{e_E}) \]  

(40)

where f_γ is a complex-valued function mapping from G^E. Since the gauge group G is compact, there is a natural measure, the Ashtekar–Lewandowski measure, which enables one to have a inner product on the set of cylindrical functions. With this, the Hilbert space of the Yang–Mills field is defined as the set of all cylindrical functions which are square integrable with respect to the above measure:

\[ \mathcal{H} := L_2(\bar{A}, d\mu_{AL,G}). \]  

(41)

In our case, G = SU(2) × U(1), so

\[ \mathcal{H}_{G,YM} := L_2(\bar{A}_{SU(2)}, d\mu_{SU(2)}) \otimes L_2(\bar{A}_{U(1)}, d\mu_{U(1)}). \]  

(42)

In order to analyse the action of the Hamiltonian and to compute its kernel, it is convenient to introduce a complete orthonormal basis on the Hilbert space (42).

In the case of L_2(\bar{A}_{U(1)}, d\mu_{U(1)}) one must simply replace SU(2) with U(1) in the above definition—these are called flux network functions [15]. Since U(1) is a commutative group, we will have the following definition: for each edge e_i of the graph there associates an
integer $n_i$. Then the flux network function is defined as

$$|F(A)\rangle_{\gamma,\vec{n}} := \prod_{i=1}^{N} (h_{e_i}(A))^{n_i}. \tag{44}$$

What remains is to define the operators corresponding to the connection and the electric field on the Hilbert space. If we want to implement the Poisson brackets in the quantum theory in a diffeomorphism covariant way, we have to use smeared versions of these fields. In the case of gauge fields the natural candidates are the holonomy and the electric flux, respectively:

$$h_e(A) = \mathcal{P} \exp \int_e A \tag{45}$$

$$E(S) = \int_S * E, \tag{46}$$

where $e$ is a path and $S$ is a surface in $\Sigma$. Then the action of the corresponding operators will be defined via the following way:

$$\hat{h}_e(A)f(A) := h_e(A)f \tag{47}$$

$$\hat{E}(S)f(A) := i\hbar \{E(S), f(A)\}. \tag{48}$$

Let us make the action of the electric flux operator a bit more explicit for both the gravitational and the Yang–Mills part. The latter case is simple, since the $U(1)$ group is commutative. On a flux network function, the action of this operator is

$$\hat{E}(S)|F\rangle_{\gamma,\vec{n}} = i\hbar \sum_{i=1}^{N} n_i \kappa(S,e_i)|F\rangle_{\gamma,\vec{n}} \tag{49}$$

where

$$\kappa(S,e) = \begin{cases} 0, & \text{if } e \cap S = 0 \text{ or } e \cap S = e \text{ modulo the endpoints} \\ +1, & \text{if } e \text{ lies above } S \\ -1, & \text{if } e \text{ lies below } S. \end{cases}$$

In the case of the gravitational part introduce the left and right invariant vector fields on $SU(2)$:

$$R_{h}(g) = \frac{d}{dt} f(e^{th}g) \text{ and } L_{h}(g) = \frac{d}{dt} f(g e^{th}). \tag{50}$$

Using these quantities the action of the flux operator on a cylindrical function is

$$\hat{E}(S)f_p(A) = \frac{1}{4} \sum_{e} \kappa(S,e) \left( \delta_{e \cap S, h(e)} R_{e}^{(e)} + \delta_{e \cap S, f(e)} L_{e}^{(e)} \right) f_p(A), \tag{50}$$

where $b(e)$ and $f(e)$ are the beginning- and end-points of the edge $e$ and $R^{(e)}$ is $R$ on the copy of $SU(2)$ labelled by $e$.

3.2. Quantization of the scalar field

Though the scalar field requires more careful treatment, the results are very similar to the Yang–Mills case. The main difference is that in this case one has to use so-called point holonomies of the form

$$U(\nu) = \exp(i\phi(x_e)). \tag{51}$$
Then a cylindrical function with respect to a $\gamma$ graph has the form

$$f_\gamma(U) := f(U_{v_1}, \ldots, U_{v_V}), \quad (52)$$

where $v_1, \ldots, v_V$ are the vertices of the graph. Then the Hilbert space will be

$$H_U := L_2(\mathcal{U}, d\mu_{\phi}). \quad (53)$$

Recently it was shown that a more general description should be used because in the original approach the configuration variables are periodic functions, which do not suffice to separate the points of configuration space (for details see [17] and [18]). So instead of $U(v)$ we define the configuration variable to be

$$U(\lambda, v) = \exp(i\lambda\phi(x_v)). \quad (54)$$

Given a graph $\gamma$ with vertices $(v_1, \ldots, v_N)$ and real numbers $\lambda_1, \ldots, \lambda_N$ associated with the vertices, let $Cyl_N$ be the set of finite linear combinations of the following functions:

$$|D(U)\rangle_{\gamma, \vec{\lambda}} := \prod_{i=1}^N (U(\lambda_i, v_i)). \quad (55)$$

Then the completion of $Cyl = \bigcup N Cyl_N$ will be the $C^*$ algebra of configuration observables.

Because the point holonomy is not smeared, the regulated version of the canonical momenta $\pi$ is smeared in three dimensions:

$$P_B = \int_B \pi, \quad (56)$$

where $B$ is an open ball in $\Sigma$.

The operator corresponding to (52) is defined the same way as was done in the case of the Yang–Mills field, but the operator version of (56) needs careful treatment, because the functional derivative of $U(v)$ with respect to $\phi$ is meaningless. The precise calculations can be found in [16], the result is

$$\hat{P}_B|D(U)\rangle_{\gamma, \vec{\lambda}} := -i\hbar \chi_B(v)X(v)|D(U)\rangle_{\gamma, \vec{\lambda}}, \quad (57)$$

where $\chi_B(v)$ is zero unless $v \in B$ and $X(v)$ is the symmetric sum of left and right invariant vector fields at $U(v)$. But since we have an Abelian group, we can write that $X(v) = X_L(v) = X_R(v)$, so we obtain

$$\hat{P}_B|D(U)\rangle_{\gamma, \vec{\lambda}} := -i\hbar \sum_{j=1}^N \chi_B(v_j)\lambda_j|D(U)\rangle_{\gamma, \vec{\lambda}}. \quad (58)$$

Now let us apply these results to the case of the Proca field. Since we have an $SU(2) \times U(1)$ Yang–Mills field and a real scalar field, the Hilbert space will be

$$\mathcal{H} = L_2(\mathcal{A}_{SU(2)}, d\mu_{SU(2)}) \otimes L_2(\mathcal{A}_{U(1)}, d\mu_{U(1)}) \otimes L_2(\mathcal{U}, d\mu_{U(1)}), \quad (59)$$

with basis (later referred to as generalized spin network functions)

$$|S\rangle_{\gamma, \vec{\lambda}} = |T(A)\rangle_{\gamma, \vec{\lambda}} \otimes |F(A)\rangle_{\gamma, \vec{\lambda}} \otimes |D(U)\rangle_{\gamma, \vec{\lambda}}. \quad (60)$$

Remark. We have seen that in the original (not symplectically embedded) case (7) we arrived at a second-class constraint algebra, thus we had to introduce Dirac brackets instead of Poisson brackets. Because of this, the above definitions of the momentum operators should be modified.
by replacing the Poisson brackets with Dirac brackets. Specifically the momentum operators should be redefined in the following way:

\[
\hat{E}(S, Df(A)) = \frac{i\hbar}{2} \{E(S), f(A)\} - i\hbar \int d^3x \, d^3y \{E(S), C_i(x)\} (M^{(-1)})_{ij}(x, y) \{C_j(y), f(A)\},
\]

(61)

where \(C_i\) are the second-class constraints \(\bar{G}\) and \(\bar{h}\). This does not modify the properties of the momentum operators, since the Dirac brackets have the same properties as the Poisson brackets. Only the action on spin network functions changes, which becomes more complicated since \(\tilde{M}(x, y)\) is not explicit and depends also on \(A_a\) and \(E_a\). In fact only the momentum operator of the Maxwell field changes, since \(\{\bar{G}, A_i\} = \{\bar{G}, E_i\} = 0\). The only question is whether the momentum operator defined above is a well defined operator on the Hilbert space, since it is not trivial if its action is a cylindrical function. The more detailed analysis of this question can be found in section 6.

4. The Hamiltonian of the Proca field

The total (non-smeared) Hamiltonian of the symplectically embedded Proca field has the form

\[
H = \int_{\Sigma} N(H_{GE} + H_{GL} + H_E + H_B + H_P + H_M)
\]

where

\[
H_{GE} = \frac{2}{\kappa} \epsilon^{abc} tr(F_{ab}[A_c, V])
\]

\[
H_{GL} = \frac{8}{\kappa^2} \epsilon^{abc} tr([A_a, K][A_b, K][A_c, V])
\]

\[
H_E = \frac{q_{ab}}{2\sqrt{q}} E^a E^b
\]

\[
H_B = \frac{q_{ab}}{2\sqrt{q}} B^a B^b
\]

\[
H_P = \frac{\pi^2}{2\sqrt{q}m^2}
\]

\[
H_M = \frac{\sqrt{q}m^2}{2} q^{ab} (A_a + \partial_a \phi)(A_b + \partial_b \phi),
\]

(62)

where \(H_{GE}\) and \(H_{GL}\) are the Euclidean and Lorentzian part of the gravitational, \(H_E\) and \(H_B\) are the electric and magnetic part of the Yang–Mills Hamiltonian, \(H_P\) is the kinetic term and \(H_M\) is the mass term (in the following we consider only the symplectically embedded Hamiltonian). To arrive at a well defined, diffeomorphism covariant Hamiltonian operator, one first has to get rid of the \(\sqrt{q}\) quantity from the denominator. This is achieved via using the identity (introduced by Thiemann):

\[
\frac{1}{\kappa} \{A'_i, V\} = 2 \text{sgn}(\det e) e^i_a.
\]

(63)

The next step is to rewrite the Hamiltonian in terms of holonomies and fluxes. To do this, one introduces a regularization scheme, a \(\Delta\) triangulation of \(\Sigma\). Each tetrahedra in \(\Delta\) has a base point \(v\) with three outgoing segments \(s_i j = 1, 2, 3\). Denote the arcs connecting the end-points of \(s_i \) and \(s_j \) by \(a_{ij}\), so we can form the loop \(a_{ij} := s_i \circ a_{ij} \circ s^{-1}_j\). After an appropriate point splitting, one approximates the quantities in the Hamiltonian by their smeared counterparts, substitutes the corresponding operators and takes the limit. This can be done with each of the
terms appearing in (62). The details of the computations are quite lengthy so here we will only present the regulated Hamiltonians and refer the reader to [9] or [13] where one can find the regularization of all terms except the mass term. Since the latter is important in our analysis we will sketch the derivation of that particular term. The regulated Hamiltonians are

\[ \hat{h}_{\text{GE}} = -\frac{8}{3i\hbar\kappa} \sum_v \frac{N(v)}{E(v)} e^{ijk} \text{tr}(\hbar_{\delta_0} h_{\alpha} [h_{\delta_0}^{-1}, \hat{V}]) \] (64)

\[ \hat{h}_{\text{GL}} = \frac{64}{3i\hbar^3\kappa^3} \sum_v \frac{N(v)}{E(v)^2} e^{ijk} \text{tr}(h_{\delta_0} [h_{\delta_0}^{-1}, K_v] h_{\delta_0} [h_{\delta_0}^{-1}, K_v] h_{\delta_0} [h_{\delta_0}^{-1}, V_v]) \] (65)

\[ \hat{h}_{\text{YM}} = -\frac{32}{9i\hbar^2\kappa^2} \sum_v \frac{N(v)}{E(v)} E(v) \sum_{v/\Delta(v) = v/\Delta(v)'} \epsilon_{JKL} \epsilon_{MNP} \times \hat{Q}^k_{sL/\Delta(v)}(v, 1/2) \hat{Q}^l_{sM/\Delta(v)'}(v, 1/2) \] (66)

where

\[ \hat{Q}^k_{s}(v, r) = \text{tr}(\tau_k h_{\delta_0}[h_{\delta_0}^{-1}, V(v) r]) \]

\[ E(v) = \frac{n(n-1)(n-2)}{6} , \]

\[ H_M = \frac{m^2}{2} \int d^3x \int d^3y N(x) \chi(x, y) e^{ijk} \epsilon_{imn} e_{abc} e_{bef} \]

\[ \times \frac{((\partial_0 + \Lambda_0) e_{ij}^k e_{lm}^m) (\partial_0 + \Lambda_0) e_{ij}^k e_{lm}^m)}{\sqrt{V(x, \epsilon)}} \frac{\sqrt{V(y, \epsilon)}}{\sqrt{V(y, \epsilon)}} \]

\[ = \frac{m^2}{2} \left( \frac{2}{3\kappa} \right)^4 \int N(x) e^{ijk} (\partial_0 + \Lambda_0) A^i(x) \wedge \{ A^j(x), V(x, \epsilon)^{3/4} \} \wedge \{ A^k(x), V(x, \epsilon)^{3/4} \} \]

\[ \times \int \chi(x, y) e^{imn} (\partial_0 + \Lambda_0) A^m(y) \wedge \{ A^n(y), V(y, \epsilon)^{3/4} \} \wedge \{ A^0(y), V(y, \epsilon)^{3/4} \} . \] (68)

Now we shall introduce the familiar triangulation (with \( v = s(0) \)) and use the following (exploiting the fact that we are dealing with Abelian fields):

\[ U(1, s(\delta t)) = \exp[i(\phi(v) + \delta t \hat{s}_0 \phi(v) + o(\delta t^2))] \]

\[ h_{\delta_0}(0, \delta t) = \exp[i(\delta t \hat{s}_0 \Lambda_0 + o(\delta t^2))] . \] (69)
With this we have that
\[ U(1, s(\delta t))h_{\gamma}(0, \delta t)U(1, v)^{-1} - 1 = i\deltaTs^a(\partial_a\phi(v) + A_a(v)) + o(\delta t^2), \] (70)
so we have
\[ \int_\Delta (\partial \phi + A)(x) \wedge \{ A^j(x), V(x, \epsilon)^{3/4} \} \wedge \{ A^k(x), V(x, \epsilon)^{3/4} \} \approx \frac{2}{3} \epsilon^{mnp} [U(1, s_m(\Delta))h_{\gamma}(\Delta)U(1, v(\Delta))^{-1} - 1] \hat{\mathcal{Q}}^{j}_{v} \left( v, \frac{3}{4} \right) \hat{\mathcal{Q}}^{k}_{\epsilon} \left( v, \frac{3}{4} \right). \] (71)

Substituting this into (68) and taking the limit \( \epsilon \to 0 \) we have
\[ \hat{h}_M = \frac{m^2}{2\hbar^4k^4} \left( \frac{4}{3} \right)^6 \sum_v \frac{N(v)}{E(v)^2} \sum_{v(\Delta)=v(\Delta')} \epsilon^{ijkl} \epsilon^{lm} \epsilon_{npq} \epsilon_{rst} \times \left[ U(1, s_m(\Delta))h_{\gamma}(\Delta)U(1, v)^{-1} - 1 \right] \left[ U(1, s_r(\Delta'))h_{\gamma}(\Delta')U(1, v)^{-1} - 1 \right] \times \hat{\mathcal{Q}}^{j}_{v} \left( v, \frac{3}{4} \right) \hat{\mathcal{Q}}^{k}_{\epsilon} \left( v, \frac{3}{4} \right) \hat{\mathcal{Q}}^{m}_{r} \left( v, \frac{3}{4} \right) \hat{\mathcal{Q}}^{n}_{s} \left( v, \frac{3}{4} \right). \] (72)
This part of the Hamiltonian looks problematic due to the open ends of the holonomies, but since \( U(1) \) gauge invariance is studied with respect to (34) and \( \{ \mathcal{G}, A^a + \partial_a\phi \} = 0 \) the term \( \hat{h}_M \) is \( U(1) \) gauge invariant (open ends of holonomies are compensated by the scalar field, since the latter is defined in the vertices). Further more this term is also diffeomorphism and \( SU(2) \) gauge invariant, thus during quantization we do not come up against any problems.

The total Hamiltonian of the (symplectically embedded) Proca field is
\[ \hat{h} = \hat{h}_{GE} + \hat{h}_{GL} + \hat{h}_P + \hat{h}_M. \]
As Thiemann noticed earlier for similar systems, the Hamiltonian is well defined, i.e. it does not suffer from UV divergences, and this is achieved not via renormalization or spontaneous symmetry breaking but treating the gravitational field dynamical.

5. Kernel of the Hamiltonian of the Proca field

5.1. Complete solution

Though this Hamiltonian is quite complicated, there is a lot of relevant information that can be extracted from it. First let us look at the action of the different terms in the Hamiltonian on a generalized spin network state.

The action of the gravitational term \( \hat{h}_G \) changes the graph (as pointed out in [10]) in a way that it adds additional edges (specifically extraordinary edges) to the graph and changes the intertwiners, but it does not affect the labels which correspond to the matter fields. The other parts of the Hamiltonian describe the matter fields. Their structure is similar: they all contain matter operators and the \( \hat{Q} \) operator in some way which encode the interaction of the fields with gravity. This operator only changes the intertwiners, it does not change either the colourings or the graph itself. The derivative operators—the electric part of the Yang–Mills and the kinetic term—do not change the graphs, only the coefficients, but the mass term and the magnetic term do.

Whether this Hamiltonian possesses a non-trivial kernel is not an obvious question but we will show that the construction of generating a solution to the Hamiltonian constraint, which was introduced by Thiemann, can be generalized to the present case. Let \( |T\rangle_{\gamma, \hat{p}, \hat{q}, \hat{s}, \hat{t}} := |T\rangle_{\epsilon} \).
be a spin-colour network state. Then $\langle \Phi |$ is in the kernel of the Hamiltonian of the Proca field if for all $|T⟩$, we have

$$\langle \Phi | \hat{h} | T⟩ = 0.$$  (73)

The key observation of Thiemann was that the Hamiltonian of gravity acts as it generates so-called extraordinary edges (see details in [10] or [11]), and with this the kernel can be constructed in the following way: denote the set of labelled graphs (spin-nets) $S_0 \in (\gamma_0, l_0)$ which contain no extraordinary edges (these are the ‘sources’). Then compute $S_{n+1}$ by acting $\hat{h}_G$ on the elements of $S_n$ and decomposing them into spin-network states. The main advantages of the sets $S_n$ are that (1) they are disjoint, i.e. $S_n \cap S_m = \delta_{nm}$ and (2) finding a general diffeomorphism invariant solution to the Hamiltonian constraint reduces to finding a solution on a finite subspace.

Since we are only interested in solutions which are diffeomorphism invariant, we use $T[\bar{s}]$ instead of $T_s$, where $[\bar{s}]$ labels the diffeomorphism invariant distribution. In particular, let the ansatz for a solution be of the form

$$\langle \Psi | := \sum_{i=1}^{N} \sum_{[\bar{s}] \in [^{S^n}]} c_{[\bar{s}]}|T[\bar{s}]⟩.$$  

Since the Euclidean part of the gravitational Hamiltonian maps from $S^{(n)}$ to $S^{n+1}$, we have that the condition

$$\sum_{i=1}^{N} \sum_{[\bar{s}] \in [^{S^n}]} c_{[\bar{s}]}|T[\bar{s}]⟩ \hat{h}_{GE}|T[\bar{s}]⟩ = 0$$  (74)

is non-trivial if and only if $[\bar{s}'] \in [^{S^{n-1}}]$. Since $\hat{h}_{GE} = \sum_v N(v)\hat{h}_{GE}(v)$ and the above equation has to hold for all possible $N$, we have

$$\sum_{i=1}^{N} \sum_{[\bar{s}] \in [^{S^n}]} c_{[\bar{s}]}|T[\bar{s}]⟩ \hat{h}_{GE}(v)|T[\bar{s}]⟩ = 0$$  (75)

for each choice of the finite number of vertices $v$ and spin nets. Thus, we arrived at a finite system of linear equations with finite number of coefficients.

In the case of the Proca field we have three fields. Since the orthonormal base is of the form $|T⟩ \otimes |F⟩ \otimes |D⟩$, we will first look for analogues of the sets $S^{(n)}$ in the case of the scalar field and the Yang–Mills field.

The case of the scalar field is simple: denote the set $S^{(0)}(U)(\gamma )$ of all coloured graphs, so that all vertices are labelled by zero. Now define $S^{(n+1)}(U)$ by acting with $U(1, v)$ (for all possible $v$) on every element of $S^{(n)}(U)$. From the simple action of $U(1, v)$ it is clear that this is equivalent to the elements of $S^{(n)}(U)$ being those coloured graphs, for which the sum of the vertex colourings are $n$. If we look at the form of $\hat{h}_M$ we find that it maps from $S^{(n)}(U)$ to $S^{(n+1)}(U) \cup S^{(n+2)}(U)$.

The case of the Yang–Mills field is a bit more complicated, because both $\hat{h}_B$ and $\hat{h}_M$ change the graph. The former adds two (Yang–Mills) loops with colour 1, while the latter increases the colour of two edges by one (non-existent edges can be treated like they were edges with colouring zero). This is why in this case the analogues of $S^{(n)}$ will have two indices. Denote by $S^{(n,m)}(YM)$ the set of labelled graphs which have $n$ loops with colour 1 and the sum of the colours on all edges are $m$. It is easy to see that these sets are disjoint and the action of the $\hat{h}_B$ and $\hat{h}_M$ operators are the following: while $\hat{h}_B$ maps from $S^{(n,m)}(YM)$ to $S^{(n+2,m+2)}(YM)$, $\hat{h}_M$ maps from $S^{(n,m)}(YM)$ to $S^{(n,m+2)}(YM) \cup S^{(n,m+1)}(YM) \cup S^{(n,m)}(YM)$. It follows from the
construction that \( S^{(0,0)}(YM) \) will contain labelled graphs with zero colourings on all edges. Also note that the set \( S^{(n,m)}(YM) \) is empty unless \( n \geq m \).

Now, let the ansatz for the solution to the kernel of the Proca field be of the form

\[
\langle \Psi | := \sum_{i,j,k,l} \sum_{[s] \in S^{(i,j,l)}} c_{[s], [f], [d]} [T]_{[s]} \otimes [F]_{[f]} \otimes [D]_{[d]}.
\]

(76)

Now (76) is in the kernel of the Hamiltonian of the Proca field, if for all \([s]’, [f]’, [d]’\)

\[
\langle \Psi | \hat{h}_0 | T |_{[s]’} \otimes | F |_{[f]’} \otimes | D |_{[d]’} = 0.
\]

(77)

With the same reasoning as before, this condition is non-trivial if

- \([s]’ \in [S^{n_i, -1}] \cup [S^{n_i, -2}] \) (the union of the two sets is necessary if one takes the action of \( \hat{h}_{GL} \) also into account, since this latter operator adds two extra edges)
- \([d]’ \in [S^{(q_i, -1)}(U)] \cup [S^{(n_i, -2)}(U)] \)
- \([f]’ \in [S^{(m_i, p_i, -2)}(YM)] \cup [S^{(m_i, p_i, -1)}(YM)] \cup [S^{(m_i, p_i)}(YM)].\)

5.2. Special solutions

Since the system of equation (77) is very complicated, it is useful to take some special solutions in order to understand the full theory. Let \([0]_{YM} \) be the ‘vacuum’ flux network state of the Yang–Mills sector, which means that it has no Yang–Mills colours on either edge (note that this is not actually the familiar vacuum state as was shown in [15], since we are not dealing with Fock spaces). Similarly, denote the vacuum dust network state and vacuum spin network state by \( |0\rangle_G \) and \( |0\rangle_M \), respectively. Now it is easy to check that the state \( \langle \Psi |_G \otimes |0\rangle_{YM} \otimes |0\rangle_U \) is a solution of (77) if \( \langle \Psi |_G \) is the solution of the gravitational part of the Hamiltonian (this is because these ‘vacuum states’ are annihilated by the corresponding derivative operators in \( \hat{h}_P \) and \( \hat{h}_{YM} \), and are orthogonal to every state created by the operators in \( \hat{h}_{YM} \) and \( \hat{h}_U \)). So these special states can be interpreted as pure gravity.

Now let us check states of the form \( |0\rangle_G \otimes \langle \Psi |_{YM} \otimes |0\rangle_U \). It is easy to show that this is in the kernel of the Hamiltonian for all \( \langle \Psi |_{YM} \). In fact the same is true for states of the form \( |0\rangle_G \otimes |\Psi \rangle_{YM} \otimes |0\rangle_U \). These states are obviously nonphysical since the expectation value of the volume, area and length operators of these states are all zero for all volumes, surfaces and curves, respectively. (It is worth mentioning that these states are in the kernel of all Hamiltonians which have density weight one and are composed only from the gravitational, Yang–Mills and scalar fields.)

Let us check whether there are solutions of the form \( |\Psi \rangle_G \otimes |\Psi \rangle_{YM} \otimes |0\rangle_U \), where \( |\Psi \rangle_G \otimes |\Psi \rangle_{YM} \) is in the kernel of \( \hat{h}_G + \hat{h}_{YM} \). The answer is yes, if \( |\Psi \rangle_G \otimes |\Psi \rangle_{YM} \) contains only flux networks that have \( U(1) \) colours only on the loops, not on the edges, since in this case these states are annihilated by the operator \( \hat{h}_M \). These states are in the subset of the kernel of the Yang–Mills field coupled to gravity. Since currently we do not have a semi-classical description of the above system it will be for future investigations to check the physical meaning of these states. But if we look at the limit \( m \to 0 \), we find that these states will be solutions, since \( \langle \Psi |_G \otimes |\Psi \rangle_{YM} \otimes |0\rangle_U |X(v)\rangle |\phi \rangle = 0 \) for all \( |\phi \rangle \) and

\[
\langle \Psi |_G \otimes |\Psi \rangle_{YM} \otimes |0\rangle_U (\hat{h} - \hat{h}_G - \hat{h}_{YM}) |\phi \rangle = \langle \Psi |_G \otimes |\Psi \rangle_{YM} \otimes |0\rangle_U \hat{h}_M |\phi \rangle \to 0.
\]

(78)
6. Gauge fixing

We used the symplectically embedded Proca field to avoid implementing the Dirac brackets in the quantum theory. This approach led to a well defined quantum theory, as shown in the previous sections. This was achieved by introducing an auxiliary scalar field to the formalism. The theory we gained is equivalent to the original one since if the constraints are solved the scalar field disappears automatically. But this equivalence is not manifest if we do not solve the constraints, thus we need to introduce gauge fixing. As we shall see this will again lead to a system with second class constraints like in the original theory (without the scalar field). Below we outline how the quantization of such systems could be handled.

The key observation is, as we pointed out in section 2, that the original Lagrangian can be obtained via the gauge fixing $D^a \phi = 0$. The strategy will be to implement this condition in the Hamiltonian formalism. If we compare the original Hamiltonian with the Hamiltonian of the symplectically embedded Proca field we find that if we substitute

$$\partial_a \varphi = 0 \quad (79)$$

$$\pi = \sqrt{qm}^2 A_0 - N^a A_a \quad (80)$$

into (32)–(34), we obtain the constraints (14)–(16). So if we introduce the two extra conditions (constraints)

$$C_a = \partial_a \varphi = 0 \quad (81)$$

$$C = \pi - \sqrt{qm}^2 A_0 - N^a A_a = 0 \quad (82)$$

we arrive at the original case. One may ask how the original theory has second-class constraints while the symplectically embedded one has only first-class constraints. The trick is that, as pointed out in [19], the conditions (81) and (82) are also constraints and if we include these in the constraint algebra, we obtain a system with second-class constraints.

(In the case of second-class constraints one must use Dirac brackets, so it is natural to ask: what did we gain with the symplectic embedding? Actually, the main advantage of this method is that we were able to quantize the theory without introducing the Dirac brackets. The Dirac brackets are only needed when one fixes the gauge.)

Now what remains is calculating the Dirac bracket and implementing the two conditions (81) and (82) in the quantum theory. First we need the Poisson brackets of the new constraints with the existing ones. If we define the same linear combinations (18), (19), (20) as for the original case, we find that these are first-class constraints. Also, with the same reasoning as before, one finds that $G_i$ and $H_a + G_a A_i + G A_a$ are also first-class constraints. Thus we have four second-class constraints: $\mathcal{H}, G, C_a, C$. The elements of the antisymmetric matrix $M_{ij}^{(P)}(x, y)$ are therefore:

$$M_{12}^{(P)} = \{\mathcal{H}(x), G(y)\} = 0 \quad (83)$$

$$M_{11}^{(P)} = \{\mathcal{H}(x), C_a(y)\} = \frac{\pi(x)}{m^2 \sqrt{q}(x)} D^a(y) \delta(x - y) \quad (84)$$

$$M_{14}^{(P)} = \{\mathcal{H}(x), C(y)\} = -m^2 \sqrt{q}(x)(A_a + \partial_a \varphi) D^a(x) \delta(x - y) + m^2 \sqrt{q}(y) N^a(y) E_a(x) \delta(x - y) \quad (85)$$
we get the following condition on the coefficients

\[ M^{(P)}_{23} = \{ G(x), C_a(y) \} = -N_3^{(a)} \delta(x - y) \]  

(86)

\[ M^{(P)}_{24} = \{ G(x), C(y) \} = m^2 \sqrt{q(y)} N^{(a)} \delta(x - y) \]  

(87)

\[ M^{(P)}_{34} = \{ C_a(x), C(y) \} = -\delta^{(a)} \delta(x - y). \]  

(88)

With the inverse matrix \((M^{(P)}_{ij})^{-1}(x, y)\) one can define the Dirac brackets in a similar way as in the first chapter, the only difference is that now we have six second-class constraints and the matrix \(M^{(P)}_{ij}(x, y)\) is much more complicated.

After this we quantize the theory in the following way. The Hilbert space and the configuration variables are defined as in the original (first-class constraint) case, but we have to redefine the momentum operators \(E(S)\) and \(P_B\) (see the Remark at the end of section 3). Now let us check how our new constraints can be interpreted in the quantum theory. Because of the complicated Dirac bracket, the precise action of the operator version of (82) is left for future studies. The constraint (81) on the other hand is much more simple. If we look at the regularization of the mass term, specifically at equations (69), we see that

\[ U(1, s(\delta t)) - U(1, v) = i\delta t^\alpha \partial_\alpha \phi(v) + o(\delta t^2), \]  

(89)

so the constraint can be implemented such that \(\langle \Psi \rangle\) is in its kernel only if

\[ \langle \Psi | (U(1, b(e)) - U(1, f(e))) | \Psi \rangle = 0 \]  

(90)

for all \(|\Psi\rangle\) and all e edge \((b(e)\) is the beginning-, \(f(e)\) is the end-point of the edge). Let \(|\psi\rangle\) be a basis element, that is

\[ |\psi\rangle = |S\rangle_{\gamma, j, \rho, \lambda, h, l} = |T(A)\rangle_{\gamma, j, \rho} \otimes |F(A)\rangle_{\gamma, j} \otimes |D(U)\rangle_{\gamma, \lambda}. \]  

(91)

It is obvious that the action of the operator in (90) will be

\[ (U(1, b(e)) - U(1, f(e))) |\psi\rangle = |S\rangle_{\gamma, j, \rho, \lambda, h, l} - |S\rangle_{\gamma, j, \rho, \lambda, h, l}, \]  

(92)

where \(\lambda_1\) and \(\lambda_2\) is obtained by increasing the value of \(\lambda\) in the appropriate vertex by one. Since the condition is implemented in all vertices, we have that the coefficient of \(|S\rangle_{\gamma, j, \rho, \lambda, h, l}\) in \(\langle \Psi |\) is the same as the coefficient of \(|S\rangle_{\gamma, j, \rho, \lambda, h, l}\) in \(\langle \Psi |\) if both \(|D(U)\rangle_{\gamma, \lambda}\) and \(|D(U)\rangle_{\gamma, \lambda}\) are elements of \(S^{(n)}(U)\) for a fixed value of \(n\). In other words if we substitute (76) into the above constraint we get the following condition on the coefficients

\[ c_{[s], [f], [d]} - c_{[s], [f], [d]} = 0 \]  

(93)

for all \([s], [f], [d]\), where \([d]_1, [d]_2\) are in the same set \(S^{(n)}(U)\). This condition has non-trivial solutions, for example the case where only those coefficients of \(|S\rangle_{\gamma, j, \rho, \lambda, h, l}\) are not zero where \(\lambda_3\) is the same for all vertices.

This is the case when one first implements gauge fixing, then quantizes the system. One may ask whether it is possible to first quantize the system, then do gauge fixing. This is a problematic issue for the following reasons. Consider the operator versions of the constraints \(h_t, \bar{G}, C\). These, when quantized, are smeared with test functions \(N, A_0, \Lambda\), respectively and have the form \(\hat{h} = \sum_v N(v) \hat{h}_v\) etc. In the case of \(C_a\) one simply uses the operator in (90). Now consider the operator version \(\hat{M}_{ij}(v, v')\)—which could be interpreted as the operator version of \(M^{(P)}_{ij}(x, y)\)—defined with the help of the commutators of the constraint operators: \(\hat{M}_{12} = [\hat{h}_v, \bar{G}_v]\) etc. The question is whether the inverse of this matrix—defined via the condition \(\sum_{v''} \hat{M}_{ijk}(v, v'')(\hat{M})^{-1}_{ij}(v, v') = \delta_{v'}\delta_{v''}\)—actually exists. If it does, then with the

\[ M^{(P)}_{12} = \{ G(x), C_a(y) \} = N_3^{(a)} \delta(x - y) \]  

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help of this matrix we can define a Dirac commutator—analogue of the Dirac bracket—the following way:

\[
\{\hat{O}_1, \hat{O}_2\}_D = \{\hat{O}_1, \hat{O}_2\} - \frac{1}{2} \sum_{v_1, v_2} (\{\hat{O}_1, \hat{C}_i(v_1)\}(\hat{M})^{-1}_{ij}(v_1, v_2)[\hat{C}_j(v_2), \hat{O}_2] - \{\hat{O}_2, \hat{C}_i(v_1)\}(\hat{M})^{-1}_{ij}(v_1, v_2)[\hat{C}_j(v_2), \hat{O}_1],
\]

(94)

where the \(\hat{C}_i\) are the constraint operators. What one now has to do is replace the commutators with this Dirac commutator. Furthermore one has to modify the action of the momentum operators. To see why, consider first a dust network state

\[
|D(U)\rangle_{\gamma, \vec{\lambda}} = \prod_{i=1}^{N} (U(\lambda_i, v_i)),
\]

(95)

which can be interpreted if the operator \(\prod_{i=1}^{N} (U(\lambda_i, v_i))\) acted on the vacuum state \(|0\rangle\). If we have first-class constraint algebra we have the following identity:

\[
\hat{P}|D(U)\rangle_{\gamma, \vec{\lambda}} = \left[ \hat{P}, \prod_{i=1}^{N} (U(\lambda_i, v_i)) \right] |0\rangle.
\]

(96)

After gauge fixing this identity is the key to defining the new momentum operator \(\hat{P}_D\):

\[
\hat{P}_D|D(U)\rangle_{\gamma, \vec{\lambda}} := \left[ \hat{P}, \prod_{i=1}^{N} (U(\lambda_i, v_i)) \right] |0\rangle.
\]

(97)

The new electric flux operator \(\hat{E}(S)_D\) can be defined the same way. The only thing one has to do is replace the momentum operators in the constraints and use Dirac commutators instead of usual commutators (note that the gravitational momentum operator does not change since the Poisson bracket of \(E_i^a\) is zero with all four second-class constraints).

The critical part of the above construction is the existence and uniqueness of \((\hat{M})^{-1}_{ij}(v, v')\). But even if it did exist, the other question is whether (94) is really the operator version of the Dirac bracket. The answers to these questions depend on whether the above construction works.

If the above operator exists then the anomalies of the constraint algebra are removed. First let us focus on the gravitational variables. Since the gravitational gauge and the diffeomorphism constraints are first class, and the momentum operator for the canonical momenta \(E_i^a\) does not change after gauge fixing, there will be no gravitational anomalies in the theory. Other anomalies will not appear since by construction \([C_i, C_j]_D = 0\) for all constraints \(C_i\), and if (94) is the operator version of the Dirac bracket, we obtain

\[
[C_i, C_j]_D = \{\hat{C}_i, \hat{C}_j\} - \frac{1}{2} \sum_{v_1, v_2} (\{\hat{C}_i, \hat{C}_k(v_1)\}(\hat{M})^{-1}_{ij}(v_1, v_2)[\hat{C}_j, \hat{C}_k] - \{\hat{C}_j, \hat{C}_k(v_1)\}(\hat{M})^{-1}_{ij}(v_1, v_2)[\hat{C}_i, \hat{C}_k])
\]

\[
= \{\hat{C}_i, \hat{C}_j\} - \frac{1}{2} (\{\hat{C}_i, \hat{C}_j\} - [\hat{C}_i, \hat{C}_j]) = 0
\]

(98)

so there are no anomalies (the above commutator does not impose a new constraint since it is identically zero). This is also true for first-class constraints that have a vanishing Poisson bracket with all constraints, since then the corresponding operators will have zero commutator, thus the previous expression is also zero. The only problem is the case of first-class constraints that have a non-zero Poisson bracket with the constraints since then the structure constants
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will appear in the Dirac bracket. This is problematic in the quantum theory since the structure constants may become operators which could cause anomalies. In all cases factor-ordering ambiguities occur because we have terms that contain the product of three non-trivial operators, but these do not cause inconsistencies but only change the results of the theory.

7. Mass

Note that in this theory, mass is a parameter, in fact it is a coupling constant which couples the scalar field, the Yang–Mills field and gravity. In the classical Hamiltonian analysis one can make the following rescaling: \( \pi \rightarrow \pi/m, \phi \rightarrow m\phi, \mathcal{A}_\mu \rightarrow \mathcal{A}_\mu/m, \mathcal{E}_\alpha \rightarrow m\mathcal{E}_\alpha \), which is a canonical transformation. But in the quantum regime, this parameter enters the Hamiltonian in a non-trivial way. In this sense it is very similar to the Immirzi parameter of the pure gravitational case. In the latter case the Hawking-entropy provided a tool that helped fix this parameter [10, 11], so there is a chance that with a similar method one might be able to make predictions on the value of \( m \).

Another way would be to define propagators in loop quantum gravity, since the poles of the propagators could be interpreted as mass. But so far the question of time remains unsolved in the theory, leaving this idea for future research. Nonetheless there are attempts which could provide a solution to the problem of time, see, e.g. [20] and references therein. But without further input, mass is an undefined parameter of the theory which has to be given from experiments.

8. Summary and outlook

In this paper we investigated the Proca field in the framework of loop quantum gravity. It turned out that the tools developed in this theory can be applied to the Proca field if one introduces a scalar field and rewrites the Lagrangian as (28). But this scalar field is quite different from that used in spontaneous symmetry breaking, since there is no need for the scalar mass term and it restores symmetry rather than breaking it. The role of this field is actually to make the constraint algebra first-class. After rewriting the Lagrangian, the (3+1) decomposition and the quantization was straightforward. The resulting Hamilton operator is well defined and diffeomorphism covariant. We provided a method to calculate the matrix elements of this Hamiltonian and some special solutions, which may be a starting point on the interpretation of the theory. We showed that the parameter \( m \) is actually a coupling constant and very similar to the Immirzi parameter. We also showed how to introduce a gauge fixing both at the classical and quantum level which gives back the original theory. This method could be useful for quantizing general systems with second class constraint algebra.

Three major questions remain. The first is the question of the mass. In our description we treated it as a given parameter and did not obtain any condition which could have given it fixed values. Research in this direction could give us values for \( m \) which could prove useful for testing loop quantum gravity.

The second question is closely related to the first and it is the relationship of theories with spontaneous symmetry breaking. This would give some more understanding of the Higgs mechanism on one hand and gauge fixing on the other.

The third is that one should give a more precise and more general method for gauge fixing in order to use it in other cases. For example the gauge fixing used in the case of the Proca field was simple in the sense that it did not affect the gravitational part of the theory (it was diffeomorphism invariant etc). But one could imagine gauges that break diffeomorphism invariance where the construction given here cannot be applied.
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