Quantum Compressed Sensing with Unsupervised Tensor Network Machine Learning

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We propose tensor-network compressed sensing (TNCS) for compressing and communicating classical information via the quantum states trained by the unsupervised tensor network (TN) machine learning. The main task of TNCS is to reconstruct as accurately as possible the full classical information from a generative TN state, by knowing as small part of the classical information as possible. In the applications to the datasets of hand-written digits and fashion images, we train the generative TN (matrix product state) by the training set, and show that the images in the testing set can be reconstructed from a small number of pixels. Related issues including the applications of TNCS to quantum encrypted communication are discussed.

I. INTRODUCTION

An important perspective of quantum information is to transfer and process classical information by taking advantages of quantum physics. Taking dense/super-dense coding protocol [1–7] as an example, the idea is to use previously shared entangled state between a sender and the receiver(s) to send more classical information than is possible without the resource of entanglement. Another example that was recently developed is the machine learning algorithms by tensor network (TN) [8–14]. The aim is to employ TN (see some reviews of TN in Refs. [15–20]) as a novel machine-learning model to learn, classify, and/or generate general quantum features in the quantum many-body Hilbert space.

In this work, we propose tensor-network compressed sensing (TNCS) by combining the idea of quantum communication [21], compressed sensing [22] (see also the book in Ref. [23]), and unsupervised TN machine learning [10]. Let us consider the following scenario. Alice wants to send an image of, e.g., a signature, to Bob in a secured way. She only sends a small number $N_f$ of pixels to Bob by classical communication which might be unsafe or even public. After Bob receives these pixels, he measures the quantum state that is previously provided by Alice, in the way determined by the received pixels, and then reconstructs the full information from the measured state. $N_f$ should be as small as possible, so that any other parties without the provided quantum state cannot access the full information from these pixels.

In the TNCS, Alice firstly trains the quantum state $|\Psi\rangle$ by the unsupervised TN machine learning algorithm. Then, she needs to determine which $N_f$ pixels should be sent to Bob so that $N_f$ can be as small as possible. We propose to choose the pixels in a specific way, dubbed as entanglement ordering (EO) that is determined by the entanglement property of $|\Psi\rangle$. With the $N_f$ pixels sent from Alice, Bob can recover the unsent pixels from $|\Psi\rangle$ in a generative process of the TN machine learning.

We testify the TNCS with the datasets of hand-written digits and fashion images (namely MNIST [24] and fashion-MNIST [25]). Three different ways of choosing the sent pixels are testified, which are the random ordering (RO), the ordering by variance (VO), and the EO. By comparing the average peak signal-noise ratio (PSNR) [see Eq. (8)] between the reconstructed images and the original ones from the testing set, the EO exhibits the highest PSNR. Discussions about several relevant issues of TNCS are given, including the potential quantum nature of the TNCS, and the necessary ambiguous correlations of the information, security for quantum encrypted communication, and etc.

II. TENSOR NETWORK COMPRESSED SENSING

Suppose Alice wants to send an image of a hand-written digit “3”. She firstly trains the quantum state $|\Psi\rangle$ as the generative model for the training set of “3” in MNIST. This can be done with the unsupervised TN machine learning algorithm [10], where $|\Psi\rangle$ has the form of matrix product state (MPS) [26]. In the training process, the tensors in the MPS are updated to minimize the negative log-likelihood, so that the whole state optimally captures the joint probability distribution of the training set. See more details in Appendix A.

With $|\Psi\rangle$, Alice needs to decide which $N_f$ pixels should be sent to Bob, so that $N_f$ can be as small as possible, or the accuracy for Bob to reconstruct the image can be as high as possible with a fixed $N_f$. This is similar to compress an image of $N$ pixels to $N_f \ll N$ pixels with the assistance of $|\Psi\rangle$. In the language of probability, the problem can be restated as the following. Knowing the $N_f$ pixels $\{x^{\text{sent}}_i\} = \{x_{p_1}, x_{p_2}, \ldots\}$ at the positions $(p_1, p_2, \ldots)$, the (conditional) probability dis-

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A simple example that helps to understand EOMP is provided in Appendix B. After Bob receives the \( N_f \) pixels (and the corresponding positions in the image), he measures \(|\Psi\rangle\) to obtain \(|\Phi\rangle\) [Eq. (2)]. The rest of the unsent pixels are then generated by \(|\Phi\rangle\).

In Ref. [10], the authors employ a probabilistic process to generate the images, where the pixels are sampled to be black or white (0 or 1). The probability \( P(x) \) of being 0 or 1 for the \( n \)-th pixel is determined from \( \rho_n \), \( P(x) = \langle x|\tilde{\rho}_n|x\rangle, x = 0, 1 \) (note \( \sum_x P(x) = Tr\tilde{\rho}_n = 1 \) due to the normalization of \(|\Psi\rangle\)). It means that one simply measures the qubit in the basis of the Pauli matrix \( \sigma^z \). Here, to generate gray-scale images (as the original images are gray-scale), we generate the pixels \( \{x^{[\text{rest}]n}\} \) that have the maximal probability, i.e.,

\[
\{x^{[\text{rest}]n}\} = \arg\max_{\{x\}} \prod_{n} \langle s(x) | \Phi \rangle^2, \tag{6}
\]

where the product \( \prod_{n} \) goes through all the pixels in \(|\Phi\rangle\). It means that each measurement basis \( |s(x)\rangle \) is the dominant eigenstate of the corresponding single-site reduced density matrix of \(|\Phi\rangle\) [Eq. (5)].

III. RESULTS

We testify the TNCS on the MNIST and fashion-MNIST datasets. Each dataset has 10 classes of images, and in total has 60000 training images and 10000 testing images. Each image contains \( 28 \times 28 = 784 \) gray-scale pixels. Fig. 1 demonstrates two original images and the reconstructed images with different numbers of known pixels \( N_f \) picked in three different orders (EO, RO, and VO). EO is determined by the EOMP with the state \(|\Psi\rangle\). Note that the MPS is trained by the training images, and the reconstructed images are from the testing dataset. For VO, the pixels are picked referring to the variance of the training images. The variance of the \( n \)-th pixel is calculated as

\[
V_n = \sum_i [x_{i,n} - (\sum_j x_{j,n}/K)]^2 / K, \tag{7}
\]

where the summation goes though the training images and \( K \) is the number of the training images. In the VO, the pixels that have larger variance in the training images are considered to convey more information, thus selected for sending. RO means to pick the pixels of each image randomly. EO and VO are determined by the state and training set, respectively, and do not change with the specific images to be sent.

For \( N_f = 0 \), \(|\Psi\rangle\) just generates the image (denoted as \( \tilde{x} \)) that has the maximal probability in \(|\Psi\rangle\) [see Eq. (6)]. We name such an image generated with no known pixel as the quantum average. One \(|\Psi\rangle\) gives one unique quantum average (we assume that all \( \tilde{\rho}_n \)s have non-degenerated eigenvalues).

As shown in Fig. 2, the quantum average is different from the simple average \( \bar{x}_n = \sum_j x_{j,n}/K \), since no correlations are considered in the simple average. Such correlations (and entanglement) are considered in the quantum average when calculating the reduced density matrix.
FIG. 1. Examples of original and generated images in MNIST and fashion-MNIST in the entanglement order (EO), random order (RO), and variance order (VO). The number of known features \( N_f \) varies from 0 to 170, while the total number of features in an image is 784. We take the bond dimension of the generative MPS as \( \chi = 40 \).

FIG. 2. The images by taking simple average of each pixel and by taking the quantum average (generated by MPS with no known pixel).

For \( N_f > 0 \), the more the known pixels there are, the closer the generated image will be to the specific image (the one to be reconstructed). Fig. 1 shows how well an image can be reconstructed from \( |\Psi\rangle \) with different numbers \( N_f \) of pixels. Take the reconstruction of a dress image as an example (last three rows in Fig. 1). The original image is quite different from the quantum average \( (N_f = 0) \). With only \( N_f \approx 5 \) known pixels picked by EO, the sleeves emerge. In contrast, the sleeves appear until 50 pixels are known if they are picked randomly. For the VO, the sleeves also emerge with 5 pixels but in a bad shape. The shape of sleeves is reconstructed with \( N_f \approx 20 \) in VO to a similar quality as \( N_f \approx 5 \) in EO. The length of the sleeves is corrected with \( N_f \approx 50 \) for EO and \( N_f \approx 110 \) for RO and VO.

Fig. 3 shows the average PSNR of reconstructing all the images of “3” in MNIST and dresses in fashion-MNIST. We take the virtual bond dimension (see Appendix A) as \( \chi = 16 \) and \( \chi = 40 \). The PSNR between two images \( \{x\} \) and \( \{y\} \) is defined as

\[
\text{PSNR}(\{x\}, \{y\}) = 10 \log_{10} \frac{784}{\sum_n (x_n - y_n)^2}.
\]

Generally, PSNR increases with \( N_f \) and \( \chi \) as expected. With the same \( \chi \), the EO achieves the highest PSNR among the three ways.

Both SEE and variance measure the amount of the carried information. Considering a pixel (labeled as \( n \)) that is always black in all the training images, such a pixel obviously carries no information, and we have \( S_n = V_n = 0 \). On the other hand, if a pixel changes dramatically with the training images, not necessarily but normally, this pixel may contain more information, and we will have large \( S_n \) and \( V_n \). One essential difference is that \( S_n \) and \( V_n \) are properties from the quantum state and classical data, respectively. In our case, the quantum quantity (EO) outperforms the classical one (VO). More discussions are given in Sec. IV.

Fig. 3 shows which pixels are selected in EO and VO with different values of \( N_f \). To illustrate the orders, we mark a pixel redder than those pixels that are behind this pixel in the order. Both EO and VO manage to capture the general shapes. Particularly, the “checker-board” pattern appears in EO with relatively large \( N_f \). This brings higher efficiency for the following reason. Since each two nearest-neighbor pixels should possess a strong correlation, the corresponding qubits are ex-
In the following, we raise several relevant questions from different aspects of TNCS.

What differences does quantum physics make in TNCS? Though the EO works better than VO, we are not stating here that the quantum advantages over classical information with these two specific methods. In fact, even RO where the pixels are picked randomly works well. That means that the TNCS is a valid and efficient scheme for transferring and reconstructing images via quantum states.

Nevertheless, TNCS indeed provides a new path of investigating quantum advantages over classical information techniques. A question is how to define new (classical or quantum) quantities that better suppress $N_f$ and/or increase PSNR. Possible choices include the (classical) correlation functions of the training data, the (quantum) correlation functions from $|\Psi\rangle$, and the $k$-site entanglement entropy with $k > 1$. The current results with SEE and variance show some signs of the differences between quantum and classical means. Indeed, the performance of both quantum and classical means need to be pushed to their limits to discuss more clearly about the quantum advantages.

Ambiguous correlations of information in TNCS. Another immediate question about TNCS is how to determine the samples (denoted by $A$) for training the quantum state $|\Psi\rangle$, and what are the relations to the information (denoted by $B$) to be transferred or reconstructed. Here we require that the reconstructed images do not belong to those for training the state. It brings flexibility, meaning that Alice does not have to know what exactly are in $B$ while preparing $|\Psi\rangle$ by $A$.

But, $B$ has to be “ambiguously” correlated to $A$ somehow. Let us consider an extreme situation, where all training samples in $A$ are formed by uncorrelated random numbers. The trained state $|\Psi\rangle$ is an entangled state. However, such a state obviously cannot be used to effectively transfer a random image as no correlations exist between the random image and the state.

In this work, we choose $A$ and $B$ as the training and testing images of the same dataset, respectively. For instance, $A$ and $B$ are handwritten digits “3” or images of dresses. Although the “microscopic information” (pixels) of all the images in $A$ and $B$ are different from each other, a human being can recognize the “macroscopic information” of each image as a digit “3” (or a dress) without any problem. This suggests that $A$ and $B$ must be correlated somehow. In other words, we here ensure the existence of the “ambiguous” correlations between $A$ and $B$ by the “macroscopic” information. How to characterize and quantify such ambiguous correlations is an important issue to TNCS, and will be also helpful to further understand and model the recognition process.

Security in encrypted communication and potential application. In the scenario depicted above, TNCS can be used to securely send information via quantum states. Since $|\Psi\rangle$ cannot be cloned, one cannot copy an unknown state to others. The information is secured under the assumption that those without $|\Psi\rangle$ cannot reconstruct the full information solely from $N_f \ll N$ pixels. Moreover, there are many ways to enhance the security to avoid that the full information be cracked from the known pixels.

For example, Alice can introduce a one-to-one (reversible) deterministic map $\{y[x]\} = F(\{x[y]\}, \{z[x]\})$ to encrypt $\{z[x]\}$. Without $F$, the $\{z[x]\}$, which might be unsafe, could contain critical information (see for example the second
and fifth rows in Fig. A1 in Appendix C, which are almost meaningful images). The purpose of \( F \) is to avoid containing any meaningful information in \( \{ x^{[\text{sent}]}, y^{[\text{sent}]}, \} \).

Such a \( F \)-encrypted TNCS will contain the following steps: 1) Alice designs the function \( F \), and trains \( \{|\Psi\rangle \} \) by the images formed by \( \{ x^{[\text{sent}]}, y^{[\text{sent}]}, \} \); 2) Alice sends \( \{|\Psi\rangle \} \) to Bob; 3) For the information to be sent, Alice sends \( \{ x^{[\text{sent}]}, y^{[\text{sent}]}, \} = F(\{ x^{[\text{sent}]}, y^{[\text{sent}]}, \}) \) and the function \( F \) to Bob through classical channels that may not be safe; 4) Bob obtains \( \{ x^{[\text{sent}]}, y^{[\text{sent}]}, \} \) by \( \{|\Psi\rangle \} \) and \( \{ y^{[\text{sent}]}, \} \) (same to the standard TNCS), and obtains \( \{ x^{[\text{sent}]}, y^{[\text{sent}]}, \} \) by \( \{ \psi^{[\text{sent}]}, \} \), \( \{ x^{[\text{sent}]}, \} \), and the inverse of \( F \). Then Bob will have the full information \( \{ x^{[\text{sent}]}, y^{[\text{sent}]}, \} \). The information will be safe since those without \( \{|\Psi\rangle \} \) cannot have \( \{ x^{[\text{sent}]}, y^{[\text{sent}]}, \} \), thus cannot obtain \( \{ x^{[\text{sent}]}, y^{[\text{sent}]}, \} \) even if they have \( F \) and \( \{ y^{[\text{sent}]}, \} \).

Since the information to be sent is not restricted to the data that train \( \{|\Psi\rangle \} \), Alice can provide previously the copies of \( \{|\Psi\rangle \} \) to multiple parties, and send any piece of “ambiguously” correlated information to each party anytime afterwards. Different pieces of information can be sent via the copies of the same state.

Meanwhile, Alice does not allow other parties to access the coefficients of \( \{|\Psi\rangle \} \), to guarantee herself as the only provider of the state. One potential risk is that Alice provides too many copies of \( \{|\Psi\rangle \} \) to others, with which the coefficients of \( \{|\Psi\rangle \} \) can be cracked by, e.g., quantum state tomography [28]. In our case, this risk is low since \( N \) is large, and it can be easily controlled by the number of the states provided to other parties. Note that schemes can be designed to avoid such a risk. See an example in Appendix D.

With TNCS, the “microscopic” information cannot be exactly reconstructed. One possible application in this case is to transfer signatures, where the exact “microscopic” information is not necessarily needed. However, it would be interesting to develop a modified version of TNCS with the aim of reconstructing exactly the full information. One may use the strings of 0’s and 1’s with a much smaller length \( (N \ll 784) \) for training and reconstructing. The images in \( A \) and \( B \) may not contain any “macroscopic” information (may not be, e.g., meaningful images), but the “ambiguous” correlations are necessary. How to construct the training dataset for sending/reconstructing a certain kind of information, and how are the efficiency, accuracy, and security, are open questions.

**Other open issues about TNCS.** The TNCS proposed in this work can be further improved from several aspects. Considerring the TN model, MPS is the simplest TN model and is a good choice for 1D data such as the time series. For images that are in fact 2D data, MPS is still usable, as shown in e.g., Refs. [8, 10, 27]. Tree TN and MERA (originally proposed in Refs. [13, 29] for physical problems) have been suggested as a more suitable TN for learning 2D images [9, 14, 30]. Other TN’s such as PEPS [15, 31] are much less investigated in the literature.

One may also consider to develop different ways of generating the information. The way we propose in this work can be easily done in classical simulations, but is surely more challenging to implement in experiments or quantum computations. The probabilistic way proposed in Ref. [10] is more feasible in experiments. As the TNCS can be utilized as a classical or method, one may choose (or develop) the generating way to serve the specific purpose.

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**Appendix A: Unsupervised tensor network machine learning algorithm**

In the generative TN machine learning algorithm proposed in Ref. [10], each image is mapped to a product state of \( N \) qubits as \( |\phi_i\rangle = \prod_n |s(x_{i,n})\rangle \) with \( |s(x_{i,n})\rangle = \cos(x_{i,n} \pi/2)|0\rangle + \sin(x_{i,n} \pi/2)|1\rangle \) and \( N \) the total number of pixels in one image. Here, \( x_{i,n} \) is the \( n \)-th pixel (gray with 0 \( \leq x_{i,n} \leq 1 \)) of the \( i \)-th image. The coefficients in the quantum state \( |\Psi\rangle \) are optimized to minimize the negative log-likelihood (NLL) defined as

\[
f = \ln |\langle \Psi | \Psi \rangle|^2 - \frac{1}{N} \sum_i \ln |\langle \Psi | \phi_i \rangle|^2.
\]

(A1)

The summation \( \sum_i \) is over all the training images. NLL characterizes the resemblance between two probability distributions.

In this work, we choose the TN to be matrix product state (MPS). The coefficients of \( \{|\Psi\rangle \} \) are in a special form satisfying

\[
|\Psi\rangle = \sum_{\{a\}} \prod_n \sum_{s_n=0,1} A[n]^{a_n,a_{n+1}} |s_n\rangle.
\]

(A2)

\( A[n] \) represents a tensor that corresponds to the \( n \)-th pixel. The indexes \( \{a\} \) are known as virtual bonds of the MPS; their dimensions are bounded by \( \dim(a_n) \leq \chi \), with \( \chi \) called virtual bond dimension. MPS is an efficient representation of quantum-many-body states where the total number of parameters scales linearly with \( N \) as \( \sim 2N \chi^2 \). Note that the dimension of the Hilbert space actually scales exponentially as \( \sim 2^N \). The tensors in the MPS are updated alternatively.
by the gradient method as $A^{[n]} \leftarrow A^{[n]} - \tau \partial f / \partial A^{[n]}$, with $\tau$ the gradient step; see Ref. [8] or [10] for more details. After converging, $|\Psi\rangle$ gives the joint probability of the pixels. The probability for any image $\{x\}$ in $|\Psi\rangle$ is given as $P(\{x\}) = |\prod_{n \in \Psi} s(x_n)|^2$.

Appendix B: A simple example to understand entanglement ordering measurement protocol

To explain why the entanglement ordering measurement protocol (EOMP) works, let us consider the following four-qubit state as an example,

$$|\Psi\rangle = \left(\frac{\sqrt{2}}{2}|01\rangle + \frac{\sqrt{2}}{2}|10\rangle\right) \otimes \left(\frac{\sqrt{3}}{2}|01\rangle + \frac{\sqrt{3}}{2}|10\rangle\right)$$

$$= \frac{\sqrt{2}}{4}|0101\rangle + \frac{\sqrt{6}}{4}|0110\rangle + \frac{\sqrt{2}}{4}|1001\rangle + \frac{\sqrt{6}}{4}|1010\rangle.$$  

(A1)

Such a state can describe a dataset of four images (0, 1, 0, 1), (0, 1, 1, 0), (1, 0, 0, 1), and (1, 1, 0, 0), with the probability $P = 1/8, 3/8, 1/8, \text{and} 3/8$, respectively.

If Alice wants to send two pixels and encode the rest two in the state, the pixel that Alice should firstly choose is obviously the first (or the second) pixel. Since the first two qubits are in the maximally entangled state, one can be determined by knowing the other pixel. The second pixel Alice chooses should be the third or the forth one. These two qubits are entangled (but not maximally), thus knowing one of them will gain certain (but not the full) information of the other. In all, Alice should send the first (or second) and the third (or the forth) pixels to Bob.

The EOMP gives the same answer. The SEE of $|\psi\rangle$ satisfies $S_1^{\text{ent}} = S_2^{\text{ent}} = \ln 2 \approx 0.693$, and $S_3^{\text{ent}} = S_4^{\text{ent}} = -\frac{1}{2} \ln \frac{1}{4} = \frac{1}{2} \ln 4 \approx 0.562$. In the step 1 of the EOMP, Alice chooses the first or the second pixel. The reduced density matrices satisfy $\rho_1 = \rho_2 = 1/2$, with $I$ the $2 \times 2$ identity. Therefore, Alice decides to measure the first qubit by $|0\rangle \langle 0|$ or $|1\rangle \langle 1|$. In either case, the resulting three-qubit state will be $|\Psi(3)\rangle = |x\rangle \otimes (\frac{1}{2}|01\rangle + \frac{\sqrt{2}}{2}|10\rangle)$ with $x = 0$ or 1. In the second iteration, Alice has $S_3^{\text{ent}} = 0$ and $S_4^{\text{ent}} = S_4^{\text{ent}} \approx 0.562$, thus she decides to send the third (or forth) pixel. In comparison, Alice will choose to send the first and second pixels according to the variance, which is not a good idea since Bob will not be able to gain any information about the third and forth pixels. Again, we would like to emphasize that this example is to help understand EOMP; it is too simple to draw any general conclusions about the advantages/disadvantages of quantum methods over classical ones.

Appendix C: More examples generated by tensor network compressed sensing

In Fig. A1, we demonstrate twenty different images from the two dataset. The first and forth rows show the original images. The second and fifth rows show $\{x^{[\text{sent}]}\}$ without being encrypted by $F$ (see the discussions in the main text). The third and sixth rows show the reconstructed images, where each image is generated from $N_f = 80$ known pixels. Although the images (from the same dataset) are reconstructed by the same state, the differences of the shapes are well recovered. The challenging part particularly for the fashion-MNIST is to recover the details, such as the shades on the dresses.

Fig. A2 demonstrates the images reconstructed from the MPS’s with different virtual bond dimensions $\chi$. Fig. A3 show the original and generated images from different classes of the MNIST and fashion-MNIST datasets. For each class, an MPS is trained by taking $\chi = 40$. The images are generated by EOMP with $N_f = 80$. In general, the quality will be improved with larger $\chi$, particularly the sharpness of the shape. However, the particular details of different images, such as the unique pictures on the coats or the stripes on the dresses, are challenging to be generated.

Appendix D: tensor network compressed sensing and quantum communication

In the scenario given in the main text, Alice sends a small part of the classical information $\dot{x}_{\text{sent}}$ and the whole state $|\Psi\rangle$ to Bob. Then generates the missing information $\dot{x}_{\text{rest}}$ from $|\Psi\rangle$ and $\dot{x}_{\text{sent}}$. This process can be re-interpreted as a more standard quantum communication scheme.

First, Alice trains and prepares $|\Psi\rangle$. Then she sends the qubits corresponding to $\dot{x}_{\text{sent}}$ to Bob, and keeps those corresponding to $\dot{x}_{\text{rest}}$ to herself. Note that these qubits of $\dot{x}_{\text{sent}}$ and $\dot{x}_{\text{rest}}$ form the whole entangled state $|\Psi\rangle$. To send the information, Alice measures her qubits according to $\dot{x}_{\text{sent}}$. Afterwards, Bob generates the $\dot{x}_{\text{rest}}$ from his qubits. In this scenario, Alice only gives a part of the qubits in $|\Psi\rangle$ to Bob or other receivers, thus it becomes impossible for others to reproduce $|\Psi\rangle$ by state tomography and etc. In addition, Alice does not need to transfer the information of $\dot{x}_{\text{sent}}$ through classical channel, which avoids the risks in communicating $\dot{x}_{\text{sent}}$ classically. But the qubits with Alice and Bob need to be kept entangled until Alice implements the measurement on her qubits.

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FIG. A1. (Color online) Images (digits “3” in the first row and dresses in the forth row), the pixels \(\{x^{[\text{sel}]}\}\) (the second and fifth rows), and the generated images (the third and sixth rows) with \(N_f = 80\) known pixels. The generated images in the same row are from a same state written in the form of MPS. We take the bond dimension of the MPS as \(\chi = 40\).

FIG. A2. The original and generated images with \(N_f = 80\) and different \(\chi\) of the MPS.

FIG. A3. Original and generated images in the MNIST and fashion-MNIST datasets. We take the bond dimension of the MPS as \(\chi = 40\) and the number of known pixels \(N_f = 80\) in the EOMP.
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