Generalization of a theorem of Carathéodory

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April 7, 2006

Abstract

Carathéodory showed that \( n \) complex numbers \( c_1, \ldots, c_n \) can uniquely be written in the form \( c_p = \sum_{j=1}^{m} \rho_j \epsilon_p \) with \( p = 1, \ldots, n \), where the \( \epsilon_j \)s are different unimodular complex numbers, the \( \rho_j \)s are strictly positive numbers and integer \( m \) never exceeds \( n \). We give the conditions to be obeyed for the former property to hold true if the \( \rho_j \)s are simply required to be real and different from zero. It turns out that the number of the possible choices of the signs of the \( \rho_j \)s are at most equal to the number of the different eigenvalues of the Hermitian Toeplitz matrix whose \( i, j \)-th entry is \( c_j - c_i \), where \( c_{-p} \) is equal to the complex conjugate of \( c_p \) and \( c_0 = 0 \). This generalization is relevant for neutron scattering. Its proof is made possible by a lemma - which is an interesting side result - that establishes a necessary and sufficient condition for the unimodularity of the roots of a polynomial based only on the polynomial coefficients.

Keywords: Toeplitz\_matrix\_factorization, unimodular\_roots, neutron\_scattering, signal\_theory, inverse\_problems

PACS: 61.12.Bt, 02.30.Zz, 89.70.+c, 02.10.Yn, 02.50.Ga

MSC2000: 11L03, 30C15, 15A23, 15A90, 42A63, 42A70, 42A82
1 Introduction

Carathéodory’s theorem\(^1\) states that \( n \) complex numbers \( c_1, \ldots, c_n \) as well as their complex conjugates, respectively denoted by \( c_{-1}, \ldots, c_{-n} \), can always and uniquely be written as

\[
c_p = \sum_{j=1}^{m} \rho_j \epsilon_j^p, \quad p = 0, \pm 1, \ldots, \pm n, \tag{1}\]

with \( \rho_j \in \mathbb{R}, \rho_j > 0; \epsilon_j \in \mathbb{C}, |\epsilon_j| = 1, \epsilon_j \neq \epsilon_k (j \neq k = 1, \ldots, m) \) and \( c_0 \) uniquely determined by the \( c_j \)’s with \( j \neq 0 \). Further, the \( \rho_j \)’s, the \( \epsilon_j \)’s and \( m \) are unique and \( m \) obeys to \( 1 \leq m \leq n \).

The practical relevance of this theorem for the inverse scattering problem\(^2\) as well as for information theory\(^3,4\) appears evident from the following remark. Writing the \( \epsilon_j \)’s as \( e^{i2\pi x_j} \) with \( 0 \leq x_j < 1 \), the \( c_p \)’s take the form \( \sum_{j=1}^{m} \rho_j e^{i2\pi x_j p} \) so as they can be interpreted as the scattering amplitudes generated by \( m \) point scatterers (with "charges" \( \rho_1, \ldots, \rho_m \) respectively located at \( x_1, \ldots, x_m \)) and relevant to the "scattering vector" values \( p = 0, \pm 1, \ldots, \pm n \). One concludes that the theorem of Carathéodory allows us to determine the positions and the charges from the knowledge of the scattering amplitudes \( c_1, \ldots, c_n \).

Further, it ensures that the solution of this inverse problem is unique. However, in the case of neutron scattering\(^5\), the charges of the scattering centers have no longer the same sign. Nonetheless, it has recently been shown\(^6\) that the so-called algebraic approach for solving the structure in the case of X-ray scattering from an ideal crystal\(^2\) can also be applied to the case of neutron scattering. This result suggests that the aforesaid Carathéodory theorem can be generalized so as to avoid the requirement that the sign of all the
scattering charges be positive. In this note we show how this generalization
is carried out.

Before proving this statement, we find it convenient to briefly sketch the proof
of Carathéodory’s theorem reported by Grenander and Szegő. The proof is
based on an enlargement of the set of the \( n \) given complex numbers \( c_p \) to a
set containing \((2n+1)\) complex numbers still denoted as \( c_p \) with \(-n \leq p \leq n\),
the \( c_p \)s with negative index being defined as the complex conjugates of the
given \( c_p \)s, \( i.e. \ c_{-p} = \overline{c_p} \) with \( p = 1, \ldots, n \). (Hereafter an overbar will always
denote the complex conjugate). The remaining value \( c_0 \), real by assumption,
is determined as follows. Consider the \((n+1) \times (n+1)\) matrix \((C)\) with its
\((r,s)\)th element defined as

\[
C_{r,s} = c_{s-r}, \quad r, s = 1, \ldots, (n+1). \tag{2}
\]

This matrix is a Hermitian Toeplitz matrix and its diagonal elements are
equal to \( c_0 \). This value is chosen in such a way that the matrix \((C)\) turns out
to be singular \( i.e. \ \det(C) = 0 \) and the associated bilinear Hermitian form

\[
u^\dagger(C)u \equiv \sum_{r,s=1}^{n+1} \overline{u_r} C_{r,s} u_s \tag{3}
\]

[where \( u \) is an \((n+1)\) dimensional complex vector] non-negative (or semi-
positive) definite. To show that \( c_0 \) can uniquely be determined, one proceeds
as follows. Consider the \((n+1) \times (n+1)\) matrix \((\hat{C})\) that has its non-diagonal
elements equal to the correspondent elements of \((C)\) and the diagonal ele-
ments equal to zero, \( i.e., \) with \( r, s = 1, \ldots, (n+1)\),

\[
\hat{C}_{r,s} = \begin{cases} 
c_{s-r}, & \text{if } r \neq s, \\
0, & \text{if } r = s.
\end{cases} \tag{4}
\]
This matrix is Hermitian. Then its eigenvalues \((\chi_j, j = 1, \ldots n + 1)\), are real and can be labeled so as to have \(\chi_1 \leq \ldots \leq \chi_{n+1}\). Further, they are such that \(\sum_{j=1}^{n+1} \chi_j = 0\) because the trace of \((\hat{C})\) is zero. Hence, \(\chi_1 < 0\) and \(\chi_{n+1} > 0\). One immediately realizes that matrix \((C)\) is obtained by setting \(c_0 = -\chi_1 > 0\) so that \((C) = (\hat{C}) - \chi_1(I), (I)\) being the unit matrix. In fact, the matrix \((\hat{C} - \chi_1 I)\) is a Toeplitz Hermitian matrix with its diagonal elements equal to \((-\chi_1)\). The secular equation of this matrix is

\[
\det((C) - z(I)) = \det((\hat{C}) - \chi_1(I) - z(I)) = \det((\hat{C}) - (\chi_1 + z)(I)) = 0.
\]

This equation is the same equation that determines the eigenvalues of \((\hat{C})\) if, instead of variable \(z\), we use the shifted variable \(z + \chi_1\). Thus, the eigenvalues of \((C)\) are: \(0 = (\chi_1 - \chi_1) \leq (\chi_2 - \chi_1) \leq \ldots \leq (\chi_{n+1} - \chi_1)\) and the matrix \((C)\) is semi-definite positive. Let \(\mu_i (\geq 1)\) denote the multiplicity of the eigenvalue \(\chi_i\) of \((\hat{C})\), then the rank of \((C)\) is \((n + 1 - \mu_i)\) and the \(m\) present in Eq. \(\text{I}\) has the same value, \(i.e. m = (n + 1 - \mu_i)\). Exploiting the non-negative definiteness of \((C)\), Grenander and Szegö showed that: I) the \(\epsilon_j\)s are distinct and are the roots of the following polynomial equation, referred to as \textit{resolvent equation} in the following,

\[
P_m(z) = D_m^{-1} \det \begin{pmatrix}
    c_0 & c_1 & \cdots & c_{m-1} & c_m \\
    c_{-1} & c_0 & \cdots & c_{m-2} & c_{m-1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    c_{-m+1} & c_{-m+2} & \cdots & c_0 & c_m \\
    1 & z & \cdots & z^{m-1} & z^m
\end{pmatrix} = 0, \quad (5)
\]

where \(D_m > 0\) is the determinant of the left principal minor contained in the first \(m\) rows; II) the \(\rho_j\)s are strictly positive and given by

\[
\rho_j = \frac{1}{P_m(\epsilon_j)} \sum_{p=0}^{m-1} \beta_{j,p} c_p, \quad (6)
\]
where the prime denotes the derivative and the $\beta_{j,p}$s are the coefficients of the following polynomial

$$P_m(z)/(z - \epsilon_j) \equiv \sum_{p=0}^{m-1} \beta_{j,p} z^p;$$

(7)

and III) that Eq. (11) holds true with $p = -n, (-n + 1), \ldots, n$.

This theorem, via (1), implies that any non-negative definite Hermitian Toeplitz matrix $(C)$, defined by (2), uniquely factorizes as $(V)(Q)(V)^\dagger$ where $(V)$ is an $(n + 1) \times m$ Vandermonde matrix with $V_{i,j} \equiv \epsilon_i^{-1}$, $(V)^\dagger$ its Hermitian conjugate and $(Q)$ an $m \times m$ positive-definite diagonal matrix with $Q_{i,j} \equiv \rho_i \delta_{i,j}$. A more general factorization was obtained by Ellis and Lay$^9$. In their theorem 3.4, they showed that a Toeplitz matrix of order $(n + 1)$ and rank $m \leq n$ can be factorized as $(V)(\Delta)(V)^\dagger$ where $(V)$ is a generalized confluent Vandermonde matrix generated by the distinct roots of the resolvent equation and $(\Delta)$ is a block-diagonal matrix with reversed upper triangular blocks. In proving this generalized factorization, the Toeplitz matrix was neither required to be Hermitian nor positive/negative semidefinite and this last property - as it will appear clear in § 2 - amounts to relax the condition that the $\rho_j$s present in (11) be strictly positive. However, in proving this theorem, it was explicitly assumed that all the roots of the resolvent equation were unimodular without stating the conditions that have to be obeyed by the Toeplitz matrix elements for the unimodularity property to occur. On the one hand, this point makes the generalization not fully proven. On the other hand, the physical motivation that underlies our attempt of generalizing Carathéodory’s theorem requires that Toeplitz matrices are Hermitian.
as well as a strictly diagonal structure of $(\Delta)$.

The generalization of Carathéodory’s theorem presented below meets the last two requirements. In particular, in appendix A we report the lemma that specifies the necessary and sufficient conditions that must be obeyed by the coefficients of a polynomial equation for all its roots to lie on the unit circle. We also stress that all the so far reported\textsuperscript{10,11} theorems, ensuring the unimodularity of the roots, are not based on the only knowledge of the polynomial coefficients, as our lemma does. Finally, appendices B, C and D report some properties of the coefficients of the resolvent equation, of Hermitian Toeplitz matrices and some numerical illustrations, respectively.

2 Generalization of Carathéodory’s theorem

The aforesaid choice of $c_0$ is the only one that yields a non negative definite matrix $(C)$. It is evident that if we choose $c_0 = -\chi_{n+1}$ the resulting $(C)$ is a non-positive definite matrix and $-(C)$ is a semi-positive definite one. We can apply to the latter the same analysis made by Grenander and Szegö with the conclusion that

$$-c_p = \sum_{j=1}^{m} \rho_j' \epsilon_j'^p, \quad p = 0, \pm 1, \ldots, \pm n$$

where $m = (n + 1 - \mu_{n+1})$, $\mu_{n+1}$ denoting the multiplicity of $\chi_{n+1}$, the $\rho_j'$s are positive and the $\epsilon_j'$s are distinct unimodular numbers.

Of course this generalization is quite trivial. The remaining choices are however more interesting. In fact, if we choose $c_0 = -\chi_j$ with $\chi_j \neq \chi_1$ and $\chi_j \neq \chi_{n+1}$, the resulting matrix $(C)$ will be indefinite, since some of its eigen-
values are negative, others positive and one, at least, equal to zero. The question is now the following: does a relation similar to (1) hold also true in this case? We shall show that the answer is affirmative iff: i) \( m \), the rank of \((C')\) smaller than the order \((n + 1)\) of \((C)\), is equal to the principal rank of \((C')\) and ii) the conditions reported in the lemma discussed in appendix A are fulfilled. In i) principal rank means the highest of the rank values of all the strictly principal left minors of \((C)\) where, by definition, an \((m \times m)\) strictly principal (left) minor of \((C)\) is a principal minor formed by \(m\) subsequent rows and columns with indices \(p + 1, \ldots, p + m\) and \(0 \leq p \leq (n + 1 - m)\).

Let \(\hat{\chi}_1 < \ldots < \hat{\chi}_\nu\) denote the distinct eigenvalues of \((\hat{C})\) (the matrix with \(c_0 = 0\)), \(\mu_1, \ldots, \mu_\nu\) their respective multiplicities (independent from the choice of \(c_0\)) and \(n_\hat{\chi}\) and \(p_\hat{\chi}\) the number of the \(\hat{\chi}\)s that respectively are negative and positive, so that either \(\nu = (n_\hat{\chi} + p_\hat{\chi} + 1)\) or \(\nu = (n_\hat{\chi} + p_\hat{\chi})\) depending on whether one of the \(\hat{\chi}\)s is equal to zero or not. If we set

\[
c_{l,0} = -\hat{\chi}_l, \quad N_{l,-} \equiv \sum_{i=1}^{l-1} \mu_i \quad \text{and} \quad N_{l,+} \equiv \sum_{i=l+1}^{\nu} \mu_i,
\]

(8)

the resulting matrix \((C_l)\) will have \(N_{l,-}\) negative eigenvalues, \(N_{l,+}\) positive eigenvalues and \(\mu_l\) eigenvalues equal to zero so that its rank is \(m_l = (n + 1 - \mu_l)\). We shall prove that the following relations

\[
c_{p} = \sum_{j=1}^{m_l} \rho_{l,j} \epsilon_{l,j}^{p}, \quad p = \pm 1, \ldots, \pm n,
\]

(9)

\[
c_{l,0} = \sum_{j=1}^{m_l} \rho_{l,j}, \quad p = 0
\]

hold true if \(m_l\) is equal to the principal rank of \((C_l)\) and the conditions stated in the lemma are obeyed. In (9) the value of \(m_l\) has been just defined,
the $\epsilon_{l,j}$s are the distinct unimodular roots of a polynomial equation of degree $m_l$ uniquely determined by the $c_p$s and $c_{l,0}$ and, finally, the number of the positive and negative $\rho_{l,j}$s respectively is $N_{l,+}$ and $N_{l,-}$.

It is remarked that we can construct $\nu$ different matrices ($C_l$) from a given ($\hat{C}$) by setting $c_{l,0} = -\hat{\chi}_l$, $l = 1, \ldots, \nu$. Among these matrices only those which obey i) and ii) allow us to write $c_{l,0}$ and the $c_p$s, with $p = \pm 1, \ldots, \pm n$, in the form (9). Hence, if the number of these matrices is denoted by $\mu$, one concludes that we have $\mu$ different ways for writing the $n$ complex numbers $c_1, \ldots, c_n$ in the form (9). In general $\mu$ is such that $2 \leq \mu \leq \nu$. Clearly this result generalizes Carathéodory’s theorem and we pass now to prove it.

The proof must be achieved by a procedure different from that followed by Grenander and Szegő because matrices ($C_l$) are no longer non-negative definite. Since our attention will focus on a particular ($C_l$), for notational simplicity we shall omit index $l$ in the following considerations. Assume that Eqs (9) are fulfilled so that

$$C_{r,s} = c_{s-r} = \sum_{j=1}^{m} \rho_j \epsilon_j^{s-r}, \quad r, s = 1, \ldots, (n+1),$$

with

$$\rho_j \in \mathbb{R}, \quad \rho_j \neq 0, \quad \epsilon_j \in \mathbb{C}, \quad |\epsilon_j| = 1, \quad \epsilon_j \neq \epsilon_i \text{ if } j \neq i, \quad j, i = 1, \ldots, m.$$ 

Then $C_{r,s} = \overline{C_{s,r}}$ and the Hermiticity of ($C$) is ensured. Introduce now two further matrices: ($V$) and ($Q$) such that

$$V_{j,r} \equiv \epsilon_j^{r-1}, \quad j = 1, \ldots, m, \quad r = 1, \ldots, (n+1),$$

$$Q_{j,i} \equiv \rho_j \delta_{j,i}, \quad j, i = 1, \ldots, m.$$
The first is an $m \times (n + 1)$ rectangular Vandermonde matrix and the second a diagonal $m \times m$ matrix. As the $\epsilon_j$s are distinct, the rows of $(V)$ are linearly independent. From the unimodularity of the $\epsilon_j$s and (10) immediately follows that

$$
\sum_{i,j=1}^{m} V_{i,j} Q_{i,j} V_{j,s} = \sum_{i=1}^{m} \rho_i \delta_{i,j} V_{i,s} = \sum_{i=1}^{m} \rho_i \epsilon_i^{s-r} = C_{r,s}
$$

so that one has

$$(C) = (V)^\dagger(Q)(V).$$

Consider now the $p \times p$ minor of $(C)$ formed by the rows $r_1 < \cdots < r_p$ and the columns $s_1 < \cdots < s_p$. By Bezout’s theorem\textsuperscript{12,13} one finds that

$$
\det(C_{r_1,\cdots,r_p}^{s_1,\cdots,s_p}) = \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq m} \prod_{j=1}^{p} \rho_{i_j} \det(V_{i_1,\cdots,i_p}^{s_1,\cdots,s_p}) \det(V_{i_1,\cdots,i_p}^{1,\cdots,m})
$$

where, adopting Gantmacher’s notation\textsuperscript{12}, the lower and upper indices inside each determinant symbol denote the rows and the columns of the considered minors of $(C)$, $(V)$ and $(\bar{V})$. This expression makes it clear that the determinant of any minor of $(C)$ of order $p > m$ is equal to zero because the order of $(Q)$ is $m$. For this reason the rank of $(C)$ cannot exceed $m$. At the same time, if $p = m$, any $(m \times m)$ strictly principal minor $\det(C_{q+1,\cdots,q+m}^{q+1,\cdots,q+m})$ (with $0 \leq q \leq n + 1 - p$) of $(C)$ will have determinant equal to

$$
\prod_{j=1}^{m} \rho_j \prod_{1 \leq i \leq j \leq m} (\epsilon_j - \epsilon_i)^2.
$$

In fact, if $p = m$, the sum present in (16) involves a single term and, due to (12), in $\det(V_{q+1,\cdots,q+m}^{1,\cdots,m})$ we can factorize $\epsilon_1^q$ in the first row, $\epsilon_2^q$ in the second row and so on. The remaining matrix is a Vandermonde matrix so that

$$
\det(V_{q+1,\cdots,q+m}^{1,\cdots,m}) = \prod_{l=1}^{m} \epsilon_l^q \prod_{1 \leq i \leq j \leq m} (\epsilon_j - \epsilon_i).
$$
In the same way one shows that
\[
\det\left(\bar{V}_{q+1,\ldots,q+m}\right) = \prod_{l=1}^{m} \epsilon_{l}^{q} \prod_{1\leq i\leq j\leq m} (\epsilon_{j} - \epsilon_{i}).
\]

Finally, the unimodularity of the \(\epsilon_{j}\)'s yields Eq. (17). In this way, it has been shown that, if the elements of \((C)\) have the expression reported on the right hand side (rhs) of (10) and conditions (11) are obeyed, matrix \((C)\) has rank \(m\) and each of its principal minors formed by \(m\) subsequent rows is nonsingular. In this way the rank of \((C)\) is equal to its principal rank and the necessity of condition i) is now clear. Since these minors are equal, the first \(m\) rows (or columns) of \((C)\) are linearly independent. One recalls that the polynomial equation having its roots equal to the \(m\) \(\epsilon_{j}\)'s, can be written as
\[
P_{m}(z) \equiv \prod_{j=1}^{m} (z - \epsilon_{j}) \equiv \sum_{l=0}^{m} a_{l} z^{l} = 0
\]
with
\[
a_{l} = (-)^{m-l} \sum_{1\leq i_{1}<\cdots<i_{m-l}\leq m} \epsilon_{i_{1}} \cdots \epsilon_{i_{m-l}}, \quad l = 0, \ldots, (m - 1) \quad (19)
\]
and \(a_{m} = 1\). From (19) and the unimodularity of the \(\epsilon_{i}\)'s follows that \(a_{0}\) is unimodular and, therefore, is different from zero. From the condition \(P_{m}(\epsilon_{i}) = 0\) follows that \(\epsilon_{i}^{m} = -\sum_{l=0}^{m-1} a_{l} \epsilon_{i}^{l}\). After multiplying the latter by \(\epsilon_{i}^{j}\), with \(j \in \mathbb{Z}\), and setting \(q = m + j\), one finds that
\[
\epsilon_{i}^{q} = -\sum_{l=0}^{m-1} a_{l} \epsilon_{i}^{l+q-m}, \quad q \in \mathbb{Z} \quad \text{and} \quad i = 1, \ldots, m.
\]

The substitution of these relations in (10) yields
\[
c_{s-r} = -\sum_{i=1}^{m} \rho_{i} \sum_{l=0}^{m-1} a_{l} \epsilon_{i}^{l+s-r-m} = -\sum_{l=0}^{m-1} a_{l} c_{s-r-(m-l)}.
\]
Taking \((s-r) = 1, 2, \ldots, m\), one obtains the following system of linear equations

\[
\begin{align*}
    c_0 a_0 + c_1 a_1 + c_2 a_2 + \ldots + c_{m-1} a_{m-1} &= -c_m \\
    c_{-1} a_0 + c_0 a_1 + c_1 a_2 + \ldots + c_{m-2} a_{m-1} &= -c_{m-1} \\
    \vdots & \vdots & \vdots & \vdots & \vdots \\
    c_{-m+1} a_0 + c_{-m+2} a_1 + c_{-m+3} a_2 + \ldots + c_0 a_{m-1} &= -c_1.
\end{align*}
\]  

(22)

This uniquely determines coefficients \(a_0, \ldots, a_{m-1}\) of (18) because the determinant of the coefficients in (22) is different from zero and equal to \(D_m\), defined below Eq. (5). Recalling that \(a_m = 1\), resolvent equation (18) is fully determined and formally coincides with Grenander & Szegö’s Eq. (5).

The solution of (22) is

\[
    a_l = (-)^{m-l} \frac{\det(C_{1,\ldots,m+1}^{1\ldots,l+2\ldots,m+1})}{D}, \quad l = 0, \ldots, m \tag{23}
\]

\[
    D \equiv \det(C_{1,\ldots,m}^{1\ldots,m}) \neq 0 \tag{24}
\]

where in (23) we let \(l\) assume the value \(m\) since it yields \(a_m = 1\) as reported below (19). Further, setting \(l = 0\) in Eq. (23) and using (58) (with \(n = m+1, p = 1\) and \(q = 2\)), one finds

\[
    a_0 = (-)^m \frac{\det(C_{1,\ldots,m}^{2\ldots,m+1})}{D} = (-)^m e^{i\theta} \tag{25}
\]

In this way one recovers that \(a_0\) is unimodular as already reported below (19). As discussed in appendix C, coefficients (23) are partly related by the relations \(a_l = a_{m-l}/a_0, \ l = 0, \ldots, m\). These imply that the roots of (18) obey to \(\epsilon_{ij} = 1/\epsilon_j\) with \(j = 1, \ldots, m\) and \(i_1, \ldots, i_m\) equal to a permutation of \(\{1, \ldots, m\}\). This result ensures the unimodularity only for those roots with \(i_j = j\). The unimodularity of all the roots is ensured by the further
conditions stated in the lemma of appendix A. Thus, the roots of (18) are unimodular if the new \((m + 1) \times (m + 1)\) Toeplitz matrix \((S)\), defined as

\[ S_{r,s} \equiv \sigma_{s-r}, \quad r, s = 1, \ldots, m + 1, \]  

is nonnegative definite and has rank \(m\). Quantities \(\sigma_p\) are easily and uniquely determined by Eq.s (39) and (40) in terms of the \(a_p\)s given by (23). Thus, after checking that the determinants of the left principal minors, contained in the first \(p\) rows of \((S)\), are strictly positive for \(p = 1, \ldots, m\) and equal to zero for \(p = m + 1\), the unimodularity of all the roots of (18) is ensured.

After solving the resolvent equation, the \(\rho_j\)s can be determined by Eq.s (6) and (7) since these also apply in the case of non-definite \((C)\). Alternatively, the \(\rho_j\)s can be determined solving the system of \(m\) linear equations

\[ \sum_{j=1}^{m} \epsilon_j^p \rho_j = c_p, \quad p = 0, \ldots, (m - 1) \]  

that follow from (9). [For notational simplicity, we still omit index \(l\) present in the definition of \(c_0\).] These equations can also be written as

\[ \sum_{j=1}^{m} V^T_{p+1,j} \rho_j = c_p, \quad p = 0, \ldots, (m - 1) \]  

where \((V^T)\) is the transpose of the \(m \times m\) upper left principal minor of matrix \((V)\) defined by (12). The formal solution of (28) is

\[ \rho_j = \sum_{p=0}^{m-1} (V^T)^{-1}_{j,p+1} c_p, \quad j = 1, \ldots, m, \]  

since \((V^T)\) is certainly non singular.

Finally it must be proven that the numbers of the \(\rho_j\)s that turn out to be
positive or negative are respectively equal to \(N_+\) and \(N_-\) (again we omit index \(l\)). To this aim consider the Hermitian bilinear form

\[
C_2[u] \equiv \sum_{r,s=1}^{n+1} \bar{u}_r C_{r,s} u_s, \quad u_s \in \mathbb{C}. \tag{30}
\]

By Eq \([15]\) this is immediately expressed in terms of the diagonal form

\[
C_2[u] = \sum_{p=1}^{m} v_p[u] \rho_p v_p[u] \tag{31}
\]

where

\[
v_p[u] \equiv \sum_{s=1}^{n+1} V_{p,s} u_s, \quad p = 1, \ldots, m. \tag{32}
\]

At the same time, since \((C)\) is Hermitian, it can be diagonalized by a unitary transformation \((U)\) and written as

\[
(C) = (U)^\dagger (\chi) (U) \tag{33}
\]

where

\[
(\chi)_{r,s} = (\chi_r - \hat{\chi}_l) \delta_{r,s}, \quad r, s = 1, \ldots, (n + 1) \tag{34}
\]

with the \(\chi_r\)s equal to the eigenvalues of \((\hat{C})\). As discussed at the beginning of this section, \(\mu_l\) of the \((\chi_r - \hat{\chi}_l)\)s are equal to zero, \(N_{l,+}\) are positive and \(N_{l,-}\) negative. Therefore we can compact \((U^\dagger)\) by eliminating \(\mu_l\) columns whose index correspond to the rows of \((\chi)\) containing the zero eigenvalues and, subsequently, \((\chi)\) eliminating the rows and the columns containing the zero eigenvalues. Hereinafter \((U)\) shall be an \(m \times (n + 1)\) rectangular matrix with orthonormal rows and \((\chi)\) an \(m \times m\) diagonal and nonsingular matrix.

We set

\[
w_p \equiv \sum_{s=1}^{n+1} U_{p,s} u_s, \quad p = 1, \ldots, m \tag{35}
\]
and consider the $w_p$ as the arbitrary independent variables. Using Eq. (15) we can write

$$(U)^\dagger (\chi) (U) = (C) = (V)^\dagger (Q) (V).$$

The row-spaces of $(U)$ and $(V)$ necessarily coincide with the $(U)$ and $(V)$ right image spaces that in turn coincide with the eigenspace of $(C)$ associated to the eigenvalue zero. There exists then a non-singular $m \times m$ matrix $(R)$ such that

$$(V) = (R)(U).$$

Now, for any complex $m$-tuple $w = (w_1, \ldots, w_m)$, we have $w^\dagger (\chi) w = w^\dagger (R)^\dagger (Q) (R) w$ which leads to $(\chi) = (R)^\dagger (Q) (R)$. Thus $(\chi)$ and $(Q)$ are related by a congruence and Sylvester’s inertia law\textsuperscript{17} applies, and the number of positive (negative) $\rho_p$s coincides with the number of positive (negative) $(\chi_p - \hat{\chi}_l)s$. In this way the generalization of the Carathéodory theorem is complete.

3 Conclusion

Summarizing, given $n$ complex numbers $c_1, \ldots, c_n$ one considers the Hermitian Toeplitz matrix $(\hat{C})$ defined by (4). One evaluates its distinct eigenvalues, denoted by $\hat{\chi}_1 < \ldots < \hat{\chi}_\nu$ with multiplicities $\mu_1, \ldots, \mu_\nu$. Setting $(C_l) \equiv (\hat{C}) - \hat{\chi}_l (I)$ with $l = 1, \ldots, \nu$, the resulting matrices with $l \neq 1$ and $l \neq \nu$ are indefinite. For each of these $l$ values, the complex numbers $-\hat{\chi}_l, c_1, \ldots, c_n$ also can uniquely be written in the form (1) with $m_l = (-\mu_l + \sum_{q=1}^{\nu} \mu_q)$ iff i) the rank and the principal rank of $(C_l)$ are equal
[its value turns out to be equal to \( m_l \)] and ii) the \((m_l + 1) \times (m_l + 1)\) matrix \((S_l)\) [defined by (26), (38), (39) and (28)] is non-negative definite and has rank \( m_l \). In proving these results it is essential to know the conditions that must be obeyed by the coefficients of a polynomial equations for all its distinct roots to lie on the unit circle. The answer to this problem is given by the lemma reported in appendix A.

As last remark we observe that, in theorems 2.2 and 3.4 of Ellis and Lay\(^9\), the assumption that the resolvent has unimodular roots can be removed by the aforesaid lemma. It can be substituted with the constructive requirements that: a) the coefficients of the resolvent equation obey Eq. (37) if the given Toeplitz matrix \((T)\) is not Hermitian (oppositely, the condition is already fulfilled), b) if the discriminant of the resolvent equation is equal to zero, one algebraically eliminates\(^{18,19}\) all the multiple roots from the resolvent obtaining the lowest degree resolvent equation \([i.e.\ the\ equation\ with\ roots\ equal\ to\ all\ the\ distinct\ roots\ of\ the\ outset\ resolvent]\), c) from the coefficients of the (new) resolvent equation one constructs matrix \((S)\) defined by (20) and one checks its positive definiteness. In the only affirmative case the Ellis-Lay generalized factorization of \((T)\) is possible. This reduces to Carathéodory’s generalized one if \((T)\) is Hermitian and the outset resolvent has no multiple roots \([i.e.\ step\ b)\ is\ not\ required]\).
A Unimodular roots’ conditions

We shall now prove a lemma that states the necessary and sufficient condition for all the zeros of a polynomial equation with complex coefficients lie on the unit circle. The so far known theorems that ensure such property leans upon the existence of other polynomial with unimodular roots, while the following lemma only involves the coefficients of the given polynomial.

In appendix B it is shown that, if the coefficients $a_l$ of the $N$th degree polynomial equation

$$P_N(z) \equiv \prod_{1 \leq j \leq N} (z - \epsilon_j) = \sum_{l=0}^{N} a_l z^l = 0,$$  

(36)

obey the following conditions

$$a_N = 1, \quad |a_0| = 1 \quad \text{and} \quad a_m = a_{N-m}/a_0 \text{ for } m = 0, \ldots, N,$$  

(37)

the roots of the equation are such that $\overline{\epsilon_j} = 1/\epsilon_{i_j}$ for $j = 1, \ldots, m$ and $i_1, \ldots, i_m$ equal to a permutation of $\{1, \ldots, m\}$. The unimodularity of all the roots being not assured by this property, it is natural to ask: which are the further conditions to be obeyed by the $a_l$s for all the roots to lie on the unit circle?

The answer is given in the lemma reported later and based on Carathéodory’s theorem.

We first observe that it is not restrictive to assume -as we do below - that the roots of Eq. (36) are distinct because possible multiple roots can algebraically be eliminated. Consider the following symmetric functions of the roots
\[ \sigma_p = \sum_{j=1}^{N} \epsilon_j^p \quad p = 0, \pm 1, \pm 2, \ldots \quad (38) \]

They exist for negative \( p \) integers because \( a_0 \neq 0 \). For non-negative \( p \)s the \( \sigma_p \)s are uniquely determined from the coefficients of (38) by the following relations (see, e.g., Ref. [18], Chap. XIII)

\[ \begin{array}{lcl}
\sigma_0 &=& N \\
a_N \sigma_1 &=& -a_{N-1} \\
a_N \sigma_2 + a_{N-1} \sigma_1 &=& -2a_{N-2} \\
a_N \sigma_3 + a_{N-2} \sigma_1 + a_{N-1} \sigma_2 &=& -3a_{N-3} \\
\vdots \\
a_N \sigma_{N-1} + \ldots + a_2 \sigma_3 + a_3 \sigma_2 + a_2 \sigma_1 &=& -(N-1)a_1 \\
a_N \sigma_p + \ldots + a_2 \sigma_{p+2} + a_2 \sigma_{p+1} + a_0 \sigma_p &=& 0, \quad p = 0, 1, \ldots \\
\end{array} \quad (39) \]

Owing to the condition \( \epsilon_j = 1/\epsilon_i \), from (38) follows that

\[ \sigma_{-p} = \overline{\sigma_p} \quad p = 1, 2, \ldots, \quad (40) \]

and the last of relations (39) holds also true for negative \( p \) integers. In fact, the complex conjugate of this relation by (37) becomes

\[ a_0 \left( \sigma_{p+N} + \ldots + a_2 \sigma_{p+2} + a_1 \sigma_{p+1} + a_0 \sigma_p \right) = \\
a_0 \sigma_{-p-N} + \ldots + a_{N-2} \sigma_{-p-2} + a_{N-1} \sigma_{-p-1} + a_0 \sigma_{-p} = 0, \quad (41) \]

and the statement is proven. The previous considerations show that all the \( \sigma_p \)s are known in terms of \( a_0, \ldots, a_N \).

Introduce now the \((N+1) \times (N+1)\) Hermitian Toeplitz matrix \( S \) having its \((i,j)\)th element defined as

\[ S_{i,j} \equiv \sigma_{j-i}, \quad i, j = 1, \ldots, N + 1. \quad (42) \]

Assume first that the \( \epsilon_j \)s are unimodular (and distinct) and, similarly to (12), introduce a further \( N \times (N+1) \) matrix \( V \) with \( V_{r,s} \equiv \epsilon_r^{s-1} \). The
assumed properties of the $\epsilon_j$s ensure that $(S) = (V^\dagger)(V)$, that $\det(S) \neq 0$ and that the rank of $(V)$ is $N$. The three properties in turn imply that $(S)$ is a non negative definite matrix of rank $N$. Then, from Carathéodory’s theorem follows that the $\sigma_p$s can uniquely be written as

$$\sigma_p = \sum_{j=1}^{N} \tau_j \omega_j^p, \quad p = 0, \pm 1, \ldots, \pm N$$

with the $\omega_j$s unimodular, distinct and roots of the resolvent equation generated by matrix $(S)$, i.e.

$$Q_N(z) = \Delta_N^{-1} \det \begin{pmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_{N-1} & \sigma_N \\ \sigma_{-1} & \sigma_0 & \cdots & \sigma_{N-2} & \sigma_{N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & z & \cdots & z^{N-1} & z^N \end{pmatrix} = 0$$

(here $\Delta_N$ denotes the determinant of the $N \times N$ upper left principal minor of $(S)$). The comparison of (43) with (38) and the uniqueness of the Carathéodory decomposition imply that $\tau_1 = \ldots = \tau_N = 1$ and $\{\omega_1, \ldots, \omega_n\} = \{\epsilon_1, \ldots, \epsilon_N\}$. From the last follows that $Q_N(z) = P_N(z)$.

At this point we can state the lemma:

the roots of an $N$ degree polynomial equation $P_N(z) = 0$ are unimodular and distinct iff its coefficients $a_i$, besides obeying conditions (37), are such that matrix $(S)$, defined by (42), is non-negative definite and has rank $N$.

# The necessity of the lemma has already been proven. To prove its sufficiency one has to show that the properties that the rank of $(S)$ is $N$ and that $(S)$ is non-negative definite ensure that the roots of $P_N(z) = 0$ are distinct and unimodular, respectively. In fact, the first property implies that $\Delta_N \neq 0$. 

18
From definition [38] and property [40] follows that

\[
\Delta_N = \det \begin{pmatrix}
N & \sum_{j=1}^{N} \epsilon_j & \sum_{j=1}^{N} \epsilon_j^2 & \cdots & \sum_{j=1}^{N} \epsilon_j^{N-1} \\
\sigma_{-1} & \sigma_0 & \sigma_1 & \cdots & \sigma_{N-2} \\
\sigma_{-N+1} & \sigma_{-N+2} & \sigma_{-N+3} & \cdots & \sigma_0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma_{-2N+1} & \sigma_{-2N+2} & \sigma_{-2N+3} & \cdots & \sigma_0 
\end{pmatrix} \tag{45}
\]

\[
= \sum_{j=1}^{N} \det \begin{pmatrix}
1 & \epsilon_j & \epsilon_j^2 & \cdots & \epsilon_j^{N-1} \\
\sigma_{-1} & \sigma_0 & \sigma_1 & \cdots & \sigma_{N-2} \\
\sigma_{-N+1} & \sigma_{-N+2} & \sigma_{-N+3} & \cdots & \sigma_0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma_{-2N+1} & \sigma_{-2N+2} & \sigma_{-2N+3} & \cdots & \sigma_0 
\end{pmatrix} 
\]

The last expression can also be written as

\[
\sum_{1 \leq j_1, \ldots, j_N \leq N} \frac{1}{\epsilon_{j_1} \epsilon_{j_2} \cdots \epsilon_{j_N-1}} \det \begin{pmatrix}
1 & \epsilon_{j_1} & \epsilon_{j_1}^2 & \cdots & \epsilon_{j_1}^{N-1} \\
1 & \epsilon_{j_2} & \epsilon_{j_2}^2 & \cdots & \epsilon_{j_2}^{N-1} \\
1 & \epsilon_{j_N} & \epsilon_{j_N}^2 & \cdots & \epsilon_{j_N}^{N-1} 
\end{pmatrix}. \tag{46}
\]

Within the sum the only terms with \( j_1 \neq j_2 \neq \ldots \neq j_N \) can differ from zero.

In other words, the possible values of \( \{j_1, \ldots, j_N\} \) correspond to the possible permutations of \( \{1, \ldots, N\} \). The values of the corresponding determinants are \((-)^P \prod_{1 \leq i < j \leq N} (\epsilon_j - \epsilon_i)\) where \( P \) is the number of the transpositions required for passing from \( \{j_1, \ldots, j_N\} \) to \( \{1, \ldots, N\} \). One concludes that

\[
\Delta_N = \prod_{1 \leq i < j \leq N} (\epsilon_j - \epsilon_i)(1/\epsilon_j - 1/\epsilon_i). \tag{47}
\]

Thus, \( \Delta_N \neq 0 \) ensures that the roots of \( P_N(z) = 0 \) are distinct. We show now that the resolvent of \((S)\), i.e. Eq. [44], coincides with \( P_N(z) \). In fact, \( Q_N(z) \) can be written as \( Q_N(z) \equiv \sum_{p=0}^{N} q_p z^p = 0 \) with

\[
q_p \equiv \frac{(-1)^{N+p}}{\Delta_N} \det \begin{pmatrix}
\sigma_0 & \cdots & \sigma_{p-1} & \sigma_{p+1} & \cdots & \sigma_{N-1} & \sigma_N \\
\sigma_{-1} & \cdots & \sigma_{p-2} & \sigma_{p+2} & \cdots & \sigma_{N-3} & \sigma_{N-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\sigma_{-N+1} & \cdots & \sigma_{-N+p} & \sigma_{-N+p-2} & \cdots & \sigma_0 & \sigma_1 
\end{pmatrix}. \tag{48}
\]
Manipulations similar to those performed in Eq.s (45-46) convert the determinant present in (48) into

\[
\sum_{1 \leq j_1, \ldots, j_N \leq N} \frac{1}{\epsilon_{j_1}^{1} \cdots \epsilon_{j_N}^{N-1}} \begin{vmatrix}
1 & \cdots & \epsilon_{j_1}^{p-1} & \epsilon_{j_1}^{p+1} & \cdots & \epsilon_{j_1}^{N-1} & \epsilon_{j_1}^{N} \\
1 & \cdots & \epsilon_{j_2}^{p-1} & \epsilon_{j_2}^{p+1} & \cdots & \epsilon_{j_2}^{N-1} & \epsilon_{j_2}^{N} \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & \cdots & \epsilon_{j_N}^{p-1} & \epsilon_{j_N}^{p+1} & \cdots & \epsilon_{j_N}^{N-1} & \epsilon_{j_N}^{N} 
\end{vmatrix},
\]

Using the property that \( \epsilon_j^N = -\sum_{p=0}^{N-1} a_p \epsilon_j^p \), the above expression becomes

\[
\sum_{1 \leq j_1, \ldots, j_N \leq N} \frac{-a_p}{\epsilon_{j_1}^{1} \cdots \epsilon_{j_N}^{N-1}} \begin{vmatrix}
1 & \cdots & \epsilon_{j_1}^{p-1} & \epsilon_{j_1}^{p+1} & \cdots & \epsilon_{j_1}^{N-1} & \epsilon_{j_1}^{N} \\
1 & \cdots & \epsilon_{j_2}^{p-1} & \epsilon_{j_2}^{p+1} & \cdots & \epsilon_{j_2}^{N-1} & \epsilon_{j_2}^{N} \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & \cdots & \epsilon_{j_N}^{p-1} & \epsilon_{j_N}^{p+1} & \cdots & \epsilon_{j_N}^{N-1} & \epsilon_{j_N}^{N} 
\end{vmatrix},
\]

and from (46) and (48) one concludes that \( q_p = a_p, p = 0, \ldots, N \). In this way, the resolvent of \( (S) \) coincides with \( P_N(z) \). Consequently, the \( \epsilon_j \)s also are unimodular because the assumed non-negativeness of \( (S) \) and Carathéodory’s theorem ensure that the unimodularity is true for the roots of resolvent \( Q_N(z) \). Thus, the lemma’s sufficiency is proven.#

From the lemma follows, for instance, that the quadratic and cubic equations have distinct unimodular roots iff their coefficients are as follows

\[
N = 2 : \quad a_1 = \rho e^{i\phi/2}, \quad a_0 = e^{i\phi} \quad \text{with} \quad 0 \leq \rho < 2 \quad \text{and} \quad \phi \in [0, 2\pi)
\]

\[
N = 3 : \quad a_2 = \rho e^{i(\phi-\psi)}, \quad a_1 = \rho e^{i\psi}, \quad a_0 = e^{i\phi} \quad \text{with}
\]

either \( 0 \leq \rho \leq 1, \quad \phi, \psi \in [0, 2\pi) \)

or \( 1 \leq \rho < 3, \quad \phi \in [0, 2\pi), \quad \left(2\phi - \Phi(\rho)\right) < 3\psi < \left(2\phi + \Phi(\rho)\right) \)

where \( \Phi(\rho) \equiv \arccos[(\rho^4 + 18\rho^2 - 27)/8\rho^3] \).

An example of Hermitian Toeplitz matrix whose resolvent does not obey the conditions required by the lemma because the rank of \( (S) \) is smaller than \( N \) is given at the end of appendix D.
B Properties of the resolvent coefficients

Given a polynomial equation of degree m

\[ P_m(z) \equiv \prod_{j=1}^{m} (z - \epsilon_j) = \sum_{l=0}^{m} a_l z^l = 0 \]

with \(a_0 \neq 0\), we have the interesting property:

the coefficients \(a_l\) obey to

\[ \overline{a_l} = a_{m-l}/a_0, \quad l = 0, \ldots, m \] (49)

iff the roots of the equation are such that \(\overline{\epsilon_j} = 1/\epsilon_{i_j}\) for \(j = 1, \ldots, m\) and \(i_1, \ldots, i_m\) equal to a permutation of \(\{1, \ldots, m\}\).

# This condition amounts to say that the polynomial is self-reciprocal\(^{10}\). To prove the necessity one starts from expression (19) of \(a_l\). Taking its complex conjugate and using the assumed property of the roots one finds

\[
\overline{a_l} = (-)^{m-l} \sum_{1 \leq j_1 < \cdots < j_{m-l} \leq N} \frac{\epsilon_{j_1} \cdots \epsilon_{j_{m-l}}}{\epsilon_{i_1} \cdots \epsilon_{i_{m-l}}}
\]

= \((-)^{m-l} \frac{1}{\epsilon_{i_1} \cdots \epsilon_{i_{m-l}}}
\]

= \((-)^{m-l} \frac{\epsilon_1 \cdots \epsilon_m}{\prod_{j=1}^{m} \epsilon_j} \sum_{1 \leq i_1 < \cdots < i_{m-l} \leq N} \frac{\epsilon_{i_1} \cdots \epsilon_{i_{m-l}}}{\epsilon_{i_1} \cdots \epsilon_{i_{m-l}}}
\]

= \((-)^{m-l} \frac{\epsilon_{i_1} \cdots \epsilon_{i_l}}{\prod_{j=1}^{m} \epsilon_j} \sum_{1 \leq i_1 < \cdots < i_{m-l} \leq m} \frac{a_{m-l}}{a_0}
\]

To prove the sufficiency one observes that

\[ \overline{P_m(z)} = \prod_{j=1}^{m} (z - \overline{\epsilon_j}) = \sum_{j=1}^{m} \overline{a_j} z^j = \sum_{j=1}^{m} \frac{a_{m-j}}{a_0} z^j
\]

= \(z^m \frac{\sum_{l=0}^{m} a_l}{a_0} \overline{z}^l = \frac{z^m}{a_0} \prod_{j=1}^{m} \frac{1}{z - \epsilon_j}\). (50)
The previous manipulations require that no root is equal to zero and this is ensured by the condition $a_0 \neq 0$. With $z = \overline{\epsilon_j}$, whatever $j$ in $\{1, \ldots, m\}$, the first product in (51) vanishes. For the second to vanish one must have that $1/\overline{\epsilon_j} = \epsilon_i$ and the property of the roots is recovered. In passing it is noted that the property is true also when some roots have multiplicity greater than one.

We show now that conditions (49) are obeyed by the $a_l$s defined by Eq.s (23) and (24). In fact, setting $l = 0$ in Eq.(23) one finds

$$a_0 = (-)^m \det(C_{1, \ldots, m}^{2, \ldots, m+1})/\det(C_{1, \ldots, m}^{1, \ldots, m}) = (-)^m e^{i\theta_1}$$

(51)

where the last equality follows putting $n = m + 1$, $p = 1$ and $q = 2$ in (50). For the remaining $l$ values we substitute (23) in (49) obtaining

$$\det(C_{1, \ldots, m-l, m-l+2, \ldots, m}^{1, \ldots, m-2}) \det(C_{1, \ldots, m}^{1, \ldots, m}) = \det(C_{1, \ldots, m}^{1, \ldots, m}) \det(C_{1, \ldots, m}^{1, \ldots, m-2, m-2+2, \ldots, m+1}), \quad l = 0, \ldots, m$$

(52)

where we let $l$ take value $m$ because $a_0 \neq 0$. Taking $n = (m + 1)$, $(j_1, \ldots, j_m) = (1, \ldots, m-l, m-l+2, \ldots, m+1)$ and $(i_1, \ldots, i_m) = (1, \ldots, m)$ in (17) one finds that

$$\det(C_{1, \ldots, m-l, m-l+2, \ldots, m}^{1, \ldots, m}) = \det(C_{1, \ldots, m}^{1, \ldots, m, m+1, 2, \ldots, m+1}).$$

(53)

Using the property that all the $m \times m$ strictly principal minors of $(C)$ coincide, the rhs of (52) becomes

$$\det(C_{1, \ldots, m}^{1, \ldots, m, m+1, 2, \ldots, m+1}) \det(C_{2, \ldots, m+1}^{2, \ldots, m+1}).$$

(54)
From [61] follows that

\[ \det(C_{2,\ldots,m+1}) = \det(\lambda_{1,\ldots,m}) \det(C_{1,\ldots,m+1}), \]

and

\[ \det(C^{1,\ldots,l,l+2,\ldots,m+1}) = \det(\lambda_{1,\ldots,m}) \det(C^{1,\ldots,l,l+2,\ldots,m+1}). \]

The substitution of the above two relations in [54] yields

\[ \left| \det(\lambda_{1,\ldots,m}) \right|^2 \det(C^{1,\ldots,l,l+2,\ldots,m+1}) \det(C_{1,\ldots,m+1}), \]

that coincides with the left hand side of [52] by [62].

C Some properties of Hermitian Toeplitz matrices

We list here a series of properties obeyed by a square Hermitian Toeplitz matrix \((C)\) of order \(n\) and partly reported in Ref. [8].

(a) - Its elements obey to

\[ C_{r,s} = C_{s,r} = c_{r-s} = c_{s-r}, \quad r, s = 1, \ldots, n, \quad (55) \]

so that all the elements of \((C)\) contained in a line parallel to the main diagonal are equal.

(b) - One has the reflection symmetry with respect to the second diagonal formalized by the condition

\[ C_{r,s} = c_{n+1-s,n+1-r}, \quad (56) \]

(c) - All the \((m \times m)\) principal minors of \((C)\), whatever the considered rows (and columns), are identical.
(d) - For any choice of \( m \) rows \((1 \leq i_1 < \cdots < i_m \leq n)\) and \( m \) columns \((1 \leq j_1 < \cdots < j_m \leq n)\) it results
\[
\det\left( C_{i_1, \ldots, i_m}^{j_1, \ldots, j_m} \right) = \det\left( C^{T}_{(n+1-j_m), \ldots, (n+1-j_1)} \right) = \overline{\det\left( C_{(n+1-i_m), \ldots, (n+1-i_1)}^{(n+1-j_m), \ldots, (n+1-j_1)} \right)}.
\]

(57)

# The first equality, where \((C^T)\) denotes the transposed of \((C)\), follows from property \((b)\) and the second from the Hermiticity of \((C)\). #

The following properties, that we think to be original, hold only true for Hermitian Toeplitz matrices having their rank equal to the principal one.

(e) - If the principal rank of a Hermitian Toeplitz matrix \((C)\) is equal to the rank \( m(\leq n) \) of \((C)\), any \((m \times m)\) strictly principal minor of \((C)\) is non-singular.

# The property immediately follows from the definition of "principal rank" of a matrix, reported above Eq. (58), and \((c)\).#

(f) - For any \((n \times n)\) Hermitian Toeplitz matrix of rank equal to its principal rank \( m(\leq n) \), the determinant of any of its minors formed by \( m \) subsequent rows and \( m \) subsequent columns is simply related by a phase factor to the determinant of the (strictly) principal minor contained in the considered rows or columns, \( i.e. \)
\[
\begin{align*}
\det\left( C_{(q+1), \ldots, q+m}^{p+1, \ldots, p+m} \right) &= e^{i\theta_{p-q}} \det\left( C_{p+1, \ldots, p+m}^{q+1, \ldots, q+m} \right), \\
\det\left( C_{(q+1), \ldots, q+m}^{p+1, \ldots, p+m} \right) &= e^{i\theta_{p-q}} \det\left( C_{q+1, \ldots, q+m}^{p+1, \ldots, p+m} \right),
\end{align*}
\]

\(\theta_{p-q} \in \mathbb{R}, \quad p, q = 0, 1, \ldots, n - m.\)
Clearly, if the first of the above two equalities is true the second also is true because of (c). To prove the first of equalities (58) one observes that (d) implies that any \( m \) distinct rows of \( (C) \) can be written as linear combinations of \( m \) other distinct rows (see, e.g., Ref. [18], Chap. III). Hence rows \((p+1), \ldots, (p+m)\) can be expressed in terms of rows \((q+1), \ldots, (q+m)\) as

\[
C_{r,s} = \sum_{t=q+1}^{q+m} \lambda_{r,t} C_{t,s}, \quad r = (p + 1), \ldots, (p + m), \quad s = 1, \ldots, n, \quad (59)
\]

where the \( \lambda_{r,t} \) are suitable numerical coefficients. From these relations follows that

\[
det\left(C_{p+1,\ldots,p+m}^{p+1,\ldots,p+m}\right) = det\left(\lambda_{q+1,\ldots,q+m}^{q+1,\ldots,q+m}\right) det\left(C_{p+1,\ldots,p+m}^{q+1,\ldots,q+m}\right). \quad (60)
\]

Due to (e) the left hand side of (60) is different from zero so that both factors on the rhs are different from zero. The complex conjugation of (60), by the Hermiticity of \( (C) \), yields

\[
det\left(C_{p+1,\ldots,p+m}^{p+1,\ldots,p+m}\right) = \overline{det\left(\lambda_{q+1,\ldots,q+m}^{q+1,\ldots,q+m}\right)} det\left(C_{p+1,\ldots,p+m}^{q+1,\ldots,q+m}\right). \quad (61)
\]

From Eq. (59) also follows that

\[
det\left(C_{q+1,\ldots,q+m}^{p+1,\ldots,p+m}\right) = det\left(\lambda_{q+1,\ldots,q+m}^{q+1,\ldots,q+m}\right) det\left(C_{q+1,\ldots,q+m}^{q+1,\ldots,q+m}\right) = det\left(\lambda_{q+1,\ldots,q+m}^{q+1,\ldots,q+m}\right) det\left(C_{p+1,\ldots,p+m}^{p+1,\ldots,p+m}\right),
\]

where the last equality follows from (c). The substitution of the last equality in Eq. (60) and the fact that, by assumption, \( det\left(C_{p+1,\ldots,p+m}^{p+1,\ldots,p+m}\right) \neq 0 \) imply that

\[
\left|det\left(\lambda_{q+1,\ldots,q+m}^{q+1,\ldots,q+m}\right)\right|^2 = 1 \quad p, q = 0, \ldots, n - m, \quad (62)
\]
and Eq. (58) is proven. That the phase factor depends on $p - q$ instead of $(p, q)$ follows from the fact that the two determinants present in (58) do not change with the two substitutions $p \rightarrow p + 1$ and $q \rightarrow q + 1$ owing to Eq. (38).

An immediate consequence of (f) is the property that

(g) - any $(m \times m)$ minor formed by $m$ subsequent rows and $m$ subsequent columns of a Hermitian Toeplitz matrix with rank equal to its principal rank $m$ is non-singular.

D Numerical examples

To illustrate the application of the results discussed above we report three numerical examples.

1 - The first shows a case where it is impossible to express a set of $c_j$s in terms of positive and negative $\rho_j$s. Assume that $c_1 = 0$, $c_2 = 0$, $c_3 = 1$. The corresponding matrix ($\hat{C}$) has eigenvalues equal to $-1$, $0$, $0$, $1$. The matrix ($C_1$), obtained by setting $c_0 = 1$, is semi-positive definite with rank $3$ and eigenvalues equal to $0$, $1$, $1$, $2$. The solutions are: $\rho_1 = \rho_2 = \rho_3 = 1/3$ and $\epsilon_1 = 1$, $\epsilon_2 = -e^{i\pi/3}$, $\epsilon_3 = e^{i2\pi/3}$. The matrix ($C_2$), obtained by setting $c_0 = -1$, is semi-negative definite with rank equal to $3$ and eigenvalues equal to $-2$, $-1$, $-1$, $0$. The solutions are: $\rho_1 = \rho_2 = \rho_3 = -1/3$ and $\epsilon_1 = -1$, $\epsilon_2 = e^{i\pi/3}$, $\epsilon_3 = -e^{i2\pi/3}$. Finally, the matrix ($C_3$) obtained by setting $c_0 = 0$ coincides with ($\hat{C}$). It is non-definite, has rank equal to $2$ and principal rank equal to $0$. For this reason it is impossible to write $0$, $0$, $0$ and $1$ in the form (1) with $m = 2$ as it is easily checked.

2 - The second example considers the case where $c_1 = 1$, $c_2 = 0$, $c_3 = 1$. The
The eigenvalues of the associated matrix \( \hat{C} \) are \(-2, 0, 0, 2\). Setting \( c_0 = 0 \) the resulting \( (C) \) matrix coincides with \( \hat{C} \). It is non-definite, its rank is two and equal to its principal rank value. The generalized Carathéodory’s theorem applies and the solution is \( \epsilon_1 = 1, \epsilon_2 = -1, \rho_1 = 1/2 \) and \( \rho_2 = -1/2 \). With these values one easily checks that \( c_0 = \rho_1 + \rho_2 = 0, c_1 = \rho_1 \epsilon_1 + \rho_2 \epsilon_2 = 1, c_2 = \rho_1 \epsilon_1^2 + \rho_2 \epsilon_2^2 = 0 \) and \( c_3 = \rho_1 \epsilon_1^3 + \rho_2 \epsilon_2^3 = 1 \).

Setting \( c_0 = 2 \) the resulting \( (C) \) is non-negative defined with rank 3. The solution is: \( \epsilon_1 = 1, \epsilon_2 = i, \epsilon_3 = -i, \rho_1 = 1, \rho_2 = 1/2 \) and \( \rho_3 = 1/2 \). The last choice \( c_0 = -2 \) defines a non-positive defined \( (C) \) matrix with rank equal to 3 with solution: \( \epsilon_1 = -1, \epsilon_2 = i, \epsilon_3 = -i, \rho_1 = -1, \rho_2 = -1/2 \) and \( \rho_3 = -1/2 \).

The last example corresponds to have \( c_0 = \delta_0, c_1 = \delta_0 + i \delta_1 e^{i \varphi} \) and \( c_2 = (\delta_0 + 2 i \delta_1) e^{2i \varphi} \) with \( \delta_0, \delta_1 \) and \( \varphi \) reals. The eigenvalues are 0 and \((3 \delta_0/2)(1 \pm \sqrt{1 + 8 \delta_1^2/(3 \delta_0^2)})\). The Hermitian matrix \( (C) \) is indefinite and has rank 2 under the assumption that no further eigenvalue is equal to zero. The resolvent equation is \( P(z) = z^2 - 2e^{i \varphi} z + e^{2i \varphi} = 0 \) so that its coefficients \( a_0 = e^{2i \varphi}, a_1 = -2e^{i \varphi} \) and \( a_2 = 1 \) obey Eq. (37). From (39) one immediately finds that \( \sigma_0 = 2, \sigma_1 = 2e^{i \varphi} \) and \( \sigma_2 = 2e^{2i \varphi} \) and the resulting matrix \( (S) \) has rank 1. Hence the previous resolvent equation has a unimodular root with multiplicity 2: in fact, \( P(z) = (z - e^{i \varphi})^2 = 0 \). According to our analysis, \( (C) \) cannot be written as \( (V)(\Delta)(V)^\dagger \) with \( (\Delta) \) equal to a diagonal matrix. In this example, however, the roots are unimodular and \( (C) \) factorizes in the
Ellis-Lay form⁹ as

\[
\begin{pmatrix}
\delta_0, & (\delta_0 + i\delta_1)e^{i\varphi}, & (\delta_0 + 2i\delta_1)e^{2i\varphi} \\
(\delta_0 - i\delta_1)e^{-i\varphi}, & \delta_0, & (\delta_0 + i\delta_1)e^{i\varphi} \\
(\delta_0 - 2i\delta_1)e^{-2i\varphi}, & (\delta_0 - i\delta_1)e^{-i\varphi}, & \delta_0
\end{pmatrix} = \\
\begin{pmatrix}
1, & 0, & e^{-i\varphi}, & e^{-2i\varphi} \\
e^{-i\varphi}, & e^{-2i\varphi}, & -i\delta_1, & 0 \\
e^{-2i\varphi}, & 2e^{-2i\varphi}, & 0, & e^{2i\varphi}
\end{pmatrix}
\begin{pmatrix}
1, & e^{i\varphi}, & e^{2i\varphi} \\
e^{i\varphi}, & 0, & e^2, & 2e^{2i\varphi}
\end{pmatrix}.
\]

(63)
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This is also evident from (15): by construction the rows of \((V)\) span the range space of \((C')\) and \((C')\) has rank at most equal to \(m\).

For the case of Hermitian non-negative Toeplitz matrices considered by Grenander and G. Szegö, their proof shows that the principal rank of these matrices is equal to their rank. An alternative proof was given by Goedkoop\(^{16}\) introducing a finite dimensional Hilbert space (see also Ref. [2]). Thus, for this kind of matrices, one has the interesting property that the rank is obtained by considering the strictly principal minors of increasing order till finding a singular minor. If the latter’s order is \(m + 1\) the rank of the matrix is \(m\).

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