Accounting for the time evolution of the equation of state parameter during reheating

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One of the important parameters in cosmology is the parameter characterizing the equation of state (EoS) of the sources driving the cosmic expansion. Epochs that are dominated by radiation, matter or scalar fields, whether they are probed either directly or indirectly, can be characterised by a unique value of this parameter. However, the EoS parameter during reheating—a phase succeeding inflation which is supposed to rapidly defrost our universe—remains to be understood satisfactorily. In order to circumvent the complexity of defining an instantaneous EoS parameter during reheating, an effective parameter \( w_{\text{eff}} \), which is an average of the EoS parameter over the duration of reheating, is usually considered. The value of \( w_{\text{eff}} \) is often chosen arbitrarily to lie in the range \(-1/3 \leq w_{\text{eff}} \leq 1\). In this work, we consider the time evolution of the EoS parameter during reheating and relate it to inflationary potentials \( V(\phi) \) that behave as \( \phi^p \) around the minimum, a proposal which can be applied to a wide class of inflationary models. We find that, given the index \( p \), the effective EoS parameter \( w_{\text{eff}} \) is determined uniquely. We discuss the corresponding effects on the reheating temperature and its implications.

I. INTRODUCTION

In order to explain the current observable universe, the conventional hot big bang model requires very fine tuned initial conditions during the radiation dominated epoch. This difficulty can be overcome if we assume that the universe went through a brief phase of nearly exponential expansion—an epoch dubbed as inflation—in its very early stages [1,3]. Apart from explaining the observed extent of isotropy of the Cosmic Microwave Background (CMB) [4,7], inflation also provides a natural mechanism to generate the small anisotropies superimposed on the nearly isotropic background [8,11]. It is these CMB anisotropies which act as the seeds for the eventual formation of the large scale structure in the universe [12].

But, due to the accelerated expansion, inflation makes the universe cold and dilute. To be consistent with big bang nucleosynthesis (BBN), the universe must consist of radiation and matter in thermal equilibrium, when its temperature is around 10 MeV [13,14]. Inflation is typically driven with the aid of scalar fields, often referred to as the inflaton. At the termination of inflation, the energy from the inflaton is supposed to be transferred to the particles constituting the standard model through a process called reheating [10,19]. During this phase of reheating, the inflaton is expected to rapidly decay producing matter and radiation in equilibrium, thereby setting the stage for the conventional hot big bang evolution.

The original mechanism for reheating, suggested soon after the idea of inflation was proposed, was based on the perturbative decay of the inflaton [16,19]. However, about a decade later, it was realized that the perturbative mechanism does not capture the complete picture, as the decay of the inflaton was found to be dominated by non-perturbative processes. Importantly, it was recognized that, immediately after the termination of inflation, the inflaton acts like a coherently oscillating condensate which leads to parametric resonance of the fields coupled to the inflaton [20,24]. In fact, the initial stage of reheating is referred to as preheating, to distinguish it from the later stage of perturbative decay.

The details of the perturbative as well as the non-perturbative processes taking place during reheating can be non-trivial and will actually depend upon the various fields that are taken into account and the nature of their interactions. Moreover, the lack of direct observables that can reveal the dynamics during this phase poses additional challenges towards understanding the mechanism of reheating. In such a situation, as a first step, it would be convenient to characterise the phase through an equation of state (EoS) parameter \( w \) which captures the background evolution and, consequently, the dilution of the energy density of the fields involved, without going into the complexity of models and interactions.

After all, the different epochs of the universe—viz. inflation, radiation, and matter domination as well as late time acceleration—are often simply characterized in terms of the corresponding EoS parameter (in this context, see Fig. 1). One widely adopted approach is to define an effective EoS parameter \( w_{\text{eff}} \), which is an average of the instantaneous EoS parameter during the period of reheating [25]. Although, the averaging washes out the details of the microphysics over the intermediate stages, it allows us to conveniently characterize the reheating phase in terms of two other vital observables, viz. the duration of the phase and the reheating temperature. While such an approach may be adequate as a first step, needless to add, it is important to characterize and understand the dynamics of reheating in further detail.

As we mentioned, at the end of inflation, the inflaton starts to oscillate about the minimum of the potential. During the initial stages of this phase, most of the energy is stored in the coherently oscillating scalar field. It can be shown that the EoS parameter of a ho-
mogeneous condensate oscillating in a potential which has a minimum of the form $V(\phi) \propto \phi^p$ is given by $w_{\text{co}} = (p-2)/(p+2)$ \cite{20, 27}. However, in the process, the homogeneous condensate fragments leading to the growth of the inhomogeneities \cite{28–30}. As a result, the EoS parameter differs from the above-mentioned form. The time when the EoS starts to change from its form during the period of coherent oscillations is referred to in the literature as the onset of the phase of backreaction \cite{31, 32}. The effects of fragmentation on the EoS can be studied using lattice simulations and one finds that the EoS parameter indeed eventually approaches that of the radiation dominated phase (i.e. $w \rightarrow 1/3$), as required \cite{32, 34}. Evidently, the average $w_{\text{eff}}$ during reheating will depend on the time evolution of the EoS parameter from the end of coherent oscillations to the start of the radiation domination epoch. Usually, the value of $w_{\text{eff}}$ during this phase is either identified to be the value $w_{\text{co}}$ during the coherent oscillation phase or chosen arbitrarily to lie in the range $-1/3 \leq w_{\text{eff}} \leq 1$ \cite{35}.

In this work, we examine the time evolution of the EoS parameter and its average $w_{\text{eff}}$ during reheating. We consider the time evolution of the EoS from the end of the coherent oscillation stage until the onset of the radiation domination epoch. We argue that the presence of gradient and/or interaction energy of the inflaton leads to the deviation of the EoS parameter from its value $w_{\text{co}}$ during the period of coherent oscillations. Not surprisingly, we find that, even after the phase of coherent oscillations, the shape the inflationary potential near its minimum plays a role in the time evolution of the EoS. We shall assume that, near their minima, the inflationary models of our interest have the following form: $V(\phi) \propto \phi^p$. We should point out here that large field models which are completely described by such power law potentials are already ruled out due to the constraints from the CMB data on the primary inflationary observables, viz. the scalar spectral index $n_s$ and tensor-to-scalar ratio $r$ \cite{36, 38}. In contrast, potentials that contain a plateau, such as the original Starobinsky model, which lead to smaller values of $r$ are favored by the CMB data. However, such potentials too can be expressed as $V(\phi) \propto \phi^p$ around the minima (in this context, see Fig. 2 wherein we have schematically illustrated the potentials for $p = 2$). To capture the microphysics during reheating and, specifically, the turbulent backreaction phase, we propose an evolving EoS parameter which asymptotically approaches its value during the radiation dominated epoch from its value at the end of the phase of coherent oscillations. With such a time evolving EoS parameter, we establish a link between the value of $w_{\text{eff}}$ and the inflationary potential parameters. This allows us to connect the reheating temperature uniquely to the inflationary parameters, while, importantly, accounting for the time evolution of the EoS.

The remainder of the paper is structured as follows. In Sec. \textsection{II} we shall provide a rapid overview of reheating and connect the parameters describing the phase with the observables in the CMB. In Sec. \textsection{III} we shall first briefly highlight the motivations for accounting for the time-dependence of the EoS during reheating. We shall then go on to consider two types of parametrizations for the EoS parameter and arrive at the associated effective EoS parameter. In Sec. \textsection{IV} we shall apply these arguments to the so-called $\alpha$-attractor model of inflation and evaluate the corresponding reheating temperatures for this model. We shall conclude with a brief summary of our results in Sec. \textsection{V}.
II. CONNECTING THE REHEATING PHASE WITH THE CMB OBSERVABLES

While the period of reheating is phenomenologically rich, as we mentioned, it is difficult to observationally constrain the dynamics due to the paucity of direct access to that epoch. Another difficulty arises due to the fact that by the time of BBN, all the particles associated with the standard model are expected to have been thermalized, thereby possibly hiding away the details of their production. Despite these limitations, one finds that reheating can still be constrained to a certain extent from the CMB and BBN observables. The upper bound on the inflationary energy scale, inferred from the constraints on the tensor-to-scalar ratio $r$ (arrived at originally from the WMAP data [35] and improved upon later by the Planck data [37, 38]), is closer to the GUT scale of about $10^{16}$ GeV, whereas BBN requires a radiation dominated universe at around 10 MeV [13,15]. The inflationary observables are either well measured or have bounds on them, while the physics of BBN have been tested with great precision. Thus, there is a huge window in energy scales of several order of magnitudes which remains unconstrained by the cosmological data.

However, as has been pointed out in the literature, a connection can be made between the reheating phase and the CMB observables measured today [35,39,40]. As it proves to be essential for our discussion later on, we shall quickly summarize the primary arguments in this section. Recall that, during inflation, a scale of interest described by the comoving wavenumber $k$ leaves the Hubble radius at the time when $k = a_k H_k$. Let this time correspond to, say, $N_k$, e-folds before the end of inflation. For instance, the Planck team choose their pivot scale to be $k = 0.05$ Mpc$^{-1}$ and assume $N_k \approx 50$ for this scale [37,38]. The physical wavenumber $k/a_k$ at the time when it exits the Hubble radius during inflation can be related to its corresponding value $k/a_0$ at the present time as follows:

$$
\frac{k}{a_k H_k} = \frac{k}{a_0 H_{k0}} = \frac{a_0}{a_0} \frac{a_{re}}{a_{re}} \frac{a_{co}}{a_{co}} \frac{a_{end}}{a_{end}} \frac{k}{a_k} \frac{H_k}{H_0}, \quad (1)
$$

where $a_{end}$, $a_{co}$ and $a_{re}$ are the values of the scale factor when inflation, the phase of coherent oscillations and reheating end, respectively. Since $e^{N_k} = a_{end}/a_k$, $e^{N_{co}} = a_{co}/a_{end}$ and $e^{N_{re}} = a_{re}/a_{co}$, we can express the above equation as

$$
N_k + N_{co} + N_{re} + \ln \left( \frac{a_0}{a_{re}} \right) + \ln \left( \frac{k}{a_0 H_k} \right) = 0, \quad (2)
$$

where, evidently, $N_{co}$ and $N_{re}$ denote the durations of the phase of coherent oscillations and the backreaction phase.

At the end of reheating, the universe is supposed to be radiation dominated and if no significant entropy is released into the primordial plasma, we can relate the reheating temperature, say, $T_{re}$, with the present CMB temperature, say, $T_0$, as follows (see, for example, Ref. [35]):

$$
\frac{T_{re}}{T_0} = \left( \frac{43}{11 g_{s, re}} \right)^{1/3} \frac{a_0}{a_{re}}, \quad (3)
$$

where $g_{s, re}$ denotes the effective number of relativistic degrees of freedom that contribute to the entropy during reheating. We should mention that, to arrive at the above expression, we have expressed the neutrino temperature in terms of the temperature $T_0$ of the CMB using the relation $T_{re} = (4/11)^{1/3} T_0$. On using Eqs. (2) and (3), we can express the reheating temperature as

$$
T_{re} = \left( \frac{43}{11 g_{s, re}} \right)^{1/3} \left( \frac{a_0 T_0}{k} \right) H_k e^{-N_k} e^{-N_{co}} e^{-N_{re}}. \quad (4)
$$

Let us now assume that the backreaction phase succeeding the period of coherent oscillations is described by the time-dependent EoS parameter $w(N)$. In such a case, from the conservation of energy, the cosmic energy density during the phase can be expressed as

$$
\rho(N) = \rho_{co} \exp \left\{ -3 \int_0^N dN' \left[ 1 + w(N') \right] \right\}, \quad (5)
$$

where $\rho_{co}$ is the energy density at the end of the coherent oscillation phase. On defining an averaged EoS parameter as

$$
w_{eff} = \frac{1}{N_{re}} \int_0^{N_{re}} dN \ w(N), \quad (6)
$$

we can rewrite the above expression as

$$
\ln \left( \frac{\rho_{co}}{\rho_{re}} \right) = 3 \left( 1 + w_{eff} \right) N_{re}, \quad (7)
$$

where $N_{re}$ denotes the number of e-folds during the backreaction phase counted from the end of the period of coherent oscillations.

If we now assume that, at the end of reheating, the dominant component of energy is radiation, then we can express the energy density of radiation in terms of $T_{re}$ as

$$
\rho_{re} \equiv \rho_r(T_{re}) = \frac{\pi^2}{30} \frac{g_{re}}{T_{re}^4} \quad (8)
$$

where $g_{re}$ is the number of effective relativistic degrees of freedom at the end of reheating. In such a case, upon using Eqs. (7) and (8), we can readily express $T_{re}$ as

$$
T_{re} = \left( \frac{30 \rho_{co}}{g_{re} \pi^2} \right)^{1/4} \exp \left[ -\frac{3}{4} (1 + w_{eff}) N_{re} \right]. \quad (9)
$$

From Eqs. (4) and (9), we can then arrive at the following
expression for the duration $N_{\text{re}}$ of the phase of reheating:

$$N_{\text{re}} = \frac{4}{3} \frac{1}{w_{\text{eff}} - 1} \left[ N_k + N_{\text{co}} + \ln \left( \frac{k}{a_0 T_0} \right) \right] + \frac{1}{4} \ln \left( \frac{30}{\pi^2 g_{\text{re}}} \right) + \frac{1}{2} \ln \left( \frac{11 g_{s,\text{re}}}{43} \right)$$

$$- \ln \left( \frac{H_k}{\rho_{\text{end}}^{1/4}} \right) - \frac{1}{4} \ln \left( \frac{\rho_{\text{end}}}{\rho_{\text{co}}} \right),$$

where $\rho_{\text{end}}$ is the energy density of the inflaton at the end of inflation. Since the EoS parameter during the phase of coherent oscillations is $w_{\text{co}} = (p - 2)/(p + 2)$, which is obviously a constant for given value of $p$, we can express $\rho_{\text{co}}$ in terms of $\rho_{\text{end}}$ as

$$\ln \left( \frac{\rho_{\text{end}}}{\rho_{\text{co}}} \right) = 3 \left( 1 + \frac{p - 2}{p + 2} \right) N_{\text{co}}.$$  

Therefore, the duration of reheating $N_{\text{re}}$ can finally be expressed as

$$N_{\text{re}} = \frac{4}{3} \frac{1}{w_{\text{eff}} - 1} \left[ N_k + N_{\text{co}} + \ln \left( \frac{k}{a_0 T_0} \right) \right] + \frac{1}{4} \ln \left( \frac{30}{\pi^2 g_{\text{re}}} \right) + \frac{1}{2} \ln \left( \frac{11 g_{s,\text{re}}}{43} \right) - \ln \left( \frac{H_k}{\rho_{\text{end}}^{1/4}} \right).$$  

During inflation, the energy density of the inflaton can be expressed in terms of the potential $V(\phi)$ and the first slow roll parameter $\epsilon = -\dot{H}/H^2$ as

$$\rho = V \left( 1 + \frac{\epsilon}{3 - \epsilon} \right).$$

Since inflation ends when $\epsilon = 1$, we have $\rho_{\text{end}} = (3/2) V_{\text{end}}$, where $V_{\text{end}}$ denotes the potential at $\phi_{\text{end}}$, viz. the value of the scalar field at which inflation is terminated. Given the potential $V(\phi)$, the value of $\phi_{\text{end}}$ can be readily determined using the condition $\epsilon \approx (M^2_{\text{pl}}/2)(V_0/V)^2 = 1$, where the subscript on the potential denote derivatives with respect to the scalar field. Also, working in the slow roll approximation, we can calculate the value of the scalar field at $N_k$. This, in turn, can be utilized to express $N_k$ in terms of the inflationary observables, viz. the scalar spectral index $n_s$ and tensor-to-scalar ratio $r$. Therefore, the bounds on the inflationary parameters from the CMB will lead to the corresponding constraints on the reheating parameters as well (in this context, see Refs. [33-42]). However, note that the quantity $N_{\text{co}}$ depends on the details of the inflationary model under investigation and, importantly, on the coupling of the inflaton to the other fields.

### III. TIME-DEPENDENT EOS

As discussed earlier, the EoS parameter for the homogeneous condensate, oscillating about the minimum of a potential behaving as $V(\phi) \propto \phi^p$, is given by $w_{\text{co}} = (p - 2)/(p + 2)$ [26-27]. However, due to the growth of inhomogeneities, the EoS parameter can be expected to differ from the above value during the backreaction phase. We can study the resulting variation in the EoS parameter by considering virialization of the inhomogeneous system in equilibrium.

Consider a situation wherein the inflaton $\phi$ decays into daughter fields collectively represented as $F$ through the interaction potential $V_1(\phi, F)$. For a potential which behaves as $V(\phi) \propto \phi^p$ near its minimum, one can show that, in equilibrium, the following virial relations between the kinetic, potential and the interaction energy densities hold (in this context, see, for example, Refs. [32-34]):

$$\frac{1}{2} \langle \dot{\phi}^2 \rangle = \frac{1}{2} \left( \frac{\left| \nabla \phi \right|^2}{a^2} \right) + \frac{p}{2} \left( V(\phi) + V_1(\phi, F) \right),$$

$$\frac{1}{2} \langle \dot{F}^2 \rangle = \frac{1}{2} \left( \frac{\left| \nabla F \right|^2}{a^2} \right) + \langle V_1(\phi, F) \rangle,$$

where the angular brackets indicate that the quantities have been averaged over space as well as the period of oscillation of the inflaton. During this backreaction phase, one can define the instantaneous EoS averaged over the spatial volume as:

$$w = \frac{1}{2} \frac{\dot{\phi}^2 + \frac{1}{2} \dot{F}^2 - \frac{1}{2a^2} \left| \nabla \phi \right|^2 - \frac{1}{2a^2} \left| \nabla F \right|^2 - V_1(\phi, F)}{\rho_{\phi} + \frac{1}{2} \left| \nabla \phi \right|^2 + \frac{1}{2a^2} \left| \nabla F \right|^2 + V_1(\phi, F)}.$$

(15)

Upon using the virial relations [14], we find that the above expression for $w$ reduces to

$$w = 1 + \frac{(p - 4)}{6} \left( \frac{p + 2}{4} + \frac{\langle \rho_{\phi} \rangle}{\langle V(\phi) \rangle} + \frac{3 \langle V_1(\phi, F) \rangle}{2 \langle V(\phi) \rangle} \right)^{-1},$$

where $\langle \rho_{\phi} \rangle = \left( \langle |\nabla \phi|^2/(2a^2) \rangle + \langle |\nabla F|^2/(2a^2) \rangle \right)$ is the total energy density associated with the spatial gradients in the fields.

It should be clear from the above equation for $w$ that, as the gradient and the interaction energies begin to dominate, the second term in the expression becomes insignificant and the EoS parameter approaches $1/3$. To explicitly demonstrate these effects of the gradient and interaction energy densities on the EoS parameter, in Fig. [3], we have plotted the contours of fixed $w$ from Eq. [16] for potentials $V(\phi)$ which behave as $\phi^p$ and $\phi^2$ around their minima. There are two points that should be evident from the figure. Firstly, even a slight increase in gradient or interaction energy densities results in a non-zero instantaneous EoS parameter. Secondly, as we pointed out above, $w$ asymptotically approaches $1/3$, as both the gradient and interaction energy densities increase. In fact, these expectations are also corroborated by lattice simulations which allows one to track the EoS from the end of the phase of coherent oscillations to the beginning of the radiation dominated epoch (in this context, see, for instance, Refs. [32-34]). These simulations suggest that,
for $p < 4$, the EoS parameter monotonically increases towards the asymptotic value of $1/3$. Similarly, for $p > 4$, one finds that it decreases monotonically towards $1/3$. From these arguments, we conclude that, in any realistic scenario, the EoS during reheating must be different than its value during the phase of coherent oscillations and that a vanishing EoS parameter is highly unlikely during this stage.

A. Parametrizing the EoS

In order to incorporate the continuous variation of the EoS from the end of coherent oscillations to radiation domination, we parametrize the instantaneous EoS parameter by hand in terms of $e$-folds. In choosing the functional form of the EoS parameter, we assume that it varies monotonically from its initial value $w_{\text{co}}$ to the final value of $1/3$. We find that this condition considerably restricts the form of the functions we can consider.

We consider two different parametrizations of the following forms:

- **Case A: Exponential form**
  \[
  w(N) = w_0 + w_1 \exp\left( -\frac{1}{\Delta} \frac{N}{N_{\text{re}}} \right),
  \]  
  \[
  (17)
  
- **Case B: tan-hyperbolic form**
  \[
  w(N) = w_0 + w_1 \tanh\left( \frac{1}{\Delta} \frac{N}{N_{\text{re}}} \right),
  \]  
  \[
  (18)
  
where $N$ is the number of $e$-folds **counted from the end of the phase of coherent oscillations**. The parameters $w_0$ and $w_1$ are fixed from the values of $w$ at the end of the coherent oscillations and the asymptotic limit which we take to be that of the radiation dominated epoch. Evidently, the parameter $\Delta$ controls the efficiency of the reheating process and determines how quickly the radiation dominated phase is attained. We further assume that the EoS parameter just at the end of reheating, say, $w_{\text{re}}$, is within 10% of the asymptotic value of $1/3$. There are two reasons for this assumption. The first is the reason that one has to account for various physical effects that can result in the deviation of the EoS parameter from $1/3$ during the initial stages of radiation domination (see Ref. [49]; in this context, also see Ref. [44, Sec. 2.11]). The second is the practical reason to set a benchmark where the energy density of radiation has formally begun to dominate the rest of the energy densities. We find that this choice of $w_{\text{re}}$ fixes the value of $\Delta$. Under these conditions, the two parametrizations take the following form:

\[
\begin{align*}
  w(N, p) &= \left\{ \begin{array}{ll}
  1 & + \frac{2}{3} \left( \frac{p-4}{p+2} \right) \exp\left( -\frac{1}{\Delta} \frac{N}{N_{\text{re}}} \right), \\
  \frac{p-2}{p+2} - \frac{2}{3} \left( \frac{p-4}{p+2} \right) \tanh\left( \frac{1}{\Delta} \frac{N}{N_{\text{re}}} \right),
  \end{array} \right. \\
  (19)
\end{align*}
\]

with

\[
\begin{align*}
  \frac{1}{\Delta} &= \left\{ \begin{array}{ll}
  \ln \left[ \left( \frac{p-4}{p+2} \right) \left( \frac{2}{3 w_{\text{re}} - 1} \right) \right], \\
  \tanh^{-1} \left[ \frac{3}{2} \left( \frac{p-2-w_{\text{re}}(p+2)}{p-4} \right) \right].
  \end{array} \right. \\
  (20)
\]
We had already pointed out that, from its initial value of \( w_0 = (p - 2)/(p + 2) \), the EoS parameter \( w \) increases or decreases monotonically towards 1/3 for \( p < 4 \) and \( p > 4 \), respectively. For \( p = 4 \), the reheating phase is indistinguishable from the radiation dominated epoch since \( w_{co} = 1/3 \). Hence, in such a case, \( \Delta \to 0 \). In Fig. 4 we have compared the two parameterizations described by Eq. (19) for different values of \( p \). As a benchmark, we take the end of reheating (denoted by the red vertical line) to be when the EoS parameter reaches within 10% of its asymptotic value of 1/3.

Thus, for a given inflationary potential, \( w_{\text{eff}} \) is fixed. In Tab. I, we compare the values of \( w_{\text{eff}} \) for the two parameterizations with the value of \( w_{co} \) for a set of values of \( p \). On substituting the expression (21) for \( w_{\text{eff}} \) in the expression (10) for \( N_{re} \), we obtain that

\[
N_{re} = \begin{cases} 
\frac{2}{p+2} N \left[ 1 - \left( \frac{p-4}{p+2} \right) \left[ e^{-1/\Delta} - 1 \right] \right], & \text{(A)} \\
\frac{2}{p+2} N \left[ 1 - \left( \frac{p-4}{p+2} \right) \left[ e^{-1/\Delta} \log \left[ \cosh \left( 1/\Delta \right) \right] \right] \right]^{-1}, & \text{(B)} 
\end{cases}
\]

where the quantity \( N \) is defined as

\[
N = N_k + \frac{4 - p}{2(p + 2)} N_{co} + \frac{k}{a_0 T_0} + \frac{1}{4} \ln \left( \frac{30}{\pi^2 g_{re}} \right) + \frac{1}{3} \ln \left( \frac{11 g_{s, re}}{43} \right) - \ln \left( \frac{H_k}{\rho_{\text{end}}} \right),
\]

while, recall that, \( \Delta \) is given by Eq. (20). It should be clear from the above equation that, barring \( g_{s, re} \) and \( N_{co} \), the duration of reheating \( N_{re} \) depends only on the inflationary parameters and the CMB observables. On substituting the above expressions for \( w_{\text{eff}} \) and \( N_{re} \), in Eq. (19), we can arrive at the corresponding reheating temperature \( T_{re} \).

### IV. APPLICATION TO AN INFLATIONARY MODEL

Let us now apply our arguments to an inflationary model which has the desired behaviour near its minima. Towards this end, we shall consider the so-called \( \alpha \)-attractor model described by potential \( \alpha \)

\[
V(\phi) = \Lambda^4 \left[ 1 - \exp \left( -\sqrt{\frac{2}{3 \alpha M_{Pl}^2}} \phi \right) \right]^p,
\]

where \( \Lambda, \alpha \) and \( p \) are, evidently, parameters that characterize the model. As we had pointed out, we can express the first slow roll parameter as \( \epsilon \simeq (M_{Pl}^2/2)(V_{\phi}/V)^2 \), where the subscript \( \phi \) denotes the derivative of the potential with respect to the field. Let us define the second slow roll parameter as \( \eta \equiv M_{Pl}^2 (V_{\phi\phi}/V) \). Then, in the slow roll approximation, the inflationary observables—viz. the scalar spectral index \( n_s \) and the tensor-to-scalar ratio \( r \)—can be expressed in terms of these parameters as (see, for instance, the reviews \( \alpha \))

\[
n_s = 1 - 6 \epsilon_k + 2 \eta_k, \quad \eta_k = 16 \epsilon_k, \label{eqns}
\]

where the subscript \( k \) indicates that these quantities have to be evaluated when the mode leaves the Hubble radius. Moreover, the scalar amplitude \( A_s \) can be expressed in terms of the value of the Hubble parameter \( H_k \) and the tensor-to-scalar ratio \( r \) as follows:

\[
A_s = \frac{H_k}{\sqrt{r A_t}} M_{Pl}, \label{eqns}
\]
The number of e-folds $N_k$ between the mode $k$ leaving the Hubble radius and the end of inflation can be expressed in the slow roll approximation as

$$N_k = \frac{1}{M_{pl}^2} \int_{\phi_k}^{\phi_{end}} d\phi \frac{H}{\dot{\phi}} \simeq \frac{1}{M_{pl}^2} \int_{\phi_k}^{\phi_{end}} d\phi \frac{V}{\dot{\phi}}.$$  \hspace{1cm} (27)

For the inflationary potential (24) of our interest, $N_k$ can be evaluated to be

$$N_k = \frac{3\alpha}{2p} \left[ \exp \left( \sqrt{\frac{2}{3\alpha}} \frac{\phi_k}{M_{pl}} \right) - \exp \left( \sqrt{\frac{2}{3\alpha}} \frac{\phi_{end}}{M_{pl}} \right) \right] - \sqrt{\frac{2}{3\alpha}} \left( \frac{\phi_k - \phi_{end}}{M_{pl}} \right).$$  \hspace{1cm} (28)

The quantity $\phi_{end}$ can be determined by the condition $\epsilon = 1$ and is given by

$$\phi_{end} = \frac{3\alpha}{2} \ln \left( 1 + \frac{p}{\sqrt{3\alpha}} \right) \sqrt{\frac{p}{p + 3\alpha}}.$$  \hspace{1cm} (29)

so that we have

$$V_{end} = V(\phi_{end}) = \Lambda^4 \left( \frac{p}{p + 3\alpha} \right).$$  \hspace{1cm} (30)

The above relations between $\phi_k$, $\phi_{end}$ and $N_k$ and the expression (25a) for the scalar spectral index allows us to write $n_s$ in terms of $N_k$. We can then invert the relation to express $N_k$ in terms of $n_s$.

Note that, near its minimum, the inflationary potential (24) can be approximated as

$$V(\phi) \simeq \Lambda^4 \left( \frac{2\phi}{3\alpha M_{pl}} \right)^p,$$  \hspace{1cm} (31)

which is of the form we desire. We should mention here that, for $\alpha = 1$ and $p = 2$, the potential (24) corresponds to the Starobinsky model, which is the most favored model according to the recent CMB observations [37, 38].

Recall that, for a given $p$, $w_{\text{eff}}$ is fixed [cf. Eq. (21)]. Hence, we have most of the required ingredients to calculate the duration of reheating $T_{re}$ and the corresponding reheating temperature $T_{re}$ using the expressions (22) and (6). However, we shall require values for $g_{\text{re}}$, $g_{s,\text{re}}$ and $N_{co}$. It seems reasonable to choose $g_{\text{re}} = g_{s,\text{re}} = 10^2$ [27].

Let us now turn to identifying a suitable choice for $N_{co}$. The duration of the phase of coherent oscillation can strongly depend on the model parameters and, importantly, on the couplings of the inflaton to other fields [31]. In particular, if non-perturbative processes dominate throughout the reheating phase, thermalization may be achieved within a few e-folds making it difficult to connect the reheating phase with the CMB observables. However, this phase can be inefficient or delayed [50] and can result in the breakdown of coherent oscillations without thermalization [51]. Due to these reasons, we consider $N_{co}$ to be small and set it to unity.

With all these necessary ingredients at hand, let us now compute the reheating temperature $T_{re}$ for the model of our interest. Note that, $T_{re}$ depends on $n_s$, $p$, $\alpha$, and $w_{\text{eff}}$. We shall set $\alpha = 1$ without any loss of generality. Since $w_{\text{eff}}$ is largely independent of the two parametrizations [cf. Tab. (1)], we choose to work with the values corresponding to the exponential form for $w(N)$. In Fig. 3 we have highlighted the dependence of $T_{re}$ on $n_s$ and $p$ in two different manner. We have first plotted $T_{re}$ as a function of $p$ for the value of $n_s$ that leads to the best-fit to the recent CMB data [37, 38]. In the figure, we have also illustrated the simultaneous dependence of $T_{re}$ on $n_s$ and $p$. Note that the lower bound on the reheating temperature comes from the BBN constraints as $T_{BBN} \sim 10$ MeV [13–15], whereas the upper limit comes from the condition of instantaneous reheating which corresponds to the inflationary energy scale of the order of the GUT scale of about $10^{16}$ GeV that arises in certain supersymmetric theories.

Let us emphasize a few more points concerning Fig. 5. It is clear that the new effective EoS parameter we have arrived at lowers the reheating temperature. This effect can be attributed to the dependence of $T_{re}$ on the ratio $(1 + w_{\text{eff}})/(3 w_{\text{eff}} - 1)$, which is always higher than the one computed with $w_{\text{eff}} = w_{\text{co}} = (p - 2)/(p + 2)$ for a given value of $p$. Thus, our proposal for the time-dependent EoS and its effect can, in principle, be tested in future experiments [53, 59]. Moreover, note that, the variation of $T_{re}$ with $p$ also depends on the value of scalar spectral index. It is evident from Fig. 5 that, for $p < 4$, an increase in the value of $n_s$ results in a larger value of $T_{re}$. This is due to the fact that for $p < 4$, $w_{\text{eff}} < 1/3$ and, hence, an increase in the value of $n_s$ leads to a smaller value of $N_{co}$ which, in turn, leads to a larger value of $T_{re}$. However, for $p > 4$, the conditions are reversed and we have a decreasing $T_{re}$ for an increasing $n_s$.

V. DISCUSSION

In this work, we have computed the effective EoS parameter during the reheating phase of the universe by taking into account the time evolution of the instantaneous EoS parameter. We have shown that the gradient and interaction energy densities force the instantaneous EoS parameter to deviate from its value during the phase of coherent oscillations which succeeds inflation. Assuming that the inflationary potential behaves as $V(\phi) \propto \phi^p$ about its minimum, we have argued that, during reheating, $w$ increases monotonically and approaches 1/3 for $p < 4$, whereas, for $p > 4$, it decreases monotonically to 1/3 (cf. Fig. 4). In order to capture such a behaviour, we have proposed two different functional forms of the time varying EoS parameter during reheating. We find that the resulting value of $w_{\text{eff}}$ depends only on the inflationary model parameter $p$ and is largely independent of the parametrization we have considered for $w(N)$ (cf. Tab. (1)).

Let us stress here a few further points concerning the results we have obtained. Note that, in our approach,
FIG. 5. The dependence of the reheating temperature $T_{re}$ on the index $p$ has been illustrated (on the left) for $n_s = 0.9649$ which leads to the best-fit to the CMB data [37, 38]. We have plotted the dependence of $T_{re}$ on $p$ for $w_{eff}$ corresponding to the exponential parametrization [cf. Eqs. (19) and (21)] (as the solid blue curve) as well as for the choice $w_{eff} = w_{co} = (p-2)/(p+2)$ (as the dashed orange curve). We have also indicated the following domains (in the figure on the left): the region above maximum possible reheating temperature of $T_{inst} = \left[30\rho_{end}/(9\pi^2)\right]^{1/3}$ corresponding instantaneous reheating or $N_{re} = 0$ (in red), the domains below the electroweak scale taken to be $T_{EW} \sim 100$ GeV (in lighter blue) and the region below 10 MeV which is the minimum temperature required for BBN (in darker blue). We have also illustrated the dependence of $T_{re}$ on $n_s$ and $p$ (on the right) for the choice of $w_{eff}$ corresponding to the exponential parametrization. Note that we have set $\alpha = 1$ in both these plots.

$w_{eff}$ is completely determined by the inflationary parameter $p$. Therefore, for a specific $p$, the reheating temperature $T_{re}$ is fixed for a given value of the scalar spectral index $n_s$. This should be contrasted with earlier studies, wherein there is an arbitrariness in choosing the value of $w_{eff}$. As we discussed earlier, often $w_{eff}$ is either assumed to lie in the range $-1/3 \leq w_{eff} \leq 1$ or simply taken to be same as that of $w_{co}$. However, various preheating studies have indicated towards time varying EoS, which has been captured efficiently with our parametrization. With such a time varying EoS parameter, we can uniquely define $w_{eff}$ which, as we highlighted, is fixed by the behaviour of the field around the minimum of the potential. It is worth stressing again that the $w_{eff}$ we have arrived at is largely independent of parametrization. Thus, this study mitigates the arbitrariness in defining the effective EoS parameter during reheating for a given inflationary model.

Lastly, note that, though we have worked with the $\alpha$-attractor model of inflation specified by the potential \cite{24}, our analysis applies to all the inflationary models which behave as $\phi^p$ around their minima. With ongoing and forthcoming CMB missions expected to constrain the inflationary parameters more accurately, we believe that our proposal for the time-dependent EoS during reheating can be well tested in the near future.

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