Long-Range Rapidity Correlations
in the Model with Independent Emitters

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Long Range Rapidity Correlations

two rapidity intervals separated by a gap

- the event multiplicity in the BACKWARD or FORWARD rapidity windows.

\[ \langle n_B \rangle_{n_F} \equiv f(n_F) \] – the correlation function (regression)
The **linear** correlation function (linear regression):

\[
\langle n_B \rangle_{n_F} = a^{abs} + b^{abs} n_F
\]

\[
\frac{\langle n_B \rangle_{n_F}}{\langle n_B \rangle} = a^{rel} + b^{rel} \frac{n_F - \langle n_F \rangle}{\langle n_F \rangle} = a^{rel} + b^{rel} \left( \frac{n_F}{\langle n_F \rangle} - 1 \right)
\]

\[
b^{rel} = \frac{\langle n_F \rangle}{\langle n_B \rangle} b^{abs}, \quad a^{rel} = \frac{\langle n_B \rangle_{n_F=\langle n_F \rangle}}{\langle n_B \rangle}
\]
For a nonlinear correlation function \( \langle n_B \rangle_{n_F} = f(n_F) \) (nonlinear regression), expanding in powers of \( n_F - \langle n_F \rangle \) we have

\[
\langle n_B \rangle_{n_F} \equiv f(n_F) = f_0 + f_1[n_F - \langle n_F \rangle] + f_2[n_F - \langle n_F \rangle]^2 + f_3[n_F - \langle n_F \rangle]^3 + \ldots
\]

\[
b_{\text{abs}} \equiv \left. \frac{d \langle n_B \rangle_{n_F}}{dn_F} \right|_{n_F = \langle n_F \rangle} = f_1 , \quad b_{\text{rel}} = \left. \frac{\langle n_B \rangle_{n_F}/\langle n_B \rangle}{dn_F/\langle n_F \rangle} \right|_{n_F = \langle n_F \rangle} = \frac{\langle n_F \rangle}{\langle n_B \rangle} b_{\text{abs}}
\]

\[
\langle n_B \rangle_{n_F = \langle n_F \rangle} = f(\langle n_F \rangle) = f_0 , \quad a_{\text{rel}} = \frac{\langle n_B \rangle_{n_F = \langle n_F \rangle}}{\langle n_B \rangle} = \frac{\langle n_F \rangle}{\langle n_B \rangle}
\]
To exclude the trivial dependence on the lengths of the forward $\Delta y_F$ and backward $\Delta y_B$ rapidity windows we define the correlation coefficient $b^{rel}$ using the scaled variables:

**Definition 1:**

$$b^{rel} \equiv \frac{d\langle n_B \rangle_{n_F}}{dn_F/\langle n_F \rangle} \bigg|_{n_F = \langle n_F \rangle} = \frac{\langle n_F \rangle d\langle n_B \rangle_{n_F}}{dn_F/\langle n_F \rangle} \bigg|_{n_F = \langle n_F \rangle} = \frac{\langle n_F \rangle}{\langle n_B \rangle} b^{abs}$$

where $\langle n_F \rangle$ and $\langle n_B \rangle$ are the mean multiplicities in the forward and backward rapidity windows. The $\langle n_B \rangle_{n_F}$ is the correlation function (regression) - the mean multiplicity in the backward window $\Delta y_B$ as a function of the multiplicity in the forward window $\Delta y_F$. 
In the framework of the model with independent emitters in paper [1] using methods developed in [2] under some very specific assumptions the following formula for the defined correlation coefficient $b^{rel}$ was obtained:

$$b^{rel} = \frac{\kappa \mu_F}{\kappa \mu_F + 1}.$$

Here the $\kappa$ is the ratio of two scaled variances:

$$\kappa = \frac{V_N}{V_{\mu_F}}, \quad V_N = \frac{D_N}{\langle N \rangle}, \quad V_{\mu_F} = \frac{D_{\mu_F}}{\mu_F},$$

$\langle N \rangle$ and $D_N = \langle N^2 \rangle - \langle N \rangle^2$ - the mean number of emitters and the event-by-event variance of the number of emitters.

$\mu_F$ and $D_{\mu_F} = \mu_F^2 - \mu_F^2$ - the mean multiplicity produced by one emitter in the forward window and the corresponding variance.

1. V.V. Vechernin, R.S. Kolevatov, hep-ph/0304295 (2003); Vestnik SPbU, ser.4, no.2, 12 (2004).
2. M.A. Braun, C. Pajares and V.V. Vechernin, Phys. Lett. B493, 54 (2000).
For Poisson distributions $V_N = V_{\mu_F} = 1$ and $\kappa = 1$. Clear that the $\bar{\mu}_F$ is depends on the length of the forward rapidity window. In a first approximation we can assume

$$\bar{\mu}_F = \mu_0 F \Delta y_F$$

where $\mu_0 F$ is the average multiplicity produced by one emitter in the forward window per a unit of rapidity.

$$b^{rel} = \frac{\kappa \mu_0 F \Delta y_F}{\kappa \mu_0 F \Delta y_F + 1}.$$ 

So the multiplicity correlation coefficient $b^{rel}$ even defined for scaled variables nevertheless depends through $\mu_F$ on the length of the forward rapidity window $\Delta y_F$ and does not depend on the length of the backward one $\Delta y_B$.

This is because the regression procedure is being made by the forward window. One can find the physical discussion of this phenomenon in ref.:

V.V. Vechernin, R.S. Kolevatov, hep-ph/0304295 (2003); Vestnik SPbU, ser.4, no.2, 12 (2004).
For a **linear** correlation function:

\[
\langle n_B \rangle_{n_F} = a^{\text{abs}} + b^{\text{abs}} n_F, \quad \frac{\langle n_B \rangle_{n_F}}{\langle n_B \rangle} = a^{\text{rel}} + b^{\text{rel}} \left( \frac{n_F}{\langle n_F \rangle} - 1 \right)
\]

we have exactly:

\[
b^{\text{abs}} = \frac{\langle n_B n_F \rangle - \langle n_B \rangle \langle n_F \rangle}{\langle n_F^2 \rangle - \langle n_F \rangle^2} = \frac{\langle n_B n_F \rangle - \langle n_B \rangle \langle n_F \rangle}{D_{n_F}}, \quad b^{\text{rel}} = \frac{\langle n_F \rangle}{\langle n_B \rangle} b^{\text{abs}},
\]

\[
a^{\text{rel}} = \frac{\langle n_B \rangle_{n_F=\langle n_F \rangle}}{\langle n_B \rangle} = 1
\]

So we can take as

**Definition 2:**

\[
b^{\text{abs}} = \frac{\langle n_B n_F \rangle - \langle n_B \rangle \langle n_F \rangle}{\langle n_F^2 \rangle - \langle n_F \rangle^2}, \quad b^{\text{rel}} = \frac{\langle n_F \rangle}{\langle n_B \rangle} b^{\text{abs}},
\]

Note that for a **nonlinear** correlation function \( \langle n_B \rangle_{n_F} = f(n_F) \)

Definition 1 \( \neq \) Definition 2
If we use the Definition 2 we can obtain the above formula for $b_{rel}$ at very general assumptions.

Because to calculate such defined correlation coefficient we need not to calculate the correlation function $\langle B \rangle_F = f(F)$, but only four averages: $\langle F \rangle, \langle B \rangle, \langle BF \rangle$ and $\langle F^2 \rangle$.

**Calculation of the correlation coefficient**

Simplified notations:

- $\langle B \rangle_F = a + bF$, $F \equiv n_F$, $B \equiv n_B$

- $P(B, F)$ - basic, $\sum_{B,F} P(B, F) = 1$, $\langle BF \rangle \equiv \sum_{B,F} BF P(B, F)$

- $P(F) = \sum_B P(B, F)$, $\sum_F P(F) = 1$, $\langle F \rangle \equiv \sum_F F P(F) = \sum_{B,F} F P(B, F)$

- $P(B) = \sum_F P(B, F)$, $\sum_B P(B) = 1$, $\langle B \rangle \equiv \sum_B BP(B) = \sum_{B,F} BP(B, F)$

- $P(B, F) = P(F)P_F(B) \Rightarrow P_F(B) = P(B, F)/P(F)$, $\langle B \rangle_F \equiv \sum_B BP_F(B)$
For independent identical emitters:

\[ P(B, F) = \sum_N w(N) \sum_{B_1, \ldots, B_N} \sum_{F_1, \ldots, F_N} \delta_{B_1 + \ldots + B_N} \delta_{F_1 + \ldots + F_N} \prod_{i=1}^{N} p(B_i, F_i) \]

For LRC:

\[ p(B_i, F_i) = p_B(B_i) \ p_F(F_i) \]

Clear that for identical emitters:

\[ \sum_{F_i} p_F(F_i) = 1 \ , \quad \sum_{F_i} F_i^2 p_F(F_i) = \bar{\mu}_F^2 \]

\[ \sum_{B_i} p_B(B_i) = 1 \ , \quad \sum_{B_i} B_i p_B(B_i) = \bar{\mu}_B \ , \quad \sum_{B_i} B_i^2 p_B(B_i) = \bar{\mu}_B^2 \]

We denote also

\[ \sum_N w(N) = 1 \ , \quad \sum_N N w(N) = \langle N \rangle \ , \quad \sum_N N^2 w(N) = \langle N^2 \rangle \]

The variances:

\[ D_N = \langle N^2 \rangle - \langle N \rangle^2 \ , \quad D_{\mu_F} = \bar{\mu}_F^2 - \mu_F^2 \]

and the scaled variances:

\[ V_N = D_N / \langle N \rangle \ , \quad V_{\mu_F} = D_{\mu_F} / \bar{\mu}_F \]
Calculation of $\langle n_F^2 \rangle \equiv \langle F^2 \rangle$ as an example

\[
\langle n_F^2 \rangle \equiv \langle F^2 \rangle \equiv \sum_F F^2 P(F) = \sum_F F^2 \sum_N w(N) \sum_{F_1,\ldots,F_N} \delta_{F,F_1+\ldots+F_N} \prod_{i=1}^N p_F(F_i) =
\]

\[
= \sum_N w(N) \sum_{F_1,\ldots,F_N} (F_1 + \ldots + F_N)^2 \prod_{i=1}^N p_F(F_i) =
\]

\[
= \sum_N w(N) \left[ \sum_{F_1,\ldots,F_N} \sum_{i=1}^N F_i^2 + \sum_{i=1}^N \sum_{j=1}^N F_i F_j \right] \prod_{i=1}^N p_F(F_i) =
\]

\[
= \sum_N w(N) [N \overline{\mu_F^2} + (N^2 - N) \overline{\mu_F^2}] = \langle N \rangle \overline{\mu_F^2} + (\langle N^2 \rangle - \langle N \rangle) \overline{\mu_F^2} =
\]

\[
= \langle N \rangle(\overline{\mu_F^2} - \overline{\mu_F^2}) + \langle N^2 \rangle \overline{\mu_F^2}
\]

So we find

\[
\langle n_F^2 \rangle = \langle N \rangle D_\mu_F + \langle N^2 \rangle \overline{\mu_F^2}
\]

and

\[
D_{n_F} \equiv \langle n_F^2 \rangle - \langle n_F \rangle^2 = \langle N \rangle D_\mu_F + \langle N^2 \rangle \overline{\mu_F^2} - \langle N \rangle^2 \overline{\mu_F^2} = \langle N \rangle D_\mu_F + D_N \overline{\mu_F^2}
\]
Gathering we find
\[ b_{\text{abs}} \equiv \frac{\langle BF \rangle - \langle B \rangle \langle F \rangle}{\langle F^2 \rangle - \langle F \rangle^2} = \frac{\langle BF \rangle - \langle B \rangle \langle F \rangle}{D_{n_F}} = \frac{D_N \bar{\mu}_B \bar{\mu}_F}{\langle N \rangle D_{\mu_F} + D_N \bar{\mu}_F^2} \]

and
\[ b_{\text{rel}} = \frac{\langle n_F \rangle}{\langle n_B \rangle} \frac{b_{\text{abs}}}{\bar{\mu}_F} = \frac{\langle N \rangle \bar{\mu}_F}{\langle N \rangle \bar{\mu}_B} \frac{\bar{\mu}_F}{\bar{\mu}_B} \frac{b_{\text{abs}}}{b_{\text{abs}}} = \frac{D_N \bar{\mu}_F^2}{\langle N \rangle D_{\mu_F} + D_N \bar{\mu}_F^2} = \frac{\kappa \bar{\mu}_F}{\kappa \bar{\mu}_F + 1}, \]

where the \( \kappa \) is the ratio of two scaled variances:
\[ \kappa = V_N, \quad V_N = \frac{D_N}{\langle N \rangle}, \quad V_{\mu_F} = \frac{D_{\mu_F}}{\bar{\mu}_F}, \]
\[ D_N = \langle N^2 \rangle - \langle N \rangle^2, \quad D_{\mu_F} = \bar{\mu}_F^2 - \bar{\mu}_F^2. \]
Comparing the definitions

In the case of \textit{nonlinear} regression:

\[
\langle n_B \rangle_{n_F} \equiv f(n_F) = f_0 + f_1[n_F - \langle n_F \rangle] + f_2[n_F - \langle n_F \rangle]^2 + f_3[n_F - \langle n_F \rangle]^3 + \ldots
\]

\textbf{Definition 1} \neq \textbf{Definition 2}

By \textbf{Definition 1}:

\[
\bar{b}^{abs} = \frac{d\langle n_B \rangle_{n_F}}{dn_F} \bigg|_{n_F = \langle n_F \rangle} = f_1, \quad \bar{b}^{rel} = \frac{\langle n_F \rangle}{\langle n_B \rangle} \bar{b}^{abs},
\]

By \textbf{Definition 2}:

\[
b^{abs} = \frac{\langle n_B n_F \rangle - \langle n_B \rangle \langle n_F \rangle}{\langle n_F \rangle^2 - \langle n_F \rangle^2} = \frac{\langle n_B n_F \rangle - \langle n_B \rangle \langle n_F \rangle}{D_{n_F}}, \quad b^{rel} = \frac{\langle n_F \rangle}{\langle n_B \rangle} b^{abs},
\]

\[
b^{abs} - \bar{b}^{abs} = \frac{f_2[n_F - \langle n_F \rangle]^3 + f_3[n_F - \langle n_F \rangle]^4 + \ldots}{D_{n_F}}.
\]
Coefficient $a_{\text{rel}}$

\[ a_{\text{rel}} = \frac{\langle n_B \rangle_{n_F=\langle n_F \rangle}}{\langle n_B \rangle} = \frac{f(\langle n_F \rangle)}{\langle f(n_F) \rangle} \]

\[
\langle n_B \rangle_{n_F=\langle n_F \rangle} \equiv f(n_F) = f_0 + f_1[n_F - \langle n_F \rangle] + f_2[n_F - \langle n_F \rangle]^2 + f_3[n_F - \langle n_F \rangle]^3 + ...
\]

\[
\langle n_B \rangle = \langle f(n_F) \rangle = f_0 + f_2\langle [n_F - \langle n_F \rangle]^2 \rangle + f_3\langle [n_F - \langle n_F \rangle]^3 \rangle + ...
\]

\[
\langle n_B \rangle - \langle n_B \rangle_{n_F=\langle n_F \rangle} = f_2 D_{n_F} + f_3 \langle [n_F - \langle n_F \rangle]^3 \rangle + ... .
\]

So $a_{\text{rel}} = 1$ for linear correlation function.

In the next (quadratic) approximation:

If $a_{\text{rel}} < 1$ - the correlation function is convex downwards: $f_2 > 0$.

If $a_{\text{rel}} > 1$ - the correlation function is convex upwards: $f_2 < 0$. 
Conclusions

♦ The formula for the long-range multiplicity correlation coefficient in the model with independent emitters is obtained at very general assumptions:

\[ b_{\text{rel}} = \frac{\kappa \mu_F}{\kappa \bar{\mu}_F + 1}, \]

where the \( \kappa \) is the ratio of two scaled variances: \( \kappa = \frac{V_N}{V_{\mu F}}, \) \( V_N = \frac{\langle N \rangle}{N}, \) \( V_{\mu F} = \frac{\mu F}{\bar{\mu}_F} \)

and \( \bar{\mu}_F \) - the mean multiplicity produced by one emitter in the forward window.

♦ The multiplicity correlation coefficient _defined for scaled variables_ nevertheless depends through \( \mu_F \) on the length of the forward rapidity window \( \Delta y_F \) and does not depend on the length of the backward one \( \Delta y_B \):

\[ b_{\text{rel}} = \frac{\kappa \mu_0 F \Delta y_F}{\kappa \mu_0 F \Delta y_F + 1}, \quad \bar{\mu}_F = \mu_0 F \Delta y_F \]

where \( \mu_0 F \) is the average multiplicity produced by one emitter in the forward window per a unit of rapidity.

♦ This is due to the regression procedure is being made by the forward window.

One can find the physical discussion of this phenomenon in ref.:

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