METRIC DENSITY RESULTS FOR THE VALUE DISTRIBUTION OF SUDLER PRODUCTS

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Abstract. We study the value distribution of the Sudler product $P_N(\alpha) := \prod_{n=1}^{N} \left|2 \sin(\pi n \alpha)\right|$ for Lebesgue-almost every irrational $\alpha$. We show that for every non-decreasing function $\psi : (0, \infty) \to (0, \infty)$ with $\sum_{k=1}^{\infty} \frac{1}{\psi(k)} = \infty$, the set $\{ N \in \mathbb{N} : \log P_N(\alpha) \leq -\psi(\log N) \}$ has upper density 1, which answers a question of Bence Borda. On the other hand, we prove that $\{ N \in \mathbb{N} : \log P_N(\alpha) \geq \psi(\log N) \}$ has upper density at least $\frac{1}{2}$, with remarkable equality if $\liminf_{k \to \infty} \psi(k)/(k \log k) \geq C$ for some sufficiently large $C > 0$.

1. Introduction and statement of results

For $\alpha \in \mathbb{R}$ and $N$ a natural number, the Sudler product is defined as

$$P_N(\alpha) := \prod_{r=1}^{N} \left|2 \sin(\pi r \alpha)\right|.$$ 

This product was first studied by Erdös and Szekeres [12]. Later, Sudler products appeared in many different areas of mathematics that include, among others, Zagier’s quantum modular forms and hyperbolic knots in algebraic topology [3, 8, 24], restricted partition functions [23], KAM theory [17] and Padé approximants [18]. Furthermore, they were used in the solution of the Ten Martini Problem [5]. Note that by 1–periodicity of $P_N(\alpha)$ and the fact that $P_N(\alpha) = 0$ for rational $\alpha$ and $N$ sufficiently large, it suffices to consider irrational numbers $\alpha \in [0, 1]$.

In [12], it was proven that

$$\liminf_{N \to \infty} P_N(\alpha) = 0, \quad \limsup_{N \to \infty} P_N(\alpha) = \infty$$

holds for almost every $\alpha$, raising the question of whether this holds for all irrationals $\alpha$. Lubinsky [19] showed that (1) remains true for all $\alpha$ that have unbounded partial quotients. On the other hand, Grepstad, Kaltenböck and Neumüller showed in [13] that $\liminf_{N \to \infty} P_N(\phi) > 0$ for $\phi$ being the Golden Ratio, answering the question negatively. This counterexample was extended in [4, 15] to certain quadratic irrationals that have only particularly small partial quotients. For more results in this area, we refer the reader to [14] and the references therein.

The asymptotic behaviour of the Sudler product depends delicately on the size of the partial quotients of $\alpha$. Since very much is known about the Diophantine properties for almost all irrationals, many results have been obtained in the metrical setting. Note that after taking logarithm, we see that $\log P_N(\alpha) = \sum_{r=1}^{N} f(n \alpha)$ is a Birkhoff sum for the irrational rotation with $f(x) = \log|2 \sin(\pi x)|$, having a logarithmic singularity. For a general overview of Birkhoff...
sums in similar settings, we refer the reader to the survey [11]. Lubinsky and Saff [20] proved that for almost all \( \alpha \), we have \( \lim_{N \to \infty} \frac{\log P_N(\alpha)}{N} = 0 \). Subsequently, Lubinsky [19] improved this result and obtained a divergence/convergence result as it is typical in metric Diophantine approximation: under a regularity condition (see [19] for the precise requirements), he showed that for a positive, non-decreasing function \( \psi \) with \( \sum_{k=1}^{\infty} \frac{1}{\psi(k)} < \infty \), almost all \( \alpha \) satisfy

\[
|\log P_N(\alpha)| \ll \psi(\log N)
\]

(2)

(\text{where} \ll \text{denotes the usual Vinogradov symbol, see Section 2.1 for a proper definition). On the other hand, if } \sum_{k=1}^{\infty} \frac{1}{\psi(k)} = \infty, \text{ then both inequalities}

\[
\log P_N(\alpha) \geq \psi(\log N), \quad \log P_N(\alpha) \leq -\psi(\log N)
\]

(3)

hold for infinitely many \( N \). These statements also follow from a more refined result obtained by Aistleitner and Borda [3], who showed that for all \( \alpha \) whose partial quotients fulfill \((a_1 + \ldots + a_K)/K \to \infty, \) we have

\[
\max_{0 \leq N < q_K} \log P_N(\alpha) = (V + o(1))(a_1 + \ldots + a_K) + O\left(\frac{\log \max_{1 \leq \ell \leq K} a_{\ell}}{a_{K+1}}\right),
\]

(4)

where \( V = \int_0^{5/6} \log |2\sin(\pi x)| \, dx \approx 0.1615 \). In a recent work, Borda [9] proved several results on the value distribution of Sudler products, both for badly approximable irrationals and for almost all \( \alpha \). In the latter context, he improved (3) in the sense that the inequalities in (3) both hold on a set of positive upper density.

**Theorem A** (Borda, [9, Theorem 6]). Let \( \psi \) be a non-decreasing, positive function such that \( \sum_{k=1}^{\infty} \frac{1}{\psi(k)} = \infty \). Then for almost all \( \alpha \), the sets

\[
\{ N \in \mathbb{N} : \log P_N(\alpha) \geq \psi(\log N) \}
\]

(5)

\[
\{ N \in \mathbb{N} : \log P_N(\alpha) \leq -\psi(\log N) \}
\]

(6)

have upper density at least \( \pi^2/(1440V^2) \approx 0.2627 \), where \( V = \int_0^{5/6} \log |2\sin(\pi x)| \, dx \).

The proof relies on (1) and the variance estimate

\[
\sqrt{\frac{1}{M} \sum_{N=1}^{M} \log^2 P_N(\alpha)} = \left(\frac{\pi}{\sqrt{720V}} + o(1)\right) \max_{0 \leq N < M} \log P_N(\alpha),
\]

which is shown to hold for infinitely many \( M \in \mathbb{N} \). Additionally, Borda makes use of the “reflection principle” of Sudler products, which will also play a main role in this paper. This principle was observed by [4] and used in the subsequent literature on Sudler products several times. We state it here in the form of [3, Propositions 2 and 3]: for any irrational \( \alpha \) and \( 0 \leq N < q_K \) (where \( q_K \) denotes the denominator of the \( k \)-th convergent of \( \alpha \), see Section 2.2 for a proper definition), we have

\[
\log P_N(\alpha) + \log P_{q_K-N-1}(\alpha) = \log q_K + O\left(\frac{1 + \log \max_{1 \leq \ell \leq K} a_{\ell}}{a_{K+1}}\right).
\]

(7)

In particular, (7) implies that for almost all \( \alpha \), the values \( \log P_N(\alpha), \; N = 1, \ldots, q_K \), distribute symmetrically around the center \( \log q_K \), which is however of negligible order for almost all \( \alpha \). Hence, the numbers \( 1 \leq N < q_K \) lie approximately as often in (5) as in (6). Borda remarked in [9] that the estimate on the upper density in Theorem A is probably not optimal, saying that it
might be possible that the union of (5) and (6) has upper density 1. Here we prove something even stronger: we show that already (6) on its own has upper density 1.

**Theorem 1.** Let \( \psi \) be a non-decreasing, positive function such that \( \sum_{k=1}^{\infty} \frac{1}{\psi(k)} = \infty \). Then for almost every \( \alpha \), the set

\[
\{ N \in \mathbb{N} : \log P_N(\alpha) \leq -\psi(\log N) \}
\]

has upper density 1.

The symmetry around the negligible center \( \log q_k \) discussed above leads to the belief that (5) has the same upper density than (6). Surprisingly, this turns out to be wrong: we prove that if \( \psi \) is as in Theorem 1 and additionally fulfills a certain regularity condition, (5) has upper density 1/2 for almost every \( \alpha \).

**Theorem 2.** Let \( \psi \) be a non-decreasing, positive function such that \( \sum_{k=1}^{\infty} \frac{1}{\psi(k)} = \infty \). Then for almost every \( \alpha \), the set

\[
\{ N \in \mathbb{N} : \log P_N(\alpha) \geq \psi(\log N) \}
\]

has upper density at least 1/2, with equality if \( \liminf_{k \to \infty} \frac{\psi(k)}{k \log k} \geq C \) for some absolute constant \( C > 0 \).

**Remarks on Theorems 1 and 2 and further research.**

- Note that the divergence criterion of \( \sum_{k=1}^{\infty} \frac{1}{\psi(k)} \) is invariant under multiplication with constant factors. Therefore, it suffices to show Theorems 1 and the first part of Theorem 2 for the sets (5) and (6) with \( \psi(\log N) \) substituted with \( C_1 \cdot \psi(C_2 \log N) \), where \( C_1, C_2 > 0 \) are arbitrary constants. We will make use of this fact several times in the subsequent proofs without explicitly stating it.

- By (2), we see that the assumption \( \sum_{k=1}^{\infty} \frac{1}{\psi(k)} = \infty \) is essential, as otherwise the upper density is trivially zero. Note that also “upper density” cannot be replaced by “lower density”: for \( \psi(k) \geq (12V/\pi^2 + \varepsilon) k \log k \), where \( V \) is the constant from Theorem A, even the union of (5) and (6) has lower density zero (see [9, Theorem 7]). It is interesting to find the minimal growth rate of \( \psi \) such that the sets (5), (6) or their union have non-zero lower density.

- Note that even in the case when the regularity condition \( \liminf_{k \to \infty} \frac{\psi(k)}{k \log k} \geq C \) is not satisfied, Theorem 2 gives an improved lower bound in comparison to Theorem A. Our approach relies on the fact that for almost every irrational, the trimmed sum of its first \( k \) partial quotients is bounded from above by \( k \log k \), with the largest partial quotient dominating the sum infinitely often. Therefore, we only need to control the Ostrowski coefficient of the largest partial quotient (see Section 5 for an overview). It remains open how far the regularity condition from Theorem 2 can be relaxed such that the upper density of (5) is still 1/2 for almost every \( \alpha \). Below we show that \( \psi \) has to fulfill \( \psi(k) \geq (1/2 - \varepsilon)k \) infinitely often for arbitrary small \( \varepsilon > 0 \). This can be deduced in the following way from [9, Theorem 9]: the theorem states (among other results) that for any \( t \geq 0 \),
\[
\lim_{M \to \infty} \lambda \left( \left\{ \alpha \in [0, 1] : \frac{10\pi}{M \log^2 M} \sum_{N=1}^{M} \left( \log P_N(\alpha) - \frac{1}{2} \log M \right)^2 \leq t \right\} \right)
\]
\[
= \int_{0}^{t} \frac{e^{-1/(2x)}}{\sqrt{2\pi x^{3/2}}} \, dx =: c(t),
\]
where \( \lambda \) denotes the 1-dimensional Lebesgue measure. By Chebyshev's inequality, we obtain that for any \( \varepsilon, y > 0, \)
\[
\liminf_{M \to \infty} \lambda \left( \left\{ \alpha \in [0, 1] : \frac{1}{M} \# \left\{ 1 \leq N \leq M : \left| \log P_N(\alpha) - \frac{1}{2} \log M \right| \geq \varepsilon \log M \right\} \leq y \right\} \right)
\]
\[
\geq c(10\pi \varepsilon^2 y).
\]
Applying Fatou's Lemma, we get that on a set of measure of at least \( c(10\pi \varepsilon^2 y) > 0, \)
\[
\frac{1}{M} \# \left\{ 1 \leq N \leq M : \left| \log P_N(\alpha) - \frac{1}{2} \log M \right| \geq \varepsilon \log M \right\} \leq y
\]
holds for infinitely many \( M. \) This implies that the upper density of
\[
\left\{ N \in \mathbb{N} : \log P_N(\alpha) > \left( \frac{1}{2} - \varepsilon \right) \log N \right\}
\]
is bounded from below by \( 1 - y, \) so choosing \( y < \frac{1}{2}, \) we can deduce that for \( \psi(k) \leq (1/2 - \varepsilon)k, \) the upper density of \( \psi \) being 1/2 fails to hold on a set of positive measure. However, it remains open whether having \( \psi(k) \geq \frac{k}{2} \) is already sufficient to deduce upper density 1/2 for almost all \( \alpha. \)

Similarly, it is interesting if there is some threshold function where the upper density of the set in \( \psi \) jumps from 1/2 to 1 for almost every \( \alpha \) (and if so, how fast does this function grow?), or if the value of the upper density attains a fixed number strictly between 1/2 and 1 for certain functions \( \psi \) and almost every irrational.

2. Notation and preliminary results

2.1. Notation. Given two functions \( f, g : (0, \infty) \to \mathbb{R}, \) we write \( f(x) = O\left(g(x)\right) \) or \( f \ll g, \)
when \( \limsup_{x \to \infty} \frac{f(x)}{g(x)} < \infty \) and \( f(x) = o\left(g(x)\right), \) when \( \limsup_{x \to \infty} \frac{f(x)}{g(x)} = 0. \) If \( f \ll g \) and \( g \ll f, \) we write \( f \asymp g \) and \( f \sim g \) for \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1. \) Given a real number \( x \in \mathbb{R}, \) we write \( \|x\| = \min\{|x - k| : k \in \mathbb{Z}\} \) for the distance of \( x \) from its nearest integer.

2.2. Continued fractions. In this subsection, we shortly recall all necessary facts about the theory on continued fraction that are used to prove Theorems \( \mathbb{1} \) and \( \mathbb{2}. \) For a more detailed introduction, we refer the reader to the classical literature, e.g. \([1, 21, 22]. \) Every irrational \( \alpha \) has a unique infinite continued fraction expansion \([a_0; a_1, \ldots] \) with convergents \( p_k/q_k = [a_0; a_1, \ldots, a_k] \) that fulfill the recursions
\[
p_{k+1} = p_{k+1}(\alpha) = a_{k+1}(\alpha)p_k + p_{k-1}, \quad q_{k+1} = q_{k+1}(\alpha) = a_{k+1}(\alpha)q_k + q_{k-1}.
\]
For shorter notation, we will just write $p_k, q_k, a_k$, although these entities depend on $\alpha$. We know that $p_k/q_k$ approximates $\alpha$ very well, which leads to the following well-known inequalities for $k \geq 1$:

\begin{equation}
\frac{1}{q_{k+1} + q_k} \leq \delta_k := \|q_k \alpha\| = |q_k \alpha - p_k| \leq \frac{1}{q_{k+1}},
\end{equation}

from where we can deduce that

\begin{equation}
\frac{1}{a_{k+1} + 2} \leq q_k \delta_k \leq \frac{q_k}{q_{k+1}} \leq \frac{1}{a_{k+1}}.
\end{equation}

Using (8), we obtain that

\begin{equation}
a_{k+1} \delta_k = \delta_{k-1} - \delta_{k+1}.
\end{equation}

Fixing an irrational $\alpha = [a_0; a_1, ...]$, the Ostrowski expansion of a non-negative integer $N$ is the unique representation

\[ N = \sum_{\ell=0}^{K-1} b_{\ell} q_{\ell} \quad \text{where} \quad b_K \neq 0, \quad 0 \leq b_{\ell} \leq a_{\ell+1}, \quad b_0 < a_1,
\]

with the additional rule that $b_{\ell-1} = 0$ whenever $b_{\ell} = a_{\ell+1}$.

**Metrical results.** Much is known about the almost sure behavior of continued fraction coefficients and convergents. Below we state all known properties of almost every $\alpha$ that are used during the proofs of Theorems 1 and 2.

- *(Bernstein [7]):* For any monotonically non-decreasing function $\psi : [1, \infty) \to [1, \infty)$, we have

\begin{equation}
\# \{k \in \mathbb{N} : a_k > \psi(k)\} \text{ is } \begin{cases} \text{infinite} & \text{if } \sum_{k=0}^{\infty} \frac{1}{\psi(k)} = \infty \\ \text{finite} & \text{if } \sum_{k=0}^{\infty} \frac{1}{\psi(k)} < \infty. \end{cases}
\end{equation}

- *(Diamond and Vaaler [10]):

\begin{equation}
\sum_{\ell \leq K} a_{\ell} - \max_{\ell \leq K} a_{\ell} \sim \frac{K \log K}{\log 2}, \quad K \to \infty.
\end{equation}

- *(Khintchine and Lévy, see e.g. [21], Chapter 5, §9, Theorem 1):

\begin{equation}
\log q_k \sim \frac{\pi^2}{12 \log^2 k} \text{ as } k \to \infty.
\end{equation}

Combining (12) and (13), the following corollary follows immediately.

**Corollary 3.** Let $\psi$ be a non-decreasing, positive function such that $\sum_{k=1}^{\infty} \frac{1}{\psi(k)} = \infty$. Then for almost every $\alpha$, there exist infinitely many $K \in \mathbb{N}$ such that the following hold.

a) $\psi(K) < a_K < K^2$.

b) $\sum_{\ell=1}^{K-1} a_{\ell} \ll K \log K$ with an absolute implied constant.
3. Heuristic behind the proofs

We start by sketching the heuristic idea behind the proof of Theorems 1 and 2. This can be compared with [2, Section 2.1]. Starting with Theorem 1, note that we can assume without loss of generality that $\psi(k)/(k \log k) \to \infty$, since this implies the statement also for slower-growing $\psi$. Let $\psi$ and $K$ be as in Corollary 3 and let $N < q_K$ be arbitrary with Ostrowski expansion $N = \sum_{\ell=0}^{K-1} b_{\ell} q_{\ell}$. We use the usual decomposition of $P_N(\alpha)$ into certain shifted Sudler products. This approach was first used in the special case for $\alpha$ being the Golden Ratio in [13] and made more explicit and general in subsequent works in this area, e.g. [3, 4, 14, 15, 16]. Defining

$$ P_N(\alpha, x) := \prod_{n=1}^{N} |2 \sin(\pi(n\alpha + x))|, \quad \alpha, x \in \mathbb{R}, $$

and

$$ \epsilon_\ell(N) := q_{\ell} \sum_{k=\ell+1}^{K-1} (-1)^{k+\ell} b_k \delta_k, $$

we can deduce (see [3, Lemma 2]) that

$$ P_N(\alpha) = \prod_{\ell=0}^{K-1} \prod_{\ell=0}^{b_{\ell}-1} P_{q_\ell} \left( \alpha, (-1)^{\ell} (b q_\ell \delta_\ell + \epsilon_\ell(N))/q_\ell \right). $$

Ignoring first the contribution of the numbers $\epsilon_\ell(N)$, and using the approximation $P_{q_\ell} (\alpha, (-1)^{\ell} x/q_\ell) \approx |2 \sin(\pi x)|$ elaborated later, we see that

$$ \log P_N(\alpha) \approx \sum_{b=1}^{b_{K-1}-1} \log |2 \sin(\pi b q_{K-1} \delta_{K-1})| + \sum_{\ell=0}^{K-2} \sum_{b=1}^{b_{\ell}-1} \log |2 \sin(\pi b q_\ell \delta_\ell)| $$

$$ \approx a_K \int_0^{b_{K-1}/a_K} \log |2 \sin(\pi x)| \, dx + \sum_{\ell=0}^{K-2} a_{\ell+1} \int_0^{b_{\ell}/a_{\ell+1}} \log |2 \sin(\pi x)| \, dx. $$

By the choice of $K$ as in Corollary 3, the value $a_K$ dominates the sum $\sum_{\ell=0}^{K-1} a_\ell$. So using $\log|2 \sin(\pi x)| \leq \log(2)$ and assuming that

$$ \int_0^{b_{K-1}/a_K} \log |2 \sin(\pi x)| \, dx $$

is bounded away from 0, we have that $\log P_N(\alpha) \ll -a_K$, provided that the integral in (17) is negative. It is easy to see that this is the case if and only if $b_{K-1}/a_K < \frac{1}{2}$, which leads to

$$ \log P_N(\alpha) \ll -\psi(K) $$

for $b_{K-1}/a_K < 1/2 - \varepsilon$. As almost all numbers $N < \left[ \frac{q_K}{2} \right]$ fulfill

$$ \log N \asymp \log q_K \asymp K, $$

(18) is equivalent to $\log P_N(\alpha) \ll -\psi(\log N)$ for most $N$, which implies Theorem 1.
By the same reasoning, we can immediately deduce that at least 50% of all numbers \( N < q_K \) fulfill (18). Using the reflection principle, we see that also
\[
\log P_N(\alpha) \gg \psi(K)
\]
is fulfilled for about 50% of all numbers \( N < q_K \), hence the first part of Theorem 2 follows immediately. For the equality in case \( \liminf_{k \to \infty} \psi(k)/(k \log k) \geq C \), we fix some integer \( q_{K-1} \leq M < q_K \) (this \( K \) does not fulfill in general the properties of Corollary 3), and show that asymptotically, at most 50% of all \( N < M \) can fulfill \( \log P_N(\alpha) \gg \psi(K) \). Defining \( a_{\ell_0} = \max_{\ell \leq K} a_{\ell} \), we can argue similar to before that for \( C \) sufficiently large and \( \log N \gg \log q_K \),
\[
\log P_N(\alpha) \lesssim a_{\ell_0} \int_0^{b_{0-1}/a_{\ell_0}} \log|2 \sin(\pi x)| \, dx + \mathcal{O} \left( \sum_{k \neq \ell_0} a_k \right)
\leq a_{\ell_0} \int_0^{b_{0-1}/a_{\ell_0}} \log|2 \sin(\pi x)| \, dx + \frac{\psi(\log N)}{2}.
\]
So in order to fulfill \( \log P_N(\alpha) \geq \psi(\log N) \), we have the necessary condition
\[
(19) \quad \int_0^{b_{0-1}/a_{\ell_0}} \log|2 \sin(\pi x)| > 0,
\]
or equivalently, \( b_{0-1}(N)/a_{\ell_0} > 1/2 \), which can be seen to be fulfilled by at most 50% of all \( N < M \). Hence, no matter how we choose \( M \in \mathbb{N} \), at most half the numbers \( N < M \) fulfill (19), so the upper density of (5) cannot exceed 1/2.

The punchline why the upper densities of (5) and (6) differ is the following: on the full period \( 1 \leq N \leq q_K \), there are about as many elements in (5) as in (6), and for \( a_K \) being large, almost all elements are in one of those sets. The criterion whether \( N \) is in (5) or in (6) is (almost) equivalent \( b_{K-1}(N) > a_K/2 \) or not. As \( b_{K-1} \) is the most significant coefficient for the size of \( N \) (since \( b_{K-1}(M) < b_{K-1}(N) \) implies \( M < N \)), we see that all elements in (6) appear before the elements in (5), causing the asymmetric result.

**Remark.** Note that all estimates in this paper only consider upper bounds. This makes the analysis much easier since we can ignore the singularities of the function \( \log|2 \sin(\pi x)| \) at \( x = 0 \) or \( x = 1 \), as we trivially bound \( \log|2 \sin(\pi x)| \leq \log(2) \) from above. The reflection principle provides the tool to use the upper bounds also to achieve Theorem 2, without having to consider that singularities.

4. PROOF OF THE THEOREMS

4.1. Preparatory results for the approximation errors. In this section, we discuss the actual errors that are made by comparing \( \log P_N(\alpha) \) with \( a_K \int_0^{b_{K-1}/a_K} \log|2 \sin(\pi x)| \, dx \) (see Lemma 7). The first step in this direction is done by [2 Proposition 12]. For the convenience of the reader, we state it below as Proposition 4.

**Proposition 4.** Let \( N = \sum_{\ell=0}^{K-1} b_{\ell} q_{\ell} \) be the Ostrowski expansion of a non-negative integer and \( \varepsilon_{\ell}(N) \) as in (15). There exists a universal constant \( C > 0 \) such that for any \( \ell \geq 1 \) with \( b_{\ell} \geq 1 \), we have
\[
\sum_{b=0}^{b-1} \log P_{q\ell}(\alpha, (-1)^{\ell}(bq\ell\delta_\ell + \varepsilon_\ell(N))/q\ell) \leq \sum_{b=1}^{b-1} \log|2\sin(\pi(bq\ell\delta_\ell + \varepsilon_\ell(N)))|
\]
\[
+ \sum_{b=0}^{b-1} V_\ell(bq\ell\delta_\ell + \varepsilon_\ell(N))
\]
\[
+ \log(2\pi(bq\ell\delta_\ell + \varepsilon_\ell(N))) + \frac{C}{a_{\ell+1}q\ell},
\]

where
\[
V_\ell(x) := \sum_{n=1}^{q\ell-1} \sin(\pi n\delta_\ell/q\ell) \cot \left( \frac{\pi n(-1)^{\ell}\ell + x}{q\ell} \right)
\]
denotes a modified cotangent sum.

We see that we need to find upper bounds on the modified cotangent sums \(V_\ell\). This is done by the following variant of \([2, \text{Lemma 8}]\).

**Lemma 5.** Let \(1 \leq k \leq K - 1\), \(a_{\ell_0} = \max_{1 \leq \ell \leq K} a_\ell\), \(x \in (-1, 1)\) and \(V_k\) as in (20). Then the following statements hold.

(i) \(V'_k(x) < 0\), \(|V_k(0)| \ll \frac{1 + \log a_{\ell_0}}{a_{k+1}}\).

(ii) \(|V_k(x)| \ll \log a_{\ell_0} + \frac{1}{1 - |x|}\),

with the implied constants independent of \(x\) and \(k\).

**Proof.** The statements in (i) are proven in \([2, \text{Lemma 8}]\). For (ii), we use the estimate \(|V_k'(x)| \ll \frac{1}{(1 - |x|)^2}\), which is also shown in \([2]\). The result now follows immediately after integration. \(\Box\)

Next, we turn our attention to controlling the size of the perturbations \(\varepsilon_\ell(N)\). It is easy to see that \(-1 < \varepsilon_\ell(N) < 1\) for any \(1 \leq \ell \leq K - 1\). By Lemma 5 we see that the error made by \(V_\ell(bq\ell\delta_\ell + \varepsilon_\ell(N))\) is particularly large when its argument is close to its singularities at \(-1\) and \(1\). The following proposition aims to bound the arguments away from those singularities and to show that the perturbation \(\varepsilon_\ell(N)\) is small if \(a_{\ell+1}\) is large, which will be the case in the main term (see Section 3).

**Proposition 6.** Let \(\varepsilon_\ell(N)\) be defined as in (15) and \(b_\ell \geq 1\). Then we have the following inequalities:

(i) \(\frac{1}{a_{\ell+1}} \leq -q_\ell\delta_\ell \leq \varepsilon_\ell(N) \leq \frac{1}{a_{\ell+1}}\).

(ii) \(1 - |\varepsilon_\ell(N)| \gg \frac{1}{a_{\ell+2}}\).
If \( b_{\ell+1} \leq \frac{a_{\ell+2}}{2} \), then

\[
\left(23\right) \quad 1 - |\varepsilon_{\ell}(N)| \gg 1,
\]

with the implied constants being absolute.

**Proof.** We argue similarly to [3, Lemma 3]. By definition of \( \varepsilon_{\ell}(N) \) and (11), we obtain

\[
\varepsilon_{\ell}(N) = q_{\ell} \sum_{k=\ell+1}^{K-1} (-1)^{k+\ell} b_k \delta_k \leq q_{\ell} \left( a_{\ell+3} \delta_{\ell+2} + a_{\ell+3} \delta_{\ell+4} + \ldots \right)
\]

\[
= q_{\ell} \left( (\delta_{\ell+1} - \delta_{\ell+3}) + (\delta_{\ell+2} - \delta_{\ell+4}) + \ldots \right)
\]

\[
= q_{\ell} \delta_{\ell+1} \leq \frac{q_{\ell}}{q_{\ell+2}} \leq \frac{1}{2},
\]

where we used (9) in the last line. Similarly, we get

\[
\varepsilon_{\ell}(N) \geq -q_{\ell} \left( b_{\ell+1} \delta_{\ell+1} + a_{\ell+4} \delta_{\ell+3} + \ldots \right)
\]

\[
= -q_{\ell} \left( (b_{\ell+1} - a_{\ell+2}) \delta_{\ell+1} + (\delta_{\ell} - \delta_{\ell+2}) + (\delta_{\ell+2} - \delta_{\ell+4}) + \ldots \right)
\]

\[
= -q_{\ell} \left( \delta_{\ell} - (b_{\ell+1} - a_{\ell+2}) \delta_{\ell+1} \right).
\]

As \( b_{\ell} \geq 1 \) implies \( b_{\ell+1} \leq a_{\ell+2} - 1 \), combining these bounds leads to

\[
\left(24\right) \quad -1 < -q_{\ell} \delta_{\ell} + q_{\ell} \delta_{\ell+1} \leq -q_{\ell} \delta_{\ell} + q_{\ell} (a_{\ell+2} - b_{\ell+1}) \delta_{\ell+1} \leq \varepsilon_{\ell}(N) \leq q_{\ell} \delta_{\ell+1} \leq \frac{1}{2}.
\]

(i): As \( \delta_{\ell+1} \leq \delta_{\ell} \), (21) follows immediately from (10) and (24).

(ii): By (24), we have \( \varepsilon_{\ell}(N) < \frac{1}{2} \), so it suffices to find lower bounds for \( \varepsilon_{\ell}(N) \). Using (9) and \( q_{\ell+1} \leq 2a_{\ell+1}q_{\ell} \), we get

\[
q_{\ell} \delta_{\ell+1} \geq \frac{q_{\ell}}{q_{\ell+2} + q_{\ell+1}} \geq \frac{q_{\ell}}{3a_{\ell+2}q_{\ell+1}} \geq \frac{1}{6a_{\ell+2}a_{\ell+1}}.
\]

Applying (10), we get

\[
-q_{\ell} \delta_{\ell} + (a_{\ell+2} - b_{\ell+1}) q_{\ell} \delta_{\ell+1} \geq \frac{1}{a_{\ell+1}} \left( -1 + \frac{a_{\ell+2} - b_{\ell+1}}{6a_{\ell+2}} \right),
\]

which in view of (24) finishes the proof. \( \square \)

The following lemma combines the preparatory results from above. It contains the main ingredients to the proof of both Theorems 1 and 2.

**Lemma 7.** Let \( N < q_K \) with Ostrowski expansion \( \sum_{\ell=0}^{K-1} b_{\ell}q_{\ell} \) and let \( 1 \leq \ell_0 \leq K \) be such that \( a_{\ell_0} = \max_{\ell \leq K} a_{\ell} \geq 2 \). Assume that \( b_{\ell_0-1} \leq \frac{a_{\ell_0}}{2} \leq \frac{K^2}{2} \) and

\[
\sum_{k=1, \ k \neq \ell_0}^{K} a_k \ll K \log K.
\]

Then we have
\[
\log P_N(\alpha) \leq \sum_{b=1}^{b_{\ell_0}-1} \log \left| 2 \sin (\pi bq_{\ell_0-1} \delta_{\ell_0-1} + \varepsilon_{\ell_0-1}(N)) \right| + O(K \log K).
\]

**Proof of Lemma 7.** Using the decomposition into shifted Sudler products from (16), we have

\[
\log P_N(\alpha) = \sum_{k=0}^{K-1} \sum_{b=0}^{b_k-1} \log P_{q_k}(\alpha, (-1)^k (bq_k \delta_k + \varepsilon_k(N)) / q_k).
\]

Next, we apply Proposition 4 for every \(1 \leq k \leq K-1\) with \(b_k \neq 0\) and obtain for some \(C > 0\) that

\[
\log P_N(\alpha) \leq \sum_{k=1}^{K-1} \left( \sum_{b=0}^{b_k-1} \log \left| 2 \sin (\pi bq_k \delta_k + \varepsilon_k(N)) \right| \\
+ \sum_{b=0}^{b_k-1} V_k(bq_k \delta_k + \varepsilon_k(N)) \right) + O(K \log K).
\]

Applying rough bounds on the arguments of the logarithms and using (25) leads to

\[
\log P_N(\alpha) \leq \sum_{b=1}^{b_{\ell_0}-1} \log \left| 2 \sin (\pi bq_{\ell_0-1} \delta_{\ell_0-1} + \varepsilon_{\ell_0-1}(N)) \right| \\
+ \sum_{k=1}^{K-1} \sum_{b=0}^{b_k-1} V_k(bq_k \delta_k + \varepsilon_k(N)) + O(K \log K).
\]

By Proposition 6(i), we see that \(b \geq 1\) implies that \(bq_k \delta_k + \varepsilon_k(N) \geq 0\). So Lemma 5(i) and \(a_{\ell_0} \leq K^2\) lead to

\[
\sum_{k=1}^{K-1} \sum_{b=1}^{b_k-1} V_k(bq_k \delta_k + \varepsilon_k(N)) \ll \sum_{k=1}^{K-1} \frac{b_k}{a_{k+1}} \log a_{\ell_0} \ll K \log K.
\]

For \(1 \leq k \neq \ell_0 - 2 \leq K - 2\), we use (22) to obtain

\[
\frac{1}{1 - |\varepsilon_k(N)|} \ll a_{k+2}.
\]

For \(k = \ell_0 - 2\), we observe that \(b_{\ell_0-1} \leq \frac{a_{\ell_0}}{2}\), hence we have by (23) that

\[
\frac{1}{1 - |\varepsilon_{\ell_0-2}(N)|} \ll 1.
\]

For \(k = \ell_0 - 1\), we apply Proposition 6(i) to obtain \(|\varepsilon_{\ell_0-1}(N)| \leq \frac{1}{2}\), and from the definition of \(\varepsilon_k(N)\), we can follow that \(\varepsilon_{K-1}(N) = 0\). Combining these observations with (26) and (27) yields

\[
\sum_{k=1}^{K-1} V_k(\varepsilon_k(N)) \ll K \log K,
\]
where we used (25) once more. This finishes the proof. □

4.2. Proof of Theorem 1. We can assume without loss of generality that \( \lim_{k \to \infty} \psi(k)/(k \log k) = \infty \), as showing this will imply the statement of Theorem 1 also for slower growing \( \psi \). Applying Corollary 3, we know that there exist infinitely many \( K \) such that

\[
\psi(K) < a_K < K^2, \quad \sum_{k=1}^{K-1} a_k \ll K \log K.
\]

Fixing an arbitrary small \( \delta > 0 \), we define for every \( K \geq 1 \) that fulfills (28), the set

\[
M_K = M_K(\delta) := \left\{ 1 \leq N \leq \left\lfloor \frac{qK}{2} \right\rfloor : \delta a_K \leq b_{K-1}(N) \leq \left( \frac{1}{2} - \delta \right) a_K \right\}.
\]

Choosing \( K \) sufficiently large, we have by (14) that for all \( N \in M_K \),

\[
\psi(\log N) \asymp \psi(\log M_K) \asymp \psi(K).
\]

As \( \#M_K(\delta)/\left\lfloor \frac{qK}{2} \right\rfloor \to 1 \), it suffices to show that for each \( N \in M_K \), we have

\[
\log P_N(\alpha) \ll -\psi(K).
\]

We apply Lemma 7 with \( \ell_0 = K \) and obtain

\[
\log P_N(\alpha) \leq \sum_{b=1}^{b_{K-1}-1} \log \left| 2 \sin \left( \pi b_{K-1} - \delta_{K-1}(N) \right) \right| + O(K \log K).
\]

Note that we have \( \varepsilon_{K-1}(N) = 0 \) and \( b_{K-1}(N) \leq \left( \frac{1}{2} - \delta \right) a_K \), so since \( \log |2\sin(\pi x)| \) is monotonically increasing on \([0, 1/2]\), we have for some \( c = c(\delta) > 0 \) that

\[
\sum_{b=0}^{b_{K-1}-1} \log \left| 2 \sin \left( \pi b_{K-1} - \delta_{K-1}(N) \right) \right| \leq a_K \int_1^{b_{K-1}/a_K} \log |2\sin(\pi x)| \, dx \leq -c \cdot a_K \ll -\psi(K),
\]

which completes the proof.

4.3. Proof of Theorem 2. By the proof of Theorem 1 we can deduce that

\[
\limsup_{K \to \infty} \frac{\#\{0 \leq N \leq q_K : \log P_N(\alpha) \leq -2\psi(K)\}}{q_K} \geq \frac{1}{2}.
\]

By the reflection principle (7), we see that at most one of the inequalities

\[
\log P_N(\alpha) \leq -2\psi(K), \quad \log P_{q_K-N-1}(\alpha) \leq -2\psi(K)
\]

can be fulfilled, hence there is equality in (29). Applying the reflection principle a second time implies

\[
\limsup_{K \to \infty} \frac{\#\{0 \leq N \leq q_K : \log P_N(\alpha) \geq \psi(K)\}}{q_K} \geq \frac{1}{2},
\]

which finishes the proof of the first part of Theorem 2
To show equality in the case where \( \lim \inf_{k \to \infty} \phi(k)/(k \log k) \geq C \), let \( q_{K-1} \leq M < q_K \) be an arbitrary integer and let \( a_{\ell_0} = \max_{\ell \leq K} a_\ell \). We define the sets

\[
M^+ := \left\{ N \leq M : b_{\ell_0-1}(N) \geq \frac{a_{\ell_0}}{2} \right\}, \quad M^- := \left\{ N \leq M : b_{\ell_0-1}(N) \leq \frac{a_{\ell_0}}{2} \right\}
\]

and the function

\[
f : M^+ \to M^-
N = \sum_{\ell=0}^{K-1} b_\ell q_\ell \mapsto \sum_{\ell=0}^{K-1} \tilde{b}_\ell q_\ell
\]

with \( \sum_{\ell=0}^{K-1} b_\ell q_\ell \) being the Ostrowski expansion of \( N \) and

\[
\tilde{b}_\ell := \begin{cases} 
    a_{\ell_0} - b_{\ell_0-1} & \text{if } \ell = \ell_0, \\
    b_\ell & \text{otherwise}.
\end{cases}
\]

It is straightforward to check that \( f \) is well-defined and injective, hence \( |M^-| \geq \frac{M}{2} \). For arbitrary \( N \in M^- \), we apply Lemma 7 to obtain

\[
(30) \quad \log P_N(\alpha) \leq \sum_{b=1}^{b_{\ell_0-1}-1} \log \left| 2 \sin \left( \pi b q_{\ell_0-1} \delta_{\ell_0-1} + \varepsilon_{\ell_0-1}(N) \right) \right| + O(K \log K).
\]

By (21), it follows that

\[
0 \leq b q_{\ell_0-1} \delta_{\ell_0-1} + \varepsilon_{\ell_0-1}(N) \leq \frac{1}{2}, \quad b = 1, \ldots, b_{\ell_0-1} - 1,
\]

so each summand on the right-hand side of (30) is negative. Thus, for \( N \geq \sqrt{M} \), we have for almost every \( \alpha \) that

\[
\log P_N(\alpha) \ll K \log K \ll \log N \log \log N.
\]

Choosing \( C \) sufficiently large, this shows \( \log P_N(\alpha) \leq \psi(\log N) \) for \( N \in M^- \cap \{ \lfloor \sqrt{M} \rfloor, \ldots, M \} \), and as

\[
\limsup_{M \to \infty} \frac{|M^- \cap \{ 1, \ldots, \lfloor \sqrt{M} \rfloor \}|}{M} = 0,
\]

the result follows.

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