Quantum effective force and Bohmian approach to time-dependent traps

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Abstract
Trajectories of a Bohmian particle confined in time-dependent cylindrical and spherical traps are computed for both contracting and expanding boxes. A quantum effective force is considered in arbitrary directions. It is seen that in contrast to the case for the problem of a particle in an infinite rectangular box with one wall in motion, if the particle is initially in an energy eigenstate of a tiny box, the force is zero in all directions. Trajectories of a two-body system confined in the spherical trap are also computed for different statistics types. Computations show that there are situations for which the distance between bosons is greater than that between fermions. However, the results on the average separation of the particles confirm our expectation as regards the statistics.

Keywords: Schrödinger equation, moving boundary, Bohmian mechanics

(Some figures may appear in colour only in the online journal)

1. Introduction
In the real quantum world, the Hamiltonian of most systems is time dependent, and thus solving the time-dependent Schrödinger equation, in the non-relativistic domain, is a necessary task. One class of such systems is that of systems with moving boundaries. The problem of a particle in a one-dimensional infinite square-well potential with one wall in uniform motion has drawn attention in various respects [1, 2]. The concept of the quantum effective force—the time derivative of the expectation value of the momentum operator—was introduced in [3] in the context of the quantum deflection of ultracold particles from mirrors. For the problem of a particle in a 1D expanding infinite well potential, it was shown that this force is, apart from a minus sign, equal to the expectation value of the gradient of the quantum potential [2].

The exact solution of the Schrödinger equation has been found and examined for a particle in a circular trap with a wall in uniform motion [4] and for a particle in a hard sphere whose wall is moving with a constant velocity [5].

The aim of the present work is to probe some aspects of the time-dependent boundary conditions for a particle confined in a circular and in a spherical box—focusing on the Bohmian interpretation of quantum mechanics—that have remained unnoticed so far. Although the Bohmian formalism does not give predictions going beyond those of quantum mechanics whenever the predictions of the latter are unambiguous, it should be used due to its interpretational advantages.

The plan of the paper is as follows. In section 2 we consider the quantum effective force in arbitrary directions for the problem of a particle confined in time-dependent cylindrical and spherical potentials, separately. In section 3 we present the essentials of the Bohmian trajectory approach to quantum mechanics; and we give the solution to the guidance equation for when the particle is initially in an energy eigenstate. This section contains numerical calculations. Section 4 gives the concluding remarks.

2. The quantum effective force
The time derivative of the expectation value of the momentum operator of a free particle in a box with volume $V$ bounded by the moving closed surface $\sigma$ along the arbitrary
fixed direction $\hat{a}_0$ is given by
\[
\frac{d}{dt}\left\langle p_{\alpha_0} \right\rangle (t) = \frac{d}{dt} \int_{\nu} \psi^* (\nu, t) \frac{\hbar}{i} \nabla_{\alpha_0} \psi (\nu, t) \psi \nu (\nu, t) \, da = \nabla_{\nu} \cdot \left( \nabla_{\nu} \psi \nu (\nu, t) \right) \cdot \mathbf{u} + \int_{\nu} \psi^* \left( \nabla_{\nu} \psi \nu (\nu, t) \right) \, da
\]
\[+ \int_{\nu} \psi \left( \frac{i \hbar}{\mu} \nabla_{\nu} \psi \nu (\nu, t) \right) \, da - \psi^* \nabla_{\nu} \left( \frac{i \hbar}{\mu} \nabla_{\nu} \psi \nu (\nu, t) \right) \right], \tag{1}
\]
where we have used the Leibniz integral rule in the second equality; and $\mathbf{u}$ shows the velocity of the surface $\sigma$. Now, using the divergence theorem and the Schrödinger equation
\[
i\hbar \frac{d}{dt} \psi (\nu, t) = -\frac{\hbar^2}{2\mu} \nabla^2 \psi (\nu, t) \tag{2}
\]
for the free particle, we get
\[
\frac{d}{dt} \left\langle p_{\alpha_0} \right\rangle (t) = -\frac{\hbar^2}{2\mu} \int_{\nu} \left[ (\hat{n} \cdot \nabla) \psi^* \right] \nabla_{\nu} \psi, \tag{3}
\]
where we have used the vanishing on the surface boundary condition. $\hat{n} \cdot \nabla$ represents the normal component of the operator $\nabla$.

2.1. The quantum particle in an infinite cylindrical-well potential and the quantum effective force

Consider a particle with mass $\mu$ inside an infinite cylindrical-well potential. The cylinder has a time-dependent radius $L(t) = a + ut$ where $u$ is a constant; and a height $Z$, the bottom and the top surfaces being at $z = 0$ and $z = Z$. The exact solutions of the Schrödinger equation (2) for this problem are given by [4]
\[
\psi_{n\alpha}(\rho, \phi, z, t) = \exp \left[ i \alpha z (t) \left( \frac{\rho}{L(t)} \right)^2 - i \alpha \frac{1}{2} \frac{\hbar^2 \pi^2}{4a} t \right]
\times \frac{\sqrt{2}}{L(t)} \sum_{m=1}^{\infty} \chi_m (x_m) \exp \left[ \frac{i \alpha \hbar^2 \pi^2}{2a} t \right]
\times \frac{\sqrt{2}}{Z} \sin \left( \frac{k \pi z}{Z} \right) \exp \left[ -i \frac{k \pi z}{2Z} t \right]
\equiv f_{n\alpha}(\rho, t) \frac{1}{\sqrt{\pi Z}} \exp \left[ -i \frac{k \pi z}{Z} t \right], \tag{4}
\]
where $m = 0, 1, 2, \ldots$; $n = 1, 2, 3, \ldots$; $k = 1, 2, 3, \ldots$; $f_m (x_m) = 0$, $\alpha = \mu a / (2\hbar)$ and $\xi (t) = L(t)/a$. The first two lines gives the solutions for a particle in a circular box rather than a cylindrical one. The general solution of equation (2) is a superposition of functions (4):
\[
\psi(\rho, \phi, z, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} c_{n\alpha k} \psi_{n\alpha k}(\rho, \phi, z, t), \tag{5}
\]
with time-independent coefficients $c_{n\alpha k}$ determined from the relation
\[
c_{n\alpha k} = \int \rho d\rho \int \phi d\phi \int z dz \psi^*_{n\alpha k}(\rho, \phi, z, 0) \psi(\rho, \phi, z, 0). \tag{6}
\]
Equation (3) transforms to
\[
\frac{d}{dt} \left\langle p_{\alpha_0} \right\rangle (t) = -\frac{\hbar^2}{2\mu} \left( -\int_{\nu} + \int_{\nu} \right) \rho d\rho \phi d\phi \psi^* \left( \frac{i \hbar}{\mu} \nabla \psi \right), \tag{7}
\]
in cylindrical coordinates $(\rho, \phi, z)$. From equations (4) and (5), one has
\[
\frac{\partial \psi}{\partial \rho} \bigg|_{\rho=0} = \frac{\partial \psi}{\partial \rho} \bigg|_{\rho=L(t)} = 0 \tag{8}
\]
For arbitrary radial direction $\hat{a}_0 = \cos \phi \hat{x} + \sin \phi \hat{y}$, one obtains
\[
\nabla_{\rho} \psi = \cos \phi \frac{\partial \psi}{\partial \rho} + \sin \phi \frac{\partial \psi}{\partial \phi} = \cos (\phi - \phi_0) \frac{\partial \psi}{\partial \rho} - \sin (\phi - \phi_0) \frac{\partial \psi}{\partial \phi}, \tag{9}
\]
which from equation (8) becomes zero at both the bottom and the top surfaces, $z = 0$ and $z = Z$; and the second term of the right-hand side is zero at the lateral surface $\rho = L(t)$. Thus, we get
\[
\frac{d}{dt} \left\langle p_{\alpha_0} \right\rangle (t) = -\frac{\hbar^2}{2\mu} L(t) \int_0^{2\pi} \frac{\partial \psi}{\partial \rho} \left( \rho = L(t), \phi \right) \cos (\phi - \phi_0) + \frac{\hbar^2}{4\mu} \int_0^{2\pi} \frac{\partial \psi}{\partial \phi} \left( \rho = L(t), \phi \right) \sin (\phi - \phi_0) \tag{10}
\]
\[
\times \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} c_{n\alpha k} \left( e^{-\psi_{n\alpha k}(L(t), \phi_0)} + e^{\psi_{n\alpha k}(L(t), \phi_0)} \right). \]
One can similarly obtain the quantum effective force
\[
\frac{d}{dt} \left\{ \psi(p_\theta) \right\}(t) = -\frac{\hbar^2}{2\mu} L(t) \int_0^{2\pi} d\phi \sin(\phi - \phi_0) \\
\times \int_0^Z dz \left[ \frac{\partial}{\partial \rho} \left( \rho = L(t), \phi \right) \right] \\
= -\frac{\hbar^2}{4\mu} L(t) \sum_{n,m,n',1} \sum_{m',k} c_{n,m,k}^* c_{n',m',k'} \left[ -1 + (-1)^{n+n'} \right] \\
\times \exp \left[ -\frac{1}{2\rho} \left( k^2 - k'^2 \right) \pi^2 \right].
\]
(11)

along arbitrary azimuthal direction \( \hat{\phi} \) and

\[
\frac{d}{dt} \left\{ p_\phi \right\}(t) = -\frac{\hbar^2}{2\mu} \left( -\int_{-\infty}^\infty + \int_{-\infty}^\infty \right) \rho d\rho d\phi \left[ \frac{\partial}{\partial \rho} \right]^2
\]
\[
= -\frac{\hbar^2\pi^2}{4\mu Z^2} \sum_{n,m,k,k'} c_{n,m,k,k'} \left[ -1 + (-1)^{n+k} \right] \\
\times \exp \left[ -\frac{1}{2\rho} \left( k^2 - k'^2 \right) \frac{\pi^2}{\rho^2} \right].
\]
(12)

along direction \( \xi \).

When the initial wavefunction is an energy eigenstate of a smaller box with radius \( a_i \), \( a_i < a \),
\[
\psi(r,0) = \frac{\sqrt{2}}{a_i} \frac{1}{L_{m+1}(x_{m+1})} L_n(x_{m+1}^\rho/a_i)
\times \exp \left[ \frac{\pi}{\sqrt{2}\pi} \frac{r}{Z} \sin \left( k\pi \frac{r}{Z} \right) \right],
\]
(13)

then
\[
\psi(r,t) = \sum_{n'} I_{m,n,n',1}^{(a)}(r) \psi_{n,m}(r,t)
\]
(14)

where
\[
I_{m,n,n',1}^{(a)} = \frac{2}{a_i^2} \frac{1}{L_{m+1}(x_{m+1})} \frac{1}{L_{m+1}(x_{m+1}^\rho/a_i)} \\
\times \int_0^{a_i} dppe^{-i\rho(a+i\pi)} J_m(x_{m+1}^\rho/a_i) J_{m'(a+i\pi)}(x_{m'}^\rho/a_i).
\]
(15)

In such a case the modulus of \( \psi \) and its derivatives are independent of the azimuthal angle \( \phi \), and thus the integral over \( \phi \) of (10), (11) and (12) is zero. Therefore, the quantum effective force is zero in any arbitrary direction for the problem of a particle in a cylindrical trap.

2.2. The quantum particle in an infinite spherical-well potential and the quantum effective force

For a particle with mass \( \mu \) inside an infinite spherical-well potential with radius \( L(t) = a + ut \), exact solutions of the Schrödinger equation (2) read [5]
\[
\psi_{mn}^{l}(r_\theta, t) = \exp \left[ i\alpha \xi (t) \left( \frac{r}{L(t)} \right)^2 - i\alpha \frac{1 - 1/\xi(t)}{4\alpha} \right] \\
\times \int_0^{2\pi} \frac{1}{L(t)} \left[ j_{l+1}(x_{m}) \right] j_l(x_{m}r/L(t)) Y_{m}(\theta, \phi),
\]
\[
\equiv g_{m}^{l}(r_\theta, t) Y_{m}(\theta, \phi),
\]
(16)

where \( l = 0, 1, 2, \ldots; m = -l, -l + 1, \ldots, 0, \ldots, l - 1, l; \)
\( n = 1, 2, 3, \ldots \) and \( j_l(x_{m}) = 0 \). The general solution of equation (2) is a superposition of functions (16)
\[
\psi(r_\theta, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=0}^{m} c_{l,n,m}^{r} \psi_{l,n,m}^{r}(r_\theta, t).
\]
(17)

with time-independent coefficients \( c_{l,n,m}^{r} \) determined from the relation
\[
c_{l,n,m}^{r} = \int_0^{\infty} \int d\Omega r^2 \left( \frac{d\psi_r^{l,n,m}^{r}}{dr} \right) \psi(r_\theta, 0). \]
(18)

In spherical coordinates \( (r, \theta, \phi) \), equation (3) transforms to
\[
\frac{d}{dt} \left\{ p_\theta \right\}(t) = -\frac{\hbar^2}{2\mu} L(t) \int d\Omega \left( \frac{d\psi_r^{l,n,m}^{r}}{dr} \right) \nabla_{\psi} \psi \bigg|_{l=n}.
\]
(19)

From equations (16) and (17), one obtains
\[
\frac{\partial \psi}{\partial \theta} \bigg|_{l=n} = 0.
\]
(20)

For an arbitrary radial direction \( \hat{r} \), \( \hat{r}_r = (\hat{x} \cos \phi_0 + \hat{y} \sin \phi_0) \sin \theta_0 + \hat{z} \cos \theta_0 \), one has
\[
\nabla_{\psi} \psi = \hat{r}_r \cdot \nabla_{\psi} \psi
\]
\[
= \left[ \sin \theta \cos \theta \cos(\phi - \phi_0) + \cos \theta \sin \phi \right] \frac{\partial \psi}{\partial r} \\
+ \left[ \sin \theta \cos \theta \cos(\phi - \phi_0) - \cos \theta \sin \phi \right] \frac{1}{r} \frac{\partial \psi}{\partial \theta} \\
- \sin \theta_0 \sin(\phi - \phi_0) \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi}.
\]
(21)

and thus from (20),
\[
\nabla_{\psi} \psi \bigg|_{l=n} = \left[ \sin \theta \cos \theta \cos(\phi - \phi_0) \\
+ \cos \theta \sin \phi \right] \frac{\partial \psi}{\partial r} \bigg|_{l=n}.
\]
(22)
Ultimately, we get
\[ \frac{d}{dt} \left( \langle p_\mu \rangle \right)(t) = \frac{\hbar^2}{2\mu} L^2(t) \sqrt{\frac{4\pi}{3}} \int d\Omega \times \left[ \sqrt{2} \sin \theta_0 \, \text{Re} \left( e^{-i\phi_0} Y_1(\Omega) \right) - \cos \theta_0 Y_0(\Omega) \right] \times \frac{\partial \psi}{\partial r} (r = L(t), \Omega) \right] \hat{r}. \] (23)

In the same way, one can show that
\[ \frac{d}{dt} \left( \langle p_\mu \rangle \right)(t) = \frac{\hbar^2}{2\mu} L^2(t) \left( \sqrt{\frac{8\pi}{3}} \int d\Omega \times \left[ \sqrt{2} \cos \theta_0 \, \text{Re} \left( e^{-i\phi_0} Y_1(\Omega) \right) + \sin \theta_0 Y_0(\Omega) \right] \times \frac{\partial \psi}{\partial r} (r = L(t), \Omega) \right] \hat{r}, \] (24)
\[ \text{Im} \left( e^{i\phi_0} \int d\Omega Y_1(\Omega) \frac{\partial \psi}{\partial r} (r = L(t), \Omega) \right) \hat{r}. \] (25)

where equations (24) and (25) display respectively the quantum effective force in arbitrary directions
\[ \hat{\theta}_0 = (\hat{x} \cos \phi_0 + \hat{y} \sin \phi_0) \cos \theta_0 - \hat{z} \sin \theta_0 \]
and
\[ \hat{\phi}_0 = -\hat{x} \sin \phi_0 + \hat{y} \cos \phi_0. \]

When the initial wavefunction is an energy eigenstate of a smaller box with radius \( a_s < a \),
\[ \psi(\mathbf{r}, 0) = \frac{2}{\sqrt{a_s}} \frac{1}{\sqrt{i_{11}(x_{1n})}} j_n \frac{r}{a_s} Y_{1n}(\theta, \phi), \] (26)
then
\[ \psi(\mathbf{r}, t) = \sum_n I_{n}(\alpha) \psi_{1n}(\mathbf{r}, t), \] (27)
where
\[ I_{n}(\alpha) = \frac{2}{\sqrt{a_s}} \frac{1}{\sqrt{i_{11}(x_{1n})}} j_n \frac{r}{a_s} \int_0^{a_s} \int_0^{a_s} \int_0^{a_s} e^{-i\alpha(r'/r)} j_n \frac{r'}{a_s} j_n \frac{r'^2}{a_s}. \] (28)

In such a case, \[ \partial \psi/\partial r \] is independent of the azimuthal angle \( \phi \), and thus the first integral of (23) and (24); and the integral of (25) would be zero. \[ Y_{1n}(\theta, \phi) \] is an even function of \( \cos \theta \), and thus the second integral of (23) and (24) is also zero. Therefore, the quantum effective force is zero in any arbitrary direction while the force is not zero for the corresponding problem in 1D [2].

3. Bohmian trajectories

In the causal interpretation of quantum mechanics [6, 7], by introducing the polar form \( \psi(\mathbf{r}, t) = R \exp \left(iS/\hbar \right) \) in the Schrödinger equation (2), one obtains
\[ \frac{\partial S}{\partial t} + \frac{(\mathbf{\nabla} S)^2}{2\mu} + V + Q = 0, \] (29)
\[ \frac{\partial R^2}{\partial t} + \mathbf{\nabla} \cdot \left( \frac{R^2 \mathbf{\nabla} S}{\mu} \right) = 0, \] (30)

where \( Q(\mathbf{r}, t) = (-\hbar^2/2\mu) \mathbf{V} \cdot \mathbf{r}/R \) is called the quantum potential energy. In this theory, the particle trajectory \( \mathbf{r}(t) \) is determined by simultaneous integration of the time-dependent Schrödinger equation and the guidance equation
\[ \frac{d\mathbf{r}(t)}{dt} = \mathbf{v}(\mathbf{r}(t), t) = \mu \frac{\mathbf{\nabla} S}{(\mu S(\mathbf{r}(t), t), t)} \] (31)
where
\[ \frac{d}{dt} \left( \mathbf{V}_{\eta S} \right)(t) = \int_\nu d\nu' \left[ \frac{\partial^2}{\partial t^2} \mathbf{V} \mathbf{v}_{\nu S} + R^2 \mathbf{V} \frac{\partial S}{\partial t} \right] = -\int_\nu d\nu' R^2 \mathbf{V} \eta Q = \langle -\mathbf{V} \eta Q \rangle(t). \] (32)

Using the polar form of the wavefunction in the second equality of equation (1) and the vanishing on the surface boundary condition, we obtain the equality
\[ \frac{\partial}{\partial \mathbf{r}} \left( \mathbf{V}_{\eta S} \right)(t) = \frac{\partial}{\partial \mathbf{r}} \left( \mathbf{V}_{\eta S} \right)(t). \] Respecting Newton’s second law, \( \frac{d}{dt} \mathbf{p} = -\mathbf{V} (Q + V) \) in Bohmian mechanics, the last equality in equation (32) shows the relation \( \langle \mathbf{p} \rangle = \frac{\mu}{\hbar} \langle \mathbf{V} \eta Q \rangle \) for the actual momentum of the particle. Although the classical potential energy \( V(\mathbf{r}, t) \) is zero inside the box, the quantum potential energy is non-zero. This is the reason for the interaction of the boundary with the confined particle.

In the following we will compute trajectories of a Bohmian particle which is initially in an energy eigenstate of the box. So, \( \alpha_0 \) in equations (15) and (28) is replaced with \( \alpha \). For numerical calculations, new quantities \( v_{mn} = \hbar x_{mn}/\mu a \) with the dimension of velocity and \( t_{mn} = a/\hbar v_{mn} = \mu a^2/\hbar x_{mn} \) with the dimension of time, and the dimensionless quantity \( \alpha_{mn} = \mu v_{mn}/2\hbar = x_{mn}/2 \) are defined for the problem of a particle in a circular trap. The corresponding quantities in the case of a particle in a spherical trap are \( v_{mn} = \hbar x_{mn}/\mu a \), \( \alpha_{mn} = x_{mn}/2 \) and \( t_{mn} = \mu a^2/\hbar x_{mn} \). In our calculations, the velocity of the moving wall is determined versus the above velocities.
Figure 1. A selection of Bohmian trajectories for a particle in a time-dependent circular trap for different values of the wall velocity: (a) $\alpha = -0.5 \alpha_{t_0}$, (b) $\alpha = -2 \alpha_{t_0}$, (c) $\alpha = 0.5 \alpha_{t_0}$, and (d) $\alpha = 2 \alpha_{t_0}$. The particle is initially in the energy eigenstate $u_{11}(\rho, \phi)$.

Figure 2. A selection of Bohmian trajectories for a particle in a time-dependent spherical trap for different values of the wall velocity: (a) $\alpha = -0.5 \alpha_{t_0}$, (b) $\alpha = -2 \alpha_{t_0}$, (c) $\alpha = 0.5 \alpha_{t_0}$, and (d) $\alpha = 2 \alpha_{t_0}$. The particle is initially in the energy eigenstate $u_{01}(r, \theta, \phi)$ and locates on the $x$-axis. The red curve shows the position of the wall.
3.1. A circular box

Using equations (31) and (14) one finds

\[ \rho = \frac{\hbar}{\mu} \text{Im} \left( \sum_{a} I_{\text{max}}^{\alpha} (\alpha \frac{\partial g_{\alpha}}{\partial \rho}) \right), \]  
\( \rho = \frac{\hbar}{\mu \rho}, \)

(33)

(34)

\( \phi = \frac{m \hbar}{\mu r \sin \theta}, \)

(38)

\( r(t) \) is found by numerically solving equation (33), and then using this result in (34) one obtains

\[ \phi(t) = \phi_0 + \frac{m \hbar}{\mu r^2} \int_{0}^{t} \frac{dt'}{r^2(t')} \]  
\( \phi(t) = \phi_0 + \frac{m \hbar}{\mu r^2 \sin^2 \theta} \int_{0}^{t} \frac{dt'}{r^2(t')} \)  
(35)

(39)

Trajectories for the case of a particle in a circular box are displayed in the xy plane in figure 1 for different rates of contraction and expansion. Here, the particle is initially in the energy eigenstate \( u_i (\rho, \phi) = \sqrt{2} J_i (x_i \rho/a) \) \( \exp \left( i \phi \right) \frac{a}{a} I_{\alpha}^{\alpha} (x_i) \). It should be noted that the trajectories cross each other at different times (different values of the \( \phi \)-coordinate). Thus, there is no problem regarding the single-valuedness of the wavefunction.

3.2. A spherical box

Equations (31) and (28) lead to

\[ \dot{r} = \frac{\hbar}{\mu} \text{Im} \left( \sum_{a} I_{\text{max}}^{\alpha} (\alpha \frac{\partial g_{\alpha}}{\partial r}) \right), \]  
\[ \dot{\theta} = \frac{\hbar}{\mu r^2} \text{Im} \left( \frac{\partial Y_{\text{max}}^{\alpha} (\theta, \phi)}{\partial \theta} \right) = 0, \]

(36)

(37)

(38)

for the velocity of the Bohmian particle in a spherical trap and initially in an energy eigenstate. From equation (37) one sees that during the motion, the polar angle \( \theta \) will not change, i.e. the motion would be on the surface of a cone with opening angle \( \theta = \theta_0 \). By numerically solving equation (36), one finds \( r(t) \) and then equation (38) is solved:

\[ \phi(t) = \phi_0 + \frac{m \hbar}{\mu \rho^2 \sin \theta} \int_{0}^{t} \frac{dt'}{r^2(t')} \]  
\( \phi(t) = \phi_0 + \frac{m \hbar}{\mu r^2 \sin^2 \theta} \int_{0}^{t} \frac{dt'}{r^2(t')} \)  
(39)

From this equation it is apparent that for \( m = 0 \) the azimuthal angle \( \phi \) does not change during the motion, which is an obvious result: the angular momentum in the \( \hat{z} \)-direction is zero in such a case. Thus, in this case the motion is along the straight line \( \phi = \phi_0 \).

Figure 2 represents trajectories for a particle in a spherical trap. Here, the particle is initially in the ground state \( u_{00} (r, \theta, \phi) = \sqrt{2 \rho} \sin \left( \frac{\pi \rho}{a} \right) Y_{00} (\Omega) / r \), which corresponds to a particle in a 1D square box [5].

So far we have studied a single-particle system. Now, we consider a two-body system which is confined in our time-dependent spherical trap. A novel feature of the causal description of a many-body system is the non-local connection of particles: the dependence of the instantaneous motion of any one particle on the coordinates of all other particles at the same time. In this case, the particle trajectories \( r_i(t) \) are determined by the following system of two simultaneous
Figure 4. A selection of two-particle trajectories $r_1(t)$ and $r_2(t)$: Maxwell–Boltzmann (first row), Fermi–Dirac (second row) and Bose–Einstein (third row). In the left column the box is in contraction with the rate $\alpha = -0.5\alpha_{st}$, while in the right column it expands with the rate $\alpha = 0.5\alpha_{st}$.

Figure 5. The relative particle separation $r_1(t) - r_2(t)$ (first row) and root mean square separation $\sqrt{\langle (r_1 - r_2)^2 \rangle}(t)$ (second row) for two particles confined in a time-dependent spherical trap obeying Maxwell–Boltzmann (black curves), Fermi–Dirac (red curves) and Bose–Einstein (green curves) statistics. In the left column the wall velocity is $\alpha = -0.5\alpha_{st}$ and in the right column it is $\alpha = 0.5\alpha_{st}$. 

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differential equations:
\[ \frac{d\mathbf{r}_i(t)}{dt} = \frac{\hbar}{\mu_i} \text{Im} \left( \frac{\mathbf{\nabla}_j \Psi^*(\mathbf{r}_i, \mathbf{r}_j, t)}{\Psi(\mathbf{r}_i, \mathbf{r}_j, t)} \right) \mid_{\mathbf{r}_i=\mathbf{r}_j}, \quad i, j = 1, 2. \quad (40) \]

If the particles are identical, the wavefunction of the system must be symmetric. There are different statistics types: Fermi–Dirac (FD) for which the total wavefunction must be antisymmetric with respect to the exchange of particles in the system and Bose–Einstein (BE) for which the total wavefunction is symmetric under the exchange of particles. When particles are distinguishable they are independent, obeying Maxwell–Boltzmann (MB) statistics. In the Bohmian perspective, the particles are always distinguishable in all cases [7]. The initial wavefunction is taken to be
\[ \Psi_i^0(\mathbf{r}_i, 0) = \frac{1}{\sqrt{2}} \begin{vmatrix} u_{010}(\mathbf{r}_1, 0) & u_{020}(\mathbf{r}_2, 0) \\ u_{200}(\mathbf{r}_1, 0) & u_{010}(\mathbf{r}_2, 0) \end{vmatrix} \pm \frac{\sqrt{2}}{\alpha \pi r_f} \sin \left( \frac{\pi r_1}{\alpha} \right) \sin \left( \frac{2\pi r_2}{\alpha} \right) \pm \sin \left( \frac{2\pi r_1}{\alpha} \right) \sin \left( \frac{\pi r_2}{\alpha} \right) \times Y_{00}(\Omega_i) Y_{00}(\Omega_j), \quad (41) \]

which is independent of the polar and azimuthal angles, and so the motion is one-dimensional. As particles are classically non-interacting, this expression will be preserved by the Schrödinger evolution. An interesting quantity which shows the density probability for particle 1 being in \( r_1 \) regardless of the position of particle 2 is \( \rho_i^0(r) = \int d\Omega_f \int_0^\infty d r_2 r_2^2 |\Psi_i(r, r_2)|^2 \). The initial one-particle distribution function \( \rho_i^0(r) \) and two-particle density \( \rho(r, r_f) = |\Psi(r, r_f)|^2 \) are depicted in figure 3 for three different statistics. Figure 4 displays a selection of two-particle trajectories \( r_i(t) \) and \( r_f(t) \) with initial positions \( r_f(0) = r_f(0) + 0.4a \). The distance between two particles is shown in figure 5 for given initial conditions. This figure shows that, depending on the initial conditions, the relative motions of the particles are determined. There are situations in which the distance between bosons is greater than that between the fermions. On the other hand, the average separation of particles confirms one’s expectations. The mean separation \( \langle r_f - r_i \rangle \) is zero for fermions and bosons due to the symmetry of the wavefunction under exchange of particles, while the root mean square separation is less (more) in the BE (FD) case than in the MB one, but these results do not reveal details of individual motions. A similar result has been reported for a two-particle system composed of two identical and independent 1D harmonic oscillators where the one-particle wavefunctions are taken to be non-dispersive packets oscillating between two symmetrical points with respect to the origin [7]. It is worth mentioning that in the case of a non-moving box, the evolved wavefunction is given by
\[ \Psi_e^0(r_1, r_2, t) = \frac{1}{\sqrt{2}} \begin{vmatrix} u_{010}(r_1, 0) & u_{020}(r_2, 0) \\ u_{020}(r_1, 0) & u_{010}(r_2, 0) \end{vmatrix} e^{-i(E_{00} - E_{00})t/\hbar}; \quad E_{00} = \frac{\hbar^2}{2\mu a^2} n_0^2. \]

Thus, both particles stay at rest. The moving wall makes the system non-stationary, as a result of which the Bohmian particles move.

4. Summary and discussion

In this work the problems of a one-body and a two-body system confined in time-dependent traps were studied for particular initial conditions. For the one-body case we gave analytical relations for the quantum effective force. It was seen that when the particle is initially in an energy eigenstate of a smaller box, irrespective of the size of the smaller box the expectation value of the momentum operator does not change with time, while for the corresponding 1D system the behavior is different. A selection of Bohmian trajectories were computed for a particle initially in an energy eigenstate of the time-dependent circular and spherical boxes for different values of the wall velocity.

For the two-particle system, the details of the motions implied by non-factorizable wavefunctions are subtle: one cannot say that fermions are repelled and bosons are attracted, in contrast with the case for distinguishable particles obeying MB statistics. The root mean square separation confirms one’s expectation: it is less (more) in the BE (FD) case than in the MB one. In both one-body and two-body systems, a moving wall makes the system non-stationary, as a result of which Bohmian particles move.

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