THE BARYONIC SELF SIMILARITY OF DARK MATTER

C. Alard
Institut d’Astrophysique de Paris, 98bis boulevard Arago, F-75014 Paris, France; alard@iap.fr

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ABSTRACT

The cosmological simulations indicate that dark matter halos have specific self-similar properties. However, the halo similarity is affected by the baryonic feedback. By using momentum-driven winds as a model to represent the baryon feedback, an equilibrium condition is derived which directly implies the emergence of a new type of similarity. The new self-similar solution has constant acceleration at a reference radius for both dark matter and baryons. This model receives strong support from the observations of galaxies. The new self-similar properties imply that the total acceleration at larger distances is scale-free, the transition between the dark matter and baryons dominated regime occurs at a constant acceleration, and the maximum amplitude of the velocity curve at larger distances is proportional to $M^{1/4}$. These results demonstrate that this self-similar model is consistent with the basics of modified Newtonian dynamics (MOND) phenomenology. In agreement with the observations, the coincidence between the self-similar model and MOND breaks at the scale of clusters of galaxies. Some numerical experiments show that the behavior of the density near the origin is closely approximated by a Einasto profile.

Key words: dark matter – galaxies: kinematics and dynamics

1. INTRODUCTION

The numerical simulations of structure formation in the cosmological context have been providing us with a large wealth of interesting information about the structure of cold dark matter (CDM) halos. Among the results from the numerical simulations, two remarkable facts stand out: a universal dark matter (DM) profile (Navarro et al. 1997, 2010), and the occurrence of a power-law behavior for the pseudo phase space density (Taylor & Navarro 2001; Ludlow et al. 2010). As an example, Ludlow et al. (2010) demonstrated that the residual to the fit of a power law to the pseudo phase space density $Q(r) = \rho/\sigma^3$ is typically about 10% to 20% and does not exceed 30%. This behavior is observed in a range of about two decades. Similar results were also derived by Navarro et al. (2010) using Aquarius data. The power-law regime for $Q(r)$ has a well-defined outer boundary, while the inner regime is more difficult to probe due to the intrinsic difficulty of reconstructing the complex evolution of the phase space density when the system experiences a large number of multi-dimensional folds in phase space. The agreement between the power-law exponent and the prediction from the Bertschinger self-similar solution (Bertschinger 1985) suggested that the CDM halos had self-similar properties. However, the Bertschinger solution is a purely radial solution which does not corresponds to the halos obtained in numerical simulations. The reason for the correct prediction of the $Q(r)$ exponent is due to the fact that Bertschinger’s solution belongs to a family of self-similar solutions with the same specific values of the constants. The actual solution belongs to the same family but with completely different density and velocity dispersion, only the pseudo phase space density and other related quantities are the same in this family of solutions (Alard 2013). Thus, Bertschinger’s solution should be seen in historical perspective as the first clue to suggest the self-similar nature of the real solution. A puzzling unsolved problem was the reason for the observed power-law behavior of the pseudo phase space density and the non-power-law behavior of other quantities. Alard (2013) demonstrated that dynamically cold self-similar solutions in a quasi equilibrium situation have pseudo phase space density with power-law behavior. This result derives from the fact that the smoothed probability distribution $P(f)$ of $f$ remains self similar (Alard 2013). A direct prediction is that the higher-order moments constructed using $P(f)$ should also be power laws with predictable exponents. These predicted power laws are effectively consistent with the measurements obtained using data from numerical simulations (Alard 2013). Other quantities like the smoothed density do not have a self-similar expectation, and thus they are not power laws near equilibrium. At this point it is important to note that the results that were obtained with an intrinsically CDM model are incompatible with the self-similar models developed in the fluid limit (see, for instance, Subramanian 2000 or Lapi & Cavaliere 2011). The cold model is not analogous to a continuous model; this fundamental difference is related to the different behavior of $Q(r)$ and other related quantities, and non-self-similar smoothed quantities like $\rho(r)$ or $\sigma(r)$. Thus, $Q(r)$ and higher order moments of the probability distribution of $f$ are fundamental and specific in the cold non-fluid approach only. As a consequence, the consistency between self similarity and the near equilibrium phase space density corresponding to the collapse of dynamically cold initial fluctuation is well established. This description of pure CDM halos has to be modified to take into account the baryon population co-existing with the DM in real galaxies. The baryonic feedback modifies the mass distribution and as a result influences the DM halo through changes in the gravitational potential. This mechanism leads to a much more complex picture. A good illustration of this picture is given by the recent observations of the rotation curves of galaxies, and in particular from the THINGS survey (Walter et al. 2008). The high-quality rotation curves from De Blok et al. (2008) and Oh et al. (2011a) show that the central density core slope is lower than the expectation from the NFW profile. These strong and indisputable discrepancies between the observations and the expectation from the CDM model have always been a serious problem for the CDM approach. Two different types of solutions have been proposed to solve this issue; first, CDM is not valid and must be replaced by another model, such as, for instance, modified Newtonian dynamics (MOND; Milgrom 1983; Milgrom 1986). Interestingly,
Gentile et al. (2011) show that the THINGS survey rotation curves are consistent with MOND. The second solution considers that the baryonic feedback is sufficient to produce the softening of the DM density cusp, and this is the solution that will be considered here. The baryonic feedback has a particularly strong effect on the central region of the DM halo by flattening the initial density cusp. Using hydrodynamical simulations, Oh et al. (2011b) show that the baryonic feedback model is consistent with the THINGS survey data. Since the baryons are coupled to the DM though gravitational interaction, the destruction of the central DM cusp is produced through rapid fluctuations of the gravitational potential due to the ejection, the destruction of the central DM cusp is produced through rapid fluctuations of the gravitational potential due to the ejection of gas and dust. This cusp-smoothing mechanism has been investigated in detail by Navarro et al. (1996), Read & Gilmore (2005), Pontzen & Governato (2012), and Teyssier et al. (2013), who demonstrated that this mechanism is efficient and offers a possibility to reconcile CDM and the observations. The process operates through a cycle of gas ejection and re-accretion, however, it is interesting to note that the re-accretion of the gas does not produce a compensation of the effects on the DM core Pontzen & Governato (2012). See Pontzen & Governato (2014) for a general review on the baryonic feedback model. Note that the observations and the numerical simulations both suggest that the central density core is not completely flat. The problem of the asymptotic slope at the origin of the central core has been studied by Oh et al. (2011a), Pontzen & Governato (2013), and Di Cintio et al. (2013). Imposing a baryonic scale length to the DM halo indicates that the scaling relations are affected by the baryons in a way that may not be compatible with the initial similarity class and could result in the development of a new type of baryonic-induced similarity. This model is supported by the results from high-resolution hydrodynamical simulations, where the baryonic feedback influences the DM halo parameters and in particular the pseudo phase space density (Butsky & Maccio 2014). The idea of an universal similarity class for the rotation curve of galaxies was already introduced by Persic et al. (1996), Salucci & Persic (1997), Salucci et al. (2007), and Donato et al. (2009). As we shall see, this new similarity class also has some relation to the MOND phenomenology and provides an explanation to the scale independent accelerations observed at a typical radius for a large number of galaxies (Gentile et al. 2009). Note that in the continuation we distinguish between “scale independence” and “self similarity.” In the forthcoming sections, scale independence will mean independent of the distance scale (basically the size of the galaxy), while self similarity will mean independent on the scale of all the variables (distance, velocity, time; see Alard 2013, Section 3).

2. THE BARYONIC SCALING

There exists extensive literature on the baryonic feedback model. As an example, Navarro et al. (1996), Gnendin & Zhao (2002), Read & Gilmore (2005), Ragone-Figueroa et al. (2012), and Ogiya & Mori (2012) developed various models and investigations to document the effect of the baryonic feedback on the central parts of the DM halos. These works use the effects of supernovae and active galactic nuclei (AGNs) to drive the baryonic feedback. The effects of supernovae tend to be dominant for low-mass galaxies (see Governato et al. 2012 for more details); however, there are several mechanisms involved in the supernova feedback. The supernova outflow may be driven through “bursty energy transfer” Governato et al. (2012), or though momentum-driven wind (Murray et al. 2005; Oppenheimer & Davé 2006, 2008; Davé et al. 2007). The model presented here will consider the momentum-driven winds as the source of baryonic feedback. Interestingly, Oppenheimer & Davé (2006) found that among 12 other models tested, the momentum-driven outflow model adjusted on local starburst data was the only one to reproduce the inter galactic medium enrichment data. In the momentum-driven wind model, the radiation from young stars impinges on dust in the outflow, which then couples to the gas and propels matter out of the galaxy. The associated outflow of baryonic matter affects the kinematics of the dark halo, with a dominant effect in the central area. This baryonic feedback leads to the suppression of the central density cusp and its replacement with a nearly constant or lower density slope central core. As a consequence, the scale length of the DM halo $r_{DM}$ is imposed by the baryonic processes, and it should be closely related to the baryonic distribution size $r_{B}$. The foundations of this model are directly derived from the observational analysis of the rotation curves of galaxies. A correlation between the gas content and the structure of the gravitational potential is observed (Alard 2011). This analysis is reinforced by the results of Lelli et al. (2013, 2014), who find similar results with the additional finding that the structure of the potential is also related to the star formation activity. These observational results show that baryonic process related to stellar formation are responsible for the modification of the gravitational potential. However, the detailed sequence of the events leading to these observational correlation is not clearly defined. A possibility is that most of the DM halo re-shaping occurs at an early epoch when the star formation is very active, or that a number of particularly strong starbursts have a major influence. However, the observational result from Kauffmann (2014) suggests that the total amount of energy released by the stellar formation is the real and fundamental parameter. It is important to note that the detailed sequence of the events do not really matter as long as the global process remains self similar.

Actually, the DM core size $r_{DM}$ is constructed by a dominant baryonic process $P$, which has a typical scale length $r_{B}$, basically $P = P(r/r_{B})$. The FWHM $r_{p}$ of $P$ reads, $r_{p} = P^{-1}(P(0)/2)_{B}$. Thus, $r_{DM} = kr_{B}$, with a constant of proportionality $k$ depending on the specific functional form for $P$. Note that if we had estimated the typical scale length of the process $P$ not by taking the width at any other fractional value, it would change the value of $k$, but we would still have $r_{DM} \propto r_{B}$. Interestingly, Donato et al. (2004) found that the baryonic and DM scale length are nearly proportional. Similarly, a correlation between the rotation curve and the baryonic distribution is also reported by Swaters et al. (2013). The strength of the baryonic influence on the DM kinematics must be compared to the gravitational force due to DM. If the baryonic force dominates, we are in a pure flat core regime, while in the case where the DM force dominates, we return to the NFW profile (Navarro et al. 1997). The intermediate regime when the DM force $F_{DM}$ and baryonic force $F_{B}$ are of the same order defines the typical size of the baryonic core induced regime. The force is proportional to the acceleration; thus, the DM core size $r_{DM}$ corresponds to the following condition:

$$a_{DM}(r_{DM}) = a_{B}(r_{DM}).$$

Assuming a Burkert profile for the DM (Burkert 1995), Gentile et al. (2009) found that at a Burkert profile scale length $r_{0}$ the respective DM and baryonic acceleration are $3.2^{+1.8}_{-1.2} \times 10^{-9}$ cm s$^{-2}$ and $5.7^{+1.8}_{-2.8} \times 10^{-10}$ cm s$^{-2}$. The radius corresponding to a half of the central value for the Burkert...
profile is \( r_{\text{DM}} \approx 0.55 r_{0} \). Assuming that most of the baryon mass is inside \( r_{\text{DM}} \), the baryons in the range \( r_{\text{DM}} < r < r_{0} \) behave like a point mass, the acceleration at \( r_{0} \) must be re-normalized by the scale factor \( (r_{\text{DM}}/r_{0})^{2} \) to obtain the acceleration at \( r_{\text{DM}} \). On the other hand, the acceleration due to the Burkert acceleration decreases by about 13%. These corrections imply that \( a_{\text{DM}}(r_{\text{DM}}) \approx 2.8^{+1.6}_{-1.0} \times 10^{-9} \text{ cm s}^{-2} \) and \( a_{\theta}(r_{\text{DM}}) \approx 1.9^{+1.3}_{-0.9} \times 10^{-9} \text{ cm s}^{-2} \). These values of the acceleration are consistent with Equation (1) within the error bars. This result shows that the scale length of the DM halo core radius is consistent with the strength of the baryonic influence. The radial distance scale of DM is imposed by the baryons, which in itself is not sufficient to impose another similarity class to the DM halo. However, as we shall see, an equilibrium condition for the gas also implies a constraint on the total force at the distance scale, which is only compatible with a specific similarity class.

### 2.1. Critical Condition for Optically Thick Gas

We consider the effect of the baryonic feedback on a galaxy composed of stellar populations gas and DM. The momentum-driven winds offer an efficient mechanism to drive winds other large distances. A potential problem in this model is that the gas and dust are ejected from the galaxy and sent to the intergalactic medium. However, to have an effect on the DM halo, there must be a cycle where matter is ejected and falls back on the galaxy. Contrary to conventional wisdom, Oppenheimer & Davé (2008) show that a cycle occurs in the momentum-driven wind model. The ejected material is far more likely to fall back rather than stay in the intergalactic medium, with a typical fall back time of 1 Gyr. Note that the critical opacity (optically thick limit) is reached very quickly in a galaxy due to the production of dust by supernovae. Murray et al. (2005) shows that the critical opacity is reached in only \( 10^{6} \) yr for a major starburst. As a consequence, we will work in the optically thick limit where the equivalent starburst luminosity \( L_{B} \) of the stellar population is entirely absorbed by the gas. The mass distribution \( M(r) \) is assumed to be spherically symmetric. Murray et al. (2005) proposed that such systems are prone to reach a critical luminosity \( L_{B} = L_{M} \). If \( L_{B} > L_{M} \) (which in general is expected), the acceleration of the gas layer is positive, and the system loses mass which in turn decreases the starburst activity. This mechanism operates until the starburst luminosity reaches the critical limit \( L_{M} \). The luminosity \( L_{M} \) corresponds to an equilibrium between the momentum deposition of the radiation in the gas and the gravitational force applied to the gas.

#### 2.1.1. Gas Distributed in a Ring

Murray et al. (2005) studied this equilibrium for a gas component distributed in a shell. In this case, the gas shell diameter is supposed to be close to the typical size of the galaxy \( r_{B} \), which is proportional to the DM core size \( r_{\text{DM}} \). Considering a total mass of gas \( M_{G} \), the equation corresponding to the equilibrium condition reads:

\[
\frac{GM_{G}}{r_{B}^{2}} = \frac{L_{M}}{c}.
\]

(2)

The starburst luminosity is associated with young stars, and the kinematics of these stars are not much different from the kinematics of the cold molecular gas; thus, the mass loss affecting this population of stars will be proportional to the mass loss of the cold molecular gas. Since we expect that the rate of new stars will be also proportional to the cold molecular gas mass, the total number of stars in the starburst population will be proportional to the cold molecular gas mass. High-quality observations by Tacconi et al. (2010) demonstrate that the fraction of cold molecular gas was much higher at the epoch of galaxy formation than what it is today. On average, the fraction of molecular gas is at about 40% when galaxies are forming, which is quite uncommon at the present time. The variability of the molecular gas fraction is also smaller at the epoch of galaxy formation, which justifies the approximation that the total mass and the cold molecular gas mass are proportional. The starburst population is similar to the cold molecular gas and that the cold molecular gas mass is approximately proportional to the gas mass in forming galaxies. Thus, we consider that the amount of stars in the starburst population will be approximately proportional to the total gas mass, which corresponds to the following relation:

\[
L_{M} \propto M_{G}.
\]

(3)

By combining Equations (2) and (3), we obtain the following equation:

\[
\frac{GM}{r_{B}^{2}} = \text{Constant}.
\]

(4)

### 2.1.2. General Gas Distribution

In the general case, assuming a density distribution \( \rho_{G}(r) \) of the gas, the equilibrium at \( r = r_{B} \) condition reads:

\[
\int_{0}^{r_{B}} \rho_{G}(r)M(r)dr \propto \frac{L_{M}}{c}.
\]

(5)

Applying Equation (3) again, we obtain:

\[
\int_{0}^{r_{B}} \rho_{G}(r)M(r)dr \propto M_{G}.
\]

(6)

In the case of a stable equilibrium, the total force must be null at the point of equilibrium but also at its first derivative. If we assume that an equilibrium is effectively realized at a typical baryonic scale \( r_{B} \), we have:

\[
\frac{GM}{r_{B}^{2}} = \text{Constant}.
\]

(7)

#### 2.1.3. Consequences of the Baryonic Critical Condition for the Self-similar Solution

Equation (7) implies that the mass scales are similar to the square of the radius of the distribution, which is not consistent with the Bertschinger self-similarity class. The nature of the baryonic scaling implies that \( r_{B} \propto r_{\text{DM}} \), adopting \( r_{\text{DM}} = \lambda r_{B} \), using Equation (7) and defining a scale-free acceleration, \( a(r) = a_{2}(r/r_{\text{DM}}) \), we have \( a(r_{B}) = \frac{GM}{r_{B}^{2}} = a_{2}(1/\lambda) \). Thus, the acceleration is constant at a fixed point in re-scaled coordinates. The self-similar regime associated with this constant acceleration corresponds to a specific scaling of the distance and velocity variables. The development of a new self-similar regime induced by conditions developing near the center of the distribution remind us of the situation observed for the Binney conjecture (Binney 2004). In the two-dimensional phase space, the Binney conjecture states that for a large variety of initial conditions the system converges to a power law with an exponent equal to \(-1/2\). It was demonstrated by Alard (2013)
that the power law is induced by a singularity developing at the
center of the system. Note that since the scale length of the DM
self-similar solution is time-dependent, the scale length of the
baryonic distribution must also be time-dependent. However,
we consider a near equilibrium situation where the temporal
variation of the baryonic scale radius is small compared to
the system typical timescale; thus, the baryonic and DM scale length
need only to be asymptotically identical. It is expected that the
variation of the baryonic scale length is due to the slow accretion
of new baryonic material.

2.2. Self-similar Solutions of the Vlasov–Poisson System

Before we relate the baryonic scaling obtained in the previous
section to a given similarity, let us first recall some of the results
obtained in Alard (2013) on the self-similar solutions of the
Vlasov–Poisson system. Given a phase space density \( f(\mathbf{x}, \mathbf{v}, t) \),
the general solution in six-dimensional phase space reads:

\[
\begin{aligned}
    f(\mathbf{x}, \mathbf{v}, t) &= t^{\alpha_0} g \left( \frac{r}{r_{\alpha_1}}, \frac{\mathbf{v}}{v_{\alpha_1}} \right) \\
    a_0 &= -2 - 3\alpha_2 \quad ; \quad \alpha_1 = 1 + \alpha_2
\end{aligned}
\]  

Equation (8) implies that the density \( \rho \) has the following time scaling:

\[
\rho(r) = \int g(r, \mathbf{v}, t) d^3v = r^{-2} \rho_2 \left( \frac{r}{r_{\alpha_1}} \right)
\]

Consequently, the time scaling of the acceleration reads:

\[
a(r) \propto \frac{M}{r^2} = \frac{1}{r^2} \int \rho(r) r^2 dr = t^{-2+\alpha_2} \alpha_2 \left( \frac{r}{r_{\alpha_1}} \right).
\]  

The self-similar growth of a given DM halo with the
constraint from Equation (8) implies that the acceleration at a given re-scaled coordinate remains constant (see Section 2.1.2). Thus, Equation (9) should depend only on the
re-scaled coordinate and not on time. Considering the growth of an individual DM halo with typical scale \( r_{DM}(t) \), Equation (8)
implies that the acceleration at \( r_{DM}(t) \) is constant. Assuming a
slow adiabatic process, the time dependence of the re-scaling factor in Equation (9) can be linearized. The same linearization can be applied to \( r_{DM}(t) \), which implies that the two expressions become compatible, and that \( r_{DM}(t) \) can be identified to the
scaling factor \( r^{\alpha_1} \). Considering this identification, Equation (9)
coupled with Equation (7) implies that \( \alpha_1 = 2 \) and \( \alpha_2 = 1 \).
Thus, the new similarity class corresponds to \( \alpha_2 = 1 \), this must be compared to the initial Bertschinger similarity class, where
the similarity was imposed by the nature of the cosmological
infall and corresponded to \( \alpha_2 = -1/9 \).

2.2.1. Correspondence between Time and Distance Scales

The halo velocity and distance scales are related to a free pa-
parameter \( t_0 \) in the definition of time for the self-similar solution.
Basically, if \( f(\mathbf{x}, \mathbf{v}, t) \) is a solution then \( f(\mathbf{x}, \mathbf{v}, t/t_0) \) is also a solution.
By introducing \( t_0 \), the scaling of \( r \) and \( v \) are trans-
formed to \((t/t_0)\alpha_1 \) and \((t/t_0)\alpha_2 \), respectively. The corresponding re-scalings of \( r \) and \( v \) are \( x_0 = t_0^{\alpha_1} \) and \( v_0 = t_0^{\alpha_2} \). For the final stage of the evolution of an individual halo, \( a_{DM} \) does not depend on time, which also implies that \( a_{DM} \) does not depend on \( t_0 \), and thus does not depend on \( x_0 \) or \( v_0 \). These results illustrate the correspondence between time and scale independence. Equation (7) implies that the total acceleration at \( r_B \) is scale-free
and since \( a_{DM} \) is scale-free at all positions, the baryonic acceler-
eration at \( r_B \), \( a_{0B} = a_B(r_B) \) is scale-free or time-independent.
The baryonic acceleration at larger distances out of the baryonic
distribution is properly approximated with the acceleration due
to a single massive point, thus,

\[
a_B \simeq a_{0B} \left( \frac{r_B}{r} \right)^2 \quad r > r_B \quad \text{with} \quad a_{0B} = a_B(r_B).
\]

Equation (10) indicates that at larger distances \( a_B \) is only a function of scale-free variables and, as a consequence, is scale-
free. Since \( a_{DM} \) is also scale-free, the total acceleration at larger
distances \( (r > r_B) \) is scale-free. The fact that the both the
baryonic and DM accelerations are independent of scale is
supported by observations. Gentile et al. (2009) found that at a
specific scale length \( r_0 \), the baryonic and DM acceleration are constant.

3. CONNECTION TO MOND

Milgrom (1983) and Bekenstein & Milgrom (1984) develop-
oped an empirical modification of Newtonian dynamics in or-
der to reproduce the rotation curves of galaxies without the
need to include a dark unseen component. The remarkable suc-
cess of this approach in reproducing the data (see, for instance,
Milgrom & Sanders 2007; Milgrom 1995, 2001) is particu-
larly compelling since the modelization is based on parameters
reconstructed directly from the distribution of visible matter.
MOND assumes that a transition from the Newtonian regime
occurs at an acceleration \( a_0 \simeq 10^{-10} \, \text{cm s}^{-2} \) and that below
this acceleration, we observe an evolution to the deep MOND
regime which represents the very low acceleration limit. Be-
tween the Newtonian and deep MOND regimes, an empirical
interpolation function is assumed. There are various models for
the interpolation function with the obvious consequence that
this intermediate regime is not a very well-defined feature of
MOND. The essential features are clearly the acceleration scale
at which the transition occurs and the properties of the deep
MOND regime at very low accelerations. These two features
derived from MOND are clearly related to universal properties
of galaxies and have to be reproduced by any theory aiming to
represent the mass distribution in galaxies. As we will see,
the baryonic induced self similar model is consistent with these
MOND features. Let us now review MOND general properties
and confront them with the self similar model. For spherically
symmetric systems, the new equation reads:

\[
\mu \left( \frac{a}{a_0} \right) a = a_B.
\]  

3.1. Scale-free Behavior of MOND

Milgrom (1986) already noticed that a similarity rela-
tion existed in the MOND approach and that Equation (11)
can be written in a dimensionless form (see Milgrom 1986,
Equation (5)). At larger distances \( r > r_B \), it is straightforward
to re-write Equation (11) using Equation (10):

\[
\mu \left( \frac{a}{a_{0B}} \right) a = a_{0B} \left( \frac{r_B}{r} \right)^2.
\]  

The constant \( a_{0B} \) is independent of scale; thus, the acceleration
in Equation (12) is only of a function of scale-free variables.
In the self-similar model, the DM acceleration is scale-free,
and the baryonic acceleration (Equation (10)) is scale-free at
larger distances; thus, the total acceleration is scale-free at larger
distances \( (r > r_B) \), which is consistent with Equation (12).
3.2. Rotation Curves in MOND and the Self-similar Model

A general feature of the MOND phenomenology is that at larger distances \( r \gg r_B \), and \( a \ll a_0 \), the Newtonian force field is a point mass field which implies that the relation between the velocity at large distances, \( v_M \), and the baryon mass, \( M_B \), is

\[
v_M^4 = a_0 GM_B. \tag{13}
\]

The velocity at \( r \gg r_B \) in the self-similar model corresponds to the maximum of the velocity curve \( v_M \), for which, a typical dominant DM profile, occurs at larger distances (see Figure 1 for an illustration corresponding to a Burkert profile). We define the position of the maximum of the self-similar velocity curve \( r_M \), with, \( r_M = \eta r_{DM} \), then \( v_M^4 / r_M = a_M \), is scale-free, and we obtain

\[
v_M^4 = \eta^2 \frac{a_M^2}{a_B(r_{DM})} GM_B. \tag{14}
\]

An identification between Equations (13) and (14) indicates that \( a_0 = a_B(r_{DM}) a_M / a_B(r_{DM}) \eta^2 \). We will adopt \( a_B(r_{DM}) \approx a_{DM}(r_{DM}) \approx 2 \times 10^{-8} \text{ cm s}^{-2} \) (see Section 2). Note that \( a_M / a_B(r_{DM}) = (a_2(\eta)/a_2(1)) \) is scale-free and thus a constant, since the acceleration is scale-free at larger distances. Assuming that an estimation of the constants can be obtained by modeling the mass distribution with a Burkert profile, \( (a_M / a_{DM}(r_{DM})) \approx (1/2) \) and \( \eta \approx 6 \), we obtain \( a_0 \approx 1.8 \times 10^{-8} \text{ cm s}^{-2} \). Milgrom (2001) estimated that \( a_0 \) is of the order of \( 10^{-7} \text{ cm s}^{-2} \), which is consistent and shows that MOND and the self-similar model have the same expectation at large distances. An additional point is that the self-similar model predicts that at a characteristic scale \( r_{DM} \) the acceleration due to the baryons is of the order of the acceleration due to DM. The scale-free behavior of the acceleration implies effectively that at the distance scale \( r_{DM} \), the acceleration is constant. The region \( r \approx r_{DM} \) corresponds to the MOND intermediate regime, where the function \( \mu(x) \) is between the Newtonian regime \( \mu = 1 \) and the deep MOND regime, \( \mu = x \). The fact that the transition between the baryon-dominated regime and the DM regime occurs at a fast acceleration in the self-similar model is a clear connection to MOND. To compare the acceleration \( \dot{a} \) at which the transition occurs in the two approaches, we have to consider the equivalent of the situation where \( \dot{a}_{DM} = a_B \) in the MOND approach. Considering Equation (11), this will correspond to \( \dot{a} = 2a_B \), which in turn implies,

\[
\mu \left( \frac{\dot{a}}{a_0} \right) = \frac{1}{2}. \tag{15}
\]

Begeman et al. (1991) showed that a sample of high quality rotation curves of galaxies could be fitted using \( \mu = (x/\sqrt{1 + x^2}) \) and \( a_0 = 1.2 \pm 0.27 \times 10^{-8} \text{ cm s}^{-2} \). Using these results the solution of Equation (15) is \( \dot{a} \approx 0.69 \pm 0.16 \times 10^{-8} \text{ cm s}^{-2} \).

The results of Section 2 imply that in the self-similar model \( \dot{a} = \dot{a}_{DM}(r_{DM}) + \dot{a}_B(r_{DM}) \approx 0.47 \pm 0.13 \times 10^{-8} \text{ cm s}^{-2} \), which is consistent with the MOND value for \( \dot{a} \) considering the error bars. We compared the large distance low acceleration and intermediate regime between MOND and the self-similar model. In the remaining domain \( (a \gg a_0) \), the dynamic is Newtonian, and since in galaxies this regime also occurs at shorter distances from the center \( (r \ll r_{DM}) \), the baryons dominate and will thus satisfy the Newtonian limit of MOND (Equation (11)). As a consequence, the asymptotic limits in the MOND and the self-similar approach are the same. The difference is only a matter of interpolation between the low acceleration and the Newtonian limits. In MOND, the interpolation function itself is not defined in the theory and is free to vary within some limited constraints. However, there is a significant difference between the self-similar model and MOND; the equilibrium Equation (7) applies to a galaxy, but some fundamental mechanisms are missing which prevent it from being applied to clusters of galaxies. Despite the fact that core formation via a feedback due to AGN has been found to operate in clusters of galaxies (Martizzi et al. 2013), the nature of the process does not include a regulation mechanism that would lead to an equilibrium condition like Equation (7). In galaxies, the regulation operates by interaction between star formation and the loss of gas. If star formation is too high, the winds are higher than the critical limit, which implies that gas is removed, and, as a result, it slows down star formation (Murray et al. 2005). Such a regulation mechanism does not exist with the AGN feedback model of Martizzi et al. (2013). As a consequence, this self-similar model and its associated phenomenology should not be present in clusters of galaxies, which is a clear difference with MOND. This break of the phenomenology is in good agreement with the observations as illustrated with the case of the Bullet cluster (Clowe et al. 2004, 2006).

4. DENSITY AND PSEUDO-PHASE SPACE DENSITY OF DARK MATTER HALOS

It was demonstrated in Alard (2013) that the pseudo-phase space density of self-similar solutions of the Vlasov–Poisson system has a power law behavior. When this result is coupled with the Jeans equation, an equation for the density can be obtained (see Dehnen et al. 2005 for a discussion in the case of the Bertschinger similarity class). This section will now discuss the solution for the density in the case of the baryonic induced similarity class. It is clear that changing the similarity class has a major influence on the solution for the density, and that the work already conducted for the similarity class \( a_2 = -1/9 \) needs to be re-conducted for the new similarity class \( a_2 = 1 \). We will assume a spherically symmetric system, thus the spatial coordinates will be reduced to the radial distance modulus \( r \). The Jeans equation reads:

\[
\frac{1}{\rho} \frac{d}{dr} \left( \rho \frac{d \sigma}{dr} \right) + \frac{2 \rho \sigma^2}{r} + \frac{G}{r^2} \int_0^r \rho u^2 du = 0. \tag{16}
\]
The pseudo-phase space density $\rho/\sigma^3$ is a power law with a predictable exponent for the self-similar solutions of the Vlasov–Poisson system (Alard 2013). The exponent is a function of the self similar solution constant $\alpha_2$, $(\rho/\sigma^3) \propto (r/r^*)^\gamma$, and $\gamma = -(2 + 3\alpha_2)/(1 + \alpha_2)$. In the regime of the imposed baryonic self similarity, $\alpha_2 = 1$, and thus, $\gamma = -5/2$. It is useful to define the density and anisotropy parameter as a function of the re-scaled variable, $u = r/r_0$:

$$\rho(r) = \rho_0\rho_s(u); \quad \beta(r) = \beta_s(u); \quad \sigma(r) = \sigma_0\sigma_s(u); \quad M(r) = \rho_0\sigma_0^3M_s(u).$$  \hspace{1cm} (17)

The Jeans equation in these variables reads:

$$5u \frac{d^2M_s}{du^2} - 5 \frac{d}{du} M_s + 6\beta \frac{dM_s}{du} + 3q_0 \left[ \frac{dM_s}{du} \right]^\gamma = 0. \hspace{1cm} (18)$$

By applying a derivative to Equation (18) an equation for the density $\rho$ is obtained.

$$\rho_s \left( 15u \frac{d^2\rho_s}{du^2} \right) + \rho_s \left( 18 \frac{d\beta_s}{du} + \frac{1}{u}(48\beta + 40) \right) - 5u \frac{d\rho_s}{du} + q_0 \rho_s^{\gamma/2} u^{-\gamma/2} = 0. \hspace{1cm} (19)$$

With the following definition for the parameter $q_0$:

$$q_0 = \frac{9G\rho_0\sigma_0^2}{\rho_s^2}.$$

Assuming that the halo is virialized at radius $r_0$, we obtain an estimation of the parameter $q_0$.

$$\int GM\rho rdr \simeq \int \rho \sigma^2 r^2 dr$$

with

$$\frac{\rho}{\sigma^3} = \frac{\rho_0}{\sigma_0} u^{-3}. \hspace{1cm} (20)$$

Equation (20) provides a direct estimation of $q_0$.

### 4.1. General Solution and Asymptotic Properties

There are two types of asymptotic regimes to consider: a power law or a constant core.

#### 4.1.1. Power Law Asymptotic Behavior

The power law solution for a similar equation was already discussed extensively by Dehnen & McLaughlin (2005). The asymptotic behavior at the origin is related to the dominant behavior of the left term in Equation (16), which implies an asymptotic solution of the type $\rho a^2 \equiv$ constant. As a consequence, with $\rho \propto r^a$ and $(\rho/\sigma^3) \propto r^\gamma$, the corresponding asymptotic behavior is $\alpha = (2/5)\gamma$. For the Bertchinger solution, $\alpha_2 = (-1/9)$, $\gamma = (-15/8)$, and $\alpha = -3/4$, which corresponds to the results obtained by Dehnen & McLaughlin (2005). For the solution discussed in this paper $\gamma = -5/2$, and $\alpha = -1$. Note that $\alpha = -1$ is the limit for the dominance of the left term in Equation (16), and that, as a consequence, we have a full solution of the equation, not just for the dominant term.

#### 4.1.2. Constant Core Asymptotic Behavior

The observations favor models with constant density core, such as the cored iso-thermal models Spano et al. (2008) or the Burkert profile (Burkert 1995). In such a case, the general solution of Equations (18) and (19) is:

$$\rho_s = \sum_{n=0}^{\infty} a_n u^n \hspace{1cm} \beta_s = \sum_{n=0}^{\infty} b_n u^n. \hspace{1cm} (21)$$

It is interesting to explicitly write the first terms of the solution series expansion:

$$b_0 = -\frac{5}{6} \quad b_1 = \frac{\frac{5}{2} a_1 + q_0 a_0^2}{6a_0} \quad b_2 = \frac{100a_2a_0 + 7q_0a_0^2a_1 - 50a_1^2}{180a_0^2}. \hspace{1cm} (22)$$

A general property of the solutions presented in Equation (22) is that the zeroth order term in the expansion of $\beta$ is constant. Another point is that in general the next terms in the expansion are of low order in $u$, unless these terms are equal to zero, the asymptotic behavior of $\beta$ at the origin will not correspond to a local minimum of the function. An approximate estimation of the functional $\beta$ is obtained by assuming a simple empirical model known for its good consistency with the observations, like the cored isothermal model:

$$\rho_s \propto \frac{1}{(1 + u^2)^2}$$

or the Burkert profile:

$$\rho_s \propto \frac{1}{(1 + u)(1 + u^2)}. \hspace{1cm} (23)$$

The functional $\beta$ is directly estimated by introducing these models of the density in Equation (18), the results are presented in Figure 2. The variable $q_0$ is estimated using Equation (20), and we find $q_0 \simeq 3.2$ for the cored isothermal density and $q_0 \simeq 3.4$ for the Burkert profile. Both profiles converge at $\beta = -5/6$ at the origin and cross the zero line near $u = 1$; at larger distances,

![Figure 2](image-url)
they increase slowly and converge to radial orbits at infinity, which is consistent with the cosmological infall. Although these profiles allow us to reproduce the general features of $\beta$, a generic problem is that in both cases the minimum of $\beta$ is not situated at the origin. Another way to consider the problem would be to assume a generic behavior for $\beta$ and to estimate $\rho$. The fixed properties of $\beta$ are the value at the origin and the crossing of the zero line at $u = 1$. If we add that the minimum of $\beta$ must be located at the origin, it is straightforward to infer a parabolic model for $\beta$.

$$\beta = \frac{5}{6} (u^2 - 1), \quad 0 < u < 1. \quad (23)$$

By introducing Equation (23) in Equation (19), we obtain an equation for $\rho$. The solution of this differential equation is obtained numerically using a Runge Kutta method. Finding the solution requires a value of $q_0$, but since $q_0$ is unknown at the initial step, a first guess is assumed for $q_0$, then a solution is found, and $q_0$ is estimated. This process is iterated until the guess for $q_0$ and the value estimated from the numerical solution are the same. We start from $q_0 = 3$ which corresponds approximately to the values obtained for the cored isothermal and Burkert profiles. Starting from this initial value, the iteration process converges to a value of $q_0 = 5.59$. The numerical solution for $\rho$ corresponding to the simple asymptotic model of $\beta$ at the origin described in Equation (23) is presented in Figure 3. It is closely approximated with a Einasto profile (Einasto 1972):

$$\rho(r) \propto \exp \left( e^{r \rho^\frac{1}{n}} \right). \quad (24)$$

Note that the nature of the expansion in Equation (21) implies that if the solution is consistent with an Einasto profile, then we necessarily have $n = 3$.

5. SYNTHESIS AND CONCLUSION

The main concept presented in this article is that the initial DM self similarity is affected by the baryonic-induced self similarity. It was demonstrated in Section 2.1 that the baryonic feedback imposes two conditions on the DM distribution. These baryonic constraints are not compatible with the initial similarity class. Provided that the solution remains self similar, the conditions from Section 2.1 imply the emergence of a new similarity class for the DM halo. An important point is that this model relies on the assumption that an equilibrium is obtained between the wind pressure and the gravity of the system, leading to a baryonic-induced similarity of the DM halo. However, to reach this equilibrium, the equivalent luminosity $L$ must be greater than some critical luminosity $L_{\text{c}}$ (Murray et al. 2005). Obviously, if the star formation in the galaxy is not sufficient to reach this critical luminosity, no universal acceleration would exist and the associated self-similarity class would not be present. In this case, it is not even clear that any self-similar properties would emerge from the baryonic feedback. However, we must keep in mind that violating the equilibrium condition would definitely go against the observations, and the universal accelerations for the galaxies observed by Gentile et al. (2009). This would also again go against the general MOND conjecture (Milgrom 1983, 1986, 1995, 2001).

Another crucial assumption is the proportionality between the luminosity and the mass of gas (see Section 2.1, Equation (3)). An open possibility is that the scale factor between mass and luminosity in Equation (3) depends on galaxy type. As a result, we would still have a baryonic induced self similarity for each galaxy, but the similarity parameter (the constant in Equation (4)), which is an acceleration would depend on the galaxy type. Interestingly, Del Popolo et al. (2013) found that the acceleration constant estimated by Gentile et al. (2009) is correlated to the mass of the galaxy, which would support the fact that the scaling in Equation (3) depends on galaxy type. A direct consequence of this finding is also to support the fact that the nearly universal relations observed for galaxies are due to internal physics within the galaxies, a category to which the baryonic feedback obviously belongs. A final and crucial assumption is that the effect of the baryonic feedback is sufficient to induce a new class of similarity in the DM halo. It is reasonable to consider that the baryonic feedback is sufficient to alter the shape of the DM core and that this process has self similar properties. However, does this means that this baryonic self similarity is transmitted to the whole DM halo? It is clear that at least self similarity should be transmitted in some domain with boundaries scaling that are typical of the scale of the baryonic distribution. However, does this means that the baryonic self similarity will be transmitted to the very central region? does self similarity breaks at some small fraction of the baryonic scale? We should also expect that self similarity breaks at some distance in the outer regions. Currently, the answer to these questions is not clear, but we hope that some new insight should come from the detailed exploration of this type of model using numerical simulations. A possible observational test of self similarity is provided in Section 4, with the prediction of an Einasto profile with index $n = 3$. However, the reconstruction of the parameters of an Einasto profile is especially difficult due to the intrinsic difficulty of subtracting the baryon contribution in the inner region (Chemin et al. 2011). One important property of this new self-similar solution is that the acceleration generated by the DM halo is scale-free. When combined with the properties of the baryonic feedback, the scale-free acceleration of the DM implies that the baryonic acceleration at one scale radius $r_{\text{DM}}$ is independent on $r_{\text{DM}}$. This self-similar model put a number of observational facts on galaxies in a coherent framework. First, it confirmed the universality of the baryon and the DM accelerations observed at a scale radius of the DM distribution for a large number of galaxies (Gentile et al. 2009). Second, this self-similar model is related to the MOND phenomenology of galaxies (Milgrom 1983, and Bekenstein & Milgrom 1984). An additional point is that the density corresponding to this self-similar model is expected to form a flat cored distribution in the
central region, and a large variety of profiles has been proposed to fit the observations, cored isothermal, Burkert, or Einasto profiles. Among the different density profiles, the Einasto profile has the best consistency near the origin with the expectation from the self similar, baryonic-induced model. From these results, it is of particular interest to point out that in the self-similar model, CDM and MOND become consistent with each other. The general features observed in the rotation curves of galaxies are properly described in the MOND framework, but we see that it can be as well represented by a DM self-similar solution.

The incompatibility of the MOND phenomenology and of the observations in general is a serious problem for the CDM model (see, for instance, Kroupa et al. 2012 for a review). Thus, the result that this new CDM model based on self similarity is consistent with the observed phenomenology is definitely a change in the CDM paradigm. A major difference between the MOND approach and the self-similar CDM model is that the self-similar model does not apply to clusters of galaxies, since the equilibrium condition (Equation (7)) does not apply to a cluster. The discrepancies between the MOND phenomenology and the observations of the Bullet cluster are thus predicted by the self-similar model.

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