SYM $\mathcal{N}=4$ in light-cone gauge and the “bridge” identities

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Abstract
The light-cone gauge allows to single out a set of “transverse” fields (TF), whose Green functions are free from UV divergences in SYM $\mathcal{N}=4$. Green functions with external lines involving the remaining fields do instead exhibit divergences: indeed those fields can be expressed, by solving their equations of motion, as composite operators in terms of “transverse” fields. A set of exact identities (bridge identities) automatically realize their insertions in a path-integral formulation.

DFPD/TH 07-07
PACS numbers: 11.15.-q, 11.30.Pb, 12.60.Jv
Keywords: Light-cone gauge, supersymmetric Yang-Mills theory $\mathcal{N}=4$.

1. Introduction
Long ago it has been proved that the maximally extended supersymmetric Yang-Mills theory SYM $\mathcal{N}=4$ [1, 2], when quantized in light-cone gauge, possesses a set of Green functions, the “transverse sector” (TS) [3], which are (perturbatively) UV-finite at any order in the loop expansion [4, 5] 1, a result which is enough to conclude that the $\beta$-function vanishes. In turn this property, being gauge invariant, entails deep consequences on important features, like for instance anomalous dimensions and ADS/CFT duality [6]. Still the theory exhibits divergences in the set of the remaining Green functions (the non-transverse sector) (NTS).

Quite generally, the properties of an arbitrary Yang-Mills theory (YMT), once correctly quantized in the light-cone gauge [7], have been extensively worked out and renormalization has been proved recursively to any order in the loop expansion, providing a full control of one-particle irreducible vertices and of Green functions [8, 9].

The way in which divergences are produced in SYM $\mathcal{N}=4$ and how they can be handled seem not to have been exhaustively examined so far. This may be relevant also in connection with some persisting uncertainties in the regularization procedure when supersymmetry Slavnov-Taylor identities are considered [10, 11].

The main result of this paper is to generalize a set of identities (BI), sketched for the pure YMT in ref. [12] and introduced for QCD in ref. [13], to the supersymmetric case $\mathcal{N}=4$. They allow to express the Green functions of the NTS in terms of integrals involving transverse Green functions; in turn the latter can be computed by means for instance of supergraphs and do not require a regularization procedure, just being UV-finite [5, 2].

These identities hold for exact amplitudes, i. e. are not confined to perturbation theory. In such a context they hold order by order in the coupling constant expansion.

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1To reach this goal, as firstly pointed out by Mandelstam [4], a correct definition of the vector propagator is essential.
2Divergences occur in the (TS) when composite operators are considered.
Regularization and renormalization of an arbitrary YMT in light-cone gauge are briefly recalled in Sect.2; in so doing the relevant notations are introduced. The lagrangian density of SYM $\mathcal{N}=4$ is given explicitly, together with the projection operators singling out the TF and the NTF. General renormalization results are particularized to this case and the argument for the finiteness of the Green functions in the TS is briefly recalled \[14\]. In Sect.3 the BI are derived and their mathematical structure is discussed, in particular their involving peculiar composite operators which are eventually responsible of the appearance of divergences. In Sect.4 the simplest BI are perturbatively checked at the one-loop level, in order to show how they operate in actual calculations. Sect.5 contains our final remarks and conclusions, while figures, figure captions and technical details are given in the Appendix.

2. The renormalization in light-cone gauge

The quantization and the renormalization of a generic YMT in light-cone gauge have been extensively studied in the eighties \[9\]\[15\]. After using the Mandelstam-Leibbrandt prescription in the vector propagator

\[
\tilde{G}^{(0),ab}_{\mu\nu}(p) = \frac{-i\delta^{ab}}{p^2 + ie} \left[ g_{\mu\nu} - \frac{n_\mu p_\nu + n_\nu p_\mu}{n \cdot p} \right], \quad n_\mu = \sqrt{2} (1, 0, 0, 1),
\]

proposed in \[4\]\[16\], and derived from a canonical quantization in \[7\], the Wick rotation is possible in Feynman diagram calculations and thereby the usual power counting criterion for convergence. The novelty concerns the appearance of non-polynomial singularities in the residues of the poles when using dimensional regularization \[3\]; their origin can be traced back to the gauge (“spurious”) singularity of the vector propagator.

In spite of the fact that non-polynomialities threaten dimensional counting, it has been proved that all non-polynomial counterterms are encoded, at any order in the loop expansion, in a unique non-local structure which provides the required subtractions \[8\]. Its presence affects the proper vertices, but non-localities cancel in the Green functions and \textit{a fortiori} in S-matrix elements where gauge and Lorentz invariance are correctly recovered. These renormalization results have been successfully tested in quite non-trivial two-loop calculations concerning Wilson loops and partonic anomalous dimensions \[17\]\[18\].

The theory is even simpler when the maximally extended supersymmetry $\mathcal{N}=4$ is introduced, although the S-matrix cannot be defined in this case. To be concrete, let us write explicitly its lagrangian density \[2\]

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a} + \frac{1}{2} (D_\mu \phi_r^a)^2 + \frac{1}{2} (D_\mu \chi_r^a)^2 + \frac{g^2}{4} \left[ f^{abc} \phi_r^b \phi_r^c \right]^2 + \frac{i}{2} \lambda_m^a (iD_\mu \lambda_m^a) - \Lambda^a n_\mu A_{\mu,a}
\]

(2)

where, as usual,

\[
F_{\mu\nu}^a = \partial_\mu A_{\nu}^a - \partial_\nu A_{\mu}^a + g f^{abc} A_{\mu,b} A_{\nu,c}
\]

(3)

and $A_{\mu}^a, \lambda_m^a, \phi_r^a, \chi_r^a$ represent respectively vector potentials, Majorana spinors, scalar and pseudoscalar fields, all lying in the adjoint representation of the gauge group. $\Lambda^a$ are Lagrange multipliers and $n_\mu$ is the gauge-fixing vector. “Color” and internal indices will be sometimes understood.

\[3\] Actually, to preserve supersymmetry, we rather use dimensional reduction \[10\]\[11\].
The matrices $\alpha^r_{mn}$ and $\beta^r_{mn}$, $m, n = 1, ..., 4$, $r = 1, 2, 3$ are
\[
\alpha^1 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 0 & -\sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix},
\beta^1 = \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}, \quad \beta^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \beta^3 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix},
\]
and satisfy
\[
\{\alpha_i, \alpha_j\} = \{\beta_i, \beta_j\} = -2\delta_{i,j}, \quad [\alpha_i, \beta_j] = 0, \quad \alpha^r\alpha^r = \beta^r\beta^r = -3. \tag{5}
\]
In light-cone gauge some components (NTF) turn out to be gaussian and can be expressed in terms of a subset of independent fields, namely
\[
A^a, \chi^a_{+}, \phi^a, \chi^a_r \tag{6}
\]
where the projection operators $P_{\pm}$ are defined as
\[
P_{\pm} = \frac{\gamma^+\gamma^\pm}{2}, \quad \gamma^\pm = \frac{\gamma^0 \pm \sqrt{2}}{2}, \tag{7}
\]
and Lorentz indices from the beginning of the Greek alphabet are taken to be transverse $\alpha, \beta = 1, 2$. Sometimes those fields are referred to as “physical” ones; we have decided to call them “transverse” (as a matter of fact the fermionic decomposition $\lambda = \lambda_+ + \lambda_-$ is obviously gauge dependent and the $\lambda_-$ are certainly not less physical than the $\lambda_+$).

The TF can be recast in a single chiral-type multiplet [5]; then, using superfield Feynman diagrams, it has been shown that all Green functions turn out to be UV convergent [4], since the Mandelstam-Leibbrandt prescription in the vector propagator guarantees the validity of the power counting, which in turn is an essential ingredient of the proof of finiteness.

Let us now rephrase the analysis of [8] where it has been proved that the most general counterterms required in the total Lagrangian can be reproduced by means of the redefined fields and charge
\[
A^{(0)}_\mu = Z_3^{1/2}[A_\mu - (1 - \tilde{Z}_3^{-1})n_\mu\Omega], \\
\chi^{(0)} = Z_2^{1/2}(P_+ + \tilde{Z}_2P_-)\lambda, \\
\phi^{(0)}_r = Z_\phi^{1/2}\phi_r, \\
\chi^{(0)}_r = Z_\chi^{1/2}\chi_r, \\
\Lambda^{(0)} = Z_3^{-1/2}\Lambda, \\
g_0 = Z_3^{-1/2}g, \tag{8}
\]
where
\[
n_\mu D^\mu\Omega \equiv n_\mu\tilde{n}_\mu F^{\mu\nu}, \tag{9}
\]
$D^\mu$ being the covariant derivative in the adjoint representation. In the present case $Z_3 = 1$ (in a minimal subtraction scheme) and $Z_2, Z_\phi, Z_\chi$ are also one, thanks to the supersymmetry of the TS, at any order in

\footnote{When performing a dimensional reduction this algebra has to be suitably deformed [10]: $\alpha^r\alpha^r = \beta^r\beta^r = \omega - 5$.}
the loop expansion. The two remaining renormalization constants \( \tilde{Z}_3 \) and \( \tilde{Z}_2 \) do instead diverge, in spite of the vanishing of the \( \beta \)-function:

\[
\tilde{Z}_2 = 1 - g^2 C_{Ad} \frac{1}{8\pi^2 (2 - \omega)} + O(g^4), \quad \tilde{Z}_3 = 1 + g^2 C_{Ad} \frac{1}{8\pi^2 (2 - \omega)} + O(g^4),
\]

\( C_{Ad} \) being the quadratic Casimir operator in the adjoint representation of the “color” group.

The results of one-loop calculation of the divergent parts of self-energy diagrams are presented in [14]. In Sect.4 and in the Appendix we generalize those results including all finite contributions as well. To our knowledge it is the first time they appear in the literature. Then we verify that the transverse components (TC) of the Green functions are indeed finite. The non-transverse components (NTC) do instead diverge.

In QCD the divergent part \( \tilde{\Gamma} \) of the regularized effective action contains the gauge dependent and Lorentz non covariant expression:

\[
\tilde{\Gamma}_{nc} = i\tilde{a}_2 \bar{\psi} \left( \frac{n \cdot D\tilde{n} - \tilde{n} \cdot Dn}{n \cdot \tilde{n}} \right) \psi + \tilde{a}_3 \frac{\Omega^a n^\mu}{n \cdot \tilde{n}} \left[ (D^\nu F_{\mu\nu})^a - g\mu^{2-\omega} (\bar{\psi} \gamma^\mu \gamma^\nu \psi) \right],
\]

where \( \tilde{a}_2 \) and \( \tilde{a}_3 \) are divergent coefficients when the number of space-time dimensions \( 2\omega \) tends to 4. It is easy to realize that the Lorentz non-covariant parts do not contribute to S-matrix elements as those extra pieces are proportional to the classical equations of motions [19].

In the present case, where the S-matrix cannot be consistently defined, one should resort to different “observable quantities”, like, for instance, Wilson loops. It is expected that singular Lorentz non covariant contributions cancel here as well, but this should be proved. We plan to investigate this interesting issue shortly.

3. The bridge identities

Now we want to exploit the consequences on the Green functions generated by the presence in the Lagrangian of field components which appear only quadratically and with the coefficient of the quadratic term being purely kinematical, namely not involving other fields.

After adding to the original Lagrangian a term containing sources

\[
\mathcal{L}_s = \frac{1}{2}(\lambda_m^a \xi_m^a + \bar{\xi}_m^a \lambda_m^a) + J^\mu A_\mu^a + h^a \phi_r^a + k_r^a \chi_r^a
\]

(12)

the generating functional \( W \) becomes

\[
W = \int \mathcal{D}[A_\mu^a, \lambda_m^a, \phi_r^a, \chi_r^a] \exp i \int (\mathcal{L} + \mathcal{L}_s) d^4x.
\]

Then the gaussian components can be explicitly integrated over leading to

\[
W = \exp \left[ \frac{i}{2} \int \left( J^+ \partial_- J^+ + \bar{\xi}^+ \gamma^+ \partial_- \xi^+ \right) d^4x \right]
\]

\[
\int \mathcal{D}[A_\alpha, \lambda_\alpha] \exp i \int (\mathcal{L}_{\text{eff}} + \mathcal{L}_{\text{mix}} + \mathcal{L}_{\text{tr}}) d^4x
\]

(14)

where

\[
\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + \partial_+ A_\alpha \partial_- A_\alpha - \frac{1}{2} K^2 + \frac{i}{4} \Theta \partial_- K^+ \Theta,
\]

(15)

\[
\mathcal{L}_{\text{mix}} = J^+ \partial_- K + \frac{i}{4} (\bar{\xi}^+ \partial_- \gamma^+ \Theta + \bar{\Theta} \partial_- \gamma^+ \xi^+),
\]

(16)

\[
\mathcal{L}_{\text{tr}} = J^\alpha A_\alpha + \frac{1}{2} (\bar{\xi}^+ \lambda_\alpha + \bar{\lambda}_+ \xi^-) + (h \phi + k \chi),
\]

(17)

The coefficient of \( \tilde{a}_2 \) can be shown to vanish taking the Dirac equation for \( \lambda \) and \( \bar{\lambda} \) into account; the coefficient of \( \tilde{a}_3 \) contains in square brackets the equation of motion for \( F_{\mu\nu} \).
and
\[ K^a = \partial^{-1} \left[ (D_i \partial_i A_i)^a - \frac{ig}{2} f^{abc} \lambda^+_{+m} \gamma^+ \lambda^+_{+m} + \frac{g}{2} f^{abc} (\phi_r \partial_\phi_r - \partial_\phi_r \phi_r) + \frac{g}{2} f^{abc} (\chi^+_r \partial_r \chi^*_r - \partial_r \chi^*_r \chi^+_r) \right], \quad (18) \]
\[ \Theta^a_m = i \gamma^a (D_\alpha \lambda^+_{+m})^a - g f^{abc} (\alpha^r_{mn} \delta^{b}_r + \gamma_5 \beta^r_{mn} \lambda^c_{+n}) \lambda^c_{+n} = \]
\[ (i \partial_\perp \lambda^a_{+m} + igf^{abc} \gamma^a \lambda^c_{+m}) - g f^{abc} (\alpha^r_{mn} \delta^{b}_r + \gamma_5 \beta^r_{mn} \lambda^c_{+n}) \lambda^c_{+n}. \quad (19) \]
Eqs. (13) and (14) provide two equivalent ways of formulating the theory (the so-called four- and two-component formulations [20]). In a perturbative context they give rise to different Feynman rules and to graphs with different topologies. In particular in the two-component formulation propagators and vertices exhibit peculiar non-polynomialities which prevent a direct proof of renormalization. The rules in the four-component formulation are instead the standard ones in light-cone gauge. Renormalization can here be proved following general theorems and then translated in the two-component formulation [13].

From a computational point of view the four-component formulation has proved in the past to be by far simpler. On the other hand, in SYM \( \mathcal{N}=4 \), calculations in the TS can be performed by supergraph techniques, i.e. considering only transverse fields.

Functional differentiation with respect to sources multiplying NTF obviously generate Green functions with non-transverse external lines. The crucial remark is now that a differentiation with respect to a non-transverse source can be traded, thanks to eq. (14), with differentiations with respect to transverse insertions with non-transverse external lines. The crucial remark is now that a differentiation with respect to functions, with non-transverse external lines in terms of purely transverse Green functions. In this context, where the TS is UV-finite and therefore needs not be renormalized, they provide a direct way of controlling all the singularities occurring in the Green functions of the theory. Technically, divergences are generated by insertions of peculiar composite operators involving TF, namely the ones obtained by solving the equations of motion for NTF. When several external lines are non-transverse, several extra loop integrations are generated; nevertheless we know that the corresponding amplitudes are renormalizable as this follows from the general treatment in the four-component formulation.

Of course Green functions with insertions of composite operators can also be computed by means of the operator algebra. However, using a path-integral approach, the procedure becomes automatic and by far simpler. To realize this point the reader should compare the situation concerning the Slavnov-Taylor identities: obviously the physics is quite different, but the mathematical context is similar. Slavnov-Taylor identities can be obtained using T-products, but the path-integral approach is more convenient. That is precisely what the BI accomplish in this case.

Since \( W \) in eq. (14) is gaussian in the sources \( J_+, \tilde{\zeta}_+, \zeta_+ \), it is easy to give an explicit formula for functional differentiation with respect to those sources. Terms in \( K, \Theta, \Theta \), which depend on TF, can in turn be represented in terms of functional derivatives as usual, thus leading to the following equivalences between functional differential operators
\[ \partial^2 \frac{\delta}{i \delta J^{+,a}} = J^{+,a} + \partial_\perp K^a \left[ \frac{\delta}{i \delta J^3}, \frac{\delta}{i \delta \zeta_m^b}, \frac{\delta}{i \delta \zeta_m^b}, \frac{\delta}{i \delta \chi^+}, \frac{\delta}{i \delta \chi^+} \right] = \]
\[ = \left[ J^{+,a} + (D_\alpha \frac{\delta}{i \delta J^3}) \partial_\perp \frac{\delta}{i \delta J^\alpha} \right]^a - \frac{ig}{2} f^{abc} \frac{\delta}{i \delta \zeta_m^b} \gamma^+ \frac{\delta}{i \delta \chi^+} + \]
\[ + g f^{abc} \frac{\delta}{i \delta \chi^+} \partial_\perp \frac{\delta}{i \delta k^c} + g f^{abc} \frac{\delta}{i \delta k^c} \partial_\perp \frac{\delta}{i \delta k^c} \right], \quad (20) \]
\[
\partial \frac{\delta}{i\delta \zeta_{a,m}^{\alpha}} = \frac{i}{2} \gamma^+ \left[ \zeta_{a,m}^{\alpha} + \Theta_{a,m}^{\alpha} \left[ \frac{\delta}{i\delta J_{\alpha}}, \frac{\delta}{i\delta \zeta_{b,n}^{\beta}}, \frac{\delta}{i\delta h_{b,n}^{\beta}}, \frac{\delta}{i\delta k_{b,n}^{\beta}} \right] \right] = 
\]
\[
= \frac{i}{2} \gamma^+ \left[ \zeta_{a,m}^{\alpha} + i\gamma^a D_\alpha \left[ \frac{\delta}{i\delta J_{\alpha}}, \frac{\delta}{i\delta \zeta_{a,m}^{\alpha}}, \frac{\delta}{i\delta h_{a,m}^{\alpha}}, \frac{\delta}{i\delta k_{a,m}^{\alpha}} \right] \right] - g^f \left[ \alpha_{mn} \frac{\delta}{i\delta h_{\beta,n}^{\beta}} + \gamma^5 \beta_{mn} \frac{\delta}{i\delta k_{\beta,n}^{\beta}} \right] \frac{\delta}{i\delta \zeta_{a,m}^{\alpha}} \gamma^+ \]
\[
= \frac{i}{2} \left[ \zeta_{a,m}^{\alpha} + i(\partial_{\alpha} \frac{\delta}{i\delta \zeta_{a,m}^{\alpha}} \gamma^a) + g^f \left[ \alpha_{mn} \frac{\delta}{i\delta h_{\beta,n}^{\beta}} + \gamma^5 \beta_{mn} \frac{\delta}{i\delta k_{\beta,n}^{\beta}} \right] \right] \gamma^+ \quad (22)
\]

when applied to the Green function generating functional \( W \) in eq. (14). We notice that the differentiations appearing at the right-hand side concern only transverse sources. Non-linearities occur with respect to TF; moreover, if \( W \) is expressed in terms of \( Z \), the generating functional of connected Green functions, the identities themselves become non-linear.

Applications to concrete low order perturbative examples are deferred to the next section.

4. Low order examples

Here we present two simple concrete examples where one can see the identities at work.

The first example we consider is the two-point Green function involving a transverse and a longitudinal component of the vector potential \( G_{\alpha+} \). The BI allows to express it in terms of Green functions with only transverse external lines, more precisely the identity relates an amplitude in the NTS (on the left side of the equality) to amplitudes in the TS (at the right side):

\[
\partial_{-x}^2 G_{\alpha+}^{ab}(x-y) = \partial_{-x} \partial_{-x} G_{\alpha+}^{ab}(x-y) + g f^{f_{\alpha+}} \partial_{-x} G_{\alpha+}^{ac} \partial_{-x} G_{\alpha+}^{bd} (x-y) + 6 g f^{f_{\alpha+}} \partial_{-x} G_{\alpha+}^{ad} (x-y) \quad (23)
\]

where the factor 6 in the last term takes into account the equal contributions of three scalars and three pseudoscalars and the trace for the spinors also acts on the \( \alpha, \beta \) matrices.

After Fourier transforming we get

\[
(r^+)^2 \tilde{G}_{\alpha+}^{ab}(r) - r^+ r^\beta \tilde{G}_{\alpha+}^{ab}(r) = i g \mu^4 2 \omega f^{f_{\alpha+}} \left[ \int \frac{d^2q}{(2\pi)^2} q^+ \tilde{G}_{\alpha+}^{ac} (r, -r - q, q) + \int \frac{d^2q}{2(2\pi)^2} \mu \tilde{G}_{\alpha+}^{ac} (r, -r - q, q) \right] \quad (24)
\]

\( \mu \) being the usual dimensional regularization scale.

Figure 1 presents the one-loop contributions to the self-energies for all transverse and non transverse fields. Singular contributions to the self-energies were explicitly known up to \( O(g^2) \) in [21] only the one-loop gluon self-energy, including finite parts, has been computed. For the purpose of performing a complete check of the BI, here we have calculated the finite parts of the fermion self-energies as well. We have used the “four-component” formulation and therefore the Feynman graph topologies are the usual ones. The results are presented in Table 1.

After suitable projections, we obtain the related two-point Green functions we summarize in Table 2. One immediately realizes that there is no divergent correction to transverse two-point Green functions. For the transverse three-point Green functions, since the check is performed up to \( O(g^2) \), the tree level is enough.
There are in principle two equivalent possibilities: to use either the two-component formulation of the theory or the four-component one performing the suitable projections afterwards. At this stage they are essentially equivalent; at higher perturbative orders the second one is simpler thanks to its traditional Feynman graph topology. Tree level three-point amplitudes are reported in Figure 2.

It is now fairly easy to perform the balance in eq. (24). First of all we realize that the contributions from integrals of three point functions involving spinors and scalar-pseudoscalar particles vanish. As a matter of fact we get (see Fig. 2)

\[
\int \frac{d^2 \omega}{(2\pi)^2} \frac{q}{q^2(q-r)^2} \left[ q^+(q-r)\alpha + (q-r)^+q_\alpha - 2(q-r)^+q_\alpha \right] = 0 \tag{25}
\]

and

\[
\int \frac{d^2 \omega}{(2\pi)^2} \frac{(q + r)^\mu}{q^2(q+r)^2} = 0. \tag{26}
\]

Moreover terms at \( \mathcal{O}(g^0) \) trivially compensate. The balance in eq. (24) involves therefore the three following contributions

\[
(r^+)G_{\alpha \lambda +}^a(r) = -\frac{2Ar^+r_\alpha}{r^2} \left[ \frac{1}{e} + 2 + \frac{\pi^2}{3} - 2L_2(1 - \frac{\zeta}{\zeta - 1}) \right], \tag{27}
\]

\[
r^+r_\beta \tilde{G}_{\alpha \beta}^{ab}(r) = -\frac{2Ar^+r_\alpha}{r^2} \left[ \frac{\pi^2}{3} - 2L_2(1 - \frac{\zeta}{\zeta - 1}) - \frac{2}{\zeta} \right], \tag{28}
\]

and

\[
ig\mu^{4-2\omega} f^{abcd} \int \frac{d^2 \omega}{(2\pi)^2} q^+\tilde{G}_{\alpha \beta}^{acbd}(r, -r, q, q) = \frac{2Ar^+r_\alpha}{r^2} \left[ 1 + \frac{2}{e} - \frac{\zeta \log \frac{\zeta}{\zeta - 1}}{\zeta - 1} \right], \tag{29}
\]

where

\[
A = \frac{ig^2 C_{Ab}}{16\pi^2}, \quad \zeta = \frac{2r^+ - r^-}{r^2}, \quad \frac{1}{e} = \frac{1}{2 - \omega} + \log\left(\frac{A\mu^2}{-\gamma^2}\right) - \gamma. \tag{30}
\]

and \( L_2(z) \) is the Euler dilogarithm.

The second example concerns the two-point Green function involving a transverse and a non-transverse component of the spinor field. The identity in this case reads

\[
2\partial_- G_{\lambda, \lambda +}^{ab}(x - y) = -\gamma^+ \partial_- G_{\lambda, \lambda +}^{ab}(x - y) - gf^{acd}_{\alpha, \gamma} G_{\lambda, \alpha \lambda +}^{abcd}(x - z, z - y)|_{z=x} + C_{Ab}^{acbd}(x - z, z - y)|_{z=x} + \tag{31}
\]

where \( G_{\lambda, \alpha \lambda +}^{abcd} \) represents the three-point Green function involving two transverse components of the spinor and a transverse component of the vector field, \( G_{\lambda, \lambda +}^{abcd} \) and \( G_{\lambda, 5r \lambda +}^{abcd} \) represent the three-point Green functions with scalars and pseudoscalars respectively.

After a Fourier transform we get

\[
2p^+ \tilde{G}_{\lambda, \lambda +}^{ab}(p) + \gamma^+ p_\perp \tilde{G}_{\lambda, \lambda +}^{ab}(p) + ig\mu^{4-2\omega} \gamma^+ f^{acd}_{\alpha, \gamma} \int \frac{d^2 \omega}{(2\pi)^2} \tilde{G}_{\lambda, \alpha \lambda +}^{acbd}(q, p, q, p) + \tag{32}
\]

\[
-\mu^{4-2\omega} f^{abcd}_{\alpha, \gamma} \int \frac{d^2 \omega}{(2\pi)^2} \left[ \gamma^+ \tilde{G}_{\lambda, \alpha \lambda +}^{acbd}(q, p, q, p) + \beta^r \gamma^5 G_{\lambda, 5r \lambda +}^{abcd}(q, p, q, p) \right] = 0.
\]

Taking into account that the integrals over three-point Green functions involving scalar and pseudoscalar fields, give vanishing contributions

\[
\int \frac{d^2 \omega}{(2\pi)^2} \frac{1}{q^2(p-q)^2} (p^+ q_\perp - q^+ p_\perp) = 0. \tag{33}
\]
and recalling the results summarized in Tables 1 and 2, the balance in eq. (32) turns out to be

\[ 2p^+ \tilde{G}^{ab}_{\lambda+\lambda_+}(p) = g^2 C_{Ad} \frac{p^+ p_+ P_+}{p^2} [4I(\omega) + 8Jp^+ - 8p^+_\perp p^\mu J_\mu + \frac{4p^+ p^-}{p^2} I(\omega)] + \]

\[ + \frac{8}{p^2} (p^+_\perp) J_+ - \frac{4}{p^2} M (2p^+_\perp + p^2_\perp) + \delta_{ab}, \]

\[ \gamma^+ p_+ \tilde{G}^{ab}_{\lambda+\lambda_+}(p) = -g^2 C_{Ad} \delta^{ab} \frac{p^+ p_+ P_+}{p^2} [4I(\omega) + 8Jp^+ - 8p^+_\perp p^\mu J_\mu + \]

\[ + \frac{4}{p^2} [2(p^+_\perp) J_+ - 2p^2_\perp M + p^2_\perp I(\omega)] \]

and

\[ ig\mu \gamma^+ \gamma^a f^{acd} \int \frac{d^{2\omega} q}{(2\pi)^{2\omega}} \tilde{G}^{abc}_{\alpha\lambda\alpha\lambda_+}(p, q, q - p) = -\frac{4g^2 C_{Ad}}{p^2} \delta^{ab} p^+_\perp P_+ [I(\omega) - M], \]

where \( M \equiv i \frac{\zeta \text{log} \zeta}{\zeta - 1}. \) It is fairly easy to check that \( \tilde{G}^{ab}_{\lambda+\lambda_+} \) is free from UV divergences.

We notice that again the presence of scalar and pseudoscalar fields affects only two-point functions.

5. Concluding remarks

We conclude by recalling that the light-cone quantization and renormalization of YMT developed in the eighties, put this gauge on an equal footing with respect to other gauge choices. In the past it has been successfully used in phenomenological partonic calculations; another peculiarity concerns the treatment of composite operators. As a matter of fact the light-cone gauge was used in the study of evolution equations for the matrix elements of composite quasi-partonic operators in the pioneering paper by A. Bukhvostov, G. Frolov, E. Kuraev and L. Lipatov [22]; those operators are the first example of integrable structures in SYM \( \mathcal{N}=4 \). In ref. [23] it has been shown that gauge invariant operators in QCD mix under renormalization only among themselves at variance with the situation occurring in covariant gauges.

The novelty in SYM \( \mathcal{N}=4 \) is that the coupling constant does not run, owing to the vanishing of the \( \beta \)-function. The advantage in using the light-cone gauge in this case mainly resides in the direct proof of the finiteness of an entire sector of the theory (the TS), where supersymmetry is realized in terms of a single chiral superfield, as firstly realized by Mandelstam [4]. Divergences are present in the Green functions of the NTS; however they are completely controlled by the identities we have hitherto proposed. Indeed they only occur in connection with non transverse external lines: each external non transverse line entails the insertion of a composite operator, quadratic in the transverse fields, and thereby a single extra loop integration over a UV-finite amplitude.

Those divergences concern the wave-function renormalization of NTF and, in the operator language, they are generated by the insertions of peculiar composite operators, namely the ones obtained by solving the equation of motion of “longitudinal” operators in terms of transverse ones. The BI automatically realize such insertions in a path-integral formulation and might be useful, for instance, in higher order perturbative calculations.

To prove finiteness of the TS in the “four component” formulation would entail more and more cancellations in higher orders.

Since the BI are exact identities, they should survive unscathed in the presence of instantons; one should however keep in mind that the light cone gauge choice is mandatory, turning the theory into a two-component formulation. We do not believe this is a convenient setting for the study of instanton effects;
in particular difficulties of a topological nature might arise when inverting the operator $\partial_-$. Nevertheless an insight to this problem may be worthwhile.

Finally our procedure could be generalized to $\beta$-deformed SYM $\mathcal{N}=4$, like for instance, the one considered in ref.\[24\]. As a matter of fact, by choosing the light cone gauge, non transverse degrees of freedom can again be functionally integrated over, turning the theory into a two-component formulation. Finiteness of the Green functions of the transverse sector however can only be proved in the planar limit.

Appendix

We begin by providing the integrals which are needed in order to evaluate the one-loop perturbative corrections.

\[
I(\omega) \equiv \int \frac{d^2q}{(2\pi)^2} \frac{(\mu^2)^{2-\omega}}{q^2(q-p)^2} = \frac{i}{16\pi^2} \frac{p^2}{4\pi \mu^2} \frac{\Gamma(2-\omega)\Gamma(\omega-1)}{\Gamma(2\omega-2)},
\]

\[
I_\mu \equiv \int \frac{d^2q}{(2\pi)^2} \frac{\mu^2(\mu^2)^{2-\omega}}{q^2(q-p)^2} = \frac{p_\mu}{2} I(\omega),
\]

\[
I_{\mu\nu} \equiv \int \frac{d^2q}{(2\pi)^2} \frac{\mu^2 q_\mu q_\nu(\mu^2)^{2-\omega}}{q^2(q-p)^2} = I(\omega) \frac{1}{4(2\omega-1)} \left[ 2\omega p_\mu p_\nu - g_{\mu\nu} p^2 \right],
\]

\[
J \equiv \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + (q-p)^2} = \frac{i}{16\pi^2 p^4} \left[ \frac{\pi^2}{6} - L_2(1 - \zeta) \right],
\]

\[
J_\alpha \equiv \int \frac{d^4q}{(2\pi)^4} \frac{q_\alpha}{q^2 + (q-p)^2} = \frac{ip_\alpha}{4(2\pi)^2} \frac{1}{p^+} \zeta \log \zeta - 1,
\]

\[
J_+ \equiv \int \frac{d^4q}{(2\pi)^4} \frac{q^+}{q^2 + (q-p)^2} = \frac{ip^2}{32\pi^2(p^+)^2} \left[ \frac{\pi^2}{6} - L_2(1 - \zeta) + \zeta \log \zeta - \pi^2 \right],
\]

\[
J_- \equiv \int \frac{d^2q}{(2\pi)^2} \frac{\mu^{4-2\omega} q_-}{q^2 + (q-p)^2} = I(\omega).
\]

Now we proceed by presenting in Fig.1 all the one-loop contributions to the self-energies for transverse and non transverse fields. Those self-energies are computed using the “four-component” formulation; therefore the Feynman graph topologies are the usual ones. The results are reported in Table 1 together with the relevant definitions.

Table 2 presents the corresponding Green functions at $O(g^0)$ and at $O(g^2)$. The three-point Green functions at tree level for the transverse fields, which are required to match at $O(g^2)$ the BI, appear in Fig. 2.
Fig. 1: One loop self-energy diagrams

- : vector meson
- : Majorana spinors
- - - - : scalars
- - - - - - : pseudoscalars
\[
\{(\frac{11}{3}e + \frac{57}{9})S_{\mu\nu}^{(1)} + 2\left[-2(\frac{\pi^2}{6} - L_2(1 - \zeta))(S_{\mu\nu}^{(1)} + S_{\mu\nu}^{(2)}) + 2\frac{\zeta \log \zeta}{\zeta - 1}S_{\mu\nu}^{(2)} + \left(\frac{1}{\zeta} + 2 - \frac{\zeta \log \zeta}{\zeta - 1}\right)S_{\mu\nu}^{(3)}\}\right] A \\
- \left[\frac{1}{\zeta} + 3\right] S_{\mu\nu}^{(1)} A \\
- \frac{8}{3}\left[\frac{1}{\zeta} + \frac{5}{3}\right] S_{\mu\nu}^{(1)} A \\
2\left[-2(\frac{\pi^2}{6} - L_2(1 - \zeta))(S_{\mu\nu}^{(1)} + S_{\mu\nu}^{(2)}) + 2\frac{\zeta \log \zeta}{\zeta - 1}S_{\mu\nu}^{(2)} + \left(\frac{1}{\zeta} + 2 - \frac{\zeta \log \zeta}{\zeta - 1}\right)S_{\mu\nu}^{(3)}\right] A
\]
\]

\[
\delta^{m_1m_2}C_Adg^2[(\omega - 3) \pi I(\omega) + 2m^2p^2J - 2m^2p^2J_\mu + 2p^2J_\mu] \\
(5 - \omega)\delta^{m_1m_2}C_Adg^2\pi I(\omega) \\
\delta^{m_1m_2}C_Adg^2[2m^2p^2J - 2m^2p^2J_\mu + 2p^2J_\mu]
\]

\[
A = ig^2C_Ad\delta^{a_1a_2}/16\pi^2 \\
C_Ad\delta^{a_1a_2} = f_{a_1b_1}f_{a_2b_2} \\
S_{\mu\nu}^{(1)} = g^{\mu\nu}p^2 - p^\mu p^\nu \\
S_{\mu\nu}^{(2)} = p^\mu p^\nu - (n^\mu p^\nu + n^\nu p^\mu)p^2/(n \cdot p) + n^\mu n^\nu(p^2)^2/(n \cdot p)^2 \\
S_{\mu\nu}^{(3)} = [2n^\mu n^\nu p^2\bar{n} \cdot p/n \cdot p + n \cdot p(\bar{n}^\mu p^\nu + \bar{n}^\nu p^\mu) - \bar{n} \cdot p(n^\mu p^\nu + n^\nu p^\mu) - p^2(n^\mu \bar{n}^\nu + n^\nu \bar{n}^\mu)]/n \cdot \bar{n} \\
\mathcal{B} = (\bar{n} \cdot p - \mathcal{N} \bar{n} \cdot p)/n \cdot \bar{n}
\]

Table 1: One loop self-energies.

\[
G_{\lambda^+\lambda^+}^{(0)} = P_+S^{(0)}P_- = \frac{ip^+\gamma}{p^2} \\
G_{\lambda^+\lambda^-}^{(0)} = P_+S^{(0)}P_+ = \frac{ip^+p^+}{p^2} \\
G_{\lambda^-\lambda^+}^{(0)} = P_-S^{(0)}P_- = \frac{ip^-p^+}{p^2} \\
G_{\lambda^-\lambda^-}^{(0)} = P_-S^{(0)}P_+ = \frac{ip^-\gamma^+}{p^2} \\
G_{\alpha^\beta}^{(2)} = \frac{4A}{p^2}g_{\alpha\beta}\left[\frac{\pi^2}{6} - L_2(1 - \zeta)\right] \\
G_{\alpha^+}^{(2)} = -\frac{24}{p^2}g_{\alpha^+}g_{\alpha^-}\left[\frac{1}{2} - \log\left(\frac{4p^2}{p^2}\right) - \gamma + 2(\frac{\pi^2}{6} - L_2(1 - \zeta)) - \frac{\zeta \log \zeta}{\zeta - 1}\right] \\
G_{\lambda^+\lambda^+}^{(2)} = g^2C_Ad\left[\frac{4(p^+)^2}{p^2}J - \frac{4(p^+)^2}{p^2}J + \left(\frac{4p^+p^2}{p^4}J\right)\gamma - \frac{4p^+p^2}{p^4}M\right]p^\perp P_- \\
G_{\lambda^-\lambda^-}^{(2)} = g^2C_Ad\left[\frac{4p^+p^2}{p^4}J - \frac{4p^+p^2}{p^4}J + \left(\frac{4p^+p^2}{p^4}J\right)\gamma - \frac{4p^+p^2}{p^4}M\right]p^\perp P_-
\]

Table 2: 0- and 1-loop two-point Green functions
\[ \tilde{G}^{(1)}_{\alpha\beta\gamma} = \frac{igf^{abc}}{p \cdot q \cdot r} \left[ \delta_{\alpha\beta}(p - q)\gamma + \delta_{\beta\gamma}(q - r)\alpha + \delta_{\gamma\alpha}(r - p)\beta + \right. \\
- \delta_{\alpha\beta} \frac{r}{r^+}(p^+ - q^+) - \delta_{\beta\gamma} \frac{p}{p^+}(q^+ - r^+) - \delta_{\gamma\alpha} \frac{q}{q^+}(r^+ - p^+) \right] \]

\[ \tilde{G}^{(1)}_{\lambda_+\alpha\lambda_+} = \frac{igf^{abc}}{p^+ q^+ r^+} \left[ q^+ \gamma p_\perp + p^+ \gamma q_\perp - 2 \frac{p^+ q^+}{r^+} r^+ \right] \gamma^- \]

\[ \tilde{G}^{(1)}_{s\alpha s\lambda_+} = \frac{igf^{abc}}{p^+ q^+ r^+} \left[ r\alpha(p^+ - q^+) - (p_\alpha - q_\alpha) \right] \gamma^- \]

\[ \tilde{G}^{(1)}_{\lambda_+ s\lambda_+} = gf^{abc} \frac{r}{r^+} \frac{1}{p^+ q^+ r^+} (p_\gamma q_\perp - q_\gamma p_\perp) \gamma^- \]

Fig. 2: Tree-level Green functions
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