THE BREZIS–NIRENBERG PROBLEM ON $\mathbb{S}^N$, IN SPACES OF FRACTIONAL DIMENSION

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Abstract. We consider the nonlinear eigenvalue problem,

$$-\Delta_{\mathbb{S}^N} u = \lambda u + |u|^{4/(n-2)}u,$$

with $u \in H^1_0(\Omega)$, where $\Omega$ is a geodesic ball in $\mathbb{S}^n$ contained in a hemisphere. In dimension 3, Bandle and Benguria proved that this problem has a unique positive solution if and only if

$$\frac{\pi^2 - 4\theta_1^2}{4\theta_1^2} < \lambda < \frac{\pi^2 - \theta_1^2}{\theta_1^2},$$

where $\theta_1$ is the geodesic radius of the ball. For positive radial solutions of this problem one is led to an ODE that still makes sense when $n$ is a real number rather than a natural number. Here we consider precisely that problem with $3 < n < 4$. Our main result is that in this case one has a positive solution if and only if $\lambda$ is such that

$$\frac{1}{4}[(2\ell_2 + 1)^2 - (n-1)^2] < \lambda < \frac{1}{4}[(2\ell_1 + 1)^2 - (n-1)^2]$$

where $\ell_1$ (respectively $\ell_2$) is the first positive value of $\ell$ for which the associated Legendre function $P_{\ell}^{(2-n)/2}(\cos \theta_1)$ (respectively $P_{\ell}^{(n-2)/2}(\cos \theta_1)$) vanishes.

1. Introduction

In 1983, Brezis and Nirenberg [5] considered the nonlinear eigenvalue problem,

$$-\Delta u = \lambda u + |u|^{4/(n-2)}u,$$

with $u \in H^1_0(\Omega)$, where $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$, with $n \geq 3$. Among other results, they proved that if $n \geq 4$, there is a positive solution of this problem for all $\lambda \in (0, \lambda_1)$ where $\lambda_1$ is the first Dirichlet eigenvalue of $\Omega$. They also proved that if $n = 3$, there is a $\mu(\Omega) > 0$, such that for any $\lambda \in (\mu, \lambda_1)$, the nonlinear eigenvalue problem has a positive solution. Moreover, if $\Omega$ is a ball, $\mu = \lambda_1/4$.

For positive radial solutions of this problem in a (unit) ball, on is led to an ODE that still makes sense when $n$ is a real number rather than a natural number. Precisely this problem with $3 \leq n \leq 4$, was considered by E. Janelli [6]. Among other things, Janelli proved that this problem has a positive solution if and only if $\lambda$ is such that

$$j_{-(n-2)/2,1} < \sqrt{\lambda} < j_{(n-2)/2,1},$$

where $j_{\nu,k}$ denotes the $k$-th positive zero of the Bessel function $J_{\nu}$.

Here we consider the nonlinear eigenvalue problem

$$-\Delta_{\mathbb{S}^n} u = \lambda u + |u|^{4/(n-2)}u,$$  (1)
acting on $H^1_0(D)$, where $D$ is a geodesic ball in $\mathbb{S}^n$ contained in a hemisphere. Here $-\Delta_{\mathbb{S}^n}$ denotes the Laplace–Beltrami operator in $\mathbb{S}^n$ and $(n + 2)/(n - 2)$ is the critical Sobolev exponent. In dimension 3, Bandle and Benguria [4] proved that this problem has a unique positive solution if and only if

$$\frac{\pi^2 - 4\theta_1^2}{4\theta_1^2} < \lambda < \frac{\pi^2 - \theta_1^2}{\theta_1^2},$$

where $\theta_1$ is the geodesic radius of the ball.

As in the Euclidean case, for positive radial solutions of this problem one is led to an ODE that still makes sense when $n$ is a real number. This is the problem we consider in this manuscript, with $2 < n < 4$.

Henceforth, we will only consider positive radial solutions of (1) defined on geodesic caps centered at the north–pole, satisfying Dirichlet boundary conditions, i.e., $u(\theta_1) = 0$. We will denote by $\theta$ the azimuthal coordinate of a point on the sphere, with $0 \leq \theta \leq \theta_1$, and $\theta_1$ being the geodesic radius of the cap. For positive radial functions, (1) reads,

$$- u''(\theta) + (n - 1) \cot \theta u' = \lambda u + |u|^{4/(n-2)}u,$$

where $u$ is such that $u(\theta_1) = 0$. Here $' \equiv d/d\theta$, etc. As said, the ODE (2) still makes sense when $n$ is not a positive integer. In what follows we will consider $n$ as just being a parameter in equation (2), taking values in $(2, 4)$.

Our main result is the following:

**Theorem 1.1.** For any $2 < n < 4$, the boundary value problem (2), in the interval $(0, \theta_1)$, with $u'(0) = u(\theta_1) = 0$ has a positive solution if and only if $\lambda$ is such that

$$\frac{1}{4}[(2\ell_2 + 1)^2 - (n - 1)^2] < \lambda < \frac{1}{4}[(2\ell_1 + 1)^2 - (n - 1)^2]$$

where $\ell_1$ (respectively $\ell_2$) is the first positive value of $\ell$ for which the associated Legendre function $P^{(2-n)/2}_\ell(\cos \theta_1)$ (respectively $P^{(n-2)/2}_\ell(\cos \theta_1)$) vanishes.

In section 2 we begin by showing that $\ell_2 < \ell_1$. That is, the range of existence we obtain above is non-empty. We then show that the upper bound corresponds to the first Dirichlet eigenvalue of the geodesic ball. That is, we show that if $\lambda_1$ is the first positive eigenvalue of the boundary value problem

$$- u''(\theta) + (n - 1) \cot \theta u' = \lambda u$$

with $u(\theta_1) = 0$, then $\lambda_1 = \frac{1}{4}[(2\ell_1 + 1)^2 - (n - 1)^2]$.

In section 3 we show that there are solutions if $\frac{1}{4}[(2\ell_2 + 1)^2 - (n - 1)^2] < \lambda < \frac{1}{4}[(2\ell_1 + 1)^2 - (n - 1)^2]$, and in section 4 we show that there are no solutions if $\lambda \leq \frac{1}{4}[(2\ell_2 + 1)^2 - (n - 1)^2]$.

2. Preliminaries

We begin by studying the order of the first positive zeroes of $P^{\alpha}_\ell(s)$ and $P^{\alpha-\nu}_\ell(s)$ respectively, where $\nu \in (0, 1)$.

**Lemma 2.1.** Let $\alpha = (2 - n)/2$, with $2 < n < 4$. Let $\theta_1 \in (0, \pi/2)$ be fixed and choose $\ell_1$ (respectively $\ell_2$) to be the first positive value of $\ell$ for which the associated Legendre function $P^{(2-n)/2}_\ell(\cos \theta_1)$ (respectively $P^{(n-2)/2}_\ell(\cos \theta_1)$) vanishes. Then $\ell_2 < \ell_1$. 
Proof. Let $y_1 = P_{\ell_1}^\alpha(\cos \theta)$ and $y_2 = P_{\ell_2}^{-\alpha}(\cos \theta)$. Then $y_1$ and $y_2$ satisfy the equations

$$y_1'' + \cot \theta y_1' + \left(\ell_1(\ell_1 + 1) - \frac{\alpha^2}{\sin^2 \theta}\right) y_1 = 0,$$

and

$$y_2'' + \cot \theta y_2' + \left(\ell_2(\ell_2 + 1) - \frac{\alpha^2}{\sin^2 \theta}\right) y_2 = 0$$

respectively.

Let $W = y_1 y_2 - y_2 y_1$ the Wronskian of $y_2$ and $y_1$. Then $W' = y_1'y_2 - y_2'y_1$. Multiplying equation (3) by $y_2$ and equation (4) by $y_1$ and substracting it follows that

$$(\sin \theta W)' + (\Delta_1 - \Delta_2) \sin \theta y_1 y_2 = 0,$$

where $\Delta_1 = \ell_1(\ell_1 + 1)$ and $\Delta_2 = \ell_2(\ell_2 + 1)$. To prove the lemma It suffices to show that $\Delta_1 > \Delta_2$.

Integrating (5) in $\theta$ between 0 and $\theta_1$, we get,

$$\sin \theta_1 W(\theta_1) - \lim_{\theta \to 0} \sin \theta W(\theta) + (\Delta_1 - \Delta_2) C = 0$$

where $C = \int_0^{\theta_1} \sin \theta y_1(\theta) y_2(\theta) d\theta > 0$ by hypothesis. Since $W(\theta_1) = 0$, it suffices to show that $\lim_{\theta \to 0} \sin \theta W(\theta) > 0$. The series expansion of the associated Legendre functions around $\theta = 0$ is given by

$$P_\ell^\nu(\cos \theta) = \frac{1}{\Gamma(1 - \nu)} \left(\cot \frac{\theta}{2}\right)^\nu 2F_1 \left(-\ell, \ell + 1, 1 - \nu, \sin^2 \frac{\theta}{2}\right),$$

in terms of the hypergeometric function,

$$2F_1(\delta, \beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\delta)\Gamma(\beta)} \sum_{n=0}^\infty \frac{\Gamma(n + \delta)\Gamma(n + \beta)}{\Gamma(n + \gamma)n!} z^n.$$ (8)

From (7) and (8), and using that $-1 < \alpha < 0$, the behavior of $y_1 y_2$, $y_1'$ and $y_2'$ in a neighborhood of the origin to leading order is given by

$$y_1 \approx \frac{1}{\Gamma(1 - \alpha)} \left(\cot \frac{\theta}{2}\right)^\alpha,$$

$$y_2 \approx \frac{1}{\Gamma(1 + \alpha)} \left(\cot \frac{\theta}{2}\right)^{-\alpha},$$

$$y_1' \approx \frac{\alpha}{\Gamma(1 - \alpha)} \left(\cot \frac{\theta}{2}\right)^{\alpha-1} \left(\frac{-1}{2 \sin^2 \frac{\theta}{2}}\right),$$

and

$$y_2' \approx -\frac{\alpha}{\Gamma(1 + \alpha)} \left(\cot \frac{\theta}{2}\right)^{-\alpha-1} \left(\frac{-1}{2 \sin^2 \frac{\theta}{2}}\right),$$
Using this behavior of \( y_1(\theta) \), \( y_2(\theta) \), \( y'_1(\theta) \), and \( y'_2(\theta) \), for small \( \theta \), after some calculations we get

\[
\lim_{\theta \to 0} \sin \theta W(\theta) = \frac{2}{\pi} \sin \left( \frac{\pi(n-2)}{2} \right) > 0, \tag{9}
\]

for all \( 2 < n < 4 \). To obtain (9) we have used that \( \alpha = (2-n)/2 \) and the fact that

\[
\Gamma(1 + \alpha) \Gamma(1 - \alpha) = \frac{\pi \alpha}{\sin(\pi \alpha)}.
\]

\[\square\]

**Lemma 2.2.** Let \( \lambda_1 \) be the first positive eigenvalue of

\[
- u''(\theta) + (n-1) \cot \theta u' = \lambda u \tag{10}
\]

in the interval \( (0, \theta_1) \) with \( u'(0) = 0 \) and \( u(\theta_1) = 0 \). Then,

\[
\lambda_1 = \frac{1}{4} [ (2\ell_1 + 1)^2 - (n-1)^2 ] ,
\]

where \( \ell_1 \) is the first positive value of \( \ell \) for which the associated Legendre function \( P_{\ell}^{(2-n)/2}(\cos \theta_1) \) vanishes.

**Proof.** Let \( \alpha = (2-n)/2 \), and set

\[
u(\theta) = (\sin \theta)^\alpha v(\theta). \tag{11}\]

Then \( v(\theta) \) satisfies the equation,

\[
v''(\theta) + \frac{\cos \theta}{\sin \theta} v'(\theta) + \left( \lambda_1 + \alpha(\alpha - 1) - \frac{\alpha^2}{\sin^2 \theta} \right) v = 0. \tag{12}\]

In the particular case when \( n = 3 \), \( \alpha = -1/2 \) and this equation becomes,

\[
v''(\theta) + \frac{\cos \theta}{\sin \theta} v'(\theta) + \left( \lambda_1 + \frac{3}{4} - \frac{1}{4 \sin^2 \theta} \right) v = 0. \tag{13}\]

whose positive regular solution is given by,

\[
v(\theta) = C \frac{\sin \left( \sqrt{1 + \lambda_1} \theta \right)}{\sqrt{\sin \theta}}. \tag{14}\]

Hence, in this case,

\[
u(\theta) = C \frac{\sin \left( \sqrt{1 + \lambda_1} \theta \right)}{\sin \theta}. \tag{15}\]

Imposing the boundary condition \( u(\theta_1) = 0 \), in the case \( n = 3 \), we find that,

\[
\lambda_1(\theta_1) = \frac{\pi^2 - \theta_1^2}{\theta_1^2}. \tag{16}\]

Now, for any \( 3 < n < 4 \) the solutions of (13) are \( P_{\ell}^\alpha(\cos \theta) \) and \( P_{\ell}^{-\alpha}(\cos \theta) \), with

\[
\alpha = (2-n)/2, \tag{17}\]

and \( \ell \) the positive root of

\[
\ell(\ell + 1) = \lambda_1 + \alpha(\alpha - 1), \tag{18}\]

that is,

\[
\ell = \frac{1}{2} \left( \sqrt{4\lambda_1 + (n-1)^2} - 1 \right). \tag{19}\]
Taking into account (7) and (8) we see that the regular solution of (10) is given by
\[ u(\theta) = \sin^n \theta P_{\ell}^n (\cos \theta). \tag{19} \]
Finally, the boundary conditions \( u(\theta_1) = 0 \) and \( u(\theta) > 0 \) if \( 0 \leq \theta < \theta_1 \) imply that \( \ell = \ell_1 \), and so
\[ \lambda_1 = \frac{1}{4} [ (2\ell_1 + 1)^2 - (n - 1)^2 ] . \]

Here, \( \ell_1 \) is the first positive value of \( \ell \) for which the associated Legendre function \( P_{\ell}^{(2-n)/2} (\cos \theta_1) \) vanishes.

\[ \square \]

3. Existence of solutions

Let \( D \) be a geodesic ball on \( S^n \). If \( n \) is a natural number, the solutions of
\[
\begin{align*}
-\Delta_{S^n} u &= \lambda u + u^p & \text{on} & \quad D \\
u &= 0 & \text{on} & \quad \partial D,
\end{align*}
\]
where \( p = \frac{n+2}{n-2} \) correspond to minimizers of
\[ Q_{\lambda}(u) = \frac{\int_D (\nabla u)^2 q^{n-2} \, dx - \lambda \int_D u^2 q^n \, dx}{\left( \int_D u^{\frac{2n}{n-2}} q^n \, dx \right)^{\frac{n-2}{n}}} . \tag{21} \]
Here \( q(x) = \frac{2}{1+|x|^2} \), so that \( ds = q(x) dx \) is the line element of \( S^n \); and \( x \in D' \), where \( D' \) is the projection of the stereographic ball.

If \( u \) is radial, then even for fractional \( n \) we can write
\[ Q_{\lambda}(u) = \frac{\omega_n \int_0^R r^{n-1} q(r)^{n-2} u'(r)^2 \, dr - \lambda \omega_n \int_0^R r^{n-1} q(r)^n u^2(r) \, dr}{\left( \omega_n \int_0^R r^{n-1} q(r)^{\frac{2n}{n-2}} u(r)^{-\frac{2n}{n-2}} \, dr \right)^{\frac{n-2}{n}}} . \tag{22} \]
Here \( R \) corresponds to the stereographic projection of \( \theta_1 \).

As in [2], let
\[ S_{p,\lambda}(D) = \inf_{u \in H^1_0} \{ ||\nabla u||_2^2 - \lambda ||u||_2^2 \} , \tag{23} \]
so that \( S_{\lambda} \leq Q_{\lambda}(u) \), and let
\[ S = \inf_{u \in H^1_0} ||\nabla u||_2^2 . \tag{24} \]

By the Brezis–Lieb compactness lemma [3], it is known that in \( \mathbb{R}^n \), if there is a function that satisfies \( Q_{\lambda}(u) < S \), then the minimizer for \( Q_{\lambda} \) is attained. The minimizer is positive and satisfies the Brezis–Nirenberg equation. Bandle and Pelletier [3] proved that for domains in \( S^n \) contained in the hemisphere, the Brezis–Lieb lemma still holds.
Lemma 3.1. Let $2n^2/4$ and
\[
\frac{1}{4}[(2\ell_2 + 1)^2 - (n-1)^2] < \lambda < \frac{1}{4}[(2\ell_1 + 1)^2 - (n-1)^2],
\]
where \(\ell_1\) (respectively \(\ell_2\)) is the first positive value of \(\ell\) for which the associated Legendre function \(P_{\ell}^{(2-n)/2}(\cos \theta_1)\) (respectively \(P_{\ell}^{(n-2)/2}(\cos \theta_1)\)) vanishes. Then there is a positive solution to
\[
-u''(\theta) + (n - 1) \cot \theta u' = \lambda u
\]
with \(u'(0) = u(\theta_1) = 0\).

Proof. It suffices to show that there exists \(u \in H_0^1(D)\) such that \(Q_\lambda(u) < S\).

Let \(\varphi\) be a smooth function such that \(\varphi(0) = 1\), \(\varphi'(0) = 0\) and \(\varphi(R) = 0\), where \(R\) is the stereographic projection of \(\theta_1\). For \(\epsilon > 0\), let
\[
u_\epsilon(r) = \frac{\varphi(r)}{\epsilon + r^2}^{\frac{n-2}{2}}.
\]
We claim that for \(\epsilon\) small enough, \(Q_\lambda(u_\epsilon) \leq S\). In the next three claims we compute \(||\nabla u_\epsilon||^2_2\), \(||u_\epsilon||^2_{p+1}\) and \(||u_\epsilon||^2_2\).

Claim 3.2.
\[
\omega_n \int_0^R r^{n-1} q(r)^{n-2} u_\epsilon'(r)^2 dr = \omega_n \int_0^R \varphi'(r)^2 r^{3-n} q^{n-2} dr - \omega_n (n-2)^2 \int_0^R \varphi(r)^2 r^{3-n} q^{n-1} dr + \omega_n n(n-2)2^n D_n \epsilon^{2-n} + O(\epsilon^{4-n}),
\]
where
\[
D_n = \frac{1}{2} \frac{\Gamma \left( \frac{n}{2} \right)^2}{\Gamma(n)};
\]
\[
\omega_n = \frac{2n^2}{\Gamma \left( \frac{n}{2} \right)}.
\]

Proof. Let
\[
I(\epsilon) = \omega_n \int_0^R r^{n-1} q(r)^{n-2} u_\epsilon'(r)^2 dr.
\]
Then
\[
I(\epsilon) = \omega_n \int_0^R r^{n-1} q^{n-2} \left( \frac{\varphi'^2}{(\epsilon + r^2)^{n-2}} \frac{2(n-2)r \varphi \varphi'}{(\epsilon + r^2)^{n-1}} + \frac{r^2 \varphi^2(n-2)^2}{(\epsilon + r^2)^{n}} \right) dr.
\]
Integrating by parts the term with \(\varphi \varphi'\), we obtain \(I(\epsilon) = I_1 + I_2 + I_3\), where
\[
I_1(\epsilon) = \omega_n \int_0^R r^{n-1} q^{n-2} \frac{\varphi'^2}{(\epsilon + r^2)^{n-2}} dr;
\]
\[
I_2(\epsilon) = \omega_n (n-2)^2 \int_0^R r^n q^{n-3} \frac{\varphi^2}{(\epsilon + r^2)^{n-1}} dr;
\]
and
\[ I_3(\epsilon) = \omega_n (n - 2)n \epsilon \int_0^R q^{n-2} r^{n-1} \frac{\varphi^2}{(\epsilon + r^2)^n}. \]

We begin by showing that
\[ I_1(\epsilon) = \omega_n \int_0^R r^{3-n} q^{n-2} \varphi^2 dr + O(\epsilon). \]

Notice that
\[ I_1(0) = \omega_n \int_0^R r^{3-n} q^{n-2} \varphi^2 dr \]
converges for \( n < 4 \). It suffices to show that \( I_1(\epsilon) - I_1(0) = O(\epsilon) \). We can write
\[ I_1(\epsilon) - I_1(0) = \omega_n \int_0^R r^{n-1} q^{n-2} \varphi^2 \int_0^\epsilon \frac{n-2}{(a+r^2)^{n-1}} da dr. \]

But
\[ \int_0^R q^{n-2} r^{n-1} \frac{\varphi^2}{(a+r^2)^{n-1}} dr \leq \int_0^R 2^{n-2} r^{3-n} dr, \]
which converges if \( n < 4 \), thus yielding the desired result.

Next let us consider \( I_2 \). We will show that
\[ I_2(\epsilon) = -\omega_n (n - 2)^2 \int_0^R q^{n-1} \varphi^2 r^{3-n} dr + O(\epsilon^{\frac{4-n}{2}}). \]

Notice that \( q' = -q^2 r \), so that
\[ I_2(\epsilon) = -\omega_n (n - 2)^2 \int_0^R q^{n-1} r^{n+1} \varphi^2 \int_0^\epsilon \frac{n-2}{(a+r^2)^{n-1}} da dr. \]

As in the previous integral, let \( I_2(\epsilon) = I_2(0) + I_2(\epsilon) - I_2(0) \). Then it suffices to show that \( I_2(\epsilon) - I_2(0) = O(\epsilon^{\frac{4-n}{2}}) \). We can write
\[ I_2(\epsilon) - I_2(0) = \omega_n (n - 2)^2 \int_0^R q^{n-1} r^{n+1} \left[ \frac{1}{r^{2n-2}} - \frac{1}{(\epsilon + r^2)^{n-1}} \right] dr \]
\[ = \omega_n (n - 2)^2 \int_0^R q^{n-1} r^{n+1} \left[ (\varphi^2 - 1) + 1 \right] \int_0^\epsilon \frac{n-1}{(a+r^2)^n} da dr. \] (30)

Let
\[ I_{21}(\epsilon) = \int_0^R q^{n-1} r^{n+1} \int_0^\epsilon \frac{n-1}{(a+r^2)^n} da dr, \]
and
\[ I_{22}(\epsilon) = \int_0^R q^{n-1} r^{n+1} (\varphi^2 - 1) \int_0^\epsilon \frac{n-1}{(a+r^2)^n} da dr. \]

Then, since \( q^n \leq 2^n \), and making the change of variables \( r = s \sqrt{a} \), it follows that
\[ I_{21}(\epsilon) \leq 2^{n-1}(n-1) \int_0^\epsilon a^{\frac{n}{2}} \int_0^\infty s^{n+1} (1 + s^2)^n ds \, da. \]

The inner integral converges if \( n > 2 \), so it follows that

\[ I_{21}(\epsilon) = \mathcal{O}(\epsilon^{\frac{4-n}{2}}). \]

Also, since by hypothesis \( \varphi(0) = 1 \) and \( \varphi'(0) = 0 \), it follows that \( \varphi^2 - 1 \leq Cr^2 \). Thus,

\[ I_{22}(\epsilon) \leq C2^{n-1}(n-1) \int_0^\epsilon \int_0^R r^{3-n} \, dr \, da. \]

The inner integral converges if \( n < 4 \), so it follows that \( I_{22}(\epsilon) = \mathcal{O}(\epsilon) \). In particular, since \( n \geq 2 \), \( I_{22}(\epsilon) = \mathcal{O}(\epsilon^{\frac{4-n}{2}}) \) and

\[ I_2(\epsilon) - I_2(0) = \mathcal{O}(\epsilon^{\frac{4-n}{2}}). \]

Finally, we must show that

\[ I_3(\epsilon) = \omega_n n(n-2)2^{n-2}D_n \epsilon^{\frac{2+n}{2}} + \mathcal{O}(\epsilon^{\frac{4-n}{2}}). \]

Writing

\[ q^{n-2}\varphi^2 = q^{n-2}(\varphi^2 - 1) + (q^{n-2} - 2^{n-2}) + 2^{n-2}, \]

we have that \( I_3 = \omega_n (n-2)n(I_{31} + I_{32} + I_{33}) \), where

\[ I_{31} = \int_0^R \epsilon r^{n-1} q^{n-2}(\varphi^2 - 1) \, dr; \]
\[ I_{32} = \int_0^R \epsilon r^{n-1} (q^{n-2} - 2^{n-2}) \, dr; \]

and

\[ I_{33} = 2^{n-2} \int_0^R \frac{\epsilon r^{n-1}}{(\epsilon + r^2)^n} \, dr. \]

As before, since \( \varphi^2 - 1 \leq Cr^2 \), it follows that

\[ I_{31} \leq C2^{n-2} \epsilon \int_0^R \frac{r^{n+1}}{(\epsilon + r^2)^n} \, dr. \]

Letting \( r = s\sqrt{\epsilon} \), it follows that

\[ \int_0^R \frac{r^{n+1}}{(\epsilon + r^2)^n} \, dr \leq \epsilon^{\frac{2-n}{2}} \int_0^\infty \frac{s^{n+1}}{(1 + s^2)^n} ds = \mathcal{O}(\epsilon^{\frac{2-n}{2}}), \]

since the integral converges for all \( n > 2 \). Thus,

\[ I_{31} = \mathcal{O}(\epsilon^{\frac{4-n}{2}}). \]

Similarly, and since if \( 0 \leq r \leq R \) then \( 2^{n-2} - q^{n-2} \leq 2^{n-2}A(R)r^2 \), with \( A(R) = (n-2)(1 + R^2)^{n-3} \), we have that
\[ |I_{32}| \leq 2^{n-2}A(R)\epsilon \int_0^R \frac{r^{n+1}}{(\epsilon + r^2)^n} dr = \mathcal{O}(\epsilon^{\frac{4-n}{2}}). \]

Finally, making the change of variables \( r = s\sqrt{\epsilon} \), it follows that

\[ I_{33} = 2^{n-2} \epsilon^{\frac{2-n}{2}} \left( \int_0^\infty \frac{s^{n-1}}{(1+s^2)^n} ds - \int_{\frac{\sqrt{\epsilon}}{\sqrt{s}}}^\infty \frac{s^{n-1}}{(1+s^2)^n} ds \right). \]

But

\[ \int_{\frac{\sqrt{\epsilon}}{\sqrt{s}}}^\infty \frac{s^{n-1}}{(1+s^2)^n} ds \leq \int_{\frac{\sqrt{\epsilon}}{\sqrt{s}}}^\infty s^{-n-1} ds = \mathcal{O}(\epsilon^{\frac{n}{2}}). \] (32)

Moreover, notice that making the change of variables \( u = s^2 \), we can write

\[ \int_0^\infty \frac{s^{n-1}}{(1+s^2)^n} ds = \frac{1}{2} \int_0^\infty \frac{u^{\frac{n}{2}-1}}{(1+u)^n} du = \frac{1}{2} \frac{\Gamma\left(\frac{n}{2}\right)^2}{\Gamma(n)} = D_n. \] (33)

Here we have used the standard integral

\[ \int_0^\infty \frac{x^{k-1}}{(1+x)^{k+m}} dx = \frac{\Gamma(k)\Gamma(m)}{\Gamma(k+m)} \] (34)

(see, e.g., [4], equation 856.11, page 213), which holds for all \( m, k > 0 \). Thus,

\[ I_{33} = 2^{n-2} \epsilon^{\frac{2-n}{2}} D_n + \mathcal{O}(\epsilon). \]

This yields the desired estimate for \( I_3 \).

\[ \square \]

Claim 3.3.

\[ \omega_n \int_0^R r^{n-1} q^n u^2 dr = \omega_n \int_0^R q^n r^{3-n} \varphi^2 dr + \mathcal{O}(\epsilon^{\frac{4-n}{2}}). \]

Proof. Let

\[ J(\epsilon) = \omega_n \int_0^R r^{n-1} q^n \frac{\varphi^2}{(\epsilon + r^2)^{n-2}} dr. \]

Then

\[ J(0) = \omega_n \int_0^R q^n r^{3-n} \varphi^2 dr. \]

Thus, it suffices to show that \( |J(\epsilon) - J(0)| = \mathcal{O}(\epsilon^{\frac{4-n}{2}}) \). We can write

\[ |J(\epsilon) - J(0)| = \omega_n \int_0^R q^n \left[(\varphi^2 - 1) + 1\right] r^{n-1} \int_0^\epsilon \frac{r^{-2}}{(a + r^2)^{n-1}} da dr. \]

Let

\[ J_1(\epsilon) = \int_0^\epsilon \int_0^R \frac{q^n r^{n-1}}{(a + r^2)^{n-1}} dr da, \] (35)

and
\[ J_2(\epsilon) = \int_0^R (\varphi^2 - 1) q_n r^{n-1} \int_0^r \frac{1}{(a + r^2)^{n-1}} da \, dr. \]

Making the change of variables \( r = s \sqrt{\epsilon} \) in the inner integral of equation (35) we have that
\[ J_1(\epsilon) \leq 2^n \int_0^\epsilon \int_0^{2^n} \int_0^\infty \frac{s^{n-1}}{(1 + s^2)^{n-1}} ds \, da. \]

Since \( 2 < n < 4 \) it follows that \( J_1(\epsilon) = O(\epsilon^{4-n}) \).

Moreover, since \( \varphi^2 - 1 \leq C r^2 \), it follows that if \( n < 4 \), then
\[ J_2(\epsilon) \leq C \int_0^R q_n r^{n+1} \int_0^\epsilon \frac{1}{(a + r^2)^{n-1}} da \, dr \leq C 2^n \epsilon \int_0^R r^{3-n} \, dr = O(\epsilon). \]

Thus, and since \( 2 < n < 4 \), it follows that \( |J(\epsilon) - J(0)| = O(\epsilon^{4-n}) \).

\[ \square \]

Claim 3.4.

\[ \left( \omega_n \int_0^R r^{n-1} q_n u_n e^{-\frac{r}{\epsilon}} \, dr \right)^{\frac{n-2}{n}} = \omega_n^{\frac{n-2}{n}} 2^{n-2} \epsilon^{\frac{2-n}{2}} D_n^\frac{n-2}{n} + O(\epsilon^{\frac{4-n}{2}}), \]

where
\[ D_n = \frac{1}{2} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}. \]

Proof. Let
\[ K(\epsilon) = \omega_n \int_0^R r^{n-1} q_n u_n e^{-\frac{r}{\epsilon}} \, dr = \omega_n \int_0^R r^{n-1} q_n \frac{\varphi^{\frac{2n}{n-2}}}{(\epsilon + r^2)^n} \, dr. \]

Then, and since \( q_n \varphi^{\frac{2n}{n-2}} = q_n (\varphi^{\frac{2n}{n-2}} - 1) + (q_n - 2^n) + 2^n \), we can write \( K(\epsilon) = \omega_n (K_1(\epsilon) + K_2(\epsilon) + K_3(\epsilon)) \), where
\[ K_1(\epsilon) = \int_0^R \frac{r^{n-1}}{(\epsilon + r^2)^n} (\varphi^{\frac{2n}{n-2}} - 1) \, dr; \]
\[ K_2(\epsilon) = \int_0^R \frac{r^{n-1} (q_n - 2^n)}{(\epsilon + r^2)^n} \, dr; \]

and
\[ K_3(\epsilon) = 2^n \int_0^\epsilon \frac{r^{n-1}}{(\epsilon + r^2)^n} \, dr. \]

Since \( \varphi(0) = 1 \) and \( \varphi'(0) = 1 \) it follows that \( \varphi^{\frac{2n}{n-2}} - 1 \leq C r^2 \). Thus, making the change of variables \( r = s \sqrt{\epsilon} \), and since \( n > 2 \), it follows that
\[ K_1(\epsilon) \leq C 2^n \int_0^R \frac{r^{n+1}}{(\epsilon + r^2)^n} \, dr \leq C 2^n \epsilon^{\frac{2-n}{2}} \int_0^\infty \frac{s^{n+1}}{(1 + s^2)^n} \, ds = O(\epsilon^{\frac{2-n}{2}}). \]
In order to obtain an estimate for $K_2(\epsilon)$, notice that if $0 \leq r \leq R$, then $0 \leq 2^n - q^n \leq 2^n A(R) r^2$, where $A(R) = n(1 + R^2)^{n-1}$. Thus,

$$|K_2(\epsilon)| \leq 2^n A(R) \int_0^R \frac{r^{n+1}}{(\epsilon + r^2)^n} \, dr.$$ 

As before, we can make the change of variables $r = s\sqrt{\epsilon}$ to obtain

$$|K_2(\epsilon)| \leq 2^n A(R) \epsilon^{\frac{n-2}{2}} \int_0^R \frac{s^{n+1}}{(1 + s^2)^n} \, ds = \mathcal{O}(\epsilon^{\frac{n}{2}}). \quad (37)$$

Finally, we will show that

$$K_3(\epsilon) = 2^n \epsilon \omega_n D_n + \mathcal{O}(1). \quad (38)$$

In fact, making the change of variables $r = s\sqrt{\epsilon}$ we have that

$$K_3(\epsilon) = 2^n \epsilon^{-\frac{n}{2}} \left( \int_0^\infty \frac{s^{n-1}}{(1 + s^2)^n} \, ds - \int_{\frac{R}{\sqrt{\epsilon}}}^\infty \frac{s^{n-1}}{(1 + s^2)^n} \, ds \right).$$

But by equations (33) and (32) it follows that

$$\int_0^\infty \frac{s^{n-1}}{(1 + s^2)^n} \, ds = D_n,$$

and

$$\int_{\frac{R}{\sqrt{\epsilon}}}^\infty \frac{s^{n-1}}{(1 + s^2)^n} \, ds = \mathcal{O}(\epsilon^{\frac{n}{2}}).$$

It follows from equations (36), (37) and (38) that

$$K(\epsilon) = 2^n \omega_n \epsilon^{-\frac{n}{2}} D_n + \mathcal{O}(\epsilon^{\frac{n}{2}}),$$

and so

$$K(\epsilon) \frac{n-2}{n} = \omega_n \frac{n-2}{n} \epsilon^{-\frac{n}{2}} \frac{1}{2} D_n \frac{n-2}{n} + \mathcal{O}(\epsilon^{\frac{n}{2}}).$$

□

Recall that our goal is to show that if $\lambda > \frac{1}{4}[(2\ell_2 + 1)^2 - (n - 1)^2]$, then

$$Q_\lambda(u_\epsilon) = \frac{\int (\nabla u_\epsilon)^2 q^{n-2} \, dx - \lambda \int u_\epsilon^2 q^n \, dx}{\left( \int u_\epsilon^{\frac{2n}{n-2}} q^n \, dx \right)^{\frac{n-2}{n}}} < S \quad (39)$$

where $S$ is the Sobolev critical constant.

From the estimates obtained in Claim 3.2, Claim 3.3 and Claim 3.4 it follows that

$$Q_\lambda(u_\epsilon) = n(n-2)(\omega_n D_n)^{\frac{2}{n}} + \epsilon^{\frac{n-2}{n}} C_n \left[ \int_0^R r^{3-n} \left( q^{n-2} \varphi^2 - (n-2)^2 q^{n-1} \varphi^2 - \lambda q^n \varphi^2 \right) \, dr \right] + \mathcal{O}(\epsilon), \quad (40)$$
where \( C_n = \omega_n^n 2^{2-n} D_n^{\frac{2-n}{n}} \).

Notice that

\[
n(n - 2)(\omega_n D_n)^{\frac{2}{n}} = \pi n(n - 2) \left( \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma(n)} \right)^{\frac{2}{n}},
\]

which is precisely the Sobolev critical constant \( S \) (see, e.g., [10], with \( p = 2, m = n \) and \( q = \frac{2n}{n-2} \)).

Let

\[
T(\varphi) = \int_0^R r^{3-n} \left( q^{n-2} \varphi'^2 - (n - 2)^2 q^{n-1} \varphi^2 - \lambda q^n \varphi^2 \right) dr.
\]

It suffices to show that \( T(\varphi) \) is negative if \( \lambda > \frac{1}{4} [(2\ell_2 + 1)^2 - (n - 1)^2] \). In order to conclude the proof we choose \( \varphi = \varphi_1 \), where \( \varphi_1 \) is the minimizer of

\[
M(\varphi) = \int_0^R r^{3-n} \left( q^{n-2} \varphi'^2 - (n - 2)^2 q^{n-1} \varphi^2 \right) dr
\]

subject to the constraint

\[
\int_0^R r^{3-n} q^n \varphi^2 dr = 1.
\]

The minimizer of \( M(\varphi) \), \( \varphi_1 \), satisfies the Euler equation

\[
- \frac{d}{dr} \left( r^{3-n} q^{n-2} \varphi' \right) - (n - 2) r^{3-n} q^{n-1} \varphi = \mu q^n r^{3-n} \varphi_1. \tag{41}
\]

Multiplying (41) by \( \varphi_1(r) \) and integrating between 0 and \( R \) we get, after integrating by parts,

\[
\int_0^R r^{3-n} q^{n-2} \varphi_1^2 dr - (n - 2) \int_0^R r^{3-n} q^{n-1} \varphi_1^2 dr = \mu \int_0^R q^n r^{3-n} \varphi_1^2 dr.
\]

Thus, since \( \int_0^R q^n r^{3-n} \varphi_1^2 dr = 1 \), \( M(\varphi_1) = \mu \); hence

\[
T(\varphi_1) = M(\varphi_1) - \lambda = \mu - \lambda < 0
\]

if \( \lambda > \mu \).

It suffices to show that \( \mu = \frac{1}{4} [(2\ell_2 + 1)^2 - (n - 1)^2] \), where \( \ell_2 \) is the first positive value for which the associated Legendre function \( P_{\ell_2}^{(n-2)/2} (\cos \theta_1) \) vanishes.

Changing coordinates (setting \( r = \tan \theta/2 \), so that \( q = 2 \cos^2 \theta/2 \)) and letting

\[
\varphi_1(\theta) = \sin^b \left( \frac{\theta}{2} \right) \sin^a (\theta) v(\theta),
\]

where \( b = 2n - 4 \) and \( a = \frac{1}{2}(6 - 3n) \) we obtain the equation for \( v \)

\[
\ddot{v}(\theta) + \cot \theta \dot{v}(\theta) + \left( \mu + \frac{n(n - 2)}{4} - \frac{(n - 2)^2}{4 \sin^2 \theta} \right) v = 0, \tag{42}
\]
with boundary condition \( v(\theta_1) = 0 \).

**Remark 3.5.** This equation is the same equation that determines the first Dirichlet eigenvalue of the original problem (equation [12]). We choose \( a \) and \( b \) precisely so that these two equations coincide.

The solutions of equation (12) are \( P_\ell^\alpha \) and \( P_\ell^{-\alpha} \), where \( \alpha = \frac{2-n}{2} \) and \( \ell(\ell + 1) = \mu + \frac{n(n-2)}{4} \).

That is, \( \ell = \frac{1}{2} \left( \sqrt{1 + 4\mu - 4\alpha + 4\alpha^2} - 1 \right) \), and so

\[
\mu = \frac{1}{4} \left[ (2\ell + 1)^2 - (n-1)^2 \right].
\]

It follows that \( \varphi_1 \) is of the form

\[
\varphi_1 = \sin^b \left( \frac{\theta}{2} \right) \sin^a \theta (AP_\ell^\alpha + BP_\ell^{-\alpha}),
\]

where the choice of \( A \) and \( B \) must ensure the regularity of the solution. Notice that from the definition of \( a \) and \( b \) we have that \( a + b = (n-2)/2 \). Moreover, \( \alpha = (2-n)/2 \). Since \( 2 < n < 4 \), we see that in order to have regular solutions at the origin we have to choose \( A = 0 \). Finally, to satisfy the boundary condition \( u(\theta_1) = 0 \) we must choose \( \ell = \ell_2 \), which finishes the proof of the lemma. \( \square \)

### 4. Nonexistence of solutions

In this section we use a Rellich–Pohozaev [8, 9] type argument to prove the nonexistence of regular positive solutions of the Boundary Value Problem

\[
-u'' - (n-1) \cot \theta u' = u^p + \lambda u
\]

in the interval \((0, \theta_1)\), with boundary conditions \( u'(0) = 0, \ u(\theta_1) = 0 \) for a sharp range of values of \( \lambda \). Here \( 2 < n < 4 \) and \( p = (n+2)/(n-2) \) is the critical Sobolev exponent. Our main result in this section is the following Lemma.

**Lemma 4.1.** Let \( \ell_2 \) be the first positive value of \( \ell \) for which the associated Legendre function \( P_\ell^{(n-2)/2}(\cos \theta_1) \) vanishes. Then if

\[
\lambda \leq \frac{1}{4} \left[ (2\ell_2 + 1)^2 - (n-1)^2 \right],
\]

there are no positive solutions of

\[
- \frac{\sin^{n-1} \theta u'}{\sin^{n-1} \theta} = u^p + \lambda u,
\]

with boundary conditions \( u'(0) = 0, \ and \ u(\theta_1) = 0 \).

**Remark 4.2.** Notice that we have recast equation (43) in the form (44) which is more suitable in our proof.

**Proof.** Multiplying equation (43) by \( g(\theta)u'(\theta) \sin^{2n-2} \theta \), where \( g(\theta) \) is a sufficiently smooth, nonnegative function defined in the interval \((0, \theta_1)\) satisfying the boundary conditions \( g(0) = g'(0) = 0 \), we obtain

\[
- \int_0^{\theta_1} (\sin^{n-1} \theta u')' u' g \sin^{n-1} \theta \ d\theta = \int_0^{\theta_1} \left( \frac{u^{p+1}}{p+1} \right)' g \sin^{2n-2} \theta \ d\theta + \mu \int_0^{\theta_1} \left( \frac{u^2}{2} \right)' g \sin^{2n-2} \theta \ d\theta.
\]
Integrating all the terms by parts, using the boundary conditions, we have that

\[
\int_0^{\theta_1} u'^2 \left( \frac{g'}{2} \sin^{2n-2} \theta \right) \, d\theta + \int_0^{\theta_1} \frac{u^{p+1}}{p+1} \left( g' \sin^{2n-2} \theta + g(2n-2) \sin^{2n-3} \theta \cos \theta \right) \, d\theta \\
+ \lambda \int_0^{\theta_1} \frac{u^2}{2} \left( g' \sin^{2n-2} \theta + g(2n-2) \sin^{2n-3} \theta \cos \theta \right) \, d\theta = \frac{1}{2} \sin^{2n-2} \theta_1 u'/(\theta_1)^2 g(\theta_1).
\]  

(45)

On the other hand, setting \( h = \frac{1}{2} g' \sin^{n-1} \theta \) and multiplying equation (45) by \( h(\theta) u(\theta) \sin^{n-1}(\theta) \) we obtain

\[
-\int_0^{\theta_1} (\sin^{n-1} \theta u')' h \, d\theta = \int_0^{\theta_1} hu^{p+1} \sin^{n-1} \theta \, d\theta + \lambda \int_0^{\theta_1} hu^2 \sin^{n-1} \theta \, d\theta.
\]

Integrating by parts we obtain

\[
\int_0^{\theta_1} u'^2 h \sin^{n-1} \theta \, d\theta = \int_0^{\theta_1} u^{p+1} h \sin^{n-1} \theta \, d\theta \\
+ \int_0^{\theta_1} u^2 \left( \lambda h \sin^{n-1} \theta + \frac{1}{2} h' \sin^{n-1} \theta + \frac{1}{2} h'/(n-1) \sin^{n-2} \theta \cos \theta \right) \, d\theta.
\]

(46)

Notice that by our choice of \( h \), the coefficient of \( u'^2 \) in equation (45) is the same as the coefficient of \( u'^2 \) in equation (46). Finally, subtracting equation (45) from equation (46) we obtain

\[
\frac{1}{2} \sin^{2n-2} \theta_1 u'/(\theta_1)^2 g(\theta_1) = \int_0^{\theta_1} B u^{p+1} \, d\theta + \int_0^{\theta_1} A u^2 \, d\theta,
\]

(47)

where

\[
A \equiv \lambda \left( h \sin^{n-1} \theta + \frac{1}{2} g' \sin^{2n-2} \theta + g(n-1) \sin^{2n-3} \theta \cos \theta \right) \\
+ \frac{1}{2} h'' \sin^{n-1} \theta + \frac{1}{2} h'(n-1) \sin^{n-2} \theta \cos \theta,
\]

and

\[
B \equiv h \sin^{n-1} \theta + \frac{g' \sin^{2n-2} \theta}{p+1} + \frac{(2n-2)g \sin^{2n-3} \theta \cos \theta}{p+1}.
\]

(48)

(49)

Since by hypothesis \( g(\theta_1) \geq 0 \), it follows that the left hand side of equation (47) is nonnegative. In the sequel (see the Claim 4.3 and the Lemma 4.4 below), we show that for any

\[
\lambda \leq \frac{1}{4} [(2n_2 + 1)^2 - (n-1)^2],
\]

there exists a choice of \( g \) so that \( A \equiv 0 \), and \( B \) is negative. That is, we will show that for that range of \( \lambda \)'s the right hand side of equation (47) is negative, thus obtaining a contradiction. □

Substituting \( h = \frac{1}{2} g' \sin^{n-1} \theta \) in equation (48) we obtain

\[
A = \sin^{2n-2} \theta \left[ \frac{g''}{4} + \frac{3}{4} g''/(n-1) \cot \theta \\
+ g' \left( (n-1)(2n-3) \cot^2 \theta - \frac{n-1}{4} + \lambda \right) + \lambda g(n-1) \cot \theta \right].
\]

(50)
Finally, making the change of variables \( g = f / \sin^2 \theta \) we obtain
\[
A = \sin^{2n-4} \theta \left[ \frac{f'''}{4} + \frac{3}{4} (n-3) \cot \theta f'' + f' \left( \frac{(n-3)(2n-11)}{4} \cot^2 \theta + \frac{7-n}{4} + \lambda \right) \right. \\
+ f \left( (n-3)(4-n) \cot^3 \theta + 2(n-3) \cot \theta + \lambda (n-3) \cot \theta \right].
\] (51)

Claim 4.3. For any \( 2 < n < 4 \), the function
\[
z(\theta) = \sin^{4-n} \theta P_\ell^\alpha (\cos \theta) P_\ell^{-\alpha} (\cos \theta),
\]
with \( \alpha = (2-n)/2 \) and \( \ell = \frac{1}{2} \left( \sqrt{4\lambda + (n-1)^2} - 1 \right) \), is a solution of
\[
\frac{f'''}{4} + \frac{3}{4} (n-3) \cot \theta f'' + f' \left( \frac{(n-3)(2n-11)}{4} \cot^2 \theta + \frac{7-n}{4} + \lambda \right) \\
+ f \left( (n-3)(4-n) \cot^3 \theta + 2(n-3) \cot \theta + \lambda (n-3) \cot \theta \right) = 0.
\] (52)

Proof. Let \( y_1(\theta) = P_\ell^\alpha (\cos \theta) \) and \( y_2(\theta) = P_\ell^{-\alpha} (\cos \theta) \). Then \( y_1 \) and \( y_2 \) are solutions to
\[
y''(\theta) + \cot \theta y'(\theta) + k(\theta)y(\theta) = 0,
\] (53)
where
\[
k(\theta) = \ell (\ell + 1) - \frac{\alpha^2}{\sin^2 \theta}.
\] (54)

Let \( v(\theta) = y_1(\theta) y_2(\theta) \). Then, it follows from (53) that
\[
y''_1 y_2 + y''_2 y_1 = -\cot \theta v' - 2kv,
\]
which in turn implies
\[
v'' = -2kv - \cot \theta v' + 2y_1 y_2'.
\]
Similarly, and since
\[
y''_1 y_2' + y''_2 y_1' = -2 \cot \theta y_1 y_2' - kv',
\]
we obtain
\[
v''' + 3 \cot \theta v'' + v' \left( 4k - \csc^2 \theta + 2 \cot^2 \theta \right) + 4v \left( \alpha^2 \cot \theta \csc^2 \theta + k \cot \theta \right) = 0.
\] (55)

Now, we make the change of variables \( v \to f \) given by
\[
f(\theta) = \sin^{4-n} \theta v(\theta)
\]
in equation (55) and multiply the resulting equation through by \( \sin^{n-4} \theta \). Setting \( \alpha = (2-n)/2 \),
\[
\ell = \frac{1}{2} \left( \sqrt{4\lambda + (n-1)^2} - 1 \right)
\]
(which is the positive solution of \( 4\ell (\ell + 1) = 4\lambda + n^2 - 2n \)) and, using (52) we see that \( f \) satisfies (52). This finishes the proof of Claim 4.3. \( \square \)

Lemma 4.4. Let \( \alpha = (2-n)/2 \), \( \ell = \frac{1}{2} \left( \sqrt{4\lambda + (n-1)^2} - 1 \right) \), and \( \mu \) be the first positive value of \( \ell \) for which \( P_\ell^\alpha (\cos \theta_1) \) vanishes. Consider
\[
B \equiv h \sin^{n-1} \theta + \frac{g' \sin^{2n-2} \theta}{p+1} + \frac{(2n-2)g \sin^{2n-3} \theta \cos \theta}{p+1},
\] (56)
where \( h(\theta) = \frac{1}{2} \theta' \sin^{n-1} \theta \), \( g(\theta) = f(\theta) \sin^{-2} \theta \) and \( f(\theta) = \sin^{4-n} \theta P_\ell^\alpha (\cos \theta) P_\ell^{-\alpha} (\cos \theta) \). Then \( B \) is negative on \([0, \mu]\).
Proof. The associated Legendre functions satisfy the following raising and lowering relations (see, e.g., [1], equation 8.1.2, pp. 332), which we will use repeatedly in the proof of this lemma.

\[ \dot{P}_\ell^\alpha(\cos \theta) = -\frac{P_{\ell+1}^\alpha}{\sin \theta} - \frac{\alpha \cos \theta P_\ell^\alpha}{\sin^2 \theta} \]  

(57)

and

\[ \dot{P}_{\ell+1}^\alpha(\cos \theta) = \frac{1}{\sin^2 \theta} \left( (\ell + \alpha + 1)(\ell - \alpha) \sin \theta P_\ell^\alpha + (\alpha + 1) \cos \theta P_{\ell+1}^\alpha \right). \]  

(58)

Notice that in the two previous equations, \( \dot{P}_\ell^\alpha \) means the derivative of \( P_\ell^\alpha \) with respect to its argument, therefore,

\[ \frac{d}{d\theta} P_\ell^\alpha(\cos \theta) = -\sin \theta \dot{P}_\ell^\alpha(\cos \theta). \]

After substituting for \( h, g \) and \( f \) we can write

\[ B = -\sin \theta \frac{n}{\sin^2 \theta} P_\ell^\alpha P_{\ell+1}^\alpha - \frac{\alpha}{\sin \theta} P_\ell^\alpha \dot{P}_{\ell+1}^\alpha. \]  

(59)

Hence, it suffices to show that \( \dot{P}_\ell^\alpha P_{\ell+1}^\alpha + P_{\ell+1}^\alpha \dot{P}_\ell^\alpha > 0 \) on \([0, \mu)\). Because of Lemma 2.1, \( P_\ell^\alpha P_{\ell+1}^\alpha \) is positive, on this interval. Thus, we can write this inequality as

\[ \frac{\dot{P}_\ell^\alpha}{P_\ell^\alpha} + \frac{\dot{P}_{\ell+1}^\alpha}{P_{\ell+1}^\alpha} > 0. \]  

(60)

It follows from equation (57) that

\[ \frac{\dot{P}_\ell^\alpha}{P_\ell^\alpha} + \frac{\dot{P}_{\ell+1}^\alpha}{P_{\ell+1}^\alpha} = -\frac{1}{\sin \theta} \frac{P_{\ell+1}^\alpha}{P_\ell^\alpha} - \frac{1}{\sin \theta} \frac{P_{\ell+1}^\alpha}{P_{\ell+1}^\alpha}. \]  

(61)

Given, the identity (59) above, in order to prove (60) it is convenient to introduce the function,

\[ y_\nu(\theta) = -\frac{1}{\sin \theta} \frac{P_{\ell+1}^\alpha}{P_\ell^\alpha(\cos \theta)} - \frac{\nu}{2 \sin^2 \frac{\theta}{2}}. \]  

(62)

In the sequel, we study the behavior of \( y_\nu(\theta) \) on \([0, \mu)\). In particular, we will show that \( y_\nu \) is positive on this interval if \(-1 < \nu < 1\). This in turn will imply that

\[ \frac{\dot{P}_\ell^\alpha}{P_\ell^\alpha} + \frac{\dot{P}_{\ell+1}^\alpha}{P_{\ell+1}^\alpha} = y_\alpha(\theta) + y_{-\alpha}(\theta) > 0. \]

The series expansion of the Legendre associated functions is given by the following expression:

\[ P_\ell^\nu(\cos \theta) = \frac{1}{\Gamma(1 - \nu)} \left( \cot \frac{\theta}{2} \right)^\nu \frac{\Gamma(\gamma)}{\Gamma(\delta) \Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(n + \delta) \Gamma(n + \beta)}{\Gamma(n + \gamma) n!} (-\nu, \ell + 1 - \nu, \sin^2 \frac{\theta}{2})_n. \]  

(63)

where

\[ 2F_1(\delta, \beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\delta) \Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(n + \delta) \Gamma(n + \beta)}{\Gamma(n + \gamma) n!} z^n. \]  

(64)

We can write this last function as

\[ 2F_1(\delta, \beta, \gamma, z) = 1 + \frac{\delta \beta}{\gamma} z + \frac{\delta(\delta + 1) \beta(\beta + 1)}{2\gamma(\gamma + 1)} z^2 + O(z^3), \]
so that
\[ P_\nu^\ell(\cos \theta) = \frac{1}{\Gamma(1-\nu)} \cot^\nu \theta \left( 1 - \frac{\ell(\ell + 1)}{1-\nu} \sin^2 \theta + \frac{\ell(\ell^2 - 1)(\ell + 2)}{2(1-\nu)(2-\nu)} \sin^4 \theta + \mathcal{O}\left(\sin^6 \theta^2\right) \right). \]

It follows that
\[ \frac{P_{\nu+1}^\ell(\cos \theta)}{P_\nu^\ell(\cos \theta)} = \frac{\Gamma(1-\nu)}{\Gamma(-\nu)} \cot^\nu \theta \left( 1 + E \sin^2 \theta + \mathcal{O}\left(\sin^4 \theta^2\right) \right), \tag{65} \]

where
\[ E = \frac{\ell(\ell + 1)}{\nu(1-\nu)} \]

(Here we used that \( \Gamma(1-\nu) = -\nu \Gamma(-\nu) \)). Thus, it follows from equations (62) and (65) that
\[ y_\nu(\theta) = \frac{\nu}{2} \left( E + F \sin^2 \theta + \mathcal{O}\left(\sin^4 \theta^2\right) \right). \]

In particular,
\[ \lim_{\theta \to 0} y_\nu(\theta) = \frac{\ell(\ell + 1)}{2(1-\nu)} > 0, \]

since we are considering \( \ell > 0 \) and \(-1 < \nu < 1 \). We will now show by contradiction that there is no point on the interval \([0, \mu)\) where \( y_\nu \) changes sign. To do so, we first derive a Riccati equation for \( y_\nu \). It follows from equation (62) that
\[ \dot{y}_\nu = \frac{\cos \theta}{\sin^2 \theta} \frac{P_{\nu+1}^\ell}{P_\nu^\ell} + \frac{\dot{P}_{\nu+1}^\ell}{P_\nu^\ell} - \frac{P_{\nu+1}^\ell \dot{P}_\nu^\ell}{(P_\nu^\ell)^2} + \frac{\nu(1 + \cos \theta)^2}{\sin^3 \theta}. \tag{66} \]

Using equations (57) and (58) in equation (66) we obtain
\[ \dot{y}_\nu = \frac{1}{\sin \theta} \left( \frac{P_{\nu+1}^\ell}{P_\nu^\ell} \right)^2 + \frac{2(\nu + 1) \cos \theta}{\sin^2 \theta} \frac{P_{\nu+1}^\ell}{P_\nu^\ell} + \frac{(\ell + \nu + 1)(\ell - \nu)}{\sin \theta} + \frac{\nu(1 + \cos \theta)^2}{\sin^3 \theta}. \tag{67} \]

Finally, using equation (62) to solve for \( P_{\nu+1}^\ell/P_\nu^\ell \) we obtain the following Riccati equation for \( y_\nu \),
\[ \dot{y}_\nu = \sin \theta y_\nu^2 + \frac{2y_\nu}{\sin \theta} (\nu - \cos \theta) + \frac{\ell(\ell + 1)}{\sin \theta}. \tag{68} \]

Since \( y_\nu(0) > 0 \), and \( y_\nu(\theta) \) is continuous in \( \theta \), If \( y_\nu(\theta) \) were to cross \( y_\nu = 0 \), there would exist a point, \( \theta^* \), such that \( y_\nu(\theta^*) = 0 \) and \( \dot{y}_\nu(\theta^*) < 0 \). But from equation (68) we would then have
\[ \dot{y}_\nu(\theta^*) = \frac{\ell(\ell + 1)}{\sin \theta^*} > 0, \]

arriving at a contradiction. We conclude that \( y_\nu \) is positive on \([0, \mu)\).

\[ \square \]

**References**

[1] M. Abramowitz and I. A. Stegun, Editors, *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York, 1965.

[2] C. Bandle and R. Benguria, *The Brézis-Nirenberg problem on \( S^3 \)*, J. Differential Equations, 178 (2002), pp. 264–279.

[3] C. Bandle and L. A. Peletier, *Best Sobolev constants and Emden equations for the critical exponent in \( S^3 \)*, Math. Ann., 313 (1999), pp. 83–93.

[4] H. Brézis and E. Lieb, *A Relation Between Pointwise Convergence of Functions and Convergence of Functionals*, Proceedings of the American Mathematical Society 88 (1983), pp. 486–490.
[5] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., 36 (1983), pp. 437–477.

[6] H. B. Dwight, *Tables of integrals and other mathematical data*, 4th ed, The Macmillan Company, New York, 1961.

[7] E. Jannelli, *The role played by space dimension in elliptic critical problems*, J. Differential Equations, 156 (1999), pp. 407–426.

[8] S. I. Pohozaev, *On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$*, Dokl. Akad. Nauk SSSR, 165 (1965), pp. 36–39 (In Russian).

[9] F. Rellich, *Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral*, Math. Z, 46 (1940), pp. 635–636.

[10] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4), 110 (1976), pp. 353–372.

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