Abelian Ideals and the Variety of Lagrangian Subalgebras

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Abstract

For a semisimple algebraic group $G$ of adjoint type with Lie algebra $g$ over the complex numbers, we establish a bijection between the set of closed orbits of the group $G \ltimes g^*$ acting on the variety of Lagrangian subalgebras of $g \ltimes g^*$ and the set of abelian ideals of a fixed Borel subalgebra of $g$. In particular, the number of such orbits equals $2^{r_{k\theta}}$ by Peterson's theorem on abelian ideals.

Contents

1 Introduction  1
2 Notation and Preliminaries  3
3 The Variety of Lagrangian Subalgebras  4
4 Closed Orbits  5

1 Introduction

In the 1990’s, Dale Peterson proved the remarkable and surprising assertion that the number of abelian ideals of a Borel subalgebra of a semisimple Lie algebra is $2^l$, where $l$ is the rank of the Lie algebra. Although Peterson never published his result, his argument was widely popularized by Kostant, who further developed the theory in [11], and the theory was further developed by many others, including in [1, 2, 3, 12, 13, 14]. In this paper, we find a new connection between the theory of abelian ideals and the variety of Lagrangian subalgebras, which is a variety that arises naturally in Poisson geometry.

Throughout this paper, we work over the complex numbers $\mathbb{C}$. Let $G$ be a semisimple algebraic group of adjoint type with Lie algebra $g$. There is a bi-vector field $\pi_{st}$, known as the standard Poisson bi-vector field, on $G$, making it a Poisson Lie group. Infinitesimal information of $(G, \pi_{st})$ near the identity element of the group $G$ is captured by the Manin triple $(g \oplus g, g_{st}, g_{st})$ (c.f. [7, 8]). Here, the symbol $g_\Delta$ stands for the diagonal Lie subalgebra of $g \oplus g$ and $g_{st}$ is defined to be the Lie subalgebra of dimension $\dim g$ equipped with a symmetric nondegenerate invariant bilinear form $\langle \ , \ \rangle$. A Lie subalgebra $l$ of $g \oplus g$ such that $\langle l, l \rangle = 0$ is called a Lagrangian subalgebra of $g \oplus g$. The variety $L(g \oplus g)$ of Lagrangian subalgebras of $g \oplus g$ has been extensively studied in algebraic and Poisson geometry:

- In [7, 8], the first author and Lu discussed a $(G \times G)$-action on $L(g \oplus g)$ and described explicitly the $(G \times G)$-orbits and their closures in $L(g \oplus g)$. Orbits of many subgroups of $G \times G$, for example the diagonal subgroup $G_\Delta$, are also explicitly described.
• It follows directly from the fundamental work of De Concini and Procesi [4] (see [6, 8]) that there is an embedding of $G$ into $L(g \oplus g)$ (as a single $(G \times G)$-orbit) so that the closure of the image of this embedding is isomorphic to the wonderful compactification of $G$.

• The variety $L(g \oplus g)$ can be thought of as a ‘universal Poisson homogeneous space’ for the Poisson Lie group $(G, \pi_{sl})$ as well as its Poisson dual group ([5, 7, 8]).

The zero bi-vector field on $G$ makes $G$ into a Poisson Lie group. The corresponding Manin triple is $(g \ltimes g^*, g, g^*)$. Following the construction from Section 2 of [7], we consider the variety $L = L(g \ltimes g^*)$, called the variety of Lagrangian subalgebras of $g \ltimes g^*$. We note that the construction of $L(g \ltimes g^*)$ is the analogue of the construction of $L(g \oplus g)$ when we substitute the Manin triple $(g \ltimes g^*, g, g^*)$ in place of $(g \oplus g, g^\Delta, g^\Delta_\ast)$. We view $g^*$ as an additive algebraic group and let $D$ denote the semidirect product $G \ltimes g^*$, viewed as an algebraic group. There is an action of $D$ on $L$ analogous to the $(G \times G)$-action on $L(g \oplus g)$, and the purpose of this paper is to classify closed $D$-orbits in $L$.

A Lie ideal $\mathfrak{a}$ of a Borel subalgebra of $g$ such that $[\mathfrak{a}, g] = 0$ is called an abelian ideal of the Borel subalgebra. The main result of this paper is the following

**Theorem 1.1.** There is a bijection between the set of closed $D$-orbits in $L$ and the set of abelian ideals of a fixed Borel subalgebra of $g$.

We will make this bijection explicit after introducing some notation. Along the way, we will see that the geometry of $L$ is much harder to understand than that of $L(g \oplus g)$. The subgroup $G \cong G \times \{0\}$ of $D$ is an analogue of the subgroup $G_\Delta$ of $G \times G$. We will see that a complete understanding of the orbits of $G$ in $L$ contains as a subquestion the question of classifying all finite dimensional Lie algebras. So our current situation is quite different than that of [7, 8]. It is for this reason that we restricted our attention to closed $D$-orbits in $L$. In a subsequent publication, we will study more general $D$-orbits and orbits of certain subgroups of $D$, using a degeneration of $L(g \oplus g)$ into $L$.

Peterson proved the remarkable result that the number of abelian ideals of a fixed Borel subalgebra of $g$ equals $2^{hk_g}$. As a consequence of Theorem 1.1 and Peterson’s result, we have the following

**Theorem 1.2.** The number of closed $D$-orbits in $L$ is exactly $2^{hk_g}$.

Denote by $(G, 0)$ the Poisson Lie group $G$ equipped with the zero Poisson bi-vector field. We note that the Poisson dual group of $(G, 0)$ is $(g^*, \pi_{KK})$, where $\pi_{KK}$ stands for the Kirillov-Kostant Poisson bi-vector field on $g^*$. In our context, the Poisson Lie groups $(G, 0)$ and $(g^*, \pi_{KK})$ play a similar role as that of $(G, \pi_{sl})$ and its Poisson dual group in [7, 8]. The theory of abelian ideals of Borel subalgebras has been initiated by Kostant in [11], and it is a well developed theory by now, c.f. [1, 2, 3, 12, 13, 14]. However, the proofs of several key results of this theory are combinatorial in nature. We hope that Poisson geometry may provide an alternative understanding of these results.

In the paper [9], a Poisson structure $\pi$ is introduced on $G$ for which the symplectic leaves are related to conjugacy classes and the Bruhat decomposition. In a rough sense, it is reasonable to regard $(g^*, \pi_{KK})$, resp. $L$, as an additive analogue of $(G, \pi)$, resp. $L(g \oplus g)$. Since many important geometric objects associated to $G$ can be realized as closed subvarieties of $L(g \oplus g)$, c.f. [6, 7, 8], one can define these objects in the additive setting as well. For example, one can study the geometry of the ‘wonderful compactification’ of $g^*$, the closure of a single $D$-orbit in $L$, and of a Cartan subalgebra of $g$. The latter is a counterpart of the wonderful compactification of a Cartan subgroup of $G$ and potentially leads to a theory of additive toric varieties. These topics will be discussed in a separate paper.

This paper is organized as follows. In Section 2 we introduce notation and define various basic objects. In Section 3, we construct the variety $L$ of Lagrangian subalgebras of $g \ltimes g^*$. We recall the algebraic parametrization of points of $L$ due to Karolinsky and Stolin [10]. In Section 4 we prove Theorem 1.1. Our proof is based on the Karolinsky-Stolin parametrization.

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2 Notation and Preliminaries

Let $G$ be a semisimple algebraic group of adjoint type with Lie algebra $\mathfrak{g}$. By $\mathfrak{g}^*$ we mean the dual space of $\mathfrak{g}$. In what follows, $\mathfrak{g}^*$ will also be viewed as an algebraic group and an abelian Lie algebra, where the group operation is the usual addition in $\mathfrak{g}^*$. The adjoint (resp. coadjoint) action of $\mathfrak{g}$ on $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) will be denoted by $\text{Ad}$ (resp. $\text{Ad}^*$). The adjoint (resp. coadjoint) action of $\mathfrak{g}$ on $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) will be denoted by $\text{ad}$ (resp. $\text{ad}^*$). Elements of $\mathfrak{g}$ will be denoted by Roman letters $x, y, \ldots$ and elements of $\mathfrak{g}^*$ will be denoted by Greek letters $\alpha, \beta, \ldots$.

We consider the semidirect product algebraic group $D := G \ltimes \mathfrak{g}^*$, which has the following structure. As a set, $D$ is the Cartesian product $G \times \mathfrak{g}^*$. Multiplication and inversion in $D$ are given by:

\begin{equation}
(g, \alpha)(g', \alpha') := (gg', \text{Ad}_{g'}^{-1}\alpha + \alpha') \quad \text{for all } g, g' \in G \text{ and } \alpha, \alpha' \in \mathfrak{g}^*
\end{equation}

\begin{equation}
(g, \alpha)^{-1} := (g^{-1}, -\text{Ad}_g^*\alpha) \quad \text{for all } g \in G \text{ and } \alpha \in \mathfrak{g}^*.
\end{equation}

The identity element of $D$ is $(e, 0)$, where $e$ stands for the identity element of $G$. It is easy to verify that the homomorphism of algebraic groups $G \to D$ (resp. $\mathfrak{g}^* \to D$) given by $g \mapsto (g, 0)$ (resp. $\alpha \mapsto (e, \alpha)$) is an embedding of $G$ (resp. $\mathfrak{g}^*$) into $D$. In what follows, we will view $G$ (resp. $\mathfrak{g}^*$) as an algebraic subgroup of $D$ via this homomorphism.

Consider the Lie algebra $\mathfrak{d}$ of $D$, which is the semidirect sum $\mathfrak{d} = \mathfrak{g} \ltimes \mathfrak{g}^*$. As a vector space, it is the Cartesian product $\mathfrak{g} \times \mathfrak{g}^*$. The Lie bracket in $\mathfrak{d}$ is given by:

\[ [(x, \alpha), (y, \beta)] := ([x, y], \text{ad}_x^*\beta - \text{ad}_y^*\alpha) \quad \text{for all } x, y \in \mathfrak{g} \text{ and } \alpha, \beta \in \mathfrak{g}^*. \]

As above, $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) will be viewed as a Lie subalgebra of $\mathfrak{d}$ via the homomorphism $\mathfrak{g} \to \mathfrak{d}$ (resp. $\mathfrak{g}^* \to \mathfrak{d}$) given by $x \mapsto (x, 0)$ (resp. $\alpha \mapsto (0, \alpha)$). The exponential map $\text{Exp} : \mathfrak{d} \to D$ is given by:

\begin{equation}
(x, \alpha) \mapsto (\text{exp}(x), \alpha),
\end{equation}

where $\text{exp} : \mathfrak{g} \to G$ stands for the exponential map for $G$.

By abuse of notation, the adjoint action of $D$ on $\mathfrak{d}$ will also be denoted by $\text{Ad}$. Using formulas (1) and (2), one easily verifies that:

\[ \text{Ad}_{(g, \alpha)}(x, \beta) = (\text{Ad}_g x, -\text{Ad}_g^*\text{ad}_x^*\alpha + \text{Ad}_g^*\beta) \quad \text{for all } g \in G, x \in \mathfrak{g} \text{ and } \alpha, \beta \in \mathfrak{g}^*. \]

In particular, for $g \in G, x \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$, we have:

\begin{equation}
\text{Ad}_g(x, \beta) = (\text{Ad}_g x, \text{Ad}_g^*\beta) \\
\text{Ad}_\alpha(x, \beta) = (x, -\text{ad}_x^*\alpha + \beta).
\end{equation}

Define a bilinear form $(\ , \ ) : \mathfrak{d} \otimes \mathfrak{d} \to \mathbb{C}$ by:

\[ ((x, \alpha), (y, \beta)) := \alpha(y) + \beta(x), \]

for all $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$. It is easily verified that $(\ , \ )$ is symmetric, nondegenerate and $D$-invariant. Here, $D$-invariance means:

\begin{equation}
(\text{Ad}_{(g, x)}(y, \beta), \text{Ad}_{(g, x)}(z, \gamma)) = ((y, \beta), (z, \gamma)),
\end{equation}

for all $g \in G, x, y, z \in \mathfrak{g}$ and $\beta, \gamma \in \mathfrak{g}^*$.

Recall that for a vector space $V$ equipped with a symmetric nondegenerate bilinear form $(\ , \ )$, a vector subspace $W$ of $V$ is called isotropic if $\langle w, w' \rangle = 0$ for all $w, w' \in W$. An isotropic subspace $W$ of $V$ is called Lagrangian if $W$ is maximal, with respect to inclusion, among all isotropic subspaces of $V$. When $V$ is even dimensional, an isotropic subspace $W$ of $V$ is Lagrangian if and only if $\dim W = \frac{1}{2} \dim V$.

A Lie subalgebra $\mathfrak{l}$ of $\mathfrak{d}$ is called a Lagrangian subalgebra if $\mathfrak{l}$ is a Lagrangian vector subspace of $\mathfrak{d}$ with respect to the bilinear form on $\mathfrak{d}$ introduced above. Although we will not use it in this paper, it is worth pointing out that $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$, together with the bilinear form $(\ , \ )$ on $\mathfrak{d}$, is a Manin triple. This amounts to saying that $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lagrangian subalgebras of $\mathfrak{d}$ and $\mathfrak{d}$ is the direct sum of $\mathfrak{g}$ and $\mathfrak{g}^*$ as vector spaces, c.f. [5, 7, 8] for details.
3 The Variety of Lagrangian Subalgebras

Let $a$ be a Lie subalgebra of $g$ and $a^*$ its dual space. We write $Z_{CE}^1(a, a^*)$ for the vector space of Chevalley-Eilenberg 1-cocycles on $a$ with coefficients in $a^*$, where $a$ acts on $a^*$ by the coadjoint action. Specifically, $Z_{CE}^1(a, a^*)$ consists of linear maps $f : a \to a^*$ such that

$$\text{ad}_x^* f(y) - \text{ad}_y^* f(x) - f([x, y]) = 0$$

for all $x, y \in a$. \hspace{1cm} (5)

We say that $f \in Z_{CE}^1(a, a^*)$ is skew if

$$f(x)(y) + f(y)(x) = 0$$

for all $x, y \in a$. \hspace{1cm} (6)

**Definition 3.1.** Let $a$ be a Lie subalgebra of $g$ and $f \in Z_{CE}^1(a, a^*)$ a skew element. We define

$$l(a, f) := \{ (x, \alpha) \in \mathfrak{d} : x \in a, f(x) = \alpha|_a \},$$

where $\alpha|_a$ stands for the restriction of $\alpha$ to $a$.

Let $a$ and $f$ be as in Definition 3.1. It follows from (5) (resp. (6)) that $l(a, f)$ is a Lie subalgebra (resp. isotropic vector subspace) of $\mathfrak{d}$. It is clear that $\dim l(a, f) = \frac{1}{2} \dim \mathfrak{d}$. Therefore, $l(a, f)$ is a Lagrangian subalgebra of $\mathfrak{d}$.

The following result of Karolinsky and Stolin establishes a bijection between points of $L$ and pairs $(a, f)$ as in Definition 3.1.

**Theorem 3.1.** [10] The Lagrangian subalgebras of $\mathfrak{d}$ are exactly the Lagrangian subalgebras $l(a, f)$, for $(a, f)$ as in Definition 3.1.

Let $(a, f)$ and $(a', f')$ be as in Definition 3.1. It is easily verified that $l(a, f) = l(a', f')$ if and only if $a = a'$ and $f = f'$.

Let $Gr(n, \mathfrak{d})$ be the Grassmannian of $n$-dimensional vector subspaces of $\mathfrak{d}$, where $n := \dim g = \frac{1}{2} \dim \mathfrak{d}$. For an $n$-dimensional vector subspace of $\mathfrak{d}$, the property of being a Lie subalgebra, resp. being an isotropic vector subspace, of $\mathfrak{d}$ is a closed condition. Therefore, the set of Lagrangian subalgebras of $\mathfrak{d}$ has a natural structure of a reduced closed subvariety of $Gr(n, \mathfrak{d})$, to be denoted by $L$. Since $Gr(n, \mathfrak{d})$ is a projective variety, the variety $L$ is projective as well. We will call $L$ the variety of Lagrangian subalgebras of $\mathfrak{d}$.

Let $S$ be a subset of $\mathfrak{d}$. For any $d \in D$, we write $Ad_d S$ for the set $\{ Ad_d s : s \in S \}$. Since the $D$-action on $\mathfrak{d}$ preserves the symmetric form $(\cdot, \cdot)$, it follows that if $l$ is a Lagrangian subalgebra of $\mathfrak{d}$, then so is $Ad_d l$ for any $d \in D$. Therefore, we obtain an action, still denoted by $Ad$, of $D$ on $L$.

We deduce some formulas for the action of $D$ on $L$. Let $a$ be a Lie subalgebra of $g$ and $f \in Z_{CE}^1(a, a^*)$ a skew element. For $g \in G$, we define a linear map $g.f : Ad_g a \to (Ad_g a)^*$ by

$$(g.f)(x)(y) := f(Ad_{g^{-1}} x)(Ad_{g^{-1}} y)$$

for all $x, y \in Ad_g a$. A simple computation shows that $g.f$ is a skew element of $Z_{CE}^1(Ad_g a, (Ad_g a)^*)$.

For $\alpha \in g^*$, we define a linear map $f_\alpha : a \to a^*$ by

$$f_\alpha(x) := - \text{ad}_x^*(\alpha|_a)$$

for all $x \in a$. Another simple computation shows that $f_\alpha$ is a skew element of $Z_{CE}^1(a, a^*)$.

**Lemma 3.2.** Let $a$ be a Lie subalgebra of $g$ and $f \in Z_{CE}^1(a, a^*)$ a skew element. Then,

1. For any $g \in G$, we have

$$Ad_g l(a, f) = l(Ad_g a, g.f).$$

2. For any $\alpha \in g^*$, we have

$$Ad_\alpha l(a, f) = l(a, f + f_\alpha).$$


Proof. We only prove the first statement. The proof of the second statement is analogous.

Note that $\text{Ad}_g l(a, f)$ and $l(\text{Ad}_g a, g, f)$ are $n$-dimensional vector subspaces of $\mathfrak{a}$, hence it suffices to show that $\text{Ad}_g l(a, f) \subseteq l(\text{Ad}_g a, g, f)$. Let $(x, \beta) \in l(a, f)$. Then we have

$$\text{Ad}_g(x, \beta) = (\text{Ad}_g x, \text{Ad}_g^\ast \beta)$$

by formula (3). For any $y \in \mathfrak{a}$, we have

$$(g.f)(\text{Ad}_g x)(\text{Ad}_g y) = f(x)(y) = \beta(y) = (\text{Ad}_g^\ast \beta)(\text{Ad}_g y).$$

Hence, we have

$$(g.f)(\text{Ad}_g x) = (\text{Ad}_g^\ast \beta)|_{\text{Ad}_g \mathfrak{a}},$$

i.e., $(\text{Ad}_g x, \text{Ad}_g^\ast \beta) \in l(\text{Ad}_g a, g, f)$, proving the desired inclusion. \qed

Remark. Let $\mathfrak{a}$ be a Lie subalgebra of $\mathfrak{g}$. For any element $g \in G$, by Lemma 3.2, we have

$$\text{Ad}_g l(a, 0) = l(\text{Ad}_g a, 0).$$

Thus the $G$-orbit through $l(a, 0)$ is

$$\{l(\text{Ad}_g a, 0) : g \in G\}.$$ 

It follows from this and the observation that $l(a, f) = l(a', f')$ if and only if $a = a'$ and $f = f'$, that for any Lie subalgebras $\mathfrak{a}, \mathfrak{a'}$ of $\mathfrak{g}$, the Lagrangian subalgebras $l(a, 0)$ and $l(a', 0)$ are in the same $G$-orbit if and only if $\mathfrak{a}$ and $\mathfrak{a'}$ are conjugate by an element of $G$. Consequently, if we would like to classify all $G$-orbits in $\mathcal{L}$, we have to first classify Lie subalgebras of $\mathfrak{g}$ up to conjugation by elements of $G$. By Ado’s theorem, every finite dimensional Lie algebra is isomorphic to a Lie subalgebra of $\mathfrak{g}(N)$ for some $N \in \mathbb{N}$. So, if we were able to classify $G$-orbits in $\mathcal{L}$ in the case where $G$ is of type $A$, we would (more or less) be able to classify all finite dimensional Lie algebras. This shows that any classification of $G$-orbits in $\mathcal{L}$ is likely very complicated. In contrast, the classification of $G_\Delta$-orbits in $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ is given in Theorem 3.7 of [8].

4 Closed Orbits

Below, we use the notation $N_G(\mathfrak{a})$ (resp. $N_\mathfrak{g}(\mathfrak{a})$) for the normalizer in $G$ (resp. $\mathfrak{g}$) of a Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$.

We first prove

Proposition 4.1. Let $\mathfrak{a}$ be an abelian ideal of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. Then the $D$-orbit $\text{Ad}_D l(a, 0)$ through $l(a, 0)$ is closed.

Proof. For any $\alpha \in \mathfrak{g}^\ast$, by Lemma 3.2, we have

$$\text{Ad}_\alpha l(a, 0) = l(a, f_\alpha).$$

Since $\mathfrak{a}$ is an abelian Lie algebra, for any $x, y \in \mathfrak{a}$, we have

$$f_\alpha(x)(y) = (-\text{ad}_\alpha^\ast(y))(x) = \alpha([x, y]) = \alpha(0) = 0.$$

Thus, we see that $f_\alpha = 0$, so that

$$\text{Ad}_\alpha l(a, 0) = l(a, 0).$$

It follows from (1) that, for any $(g, \alpha) \in D$, we have $(g, \alpha) = (g, 0)(e, \alpha)$, so

$$\text{Ad}_{(g,\alpha)} l(a, 0) = \text{Ad}_g \text{Ad}_\alpha l(a, 0) = \text{Ad}_g l(a, 0) = l(\text{Ad}_g a, 0),$$

again by Lemma 3.2. Since $l(\text{Ad}_g a, 0) = l(a, 0)$ if and only if $\text{Ad}_g a = a$, i.e., $g \in N_G(\mathfrak{a})$, the $D$-orbit $\text{Ad}_D l(a, 0)$ is isomorphic to $G/N_G(\mathfrak{a})$ as a variety. Since $\mathfrak{a}$ is an ideal of $\mathfrak{b}$, the normalizer $N_G(\mathfrak{a})$ contains the Borel subgroup of $G$ corresponding to $\mathfrak{b}$. Hence $N_G(\mathfrak{a})$ is a parabolic subgroup of $G$ and, therefore, $G/N_G(\mathfrak{a})$ is a projective variety. It follows that the orbit $\text{Ad}_D l(a, 0)$ is closed. \qed
For the converse we need a simple lemma. Let \( g = n \oplus \mathfrak{h} \oplus n^- \) be a triangular decomposition of \( g \).

Write \( \Phi = \Phi^+ \cup \Phi^- \) for the root system of \((g, \mathfrak{h})\) so that roots in \( \Phi^+ \) correspond to \( n \) and roots in \( \Phi^- \) correspond to \( n^- \). For \( \lambda \in \Phi \), the \( \lambda \)-root space is denoted by \( g_\lambda \). Let \( p \) be a parabolic subalgebra of \( g \) containing \( n \oplus \mathfrak{h} \) and \( i \) a Lie ideal of \( p \). Define \( i_0 \) to be \( i \cap \mathfrak{h} \). Since \( i \) is an ideal of \( p \), \( p \) acts on \( i \) by the adjoint action. In particular, the Lie subalgebra \( \mathfrak{h} \) of \( p \) acts on \( i \) by the adjoint action. Hence \( i \) decomposes into a direct sum of \( \mathfrak{h} \)-weight spaces. Since an \( \mathfrak{h} \)-weight on \( g \) is either 0 or a root, we have proved the following

**Lemma 4.1.** With the above notation, we have
\[
i = i_0 \oplus \bigoplus_{\lambda \in \Lambda} \mathfrak{h}_\lambda,
\]
where \( \Lambda \) is some subset of \( \Phi \).

Now we are ready to prove

**Proposition 4.2.** Let \( a \) be a Lie subalgebra of \( g \) and \( f \in Z_{CE}^1(a, a^*) \) a skew element. If the \( D \)-orbit \( \text{Ad}_D l(a, f) \) is closed, then \( a \) is an abelian ideal of some Borel subalgebra of \( g \) and \( f = 0 \).

**Proof.** Write \( k \) for the dimension of \( a \). Consider the first projection map \( \text{pr}_1 : \mathfrak{d} \to g \) sending \((x, \alpha)\) to \( x \). For any \((g, \alpha) \in D\), by Lemma 3.2 and formula (1), we have
\[
\text{Ad}_{(g, \alpha)} l(a, f) = \text{Ad}_g \text{Ad}_\alpha l(a, f) = \text{Ad}_g l(a, f + f_\alpha) = l(\text{Ad}_g a, g.(f + f_\alpha)).
\]

It follows that \( \text{pr}_1(\text{Ad}_{(g, \alpha)} l(a, f)) = \text{Ad}_g a \). Let \( \text{Gr}(k, g) \) be the Grassmannian of \( k \)-dimensional vector subspaces of \( g \). From the computation above, we see that \( \text{pr}_1 \) induces a morphism of varieties \( p : \text{Ad}_D l(a, f) \to \text{Gr}(k, g) \), sending \( \text{Ad}_{(g, \alpha)} l(a, f) \) to \( \text{Ad}_g a \).

By hypothesis, the orbit \( \text{Ad}_D l(a, f) \) is closed in the projective variety \( \mathcal{L} \). Hence \( \text{Ad}_D l(a, f) \) is itself a projective variety. In particular, \( \text{Ad}_D l(a, f) \) is complete. So, the image \( p(\text{Ad}_D l(a, f)) \) of \( \text{Ad}_D l(a, f) \) must also be complete. But, by the analysis of the previous paragraph, the image \( p(\text{Ad}_D l(a, f)) \) is isomorphic as a variety to \( G/\text{N}_G(a) \). Hence \( \text{N}_G(a) \) is a parabolic subgroup of \( G \).

The fiber of \( p \) over \( a \in \text{Gr}(k, g) \) is easily seen to be
\[
\{l(a, g.(f + f_\alpha)) : g \in \text{N}_G(a), \alpha \in g^*\}.
\]

Write \( f' \) for \( g.(f + f_\alpha) \) \((g \in \text{N}_G(a), \alpha \in g^*)\). Choose a basis \( \{v_1, \cdots, v_k\} \) of \( a \) and extend it to a basis \( \{v_1, \cdots, v_k, v_{k+1}, \cdots, v_n\} \) of \( g \). Let \( \{v_1^*, \cdots, v_n^*\} \) be the dual basis of \( g^* \). For all \( 1 \leq i \leq k \), we write
\[
f'(v_i) = \sum_{j=1}^{k} a_{ij} (v_j^*|a),
\]
for some \( a_{ij} \in \mathbb{C} \). Hence, by definition, we see that \( l(a, g.(f + f_\alpha)) \) is spanned by \((v_1, \sum_{j=1}^{k} a_{1j} v_j^*), \cdots, (v_k, \sum_{j=1}^{k} a_{kj} v_j^*), (0, v_{k+1}^*), \cdots, (0, v_n^*)\).

Recall that, for any basis \( \{w_1, \cdots, w_{2n}\} \) of \( \mathfrak{d} \), the set
\[
\{\text{Span}(w_1 + \sum_{j=n+1}^{2n} b_{1j} w_j, \cdots, w_n + \sum_{j=n+1}^{2n} b_{nj} w_j) : b_{ij} \in \mathbb{C}\}
\]
is an affine chart on \( \text{Gr}(n, \mathfrak{d}) \). We consider the affine open set given by the basis \( \{w_1, \cdots, w_{2n}\} \) with
\[
\{w_1, \cdots, w_n\} := \{v_1, \cdots, v_k, v_{k+1}^*, \cdots, v_n^*\}, \{w_{n+1}, \cdots, w_{2n}\} := \{v_{k+1}, \cdots, v_n, v_1^*, \cdots, v_k^*\},
\]
and note that the fiber of \( p \) over \( a \) is contained in the corresponding affine open set. But this fiber is closed in \( L \), hence it is a projective variety. Therefore, the fiber of \( p \) over \( a \) must be a finite set. In particular,

\[ \{ l(a, f + f_a) : \alpha \in \mathfrak{g}^* \} \]

is a finite set. Suppose \( f_\alpha \neq 0 \) for some \( \alpha \in \mathfrak{g}^* \). Then we have \( f \neq f + f_\alpha \), so \( l(a, f) \neq l(a, f + f_\alpha) \). We deduce that the subset \( \{ l(a, f + f_\alpha) : s \in \mathbb{C} \} \) is a curve connecting the distinct points \( l(a, f) \) and \( l(a, f + f_\alpha) \), a contradiction. Therefore, we have \( f_\alpha = 0 \) for all \( \alpha \in \mathfrak{g}^* \). Then, by the definition of \( f_\alpha \), we have \( \alpha([x, y]) = 0 \) for all \( \alpha \in \mathfrak{g}^* \) and \( x, y \in a \). It follows that \( [x, y] = 0 \) for all \( x, y \in a \), i.e., \( a \) is an abelian Lie algebra.

Since \( N_G(a) \) is a parabolic subgroup of \( G \), we see that \( N_G(a) \) is a parabolic Lie subalgebra of \( \mathfrak{g} \) which contains \( a \) as a Lie ideal. Choose a triangular decomposition \( \mathfrak{g} = n \oplus \mathfrak{h} \oplus n^- \) of \( \mathfrak{g} \) such that \( n \oplus \mathfrak{h} \subseteq N_G(a) \). Then, by Lemma 4.1, we have

\[ a = a_0 \oplus \bigoplus_{\lambda \in \Lambda} \mathfrak{g}_\lambda \]

for some subset \( \Lambda \) of the set \( \Phi = \Phi^+ \cup \Phi^- \) of roots for \( (\mathfrak{g}, \mathfrak{h}) \). Suppose that there exists \( \lambda \in \Phi^+ \) such that \( -\lambda \in \Lambda \). Choose an \( \mathfrak{sl}_2 \)-triple \( \{ e_\lambda, f_\lambda, h_\lambda \} \) such that \( e_\lambda \in \mathfrak{g}_\lambda, f_\lambda \in \mathfrak{g}_{-\lambda} \subseteq a \) and \( h_\lambda \in \mathfrak{h} \). Since \( a \) is a Lie ideal of \( N_G(a) \) and \( n \subseteq N_G(a) \), we have \( h_\lambda = [e_\lambda, f_\lambda] \in a \). But then \( [h_\lambda, f_\lambda] = -2f_\lambda \neq 0 \), contradicting the fact that \( a \) is an abelian Lie algebra. From this it follows that \( a \subseteq n \oplus \mathfrak{h} \). Since \( a \) is a Lie ideal of \( N_G(a) \) and \( n \oplus \mathfrak{h} \subseteq N_G(a) \), we conclude that \( a \) is a Lie ideal of the Borel subalgebra \( n \oplus \mathfrak{h} \) of \( \mathfrak{g} \). It is now well-known and very easy to prove that \( a \subseteq n \) [11].

Let \( H \) be the Cartan subgroup of \( G \) corresponding to \( \mathfrak{h} \). It is clear that \( H \) normalizes \( a \), hence is contained in \( N_G(a) \). Therefore, the subset

\[ \{ l(a, t, f) : t \in H \} \]

of the fiber of the morphism \( p \) over \( a \) must be discrete. For any \( \lambda, \mu \in \Phi^+ \) such that \( \mathfrak{g}_\lambda, \mathfrak{g}_\mu \subseteq a \), choose nonzero elements \( e_\lambda \in \mathfrak{g}_\lambda \) and \( e_\mu \in \mathfrak{g}_\mu \). Then we have

\[ (\exp(s, h), f)(e_\lambda)(e_\mu) = e^{-s(\lambda(h) + \mu(h))}f(e_\lambda)(e_\mu) \]

for all \( s \in \mathbb{C} \) and \( h \in \mathfrak{h} \). From this we see that, for any \( h \in \mathfrak{h} \) with \((\lambda + \mu)(h) \neq 0\), the subset \( \{ l(a, \exp(s, h), f) : s \in \mathbb{C} \} \) is a curve connecting distinct points unless \( f = 0 \). Thus \( f = 0 \), as desired. 

From Proposition 4.1 and Proposition 4.2 we obtain

**Theorem 4.2.** Let \( a \) be a Lie subalgebra of \( \mathfrak{g} \) and \( f \in Z^1_{CE}(a, a^*) \) a skew element. Then the D-orbit \( \text{Ad}_D \{ l(a, f) \} \) is closed if and only if \( a \) is an abelian ideal of some Borel subalgebra of \( \mathfrak{g} \) and \( f = 0 \).

Fix a Borel subalgebra \( \mathfrak{b} \) of \( \mathfrak{g} \).

**Theorem 4.3.** The assignment \( F : a \mapsto \text{Ad}_D \{ l(a, 0) \} \) provides a bijection between the set of abelian ideals of \( \mathfrak{b} \) and the set of closed D-orbits in \( L \). In particular, the number of closed D-orbits in \( L \) is exactly \( 2^{\text{rk} \mathfrak{g}} \).

**Proof.** Let \( a' \) be an abelian ideal of some Borel subalgebra \( \mathfrak{b}' \) of \( \mathfrak{g} \). Since all Borel subalgebras of \( \mathfrak{g} \) are conjugate, there exists \( g \in G \) such that \( \text{Ad}_g \mathfrak{b}' = \mathfrak{b} \). Then \( a := \text{Ad}_g a' \) is an abelian ideal of \( \mathfrak{b} \). It follows from Lemma 3.2 that

\[ \text{Ad}_g \{ l(a', 0) \} = \{ l(a, 0) \}. \]
Hence, we have
\[ \text{Ad}_D l(a', 0) = \text{Ad}_D l(a, 0) = F(a). \]
This, together with Theorem 4.2, proves that $F$ is surjective.

Let $a_1, a_2$ be abelian ideals of $b$. Assume that $F(a_1) = F(a_2)$. Again by Lemma 3.2, we can find $g \in G$ such that $\text{Ad}_g l(a_1, 0) = l(a_2, 0)$, i.e., $l(\text{Ad}_g a_1, 0) = l(a_2, 0)$. It follows that $\text{Ad}_g a_1 = a_2$. Define $P_i$ to be the normalizer of $a_i$ in $G$, for $i = 1, 2$. It is easy to see that $gP_1g^{-1} = P_2$. For $i = 1, 2$, since $a_i$ is a Lie ideal of $b$, we see that $P_i$ contains the Borel subgroup $B$ corresponding to $b$. But any parabolic subgroup of $G$ is conjugate to a unique parabolic subgroup containing $B$, hence we have $P_1 = P_2$. Since the normalizer of a parabolic subgroup is itself, we deduce from $gP_1g^{-1} = P_2 = P_1$ that $g \in P_1$. It follows that $a_1 = \text{Ad}_g a_1 = a_2$. This proves that $F$ is injective.

By Peterson’s theorem on abelian ideals, the cardinality of the set of abelian ideals of $b$ is $2^{rk g}$ (see [1, 2, 3, 11, 12, 13, 14]). Now the second statement follows easily.

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