COMPLEX MONGE-AMPÈRE EQUATIONS FOR PLURIFINELY PLURISUBHARMONIC FUNCTIONS

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ABSTRACT. This paper studies the complex Monge-Ampère equations for \( F \)-plurisubharmonic functions in bounded \( F \)-hyperconvex domains. We give sufficient conditions for this equation to solve for measures with a singular part.

1. INTRODUCTION AND RESULTS

The plurifinely topology \( F \) on a Euclidean open set \( \Omega \subset \mathbb{C}^n \) is the weakest topology that makes all plurisubharmonic functions on \( \Omega \) continuous. Notions pertaining to the plurifinely topology are indicated with the prefix \( F \) to distinguish them from notions pertaining to the Euclidean topology on \( \mathbb{C}^n \). The notion \( F \)-plurisubharmonic functions in \( F \)-open subsets \( \Omega \) of \( \mathbb{C}^n \) and basic properties of these functions are introduced in [9]. Recall that an \( F \)-upper semicontinuous function \( u \) defined on an \( F \)-open set \( \Omega \) is \( F \)-plurisubharmonic if for every complex line \( l \) in \( \mathbb{C}^n \), the restriction of \( u \) to any \( F \)-component of the finely open subset \( l \cap \Omega \) of \( l \) is either finely subharmonic or \( \equiv -\infty \) (see [26]).

When \( \Omega \) is Euclidean open, the class of \( F \)-plurisubharmonic functions is identical to the class of plurisubharmonic functions on \( \Omega \) (see [10]). The Monge-Ampère operator of a smooth plurisubharmonic function \( u \) can be defined as

\[
(dd^c u)^n = n! 4^n \det \left( \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} \right) dV_{2n},
\]

where \( dV_{2n} \) is the volume form in \( \mathbb{C}^n \). In 1982, E. Bedford and B. A. Taylor [3] gave the definition of the complex Monge-Ampère operator for the class of the locally bounded plurisubharmonic functions (also see [2, 4]). After that, the Monge-Ampère operator for finite \( F \)-plurisubharmonic functions in \( F \)-domain is defined by M. El Kadiri and J. Wiegerinck [12]. They used the fact that any finite \( F \)-plurisubharmonic function \( u \) on an \( F \)-domain can \( F \)-locally at \( z \in \Omega \) be written as \( f - g \) where \( f, g \) are bounded plurisubharmonic functions defined on a ball about \( z \). Therefore, the non-polar part \( NP(dd^c u)^n \) is \( F \)-locally defined by

\[
NP(dd^c u)^n := \sum_{p=0}^{n} \binom{n}{p} (-1)^p (dd^c f)^{n-p} \wedge (dd^c g)^p.
\]

Recently, the second author and the fourth author studied the pluripolar part \( P(dd^c u)^n \) of complex Monge-Ampère measures of a \( F \)-plurisubharmonic function \( u \) defined in a bounded \( F \)-hyperconvex domain (see [6]).

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This paper aims to establish the existing solutions of the complex Monge-Ampère equations in $\mathcal{F}$-hyperconvex domain of $\mathbb{C}^n$. Firstly, we recall the definition of the $\mathcal{F}$-hyperconvex domain from [25].

**Definition 1.1.** A bounded, connected, $\mathcal{F}$-open set $\Omega$ is called $\mathcal{F}$-hyperconvex if there exist a negative bounded plurisubharmonic function $\gamma_\Omega$ defined on a bounded hyperconvex domain $\Omega' \supset \Omega$ such that $\Omega = \{ \gamma_\Omega > -1 \}$ and $-\gamma_\Omega$ is $\mathcal{F}$-plurisubharmonic in $\Omega$.

Observe that every bounded hyperconvex domain is $\mathcal{F}$-hyperconvex. Moreover, the first author and his co-author gave in [25] an example to show that a bounded $\mathcal{F}$-hyperconvex domain with no Euclidean interior point exists. Our first main theorem is the following theorem about the relationship of Cegrell’s classes in the hyperconvex domain and the $\mathcal{F}$-hyperconvex one.

**Theorem 1.2.** Let $\Omega$ be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$ and let $D$ be a bounded hyperconvex domain containing $\Omega$. If $u \in \mathcal{E}(D)$ then $u|\Omega \in \mathcal{E}(\Omega)$.

The above result shows that Cegrell’s classes can be naturally extended to bounded $\mathcal{F}$-hyperconvex domains. Therefore, we can surmise that the complex Monge-Ampère equation is also solvable on pluripolar sets for the $\mathcal{F}$-plurisubharmonic functions. Moreover, the pluripolar part of the complex Monge-Ampère measures of the $\mathcal{F}$-plurisubharmonic functions is defined as follows (see Theorem 1.3 and 1.4 in [6]).

**Definition 1.3.** Let $D$ and $\Omega$ be as in Theorem 1.2. Assume that $u \in \mathcal{F}(\Omega)$ and define

$$\hat{u} = \hat{u}_D := \sup\{ \varphi \in \mathcal{F} \text{-PSH}^-(D) : \varphi \leq u \text{ on } \Omega \}.$$  

The pluripolar part $P(dd^c u)^n$ is defined as follows

$$P(dd^c u)^n := 1_{D \cap \{ \hat{u} = -\infty \}} (dd^c \hat{u})^n.$$  

The complex Monge-Ampère equations for $\mathcal{F}$-plurisubharmonic functions is the problem of finding a function $u \in \mathcal{F}(\Omega)$ satisfying:

$$\mathcal{M}\mathcal{A}(\Omega, \mu, \nu) : \begin{cases} P(dd^c u)^n = \mu & \text{in } \mathbb{C}^n, \\ NP(dd^c u)^n = \nu & \text{on } QB(\Omega). \end{cases}$$

Here, $\mu$ is a Borel measure in $\mathbb{C}^n$ and $\nu$ is a non-negative measure on $QB(\Omega)$. Moreover, a function $w \in \mathcal{F}(\Omega)$ is called sub-solution to $\mathcal{M}\mathcal{A}(\Omega, \mu, \nu)$ if

$$\begin{cases} P(dd^c w)^n \geq \mu & \text{in } \mathbb{C}^n, \\ NP(dd^c w)^n \geq \nu & \text{on } QB(\Omega). \end{cases}$$

When $\Omega$ is a bounded hyperconvex domain, U. Cegrell [7] proved the existence of classical plurisubharmonic solutions of the problem in the case $\mu = 0$ and $\nu(\Omega) < +\infty$. In 2009, P. Åhag, U. Cegrell, R. Czyż and P. H. Hiep [11] showed that the problem can be solved if it has a sub-solution. Later, some authors investigated the existence of the problem (see [8, 14, 15, 17]).

When $\mu = 0$, the first author gave in [18] sufficient conditions for which the problem can be solved. Later on, the first and the second authors showed in [20] that the problem has solutions to a class of measures $\nu$ (also see [22, 23]).

Our second main result is a result about the solvability of the problem. It is not surprising that we need to add the geometry property of $\Omega$. Specifically, we require it has the $\mathcal{F}$-approximation
property. Here, a bounded $F$-hyperconvex domain $\Omega$ in $\mathbb{C}^n$ has the $F$-approximation property if there exists a sequence of bounded hyperconvex domains $\Omega_j$ such that $\Omega \subset \Omega_{j+1} \subset \Omega_j$ and an increasing sequence of functions $\psi_j \in E_0(\Omega_j)$ that converges a.e. to a function $\psi \in E_0(\Omega)$ on $\Omega$ (see [5, 19, 25]). We prove the following.

**Theorem 1.4.** Let $\Omega$ be a bounded $F$-hyperconvex domain in $\mathbb{C}^n$ that has the $F$-approximation property. If the problem $MA(\Omega, \mu, \nu)$ has a sub-solution $w$ then it can be solved.

Note that we haven’t controlled the pluripolar part of the complex Monge-Ampère measures of the $F$-plurisubharmonic functions before. The above Theorem also brings us more information about it. Now, let $\Omega$ be a bounded $F$-hyperconvex domain without Euclidean interior points. Such domains exist. Assume that $a \in \Omega$ and $\varphi \in E_0(\Omega)$ such that $\varphi(a) < -1$. Let $r > 0$ be such that

$$\Omega \subsetneq B(a, r) := \{z \in \mathbb{C}^n : \|z - a\| < r\}.$$

Since $f(z) := \log \|z - a\| - \log r \in F(B(a, r))$, Theorem [1.2] tells us that $f \in E(\Omega)$. By Definition [2.2] we can find $w \in F(\Omega)$ such that $w \geq f$ in $\Omega$ and

$$w = f \text{ on } \Omega \cap \{\varphi < -1\}.$$

Because $\hat{w}_{B(a, r)} = f$ in $B(a, r)$, we have

$$P(dd^c w)^n = 1_{\{f = -\infty\}}(dd^c f)^n.$$

Hence, $w$ will satisfy all the assumptions of Theorem [1.4] with

$$\mu := 1_{\{f = -\infty\}}(dd^c f)^n \neq 0 \text{ and } \nu := 0.$$

Thus, Theorem [1.4] is a generalization of the results of P. Åhag, U. Cegrell, R. Czyż and P. H. Hiep results in [11].

Our paper is organized as follows. In Section 2, we study Cegrell’s classes of the $F$-plurisubharmonic functions and prove Theorem [1.2]. Section 3 is devoted to the solvability of the Monge-Ampère equations in the class $F(\Omega)$.

2. Cegrell’s Classes of $F$-Plurisubharmonic Functions

Firstly, we recall the definition of the non-polar part of $F$-plurisubharmonic functions from [12].

**Definition 2.1.** Let $\Omega \subset \mathbb{C}^n$ be an $F$-open set and let $F-PSH(\Omega)$ be the $F$-plurisubharmonic functions in $\Omega$. Denote by $QB(\mathbb{C}^n)$ the measurable space on $\mathbb{C}^n$ generated by the Borel sets and the pluripolar subsets of $\mathbb{C}^n$ and $QB(\Omega)$ is the trace of $QB(\mathbb{C}^n)$ on $\Omega$. Assume that $u \in F-PSH(\Omega)$. Then, we can find a pluripolar set $E$ and bounded plurisubharmonic functions $f_j, g_j$ defined in Euclidean neighborhoods of $F$-open sets $O_j$ such that

$$\Omega = E \cup \bigcup_{j=1}^{\infty} O_j \text{ and } u = f_j - g_j \text{ on } O_j.$$

The non-polar part $NP(dd^c u)^n$ of $F$-plurisubharmonic function $u$ is defined by

$$\int_A NP(dd^c u)^n := \sum_{j=1}^{\infty} \int_{A \cap (O_j \backslash \bigcup_{k=1}^{j-1} O_k)} (dd^c (f_j - g_j))^n, \quad A \in QB(\Omega).$$
The following definition of Cegrell’s classes for $\mathcal{F}$-plurisubharmonic functions was given in [25] (also in [1],[7],[8],[27]).

**Definition 2.2.** Let $\Omega$ be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$ and let $\gamma_\Omega$ be a negative bounded plurisubharmonic function defined in a bounded hyperconvex domain $\Omega'$ such that $-\gamma_\Omega$ is $\mathcal{F}$-plurisubharmonic in $\Omega$ and

$$\Omega = \Omega' \cap \{\gamma_\Omega > -1\}.$$  

(a) We say that a bounded, negative $\mathcal{F}$-plurisubharmonic function $u$ defined on $\Omega$ belongs to $\mathcal{E}_0(\Omega)$ if

$$\int_{\Omega} (dd^c u)^n < +\infty$$

and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\Omega \cap \{u < -\varepsilon\} \subset \Omega' \cap \{\gamma_\Omega > -1 + \delta\}.$$  

(b) Denote by $\mathcal{F}(\Omega)$ the family of $\mathcal{F}$-plurisubharmonic functions $u$ in $\Omega$ such that there exists a decreasing sequence $\{u_j\} \subset \mathcal{E}_0(\Omega)$ that converges pointwise to $u$ on $\Omega$ and satisfies

$$\sup_{j \geq 1} \int_{\Omega} (dd^c u_j)^n < +\infty.$$  

(c) Let $\mathcal{E}(\Omega)$ be the set of $\mathcal{F}$-plurisubharmonic functions $u$ in $\Omega$ such that for each $\varphi \in \mathcal{E}_0(\Omega)$, there exists a function $v \in \mathcal{F}(\Omega)$ satisfying $v \geq u$ in $\Omega$ and $v = u$ in $\Omega \cap \{\varphi < -1\}$.

**Lemma 2.3.** Let $\Omega$ be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$ and let $\varphi \in \mathcal{E}_0(\Omega)$. Then, for every $\varepsilon > 0$, there exist bounded plurisubharmonic functions $\tilde{\varphi}$ and $\tilde{\varphi}_\varepsilon$ defined in bounded hyperconvex domain $\Omega' \supset \Omega$ such that

$$\Omega \cap \{\varphi < -\varepsilon\} = \Omega' \cap \{\tilde{\varphi}_\varepsilon < \tilde{\varphi}\}.$$  

**Proof.** Without loss of generality we can assume that $-1 < \varphi < 0$ in $\Omega$. We set

$$\varphi_\varepsilon := \max(\varphi, -\varepsilon).$$  

Let $\gamma$ be a negative bounded plurisubharmonic function defined on a bounded hyperconvex domain $\Omega'$ such that $-\gamma$ is $\mathcal{F}$-plurisubharmonic in $\Omega$ and satisfies

$$\Omega = \Omega' \cap \{\gamma > -1\}.$$  

Let $\delta \in (0, 1)$ be such that

$$\Omega \cap \{\varphi \leq -\varepsilon\} \subset \Omega' \cap \{\gamma > -1 + 2\delta\}.$$  

Assume that $f \in \mathcal{F}-\text{PSH}(\Omega)$ with $-1 < f < 0$ and define

$$\tilde{f} := \begin{cases} 
\max(-\delta^{-1}, f + \delta^{-1}\gamma) & \text{in } \Omega, \\
-\delta^{-1} & \text{in } \Omega' \setminus \Omega.
\end{cases}$$  

Proposition 2.3 in [11] tells us that $\tilde{f}$ is a $\mathcal{F}$-plurisubharmonic function in $\Omega'$, and hence, Proposition 2.14 in [10] implies that $\tilde{f}$ is plurisubharmonic in $\Omega'$ because $\Omega'$ is a Euclidean open set. It is easy to see that

$$f = \tilde{f} - \delta^{-1}\gamma \text{ on } \Omega' \cap \{\gamma > -1 + \delta\}.$$
We deduce by (2.1) that
\[ \Omega \cap \{ \varphi < -\varepsilon \} = \Omega \cap \{ \varphi < \varphi_\varepsilon \} = \Omega' \cap \{ \varphi < \tilde{\varphi}_\varepsilon \}. \]

The proof is complete. \(\square\)

**Lemma 2.4.** Let \( \Omega \) be a bounded \( \mathcal{F} \)-hyperconvex domain in \( \mathbb{C}^n \) and let \( \varphi \in \mathcal{E}_0(\Omega) \). Assume that \( u \) is a \( \mathcal{F} \)-plurisubharmonic function in \( \Omega \). Then, \( u \in \mathcal{E}(\Omega) \) if and only if there exists a decreasing sequence \( \{ u_j \} \subset \mathcal{F}(\Omega) \) such that
\[ u_j = u \text{ on } \Omega \cap \{ j \varphi < -1 \}. \]

**Proof.** Since \( j \varphi \in \mathcal{E}_0(\Omega) \), there exists a function \( v_j \in \mathcal{F}(\Omega) \) such that
\[ v_j = u \text{ on } \Omega \cap \{ j \varphi < -1 \}. \]

Let \( u_j \) be the \( \mathcal{F} \)-upper semi-continuous majorant of \( \sup_{k \geq j} v_k \) in \( \Omega \). It is easy to see that \( \{ u_j \} \) is a decreasing sequence of \( \mathcal{F} \)-plurisubharmonic functions in \( \Omega \). Since
\[ \{ j \varphi < -1 \} \subset \{ k \varphi < -1 \}, \quad \forall k \geq j, \]
it follows that
\[ v_k = u \text{ on } \{ j \varphi < -1 \}, \quad \forall k \geq j, \]
and therefore,
\[ u_j := u \text{ on } \Omega \cap \{ j \varphi < -1 \}. \]

We now deduce by the definition of \( u_j \) that.
\[ v_j \leq u_j < 0 \text{ in } \Omega, \]
and hence, \( u_j \in \mathcal{F}(\Omega) \). This proves the lemma. \(\square\)

**Lemma 2.5.** Let \( \Omega \) be a bounded \( \mathcal{F} \)-hyperconvex domain in \( \mathbb{C}^n \) and let \( \{ u_j \} \subset \mathcal{F}(\Omega) \) be a decreasing sequence such that
\[ \sup_{j \geq 1} \int_{\Omega} (dd^c \max(u_j, -1))^n < +\infty. \]

Then, \( u := \lim_{j \to +\infty} u_j \in \mathcal{F}(\Omega) \).

**Proof.** Let \( \varphi \in \mathcal{E}_0(\Omega) \) such that \( \varphi < 0 \) in \( \Omega \). Put \( \varphi_j = \max\{ u_j, j \varphi \} \), then \( \varphi_j \in \mathcal{E}_0(\Omega) \) for all \( j \geq 1 \). Because of the fact that \( u_j \searrow u \) as \( j \to +\infty \), \( \varphi_j \searrow \varphi \) as \( j \to +\infty \), too. Moreover, we infer from Proposition 4.2 in [25] and Proposition 4.3 in [25] that
\[ \sup_{j \geq 1} \int_{\Omega} (dd^c \varphi_j)^n = \sup_{j \geq 1} \int_{\Omega} (dd^c \max(u_j, j \varphi, -1))^n \leq \sup_{j \geq 1} \int_{\Omega} (dd^c \max(u_j, -1))^n < +\infty. \]

This implies that \( u := \lim_{j \to +\infty} u_j \in \mathcal{F}(\Omega) \). This proves the lemma. \(\square\)
Lemma 2.6. Let $\Omega$ be a bounded $\mathcal{F}$-hyperconvex domain and let $\gamma$ be a negative bounded plurisubharmonic function defined on a bounded hyperconvex domain $\Omega'$ such that $-\gamma$ is $\mathcal{F}$-plurisubharmonic in $\Omega$ and satisfies

$$\Omega = \Omega' \cap \{ \gamma > -1 \}.$$

Assume that $\varphi \in \mathcal{E}_0(\Omega)$ and $\varepsilon, \delta \in (0, 1)$ such that

$$\Omega \cap \{ \varphi \leq -\frac{\varepsilon}{4} \} \subset \Omega' \cap \{ \gamma > -1 + 2\delta \}. \quad (2.2)$$

Then, there exists $\tilde{\varphi}_1, \tilde{\varphi}_2 \in \text{PSH}(\Omega') \cap L^\infty(\Omega')$ such that for every $p \geq 0$ and for every $u, v, w_1, \ldots, w_{n-1} \in \mathcal{F}\text{-PSH}(\Omega) \cap L^\infty(\Omega)$ with $u \leq v < 0$ in $\Omega$, we have

$$\frac{p}{2} \int_{\{ \varphi \leq -\varepsilon \}} (-u)^{p-1} du \wedge d^c u \wedge T \leq \frac{1}{p+1} \int_{\{ \varphi \leq -\frac{\varepsilon}{2} \}} (-u)^{p+1} d^c \tilde{\varphi}_1 \wedge T + \int_{\{ \varphi \leq -\frac{\varepsilon}{2} \}} (-u)^{p} d\nu \wedge T, \quad (2.3)$$

and

$$\int_{\{ \varphi \leq -\varepsilon \}} (-u)^p d\nu \wedge T \leq (p + 2) \int_{\{ \varphi \leq -\frac{\varepsilon}{2} \}} (-u)^p d\nu \wedge T + \left( \frac{4p(1 + \delta)e^{\frac{\varepsilon}{\delta}}}{{\delta}} + \frac{2e^{\frac{\varepsilon}{\delta}}}{p+1} \right) \int_{\{ \varphi \leq -\frac{\varepsilon}{2} \}} (-u)^{p+1} d\nu (e^{\tilde{\varphi}_1} + e^{\tilde{\varphi}_2}) \wedge T. \quad (2.4)$$

Here,

$$T := d\nu w_1 \wedge \ldots \wedge d\nu w_{n-1}.$$

Proof. Since $\tilde{\varphi} \in \mathcal{E}_0(\Omega)$, by replacing $\varphi$ with $\tilde{\varphi}$ if necessary, we can assume that $\varepsilon = 1$. Moreover, without loss of generality we can assume that

$$-1 \leq u, v, w_1, \ldots, w_{n-1} < 0 \text{ in } \Omega.$$

Now assume that $-1 \leq f \leq 0$ is a $\mathcal{F}$-plurisubharmonic function in $\Omega$ and define

$$\tilde{f} := \begin{cases} \max(-\delta^{-1}, f + \tilde{\gamma}) & \text{in } \Omega, \\ -\delta^{-1} & \text{in } \Omega' \setminus \Omega. \end{cases}$$

Here, $\tilde{\gamma} := \delta^{-1} \gamma$. Proposition 2.3 in [11] tells us that $\tilde{f}$ is a $\mathcal{F}$-plurisubharmonic function in $\Omega'$, and hence, Proposition 2.14 in [10] implies that $\tilde{f}$ is plurisubharmonic in $\Omega'$ because $\Omega'$ is a Euclidean open set. Moreover,

$$f = \tilde{f} - \tilde{\gamma} \text{ on } \Omega' \cap \{ \gamma > -1 + \delta \}. \quad (2.5)$$

Put

$$\varphi_1 := \max(\varphi, -1) \text{ and } \varphi_2 := \max(\varphi, -\frac{1}{2}).$$

We deduce by (2.2) that

$$\Omega \cap \{ \gamma < -1 + 2\delta \} \subset \Omega \cap \{ \varphi > -\frac{1}{2} \}. \quad (2.5)$$
Hence,
\begin{equation}
\tilde{\varphi}_1 = \tilde{\varphi}_2 \text{ on } (\Omega' \cap \{ \gamma < -1 + 2\delta \}) \cup (\Omega \cap \{ \varphi > \frac{1}{2} \}).
\end{equation}
It is easy to see that \( \varphi_1 \leq \varphi_2 \) in \( \Omega \) and
\begin{equation}
\varphi_2 - \varphi_1 = \frac{1}{2} \text{ on } \Omega \cap \{ \varphi \leq -1 \}.
\end{equation}
Set
\[ T := dd^c(\tilde{w}_1 - \tilde{\gamma}) \land \ldots \land dd^c(\tilde{w}_{n-1} - \tilde{\gamma}). \]
We obtain from (2.2), (2.5) and (2.6) that
\begin{align*}
p(\varphi_2 - \varphi_1)(-u)^{p-1}du \land d^c u \land T \\
= -(\tilde{\varphi}_2 - \tilde{\varphi}_1)d(\tilde{u} - \tilde{\gamma})^p \land d^c(\tilde{u} - \tilde{\gamma}) \land \tilde{T} \text{ on } \Omega.
\end{align*}
Therefore, we infer by (2.6) and (2.7) that
\begin{align*}
\frac{p}{2} \int_{\Omega \cap \{ \varphi \leq -1 \}} (-u)^{p-1}du \land d^c u \land T \\
\leq \int_{\Omega'} -(\tilde{\varphi}_2 - \tilde{\varphi}_1)d(\tilde{u} - \tilde{\gamma})^p \land d^c(\tilde{u} - \tilde{\gamma}) \land \tilde{T}
\end{align*}
because
\[ \tilde{\varphi}_2 - \tilde{\varphi}_1 = 0 \text{ on } \Omega' \setminus \Omega. \]
Integration by parts tells us that
\[ \int_{\Omega'} -(\tilde{\varphi}_2 - \tilde{\varphi}_1)d(\tilde{u} - \tilde{\gamma})^p \land d^c(\tilde{u} - \tilde{\gamma}) \land \tilde{T} \\
= \int_{\Omega'} (\tilde{\gamma} - \tilde{u})^p d(\tilde{\varphi}_2 - \tilde{\varphi}_1) \land d^c(\tilde{u} - \tilde{\gamma}) \land \tilde{T} + \int_{\Omega'} (\tilde{\varphi}_2 - \tilde{\varphi}_1)(\tilde{\gamma} - \tilde{u})^p dd^c(\tilde{u} - \tilde{\gamma}) \land \tilde{T} \\
= \frac{-1}{p + 1} \int_{\Omega'} (\tilde{\gamma} - \tilde{u})^{p+1} dd^c(\tilde{\varphi}_2 - \tilde{\varphi}_1) \land \tilde{T} + \int_{\Omega'} (\tilde{\varphi}_2 - \tilde{\varphi}_1)(\tilde{\gamma} - \tilde{u})^p dd^c(\tilde{u} - \tilde{\gamma}) \land \tilde{T}.
\]
Hence, we deduce from (2.6) and (2.8) that
\[ \frac{p}{2} \int_{\{ \varphi \leq -1 \}} (-u)^{p-1}du \land d^c u \land T \\
\leq \frac{-1}{p + 1} \int_{\{ \varphi \leq -\frac{1}{2} \}} (-u)^{p+1} dd^c(\varphi_2 - \varphi_1) \land T + \int_{\{ \varphi \leq -\frac{1}{2} \}} (\varphi_2 - \varphi_1)(-u)^p dd^c u \land T \\
\leq \frac{1}{p + 1} \int_{\{ \varphi \leq -\frac{1}{2} \}} (-u)^{p+1} dd^c \varphi_1 \land T + \int_{\{ \varphi \leq -\frac{1}{2} \}} (-u)^p dd^c u \land T.
\]
This proves (2.3). We now give the proof of (2.4). Observe that
\[ (\varphi_2 - \varphi_1)(-u)^p dd^c v \land T = (\tilde{\varphi}_2 - \tilde{\varphi}_1)(\tilde{u} - \tilde{\gamma})^p dd^c(\tilde{u} - \tilde{\gamma}) \land \tilde{T} \text{ on } \Omega. \]
We deduce from (2.6) that
\[ dd^c((\tilde{\varphi}_2 - \tilde{\varphi}_1)(\gamma - \tilde{u})^p) \wedge \tilde{T} = 0 \text{ on } (\Omega' \cap \{\gamma < -1 + 2\delta\}) \cup (\Omega \cap \{\varphi > -\frac{1}{2}\}). \]

Using integration by parts we have
\[
\frac{1}{2} \int_{\{\varphi \leq -1\}} |u|^p dd^cv \wedge T \leq \int_{\Omega'} ((\tilde{\varphi}_2 - \tilde{\varphi}_1)(\gamma - \tilde{u})^p) dd^c(\tilde{v} - \gamma) \wedge \tilde{T} = \int_{\Omega'} (\tilde{v} - \gamma) dd^c((\tilde{\varphi}_2 - \tilde{\varphi}_1)(\gamma - \tilde{u})^p) \wedge \tilde{T} = \int_{\Omega \cap (\varphi \leq -\frac{1}{2})} (\tilde{v} - \gamma) dd^c((\tilde{\varphi}_2 - \tilde{\varphi}_1)(\gamma - \tilde{u})^p) \wedge \tilde{T}.
\]

By computation we have
\[
dd^c((\tilde{\varphi}_2 - \tilde{\varphi}_1)(\gamma - \tilde{u})^p) = (\gamma - \tilde{u})^p dd^c(\tilde{\varphi}_2 - \tilde{\varphi}_1) + (\tilde{\varphi}_2 - \tilde{\varphi}_1) dd^c(\gamma - \tilde{u})^p + d(\tilde{\varphi}_2 - \tilde{\varphi}_1) \wedge d^c(\gamma - \tilde{u})^p + d(\gamma - \tilde{u})^p \wedge d^c(\tilde{\varphi}_2 - \tilde{\varphi}_1)
\[
\geq -p(\gamma - \tilde{u})^{p-1} (d(\tilde{\varphi}_2 - \tilde{\varphi}_1) \wedge d^c(\gamma - \tilde{u}) + d(\tilde{u} - \gamma) \wedge d^c(\tilde{\varphi}_2 - \tilde{\varphi}_1))
\]
\]
on \Omega' \cap \{\gamma > -1 + \delta\}. We infer by (2.6) that
\[
(\tilde{\varphi}_2 - \tilde{\varphi}_1)(\gamma - \tilde{u})^{p-2} d(\tilde{u} - \gamma) \wedge d^c(\tilde{u} - \gamma) \geq 0 \text{ on } \Omega'.
\]

Using the basic inequality
\[
\pm (f + a)^k (df \wedge d^c f + dg \wedge d^c g) \geq -|f + a|^{k-1} df \wedge d^c f - |f + a|^{k+1} dg \wedge d^c g, \forall a \in \mathbb{R},
\]
we get
\[
(\gamma - \tilde{u})^{p-1} [d(\tilde{\varphi}_2 - \tilde{\varphi}_1) \wedge d^c(\tilde{u} - \gamma) + d(\tilde{u} - \gamma) \wedge d^c(\tilde{\varphi}_2 - \tilde{\varphi}_1)] \geq -|\gamma - \tilde{u}|^{p-2} d(\tilde{u} - \gamma) \wedge d^c(\tilde{u} - \gamma) - |\gamma - \tilde{u}|^p d(\tilde{\varphi}_2 - \tilde{\varphi}_1) \wedge d^c(\tilde{\varphi}_2 - \tilde{\varphi}_1).
\]

Since $-\frac{1}{\delta} \leq \tilde{\varphi}_1 \leq \tilde{\varphi}_2 \leq 0$ in $\Omega'$, again using (2.12) we obtain that
\[
-d(\tilde{\varphi}_2 - \tilde{\varphi}_1) \wedge d^c(\tilde{\varphi}_2 - \tilde{\varphi}_1)
\]
\[
= -d\tilde{\varphi}_1 \wedge d^c\tilde{\varphi}_1 - d\tilde{\varphi}_2 \wedge d^c\tilde{\varphi}_2 + d\tilde{\varphi}_1 \wedge d^c\tilde{\varphi}_2 + d\tilde{\varphi}_2 \wedge d^c\tilde{\varphi}_1
\]
\[
\geq -d\tilde{\varphi}_1 \wedge d^c\tilde{\varphi}_1 - d\tilde{\varphi}_2 \wedge d^c\tilde{\varphi}_2 - \frac{1}{|\tilde{\varphi}_1 + \frac{2}{\delta}|} d\tilde{\varphi}_1 \wedge d^c\tilde{\varphi}_1 - |\tilde{\varphi}_1 + \frac{2}{\delta}| d\tilde{\varphi}_2 \wedge d^c\tilde{\varphi}_2
\]
\[
\geq -\frac{2(1 + \delta)e^\frac{\gamma}{\delta}}{\delta} dd^c(e^{\tilde{\varphi}_1} + e^{\tilde{\varphi}_2}).
\]
Combining this with (2.10), (2.11) and (2.13) we arrive that
\[
dd^c((\tilde{\varphi}_2 - \tilde{\varphi}_1)(\gamma - \tilde{u})) \\
\geq -((\gamma - \tilde{u})^p dd^c \phi_1 - p(\tilde{\varphi}_2 - \tilde{\varphi}_1)(\gamma - \tilde{u})^pp^{-1}dd^c(\tilde{u} - \gamma)) \\
- p(\gamma - \tilde{u})^p d(\tilde{u} - \gamma) \cap d^c(\tilde{u} - \gamma) - \frac{2p(1 + \delta)e^{\frac{1}{p}}}{\delta}((\gamma - \tilde{u})^p dd^c(e^{\tilde{\varphi}_1} + e^{\tilde{\varphi}_2})) \\
\geq -p(\tilde{\varphi}_2 - \tilde{\varphi}_1)(\gamma - \tilde{u})^p dd^c(\tilde{u} - \gamma) - p(\gamma - \tilde{u})^p d(\tilde{u} - \gamma) \cap d^c(\tilde{u} - \gamma) \\
- \frac{4p(1 + \delta)e^{\frac{1}{p}}}{\delta}((\gamma - \tilde{u})^p dd^c(e^{\tilde{\varphi}_1} + e^{\tilde{\varphi}_2})) \\
\leq -p(-u)^p d^c u - p(-u)^p d^2 u \cap d^c u - \frac{4p(1 + \delta)e^{\frac{1}{p}}}{\delta}(-u)^p dd^c(e^{\tilde{\varphi}_1} + e^{\tilde{\varphi}_2})
\] on $\Omega' \cap \{\gamma > -1 + \delta\}$. On the other hand, we obtain from (2.3) that
\[
p \int_{\{\varphi \leq -1/2\}} (-u)^p d^2 u \cap T \\
\leq \frac{2}{p + 1} \int_{\{\varphi \leq -1/2\}} (-u)^p d^c \phi_1 \cap T + 2 \int_{\{\varphi \leq -1/2\}} (-u)^p d^c u \cap T. \\
\leq \frac{2e^{\frac{1}{p}}}{p + 1} \int_{\{\varphi \leq -1/2\}} (-u)^p d^c \phi_1 \cap T + 2 \int_{\{\varphi \leq -1/2\}} (-u)^p d^c u \cap T.
\]
Since $\{\varphi \leq -1/2\} \subset \{\varphi \leq -1/4\} \subset \Omega' \cap \{\gamma > -1 + \delta\}$ and $u \leq v < 0$ on $\Omega$, we deduce by (2.9), (2.14) and (2.15) that
\[
\int_{\{\varphi \leq -1\}} (-u)^p d^c v \cap T \\
\leq \int_{\{\varphi \leq -1/2\}} v \left[ -p(-u)^p d^c u - p(-u)^p d^2 u \cap d^c u - \frac{4p(1 + \delta)e^{\frac{1}{p}}}{\delta}(-u)^p dd^c(e^{\tilde{\varphi}_1} + e^{\tilde{\varphi}_2}) \right] \cap T \\
\leq \int_{\{\varphi \leq -1/2\}} \left[ p(-u)^p d^c u + p(-u)^p d^2 u \cap d^c u + \frac{4p(1 + \delta)e^{\frac{1}{p}}}{\delta}(-u)^p d^c(e^{\tilde{\varphi}_1} + e^{\tilde{\varphi}_2}) \right] \cap T \\
\leq \int_{\{\varphi \leq -1/2\}} \left[ (p + 2)(-u)^p d^c u + \left( \frac{4p(1 + \delta)e^{\frac{1}{p}}}{\delta} + \frac{2e^{\frac{1}{p}}}{p + 1} \right)(-u)^p d^c(e^{\tilde{\varphi}_1} + e^{\tilde{\varphi}_2}) \right] \cap T,
\]
which completes the proof. 

We now able to give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Let $\gamma$ be a negative bounded plurisubharmonic function defined on a bounded hyperconvex domain $\Omega'$ such that $-\gamma$ is $F$-plurisubharmonic in $\Omega$ and satisfies
\[
\Omega = \Omega' \cap \{\gamma > -1\}.
\]
By replacing $\Omega'$ with $D \cap \Omega'$, we can assume that $\Omega' \subset D$, and hence, $u \in \mathcal{E}(\Omega')$. Let $\varphi \in \mathcal{E}_0(\Omega)$ and let $G$ be an open set such that 

$$\Omega \cap \{ \varphi \leq -\frac{1}{4n} \} \subset G \subset \Omega'. $$

Since $u \in \mathcal{E}(\Omega')$, there exists a decreasing sequence of functions $\{ u_j \} \subset \mathcal{E}_0(\Omega')$ such that $u_j \to u$ on $G$ and

$$\sup_{j \geq 1} \int_{\Omega'} (dd^c u_j)^n < +\infty. $$

Define 

$$v_j := \sup \{ \psi \in \mathcal{F}-\text{PSH}^-(\Omega) : \psi \leq u_j \text{ on } \Omega \cap \{ \varphi < -1 \} \}. $$

Since $\Omega \cap \{ \varphi < -1 \}$ is $\mathcal{F}$-open, so $v_j$ is $\mathcal{F}$-upper semi-continuous on $\Omega$ and 

$$\| u_j \|_{L^\infty(D)} \varphi \leq v_j \text{ on } \Omega. $$

It follows that $v_j \in \mathcal{E}_0(\Omega)$, and hence, $\{ v_j \} \subset \mathcal{E}_0(\Omega)$ is decreasing. Proposition 3.2 in [11] tells us that $v_j$ is $\mathcal{F}$-maximal on $\Omega \cap \{ \varphi > -1 \}$, and hence, Theorem 4.8 in [11] implies that

$$(dd^c v_j)^n = 0 \text{ on } \Omega \cap \{ \varphi > -1 \}. $$

Since $u_j \leq v_j < 0$ in $\Omega$, Lemma 2.6 tells us that for every $p, q \geq 0$, there exist a positive constant $c_{p,q}$ and bounded plurisubharmonic functions $\psi_{p,q}, \varphi_{p,q}$ in $\Omega'$ such that

$$\int_{\{ \varphi \leq -\frac{1}{4n} \}} (-u_j)^p dd^c v_j \wedge dd^c w_1 \wedge \ldots \wedge dd^c w_{n-1}$$

$$\leq c_{p,q} \int_{\{ \varphi \leq -\frac{1}{4n} \}} (-u_j)^p dd^c u_j \wedge dd^c w_1 \wedge \ldots \wedge dd^c w_{n-1}$$

$$+ c_{p,q} \int_{\{ \varphi \leq -\frac{1}{4n} \}} (-u_j)^{p+1} dd^c(e^{\psi_{p,q}} + e^{\varphi_{p,q}}) \wedge dd^c w_1 \wedge \ldots \wedge dd^c w_{n-1}$$

for all $j \geq 1$ and for all bounded $\mathcal{F}$-plurisubharmonic functions $w_1, \ldots, w_{n-1}$ in $\Omega$. Let $\psi \in \mathcal{E}_0(\Omega')$ be such that 

$$\psi := \sum_{p,q=0}^{n} (e^{\psi_{p,q}} + e^{\varphi_{p,q}}) \text{ on } G. $$

Set

$$c_1 := \sum_{p,q=0}^{n} c_{p,q}.$$ 

From (2.17) we arrive that

$$\int_{\{ \varphi \leq -\frac{1}{4n} \}} (-u_j)^p (dd^c u_j)^{s-p} \wedge (dd^c \psi)^p \wedge (dd^c v_j)^{n-s}$$

$$\leq c_1 \sum_{q=0}^{1} \int_{\{ \varphi \leq -\frac{1}{4n} \}} (-u_j)^{p+q} (dd^c u_j)^{s+1-p-q} \wedge (dd^c \psi)^{p+q} \wedge (dd^c v_j)^{n-s-1}$$

(2.18)
for all \( j \geq 1 \) and for all \( 0 \leq p \leq s \leq n - 1 \). By applying (2.18) many times, we infer that

\[
\int_{\{\varphi \leq -1\}} (dd^c v_j)^n \leq c_1 \sum_{p=0}^{1} \int_{\{\varphi \leq -\frac{1}{2}\}} (-u_j)^p (dd^c u_j)^{1-p} \land (dd^c \psi)^p \land (dd^c v_j)^{n-1} \\
\leq c_1^2 \sum_{p=0}^{1} \sum_{q=0}^{1} \int_{\{\varphi \leq -\frac{1}{2}\}} (-u_j)^{p+q} (dd^c u_j)^{2-p-q} \land (dd^c \psi)^{p+q} \land (dd^c v_j)^{n-2} \\
\leq c_2 \sum_{p=0}^{2} \int_{\{\varphi \leq -\frac{1}{2}\}} (-u_j)^p (dd^c u_j)^{2-p} \land (dd^c \psi)^p \land (dd^c v_j)^{n-2} \\
\leq \ldots \leq c_n \sum_{p=0}^{n} \int_{\{\varphi \leq -\frac{1}{2}\}} (-u_j)^p (dd^c u_j)^{n-p} \land (dd^c \psi)^p.
\]

Combining this with (2.16) we obtain that

\[
(2.19) \quad \int_{\Omega} (dd^c v_j)^n \leq c_n \sum_{p=0}^{n} \int_{\Omega'} (-u_j)^p (dd^c u_j)^{n-p} \land (dd^c \psi)^p.
\]

Now, using integration by parts we have

\[
I_p := \int_{\Omega'} (dd^c u_j)^{n-p} \land (dd^c \psi)^p \\
= \int_{\Omega'} \psi (dd^c u_j)^{n-p} \land (dd^c \psi)^{p-1} \land dd^c (-u_j)^p \\
= \int_{\Omega'} \psi (dd^c u_j)^{n-p} \land (dd^c \psi)^{p-1} \land [p(p-1)(-u_j)^{p-2} du_j \land dd^c u_j - p(-u_j)^{p-1} dd^c u_j] \\
\leq p \int_{\Omega'} (-\psi)(-u_j)^{p-1}(dd^c u_j)^{n-p+1} \land (dd^c \psi)^{p-1} \\
\leq p \|\psi\|_{L^\infty(\Omega')} \int_{\Omega'} (-u_j)^{p-1}(dd^c u_j)^{n-p+1} \land (dd^c \psi)^{p-1} \\
= p \|\psi\|_{L^\infty(\Omega')} I_{p-1}.
\]

This implies that

\[
I_p \leq p! \|\psi\|_{L^\infty(\Omega')}^p \int_{\Omega'} (dd^c u_j)^n.
\]

Hence, we deduce by (2.19) that

\[
\sup_{j \geq 1} \int_{\Omega} (dd^c v_j)^n \leq c_n \left( \sum_{p=0}^{n} p! \|\psi\|_{L^\infty(\Omega')}^p \right) \sup_{j \geq 1} \int_{\Omega'} (dd^c u_j)^n < +\infty.
\]

Therefore,

\[
v := \lim_{j \rightarrow +\infty} v_j \in \mathcal{F}(\Omega).
\]

Since \( u = v \) on \( \Omega \cap \{\varphi < -1\} \), we conclude that \( u \in \mathcal{E}(\Omega) \). The proof is complete. \( \square \)
3. Complex Monge-Ampère equations

In this section, we give proof of our main result, Theorem 1.4. Firstly, we prove the solvability of the problem stated in Theorem 1.4 in the case \( \nu = 0 \). Then, we solve this problem with \( \nu \) arbitrary.

First of all, we need the following lemma.

**Lemma 3.1.** Let \( \Omega \) be a bounded \( \mathcal{F} \)-hyperconvex domain in \( \mathbb{C}^n \) that has the \( \mathcal{F} \)-approximation property. Assume that \( u \in \mathcal{F} \)-PSH \( (\Omega) \) and \( v \in \mathcal{F}(\Omega) \) such that

\[
(3.1) \quad w + v \leq u \leq v \text{ on } \Omega, \text{ for some } w \in \mathcal{F}^a(\Omega).
\]

Then, \( u \in \mathcal{F}(\Omega) \) and

\[
P(\ddc u)^n = P(\ddc v)^n \text{ in } \mathbb{C}^n.
\]

**Proof.** Since \( \Omega \) has the \( \mathcal{F} \)-approximation property, we can find a decreasing sequence of bounded hyperconvex domains \( \{\Omega_j\} \) and a sequence of functions \( \psi_j \in E_0(\Omega_j) \) such that \( \Omega \subset \Omega_{j+1} \subset \Omega_j \) and

\[
\psi_j \nearrow \psi \in E_0(\Omega) \text{ a.e. on } \Omega.
\]

For \( f \in \mathcal{F} \)-PSH \( (\Omega) \), we use the symbol

\[
\hat{f}_j := \sup\{\varphi \in \mathcal{F} \text{-PSH}(\Omega_j) : \varphi \leq f \text{ on } \Omega\}.
\]

Theorem 1.2 in [19] tells us that

\[
\hat{w}_j \nearrow w \text{ and } \hat{v}_j \nearrow v \text{ a.e. on } \Omega.
\]

Observed that

\[
(3.2) \quad \mathcal{F}(\Omega_j) \ni \max(\hat{w}_j + \hat{v}_j, k\psi_j) \nearrow \max(w + v, k\psi) \in E_0(\Omega) \text{ a.e. in } \Omega, \forall k \geq 1.
\]

Since \( \hat{v}_j \) and \( \hat{w}_j \) belong to the class \( \mathcal{F}(\Omega_j) \) so

\[
\left[ \int_{\Omega_j} (\ddc(\hat{w}_j + \hat{v}_j))^n \right]^{1/n} \leq \left[ \int_{\Omega_j} (\ddc \hat{w}_j)^n \right]^{1/n} + \left[ \int_{\Omega_j} (\ddc \hat{v}_j)^n \right]^{1/n}.
\]

Hence, by Proposition 2.7 in [25] and Theorem 1.2 in [6] we obtain that

\[
\int_{\Omega} (\ddc \max(w + v, k\psi))^n \leq \limsup_{j \to +\infty} \int_{\Omega_j} (\ddc \max(\hat{w}_j + \hat{v}_j, k\psi_j))^n
\]

\[
\leq \limsup_{j \to +\infty} \int_{\Omega_j} (\ddc (\hat{w}_j + \hat{v}_j))^n
\]

\[
\leq \limsup_{j \to +\infty} \left( \left[ \int_{\Omega_j} (\ddc \hat{w}_j)^n \right]^{1/n} + \left[ \int_{\Omega_j} (\ddc \hat{v}_j)^n \right]^{1/n} \right)^n
\]

\[
\leq \left( \left[ \int_{\Omega} (\ddc \max(w, -1))^n \right]^{1/n} + \left[ \int_{\Omega} (\ddc \max(v, -1))^n \right]^{1/n} \right)^n.
\]

This implies that

\[
\sup_{k \geq 1} \int_{\Omega} (\ddc \max(w + v, k\psi))^n
\]
\[ \leq \left( \left[ \int_{\Omega} (dd^c \max(w, -1))^n \right]^{1/n} + \left[ \int_{\Omega} (dd^c \max(v, -1))^n \right]^{1/n} \right)^n < +\infty. \]

Therefore, \( w + v \in \mathcal{F}(\Omega) \), and thus, we conclude by the hypotheses (3.1) that \( u \in \mathcal{F}(\Omega) \).

Now, it is easy to see that \( \hat{w}_j + \hat{v}_j \leq \hat{u}_j \leq \hat{v}_j \) in \( \Omega_j \).

By Lemma 4.1 in [1] and Theorem 1.2 in [6] we have

\[ P(\ddc v)^n = 1_{\Omega_j \cap \{ \hat{u}_j = -\infty \}} (\ddc \hat{u}_j)^n \]

\[ \leq 1_{\Omega_j \cap \{ \hat{v}_j = -\infty \}} (\ddc \hat{v}_j)^n = P(\ddc u)^n \]

(3.3)

Since \( w \in \mathcal{F}^a(\Omega) \) so \( \hat{w}_j \in \mathcal{F}^a(\Omega_j) \) and hence, we infer by Lemma 4.12 in [1] that

\[ 1_{\Omega_j \cap \{ \hat{v}_j = -\infty \}} (\ddc (\hat{w}_j + \hat{v}_j))^n = 1_{\Omega_j \cap \{ \hat{v}_j = -\infty \}} (\ddc \hat{v}_j)^n = P(\ddc v)^n. \]

Combining this with (3.3) we obtain that

\[ P(\ddc u)^n = P(\ddc v)^n. \]

The proof is complete. \( \square \)

**Lemma 3.2.** Let \( \Omega \) be a bounded \( \mathcal{F} \)-hyperconvex domain in \( \mathbb{C}^n \) that has the \( \mathcal{F} \)-approximation property. Assume that \( u \in \mathcal{F}(\Omega) \) and define

\[ v_j := \sup\{ \psi \in \mathcal{F}^{PSH^{-}}(\Omega) : \psi \leq u + j \text{ on } \Omega \}, \quad j \in \mathbb{N}. \]

Then, the \( \mathcal{F} \)-upper semi-continuous majorant \( v \) of \( \sup_{j \geq 1} v_j \) in \( \Omega \) belongs to the class \( \mathcal{F}(\Omega) \) and satisfies

\[ \begin{cases} P(\ddc v)^n = P(\ddc u)^n & \text{in } \mathbb{C}^n, \\ NP(\ddc v)^n = 0 & \text{on } QB(\Omega). \end{cases} \]

**Proof.** Since \( \Omega \) is \( \mathcal{F} \)-open, so \( v_j \) are \( \mathcal{F} \)-plurisubharmonic functions in \( \Omega \), and hence, \( v_j \in \mathcal{F}(\Omega) \) because

\[ u \leq v_j \leq 0 \text{ in } \Omega. \]

This implies that \( v \in \mathcal{F}(\Omega) \). By Theorem 1.2 in [19], we can find sequences of plurisubharmonic functions \( \varphi_{j,s} \) defined on bounded hyperconvex domains \( \Omega_s \) such that

\[ \Omega_s \supset \Omega_{s+1} \supset \Omega = \bigcap_{s=1}^{+\infty} \Omega_s \]

and

\[ \varphi_{j,s} \nearrow v_j \text{ a.e. in } \Omega \text{ as } s \nearrow +\infty. \]

Now, Lemma 5.14 in [7] tells us that there exists a sequence \( \{g_j\} \subset \mathcal{F}^a(\Omega_1) \) satisfying

\[ (\ddc g_j)^n = 1_{\Omega \cap (-\infty < u < -j + 1)} (\ddc u)^n \text{ on } \Omega_1. \]

Obviously that the measure sequence \( \{(\ddc g_j)^n\} \) is decreasing and converges to 0 in \( \Omega_1 \). Hence, Theorem 5.15 in [7] shows that

\[ g_j \nearrow 0 \text{ a.e. in } \Omega_1. \]
Since
\[ f_{j,s} := \sup\{ \varphi \in PSH^- (\Omega_s) : \varphi \leq v_j \text{ on } \Omega \} \]
\[ = \sup\{ \varphi \in PSH^- (\Omega_s) : \varphi \leq u + j \text{ on } \Omega \}, \]

Theorem 1.2 in [6] tells us that \( f_{j,k} \in \mathcal{F}(\Omega_k) \) and
\[
(3.4) \quad P(\partial^c u)^n = 1_{\Omega_1 \cap \{ f_{j,s} = -\infty \}}(\partial^c f_{j,s})^n \text{ in } \mathbb{C}^n.
\]

Let \( j \geq k \) be positive integer numbers. Since \( f_{k,s} \leq f_{j,s} \) on \( \Omega_s \), the inequality \( (3.4) \) shows that \( f_{k,s} \in \mathcal{N}^\alpha(\Omega_s, f_{j,k}) \). Hence, using Theorem 1.1 in [6] and Theorem 1.2 in [6] we infer by \( (3.4) \) that
\[
(\partial^c (f_{j,s} + g_k))^n \geq (\partial^c f_{j,s})^n + (\partial^c g_k)^n \]
\[ \geq P(\partial^c u)^n + 1_{\Omega \cap \{-\infty < u < -k + 1\}}(\partial^c u)^n \]
\[ \geq (\partial^c f_{k,s})^n. \]

Therefore, Proposition 2.2 in [24] that
\[
f_{j,s} + g_k \leq f_{k,s} \text{ on } \Omega_s.
\]

This implies that
\[
\varphi_{j,s} + g_k \leq f_{j,s} + g_k \leq f_{k,s} \leq v_k \text{ on } \Omega.
\]

Letting \( s \to +\infty \), we obtain that
\[
v_j + g_k \leq v_k \text{ on } \Omega, \quad \forall j \geq k \geq 1,
\]

and thus,
\[
(3.5) \quad v + g_k \leq v_k \text{ on } \Omega, \quad \forall k \geq 1.
\]

We set
\[
f := \sup\{ \varphi \in PSH^- (\Omega_1) : \varphi \leq v \text{ on } \Omega \}.
\]

Since \( v \geq v_k \) in \( \Omega \), we infer by \( (3.5) \) that
\[
f + g_k \leq f_{k,1} \leq f \text{ on } \Omega_1.
\]

Hence, Theorem 1.2 in [6] implies that
\[
P(\partial^c u)^n = 1_{\Omega_1 \cap \{ f = -\infty \}}(\partial^c f)^n
\]
\[ = 1_{\Omega_1 \cap \{ f_{k,1} = -\infty \}}(\partial^c f_{k,1})^n
\]
\[ = P(\partial^c u)^n.
\]

On the other hand, Theorem 1.1 in [6] tells us that
\[
NP(\partial^c v_j)^n = 0 \text{ on } \Omega \cap \{ u > -k \}, \quad \forall j \geq k \geq 1.
\]

Letting \( j \to +\infty \) we conclude by Theorem 4.5 in [11] that
\[
NP(\partial^c u)^n = 0 \text{ on } \Omega,
\]

and thus, Lemma is proved. \( \square \)

We now able to give the proof of Theorem 1.4.
Proof of Theorem 1.4. Let \( \hat{\Omega} \supset \Omega \) be a bounded hyperconvex domain in \( \mathbb{C}^n \). For \( f \in \mathcal{F}(\Omega) \), we set
\[
\hat{f} := \sup\{ g \in PSH^-(\hat{\Omega}) : g \leq f \text{ on } \Omega \}.
\]
Theorem 1.2 in [6] tells us that \( \hat{f} \in \mathcal{F}(\hat{\Omega}) \) and
\[
P(\ddc f)^n = 1_{\Omega \cap \{ \hat{f} = -\infty \}}(\ddc \hat{f})^n \text{ in } \mathbb{C}^n.
\]
The proof is split into three steps.

Step 1. We prove that there exists \( g \in \mathcal{F}(\Omega) \) such that
\[
\left\{ \begin{array}{ll}
P(\ddc g)^n = \mu & \text{in } \mathbb{C}^n, \\
NP(\ddc g)^n = 0 & \text{on } QB(\Omega).
\end{array} \right.
\]
Indeed, by the hypotheses, we infer by (3.6) that
\[
\mu \leq P(\ddc w)^n \leq (\ddc \hat{w})^n \text{ in } \hat{\Omega}.
\]
Theorem 4.14 in [1] tells us that there exists a function \( h \in \mathcal{F}(\hat{\Omega}) \) such that
\[
(\ddc h)^n = \mu \text{ on } \hat{\Omega}.
\]
Therefore, Theorem 1.2 states that \( v := h|\Omega \in \mathcal{E}(\Omega) \).

Let \( \varphi \in \mathcal{E}_0(\Omega) \). By Lemma 2.4 we can find a decreasing sequence \( \{v_j\} \subset \mathcal{F}(\Omega) \) such that
\[
v_j = v \text{ on } \Omega \cap \{ j \varphi < -1 \}.
\]
It follows that
\[
h \leq \hat{v}_{j+1} \leq \hat{v}_j \text{ on } \hat{\Omega}
\]
and
\[
h = \hat{v}_j \text{ on } \Omega \cap \{ j \varphi < -1 \}.
\]
Now, by Lemma 2.3 we can find plurisubharmonic functions \( \varphi_{j,1} \) and \( \varphi_{j,2} \) defined on Euclidean neighborhood \( \Omega' \) of \( \Omega \) such that
\[
\Omega \cap \{ j \varphi < -1 \} = \Omega' \cap \{ \varphi_{j,1} < \varphi_{j,2} \}.
\]
Hence, using Theorem 1.1 in [16] we deduce by (3.10) that
\[
(\ddc h)^n = (\ddc \hat{v}_j)^n \text{ on } \Omega \cap \{ j \varphi < -1 \}.
\]
On the other hand, we infer by (3.9) and Lemma 4.1 in [1] that
\[
1_{\{\hat{v}_j = -\infty\}}(\ddc \hat{v}_j)^n \leq 1_{\{h = -\infty\}}(\ddc h)^n \text{ in } \hat{\Omega}.
\]
Combining this with (3.6), (3.8) and (3.11) that
\[
1_{\Omega \cap \{ j \varphi < -1 \}} \mu \leq P(\ddc v_j)^n \leq \mu \text{ in } \hat{\Omega}.
\]
We set
\[
g_{j,k} := \sup\{ \psi \in \mathcal{F}-PSH(\Omega) : \psi \leq v_j + k \text{ on } \Omega \}, \ k \in \mathbb{N}.
\]
Let $g_j$ be the $\mathcal{F}$-upper semi-continuous majorant of $\sup_{k \geq 1} g_{j,k}$ in $\Omega$. Lemma 3.2 tells us that $g_j \in \mathcal{F}(\Omega)$ and

\[
\begin{cases}
P(\ddc g_j)^n = P(\ddc v_j)^n & \text{in } \mathbb{C}^n, \\
NP(\ddc g_j)^n = 0 & \text{on } QB(\Omega).
\end{cases}
\]  

(3.13)

Since $\{v_j\}$ is decreasing, so $g_{j+1,k} \leq g_{j,k}$ on $\Omega$, and thus, $\{g_j\}$ is a decreasing sequence. Moreover, using Theorem 1.3 in [6] we obtain by (3.13) that

\[
\sup_{j \geq 1} \int_{\Omega} (\ddc \max(g_j, -1))^n = \sup_{j \geq 1} \int_{\Omega} P(\ddc g_j)^n \leq \int_{\Omega} d\mu \leq \int_{\Omega} P(\ddc w)^n < +\infty.
\]

Hence, Lemma 2.5 tells us that $g := \lim_{j \to +\infty} g_j \in \mathcal{F}(\Omega)$.

Since $\Omega \cap \{j \varphi < -1\} \nearrow \Omega$ as $j \nearrow +\infty$, we conclude by (3.11), (3.12) and (3.13) that

\[
\begin{cases}
NP(\ddc u)^n = \nu & \text{in } QB(\Omega), \\
w_1 + g \leq u \leq g & \text{on } \Omega.
\end{cases}
\]  

(3.14)

This proves step 1.

Step 2. We prove that there exist $w_1 \in \mathcal{F}^a(\Omega)$ and $u \in \mathcal{F}-PSh^-(\Omega)$ such that

\[
\begin{cases}
NP(\ddc u)^n = \nu & \text{in } QB(\Omega), \\
w_1 + g \leq u \leq g & \text{on } \Omega.
\end{cases}
\]  

Indeed, by the definition of $g_j$ we have

\[
g_j \geq g_{j,k} \geq v_j \geq h \text{ on } \Omega, \ \forall j, k \geq 1,
\]

and hence,

\[
g = \lim_{j \to +\infty} g_j \geq h \text{ in } \Omega.
\]

We set

\[
U_j := \Omega \cap \{h > -j\}.
\]

Since $h$ is plurisubharmonic function in $\hat{\Omega}$, so $U_j$ is $\mathcal{F}$-open. From $g$ is bounded on $U_j$, we infer by (3.7) and Theorem 1 in [21] that $g$ is $\mathcal{F}$-maximal on $U_j$. Therefore, the proof of Theorem 1.1 in [18] tells us that

\[
\psi_j := \sup\{ f \in \mathcal{F}-PSh^-(U_j) : f \leq g \text{ on } U_j \text{ and } NP(\ddc f)^n \geq \nu \text{ on } QB(U_j)\}
\]

is $\mathcal{F}$-plurisubharmonnic in $U_j$ and satisfies

\[
NP(\ddc \psi_j)^n = \nu \text{ on } QB(U_j).
\]  

(3.15)

Let $w_1 \in \mathcal{F}^a(\Omega)$ be such that

\[
(\ddc w_1)^n = NP(\ddc w)^n \text{ on } \Omega.
\]
Since $U_j \subset U_{j+1}$ and $NP(dd^c(w_1 + g))^n \geq \nu$ in $QB(U_j)$, we deduce by the definition of $\{\psi_j\}$ that

\begin{equation}
(3.16) \quad w_1 + g \leq \psi_{j+1} \leq \psi_j \leq g \text{ on } U_j.
\end{equation}

Hence, the $F$-upper semi-continuous majorant $\psi$ of

\[ \limsup_{j \to +\infty} \psi_j(z), \quad z \in \Omega \cap \{h > -\infty\} \]

is a finite $F$-plurisubharmonic function on $\Omega \cap \{h > -\infty\}$. Using Theorem 4.5 in [11] we deduce by (3.15) and (3.16) that

\begin{equation}
(3.17) \quad \begin{cases} 
NP(dd^c\psi)^n = \nu & \text{on } QB(\Omega \cap \{h > -\infty\}), \\
w_1 + g \leq \psi \leq g & \text{on } \Omega \cap \{h > -\infty\}.
\end{cases}
\end{equation}

Now, using Theorem 3.7 in [10] we obtain that

\[ u(z) := \begin{cases} 
\psi(z) & \text{if } z \in \Omega \cap \{h > -\infty\} \\
F\lim_{\Omega \cap \{h > -\infty\} \ni \xi \to z} \psi(\xi) & \text{if } z \in \Omega \cap \{h = -\infty\}
\end{cases} \]

is a $F$-plurisubharmonic function in $\Omega$. Since $\Omega \cap \{h = -\infty\}$ is a pluripolar set, we deduce by (3.17) and Corollary 3.2 in [13] that

\[ w_1 + g \leq u \leq g \text{ on } \Omega. \]

Combining this with (3.17) we conclude that (3.14) has been proven.

**Step 3.** The inequality (3.14) tells us that $w_1 + g \leq u \leq g$ in $\Omega$.

Lemma 3.1 states that $u \in F(\Omega)$ and

\[ P(dd^c u)^n = P(dd^c g)^n \text{ in } \mathbb{C}^n. \]

Therefore, we conclude by (3.7) and (3.14) that

\begin{equation}
(3.18) \quad \begin{cases} 
P dd^c u)^n = P(dd^c g)^n = \mu & \text{in } \mathbb{C}^n, \\
NP(dd^c u)^n = \nu & \text{on } QB(\Omega).
\end{cases}
\end{equation}

The proof is complete. \[\Box\]

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