ORBIFOLD QUANTUM COHOMOLOGY OF THE CLASSIFYING
SPACE OF A FINITE GROUP

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ABSTRACT. We work through, in detail, the quantum cohomology, with gravitational descendants, of the orbifold $BG$, the point with action of a finite group $G$. We provide a simple description of algebraic structures on the state space of this theory. As a consequence, we find that multiple copies of commuting Virasoro algebras appear which completely determine the correlators of the theory.

1. INTRODUCTION

W. Chen and Y. Ruan [CR1, CR2] introduced the notion of the Gromov-Witten invariants of an orbifold $V$. Their construction reduces to the usual Gromov-Witten invariants when $V$ is a smooth, projective variety. When $V = [Y/G]$, where $Y$ is a smooth, projective variety, and $G$ is a finite group, the state space of this theory is generally larger than the $G$-invariant part of the cohomology of $Y$; indeed, it has additional direct summands associated to loci in $Y$ with nontrivial isotropy. These loci are called the twisted sectors of the theory, and their presence should be part of the proper notion of the cohomology of an orbifold.

The purpose of this paper is to provide a detailed treatment of this theory for the simplest case, namely, when $V$ is the classifying stack $BG$ of a finite group $G$. Many of the features and subtleties of the Gromov-Witten invariants of orbifolds are present even here. The state space of this theory contains twisted sectors and the correlators in this theory are intersection numbers on $\overline{M}_{g,n}(BG)$, the moduli space of genus-$g$, $n$-pointed orbifold stable maps into $BG$.

The correlators in this theory can be described in purely group theoretic terms, and we recover the result [Rh] that the algebraic structure on the state space $H$ is isomorphic to the center of the group algebra, $\mathbb{Z}C[G]$, together with an invariant metric.

Furthermore, we show that on the large phase space of this theory, there are $r$ commuting copies of “half” the Virasoro algebra in this theory, where $r$ is the dimension of $H$, all of which annihilate, and completely determine, the exponential of the large phase space potential function. We obtain a proof of the usual Virasoro conjecture [EHX] as the special case where a diagonal action (after a variable rescaling) of these Virasoro algebras is considered. Similarly, the relevant integrable hierarchy consists of $r$ commuting copies of the KdV hierarchy.

Finally, it is worth pointing out that the moduli spaces $\overline{M}_{g,n}(BG)$ have many features similar to $\overline{M}_{g,n}^{1/r}$, the moduli space of $r$-spin curves [JKV]. The moduli spaces $\overline{M}_{g,n}(BG)$ have boundary strata indexed by stable graphs whose tails and

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half-edges are decorated by elements of $G$ (up to conjugation) while $\overline{M}_{g,n}$ have boundary strata indexed by stable graphs whose tails and half-edges are decorated by elements of $\mathbb{Z}_r$. The construction of the correlators in both theories is also analogous. However, the ring structures on the state spaces are distinctly different. It would be interesting to find the analogs of the above results for the spaces $\overline{M}_{g,n}$.

1.1. Conventions and notation. Throughout this paper, except where otherwise specified, we will work only over the complex numbers $\mathbb{C}$, and all groups will be finite.

For a given group $G$ we denote the classifying stack of $G$ by $B\mathbb{G}$; namely, $B\mathbb{G}$ is the stack quotient $B\mathbb{G} := [pt/G]$ of a point modulo a trivial $G$ action.

Group elements will always be denoted by lower-case Greek letters, and the conjugacy class of an element $\gamma \in G$ is denoted $[\gamma]$. An $n$-tuple of elements $(\gamma_1, \ldots, \gamma_n)$ will generally be denoted by a boldfaced $\gamma$. The centralizer in $G$ of an element $\gamma$ is denoted $C(\gamma)$, and the intersection $\bigcap_{i=1}^n C(\gamma_i)$ of several centralizers is denoted $C(\gamma_1, \ldots, \gamma_n) = C(\gamma)$. The commutator $[\alpha, \beta]$ of two group elements $\alpha$ and $\beta$ is denoted $[\alpha, \beta]$. Finally, the center of an algebra $A$ will be denoted $ZA$.

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2. Orbifold stable maps into $BG$

2.1. The stack. Our chief objects of study in this paper are Gromov-Witten invariants arising from the stack $\overline{M}_{g,n}(BG)$ of $n$-pointed orbifold stable maps into $BG$, in the sense of W. Chen and Y. Ruan [CR1, CR2] (called balanced twisted stable maps by Abramovich and Vistoli [AV]).

These are maps $f : \Sigma \rightarrow BG$ from an orbifold Riemann surface (orbicurve) $\Sigma$ into the classifying stack $BG$ of a finite group $G$. Here $\Sigma$ has non-trivial orbifold structure only at marked points $p_1, \ldots, p_n$ and at nodes, and the orbifold structure at the nodes is balanced, meaning that the action of the stabilizer $G_q \simeq \mathbb{C}_l$ at a nodal geometric point $q$ of $\Sigma$ has complementary eigenvalues on the tangent spaces of the two branches of $\Sigma$ at $q$.

The stack $\overline{M}_{g,n}(BG)$ is a smooth, proper Deligne-Mumford stack with projective coarse moduli space $[ACV$, Thm 3.0.2]. This stack has a number of important connections to other moduli problems. For example, in the case that $G$ is the symmetric group $S_d$ on $d$ letters, the stack $\overline{M}_{g,n}(BS_d)$ is the normalization of the stack of admissible covers $[ACV$, §4].

Recall that an orbifold stable map from a smooth, $n$-pointed orbicurve $(\Sigma, p_1, \ldots, p_n)$ into $BG$ determines a principal $G$ bundle on the complement $\Sigma - \{p_1, \ldots, p_n\}$. This, in turn, is determined by its holonomy, that is by a homomorphism $\pi_1(\Sigma - \{p_1, \ldots, p_n\}) \rightarrow G$. Moreover, the stabilizer $G_{p_i}$ of a marked point $p_i$ of $\Sigma$ is always cyclic, and the order of $G_{p_i}$ is equal to the order of the holonomy around that marked point. Conversely, $G$ acts by conjugation on the homomorphisms $\pi_1(\Sigma - \{p_1, \ldots, p_n\}) \rightarrow G$, and two such homomorphisms determine the same $G$-bundle on $\Sigma - \{p_1, \ldots, p_n\}$ precisely when they differ by this adjoint action of $G$. 
Finally, since $BG$ is a proper, separated stack, any principal $G$ bundle on $\Sigma - \{p_1, \ldots, p_n\}$ extends uniquely, after suitable base extension, to a principal bundle on some proper curve $\Sigma$, with an isomorphism $\Sigma - \{p_1, \ldots, p_n\} \to \{p_1, \ldots, p_n\}$. The data of this cover (up to the obvious notion of equivalence of such data) is exactly equivalent to the data of a principal bundle on an orbicurve. Moreover, the adjoint action of $G$ on these homomorphisms exactly corresponds to the natural action of $G$ on the orbifold stable maps. Thus we have the following:

**Proposition 2.1.** For a given curve $[C, p_1, \ldots, p_n]$ corresponding to a point of the smooth locus $\mathcal{M}_{g,n}$, the fiber of $\mathcal{M}_{g,n}(BG)$ over the point $[C]$ corresponds to the quotient $\Hom(\pi_1(C - \{p_1, \ldots, p_n\}), G)/\text{ad } G$.

2.2. **Morphisms.** There are several natural morphisms from $\mathcal{M}_{g,n}(BG)$.

First, recall that for $X = BG$ the twisted sectors $X_{[\gamma]} = \{(x, [\gamma]) | x \in X, \gamma \in G_x\}$ of $[\text{CR1}]$ are simply $BG_{[\gamma]} \cong [pt/C(\gamma)]$.

There are evaluation morphisms

$$ev_i : \mathcal{M}_{g,n}(BG) \to BG,$$

where $BG = \bigsqcup_{[\gamma]} BG_{[\gamma]}$ is the disjoint union of twisted sectors. We can describe the evaluation morphism $ev_i$ as follows: Any stable map $f : \Sigma \to BG$ must be representable, and hence must induce an injective homomorphism $f_* : G_{p_i} \to G$ from the local group $G_{p_i} = \mathbb{Z}/m_i$ of the $i$th marked point $p_i$ of $\Sigma$ into $G$. The group $G$ acts by conjugation on this homomorphism $f_*$, and so the image of $1 \in \mathbb{Z}/m_i$ in $G$ is defined only up to conjugacy. The evaluation morphism $ev_i$ is the morphism taking $[\Sigma, p_1, \ldots, p_n, f]$ to the point $(f(p_i), [f_*(1)]) \in BG_{[f_*(1)]} \subseteq BG$. Alternatively, the image $f_*(1)$ is simply the holonomy of the induced $G$-bundle on $\Sigma - \{p_1, \ldots, p_n\}$ around the marked point $p_i$.

We can use the evaluation morphism to see that the stack $\mathcal{M}_{g,n}(BG)$ breaks up as the disjoint union of open and closed substacks

$$\mathcal{M}_{g,n}(BG) = \bigsqcup_{(\gamma_1, \ldots, \gamma_n)} \mathcal{M}_{g,n}(BG, [\gamma_1], \ldots, [\gamma_n]),$$

where $\mathcal{M}_{g,n}(BG, [\gamma_1], \ldots, [\gamma_n]) = ev_1^{-1}(BG_{[\gamma_1]}) \cap \cdots \cap ev_n^{-1}(BG_{[\gamma_n]})$ is the sub-stack of points of $\mathcal{M}_{g,n}(BG)$ mapped by $ev_i$ to $BG_{\gamma_i}$ for every $i \in \{1, \ldots, n\}$. Of course, $\mathcal{M}_{g,n}(BG, [\gamma_1], \ldots, [\gamma_n])$ may be empty for some choices of conjugacy classes $([\gamma_1], \ldots, [\gamma_n])$.

In the special case of $\mathcal{M}_{0,3}(BG)$, since there is only one 3-pointed, genus-zero stable curve (call it $\Sigma$), we may fix, once and for all a base point $q$ (distinct from the three marked points $p_1, p_2$, and $p_3$) and a basis $\{s_1, s_2, s_3\}$ of $\pi_1(\Sigma - \{p_1, p_2, p_3\}, q)$ such that $\prod_{i=1}^3 s_i = 1$. In this case, Proposition 2.1 shows that each component of $\mathcal{M}_{0,3}(BG)$ is uniquely determined up to simultaneous (diagonal) adjoint action of $G$ by a choice of three elements $\gamma_1, \gamma_2, \gamma_3 \in G$ such that $\prod \gamma_i = 1$. That is, if we let $[\gamma]$ denote the diagonal conjugacy class of the triple $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, and we let $T^3$ denote the set of all such triple conjugacy classes, whose product is trivial (i.e., $\prod \gamma_i = 1$), then

$$\mathcal{M}_{0,3}(BG) = \bigsqcup_{[\gamma] \in T^3} \mathcal{M}_{0,3}(BG, [\gamma]).$$
and for a given $[γ] ∈ T^3$, we have

$$\overline{M}_{0,3}(BG, [γ]) ≅ BC(γ).$$

For $g > 0$ or $n > 3$ one cannot index the moduli space so easily, since deformations in the moduli space may act on the holonomies in non-diagonal ways. Nevertheless, for any fixed, smooth, $n$-pointed curve, Proposition 2.1 gives a complete description of the points of $\overline{M}_{g,n}(BG)$ that lie over it. This shows that the forgetful morphism $π : \overline{M}_{g,n}(BG) → \overline{M}_{g,n}$, is quasi-finite. In fact $π$ is also proper, but it is not generally representable [ACV, Cor 3.0.5].

Finally we have the forgetting-tails morphism, defined as follows. When the holonomy $γ_i$ around a marked point $p_i$ is trivial, then we may forget the data of that marked point. This gives a morphism

$$\overline{M}_{g,n+1}(BG, [γ_1], \ldots, [1], \ldots, [γ_n]) → \overline{M}_{g,n}(BG, [γ_1], \ldots, [1], \ldots, [γ_n]).$$

Note that the forgetting-tails morphism is not defined for all of $\overline{M}_{g,n+1}(BG)$, but rather only for the components corresponding to marked points with trivial holonomy.

3. Gromov-Witten invariants, cohomological field theory and K-theory

We define the classes $ψ_i$ on $\overline{M}_{g,n}(BG)$ to be the pullbacks $ψ_i = π^*(ψ_i)$ of the $ψ_i$ classes on $M_{g,n}$.

The tangent bundle of $BG$ is trivial, and thus the virtual fundamental class of $\overline{M}_{g,n}(BG)$ is just the usual fundamental class.

The orbifold cohomology of $BG$ is, as a vector space,

$$H := H^*_\text{orb}(BG, \mathbb{C}) := H^*(BG, \mathbb{C}) = \bigoplus_{[γ]} \mathbb{C}. $$

For each conjugacy class $[γ]$ in $G$, let $e_1$ denote the class $1 ∈ H^0(BG_{[γ]}), \subseteq H$. The $e_1$’s form a basis of $H$ and we may form $n$-point correlators

$$\langle τ_{a_1}(e_1) \cdots τ_{a_n}(e_1) \rangle_g := \int_{\overline{M}_{g,n}(BG)} \prod_i \psi_i^{a_i} e_i^{a_i} (e_1),$$

3.1. Cohomological field theory. The three-point, genus-zero correlators play a special role, since they define the metric on $H$ and the quantum (orbifold) product. They vanish for dimensional reasons unless $\sum_{i=1}^n a_i = 0$.

Proposition 3.1. We have

$$\langle τ_{a_1}(e_1) τ_{a_2}(e_2) τ_{a_3}(e_3) \rangle_g = \sum_{\sigma_1, \sigma_2, \sigma_3} \frac{1}{|G|} \prod_{\sigma_i = 1} [\sigma_i]_{e_1, [γ]} = \sum_{\sigma_1, \sigma_2, \sigma_3} \frac{1}{|C(σ_1, σ_2)|}. $$
Proof. For a given component $\overline{\mathcal{M}}_{0,3}(BG, [\sigma_1, \sigma_2, \sigma_3])$ of the moduli space $\overline{\mathcal{M}}_{0,3}(BG)$, the evaluation map $ev_i$ maps this component to $BG[\sigma_i]$, so

$$ev_i^*(e_{[\gamma_i]}) = \begin{cases} 0 & \text{if } \sigma_i \notin \gamma_i \\ 1 & \text{if } \sigma_i \in \gamma_i \end{cases}.$$ 

Moreover,

$$\int_{\overline{\mathcal{M}}_{0,3}(BG, [\sigma_1, \sigma_2, \sigma_3])} 1 = \text{deg}(\overline{\mathcal{M}}_{0,3}(BG, [\sigma_1, \sigma_2, \sigma_3])) = \frac{1}{|C(\sigma_1, \sigma_2, \sigma_3)|} = \frac{1}{|C(\sigma_1, \sigma_2)|}.$$ 

So

$$\langle \tau_0(e_{[\gamma_1]})\tau_0(e_{[\gamma_2]})\tau_0(e_{[\gamma_3]}) \rangle^G_{\overline{\mathcal{M}}_{0,3}(BG, [\sigma_1, \sigma_2, \sigma_3])} = \sum_{\sigma_i, \sigma_j, \sigma_k \in \gamma_i} \frac{1}{|C(\sigma_1, \sigma_2)|},$$

and it is easy to see that the number of elements in a given non-empty conjugacy class $[\sigma_1, \sigma_2, \sigma_3]$ of triples is exactly $\frac{1}{|C(\sigma_1, \sigma_2)|}$, so the expression (3.1) gives the rest of the proposition. \hfill $\square$

**Corollary 3.2.** The metric on $\mathcal{H}$ induced by the 3-point correlators is

$$\eta_{[\gamma_1][\gamma_2]} := \eta(e_{[\gamma_1]}, e_{[\gamma_2]}) = \frac{1}{|G|}\delta_{[\gamma_1][\gamma_2^{-1}]}(\gamma_1 \sigma_1 \sigma_2 \gamma_2^{-1})^{-1},$$

which is non-degenerate on $\mathcal{H}$. The quantum product is given by

$$e_{[\gamma_1]} * e_{[\gamma_2]} = \sum_{\sigma_1, \sigma_2 \in \overline{\gamma}_i} \frac{|C(\sigma_1, \sigma_2)|}{|G|} e_{[\sigma_1, \sigma_2]}.$$ 

It is clear that $\mathcal{H}$ is additively isomorphic to the ring $\text{Class}_C(G)$ of class functions of $G$, where $e_{[\gamma]}$ is the class function $e_{[\gamma]}(\sigma) = \delta_{[\gamma],[\sigma]}$. Moreover, $\text{Class}_C(G)$ has a natural metric on it: $(f, g) = \frac{1}{|G|} \sum_{\sigma \in G} f(\sigma)g(\sigma^{-1})$. Let $\Phi$ denote the standard additive isomorphism

$$\Phi: \text{Class}_C(G) \longrightarrow ZG[G]$$
defined by linearly extending $\Phi(e_{[\gamma]}) = \sum_{\alpha \in \gamma} \alpha$. The standard group-algebra product $\star$ in $ZG[G]$ pulls back via $\Phi$ to convolution $\star$ of functions. And the push-forward $\Phi_*(\cdot, \cdot)$ to $ZG[G]$ of the metric on $\text{Class}_C(G)$ agrees with the metric $(\cdot, \cdot)$ on $ZG[G]$ defined as $(\alpha, \beta) = \frac{1}{|G|} \delta_{\alpha, \beta^{-1}}$, when $\alpha$ and $\beta$ are in $G$. With respect to this metric, $ZG[G]$ is a Frobenius algebra.

**Corollary 3.3.** The homomorphisms

$$\begin{array}{ccc} (\mathcal{H}, \star, \eta) & \longrightarrow & (\text{Class}_C(G), \star, \langle \cdot, \cdot \rangle) \phi \longrightarrow (ZG[G], \cdot, \langle \cdot, \cdot \rangle) \end{array}$$

are isomorphisms of Frobenius algebras.

**Proof.** It is well-known, and clear, that these maps are isomorphisms of vector spaces. The rest is a straightforward computation using the definitions

$$\eta_{[\gamma_1][\gamma_2]} = \langle \tau_0(e_{[\gamma_1]})\tau_0(e_{[\gamma_2]})\tau_0(e_{[\gamma_1]}) \rangle^G_0$$

and

$$e_{[\alpha]} \star e_{[\beta]} := \sum_{[\gamma] \in \gamma} \langle \tau_0(e_{[\alpha]})\tau_0(e_{[\beta]})\tau_0(e_{[\gamma]}) \rangle^G_0 \eta_{[\gamma][\gamma_1]} e_{[\gamma_1]}.$$
These definitions are essentially the same as those given in [AGV], but they differ a priori from those of Chen and Ruan [CR1, CR2] for the “orbifold Poincaré pairing” and “orbifold cup product.” Nevertheless, it is easy to check that both the geometry and the final calculations are identical to those in [CR1, Ru]. For example, the three-fold multisectors $X_{[\gamma]}$ of [Ru, Defn 3.1.3] are, in our case, precisely the components $M_{[\gamma]}(BG, [\gamma])$.

We are interested now in the corresponding cohomological field theory (CohFT) and the large phase space (when $a_i > 0$).

**Proposition 3.4.** The correlators $\langle \tau_{a_1}(e_{[\sigma_1]}) \cdots \tau_{a_n}(e_{[\sigma_n]}) \rangle^G$ are related to the usual correlators $\langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g$ (corresponding to the case of $G = \{1\}$) by

$$\langle \tau_{a_1}(e_{[\sigma_1]}) \cdots \tau_{a_n}(e_{[\sigma_n]}) \rangle^G_g = \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g \Omega^G_g(\gamma)$$

where

$$\Omega^G_g(\gamma) = \frac{|\mathcal{X}^G_g(\gamma)|}{|G|},$$

and

$$\mathcal{X}^G_g(\gamma) := \{(a_1, \ldots, a_g, \beta_1, \ldots, \beta_g, \sigma_1, \ldots, \sigma_n) | \prod_{i=1}^g [\alpha_i, \beta_i] = \prod_{j=1}^n \sigma_j, \sigma_j \in [\gamma_j] \text{ for all } j\}.$$

**Proof.** The sublocus of $\mathcal{M}_{g,n}(BG)$ where $\prod_{i=1}^g ev^*_i(e_{[\gamma_i]})$ is non-zero is $\mathcal{M}_{g,n}(BG, [\gamma_1], \ldots, [\gamma_n])$. Proposition [2] shows that the degree of the forgetful map $\pi : \mathcal{M}_{g,n}(BG, [\gamma_1], \ldots, [\gamma_n]) \to \mathcal{M}_{g,n}$ is exactly $\Omega^G_g(\gamma)$. The proof now follows from the projection formula, since $\psi_i$ on $\mathcal{M}_{g,n}(BG)$ is just the pullback $\pi^* \psi_i$ of the corresponding class on $\mathcal{M}_{g,n}$. \hfill $\Box$

**Lemma 3.5.** The numbers $\Omega^G_g(\gamma)$ depend only on the conjugacy classes $[\gamma_i]$, are independent of the ordering of the $\gamma_i$ in $\gamma$, and satisfy the following relations:

1. **Cutting trees:** For $g = g_1 + g_2$ and $I \bigsqcup J = \{1, \ldots, n\}$, let $\gamma_I = (\gamma_{i_1}, \ldots, \gamma_{i_{|I|}})$ and $\gamma_J = (\gamma_{j_1}, \ldots, \gamma_{j_{|J|}})$

$$\Omega^G_g(\gamma) = \Omega^G_{g_1}(\gamma_I, \zeta) \eta^{[k] \xi_{[\gamma]}} \Omega^G_{g_2}(\zeta, \gamma_J)$$

2. **Cutting loops:**

$$\Omega^G_g(\gamma) = \eta^{[k] \xi_{[\gamma]}} \Omega^G_{g-1}(\zeta, \xi, \gamma)$$

3. **Forgetting tails:**

$$\Omega^G_g(\gamma) = \Omega^G_g(1, \gamma)$$

**Proof.** Independence of conjugacy class representative and of order are immediate from the definition, as is relation 3 (Forgetting tails). To prove relation 2, note that we may assume that $[\zeta] = [\xi^{-1}]$. Let $\mathcal{X}^G_g(\gamma)$ be as in Proposition 3.4 and let $\mathcal{Y}$ be the set

$$\mathcal{Y} := \{(\alpha'_{i_1}, \alpha'_{i_2}, \beta'_{i_2}, \ldots, \beta'_{i_{|I|}}, \sigma'_{i_1}, \ldots, \sigma'_{i_{|I|}}, \sigma'_{n+1}) | \prod_{i=1}^{|I|} [\alpha'_i, \beta'_i] = \prod_{j=1}^{|I|} \sigma'_{i_j}, \sigma'_{i_j} \in [\gamma_{i_j}] \text{ for } j \leq n, \sigma'_{n+2} \in [\sigma'_{n+2}] \}.$$

Define a map $f : \mathcal{X}^G_g(\gamma) \to \mathcal{Y}$, taking $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \sigma_1, \ldots, \sigma_n)$ in $\mathcal{X}^G_g(\gamma)$ to $(\alpha_{i_1}, \ldots, \alpha_{i_{|I|}}, \beta_{i_2}, \ldots, \beta_{i_{|I|}}, \sigma_{i_1}, \ldots, \sigma_{i_{|I|}}, \sigma_{n+1})$ in $\mathcal{Y}$.

For a given conjugacy class $[\psi]$ in $G$, the map $f$, restricted to the subset of $\mathcal{X}^G_g(\gamma)$ where $\beta_g \in [\psi]$, takes $[C(\psi)]$ elements of $\mathcal{X}^G_g(\gamma)$ to one element of $\mathcal{Y}$. Moreover,
Tensor products arise in this theory in at least two ways. Tensor products.

3.3.2. character (see [AR]) ∗ orbifold product $\chi$ of the trace map corresponding to the obvious homomorphism of representation rings. But the center $B$ is interesting to note that K-theory has functoriality properties that $H^*_orb$ does not enjoy. In particular, a homomorphism of groups $G \longrightarrow H$ gives a morphism $BG \longrightarrow BH$ which induces a ring homomorphism

$$K^*_orb(BH) \longrightarrow K^*_orb(BG),$$

corresponding to the obvious homomorphism of representation rings. But the center $ZC[H]$ generally has no ring homomorphism to $ZC[G]$.

3.3.2. Tensor products.

Tensor products arise in this theory in at least two ways.

- $B(G \times H)$

For any two finite groups $G$ and $H$, the classifying stack $B(G \times H)$ splits up as a product

$$B(G \times H) = BG \times BH,$$

and although the moduli stack of the product is not quite the product of the moduli stacks of the factors, it is easy to see that the corresponding CohFT
is the tensor product of the two components. That is, a straightforward check shows that if \( \gamma = (\gamma_1, \ldots, \gamma_n) \in G^n \) and \( \sigma = (\sigma_1, \ldots, \sigma_n) \in H^n \) then
\[
\Omega^G_{g \times H}((\gamma_1, \sigma_1), \ldots, (\gamma_n, \sigma_n)) = \Omega^G_g(\gamma) \Omega^H_g(\sigma).
\]

• \([X/G]\) with trivial \( G \) action

If \( X \) is a smooth projective variety with trivial \( G \) action, then the quotient stack \([X/G]\) is isomorphic to the product \( X \times B G \).

**Proposition 3.7.** For a smooth projective variety \( X \) with finite group \( G \) acting trivially, the CohFT arising from stable maps into the orbifold \([X/G]\) is simply the tensor product of the CohFT arising from stable maps into \( X \) and the CohFT arising from stable maps into \( B G \).

**Proof.** This follows from the fact that the degree of the forgetful maps \( \overline{M}_{g,n}(X \times B G, [\gamma_1], \ldots, [\gamma_n]) \to \overline{M}_{g,n}(X) \) and \( \overline{M}_{g,n}(B G, [\gamma_1], \ldots, [\gamma_n]) \to \overline{M}_{g,n} \) are both equal to \( \Omega_g(\gamma) \). \(\square\)

4. Semisimplicity and Virasoro Algebras

In this section, the summation convention is NOT used on any subscripts or superscripts \( \alpha \) or \( \alpha_i \), although it is applied to all other variables.

4.1. Semisimple Frobenius algebras. Let \( V \) be any \( r \)-dimensional Frobenius algebra with multiplication \( \ast \), metric \( \eta \), and identity element \( 1 \). It is said to be a semisimple Frobenius algebra if there exists a canonical basis \( \{ f_\alpha \}_{\alpha = 1}^r \) such that for all \( \alpha_1, \alpha_2 = 1, \ldots, r \),

\[
(f_\alpha \ast f_\beta) = \delta_{\alpha_1, \alpha_2} f_\alpha
\]

and

\[
\eta(f_\alpha, f_\beta) = \delta_{\alpha_1, \alpha_2} \nu_\alpha
\]

for some nonzero numbers \( \nu_\alpha \). The identity element satisfies

\[
1 = \sum_{\alpha = 1}^r f_\alpha.
\]

As discussed before, the Frobenius algebra \((\mathcal{H}, \eta, \ast, e_{[1]})\) can be identified with \( \mathbb{C}[G] \) and the latter is a semisimple Frobenius algebra with canonical basis given as follows.

**Proposition 4.1.** Let \( \{ V_\alpha \}_{\alpha = 1}^r \) be the set of irreducible representations of \( G \) and let \( \chi_\alpha \) denote the character of \( V_\alpha \). For all \( \alpha = 1, \ldots, r \), the elements

\[
f_\alpha := \dim V_\alpha \frac{\chi_\alpha(g^{-1})g}{|G|} \sum_{g \in G} \chi_\alpha(g^{-1})g
\]

form a basis of \( \mathbb{C}[G] \) and satisfy equations \((4.1)\) and \((4.2)\), where for all \( \alpha = 1, \ldots, r \),

\[
\nu_\alpha = \left( \frac{\dim V_\alpha}{|G|} \right)^2.
\]

**Proof.** It is clear from the definition that the \( f_\alpha \) lie in \( \mathbb{C}[G] \). The fact that they satisfy \((4.1)\) follows from [FH, §2.4]. That equation \((4.2)\) holds for the given values of \( \nu_\alpha \) is a straightforward computation. \(\square\)
The results in the remainder of this section hold for any semi-simple Frobenius algebra, since any Frobenius algebra is a CohFT.

4.2. The Potential Function. We will now calculate the correlators for our theory in the canonical coordinates.

Proposition 4.2. Let \((g, n)\) be any stable pair, where \(n \geq 1\). If \(\alpha_i = \alpha\) for all \(i = 1, \ldots, n\) then for all \(a_1, \ldots, a_n \geq 0\), we have
\[
\langle \tau_{a_1}(f_{\alpha_1}) \cdots \tau_{a_n}(f_{\alpha_n}) \rangle^G_g = \nu^{1-g}_\alpha \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g;
\]
otherwise, we have \(\langle \tau_{a_1}(f_{\alpha_1}) \cdots \tau_{a_n}(f_{\alpha_n}) \rangle^G_g = 0\). Furthermore, when \(n = 0\), we have \(\langle \rangle^G_g = 0\).

Proof. Of course, \(\langle \rangle^G_g = 0\) holds for dimensional reasons.

The proof for the rest of the proposition follows by degenerating to curves whose irreducible components are all three-pointed, genus-zero curves, where the proposition is easily verified, and then calculating the general correlators from the cutting axioms for CohFTs.

More explicitly, the correlator is simply
\[
\langle \tau_{a_1}(f_{\alpha_1}) \cdots \tau_{a_n}(f_{\alpha_n}) \rangle^G_g = \Lambda^{G}_{g,n}(f_{\alpha_1} \otimes \cdots \otimes f_{\alpha_n}) \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g,
\]
where \((\mathcal{H}, \eta, \Lambda^G)\) is our CohFT. The definition of \(*\) gives
\[
\Lambda^{G}_{0,3}(f_{\alpha_1} \otimes f_{\alpha_2} \otimes f_{\alpha_3}) = \eta(f_{\alpha_1} * f_{\alpha_2}, f_{\alpha_3}) = \delta_{\alpha_1, \alpha_2} \delta_{\alpha_1, \alpha_3} \nu_\alpha.
\]

Now proceed by induction on the genus and number of marked points. Each application of the cutting trees axiom (for a 3-pointed, genus-zero vertex) leaves the genus unchanged, and reduces the number of marked points by one, but contributes nothing to the final result, since the node (cut edge) contributes the inverse metric—a factor of \(\nu^{-1}_\alpha\)—and the 3-pointed, genus-zero, irreducible component (vertex of the dual graph) contributes a factor of \(\nu_\alpha\).

Each application of the cutting loops axiom increases the number of marked points by 2, reduces the genus by 1, and contributes the inverse of the metric—namely, \(\nu^{-1}_\alpha\)—to the final result.

\[\square\]

The large phase space potential is defined by
\[
\Phi^G(t) = \sum_g \Phi_g^G(t) \lambda^{2g-2} \text{in } \lambda^{-2} \mathbb{C}[[t, \lambda]],
\]
where \(\Phi_g^G(t) := (\exp(t \cdot \mathbf{\tau}))^G_g\), where \(t \cdot \mathbf{\tau} = \sum_{a,m} \lambda^m \tau_a(h_m)\), and where \(\{h_m\}\) is any basis for \(\mathcal{H}\). Let \(Z^G := \exp(\Phi^G)\).

When \(G = \{1\}\) is the trivial group, we denote by \(\Phi := \Phi^{(1)}\), the potential of the Gromov-Witten invariants of a point. Similarly, we let \(Z := Z^{(1)}\).

Proposition 4.3. Let \(u\) be formal variables \(\{u^a_{\alpha}\}\) for all integers \(a \geq 0\) and \(\alpha = 1, \ldots, r\) associated to the canonical basis \(\{f_\alpha\}\). For each \(\alpha = 1, \ldots, r\), let \(\mathbf{u}^\alpha\)
be formal variables \(\{\mathbf{u}_{\alpha}^a\}\), where \(a \geq 0\) and \(\mathbf{u}_{\alpha}^a := (\nu_\alpha)^{1/a} u^a_{\alpha}\). Then we have
\[
\Phi^G(u) = \sum_{\alpha=1}^r \Phi(\mathbf{u}^\alpha).
\]

Proof. This follows from Proposition 4.2 and dimensional considerations. \(\square\)
4.3. Virasoro algebras. For each \( \alpha = 1, \ldots, r \) and \( n \geq -1 \), let

\[
L_{n}^{(\alpha)} := -(2n + 3)!! \frac{\partial}{\partial \tilde{u}_{n+1}^{\alpha}} + \sum_{i=0}^{\infty} (2i + 2n + 1)!! \frac{\partial}{\partial \tilde{u}_{i+n}^{\alpha}} + \frac{\lambda^2}{2} \sum_{i=0}^{n-1} (2i + 1)!! \left( \sum_{m=1}^{n} \eta_{m}^{\alpha} \frac{\partial^2}{\partial \tilde{u}_{i+n}^{\alpha}} \right) + \delta_{n,-1} \frac{\lambda}{2} \tilde{u}_{n}^{\alpha} \tilde{u}_{0}^{\alpha} + \delta_{n,0} \frac{1}{16},
\]

where \((2n - 1)! := 1 \cdot 3 \cdot 5 \cdots (2n - 1)\). These operators satisfy

\[
[L_{m}^{(\alpha)}, L_{n}^{(\beta)}] = (m - n) L_{m+n}^{(\alpha, \beta)}
\]

for all \( m, n \geq -1 \) and \( \alpha, \beta \in \{1, \ldots, r\} \), forming \( r \) commuting copies of “half” of the Virasoro algebra.

Moreover, if \( \{b_m\} \) is any basis for \( \mathcal{H} \), where \( m = 0, \ldots, r - 1 \) such that \( b_0 = 1 \), and if \( t = \{t_m^n\} \) are the associated formal parameters, then there are also operators for all \( n \geq -1 \) given by

\[
L_{n} := -(2n + 3)!! \frac{\partial}{\partial t_{n+1}^{0}} + \sum_{i=0}^{\infty} (2i + 2n + 1)!! \left( \sum_{m=1}^{n} t_{m}^{m} \frac{\partial}{\partial t_{i+n}^{m}} \right) + \frac{\lambda^2}{2} \sum_{i=0}^{n-1} (2i + 1)!! \left( \sum_{m_1, m_2} \eta_{m_1 m_2}^{\alpha} t_{i+n}^{m_1} t_{i+n}^{m_2} \right) + \delta_{n,0} \frac{r}{16},
\]

satisfying \([L_k, L_n] = (k - n) L_{k+n}\) for any \( k, n \geq -1 \).

**Proposition 4.4.** For all \( \alpha \in \{1, \ldots, r\} \) and \( n \geq -1 \),

\[
(4.5) \quad L_{n}^{(\alpha)} Z^G = 0.
\]

These equations completely determine \( Z^G \). Furthermore, for all \( n \geq -1 \),

\[
(4.6) \quad L_{n} Z^G = 0.
\]

**Proof.** Equation (4.5) follows from Proposition 4.3 and the Kontsevich-Witten theorem [Ko, Wi] for the case of \( G = \{1\} \). Equation (4.6) follows from (4.5) and the identity

\[
L_{m} = \sum_{\alpha=1}^{r} (\nu_{\alpha})^{-\frac{\lambda}{2}} L_{m}^{(\alpha)}.
\]

\( \square \)

**Remark 4.5.** Equation (4.6) is a verification of the Virasoro conjecture for \( BG \) [EHX] and can also be regarded as an example of [G3].
4.4. KdV hierarchies. Let \( \langle A \rangle^G := \langle A \exp(t \cdot \tau) \rangle^G \) and \( \langle A \rangle^G := \sum_g \langle A \exp(t \cdot \tau) \rangle^G \lambda^{2g-2} \).

The superscript \( G \) will be suppressed when \( G = \{1\} \), the trivial group.

**Proposition 4.6.** Let \( \{e_1, \ldots, e_m\} \) be any basis for \( \mathcal{H} \). For all \( v \in \mathcal{H} \) and \( a \geq 0 \), the following equation holds:

\[
(2a + 1) \lambda^{-2} \langle \tau_a v \rangle = \langle \tau_a \rangle^{G} \eta^{e_1 m_1} m_2 = \langle \tau_a \rangle^{G} \eta^{e_1 m_1} m_2 + \langle \tau_a (v) \rangle^{G} \eta^{e_1 m_1} m_2 \langle \tau_0 e_m (v) \rangle^{G} \eta^{m_3 m_4} + \frac{1}{4} \langle \tau_a (v) \rangle^{G} \eta^{m_1 m_2} \eta^{m_3 m_4}.
\]

Equation (4.7) and the fact that \( L_{-1} Z^G = 0 \) for all \( \alpha = 1, \ldots, r \) completely determine \( \Phi^G \).

**Proof.** When \( G = \{1\} \), the trivial group, equation (4.7) reduces to

\[
(2a + 1) \lambda^{-2} \langle \tau_a \rangle = \langle \tau_a \rangle + 2 \langle \tau_{a-1} \rangle \langle \tau_0 \rangle + \frac{1}{4} \langle \tau_{a-1} \rangle \langle \tau_0 \rangle \langle \tau_0 \rangle.
\]

This equation is the Kontsevich-Witten theorem [Ko, Wi]. Witten also showed that this equation together with \( L_{-1} Z = 0 \) completely determines \( \Phi \).

To prove the formula for general \( G \), choose a canonical basis \( \{f_0\}_{a=1}^r \) and let \( v = f_a \). Consider the terms of equation (4.7) proportional to \( \lambda^{2g-4} \). By Proposition 4.2, one obtains the terms of equation (4.7) proportional to \( \lambda^{2g-4} \) up to an overall scalar factor. \( \square \)

**Remark 4.7.** Equation (4.7) is a simultaneous solution of \( r \) commuting KdV hierarchies with time parameters \( \tilde{u}_a^\alpha \) for \( a \geq 0 \) and \( \alpha = 1, \ldots, r \).

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