Hiding Quantum Data

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Recent work has shown how to use the laws of quantum mechanics to keep classical and quantum bits secret in a number of different circumstances. Among the examples are private quantum channels, quantum secret sharing and quantum data hiding. In this paper we show that a method for keeping two classical bits hidden in any such scenario can be used to construct a method for keeping one quantum bit hidden, and vice-versa. In the realm of quantum data hiding, this allows us to construct bipartite and multipartite hiding schemes for qubits from the previously known constructions for hiding bits. Our method also gives a simple proof that two bits of shared randomness are required to construct a private quantum channel hiding one qubit.

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Dedication to David Mermin

It is a pleasure to have an opportunity to include our work in this tribute to our friend David Mermin. We hope that he will enjoy it, as it is a brand new result flowing from the teleportation/dense coding mindset that has been so fantastically productive in quantum information theory over the past few years. As the now-admitted midwife of teleportation, he will recognize the usual elements (the ensemble of Pauli rotations, the one-qubit-gets-you-two-bits-and-vice-versa structure) of these kinds of arguments, twisted though they may be in the service of some new cryptographic application. It seems that the metamorphosis of the teleportation game of 1993 into a myriad of different serious constructions in the service of cryptography, secure and fault tolerant computation, and communication complexity has not yet come to an end. We are grateful to David Mermin for helping set this process in motion, and for continuing to observe the resulting flowering with a generous, interested, and critical eye.

I. INTRODUCTION

Work in recent years has shown how to use the laws of quantum mechanics to keep classical and quantum bits secret in a number of different circumstances. In some scenarios, the bits are kept secret from an eavesdropper while in others, they are kept secret from the participants themselves. Perhaps the simplest such example is the quantum generalization of the one-time pad, known as a private quantum channel [1, 2]. In this setting, two parties make use of shared random bits to create a secure quantum channel between them. In this case, the message is kept secret from an eavesdropper with access to the output of the quantum channel. In contrast, the goal in quantum secret sharing [3] is to share a secret, in the form of classical or quantum bits, between many parties. Certain prescribed combinations of the parties, known as authorized sets, are
capable of fully reconstructing the secret using quantum communication while the other unauthorized combinations of parties can learn nothing at all about the secret, even if they act jointly on their shares. A third example, which will be the focus of this paper, is known as quantum data hiding. This task, introduced in Refs. [4, 5] for the bipartite setting and generalized to multiple parties in Ref. [6], imposes a stronger security criterion than quantum secret sharing. Whereas in quantum secret sharing an authorized set may be able to extract information about the secret by performing local operations in addition to classical communication, in quantum data hiding the authorized set needs to communicate quantum data in order to get substantive information about the secret. So in quantum data hiding one allows all parties to communicate classical data to one another in an effort to reveal the secret. Quantum communication within an unauthorized set, supplemented with classical communication between all parties, reveals nothing. (Or, rather, next to nothing; one of the results of Ref. [5] is that perfect quantum data hiding is impossible.)

The main result of this paper is the first construction of quantum data hiding protocols for hiding qubits; the protocols of Refs. [5] and [6] only work for hiding classical bits. Our method is to build on top of the earlier work, converting any method for hiding $2^n$ bits into a method for hiding $n$ qubits. For symmetry, we will also demonstrate how any $n$-qubit data hiding scheme can be converted into a $2^n$-bit hiding scheme. The connection, which is closely related to the duality between superdense coding [7] and teleportation [8], is largely independent of the setting of the problem. Indeed, the basic idea, as sketched in Fig. 1, can actually be applied just as well to the private quantum channel and quantum secret sharing as to quantum data hiding.

We begin section II by defining bipartite data hiding and describing the method for converting between bit and qubit hiding schemes. In sections II A and II B we provide security proofs for the resulting schemes under the idealized assumption that the original schemes were perfectly secure before relaxing to approximate hiding in sections II C and II D. The additional complications that arise in multipartite hiding are dealt with in section III, where our main result is that the only constraints on the authorized sets are the same as those for quantum secret sharing [9]. Section IV demonstrates how the duality imposes limits on the resources required for hiding. As an application, we provide a simple, conceptual proof that $2^n$ bits of shared key are required for an $n$-qubit private quantum channel.

As the reader has probably already noticed, the term quantum data hiding refers to the methods used rather than the data stored. Rather than resorting to contorted phrases like ‘quantum hiding of quantum data’, we will henceforth omit the first ‘quantum’ and refer to qubit-hiding schemes or bit-hiding schemes. Hopefully, this will simultaneously keep both confusion and redundancy to a minimum. In this paper we will denote the density operator corresponding to a pure state $|\varphi\rangle$ as $\varphi$. The phrase ‘Trace-preserving Completely Positive map’ will be abbreviated to ‘TCP map’. A TCP map that can be implemented by Local Operations supplemented by Classical Communication is called an LOCC map or operation. The trace norm $||A||_1$ of an operator $A$ is defined as $||A||_1 = \text{Tr} \sqrt{A^\dagger A}$.

II. BIPARTITE HIDING

Formalizing the description of data hiding used in the introduction, we define an $n$-bit data hiding scheme for two parties Alice ($A$) and Bob ($B$) to consist simply of a set of orthogonal bipartite hiding states $\{\rho^I_{AB}\}$, where $I = i_1i_2\ldots i_n$ is an $n$-bit string. Since these states are orthogonal, we can define a physical encoding map $E_\varphi(I)|I\rangle\langle I| = \rho_I$. The orthogonality condition guarantees that, if allowed quantum communication, Alice and Bob can perfectly recover $I$ by some decoding operations $D$. The scheme is said to be perfectly secure if Alice and Bob are incapable of learning anything using only LOCC operations. Equivalently, the scheme is perfectly secure if, for all $I, J$
and LOCC operations $L$,

$$\text{Tr}_A L(\rho_I) = \text{Tr}_A L(\rho_J).$$  \hspace{1cm} (1)

As noted in the introduction, this perfect security is not actually possible. We say that the scheme is $\epsilon$-secure if, for all $I$, $J$ and LOCC operations $L$,

$$\|\text{Tr}_A L(\rho_I) - \text{Tr}_A L(\rho_J)\|_1 < \epsilon.$$  \hspace{1cm} (2)

If $\epsilon = 0$ this definition reduces to perfect security.

Extending this approach to the case of quantum data, we say that a bipartite $n$-qubit hiding scheme consists of an encoding map $E_q$ taking $n$-qubit states $\varphi$ to bipartite hiding states $E_q(\varphi)$ on $AB$ such that there exists a TCP decoding map $D$ satisfying $D(E_q(\varphi)) = \varphi$ for all $\varphi$. The scheme is $\delta$-secure if for all $\varphi_0$ and $\varphi_1$ as well as LOCC operations $L$,

$$\|\text{Tr}_A L(E_q(\varphi_0)) - \text{Tr}_A L(E_q(\varphi_1))\|_1 < \delta.$$  \hspace{1cm} (3)

Henceforth we will restrict our attention to pure state inputs $\varphi_0$ and $\varphi_1$. This is sufficient because the convexity of the trace norm ensures that the most distinguishable states will always be pure. For the rest of the paper, we will also impose the additional requirement that the map $E_q$ correspond to a physical operation, meaning that it will be TCP. For the qubit-hiding schemes we construct, the condition will be satisfied automatically. When we attempt to construct bit hiding schemes from qubit-hiding schemes however, our method would fail without the extra condition.

Now we are ready to explain how to construct secure hiding schemes for sets of $n$ qubits starting from secure hiding schemes for $2n$ classical bits.

Assume we have a classical hiding scheme for $2n$ bits with encoding map $E_c$. The hiding states are $\rho_I$, where $I = i_1 i_2 \cdots i_n$ is a string of length $n$, each position taking an integer value between $0$
and 3. Let $\varphi$ be an $n$-qubit state. We define a TCP encoding map $E_q$ by

$$E_q(\varphi) = \frac{1}{2^n} \sum I E_c(|I\rangle\langle I|)^{AB_1} \otimes \sigma_I \varphi^{B_2} = \frac{1}{2^n} \sum I \rho_I^{AB_1} \otimes \sigma_I \varphi^{B_2},$$

where $\sigma_I = \sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_n}$ is a tensor product of Pauli operators, adopting the convention that $\sigma_0 = I$. It is clear that if $\rho_I$ are a set of orthogonal states, there exists a decoding operation $D$ such that $D \circ E_q(\varphi) = \varphi$. Notice that Bob’s register is divided into two parts, the first storing his half of the bit-hiding states $\rho_I$ and the second a Pauli-conjugated version of $\varphi$. Equivalently, a more operational way of thinking about $E_q(\varphi)$ is as the output of the circuit illustrated in Fig. 1A.

Using a dual construction, any perfectly secure $n$-qubit hiding scheme can be used to build a perfectly secure $2n$-bit hiding scheme. Let $E_q$ be some TCP encoding map, not necessarily of the form of Eq. (4), hiding states of an $n$-qubit register $B_1$ on a bipartite system $AB_1$. (We will assume without loss of generality when discussing this ‘superdense coding scheme’ that the initial state is stored on Bob’s system.) Our method of hiding $2n$ classical bits combines the quantum hiding scheme and superdense coding, see Fig. 1B. Let

$$|\Phi_I\rangle^{B_1B_2} = (\sigma_I \otimes I^{B_2})|\Phi\rangle^{B_1B_2},$$

(5)

where $|\Phi\rangle^{B_1B_2} = 2^{-n/2} \sum_{k=1}^{2^n} |k\rangle^{B_1} |k\rangle^{B_2}$ is a maximally entangled state between the registers $B_1$ and $B_2$. We define the hiding states for the classical bits to be

$$\rho_I = (E_q \otimes I^{B_2}) (\Phi_I^{B_1B_2}).$$

(6)

Since $E_q$ is a qubit-hiding scheme there exists a decoding operation $D$ such that $D \circ E_q = I$. This implies that the classical hiding states $\rho_I$ are orthogonal since they can be mapped by an operation $D \otimes I$ onto the orthogonal states $\Phi_I$.

In the next few sections we will prove the security of a qubit hiding scheme from the security of the bit-hiding scheme and vice-versa.

A. Perfect Hiding: Classical $\rightarrow$ Quantum

Our goal is to show that if the $\rho_I$ are perfectly secure $2n$-bit hiding states then the states $E_q(\varphi)$, defined in Eq. (4), are, likewise, perfectly secure $n$-qubit hiding states. By construction, some simple-minded approaches to cheating by Alice and Bob will fail to yield any information about $\varphi$. First, because the reduced state on the $B_2$ register is always maximally mixed, no measurement by Bob on $B_2$ alone will yield any information about the input state. Similarly, since the $AB_1$ register starts independent of $\varphi$, any LOCC protocol applied to it alone will have output independent of $\varphi$. Moreover, since the $\rho_I$ form a set of perfect hiding states, Bob’s final reduced density operator on $B_1$ will be independent of $I$ so he can’t learn anything that would help him to undo the $I$-dependent Pauli rotations on $B_2$. This doesn’t prove security, however. It is conceivable that by acting on $B_1$ and $B_2$ together in an LOCC protocol that Bob might be able cheat by a strategy we haven’t yet considered. We now give a formal proof that this is not possible.

Suppose, on the contrary, that the proposed $n$-qubit hiding scheme is not secure against arbitrary LOCC cheating. That is, there is a choice of input states $\varphi_0$ and $\varphi_1$ and an LOCC operation $L$ such that

$$\text{Tr}_A L(E_q(\varphi_0)) \neq \text{Tr}_A L(E_q(\varphi_1)).$$

(7)

In words, Bob’s output density operator at the end of the LOCC protocol depends on the input state, meaning that he can perform a local measurement that will distinguish to some degree...
between inputs $\varphi_0$ and $\varphi_1$. Our goal in what follows will be to prove that if this were true, the $\rho_I$ could not be perfect hiding states.

For convenience, we’ll adopt the more compact notation $L = \text{Tr}_A \circ L$. We can then introduce the operations

$$L_I(\tau^{B_2}) = L(\rho_I^{AB_1} \otimes \tau^{B_2})$$

which represent the action of the LOCC operation $L$ given a particular value of the hiding state $\rho_I$. Note that $L_I$, while it is TCP, is not necessarily itself an LOCC operation because it involves the preparation of a potentially entangled ancilla $\rho_I^{AB_1}$. We can then write

$$L(E_q(\varphi)) = \frac{1}{2^n} \sum_I L_I(\sigma_I \varphi \sigma_I)$$

for the output of the ‘cheating’ operation on an $n$-qubit hiding state. From this identity, we can conclude that not all the $L_I$ are identical to $L_0$, however; if they were, then by linearity,

$$L(E_q(\varphi)) = L_0 \left( \frac{1}{2^n} \sum_I \sigma_I \varphi \sigma_I \right)$$

would be independent of the input state $\varphi$, violating Eq. (7).

The non-constancy of the $L_I$ can then be converted into a method for breaking the $2n$-bit hiding scheme. Supplied with a state $\rho_I^{AB_1}$ from which they would like to learn about $I$, Alice and Bob implement the following LOCC protocol. First, Bob prepares a maximally entangled state $|\Phi\rangle^{B_2 B_3} = \frac{1}{\sqrt{2^n}} \sum_{k=1}^{2^n} |k\rangle^{B_2} |k\rangle^{B_3}$, resulting in the outcome $(L_I \otimes I_{B_3})(\Phi)$ on Bob’s system alone. By the Jamiołkowski isomorphism between operations and states [10], the non-constancy of the $L_I$ implies that the outcome cannot be independent of $I$. Hence, Bob can perform a local measurement whose outcome will be $I$-dependent and the $2n$ bit hiding scheme based on the $\rho_I$ cannot be secure.

### B. Perfect Hiding: Quantum $\rightarrow$ Classical

Consider the definition of the bit-hiding states in Eq. (6). Because we can choose an operator basis consisting of density operators $\tau_J$, there is an expansion

$$|\Phi\rangle^B = \sum_{JK} \alpha_{JK} \tau_J \otimes K$$

of the projector for the maximally entangled state in terms of density operators for product states. (The $\alpha_{JK}$, of course, will not all be positive.)

Now let’s try cheating on our $2n$-bit hiding scheme. If $L = \text{Tr}_A \circ L$ is again an LOCC operation with output on Bob’s system, then substituting Eq. (12) into (9) shows that

$$L(\rho_I) = L((E_q \otimes I_{B_2})(\Phi^{B_1 B_2}))$$

$$= \sum_{JK} \alpha_{JK} L(E_q(\sigma_I \tau_J \sigma_I)^{AB_1} \otimes \tau_K^{B_2}).$$

(13)
The operation of first preparing $\tau_K$ on $B_2$ and then applying $\mathcal{L}$ is itself LOCC so by the perfect security of the $n$-qubit hiding scheme, we can conclude that $\mathcal{L}(E_q(\sigma_I\tau_I\sigma_J)^{AB_1}\otimes \tau^{B_2}_K)$ is independent of $\sigma_I\tau_J\sigma_I$ for all $I$ and $J$. Consequently, $\mathcal{L}(\rho_I)$ is independent of $I$, meaning that the $2n$-bit hiding scheme is perfectly secure.

C. Imperfect Hiding: Classical $\to$ Quantum

As was shown in Ref. [3], while a bit-hiding scheme can be made $\epsilon$-secure for all $\epsilon > 0$, perfect security is not possible. So, we need to investigate whether a nearly secure bit-hiding scheme can be converted into a $\delta$-secure $n$-qubit hiding scheme, for $\delta = \epsilon 2^{n+1}$. The exponential factor $2^{n+1}$, while undesirable, needn’t cause practical difficulties: for the bit-hiding schemes presented in Refs. [5] and [6], $\epsilon$ decreases exponentially with the size of the hiding state. Therefore, the factor $2^{n+1}$ can be suppressed at a cost of increasing the size of the hiding state by a factor polynomial in $n$. Furthermore, it could very well be that the estimates we present here are not tight and that the factor is only an artifact of our analysis. In any case, the idea behind the proof of security is the same as in the perfect case but the details, unfortunately, become significantly more technical. So as not to repeat ourselves, we will adopt the notation of section II A.

In the perfect case, we proceeded by making a connection between the behavior of $\mathcal{L} \circ E_q$ and $(\mathcal{L}_I \otimes I_{B_3})(\Phi^{B_2B_3})$. Name this last operator $\omega_f$ and introduce $\Delta_f = \omega_f - \omega_0$. Recall also that the encoding operation $E_q$ takes $B_2$, an $n$-qubit system, to $AB_1B_2$ while the LOCC $\mathcal{L}$ takes $AB_1B_2$ to a Bob-only system. Since we don’t want to make any assumptions yet about it’s structure, we will call this system $B_f$. We define the state

$$\xi^{B_fB_3} = ((\mathcal{L} \circ E_q) \otimes I_{B_3})(\Phi^{B_2B_3}).$$

The state $\xi$ can be related to the action of the map $\mathcal{L} \circ E_q$ on a state $\varphi$ by the identity

$$\varphi = 2^n \text{Tr}_2((I \otimes \varphi^*)\Phi^{12}),$$

where $\varphi^*$ is the complex conjugate of the density matrix of $\varphi$ and the numbers 1 and 2 are general system labels. Now, as in the perfect hiding case, assume that the $n$-qubit hiding scheme is not $\delta$-secure, meaning that there exist states $\varphi_0$ and $\varphi_1$ such that

$$\delta < \|((\mathcal{L} \circ E_q)(\varphi_0) - (\mathcal{L} \circ E_q)(\varphi_1))\|_1.$$  

(16)

Using the identity in Eq. (15) and the definition of $\xi$, we find that

$$\delta < 2^n \|\text{Tr}_{B_3}(I \otimes \varphi_0^*)\xi^{B_fB_3} - \text{Tr}_{B_3}(I \otimes \varphi_1^*)\xi^{B_fB_3}\|_1.$$  

(17)

In order to relate $\delta$ to $\Delta_f$, we rewrite $\xi$ in the following manner:

$$\xi^{B_fB_3} = ((\mathcal{L} \circ E_q) \otimes I_{B_3})(\Phi^{B_2B_3})$$

$$= \frac{1}{2^{2n}} \sum_I (\mathcal{L}_I \otimes I_{B_3})((\sigma_I \otimes I_{B_3})(\Phi^{B_2B_3}(\sigma_I \otimes I_{B_3})))$$

$$= \frac{1}{2^{2n}} \sum_I (\mathcal{L}_I \otimes \sigma_I)(\Phi^{B_2B_3}(\sigma_I \otimes I_{B_3})))$$

$$= \text{Tr}_{B_3} \omega_0 \otimes \frac{1}{2^n} I_{B_3} + \frac{1}{2^{2n}} \sum_I (\mathcal{L}_I \otimes \sigma_I)(\Phi^{B_2B_3}(\sigma_I \otimes I_{B_3})))$$

$$= \text{Tr}_{B_3} \omega_0 \otimes \frac{1}{2^n} I_{B_3} + \frac{1}{2^{2n}} \sum_I (\mathcal{L}_I \otimes \sigma_I)(\Phi^{B_2B_3}(\sigma_I \otimes I_{B_3})))$$

(18)
where we have used the fact that $(\mathbb{I} \otimes \sigma_I)\Phi = \pm (\sigma_I \otimes \mathbb{I})\Phi$. When inserting this in Eq. (17) we observe that the term involving $\text{Tr}_{B_3}\omega_0 \otimes \frac{1}{2^{n-1}} \mathbb{I}_{B_3}$ makes no contribution so we need only keep the sum over $(\mathbb{I} \otimes \sigma_I)\Delta_I (\mathbb{I} \otimes \sigma_I)$. In the following derivation we will need the inequality

$$
\|PA\|_1 \leq \|A\|_1, \tag{19}
$$

where $P$ is a projector. This can be proved as follows. Let $\lambda_1 \geq \lambda_2 \geq \ldots$ be the singular values of $A$. Let $P$ be a $k$-dimensional projector. We have

$$
\|PA\|_1 = \max_U \text{Tr}(PAU) \leq \max_U \text{Tr}(QAU) = k \sum_{i=1}^k \lambda_i \leq \|A\|_1, \tag{20}
$$

where we used that $Q$ is $k$-dimensional projector.

We insert the result of Eq. (18) in Eq. (17) and apply Eq. (19), the monotonicity under partial trace and subadditivity of the trace norm to find

$$
\delta < \frac{1}{2^n} \sum_I \|(\mathbb{I} \otimes \varphi_0^I)(\mathbb{I} \otimes \sigma_I)\Delta_I (\mathbb{I} \otimes \sigma_I)\|_1
$$

$$
+ \frac{1}{2^n} \sum_I \|(\mathbb{I} \otimes \varphi_1^I)(\mathbb{I} \otimes \sigma_I)\Delta_I (\mathbb{I} \otimes \sigma_I)\|_1
$$

$$
\leq \frac{1}{2^{n-1}} \sum_I \|(\mathbb{I} \otimes \sigma_I)\Delta_I (\mathbb{I} \otimes \sigma_I)\|_1
$$

$$
= \frac{1}{2^{n-1}} \sum_I \|\Delta_I\|_1. \tag{21}
$$

Reading this inequality as an average over the $2^{2n}$ possible values of $I$, there must exist a particular choice for $I$ for which $\|\omega_I - \omega_{I=0}\|_1 = \|\Delta_I\|_1 > \delta/2^{n+1}$. As in the perfect hiding argument, this provides a cheating operation for the classical scheme that will distinguish the hiding states of the particular $I$ and $I = 0$. The classical scheme, therefore, cannot be $\delta/2^{n+1}$-secure.

**D. Imperfect Hiding: Quantum $\rightarrow$ Classical**

We suppose that there is some quantum hiding scheme $E_q$ that is $\delta$-secure, i.e. for all pairs of quantum states $\varphi_0$ and $\varphi_1$ and LOCC operations $L$ we have

$$
\|(L \circ E_q)(\varphi_0) - (L \circ E_q)(\varphi_1)\|_1 \leq \delta. \tag{22}
$$

¿From this we will deduce the quality of the derived bit-hiding scheme, that is, we study

$$
\|L(\rho_I) - L(\rho_J)\|_1, \tag{23}
$$

where $\rho_I$ and $\rho_J$ are given in Eq. (8). We will use an explicit operator expansion of the maximally entangled projector Eq. (12) [5, Eq. (24)]

$$
\Phi = \frac{1}{4^n} \sum_{M=0}^{4^n-1} (-1)^{N(11)} \sigma_M \otimes \sigma_M. \tag{24}
$$

Here $N(11)$ counts the number of $\sigma_y$ operators in the product $\sigma_M$. Note also that $\sigma_M$, for any $M$, has $2^{n-1}$ positive eigenvalues ($\lambda = +1$), and the same number of negative eigenvalues ($\lambda = -1$); therefore, it can be written as the difference of two density operators using

$$
\sigma_M = 2^{n-1}(\rho^+_M - \rho^-_M), \tag{25}
$$
where $\rho_{\pm}^M$ are separable. In the following, we will use the shorthand $\sigma_{IMI} = \sigma_I \sigma_M \sigma_I$, and $\rho_{\pm}^{IMI} = \sigma_I \rho_{\pm}^M \sigma_I$. With all this, we can write for Eq. (23):

$$\| \mathcal{L}(\rho_I) - \mathcal{L}(\rho_J) \|_1 = \| \mathcal{L}((E_q \otimes Id_2)(|\Phi_I\rangle\langle \Phi_I| - |\Phi_J\rangle\langle \Phi_J|)) \|_1$$

$$= \left\| \frac{1}{4^n} \sum_M (-1)^{N(1)} [\mathcal{L}(E_q(\sigma_{IMI}) \otimes \sigma_M) - \mathcal{L}(E_q(\sigma_{IJ}) \otimes \sigma_M)] \right\|_1$$

$$\leq \frac{1}{4^n} \sum_M \| \mathcal{L}(E_q(\sigma_{IMI}) \otimes \sigma_M) \|_1 + \| \mathcal{L}(E_q(\sigma_{IJ}) \otimes \sigma_M) \|_1$$

$$\leq \frac{2}{4^n} \max_K \sum_M \| \mathcal{L}(E_q(\sigma_{KMK}) \otimes \sigma_M) \|_1$$

$$\leq \frac{2 \cdot 2^{2(n-1)}}{4^n} \max_K \sum_M \| \mathcal{L}(E_q(\rho_{\pm}^{KMK}) \otimes \rho_{\pm}^M) - \mathcal{L}(E_q(\rho_{\pm}^{KMK}) \otimes \rho_{\pm}^M) \|_1$$

$$\leq \frac{1}{2} \max_K \sum_M \| \mathcal{L}(E_q(\rho_{+}^{KMK}) \otimes \rho_{+}^M) - \mathcal{L}(E_q(\rho_{+}^{KMK}) \otimes \rho_{+}^M) \|_1$$

$$\leq \frac{1}{2} \max_K \sum_M \| \mathcal{L}(E_q(\rho_{-}^{KMK}) \otimes \rho_{-}^M) - \mathcal{L}(E_q(\rho_{-}^{KMK}) \otimes \rho_{-}^M) \|_1$$

$$\leq \frac{4^n}{2} \max_K \sum_M \| \mathcal{L}(E_q(\rho_{+}^{KMK}) \otimes \rho_{+}^M) - \mathcal{L}(E_q(\rho_{+}^{KMK}) \otimes \rho_{+}^M) \|_1$$

$$\leq \frac{4^n}{2} \max_K \sum_M \| \mathcal{L}(E_q(\rho_{-}^{KMK}) \otimes \rho_{-}^M) - \mathcal{L}(E_q(\rho_{-}^{KMK}) \otimes \rho_{-}^M) \|_1$$

$$\leq \frac{4^n}{2} (2\delta) = 4^n \delta. \quad (26)$$

In the last inequality, we used the fact that $\rho_{\pm}^M$ are separable density matrices, independent of $K$, which implies that Alice and Bob can prepare them by LOCC operations. Thus to distinguish, say, $E_q(\rho_{+}^{KMK}) \otimes \rho_{+}^M$ from $E_q(\rho_{-}^{KMK}) \otimes \rho_{-}^M$ by LOCC should not be easier than to distinguish $E_q(\rho_{+}^{KMK})$ from $E_q(\rho_{-}^{KMK})$ by LOCC, for which the distinguishability is bounded as in Eq. (22).

So, we get a bound on the quality of the bit-hiding scheme, although one suffering the same exponential deficiency as the bound of section III C.

### III. MULTIPARTY HIDING & QUANTUM SECRET SHARING

We now consider the task of hiding quantum data in a multiparty setting. Generalizations of bipartite bit-hiding schemes to multipartite situations have been developed by Eggeling and Werner [3]. Unfortunately, there is a problem with combining these bit-hiding schemes with the qubit-hiding construction of Eq. (3): the qubit-hiding scheme places the hidden quantum state entirely in the possession of a single party since register $B_2$ belongs to Bob. In the direct generalization of the scheme to the multiparty setting, the privileged holder of the hidden quantum state would, therefore, necessarily have to be a member of every authorized set. This would eliminate the possibility of threshold schemes, for example, in which any sufficiently large subset of the parties should be able reconstruct the secret. The solution is to hide distributed quantum data, using quantum error correcting codes to share the hidden quantum state between the parties in a more symmetrical fashion.

An application involving such distributed hiding has been considered in the literature: quantum secret sharing [3]. In quantum secret sharing, the identity of a distributed quantum state,
The quantum state, and there are unauthorized sets, for whom no reconstruction is possible even with quantum communication. Such quantum secret sharing schemes are implementable with quantum error correcting codes; for example, there is a five-qubit error correcting code for which any three out of the five parties constitute an authorized set, while any set of two is unauthorized. With such error correction codes, quantum secret sharing schemes with any “access structure” are realizable. This access structure need only be consistent with monotonicity and the quantum no-cloning theorem, meaning that any superset of an authorized set is authorized, and the complement of an authorized set is unauthorized [9].

The capabilities of quantum secret sharing can be strengthened by the techniques of this paper. The quantum secret sharing protocol does not specify the status of the secret if the parties can perform LOCC operations, rather than just local operations. In fact, for the implementation of quantum secret sharing using quantum error correcting codes, LOCC operations between the parties can, and often do, result in the parties obtaining partial information about the secret. However, by wrapping the quantum state of quantum secret sharing inside a multipartite version of our qubit-hiding protocol, we can guarantee that the quantum secret is impervious to attack by LOCC operations of the parties; we illustrate the idea in Fig. 2. This requires some generalization of the protocol given above, and of its security analysis.

Suppose we have a $p$-party quantum secret sharing state $\varphi$, and that $k$ qubits distributed among these parties are sufficient to hold this state. (For quantum secret sharing schemes, $k$ is polynomially related to the number of logical qubits $n$ that can be hidden in such a state, $k = \text{poly}(n, p)$.) Now, we can create a new state with stronger security properties using the map

$$E(\varphi) = \frac{1}{2^k} \sum I \rho_I \otimes (\sigma_I \varphi \sigma_I)^2$$

(27)
Here the quantum secret lives in subsystem “2”, which is an $k$-qubit multipartite Hilbert space distributed among the $p$ parties $\mathcal{H}_{A1} \otimes \mathcal{H}_{B2} \otimes \mathcal{H}_{C3} \otimes \ldots$. The dimensions of these local spaces need not be the same, since some parties may get larger shares of the secret than others. Each party also has a register of subsystem “1”, comprising $s$ qubits in total, which contains the data hiding state $\rho_I$, and which is capable of hiding the $2k$-bit string $I$. Note that, although the secret-sharing state $\varphi$ may only occupy a subspace of the “2” subsystem (as when it is a quantum error correcting code state), $\sigma_I$ acts on the entire “2” Hilbert space, not just on the code subspace in which the quantum secret may be contained.

As Eggeling and Werner have recently shown, there exist multipartite data-hiding states $\rho_I$ with any desired access structure, and with hiding security that is exponential in $s/k$. (Unlike the bipartite states used in Ref. [3], however, these states are not orthogonal, just nearly so. This gives rise to a small probability of error when authorized sets reconstruct the secret but otherwise has no effect on the analysis for our purposes.) If we choose the access structure of the $\rho_I$ states and the $\varphi$ quantum secret sharing states to be identical, then the quantum state can obviously still be reconstructed by quantum communication within an authorized set; first $\rho_I$ is measured to identify $I$, then the Pauli rotation $\sigma_I$ is done by the authorized parties, so that they have the state $\varphi$ “in the clear”, permitting them to reconstruct it by whatever operations the original quantum secret sharing protocol prescribed. Of course, it does not matter whether the Pauli rotations are done by the parties outside the authorized set, since these parts of the quantum state $\varphi$ are not needed for the reconstruction anyway.

In the rest of this section, we demonstrate the other part of the desired security of the protocol: an unauthorized set cannot reveal the quantum state even when all parties can perform LOCC and quantum communication can be performed within the unauthorized set. The proof relies on the fact, as in the bipartite case, that with these resources, the parties cannot decode the classical hiding states $\rho_I$.

So, we suppose that in the given “unauthorized” setting, the $2k$-bit hiding scheme is $\epsilon$-secure; we will show that the quantum-state hiding is guaranteed to be $\delta$-secure, for $\delta = e^{2^{3k+5}}$. The first part of the demonstration closely follows the reasoning of Section II C. We consider “cheating” operations $\mathcal{L}^+ = \text{Tr}' \circ L^+$, where $L^+$ is a member of the set of LOCC operations + quantum operations among members of the unauthorized set, and $\text{Tr}'$ indicates a tracing out of all parties except one. We will also need the multipartite version of Eq. (3), $\mathcal{L}^+_I(\tau^2) = \mathcal{L}^+(\rho^1 \otimes \tau^2)$. Given that the parties are supplied with the state $\rho^1$, $\mathcal{L}^+_I$ can be implemented with the same limited communication resources as $\mathcal{L}^+$ can. We introduce ancilla subsystem “3”, which has the same dimension and the same multipartite structure as subsystem “2”. Each party locally creates a maximally entangled state $\Phi$ between its part of system 2 and 3. Let us denote the tensor product of all these local maximally entangled states as a big maximally entangled state $\Phi^{23}$. Then the parties can create the state

$$\omega_I = (\mathcal{L}^+_I \otimes I_3)(\Phi^{23}). \quad (28)$$

Now the proof begins by contradiction as in Section II C: suppose the qubit-hiding scheme is not $\delta$-secure, meaning that there are secret sharing states $\varphi_0$ and $\varphi_1$ and a cheating operation $\mathcal{L}^+$ such that Eq. (2) is true. Then by following without change the analysis after Eq. (16), we conclude that there must exist a bit string $I$ for which (cf. below Eq. (21))

$$\|\omega_{I=0} - \omega_I\|_1 = \|\Delta_I\|_1 > \delta/2^{k+1}. \quad (29)$$

Unlike in the previous case, we cannot use this equation immediately to bound $\epsilon$ and end the argument; the trace norm is only directly related to the distinguishability when any quantum operation can be done on the state. In this case $\omega_I$ is a state shared by all parties who can perform...
LOCC operations and some additional quantum communication depending on the protocol. In other words, we can only bound the distinguishability $\epsilon$ of the classical message by (see Eq. (2))

$$
\epsilon \geq \max_{I,K} \text{Dist}_{L^+}(\omega_I, \omega_K) \geq \max_{I,K} \text{Dist}_{LOCC}(\omega_I, \omega_K) = \max_{I,K} \max_{\mathcal{L}_{LOCC}} \|\mathcal{L}_{LOCC}(\omega_I) - \mathcal{L}_{LOCC}(\omega_K)\|_1.
$$

(30)

Here $\text{Dist}_X$ denotes the distinguishability of two states under the set of operations $X$. We restrict to only LOCC operations here because we can use a known relationship between the LOCC distinguishability of two states and their trace-norm distance, using the tomography arguments of Ref. [5]. Of course, this LOCC distinguishability may be very much less than $\|\omega_{I=0} - \omega_I\|_1$, precisely because of the data-hiding effect that is the subject of this paper, which sometimes prevents the distinguishability of states from being detectable by LOCC operations. The effect is never perfect, however, a fact which we now use.

Suppose the states $\omega_I$ are written as [5, Eq. (105)]

$$
\omega_I = \frac{1}{d} \sum_J a_{IJ} \sigma_J,
$$

(31)

where $d$ is the dimension of the space supporting $\omega_I$. The “3” register in Eq. (28) supports $k$ qubits whereas the map $\mathcal{L}_I$ can be taken to output a single qubit [13]. Therefore, $d = 2^{k+1}$. Then Appendix B of Ref. [5, Eq. (110)] shows that

$$
\text{Dist}_{LOCC}(\omega_I, \omega_K) \geq \frac{1}{2} \max_J |a_{IJ} - a_{KJ}|.
$$

(32)

We apply a chain of inequalities:

$$
\|\omega_I - \omega_K\|_1 \leq \frac{1}{d} \sum_J \| (a_{IJ} - a_{KJ}) \sigma_J \|_1 \leq \frac{1}{d} \sum_J \| a_{IJ} - a_{KJ} \|_1 \| \sigma_J \|_1
$$

$$
= \sum_J |a_{IJ} - a_{KJ}| \leq d^2 \max_J |a_{IJ} - a_{KJ}|.
$$

(33)

We have used $\|\sigma_J\|_1 = d$. Note also that there are $d^2$ terms in the $J$-sum. Combining (30), (32) and (33):

$$
\epsilon \geq \frac{1}{2d^2} \|\omega_{I=0} - \omega_I\|_1.
$$

(34)

So, combining this with Eq. (29), we find that

$$
\epsilon \geq \frac{\delta}{4d^3} = \frac{\delta}{2^{3k+5}}.
$$

(35)

But since the classical hiding can be chosen such that $\epsilon = c_1 2^{-c_2 s/k}$, there is always a choice of $s$ that will guarantee that $\delta$ is as small as desired. So, at the price of a worse bound (but only polynomially worse), we prove security of the multipartite case.

### IV. HOW MANY CLASSICAL BITS ARE NEEDED IN QUANTUM HIDING

In the previous sections we have seen that any $2n$-bit hiding scheme can be used to construct an $n$-qubit hiding scheme and vice-versa. This duality has immediate implications for the resource
requirements of quantum data hiding schemes. In particular, suppose that \( \{ \rho_I \} \) is a set of perfectly secure hiding states for the string \( I \), representing \( k \) bits of data and that

\[
E_q(\varphi) = \frac{1}{2^k} \sum_I \rho_I^{AB_1} \otimes T_I(\varphi)^{B_2},
\]

where \( T_I \) is a TCP map and \( \varphi \) is an \( n \)-qubit state. We do not know whether all \( n \)-qubit hiding schemes will have this form but it is a significant generalization of the construction we described in section II. We will show that in order for this to be a secure qubit hiding scheme \( k \geq 2n \).

Let us assume that this provides a perfectly secure \( n \)-qubit hiding scheme for \( \varphi \), and use the encoding \( E_q \) to hide bits by means of our superdense coding construction. We will get a secure \( 2n \)-bit hiding scheme by applying \( E_q \otimes \mathbb{I} \) to the appropriate maximally entangled states. We could interpret this construction as a way of hiding a message of \( 2n \) bits by means of a key \( I \) of \( k \) bits. We will now prove that this implies that \( k \geq 2n \), using an argument nearly identical to the one Shannon used to show that one-time pad encryption of a \( 2n \)-bit message requires \( 2n \) shared random key bits \([11]\). The only difference here is that we substitute quantum entropy functions for their classical counterparts and then have to verify in a couple of places that these quantum functions are nonnegative, a property guaranteed for their classical versions. For definitions of the functions we use below, see, for example, Ref. \([12]\).

Consider the density operator

\[
\sum_{m,I} p_m |m\rangle \langle I|^{M} \otimes |I\rangle \langle I|^K \otimes \rho_I^{AB_1} \otimes (T_I \otimes \mathbb{I})(|\Phi_m\rangle \langle \Phi_m|)^{B_2B_3}. \tag{37}
\]

Here \( M \) is a register storing the message \( m \), \( K \) a register storing the key \( I \) and the set \( \{ \Phi_m \} \) is a set of mutually orthogonal maximally entangled states. Because the message and key are independent, \( S(M : K) = 0 \). Likewise, because the bit-hiding scheme is perfectly secure, \( S(M : B_2B_3) = 0 \). Once the key is known, however, the classical message can be reconstructed from register \( B_2B_3 \) so that

\[
S(M : B_2B_3|K) = S(M : K|B_2B_3) = S(M). \tag{38}
\]

Equivalently, \( S(K|B_2B_3) - S(K|B_2B_3M) = S(M) \). Because the multipartite density operator is separable across the \( M/K/AB_1/B_2B_3 \) cuts, \( S(K|B_2B_3M) \geq 0 \), and we can conclude that \( S(K) \geq S(M) \). In particular, applying this inequality for the uniform distribution over \( 2n \) bit messages yields \( k \geq 2n \). Our conclusion is that if an \( n \)-qubit hiding scheme is constructed from a \( k \)-bit hiding scheme in the manner of Eq. \((36)\) then \( k \), the number of bits, must be at least twice the number of qubits being hidden.

These arguments can also be applied to the case of a private quantum channel. In this scenario, the analog of an \( k \)-bit hiding scheme is just \( k \) secret random bits shared between two parties Alice and Bob. A general private quantum channel then has the form

\[
E(\varphi) = \frac{1}{2^k} \sum_I |I\rangle \langle I|^K \otimes T_I(\varphi)^C, \tag{39}
\]

where \( T_I \) is a general TCP map with output on the channel system \( C \). The requirements for the task are that, using their access to \( I \), Bob (or Alice) can reconstruct \( \varphi \) but an eavesdropper with access only to the channel \( C \) can learn nothing. Assume that one can encrypt an \( n \)-qubit quantum state \( \varphi \) in this manner. Such a private quantum channel can be converted into a secure \( 2n \)-bit one-time pad using the superdense coding construction of section II B – the proof goes through unchanged. Likewise, the resource considerations developed above imply that \( k \geq 2n \) bits of shared secret key are necessary and sufficient to build the \( n \)-qubit private quantum channel, confirming the main result of Refs. \([11]\) and \([12]\).
V. DISCUSSION AND CONCLUSIONS

Our main goal in this paper was to show how the duality between superdense coding and teleportation can be used to construct new cryptographic protocols. From a constructive point of view, our main results are a protocol for hiding qubits given a protocol for hiding twice as many bits and a method for strengthening quantum secret sharing protocols such that they are not vulnerable to cheating by LOCC.

In our analyses of imperfect hiding, however, we could only guarantee a quality of hiding decreasing exponentially in the number of qubits, or bits, being hidden. Because the quality typically improves exponentially with the size of the hiding state measured in qubits, our security proofs could still be useful. Nonetheless, it seems possible that a more careful analysis of the hiding quality of the new protocols would reveal that the exponential factor, in fact, disappears. We leave that possibility open for future work.

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