SUPERLOOP EQUATIONS AND
TWO DIMENSIONAL SUPERGRAVITY

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ABSTRACT

We propose a discrete model whose continuum limit reproduces the string susceptibility and the scaling dimensions of (2, 4m)-minimal superconformal models coupled to 2D-supergravity. The basic assumption in our presentation is a set of super-Virasoro constraints imposed on the partition function. We recover the Neveu-Schwarz and Ramond sectors of the theory, and we are also able to evaluate all planar loop correlation functions in the continuum limit. We find evidence to identify the integrable hierarchy of non-linear equations describing the double scaling limit as a supersymmetric generalization of KP studied by Rabin.

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1. INTRODUCTION

The discrete formulation of $N = 1$ Superconformal Field Theories [1] coupled to world-sheet supergravity lags far behind its purely bosonic counterpart. Some of the results in [2] were obtained previously in terms of discrete models of 2D-gravity (see for example [3]). The supersymmetric extension of the analysis in [2] carried out in [4], [5] has no analogues in terms of random (super)-surfaces. In the double scaling limit [6], the theory of the KP (Kadomtsev-Petviashvili) hierarchy was shown to play a central role [7]. Motivated by this connection, the authors in [8] proposed an approach to the double scaling limit of 2D-supergravity coupled to superconformal matter using a supersymmetric generalization of the KP hierarchy due to Manin and Radul [9]. An important feature of the one-matrix model which will be central in our arguments is the fact that its partition function satisfies a set of Virasoro constraints [10],[11].

In this paper we propose a discrete model of 2D-supergravity with superconformal matter following the more “phenomenological” approach in [12]. Reasoning by analogy with the Virasoro constraints we define a discrete analogue of the Hermitean one-matrix model. Our basic postulate is to begin with a set of super-Virasoro constraints in the Neveu-Schwarz (NS) sector satisfied by the partition function. From them we derive the explicit form of our model and a set of super-loop equations. In the planar limit these equations can be solved exactly. In this way we compute the spectrum of anomalous dimensions which coincide with the (super)-gravitationally dressed dimensions of the $(2,4m)$ minimal superconformal models in the NS and the Ramond (R) sectors. We compute arbitrary multiloop correlation functions on spherical topologies, and our results agree with those obtained in the continuum limit [13] using the super-Liouville formulation of the problem. Since there are no higher point functions computed in the continuum we cannot compare our results further. In spite of these encouraging properties, a derivation of our model in terms of “triangulated super-surfaces”, orthogonal polynomials and generalized matrices is still lacking. Thus, the identification of
our model as a discrete version of 2D-supergravity should be taken as preliminary.

We take as guiding principle in our work a set of super-Virasoro constraints satisfied by the partition function. One reason why we believe this to be a correct starting point is the prominent role the Virasoro constraints play in the description of the geometry of the moduli space of stable Riemann surfaces after Witten’s work [14] on 2D-gravity and matrix models, and the proof of Witten’s conjecture by Kontsevich [15] (see also [16]). According to Witten’s theory, the intersection theory of certain line bundles on the moduli space $\mathcal{M}_{g,n}$ of genus $g$ surfaces with $n$ distinguished points is captured by the Virasoro constraints satisfied by the partition function. Any discrete version of $N = 1$ supergravity on the world-sheet should necessarily have to address similar issues for super-surfaces. In this case, however, the mathematical theory is not sufficiently well developed to allow us to borrow results which could shed light on the problem. We take a more radical point of view in expecting many properties of super-moduli spaces to be captured by the super-Virasoro constraints $G_{n-\frac{1}{2}}Z = 0$, $n = 0, 1, \ldots$. Since $\{G_n, G_m\} \sim L_{n+m}$, the Virasoro constraints are automatically satisfied. The results of this paper can be interpreted as giving support to the validity of this basic assumption.

We also find evidence that the generalized KP hierarchy appearing in our model is not the super-KP hierarchy of Manin-Radul [9] but rather a super-hierarchy defined by Rabin [17]. The basic difference between the two is related to the fact that the latter does not admit a simple presentation in terms of a Lax pair. We will comment on these issues at the end of the paper.

This paper is organized as follows. In section two we collect several results concerning matrix models, loop equations and Virasoro constraints and present them in a way which will be easily generalized later. In section three we study the super-Virasoro constraints, we use them to derive the exact form of our model and then we derive the super-loop equation in its planar approximation. Section four analyzes the solution to the planar loop equations, the scaling limit and the spectrum of scaling operators. We obtain the dressed gravitational dimensions
expected in the Neveu-Schwarz and Ramond sectors of the $(2, 4m)$ minimal superconformal models coupled to 2D-supergravity. As an application we compute in section five correlators of arbitrary numbers of loops (bosonic and fermionic) in planar topologies. Section six contains our conclusions and outlook, and our remarks on the connection between our model and the super-KP hierarchy described by Rabin [17].

2. VIRASORO CONSTRAINTS AND LOOP EQUATIONS

We review succinctly in this section some properties of the one-matrix models and their loop equations. The starting point of Kazakov’s analysis of multicritical points [12] was the planar loop equation*

$$\sum_{k \geq 1} k g_k \frac{\partial^{k-1}}{\partial l^{k-1}} w(l) = \int_0^l dt' w(l - t') w(l')$$ (2.1)

which describes a loop of length $l$ bounding a surface with the topology of a disk. This equation follows from some simple heuristics, but it can also be derived from a Hermitean matrix model [18]. We take the partition function to be

$$Z = \int d^{N^2} \Phi \exp\left[-\frac{N}{\Lambda} tr V(\Phi)\right]$$

$$V(\Phi) = \sum_{k \geq 0} g_k \Phi^k \quad \Lambda = e^{-\mu_B}$$ (2.2)

where $\Phi$ is a Hermitean $N \times N$ matrix and $\mu_B$ is the bare cosmological constant. The loop operator is represented by

$$w(l) = \frac{\Lambda}{N} tr e^{l \Phi} = \sum_{n=0}^\infty \frac{l^n}{n!} \frac{\Lambda}{N} tr \Phi^n = \sum_{n=0}^\infty \frac{l^n}{n!} w^{(n)}$$ (2.3)

Writing the partition function in terms of the free energy $Z = e^{N^2 F}$, $F = F_0 +$
$N^{-2}F_1 + N^{-4}F_2 + ..., \text{ the (expectation values) moments } w^{(n)} \text{ are given by}$

$$w^{(0)} = \Lambda \quad w^{(n)} = -\Lambda^2 \frac{\partial F}{\partial g_n} \quad (2.4)$$

Near the critical point $\mu_c$, the genus expansion of (2.2) behaves according to

$$Z(\mu_B) = \sum_h Z_h [N^2(\mu_B - \mu_c)^{2-\gamma_{st}}]^{1-h} \quad (2.5)$$

where $h$ is the handle counting parameter and $\gamma_{st}$ is the string susceptibility. Introducing the renormalized cosmological constant $\mu_B - \mu_c = a^2 t$, where $a$ is a cut-off with units of length, the double scaling limit \cite{6} is obtained by taking $N \to \infty$, $a \to 0$, and keeping fixed the combination

$$Na^{2-\gamma_{st}} = \kappa^{-1} \quad (2.6)$$

which is the string coupling constant. (More details and references to the literature can be found in the review articles \cite{19}, \cite{20}). For later convenience we derive the planar-loop equations (2.1) through the Virasoro constraints satisfied by (2.2) \cite{21}. They are obtained by making the change of variables $\Phi \to \Phi + \epsilon \Phi^{n+1}$. After some simple manipulations we obtain

$$L_n Z = 0 \quad n \geq -1 \quad (2.7)$$

with

$$L_n = \Lambda^2 \frac{\sum_{k=0}^{n} \partial^2}{N^2} \frac{g_{n-k}}{\partial g_k} + \sum_{k \geq 0} kg_k \frac{\partial}{\partial g_{k+n}} \quad (2.8)$$

To leading order in $\frac{1}{N}$ (the planar limit) we write $Z = e^{N^2 F_0}$ and obtain

$$\lambda^2 \sum_{k=0}^{n} \frac{\partial F_0}{\partial g_k} \frac{\partial F_0}{\partial g_{n-k}} + \sum_{k \geq 1} kg_k \frac{\partial F_0}{\partial g_{k+n}} = 0 \quad (2.9)$$

Using (2.4) it is easy to see that (2.9) is exactly identical to (2.1). Hence the planar limit of the Virasoro constraint (2.7) is nothing but the planar loop equation. The
equations (2.1) (2.9) are solved by introducing the Laplace transform of the loop operator

\[ w(p) = \int_0^\infty e^{-pl} w(l) dl = \sum_{k=0}^{\infty} \frac{w^{(k)}}{p^{k+1}} \] (2.10)

with the assumption that \( w(l) \) behaves well at \( l = 0 \) and \( \infty \). With the definition (2.10), (2.1) becomes an algebraic equation

\[ w(p)^2 - V'(p)w(p) + Q(p) = 0 \]

\[ Q(p) = \sum_{k \geq 1} k g_k \sum_{r=1}^{k-1} p^{r-1} w^{(k-r-1)} \] (2.11)

Two remarks should be made at this point. First, if we make the identification

\[ \alpha_n = -\frac{\Lambda \sqrt{2}}{N} \frac{\partial}{\partial g_n}, \ n \geq 0 \ ; \ \alpha_{-n} = -\frac{N}{\Lambda \sqrt{2}} n g_n, \ n > 0 \] (2.12)

the \( L_n \)'s can be rewritten as the components of the energy-momentum tensor of a free massless scalar field

\[ L_n = \frac{1}{2} \sum : \alpha_{-k} \alpha_{k+n} : , \ T(p) = \sum_{n \in \mathbb{Z}} L_n p^{-n-2} \] (2.13)

Defining

\[ \partial X(p) = \sum_{n \in \mathbb{Z}} \alpha_n p^{-n-1} = \partial X^+ + \partial X^- \]

\[ \partial X^+ = \sum_{n \geq 0} \alpha_n p^{-n-1} \] (2.14)

we obtain

\[ \lim_{N \to \infty} \frac{1}{N} Z^{-1} \partial X^+ Z \propto w(p) = \sum \frac{w^{(n)}}{p^{n+1}} \]

\[ \lim_{N \to \infty} \frac{1}{N} Z^{-1} \partial X^- Z \propto V'(p') = \sum k g_k p^{k-1} \] (2.15)
and the Laplace transformed equation (2.11) becomes

$$\lim_{N \to \infty} \frac{1}{N^2} Z^{-1} T(p) Z = 0$$  \quad (2.16)$$
in the limit $p \to 0$. This is important because with the identification (2.15) we can find the potential for the matrix model. The planar loop equations are therefore equivalent to (2.16). The second remark has to do with the form of $Z$ in terms of eigenvalues. If the eigenvalues of $\Phi$ are $\lambda_1, \ldots, \lambda_N$, (2.2) can be written as [22]

$$Z = \text{const} \cdot \int \prod d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp\left[ -\frac{N}{\Lambda} \sum_i V(\lambda_i) \right]$$  \quad (2.17)$$
We could instead write

$$Z = \text{const} \cdot \int \prod d\lambda_i \Delta^2(\lambda) \exp\left[ -\frac{N}{\Lambda} \sum_i V(\lambda_i) \right]$$  \quad (2.18)$$
and use the Virasoro constraints (2.7) (2.8) to determine the form of $\Delta$. Indeed, if we act with (2.8) in (2.18) and perform some simple integrations by parts, we obtain a set of differential equations satisfied by $\Delta$

$$\sum_i \lambda_i^{n+1} \frac{\partial \Delta}{\partial \lambda_i} = \Delta \sum_{i \neq j} \frac{\lambda_i^{n+1}}{\lambda_i - \lambda_j}$$  \quad (2.19)$$
whose solution up to an irrelevant constant is $\Delta = \prod_{i<j} (\lambda_i - \lambda_j)$. The solution to the planar equations (2.11) takes the form [12]

$$w(p) = \frac{1}{2} \left( V'(p) - M(p^2) \sqrt{p^2 - R} \right)$$  \quad (2.20)$$
in the case when the potential is even $V(p) = V(-p)$. Since

$$w(p) = \frac{\Lambda(R)}{p} + \frac{w^{(2)}(p)}{p^3} + \ldots, \quad w^{(0)} \equiv \Lambda(R)$$  \quad (2.21)$$
$M(p^2)$ is completely determined by requiring the right hand side of (2.20) to have only negative powers of $p$ in the large $p$ expansion, and $R$ is completely determined.
in terms of \( \Lambda \). Kazakov showed \cite{12} that

\[
\frac{\partial w(p)}{\partial \Lambda} = \frac{1}{\sqrt{p^2 - R}} \tag{2.22}
\]

\[
\Lambda = e^{-\mu_B} = \Lambda(R) = \sum_{k \geq 1} k g_{2k} R^k \left( \binom{2k}{k} \right) \frac{1}{4k} = \int_0^\infty dx V'(x + \frac{R}{4x}) \tag{2.23}
\]

At the \( m \)-th critical point

\[
\Lambda = 1 - (1 - R)^m = e^{-\mu_B} = 1 - a^2 t
\]

\[
1 - R = a^{2/m} u \tag{2.24}
\]

and \( t \) and \( u \) are scaling variables. The variable \( t \) is the renormalized cosmological constant and \( u \) is the “heat capacity” of the theory.

To compute loop correlators notice from (2.22) that

\[
\frac{\partial w(2k)}{\partial \Lambda} = \frac{R^k}{4^k} \left( \binom{2k}{k} \right) \tag{2.25}
\]

Taking the limit as \( k \to \infty \), and defining the renormalized length according to

\[
l = ka^{2/m} \tag{2.26}
\]

(\( l \) kept fixed as \( a \to 0, k \to \infty \)), using the scaling limit (2.6) with \( \gamma_{st} = -\frac{1}{m} \) and using the scaling variables (2.24) we obtain

\[
w(l) \equiv \sqrt{\pi} \langle tr \Phi^{2k} \rangle = \frac{1}{\kappa \sqrt{l}} \int_t^\infty dt' e^{-lu(t')} \tag{2.27}
\]

To compute multiloop correlation functions all we need to know is \( \frac{\partial u}{\partial g_{2k}} \)

\[
\langle w(l_1)w(l_2) \rangle = \pi \langle tr \Phi^{2k_1} tr \Phi^{2k_2} \rangle
\]

\[
= -\frac{\Lambda}{N} \sqrt{\pi} \frac{\partial}{\partial g_{2k_2}} \langle w(l_1) \rangle , \quad k_i = l_i a^{-2/m} \tag{2.28}
\]

Perturbing the string equation (2.24) with the \( g_{2k_2} \) coupling and taking \( k_2 \to \infty \)
as in (2.26) we obtain

\[ \frac{\partial u}{\partial \theta_{2k}} = -a^{-2-1/m} \sqrt{\frac{l_2}{\pi}} e^{-l_2 u} \tag{2.29} \]

which together with (2.27) , (2.28) yields

\[ \langle w(l_1)w(l_2) \rangle = \sqrt{l_1 l_2} e^{-u(l_1+l_2)} \frac{1}{l_1 + l_2} \tag{2.30} \]

Repeating the same arguments we obtain

\[ \langle w(l_1) \ldots w(l_n)w(l_{n+1}) \rangle = -\kappa \sqrt{l_{n+1}} e^{-l_{n+1} u} \frac{\partial}{\partial t} \langle w(l_1) \ldots w(l_n) \rangle \tag{2.31} \]

in agreement with the results in [23].

In the generalization to the supersymmetric case in later sections we will follow closely the arguments in this section.

3. SUPER-VIRASORO CONSTRAINTS AND SUPERLOOP EQUATIONS

We now introduce the analogue of \( w(l) \) which we take to depend on two variables, \( w(l, \theta) \), a bosonic and a fermionic length \( l \) and \( \theta \) respectively. We can imagine these two parameters as characterizing the boundary of a super-disk. As in the previous case we can introduce the super-Laplace transform

\[ w(p, \Pi) \equiv v(p) + \Pi u(p) \equiv \int_0^\infty dl \int d\theta e^{-pl-\Pi \theta} w(l, \theta) \tag{3.1} \]

Some properties of the Laplace transform which are useful are the following:

\[ \mathcal{L}[f] \equiv f(p, \Pi) = \int_0^\infty dz \int d\theta e^{-pz-\Pi \theta} (f_0(z) + \theta f_1(z)) \tag{3.2} \]

\[ = f_1(p) + \Pi f_0(p) \]

\[ f_i(p) \equiv \int_0^\infty dz e^{-pz} f_i(z) , \quad i = 0, 1 \tag{3.2a} \]
\[
\mathcal{L}[Df] = (\Pi + p \frac{\partial}{\partial \Pi})\mathcal{L}[f] - f_0(0) \quad , \quad D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}
\]

(3.2b)

\[
\mathcal{L}\left[(\theta - z \frac{\partial}{\partial \theta})f\right] = \left(\frac{\partial}{\partial \Pi} + \Pi \frac{\partial}{\partial p}\right)\mathcal{L}[f]
\]

(3.2c)

\[
\mathcal{L}[f \circ g] = -(-)^{\partial f} \mathcal{L}[f] \mathcal{L}[g]
\]

(3.2d)

\[
(f \circ g)(z, \theta) \equiv \int d\theta' \int dz' f(z', \theta') g(z - z', \theta - \theta')
\]

\[
= f_1 \circ g_0 - f_0 \circ g_1 - \theta(f_1 \circ g_1)
\]

(3.2e)

where \(\partial f\) is the grading of \(f\) (\(\partial f = 0\) if \(f\) is even and \(\partial f = 1\) if it is odd). In the last line \(f_i \circ g_j\) is the standard convolution of functions. We introduce also the symbol

\[
P = \theta - z \frac{\partial}{\partial \theta}
\]

(3.3)

related to \(D = \frac{\partial}{\partial \Pi} + \Pi \frac{\partial}{\partial p}\) via the Laplace transform.

The first (of three) derivation of the planar loop equations is based on the analogy with the \(c = 1\) energy-momentum tensor. Assuming again that the loop \(w(l, \theta)\) behaves well as \(l\) is near 0 or \(\infty\), we can expand the Laplace transform \(w(p, \Pi)\) in inverse powers of \(p\)

\[
v(p) = \sum_{k \geq 0} \frac{v^{(k)}}{p^{k+1}} \quad , \quad u(p) = \sum_{k \geq 0} \frac{u^{(k)}}{p^{k+1}}
\]

(3.4)

Consider a \(\hat{c} = 1\) free massless superfield \(X(p, \Pi) = X(p) + \Pi \psi(p)\). Its super-energy-momentum tensor is

\[
T(p, \Pi) \propto D X \partial X := \psi \partial_p X + \Pi : (\partial_p X \partial_p X + \partial_p \psi \psi) :
\]

(3.5)

Writing

\[
\partial X(p) = \sum_{n \in \mathbb{Z}} \alpha_n p^{-n-1} \quad , \quad \psi(p) = \sum_{r \in \mathbb{Z}+1/2} b_r p^{-r-1/2}
\]

(3.6)
we can identify \( \alpha_n, b_r \) with bosonic and fermionic couplings

\[
\alpha_p = -\frac{\Lambda}{N} \frac{\partial}{\partial g_p}, \quad \alpha_{-p} = -\frac{N}{\Lambda} g_p, \quad p = 0, 1, 2, \ldots
\] (3.7)

\[
b_{p+1/2} = -\frac{\Lambda}{N} \frac{\partial}{\partial \xi_{p+1/2}}, \quad b_{-p-1/2} = -\frac{N}{\Lambda} \xi_{p+1/2}, \quad p = 0, 1, 2, \ldots
\] (3.8)

The Laplace transformed loop \( w(p, \Pi) \) is identified with the positive frequency part of \( w(p, \Pi) \sim DX^+ \) and the analogue of the potential in the one-matrix model is identified with \( DX^-, \) more precisely \( DV(p, \Pi) \sim DX^- \). This leads to

\[
V(p, \Pi) = \sum_{k \geq 0} (g_k p^k + \xi_{k+1/2} \Pi p^k)
\] (3.9)

Writing \( Z = e^{N^2 F} \) we can also identify the moments \( u^{(k)}, v^{(k)} \) with derivatives of \( F \):

\[
u^{(0)} = \Lambda, \quad u^{(n)} = -\Lambda^2 \frac{\partial F}{\partial g_n}; \quad v^{(n)} = -\Lambda^2 \frac{\partial F}{\partial \xi_{n+1/2}}
\] (3.10)

With these identifications, we can take the planar limit as in the pure gravity case

\[
\lim_{N \to \infty} N^{-2} Z^{-1} T(p, \Pi) Z = 0
\] (3.11)

with \( Z \sim e^{N^2 F_0} \). Some simple algebra yields what should be considered as the Laplace transform of the superloop equations:

\[
KDw + DKw = w Dw + Q
\]

\[
K = DV = \sum_{k \geq 1} (\xi_{k-1/2} + \Pi k g_k) p^{k-1}, \quad Q = Q_1 + \Pi Q_0
\]

\[
Q_0 = 2 \sum_{k \geq 1} \sum_{j=1}^{k-1} (k g_k u^{(k-j-1)} + (k - 1 - j/2) \xi_{k-1/2} v^{(k-j-2)}) p^{j-1}
\] (3.12)

\[
Q_1 = \sum_{k \geq 1} \sum_{j=1}^{k-1} (k g_k v^{(k-j-1)} + \xi_{k-1/2} u^{(k-j-1)}) p^{j-1}
\]

Using the Laplace transform formulae in (3.2) one shows that (3.12) is the Laplace
transform of the superloop equations

\[ \mathcal{P}K_{w}(l, \theta) + 2\mathcal{K}P_{w}(l, \theta) = (w \circ \mathcal{P}w)(l, \theta) \]

\[ \mathcal{K} \equiv \sum_{k \geq 1} (kg_{k} \partial_{\theta} + \xi_{k+1/2}) \partial_{l}^{k-1} \quad (3.13) \]

which is strongly reminiscent of (2.1) the starting point of Kazakov’s analysis. It should be quite interesting to derive (3.13) heuristically in terms of gluing superdisks through their boundaries.

In the second and more fundamental derivation of the superloop equations (3.12), (3.13) we begin by constructing an “eigenvalue model” similar to (2.18) and use super-Virasoro to determine the measure. The form of the potential (3.9) suggests the introduction of \( N \) pairs of eigenvalues \( (\lambda_i, \theta_i) \), one bosonic and the other fermionic. The potential for this eigenvalue model is taken to be

\[ V(\lambda, \theta) = \sum_{k \geq 0} \sum_{i} (g_{k}\lambda_{i}^{k} + \xi_{k+1/2}\theta_{i}\lambda_{i}^{k}) \quad (3.14) \]

and the partition function is written as

\[ Z(g, \xi) = \int \prod_{i} d\lambda_{i}d\theta_{i}\Delta(\lambda, \theta) \exp\left[-\frac{N}{\Lambda}V(\lambda, \theta)\right] \quad (3.15) \]

The explicit form of the super-Virasoro generators in the \( \hat{c} = 1 \) case with the oscillators (3.7), (3.8) is given by

\[ G_{r} = \frac{\Lambda^{2}}{N^{2}} \sum_{s=1/2}^{r} \frac{\partial^{2}}{\partial \xi_{s} \partial g_{r-s}} + \sum_{s=1/2}^{\infty} \xi_{s} \frac{\partial}{\partial g_{r+s}} + \sum_{k=1}^{\infty} kg_{k} \frac{\partial}{\partial \xi_{k+r}} \quad (3.16) \]

\[ L_{n} = \frac{\Lambda^{2}}{2N^{2}} \sum_{k=0}^{n} \frac{\partial^{2}}{\partial g_{k} \partial g_{n-k}} + \sum_{k=1}^{\infty} kg_{k} \frac{\partial}{\partial g_{k+n}} \]

\[ + \frac{\Lambda^{2}}{2N^{2}} \sum_{r=1/2}^{n-1/2} \left( \frac{n}{2} - r \right) \frac{\partial}{\partial \xi_{r}} \frac{\partial}{\partial \xi_{n-r}} + \sum_{r=1/2}^{\infty} \left( \frac{n}{2} + r \right) \xi_{r} \frac{\partial}{\partial \xi_{r+n}} \quad (3.17) \]

Since \( L_{n} \) is obtained in terms of anticommutators of \( G_{r} \)'s, it suffices to impose on
only the fermionic constraints $G_{n-1/2}Z = 0$. It is convenient to write $G_{n-1/2}$ as

$$G_{n-1/2} = \sum_{k=0}^{\infty} \xi_{k+1/2} \frac{\partial}{\partial g_{k+n}} + \sum_{k=0}^{\infty} k g_k \frac{\partial}{\partial \xi_{k+n-1/2}} + \frac{\Lambda^2}{N^2} \sum_{k=0}^{n-1} \frac{\partial}{\partial \xi_{k+1/2}} \frac{\partial}{\partial g_{n-1-k}}$$ (3.18)

If we recall the explicit representation of the action of the algebra of supervector fields (super-Virasoro without central extension) on the space of functions of $(\lambda, \theta)$:

$$g_{n-1/2} = -\theta \lambda^n \frac{\partial}{\partial \lambda} + \lambda^n \frac{\partial}{\partial \theta}$$ (3.19)

$$l_n = -\lambda^{n+1} \frac{\partial}{\partial \lambda} - \frac{1}{2} (n + 1) \lambda^n \theta \frac{\partial}{\partial \theta}$$ (3.20)

we find that the action of $G_{n-1/2}$ on $Z$ can be traded off by the action of

$$\sum_i (-\theta_i \lambda_i^n \frac{\partial}{\partial \lambda_i} + \lambda_i^n \frac{\partial}{\partial \theta_i})$$ (3.21)

on the exponential term. Integrating by parts and using identities like

$$\sum_{k=0}^{n-1} \lambda_i^k \lambda_j^{n-1-k} = \frac{\lambda_i^n - \lambda_j^n}{\lambda_i - \lambda_j}, \quad i \neq j$$ (3.22)

we obtain a set of differential constraints on $\Delta$

$$\sum_i \lambda_i^n (-\frac{\partial}{\partial \theta_i} + \theta_i \frac{\partial}{\partial \lambda_i}) \Delta = \Delta \sum_{i \neq j} \theta_i \frac{\lambda_i^n - \lambda_j^n}{\lambda_i - \lambda_j}$$ (3.23)

whose unique solution (up to an irrelevant multiplicative constant) is

$$\Delta = \prod_{i<j} (\lambda_i - \lambda_j - \theta_i \theta_j)$$ (3.24)
Hence our model is explicitly given by

\[ Z = \int \prod_i d\lambda_id\theta_i \prod_{i<j}(\lambda_i - \lambda_j - \theta_i\theta_j) \exp\left[ -\frac{N}{\Lambda} V(\lambda, \theta) \right] \]  

(3.25)

Introduce now the explicit representation of the superloop operator

\[ w(l, \theta) \equiv \frac{\Lambda}{N} \sum_i e^{i\lambda_i + \theta_i} \]  

(3.26)

and its expectation value with respect to \( Z \). Acting on \( \langle w(l, \theta) \rangle \) with the operator \((\mathcal{PK} + 2\mathcal{KP})\) appearing in (3.13), using the super-Virasoro constraints and the factorization of amplitudes in the large \( N \) limit we obtain (3.13) after some computation.

Finally we could obtain (3.12), (3.13) by using (3.10), and the explicit formula for \( Z^{-1}G_{n-\frac{1}{2}}Z = 0, n \geq 0 \), and \( Z^{-1}L_nZ = 0, n \geq -1 \), with \( Z \sim e^{N^2F_0} \) and taking the large \( N \) limit. This gives the third derivation of the planar loop equations (3.12) which we can write in components according to

\[ a) \quad (u(p) - V'(p))^2 + (v(p) - \xi(p))'(v(p) - \xi(p)) = \Delta_0 \]
\[ b) \quad (v(p) - \xi(p))(u(p) - V'(p)) = \Delta_1 \]  

(3.27)

where

\[ \xi(p) = \sum_{k \geq 0} \xi_k + \frac{1}{2}p^k \]

\[ V'(p) = \sum_{k \geq 1} kgkp^{k-1} \]  

(3.28)

and

\[ \Delta_0 = V'(p)^2 + \xi'(p)\xi(p) - Q_0 \]
\[ \Delta_1 = \xi(p)V'(p) - Q_1 \]  

(3.29)

\( Q_0 \) and \( Q_1 \) are given in (3.12) although their explicit form is unnecessary except for the fact that they are polynomials in \( p \). The primes denote differentiation with respect to \( p \).
After the three derivations presented of (3.27) the next step is to solve the superloop equations, take the scaling limit and find the critical points and critical dimensions. This we do in the next section.

4. SOLVING THE SUPERLOOP EQUATIONS. SPECTRUM OF THE MODEL

In solving the superloop equations we will make the simplifying assumption that the bosonic part of the potential is even

\[ V(\lambda, \theta) = \sum_{k \geq 0} \sum_{i} (g_{2k} \lambda_i^{2k} + \xi_{k+1/2} \theta_i \lambda_i^k) \] (4.1)

Before attempting the solution of (3.27) there are a number of useful remarks that should be made concerning the solution of the 2D-gravity case (2.11), (2.20). As written in (2.11) it seems that the polynomial \( Q \) contains a number of “initial conditions” for some of the loop moments \( w^{(k)} \). If the potential is of order \( k \), apparently the first \( k-1 \) moments are completely arbitrary, and all other moments are computed in terms of them.

This is an incorrect impression if we think in terms of the original matrix model. Once the couplings \( g_k \) and the cosmological constant \( \Lambda \) are given there are no ambiguities. If we think in terms of the formal perturbative evaluation, we expand the potential in powers of all couplings \( g_k \) with \( k \neq 2 \), i.e. we keep the quadratic term \( g_2 \phi^2 \) in the exponent and evaluate the free energy \( F \) using Wick contractions. The different correlators are then formally analytic in all couplings \( g_k , k \neq 2 \), and there are no free constants. This can also be shown by solving the Virasoro constraints perturbatively in \( g_k , k \neq 2 \). This uniqueness of the formal loop correlators when written in terms of \( (g_k, \Lambda) \) implies that the solution to (2.11) is unique provided: i) it is parametrized by \( \Lambda \) and it is perturbative in the couplings \( g_k , k \neq 2 \); ii) \( w(p) \) only contains negative powers of \( p \) for \( |p| \) large; and of course iii) \( w^2(p) - V'(p)w(p) \) is a polynomial, which is another way of expressing (2.11).
Thus although different ways of parametrizing the cut in (2.20) and the function $M(p^2)$ might seem to yield inequivalent answers, when we express $R$ in terms of $\Lambda$ through the condition

$$w(p) = \frac{\Lambda}{p} + O\left(\frac{1}{p^2}\right), \quad \Lambda = \Lambda(R)$$

(4.2)

all the answers will be the same.

We can take the previous remarks and apply them in our situation. It is again true that the perturbative evaluation of $Z$, or the perturbative solution to the super-Virasoro constraints yields unambiguous answers for the superloop moments $u^{(k)}, v^{(k)}$. Thus we stress the fact that if we find a superloop $w(p, \Pi) = v(p) + \Pi u(p)$ satisfying: i) at large $|p|$ it only contains inverse powers of $p$; ii) it is perturbative in all couplings except the quadratic even coupling; iii) the left hand sides of (3.27) computed with the proposed solution $w(p, \Pi)$ should be polynomials ($\Delta_0$ and $\Delta_1$); then when we use the string equation

$$u(p) = \frac{\Lambda}{p} + O\left(\frac{1}{p^2}\right)$$

(4.3)

to express the auxiliary parameters in terms of $\Lambda$, the solution is unique. This uniqueness property of $w(p, \Pi)$ is very important and it will be used presently. The solution may be parametrized differently but the answers in terms of $\Lambda$, $g_k$ $k \neq 2$, $\xi_k$ will be the same.

To obtain a preliminary idea of the possible analytic structure of $u(p), v(p)$, we solve (3.27) explicitly in terms of $\Delta_0, \Delta_1$. The second equation in (3.27) gives

$$v(p) - \xi(p) = \frac{\Delta_1}{u(p) - V'(p)}$$

(4.4)

which turns the first equation in (3.27) into a quartic equation in $u - v'$. However
since $\Delta_1$ is odd, $\Delta_1^2 = 0$, we obtain after some simple manipulations

$$u(p) - V'(p) = \sqrt{\Delta_0} - \frac{\Delta_1^2}{2\Delta_0^{3/2}}$$  \hspace{1cm} (4.5)$$

$$v(p) - \xi(p) = \frac{\Delta_1}{\sqrt{\Delta_0}}$$  \hspace{1cm} (4.6)$$

In principle $\Delta_0$, $\Delta_1$ are respectively even and odd polynomials in the fermionic couplings $\xi_{k+\frac{1}{2}}$, and a similar conclusion applies to $u(p)$ and $v(p)$. Among the two non-trivial solutions to the quartic equation defining $u - V'$ we choose the one which gives a contribution to $u - V'$ of order zero in the fermionic variables. Imitating the solution to the planar bosonic case we choose $u(p)$ to have a single cut

$$u(p) = V'(p) - M(p^2)\sqrt{p^2 - R} + \ldots$$  \hspace{1cm} (4.7)$$

and for the fermionic operator $v(p)$ we take the starting ansatz

$$v(p) = \xi_+(p^2) + \xi_-(p) - N_-(p^2)\sqrt{p^2 - R} - \frac{pN_+(p^2)}{\sqrt{p^2 - R}} + \ldots$$

$$\xi_+(p^2) = \sum_{k \geq 0} \xi_{2k+1/2}p^{2k}, \hspace{0.5cm} \xi_-(p) = \sum_{k \geq 0} \xi_{2k+3/2}p^{2k+1}$$  \hspace{1cm} (4.8)$$

From the form of (4.5) we easily guess that $u(p)$ must also have a contribution proportional to $(p^2 - R)^{-\frac{3}{2}}$. The polynomials $M$, $N_\pm$ are chosen to cancel the positive powers in $g, \xi_+, \xi_-$. Substituting (4.8) into the left hand side of (3.27b) we see that we find a polynomial on the right hand side. As expected, with (4.7) we find that we need to modify the ansatz. It is easy to check that it suffices to add to $u(p)$ the piece

$$- \frac{R}{M(R)} n_0^\pm(R)\left(n_0^-(R) + p n_1^+(R)\right)$$

$$\frac{\left(n_0^+(R) + p n_1^-(R)\right)}{(p^2 - R)^{3/2}}$$  \hspace{1cm} (4.9)$$

to guarantee that the left hand side of (3.27a) is a polynomial. In (4.9) $n_0^\pm(R)$,
$n_1^+(R)$ and $M(R)$ are obtained by expanding $N_\pm, M$ in powers of $p^2$ about $p^2 = R$:

\begin{align*}
M(R) &= M(p^2)|_{p^2=R} \\
N_\pm(p^2) &= n_0^\pm(R) + n_1^\pm(R)(p^2 - R) + \ldots
\end{align*}
(4.10)

To summarize, our solution to the planar superloop equations is given by

\begin{align*}
u(p) &= V'(p) - M(p^2)\sqrt{p^2 - R} - \frac{R}{M(R)} \frac{n_0^+(n_0^- + pn_1^+)}{(p^2 - R)^{3/2}} \quad \text{(4.11)} \\
u(p) &= \xi_+(p^2) + \xi_-(p) - N_-(p^2)\sqrt{p^2 - R} - \frac{pN_+(p^2)}{\sqrt{p^2 - R}} \quad \text{(4.12)}
\end{align*}

As we will see below, $N_\pm$ are linear in the fermionic couplings and $u(p)$ is at most bilinear in fermions. This is rather surprising and one might be tempted to add higher order terms in the $\xi$-couplings. Note however that (as soon as we give the explicit form of $M, N_\pm$) $u(p), v(p)$ have only negative powers of $p$ as $|p| \to \infty$, that the solution is perturbative in the couplings $g_k, k \neq 2$ and furthermore, by construction the left hand side of (3.27) are polynomials. If the reader has accepted the uniqueness arguments given at the beginning of this section he or she is unavoidably led to conclude that after $R$ is traded by the cosmological constant the same conclusion holds. Therefore any other (more complicated) dependence in the fermionic couplings is spurious and it will be redefined away when the correlators are expressed in terms of $\Lambda$. This is purely a planar phenomenon and it is expected not to hold to higher orders in the large $N$ expansion in our model (3.25). We will offer a more geometrical explanation of this fact in the next section, after we evaluate correlation functions of loop operators. Nevertheless this feature of the planar superloop equations is rather surprising.

The determination of $M(p^2)$ is exactly as in the bosonic case, and this implies therefore that the critical points will be labelled by a positive integer $m$ with a string susceptibility $\gamma_{st} = -\frac{1}{m}$. Decompose $u(p)$ into $u_0(p) + u_2(p)$, $u_0(p)$ (resp.
$u_2(p)$ is of order zero (resp. two) in the fermionic couplings

$$u_0(p) = V'(p) - M(p^2)\sqrt{p^2 - R}$$ (4.13)

This form of $u_0(p)$ automatically implies the expansion

$$u_0(p) = \frac{\Lambda(R)}{p} + \frac{u^{(2)}}{p^3} + \ldots + \frac{u^{(2k)}}{p^{2k+1}} + \ldots$$ (4.14)

We can arrive at the same result directly from our model (when $V(p)$ is even) by showing that all odd expectation values $\langle \sum_i \lambda_i^{2k+1} \rangle$ vanish when all fermionic couplings are set to zero. In this case, the partition function of our model takes the form

$$Z = \int \prod_{i=1}^{2N} d\lambda_i d\theta_i \prod_{i<j} (\lambda_i - \lambda_j - \theta_i \theta_j) \exp[-\frac{N}{\Lambda} \sum_k \sum_i g_{2k} \lambda_i^{2k}]$$ (4.15)

The integration over $\theta$'s yields $\Delta(\lambda) Pf(\lambda_i^{-1})$, where $\Delta$ is the standard Vandermonde determinant and $Pf(\lambda_i^{-1})$ is the Pfaffian of the antisymmetric matrix $M_{ij} = (\lambda_i - \lambda_j)^{-1} i \neq j$, $M_{ii} = 0$. Since $\Delta(\lambda) Pf(\lambda_i^{-1})$ is even under $\lambda_i \to -\lambda_i$ it is clear that any correlators of the form $\langle \sum_i \lambda_i^{2k+1} \rangle$ vanishes.

Multiplying (4.13) by $p$ and introducing the new variable $t = p^{-2}$ we have

$$F(t) \equiv t^{-1/2} u_0(t^{-1/2}) = g(t^{-1}) - M(t^{-1}) t^{-1} \sqrt{1 - Rt}$$

$$g(p) \equiv pV'(p)$$ (4.16)

The left hand side $F(t)$ is analytic near $t = 0$: $F(t) = \Lambda(R) + u^{(2)} t + \ldots$. Similarly, for $t$ small $F(t)(1 - Rt)^{-\frac{1}{2}}$ is also analytic near $t = 0$, hence $g(t^{-1})(1 - Rt)^{-\frac{1}{2}}$ -
$M(t^{-1})t^{-1}$ should be analytic as well and we can determine $M$

$$M(t^{-1}) = \sum_{n \geq 0} M_n t^{-n},$$

$$\oint_0 M(t^{-1})t^{-1} dt = \oint_0 \frac{g(t^{-1})}{\sqrt{1-tR}} t dt$$

(4.17)

We need in particular

$$M_0 = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{2k+2}{4^k} R^k g_{2k+2} \equiv f(R)$$

Furthermore

$$\frac{\partial F(t, R)}{\partial R} = \frac{1}{\sqrt{1-Rt}} \left( \frac{1}{2} M(t^{-1}) - \frac{\partial M(t^{-1})}{\partial R} (t^{-1} - 1) \right)$$

(4.18)

by analyticity the term in parenthesis in (4.18) is independent of $t$ and can be computed using $f(R)$. From (4.14), (4.16) we conclude

$$\frac{\partial F(t, R)}{\partial R} = \frac{\partial R \Lambda(R)}{\sqrt{1-Rt}}$$

(4.19)

$$\partial R \Lambda(R) = \frac{1}{2} f(R) + R \frac{\partial f(R)}{\partial R}$$

(4.20)

$$\Lambda(R) = \sum_{k \geq 1} k g_{2k} \frac{R^k}{4^k} \left( \frac{2k}{k} \right) = \int_0 dx V'(x + \frac{R}{4x})$$

(4.21)

Evaluating the term in parenthesis in (4.18) (which is independent of $t$) at $t^{-1} = R$, we obtain the useful identity

$$M(p^2)|_{p^2=R} = 2 \partial R \Lambda(R)$$

(4.22)

The determination of the fermionic functions $N_{\pm}(p^2)$ goes along the same lines.
Defining
\[ v_+(p^2) = \sum_{k \geq 0} \frac{v(2k+1)}{p^{2k+2}} , \quad v_-(p) = \sum_{k \geq 0} \frac{v(2k)}{p^{2k+1}} \] (4.23)

(4.12) splits into two equations
\[ v_+(p^2) = \xi_+ - \frac{pN_+}{\sqrt{p^2 - R}} , \quad v_-(p) = \xi_- - N_- \sqrt{p^2 - R} \] (4.24)

The equation for \( v_-(p) \) is identical to the bosonic case. Introducing \( t = p^{-2} \)
\[ v_+(t^{-1}) = \xi_+(t^{-1}) - \frac{N_+(t^{-1})}{\sqrt{1 - Rt}} \] (4.25)

since \( \sqrt{1 - Rt} v_+(t^{-1}) \) is analytic near \( t = 0 \), the analogue of (4.16) becomes
\[ \oint_0^{\xi_+(t^{-1})} \sqrt{1 - Rt} t^{t^{-1} - 1} \, dt = \oint_0^{N_+(t^{-1})} t^{t^{-1} - 1} \, dt \] (4.26)

and
\[ \frac{\partial v_+(t, R)}{\partial R} = -\frac{1}{(1 - Rt)^{3/2}} \left( \frac{1}{2} N_+(t^{-1}) + \frac{\partial N_+(t^{-1})}{\partial R} (t^{-1} - 1) \right) \] (4.27)

As \( v_+(t, R) \) is divisible by \( t \), the term in parenthesis is independent of \( t \), and only the zeroth order in \( N_+ = \sum_{n=0} N_+ t^{-n} \) is relevant
\[ \rho'_+(R) \equiv -\frac{1}{2} N_+ + R \frac{\partial N_+}{\partial R} = -\frac{1}{2} \sum_{k \geq 0} \xi_{2k+1/2} \frac{R^k}{4^k} \binom{2k}{k} \] (4.28)

Expanding \( N_+ \) about \( p^2 = R \)
\[ N_+(p^2) = n^+_0(R) + n^+_1(R)(p^2 - R) + \ldots \] (4.29)

and using the \( t \)-independence of the term in parenthesis in (4.27) we obtain
\[ n^+_0(R) = -2\rho'_+(R) , \quad n^+_1(R) = -4\rho''_+(R) \] (4.30)

We can carry out similar computations for \( v_-(p) \), and determine as well \( u_2(p) \). The
The final results can be summarized as follows:

\[
\frac{\partial u_0(p)}{\partial \Lambda} = \frac{1}{\sqrt{p^2 - R}} \quad , \quad \Lambda(R) = \sum_{k \geq 1} 2k g_{2k} \frac{R^k}{4^k} \binom{2k}{k} \quad (4.31)
\]

\[
\frac{\partial v_+(p)}{\partial R} = \frac{p \rho'_+(R)}{(p^2 - R)^{3/2}} \quad , \quad \rho'_+(R) = -\frac{1}{2} \sum_{k \geq 0} \xi_{2k+1/2} \frac{R^k}{4^k} \binom{2k}{k} \quad (4.32)
\]

\[
\frac{\partial v_-(p)}{\partial R} = \frac{\rho'_-(R)}{\sqrt{p^2 - R}} \quad , \quad \rho'_-(R) = \frac{1}{2} \sum_{k \geq 0} (2k + 1) \xi_{2k+3/2} \frac{R^k}{4^k} \binom{2k}{k} \quad (4.33)
\]

\[
u_2(p) = \frac{R}{\partial R \Lambda(R)} \frac{2 \rho'_+(R)(\rho'_-(R) - 2p \rho''_+(R))}{(p^2 - R)^{3/2}} \quad (4.34)
\]

Expanding in powers of \( \frac{1}{p} \), we can write equivalently

\[
\frac{\partial u_0^{(2k)}(p)}{\partial \Lambda} = \frac{\partial}{\partial \Lambda} \langle \Lambda N \sum_i \lambda_i^{2k} \rangle_0 = \frac{R^k}{4^k} \binom{2k}{k} \quad (4.35)
\]

\[
\frac{\partial v^{(2k+1)}(p)}{\partial \Lambda} = \frac{\partial}{\partial \Lambda} \langle \Lambda N \sum_i \theta_i \lambda_i^{2k+1} \rangle = (2k + 1) \frac{R^k}{4^k} \binom{2k}{k} \rho'_+(R) \partial \Lambda R \quad (4.36)
\]

\[
\frac{\partial v^{(2k)}(p)}{\partial \Lambda} = \frac{\partial}{\partial \Lambda} \langle \Lambda N \sum_i \theta_i \lambda_i^{2k} \rangle = \frac{R^k}{4^k} \binom{2k}{k} \rho'_-(R) \partial \Lambda R \quad (4.37)
\]

\[
\nu_2(2k) = \langle \Lambda N \sum_i \lambda_i^{2k} \rangle_2 = 4k \frac{R^k}{4^k} \binom{2k}{k} \rho'_+(R) \rho'_-(R) \partial \Lambda R \quad , \quad k \geq 1 \quad (4.38)
\]

\[
\nu_2^{(2k+1)} = \langle \Lambda N \sum_i \lambda_i^{2k+1} \rangle = -4(2k + 1) \frac{R^k}{4^k} \binom{2k}{k} \rho'_+(R) \rho''_+(R) \partial \Lambda R \quad (4.39)
\]

The subscripts 0 or 2 in \( \langle \ldots \rangle \) indicate orders 0 or 2 respectively with respect to the fermionic couplings. Equations (4.35) - (4.39) are the basis for the computation of loop correlators in section five.
To take the scaling limit of the previous expressions we notice that the zeroth order equations in fermionic couplings are those of the purely bosonic theory. Furthermore $\Lambda(R)$ in (4.31) is independent of any fermionic couplings. In analogy with the analysis in [12], near the $m$-th critical point $\Lambda(R)$ is given by

$$\Lambda(R) = 1 - (1 - R)^m + \sum_{n} t^n_B (1 - R)^n$$

$$\Lambda = e^{-\mu_B} \sim 1 - a^2 t$$

where $t$ is the renormalized cosmological constant and $t^n_B$ is the bare coupling of the $n$-th scaling operator. Exactly at the $m$-th critical point all $t^n_B = 0$. Introducing the scaling variable

$$R = 1 - a^{2/m} u$$

The string equation and the renormalized couplings are as in the bosonic case

$$u^m - \sum t_n u^n = t$$

$$t^n_B = a^{2(1-n/m)} t_n$$

with string susceptibility at the $m$-th critical point given by

$$\gamma_{st} = -1/m$$

and in the double scaling limit we again keep

$$N a^{2+1/m} = \kappa^{-1}$$

fixed. From (4.35) and (4.40) we can determine the form of the bare scaling bosonic
operators following Kazakov [12]:

\[
\sigma_{n}^{B+} = -\frac{1}{2(n+1)} N \sum_{k=0}^{n+1} \frac{(-4)^k}{2k^k} \binom{n+1}{k} \sum_{i=1}^{N} \lambda_i^{2k} \tag{4.46}
\]

There are two contributions to the planar expectation value:

\[
\langle \sigma_{n}^{B+} \rangle = \langle \sigma_{n}^{B+} \rangle_0 + \langle \sigma_{n}^{B+} \rangle_2 \tag{4.47}
\]

Notice that the contribution to \( \Lambda(R) \) of adding the operator (4.46) to the potential is

\[
\delta \Lambda(R) = -\frac{R}{n+1} \partial_R (1 - R)^{n+1} = (1 - R)^n - (1 - R)^{n+1} \tag{4.48}
\]

and in the continuum limit only the first term survives. This agrees with the identification of scaling operators in [24]. From (4.43) the renormalized operator is

\[
\sigma_{n}^{+} = a^{2(1-n/m)} \sigma_{n}^{B+} \tag{4.49}
\]

Since \( \partial / \partial \Lambda = -a^{-2} \partial / \partial t \) we obtain

\[
\partial_t \langle \sigma_{n}^{+} \rangle_0 = -\frac{1}{2\kappa^2} \frac{w^{n+1}}{n+1} \tag{4.50}
\]

and

\[
\langle \sigma_{n}^{+} \rangle \sim t^{1+1/m+n/m} = t^{2\gamma_{st}+(d_n-1)} \tag{4.51}
\]

We can identify the gravitational dimension of \( \sigma_{n}^{+} \) as

\[
d_n = \frac{n}{m} \tag{4.52}
\]

This result coincides with the scaling dimensions in the NS-sector of a \((2,4m)\) \(N = 1\) superconformal minimal model coupled to 2D-supergravity.
For the fermionic operators, in analogy with (4.40) we introduce the fermionic scaling variables according to

\[
\frac{\partial \rho \pm}{\partial R} = \mp \frac{1}{2} \sum_n \tau_n^{B \pm} (1 - R)^n
\]  

(4.53)

This definition determines the form of the fermionic operators to be

\[
\nu_n^{B+} = -\frac{N}{\Lambda} \sum_{k=0}^{n} \frac{(-4)^k}{2k \choose k} \left( \frac{n}{k} \right) \sum_{i=1}^{N} \theta_i \lambda_i^{2k}
\]  

(4.54)

\[
\nu_n^{B-} = -\frac{N}{\Lambda} \sum_{k=0}^{n} \frac{(-4)^k}{(2k + 1) \choose 2k} \left( \frac{n}{2k} \right) \left( \frac{k}{k} \right) \sum_{i=1}^{N} \theta_i \lambda_i^{2k+1}
\]  

(4.55)

Hence

\[
\frac{\partial}{\partial \Lambda} \left( \frac{\Lambda^2}{N^2} \langle \nu_n^{B\pm} \rangle \right) = \rho_+^l(R) \partial_R R(1 - R)^n
\]  

(4.56)

To unambiguously determine the scaling behavior of the fermionic variables \( \tau_n^{B \pm} \), we find it convenient to work first with the odd bosonic operators i.e. those determined by the odd expectation values \( \sum \lambda_i^{2k+1} \). From (4.39) we see that we have three point functions involving one odd bosonic operator and two fermionic operators \( \nu^+ \). The explicit form of the odd bosonic operators is

\[
\sigma_n^{B-} = -\frac{N}{\Lambda} \sum_{k=0}^{n} \frac{(-4)^k}{(2k + 1) \choose 2k} \left( \frac{n}{2k} \right) \left( \frac{k}{k} \right) \sum_{i=1}^{N} \lambda_i^{2k+1}
\]  

(4.57)

To derive this expression we must go back to the original bosonic loop equation
(3.27a) and add a small odd perturbation to the bosonic potential:

\[
V'(p) = \sum_{k \geq 1} k g_k p^{k-1} \equiv V'_+(p) + V'_-(p^2)
\]  

(4.58)

\[
V'_-(p^2) = \sum_{k \geq 0} (2k + 1) g_{2k+1} p^{2k}
\]  

(4.59)

The loop equation has a solution (to first order in \( V'_- \)):

\[
u_0(p) = V'_+(p) + V'_-(p^2) - M(p^2) \sqrt{p^2 - R} - \frac{p M_-(p^2)}{\sqrt{p^2 - R}} + O(V'_2)
\]  

(4.60)

Following arguments similar to those explained previously, and defining \( \Lambda_-(R) \) by

\[
u(p) = \frac{\Lambda(R)}{p} + \frac{\Lambda_-(R)}{p^2} + O\left(\frac{1}{p^3}\right)
\]  

(4.61)

we obtain:

\[
\partial_R u_0 = \frac{\partial_R \Lambda}{\sqrt{p^2 - R}} + \frac{p \partial_R \Lambda_-}{(p^2 - R)^{3/2}} \quad , \quad M_-(p^2)|_{p^2=R} = -2 \partial_R \Lambda_-
\]  

(4.62)

and

\[
\partial_R \Lambda_- = -\frac{1}{2} \sum_{k \geq 0} g_{2k+1} \frac{2k + 1}{4^k} \binom{2k}{k} R^k
\]  

(4.63)

Therefore

\[
\partial_\Lambda \left( \frac{\Lambda}{N} \sum_i \chi_i^{2k+1} \right) = \frac{2k + 1}{4^k} \binom{2k}{k} R^k \frac{\partial \Lambda_-}{\partial R} \partial_R R
\]  

(4.64)

We see that the operator (4.57) behaves according to

\[
\frac{\partial}{\partial \Lambda} \left( \frac{\Lambda^2}{N^2} \langle \sigma_{B^-} \rangle \right) = \partial_R \Lambda_- (R) \partial_R (1 - R)^n
\]  

(4.65)
\[ \partial_R \Lambda_\pm = -\frac{1}{2} \sum_n t_n B^- (1 - R)^n \]  
(4.66)

We can now derive the scaling dimension of \( \sigma_n B^- \) by computing two-point functions:

\[ \langle \sigma_n B^- \sigma_p B^- \rangle = \frac{\partial \langle \sigma_n B^- \rangle}{\partial t_p B^-} \]

\[ \frac{\partial}{\partial \Lambda} \left( \frac{\Lambda^2}{N^2} \langle \sigma_n B^- \sigma_p B^- \rangle \right) = \frac{1}{2} (1 - R)^{n+p} \partial_\Lambda R \]  
(4.67)

Hence

\[ \partial_t \langle \sigma_n B^- \sigma_p B^- \rangle = -\frac{1}{2m\kappa^2} t^{1/m+n/m+p/m-1} a^{2(n/m-1)} a^{2(p/m-1)} \]  
(4.68)

and

\[ t_n B^- = t_n a^{2(1-n/m)} \]
\[ \sigma_n B^- = a^{2(1-n/m)} \]  
(4.69)

Therefore the dimensions of the odd operators can be read off from

\[ \langle \sigma_n \sigma_p \rangle \sim t^{1/m+n/m+p/m} \]  
(4.70)

coinciding with the dimensions of the even operators (4.52). This duplication of operators is familiar from the bosonic case.

The bosonic loop \( u(p) \) in our solution contains a term bilinear in \( \rho_+ \). Equation (4.39) together with (4.57) implies:

\[ \frac{\Lambda}{N} \left( \sum_i \lambda_i^{2k+1} \right) = -4 R \rho'_+(R) \rho''_+(R) \partial_\Lambda R \frac{2k + 1}{4^k} \binom{2k}{k} R^k \]  
(4.71)

\[ \langle \sigma_n B^- \rangle = -4 \frac{N^2}{\Lambda^2} R \rho'_+(R) \rho''_+(R) \partial_\Lambda R (1 - R)^n \]
From the scaling behavior of $\rho'_\pm$ (4.53) we obtain

$$\langle \sigma_n^{B-} - \nu_p^{B+} + \nu_q^{B+} \rangle = \frac{N^2}{\Lambda^2} (q - p) (1 - R)^{p+q+n-1} \partial_\Lambda R \tag{4.72}$$

hence

$$\langle \sigma_n^{B-} - \nu_p^{B+} + \nu_q^{B+} \rangle = \frac{(q - p)}{mn^2} t^{(1/m-1)+n/m+p/m+q/m-1/m} a^{2(n/m+p/m+q/m-1/m-3)} \tag{4.73}$$

Since for scaling operators we expect

$$\langle \prod_i O_i \rangle \sim t^{2-\gamma_{st}+\sum(d_i-1)} \tag{4.74}$$

we conclude

$$\nu_n^+ = \nu_n^{B+} a^{2(1-\frac{n}{m}+\frac{1}{2m})} \tag{4.75}$$

and

$$d_p^+ + d_q^+ = \frac{1}{m}(p + q - 1) \tag{4.76}$$

We therefore identify

$$d_p^+ = \frac{p}{m} - \frac{1}{2m} \tag{4.77}$$

Finally, from (4.36,37) and (4.54,55):

$$\frac{\partial}{\partial \Lambda} \left( \frac{\Lambda^2}{N^2} \langle \nu_q^{B-} \rangle \right) = -\rho'_+(R) \partial_\Lambda R (1-R)^q \tag{4.78}$$

implying

$$\partial_t \langle \nu_p^{B+} \nu_q^{B-} \rangle = -\frac{1}{2m\kappa^2} t^{(1/m-1)+n/m+p/m+q/m} a^{2(p/m-1)+2(q/m-1)} \tag{4.79}$$

Thus

$$\nu_n^- = \nu_n^{B-} a^{2(1-\frac{n}{m}+\frac{1}{2m})} \tag{4.80}$$
and

\[ d_p^- = \frac{p}{m} + \frac{1}{2m} \]  

(4.81)

Summarizing

\[ \nu_n^\pm = r_n^{B\pm} a^2(1 - \frac{n}{m} \pm \frac{1}{2m}) \quad , \quad d_n^\pm = \frac{n}{m} \mp \frac{1}{2m} \]  

(4.82)

and these are the gravitational scaling dimensions of the operators in the Ramond sector for the \((2, 4m)\)-minimal superconformal model coupled to 2D-supergravity. Notice that in the spectrum (4.82) all states are doubled except for the state with dimension \(d_0^-\). This is a state in the boundary of the Kac table analogous to the redundant operator \(\sigma_m\) in the \(m\)-th critical bosonic model. It is tempting to interpret \(\nu_0^-\) as related to the ground state of the theory. We also obtain the precise scaling behavior of \(\rho^\pm\):

\[ \rho^\pm = \mp \frac{1}{2} a^{\frac{1}{2} \pm \frac{1}{2m}} r^\pm(u) \]  

(4.83)

\[ r^\pm(u) = \sum_n r_n^\pm u^n \quad , \quad r_n^{B\pm} = r_n^{B\pm} a^{2(1 - \frac{n}{m} \pm \frac{1}{2m})} \]  

(4.84)

The reason why we had to go through such a long argument to compute the fermionic scaling dimensions is related to the extra term \(\pm 1/2m\) in (4.82). If we had only considered (4.79) the identification of the Ramond sector would have been ambiguous. To distinguish between different possible assignments we had to study carefully the coupling of \(\sigma_n^-\) to \(\nu_p^+\nu_q^+\). We have therefore shown that our model in the planar limit has critical points labeled by \(m = 1, 2, 3, \ldots\) and with scaling dimensions in agreement with the NS- and R-sectors of the \((2, 4m)\) superminimal models. It is also easy to see that in the NS-sector the one-, two- and three-point functions of scaling operators agree with the results obtained in the continuum super-Liouville theory [13]. We can compute arbitrary correlators as well. In the next section we compute the correlation functions for an arbitrary number of loop operators.
5. PLANAR LOOP AND SUPERLOOP CORRELATION FUNCTIONS

As an application of the result in the previous section we compute correlation functions of loop operators. We start with bosonic loops. We rescale for convenience by a factor of \( l^{-1/2} \) the definition of the loop operator in (2.27). This section is a translation of (2.25-31) to the present situation. Define

\[
\tilde{U}(l) \equiv \sqrt{\frac{\pi}{l}} \sum_{i=1}^{N} x_i^{2k} \quad l = ka^\frac{1}{2}
\]  

(5.1)

We will take \( a \to 0 \) and \( k \to \infty \) keeping \( l \) fixed. There are two contributions in the planar case to (5.1) coming from (4.35) and (4.38)

\[
\tilde{U}(l) = \tilde{U}_0(l) + \tilde{U}_2(l)
\]  

(5.2)

For \( \tilde{U}_0(l) \) the result is as given in (2.27) apart from a trivial rescaling

\[
\langle \tilde{U}_0(l) \rangle = \frac{1}{\kappa l} \int_{t}^{\infty} dt' e^{-lu(t')}\]

(5.3)

The other contribution comes from

\[
u_2^{(2k)} = \binom{2k}{k} \frac{k}{4^{k-1}} \rho_+(R) \rho_-(R) R^k \partial \Lambda R
\]  

(5.4)

Using (4.41,83) we obtain

\[
\langle \tilde{U}_2(l) \rangle = -\frac{1}{\kappa} \tau^+(u) \tau^-(u) e^{-lu} \partial_t u
\]  

(5.5)

In total

\[
\langle \tilde{U}(l) \rangle = \frac{1}{\kappa l} \int_{t}^{\infty} dt' e^{-lu(t')} - \frac{1}{\kappa} e^{-lu} \partial_t u \tau^+(u) \tau^-(u)
\]  

(5.6)
Similarly we can introduce two fermion loop operators

\[
\tilde{V}_+(l) \equiv 2\sqrt{\pi} l^2 \sum_{i=1}^{N} \theta_i \lambda_i^{2k} \quad \quad \tilde{V}_-(l) \equiv \sqrt{\pi} l^2 \sum_{i=1}^{N} \theta_i \lambda_i^{2k+1}
\]  
(5.7)

After some simple computations using (4.40,41,45,83) we arrive at

\[
\langle \tilde{V}_\pm(l) \rangle = \pm \frac{a^{1/m}}{\kappa} \int_0^\infty dt' e^{-lu(t')} \tau^\pm(u) \partial_u u
\]  
(5.8)

It may seem surprising that \( \langle \tilde{V}_\pm(l) \rangle \) contains a factor \( a^{1/m} \). However, when we think of superloops with length \( l \) and (bare) superlengths \( \theta^B_\pm \):

\[
\tilde{W}_\pm(l, \theta) \equiv \tilde{U}(l) + \theta^B_\pm \tilde{V}_\pm(l)
\]  
(5.9)

we define renormalized superlengths

\[
\theta_\pm = \theta^B_\pm a^{1/m}
\]  
(5.10)

and they behave dimensionally as \( l^{1/2} \), as we might expect. It is possible to show that \( \tilde{U}(l) \) and \( \tilde{V}_\pm(l) \) admit an expansion in terms of microscopic scaling operators. The bosonic scaling operators (4.46) have expectation values given by (4.50) and

\[
\frac{\Lambda^2}{N^2} \langle \sigma^B_n \rangle_2 = -2[(1 - R)^n - (1 - R)^{n+1}] \rho'_+(R) \rho'_-(R) \partial_R R
\]

as one can show using (4.38). In the continuum limit at the \( m \)-th critical point:

\[
\partial_t \langle \sigma_n \rangle_0 = -\frac{1}{2\kappa^2} \frac{u^{n+1}}{n+1} \quad \quad \langle \sigma_n \rangle_2 = \frac{u^n}{2\kappa^2} \tau^+(u) \tau^-(u) \partial_t u
\]  
(5.11)

This together with (5.6) yields

\[
\tilde{U}(l) = 2\kappa \sum_{n=0}^\infty (-1)^{n+1} \frac{u^n}{n!} \sigma_n + \text{(singular as } l \to 0) \]

(5.12)

The singular terms are analytic in \( t \) and they correspond to contributions from
microscopic loops. Similarly in the fermionic case:

$$\partial_\Lambda \left( \frac{\Lambda^2}{N^2} \langle \nu_n^\pm \rangle \right) = -\rho'_\mp (R) \partial_\Lambda R (1 - R)^n$$

(5.13)

In the continuum limit

$$\partial_t \langle \nu_n^\pm \rangle = \pm \frac{u^n}{2\kappa^2} \tau^\pm (u) \partial_t u$$

(5.14)

and comparing with (5.8) we obtain

$$\tilde{V}_\pm (l) = 2\kappa a^{1/m} \sum_{n=0}^{\infty} \frac{(-)^{n+1}}{n!} l^n \nu_n^\pm + (\text{singular terms as } l \to 0)$$

(5.15)

Thus

$$\tilde{W}_\pm (l, \theta) = \tilde{U}(l) + \theta^B \tilde{V}_\pm (l) = 2\kappa \sum_{n=0}^{\infty} \frac{(-)^{n+1}}{n!} l^n (\sigma_n + \theta \nu_n^\pm)$$

(5.16)

This result is dimensionally consistent with the dimensions computed for $\sigma_n, \nu_n^\pm, \tilde{U} \sim l^{-1/2}$, $\tilde{V}_+ \sim l^{1/2}, \tilde{V}_- \sim l^{-1/2}$. Then we use $\tilde{W}_\pm = \tilde{U}(l) + \theta^B \tilde{V}_\pm (l)$ where $\theta^B$ is dimensionless but $\theta^B \sim l^{-1}$. Hence $[\theta_+] \sim l^{-1/2}$ and $[\theta_-] \sim l^{1/2}$. Since $[l] = -1/m$ in units of $t$, $[\theta_{\pm}] = \pm 1/2m$ and

$$\dim \nu_n^\pm = \dim \sigma_n \mp \frac{1}{2m}$$

(5.17)

as expected.

Finally we compute multiloop correlation functions. We need to use equation (2.29), and similar formulae for $\tau^\pm$. It is not difficult to show that when $k \to \infty$
as in (2.26),
\[
\frac{\Lambda}{N} \frac{\partial u}{g_{2\nu}} = -2\kappa \sqrt{\frac{t}{\pi}} e^{-tu}
\]
(5.18)
\[
\frac{\Lambda}{N} \frac{\partial \tau^+}{\xi_{2k+\frac{1}{2}}} = \kappa a^{1/m} \frac{e^{-tu}}{\sqrt{\pi l}}
\]
(5.19)
\[
\frac{\Lambda}{N} \frac{\partial \tau^-}{\xi_{2k+\frac{3}{2}}} = 2\kappa a^{1/m} \sqrt{\frac{t}{\pi}} e^{-tu}
\]
(5.20)

After some simple algebra one finds
\[
\langle \tilde{U}(l_1) \cdots \tilde{U}(l_n) \rangle = (-2\kappa \partial_t)^{n-1} \langle \tilde{U}(l_1 + \cdots + l_n) \rangle
\]
(5.21)
\[
\langle \tilde{V}_+(l_1) \tilde{U}(l_2) \cdots \tilde{U}(l_n) \rangle = (-2\kappa \partial_t)^{n-1} \frac{\partial}{\partial \tau^\pm} \langle \tilde{U}(l_1 + \cdots + l_n) \rangle
\]
(5.22)
\[
\langle \tilde{V}_+(l_1) \tilde{V}_-(l_2) \cdots \tilde{U}(l_n) \rangle = (-2\kappa)^{n-1} \partial_t^{n-3} \frac{\partial}{\partial \tau^+} \frac{\partial}{\partial \tau^-} \langle \tilde{U}(l_1 + \cdots + l_n) \rangle
\]
(5.23)

From these expressions we can derive the correlator of \(n\) superloops with \(n^+ \tilde{W}_+\) operators and \(n^- \tilde{W}_-\) operators. Define the differential operators
\[
D_t^\pm \equiv \partial_t + \theta \pm i \frac{\partial}{\partial \tau^\pm}
\]
(5.24)

Then
\[
\langle \tilde{W}_+(l_1, \theta_1) \cdots \tilde{W}_+(l_{n^+}, \theta_{n^+}) \tilde{W}_-(l_{n^++1}, \theta_{n^++1}) \cdots \tilde{W}_-(l_n, \theta_n) \rangle = \]
\[
= (-2\kappa)^{n-1} \prod_{i=1}^{n^+} D_t^+ \prod_{j=n^++1}^{n} D_j^- \partial_t^{-1} \langle \tilde{U}(l_1 + l_2 + \cdots + l_n) \rangle
\]
(5.25)
where

\[ \partial_t^{-1} \equiv \int_t^\infty dt \]  \hspace{1cm} (5.26)

With this definition \( \tilde{U}(l) \) in (5.6) becomes

\[ \langle \tilde{U}(l) \rangle = \frac{1}{\kappa l} \left(-\partial_t^{-1} + \tau^+ \tau^- \right) e^{-lu} \]  \hspace{1cm} (5.27)

Notice also that

\[ \prod_i D_i^{n_i} = \partial_t^{n_\pm} + \theta_\pm \partial_{\tau^\pm}^{n_\pm-1} \frac{\partial}{\partial \tau^\pm} \]  \hspace{1cm} (5.28)

with

\[ \theta_\pm = \sum_{i=1}^{n_\pm} \theta_{\pm i} \]  \hspace{1cm} (5.29)

Therefore the superloop correlators (5.25) depend only on the total length \( L = l_1 + l_2 + \ldots + l_n \) and the total superlengths \( \theta_\pm = \sum \theta_{\pm i} \). A posteriori this explains why our solution to the loop equations contains at most terms of order two in the fermionic couplings. This is also reminiscent of the bosonic case [25]. At the \( m \)-th critical point the relevant operators are \( \sigma_0, \sigma_1, \ldots \sigma_{m-2} \). The operator \( \sigma_{m-1} \) is redundant and the \( L_1 \) Virasoro condition in the scaling limit can be used to compute correlators with one insertion of \( \sigma_{m-1} \). In particular, if in the planar limit we evaluate \( \langle \sigma_{m-1} \prod w(l_i) \rangle \) we obtain simply total length \( L \) times \( \langle \prod w(l_i) \rangle \) and this loop length counting operator could be used to argue about the dependence of planar multiloop correlators on their total length. We believe that similar arguments should carry through in our case but using instead the fermionic partners of \( \sigma_{m-1} \).

To conclude we should mention that the planar limit of the theory is determined
by the following set of equations

\[-2\kappa^2 \partial_t^2 F = u - \partial_t (\tau^+ \tau^- \partial_t u)\]

\[t = u^m - \sum_n t_n u^n \quad \text{(5.30)}\]

\[\tau^\pm = \sum_n \tau^\pm_n u^n\]

Rederiving the one- and higher-point functions from (5.30) is left as an exercise to the reader.

6. CONCLUSIONS AND OUTLOOK

In our approach to the coupling of minimal \(N = 1\) superconformal models to \(2D\)-supergravity we have taken as our guiding principle a set of super-Virasoro constraints satisfied by the partition function (3.18). This led to our explicit representation of our model in (3.25). From it we derived the superloop equations and their planar approximation (3.27), whose solution led to a set of critical points labeled by an integer \(m = 1, 2, 3, \ldots\) with string susceptibility \(\gamma_{st} = -1/m\) and with scaling operators \(\sigma_n, \nu_n^\pm\) with scaling dimensions identical to those expected in the NS- and R-sectors of the \((2, 4m)\)-minimal superconformal model. We were also able to compute all multiloop correlation functions. There are at least two outstanding problems with regard to our model. The first and most important is the fact that (3.25) is only an “eigenvalue model” with pairs of eigenvalues \((\lambda_i, \theta_i)\) as basic variables. What kind of generalized matrix model should lead to this eigenvalue model is an important open question, and we are quite certain its solution will not involve supermatrices [26]. The second and easier problem consists of proving the fact that the superloop correlators depend only on the total length and superlengths by using the properties of redundant fermionic operators.
A physically more interesting question is to find out the properties of supersymmetry breaking to all orders in the $1/N$-expansion and non-perturbatively. This requires finding the replacement of the Painlevé-I equation found in [6] and the generalization of Douglas’s formulation [7] of the double scaling limit in terms presumably of operators realizing a Heisenberg superalgebra. The resolution of this problem should also shed some light on how to generalize our model to the coupling of $(p,p')$-minimal superconformal theories to $2D$-supergravity and also the explicit form of the $\hat{c} = 1$ theory.

The results presented in this paper lead us to believe that the integrable system replacing the KP-hierarchy in our case is not the super-KP-hierarchy constructed by Manin and Radul [9] but rather the hierarchy (SKP) found by Rabin [17]. Both are defined by starting with a pseudodifferential operator

$$L = D + u_1 + u_2 D^{-1} + u_3 D^{-2} + ...$$
$$D = \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial x}$$

where we can identify $x$ with the cosmological constant and $\xi$ with $\xi_{1/2}$. Since $D^2 = \partial/\partial x$, $D^{-1} = \partial_x^{-1} D$. Conjugating $L$ if necessary $L \rightarrow hLh^{-1}$ in order to satisfy $Du_1 + 2u_2 = 0$, there exists a unique pseudodifferential operator $S$ such that

$$S = 1 + s_1 D^{-1} + s_2 D^{-2} + ...$$

satisfying

$$S^{-1}LS = D$$

The Manin-Radul hierarchy is obtained in terms of a Lax pair constructed in terms of $L$. If the variables $t_{2n}$ (resp. $t_{2n-1}$) represent the even (resp. odd) flow parameters, and $(L^n)_+$ denotes the differential part of $L^n$, the Manin-Radul
hierarchy is given by
\[ \frac{\partial L}{\partial t_{2n}} = [L_{2n}^2, L] \]
\[ \frac{\partial L}{\partial t_{2n-1}} = [L_{2n-1}^2, L] - 2L_{2n}^2 + \sum_{k=1}^{\infty} t_{2k-1} \left[ L_{2n+2k-2}^2, L \right] \] (6.4)

Note that in the odd flows we have an explicit dependence on the odd parameters.
In Rabin’s case the hierarchy is defined according to the equations
\[ \frac{\partial S}{\partial t_{2n}} = -\left( S \partial_x^n S^{-1} \right)_- S \]
\[ \frac{\partial S}{\partial t_{2n-1}} = -\left( S \partial_x^{n-1} S^{-1} \right)_- S \] (6.5)

where \((L^n)_- = L^n - (L^n)_+\). This hierarchy is integrable but it does not admit a representation exclusively in terms of \(L\) as a Lax pair. The wave function or Baker-Akhiezer function takes the form
\[ w(z, \theta, x, \xi, t) = z^{-1} \exp \left[ \sum_{n=1}^{\infty} \left( t_{2n} z^{-n} + t_{2n-1} \theta z^{-n+1} \right) + x z^{-1} + \xi \theta \right] \] (6.6)

Identifying \(\lambda\) with \(z^{-1}\) and taking \(\theta\) to be the same as ours, we see that the expression in the exponential is the potential of our model. We believe it should be possible to show that our model is equivalent in the continuum limit to this hierarchy or some minor modification of it. All these questions are presently under investigation.

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