Communication Complexity, Corner-Free Sets and the Symmetric Subrank of Tensors

Matthias Christandl∗ Omar Fawzi† Hoang Ta‡ Jeroen Zuiddam§

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Abstract

We develop and apply new combinatorial and algebraic tools to understand multiparty communication complexity in the Number On the Forehead (NOF) model, and related Ramsey type problems. We identify barriers for progress and propose new techniques to circumvent these.

• We introduce a technique for constructing independent sets in hypergraphs via combinatorial degeneration. In particular, we make progress on the corner problem by proving the existence of a corner-free subset of $\mathbb{F}_2^n \times \mathbb{F}_2^n$ of size $\Omega(3^{1.16n}/\text{poly}(n))$, which improves the previous lower bound $\Omega(2^{0.82n})$ of Linial, Pitassi and Shraibman (ITCS 2018).

In the Eval problem over a group $G$, three players need to determine whether their inputs $x_1, x_2, x_3 \in G$ sum to zero. As a consequence of our construction of corner-free sets, the communication complexity of $\text{Eval}_{\mathbb{F}_2}$ is at most $0.34n + O(\log n)$, which improves the previous upper bound $0.5n + O(\log n)$.

• We point out how induced matchings in hypergraphs pose a barrier for existing tensor tools (like slice rank, subrank, analytic rank, etc.) to effectively upper bound the size of independent sets in hypergraphs. On the communication side of the story, this implies a barrier for these tools to effectively lower bound the communication complexity of the Eval problem over any group $G$.

• We propose to circumvent this barrier via a new natural notion called the symmetric subrank of tensors, the symmetric version of Strassen’s subrank, and we prove relations and separations for the symmetric subrank. We carry out a representation-theoretic study that leads to the symmetric quantum functional, advancing the theory of quantum functionals (STOC 2018) for symmetric tensors. Finally we prove that “Comon’s conjecture” about the equality of the rank and symmetric rank of symmetric tensors holds asymptotically for the tensor rank, the subrank as well as the restriction preorder. This implies a strong connection between Strassen’s asymptotic spectrum of tensors and the asymptotic spectrum of symmetric tensors.

∗Department of Mathematical Sciences, University of Copenhagen, christandl@math.ku.dk
†Univ. Lyon, ENS Lyon, UCBL, CNRS, Inria, LIP, omar.fawzi@ens-lyon.fr
‡Univ. Lyon, ENS Lyon, UCBL, CNRS, Inria, LIP, duy-hoang.ta@ens-lyon.fr
§Courant Institute, NYU and KdVI, University of Amsterdam, jzuiddam@nyu.edu
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1 Introduction

In the Number On the Forehead (NOF) model of communication complexity [CFL83], $k$ players need to design a protocol to compute a fixed function $F : X_1 \times \cdots \times X_k \rightarrow \{0, 1\}$ on all inputs $(x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k$. In this model, player $i$ has access to input $x_i$ for all $j \neq i$ but does not see input $x_i$. When $k = 2$, this model corresponds to the standard two-party communication model [Yao79], but when $k \geq 3$, the shared information between the players makes this model surprisingly powerful [Gro94, BGKL04, ACFN15, CS14]. The NOF model turned out to be very rich both in terms of connections to Ramsey theory and additive combinatorics [CFL83, BGG06, Shr18, LS21], as well as applications to boolean models of computation such as branching programs and boolean circuits [CFL83, BT94]. For instance, a sufficiently strong lower bound for an explicit function $F$ for $k \geq \text{polylog}(n)$ players with $n = \log |X_i|$ implies a breakthrough result in complexity theory, namely a lower bound on the complexity class $\text{ACC}^0$.

**Eval problem.** The overlap in information between the players increases as the number of players grows, and even for $k = 3$ players fundamental problems remain open. For instance, we do not know of an explicit function for which randomized protocols are significantly more efficient than deterministic ones [BDPW07]. A candidate for this separation is the function $\text{Eval}_{F_2}$, a natural generalization of the equality problem, defined by $\text{Eval}_{F_2}(x_1, x_2, x_3) = 1$ if and only if $x_1 + x_2 + x_3 = 0$, where the additions are all in $F_2$. In the randomized setting, the standard protocol for the two-party equality problem that uses $O(1)$ bits of communication works in the same way for three parties for the Eval problem. However, in the deterministic setting, the communication complexity $D_3(\text{Eval}_{F_2})$ remains wide open: the best known lower bound $\Omega(\log \log n)$ follows from [LM07] and (before this work) the best upper bound was $0.5n + O(\log n)$ [ACFN15].

**Corner problem.** It was observed already in [CFL83] that the deterministic communication complexity of many problems in the NOF model can be recast as Ramsey theory problems. In particular, the communication complexity of the Eval problem $\text{Eval}_{F_2}$ can be characterized in terms of corner-free subsets of $F_2^n \times F_2^n$. A triple of elements $(x, y, \lambda) \in F_2^n$ is called a corner and a subset $S \subseteq F_2^n \times F_2^n$ is called corner-free if it does not contain any nontrivial corners (where nontrivial means that $\lambda \neq 0$). Denoting by $r_c(F_2^n)$ the size of the largest corner-free set in $F_2^n \times F_2^n$, the communication complexity of $\text{Eval}_{F_2}$ is $\log(4^n/r_c(F_2^n))$ up to a $O(\log n)$ additive term.

**Shannon capacity problem.** The size $r_c(F_2^n)$ of the largest corner-free set in $F_2^n \times F_2^n$ can in turn be characterized as the independence number of a 3-uniform hypergraph with $4^n$ vertices.\footnote{An independent set of a hypergraph is a subset $S$ of vertices such that no hyperedge has all its vertices $S$.} In fact, this hypergraph has a recursive form: it is obtained by taking the $n$-th power of a fixed hypergraph $H_{\text{cor}, F_2}$ on 4 vertices (see Section 2 for more details). The asymptotic growth of $r_c(F_2^n)$ as $n \rightarrow \infty$ is characterized by the Shannon capacity $\Theta(H_{\text{cor}, F_2})$ of the hypergraph $H_{\text{cor}, F_2}$.\footnote{In the setting of directed graphs, the term Sperner capacity (applied to the complement graph) [GKV92, GKV93] is sometimes used for the Shannon capacity.} That is, we have $r_c(F_2^n) = \Theta((H_{\text{cor}, F_2})^{n-o(1)})$. Thus, proving $\Theta(H_{\text{cor}, F_2}) < 4$ is equivalent to proving a linear lower bound on the communication complexity of $\text{Eval}_{F_2}$. Many other Ramsey type problems can be expressed as the Shannon capacity of some fixed problem.
hypergraph, such as the capset problem that saw a recent breakthrough in [CLP17, EG17], and the USP problems that arise in the study of matrix multiplication [CKSU05, ASU13].

1.1 Our contributions

**Problem 1.1.** The open question that motivates our work asks whether the following three equivalent statements are true:

- $D_3(\text{Eval}_{\mathbb{F}_2^n}) = \Omega(n)$
- $r_\angle(\mathbb{F}_2^n) \leq O(c^n)$ for some $c < 4$
- $\Theta(H_{\text{cor},\mathbb{F}_2^n}) < 4$.

Generalizing from $\mathbb{F}_2^n$ to $G^n$, where $G$ is an arbitrary Abelian group $G$ (see Section 2 for formal definitions), the question can be formulated more generally as whether the following three equivalent statements are true:

- $D_3(\text{Eval}_{G^n}) = \Omega(n)$
- $r_\angle(G^n) \leq O(c^n)$ for some $c < |G|^2$
- $\Theta(H_{\text{cor},G}) < |G|^2$.

We will now discuss our contributions to this problem.

1.1.1 Improved lower bounds on the Shannon capacity of hypergraphs

Our first contribution consists of new lower bounds for the corner problem via a new technique to lower bound the Shannon capacity of hypergraphs. In this way we obtain improved protocols for $\text{Eval}_{G^n}$ for several groups $G$.

For a hypergraph $H$ and any $m \in \mathbb{N}$, if the $m$-th power $H^{\boxtimes m}$ of a hypergraph $H$ contains an independent set of size $s$, then the capacity $\Theta(H)$ is at least $s^{1/m}$. This was used for example in [LPS18] with $m = 2$ on $H_{\text{cor},\mathbb{F}_2}$ and they found an independent set of size $s = 8$. We improve on this simple bound by observing that it is actually sufficient to construct a set of size $s$ which does not contain “cycles”. In the context of graphs, the notion of cycle is clear but for hypergraphs there are many possible definitions. Here, to get new bounds we use the notion of *combinatorial degeneration* to model such an “acyclic set”. Combinatorial degeneration is a concept from algebraic complexity theory [Str91], where they are used to construct fast matrix multiplication algorithms.

Applying the combinatorial degeneration method to the corner hypergraph $H_{\text{cor},\mathbb{F}_2}$ and $H_{\text{cor},\mathbb{F}_3}$ leads to the following improved bounds which we state in the three equivalent forms.

**Theorem.** *For the corner and eval problem over $\mathbb{F}_2^n$ we have:*

- $D_3(\text{Eval}_{\mathbb{F}_2^n}) \leq 0.34n + O(\log n)$

---

3The equivalence among the three formulations is standard and follows from Lemma 2.4, Proposition 2.3 and Lemma 2.1. In this paper, we will mainly use the formulation in terms of Shannon capacity (see Definition 1.5 below for a precise definition).
\[ r_\geq(F_{n2}) \geq \sqrt{10n^{\text{poly}(n)}} \]
\[ \Theta(H_{\text{cor},F_{n2}}) \geq \sqrt{10} \]

For the corner and eval problem over \( F_{n3} \) we have:
\[ D_3(\text{Eval}_{F_{n3}}) \leq 0.37n + O(\log n) \]
\[ r_\geq(F_{n3}) \geq \frac{7n}{\text{poly}(n)} \]
\[ \Theta(H_{\text{cor},F_{n3}}) \geq 7. \]

For the corner and eval problem over an arbitrary Abelian group \( G \) we have
\[ D_3(\text{Eval}_G) \leq \log |G|/2 n + O(\log n) \]
\[ r_\geq(G^n) \geq |G|^{3n/2}/\text{poly}(n) \]
\[ \Theta(H_{\text{cor},G}) \geq |G|^{3/2}. \]

The bounds over \( F_{n2} \) and \( F_{n3} \) are proved in Corollary 2.10 and 2.11. The bound for arbitrary \( G \) is proved in Proposition 2.6 via a simple probabilistic argument. Note that this general bound applied to \( G = F_2 \) gives \( \sqrt{8} \) which matches the previous best bound of [LPS18]. We also introduce a notion of acyclic set of a hypergraph (Section 2.4) which puts a stronger requirement but might be simpler to check than combinatorial degeneration.

1.1.2 Upper bounds on Shannon capacity and limitations of current methods

Our second contribution is that we point out an important limitation of current methods to effectively upper bound the Shannon capacity of hypergraphs.

The general question of upper bounds on the Shannon capacity of hypergraphs is particularly well-studied in the special setting of undirected graphs, from which the name “Shannon capacity” comes: it in fact corresponds to the zero-error capacity of a channel [Sha56]. Even for undirected graphs, it is not clear how to compute the Shannon capacity in general, but some tools were developed to give upper bounds. The difficulty is to find a good upper bound on the largest independent set that behaves well under the product \( \boxtimes \). For undirected graphs, the best known methods are the Lovász theta function [Lov79] and the Haemers bound which is based on the matrix rank [Hac79]. For hypergraphs, we only know of algebraic methods that are based on various notions of tensor rank, and in particular the slice rank [TS16], and similar notions like the analytic rank [GW11, Lov19], the geometric rank [KMZ20], and the G-stable rank [Der20]. Even though the slice rank is not multiplicative under \( \boxtimes \) it is possible to give good upper bounds on the asymptotic slice rank via an asymptotic analysis [TS16], which is closely related to the Strassen support functionals [Str91] or the more recent quantum functionals [CVZ18].

Most of the rank-based bounds actually give upper bounds on induced matchings and not only on independent sets. It is simple and instructive to see this argument in the setting of undirected graphs. For a given graph \( H = (V,E) \), let \( A \) be the adjacency matrix in which we set all the diagonal coefficients to 1. Then for any independent set \( I \subseteq V \), the submatrix \( (A_{i,j})_{i,j \in I} \) of \( A \) is the identity matrix and as a result \( |I| \leq \text{rank}(A) \). As the matrix rank is
multiplicative under tensor product, we get \( \Theta(H) \leq \text{rank}(A) \). Observe that this argument works equally well if we consider an induced matching instead of an independent set. An induced matching of size \( s \) of the graph \( H = (V, E) \) can be defined by two lists of vertices \( I_1(1), \ldots, I_1(s) \) and \( I_2(1), \ldots, I_2(s) \) of size \( s \) such that for any \( \alpha, \beta \in \{1, \ldots, s\} \) we have

\[
((I_1(\alpha), I_2(\beta)) \in E \text{ or } I_1(\alpha) = I_2(\beta)) \iff \alpha = \beta.
\]

In other words, the submatrix \( (A_{i,j})_{i \in I_1, j \in I_2} \) is an identity matrix, which also implies that \( s \leq \text{rank}(A) \). As such, the matrix rank is an upper bound on the asymptotic maximum induced matching. Tensor rank methods such as the subrank, slice rank, analytic rank, geometric rank and G-stable rank also provide upper bounds on the asymptotic maximum induced matching.

Using a result of Strassen [Str91], we show that there is an induced matching of the \( n \)-th power of \( H_{\text{cor}, \mathbb{F}_2} \) of size \( 4^{n-o(1)} \). This establishes a barrier on many existing tensor tools (such as slice rank, subrank, analytic rank, etc.) to make progress on Problem 1.1. In fact, this result holds more generally for any Abelian group \( G \):

**Theorem.** For any Abelian group \( G \), the hypergraph \( H_{\text{cor}, G}^{2n} \) has an induced matching of size \( |G|^{2n-o(n)} \). In other words, for any \( n \geq 1 \), there exist lists \( I_1, I_2, I_3 \subseteq G^n \times G^n \) of size \( s(n) = |G|^{2n-o(n)} \) such that the following holds. For any \( \alpha, \beta, \gamma \in \{1, \ldots, s(n)\} \)

\[
(I_1(\alpha), I_2(\beta), I_3(\gamma)) \text{ forms a corner} \iff \alpha = \beta = \gamma.
\]

We prove this result by establishing in Theorem 3.4 that the adjacency tensor of the hypergraph \( H_{\text{cor}, G} \) is tight (see Definition 3.3). Strassen showed in [Str91] that for tight sets, the asymptotic induced matching is characterized by the support functionals. By computing the support functionals for the relevant tensors, we establish the claimed result in Corollary 3.6. Note that if we could ensure that \( I_1 = I_2 = I_3 \), this would solve Problem 1.1. We computed the maximum independent set and maximum induced matching for \( H_{\text{cor}, \mathbb{F}_2}^{2n} \) for small powers \( n = 1, 2, 3 \) (see Table 1) and we found that the maximum independent set is strictly smaller than the maximum induced matching for \( n = 2 \) and \( n = 3 \). This motivates the search for methods that go beyond the maximum induced matching barrier.

In Section 3.4, we propose a simple and generic method based on fractional coverings that in principle does not suffer from the induced matchings barrier. For the corner problem, however, it gives the trivial bound. We use it to give a simple example of a graph for which the asymptotic induced matching is arbitrarily larger than the Shannon capacity (see Example 3.12).

### 1.1.3 The symmetric subrank of tensors

In order to go beyond the induced matching barrier, we propose a natural notion of tensor rank called the **symmetric subrank**.

**Definition.** We define the **symmetric subrank** of a tensor \( f = (f_{i_1, \ldots, i_k})_{i_1, \ldots, i_k \in [d]} \in (\mathbb{F}^d)^{\otimes k} \) as

\[
Q_s(f) = \max \{ r \in \mathbb{N} : \langle r \rangle \leq_s f \},
\]

where \( \langle r \rangle \leq_s f \) means that there exists a matrix \( (A_{i,j})_{i \in [r], j \in [d]} \in \mathbb{F}^{r \times d} \) such that for any \( i_1, \ldots, i_k \in [r] \),

\[
\sum_{j_1, \ldots, j_k \in [d]} A_{i_1,j_1} \cdots A_{i_k,j_k} f_{j_1,\ldots,j_k} = \begin{cases} 1 & \text{if } i_1 = i_2 = \cdots = i_k \\ 0 & \text{otherwise}. \end{cases}
\]
The symmetric subrank is the symmetric variation on Strassen’s subrank [Str87]. In the
definition of the subrank $Q(f)$, instead of using $k$ times the same matrix $A$ in equation (2), we
may choose $k$ possibly different matrices $A^{(1)}, \ldots, A^{(k)}$ (see Section 3.1 for a general definition).
The relation between the symmetric subrank and the subrank is analogous to the relation
between the symmetric rank and the rank [CGLM08]. Note though that unlike the symmetric
rank which only makes sense for symmetric tensors, the symmetric subrank can be defined for
any tensor. Another simple observation about the symmetric subrank is that it can never be
larger than the other relevant notions of rank: $Q_s(f) \leq Q(f) \leq SR(f) \leq d$ for any tensor $f$ in
dimension $d$, where $SR(f)$ denotes the slice rank $f$ (see Section 3.1 for a definition).

It is simple to see that for a hypergraph $H$ with adjacency tensor $A_H$ where the diagonal
entries are set to 1, $Q_s(A_H)$ provides an upper bound on the maximum independent set of $H$.

**Proposition.** In general, the symmetric subrank $Q_s(A_H)$ leads to a better bound compared to
the subrank:

- There exists a directed graph $H$ such that over $\mathbb{F}_2$, $Q_s(A_H)$ can be smaller than the
  maximum induced matching (Example 4.6)

- There exists a directed graph $H$ such that over $\mathbb{C}$, $Q_s(A_H) < Q(A_H)$ (Example 4.4)

However, in some settings, we can show they are equal

- For any undirected hypergraph $H$ on $d$ vertices, then over $\mathbb{C}$, $Q(A_H) = d$ implies that
  $Q_s(A_H) = d$ (Theorem 4.9).

We leave as a natural open question here the subrank analog of the recent result of Shi-
tov [Shi18] disproving Comon’s conjecture [CGLM08]: is there a *symmetric* tensor over the
complex numbers $f \in (\mathbb{C}^d)^{\otimes k}$ such that $Q_s(f) < Q(f)$?

For symmetric tensors $f$, i.e., tensors such that $f_{i_1, \ldots, i_k} = f_{\sigma(i_1), \ldots, \sigma(i_k)}$ for any permu-
tation $\sigma$, the symmetric subrank also has a natural interpretation in terms of homogeneous
polynomials. This is analogous to the interpretation of the symmetric rank of a symmet-
ric tensor as the Waring rank of a homogeneous polynomial. The symmetric subrank of a
homogeneous polynomial $F \in \mathbb{F}[x_1, \ldots, x_d]$ of degree $k$ is the maximum $r$ such that

$$F(\ell_1(y_1, \ldots, y_r), \ldots, \ell_d(y_1, \ldots, y_r)) = \sum_{i=1}^{r} y_i^k,$$

where $\ell_1, \ldots, \ell_d$ are linear forms and $y_1, \ldots, y_r$ are variables. We refer to Section 4 for more
details on this.

**Symmetric quantum functional** Recall that our objective was to obtain upper bounds
on the Shannon capacity of hypergraphs. For such applications, we need to analyze the
asymptotic properties of the symmetric subrank for the powers of a tensor. As the first step
to developing tools for this, we introduce the natural symmetric analogue of the quantum
functionals of [CVZ18] using the diagonal action of the group $GL_d$ on $(\mathbb{C}^d)^{\otimes k}$ instead of the
action of the group $GL_d^{\times k}$ on $(\mathbb{C}^d)^{\otimes k}$. The symmetric quantum functional $F$ applied to a

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*By undirected we mean that if $e = (v_1, \ldots, v_k)$ is in the edge set, then also any permutation of $e$ is in the edge set.*
tensor $f \in (\mathbb{C}^d)^{\otimes k}$ is obtained by constructing the $k$-partite density operator $\rho(f) = \frac{ff^\dagger}{\|f\|^2}$ and computing the von Neumann entropy of the average of the $k$ marginals. We refer to Section 4.6 for a precise definition. The symmetric quantum functional does give an upper bound on the asymptotic symmetric subrank, and thus also on the Shannon capacity of hypergraphs, but unfortunately it gives trivial bounds for $H_{\text{cor},G}$:

**Theorem.** For any tensor $f$, the asymptotic symmetric subrank is bounded by the symmetric quantum functional:

$$\limsup_{n \to \infty} Q_s(f^{\otimes n})^{1/n} \leq F(f).$$

The symmetric quantum functional $F$ cannot overcome the induced matching barrier as:

$$\limsup_{n \to \infty} SR(f^{\otimes n})^{1/n} \leq F(f).$$

In fact, we can show that for symmetric tensors $f$, the asymptotic slice rank is equal to symmetric quantum functional:

$$\limsup_{n \to \infty} SR(f^{\otimes n})^{1/n} = F(f).$$ (3)

Equation (3) also gives an alternative symmetric description of the quantum functional with uniform weight $\theta = (1/k, \ldots, 1/k)$ from [CVZ18] on symmetric tensors. This description may be advantageous in the development of numerical algorithms.

However, this result reveals that for general tensors, the symmetric quantum functional cannot give better bounds than the quantum functionals which itself suffers from the induced matching barrier and cannot be used to make progress on Problem 1.1. But we hope that future improved asymptotic upper bounds on the symmetric subrank can still overcome the induced matching barrier. In particular, we leave it as an open question to define a good symmetric version of Strassen’s support functionals.

**Asymptotic symmetric subrank versus asymptotic subrank for symmetric tensors**

Finally, we prove a strong asymptotic relation between the symmetric subrank and the subrank for symmetric tensors (over appropriate fields). Namely we prove that on symmetric tensors the asymptotic symmetric subrank $\tilde{Q}_s(f) = \lim_{n \to \infty} Q_s(f^{\otimes n})^{1/n}$ and the asymptotic subrank $\tilde{Q}(f) = \lim_{n \to \infty} Q(f^{\otimes n})^{1/n}$ are equal,

$$\tilde{Q}_s(f) = \tilde{Q}(f).$$

In fact we prove the much stronger result that the asymptotic restriction preorder and the asymptotic symmetric restriction preorder are the same for symmetric tensors. From the same proof ideas it follows that the symmetric rank is at most $2^{k-1}$ times the rank of any symmetric $k$-tensor, and hence asymptotic rank and asymptotic symmetric rank coincide for symmetric tensors.

Comon [CGLM08] conjectured that rank and symmetric rank coincide on symmetric tensors. Shitov [Shi18] gave a counterexample to Comon’s conjecture. Our results can be interpreted as saying that Comon’s conjecture is true asymptotically for rank, subrank and the restriction preorder. This is discussed further in Section 4.4.
Our result that the asymptotic restriction preorder and the asymptotic symmetric restriction preorder coincide on symmetric tensor has a strong implication in the theory of Strassen’s asymptotic spectra of tensors developed in [Str86, Str88, Str88, Str91, Tob91, Bir90] (see also [CVZ18] and [Zui18]). This theory, developed to study the complexity of matrix multiplication, provides a dual formulation for the asymptotic subrank and asymptotic rank of tensors. Our result implies that the symmetric version of the theory is completely determined by the ordinary version. This is further discussed in Section 4.5.

1.2 Related work

There has been much recent work on the rich connections between NOF communication complexity and problems in combinatorics, including the works by Shraibman [Shr18], Linial, Pitassi and Shraibman [LPS18], Alon and Shraibman [AS20], and Linial and Shraibman [LS21]. This last very recent result [LS21] constructs large corner-free sets in $[N] \times [N]$ by improving the best known NOF communication protocols for the $\text{EXACT}_T$ problem by a constant factor. We note that, as far as we know, these constructions do not carry over to the groups of the form $G^n$ that we consider in this paper.

Among the tensor tools that suffer from the induced matching barrier are: the slice rank [Tao16], analytic rank [GW11, Lov19, Bri19], geometric rank [KMZ20], and G-stable rank [Der20]. Slice rank was used and studied extensively in combinatorics, in the context of cap sets [Tao16, KSS16], sunflowers [NS17] and right-corners [Nas20].

1.3 Tensor basics

We recall some standard tensor notation and definitions that we will use in the rest of the paper.

For $d \in \mathbb{N}$ let $[d] = \{1, \ldots, d\}$. Let $\mathcal{P}([d])$ be the set of all probability distributions on $[d]$.

Let $f \in \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k}$ be a $k$-tensor over a field $\mathbb{F}$. Let $\{e_1, \ldots, e_{d_j}\}$ denote the standard basis of $\mathbb{F}^{d_j}$. We may then write $f$ as

$$f = \sum_{i_1, \ldots, i_k} f_{i_1, \ldots, i_k} e_{i_1} \otimes \cdots \otimes e_{i_k},$$

where the sum goes over $i \in [d_1] \times \cdots \times [d_k]$. In this way $f$ corresponds to a $k$-way array $f \in \mathbb{F}^{d_1 \times \cdots \times d_k}$. For $f \in \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k}$ and $f' \in \mathbb{F}^{d_1'} \otimes \cdots \otimes \mathbb{F}^{d_k'}$, we define the tensor product as $(f \otimes f')(i_1, j_1, \ldots, i_k, j_k) = f_{i_1, \ldots, i_k} \cdot f'_{j_1, \ldots, j_k}$. We define the support of $f$ as the set

$$\text{supp}(f) := \{(i_1, \ldots, i_k) : f_{i_1, \ldots, i_k} \neq 0\} \subseteq [d_1] \times \cdots \times [d_k].$$

For $r \in \mathbb{N}$, we call $(r) := \sum_{i=1}^r e_i \otimes e_i$ the unit tensor of size $r$. For $f \in \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k}$ and $i \in [k]$, we denote by flatten$_i(f)$ the image of $f$ under the grouping $\mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k} \to \mathbb{F}^{d_i} \otimes \left(\bigotimes_{j \neq i} \mathbb{F}^{d_j}\right)$, which we call a flattening. We can think of flatten$_i(f)$ as a $d_i$ by $\prod_{j \neq i} d_j$ matrix.

Let $\mathfrak{S}_k$ be the symmetric group on $k$ symbols. A $k$-tensor $f \in (\mathbb{F}^d)^{\otimes k}$ is said to be symmetric if $f_{i_1, \ldots, i_k} = f_{\sigma(i_1), \ldots, \sigma(i_k)}$ for any $i_1, \ldots, i_k \in [d]$ and any permutation $\sigma \in \mathfrak{S}_k$. For example, a tensor $f \in (\mathbb{F}^d)^{\otimes 3}$ is symmetric if $f_{ijk} = f_{ikj} = f_{jik} = f_{jki} = f_{kij} = f_{kji}$, for all $i, j, k \in [d]$. 

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1.4 Hypergraph basics

We recall the definition of directed $k$-uniform hypergraphs and basic properties of Shannon capacity on directed $k$-uniform hypergraphs.

**Definition 1.2.** A directed $k$-uniform hypergraph $H$ is a pair $H = (V, E)$ where $V$ is a finite set of elements called vertices, and $E$ is a set of $k$-tuples of elements of $V$ which are called hyperedges or edges. If the set of edges $E$ is invariant under permuting the $k$ coefficients of its elements, then we may also think of $H$ as an undirected $k$-uniform hypergraph.

Let $H = (V, E)$ be a directed $k$-uniform hypergraph with $n$ vertices. The adjacency tensor $A$ of $H$ is defined as

$$A_{i_1, \ldots, i_k} = \begin{cases} 1 & \text{if } i_1 = i_2 = \cdots = i_k \text{ or } (i_1, \ldots, i_k) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 1.3.** The strong product of a pair of directed $k$-uniform hypergraphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ is denoted $G \boxtimes H$ and defined as follows. It is a directed $k$-uniform hypergraph with vertex set $V_G \times V_H$ and the following edge set: Any $k$ vertices $(g_1, h_1), \ldots, (g_k, h_k) \in V_G \times V_H$ form an edge $((g_1, h_1), \ldots, (g_k, h_k))$ if one of the following three conditions holds:

1. $g_1 = \cdots = g_k$ and $(h_1, \ldots, h_k) \in E_H$
2. $(g_1, \ldots, g_k) \in E_G$ and $h_1 = \cdots = h_k$
3. $(g_1, \ldots, g_k) \in E_G$ and $(h_1, \ldots, h_k) \in E_H$

**Definition 1.4.** An independent set in a directed $k$-uniform hypergraph $H = (V, E)$ is a subset $S$ of the vertices $V$ that induces no edges, meaning for every $(e_1, \ldots, e_k) \in E$ there is an $i \in [k]$ such that $e_i \notin S$. The independence number of $H$, denoted by $\alpha(H)$, is the maximal size of an independent set in $H$.

If $S$ and $T$ are independent sets in two directed $k$-uniform hypergraphs $G$ and $H$, respectively, then $S \times T$ is an independent set in the strong product $G \boxtimes H$. Therefore, we have $\alpha(G)\alpha(H) \leq \alpha(G \boxtimes H)$. For any directed $k$-uniform hypergraph $H$, let $H^{\otimes n}$ denote the $n$-fold product of $H$ with itself.

**Definition 1.5.** The Shannon capacity of a directed $k$-uniform hypergraph $H$ is defined as

$$\Theta(H) := \lim_{n \to \infty} (\alpha(H^{\otimes n}))^{1/n}.$$ 

By Fekete’s lemma we can write $\Theta(H) = \sup_m (\alpha(H^{\otimes n}))^{1/n}$. The following proposition can be deduced directly from the definition of Shannon capacity.

**Proposition 1.6.** Suppose $H$ is a directed $k$-uniform hypergraph with $m$ vertices and there is an independent set of size $s$ in $H^{\otimes n}$. Then $s^{1/n} \leq \Theta(H) \leq m$.

**Organization.** In Section 2 we introduce combinatorial degenerations and we use this technique to construct a better NOF protocol for the Eval functions. In Section 3 we establish the induced matching barrier for proving lower bounds on the Eval functions. In Section 4 we propose the symmetric subrank as a method to upper bound the independence number which circumvents the problematic induced matching number.
2 Lower bounds on Shannon capacity from combinatorial degenerations

We discuss three methods to obtain lower bounds on the Shannon capacity of directed 3-uniform hypergraphs: the probabilistic method, the combinatorial degeneration method and the acyclic set method. We apply these methods to the corner problem—the problem of constructing large corner-free sets—which as a consequence gives new NOF communication protocols for the Eval problem. We must begin by discussing the corner problem and its relation to NOF communication complexity.

2.1 Corner problem, cap set problem and number on the forehead communication

Let \((G, +)\) be a finite Abelian group.

Corner problem

A corner in \(G \times G\) is a three-element set of the form \(((x, y), (x + \lambda, y), (x, y + \lambda))\) for some \(x, y, \lambda \in G\) and \(\lambda \neq 0\). The element \((x, y)\) is called the center of this corner. Let \(r_{\angle}(G)\) be the size of the largest subset \(S \subseteq G \times G\) such that no three elements in \(S\) form a corner. The corner problem asks to determine \(r_{\angle}(G)\) given \(G\).

Trivially, we have the upper bound \(r_{\angle}(G) \leq |G|^2\). The best-known general upper bound on \(r_{\angle}(G)\) comes from [Shk06a, Shk06b], and reads

\[
r_{\angle}(G) \leq \frac{|G|^2}{(\log \log |G|)^c},
\]

where \(0 < c < \frac{1}{73}\) is an absolute constant. In the finite field setting, in [LM07] the following better upper bound for \(r_{\angle}(G)\) with \(G = \mathbb{F}_2^n\) was obtained:

\[
r_{\angle}(\mathbb{F}_2^n) \leq \mathcal{O}\left(|G|^2 \frac{\log \log \log |G|}{\log \log |G|}\right).
\]

We may phrase the corner problem as a hypergraph independence problem. We define \(H_{\text{cor}, G} = (V, E)\) to be the directed 3-uniform hypergraph with \(V = \{(g_1, g_2) : g_1, g_2 \in G\}\) and \(E = \{(g_1, g_2), (g_1 + \lambda, g_2), (g_1, g_2 + \lambda) : g_1, g_2, \lambda \in G, \lambda \neq 0\}\). Then by construction:

Lemma 2.1. \(r_{\angle}(G^n) = \alpha(H_{\text{cor}, G}^{\otimes n})\).

As a consequence, \(r_{\angle}(G^n) = \Theta(H_{\text{cor}, G}^{\otimes n})^{n-o(n)}\).

Example 2.2. Let \(G\) correspond to addition in \(\mathbb{F}_2\). Then \(H_{\text{cor}, G} = (V, E)\) with

\[
E = \{((0, 0), (1, 0), (0, 1)), ((0, 1), (1, 1), (0, 0)), ((1, 0), (0, 0), (1, 1)), ((1, 1), (0, 1), (1, 0))\}.
\]

Under the labeling \((0, 0) = 0, (0, 1) = 1, (1, 0) = 2\) and \((1, 1) = 3\) we will think of \(H_{\text{cor}, \mathbb{F}_2}\) as the hypergraph \(H_{\text{cor}, \mathbb{F}_2} = (V, E)\) with \(V = \{0, 1, 2, 3\}\) and \(E = \{(0, 2, 1), (1, 3, 0), (2, 0, 3), (3, 1, 2)\}\).

Closely related to \(r_{\angle}(G)\) is the minimum number of colors needed to color \(G \times G\) so that no corner is monochromatic, which we denote by \(c_{\angle}(G)\). Then:
Proposition 2.3 ([CFL83, LPS18]). Let \((G, +)\) be a finite Abelian group. There is a constant \(c\), such that for every \(n \in \mathbb{N}\),
\[
\frac{|G|^{2n}}{r_\perp(G^n)} \leq c \cdot \log |G| \leq c \cdot \frac{n |G|^{2n} \log |G|}{r_\perp(G^n)}.
\]
For \(G = \mathbb{F}_2\), the current upper bound in the literature is \(c_\perp(\mathbb{F}_2^n) \leq O(n^{2n/2})\) [LPS18], which we will improve on.

Number on the forehead communication

The corner problem is closely related to the Number On the Forehead (NOF) communication model [CFL83]. In this model, \(k\) players wish to evaluate a function \(F : \mathcal{X}_1 \times \cdots \times \mathcal{X}_k \to \{0, 1\}\) on a given input \(x_1, \ldots, x_k\). The input is distributed among the players in a way that player \(i\) sees every \(x_j\) for \(j \neq i\). This scenario is visualized as \(x_i\) being written on the forehead of Player \(i\). The computational power of everyone is unlimited, but the number of exchanged bits has to be minimized. Let \(D_k(F)\) be the minimum number of bits they need to communicate to compute the function \(F\) in the NOF model with \(k\) players. Many questions that have been thoroughly analyzed for the two-player case remain open in the general case of \(3\) or more players, where lower bounds on communication complexity are much more difficult to prove. The difficulty in proving lower bounds arises from the overlap in the inputs known to different players.

One interesting function in this context is the family of \(\text{Eval}\) functions. The function \(\text{Eval}_{G^n} : (\mathbb{G}^n)^3 \to \{0, 1\}\) outputs \(1\) on inputs \(x_1, x_2, x_3 \in G^n\) if and only if \(x_1 + x_2 + x_3 = 0\). The trivial algorithm gives that \(D_3(\text{Eval}_{G^n}) \leq n \log (|G|) + 1\). For two players Yao [Yao79] proved that \(D_2(\text{Eval}_{G^n}) = \Omega(n)\) (for nontrivial \(G\)). But, for three players it is an open problem whether \(D_3(\text{Eval}_{G^n}) = \Omega(n)\).

Lemma 2.4 ([BGG06]). \(\log(c_\perp(G^n)) \leq D_3(\text{Eval}_{G^n}) \leq 2 + \log(c_\perp(G^n))\).

From Lemma 2.4 and Proposition 2.3 it follows that \(\Theta(H_{\text{cor}}, G) < |G|^2\) would imply that \(D_3(\text{Eval}_{G^n}) = \Omega(n)\), and also that lower bounds on \(r_\perp(G^n)\) give upper bounds on \(D_3(\text{Eval}_{G^n})\).

For \(G = \mathbb{F}_2\), the best-known upper bound on \(D_3(\text{Eval}_{\mathbb{F}_2^n})\) is \(0.5n + \mathcal{O}(\log n)\) [ACFN15] which we improve on.

Three-term arithmetic progressions and the cap set problem

A three-term arithmetic progression in \(G\) is a three-element set of the form \(\{x, x + \lambda, x + 2\lambda\}\) for some \(x, \lambda \in G\) and \(\lambda \neq 0\). Let \(r_3(G)\) be the size of the largest subset \(S \subseteq G\) such that no three elements in \(S\) form a three-term arithmetic progression.

Following [Zha19, Corollary 3.24] there is a simple relation between corner-free sets and three-term-arithmetic-progression-free sets:

Lemma 2.5. \(p^n r_3(\mathbb{F}_p^n) \leq r_\perp(\mathbb{F}_p^n)\)

Proof. Let \(S \subseteq \mathbb{F}_p^n\) be a subset that is free of three-term arithmetic progressions. Define the subset \(T = \{(x, y) : x - y \in S\}\). Then \(T\) is a corner-free set of size \(p^n|S|\). Indeed, if \((x, y), (x + \lambda, y), (x, y + \lambda)\) are elements of \(T\), then \(x - y, x + \lambda - y, x - y - \lambda\) are in \(S\) and these elements form a three-term arithmetic progression. \(\square\)
A three-term-arithmetic-progression-free subset of $\mathbb{F}_3^n$ is also called a \textit{cap set}. The notorious cap set problem is to determine how $r_3(\mathbb{F}_3^n)$ grows when $n$ goes to infinity. A priori we have that $2^n \leq r_3(\mathbb{F}_3^n) \leq 3^n$. Using Fourier methods and the density increment argument of Roth, the upper bound $r_3(\mathbb{F}_3^n) \leq O(3^{n/2})$ was obtained by Meshulam [Mes95], and improved only as late as 2012 to $O(3^n/n^{1+\epsilon})$ for some positive constant $\epsilon$ by Michael Bateman and Nets Hawk Katz in [BK12]. Until recently it was not known whether $r_3(\mathbb{F}_3^n)$ grows like $3^{n-o(n)}$ or like $c^n$ for some $c < 3$. Gijswijt and Ellenberg solved this question in 2017, showing that $r_3(\mathbb{F}_3^n) \leq 2.755^n + o(n)$ [EG17]. The best lower bound is $2.2174^n \leq r_3(\mathbb{F}_3^n)$ by Edel [Ede04]. In particular, using Lemma 2.5, this implies the lower bound $3^n \cdot 2.2174^n = 6.6522^n \leq r_3(\mathbb{F}_3^n)$ for the corner problem. We will improve this lower bound in Theorem 2.11.

We may phrase the cap set problem as a hypergraph independence problem by defining the undirected 3-uniform hypergraph $H_{\text{cap}}$ consisting of three vertices $\{0,1,2\}$ and a single edge $e = \{0,1,2\}$. The independence number $\alpha(H_{\text{cap}})$ equals $r_3(\mathbb{F}_3^n)$, and thus the Shannon capacity of $H_{\text{cap}}$ determines the rate of growth of $r_3(\mathbb{F}_3^n)$.

### 2.2 Probabilistic method

This method is a very simple and generic way of obtaining lower bounds on the Shannon capacity. For any element $g \in G$, the set $\{(g,g+\lambda) : \lambda \in G\}$ is an independent set of $H_{\text{cor},G}$, and therefore we have $\Theta(H_{\text{cor},G}) \geq |G|$, which we think of as the trivial lower bound. By using a simple probabilistic argument (which does not use much of the structure of $H_{\text{cor},G}$), we show the following nontrivial lower bound for $\Theta(H_{\text{cor},G})$.

**Proposition 2.6.** For any finite Abelian group $G$, we have $\Theta(H_{\text{cor},G}) \geq |G|^{3/2}$.

**Proof.** Let $|G| = m$ and $n \in \mathbb{N}$. Recall that the hypergraph $H_{\text{cor},G}^{\boxtimes n}$ has vertices given by the elements of $G^n \times G^n$ and edges given by the corners in $G^n \times G^n$. Let $p = 1/\sqrt{3(m^n-1)}$ and choose the subset $A$ of $V(H_{\text{cor},G}^{\boxtimes n})$ randomly by choosing any element $(g_1,g_2) \in G^n \times G^n$ to be in the set $A$ with probability $p$. Let $H_A$ be the directed subhypergraph of $H_{\text{cor},G}^{\boxtimes n}$ induced by $A$. We have $E[|V(H_A)|] = m^{2n}p$. Let $e$ be any edge of $H_{\text{cor},G}^{\boxtimes n}$. Then $e$ is of the form 

$$e = ((g_1,g_2),(g_1+\lambda,g_2),(g_1,g_2+\lambda))$$

for some $g_1,g_2, \lambda \in G^n$ and $\lambda \neq 0$. Since $(g_1,g_2)$, $(g_1+\lambda,g_2)$ and $(g_1,g_2+\lambda)$ are different, and for each the probability of being in $A$ is $p$, we have that $Pr[e \in E(H_A)] = p^3$. Therefore, since $|E(H_{\text{cor},G}^{\boxtimes n})| = m^{2n}(m^n-1)$, we have $E[|E(H_A)|] = m^{2n}(m^n-1)p^3$. On the other hand, for any hypergraph $H$ we have $\alpha(H) \geq |V(H)| - |E(H)|$. Therefore 

$$\alpha(H_{\text{cor},G}^{\boxtimes n}) \geq E[|V(H_A)|] - E[|E(H_A)|] = \frac{2m^{2n}}{3\sqrt{3(m^n-1)}}.$$ 

Thus find the lower bound $\Theta(H_{\text{cor},G}) = \lim_{n \to \infty} \alpha(H_{\text{cor},G}^{\boxtimes n})^{1/n} \geq m^{3/2}$. \qed

For $G = \mathbb{F}_2$, using Proposition 2.6 and Proposition 2.3 we get $c_{\angle}(\mathbb{F}_3^n) \leq O(n^{2/n})$. This upper bound is similar to the bound provided in [LPS18, Corollary 26 in the ITCS version].
### 2.3 Combinatorial degeneration method

We now introduce the combinatorial degeneration method for lower bounding Shannon capacity. Combinatorial degeneration is an existing concept from algebraic complexity theory, first introduced by Strassen in [Str87, Theorem 6.1]\(^5\). In that original setting it was used as part of the construction of fast matrix multiplication algorithms [BCS97, Definition 15.29 and Lemma 15.31], and, in a broader setting, combinatorial degeneration was used to construct large induced matchings in [ASU13, Lemma 3.9], [AW18, Lemma 5.1] and [CVZ18, Theorem 4.11]. However, we will be using it in a novel manner in order to construct independent sets instead of induced matchings. We will subsequently apply the combinatorial degeneration method to get new bounds for the corner problem. We expect the method to be useful in the study of other problems besides the corner problem as well. First we must define combinatorial degeneration.

**Definition 2.7** (Combinatorial degeneration). Let \(I_1, \ldots, I_k\) be finite sets. Let \(\Phi \subseteq \Psi \subseteq I_1 \times \cdots \times I_k\). We say that \(\Phi\) is a **combinatorial degeneration** of \(\Psi\), and write \(\Psi \succeq \Phi\), if there are maps \(u_i : I_i \to \mathbb{Z} \ (i \in [k])\) such that for every \(x = (x_1, \ldots, x_k) \in I_1 \times \cdots \times I_k\), if \(x \in \Psi \setminus \Phi\), then \(\sum_{i=1}^{k} u_i(x_i) > 0\), and if \(x \in \Phi\), then \(\sum_{i=1}^{k} u_i(x_i) = 0\).

We apply combinatorial degeneration in the following fashion to get Shannon capacity lower bounds:

**Theorem 2.8** (Combinatorial degeneration method). Let \(H = (V, E)\) be a directed \(k\)-uniform hypergraph. Let \(S \subseteq V\). Let \(\Psi = E \cup \{(v, \ldots, v) : v \in V\}\) and let \(\Phi = \{(v, \ldots, v) : v \in S\}\) and suppose that \(\Psi \succeq \Phi\). Then \(\Theta(H) \geq |S|\).

**Proof.** Let \(u_i\) be the maps given by the combinatorial degeneration \(\Psi \succeq \Phi\). Let \(n\) be any multiple of \(|S|\). Let \((x^{(1)}, \ldots, x^{(k)}) \in \Psi^{\otimes n}\). Suppose for every \(i \in [k]\) that the \(n\) elements in the tuple \(x^{(i)} = (x^{(i)}_1, \ldots, x^{(i)}_n)\) are uniformly distributed over \(S\), so that every element of \(S\) appears \(n/|S|\) times in \(x^{(i)}\). Then, using that \(\sum_{i=1}^{k} u_i(s) = 0\) for every \(s \in S\) and the uniformity of \(x^{(i)}\), we have

\[
\sum_{i=1}^{k} \sum_{j=1}^{n} u_i(x^{(i)}_j) = \frac{n}{|S|} \sum_{s \in S} \sum_{i=1}^{k} u_i(s) = 0. \tag{4}
\]

For every \(j \in [n]\), since \((x^{(1)}_j, \ldots, x^{(k)}_j) \in \Psi\), we have \(\sum_{i=1}^{k} u_i(x^{(i)}_j) \geq 0\). Suppose that there is an index \(j \in [n]\) such that \((x^{(1)}_j, \ldots, x^{(k)}_j) \notin \Phi\). Then

\[
\sum_{i=1}^{k} u_i(x^{(i)}_j) > 0.
\]

As a consequence, \(\sum_{j=1}^{n} \sum_{i=1}^{k} u_i(x^{(i)}_j) > 0\), which contradicts (4). Thus the uniform strings in \(S^n\) form an independent set in \(H_0^{\otimes n}\). There are

\[
\binom{|S| \cdot n}{|S|, \ldots, |S|} \geq \frac{|S|^n}{(n + 1)|S|}.
\]

---

\(^5\)**Degeneration** of tensors is a powerful approximation notion in the theory of tensors. **Combinatorial degeneration** is the “combinatorial” or “torus” version of this kind of approximation.
such strings. The inequality follows from the fact that the largest multinomial coefficient is the central one, i.e., \((n, n, \ldots, n) \leq \binom{n}{n, \ldots, n}\) and the number of possible partitions of \(n\) into \(|S|\) parts is at most \((n + 1)^{|S|}\).

\[\sum_{i=1}^{k} p_i = 0, \quad \sum_{i=1}^{k} u_i(x_{j(i)}) = 0, \quad \text{for all } x \in \bigcap_{i=1}^{k} [V^n]_{p_i} \]

\[\Phi^{\otimes n} = \bigcup_{p_1, \ldots, p_k: \sum_{i=1}^{k} p_i = 0} [\Psi^{\otimes n}]_{p_1, \ldots, p_k} \]

Remark 2.9. The above proof gives in fact the precise lower bound

\[\alpha(H^{\otimes n}) \geq \frac{|S|^n}{(n + 1)^{|S|}}. \quad (5)\]

This lower bound is optimal up to a poly\((n)\) factor. The following more careful analysis improves this poly\((n)\) factor, but may safely be skipped when the reader is satisfied by the lower bound of \((5)\).

We may without loss of generality assume that \(S = V\). For \(p \in \mathbb{Z}\), let \([V^n]_p \subseteq V^n\) be the subset of all elements \((x_1, \ldots, x_n) \in V^n\) such that \(\sum_{j=1}^{n} u_i(x_j) = p\). For \(p_1, \ldots, p_k \in \mathbb{Z}\), we let \([\Psi^{\otimes n}]_{p_1, \ldots, p_k} \subseteq \Psi^{\otimes n}\) denote the subset of all elements \((x^{(1)}, \ldots, x^{(k)}) \in \Psi^{\otimes n}\) such that for every \(i \in [k]\) we have \(\sum_{j=1}^{n} u_i(x_{j(i)}) = p_i\). Thus \([\Psi^{\otimes n}]_{p_1, \ldots, p_k} = \Psi^{\otimes n} \cap ([V^n]_{p_1} \times \cdots \times [V^n]_{p_k})\).

Then

\[\Psi^{\otimes n} = \bigcup_{p_1, \ldots, p_k: \sum_{i=1}^{k} p_i = 0} [\Psi^{\otimes n}]_{p_1, \ldots, p_k} \]

and from the definition of a combinatorial degeneration we get

\[\Phi^{\otimes n} = \bigcup_{p_1, \ldots, p_k: \sum_{i=1}^{k} p_i = 0} [\Psi^{\otimes n}]_{p_1, \ldots, p_k}. \quad (6)\]

Since \(\Phi^{\otimes n}\) only contains elements of the form \((x, \ldots, x)\), we see that if \([\Psi^{\otimes n}]_{p_1, \ldots, p_k} \neq \emptyset\) and \(\sum_{i=1}^{k} p_i = 0\), then the elements of \([\Psi^{\otimes n}]_{p_1, \ldots, p_k}\) are all the elements \((x, \ldots, x)\) going over all \(x \in \bigcap_{i=1}^{k} [V^n]_{p_i}\). Thus \(\alpha(H^{\otimes n}) \geq |[\Psi^{\otimes n}]_{p_1, \ldots, p_k}|\) for any choice of \(p_1, \ldots, p_k\) such that \(\sum_{i=1}^{k} p_i = 0\).

One good choice of \(p_1, \ldots, p_k\) is obtained as follows, and lets us recover the lower bound in \((5)\). For notational simplicity we are still assuming \(S = V\). Let \((x_1, \ldots, x_n) \in V^n\) be any element that is uniform on \(S\). For every \(i \in [k]\) let \(p_i = \sum_{j=1}^{n} u_i(x_j)\). Note that for every \(i \in [k]\) the value of \(p_i\) remains the same if we had picked another uniform element \((x_1, \ldots, x_n) \in V^n\). We claim that \(\sum_{i=1}^{k} p_i = 0\). To prove this, let \((x^{(1)}, \ldots, x^{(k)}) \in \Psi^{\otimes n}\) be any element for which every \(x^{(i)}\) is uniform on \(S\). Then in the same way as in \((4)\) we have \(p_1 + \cdots + p_k = \sum_{i=1}^{k} \sum_{j=1}^{n} u_i(x_{j(i)}) = 0\), using that for every \(s \in S\) we have \(\sum_{i} u_i(s) = 0\).

Finally, note that \([\Psi^{\otimes n}]_{p_1, \ldots, p_k}\) contains all elements \((x^{(1)}, \ldots, x^{(k)}) \in \Psi^{\otimes n}\) for which every \(x^{(i)}\) is uniform. Therefore, with this choice we recover a bound that is at least as good as \((5)\).

Another choice of \(p_1, \ldots, p_k\) (that leads to an incomparable lower bound) is obtained as follows. Note that if \([\Psi^{\otimes n}]_{p_1, \ldots, p_k} \neq \emptyset\), then \(n \min_{x \in V} u_i(x) \leq p_i \leq n \max_{x \in V} u_i(x)\). Thus the number of nonzero summands in \((6)\) is at most \(c_{|S|} n^{k-1}\) for a constant \(c_{|S|}\) that depends only on \(|S|\). Therefore, there is a choice of \(p_1, \ldots, p_k\) with \(\sum_{i=1}^{k} p_i = 0\) such that

\[\alpha(H^{\otimes n}) \geq |[\Psi^{\otimes n}]_{p_1, \ldots, p_k} | \geq \frac{|\Phi^{\otimes n}|}{c_{|S|} n^{k-1}} = \frac{|S|^n}{c_{|S|} n^{k-1}}\]

which improves on \((5)\) in some parameter regimes.
Using Theorem 2.8 we obtain the following new lower bound for corners over $\mathbb{F}_2^2$:

**Theorem 2.10.** $\Theta(H_{\text{cor},\mathbb{F}_2}) \geq \sqrt{10}$.

In other words, $(\sqrt{10})^n / \text{poly}(n) \leq r_2(\mathbb{F}_2^2)$. As a consequence, we have the upper bound $c_2(\mathbb{F}_2^2) \leq \mathcal{O}(\text{poly}(n)(\frac{4}{\sqrt{10}})^n) \leq \mathcal{O}(\text{poly}(n)1.27^n)$ for the corner problem and the upper bound $D_3(\text{Eval}_{\mathbb{F}_2}) \leq \log(\frac{4}{\sqrt{10}})n + \mathcal{O}(\log n) \leq 0.34n + \mathcal{O}(\log n)$ for the eval problem.

**Proof.** Let $H = H_{\text{cor},\mathbb{F}_2} \boxtimes H_{\text{cor},\mathbb{F}_2}$. We will show $\Theta(H) \geq 10$, which implies $\Theta(H_{\text{cor},\mathbb{F}_2}) \geq \sqrt{10}$. Let $\Psi$ be the support of the adjacency tensor of $H$. Then $\Psi$ is this rather large set of 64 triples:

$$\Psi = \{((0,0),(0,0),(0,0)), ((0,1),(0,1),(0,1)), ((0,2),(0,2),(0,2)), ((0,3),(0,3),(0,3)), ((1,0),(1,0),(1,0)), ((1,1),(1,1),(1,1)), ((1,2),(1,2),(1,2)), ((1,3),(1,3),(1,3)), ((1,0),(1,2),(1,1)), ((1,1),(1,3),(1,0)), ((1,2),(1,0),(1,3)), ((1,3),(1,1),(1,2)), ((2,0),(2,0),(2,0)), ((2,1),(2,1),(2,1)), ((2,2),(2,2),(2,2)), ((2,3),(2,3),(2,3)), ((2,0),(2,2),(2,1)), ((2,1),(2,3),(2,0)), ((2,2),(2,0),(2,3)), ((2,3),(2,1),(2,2)), ((3,0),(3,0),(3,0)), ((3,1),(3,1),(3,1)), ((3,2),(3,2),(3,2)), ((3,3),(3,3),(3,3)), ((3,0),(3,2),(3,1)), ((3,1),(3,3),(3,0)), ((3,2),(3,0),(3,3)), ((3,3),(3,1),(3,2)), ((0,0),(2,0),(1,0)), ((0,1),(2,1),(1,1)), ((0,2),(2,2),(1,2)), ((0,3),(2,3),(1,3)), ((0,0),(2,2),(1,1)), ((0,1),(2,3),(1,0)), ((0,2),(2,0),(1,3)), ((0,3),(2,1),(1,2)), ((1,0),(3,0),(0,0)), ((1,1),(3,1),(0,1)), ((1,2),(3,2),(0,2)), ((1,3),(3,3),(0,3)), ((1,0),(3,2),(0,1)), ((1,1),(3,3),(0,0)), ((1,2),(3,0),(0,3)), ((1,3),(3,1),(0,2)), ((2,0),(0,0),(3,0)), ((2,1),(0,1),(3,1)), ((2,2),(0,2),(3,2)), ((2,3),(0,3),(3,3)), ((2,0),(0,2),(3,1)), ((2,1),(0,3),(3,0)), ((2,2),(0,0),(3,3)), ((2,3),(0,1),(3,2)), ((3,0),(1,0),(2,0)), ((3,1),(1,1),(2,1)), ((3,2),(1,2),(2,2)), ((3,3),(1,3),(2,3)), ((3,0),(1,2),(2,1)), ((3,1),(1,3),(2,0)), ((3,2),(1,0),(2,3)), ((3,3),(1,1),(2,2))\}.$

Let $S \subseteq V(H)$ be the subset consisting of the following ten vertices:

$$S := \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,1),(2,2),(2,3),(3,0)\}.$$

One directly verifies that the maps $u_i : \{0,1,2,3\}^2 \to \mathbb{Z}$ provided in the following table give a combinatorial degeneration from $\Psi$ to $\Phi_S := \{(v,v) : v \in S\}$.
Indeed, we see that therefore by Theorem 2.8 we obtain 

\[
\sum_i u_i(v_i) \leq \sqrt{10}.
\]

We make it easier to verify this by listing every element 

\[ e = (v_1, v_2, v_3) \in \Psi \]

again, together with the evaluation 

\[ (u_1(v_1), u_2(v_2), u_3(v_3)) \]

and whether it is in \( \Phi \) or in \( \Psi \setminus \Phi \):
We also use combinatorial degeneration to obtain the following new bound for corners over $\mathbb{F}_3^n$. Contrary to the previous construction, here we consider the first power of the hypergraph rather than the second power.

**Theorem 2.11.** $\Theta(H_{\text{cor}, \mathbb{F}_3}) \geq 7$.

In other words, $7^n / \text{poly}(n) \leq r_\Delta(\mathbb{F}_3^n)$. This improves on the lower bound $6.6522^n \leq r_\Delta(\mathbb{F}_3^n)$ that can be obtained from Edel’s construction of cap sets [Ede04] and Lemma 2.5. As a consequence of Theorem 2.10 and Theorem 2.11, it has the merits of being transparent and simple to apply.

Proof. Let $\Psi$ be the support of the adjacency tensor of $H_{\text{cor}, \mathbb{F}_3}$. We label each pair $(a, b)$ for $a, b \in \{0, 1, 2\}$ by the integer number $3a + b$. The hypergraph $H_{\text{cor}, \mathbb{F}_3}$ has vertex set $V = \{0, 1, 3, 4, 5, 6, 7, 8\}$ and the set $\Psi$ is given by

$$\Psi = \{(0, 0, 0), (1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 5), (6, 6, 6), (7, 7, 7), (8, 8, 8),$$

$$0, 3, 1), (0, 6, 2), (1, 4, 2), (1, 7, 0), (2, 5, 0), (2, 8, 1), (3, 6, 4), (3, 0, 5), (4, 7, 5),$$

$$(4, 1, 3), (5, 8, 3), (5, 2, 4), (6, 0, 7), (6, 3, 8), (7, 1, 8), (7, 4, 6), (8, 2, 6), (8, 5, 7)\}.$$

Let $S \subseteq V(H_{\text{cor}, \mathbb{F}_3})$ be the subset consisting of the following seven vertices:

$$S := \{0, 1, 2, 3, 4, 7, 8\}.$$

As before one directly verifies that the maps $u_i : V \to \mathbb{Z}$ provided in the following table give a combinatorial degeneration from $\Psi$ to $\Phi_S := \{(v, v, v) : v \in S\}$.

| vertex | $u_1$ | $u_2$ | $u_3$ |
|--------|-------|-------|-------|
| 0      | -5    | 1     | 4     |
| 1      | -5    | 4     | 1     |
| 2      | -5    | 1     | 4     |
| 3      | -5    | 5     | 0     |
| 4      | -3    | 2     | 1     |
| 5      | -1    | 5     | 5     |
| 6      | -1    | 5     | 5     |
| 7      | -3    | 2     | 1     |
| 8      | -5    | 5     | 0     |

We conclude that $\Theta(H_{\text{cor}, \mathbb{F}_3}) \geq 7$. 

At this point we have no structural understanding or explanation of how the above combinatorial degenerations that exhibit the new capacity lower bounds arise, and leave the investigation of further generalizations and improvements to future work. As a partial remedy to our limited understanding, we introduce in the next section the *acyclic method* as a tool to construct combinatorial degenerations. While the acyclic method does not recover the bounds of Theorem 2.10 and Theorem 2.11, it has the merits of being transparent and simple to apply.
2.4 Acyclic set method

The acyclic set method that we are about to introduce is modeled on the fact that the Shannon capacity of a directed graph $G$ is at least the size of any induced acyclic subgraph of $G$ [BM85]. We introduce the concept of an acyclic set in a directed $k$-uniform hypergraph as an extension of the notion of an induced acyclic subgraph.

**Definition 2.12.** Let $H$ be a directed $k$-uniform hypergraph. We associate to $H$ the directed graph $G_H$ with vertices $V(G) = V(H)$ and edges $E(G) = \{(a_1, a_2) : (a_1, a_2, \ldots, a_k) \in E \text{ for some } a_3, \ldots, a_k\}$. For any subset $A \subseteq V$ let $H[A]$ denote the subhypergraph of $H$ induced by $A$, that is, $H[A]$ is the directed $k$-uniform hypergraph with vertices $S$ and edges $E \cap A^x_k$. We call a subset $A \subseteq V$ an acyclic set of $H$ if the directed graph $G_{H[A]}$ is a directed acyclic graph.

Note that, if $A$ is an independent set of $H$, then $E(H[A]) = \emptyset$ and thus $E(G_{H[A]}) = \emptyset$, and in particular $A$ is an acyclic set of $H$. On the other hand, acyclic sets are not necessarily independent sets. However, the existence of an acyclic set does imply strong lower bounds on the Shannon capacity (via combinatorial degeneration, as we will see):

**Theorem 2.13.** Let $H$ be a directed $k$-uniform hypergraph. For any acyclic set $A$ of $H$, we have $\Theta(H) \geq |A|$.

Theorem 2.13 follows directly from the combinatorial degeneration method (Theorem 2.8) and the following lemma:

**Lemma 2.14.** Let $H = (V, E)$ be a directed $k$-uniform hypergraph. Let $A$ be an acyclic set of $H$. Then there is a combinatorial degeneration from $E \cup \{(v, \ldots, v) : v \in V\}$ to $\Phi = \{(v, \ldots, v) : v \in A\}$.

**Proof.** We may assume that $A = V = [n]$. The proof for the case that $A \subseteq V$ is a simple adaptation. Recall that we construct the directed graph $G$ associated to $H$ with the same vertex set as $H$ and the edges as follows: for every edge $e = (a_1, a_2, \ldots, a_k)$ in $H$ we add the edge $(a_1, a_2)$ to $G$. Since $V$ is an acyclic set we have that $G$ is a directed acyclic graph. Therefore, we have a topological ordering on the vertices of $G$. A topological ordering is a total ordering $> \mid$ on the vertices such that if $(u, v)$ forms an edge then $u < v$. Assume that this ordering is $1 > 2 > \cdots > n$. For each vertex $i \in [n]$, we define $u_1(i) = -i$, $u_2(i) = i$, $u_3(i) = \cdots = u_k(i) = 0$. For every $i \in [n]$ we clearly have $u_1(i) + u_2(i) + \cdots + u_k(i) = 0$. For each edge $e = (a_1, a_2, \ldots, a_k)$ in $H$ we have $u_1(a_1) + u_2(a_2) + \cdots + u_k(a_k) > 0$ because of the topological ordering and since we have the edge $(a_1, a_2)$ in $G$. Therefore we have a combinatorial degeneration from $E \cup \{(v, \ldots, v) : v \in V\}$ to $\{(v, \ldots, v) : v \in V\}$. For the case $A \subseteq V$ the proof is similar except that we define $u_1(i), u_2(i), \ldots, u_k(i)$ to be some large integer number for each $i \in V \setminus A$.

As can be seen from the proof of Lemma 2.14, the combinatorial degenerations that result from acyclic sets have a special form, and in particular the acyclic set method does not recover the full power of the combinatorial degeneration method. However the acyclic set method is much easier to apply than the combinatorial degeneration method. For example, we can use the acyclic set method to quickly see that $\Theta(H_{\text{cor},F_2}) \geq 3$. Namely, it is verified directly that the set $S = \{0, 1, 2\}$ of size three is an acyclic set in $H_{\text{cor},F_2}$, which implies the claim by Theorem 2.13.
3 Upper bounds on Shannon capacity and induced matching barrier

In this section we briefly discuss methods to obtain upper bounds on the Shannon capacity of directed \( k \)-uniform hypergraphs and we discuss limitations of these methods for hypergraphs like \( H_{\text{cor}, G} \). We will not be discussing all available tools, but rather some of the main ones: subrank and slice rank. The main point is to describe the induced matching barrier and apply it to the corner problem.

3.1 Tensor tools: subrank and slice rank

We focus on two tensor tools here: subrank and slice rank. We begin by defining subrank, for which we need the notion of restriction of tensors [Str87]. We say that the tensor \( f \in \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k} \) restricts to \( f' \in \mathbb{F}^{d'_1} \otimes \cdots \otimes \mathbb{F}^{d'_k} \), and write \( f' \leq f \) if there exist linear maps \( A^{(i)} : \mathbb{F}^{d_i} \rightarrow \mathbb{F}^{d'_i} \) such that

\[
    f' = A^{(1)} \otimes \cdots \otimes A^{(k)} \cdot f.
\]

Written in the standard basis, this corresponds to having for all \( i_1 \in [d'_1], \ldots, i_k \in [d'_k] \) that

\[
    f'_{i_1, \ldots, i_k} = \sum_{j_1 \in [d_1], \ldots, j_k \in [d_k]} A^{(1)}_{i_1, j_1} \cdots A^{(k)}_{i_k, j_k} f_{j_1, \ldots, j_k}.
\]

**Example 3.1.** Here we see restriction in action in a small example. For the tensors

\[
    f = e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1, \\
    f' = e_0 \otimes (e_0 \otimes e_0 + e_1 \otimes e_1),
\]

we have \( f' \leq f \) by letting \( A^{(1)} : e_0 \mapsto e_0, e_1 \mapsto e_0 \) and letting \( A^{(2)} \) and \( A^{(3)} \) be the identity map.

Let \( \langle n \rangle = \sum_{i \in [n]} e_i \otimes \cdots \otimes e_i \) be the unit tensor of rank \( n \). Strassen [Str87] defined the **subrank** of \( f \) as

\[
    Q(f) := \max \{ r \in \mathbb{N} : \langle r \rangle \leq f \}.
\]

Similarly, one may define the “opposite” of the subrank as \( R(f) := \min \{ r \in \mathbb{N} : f \leq \langle r \rangle \} \), which is called the rank and which coincides with the usual notion of tensor rank in terms of a rank-one decomposition. For \( k = 2 \), the subrank and rank of \( f \) are the usual matrix rank: \( Q(f) = R(f) = \text{rank}(f) \). When \( k \geq 3 \), however, we generally have \( Q(f) < R(f) \). In fact, the tensor rank can be larger than the dimensions \( d_1, \ldots, d_k \), whereas the subrank cannot exceed \( \min_i d_i \).

Applications require us to understand the rate of growth of the symmetric subrank as we take tensor product powers of a fixed tensor. Strassen [Str87] defined the **asymptotic subrank** of \( f \in \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k} \) as

\[
    \tilde{Q}(f) := \lim_{n \to \infty} Q(f^{\otimes n})^{1/n}.
\]

Since the subrank is super-multiplicative, we can, by Fekete’s lemma, replace the limit by a supremum.
The second tool we focus on is slice rank. Slice rank was introduced by Tao [Tao16] and developed further in [TS16] and [BCC+17] as a variation on tensor rank to study cap sets and approaches to fast matrix multiplication algorithms. A tensor in $\mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k}$ has slice rank one if it has the form $u \otimes v$ for $u \in \mathbb{F}^{d_i}$ and $v \in \bigotimes_{j \neq i} \mathbb{F}^{d_j}$ for some $i \in [k]$. The slice rank of $f$, denoted by $\text{SR}(f)$, is the smallest number $r$ such that $f$ can be written as sum of $r$ slice rank one tensors. Since slice rank is not sub-multiplicative and not super-multiplicative, the limit $\lim_{n \to \infty} \text{SR}(f^{\otimes n})^{1/n}$ might not exist [CVZ18]. We define

$$\tilde{\text{SR}}(f) = \limsup_{n \to \infty} \text{SR}(f^{\otimes n})^{1/n}.$$ 

Since slice rank is monotone under the restriction order and normalized on $\langle n \rangle$ [Tao16], it follows that $Q(f) \leq \tilde{\text{SR}}(f)$ and $\tilde{Q}(f) \leq \text{SR}(f)$.

### 3.2 Induced matchings and tightness

Now we discuss the notion of induced matchings, and we will discuss Strassen’s theorem that gives a construction of large induced matchings under a tightness condition.

Let $H = (V,E)$ be a directed $k$-uniform hypergraph with adjacency tensor $A$. Let $\Phi_H$ be the support of $A$. A subset $D \subseteq \Phi_H$ is called a matching if any two distinct elements $a, b \in D$ differ in all $k$ coordinates, that is, $a_i \neq b_i$ for all $i \in [k]$. We call a matching $D \subseteq \Phi_H$ an induced matching if $D = \Phi_H \cap (D_1 \times \cdots \times D_k)$, where $D_i = \{a_i : a \in D\}$ is the projection of $D$ onto the $i$-th coordinate. We denote by $Q_{\text{IM}}(\Phi_H)$ the maximum size of an induced matching $D \subseteq \Phi_H$.

For two directed $k$-uniform hypergraphs $G = (V_G,E_G)$ and $H = (V_H,E_H)$, let $\Phi_G$ and $\Phi_H$ be the support of the adjacency tensors of $G$ and $H$, respectively. We define the product $\Phi_G \times \Phi_H \subseteq (V_G \times V_H) \times \cdots \times (V_G \times V_H)$ by $\Phi_G \times \Phi_H := \{(a_1,b_1),\ldots,(a_k,b_k) : a \in \Phi_G, b \in \Phi_H\}$. The asymptotic induced matching number of $H$ is defined as $\tilde{Q}_{\text{IM}}(\Phi_H) := \lim_{n \to \infty} Q_{\text{IM}}(\Phi_H^{\otimes n})^{1/n} = \sup_n Q_{\text{IM}}(\Phi_H^{\otimes n})^{1/n}$.

The induced matching number should be thought of as the combinatorial version of the subrank, as follows. Let $\Phi_H$ be the support of the adjacency tensor $A_H$ of a directed $k$-uniform hypergraph $H$. Then the induced matching number $Q_{\text{IM}}(\Phi_H)$ is the largest number $n$ such that $\langle n \rangle$ can be obtained from $A_H$ using a restriction that consists of matrices that have at most one nonzero entry in each row and in each column. Therefore, $Q_{\text{IM}}(\Phi_H) \leq Q(A_H)$.

**Lemma 3.2.** Let $H$ be a directed $k$-uniform hypergraph and $A_H$ its adjacency tensor with support $\Phi_H = \text{supp}(A_H)$. Then

$$\Theta(H) \leq \tilde{Q}_{\text{IM}}(\Phi_H) \leq \tilde{Q}(A_H).$$

**Proof.** We begin with the first inequality. Let $S$ be an independent set of $H^{\otimes n}$. We have $\Phi^\otimes n_H = \text{supp}(A_H^{\otimes n})$. Thus $\Phi^\otimes n_H \cap (S \times S \times \cdots \times S) = \{(a,\ldots,a) : a \in S\}$. This means that $|S| \leq Q_{\text{IM}}(\Phi_H^{\otimes n})$. We conclude $\Theta(H) \leq Q_{\text{IM}}(\Phi_H)$. The second inequality follows from the already established inequality $Q_{\text{IM}}(\Phi_H) \leq Q(A_H)$. \hfill \Box

Next, we discuss tight sets, a notion introduced by Strassen [Str91].

**Definition 3.3** ([Str91], see also [CVZ18]). Let $I_1,\ldots,I_k$ be finite sets. We call any subset $\Phi \subseteq I_1 \times \cdots \times I_k$ tight if there are injective maps $u_i : I_i \to \mathbb{Z}$ for every $i \in [k]$ such that:

$$u_1(a_1) + \cdots + u_k(a_k) = 0 \text{ for every } (a_1,\ldots,a_k) \in \Phi.$$
When \( \Phi_H \) is tight, the asymptotic induced matching number is essentially known, and can be described as a simple optimization. To explain the precise formula we recall some definitions.

For any finite set \( X \), let \( \mathcal{P}(X) \) be the set of all distributions on \( X \). For any probability distribution \( P \in \mathcal{P}(X) \) the Shannon entropy of \( P \) is defined as

\[
H(P) := - \sum_{x \in X} P(x) \log_2 P(x)
\]

with \( 0 \log_2 0 = 0 \). Given finite sets \( I_1, \ldots, I_k \) and a probability distribution \( P \in \mathcal{P}(I_1 \times \cdots \times I_k) \) on the product set \( I_1 \times \cdots \times I_k \) we denote the marginal distribution of \( P \) on \( I_i \) by \( P_i \), that is, \( P_i(a) = \sum_{x : x_i = a} P(x) \) for any \( a \in I_i \).

**Theorem 3.4** ([Str91]). Let \( H \) be a directed \( k \)-uniform hypergraph. If \( \Phi_H \) is tight, then

\[
\tilde{Q}_{IM}(\Phi_H) = \max_{P \in \mathcal{P}(\Phi_H)} \min_{i \in [k]} 2^{H(P_i)}.
\]

In particular, Theorem 3.4 implies that, for \( H = (V,E) \) if there is a distribution \( P \) on \( \Phi_H \) such that every marginal distribution \( P_i \) is uniform on \( V \), then \( \Phi_H \) has asymptotically maximal induced matchings.

### 3.3 The corner hypergraph is tight

We will now apply Theorem 3.4 to the corner problem. First we see how the tightness property is satisfied by the corner problem.

**Theorem 3.5.** For any finite Abelian group \( (G,+), \) let \( \Phi_{H_{cor,G}} \) be the support of the adjacency tensor of \( H_{cor,G} \). Then the set \( \Phi_{H_{cor,G}} \) is tight.

**Proof.** Let \( m = |G| \) and \( \phi \) be a bijection between \( G \) and \( \{0,1,\ldots,m-1\} \). We define

\[
\begin{align*}
    u_1((g_1,g_2)) &= \phi(g_1) + m\phi(g_2) \\
    u_2((g_1,g_2)) &= m^2\phi(g_1 + g_2) - m\phi(g_2) \\
    u_3((g_1,g_2)) &= -m^2\phi(g_1 + g_2) - \phi(g_1).
\end{align*}
\]

It is easy to check that the maps \( u_1, u_2, u_3 \) are injective and that for every triple of pairs \((g_1,g_2), (g_1 + \lambda, g_2), (g_1, g_2 + \lambda)\), it holds that

\[ u_1((g_1,g_2)) + u_2((g_1 + \lambda,g_2)) + u_3((g_1,g_2 + \lambda)) = 0. \]

This proves the claim.

As a result of Theorem 3.5 and Theorem 3.4, we have that the asymptotic induced matching number of the corner hypergraph is maximal:

**Corollary 3.6.** For any group \( G \), \( \tilde{Q}_{IM}(H_{cor,G}) = |G|^2 \).

**Proof.** We know that \( \Phi_{H_{cor,G}} \) is tight by Theorem 3.5, and so we may apply Theorem 3.4. We take \( P \in \mathcal{P}(\Phi_{H_{cor,G}}) \) to be the uniform probability distribution. It then suffices to observe that every marginal distribution \( P_i \) is also uniform to obtain the claim.
In particular, Corollary 3.6 implies that no better upper bound on $\Theta(\mathcal{H}_{\text{cor},G})$ can be obtained via tools that also upper bound $\tilde{Q}_{\text{IM}}(\mathcal{H}_{\text{cor},G})$. Such tools include the slice rank, the analytic rank, the geometric rank and the G-stable rank.

We finish this part with the observation that already for small powers of the corner hypergraph $\mathcal{H}_{\text{cor},F_2}$ the independence number is strictly smaller than the induced matching number, as we record in Table 1. For comparison, we also give the analogous numbers for the cap set hypergraph $\mathcal{H}_{\text{cap}}$, where, interestingly, this is not the case. Corollary 3.6 and the above numerical observation suggests to actively look for upper bounds on the Shannon capacity that go below the induced matching number.

### 3.4 Fractional cover method

We discuss one more upper bound method for the Shannon capacity.

The fractional cover method was introduced by [FK00] for finding upper bounds for the Sperner capacity of directed graphs. For hypergraphs, we generalize the method to seek an upper bound for Shannon capacity by introducing the concept of upper-function on the set of directed $k$-uniform hypergraphs.

**Definition 3.7.** Let $G,H$ be directed $k$-uniform hypergraphs. Let $\gamma$ be a function mapping any subset of hypergraphs to real numbers. The function $\gamma$ is submultiplicative if $\gamma(G \boxtimes H) \leq \gamma(G)\gamma(H)$. The function $\gamma$ is supermultiplicative if $\gamma(G \boxtimes H) \geq \gamma(G)\gamma(H)$. The function $\gamma$ is multiplicative if $\gamma(G \boxtimes H) = \gamma(G)\gamma(H)$.

**Definition 3.8 (upper-function).** Let $\gamma$ be a map from directed $k$-uniform hypergraphs to real nonnegative numbers. We say that $\gamma$ is an upper-function on directed hypergraphs if the following conditions are satisfied

1. $\gamma(H) \geq \alpha(H)$ for all directed uniform hypergraphs $H$, and
2. $\gamma$ is submultiplicative on the strong product.

**Lemma 3.9.** Let $H_1, \ldots, H_n$ be directed $k$-uniform hypergraphs. Let $H = H_1 \boxtimes H_2 \boxtimes \cdots \boxtimes H_n$. Let $\gamma$ be an upper-function on directed $k$-uniform hypergraphs. Then

$$\alpha(H) \leq \prod_{i=1}^{n} \gamma(H_i).$$

**Proof.** This follows immediately from Definition 3.8 of upper-function.

| $n$ | $\alpha(H_{\text{cap}}^{2n})$ | $Q_{\text{IM}}(\Phi_{\mathcal{H}_{\text{cap}}^{2n}})$ |
|-----|-------------------------------|----------------------------------|
| 1   | 2                             | 2                                |
| 2   | 4                             | 4                                |
| 3   | 9                             | 9                                |

| $n$ | $\alpha(H_{\text{cor},F_2}^{2n})$ | $Q_{\text{IM}}(\Phi_{\mathcal{H}_{\text{cor},F_2}^{2n}})$ |
|-----|----------------------------------|----------------------------------|
| 1   | 2                               | 2                                |
| 2   | 8                               | 9                                |
| 3   | 24                              | 32                               |

Table 1: Small values of independence number and induced matching number for cap sets and corners.
**Definition 3.10.** (Fractional cover). Let $H$ be a directed $k$-uniform hypergraph with vertex set $V(H)$. A function $g : 2^{V(H)} \to \mathbb{R}_{\geq 0}$ is called a fractional cover of $V(H)$ if

$$
\sum_{U \in \mathcal{F}, v \in U} g(U) \geq 1 \text{ for all } v \in V(H),
$$

where $\mathcal{F}$ is the family of all subsets of $V(H)$.

**Theorem 3.11.** For any directed $k$-uniform hypergraph $H$, and any upper-function $\gamma$ on directed $k$-uniform hypergraphs, we have

$$
\Theta(H) \leq \min_{g} \sum_{U \subseteq V(H)} g(U)\gamma(H[U]),
$$

where the minimization is taken over all fractional covers $g$ of $V(H)$, and $H[U]$ is the directed $k$-uniform hypergraph induced by the subset $U$ of $V(H)$.

**Proof.** Let $h$ be a nonnegative integer function from $2^{V(H)}$ to $\mathbb{Z}_{\geq 0}$. For $s \in \mathbb{Z}_{\geq 0}$, $h$ is called an $s$-cover of $V(H)$ if \( \sum_{U : v \in U} h(U) \geq s \) hold for all $v \in V(H)$. Then we have

$$
\min_{g} \sum_{U \subseteq V(H)} g(U)\gamma(H[U]) = \inf_{s} \min_{h} \sum_{U \subseteq V(H)} h(U)\gamma(H[U]),
$$

where the minimization on the right-hand side is taken over all $s$-covers $h$ and the minimization on the left-hand side is taken over all fractional covers $g$. Indeed, there is a fractional cover $g$ of $V(H)$ that takes rational values and achieves the minimum on the left-hand side of (7). Therefore, there exists an integer number $s$ such that $h(U) = sg(U)$ is an integral $s$-cover of $V(H)$. In the other direction, if $h$ is an $s$-cover of $V(H)$ then the function $g(U) = h(U)/s$ is a fractional cover of $V(H)$.

Let $h$ be an $s$-cover of $V(H)$ and denote $\mathcal{U} = \{U_1, \ldots, U_m\}$ the multiset of subsets of $V(H)$ with $U \subseteq V(H)$ appearing $h(U)$ times in $\mathcal{U}$. For any independent set $I$ of $H$, we have

$$
\sum_{i=1}^{m} \alpha(H[U_i]) \geq \sum_{i=1}^{m} \alpha(H[U_i \cap I]) = \sum_{i=1}^{m} |U_i \cap I| \geq s|I|.
$$

Fix $s$ and let $h$ be a nonnegative $s$-cover attained by the minimum on the right-hand side of the equation (7). For $n \in \mathbb{N}$, let $\mathcal{U}^n$ be the multiset of all $n$-fold Cartesian products of sets from $\mathcal{U}$. For any $A = U_1 \times U_2 \times \cdots \times U_n$, define a function $h^{(n)}(A) = h(U_1) \cdot h(U_2) \cdots h(U_n)$ then $h^{(n)}$ is an $s^n$-cover of $V(H^{\otimes n})$ and the set $A = U_1 \times \cdots \times U_n$ appear in $\mathcal{U}^n$ with $h(A)$ times. Let $I^{(n)}$ be a maximum independent set in $H^{\otimes n}$, we have

$$
s^n|I^{(n)}| \leq \sum_{x_{i=1}^{n}U_i \in \mathcal{U}^n} \alpha \left( H^{\otimes n}[\times_{i=1}^{n}U_i] \right)
= \sum_{x_{i=1}^{n}U_i \in \mathcal{U}^n} \alpha \left( H[U_1] \Box \cdots \Box H[U_n] \right)
\leq \sum_{x_{i=1}^{n}U_i \in \mathcal{U}^n} \prod_{i=1}^{n} \gamma(H[U_i])
= \left[ \sum_{U_i \in \mathcal{U}} \gamma(H[U_i]) \right]^n.
$$
Recall that $I^{(n)}$ is a maximum independent set of $H^\otimes n$, by the definition of $U$, we have

$$\alpha(H^\otimes n) = |I^{(n)}| \leq \frac{1}{s^n} \left[ \sum_{U \subseteq V(H)} h(U) \gamma(H[U]) \right]^n.$$ 

This implies

$$\Theta(H) \leq \inf s \frac{1}{s} \left[ \sum_{U \subseteq V(H)} h(U) \gamma(H[U]) \right] = \min g \sum_{U \subseteq V(H)} g(U) \gamma(H[U]),$$

completing the proof.

We give a quick example using the above method of an undirected graph $G$ with Shannon capacity strictly smaller than the asymptotic maximum induced matching of $G$.

**Example 3.12.** Let $G$ be the undirected graph with adjacency matrix

$$
\begin{pmatrix}
J & I \\
I & J
\end{pmatrix}
$$

where $I$ is the $n \times n$ identity matrix and $J$ is the $n \times n$ all-ones matrix with $n \geq 2$. Clearly $\Theta(G) \geq 2$, since $\{1, n+2\}$ is an independent set and $Q_{IM}(G) \geq n$, since $\{(i, n+i) : i \in [n]\}$ is an induced matching. Therefore, $Q_{IM}(G) \geq n$. It remains to to upper bound $\Theta(G)$.

For any graph $H$ define $\gamma(H)$ as the matrix rank of the adjacency matrix of $H$ (over some arbitrary but fixed field). Then $\gamma$ is an upper-function, because matrix rank is multiplicative under the tensor product. Setting $g(V_1) = 1$ and $g(V_2) = 1$ we have that $g$ is a fractional cover of $G$. By Theorem 3.11, we have

$$\Theta(G) \leq g(V_1) \operatorname{rank}(G[V_1]) + g(V_2) \operatorname{rank}(G[V_2]) \leq 2,$$

because $\operatorname{rank}(G[V_1]) = \operatorname{rank}(G[V_2]) = 1$. Therefore, we have $\Theta(G) = 2$.

## 4 Circumventing induced matchings via symmetric subrank of tensors

Existing tensor tools for upper bounding the independence number of hypergraphs share the problematic property that they also upper bound the induced matching number. As we have seen, for the corner problem the induced matching number is asymptotically maximal. Thus, existing tensor tools give only trivial upper bounds for the corner problem. Also for the cap set problem, existing tensor tools cannot improve the current results due to induced matchings of large size.

To circumvent the problematic induced matchings, we define a natural stronger version of subrank called the **symmetric subrank**, which still upper bounds the independence number of hypergraphs, but which may go below the induced matching number.

Besides introducing the symmetric subrank and investigating the basic properties, separations and asymptotic equalities, we introduce the symmetric quantum functional as the representation-theoretic upper bound on the symmetric subrank of powers of tensors, and we relate this to the existing quantum functionals.
4.1 Symmetric subrank

The subrank of a \( k \)-tensor \( f \) was defined as the size of the largest diagonal tensor that can be obtained from \( f \) by acting with \( k \) linear maps \( A^{(1)}, \ldots, A^{(k)} \) on the \( k \) dimensions of \( f \). The symmetric subrank of a tensor is defined in the same way with the extra requirement that all linear maps \( A^{(i)} \) must be the same.

**Definition 4.1** (Symmetric restriction and symmetric subrank). For any two (not necessarily symmetric) tensors \( f \in (\mathbb{F}^d)^{\otimes k} \) and \( g \in (\mathbb{F}^e)^{\otimes k} \), we say that \( f \) symmetrically restricts to \( g \), and we write \( g \leq_s f \), if there exists a linear map \( A : \mathbb{F}^d \to \mathbb{F}^e \) such that \( g = A^{\otimes k} f \). Thus \( g \leq_s f \) if and only if there is an \( e \times d \) matrix \( A \) such that for all \( i_1, \ldots, i_k \in [e] \) we have that

\[
g_{i_1,\ldots,i_k} = \sum_{j_1,\ldots,j_k \in [d]} A_{i_1,j_1} \cdots A_{i_k,j_k} f_{j_1,\ldots,j_k},
\]

We define the **symmetric subrank** of \( f \) as the largest number \( r \) such that the diagonal tensor \( \langle r \rangle \) is a symmetric restriction of \( f \), that is,

\[
Q_s(f) = \max\{r \in \mathbb{N} : \langle r \rangle \leq_s f \}.
\]

Our main motivation for introducing the symmetric subrank is to upper bound the independence number:

**Proposition 4.2.** Let \( H = (V, E) \) be a directed \( k \)-uniform hypergraph with \( n \) vertices with adjacency tensor \( A_H \). Then, \( \alpha(H) \leq Q_s(A_H) \), where the symmetric subrank can be understood over any field \( \mathbb{F} \). In fact, for any field \( \mathbb{F} \) and tensor \( f \in (\mathbb{F}^n)^{\otimes k} \) with support included in the support of \( A_H \) and with diagonal entries \( f_{i_1,\ldots,i_k} = 1 \) for all \( i_1, \ldots, i_k \in [n] \), we have \( \alpha(H) \leq Q_s(f) \).

**Proof.** Let \( S = \{i_1, \ldots, i_r\} \) be an independent set of \( H \) with size \( r \leq n \). Then take a matrix \( B \) that has size \( r \times n \) such that \( B_{j,i} = 1 \) for all \( j \in [r] \), and other entries equal to 0. Then the tensor \( t = (B^{\otimes k}) \cdot f \) can be written as \( t_{j_1,\ldots,j_k} = f_{j_1,\ldots,j_k} \). As \( S \) is an independent set, we have \( (A_H)_{i_1\ldots,i_r} = 1 \) if and only if \( j_1 = j_2 = \cdots = j_k \). This means that \( t = \langle r \rangle \). Moreover, if any hypergraph is obtained from \( H \) by deleting some edges, then its independent number is at least \( \alpha(H) \), which proves the desired result. \( \square \)

For symmetric tensors, the symmetric rank \( R_s(f) = \min\{r \in \mathbb{N} : f \leq_s \langle r \rangle \} \), the “opposite” of the symmetric subrank, is well-studied [CGLM08]. For a symmetric tensor \( f \) the symmetric rank is the smallest number \( r \) such that there are \( r \) vectors \( v_i \) so that \( f = \sum_{i=1}^r v_i^{\otimes k} \). For any symmetric tensor \( f \in (\mathbb{C}^d)^{\otimes k} \), we have \( R_s(f) \leq \binom{d+k-1}{k} \) [CGLM08].

**Remark 4.3.** There is a natural identification between symmetric tensors on the one hand and homogeneous polynomials on the other hand. A homogeneous polynomial is a polynomial whose monomials all have the same total degree \( k \). Any symmetric \( k \)-tensor \( f \in (\mathbb{F}^d)^{\otimes k} \) corresponds uniquely to a homogeneous polynomial of degree \( k \) in \( d \) variables \( F \in \mathbb{F}[x_1, \ldots, x_d]_k \) via the expression:

\[
F(x_1, \ldots, x_d) = \sum_{j_1, \ldots, j_k \in [d]} f_{j_1,\ldots,j_k} \cdot x_{j_1} \cdot x_{j_2} \cdots x_{j_k}.
\]
We define the \textit{symmetric subrank} of $F$, written $Q_s(F)$, as the largest number $r \in \mathbb{N}$ such that there are $d$ linear forms $\ell_1(y_1, \ldots, y_r), \ldots, \ell_d(y_1, \ldots, y_r)$ in $r$ variables $y_1, \ldots, y_r$, such that

$$F(\ell_1(y_1, \ldots, y_r), \ldots, \ell_d(y_1, \ldots, y_r)) = \sum_{i=1}^{r} y_i^k.$$

To phrase it differently, the symmetric subrank $Q_s(F)$ is the largest $r \in \mathbb{N}$ such that there is a matrix $A = (a_{ij})_{i,j} \in \mathbb{F}^{d \times r}$ such that $A \cdot Y = \sum_{i=1}^{r} y_i^k$, where $Y = (y_1, \ldots, y_r)$ and $A \cdot Y = (a_{11}y_1 + \cdots + a_{1r}y_r, \ldots, a_{d1}y_1 + \cdots + a_{dr}y_r)$. The symmetric subrank for homogeneous polynomials and for symmetric tensors coincide via the above identification, in the sense that $Q_s(F) = Q_s(f)$.

In a similar way, the symmetric rank of a symmetric tensor has a natural interpretation in terms of the associated homogeneous polynomial [IK99]. This notion is also called the Waring rank.

Also the notion of symmetric restriction of symmetric tensors carries over to homogeneous polynomials, as follows. Let $F \in \mathbb{F}[x_1, \ldots, x_d]_k$ and $G \in \mathbb{F}[y_1, \ldots, y_d]_k$ be homogeneous polynomials of degree $k$ in $d$ and $d'$ variables, respectively. We say that $F$ is a symmetric restriction of $G$, and write $F \leq_s G$, if there is a matrix $A \in \mathbb{F}^{d' \times d}$ such that $F = G(A \cdot X)$, where, as before, $X = (x_1, \ldots, x_d)$ and $A \cdot X$ is defined as before. The symmetric restriction of symmetric tensors and for homogeneous polynomials coincide, in the sense that $F \leq_s G$ if and only if $f \leq_s g$, where $f$ and $g$ are the symmetric tensors associated to $F$ and $G$, respectively.

### 4.2 Subrank versus symmetric subrank

We have proposed the symmetric subrank as a new tool to upper bound the independence number of hypergraphs. Does it lead to better upper bounds? Here we will discuss relations and separations with the ordinary subrank. We obtain precise results under assumptions about the order, ground field and symmetry of the tensors.

For any $k$-tensor $f$ we have by definition that $Q_s(f) \leq Q(f)$. First of all, we observe that this inequality can be strict even when $k = 2$:

**Example 4.4.** Let $f$ be the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ over an arbitrary field. Then $\text{rank}(f) = Q(f) = 2$, while $Q_s(f) = 0$. More generally, for any full-rank non-symmetric matrix $f \in \mathbb{F}^d \otimes \mathbb{F}^d$, we have $Q_s(f) < Q(f) = d$. To see this, it suffices to observe that if $\langle d \rangle \leq_s f$, then, using matrix notation, $\langle d \rangle = AfA^T$, which implies that $f$ is symmetric.

The inequality can also be strict for the adjacency tensors of some hypergraphs.

**Proposition 4.5.** Let $k \in \mathbb{N}_{\geq 1}$ and let $C_{2k+1}$ be the directed cycle graph of length $2k + 1$. Then, $Q_s(A_{C_{2k+1}}) < Q(A_{C_{2k+1}})$ on the complex field, where $A_{C_{2k+1}}$ is the adjacency tensor (adjacency matrix) of $C_{2k+1}$.

**Proof.** First, we prove $\text{rank}(A_{C_{2k+1}}) = 2k + 1$. Denote by $v_1, v_2, \ldots, v_{2k+1}$ the rows of $A_{C_{2k+1}}$.

$$A_{C_{2k+1}} = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$
Suppose that there are \( \alpha_1, \alpha_2, \ldots, \alpha_{2k+1} \) such that \( \alpha_1 v_1 + \cdots + \alpha_{2k+1} v_{2k+1} = 0 \), this equivalent to \( \alpha_1 + \alpha_2 = 0, \alpha_2 + \alpha_3 = 0, \ldots, \alpha_{i} + \alpha_{i+1} = 0, \ldots, \alpha_{2k+1} + \alpha_1 = 0 \). This implies \( \alpha_1 = \cdots = \alpha_{2k+1} = 0 \). Thus, \( \text{rank}(A_{2k+1}) = Q(A_{2k+1}) = 2k + 1 \).

Assume that \( Q(A_{2k+1}) = 2k + 1 \), thus there is a square matrix \( B \) of size \((2k+1) \times (2k+1)\) such that \( I_{2k+1} = B A_{2k+1} B^T \), since \( A_{2k+1} \) is an invertible matrix, thus \( B \) also is an invertible matrix. Therefore, \( A_{2k+1} = B^{-1} (B^T)^{-1} = B^{-1} (B^{-1})^T \), this implies \( A_{2k+1} \) is symmetric matrix, a contradiction with the construction of \( A_{2k+1} \), completing the proof.

In fact, the symmetric subrank can be strictly smaller than the largest induced matching for some hypergraphs.

**Example 4.6.** On the finite field \( \mathbb{F}_2 \), consider a directed graph \( C_5 \) (a directed cycle with length 5) then we have \( Q_{f,M}(\Phi_{C_5}) = 3 \) while \( Q_s(A_{C_5}) = 2 \), where \( A_{C_5} \) is the adjacency tensor of \( C_5 \) and \( \Phi_{C_5} \) is the support of \( A_{C_5} \).

For symmetric complex matrices, however, it follows from the symmetric singular value decomposition that there can be no gap between the symmetric subrank and the subrank:

**Lemma 4.7.** For any symmetric complex matrix \( f \), \( Q_s(f) = Q(f) = \text{rank}(f) \).

**Proof.** For the case of matrices, the (not necessarily symmetric) restriction property \( f' = (A \otimes B) \cdot f \) can be written in terms of matrix multiplication as \( f' = Af B^T \), therefore \( Q(f) = \text{rank}(f) \). In terms of symmetric restriction, \( f' \leq_s f \) if and only if there exists a matrix \( A \) such that \( f' = Af A^T \). Since \( f \) is a symmetric complex matrix, therefore it has a symmetric singular value decomposition \( f = V \Sigma V^T \) (see [BGG88] for more details), where \( V \) is a unitary matrix and \( \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n) \). Since \( V \) is a unitary matrix, we have \( \Sigma \leq_s f \). Suppose \( \text{rank}(f) = t \), then there exists \( t \) such that \( \sigma_1, \ldots, \sigma_t \) are the nonzero, we choose \( B \) to be the \( t \times n \) matrix such that for all \( j \in [t] \) then \( B_{jj} = 1/\sqrt{\sigma_j} \) and other entries are equal zero. We get \( (t) = B \Sigma B^T \), thus we have \( (t) \leq_s \Sigma \), it implies \( Q_s(f) \geq Q_s(\Sigma) \geq t \). We conclude that \( Q_s(f) = Q(f) = \text{rank}(f) \). \( \square \)

When we go to symmetric \( k \)-tensors \( f \) with \( k \geq 3 \), there are again examples of strict inequality \( Q_s(f) < Q(f) \), over a finite field:

**Example 4.8.** Let \( f = e_1 \otimes e_2 \otimes e_3 + e_1 \otimes e_3 \otimes e_2 + e_2 \otimes e_1 \otimes e_3 + e_2 \otimes e_3 \otimes e_1 + e_3 \otimes e_1 \otimes e_2 + e_3 \otimes e_2 \otimes e_1 + e_1 \otimes e_1 \otimes e_1 \), where \( e_1, e_2, e_3 \in \mathbb{F}_2^3 \) is the standard basis of \( \mathbb{F}_2^3 \). It is not hard to verify that \( Q_s(f) = 1 \) while \( Q(f) = 2 \).

We leave the construction of a symmetric tensor \( f \) satisfying \( Q_s(f) < Q(f) \) over the complex field \( \mathbb{C} \) as an open problem. This problem is the subrank analog of Comon’s question about tensor rank, which was recently answered negatively by Shitov [Shi18].

Next, we prove that in some settings \( Q_s(f) = Q(f) \). Namely, for complex symmetric tensors, if the subrank is maximal, then also the symmetric subrank is maximal:

**Theorem 4.9.** Let \( f \in (\mathbb{C}^d)^\otimes k \) be a symmetric tensor. If \( Q(f) = d \) then \( Q_s(f) = d \).

To prove Theorem 4.9 we use the following result.

**Theorem 4.10 ([BS06]).** Let \( f, f' \in (\mathbb{C}^d)^\otimes k \) be tensors of order \( k \). If \( A^1, \ldots, A^k \) are invertible matrices of size \( d \) such that \( f' = (A^{\pi(1)} \otimes \cdots \otimes A^{\pi(k)}) f \) for all permutations \( \pi \in \mathfrak{S}_k \). Then there is an invertible matrix \( B \) of size \( d \) such that \( f' = (B \otimes \cdots \otimes B) f \).
Corollary 4.11 (Corollary of Theorem 4.10). Let \( f', f \in (\mathbb{C}^d)^\otimes k \) be symmetric tensors. If there are \( k \) invertible matrices \( A^1, \ldots, A^k \) of size \( d \) such that \( f' = (A^1 \otimes \cdots \otimes A^k)f \). Then there is an invertible matrix \( B \) of size \( d \) such that \( f' = (B \otimes \cdots \otimes B)f \).

Proof of Corollary 4.11. For any permutation \( \pi \in \mathfrak{S}_k \). We have

\[
\sum_{j_1, \ldots, j_k \in [d]} A^\pi_{i_1, \ldots, i_k} \cdot A^\pi_{i_{k+1}, \ldots, i_{2k}} f_{j_1, \ldots, j_k} = \sum_{j_1, \ldots, j_k \in [d]} A_{i_{k+1}, \ldots, i_{2k}}^\pi f_{j_1, \ldots, j_k} = f'_{i_1, \ldots, i_k}.
\]

Therefore \( f' = (A^\pi \otimes \cdots \otimes A^\pi)f \) for all \( \pi \in \mathfrak{S}_k \). By using Theorem 4.10, the proof is completed.

Proof of Theorem 4.9. Since \( \tilde{Q}(f) = d \), thus there are \( k \) matrices \( A^1, \ldots, A^k \) of size \( d \times d \) such that \( \langle d \rangle = (A^1 \otimes \cdots \otimes A^k)f \). Suppose that there is a matrix \( A^i \) which is not invertible, then the rank of \( i \)-th flattening matrix of \( (A^1 \otimes \cdots \otimes A^k)f \) is smaller than \( d - 1 \), that is, \( \text{rank} \langle \text{flatten}_i((A^1 \otimes \cdots \otimes A^k)f) \rangle \leq d - 1 \). But the rank of all flattenings of \( \langle d \rangle \) are equal to \( d \). Therefore all \( A^1, \ldots, A^k \) are invertible matrices. By the above corollary, there is an invertible matrix \( B \) such that \( \langle d \rangle = (B \otimes \cdots \otimes B)f \), this implies \( \tilde{Q}_s(f) = d \).

4.3 Asymptotic symmetric subrank

Applications ask for understanding the asymptotic growth rate of the symmetric subrank as we take tensor product powers of a fixed tensor. We capture this as follows. We define the asymptotic symmetric subrank of a tensor \( f \in (\mathbb{F}^d)^\otimes k \) as

\[
\tilde{Q}_s(f) := \limsup_{n \to \infty} Q_s(f^\otimes n)^{1/n}.
\]

For any tensor \( f \in (\mathbb{F}^d)^\otimes k \), since \( Q_s(f) \leq Q(f) \leq d \), we also have that \( \tilde{Q}_s(f) \leq \tilde{Q}(f) \leq d \).

Note that, because of the earlier Example 4.4, this lim sup cannot generally be replaced by a limit. However, we will be interested in the adjacency tensors of hypergraphs which have the special property that the coefficients on the main diagonal are all one. In that case we can replace the lim sup by a limit as follows:

Proposition 4.12. Let \( f \in (\mathbb{F}^d)^\otimes k \) be a tensor such that there is an \( i \in [d] \) with \( f_{i, \ldots, i} = 1 \). Then \( \tilde{Q}_s(f) = \sup_n Q_s(f^\otimes n)^{1/n} \).

Proof. Since there is \( i \in [d] \) such that \( f_{i, \ldots, i} = 1 \), then take a vector \( B \in \mathbb{R}^d \) such that \( B_i = 1 \), and other entries equal to 0, we have \( \langle 1 \rangle = (B \otimes \cdots \otimes B)f \), therefore \( Q_s(f) \geq 1 \). Moreover, the symmetric subrank has supermultiplicative property under tensor product. Thus, by Fekete’s lemma, we can write \( \tilde{Q}_s(f) = \sup_n Q_s(f^\otimes n)^{1/n} \).

\footnote{For the usual subrank, the asymptotic subrank of the tensor \( f \in (\mathbb{F}^d)^\otimes k \) was defined by Strassen as the limit \( \tilde{Q}(f) = \lim_{n \to \infty} Q(f^\otimes n)^{1/n} \), which, since \( Q \) is supermultiplicative and \( Q(f) \geq 1 \) if \( f \neq 0 \), equals the supremum \( \sup_n Q(f^\otimes n)^{1/n} \) (Fekete’s lemma). For the symmetric subrank, we have to be more careful about how we define the asymptotic symmetric subrank. For example, in Example 4.4 we gave a matrix \( f \) for which \( f^\otimes n \) is symmetric if \( n \) is even and skew-symmetric if \( n \) is odd, and so \( Q_s(f^\otimes n) = 2^n \) if \( n \) is even, and \( Q_s(f^\otimes n) = 0 \) when \( n \) is odd. Thus, the limit \( \lim_{n \to \infty} Q_s(f^\otimes n)^{1/n} \) might not exist.}
The important property of \( \tilde{Q}_s \) is that it directly gives an upper bound on the Shannon capacity of hypergraphs.

**Proposition 4.13.** Let \( H = (V,E) \) be directed \( k \)-uniform hypergraph with \( A_H \) is its adjacency tensor. Then \( \Theta(H) \leq \tilde{Q}_s(A_H) \).

**Proof.** By the definition of \( A_H \), we have that \( A \otimes_n H \) is the adjacency tensor of \( H \oplus_n H \), and therefore we have that \( \Theta(H) = \sup_n (\alpha(H \oplus_n H))^{1/n} \leq \sup_n (Q_s(A \otimes_n H))^{1/n} = \tilde{Q}_s(A_H) \) using Proposition 4.12. \( \square \)

We conjecture that the asymptotic symmetric subrank of a \( k \)-tensor with \( k \geq 3 \) can be strictly smaller than the asymptotic subrank. This cannot happen for \( k = 2 \). In that case we prove that there is no strict inequality:

**Theorem 4.14.** For any matrix \( f \) over \( F \neq F_2 \) we have \( \tilde{Q}(f) = \tilde{Q}_s(f) \).

To prove Theorem 4.14, we use the following result about matrices:

**Theorem 4.15 ([Bal68]).** Let \( F \) be a field with size at least 3 and \( f \) be a square matrix of size \( d \) over \( F \) that is not a nonzero skew-symmetric matrix. There is an invertible matrix \( B \) of size \( d \) such that \( BfB^T \) is lower triangular matrix and has exactly \( \text{rank}(f) \) elements on its diagonal are nonzero.

**Proof of Theorem 4.14.** We may assume that \( f \) is a \( d \times d \) matrix. Let \( \text{rank}(f) = r \). Then \( Q_s(f) = \text{rank}(f) = r \). If \( f \) is a skew-symmetric matrix. Then we have \( Q_s(f) = 0 \). Moreover, \( f \otimes_n \) is symmetric matrix if \( n \) even number and skew-symmetric matrix if \( n \) old number. Hence by the Lemma 4.7 we have

\[
Q_s(f \otimes_n) = \begin{cases} r^n & \text{if } n \text{ is even number,} \\ 0 & \text{otherwise.} \end{cases}
\]

Therefore \( \tilde{Q}_s(f) = r \). If \( f \) is not skew-symmetric matrix. By the above theorem, there is an invertible matrix \( B \) of size \( d \) and a lower triangular matrix \( L \) such that \( BF \) is lower triangular matrix and has exactly \( r \) elements nonzero element on its diagonal. Then \( A \otimes_n \) also a submatrix of \( L \otimes_n \). We choose \( n = rk \) for some \( k \in \mathbb{N}_1 \). Then the submatrix of \( A \otimes_n \) with rows and columns indexed by the elements in \( [r] \) of type \((n/r,\ldots,n/r)\) is diagonal and has size

\[
\binom{n}{n/r,\ldots,n/r} = r^{n-o(n)}.
\]

Thus, \( \tilde{Q}_s(L) \geq r \). \( \square \)

### 4.4 Asymptotic subrank versus asymptotic symmetric subrank on symmetric tensors

For symmetric tensors we prove that the asymptotic symmetric subrank is equal the asymptotic subrank (as long as the field satisfies a mild condition):
Theorem 4.16. Let \( f \) be a symmetric \( k \)-tensor over an algebraically closed field of characteristic at least \( k + 1 \). Then \( \tilde{Q}(f) = \tilde{Q}_s(f) \).

In particular, Theorem 4.16 holds for any tensor over the field of complex numbers.

In fact we prove a much more general asymptotic statement about the restriction preorder \( \preceq \) and the symmetric restriction preorder on symmetric tensors. We define the asymptotic restriction preorder \( \preceq \) on tensors \( f, g \) by writing \( f \preceq g \) if and only if \( f^\otimes n \preceq g^\otimes n + o(n) \). Similarly we define the asymptotic symmetric restriction preorder \( \preceq_s \) on tensors \( f, g \) by writing \( f \preceq_s g \) if and only if \( f^\otimes n \preceq_s g^\otimes n + o(n) \).

Theorem 4.17. For symmetric \( k \)-tensors \( f, g \) over an algebraically closed field of characteristic at least \( k + 1 \) we have \( f \preceq g \) if and only if \( f \preceq_s g \).

It will also follow from our proof that on symmetric tensors (over an appropriate field) the asymptotic rank and symmetric asymptotic rank are equal:

Theorem 4.18. Let \( f \) be a symmetric \( k \)-tensor over an algebraically closed field of characteristic at least \( k + 1 \). Then \( R_s(f) \leq 2^k - 1 R(f) \) and in particular \( \tilde{R}(f) = \tilde{R}_s(f) \).

For \( k = 3 \) the same relation between symmetric rank and rank for symmetric tensors was found in [Kay12].

The above three theorems are related to Comon’s conjecture [CGLM08], which says that rank and symmetric rank coincide on symmetric tensors. Shitov [Shi18] gave a counterexample to Comon’s conjecture. Our Theorem 4.16, Theorem 4.17 and Theorem 4.18 can be interpreted as saying that “Comon’s conjecture” is true asymptotically, not only for rank (Theorem 4.18), but also for subrank (Theorem 4.16) and the restriction preorder (Theorem 4.17).

The proofs for all of the above will follow from three basic lemmas that we will discuss now. A crucial role will be played by the following \( k \)-tensor.

Definition 4.19 (fully symmetric \( k \)-tensor). For any \( k \in \mathbb{N} \) let \( \mathcal{S}_k \) be the symmetric group on \( k \) elements and define the \( k \)-tensor \( h = \sum_{\pi \in \mathcal{S}_k} e_{\pi(1)} \otimes \cdots \otimes e_{\pi(k)} \). We will call \( h \) the fully symmetric \( k \)-tensor.

For example, for \( k = 3 \), the tensor \( h \) is given by \( h = e_1 \otimes e_2 \otimes e_3 + e_1 \otimes e_3 \otimes e_2 + e_2 \otimes e_1 \otimes e_3 + e_2 \otimes e_3 \otimes e_1 + e_3 \otimes e_1 \otimes e_2 + e_3 \otimes e_2 \otimes e_1 \). The tensor \( h \) allows us to transform any restriction to a symmetric restriction:

Lemma 4.20. Let \( f \) and \( g \) be symmetric \( k \)-tensors over a field of characteristic at least \( k + 1 \). If \( f \geq g \), then \( f \otimes h \geq_s g \otimes h \), and hence also \( f \otimes h \geq_s g \), where \( h \) is the fully symmetric tensor.

Proof. Let \( A_1, \ldots, A_k \) be linear maps such that \( (A_1 \otimes \cdots \otimes A_k)f = g \). Let \( e_i^* \) denote the elements of the basis dual to the standard basis \( e_i \). Define the linear map \( B = \sum_i A_i \otimes e_i e_i^* \). Then

\[
(B^\otimes k)(f \otimes h) = k!((A_1 \otimes \cdots \otimes A_k)f) \otimes h.
\]

Dividing by \( k! \) proves the claim. \( \square \)
Lemma 4.21. Let $f$ be a symmetric k-tensor over an algebraically closed field. Suppose that some flattening rank of $f$ is at least 2. Then there is a $c \in \mathbb{N}$ such that $f^\otimes c \geq h$.

To prepare for the proof of Lemma 4.21 we prove the following lemma.

Lemma 4.22. Let $f$ be a symmetric k-tensor over an algebraically closed field. There exists a basis transformation $A \in \mathbb{F}^{d \times d}$ such that the support $S = \text{supp}(A^\otimes k f) \subseteq [d]^k$ of $f$ after applying the transformation $A$ satisfies $(i, \ldots, i) \notin S$ for every $1 \leq i \leq d - 1$.

Proof. Suppose that $f \in (\mathbb{F}^d)^{\otimes k}$. If no element of the form $(i, \ldots, i)$ appears in $S$, then we are done. Otherwise, we may assume that $(d, \ldots, d)$ appears, so that the tensor $f$ is of the form $f = f_1 e_1^\otimes k + f_2 e_2^\otimes k + \cdots + f_d e_d^\otimes k + \text{other terms}$, for some coefficients $f_i$ with $f_d \neq 0$.

We apply to $f$ the invertible linear map that maps $e_i$ to $e_i$ for $1 \leq i \leq d - 1$ and maps $e_d$ to $e_d + \varepsilon_1 e_1 + \cdots + \varepsilon_{d-1} e_{d-1}$ for some $\varepsilon_i \in \mathbb{F}$. This gives a tensor $g \in (\mathbb{F}^d)^{\otimes k}$ that is isomorphic to $f$ and of the form $g = (f_1 + \varepsilon_1 f_d) e_1^\otimes k + \cdots + (f_{d-1} + \varepsilon_{d-1} f_d) e_{d-1}^\otimes k + \text{other terms}$. Since $f_d$ is nonzero and the ground field is algebraically closed, there are values for the $\varepsilon_i$ such that $f_i + \varepsilon_i^k f_i$ is zero for every $1 \leq i \leq d - 1$, in which case $(i, \ldots, i)$ does not appear in the support of $g$ for every $1 \leq i \leq d - 1$.

Proof of Lemma 4.21. Let $f \in (\mathbb{F}^d)^{\otimes k}$. By Lemma 4.22 we may assume that $(i, \ldots, i)$ does not appear in the support $S = \text{supp}(f) \subseteq [d]^k$ of $f$ for $1 \leq i \leq d - 1$. For every element $s \in S$ we define its type $(y_1, \ldots, y_d)$ by letting $y_i$ be the number of times that $i$ appears in $s$. Let $Y$ be the set of types of elements of $S$. Since some flattening rank of $f$ is at least 2, we cannot have that $S = \{ (d, \ldots, d) \}$. Thus without loss of generality there is a type $y \in Y$ that satisfies $1 \leq y_1 \leq k - 1$ and such that for ever type $y' \in Y$ it holds that $y'_1 \leq y_1$ (maximality assumption). Let $R \subseteq [d]^k$ be the set of all $k$-tuples in $[d]^k$ of type $y$. Let $A$ be the $|R| \times k$ matrix with rows given by the elements of $R$, in some arbitrary order. Let $C$ be the set of columns of $A$. Note that in any $s \in S$ the element 1 can appear at most $y_1$ times by our maximality assumption.

We claim that $f^\otimes |R|$ restricts symmetrically to the fully symmetric $k$-tensor $h$ by zeroing out all basis elements that are not in $C$. To prove this we need to show that for any choice of $k$ elements $v_1, \ldots, v_k$ in $C$, if for every $i$ we have that $((v_1)_i, \ldots, (v_k)_i) \in S$, then $v_1, \ldots, v_k$ are all different.

By construction of $C$, for any $y_1$ distinct elements $v_1, \ldots, v_{y_1}$ of $C$ there is an $1 \leq i \leq |R|$ such that $(v_1)_i = \cdots = (v_{y_1})_i = 1$. Thus also for any $y_1$ (not necessarily distinct) elements $v_1, \ldots, v_{y_1}$ of $C$ there is an $1 \leq i \leq |R|$ such that $(v_1)_i = \cdots = (v_{y_1})_i = 1$. Let $v_1, \ldots, v_k$ be an arbitrary collection of elements of $C$. Suppose that $v_1 = v_2$. By the previous argument we know that there is an $1 \leq i \leq |R|$ such that $(v_1)_i = \cdots = (v_{y_1+1})_i = 1$. From the assumption $v_1 = v_2$ it follows that $(v_1)_i = (v_2)_i = \cdots = (v_{y_1+1})_i = 1$. However, we
picked the type \((y_1, \ldots, y_d)\) such that \(y_1\) is maximal and \(y_1 \leq k - 1\). The element 1 appears at least \(y_1 + 1\) times in \(((v_1)_i, \ldots, (v_k)_i)\). Therefore \(((v_1)_i, \ldots, (v_k)_i)\) is not in \(S\).

\textbf{Proof of Theorem 4.17.} Suppose that \(f \succeq g\). This means that \(f^{\otimes m + o(m)} \geq g^{\otimes m}\). We know from Lemma 4.21 that there is a constant \(c \in \mathbb{N}\), depending only on \(f\), such that \(f^{\otimes c} \succeq h\). By Lemma 4.20 we then have
\[
f^{\otimes m + o(m)} \otimes f^{\otimes c} \succeq_s f^{\otimes m + o(m)} \otimes h \succeq_s g^{\otimes m}.
\]
This means \(f \succeq_s g\), which proves the claim.

Although essentially Theorem 4.16 and Theorem 4.18 can be proven abstractly from Theorem 4.17, we will give the (simple) proofs separately in terms of the above lemmas for the convenience of the reader and to get the precise statement of Theorem 4.18:

\textbf{Proof of Theorem 4.16.} Suppose that \(Q(f^{\otimes n}) \geq r\). Then \(f^{\otimes n} \geq (r)\). By Lemma 4.21 there is a constant \(c \in \mathbb{N}\), depending only on \(f\), such that \(f^{\otimes c} \succeq h\). By Lemma 4.20 we then have
\[
f^{\otimes n + c} \succeq_s f^{\otimes n} \otimes f^{\otimes c} \succeq_s f^{\otimes n} \otimes h \succeq_s (r).
\]
Thus \(Q_s(f^{\otimes n + c}) \geq r\), which implies the claim.

\textbf{Proof of Theorem 4.18.} Suppose that \(R(f) \leq r\). Then \(f \leq (r)\). Let \(s = R_s(h)\) be the symmetric rank of the fully symmetric tensor \(h\) and note that \(s\) is a constant depending only on \(k\), the order of \(f\). In fact, \(s \leq 2^k - 1\), which follows from the known identity
\[
h = \frac{1}{2^{k-1}} \sum_{\varepsilon_i = \pm 1} \left( \prod_{i=2}^{k} \varepsilon_i \right) (e_1 + \varepsilon_2 e_2 + \varepsilon_3 e_3 + \cdots + \varepsilon_k e_k)^{\otimes k}
\]
in which the sum goes over \(\varepsilon_2, \ldots, \varepsilon_k = \pm 1\). We refer to [GW09, Lemma B.2.3] for a proof of this identity. See also [LT10, Proposition 11.6]. Then
\[
(r_s) = (r) \otimes (s) \succeq_s (r) \otimes h \succeq_s f.
\]
Thus \(R_s(f) \leq rs\), which implies the first claim. Then, since \(s\) is constant, it follows that \(R(f^{\otimes n}) \leq R_s(f^{\otimes n}) \leq R(f^{\otimes n})s\) for every \(n \in \mathbb{N}\), which implies the second claim.

\section{4.5 Asymptotic spectrum of symmetric tensors}

The duality theory of Strassen introduced and studied in [Str86, Str88, Str88, Str91, Tob91, Bür90] (see also [CVZ18] and [Zui18]) gives a dual formulation for the asymptotic subrank, asymptotic rank and asymptotic restriction preorder in terms of the asymptotic spectrum of tensors. As an application of the results of Section 4.4 we prove a strong connection between this theory and the natural symmetric variation.

The asymptotic spectrum of tensors (for any fixed \(k \in \mathbb{N}\) and field \(\mathbb{F}\)) is defined as the set \(X\) of all real-valued maps from \(k\)-tensors over \(\mathbb{F}\) to the nonnegative reals that are additive under the direct sum, multiplicative under the tensor product, monotone under the restriction preorder and normalized to 1 on the diagonal tensor (1) of size one. The duality theory says that: the asymptotic rank equals the pointwise maximum over all elements in the asymptotic spectrum of tensors.
spectrum of tensors, the asymptotic subrank equals the pointwise minimum over all elements in the asymptotic spectrum of tensors, and the asymptotic restriction preorder is characterized by \( f \preceq g \) if and only if for every \( \phi \) in the asymptotic spectrum \( X \) it holds that \( \phi(f) \leq \phi(g) \).

We introduce the asymptotic spectrum of symmetric tensors as the natural symmetric variation on Strassen’s asymptotic spectrum of tensors. We define the asymptotic spectrum of symmetric tensors (for any fixed \( k \in \mathbb{N} \) and field \( \mathbb{F} \)) as the set \( X_s \) of all real-valued maps from symmetric \( k \)-tensors over \( \mathbb{F} \) to the nonnegative reals that are additive under the direct sum, multiplicative under the tensor product, monotone under the symmetric restriction preorder, and normalized to 1 on the diagonal tensor \( \langle 1 \rangle \). It follows readily from the general part of the theory in [Str88] (see also [Zui18]) that the asymptotic spectrum of symmetric tensors \( X_s \) gives a dual formulation for the asymptotic symmetric subrank, asymptotic symmetric rank and asymptotic symmetric restriction preorder:

**Theorem 4.23.** Let \( \mathbb{F} \) be an algebraically closed field of characteristic at least \( k + 1 \). Let \( X_s \) be the asymptotic spectrum of symmetric \( k \)-tensors. Let \( f \) and \( g \) be symmetric \( k \)-tensors. Then

\[
\hat{Q}_s(f) = \min_{\phi \in X_s} \phi(f), \\
\hat{R}_s(f) = \max_{\phi \in X_s} \phi(f), \\
f \preceq_s g \iff \forall \phi \in X_s, \phi(f) \leq \phi(g).
\]

We will not give the proof of Theorem 4.23 as it follows along the same lines as the original proof in [Str88] (see also [Zui18]). The bulk of the proof is to show that the symmetric restriction preorder is a so-called “good preorder” ([Str88]) or Strassen preorder ([Zui18]). The only non-standard ingredient for the proof is the fact that for every nonzero symmetric \( k \)-tensor \( f \) either \( f \) is equivalent to \( \langle 1 \rangle \) or \( \hat{Q}_s(f) > 1 \), which follows from Theorem 4.16 and the fact that this property holds for \( \hat{Q} \).

The results of Section 4.4 answer a structural question: how are the asymptotic spectrum of tensors \( X \) and the asymptotic spectrum of symmetric tensors \( X_s \) related? One relation is clear: for every element \( \phi \in X \) the restriction of \( \phi \) to symmetric tensors is an element of \( X_s \). We thus have the restriction map \( r : X \to X_s \) that maps \( \phi \in X \) to the restriction of \( \phi \) to symmetric tensors. We prove:

**Theorem 4.24.** The restriction map \( r : X \to X_s \) is surjective.

Theorem 4.24 has two readings: (1) if we understand what the elements are of the asymptotic spectrum of tensors \( X \), then we also understand what the elements are of the asymptotic spectrum of symmetric tensors \( X_s \) by restriction, and (2) for any element \( \psi \in X_s \) there is an extension \( \phi \in X \) such that \( \phi \) restricts to \( \psi \).

Theorem 4.24 follows from our Theorem 4.17 together with an application of the following powerful theorem from the theory of asymptotic spectra. The theorem uses the notion of a good preorder or Strassen preorder for which we refer the reader to the literature.

**Theorem 4.25** ([Str88], [Zui18, Corollary 2.18]). Let \( S \) be a semiring with a Strassen preorder \( P \). Let \( T \) be a subsemiring of \( S \). Then the restriction map from the asymptotic spectrum of \( S \) to the asymptotic spectrum of \( T \) is surjective.
Proof of Theorem 4.24. We give a sketch of the proof. The proof is an application of Theo-rem 4.25. Let $S$ be the semiring of $k$-tensors and let $P$ be the asymptotic restriction preorder. This is a Strassen preorder. Let $T$ be the subsemiring of $S$ of symmetric $k$-tensors. Then Theorem 4.25 implies that the restriction map from the asymptotic spectrum of $S$ with the asymptotic restriction preorder to the asymptotic spectrum of $T$ with the asymptotic restriction preorder is surjective. Since the asymptotic restriction preorder on symmetric tensors coincides with the asymptotic symmetric restriction preorder by Theorem 4.24, the claim follows.

To summarize what we have just seen, the asymptotic spectrum of tensors $X$ and the asymptotic spectrum of symmetric tensors $X_s$ are tightly related since the restriction map from the first to the second is surjective. What are the elements of $X$ and $X_s$? A long line of work [Str86, Str88, Str91, Str05, Tob91, Bür90, CVZ18, CLZ20] has been devoted to this question. Our best understanding is for the case that the ground field $F$ is the complex numbers\(^7\) and that is what we will focus our discussion on here and in the next section.

The known elements in $X$ (over the complex numbers) are a family of functions called the quantum functionals. These were introduced in [CVZ18] and are based on an information-theoretic and representation-theoretic study of powers of tensors. The quantum functionals more precisely form a continuous family $F^\theta$ indexed by probability distributions $\theta$ on $[k]$. This family includes the flattening ranks, but also includes more interesting functions that are properly real-valued which reveal asymptotic information that the flattening ranks do not reveal. It is possible but not known whether the quantum functionals are all elements of $X$. Proving this is a central open problem of the theory. In particular, the quantum functionals being all elements of $X$ would imply that the matrix multiplication exponent $\omega$ equals 2, which would be a breakthrough result in complexity theory.

We may restrict the quantum functionals to symmetric tensors to find an infinite family of elements in $X_s$. Since we do not know whether the quantum functionals are all elements of $X$, we can, however, not conclude from Theorem 4.24 that their restriction gives all elements of $X_s$.

What we will do in the next section is give a natural construction of a single element in $X_s$ following the same ideas as for the construction of the quantum functionals but applied directly to the symmetric restriction preorder. This single element we call the symmetric quantum functional. What we then find is that this symmetric quantum functional on symmetric tensors in fact equals the uniform quantum functional $F^{(1/k,\ldots,1/k)}$. Thus we do not find a new element in $X_s$, but we do find a different description of the uniform quantum functional restricted to symmetric tensors, and this might be algorithmically beneficial. This symmetric quantum functional is the pointwise smallest element among all elements in $X_s$ that we currently know, and from previous work it follows that it equals the asymptotic slice rank (on symmetric tensors). Having discussed the plan we will now go into the details in the next section.

4.6 Symmetric quantum functional

We use the ideas of the construction of the quantum functionals $F^\theta \in X$ from [CVZ18] to construct the symmetric quantum functional $F \in X_s$ over the field of complex numbers. Let us from now on fix the base field to be the field of complex numbers. In fact we will take a

\(^7\)It is known that the asymptotic spectrum can only depend on the characteristic of the field [Str88].
more general approach and define the symmetric quantum functional not just for symmetric
tensors but for arbitrary tensors.

Before recalling the definition of the quantum functionals $F^\theta$ and giving the new definition
of the symmetric quantum functional $F$, here is what we will find. For symmetric tensors we
will show that:

**Theorem 4.26.** On symmetric tensors $F = F^{(1/k,...,1/k)}$.

This gives an alternative description of the uniform quantum functional $F^{(1/k,...,1/k)}$, which
may have algorithmic benefits.

In particular on symmetric tensors the symmetric quantum functional is in the asymptotic
spectrum of symmetric tensors $X_s$:

**Theorem 4.27.** On symmetric tensors $F \in X_s$.

For general tensors we find that:

**Theorem 4.28.** On arbitrary tensors $F \geq F^{(1/k,...,1/k)}$.

In particular, since $F^{(1/k,...,1/k)} \geq \tilde{Q}$ (because every quantum functional $F^\theta$ is in the
asymptotic spectrum of tensors $X$), we also find $F \geq \tilde{Q}$ on arbitrary tensors. However, via
a known connection from [CVZ18] between the quantum functionals and the asymptotic slice
rank (the pointwise minimum $\min \theta F^\theta$ equals the asymptotic slice rank), we find that $F$, as a
tool to upper bound the Shannon capacity of hypergraphs, suffers from the induced matching
barrier.

Before defining the quantum functionals and symmetric quantum functional and giving the
proofs of the above, we must introduce some standard notation. Let $\mathcal{H}$ be a complex finite-
dimensional Hilbert space with dimension $\dim(\mathcal{H}) = d$. Thus $\mathcal{H} \cong \mathbb{C}^d$. A state or density
operator on $\mathcal{H}$ is a positive semidefinite linear map $\rho : \mathcal{H} \to \mathcal{H}$ with $\text{tr}(\rho) = 1$. Let $S(\mathcal{H})$ be the
set of states on $\mathcal{H}$. For $\rho \in S(\mathcal{H})$, let $\text{spec}(\rho) = (\lambda_1, \ldots, \lambda_d)$ be the sequence of eigenvalues of $\rho$,
ordered non-increasingly, that is, $\lambda_1 \geq \cdots \geq \lambda_d$. Since $\text{tr}(\rho) = 1$, the sequence of eigenvalue
of $\rho$ is a probability distribution. It thus makes sense to define $H(\text{spec}(\rho)) := -\sum_{j=1}^d \lambda_j \log \lambda_j$.
The von Neumann entropy of $\rho$ is defined as $H(p) = -\text{tr}(\rho \log \rho) = H(\text{spec}(\rho))$.

Given a state $\rho$ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$, the $j$th marginal is the element $\rho_j = \text{tr}_{\mathcal{H}_1 \cdots \mathcal{H}_{j-1} \mathcal{H}_{j+1} \cdots \mathcal{H}_k}(\rho)$
obtained from $\rho$ by a partial trace. The $j$th marginal is itself a state, that is, $\rho_j \in S(\mathcal{H}_j)$.
Consider a nonzero element $f \in \mathcal{H}^{\otimes k}$. Then $\rho(f) = \frac{f f^\dagger}{\|f\|^2} \in S(\mathcal{H}^{\otimes k})$, where $f^\dagger$ denotes the
conjugate transpose of $f$, and we can consider the $j$th marginal $\rho_j(f) \in S(\mathcal{H}_j)$. Let $\text{GL}(d)$
denote the set of invertible matrices acting on $\mathcal{H}$. For a tensor $f \in \mathcal{H}^{\otimes k}$, let $\text{GL}(d) \cdot f$ be the
Euclidean closure (or equivalently Zariski closure) of the orbit $\{ (g \otimes \cdots \otimes g) f : g \in \text{GL}(d) \}$.

We begin with the definition of the symmetric quantum functional.

**Definition 4.29 (Symmetric quantum functional).** Let $f \in \mathcal{H}^{\otimes k}$ be nonzero. We define the
symmetric quantum functional $F$ by

$$F(f) = 2^{E(f)},$$

$$E(f) = \max \{ H(p) : p \in \Delta(f) \},$$

where we define the subset $\Delta(f) \subseteq \mathbb{R}_d$, for $d = \dim(\mathcal{H})$, as

$$\Delta(f) = \left\{ \text{spec} \left( \frac{\rho_1(s) + \cdots + \rho_k(s)}{k} \right) : s \in \text{GL}(d) \cdot f \setminus \{0\} \right\}.$$
Lemma 4.30. \(\Delta(f)\) is a convex polytope.

Proof. See Appendix A.1.

The definition of the symmetric quantum functional \(F\) is inspired by the family of quantum functionals \(F^\theta\). Our main results about the symmetric quantum functional give precise relations between \(F\) and \(F^\theta\).

Definition 4.31 (Quantum functionals). Let \(\theta \in \mathcal{P}([k])\) and let \(f \in \mathcal{H}^\otimes k\). The quantum functionals are defined by

\[
F^\theta(f) = 2^E^\theta(f), \\
E^\theta(f) = \max \left\{ \sum_{i=1}^s \theta(i)H(p_i(s)) : s \in \text{GL}(d)^\times k \cdot f \setminus \{0\} \right\},
\]

where \(\text{GL}(d)^\times k \cdot f = \{(g_1 \otimes \cdots \otimes g_k) : g_1, \ldots, g_k \in \text{GL}(d)\}\).

There is an asymptotic connection between the quantum functionals and the slice rank, which we will be using.

Theorem 4.32 ([CVZ18]). For any \(f \in \mathcal{H}^\otimes k\) the limit \(\lim_{n \to \infty} \text{SR}(f^\otimes n)^{1/n}\) exists and equals the minimization \(\min_{\theta \in \mathcal{P}([k])} F^\theta(f)\).

Now we are ready to state the precise results on the symmetric quantum functional. These results in particular imply the three main results that we stated above in Theorem 4.26, Theorem 4.27 and Theorem 4.28.

First of all, we prove that the symmetric quantum functional is at least the uniform quantum functional, and we show that the latter can be obtained as the regularization of the former:

Theorem 4.33. Let \(f \in \mathcal{H}^\otimes k\) be any tensor. Let \(\theta = (\frac{1}{k}, \ldots, \frac{1}{k})\). Then

\[
\lim_{n \to \infty} \text{SR}(f^\otimes n)^{1/n} \leq F^\theta(f) \leq F(f) \text{ and } \lim_{n \to \infty} F(f^\otimes n)^{1/n} = \inf_n F(f^\otimes n)^{1/n} = F^\theta(f).
\]

Second, on symmetric tensors we prove the following even stronger connection between the symmetric quantum functional and the uniform quantum functional:

Theorem 4.34. Let \(f \in \mathcal{H}^\otimes k\) be a symmetric tensor. Then

\[
\lim_{n \to \infty} \text{SR}(f^\otimes n)^{1/n} = F^{(1/k, \ldots, 1/k)}(f) = F(f).
\]

Third, from the equality \(F = F^{(1/k, \ldots, 1/k)}\) on symmetric tensors (Theorem 4.34), and the known properties of \(F^{(1/k, \ldots, 1/k)}\), we directly obtain all of the following properties of the symmetric quantum functional \(F\):

Corollary 4.35. For any symmetric \(f \in (\mathbb{C}^d)^\otimes k\) and \(g \in (\mathbb{C}^e)^\otimes k\), and any \(r \in \mathbb{N}\), we have

1. \(F(\langle r \rangle) = r\)
2. \(F(f \oplus g) = F(f) + F(g)\)
3. \(F(f \otimes g) = F(f)F(g)\)

4. if \(f \leq g\) then \(F(f) \leq F(g)\).

Therefore, the symmetric quantum functional belongs to the asymptotic spectrum of symmetric tensors \(X_s\), which we discussed in Section 4.5.

We will now give the proofs of the above Theorem 4.33 and Theorem 4.34. We will need another characterization of \(\Delta(f)\) from representation theory. Let \(\lambda\) be a partition of \(nk\) into at most \(d\) parts. We denote this by \(\lambda \vdash_d nk\). Then \(\lambda := \lambda/nk = (\lambda_1/nk, \ldots, \lambda_d/nk)\) is a probability distribution on \([d]\). The symmetric group \(S_nk\) acts on \((\mathcal{H}^\otimes k)^\otimes n\) by permuting the tensor legs, that is, \(\pi \cdot (v_1 \otimes \cdots \otimes v_{nk}) = v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(nk)}\) for \(\pi \in S_nk\). The general linear group \(GL(d)\) acts on \((\mathcal{H}^\otimes k)^\otimes n\) via the diagonal embedding \(GL(d) \to GL(d)^\otimes nk : g \mapsto (g, \ldots, g)\), that is, \(g \cdot v = (g \otimes \cdots \otimes g)v\) for \(g \in GL(d), v \in (\mathcal{H}^\otimes k)^\otimes n\). The Schur–Weyl duality gives a decomposition of the space \((\mathcal{H}^\otimes k)^\otimes n\) into direct sum of irreducible \(S_nk \times GL(d)\) representations. More precisely,

\[
(\mathcal{H}^\otimes k)^\otimes n \cong \bigoplus_{\lambda \vdash nk} [\lambda] \otimes S_\lambda(\mathcal{H}),
\]

where \(S_\lambda(\mathcal{H})\) is an irreducible representation of \(GL(d)\) and \([\lambda]\) is an irreducible representation of \(S_nk\). Let \(P_\lambda : (\mathcal{H}^\otimes k)^\otimes n \to (\mathcal{H}^\otimes k)^\otimes n\) be the equivariant projector onto the isotypical component of type \(\lambda\), that is, onto the subspace of \((\mathcal{H}^\otimes k)^\otimes n\) which isomorphic to \(S_\lambda(\mathcal{H}) \otimes [\lambda]\).

Based on [Bri87], [Fra02], [Str05] and ([Wal14] in Section 2.1, [Zui18] in Chapter 6) we have another characterization of polytope \(\Delta(f)\) as follows.

**Lemma 4.36.** The polytope \(\Delta(f)\) is the Euclidean closure of the set

\[
\left\{ \frac{\lambda}{nk} : \exists n \in \mathbb{N}_{\geq 1}, \lambda \vdash_d nk, P_\lambda f^\otimes n \neq 0 \right\}.
\]

**Proof.** See Appendix A.1. \(\square\)

**Proof of Theorem 4.33.** We decompose \(\mathcal{H}^\otimes n\) into a direct sum of irreducible \(S_n \times GL(d)\) representations as

\[
\mathcal{H}^\otimes n \cong \bigoplus_{\lambda \vdash nk} [\lambda] \otimes S_\lambda(\mathcal{H}). \tag{8}
\]

Let \(P_\lambda\) be the equivariant projector onto the isotypical component of type \(\lambda\). The uniform quantum functional \(F^{(\frac{1}{k}, \ldots, \frac{1}{k})}(f)\) has another characterization as follows [CVZ18]:

\[
F^{(\frac{1}{k}, \ldots, \frac{1}{k})}(f) = \sup \left\{ \left( \prod_{i=1}^k \dim[\lambda^i] \right)^{1/kn} : \exists n \in \mathbb{N}_{\geq 1}, \lambda^i \vdash_d n, (P_{\lambda^1} \otimes \cdots \otimes P_{\lambda^k})f^\otimes n \neq 0 \right\}.
\]

For the symmetric quantum functional, using the characterization of \(\Delta(f)\) from representation theory, we have

\[
F(f) = \sup \left\{ \left( \dim[\lambda] \right)^{1/kn} : \exists n \in \mathbb{N}_{\geq 1}, \lambda \vdash kn, P_\lambda f^\otimes n \neq 0 \right\}.
\]
We may write \((\mathcal{H}^\otimes n)^\otimes k\) as a direct sum of irreducibles under the action of \(\mathfrak{S}_{nk}\) as

\[
(\mathcal{H}^\otimes n)^\otimes k \cong \bigoplus_{\lambda^d \vdash d^{kn}} ([\lambda])^{\oplus m_\lambda}
\]

(9)

where \(m_\lambda = \dim (\mathfrak{S}_\lambda(\mathcal{H}))\). We view \(\mathfrak{S}_{nk}^x\) naturally as a subgroup of \(\mathfrak{S}_{nk}\). For any \(\lambda \vdash_d kn\) the restriction of \([\lambda]\) to the action of \(\mathfrak{S}_{nk}^x\) decomposes further as a direct sum of irreducibles under the action of \(\mathfrak{S}_{nk}^x\), so that

\[
[\lambda] \cong \bigoplus_{\lambda^1 \vdash_d n, \ldots, \lambda^k \vdash_d n} ([\lambda^1] \otimes \cdots \otimes [\lambda^k])^{\oplus c_{\lambda^1, \ldots, \lambda^k}}
\]

(10)

where \(c_{\lambda^1, \ldots, \lambda^k}\) are multiplicities. Let \(\lambda^1 \vdash_d n, \ldots, \lambda^k \vdash_d n\). Then \([\lambda^1] \otimes \cdots \otimes [\lambda^k]\) is irreducible representation of \(\mathfrak{S}_n \times \cdots \times \mathfrak{S}_n\). This gives us the finer decomposition into irreducibles under the action of \(\mathfrak{S}_{nk}^x\) as

\[
(\mathcal{H}^\otimes n)^\otimes k \cong \bigoplus_{\lambda^1 \vdash_d n, \ldots, \lambda^k \vdash_d n} ([\lambda^1] \otimes \cdots \otimes [\lambda^k])^{\oplus m_{\lambda^1, \ldots, \lambda^k}}
\]

(11)

where \(m_{\lambda^1, \ldots, \lambda^k} = \prod_{i=1}^k \dim (\mathfrak{S}_{\lambda_i}(\mathcal{H}))\).

For any \(n\) and \(\lambda^1 \vdash_d n, \ldots, \lambda^k \vdash_d n\) such that \((P_{\lambda^1} \otimes \cdots \otimes P_{\lambda^k}) f^\otimes n \neq 0\) the equivariant projection of \(f^\otimes n\) on

\[
([\lambda^1] \otimes \cdots \otimes [\lambda^k])^{\oplus m_{\lambda^1, \ldots, \lambda^k}}
\]

is non-zero. From (10) we know that there is a \(\lambda \vdash_d kn\) such that \([\lambda^1] \otimes \cdots \otimes [\lambda^k]\) is a subspace of \([\lambda]\). For this \(\lambda\) it holds that \(P_{\lambda} f^\otimes n \neq 0\) and \(\dim [\lambda] \geq \prod_{i=1}^k \dim ([\lambda^i])\). This implies \(F(f) \geq F(\frac{1}{k} \cdots \frac{1}{k})(f)\).

For any tensor \(s \in \mathcal{H}^\otimes k\), it follows from a standard property of the von Neumann entropy [NC11, Theorem 11.10] that

\[
H\left(\frac{\rho_1(s) + \cdots + \rho_k(s)}{k}\right) \leq \frac{H(\rho_1(s)) + \cdots + H(\rho_k(s))}{k} + \log k.
\]

This implies \(F(f) \leq k F(\frac{1}{k} \cdots \frac{1}{k})(f)\). Thus we have proven that

\[
F(\frac{1}{k} \cdots \frac{1}{k})(f) \leq F(f) \leq k F(\frac{1}{k} \cdots \frac{1}{k})(f)
\]

holds for every tensor \(f\). In particular, applying this to the tensor power \(f^\otimes n\) we have

\[
F(\frac{1}{k} \cdots \frac{1}{k})(f^\otimes n) \leq F(f^\otimes n) \leq k F(\frac{1}{k} \cdots \frac{1}{k})(f^\otimes n).
\]

Since \(F(\frac{1}{k} \cdots \frac{1}{k})\) is multiplicative [CVZ18], we have

\[
F(\frac{1}{k} \cdots \frac{1}{k})(f) \leq F(f^\otimes n)^{1/n} \leq k^{1/n} F(\frac{1}{k} \cdots \frac{1}{k})(f).
\]

Taking \(n \to \infty\), we obtain \(\lim_{n \to \infty} F(f^\otimes n)^{1/n} = F(\frac{1}{k} \cdots \frac{1}{k})(f)\).

Finally, since \(F\) is submultiplicative (Appendix A.3), the limit \(\lim_{n \to \infty} F(f^\otimes n)^{1/n}\) equals the infimum \(\inf_n F(f^\otimes n)^{1/n}\) by Fekete’s lemma.

\(\square\)
Proof of Theorem 4.34. Let $S$ be the set of symmetric tensors in $(\text{GL}(d))^k \setminus \{0\}$. Since $f$ is a symmetric tensor, for any matrix $A$ the tensor $(A \otimes \cdots \otimes A)f$ is also a symmetric tensor. Therefore $\text{GL}(d) \cdot f \setminus \{0\} \subseteq S$. Moreover, if $s$ is a symmetric tensor then all marginal density matrices are equal: $\rho_1(s) = \cdots = \rho_k(s)$. Thus, for any $\theta \in P([k])$, we have $E^\theta(s) = \rho_1(s)$. This implies $F(f) \leq F^\theta(f)$ since both $F(f)$ and $F^\theta(f)$ are given by the supremum of the same function and for $F(f)$ the supremum is taken over a smaller set than for $F^\theta(f)$. By Theorem 4.33 we have $F(f) = F^\theta(f)$ with $\theta = (\frac{1}{k}, \ldots, \frac{1}{k})$. Moreover, from the Proposition 4.32 we have $\lim_{n \to \infty} \text{SR}(f^\otimes n)^{1/n} = \min_{\theta \in P([k])} F^\theta(f) \geq F(f)$, which implies $\lim_{n \to \infty} \text{SR}(f^\otimes n)^{1/n} = F(f)$. This proves the claim.

Finally, a natural question we can ask is: Given a nonzero tensor $f \in \mathcal{H}^\otimes k$, is $F(f)$ equal to $d$ or not? We will give a sufficient condition. We first recall a concept from geometric invariant theory [KN79]. Let $\text{SL}(d) = \{M \in \text{GL}(d) : \det(M) = 1\}$ be the special linear group. Consider the representation $\pi$ of $\text{SL}(d)$ on $\mathcal{H}^\otimes k$ by $\pi(g)f := (g \otimes \cdots \otimes g)f$ for all $g \in \text{SL}(d)$ and $f \in \mathcal{H}^\otimes k$. We define the null cone of this representation as

$$N(\text{SL}(d)) := \{f \in \mathcal{H}^\otimes k : 0 \in \text{SL}(d) \cdot f\}.$$ 

Theorem 4.37. For any nonzero tensor $f \in \mathcal{H}^\otimes k$, if $f \notin N(\text{SL}(d))$, then $F(f) = d$.

Proof. See Appendix A.2.

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A Various results

A.1 Properties of the moment polytope

In this section, we present the proof of Lemma 4.30 and Lemma 4.36. We first recall some concepts and results of geometric invariant theory and representation theory. We refer to [NM84], [Bri87], [Fra02], [Wal14], and [BFG+19] for more information. Let $\text{GL}(d)$ be the group of $d \times d$ invertible matrices over the complex numbers. Let $\mathcal{H}$ be a complex finite-dimensional vector space, with $\dim(\mathcal{H}) = d$. Denote by $\mathcal{M}(d)$ the set of complex $d \times d$ matrices, and denote by $\text{Herm}(d)$ the set of $d \times d$ Hermitian matrices. We define the representation $\pi$ of $\text{GL}(d)$ on $\mathcal{H} \otimes k$ by $\pi(g)f := (g \otimes \cdots \otimes g)f$ for all $g \in \text{GL}(d)$ and $f \in \mathcal{H} \otimes k$. Let $\text{GL}(d) \cdot f := \{ \pi(g)f : g \in \text{GL}(d) \}$.

For any nonzero vector $f \in \mathcal{H} \otimes k$, we define the function:

$$F_f : \text{GL}(d) \to \mathbb{R}$$

$$g \mapsto \frac{1}{2} \log \| \pi(g)f \|^2.$$  

The following definition defines the gradient of $F_f$ at $g = I$.

**Definition A.1.** The moment map is the function $\mu : \mathcal{H} \otimes k \setminus \{0\} \to \text{Herm}(d)$ defined by the property that, for all $H \in \text{Herm}(d)$,

$$\text{tr}[\mu(f)H] = \partial_{t=0} F_f(e^{tH}).$$

For any $H \in \text{Herm}(d)$, we have $\partial_{t=0} F_f(e^{tH}) = \partial_{t=0} \frac{\langle f, \pi(e^{tH})f \rangle}{\|f\|^2}$. Therefore:

$$\text{tr}[\mu(f)H] = \partial_{t=0} \frac{\langle f, \pi(e^{tH})f \rangle}{\|f\|^2}$$

$$= \langle f, (\sum_{j=1}^k I^{\otimes j-1} \otimes H \otimes I^{\otimes n-j})f \rangle$$

$$= \sum_{j=1}^k \text{tr} \left[ \frac{f f^\dagger}{\|f\|^2} (I^{\otimes j-1} \otimes H \otimes I^{\otimes n-j}) \right]$$

$$= \sum_{j=1}^k \text{tr}[\rho_j(f)H],$$

where $\rho_j(f)$ is $j$th reduced density matrix of $\rho(f) = \frac{f f^\dagger}{\|f\|^2}$. Thus, $\mu(f) = \sum_{j=1}^k \rho_j(f)$.

**Lemma A.2 ([NM84] and [Bri87]).** For any nonzero vector $f \in \mathcal{H} \otimes k$,

$$\Delta(f) := \{ \text{spec}(\mu(s)) : s \in \text{GL}(d) \cdot f \setminus \{0\} \}$$

is a convex polytope with rational vertices, which is called the moment polytope.

By the above lemma, for any $0 \neq f \in \mathcal{H} \otimes k$ we have $\{ \text{spec}(\rho_1(s) + \cdots + \rho_k(s)) : s \in \text{GL}(d) \cdot f \setminus \{0\} \}$ is also a convex polytope, which completes the proof of Lemma 4.30.

Following [FH91], any rational irreducible representations of $\text{GL}(d)$ can be labeled by highest weight $\lambda \in \mathbb{N}^d$ such that $\lambda_1 \geq \cdots \geq \lambda_d$. For any natural number $n \geq 1$, consider the
representation $\Pi$ of $GL(d)$ on $(\mathcal{H} \otimes k)^{\otimes n}$ by $\Pi(g) \cdot v := (\pi(g) \otimes \cdots \otimes \pi(g))v$ for all $v \in (\mathcal{H} \otimes k)^{\otimes n}$. Let $V$ be a finite-dimensional rational representation of $GL(d)$. For each highest weight $\lambda$ of $GL(d)$, we denote by $V_{\lambda}$ the $\lambda$-isotypical component of $V$. Let $Z \subseteq V$ be a Zariski closed set. We denote by $C[Z]_n$ the degree-$n$ part of the coordinate ring of $Z$. Letting $\lambda = (\lambda_1, \ldots, \lambda_d)$ be a highest weight of $GL(d)$, we define $\lambda^* = (-\lambda_d, \ldots, -\lambda_1)$. For any nonzero vector $f \in \mathcal{H} \otimes k$,

the following lemma says that the moment polytope $\Delta(f)$ has another representation theoretic description.

**Lemma A.3** ([Bri87], [Fra02], [Str05, Theorem 11] or [Zui18, Chapter 6]). Let $f \in \mathcal{H} \otimes k$ be nonzero. Then

$$\Delta(f) = \left\{ \frac{\lambda}{n} : \exists n \in \mathbb{N}_{\geq 1}, (C[GL(d) \cdot f]_n)_{\lambda^*} \neq 0 \right\}$$

$$= \left\{ \frac{\lambda}{n} : \exists n \in \mathbb{N}_{\geq 1}, P_{\lambda} f^{\otimes n} \neq 0 \right\},$$

where $P_{\lambda}$ is the projector from $(\mathcal{H} \otimes k)^{\otimes n}$ onto the $\lambda$-isotypical component in the decomposition of $(\mathcal{H} \otimes k)^{\otimes n}$ with respect to $\Pi$.

More precisely, by Schur–Weyl duality we have a decomposition of the space $(\mathcal{H} \otimes k)^{\otimes n}$ as

$$(\mathcal{H} \otimes k)^{\otimes n} \cong \bigoplus_{\lambda \vdash d \cdot k} S_{\lambda}(\mathcal{H}) \otimes [\lambda].$$

For $\lambda \vdash_{d \cdot k} kn$, let $P_{\lambda}$ be the projector onto the isotypical component of type $\lambda$, that is, onto the subspace of $(\mathcal{H} \otimes k)^{\otimes n}$ which isomorphic to $S_{\lambda}(\mathcal{H}) \otimes [\lambda]$, since all irreducible representations of $\Pi$ are labeled by the partitions of $kn$ in at most $d$ parts. Therefore,

$$\Delta(f) = \left\{ \frac{\lambda}{n} : \exists n \geq 1, \lambda \vdash_{d \cdot k} kn, P_{\lambda} f^{\otimes n} \neq 0 \right\},$$

completing the proof of Lemma 4.36.

**A.2 Null cone condition for maximality of the symmetric quantum functional**

The goal of this section is to prove Theorem 4.37. We need some concepts from geometric invariant theory. Let $SL(d) := \{ M \in GL(d) : \text{det}(M) = 1 \}$. Consider the representation $\pi$ of $SL(d)$ on $\mathcal{H} \otimes k$ by $\pi(g)f := (g \otimes \cdots \otimes g)f$ for all $g \in SL(d)$ and $f \in \mathcal{H} \otimes k$. We define the null cone as

$$\mathcal{N}(SL(d)) := \{ f \in \mathcal{H} \otimes k : 0 \in SL(d) \cdot f \}.$$ 

For a nonzero vector $f \in \mathcal{H} \otimes k$, define the function:

$$F'_f : SL(d) \to \mathbb{R}$$

$$g \mapsto \frac{1}{2} \log \| \pi(g)f \|^2.$$ 

Let $Herm_{al}(d) := \{ M \in Herm(d) : \text{tr}(M) = 0 \}$. The following definition defines the gradient of $F'_f$ at $g = I$. 

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Definition A.4. The moment map (in terms of $F'_f$) is the function $\mu_{sl} : \mathcal{H}^\otimes k \setminus \{0\} \to \text{Herm}_{sl}(d)$ defined by the property that, for all $H \in \text{Herm}_{sl}(d)$,

$$\text{tr}[\mu_{sl}(f)H] = \partial_{t=0} F'_f(e^t H).$$

Similar to the case of $\text{GL}(d)$, we have for any $H \in \text{Herm}_{sl}(d)$, that $\partial_{t=0} F'_f(e^t H) = \partial_{t=0} \frac{(f, \pi(e^t H)f)}{\|f\|^2}$. Therefore:

$$\text{tr}[\mu_{sl}(f)H] = \partial_{t=0} \frac{\langle f, \pi(e^t H)f \rangle}{\|f\|^2} = \frac{\langle f, (\sum_{j=1}^k I^\otimes j-1 \otimes H \otimes I^\otimes n-j) f \rangle}{\|f\|^2} = \sum_{j=1}^k \text{tr} \left[ \frac{f f^\dagger}{\|f\|^2} (I^\otimes j-1 \otimes H \otimes I^\otimes n-j) \right] = \sum_{j=1}^k \text{tr} \rho_j(f)H = \sum_{j=1}^k \text{tr} [\rho_j(f) - c I_d]H$$

for all $c$, where $\rho_j(f)$ is $j$th reduced density matrix of $\rho(f) = \frac{f f^\dagger}{\|f\|^2}$. Thus, $\mu_{sl}(f) = \sum_{j=1}^k \rho_j(f) - \frac{k}{d} I_d$ (note that we choose $c = \frac{k}{d}$ since $\text{tr}(\sum_{j=1}^k \rho_j(f) - \frac{k}{d} I_d) = 0$).

Proposition A.5 ([KN79]). Let $f \in \mathcal{H}^\otimes k$. Then $f \notin \mathcal{N}(\text{SL}(d))$ if and only if there is a non-zero $s \in \text{SL}(d) \cdot \bar{v}$ such that $\mu_{sl}(s) = 0$.

Proof of Theorem 4.37. Let $f \in \mathcal{H}^\otimes k$ be not in the null cone $\mathcal{N}(\text{SL}(d))$. By Proposition A.5 there is a nonzero tensor $w \in \text{SL}(d) \cdot \bar{f} \subseteq \text{GL}(d) \cdot \bar{f}$ such that $\mu_{sl}(w) = 0$, which is equivalent to $\sum_{j=1}^k \rho(w) = \frac{k}{d} I_d$. This implies $F(f) = d$, which completes the proof of Theorem 4.37.

A.3 Submultiplicativity of the symmetric quantum functional

In this section we prove that the symmetric quantum functional is submultiplicative on (not necessarily symmetric) tensors.

Lemma A.6. For tensors $s \in V^\otimes k$ and $t \in W^\otimes k$, we have

$$\Delta(s \otimes t) \subseteq \Delta(s) \otimes_{\text{Kron}} \Delta(t),$$

where

$$\Delta(s) \otimes_{\text{Kron}} \Delta(t) := \text{closure}\left\{ \tilde{\mu} : \tilde{\lambda} \in \Delta(s), \tilde{\lambda'} \in \Delta(t), P_{\tilde{\mu}}(P_{\lambda} \otimes P_{\lambda'}) \neq 0 \right\}.$$

Proof. Let $\dim(V) = d$ and $\dim(W) = d'$. If $\tilde{\mu} \in \Delta(s \otimes t)$, then for some $n$, we have $P_{\tilde{\mu}}(s \otimes t)^\otimes n \neq 0$. We have $\sum_{\lambda \in d^k n} P_{\lambda} = I_d^V \otimes_{\lambda} n$ and $\sum_{\lambda' \in d'^k n} P_{\lambda'} = I_d^W \otimes_{\lambda'} n$. Thus, we can write

$$P_{\tilde{\mu}}(s \otimes t)^\otimes n = P_{\tilde{\mu}} \left( \sum_{\lambda, \lambda'} P_{\lambda} \otimes P_{\lambda'} \right) (s \otimes t)^\otimes n.$$
So there exists \( \lambda, \lambda' \) such that \( P_\mu(P_\lambda \otimes P_{\lambda'}) (s \otimes t)^{\otimes n} \neq 0 \). But this implies that

\[
\begin{align*}
P_\lambda s^{\otimes n} &\neq 0 \\
P_{\lambda'} t^{\otimes n} &\neq 0 \\
P_\mu(P_\lambda \otimes P_{\lambda'}) &\neq 0,
\end{align*}
\]

which completes the proof. \( \square \)

**Lemma A.7** (Submultiplicativity of the symmetric quantum functional). *For tensors* \( s \in V^{\otimes k} \) *and* \( t \in W^{\otimes k} \), *we have*

\[
F(s \otimes t) \leq F(s)F(t).
\]

**Proof.** Let \( \dim(V) = d \) and \( \dim(W) = d' \). Let \( E = \log_2 F \). We need to prove \( E(s \otimes t) \leq E(s) + E(t) \). By definition

\[
E(s \otimes t) = \max_{p \in \Delta(s \otimes t)} H(p)
\leq \max_{p \in \Delta(s) \otimes \text{Kron} \Delta(t)} H(p).
\]

But if \( p \in \Delta(s) \otimes \text{Kron} \Delta(t) \), then there exists \( \mu \) a partition of \( kn \) in at most \( dd' \) parts such that \( P_\mu(P_\lambda \otimes P_{\lambda'}) \neq 0 \) with \( \lambda \in \Delta(s) \) and \( \lambda' \in \Delta(t) \) by Lemma A.6. It is shown in [CM06, Proposition 3] that if \( P_\mu(P_\lambda \otimes P_{\lambda'}) \neq 0 \), then \( H(\bar{\mu}) \leq H(\bar{\lambda}) + H(\bar{\lambda'}) \). This shows that \( E(s \otimes t) \leq E(s) + E(t) \). \( \square \)