Converse theorem on a contraction metric for a periodic orbit

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Abstract

Contraction analysis uses a local criterion to prove the long-term behaviour of a dynamical system. A contraction metric is a Riemannian metric with respect to which the distance between adjacent solutions contracts. If adjacent solutions in all directions perpendicular to the flow are contracted, then there exists a unique periodic orbit, which is exponentially stable and we obtain a bound on the rate of exponential attraction.

In this paper we study the converse question and show that, given an exponentially stable periodic orbit, a contraction metric exists on its basin of attraction and we can recover the bound on the rate of exponential attraction.

Keywords: Periodic orbit; Basin of attraction; Contraction metric; Converse theorem; Floquet theory.

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1 Introduction

The stability and basin of attraction of periodic orbits is an important problem in many applications. Already the determination of a periodic orbit is a non-trivial task as it involves solving the differential equation. The classical definition of stability, as well as its study using a Lyapunov function require the knowledge of the position of the periodic orbit which in many applications can only be approximated. An alternative way to study the stability and basin of attraction is contraction analysis, which is a local criterion and does not require us to know the location of the periodic orbit.

Throughout the paper we will study the autonomous ODE

\[ \dot{x} = f(x) \]  

where \( f \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n) \) with \( \sigma \geq 1 \). We denote the solution \( x(t) \) with initial condition \( x(0) = x_0 \) by \( S_t x_0 = x(t) \) and assume that it exist for all \( t \geq 0 \).
In the next definition we will define a contraction metric on \( \mathbb{R}^n \). Note that \( M(x) \) defines a point-dependent scalar product through \( \langle v, w \rangle_{M(x)} = v^T M(x) w \) for all \( v, w \in \mathbb{R}^n \).

**Definition 1.1 (Contraction metric)** A Riemannian metric is a function \( M \in C^0(G, S^n) \), where \( G \subset \mathbb{R}^n \) is open and \( S^n \) denotes the symmetric \( n \times n \) matrices, \( M(x) \) is positive definite for all \( x \in G \) and the orbital derivative of \( M \) exists for all \( x \in G \) and is continuous, i.e.

\[
M'(x) = \left. \frac{d}{dt} M(S_t x) \right|_{t=0}
\]

exists and is continuous. A sufficient condition for the latter is that \( M \in C^1(G, S^n) \); then \( M_{ij}'(x) = \nabla M_{ij}(x) \cdot f(x) \) for all \( i, j \in \{1, \ldots, n\} \).

Define

\[
L_M(x; v) := \frac{1}{2} v^T \left( M(x) Df(x) + Df(x)^T M(x) + M'(x) \right) v. \quad (1.2)
\]

The Riemannian metric \( M \) is called **contraction metric in** \( K \subset G \) **with exponent** \(-\nu < 0\) if \( L_M(x) \leq -\nu \) for all \( x \in K \), where

\[
L_M(x) := \max_{v^T M(x)v=1, v^T M(x)f(x)=0} L_M(x; v). \quad (1.3)
\]

The following theorem shows the implications of the existence of such a contraction metric on a certain set in the phase space.

**Theorem 1.2** Let \( \emptyset \neq K \subset \mathbb{R}^n \) be a compact, connected and positively invariant set which contains no equilibrium. Let \( M \) be a contraction metric in \( K \) with exponent \(-\nu < 0\), see Definition 1.1.

Then there exists one and only one periodic orbit \( \Omega \subset K \). This periodic orbit is exponentially asymptotically stable, and the real parts of all Floquet exponents – except the trivial one – are less than or equal to \(-\nu\). Moreover, the basin of attraction \( A(\Omega) \) contains \( K \).

This theorem goes back to Borg \cite{Borg1954} with \( M(x) = I \), and has been extended to a general Riemannian metric \cite{LaC1969}. For more results on contraction analysis for a periodic orbit see \cite{LaC1969, Hara1985, Hara1986, Hara1987}.

Note that a similar result holds with an equilibrium if the contraction takes place in all directions \( v \), i.e. if \( L_M(x) \leq -\nu \) in (1.3) is replaced by \( L_M(x) := \max_{v^T M(x)v=1} L_M(x; v) \leq -\nu \). For more references on contraction analysis see \cite{Hara1986}, and for the relation to Finsler-Lyapunov functions see \cite{Hara1985}.

Note that \( L_M(x) \) is a continuous with respect to \( x \) and, as we will show in the paper, also locally Lipschitz-continuous. Due to the maximum, however, it is not differentiable in general.

In this paper we are interested in converse results, i.e. given an exponentially stable periodic orbit, does a Riemannian contraction metric as in Definition 1.1 exist?
[12] gives a converse theorem, but here $M(t, x)$ depends on $t$ and will, in general, become unbounded as $t \to \infty$. In [9] the existence of such a contraction metric was shown on a given compact subset of $A(\Omega)$, first on the periodic orbit, using Floquet theory, and then on $K$, using a Lyapunov function. The local construction, however, neglected the fact that the Floquet representation of solutions of the first variation equation along the periodic orbit is in general not real, but complex. We will show in this paper, that, by choosing the complex Floquet representation appropriately, the constructed Riemannian metric is real-valued, thus justifying the arguments in [9]. Moreover, we will show the existence of a Riemannian metric on the whole, possibly unbounded basin of attraction by using a new construction. The Riemannian metric will be arbitrarily close to the true rate of exponential attraction. Let us summarize the main result of the paper in the following theorem.

**Theorem 1.3** Let $\Omega$ be an exponentially stable periodic orbit of $\dot{x} = f(x)$, let $-\nu$ be the largest real part of all its non-trivial Floquet exponents and $f \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$ with $\sigma \geq 3$.

Then for all $\epsilon \in (0, \nu/2)$ there exists a contraction metric $M \in C^{\sigma-1}(A(\Omega), S^n)$ in $A(\Omega)$ as in Definition 1.1 with exponent $-\nu + \epsilon < 0$, i.e.

$$L_M(x) = \frac{1}{2} \max_{v^TM(x)v=1, v^Tf(x)=0} v^T \left( M(x)Df(x) + Df(x)^TM(x) + M'(x) \right) v \leq -\nu + \epsilon$$

holds for all $x \in A(\Omega)$.

The metric is constructed in several steps: first on the periodic orbit, then in a neighborhood, and finally in the whole basin of attraction. In the proof, we define a projection of points $x$ in a neighborhood of the periodic orbit onto the periodic orbit, namely onto $p \in \Omega$, such that to $(x - p)^TM(p)f(p) = 0$. This is then used to synchronize the times of solutions through $x$ and $p$, and to define a time-dependent distance between these solutions, which decreases exponentially.

Let us compare our result with converse theorems for a contraction metric for an equilibrium. In [7], three converse theorems were obtained: Theorem 4.1 constructs a metric on a given compact subset of the basin of attraction (see [5] for the case of a periodic orbit), Theorem 4.2 constructs a metric on the whole basin of attraction (see this paper for the case of a periodic orbit), while Theorem 4.4 constructs a metric as solution of a linear matrix-valued PDE (see [5] for the case of a periodic orbit). The latter construction is beneficial for its computation by solving the PDE, and it also constructs a smooth function; however, the exponential rate of attraction cannot be recovered, which is an advantage of the approach in this paper.

Let us give an overview over the paper: In Section 2 we prove a special Floquet normal form to ensure that the contraction metric that we later construct on the periodic orbit is real-valued. In Section 3 we prove the main result of the paper, Theorem 1.3 showed the existence of a Riemannian metric on the whole basin of attraction. The section also contains Corollary 3.6 defining a projection onto the
periodic orbit and related estimates. In the appendix we prove that \(L_M\) is locally Lipschitz-continuous.

2 Floquet normal form

Before we consider the Floquet normal form, we will prove a lemma which calculates \(L_M(x)\) for the Riemannian metric \(M(x) = e^{2V(x)}N(x)\).

**Lemma 2.1** Let \(N: \mathbb{R}^n \to S^n\) be a Riemannian metric and \(V: \mathbb{R}^n \to \mathbb{R}\) a continuous and orbitally continuously differentiable function.

Then \(M(x) = e^{2V(x)}N(x)\) is a Riemannian metric and

\[
L_M(x) = L_N(x) + V'(x).
\]

**Proof:** It is clear that \(M(x)\) is a positive definite for all \(x\) since \(e^{2V(x)} > 0\). We have

\[
L_M(x; v) = \frac{1}{2}v^T (M(x)Df(x) + Df(x)^T M(x) + M'(x)) v
\]

\[
= \frac{1}{2}v^T \left( e^{2V(x)}N(x)Df(x) + e^{2V(x)}Df(x)^T N(x) + e^{2V(x)}(2V'(x)) N(x) + N'(x) \right) v
\]

\[
= \frac{1}{2}w^T (N(x)Df(x) + Df(x)^T N(x) + N'(x)) w + w^T N(x)w V'(x)
\]

with \(w = e^{V(x)}v\), so \(L_M(x; v) = L_N(x; w) + w^T N(x)w V'(x)\).

This shows the lemma. \(\square\)

In order to show later that our constructed Riemannian metric \(M\) is real-valued, we will construct a special Floquet normal form in Proposition 2.2 such that the matrix in (2.2) is real-valued. In Corollary 2.3 we will show estimates in the case that (2.1) is the first variation equation of a periodic orbit. The proof of the following proposition is inspired by [3].

**Proposition 2.2** Consider the periodic differential equation

\[
\dot{y} = F(t)y
\]

(2.1)
where \( F \in C^s(\mathbb{R}, \mathbb{R}^{n \times n}) \) is \( T \)-periodic, \( s \geq 1 \) and denote by \( \Phi \in C^s(\mathbb{R}, \mathbb{R}^{n \times n}) \) its principal fundamental matrix solution with \( \Phi(0) = I \).

Then there exists a \( T \)-periodic function \( P \in C^s(\mathbb{R}, \mathbb{C}^{n \times n}) \) with \( P(0) = P(T) = I \) and a matrix \( B \in \mathbb{C}^{n \times n} \) such that for all \( t \in \mathbb{R} \)
\[
\Phi(t) = P(t)e^{Bt}.
\]

Denote by \( \lambda_1, \ldots, \lambda_r \in \mathbb{R} \setminus \{0\} \) the pairwise distinct real eigenvalues and by \( \lambda_{r+1}, \lambda_{r+1}, \ldots, \lambda_{r+r_c} \in \mathbb{C} \setminus \mathbb{R} \) the pairwise distinct pairs of complex conjugate complex eigenvalues of \( \Phi(T) \) with algebraic multiplicity \( m_j \) of \( \lambda_j \). For \( \epsilon > 0 \) there exists a non-singular matrix \( S \in \mathbb{R}^{n \times n} \) such that \( B = SAS^{-1} \) with \( A = \text{blockdiag}(K_1, K_2, \ldots, K_{r+c}) \) and \( K_j \in \mathbb{C}^{m_j \times m_j} \) for \( j = 1, \ldots, r \) and \( K_j \in \mathbb{R}^{2m_j \times 2m_j} \) for \( j = r + 1, \ldots, r + c \) as well as
\[
\frac{1}{2} w^*(A^* + A)w \leq \sum_{j=1}^{r+c} c_j \sum_{i=1}^{m_j} |w_{i+n-j-1}m_k|^2 \text{ for all } w \in \mathbb{C}^n,
\]
where \( c_j = \left( \frac{\ln |\lambda_j|}{\epsilon} + \epsilon \right) \) if \( m_j \geq 2 \) and \( c_j = \frac{\ln |\lambda_j|}{\epsilon} \) if \( m_j = 1 \).

Moreover, we have
\[
(P^{-1}(t))^*(S^{-1})^*S^{-1}P^{-1}(t) \in \mathbb{R}^{n \times n}
\]
for all \( t \in \mathbb{R} \).

**Proof:** Since \( F \in C^s \), we also have \( \Phi \in C^s(\mathbb{R}, \mathbb{R}^{n \times n}) \). Noting that \( \Psi(t) := \Phi(t+T) \) solves (2.1) with \( \Psi(0) = \Phi(T) \), we obtain from the uniqueness of solutions that
\[
\Phi(t+T) = \Psi(t) = \Phi(t)\Phi(T) \text{ for all } t \in \mathbb{R}.
\]

Consider \( C := \Phi(T) \in \mathbb{R}^{n \times n} \) which is non-singular and hence all eigenvalues of \( \Phi(T) \) are non-zero. Let \( \epsilon' := \frac{1}{2} \min \left( \frac{\epsilon}{T}, 1 \right) \) and \( S \in \mathbb{R}^{n \times n} \) be such that \( S^{-1}CS =: J \) is in real Jordan normal form with the 1 replaced by \( \epsilon' |\lambda_j| \) for each eigenvalue \( \lambda_j \), i.e. \( J \) is a block-diagonal matrix with blocks \( J_j \) of the form
\[
J_j = \begin{pmatrix}
\lambda_j & \epsilon' |\lambda_j| & & \\
\lambda_j & \epsilon' |\lambda_j| & & \\
& \ddots & \ddots & \\
& & \ddots & \epsilon' |\lambda_j|
\end{pmatrix} \in \mathbb{R}^{m_j \times m_j} \text{ for real eigenvalues } \lambda_j \text{ of } C \text{ and}
\]
\[
J_j = \begin{pmatrix}
\alpha_j & -\beta_j & \epsilon' r_j & \\
\beta_j & \alpha_j & \epsilon' r_j & \\
& \ddots & \ddots & \\
& & \alpha_j & -\beta_j & \epsilon' r_j & \\
& & & \beta_j & \alpha_j & \epsilon' r_j & \\
& & & & \alpha_j & -\beta_j & \epsilon' r_j & \\
& & & & & \beta_j & \alpha_j
\end{pmatrix} \in \mathbb{R}^{2m_j \times 2m_j} \text{ for each pair of complex eigenvalues } \alpha_j \pm i\beta_j \text{ of } C, \text{ where } r_j = \sqrt{\alpha_j^2 + \beta_j^2} \text{ and } m_j \text{ denotes the dimension}.
\]
of the generalized eigenspace of one of them; note we have pairs of complex conjugate eigenvalues since $C$ is real.

This can be achieved by letting $S_1 \in \mathbb{R}^{n \times n}$ be an invertible matrix such that $S_1^{-1}CS_1$ is the standard real Jordan Normal Form with 1 on the super diagonal. Then define $S_2$ to be a matrix of blocks

$$\text{diag}(1, \epsilon|\lambda_j|, (\epsilon')^2|\lambda_j|^2, \ldots, (\epsilon')^{m_j-1}|\lambda_j|^{m_j-1})$$

for real $\lambda_j$ and

$$\text{diag}(1, 1, \epsilon|\lambda_j|, \epsilon'|\lambda_j|, \ldots, (\epsilon')^{m_j-1}|\lambda_j|^{m_j-1}, (\epsilon')^{m_j-1}|\lambda_j|^{m_j-1})$$

for a pair of complex conjugate eigenvalues $\lambda_j$ and $\overline{\lambda_j}$. Setting $S = S_1S_2$ yields the result.

For each of the blocks, we will now construct a matrix $K_j \in \mathbb{C}^{m_j \times m_j}$ for real eigenvalues $\lambda_j$ and $K_j \in \mathbb{R}^{2m_j \times 2m_j}$ for each pair of complex eigenvalues $\alpha_j \pm i\beta_j$ such that

$$e^{K_jT} = J_j,$$

which shows with $B = SAS^{-1}$, where $A := \text{blockdiag}(K_1, \ldots, K_r)$,

$$e^{BT} = S e^{AT} S^{-1} = S \text{blockdiag}(e^{K_1T}, \ldots, e^{K_rT}) S^{-1} = SJS^{-1} = C = \Phi(T). \quad (2.4)$$

We distinguish between three cases: $\lambda_j$ being real positive, real negative or complex. Using the series expansion of $\ln(1 + x)$ we obtain for a nilpotent matrix $M \in \mathbb{R}^{n \times n}$

$$\exp \left( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} M^{k} \right) = I + M; \quad (2.5)$$

note that the sum is actually finite.

**Case 1: $\lambda_j \in \mathbb{R}^+$**

Writing $J_j = \lambda_j(I + \epsilon'N)$ with the nilpotent matrix $N = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{R}^{m_j \times m_j}$, we define

$$K_j = \frac{1}{T} \left( (\ln \lambda_j) I + \sum_{k=1}^{m_j-1} \frac{(-1)^{k+1}}{k} (\epsilon')^k N^k \right) \in \mathbb{R}^{m_j \times m_j}.$$

Since $I$ and $N$ commute, we have with $N^k = 0$ for $k \geq m_j$

$$\exp(K_jT) = \lambda_j \left( I + \epsilon'N \right) = J_j.$$
Case 2: \( \lambda_j \in \mathbb{R}^- \)

With the nilpotent matrix \( N = \begin{pmatrix} 0 & 1 \\ \vdots & \ddots & \ddots \\ 0 & 1 \\ 0 \end{pmatrix} \in \mathbb{R}^{m_j \times m_j} \) we write \( J_j = -|\lambda_j|(I - \epsilon'N) \) and define
\[
K_j = \frac{1}{T} \left( (i\pi + \ln|\lambda_j|)I + \sum_{k=1}^{m_j-1} \frac{(-1)^k+1}{k}(-\epsilon')^kN^k \right) \in \mathbb{C}^{m_j \times m_j}.
\]
Since \( I \) and \( N \) commute, and \( N^k = 0 \) for \( k \geq m_j \) we have with (2.5)
\[
\exp(K_jT) = -|\lambda_j|(I - \epsilon'N) = J_j.
\]

Case 3: \( \lambda_j = \alpha_j + i\beta_j \) with \( \beta_j \neq 0 \)

We only consider one of the two complex conjugate eigenvalues \( \lambda_j \) and \( \overline{\lambda}_j \) of \( \Phi(T) \).

Writing \( \lambda_j \) in polar coordinates gives
\[
\lambda_j = \alpha_j + i\beta_j = r_j e^{i\theta_j} = r_j \cos \theta_j + ir_j \sin \theta_j
\]
with \( r_j > 0 \) and \( \theta_j \in (0, 2\pi) \). Then, defining \( R_j = r_j \begin{pmatrix} \cos \theta_j & -\sin \theta_j \\ \sin \theta_j & \cos \theta_j \end{pmatrix} \),
\[
\mathcal{R} = \text{blockdiag}(R_j, R_j, \ldots, R_j) \in \mathbb{R}^{2m_j \times 2m_j}
\]
and the nilpotent matrix \( N \in \mathbb{R}^{2m_j \times 2m_j} \) having \( 2 \times 2 \) blocks of
\[
\begin{pmatrix} \cos \theta_j & \sin \theta_j \\ -\sin \theta_j & \cos \theta_j \end{pmatrix} = \begin{pmatrix} \cos \theta_j & -\sin \theta_j \\ \sin \theta_j & \cos \theta_j \end{pmatrix}^{-1}
\]
above its diagonal, we have \( J_j = \mathcal{R}(I + \epsilon'N) \). We define \( \Theta = \begin{pmatrix} 0 & -\theta_j \\ \theta_j & 0 \end{pmatrix} \) and
\[
K_j = \frac{1}{T} \left( (\ln r_j)I + \text{blockdiag}(\Theta, \Theta, \ldots, \Theta) + \sum_{k=1}^{2m_j-2} \frac{(-1)^{k+1}}{k}(-\epsilon')^kN^k \right) \in \mathbb{R}^{2m_j \times 2m_j}.
\]
Since \( I \), \( \text{blockdiag}(\Theta, \Theta, \ldots, \Theta) \) and \( N \) commute, we have, using \( N^k = 0 \) for \( k \geq 2m_j - 1 \) and (2.5)
\[
\exp(K_jT) = r_j \text{blockdiag} \left( \begin{pmatrix} \cos \theta_j & -\sin \theta_j \\ \sin \theta_j & \cos \theta_j \end{pmatrix}, \ldots, \begin{pmatrix} \cos \theta_j & -\sin \theta_j \\ \sin \theta_j & \cos \theta_j \end{pmatrix} \right) (I + \epsilon'N)
\]
\[
= J_j.
\]

We can now define \( P \in C^\infty(\mathbb{R}, \mathbb{C}^{n \times n}) \) by \( P(t) = \Phi(t)e^{-Bt} \), which satisfies \( P(0) = I \) and
\[
P(t + T) = \Phi(t + T)e^{-BT}e^{-Bt} \]
\[
= \Phi(t)\Phi(T)e^{-BT}e^{-Bt} \text{ by (2.5)}
\]
\[
= P(t) \text{ by (2.4)}
\]
for all \( t \geq 0 \), so in particular \( P(T) = P(0) = I \). We can now write

\[
\Phi(t) = P(t)e^{Bt}.
\]

This shows the first statement of the proposition.

We now evaluate \( A^* + A = \text{blockdiag}(K_1^* + K_1, \ldots, K_r^* + K_r) \). Let us consider \( K_j \) as in the three cases above. If \( m_j = 1 \), then \( K_j \) below does not contain the last sum with \( \varepsilon' \) and the form of \( c_j \) is immediately clear.

**Case 1:** \( \lambda_j \in \mathbb{R}^+ \)

\[
K_j = \frac{1}{T} \left( (\ln \lambda_j)I + \sum_{k=1}^{m_j-1} \frac{(-1)^k + 1}{k} (\varepsilon')^k N^k \right) \in \mathbb{R}^{m_j \times m_j};
\]

hence, for \( w \in \mathbb{C}^{m_j} \)

\[
\frac{1}{2} w^*(K_j^* + K_j)w = \frac{\ln \lambda_j}{T} \sum_{i=1}^{m_j} |w_i|^2 + \varepsilon' \frac{1}{2T} (w_1 w_2 + w_1 w_3 + w_2 w_3 + \ldots + w_{m_j-1} w_{m_j} + w_{m_j-1} w_{m_j})
\]

\[
-\frac{(\varepsilon')^2}{2} \frac{1}{2T} (w_1 w_3 + w_1 w_4 + w_2 w_4 + \ldots + w_{m_j-2} w_{m_j} + w_{m_j-2} w_{m_j})
\]

\[
+ \ldots
\]

\[
+(-1)^{m_j} (\varepsilon')^{m_j-1} \frac{1}{m_j - 1} \frac{1}{2T} (w_1 w_{m_j} + w_1 w_{m_j}) .
\]

Note that the Cauchy–Schwarz inequality implies \( \mathbb{R} \ni \xi \eta + \xi \eta \leq |\xi|^2 + |\eta|^2 \), which yields that, using \( \varepsilon' = \frac{1}{2} \min \left( \frac{T}{2}, 1 \right) \)

\[
\frac{1}{2} w^*(K_j^* + K_j)w \leq \frac{\ln \lambda_j}{T} \sum_{i=1}^{m_j} |w_i|^2
\]

\[
+ \varepsilon' + (\varepsilon')^2 + \ldots + (\varepsilon')^{m_j-1} \frac{1}{T} \sum_{i=1}^{m_j} |w_i|^2
\]

\[
\leq \left( \frac{\ln \lambda_j}{T} + \varepsilon \left( \frac{1}{4} + \frac{1}{8} + \ldots \right) \right) \sum_{i=1}^{m_j} |w_i|^2
\]

\[
\leq \left( \frac{\ln \lambda_j}{T} + \varepsilon \right) \sum_{i=1}^{m_j} |w_i|^2 .
\]
**Case 2:** $\lambda_j \in \mathbb{R}^-$

\[ K_j = \frac{1}{T} \left( (i\pi + \ln |\lambda_j|)I + \sum_{k=1}^{m_j-1} \frac{(-1)^{k+1}}{k} (-\epsilon')^k N^k \right) \in \mathbb{C}^{m_j \times m_j}; \]

hence, for $w \in \mathbb{C}^{m_j}$

\[ \frac{1}{2} w^*(K_j^* + K_j)w = \frac{\ln |\lambda_j|}{T} \sum_{i=1}^{m_j} |w_i|^2 \]

\[ + \epsilon' \frac{1}{2T} \left( w_1 w_2 + w_1 w_3 + w_2 w_3 + \ldots + w_{m_j-1} w_{m_j} + w_{m_j-1} \right) \]

\[ - \left( \frac{\epsilon'}{2} \right)^2 \frac{1}{2T} \left( w_1 w_3 + w_1 w_4 + w_2 w_4 + \ldots + w_{m_j-2} w_{m_j} + w_{m_j-2} \right) \]

\[ + \ldots \]

\[ \leq \left( \frac{\ln |\lambda_j|}{T} + \epsilon \right) \sum_{i=1}^{m_j} |w_i|^2 \]

similarly to case 1.

**Case 3:** $\lambda_j = \alpha_j + i\beta_j$ with $\beta_j \neq 0$

Recall that

\[ K_j = \frac{1}{T} \left( \ln r_j I + \text{blockdiag}(\Theta, \Theta, \ldots, \Theta) + \sum_{k=1}^{2m_j-2} \frac{(-1)^{k+1}}{k} (\epsilon')^k N^k \right) \in \mathbb{R}^{2m_j \times 2m_j}; \]

where $\Theta = \begin{pmatrix} 0 & -\theta_j \\ \theta_j & 0 \end{pmatrix}$ and the nilpotent matrix $N$ has $2 \times 2$ blocks of

\( \begin{pmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{pmatrix} \)

on its super diagonal. Note that all entries of $N^k$, $k \in \mathbb{N}$ are real and have an absolute value of $\leq 1$ as they are of the form $\cos(k\theta_j)$ and $\pm \sin(k\theta_j)$ for $k = 1, 2, \ldots$. Hence, for $w \in \mathbb{C}^{2m_j}$

\[ \frac{1}{2} w^*(K_j^* + K_j)w = \frac{\ln r_j}{T} \sum_{i=1}^{2m_j} |w_i|^2 \]

\[ + \epsilon \frac{1}{2T} \left( \cos\theta_j(w_1 w_2 + w_1 w_3) + \sin\theta_j(w_1 w_4 + w_1 w_1) \right) \]

\[ - \sin\theta_j(w_2 w_3 + w_2 w_3) + \cos\theta_j(w_2 w_4 + w_2 w_4) + \ldots \]
\[
\leq \frac{\ln r_j}{T} \sum_{i=1}^{2m_j} |w_i|^2 + 2\epsilon + (\epsilon')^2 + \ldots + (\epsilon')^{2m_j} \sum_{i=1}^{2m_j} |w_i|^2
\]
\[
\leq \left( \frac{\ln r_j}{T} + \epsilon \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \right) \right) \sum_{i=1}^{2m_j} |w_i|^2
\]
\[
\leq \left( \frac{\ln r_j}{T} + \epsilon \right) \sum_{i=1}^{2m_j} |w_i|^2
\]

since \( \epsilon' = \min \left( \frac{T}{T}, 1 \right) \). This shows the second statement of the proposition.

To show that \((P^{-1}(t))^*(S^{-1})^*S^{-1}P^{-1}(t)\) has real entries, note that

\[
P^{-1}(t) = e^{Bt} \Phi^{-1}(t)
\]
\[
= Se^{A}S^{-1} \Phi^{-1}(t)
\]

so that

\[
(P^{-1}(t))^*(S^{-1})^*S^{-1}P^{-1}(t) = (\Phi^{-1}(t))^*(S^{-1})^*e^{A}s^{-1} \Phi^{-1}(t)
\]

It is thus sufficient to show that \((e^{A})^*e^{A}\) is real-valued, since all other matrices are real-valued. Note that since \(A = \text{blockdiag}(K_1, \ldots, K_r)\), we have

\[
e^{A} = \text{blockdiag}(e^{tK_1}, \ldots, e^{tK_r})
\]
\[
(e^{tA})^*e^{tA} = \text{blockdiag}((e^{tK_1})^*e^{tK_1}, \ldots, (e^{tK_r})^*e^{tK_r})
\]

and the blocks where \(K_j\) have only real entries are trivially real-valued (cases 1 and 3). In case 2, \(K_j = \frac{1}{T}((i\pi + \ln |\lambda_j|)I + N')\), where \(N' \in \mathbb{R}^{m_j \times m_j}\) is a nilpotent, upper triangular matrix. Then, noting that \(I\) and \(N'\) commute,

\[
e^{tK_j} = e^{\frac{t}{T}(i\pi + \ln |\lambda_j|)I} \exp \left( \frac{t}{T}N' \right)
\]
\[
(e^{tK_j})^* = e^{\frac{t}{T}(-i\pi + \ln |\lambda_j|)I} \exp \left( \frac{t}{T}(N')^T \right)
\]
\[
(e^{tK_j})^*e^{tK_j} = e^{\frac{2t}{T} \ln |\lambda_j|} \exp \left( \frac{t}{T}(N')^T \right) \exp \left( \frac{t}{T}N' \right)
\]

which has real entries. \(\square\)

**Corollary 2.3** Consider the ODE \(\dot{x} = f(x)\) with \(f \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n), \sigma \geq 2\) and let \(S_tq\) be an exponentially stable periodic solution with period \(T\) and \(q \in \mathbb{R}^n\). Then the first variation equation \(\dot{y} = Df(S_tq)y\) is of the form as in the previous proposition with \(s = \sigma - 1\); \(I\) is a single eigenvalue of \(\Phi(T)\) with eigenvector \(f(q)\) and all other
eigenvalues of $\Phi(T)$ satisfy $|\lambda| < 1$. More precisely, if $-\nu < 0$ is the maximal real part of all non-trivial Floquet exponents, we have $\frac{\ln|\lambda|}{T} \leq -\nu$. With the notations of Proposition 2.2 we can assume that $\lambda_1 = 1$ and $Se_1 = f(q)$.

Then we have for all $\epsilon > 0$

$$f(S_t q) = P(t)Se_1 \text{ for all } t \in \mathbb{R}$$

and $\frac{1}{2}w^*(A^* + A)w \leq (-\nu + \epsilon)(\|w\|^2 - |w_1|^2)$.

for all $w \in \mathbb{C}^n$, where $\|w\| = \sqrt{w^*w}$.

**Proof:** Since $f(S_t q)$ solves (2.1), we have $f(S_t q) = P(t)e^{Bt}f(q)$ and, in particular for $t = T$, $f(q) = f(S_T q) = e^{BT}f(q)$. Hence,

$$f(S_t q) = P(t)Se^{At}S^{-1}f(q) = P(t)Se^{At}e_1 = P(t)Se_1$$

since $K_1 = 0$ in the definition of $A$. Proposition 2.2 shows the result taking $\lambda_1 = 1$ and $m_1 = 1$ into account. \hfill \Box

### 3 Converse theorem

We will prove Theorem 1.3 showing that a contraction metric exists for an exponentially stable periodic orbit in the whole basin of attraction. Moreover, we can achieve the bound $-\nu + \epsilon$ for $L_M$ for any fixed $\epsilon > 0$, where $-\nu$ denotes the largest real part of all non-trivial Floquet exponents.

Note that we consider contraction in directions $v$ perpendicular to $f(x)$ with respect to the metric $M$, i.e. $v^TM(x)f(x) = 0$. One could alternatively consider directions perpendicular to $f(x)$ with respect to the Euclidean metric, i.e. $v^Tf(x) = 0$, but then the function $L_M$ needs to reflect this, see [5, 1].

In the proof we will first construct $M = M_0$ on the periodic orbit $\Omega$ using Floquet theory. Then, we define a projection $\pi$ of points in a neighborhood $U$ of $\Omega$ onto $\Omega$ such that $(x - \pi(x))^TM_0(\pi(x))f(\pi(x)) = 0$, which will be used to synchronize the time of solutions such that $\pi(S_\tau x) = S_{\theta_\tau}(\pi x)$. Finally, $M$ will be defined through a scalar-valued function $V$ by $M(x) = M_1(x)e^{2V(x)}$, where $M_1 = M_0$ on the periodic orbit.

**Proof:** [of Theorem 1.3] Note that we assume $f \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$ to achieve more detailed results concerning the smoothness and assume lower bounds on $\sigma$ as appropriate for each result; we always assume $\sigma \geq 2$. 

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I. Definition and properties of $M_0$ on $\Omega$

We fix a point $q \in \Omega$ and consider the first variation equation

$$\dot{y} = Df(S_t q) y$$

(3.1)

which is a $T$-periodic, linear equation for $y$, and $Df \in C^{\sigma-1}$. By Proposition 2.2 and Corollary 2.3 the principal fundamental matrix solution $\Phi \in C^{\sigma-1}(\mathbb{R}, \mathbb{R}^{n \times n})$ of (3.1) with $\Phi(0) = I$ can be written as

$$\Phi(t) = P(S_t q) e^{Bt},$$

where $B \in \mathbb{C}^{n \times n}$; note that $P \in C^{\sigma-1}(\mathbb{R}, \mathbb{C}^{n \times n})$ can be defined on the periodic orbit as it is $T$-periodic. By the assumptions on $\Omega$, the eigenvalues of $B$ are $0$ with algebraic multiplicity one and the others have a real part $\leq -\nu < 0$.

We define $S$ as in Proposition 2.2 and define the $C^{\sigma-1}$-function

$$M_0(S_t q) = P^{-1}(S_t q)^* (S^{-1})^* S^{-1} P^{-1}(S_t q) \in \mathbb{R}^{n \times n}. \quad (3.2)$$

Note that $M_0(S_t q)$ is real by Proposition 2.2, symmetric, since it is Hermitian and real, and positive definite by

$$v^T M_0(S_t q) v = \| S^{-1} P^{-1}(S_t q) v \|^2$$

for all $v \in \mathbb{R}^n$.

(3.3)

and since $S^{-1} P^{-1}(S_t q)$ is non-singular.

We will now calculate $L_{M_0} (S_t q; v)$. First, we have for the orbital derivative

$$M'_0(S_t q) = (P^{-1}(S_t q)^* (S^{-1})^* S^{-1} P^{-1}(S_t q) + P^{-1}(S_t q)^* (S^{-1})^* S^{-1} (P^{-1}(S_t q))^').$$

Furthermore, by using $(P^{-1}(S_t q) P(S_t q))^' = 0$, we obtain

$$(P^{-1}(S_t q))^' = -P^{-1}(S_t q) (P(S_t q))^' P^{-1}(S_t q).$$

In addition, since $t \mapsto P(S_t q)e^{Bt}$ is a solution of (3.1), we have $(P(S_t q))^' = Df(S_t q) P(S_t q) - P(S_t q) B$. Altogether, we get

$$(P^{-1}(S_t q))^' = -P^{-1}(S_t q) Df(S_t q) + BP^{-1}(S_t q). \quad (3.4)$$

Hence,

$$M'_0(S_t q) = -Df(S_t q)^T P^{-1}(S_t q)^* (S^{-1})^* S^{-1} P^{-1}(S_t q)$$

$$+ P^{-1}(S_t q)^* B^* (S^{-1})^* S^{-1} P^{-1}(S_t q)$$

$$- P^{-1}(S_t q)^* (S^{-1})^* S^{-1} P^{-1}(S_t q) Df(S_t q)$$

$$+ P^{-1}(S_t q)^* (S^{-1})^* S^{-1} B P^{-1}(S_t q).$$

Thus, we obtain

$$M_0(S_t q) Df(S_t q) + Df(S_t q)^T M_0(S_t q) + M'_0(S_t q)$$

$$= P^{-1}(S_t q)^* B^* (S^{-1})^* S^{-1} P^{-1}(S_t q) + P^{-1}(S_t q)^* (S^{-1})^* S^{-1} B P^{-1}(S_t q).$$
Furthermore, we have for $\mathbf{v} \in \mathbb{R}^n$

$$L_{M_0}(S_t\mathbf{q};\mathbf{v}) = \frac{1}{2} \mathbf{v}^T M_0(S_t\mathbf{q}) Df(S_t\mathbf{q}) + Df(S_t\mathbf{q})^T M_0(S_t\mathbf{q}) + M_0(S_t\mathbf{q}) \mathbf{v}$$

$$= \mathbf{v}^T P^{-1}(S_t\mathbf{q})^* (S^{-1})^* \left( \frac{1}{2} (S^* B^* (S^{-1})^* + S^{-1} BS) \right) S^{-1} P^{-1}(S_t\mathbf{q}) \mathbf{v}$$

$$= \mathbf{w}^* \left( \frac{1}{2} (A^* + A) \right) \mathbf{w},$$

(3.5)

where $\mathbf{w} := S^{-1} P^{-1}(S_t\mathbf{q}) \mathbf{v} \in \mathbb{C}^n$ and $A = S^{-1} BS$.

For $\mathbf{v} \in \mathbb{R}^n$ with $\mathbf{v}^T M_0(S_t\mathbf{q}) \mathbf{v} = 1$ and $f(S_t\mathbf{q})^T M_0(S_t\mathbf{q}) \mathbf{v} = 0$ we have

$$\mathbf{w}^* \mathbf{w} = \mathbf{v}^T P^{-1}(S_t\mathbf{q})^* (S^{-1})^* S^{-1} P^{-1}(S_t\mathbf{q}) \mathbf{v}$$

$$= \mathbf{v}^T M_0(S_t\mathbf{q}) \mathbf{v}$$

$$= 1$$

and, using $\mathbf{e}_1 = S^{-1} P^{-1}(S_t\mathbf{q}) f(S_t\mathbf{q})$ from Corollary 2.3

$$w_1 = \mathbf{e}_1^* \mathbf{w}$$

$$= f(S_t\mathbf{q})^T P^{-1}(S_t\mathbf{q})^* (S^{-1})^* S^{-1} P^{-1}(S_t\mathbf{q}) \mathbf{v}$$

$$= f(S_t\mathbf{q})^T M_0(S_t\mathbf{q}) \mathbf{v}$$

$$= 0.$$

This shows with Corollary 2.3 and (3.5)

$$L_{M_0}(S_t\mathbf{q}) = \mathbf{v}^T M_0(S_t\mathbf{q}) \mathbf{v} = 1, \mathbf{v}^T M_0(S_t\mathbf{q}) f(S_t\mathbf{q}) = 0$$

$$\leq \max_{\mathbf{w} \in \mathbb{C}^n, |w_1| = 0, \|\mathbf{w}\| = 1} (-\nu + \epsilon)(\|\mathbf{w}\|^2 - |w_1|^2)$$

$$\leq -\nu + \epsilon.$$  

(3.6)

II. Projection

Fix a point $\mathbf{q} \in \Omega$ on the periodic orbit. For $\mathbf{x}$ near the periodic orbit we define the projection $\pi(\mathbf{x}) = S_0\mathbf{q}$ on the periodic orbit orthogonal to $f(S_0\mathbf{q})$ with respect to the scalar product $\langle \mathbf{v}, \mathbf{w} \rangle_{M_0(S_0\mathbf{q})} = \mathbf{v}^T M_0(S_0\mathbf{q}) \mathbf{w}$ implicitly by (3.7) below. The following lemma is based on the implicit function theorem and shows that the projection can be defined in a neighborhood of the periodic orbit, not just locally.

**Lemma 3.1** Let $\Omega$ be an exponentially stable periodic orbit of $\dot{\mathbf{x}} = f(\mathbf{x})$ where $f \in C^{\sigma}(\mathbb{R}^n, \mathbb{R}^n)$ with $\sigma \geq 2$.

Then there is a compact, positively invariant neighborhood $U$ of $\Omega$ with $U \subset A(\Omega)$ and a function $\pi \in C^{\sigma-1}(U, \Omega)$ such that $\pi(\mathbf{x}) = \mathbf{x}$ if and only if $\mathbf{x} \in \Omega$. Moreover, for all $\mathbf{x} \in U$ we have

$$(\mathbf{x} - \pi(\mathbf{x}))^T M_0(\pi(\mathbf{x})) f(\pi(\mathbf{x})) = 0.$$  

(3.7)
Proof: Fix a point $q \in \Omega$ and define $M_0$ by (3.2). Define the $C$ function

$$G(x, \theta) = (x - S_\theta q)^TM_0(S_\theta q)f(S_\theta q)$$

for $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}$.

Define the following constants:

$$\begin{align*}
\min_{p \in \Omega} \|f(p)\| &= c_1 > 0 \\
\max_{p \in \Omega} \|f(p)\| &= c_2 \\
\max_{p \in \Omega} \|Df(p)\| &= c_3 \\
\max_{p \in \Omega} \|P(p)\| &= p_1 \\
\max_{p \in \Omega} \|P^{-1}(p)\| &= p_2 \\
\min_{p \in \Omega} \|M_0(p)\| &= m_1 > 0 \\
\max_{p \in \Omega} \|M_0(p)\| &= m_2 \\
\max_{p \in \Omega} \|M'_0(p)\| &= m_3,
\end{align*}$$

with the matrix norm $\|\cdot\| = \|\cdot\|_2$, which is induced by the vector norm $\|\cdot\| = \|\cdot\|_2$ and is sub-multiplicative. We will first prove the following quantitative version of the local implicit function theorem, using that $\theta$ is one-dimensional.

**Lemma 3.2** There are constants $\delta, \epsilon > 0$ such that for each point $x_0 = S_{\theta_0} q \in \Omega$, there is a function $p_{x_0} \in C^{\sigma-1}(B_\delta(x_0), B_\epsilon(\theta_0))$ such that for all $(x, \theta) \in B_\delta(x_0) \times B_\epsilon(\theta_0)$

$$G(x, \theta) = 0 \iff \theta = p_{x_0}(x).$$

If $x \in B_\delta/2(x_0)$, then $p_{x_0}(x) \in B_\epsilon/2(\theta_0)$.

**Proof:** We have

$$G_{\theta}(x, \theta) = \frac{d}{d\theta}(x - S_\theta q)^TM_0(S_\theta q)f(S_\theta q)$$

$$= -f(S_\theta q)^TM_0(S_\theta q)f(S_\theta q)$$

$$+ (x - S_\theta q)^TM'_0(S_\theta q)f(S_\theta q)$$

$$+ (x - S_\theta q)^TM_0(S_\theta q)Df(S_\theta q)f(S_\theta q).$$

With $\min_{\theta \in [0, T]} f(S_\theta q)^TM_0(S_\theta q)f(S_\theta q) \geq c_1^2 m_1 > 0$ we have for all $\|x - S_\theta q\| < \delta_2 := \frac{c_1^2 m_1}{2c_2(m_3 + m_1 c_3)}$

$$G_{\theta}(x, \theta) < -c_1^2 m_1 + \delta_2 c_2 (m_3 + m_1 c_3) = -\frac{c_1^2 m_1}{2} < 0.$$
Let $\delta_1 := \frac{\delta_2}{2}$ and $\epsilon_1 := \frac{\delta_2}{2c_2}$. For any $x_0 = S_{\theta_0}q \in \Omega$ we have for all $x \in \mathbb{R}^n$ with $\|x - x_0\| < \delta_1$ and all $\theta \in \mathbb{R}$ with $|\theta - \theta_0| < \epsilon_1$

$$G(\theta, x) < -\frac{c_1^2m_1}{2} < 0 \quad (3.8)$$

since

$$\|x - S_{\theta}q\| \leq \|x - x_0\| + \|S_{\theta_0}q - S_{\theta}q\| < \delta_1 + |\theta_0 - \theta|c_2 < \delta_2.$$

Since $G(x_0, \theta_0) = 0$ we have with $\epsilon := \epsilon_1/2$ by (3.8)

$$G(x_0, \theta_0 + \epsilon) < -\frac{c_1^2m_1}{2}\epsilon,$$

$$G(x_0, \theta_0 - \epsilon) > \frac{c_1^2m_1}{2}\epsilon.$$

Furthermore, we have

$$\nabla_x G(x, \theta) = f(S_{\theta}q)^T M_0(S_{\theta}q),$$

$$\|\nabla_x G(x, \theta)\| \leq c_2m_1$$

for all $x \in \mathbb{R}^n$ and $\theta \in \mathbb{R}$.

Define $\delta := \min\left(\delta_1, \frac{c_1^2m_1}{2c_2m_1}\epsilon\right)$. For $\|x - x_0\| < \delta$ we have

$$G(x, \theta_0 + \epsilon) \leq G(x_0, \theta_0 + \epsilon) + \int_0^1 \nabla_x G(x_0 + \lambda(x - x_0), \theta_0 + \epsilon) \, d\lambda \cdot (x - x_0)$$

$$< -\frac{c_1^2m_1}{2}\epsilon + c_2m_1\delta$$

$$\leq -\frac{c_1^2m_1}{4}\epsilon < 0$$

and

$$G(x, \theta_0 - \epsilon) > \frac{c_1^2m_1}{4}\epsilon > 0.$$

Since $G(x, \theta)$ is strictly decreasing with respect to $\theta$ in $B_\epsilon(\theta_0)$ by (3.8), the intermediate value theorem implies that there is a unique $\theta^* \in (\theta_0 - \epsilon, \theta_0 + \epsilon)$ such that $G(x, \theta^*) = 0$, which defines a function $p_{x_0}(x) = \theta^*$. The statement for $\epsilon/2$ and $\delta/2$ follows similarly. The smoothness of $p_{x_0}$ follows by the classical implicit function theorem, since $G \in C^{\sigma - 1}$.

Now we want to show the uniqueness of the function in a suitable neighborhood $\tilde{U}$ of $\Omega$. Denote the minimal period of the periodic orbit by $T$; we can assume that $\epsilon < T$. Define

$$c := \min_{p \in \Omega} \min_{\theta \in [-T/2, T/2]} \| S_{\theta}p - p \| > 0.$$ 

We can conclude that if $\|S_{\theta}p - p\| \leq c/2$ with $p \in \Omega$ and $|\theta| \leq T/2$, then $|\theta| < \epsilon/2$. 

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Let \( \delta' = \min(\delta/2, c/4) \). Since \( \Omega \) is compact and \( \Omega \subset \bigcup_{x_0 \in \Omega} B_{\delta'}(x_0) \), there is a finite number of \( x_i = S_{\theta_i}q \in \Omega, \ i = 1 \ldots, N \), with
\[
\Omega \subset \bigcup_{i=1}^{N} B_{\delta'}(x_i) =: \tilde{U},
\]
such that \( \tilde{U} \) is an open neighborhood of \( \Omega \). We want to show that the \( p_{x_i} = p_i \) define a unique function \( p : \tilde{U} \to S^1_{\theta} \), where \( S^1_{\theta} \) are the reals modulo \( T \) such that \( p = p_i \) on \( B_{\delta'}(x_i) \). We need to show that if \( x \in B_{\delta'}(x_i) \cap B_{\delta'}(x_j) \), then \( p_i(x) = p_j(x) \).

Let \( x \in B_{\delta/2}(x_i) \cap B_{\delta/2}(x_j) \) and, without loss of generality, \( |\theta_j - \theta_i| < T/2 \) since the \( \theta_i \) and \( \theta_j \) are uniquely defined only modulo \( T \). Then
\[
|x_i - S_{\theta_j - \theta_i}x_i| = |x_i - x_j| 
\leq |x_i - x| + |x - x_j| 
< 2\delta' \leq \min(\delta, c/2).
\]
Hence, \( |\theta_j - \theta_i| < \epsilon/2 \).

Since \( x \in B_{\delta/2}(x_i) \cap B_{\delta/2}(x_j) \), we have \( p_i(x) \in B_{\epsilon/2}(\theta_i) \) and \( p_j(x) \in B_{\epsilon/2}(\theta_j) \) by Lemma 3.2. Then
\[
|p_i(x) - \theta_j| \leq |p_i(x) - \theta_i| + |\theta_i - \theta_j| < \epsilon
\]
and similarly \( p_j(x) \in B_{\epsilon/2}(\theta_i) \). Moreover, \( x \in B_{\delta}(x_j) \cap B_{\delta}(x_j) \). Lemma 3.2 implies that \( \theta = p_i(x) \) if and only if \( G(x, \theta) = 0 \) if and only if \( \theta = p_j(x) \), which shows \( p_i(x) = p_j(x) \).

Since \( \Omega \) is stable, we can choose \( \Omega \subset U^0 \subset U \subset \tilde{U} \) such that \( U \) is compact and positively invariant. For \( x \in U \) define \( \pi(x) = S_{p(x)}q \). Since \( p \) is defined by \( p_{x_i} \), we have by Lemma 3.2 \( 0 = G(x, p(x)) = (x - \pi(x))^T M_0(\pi(x)) f(\pi(x)) \).

If \( x = S_\theta q \in \Omega \), then there is a \( x_i = S_{\theta_i}q \in \Omega \) by (3.9) such that \( x \in B_{\delta'}(x_i) \) and thus, as above, \( |\theta - \theta_i| < \epsilon/2 \). Hence, by Lemma 3.2, as \( x \in B_{\delta}(x_i) \) and \( \theta \in B_{\epsilon}(\theta_i) \), \( p_i(x) = \theta \) and thus \( \pi(x) = x \), as this satisfies \( 0 = G(x, \theta) \). If \( x \notin \Omega \), then, since \( \pi(x) \in \Omega \), \( x \neq \pi(x) \). This shows the lemma. \( \square \)

### III. Synchronization

In this step we synchronize the time between the solution \( S_t x \) and the solution on the periodic orbit \( S_\theta \pi(x) \) such that (3.10) holds. This will enable us to later define a distance between \( S_t x \) and \( \Omega \) in Step IV.

**Definition 3.3** For \( x \in U \) we can define \( \theta_x \in C^{\sigma-1}(\mathbb{R}_{0}^{+}, \mathbb{R}) \) by \( \theta_x(0) = 0 \) and
\[
S_{\theta_x(t)}(x) = \pi(S_t x)
\]
for all \( t \geq 0 \).
We have
\[
\dot{x}(t) = \left( f(S_t x)^T M_0(S_{\theta(t)} \pi(x)) f(S_{\theta(t)} \pi(x)) \right)
\left[ f(S_{\theta(t)} \pi(x))^T M_0(S_{\theta(t)} \pi(x)) f(S_{\theta(t)} \pi(x)) \right]^{-1} \left[ - (S_t x - S_{\theta(t)} \pi(x))^T M_0(S_{\theta(t)} \pi(x)) f(S_{\theta(t)} \pi(x)) \right.
\]
\[+ M_0(S_{\theta(t)} \pi(x)) Df(S_{\theta(t)} \pi(x)) f(S_{\theta(t)} \pi(x)) \right].
\] (3.11)

The denominator of (3.11) is strictly positive for all \( t \geq 0 \) and \( x \in U \).

**Proof:** Denote \( \pi(x) =: p \in \Omega \). Observe, that both sides of (3.11) equal for \( t = 0 \).

For any \( t \geq 0 \), \( S_t x \in U \) and \( \pi(S_t x) \) denotes a point on the periodic orbit, so we can write it as \( \pi(S_t x) = S_{\theta(t)} p \). Note that \( \theta_x(t) \) is only uniquely defined modulo \( T \), however, it is uniquely defined by the requirement that \( \theta_x \) is a continuous function.

By (3.7), we have
\[
(S_t x - S_{\theta(t)} p)^T M_0(S_{\theta(t)} p) f(S_{\theta(t)} p) = 0.
\]

Hence, \( \theta_x(t) \) is implicitly defined by
\[
Q(t, \theta) = (S_t x - S_{\theta} p)^T M_0(S_{\theta} p) f(S_{\theta} p) = 0.
\] (3.12)

Note that \( \theta_x \in C^{s-1}(\mathbb{R}_{\geq 0}, \mathbb{R}) \) by the Implicit Function Theorem which implies
\[
\frac{d\theta_x}{dt} = - \frac{\partial Q}{\partial \theta} \bigg|_{\theta = \theta_x(t)} = (f(S_t x)^T M_0(S_{\theta(t)} p) f(S_{\theta(t)} p))
\left[ f(S_{\theta(t)} p)^T M_0(S_{\theta(t)} p) f(S_{\theta(t)} p) \right]^{-1} \left[ -(S_t x - S_{\theta(t)} p)^T M_0(S_{\theta(t)} p) f(S_{\theta(t)} p) \right.
\]
\[+ M_0(S_{\theta(t)} p) Df(S_{\theta(t)} p) f(S_{\theta(t)} p) \right].
\]

With the notations of the proof of Lemma 3.1 for \( S_t x \in U \) there is a point \( x_i = S_{\theta_i} q \in \Omega \) such that \( S_i x \in B_{\delta'}(x_i) \). We have \( S_t x \in B_{\delta}(x_i) \) and, modulo \( T \), we have \( p_i(S_t x) = \theta_x(t) \in B_{\epsilon}(\theta_i) \). Hence, the denominator is \( > \frac{c^2 \delta_1}{2} \) by (3.3). \( \square \)

**Lemma 3.4** For \( x \in U \) we have
\[
S_{\theta_{x}(t)} \pi(S_{\tau} x) = S_{\theta_{x}(t+\tau)} \pi(x)
\]
for all \( t, \tau \geq 0 \).
Proof: We apply (3.10) to the point \( S^\tau x \) and the time \( t \), obtaining
\[
S_{\theta_{S^\tau x}(t)} \pi(S^\tau x) = \pi(S_t S^\tau x).
\]
Now we apply (3.10) to the point \( x \) and the time \( t + \tau \), obtaining
\[
S_{\theta_{x}(t+\tau)} \pi(x) = \pi(S_{t+\tau} x).
\]
As both right-hand sides are equal by the semi-flow property, this proves the statement. \( \square \)

IV. Distance to the periodic orbit

In the following lemma we define a distance of points in \( U \) to the periodic orbit, and we show that it decreases exponentially.

Lemma 3.5 Let \( \epsilon < \min(1, \nu/2) \) and \( \sigma \geq 2 \). Then there is a positively invariant, compact neighborhood \( U \) of the periodic orbit \( \Omega \) such that the function \( d \in C^{\sigma-1}(U, \mathbb{R}_+^0) \), defined by
\[
d(x) = (x - \pi(x))^T M_0(\pi(x))(x - \pi(x))
\]
satisfies \( d(x) = 0 \) if and only if \( x \in \Omega \). Moreover, \( d'(x) < 0 \) for all \( x \in U \setminus \Omega \) and
\[
d(S_t x) \leq e^{2(-\nu+2\epsilon)t} d(x) \quad \text{for all } x \in U \text{ and all } t \geq 0,
\]
\[
1 - \epsilon \leq \dot{\theta}_x(t) \leq 1 + \epsilon \quad \text{for all } t \geq 0.
\]

Proof: Note that \( d \) is \( C^{\sigma-1} \) as all of its terms are. As \( M_0(x) \) is positive definite, \( d(x) = 0 \) if and only if \( x = \pi(x) \), i.e. \( x \in \Omega \) by Lemma 3.1. Define
\[
c^* := \frac{\epsilon}{2 p_1 p_2 \|S^{-1}\| \|S\|} > 0, \quad (3.13)
\]
\[
c_4 = \frac{2 c_2 c_3 m_2 + m_3 + c^* m_2}{c_1^2 m_1}, \quad (3.14)
\]
\[
v^* := \frac{\epsilon}{2 p_1 p_2 \|S^{-1}\| \|S\| c_4 (c_3 + \|B\|)}. \quad (3.15)
\]
where the constants were defined in Step II, proof of Lemma 3.1.

For \( y \in U \) we use the Taylor expansion around \( \pi(y) \in \Omega \). Hence, there is a function \( \psi(y) \) satisfying
\[
f(y) = f(\pi(y)) + Df(\pi(y))(y - \pi(y)) + \psi(y) \quad (3.16)
\]
with \( \|\psi(y)\| \leq c^* \|y - \pi(y)\| \) for all \( y \in U \), noting that \( \Omega \) is compact, where we choose \( U \) still to be a positively invariant, compact neighborhood of \( \Omega \), possibly smaller than before and such that also have
\[
\|y - \pi(y)\| \leq \delta' = \min \left( \delta^*, \frac{c_1^2 m_1}{2 c_2 (m_3 + m_2 c_3)}, \frac{\epsilon}{c_4}, 1 \right) \quad \text{for all } y \in U. \quad (3.17)
\]
Recall that, due to the definition of $M_0$ and (3.10) we have
\[
\begin{align*}
\frac{d}{dt} x &= (x - \pi(x))^T (P^{-1}(\pi(x)))^*(S^{-1})^* S^{-1}(\pi(x)) (x - \pi(x)) \\
\frac{d}{dt} Sx &= (S_x x - S_{\theta(t)}(\pi(x)))^T (P^{-1}(S_{\theta(t)}(\pi(x))))^*(S^{-1})^* \\
S^{-1}P^{-1}(S_{\theta(t)}(\pi(x)))(S_x x - S_{\theta(t)}(\pi(x))).
\end{align*}
\]

Now let us calculate the orbital derivative, denoting $\theta(t) := \theta_0(t)$.
\[
\begin{align*}
d'(Sx) &= \left[ \frac{d}{dt} \frac{d}{dt} (P^{-1}(S_{\theta(t)}(\pi(x)))) (S_x x - S_{\theta(t)}(\pi(x))) \\
&+ P^{-1}(S_{\theta(t)}(\pi(x)))[f(S_x x) - \dot{\theta}(t)f(S_{\theta(t)}(\pi(x)))] \\
&+ (S_{\theta(t)}(\pi(x)))^T (P^{-1}(S_{\theta(t)}(\pi(x))))^*(S^{-1})^* \\
&\left[ \frac{d}{dt} \frac{d}{dt} (P^{-1}(S_{\theta(t)}(\pi(x)))) (S_x x - S_{\theta(t)}(\pi(x))) \\
&+ P^{-1}(S_{\theta(t)}(\pi(x)))[f(S_x x) - \dot{\theta}(t)f(S_{\theta(t)}(\pi(x)))] \right].
\end{align*}
\]

We denote $p := \pi(x)$ and $v(t) := S_x x - S_{\theta(t)}(\pi(x)) = S_x x - \pi(S_tx)$ by (3.10). Hence, using (3.17) for $y = S_t x \in U$ since $x \in U$, which is positively invariant, we have
\[
\|v(t)\| \leq \delta' = \min\left( v^*, \frac{c_1 m_1}{2 c_2 [m_3 + 2 c_3]}, \frac{\epsilon}{c_4}, 1 \right) \tag{3.18}
\]
for all $t \geq 0$. We have $\frac{\frac{d}{dt} (P^{-1}(S_{\theta(t)}(\pi(x))))}{\frac{d}{dt} (P^{-1}(S_{\theta(t)}(\pi(x))))} = \dot{\theta}(t)(-P^{-1}(S_{\theta(t)}(\pi(x)))Df(S_{\theta(t)}(\pi(x))) + BP^{-1}(S_{\theta(t)}(\pi(x))))$ by (3.11). Thus,
\[
\begin{align*}
d'(Sx) &= \left[ \dot{\theta}(t)(-P^{-1}(S_{\theta(t)}(\pi(x)))Df(S_{\theta(t)}(\pi(x))) + BP^{-1}(S_{\theta(t)}(\pi(x))))v(t) \\
&+ P^{-1}(S_{\theta(t)}(\pi(x)))[f(S_x x) - \dot{\theta}(t)f(S_{\theta(t)}(\pi(x)))] \\
&+ (S_{\theta(t)}(\pi(x)))^T (P^{-1}(S_{\theta(t)}(\pi(x))))^*(S^{-1})^* \\
&\left[ \dot{\theta}(t)(-P^{-1}(S_{\theta(t)}(\pi(x)))Df(S_{\theta(t)}(\pi(x))) + BP^{-1}(S_{\theta(t)}(\pi(x))))v(t) \\
&+ P^{-1}(S_{\theta(t)}(\pi(x)))[f(S_x x) - \dot{\theta}(t)f(S_{\theta(t)}(\pi(x)))] \right]. \tag{3.19}
\end{align*}
\]

Using the Taylor expansion (3.16) for $y = S_t x$, we obtain with $\pi(S_t x) = S_{\theta(t)}(\pi(x))$,
\[
\begin{align*}
f(S_t x) &= f(S_{\theta(t)}(\pi(x))) + Df(S_{\theta(t)}(\pi(x)))v(t) + \psi(S_t x) \tag{3.20}
\end{align*}
\]
and thus with (3.11)
\[ \dot{\theta}(t) - 1 = \]
\[ \begin{align*}
&= \left( f(S_t x)^T M_0(S_{\theta(t)} p) f(S_{\theta(t)} p) - f(S_{\theta(t)} p)^T M_0(S_{\theta(t)} p) f(S_{\theta(t)} p) \\
&\quad + v(t)^T M'_0(S_{\theta(t)} p) f(S_{\theta(t)} p) + v(t)^T M_0(S_{\theta(t)} p) D f(S_{\theta(t)} p) f(S_{\theta(t)} p) \\
&- v(t)^T M_0(S_{\theta(t)} p) D f(S_{\theta(t)} p) f(S_{\theta(t)} p) \right)^{-1} \\
&= \left( v(t)^T D f(S_{\theta(t)} p)^T M_0(S_{\theta(t)} p) f(S_{\theta(t)} p) + \psi(S_t x)^T M_0(S_{\theta(t)} p) f(S_{\theta(t)} p) \\
&+ v(t)^T M'_0(S_{\theta(t)} p) f(S_{\theta(t)} p) + v(t)^T M_0(S_{\theta(t)} p) D f(S_{\theta(t)} p) f(S_{\theta(t)} p) \\
&- v(t)^T M_0(S_{\theta(t)} p) D f(S_{\theta(t)} p) f(S_{\theta(t)} p) \right)^{-1}
\end{align*} \]

which shows, using (3.13) and (3.14),
\[
|\dot{\theta}(t) - 1| \leq \left| \frac{\|v(t)\| c_2 [2c_3 m_2 + m_3] + \|\psi(S_t x)\| m_2 c_2}{c_1^2 m_1 - \|v(t)\| c_2 [m_3 + m_2 c_3]} \right| \\
\leq 2 c_2 \frac{2c_3 m_2 + m_3 + c_1^2 m_2}{c_1^2 m_1} \|v(t)\| = c_4 \|v(t)\| \leq \varepsilon.
\]

In particular, we have \( 1 - \varepsilon \leq \dot{\theta}(t) \leq 1 + \varepsilon \), which shows the existence of \( \theta(t) \) for all \( t \geq 0 \), \( \dot{\theta}(t) > 0 \) for all \( t \geq 0 \), that \( \theta(t) \) is a bijection function from \( [0, \infty) \) to \( [0, \infty) \) and \( \lim_{t \to \infty} \theta(t) = \infty \).

Hence, we have from (3.14) and (3.20)
\[
d'(S_t x) = \left[ (1 - \dot{\theta}(t)) P^{-1}(S_{\theta(t)} p) D f(S_{\theta(t)} p) v(t) + B P^{-1}(S_{\theta(t)} p) v(t) \\
- (1 - \dot{\theta}(t)) B P^{-1}(S_{\theta(t)} p) v(t) + (1 - \dot{\theta}(t)) P^{-1}(S_{\theta(t)} p) f(S_{\theta(t)} p) \\
\quad + P^{-1}(S_{\theta(t)} p) \psi(S_t x))^*(S^{-1})^* S^{-1} P^{-1}(S_{\theta(t)} p) v(t) \\
\quad + v(t)^T (P^{-1}(S_{\theta(t)} p))^*(S^{-1})^* S^{-1} \left[ (1 - \dot{\theta}(t)) P^{-1}(S_{\theta(t)} p) D f(S_{\theta(t)} p) v(t) \\
\quad + B P^{-1}(S_{\theta(t)} p) v(t) - (1 - \dot{\theta}(t)) B P^{-1}(S_{\theta(t)} p) v(t) \\
\quad + (1 - \dot{\theta}(t)) P^{-1}(S_{\theta(t)} p) f(S_{\theta(t)} p) + P^{-1}(S_{\theta(t)} p) \psi(S_t x) \right] \right] \\
\leq 2 \|S^{-1} P^{-1}(S_{\theta(t)} p) v(t)\| \|S^{-1}\| \|P^{-1}(S_{\theta(t)} p)\| \\
\quad \left[ (1 - \dot{\theta}(t)) \|D f(S_{\theta(t)} p)\| + \|B\| \|v(t)\| + \|\psi(S_t x)\| \right] \\
+ v(t)^* (P^{-1}(S_{\theta(t)} p))^* \left[ (S^{-1})^* S^{-1} B + B^*(S^{-1})^* S^{-1} \right] P^{-1}(S_{\theta(t)} p) v(t)
\]
using
\[ 0 = f(S_{\theta(t)} p)^* M_0 (S_{\theta(t)} p) v(t) = f(S_{\theta(t)} p)^* (P^{-1} (S_{\theta(t)} p))^* (S^{-1})^* S^{-1} P^{-1} (S_{\theta(t)} p) v(t) \]
by (5.12).

Setting \( w(t) = S^{-1} P (S_{\theta(t)} p)^{-1} v(t) \), we obtain, using (3.21) and (3.13),
\[
d'(S_t x) \leq 2 p_2 \| w(t) \| \| S^{-1} \| \| v(t) \| [c_4 (c_3 + \| B \|) \| v(t) \| + c^*] + w(t)^* [S^{-1} BS + S^* B^* (S^{-1})^*] w(t)
\]
\[
\leq 2 p_1 p_2 S \| S^{-1} \| [c_4 (c_3 + \| B \|) \| v(t) \| + c^*] \| w(t) \|^2 + w(t)^* [A + A^*] w(t)
\]
\[
\leq 2 \epsilon \| w(t) \|^2 + w(t)^* [A + A^*] w(t)
\]
by (3.13) and (3.18). Noting that
\[ w_1(t) = e_t^* w(t) = f(S_{\theta(t)} p)^* (P^{-1} (S_{\theta(t)} p))^* (S^{-1})^* S^{-1} P^{-1} (S_{\theta(t)} p) v(t) = 0 \]
we have with Corollary 2.3
\[ w(t)^* [A + A^*] w(t) \leq 2 (-\nu + \epsilon) \| w(t) \|^2. \]

Altogether, we have
\[
d'(S_t x) \leq \| 2 \epsilon - 2 \nu + 2 \epsilon \| w(t) \|^2
\]
\[
= 2 (-\nu + 2 \epsilon) d(S_t x),
\]
which shows \( d(S_t x) \leq e^{2(-\nu + 2 \epsilon) t} d(x) \) for all \( x \in U \) and \( t \geq 0 \).

Let us summarize the results obtained so far in the following corollary.

**Corollary 3.6** Let \( \Omega \) be an exponentially stable periodic orbit of \( \dot{x} = f(x) \) with \( f \in C^\sigma (\mathbb{R}^n, \mathbb{R}^n) \) and \( \sigma \geq 2 \), such that \( -\nu < 0 \) is the maximal real part of all non-trivial Floquet exponents.

For \( \epsilon_0 \in (0, \min(\nu, 1)) > 0 \) there is a compact, positively invariant neighborhood \( U \) of \( \Omega \) with \( U \subset \Omega^0 \) and \( U \subset A(\Omega) \), and a map \( \pi \in C^{\sigma-1} (U, \Omega) \) with \( \pi(x) = x \) if and only if \( x \in \Omega \).

Furthermore, for a fixed \( x \in U \), there is a bijective \( C^{\sigma-1} \) map \( \theta_x : [0, \infty) \to [0, \infty) \) with inverse \( t^*_x = \theta^{-1}_x \in C^{\sigma-1} ([0, \infty), [0, \infty)) \) such that \( \theta_x(0) = 0 \) and
\[ \pi(S_t x) = S_{\theta_x(t)} \pi(x) \]
for all \( t \in [0, \infty) \). We have \( \dot{\theta}_x(t) \in [1 - \epsilon_0, 1 + \epsilon_0] \) for all \( t \geq 0 \) and \( \dot{t}_x(\theta) \in [1 - \epsilon_0, 1 + \epsilon_0] \) for all \( \theta \geq 0 \).

Finally, there is a constant \( C > 0 \) such that
\[
|\dot{t}_x(\theta) - 1| \leq C e^{-\nu + \epsilon_0} \theta
\]
\[
\| S_{\theta_x(\theta)} x - S_{\theta} \pi(x) \| \leq C e^{-\nu + \epsilon_0} \theta \| x - \pi(x) \|
\]
for all \( \theta \geq 0 \) and all \( x \in U \).
PROOF: Setting $\epsilon := \frac{c_0}{2(1+\nu)} \leq \min \left( \frac{\nu}{2}, \frac{1}{2} \right) \leq \min \left( \frac{\nu}{2}, 1 \right)$, all results follow directly from Lemma 3.5 by using the inverse $t(\theta)$ of $\theta(t)$. Indeed, we have

$$|\hat{t}_x(\theta) - 1| = \left| \frac{1 - \hat{t}_x(t(\theta))}{\theta_x(t(\theta))} \right| \leq \frac{\epsilon}{1 - \epsilon} \leq 2\epsilon \leq \epsilon_0.$$  

Furthermore, we have by (3.21) and noting that $m_1||S_{t_\theta}(\theta)x - S_\theta\pi(x)||^2 \leq d(S_{t_\theta}(\theta)x) \leq m_2||S_{t_\theta}(\theta)x - S_\theta\pi(x)||^2$

$$|\hat{t}_x(\theta) - 1| \leq \left| \frac{1 - \hat{t}_x(t(\theta))}{1/2} \right| \leq 2c_4\|v(t(\theta))\| \leq \frac{2c_4}{\sqrt{m_1}}\sqrt{d(S_{t_\theta}(\theta)x)} \leq Ce^{(-\nu+2\epsilon)t(\theta)}\sqrt{d(x)} \leq Ce^{(-\nu+2\epsilon)(1-2\epsilon)} \leq Ce^{(-\nu+\epsilon\theta)} \leq Ce^{(-\nu+\epsilon\theta)} ,$$

using $t(\theta) = \int_0^\theta \hat{t}(\tau) d\tau \geq \theta(1-2\epsilon)$ and that $d(x)$ is bounded in $U$. Similarly, we can prove (3.30) from Lemma 3.5.

V. Definition of $M_1$ and $M$ in $A(\Omega)$

For all $x \in U$ we have defined the distance

$$d(x) = (x - \pi(x))^T M_0(\pi(x))(x - \pi(x))$$

in Lemma 3.5 which is $C^{\sigma-1}$. Let $\iota > 0$ be so small that the set $\Omega_{2\iota} := \{x \in U: d(x) \leq 2\iota\}$ satisfies $\Omega_{2\iota} \subset U^\circ$. Define the $C^\infty$ functions $h_1: \Omega_{\iota} \to [0, 1]$, $h_2: \Omega_{2\iota} \to [0, 1]$ such that $h_1(x) = 1$ for all $d(x) \leq 1 \iota$ and $h_1(x) = 0$ for all $d(x) \geq 2/3\iota$, and $h_2(x) = 1$ for all $d(x) \leq 1/3\iota$ and $h_2(x) = 0$ for all $d(x) \geq 2/3\iota$. Set

$$M_1(x) := \begin{cases} I & \text{if } x \not\in \Omega_{2\iota}, \\ (1 - h_2(x))I + h_2(x)M_0(\pi(x)) & \text{if } x \in \Omega_{2\iota}. \end{cases}$$

It is clear that $M_1(x)$ is positive definite for all $x \in \mathbb{R}^n$, $M_1$ is $C^{\sigma-1}$ and $M_1(\pi(x)) = M_0(\pi(x))$ for all $x \in \Omega_{2\iota}$.

We will define the Riemannian metric $M$ through $M_1$ and a scalar-valued function $V: A(\Omega) \to \mathbb{R}$, which will be defined later. Let us denote $\mu := \nu - \epsilon > 0$. The
function $V$ will be continuous and continuously orbitally differentiable and satisfy

$$V'(x) = -L_{M_1}(x) + r(x), \quad \text{where}$$

$$r(x) = \begin{cases} -\mu & \text{if } x \notin \Omega_i, \\ -\mu(1 - h_1(x)) + h_1(x)L_{M_i}(\pi(x)) & \text{if } x \in \Omega_i. \end{cases} \quad (3.25)$$

Note that $r(x) \leq -\mu$ for all $x \in \mathbb{R}^n$. Indeed, for $x \in \Omega_i$ we have $L_{M_1}(\pi(x)) = L_{M_0}(\pi(x)) \leq -\mu$ as $\pi(x) \in \Omega$, see (3.6), and thus

$$r(x) = -\mu + h_1(x)(\mu + L_{M_1}(\pi(x))) \leq -\mu.$$

Then we define

$$M(x) = e^{2V(x)}M_1(x).$$

We obtain by Lemma 2.1

$$L_M(x) = L_{M_1}(x) + V'(x) = L_{M_1}(x) - L_{M_1}(x) + r(x) \leq -\mu.$$

This shows the theorem. In the last steps we will define the function $V$ and prove the properties stated above.

**VI. Definition of $V_{loc}$**

We define $V_{loc}(x)$ for $x \in \Omega_i$. Note that $\Omega_i$ is positively invariant by Lemma 3.5 so $S_t x \in \Omega_i$ for all $t \geq 0$. We define

$$V_{loc}(x) = \int_0^\infty [L_{M_1}(S_t x) - L_{M_1}(S_{\theta_t}(\pi(x)))] dt. \quad (3.26)$$

We have $V_{loc}(x) = 0$ for all $x \in \Omega$. We will show that the $V_{loc}$ is well-defined, continuous and orbitally continuously differentiable for all $x \in \Omega_i$ and that holds for all $x \in \Omega_i/\beta$.

For $x \in U$, define

$$g_T(\tau, x) = \int_\tau^{T+\tau} [L_{M_1}(S_t x) - L_{M_1}(S_{\theta_t}(\pi(x)))] dt.$$ 

By Lemma 3.5 there is a constant $C > 0$ such that, defining $p := \pi(x) \in \Omega$,

$$\|S_t x - S_{\theta_t}(p)\| \leq C e^{-\mu_0 t} \quad (3.27)$$

for all $t \geq 0$ and all $x \in U$ with $\mu_0 := \nu - 2\epsilon > 0$; note that $S_{\theta_t}(p) = \pi(S_t x)$ by (3.10).
Now, we use Lemma A.11 and \( \sigma \geq 3 \), showing that \( L_{M_1} \) is Lipschitz-continuous on the compact set \( U \); note that \( \sigma - 1 \geq 2 \). Hence,

\[
|L_{M_1}(S_t x) - L_{M_1}(S_{\theta_\kappa(t)} \pi(x))| \leq LC_1 \| S_t x - S_{\theta_\kappa(t)} p \| \leq L C_2 e^{-\mu t}
\]

by (3.27), which is integrable over \([0, \infty)\). Hence, by Lebesgue’s dominated convergence theorem, the function \( g_T(\tau, x) \) converges point-wise for \( T \to \infty \) for all \( \tau \geq 0 \) and \( x \in U \).

Choose \( \theta_0 > 0 \) so small that \( S_{-\theta_0} \Omega_t \subset U \). We have that

\[
\frac{\partial}{\partial \tau} g_T(\tau, x)
\]

\[
= [L_{M_1}(S_{T+x}) - L_{M_1}(S_{\theta_\kappa(T+x)} \pi(x))] - (L_{M_1}(S_t x) - L_{M_1}(S_{\theta_\kappa(t)} p))
\]

\[
= [L_{M_1}(S_T x) - L_{M_1}(S_{\theta_\kappa(T)} \pi(x))] - (L_{M_1}(S_t x) - L_{M_1}(S_{\theta_\kappa(t)} p))
\]

by Lemma 3.4. For \( x \in \Omega_t \), the right-hand side converges uniformly in \( \tau \in (-\theta_0, \theta_0) \) as \( T \to \infty \) to \( -(L_{M_1}(S_t x) - L_{M_1}(S_{\theta_\kappa(t)} p)) \) by the same estimate as above. Hence, we can exchange \( \frac{d}{d\tau} \) and \( \lim_{T \to \infty} \). Altogether, we thus have for all \( x \in \Omega_t \), using Lemma 3.4

\[
V_{\text{loc}}'(x) = \left. \frac{d}{d\tau} V_{\text{loc}}(S_{\tau} x) \right|_{\tau=0}
\]

\[
= \frac{d}{d\tau} \int_0^\infty [L_{M_1}(S_{t+\tau} x) - L_{M_1}(S_{\theta_\kappa(\tau)} \pi(x))] \, dt \bigg|_{\tau=0}
\]

\[
= \frac{d}{d\tau} \lim_{T \to \infty} \int_0^T [L_{M_1}(S_{t+\tau} x) - L_{M_1}(S_{\theta_\kappa(\tau)} \pi(x))] \, dt \bigg|_{\tau=0}
\]

\[
= \frac{d}{d\tau} \lim_{T \to \infty} \int_0^{T+\tau} [L_{M_1}(S_t \pi(x)) - L_{M_1}(S_{\theta_\kappa(\tau)} \pi(x))] \, dt \bigg|_{\tau=0}
\]

\[
= \frac{d}{d\tau} \lim_{T \to \infty} g_T(\tau, x) \bigg|_{\tau=0}
\]

\[
= \lim_{T \to \infty} \frac{d}{d\tau} g_T(\tau, x) \bigg|_{\tau=0}
\]

\[
= -L_{M_1}(x) + L_{M_1}(p)
\]

and in particular, that \( V_{\text{loc}} \) is continuously orbitally differentiable. Note that \( V_{\text{loc}}'(x) = -L_{M_1}(x) + r(x) \) for all \( x \in \Omega_t/3 \).

**VII. Definition of \( V_{\text{glob}} \) in \( A(\Omega) \)**

For the global part note that \( V_{\text{loc}} \) is defined and smooth in \( \Omega_t/3 \) and we have \( V_{\text{loc}}'(x) = -L_{M_1}(x) + r(x) \) for all \( x \in \Omega_t/3 \). The global function \( V_{\text{glob}}: A(\Omega) \setminus \Omega \to \mathbb{R} \) is defined as the solution of the non-characteristic Cauchy problem

\[
\nabla V_{\text{glob}}(x) \cdot f(x) = -L_{M_1}(x) + r(x) \text{ for } x \in A(\Omega) \setminus \Omega
\]

\[
V_{\text{glob}}(x) = V(x) \text{ for } x \in \Gamma,
\]

(3.28)
where \( \Gamma = \{ x \in U \mid d(x) = \iota/3 \} \).

In particular, we can construct the solution by first defining the function \( \tau \in C^\infty(A(\Omega) \setminus \Omega, \mathbb{R}) \) implicitly by
\[
d(S_\tau x) = \iota/3.
\]
Since \( x \in A(\Omega) \setminus \Omega \), there exists a \( \tau \) satisfying the equation, and since \( d'(x) < 0 \) for all \( x \in \Gamma \), \( \tau(x) \) is unique. The function \( \tau \) is \( C^{\sigma-1} \), since \( d \) and \( S_\tau \) are. We have \( \tau'(x) = -1 \). Then the function
\[
V_{\text{glob}}(x) = \int_0^{\tau(x)} q(S_t x) \, dt + V_{\text{loc}}(S_{\tau(x)}(x))
\]
with \( q(x) := L_M(x) - r(x) \) is continuous and orbitally continuously differentiable and satisfies (3.28), noting that \( S_\tau(x)(x) = S_{\tau(S_\theta x)}(S_\theta x) \) for all \( \theta \geq 0 \). Indeed, for \( x \in \Gamma \) we have \( V_{\text{glob}}(x) = V_{\text{loc}}(x) \) and we have
\[
V'_{\text{glob}}(x) = \frac{d}{d\theta} \left( \int_0^{\tau(S_\theta x)} q(S_t x) \, dt + V(S_{\tau(S_\theta x)}(x)) \right) \bigg|_{\theta=0} = \frac{d}{d\theta} \left( \int_0^{\tau(S_\theta x)+\theta} q(S_t x) \, dt + V(S_{\tau(x)}(x)) \right) \bigg|_{\theta=0} = (q(S_{\tau(S_\theta x)+\theta} x)(\tau'(x) + 1) - q(S_\theta x)) \bigg|_{\theta=0} = -q(x)
\]
since \( \tau'(x) = -1 \).

Note that we have \( V_{\text{glob}}(x) = V_{\text{loc}}(x) \) for \( x \in \overline{\Omega_{\iota/3}} \setminus \Omega \), and hence \( V_{\text{glob}} \) can be extended to a continuous and orbitally continuously differentiable function \( V \) on \( A(\Omega) \) satisfying (3.24) by setting \( V_{\text{glob}}(x) := V_{\text{loc}}(x) = 0 \) for all \( x \in \Omega \). This proves the theorem. \( \square \)

Conclusions

In this paper we have proven a converse theorem, showing the existence of a contraction metric for an exponentially stable periodic orbit. The metric is defined in its basin of attraction and the bound on the function \( L_M \) is arbitrarily close to the true exponential rate of attraction.

A  Local Lipschitz-continuity of \( L_M \)

In the appendix we prove that the function \( L_M \) is locally Lipschitz continuous.

Lemma A.1 Let \( f \in C^2(\mathbb{R}^n, \mathbb{R}^n) \) and \( M \in C^2(\mathbb{R}^n, S^n) \) such that \( M(x) \) is positive definite for all \( x \in \mathbb{R}^n \).

Then \( L_M \) is locally Lipschitz continuous on \( D = \{ x \in \mathbb{R}^n \mid f(x) \neq 0 \} \).
Proof: For \( y \in D \) we define a projection \( P_y : \mathbb{R}^n \rightarrow \mathbb{R}^n \) onto the \((n-1)\)-dimensional space of vectors \( w \in \mathbb{R}^n \) with \( f(y)^T M \langle y \rangle w = 0 \) by

\[
P_y v = v - \frac{f(y)^T M \langle y \rangle v}{f(y)^T M \langle y \rangle f(y)} f(y)
\]

for all \( y \in D \) and all \( v \in \mathbb{R}^n \). Note that indeed

\[
f(y)^T M \langle y \rangle P_y v = f(y)^T M \langle y \rangle v - \frac{f(y)^T M \langle y \rangle v}{f(y)^T M \langle y \rangle f(y)} f(y)^T M \langle y \rangle f(y) = 0.
\]

Fix \( x \in D \) and choose a basis \( v_1 = f(x), v_2, \ldots, v_n \) of \( \mathbb{R}^n \) such that \( M \langle x \rangle v_j = 0 \) for \( i \neq j \). Choose \( \epsilon > 0 \) such that

\[
f(y)^T M \langle x \rangle f(x) \neq 0 \quad (A.1)
\]

holds for all \( y \in B_\epsilon(x) \); note that for \( y = x \) we have \( f(x)^T M \langle x \rangle f(x) \neq 0 \).

For \( y \in B_\epsilon(x) \) we define \( w_1 = f(y) \) and \( w_i = P_y v_i \) for \( i = 2, \ldots, n \). We show that \( (w_1, \ldots, w_n) \) is a basis of \( \mathbb{R}^n \).

Let us first show that \( w_i \neq 0 \) for \( i = 2, \ldots, n \). Assuming the opposite, we have

\[
v_i = \frac{f(y)^T M \langle y \rangle v_i}{f(y)^T M \langle y \rangle f(y)} f(y)
\]

0 = \( \frac{f(y)^T M \langle y \rangle v_i}{f(y)^T M \langle y \rangle f(y)} f(y) \)

multiplying by \( f(x)^T M \langle x \rangle \) from the left as \( f(x)^T M \langle x \rangle f(y) \neq 0 \) by \( (A.1) \). This, however, implies by \( (A.2) \) that \( v_i = 0 \) which is a contradiction. \( w_1 \neq 0 \) follows directly from \( (A.1) \).

We express \( f(y) = \sum_{j=1}^n \beta_j v_j \) and note that multiplying this equation by \( f(x)^T M \langle x \rangle \) from the left gives

\[
0 \neq f(x)^T M \langle x \rangle f(y) = \beta_1 f(x)^T M \langle x \rangle f(x)
\]

by \( (A.1) \), i.e. in particular \( \beta_1 \neq 0 \).

To show that the \( w_i \) form a basis, we assume \( \sum_{i=1}^n \alpha_i w_i = 0 \). Multiplying this equation by \( f(y)^T M \langle y \rangle \) from the left gives \( \alpha_1 f(y)^T M \langle y \rangle f(y) = 0 \) by the projection property, hence \( \alpha_1 = 0 \).

Hence,

\[
0 = \sum_{i=2}^n \alpha_i \left[ v_i - \frac{f(y)^T M \langle y \rangle v_i}{f(y)^T M \langle y \rangle f(y)} f(y) \right]
\]

\[
= \sum_{i=2}^n \alpha_i v_i - \sum_{i=2}^n \sum_{j=1}^n \frac{f(y)^T M \langle y \rangle v_i}{f(y)^T M \langle y \rangle f(y)} \beta_j v_j.
\]

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Using that \( v_j \) is a basis, we can conclude that the coefficient in front of \( v_1 \) is zero, namely
\[
\sum_{i=2}^{n} \frac{f(y)^T M(y)v_i}{f(y)^T M(y)f(y)} \beta_1 = 0.
\]
Since \( \beta_1 \neq 0 \), we have \( \sum_{i=2}^{n} \frac{f(y)^T M(y)v_i}{f(y)^T M(y)f(y)} = 0 \). Plugging this back in, we obtain \( \sum_{i=2}^{n} \alpha_i v_i = 0 \), which shows \( \alpha_2 = \ldots = \alpha_n = 0 \) as the \( v_i \) are linearly independent.

Now define the matrix-valued function \( Q: B_n(x) \to \mathbb{R}^{n \times n} \) by the columns
\[
Q(y) = (w_1(y), \ldots, w_n(y)).
\]
Note that \( Q \in C^2(B_n(x), \mathbb{R}^{n \times n}) \) due to the smoothness of \( f \) and \( M \), and \( Q \) is invertible for every \( y \). We have \( w^T M(y)f(y) = 0 \) if and only if \( w \in \text{span}(w_2(y), \ldots, w_n(y)) \), which in turn is equivalent to \( u \in \text{span}(e_2, \ldots, e_n) =: E_{n-1} \), where \( u = Q(y)^{-1}w \) and \( e_1, \ldots, e_n \) denotes the standard basis in \( \mathbb{R}^n \).

Now we write
\[
L_M(y) = \max_{w^T M(y)w = 1, w^T M(y)f(y) = 0} \frac{1}{2} w^T \left[ M(y)Df(y) + Df(y)^T M(y) + M'(y) \right] w
\]
\[
= \max_{u^T Q(y)^T M(y)Q(y)u = 1, u \in E_{n-1}} \frac{1}{2} u^T Q(y)^T \left[ M(y)Df(y) + Df(y)^T M(y) + M'(y) \right] Q(y)u.
\]
Denoting by \([A]_{n-1} \in \mathbb{S}^{n-1}\) the lower-right square \((n-1)\) matrix of \( A \in \mathbb{S}^n \) and with \( u = \begin{pmatrix} 0 \\ \tilde{u} \end{pmatrix} \), where \( \tilde{u} \in \mathbb{R}^{n-1} \) we get
\[
L_M(y) = \max_{\tilde{u}^T Q(y)^T M(y)Q(y)\tilde{u} = 1, \tilde{u} \in \mathbb{R}^{n-1}} \frac{1}{2} \tilde{u}^T \left[ Q(y)^T \left[ M(y)Df(y) + Df(y)^T M(y) + M'(y) \right] Q(y) \right] \tilde{u}.
\]
Now denote by \( \text{Chol}(A) \) the unique Cholesky decomposition of the symmetric, positive definite matrix \( A \in \mathbb{S}^{n-1} \), such that \( \text{Chol}(A) \) is an invertible, upper triangular matrix with \( \text{Chol}(A)^T \text{Chol}(A) = A \). Denoting \( C(y) := \text{Chol}([Q(y)^T M(y)Q(y)]_{n-1}) \in \mathbb{R}^{(n-1) \times (n-1)} \) and \( \tilde{v} = C(y)\tilde{u} \in \mathbb{R}^{n-1} \) we have
\[
L_M(y) = \max_{\|\tilde{v}\| = 1, \tilde{v} \in \mathbb{R}^{n-1}} \frac{1}{2} \tilde{v}^T \left[ C^{-1}(y) \right]^T \left[ Q(y)^T \left[ M(y)Df(y) + Df(y)^T M(y) + M'(y) \right] Q(y) \right] \tilde{v} \]
\[
= \max_{\|\tilde{v}\| = 1, \tilde{v} \in \mathbb{R}^{n-1}} \tilde{v}^T H(y)\tilde{v}
\]
\[
= \lambda_{\text{max}}(H(y))
\]
where $H(y) \in S^{n-1}$ is defined by

$$H(y) = \frac{1}{2}(C^{-1}(y))^T [Q(y)^T [M(y)Df(y) + Df(y)^T M(y) + M'(y)] Q(y)]^{-1} C^{-1}(y).$$

The function $y \rightarrow H(y)$ is continuously differentiable as the Cholesky decomposition, the inverse, the operation $[\cdot]_{n-1}$, $Q$, $M$, $Df$ and $M'$ are continuously differentiable by the assumptions. Hence, the function $H(y)$ is locally Lipschitz-continuous.

The function $\lambda_{\text{max}}$ is globally Lipschitz-continuous, hence, $L_M$ is locally Lipschitz-continuous.

\[\square\]

References

[1] V. A. Boichenko and G. A. Leonov. Lyapunov orbital exponents of autonomous systems. Vestnik Leningrad. Univ. Mat. Mekh. Astronom., 3:7–10, 123, 1988.

[2] G. Borg. A condition for the existence of orbitally stable solutions of dynamical systems, volume 153. Elander, 1960.

[3] C. Chicone. Ordinary Differential Equations with Applications. New York: Springer-Verlag, 2006.

[4] F. Forni and R. Sepulchre. A differential Lyapunov framework for Contraction Analysis. IEEE Trans. Automat. Control, 59(3):614–628, 2014.

[5] P. Giesl. On a matrix-valued PDE characterizing a contraction metric for a periodic orbit. submitted.

[6] P. Giesl. Necessary conditions for a limit cycle and its basin of attraction. Nonlinear Anal., 56:643–677, 2004.

[7] P. Giesl. Converse theorems on contraction metrics for an equilibrium. J. Math. Anal. Appl., 424:1380–1403, 2015.

[8] P. Hartman. Ordinary Differential Equations. Wiley, New York, 1964.

[9] P. Hartman and C. Olech. On global asymptotic stability of solutions of differential equations. Trans. Amer. Math. Soc., 104:154–178, 1962.

[10] A. Yu. Kravchuk, G. A. Leonov, and D. V. Ponomarenko. Criteria for strong orbital stability of trajectories of dynamical systems. I. Differentsiaalnye Uravneniya, 28(9):1507–1520, 1652, 1992.

[11] G. A. Leonov, I. M. Burkin, and A. I. Shepelyavyi. Frequency Methods in Oscillation Theory. Ser. Math. and its Appl.: Vol. 357, Kluwer, 1996.

[12] W. Lohmiller and J.-J. Slotine. On contraction analysis for non-linear systems. Automatica, 34:683–696, 1998.

[13] B. Stenström. Dynamical systems with a certain local contraction property. Math. Scand., 11:151–155, 1962.