Universal Behaviors of Speed of Sound from Holography

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We investigate the speed of sound in \((d+1)\)-dimensional field theory by studying its dual \((d+2)\)-dimensional gravity theory from gauge/gravity correspondence. Instead of the well known conformal limit \(c_s^2 \to 1/d\) at high temperature, we reveal two more universal quantities in various limits: \(c_s^2 \to (d-1)/16\pi\) at low temperature and \(c_s^2 \to (d-1)/16\pi d\) at large chemical potential.

I. INTRODUCTION

Gauge/gravity correspondence, as an useful tool to explore the field theories with strong interaction, e.g. QCD, has been widely studied during the last decade. However, one might hope to learn some qualitative lessons by searching for quantities that do not depend on the details of the particular gravity dual. Such ‘universal’ properties may apply to field theories without knowing their gravity dual. An elegant example of such universal quantity is the ratio \(\eta/s\) of shear viscosity to entropy density. This takes the value \(1/4\pi\) in all theories with gravity dual. In \([1,2]\), it is showed that the speed of sound approaches the conformal value \(c_s^2 = 1/3\) universally from below in a general class of strongly interacting \((3+1)\)-dimensional theories at high temperatures and zero chemical potential. This result is consistent with the Monte Carlo lattice QCD calculations \([3,4]\). A number of string theory examples of holographically dual theories, including both bottom-up models \([1,2,5]\) and top-down models \([6,11]\) do indeed consistently show that \(c_s^2 \leq 1/d\) case by case. The speed of sound in QCD with isospin chemical potential has been studied in \([15]\), with the conjecture that the transition from hadron to quark matter is smooth, it is showed that the speed of sound raises from 0 to some value close to 1 (speed of light), then drops to some minimal value, and then approaches 1/3 from below at large isospin chemical potential. While for baryon chemical potential, no physical systems in a deconfined phase with a speed of sound exceeding the conformal value has prompted a conjecture that this might represent a theoretical upper limit for the quantity.

In this work, we study \((d+2)\)-dimensional Einstein-Maxwell-Scalar (EMS) system, which has been widely studied as a successful class of holographic QCD models by gauge/gravity correspondence. We analytically obtained a general class of back-reacted black hole solutions and study their dual \((d+1)\)-dimensional field theories. We focused on the behaviors of the speed of sound at arbitrary temperature and baryon chemical potential. Instead of the well known conformal limit \(c_s^2 \to 1/d\) at high temperature, we reveal two more universal quantities in various limits: \(c_s^2 \to (d-1)/16\pi\) at low temperature and \(c_s^2 \to (d-1)/16\pi d\) at large chemical potential.

We briefly review the EMS system and the solutions in section II. In section III, we investigate the behavior of speed of sound in various limits. We summarize our results in section IV.

II. EINSTEIN-MAXWELL-SCALAR BACKGROUND

To study holographic QCD theory in \((d+1)\)-dimensional spacetime, we consider a \((d+2)\)-dimensional gravitational background coupled to a Maxwell field and a neutral scalar field, i.e. the Einstein-Maxwell-scalar system. In Einstein frame, the action is

\[
S = \frac{1}{16\pi G_{d+2}} \int d^{d+2}x \sqrt{-g} \left[ R - \frac{f(\phi)}{4} F^2 - \frac{1}{2} (\partial\phi)^2 - V(\phi) \right],
\]

where \(f(\phi)\) is a positive defined gauge kinematic function. The equations of motion is derived as

\[
\nabla^2 \phi = V_\phi + \frac{1}{4} f_\phi F^2,
\]

\[
\nabla_\mu [F^{\mu\nu}] = 0.
\]

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{f}{2} \left( F_{\mu\rho} F^{\rho\nu} - \frac{1}{4} g_{\mu\nu} F^2 \right) + \frac{1}{2} \left( \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial \phi)^2 - g_{\mu\nu} V \right).
\]

To study the general asymptotic AdS black hole backgrounds with spherical symmetry, we use the follow-
ing ansatz for the fields,
\[ ds^2 = \frac{e^{2A(z)}}{z^2} \left[ -g(z)dt^2 + \frac{dz^2}{g(z)} + dx^2 \right], \]
\[ \phi = \phi(z), \quad A_\mu = A_\mu(z) dt. \]

(5)

(6)

The equations of motion reduce to
\[ \phi'' + \left( \frac{g'}{g} + \frac{dw'}{2gw} \right) \phi' + \frac{A''_t f_\phi - e^{2w} V_\phi}{2g e^{2w}} = 0, \]
\[ A''_t + \left( \frac{f'}{f} + (d - 2) \frac{w'}{w} \right) A'_t = 0, \]
\[ w'' - w'^2 + \frac{\phi'^2}{2 d} = 0, \]
\[ g'' + dw' g' - \frac{f A''_t}{e^{2w}} = 0, \]
\[ w'' + dw'^2 + \frac{3g'}{2g} w' + \frac{g'' + 2e^{2w} V}{2 dw} = 0, \]

(7)

(8)

(9)

(10)

(11)

where we defined \( w(z) = A(z) - \ln z \).

Given the boundary conditions of regularity at the horizon \( z = z_H \),
\[ g(z_H) = A_t(z_H) = 0, \]
and asymptotic AdS spacetime at the boundary \( z = 0 \),
\[ g(0) = f(0) = 1, A(0) = A'(0) = 0, \]
the most general black hole solutions can be analytically obtained as
\[ \phi = \int_0^z \sqrt{2d(w'^2 - w'') dw} dy, \]
\[ A_t = \mu \frac{\int_0^{z_H} \frac{e^{2(2-d)w}}{f} dw}{\int_0^{z_H} \frac{e^{2(2-d)w}}{f} dw} = \mu - \rho z^{-1} + \cdots, \]
\[ g = 1 - \left. \frac{\int_0^z \frac{e^{-dw}}{e^{-dw}} dw}{\int_0^{z_H} \frac{e^{-dw}}{e^{-dw}} dw} \right|_{z_H} \]
\[ = \mu^2 \left[ \int_0^{z_H} e^{-dw} dy \int_0^{z_H} e^{2(2-d)w} dy \right] \frac{f}{f} \left[ \int_0^{z_H} e^{-dw} dy \right]^2 \]
\[ V = -\frac{e^{-2w}}{2} \left( 2dw'w'' + 2d^2 gw'^2 + 3dg'w' + g'' \right), \]

(14)

(15)

(16)

(17)

where \( \mu \) is chemical potential and \( \rho \) is the baryon density
\[ \rho = \frac{\mu}{(d - 1) f_0 \int_0^{z_H} e^{(2-d)w} dw dy} \]
\[ \frac{f}{f} \]

(18)

In the solution Eq. (14 17), the warped factor \( w(z) \) and the gauge kinetic function \( f(\phi) \) are two arbitrary functions. We should note that, to guarantee the scalar field \( \phi \) to be real, the warped factor \( w(z) \) need to satisfy the condition \( w'^2 \geq w'' \), which leads to \( A''(0) \leq 0 \).

The entropy density and the temperature of the black hole can be obtained from the background as
\[ s = \frac{e^{dw(z_H)}}{4}. \]
\[ T = \left| \frac{g'(z_H)}{4\pi} \right| = T_0 \left[ 1 - \frac{\mu^2 \int_0^{z_H} e^{-dw} dz \int_0^{z_H} e^{(2-d)w} dy}{\int_0^{z_H} e^{(2-d)w} dy} \right], \]

(19)

(20)

where
\[ T_0 = \frac{e^{-dw(z_H)}}{4\pi \int_0^{z_H} e^{-dw} dz}. \]

(21)

is the black hole temperature at \( \mu = 0 \). It is easy to see that the high temperature limit corresponds to \( z_H \to 0 \).

### III. SPEED OF SOUND

In grand canonical ensemble with fixed chemical potential, the squared speed of sound can be calculated as
\[ c_s^2 = \frac{s}{T \left( \frac{\partial s}{\partial \mu} \right)_\mu + \mu \left( \frac{\partial T}{\partial \mu} \right)_\mu}, \]

(22)

Plugging Eqs. (19 20) into Eq. (22), it is straightforward to obtain
\[ c_s^2 = \frac{c_{s0}^2 + a \left( 1 + c_{s0}^2 \right) \left[ 1 - b (T_0 - T) \right] c_{s0}^2}{1 + a \left( 1 + c_{s0}^2 \right)} \]

(23)

where
\[ c_{s0}^2 = \frac{d - 1}{16\pi}, \]
\[ a = \frac{(d - 1) \rho^2}{\pi T_0 f(z_H) e^{2(d-1)w(z_H)}} \geq 0, \]
\[ b = \frac{8\pi}{(d - 1) \mu \rho \ e^{-dw(z_H)}} \geq 0. \]

(24)

(25)

(26)
and
\[ c_{s0}^2 = -1 - \frac{e^{-dw(z_H)}}{dw' (z_H) \left( \int_0^{z_H^*} e^{-dw}dz \right)^2}, \] (27)
is the squared speed of sound at \( \mu = 0 \).

**Zero Chemical Potential.** First, we will study the properties for the speed of sound at zero chemical potential, i.e. \( \mu = 0 \). It is well known that the squared speed of sound in QCD approaches the conformal limit 1/3 in high temperature limit. From Eq. (27), it is easy to show that,
\[ \lim_{T \to \infty} c_{s0}^2 = \lim_{z_H \to 0} \left(-1 - \frac{e^{-dw(z_H)}}{dw' (z_H) \left( \int_0^{z_H^*} e^{-dw}dz \right)^2} \right) = \frac{1}{d}, \] (28)
is an universal quantity that is model independent, i.e. independent of the choice of the functions \( w(z) \) and \( f(z) \) in the above solution Eqs. (14-17). Eq. (28) applies to \( (d + 1) \)-dimensional QCD, which is the natural generalization of the \((3 + 1)\)-dimensional QCD.

To further investigate the behavior of the speed of sound in high temperature limit, we expand \( c_{s0}^2 \) at \( z_H = 0 \),
\[ c_{s0}^2 = \frac{1}{d} + \frac{3 (d + 1)}{d (d + 3)} A''_0 z_H^2 + O (z_H^4). \] (29)
Since \( A'' (0) \leq 0 \), the coefficient of \( z_H^2 \) in Eq. (29) is always negative. It indicates that, in high temperature limit, the squared speed of sound approaches to \( 1/d \) from below.

To obtain the bound for the speed of sound at arbitrary temperature, we calculate the extreme of speed of sound by differentiating it with respect to \( z_H \),
\[ \frac{dc_{s0}^2}{dz_H} = - \frac{e^{-dw(z_H)}}{dw' (z_H) \left( \int_0^{z_H^*} e^{-dw}dz \right)^2} \left( \frac{w'' (z_H)}{dw'^2 (z_H)} - c_{s0}^2 \right) = 0, \] (30)
which leads to
\[ c_{s0}^2 = \frac{w'' (z_H)}{dw'^2 (z_H)} \leq \frac{1}{d}, \] (31)
in which we have used the condition \( w''^2 \geq 0 \).

We therefore conclude that, at zero chemical potential \( \mu = 0 \), the squared speed of sound approaches to its maximum value, the conformal limit \( 1/d \), in high temperature limit for \((d + 1)\)-dimensional QCD. This is consistent with the recent results from lattice QCD [16].

**Finite Chemical Potential.** Next, we will study the properties for the speed of sound at finite chemical potential, i.e. \( 0 < \mu < \infty \). In high temperature limit, \( T, T_0 \to \infty, a \to 0 \). By using Eq. (23), it is easy to verify that
\[ \lim_{T \to \infty} c_s^2 = \lim_{T \to \infty} c_{s0}^2 = \frac{1}{d}. \] (32)
Thus the speed of sound in \((d + 1)\)-dimensional QCD approaches to the same universal value \( 1/d \) in high temperature limit even for finite chemical potential.

Similarly, we expand \( c_s^2 \) at \( z_H = 0 \),
\[ c_s^2 = \frac{1}{d} + \frac{3 (d + 1)}{d (d + 3)} A''_0 z_H^2 + O (z_H^4) \] (33)
Since \( A'' (0) \leq 0 \), the coefficient of \( z_H^2 \) in Eq. (33) is always negative for \( d < 1 + 16\pi \simeq 52 \). This indicates that, in high temperature limit, the speed of sound also approaches to \( 1/d \) from below for finite chemical potential.

Furthermore, it is easy to see that \( T < T_0 \) from Eq. (20), which implies, from Eq. (23), that
\[ c_s^2 \leq \frac{c_{s0}^2}{1 + a \left( 1 + c_{s0}^2 \right)} \leq \frac{c_{s0}^2}{1 + a \left( 1 + c_{s0}^2 \right)} = \max \left( c_{s0}^2, c_s^2 \right). \] (34)
In Eq. (31), we have proved \( c_{s0}^2 < 1/d \), and it is easy to show that \( c_s^2 = (d - 1)/16\pi < 1/d \) for \( d \leq 7 \). We thus conclude that, at finite chemical potential, the squared speed of sound approaches to its maximum value, the conformal limit \( 1/d \), in high temperature limit for \((d + 1)\)-dimensional QCD at least for \( d \leq 7 \).

**Large Chemical Potential.** For large chemical potential, a new phase, color-flavor-locking or color superconductivity, has been conjectured in QCD. It is thus interesting to investigate the behavior of speed of sound at large chemical potential. By using Eq. (18) and Eq. (20), we rewrite the speed of sound Eq. (23) in the following form,
\[ c_s^2 = \frac{1}{1 + a \left( 1 + c_{s0}^2 \right)} \left[ \frac{f_{s0}^H e^{-dw}dz f_{s0}^H e^{(2 - d)w}}{f_{s0}^H e^{-dw}dz f_{s0}^H e^{(2 - d)w}} \right] c_{s0}^2 \] (35)
We should be more careful to take the large chemical potential limit because that, from Eq. (20), the temperature become negative when \( \mu \) exceed a critical value, which certainly does not make sense physically. To take the large chemical potential limit at a fixed temperature, the correct process is to take the double limits of \( \mu \to \infty \) with \( z_H \to 0 \) together.
double limits, \( a \to \infty \) and the speed of sound reduces to
\[
\lim_{\mu \to -\infty} c_s^2 = \frac{c_s^2}{d} = \frac{d-1}{16\pi d}.
\] (36)

In stead of the conformal limit \( c_s^2 \to 1/d \) in high temperature limit, we find another universal quantity \( c_s^2 \to (d-1)/16\pi d \) in the limit of infinity chemical potential. Eq. (36) shows that the speed of sound does not approach to the conformal limit at large chemical potential. This implies that the theory at large chemical potential is in a new phase which is different from the phase at high temperature as people conjectured.

**Low Temperature.** We further analyze the behavior of speed of sound in low temperature limit \( T \to 0 \) which leads to \( a \to \infty \). From Eq. (23), we have
\[
\lim_{T \to 0} c_s^2 = c_s^2 = \frac{d-1}{16\pi}.
\] (37)

Remarkably, we find one more universal quantity \( c_s^2 \to (d-1)/16\pi \) in low temperature limit.

Finally, we present an explicit example of \((3 + 1)\)-dimensional holographic QCD model by taking the warped factor \( w(z) \) and gauge kinetic function \( f(\phi) \) in [17]. For \( d = 3 \), the universal quantities that we found in Eqs. (32,36,37) reduce to
\[
\begin{align*}
\text{at large } T : c_s^2 &= 1/d \to 1/3, \\
\text{at small } T : c_s^2 &= \frac{d-1}{16\pi} \to \frac{1}{8\pi}, \\
\text{at large } \mu : c_s^2 &= \frac{d-1}{16\pi d} \to \frac{1}{24\pi}.
\end{align*}
\] (38, 39, 40)

We plot the squared speed of sound v.s. temperature and chemical potential in Fig. 1 and Fig. 2 respectively. The behaviors of the sound speed in the figures perfectly match the analysis in this work.

**IV. SUMMARY**

In this work, we studied gauge/gravity correspondence by considering the \((d+2)\)-dimensional Einstein-Maxwell-Scalar system, which has been widely studied as a successful class of holographic QCD models. We analytically obtained a general class of back-reacted solutions and focused on the behaviors of speed of sound at arbitrary temperature and chemical potential in \((d+1)\)-dimensional holographic QCD. We found that, in various limits, the speed of sound approaches certain universal quantities which are not dependent on the details of the models. The universal behaviors of speed of sound we have found in this work are follows:

- \( c_s^2 \to 1/d \), as \( T \to \infty \) with fixed \( \mu \);
- \( c_s^2 \to \frac{d-1}{16\pi} \), as \( T \to 0 \) with fixed \( \mu \);
- \( c_s^2 \to \frac{d-1}{16\pi d} \), as \( \mu \to \infty \) with fixed \( T \).

We also proved that \( c_s^2 \leq 1/d \) for all temperature and chemical potentials (at least for \( d \leq 7 \)) provided that the solution of scalar \( \phi \) in Eq. (14) is real.

To investigate these universal quantities in further details is not only attractive in QCD theory, such as the new phase in large chemical potential, but also important to understand the deep structure of the gauge/gravity correspondence.
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