The problem about a symmetric convex body that is lifted from shallow water

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Abstract. We studied wave currents arising from a vertical lifting of a symmetric convex body, partially submerged in shallow water, filling a rectangular prismatic channel with a horizontal bottom. The simulation of such flows was carried out within the framework of the first approximation of the shallow water theory without taking into account the influence of the friction, the viscosity of the liquid and its surface tension. The flow of liquid in the domain adjacent to the lower surface of the body was obtained analytically, and outside this domain by numerically solving shallow water equations. We obtained the equations that determine the motion of the boundary of the contact area of the liquid with the lower surface of the body. We showed that the form of these equations is depend of the pressure spatial derivative sign at this boundary. Numerical calculations are presented that demonstrate the rise of the liquid after the body exiting the liquid.

1. Introduction

The lifting of the body from the surface of the liquid and the current arising from this are of considerable interest from both theoretical and applied points of view. Theoretical, experimental and numerical studies of lifting bodies from the surface of a deep fluid without taking into account the effect of the bottom were carried out in [1–4]. In [1] was carried out an experimental study of the rise of a glass circular disk from the surface of the water, with the purpose to explain the feline lapping process, and it was shown that basically the process of raising the liquid after the disk is determined by gravity and inertia, and the viscosity of the liquid and its surface tension has negligible effect. Entry into the water with constant negative acceleration and subsequent outflow from water was studied numerically for a wedge [2] and a parabolic contour [3]. Linear model of a flat body lifting from the surface of an infinitely deep fluid was sugested in [4]. Hydrodynamic forces calculated by the help of the model proposed in [4] for a wedge and a parabolic contour are in good agreement with the results of numerical calculations given in [2,3].

Wave flows induced by lifting a rectangular beam, partially immersed in shallow water, were studied in [6–8]. Currents arising at the first stage of the beam lifting from a channel of finite length were considered in [6] and currents arising when the rectangular beam emerged from the water filling the channel of infinite length were considered in [7,8]. In this paper, we continue the studies began in [7,8]. We have studied wave currents arising from a vertical lifting of a symmetric convex body, partially submerged in shallow water, filling a rectangular prismatic channel with a horizontal bottom. The simulation of such flows was carried out within the framework of the
first approximation of the shallow water theory [9] without taking into account the influence of the friction, the viscosity of the liquid and its surface tension.

2. Formulation of the problem
Let us assume that an infinite rectangular prismatic channel with a horizontal bottom is filled by an ideal incompressible liquid with a partly submerged symmetric convex body of length $2L$, whose width $2b$ coincides with the channel width, the end faces are perpendicular to the channel bottom, and the lower downward-convex surface is completely submerged in liquid at the initial time. At the time $t = 0$, the liquid and body are at rest. Let us introduce a Cartesian coordinate system whose $x$ axis is aligned on the channel bottom parallel to the side walls at identical distances $b$ from these walls; the $z$ axis is directed vertically upward, normal to the channel bottom. The body is assumed to be located symmetrically with respect to the coordinate plane $y0$; therefore, its edges have the coordinates $\pm L$ on the $x$ axis. The initial depth of the liquid outside the body, i.e. at $|x| > L$ is $h_0 \ll L$, and the pressure on the free surface of the liquid is equal to the atmospheric value, which is assumed to be zero.

At the time $t = 0$, the body occupies the domain

$$V = \{(x, y, z) : |x| \leq L, |y| \leq b, f(x) + h_1 \leq z \leq z_0\},$$

where

$$z_0 > h_0 \geq h_2 \geq h_1 > 0, \quad h_2 = f(L) + h_1.$$  

The continuous function $f(x)$ defining the lower surface of the body is even and satisfies the conditions $f(0) = 0$, $f'(x) \geq 0$, $f''(x) \geq 0 \quad \forall x \in (0, L)$.

At the time $t = 0+0$, the body moves vertically upward with a given acceleration $H''(t) \geq 0$ and with zero initial velocity. Assuming that $H(0) = h_1$, we can uniquely determine the body lifting law $H(t)$, from which it follows that the lower surface of the body at $t \geq 0$ is determined by the formula $z = f(x) + H(t)$ for all $|x| \leq L$. The flow of the liquid that occurs in the framework of the first approximation of the shallow water theory [9], without taking into account the influence of friction, is described by the equations

$$h_t + q_x = 0, \quad q_t + (qu + gh^2/2)_x = -hp_x, \quad (1)$$

where $h(x, t)$, $q(x, t)$, $u = q/h$ — the liquid depth, the flow rate and the velocity; $p(x, t)$ — specific pressure on the liquid surface; $g$ — gravity-induced acceleration.

When the body is lifted, the edges of its lower surface leave the liquid medium at a certain time $T_1 \geq 0$. The first stage of the process occurs at $T_1 > 0$, in the time interval $(0, T_1)$, when the lower surface of the body is still completely submerged in the liquid and the distance $H_L(t) = f(L) + H(t)$ is smaller than the depth $h_L(t) = h(L + 0, t)$ of the liquid adjacent to the body end faces, i.e. $H_L(t) < h_L(t)$. We assume that at this stage, as a result of the action of hydrostatic pressure, the depth $h(x, t)$ of the liquid under the body increases during body lifting so that $h(x, t) = f(x) + H(t)$ for all $|x| \leq L$. As a result, a liquid flow directed toward the origin is formed; outside the body, this flow consists of depression waves of the initial level $h_0$. The first stage is finalized when the depth of the liquid adjacent to the end faces of the body becomes equal to the height of lifting of the edges of the lower surface of the body, i.e. $h_L(T_1) \leq H_L(T_1)$.

Separation of the body from the liquid occurs at a certain time $T_2 \geq T_1$. The second stage of the process occurs in the time interval $(T_1, T_2)$: the edges of the lower surface of the body start to move out of the liquid, while some part of the lower surface still contacts the liquid. We assume that this part of the lower surface of the body is defined by the interval $|x| \leq a(t)$ whose length $a(t)$ at $t \in (T_1, T_2)$ is a rigorously monotonically decreasing function. At the third stage, at $t > T_2$, the body becomes separated from the liquid, after which liquid lifting that occurred at the second stage leads to the formation of two diverging waves.
3. Liquid flow outside the body at the first stage
At the first two stages, when \( t \in (0, T_2) \) the flow region is divided into two parts: a subdomain \(|x| \leq a(t)\), where the liquid is adjacent to the body, and a doubly connected subdomain \(|x| > a(t)\) with a free upper surface of the liquid, where \( a(t) = L \) at \( t \in [0, T_1) \) and \( a(t) \in [0, L] \) at \( t \in [T_1, T_2] \). In the first subdomain, the depth of the liquid \( h(x, t) = f(x) + H(t) \) whereby from the first equation of the system (1) we get \( H' + (hu)_{x} = 0 \). Integrating this equation over \( x \) and taking into account the boundary condition \( u(0, t) = 0 \), which follows from the symmetry of the problem with respect to the point \( x = 0 \), we find

\[
u(x, t) = -xH'/(h(x, t)), \quad |x| \leq a(t).
\]

In particular, at the boundary point \( x = a(t) \) we have

\[
U(t) = u(a(t), t) = -a(t)H'(t)/h(a(t), t).
\]

At the first stage at \( t \in (0, T_1) \) the flow in subdomain \(|x| \leq L\) is uniquely determined by the formula (2), where \( a(t) = L \). To characterize the flow outside the body at this stage (in view of the symmetry of the problem and of the fact that \( p = 0 \) at \(|x| > L\) ), it is sufficient to solve the following initial-boundary-value problem in the domain \( x > L, t > 0 \) for the system (1)

\[
h(x, 0) = h_0, \quad q(x, 0) = 0; \quad q(L, t) = Q(t) = U(t)H_L(t) = -LH'(t).
\]

For the correctness of this problem in the case with \( h_2 < h_c = 4h_0/9 \) in the time interval \((0, T_c)\), where \( H(T_c) = h_c \) and \( T_c \leq T_1 \), as it was shown in [6–8], it is sufficient to satisfy the condition

\[
|Q| \leq 8h_0c_0/27 \iff H' \leq 8h_0c_0/(27L),
\]

where \( c_0 = \sqrt{gh_c} \). If these conditions are satisfied, the depth of the liquid adjacent to the body end faces is calculated by the formula

\[
h_L = \frac{h_0}{9} \left( 1 + 2 \cos \left( \frac{1}{3} \arccos \left( 1 - \frac{27LH'}{4h_0c_0} \right) \right) \right)^2 \geq h_c,
\]

where the minimum depth \( h_c \) corresponds to the liquid flow with a critical velocity \( u_c = \sqrt{gh_c} = 2c_0/3 \).

4. Calculation of the pressure between the liquid and the lower surface of the body
In the subdomain \(|x| \leq a(t)\) the second equation of system (1) can be used to find the pressure \( p(x, t) \) on the upper boundary of domain adjacent to the lower surface of the body. Substituting the values \( h = f + H \) and \( u = -xH'/h \) in the left side of this equation, we obtain

\[
p_x = F(x, t) = -\frac{q(t) + (qu + gh^2/2)_x}{h} = \frac{xH'' + (u^2 - gh)f' + 2uH'}{h}.
\]

Integrating this equation we find

\[
p(x, t) = P(t) - \int_x^{a(t)} F(\xi, t)d\xi,
\]

where \( P(t) = 0 \) at \( t \in [T_1, T_2] \) and \( P(t) = g(h_L - H_L) \) at \( t \in [0, T_1) \), when \( a(t) = L \). In the simplest case, where the lifted body is a rectangular beam [7,8], the equation (5) takes the form

\[
p(x, t) = P(t) + \frac{G(t)(a^2 - x^2)}{2}, \quad G(t) = \frac{2(H')^2 - HH''}{H^2}.
\]
At the second stage, the movement of the boundary line \( x = a(t) \) essentially depends on the sign of the left one-sided spatial derivative of the pressure on this line
\[
P_x^-(t) = p_x(a(t) - 0, t).
\]
The sign of this derivative determines the sign of the surface pressure \( p \) in a certain left one-sided neighborhood \( (a(t) - \varepsilon, a(t)) \) of the boundary line. As a result, the body decelerates lifting of the liquid adjacent to the body in this neighborhood at \( P_x^- < 0 \) and accelerates the lifting process at \( P_x^- > 0 \).

Under the condition \( P_x^- < 0 \), the position of the boundary line \( x = a(t) \) is determined from the algebraic equation
\[
aH' + H_a (r + 2C) = 0,
\]
where \( r = u - 2c \) is the value of the \( r \)-invariant which is carried to this line from the outer region \( x > a(t) \), where \( c = \sqrt{gH}, \ C = \sqrt{gH_a}, \ H_a = f(a) + H \). Under the condition \( P_x^- \geq 0 \) the boundary line propagates with the critical velocity \( a' = \lambda_r \) and its position is determined by integrating equation
\[
a' + aH'/H_a + \sqrt{gH_a} = 0.
\]

5. On motion of the boundary line \( x = a(t) \)

![Figure 1](image-url)

**Figure 1.** Wave profiles for six consecutive time moments in the case of parabolic body \( f(x) = 10^{-5}|x|^3 \) lifting with the acceleration \( H'' = 15 \text{ cm}/\text{c}^2 \) at the initial depths \( h_0 = 20 \text{ cm}, \ h_1 = 5 \text{ cm}, \ h_2 = 15 \text{ cm} \). The surface of the lifted body at the different time moments are shown by --- ---, the depth at \( t = 0 \) is shown by - - - - - - , for other time moments wave profiles are shown by numbered solid lines: \( t = T_1 = 0.33 \text{ c} \) (1), \( t = 0.8 \text{ c} \) (2), \( t = 1.2 \text{ c} \) (3), \( t = T_2 = 1.41 \text{ c} \) (4), \( t = 2 \text{ c} \) (5), \( t = 2.5 \text{ c} \) (6).

In order to the boundary line can propagate with subcritical velocity
\[
\lambda_r = U - C < a' < \lambda_s = U + C
\]
it is necessary to fulfill the inequality $P_x^- < 0$, and in order to it can propagate along the $r$-characteristic of system (1) it is necessary to satisfy the equality $P_x^- = 0$. If the inequality $P_x^- > 0$ is satisfied, then the boundary line propagates with the critical velocity $a' = \lambda_r$ and is an envelope of the family of $r$-characteristics moving away from this line to the external domain $x > a(t)$. If the inequality $P_x^- < 0$ is fulfilled and the boundary line propagates with the critical velocity $a' = \lambda_r$, then it is an envelope of the family of $r$-characteristics, coming to it from the domain $x > a(t)$.

Since under the condition $P_x^- > 0$ the liquid flow in the external domain $|x| > a(t)$ does not affect the flow in the domain $|x| \leq a(t)$, where the liquid is adjacent to the body, so the boundary line $x = a(t)$ motion cannot be unambiguously determined within the framework of the shallow water theory. In this case, it is necessary to use the additional relation obtained from a fundamental physical law to which this flow satisfies. Following [8], as such the physical law we apply the total energy conservation law in the domain $|x| \leq a(t)$, where the liquid is adjacent to the body. In this case, the main criteria for determining the velocity of propagation of the boundary line is the assumption that the smallest possible part of the total energy of the liquid transforms into the energy of vortex motion when the liquid flows from the domain $|x| \leq a(t)$. As a result of applying this criteria, it turns out that the boundary line propagates with a critical velocity $a' = \lambda_r$.

### Table 1.
The dependence of the maximum height of the liquid $h_m = h(0, T_2)$ on the coefficient of flatness $k$ when the parabolic body $f(x) = k \cdot 10^{-6} |x|^3$ is lifted from the liquid with acceleration $H'' = 250$ cm/s$^2$ at the initial depths $h_2 = h_0 = 20$ cm.

| $k$  | 15   | 10   | 8    | 6    | 4    | 2    | 0.1  |
|------|------|------|------|------|------|------|------|
| $T_2$, c | 0.450 | 0.445 | 0.442 | 0.440 | 0.437 | 0.434 | 0.432 |
| $h_m$, cm  | 30.293 | 34.765 | 36.468 | 38.186 | 39.893 | 41.596 | 43.224 |

If at the second stage the inequality $P_x^- \leq 0$ is satisfied, and consequently the boundary line propagates with subcritical or critical velocity $a' \in [\lambda_r, \lambda_0]$, and $r$-characteristics bring the initial the value of the $r$-invariant $r_0 = -2c_0 = -2\sqrt{gh_0}$ to this line, then in the domain $|x| \leq a(t)$, where the liquid adjacent to the body, the liquid level does not rise above the initial level $h_0$ outside the body (see Fig. 1), whereby the body is separated from the liquid at the depth $h(0, T_2) = h_0$. If at the beginning of the second stage the inequality $P_x^- > 0$, is satisfied, then the maximum rise of the liquid can significantly exceed its initial level outside the body (see Fig. 2 and table 1).

### 6. Results of numerical simulations

Figures 1, 2 and table 1 show the results of numerical simulation of the problem when the body with length $2L = 2$ m is lifting with constant acceleration at the initial liquid depth outside the body is $h_0 = 20$ cm. The calculations were carried out by the explicit combined finite-difference scheme proposed in [10]. This scheme localizes the shocks with high accuracy and at the same time conserves a high order of convergence in the areas of their influence. In Figures 1, 2 the depths of the liquid are shown for six consecutive time moments, when the parabolic body is lifted from the liquid. The lower surface of this body is given by the formula $z = f(x) + H(t)$, where $f(x) = 10^{-5} |x|^3$. In these figures the initial levels of the depth are shown by the dashed lines, and the bottom surface of the lifted body is depicted by the dot-and-dashed lines.

Figure 1 illustrate the situation when the body is lifted from the liquid with the constant acceleration $H'' = 15$ cm/s$^2$ at the initial depth of the liquid under the body being $h_1 = 5$ cm.
Figure 2. Wave profiles for six consecutive time moments in the case of parabolic body $f(x) = 10^{-5}|x|^3$ lifting with the acceleration $H'' = 250 \text{ cm}/\text{c}^2$ at the initial depths $h_1 = 10 \text{ cm}$, $h_2 = h_0 = 20 \text{ cm}$. The surface of the lifted body at the different time moments are shown by $\cdot \cdot \cdot$, the depth at $t = 0$ is shown by $\cdot \cdot \cdot \cdot \cdot$, for other time moments wave profiles are shown by numbered solid lines: $t = 0.1 \text{ c (1), } t = 0.2 \text{ c (2), } t = 0.4 \text{ c (3), } t = T_2 = 0.45 \text{ c (4), } t = 0.8 \text{ c (5), } t = 1.2 \text{ c (6).}$

and $h_2 = 15 \text{ cm}$. For this solution, condition (4) is satisfied, which ensures the correctness of the initial-boundary-value problem (1), (3) at the first stage. The first stage is finalized at the time moment $T_1 \approx 0.33 \text{ c (curve 1 in Figure 1a),}$ and the second stage (curves 1 and 3 in Figure 1a) is finalized at the time moment $T_2 \approx 1.41 \text{ c (curve 4 in Figure 1b).}$ Since at the second stage the derivative of pressure $P_x^- < 0,$ the liquid does not rise above its initial level $h_0 = 20 \text{ cm},$ so $h(0,T_2) = 20 \text{ cm.}$ This rise of the liquid in the vicinity of the origin leads to the formation of two diverging elevation waves (curves 5 and 6 in Figure 1b).

Figure 2 illustrate the situation when the body is lifted from the liquid with the constant acceleration $H'' = 250 \text{ cm}/\text{c}^2$ at the initial depth of the liquid under the body being $h_1 = 10 \text{ cm}$ and $h_2 = h_0 = 20 \text{ cm}$. As the end faces of the body are not immersed in the liquid at the initial time moment, in this solution the first stage of the process is absent and the flow begins with the second stage (curves 1–3 in Figure 2a). In this variant of body lifting the pressure derivative is $P_x^- > 0$ in a certain time interval $(T_1,t_2) \subset (T_1,T_2)$ and $P_x^- < 0$ in the time interval $(t_2,T_2).$ In the time interval $(T_1,t_2),$ the boundary line $x = a(t)$ propagates with the critical velocity $a' = \lambda_r$ and is an envelope of the family of $r$-characteristics moving away from this line to the external domain $x > a(t).$ And in the time interval $(t_2,T_2)$ this line propagates with the subcritical velocity (6) and $r$-characteristics come to it from the external domain $x > a(t).$ In
this case these $r$-characteristics emerge of the boundary line at $t \in (T_1, t_2)$ and therefore do not bring any information from the unperturbed part of the solution located in the domain $x > L + c_0 t$. In view of this fact, the body separates from the liquid (curve 4 in Figure 2b) at the depth $h(0, T_2) > h_0$, where $T_2 \approx 0.45$ c. It leads to formation of two diverging waves, in the head part of which a shock wave is formed as a result of a gradient catastrophe (curves 5 and 6 in Figure 2b). The maximum height of liquid lifting increases significantly with increasing acceleration of body lifting and also with increasing the flatness of the body (Table 1). We can see from the Table 1 that $h_m$ can be greater than $2h_0$.

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