Partonic description of a supersymmetric $p$-brane

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Abstract

We consider supersymmetric extensions of a recently proposed partonic description of a bosonic $p$-brane which reformulates the Nambu-Goto action as an interacting multi-particle action with Filippov-Lie algebra gauge symmetry. We construct a worldline supersymmetric action by postulating, among others, a $p$-form fermion. Demanding a local worldline supersymmetry rather than the full worldvolume supersymmetry, we circumvent a known no-go theorem against the construction of a Ramond-Neveu-Schwarz supersymmetric action for a $p$-brane of $p > 1$. We also derive a spacetime supersymmetric Green-Schwarz extension from the preexisting kappa-symmetric action.

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1 Introduction

Supersymmetry in string theory is two fold: one on the string worldsheet by the Ramond-Neveu-Schwarz formalism and the other one on the spacetime by the Green-Schwarz formalism. It is known that these two approaches are equivalent, at least for ten-dimensional Minkowskian spacetime. One may attempt to extend the two formalisms to a $p$-brane with $p > 1$ i.e. an extended object over $p$-spatial dimensions. Indeed, the extension of the Green-Schwarz covariant superstring action to a $p$-brane is possible for $p \leq 5$ [1]. The resulting action is invariant under not only the spacetime supersymmetry but also a fermionic gauge symmetry called kappa-symmetry, such that the on-shell Bose and Fermi degrees of freedom are equal. On the other hand, for a $p$-brane with $p > 1$, the Ramond-Neveu-Schwarz extension to the corresponding Nambu-Goto action reformulated by an auxiliary worldvolume metric [2,4,9] is known impossible: in Ref. [5] it was shown that the worldvolume supersymmetric extension requires the existence of the Einstein-Hilbert term for the worldvolume metric such that the metric is no longer auxiliary and the connection to the Nambu-Goto action is lost.

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1This action is often dubbed “Polyakov” action.
Recently, a partonic description of a bosonic $p$-brane was proposed in Ref. [6]. With an embedding of $(p+1)$-dimensional worldvolume coordinates into $D$-dimensional target spacetime, $X^M(\tau, \sigma^i)$ where and henceforth $i = 1, \cdots, p$ and $M = 0, 1, \cdots, D-1$, the proposed action assumes the form:

$$S_{\text{bosonic}} = \int d\tau \text{Tr} \left( \frac{1}{2} D_\tau X^M D_\tau X_M - \frac{1}{2p!} \{X^{M_1}, X^{M_2}, \cdots, X^{M_p}\}_{\text{N.B.}} \{X_{M_1}, X_{M_2}, \cdots, X_{M_p}\}_{\text{N.B.}} \right).$$

(1.1)

The action contains two kinds of auxiliary fields: the inverse of an einbein $\varphi$ and a gauge connection $A^i_\tau$. The former defines the trace inside the action,

$$\text{Tr}(\cdot) := \int d^p\sigma \ (\varphi \cdot),$$

(1.2)

and the Nambu bracket,

$$\{X^{M_1}, X^{M_2}, \cdots, X^{M_p}\}_{\text{N.B.}} := \varphi^{-1} \epsilon^{i_1 i_2 \cdots i_p} \partial_{i_1} X^{M_1} \partial_{i_2} X^{M_2} \cdots \partial_{i_p} X^{M_p},$$

(1.3)

while the latter sets the covariant derivative to be

$$D_\tau X^M := \partial_\tau X^M - A^i_\tau \partial_i X^M.$$  

(1.4)

In fact, $\varphi$ and $A^i_\tau$ may be identified with the “lapse” and “shift” Lagrange multipliers of the Nambu-Goto action in the canonical formalism [8]. Characteristic features of the above action are [6]:

- Integrating out the auxiliary fields, i.e. replacing them by their on-shell values, reduces the action to the standard Nambu-Goto action, as in [2-4] or [9].

- The action is manifestly spacetime Lorentz invariant, despite the similarity to the light-cone gauge fixed actions in [10, 11].

- Though not manifest, the action enjoys the full $(p+1)$-dimensional worldvolume diffeomorphism.

- The number of auxiliary component fields is $p + 1$, and hence the worldvolume diffeomorphism can fix them completely, such as $\varphi \equiv 1$ and $A^i_\tau \equiv 0$.\footnote{As usual, $\epsilon^{i_1 i_2 \cdots i_p}$ is the totally anti-symmetric $p$-dimensional tensor density with the normalization $\epsilon^{12 \cdots p} = 1$.}

\footnote{c.f. “Polyakov” action where the number of auxiliary component fields is $\frac{1}{2}(p+1)(p+2)$ such that for $p > 1$ they can not be gauge fixed completely.}
• Partial gauge fixing as \( \varphi \equiv 1 \) and \( \partial_i A_i^\tau \equiv 0 \) breaks the worldvolume diffeomorphism to a volume preserving \( p \)-dimensional diffeomorphism. For a compact \( p \)-brane this leads to a quantum mechanical system based on Filippov-Lie \( p \)-algebra.\(^4\)

• A physical picture behind the reformulation is to describe a single (compact) brane as a collection of interacting multi-particles, and hence the title of this paper: *partonic description of a \( p \)-brane*.\(^5\)

• One may consider implementing a *worldline* supersymmetry, rather than the full \((p+1)\)-dimensional worldvolume supersymmetry.

In the present paper, we focus on exploring the last property. In the multi-particle description of a (compact) \( p \)-brane, the temporal worldline direction is singled out from the full \((p+1)\)-dimensional worldvolume. Demanding a local supersymmetry along the worldline we may circumvent the aforementioned no-go theorem against the construction of a Ramond-Neveu-Schwarz supersymmetric action for a \( p \)-brane with \( p > 1 \).\(^6\)

Filippov-Lie \( p \)-algebra appears as a natural generalization of Lie-algebra *i.e.* two-algebra. While Lie algebra has been extensively studied ever since the inception of Yang-Mills theory, Filippov-Lie \( p \)-algebra with \( p > 2 \) had not been much explored until 2007 when Bagger-Lambert and Gustavsson employed Filippov-Lie three-algebra with the aim to describe multiple M2-branes \([12,13]\). In the present paper we shall construct supersymmetric gauge models based on arbitrary Filippov-Lie \( p \)-algebras.

The organization of the rest of the paper is as follows. In section 2 we review the bosonic action (1.1) with some details, including the full \((p+1)\)-dimensional worldvolume diffeomorphism and the Filippov-Lie \( p \)-algebra regularization. The worldline supersymmetric action is constructed in section 3. We first present a foliation preserving, diffeomorphism invariant and locally supersymmetric Ramond-Neveu-Schwarz action. After gauge fixing we also obtain an action with a global supersymmetry. A crucial ingredient in our worldline supersymmetric extension is to postulate a \( p \)-form fermion, in addition to a one-form fermion and a gravitino. In section 4 we derive a spacetime supersymmetric Green-Schwarz extension from the known kappa-symmetric action. We write down the proper transformation of the auxiliary fields which will ensure all the symmetries of the spacetime supersymmetric Nambu-Goto action to persist in our refor-

\(^4\)Relates works are the Bagger-Lambert-Gustavsson description of multiple M2-branes via Filippov-Lie three-algebra \([12,13]\).

\(^5\)Related works include BFSS, \( \mathcal{M} \)-theory matrix model \([15]\) and Myers’ effect \([16]\) *etc.* See also \([11,17]\).

\(^6\)For earlier proposals to circumvent the no-go theorem, we refer \([18,19]\).
2 More on the bosonic action \((1.1)\)

In this section we review, from Ref. \([6]\), some properties of the bosonic action \((1.1)\) which are relevant to our main results of the supersymmetrization.

With a \(p \times p\) matrix defined by

\[
V_{ij} := \partial_i X_M \partial_j X_M ,
\]

utilizing an identity,

\[
\varphi^{-2} \det V = \frac{1}{p!} \{X_{M_1}, X_{M_2}, \cdots, X_{M_p}\}_{N.B.} \{X_{M_1}, X_{M_2}, \cdots, X_{M_p}\}_{N.B.} ,
\]

the bosonic action \((1.1)\) can be rewritten as

\[
S_{\text{bosonic}} = \int d\tau dp \sigma \left( \frac{1}{2} \varphi D_{\tau} X_M D_{\tau} X_M - \frac{1}{2} \varphi^{-1} \det V \right) .
\]

The on-shell values of the auxiliary fields are

\[
A^i_\tau \equiv \partial_{\tau} X_M \partial_j X_M V^{-1 ji} , \quad \varphi \equiv \sqrt{- \det V / (D_{\tau} X_M D_{\tau} X_M)} .
\]

Substituting these into the action \((2.3)\), one recovers the Nambu-Goto action \([20]\),

\[
S_{\text{bosonic}} \implies S_{\text{N.G.}} = - \int d\tau dp \sigma \sqrt{- \det (\partial_\mu X_M \partial_\nu X_M)} ,
\]

where and henceforth \(\mu, \nu\) are the full \((p+1)\)-dimensional worldvolume coordinate indices running from zero to \(p\). The worldvolume diffeomorphism is realized in rather nontrivial fashion:

\[
\delta X^M = \nu^\mu \partial_\mu X^M ,
\]

\[
\delta \varphi = \partial_\mu (\varphi \nu^\mu) - 2 \varphi D_{\tau} \nu^\tau ,
\]

\[
\delta A^i_\tau = D_{\tau} \nu^i - \varphi^{-2} \partial_j \nu^\tau V^{-1 ji} \det V + (D_{\tau} \nu^\tau + \nu^\mu \partial_\mu) A^i_\tau ,
\]

where \(\nu^\mu\) is a local parameter having an arbitrary dependence on \(\tau\) and \(\sigma^j\). In general, a symmetry of a given action persists after any reformulation by auxiliary fields: we can always assign transformations to
the auxiliary fields such that the symmetry is preserved [5]. The above transformation (2.6) is an explicit example of this general statement.

From

$$\varphi(D_\tau YZ + YD_\tau Z) = YZ [\partial_i (\varphi A^i) - \partial_\tau \varphi] + \partial_\tau (\varphi YZ) - \partial_i (\varphi A^i YZ), \quad (2.7)$$

the vanishing of the following quantity,

$$\partial_i (\varphi A^i) - \partial_\tau \varphi = \varphi (\partial_i A^i - D_\tau \ln \varphi) \equiv 0, \quad (2.8)$$

is the sufficient and necessary condition of the integration by part for the covariant derivative:

$$\int d\tau \text{Tr}( D_\tau YZ) = - \int d\tau \text{Tr}( YD_\tau Z), \quad (2.9)$$

with arbitrary $Y$ and $Z$. Under the transformation (2.6),

$$\delta(\partial_i A^i - D_\tau \ln \varphi) = D^2_\tau v^\tau - \frac{1}{(p-1)!} \{X^M_1, \ldots, X^{M_{p-1}}, \{X_{M_1}, \ldots, X_{M_{p-1}}, v^\tau\}\}_{\text{N.B.}} \quad \text{N.B.}$$

$$+ (D_\tau v^\tau + \nu^\mu \partial_\mu)(\partial_i A^i - D_\tau \ln \varphi). \quad (2.10)$$

Thus, fixing the gauge (2.8) generically breaks the worldline reparametrization to the global transformation, $v^\tau = \alpha \tau + \beta$ with constant parameters $\alpha, \beta$, and reduces the $(p+1)$-dimensional worldvolume diffeomorphism to the $p$-dimensional diffeomorphism on the ‘space’ part of the worldvolume.

On the other hand, fixing the gauge $\varphi \equiv 1$ and $\partial_i A^i \equiv 0$, reduces the worldvolume diffeomorphism down to the $p$-dimensional volume preserving diffeomorphism that is subject to the divergence free condition, $\partial_\tau v^i = 0$. Consequently, the volume preserving gauge symmetry generator as well as the covariant derivative can be represented by the Nambu $p$-bracket: with a functional basis $T^a(\sigma^i)$, $a = 1, 2, 3, \cdots$ for the $p$-dimensional manifold which we assume to be compact, we have

$$v^i \partial_i = v_{a_1 a_2 \cdots a_{p-1}} \{T^{a_1}, T^{a_2}, \ldots, T^{a_{p-1}}\}, \quad \text{N.B.}$$

$$D_\tau = \partial_\tau - A^{a_1 a_2 \cdots a_{p-1}} \{T^{a_1}, T^{a_2}, \ldots, T^{a_{p-1}}\}, \quad \text{N.B.} \quad (2.11)$$

Note that here $v_{a_1 a_2 \cdots a_{p-1}}$ and $A^{a_1 a_2 \cdots a_{p-1}}$ depend on $\tau$ only being independent of the $\sigma^i$ coordinates.

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As is well known (see e.g. [21]), Nambu $p$-bracket provides an explicit realization of the bracket of the Filippov-Lie $p$-algebra [22], satisfying the totally anti-symmetric property:

\[ [X_1, \cdots, X_i, \cdots, X_j, \cdots, X_p] = -[X_1, \cdots, X_j, \cdots, X_i, \cdots, X_p], \quad (2.12) \]

and the Leibniz rule, also known as a fundamental identity:

\[ [X_1, \cdots, X_{p-1}, [Y_1, \cdots, Y_p]] = \sum_{j=1}^{p} [Y_1, \cdots, [X_1, \cdots, X_{p-1}, Y_j], \cdots, Y_p]. \quad (2.13) \]

In the Nambu bracket realization of the Filippov-Lie algebra, we may employ the structure constant through

\[ \{T^a_1, T^a_2, \cdots, T^a_p\}_\text{N.B.} = f^{a_1a_2\cdots a_p}_{b} T^b. \quad (2.14) \]

The structure constant is then totally anti-symmetric for the upper indices and satisfies from the Leibniz rule (2.13):

\[ f^{a_1a_2\cdots a_p}_{c_1c_2\cdots c_{p-1}} f^{b_1b_2\cdots b_p}_{a} = \sum_{j=1}^{p} f^{a_1a_2\cdots a_{p-1}b_j}_{c_1c_2\cdots c_{p-1}} f^{b_1\cdots b_{j-1}eb_{j+1}\cdots b_p}_{c}. \quad (2.15) \]

Now from (2.11) and (2.14), expanding the dynamical variables by the functional basis, $X^M(\tau, \sigma) = X^M_a(\tau) T_a(\sigma)$, the covariant derivative can be rewritten as

\[ D_\tau X^M = (D_\tau X^M)_a T^a, \quad (D_\tau X^M)_a = \frac{d}{d\tau} X^M_a - X^M_b \tilde{A}^b_\tau a, \quad (2.16) \]

where we set

\[ \tilde{A}^b_\tau a := A_{\tau c_1c_2\cdots c_{p-1}} f^{c_1c_2\cdots c_{p-1}}_{a} b. \quad (2.17) \]

In this way, the bosonic action (1.1), (2.3) reduces to a genuine quantum mechanical system with gauge symmetry based on an arbitrary Filippov-Lie $p$-algebra. It is worthwhile to note that for $p \geq 3$ the only nontrivial irreducible finite dimensional Filippov-Lie $p$-algebra is, up to signature, $\text{so}(p+1)$ [23,26]. With $\tilde{v}^b_a := v_{c_1c_2\cdots c_{p-1}} f^{c_1c_2\cdots c_{p-1}}_{a} b$, from (2.6) and (2.11), the Filippov-Lie $p$-algebra gauge transformation is given by

\[ \delta X^M_a = X^M_b \tilde{v}^b_a, \quad \delta A_{\tau a_1a_2\cdots a_{p-1}} = \partial_\tau v_{a_1a_2\cdots a_{p-1}} + (-1)^p (p-1) A_{\tau c[a_1a_2\cdots a_{p-2}} \tilde{v}^c_{a_{p-1}]}, \quad (2.18) \]

of which the latter induces, from (2.15),

\[ \delta \tilde{A}^b_\tau a = \partial_\tau \tilde{v}^b_a - \tilde{v}^b_c \tilde{A}^c_\tau a + \tilde{A}^b_\tau \tilde{v}^c_a. \quad (2.19) \]

In fact, the case of $p = 3$ matches with the Bagger-Lambert-Gustavsson formalism [12,13]. Other useful relations for the bosonic action are written in Appendix.
3 Worldline supersymmetry

3.1 Action with foliation preserving local supersymmetry

The action for the partonic description of a $p$-brane with a local worldline supersymmetry we propose is, with the trace defined in Eq. (1.2):

$$S_{\text{worldline}} = \int d\tau \ Tr(\hat{\mathcal{L}}),$$

where

$$\hat{\mathcal{L}} = \frac{1}{2} D_\tau X^M D_\tau X_M - \frac{1}{2 p!} \{X^{M_1}, X^{M_2}, \ldots, X^{M_p}\}_{\text{N.B.}} \{X_{M_1}, X_{M_2}, \ldots, X_{M_p}\}_{\text{N.B.}}$$

$$+ i \frac{1}{2} \psi^M D_\tau \psi_M + i \frac{1}{2 p!} \psi^{M_1 M_2 \ldots M_p} D_\tau \psi_{M_1 M_2 \ldots M_p}$$

$$- i \frac{1}{(p-1)!} \psi^{M_1 M_2 \ldots M_p} \{X_{M_1}, X_{M_2}, \ldots, X_{M_{p-1}}, \psi_{M_p}\}_{\text{N.B.}}$$

$$+ i \chi \left( D_\tau X^M \psi_M + \frac{1}{p!} \{X^{M_1}, X^{M_2}, \ldots, X^{M_p}\}_{\text{N.B.}} \psi_{M_1 M_2 \ldots M_p} \right).$$

(3.2)

In addition to the bosonic fields in (1.1) which are $X^M, \varphi, A^i_\tau$, the above supersymmetric action contains three kinds of fermions: one-form $\psi^M$, $p$-form $\psi^{M_1 M_2 \ldots M_p}$, and one-dimensional gravitino $\chi$.

The action is invariant under the following foliation preserving diffeomorphism:

$$\delta X^M = v^\lambda \partial_\lambda X^M,$$

$$\delta \varphi = \partial_\lambda (v^\lambda \varphi) - 2 \varphi D_\tau v^\tau,$$

$$\delta A^i_\tau = D_\tau v^i + (D_\tau v^\tau + v^\lambda \partial_\lambda) A^i_\tau,$$

$$\delta \psi^M = v^\lambda \partial_\lambda \psi^M + \frac{1}{2} (D_\tau v^\tau) \psi^M,$$

$$\delta \psi^{M_1 M_2 \ldots M_p} = v^\lambda \partial_\lambda \psi^{M_1 M_2 \ldots M_p} + \frac{1}{2} (D_\tau v^\tau) \psi^{M_1 M_2 \ldots M_p},$$

$$\delta \chi = v^\lambda \partial_\lambda \chi + \frac{1}{2} (D_\tau v^\tau) \chi,$$

(3.3)

where

$$\partial_\tau v^\tau = 0,$$

(3.4)

\footnote{Under the transformations (3.3), (3.5), the corresponding Lagrangian, $\mathcal{L}_{\text{worldline}} = \varphi \hat{\mathcal{L}}$, transforms to a total derivative.}
such that in fact, $D_\tau \nu^\tau = \frac{d\nu^\tau}{d\tau}$. Namely while $\nu^i(\tau, \sigma^j)$ is an arbitrary local parameter on the $p$-brane worldvolume, $\nu^\tau(\tau)$ is arbitrary only over the worldline direction and independent of the spatial coordinates $\sigma^j$. From the quantum mechanical point of view, the former generates a gauge symmetry, while the latter corresponds to the genuine worldline diffeomorphism. The action is also invariant under a foliation preserving local supersymmetry:

$$\delta X^M = i\psi^M \varepsilon,$$

$$\delta \varphi = -2i\varphi \chi \varepsilon,$$

$$\delta A^i_\tau = 0,$$

$$\delta \psi^M = D_\tau X^M \varepsilon,$$

$$\delta \psi^{M_1 M_2 \cdots M_p} = \{X^{M_1}, X^{M_2}, \ldots, X^{M_p}\}_\text{N.B.} \varepsilon + i\psi^{M_1 M_2 \cdots M_p} \chi \varepsilon,$$

$$\delta \chi = D_\tau \varepsilon + \frac{i}{2}(\partial_i A^i_\tau - \varphi^{-1} D_\tau \varphi) \varepsilon,$$

where $\varepsilon$ is a local fermionic parameter which has arbitrary dependence on the worldline but is independent of the worldvolume spatial coordinates,

$$\partial_i \varepsilon = 0. \quad (3.6)$$

The supersymmetry algebra reads

$$\delta_{\varepsilon_1} \delta_{\varepsilon_2} - \delta_{\varepsilon_2} \delta_{\varepsilon_1} = \delta_\nu, \quad (3.7)$$

where the right hand side is given by the diffeomorphism parameter,

$$\nu^\tau = 2i\varepsilon_1 \varepsilon_2, \quad \nu^i = -2i\varepsilon_1 \varepsilon_2 A^i, \quad (3.8)$$

such that the foliation structure is preserved. From the quantum mechanical point of view, the anticommutator of the one-dimensional local supersymmetry amounts to a one-dimensional diffeomorphism plus a gauge symmetry, as usual for supersymmetric gauge theories. Compared to the bosonic action (1.1), the action (3.1) lacks the full $(p+1)$-dimensional worldvolume diffeomorphism, but this is consistent with the no-go theorem against the construction of a worldvolume supersymmetric action for a $p$-brane of $p > 1$ [5].
3.2 Action with global worldline supersymmetry

A gauge fixed ($\varphi \equiv 1$ and $\chi \equiv 0$) action follows: in terms of the Filippov-Lie $p$-bracket,

$$\mathcal{L}'_{\text{worldline}} = \frac{1}{2} D_\tau X^M D_\tau X_M - \frac{1}{2^{p!}} \left[ X^{M_1}, X^{M_2}, \ldots, X^{M_p} \right] \left[ X_{M_1}, X_{M_2}, \ldots, X_{M_p} \right]$$

$$+ i \frac{1}{2} \psi^M D_\tau \psi_M + i \frac{1}{2^{p!}} \psi^{M_1 \cdots M_p} D_\tau \psi_{M_1 \cdots M_p} - i \frac{1}{(p-1)!} \psi^{M_1 \cdots M_{p-1} M_p} \left[ X_{M_1}, \ldots, X_{M_{p-1}}, \psi_{M_p} \right] .$$

(3.9)

The action is clearly invariant under the Filippov-Lie $p$-algebra gauge transformation (2.18), and further enjoys one global worldline supersymmetry: with a constant parameter $\varepsilon_0$,

$$\delta X^M = i \psi^M \varepsilon_0 ,$$

$$\delta A^i_\tau = 0 ,$$

$$\delta \psi^M = D_\tau X^M \varepsilon_0 ,$$

$$\delta \psi^{M_1 M_2 \cdots M_p} = \left[ X^{M_1}, X^{M_2}, \ldots, X^{M_p} \right] \varepsilon_0 .$$

(3.10)

Especially, when $p = 1$ i.e. string, both fermions $\psi^M, \psi^{M_1 M_2 \cdots M_p}$ are on the equal footing carrying only one spacetime index, and there appears an additional $\text{SO}(2)$ $R$-symmetry in the action. Consequently the supersymmetry is doubled and the above action reduces, after putting $A^i_\tau \equiv 0$, to the well-known conformal gauge fixed Ramond-Neveu-Schwarz superstring action [14]. The case of $p = 2$ is similar to the BFSS $M$-theory matrix model [15], but different points in our action are the types of fermions, the worldline supersymmetry and the full target spacetime Lorentz invariance.
4 Spacetime supersymmetry

4.1 Action with kappa-symmetry

The spacetime supersymmetric Green-Schwarz covariant p-brane Lagrangian reads \[1\]
\[- \sqrt{-\det (\Pi^M \Pi^M_M)} + \mathcal{L}_{\text{Wess-Zumino}}, \tag{4.1}\]
where \(\Pi^M_M = \partial_\mu X^M - i \tilde{\theta} \Gamma^M \partial_\mu \theta\) and \(\mathcal{L}_{\text{Wess-Zumino}}\) corresponds to the Wess-Zumino term necessary for the kappa-symmetry. The super p-brane action exists if and only if the Bose and Fermi degrees of freedom match, such that the possible values of \(p\) and the spacetime dimension \(D\) are
\[
\begin{align*}
   p = 1 : & \quad D = 3, 4, 6, 10 \\
   p = 2 : & \quad D = 4, 5, 7, 11 \\
   p = 3 : & \quad D = 6, 8 \\
   p = 4 : & \quad D = 9 \\
   p = 5 : & \quad D = 10. \tag{4.2}
\end{align*}
\]

Our partonic reformulation of the spacetime supersymmetric Green-Schwarz p-brane Lagrangian is then:
\[
\mathcal{L}_{\text{spacetime}} = \mathcal{L}(\varphi, A^i_\tau, \Pi^M_\mu) + \mathcal{L}_{\text{Wess-Zumino}}, \tag{4.3}
\]
where, with \(\mathcal{V}_{ij} := \Pi^M_1 \Pi^M_i M \Pi^M_2 \Pi^M_j M\),
\[
\mathcal{L}(\varphi, A^i_\tau, \Pi^M_\mu) = \frac{1}{2} \varphi (\Pi^M_\tau - A^i_\tau \Pi^M_i)(\Pi^M_\mu - A^j_\mu \Pi^M_j) - \frac{1}{2} \varphi^{-1} \det \mathcal{V}. \tag{4.4}
\]
In particular, in a similar fashion to (2.2), we may write [27]
\[
\varphi^{-2} \det \mathcal{V} = \frac{1}{p!} \langle \Pi^M_1, \Pi^M_2, \cdots, X^{M_p} \rangle \langle \Pi^M_1, \Pi^M_2, \cdots, \Pi^M_{M_p} \rangle, \tag{4.5}
\]
\[
\langle \Pi^M_1, \Pi^M_2, \cdots, X^{M_p} \rangle := \varphi^{-1} \epsilon^{i_1 i_2 \cdots i_p} \Pi^M_{i_1} \Pi^M_{i_2} \cdots \Pi^M_{i_p}. \tag{4.5}
\]

The auxiliary fields assume the following on-shell values,
\[
A^i_\tau \implies \hat{A}^i_\tau := \Pi^M_\mu \Pi^M_{j \mu} \varphi^{-1 j i}, \quad \varphi \implies \hat{\varphi} := \sqrt{\frac{\det \mathcal{V}}{(\Pi^M - \hat{A}^i_\mu \Pi^M_i)(\Pi^M_\tau - \hat{A}^j_\mu \Pi^M_j)}}. \tag{4.6}
\]
Integrating them out reduces $\mathcal{L}(\varphi, A^i_\tau, \Pi^M_\mu)$ to the supersymmetric Nambu-Goto term in (4.1),

$$\mathcal{L}(\varphi, A^i_\tau, \Pi^M_\mu) \implies \mathcal{L}(\hat{\varphi}, \hat{A}^i_\tau, \Pi^M_\mu) = -\sqrt{-\det(\Pi^M_\mu \Pi^\nu_M)}.$$  \hfill (4.7)

Furthermore, along with an arbitrary transformation $\delta \Pi^M_\mu$, if we let the auxiliary fields transform as

$$\delta A^i_\tau = \delta \hat{A}^i_\tau + \frac{1}{2}(\hat{A}^i_\tau - A^i_\tau)\delta \ln \varphi + (\hat{A}^k_\tau - A^k_\tau)\Pi^M_\mu \delta \Pi^M_j \mathcal{V}^{1ji},$$

$$\delta \varphi = \frac{2\varphi^2}{(\hat{\varphi} + \varphi)\varphi} \delta \hat{\varphi} + \frac{(\hat{\varphi} - \varphi)\varphi}{\hat{\varphi} + \varphi} \delta \ln \det \mathcal{V},$$  \hfill (4.8)

the variation of $\mathcal{L}(\varphi, A^i_\tau, \Pi^M_\mu)$ becomes independent of the auxiliary fields and, moreover, coincides with that of the spacetime supersymmetric Nambu-Goto term:

$$\delta \mathcal{L}(\varphi, A^i_\tau, \Pi^M_\mu) = \delta \mathcal{L}(\hat{\varphi}, \hat{A}^i_\tau, \Pi^M_\mu) = -\delta \sqrt{-\det(\Pi^M_\mu \Pi^\nu_M)}.$$  \hfill (4.9)

Therefore, all the symmetries of the Green-Schwarz super $p$-brane Lagrangian (4.1) survive in our partonic reformulation (4.3), which include the spacetime supersymmetry, the spacetime Lorenz symmetry, the kappa-symmetry and the worldvolume diffeomorphism.\footnote{See [27, 28] for similar works, and recall the general phenomenon that no symmetry is lost under an arbitrary reformulation of a given action by auxiliary fields [5].}

### 4.2 Action with global spacetime supersymmetry

The light-cone gauge fixed actions are ready to be read-off from an earlier work by Bergshoeff, Sezgin, Tanii and Townsend [11]. In its appendix the authors listed light-cone gauge fixed supersymmetric actions for various $p$-branes in diverse spacetime dimensions. Utilizing the identity (2.2), in terms of Filippov-Lie algebra $p$-bracket, their light-cone gauge fixed spacetime supersymmetric $p$-brane actions can be rewritten in a compact form:

$$\mathcal{L}'_{\text{spacetime}} = \frac{1}{2}(D_t X^I)^2 - \frac{1}{2p!} \left[ X^{I_1}, X^{I_2}, \ldots, X^{I_p} \right]^2 + \frac{1}{2} \hat{\theta} D_t \theta + \frac{1}{2(p-1)!} \hat{\theta} \Gamma^{I_1 I_2 \cdots I_{p-1}} \left[ X_{I_1}, \cdots, X_{I_{p-1}}, \theta \right].$$  \hfill (4.10)

Here the spacetime index $I$ runs from one to $D - 2$ with the possible values of $p$ and $D$ in (4.2). For details of the supersymmetry transformation we refer to Ref. [11].\footnote{See also Ref. [29] for a zero-dimensional analogy.}
5 Discussion

In this paper we have constructed supersymmetric extensions of a bosonic $p$-brane action which reformulates the Nambu-Goto action as an interacting multi-particle action with Filippov-Lie $p$-algebra gauge symmetry. We obtained a worldline supersymmetric action by postulating, among others, a $p$-form fermion. We also derived a spacetime supersymmetric Green-Schwarz extension from the preexisting kappa-symmetric action.

Compared to the ordinary Lie algebra, one limited feature of Filippov-Lie $p$-algebra for $p \geq 3$ is that, finite dimensional irreducible Filippov-Lie algebra is essentially unique, i.e. $so(p+1)$ \cite{23,26}. Consequently there is no arbitrary tunable parameter as for the number of finite degrees of freedom. One should deal with infinite dimensional Filippov-Lie $p$-algebras or $so(p+1)$. In the latter case the $p$-brane corresponds to a fuzzy $p$-sphere.

For $p = 2$, the situation is different. We may safely adopt matrices of arbitrary size. With the usual matrix commutator, our actions read from (3.9),

$$L_{\text{worldline}} = \frac{1}{2} D_{\tau} X^M D_{\tau} X_M - \frac{1}{4} \left[ X^M, X^N \right]^2 + i \frac{1}{2} \psi^M D_{\tau} \psi_M + i \frac{1}{4} \psi^{MN} D_{\tau} \psi_{MN} - i \psi^{MN} \left[ X_M, \psi_N \right],$$

and from (4.10),

$$L_{\text{spacetime}} = \frac{1}{2} D_t X^I D_t X_I - \frac{1}{4} \left[ X^M, X^N \right]^2 + i \frac{1}{2} \bar{\theta} D_t \theta + \frac{1}{2} \bar{\theta} \Gamma^I \left[ X_I, \theta \right].$$

Of course, the latter corresponds to the well-known BFSS $M$-theory matrix model \cite{15} where the fermion is a target spacetime spinor. In analogy with the equivalence between the RNS and the GS superstring actions, it is crucial to check the connection between the above two actions for M2-brane.

In the case of $p = 0$, with vanishing Nambu bracket, our worldline supersymmetric action (5.1) reduces to the well-known action for a massless supersymmetric particle (see e.g. \cite{30}). Our action then corresponds to non-Abelian or Filippov-Lie algebra generalization of it, where a ‘mass’ term appears as the square of Nambu bracket. We recall the known difficulty that a massive point-particle does not allow a worldline supersymmetric extension, as there is no supersymmetric counter part to its mass term,

$$S = \frac{1}{2} \int d\tau \left( e \partial_{\tau} X^M \partial_{\tau} X_M - e^{-1} m^2 \right).$$

While a compact $p$-brane should look like a point-particle at far distance, our result delivers a novel way of introducing a supersymmetric mass term: the Filippov-Lie algebra based interaction of the partons of the
compact $p$-brane gives rise to the mass term. Namely, mass originates from the internal interaction, like a proton made of light quarks. Further, we expect that the constraint form the equation of motion for $\chi$ also leads, after quantization, to a certain massive Dirac equation. We leave the quantization of the supersymmetric $p$-brane in our formulation for future work. In the present paper we focused on the partonic description of a super $p$-brane. Since the aforementioned no-go theorem prohibits the construction of a worldvolume supersymmetric $p$-brane action for $p > 1$, the other alternative possibility worth trying is the supersymmetric extensions of the ‘multi-string description of a $p$-brane,’ starting from the bosonic action [6].

\[
S_{\text{string}} = \int d^2 \tau \text{ Tr} \left( \sqrt{-h} L_{\text{string}} \right), \quad \text{Tr} := \int d^{p-1} \sigma, \\
L_{\text{string}} = -e^{-\phi} h^{ab} D_a X^M D_b X_M - \frac{1}{4} e^\phi \det V + e^{-\phi},
\]

where $a, b = 0, 1$ are the two-dimensional worldsheet indices [34]. The resulting action may correspond to a RNS version of the well-known matrix string action [35].

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\[10\text{c.f. [31, 32] and some BPS equations in the Bagger-Lambert-Gustavsson theory [33].}\]
A Useful relations

Here we write some useful identities:

\[ \{X^{[M_1, \ldots, M_i, \ldots, M_p]}\}_{N.B.} \partial_i X^{M_{p+1}} = 0 \quad \text{for arbitrary } i, \]
\[ \{X^{[M_1, \ldots, M_{p-1}, u^i]}\}_{N.B.} \partial_i X^{M_p} = \frac{1}{p} \partial_i u^i \{X^{M_1, \ldots, M_p}\}_{N.B.}, \]
\[ \varphi^{-2} \det V V^{-1ij} \partial_j Y \partial_j Z = \frac{1}{(p-1)!} \{X^{M_1, M_2, \ldots, M_{p-1}, Y}\}_{N.B.} \{X^{M_1, M_2, \ldots, M_{p-1}, Z}\}_{N.B.}, \]
\[ \varphi^{-1} \partial_i (\varphi^{-1} \det V V^{-1ij} \partial_j Y) = \frac{1}{(p-1)!} \{X^{M_1, M_2, \ldots, M_{p-1}, \{X^{M_1, M_2, \ldots, M_{p-1}, Y}\}}_{N.B.}\}_{N.B.}, \]

\[ \det V = \frac{1}{p!} e^{\epsilon_{r_1 r_2 \ldots r_p} s_1 s_2 \ldots s_p} V_{r_1 s_1} V_{r_2 s_2} \ldots V_{r_p s_p}, \]
\[ \frac{\partial^p \det V}{\partial V_{i_1 j_1} \partial V_{i_2 j_2} \ldots \partial V_{i_n j_n}} = \frac{1}{(p-1)!} \epsilon^{i_1 i_2 \ldots i_p} \epsilon^{j_1 j_2 \ldots j_p} V_{r_1 s_1} \ldots V_{r_p s_p} \psi^n. \]

In particular,

\[ V^{-1ij} \det V = \frac{1}{(p-1)!} e^{i r_1 \ldots r_p \epsilon^{j s_1 \ldots s_p}} V_{r_1 s_1} \ldots V_{r_p s_p}, \]
\[ (V^{-1ij} V^{-1kl} - V^{-1ik} V^{-1lj}) \det V = \frac{1}{(p-2)!} e^{i k r_1 \ldots r_p \epsilon^{s_1 \ldots s_p}} V_{r_1 s_1} \ldots V_{r_p s_p}. \]

Under the worldvolume diffeomorphism (2.6),

\[ \delta D_\tau X^M = \frac{1}{(p-1)!} \{X^{N_1, \ldots, N_{p-1}, u^\tau}\}_{N.B.} \{X^{N_1, \ldots, N_{p-1}, X^M}\}_{N.B.}, \]
\[ + (D_\tau u^\tau + u^\mu \partial_\mu) D_\tau X^M, \]
\[ \delta \{X^{M_1, \ldots, M_p}\}_{N.B.} = \sum_{k=1}^p D_\tau X^{M_k} \{X^{M_1, \ldots, M_{k-1}, u^\tau, X^{M_{k+1}, \ldots, M_p}}\}_{N.B.}, \]
\[ + (D_\tau u^\tau + u^\mu \partial_\mu) \{X^{M_1, \ldots, M_p}\}_{N.B.}, \]

Under the worldvolume diffeomorphism (2.6),

\[ \delta D_\tau X^M = \frac{1}{(p-1)!} \{X^{N_1, \ldots, N_{p-1}, u^\tau}\}_{N.B.} \{X^{N_1, \ldots, N_{p-1}, X^M}\}_{N.B.}, \]
\[ + (D_\tau u^\tau + u^\mu \partial_\mu) D_\tau X^M, \]
\[ \delta \{X^{M_1, \ldots, M_p}\}_{N.B.} = \sum_{k=1}^p D_\tau X^{M_k} \{X^{M_1, \ldots, M_{k-1}, u^\tau, X^{M_{k+1}, \ldots, M_p}}\}_{N.B.}, \]
\[ + (D_\tau u^\tau + u^\mu \partial_\mu) \{X^{M_1, \ldots, M_p}\}_{N.B.}. \]
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