A Scaling Limit With
Many Noncommutativity Parameters

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We derive the worldsheet propagator for an open string with different magnetic fields at the two ends, and use it to compute two distinct noncommutativity parameters, one at each end of the string. The usual scaling limit that leads to noncommutative Yang-Mills can be generalized to a scaling limit in which both noncommutativity parameters enter. This corresponds to expanding a theory with $U(N)$ Chan-Paton factors around a background $U(1)^N$ gauge field with different magnetic fields in each $U(1)$.

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1. Introduction and Conventions

The propagator for the open neutral string in a slowly varying background $U(1)$ gauge field was computed in [1] by solving the equations of motion

$$\Box < x^i(z, \bar{z}) x^j(\zeta, \bar{\zeta}) > = -2\pi\alpha' \delta^2(z - \zeta) G^{ij}$$

with the boundary conditions

$$(\partial_z - \partial_{\bar{z}}) < x^i(z, \bar{z}) x^j(\zeta, \bar{\zeta}) > + 2\pi\alpha' g^{ik} B_{kl}(\partial_z + \partial_{\bar{z}}) < x^\ell(z, \bar{z}) x^j(\zeta, \bar{\zeta}) > |_{z=\bar{z}} = 0$$

where $\Box \equiv 4\partial_z \partial_{\bar{z}}$. Here the open string worldsheet $\Sigma$ will denote the disk with Euclidean metric $\gamma^{\alpha\beta} = \delta^{\alpha\beta}$. The complex coordinates $z, \bar{z}$ are related to the original strip coordinates $\sigma, \tau$ with $\tau$ rotated to $t \equiv i\tau$ and $-\infty \leq t \leq \infty, 0 \leq \sigma \leq \pi$, by $z \equiv e^{t+i\sigma}, \bar{z} \equiv e^{t-i\sigma}$ with $\text{Im}z \geq 0$.

The propagator was found to be

$$< x^i(z, \bar{z}) x^j(\zeta, \bar{\zeta}) > = -\alpha' \left[ \frac{1}{2} g^{ij} \ln(z - \zeta) + \frac{1}{2} g^{ij} \ln(\bar{z} - \bar{\zeta}) + (-\frac{1}{2} g^{ij} + G^{ij} + \frac{\theta^{ij}}{2\pi\alpha'}) \ln(z - \zeta) \right. \left. + (-\frac{1}{2} g^{ij} + G^{ij} - \frac{\theta^{ij}}{2\pi\alpha'}) \ln(\bar{z} - \bar{\zeta}) - \frac{i}{2\alpha'} \theta^{ij} \right]$$

where

$$G^{ij} \equiv [(g + 2\pi\alpha' B)^{-1} g (g - 2\pi\alpha' B)^{-1}]^{ij}$$

$$G_{ij} \equiv g_{ij} - (2\pi\alpha')^2 (B g^{-1} B)_{ij} = [(g - 2\pi\alpha' B) g^{-1} (g + 2\pi\alpha' B)]_{ij}$$

$$\theta^{ij} \equiv -(2\pi\alpha')^2 [(g + 2\pi\alpha' B)^{-1} B (g - 2\pi\alpha' B)^{-1}]^{ij}.$$
parameter. This scaling limit described \textit{massless} charged and neutral non-abelian gluons living on a noncommutative geometry with one noncommutative parameter $\theta^{ij}$.

In this paper we start with a string theory in a non-abelian $U(N)$ background and study its scaling limit. Actually, we will restrict ourselves to backgrounds that reside in the $U(1)^N$ Cartan subgroup, but with different constant background $U(1)$ fields on each brane. The novelty compared to the $U(1)$ case is the presence of charged strings.

The first step will be to compute the propagator on the disk, which we do by starting with the mode expansion for the charged string originally derived in [1]. Hence, in sect. 2 we will rederive it in a more convenient notation, and pay special attention to the zero modes so to ensure that the the charged string mode expansion converges to the neutral one when the background fields at the two ends become the same. In sect. 4, we will then use this mode expansion together with the commutation relations to compute the charged string propagator. We will have to pay attention to which string states we use to define the propagator, as we want to recover the usual neutral string propagator [1.3], when the backgrounds are the same at the two ends. This will require evaluating the propagator between coherent states of the Landau levels. In sect. 3, we therefore discuss the spectrum of the charged string.

In sect. 5, we repeat the usual argument [3], but now compute two noncommutativity parameters, one at each end of the string. This emphasizes the interpretation that the noncommutativity of the D-brane worldvolume in the presence of a background $B$-field along the brane is really a property of the endpoint of the string, rather than a feature of the worldvolume itself. For a $U(1)^N$ background, there are $N$ different noncommutativity parameters. In sect.6, we compute the short distance behavior of the operator products of tachyon vertex operators which are inserted on the boundaries, and show they reduce to star products, with two different noncommutativity parameters. These enter into the computation of the scattering amplitudes of the scaling limit theory.

In sect.7, we discuss the spectrum of the theory and show that there is a scaling limit in this more general background. In addition to the $U(1)^N$ massless gauge bosons, charged vector states survive for each Landau level. As in the string theory we start from, the states of the limiting non-abelian noncommutative gauge theory are no longer massless, but rather tachyonic or massive. An obvious question is what is the generalization of the $\hat{F}^2$ noncommutative Lagrangian. Our construction provides an example of the bimodules discussed in [2].
2. Charged Open String Normal Mode Expansion

The complete worldsheet action for the “charged” string with different magnetic fields at each end is

\[ S = \frac{1}{4\pi\alpha'} \int_{\Sigma} g_{\mu\nu} \partial_{\alpha} X^{\mu} \partial^{\alpha} X^{\nu} - \frac{i}{2} \int_{-\infty}^{\infty} dt \left( B^{(1)}_{ij}(x^i \partial X^j |_{\sigma=0} + B^{(2)}_{ij}(x^i \partial X^j |_{\sigma=\pi}) \right). \]  

(2.1)

Here \( 0 \leq \mu, \nu \leq 25 \) and the open strings end on \( Dp \)-branes in the \((0, i)\) directions for \( 1 \leq i \leq p \). Variation of (2.1) gives the equations of motion for the worldsheet field

\[ (\partial^2_{\sigma} + \partial^2_{t}) X^{\mu}(\sigma, t) = 0 \]  

(2.2)

and the boundary conditions at each end of the string are

\[ g_{ij} \partial_{\sigma} X^{j} + 2\pi i \alpha' B^{(1)}_{ij}(x^i \partial X^j |_{\sigma=0} = 0 \]  

\[ g_{ij} \partial_{\sigma} X^{j} - 2\pi i \alpha' B^{(2)}_{ij}(x^i \partial X^j |_{\sigma=\pi} = 0 \]  

(2.3)

\[ \partial_{\sigma} X^{0} |_{\sigma=0, \sigma=\pi} = 0 \]  

\[ X^{I} |_{\sigma=0, \sigma=\pi} = 0, \quad p + 1 \leq I \leq 25 \]  

(2.4)

(Note that the worldsheet action for the neutral string, whose propagator in the directions along the \( Dp \)-brane is given in (1.3), is a special case of (2.1) with \( B^{(1)}_{ij} = -B^{(2)}_{ij} \).

For simplicity, we now specialize to the case where the ends of the string live on \( D2 \)-branes, \textit{i.e.} \( i = 1, 2 \). In this case the magnetic fields have only one component and we relabel them as \( B^{(1)}_{12} = q_1 B_{12} \) and \( B^{(2)}_{12} = q_2 B_{12} \), where \( q_1 + q_2 \neq 0 \). We choose a diagonal metric in the 1, 2 directions with equal components \( g_{ij} \equiv g^{-1} \delta_{ij} \), and retain an overall factor since eventually we will scale the metric. The open string metric defined in (2.8) is thus also diagonal \( G_{ij} = \delta_{ij} G^{-1} \).

In a basis given by \( X^{\pm}(\sigma, t) \equiv X^{1}(\sigma, t) \pm iX^{2}(\sigma, t) \), the charged string normal mode expansion \([4, 7] \) can be written as

\[ X^{+}(z, \bar{z}) = x^{+} + \frac{i}{2} \sqrt{2\alpha'} \sum_{r \in Z + A} \frac{a_r}{r} (z^{-r} + \bar{z}^{-r}) - \frac{1}{2} \sqrt{2\alpha'} B \sum_{r \in Z + A} \frac{a_r}{r} (z^{-r} - \bar{z}^{-r}) \]  

(2.5)

\[ X^{-}(z, \bar{z}) = x^{-} + \frac{i}{2} \sqrt{2\alpha'} \sum_{s \in Z - A} \frac{\tilde{a}_s}{s} (z^{-s} + \bar{z}^{-s}) + \frac{1}{2} \sqrt{2\alpha'} B \sum_{s \in Z - A} \frac{\tilde{a}_s}{s} (z^{-s} - \bar{z}^{-s}) \]  

(2.6)
with commutation relations\footnote{The $[x^+, x^-]$ commutation relation, which provided the first indication of noncommutativity of spacetime in the context of strings in background gauge fields, was derived in \cite{9}, and re-examined in \cite{10} and \cite{11}.}

\[ [a_r, \tilde{a}_s] = 2G r \delta_{r,-s} ; \quad [a_r, a_{r'}] = 0 = [\tilde{a}_s, \tilde{a}_{s'}] ; \]
\[ [x^+, x^-] = -\frac{2}{(q_1 + q_2)B_{12}} ; \quad [a_r, x^\pm] = 0 = [\tilde{a}_s, x^\pm] ; \]

where

\[
G = \frac{g}{1 + B^2} \quad B \equiv g q_1 2\pi \alpha' B_{12} , \quad A = \frac{1}{\pi} (\text{arctan} B + \text{arctan} \frac{q_2}{q_1} B). \tag{2.8}
\]

The oscillators $a_r, \tilde{a}_s$ are non-integrally moded with $r = n + A, s = n - A$, for $n \in \mathbb{Z}$. The operators have hermiticity $a_r^\dagger = \tilde{a}_{-r}$ and $(x^+)^\dagger = x^-$. The comparison with the neutral string case is given by the limit as $A \to 0$, and it requires care with the zero modes:

\[
\lim_{A \to 0} X^+(z, \bar{z}) = \lim_{A \to 0} (x^+ + i \sqrt{2\alpha'} \frac{a_A}{A} - \frac{i}{2} \sqrt{2\alpha'} a_0 \ln z\bar{z} + \frac{1}{2} \sqrt{2\alpha'} B a_0 \ln \frac{z}{\bar{z}} + \frac{i}{2} \sqrt{2\alpha'} \sum_{n \neq 0} \frac{a_n}{n} (z^{-n} + \bar{z}^{-n}) - \frac{1}{2} \sqrt{2\alpha'} B \sum_{n \neq 0} \frac{a_n}{n} (z^{-n} - \bar{z}^{-n})) \tag{2.9}
\]

\[
\lim_{A \to 0} X^-(z, \bar{z}) = \lim_{A \to 0} (x^- - i \sqrt{2\alpha'} \frac{\tilde{a}_{-A}}{A} - \frac{i}{2} \sqrt{2\alpha'} a_0 \ln z\bar{z} - \frac{1}{2} \sqrt{2\alpha'} B \tilde{a}_0 \ln \frac{z}{\bar{z}} + \frac{i}{2} \sqrt{2\alpha'} \sum_{n \neq 0} \tilde{a}_n (z^{-n} + \bar{z}^{-n}) + \frac{1}{2} \sqrt{2\alpha'} B \sum_{n \neq 0} \tilde{a}_n (z^{-n} - \bar{z}^{-n})). \tag{2.10}
\]

To make contact with the neutral string zero mode operators, we need therefore to identify the limits

\[
\lim_{A \to 0} (x^+ + i \sqrt{2\alpha'} \frac{a_A}{A}) = x^+_0 \tag{2.11}
\]
\[
\lim_{A \to 0} (x^- - i \sqrt{2\alpha'} \frac{a_{-A}}{A}) = x^-_0 .
\]

The neutral string commutation relations in the $\pm$ basis are

\[
[a_n, \tilde{a}_m] = 2G n \delta_{n,-m} ; \quad [a_n, a_{n'}] = 0 = [\tilde{a}_n, \tilde{a}_{n'}] ; \quad [x^+_0, x^-_0] = 2 \Theta^{12} ; \quad [a_n, x^0_\pm] = 0 = [\tilde{a}_n, x^0_\pm] \text{ for } n \neq 0 ; \tag{2.12}
\]
\[
[x^+_0, \tilde{a}_0] = i \sqrt{2\alpha'} 2G = [x^-_0, a_0] ; \quad [x^+_0, a_0] = 0 = [x^-_0, \tilde{a}_0].
\]

The neutral string commutation relations in the $\pm$ basis are
with $\Theta^{12} = -2\pi\alpha'BG$. Indeed using

$$
\lim_{A \to 0} (\pi A)^{-1} = \lim_{(Q=q_1+q_2) \to 0} \left[ \arctan \left( \frac{q_1^{-1}BQ}{1 + B^2 - q_1^{-1}QB^2} \right) \right]^{-1} \\
\sim \left( \frac{1 + B^2}{q_1^{-1}BQ} \right) (1 - \frac{q_1^{-1}QB^2}{1 + B^2}).
$$

(2.13)

one can show that the identification in (2.11) is consistent since

$$
\lim_{A \to 0} \left[ x^+ + i\sqrt{2\alpha'} a_A, x^- - i\sqrt{2\alpha'} \tilde{a}_{-A} \right] \\
= \lim_{A \to 0} \left[ x^+, x^- \right] + 2\alpha' \frac{1}{A^2} [a_A, \tilde{a}_{-A}] \\
= \lim_{A \to 0} \left( \frac{-4\alpha' g\pi}{B + \frac{4\pi}{q_1} B} + 4\alpha' \frac{1}{A} G \right) \\
= \frac{-4\alpha' gB\pi}{1 + B^2} = 2\Theta^{12},
$$

and $[x^+_0, x^-_0] = 2\Theta^{12}$ from (2.12).

3. Charged String Spectrum and Coherent States

We briefly review the charged string spectrum\cite{1} in our notation, so that we can study its scaling limit in sect.7. We will also consider coherent states in order to introduce the states we use to define the charged string propagator. The commutation relations (2.7) correspond to the operator product expansion

$$
a(z) \tilde{a}(\zeta) = (z - \zeta)^{-2} 2G [A (\frac{\zeta}{z})^{A-1} + (1 - A) (\frac{z}{\zeta})^A] \\
+ N a(z) \tilde{a}(\zeta),
$$

(3.1)

where we define complex worldsheet bosons $a(z) = \sum_{r \in \mathbb{Z} + A} a_r z^{-r-1}$, $\tilde{a}(z) = \sum_{s \in \mathbb{Z} - A} \tilde{a}_s z^{-s-1}$, and the normal ordering as

$$
N a_r \tilde{a}_s \equiv : a_r \tilde{a}_s : \equiv \tilde{a}_s a_r \quad \text{for } r \geq A, \ s \leq -A, \\
\quad \quad a_r \tilde{a}_s \quad \text{otherwise}.
$$

(3.2)

The Virasoro current is $L(z) = \frac{1}{2} G^{-1} : a(z) \tilde{a}(z) : + z^{-2} \frac{1}{4} A(1 - A)$ with central charge equal to 2 corresponding to the two bosonic degrees of freedom. The commutation relations with the oscillators are

$$
[L_n, a_r] = (n - r) a_{r+n}, \quad [L_n, \tilde{a}_s] = (n - s) \tilde{a}_{s+n}.
$$

(3.3)
As in [1] we take the states to be eigenstates of the position operator \( x^+ \) and label them with the continuous eigenvalue \( x_+ \) as \( |x_+\rangle \equiv e^{-\frac{1}{2}(q_1+q_2)B_{12}x^+} |0\rangle \), since from (2.7) we have \( x^+|x_+\rangle = x_+|x_+\rangle \). Notice that we have \( a_r|x_+\rangle = 0 \) for \( r \geq A \), and \( \tilde{a}_s|x_+\rangle = 0 \) for \( s \geq 1 - A \). In (2.8) it is sufficient to consider \( 0 \leq A \leq \frac{1}{2} \). Then

\[
L_0|x_+\rangle = \frac{1}{2}A(1 - A)|x_+\rangle \tag{3.4}
\]

for each of the infinite number of states \( |x_+\rangle \). So they are degenerate all with the same tachyonic mass

\[
\alpha' m^2 = -1 + \frac{1}{2}A(1 - A) < 0 \tag{3.5}
\]

since the contribution from these degrees of freedom to the mass operator is \( \alpha' m^2 = L_0 - 1 \). There is a tower of oscillator states built from \( |x_+\rangle \) starting with

\[
a_{-1+A}|x_+\rangle, \quad \tilde{a}_{-1-A}|x_+\rangle \tag{3.6}
\]

which have masses \( \alpha' m^2 = -\frac{1}{2}A(1 + A) < 0 \) and \( \alpha' m^2 = \frac{1}{2}A(3 - A) > 0 \) respectively. Although \( a_A|x_+\rangle = 0 \), we can also consider the set of states

\[
\tilde{a}_{-A}|x_+\rangle, \quad \tilde{a}_{-A}\tilde{a}_{-A}|x_+\rangle, \quad \text{etc.} \tag{3.7}
\]

with masses \(-1 + \frac{1}{2}A(1 - A) + A; -1 + \frac{1}{2}A(1 - A) + 2A\), etc. The states in (3.7) together with \( |x_+\rangle \) are separated from each other in \( \alpha' m^2 \) values by \( A \). This is reminiscent of the equally spaced Landau levels with frequency separation \( qB \) of a charged particle in a constant magnetic field where here \( A \) plays the role of \( qB \); so \( |x_+\rangle, \tilde{a}_{-A}|x_+\rangle, \tilde{a}_{-A}\tilde{a}_{-A}|x_+\rangle \), etc. form equally spaced Landau levels each of infinite degeneracy and each is the lowest state of an oscillator tower such as that described in (3.6).

To compute the charged string propagator we introduce the states

\[
|\alpha\rangle \equiv e^{-\frac{1}{2}(q_1+q_2)B_{12}x^+} |0\rangle \tag{3.8}
\]

\[
|\beta\rangle \equiv |0\rangle e^{-\frac{1}{2}(x^++i\sqrt{2\alpha'}a_A)A} \tag{3.9}
\]

since they correspond in the \( A \to 0 \) limit to the vacuum states defined by the normal ordering \( Nx_0^+x_0^- \equiv \frac{1}{2}(x_0^+x_0^- + x_0^-x_0^+) \) in the neutral string case, i.e. these states have the
property
\[
\lim_{A \to 0} \langle \beta | (x^+ + \frac{i\sqrt{2\alpha'}}{A} a_A)(x^- - \frac{i\sqrt{2\alpha'}}{A} \bar{a}_{-A}) | \alpha \rangle \\
= \lim_{A \to 0} \left( [x^+, x^-] + \frac{2\alpha'}{A^2} [a_A, \bar{a}_{-A}] + \langle \beta | (x^- - \frac{i\sqrt{2\alpha'}}{A} \bar{a}_{-A}) x^+ | \alpha \rangle \right) \\
= 2\Theta^{12} - \frac{1}{2} \lim_{A \to 0} \left( -\frac{4\pi\alpha' g}{B + \frac{2z}{q_1} B} + \frac{4\alpha' G}{A} \right) \\
= \Theta^{12} = \lim_{A \to 0} \langle \beta | x^+_0 x^-_0 | \alpha \rangle ,
\]
and
\[
\lim_{A \to 0} \langle \beta | (x^- - \frac{i\sqrt{2\alpha'}}{A} \bar{a}_{-A}) (x^+ + \frac{i\sqrt{2\alpha'}}{A} a_A) | \alpha \rangle \\
= -\Theta^{12} = \lim_{A \to 0} \langle \beta | x^-_0 x^+_0 | \alpha \rangle .
\]

The states have been normalized \( \langle \beta | \alpha \rangle = 1\). Note that the state \( \langle \beta | \) is a coherent state made up from the Landau level \( \bar{a}_{-A} | 0 \rangle \).

4. Charged String Propagator

In order to compute a charged propagator that reduces to the neutral expression \( \text{(1.3)} \) as \( A \) goes to zero, we consider the charged string propagator on the disk evaluated between the states \( |\alpha \rangle \) and \( |\beta \rangle \). For \( |z| > |\zeta| \), we find it is given by

\[
< X^+(z, \bar{z}) X^-(\zeta, \bar{\zeta}) > \equiv \langle \beta | X^+(z, \bar{z}) X^-(\zeta, \bar{\zeta}) | \alpha \rangle \\
= -\frac{2\alpha' \pi g}{B + \frac{2z}{q_1} B} - 2\alpha' G \frac{1}{A}(\zeta^A + \bar{\zeta}^A - 1) + 2i\alpha' G B \frac{1}{A} \frac{1}{4} (\zeta^A - \bar{\zeta}^A) \\
+ \alpha' G \left[ f(\zeta) + f(\bar{\zeta}) + f(\zeta^\frac{1}{2}) + f(\bar{\zeta}^\frac{1}{2}) \right] \\
+ \alpha' G B^2 \left[ f(\zeta^\frac{1}{2}) + f(\bar{\zeta}^\frac{1}{2}) - f(\zeta^\frac{1}{2}) - f(\bar{\zeta}^\frac{1}{2}) \right] \\
+ 2i\alpha' G B \left[ -f(\zeta^\frac{1}{2}) + f(\bar{\zeta}^\frac{1}{2}) \right]
\]
where
\[
f(\rho) \equiv \sum_{r=n+A, n \geq 0} \frac{\rho^r}{r}; \quad \lim_{A \to 0} f(\rho) = -\ln(1 - \rho) + \lim_{A \to 0} \frac{\rho^A}{A} .
\]
The other non-zero component of the charged string propagator, for \( |z| > |\zeta| \), is

\[
< X^-(z, \bar{z}) X^+(\zeta, \bar{\zeta}) > \equiv \langle \beta | X^-(z, \bar{z}) X^+(\zeta, \bar{\zeta}) | \alpha \rangle
\]

\[
= \frac{2\alpha' \pi g}{B + \frac{2a}{\alpha_1} B} - 2\alpha' G^1_A (z^A + \bar{z}^A - 1) + 2i\alpha' GB^1_A (z^A - \bar{z}^A)
\]

\[
+ \alpha' G [g(\bar{\bar{\zeta}}) + g(\bar{\zeta}) + g(\bar{\bar{\zeta}}) + g(\bar{\zeta})]
\]

\[
+ \alpha' GB^2 [g(\bar{\bar{\zeta}}) + g(\bar{\zeta}) - g(\bar{\bar{\zeta}}) - g(\bar{\zeta})]
\]

\[
- 2i\alpha' GB [-g(\bar{\bar{\zeta}}) + g(\bar{\bar{\zeta}})]
\]

(4.3)

and

\[
g(\rho) \equiv \sum_{s=n-A; n \geq 1} \frac{\rho^s}{s}; \quad \lim_{A \rightarrow 0} g(\rho) = -\ln(1 - \rho).
\]

(4.4)

To compute (4.1), (4.3) we used the normal mode expansion (2.5), (2.6), the commutation relations (2.7), the normal ordering defined in (3.2), and the states \(|\alpha\rangle, |\beta\rangle\) described in (3.8), (3.9). For all \( z, \zeta \), the charged string propagator is

\[
G^+-(z, \bar{z}; \zeta, \bar{\zeta}) \equiv \langle \beta | TX^+(z, \bar{z}) X^-(\zeta, \bar{\zeta}) | \alpha \rangle
\]

\[
= \theta(|z| - |\zeta|) \cdot < X^+(z, \bar{z}) X^-(\zeta, \bar{\zeta}) > + \theta(|\zeta| - |z|) \cdot < X^-(\zeta, \bar{\zeta}) X^+(z, \bar{z}) > .
\]

(4.5)

The other component is given from the symmetry property

\[
G^-+(z, \bar{z}; \zeta, \bar{\zeta}) = G^+-(\zeta, \bar{\zeta}; z, \bar{z}).
\]

(4.6)

These are the propagators that reduce to the neutral string expressions since we can show

\[
\lim_{A \rightarrow 0} G^+-(z, \bar{z}; \zeta, \bar{\zeta}) = \langle x^+(z, \bar{z}) x^-(\zeta, \bar{\zeta}) >= 0,
\]

(4.7)

\[
\lim_{A \rightarrow 0} G^-+(z, \bar{z}; \zeta, \bar{\zeta}) = \langle x^-(z, \bar{z}) x^+(\zeta, \bar{\zeta}) >= 0.
\]

(4.8)

The right hand sides in (4.7), (4.8) are equivalent to the neutral propagators (4.3) in the \( \pm \) basis. For completeness, we mention that other limits are also consistent. For example, as \( \zeta \rightarrow 0 \), we have

\[
G^+-(z, \bar{z}; 0) = \frac{-2\alpha' \pi g}{B + \frac{2a}{\alpha_1} B} + \frac{2\alpha' G}{A}, \text{for} A > 0;
\]

\[
G^-+(z, \bar{z}; 0) = \frac{2\alpha' \pi g}{B + \frac{2a}{\alpha_1} B} - 2\alpha' G^1_A (z^A + \bar{z}^A - 1) + 2i\alpha' GB^1_A (z^A - \bar{z}^A).
\]

(4.9)
If we consider the simultaneous limit $\zeta \to 0$, $A \to 0$, in the expression for $G^+ - (z, \bar{z}; \zeta, \bar{\zeta})$ we must be careful to use $\lim_{\zeta \to 0} \zeta^A = \lim_{A \to 0} e^{A \ln \zeta} = \lim_{x \to 0} e^{x \ln x} = 1$. Then

$$\lim_{\zeta \to 0} G^+ (z, \bar{z}; \zeta, \bar{\zeta}) = \lim_{A \to 0} G^+ (z, \bar{z}; 0) = \lim_{x \to 0} e^{x \ln x} = 1.$$ (4.10)

5. The Set of Noncommutativity Parameters

We are going to compute the “equal time” commutator of the string operators at coincident points on the boundary $[3,2]$. We distinguish the two boundary regions of the open string disk as follows. On the boundary $\sigma = 0$, we have $z = |z| = \tau$ and $\zeta = |\zeta| = \tau'$ so $\tau, \tau' > 0$; while on $\sigma = \pi$, then $z = |z| e^{i\pi} = \tau$ and $\zeta = |\zeta| e^{i\pi} = \tau'$ so here $\tau, \tau' < 0$. We will evaluate both of the propagators on the boundaries $\sigma = 0$ and $\sigma = \pi$, and find a different noncommutativity parameter at each end of the string. For $|z| > |\zeta|$, and on the boundary $\sigma = 0$, we have

$$< X^+ (z, \bar{z}) X^- (\zeta, \bar{\zeta}) > |_{\sigma = 0} = \frac{-2\alpha' \pi g}{B + \frac{2g}{q_1} B} + 4\alpha' G \sum_{n=0}^{\infty} \frac{1}{n + A} (\frac{\zeta}{z})^{n + A}$$

$$+ \frac{2\alpha'}{A} - \frac{4\alpha'}{A} \zeta^A,$$

$$< X^- (z, \bar{z}) X^+ (\zeta, \bar{\zeta}) > |_{\sigma = 0} = \frac{2\alpha' \pi g}{B + \frac{2g}{q_1} B} + 4\alpha' G \sum_{n=1}^{\infty} \frac{1}{n - A} (\frac{\zeta}{z})^{n - A}$$

$$+ \frac{2\alpha'}{A} - \frac{4\alpha'}{A} z^A.$$ (5.1)

We now compute the commutator that defines the noncommutativity parameter at $\sigma = 0$.

5 After completion of our paper a preprint [10] appeared which derives these parameters from a charged string annulus propagator. Since the result is a short distance effect, it is independent of the topology of the worldsheet.
\[ [X^+(\tau), X^-(\tau)] = T(X^+(\tau) X^- (\tau^-) - X^+(\tau) X^- (\tau^+)) \]

\[ \equiv \lim_{\epsilon \to 0} (< X^+(\tau) X^- (\tau - \epsilon) > - < X^- (\tau + \epsilon) X^+(\tau) >) \text{, (for } \epsilon > 0) \]

\[ = \lim_{\epsilon \to 0} \left( \frac{-4\alpha' \pi g}{B + \frac{q_1}{q_2} B} + 4\alpha' G \left[ \sum_{n=0}^{\infty} \frac{1}{n + A} (\frac{\tau - \epsilon}{\tau})^{n + A} - \sum_{n=0}^{\infty} \frac{1}{n - A} (\frac{\tau + \epsilon}{\tau})^{n - A} \right] \right) \]

\[ - \frac{4\alpha' G}{A} (\tau - \epsilon)^A + \frac{4\alpha' G}{A} (\tau + \epsilon)^A \]

\[ = -\frac{4\alpha' \pi g}{B + \frac{q_1}{q_2} B} + 4\alpha' G \pi \cot \pi A \]

\[ = -4\alpha' \pi (g)^2 \frac{q_1 2\pi \alpha' B_{12}}{1 + (g)^2 (q_1 2\pi \alpha' B_{12})^2} \]

\[ = 2\Theta^{12}. \] (5.2)

Notice that \( \Theta^{12} \) is the same expression that appears in the neutral string commutation relations (2.12). For \(|z| > |\zeta|\), and on the boundary \( \sigma = \pi \), we have

\[ < X^+(z, \bar{z}) X^- (\zeta, \bar{\zeta}) > |_{\sigma = \pi} = \frac{-2\alpha' \pi g}{B + \frac{q_1}{q_2} B} + \frac{2\alpha' G}{A} \]

\[ + 2\alpha' G (1 + B^2) \sum_{n=0}^{\infty} \frac{1}{n + A} (\frac{|\zeta|}{|z|})^{n + A} \]

\[ + 2\alpha' G (1 - B^2) \cos 2\pi A \sum_{n=0}^{\infty} \frac{1}{n + A} (\frac{|\zeta|}{|z|})^{n + A} \] (5.3)

\[ + i2\alpha' GB (-2i \sin 2\pi A) \sum_{n=0}^{\infty} \frac{1}{n + A} (\frac{|\zeta|}{|z|})^{n + A} \]

\[ - \frac{4\alpha' G}{A} |\zeta|^A \cos \pi A - \frac{2\alpha' GB}{A} |\zeta|^A \sin \pi A. \]

\[ < X^- (z, \bar{z}) X^+ (\zeta, \bar{\zeta}) > |_{\sigma = \pi} = \frac{2\alpha' \pi g^{11}}{B + \frac{q_1}{q_2} B} + \frac{2\alpha' G}{A} \]

\[ + 2\alpha' G (1 + B^2) \sum_{n=1}^{\infty} \frac{1}{n - A} (\frac{|\zeta|}{|z|})^{n - A} \]

\[ + 2\alpha' G (1 - B^2) \cos 2\pi A \sum_{n=0}^{\infty} \frac{1}{n - A} (\frac{|\zeta|}{|z|})^{n - A} \] (5.4)

\[ + i2\alpha' GB (2i \sin 2\pi A) \sum_{n=0}^{\infty} \frac{1}{n - A} (\frac{|\zeta|}{|z|})^{n - A} \]

\[ - \frac{4\alpha' G}{A} |z|^A \cos \pi A - \frac{2\alpha' GB}{A} |z|^A \sin \pi A. \]
The noncommutativity parameter at the $\sigma = \pi$ end of the string is defined from the propagators on the boundary $\tau = \text{Re} z < 0, \tau' = \text{Re} \zeta < 0$, to be

$$[X^+(\tau), X^-(\tau)] = T(X^+(\tau) X^-(\tau) - X^+(\tau) X^-(\tau^+))$$

$$= \equiv \lim_{\epsilon \to 0} (\langle X^+(\tau) X^-(\tau + \epsilon) > - \langle X^-(\tau - \epsilon) X^+(\tau) >), \quad (\text{for } \epsilon > 0)$$

$$= \lim_{\epsilon \to 0} \left( \frac{-4\alpha' \pi g}{B + \frac{q_2}{q_1} B} \right)$$

$$+ 2\alpha' G (1 + B^2) \left[ \sum_{n=0}^{\infty} \frac{1}{n + A} \left(\frac{|\tau + \epsilon|}{|\tau|}\right)^{n+1} - \sum_{n=1}^{\infty} \frac{1}{n - A} \left(\frac{|\tau - \epsilon|}{|\tau|}\right)^{n} \right]$$

$$+ 2\alpha' G (1 - B^2) (1 - 2 \sin^2 \pi A) \left[ \sum_{n=0}^{\infty} \frac{1}{n + A} \left(\frac{|\tau + \epsilon|}{|\tau|}\right)^{n+1} - \sum_{n=1}^{\infty} \frac{1}{n - A} \left(\frac{|\tau - \epsilon|}{|\tau|}\right)^{n} \right]$$

$$+ i2\alpha' GB (-2i \sin 2\pi A) \left[ \sum_{n=0}^{\infty} \frac{1}{n + A} \left(\frac{|\tau + \epsilon|}{|\tau|}\right)^{n+1} - \sum_{n=1}^{\infty} \frac{1}{n - A} \left(\frac{|\tau - \epsilon|}{|\tau|}\right)^{n} \right]$$

$$- \frac{4\alpha' G}{A} |\tau + \epsilon|^A \cos \pi A - \frac{2\alpha' GB}{A} |\tau + \epsilon|^A \sin \pi A$$

$$+ \frac{4\alpha' G}{A} |\tau - \epsilon|^A \cos \pi A + \frac{2\alpha' GB}{A} |\tau - \epsilon|^A \sin \pi A$$

$$= -4\alpha' \pi (g)^2 \frac{q_2 \pi \alpha' B_{12}}{1 + (g)^2 (q_2 \pi \alpha' B_{12})^2}$$

$$= 2\tilde{\Theta}^{12}.$$

(5.5)

In the limit $q_1 \to -q_2$, then $\tilde{\Theta}^{12} \to -\Theta^{12}$. Indeed in the neutral string case, where both ends of the string are on the same $D$-brane, the noncommutativity parameter at one end of the string is equal to minus that of the other end. For $U(N)$ Chan-Paton factors, the background magnetic fields can take on $N$ possible values, giving rise to $N$ noncommutativity parameters.

We remark that had we computed the charged string propagator between the states $\langle x_- |$ and $| x_+ \rangle$, rather than $\langle \beta |, | \alpha \rangle$, we would have found the same noncommutativity parameters as derived in (5.2) and (5.5), but the $A \to 0$ limit of this propagator would be different from the neutral string expression.
6. Short Distance Behavior and Star Products

For \( \tau > \tau' \) and at \( \sigma = 0 \), the leading short distance singularity in the product of tachyon vertex operators is

\[
e^{ip\cdot X(\tau)} e^{iq\cdot X(\tau')} = e^{i(p+X^+(\tau')+i(p+X^-(\tau'))} \sim e^{i(p+X^+(\tau)+i(p+X^-)(\tau')} \cdot e^{i(p+X^+(\tau')+i(p+X^-)(\tau')}
\]

\[
= e^{-4\alpha'(Gp+q)n \sum_{n \geq 0} \frac{1}{n-A}(\frac{\tau'}{n+1})^{n+A}} e^{-4\alpha'(Gp+q)n \sum_{n > 0} \frac{1}{n-A}(\frac{\tau'}{n})^{n-A}}
\]

\[
\sim e^{(\tau - \tau')^4\alpha'(Gp+q-i(p+q)\cdot X(\tau')}
\]

where we have defined

\[
X^+_<(\tau) \equiv i\sqrt{2\alpha'} \sum_{n \geq 0} \frac{a_{n+A}}{n+A} \tau^{-n-A}, \quad X^-<_<(\tau) \equiv i\sqrt{2\alpha'} \sum_{n > 0} \frac{\bar{a}_{n-A}}{n-A} \tau^{-n-A}.
\]

In order to derive the next to the last line in (6.1) we need to use the same identities used in computing (5.2). Note that (5.1) holds for fixed \( A \). In the limit scaling limit \([2]\), \( (\alpha' \to 0, \text{keeping } G \text{ and } \Theta^{12} \text{ fixed}) \), the OPE reduces to the star product.

\[
e^{ip\cdot X(\tau)} e^{iq\cdot X(\tau')} \sim e^{-\Theta^{12}(p+q-p\cdot X(\tau'))} \cdot e^{i(p+q)\cdot X(\tau')}
\]

\[
\equiv e^{ip\cdot X(\tau')} \ast e^{iq\cdot X(\tau')}.
\]

For \( \sigma = \pi \) the same equations (5.1),(5.3), will hold with \( \Theta^{12} \) replaced by \( \tilde{\Theta}^{12} \).

To accommodate the existence of many noncommutativity parameters, the non-abelian \( U(N) \) gauge theory (whose expansion around a \( U(1)^N \) background we are considering here), must require a generalization of the usual noncommutative star product. It would be interesting to figure out the Lagrangian for our noncommutative gauge theory as well as the underlying \( U(N) \) gauge transformations.
7. The Spectrum in the Scaling Limit

As in [2], we consider the scaling limit $g^{-1} \to \epsilon$ and $\alpha' \to \sqrt{\epsilon}$, for $\epsilon \to 0$. The other quantities $B_{12}, q_1, q_2$ are held fixed. Actually this limit means letting the dimensionless quantity $\alpha' B_{12} \to \sqrt{\epsilon}$, while keeping $B_{12}$ fixed. Then we have that the noncommutativity parameters are finite in the scaling limit and are given by $\Theta^{12} \to (q_1 B_{12})^{-1}$, $\tilde{\Theta}^{12} \to (q_2 B_{12})^{-1}$.

Starting from (2.8), we have $\tan \pi A = B + \frac{q_2}{q_1} B_{12} - \frac{q_2}{q_1} B_{2}$. So in the scaling limit,

$$\tan \pi A \to -\frac{(q_1 + q_2)\sqrt{\epsilon}}{2\pi q_1 q_2}, \quad \text{(7.1)}$$

and

$$A \to -\frac{(q_1 + q_2)\sqrt{\epsilon}}{2\pi^2 q_1 q_2} \quad \text{(7.2)}$$

Therefore the mass formulae for the two polarization states of the charged vector we have listed in (3.6) are also finite in the scaling limit and behave as

$$\text{mass}^2 = -\frac{1}{2\alpha'} A(1 + A) \to \frac{1}{2} \frac{(q_1 + q_2)B_{12}}{2\pi^2 q_1 q_2} = \frac{(q_1 + q_2)G}{q_2 \Theta^{12}}, \quad \text{(7.3)}$$

$$\text{mass}^2 = -\frac{3}{2\alpha'} A(3 - A) \to -\frac{3}{2} \frac{(q_1 + q_2)B_{12}}{2\pi^2 q_1 q_2} = -3 \frac{(q_1 + q_2)G}{q_2 \Theta^{12}}.$$

Also in the spectrum are the charged vectors at different Landau levels. For each charged boson, the two polarizations have different masses. They differ from those of (7.3) by integer multiples of $\frac{2(q_1 + q_2)G}{q_2 \Theta^{12}}$. (Notice that each Landau level has infinite degeneracy labeled by $|x_+\rangle$). All other states in the charged string spectrum become infinitely heavy and decouple in the scaling limit. So the complete spectrum of our $U(1)^N$ noncommutative field theory, which is derived from $N$ neutral and $N^2 - N$ charged string sectors, is described by $N$ massless neutral gluons and the charged vectors above.

We have shown that our theory, which is a $U(N)$ gauge theory expanded around a $U(1)^N$ background, has a scaling limit. It would be of interest to work out the scaling limit of the full non-abelian gauge theory and its noncommutativity parameters.

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