We introduce two independent quantifications for 3-mode and 4-mode entanglement. We investigate the conversion of one type of nonclassicality, i.e. single-mode nonclassicality, into another type of nonclassicality, i.e. multi-mode entanglement, in beam-splitters. We observe parallel behavior of the two quantifications. The methods can be generalized to the quantification of any multi-mode entanglement.

Multi-mode continuous variable entanglement is a key element for quantum information processing. Quantum teleportation among multiple parties [1], quantum computation with clusters [2] [3], e.g. quantum networks, [4] and quantum internet [5], all, necessitate the presence of multi-mode entanglement. Multi-mode entanglement can be generated using several number of beam-splitters which converts the single-mode nonclassicality (SMNc), e.g. quadrature squeezing, of input beam(s) into multi-mode entanglement at the outputs of the beam splitters [6]. Hence, mechanism of the conversion of nonclassicality into multi-mode entanglement, via beam splitters, is important for distribution of the entanglement among multi-parties used in quantum information.

One can notice that quantum nonclassicality is like "energy". That is, it can be converted into different forms. For instance, single-mode nonclassicality (SMNc), two-mode entanglement (TME) and multiparticle entanglement (MPE) are different forms nonclassicality (Nc) and can be converted to each other via linear interactions. A quadrature-squeezed light (SMNc) can transfer its squeezing to an ensemble of atoms [8] [10] as spin-squeezing [11]. This creates MPE [7] [12] in the ensemble. Similarly, interaction of an ensemble with two entangled beams (TME) also creates MPE in the ensemble [13] [14]. It is also possible to convert the nonclassicality of a single-mode, e.g. quadrature-squeezing, into two-mode entanglement at the output of a beamsplitter (BS) [15] [16]. Generation of TME in a BS necessitates a nonclassical input light [15].

Recent works [17] [20] show that there appears a conservation-like relation between the generated TME and the remaining SMNc in a BS. A BS cannot convert all of the nonclassicality of the input beam into TME at the output. There still remains some SMNc in the two output modes [17]. However, the total SMNc decreases with respect to the input one. The form of the analytical expressions for the maximum TME extracted at the BS output, $E_N = \max\{0, -\frac{1}{2} \log_2(\lambda_1^{\text{sm}} \lambda_2^{\text{sm}})\}$ [21] [22], suggests us to quantify the SMNc in terms of a noise-area $\Omega = \lambda_{1,\text{sm}} \lambda_{2,\text{sm}}$. Here, $\lambda_{i,\text{sm}}$ is the minimum noise of the $i$th beam input into the BS, where $\lambda_{1,\text{sm}} < 1$ implies the presence of squeezing in the $i$th beam. For mixing with vacuum or a coherent state, one of the modes becomes $\lambda_{1,\text{sm}} = 1$.

As an illuminating example: if a squeezed beam is mixed with a thermal noise [21] at the input channels of a BS, the output modes are entangled only if the noise-area $\Omega = \lambda_{\text{sq}} (1 + 2\bar{n}) < 1$, where $\lambda_{\text{sq}} < 1$ is the reduced noise of the squeezed beam and $\bar{n}$ is the mean number of photons in the thermal noise. Also Ref. [22] shows that the maximum amount of entanglement extractable at the BS output is $E_N = -\frac{1}{2} \log_2(2\bar{n}) = -\frac{1}{4} \log_2(\lambda_{1,\text{sm}} \lambda_{2,\text{sm}})$ if any two Gaussian states are mixed in the BS input. Here, one can observe that the smallest symplectic eigenvalue of an inseparable system [23] [24] becomes $\mu = 2\nu = \lambda_{1,\text{sm}} \lambda_{2,\text{sm}}$, which is actually the input noise-area. In Ref. [17], we further show that the output TME is proportional to the change (increase) in the noise-area $S_N = \log_2(\lambda_{1,\text{sm}}^{\text{out}} \lambda_{2,\text{sm}}^{\text{out}}) - \log_2(\lambda_{1,\text{sm}}^{\text{in}} \lambda_{2,\text{sm}}^{\text{in}})$ of the out beams with respect to the input beams. A geometric demonstration of this SMNc→TME swap can be found in Fig. 1 of Ref. [25]. More interestingly, well-known TME criteria, like Duan-Giedke-Cirac-Zoller [26] and Hillery-Zubairy [27], actually do search for a noise-area below unity in BS-like rotations [25].

In the present work, we investigate the swapping of single-mode nonclassicality into multi-mode entanglement (MME). In particular, we study three-mode entanglement (3ME) and four-mode entanglement (4ME) after two (Fig. 1) and three beam-splitters (Fig. 3), respectively. We define the remaining SM nonclassicalities in the 3 modes (or 4 modes) as a noise-volume (or a 4D noise-volume), $\Omega = \lambda_{1,\text{sm}}^{\text{out}} \lambda_{2,\text{sm}}^{\text{out}} \lambda_{3,\text{sm}}^{\text{out}}$. Here, $\lambda_{1,\text{sm}}^{\text{out}} = 1 - \lambda_{1,\text{sm}}^{\text{in}}$ is the degree of the remaining SMNc, i.e. the nonclassical depth [28], of each mode after the BSs. It is calculated by wiping out the entanglement (correlations) between the modes [17]. We show that the SMNc decreases (noise-volume $\Omega$ increases) after the BSs while the the 3ME increases.

In the quantification of 3ME we use two different (independent) approaches. (i) We multiply the three noises-
associated with the symplectic eigenvalues \((\nu_1, \nu_2, \nu_3)\) of the partial transposed system. \(\nu_1\), for instance, is the smallest symplectic eigenvalue of the 3-mode system when the 1st mode is partial transposed \([23, 24]\). That is, smaller values of the eigenvalue compared to unity, i.e. \(2\nu_1 < 1\), imply stronger entanglement of the 1st mode with the system composed of \((2\text{nd} + 3\text{rd})\) modes \([29]\).

Similarly, \(2\nu_2 < 1\) refers to the inseparability of the 2nd mode from the system of \((1\text{st} + 3\text{rd})\) modes. Hence, \(\mu = 2\nu_1 2\nu_2 2\nu_3\) refers to a kind of 3-mode entanglement strength, where \(2\nu_1 > 1\) implies the absence of a genuine 3-mode entanglement.

(ii) Second, we use an alternative method given in Sec. II.3.2 of Ref. \([25]\). The idea is very simple: Nonclassical depth \(\tau\) quantifies the whole nonclassicality, i.e. SMNC + entanglement, in a multi-mode system \([22]\). If we remove the unconverted (unused) SM nonclassicalities (after the BSs) from the noise-matrix, then the remaining nonclassicality is due to the entanglement only\(^2\).

Nonclassicality of a single-mode state can be determined by introducing a Gaussian filter function transformation on the Glauber-Sudarshan \(P\)-function, i.e. \(P(\alpha, \tau) = \int d^2 \alpha' \exp\{-|\alpha - \alpha'|^2/\tau\} P(\alpha') / \tau \pi\). So that, the new \(P\)-function is non-negative \([28]\). This corresponds to injecting noise which destroys the nonclassicality \([30]\). A similar method can be used also to determine the nonclassicality of a multi-mode system \([22]\). Here, in difference to Ref. \([22]\), we introduce different \(\tau\)'s for each mode. This way, we prevent the injection of unnecessary noise \((\tau)\) by constraining \(\tau_1 = \tau_2 = \tau_3 = \tau\).

We emphasize that, our aim, in this short manuscript, is to introduce the basics of two possible quantifications for the multi-mode entanglement (MME). We do not aim to present a detailed analysis on MME, but we suffice with demonstrating that SMNC, quantified as a noise-volume, is converted into MME.

\section*{3-mode entanglement}

We study the system depicted in Fig. 1. A single-mode \((\text{SM})\) nonclassical state \(\hat{a}\) is mixed with vacuum noise in a BS with two output states \(\hat{a}_1\) and \(\hat{a}_2\). One of the output modes, \(\hat{a}_2\), is input to a second BS, mixed with vacuum, resulting two new output modes \(\hat{b}_1\) and \(\hat{b}_2\). We examine the 3-mode entanglement \((\text{3ME})\) of \(\hat{a}_1, \hat{b}_1,\) and \(\hat{b}_2\) modes.

In Fig. 2a and Fig. 2b, we examine the 3ME of the \(\hat{a}_1, \hat{b}_1, \hat{b}_2\) modes using the two methods, (i) and (ii), respectively. We observe a similar behavior for the two quantifications. In Fig. 2c, one can observe that the SM nonclassicalities remaining in the \(\hat{a}_1, \hat{b}_1, \hat{b}_2\) modes decreases (noise-volume increases), while the 3ME increases. In all Fig. 2a-2c, smaller \(\mu, \eta\) and \(N\) imply stronger 3ME or SMNC. The first BS fed with a nonclassical light of squeezing parameter \([31]\) \(r = 0.1\). We fix the angle of the first BS to \(\theta_{\text{BS}_1} = \pi/4\) and vary the angle of the second BS \(\theta_{\text{BS}_2}\). A similar behavior is obtained for varying \(\theta_{\text{BS}_1}\) with a fixed \(\theta_{\text{BS}_2}\).

\(^2\) In Ref. \([17]\), we do the reverse. We remove the correlations in the noise matrix and examine the remaining SM nonclassicalities.
matrix as follows.

**Calculation of $\mu$.—** We compose the $6 \times 6$ noise-matrix $V_{ij} = (\xi_i \xi_j + \xi_i \xi_j) / 2 - (\xi_i) (\xi_j)$ for the mode system in the real representation [32] by introducing the operator $\hat{\xi} = [\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \hat{x}_3, \hat{p}_3]$, where $\hat{x}_i = (\hat{c}_i^\dagger + \hat{c}_i) / \sqrt{2}$ and $\hat{p}_i = i(\hat{c}_i^\dagger - \hat{c}_i) / \sqrt{2}$ are obtained using $\hat{c}_i = a_1 b_1 a_2 b_2$ respectively. Partial transpose (PT) operation, e.g., on the 1st mode is equivalent to $\hat{p}_1 \rightarrow -\hat{p}_1$ in the noise-matrix [23]. If the 1st mode is separable from the system of (2+3) modes, symplectic eigenvalues of the partial transposed noise-matrix must be all larger than $1 / 2$ [23, 24]. (So, smallest eigenvalue $\nu_1$ also satisfies $2 \nu_1 > 1$.) The symplectic eigenvalues of the partial transposed noise-matrix can alternatively be calculated as $\text{eig}[\chi_1 V]$ where $\chi_1$ is the $6 \times 6$ matrix $\chi_1 = [J, 0_{2 \times 2}, 0_{2 \times 2}; 0_{2 \times 2}, J, 0_{2 \times 2}, 0_{2 \times 2}, J]$ with $0_{2 \times 2}$ is $2 \times 2$ matrix of zeros and $J = [0, 1; -1, 0]$. For $\chi_2$ and $\chi_3$ the "\(\pm\)" sign must be in the second and third $J$, respectively. $\text{eig}[\chi_1 V]$ yields only a single eigenvalue with $2 \nu_1 < 1$. Similarly, $\nu_2 (\nu_3)$ is the only $2 \nu_2 < 1 (2 \nu_3 < 1)$ eigenvalue from the partial transposition of the 2nd (3rd) mode.

We remind one more time that introduction of the noise-volume $\mu = 2 \nu_1 2 \nu_2 2 \nu_3$ via the symplectic eigenvalues follows from the observation $\text{log}_2 2 \nu_1 = \text{log}_2 (\lambda_1^{(in)} \lambda_2^{(in)})$ for a single BS [17, 21, 22], where $2 \nu_1 = \lambda_1^{(in)} \lambda_2^{(in)}$ determines the maximum entanglement extractable from the input noise-area $\lambda_1^{(in)} \lambda_2^{(in)}$.

**Calculation of $\eta_{\text{ent}}$.—** The nonclassical depth, associated only with the entanglement, is calculated as follows. We first transform the real noise-matrix $V$ into the complex representation [32] $V^{(c)} = CV^\dagger C^\dagger$, where $C = [C_1, 0_{2 \times 2}, 0_{2 \times 2}; 0_{2 \times 2}, C_1, 0_{2 \times 2}, C_1, 0_{2 \times 2}, 0_{2 \times 2}]$ with $C_1 = [1, i; 1, -i] / \sqrt{2}$. In the $6 \times 6$ complex noise-matrix $V^{(c)}$, we wipe out the SM nonclassicalities of the 3 modes, e.g. in the 1st mode, by replacing the $[V_{11}, V_{12}, V_{21}, V_{22}]$ with $[0, 1/2, 0, 1/2]$, the noise-matrix for vacuum or a coherent state [24]. We wipe out the SMnc of the other $2 \times 2$ block-diagonals similarly. Then, we obtain the new noise-matrix $V^{(c)}_{\text{ent}}$, where the nonclassical depths $\tau_1, \tau_2, \tau_3$ accounts the entanglement only. For Gaussian states, we consider here, this can be performed by calculating the $\tau_1, \tau_2, \tau_3$ which makes all $\text{eig}[V^{(c)}_{\text{ent}} + \tau]$ positive where $\tau = \text{diag}(\tau_1, \tau_2, \tau_2, \tau_3, \tau_3)$. Ref. [22] assigns a single $\tau_1 = \tau_2 = \tau_3 = \tau$ for the $\tau$ matrix which certainly increases the injected noise. We quantify the nonclassicality of $V^{(c)}_{\text{ent}}$, or the entanglement, by choosing the minimum of $\tau = [\tau_1 \tau_2 \tau_3]_{\text{min}}$, or $\tau_{\text{ent}} = [(1 - 2 \tau_1)(1 - 2 \tau_2)(1 - 2 \tau_3)]_{\text{max}}$ in terms of the injected noise-volume. We note that, for a single-mode, $\lambda_{\text{sm}} = (1 - 2 \tau)$ corresponds to the reduced noise of that particular mode.

**Calculation of $N_{\text{SMnc}}$.—** In the calculation of SMnc $N_{\text{SMnc}}$, this time, we wipe out the correlations in the noise-matrix and left with the three $2 \times 2$ block-diagonals. $2 \times 2$ matrices give the SMnc associated with each mode [17]. Then, we introduce the SMnc noise-volume $N_{\text{SMnc}} = (1 - 2 \tau^{(SMnc)}_1)(1 - 2 \tau^{(SMnc)}_2)(1 - 2 \tau^{(SMnc)}_3)$.

**4-mode entanglement**

![Fig. 3. Creation of 4-mode entanglement (4ME) with 3 beam-splitters. We examine the 4ME of $b_1, b_2, c_1, c_2$ modes. We also investigate the conversion of the single-mode nonclassicality into 4ME, see Fig. 4]  

We perform similar calculations also for a 4-mode system given in Fig. 3. We obtain the same behavior depicted in Fig. 4.

**Summary**

In summary, we introduce quantifications for 3-mode and 4-mode entanglement via two independent methods. We demonstrate how single-mode nonclassicality is converted into 3-mode and 4-mode entanglement. We quantify all nonclassicalities in terms of noise-volume. A
smaller noise-volume implies a stronger nonclassicality or entanglement. The method we introduce here can be generalized to other multi-mode entanglement which has fundamental importance in quantum communication.

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