On the Convergence of Momentum-Based Algorithms for Federated Stochastic Bilevel Optimization Problems

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Abstract

In this paper, we studied the federated stochastic bilevel optimization problem. In particular, we developed two momentum-based algorithms for optimizing this kind of problem. In addition, we established the convergence rate of these two algorithms, providing their sample and communication complexities. To the best of our knowledge, this is the first work achieving such favorable theoretical results.

1 Introduction

In recent years, Federated Learning has attracted a surge of attention due to its great potential application in numerous real-world computer vision and machine learning tasks. As such, a wide variety of federated optimization algorithms have been proposed under various settings. However, most of them only focus on the standard minimization problem, which are incapable of solving the emerging machine learning problems, such as bilevel optimization problems in meta learning [4], minimax optimization problems in adversarial learning [14, 6]. In this paper, we aim to develop new optimization algorithms for federated stochastic nested optimization problems, which is defined as follows:

$$\min_x \frac{1}{K} \sum_{k=1}^{K} f^{(k)}(x, y^*(x))$$

s.t. \( y^*(x) = \arg\min_y \frac{1}{K} \sum_{k=1}^{K} g^{(k)}(x, y) \),

(1.1)

where \( K \) is the total number of devices, \( f^{(k)}(x, y^*(x)) = \mathbb{E}_{\xi \sim A^{(k)}} [f^{(k)}(x, y^*(x); \xi)] \) denotes the loss function of the upper-level optimization problem on the \( k \)-th device, \( g^{(k)}(x, y) = \mathbb{E}_{\zeta \sim B^{(k)}} [g^{(k)}(x, y; \zeta)] \) represents the loss function of the lower-level optimization problem on the \( k \)-th device, \( A^{(k)} \) and \( B^{(k)} \) are the data distribution on the \( k \)-th device. It can be observed that the upper-level loss function depends on the optimal solution of the lower-level optimization problem. Regarding Eq. (1.1), this kind of stochastic nested optimization problem is typically called stochastic bilevel optimization problem [1], which covers numerous computer vision and machine learning models, such as meta-learning, hyperparameter optimization.

To solve the stochastic bilevel optimization problem, a wide variety of stochastic gradient based algorithms under the single-machine setting have been developed in the past few years. Specifically, [5] developed a double-loop stochastic gradient algorithm where the model parameter \( y \) of the lower-level optimization problem is updated by stochastic gradient descent (SGD) for multiple iterations in the inner-loop and then the model parameter \( x \) is updated. The convergence rate of this algorithm is established for nonconvex-strongly-convex problems. Later, the convergence rate is further improved in [9, 12] by new algorithm design. Recently, a couple of single-loop algorithms have been proposed where the model parameter \( x \) and \( y \) are updated simultaneously. Among them, the momentum-based algorithm has attracted much attention. For instance, [7] applied the moving-average momentum to the stochastic bilevel optimization problem and established its convergence rate. [14, 8, 13] applied the momentum-based variance reduction technique to optimize the stochastic bilevel problem and obtained a better convergence rate. However, all these algorithms just focus on the single-machine setting. When the data is distributed on multiple devices, new optimization algorithms should be designed to coordinate the collaboration among multiple devices.

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To handle the distributed learning setting, federated learning has been developed and extensively studied in recent years. Essentially, in federated learning, the model is updated on each device for multiple iterations and then shared across devices. Based on this learning schema, numerous federated optimization algorithms [15, 16, 18] have been studied. For instance, [15] studied the convergence rate of local SGD for strongly-convex problems. [18] established the convergence rate of momentum local SGD for nonconvex problems. However, all these studies only focus on the standard minimization problem. As far as we know, the federated stochastic bilevel optimization problem has not been studied yet. On the other hand, numerous machine learning models can be formulated as a stochastic bilevel optimization problem. Thus, it is of importance to study how the stochastic bilevel optimization algorithm converges under the federated learning setting to establish its convergence rate.

1.1 Contribution

In this paper, we studied the momentum-based optimization algorithms for the federated stochastic bilevel optimization problem. In particular, we developed a local bilevel stochastic gradient with momentum (LocalBSGM) algorithm where each device leverages the momentum stochastic gradient to update the model parameter $x$ and $y$ locally for multiple iterations and then communicates the updated model parameters with the central server. However, it is challenging to establish its convergence rate due to the interaction between the bilevel structure and the gradient momentum. In this paper, we addressed this challenging problem by a novel theoretical analysis strategy, establishing the convergence rate of our algorithm. To the best of our knowledge, this is the first federated optimization algorithm with theoretical guarantees for Eq. (1.1).

Moreover, to further improve the convergence rate, we developed a new local bilevel stochastic gradient with momentum-based variance reduction (LocalBSGVRM) algorithm where each device employs a momentum-based stochastic variance-reduced gradient to update the model parameter $x$ and $y$ simultaneously to reduce the computational cost and conducts the communication with the central server periodically to save the communication cost. Compared with LocalBSGM, the variance-reduced gradient estimator makes LocalBSGVRM converge faster. In particular, our theoretical analysis demonstrates that LocalBSGVRM enjoys better sample and communication complexities than LocalBSGM. To the best of our knowledge, this is the first algorithm achieving such sample and communication complexities for federated bilevel optimization problems. To sum up, we made the following contributions in this paper.

- We developed a novel local bilevel stochastic gradient with momentum algorithm for the federated bilevel optimization problem and established its convergence rate.
- We proposed a novel local bilevel stochastic gradient with momentum-based variance reduction algorithm to improve the convergence rate of our first algorithm, which can achieve better sample and communication complexities.

2 Related Work

To solve the bilevel optimization problem in machine learning, a large number of gradient-based bilevel optimization algorithms have been proposed in recent years. Especially, [5] developed a stochastic gradient based algorithm and established its convergence rate. After that, a series of algorithms were proposed to improve the convergence rate. For instance, [9] developed a two-timescale based algorithm to coordinate the updates of the model parameters in the upper-level problem and the lower-level problem. [12] proposed to use a large batch size to improve the convergence rate. Recently, inspired by the variance reduction technique in the standard minimization problem, a couple of variance-reduced algorithms have been proposed to improve the convergence rate. For instance, [7] incorporated the momentum technique into bilevel stochastic gradient descent and established its convergence rate. However, its theoretical convergence rate is in the same order with that of [12]. [17] incorporated the momentum-based variance reduction technique to bilevel stochastic gradient descent, which orderwisely improves the convergence rate of other algorithms. In addition, [17] employed the SPIDER gradient estimator to reduce the gradient variance. As such, as [17], this algorithm also enjoys a better convergence rate than [12]. More recently, a couple of gradient-based bilevel optimizers have been proposed to deal with the nonsmooth problem and the adaptive learning rate. However, all these algorithms only studied the single-machine setting. Thus, it is of importance to develop federated optimization algorithms for solving Eq. (1.1).
3 Momentum-Based Algorithms for Federated Bilevel Optimization Problems

3.1 Preliminaries

For Eq. (1.1), to compute the stochastic gradient of the upper-level function regarding the model parameter \( x \), we first introduce the auxiliary function \( \Phi(k)(x) = f'(k)(x, y^*(x)) \) and \( \Phi(x) = \frac{1}{K} \sum_{k=1}^{K} \Phi(k)(x) \) where \( y^*(x) \) denotes the optimal solution of the lower-level optimization problem. Then, we can compute the gradient regarding \( x \) as follows:

\[
\nabla \Phi(k)(x) = \nabla_x f(k)(x, y^*(x)) - \nabla^2_{xy} g(k)(x, y^*(x)) H \nabla_y f(k)(x, y^*) \, , \tag{3.1}
\]

where \( H = [\nabla^2_{yy} g(k)(x, y^*)]^{-1} \). However, this gradient is typically infeasible because the optimal solution \( y^*(x) \) is expensive to obtain in practice. Thus, a commonly strategy \cite{17} is to use the following gradient to approximate it:

\[
\nabla \Phi(k)(x, y) = \nabla_x f(k)(x, y) - \nabla^2_{xy} g(k)(x, y) \hat{H} \nabla_y f(k)(x, y) \, , \tag{3.2}
\]

where \( \hat{H} = \theta \sum_{q=1}^{Q} \prod_{q=Q}^{Q} (I - \theta \nabla^2_{yy} g(k)(x, y)) \) and \( \theta > 0 \) and \( Q > 0 \) are hyperparameters. Based on this approximation, we can compute the stochastic gradient regarding \( x \) as follows:

\[
\nabla \Phi(k)(x, y; \hat{\xi}) = \nabla_x f(k)(x, y; \xi) - \nabla^2_{xy} g(x, y; \xi) \hat{H} \nabla_y f(k)(x, y; \xi) \, , \tag{3.3}
\]

where \( \hat{H} = \theta \sum_{q=1}^{Q} \prod_{q=Q}^{Q} (I - \theta \nabla^2_{yy} g(x, y; \xi)) \) and \( \hat{\xi} = \{\xi, \hat{\xi}_t\} \).

To investigate the convergence rate of the federated bilevel optimization algorithm, we introduce the following common used assumptions.

Assumption 1. For any \( k \in \{1, 2, \ldots, K\} \), the function \( g^{(k)}(x, y; \xi) \) is \( \mu \)-strongly convex regarding \( y \) for any \( x \).

Assumption 2. For any \( k \in \{1, 2, \ldots, K\} \), the function \( f^{(k)}(x, y; \xi) \) and \( g^{(k)}(x, y; \xi) \) satisfy:

- \( f^{(k)}(x, y; \xi) \) is \( L_0 \)-Lipschitz continuous.
- \( \nabla f^{(k)}(x, y; \xi) \) and \( \nabla g^{(k)}(x, y; \xi) \) are \( L_1 \)-Lipschitz continuous.
- \( \nabla^2_{xy} g^{(k)}(x, y; \xi) \) is \( L_2 \)-Lipschitz continuous. \( \nabla^2_{yy} g^{(k)}(x, y; \xi) \) is \( L_2 \)-Lipschitz continuous.

Assumption 3. For any \( k \in \{1, 2, \ldots, K\} \), \( \| \nabla g^{(k)}(x, y) - \nabla g^{(k)}(x, y; \xi) \|^2 \leq \sigma^2 \).

Assumption 4. The devices have heterogeneous data distributions such that

\[
\frac{1}{K} \sum_{k=1}^{K} \| \nabla \Phi(k)(x, y) - \frac{1}{K} \sum_{k=1}^{K} \nabla \Phi(k)(x, y) \| \leq \delta^2 \quad \text{and} \quad \frac{1}{K} \sum_{k=1}^{K} \| \nabla g^{(k)}(x, y) - \frac{1}{K} \sum_{k=1}^{K} \nabla g^{(k)}(x, y) \| \leq \delta^2 .
\]

Based on these assumptions, one can get that the condition number is \( \kappa = \frac{L_2}{\mu} \). In addition, from \cite{17}, one can know that \( \Phi(k)(x) \) is \( L_3 \)-smooth where \( L_3 = L_1 + \frac{2L_1^2 + L_1^3}{\mu} + \frac{L_2^3}{L_3^2} + \frac{L_2^3}{L_3} + \frac{L_3}{L_2} \).

Throughout this paper, we denote \( \bar{a} = \frac{1}{K} \sum_{k=1}^{K} a^{(k)} \).

3.2 Local Bilevel Stochastic Gradient Descent with Momentum

In this subsection, we develop a novel local bilevel stochastic gradient descent with momentum algorithm in Algorithm\textsuperscript{[11]} In detail, each device \( k \) computes the momentum for stochastic gradients \( \nabla \Phi(k)(x^t_k, y^t_k; \xi^t_k) \) and \( \nabla_y g^{(k)}(x^t_k, y^t_k; \xi^t_k) \) as follows:

\[
u^t_k = (1 - \alpha \eta) u^t_{k-1} + \alpha \eta \nabla \Phi(k)(x^t_k, y^t_k; \xi^t_k) \, , \tag{3.4}
\]

where \( \alpha > 0, \beta > 0, \eta > 0 \) are hyperparameters, \( \alpha \eta < 1, \beta \eta < 1 \). It can be observed that \( u^t_k \) and \( v^t_k \) are the moving-average estimation for those two stochastic gradients. Then, the \( k \)-th device leverages those two momentum to update its local model parameters as follows:

\[
x^t_{k+1} = x^t_k - \rho_1 \eta u^t_k \, , \tag{3.5}
\]

\[
y^t_{k+1} = y^t_k - \rho_2 \eta v^t_k ,
\]
Algorithm 1 LocalBSGM

Input: \( x_0^{(k)} = x_0, y_0^{(k)} = y_0, p > 1, \eta > 0, \alpha > 0, \beta > 0, \rho_1 > 0, \rho_2 > 0. \)

Conduct following steps for all devices.
1: for \( t = 0, \cdots, T - 1 \) do
2: if \( t == 0 \) then
3: With the batch size being 1, compute:
4: \[ u_t^{(k)} = \nabla \Phi^{(k)}(x_t^{(k)}, y_t^{(k)}; \xi_t^{(k)}), \]
5: else
6: With the batch size being 1, compute:
7: \[ u_t^{(k)} = (1 - \alpha \eta) u_t^{(k)} + \alpha \eta \nabla \Phi^{(k)}(x_t^{(k)}; \xi_t^{(k)}), \]
8: \[ v_t^{(k)} = (1 - \beta \eta) v_t^{(k)} + \beta \eta \nabla y g^{(k)}(x_t^{(k)}) \]
9: end if
10: \[ x_{t+1}^{(k)} = x_t^{(k)} - \rho_1 \eta u_t^{(k)}, \]
11: if \( \text{mod}(t+1, p) = 0 \) then
12: Upload \( u_{t+1}^{(k)}, v_{t+1}^{(k)}, x_{t+1}^{(k)}, y_{t+1}^{(k)} \) to server and reset
13: end if
14: end for

where \( \rho_1 > 0 \) and \( \rho_2 > 0 \) are two hyperparameters. As the standard federated optimization algorithm, each device communicates both local momentum and model parameters with the central server at every \( p \) (where \( p > 1 \) iterations, which is shown in Line 12 of Algorithm 1).

In Theorem 1, we establish the convergence rate of Algorithm 1.

**Theorem 1.** Suppose Assumptions [1][2][4] hold, by setting \( \alpha > 0, \beta > 0, \theta < \frac{1}{L} \), \( \rho_1 \leq \left\{ \frac{3 \rho_2 \mu}{50 \alpha L}, \frac{1}{\eta T} \right\}, \rho_2 \leq \left\{ \frac{6 L^2}{\mu} \left( \frac{4 L^2}{\alpha^2} + \frac{400 L^2 \sigma^2}{3 \beta^2 \mu^2} \right), \frac{1}{6 L} \right\} \), \( \eta \) is such \( \left\{ \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{2 \rho_1 L \sigma}, \frac{1}{3 \rho_2 (\alpha^2 + \beta^2)^{1/4} (\rho_1^2 + \rho_2^2)^{1/4} L^{1/2}} \right\} \), Algorithm 1 has the following convergence rate:

\[
\frac{1}{T} \sum_{t=0}^{T-1} \left( \| \nabla \Phi(\tilde{x}_t) \|^2 + \tilde{L}^2 \| y_t - y^*(\tilde{x}_t) \|^2 \right) \\
\leq \frac{2 (\Phi(x_0) - \Phi(x_*))}{\eta \rho_1 T} + \frac{20 \tilde{L}^2}{\eta T \rho_2 \mu} \| y_0 - y^*(\tilde{x}_0) \|^2 \\
+ \frac{4 G^2}{\alpha T} + \frac{500 \tilde{L}^2 \sigma^2}{3 \eta T \beta \mu^2} + \frac{4 \alpha \eta \sigma^2}{K} + \frac{500 \beta \eta \sigma^2 \tilde{L}^2}{3 K \mu^2} + 4 \Delta_0^2 \\
+ 24 \alpha \eta \sigma^2 p (\rho_1^2 + \rho_2^2) \left( 2 \tilde{L}^2 + \frac{500 L_4^2 \tilde{L}^2}{3 \mu^2} \right) (2 G^2 + 3 \delta^2) \\
+ 24 \beta \eta \sigma^2 q (\rho_1^2 + \rho_2^2) \left( 2 \tilde{L}^2 + \frac{500 L_4^2 \tilde{L}^2}{3 \mu^2} \right) (2 \sigma^2 + 3 \delta^2) \\
+ 24 \alpha^2 \eta \sigma^2 \rho_1^2 (\rho_1^2 + \rho_2^2) \left( 4 \tilde{L}^2 + \frac{500 L_4^2 \tilde{L}^2}{3 \mu^2} \right) (2 G^2 + 3 \delta^2) \\
+ 24 \beta^2 \eta \sigma^2 \rho_1^2 (\rho_1^2 + \rho_2^2) \left( 4 \tilde{L}^2 + \frac{500 L_4^2 \tilde{L}^2}{3 \mu^2} \right) (2 \sigma^2 + 3 \delta^2),
\]

where \( \tilde{L}^2 = 2 L_4^2 + 4 \theta^2 L_5^2 L_2^2 (Q+1)^2 + 8 \theta^2 L_1^2 (Q+1)^2 + 2 \theta^4 L_0^2 L_1^2 L_2^2 Q^2 (Q+1)^2, \tilde{L} = L_1 + \frac{L^2}{\mu} + \frac{L_0 L_2}{\mu} + \frac{L_0 L_1 L_2}{\mu^2} \).
\[ \Delta_Q = \frac{(1-\theta_0)\bar{O}(\kappa)}{\mu} L_t. \]

**Remark 1.** From Theorem 4, it can be observed that \( \rho_1 = O(\frac{1}{\kappa}) \) and \( \rho_2 = O(\frac{1}{\kappa^2}). \)

**Corollary 1.** Suppose Assumptions 1-4 hold, by setting \( \alpha > 0, \beta > 0, T = O(\frac{1}{\kappa^2}), p = O(\frac{1}{\kappa^2}), \n = O(K\epsilon), \) and \( Q = O(\log(\frac{1}{\epsilon})), \) one can get

\[ \frac{1}{T} \sum_{t=0}^{T-1} \left( \| \nabla \Phi(x_t) \|^2 + \tilde{L}^2 \| y_t - y^*(x_t) \|^2 \right) \leq \epsilon. \] (3.7)

**Remark 2.** From Corollary 4, one can know that the iteration complexity is \( O(\frac{K^2}{L^2}) \), which indicates that our LocalBSGM can achieve linear speedup with respect to the number of devices. In addition, the gradient complexity and the Jacobian-vector complexity is \( O(\frac{K^2}{L^2}) \) and the Hessian-vector complexity is also \( \tilde{O}(\frac{K^2}{L^2}) \). Note that when \( K = 1 \), these complexities can match those of the algorithms for the single-machine setting without using variance-reduction techniques, such as [12, 13]. Moreover, the communication complexity of our algorithm is \( T/p = O(\frac{K^2}{L^2}). \)

**Algorithm 2 LocalBSGVRM**

**Input:** \( x_1^{(k)} = x_1, y_1^{(k)} = y^*(x_1), p > 1, \alpha > 0, \beta > 0, \rho_1 > 0, \rho_2 > 0. \)

Conduct following steps for all devices.

1. for \( t = 1, \ldots, T \) do
2. if \( t == 1 \) then
3. With the batch size being \( S > 1 \), compute:
4. \( u_t^{(k)} = \nabla \Phi(x_t^{(k)}, y_t^{(k)}, \xi_t^{(k)}), \)
5. \( v_t^{(k)} = \nabla y_t^{(k)}(x_t^{(k)}, y_t^{(k)}; \gamma_t^{(k)}) \)
6. else
7. With the batch size being 1, compute:
8. \( u_t^{(k)} = (1 - \alpha \eta_t^{(k)})(u_t^{(k)} - \nabla \Phi(x_t^{(k)}, y_t^{(k)}; \xi_t^{(k)})) + \nabla \Phi(x_t^{(k)}, y_t^{(k)}; \xi_t^{(k)}), \)
9. \( v_t^{(k)} = (1 - \beta \eta_t^{(k)})(v_t^{(k)} - \nabla y_t^{(k)}(x_t^{(k)}, y_t^{(k)}; \gamma_t^{(k)})) + \nabla y_t^{(k)}(x_t^{(k)}, y_t^{(k)}; \gamma_t^{(k)}), \)
10. end if
11. if \( \text{mod}(t + 1, p) = 0 \) then
12. Upload \( u_{t+1}^{(k)}, v_{t+1}^{(k)}, x_{t+1}^{(k)}, y_{t+1}^{(k)} \) to server and reset
13. \( u_{t+1}^{(k)} = u_{t+1}^{(k)} = \frac{1}{K} \sum_{k'=1}^{K} u_t^{(k')}, \)
14. \( v_{t+1}^{(k)} = v_{t+1}^{(k)} = \frac{1}{K} \sum_{k'=1}^{K} v_t^{(k')}, \)
15. \( x_{t+1}^{(k)} = x_{t+1}^{(k)} = \frac{1}{K} \sum_{k'=1}^{K} x_t^{(k')}, \)
16. \( y_{t+1}^{(k)} = y_{t+1}^{(k)} = \frac{1}{K} \sum_{k'=1}^{K} y_t^{(k')}, \)
17. end if
18. end for

3.3 Local Bilevel Stochastic Gradient Descent with Momentum-Based Variance Reduction

In Algorithm 2, we further developed a local bilevel stochastic gradient descent with momentum-based variance reduction (LocalBSGVRM) algorithm. Compared with Algorithm 1, LocalBSGVRM employs a variance-reduced gradient estimator, which was first proposed in [2] for the standard minimization problem, to accelerate the convergence rate. Specifically, the \( k \)-th device computes the momentum-based variance-reduced gradient as follows:

\[ u_t^{(k)} = (1 - \alpha \eta_t^{(k)})(u_t^{(k)} - \nabla \Phi(x_t^{(k)}, y_t^{(k)}; \xi_t^{(k)})) + \nabla \Phi(x_t^{(k)}, y_t^{(k)}; \xi_t^{(k)}), \]
\[ v_t^{(k)} = (1 - \beta \eta_t^{(k)})(v_t^{(k)} - \nabla y_t^{(k)}(x_t^{(k)}, y_t^{(k)}; \gamma_t^{(k)})) + \nabla y_t^{(k)}(x_t^{(k)}, y_t^{(k)}; \gamma_t^{(k)}), \] (3.8)

1Throughout this paper, \( \tilde{O} \) denotes that the log term is ignored.
where $\alpha \eta_t^2 < 1$ and $\beta \eta_t^2 < 1$. With this new gradient estimator, LocalBSGVRM updates and communicates local model parameters in the same way as LocalBSGM.

In Theorem 2, we established the convergence rate of LocalBSGVRM.

**Theorem 2.** Suppose Assumptions 1-4 hold, by setting $\alpha = \frac{L^2}{3pK}, \beta = \frac{L^2}{3pK^2} + \frac{L^2}{K}$, $\eta_t = \frac{K^{2/3}L}{(\mu_1 + t)^{1/3}}, w_t = \max\{2, K^2p^3 - t, \frac{8\rho^2_1L^2K^2}{L^2} - t\}, \rho_1 \leq \left\{\frac{\rho_1^2}{200\sigma^2} - \frac{1}{1500\sqrt{\mu_1}}, 1\right\}, \rho_2 \leq \left\{\frac{\rho_2^2}{19900\sigma^2} - \frac{1}{200\mu_1}, 1\right\}, \theta < \frac{1}{\sqrt{\eta_t}}, p \leq \frac{1}{\sqrt{288(2+4L^2/K)(\rho_1^2 + \rho_2^2)}}, S = p$, Algorithm 4 has the following convergence rate:

$$
\frac{1}{T} \sum_{t=1}^{T} \left(\|\nabla \Phi(\bar{x}_t)\|^2 + \tilde{L}^2\|\bar{y}_t - y^*(\bar{x}_t)\|^2\right) \leq \frac{2(\Phi(x_1) - \Phi(x_*))}{\rho_1}(\frac{p}{T} + \frac{1}{K^{2/3}T^{2/3}})
$$

$$
+ \left(\frac{p}{T} + \frac{1}{K^{2/3}T^{2/3}}\right)4\Delta_t^2 + \left(\frac{p}{T} + \frac{1}{K^{2/3}T^{2/3}}\right)\frac{500\tilde{L}^2\sigma^2}{3\mu^2}
$$

$$
+ \left(\frac{p}{T} + \frac{1}{K^{2/3}T^{2/3}}\right)32\sigma^2 \ln(T + 1) + \left(\frac{p}{T} + \frac{1}{K^{2/3}T^{2/3}}\right)\frac{4000\sigma^2\tilde{L}^2}{3\mu^2} \ln(T + 1)
$$

$$
+ \left(\frac{p}{T} + \frac{1}{K^{2/3}T^{2/3}}\right)\frac{510(\rho_1^2 + \rho_2^2) \times 96 \times 8\sigma^2\tilde{L}^2}{\mu^2} \ln(T + 1)
$$

$$
+ \left(\frac{p}{T} + \frac{1}{K^{2/3}T^{2/3}}\right)\frac{510(\rho_1^2 + \rho_2^2) \times 144 \times 8\sigma^2\tilde{L}^2}{\mu^2} \ln(T + 1)
$$

(3.9)

**Remark 3.** From Theorem 2, it can be observed that $\rho_1 = O(\frac{1}{\sqrt{T}})$ and $\frac{L^2}{p^2} = O(\kappa^3)$.

**Corollary 2.** Suppose Assumptions 1-4 hold, by setting $\alpha = O(\frac{L^2}{K^2}), \beta = O(\frac{L^2}{K^2}), T = O(\frac{\kappa^3}{K^2}), p = O(\frac{1}{K^2}), \mu = O(\log(\frac{1}{\epsilon}))$, one can get

$$
\frac{1}{T} \sum_{t=0}^{T-1} \left(\|\nabla \Phi(\bar{x}_t)\|^2 + \tilde{L}^2\|\bar{y}_t - y^*(\bar{x}_t)\|^2\right) \leq \epsilon.
$$

(3.10)

**Remark 4.** From Corollary 2, one can know that the iteration complexity is $O(\frac{\kappa^3}{K^2\epsilon^3})$, which is much better than that of our Algorithm 4. In addition, this iteration complexity also indicates that our LocalBSGVRM can achieve linear speedup with respect to the number of devices. Moreover, it is easy to obtain that the gradient complexity and the Jacobian-vector complexity is $O(\frac{\kappa^3}{K^2\epsilon^3})$ and the Hessian-vector complexity is also $O(\frac{\kappa^3}{K^2\epsilon^3})$. Similarly, when $K = 1$, these sample complexities match those of the methods for single-machine settings, such as 1, 2, 3, 4, 5. Furthermore, the communication complexity is $T/p = O(\frac{\kappa^3}{\epsilon})$, which is also much better than $O(\frac{\kappa^3}{\epsilon^2})$ of Algorithm 4.

In summary, we established the convergence rate of our proposed LocalBSGM and LocalBSGVRM. Both of them achieve linear speedup regarding the number of workers. Additionally, we provided the communication complexity of our two algorithms. To the best of our knowledge, this is the first work achieving such theoretical results for federated bilevel optimization algorithms.

4 Conclusion

In this paper, we developed two novel momentum-based algorithms for the federated stochastic bilevel optimization problems. To the best of our knowledge, this is the first work studying this kind of problem. More importantly, we established the convergence rate of our two algorithms, which enjoy superior sample complexity and communication complexities.

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