Bohr Radius Problems for Some Classes of Analytic Functions Using Quantum Calculus Approach

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Abstract: The main purpose of this investigation is to use quantum calculus approach and obtain the Bohr radius for the class of $q$-starlike ($q$-convex) functions of order $\alpha$. The Bohr radius is also determined for a generalized class of $q$-Janowski starlike and $q$-Janowski convex functions with negative coefficients.

Keywords: $q$-Bohr radius; $q$-Janowski starlike functions; $q$-Janowski convex functions; $q$-starlike functions of order $\alpha$; $q$-convex functions of order $\alpha$; $q$-derivative (or $q$-difference) operator; quantum calculus approach

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1. Introduction

Let $D := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in $\mathbb{C}$. Suppose $A$ denote the class of analytic functions in $D$ normalized by $f(0) = 0 = f'(0) - 1$. Also, let $S$ be the subclass of $A$ consisting of univalent functions in $D$.

Suppose $H(D, \Omega)$ is the class of analytic functions mapping open unit disc $D$ into a domain $\Omega$. Harald Bohr [1] in 1914 proved that if a function $f$ of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belong to $H(D, D)$, then $\sum_{n=0}^{\infty} |a_n z^n| \leq 1$ in the disc $|z| \leq k$, where $k \geq 1/6$. As reported by Bohr in [1], Riesz, Schur and Wiener discovered that $|z| \leq k$ is actually true for $0 \leq k \leq 1/3$ and that $1/3$ is the best possible. The number $1/3$ is commonly called the "Bohr radius" for the class of analytic self-maps $f$ in $D$, while the inequality $\sum_{n=0}^{\infty} |a_n z^n| \leq 1$ is known as the "Bohr inequality". Later on, extensions of Bohr inequality and their proofs were given in [2–4]. Note that Bohr Radius is somewhat whimsical, for physicists consider the Bohr Radius of the hydrogen atom to be a fundamental constant, that is, $4\pi\hbar^2 / m_e e^2$, or about 0.529A. The physicists Bohr Radius is named for Niels Bohr, a founder of the Quantum Theory and 1922 recipient of the Nobel Prize for physics.
The Bohr inequality has emerged as an active area of research after Dixon [5] used it to disprove a conjecture in Banach algebra. Using the Euclidean distance, denoted by \( d \), the Bohr inequality \( \sum_{n=0}^{\infty} |a_n z^n| \leq 1 \) for a function \( f \) of the form \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) can be written as

\[
\sum_{n=0}^{\infty} |a_n z^n| \leq 1 \iff \sum_{n=1}^{\infty} |a_n z^n| \leq 1 - |a_0|
\]

\[
\iff d \left( \sum_{n=0}^{\infty} |a_n z^n|, |a_0| \right) = \sum_{n=1}^{\infty} |a_n z^n| \leq 1 - |a_0| = 1 - |f(0)|
\]

\[
\iff d \left( \sum_{n=0}^{\infty} |a_n z^n|, |a_0| \right) \leq d(f(0), \partial D).
\]

where \( \partial D \) is the boundary of the disc \( D \). Thus, the concept of the Bohr inequality for a function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), defined in \( D \), can be generalized by

\[
d \left( \sum_{n=0}^{\infty} |a_n z^n|, |f(0)| \right) = \sum_{n=1}^{\infty} |a_n z^n| \leq d(f(0), \partial D).
\]  

(1)

Accordingly, the Bohr radius for a class \( \mathcal{M} \) consisting of analytic functions \( f \) of the form \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) in the disc \( D \) is the largest \( r^* > 0 \) such that every function \( f \in \mathcal{M} \) satisfies the inequality (1) for all \( |z| = r \leq r^* \). In this case, the class \( \mathcal{M} \) is said to satisfy a Bohr phenomenon.

Quantum calculus (or \( q \)-calculus) is an approach or a methodology that is centered on the idea of obtaining \( q \)-analogues without the use of limits. This approach has a great interest due to its applications in various branches of mathematics and physics, such as, the areas of ordinary fractional calculus, optimal control problems, \( q \)-difference, \( q \)-integral equations and \( q \)-transform analysis. Jackson [6] introduced the \( q \)-derivative (or \( q \)-difference, or Jackson derivative) denoted by \( D_q \), \( q \in (0, 1) \), which is defined in a given subset of \( \mathbb{C} \) by

\[
(D_q f)(z) = \begin{cases} 
\frac{f(z)-f(qz)}{(1-q)z}, & \text{if } z \neq 0 \\
 f'(0), & \text{if } z = 0 
\end{cases}
\]  

(2)

provided \( f'(0) \) exists. If \( f \) is a function defined in a subset of the complex plane \( \mathbb{C} \), then (2) yields

\[
\lim_{q \to 1} (D_q f)(z) = \lim_{q \to 1} \frac{f(z)-f(qz)}{(1-q)z} = f'(z).
\]

It is easy to see that if \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \), then by using (2) we have

\[
(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},
\]

\[
D_q (zD_q f(z)) = 1 + \sum_{n=2}^{\infty} [n]_q^2 a_n z^{n-1},
\]

\[
D_q^2 f(z) = D_q(D_q f(z)) = \sum_{n=2}^{\infty} [n]_q^2 a_n z^{n-2},
\]

where \([n]_q \) is given by

\[
[n]_q = \frac{1-q^n}{1-q}, q \in (0, 1).
\]

It is a routine to check that

\[
D_q (zD_q f(z)) = D_q f(z) + zD_q^2 f(z).
\]
In 1869, Thomae introduced the particular $q$-integral [7] which is defined as
\[ \int_0^1 f(t)d_qt = (1-q) \sum_{n=0}^{\infty} q^n f(q^n), \]
provided the $q$-series converges. Later on, Jackson [8] defined the general $q$-integral as follows:
\[ \int_a^b f(t)d_qt = \int_a^b f(t)d_qt - \int_0^a f(t)d_qt, \]
where
\[ \int_0^a f(t)d_qt = a(1-q) \sum_{n=0}^{\infty} q^n f(a^n), \]
provided the $q$-series converges. Also note that
\[ D_q \int_0^x f(t)d_qt = f(x) \quad \text{and} \quad \int_0^x D_q f(t)d_qt = f(x) - f(0), \]
where the second equality holds if $f$ is continuous at $x = 0$.

The $q$-calculus plays an important role in the investigation of several subclasses of $A$. A firm footing of the $q$-calculus in the context of geometric function theory and its usages involving the basic (or $q$-) hypergeometric functions in geometric function theory was actually made in a book chapter by Srivastava (see, for details [9]; see also [10]). In 1990, Ismail et al. [11] introduced a connection between starlike (convex) functions and the $q$-calculus by introducing a $q$-analogue of starlike (convex) functions. They generalized a well-known class of starlike functions, called the class of $q$-starlike functions denoted by $S_q^*$, consisting of functions $f \in A$ satisfying the inequality
\[ \left| \frac{z(D_qf)(z)}{f(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad z \in \mathbb{D}. \]

Baricz and Swaminathan [12] introduced a $q$-analogue of convex functions, denoted by $C_q$, satisfying the relation
\[ f \in C_q \quad \text{if and only if} \quad z(D_qf) \in S_q^*. \]

Recently Srivastava et al. [13] (see also [14]) successfully combined the concept of Janowski [15] and the above mentioned $q$-calculus and introduced the class $S_q^*[A,B]$ and $C_q[A,B], -1 \leq B < A \leq 1, q \in (0,1)$, given by
\[ S_q^*[A,B] := \left\{ f \in A : \frac{zf''(z)}{f'(z)} \prec \frac{(A+1)z+2+(A-1)qz}{(B+1)z+2+(B-1)qz} \right\}, \]
and
\[ C_q[A,B] := \left\{ f \in A : 1 + \frac{zf'''(z)}{f'(z)} \prec \frac{(A+1)z+2+(A-1)qz}{(B+1)z+2+(B-1)qz} \right\} \]
respectively, where $\prec$ denotes subordination. As $q \to 1^-$, $S_q^*[A,B]$ and $C_q[A,B]$ yield respectively the classes $S^*[A,B]$ and $C[A,B]$ defined by Janowski [15]. For various choices of $A$ and $B$, these classes reduce to well-known subclasses of $q$-starlike and $q$-convex functions. For instance, with $0 \leq \alpha < 1, S_q^*(\alpha) := S_q^*[1-2\alpha, -1]$ is the class of $q$-starlike functions of order $\alpha$, introduced by Agrawal and
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Sahoo [16]. Motivated by the authors in [16], Agrawal [17] defined a $q$-analog of convex functions of order $a$, $0 \leq a < 1$, $C_q(a) := C_q[1 - 2a, -1]$, satisfying

$$f \in C_q(a) \text{ if and only if } z(D_qf) \in S_q^+(a).$$

(3)

Note that $S_q^*[1, -1] \equiv S_q^*$ and $C_q[1, -1] \equiv C_q$.

In recent years, there is a great development of geometric function theory because of using quantum calculus approach. In particular, Srivastava et al. [18] found distortion and radius of univalence and starlikeness for several subclasses of $q$-starlike functions with negative coefficients. They [19] also determined sufficient conditions and containment results for the different types of $k$-uniformly $q$-starlike functions. Srivastava et al. [18] found distortion and radius of univalence and starlikeness for several subclasses of $q$-starlike functions with negative coefficients. They [19] also determined sufficient conditions and containment results for the different types of $k$-uniformly $q$-starlike functions. Naeem et al. [20] investigated subfamilies of $q$-convex functions and $q$-close to convex functions with respect to the Janowski functions connected with $q$-conic domain which explored some important geometric properties such as coefficient estimates, sufficiency criteria and convolution properties of these classes. For a survey on the use of quantum calculus approach in mathematical sciences and its role in geometric function theory, one may refer to [21]. In addition, one may refer to a survey-cum-expository article written by Srivastava [22] where he explored the mathematical application of $q$-calculus, fractional $q$-calculus and fractional $q$-differential operators in geometric function theory.

In this paper, we investigate Bohr radius problems for the classes $S_q^*(a)$ and $C_q(a)$, respectively, in Sections 2 and 3. In Section 4, we define and investigate the Bohr radius problem for a generalized class, $TP_q(\lambda, A, B)$, of functions with negative coefficients, where $q \in (0, 1)$, $\lambda \in [0, 1]$ and $-1 < B < A \leq 1$. In particular, we also define and obtain sharp Bohr radius for the class of the $q$-Janowski functions with negative coefficients in Section 4.

2. The Bohr Radius for the Class $S_q^*(a)$

To find the Bohr radius for the class $S_q^*(a)$, we first need the following four lemmas.

Lemma 1 ([23] (Theorem 2.5, p. 1511)). For $q \in (0, 1)$, suppose $a, b, c$ are non-negative real numbers satisfying $0 \leq 1 - aq \leq 1 - cq$ and $0 < 1 - b \leq 1 - c$. Then there exists a non-decreasing function $\mu : [0, 1] \to [0, 1] \text{ with } \mu(1) - \mu(0) = 1$ such that

$$\frac{w\phi(q, q, q^2, q, w)}{\phi(q^2, q, q^2, q, w)} = \int_0^1 \frac{w}{1 - tw} dt,$$

where $\phi(a, b; c; q, z)$ is a hypergeometric function (see [24, 25]) given by

$$\phi(a, b; c; q, z) = \sum_{n=0}^{\infty} \frac{(a; q)_n(b; q)_n}{(c; q)_n(q; q)_n} z^n$$

and $(a; q)_0 = 1, (a; q)_n = (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1})$, which is analytic in the cut-plane $\mathbb{C} \setminus [1, \infty]$ and maps both the unit disc and the half-plane $\{z \in \mathbb{C} : \text{Re } z < 1\}$ univalently onto domains convex in the direction of the imaginary axis.

Lemma 2 ([16] (Theorem 1.1, p. 17)). If $f \in A$, then $f \in S_q^*(a)$ if and only if there exists a probability measure $\mu$ supported on the circle such that

$$\frac{zf'(z)}{f(z)} = 1 + \int_{|\sigma| = 1} \sigma F_{q,a}(\sigma z) d\mu(\sigma),$$

where

$$F_{q,a}(z) = \sum_{n=1}^{\infty} \frac{-2}{1 - q^n} \ln \left(\frac{q}{1 - a(1 - q)}\right) z^n, \quad z \in \mathbb{D}.$$
Lemma 3 (Distortion theorem). Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n = zh(z) \in S^*_q(a) \). Then

\[
\exp(F_{q,a}(-r)) \leq |h(z)| \leq \exp(F_{q,a}(r)).
\]

Proof. Let \( f \in S^*_q(a) \). By Lemma 2, there exists a probability measure \( \mu \) supported on the unit circle such that

\[
\frac{zf'(z)}{f(z)} = 1 + \int_{|\sigma|=1} \sigma z F'_{q,a}(\sigma z) d\mu(\sigma),
\]

where

\[
F_{q,a}(z) = \sum_{n=1}^{\infty} -2 \frac{\ln \left( \frac{q}{1 - a (1 - q)} \right)}{1 - q^n} z^n, \quad z \in \mathbb{D}.
\]

Integrating and then taking exponential on both sides, we have

\[
f(z) = z \exp \left( \int_{|\sigma|=1} F_{q,a}(\sigma z) d\mu(\sigma) \right).
\]

Since \( f(z) = zh(z) \in S^*_q(a) \), it follows that

\[
|h(z)| = \exp \left( \text{Re} \int_{|\sigma|=1} F_{q,a}(\sigma z) d\mu(\sigma) \right).
\]

Thus

\[
\ln |h(z)| = \text{Re} \int_{|\sigma|=1} F_{q,a}(\sigma z) d\mu(\sigma)
\]

\[
= -2 \ln \left( \frac{q}{1 - a (1 - q)} \right) \text{Re} \int_{|\sigma|=1} \sum_{n=1}^{\infty} \left( \frac{\sigma z}{1 - q^n} \right)^n d\mu(\sigma)
\]

\[
= -2 \frac{1}{1 - q} \ln \left( \frac{q}{1 - a (1 - q)} \right) \text{Re} \int_{|\sigma|=1} (\sigma z \phi(q, q, q^2, q, \sigma z)) d\mu(\sigma)
\]

\[
= -2 \frac{1}{1 - q} \ln \left( \frac{q}{1 - a (1 - q)} \right) \text{Re} \int_0^{2\pi} ((e^{i\theta} z) \phi(q, q, q^2, q, e^{i\theta} z)) d\mu(\theta), \quad w = e^{i\theta} z \in \mathbb{D}
\]

\[
= -2 \frac{1}{1 - q} \ln \left( \frac{q}{1 - a (1 - q)} \right) \text{Re} \int_0^{2\pi} w \phi(q, q, q^2, q, w) \phi(q^2, q, q^2, q, w) d\mu(\theta), \quad w = e^{i\theta} z \in \mathbb{D}
\]

(4)

where \( \phi(a, b; c; q, z) \) is the hypergeometric function defined in Lemma 1. By Lemma 1, we have

\[
\frac{w \phi(q, q, q^2, q, w)}{\phi(q^2, q, q^2, q, w)} = \int_0^1 \frac{w}{1 - tw} d\mu(t).
\]

(5)

Let

\[
g(re^{i\psi}) = \text{Re} \frac{w}{1 - tw}, \quad w = re^{i\psi}
\]

\[
= \text{Re} \frac{r(\cos \psi + i \sin \psi)}{1 - tr(\cos \psi + i \sin \psi)}
\]

\[
= \frac{r \cos \psi (1 - tr \cos \psi) - tr^2 \sin^2 \psi}{1 + r^2t^2 - 2rt \cos \psi}.
\]
A routine calculation shows that
\[
\min_{\psi} g(re^{i\psi}) = g(-r) \quad \text{and} \quad \max_{\psi} g(re^{i\psi}) = g(r).
\]
Thus
\[
\min_{|w| \leq r} \frac{w}{1 - lw} = \frac{-r}{1 + rl} \quad \text{and} \quad \max_{|w| \leq r} \frac{w}{1 - lw} = \frac{r}{1 - rl}.
\]
(6)

By (4)–(6), it follows that
\[
\ln |h(z)| \geq -\frac{2}{1 - q} \ln \left(\frac{q}{1 - \alpha(1 - q)}\right) \int_{|\sigma| = 1} (-r\phi(q, q, q^2, q, -r)) d\mu(\sigma)
\]
(7)
\[
= F_{q,\alpha}(-r)
\]
and
\[
\ln |h(z)| \leq \int_{|\sigma| = 1} F_{q,\alpha}(r) d\mu(\sigma)
\]
(8)

By (7) and (8), we have \(\exp(F_{q,\alpha}(-r)) \leq |h(z)| \leq \exp(F_{q,\alpha}(r))\).

**Remark 1.** As \(q \to 1^-\), Lemma 3 yields the corresponding distortion theorem [26] (Theorem 8, p. 117) for the class \(S^*(\alpha)\).

**Lemma 4** ([16] (Theorem 1.3, p. 8)). Let
\[
G_{q,\alpha}(z) = z \exp(F_{q,\alpha}(z)) = z + \sum_{n=2}^{\infty} c_n z^n.
\]
Then \(G_{q,\alpha}(z) \in S^*_q(\alpha)\). However, if \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*_q(\alpha)\), then \(|a_n| \leq c_n\) with equality holding for all \(n\) if and only if \(f\) is a rotation of \(G_{q,\alpha}\).

**Theorem 1.** Let \(\phi(z) = \sum_{n=1}^{\infty} \phi_n z^n\) and \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z \exp(\phi(z)) \in S^*_q(\alpha)\). Then
\[
|z| + \sum_{n=2}^{\infty} |a_n| |z|^n \leq d(0, \partial f(\mathbb{D}))
\]
for \(|z| \leq r^*\), where \(r^* \in (0, 1)\) is the unique root of the equation
\[
r \exp(F_{q,\alpha}(r)) = \exp(F_{q,\alpha}(-1)).
\]
The radius is sharp.

**Proof.** Let \(f \in S^*_q(\alpha)\). Proceeding as in proof of [16] (Theorem 1.3, p. 8), it is easy to see that coefficients bound for the function \(\phi(z) = \sum_{n=1}^{\infty} \phi_n z^n\) are given by
\[
|\phi_n| \leq -\frac{2 \ln \left(\frac{q}{1 - \alpha(1 - q)}\right)}{1 - q^n}.
\]
(9)
For $|z| = r \leq r^*$, using Lemma 3 and inequality (9), it follows that

$$d(0, \partial f(D)) = \lim_{|z| \to 1^-} \inf_{|z| = 1} |f(z) - f(0)| = \lim_{|z| \to 1^-} \inf_{|z| = 1} \frac{|f(z)|}{|z|} \geq \exp F_{q,a}(-1)$$

$$\geq r \exp F_{q,a}(r)$$

$$= r \exp \left( \sum_{n=1}^{\infty} -\frac{2}{1-q^n} \ln \left( \frac{q}{1-a(1-q)} \right) r^n \right)$$

$$\geq |z| + \sum_{n=2}^{\infty} |a_n||z|^n$$

if and only if

$$r \exp(F_{q,a}(r)) \leq \exp F_{q,a}(-1).$$

In order to prove that the radius is sharp, let

$$G_{q,a}(z) := z \exp(F_{q,a}(z)),$$

where

$$F_{q,a}(z) = \sum_{n=1}^{\infty} -\frac{2}{1-q^n} \ln \left( \frac{q}{1-a(1-q)} \right) z^n, \quad z \in \mathbb{D}.$$ 

By Lemma 4, it follows that $G_{q,a} \in S^*_q(\alpha)$. For $|z| = r^*$, we obtain

$$|z| + \sum_{n=2}^{\infty} |a_n||z|^n = r^* \exp \left( \sum_{n=1}^{\infty} -\frac{2}{1-q^n} \ln \left( \frac{q}{1-a(1-q)} \right) (r^*)^n \right)$$

$$= r^* \exp F_{q,a}(r^*)$$

$$= \exp F_{q,a}(-1)$$

$$= \lim_{|z| \to 1^-} \inf_{|z| = 1} \frac{|G_{q,a}(z)|}{|z|}$$

$$= \lim_{|z| \to 1^-} \inf_{|z| = 1} |G_{q,a}(z) - f(0)|$$

$$= d(0, G_{q,a}(D)). \quad \square$$

Remark 2. For $\alpha = 0$, Theorem 1 yields the corresponding results found in [27] for the class $S^*_q$.

Remark 3. Theorem 1 with letting $q \to 1^-$ leads to the Bohr radius for the class of starlike functions of order $\alpha$, $0 \leq \alpha < 1$. Bhowmik and Das [28] (Theorem 3, p. 1093) found the Bohr radius for $S^*(\alpha)$ with $\alpha \in [0, 1/2]$.

3. The Bohr Radius for the Class $C_q(\alpha)$

In the present section, we obtain the sharp Bohr radius for the class of $q$-convex functions of order $\alpha$, $0 \leq \alpha < 1$.

Lemma 5 ([17] (Theorem 2.9, p. 5)). Let

$$E_q(z) := \int_0^z \exp(F_{q,a}(t)) dt = z + \sum_{n=2}^{\infty} \left( \frac{1-q}{1-q^n} c_n z^n \right),$$

where $c_n$ is the $n$th coefficient of the function $z \exp(F_{q,a}(z))$. Then $E_q \in C_q(\alpha)$ for $0 \leq \alpha < 1$. Moreover, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C_q(\alpha)$, then $|a_n| \leq ((1-q)/(1-q^n)) c_n$, with equality holding for all $n$ if and only if $f$ is a rotation of $E_q$. 

Theorem 2. The Bohr radius for the class $C_q(a)$ is $r^*$, where $r^* \in (0, 1]$ is the unique root of the equation

$$\int_0^r \exp(F_{q,a}(t))d_qt = \int_0^1 \exp(F_{q,a}(-t))d_qt.$$ 

The radius is sharp.

Proof. Let $f \in C_q(a)$. Then, by (3), $z(D_qf)(z) \in S^*_q(a)$. It follows from Lemma 3 that

$$\exp(F_{q,a}(-r)) \leq |(D_qf)(z)| \leq \exp(F_{q,a}(r)).$$

Taking $q$-integral of all the inequalities, we have

$$\int_0^r \exp(F_{q,a}(-t))d_qt \leq |f(z)| \leq \int_0^r \exp(F_{q,a}(t))d_qt. \tag{10}$$

Since $f(z) = z + \sum_{n=2}^\infty a_n z^n \in C_q(a)$, Lemma 5 yields the coefficients bound for the function $f$ given by

$$|a_n| \leq \frac{1-q}{1-q^n} c_n r^n \tag{11}$$

where inequality holds for all $n$ if and only if $f$ is a rotation of

$$E_q(z) = \int_0^r \exp(F_{q,a}(t))d_qt = z + \sum_{n=2}^\infty \left(\frac{1-q}{1-q^n}\right) c_n z^n$$

and where $c_n$ is the $n$th coefficient of $z \exp(F_{q,a}(z))$.

By (10) and (11), we have

$$r + \sum_{n=2}^\infty |a_n|r^n \leq r + \sum_{n=2}^\infty \frac{1-q}{1-q^n} c_n r^n$$

if and only if

$$\int_0^r \exp(F_{q,a}(t))d_qt \leq \int_0^1 \exp(F_{q,a}(t))d_qt.$$

Now, consider the function

$$E_q(z) := \int_0^r \exp(F_{q,a}(t))d_qt = z + \sum_{n=2}^\infty \left(\frac{1-q}{1-q^n}\right) c_n z^n.$$

It follows from Lemma 5 that the function $E_q(z) \in C_q(a)$. At $|z| = r^*$, we have

$$r^{*^2} + \sum_{n=2}^\infty |a_n|(r^*)^n = r^{*^2} + \sum_{n=2}^\infty \frac{1-q}{1-q^n} c_n (r^*)^n$$

if and only if

$$\int_0^r \exp(F_{q,a}(t))d_qt = \int_0^1 \exp(F_{q,a}(t))d_qt = d(0, \partial E_q(D)).$$
which shows that the Bohr radius $r^*$ is sharp for the class $C_q(\alpha)$. □

Putting $\alpha = 0$ in Theorem 2, we obtain the Bohr radius for the class $C_q$ of $q$-convex functions.

**Corollary 1** ([27] (Theorem 2, p. 111)). The Bohr radius for the class $C_q$ is $r^*$, where $r^* \in (0, 1]$ is the unique root of

$$
\int_0^r \exp(F_{q,0}(t))dt = \int_0^1 \exp(F_{q,0}(-t))dt.
$$

The radius is sharp.

If $q \to 1^-$, then Corollary 1 yields the Bohr radius for the class $C$ of convex functions, that is, $r^* = 1/3$. The same Bohr radius for general convex functions had been earlier obtained by Aizenberg in [29] (Theorem 2.1).

4. The Bohr Radius Problems for the Class $TP_q(\lambda, A, B)$

In 1975, Silverman [30] investigated two new subclasses of the family $T$, where

$$
T = \{f \in S : f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, z \in \mathbb{D}\}.
$$

Recently, Altıntaş and Mustafa [31] introduced a generalized class, $TP_q(\lambda, A, B), q \in (0, 1), \lambda \in [0, 1], -1 \leq B < A \leq 1$, given by

$$
TP_q(\lambda, A, B) = \left\{ f \in T : \frac{zD_qf(z) + \lambda z^2D_q^2f(z)}{AzD_qf(z) + (1 - \lambda)f(z)} \prec \frac{1 + Az}{1 + Bz}, z \in \mathbb{D} \right\}.
$$

For $\lambda = 0$, this class reduces to the class $TS_q^+[A, B]$ of $q$-Janowski starlike functions with negative coefficients defined by

$$
TS_q^+[A, B] = \left\{ f \in T : \frac{zD_qf(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, z \in \mathbb{D} \right\}.
$$

On the other hand, the case $\lambda = 1$ yields the class $TC_q[A, B]$ of $q$-Janowski convex functions, defined by

$$
TC_q[A, B] = \left\{ f \in T : 1 + \frac{zD_q^2f(z)}{D_qf(z)} \prec \frac{1 + Az}{1 + Bz}, z \in \mathbb{D} \right\}.
$$

As $q \to 1^-$, $TS_q^+[A, B]$ and $TC_q[A, B]$ reduce respectively to $TS^+[A, B]$ and $TC[A, B]$ studied initially in [32]. Note that the classes $TS^+(a) = \lim_{q \to 1^-} TS_q^+[1 - 2a, -1]$ and $TC(a) = \lim_{q \to 1^-} TC_q[1 - 2a, -1]$ were defined and studied by Silverman [30] in 1975.

In the present section, we will first investigate the sharp Bohr radius for the class $TP_q(\lambda, A, B), q \in (0, 1), \lambda \in [0, 1]$ which in particular gives the Bohr radius for the classes $TS_q^+[A, B]$ and $TC_q[A, B]$. However, in order to obtain Bohr radius, we first need some results given here in two lemmas.

Note that there is a typing error in the statement of [31] (Theorem 3.1, p. 993) (replace $a$ by $\beta$).

The correct statement in Lemma 6 is as follows:

**Lemma 6** ([31] (Theorem 3.1, p. 993)). If $f \in TP_q(\lambda, A, B), q \in (0, 1), \lambda \in [0, 1]$, then

$$
1 - \beta
\frac{1 - \beta}{(2q - \beta)(1 + (2q - 1)q)}^2 \leq |f(z)| \leq r + \frac{1 - \beta}{(2q - \beta)(1 + (2q - 1)q)}^2
$$
where $\beta = (1 - A)/(1 - B)$, $-1 \leq B < A \leq 1$, with equality for the function
\[
f(z) = z - \frac{1 - \beta}{(2|q| - \beta)(1 + (2|q| - 1)\lambda)}z^2, |z| = r.
\]

**Lemma 7** ([31] (Theorem 2.8, p. 991)). If $f \in TP_q(\lambda, A, B)$, $q \in (0, 1)$, $\lambda \in [0, 1]$, then the following conditions are satisfied:
\[
\sum_{n=2}^{\infty} |a_n| \leq \frac{1 - \beta}{(|n|q - \beta)(1 + (|n|q - 1)\lambda)}
\]
\[
\sum_{n=2}^{\infty} |n|q|a_n| \leq \frac{(1 - \beta)|n|q}{(|n|q - \beta)(1 + (|n|q - 1)\lambda)}, n = 2, 3, \ldots,
\]
where $\beta = (1 - A)/(1 - B)$, $-1 \leq B < A \leq 1$. The results obtained here are sharp.

**Theorem 3.** If $f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n \in TP_q(\lambda, A, B)$ where $q \in (0, 1)$, $\lambda \in [0, 1]$, $\beta = (1 - A)/(1 - B)$ and $c = q(\lambda + 1 + q\lambda - \beta\lambda)$, then
\[
|z| + \sum_{n=2}^{\infty} |a_nz^n| \leq d(0, \partial f(D))
\]

for $|z| < r^*$, where
\[
r^* = \frac{2c}{1 - \beta + c + \sqrt{(1 - \beta)c + (1 - \beta + c)^2}}.
\]
The radius $r^*$ is the sharp Bohr radius for class $TP_q(\lambda, A, B)$.

**Proof.** It follows from Lemma 6 that the distance between the origin and the boundary of $f(D)$ satisfies the inequality
\[
d(0, \partial f(D)) \geq 1 - \frac{1 - \beta}{(1 + q - \beta)(1 + q\lambda)}.
\]
The given $r^*$ is the root of the equation
\[
r^* + \frac{(1 - \beta)(r^*)^2}{(1 + q - \beta)(1 + q\lambda)} = 1 - \frac{1 - \beta}{(1 + q - \beta)(1 + q\lambda)}.
\]
For $0 < r \leq r^*$, we have
\[
r + \frac{(1 - \beta)r^2}{(1 + q - \beta)(1 + q\lambda)} \leq r^* + \frac{(1 - \beta)(r^*)^2}{(1 + q - \beta)(1 + q\lambda)} = 1 - \frac{1 - \beta}{(1 + q - \beta)(1 + q\lambda)}.
\]
Using Lemma 7, it is easy to show that
\[
\sum_{n=2}^{\infty} |a_n| \leq \frac{1 - \beta}{(1 + q - \beta)(1 + q\lambda)}.
\]
The above inequality together with inequality (12) yield
\[
|z| + \sum_{n=2}^{\infty} |a_nz^n| \leq r + \frac{1 - \beta}{(1 + q - \beta)(1 + q\lambda)}r^2 \leq 1 - \frac{1 - \beta}{(1 + q - \beta)(1 + q\lambda)} \leq d(0, \partial f(D)).
\]

For sharpness, consider the function $f : D \rightarrow \mathbb{C}$ defined by
\[
f(z) = z - \frac{1 - \beta}{(1 + q - \beta)(1 + q\lambda)}z^2.
\]
This function clearly belongs to $TP_\varphi(\lambda, A, B)$. For $|z| = r^*$, we find

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| = r^* + \frac{1 - \beta}{(1 + q - \beta)(1 + q \lambda)} (r^*)^2 = 1 - \frac{1 - \beta}{(1 + q - \beta)(1 + q \lambda)} = d(0, \partial f(\mathbb{D})). \quad \Box$$

Putting $\lambda = 0$ in Theorem 3, we get the sharp Bohr radius for the class $TS_\varphi^* [A, B]$.

**Theorem 4.** If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \in TS_\varphi^* [A, B]$, $\beta = (1 - A) / (1 - B)$ and $-1 \leq B < A \leq 1$, then

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{D}))$$

for $|z| < r^*$, where

$$r^* = \frac{2q}{1 + q - \beta + \sqrt{1 + 6q + q^2 - 2\beta - 6q\beta + \beta^2}}.$$

The radius $r^*$ is sharp.

Letting $A = 1 - 2\alpha$ and $B = -1$ in Theorem 4, we obtain the sharp Bohr radius for the class of $q$-starlike functions of order $\alpha$, $0 \leq \alpha < 1$, with negative coefficients.

**Corollary 2.** Let $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \in TS_\varphi^* (\alpha)$. Then

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{D}))$$

for $|z| < r^*$, where

$$r^* = \frac{2q}{1 + q - \alpha + \sqrt{q^2 + 6q(1 - \alpha) + (1 - \alpha)^2}}.$$

When $q \to 1^-$ in Corollary 2, we obtain the following sharp Bohr radius for the class of starlike functions of order $\alpha$, $0 \leq \alpha < 1$, with negative coefficients obtained by Ali et al. [33].

**Corollary 3 ([33] Theorem 2.3).** If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \in TS^* (\alpha)$, then

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{D}))$$

for $|z| < r^*$, where

$$r^* = \frac{2}{2 - \alpha + \sqrt{8 - 8\alpha + \alpha^2}}.$$

The radius $r^*$ is the Bohr radius for $TS^* (\alpha)$.

When $A = 1$ and $B = -1$, Theorem 4 gives the following sharp Bohr radius for the class of $q$-starlike functions with negative coefficients.

**Corollary 4.** If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \in TS_\varphi^* [q, \lambda]$, then

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{D}))$$

for $|z| < r^*$, where

$$r^* = \frac{2q}{1 + q + \sqrt{1 + 6q + q^2}}.$$
When \( A = 1, B = -1 \) and \( q \to 1^- \), Theorem 4 gives the following sharp Bohr radius for the class of starlike functions with negative coefficients obtained by Ali et al. [33].

**Corollary 5 ([33]).** The sharp Bohr radius for the class \( T S^* \) is
\[
\sqrt{2} - 1 \approx 0.414214.
\]

When \( \lambda = 1 \), Theorem 3 gives the following sharp Bohr radius for the class of \( TC_q[A, B] \).

**Theorem 5.** If \( f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n \in TC_q[A, B], \beta = (1 - A)/(1 - B) \) and \(-1 \leq B < A \leq 1\), then
\[
|z| + \sum_{n=2}^{\infty} |a_nz^n| \leq d(0, \partial f(D))
\]
for \( |z| < r^* \), where
\[
r^* = \frac{2q(2 + \beta) + 2q + q^2 - \beta - q\beta + \sqrt{4(1 - \beta)(2q + q^2 - q\beta) + (q\beta - 1 - 2q - q^2 + \beta)^2}}{1 + 2q + q^2 - \beta - q\beta + \sqrt{4(1 - \beta)(2q + q^2 - q\beta) + (q\beta - 1 - 2q - q^2 + \beta)^2}}.
\]
The result is sharp for the function
\[
f(z) = z - \frac{1 - \beta}{(1 + q - \beta)(1 + q)} z^2.
\]

When \( A = 1 - 2\alpha \) and \( B = -1 \), Theorem 5 gives the sharp Bohr radius for the class of \( q \)-convex functions with negative coefficients.

**Corollary 6.** The sharp Bohr radius for the class \( TC_q(\alpha) \) is
\[
\frac{2q(2 + \alpha)}{1 + 2q + q^2 - \alpha - q\alpha + \sqrt{(1 + q)^2(1 + q - \alpha)^2 + 4q(2 + q - \alpha)(1 - \alpha)}}.
\]

Letting \( q \to 1^- \) in Corollary 6, we get the following sharp Bohr radius for the class of convex functions of order \( \alpha, 0 \leq \alpha < 1 \), with negative coefficients obtained by Ali et al. [33].

**Corollary 7 ([33] (Theorem 2.4)).** If \( f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n \in TC(\alpha), \) then
\[
|z| + \sum_{n=2}^{\infty} |a_nz^n| \leq d(0, \partial f(D))
\]
for \( |z| < r^* \), where
\[
r^* = \frac{3 - \alpha}{2 - \alpha + \sqrt{7 - 8\alpha + 2\alpha^2}}.
\]
The radius \( r^* \) is the Bohr radius for \( TC(\alpha) \).

For \( A = 1 \) and \( B = -1 \), Theorem 5 yields the sharp Bohr radius for the class of \( q \)-convex functions with negative coefficients.

**Corollary 8.** The sharp Bohr radius for the class \( TC_q \) is
\[
\frac{2q(2 + q)}{1 + 2q + q^2 + \sqrt{1 + 12q + 10q^2 + 4q^3 + q^4}}.
\]

Letting \( q \to 1^- \), \( A = 1 \) and \( B = -1 \), Theorem 5 gives the sharp Bohr radius for the class of convex functions with negative coefficients by Ali et al. [33].
Corollary 9 ([33]). The sharp Bohr radius for the class $TC$ is $\sqrt{7} - 2 \approx 0.645751$.

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