Reducible systems and embedding procedures in the canonical formalism

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Abstract

We propose a systematic method of dealing with the canonical constrained structure of reducible systems in the Dirac and symplectic approaches which involves an enlargement of phase and configuration spaces, respectively. It is not necessary, as in the Dirac approach, to isolate the independent subset of constraints or to introduce, as in the symplectic analysis, a series of lagrange multipliers-for-lagrange multipliers. This analysis illuminates the close connection between the Dirac and symplectic approaches of treating reducible theories, which is otherwise lacking. The example of $p$- form gauge fields ($p = 2, 3$) is analyzed in details.

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1 Introduction

The problem of giving a proper formulation for reducible constrained systems, be it in the canonical Hamiltonian, symplectic, or path integral approaches, is quite involved. They occur whenever the set of constraints found by the usual canonical prescription is not linearly independent. Perhaps their most popular occurrence is in the theory of p-form gauge fields involving completely antisymmetric p-rank tensors, but these are also present in other examples. Different proposals exist to account for the reducibility property depending on the manifestation of the problem. In the canonical approach, for instance, the Dirac brackets cannot be computed since the corresponding matrix is singular and hence noninvertible. A possible remedy is to isolate the independent subset of constraints. Then the Dirac brackets are computed within this set following the normal procedure. The process is then extended to include the complete set of constraints. This was the approach adopted in [3] for analyzing the 2-form gauge theory. An alternative canonical method based on the symplectic form also suffers from an identical problem. As is known, the constraints are obtained from the zero modes of the symplectic matrix and inserted back in the Lagrangian through multipliers, in analogy with the usual Dirac method of introducing constraints in the Hamiltonian. For reducible systems the symplectic matrix is noninvertible. It is cured by imposing additional conditions on the Lagrange multipliers. Therefore a series of Lagrange multipliers-for-Lagrange multipliers is involved. Details of this approach can be found in [5]. In the BRST path integral formulation, on the other hand, the functional measure is ill defined because reducibility leads to presence of $\delta(0)$ terms. This is usually avoided by introducing extra ghost fields. Depending on the degree of reducibility, a tower of ghosts-for-ghosts may be necessary.

It is quite clear that the problem of reducibility manifests in different ways leading to different suggestions for their treatment. But several unanswered and unpleasant issues prevail. For instance, there is no unique and systematic way of identifying the independent subset of constraints in the Dirac approach. Additionally, such an abstraction may lead to the loss of important symmetries of the problem, as has been pointed out recently. In both the symplectic and BRST formalisms, on the contrary, it is not clear whether the tower of extra fields is really necessary or an artifact of the prescription. Furthermore, apparently there does not seem to be any correlation among the different available resolutions of the same problem. This is all the more surprising since the Dirac and symplectic approaches are known to be completely equivalent while the path integral can always be derived from the canonical formalism.

The preceding comments show that the problem of reducibility merits a closer examination. This is the motivation of the present paper where we attempt to provide answers to some of these basic issues. A systematic canonical formalism for reducible systems in both the Dirac and symplectic viewpoints has been developed in details, while the path integral has been left for a forthcoming work.
approach it is shown that a suitable phase space extension, involving a single pair of canonical fields, accounts for the reducibility. This is true irrespective of the degree of reducibility. It is not necessary to isolate an independent subset of constraints. In the symplectic approach, on the contrary, the constraints are embedded in an extended configuration space. Apart from the Lagrange multipliers, which occur even for irreducible systems, there are two extra fields which can be identified with the additional canonical pair in the Dirac approach. This properly accounts for the reducibility. The generalized brackets following from the symplectic matrix agree with the Dirac brackets.

Our ideas are introduced in a simple setting by discussing a quantum mechanical toy model in section II. These ideas are elaborated, in section III, to cope with reducibility in Dirac’s hamiltonian formalism. The examples of the 2-form and 3-form gauge field theories are worked out in details. The analysis is next repeated in section IV using the symplectic lagrangian formulation. Section V contains our concluding remarks.

2 A toy model

In this section a quantum mechanical model is considered to introduce the ideas in a simple setting. The full power and utility of the approach will be elaborated in the subsequent sections.

Consider the following set of reducible constraints,

\[
\begin{align*}
T_a &= p_a + \epsilon_{ab} q_b \approx 0 \quad (\epsilon_{12} = 1 \quad a, b = 1, 2) \\
T_3 &= p_1 + p_2 - q_1 + q_2 \approx 0 \\
T_4 &= p_1 - p_2 + q_1 + q_2 \approx 0
\end{align*}
\]

(2.1)

where \((g_a, p_a)\) is a canonical set of variables. It is clear that only two of these constraints are independent. For convenience, choose them to be \(T_a\). Then the other constraints are expressed by the combinations,

\[
\begin{align*}
T_3 &= T_1 + T_2 \approx 0 \\
T_4 &= T_1 - T_2 \approx 0
\end{align*}
\]

(2.2)

It is simple to see that the usual Poisson brackets (PB) among the canonical variables are incompatible with the above constraints. A standard way to overcome this in the canonical formalism is to work with the Dirac brackets (DB). For computing these brackets it is necessary to obtain the inverse of the matrix formed by the PB
of the complete set of constraints. In this case although the constraints (2.1) are second-class, the inverse does not exist because of the reducibility condition (2.2). The usual approach [1, 3] is to isolate the independent set of constraints and evaluate the DB. These brackets will then strongly enforce (2.1). The matrix elements of the PB of the independent constraints is given by,

\[ S_{ab} = \{T_a, T_b\} = 2\epsilon_{ab} \]  

(2.3)

which has the following inverse

\[ S_{ab}^{-1} = -\frac{1}{2}\epsilon_{ab} \]  

(2.4)

Then the DB defined by the general formula [1],

\[ \{Q, P\}^* = \{Q, P\} - \{Q, T_a\} S_{ab}^{-1}\{T_b, P\} \]  

(2.5)

are found to be,

\[ \{q_a, p_b\}^* = \frac{1}{2}\delta_{ab} \]

\[ \{q_a, q_b\}^* = \{p_a, p_b\}^* = -\frac{1}{2}\epsilon_{ab} \]  

(2.6)

which strongly imposes the constraint sector (2.1). This completes the conventional treatment.

In our approach, on the other hand, it is possible to work with the full set of reducible constraints by first extending the phase space, introducing a canonical pair of variables \((\eta, \pi)\),

\[ \{\eta, \pi\} = 1 \]

\[ \{\eta, \eta\} = \{\pi, \pi\} = 0 \]  

(2.7)

These variables have vanishing brackets with \(q_a\) and \(p_a\). In the extended space the dependent constraints are modified as [1],

\[ \hat{T}_3 = p_1 + p_2 - q_1 + q_2 + 2c\eta \approx 0 \]

\[ \hat{T}_4 = p_1 - p_2 + q_1 + q_2 + c\pi \approx 0 \]  

(2.8)

\footnote{In those cases when the dependent constraints cannot be easily separated, all the constraints have to be modified. Later on we consider such examples.}
where $c$ is an arbitrary parameter and the factor 2 is included only for computational ease. The matrix of the PB of the complete set of constraints $(T_a, \tilde{T}_3, \tilde{T}_4)$, which are now independent in the extended space, is given by,

$$\tilde{S} = 2 \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & 1 & 0 & -2 + c^2 \\ 1 & 1 & 2 - c^2 & 0 \end{pmatrix}$$ (2.9)

The inverse is,

$$\tilde{S}^{-1} = \frac{1}{2c^2} \begin{pmatrix} 0 & 2 - c^2 & -1 & 1 \\ c^2 - 2 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$$ (2.10)

Expectedly, $\tilde{S}^{-1}$ does not exist for $c = 0$. The DB are now modified as,

$$\{Q, P\}^* = \{Q, P\} - \{Q, \tilde{T}\} \tilde{S}^{-1} \{\tilde{T}, P\}$$ (2.11)

where $\tilde{T}$ generically denotes the constraints $(T_a, \tilde{T}_3, \tilde{T}_4)$. A simple algebra reproduces (2.6). It is interesting to point out that the parameter $c$ is canceled in the evaluation of these DB. This is related to the vanishing of $\eta$ and $\pi$ if $T_a$ are imposed in (2.8). In other words, the phase space extension removes the reducibility but retains the original constraint sector independent of the value of $c$. This is the reason that the DB of the reducible system were reproduced without the need of taking any limit like $c \to 0$ at the end of the computations.

3 The Dirac formalism

The Dirac formalism provides a systematic way of discussing the canonical constrained structure of different systems. If such systems are reducible, however, the usual analysis must be modified, as already elaborated in the toy model. In this section we extend our approach to the examples of 2-form and 3-form gauge field theories. Two schemes will be developed, with and without a parameter in the extended phase space.
3.1 Two-form with a mass parameter

The Lagrangian density is defined by,

\[ \mathcal{L} = \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \]  

(3.1)

where,

\[ H_{\mu\nu\rho} = \partial_\mu A_{\nu\rho} + \partial_\rho A_{\mu\nu} + \partial_\nu A_{\rho\mu} \]  

(3.2)

is the field tensor corresponding to the 2 form gauge field \( A_{\mu\nu} \).

The canonical momenta are given by

\[ T_0 = \pi_{00} \approx 0 \]  

(3.3)

\[ \pi_{ij} = \dot{A}_{ij} + \partial_i A_{j0} - \partial_j A_{i0} \]  

(3.4)

from which the total Hamiltonian density is obtained

\[ \mathcal{H} = \frac{1}{4} \pi_{ij} \pi^{ij} + \partial_i A_{0j} \pi^{ij} - \frac{1}{4} A_{ij} \nabla^2 A^{ij} + \frac{1}{2} A_{ij} \partial^j \partial_k A^{ik} + \lambda_0 T_0 \]  

(3.5)

Persistence in time of the primary constraint \( T_0 \approx 0 \) leads to a secondary constraint

\[ T_i = \partial^j \pi_{ji} \approx 0 \]  

(3.6)

The constraints \( T_i \) are reducible since,

\[ \partial^i T_i = \partial^j \partial^i \pi_{ji} = 0 \]  

(3.7)

implies that all \( T_i \) are not independent. This is related to the fact that the original gauge transformations,

\[ \delta A_{ij} = \partial_i \xi_j - \partial_j \xi_i \]  

(3.8)

are not independent since \( \delta A_{ij} = 0 \) if the parameters are \( \xi_i = \partial_i \theta \) for any \( \theta \). The conventional gauge-fixing in the Dirac procedure is to choose \( \delta \),

\[ \chi_i = \partial^j A_{ji} \]  

(3.9)

which also satisfies a reducibility condition like (3.7). Due to this condition,
\[ S_{ij}(\vec{x}, \vec{y}) = \{T_i(\vec{x}), \chi_j(\vec{y})\} \]
\[ = -(\eta_{ij} \nabla^2 + \partial_i \partial_j) \delta(\vec{x} - \vec{y}) \] (3.10)

does not possess an inverse \[\footnote{\text{We work with the Bjorken-Drell metric } \eta_{ij} = -\delta_{ij}.}]. Hence the DB cannot be computed in the usual way. The orthodox method \[\footnote{\text{Hence the DB cannot be computed in the usual way. The orthodox method \textbf{is} to isolate the independent subset of constraints. Apart from a lack of uniqueness in the procedure, the ensuing algebra is quite messy \textbf{is}. \text{In the present approach we proceed, as before, by modifying the constraints in an enlarged phase space,}}.} \text{is to isolate the independent subset of constraints. Apart from a lack of uniqueness in the procedure, the ensuing algebra is quite messy \textbf{is}.} \]

In the present approach we proceed, as before, by modifying the constraints in an enlarged phase space,

\[ \tilde{T}_i = \partial^j \pi_{ji} + m p_i \approx 0 \]
\[ \tilde{\chi}_i = \partial^j A_{ji} + m \phi_i \approx 0 \] (3.11)

where \((\phi_i, p^j)\) is a canonical pair,

\[ \{\phi_i(\vec{x}), p^j(\vec{y})\} = \delta^i_j \delta(\vec{x} - \vec{y}) \] (3.12)

and \(m\) is some parameter having the dimensions of mass. The matrix involving the PB of the modified constraints is,

\[ \tilde{S}_{ij}(\vec{x}, \vec{y}) = \begin{pmatrix} \{\tilde{T}_i(\vec{x}), \tilde{T}_j(\vec{y})\} & \{\tilde{T}_i(\vec{x}), \tilde{\chi}_j(\vec{y})\} \\ \{\tilde{\chi}_i(\vec{x}), \tilde{T}_j(\vec{y})\} & \{\tilde{\chi}_i(\vec{x}), \tilde{\chi}_j(\vec{y})\} \end{pmatrix} \]
\[ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[ (\nabla^2 + m^2) \eta_{ij} + \partial_i \partial_j \right] \delta(\vec{x} - \vec{y}) \] (3.13)

whose inverse is given by,

\[ \tilde{S}^{-1}_{ij}(\vec{x}, \vec{y}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left( \eta_{ij} - \frac{1}{m^2} \partial_i \partial_j \right) \frac{1}{\nabla^2 + m^2} \delta(\vec{x} - \vec{y}) \] (3.14)

which will be used for computing the DB. These brackets are,

\[ \{A_{ij}(\vec{x}), \pi_{kl}(\vec{y})\}^* = \{A_{ij}(\vec{x}), \pi_{kl}(\vec{y})\} \]
\[ = \int d\vec{z} d\vec{w} \{A_{ij}(\vec{x}, \vec{z}), \tilde{T}_n(\vec{z})\} \{\tilde{S}^{-1}_{ij}(\vec{z}, \vec{w})\}_{12} \{\tilde{\chi}_r(\vec{w}), \pi_{kl}(\vec{y})\} \]
\[ = \left( \eta_{ij,kl} + \frac{\partial^2_{ij,kl}}{\nabla^2 + m^2} \right) \delta(\vec{x} - \vec{y}) \] (3.15)
where we have used some of the definitions below that shall be considered throughout this paper

\[ \eta_{ij,kl} = \eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk} \]  
(3.16)

\[ \eta_{ijk,lmn} = \eta_{il}\eta_{jm}\eta_{kn} + \eta_{in}\eta_{jm}\eta_{lk} - \eta_{in}\eta_{jl}\eta_{km} - \eta_{im}\eta_{jn}\eta_{kl} \]  
(3.17)

\[ \partial_{i,jk} = \eta_{ij}\partial_{k} - \eta_{ik}\partial_{j} \]  
(3.18)

\[ \partial_{ij,klm} = \eta_{ij,kl}\partial_{m} + \eta_{ij,mk}\partial_{l} + \eta_{ij,lm}\partial_{k} \]  
(3.19)

\[ \partial^2_{ij,kl} = \eta_{ik}\partial_{j}\partial_{l} + \eta_{jl}\partial_{i}\partial_{k} - \eta_{jk}\partial_{i}\partial_{l} - \eta_{il}\partial_{j}\partial_{k} \]  
(3.20)

In the limit \( m \to 0 \), this reproduces the standard result \[4\]. It is worthwhile to mention that the final result for the DB is obtained only after taking the limit \( m \to 0 \) because this is essential for mapping \[1\] to the original constraint shell. Naturally, it would be conceptually cleaner if, as was done in the toy model, the phase space extension accounts for the reducibility but does not deform the original constraint sector. Then the relevant DB will be obtained directly. This formulation is presented now in the next section.

### 3.2 Two-form without a mass parameter

As an alternative approach which also illuminates the gauge-fixing in the path integral quantization of reducible systems, the original constraints are extended as

\[ T_i = \partial^j\pi_{ji} + \partial_i p \approx 0 \]
\[ \chi_i = \partial^j A_{ji} + \partial_i \phi \approx 0 \]  
(3.21)

where \((\phi, p)\) denote a canonical set in the enlarged phase space. It is important to note that on the new constraint surface \( \partial^j T_i = \partial^j \chi_i \approx 0 \) imply \( \phi = p \approx 0 \), provided reasonable boundary conditions are assumed. This phase space extension, therefore, simultaneously avoids the reducibility and enforces the original constraints \( T_i = \chi_i \approx 0 \), irrespective of any limiting procedure, and the correct DB of the original theory ought to be reproduced. This is reminiscent of the quantum mechanical example. Incidentally, the extended gauge condition \( \chi_i \) is precisely used for gauge-fixing in the path integral BRST analysis \[3\] where \( \phi \) plays the role of the ghost field. By eliminating the reducibility, terms like \( \delta(0) \) no longer occur in the path integral. A modified constraint like \( T_i \) was suggested earlier \[8\], though in a different context.

The new constraint matrix analogous to \(8.13\) is now given by,

\[ S_{ij}(\vec{x}, \vec{y}) = \eta_{ij} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla^2 \delta(\vec{x} - \vec{y}) \]  
(3.22)
The inverse matrix is,

\[ \tilde{S}^{-1}_{ij}(\vec{x}, \vec{y}) = \eta_{ij} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{1}{V^2} \delta(\vec{x} - \vec{y}) \] (3.23)

The only nonvanishing DB is easily computed,

\[ \{ A_{ij}(\vec{x}), \pi_{kl}(\vec{y}) \} = \{ A_{ij}(\vec{x}), \pi_{kl}(\vec{y}) \} - \int d\vec{z} d\vec{w} \{ A_{ij}(\vec{x}), \tilde{T}_n(\vec{z}) \} (\tilde{S}^{-1}_{nr}(\vec{z}, \vec{w}))_{12} \{ \tilde{\chi}_r(\vec{w}), \pi_{kl}(\vec{y}) \} = \left( \eta_{ij,kl} + \frac{\partial^2_{ij,kl}}{V^2} \right) \delta(\vec{x} - \vec{y}) \] (3.24)

which reproduces the familiar expression. Both the elegance and algebraic simplification in obtaining the result are noteworthy. Therefore, from now on we shall only consider extensions analogous to (3.21) which avoid the necessity of introducing any parameter and a subsequent limiting prescription.

It is interesting to observe that the originally canonical pair \((\phi, p)\) now has vanishing DB either among them or with the other fields \(A_{ij}, \pi_{ij}\). A similar phenomenon occurs in the symplectic analysis discussed later.

### 3.3 Three-form case

The theory of a three form gauge field presents features that are peculiar and representative of higher form examples. Consequently, the analysis given here can be easily implemented to such examples. The Lagrangian is now given by,

\[ \mathcal{L} = \frac{1}{48} H_{\mu\nu\rho\lambda} H^{\mu\nu\rho\lambda} \] (3.25)

where

\[ H_{\mu\nu\rho\lambda} = \partial_\mu A_{\nu\rho\lambda} - \partial_\lambda A_{\mu\nu\rho} + \partial_\rho A_{\lambda\mu\nu} - \partial_\nu A_{\rho\lambda\mu} \] (3.26)

is the fully antisymmetric field tensor written in terms of the 3-form field. By following the canonical Dirac procedure, the reducible constraint is easily obtained,

\[ T_{ij} = \partial^k \pi_{kij} \approx 0 \] (3.27)

where the momentum conjugate to \(A^{ijk}\) is given by

\[ \pi_{ijk} = \delta^{lmn}_{ijk} \left( \frac{1}{6} \dot{A}_{lmn} - \frac{1}{2} \partial_l A_{mn0} \right) \] (3.28)
and we have used the notation given by Eq. (3.17).

The corresponding Coulomb-like gauge fixing condition reads,

$$\chi_{ij} = \partial^k A_{kij} \approx 0$$  \hspace{1cm} (3.29)

The PB matrix among these constraints, defined analogously to (3.13), is,

$$S_{ijkl}(\vec{x}, \vec{y}) = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \left( \eta_{ij,kl} \nabla^2 + \partial^2_{ij,kl} \right) \delta(\vec{x} - \vec{y})$$  \hspace{1cm} (3.30)

This quantity does not have an inverse which is defined as \footnote{The factor 1/2 is necessary for avoiding double counting due to the antisymmetry in the repeated indices $k, l$.}

$$\frac{1}{2} \int dy \ S_{ijkl}(x, y) S^{-1}_{klmn}(y, z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta_{ij} \delta(x - z)$$  \hspace{1cm} (3.31)

Exactly as was done for the two-form gauge field, it is possible to extend the phase space such that the constraints are now modified as,

$$\bar{T}_{ij} = \partial^k \pi_{kij} + \partial_j p_i - \partial_i p_j \approx 0$$

$$\bar{\chi}_{ij} = \partial^k A_{kij} + \partial_i \phi_j - \partial_j \phi_i \approx 0$$  \hspace{1cm} (3.32)

where, as usual, the canonical set $(\phi_i, p^j)$ in the enlarged space has vanishing PB with the original variables. There is, however, an important distinction from the analysis in the two form example. In that case, the enlargement (3.21) implied the vanishing of the extra fields on the constraint surface thereby reproducing the original constraint sector. Here, on the contrary, $\partial^i \bar{T}_{ij} = \partial^i \bar{\chi}_{ij} \approx 0$ leads to,

$$(\eta_{ij} \nabla^2 + \partial_i \partial_j) p_j = (\eta_{ij} \nabla^2 + \partial_i \partial_j) \phi_j \approx 0$$  \hspace{1cm} (3.33)

Clearly, it is not possible to set $p_i = \phi_i \approx 0$ since these are multiplied by a noninvertible operator. Hence by itself the extension (3.32) does not reduce to the original constraint sector. There are two ways to overcome this situation. The phase space is extended further by introducing more fields and performing a fresh analysis. This is the analog of the BRST analysis where a tower of ghosts-for-ghosts etc. has to be inserted \footnote{The factor 1/2 is necessary for avoiding double counting due to the antisymmetry in the repeated indices $k, l$.}. Alternatively, the symplectic structure can be altered so that the new fields are no longer canonical but satisfy,

$$\{ \phi_i(\vec{x}), p_j(\vec{y}) \} = \left( \eta_{ij} + \frac{\partial_i \partial_j}{\nabla^2} \right) \delta(\vec{x} - \vec{y})$$  \hspace{1cm} (3.34)
This means that these fields are transversal so that desired inversion in (3.33) is possible. The new fields vanish on the constraint surface and the original constraint set is reproduced. A related observation is that (3.34) just corresponds to the DB evaluated for the transversal constraints $\partial_i \phi^i = \partial_i p^i \approx 0$. It is now crucial to note that the algebra of the antisymmetric combination of new fields in (3.32) is, however, unaffected by the deformation (3.34) and yields the same result as if the fields were canonical. We find,

$$\{\partial_i p_j(\vec{x}) - \partial_j p_i(\vec{x}), \partial_k \phi_i(\vec{y}) - \partial_i \phi_k(\vec{y})\} = \partial^2_{ij,kl} \delta(\vec{x} - \vec{y})$$ (3.35)

irrespective of whether the Poisson algebra or the modified (3.34) is used. This is understandable since (3.35) involves the brackets among gauge invariant variables in which case there should be no difference between the Poisson and modified (Dirac) algebras [3]. In the evaluation of PB among $\bar{T}_{ij}$ and $\bar{\chi}_{ij}$, therefore, the new fields, which vanish on the constraint surface, can still be chosen as canonical. The DB of the original reducible theory will be obtained by working with the modified constraints (3.32) without the necessity of introducing either additional tower of fields, as in the BRST approach, or deforming the canonical structure.

The nontrivial PB among the constraints (3.32) is

$$(\bar{S}_{ijkl}(\vec{x}, \vec{y}))_{12} = \{\bar{T}_{ij}(\vec{x}), \bar{\chi}_{kl}(\vec{y})\} = -\eta_{ij,kl} \nabla^2 \delta(\vec{x} - \vec{y})$$ (3.36)

which has the following inverse,

$$[(\bar{S}_{ijkl}(x, y))^{-1}]_{21} = -\eta_{ij,kl} \frac{1}{\nabla^2} \delta(\vec{x} - \vec{y})$$ (3.37)

The complete inverse analogous to (3.23) follows trivially. Now the nontrivial brackets are easily computed,

$$\{A_{ijk}(\vec{x}), \pi_{lmn}(\vec{y})\}^* = \left(\eta_{ijk,lmn} + \frac{\partial^2_{ijk,lmn}}{\nabla^2}\right) \delta(\vec{x} - \vec{y})$$ (3.38)

where the first term is the PB and the second is generated by (3.37). This yields the DB of the original reducible theory. It is simple to generalize this approach to arbitrary $p$-form gauge fields. All the arguments given above are applicable and the DB can be evaluated by enlarging the constraint sector with a single canonical pair of $(p - 1)$-form field.
4 The symplectic formalism

The symplectic formalism is a geometrical manner of dealing with canonical sys-

tems. Although it existed in the contemporary literature [10], it was resurrected

by Faddeev-Jackiw [2] from a physicist’s point of view. Since the subject is still

evolving [7] it is reasonable to provide a brief overview before proceeding with the

actual computations.

The symplectic formalism deals with first-order Lagrangians. It is opportune to

mention that this is not a serious restriction because all systems we know, described

by quadratical Lagrangians, can always be set in the first-order formulation. This

is achieved by extending the configuration space with the introduction of auxiliary

fields. For algebraic simplifications, these are usually the momenta, but this is not

necessary. The symplectic formalism is, therefore, basically a Lagrangian approach

complementing the Hamiltonian formulation of Dirac.

Let us consider a system described by a first-order Lagrangian such as

\[ L = a_\alpha(y) \dot{y}\alpha - V(y) \]  

(4.1)

where \( y^\alpha \) is a set of 2\( N \) coordinates. The momenta or other auxiliary quantities

required to render the Lagrangian in the first-order form will be denoted by \( \dot{y}^{i+N} \).

We develop the formalism by using discrete degrees of freedom. The extrapolation

for fields can be done in a straightforward way.

From (4.1), the Euler-Lagrange equation of motion reads

\[ f_{\alpha\beta} \dot{y}^\beta = \partial_\alpha V \]  

(4.2)

where \( \partial_\alpha \equiv \partial/\partial y^\alpha \) and

\[ f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha \]  

(4.3)

is the symplectic 2-form. If \( \det(f_{\alpha\beta}) \neq 0 \), the system is unconstrained and one can

solve eq. (4.2) for velocities \( \dot{y}^\alpha \), i.e.,

\[ \dot{y}^\alpha = f^{\alpha\beta} \partial_\beta V \]  

(4.4)

where \( f^{\alpha\beta} \) is the inverse of \( f_{\alpha\beta} \). The generalized brackets, which are the Poisson

brackets, between the coordinates \( y^\alpha \) and \( y^\beta \) are given by \( f^{\alpha\beta} \), i.e.

\[ \{y^\alpha, y^\beta\} = f^{\alpha\beta} \]  

(4.5)

An interesting and instructive point occurs when the quantity \( f_{\alpha\beta} \) is singular. In

this case one cannot identify the symplectic tensor and, consequently, the brackets of
the theory cannot be consistently defined. This means that the system is constrained from a symplectic point of view. To identify and incorporate the constraints in this approach, we proceed as discussed below.

Let us denote the above mentioned singular quantity by $f^{(0)}_{\alpha\beta}$, and suppose that it has, for example, $M$ ($M < 2N$) zero modes $v^{(0)}_m$, $m = 1, \ldots, M$, i.e.,

$$f^{(0)}_{\alpha\beta} v^{(0)}_m = 0$$ \hfill (4.6)

The combination of (4.2) and (4.6) gives

$$\tilde{v}^{(0)}_m \partial_\alpha V^{(0)} = 0$$ \hfill (4.7)

This may lead to a constraint. Let us suppose that this actually occurs (we shall discuss the opposite case soon). Usually, as for instance in the Dirac approach, the constraints are introduced in the potential part of the Lagrangian by means of Lagrange multipliers. Here, in order to get a deformation in the tensor $f^{(0)}_{\alpha\beta}$ we introduce them instead into the kinetic part. This is done by taking the time derivative of the constraint and putting them in the Lagrangian by means of multipliers. These multipliers, which we denote by $\lambda^{(0)}_m$, enlarge the configuration space of the theory. This permits us to identify new vectors $a^{(1)}_\alpha$ and $a^{(1)}_m$ as

$$a^{(1)}_\alpha = a^{(0)}_\alpha + \lambda^{(0)}_m \partial_\alpha \Omega^{(0)}_m$$

$$a^{(1)}_m = 0$$ \hfill (4.8)

where $\Omega^{(0)}_m$ are the constraints obtained from (4.7). Consequently, one can now introduce the quantities defining the elements of the deformed symplectic matrix in the extended configuration space $(y^{(0)} \alpha, \lambda^{(0)}_m)$,

$$f^{(1)}_{\alpha\beta} = \partial_\alpha a^{(1)}_\beta - \partial_\beta a^{(1)}_\alpha$$

$$f^{(1)}_{\alpha m} = \partial_\alpha a^{(1)}_m - \partial_m a^{(1)}_\alpha = -\partial_m a^{(1)}_\alpha$$

$$f^{(1)}_{mn} = \partial_m a^{(1)}_n - \partial_n a^{(1)}_m$$ \hfill (4.9)

where $\partial_m = \partial/\partial\lambda^m$. If $f^{(1)} \neq 0$, then the process of finding the constraints terminates. If not, one should repeat the above iterative procedure as many times as necessary.

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4We mention that the number of constraints in the symplectic formalism is equal to or lesser than in the Dirac case. However, the functional form is the same.

5Since constraints do not evolve in time, a time derivative of a constraint is also a constraint. Another point is that one could, instead, take the time derivative of the Lagrange multiplier. The difference, being a total derivative, does not affect the equation of motion.
It may also occur that we arrive at a stage where the zero modes of the singular matrix do not lead to any new constraint. This is the case, for example, of gauge theories. At this point, in order to define the symplectic tensor, some gauge condition has to be imposed. For details, see Ref. [2]. For reducible systems, further complications arise. The usual technique is to mimic the BRST analysis [3] and introduce a series of Lagrange multiplier-for-Lagrange multipliers [5]. In this section we develop in details, both for two and three form gauge fields, the analogous procedure already discussed in Dirac’s formalism.

4.1 The two-form case

To use the symplectic formalism, it is necessary to express the Lagrangian (3.1) in the first order notation. For convenience, this Lagrangian is rewritten as,

\[
\mathcal{L} = \frac{1}{4} \dot{A}_{ij} \dot{A}^{ij} + \partial^i A^0_0 \dot{A}_{ij} + \frac{1}{2} \partial_i A_{0j} \partial^i A^{0j} + \frac{1}{4} \partial_i A_{jk} \partial^j A^{ik} - \frac{1}{2} \partial_i A_{j0} \partial^j A^{0i} + \frac{1}{2} \partial_i A_{jk} \partial^k A^{ji} \tag{4.10}
\]

Using the momentum conjugate to \( A_{ij} \) as the auxiliary field, we can put (4.10) in the desired first order form,

\[
\mathcal{L} = -\frac{1}{4} \pi_{ij} \dot{\pi}^{ij} + \frac{1}{2} \left( \dot{A}_{ij} + \partial_i A_{j0} - \partial_j A_{i0} \right) \pi^{ij} + \frac{1}{4} \partial_i A_{jk} \partial^j A^{ik} + \frac{1}{2} \partial_i A_{jk} \partial^k A^{ji} \tag{4.11}
\]

This is conveniently expressed as,

\[
\mathcal{L}^{(0)} = \frac{1}{2} \pi_{ij} \dot{A}^{ij} - V^{(0)} \tag{4.12}
\]

where,

\[
V^{(0)} = \frac{1}{4} \pi_{ij} \pi^{ij} + \partial_i A_{0j} \pi^{ij} - \frac{1}{4} A_{ij} \nabla^2 A^{ij} + \frac{1}{2} A_{ij} \partial^j \partial_k A^{ik} \tag{4.13}
\]

A comparison with the general structure (4.1) leads to the following identifications

\[
\begin{align*}
  a^{(0)}_{ij} A & = \pi_{ij} \\
  a^{(0)}_i A & = 0 \\
  a^{(0)}_{ij} \pi & = 0
\end{align*} \tag{4.14}
\]

Nonvanishing elements of the symplectic matrix are therefore given by,
\[
f^{(0)\pi}_{ijkl}(\vec{x}, \vec{y}) = \frac{\delta a^{(0)\pi}_{kl}(\vec{y})}{\delta A^{ij}(\vec{x})} - \eta_{ijkl} \delta(\vec{x} - \vec{y})
\]

(4.15)

Correspondingly, the matrix \( f^{(0)} \), whose general form reads

\[
f^{(0)} = \begin{pmatrix}
    f^{(0)\pi}_{ik} & f^{(0)\pi}_{kl} & f^{(0)\pi}_{ijk} & f^{(0)\pi_{ijkl}} \\
    f^{(0)\pi}_{ikl} & f^{(0)\pi_{ijkl}} & f^{(0)\pi_{ijkl}} & f^{(0)\pi_{ijkl}} \\
    f^{(0)\pi_{ijkl}} & f^{(0)\pi_{ijkl}} & f^{(0)\pi_{ijkl}} & f^{(0)\pi_{ijkl}} \\
    f^{(0)\pi_{ijkl}} & f^{(0)\pi_{ijkl}} & f^{(0)\pi_{ijkl}} & f^{(0)\pi_{ijkl}} \\
\end{pmatrix}
\]

(4.16)

can be written as,

\[
f^{(0)} = \begin{pmatrix}
    0 & 0 & 0 & \delta(\vec{x} - \vec{y}) \\
    0 & 0 & -\eta_{ijkl} & 0 \\
    0 & \eta_{ijkl} & 0 & \delta(\vec{x} - \vec{y}) \\
\end{pmatrix}
\]

(4.17)

This is clearly a singular matrix, which is exactly the way constraints are manifested in the symplectic formalism. Let us consider that a zero mode of (4.17) has the general form: \((v^k, u^{kl}, \omega^{kl})\), where \(u^{kl}\) and \(\omega^{kl}\) are antisymmetric quantities. Possible new constraints might appear from

\[
\int d\vec{x} \left( v^k \frac{\delta}{\delta A^{0k}} + \frac{1}{2} u^{kl} \frac{\delta}{\delta A^{kl}} + \frac{1}{2} \omega^{kl} \frac{\delta}{\delta \pi^{kl}} \right) \int d\vec{y} V^{(0)} = 0
\]

(4.18)

For \((v^k, u^{kl}, \omega^{kl})\) to be a zero mode of \(f^{(0)}\), we have

\[
\eta_{ijkl} \omega^{kl} = 0 \Rightarrow \omega_{ij} = 0 \\
\eta_{ijkl} u^{kl} = 0 \Rightarrow u_{ij} = 0
\]

(4.19)

and the quantity \(v^k\) remains undetermined. Using these results and the expression for \(V^{(0)}\) given in (4.13), it is found that (4.18) yields,

\[
\int d\vec{x} v^k(\vec{x}) \frac{\delta}{\delta A^{0k}(\vec{x})} \int d\vec{y} V^{(0)} = \int d\vec{x} v_j \partial_i \pi^{ij} = 0
\]

(4.20)

Since \(v_j\) is a generic function of \(\vec{x}\), we obtain the constraint

\[
T^j = \partial_i \pi^{ij}
\]

(4.21)

This is how the secondary constraint of the Dirac formalism manifests in the symplectic version. We mention that primary constraints in the Dirac approach are not
constraints in the symplectic case. Now, proceeding to the next step in the iterative process,
\[\mathcal{L}^{(1)} = \frac{1}{2} \pi_{ij} \dot{A}^{ij} + \lambda_j (\partial_i \pi^{ij} + \partial^j \dot{p}) - V^{(0)}\]
\[= \frac{1}{2} \pi_{ij} \dot{A}^{ij} - \partial_i \lambda_j \pi^{ij} - \partial^j \lambda_j \dot{p} - V^{(1)}\]  
(4.22)

where
\[V^{(1)} = \frac{1}{4} \pi_{ij} \pi^{ij} - \frac{1}{4} A_{ij} \nabla^2 A^{ij} + \frac{1}{2} A_{ij} \partial^j \partial_k A^{ik}\]  
(4.23)

The term \(\partial_i A_{0j} \pi^{ij}\) in \(V^{(0)}\) was absorbed by the term \(\partial_i \lambda_j \pi^{ij}\). Hence, the \(A_{0i}\) field has disappeared from the theory. Note also that the reducible constraint \((4.21)\) has been modified following the same pattern as in \((3.21)\). The new coefficients are
\[a^{(1)A}_{ij} = \pi_{ij}\]
\[a^{(1)\pi}_{ij} = -\partial_i \lambda_j + \partial_j \lambda_i\]
\[a^{(1)\lambda}_i = 0\]
\[a^{(1)p} = -\partial_i \lambda^i\]  
(4.24)

The matrix \(f^{(1)}\), introduced in \((4.9)\), now has the general form
\[f^{(1)} = \begin{pmatrix}
  f_{ijkl}^{(1)AA} & f_{ijkl}^{(1)A\pi} & f_{ijkl}^{(1)A\lambda} & f_{ijkl}^{(1)Ap} \\
  f_{ijkl}^{(1)\pi A} & f_{ijkl}^{(1)\pi \pi} & f_{ijkl}^{(1)\pi \lambda} & f_{ijkl}^{(1)\pi p} \\
  f_{ijkl}^{(1)\lambda A} & f_{ijkl}^{(1)\lambda \pi} & f_{ijkl}^{(1)\lambda \lambda} & f_{ijkl}^{(1)\lambda p} \\
  f_{ijkl}^{(1)pA} & f_{ijkl}^{(1)p\pi} & f_{ijkl}^{(1)p\lambda} & f_{ijkl}^{(1)p p}
\end{pmatrix}\]  
(4.25)

where the notation follows \((1.13)\), except that the coefficients \(a^{(0)}\) are replaced by \(a^{(1)}\). A simple algebra leads to the result,
\[f^{(1)} = \begin{pmatrix}
  0 & -\eta_{ij,kl} & 0 & 0 \\
  \eta_{ij,kl} & 0 & \partial_{k,ij} & 0 \\
  0 & \partial_{k,ik} & 0 & \partial_i \\
  0 & 0 & \partial_k & 0
\end{pmatrix} \delta(x - y)\]  
(4.26)

This matrix is still singular so that the iterative process has to be continued.
Let us consider that the zero mode of (4.26) is given by $(v^{kl}, u^{kl}, \omega^k, h)$, where $v^{kl}$ and $u^{kl}$ are antisymmetric quantities. This implies

$$\eta_{ij,kl} u^{kl} = 0 \implies u_{ij} = 0$$
$$\eta_{ij,kl} v^{kl} + \partial_k i,j \omega^k = 0 \implies v_{ij} = \frac{1}{2} \left( \partial_i \omega_j - \partial_j \omega_i \right)$$
$$\partial_i h = 0$$
$$\partial_k \omega^k = 0$$

(4.27)

In order to look for new constraints, we write

$$\int d\vec{x} \left( \frac{1}{2} v^{kl} \frac{\delta}{\delta A^{kl}} + \omega^k \frac{\delta}{\delta \lambda^k} + h \frac{\delta}{\delta \pi} \right) \int d\vec{y} V^{(1)} = 0$$

(4.28)

The l.h.s. of this equation vanishes identically. Thus, the zero mode of $f^{(1)}$ does not lead to a new constraint. This fact means that we are in the presence of a gauge theory. Let us then choose the Coulomb-like gauge condition already considered in Eq. (3.21),

$$\partial_i A^{ij} + \partial^j \phi = 0$$

(4.29)

Introducing this constraint into the kinetic part of the Lagrangian, we have

$$\mathcal{L}^{(2)} = \frac{1}{2} \pi_{ij} \dot{A}^{ij} - \partial_i \lambda_j \dot{\pi}^{ij} - \partial_i \lambda^i \dot{\phi} + \eta_j (\partial_i \dot{A}^{ij} + \partial^j \dot{\phi}) - V^{(1)}$$
$$= \frac{1}{2} \left( \pi_{ij} - \partial_i \eta_j + \partial_j \eta_i \right) \dot{A}^{ij} - \partial_i \lambda_j \dot{\pi}^{ij} - \partial_i \lambda^i \dot{\pi} - \partial_i \eta^j \dot{\phi} - V^{(2)}$$

(4.30)

where

$$V^{(2)} = \frac{1}{4} \pi_{ij} \pi^{ij} - \frac{1}{4} A_{ij} \nabla^2 A^{ij}$$

(4.31)

The term $\frac{1}{2} A_{ij} \partial^i \partial_k A^{ik}$ was absorbed by $\partial_i \eta_j \dot{A}^{ij}$. The new coefficients are

$$a_{ij}^{(2)A} = \pi_{ij} - \partial_i \eta_j + \partial_j \eta_i$$
$$a_{ij}^{(2)\pi} = -\partial_i \lambda_j + \partial_j \lambda_i$$
$$a_{ij}^{(2)p} = -\partial_i \lambda^i$$
$$a_{ij}^{(2)\phi} = -\partial_i \eta^i$$
$$a_i^{(2)\lambda} = 0$$
$$a_i^{(2)\eta} = 0$$

(4.32)
The second iterated matrix $f^{(2)}$ is now calculated from $a^{(2)}$, just as $f^{(1)}$ was done from $a^{(1)}$. We find

$$f^{(2)} = \begin{pmatrix}
0 & -\eta_{ij,kl} & 0 & 0 & \partial_k & 0 \\
\eta_{ij,kl} & 0 & \partial_k & 0 & 0 & 0 \\
0 & \partial_{ik} & 0 & \partial_i & 0 & 0 \\
0 & 0 & \partial_k & 0 & 0 & 0 \\
\partial_{ik} & 0 & 0 & 0 & \partial_i & 0 \\
0 & 0 & 0 & 0 & \partial_k & 0 \\
\end{pmatrix} \delta(\vec{x} - \vec{y}) \quad (4.33)$$

where rows and columns follow the order $A_{ij}$, $\pi_{ij}$, $\lambda_i$, $\pi$, $\eta_i$, and $\phi$. The above matrix is not singular. Hence, it can be identified as the symplectic tensor of the constrained theory. Its inverse will gives us the generalized brackets of the physical fields of the theory. The calculation of the inverse is done in Appendix A. We simply write the final result for the symplectic matrix

$$f^{(2)^{-1}} = \begin{pmatrix}
0 & -\frac{\delta_m^m}{\sqrt{\gamma}} & 0 & 0 & \frac{\delta_m^m}{\sqrt{\gamma}} & 0 \\
\frac{\delta^m_m}{\sqrt{\gamma}} & 0 & \frac{\partial_k}{\sqrt{\gamma}} & 0 & 0 & 0 \\
0 & \frac{\partial_k}{\sqrt{\gamma}} & 0 & -\frac{\partial_m}{\sqrt{\gamma}} & -\frac{1}{\sqrt{\gamma}}(\delta_i^m + \partial_i \frac{\partial_m}{\sqrt{\gamma}}) & 0 \\
0 & 0 & -\frac{\partial_i}{\sqrt{\gamma}} & 0 & 0 & 0 \\
\frac{\partial^m_m}{\sqrt{\gamma}} & 0 & \frac{1}{\sqrt{\gamma}}(\delta^m_i + \partial_i \frac{\partial^m}{\sqrt{\gamma}}) & 0 & 0 & -\frac{\partial^m}{\sqrt{\gamma}} \\
0 & 0 & 0 & 0 & \frac{\partial_i}{\sqrt{\gamma}} & 0 \\
\end{pmatrix} \times \delta(\vec{x} - \vec{y}) \quad (4.34)$$

Recalling the ordering of the fields, it is easy to read-off the bracket between $A_{ij}$ and $\pi_{mn}$ from the (12) element of the above matrix. This reproduces the DB given in (3.24). Likewise, other brackets are easily obtained.

Incidentally, the symplectic brackets between the set $(\phi, p)$ reproduce the vanishing algebra found earlier in the Dirac formalism. This illuminates the close connection between the embedding procedures adopted in these two approaches.

### 4.2 The three-form case

The initial step in the symplectic formalism is to rewrite the Lagrangian (3.25) in its first order form. It is convenient to choose the momenta conjugate to $A_{ijk}$ as the auxiliary variables in analogy with the 2-form example. We find

$$\mathcal{L} = \frac{1}{6} \pi_{ijk} \dot{A}^{ijk} - V^{(0)} \quad (4.35)$$

where
\[ V^{(0)} = \frac{1}{12} \pi_{ijk} \pi^{ijk} + \frac{1}{2} \partial_i A_{jkl} \pi^{ijk} - \frac{1}{12} A_{ijk} \nabla^2 A^{ijk} \]

\[ + \frac{1}{4} \partial_i A^{ijkl} \partial_j A_{jkl} \]

Following a similar procedure as in the previous subsection, we find the constraint \((3.27)\). Adopting the same logic, this constraint is modified by introducing new fields. Its structure is now identical to the first relation in \((3.32)\). At this point, there is an important difference from the Dirac analysis. The transversality condition on the new field,

\[ \partial_i p^i = 0 \]

is explicitly taken as an additional constraint.

The corresponding gauge condition is given by the second relation in \((3.32)\), together with a condition akin to \((4.37)\),

\[ \partial_i \phi^i = 0 \]

All these constraints are now introduced into the kinetic part of the Lagrangian by means of Lagrange multipliers. We thus have

\[ \mathcal{L} = \frac{1}{6} \pi_{ijk} \dot{A}^{ijk} + \frac{1}{2} \lambda_{jk} \left( \partial_i \pi^{ijk} + \partial^i p^j - \partial^k \dot{p}^i \right) \]

\[ + \frac{1}{2} \eta_{jk} \left( \partial_i \dot{A}^{ijk} + \partial^i \phi^j - \partial^k \dot{\phi}^i \right) \]

\[ + \rho \partial_i p^i + \zeta \partial_i \dot{p}^i - V^{(0)} \]

\[ = \frac{1}{6} \left( \pi_{ijk} - \partial_i \eta_{jk} - \partial_j \eta_{ki} - \partial_k \eta_{ij} \right) \dot{A}^{ijk} - \frac{1}{6} \left( \partial_i \lambda_{jk} + \partial_j \lambda_{ki} + \partial_k \lambda_{ij} \right) \pi^{ijk} \]

\[ - \left( \partial_i \rho + \partial^j \lambda_{ji} \right) \dot{p}^i - \left( \partial_i \zeta + \partial^j \eta_{ji} \right) \dot{\phi}^i - V^{(2)} \]

where

\[ V^{(2)} = \frac{1}{12} \pi_{ijk} \pi^{ijk} - \frac{1}{12} A_{ijk} \nabla^2 A^{ijk} \]

The symplectic coefficients are easily identified as,
\[ a^{(2)\pi}_{ijk} = \pi_{ijk} - \partial_i \eta_{jk} - \partial_j \eta_{ki} - \partial_k \eta_{ij} \]
\[ a^{(2)\gamma}_{ijk} = -\partial_i \gamma_{jk} - \partial_j \gamma_{ki} - \partial_k \gamma_{ij} \]
\[ a^{(2)\rho}_{i} = -\partial_i \rho - \partial^j \lambda_{ji} \]
\[ a^{(2)\phi}_{i} = -\partial_i \phi - \partial^j \eta_{ji} \]
\[ a^{(2)\rho}_{i} = 0 \]
\[ a^{(2)\zeta}_{i} = 0 \]
\[ a^{(2)\lambda}_{i} = 0 \]
\[ a^{(2)\eta}_{i} = 0 \]

(4.41)

The nonvanishing elements of the matrix \( f^{(2)} \) are now computed. Two elements are explicitly furnished to clarify the definitions and notations.

\[ f^{(2)\pi}_{ijklmn}(\vec{x}, \vec{y}) = \frac{\delta a^{(2)\pi}_{lmn}(\vec{y})}{\delta A^{ijk}(\vec{x})} - \frac{\delta a^{(2)\pi}_{ijk}(\vec{x})}{\delta \pi^{lmn}(\vec{y})} = -\eta_{jk,lmn} \delta(\vec{x} - \vec{y}) \]
\[ f^{(2)\eta}_{ijklmn}(\vec{x}, \vec{y}) = \frac{\delta a^{(2)\eta}_{lm}(\vec{y})}{\delta A^{ijk}(\vec{x})} - \frac{\delta a^{(2)\eta}_{ijk}(\vec{x})}{\delta \eta^{lmn}(\vec{y})} = (\eta_{jk,lm} \partial_i + \eta_{ki,lm} \partial_j + \eta_{ij,lm} \partial_k) \delta(\vec{x} - \vec{y}) \]

(4.42)

Likewise, all the entries are obtained to yield

\[ f^{(2)} = \begin{pmatrix}
0 & -\eta_{jk,lmn} & 0 & \partial_{lm,ijk} & 0 & 0 & 0 & 0 \\
\eta_{jk,lmn} & 0 & \partial_{lm,ijk} & 0 & 0 & 0 & 0 & 0 \\
0 & \partial_{ij,lmn} & 0 & 0 & 0 & \eta_{kl,ij} \partial^k & 0 & 0 \\
\partial_{ij,lmn} & 0 & 0 & 0 & 0 & \eta_{kl,ij} \partial^k & 0 & 0 \\
0 & 0 & \eta_{ji,lm} \partial^j & 0 & 0 & 0 & \partial_i & 0 \\
0 & 0 & \eta_{ji,lm} \partial^j & 0 & 0 & 0 & \partial_i & 0 \\
0 & 0 & 0 & \partial_l & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \partial_l & 0 & 0 & 0 \\
\end{pmatrix} \times \delta(\vec{x} - \vec{y}) \]

(4.43)

where the matrix is arrayed in the sequence followed in (4.41). Proceeding as in the 2-form theory, the inverse of this matrix is calculated. From the appropriate entry in this symplectic matrix the nontrivial bracket between \( A_{ijk} \) and \( \pi^{lmn} \) is easily obtained. It reproduces the algebra (3.38) found in the Dirac method.
5 Conclusion

A general scheme for treating reducible systems has been developed in the canonical formalism, both from the Hamiltonian (Dirac) and Lagrangian (symplectic) viewpoints. This avoids either the abstraction of the independent subset of constraints, or the introduction of an infinite set of new fields - manipulations that are essential in the conventional analysis of reducible theories [1, 3, 4, 5]. Apart from being systematic, a significant feature was the algebraic simplicity of the method. In this connection it may be recalled that the usual Dirac method [1, 4] of obtaining the algebra (3.24) appears quite involved compared to this calculation.

Within the Dirac approach, the present scheme consisted in a phase space extension involving only a single canonical pair. There are two interesting aspects of this procedure. The first is that the present phase space extension achieves exactly the opposite of a similar approach [11] designed to convert second class constraints into their first class forms. The point is that the second class reducible constraints discussed here yield a vanishing determinant for the constraint Poisson bracket matrix. Roughly speaking, therefore, these constraints continue to display a first class character. The phase space embedding changes this character into true second class, enabling a simple and direct evaluation of the Dirac brackets. The second aspect is that by imposing restrictions on the single extra canonical pair of variables, the need for more fields was avoided. This was explicitly demonstrated for the three form gauge theory.

The symplectic approach, in contrast to the Dirac procedure, is a Lagrangian formulation. Nevertheless, the reducible constraints were identified and then modified as in the Dirac treatment. The generalized brackets obtained from the symplectic matrix agreed with the Dirac brackets. An important observation concerns the distinct roles played by the extra fields in the two approaches. In the Dirac approach these fields, to begin with, formed a canonical pair. However, at the end, the same fields were found to have vanishing Dirac brackets either among them or with the other fields. In the symplectic case, the new fields were just some multipliers, which were obviously not canonical pairs since there is only an extension of the configuration space. However, the symplectic matrix revealed that these multipliers had vanishing brackets. Thus, although the extra fields have different interpretations, they are algebraically equivalent. This also shows, in a precise fashion, that the same basic principle works for dealing with reducibility either in the Dirac or symplectic formalism.

A final question remains regarding the application of these ideas to provide a path integral formulation for reducible systems. Some hints can already be obtained from the present analysis. Recall that the usual symplectic formalism requires a tower of Lagrange multipliers [3] in analogy with a series of ghosts-for-ghosts necessary in the BRST path integral formulation [3]. Since the tower of Lagrange multipliers was avoided in this work, it suggests the possibility that the corresponding situation in the BRST framework is also redundant. Indeed, we shall explicitly show in a future publication [3] that the path integral following from the
present canonical prescription eliminates the reducibility without the necessity of any ghosts-for-ghosts.

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**Appendix A**

Here we briefly outline the computation of the inverse of the matrix $f^{(2)}$ given by (4.32). We write the general form of the inverse, which will give the symplectic matrix, as

$$f^{(2)-1} = \begin{pmatrix}
(AA)^{k\ell m n} & (A\pi)^{k\ell m n} & (A\lambda)^{k\ell m} & (Ap)^{k l} & (A\eta)^{k l m} & (A\phi)^{k l} \\
(\pi A)^{k\ell m n} & (\pi\pi)^{k\ell m n} & (\pi\lambda)^{k\ell m} & (\pi p)^{k l} & (\pi\eta)^{k l m} & (\pi\phi)^{k l} \\
(\lambda A)^{k m n} & (\lambda\pi)^{k m n} & (\lambda\lambda)^{k m} & (\lambda p)^{k} & (\lambda\eta)^{k m} & (\lambda\phi)^{k} \\
(p A)^{m n} & (p p)^{m n} & (p p)^{m} & (p\eta)^{m} & (p\phi)^{m} & \\
(\eta A)^{k m n} & (\eta\pi)^{k m n} & (\eta\lambda)^{k m} & (\eta p)^{k} & (\eta\eta)^{k m} & (\eta\phi)^{k} \\
(\phi A)^{m n} & (\phi\pi)^{m n} & (\phi\lambda)^{m} & (\phi p)^{k} & (\phi\eta)^{m} & (\phi\phi)^{k} \\
\end{pmatrix} \quad (A.1)$$

This inverse is defined in such a way that

$$\int d\vec{y} f^{(2)}(\vec{x}, \vec{y}) f^{(2)-1}(\vec{y}, \vec{z}) = \int d\vec{y} \begin{pmatrix}
\delta^{m n} & 0 & 0 & 0 & 0 & 0 \\
0 & \delta^{m n} & 0 & 0 & 0 & 0 \\
0 & 0 & \delta^{m} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \delta^{m} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \delta(\vec{x} - \vec{z}) \quad (A.2)$$

By a straightforward matrix algebra a set of equations is obtained, the solutions of which yield the desired entries in (A.1). The calculations are simplified by noting that the symplectic matrix must possess the same symmetry structure as $f^{(2)}$. The final result is explicitly displayed in (4.34).
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