Constraining Maximal Supersymmetric Membrane Actions

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Abstract: We study the recent construction of maximal supersymmetric field theory Lagrangians in three spacetime dimensions that are based on algebras with a triple product. Assuming that the algebra has a positive definite metric compatible with the triple product, we prove that the only non-trivial examples are either the well known case based on a four dimensional algebra or direct sums thereof.

Keywords: M-Theory, Supersymmetric gauge theory.
1. Introduction

A better understanding of the three-dimensional superconformal field theory that arises on multiple membranes in flat space is an important outstanding issue in M-theory. Building on earlier work [1, 2], an interesting Lagrangian description of a maximally supersymmetric conformal field theory in three dimensions was constructed in [3, 4, 5] which has been further studied in [6] - [19]. The construction relies on an algebra with a skew triple product whose structure constants $f_{\mu_1 \mu_2 \mu_3 \nu} = f^{[\mu_1 \mu_2 \mu_3]} \nu$ satisfy

$$f_{\mu_1 \mu_2 \mu_3 \nu} f^{\mu_4 \mu_5 \nu \mu_6} = 3 f^{[\mu_1 \mu_2 \mu_3]} \nu f^{\mu_4 \mu_5 \nu \mu_6}$$

or equivalently

$$f^{[\mu_1 \mu_2 \mu_3 \nu} f^{\mu_4 \mu_5 \nu \mu_6} = 0 .$$

The construction of the Lagrangian requires a compatible metric and, after raising an index on $f$ using this metric, $f$ is totally antisymmetric $f_{\mu_1 \mu_2 \mu_3 \mu_4} = f^{[\mu_1 \mu_2 \mu_3 \mu_4]}$. Since the metric appears in the kinetic terms of the Lagrangian, it is natural to demand that the metric is positive definite. In this case, after a suitable change of basis, we can assume that the metric is simply $\delta_{\mu \nu}$. The basic non-trivial solution [4] corresponds to a four dimensional algebra with $f^{[\mu_1 \mu_2 \mu_3 \mu_4]} = \epsilon^{[\mu_1 \mu_2 \mu_3 \mu_4]}$. One can also consider direct sums of this basic example, but this simply leads to three-dimensional supersymmetric field theories which are non-interacting copies of the basic example.

We started this work by trying to construct additional solutions to (1.2) with totally antisymmetric $f$. However, as also noticed by others, obvious generalisations fail and simple computer searches are fruitless. It has also been shown [20] that in up to seven dimensions, a 4-form whose components satisfy (1.2) must be proportional...
to \(dx^{1234}\) (in some appropriately chosen co-ordinates), and in eight dimensions, the solution is a linear combination \(dx^{1234}\) and \(dx^{5678}\).

Here we will prove the general result, that all solutions of (1.2), in any dimension, can be written as a linear combination 4-forms, each of which is the wedge product of four 1-forms, which are all mutually orthogonal. This then proves conjectures made in [20] and [16].

**Note added:** Concurrent with the posting of this work to the ArXive, a proof of this result also appeared in [21]. After this paper was submitted for publication, we became aware of [22], which claims the same result using a different approach.

### 2. Analysis

We are interested in solutions to (1.2) for totally anti-symmetric and real \(f\) with indices raised and lowered using the metric \(\delta_{\mu\nu}\). Let us assume that we have a \(D+1\) dimensional algebra and write the indices as \(\mu = (q, D+1)\) where \(q = 1, \ldots, D\). We can write

\[
f = dx^{D+1} \wedge \psi + \phi
\]

where \(\psi\) is a 3-form on \(\mathbb{R}^D\), and \(\phi\) is a 4-form on \(\mathbb{R}^D\). We can demand that \(\psi \neq 0\) (otherwise we end up in \(D\) dimensions). The constraint (1.2) is equivalent to

\[
\phi^{[q_1 q_2 q_3}_m \phi^{q_4]q_5 q_6} + \psi^{[q_1 q_2 q_3} \psi^{q_4]q_5 q_6} = 0
\]

(2.2)

\[
\phi^{[q_1 q_2 q_3}_m \psi^{q_4]q_5 q_6} = 0
\]

(2.3)

\[
\phi^{q_1 q_2 q_3}_m \psi^{q_4]q_5 q_6} - 3 \psi^{[q_1 q_2}_m \phi^{q_3]q_4 q_5 q_6} = 0
\]

(2.4)

\[
\psi^{[q_1 q_2}_m \psi^{q_3]q_4 q_5 q_6} = 0
\]

(2.5)

where indices on \(\psi, \phi\) are raised/lowered with \(\delta_{mn}\). Observe that (2.5) is the Jacobi identity. This identity implies that \(\psi_{mnp}\) are the structure constants of a Lie algebra \(\mathcal{L}\). The Killing form of this Lie algebra has components

\[
\kappa_{mn} = \psi_{mrl} \psi_{nlp} \ell.
\]

(2.6)

As \(\psi\) is totally antisymmetric, note that \(\kappa\) is negative semi-definite. There are two possibilities: \(\kappa\) is non-degenerate and \(\mathcal{L}\) is semi-simple or \(\kappa\) is degenerate.

Suppose that \(\mathcal{L}\) is semi-simple. By making a \(SO(D)\) rotation, one can diagonalize the Killing form and set

\[
\kappa_{mn} = -\lambda_n \delta_{mn}
\]

(2.7)

(no sum over \(n\)), and \(\lambda_n > 0\) for all \(n\).

On the other hand if \(\kappa\) is degenerate, then \(\mathcal{L} = u(1)^p \varoplus \mathcal{L}'\) where \(p > 0\) and \(\mathcal{L}\) is semi-simple. To see this we first note that \(X^m \kappa_{mn} = 0\) for some non-zero vector \(X^n\). Then it follows that

\[
X^m X^n \psi_{mpq} \psi_{n}^{pq} = 0
\]

(2.8)
which implies that \( X^n \psi_{npq} = 0 \). Without loss of generality, one can make an \( SO(D) \) rotation so that the only non-vanishing component of \( X^n \) is \( X^1 \) and then \( \psi_{1mn} = 0 \) for all \( m, n \), and \( \kappa_{1m} = 0 \) for all \( m \). By repeating this process in the directions \( 2, \ldots, D \) one finds after a finite number of steps, either that \( \mathcal{L} = u(1)^p \oplus \mathcal{L}' \) where \( p > 0 \) and \( \mathcal{L}' \) is semi-simple, or \( \psi = 0 \) which we have assumed not to be the case.

We will analyse the two cases in turn, but we first establish some useful identities arising from (2.3)-(2.5) that are valid in both cases. We define \( h = -\kappa \) i.e.

\[
h_{mn} = \psi_{mab} \psi_{nab}.
\]  

First contract (2.3) with \( \psi_{q5\ell} \) so that one obtains

\[
\phi^{q1q2q3m} h_{m\ell} - \phi^{q1q2q3m} \psi^{q5q1_m} \psi_{q5q4\ell} - \phi^{q1q2q3m} \psi^{q5q2_m} \psi_{q5q4\ell} - \phi^{q1q2q3m} \psi^{q5q3_m} \psi_{q5q4\ell} = 0.
\]  

However, note that the Jacobi identity implies that

\[
\phi^{q1q2q3m} \psi^{q5q1_m} \psi_{q5q4\ell} = \frac{1}{2} \phi^{q2q3mn} \psi^{r5q1_m} \psi_{r5q4\ell}.
\]  

Using this identity one can rewrite (2.10) as

\[
\phi^{q1q2q3m} h_{m\ell} - \frac{1}{2} \phi^{q2q3mn} \psi^{r5q1_m} \psi_{r5q4\ell} - \frac{1}{2} \phi^{q1q2q4mn} \psi^{r5q2_m} \psi_{r5q4\ell} - \frac{1}{2} \phi^{q1q2q3n} \psi^{r5q3_m} \psi_{r5q4\ell} = 0.
\]  

Also, contracting (2.3) with \( \delta_{q5q5} \) gives

\[
\phi^{q1q2q4mn} \psi_{q4q1_m} + \phi^{q2q4mn} \psi_{q4q1_m} + \phi^{q4q1mn} \psi_{q4q2_m} = 0.
\]  

Next, contract (2.4) with \( \psi_{q1q2\ell} \) to obtain

\[
-\phi^{q1q2q5m} h_{m\ell} + \phi^{q1q2q5m} \psi_{q1q2\ell} \psi_{q5q3_m} - 2 \phi^{q2q4q5m} \psi_{q4q1_m} \psi_{q1q2\ell} = 0.
\]  

This can be rewritten (using (2.11) to simplify the last term) as

\[
-\phi^{q1q2q5m} h_{m\ell} + \phi^{mnr} \psi_{rne} \psi_{q2q3r} + \phi^{q2q3mn} \psi^{r5q1_m} \psi_{r5q4\ell} = 0.
\]  

On contracting this expression with \( \delta_{q1q3} \), the first and the third term vanish (the third term vanishes as a consequence of the Jacobi identity), and we find

\[
\phi^{n1n2m1m2} \psi_{n1n2\ell} \psi_{m1m2r} = 0.
\]  

Next, contract (2.13) with \( \psi_{q2q3s} \). The last term vanishes as a consequence of (2.16), and we obtain

\[
-\phi^{mnr} h_{r\ell} \psi_{mns} + \phi^{mnr} h_{rs} \psi_{m\ell} = 0.
\]  

\[
-\phi^{mnr} h_{r\ell} \psi_{mns} + \phi^{mnr} h_{rs} \psi_{m\ell} = 0.
\]
2.1 Solutions when $\mathcal{L}$ is semi-simple

We now assume that $\mathcal{L}$ is semi-simple. As we have already observed, we can make a rotation and work in a basis for which

$$h_{mn} = \lambda_n \delta_{mn}$$  \hspace{1cm} (2.18)

(no sum over $n$), with $\lambda_n > 0$ for all $n$.

Then (2.17) implies

$$-\phi^{mnq} \lambda_\ell \psi_{mn\ell} + \phi^{mnq} \lambda_s \psi_{m\ell\ell} = 0$$  \hspace{1cm} (2.19)

with no sum over $\ell$ or $s$. On substituting this expression back into (2.13) we obtain

$$(\lambda_{q_4} - \lambda_{q_1} - \lambda_{q_2}) \phi^{q_1 q_2 q_4} \psi_{q_4 mn} = 0$$  \hspace{1cm} (2.20)

(no sum on $q_1, q_2, q_4$). Hence $\phi^{q_1 q_2 q_4} \psi_{q_4 mn} = 0$, or $\lambda_{q_4} - \lambda_{q_1} - \lambda_{q_2} = 0$ for some choice of $q_1, q_2, q_4$. Now, it is not possible to have $\lambda_{q_1} - \lambda_{q_2} = \lambda_{q_1} - \lambda_{q_2} - \lambda_{q_4} = \lambda_{q_2} - \lambda_{q_1} - \lambda_{q_4} = 0$ simultaneously. Hence, at least one of $\phi^{q_1 q_2 q_4} \psi_{q_4 mn}, \phi^{q_1 q_4 q_2} \psi_{q_2 mn}, \phi^{q_2 q_4 q_1} \psi_{q_1 mn}$ must vanish. However, (2.19) then implies that all these terms vanish. Hence we conclude that

$$\phi^{q_1 q_2 q_4} \psi_{q_4 mn} = 0$$  \hspace{1cm} (2.21)

for all $q_1, q_2, q_4$. Finally, on substituting (2.21) back into (2.12), the last three terms are constrained to vanish, hence

$$\phi^{q_1 q_2 q_3 q_4} = 0$$  \hspace{1cm} (2.22)

Now consider (2.2). This implies that

$$\psi_{[q_1 q_2 q_3]}_{q_4 q_5} = 0$$  \hspace{1cm} (2.23)

which implies (see e.g. [20]) that $\psi$ is simple i.e. it can be written as the wedge product of three one forms. Hence one can chose a basis for which

$$\psi = \lambda dx^1 \land dx^2 \land dx^3$$  \hspace{1cm} (2.24)

Furthermore, as $\mathcal{L}$ is compact, this implies that $\mathcal{L}$ must be 3-dimensional i.e. $\mathcal{L} = su(2)$. We have thus recovered the basic four-dimensional case with $f^{\mu_1 \mu_2 \mu_3 \mu_4} = \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4}$.

2.2 Solutions when $\mathcal{L}$ is not semi-simple

Set $\mathcal{L} = u(1)^p \oplus \mathcal{L}'$ where $p > 0$ and $\mathcal{L}'$ is semi-simple. It will be useful to split the indices $m$ into “semi-simple” directions $\hat{m}$ and “$u(1)$” directions $A$, so $m = (\hat{m}, A)$.\hspace{1cm}
Note that \( \psi_{Amn} = 0 \) for all \( m, n \), and \( h_{Am} = 0 \) for all \( m \), but \( h_{\hat{m}\hat{n}} = \lambda_{\hat{n}} \delta_{\hat{m}\hat{n}} \) (no sum on \( \hat{n} \)). Recall the identity \((2.12)\). Setting \( q_1 = A \), \( q_2 = B \), \( q_3 = C \) one finds

\[
\phi_{ABC\hat{m}} = 0 .
\]

(2.25)

Also, setting \( q_1 = A \), \( q_2 = B \), \( q_3 = \hat{m} \) one finds

\[
\phi^{AB\hat{m}\hat{n}} h_{\hat{s}\hat{\ell}} - \frac{1}{2} \phi^{AB\hat{p}\hat{q}} \psi^{\hat{s}\hat{m}} \psi_{\hat{s}\hat{p}\hat{q}} = 0 .
\]

(2.26)

However, \((2.13)\) implies that

\[
\phi^{AB\hat{p}\hat{q}} \psi_{\hat{s}\hat{p}\hat{q}} = 0
\]

(2.27)

and so on substituting this back into \((2.26)\) one finds

\[
\phi_{AB\hat{m}\hat{n}} = 0 .
\]

(2.28)

Returning to the general conditions \((2.3)\), \((2.4)\) and \((2.5)\) with all free indices hatted, we can follow the same steps in the last subsection to conclude that

\[
\phi_{\hat{m}\hat{n}\hat{p}\hat{q}} = 0 .
\]

(2.29)

Thus the only non-zero components of \( \phi \) are of the form \( \phi^{A\hat{q}_1\hat{q}_2\hat{q}_3} \) and \( \phi^{ABCD} \).

Considering other indices in \((2.3)\), \((2.4)\) and \((2.5)\) we conclude that

\[
\psi^{[\hat{q}_1\hat{q}_2\hat{m}\hat{n}\hat{q}_3\hat{q}_4]} = 0
\]

(2.30)

\[
\phi^{A\hat{q}_1\hat{q}_2\hat{m}\hat{n}\hat{q}_3\hat{q}_4} = \phi^{A\hat{q}_1\hat{q}_4\hat{m}\hat{n}\hat{q}_3\hat{q}_2}
\]

(2.31)

\[
\phi^{A\hat{q}_1[\hat{q}_2\hat{m}\hat{n}\hat{q}_3\hat{q}_4]} = 0
\]

(2.32)

From \((2.2)\) we also get

\[
\phi^{[A_1A_2A_3B\phi A_4])A_5A_6B} = 0
\]

(2.33)

\[
\phi^{A\hat{q}_1\hat{q}_2\hat{m}\hat{n}\phi\hat{q}_3\hat{q}_4} = 0
\]

(2.34)

\[
\phi^{A[\hat{q}_1\hat{q}_2\hat{m}\phi\hat{q}_3\hat{q}_4\hat{B}\hat{m}]} = 0
\]

(2.35)

\[
\phi^{A[\hat{q}_1\hat{q}_2\hat{m}\phi\hat{q}_3\hat{q}_4\hat{m}]} = 0
\]

(2.36)

\[
\psi^{[\hat{q}_1\hat{q}_2\hat{m}\phi\hat{q}_3\hat{q}_4\phi\hat{q}_6\hat{A}]} + \phi^{[\hat{q}_1\hat{q}_2\hat{q}_3\phi\hat{q}_4\hat{A}]} = 0 .
\]

(2.37)

To proceed with the analysis, it is convenient to define the matrices \( T^A \) by

\[
(T^A)^{\hat{m}}_{\hat{n}} = \phi^{A\hat{q}_1\hat{q}_2\hat{m}\hat{n}\hat{q}_3\hat{q}_4}.
\]

(2.38)

On contracting \((2.31)\) with \( \delta_{\hat{q}_1\hat{q}_4} \), we observe that \( T^A \) are all symmetric matrices. Furthermore, on contracting \((2.36)\) with \( \delta_{\hat{q}_1\hat{q}_4} \) and making use of \((2.31)\), it is straightforward to show that the matrices \( T^A \) commute with each other. Also, \((2.31)\) implies that the \( T^A \) commute with \( h \).
Next, note that the Jacobi identity \((2.30)\) implies that
\[
(T^A)_{\tilde{m}\tilde{t}}\psi^\ell_{\tilde{p}\tilde{q}} = \phi^{A\tilde{s}_i} \tilde{m}_{\tilde{s}_i} \psi^\ell_{\tilde{p}\tilde{q}} = -2\phi^{A\tilde{s}_i} \tilde{m}_{\tilde{s}_i} \psi^\ell_{\tilde{p}\tilde{q}} = -2\phi^{A\tilde{s}_i} \tilde{m}_{\tilde{s}_i} \psi^\ell_{\tilde{p}\tilde{q}}.
\]
(2.39)

However, now using \((2.31)\) and then the Jacobi identity again, we get
\[
-2\phi^{A\tilde{s}_i} \tilde{m}_{\tilde{s}_i} \psi^\ell_{\tilde{p}\tilde{q}} = -2\phi^{A\tilde{s}_i} \tilde{m}_{\tilde{s}_i} \psi^\ell_{\tilde{p}\tilde{q}} = -2\phi^{A\tilde{s}_i} \tilde{m}_{\tilde{s}_i} \psi^\ell_{\tilde{p}\tilde{q}}.
\]
(2.40)

Thus
\[
(T^A)_{\tilde{m}\tilde{t}}\psi^\ell_{\tilde{p}\tilde{q}} = -(T^A)_{\tilde{p}\tilde{t}}\psi^\ell_{\tilde{m}\tilde{q}}.
\]
(2.41)

Next, decompose semi-simple \(L' = L_1 \oplus \cdots \oplus L_m\) where \(L_i\) are simple ideals such that \(L_i \perp L_j\) (with respect to \(h\)), and \([L_i, L_j] = 0\) if \(i \neq j\), and the restriction of the adjoint rep. to \(L_i\) is irreducible; furthermore, \(h\mid_{L_i} = 2\mu_i^2\) for \(\mu_i \neq 0\). Contract \((2.31)\) with \(\psi_{\tilde{q}_{3\tilde{3}\tilde{4}\tilde{4}}\tilde{i}}\) to obtain
\[
\phi^{A\tilde{i}_1\tilde{i}_2\tilde{i}_3\tilde{i}_4} h_{\tilde{m}\tilde{i}} = \phi^{A\tilde{i}_1\tilde{i}_2\tilde{i}_3\tilde{i}_4} h_{\tilde{m}\tilde{i}} = \phi^{A\tilde{i}_1\tilde{i}_2\tilde{i}_3\tilde{i}_4} h_{\tilde{m}\tilde{i}}.
\]
(2.42)

Suppose that the indices \(\tilde{q}_1, \tilde{q}_2\) lie in two different ideals \(L_i, L_j\) for \(i \neq j\). Then the RHS of the above expression vanishes, hence for these indices, \(\phi^{A\tilde{i}_1\tilde{i}_2\tilde{i}_3\tilde{i}_4} = 0\), for all \(\tilde{m}\). Similarly, for these indices \((T^A)_{\tilde{q}_1\tilde{q}_2}\) = \(\phi^{A\tilde{i}_1\tilde{i}_2\tilde{i}_3\tilde{i}_4} h_{\tilde{m}\tilde{i}} = 0\).

Consider \(T^A_i\), the restriction of \(T^A\) to \(L_i\). Then \((2.41)\) implies that \(T^A_i\) commutes with the restriction of the adjoint rep. to \(L_i\). However, as this restriction of the adjoint rep. is irreducible, it follows by Schur’s Lemma that
\[
T^A_i = \lambda_i A^A_i.
\]
(2.43)

As the \(T^A\) all commute, this can be achieved for all \(T^A\).

Next, consider \((2.37)\) with all \(\tilde{q}\) indices restricted to \(L_i\). Contracting this expression with \(\psi_{\tilde{q}_1\tilde{q}_2\tilde{q}_3}\) restricted to \(L_i\) gives
\[
\left( \sum_A (\lambda_i^A)^2 + 4(\mu_i)^4 \right) \left( \dim L_i - 3 \right) s_{\tilde{m}} = 0
\]
(2.44)

which implies that \(\dim L_i = 3\) for all \(i\), so \(L_i = su(2)\). It follows that
\[
\psi = \sum_i \mu_i \theta_i
\]
(2.45)

with \(\mu_i \neq 0\), where
\[
\theta_i = dy_1^i \wedge dy_2^i \wedge dy_3^i.
\]
(2.46)

If the \(\tilde{q}\) indices are restricted to \(L_i\), since \(\dim L_i = 3\), \(\phi_{A\tilde{i}_1\tilde{i}_2\tilde{i}_3}\) must be proportional to \(\theta_i\). The proportionality constant can be fixed from \((2.43)\) and we find
\[
\phi_{A\tilde{i}_1\tilde{i}_2\tilde{i}_3} = \frac{\lambda_i^A}{2\mu_i} (\theta_i)_{\tilde{q}_1\tilde{q}_2\tilde{q}_3}.
\]
(2.47)
It is convenient to re-define $\lambda_i^A = 2\mu_i\chi_i^A$, so that

$$f = dx^{d+1} \land \psi + \sum_{i,A} \chi_i^A dz^A \land \theta_i + \Phi$$

(2.48)

where $\Phi$ lies entirely in the $u(1)$ directions, whose directions we have denoted by $z^A$. The remaining content of (2.37) is obtained by restricting the indices $\hat{q}_1, \hat{q}_2, \hat{q}_3$ to $\mathcal{L}_i$, and $\hat{q}_4, \hat{q}_5, \hat{q}_6$ to $\mathcal{L}_j$ for $i \neq j$; we find

$$\mu_i\mu_j + \sum_A \chi_i^A \chi_j^A = 0 \ .$$

(2.49)

Note that the form $\Phi$ satisfies the quadratic constraint (2.33), whereas (2.34) is equivalent to

$$\chi_i^A \Phi_{AMNP} = 0$$

(2.50)

for all $i$.

There are then two cases to consider. In the first case, $\chi_i^A = 0$ for all $A, i$. Then (2.49) implies that $\mathcal{L}' = su(2)$, and hence

$$f = \mu_1 dx^{d+1} \land dy_1^1 \land dy_2^1 \land dy_3^1 + \Phi$$

(2.51)

where $\Phi$ has no components in the $x^{d+1}, y_1^1, y_2^1, y_3^1$ directions.

In the second case, there exists some $A, i$ with $\chi_i^A \neq 0$. Without loss of generality, take $i = 1$. By making an $SO(p)$ rotation entirely in the $u(1)$ directions, without loss of generality set

$$\chi_1^1 = \tau, \quad \chi_1^A = 0 \quad \text{if } A > 1$$

(2.52)

where $\tau \neq 0$. Then, if $j \neq 1$, (2.49) implies that

$$\chi_j^1 = -\frac{\mu_1}{\tau} \mu_j \ .$$

(2.53)

Substituting these constraints back into (2.48), and rearranging the terms, one finds

$$f = (\mu_1 dx^{d+1} + \tau dz^1) \land \theta_1 + \tau^{-1}(\tau dx^{d+1} - \mu_1 dz^1) \land \sum_{j>1} \mu_j \theta_j$$

$$+ \sum_{j>1, A>1} \chi_j^A dz^A \land \theta_j + \Phi \ .$$

(2.54)

Writing

$$f_1 = (\mu_1 dx^{d+1} + \tau dz^1) \land \theta_1$$
$$\tilde{f} = \tau^{-1}(\tau dx^{d+1} - \mu_1 dz^1) \land \sum_{j>1} \mu_j \theta_j + \sum_{j>1, A>1} \chi_j^A dz^A \land \theta_j + \Phi$$

(2.55)

we have found $f = f_1 + \tilde{f}$ where, as a consequence of (2.50) and (2.52), it follows that $\Phi$ has no components in the $z^1$ direction.
So, in both cases, we have the decomposition

\[ f = f_1 + \tilde{f} \]  

(2.56)

where \( f_1 \) is a simple 4-form, and \( f_1, \tilde{f} \) are totally orthogonal i.e. \( f_1^{\mu_1 \mu_2 \mu_3 \nu} \tilde{f}^{\mu_4 \mu_5 \mu_6 \nu} = 0 \).

Having obtained this result, it is straightforward to prove that if such an \( f \) satisfies (1.2), then

\[ f = \sum_{s=1}^{N} f_s \]  

(2.57)

where \( f_s \) are totally orthogonal simple 4-forms. The proof proceeds by induction on the spacetime dimension \( D \) (\( D \geq 4 \)). The result is clearly true for \( D = 4 \). Suppose it is true for \( 4 \leq D \leq d \). Suppose that \( D = d + 1 \). Then by the previous reasoning, one has the decomposition \( f = f_1 + \tilde{f} \), where \( f_1 \) is a simple 4-form, and \( f_1, \tilde{f} \) are totally orthogonal. It follows that \( \tilde{f} \) must satisfy (1.2). Then either \( \tilde{f} = 0 \) and we are done, or \( \tilde{f} \) is a nonzero 4-form in dimension \( d - 3 \), in which case it follows that one can decompose \( \tilde{f} \) into a finite sum of orthogonal simple 4-forms, each of which is also orthogonal to \( f_1 \).

Hence we conclude that the decomposition (2.57) holds for all 4-forms \( f \) satisfying (1.2).

3. Discussion

Given the results presented here, the maximally supersymmetric field theory Lagrangian based on the four-dimensional algebra with \( f^{\mu_1 \mu_2 \mu_3 \mu_4} = \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \) is rather enigmatic. If it is not to be an isolated curiosity, the assumptions going into the general constructions of [3, 4, 5] need to be relaxed. One possibility is to relax the condition that the metric living on the algebra is positive definite and some discussion recently appeared in [16]. A different possibility is to not demand a Lagrangian description, but to work instead at the level of the field equations and this was recently discussed in [15]. Another possibility, which also does not use totally antisymmetric structure constants, was considered in [12].

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