Two-qubit catalysis in a four-state pure bipartite system

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Abstract

We consider a four-state pure bipartite system consisting of four qubits shared among two parties using the same Schmidt basis, two qubits per party. In some cases, transformation between two known pure states may not be possible using LOCC transformations but may be possible with the addition of a two-qubit catalyst. We provide a necessary and sufficient condition for this to occur.

1 Introduction

Consider the case of two people, Alice and Bob, who live far apart. Some time ago, Alice boarded a plane with some of her qubits, visited Bob in his laboratory, entangled her qubits with his, and returned home with her qubits. Together, their qubits form the known pure state $|\psi\rangle$. Now they wish to perform an experiment, but one which requires the joint system to be in the known pure state $|\phi\rangle$. It would be very inconvenient for Alice to fly back to Bob’s lab, but it is easy for her to phone him. It would be nice if they could change the state $|\psi\rangle$ to the state $|\phi\rangle$ by each of them operating locally on their portion of the system and exchanging classical information as necessary. Such a transformation is called an LOCC transformation (Local Operations and Classical Communication). Note that the local operations are not necessarily unitary nor are they confined to the qubits which form the entangled state. Both Alice and Bob may bring in ancilla qubits in various
states of entanglement and entangle them with their local systems. However, they do not have any ancillary quantum communication channels. In summary, Alice and Bob may perform arbitrary local operations on their local systems, but may only communicate using classical communication channels.

Majorization is a useful concept which relates to convexity. Given a vector $\vec{\alpha}$, let $\alpha[k]$ be the largest component of the vector, $\alpha[k]$ the second largest component, and so on. The $n$-long vector $\vec{\alpha}$ is majorized by the $n$-long vector $\vec{\alpha}'$ iff

$$\sum_{i=1}^{k} \alpha[i] \leq \sum_{i=1}^{k} \alpha'[i], \quad k = 1, \ldots, n - 1$$

and

$$\sum_{i=1}^{n} \alpha[i] = \sum_{i=1}^{n} \alpha'[i].$$

Nielsen’s Theorem [1] gives a necessary and sufficient condition for an LOCC to be possible. We will assume that all states have the same basis for their Schmidt decompositions. This assumption is not necessary for obtaining any of the results, but it simplifies the notation and computation enormously. With this assumption, Nielsen’s Theorem reduces to the following:

Let $|\psi\rangle = \sum_{i=1}^{n} \sqrt{\alpha_i} |i_A\rangle|i_B\rangle$ and $|\phi\rangle = \sum_{i=1}^{n} \sqrt{\alpha'_i} |i'_A\rangle|i'_B\rangle$ be pure bipartite states with respective Schmidt coefficients $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 0$ and $\alpha'_1 \geq \alpha'_2 \geq \ldots \alpha'_n \geq 0$. Then the transformation $|\psi\rangle \rightarrow |\phi\rangle$ can be performed under LOCC iff the vector $\vec{\alpha}$ is majorized by the vector $\vec{\alpha}'$.

2 Catalysis

There are clearly cases where LOCC transformations cannot occur, i.e. when $\vec{\alpha}$ is not majorized by $\vec{\alpha}'$. However all is not necessarily lost for Alice and Bob; there is still the possibility of catalysis. Suppose that the Quantum Entanglement Savings Bank has branches in both Alice’s and Bob’s hometown. The branch in Alice’s town has a qubit which is entangled with a qubit held by the branch in Bob’s town; we will assume that this is also a pure bipartite state of the form $|\kappa\rangle = \sqrt{p} |00\rangle + \sqrt{1-p} |11\rangle$. Alice and Bob may borrow these qubits from the bank; however, the bank demands that
the qubits be returned to them in exactly the same state $|\kappa\rangle$. (How exactly the bankers manage to verify this is not our problem.)

At first, this seems to be of no help. Certainly if Alice and Bob were allowed to tamper with the entanglement of $|\kappa\rangle$, they could use this to transmit quantum information and thus perform a larger class of operations. But such a process would reduce the Von Neumann entropy of the pair. Requiring the state $|\kappa\rangle$ at the end of the process effectively means that there can be no net transfer of entanglement from these ancillary particles to the original system (however one chooses to measure entanglement). Nevertheless, the resource of entanglement can be borrowed provided it is returned at the end.

In [2], Johnathan and Plenio examine the case when $n = 4$. They show that the only case in which a transformation cannot be performed under LOCC but can be performed under LOCC with a catalyst is when

$$(*) \quad \alpha_1 \leq \alpha'_1, \quad \alpha_1 + \alpha_2 > \alpha'_1 + \alpha'_2, \quad \alpha_4 \geq \alpha'_4$$

and they provide an example:

$$|\psi\rangle = \sqrt{0.4} |00\rangle + \sqrt{0.4} |11\rangle + \sqrt{0.1} |22\rangle + \sqrt{0.1} |33\rangle$$

$$|\phi\rangle = \sqrt{0.5} |00\rangle + \sqrt{0.25} |11\rangle + \sqrt{0.25} |22\rangle + 0 |33\rangle$$

They show the transformation cannot be performed under LOCC, but can be done with the addition of the catalyst

$$|\kappa\rangle = \sqrt{0.6} |00\rangle + \sqrt{0.4} |11\rangle.$$ 

To verify this example, we will note the following fact: If we let $\beta_i$ and $\beta'_i$ be the pure bipartite state coefficients of the augmented systems $|\psi\rangle|\kappa\rangle$ and $|\phi\rangle|\kappa\rangle$, then $\{\beta_i\} = \{\alpha_i p, \alpha_i (1-p)\}$ and $\{\beta'_i\} = \{\alpha'_i p, \alpha'_i (1-p)\}$. By Nielsen’s Theorem, a catalytic conversion is possible iff $\tilde{\beta}$ is majorized by $\tilde{\beta}'$. It is thus easy to calculate the components of the augmented systems and verify that majorization occurs.

The proof of $(*)$ is fairly easy and straightforward. We will paraphrase Jonathan and Plenio’s proof here. Suppose transformation under catalysis is possible. Let $K$ be the largest component of the catalyst, and $k$ the smallest component. Then the first partial sums of the new source and target states will be $K\alpha_1$ and $K\alpha'_1$. Because LOCC is now possible, we have majorization and $K\alpha_1 \leq K\alpha'_1$ and hence $\alpha_1 \leq \alpha'_1$. Similarly, the penultimate partial sums will be $1 - k\alpha_4$ and $1 - k\alpha'_4$. Since majorization occurs, we
have $1 - k\alpha_4 \leq 1 - k\alpha'_4$ and hence $\alpha_4 \geq \alpha'_4$. Now, we are also supposing that transformation is not possible under LOCC without the presence of a catalyst. Hence we must have $\alpha \not\preceq \alpha'$. We have just shown $\alpha_1 \leq \alpha'_1$. Since $\alpha_4 \geq \alpha'_4$, we have

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 - \alpha_4 \leq 1 - \alpha'_4 = \alpha'_1 + \alpha'_2 + \alpha'_3.$$ 

Finally, the $\alpha_i$'s and $\alpha'_i$'s must sum to 1, so the fourth partial sums are equal. The only way majorization can fail to occur is in the second partial sum. Hence, we must have $\alpha_1 + \alpha_2 > \alpha'_1 + \alpha'_2$. This completes the proof.

Thus, when in the $n = 4$ case, (1) is a necessary condition for catalysis to be effective with a pure bipartite two-qubits catalyst. However, if we restrict ourselves to catalysts of this form, the condition is not both necessary and sufficient. Zhou and Guo [3] give an example of a five-state system in which no two-qubit catalyst can effect an LOCC transformation. Here we will give an example with a four-state system. Consider the states

$$|\psi\rangle = \sqrt{0.45} |00\rangle + \sqrt{0.45} |11\rangle + \sqrt{0.05} |22\rangle + \sqrt{0.05} |33\rangle$$
$$|\phi\rangle = \sqrt{0.5} |00\rangle + \sqrt{0.35} |11\rangle + \sqrt{0.15} |22\rangle + 0 |33\rangle$$

They satisfies (1); however if we apply a catalyst of the form

$$|\kappa\rangle = \sqrt{p} |00\rangle + \sqrt{1 - p} |11\rangle,$$

the pure state bipartite coefficients will be

$$\vec{\beta} = (.45p, .45p, .05p, .05p, .45(1 - p), .45(1 - p), .05(1 - p), .05(1 - p))$$
$$\vec{\beta}' = (.50p, .35p, .15p, 0, .50(1 - p), .35(1 - p), .15(1 - p), 0)$$

where $\beta$ and $\beta'$ are the vectors of eigenvalues of the joint system obtained by pairing $|\psi\rangle$ and $|\phi\rangle$ respectively with the catalyst. Let $\lambda_i$ be the sum of the $i$ largest eigenvalues in the vector $\vec{\beta}$ and $\lambda'_i$ be the sum of the $i$ largest eigenvalues in the vector $\vec{\beta}'$. The condition of $\vec{\beta}$ being majorized by $\vec{\beta}'$ then becomes simply $\lambda_i \leq \lambda'_i$ for all $i$.

The two largest eigenvalues in $\vec{\beta}$ are both $0.45p$ and hence $\lambda_2 = 0.9p$. Let us suppose $p > 10/17$. Then $0.35p > 0.50(1 - p)$ and $\lambda'_2 = 0.85p$. Thus, $\lambda_2 > \lambda'_2$ and we cannot perform an LOCC. On the other hand, suppose $1/2 \leq p \leq 10/17$. In this case, the four largest eigenvalues of $\vec{\beta}$ are $0.45p, 0.45p, 0.45(1 - p)$, and $0.45(1 - p)$ yielding $\lambda_4 = 0.9$. The four largest eigenvalues of $\vec{\beta}'$ are $0.50p,
.50(1 - p), .35p, and .35(1 - p) yielding $\lambda'_4 = .85$. Thus, $\lambda_4 > \lambda'_4$ and again we cannot perform an LOCC.

This naturally raises the question “What is a necessary and sufficient condition to perform catalytic conversion under this set-up?” We have already seen that (∗) is necessary. It is easy to see that (∗) is equivalent to the following:

$$\exists \epsilon_1 \geq 0, \epsilon_2 > 0, \epsilon_3 \geq 0 \text{ such that }$$

$$\alpha'_1 = \alpha_1 + \epsilon_1$$
$$\alpha'_2 = \alpha_2 - \epsilon_1 - \epsilon_2$$
$$\alpha'_3 = \alpha_3 + \epsilon_2 + \epsilon_3$$
$$\alpha'_4 = \alpha_4 - \epsilon_3.$$

In illustration, let us look back at Jonathan and Plenio’s example:

$$|\psi\rangle = \sqrt{0.4} |00\rangle + \sqrt{0.4} |11\rangle + \sqrt{0.1} |22\rangle + \sqrt{0.1} |33\rangle$$
$$|\phi\rangle = \sqrt{0.5} |00\rangle + \sqrt{0.25} |11\rangle + \sqrt{0.25} |22\rangle + 0 |33\rangle$$
$$|\kappa\rangle = \sqrt{0.6} |00\rangle + \sqrt{0.4} |11\rangle$$

Here we have $\alpha_1 = .4$, $\alpha_2 = .4$, $\alpha_3 = .1$, $\alpha_4 = .1$, $\epsilon_1 = .1$, $\epsilon_2 = .05$, and $\epsilon_3 = .1$. Also, $p = .6$.

**Theorem:** Let

$$m = \max \left( \frac{\alpha_2 - \epsilon_1}{\alpha_1 + \epsilon_1}, \frac{\alpha_4 - \epsilon_3}{\alpha_3 + \epsilon_3}, \frac{\epsilon_2}{\epsilon_1} \right)$$

and

$$M = \min \left( \frac{\alpha_3 + \epsilon_3}{\alpha_2 - \epsilon_1}, \frac{\epsilon_3}{\epsilon_2} \right)$$

(We take $m = +\infty$ if $\epsilon_1 = 0$.) The transformation $|\psi\rangle \rightarrow |\phi\rangle$ cannot be performed under LOCC by itself but can be performed under LOCC with a two-qubit pure bipartite state catalyst $|\kappa\rangle$ if and only if (**) holds as above and $m \leq M$. Moreover if this is the case, then $|\kappa\rangle$ will be a valid catalyst for $|\psi\rangle \rightarrow |\phi\rangle$ iff

$$m \leq \frac{1 - p}{p} \leq M$$
3 The Proof

The proof itself is rather cumbersome. Because of this, we will give a pre-
liminary overview to help the reader follow the logic. We will start by using
a few simple observations to reduce the requirements of the proof. We will
then start out assuming that we are in the situation where catalysis occurs,
and use this fact to gain information about \( \lambda \) and \( \lambda' \). It will turn out that the
condition of majorization forces only one possible choice for the \( \lambda'_i \)'s, and we
will then compute them. There will, unfortunately, be many choices for \( \lambda \);
however, each \( \lambda_i \) will be limited. By breaking each one up into a few cases,
we can determine conditions that work for each case. (This is the bulk of
the proof, Sections 3.5 through 3.12. They are, frankly, tedious to check
and the reader may wish to skip them.) It is useful to note that there is a
certain symmetry among the cases; in general, the \( \lambda_i \) case is the mirror of
the \( \lambda_{8-i} \) case. (The \( \lambda_8 \) case itself is merely \( 1 = 1 \).) Finally, we will show that
the argument used for the forward direction of the theorem is com pletely
reversible and provides a proof for the backward direction as well.

We will now begin the proof with a few preliminaries. First, the quantity
\((1 - p)/p\) occurs frequently in our calculations; it will be convenient to refer
to it as \( r \). Note that \( p = 1/(1+r) \). Since all the arguments of \( m \) are positive,
\( m \geq 0 \). Since \( \alpha_2' \geq \alpha_3' \), \( \alpha_2 - \epsilon_1 - \epsilon_2 \geq \alpha_3 + \epsilon_2 + \epsilon_3 \), we have \( \alpha_2 - \epsilon_1 \geq \alpha_3 + \epsilon_3 \). Thus
\( (\alpha_3 + \epsilon_3)/(\alpha_2 - \epsilon_1) \leq 1 \). This implies \( M \leq 1 \). Therefore, if \( m \leq r \leq M \),
\( 0 \leq r \leq 1 \) and so \( 1/2 \leq p \leq 1 \). Therefore, every \( r \) between \( m \) and \( M \) will
produce a \( p \) in the valid range between \( 1/2 \) and \( 1 \).

Also, let us look at the vectors \( \vec{\beta} \) and \( \vec{\beta}' \). We have noted that the com-
ponents of \( \vec{\beta} \) are \( \alpha_1 p, \alpha_2 p, \alpha_3 p, \alpha_4 p, \alpha_1(1 - p), \alpha_2(1 - p), \alpha_3(1 - p) \) and
\( \alpha_4(1 - p) \) and similarly for \( \vec{\beta}' \). We know \( \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \) and \( p \geq (1 - p) \).
Therefore, we have

\[
\alpha_1 p \geq \alpha_2 p \geq \alpha_3 p \geq \alpha_4 p
\]

\[
\alpha_1(1 - p) \geq \alpha_2(1 - p) \geq \alpha_3(1 - p) \geq \alpha_4(1 - p).
\]

We also know that \( \alpha_i p \geq \alpha_i(1 - p) \) for \( i = 1, \ldots, 4 \). However, we do not
\textit{a priori} have any additional knowledge about the ordering of the components.

We will now prove a simple lemma:
Lemma: Suppose
\[ \frac{a}{b} \geq \frac{c}{d}. \]
Then
\[ \frac{a}{b} \geq \frac{a + c}{b + d} \geq \frac{c}{d}. \]

The proof is completely trivial. Multiplying out the inequality in the hypothesis yields \( ad \geq bc \). Adding \( ab \) to both sides yields the first inequality and adding \( cd \) to both sides yields the second. Despite the simplicity of this lemma, it will frequently prove useful.

Our next observation is that the first part of the theorem follows immediately from the second part. If \( m > M \), then it is impossible to pick \( r \) such that \( m \leq r \leq M \). Therefore there will be no valid catalysts of the appropriate form and thus catalysis cannot occur. On the other hand, if \( m \leq M \), we simply choose \( r \) such that \( m \leq r \leq M \). In that case \( |\kappa\rangle \) will allow catalytic conversion to take place, and hence catalysis is possible.

To prove the second part of the theorem, fix two pure bipartite states \( |\psi\rangle \) and \( |\phi\rangle \) which satisfy (**) and fix a catalyst \( |\kappa\rangle \). Let \( m, M, \) and \( r \) be as specified above. By Nielsen’s Theorem, \( |\kappa\rangle \) is a valid catalyst for an LOCC transformation iff \( \vec{\beta} \) is majorized by \( \vec{\beta}' \). This reduces the argument to proving the following statement: \( \vec{\beta} \) is majorized by \( \vec{\beta}' \) if and only if \( m \leq r \leq M \).

We begin with the forward direction: Assume that \( \vec{\beta} \) is majorized by \( \vec{\beta}' \). We will use this assumption to find a set of restrictions on \( r \).

### 3.1 The first two components: \( r \geq \frac{\alpha'_2}{\alpha'_1} \)

Suppose \( r < \alpha'_2/\alpha'_1 \). Then \( \alpha'_2p > \alpha'_1(1 - p) \), i.e. \( (\alpha_2 - \epsilon_1 - \epsilon_2)p > (\alpha_1 + \epsilon_1)(1 - p) \). This implies \( \alpha_2p > \alpha_1(1 - p) \). We know that \( \alpha_1p \) is the largest component of \( \vec{\beta} \). Since \( \alpha_2p > \alpha_1(1 - p) \), we know that \( \alpha_2p \) is the second largest component. Therefore, the sum of the two largest components of \( \vec{\beta} \) is

\[ \lambda_2 = \alpha_1p + \alpha_2p. \]
But we also know that $\alpha_2'p > \alpha_1'(1 - p)$ and so the two largest components $\vec{\beta}'$ are $\alpha_1'p$ and $\alpha_2'p$. Thus

$$
\lambda'_2 = \alpha_1'p + \alpha_2'p
= \alpha_1p + \epsilon_1p + \alpha_2p - \epsilon_1p - \epsilon_2p
= \alpha_1p + \alpha_2p - \epsilon_2p
< \alpha_1p + \alpha_2p
= \lambda_2
$$

But $\vec{\beta}'$ majorizes $\vec{\beta}$ and so $\lambda'_2 \geq \lambda_2$. This is a contradiction, so we must have $r \geq \alpha_2'/\alpha_1'$.

### 3.2 The second and third components: $r \leq \alpha_3'/\alpha_2'$

Proof: Suppose $r > \alpha_3'/\alpha_2'$. Then $\alpha_2'(1 - p) > \alpha_3'p$. From the proof of Step 1, we know the two largest components of $\vec{\beta}'$ are $\alpha_1'p$, $\alpha_1'(1 - p)$. The component $\alpha_3'p$ is larger than any of the remaining components, so it is the third largest. The preceding inequality shows that $\alpha_2'(1 - p)$ is the fourth largest component. Hence,

$$
\lambda'_4 = \alpha_1' + \alpha_2' = \alpha_1 + \alpha_2 - \epsilon_2.
$$

Consider $\vec{\beta}$. We know that $\alpha_1p$, $\alpha_1(1 - p)$, $\alpha_2p$, and $\alpha_2(1 - p)$ are four components of this vector. If they are the four largest, then $\lambda_4 = \alpha_1 + \alpha_2$. If they are not the four largest, then $\lambda_4 > \alpha_1 + \alpha_2$. In either case, we have

$$
\lambda_4 \geq \alpha_1 + \alpha_2
> \alpha_1 + \alpha_2 - \epsilon_2
= \lambda'_4
$$

But again, $\vec{\beta}'$ majorizes $\vec{\beta}$ and so $\lambda'_4 \geq \lambda_4$. This is another contradiction, so we must have $r \leq \alpha_3'/\alpha_2'$.

### 3.3 The last two components: $r \geq \alpha_4'/\alpha_3'$

Proof: This argument exactly mirrors the argument of 3.1. If the condition on $r$ were false, then $\alpha_3(1 - p)$ and $\alpha_4(1 - p)$ would be the smallest components of $\vec{\beta}$, and $\alpha_3'(1 - p)$ and $\alpha_4'(1 - p)$ the smallest components of $\vec{\beta}'$. Hence

$$
\lambda'_6 = \alpha_1 + \alpha_2 + \alpha_3p + \alpha_4p - \epsilon_2(1 - p)
= \alpha_1 + \alpha_2 + \alpha_3p + \alpha_4p.
$$

Once again we would have $\lambda'_6 < \lambda_6$, which cannot occur because $\vec{\beta}'$ majorizes $\vec{\beta}$. Therefore, we must have $r \geq \alpha_4'/\alpha_3'$. 

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3.4 Computing the $\lambda'_i$’s.

We now know the ordering of the components of $\vec{\beta}'$ for valid catalysts: We know $\alpha'_i p \geq \alpha'_i (1 - p)$ since $p \geq (1 - p)$. We have $\alpha'_1 (1 - p) \geq \alpha'_2 p$ by 3.1. We know $\alpha'_2 p \geq \alpha'_3 p$ since $\alpha'_2 \geq \alpha'_3$. We have $\alpha'_3 p \geq \alpha'_4 (1 - p)$ by 3.2. We know $\alpha'_2 (1 - p) \geq \alpha'_3 (1 - p)$ since $\alpha'_2 \geq \alpha'_3$. We have $\alpha'_3 (1 - p) \geq \alpha'_4 p$ by 3.3. And we have $\alpha'_4 p \geq \alpha'_4 (1 - p)$ since $p \geq (1 - p)$. This means we can calculate the $\lambda'_i$’s:

\[
\begin{align*}
\lambda'_1 &= \alpha_1 p + \epsilon_1 p \\
\lambda'_2 &= \alpha_1 + \epsilon_1 \\
\lambda'_3 &= \alpha_1 + \alpha_2 p + \epsilon_1 (1 - p) - \epsilon_2 p \\
\lambda'_4 &= \alpha_1 + \alpha_2 p + \alpha_3 p + \epsilon_1 (1 - p) + \epsilon_3 p \\
\lambda'_5 &= \alpha_1 + \alpha_2 + \alpha_3 p - \epsilon_2 (1 - p) + \epsilon_3 p \\
\lambda'_6 &= \alpha_1 + \alpha_2 + \alpha_3 + \epsilon_3 \\
\lambda'_7 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 p + \epsilon_3 (1 - p) \\
\lambda'_8 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1
\end{align*}
\]

Since we are assuming that $\vec{\beta}$ is majorized by $\vec{\beta}'$, we know that $\lambda'_i \geq \lambda_i$ for $i = 1, \ldots, 8$ with equality holding for $i = 8$. We will analyze each of these inequalities and see what restrictions they place on $r$.

3.5 The first sum: $\lambda'_1 \geq \lambda_1$ always holds.

We know that $\alpha_1 p$ is the largest component in $\vec{\beta}$ so $\lambda_1 = \alpha_1 p$. Thus $\lambda'_1 = \lambda_1 + \epsilon_1 p$ and we have $\lambda'_1 \geq \lambda_1$. This inequality always holds and there is no further restriction on $r$.

3.6 The second sum: $\lambda'_2 \geq \lambda_2$ iff $r \geq (\alpha_2 - \epsilon_1)/(\alpha_1 + \epsilon_1)$.

For this step, we will divide the proof into two cases. First, suppose that $r \leq \alpha_2/\alpha_1$. This implies that $\alpha_1 (1 - p) \leq \alpha_2 p$. Thus $\lambda_2 = \alpha_1 p + \alpha_2 p$. Therefore, the following inequalities are equivalent:

\[
\begin{align*}
\lambda'_2 &\geq \lambda_2 \\
\alpha_1 + \epsilon_1 &\geq \alpha_1 p + \alpha_2 p \\
(\alpha_1 + \epsilon_1)/(\alpha_1 + \alpha_2) &\geq p = 1/(1 + r) \\
1 + r &\geq (\alpha_1 + \alpha_2)/(\alpha_1 + \epsilon_1) \\
r &\geq (\alpha_2 - \epsilon_1)/(\alpha_1 + \epsilon_1)
\end{align*}
\]
which is the desired inequality.

For the second case, suppose that \( r > \alpha_2/\alpha_1 \). This implies that \( \alpha_1(1-p) > \alpha_2p \). Thus \( \lambda_2 = \alpha_1 \). Since \( \lambda_2' = \alpha_1 + \epsilon_1 \), we have \( \lambda_2' \geq \lambda_2 \). But
\[
\frac{\alpha_2}{\alpha_1} > \frac{\alpha_2 - \epsilon_1}{(\alpha_1 + \epsilon_1)},
\]
so \( r > \alpha_2/\alpha_1 \) implies \( r \geq (\alpha_2 - \epsilon_1)/(\alpha_1 + \epsilon_1) \). Therefore both inequalities are always true in this case.

We have therefore shown the desired equivalence.

**3.7 The third sum:** \( \lambda_3' \geq \lambda_3 \) iff \( r \geq (\alpha_3 + \epsilon_2)/(\alpha_1 + \epsilon_1) \) and \( r \geq \epsilon_2/\epsilon_1 \).

Again, we will split this into two cases. First, suppose \( r \leq \alpha_3/\alpha_1 \). This implies that \( \alpha_1(1-p) \leq \alpha_3p \). Then by the same reasoning as above, \( \alpha_1p, \alpha_2p, \) and \( \alpha_3p \) are the three largest components of \( \beta' \). Hence \( \lambda_3 = \alpha_1p + \alpha_2p + \alpha_3p \). Therefore the following inequalities are equivalent:
\[
\begin{align*}
\lambda_3' &\geq \lambda_3 \\
\alpha_1 + \alpha_2p + \epsilon_1(1-p) - \epsilon_2p &\geq \alpha_1p + \alpha_2p + \alpha_3p \\
r &\geq (\alpha_3 + \epsilon_2)/(\alpha_1 + \epsilon_1)
\end{align*}
\]

Notice also, that we have assumed \( r \leq \alpha_3/\alpha_1 \). Since \( r \geq (\alpha_3 + \epsilon_2)/(\alpha_1 + \epsilon_1) \), our preliminary lemma implies \( r \geq \epsilon_2/\epsilon_1 \). Conversely, if \( r \geq (\alpha_3 + \epsilon_2)/(\alpha_1 + \epsilon_1) \), then \( \lambda_3' \geq \lambda_3 \).

The second case, \( r > \alpha_3/\alpha_1 \), is the opposite of the first. Here \( \lambda_3 = \alpha_1 + \alpha_2p \). This leads to \( \lambda_3' \geq \lambda_3 \iff r \geq \epsilon_2/\epsilon_1 \). Also, we have assumed \( r > \alpha_3/\alpha_1 \) and shown \( r \geq \epsilon_2/\epsilon_1 \), so by the preliminary lemma \( r \geq (\alpha_3 + \epsilon_2)/(\alpha_1 + \epsilon_1) \). Conversely, if \( r \geq \epsilon_2/\epsilon_1 \) and \( r \geq (\alpha_3 + \epsilon_2)/(\alpha_1 + \epsilon_1) \), then \( \lambda_3' \geq \lambda_3 \).

We have therefore shown the desired equivalence.

**Remark:** Notice that this step shows that \( \epsilon_1 > 0 \) is necessary for catalysis to occur.

**3.8 The fourth sum:** \( \lambda_4' \geq \lambda_4 \) iff \( r \geq (\alpha_4 - \epsilon_3)/(\alpha_1 + \epsilon_1) \) and \( r \leq (\alpha_3 + \epsilon_3)/(\alpha_2 - \epsilon_1) \).

The argument for this is practically the same as Step 7, only we have three cases. For the first case, suppose \( r \leq \alpha_4/\alpha_1 \). This implies that \( \alpha_1(1-p) \leq \)
 Remark: Notice that we require \( \alpha_4 p \). Hence \( \lambda_4 = \alpha_1 p + \alpha_2 p + \alpha_3 p + \alpha_4 p \). In this case the condition \( \lambda'_4 \geq \lambda_4 \) is true iff \( r \geq (\alpha_4 - \epsilon_3)/(\alpha_1 + \epsilon_1) \). Also, we have assumed \( r \leq \alpha_4/\alpha_1 \) and we know that \( \alpha_4 \leq \alpha_3 \leq \alpha_3 + \epsilon_3 \) and \( \alpha_1 \geq \alpha_2 \geq \alpha_2 - \epsilon_1 \). Thus \( r \leq (\alpha_3 + \epsilon_3)/(\alpha_2 - \epsilon_1) \). Conversely, if both these inequalities hold, then \( \lambda'_4 \geq \lambda_4 \).

For our second case, suppose \( \alpha_4/\alpha_1 < r \leq \alpha_3/\alpha_2 \). (We are only guaranteed that \( \alpha_4/\alpha_1 \leq \alpha_3/\alpha_2 \), so this case may not occur.) Then we get \( \lambda_4 = \alpha_1 + \alpha_2 p + \alpha_3 p \) and \( \lambda'_4 \geq \lambda_4 \iff \epsilon_1 + \epsilon_3 p \geq \epsilon_1 p \). But \( p \leq 1 \) so \( \epsilon_1 > \epsilon_1 p \) and hence we always have \( \lambda'_4 \geq \lambda_4 \). Likewise, since \( \alpha_4/\alpha_1 < r \leq \alpha_3/\alpha_2 \), both \( r \leq (\alpha_4 - \epsilon_3)/(\alpha_1 + \epsilon_1) \) and \( r \leq (\alpha_3 + \epsilon_3)/(\alpha_2 - \epsilon_1) \) will always hold.

For the final case, suppose \( r > \alpha_3/\alpha_2 \). This implies \( \alpha_2 (1 - p) > \alpha_3 p \) and thus \( \lambda_4 = \alpha_1 + \alpha_2 \). Therefore, \( \lambda'_4 \geq \lambda_4 \iff r \leq (\alpha_3 + \epsilon_3)/(\alpha_2 - \epsilon_1) \). Additionally, \( r > \alpha_3/\alpha_2 \geq (\alpha_4 - \epsilon_3)/(\alpha_1 + \epsilon_1) \). Conversely, if both these inequalities hold, then \( \lambda'_4 \geq \lambda_4 \).

We have therefore shown the desired equivalence.

3.9 The fifth sum: \( \lambda'_5 \geq \lambda_5 \) iff \( r \geq (\alpha_4 - \epsilon_3)/(\alpha_2 - \epsilon_2) \) and \( r \leq \epsilon_3/\epsilon_2 \).

Fortunately we are back down to considering just two cases. For the first, suppose \( r \leq \alpha_4/\alpha_2 \). Then \( \alpha_2 (1 - p) \leq \alpha_4 p \) and thus \( \lambda_5 = \alpha_1 + \alpha_2 p + \alpha_3 p + \alpha_4 p \). Again we can manipulate the inequality to get, \( \lambda'_5 \geq \lambda_5 \iff r \geq (\alpha_4 - \epsilon_3)/(\alpha_2 - \epsilon_2) \). Notice also, that we have assumed \( r \leq \alpha_4/\alpha_2 \). Since \( r \geq (\alpha_4 - \epsilon_3)/(\alpha_2 - \epsilon_2) \), and our preliminary lemma implies \( r \leq \epsilon_3/\epsilon_2 \). Conversely, if \( r \geq (\alpha_4 - \epsilon_3)/(\alpha_2 - \epsilon_2) \), then \( \lambda'_5 \geq \lambda_5 \).

For the second case \( r > \alpha_4/\alpha_2 \). This implies \( \lambda_5 = \alpha_1 + \alpha_2 + \alpha_3 p \) and \( \lambda'_5 \geq \lambda_5 \iff r \leq \epsilon_3/\epsilon_2 \). Also, we have assumed \( r > \alpha_4/\alpha_2 \). Since \( r \leq \epsilon_3/\epsilon_2 \), the preliminary lemma again implies \( r \geq (\alpha_4 - \epsilon_3)/(\alpha_2 - \epsilon_2) \). Conversely, if \( r \geq \epsilon_3/\epsilon_2 \), then \( \lambda'_5 \geq \lambda_5 \).

We have therefore shown the desired equivalence.

Remark: Notice that we require \( r \geq (\alpha_4 - \epsilon_3)/(\alpha_2 - \epsilon_2) \). Since both the numerator and denominator are positive, this requires \( r \) to be positive. But we also require \( r \leq \epsilon_3/\epsilon_2 \). If \( \epsilon_3 = 0 \), this is not possible. Therefore this step shows that \( \epsilon_3 > 0 \) is necessary for catalysis to occur. Hence the weak inequalities in (**) may be replaced by strict inequalities. (Strictly speaking, we have only shown this for a 2-state catalyst. However it is easy to generalize this argument to an arbitrary \( n \)-state catalyst by considering the sums of the \( n + 1 \) and \( 3n - 1 \) largest components of the \( 4n \)-long vectors.)
3.10  The sixth sum: $\lambda'_6 \geq \lambda_6$ iff $r \geq (\alpha_4 - \epsilon_3)/(\alpha_3 + \epsilon_3)$.

Again, we consider two cases. First, suppose $r \leq \alpha_4/\alpha_3$. This implies $\alpha_3(1 - p) \leq \alpha_4 p$ and $\lambda_6 = \alpha_1 + \alpha_2 + \alpha_3 p + \alpha_4 p$. Therefore, $\lambda'_6 \geq \lambda_6 \iff r \geq (\alpha_4 - \epsilon_3)/(\alpha_3 + \epsilon_3)$.

Now suppose $r > \alpha_4/\alpha_3$. This implies $\lambda_6 = \alpha_1 + \alpha_2 + \alpha_3$. Hence $\lambda'_6 \geq \lambda_6 \iff \alpha_1 + \alpha_2 + \alpha_3 + \epsilon_3 \geq \alpha_1 + \alpha_2 + \alpha_3$ which is always true since $\epsilon_3 \geq 0$. Moreover, since we are assuming $r > \alpha_4/\alpha_3$, we always have $r \geq (\alpha_4 - \epsilon_3)/(\alpha_3 + \epsilon_3)$.

We have therefore shown the desired equivalence.

3.11  The seventh sum: $\lambda'_7 \geq \lambda_7$ always holds.

We know that $\alpha_4(1 - p)$ is the smallest component in $\vec{\beta}$ so $\lambda_7 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 p$. Thus $\lambda'_7 = \lambda_7 + \epsilon_3(1 - p)$ and we have $\lambda'_7 \geq \lambda_7$. This inequality always holds and there is no further restriction on $r$.

3.12  The last sum: $\lambda'_8 = \lambda_8$ always holds.

Since $\lambda_8 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \lambda'_8$, this is automatically true and no further restrictions are placed on $r$.

3.13  Combining the restrictions: $m \leq r \leq M$.

If we check back through 3.1-3.12, we see that if $\vec{\beta}$ is majorized by $\vec{\beta}'$, then $r$ must be greater than or equal to each of the following terms:
\[
\frac{\alpha'_2}{\alpha'_1}, \frac{\alpha'_4}{\alpha'_3}, \frac{\alpha_2 - \epsilon_1}{\alpha_3 + \epsilon_1}, \frac{\alpha_3 + \epsilon_2}{\epsilon_1}, \frac{\alpha_4 - \epsilon_3}{\alpha_3 + \epsilon_3}, \frac{\alpha_4 - \epsilon_3}{\alpha_3 + \epsilon_3}.
\]

However $\alpha'_2/\alpha'_1$, $(\alpha_3 + \epsilon_2)/(\alpha_1 + \epsilon_1)$, and $(\alpha_4 - \epsilon_3)/(\alpha_1 + \epsilon_1)$ are less than $(\alpha_2 - \epsilon_1)/(\alpha_1 + \epsilon_1)$; and $\alpha'_4/\alpha'_3$ and $(\alpha_4 - \epsilon_3)/(\alpha_2 - \epsilon_2)$ are less than $(\alpha_4 - \epsilon_3)/(\alpha_3 + \epsilon_3)$. Therefore, it is only necessary to require
\[
r \geq \frac{\alpha_2 - \epsilon_1}{\alpha_1 + \epsilon_1}, \frac{\alpha_4 - \epsilon_3}{\alpha_3 + \epsilon_3}, \frac{\epsilon_2}{\epsilon_1}.
\]

In other words, we require $r \geq m$. 

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Similarly, from 3.1-3.12, \( r \) must be less than or equal to the following terms:

\[
\frac{\alpha'_3}{\alpha'_2}, \frac{\alpha_3 + \epsilon_3}{\alpha_2 - \epsilon_1}, \frac{\epsilon_3}{\epsilon_2}
\]

However \( \alpha'_3/\alpha'_2 \) is greater than \( (\alpha_3 + \epsilon_3)/(\alpha_2 - \epsilon_1) \). Therefore, it is only necessary to require

\[
r \leq \frac{\alpha_3 + \epsilon_3}{\alpha_2 - \epsilon_1}, \frac{\epsilon_3}{\epsilon_2}
\]

In other words, we require \( r \leq M \).

We have now completed the first direction of the proof, \( \vec{\beta} \) majorized by \( \vec{\beta}' \) implies \( m \leq r \leq M \).

3.14 The other direction: \( m \leq r \leq M \) implies \( \vec{\beta} \) is majorized by \( \vec{\beta}' \).

Assume that \( m \leq r \leq M \). Note that the if and only if statements of 3.5 through 3.12 depend upon the results derived in 3.1 through 3.4. We will now derive them for this direction. Since \( \alpha'_2/\alpha'_1 \leq m \leq r \), we have \( \alpha'_2 p \leq \alpha'_1 (1 - p) \). Since \( \alpha'_3/\alpha'_2 \geq M \geq r \), we have \( \alpha'_3 p \geq \alpha'_2 (1 - p) \). Since \( \alpha'_4/\alpha'_3 \leq m \leq r \), we have \( \alpha'_4 p \leq \alpha'_3 (1 - p) \). This fixes the ordering of the components and thus the calculation of the \( \lambda'_i \)'s done in 3.4 holds in this particular case.

We now know the results of 3.1-3.4 hold. Also, we know that all of the inequalities listed in 3.13 are true. Thus we may use the equivalences shown in 3.5-3.12 and conclude that \( \lambda'_i \geq \lambda_i \) for \( i = 1, \ldots, 8 \) with equality holding for \( i = 8 \). Therefore, \( \vec{\beta} \) is majorized by \( \vec{\beta}' \).

This completes the proof of the theorem.

4 Conclusions

Note that the arguments of the min and max functions which determine \( m \) and \( M \) are of a special form. Three of them,

\[
\frac{\alpha_2 - \epsilon_1}{\alpha_1 + \epsilon_1}, \frac{\alpha_3 + \epsilon_3}{\alpha_2 - \epsilon_1}, \frac{\alpha_4 - \epsilon_3}{\alpha_3 + \epsilon_3}
\]

are ratios of the form \( \alpha'_i/\alpha'_{i+1} \) with \( \epsilon_2 \) replaced by 0. The other two ratios,

\[
\frac{\epsilon_2}{\epsilon_1}, \frac{\epsilon_3}{\epsilon_2}
\]
compare the sizes of $\epsilon_1$ and $\epsilon_3$ to $\epsilon_2$.

The quantity $\epsilon_2$ can be considered a measure of how much majorization is violated in the uncatalyzed system. Similarly, $\epsilon_1$ and $\epsilon_3$ are the amount of “slack” we are given to work with in the other components. If $\epsilon_2$ is large with respect to $\epsilon_1$, $m$ becomes large; if it is large with respect to $\epsilon_3$, $M$ becomes small. We thus require enough “slack” on both ends to make up for the “bulge” in the middle. These notions of “bulge” and “slack” can be formalized by looking at the areas bounded between the Lorenz curves generated by $\vec{\alpha}$ and $\vec{\alpha}'$. See [4] for further details.

The other quantities are ratios of the components in the limiting case where no catalysis is necessary. This corresponds to looking at the slopes of the two Lorenz curves.

Finally, we note that this paper deals only with the case of a 4-particle system evenly divided between two parties and a 2-particle catalyst similarly divided. Moreover, we assume that all pieces of this system have the same Schmidt basis. This is clearly not the most general case one could consider. The next logical generalization would be to analyze the case of a 4-particle system and $2n$-particle catalyst, all with the same Schmidt basis.

5 Some Examples

We will now return to the examples discussed at the beginning of the paper. Jonathan and Plenio’s catalysis example had

$$|\psi\rangle = \sqrt{0.4} |00\rangle + \sqrt{0.4} |11\rangle + \sqrt{0.1} |22\rangle + \sqrt{0.1} |33\rangle$$

$$|\phi\rangle = \sqrt{0.5} |00\rangle + \sqrt{0.25} |11\rangle + \sqrt{0.25} |22\rangle + 0 |33\rangle$$

$$|\kappa\rangle = \sqrt{0.6} |00\rangle + \sqrt{0.4} |11\rangle$$

This becomes $\alpha_1 = .4$, $\alpha_2 = .4$, $\alpha_3 = .1$, $\alpha_4 = .1$, $\epsilon_1 = .1$, $\epsilon_2 = .05$, and $\epsilon_3 = .1$. Thus

$$m = \max \left( \frac{3}{5}, \frac{0}{2}, \frac{.05}{.1} \right) = \frac{3}{5}$$

and

$$M = \min \left( \frac{2}{3}, \frac{1}{.05} \right) = \frac{2}{3}$$

Since $m \leq M$, catalysis is possible and any $|\kappa\rangle$ with $3/5 \leq r \leq 2/3$ will be a valid catalyst – in other words, $3/5 \leq p \leq 5/8$. In this example, $p = 3/5$. 

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In the second example above, we had

\[
|\psi\rangle = \sqrt{0.45} |00\rangle + \sqrt{0.45} |11\rangle + \sqrt{0.05} |22\rangle + \sqrt{0.05} |33\rangle
\]

\[
|\phi\rangle = \sqrt{0.5} |00\rangle + \sqrt{0.35} |11\rangle + \sqrt{0.15} |22\rangle + 0 |33\rangle
\]

\[
|\kappa\rangle = \sqrt{p} |00\rangle + \sqrt{1-p} |11\rangle
\]

This becomes \(\alpha_1 = .45, \alpha_2 = .45, \alpha_3 = .05, \alpha_4 = .05, \epsilon_1 = .05, \epsilon_2 = .05, \) and \(\epsilon_3 = .05.\) Thus

\[
m = \max \left( \frac{4}{5}, \frac{0}{1}, \frac{.05}{.05} \right) = 1
\]

and

\[
M = \min \left( \frac{1}{.4}, \frac{.05}{.05} \right) = \frac{1}{4}
\]

Since \(m > M,\) catalysis is not possible for any value of \(p,\) as we have already seen.

### 6 Existence of Specific Values

We have seen above that the value of \(m\) must be positive and the value of \(M\) must be less than 1. Also, since \(M\) is the minimum of two positive quantities, \(M\) must be positive. Therefore, we can pose the question:

*Given numbers \(m_0\) and \(M_0\) with \(0 < m_0\) and \(0 < M_0 < 1,\) do there exist states \(|\psi\rangle\) and \(|\phi\rangle\) satisfying (**)) for which \(m = m_0\) and \(M = M_0?\)*

The answer to this question is yes, and we will proceed to give a construction. We first consider the case where \(m_0 \leq 1.\) Choose a positive number \(\mu\) with

\[
\mu < \min \left( \frac{1}{2} \frac{1}{1+M_0}, \frac{1}{2} \frac{1-m_0/2}{1+2M_0} \right)
\]

and let

\[
a = \left( \frac{2}{m_0 + 2} \right)^2.
\]
Let $|\psi\rangle$ and $|\phi\rangle$ be the states given by

$$
\begin{align*}
\alpha_1 &= a(1 - \mu) \\
\alpha_2 &= a(m_0/2 + (m_0 + 1)\mu) \\
\alpha_3 &= a(m_0/2 - (M_0 + 1)m_0\mu) \\
\alpha_4 &= a(m_0^2/4 + M_0m_0\mu) \\
\alpha_1' &= a \\
\alpha_2' &= am_0/2 \\
\alpha_3' &= am_0/2 \\
\alpha_4' &= am_0^2/4 \\
\end{align*}
$$

Since $m_0 \leq 1$, it is clear that the $\alpha_i'$'s are in decreasing order and one can easily verify that they sum to 1. The fact that $\mu < 1/2(1 - m/2)/(1 + 2M)$ implies that the $\alpha_i$'s are in decreasing order and it is easy to verify that they also sum to 1. Computing the $\epsilon_i$'s, we get

$$
\epsilon_1 = \mu a, \quad \epsilon_2 = m_0\mu a, \quad \epsilon_3 = M_0m_0\mu a.
$$

Performing the calculation of $m$ and $M$, we obtain

$$
\begin{align*}
m &= \max \left( \frac{m_0}{2}(1 + 2\mu), \frac{m_0}{2} \frac{1}{1 - 2\mu}, m_0 \right) = m_0 \\
M &= \min \left( \frac{1 - 2\mu}{1 + 2\mu}, M_0 \right) = M_0.
\end{align*}
$$

Here, the fact that $m_0$ is the largest of the three values is straight-forward, while the fact that $M_0$ is the smaller of the two values follows from the fact that $\mu < 1/2(1 - M_0)/(1 + M_0)$. We have therefore produced two states which yield the desired $m$ and $M$.

For the case of $m > 1$, we set

$$
\mu < \min \left( \frac{1}{2}, \frac{1 - M_0}{1 + M_0}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{(1/2)}{1 + 2M_0} \right)
$$

and define the states $|\phi\rangle$ and $|\psi\rangle$ by

$$
\begin{align*}
\alpha_1 &= a(1 - \mu) \\
\alpha_2 &= a(1/2 + (m_0 + 1)\mu) \\
\alpha_3 &= a(1/2 - (M_0 + 1)m_0\mu) \\
\alpha_4 &= a(1/4 + M_0m_0\mu) \\
\alpha_1' &= a \\
\alpha_2' &= a/2 \\
\alpha_3' &= a/2 \\
\alpha_4' &= a/4
\end{align*}
$$

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Note that if we compute the values of the $\epsilon$’s, they remain unchanged. Again, the various terms are ordered properly because of the choice of $\mu$. Also,

\[
m = \max \left( \frac{1}{2} (1 + 2\mu), \frac{1}{2} \frac{1}{1 - 2\mu}, m_0 \right) = m_0
\]
\[
M = \min \left( \frac{1 - 2\mu}{1 + 2\mu}, M_0 \right) = M_0.
\]

Here, the minimality of $M_0$ is the same as before and the maximality of $m_0$ follows from the fact that $m_0 > 1$ and the definition of $\mu$.

We will now provide a concrete example. Let us choose $m_0 = 2/3$ and $M_0 = 1/3$. (So we have a case in which catalysis cannot occur.) We require

\[
0 < \mu < \min \left( \frac{1}{2} \frac{2/3}{4/3}, \frac{1}{2} \frac{2/3}{5/3} \right) = 1/5
\]

so let us arbitrarily choose $\mu = 1/10$. We set

\[
a = \left( \frac{2}{2/3 + 2} \right)^2 = \left( \frac{3}{4} \right)^2 = \frac{9}{16}.
\]

Since $m_0 < 1$, we set

\[
\alpha_1 = 81/160, \quad \alpha_2 = 45/160, \quad \alpha_3 = 22/160, \quad \alpha_4 = 12/160
\]
\[
\alpha'_1 = 90/160, \quad \alpha'_2 = 30/160, \quad \alpha'_3 = 30/160, \quad \alpha'_4 = 10/160
\]

This yields

\[
\epsilon_1 = 9/160, \quad \epsilon_2 = 6/160, \quad \epsilon_3 = 2/160.
\]

and we have

\[
m = \max \left( \frac{36}{90}, \frac{10}{22}, \frac{6}{9} \right) = \frac{2}{3}
\]
\[
M = \min \left( \frac{24}{30}, \frac{2}{3} \right) = \frac{1}{3}
\]

as desired.

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