PLANE AUTOMORPHISMS GIVEN BY POLYNOMIALS OF SCATTERED DEGREES

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Abstract. We study the plane automorphisms given by polynomials with certain degree decompositions.

1. Introduction

The jacobian conjecture, raised by Keller [15], has been studied by many mathematicians: a partial list of related results includes [1], [3], [5], [7], [9], [12], [13], [14], [17], [18], [19], [21], [22], [23], [24], [25]. A survey is given in [11]. In this paper we exclusively deal with the plane case. This is the first in a series of papers that introduce a new computational approach, which is completely elementary.

Let $k$ be a field of characteristic 0, and let $R = k[x, y]$. Throughout the paper, let $f, g \in R$ be polynomials satisfying the following:

Assumptions 1. Let $n$ be any positive integer, and let \{ $d_1, ..., d_n$ \} be any set of $n$ distinct positive integers with the following property:

(1.1) if $d_i + d_j = d_p + d_q$ for (not necessarily distinct) $i, j, p, q$, then \{ $d_i, d_j$ \} = \{ $d_p, d_q$ \}.

For each $d_i \in \{ d_1, ..., d_n \}$, let $f_{d_i}$ and $g_{d_i}$ be homogeneous polynomials of degree $d_i$ in $R$, and assume that at least one of $f_{d_i}$ and $g_{d_i}$ is nonzero. Let $f = \sum_{i=1}^{n} f_{d_i}$ and $g = \sum_{i=1}^{n} g_{d_i}$. Let

$$J = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} = \sum_{1 \leq i, j \leq n} \left( \frac{\partial f_{d_i}}{\partial x} \frac{\partial g_{d_j}}{\partial y} - \frac{\partial f_{d_i}}{\partial y} \frac{\partial g_{d_j}}{\partial x} \right).$$

Denote by $J_{i,j}$ the coefficient of $x^i y^j$ in $J$.

Theorem 2. If $J \in k \setminus \{0\}$, then $k[x, y] = k[f, g]$.

If $J \in k \setminus \{0\}$, then $f$ and $g$ must have linear parts, i.e., $1 \in \{ d_1, ..., d_n \}$. Assume $d_n = 1$. By linear change of variables, we can assume that $f_{d_n} = x$ and $g_{d_n} = y$. The case of $n = 2$, where (1.1) is trivially satisfied, has been proved in [2], [6, Corollary 6], [8], [10], [20] for the plane case under the assumption

(1.2) \[
\begin{pmatrix}
\frac{\partial (f-x)}{\partial x} & \frac{\partial (f-x)}{\partial y} \\
\frac{\partial (g-y)}{\partial x} & \frac{\partial (g-y)}{\partial y}
\end{pmatrix}^2 = 0,
\]

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and in [10 Corollary 2.2] without assuming [12]. The case of \(d_1, \ldots, d_n \leq 100\) is obtained as a special case of a result of Moh [21].

2. Proof

For any \(d_i \in \{d_1, \ldots, d_{n-1}\}\), let \(f_{d_i} = \sum_{j=0}^{d_i} s_{d_i-j,j} x^{d_i-j} y^j\) and \(g_{d_i} = \sum_{j=0}^{d_i} t_{d_i-j,j} x^{d_i-j} y^j\). Thanks to (1.1), the coefficients of \(x^{d_i-j} y^j\) in \(\left( \frac{\partial f_{d_i}}{\partial x} \frac{\partial g_{d_i}}{\partial y} - \frac{\partial f_{d_i}}{\partial y} \frac{\partial g_{d_i}}{\partial x} \right)\) are the only ones that contribute to the coefficient \(J_{d_i-j,j-1}\), which is equal to

\[(d_i - j + 1)s_{d_i-j+1,j-1} + j t_{d_i-j,j}\]

for \(1 \leq j \leq d_i\). Since \(J \in k \setminus \{0\}\) implies (2.1) = 0, there is an element, say \(c_{d_i-j+1,j}\), in \(k\) such that \(s_{d_i-j+1,j-1} = j c_{d_i-j+1,j}\) and \(t_{d_i-j,j} = -(d_i - j + 1)c_{d_i-j+1,j}\).

By letting \(s_{0,d_i} = (d_i + 1)c_{0,d_i+1}\) and \(t_{d_i,0} = (d_i + 1)c_{d_i+1,0}\), we have

\[f = \sum_{i=1}^{n-1} \sum_{j=1}^{d_i} j c_{d_i-j+1,j} x^{d_i-j+1} y^j + x, \quad \text{and} \]

\[g = -\sum_{i=1}^{n-1} \sum_{j=0}^{d_i} (d_i - j + 1)c_{d_i-j+1,j} x^{d_i-j} y^j + y.\]

In section 3, we will show that any \(2 \times 2\) minor of the matrix

\[
\begin{pmatrix}
  c_{d_i,1} & 2c_{d_i-1,2} & \cdots & (d_i + 1)c_{0,d_i+1} & c_{d_j,1} & 2c_{d_j-1,2} & \cdots & (d_j + 1)c_{0,d_j+1} \\
  (d_i + 1)c_{d_i+1,0} & d_i c_{d_i,1} & \cdots & c_{1,d_i} & (d_j + 1)c_{d_j+1,0} & d_j c_{d_j,1} & \cdots & c_{1,d_j}
\end{pmatrix}
\]

is equal to 0 for any \(d_i, d_j \in \{d_1, \ldots, d_{n-1}\}\).

Suppose that all \(2 \times 2\) minors of (2.2) are 0. Then it is straightforward to prove Theorem 2 as follows.

Case 1. Suppose that \(c_{d_i+1,0} = 0\) for some \(d_i\). Then \(c_{d_i,1} = \cdots = c_{1,d_i} = 0\). Since at least one of \(f_{d_i}\) and \(g_{d_i}\) is nonzero, \(c_{0,d_i+1} \neq 0\). Then \(c_{d_j+1,0} = c_{d_j,1} = \cdots = c_{1,d_j} = 0\) for any \(d_j \in \{d_1, \ldots, d_{n-1}\}\). Again since \(f_{d_j} \neq 0\) or \(g_{d_j} \neq 0\), we get \(c_{0,d_j+1} \neq 0\). So we have

\[f = \sum_{i=1}^{n-1} (d_i + 1)c_{0,d_i+1} y^{d_i} + x \quad \text{and} \quad g = y.\]

Then \(x = f - \sum_{i=1}^{n-1} (d_i + 1)c_{0,d_i+1} y^{d_i}\) and \(y = g\), hence \(k[f, g] = k[x, y]\).

Case 2. Suppose that \((d_i + 1)c_{d_i+1,0} \neq 0\) and \(c_{d_i,1} = 0\) for some \(d_i\). Then we use the same argument as in Case 1, and get

\[g = -\sum_{i=1}^{n-1} (d_i + 1)c_{d_i+1,0} x^{d_i} + y \quad \text{and} \quad f = x.\]
Case 3. Suppose that \((d_i + 1)c_{d_i+1,0} \neq 0\) and \(c_{d_i,1} \neq 0\) for any \(d_i \in \{d_1, \ldots, d_{n-1}\}\). Then we obtain
\[
f_{d_i} = c_{d_i,1} \left( x + \frac{c_{d_i,1}}{(d_i + 1)c_{d_i+1,0}} y \right)^{d_i} \quad \text{and} \quad g_{d_i} = -(d_i + 1)c_{d_i+1,0} \left( x + \frac{c_{d_i,1}}{(d_i + 1)c_{d_i+1,0}} y \right)^{d_i}.
\]
Note that \(\frac{c_{d_i,1}}{(d_i + 1)c_{d_i+1,0}} = \frac{c_{d_j,1}}{(d_j + 1)c_{d_j+1,0}}\) for any \(d_i, d_j \in \{d_1, \ldots, d_{n-1}\}\). So
\[
f = x + \sum_{j=1}^{n-1} c_{d_j,1} \left( x + \frac{c_{d_j,1}}{(d_j + 1)c_{d_j+1,0}} y \right)^{d_j}, \quad \text{and} \quad g = y - \sum_{j=1}^{n-1} (d_j + 1)c_{d_j+1,0} \left( x + \frac{c_{d_j,1}}{(d_j + 1)c_{d_j+1,0}} y \right)^{d_j}.
\]
Then it is easy to check that
\[
x = f - \sum_{j=1}^{n-1} c_{d_j,1} \left( f + \frac{c_{d_j,1}}{(d_j + 1)c_{d_j+1,0}} g \right)^{d_j}, \quad \text{and} \quad y = g + \sum_{j=1}^{n-1} (d_j + 1)c_{d_j+1,0} \left( f + \frac{c_{d_j,1}}{(d_j + 1)c_{d_j+1,0}} g \right)^{d_j}.
\]

3. The vanishing of \(2 \times 2\) minors of \((2.2)\)

For any \(p \in \{1, \ldots, n-1\}\), let
\[
A_p = \begin{pmatrix} c_{d_p,1} & 2c_{d_p-1,2} & \cdots & (d_p + 1)c_{0,d_p+1} \\ (d_p + 1)c_{d_p+1,0} & d_p c_{d_p,1} & \cdots & c_{1,d_p} \end{pmatrix}.
\]
Let \((A_p)_{i,j}\) be the determinant of the submatrix of \(A_p\) obtained by concatenating the \(i\)-th and \(j\)-th columns, that is,
\[
(A_p)_{i,j} = \det \begin{pmatrix} \ic_{d_p-i+1,i} & jc_{d_p-j+1,j} \\ (d_p + 2 - i)c_{d_p-i+2,j-1} & (d_p + 2 - j)c_{d_p-j+2,j-1} \end{pmatrix}.
\]

For simplicity, let \(d = d_p\), \(A = A_p\) and \(A_{(i,j)} = (A_p)_{i,j}\).

**Proposition 3.** Any \(2 \times 2\) minor of \(A\) is equal to 0.

**Proof.** This is an immediate consequence of the following two lemmas. \(\square\)

**Lemma 4.** Let \(m\) be any positive integer \(\leq d + 1\). If \(A_{(1,j)} = 0\) for \(j \in \{1, \ldots, m\}\), then \(A_{(i,j)} = 0\) for \(i, j \in \{1, \ldots, m\}\).

**Proof.** We use induction on \(m\). If \(m = 1\) then trivial. Suppose that the statement holds for \(m - 1\), and that \(A_{(1,m)} = 0\). If at least one of \(c_{d_1}, c_{d+1,0}, c_{d-m+1,m}, c_{d-m+2,m-1}\) is equal to 0, then it is easy to show that \(c_{d,1} = \cdots = c_{d-m+2,m-1} = 0\) and that \(c_{d+1,0} = 0\) or \(c_{d-m+1,m} = 0\). If not, \(\begin{pmatrix} m c_{d-m+1,m} \\ (d + 2 - m)c_{d-m+2,m-1} \end{pmatrix}\) is a multiple of \(\begin{pmatrix} c_{d,1} \\ (d + 1)c_{d+1,0} \end{pmatrix}\), so \(A_{(2,m)} = \cdots = A_{(m-1,m)} = 0\) follows from \(A_{(2,1)} = \cdots = A_{(m-1,1)} = 0\). \(\square\)
Lemma 5. Let \( m \) be any positive integer \( \leq d + 1 \). Then \( A_{(1,j)} = 0 \) for \( j \in \{1, \ldots, m\} \).

Proof. We use induction on \( m \). If \( m = 1 \) then trivial. Suppose that the statement holds for \( m - 1 \). Due to (1.1), \( J_{2d-m,m-2} \) is the coefficient of \( x^{2d-m}y^{m-2} \) in \( \frac{\partial f_d}{\partial x} \frac{\partial g_d}{\partial y} \). Looking at

\[
\frac{\partial f_d}{\partial x} = 1 \cdot dc_{d,1}x^{d-1} + 2(d-1)c_{d-1,2}x^{d-2}y + 3(d-2)c_{d-2,3}x^{d-3}y^2 + \cdots ,
\]

\[
-\frac{\partial g_d}{\partial y} = 1 \cdot dc_{d,1}x^{d-1} + 2(d-1)c_{d-1,2}x^{d-2}y + 3(d-2)c_{d-2,3}x^{d-3}y^2 + \cdots ,
\]

\[
\frac{\partial f_d}{\partial y} = 1 \cdot 2c_{d-1,2}x^{d-1} + 2 \cdot 3c_{d-2,3}x^{d-2}y + 3 \cdot 4c_{d-3,4}x^{d-3}y^2 + \cdots ,
\]

\[
-\frac{\partial g_d}{\partial x} = (d+1)c_{d+1,0}x^{d-1} + (d-1)dc_{d,1}x^{d-2}y + (d-2)(d-1)c_{d-1,2}x^{d-3}y^2 + \cdots ,
\]

we see that

\[
J_{2d-m,m-2} = -\sum_{i=1}^{m-1} (d - i + 1)(m - i)A_{(i,m-i+1)}
\]

By induction and Lemma 4, we have \( A_{(2,m-1)} = \cdots = A_{(m-1,2)} = 0 \). Since \( J_{2d-m,m-2} = 0 \), we obtain \( A_{(1,m)} = 0 \).

Now we will prove that \( 2 \times 2 \) minors of \( (2.2) \) are all 0. Fix two distinct integers \( p, q \in \{1, \ldots, n-1\} \). Let \( B_{(i,j)} \) be the determinant of the matrix obtained by concatenating the \( i \)-th column in \( A_p \) and \( j \)-th column in \( A_q \), that is,

\[
B_{(i,j)} = \det \left( \begin{array}{cccc}
(c_{d-p-i+1,i}^{i}) & \cdots & \cdots & \cdots \\
(d+p-2-i)c_{d-p-i+2,i-1} & \cdots & \cdots & \cdots \\
\end{array} \right).
\]

For simplicity, let \( d = d_p \) and \( e = d_q \).

Lemma 6. \( B_{(d+1,e+1)} = B_{(1,1)} = 0 \).

Proof. This is because

\[
B_{(d+1,e+1)}^2 = ((d+1)c_{0,d+1,c_1,e} - (e+1)c_{1,d}c_{0,e+1})^2
\]

\[
= \left( ec^{2}_{1,e} - 2(e + 1)c_{0,e+1}c_{2,e-1} \right) \frac{(d+1)^2}{e} c^{2}_{0,d+1}
\]

\[
+ \left( dc^{2}_{1,d} - 2(d+1)c_{0,d+1}c_{2,d-1} \right) \frac{(e+1)^2}{d} c^{2}_{0,e+1}
\]

\[
+ (2(d + 1))dc_{0,d+1}c_{2,e-1} - 2dec_{1,d}c_{1,e} + 2(e+1)ec_{2,d-1}c_{0,e+1} \frac{(d+1)(e+1)}{de} c_{0,d+1}c_{0,e+1}
\]

\[
= (A_p)_{(e+1)} \frac{(d+1)^2}{e} c^{2}_{0,d+1} + (A_p)_{(d+1)} \frac{(e+1)^2}{d} c^{2}_{0,e+1} + J_{0,d+e-2} \frac{(d+1)(e+1)}{de} c_{0,d+1}c_{0,e+1}
\]

\[
= 0,
\]

where the last equality follows from Proposition 3 and \( J_{0,d+e-2} = 0 \). Similarly (by symmetry of indices), we obtain \( B_{(1,1)} = 0 \).
For any \( m \in \{1, \ldots, e+1\} \), it is elementary to check that all \( 2 \times 2 \) minors of

\[
\begin{pmatrix}
 dB_{(d+1,m)} & B_{(d,m)} \\
 (d-1)B_{(d,m)} & 2B_{(d-1,m)} \\
 (d-2)B_{(d-1,m)} & 3B_{(d-2,m)} \\
 \vdots & \vdots \\
 B_{(2,m)} & dB_{(1,m)}
\end{pmatrix}
\]

are equal to 0. For example,

\[
2dB_{(d+1,e+1)}B_{(d-1,e+1)}
= 2d((d+1)c_{0,d+1}c_{1,e} - (e+1)c_{1,d}c_{0,e+1})((d-1)\cdots - (e+1)c_{3,d-2}c_{0,e+1})
= d(d-1)(2d+1)c_{0,d+1}c_{2,d-1} - dc_{1,d}c_{1,e}c_{1,e}
+ (d-1)(dc_{1,d}c_{1,e} - 2(e+1)c_{2,d-1}c_{0,e+1})(dc_{1,d}c_{1,e} - 2(e+1)c_{2,d-1}c_{0,e+1})
+ 2d(e+1)(d-1)c_{2,d-1}c_{1,d} - 3(d+1)c_{3,d-2}c_{0,e+1}+1c_{1,e}
+ 2(e+1)^2(3dc_{1,d}c_{3,d-2} - 2(d-1)c_{2,d-1}c_{2,d-1})c_{0,e+1}c_{0,e+1}
= -d(d-1)(A_p)(d,d+1)c_{1,e}c_{1,e}
+ 2d(e+1)B_{(d,e+1)}B_{(d,e+1)}
+ 2d(e+1)^2(A_p)(d-1,d)c_{0,e+1}c_{1,e}
- 2(e+1)^2A_p(d-1,d)c_{0,e+1}c_{0,e+1}
\]

implies that \( 2dB_{(d+1,e+1)}B_{(d-1,e+1)} - (d-1)B_{(d,e+1)}B_{(d,e+1)} = 0 \), which is a consequence of Proposition 3.

Then \( B_{(d+1,e+1)} = 0 \) implies that \( B_{(i,e+1)} = 0 \) for all \( 2 \leq i \leq d \). Note that the coefficients of \( y^{d+e-2} \) in \( \left( \frac{\partial f_i}{\partial x} \frac{\partial g_j}{\partial y} - \frac{\partial f_j}{\partial y} \frac{\partial g_i}{\partial x} \right) \) and \( \left( \frac{\partial f_i}{\partial y} \frac{\partial g_j}{\partial y} - \frac{\partial f_j}{\partial y} \frac{\partial g_i}{\partial y} \right) \) are the only ones that contribute to \( J_{0,d+e-2} \), because of \( \text{[1]} \). Then we can see that \( J_{0,d+e-2} \) is a linear combination of \( B_{(d+1,e)} \) and \( B_{(d,e+1)} \) with nonzero coefficients, so we get \( B_{(d+1,e)} = 0 \). Then this implies that \( B_{(i,e)} = 0 \) for all \( 2 \leq i \leq d \). Since \( J_{1,d+e-3} \) is a linear combination of \( B_{(d+1,e-1)}, B_{(d,e)} \) and \( B_{(d-1,e+1)} \) with nonzero coefficients, we get \( B_{(d+1,e-1)} = 0 \). Repeating this argument, we get \( B_{(i,j)} = 0 \) for all \( 2 \leq i \leq d + 1 \) and \( 1 \leq j \leq e + 1 \). Similarly (by symmetry of indices), we obtain \( B_{(1,j)} = 0 \) for all \( 1 \leq j \leq e + 1 \). The proof is completed.

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