On the exact learnability of graph parameters: The case of partition functions

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Abstract

We study the exact learnability of real valued graph parameters \( f \) which are known to be representable as partition functions which count the number of weighted homomorphisms into a graph \( H \) with vertex weights \( \alpha \) and edge weights \( \beta \). M. Freedman, L. Lovász and A. Schrijver have given a characterization of these graph parameters in terms of the \( k \)-connection matrices \( C(f,k) \) of \( f \). Our model of learnability is based on D. Angluin’s model of exact learning using membership and equivalence queries. Given such a graph parameter \( f \), the learner can ask for the values of \( f \) for graphs of their choice, and they can formulate hypotheses in terms of the connection matrices \( C(f,k) \) of \( f \). The teacher can accept the hypothesis as correct, or provide a counterexample consisting of a graph. Our main result shows that in this scenario, a very large class of partition functions, the rigid partition functions, can be learned in time polynomial in the size of \( H \) and the size of the largest counterexample in the Blum-Shub-Smale model of computation over the reals with unit cost.

1 Introduction

A graph parameter \( f : \mathcal{G} \rightarrow \mathcal{R} \) is a function from all finite graphs \( \mathcal{G} \) into a ring or field \( \mathcal{R} \), which is invariant under graph isomorphisms.

In this paper we initiate the study of exact learnability of graph parameters with values in \( \mathcal{R} \), which is assumed to be either \( \mathbb{Z}, \mathbb{Q} \) or \( \mathbb{R} \). As this question seems new, we focus here on the special case of graph parameters given as partition functions, \[10, 13]. We adapt the model of exact learning introduced by D. Angluin \[1\]. Our research extends the work of \[3, 11\], where exact learnability of languages (set of words or labeled trees) recognizable by multiplicity automata (aka weighted automata) was studied, to graph parameters with values in \( \mathcal{R} \).

1.1 Exact learning

In each step, the learner may make membership queries \( \text{VALUE}(x) \) in which they ask for the value of the target \( f \) on specific input \( x \). This is the analogue of the membership queries

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1.2 Formulating a hypothesis

To make sense one has to specify the formalism (language) \( \Sigma \) in which a hypothesis has to be formulated. It will be obvious in the sequel, that the restriction imposed by the choice of \( \Sigma \) will determine whether \( f \) is learnable or not.

Let us look at the seemingly simpler case of learning integer functions \( f : \mathbb{Z} \to \mathbb{Z} \) or integer valued functions of words \( w \in \Sigma^* \) over an alphabet \( \Sigma \).

(i) If \( f \) can be any function \( f : \mathbb{Z} \to \mathbb{Z} \) or \( f : \Sigma^* \to \mathbb{Z} \), there are uncountably many candidate functions as hypotheses, and no finitary formalism \( \Sigma \) is suitable to formulate a hypothesis.

(ii) If \( f \) is known to be a polynomial \( p(X) = \sum_i a_i X^i \in \mathbb{Z}[X] \), we can formulate the hypothesis as a vector \( a = (a_1, \ldots, a_m) \) in \( \mathbb{Z}^m \). Learning is successful if the learner finds the hypothesis \( h = a \) in the required time. Here Lagrange interpolation will be used to formulate the hypotheses.

(iii) If \( f \) is known to satisfy some recurrence relation, the hypothesis will consist of the coefficients and the length of the recurrence relation, and exact learnability will depend on the class of recurrence relations one has in mind.

(iv) If \( f : \Sigma^* \to \mathbb{Z} \) is a word function recognizable by a multiplicity automaton \( MA \), the hypotheses are given by the weighted transition tables of \( MA \), cf. [3].

Looking now at a graph parameter \( f : G \to R \) what can we expect? Again we have to restrict our treatment to a class of parameters where each member can be described by a finite string in a formalism \( \Sigma \).

We illustrate the varying difficulty of the learning problem with the example of the chromatic polynomial \( \chi(G; X \in \mathbb{N}[X]) \) for a graph \( G \). For \( X = k \), the evaluation of \( \chi(G; k) \) counts the number of proper colorings of \( G \) with at most \( k \) colors. It is well known that for fixed \( G \), \( \chi(G; k) \) is indeed a polynomial in \( k \); [4, 7]. A graph parameter \( f \) is a chromatic invariant over \( R \) if

(i) it is multiplicative, i.e., for the disjoint union \( G_1 \sqcup G_2 \) of \( G_1 \) and \( G_2 \), it holds that \( f(G_1 \sqcup G_2) = f(G_1) \cdot f(G_2) \), and

(ii) there are \( \alpha, \beta, \gamma \in R \) such that \( f(G) = \alpha \cdot f(G-\epsilon) + \beta \cdot f(G/\epsilon) \) and \( f(K_1) = \gamma \).

\( K_n \) denotes the complete graph on \( n \) vertices, and \( G-\epsilon \) and \( G/\epsilon \) are, respectively, the graphs obtained from deleting the edge \( e \) from \( G \) and contracting \( e \) in \( G \).

The parameter \( \chi(G; k) \) is a chromatic invariant with \( \alpha = 1, \beta = -1 \) and \( \gamma = k \). Finally, \( \chi(G; k) \) has an interpretation by counting homomorphisms:

\[ \chi(G; m) = \sum_{t : G \to K_m} 1, \]

This is a special case of the homomorphism counting function for a fixed graph \( H \):

\[ \text{hom}(G, H) = \sum_{t : G \to H} 1, \]

where \( t \) is a homomorphism \( t : G \to H \).

Now, let a graph parameter \( f : G \to R \) be the target of a learning algorithm.
Theorem 1

(i) If \( f \) is known to be an instance of \( \chi(G;X) \), a hypothesis consists of a value \( X = a \).
But in this case we know that \( \chi(K_1;X) = X \), so it suffices to ask for \( f(K_1) = a \).

(ii) If \( f \) is known to be a chromatic invariant, the hypothesis consists of the triple \( (\alpha, \beta, \gamma) \).
In this case a hypothesis can be computed from the values of \( f(P_m) \) for undirected paths \( P_m \) for sufficiently many values of \( m \).

(iii) If \( f \) is known to be an instance of \( \text{hom}(\cdot;H) \), a hypothesis would consist of a target graph \( H \).

1.3 Counting weighted homomorphisms aka partition functions

A weighted graph \( H(\alpha, \beta) \) is a graph \( H = (V(H), E(H)) \) on \( n = |V(H)| \) vertices together
with a vertex weight function \( \alpha : V(H) \to \mathbb{R} \), viewed as a vector of length \( n \), and an edge
weights function \( \beta : V(H)^2 \to \mathbb{R} \) viewed as an \( n \times n \) matrix, with \( \beta(u, v) = 0 \) if \( (u, v) \not\in E(H) \).

A partition function \( \text{hom}(\cdot,H(\alpha, \beta)) \) is the generalization of \( \text{hom}(\cdot,H) \) to weighted
graphs, whose value on a graph \( G \) is defined as follows:

\[
\text{hom}(G,H(\alpha, \beta)) = \sum_{t : G \to H} \prod_{v \in V(G)} \alpha(t(v)) \prod_{(u,v) \in E(G)^2} \beta(t(u), t(v))
\]

To illustrate the notion of a partition function, let \( H_{\text{indep}} \) be the graph with two vertices
\( \{u, v\} \) and the edges \( \{(u, v), (u, u)\} \), shown in Figure 1. Let \( \alpha(u) = 1, \alpha(v) = X \) and
\( \beta(u, v) = 1, \beta(u, u) = 1 \). Then \( \text{hom}(\cdot,H_{\text{indep}}(\alpha, \beta)) \) is the independence polynomial,

\[
\text{hom}(G,H_{\text{indep}}(\alpha, \beta)) = I(G;X) = \sum_j \text{ind}_j(G) X^j
\]

where \( \text{ind}_j(G) \) is the number of independent sets of size \( j \) in the graph \( G \).

We say a partition function \( \text{hom}(\cdot,H(\alpha, \beta)) \) is rigid aka asymmetric \footnote{In the literature \( \text{hom}(\cdot,H(\alpha, \beta)) \) is also denoted by \( Z_{H(\alpha, \beta)}(G) \), e.g., in \cite{15}. We follow the notation \cite{13}.}, if \( H \) has no
proper automorphisms. Note that automorphisms in a weighted graph also respect vertex
and edge weights. In our examples above, the evaluations of the independence polynomial are rigid partition functions, whereas the evaluations of the chromatic polynomial are not.

It is known that almost all graphs are rigid:

**Theorem 1** \cite{9,14}. Let \( G \) be a uniformly selected graph on \( n \) vertices. The probability
that \( G \) is rigid tends to 1 as \( n \to \infty \).

If the target \( f \) is known to be a (rigid) partition function \( \text{hom}(\cdot,H(\alpha, \beta)) \) then the
hypothesis consists of a (rigid) weighted graph \( H(\alpha, \beta) \).

In Section 2 we give the characterization of rigid and non-rigid partition functions from
\cite{10,14,13} in terms of connection matrices.

For technical reasons discussed in Section 3 in this paper we deal only with the learnability of rigid partition functions, and leave the general case to future work.

\footnote{Some authors say \( G \) is asymmetric if \( G \) has no proper automorphisms, and \( G \) is rigid if \( G \) has no proper endomorphisms, \cite{15}. Wikipedia uses rigid as we use it here.}
1.4 Main result

Our main result can now be stated:

**Theorem 2.** Let $f$ be a graph parameter which is known to be a rigid partition function $f(G) = \text{hom}(G,H(\alpha,\beta))$. Then $f$ can be learned in time polynomial in the size of $H$ and the size of the largest counterexample in the Blum-Shub-Smale model of computation over the reals with unit cost.

**Remark 3.** If $f$ takes values in $\mathbb{Q}$ rather than in $\mathbb{R}$ we can also work in the Turing model of computation with logarithmic cost for the elements in $\mathbb{Q}$.

To prove Theorem 2 we will use the characterization of rigid partition functions in terms of connection matrices, [13, Theorem 5.54], stated as Theorems 4 and Corollary 6 in Section 2. The difficulty of our result lies not in finding a learning algorithm by carefully manipulating the counterexamples to meet the complexity constraints, but in proving the algorithm correct. In order to do this we had to identify and extract the suitable algebraic properties underlying the proof of Theorem 4 and Corollary 6.

The learning algorithm is given in pseudo-code as Algorithm 1. It maintains a matrix $M$ used in the generation of the hypothesis $h$ from $\text{VALUE}$ and $\text{EQUIVALENT}$ query results. After an initial setup of $M$, in each iteration the algorithm generates a hypothesis $h$, queries the teacher for equivalence between $h$ and the target and either terminates, or updates $M$ accordingly and moves on to the next iteration.

**Algorithm 1 Learning algorithm for rigid partition functions**

1: $n = 1$
2: while True do
3:   augment $M$ with $(B_n)$
4:   $P = \text{find basis}(M)$
5:   $h = \text{generate hypothesis}(P)$
6:   if $\text{EQUIVALENT}(h) = \text{YES}$ then
7:     return $h$
8:   else
9:     $n = n + 1$
10:    $B_n = \text{EQUIVALENT}(h)$ \> $B_n$ receives a counterexample
11:  end if
12:  end while

It uses three black-boxes; $\text{find basis}$ which uses $M$ to find a certain basis $P$ of a graph algebra associated with the target function (see Section 2), $\text{generate hypothesis}$ which uses this basis and $\text{VALUE}$ queries to construct a hypothesis $h$, and $\text{augment M}$ which augments the matrix $M$ after a counterexample is received, using $\text{VALUE}$ queries.

We briefly overview the complexity of the algorithm to illustrate that rigid partition functions are indeed exactly learnable. Proofs of validity and detailed analysis of the complexity are given in later sections. For a target $H(\alpha,\beta)$ on $q$ vertices, the procedure $\text{find basis}$ solves $O(q)$ systems of linear equations, and systems of linear matrix equations, all of dimension $O(\text{poly}(q))$. The procedure $\text{generate hypothesis}$ performs $O(q)$ graph operations of polynomial time complexity on graphs of size $O(\text{poly}(q,|x|))$, where $|x|$ is the size of the largest counterexample, and $O(q^2)$ $\text{VALUE}$ queries. The procedure $\text{augment M}$ performs $O(q)$ $\text{VALUE}$ queries. Thus, each iteration takes time $O(\text{poly}(q,|x|))$. Lemma 18 will show that there are $O(q)$ iterations, so the total run time of the algorithm is polynomial in the size $q$ of $H(\alpha,\beta)$ and the size $|x|$ of the largest counterexample.

**Organization** In Section 2 we give the necessary background on partition functions and the graph algebras induced by them. Section 3 presents the algorithm in detail and in Section 4 we prove its validity and analyze its time complexity. We discuss the results and future work in Section 5. Some of the more technical proofs appear in Appendix A.
2 Preliminaries

Let $k \in \mathbb{N}$. A $k$-labeled graph $G$ is a finite graph in which $k$ vertices, or less, are labeled with labels from $[k] = \{1, \ldots, k\}$. We denote the class of $k$-labeled graphs by $\mathcal{G}_k$. The $k$-connection of two $k$-labeled graphs $G_1, G_2 \in \mathcal{G}_k$ is given by taking the disjoint union of $G_1$ and $G_2$ and identifying vertices with the same label. This produces a $k$-labeled graph $G = G_1G_2$. Note that $k$-connections are commutative.

2.1 Quantum graphs

A formal linear combination of a finite number of $k$-labeled graphs $F_i \in \mathcal{G}_k$ with coefficients from $\mathbb{R}$ is called a $k$-labeled quantum graph. $\mathcal{Q}_k$ denotes the set of $k$-labeled quantum graphs.

Let $x, y$ be $k$-labeled quantum graphs: $x = \sum_{i=1}^n a_i F_i$, and $y = \sum_{i=1}^n b_i F_i$. Note that some of the coefficients may be zero. $\mathcal{Q}_k$ is an infinite dimensional vector space, with the operations: $x + y = (\sum_{i=1}^n a_i F_i) + (\sum_{i=1}^n b_i F_i) = \sum_{i=1}^n (a_i + b_i) F_i$, and $\alpha \cdot x = \sum_{i=1}^n (\alpha a_i) F_i$.

$k$-connections extend to $k$-labeled quantum graphs by $xy = \sum_{i,j=1}^n (a_i b_j) (F_i, F_j)$. Any graph parameter $f$ extends to $k$-labeled quantum graphs linearly: $f(x) = \sum_{i=1}^n a_i f(F_i)$.

2.2 Equivalence relations for quantum graphs

The $k$-connection matrix $C(f, k)$ of a graph parameter $f : G \to \mathbb{R}$ is a bi-infinite matrix over $\mathbb{R}$ whose rows and columns are labeled with $k$-labeled graphs, and its entry at the row labeled with $G_1$ and the column labeled with $G_2$ contains the value of $f$ on $G_1G_2$:

$$C(f, k)_{G_1, G_2} = f(G_1G_2).$$

Given a connection matrix $C(f, k)$, we associate with a $k$-labeled graph $G \in \mathcal{G}_k$, the (infinite) row vector $R_G \in \mathcal{Q}_k$ appearing in the row labeled by $G$ in $C(f, k)$. If $k$ is clear from context we write $R_G$. Similarly, we associate an infinite row vector $R_F$ with $k$-labeled quantum graphs $x = \sum_{i=1}^n a_i F_i$, defined as $R_x = \sum_{i=1}^n a_i R_{F_i}$, where $R_{F_i}$ is the row in $C(f, k)$ labeled by the $k$-labeled graph $F_i$.

We say $C(f, k)$ has finite rank if there are finitely many $k$-labeled graphs $\mathcal{B}_{C(f, k)} = \{B_1, \ldots, B_n\}$ whose rows $R_{C(f, k)} = \{R_{B_1}, \ldots, R_{B_n}\}$ linearly span $C(f, k)$. Meaning, for any $k$-labeled graph $G$, there exists a linear combination of the rows in $\mathcal{R}_{C(f, k)}$ which equals the row vector $R_G$. We say that $C(f, k)$ has rank $n$ and denote $r(f, k) = n$ if any set of less than $n$ graphs does not linearly span $C(f, k)$.

The main result we use is the characterization of partition functions in terms of connection matrices. We do not need its complete power, so we state the relevant part:

**Theorem 4** (Freedman, Lovász, Schrijver, 1984). Let $f$ be a graph parameter that is equal to $\text{hom}(-, H(\alpha, \beta))$ for some $H(\alpha, \beta)$ on $q$ vertices. Then $r(f, k) \leq q^k$ for all $k \geq 0$.

The exact rank $r(f, k)$ was characterized in 1984, but first we need some definitions. A weighted graph $H(\alpha, \beta)$ is said to be twin-free if $\beta$ does not contain two separate rows that are identical to each other $^3$ Let $H(\alpha, \beta)$ be a weighted graph on $q$ vertices, and let $\text{Aut}(H(\alpha, \beta))$ be the automorphism group of $H(\alpha, \beta)$. $\text{Aut}(H(\alpha, \beta))$ acts on ordered $k$-tuples of vertices $[q]^k = \{\phi : [k] \to [q]\}$ by $(\sigma \circ \phi)(i) = \sigma(\phi(i))$ for $\sigma \in \text{Aut}(H(\alpha, \beta))$. The orbit of $\phi$ is the set of ordered $k$-tuples $\psi$ of vertices such that $\sigma \circ \phi = \psi$ for an automorphism $\sigma \in \text{Aut}(H(\alpha, \beta))$. The number of orbits of $\text{Aut}(H(\alpha, \beta))$ on $[q]^k$ is the number of different orbits for elements $\phi \in [q]^k$.

**Theorem 5** (Lovász, 1984). Let $f = \text{hom}(-, H(\alpha, \beta))$ for a twin-free weighted graph $H(\alpha, \beta)$ on $q$ vertices. Then $r(f, k)$ is equal to the number of orbits of $\text{Aut}(H(\alpha, \beta))$ on $[q]^k$ for all $k \geq 0$.

$^3$ If $H(\alpha, \beta)$ has twin vertices, they can be merged into one vertex by adding their vertex weights without changing the partition function. As the size of the target representation is the smallest possible, we assume all targets are twin-free.
We use the special case:

**Corollary 6.** Let \( f = \text{hom}(\beta, H(\alpha, \beta)) \) for a rigid twin-free weighted graph \( H(\alpha, \beta) \) on \( q \) vertices. Then \( r(f, k) = q^k \) for all \( k \geq 0 \).

We define an equivalence relation \( \equiv_{f,k} \) over \( \mathcal{Q}_k \) where two \( k \)-labeled quantum graphs \( x \) and \( y \) are in the same equivalence class if and only if the infinite vectors \( R_x \) and \( R_y \) are identical: \( x \equiv_{f,k} y \leftrightarrow R_x^k = R_y^k \). Note that the set \( \mathcal{Q}_k / f \) of equivalence classes of \( x \equiv_{f,k} y \) is exactly the vector space \( \text{span}(C(f, k)) \) generated by linear combinations of rows in \( C(f, k) \).

Thus, if \( r(f, k) = n \) with spanning rows \( \mathcal{R}_{C(f,k)} = \{R_{B_1}, \ldots, R_{B_n}\} \), they form a basis of \( \mathcal{Q}_k / f = \text{span}(C(f, k)) \). For brevity, we occasionally also refer to \( \mathcal{B}_{C(f,k)} \) as a basis.

Let \( x \) be a \( k \)-labeled quantum graph whose equivalence class \( R_x \) is given as the linear combination \( R_x = \sum_{i=1}^n \gamma_i R_{B_i} \). We call the column vector \( \bar{e}_x = (\gamma_1, \ldots, \gamma_n)^T \) the coefficients vector of \( x \), or representation of \( x \) using \( \mathcal{B}_{C(f,k)} \).

### 3 The learning algorithm in detail

In this section we present the learning algorithm in full detail. The commentary in this exposition foreshadows the arguments in Section 4, but otherwise validity is not considered here. We do not address complexity concerns in this section either, however, we reiterate for the sake of clarity that the algorithm runs on a Blum-Shub-Smale machine, \([6, 5]\), over the reals. In such a machine, real numbers are treated as atomic objects; they are stored in single cells, and arithmetic operations are performed on them in a single step.

The objects the algorithm primarily works with are real matrices. In a context containing a basis \( \mathcal{B}_{C(f,k)} \), we associate a real matrix \( A_x \) with each quantum graph \( x \) such that the following holds.

The coefficients vector \( \bar{e}_{xy} \) of \( xy \) using \( \mathcal{B}_{C(f,k)} \) is given by \( A_x \bar{e}_y \). (*)

This device, as we will see in Section 4, will allow the algorithm to search for, and find, special quantum graphs that provide a translation of the answers of VALUE and EQUIVALENT queries into a hypothesis.

As mentioned earlier, Algorithm 1 maintains a matrix \( M \) which is a submatrix of \( C(f, 1) \). In each iteration the algorithm generates a hypothesis \( h = (\alpha^{(h)}, \beta^{(h)}) \) using \( M \), and queries the teacher for equivalence between \( h \) and the target \( f \). If the hypothesis is correct, the algorithm returns \( h \), otherwise it augments \( M \) with a 1-labeled version of the counterexample, and moves on to the next iteration.

**Remark 7.** Strictly speaking, the teacher may be asked VALUE queries on (unlabeled) graphs, however, we freely write VALUE(G) for \( k \)-labeled graphs \( G \in \mathcal{Q}_k \). Additionally, the algorithm will need to know the value of the target on some quantum graphs. Since any graph parameter extend to quantum graphs linearly, for a quantum graph \( x = \sum_{i=1}^n a_i F_i \) we write VALUE(\( x \)) as shorthand for \( \sum_{i=1}^n a_i \cdot \text{VALUE}(F_i) \) throughout the presentation.

**Incorporating counterexamples** The objective is to keep a non-singular submatrix \( M \) of \( C(f, 1) \). The first 1-labeled graph \( B_1 \) with which \( M \) is augmented is some arbitrarily chosen 1-labeled graph.

Upon receiving a \( B_n \) graph as counterexample, the 1-label is arbitrarily assigned to one of its vertices, making it a 1-labeled graph. Then augment \( M \) with \( B_n \) adds a row and a column to \( M \) labeled with the (now) 1-labeled graph \( B_n \), and fills their entries with the values \( f(B_n B_i) = f(B_i B_n) \), for \( i \in [n] \), using VALUE queries.

The other functions are slightly more complex.
Finding an idempotent basis  The function find basis, given in pseudo-code as Algorithm 2, receives as input the matrix $M$. For reasons which will become apparent later, we are interested in finding a certain (idempotent) basis of the linear space generated by the rows of $C(f, 1)$. For this purpose, in its first part find basis iteratively, over $k = 1, \ldots, n$, computes the entries of matrices $A_k$ as in (*), where $x$ are $B_i$, $i \in [n]$, by solving multiple systems $Mx = b$ of linear equations, and using the solutions $\Gamma$ of those systems to fill the entries of the matrices $A_{B_i}$, where the $(k,j)$ entry of $A_{B_i}$ is $\gamma^j(k)$. Let $p_i$, $i \in [n]$ be those quantum graphs for which $A_{p_i}$ is the $n \times n$ matrix with the value 1 in the entry $(i,i)$ and zero in all other entries. Note that the matrices $A_{p_i}$, $i \in [n]$ are linearly independent. We will see that $p_i$, $i \in [n]$ are the idempotent basis, now we wish to find their representation using $B_i$, $i \in [n]$.

For $i \in [n]$, the representation $c_{p_i}$ of the basic idempotent $p_i$ using the basis elements $B_i$, $i \in [n]$ is found by solving a system $AX = A_{p_i}$ of linear matrix equations, where $A$ is a block matrix whose blocks are the matrices $A_{B_i}$, $i \in [n]$. Each solution is added to $\Delta$.

Finally, find basis outputs the set $\Delta$ of these representations $c_{p_i}$, $i \in [n]$. Then we have that $B_{p_i} = \sum_{k=1}^n c_{p_i}(k)B_{B_k}$, where $c_{p_i}$ is the coefficients vector of $p_i$ using $B_i$, $i \in [n]$. The representations $c_{p_i} \in \Delta$ of the elements $p_i$, $i \in [n]$, are what will provide a translation from results of VALUE queries to weights.

Algorithm 2 find basis function

1: $\Gamma = \emptyset$
2: for each $i, j \in [n]$ do
3:   for $k = 1, \ldots, n$ do
4:     $b(k) = \text{VALUE}(B_iB_jB_k)$
5:   end for
6:   $\gamma^j = \text{solve linear system}(MX = b)$
7: $\Gamma = \Gamma \cup \{\gamma^j\}$
8: end for
9: for $i \in [n]$ do
10:   $A_{B_i} = \text{fill matrix}(i, \Gamma)$
11:   $A = \text{add block}(A, i, A_{B_i})$ /* $A$ is a block matrix with $A_{B_i}$ on its $i$th block*/
12: end for
13: $\Delta = \emptyset$
14: for $i \in [n]$ do
15:   $c_{p_i} = \text{solve linear matrix system}(AX = A_{p_i})$
16:   $\Delta = \Delta \cup \{c_{p_i}\}$
17: end for
18: return $\Delta$

Generating a hypothesis  The function generate hypothesis, given in pseudo-code as Algorithm 3, receives as input the representations $c_{p_i}$ of the 1-labeled quantum graphs $p_i$, $i \in [n]$, which it uses to find the entries of the vertex weights vector $\alpha^{(h)}$ directly through VALUE queries.

Then generate hypothesis finds the 2-labeled analogues of these 1-labeled quantum graphs. Those 2-labeled analogues form a basis of of $Q_2/f$.

Denote by $K_2$ the 2-labeled graph composed of a single edge with both vertices labeled. Next, generate hypothesis finds the representation of $R_{K_2}$, that is the row labeled with $K_2$ in $C(f, 2)$, using the basis $R_{C(f, 2)}$. We find the representation of this specific graph $K_2$ as the coefficients in $c_{K_2}$ constitute the entries of the edge weights matrix $\beta^{(h)}$ (see Section 4).

This representation is found by solving a linear system of equations, similarly to how find basis uses solve linear system, but here we use the diagonal matrix $N$ whose entries correspond to the elements of $B_{C(f, 2)}$.

The solution of said system, i.e., the coefficients vector $c_{K_2}$ of $K_2$, is used to fill the edge
weights matrix $\beta^{(h)}$. If needed, $\beta^{(h)}$ is made twin-free by contracting the twin vertices into one and summing their weights in $\alpha^{(h)}$.

Finally, generate hypothesis returns the hypothesis $h = (\alpha^{(h)}, \beta^{(h)})$ as output.

**Algorithm 3 generate hypothesis function**

1: for each $i \in [n]$ do
2: $\alpha^{(h)}(i) = \text{VALUE}(p_i)$
3: end for
4: $N = 0^{n^2 \times n^2}$ → $N$ is a zero matrix of dimensions $n^2 \times n^2$.
5: for $i = 1, \ldots, n$ do
6: for $j = 1, \ldots, n$ do
7: $p_{ij} = p_i \otimes p_j$ → See Remark 8
8: $N_{p_{ij}, p_{ij}} = \text{VALUE}(p_{ij} p_{ij})$
9: $b(ij) = \text{VALUE}(K_2 p_{ij})$
10: end for
11: end for
12: $\beta^{(h)} = \text{solve linear system}(N x = b)$
13: make twin-free($\alpha^{(h)}, \beta^{(h)}$)
14: $h = (\alpha^{(h)}, \beta^{(h)})$
15: return $h$

**Remark 8 (Algorithm 3).** Let $q_i$ be the 1-labeled quantum graph $p_i$ interpreted as a 2-labeled quantum graph, and let $q_j$ be $p_j$ with the labels of its components renamed to 2, and also interpreted as a 2-labeled quantum graph. The result of $p_i \otimes p_j$ is the 2-labeled quantum graph $q_i \sqcup 2 q_j$.

4 Validity and complexity

As stated earlier, a class of functions is exactly learnable if there is a learner that for each target function $f$, outputs a hypothesis $h$ such that $f$ and $h$ identify on all inputs, and does so in time polynomial in the size of a shortest representation of $f$ and the size of a largest counterexample returned by the teacher. The proof of Theorem 2 argues that Algorithm 1 is such a learner for the class of rigid partition functions, through Theorem 9, which proves validity, and Theorem 22, which proves the complexity constraints are met.

To prove validity, we first state existing results on properties of graph algebras induced by partition functions, then show, through somewhat technical algebraic manipulations, how our algorithm successfully exploits these properties to generate hypotheses. We then show our algorithm eventually terminates with a correct hypothesis.

For the rest of the section, let $H(\alpha, \beta)$ be a rigid twin-free weighted graph on $q$ vertices, and denote $f = \text{hom}(\cdot, H(\alpha, \beta))$.

**Theorem 9.** Given access to a teacher for $f$, Algorithm 1 outputs a hypothesis $h$ such that $f(G) = h(G)$ for all graphs $G \in G$.

The proof of the theorem follows from arguing that:

**Theorem 10.** If $M$ is of rank $q$, then generate hypothesis outputs a correct hypothesis.

and that the rank of $M$ is incremented with every counterexample:

**Theorem 11.** In the $n^{th}$ iteration of Algorithm 1 on $f$, $M$ has rank $n$.

First we confirm the hypotheses Algorithm 1 generates are indeed in the class of graph parameters we are trying to learn, namely, rigid partition functions $\text{hom}(\cdot, H(\alpha, \beta))$ for twin-free weighted graphs $H(\alpha, \beta)$.  

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Given Theorem 11 for the hypothesis $h$ returned in the $n^{th}$ iteration, the rank of $C(h, 1)$ is at least $n$, since $M$ is a submatrix of $C(h, 1)$. Thus, from Theorem 5 $h$ cannot have proper automorphisms, as it would imply that the rank of $C(h, 1) < n$. The fact that $h$ is twin-free is immediate from the construction in generate hypothesis.

4.1 From the idempotent bases to the weights - proof of Theorem 10

Let $Q_{k}/f$ be of finite dimension $n$. The idempotent basis $p_1, \ldots, p_n$ of $Q_{k}/f$ consists of those $k$-labeled quantum graphs $p_i$ for which $p_i p_i \equiv f, k p_i$ and $p_i p_j \equiv f, k 0$ for $i, j \in [n], i \neq j$. Recall how find basis found those 1-labeled quantum graphs $p_i, i \in [n]$ whose matrices $A_p$, behaved in this way.

In our setting of rigid twin-free weighted graphs, by Chapter 6, we have that if $p_1, \ldots, p_n$ are the idempotent basis of $Q_1/f$, then the idempotent basis of $Q_2/f$ is given by $p_i \otimes p_j$, $i, j \in [q]$. These are the 2-labeled analogues mentioned in the description of generate hypothesis.

Furthermore by Chapter 6, the vertex weights $\alpha$ of $H$ are given by $\alpha(i) = f(p_i), i \in [q]$, and if the representation of $K_j$ using $p_i \otimes p_j$, $i, j \in [q]$ is $\sum_{i,j \in [q]} \beta_{ij}(p_i \otimes p_j)$, then the edge weights matrix $\beta$ is given by $\beta_{i,j} = \beta_{ij}$.

Equipped with these useful facts, we show that:

**Lemma 12.** If $M$ is of rank $q$, then find basis outputs the idempotent basis of $Q_1/f$.

Then obtain Theorem 10 by showing how, if generate hypothesis receives the idempotent basis of $Q_1/f$ as input, it outputs a correct hypothesis.

**Finding the idempotent basis - proof of Lemma 12** Recall that in the presence of a basis $B_{C(f,k)}$ we associate a real matrix $A_x$ with each quantum graph $x$ such that the following holds.

The coefficients vector $\bar{e}_{xy}$ of $xy$ using $B_{C(f,k)}$ is given by $A_x \bar{e}_y$.

Let $B_i, B_j \in B_{C(f,1)}$, and denote by $\sum_{i=1}^{\mu} \gamma^i_j B_i$ the representation of the row $R_{B_i} B_j$ using $R_{C(f,1)}$, i.e., the row in $C(f,1)$ labeled with the graph resulting from the product $B_i B_j$.

**Claim 13.** Let $x$ be some 1-labeled quantum graph such that $R_x = \sum_{i=1}^{\mu} a_i R_{B_i}$. The matrix $A_x$ is given by $(A_x)_{t,m} = \sum_{i=1}^{\mu} a_i \gamma^i_j m$.

Note that for a basis graph $B_k \in B_{C(f,1)}$, we have that $(A_{B_k})_{i,j} = \gamma^k_{i,j}$. The proof of this claim appears in Appendix A.

**Proposition 14.** The matrices $A_{B_1}, \ldots, A_{B_n}$ of the graphs in $B_{C(f,1)}$ are linearly independent and span all matrices of the form $A_x$ for a quantum graph $x$.

If we know what are the matrices $A_{p_1}, \ldots, A_{p_n}$ of the idempotent basis $p_1, \ldots, p_n$, we can find their representation using $A_{B_1}, \ldots, A_{B_n}$ by solving systems of linear matrix equations. Then, given a representation $A_{p_i} = \sum_{k=1}^{n} \delta_{k}^{(n)} A_{B_k}$, we will have the representation of the basic idempotents using $B_{C(f,1)}$ as $p_i = \sum_{k=1}^{n} \delta_{k}^{(n)} B_k$.

The definitions of $A_x$ and idempotence lead to the observation that for idempotent basis $p_i, p_j$, it holds that $A_{p_i} A_{p_j} = A_{p_i}$ and $A_{p_i} A_{p_j} = 0$. From Corollary 5 we know the dimension of $Q_1/f$ is $q$, so we conclude:

**Proposition 15.** The idempotent basis for $Q_1/f$ consists of the quantum graphs $p_i, i \in [q]$ for which $A_{p_i}$ is the $q \times q$ matrix with the value 1 in the entry $(i, i)$ and zero in all other entries. That is,

$$A_{p_i}(k,j) = \begin{cases} 1, & \text{if } (k,j) = (i,i) \\ 0, & \text{otherwise} \end{cases}$$
As \textbf{find basis} solves the systems of linear matrix equations for these matrices, it remains to show that \textbf{find basis} correctly computes the matrices \(A_{B_i}, \ i \in [q]\).

Since \(M\) is of full rank, the representations \(\sum_{k=1}^{n} \gamma_k^{ij} R_{B_k}\) of graphs \(B_i, B_j, \ i, j \in [q]\) using \(B_{C(f,1)}\) are correctly computed by the \textit{solve linear system} calls. And as noted before, the coefficients \(\gamma_k^{ij}\) are the entries of the matrices \(A_{B_i}, \ i \in [q]\). Thus they indeed are correctly computed, and we have Lemma 12.

Since \textbf{generate hypothesis} directly queries the teacher for the values of \(\alpha^{(k)}\), we have:

**Corollary 16.** If \(M\) is of rank \(q\), then \textbf{generate hypothesis} outputs a correct vertex weights vector \(\alpha^{(k)}\).

It remains to show this is true also for the edge weights:

**Proposition 17.** If \(M\) is of rank \(q\), then \textbf{generate hypothesis} outputs a correct edge weights matrix \(\beta^{(k)}\).

\textit{Proof.} As \(p_{ij} = p_i \oplus p_j, \ i, j \in [q]\) are the idempotent basis for \(Q_2 / f\) we have that \(p_{ij} p_{ij} \neq f, 2 0\), so the matrix \(N\) is a diagonal matrix of full rank, and \textit{solve linear system} indeed finds the representation of \(K_2\) using \(p_{ij}, \ i, j \in [q]\). \hfill \Box

From Corollary 16 and Proposition 17 we have Theorem 10.

Now we show that Algorithm 1 reaches that point in the first place.

### 4.2 Augmentation results in larger rank - proof of Theorem 11

Theorem 11 is proved using the fact that \(A_x\) are linearly independent for \(k\)-labeled quantum graphs which are not equivalent in \(\equiv_{f,k}\).

**Lemma 18.** In the \(n^\text{th}\) iteration of Algorithm 1 if the teacher returns a counterexample \(x\), then \(R_x\) is not spanned by \(R_{B_1}, \ldots, R_{B_n}\) where \(B_1, \ldots, B_n\) are the graphs associated with the rows and columns of \(M\).

\textit{Proof.} If \(n = 1\), \(M\) has rank \(n\). Now let \(M\) have rank \(n\).

For contradiction, assume that \(R_x = \sum_{i=1}^{n} a_i R_{B_i}\). Then \(x \equiv_{f,1} \sum_{i=1}^{n} a_i B_i\) and we have that \(\text{hom}(x, H) = \sum_{i=1}^{n} a_i \text{hom}(B_i, H)\) for the target graph \(H\). Denote by \(h^{(n)}\) the hypothesis generated in this iteration. If \(x\) is a counterexample, it must hold that

\[
\text{hom}(x, h^{(n)}) \neq \text{hom}(x, H) = \sum_{i=1}^{n} a_i \text{hom}(B_i, H)
\]

The solution of the system of equations for \(b_x\) would give

\[
\text{hom}(x, h^{(n)}) = \sum_{i=1}^{n} a_i \text{hom}(B_i, h^{(n)}) = \sum_{i=1}^{n} a_i \text{hom}(B_i, H)
\]

So we conclude that \(\sum_{i=1}^{n} a_i \text{hom}(B_i, h^{(n)}) \neq \sum_{i=1}^{n} a_i \text{hom}(B_i, H)\).

Since \(M\) is of full rank, one can solve a system of linear equations using \(M\) for \(b_x\) defined as \(b_x(k) = \text{VALUE}(x B_k), \ k \in [n]\). Now recall that the matrix \(M\) contains correct values \(\text{hom}(B_i, B_j, H(\alpha, \beta))\), as it was augmented using \textit{VALUE} queries, therefore \(M\) is a submatrix of \(C(f,1)\). Thus the coefficients of the solution \(a\) of \(M a = b_x\) equal \(a_i, \ i \in [k]\), and we reach a contradiction. Therefore we conclude \(x \neq f, 1 \sum_{i=1}^{n} a_i B_i\) and its row \(R_x\) is linearly independent from \(R_{B_1}, \ldots, R_{B_n}\). \hfill \Box

This also implies that the matrix \(A_x\) associated with \(x\) is not spanned by \(A_{B_1}, \ldots, A_{B_n}\). Therefore the submatrix of \(C(f,1)\) composed of the entries of the rows and columns of \(B_1, \ldots, B_n, x\) is of full rank \(n + 1\). This is exactly the matrix \(M\) augmented with \(x\), and we have Theorem 11. Combining this with Corollary 16 we have:

**Corollary 19.** Let \(f\) be a rigid partition function of a twin-free weighted graph on \(q\) vertices. Then Algorithm 1 terminates in \(q\) iterations.
4.3 Complexity analysis

As the algorithm runs on a Blum-Shub-Smale machine for the reals and mostly solves systems of linear equations, it is not difficult to show that it runs in time polynomial in the size of target and the largest counterexample. First we observe:

**Proposition 20.** Let $G_1, G_2 \in G_1$. Then $G_1G_2$ can be computed in time $O(\text{poly}(|G_1|, |G_2|))$.

**Remark 21.** $B_1$ is of fixed size, and all other $B_i$, $i = 2, \ldots, n$, used in Algorithm 4 are counterexamples provided by the teacher, therefore they are all of size polynomial in the size $|x|$ of the graph $x$.

**Theorem 22.** Let $H(\alpha, \beta)$ be a rigid twin-free weighted graph on $q$ vertices and denote $f = \text{hom}(\cdot, H(\alpha, \beta))$. Given access to a teacher for $f$, Algorithm 4 terminates in time $O(\text{poly}(q, |x|))$, where $|x|$ is the size of the largest counterexample provided by the teacher.

**Proof.** From Corollary 19 it is enough to show that each iteration of Algorithm 1 does not take too long (Lemma 23).

**Lemma 23.** In the $n^{th}$ iteration of Algorithm 1, augment $M$, find basis, and generate hypothesis all run in time $O(\text{poly}(n, |x|))$.

The easy proof is given in Appendix A.

**Remark 24.** We note that, from [13, Theorem 6.45], the counterexamples provided by the teacher may be chosen to be of size at most $2(1 + q^2)q^6$ where $q$ is the size of the target weighted graph.

5 Conclusion and future work

This paper presented an adaptation of the exact model of learning of Angluin, [1], to the context of graph parameters $f$ representable as partition functions of weighted graphs $H(\alpha, \beta)$. We presented an exact learning algorithm for the class of rigid partition functions defined by twin-free $H(\alpha, \beta)$.

If a weighted graph has proper automorphisms, its connection matrices $C(f, k)$ may have rank smaller than $q^k$. In this case, the translation from query results to a weighted graph would involve the construction of a submatrix of $C(f, k)$ for a sufficiently large $k$, and then find an idempotent basis for $Q_{k+1}/f$. We will study the learnability of non-rigid partition functions in a sequel to this paper.

Theorems similar to Theorem 4 have been proved for variants of partition functions and connection matrices, [15, 8, 16, 17]. It seems reasonable to us that similar exact learning algorithms exist for these settings, but it is unclear how to modify our proofs here for this purpose.

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**References**

[1] D. Angluin. On the complexity of minimum inference of regular sets. *Information and Control*, 39(3):337–350, 1978.

[2] D. Angluin. Queries and concept learning. *Machine Learning*, 2(4):319–342, 1987.
A. Beimel, F. Bergadano, N.H. Bshouty, E. Kushilevitz, and S. Varricchio. Learning functions represented as multiplicity automata. *Journal of the ACM (JACM)*, 47(3):506–530, 2000.

G.D. Birkhoff. A determinant formula for the number of ways of coloring a map. *Annals of Mathematics*, 14:42–46, 1912.

L. Blum, F. Cucker, M. Shub, and S. Smale. *Complexity and real computation*. Springer Science & Business Media, 2012.

L. Blum, M. Shub, S. Smale, et al. On a theory of computation and complexity over the real numbers: NP-completeness, recursive functions and universal machines. *Bulletin (New Series) of the American Mathematical Society*, 21(1):1–46, 1989.

B. Bollobás. *Modern Graph Theory*. Springer, 1999.

J. Draisma, D.C. Gijswijt, L. Lovász, G. Regts, and A. Schrijver. Characterizing partition functions of the vertex model. *Journal of Algebra*, 350(1):197–206, 2012.

P. Erdős and A. Rényi. Asymmetric graphs. *Acta Mathematica Hungarica*, 14(3-4):295–315, 1963.

M. Freedman, L. Lovász, and A. Schrijver. Reflection positivity, rank connectivity, and homomorphism of graphs. *Journal of the American Mathematical Society*, 20(1):37–51, 2007.

A. Habrard and J. Oncina. Learning multiplicity tree automata. In *Grammatical Inference: Algorithms and Applications*, pages 268–280. Springer, 2006.

J. Kötters. Almost all graphs are rigid - revisited. *Discrete Mathematics*, 309(17):5420–5424, 2009.

L. Lovász. *Large Networks and Graph Limits*, volume 60 of *Colloquium Publications*. AMS, 2012.

L. Lovsz. The rank of connection matrices and the dimension of graph algebras. *European Journal of Combinatorics*, 27(6):962 – 970, 2006.

A. Schrijver. Graph invariants in the spin model. *J. Comb. Theory, Ser. B*, 99(2):502–511, 2009.

A. Schrijver. Characterizing partition functions of the spin model by rank growth. *Indagationes Mathematicae*, 24.4:1018–1023, 2013.

A. Schrijver. Characterizing partition functions of the edge-coloring model by rank growth. *Journal of Combinatorial Theory, Series A*, 136:164 – 173, 2015.

A.D. Sokal. The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. In *Survey in Combinatorics, 2005*, volume 327 of *London Mathematical Society Lecture Notes*, pages 173–226, 2005.

## A Proofs omitted from paper

### A.1 Proof of Claim 13

For two graphs $B_i, B_j \in \mathcal{B}_{C(f,1)}$, denote by $\sum_{k=1}^n \gamma_k^{i,j} R_{B_k}$ the representation of the row $R_{B_iB_j}$ using $\mathcal{R}_{C(f,1)}$, i.e., the row labeled with the 1-labeled graph resulting from the product $B_iB_j$. 

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Let \( x, y \) be some 1-labeled quantum graphs whose infinite row vectors are represented using \( R_{C(f,1)} \) as
\[
R_x = \sum_{i=1}^{n} a_i R_{B_i} \quad R_y = \sum_{j=1}^{n} b_j R_{B_j}
\]

Then the representation of the row \( R_{xy} \) of their product \( xy \) is
\[
R_{xy} = \sum_{1 \leq i,j \leq n} a_i b_j R_{B_i B_j} = \sum_{1 \leq i,j \leq n} a_i b_j \left( \sum_{k=1}^{n} \gamma_{i,j}^{k} R_{B_k} \right)
\]
\[
= \sum_{1 \leq i,j,k \leq n} a_i b_j \gamma_{i,j}^{k} R_{B_k}
\]

Thus the entry corresponding to the basis graph \( B_k \in B_{C(f,1)} \) in the coefficients vector \( c_{xy} \) is the scalar
\[
\sum_{1 \leq i,j \leq n} a_i b_j \gamma_{i,j}^{k}.
\]

This scalar should equal the result of multiplying the \( k \)-th row of \( A_x \) with the coefficients vector of \( y \). Therefore the \( k \)-th row of \( A_x \) would be
\[
\left( \sum_{i=1}^{n} a_i \gamma_{i,1}^{k}, \sum_{i=1}^{n} a_i \gamma_{i,2}^{k}, \ldots, \sum_{i=1}^{n} a_i \gamma_{i,n}^{k} \right),
\]

Since then we would have:
\[
\left( \sum_{i=1}^{n} a_i \gamma_{i,1}^{k}, \sum_{i=1}^{n} a_i \gamma_{i,2}^{k}, \ldots, \sum_{i=1}^{n} a_i \gamma_{i,n}^{k} \right) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \sum_{j=1}^{n} b_j \left( \sum_{i=1}^{n} a_i \gamma_{i,j}^{k} \right) = \sum_{1 \leq i,j \leq n} a_i b_j \gamma_{i,j}^{k}
\]

Therefore the matrix \( A_x \) is given by
\[
A_x = \begin{pmatrix}
\sum_{i=1}^{n} a_i \gamma_{i,1}^{1} & \cdots & \sum_{i=1}^{n} a_i \gamma_{i,n}^{1} \\
\vdots & \ddots & \vdots \\
\sum_{i=1}^{n} a_i \gamma_{i,1}^{n} & \cdots & \sum_{i=1}^{n} a_i \gamma_{i,n}^{n}
\end{pmatrix}
\]

### A.2 Detailed complexity analysis - proof of Lemma 23

Let \( H(\alpha, \beta) \) be a rigid twin-free weighted graph on \( q \) vertices, and denote \( f = \text{hom}(\cdot, H(\alpha, \beta)) \). Let \( |x| \) denote the size of the largest counterexample Algorithm 1 receives from the teacher.

**Lemma 25.** In the \( n \)\( \text{th} \) iteration of Algorithm 7 augment \( M \) runs in time \( O(\text{poly}(n, |x|)) \).

**Proof.** In the \( n \)\( \text{th} \) iteration, augment \( M \) performs \( O(n) \) value queries as it adds a new row and column labeled with \( B_n \) to \( M \). For this it performs value queries on graphs that are 1-connections between \( B_n \) and \( B_i, i \in [n] \). From Proposition 20 and Remark 21 it runs in time \( O(\text{poly}(n, |x|)) \). \( \square \)

**Lemma 26.** In the \( n \)\( \text{th} \) iteration of Algorithm 7 find basis runs in time \( O(\text{poly}(n, |x|)) \).

**Proof.** find basis has three for loops. In the first for loop, it repeats \( O(n^2) \) times:
1. Fills an \( n \)-length vector \( b \) by making value(\( B_i B_j B_k \)) queries, for which it computes \( B_i B_j B_k \). Again from Proposition 20 and Remark 21 we have that the computation of \( b \) in each of the \( O(n^2) \) iterations is in time \( O(\text{poly}(n, |x|)) \).
2. Solves a linear system of equation of dimension \( n \). This is in time \( O(n^3) \).
In each iteration of its second for loop, find basis fills an \( n \times n \) matrix and adds it to a block matrix, in time \( O(n^2) \). This is repeated \( n \) times.

In each iteration of its third for loop, find basis solves a linear system of matrix equations involving \( n \times n \) matrices, of dimension \( n \). Such a system can be solved as a usual linear system of equations at the cost of a polynomial blowup where each matrix is replaced by \( n^2 \) variables, giving us time \( O(n^6) \) for each of the \( n \) iterations.

In total, in the \( n^{th} \) iteration of Algorithm 1, find basis runs in time \( O(\text{poly}(n, |x|)) \).

**Lemma 27.** In the \( n^{th} \) iteration of Algorithm 1, generate hypothesis runs in time \( O(\text{poly}(n, |x|)) \).

**Proof.** Recall that for a quantum graph \( x = \sum_{i=1}^{n} a_i F_i \), we wrote \( \text{value}(x) \) as shorthand for \( \sum_{i=1}^{n} a_i \text{value}(F_i) \).

All quantum graphs in the run are linear combinations of at most \( n \) graphs, thus any linear combination requires \( O(n) \) arithmetic operations.

For the extraction of \( \alpha^{(h)} \), generate hypothesis computes linear combinations of the results of \( \text{value} \) queries, \( n \) times.

For the extraction of \( \beta^{(h)} \), generate hypothesis:

1. Computes \( p_{ij} = p_i \otimes p_j \) for \( i, j \in [n] \). Each of these requires performing 2-connections between \( O(n^2) \) pairs of graphs of size \( O(|x|) \), and the computation is performed for \( O(n^2) \) indices \( i, j \).

2. Computes \( p_{ij} p_{ij} \) for \( i, j \in [n] \) and \( \text{value}(p_{ij} p_{ij}) \), and computes \( p_{ij} K_2 \) and \( \text{value}(p_{ij} K_2) \). Each of these requires \( O(n^4) \) operations on graphs of size \( O(\text{poly}(|x|)) \). The computation is performed for \( O(n^2) \) indices \( i, j \).

In total, in the \( n^{th} \) iteration of Algorithm 1, generate hypothesis runs in time \( O(\text{poly}(n, |x|)) \).