Isoperimetric bounds for eigenvalues of the Wentzell-Laplace, the Laplacian and a biharmonic Steklov problem

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Abstract

In this paper, we prove some isoperimetric bounds for lower order eigenvalues of the Wentzell-Laplace operator on bounded domains of a Euclidean space or a Hadamard manifold, of the Laplacian on closed hypersurfaces of a Euclidean space or a Hadamard manifold, and of a biharmonic Steklov problem on bounded domains of a Euclidean space. Especially, interesting rigidity results can be obtained if sharp bounds were achieved.

1 Introduction

Let $M$ be an $n$-dimensional compact Riemannian manifold with boundary $\partial M$ and denote by $\bar{\Delta}$ and $\Delta$ the Laplace-Beltrami operators on $M$ and $\partial M$, respectively. Assume that $\beta$ is a real number and consider the following eigenvalue problem with the Wentzell boundary condition

\[
\begin{cases}
\bar{\Delta} u = 0 & \text{in } M, \\
-\beta \Delta u + \partial_{\nu} u = \lambda u & \text{on } \partial M,
\end{cases}
\]

1 For the eigenvalue problem (1.1), the regularity assumption for $\partial M$ should be made such that the embedding $H(M) \subset L^2(M) \cup L^2(\partial M)$ is compact, which is the essential requirement so that the Laplacian in (1.1) has a discrete spectrum. Here, $H(M)$ defined by (1.4) below is the admissible space of the eigenvalue problem (1.1). In fact, for the eigenvalue problem (1.1), the regularity assumption “$\partial M$ is Lipschitz continuous” can already make sure that the embedding $H(M) \subset L^2(M) \cup L^2(\partial M)$ is compact. Generally speaking, different eigenvalue problems might have different regularity requirements on the boundary (if nonempty) to have discrete spectrum. In order to avoid the repeated argument on regularity of the boundary (which is not the main part of this paper), without specification we always assume that the boundary regularity is smooth for all eigenvalue problems considered in this paper. Moreover, this smooth assumption would be mentioned in the sequel if necessary.
where $\partial_{\nu}$ denotes the derivative w.r.t. the outward unit normal vector $\nu$ to $\partial M$. This problem has already been studied in [6, 13] when $M$ is a bounded domain in a Euclidean space. Recently, some interesting estimates for the nonzero eigenvalues of the problem (1.1) and its weighted version have been obtained (see, e.g., [8, 26, 27]). Besides, following the convention in [27], we call (1.1) the Wentzell eigenvalue problem of the Laplacian. Note that when $\beta = 0$, the eigenvalue problem (1.1) becomes the classical Steklov problem

$$\begin{cases}
\Delta u = 0 & \text{in } M, \\
\partial_{\nu} u = pu & \text{on } \partial M,
\end{cases}$$

which has a discrete spectrum consisting in a non-decreasing sequence

$$p_0 < p_1 \leq p_2 \leq \cdots \to +\infty.$$  

For the Steklov eigenvalue problem (1.2), many interesting results have been obtained and one can find some of them – see, e.g., [5, 10, 11, 12, 14, 16, 17, 23, 24, 25, 27, 28] and the references therein.

When $\beta \geq 0$, the spectrum of the Laplacian with Wentzell boundary condition, i.e. the spectrum of the eigenvalue problem (1.1), consists in an non-decreasing countable sequence of eigenvalues

$$\lambda_{0,\beta} = 0 < \lambda_{1,\beta} \leq \lambda_{2,\beta} \leq \cdots \to +\infty,$$

with corresponding real orthonormal (in $L^2(\partial M)$ sense) eigenfunctions $u_0, u_1, u_2, \cdots$. We adopt the convention that each eigenvalue is repeated according to its multiplicity. Consider the Hilbert space

$$H(M) = \{ u \in H^1(M), \text{Tr}_{\partial M}(u) \in H^1(\partial M) \},$$

where $\text{Tr}_{\partial M}$ is the trace operator on $\partial M$. We define on $H(M)$ the two bilinear forms

$$A_\beta(u, v) = \int_M \nabla u \cdot \nabla v + \beta \int_{\partial M} \nabla u \cdot \nabla v, \quad B(u, v) = \int_{\partial M} uv,$$

where, as before, $\nabla$ and $\nabla$ are the gradient operators on $M$ and $\partial M$, respectively. Here, volume elements in the above integrals have been dropped. Since we assume that $\beta$ is nonnegative, two bilinear forms defined by (1.5) are positive and the variational characterization of the $k$-th eigenvalue is

$$\lambda_{k,\beta} = \min \left\{ \frac{A_\beta(u, u)}{B(u, u)} \right\} \quad u \in H(M), u \neq 0, \quad \int_{\partial M} uu_i = 0, i = 0, \cdots, k - 1.$$  

Clearly, when $k = 1$, the minimum is taken over the functions orthogonal (in $L^2(\partial M)$ sense) to the eigenfunctions associated to $\lambda_{0,\beta} = 0$, i.e., nonzero constant functions.

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2 Of course, each element in the discrete spectrum is called eigenvalue.

3 Clearly, eigenvalues $p_i$ in the sequence (1.3) should be written as $p_i(M)$ for accuracy. However, for convenience, in the sequel, we prefer to simplify the notations for eigenvalues (of different type) discussed in this paper, that is, we separately write $p_i(M), \lambda_{i,\beta}(\Omega), \lambda_i(M), \lambda_{i,\tau}(\Omega)$ as $p_i$, $\lambda_{i,\beta}$, $\lambda_i$ and $\lambda_{i,\tau}$. We also make an agreement that these notations would be written completely if necessary.

4 For convenience, in the sequel we will drop the integral measures for all integrals except it is necessary.
If $M$ is an $n$-dimensional Euclidean ball of radius $R$, then (cf. [6])

$$\lambda_1, \beta = \lambda_2, \beta = \cdots = \lambda_n, \beta = \frac{(n - 1)\beta + R}{R^2} \tag{1.7}$$

and the corresponding eigenfunctions are the coordinate functions $x_i, i = 1, \cdots, n$. For the Steklov problem (1.2), Brock [1] proved the following well-known result:

**Theorem 1.1** ([1]) Let $\Omega$ be a bounded domain with smooth boundary in the Euclidean $n$-space $\mathbb{R}^n$ and denote by $p_1(\Omega), \cdots, p_n(\Omega)$ the first $n$ nonzero Steklov eigenvalues of $\Omega$. Then

$$\sum_{i=1}^{n} \frac{1}{p_i(\Omega)} \geq n \left( \frac{|\Omega|}{\omega_n} \right)^{\frac{1}{n}},$$

where $\omega_n$ and $|\Omega|$ denote the volume of the unit ball in $\mathbb{R}^n$ and of $\Omega$, respectively.

The proof of Brock’s theorem is a nice application of an inequality for sums of reciprocals of eigenvalues shown by Hile-Xu in [15] and a weighted isoperimetric inequality obtained by Betta-Brock-Mercaldo-Posteraro in [2]. Brock’s method has been used by Dambrine-Kateb-Lamboley in [6] to obtain an estimate for eigenvalues of the Wentzell eigenvalue problem of the Laplacian (1.1).

In the first part of this paper, we use the variational characterization (1.6) to get the following Brock-type isoperimetric bound.

**Theorem 1.2** Let $\beta \geq 0$ and $\Omega$ be a bounded domain with smooth boundary $\partial \Omega$ in $\mathbb{R}^n$. Let $|\partial \Omega|$ be the area of $\partial \Omega$ and denote by $\lambda_{1, \beta} \leq \lambda_{2, \beta} \leq \cdots \leq \lambda_{n, \beta}$ the first $n$ nonzero eigenvalues of the following eigenvalue problem with the Wentzell boundary condition

$$\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
-\beta \Delta u + \partial_{\nu} u = \lambda u & \text{on } \partial \Omega.
\end{cases} \tag{1.8}$$

Then we have

$$\sum_{i=1}^{n} \frac{|\Omega|}{\lambda_{i, \beta}} + \sum_{i=1}^{n-1} \frac{\beta |\partial \Omega|}{\lambda_{i, \beta}} \geq n\omega_n \left( \frac{|\Omega|}{\omega_n} \right)^{\frac{1}{n}} \left[ 1 + \frac{(n + 1)(2^{1/n} - 1)}{4n} \left( \frac{|\Omega \Delta B|}{|B|} \right)^2 \right], \tag{1.9}$$

where $B$ is the ball of volume $|\Omega|$ and with the same center of mass than $\partial \Omega$, and $\Omega \Delta B$ is the symmetric difference of $\Omega$ and $B$. Equality holds in (1.9) if and only if $\Omega$ is a ball.

**Remark 1.3** Taking $\beta = 0$ in Theorem 1.2 one gets Theorem 1.1 directly.

Using the monotonicity of eigenvalues $\lambda_{i, \beta}$’s and Theorem 1.2 immediately, we get

$$\frac{1}{\lambda_{1, \beta}} \left[ n|\Omega| + (n - 1)\beta |\partial \Omega| \right] \geq$$

$$\sum_{i=1}^{n} \frac{|\Omega|}{\lambda_{i, \beta}} + \sum_{i=1}^{n-1} \frac{\beta |\partial \Omega|}{\lambda_{i, \beta}} \geq n\omega_n \left( \frac{|\Omega|}{\omega_n} \right)^{\frac{1}{n}} \left[ 1 + \frac{(n + 1)(2^{1/n} - 1)}{4n} \left( \frac{|\Omega \Delta B|}{|B|} \right)^2 \right],$$

which directly implies the following eigenvalue estimate.

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5In the sequel, without specification, for a given geometric subject, $| \cdot |$ would denote its Hausdorff measure.
Corollary 1.4 Under the assumptions in Theorem 1.2, we have
\[
\lambda_{1,\beta}(\Omega) \leq \left[ 1 + \beta \frac{(n-1)|\partial\Omega|}{n|\Omega|} \right] \left( \frac{\omega_n}{|\Omega|} \right)^{\frac{1}{n}} \left[ 1 + \frac{(n+1)(2^{1/n} - 1)}{4n} \left( \frac{|\Omega\Delta B|}{|B|} \right)^2 \right]^{-1},
\]
where equality holds if and only if \( \Omega \) is a ball.

Corollary 1.4 and the truth (1.7), together with the classical isoperimetric inequality for bounded Euclidean domains with fixed volume (see, e.g., [21, Chapter 1]), imply the following estimate:

Corollary 1.5 Under the assumptions in Theorem 1.2, we have
\[
\lambda_{1,\beta}(\Omega) \leq \lambda_{1,\beta}(B) \cdot \frac{n|\Omega| + (n-1)\beta|\partial\Omega|}{n|\Omega| + (n-1)\beta|\partial\Omega|} \left[ 1 + \frac{(n+1)(2^{1/n} - 1)}{4n} \left( \frac{|\Omega\Delta B|}{|B|} \right)^2 \right]^{-1},
\]
where similarly \(|\partial B|\) denotes the area of the sphere \(\partial B\) (i.e., the boundary of the ball \(B\) of volume \(|\Omega|\)). Equality holds if and only if \(\Omega\) is a ball, and moreover, in this situation,
\[
\lambda_{1,\beta}(\Omega) = \lambda_{1,\beta}(B) = \frac{(n-1)\beta + R}{R^2} \text{ with } R = \left( \frac{|\Omega|}{\omega_n} \right)^{1/n}.
\]

Remark 1.6 (1) If \(\beta = 0\), then the conclusion of Corollary 1.5 degenerates into
\[
p_{1}(\Omega) \leq p_{1}(B) \cdot \left[ 1 + \frac{(n+1)(2^{1/n} - 1)}{4n} \left( \frac{|\Omega\Delta B|}{|B|} \right)^2 \right]^{-1},
\]
where equality holds if and only if \(\Omega\) is a ball (i.e., in this case, \(\Omega\) is the ball \(B\)). This result indicates:

- Among all bounded Euclidean domains of smooth boundary with fixed volume, the ball maximizes the first nonzero Steklov eigenvalue.\(^6\)

This spectral isoperimetric inequality can be obtained directly by Theorem 1.1 and has already been pointed out by Brock [1].

(2) When \(\beta > 0\), by the classical isoperimetric inequality, we know that under the constraint \(|\Omega| = |B|\), one has
\[
1 \leq \frac{n|\Omega| + (n-1)\beta|\partial\Omega|}{n|\Omega| + (n-1)\beta|\partial\Omega|} < 1 + \beta \frac{(n-1)|\partial\Omega|}{n|\Omega|},
\]
where equality holds if and only if \(\Omega\) is the ball \(B\). Hence, Corollary 1.5 gives a partial answer to Dambrine-Kateb-Lamboley’s conjecture proposed in [6, page 412].

(3) For the eigenvalue problem (1.8), Dambrine, Kateb and Lamboley [6, Theorem 1.1] gave the estimate:
\[
(\Omega + \beta A[\Omega]) \sum_{i=1}^{n} \frac{1}{\lambda_{i,\beta}} \geq n \omega_n \left( \frac{|\Omega|}{\omega_n} \right)^{\frac{1}{n}} \left[ 1 + \frac{(n+1)(2^{1/n} - 1)}{4n} \left( \frac{|\Omega\Delta B|}{|B|} \right)^2 \right], \quad (1.10)
\]

\(^6\)As shown in the footnote of 1st page, the regularity of the boundary can be weaken to Lipschitz continuity. However, by (1) of Remark 1.6, we do not know whether the ball \(B\) is the only domain such that the functional \(p_{1}(\cdot) : \Omega \rightarrow p_{1}(\Omega)\) attains its maximum value or not. But Brock [1] has given an affirmative answer already – the ball is the only possibility to get the maximum value.
where \( \Lambda[\Omega] \) is spectral radius of the symmetric and positive semidefinite matrix \( p(\Omega) = (p_{ij})_{n \times n} \) defined as

\[
p_{ij} = \int_{\partial \Omega} (\delta_{ij} - \nu_i \nu_j)
\]

with, similarly, \( \nu \) is the outward unit normal vector to \( \partial \Omega \). They [6, Lemma 2.4] also proved that the matrix \( p(\Omega) = (p_{ij})_{n \times n} \) defined as above should be symmetric, positive definite, and its spectral radius \( \Lambda[\Omega] \) satisfies

\[
(n - 1) |\partial \Omega| \geq \Lambda[\Omega] \geq \frac{n - 1}{n} |\partial \Omega|.
\]

In particular, among sets of given volume, the spectral radius is minimal for the ball. On the other hand, by direct calculation, one has

\[
\frac{n - 1}{n} \leq \sum_{i=1}^{n-1} \frac{1}{\lambda_{i,\beta}} < 1,
\]

with equality if and only if \( \lambda_{1,\beta} = \lambda_{2,\beta} = \cdots = \lambda_{n,\beta} \). Hence, combining (1.10), (1.11) with (1.12), it is easy to know that:

- If \( \Lambda[\Omega] \in \left( \frac{n - 1}{n} |\partial \Omega|, |\partial \Omega| \right) \), for our estimate (1.9) and Dambrine-Kateb-Lamboley’s estimate (1.10), one does not know which one is better;
- If \( \Lambda[\Omega] \in \left( |\partial \Omega|, (n - 1) |\partial \Omega| \right) \), then our estimate (1.9) is sharper than Dambrine-Kateb-Lamboley’s estimate (1.10).

(4) By (1.11), it is easy to know that the estimate in Corollary 1.4 here is sharper than the one given in [6, Corollary 1.2].

For the Wentzell eigenvalue problem of the Laplacian, we also have:

**Theorem 1.7** Let \( Q^n \) be a Hadamard manifold and let \( \Omega \) be a bounded domain with smooth boundary \( \partial \Omega \) in \( Q^n \). Let \( \beta \geq 0 \) and \( \rho \) be a continuous positive function on \( \partial \Omega \). Then the first \( n \) nonzero eigenvalues of the eigenvalue problem

\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega, \\
-\beta \Delta u + \partial_\nu u &= \lambda \rho u \quad \text{on } \partial \Omega
\end{align*}
\]

satisfy

\[
\sum_{i=1}^{n} \frac{1}{\lambda_{i,\beta}} + \sum_{i=1}^{n-1} \frac{\beta |\partial \Omega|}{\lambda_{i,\beta}[\Omega]} \geq \frac{n^2 |\Omega|}{\int_{\partial \Omega} \rho^{-1}}.
\]

Moreover, when \( \rho = \rho_0 \) is constant, equality in (1.14) holds if and only if \( \Omega \) is isometric to an \( n \)-dimensional Euclidean ball.

Using the monotonicity of eigenvalues \( \lambda_{i,\beta} \)’s and Theorem 1.7 immediately, we can obtain:
Corollary 1.8 Under the assumptions of Theorem 1.7, we have

\[
\lambda_1,\beta(\Omega) \leq \left[ 1 + \beta \frac{(n-1)|\partial \Omega|}{n|\Omega|} \right] \cdot \frac{\int_{\partial \Omega} \rho^{-1}}{n|\Omega|}.
\] (1.15)

Moreover, when \( \rho = \rho_0 \) is constant, equality in (1.15) holds if and only if \( \Omega \) is isometric to an \( n \)-dimensional Euclidean ball.

In the second part of the present paper, we consider eigenvalues of the Laplacian acting on functions on closed hypersurfaces in a Euclidean space or a Hadamard manifold, and can prove the following two conclusions.

Theorem 1.9 Let \( M \) be a connected closed hypersurface in \( \mathbb{R}^n \) with \( n \geq 3 \).

(i) The first \( (n-1) \) nonzero eigenvalues of the Laplacian on \( M \) satisfy

\[
\sum_{j=1}^{n-1} \lambda_j^+ \leq (n-1)^2 \left( \frac{\int_M |H|^2}{|M|} \right)^{\frac{1}{2}},
\] (1.16)

where \( H \) denotes the mean curvature vector of \( M \) in \( \mathbb{R}^n \) and, as before, \( |M| \) is the area of \( M \). The equality holds in (1.16) if and only if \( M \) is a hypersphere.

(ii) If \( M \) is embedded and bounds a region \( \Omega \), then

\[
\sum_{i=1}^{n-1} \frac{1}{\lambda_i} \geq \frac{n \omega_n}{|M|} \left( \frac{|\Omega|}{\omega_n} \right)^{1+\frac{1}{n}} \left[ 1 + \frac{(n+1)(2^{1/n} - 1)}{4n} \left( \frac{|\Omega \Delta B|}{|B|} \right)^2 \right],
\] (1.17)

where \( B \) is the ball of volume \( |\Omega| \) and with the same center of mass than \( M \). Equality holds in (1.17) if and only if \( M \) is a hypersphere.

Theorem 1.10 Let \( Q^n \) be a Hadamard \( n \)-manifold \((n \geq 3)\). Let \( M \) be a connected closed hypersurface embedded in \( Q^n \) and \( \Omega \) be the region bounded by \( M \). Then the first \( (n-1) \) nonzero eigenvalues of the Laplacian of \( M \) satisfy

\[
\sum_{i=1}^{n-1} \frac{1}{\lambda_i} \geq \frac{n^2|\Omega|^2}{|M|^2}.
\] (1.18)

Equality holds in (1.18) if and only if \( \Omega \) is isometric to an \( n \)-dimensional Euclidean ball.

Remark 1.11 (1) By applying Theorem 1.7 and the monotonicity of eigenvalues \( \lambda_i \)'s, we separately have

\[
\begin{align*}
(i) \quad & \lambda_1(M) \leq (n-1) \frac{\int_M |H|^2}{|M|}; \\
(ii) \quad & \lambda_1(M) \leq \frac{(n-1)|M|}{n|\Omega|} \left( \frac{\omega_n}{|\Omega|} \right)^{\frac{1}{n}} \left[ 1 + \frac{(n+1)(2^{1/n} - 1)}{4n} \left( \frac{|\Omega \Delta B|}{|B|} \right)^2 \right]^{-1},
\end{align*}
\]

and moreover, the equality holds implies the rigidity described as in Theorem 1.9. The eigenvalue estimate (1.19) is actually the well-known Reilly’s inequality. In fact, estimate

\[7\) Of course, it should be “under the assumptions of Theorem 1.7".
(1.19), together with the corresponding rigidity, can be extended to the case of codimension \(\geq 2\), and this is actually the main result of the influential paper [20]. Besides, there is one more thing we prefer to mention here, that is, Mao and his collaborators [9, Theorem 1.11] successfully gave a sharp lower bound for the sum of the reciprocals of the first \(n\) nonzero eigenvalues of the Laplacian on a closed \(n\)-submanifold immersed in a Euclidean space, and then Reilly’s inequality follows as a direct consequence.

(2) By applying Theorem 1.10 and the monotonicity of eigenvalues \(\lambda_i\)'s, we get

\[
\lambda_1(M) \leq \frac{(n - 1)|M|^2}{n^2|\Omega|^2},
\]

and equality holds implies the rigidity described as in Theorem 1.10.

Let \(\tau > 0\) be a positive constant, and let \(\Omega\) be a bounded domain in \(\mathbb{R}^n\) with boundary \(\partial \Omega\). As before, let \(\nabla, \Delta\) the gradient and the Laplace operators in \(\mathbb{R}^n\), respectively. Consider the following Steklov-type eigenvalue problem

\[
\begin{cases}
\Delta^2 u - \tau \Delta u = 0 & \text{in } \Omega, \\
\frac{\partial^2 u}{\partial n^2} = 0 & \text{on } \partial \Omega, \\
\tau \frac{\partial u}{\partial n} - \text{div}_{\partial \Omega} (\nabla^2 u \cdot \nu) - \frac{\partial^2 u}{\partial n} = \lambda u & \text{on } \partial \Omega,
\end{cases}
\]

(1.20)

where \(\text{div}_{\partial \Omega}\) denotes the tangential divergence operator on \(\partial \Omega\), \(\nabla^2 u\) is the Hessian of \(u\), \((\nabla^2 u \cdot \nu)_{\partial \Omega}\) stands for the projection of \(\nabla^2 u \cdot \nu\) to the tangent bundle of \(\partial \Omega\). This problem (1.20) has a discrete spectrum whose elements (i.e., eigenvalues) can be listed non-decreasingly as follows

\[
0 = \lambda_{0,\tau} < \lambda_{1,\tau} \leq \lambda_{2,\tau} \leq \cdots \leq \lambda_{k,\tau} \leq \cdots \to \infty.
\]

The eigenvalue 0 is simple and its eigenfunctions are nonzero constant functions. By the variational principal, the \(k\)-th \((k \geq 1)\) eigenvalue \(\lambda_{k,\tau}\) can be characterized as follows

\[
\lambda_{k,\tau} = \min \left\{ \frac{\int_{\Omega} \left( |\nabla^2 u|^2 + \tau |\nabla u|^2 \right)}{\int_{\partial \Omega} u^2} \ | u \in H^2(\Omega), u \neq 0, \right. \\
\left. \int_{\partial \Omega} u = 0, j = 0, \cdots, k - 1 \right\},
\]

where \(u_j\) is the eigenvalue function of \(\lambda_{j,\tau}\). We can prove the following sharp upper bound estimate:

**Theorem 1.12** For the eigenvalue problem (1.20), we have

\[
\frac{1}{n - 1} \sum_{j=1}^{n-1} \lambda_{j,\tau}' \leq \left\{ \frac{n\tau |\Omega|}{|\partial \Omega|} \right\}^{\frac{1}{2}} |\text{H}|^2
\]

where \(\text{H}\) is the mean curvature vector of \(\partial \Omega\) in \(\mathbb{R}^n\). Equality in (1.22) holds if and only if \(\Omega\) is a ball.

\[^8\text{Of course, it should be “under the assumptions of Theorem 1.10.”}\]
Remark 1.13  (1) In [18], the authors therein used the operator \( \text{Proj}_{\partial \Omega} \left[ (\nabla^2 u) \nu \right] \) to denote the projection of \( (\nabla^2 u) \nu \) onto the space tangent to \( \partial \Omega \), which obviously has the same meaning as \( (\nabla^2 u \cdot \nu)_{\partial \Omega} \) here.

(2) By applying Theorem 1.12 and the monotonicity of eigenvalues \( \lambda_{i,\tau}'s \), one has

\[
\lambda_{1,\tau}(\Omega) \leq \frac{n \tau |\Omega| \int_{\partial \Omega} |H|^2}{|\partial \Omega|^2},
\]

with equality if and only if \( \Omega \) is a ball.

(3) The Steklov-type eigenvalue problem (1.20) has been studied by Buoso and Provenzano [4] already, and when \( n = 2 \), it can be used to describe transverse vibrations of a thin plate, which arises in the theory of linear elasticity (see [4] Section 2 for details). One can also check [4, Section 3] for a detailed explanation about the existence of discrete spectrum and the characterization (1.21) of the \( k \)-th eigenvalue of the eigenvalue problem (1.20). Besides, [4, Corollary 5.20] tells us that:

- Among all bounded domains of class \( C^1 \) with fixed measure, the ball maximizes the first nonnegative eigenvalue of problem (1.20), that is, \( \lambda_{1,\tau}(\Omega) \leq \lambda_{1,\tau}(B) \) with \( B \) is the Euclidean ball having the same measure as \( \Omega \).

Very recently, this interesting spectral isoperimetric inequality has been improved by Li and Mao – see [19, Theorem 1.1].

2 Proofs of Theorems 1.2 and 1.7

In this section, we will give the proofs of Theorems 1.2 and 1.7. First, we recall the following lemma from [3].

Lemma 2.1 Let \( \Omega \) be a bounded, open Lipschitz set in \( \mathbb{R}^n \). Then

\[
\int_{\partial \Omega} |x|^2 \geq R^2 |\partial B_R| \left( 1 + \frac{(n+1)(2^{1/n} - 1)}{4n} \left( \frac{|\Omega \Delta B_R|}{|B_R|} \right)^2 \right),
\]

where \( B_R \) is the ball centered at the origin such that \( |B_R| = |\Omega| \).

Proof of Theorem 1.2 Let \( u_0, u_1, u_2, \ldots \) be the orthonormal (in \( L^2(\partial \Omega) \) sense) eigenfunctions corresponding to the eigenvalues \( 0 = \lambda_{0,\beta} < \lambda_{1,\beta} \leq \lambda_{2,\beta} \leq \cdots \) of the eigenvalue problem (1.3), that is,

\[
\begin{aligned}
\Delta u_i &= 0 \quad &\text{in } \Omega, \\
-\beta \Delta u + \partial_{\nu} u_i &= \lambda_{i,\beta} u_i \quad &\text{on } \partial \Omega, \\
\int_{\partial \Omega} u_i u_j &= \delta_{ij}.
\end{aligned}
\]

By (1.6), the eigenvalues \( \lambda_{i,\beta}, i = 1, 2, \cdots \), are characterized by

\[
\lambda_{i,\beta} = \min_{u \in H(\Omega) \setminus \{0\}, u \perp \text{span} \{ u_0, \ldots, u_{i-1} \}} \frac{\int_{\Omega} |\nabla u|^2 + \beta \int_{\partial \Omega} |\nabla u|^2}{\int_{\partial \Omega} u^2}. \quad (2.2)
\]
We need to choose nice trial functions $\phi_i$ for each of the eigenfunctions $u_i$ and insure that these are orthogonal to the preceding eigenfunctions $u_0, \cdots, u_{i-1}$. For the $n$ trial functions $\phi_1, \phi_2, \cdots, \phi_n$, we simply choose the $n$ coordinate functions 

$$\phi_i = x_i, \quad \text{for } i = 1, \cdots, n,$$

but before we can use these we need to make adjustments so that $\phi_i \perp \text{span}\{u_0, \cdots, u_{i-1}\}$ in $L^2(\partial \Omega)$. Simply, by translating the origin appropriately we can assume that 

$$\int_{\partial \Omega} x_i = 0, \quad i = 1, \cdots, n,$$

that is, $x_i \perp u_0$ (which is actually just the constant function $1/|\partial \Omega|$). Nextly we show that a suitable rotation of axes can be made so as to insure that 

$$\int_{\partial \Omega} \phi_j u_i = \int_{\partial \Omega} x_j u_i = 0,$$

for $j = 2, 3, \cdots, n$ and $i = 1, \cdots, j - 1$. To see this, we define an $n \times n$ matrix $Q = (q_{ji})_{n \times n}$, where $q_{ji} = \int_{\partial \Omega} x_j u_i$, for $i, j = 1, 2, \cdots, n$. Using the orthogonalization of Gram and Schmidt (QR-factorization theorem), we know that there exist an upper triangle matrix $T = (T_{ji})_{n \times n}$ and an orthogonal matrix $U = (a_{ji})_{n \times n}$ such that 

$$T_{ji} = \sum_{k=1}^{n} a_{jk} q_{ki} = \int_{\partial \Omega} \sum_{k=1}^{n} a_{jk} x_k u_i = 0, \quad 1 \leq i < j \leq n.$$

Letting $y_j = \sum_{k=1}^{n} a_{jk} x_k$, we get 

$$\int_{\partial \Omega} y_j u_i = \int_{\partial \Omega} \sum_{k=1}^{n} a_{jk} x_k u_i = 0, \quad 1 \leq i < j \leq n.$$

Since $U$ is an orthogonal matrix, $y_1, y_2, \cdots, y_n$ are also coordinate functions on $\mathbb{R}^n$. Therefore, denoting these coordinate functions still by $x_1, x_2, \cdots, x_n$, we arrive at the condition (2.3).

It then follows from (2.2) that for each fixed $i = 1, \cdots, n$,

$$\lambda_{i, \beta} \int_{\partial \Omega} x_i^2 \leq \int_{\Omega} |\nabla x_i|^2 + \beta \int_{\partial \Omega} |\nabla x_i|^2 = |\Omega| + \beta \int_{\partial \Omega} |\nabla x_i|^2,$$

with equality holding if and only if 

$$\beta \Delta x_i + \partial \nu x_i = -\lambda_{i, \beta} x_i \quad \text{on } \partial \Omega. \quad (2.4)$$

Hence,

$$\int_{\partial \Omega} x_i^2 \leq \frac{|\Omega|}{\lambda_{i, \beta}} + \frac{\beta}{\lambda_{i, \beta}} \int_{\partial \Omega} |\nabla x_i|^2. \quad (2.5)$$

Observing 

$$\sum_{i=1}^{n} |\nabla x_i|^2 = n - 1, \quad |\nabla x_i|^2 \leq 1,$$
we get
\[
\sum_{i=1}^{n} \frac{\left| \nabla x_i \right|^2}{\lambda_{i,\beta}} = \sum_{i=1}^{n-1} \frac{\left| \nabla x_i \right|^2}{\lambda_{i,\beta}} + \frac{\left| \nabla x_n \right|^2}{\lambda_{n,\beta}}
\]
\[
= \sum_{i=1}^{n-1} \frac{\left| \nabla x_i \right|^2}{\lambda_{i,\beta}} + \frac{1}{\lambda_{n,\beta}} \sum_{i=1}^{n-1} \left( 1 - \left| \nabla x_i \right|^2 \right)
\]
\[
\leq \sum_{i=1}^{n-1} \frac{\left| \nabla x_i \right|^2}{\lambda_{i,\beta}} + \sum_{i=1}^{n-1} \frac{1}{\lambda_{i,\beta}} \left( 1 - \left| \nabla x_i \right|^2 \right)
\]
\[
= \sum_{i=1}^{n-1} \frac{1}{\lambda_{i,\beta}}. \tag{2.6}
\]
Combining (2.5) and (2.6), we have
\[
\sum_{i=1}^{n} \int_{\partial \Omega} x_i^2 \leq \sum_{i=1}^{n-1} \frac{\left| \Omega \right|}{\lambda_{i,\beta}} + \sum_{i=1}^{n-1} \frac{\beta \left| \partial \Omega \right|}{\lambda_{i,\beta}}. \tag{2.7}
\]
Substituting (2.1) into (2.7), one gets (1.9). If the equality holds in (1.9), then the inequality (2.6) must take equality sign and (2.4) holds. It is easy to see from the equality case of (2.6) at any point of \( \partial \Omega \) that
\[
\lambda_{1,\beta} = \lambda_{2,\beta} \cdots = \lambda_{n,\beta}.
\]
It then follows that the position vector \( x = (x_1, \cdots, x_n) \) when restricted on \( \partial \Omega \) satisfies
\[
\Delta x = (\Delta x_1, \cdots, \Delta x_n) = -\frac{1}{\beta} \nu - \frac{\lambda_{1,\beta}}{\beta} (x_1, \cdots, x_n). \tag{2.8}
\]
On the other hand, it is well known that
\[
\Delta x = (n - 1)H, \tag{2.9}
\]
where \( H \) is the mean curvature vector of \( \partial \Omega \) in \( \mathbb{R}^n \). Combining (2.8) and (2.9), we have
\[
x = -\frac{1}{\lambda_{1,\beta}} \nu - \frac{(n - 1)\beta}{\lambda_{1,\beta}} H \quad \text{on} \quad \partial \Omega. \tag{2.10}
\]
Consider the function \( g = |x|^2 : M \rightarrow \mathbb{R} \). It is easy to see from (2.10) that
\[
Zg = 2(Z, x) = 0, \quad \forall Z \in \mathcal{X}(\partial \Omega),
\]
where \( \mathcal{X}(\partial \Omega) \) is the tangent bundle of \( \partial \Omega \). Thus \( g \) is a constant function and so \( \partial \Omega \) is a hypersphere. Theorem 1.2 follows. \( \square \)

**Proof of Theorem 1.7** Let \( \{u_i\}_{i=0}^{+\infty} \) be an orthonormal set of eigenfunctions corresponding to the eigenvalues \( \{\lambda_i\}_{i=0}^{+\infty} \) of the eigenvalue problem (1.13), that is,
\[
\begin{aligned}
\sum_{j} u_j &= 0 \quad \text{in} \quad \Omega, \\
-\beta \Delta u + \partial_n u_i &= \lambda_{i,\beta} u_i \quad \text{on} \quad \partial \Omega, \\
\int_{\partial \Omega} \rho u_i u_j &= \delta_{ij}.
\end{aligned}
\]
By (1.6), we have
\[ \lambda_{i,j} = \min_{u \in H(\Omega) \setminus \{0\}} \lambda_{i,j} = \frac{\int_{\partial \Omega} |\nabla u|^2 + \beta \int_{\partial \Omega} u^2}{\int_{\partial \Omega} |u|^2}. \]  

(2.11)

The idea is to use globally defined coordinate functions suitably chosen on $Q^n$ as trial functions for the first $n$ nonzero eigenvalues of the problem (1.13). To do this, let us denote by $\langle \cdot, \cdot \rangle$ the Riemannian metric on $Q^n$. For any $p \in Q^n$, let $\exp_p$ and $UQ^n_p$ be the exponential map and unit tangent space of $Q^n$ at $p$, respectively. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of $T_pQ^n$, the tangent space of $Q^n$ at $p$, and $y : M \rightarrow \mathbb{R}^n$ be the Riemannian coordinates on $Q^n$ determined by $(p; e_1, \cdots, e_n)$. It follows from the Cartan-Hadamard theorem (see, e.g., [7]) that $y$ is well-defined on all of $Q^n$ and is a diffeomorphism of $Q^n$ onto $\mathbb{R}^n$. We can choose $p$ and $\{e_1, \cdots, e_n\}$ so that the respective coordinate functions $y^i : M \rightarrow \mathbb{R}, i = 1, \cdots, n$, of $y : Q^n \rightarrow \mathbb{R}^n$ satisfy
\[ \int_{\partial \Omega} \rho y_i = 0. \]  

(2.12)

In fact, parallelly translate the frame $\{e_1, \cdots, e_n\}$ along every geodesic emanating from $p$ and thereby obtain a differentiable orthonormal frame field $\{E_1, \cdots, E_n\}$ on $Q^n$. Let $y^q : M \rightarrow \mathbb{R}^n$ denote the Riemannian normal coordinates of $Q^n$ determined by $\{E_1, \cdots, E_n\}$ at $q$, and let $y^q_i, i = 1, \cdots, n$, be the coordinate functions of $y^q$. By definition, $y^q_i : M \rightarrow \mathbb{R}$ is given by $y^q_i(z) = \langle \exp_{y^q}^{-1}(z), E_i(q) \rangle, i = 1, \cdots, n$. Then
\[ Y(q) = \sum_{i=1}^n \left\{ \int_{\partial \Omega} \rho y^q_i \right\} E_i(q) \]

is a continuous vector field on $Q^n$. If we restrict $Y$ to a geodesic ball $B$ containing $\Omega$ then the convexity of $B$ implies that on the boundary of $B$, $Y$ points into $B$. The Brouwer fixed point theorem then implies that $Y$ has a zero. So we may assume that $p$ and $\{e_1, \cdots, e_n\}$ actually satisfy (2.12).

For any $w \in UQ^n_p$, let $\theta_w$ be the function on $Q^n$ defined by $\theta_w(z) = \langle \exp_{y^q}^{-1}(z), w \rangle$. Then (2.12) is equivalent to say that for $i = 1, \cdots, n$,
\[ \int_{\partial \Omega} \rho \theta_{e_i} = 0. \]

Thus for any $\sigma \in UQ^n_p$,
\[ \int_{\partial \Omega} \rho \theta_{\sigma} = 0. \]  

(2.13)

Next, we show that there exists an orthonormal basis $\{\sigma_j\}_{j=1}^n$ of $T_pQ^n$ such that the coordinate functions $x_\alpha = \theta_{\sigma_\alpha}, \alpha = 1, \cdots, n$ of the Riemannian normal coordinate system of $Q^n$ determined by $(p; \sigma_1, \cdots, \sigma_n)$ satisfy
\[ \int_{\partial \Omega} \rho x_i u_j = 0, \quad \text{for } i = 2, 3, \cdots, n, \text{ and } j = 1, \cdots, i - 1. \]  

(2.14)
In fact, let us consider the \( n \times n \) matrix \( P = (p_{\alpha \beta})_{n \times n} \), where \( p_{\alpha \beta} = \int_{\partial \Omega} \rho y_\alpha u_\beta = \int_{\partial \Omega} \theta e_\alpha u_\beta \), for \( \alpha, \beta = 1, 2, \ldots, n \). Using the same discussion as that in the proof of Theorem 1.2, we can find an orthogonal matrix \( U = (a_{\alpha \beta})_{n \times n} \) such that

\[
\int_{\partial \Omega} \sum_{\gamma=1}^{n} a_{\alpha \gamma} \rho y_\gamma u_\beta = 0, \quad 1 \leq \beta < \alpha \leq n.
\]

Setting \( \sigma_\alpha = \sum_{\gamma=1}^{n} a_{\alpha \gamma} e_\gamma \), we know that \( \{\sigma_\alpha\}_{\alpha=1}^{n} \) is an orthonormal basis of \( T_p Q_n \) and that the condition (2.14) is satisfied. Therefore, we have from (2.11), (2.13) and (2.14) that

\[
\lambda_{i,\beta} \int_{\partial \Omega} \rho x_i^2 \leq \int_{\partial \Omega} |\nabla x_i|^2 = \beta \int_{\partial \Omega} |\nabla x_i|^2, \quad i = 1, \ldots, n.
\]

Let \( \{\frac{\partial}{\partial x_k}, k = 1, \ldots, n\} \) be the natural basis of the tangent spaces associated with the coordinate chart \( x \) and let \( g_{k\ell} = \langle \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_\ell} \rangle \), \( k, l = 1, \ldots, n \). Since \( Q_n \) has non-positive sectional curvature, the Rauch comparison theorem (see, e.g., [7]) implies that the eigenvalues of the matrix \( (g_{k\ell})_{n \times n} \) are all \( \geq 1 \). Thus the eigenvalues of \( (g_{k\ell}^{-1})_{n \times n} =: (g_{k\ell})_{n \times n} \) are \( \leq 1 \) and so we have \( g^{k\ell} \leq 1, \quad k, l = 1, \ldots, n \). Thus we have for \( i = 1, \ldots, n \),

\[
|\nabla x_i|^2 \leq |\nabla x_i|^2 = g^{ii} \leq 1.
\]

We claim that

\[
\sum_{i=1}^{n} |\nabla x_i|^2 \leq n - 1 \quad \text{on} \quad \partial \Omega.
\]

In fact, for a fixed point \( z \in \partial \Omega \), let \( \{v_1, \ldots, v_{n-1}\} \) be an orthonormal basis of \( T_z(\partial \Omega) \). From the fact that the eigenvalues of \( (g_{ij})_{n \times n} \) are all \( \geq 1 \), we conclude that

\[
\sum_{i=1}^{n} |\nabla x_i|^2 = \sum_{i=1}^{n} \sum_{a=1}^{n-1} (v_a x_i)^2
\]

\[
= \sum_{a=1}^{n-1} \sum_{i,j=1}^{n} (v_a x_i) \delta_{ij} (v_a x_j)
\]

\[
\leq \sum_{a=1}^{n-1} \sum_{i,j=1}^{n} (v_a x_i) g_{ij} (v_a x_j)
\]

\[
= \sum_{a=1}^{n-1} |v_a|^2 = n - 1.
\]

Using (2.15)-(2.17) and the same arguments as in the proof of Theorem 1.2, we obtain

\[
\int_{\partial \Omega} \rho r^2 = \int_{\partial \Omega} \rho \sum_{i=1}^{n} x_i^2 \leq \int_{\partial \Omega} \frac{d(\sigma, \rho \cdot \partial \Omega)}{\lambda_{i,\beta}} + \int_{\partial \Omega} \frac{\beta |\partial \Omega|}{\lambda_{i,\beta}},
\]

where \( r = d(p, \cdot) : \Omega \to \mathbb{R} \) denotes the distance function from \( p \). The Cauchy-Schwarz inequality implies that

\[
\int_{\partial \Omega} \rho r^2 \geq \frac{(\int_{\partial \Omega} r)^2}{\int_{\partial \Omega} \rho^{-1}}.
\]
with equality holding if and only if \( r\rho = \text{const.} \) on \( \partial \Omega \). From the Laplace comparison theorem (cf. [22]), we have
\[
\Delta r^2 \geq 2n. \tag{2.20}
\]
Integrating (2.20) on \( \Omega \) and using the divergence theorem, we get
\[
n|\Omega| \leq \int_{\partial \Omega} r \langle \nabla r, \nu \rangle \leq \int_{\partial \Omega} r |\nabla | = \int_{\partial \Omega} r. \tag{2.21}
\]
Combining (2.18), (2.19) and (2.21), we get (1.14).

Assume now that \( \rho = \rho_0 \) is constant and that the equality holds in (1.14). In this case, we have \( r|_{\partial \Omega} = \text{const.} \) and so \( \Omega \) is a geodesic ball with center \( p \). Also, we have
\[
\Delta r^2 |_{\Omega} = 2n. \tag{2.22}
\]
It then follows from the equality case in the Laplace comparison theorem (cf. [22]) and the Cartan’s theorem (see, e.g., [7]) that \( \Omega \) is isometric to an \( n \)-dimensional Euclidean ball. This completes the proof of Theorem 1.7. \( \square \)

3 Proofs of Theorems 1.9-1.12

In this section, we will give the proofs of Theorems 1.9-1.12 in detail.

Proof of Theorem 1.9. Let \( \{u_i\}_{i=0}^{+\infty} \) be the orthonormal system of eigenfunctions corresponding to the eigenvalues
\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty
\]
of the Laplacian of \( M \), that is,
\[
\Delta u_i = -\lambda_i u_i, \quad \int_M u_i u_j = \delta_{ij}. \tag{3.1}
\]
We have \( u_0 = 1/\sqrt{|M|} \) and for each \( i = 1, \cdots \),
\[
\lambda_i = \min_{u \neq 0, \int_M u u_j = 0, j = 0, \cdots, i-1} \frac{\int_M |\nabla u|^2}{\int_M u^2}. \tag{3.2}
\]
Let \( x_1, \cdots, x_n \) be the coordinate functions on \( \mathbb{R}^n \). By using the same arguments as in the proof of Theorem 1.2, we can assume that
\[
\int_{\Omega} x_i u_j = 0, \quad i = 1, \cdots, n; \quad j = 0, \cdots, i-1
\]
and so we have
\[
\lambda_i \int_M x_i^2 \leq \int_M |\nabla x_i|^2, \quad i = 1, \cdots, n, \tag{3.3}
\]
with equality holding if and only if
\[
\Delta x_i = -\lambda_i x_i.
\]
Since
\[ |\nabla x_i|^2 \leq 1, \quad \sum_{i=1}^{n} |\nabla x_i|^2 = n - 1, \quad (3.4) \]
we have
\[ \sum_{i=1}^{n} \lambda_i^{\frac{1}{2}} |\nabla x_i|^2 = \sum_{i=1}^{n-1} \lambda_i^{\frac{1}{2}} |\nabla x_i|^2 + \lambda_n^{\frac{1}{2}} |\nabla x_n|^2 \]
\[ = \sum_{i=1}^{n-1} \lambda_i^{\frac{1}{2}} |\nabla x_i|^2 + \lambda_n^{\frac{1}{2}} \sum_{i=1}^{n-1} (1 - |\nabla x_i|^2) \]
\[ \geq \sum_{i=1}^{n-1} \lambda_i^{\frac{1}{2}} |\nabla x_i|^2 + \sum_{i=1}^{n-1} \lambda_i^{\frac{1}{2}} (1 - |\nabla x_i|^2) \]
\[ = \sum_{i=1}^{n-1} \lambda_i^{\frac{1}{2}}, \]
which gives
\[ \sum_{i=1}^{n} \lambda_i^{\frac{1}{2}} \int_M |\nabla x_i|^2 \geq |M| \sum_{i=1}^{n-1} \lambda_i^{\frac{1}{2}}. \quad (3.5) \]

For any positive constant \( \delta \), we have from Schwarz inequality and (3.3) that
\[ \sum_{i=1}^{n} \lambda_i^{\frac{1}{2}} \int_M |\nabla x_i|^2 = \sum_{i=1}^{n} \lambda_i^{\frac{1}{2}} \int_M (-x_i \Delta x_i) \]
\[ \leq \frac{1}{2} \sum_{i=1}^{n} \left\{ \delta \lambda_i \int_M x_i^2 + \frac{1}{\delta} \int_M (\Delta x_i)^2 \right\} \]
\[ \leq \frac{1}{2} \left\{ (n - 1) \delta |M| + \frac{1}{\delta} \int_M (n - 1)^2 |H|^2 \right\}. \quad (3.6) \]

Taking
\[ \delta = \left\{ \frac{\int_M (n - 1) |H|^2}{|M|} \right\}^{\frac{1}{2}} \]
in (3.6) and using (3.3), we get (1.16) with equality holding if and only if \( M \) is a hypersphere.

On the other hand, one can use (3.3), (3.4) and a similar argument as that in the proof of Theorem 1.2 to obtain
\[ \sum_{i=1}^{n} \int_M x_i^2 \leq |M| \sum_{i=1}^{n-1} \frac{1}{\lambda_i}, \quad (3.7) \]
which, combining with (2.4) gives (1.17). Also, one can deduce as in the proof of the final part of Theorem 1.2 that equality holds in (1.16) if and only if \( M \) is a sphere of \( \mathbb{R}^n \).

This completes the proof of Theorem 1.9. \( \square \)
Proof of Theorem 1.10. From the proof of Theorem 1.7, we know that there exist a point \( p \in Q^n \) and an orthonormal basis \( \{ e_i \}_{i=1}^n \) of \( T_p Q^n \) such that the coordinate functions of the Riemannian normal coordinate system determined by \( \{ p; (e_1, \cdots, e_n) \} \) satisfy
\[
\int_M x_i \phi_j = 0, \quad i = 1, \cdots, n, \quad j = 0, \cdots, i - 1,
\]
where \( \{ \phi_j \}_{j=0}^{+\infty} \) are orthonormal eigenfunctions corresponding to the eigenvalues
\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty
\]
of the Laplacian of \( M \). Thus we have
\[
\lambda_i \int_M x_i^2 \leq \int_M |\nabla x_i|^2, \quad i = 1, \cdots, n.
\]
Using a similar argument as that in the proof of Theorem 1.7, we conclude
\[
\sum_{i=1}^{n-1} \frac{1}{\lambda_i} \geq \frac{\int_M k^2}{|M|} \geq \frac{n^2 |\Omega|^2}{|M|^2},
\]
with equality holding if and only if \( \Omega \) is isometric a ball in \( \mathbb{R}^n \), where \( r = d(p, \cdot) : \Omega \to \mathbb{R} \) is the distance function from the point \( p \). The proof of Theorem 1.10 is finished.

Proof of Theorem 1.12. Let \( x = (x_1, \cdots, x_n) \) be the coordinate functions on \( \mathbb{R}^n \). Since \( \Omega \) is a bounded domain with smooth boundary \( \partial \Omega \) in \( \mathbb{R}^n \), we can regard \( \partial \Omega \) as a closed hypersurface of \( \mathbb{R}^n \) without boundary. Let \( \Delta, \nabla \) be the Laplace operator and the gradient operator on \( \partial \Omega \), respectively. The position vector \( x = (x_1, \cdots, x_n) \) when restricted on \( \partial \Omega \) satisfies
\[
\sum_{\alpha=1}^n (\Delta x_\alpha)^2 = (n-1)^2 |\mathbf{H}|^2,
\]
where \( |\mathbf{H}| \) is the mean curvature of \( \partial \Omega \) in \( \mathbb{R}^n \).

Let \( u_i \) be the eigenfunction corresponding to eigenvalue \( \lambda_{i,\tau} \) such that \( \{ u_i \}_{i=0}^{\infty} \) becomes an orthonormal basis of \( L^2(\partial \Omega) \), that is,
\[
\begin{cases}
\Delta^2 u_i - \tau \Delta u_i = 0, & \text{in } \Omega \\
\frac{\partial^2 u_i}{\partial \nu^2} = 0 & \text{on } \partial \Omega, \\
\tau \frac{\partial u_i}{\partial \nu} - \text{div}_{\partial \Omega} \left( \nabla u_i \cdot \nu \right) - \frac{\partial \Delta u_i}{\partial \nu} = \lambda_{i,\tau} u_i & \text{on } \partial \Omega, \\
\int_{\partial \Omega} u_i u_j = \delta_{ij}. & \text{on } \partial \Omega
\end{cases}
\]
Observe that \( u_0 = 1/\sqrt{|\partial \Omega|} \). We can assume as before that
\[
\int_{\partial \Omega} x_i u_j = 0, \quad i = 1, \cdots, n, \quad j = 0, \cdots, i - 1.
\]
It follows from the variational characterization \((1.21)\) that
\[
\lambda_{i,\tau} \int_{\partial\Omega} x_i^2 \leq \int_{\Omega} (|\nabla^2 x_i|^2 + \tau |\nabla x_i|^2) = \tau |\Omega|.
\]

Thus we have
\[
\sum_{i=1}^{n} \lambda_{i,\tau} \int_{\partial\Omega} x_i^2 \leq n\tau |\Omega|,
\]
which implies that
\[
\sum_{i=1}^{n} \lambda_{i,\tau}^2 \int_{\partial\Omega} |\nabla x_i|^2 = \sum_{i=1}^{n} \lambda_{i,\tau} \int_{\partial\Omega} (-x_i \Delta x_i)
\leq \frac{1}{2} \sum_{i=1}^{n} \left\{ \delta \lambda_{i,\tau} \int_{\partial\Omega} x_i^2 + \frac{1}{\delta} \int_{\partial\Omega} (\Delta x_i)^2 \right\}
\leq \frac{1}{2} \left\{ \delta n\tau |\Omega| + \frac{1}{\delta} \int_{\partial\Omega} (n-1)^2 |H|^2 \right\}
\]
for any \(\delta > 0\).

Using a similar argument as that in the proof of Theorem \(1.9\) we know that
\[
\sum_{i=1}^{n} \lambda_{i,\tau}^2 \int_{\partial\Omega} |\nabla x_i|^2 \geq |\partial\Omega| \sum_{i=1}^{n} \lambda_{i,\tau}^2.
\]
(3.9)

Combining (3.8) with (3.9) and taking
\[
\delta = \left\{ \frac{\int_{\partial\Omega} (n-1)^2 |H|^2}{n\tau |\Omega|} \right\}^{\frac{1}{2}},
\]
we obtain
\[
\frac{1}{n-1} \sum_{j=1}^{n-1} \lambda_{j,\tau}^2 \leq \frac{n\tau |\Omega| \int_{\partial\Omega} |H|^2}{n\tau |\Omega|}.
\]

If the equality holds in above inequality, we know that
\[
\lambda_{1,\tau} = \cdots = \lambda_{n,\tau} = \Lambda
\]
and
\[
\Delta x_i = -\delta \sqrt{\lambda_{n,\tau}} x_i, \quad i = 1, \cdots, n, \text{ on } \partial\Omega.
\]

Since \(\partial\Omega\) is a closed hypersurface of \(\mathbb{R}^n\), we conclude that \(\partial\Omega\) is a round sphere. This completes the proof of Theorem \(1.12\).
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