Convergence of Ramanujan expansions, I

Multiplicativity on Ramanujan clouds

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Abstract. We call \( R_G(a) := \sum_{q=1}^{\infty} G(q)c_q(a) \) the Ramanujan series, of coefficient \( G : \mathbb{N} \to \mathbb{C} \), where \( c_q(a) \) is the well-known Ramanujan sum. We study the convergence of this series (a preliminary step, to study Ramanujan expansions) and define \( G \) a Ramanujan coefficient when \( R_G(a) \) converges pointwise, in all natural \( a \). Then, \( R_G : \mathbb{N} \to \mathbb{C} \) is well defined (W-D). The Ramanujan cloud of a fixed \( F : \mathbb{N} \to \mathbb{C} \) is \( < F > := \{ G : \mathbb{N} \to \mathbb{C} \mid R_G \) w-d, \( F = R_G \} \). (See the Appendix.) We study in detail the multiplicative Ramanujan coefficients \( G \): their \( < F > \) subset is called the multiplicative Ramanujan cloud, \( < F >_M \).

Our first main result, the “finiteness convergence Theorem”, for \( G \) multiplicative, among other properties equivalent to “\( R_G \) well defined”, reduces the convergence test to a finite set, i.e., \( R_G \) W-D is equivalent to: \( R_G(a) \) converges for all a dividing \( N(G) \in \mathbb{N} \), that we call the “Ramanujan conductor”.

Our second main result, the “finite Euler product explicit formula”, for multiplicative Ramanujan coefficients \( G \), writes \( F = R_G \) as a finite Euler product; thus, \( F \) is a semi-multiplicative function (following Rearick definition) and this product is the Selberg factorization for \( F \). In particular, we have: \( F(a) = R_G(a) \) converges absolutely, being finite (of length depending on non-zero \( p \)-adic valuations of \( a \)).

Our third main result, the “multiplicative Ramanujan clouds”, studies the important subsets of \( < F >_M \); also giving, for all multiplicative \( F \), the canonical Ramanujan coefficient \( G_F \in < F >_M \), proving: any multiplicative \( F \) has a finite Ramanujan expansion with multiplicative coefficients.

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1. Introduction and Main results

We began studying the Ramanujan expansions \( \mathcal{R} \) from a new point of view. In fact, usually, we have a fixed arithmetic function \( F : \mathbb{N} \to \mathbb{C} \) (as usual, \( \mathbb{N} \) and \( \mathbb{C} \) are natural and complex numbers) and we wish to, if possible, find a sequence of “Ramanujan coefficients”, \( G(1), G(2), \ldots, G(q), \ldots \), say a Ramanujan coefficient \( G : \mathbb{N} \to \mathbb{C} \) such that

\[
F(n) = \sum_{q=1}^{\infty} G(q)c_q(n),
\]

say, for all the natural numbers \( n \). Here, as usual, the trigonometric sum \( c_q(n) \) is the Ramanujan sum \( \mathcal{R} \):

\[
c_q(n) \overset{def}{=} \sum_{j \leq q, (j,q)=1} \cos \frac{2\pi jn}{q},
\]

where we write \( (a,b) \) as an abbreviation for their g.c.d. : so, \( (j,q) = 1 \) means they’re coprime.
First of all, we know that, for each fixed $F : \mathbb{N} \to \mathbb{C}$, there are a lot (uncountably many!) Ramanujan coefficients, compare [C3].

Second, the problem of finding, once fixed $F$, all $G$ such that the above expansion holds, needs of course, first, to check the convergence! However even this (seemingly) easier problem, in absence of properties for $G$, could be a hopeless, at least huge, task. So, in the present paper we will confine ourselves with multiplicative Ramanujan coefficients $G$. (This general rule will be relaxed in the Appendix.)

In the following, we'll abbreviate "if and only if", with "iff".

We define $G$ multiplicative iff $G(ab) = G(a)G(b)$, for all coprime $a, b \in \mathbb{N}$; so, including the constant $0$–function, or null-function, say $0(a) \overset{\text{def}}{=} 0, \forall a \in \mathbb{N}$ (however, $G \neq 0$ multiplicative $\Rightarrow G(1) = 1$).

In future papers, but with a glance already in this (compare the Appendix), we will face the problem of finding, once fixed $F : \mathbb{N} \to \mathbb{C}$, ALL its Ramanujan coefficients $G$.

In all, our approach will be to fix a $G : \mathbb{N} \to \mathbb{C}$, in this paper (except Appendix) multiplicative, and first check for the convergence in $\mathbb{N}$ of the corresponding Ramanujan series (compare [C3])

$$R_G(a) \overset{\text{def}}{=} \sum_{q=1}^{\infty} G(q)c_q(a), \forall a \in \mathbb{N};$$

then, try to describe, from the hypotheses on $G$, the properties of its sum, say $F(a) = R_G(a)$.

An important warning: hereafter, when we say "Ramanujan coefficient", we mean any arithmetic function $G : \mathbb{N} \to \mathbb{C}$, such that Ramanujan series $R_G(a)$ converges pointwise in all natural numbers $a$. (For example, in [La] the Author considers also convergence in subsets of $\mathbb{N}$.)

In other words, we wish to study the so-called Ramanujan Cloud [C3] of a fixed $F : \mathbb{N} \to \mathbb{C}$, that is the set of its Ramanujan coefficients $\overset{\text{def}}{=} < F >$; but, since this is too difficult (at least nowadays), we study parts of a cloud, namely we look for subsets, of this $< F >$.

Let us say, the most beautiful subset of a cloud (up to now, also considering [C3] subsets, like $< F >_*$, $< F >_{**}$ and $< F >_{##}$, compare the Appendix) is the so-called Multiplicative Ramanujan Cloud, i.e.:

$$< F >_M \overset{\text{def}}{=} \{ G \in < F > : G \text{ multiplicative} \}.$$

Our main concern is the study of this $< F >_M$ and, also, of how the $F$ properties and the $G$ properties influence each other. One typical question, for example: what properties, of a fixed $F$, ensure $< F >_M \neq \emptyset$?

The first thing to check is the convergence of $R_G$ and so our first result gives a set of properties, all equivalent to: $G$ is a Ramanujan coefficient (recall, one last time, meaning: $R_G(a)$ converges $\forall a \in \mathbb{N}$).

A kind of first elementary approach to this problem began in [C2], in which the first Author studied the “finite Euler products”, of Ramanujan series. There, we encounter the coprime series, with the usual Möbius function $\mu$ (see §2)

$$S_G(a) \overset{\text{def}}{=} \sum_{(r,a)=1} G(r)\mu(r), \forall a \in \mathbb{N}.$$

Then, the Authors started to study the multiplicative cloud of 0, the null-function (see our [CG]), but under hypotheses of convergence, for these coprime series. Here, thanks to a wider view on convergence conditions (compare next Theorem 1.1), we are able to complete this $< 0 >_M$ Classification, see §9.1.

Then, we realized that a much better understanding of Ramanujan expansions convergence might come from classic general results (of pointwise convergence), due mainly to Lucht [Lu1], compare the survey [Lu2]. In building upon Lucht’s insight, compare Corollary 3.7 and see §4, we define a new series: the Lucht series

$$L_G(d) \overset{\text{def}}{=} \sum_{K=1}^{\infty} \mu(K)G(dK), \forall d \in \mathbb{N}.$$

Notice an important link, namely (in case of common convergence in 1)

$$R_G(1) = L_G(1) = S_G(1).$$
Hence, we have trivially:
\[ R_N(1) = \mathcal{L}_G(1) \]

After we know how to characterize the convergence in \( \mathcal{R}_G \), we prove in Section 6.

**Theorem 1.1. (Finiteness convergence theorem)** Let \( G \) be a multiplicative function, such that \( \mathcal{R}_G(1) = \mathcal{S}_G(1) = \mathcal{L}_G(1) \) converges. Then the following are equivalent:

1. \( \mathcal{R}_G(a) \) converges for all \( a \in \mathbb{N} \);
2. \( \mathcal{R}_G(a) \) converges for all \( a \mid N(G) \);
3. \( \mathcal{S}_G(N(G)) \) converges;
4. \( \mathcal{S}_G(b) \) converges for all \( b \) not divisible by any hyperbad prime \( p \in \mathcal{B}^\infty \).

We prove this theorem in Section 7, via the sequence of implications: \( (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1) \).

The implication \( (1) \Rightarrow (2) \) is trivial, while the implication \( (3) \Rightarrow (4) \) will follow from the “Converse convergence Theorem”, that we prove in Section 6.

The formula, for Ramanujan series \( \mathcal{R}_G \) and coprime series \( \mathcal{S}_G \), provided in Section 3 will prove, in particular from Corollaries 3.3 and 3.4, the implication \( (4) \Rightarrow (1) \).

The implication \( (2) \Rightarrow (3) \) is the “most difficult” part of the proof; in fact, it is based on a formula (see §7) at the heart of the links in between \( \mathcal{R}_G, \mathcal{S}_G \) and \( \mathcal{L}_G \), involving the three variables function \( \mathcal{T}_G \), see §2.

This key formula is stated in Theorem 7.1.

The complete proof of Theorem 1.1 is given in §7.

After we know how to characterize the convergence in \( \mathbb{N} \), with previous result, we wish to know which kind of function is \( \mathcal{R}_G : \mathbb{N} \to \mathbb{C} \) (always well-defined in \( \mathbb{N} \) hereafter).

For this, we give Euler products formulae, that might seem expansions into infinite products, but at a closer inspection reveal to be, actually, **finite Euler products**! A kind of unexpected result is that multiplicative Ramanujan coefficients \( G \) give Ramanujan expansions \( \mathcal{R}_G \) that are not multiplicative functions, but only slightly more general arithmetic functions: the \( \mathcal{R}_G \) are semi-multiplicative. (Compare section end, where semi-multiplicative functions are defined and also characterized in terms of finite Euler products.)

The “Ramanujan series factorization”, Theorem 8.1 in §8, implies in fact a kind of explicit form in finite Euler products. Recall, from §2, the \( p \)-adic valuations \( v_p(a) \) and \( v_{p,G} \), like Definitions A and B.

**Corollary 1.2. (Finite Euler products explicit formula)**

Let \( G : \mathbb{N} \to \mathbb{C} \) be a multiplicative function such that \( \mathcal{R}_G \) converges in \( \mathbb{N} \). Then \( \mathcal{R}_G(a) = 0 \) for every \( a \neq 0 \) mod \( N_T(G) \). Moreover, there is a multiplicative function \( \mathcal{M}_G : \mathbb{N} \to \mathbb{C} \) such that for all \( a \in \mathbb{N} \):

\[
\mathcal{R}_G(aN_T(G)) = \mathcal{R}_G(N_T(G)) \cdot \mathcal{M}_G(a) = \mathcal{R}_G(N_T(G)) \cdot \prod_{p \mid a} \mathcal{M}_G(p^{v_{p,G}(a)}).
\]

Hence, we have trivially:

\[
\mathcal{R}_G(N_T(G)) = 0 \Rightarrow \mathcal{R}_G = 0.
\]

If \( \mathcal{R}_G(N_T(G)) \neq 0 \), then the function \( \mathcal{M}_G \) is uniquely determined, with the following values at prime powers:

\[
\mathcal{M}_G(p^K) \overset{\text{def}}{=} 1 + p + \ldots + p^K, \quad \text{if } p \text{ is hypertransparent};
\]

\[
\mathcal{M}_G(p^K) \overset{\text{def}}{=} (1 - G(p^{v_{p,G}+1}))^{-1} \sum_{k=0}^{K} p^k (G(p^{v_{p,G}+k}) - G(p^{v_{p,G}+k+1})), \quad \text{if } p \text{ is simply transparent};
\]

\[
\mathcal{M}_G(p^K) \overset{\text{def}}{=} (1 - G(p))^{-1} \sum_{k=0}^{K} p^k (G(p^k) - G(p^{k+1})), \quad \text{otherwise}.
\]

Furthermore, in the hypothesis \( G \) multiplicative Ramanujan coefficient, we get the general formula:

\[
\mathcal{R}_G(a) = \mathcal{R}_G(N_T(G)) \cdot \prod_{v_{p,G} = \infty} \left( \sum_{v=0}^{v_{p,G}} p^v \right) \cdot \prod_{v_{p,G} \neq \infty} \left( \sum_{v=0}^{v_{p,G}} p^v - v_{p,G} G(p^v) - G(p^{v+1}) \right), \quad \forall a \in \mathbb{N}.
\]
Note that all the above products are finite: all but finitely many factors are equal to 1, because \(v_{p,G} = v_p(a) = 0\) for all but finitely many primes. Also, when \(\mathcal{R}_G(N_T(G)) = 0\), the value of this right-hand side is unimportant, because of previous trivial implication.

We prove this Corollary in §8, after having proved its, say, relative Theorem 8.1 (in section 8).

We give here the necessary definitions and properties of semi-multiplicative and Selberg-multiplicative arithmetic functions, that we apply to interpret our results. Recall, from §2, that \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\).

The semi-multiplicative arithmetic functions \(F : \mathbb{N} \rightarrow \mathbb{C}\) where defined by Rearick [Re] as those satisfying, for a certain complex \(c \neq 0\) and an integer \(a_F > 0\),

\[
F(n) = c M_F(n/a_F) \; \forall n \in \mathbb{N},
\]

where the function \(M_F\), vanishing outside \(\mathbb{N}\) (so \(F(n) = 0\), on all \(n \neq 0 \mod a_F\), is multiplicative. Notice that \(F = 0\) iff \(M_F = 0\) and that, whenever \(F \neq 0\), we have \(c = F(a_F)\), where \(a_F = \min \{v \in \mathbb{N} : F(v) \neq 0\}\) is called the threshold. Of course, \(F \neq 0\) multiplicative implies \(F\)-semi-multiplicative, with \(a_F = 1\) and \(c = 1\).

Another generalization of “\(F : \mathbb{N} \rightarrow \mathbb{C}\) multiplicative” is when \(F\) is Selberg-multiplicative:

\[
F(a) = \prod_{p \in P} F_p(v_p(a)), \forall a \in \mathbb{N}
\]

(recall: \(p\)-adic valuation \(v_p(a)\) in §2), where the functions \(F_p : \mathbb{N}_0 \rightarrow \mathbb{C}\) have \(F_p(0) = 1\) for all but finitely many primes \(p\). The importance of this Selberg factorization comes from the explicit form we can give to its factors \(F_p\) and, also, that Selberg multiplicativity is the same as semi-multiplicativity: both these properties were proved, by Haukkanen, see [Hau], Theorem 2.8, that we quote, here.

**Proposition 1.3 (Semi-multiplicative \(F\) are Selberg-multiplicative : explicit factorization)**

An arithmetic function \(F : \mathbb{N} \rightarrow \mathbb{C}\) is Selberg multiplicative iff it is semi-multiplicative. In this case,

\[
F(a) = F(a_F) \prod_{p \in P} \left( \frac{F(a_F p^{v_p(a)} - v_p(a_F))}{F(a_F)} \right)
\]

is the Selberg factorization of our \(F\).

In fact: take \(F = \mathcal{R}_G\), \(a_F = N_T(G) \geq 1\) and \(c = F(a_F) = \mathcal{R}_G(N_T(G)) \neq 0\) (otherwise, \(F = 0\)) in Corollary 1.2; so, it proves, in particular, that, for multiplicative Ramanujan coefficients \(G\), the Ramanujan expansion \(\mathcal{R}_G\) is a semi-multiplicative function, with above Selberg factorization. Notice, once again, that it is, of course, a finite Euler product. Sometimes we’ll call it the Euler-Selberg product of our Ramanujan expansion.

We arrive to the structure of multiplicative Ramanujan clouds. Next Corollary is a kind of summary of all our results on multiplicative Ramanujan clouds (see §9) and its important subsets (§4, §5), for which see §2 definitions. Thus we will not prove it.

**Corollary 1.4 (Multiplicative Ramanujan Clouds)**

We start with the easiest multiplicative Ramanujan cloud: that of \(0\), the null-function:

\[
< 0 >_M = \{G : \mathbb{N} \rightarrow \mathbb{C} \mid G \text{ multiplicative, with } S_G(N(G)) = 0\}.
\]

Let \(F \neq 0\) hereafter.

Then \(G\) is multiplicative \(\iff\) \(\mathcal{R}_G = F\) is semi-multiplicative. (The \(\Rightarrow\) from Corollary 1.2 and the \(\Leftarrow\) from Proposition 9.8.) In this case, \(\mathcal{R}_G(N_T(G)) \neq 0\); otherwise, by previous Corollary 1.2, \(F = 0\).

Coming to remarkable subsets of \(< F >_M\) (recall their definitions in §2), consider first the Euler separating multiplicative Ramanujan cloud of a fixed, non-null multiplicative \(F : \mathbb{N} \rightarrow \mathbb{C}\), i.e.

\[
F \neq 0 \text{ IS multiplicative } \iff < F >_{ESM} = \{G_F\},
\]
where the $G_F$ is the canonical Ramanujan coefficient of this $F$, see §2. In particular, (compare §4.A.6) 

$F \neq 0$ is multiplicative $\iff$ $F = R_G$ is finite and pure with multiplicative $G = G_F$.

The separating multiplicative Ramanujan cloud of a fixed $F : \mathbb{N} \rightarrow \mathbb{C}$ is, as follows,

\[
<F>_{SM} \neq \emptyset \iff F \text{ is quasi-multiplicative.}
\]

In fact, the multiplicative Ramanujan coefficients in it, say $G$, have $S_G/F(1)$, $F/F(1) = R_G/F(1)$ and $L_G/F(1)$ that are all well defined (with $S_G(1) = R_G(1) = L_G(1) = F(1) \neq 0$, of course) and multiplicative. Thus, in particular,

\[
<F>_{NSM} \neq \emptyset \iff F \text{ is multiplicative.}
\]

The Ramanujan coefficients, say $G$, in the multiplicative Ramanujan cloud of a semi-multiplicative $F$, i.e., $G \in <F>_M$, are determined (inside $<F>_M$) by a square-free supported arithmetic function, say $H_G$, called the "opacity core" of $G$ (see in §9.2, Theorem 9.3).

Finally, the complete Classification for the $<F>_M$, when $F \neq 0$, is given in Proposition 9.8.
2. Notation, definitions & basic formulæ

Typographic definitions: ◦ is a Remark’s end, “QED” is to separate parts of a proof, which ends with □

We abbreviate \(\mathbb{N}_0 := \mathbb{N} \cup \{0\}\), with \(\mathbb{N}\) set of natural numbers; while \(\mathbb{P}\) set of primes, usually denoted \(p\).

An arithmetic function is any \(F : \mathbb{N} \rightarrow \mathbb{C}\) (“a sequence of complex numbers”, outside Number Theory). The characteristic function of a subset \(\mathcal{H} \subseteq \mathbb{N}\) of natural numbers, an arithmetic function, will be as usual: \(\mathbbm{1}_\mathcal{H}(n) \overset{\text{def}}{=} 1\), iff \(n \in \mathcal{H}\) \((\overset{\text{def}}{=} 0\), otherwise); in particular \(\mathbbm{1} \overset{\text{def}}{=} \mathbbm{1}_{\mathbb{N}}\) is the constant \(1\) function.

The completely multiplicative arithmetic functions are \(f : \mathbb{N} \rightarrow \mathbb{C}\) with \(f(ab) = f(a)f(b), \forall a, b \in \mathbb{N}\). If this happens when \(a, b \in \mathbb{N}\) are coprime, we call it multiplicative. The null-function (or \(0\)-constant function) \(0(a) \overset{\text{def}}{=} 0, \forall a \in \mathbb{N}\), is (here) a multiplicative function: the only one with \(f(1) = 0\), all others have \(f(1) = 1\).

Also, \(1_{\mathbb{N}}\) will denote the function which is always 0, except in 1: \(1_{\mathbb{N}}(1) \overset{\text{def}}{=} 1\), a completely multiplicative arithmetic function. (For \(0, 1, 1_{\mathbb{N}}\), compare [La].) Once fixed any \(p\), the classic \(p\)-adic valuation is:

\[ v_p(a) \overset{\text{def}}{=} \max\{k \in \mathbb{N}_0 : p^k | a\}, \forall a \in \mathbb{N}, \] a completely additive function: \(v_p(ab) = v_p(a) + v_p(b), \forall a, b \in \mathbb{N}\).

The Ramanujan sum of modulus \(q \in \mathbb{N}\) and argument \(a \in \mathbb{N}\), \(c_q(a)\), defined above, has, resp., Kluyver’s formula [K], (3) in [M], and Hölder’s formula [Hö], [M] (we’ll use both extensively throughout the paper):

\[ c_q(a) = \sum_{d | q, d | a} d\mu(q/d), \quad c_q(a) = \varphi(q)\frac{\mu(q/(q,a))}{\varphi(q/(a,q))} \quad \forall q, a \in \mathbb{N}. \]

Here \(\mu\) is Möbius function, a multiplicative function with \(\mu(1) \overset{\text{def}}{=} 1, \mu(p) \overset{\text{def}}{=} -1, \forall p \in \mathbb{P}\), while \(\exists p \in \mathbb{P}: p^2 | q\) implies \(\mu(q) \overset{\text{def}}{=} 0\). In fact, \(\mu = 0\) outside the square-free numbers: these have characteristic function \(\mu^2\).

Also, the Euler function is \(\varphi(q) \overset{\text{def}}{=} |\{n \leq q : (n, q) = 1\}|\), the cardinality (we’ll use \(|A| \overset{\text{def}}{=} \text{number of elements, for finite sets } A\) of naturals, below \(q\), coprime with \(q\). Like \(\mu\), it is multiplicative.

Given \(G : \mathbb{N} \rightarrow \mathbb{C}\) multiplicative and \(p\) prime, we introduce the quantity \(w_{p,G} \in \mathbb{N} \cup \{\infty\}\) given by

\[ w_{p,G} \overset{\text{def}}{=} \max\{w \in \mathbb{N} : G(p^k) = G(p)^k, \forall k \leq w\}, \]

with the convention that \(w_{p,G} = \infty\) if this set is empty (that is, if \(G\) is hypertransparent). In general, the quantity \(v_{p,G}\) should not be confused with \(w_{p,G}\): if \(G(p) \neq 1\) we have \(v_{p,G} = 0\), whereas \(w_{p,G} \geq 1\) for all \(p\) prime. Note however that \(v_{p,G} = w_{p,G}\) if \(G(p) = 1\) (that is, when \(p\) is a transparent prime). Recall from [CG]: an arithmetic function \(G\) is normal iff (by definition) it’s multiplicative and has no transparent primes. Last but not least, see that we sometimes call \(v_{p,G}\) the \(p\)-adic valuation of \(G\) (on transparent primes).

Next, we introduce two notions of “conductors”, which encode the information coming from simply bad and simply transparent primes.
Definition B. (Ramanujan Conductor and Transparency Conductor)
If $G$ is a multiplicative function with finitely many bad primes, we define its **Ramanujan conductor** $N(G)$ and its **transparency conductor** $N_T(G)$ via the C.M. index $w_{p,G}$:

$$N(G) \overset{\text{def}}{=} \prod_{p \in \mathcal{B}} p^{w_{p,G}} \quad \text{and} \quad N_T(G) \overset{\text{def}}{=} \prod_{p \in \mathcal{T}} p^{w_{p,G}} = \prod_{p \in \mathcal{T}} p^{v_{p,G}}.$$  

Note that the products above are restricted to simply bad and simply transparent primes respectively. If $G$ has infinitely many bad primes, we agree that $N(G) \overset{\text{def}}{=} N_T(G) \overset{\text{def}}{=} 0$.

We recall a classical object, in Number Theory, whose name is due to Wintner.

Definition C. (Eratosthenes Transforms of Arbitrary Arithmetic Functions)
Given any arithmetic function $f : \mathbb{N} \to \mathbb{C}$, its **Eratosthenes Transform** (Wintner’s [Wi] terminology) is

$$f' \overset{\text{def}}{=} f * \mu,$$

where $*$ is the usual Dirichlet product (see [T]) between arithmetic functions; by Möbius inversion [T], this is equivalent to:

$$f = f' * 1, \text{ i.e., } f(n) = \sum_{d|n} f'(d), \forall n \in \mathbb{N}.$$  

Recall: $f$ is multiplicative iff $f'$ is multiplicative [T]. By the way, every divisor is a natural number, here.

A set (of natural numbers) may be very useful, when passing from some naturals to their divisors.

Definition D. (Divisor-Closed Subsets of Natural Numbers)
We call $\mathcal{D} \subseteq \mathbb{N}, \mathcal{D} \neq \emptyset$ a divisor-closed set iff

$$a \in \mathcal{D}, d|a \implies d \in \mathcal{D}.$$  

Throughout the paper it will be useful to use truncated versions of the above series (we’ll also drop, at some point, the subscript “$G$”, when the multiplicative function $G$ is implicit): we define, $\forall a \in \mathbb{N}, \forall x \geq 0$,

$$\mathcal{R}_G(a,x) \overset{\text{def}}{=} \sum_{q \leq x} G(q)c_q(a); \quad \mathcal{L}_G(a,x) \overset{\text{def}}{=} \sum_{K \leq x} \mu(K)G(aK); \quad \mathcal{S}_G(a,x) \overset{\text{def}}{=} \sum_{r \leq x \atop (r,a)=1} G(r)\mu(r).$$

(Of course, $0 \leq x < 1$ gives empty sums, which are 0: however, we consider all real $x \geq 0$, for technical reasons in the Converse Convergence Theorem §6. Also, this convention simplifies details in formulæ.)

A kind of unification of these partial sums is, for $G : \mathbb{N} \to \mathbb{C}$ and the triple $(a,b,c) \in \mathbb{N}^3$ both fixed:

$$\mathcal{F}_G(a,b,c)(x) \overset{\text{def}}{=} \sum_{q \leq x \atop (q,b)=1} G(cq)c_q(a),$$

which, if in the limit $x \to \infty$ converges (in $\mathbb{C}$, for the same $(a,b,c) \in \mathbb{N}^3$ and $G$ fixed), gives the “$\mathcal{RSL}$ series”:

$$\mathcal{F}_G(a,b,c) \overset{\text{def}}{=} \sum_{q=1 \atop (q,b)=1}^{\infty} G(cq)c_q(a).$$

Notice: this case (i.e., of convergence) entails that all of our three series,

$$\mathcal{F}_G(a,1,1) = \mathcal{R}_G(a), \quad \mathcal{F}_G(1,b,1) = \mathcal{S}_G(b), \quad \mathcal{F}_G(1,1,c) = \mathcal{L}_G(c).$$
Definition E. \(F\) normalized separating multiplicative cloud is an important subset of previous \(F\), which may be empty; however, for multiplicative \(F\), dilating factor see \(F\).

We pass now to the definition of finite sums, playing the part of finite factors (in Euler products formulae).

We define two sums: \(E_G(a)\), the Euler-Ramanujan factor, and, resp., \(U_G(a)\), the Lucht-Ramanujan factor, that appear in the factorization of \(R_G(a)\) and, resp., \(L_G(a)\). For any multiplicative \(G : \mathbb{N} \to \mathbb{C}\), given any \(a \in \mathbb{N}\), \(\forall p \in \mathbb{P}\), recall \(v_p(a)\) is the \(p\)-adic valuation of \(a\); the \(p\)-Euler factor of \(G(q)c_q(a)\) is:

\[
E_{p,G}(a) \overset{\text{def}}{=} \sum_{K=0}^{v_p(a)+1} G(p^K)c_{p^K}(a) = \sum_{d|a \text{ rad } a} G(d)c_d(a) = \prod_{p|a} E_{p,G}(a)
\]

(we leave this equation as a kind of exercise for the reader), for which see [Main Lemma, C2] formulae:

\[
E_{p,G}(a) = \sum_{K=0}^{\infty} G(p^K)c_{p^K}(a) = \sum_{K=0}^{v_p(a)} G(p^K)c_{p^K}(a) = \sum_{K=0}^{v_p(a)} p^K (G(p^K) - G(p^{K+1}))
\]

and (for next equation, see Corollary 3.5 proof):

\[
U_{p,G}(a) \overset{\text{def}}{=} G(p^{v_p(a)}) - G(p^{v_p(a)+1}) \Rightarrow U_G(a) \overset{\text{def}}{=} \sum_{d|a} \mu(d)G(da) = \prod_{p|a} U_{p,G}(a).
\]

Last but not least, we define the other two sums, relative to \(G\), for all \(a \in \mathbb{N}\):

\[
E_G(a) \overset{\text{def}}{=} \sum_{d|a} G(d)d, \quad D_G(a) \overset{\text{def}}{=} \sum_{d|a} G(d)d,
\]

say, the dilating factor (for \(G\) entails a kind of dilation on \(a\) divisors, whenever \(|G(p)| > 1\) on primes).

We come, of course, to clouds, now. The Ramanujan cloud, of an arbitrary \(F : \mathbb{N} \to \mathbb{C}\) is

\[
< F > \overset{\text{def}}{=} \{ G : \mathbb{N} \to \mathbb{C} \mid R_G \text{ is well-defined and } F = R_G \},
\]

like defined in [CG], the set of Ramanujan coefficients of this \(F\). (See Appendix.A.6: for each \(F\), \(< F > \neq \emptyset\).)

Its most important subset, in this paper, is the multiplicative Ramanujan cloud of the same \(F\) above:

\[
< F >_M \overset{\text{def}}{=} \{ G \in < F > \mid G \text{ is multiplicative} \},
\]

which may be empty; however, for multiplicative \(F \neq 0\), \(< F >_M \neq \emptyset\), see §5: in which we study the important subset of previous \(< F >_M\), that we call the separating multiplicative cloud, of our \(F\), i.e.:

\[
< F >_{SM} \overset{\text{def}}{=} \{ G \in < F >_M \mid S_G \text{ is well-defined and } S_G(1) \neq 0 \},
\]

having in turn the subset which we call the normalized separating multiplicative cloud

\[
< F >_{NSM} \overset{\text{def}}{=} \{ G \in < F >_M \mid S_G \text{ is well-defined and } S_G(1) = 1 \},
\]

to which we give a glance in §5. Finally, its notable subset, the Euler separating multiplicative cloud

\[
< F >_{ESM} \overset{\text{def}}{=} \{ G \in < F >_M \mid S_G \text{ is well-defined and } S_G = 1 \},
\]

is, for multiplicative \(F \neq 0\), a singleton (compare Appendix.A.6), \(\{G_F\} : \text{ Theorem 5.1 and next definition.}

Last but not least, the construction in Theorem 5.1 gives the following Ramanujan coefficient.

**Definition E.** (Canonical Ramanujan Coefficient of a multiplicative \(F \neq 0\))

\[
G_F(q) \overset{\text{def}}{=} \prod_{p|q} \left( 1 - \sum_{K=0}^{v_p(a)-1} \frac{F^*(p^K)}{p^K} \right), \quad \forall q \in \mathbb{N}.
\]
3. Ramanujan, Lucht and coprime series: from infinite to finite Euler products

We start quoting a result in our previous paper, [CG], for which we may assume absolute convergence of $\mathcal{R}_G(a)$, for fixed $a$, from absolute convergence of the series of $G(q)|\mu(q)$, see Lemma 4 in §7.1. We recognize, in passing, that this is exactly the absolute convergence of $S_G(1)$, whence that of every $S_G(d)$. Then, with this absolute convergence hypothesis, we may write (compare [C2] Proposition):

$$
\mathcal{R}_G(a) = \prod_{p \in \mathcal{D}} \sum_{K=0}^{\infty} G(p^K)c_{p^K}(a) = \prod_{p \nmid a} \sum_{K=0}^{\infty} G(p^K)c_{p^K}(a) \prod_{p \mid a} (1 - G(p)),
$$

where the Euler product over $p \nmid a$ is an infinite, but absolutely convergent one. (Compare §3 of [C2].)

This shows clearly that the problem of pointwise convergence of coprime series, without the absolute convergence hypothesis, is a delicate one. Instead, the absolute convergence for $S_G$ ensures:

$$
S_G(a) = \prod_{p \nmid a} \sum_{K=0}^{\infty} \mu(p^K)c_{p^K}(a) = \prod_{p \nmid a} (1 - G(p)),
$$

thus it’s a triviality to recover the factorization $\mathcal{R}_G = \mathcal{E}_G \cdot S_G$ (here $\cdot$ is pointwise product, recall $\mathcal{E}_G$ definition in §2). Also, absolute convergence makes (set $K = tm$ here)

$$
\sum_{K=1}^{\infty} |\mu(K)G(dK)| \leq \sum_{t|d} \sum_{K=1}^{\infty} |G(dK)| |\mu^2(K)| \leq \sum_{t|d} |\mu^2(t)||G(dt)| \sum_{m\geq 1} \sum_{\left\{m, d\right\} \neq 1} |G(m)||\mu^2(m) < \infty,
$$

ensuring, also, the absolute convergence of $\mathcal{L}_G(a)$, so we may pass to the limit $x \to \infty$ inside:

$$
\sum_{K \leq x} \mu(K)G(dK) = \sum_{t|d} \sum_{K \leq x} G(dK)\mu(K) = \sum_{t|d} \mu(t)G(dt) \sum_{m \leq x/t} \sum_{\left\{m, d\right\} \neq 1} G(m)\mu(m)
$$

getting: (recall $\mathcal{U}_G$ formulæ, in §2)

$$
\mathcal{L}_G(d) = \prod_{p|d} \left(G(p^{v_p(d)}) - G(p^{v_p(d)+1})\right) \prod_{p|d} \sum_{K=0}^{\infty} \mu(p^K)G(p^K) = \mathcal{U}_G(d) \cdot \prod_{p|d} (1 - G(p)),
$$

thus it’s again a triviality to factor $\mathcal{L}_G = \mathcal{U}_G \cdot S_G$.

Of course, absolute convergence (of $S_G$, meaning : of $S_G(1)$), entailing that of all $S_G(d)$ is a very powerful and useful hypothesis; but we wish not to confine the general frame, so to speak, of our series : in the following, we wish to reach the same conclusions (i.e., factorizations) in the less restrictive assumption, on $S_G$ series, of pointwise convergence.

Thus, in the following we will assume convergence of $S_G(a)$ to factor, first $\mathcal{R}_G(a)$ (next Corollary 3.3) and then $\mathcal{L}_G(a)$ (Corollary 3.5).

Also, the same philosophy, that we explain in the following (soon before Corollary 3.1), allows to give a kind of “recursive formulæ”, for $S_G$ (Corollary 3.2), for $\mathcal{R}_G$ (Corollary 3.4) and for $\mathcal{L}_G$ (Corollary 3.6). These involve only one series at a time, on which convergence hypotheses, “ad hoc”, are required; and factors them, according to the corresponding factorization of their argument $a$. This strategy is a precious tool, when proving Theorem 1.1 and Theorem 8.1.

Last but not least, we’ll connect Ramanujan and Lucht series : $G$ is a Ramanujan coefficient (see the above) iff it is a LUCHT COEFFICIENT, namely (by definition) $\mathcal{L}_G(a)$ is pointwise converging in all $a \in \mathbb{N}$. (Actually, this equivalence was introduced and proved by Professor Lucht in [Lu1], justifying the names, here.) This is proved in Corollary 3.7; that, actually, proves even more: recalling the Definition D §2 of divisor-closed set, $\mathcal{L}_G$ converges in a divisor-closed set $\mathcal{D}$ iff $\mathcal{R}_G$ converges in $\mathcal{D}$. 
When dealing with an infinite sum of the form $\sum_{q=1}^{\infty} h(q)$, where $h : \mathbb{N} \to \mathbb{C}$ is a multiplicative function, a basic technique is to try to write such sum as an infinite Euler product. We can always extract the Euler factors one at at time, to reduce the issues related to convergence: at least formally, we have, for fixed $p$,

$$\sum_{q=1}^{\infty} h(q) = \sum_{k=0}^{\infty} h(p^k) \sum_{(r,p)=1} h(r).$$

In fact, this factorization is justified as long as the two factors on the RHS (right-hand side) converge. More generally, we have the following result, which may be considered (like all §3 Corollaries) a “consequence”, of the arguments proving MAIN LEMMA in [C2]: mainly, the vertical limit $c_{p,K}(a) = 0$, for all $K > v_p(a) + 1$.

**Corollary 3.1.** (Prototype of finite Euler products extraction)

Let $h : \mathbb{N} \to \mathbb{C}$ be a multiplicative function and let $b \in \mathbb{N}$. Suppose that $\sum_{(r,b)=1} h(r)$ converges and assume that, for each $p|b$, $\exists m_h(p) \in \mathbb{N} : h(p^k) = 0$, for all large $k$. Then also $\sum_{q=1}^{\infty} h(q)$ converges and moreover we have

$$\sum_{q=1}^{\infty} h(q) = \sum_{d \leq x} h(d) \sum_{(r,b)=1} h(r) = \prod_{p|b} \left( \sum_{k=0}^{\infty} h(p^k) \right) \sum_{(r,b)=1} h(r).$$

Proof. Since all $q \in \mathbb{N}$ may be written $q = dr$, where $\text{rad} d | b$ and $(r,b) = 1$, entailing $(d,r) = 1$, if we set $D \overset{\text{def}}{=} \prod_{p|b} p^{m_h(p)}$, we get

$$x > D \Rightarrow \sum_{q \leq x} h(q) = \sum_{d \leq x \text{ rad} d | b} h(d) \sum_{r \leq x/d} h(r) = \sum_{d \leq x \text{ rad} d | b} h(d) \sum_{r \leq x/d} h(r) = \sum_{d \leq x \text{ rad} d | b} h(d) \sum_{(r,b)=1} h(r),$$

so, passing to the limit $x \to \infty$, we obtain first the convergence of LHS (left-hand side) and then, from

$$\sum_{d \leq x \text{ rad} d | b} h(d) = \prod_{p|b} \left( \sum_{k=0}^{\infty} h(p^k) \right) = \prod_{p|b} \left( \sum_{k=0}^{\infty} h(p^k) \right),$$

also the formula in the thesis. \hfill \Box

The convergence of the Euler factor $\sum_{k=0}^{\infty} h(p^k)$ is, in fact, automatic if $h(p^k) = 0$ for all large $k$. This happens for instance if $h$ is supported on the square-free numbers, such as in the case of coprime series, for which $h = G_\mu$ (µ vertical limit’s 1) and we have the following. (Recall $\mathcal{C}_G$ formulae, in §2.)

**Corollary 3.2.** (The $\mathcal{S}_G$–recursive formula)

Let $G : \mathbb{N} \to \mathbb{C}$ be a multiplicative function. Let $b,c \in \mathbb{N}$ be such that $(b,c) = 1$. Then

$$\mathcal{S}_G(b,x) = \sum_{d|c} G(d) \mu(d) \mathcal{S}_G(bc,x/d), \quad \forall x \geq 0;$$

whence, in case $\mathcal{S}_G(bc)$ converges, then also $\mathcal{S}_G(b)$ converges and moreover

$$\mathcal{S}_G(b) = \left( \sum_{d|c} G(d) \mu(d) \right) \cdot \mathcal{S}_G(b) = \prod_{p|c} (1 - G(p)) \cdot \mathcal{S}_G(bc) = \mathcal{C}_G(c) \cdot \mathcal{S}_G(bc).$$

Proof. Since we may write, for any fixed $b \in \mathbb{N}$, all $r \in \mathbb{N}$ with $(r,b) = 1$ as $r = du$, with $\text{rad} d | c$ and $(u, bc) = 1$, so that $(d,u) = 1$, we get, recalling that $x > d$ makes the partial sums $\mathcal{S}_G(bc,x/d)$ vanish:

$$\mathcal{S}_G(b,x) = \sum_{d|c} G(d) \mu(d) \sum_{(u, bc) = 1} G(u) \mu(u) = \sum_{d|c} G(d) \mu(d) \mathcal{S}_G(bc,x/d) = \sum_{d|c} G(d) \mu(d) \mathcal{S}_G(bc,x/d), \quad \forall x \geq 0.$$

Passing to the limit $x \to \infty$, we get second formula. \hfill \Box

A similar observation holds for Ramanujan sums, in fact having a vertical limit (like Möbius function). Then, we get the following formula relating Ramanujan series and coprime series, compare the PROPOSITION in [C2]. (Recall $\mathcal{C}_G$ formulae, in §2.)
Corollary 3.3. (The RS Formula)
Let $G : \mathbb{N} \to \mathbb{C}$ be a multiplicative function and $a \in \mathbb{N}$. Then

$$\mathcal{R}_G(a, x) = \sum_{d \mid \text{rad } a} G(d)c_d(a)S_G(a, x/d), \quad \forall x \geq 0;$$

whence, in case $S_G(a) = S_G(\text{rad } a)$ converges, then also $\mathcal{R}_G(a)$ converges and moreover

$$\mathcal{R}_G(a) = \left( \sum_{d \mid \text{rad } a} G(d)c_d(a) \right) \cdot S_G(a) = \mathcal{E}_G(a) \cdot S_G(a).$$

Proof. Similarly, any $a \in \mathbb{N}$ admits the factorization $a = dr$, with rad $d \mid a$ and $(r, a) = 1$, implying again that $(d, r) = 1$, so we obtain, again by previous remark on $S_G$ partial sums, $\forall x \geq 0$,

$$\mathcal{R}_G(a, x) = \sum_{d \mid \text{rad } a} G(d)c_d(a) \sum_{(r, a) \leq x \leq d} G(r)\mu(r) = \sum_{d \mid \text{rad } a} G(d)c_d(a)S_G(a, x/d) = \sum_{d \mid \text{rad } a} G(d)c_d(a)S_G(a, x/d).$$

Passing to the limit $x \to \infty$ we get second formula.

In order to get next recursive formula for $\mathcal{R}_G$ we apply, together with Corollary 3.2, twice the RS formula, namely previous Corollary. (Recall $\mathcal{D}_G$ formulæ, in §2.)

Corollary 3.4. (The $\mathcal{R}_G$–recursive Formula)
Fix a multiplicative $G : \mathbb{N} \to \mathbb{C}$. Let $a, b, c \in \mathbb{N}$ with $a = bc$ and $(b, c) = 1$ be such that $w_{p, G} = \infty$ for all $p | c$. Then

$$\mathcal{R}_G(a, x) = \sum_{h \mid c} G(h)h \mathcal{R}_G(b, x/h), \quad \forall x \geq 0.$$

Consequently, if $\mathcal{R}_G(b)$ converges, then $\mathcal{R}_G(a)$ converges, as well, and $\mathcal{R}_G(a) = \mathcal{D}_G(c) \cdot \mathcal{R}_G(b)$.

Proof. First, by Corollary 3.3 we have

$$\mathcal{R}_G(a, x) = \sum_{d \mid \text{rad } a} G(d)c_d(a)S_G(a, x/d).$$

For every $d \mid a \text{rad } a$ we may write $d = qr$ where $q \mid b \text{rad } b$ and $r \mid c \text{rad } c$, so

$$\mathcal{R}_G(a, x) = \sum_{q \mid b \text{rad } b} G(q)c_q(a) \sum_{r \mid c \text{rad } c} G(r)c_r(a)S_G(a, x/qr).$$

We now recall Kluyver’s formula (see §2):

$$c_r(a) = \sum_{h \mid (r, a)} \mu(r/h)h$$

and note that $(r, a) = (r, c)$. We then operate the change of variable $t = r/h$ and make the following change of the order of summation:

$$\sum_{r \mid c \text{rad } c} G(r) \sum_{h \mid (r, c)} \mu(r/h)hS_G(a, x/qr) = \sum_{h \mid c} h \sum_{t \mid \text{rad } c} \mu(t)G(ht)S_G(a, x/qht).$$

Since $w_{p, G} = \infty$ for all $p | c$, we have that $G$ is completely multiplicative when restricted on the divisors of $c \text{rad } c$. In other words, we have $G(ht) = G(h)G(t)$. Thus, we arrive at the following formula:

$$\mathcal{R}_G(a, x) = \sum_{q \mid b \text{rad } b} G(q)c_q(a) \sum_{h \mid c} G(h)h \sum_{t \mid \text{rad } c} G(t)\mu(t)S_G(a, x/qht).$$
Let $G$ be a multiplicative function and assume $c \in \mathbb{N}$ is such that $p|c$ implies $w_{p,G} = \infty$. Then
\[
\mathcal{L}_G(bc, x) = G(c)\mathcal{L}_G(b, x), \quad \forall x \geq 0.
\]

Suppose, furthermore, that $\mathcal{L}_G(b)$ converges. Then also $\mathcal{L}_G(bc)$ converges and
\[
\mathcal{L}_G(bc) = G(c) \cdot \mathcal{L}_G(b).
\]

Our next result is for Lucht series. (Recall $\mathcal{U}_G$ definition, in §2. We prove in next (⋆) its formulæ.)

\begin{corollary} \textbf{(The $\mathcal{L}^S$ Formula)} \end{corollary}

Let $G : \mathbb{N} \to \mathbb{C}$ be a multiplicative function. Fix any $d \in \mathbb{N}$. Then
\[
\mathcal{L}_G(d, x) = \sum_{\ell | d} \mu(\ell)G(\ell d)S_G(d, x/\ell), \quad \forall x \geq 0.
\]

Suppose, furthermore, that $S_G(d) = S_G(\text{rad } d)$ converges. Then also $\mathcal{L}_G(d)$ converges and moreover
\[
\mathcal{L}_G(d) = \mathcal{U}_G(d) \cdot S_G(d).
\]

\begin{proof}
Since we may gather, depending on greatest common divisor with any fixed $d \in \mathbb{N},$
\[
\mathcal{L}_G(d, x) = \sum_{\ell | d} \sum_{(K,d) = \ell} \mu(K)G(dK) = \sum_{\ell | d} \sum_{m | \frac{d}{\ell}} \mu(\ell m)G(\ell dm) = \sum_{\ell | d} \mu(\ell)G(\ell d) \sum_{m \leq \frac{x}{\ell}} \mu(m)G(m)
\]
we get first formula for all $x \geq 0$; passing to the limit $x \to \infty$ we get second formula.

We profit here to show that, for multiplicative $G,$ we have the following product formula:
\[\tag{⋆}
\mathcal{U}_G(d) = \mathcal{L}_G(d) = \prod_{p \mid d} \left( G(p^{v_p(d)}) - G(p^{v_p(d)+1}) \right).
\]

In order to prove this, we first prove that $G$ multiplicative implies $\mathcal{U}_G$ multiplicative. This follows from the definition:
\[
(a,b) = 1 \Rightarrow \sum_{\ell \mid ab} \mu(\ell)G(\ell ab) = \sum_{\ell \mid a} \sum_{\ell \mid b} \mu(\ell_a)\mu(\ell_b)G(\ell_a \ell_b) = \sum_{\ell \mid a} \mu(\ell)G(\ell a) \cdot \sum_{\ell \mid b} \mu(\ell)G(\ell b).
\]
Then, the property “$\mathcal{U}_G$ multiplicative”, just proved, gives the following factorization, that settles (⋆):
\[
\sum_{\ell \mid d} \mu(t)G(\ell t d) = \prod_{p \mid d} \sum_{\ell \mid p^{v_p(d)}} \mu(\ell)G(\ell p^{v_p(d)}) = \prod_{p \mid d} \left( G(p^{v_p(d)}) - G(p^{v_p(d)+1}) \right), \quad \forall d \in \mathbb{N}.
\]
\end{proof}

Our next formula is the recursion for Lucht series.

\begin{corollary} \textbf{(The $\mathcal{L}_G$-Recursive Formula)} \end{corollary}

Let $G : \mathbb{N} \to \mathbb{C}$ be a multiplicative function and assume $c \in \mathbb{N}$ is such that $p\mid c$ implies $w_{p,G} = \infty$. Then
\[
\mathcal{L}_G(bc, x) = G(c)\mathcal{L}_G(b, x), \quad \forall x \geq 0.
\]

Suppose, furthermore, that $\mathcal{L}_G(b)$ converges. Then also $\mathcal{L}_G(bc)$ converges and
\[
\mathcal{L}_G(bc) = G(c) \cdot \mathcal{L}_G(b).
\]
Proof. The assumption on \(c\) and \(G\) gives \(G(cd) = G(c)G(d)\), for all \(d \in \mathbb{N}\), whence

\[
\mathcal{L}_G(bc, x) = \sum_{K \leq x} \mu(K)G(bcK) = G(c) \sum_{K \leq x} \mu(K)G(bK) = G(c)\mathcal{L}_G(b, x), \quad \forall x \geq 0.
\]

When \(\mathcal{L}_G(b)\) converges, this entails (passing to \(\lim_{x \to \infty}\)) convergence of \(\mathcal{L}_G(bc)\) and

\[
\mathcal{L}_G(bc) = G(c) \cdot \mathcal{L}_G(b).
\]

\[\square\]

Notice: the property, that we have used many times in previous recursive formulæ, namely the existence of a factor, say \(c \in \mathbb{N}\), having all prime-divisors \(p\) such that \(wp_G = \infty\), when considered in the case of bad (see \(\S 2\) definitions) primes \(p|c\), is equivalent in saying that \(c\) is made of hyperbad (see \(\S 2\) again) prime factors; this, in turn, see the definition of Ramanujan factorization of a natural number in \(\S 8.1\), amounts to saying that \(h(c) = c\). In fact, the recursive formulæ, as we will see, are vital when proving Theorem 8.1.

Now, we may give our last Corollary; that is, also, the last possible connection in between \(\mathcal{S}_G, \mathcal{R}_G, \mathcal{L}_G\) (including the recursions): the \(\mathcal{LR}\) link. A great difference with ALL previous formulæ, here, is the generality: \(G\) is any arithmetic function. We are grateful to Professor Lucht.

**Corollary 3.7. (The \(\mathcal{LR}\) formula)** Let \(G\) be any arithmetic function. Then

\(\mathcal{L}_G(d)\) CONVERGES \(\forall d|a\) \(\Rightarrow\) \(\mathcal{R}_G(a)\) CONVERGES

and

\(\mathcal{R}_G(d)\) CONVERGES \(\forall d|a\) \(\Rightarrow\) \(\mathcal{L}_G(a)\) CONVERGES,

whence

\(G\) is a Ramanujan coefficient \(\iff\) \(G\) is a Lucht coefficient.

In either case,

\[
\mathcal{R}_G(a) = \sum_{d|a} d \mathcal{L}_G(d), \quad \forall a \in \mathbb{N} \quad \text{and} \quad \mathcal{L}_G(d) = \frac{1}{d} \sum_{t|d} \mathcal{R}_G(t)\mu(t), \quad \forall d \in \mathbb{N}.
\]

Proof. Clearly, it suffices, here, to prove only the following two partial sums formulæ.

First one follows from writing, from Kluyver’s formula (see \(\S 2\)):

\[
\mathcal{R}_G(a, x) = \sum_{d|a} d \sum_{q \leq x \mod d} \mu(q/d)G(q) = \sum_{d|a} d \sum_{K \leq x/d} \mu(K)G(dK) = \sum_{d|a} d \mathcal{L}_G(d, x/d), \quad \forall x \geq 0;
\]

whence, by Möbius inversion, see [T], also second formula follows:

\[
\mathcal{L}_G(d, x/d) = \frac{1}{d} \sum_{t|d} \mathcal{R}_G(t, x)\mu(d/t), \quad \forall x \geq 0.
\]

\[\square\]

This Corollary regards the wonderful link between Ramanujan series and Lucht series; actually, it is a kind of writing explicitly an idea going back to Lucht [Lu1] (and compare [Lu2] for a better explanation), namely of using the Eratosthenes transform (see \(\S 2\) & \(\S 4\)) to connect these two. It’s because of this, that we called Lucht series the series introduced in this paper. Actually, see that it is exactly the last parts of previous Corollary, that were proved by Lucht; however, his elementary argument proves even the formulæ, on divisors, of this result. By these formulæ: \(\mathcal{R}_G\) is well-defined on a divisor-closed \(D\) (Definition D,\(\S 2\)) if and only if \(\mathcal{L}_G\) is well-defined on \(D\).

We’ll apply this last Corollary in our next section for completely multiplicative clouds.
4. Eratosthenes transforms and completely multiplicative clouds

In the following, the Completely Multiplicative Ramanujan cloud of a fixed $F : \mathbb{N} \to \mathbb{C}$ is

$$< F >_{CM} \overset{def}{=} \{ G \in < F > \mid G \text{ is completely multiplicative} \}.$$  

Recall the Eratosthenes Transform $f'$ of any $f : \mathbb{N} \to \mathbb{C}$, in Definition C of §2, and that $f'$ is multiplicative iff this $f$ is multiplicative. 

From Corollary 3.7 above, for all $G : \mathbb{N} \to \mathbb{C}$ we may calculate $R'_G$, the Eratosthenes Transform of $R_G$:

$$R_G(a) = \sum_{d|a} R'_G(d), \ \forall a \in \mathbb{N} \implies R'_G(d) = d \mathcal{L}_G(d), \ \forall d \in \mathbb{N},$$

showing a rôle of Lucht series, as a kind of (normalized) Eratosthenes transform of Ramanujan series.

The Lucht series of a completely multiplicative coefficient $G$ is very easy.

**Lemma 4.1.** Let $G : \mathbb{N} \to \mathbb{C}$ be completely multiplicative, for which $\mathcal{L}_G(1)$ converges. Then

$$\mathcal{L}_G(a) = G(a)\mathcal{L}_G(1), \ \forall a \in \mathbb{N}.$$  

**Proof.** It follows from $\mathcal{L}_G$ definition. \hfill $\Box$

We describe $< F >_{CM}$ now. We write “C.M.”, for “is completely multiplicative”.

**Theorem 4.2.** Let $F : \mathbb{N} \to \mathbb{C}$ be any arithmetic function, not the null-function (i.e., $F \neq 0$). Then

$$< F >_{CM} \neq \emptyset \iff \sum_{q=1}^{\infty} \frac{F(q)}{q} \mu(q) \text{ converges to } F'(1) \neq 0 \text{ and } F'/F(1) \text{ C.M.}$$

When this is the case, $< F >_{CM}$ is the singleton $\left\{ G : \mathbb{N} \to \mathbb{C} \mid G(q) = \frac{F'(q)}{qF(1)}, \forall q \in \mathbb{N} \right\}$.  

**Proof.** First of all, hereafter $R_G(1) = F(1) \neq 0$; otherwise, for all $G$ C.M., next Lemma gives $R_G = F = 0$.

We prove, first, the $(\Rightarrow)$. From Corollary 3.7, $G \in < F > \Rightarrow F'(d) = R'_G(d) = d \mathcal{L}_G(d)$ and then from Lemma 4.1, G C.M. $\Rightarrow \mathcal{L}_G(d) = G(d)\mathcal{L}_G(1) = G(d)F(1)$, whence $F'(d) = dG(d)F(1)$, $\forall d \in \mathbb{N}$. QED

While, $(\Leftarrow)$ follows from choosing $G(d) = \frac{F'(d)}{dF(1)}$, that’s C.M. from $F'/F(1)$ C.M., so that following Lemma with $F'$ convergence hypothesis give the convergence of

$$R_G(a) = D_G(a) \cdot \sum_{q=1}^{\infty} G(q)\mu(q) = \sum_{d|a} F'(d) \cdot \sum_{q=1}^{\infty} \frac{F'(q)}{F(1)q} \mu(q) = \sum_{d|a} F'(d) = F(a), \ \forall a \in \mathbb{N},$$

whence: $G \in < F >_{CM}$. \hfill $\Box$

**Remark 1.** See that $F'/F(1) = (F/F(1))'$, by linearity of Eratosthenes transform, so, if $F'/F(1)$ is C.M., in particular it’s multiplicative and then: $F'/F(1) = (F/F(1))'$ is multiplicative, implying (Eratosthenes Transform and its inverse preserve multiplicativity) $F/F(1)$ multiplicative, whence $F$ is quasi-multiplicative.

An arithmetic function $F$ is quasi-multiplicative, in fact, by definition, iff there’s a constant $c \neq 0$ (to avoid trivialities), with $F/c$ multiplicative; and, in turn (avoiding trivial cases), this implies $c = F(1)$. For this definition and properties of quasi-multiplicative functions, see [Lah].  

This Theorem allows us to describe completely, once given any $F : \mathbb{N} \to \mathbb{C}$ (which is not the null-function, otherwise see Theorem 9.1 and compare [CG]), its C.M. Ramanujan cloud $< F >_{CM};$ in particular, depending on $F$, it may only be empty or a singleton, a kind of “uniqueness”.

Theorem 4.2 follows the general philosophy of “uniqueness results”, for Ramanujan expansions. As explained in [C3], if, for a fixed $F : \mathbb{N} \to \mathbb{C}$, we want a unique Ramanujan expansion, from the features of this $F$, this is impossible: we can always add to a fixed Ramanujan coefficient $G : \mathbb{N} \to \mathbb{C}$ of our $F$, say, $G_0 \in < 0 >$, i.e. any coefficient of null-function $0$. Instead, we have to impose conditions on $G$, not on $F$ (compare [C3, §3], Theorems 1,2,3). In Theorem 4.2, the condition is: $G$ is completely multiplicative.

Last but not least, the Ramanujan series with completely multiplicative coefficient $G$ is very easy, too.

**Lemma 4.3.** Let $G : \mathbb{N} \to \mathbb{C}$ be completely multiplicative, for which $R_G(1)$ converges. Then

$$R_G(a) = D_G(a)R_G(1), \ \forall a \in \mathbb{N}.$$  

**Proof.** It follows from the $R_G -$recursive formula, i.e. Corollary 3.4, taking $b = 1$ and $c = a$. \hfill $\Box$
5. Canonical Ramanujan coefficients and the separating multiplicative clouds

We start, in order to study the “separating multiplicative clouds”, recalling their definition (in §2):

\[<F>_{SM}^{def} = \{ G \in <F>_M : S_G \text{ well-defined and } S_G(1) \neq 0 \}, \]

which we don’t discuss, for the moment. In fact, even the reason why we chose the name “separating” will be explained later.

Our strategy will be to start with “simpler”, say, subsets of \( <F>_{SM} \): actually, these are much smaller than it (but this will be clear a posteriori) and, in fact, we’ll start with the smallest, following; then, we’ll take a glance at an intermediate one; and, finally, at last in this section, we’ll explain (the name origin and) the properties of \( <F>_{SM} \). Luckily enough, we can introduce the canonical Ramanujan coefficients, with the smallest, \( <F>_{ESM} \), following.

We start, in order to arrive to the study of separating multiplicative clouds, with the following subset, for a fixed \( F : \mathbb{N} \to \mathbb{C} \):

\[<F>_{ESM}^{def} = \{ G \in <F>_M : S_G = 1 \} \]

(notice that \( F(1) = S_G(1) = 1 \) excludes \( F = 0 \), namely, \( <0>_{ESM} = \emptyset \), which is the Euler separating multiplicative cloud of this \( F \) (compare the following, for the reason why the name “separating”). For a fixed \( F : \mathbb{N} \to \mathbb{C} \), assumes that the coprime series is the \( 1 \) function (constantly \( 1 \) on \( \mathbb{N} \), §2), so that, from the RS formula (Corollary 3.3), we get \( \mathcal{R}_G = \mathcal{E}_G \), whence name “Euler”: the Ramanujan series, here, is the same Euler-Ramanujan factor. In other words, \( F \) has a finite Ramanujan expansion, with explicit finite Euler product and is multiplicative, because \( \mathcal{E}_G = \mathcal{R}_G = F \) is multiplicative! Needless to say, for \( F \) not multiplicative, \( <F>_{ESM} = \emptyset \).

Furthermore, \( G \in <F>_{ESM} \) implies, in particular, that \( G \) vanishes on primes (which is easily proved by \( iii \)) in Lemma 5.3, holding in \( <F>_{SM} \); more in general, when \( G \) is not multiplicative, we say that

\[G : \mathbb{N} \to \mathbb{C} \text{ is square-free vanishing } \Longleftrightarrow G(s) = 0, \forall s > 1, \mu^2(s) = 1\]

which can be stated, in fact, for multiplicative \( G \) as: \( G(p) = 0 \) for all primes \( p \).

Then we are, actually, building one possible multiplicative Ramanujan coefficient, of a multiplicative, fixed \( F : \mathbb{N} \to \mathbb{C} \). We know it has to vanish on primes; but we still don’t know how it’s made on prime-powers: notice that this information is all built-in the Euler-Ramanujan factor! An explicit construction, for the canonical Ramanujan coefficient \( G_F \) (§2) of any multiplicative \( F \neq 0 \), is given in next result proof.

Thus any Euler separating multiplicative cloud of a multiplicative \( F \neq 0 \) is a singleton.

**Theorem 5.1.** Let \( F \neq 0 \) be multiplicative. Then \( <F>_{ESM} = \{ G_F \} \), whence \( <F>_M = \emptyset \).

**Proof.** Once fixed \( F : \mathbb{N} \to \mathbb{C} \), multiplicative, since we exclude \( F = 0 \) (the null-function, see [CG]), hereafter \( a = 1 \) has empty product on \( p|a \), with value 1 in the following:

\[F(a) = \prod_{p|a} F(p^{v_p(a)}), \forall a \in \mathbb{N}; \]

and we’ll build a multiplicative \( G : \mathbb{N} \to \mathbb{C} \), say, \( G = G_F \) (\( G \) depends on \( F \)), then proving \( <F>_{ESM} = \{ G_F \} \).

We start from the hypothesis that \( G \) is multiplicative with \( G(1) = 1 \) (to avoid \( G = 0 \)) and square-free vanishing. Then, (see the above)

\[\mathcal{R}_G(a) = \sum_{q=1}^{\infty} G(q)c_q(a) = \mathcal{E}_G(a) = \prod_{p|a} \sum_{K=0}^{v_p(a)} p^K (G(p^K) - G(p^{K+1})).\]

Up to now we have only imposed \( G_F \) multiplicative and \( G_F(1) = \mu(1), G_F(p) = 0 \) on all primes \( p \). (So we know \( G_F \in <F>_{ESM} \).)
Thus, one natural step is to check if $R_G(a) = F(a)$, for all natural $a$ (both sides are defined globally); so, we make, once fixed $p | a$,

$$F(p^{v_p(a)}) = \sum_{K=0}^{v_p(a)} p^K (G(p^K) - G(p^{K+1}))$$

an assumption that, made $\forall a \in \mathbb{N}$, with $p | a$, is equivalent to the following:

$(*)_{GF}$

$$F(p^v) = \sum_{K=0}^{v} p^K (G(p^K) - G(p^{K+1})) , \ \forall v \in \mathbb{N}.$$  

Let’s see our first definition, setting $v = 1$ in $(*)_{GF}$ and using $G(1) = 1, G(p) = 0$:

$$G_F(p^2) \overset{\text{def}}{=} \frac{1 - F(p)}{p}, \ \forall p \in \mathbb{P}.$$  

More in general, by induction on $v \in \mathbb{N}$, equation $(*)_{GF}$ above allows us to build $G_F$ on some power, once defined on lower powers ($K = 1$ has square-free vanishing since $K = 0$ forces $G_F(1)$ to be 1).

Calculating $F(p^v) - F(p^{v-1})$ from $(*)_{GF}$, we get $G(p^v) - G(p^{v+1})$, whence, in fact, by induction and also abbreviating $F(p^v) - F(p^{v-1}) = F'(p^v)$, where $F'$ is the Eratosthenes transform [Wi] of our $F$ (see §2, §4), we define (here for $v = 1$ the empty sum over $1 \leq K \leq v - 1$ gives, in fact, $G_F(p) = 0$)

$$G_F(p^v) \overset{\text{def}}{=} -\sum_{K=1}^{v-1} \frac{F(p^K)}{p^K} = 1 - \sum_{K=0}^{v-1} \frac{F'(p^K)}{p^K}, \ \forall v \in \mathbb{N}.$$  

and, recalling: $f$ multiplicative $\Rightarrow \prod_{p \mid n} (1 - f(p)) = \sum_{t \mid n} \mu(t)f(t)$, with $f(t) = \prod_{p \mid t} f(p)$, we give, compare §2 definition, the formula (entailing $G_F(1) = 1$, from: empty products are 1 and $G_F(p) = 0$, from: $v_p(p) = 1$)

$$G_F(q) = \sum_{t \mid q} \mu(t) \prod_{p \mid t} \sum_{K=0}^{v_p(q) - 1} \frac{F'(p^K)}{p^K}, \ \forall q \in \mathbb{N}.$$  

Thus

$$G = G_F \text{ inside } (*)_{GF} \Rightarrow F(a) = \prod_{p \mid a} F(p^{v_p(a)}) = \prod_{p \mid a} \sum_{K=0}^{v_p(a)} p^K (G(p^K) - G(p^{K+1})) = E_G(a) = R_G(a), \forall a \in \mathbb{N},$$  

having on $a = 1$ the convention of empty products being 1. This procedure proves $< F >_{ESM} = \{G_F\}$.

Hereafter we need a name for this $G_F$, since it’s the unique Ramanujan coefficient inside $< F >_{ESM}$: we call it (see §2) the canonical Ramanujan coefficient of $F$; enlarging this set, uniqueness is lost.

We relax the condition $S_G = 1$ with a more general one, namely $S_G(1) = 1$, getting the following normalized separating multiplicative clouds:

$$< F >_{NSM} \overset{\text{def}}{=} \{ G \in < F >_M : S_G \text{well-defined and } S_G(1) = 1 \},$$  

which is, still, a subset of $< F >_{SM}$, better, $< F >_{ESM} \subset< F >_{NSM} \subset< F >_{SM}$ (and $N$ stands for “normalized”, somehow imagining to divide by $S_G(1) \neq 0$). A simple, but discouraging, remark is that, now, a priori, we may lose $F$ multiplicativity ! However, from Corollary 1.2 and the $G$ multiplicativity, we always have a semi-multiplicative $F$: then, noticing $F(1) = S_G(1) = 1$, we recover that $F$ is multiplicative (since has threshold $a_F = 1$ and $c = F(a_F) = 1$ ! Even better, $< F >_{NSM} \neq \emptyset \iff F$ is multiplicative.

Actually, a kind of “raison d’être”, for this subset, is just to keep the property: $F$ multiplicative. In fact, we’ll not give, for the time being, any further property of $< F >_{NSM}$. Time to start the study of $< F >_{SM}$ now, giving a kind of “coming soon”: this time (in order to avoid $< F >_{SM} = \emptyset$) $F$ is quasi-multiplicative (more general than “$F$ is multiplicative”, compare the definition in Remark 1, soon after Theorem 4.2).

One good idea is to start from the reason why: “separating”. Actually, we thought to mean: this $< F >_{SM}$ somehow separates, for each $S_G(a)$, the primes dividing $a$ from all the others and it will be clear (compare Lemma 5.3) from the finite product (and infinite product, too) formulæ, coming next. However, other properties, in Lemma 5.3, may justify the name as well.
Before we give the important Lemma 5.3, encoding the main features of separating multiplicative coefficients, we wish to underline (in next Lemma) the importance of absolute convergence, for coprime series, in order to simplify all the formulæ, giving infinite products.

Recall $G$ is normal iff, by definition, it’s multiplicative and has no transparent primes, see §2 and [CG]. In the following, we’ll use the adjective “separating multiplicative”, not only for clouds, but also for arithmetic functions (typically, $G$).

The normal arithmetic functions $G$ with an absolute convergence condition are separating.

**Lemma 5.2. (Normal $G$ with absolutely convergent $S_G(1)$ are separating multiplicative)**

Let $G : \mathbb{N} \to \mathbb{C}$ be normal, with $S_G(1)$ absolutely convergent. Then

$S_G$ converges absolutely and is never-vanishing;

in formulæ,

$$S_G(a) = \prod_{p \mid a} (1 - G(p))^{-1} \neq 0 \text{ converges absolutely} \quad \forall a \in \mathbb{N};$$

in particular, $G$ is separating multiplicative.

**Proof.** The absolute convergence of $S_G(1)$, by positivity (recall $S_G$ definition, §1), implies that of any $S_G(a)$, at once.

The passage from $S_G(a)$ absolute convergence to the absolute convergence of its Euler product above, then, is standard (details in [C2, §3]).

A classical result (dating back to “old times”), then, ensures that an absolutely convergent infinite product which has no vanishing factor (as $G$ is normal here) doesn’t vanish (details, again, in [C2, §3], quoting Ahlfors Book).

We come to the main features of coprime series, when $G$ is separating multiplicative.

**Lemma 5.3. (Coprime series with separating multiplicative $G$)**

Let $G : \mathbb{N} \to \mathbb{C}$ be multiplicative, with $S_G$ well-defined and $S_G(1) \neq 0$. Then

i) $G$ is normal;

ii) $S_G(a) = \prod_{p \mid a} (1 - G(p))^{-1} S_G(1) = \sum_{d \mid a} \frac{\mu(d)G(d)}{\prod_{p \mid d}(G(p) - 1)} S_G(1), \quad \forall a \in \mathbb{N};$

iii) in particular, $G(p) = 1 - S_G(1)/S_G(p), \forall p \in \mathbb{P}$, so $S_G$ on $\{1\} \cup \mathbb{P}$ determines $G$ on $\mathbb{P}$;

iv) $S_G(1)$ converges absolutely iff $\sum_{p \in \mathbb{P}} |1 - S_G(1)/S_G(p)| < \infty$;

v) $S_G$ is never-vanishing.

**Proof.** The $S_G$—recursive formula (Corollary 3.2) gives

$$S_G(a) \prod_{p \mid a} (1 - G(p)) = S_G(1), \quad \forall a \in \mathbb{N},$$

whence $S_G(1) \neq 0$ proves i), then iii) and the first equation in ii). Since (compare Theorem 5.1 proof)

$$\prod_{p \mid a} (1 - G(p))^{-1} = \prod_{p \mid a} \frac{1}{1 - G(p)} = \prod_{p \mid a} \left(1 - \frac{G(p)}{G(p) - 1}\right) = \sum_{d \mid a} \frac{\mu(d)}{\prod_{p \mid d}(G(p) - 1)}, \quad \forall a \in \mathbb{N},$$

we get, as $\mu(d) \neq 0$ implies $\prod_{p \mid d} G(p) = G(d)$, the second equation in ii).

The $S_G(1)$ absolute convergence, thanks to Lemma 4 in §7.1 of [CG], is equivalent to $\sum_{p \in \mathbb{P}} |G(p)| < \infty$; then, we conclude iv) proof, by iii), proved above.

Finally, v) follows immediately, from ii) and the hypothesis $S_G(1) \neq 0$. 

Remark 2. Hence, abbreviating w-d for well-defined (recall, pointwise convergence in all $\mathbb{N}$)

$$S_G \text{ w-d and } S_G(1) \neq 0 \iff S_G \text{ w-d and never-vanishing},$$

rendering RHS condition an equivalent, possible alternative definition of “$G$ separating multiplicative”.

We conclude with a property regarding separating multiplicative clouds i.e. $<F>_{SM} \neq \emptyset$ implies that, in general (since $S_G(1) \neq 0$) $F$ is quasi-multiplicative (see Remark 1), while adding the condition $S_G(1) = 1$, i.e. $<F>_{NSM} \neq \emptyset$, implies that $F$ is multiplicative (see the above).

The quasi-multiplicativity alone, for the $R_G$, in case the multiplicative Ramanujan coefficients $G$ have $S_G$ well-defined (in [La], see Definition 1.2) and $R_G(1) \neq 0$ (equivalent to $S_G(1) \neq 0$), namely for separating multiplicative $G$, was already recognized in Corollary 1.2 of [La] (proved with same ideas, different details).
6. Converse convergence theorem

We may wonder if a converse of Corollary 3.2 holds, that is, if the convergence of \( S_G(b) \) for some \( b \in \mathbb{N} \) implies the convergence of \( S_G(bc) \) for all \( c \in \mathbb{N} \) coprime with \( b \). This is false in general: we give some counterexamples in Appendix, A.2. However, it is true as long as \( c \) has no prime divisors with \( 1 \leq |G(p)| \leq p \). We prove this result in Corollary 6.2 below. The key tool is the following “converse convergence theorem”. We remark that the results of this section are the motivation for the introduction of the notion of “bad primes” in Section 2.

We see the result proving Corollary 6.2; however, it also has an interest of his own.

**Theorem 6.1.** (Converse convergence theorem)

Let \( \alpha \in \mathbb{C}\setminus\{-1\} \) and \( \rho > 1 \). Assume that the function \( H : \mathbb{R}_{\geq 0} \to \mathbb{C} \) satisfies \( H(x) = H([x]) \), \( \forall x \geq 0 \) and \( K(x) := H(x) + \alpha H(x/\rho) - \ell \in \mathbb{C} \), as \( x \to \infty \). Then, \( \lim_{x \to \infty} H(x) \) exists in \( \mathbb{C} \) and equals \( \ell/(1 + \alpha) \), in both of the distinct hypotheses

(contraction) \quad |\alpha| < 1;

(dilation) \quad |\alpha| > \rho \quad \text{and} \quad H(x) \ll x, \text{ as } x \to \infty.

**Remark 3.** We may relax the hypothesis on \( H([x]) \), assuming only \( H \) bounded on \( \mathbb{R}_{\geq 0} \) compact subsets. \( \diamond \)

**Proof.** First, the “contractive case”, \( 0 < |\alpha| < 1 \) (since \( \alpha = 0 \) is trivial).

We start proving that \( H \) is bounded on \( \mathbb{R}_{\geq 0} \). Then, this will imply \( \lim_{x \to \infty} H(x) = \ell/(1 + \alpha) \).

Since \( K(x) \) has a finite limit, as \( x \to \infty \), and \( H \) (from hypothesis on \([x]\), above) is bounded on compact subsets of \( \mathbb{R}_{\geq 0} \), we derive

\[
M := \sup_{x \geq 0} |H(x) + \alpha H(x/\rho)| < \infty.
\]

The triangle inequality gives from this

\[
|H(x)| = |H(x) + \alpha H(x/\rho) - \alpha H(x/\rho)| \leq M + |\alpha| \cdot |H(x/\rho)|, \quad \forall x \geq 0,
\]

whence, iterating,

\[
|H(x)| \leq M(1 + |\alpha| + \cdots + |\alpha|^{i-1}) + |\alpha|^i \cdot |H(x/\rho^i)|, \quad \forall x \geq 0, \forall j \in \mathbb{N}.
\]

Fix \( n \in \mathbb{N} \) and \( 0 \leq x \leq n \). Completing to the geometric series of ratio \( |\alpha| \) and using \( \rho > 1 \),

\[
|H(x)| \leq \frac{M}{1 - |\alpha|} + |\alpha|^j \cdot \sup_{0 \leq x \leq n} |H(x/\rho^j)| \leq \frac{M}{1 - |\alpha|} + |\alpha|^j \cdot \sup_{0 \leq x \leq n} |H(x)|, \quad \forall j \in \mathbb{N}.
\]

Then, once fixed \( \varepsilon > 0 \), we may choose \( j \in \mathbb{N} \), depending on \( \varepsilon \) and \( n, H \), (esp., \( j > (\log \frac{J(n)}{\ell})/(\log |\alpha|) \), for \( J(n) := \sup_{0 \leq x \leq n} |H(x)| \), of course \( J(n) > 0 \) here) such that

\[
0 \leq x \leq n \quad \implies \quad |H(x)| \leq \frac{M}{1 - |\alpha|} + \varepsilon.
\]

Since this is true \( \forall n \in \mathbb{N} \) (and notice that \( \varepsilon \) doesn’t depend on \( n \)), \( H \) is bounded on all \( \mathbb{R}_{\geq 0} \). (QED)

Once proved this, we apply it now, assuming \( S_H := \sup_{x \geq 0} |H(x)| \) positive to avoid trivialities.

From the hypothesis, for all \( n \in \mathbb{N} \):

\[
(-\alpha)^j(H(x/\rho^j) + \alpha H(x/\rho^{j+1})) \to (-\alpha)^j \ell, \quad \forall j = 0, \ldots, n - 1;
\]

summing up,

\[
\lim_{x \to \infty} \left( H(x) - \alpha^n H(x/\rho^n) \right) = \frac{1 - (-\alpha)^n}{1 + \alpha} \ell = \frac{\ell}{1 + \alpha} - \frac{(-\alpha)^n}{1 + \alpha} \ell.
\]
Since hold, so the convergence of $S$

\[ \text{Proof.} \]
Let us first suppose $S$ and we may choose $H$ (i.e., from the definition of limit, $c$ converges for all $n$, and we may choose $H$) whence an immediate triangle inequality gives $\alpha$ as above, $K$.

The Theorem has the following, interesting consequence. $\alpha(x) = (1 - S) = 1 - S$. The result follows by induction on the number of prime factors of $c$.

\[ \text{QED.} \]
7. Finiteness convergence Theorem: proof of Theorem 1.1

The implication (1)⇒(2) is trivial, while the implication (3)⇒(4) is a direct consequence of Corollary 6.2. We prove (4)⇒(1) in next subsection. Last but not least, (2)⇒(3) follows in §7.2 from an important formula.

7.1 Proof of the implication (4)⇒(1) in Theorem 1.1

Proof of (4)⇒(1). By assumption (4) we have that \(S_G(b)\) converges for each \(b \in \mathbb{N}\) coprime with the hyperbad primes of \(G\). By Corollary 3.3 we deduce that \(\mathcal{R}_G(b)\) converges for the same \(b\). In order to prove that \(\mathcal{R}_G(a)\) converges for all \(a \in \mathbb{N}\) we need to deal with hyperbad prime factors. The Corollary 3.4 can do that, because \(w_{p,G} = \infty\) for all \(p \in \mathcal{B}^\infty\).

7.2 Proof of the implication (2)⇒(3) in Theorem 1.1

The proof of the implication (2)⇒(3) relies on a special formula for the \(\mathcal{R}\mathcal{L}\) series, namely \(\mathcal{F}_G\).

**Theorem 7.1.** (The \(\mathcal{F}_G\)-transformation formula) Let \(a, b, c \in \mathbb{N}\) and fix a prime \(p\), not dividing \(abc\).

Let \(G : \mathbb{N} \to \mathbb{C}\) be multiplicative and \(w \in \mathbb{N}\) be such that, say, \(\Delta := G(p^w)G(p) - G(p^{w+1}) \neq 0\). Then

\[
\mathcal{F}_G(a, pb, c)(x) = \frac{G(p)\mathcal{F}_G(a, pb, c)(x) - G(p^{w+1})\mathcal{F}_G(a, b, c)(x)}{\Delta}, \quad \forall x \geq 0.
\]

Assuming, also, that both \(\mathcal{F}_G(a, b, c)\) and \(\mathcal{F}_G(a, b, p^w c)\) converge, then \(\mathcal{F}_G(a, pb, c)\) converges, too.

**Proof.** Since \(p\) does not divide \(a\), we have that \(c_r(a) \neq 0\) only if \(v_p(r) \leq 1\). Then for every \(v \in \mathbb{N}\) the truncated \(\mathcal{R}\mathcal{L}\) series with arguments \(a, b, p^v c\) splits into two sums as follows

\[
\mathcal{F}_G(a, b, p^v c)(x) = \sum_{r \leq x \atop (r, b) = 1} G(p^v c_r) c_r(a) = \sum_{r \leq x \atop (r, pb) = 1} G(p^v c_r) c_r(a) + \sum_{r \leq x \atop (r, pb) = 1} G(p^{v+1} c_r) c_{pr}(a).
\]

By basic properties of Ramanujan sums we have that \(c_{pr}(a) = -c_r(a)\). Therefore specializing the above equation for \(v = 0\) and \(v = w\) and using the multiplicativity of \(G\) we get

\[
\mathcal{F}_G(a, b, c)(x) = \mathcal{F}_G(a, pb, c)(x) - G(p)\mathcal{F}_G(a, pb, c)(x/p),
\]

\[
\mathcal{F}_G(a, b, p^w c)(x) = G(p^w)\mathcal{F}_G(a, pb, c)(x) - G(p^{w+1})\mathcal{F}_G(a, pb, c)(x/p).
\]

Since

\[
\Delta \overset{\text{def}}{=} \det \begin{pmatrix} 1 & -G(p) \\ G(p^w) & -G(p^{w+1}) \end{pmatrix} = G(p^w)G(p) - G(p^{w+1}) \neq 0,
\]

we can solve for \(\mathcal{F}_G(a, pb, c)(x)\):

\[
\mathcal{F}_G(a, pb, c)(x) = \frac{(G(p)\mathcal{F}_G(a, b, p^w c)(x) - G(p^{w+1})\mathcal{F}_G(a, b, c)(x))}{\Delta}.
\]

We are now ready to finish the proof of Theorem 1.1.

Proof of (2)⇒(3). By assumption (2) we know that \(\mathcal{R}_G(a)\) converges for all \(a \mid N(G)\). By the \(\mathcal{L}\mathcal{R}\) formula, Corollary 3.7, \(\mathcal{L}_G(a) = \mathcal{F}_G(1, 1, a)\) converges for all \(a \mid N(G)\). In particular, \(\mathcal{F}_G(1, 1, a)\) converges for all \(a\) of the form \(a = \prod_{p \in J} p^{w_{p,G}}\), for each subset \(J \subseteq \mathcal{P}(N(G))\) of simply bad primes of \(G\). We now prove the following statement by induction on the cardinality of \(I\): for every disjoint subsets \(I, J \subseteq \mathcal{P}(N(G))\) we get

\[
\psi(I, J) \overset{\text{def}}{=} \mathcal{F}_G\left(1, \prod_{p \in I} p, \prod_{p \in J} p^{w_{p,G}}\right)
\]

convergence. If \(I = \emptyset\), this is what we just proved a few lines above. For the induction step, first note that

\[
G(p^{w_{p,G}+1}) \neq G(p)G(p^{w_{p,G}}),
\]

by the definition of \(w_{p,G}\). Then Theorem 7.1 tells us that the convergence of \(\psi(I \cup \{p\}, J)\) follows from the convergence of both \(\psi(I, J)\) and \(\psi(I, J \cup \{p\})\), whenever \(p \notin I \cup J\). By induction on the cardinality of \(I\) we deduce that \(\psi(I, J)\) converges for all disjoint \(I, J \subseteq \mathcal{P}(N(G))\). In particular

\[
\psi(\mathcal{P}(N(G)), \emptyset) = \mathcal{F}_G(1, \text{rad } N(G), 1) = S_G(N(G))
\]

converges.

Hence, the proof of Theorem 1.1 is complete. QED
8. Ramanujan series factorization and Euler-Selberg products

The formula used in the proof of Theorem 1.1 will be used again, in their limiting case, to prove the following general formula for the values of Ramanujan expansions with multiplicative coefficients.

8.1 The Ramanujan series factorization and an auxiliary integers’ factorization

In order to state this result, we introduce, once fixed the multiplicative $G$, a kind of unique factorization for all natural numbers $a$, that we call the Ramanujan factorization (or $G$—Ramanujan factorization):

$$a = h \cdot t \cdot \overline{a} = h(a)t(a)\overline{a}(a),$$

where the say hyperbad-factor $h : \mathbb{N} \to \mathbb{C}$, the say simply-transparent-factor $t : \mathbb{N} \to \mathbb{C}$ and, say, the regular-factor $\overline{} : \mathbb{N} \to \mathbb{C}$ are the C.M.(completely multiplicative) functions defined as $(\oplus \overset{\text{def}}{=} \text{disjoint union})$

$$h(a) \overset{\text{def}}{=} \prod_{p \mid a, p \notin B} p^{v_p(a)}, \quad t(a) \overset{\text{def}}{=} \prod_{p \mid a, p \notin B^*} p^{v_p(a)}, \quad \overline{a} \overset{\text{def}}{=} \prod_{p \mid a, p \notin B^*} p^{v_p(a)}.$$

See that, in the following, we’ll use the property:

$$(a, b) = 1 \Rightarrow (h(a), h(b)) = 1, \quad (t(a), t(b)) = 1, \quad (\overline{a}, \overline{b}) = 1.$$

We are ready to see how this factorization of natural numbers gives a $R_n$ factorization.

**Theorem 8.1. (Ramanujan series factorization)**

Let $G : \mathbb{N} \to \mathbb{C}$ be a multiplicative function such that $R_{G}(a)$ converges for all $a \in \mathbb{N}$. Then, writing each $a \in \mathbb{N}$ uniquely, following its $G$—Ramanujan factorization, $a = h \cdot t \cdot \overline{a}$, we get, $\forall a \in \mathbb{N}$,

$$R_{G}(a) = D_{G}(h) \cdot E_{G} \left( \varphi \left( \frac{\text{rad } N_{T}(G)}{\text{rad } t} \right) \right) \frac{E_{G}(t)}{E_{G}(N_{T}(G))} \frac{E_{G}(\overline{a})}{E_{G}(a)} \cdot R_{G}(N_{T}(G)).$$

**Remark 4.** Notice that $E_{G}(N_{T}(G)) \neq 0$ and $E_{G}(\overline{a}) \neq 0$, so the formula makes sense. We have, in fact,

$$E_{G}(\overline{a}) = \prod_{p \mid a} (1 - G(p))$$

which is nonzero because $G(p) \neq 1$ for all $p \mid \overline{a}$ (that is, no prime factor of $\overline{a}$ is transparent). Next, we have

$$E_{G}(N_{T}(G)) = \prod_{p \mid N_{T}(G)} \sum_{k=0}^{v_{p,G}} p^{k} \left( G(p^{k}) - G(p^{k+1}) \right).$$

For every $p \mid N_{T}(G)$ we have $G(p^{k}) = G(p)^{k} = 1$ for all $k \leq v_{p,G}$ (likewise, for $k = 0$) and $G(p^{v_{p,G}+1}) \neq 1$, so

$$E_{G}(N_{T}(G)) = \prod_{p \mid N_{T}(G)} p^{v_{p,G}} \left( 1 - G(p^{v_{p,G}+1}) \right)$$

is nonzero.

Also, trivially, any multiplicative Ramanujan coefficient $G$ with $R_{G}(N_{T}(G)) = 0$ has $R_{G} = 0$.

The Ramanujan series factorization, in particular, proves that:

$G$ mult.-Ramanujan coefficient, $R_{G}(N_{T}(G)) \neq 0 \Rightarrow R_{G}/R_{G}(N_{T}(G))$ finite Euler product.

The Proof of Ramanujan series factorization Formula simply descends from Theorem 1.1 and §3 Corollaries, about factorizations of $R_{G}$ and $S_{G}$.
Proof of Theorem 8.1. Let us recall some results that we proved in the preceding sections. Corollary 3.4 tells us that \( R_G(bc) = R_G(b)D_G(c) \) whenever \( (b, c) = 1 \) and \( w_{p,G} = \infty \) for all \( p \mid c \). In particular we have
\[
R_G(a) = D_G(h)R_G(t\tilde{a}).
\]
Next, by Theorem 1.1 we have that \( S_G(b) \) converges for any \( b \) without hyperbad prime factors. In particular \( S_G(t\tilde{a}) \) converges, so Corollary 3.3 and Corollary 3.2 imply that
\[
R_G(t\tilde{a}) = E_G(t\tilde{a})S_G(t\tilde{a}) \quad \text{and} \quad S_G(t) = E_G(\tilde{a})S_G(t\tilde{a}).
\]
Therefore, noticing that \( S_G(t) = S_G(\text{rad } t) \) for trivial reasons, we get
\[
R_G(a) = D_G(h)E_G(t\tilde{a})(E_G(\tilde{a}))^{-1}S_G(\text{rad } t).
\]
Since all prime factors of \( t \) are simply transparent, we have \( \text{rad } t \mid \text{rad } N_T(G) \). Then Corollary 3.2 gives
\[
S_G(\text{rad } t) = E_G(\frac{\text{rad } N_T(G)}{\text{rad } t})S_G(\text{rad } N_T(G)).
\]
By Corollary 3.3 we have \( R_G(N_T(G)) = E_G(N_T(G))S_G(\text{rad } N_T(G)) \), whence finally the sought formula. \( \square \)

8.2 The Euler-Selberg factors
We prove now all the Corollaries of Theorem 8.1, needed in order to prove Corollary 1.2.

We start from the first statement in Corollary 1.2.

Corollary 8.2. Let \( G \) be a multiplicative Ramanujan coefficient with transparency conductor \( N_T(G) \). Then \( R_G(a) = 0 \) for every \( a \) that is not a multiple of \( N_T(G) \).

Proof. Let us write \( a = h \cdot t \cdot \tilde{a} \) as we did in Theorem 8.1.

We distinguish two cases, according to whether \( \text{rad } t \) is equal to \( \text{rad } N_T(G) \) or not. In each case, we prove that some factor in the main formula of Theorem 8.1 vanishes.

Suppose first that \( \text{rad } t \neq \text{rad } N_T(G) \). Then the ratio, say, \( r := (\text{rad } N_T(G))/(\text{rad } t) \) is a square-free integer \( r > 1 \) divisible only by transparent primes. Thus
\[
E_G(r) = \prod_{p \mid r} (1 - G(p)) = 0.
\]
Using the formula in Theorem 8.1 we deduce that \( R_G(a) = 0 \) in case \( \text{rad } t \neq \text{rad } N_T(G) \).

Let us now assume that \( \text{rad } t = \text{rad } N_T(G) \) but \( a \) is a not a multiple of \( N_T(G) \). Then, there exists a prime \( p \) which is simply transparent and such that
\[
1 \leq v_p(t) = v_p(a) < v_p(N_T(G)) = v_p,G.
\]
Since the function \( E_G \) is multiplicative and \( (t, \tilde{a}) = 1 \), we may write
\[
E_G(t\tilde{a}) = E_G(\tilde{a})E_G(t) = E_G(\tilde{a})E_G(p^{v_p(t)})E_G(p^{-v_p(t)}t).
\]
The middle term in the right-most expression can be written as
\[
E_G(p^{v_p(t)}) = \sum_{v=0}^{v_p(t)} p^v(G(p^v) - G(p^{v+1})).
\]
Now, \( p \) is a simply transparent prime, so \( G(p^v) = 1 \) for all \( v \leq v_p,G \). Since \( v_p(t) < v_p,G \) by our choice of \( p \), we deduce that \( G(p^v) - G(p^{v+1}) = 0 \) for each \( v \leq v_p(t) \). This proves that \( E_G(p^{v_p(t)}) = 0 \) and so \( E_G(t\tilde{a}) = 0 \). By the formula in Theorem 8.1 we deduce that \( R_G(a) = 0 \) also in this case. \( \square \)
Then, next property is an important step in the proof of Corollary 1.2.

**Corollary 8.3.** Let $G$ be a multiplicative Ramanujan coefficient with transparency conductor $N_T(G)$. For every $a \in \mathbb{N}$ write the Ramanujan factorization $a = h \cdot t \cdot \tilde{a}$ and define the function

$$M_G(a) \overset{def}{=} D_G(h) \cdot \frac{E_G(tN_T(G))}{E_G(N_T(G))} \cdot \frac{E_G(\tilde{a})}{E_G(\tilde{a})}.$$ 

Then $M_G(a)$ is a multiplicative function of $a \in \mathbb{N}$.

*Proof.* First of all, note that the denominators in the definition of $M_G$ are nonzero, by Remark 1, so the function $M_G$ is well-defined. Moreover, $M_G$ is not equal to the constant zero function, because $M_G(1) = 1$.

We use now the definition and the property of Ramanujan factorization (see soon before above Theorem 8.1).

We already know that $E_G$, $D_G$ and $E_G$ are multiplicative functions, therefore also the functions

$$a \mapsto D_G(h(a)) \quad \text{AND} \quad a \mapsto \frac{E_G(\tilde{a})}{E_G(\tilde{a})}$$

are multiplicative. It remains to show that the function $a \mapsto \frac{E_G(t(a)N_T(G))/E_G(N_T(G))}{E_G(t(b)N_T(G)/E_G(N_T(G))}$ is multiplicative. Let $a, b \in \mathbb{N}$ with $(a, b) = 1$. Let $N_1$ be the largest divisor of $N_T(G)$ that is coprime with $t(b)$, and let $N_2 = N_T(G)/N_1$. In other words, we decompose $N_T(G) = N_1 \cdot N_2$ so that $(N_1, t(b)) = 1$ and $\operatorname{rad} N_2 | t(b)$. Since $t(a)$ and $t(b)$ are coprime, it follows that $(t(a), N_2) = 1$. Then by multiplicativity of $E_G$ we have

$$\frac{E_G(t(ab)N_T(G))}{E_G(N_T(G))} = \frac{E_G(t(a)N_1 N_2)}{E_G(N_1 N_2)} \cdot \frac{E_G(t(b)N_2)}{E_G(N_2)} = \frac{E_G(t(a)N_T(G))}{E_G(N_T(G))} \cdot \frac{E_G(t(b)N_T(G))}{E_G(N_T(G))}.$$ 

Note that $E_G(N_T(G)) \neq 0$ implies $E_G(N_i) \neq 0$ for $i = 1, 2$ as well.

Hence, last expression is

$$\frac{E_G(t(a)N_1)}{E_G(N_1)} \cdot \frac{E_G(N_2)}{E_G(N_2)} \cdot \frac{E_G(t(b)N_2)}{E_G(N_2)} \cdot \frac{E_G(N_1)}{E_G(N_1)} = \frac{E_G(t(a)N_T(G))}{E_G(N_T(G))} \cdot \frac{E_G(t(b)N_T(G))}{E_G(N_T(G))}.$$ 

Thus the function $M_G(a)$ is a product of three multiplicative functions, so it’s multiplicative, too. 

Another important step to prove Corollary 1.2, now.

**Corollary 8.4.** Let $G$ be a multiplicative Ramanujan coefficient with transparency conductor $N_T(G)$. Then, there exists a multiplicative function $M_G : \mathbb{N} \to \mathbb{C}$ such that

$$R_G(aN_T(G)) = R_G(N_T(G))M_G(a), \quad \forall a \in \mathbb{N}.$$ 

If $R_G(N_T(G)) \neq 0$, then the function $M_G$ is unique, being given by the formula in Corollary 8.3.

*Proof.* We let $M_G$ be the function defined in Corollary 8.3 and we are going to compare it with the formula in Theorem 8.1. We apply again the Ramanujan factorization, $\forall a \in \mathbb{N}$.

For all natural $a$, $\operatorname{rad}(t(a)N_T(G)) = \operatorname{rad} N_T(G)$ and so

$$E_G \left( \frac{\operatorname{rad} N_T(G)}{\operatorname{rad}(t(a)N_T(G))} \right) = 1.$$ 

Moreover, for every $a \in \mathbb{N}$, we have, from $t$ is a C.M. arithmetic function,

$$t(aN_T(G)) = t(a)t(N_T(G)) = t(a)N_T(G)$$

and so, from $\sim$ is C.M.,

$$E_G(t(aN_T(G)\sim(aN_T(G)))) = E_G(t(a)N_T(G)) \cdot E_G(\tilde{a}),$$

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by multiplicativity of $E_G$. When we pass from $a$ to $aN_T(G)$, the not-simply-transparent factors don’t change. It follows from the formula of Theorem 8.1 that $R_G(aN_T(G)) = R_G(N_T(G))M_G(a)$. The statement about the uniqueness is clear. 

Finally, we prove Corollary 1.2.

**Proof.** By Corollary 8.2 we have $R_G(a) = 0$ for every $a$ that is not a multiple of $N_T(G)$. Let now $a$ be a fixed natural number. From Corollary 8.4 we know that $R_G(aN_T(G)) = R_G(N_T(G))M_G(a)$, where $M_G$ is some multiplicative function; also, if $R_G(N_T(G)) \neq 0$, it is in Corollary 8.3. We now compute the values of $M_G$, at the prime powers dividing $a$, and check that they correspond to the Euler factors displayed in Corollary 1.2.

The fact that we have two left-hand sides (first $R_G(aN_T(G))$, then $R_G(a)$) in the formulæ, for Euler products, will be a real difference only when considering the factors with simply transparent primes (as $aN_T(G)$ and $a$ have the same not-simply-transparent factors, in their Ramanujan factorization).

**Case $p$ hypertransparent.** If $p$ is a hypertransparent prime, then $v_{p,G} = \infty$ and $G(p^v) = 1$ for all $v \in \mathbb{N}_0$. Thus

$$M_G(p^{v_p(a)}) = D_G(p^{v_p(a)}) = \sum_{v=0}^{v_p(a)} p^v$$

and this agrees, in Corollary 1.2, both with $M_G$ definition and the corresponding factor (with $v_{p,G} = \infty$).

**Case $p$ hyperbad, not hypertransparent.** In this case $G(p) \neq 1$ and $G(p^v) = G(p)^v$, for all $v \in \mathbb{N}_0$. Then

$$M_G(p^{v_p(a)}) = D_G(p^{v_p(a)}) = \sum_{v=0}^{v_p(a)} p^v G(p^v).$$

This agrees, in Corollary 1.2, both with $M_G$ definition and with the corresponding factor because $v_{p,G} = 0$ (since $G(p) \neq 1$) and, of course,

$$G(p^v) = \frac{G(p^v) - G(p^{v+1})}{1 - G(p)}.$$

**Case $p$ neither hyperbad nor transparent.** Let $p$ be a prime that is not transparent (so $G(p) \neq 1$ and $v_{p,G} = 0$) and not hyperbad. Then

$$E_G(p^{v_p(a)}) \frac{E_G(p^{v_p(a)})}{E_G(p^{v_p(a)})} = \frac{1}{1 - G(p)} \sum_{v=0}^{v_p(a)} p^v (G(p^v) - G(p^{v+1}))$$

and this agrees, in Corollary 1.2, both with $M_G$ definition and with the corresponding factor.

**Case $p$ simply transparent.** Finally, let $p$ be a simply transparent prime, so that $1 \leq v_{p,G} < \infty$. We have

$$v_p(aN_T(G)) = v_p(a) + v_{p,G},$$

from the complete additivity of $v_p$, because $v_p(N_T(G)) = v_{p,G}$ by definition of transparency conductor. Now, for all natural $V$ (eventually following sums are empty, giving 0):

$$E_G(p^V) = \sum_{K=v_{p,G}}^{V} p^K (G(p^K) - G(p^{K+1})),$$

by the definition of $v_{p,G}$, whence

$$E_G(p^{v_p(a)+v_{p,G}}) = \sum_{K=v_{p,G}}^{v_p(a)+v_{p,G}} p^K (G(p^K) - G(p^{K+1})) = \sum_{v=0}^{v_p(a)} p^{v+v_{p,G}} (G(p^{v+v_{p,G}}) - G(p^{v+v_{p,G}+1})).$$

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and, for the same reason,

\[ \mathcal{E}_G(p^{v_p,G}) = p^{v_p,G} \left( 1 - G(p^{v_p,G+1}) \right). \]

Since we may assume \( \mathcal{R}_G(N_T(G)) \neq 0 \) (otherwise formulæ’s both sides vanish), applying Corollary 8.4, we get \( \mathcal{M}_G(p^{v_p(a)}) \), as defined in Corollary 8.3, is

\[
\mathcal{M}_G(p^{v_p(a)}) = \frac{\mathcal{E}_G(p^{v_p(a) + v_p,G})}{\mathcal{E}_G(p^{v_p,G})} = (1 - G(p^{v_p,G+1}))^{-1} \sum_{v=0}^{v_p(a)} p^v(G(p^v) - G(p^{v+1})).
\]

This agrees with the definition of \( \mathcal{M}_G \) in Corollary 1.2.

The Euler factor made of simply transparent primes is calculated, this time, distinguishing in last formula for \( \mathcal{R}_G(a) \), in Corollary 1.2, in two sub-cases: \( a \not\equiv 0 \mod N_T(G) \) and \( a \equiv 0 \mod N_T(G) \).

In sub-case \( a \not\equiv 0 \mod N_T(G) \), the LHS \( \mathcal{R}_G(a) = 0 \), see Corollary 8.2, implies that the RHS contains a vanishing factor; in fact, \( a \not\equiv 0 \mod N_T(G) \) implies the existence of a simply transparent prime \( p \), for which \( v_p(a) < v_p(N_T(G)) = v_{p,G} \): since \( G(p^v) = G(p^{v+1}) = 1 \) for all \( v \leq v_p(a) \), in this sub-case we have

\[
\sum_{v=0}^{v_p(a)} p^v G(p^v) - G(p^{v+1}) \frac{1}{1 - G(p^{v_p,G+1})} = 0.
\]

In sub-case \( a \equiv 0 \mod N_T(G) \), write it \( a = bN_T(G) \), so that Corollary 8.4 gives

\[
\mathcal{R}_G(a) = \mathcal{R}_G(bN_T(G)) = \mathcal{R}_G(N_T(G))\mathcal{M}_G(b),
\]

where now \( \mathcal{M}_G(b) = \mathcal{M}_G(a/N_T(G)) \) has Euler \( p \)–simply transparent factor (compare formulæ, from \( \mathcal{E}_G(p^v) \) above):

\[
\frac{\mathcal{E}_G(p^{v_p(b) + v_p,G})}{\mathcal{E}_G(p^{v_p,G})} = \frac{\mathcal{E}_G(p^{v_p(a)})}{\mathcal{E}_G(p^{v_p,G})} = \frac{1}{p^{v_p,G} (1 - G(p^{v_p,G+1}))} \sum_{K=v_{p,G}}^{v_p(a)} p^K (G(p^K) - G(p^{K+1}))
\]

\[
= \sum_{v=0}^{v_p(a)} p^{v-v_p,G} G(p^v) - G(p^{v+1}) \frac{1}{1 - G(p^{v_p,G+1})},
\]

appearing in second Euler product formula in Corollary 1.2. \(\)
9. Multiplicative Ramanujan clouds

We now give a description of the full multiplicative Ramanujan cloud $< F >_M$ of a given arithmetic function $F$. This is, by definition, the set of multiplicative $G : \mathbb{N} \to \mathbb{C}$ such that $F(a) = \mathcal{R}_G(a)$ for all $a \in \mathbb{N}$.

9.1 Multiplicative Ramanujan cloud of the null function

We start with the important special case $F = 0$: we show that multiplicative Ramanujan coefficients $G$ of the constant null function are characterized by the convergence and vanishing of a single series that involves squarefree values of $G$. This result extends and completes the previous work [CG] of the authors, where a description of the special multiplicative Ramanujan cloud $< 0 >_{SM}$ was given. Recall $N(G)$ in Definition B, §1.

**Theorem 9.1** Let $G$ be a multiplicative function with Ramanujan conductor $N(G)$. We have

$$ G \in < 0 >_M \iff \sum_{(r,N(G))=1} G(r)\mu(r) = 0. $$

**Proof.** Let $G$ be a multiplicative Ramanujan coefficient of $0$ and let $N_T(G)$ be its transparency conductor. By the Finiteness Convergence Theorem we have that $S_G(N(G))$ and $S_G(N_T(G))$ converge. Furthermore, we have that $\mathcal{E}_G(N_T(G)) \neq 0$ by Remark 4. Thus by the $\mathcal{R}$S and the $S_G$–recursive formulæ, we have

$$ S_G(N(G)) = \mathcal{E}_G(N(G)/N_T(G))S_G(N_T(G)) = \frac{\mathcal{E}_G(N(G)/N_T(G))}{\mathcal{E}_G(N_T(G))}\mathcal{R}_G(N_T(G)) = 0. $$

Since $S_G(N(G)) \overset{\text{def}}{=} \sum_{(r,N(G))=1} G(r)\mu(r)$, the implication $\Rightarrow$ is proved. Now let $G$ be a multiplicative function such that $S_G(N(G))$ converges to zero. By the Finiteness Convergence Theorem we have that $\mathcal{R}_G(a)$ converges for all $a \in \mathbb{N}$. Note that

$$ \mathcal{E}_G(N(G)/N_T(G)) = \prod_{p \in \mathcal{B} \setminus \mathcal{F}} (1 - G(p)) \neq 0. $$

Hence the previous computation gives

$$ \mathcal{R}_G(N_T(G)) = \frac{\mathcal{E}_G(N_T(G))}{\mathcal{E}_G(N(G)/N_T(G))}S_G(N(G)) = 0. $$

But then the Euler-Selberg product formula (Corollary 1.2) proves that $\mathcal{R}_G(a) = 0$ for all $a \in \mathbb{N}$. \hfill $\Box$

**Remark 5.** By the $S_G$–recursive formula, we note that to have $S_G(N(G)) = 0$ it is sufficient that the condition $S_G(N) = 0$ holds for some arbitrary multiple $N$ of $\text{rad}(N(G))$. \hfill $\Diamond$

**Remark 6.** Summarizing the conclusions of Theorem 9.1 and Remark 9.1.1, we have that each multiplicative Ramanujan coefficient $G$ of the constant null function can be constructed via the following two-step procedure:

(i) choose a natural number $N$ and complex numbers $G(p)$ for each $p$ prime, so that, defining $G$ at squarefree arguments by multiplicativity, we have $S_G(N) = 0$;

(ii) choose the values of $G$ at all other natural arguments ensuring that $G$ is multiplicative and that $\text{rad}(N(G))$ divides $N$. \hfill $\Diamond$

**Remark 7.** The condition $\text{rad}(N(G)) \mid N$ in step (ii) above can be restated as follows: $w_{p,G} = \infty$ for each prime $p$ coprime with $N$ and such that $1 \leq |G(p)| \leq p$. This is essentially a condition imposed on $G \in < 0 >_M$ at all powers of these (finitely many) primes. The values of $G$ at prime powers (with exponent greater than 1) at all other primes can be chosen arbitrarily. \hfill $\Diamond$
9.2 Multiplicative Ramanujan clouds of non-null functions

Finally, we are going to describe the Ramanujan coefficients of a semi-multiplicative function $F \neq 0$. As we shall see in the following theorems, there is an important difference with respect to the case of the null function $0(a) \overset{def}{=} 0, \forall a \in \mathbb{N}$.

As we discovered in the previous paragraph, the values of $G \in <0>_{M}$ at prime arguments are subject to a single linear equation (namely, $S_G(N(G)) = 0$), whilst the values of $G$ at higher prime powers can be essentially arbitrary (there is only an additional technical condition on the complete multiplicative index of bad primes). Instead, if $G \in <F>_{M}$ is a multiplicative Ramanujan coefficient of a non-null function $F \neq 0$, the Euler-Selberg product formula (Corollary 1.2) gives a recursion for the values $G(p^k)$ for fixed $p$ and varying $k$. This implies that the values of $G$ at high powers of primes are not arbitrary: on the contrary, they are completely determined by the values at small powers of primes.

In order to make this observation precise, we shall use the following notion.

**Definition 9.2 (Opacity Core)** Let $G : \mathbb{N} \to \mathbb{C}$ be a multiplicative function with finitely many transparent primes (Definition A, §2) and transparency conductor equal to $N_T(G)$. We say that the opacity core of $G$ is the function $H_G : \mathbb{N} \to \mathbb{C}$ given by the formula

$$H_G(q) \overset{def}{=} G(q N_T(G)) \mu^2(q).$$

**Remark 8.** We recall that the transparency index of $G$ at a prime $p$ is the largest $v_{p,G} \in \mathbb{N} \cup \{0, \infty\}$ such that $G(p^v) = 1$ for each natural number $v \leq v_{p,G}$. If $p$ is not hypertransparent, we see that

$$(*)_H \quad H_G(p) = G(p^{v_{p,G}(N_T(G))+1}) = G(p^{v_{p,G}+1})$$

is the first value of $G$ at a power of $p$ that is not equal to 1. In fact, $H_G$ is the unique multiplicative function supported on the squarefree numbers that satisfies $H_G(p) = 1$ for each hypertransparent prime $p$ and that satisfies $(*_H)$ for all other primes.

It turns out that a multiplicative Ramanujan coefficient of $F \neq 0$ is completely determined by its opacity core. This is expressed qualitatively by the following rigidity statement, which we prove in section 9.3.

**Theorem 9.3 (Rigidity)** Let $F : \mathbb{N} \to \mathbb{C}$ be a non-null function and let $G_1, G_2 \in <F>_{M}$ be two multiplicative Ramanujan coefficients of $F$. Suppose that $G_1$ and $G_2$ have the same opacity core $H_{G_1} = H_{G_2}$. Then $G_1 = G_2$.

**Remark 9.** In addition to this, we shall prove in Proposition 9.6 an explicit formula that recovers all values of $G \in <F>_{M}$ from its opacity core $H_G$.

Thanks to Theorem 9.3, we see that the problem of describing the full multiplicative Ramanujan cloud of $F$ is reduced to the problem of characterizing the possible opacity cores that occur. The precise description is quite technical but ultimately (see Theorem 9.5 below for details) it boils down to a single linear equation, namely

$$S_{H_G}(1) = R_G(N_T(G)).$$

In order to state the necessary additional technical conditions we need to introduce one final notion, that of “relative simple badness”.

**Definition 9.4 (Relative Simply Bad Primes)** Let $M, H : \mathbb{N} \to \mathbb{C}$ be multiplicative functions and let $p$ be prime. We say that $p$ is a relative simply bad prime for $(M, H)$ if and only if $1 \leq |H(p)| \leq p$ and at least one of the following two conditions holds:

$$(i) \quad w_{p,M*\mu} \neq \infty, \quad \text{or} \quad (ii) \quad H(p) \neq (M(p) - 1)/p.$$
Remark 10. In the previous definition we should think $H = H_G$ as the opacity core of some multiplicative function $G \in \langle F \rangle$, and $M = M_F$ as the multiplicative function associated to the non-null semi-multiplicative function $F$ (see Proposition 1.3, §1). In this case the conditions (i) and (ii) are naturally equivalent to the statement that $p$ is not a completely multiplicative prime of $G$ (see Lemma 9.7 below).

Finally, we are able to state the classification theorem for the multiplicative Ramanujan coefficients of non-null functions.

**Theorem 9.5 (Classification)** Let $F$ be a non-null semi-multiplicative function and write $F$ in the standard form $F(a) = F(a_F)M_F(a/a_F)$ for all $a \in \mathbb{N}$, where $F(a_F) \neq 0$ and $M_F : \mathbb{N} \to \mathbb{C}$ is multiplicative. Let $H : \mathbb{N} \to \mathbb{C}$ be a multiplicative function supported on the squarefree numbers and let $N \in \mathbb{N}$ be a natural number divisible by $a_F$ and by all the relative simply bad primes of $(M_F, H)$. If

$$\sum_{(r,N)=1} H(r)\mu(r) \text{ converges} \quad \text{AND} \quad \sum_{q=1}^{\infty} H(q)\mu(q) = F(a_F),$$

then there exists a (unique) multiplicative Ramanujan coefficient $G \in \langle F \rangle$ of $F$, such that $H_G = H$.

Remark 11. An explicit formula for $G$ in terms of $H$ and $F$ is given in Propositions 9.6. We then state and prove a more precise version of Theorem 9.5 in Proposition 9.8 below.

Remark 12. Viceversa, we show in Lemma 9.9 that the statement in display in Theorem 9.5 holds if $H = H_G$ is the opacity core of a multiplicative Ramanujan coefficient $G$ of $F$. Therefore Theorem 9.5 gives a complete description of the multiplicative Ramanujan cloud of $F$.

### 9.3 Proofs of the rigidity theorem and of the classification

We shall deduce the rigidity theorem (Theorem 9.3) from the following more precise statement, that recovers all values of $G \in \langle F \rangle$ from its opacity core.

**Proposition 9.6** Let $F$, $a_F$ and $M_F$ be as in Theorem 9.5 and let $G \in \langle F \rangle$. Then for every prime $p$ and all $v \in \mathbb{N}$ we have the explicit formula

$$(*_G) \quad G(p^{v_p(a_F)+1}) = H_G(p) + (1 - H_G(p))G_M(p^v)$$

where $H_G$ is the opacity core of $G$ and $G_{M_F}$ is the canonical Ramanujan coefficient of $M_F$ (see Definition E, §2).

**Proof.** If $p$ is hypertransparent, then $H_G(p)$ and $G(p^k) = 1$ for all $k \in \mathbb{N}$, so the formula $(*_G)$ holds. If $p$ is not hypertransparent, then $v_{p,G} = v_F(a_F)$ because by the Euler-Selberg product formula (Corollary 1.2) we know that $a_F$ is the transparency conductor of $G$. Therefore $(*_H)$ tells us that

$$H_G(p) = G(p^{v_{p,F}(a_F)+v^1}) = G(p^{v_{p,G}+1}) \neq 1. \quad (eq : \text{hp2})$$

A simple property of the canonical Ramanujan coefficient is that $G_{M_F}(p) = 0$ for each prime $p$, therefore $(*_H)$ is equivalent to the case $v = 1$ of $(*_G)$. To prove the formula for $v \geq 1$ we use the Euler-Selberg product formula (Corollary 1.2) of $F = \mathbb{R}_G$. An inspection of this factorization formula reveals that

$$M_F(p^v) - M_F(p^{v-1}) = p^v G(p^{v_F(a_F)+v}) - G(p^{v_F(a_F)+v+1}) \quad \frac{1 - H_G(p)}{1 - H_G(p)}$$

for all $v \in \mathbb{N}$. We deduce that

$$G(p^{v_F(a_F)+v+1}) = G(p^{v_F(a_F)+v}) - (1 - H(p))p^{-v}M_F(p^v)$$

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for all \( v \in \mathbb{N} \), where \( M'_F \) denotes the Eratosthenes transform (Definition C, §2) of \( M_F \). Since the canonical Ramanujan coefficient \( G_{M'_F} \) of \( M_F \) is given explicitly by

\[
G_{M'_F}(p^v) = - \sum_{k=1}^{v-1} \frac{M'_F(p^v)}{p^v},
\]

the proposition is proved. \( \square \)

The rigidity theorem is now a corollary:

**Proof of Theorem 9.3.** From Proposition 9.6 we get \( G_1(p^v) = G_2(p^v) \) for every \( v > v_p(a_F) \). On the other hand, we have that \( G_1(p^v) = G_2(p^v) = 1 \) for each \( v = 1, \ldots, v_p(a_F) \). Therefore \( G_1 \) and \( G_2 \) have the same value at all powers of primes; since they are multiplicative, they coincide. \( \square \)

Before we embark the proof of the classification theorem (Theorem 9.5), we state and prove an easy lemma that illustrates the rationale behind the notion of relative simply bad primes (Definition 9.4). Briefly, we know from Proposition 9.6 that a multiplicative Ramanujan coefficient \( G \) can be fully recovered from \( F = R_G \) and \( H = H_G \). This means that all the relevant set of primes (transparent, hypertransparent, bad and hyperbad) attached to \( G \) should admit a direct description in terms of \( F \) and \( H \) only. By the structure theorem, the simply transparent primes of \( G \) are recovered as the prime divisors of \( a_F \) defined as \( a_F(p) = \min\{a \in \mathbb{N} : F(a) \neq 0\} \). Moreover Remark 9.2.1 tells us that the hypertransparent primes \( p \) are characterized by the property \( H(p) = 1 \). If \( p \) is a prime that does not divide \( a_F \), we have \( H(p) = G(p) \), so badness is detected by the inequality \( 1 \leq |H(p)| \leq p \). It remains only to differentiate between simply bad and hyperbad primes. This is accomplished by Definition 9.4 (see also Remark 9.4.1) and by the following lemma.

**Lemma 9.7** Let \( G, M : \mathbb{N} \to \mathbb{C} \) be multiplicative functions and let \( G_M \) be the canonical Ramanujan coefficient of \( M \). Let \( p \) be some prime number such that \( G(p) \neq 1 \) and suppose that

\[
G(p^v) = G(p) + (1 - G(p))G_M(p^v)
\]

for all \( v \in \mathbb{N} \). Then the following are equivalent:

1. \( w_{p,G} = \infty \);
2. \( w_{p,M*\mu} = \infty \) and \( G(p) = (M(p) - 1)/p \).

**Proof.** First we recall that the values of the canonical Ramanujan coefficient at prime powers of \( p \) are

\[
G_M(p^v) = - \sum_{k=1}^{v-1} \frac{M'(p^v)}{p^v},
\]

where \( M' \) is the Eratosthenes transform of \( M \). Then for all \( v \in \mathbb{N} \) we have

\[
\frac{G(p^v) - G(p^{v+1})}{1 - G(p)} = \frac{M'(p^v)}{p^v}.
\]

If (i) holds, then \( G(p^v) = G(p)^v \) for all \( v \in \mathbb{N} \). Therefore

\[
M'(p^v) = p^v \frac{G(p)^v - G(p)^{v+1}}{1 - G(p)} = p^v G(p)^v
\]

for all \( v \in \mathbb{N} \). This means that \( M'(p) = pG(p) \) and that \( M' \) is completely multiplicative along \( p \). Since \( M'(p) = M(p) - 1 \), we get (ii).

Suppose now that (ii) holds. Then \( M'(p^v) = (M'(p))^v = p^v G(p)^v \) for all \( v \in \mathbb{N} \) and so

\[
G(p^v) = G(p) - (1 - G(p)) \sum_{k=1}^{v-1} G(p)^v.
\]
Using the formula for the sum of a geometric progression we verify that this simplifies to $G(p^v) = G(p)^v$, and therefore we get (i).

We now present the proof of the classification theorem, in the following more precise form:

**Proposition 9.8** Let $F$, $a_F$, $M_F$ and $H$ be as in Theorem 9.5. Let $G_{M_F}$ be the canonical Ramanujan coefficient of $M_F$ and let $G : \mathbb{N} \to \mathbb{C}$ be the unique multiplicative function such that

$$G(p^{v_{p}(a_F)}) = H(p) + (1 - H(p))G_{M_F}(p^v)$$

for each $p$ prime and all $v \in \mathbb{N}$, and such that $G(p^v) = 1$ for all prime $p \mid a_F$ and all integers $v \in [1, v_p(a_F)]$. Then $H$ is the opacity core of $G$ and $G \in < F >_{M_F}$.

**Proof.** We divide the proof in 7 steps.

**Step 1:** $H = H_G$. We recall that the canonical Ramanujan coefficient satisfies $G_M(p) = 0$ for every prime $p$, so putting $v = 1$ in the definition of $G$ we obtain

$$H(p) = G(p^{v_p(a_F)+1}).$$

If $H(p) \neq 1$ then $H(p)$ is the first value of $G$ at a power of $p$ that is not equal to 1. If $H(p) = 1$ instead, then $G(p^v) = 1$ for all $v \in \mathbb{N}$. Since moreover $H$ is multiplicative and supported on squarefree numbers, we deduce that $H$ is the opacity core of $G$ (see Remark 9.2.1).

**Step 2:** $a_F = N_T(G)$. By hypothesis we have that $S_H(N)$ converges and that $S_H(1) = F(a_F) \neq 0$. By the $S_G$-recursive formula we also know that

$$S_H(N) \prod_{p \mid N}(1 - H(p)) = S_H(1),$$

hence $H(p) \neq 1$ for all prime divisor $p$ of $N$. In particular, for every $p \mid a_F$ we have $G(p^{v_p(a_F)+1}) = H(p) \neq 1$ and $G(p^v) = 1$ for all $v \leq v_p(a_F)$. This means that for each $p \mid a_F$ we have

$$v_p(N_T(G)) = v_{p,G} = v_p(a_F).$$

If instead $p$ does not divide $a_F$, then $G(p) = H(p)$ and either $v_{p,G} = 0$ (if $H(p) \neq 1$) or $v_{p,G} = \infty$ (when $H(p) = 1$). In both these cases we get $v_p(N_T(G)) = v_p(a_F) = 0$. Therefore $v_p(N_T(G)) = v_p(a_F)$ holds for every prime $p$, which implies $a_F = N_T(G)$.

**Step 3:** $\text{rad } N(G) \mid N$. As we have already seen, all simply transparent primes of $G$ are divisors of $a_F$, and so they divide $N$. If $p$ is a simply bad prime of $G$ that is not simply transparent, then $v_p(a_F) = 0$ and $G(p) = H(p) \neq 1$. Therefore we may apply Lemma 9.7 with $M = M_F$. Since $p$ is simply bad, we have that $v_{p,G} \neq \infty$ and so Lemma 9.7 implies that $p$ is a *relative simply bad prime* (see Definition 9.4) for the pair $(M_F, H)$. We therefore conclude that $p \mid N$ for every simply bad prime $p$ of $G$.

**Step 4:** $\mathcal{R}_G(a)$ converges $\forall a \in \mathbb{N}$. For every prime $p$ coprime with $a_F$ we have $H(p) = G(p)$, therefore for each $b$ multiple of $a_F$ the series $S_G(b)$ coincides with $S_H(b)$. By assumption we know that $S_H(N)$ converges, so by the $S_G$-recursive formula we have that $S_H(d)$ converges for all $d \mid N$. Since $N$ is a multiple of $\text{rad } N(G)$, and $N(G)$ is a multiple of $a_F = N_T(G)$, we deduce that

$$S_G(N(G)) = S_H(N(G)) = S_H(\text{rad } N(G))$$

converges. The convergence of $\mathcal{R}_G(a)$ for all $a \in \mathbb{N}$ follows from the finiteness convergence theorem (Theorem 1.1).

**Step 5:** $F(a_F) = \mathcal{R}_G(N_T(G))$. Since $a_F = N_T(G)$, we note the following equality:

$$\mathcal{E}_G(N_T(G)) = \prod_{p \mid N_T(G)}(1 - G(p^{v_p(a_F)+1})) = \prod_{p \mid N_T(G)}(1 - H(p)) = \mathcal{E}_H(N_T(G)).$$
By the RS–formula and by the $S_G$–recursive formula we then obtain

$$
\mathcal{R}_G(N_T(G)) = \mathcal{E}_G(N_T(G)) \delta_G(N_T(G)) = \mathcal{E}_H(N_T(G)) \delta_H(N_T(G)) = \delta_H(1).
$$

Moreover, $\delta_H(1)$ is equal to $F(a_T)$ by the assumption on $H$.

**Step 6:** $v_{p,G} = \infty \Rightarrow M'_p(p^v) = p^v$ for all $v \in \mathbb{N}$. We already noted (see Steps 1 and 2) that for all hypertransparent prime $p$ we have $H(p) = 1$ and that $p$ does not divide $N$. In particular, such $p$ is not a relative simply bad prime of $(M_F,H)$. In other words, we have

$$
w_{p,M'_p} = \infty \quad \text{and} \quad H(p) = (M_F(p) - 1)/p.
$$

We recall that $w_{p,M'_p} = \infty$ means that $M'_p$ is completely multiplicative along $p$. Since $H(p) = 1$ and $M'_p(p) = M_F(p) - 1$, we conclude that $M'_p(p^v) = (M_F(p))^v = p^v$ for all $v \in \mathbb{N}$.

**Step 7:** $F(a) = \mathcal{R}_G(a)$ for all $a \in \mathbb{N}$. By the Euler-Selberg product formula (Corollary 1.2) we have that

$$
\mathcal{R}_G(a) = \mathcal{R}_G(N_T(G)) \prod_{v_{p,G} = \infty} \left( \sum_{v=0}^{v_{p}(a)} \right) \prod_{v_{p,G} \neq \infty} \left( \sum_{v=0}^{v_{p}(a) - v_{p,G}} \frac{p^v G(p^{v_{p,G} + v}) - G(p^{v_{p,G} + v + 1})}{1 - G(p^{v_{p,G} + 1})} \right)
$$

for all $a \in \mathbb{N}$. We now aim to compare this formula with $F(a_T)M_F(a/a_T)$. First note that $\mathcal{R}_G(N_T(G)) = F(a_T)$ by Step 5. Next, for every prime $p$ with $v_{p,G} = \infty$ we have

$$
\sum_{v=0}^{v_{p}(a_T)} p^v = \sum_{v=0}^{v_{p}(a_T)} M'_p(p^v) = M_F(p^{v_{p}(a_T)}).
$$

by Step 6 and the formula $M_F = M'_F \ast 1$. Finally, when $v_{p,G} \neq \infty$ we have $G(p^{v_{p,G} + 1}) = H(p)$, so

$$
\sum_{v=0}^{v_{p}(a) - v_{p,G}} \frac{p^v G(p^{v_{p,G} + v}) - G(p^{v_{p,G} + v + 1})}{1 - G(p^{v_{p,G} + 1})} = \sum_{v=0}^{v_{p}(a) - v_{p,G}} p^v (G_{M_F}(p^v) - G_{M_F}(p^{v+1})) = \sum_{v=0}^{v_{p}(a) - v_{p,G}} M'_F(p^v)
$$

which is equal to $M_F(p^{v_{p}(a) - v_{p,G}})$ since $v_{p,G} = v_{p,a_T}$. Thus the Euler-Selberg product formula (Corollary 1.2) of $\mathcal{R}_G$ coincides with the canonical Selberg factorization of $F$.

Reasoning like in Steps 4 and 5 of the above proof we get the following result.

**Lemma 9.9** Let $G : \mathbb{N} \to \mathbb{C}$ be a multiplicative Ramanujan coefficient with Ramanujan conductor $N(G)$, transparency conductor $N_T(G)$ and opacity core $H_G$. Then

$$
\delta_G(N(G)) = \delta_{H_G}(N(G)) \quad \text{and} \quad \mathcal{R}_G(N_T(G)) = \delta_{H_G}(1).
$$

Thus we see that the series in display in Theorem 9.5 correspond to the notable series $\delta_G(N(G))$ and $\mathcal{R}_G(N_T(G))$ attached to $G$.
Appendix. Odds & ends (more details, alternative proofs, generalizations)

In order to give, in next §A.2, some examples of not-converging coprime series, we supply, in next §A.1, a short parade of elementary analytic results for the square-free numbers (as our §A.2 counterexamples need it).

A.1. Estimates for square-free numbers

We start with a classic: the asymptotic estimate for the number of square-free numbers up to $x$, as $x \to \infty$.

**Lemma A.1.1. (Square-free numbers asymptotic count)**

For all large $x$ we have

$$
\sum_{q \leq x} \mu^2(q) = \frac{x}{\zeta(2)} + O(\epsilon(x) \sqrt{x}),
$$

where $\epsilon(x) = \exp \left( -c(\log x)^{3/5}(\log \log x)^{-1/5} \right)$ for some constant $c > 0$.

**Proof.** This estimate was proved by Walfisz [Wa] using the classical zero-free region estimates due to Vinogradov and Korobov through exponential sums. (This result is quoted also in the survey paper of Pappalardi [Pa] but with a mistake : $1/5$ instead of $-1/5$).

**Lemma A.1.2. (Dirichlet series for square-free integers)**

For all $s \in \mathbb{C}$ with $\Re s > 1$ we have

$$
\sum_{q=1}^{\infty} \mu^2(q) q^{-s} = \prod_{p \in \mathbb{P}} (1 + p^{-s}) = \frac{\zeta(s)}{\zeta(2s)}.
$$

**Proof.** This is the half plane of absolute convergence, so everything works fine.

**Lemma A.1.3. (Truncated Dirichlet series for square-free)**

For all $s \neq 1$ with $\Re s \geq 1/2$ we have

$$
\sum_{q \leq x} \mu^2(q) q^{-s} = \frac{6}{\pi^2} \frac{x^{1-s}}{1-s} + \frac{\zeta(s)}{\zeta(2s)} + o(1).
$$

as $x \to \infty$.

**Proof.** First use summation by parts, and Cauchy’s criterion of convergence to have the main term and a constant dependent on $s$. Then as $x$ goes to infinity prove that the convergence is uniform on compact sets, which implies that the constant is holomorphic in $s$. Then go to the plane of absolute convergence to see that the constant coincides with $\zeta(s)/\zeta(2s)$. Another approach is to use Perron’s formula.

**Lemma A.1.4. (Truncated Dirichlet series for square-free with coprimality conditions)**

Let $b \in \mathbb{N}$. Then for all $s \neq 1$ with $\Re s \geq 1/2$ we have

$$
\sum_{(q,b)=1} \mu^2(q) q^{-s} = C_1(b) \frac{x^{1-s}}{1-s} + C_2(b) \frac{\zeta(s)}{\zeta(2s)} + o(1).
$$

as $x \to \infty$, where $C_1(b) \overset{\text{def}}{=} 6\pi^{-2} \prod_{p \mid b} (1 + p^{-1})^{-1}$ and $C_2(b) \overset{\text{def}}{=} \prod_{p \mid b} (1 + p^{-s})^{-1}$.

**Proof.** Use the Lemma A.1.3 together with the abstract lemmas for convergence of truncated Euler factors. More precisely, for every square-free $b \in \mathbb{N}$ we define the following function defined on the positive real numbers:

$$
H_b(x) \overset{\text{def}}{=} \sum_{(q,b)=1} \mu^2(q) q^{-s} - C(b) \frac{x^{1-s}}{1-s}.
$$
This consists of two parts, let \( H'_m(x) \equiv \sum_{n \leq x} \frac{\mu(n)}{n} \) and \( H''_m(x) \equiv (6C(b) / \pi^2) \cdot x^{1-s} / (1-s) \). By the truncated partial Euler product formula for multiplicative functions we have

\[
H'_1(x) = \sum_{d \mid b} A(d)H'_b(x/d),
\]

where \( A(d) = \mu^2(d)d^{-s} \). Interestingly, we also have the same formula for \( H''_b(x) \), for example for \( b = p \) prime we have

\[
H''_1(x) = C(1) \cdot \frac{x^{1-s}}{1-s} = \frac{C(1)}{1 + p^{-1}} \cdot \frac{x^{1-s}}{1-s} + p^{-s} \cdot \frac{C(1)}{1 + p^{-1}} \cdot \frac{(x/p)^{1-s}}{1-s},
\]

which is \( H''_b(x) + A(p)H''_p(x/p) \). The more general formula \( H'_1(x) = \sum_{d \mid b} A(d)H'_b(x/d) \) for every \( b \) coprime is obtained similarly. Putting all together, we get

\[
H_1(x) = \sum_{d \mid b} A(d)H_b(x/d)
\]

for every \( b \in \mathbb{N} \) square-free. However, we already know by the previous lemma that \( \lim_{x \to \infty} H_1(x) = \zeta(s)/\zeta(2s) \). By the iterated converse convergence theorem for linear combinations of multiplicative shifts, it follows that also \( \lim_{x \to \infty} H_b(x) \), exists, and it is equal to

\[
\lim_{x \to \infty} H_b(x) = \prod_{p \mid b} \frac{1}{1 + p^{-s}} : \zeta(s)/\zeta(2s).
\]

This is exactly what we wanted to prove.

\[\square\]

A.2. A class of coprime series with convergence problems

We give a first result, in the direction of the necessity of assuming some convergence hypotheses, in our Classification Theorem in [CG]. It gives a class of multiplicative \( G : \mathbb{N} \to \mathbb{C} \) whose coprime series have convergence problems.

**Proposition A.2.1. (Necessity of convergence hypotheses in Classification theorem)**

Let \( s \) be a complex number with \( 1/2 \leq \Re(s) < 1 \) and fix two distinct primes \( p_1, p_2 \in \mathbb{P} \). Let \( G \) be a multiplicative function such that

\[
G(p_1) = p_1^{1-s}, \quad G(p_2) = p_2^{1-s}
\]

and \( G(p) = -p^{-s} \) for every \( p \in \mathbb{P} \) with \( p \neq p_1 \) and \( p \neq p_2 \). Then for every \( b \in \mathbb{N} \) with \( (b, p_1) = 1 \) we have

\[
\sum_{(q, b) = 1} G(q)\mu(q) \text{ converges}, \quad \text{but} \quad \sum_{(r, p_1b) = 1} G(r)\mu(q) = \infty.
\]

If moreover \( G(p_2^2) = 0 \), then for every \( b \in \mathbb{N} \) we also have that

\[
\sum_{(r, b) = 1} G(r)c_r(p_1) = \infty.
\]

**Proof.** Recall \( S_G(b, x) \) definition in §2. First of all, for every \( m \) coprime with \( p_1p_2 \) we have that \( S_G(p_1p_2m, x) \) is just a truncated Dirichlet series for square-free numbers with coprimality conditions. Thus, by previous §A.1 results,

\[
S_G(p_1p_2m, x) = a_m(s)x^{1-s} + F_m(s) + o_m(1),
\]

where \( F_m(s) \) and \( a_m(s) \) are constant with respect to \( x \) and more precisely

\[
a_m(s) \equiv \frac{C_1(p_2p_2m)}{1-s} \quad \text{and} \quad F_m(s) = \frac{C_2(p_2p_2m)}{\zeta(2s)} \frac{\zeta(s)}{\zeta(2s)}.
\]
This immediately implies that the series $S_G(p_1p_2m)$ diverges. In order to deal with the other series we are going to use the truncated partial Euler factorization formula for $S$-series. First we finish up establishing the divergence over the numbers coprime with $p_1m$, where again $m$ is coprime with $p_1p_2$:

$$S_G(p_1m, x) = S_G(p_1p_2m, x) - G(p_2)S_G(p_1p_2m, x/p_2) \sim a_m(s)x^{1-s}\left(1 - \frac{G(p_2)}{p_2^{-s}}\right).$$

So it diverges. Now we deal with the convergence in the cases of the form $b = p_2m$:

$$S_G(p_2m, x) = S_G(p_1p_2m, x) - G(p_1)S_G(p_1p_2m, x/p_1) = F_m(s) - G(p_1)F_m(s) + o(1).$$

Then we establish the convergence of $S_G(m)$ when $m$ is coprime with $p_1p_2$:

$$S_G(m, x) = F_m(s)(1 - p_1^{-s})(1 - G(p_2)).$$

Finally, let us deal with the last assertion. First, for $b$ multiple of $p_1$ we have

$$\sum_{(r, b) = 1} G(r)c_r(p_1) = \sum_{(r, b) = 1} G(r)\mu(r)$$

and this diverges by the previous results in this Proposition. More precisely

$$R_{p_1, m}(p_1, x) \overset{\text{def}}{=} \sum_{(r, p_1m) = 1, \alpha \geq 2} G(r)c_{r\alpha}(p_1) = \alpha_m x^{1-s} + O(1)$$

for some nonzero $\alpha_m$ dependent on $m$, where $m$ is any number coprime with $p_1$. To finish, we need to consider the case $b$ coprime with $p_1$. Since $G(p_1^2) = 0$,

$$R_b(p_1, x) = R_{p_1, b}(p_1, x) + G(p_1)c_{p_1}(p_1)R_{p_1, m}(p_1, x/p_1).$$

We recall that $G(p_1)c_{p_1}(p_1) = (p - 1)p^1-s > 0$. Then, the following diverges:

$$R_b(p_1, x) \sim \alpha_m x^{1-s}(1 + (p - 1)p^{1-s})$$

\[\square\]

A.3. A longer Theorem 1.1: two equivalent properties more

We remark that our Theorem 1.1 can be completed, with other two equivalent properties, namely

1. $L_G(b)$ converges for all $b$ not divisible by any hyperbad prime $p \in B_\infty$;
2. $L_G(a)$ converges for all $a \in \mathbb{N}$.

We prove now the implications (4)$\Rightarrow$(5), (5)$\Rightarrow$(6) and (6)$\Rightarrow$(1) and “QED”, here, separates each proof.

Proof of (4)$\Rightarrow$(5). By assumption (4) we have that $S_G(b)$ converges for each $b \in \mathbb{N}$ coprime with the hyperbad primes of $G$; by Corollary 3.5 we deduce that $L_G(b)$ converges for the same $b$, namely (5). QED

Proof of (5)$\Rightarrow$(6). In order to prove (6), it suffices to apply Corollary 3.6 to the $b \in \mathbb{N}$ without hyperbad primes (for these $b$, (5) ensures $L_G(b)$ convergence) and to the $c \in \mathbb{N}$ made only of hyperbad primes. QED

Proof of (6)$\Rightarrow$(1). We already know that $G$ is a Ramanujan coefficient iff it is a Lucht coefficient; this is given by Corollary 3.7, actually proving, in particular, that (6) implies (1). QED

See that however Professor L.G. Lucht has full credit for having already proved (at least, 25 years ago), see [Lu1], the equivalence between the convergence of His series and Ramanujan series, namely, in Theorem 1.1 this (1)$\Leftrightarrow$(6) here.

The same result is quoted also in [La], of course. By the way, the more flexible result on divisor-closed sets (implied by Corollary 3.7), maybe, could be of interest, as [La] needs, in the convergence of Ramanujan expansions in subsets of natural numbers.
A.4. A shorter, alternative proof of Corollary 3.4

We sketch an alternative, slightly more direct proof of Corollary 3.4.

Proof. Given \(n, m \in \mathbb{N}\), we define (assuming, here, the multiplicative \(G : \mathbb{N} \to \mathbb{C}\) is implicit, in notation)

\[
\mathcal{R}_{G,n}(m, x) = \mathcal{R}_n(m, x) \overset{\text{def}}{=} \sum_{r \leq x \atop (r,n)=1} G(r)c_r(m).
\]

Then we have \(\mathcal{R}_c(a, x) = \mathcal{R}_c(b, x)\) and the following formulæ: (with \(c_t(b) = \mu(t)\), in next second)

\[
\mathcal{R}_G(a, x) = \sum_{d \mid \text{rad} c} G(d)c_d(a)\mathcal{R}_c(a, x/d) \quad \text{AND} \quad \mathcal{R}_G(b, x) = \sum_{t \mid \text{rad} c} G(t)c_t(b)\mathcal{R}_c(b, x/t).
\]

Writing \(d = ht\) and using Kluyver’s formula (§2), like in above original proof, this time for \(c_d(a)\), it is now easy to show the relation between \(\mathcal{R}_G(a)\) and \(\mathcal{R}_G(b)\) stated in Corollary 3.4. \(\blacksquare\)
A.5. A general approach to our main argument: M–expansions

We give a kind of general approach to the arguments we used to study Ramanujan expansions, modeling on them the following, more general (but, trying to catch main features) with respect to: \( F = \mathbb{R}_G \). We call it an M–expansion, say, \( F(a) = \sum_{q=1}^{\infty} M_q(a) \), where the \( M_q(a) \) is called an M–term, by definition, when it is an arithmetic function of two arguments \((a, q \in \mathbb{N})\), satisfying the axioms:

(0) (multiplicativity) : \( M_q(a) \) is multiplicative w.r.t. \( q \in \mathbb{N} \);
(1) (independence) : \( \forall a \in \mathbb{N}, \forall p \in \mathbb{P}, \forall k \in \mathbb{N}_0, M_{p^k}(a) = M_{p^k}(p^{v_p(a)}) \Rightarrow M_q(a) = M_q(1), \forall (a, q) = 1 \);
(2) (vertical limit) : \( \forall a \in \mathbb{N}, \forall p \in \mathbb{P}, \exists K_p(a) \in \mathbb{N} : M_{p^k}(a) = 0, \forall k > K_p(a) \).

Here \( V_p(a) \stackrel{\text{def}}{=} \min \{ K_p(a) \text{ in } (2) \} \) is the vertical limit of \( q = p^k \) with respect to \( a \), esp., for Ramanujan expansions, \( V_p(a) = v_p(a) + 1 \): this is the Ramanujan vertical limit. We used, implicitly, this in all of this paper’s results. (The results of following §A.6 depend strongly on it.) From (1) it is clear that, in general, \( V_p(a) \) depends upon \( v_p(a) \).

With these three axioms, in this order, we get, formally, the Euler product

\[
\sum_{q=1}^{\infty} M_q(a) = \prod_p \sum_{K=0}^{\infty} M_{p^K}(a) = \prod_p \sum_{K=0}^{\infty} M_{p^K}(p^{v_p(a)}) = \prod_p \sum_{K=0}^{V_p(a)} M_{p^K}(p^{v_p(a)})
\]

and we usually separate

\[
= \prod_{p|a} V_p(a) \cdot \prod_{p \nmid a} \left( M_1(1) + M_p(1) + \ldots + M_{p^{v_p(a)}}(1) \right),
\]

but we wish to have, in RHS, as infinite product (over \( p \nmid a \)), say : \( \sum_{(r,a)=1} \mu^2(r) M_r(1) \).

We may add other two axioms with which the M–term \( M_q(a) \) becomes a simple M–term, i.e.

(3) (square-free support) : \( M_r(1) = \mu^2(r) M_r(1) \), \( \forall r \in \mathbb{N} \);
(4) (normalization) : \( M_1(1) = 1 \).

With these two more, previous formal calculation (esp., under absolute convergence hypothesis, see above) becomes

\[
\sum_{q=1}^{\infty} M_q(a) = \prod_{p|a} V_p(a) \cdot \prod_{p \nmid a} (1 + M_p(1)) = \prod_{p|a} V_p(a) \cdot \prod_{p \nmid a} M_{p^K}(p^{v_p(a)}) \cdot \sum_{(r,a)=1} \mu^2(r) M_r(1),
\]

which gets closer, to our calculations for Ramanujan expansions.

In case of convergence in \( a \in \mathbb{N} \), let’s define the M–series at \( a \) (no confusion should arise with the notation in Corollary 1.2), generalizing \( \mathbb{R}_G(a) \), like the following coprime series \( S(a) \) generalizes \( S_G(a) \) (the above finite factor is the Euler–M–factor, but the convergence problems are with these two series):

\[
M(a) \overset{\text{def}}{=} \sum_{q=1}^{\infty} M_q(a), \quad S(a) \overset{\text{def}}{=} \sum_{(r,a)=1} \mu^2(r) M_r(1).
\]

Notice that \( M(a) \), resp., \( S(a) \), reduces to \( \mathbb{R}_G(a) \), resp., \( S_G(a) \), when \( M_q(a) = G(q)c_q(a) \).

Since the “easy”, say, properties (like in §3) should generalize at once, do these 5 axioms prove any of our main results (see §1)? If not, what are the necessary axioms for that aim and what are sufficient? We don’t have time (though margin, yes, it’s not too narrow) to answer.
A.6. Finite Ramanujan expansions, purity and Hildebrand’s coefficients

As given, in the book [ScSp] (see Theorem 1.1 of V.1), from Hildebrand’s 1984 paper [Hi], each arithmetic function $F$ has the following, say, **Hildebrand-Ramanujan expansion**

$$F(a) = \sum_{q|a} H_i F(q \text{ rad } q) c_{q \text{ rad } q}(a), \quad \forall a \in \mathbb{N}.$$

(Set, in quoted result, their $r^* = r \prod_{p|q} p = r \text{ rad } q$, changing letter from $r$ to $q$, in our notation.)

We call, in fact, this $H_i F$ : $\mathbb{N} \rightarrow \mathbb{C}$ the **Hildebrand coefficient** of our $F$. It can be given explicitly, in a recursive manner [ScSp], as (recall $1 \text{ rad } 1 = 1$)

$$H_i F(1) \overset{\text{def}}{=} F(1), \quad H_i F(q \text{ rad } q) \overset{\text{def}}{=} \frac{1}{c_{q \text{ rad } q}(q)} \left( F(q) - \sum_{d|q, d < q} H_i F(d \text{ rad } d) c_{d \text{ rad } d}(q) \right), \quad \forall q > 1.$$

Before to proceed, see that previous denominator doesn’t vanish, as from Hölder’s formula (see §2):

$$c_{q \text{ rad } q}(q) = \varphi(q \text{ rad } q) \frac{\mu(\text{ rad } q)}{\varphi(\text{ rad } q)} \neq 0.$$

By the product formula [T] for $\varphi$, i.e.,

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

in fact,

$$\varphi(q \text{ rad } q)/\varphi(\text{ rad } q) = q,$$

whence:

$$c_{q \text{ rad } q}(q) = q \mu(\text{ rad } q),$$

not vanishing, from $\text{ rad } q$ being square-free (by definition). Notice, also, that $H_i 0 = 0$.

How can be proved Hildebrand’s (finite Ramanujan expansion) formula? Well, we’ll recover it, from the vertical limit of Ramanujan sums:

$$c_{p^k}(p^{v_p(a)}) \neq 0 \quad \Rightarrow \quad 0 \leq K \leq v_p(a) + 1.$$

Notice: the Hildebrand coefficients are, by definition, supported over the square-full numbers $n$:

$$n \text{ is SQUARE-FULL } \overset{\text{def}}{=} p|n \Rightarrow p^2|n.$$

It is a matter of convenience, then, to define 1 as a square-full number; it’s strange, since it is also square-free (both coming from $\omega(1) = 0$, no prime-divisors!), but we’ll do it, in order to simplify many details. In fact, $H_i F(1) = F(1)$, of course, might be non-zero (thus confirming 1 in the $H_i F$ support, of square-full numbers).

The vertical limit of $c_{p^k}(a) \overset{\text{recall}}{=} c_q(a)$ is $q$–multiplicative, so may consider the vanishing of this factor), when combined to a square-full supported coefficient (with same prime $p|q$) $G(q) = G(p^{v_p(q)}, q/p^{v_p(q)})$, gives:

$$2 \leq v_p(q) \leq v_p(a) + 1, \quad \forall p|q,$$

first inequality from assuming $G$ supported on square-full (and this $p$ dividing modulus $q$), second one to avoid $c_q(a) = 0$ (from Ramanujan vertical limit).
Thus, in turn, implies \( v_p(a) \geq 1 \), i.e. \( p|a \) : we are proving that \( p|q \) implies \( p|a \) and so (assuming \( G \), even not multiplicative, vanishing outside the square-full numbers)

\[
\mathcal{R}_G(a) = \sum_{q=1}^{\infty} G(q)c_q(a) = \sum_{p|q \Rightarrow p|a} G(q)c_q(a).
\]

However, previous inequalities for \( p \)-adic valuations may be written

\[ 2 \leq v_p(q) \leq v_p(a \text{ rad } a), \quad \forall p|q \]

which, if combined with: \( p|q \Rightarrow p|a \), gives \( q|a \text{ rad } a \):

\[
\mathcal{R}_G(a) = \sum_{q|a \text{ rad } a} G(q)c_q(a), \quad \forall a \in \mathbb{N},
\]

but also, from our hypothesis \( G \) is square-full supported, that \( q = m \text{ rad } m \), with \( m|a \), giving

\[
\mathcal{R}_G(a) = \sum_{m|a} G(m \text{ rad } m)c_{m \text{ rad } m}(a), \quad \forall a \in \mathbb{N}.
\]

Thus getting Hildebrand’s finite Ramanujan expansion. Notice that we started from a generic Ramanujan expansion representation for our \( F \) and assuming ONLY its coefficient \( G \), as for \( H_{F'} \), supported on square-full numbers, we proved it’s actually a finite Ramanujan expansion abbrev. F.R.E. henceforward.

In case: \( F \neq 0 \) is multiplicative, we have (apparently) ANOTHER F.R.E., namely that of coefficient \( G = G_F \), the canonical Ramanujan coefficient of our \( F \).

Now, before proceeding further: the canonical Ramanujan coefficients, as may be easily checked, of course DO NOT DEPEND on \( a \) (say, outer variable) neither their support does depend on \( a \). We call in [CM], [C1] and [C3] an expansion, with this property on the coefficients, a pure Ramanujan expansion; there the first Author THOUGHT that this coefficient \( H_{F'} \) was not pure! Thanks to the second Author (that scrutinized much better Hildebrand’s result, through its [ScSp] reproduction) a conclusion, on the purity of Hildebrand’s coefficient, is now in order!

(Also, see that neither Theorem 3 in [C3] nor its proof have problems; whereas, the only problem is that, even if we have a pure & finite Ramanujan expansion, a bound on its length isn’t always possible! In fact, a priori all the pure F.R.E in the argument \( a \in \mathbb{N} \) have a length depending on all non-zero \( p \)-adic valuations of \( a \); so, the length may be, and usually is, unbounded with \( a \), i.e., as \( a \to \infty \).

In case: \( F \neq 0 \) is multiplicative, then we know it has an expansion with coefficient \( G = G_F \), the canonical Ramanujan coefficient of our \( F \) (and it’s a F.R.E.). Well, our \( G_F \), too, like \( H_{F'} \), is supported on square-full numbers (because it’s multiplicative and vanishes on primes, so the \( q \) in its support have \( v_p(q) \geq 2 \), for non-vanishing \( p \)-adic valuations). Let’s see how we got \( G_F \). We started from \( G_F(1) = 1 \), \( G_F(p) = 0 \) on all primes \( p \), so \( S_G = 1 \) and this means:

\[
\mathcal{R}_{G_F}(a) = \sum_{q=1}^{\infty} G_F(q)c_q(a) = \sum_{d|a \text{ rad } a} G_F(d)c_d(a) \cdot \sum_{(r,a)=1} G_F(r)\mu(r) = \sum_{d|a \text{ rad } a} G_F(d)c_d(a), \quad \forall a \in \mathbb{N},
\]

from the \( RS \)-formula (Corollary 3.3). We may proceed as above, since \( G_F(d) \neq 0 \) implies \( d \) square-full; hence, \( p|d \) implies \( 2 \leq v_p(d) \leq v_p(a) + 1 \), like above, which is equivalent to: \( 1 \leq v_p(d/\text{ rad } d) \leq v_p(a) \), \( \forall p|d \); in which, now, setting \( q := d/\text{ rad } d \), whence (from \( d \) square-full) \( \text{ rad } q = \text{ rad } d \) transforms \( d|a \text{ rad } a \) into \( q|a \) and \( d = q \text{ rad } q \):

\[
\mathcal{R}_{G_F}(a) = \sum_{q|a} G_F(q \text{ rad } q)c_{q \text{ rad } q}(a), \quad \forall a \in \mathbb{N}.
\]

Thus, any multiplicative and square-free vanishing Ramanujan coefficient of our multiplicative \( F \neq 0 \) is square-full supported, whence it’s Hildebrand’s coefficient of this \( F \) : \( H_{F'} \). A kind of astonishing, \( G_F = H_{F'} \).
However, not so strange, if we think about HOW we defined $G_F$: recursively, on the prime-powers; but this, in turn, adding multiplicativity, is the same as building $H_F$ recursively on divisor sums!

Summarizing, we have proved the following Lemmas, for finite Ramanujan expansions.

**Lemma A.6.1.** (UNIQUENESS FOR THE HILDEBRAND COEFFICIENT)

*Let $F: \mathbb{N} \to \mathbb{C}$ be any arithmetic function and let $G \in <F>$ be square-full supported. Then $G = H_F$. In case $F \neq 0$ is multiplicative, we get: $G_F = H_{F}$. Furthermore, adopting the convention $G_0 \overset{def}{=} 0$, we have that $F$ is multiplicative IFF $H_F$ is multiplicative.*

Notice: if $F = 0$, then $G$ square-full supported implies however $G = H_0 = 0$, since its support is empty (however, contained in the square-full numbers set)!

**Lemma A.6.2.** (RAMANUJAN CLOUDS ARE ALL NON-EMPTY, LIKE THEIR PURE AND FINITE SUBSETS)

*Let $F: \mathbb{N} \to \mathbb{C}$ be any arithmetic function. Then $H_F \in <F>$. In particular, all Ramanujan clouds are non-empty. In case $F \neq 0$, we have a PURE AND FINITE RAMANUJAN EXPANSION for $F$, namely that with Ramanujan coefficient $H_F$; in other words, $F \neq 0 \Rightarrow <F>_{\text{PURE&FIN}} \neq \emptyset$. For $F = 0$, if we consider the trivial Ramanujan expansion with $0$ Ramanujan coefficient a pure & finite one, then $<0>_{\text{PURE&FIN}} \neq \emptyset$. However, Ramanujan’s $G_R(q) \overset{def}{=} 1/q [R]$ proves that the pure part of $0$ cloud is, say, $<0>_{\text{PURE}} \neq \emptyset$.

(Notation $<F>_{\text{PURE}}$ here is $<F>_{\ast}$ in [C3], where $<F>_{\ast} \cap <F>_{\#}$ is written $<F>_{\text{PURE&FIN}}$ here.)

This Lemma makes a natural question arise: are there Ramanujan expansions of $0$ which are pure, finite and non-trivial (i.e., with coefficient $G \neq 0$)?

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