Elliptic families of solutions to constrained Toda hierarchy

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To Andrey K. Pogrebkov on his 75th birthday

Abstract

We study elliptic families of solutions to the recently introduced constrained Toda hierarchy, i.e., solutions which are elliptic functions of some linear combination of the hierarchical times. Equations of motion for poles of such solutions are obtained.

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1 Introduction

The Toda lattice hierarchy \cite{1} is an infinite set of evolution equations for two Lax operators $L, \bar{L}$ which are pseudo-difference operators in the variable $x$. Let $\{t_k\}_{k \in \mathbb{Z}}$ be the infinite set of independent variables (times) indexed by integer numbers. The hierarchy is defined by the infinite set of Lax equations (evolution equations for the Lax operators in the times $t_k$). They are equivalent to differential-difference equations for the coefficient functions of the Lax operators. These equations are differential with respect to the times $t_k$ with $k \neq 0$ and difference with respect to the time $t_0 = x/\eta$ ($\eta$ plays the role of the lattice spacing). An equivalent formulation is via the tau-function $\tau$ \cite{2,3} which is a function of the infinite set of independent variables satisfying bilinear functional relations.

The constrained Toda hierarchy was recently introduced in \cite{4}. It is a subhierarchy of the Toda lattice hierarchy defined by the constraint

$$\bar{L} = L^\dagger$$

(in the symmetric gauge). As is shown in \cite{4}, the constraint is preserved by the flows $\partial_{t_k} - \partial_{t_{-k}}$ and is destroyed by the flows $\partial_{t_k} + \partial_{t_{-k}}$, so one has to put $t_k + t_{-k} = 0$.

The investigation of dynamics of poles of singular solutions to nonlinear integrable equations was initiated in the seminal paper \cite{5}, where elliptic and rational solutions to the Korteweg-de Vries and Boussinesq equations were studied. As it was proved later in \cite{6,7}, poles of solutions to the Kadomtsev-Petviashvili (KP) equation which are rational functions of $t_1$, as functions of the second hierarchical time $t_2$, move as particles of the integrable Calogero-Moser system \cite{8,9,10,11}. This correspondence was extended to elliptic solutions in \cite{12}. Dynamics of poles of elliptic solutions to the 2D Toda lattice and modified KP (mKP) equations was studied in \cite{13}, see also \cite{14}. It was proved that the poles move as particles of the integrable Ruijsenaars-Schneider many-body system \cite{15,16} which is a relativistic generalization of the Calogero-Moser system.

The study of more general elliptic families of solutions to nonlinear integrable hierarchies, i.e. solutions which are elliptic functions of a general linear combination of the higher times of the hierarchy, was initiated in \cite{17}. It was shown that poles of such solutions as functions of $t_1$ and $t_2$ move according to equations of motion of the field generalization of the Calogero-Moser system. Recently, similar results for elliptic families of solutions to the Toda hierarchy were obtained in \cite{18}, where the field generalization of the Ruijsenaars-Schneider model was introduced.

The aim of this paper is to study elliptic families of solutions to the constrained Toda hierarchy. We will derive equations of motion for poles of these solutions. This system can be regarded as a field generalization of the system obtained in \cite{4}.
2 Constrained Toda hierarchy

2.1 The Toda hierarchy

We begin with the Toda lattice hierarchy [1] in the symmetric gauge [19, 20]. The two Lax operators are the pseudo-difference operators

\[ L = c(x) e^{\eta \partial_x} + \sum_{k \geq 0} U_k(x) e^{-k \eta \partial_x}, \quad \bar{L} = c(x - \eta) e^{-\eta \partial_x} + \sum_{k \geq 0} \bar{U}_k(x) e^{k \eta \partial_x}, \]

where \( e^{k \eta \partial_x} \) are shift operators acting on functions of \( x \) as \( e^{k \eta \partial_x} f(n) = f(x + k \eta) \). Given the Lax operators, one can introduce the difference operators

\[ B_m = (L_m)_{>0} + \frac{1}{2} (L_m)_0, \quad B_{-m} = (\bar{L}_m)_{<0} + \frac{1}{2} (\bar{L}_m)_0, \quad m = 1, 2, 3, \ldots, \]

where for a subset \( S \subset \mathbb{Z} \), we denote \( \left( \sum_{k \in \mathbb{Z}} U_k e^{k \partial_n} \right)_S = \sum_{k \in S} U_k e^{k \partial_n} \). The Toda lattice hierarchy is given by the Lax equations

\[ \partial_t m L = [B_m, L], \quad \partial_t m \bar{L} = [B_m, \bar{L}] \]

which define the hierarchical flows parametrized by the times \( t_m \) for any non-zero integer \( m \). An equivalent formulation is through the zero curvature (Zakharov-Shabat) equations

\[ \partial_{t_k} B_m - \partial_t m B_k + [B_m, B_k] = 0. \]

One of the main objects related to the hierarchy is the tau-function \( \tau(x, t) \) which we denote below simply as \( \tau(x) \) skipping the dependence on the times. The coefficient \( c(x) \) is expressed through the tau-function by the formula

\[ c(x) = \left( \frac{\tau(x + 2 \eta) \tau(x)}{\tau^2(x + \eta)} \right)^{1/2}. \]

2.2 Specialization to the constrained Toda hierarchy

The constrained Toda hierarchy is obtained by imposing the constraint

\[ \bar{L} = L^\dagger, \]

where the \( ^\dagger \) operation is defined as \( (f(x) \circ e^{k \eta \partial_x})^\dagger = e^{-k \eta \partial_x} \circ f(x) \). This implies that \( \bar{U}_k(x) = U_k(x + k \eta) \). It is easy to see that the constraint is preserved by the flows \( \partial_{t_k} - \partial_{t_{-k}} \) and is destroyed by the flows \( \partial_{t_k} + \partial_{t_{-k}} \), so one has to put \( t_k + t_{-k} = 0 \). In this way all the coefficient functions can be regarded as functions of \( t_k \) with \( k > 0 \) (and of \( x \)) only. Introducing difference operators

\[ A_m = B_m - B_{-m}, \]

we can write the Lax and Zakharov-Shabat equations of the constrained hierarchy in the form

\[ \partial_{t_m} L = [A_m, L], \quad [\partial_{t_m} - A_m, \partial_{t_k} - A_k] = 0, \quad m > 0. \]
In particular,
\[
A_1 = c(x)e^{\eta \partial_x} - c(x - \eta)e^{-\eta \partial_x},
\]
\[
A_2 = c(x)c(x + \eta)e^{2\eta \partial_x} + c(x)(v(x) + v(x + \eta))e^{\eta \partial_x} - c(x - \eta)(v(x) + v(x - \eta))e^{-\eta \partial_x} - c(x - \eta)c(x - 2\eta)e^{-2\eta \partial_x},
\]
where \( v(x) = U_0(x) \).

The Zakharov-Shabat and Lax equations are compatibility conditions for the linear problems
\[
\partial_t \psi = A_m \psi, \quad L \psi = z \psi \tag{10}
\]
for the wave function \( \psi = \psi(x, t, z) \) depending on the spectral parameter \( z \in \mathbb{C} \) (and on all the times \( t = \{t_1, t_2, t_3, \ldots \} \)). In particular, we have the linear problem
\[
\partial_t \psi(x) = c(x)\psi(x + \eta) - c(x - \eta)\psi(x - \eta), \tag{11}
\]
where we do not indicate the dependence on \( t \) for brevity.

Introducing the wave function \( \Psi(x) \) by means of the relation
\[
\Psi(x) = \left( \frac{\tau(x + \eta)}{\tau(x)} \right)^{1/2} \psi(x), \tag{12}
\]
we represent the linear problem (11) in the form
\[
\partial_t \Psi(x) = \Psi(x + \eta) + b(x)\Psi(x) - a(x)\Psi(x - \eta), \tag{13}
\]
where
\[
b(x) = \frac{1}{2} \partial_t \log \frac{\tau(x + \eta)}{\tau(x)}, \quad a(x) = \frac{\tau(x + \eta)\tau(x - \eta)}{\tau^2(x)}. \tag{14}
\]

### 3 Elliptic families

#### 3.1 Elliptic families among general algebraic-geometrical solutions

We are going to consider solutions that are elliptic functions of a linear combination \( \lambda = \beta_0 x + \sum \beta_k t_k \) of higher times of the hierarchy. We call them elliptic families. The elliptic families form a particular class of algebraic-geometrical solutions associated with an algebraic curve \( \Gamma \) of genus \( g \) with some additional data. An algebraic-geometrical solution is said to be elliptic with respect to some variable \( \lambda \) if there exists a \( g \)-dimensional vector \( \mathbf{W} \) such that it spans an elliptic curve \( \mathcal{E} \) embedded in the Jacobian of the curve \( \Gamma \). The tau-function of such solution has the form
\[
\tau(x, t, \lambda) = e^{Q(x, t)} \Theta \left( V_0 x/\eta + \sum_{k \geq 1} V_k t_k + \mathbf{W} \lambda + Z \right), \tag{15}
\]
where \( \Theta \) is the Riemann theta-function with the Riemann matrix being the matrix of \( b \)-periods of normalized holomorphic differentials on \( \Gamma \), and \( Q(x, t) \) is a quadratic form in
the variable $x$ and the hierarchical times $t = \{t_1, t_2, t_3, \ldots\}$. The vectors $V_k$ are related to $b$-periods of certain normalized meromorphic differentials on $\Gamma$. The existence of a $g$-dimensional vector $W$ such that it spans an elliptic curve $E$ embedded in the Jacobian is a nontrivial transcendental condition. If such a vector $W$ exists, then the theta-divisor intersects the shifted elliptic curve $E + V_0 x/\eta + \sum_{k} V_k t_k$ at a finite number of points $\lambda_i = \lambda_i(x, t)$. Therefore, for elliptic families we can write:

$$\Theta \left( V_0 x/\eta + \sum_{k \geq 1} V_k t_k + W \lambda + Z \right) = f(x, t) e^{\gamma_1 \lambda + \gamma_2 \lambda^2} \prod_{i=1}^{N} \sigma(\lambda - \lambda_i(x, t))$$

(16)

with a function $f(x, t)$ and some constants $\gamma_1, \gamma_2$. Here $\sigma(\lambda)$ is the Weierstrass $\sigma$-function with quasi-periods $2\omega_1, 2\omega_2$ (such that $\text{Im}(\omega_2/\omega_1) > 0$) defined by the infinite product

$$\sigma(x) = \sigma(x | \omega_1, \omega_2) = \prod_{s \neq 0} \left(1 - \frac{x}{s}\right) e^{\frac{x}{s} + \frac{x^2}{2s^2}}, \quad s = 2\omega_1 m_1 + 2\omega_2 m_2 \quad \text{with integer } m_1, m_2.$$  

(17)

Below we also use the Weierstrass $\zeta$-function $\zeta(x) = \sigma'(x)/\sigma(x)$. The form of the exponential factor in the right hand side of (16) follows from monodromy properties of the theta-function. The zeros $\lambda_i$ of the tau-function are poles of the elliptic solutions.

From (14), (15), (16) we conclude that for an elliptic family the coefficients $b(x) = b(x, \lambda), a(x) = a(x, \lambda)$ have the form

$$b(x, \lambda) = \frac{1}{2} \sum_{i=1}^{N} \left( \dot{\lambda}_i(x) \zeta(\lambda - \lambda_i(x)) - \dot{\lambda}_i(x + \eta) \zeta(\lambda - \lambda_i(x + \eta)) \right) + c(x, t),$$

(18)

$$a(x, \lambda) = g(x, t) \prod_{i=1}^{N} \frac{\sigma(\lambda - \lambda_i(x + \eta)) \sigma(\lambda - \lambda_i(x - \eta))}{\sigma^2(\lambda - \lambda_i(x))},$$

(19)

where dot means the $t_1$-derivative and $c(x, t), g(x, t)$ are some functions.

### 3.2 Double-Bloch functions

Our strategy is to find $b(x, \lambda), a(x, \lambda)$ such that the equation (13) has sufficiently many double-Bloch solutions. The existence of double-Bloch solutions turn out to be a rather restrictive condition.

A meromorphic function $f(\lambda)$ is called a double-Bloch function if it satisfies the following monodromy properties:

$$f(\lambda + 2\omega_\alpha) = B_\alpha f(\lambda), \quad \alpha = 1, 2.$$  

(20)

The complex constants $B_\alpha$ are called Bloch multipliers. Let the function $\Phi(\lambda, z)$ be defined by

$$\Phi(\lambda, z) = \frac{\sigma(\lambda + z)}{\sigma(z) \sigma(\lambda)} e^{-\zeta(z) \lambda}.$$  

(21)

It has a simple pole at $\lambda = 0$ with residue 1. The quasiperiodicity properties of the function $\Phi$ in the variable $\lambda$ are

$$\Phi(\lambda + 2\omega_\alpha, z) = e^{2(\zeta(\omega_\alpha) z - \zeta(z) \omega_\alpha)} \Phi(\lambda, z),$$

(22)
In order to derive equations of motion for the zeros of the tau-function, we start with the equation of motion:

3.3 Derivation of the equations of motion

Comparing (24) and (25), we obtain the following equations of motion:

\[ c \] where \( \left( \right) \) into the linear problem (13) and cancel the poles. The substitution gives:

\[ \lambda \] The cancellation of simple poles at \( \lambda \) yields the equation

\[ c_i(x + \eta) = \frac{1}{2} \hat{\lambda}_i(x + \eta) \sum_j c_j(x) \Phi(\lambda_i(x + \eta) - \lambda_j(x)). \] (24)

The cancellation of double poles at \( \lambda = \lambda_i(x) \) yields the equation

\[ c_i(x + \eta) = -2 \frac{g(x + \eta, t) \prod_i \sigma(\lambda_i(x + \eta) - \lambda_i(x + 2\eta))\sigma(\lambda_i(x + \eta) - \lambda_i(x))}{\lambda_i(x + \eta) \prod_{i \neq i} \sigma^2(\lambda_i(x + \eta) - \lambda_i(x + \eta))} \times \sum_j c_j(x) \Phi(\lambda_i(x + \eta) - \lambda_j(x)). \] (25)

Comparing (24) and (25), we obtain the following equations of motion:

\[ \dot{\lambda}_i(x) = 2g^{1/2}(x, t)\sigma^{1/2}(\lambda_i(x + \eta) - \lambda_i(x))\sigma^{1/2}(\lambda_i(x) - \lambda_i(x - \eta)) \]

\[ \times \prod_{j \neq i} \frac{\sigma^{1/2}(\lambda_i(x) - \lambda_j(x + \eta))\sigma^{1/2}(\lambda_i(x) - \lambda_j(x - \eta))}{\sigma(\lambda_i(x) - \lambda_j(x))} \] (26)

These are field analogue of the equations of motion

\[ \dot{x}_i = 2\sigma(\eta) \prod_{j \neq i} \frac{\sigma^{1/2}(x_i - x_j + \eta)\sigma^{1/2}(x_i - x_j - \eta)}{\sigma(x_i - x_j)} \] (27)
from [4] which are obtained from (26) as a particular case when one sets \( \lambda_i(x) = x + x_i \) (and \( g(x, t) = 1 \)). The function \( g(x, t) \) in (26) can be fixed by multiplying the equations over \( i \) from 1 to \( N \):

\[
g^{N/2}(x, t) = 2^{-N} \prod_{i=1}^{N} \frac{\dot{\lambda}_i(x)}{\sigma^{1/2}(\lambda_i(x + \eta) - \lambda_i(x)) \sigma^{1/2}(\lambda_i(x) - \lambda_i(x - \eta))} \prod_{i \neq j} \frac{\sigma(\lambda_i(x) - \lambda_j(x))}{\sigma^{1/2}(\lambda_i(x) - \lambda_j(x + \eta)) \sigma^{1/2}(\lambda_i(x) - \lambda_j(x - \eta))}. \tag{28}
\]

### 3.4 Continuum limit

The continuum limit is the limit \( \eta \to 0 \). We write:

\[
\lambda_i(x \pm \eta) = \lambda_i(x) \pm \eta \lambda'(x) + \frac{\eta^2}{2} \lambda''(x) + O(\eta^3),
\]

where prime means the \( x \)-derivative. We assume that

\[
g(x, t) = 1 + \eta^2 h(x, t) + O(\eta^3).
\]

This assumption is consistent with (28). Expanding the right hand side of equations (26), we have:

\[
\dot{\lambda}_i = 2\eta \lambda'_i \left( 1 + \frac{\eta^2}{2} h(x, t) - \frac{\eta^2}{8} \frac{\lambda''}{\lambda'_i} - \frac{\eta^2}{2} \sum_{j \neq i} \left( \lambda''_j \zeta(\lambda_i - \lambda_j) + \lambda'_j^2 \varphi(\lambda_i - \lambda_j) \right) + O(\eta^3) \right),
\]

where \( \varphi(x) = -\zeta'(x) \) is the Weierstrass \( \varphi \)-function. The naive \( \eta \to 0 \) limit is

\[
\dot{\lambda}_i = 2\eta \lambda'_i.
\]

The solution is \( \lambda_i = \varphi_i(x + 2\eta t) \), where \( t = t_1 \). However, a more meaningful limit consists in passing to the function \( y_i(x, t) \) connected with \( \lambda_i(x, t) \) by the relation

\[
\lambda_i(x, t) = y_i(x + 2\eta t, \eta^3 t).
\]

Then from (29), in the limit \( \eta \to 0 \), we get the equation

\[
\dot{y}_i = \frac{1}{4} y''_i - \sum_{j \neq i} \left( y'_i y''_j \zeta(y_i - y_j) + y'_i y'_j \varphi(y_i - y_j) \right) + y'_i h(x, t), \tag{30}
\]

which can be regarded as a field generalization of the equations of motion for zeros of the CKP tau-function obtained in [21].

### 4 Conclusion

We have studied solutions to the constrained Toda hierarchy which are elliptic functions of a general linear combination \( \lambda \) of higher times of the hierarchy. For such solutions, the
tau-function is essentially a product of the Weierstrass sigma-functions of \( \lambda \) with zeros \( \lambda_i, i = 1, \ldots, N \). We have investigated how these zeros (poles of the solutions) depend on \( x \) and \( t_1 \) and derived equations of motion (26) for them which are first order differential equations in the time \( t_1 \) and difference equations in the space variable \( x \). These equations can be regarded as a field generalization of the equations of motion for zeros of the tau-function of the constrained Toda hierarchy obtained in [4]. The continuum limit of these equations is a field generalization of the equations of motion for zeros of the CKP tau-function obtained in [21]. It is an open problem to clarify whether these equations are Hamiltonian.

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Conflict of Interest

The author declares that he has no conflicts of interest.

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