K-stable Fano threefolds of rank 2 and degree 30

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Received: 27 October 2021 / Revised: 5 July 2022 / Accepted: 13 July 2022 / Published online: 24 August 2022
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Abstract
We find all K-stable smooth Fano threefolds in the family No. 2.22.

Keywords Fano threefolds · K-stability · Kähler–Einstein metric

Mathematics Subject Classification 14J45 · 14J30 · 32Q20

Let $X$ be a smooth Fano threefold. Then $X$ belongs to one of the 105 families, which are labeled as No. 1.1, No. 1.2, . . . , No. 9.1, No. 10.1. See [2], for the description of these families. If $X$ is a general member of the family No. $N$, then [2, Main Theorem] gives

$$X \text{ is K-polystable } \iff N \not\in \left\{ \begin{array}{l}
2.23, 2.26, 2.28, 2.30, 2.31, 2.33, 2.35, 2.36, \\
3.14, 3.16, 3.18, 3.21, 3.22, 3.23, \\
3.24, 3.26, 3.28, 3.29, 3.30, 3.31, \\
4.5, 4.8, 4.9, 4.10, 4.11, 4.12, \\
5.2
\end{array} \right\}.$$  

The goal of this note is to find all K-polystable smooth Fano threefolds in the family No. 2.22. This family contains both K-polystable and non-K-polystable smooth

Cheltsov has been supported by EPSRC Grant EP/V054597/1. Park has been supported by IBS-R003-D1, Institute for Basic Science in Korea.

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Fano threefolds, and a conjectural characterization of all K-polystable members has been given in [2, Section 7.4]. We will confirm this conjecture—this will complete the description of all K-polystable smooth Fano threefolds of Picard rank 2 and degree 30 started in [2].

Starting from now, we suppose that $X$ is a smooth Fano threefold in the family No. 2.22. Then $X$ can be described both as the blow-up of $\mathbb{P}^3$ along a smooth twisted quartic curve, and the blow-up of $V_5$, the unique smooth threefold No. 1.15, along an irreducible conic. More precisely, there are a smooth twisted quartic curve $C_4 \subset \mathbb{P}^3$, a smooth conic $C \subset V_5$, and a commutative diagram

$$
\begin{array}{ccc}
\mathbb{P}^3 & \xrightarrow{\psi} & V_5 \\
\downarrow{\pi} & & \downarrow\phi \\
X & \rightarrow & \\
\end{array}
$$

where $\pi$ is the blow-up of $\mathbb{P}^3$ along $C_4$, $\phi$ is the blow-up of $V_5$ along $C$, and $\psi$ is given by the linear system of cubic surfaces containing $C_4$. Here, $V_5$ is embedded in $\mathbb{P}^6$ as described in [2, Section 5.10]. All smooth Fano threefolds in the family No. 2.22 can be obtained in this way.

The curve $C_4$ is contained in a unique smooth quadric surface $Q \subset \mathbb{P}^3$, and $\phi$ contracts the proper transform of this surface. Note that

$$
\text{Aut} (X) \cong \text{Aut} (\mathbb{P}^3, C_4) \cong \text{Aut} (Q, C_4).
$$

Choosing appropriate coordinates on $\mathbb{P}^3$, we may assume that $Q$ is given by $x_0x_3 = x_1x_2$, where $[x_0 : x_1 : x_2 : x_3]$ are coordinates on $\mathbb{P}^3$. Fix the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$
([u : v], [x : y]) \mapsto [xu : xv : yu : yv],
$$

where $([u : v], [x : y])$ are coordinates in $\mathbb{P}^1 \times \mathbb{P}^1$. Swapping $[u : v]$ and $[x : y]$ if necessary, we may assume that $C_4$ is a divisor of degree $(1, 3)$ in $Q$, so that $C_4$ is given in $Q$ by

$$
u f_3(x, y) = v g_3(x, y)
$$

for some non-zero cubic homogeneous polynomials $f_3(x, y)$ and $g_3(x, y)$.

Let $\sigma : C_4 \rightarrow \mathbb{P}^1$ be the map given by the projection $([u : v], [x : y]) \mapsto [u : v]$. Then $\sigma$ is a triple cover, which is ramified over at least two points. After an appropriate change of coordinates $[u : v]$, we may assume that $\sigma$ is ramified over $[1 : 0]$ and $[0 : 1]$. Then both $f_3$ and $g_3$ have multiple zeros in $\mathbb{P}^1$. Changing coordinates $[x : y]$, we may assume that these zeros are $[0 : 1]$ and $[1 : 0]$, respectively. Keeping in mind that the curve $C_4$ is smooth, we see that $C_4$ is given by

$$
u(x^3 + ax^2y) = v(y^3 + by^2x)
$$
for some complex numbers $a$ and $b$, after a suitable scaling of the coordinates. If $a = b = 0$, then the curve $C_4$ is given by $ux^3 = vy^3$, which gives $\text{Aut}(X) \cong \text{Aut}(Q, C_4) \cong \mathbb{G}_m \rtimes \mu_2$. In this case, the threefold $X$ is known to be K-polystable [2, Section 4.4].

**Example** Suppose that $ab = 0$, but $a \neq 0$ or $b \neq 0$. We can scale the coordinates further and swap them if necessary, and assume that the curve $C_4$ is given by

$$ux^3 = vy^3.$$ 

In this case, the threefold $X$ is not K-polystable [2, Section 7.4].

A conjecture in [2, Section 7.4] says that the non-K-polystable Fano threefold described in this example is the unique non-K-polystable smooth Fano threefold in the family No. 2.22. Let us show that this is indeed the case. To do this, we may assume that $a \neq 0$ and $b \neq 0$. Then, scaling the coordinates, we may assume that $C_4$ is given by

$$u(x^3 + \lambda x^2 y) = v(y^3 + \lambda y^2 x) \quad (\star)$$

for some non-zero complex number $\lambda$. Since the curve $C_4$ is smooth, we must have $\lambda \neq \pm 1$. Moreover, if $\lambda = \pm 3$, then we can change the coordinates on $Q$ in such a way that $C_4$ would be given by $ux^3 = vy^3 + y^2 x$, so that $X$ is not K-polystable in this case.

We know from [2] that $X$ is K-stable if $C_4$ is given by $(\star)$ with $\lambda$ general. In particular, we know from [2, Section 4.4] that the threefold $X$ is K-stable when $\lambda = \pm \sqrt{3}$. Our main result is the following theorem.

**Theorem** Suppose that $C_4$ is given in $(\star)$ with $\lambda \notin \{0, \pm 1, \pm 3\}$. Then $X$ is K-stable.

Let us prove this theorem. We suppose that $C_4$ is given by $(\star)$ with $\lambda \notin \{0, \pm 1, \pm 3\}$. Then the triple cover $\sigma : C_4 \to \mathbb{P}^1$ is ramified in four distinct points $P_1, P_2, P_3, P_4$, which implies that $\text{Aut}(Q, C_4)$ is a finite group, since

$$\text{Aut}(Q, C_4) \subset \text{Aut}(C_4, P_1 + P_2 + P_3 + P_4).$$

Without loss of generality, we may assume that

$$P_1 = ([1 : 0], [0 : 1]) = [0 : 1 : 0 : 0],$$

$$P_2 = ([0 : 1], [1 : 0]) = [0 : 0 : 1 : 0],$$

where we use both the coordinates on $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^3$ simultaneously.

Observe that the group $\text{Aut}(Q, C_4)$ contains an involution $\tau$ that is given by

$$([u : v], [x : y]) \mapsto ([v : u], [y : x]).$$
Let us identify $\text{Aut}(\mathbb{P}^3, C_4) \equiv \text{Aut}(Q, C_4)$ using the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ fixed above. Then $\tau$ is given by $[x_0 : x_1 : x_2 : x_3] \mapsto [x_3 : x_2 : x_1 : x_0]$. Note that $\tau$ swaps $P_1$ and $P_2$, and the $\tau$-fixed points in $C_4$ are $\{[1 : 1], [1 : 1]\}$ and $\{[1 : -1], [1 : -1]\}$, which are not ramification points of the triple cover $\sigma$. This shows that $\tau$ swaps the points $P_3$ and $P_4$. In fact, the group $\text{Aut}(Q, C_4)$ is larger than its subgroup $\langle \tau \rangle \cong \mu_2$. Indeed, one can change coordinates $([u : v], [x : y])$ on $Q$ so that

\[
P_1 = ([1 : 0], [0 : 1]), \quad P_4 = ([0 : 1], [1 : 0]),
\]

and the curve $C_4$ is given by

\[
u(x^3 + \lambda x^2 y) = v(y^3 + \lambda y^2 x)
\]

for some complex number $\lambda' \notin \{0, \pm 1, \pm 3\}$. This gives an involution $\iota \in \text{Aut}(Q, C_4)$ such that $\iota(P_1) = P_4$ and $\iota(P_2) = P_3$. Let $G$ be the subgroup $\langle \tau, \iota \rangle \subset \text{Aut}(Q, C_4) = \text{Aut}(\mathbb{P}^3, C_4)$. Then $G \cong \mu_2^3$. Note that the group $\text{Aut}(\mathbb{P}^3, C_4)$ can be larger for some $\lambda \in \mathbb{C}\setminus\{0, \pm 1, \pm 3\}$. For instance, if $\lambda = \pm \sqrt{3}$, then $\text{Aut}(\mathbb{P}^3, C_4) \cong \mathfrak{A}_4$, c.f. [2, Example 4.4.6].

The $G$-action on $C_4$ is faithful, so that the curve $C_4$ does not contain $G$-fixed points. Hence, the quadric $Q$ does not contain $G$-fixed points, since otherwise $Q$ would contain an $G$-invariant curve of degree $(1, 0)$, which would intersect $C_4$ by a $G$-fixed point. So, in particular, we see that $\mathbb{P}^3$ contains finitely many $G$-fixed points. Since the $G$-action on $\mathbb{P}^3$ is given by 4-dimensional linear representation of the group $G$, we conclude this representation splits as a sum of four distinct one-dimensional representations, which implies that the space $\mathbb{P}^3$ contains exactly four $G$-fixed points. Denote these points by $O_1, O_2, O_3, O_4$. These four points are not co-planar. For every $1 \leq i < j \leq 4$, let $L_{ij}$ be the line in $\mathbb{P}^3$ that passes through $O_i$ and $O_j$. Then the lines $L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34}$ are $G$-invariant, and they are the only $G$-invariant lines in $\mathbb{P}^3$. For each $1 \leq i < j \leq 4$, let $\Pi_i$ be the plane in $\mathbb{P}^3$ determined by the three points $\{O_1, O_2, O_3, O_4\}\setminus\{O_i\}$. Then the four planes $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ are the only $G$-invariant planes in $\mathbb{P}^3$.

**Remark** Each plane $\Pi_i$ intersects $C_4$ at four distinct points. Indeed, if $|\Pi_i \cap C_4| < 4$, then $\Pi_i \cap C_4$ is a $G$-orbit of length 2, and $\Pi_i$ is tangent to $C_4$ at both the points of this orbit. Therefore, without loss of generality, we may assume that the intersection $\Pi_i \cap C_4$ is just the fixed locus of the involution $\tau$. Then $\Pi_i \cap C_4 = ([1 : 1], [1 : 1]) \cup ([1 : -1], [1 : -1])$, so that $|\Pi_i \cap C_4|$ is a smooth conic that is given by

\[
a(vx - uy) = b(ux - vy)
\]

for some $[a : b] \in \mathbb{P}^1$. But the conic $\Pi_i \cap C_4$ cannot tangent $C_4$ at the points $([1 : 1], [1 : 1])$ and $([1 : -1], [1 : -1])$, so that $|\Pi_i \cap C_4| = 4$.

The curve $C_4$ contains exactly three $G$-orbits of length 2, and these $G$-orbits are just the fixed loci of the involutions $\tau, \iota, \tau \circ \iota$ described earlier. Let $L, L'$ and $L''$ be
the three lines in \( \mathbb{P}^3 \) such that \( L \cap C_4, L' \cap C_4 \) and \( L'' \cap C_4 \) are the fixed loci of the involutions \( \tau, \iota \) and \( \tau \circ \iota \), respectively. Then \( L, L' \) and \( L'' \) are \( G \)-invariant lines, so that they are three lines among \( L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34} \). In fact, it easily follows from Remark that the lines \( L, L', L'' \) meet at one point. Therefore, we may assume that \( L \cap L' \cap L'' = O_4 \) and \( L = L_{14}, L' = L_{24}, L'' = L_{34} \). Then

\[
\begin{align*}
\Pi_1 \cap C_4 &= (L' \cap C_4) \cup (L'' \cap C_4), \\
\Pi_2 \cap C_4 &= (L \cap C_4) \cup (L'' \cap C_4), \\
\Pi_3 \cap C_4 &= (L \cap C_4) \cup (L' \cap C_4).
\end{align*}
\]

On the other hand, the intersection \( \Pi_4 \cap C_4 \) is a \( G \)-orbit of length 4.

Since \( C_4 \) is \( G \)-invariant, the action of the group \( G \) lifts to the threefold \( X \), so that we also identify \( G \) with a subgroup of the group \( \text{Aut}(X) \). Let \( E \) be the \( \pi \)-exceptional surface, let \( \tilde{Q} \) be the proper transform of the quadric \( Q \) on the threefold \( X \), let \( H_1, H_2, H_3 \) and \( H_4 \) be the proper transforms on \( X \) of the \( G \)-invariant planes \( \Pi_1, \Pi_2, \Pi_3 \) and \( \Pi_4 \), respectively, and let \( H \) be the proper transform on \( X \) of a general hyperplane in \( \mathbb{P}^3 \). Then

\[
-K_X \sim 2 \tilde{Q} + E \sim \tilde{Q} + 2H_1 \sim \tilde{Q} + 2H_2 \sim \tilde{Q} + 2H_3 \sim \tilde{Q} + 2H_4 \sim 4H - E,
\]

and the surfaces \( E, \tilde{Q}, H_1, H_2, H_3, H_4 \) are \( G \)-invariant. Observe that \( \tilde{Q} \cong Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \), and \( H_1, H_2, H_3, H_4 \) are smooth del Pezzo surfaces of degree 5.

**Claim** Let \( S \) be a possibly reducible \( G \)-invariant surface in \( X \) such that \( -K_X \sim \tilde{Q} \mu S + \Delta \), where \( \Delta \) is an effective \( \mathbb{Q} \)-divisor, and \( \mu \) is a positive rational number such that \( \mu > 4/3 \). Then \( S \) is one of the surfaces \( \tilde{Q}, H_1, H_2, H_3, H_4 \).

**Proof** This follows from the fact that the cone \( \text{Eff}(X) \) is generated by \( E \) and \( \tilde{Q} \). \( \square \)

Suppose \( X \) is not K-stable. Since \( \text{Aut}(X) \) is finite, the threefold \( X \) is not K-polystable. Then, by [3, Corollary 4.14], there is a \( G \)-invariant prime divisor \( F \) over \( X \) with \( \beta(F) \leq 0 \), see [2, Section 1.2] for the precise definition of \( \beta(F) \). Let us seek for a contradiction.
Let $Z$ be the center of $F$ on $X$. Then $Z$ is not a surface by \[2, \text{Theorem 3.7.1}\], so that $Z$ is either a $G$-invariant irreducible curve or a $G$-fixed point. In the latter case, the point $\pi(Z)$ must be one of the $G$-fixed points $O_1, O_2, O_3, O_4$, so that the point $Z$ is not contained in $\widetilde{Q} \cup E$. Let us use the Abban–Zhuang theory \[1\] to show that $Z$ does not lie on $\widetilde{Q} \cup E$ in the former case.

**Lemma** The center $Z$ cannot be contained in $\widetilde{Q} \cup E$.

**Proof** We suppose that $Z \subset \widetilde{Q} \cup E$. Then $Z$ is an irreducible $G$-invariant curve, because neither $\widetilde{Q}$ nor $E$ contains $G$-fixed points. Let us use notations introduced in \[2, \text{Section 1.7}\]. Namely, we fix $u \in \mathbb{R}_{\geq 0}$. Then

$$-K_X-u\widetilde{Q} \sim_{\mathbb{R}} (4-2u)H+(u-1)E \sim_{\mathbb{R}} (1-u)\widetilde{Q} + 2H,$$

so that $-K_X-u\widetilde{Q}$ is nef for $0 \leq u \leq 1$, and not pseudo-effective for $u > 2$. Thus, we have

$$P(-K_X-u\widetilde{Q}) = \begin{cases} -K_X-u\widetilde{Q} & \text{if } 0 \leq u \leq 1, \\ (4-2u)H & \text{if } 1 \leq u \leq 2, \end{cases}$$

and

$$N(-K_X-u\widetilde{Q}) = \begin{cases} 0 & \text{if } 0 \leq u \leq 1, \\ (u-1)E & \text{if } 1 \leq u \leq 2. \end{cases}$$

If $Z \subset \widetilde{Q}$, then \[2, \text{Corollary 1.7.26}\] gives

$$1 \geq \frac{AX(F)}{SX(F)} \geq \min \left\{ \frac{1}{S_X(\widetilde{Q})}, \frac{1}{S(W_{\widetilde{Q}}, Z)} \right\},$$

where

$$S_X(\widetilde{Q}) = \frac{1}{(-K_X)^3} \int_0^2 \text{vol}(-K_X-u\widetilde{Q}) \, du = \frac{1}{(-K_X)^3} \int_0^2 (P(-K_X-u\widetilde{Q}))^3 \, du$$

and

$$S(W_{\widetilde{Q}}, Z) = \frac{3}{(-K_X)^3} \left\{ \int_0^2 (P(-K_X-u\widetilde{Q})^2 \cdot \widetilde{Q}) \cdot \text{ord}_Z(N(-K_X-u\widetilde{Q})|_{\widetilde{Q}}) \, du \\ + \int_0^\infty \int_0^\infty \text{vol}(P(-K_X-u\widetilde{Q})|_{\widetilde{Q}} - vZ) \, dv \, du \right\}.$$
Therefore, we conclude that $S(W_{\bullet, \bullet}; Z) \geq 1$, because $S_X(\tilde{Q}) < 1$, see [2, Theorem 3.7.1]. Similarly, if $Z \subset E$, then we get $S(W_{\bullet, \bullet}; Z) \geq 1$.

Fix an isomorphism $\tilde{Q} \cong P_1 \times P_1$ such that $E|_{\tilde{Q}}$ is a divisor in $\tilde{Q}$ of degree $(1, 3)$. For $(a, b) \in \mathbb{R}^2$, let $O_{\tilde{Q}}(a, b)$ be the class of a divisor of degree $(a, b)$ in $\text{Pic}(\tilde{Q}) \otimes \mathbb{R}$. Then

$$P(-K_X - u\tilde{Q})|_{\tilde{Q}} \sim_{\mathbb{R}} \begin{cases} O_{\tilde{Q}}(3 - u, u + 1) & \text{if } 0 \leq u \leq 1, \\ O_{\tilde{Q}}(4 - 2u, 4 - 2u) & \text{if } 1 \leq u \leq 2. \end{cases}$$

Therefore, if $Z = E \cap \tilde{Q}$, then

$$S(W_{\bullet, \bullet}; Z) = \frac{1}{10} \left\{ \int_1^2 2(4 - 2u)^2(u - 1) \, du \\
+ \int_0^1 \int_0^\infty \text{vol}(O_{\tilde{Q}}(3 - u - v, u + 1 - 3v)) \, dv \, du \\
+ \int_0^{4/3u} \int_0^{4 - 2u} \text{vol}(O_{\tilde{Q}}(4 - 2u - v, 4 - 2u - 3v)) \, dv \, du \right\}$$

$$= \frac{2}{30} + \frac{1}{10} \left\{ \int_0^{u+1} \int_0^{3v} 2(u + 1 - 3v)(3 - u - v) \, dv \, du \\
+ \int_0^{4 - 2u} \int_0^{4 - 2u - 3v} 2(4 - 2u - 3v)(4 - 2u - v) \, dv \, du \right\}$$

$$= \frac{161}{540}.$$  

To estimate $S(W_{\bullet, \bullet}; Z)$ in the case when $Z \subset \tilde{Q}$ and $Z \neq E \cap \tilde{Q}$, observe that $|Z - \Delta| \neq \emptyset$, where $\Delta$ is the diagonal curve in $\tilde{Q}$. Indeed, this follows from the fact that $\tilde{Q}$ contains neither $G$-invariant curves of degree $(0, 1)$ nor $G$-invariant curves of degree $(1, 0)$, which in turns easily follows from the fact that the curve $C_4 \cong P_1$ does not have $G$-fixed points. Thus, if $Z \subset \tilde{Q}$ and $Z \neq E \cap \tilde{Q}$, then
\[
S(W_{\bullet, \bullet}^E, Z) \leq \frac{1}{10} \left\{ \int_{0}^{2} \int_{0}^{\infty} \text{vol} \left( P(-K_X - u\widetilde{Q})|\widetilde{Q} - v\Delta \right) dv du \right. \\
= \frac{1}{10} \left\{ \int_{0}^{1} \int_{0}^{\infty} \text{vol} \left( O_{\widetilde{Q}}(3 - u - v, u + 1 - v) \right) dv du \right. \\
+ \int_{1}^{2} \int_{0}^{\infty} \text{vol} \left( O_{\widetilde{Q}}(4 - 2u - v, 4 - 2u - v) \right) dv du \right. \\
= \frac{1}{10} \left\{ \int_{0}^{1} \int_{0}^{u+1} 2(u + 1 - v)(3 - u - v) dv du \right. \\
+ \int_{1}^{2} \int_{0}^{4 - 2u} 2(4 - 2u - v)^2 dv du \right. \\
= \frac{17}{30}.
\]

Therefore, \( Z \not\subset \widetilde{Q} \), and hence \( Z \subset E \) and \( Z \neq \widetilde{Q} \cap E \).

One has \( E \cong \mathbb{P}_n \) for some integer \( n \geq 0 \). It follows from the argument as in the proof of \([2, \text{Lemma 4.4.16}]\) that \( n \) is either 0 or 2. Indeed, let \( s \) be the section of the projection \( E \to C_4 \) such that \( s^2 = -n \), and let \( l \) be its fiber. Then \(-E|_E \sim s + kl\) for some integer \( k \). But

\[-n + 2k = E^3 = -c_1(N_{C_4/P^3}) = -14,
\]

so that \( k = (n - 14)/2 \). Then

\[\widetilde{Q}|_E \sim (2H - E)|_E \sim s + (k + 8)l = s + \frac{n + 2}{2} l,\]

which implies that \( \widetilde{Q}|_E \sim s \). Moreover, we know that \( \widetilde{Q}|_E \) is a smooth irreducible curve, since the quadric surface \( Q \) is smooth. Thus, since \( \widetilde{Q}|_E \neq s \), we have

\[0 \leq \widetilde{Q}|_E \cdot s = \left( s + \frac{n + 2}{2} l \right) \cdot s = -n + \frac{n + 2}{2} = \frac{2 - n}{2}\]

so that \( n = 0 \) or \( n = 2 \). Now, let us show that \( S(W_{\bullet, \bullet}^E, Z) < 1 \) in both cases.

For \( u \geq 0 \),

\[-K_X - uE \sim 2\widetilde{Q} + (1 - u)E,\]
so that $-K_X - uE$ is pseudo-effective if and only if $u \leq 1$, and it is nef if and only if $u \leq 1/3$. Furthermore, if $1/3 \leq u \leq 1$, then

$$P(-K_X - uE) = (2 - 2u)(3H - E)$$

and $N(-K_X - uE) = (3u - 1)\tilde{Q}$. Thus, if $n = 0$, we have

$$P(-K_X - uE)|_E = \begin{cases} (1 + u)s + (9 - 7u)l & \text{if } 0 \leq u \leq 1/3, \\ (2 - 2u)s + (10 - 10u)l & \text{if } 1/3 \leq u \leq 1. \end{cases}$$

Similarly, if $n = 2$, then

$$P(-K_X - uE)|_E = \begin{cases} (1 + u)s + (10 - 6u)l & \text{if } 0 \leq u \leq 1/3, \\ (2 - 2u)s + (12 - 12u)l & \text{if } 1/3 \leq u \leq 1. \end{cases}$$

Recall that $Z \neq \tilde{Q} \cap E$. Moreover, we have $Z \sim 1$, since $\pi(Z)$ is not one of the $G$-fixed points $O_1, O_2, O_3, O_4$. Thus, using [2, Corollary 1.7.26], we get

$$S(W^E_E; Z) = \frac{1}{10} \int_0^1 \int_0^\infty \text{vol}(P(u)|_E - vZ) \, dv \, du \leq \frac{1}{10} \int_0^1 \int_0^\infty \text{vol}(P(u)|_E - vs) \, dv \, du,$$

because the divisor $|Z - s| \neq \emptyset$.

Consequently, if $n = 0$, then

$$S(W^E_E; Z) \leq \frac{1}{10} \left\{ \int_0^{1/3} \int_0^\infty \text{vol}((1 + u)s + (9 - 7u)l - vs) \, dv \, du \\ + \int_1^{1/3} \int_0^\infty \text{vol}((2 - 2u)s + (10 - 10u)l - vs) \, dv \, du \right\}$$

$$= \frac{1}{10} \left\{ \int_0^{1/3} \int_0^{1+u} 2(1 + u - v)(9 - 7u) \, dv \, du \\ + \int_{1/3}^{1} \int_0^{2-2u} 2(2 - 2u - v)(10 - 10u) \, dv \, du \right\}$$

$$= \frac{1783}{3240}.$$
Similarly, if \( n = 2 \), then
\[
S(W^E\cdot\cdot; Z) \leq \frac{1}{10} \left\{ \int_0^{\frac{1}{3}} \int_0^1 \text{vol}((1+u)s + (10-6u)l - vs) \, dvdu \\
+ \int_0^{\frac{1}{3}} \int_0^1 \text{vol}((2-2u)s + (12-12u)l - vs) \, dvdu \right\}
\]
\[
= \frac{1}{10} \left\{ \int_0^{1+u} \int_0^{1+u} 2(1+u-v)(9+v-7u) \, dvdu \\
+ \int_{\frac{1}{3}}^{2-2u} \int_0^{2-2u} 2(2-2u-v)(10+v-10u) \, dvdu \right\}
\]
\[
= \frac{157}{270}.
\]
In both cases, we have \( S(W^E\cdot\cdot; Z) < 1 \), which is a contradiction. \( \square \)

Now, we prove our main technical result using the Abban–Zhuang theory, see also [2, Section 1.7].

**Proposition** The center \( Z \) is not contained in \( H_1 \cup H_2 \cup H_3 \cup H_4 \).

**Proof** We first suppose that \( Z \subset H_1 \cup H_2 \cup H_3 \). Without loss of generality, we may assume that \( Z \subset H_1 \). Then \( \pi(Z) \subset \Pi_1 \). Therefore, we see that one of the following two subcases are possible:

- either \( \pi(Z) \) is one of the \( G \)-fixed points \( O_2, O_3, O_4 \), or
- \( Z \) is a \( G \)-invariant irreducible curve in \( H_1 \).

We will deal with these subcases separately. In both subcases, we let \( S = H_1 \) for simplicity. Recall that \( S \) is a smooth del Pezzo surface of degree 5, the surface \( S \) is \( G \)-invariant, and the action of the group \( G \) on the surface \( S \) is faithful. Note also that \( Z \not\subset \tilde{Q} \) by Lemma.

Let us use notations introduced in [2, Section 1.7]. Take \( u \in \mathbb{R}_{\geq 0} \). Then
\[
-K_X - uS \sim_{\mathbb{R}} (4-u)H - E \sim_{\mathbb{R}} \tilde{Q} + (2-u)H \sim_{\mathbb{R}} (u-1)\tilde{Q} + (2-u)(3H - E).
\]

Let \( P(u) = P(-K_X - uS) \) and \( N(u) = N(-K_X - uS) \). Then
\[
P(u) = \begin{cases} 
-K_X - uS & \text{if } 0 \leq u \leq 1, \\
(2-u)(3H - E) & \text{if } 1 \leq u \leq 2,
\end{cases}
\]
and

\[ N(u) = \begin{cases} 
0 & \text{if } 0 \leq u \leq 1, \\
(u - 1) & \text{if } 1 \leq u \leq 2.
\end{cases} \]

Note that \( S_X(S) < 1 \), see [2, Theorem 3.7.1]. In fact, one can compute \( S_X(S) = 17/30 \).

Let \( \varphi : S \to \Pi_1 \) be the birational morphism induced by \( \pi \). Then \( \varphi \) is a \( G \)-equivariant blow-up of the four intersection points \( \Pi_1 \cap C_4 \). Let \( \ell \) be the proper transform on \( S \) of a general line in \( \Pi_1 \), and let \( e_1, e_2, e_3, e_4 \) be \( \varphi \)-exceptional curves, and let \( \ell_{ij} \) be the proper transform on the surface \( S \) of the line in \( \Pi_1 \) that passes through \( \varphi(e_i) \) and \( \varphi(e_j) \), where \( 1 \leq i < j \leq 4 \). Then the cone \( \overline{NE}(S) \) is generated by the curves \( e_1, e_2, e_3, e_4, \ell_{12}, \ell_{13}, \ell_{14}, \ell_{23}, \ell_{24}, \ell_{34} \). Recall also that

\[ \Pi_1 \cap C_4 = (L_{24} \cap C_4) \cup (L_{34} \cap C_4). \]

Therefore, we may assume that \( L_{24} \cap C_4 = \varphi(e_1) \cup \varphi(e_2) \) and \( L_{34} \cap C_4 = \varphi(e_3) \cup \varphi(e_4) \), so that we have \( \varphi(\ell_{12}) = L_{24} \) and \( \varphi(\ell_{34}) = L_{34} \).

Observe that, the group \( \text{Pic}^G(S) \) is generated by the divisor classes \( \ell, e_1 + e_2, e_3 + e_4 \), because both \( L_{24} \cap C_4 \) and \( L_{34} \cap C_4 \) are \( G \)-orbits of length 2. Therefore, if \( Z \) is a curve, then \( \varphi(Z) \) is a curve of degree \( d \geq 1 \), so that

\[ Z \sim d\ell - m_{12}(e_1 + e_2) - m_{34}(e_3 + e_4) \]

for some non-negative integers \( m_{12} \) and \( m_{34} \), which gives

\[
Z \sim (d - 2m_{12})\ell + m_{12}(2\ell - e_1 - e_2 - e_3 - e_4) + (m_{12} - m_{34})(e_3 + e_4) \\
\sim (d - 2m_{12})(\ell_{12} + e_1 + e_2) + m_{12}(\ell_{12} + \ell_{34}) + (m_{12} - m_{34})(e_3 + e_4)
\]

and

\[
Z \sim (d - 2m_{34})\ell + m_{34}(2\ell - e_1 - e_2 - e_3 - e_4) + (m_{34} - m_{12})(e_1 + e_2) \\
\sim (d - 2m_{34})(\ell_{34} + e_3 + e_4) + m_{34}(\ell_{34} + \ell_{12}) + (m_{34} - m_{12})(e_1 + e_2).
\]

Moreover, if \( Z \neq \ell_{12} \) and \( Z \neq \ell_{34} \), then \( d - 2m_{12} = Z \cdot \ell_{12} \geq 0 \) and \( d - 2m_{34} = Z \cdot \ell_{34} \geq 0 \). Hence, if \( Z \) is a curve, then \( |Z - \ell_{12}| \neq \emptyset \) or \( |Z - \ell_{34}| \neq \emptyset \).

On the other hand, if \( Z \) is a curve, then [2, Corollary 1.7.26] gives

\[
1 \geq \frac{A_X(F)}{S_X(F)} \geq \min \left\{ \frac{1}{S_X(S)}, \frac{1}{S(W^S_{\bullet, \bullet}; Z)} \right\} = \min \left\{ \frac{30}{17}, \frac{1}{S(W^S_{\bullet, \bullet}; Z)} \right\},
\]

where

\[
S(W^S_{\bullet, \bullet}; Z) = \frac{3}{(-K_X)^3} \int_0^2 \int_0^\infty \text{vol}(P(u)|_S - vZ) \, dv \, du.
\]
because $Z \not\subset \tilde{Q}$. Moreover, if $S(W_{\bullet}; Z) = 1$, then [2, Corollary 1.7.26] gives

$$1 \geq \frac{A_X(E)}{S_X(E)} = \frac{1}{S_X(S)} = \frac{30}{17},$$

which is absurd. Thus, if $Z$ is a curve, then $S(W_{\bullet}; Z) > 1$, which gives

$$1 < S(W_{\bullet}; Z) = \frac{1}{10} \int_0^2 \int_0^\infty \text{vol}(P(u)|_S - vZ) \, dv \, du$$

$$\leq \max \left\{ \frac{1}{10} \int_0^2 \int_0^\infty \text{vol}(P(u)|_S - v\ell_{12}) \, dv \, du, \frac{1}{10} \int_0^2 \int_0^\infty \text{vol}(P(u)|_S - v\ell_{34}) \, dv \, du \right\},$$

because $|Z - \ell_{12}| \not= \emptyset$ or $|Z - \ell_{34}| \not= \emptyset$. Note also that

$$S(W_{\bullet}; \ell_{12}) = \frac{1}{10} \int_0^2 \int_0^\infty \text{vol}(P(u)|_S - v\ell_{12}) \, dv \, du$$

$$= \frac{1}{10} \int_0^2 \int_0^\infty \text{vol}(P(u)|_S - v\ell_{34}) \, dv \, du.$$

Hence, if $Z$ is a curve, then the second statement in [2, Corollary 1.7.26] gives

$$1 < S(W_{\bullet}; Z) \leq S(W_{\bullet}; \ell_{12}) = \frac{1}{10} \int_0^2 \int_0^\infty \text{vol}(P(u)|_S - v\ell_{12}) \, dv \, du.$$

Let us compute $S(W_{\bullet}; \ell_{12})$. For $0 \leq u \leq 1$ and $v \geq 0$, we have

$$P(u)|_S - v\ell_{12} = (-K_X - uS)|_S - v\ell_{12} \sim_{\mathbb{R}} (4 - u - v)\ell - (1 - v)(e_1 + e_2) - e_3 - e_4.$$

Therefore, if $0 \leq v \leq 1$, then this divisor is nef, and its volume is $u^2 + 2uv - v^2 - 8u - 4v + 12$. Similarly, if $1 \leq v \leq 2 - u$, then its Zariski decomposition is

$$P(u)|_S - v\ell_{12} \sim_{\mathbb{R}} (4 - u - v)(\ell) - e_3 - e_4 + (v - 1)(e_1 + e_2),$$

so that its volume is $u^2 + 2uv + v^2 - 8u - 8v + 14$. Likewise, if $2 - u \leq v \leq 3 - u$, then the Zariski decomposition of the divisor $P(u)|_S - v\ell_{12}$ is

$$P(u)|_S - v\ell_{12} \sim_{\mathbb{R}} (3 - u - v)(2\ell - e_3 - e_4) + (v - 1)(e_1 + e_2) + (v - 2 + u)\ell_{34},$$
so that its volume is \(2(3 - u - v)^2\). If \(v > 3 - u\), then \(P(u)\) is not pseudo-effective, so that the volume of this divisor is zero. Thus, we have

\[
\frac{1}{10} \int_0^1 \int_0^{3-u} \text{vol}(P(u) - v\ell_{12}) \, dv \, du = \frac{1}{10} \int_0^1 \int_0^{3-u} \text{vol}(P(u) - v\ell_{12}) \, dv \, du
\]

\[
= \frac{1}{10} \left\{ \int_0^{2-u} \int_0^{2-u} (u^2 + 2uv - v^2 - 8u - 4v + 12) \, dv \, du + \int_0^1 \int_0^{2-u} (u^2 + 2uv + v^2 - 8u - 8v + 14) \, dv \, du + \int_0^1 \int_0^{2-u} (2(3 - u - v)^2) \, dv \, du \right\}
\]

\[
= \frac{107}{120}.
\]

Similarly, if \(1 \leq u \leq 2\), then

\[P(u) \sim (6 - 3u - v)\ell + (v + u - 2)(e_1 + e_2) + (u - 2)(e_3 + e_4)\]

If \(0 \leq v \leq 2 - u\), this divisor is nef, and its volume is \(5u^2 + 2uv - v^2 - 20u - 4v + 20\). Likewise, if \(2 - u \leq v \leq 4 - 2u\), then its Zariski decomposition is

\[P(u) \sim (4 - 2u - v)(2\ell - e_3 - e_4) + (v - 2 + u)(e_1 + e_2 + \ell_{34}),\]

and its volume is \(2(4 - 2u - v)^2\). If \(v > 4 - 2u\), this divisor is not pseudo-effective,
so that

\[
\frac{1}{10} \int_1^\infty \int_0^\infty \text{vol}(P(u)|_S - v\ell_{12}) \, dv \, du = \frac{1}{10} \int_1^\infty \int_0^\infty \text{vol}(P(u)|_S - v\ell_{12}) \, dv \, du \\
= \frac{1}{10} \left\{ \int_1^\infty \int_0^\infty (5u^2 + 2uv - v^2 - 20u - 4v + 20) \, dv \, du + \int_1^\infty \int_0^\infty 2(4 - 2u - v)^2 \, dv \, du \right\} \\
= \frac{13}{120}.
\]

Therefore, we see that

\[
\frac{1}{10} \int_1^\infty \int_0^\infty \text{vol}(P(u)|_S - v\ell_{12}) \, dv \, du = \frac{1}{10} \int_1^\infty \int_0^\infty \text{vol}(P(u)|_S - v\ell_{12}) \, dv \, du \\
+ \frac{1}{10} \int_1^\infty \int_0^\infty \text{vol}(P(u)|_S - v\ell_{12}) \, dv \, du \\
= \frac{107}{120} + \frac{13}{120} = 1,
\]

which implies, in particular, that \(Z\) is not a curve.

Hence, we see that \(\pi(Z)\) is one of the points \(O_2, O_3, O_4\). Without loss of generality, we may assume that either \(\pi(Z) = O_2\) or \(\pi(Z) = O_4\), so that \(Z \in \ell_{12}\) in both subcases.

Now, using [2, Theorem 1.7.30], we see that

\[
1 \geq \frac{AX(F)}{S_X(F)} \geq \min \left\{ \frac{1}{S(W_{\bullet, \bullet}^{S, \ell_{12}}; Z)}, \frac{1}{S(W_{\bullet, \bullet}^{S}; \ell_{12})}, \frac{1}{S_X(S)} \right\} \\
= \min \left\{ \frac{1}{S(W_{\bullet, \bullet}^{S, \ell_{12}}; Z)}, 1 \right\},
\]
where $S(W_{S,\cdot,\cdot}^{S,\ell_{12}}; Z)$ is defined in [2, Section 1.7]. In fact, [2, Theorem 1.7.30] implies the strict inequality $S(W_{S,\cdot,\cdot}^{S,\ell_{12}}; Z) < 1$, because $S_X(S) < 1$. Let us compute $S(W_{S,\cdot,\cdot}^{S,\ell_{12}}; Z)$.

For $0 \leq u \leq 2$ and $v \geq 0$, let $P(u, v)$ be the positive part of the Zariski decomposition of the divisor $P(u)|_S - v\ell_{12}$, and let $N(u, v)$ be its negative part.

If $0 \leq u \leq 1$, then

$$P(u, v) = \begin{cases} 
(4 - u - v)\ell - (1 - v)(e_1 + e_2) - e_3 - e_4 & \text{if } 0 \leq v \leq 1, \\
(4 - u - v)\ell - e_3 - e_4 & \text{if } 1 \leq v \leq 2 - u, \\
(3 - u - v)(2\ell - e_3 - e_4) & \text{if } 2 - u \leq v \leq 3 - u,
\end{cases}$$

and

$$N(u, v) = \begin{cases} 
0 & \text{if } 0 \leq v \leq 1, \\
(v - 1)(e_1 + e_2) & \text{if } 1 \leq v \leq 2 - u, \\
(v - 1)(e_1 + e_2) + (v - 2 + u)e_{34} & \text{if } 2 - u \leq v \leq 3 - u.
\end{cases}$$

Similarly, if $1 \leq u \leq 2$, then

$$P(u, v) = \begin{cases} 
(6 - 3u - v)\ell + (v + u - 2)(e_1 + e_2) + (u - 2)(e_3 + e_4) & \text{if } 0 \leq v \leq 2 - u, \\
(4 - 2u - v)(2\ell - e_3 - e_4) & \text{if } 2 - u \leq v \leq 4 - 2u,
\end{cases}$$

and

$$N(u, v) = \begin{cases} 
0 & \text{if } 0 \leq v \leq 2 - u, \\
(v - 2 + u)(e_1 + e_2 + e_{34}) & \text{if } 2 - u \leq v \leq 4 - 2u.
\end{cases}$$

Recall from [2, Theorem 1.7.30] that

$$S(W_{S,\cdot,\cdot}^{S,\ell_{12}}; Z) = F_Z(W_{S,\cdot,\cdot}^{S,\ell_{12}}) + \frac{3}{(-K_X)^3} \int_0^\infty \int_0^\infty (P(u, v) \cdot \ell_{12})^2 dvdu$$

for

$$F_Z(W_{S,\cdot,\cdot}^{S,\ell_{12}}) = \frac{6}{(-K_X)^3} \int_0^\infty \int_0^\infty (P(u, v) \cdot \ell_{12}) \operatorname{ord}_Z(N'_S(u)|_{\ell_{12}} + N(u, v)|_{\ell_{12}}) dvdu,$$

where $N'_S(u)$ is the part of the divisor $N(u)|_S$ whose support does not contain $\ell_{12}$, so that $N'_S(u) = N(u)|_S$ in our case, which implies that $\operatorname{ord}_Z(N'_S(u)|_{\ell_{12}}) = 0$ for $0 \leq u \leq 2$, because $Z \not\in \overline{Q}$. Thus, if $\pi(Z) = O_2$, then $Z \not\in \ell_{34} \cup e_1 \cup e_2$, which
gives $F_Z(W_{S, \ell_{12}}) = 0$. On the other hand, if $\pi(Z) = O_4$, then $Z = \ell_{12} \cap \ell_{34}$ and $Z \notin e_1 \cup e_2$, so that

\[
F_Z(W_{S, \ell_{12}}) = \frac{1}{5} \int_0^2 \int_0^\infty (P(u, v) \cdot \ell_{12}) \text{ord}_Z(N(u, v)|\ell_{12}) \, dv \, du
\]

\[
= \frac{1}{5} \left\{ \int_0^1 \int_0^{3-u} (6 - 2u - 2v + 6)(v - 2 + u) \, dv \, du + \int_1^2 \int_{2-u}^{4-2u} (8 - 4u - 2v + 8)(v - 2 + u) \, dv \, du \right\}
\]

\[
= \frac{1}{12}.
\]

Therefore, we see that

\[
S(W_{S, \ell_{12}}; Z) \leq \frac{1}{12} + \frac{1}{10} \int_0^2 \int_0^\infty (P(u, v) \cdot \ell_{12})^2 \, dv \, du
\]

\[
= \frac{1}{12} + \frac{1}{10} \left\{ \int_0^1 \int_0^1 (2 - u + v)^2 \, dv \, du + \int_0^1 \int_0^{2-u} (4 - u - v)^2 \, dv \, du
\]

\[
+ \int_0^{3-u} \int_{2-u}^{4-2u} (6 - 2u - 2v)^2 \, dv \, du
\]

\[
+ \int_1^2 \int_{2-u}^{4-2u} (2 - u + v)^2 \, dv \, du
\]

\[
+ \int_1^2 \int_{2-u}^{4-2u} (8 - 4u - 2v)^2 \, dv \, du \right\}
\]

\[
= 1.
\]

However, as we already mentioned, one has $S(W_{S, \ell_{12}}; Z) < 1$ by \cite[Theorem 1.7.30]{2}. The obtained contradiction concludes that $Z \subset H_4$.

Since $Z \notin H_1 \cup H_2 \cup H_3$, the center $Z$ must be a $G$-invariant curve on $H_4$. Moreover, $\pi(Z)$ cannot be one of the lines determined by the points $O_1, O_2, O_3$ on $\Pi_4$. This implies that $\pi(Z)$ is a curve of degree $d \geq 2$ on $\Pi_4$.

We keep the same notations as in the beginning of the proof, i.e., put $S = H_4$ and let $\varphi: S \to \Pi_1$ be birational morphism induced by $\pi$. As before, $\varphi$ is a $G$-equivariant blow-up of the four intersection points $\Pi_4 \cap C_4$ which consist of a $G$- orbit of length
4. We also denote by $\ell$ the proper transform on $S$ of a general line in $\Pi_4$ and by $e_1$, $e_2$, $e_3$, $e_4$ the four $\varphi$-exceptional curves. In addition, denote by $C$ the proper transform of a general conic passing through the four points $\Pi_4 \cap C_4$.

Since the group $\text{Pic}^G(S)$ is generated by the divisor classes $\ell$, $e_1 + e_2 + e_3 + e_4$, we have

$$Z \sim d\ell - m(e_1 + e_2 + e_3 + e_4),$$

where $m$ is a non-negative integer. By taking intersection with the proper transforms of the lines on $\Pi_4$ passing through $\varphi(e_i)$, $\varphi(e_j)$, we obtain $d \geq 2m$. Since $d \geq 2$, this implies that $|Z - C| \neq \emptyset$. Note that $C \not\subset \tilde{Q}$. By the same argument as before, we obtain

$$1 < S(W^S_{\bullet, \bullet}; Z) = \frac{1}{10} \int_0^2 \int_0^2 \text{vol}(P(u)|_S - vZ) \, dv \, du \\ \leq \frac{1}{10} \int_0^2 \int_0^2 \text{vol}(P(u)|_S - vC) \, dv \, du = S(W^S_{\bullet, \bullet}; C),$$

where $P(u)$ is the positive part of $-K_X - uS$ as before. Let us compute $S(W^S_{\bullet, \bullet}; C)$.

Similarly to the notations used earlier in the proof, we denote by $P(u, v)$ the positive part of the Zariski decomposition of the divisor $P(u)|_S - vC$ for $0 \leq u \leq 2$ and $v \geq 0$, and we denote by $N(u, v)$ its negative part. If $0 \leq u \leq 1$, then

$$P(u, v) = \begin{cases} (4 - u - 2v)\ell - (1 - v)(e_1 + e_2 + e_3 + e_4) & \text{if } 0 \leq v \leq 1, \\ (4 - u - 2v)\ell & \text{if } 1 \leq v \leq \frac{4-u}{2}, \end{cases}$$

and

$$N(u, v) = \begin{cases} 0 & \text{if } 0 \leq v \leq 1, \\ (v - 1)(e_1 + e_2 + e_3 + e_4) & \text{if } 1 \leq v \leq \frac{4-u}{2}. \end{cases}$$

Similarly, if $1 \leq u \leq 2$, then

$$P(u, v) = \begin{cases} (6 - 3u - 2v)\ell + (v + u - 2)(e_1 + e_2 + e_3 + e_4) & \text{if } 0 \leq v \leq 2 - u, \\ (6 - 3u - 2v)\ell & \text{if } 2 - u \leq v \leq \frac{6-3u}{2}, \end{cases}$$

and

$$N(u, v) = \begin{cases} 0 & \text{if } 0 \leq v \leq 2 - u, \\ (v + u - 2)(e_1 + e_2 + e_3 + e_4) & \text{if } 2 - u \leq v \leq \frac{6-3u}{2}. \end{cases}$$
This gives

\[ 1 < S(W_\bullet^S; \mathbb{C}) = \frac{1}{10} \left\{ \int_0^1 \int_0^1 (P(u)|_S - v\mathbb{C})^2 dv du + \int_0^1 \int_0^1 ((4 - u - 2v)\ell)^2 dv du \right. \]

\[ + \int_0^1 \int_0^1 (P(u)|_S - v\mathbb{C})^2 dv du \]

\[ + \int_2^{2-\mu} \int_1^0 \left( (6 - 3u - 2v)\ell \right)^2 dv du \}

\[ = \frac{1}{10} \left\{ \int_0^1 \int_0^1 (4 - u - 2v)^2 - 4(1 - v) dv du \right. \]

\[ + \int_0^{\frac{4-u}{2}} \int_0^1 (4 - u - 2v)^2 dv du \]

\[ + \int_0^{\frac{4-u}{2}} \int_0^1 (6 - 3u - 2v)^2 - 4(2 - u - v) dv du \]

\[ + \int_0^{\frac{6-3u}{2}} \int_0^1 (6 - 3u - 2v)^2 dv du \}

\[ = \frac{23}{40}, \]

which is a contradiction. This completes the proof of the proposition. \qed

**Corollary** Both \( Z \) and \( \pi(Z) \) are irreducible curves, and \( \pi(Z) \) is not entirely contained in \( \Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4 \cup Q \).

Using \cite[Lemma 1.4.4]{2}, we see that \( \alpha_{G,Z}(X) < 3/4 \). Now, using \cite[Lemma 1.4.1]{2}, we see that there are a \( G \)-invariant effective \( \mathbb{Q} \)-divisor \( D \) on the threefold \( X \) and a positive rational number \( \mu < 3/4 \) such that \( D \sim \mathbb{Q} - K_X \) and \( Z \) is contained in the locus \( \text{Nklt}(X, \mu D) \). Moreover, it follows from Claim that \( \text{Nklt}(X, \mu D) \) does not contain \( G \)-irreducible surfaces except maybe for \( \tilde{Q}, H_1, H_2, H_3, H_4 \). Now, applying \cite[Corollary A.1.13]{2} to \((\mathbb{P}^3, \mu \pi(D))\), we see that \( \pi(Z) \) must be a \( G \)-invariant line in \( \mathbb{P}^3 \). But this is impossible by Corollary, since all \( G \)-invariant lines in \( \mathbb{P}^3 \) are contained in \( \Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4 \).

The obtained contradiction completes the proof of our Theorem.
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