Topological B-Model and $\hat{c} = 1$ Fermionic Strings

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Abstract: We construct topological B-model descriptions of $\hat{c} = 1$ strings, and corresponding Dijkgraaf-Vafa type matrix models and quiver gauge theories.

1 Introduction

In this lecture we describe the relations between four types of systems: non-critical $\hat{c} = 1$ fermionic strings, topological strings (B-models), matrix models and $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions [1]. The canonical model that exhibits these relations is the non-critical $c = 1$ bosonic string at the self-dual radius. It has been argued to be equivalent to the topological B-model on the deformed conifold, and the corresponding matrix model and gauge theory are described by the $A_1$ quiver diagram [2,3,4]. Since the $c = 1$ bosonic string is well defined only perturbatively, the above connections are perturbative. It is natural to look for examples, where the non-critical string is well defined non-perturbatively. Natural candidates are the fermionic $\hat{c} = 1$ strings (see [5,6] for a matrix model discussion).

A basic ingredient in establishing the connections between these different systems is a commutative and associative ring structure of the BRST invariant operators with zero dimension and zero ghost number. Its defining relation, for the $c = 1$ bosonic string, is the conifold equation [7]. For the fermionic $\hat{c} = 1$ strings with a non-chiral GSO projection, the ring structure corresponds to certain $\mathbb{Z}_2$ quotients of the conifold [4], while with a chiral GSO projection one gets a Calabi-Yau 3-fold suspended pinched point singularity [8].

The partition functions of the non-critical strings are matched with the partition functions of the topological B-model on the (deformed) Calabi-Yau singularities, with the partition functions of the quiver matrix models and with the glueball F-terms of four-dimensional gauge theories.

2 $\hat{c} = 1$ Fermionic Strings

Fermionic strings are described by $N = 1$ supersymmetric worldsheet field theories coupled to worldsheet supergravity. In the superconformal gauge it is described by a matter
superfield $X$ and the super Liouville field $\Phi$. In components they take the form

$$
X = x + \theta \psi_x + \bar{\theta} \bar{\psi_x} + \theta \bar{\theta} F_x,
\Phi_l = \phi_l + \theta \psi_l + \bar{\theta} \bar{\psi}_l + \theta \bar{\theta} F_l.
$$

(1)

The field $X$ is free, while $\Phi$ is described by the super Liouville Lagrangian.

We consider the theory compactified on a circle of radius $R$. One can perform a chiral GSO projection giving Type II strings, and a nonchiral GSO projection which gives type 0 string theory. In this lecture we will consider the latter case. There are two distinct choices, depending on how the worldsheet fermion number $(-1)^F$ symmetry is realized in the closed string R-R sector, they are called (circle line) 0A and 0B. In both theories there are no $(NS,R)$ or $(R,NS)$ sectors, and therefore no spacetime fermions. The ‘super affine’ theories are obtained from them by modding out by a $Z_2$ symmetry $(-\tau)^F e^{i\pi p}$ where $(-\tau)^F = 1$ $(-1)$ on NS-NS (R-R) momentum states, and $p$ is the momentum $k = p/\alpha'$. Type 0A and Type 0B are T-dual to each other under $R \to \alpha' \alpha'$. The super affine theories are self dual under $R \to 2\alpha' \alpha'$. We will consider the theories at the special radii

$$
R_{0A} = \frac{l_s}{\sqrt{2}}, \quad R_{0B} = l_s \sqrt{2}, \quad R_{\text{super- affine}} = l_s \sqrt{2},
$$

(2)

and argue that at these points the non-critical $\hat{c} = 1$ string theories have a description as a topological B-model on a Calabi-Yau 3-fold. At these radii the winding modes contribute exactly as the momentum modes to the torus partition function, and we get the results of table 1.

In the following we will use the convention $\alpha' = 2$.

### 3 The Ground Ring and Topological B-model

The spin zero ghost number zero BRST invariant operators generate a commutative, associative ring

$$
\mathcal{O}(z)\mathcal{O}'(0) \sim \mathcal{O}''(0) + \{Q, \ldots\} ,
$$

(3)

called the ground ring, where $Q$ is the BRST operator.

The chiral BRST cohomology of dimension zero and ghost number zero is given by the infinite set of states $\Psi_{(r,s)}$ with $r, s$ negative integers [9] [10] [11]

$$
\Psi_{(r,s)} \sim O_{r,s} e^{(ik_{r,s} x_L - p_{r,s} \Phi_L)}.
$$

(4)

The Liouville and matter momentum are given by

$$
k_{r,s} = \frac{1}{2}(r - s), \quad p_{r,s} = \frac{1}{2}(r + s + 2).
$$

(5)
The operators $\Psi_{(r,s)}$ are in the NS-sector if $k_{r,s} = (r - s)/2$ takes integer values, and in the R-sector if it takes half integer values.

The basic elements for the construction of the ring are the R-sector operators

$$x(z) \equiv \Psi_{(-1,-2)}(z), \quad y(z) \equiv \Psi_{(-2,-1)}(z),$$

and the NS-sector operators

$$u(z) \equiv \Psi_{(-1,-3)}(z) = x^2, \quad v(z) \equiv \Psi_{(-3,-1)}(z) = y^2, \quad w(z) \equiv \Psi_{(-2,-2)}(z) = xy.$$  \hfill (7)

In order to construct the ground ring, we combine the left and right sectors with the same left and right Liouville momenta. Denote:

$$a_{ij} = \begin{pmatrix} x\bar{x} & xy \bar{x} \\ y\bar{x} & y\bar{y} \end{pmatrix}, \quad b_{ij} = \begin{pmatrix} u\bar{u} & u\bar{w} & u\bar{v} \\ w\bar{u} & w\bar{w} & w\bar{v} \\ v\bar{u} & v\bar{w} & v\bar{v} \end{pmatrix}. \hfill (8)$$

Note that

$$\det(a_{ij}) = 0, \hfill (9)$$

which is the conifold equation.

**The ground rings**

Imposing the GSO projection we get:

- Circle line theories: The ring is generated by four elements $a_{12}, a_{21}, b_{11}, b_{33}$ with the relation

$$ (a_{12})^2(a_{21})^2 - b_{11}b_{33} = 0. \hfill (10)$$

The complex 3-fold is the $Z_2$ quotient of the conifold, with the $Z_2$ action being

$$Z_2: \quad (a_{11}, a_{22}) \rightarrow -(a_{11}, a_{22}) \quad \text{and} \quad (a_{12}, a_{21}) \rightarrow (a_{12}, a_{21}).$$ \hfill (11)

When the cosmological constant $\mu$ and the background RR charge $q$ are nonzero we get a deformation of the ground ring relation

$$ (a_{12}a_{21} + \mu)^2 = b_{11}b_{33} - \frac{q^2}{4}. \hfill (12)$$

- Super affine 0A: The ground ring is generated by the invariant elements $b_{ij}$ subject to the conditions and the projection $a_{ij} \rightarrow -a_{ij}$. The complex 3-fold described by the ground ring is the $Z_2$ quotient of the conifold.

$$Z_2: \quad a_{ij} \rightarrow -a_{ij}. \hfill (13)$$

The singular geometry described by the ground ring is that of a Calabi-Yau space, where a three cycle of the form $S^3/Z_2$ shrinks to zero size. When $\mu \neq 0$ the deformed space is such that $Vol(S^3/Z_2) \sim \mu/2$, and is locally $T^*(S^3/Z_2)$.

- Super affine 0B: The ground ring is generated by the invariant elements $a_{ij}$ subject to the conditions. The complex 3-fold described by the ground ring is the conifold. When $\mu \neq 0$ the ground ring is described by the deformed conifold.
Topological B-models

Based on the ground rings structures we predict the following dualities:

- Type 0A (0B) at the radius $R = 1$ ($R = 2$) is equivalent to the topological B-model on the deformed $Z_2$ quotient of the conifold (11).

- Super affine 0A at the radius $R = 2$ is equivalent to the topological B-model on the deformed $Z_2$ quotient of the conifold (13).

- Super affine 0B at the radius $R = 2$ is equivalent to the topological B-model on the deformed conifold.

Note that since the $\hat{c} = 1$ non-critical strings are well defined non-perturbatively, the above dualities provide a non-perturbative definition of the topological strings.

Partition functions

We find at the special radii:

- Circle line theories:

  $$ F_{0A}(R = 1) = F_{c=1} \left( \mu + \frac{iq}{2} \right) + F_{c=1} \left( \mu - \frac{iq}{2} \right), $$

  (14)

  where $F_{c=1}$ is the partition function of the $c = 1$ bosonic string at the self-dual radius, or equivalently the partition function of the topological B-model on the deformed conifold. The relation (14) is perturbative. When $q = 0$ we have

  $$ F_{0A}(R = 1) = 2F_{c=1}(R_{\text{self-dual}}), $$

  (15)

  where

  $$ F_{c=1}(R_{\text{self-dual}}) = \frac{1}{2} \mu^2 \log \mu - \frac{1}{240} \mu^{-2} + \sum_{g>2} a_g \mu^{2-2g}. $$

  (16)

  $a_g = \frac{B_{2g}}{2g(2g-2)}$ is the Euler class of the moduli space of Riemann surfaces of genus $g$.

- Super affine 0A:

  We find that non-perturbatively at the special radii (2)

  $$ F_{0A}^{\text{super-aff.}}(R = 1) = F_{0A}(R = 2), $$

  (17)

  when the RR charge $q = 0$.

- Super affine 0B:

  We suggest that while perturbatively

  $$ F_{0B}^{\text{super-aff.}}(R = 2) = F_{c=1}(R_{\text{self-dual}}), $$

  (18)

  it provide a non-perturbative completion of topological B-model on the deformed conifold.
4 Matrix Models and Quiver Gauge theories

Consider D-branes wrapping holomorphic cycles in Calabi-Yau 3-folds. In the language of four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory, the D-branes wrapping in the resolved geometry provide the UV description, while the IR physics is described by the deformed geometry after the transition. For confining gauge theories, one assumes that the relevant IR degrees of freedom are the glueball superfields $S_i$. The partition function of the topological field theory, as a function of the deformation parameters, computes the holomorphic F-terms of the gauge theory as a function of the glueball superfields. This has also a matrix model description [12, 13, 14, 2].

Consider as an example the super affine 0A theory at the self-dual radius. In order to construct the quiver gauge theory, we work in the semi classical regime, where the resolved geometry is described by the $Z_2$ quotient of the resolved conifold, and perform a $Z_2$ quotient of a system of $N_0$ D3-branes and $2N_1$ D5-branes wrapping the resolved conifold. The gauge and matter content of the resulting quiver theory reads

\[
\begin{array}{cccc}
SU(N) & SU(N) & SU(K) & SU(K) \\
A_i & N & K \\
\tilde{A}_i & N & \tilde{K} \\
B_j & \bar{N} & K \\
\tilde{B}_j & \bar{N} & \tilde{K} \\
\Phi^+_i & N & \bar{N} \\
\Phi^+_i & \bar{N} & N \\
\Phi^-_1 & K & K \\
\Phi^-_2 & K & K
\end{array}
\]  

(19)

with $i = 1, 2$, and $N = N_0, K = N_0 + N_1$. The superpotential reads

\[
W_{\text{tree}} = W_0 + W_1
\]

\[
W_0 = m tr(\Phi^+_i \Phi^+_j) - m tr(\Phi^-_i \Phi^-_j), \\
W_1 = -tr(A_i \Phi^+_i \tilde{B}_j) - tr(A_i \Phi^+_2 B_i) - tr(\tilde{B}_j \Phi^-_1 A_i) - tr(B_i \Phi^-_2 \tilde{A}_i).
\]  

(20)

Consider next the IR dynamics of the quiver gauge theory. Of particular interest for us is the case of $N_0$ being an integer multiple of $N_1$. In this case, after the duality cascades, one ends up with a confining gauge theory with the gauge group $SU(N_1)^2$, and the massive bifundamentals $\Phi^+_i, i = 1, 2$. In the deformed geometry $N_1$ sets the size of $S^3$ which we identify with the glueball superfield $S$. The holomorphic F-terms of this theory $F(S)$ as a function of $S$ are twice that of $SU(N_1)$ SYM and are related to the perturbative super affine 0A free energy as

\[
F_{\text{SYM}}(S) = F_{0A}^{\text{super-aff.}}(R_{\text{self-dual}})(\mu),
\]  

(21)

where $S \sim \mu$.

One can also write a DV matrix integral description of the quiver gauge theory F-terms. It takes the form:

\[
Z = \frac{1}{V} \int d\Phi^+_i d\Phi^-_i dA_i dB_j d\tilde{A}_i d\tilde{B}_j \exp \left[ -\frac{1}{g_s} W_{\text{tree}}(\Phi^+_i, \Phi^-_i, A_i, B_j, \tilde{A}_i, \tilde{B}_j) \right],
\]  

(22)

with $W_{\text{tree}}$ given by (20), and $V$ is the volume of the groups $SU(\tilde{N}_0)^2 \times SU(\tilde{N}_0 + \tilde{N}_1)^2$. Similar cascade arguments lead to the matrix integral

\[
Z = \frac{1}{V(SU(\tilde{N}) \times SU(\tilde{N}))} \int d\Phi^+_1 d\Phi^+_2 e^{-\frac{g_s}{2 \pi} tr(\Phi^+_1 \Phi^+_2)},
\]  

(23)
which gives perturbatively
\[
\mathcal{F}_{\text{matrix model}}(S) = 2 \times \mathcal{F}_{c=1}(R_{\text{self-dual}})(\mu) = \mathcal{F}^{\text{super-affine}}_{0A}(R_{\text{self-dual}})(\mu),
\]
with \( S = g_s \hat{N} \).

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References

[1] H. Ita, H. Nieder, Y. Oz and T. Sakai, “Topological B-model, matrix models, c-hat = 1 strings and quiver gauge theories,” JHEP 0405 (2004) 058 [arXiv:hep-th/0403256].

[2] R. Dijkgraaf and C. Vafa, “N = 1 supersymmetry, deconstruction, and bosonic gauge theories,” [arXiv:hep-th/0302011]

[3] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino and C. Vafa, “Topological strings and integrable hierarchies,” [arXiv:hep-th/0312085]

[4] D. Ghoshal and C. Vafa, “C = 1 string as the topological theory of the conifold,” Nucl. Phys. B 453 (1995) 121 [arXiv:hep-th/9506122].

[5] T. Takayanagi and N. Toumbas, “A matrix model dual of type 0B string theory in two dimensions,” JHEP 0307 (2003) 064 [arXiv:hep-th/0307083].

[6] M. R. Douglas, I. R. Klebanov, D. Kutasov, J. Maldacena, E. Martinec and N. Seiberg, “A new hat for the c = 1 matrix model,” [arXiv:hep-th/0307195]

[7] E. Witten, “Ground ring of two-dimensional string theory,” Nucl. Phys. B 373 (1992) 187 [arXiv:hep-th/9108004].

[8] H. Ita, H. Nieder and Y. Oz, “Topological Aspects of Two-Dimensional Type II Strings”, to appear.

[9] P. Bouwknegt, J. G. McCarthy and K. Pilch, “Ground ring for the 2-D NSR string,” Nucl. Phys. B 377 (1992) 541 [arXiv:hep-th/9112036].

[10] P. Bouwknegt, J. G. McCarthy and K. Pilch, “BRST analysis of physical states for 2-D (super)gravity coupled to (super)conformal matter,” [arXiv:hep-th/9110031]

[11] K. Itoh and N. Ohta, “BRST cohomology and physical states in 2-D supergravity coupled to c \( \leq 1 \) matter,” Nucl. Phys. B 377 (1992) 113 [arXiv:hep-th/9110013].

[12] R. Dijkgraaf and C. Vafa, “Matrix models, topological strings, and supersymmetric gauge theories,” Nucl. Phys. B 644 (2002) 3 [arXiv:hep-th/0206255].

[13] R. Dijkgraaf and C. Vafa, “On geometry and matrix models,” Nucl. Phys. B 644 (2002) 21 [arXiv:hep-th/0207106].

[14] R. Dijkgraaf and C. Vafa, “A perturbative window into non-perturbative physics,” [arXiv:hep-th/0208048]