The study of temperature field impact on velocity of fluid in streamlines coordinates in free convection problem

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Abstract

A transition to coordinates formed by streamlines and orthogonal ones are convenient to simplify the Navier-Stokes and Fourier-Kirchoff system. We derive transformation to such coordinates, taking into account a necessity to introduce integrating factor that is not equal to unity in a viscous flow. The transition allows to express approximately velocity module and the velocity vector inclination to vertical direction in terms of temperature gradient in explicit form.

1 Introduction

A problem of stationary convective flow theoretical description is intriguing but complicated. The necessity to include both momentum and energy equations with account viscosity and thermoconductivity leads to extra terms in the basic system that do not allow to introduce velocity potential \( \Pi \). The results of and experimental study of free convective flows from heating objects are widely published (see \( \Pi \) and refs therein) and they are useful to determine convective heat losses and tangent forces by engineers and designers.

The presented paper is devoted to general theoretical study of problem of the description of a stationary two-dimensional flow near the isothermal surface. We consider approximate analytical solution of the equations of a convective flow induced by an isothermal body. The choice of the coordinate system in the frame of the typical for laminar natural convection simplifications and for Pr \( \approx 1 \) allows to diminish the number of basic equations.

As the novel element of the approach we use a transition to coordinates formed by streamlines and spatially built orthogonal lines. It is used further to simplify the plane version of Navier-Stokes and Fourier-Kirchoff system. We
derive transformation to such coordinates by means of differential geometry, taking into account a necessity to introduce integrating factor that is not equal to unity in a viscous flow. The transition allows to express approximately the mentioned integrating factor, velocity module and the velocity vector inclination angle with respect to vertical direction in terms of temperature gradient in explicit form.

The first section contains the basic equations, written similar to [5, 6], the second one defines streamline coordinate system, integrating the equations for orthogonal line via integrating factor introduction. The next section contains a description of nonsingular perturbation theory that allows to split the system and represent the elements of the novel geometry in terms of the temperature field derivatives. The further section formulate algorithm of the transition of from Cartesian variables to the streamline coordinate system with an example of velocity field in analytic form. The final section contains an attempt to link the theory with a conventional boundary layer description with all necessary ingredients of the flow streamlines geometry.

2 The basic equations

Let us consider a two dimensional stationary flow of incompressible fluid in the gravity field. The flow is generated by a convective heat transfer from solid plate to the fluid. The plate is isothermal and lies at the half plane \( y \in [0, \infty) \).

We follow the notations of [5], writing the Navier-Stokes system of equations in the Cartesian coordinates \( x, y \)

\[
W_x \frac{\partial W_y}{\partial x} + W_y \frac{\partial W_y}{\partial y} = -g\beta (T - T_\infty) - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 W_y}{\partial y^2} + \frac{\partial^2 W_y}{\partial x^2} \right), \tag{1}
\]

\[
W_x \frac{\partial W_x}{\partial x} + W_y \frac{\partial W_x}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 W_x}{\partial y^2} + \frac{\partial^2 W_x}{\partial x^2} \right). \tag{2}
\]

In the above equations the pressure terms are divided in two parts. The first of them is the hydrostatic one that is equal to mass force \(-\rho g\), where \( \rho \) is the density of a liquid at the temperature at the non-disturbed area \( T_\infty \). The second one \(-g\beta (T - T_\infty)\) arises from dependence of the extra density on temperature, \( \beta \) is a coefficient of thermal expansion of the fluid. The last terms of the above equations represents the friction forces with the kinematic coefficient of viscosity \( \nu \). I the equations \( W_x \) and \( W_y \) are the components of the fluid velocity \( \mathbf{W} \) that are shown on the Fig.1; \( T, p \) - temperature and pressure disturbances correspondingly.

The mass continuity equation in the conditions of natural convection of incompressible fluid in the steady state [1] has the form:

\[
\frac{\partial W_x}{\partial x} + \frac{\partial W_y}{\partial y} = 0. \tag{3}
\]

The temperature field is described by the stationary Fourier-Kirchhoff equation:
\[ W_x \frac{\partial T}{\partial x} + W_y \frac{\partial T}{\partial y} = a \left( \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2} \right), \]  

(4)

After introducing nondimensional variables: \( x' = x/l, y' = y/l, T' = (T - T_\infty)/\Delta T, p' = p/p_\infty, W'_x = W_x/W_o, W'_y = W_y/W_o \) we obtain:

\[ W'_x \frac{\partial W'_x}{\partial x'} + W'_y \frac{\partial W'_x}{\partial y'} = -\frac{g\beta T' \Delta T l}{W_o^2} - \frac{p_\infty}{\rho W_o^2} \frac{\partial p'}{\partial y'} + \nu' \left( \frac{\partial^2 W'_x}{\partial y'^2} + \frac{\partial^2 W'_x}{\partial x'^2} \right), \]  

(5)

\[ W'_x \frac{\partial W'_y}{\partial x'} + W'_y \frac{\partial W'_y}{\partial y'} = -\frac{p_\infty}{\rho W_o^2} \frac{\partial p'}{\partial x'} + \nu' \left( \frac{\partial^2 W'_x}{\partial y'^2} + \frac{\partial^2 W'_y}{\partial x'^2} \right), \]  

(6)

\[ \frac{\partial W'_x}{\partial x'} + \frac{\partial W'_y}{\partial y'} = 0. \]  

(7)

\[ W'_x \frac{\partial T'}{\partial x'} + W'_y \frac{\partial T'}{\partial y'} = a' \left( \frac{\partial^2 T'}{\partial y'^2} + \frac{\partial^2 T'}{\partial x'^2} \right), \]  

(8)

where \( \frac{\nu'}{W_o} = \nu', \frac{a'}{W_o} = a' \).

Next we would formulate the problem of free convection over the heated inclined isothermal plate \( x = 0, y \in [0, \infty) \), dropping the primes, see Fig 1.

The form of the continuity equation (3) allows to introduce the stream function \( \psi \), so as:

\[ W_x = -\frac{\partial \psi}{\partial y}, \quad W_y = \frac{\partial \psi}{\partial x}. \]  

(9)

### 3 The streamline coordinate system

A stream line of the flow is determined by the equation:

\[ \psi(x, y) = n. \]  

(10)

It means that velocity \( \mathbf{W} \) is tangent to the the streamline curve, inclined to the \( x- \) axis by the angle \( \theta \).

We introduce tangent \( \tau \) and normal \( n \) unit vectors to the curve (Fig.1), it means that the normal component of the velocity \( W_n = 0 \) and the tangent one \( W_\tau = W \). We accept in the traditional point of view that models real processes on the base of the time independent form of the streamlines only. Eventual time dependence we would consider as perturbations with zero mean values.

On the base of the streamlines definition (10) we have:

\[ y = f(x, n) \]  

(11)

and the family of curves to be orthogonal to the streamlines:

\[ y = h(x, \tau) \]  

(12)
we define new curvilinear coordinate system with the variables $(\tau, n)$. The variables are connected with the Cartesian as:

$$\tau = \varphi(x, y), \quad n = \psi(x, y).$$  \hspace{1cm} (13)

The equation for the function $h$ (12) may be derived from the equation for a straight line, orthogonal to the streamline (11) in the point $X, Y$:

$$\frac{\partial \psi}{\partial y} (X - x) - \frac{\partial \psi}{\partial x} (Y - y) = 0.$$  \hspace{1cm} (14)

Therefore the equation for the function $h(x, \tau)$ (12) has the form (see also (9)):

$$\frac{dh}{dx} = \left[ \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial x} \right]_{y=h} = - \cot \theta.$$  \hspace{1cm} (15)

This differential equation is equivalent to one in the total form

$$\frac{\partial \psi}{\partial x} dy - \frac{\partial \psi}{\partial y} dx = 0$$  \hspace{1cm} (16)

The Pfaff form in the l.h.s. of the last equation is exact iff $\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = 0$, that means $rot_z \vec{V} = 0$ such condition strongly restricts the choice of velocity field (the velocity potential existence which coincides with $\varphi$) [1].

It is known that in the two-dimensional case there exist integrating factor $\mu(x, y)$ to be considered as a new variable of the theory. We identify the constant
of integration of the differential equation (15) with the variable \( \tau \).

\[
\tau = \varphi(x, y) = -\int_{x_0}^{x} \mu(x', y_0) \psi_y(x', y_0) \, dx' + \int_{y_0}^{y} \mu(x, y') \psi_x(x, y') \, dy' \quad (17)
\]

The equation that connects the integrating factor \( \mu \) and \( \psi \) is the direct corollary of the integrability condition:

\[
\frac{\partial \mu}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \mu}{\partial y} \frac{\partial \psi}{\partial y} + \mu \Delta \psi = 0 \quad (18)
\]

The partial derivatives the functions \( \psi \) and \( \varphi \) determine the matrix \( \widehat{h} \)

\[
\widehat{h} = \left( \begin{array}{cc} \frac{\partial \psi}{\partial y} & \frac{\partial \psi}{\partial \varphi} \\ \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial \varphi} \end{array} \right) \quad (19)
\]

The orthogonality condition (16) yields:

\[
\frac{\partial \psi}{\partial y} \frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} \frac{\partial \varphi}{\partial x} = -\psi_y W_x + \varphi_x W_y = 0 \quad (20)
\]

and the definition of the stream function (9) gives:

\[
\widehat{h} = \left( \begin{array}{cc} W_y & -W_x \\ \varphi_x & \varphi_x \end{array} \right), \quad \widehat{h}^{-1} = \left( \begin{array}{cc} \frac{W_y}{W_x} & W^2_{x\varphi x} \\ \frac{-W_x}{W_y} & W_{x\varphi y} \end{array} \right) 
\]

where:

\[
\varphi_x = -\mu(x, y_0) \psi_y (x, y_0) + \int_{y_0}^{y} (\mu_x(x, y') \psi_x(x, y') + \mu(x, y') \psi_{xx}(x, y')) \, dy' \quad (21)
\]

In the case of \( \mu = 1 \) (\( \Delta \psi = 0 \)), \( \frac{\partial \varphi}{\partial x} = \varphi_x = -\psi_y (x, y_0) + \int_{y_0}^{y} \psi_{yy}(x, y') \, dy' = -\psi_y (x, y') + \int_{y_0}^{y} \psi_{yy}(x, y') \, dy' = -\psi_y (x, y) = W_x \), that is equivalent to the equation introducing velocity potential. This case of \( \text{rot}_z \mathbf{W} = 0 \) means that the terms of viscosity vanish at both Navier-Stokes equations (1), (2).

In general case of \( \mu \neq 1 \) we propose to consider the integrating factor \( \mu \) as an auxiliary variable.

The components of the metric tensor \( G_{ik} = G_{ki} \) of the curvilinear coordinates are as follows:

\[
G_{nn} = \left( \frac{\partial x}{\partial n} \right)^2 + \left( \frac{\partial y}{\partial n} \right)^2 \quad (22)
\]

\[
G_{n\tau} = \frac{\partial x}{\partial n} \frac{\partial x}{\partial \tau} + \frac{\partial y}{\partial n} \frac{\partial y}{\partial \tau} \quad (23)
\]

\[
G_{\tau\tau} = \left( \frac{\partial x}{\partial \tau} \right)^2 + \left( \frac{\partial y}{\partial \tau} \right)^2 \quad (24)
\]
with the determinant:
\[ G = G_{nn}G_{\tau\tau} - G_{n\tau}^2. \quad (25) \]

It is easy to verify that
\[ \hat{h}^{-1} = \left( \frac{\partial x}{\partial n} \frac{\partial x}{\partial \tau} \frac{\partial x}{\partial \nu} \right), \quad (26) \]

where \( x = x(n, \tau) \) and \( y = y(n, \tau) \) define the inverse transformation of (13). Hence the derivatives in (26) and, therefore in (19), as well as in (25) are defined by the velocity components (see (9)). Finally the nonzero components of the metric tensor (22) and (24):

\[ G_{nn} = \left( \frac{W_x}{W_x^2 + W_y^2} \right)^2 + \left( \frac{W_y}{W_x^2 + W_y^2} \right)^2 = \frac{1}{\bar{W}}, \]
\[ G_{\tau\tau} = \left( \frac{W^2_{\varphi_x}}{W_x^2 + W_y^2} \right)^2 + \left( \frac{W_{\varphi_y}}{W_x^2 + W_y^2} \right)^2 = \left( \frac{W_{\varphi_x}}{\bar{W}} \right)^2 \frac{1}{\bar{W}}, \]

with the determinant
\[ G = \left( \frac{W_{\varphi_x}}{\bar{W}} \right)^2 \frac{1}{\bar{W}}, \quad (27) \]
defines the differential operators of the governing equations in vector form.

The correspondent relations are:
\[ W_x = W \cos \theta, \quad W_y = W \sin \theta, \]
\[ \bar{\varphi} = \bar{\varphi_x} + \pi \bar{\varphi_n} = -\pi \bar{\varphi_x} \sin \theta - \pi \bar{\varphi_x} \cos \theta, \]
\[ \nabla p = \frac{\pi}{\sqrt{G_{\tau\tau}}} \frac{\partial p}{\partial \tau} + \frac{\pi}{\sqrt{G_{nn}}} \frac{\partial p}{\partial n} = \frac{\pi}{\sqrt{G_{\tau\tau}}} \frac{W_x}{W_x^2 + W_y^2} \frac{\partial p}{\partial \tau} + \frac{\pi}{\sqrt{G_{nn}}} \frac{W_y}{W_x^2 + W_y^2} \frac{\partial p}{\partial n}, \]
\[ \Delta T = \frac{1}{\sqrt{G_{\tau\tau}}} \left[ \frac{\partial}{\partial \tau} \left( \frac{\sqrt{G_{\tau\tau}}}{\sqrt{G_{nn}}} \right) \frac{1}{\sqrt{G_{nn}}} \frac{\partial}{\partial n} \left( \frac{\sqrt{G_{\tau\tau}}}{\sqrt{G_{nn}}} \right) \right] + \frac{1}{\sqrt{G_{nn}}} \frac{\partial}{\partial n} \left( \frac{\sqrt{G_{\tau\tau}}}{\sqrt{G_{nn}}} \right) = \frac{\pi}{\sqrt{G_{\tau\tau}}} \frac{W_{\varphi_x}}{W_x^2 + W_y^2} \frac{\partial}{\partial \tau} \left( \frac{W_{\varphi_x}}{\cos \theta} \right) + \frac{\pi}{\sqrt{G_{nn}}} \frac{W_{\varphi_x}}{\cos \theta} \frac{\partial}{\partial n} \left( \frac{W_{\varphi_x}}{\cos \theta} \right), \]
\[ \nabla \bar{W} = \frac{1}{\sqrt{G_{\tau\tau}}} \left[ \frac{\partial}{\partial \tau} \left( \frac{\sqrt{G_{\tau\tau}}}{\sqrt{G_{nn}}} \right) \frac{1}{\sqrt{G_{nn}}} \frac{\partial}{\partial n} \left( \frac{\sqrt{G_{\tau\tau}}}{\sqrt{G_{nn}}} \right) \right] = \frac{1}{\sqrt{G_{\tau\tau}}} \frac{\partial}{\partial \tau} \left( \frac{\sqrt{G_{\tau\tau}}}{\sqrt{G_{nn}}} \right) \frac{\partial}{\partial n} \left( \frac{\sqrt{G_{\tau\tau}}}{\sqrt{G_{nn}}} \right) = \tau \nabla \bar{W} = \tau W_x \frac{\partial W_x}{\partial \tau} + \tau W_y \frac{\partial W_y}{\partial \tau} + \tau W_{\varphi_x} \frac{\partial W_{\varphi_x}}{\partial \tau} + \tau W_{\varphi_y} \frac{\partial W_{\varphi_y}}{\partial \tau}, \]
\[ W_x \frac{\partial x}{\partial x} + W_y \frac{\partial x}{\partial y} = W_x \left( \frac{\partial x}{\partial \tau} + W_{\varphi_x} \frac{\partial \varphi_x}{\partial \tau} + W_{\varphi_y} \frac{\partial \varphi_y}{\partial \tau} \right), \]
\[ \text{where: } A = \text{rot}_x \bar{W} = -\frac{1}{\sqrt{G_{\tau\tau}}} \left[ \frac{\partial}{\partial \tau} \left( \frac{\sqrt{G_{\tau\tau}}}{\sqrt{G_{nn}}} \right) \frac{\partial}{\partial n} \left( \frac{\sqrt{G_{\tau\tau}}}{\sqrt{G_{nn}}} \right) \right] = \frac{1}{\sqrt{G_{\tau\tau}}} \frac{\partial}{\partial \tau} \left( \frac{\sqrt{G_{\tau\tau}}}{\sqrt{G_{nn}}} \right) \frac{\partial}{\partial n} \left( \frac{\sqrt{G_{\tau\tau}}}{\sqrt{G_{nn}}} \right), \]
\[ \frac{W_{\varphi_x}}{\cos \theta} \frac{\partial (\bar{W})}{\partial n} = \frac{W_{\varphi_x}}{\cos \theta} \frac{\partial (\bar{W})}{\partial \tau} + \frac{W_{\varphi_x}}{\cos \theta} \frac{\partial (\bar{W})}{\partial n}, \]
\[ \text{hence: } \Delta \bar{W} = \tau W_x \frac{\partial}{\partial \tau} \left( \frac{W_{\varphi_x}}{\cos \theta} \right) - \pi \frac{\varphi_x}{\cos \theta} \frac{\partial}{\partial n} \left( \frac{W_{\varphi_x}}{\cos \theta} \right), \]
\[ \left( \nabla \bar{W} \right) = \frac{\partial}{\partial \tau} \left( \frac{W_{\varphi_x}}{\cos \theta} \right) - W_x \left( \frac{\varphi_x}{\cos \theta} \right) - W_y \left( \frac{\varphi_y}{\cos \theta} \right) - \pi \frac{\varphi_x}{\cos \theta} \frac{\partial}{\partial n} \left( \frac{W_{\varphi_x}}{\cos \theta} \right), \]
\[ \left( \nabla \bar{W} \right) = \frac{\partial}{\partial \tau} \left( \frac{W_{\varphi_x}}{\cos \theta} \right) - W_x \left( \frac{\varphi_x}{\cos \theta} \right) - W_y \left( \frac{\varphi_y}{\cos \theta} \right) - \pi \frac{\varphi_x}{\cos \theta} \frac{\partial}{\partial n} \left( \frac{W_{\varphi_x}}{\cos \theta} \right), \]
\[ \text{In the new coordinate system the equations (20), (19) and (18) go to the form: } \]
\[ 1 \frac{\partial W^2}{\partial \tau} = - \frac{\rho \Delta T}{W_x^2} \frac{\varphi_x}{\cos \theta} - \frac{p}{\rho W_x^2} \frac{\partial p}{\partial \tau} + \frac{\nu}{\partial n} \left( \frac{W_{\varphi_x}}{\cos \theta} \right), \quad (28) \]
We have introduced a new variable $\Pi = W^2/2 + \frac{p}{\rho}$ instead of $p$ and cross-

differentiate (28), (29) that yields

$$\frac{\partial W}{\partial x}^2 \frac{\partial}{\partial W} \left( \frac{W \cos \theta}{\cos \varphi_x} \right) = \frac{g \bar{T}}{W} \left( T - \frac{p}{\rho W^2} \frac{T}{\cos \theta} - \nu \frac{\partial}{\partial n} \frac{\partial}{\partial W} \left( \frac{W \cos \theta}{\cos \varphi_x} \right) \right),$$

(29)

$$\frac{\partial T}{\partial \tau} = a \left[ \frac{\partial}{\partial \tau} \left( \frac{\varphi_x}{W \cos \theta} \frac{\partial T}{\partial \tau} \right) + \frac{\partial}{\partial n} \left( \frac{W \cos \theta}{\cos \varphi_x} \frac{\partial T}{\partial n} \right) \right]$$

(30)

We have introduced a new variable $\Pi = W^2/2 + \frac{p}{\rho}$ instead of $p$ and cross-

differentiate (28), (29) that yields

$$\frac{\partial}{\partial \tau} \left( \frac{g \bar{T}}{W} \frac{\partial}{\partial x} \left( \frac{W \cos \theta}{\cos \varphi_x} \right) \right) = \frac{\partial}{\partial n} \left( \frac{W \cos \theta}{\cos \varphi_x} \frac{\partial}{\partial n} \left( \frac{W \cos \theta}{\cos \varphi_x} \right) \right),$$

(31)

So, the problem is formulated on a base of four equations (30), (31), (3),

(18), for three thermodynamical variables $W$, $\theta$, $T$ and one connected with

generalized potential $\varphi$.

Let us underline that the first two equations are already written in new

variables $n$, $\tau$ but the last two in Cartesian ones. Therefore we should transform

them to the same coordinates and the variables: $W$, $\theta$.

The continuity equation (3) in new variables yields

$$\frac{\partial W}{\partial x} \cos \theta + \frac{\partial W}{\partial y} \sin \theta = \frac{\partial W}{\partial x} \cos \theta - \frac{\partial W}{\partial y} \sin \theta + \frac{\partial W}{\partial y} \sin \theta + \frac{\partial W}{\partial y} \cos \theta = \left( \frac{\tau}{\varpi}, \nabla W \right) + W \left( \frac{n}{\varpi}, \nabla \theta \right) = 0,$$

where:

$$\tau = (\cos \theta, \sin \theta), \quad \nabla = (- \sin \theta, \cos \theta).$$

(32)

and in curvilinear coordinates the gradients are:

$$\nabla W = \frac{\tau}{\sqrt{G_{rr}}} \frac{\partial W}{\partial \tau} + \frac{\pi}{\sqrt{G_{nn}}} \frac{\partial W}{\partial n} = \frac{\tau}{\cos \theta} \frac{\partial W}{\partial \tau} + \frac{\pi}{\cos \theta} \frac{\partial W}{\partial n} = \frac{\tau}{\cos \theta} \frac{\partial W}{\partial \tau} + \frac{\pi}{\cos \theta} \frac{\partial W}{\partial n},$$

(33)

and

$$\nabla \theta = \frac{\tau}{\cos \theta} \frac{\partial \theta}{\partial \tau} + \frac{\pi}{\cos \theta} \frac{\partial \theta}{\partial n}.$$}

(34)

That finally gives

$$\frac{\tau}{\cos \theta} \frac{\partial W}{\partial \tau} + W^2 \frac{\partial \theta}{\partial n} = 0.$$}

(35)

The last equation of integrability (18) reads

$$\frac{\partial}{\partial x} \frac{\varphi_x}{\cos \theta} + \frac{\partial}{\partial y} \frac{\varphi_y}{\cos \theta} + \mu \Delta \psi = \frac{\partial}{\partial x} W \sin \theta - \frac{\partial}{\partial y} W \cos \theta + \mu \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) = \frac{\partial}{\partial x} W \sin \theta - \frac{\partial}{\partial y} W \cos \theta +$$

$$\frac{\partial}{\partial y} W \cos \theta +$$

$$\frac{\partial}{\partial x} \frac{\varphi_x}{\cos \theta} + \frac{\partial}{\partial y} \frac{\varphi_y}{\cos \theta} + \mu \Delta \psi = \frac{\partial}{\partial x} W \sin \theta - \frac{\partial}{\partial y} W \cos \theta + \mu \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) = \frac{\partial}{\partial x} W \sin \theta - \frac{\partial}{\partial y} W \cos \theta +$$
\[ +\mu \left( -\frac{\partial W \cos \theta}{\partial x} + \frac{\partial W \sin \theta}{\partial y} \right) = \frac{\partial(\mu W \sin \theta)}{\partial y} - \frac{\partial(\mu W \cos \theta)}{\partial y} = \text{rot}_z \left( \nabla \mu \right) = 0. \]

In new variables the \( z \)-component of the operator curl takes the form

\[ \text{rot}_z \left( \nabla \mu \right) = -\frac{\mu W \cos \theta}{\sqrt{G}} \left( \frac{\partial (\mu W \sqrt{G} \frac{\partial}{\partial n})}{\partial \tau} - \frac{\partial (\mu W \sqrt{G} \frac{\partial}{\partial n})}{\partial n} \right) = \frac{\partial \left( \frac{\mu W \cos \theta}{\varphi_s} \right)}{\partial n} = 0. \]  

(36)

The last relation for assumption \( \frac{\mu W \cos \theta}{\varphi_s} = 1 \) gives the link between \( \mu \) and \( \varphi_s \):

\[ \mu = \frac{\varphi_s}{W \cos \theta}. \]  

(37)

The constant of integration of (36) is chosen as unit on the base of the freedom in integration factor.

Returning to the equation (36) in vector form and using (32) one can rewrite it as

\[ -\frac{1}{\mu} \frac{\partial}{\partial x} \sin \theta + \frac{1}{\mu} \frac{\partial}{\partial y} \cos \theta - \frac{1}{W} \left( -\frac{\partial W \cos \theta}{\partial y} + \frac{\partial W \sin \theta}{\partial x} \right) = 0 \]

\( -\left( \nabla \ln \mu \right) - \left( \nabla \ln W \right) + (\nabla \vartheta) = 0, \]  

(38)

on the base of (34), having in mind \( \nabla \ln W = \frac{\nabla W}{W} \). Finally, in the curvilinear coordinates

\[ -\frac{\partial (\ln \mu W)}{\partial n} + \frac{\mu}{\partial \tau} = 0. \]  

(39)

Let us plug the relation (37) into (35) arriving at

\[ \mu \frac{\partial (\ln W)}{\partial \tau} + \frac{\partial \theta}{\partial n} = 0, \]  

(40)

and, next

\[ \mu = -\frac{\partial \theta}{\partial \tau}. \]  

(41)

To solve the problem for the equations (31), (30) and (39) in which (41) is implied.

Transforming the N-S equations one has

\[ \frac{\partial}{\partial \tau} \left( \mu W^2 \frac{\partial}{\partial n} - \frac{g \beta T \cos \theta}{W W^2} \frac{\partial}{\partial n} \left( W^2 \partial \frac{\partial}{\partial n} \right) \right) = \frac{\partial}{\partial \tau} \left( -W^2 \left( \frac{\partial}{\partial \mu} - \frac{g \beta T \cos \theta}{W W^2} \frac{\partial}{\partial \mu} \right) \left( W^2 \partial \frac{\partial}{\partial n} \right) \right), \]

one arrives at

\[ \frac{\partial}{\partial \tau} \left( -W^2 \frac{\partial}{\partial n} - \frac{g \beta T \cos \theta}{W W^2} \frac{\partial}{\partial n} \left( W^2 \partial \frac{\partial}{\partial n} \right) \right) = \frac{\partial}{\partial \tau} \left( -\frac{g \beta T \sin \theta}{W W^2} \frac{\partial}{\partial n} \left( W^2 \partial \frac{\partial}{\partial n} \right) \right), \]

(42)
Figure 2: The sketch of $\theta$ profiles along streamlines and along orthogonal ones.

Introducing the Rayleigh number $Ra = \frac{2 \beta \Delta T l^3}{\nu a}$ we obtained

$$\frac{\partial}{\partial \tau} \left( -\frac{W^2}{\nu} \frac{\partial \mu}{\partial n} - aRa \frac{T \cos \theta}{W} + \mu \frac{\partial}{\partial \tau} \left( W^2 \frac{\partial \ln \mu}{\partial n} \right) \right) = \frac{\partial}{\partial n} \left( -aRa \frac{T \sin \theta}{\mu W} - \frac{1}{\mu \partial n} \left( W^2 \frac{\partial \ln \mu}{\partial n} \right) \right)$$

The equation (30) is transformed as:

$$\frac{\partial T}{\partial \tau} = a \left[ \frac{\partial}{\partial \tau} \left( \frac{\mu}{\nu} \frac{\partial T}{\partial \tau} \right) + \frac{\partial}{\partial n} \left( \frac{1}{\mu} \frac{\partial T}{\partial n} \right) \right].$$

The equations (42), (30) and

$$-\frac{\partial \left( \ln \mu W \right)}{\partial n} + \mu \frac{\partial \theta}{\partial \tau} = 0.$$ 

(45)

together with the expression for $\mu$ (41) form the system of three equations for three variables $W$, $\theta$ and $T$ that is equivalent to the basic one. This system, with boundary conditions accounted, we consider as the formulation of the problem to be solved in new independent coordinates $n$, $\tau$.

We should formulate boundary conditions. It is helpful to draw the coordinate system in terms of new variables and sketch the unknown function behavior as, for example at Fig.2.

4 Nonsingular perturbation theory

We apply the nonsingular perturbation theory to the system of equations (42), (30) and (39) in which a small parameter indicates slow changes of dynamics.
variables as a function of the correspondent independent variable \([8]\). Let the variables depend on small parameter \(\varepsilon \) \([7]\).

Transport of mass of fluid particles take place along the stream lines \(\tau \) \((n = \text{const})\). It means that the main contribution to the heat transport is realized by such particles. Then the gradient component of temperature \(\partial T / \partial n\) is large compared to \(\partial T / \partial \tau\). Moreover the heat exchange between fluid particles at neighbor stream lines is defined by thermal conductivity that is characterized by second derivative by \(\tau\). This assumption may be expressed by small parameter \(\varepsilon\) introduction in temperature field as:

\[
T = T(n, \varepsilon^2 \tau).
\]  

(46)

The transport of the fluid particles momentum is similar but it is determined by buoyancy and viscosity forces that act in different directions. The module of velocity is changed essentially along the perpendicular direction to stream lines, while its angle of inclination changes opposite:

\[
W = W(n, \varepsilon \tau), \quad \theta = \theta(\varepsilon n, \tau).
\]  

(47)

The heat transfer equation \([30]\) in the first order of the parameter \(\varepsilon^2\) gives

\[
\frac{\partial T(n, \tau)}{\partial \tau} = a \left[ \frac{\partial}{\partial n} \left( \frac{1}{\mu} \frac{\partial T(n, \tau)}{\partial n} \right) \right].
\]  

(48)

After approximations the parameter is chosen conventionally as \(\varepsilon = 1\). by \(n\) it is solved with respect to \(\mu\)

\[
\mu = \frac{a \frac{\partial T(n, \tau)}{\partial n}}{\int \frac{\partial T(n, \tau)}{\partial \tau} \, dn}.
\]  

(49)
or using shorthands for derivatives:

\[ \frac{\partial T(n, \tau \varepsilon^2)}{\partial \tau} = \varepsilon^2 D_2 T(n, \tau \varepsilon^2) = \varepsilon^2 D_2 T(n, \tau \varepsilon^2) , \]

\[ \frac{\partial^2 T(n, \tau \varepsilon^2)}{\partial \tau^2} = \varepsilon^4 D_{2,2} T(n, \tau \varepsilon^2) , \]

\[ \frac{\partial^3 W(n, \tau \varepsilon^2)}{\partial n \partial \tau^2} = \varepsilon D_{1,2} W(n, \tau \varepsilon^2) . \]

The continuity equation (45) with (49) yields:

\[ \frac{\partial \theta(n, \tau)}{\partial n} - \frac{\partial \theta(n, \tau)}{\partial \tau} W(n, \tau) + \frac{\partial^2 W(n, \tau)}{\partial n \partial \tau} + 2 \frac{\partial \theta(n, \tau)}{\partial n} \frac{\partial W(n, \tau)}{\partial \tau} = 0 . \]

(50)

In the first approximation we write

\[ -W(n, \tau) \frac{\partial^2 W(n, \tau)}{\partial n \partial \tau} + 2 \frac{\partial W(n, \tau)}{\partial n} \frac{\partial W(n, \tau)}{\partial \tau} + \frac{\partial \theta(n, \tau)}{\partial n} \frac{\partial \theta(n, \tau)}{\partial \tau} (W(n, \tau))^2 = 0 . \]

(51)

From the equalities (49) and (45) we have

\[ - \partial \left( \ln \left( \frac{\partial T(n, \tau)}{\partial n} \right) W \right) + \frac{a \partial T(n, \tau)}{\partial n} \frac{\partial T(n, \tau)}{\partial \tau} W = 0 , \]

\[ \partial \ln \left( \frac{\partial T(n, \tau)}{\partial n} \right) = \frac{\partial W(n, \tau)}{\partial n} \frac{\partial T(n, \tau)}{\partial \tau} \left( \frac{\partial T(n, \tau)}{\partial n} \right) W(n, \tau) + \frac{\partial \theta(n, \tau)}{\partial n} \frac{\partial \theta(n, \tau)}{\partial \tau} W(n, \tau) = 0 . \]

(52)

Equalizing the expressions (41) and (49) for \( \mu \) yields

\[ - \frac{\partial \theta(n, \tau)}{\partial n} = \frac{a \partial T(n, \tau)}{\partial n} \frac{\partial \theta(n, \tau)}{\partial \tau} , \]

(53)

or using shorthands for derivatives:

\[ \ln \left( \frac{\partial T(n, \tau)}{\partial n} \right) W = - \frac{\theta(n, \tau)}{\partial n} \int T \ dn . \]

(54)

Integrating by \( \tau \) we have

\[ W = e ^ { \int \frac{\partial \theta(n, \tau)}{\partial n} \left( \frac{\partial T(n, \tau)}{\partial n} \right) W(n, \tau) \ d\tau} . \]

(55)

Let us analyse the contribution of the third term of (52) on the base of (55):

\[ \frac{\partial W(n, \tau)}{\partial \tau} \left( \frac{\partial T(n, \tau)}{\partial n} \right) W(n, \tau) = \frac{\partial (W(n, \tau))}{\partial \tau} \left( - \int \frac{\partial W(n, \tau)}{\partial n} \left( \frac{\partial T(n, \tau)}{\partial n} \right) W(n, \tau) \ d\tau \right) \]

\[ = - \frac{1}{a} \int \left( \frac{\partial T(n, \tau)}{\partial \tau} \frac{\partial \theta(n, \tau)}{\partial n} \right) W(n, \tau) d\tau + \frac{1}{a} \int \frac{\partial T(n, \tau)}{\partial \tau} d\tau \left( \frac{\partial T(n, \tau)}{\partial n} \right) W(n, \tau) - \frac{1}{a} \frac{\partial \theta(n, \tau)}{\partial n} \left( \frac{\partial T(n, \tau)}{\partial \tau} \right) \left( \frac{\partial T(n, \tau)}{\partial n} \right) W(n, \tau) . \]

(56)
From basic estimations expressed by small parameters entrance at (46) and (47) it follows that all terms are of the order $\varepsilon^2$. Hence the equation (52) is reduced to:

$$
\frac{1}{\partial T(n, \tau)} \frac{\partial^2 T(n, \tau)}{\partial n \partial n} + \frac{\partial T(n, \tau)}{\partial n} \left( a \frac{\partial \theta(n, \tau)}{\partial \tau} - 1 \right) = 0. \quad (56)
$$

The result allows to express the field of velocity angles $\theta(n, \tau)$ as a function of temperature field $T(n, \tau)$.

$$
\theta(n, \tau) = \int \left( -\int \frac{\partial T(n, \tau)}{\partial \tau} dn \frac{\partial^2 T(n, \tau)}{\partial n^2} \frac{1}{a} \right) d\tau. \quad (57)
$$

Under such assumptions the Navier-Stokes equations (42) in first approximation with respect to the small parameter $\varepsilon$ may derived in similar way. We however in this work concentrate our efforts on the problem of velocity field determination on the base of temperature fields by means of equations (55) and (56), see the typical $W, \theta, T$ profiles across and along streamlines in new coordinates at Fig.4.

5 On the algorithm of transition from Cartesian to coordinates $n, \tau$ and vice versa

Let us recall that $W_x = -\frac{\partial \psi}{\partial y}$ and $W_y = \frac{\partial \psi}{\partial x}$, then
\[ dn = \frac{\partial \psi}{\partial y} dx + \frac{\partial \psi}{\partial x} dy = W_y dx - W_x dy, \]

where \( W_x = W \cos \theta \) and \( W_y = W \sin \theta \), while by the definition of velocity potential
\[ d\tau = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy. \]

From (20) it follows
\[ -\varphi_y W_x + \varphi_x W_y = 0. \]

Next, (37) and (49) give \( \mu = W_x \frac{\varphi_x}{W} = \frac{\partial T(n,\tau)}{\partial n} \int \frac{\partial T(n,\tau)}{\partial \tau} dn \), that allow to express
\[ \frac{\partial \varphi}{\partial x} = \varphi_x = W \cos \theta \frac{a}{\mu} \int \frac{\partial T(n,\tau)}{\partial n} dn, \]

and
\[ \frac{\partial \varphi}{\partial y} = \varphi_y = \frac{\varphi_x W_y}{W_x} = \varphi_x \tan \theta, \]

where \( W \) is evaluated from (55) and \( \theta \) is found from (57).

\[ dn = Adx + Bdy, \quad d\tau = Cdx + Gdy. \]

The coefficients are
\[ A = W \sin(\theta), \quad B = -W \cos(\theta), \quad C = W \cos(\theta) \frac{a}{\mu} \int \frac{\partial T(n,\tau)}{\partial n} dn = \mu W \cos(\theta), \quad G = W \sin(\theta) \frac{a}{\mu} \int \frac{\partial T(n,\tau)}{\partial n} dn = \mu W \sin(\theta). \]

For differentials in Cartesian coordinates we arrive at
\[ dy = -A\frac{dx - Cdn}{BC - AG} = \frac{1}{\mu W} \left(\sin(\theta) d\tau - \mu \cos(\theta) dn\right), \quad dx = B\frac{d\tau - Gdn}{BC - AG} = \frac{1}{\mu W} \left(\cos(\theta) d\tau + \mu \sin(\theta) dn\right). \]

In conditions \( d\tau = 0 \):
\[ dy = -A\frac{dx - Cdn}{BC - AG} = \frac{1}{W} \left(-\cos(\theta) dn\right) = -\frac{W dy}{\cos(\theta)}, \quad dx = \frac{1}{W} \left(\sin(\theta) dn\right), \quad dn = \frac{W dy}{\sin(\theta)}. \]

The gradients of temperature and velocity module are rescaled as
\[ \frac{dT}{dn} = \frac{\sin(\theta)}{W} \frac{dT}{dx}, \quad \frac{dT}{dn} = -\frac{\cos(\theta)}{W} \frac{dT}{dy}, \quad \frac{dW}{dn} = \frac{\sin(\theta)}{W} \frac{dW}{dx}, \quad \frac{dW}{dn} = -\frac{\cos(\theta)}{W} \frac{dW}{dy}. \]

If the function \( W \) exponentially decays at a vicinity of the layer boundary, asymptotically we have:
\[ W = W_0 e^{-\alpha x}, \quad \frac{dW}{dx} = -\alpha W_0 e^{-\alpha x}, \]

the gradient by \( n \)-variable do not decay:
\[ \frac{\sin(\theta) dW}{W} = -\frac{\sin(\theta)}{W_0 e^{-\alpha x}} \alpha W_0 e^{-\alpha x} = -\alpha \sin(\theta). \]
Similar behavior demonstrates the temperature
Next, if $dn = 0$,
$$dy = \frac{1}{\mu W} \sin (\theta) d\tau,$$
$$dx = \frac{1}{\mu W} \cos (\theta) d\tau.$$
We have similar relations, with account of $\mu$ behavior
$$\frac{dT}{d\tau} = \frac{dT}{dx} \frac{\mu W}{\cos(\theta)} = \frac{dT}{dx} \frac{\mu W}{\sin(\theta)}.$$
This is the nonlinear system with respect to that gives the link between the temperature gradients components in both coordinate systems. A direct application is possible in its discrete version (see Fig.5).

6 The boundary layer approximation

In the theory of boundary layer the models for temperature and velocity fields are expressed as \[1, 3\]: \((T - T_\infty) = \Delta T(1-x/\delta(y))^2\), where $\delta(y)$ is the boundary layer thickness.

By the similarity of descriptions in both coordinate system we assume:
\((T - T_\infty) = (T_w - T_\infty)(1-n/\delta(\tau))^2\).

Such model is based on the concept of boundary layer. It means that boundary layer thickness $\delta(\tau)$ is defined by the equation of layer boundary $n = \delta(\tau)$ with condition $W \sim 0$ at the boundary and the angle $\theta(\delta(\tau), \tau)$ determines the form of the boundary curve (see Fig.4).

Looking for the necessary elements of geometrical description we should write the coefficients from (61) we start from (57):
$$\theta (n, \tau) = \int \left( -\frac{\partial T(n, \tau)}{\partial n} \frac{dn}{\partial n} + \frac{1}{a} \right) d\tau.$$
Evaluating $\int \frac{\partial T(n, \tau)}{\partial \tau} \, dn$, one have

$$(T_w - T_\infty) \frac{\partial (1 - n/\delta(\tau))^2}{\partial \tau} = -2 \frac{n}{\delta(\tau)^2} (T_w - T_\infty) \frac{\partial \delta(\tau)}{\partial \tau} (n - \delta(\tau)),$$

and

$$\int \frac{\partial T(n, \tau)}{\partial \tau} \, dn = \int \left(-2 \frac{n}{\delta(\tau)^2} (T_w - T_\infty) \frac{\partial \delta(\tau)}{\partial \tau} (n - \delta(\tau))\right) \, dn = \frac{1}{3} \frac{n^2}{\delta(\tau)^2} (T_w - T_\infty) (3\delta(\tau) - 2n) \frac{\partial \delta(\tau)}{\partial \tau},$$

$$\left(\frac{\partial T(n, \tau)}{\partial n}\right) = \frac{\partial ((T_w - T_\infty)(1 - n/\delta(\tau))^2)}{\partial n} = -\frac{1}{\delta(\tau)^2} (T_w - T_\infty) (2\delta(\tau) - 2n),$$

$$\frac{\partial^2}{\partial n \partial \tau} ((T_w - T_\infty)(1 - n/\delta(\tau))^2) = \frac{2}{a} (T_w - T_\infty),$$

$$\frac{\partial^2 T(n, \tau)}{\partial n \partial \tau} + \frac{1}{a} = \frac{2}{a} (T_w - T_\infty) \frac{\partial \delta(\tau)}{\partial \tau},$$

having

$$\theta(n, \tau) = \frac{1}{a} - \frac{n^2}{\delta(\tau)(\delta(\tau) - n)} \frac{1}{a} (3\delta(\tau) - 2n) \frac{\partial \delta(\tau)}{\partial \tau}.$$

Differentiating yields:

$$\frac{\partial \theta}{\partial n} = \frac{1}{a} \frac{n}{\delta(\tau)(\delta(\tau) - n)} \frac{\partial \delta(\tau)}{\partial \tau} = \frac{1}{3a} \frac{n^2}{\delta(\tau)(\delta(\tau) - n)} \frac{\partial \delta(\tau)}{\partial \tau} \left(n^2 - 3n\delta(\tau) + 3(\delta(\tau))^2\right).$$

Next we go to the

$$\ln W = - \int \frac{\partial \theta}{\partial n} \sqrt{\frac{\partial T(n, \tau)}{\partial n}} \, d\tau = \int \left[\frac{1}{3a} \frac{n}{\delta(\tau)(\delta(\tau) - n)} \frac{\partial \delta(\tau)}{\partial \tau} \left(n^2 - 3n\delta(\tau) + 3(\delta(\tau))^2\right) \right] \left(\frac{1}{a} \frac{n^2}{\delta(\tau)(\delta(\tau) - n)} \frac{\partial \delta(\tau)}{\partial \tau}\right) \, d\tau + \frac{1}{a} \frac{n}{\delta(\tau)(\delta(\tau) - n)} \frac{\partial \delta(\tau)}{\partial \tau} \left(n^2 - 3n\delta(\tau) + 3(\delta(\tau))^2\right) \left(\frac{1}{a} \frac{n^2}{\delta(\tau)(\delta(\tau) - n)} \frac{\partial \delta(\tau)}{\partial \tau}\right) \, d\tau - \frac{1}{a} \frac{n}{\delta(\tau)(\delta(\tau) - n)} \frac{\partial \delta(\tau)}{\partial \tau} \left(n^2 - 3n\delta(\tau) + 3(\delta(\tau))^2\right) \left(\frac{1}{a} \frac{n^2}{\delta(\tau)(\delta(\tau) - n)} \frac{\partial \delta(\tau)}{\partial \tau}\right) \, d\tau.$$

The geometry of the flow model is defined by

$$dy = \frac{\mu W}{\mu} \sin(\theta) \, d\tau - \mu \cos(\theta) \, d\tau = \frac{\sin(\theta)}{W} \, d\tau - \frac{\cos(\theta)}{W} \, d\tau,$$

$$dx = \frac{1}{\mu W} \cos(\theta) \, d\tau + \mu \sin(\theta) \, d\tau = \frac{\cos(\theta)}{W} \, d\tau + \frac{\sin(\theta)}{W} \, d\tau,$$

where

$$\mu = \frac{\int \frac{\partial T(n, \tau)}{\partial \tau} \, dn}{\int \frac{\partial T(n, \tau)}{\partial n} \, dn} = -\frac{a}{\delta(\tau) (T_w - T_\infty) (3\delta(\tau) - 2n) \frac{\partial \delta(\tau)}{\partial \tau}} = -3 \frac{a}{\delta(\tau) (2\delta(\tau) - 2n) \frac{\partial \delta(\tau)}{\partial \tau}},$$

$$dy = \frac{\sin(\theta)}{W} \, d\tau - \frac{\cos(\theta)}{W} \, d\tau,$$

$$dx = \frac{\cos(\theta)}{W} \, d\tau + \frac{\sin(\theta)}{W} \, d\tau.$$

7 Conclusions

We conclude that the mathematical modelling of the convective heat transfer may be realized in terms of natural for a flow coordinate system via streamline definition. We consider this partial problem that link temperature and velocity.
fields as a verification of the proposed approach including the mathematical aspects of the model.

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