We consider the role of $\mathcal{N} = 4$ conformal supergravity in the duality relation between $\mathcal{N} = 4$ SYM theory and $D = 5$ gauged supergravity expanded near the Anti de Sitter background. We discuss the structure of the SYM effective action in the conformal supergravity background, in particular, terms related to conformal anomaly. Solving the leading-order Dirichlet problem for the metric perturbation in AdS background we explicitly compute the bilinear graviton term in the $D = 5$ Einstein action, demonstrating its equivalence to the linearized Weyl tensor squared part of the gravitational effective action induced by SYM theory. We also compute the graviton-dilaton-dilaton 3-point function which is found to have the form consistent with conformal invariance of the boundary theory.
1. Introduction

The aim of this paper is to try to check and clarify further the recent proposal [1,2] about the relation between the generating functional for correlators of marginal operators of large $N$ four-dimensional $\mathcal{N} = 4$ super Yang-Mills (SYM) theory and the classical action of five-dimensional $\mathcal{N} = 8$ gauged supergravity (GSG) expanded near AdS$_5$ vacuum and evaluated on the solutions with the Dirichlet boundary conditions. This is a specific realisation of the duality conjecture of [3] (based on earlier work of [4,5,6,7,8]).

One of the important points of our discussion will be the role of $D = 4$, $\mathcal{N} = 4$ conformal supergravity (CSG) [9,10,11] in the relation between the $D = 4$, $\mathcal{N} = 4$ SYM and $D = 5$, $\mathcal{N} = 8$ GSG theories. The relevance of the conformal supergravity in this context was already noted in [12].

Coupling SYM theory to CSG multiplet and integrating over the SYM fields in a way preserving general covariance one finds the effective action $W$ which depends on the CSG fields as well as on the fields of the conformal anomaly multiplet. If one is interested only in relating the derivatives of $W$ to the correlation functions of marginal operators of SYM theory viewed as a conformal theory in flat space, the terms involving anomalous degrees of freedom may be separated out and ignored. We believe, however, that there is a broader picture which goes beyond the correspondence between correlators of the boundary conformal theory and GSG action in the AdS background in which the partition function $Z = e^{-W}$ of the SYM theory in a supergravity background should be given a more fundamental interpretation than just a formal sum of conformal field theory correlators multiplied by auxiliary sources. In particular (by analogy with familiar 2d case) $Z$ should be computed in a way preserving general (super)covariance and thus including non-linear couplings to the supergravity fields (corresponding to contact terms in the correlators).

Indeed, in type IIB string theory the $\mathcal{N} = 4$ SYM theory appears (as the leading term in the Born-Infeld action) in the description of D3-branes and thus, in general, is coupled to the fields of $D = 10$ type IIB supergravity multiplet. The fields of the $D = 4$ CSG which naturally couple to $D = 4$ SYM theory may be interpreted as a particular truncation of the $D = 10$ type IIB multiplet.

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1 Though the conformal supergravity does not appear directly from string theory and so far is only a formal ‘bridge’ between SYM and GSG theories, let us still recall that the $\mathcal{N} = 4$ CSG (coupled to four $\mathcal{N} = 4$ SYM multiplets) is remarkable in being a unique locally superconformal 4-dimensional theory which is ultraviolet-finite [12,13] and thus (conformal and axial) anomaly free [14,15].
We shall mostly concentrate on the term in the SYM effective action $W$ which is quadratic in the conformal supergravity fields. The imaginary part of this term is related to the classical absorption cross-section of certain type IIB supergravity modes by D3-brane. The conformal supergravity gives a universal supersymmetric description of different marginal (or ‘minimal’, i.e. dilaton, longitudinal graviton, etc.) cases discussed in [4,5]. Non-marginal cases [17,18] (see also [19]) are not directly described by coupling of CSG to SYM. It is natural to expect that the conformal supergravity multiplet (supplemented by the anomalous degrees of freedom) can be coupled to the full D3-brane Born-Infeld action since the latter can be coupled to the type IIB multiplet.

We shall show that the leading gravitational term in $W$ (quadratic in the linearised Weyl tensor) does indeed emerge from the $D = 5$ Einstein action when it is evaluated on the solution of the Dirichlet problem in the $\text{AdS}_5$ space. Namely, we shall explicitly compute the term bilinear in the graviton perturbation in the Einstein part of the GSG action, clarifying some subtleties (absent in simpler cases of scalar [1,2] and vector [2] perturbations) involved in establishing its relation with the correlator of the two energy-momentum tensors of SYM theory. In particular, we shall find that in order to guarantee the conformal invariance of the supergravity expression one should start with the $D = 5$ action containing an additional boundary (counter)term proportional to the volume of the boundary. The leading-order solution of the Dirichlet problem for the graviton we shall find allows one to compute also the 3-point functions involving gravitons. In particular, we shall determine the graviton-dilaton-dilaton function complementing the recently obtained 3-scalar [23,24,25] and 3-vector and scalar-scalar-vector [25] results.

In section 2 we shall review some aspects of the SYM effective action in the conformal supergravity background. In section 3 we shall discuss the correspondence between the SYM effective action in the CSG background and the on-shell value of the $D = 5$ gauged supergravity action suggested in [1,2] and its possible tests. In section 4 we shall compute the term bilinear in the gravitational perturbation in the $D = 5$ Einstein action and demonstrate its equivalence with the corresponding term in the $D = 4$ effective action. In section 5 we shall apply the results of section 4 to compute the dilaton-dilaton-graviton term in the $D = 5$ action which is found to be consistent with the conformal invariance of the boundary theory.

\[\text{2}\] The BI action for a D3-brane probe in a background of large $N$ D3-brane source (or in $\text{AdS}_5 \times S^5$ space) has (spontaneously broken) conformal invariance [20] so it is likely that it may be coupled to the CSG fields (viewed, e.g., as boundary values of the $D = 5$ GSG fields). Alternatively, this $D = 4$ BI action ($F^2 + \frac{1}{X^4}F^4 + \ldots$) can be interpreted [21,22] as the leading large $N$ part of the quantum SYM effective action in the background with a non-zero value of the scalar field $X$. If one includes in addition the conformal supergravity background the resulting quantum large $N$ SYM effective action may represent the coupling of CSG to BI action.

\[\text{3}\] One component of this correlator was already discussed in [1]. Our aim will be to see how the full expression comes out of the GSG action in a direct way.
2. Effective action of $\mathcal{N} = 4, D = 4$ SYM theory in $\mathcal{N} = 4, D = 4$ conformal supergravity background

The conformally invariant $\mathcal{N} = 4$ super Yang-Mills theory can be coupled in a natural way to $\mathcal{N} = 4$ conformal supergravity (or Weyl) multiplet. The action describing $\mathcal{N} = 4$ SYM in a background of $\mathcal{N} = 4$ CSG was found at the linearised level in [10] and at the full non-linear level in [26]. The leading terms in the Lagrangian for a single $\mathcal{N} = 4$ vector multiplet ($A_m, \psi_i, X_{ij}$) coupled in a $SU(1,1)$ covariant way to the fields of CSG (bosons $e^a_m, V^i_{m|j}, \varphi, T_{mn}^{ij}, E_{ij}, D_{kl}$ and fermions $\psi^i_m, \Lambda_i, \chi_{ij}$) can be written in the following schematic form

$$L_{SYM} = -\frac{1}{4}(e^{-\phi} F^{mn} F_{mn} + C F^{mn} F^{*}_{mn}) - \frac{1}{2} \bar{\psi}^i \gamma^m D_m \psi_i - \frac{1}{4} X_{ij} (-D^2 + \frac{1}{6} R) X^{ij}$$

$$- X_{ij} F^{+mn} T_{mn}^{ij} + D_{kl}^{ij}(X^{ij} X_{kl} - \frac{1}{6} \delta_i^i \delta_j^j |X|^2) + ... + h.c. . \quad (2.1)$$

Here the complex scalar $\varphi$ was set equal to $C + ie^{-\phi}$ (this scalar is present also in the $D = 4, \mathcal{N} = 4$ Poincare and $D = 10, \mathcal{N} = 2$ type IIB supergravity, and, from the string theory point of view, is a combination of the dilaton and the RR scalar).

The resulting action $I_{SYM}(A, G) = \int d^4x \sqrt{g} L_{SYM}$, where $A$ stands for the fields of the SYM multiplet and $G$ for the off-shell fields of the CSG multiplet, may be viewed as the $\mathcal{N} = 4$ locally superconformal invariant generalisation of the standard coupling of the YM theory to the metric $\sqrt{g} g^{mn} g^{kl} F_{mn} F_{kl}$.

Let us consider the SYM partition function in external CSG background

$$Z(G) = e^{-W(G)} = \int dA \ e^{-I_{SYM}(A, G)} . \quad (2.2)$$

---

4 In sections 2 and 3 $m, n = 0, 1, 2, 3$ are the space-time and $i, j = 1, 2, 3, 4$ are the $SU(4)$ indices.

5 $SU(1,1)$ or $SL(2, R)$ is a symmetry of the SYM-CSG coupling if the transformation of the CSG scalar $\varphi$ is accompanied by the duality rotation of the SYM vectors [10, 26]. This symmetry should be a manifest off-shell symmetry of the CSG action under which only the scalar $\varphi$ is transforming. Let us recall also that the CSG theory in 10 dimensions [27] contains the same fields as the (dual version of) $\mathcal{N} = 1, D = 10$ supergravity ($e^A_M, A_{M_1...M_6}, \Phi, \psi_M, \chi$) but the real scalar $\Phi$ and the Majorana spinor $\chi$ are subject to differential constraints. The $D = 4$ CSG scalar $\varphi$ originates upon dimensional reduction from the real $D = 10$ scalar $\Phi$ and a component of $A_{M_1...M_6}$.

6 We assume that $G$ include also the conformal ‘gauge’ or ‘anomalous’ degrees of freedom (the conformal factor of the metric and its superpartners) which decouple at the classical level.
The classical action $I_{SYM}(A, G)$ is invariant under the local superconformal transformations but this is no longer true for the effective action $W(G)$ (computed in a generally covariant and supersymmetric way): it will also depend on the ‘anomalous’ parts of the CSG fields $G$. While the $\mathcal{N} = 4$ SYM theory is finite in flat space, it may still have divergences when coupled to external fields.\footnote{Equivalently, the correlators of composite operators may contain divergences. They may be defined using special (normal-ordering, etc.) prescriptions to be consistent with conformal invariance of the flat-space SYM theory, but this may break the general covariance of the SYM-CSG coupling. The analogy with the 2d case is only a partial one since in $D = 4$ the conformal group is finite-dimensional and thus is very different from general covariance.}

The divergent part of $W$ which is superconformally invariant is nothing but the action of the $\mathcal{N} = 4$ conformal supergravity\footnote{Here $\Lambda$ is an UV cutoff. Because of supersymmetry the only non-vanishing divergence is logarithmic one. It is natural to include the total derivative (Euler density) term $R^* R^*$ in the CSG action since then the full divergence and anomaly of the SYM theory is determined simply by the CSG action. The coefficient of the $D^2 R$ term in the conformal anomaly (see below) is ambiguous. If one defines the Weyl tensor in 4 dimensions (as we are assuming here) this coefficient automatically cancels out for the $\mathcal{N} = 4$ SYM multiplet. It is non-zero if one defines the Weyl tensor in $4 - \epsilon$ dimensions as in \cite{28}. In that case one is to add the term $\frac{2}{3} D^2 R$ to the $C^2 - R^* R^* = 2(R^2_{mn} - \frac{1}{3} R^2)$ combination in the CSG action.}

$$W = W_\infty + W_{\text{fin}},$$

$$W_\infty = - \beta \ln \Lambda \ G_{\text{CSG}}, \quad \beta = \frac{\nu}{4(4\pi)^2}, \quad \nu = N^2,$$

$$I_{\text{CSG}} = \int d^4 x \sqrt{g} \ L_{\text{CSG}},$$

$$L_{\text{CSG}} = C_{mnkl} C^{mnkl} - R^* R^* + 2 F_{jm}^i (V) F^j_{imn} (V)$$

$$+ 4 [D^2 \varphi^* D^2 \varphi - 2(R_{mn} - \frac{1}{3} g_{mn} R) D_m \varphi^* D_n \varphi]$$

$$+ 16(D^m T_{ nip} + D_n T_{ mij} - \frac{1}{2} R_{mn} T_{ nip} + T_{ nip}) - E_{ij} (-D^2 + \frac{1}{6} R) E_{ij}$$

$$+ \ldots + D^i_{kl} D^j_{kl} + \text{fermionic terms}.$$
... \( \nu \) is the number of vector multiplets, i.e. \( N^2 \) in the \( U(N) \) or large \( N \) \( SU(N) \) SYM case. The UV divergence \((2.4)\) is directly related to the conformal anomaly (assuming, of course, that the UV cutoff, e.g., \( g_{mn} \Delta x^m \Delta x^n > \Lambda^{-2} \), preserves general covariance),

\[
<T_m> = \frac{2g_{mn}}{\sqrt{g}} \frac{\delta W}{\delta g_{mn}} = -\beta L_{CSG} . \tag{2.6}
\]

The one-loop coefficient here can be found by summing the contributions \((2.5)\) of the fields of the SYM multiplet. Since the conformal anomaly is in the same supersymmetry multiplet with the axial \( SU(4) \) anomaly

\[
D_m \frac{\delta W}{\delta V_m} = \beta F_{mn} (V) F^*_m (V) , \tag{2.7}
\]

which should receive only the one-loop contribution, it is natural to expect that in the present \( \mathcal{N} = 4 \) SYM case the above one-loop expression for \( W_\infty \) \((2.5)\) (and thus for the anomalous part of \( W \) discussed below) is actually exact to all loop orders \((2.5)\).

The dependence of the finite part \( W_{\text{fin}} \) of the effective action

\[
W_{\text{fin}} = W_{\text{anom}} + W_{\text{inv}}
\]

on the anomalous degrees of freedom can be determined by integrating the anomaly relations as in the 2d case \((2.5)\). The dependence of \( W_{\text{anom}} \) on the conformal factor of the metric \( \sigma \) was found in \((14)\). This part of the action (which is non-local when expressed in terms of the original unconstrained metric) takes the following explicit form

\[
W_{\text{anom}} (g) = -2\beta \int d^4x \sqrt{\tilde{g}} \left[ (\tilde{R}_{mn} - \frac{1}{3} \tilde{R}^2 + F^2_{mn} + 2 \tilde{D}^2 \varphi^* \tilde{D}^2 \varphi + \ldots) \sigma \right] \tag{2.8}
\]

\text{9} The action of \( \mathcal{N} = 4 \) CSG was originally found only to the quadratic order in the fields \((10)\). Its dependence on the metric can be determined exactly from the condition of local Weyl invariance \((12,11)\). The \( SU(1,1) \) invariant version of \( \mathcal{N} = 4 \) CSG is the ‘minimal’ one, i.e. its action cannot contain scalar-vector \( h(\varphi)F^2 \) and scalar-Weyl \( f(\varphi)C^2 \) couplings (which are not ruled out on the basis of Weyl invariance only). Here we ignore higher-order terms in the fields (since the explicit \( SU(1,1) \) invariant form of the scalar term is not known) and so that \( \varphi \) is assumed to be equal simply to \( C - i\phi \).

\text{10} We set \( g_{mn} = e^{2\sigma} \tilde{g}_{mn} \), with \( \tilde{g} \) subject to the conformal gauge condition \((13)\) \( R(\tilde{g}) = 0 \), i.e. \( e^\sigma = 1 - \frac{1}{6} (-D^2 + \frac{1}{3} R)^{-1} R \). We specify the expression in \((14)\) to the present \( \mathcal{N} = 4 \) SYM case where we do not introduce the \( D^2 R \) term in the conformal anomaly. Equivalent (but corresponding to a different splitting of the full \( W \) into the anomalous and conformally-invariant parts) non-local expression for \( W_{\text{anom}} \) has the form \((21)\) \( \int (C^2 - R^* R^* + \frac{2}{3} D^2 R + F^2 + \ldots) \Delta^{-1}_{\Delta^4} (R^* R^* - \frac{2}{3} D^2 R) \), where \( \Delta_4 \) is the 4-th order conformally-invariant scalar operator \((12)\), i.e. the kinetic operator of \( \varphi \) in \((2.3)\).
\[
+ 2\tilde{R}^{mn}\partial_m\sigma\partial_n\sigma + 2\tilde{D}^m\sigma\tilde{D}_m\sigma\tilde{D}^2\sigma + (\tilde{D}^m\sigma\tilde{D}_m\sigma)^2 \]

Here dots stand for terms depending on other CSG fields (which are the same as in the CSG Lagrangian as can be seen by replacing $\Lambda$ by $\Lambda e^\sigma$ in the divergent part of the effective action in (2.3)).

Similarly, integrating the $SU(4)$ axial anomaly relation one finds that $W_{\text{anom}}$ contains the term

\[
W_{\text{anom}}(V) = \beta \int F^{kn}(V)F^*_n(V) D^{-2}D^mV_m + \ldots .
\]

(2.9)

The terms quadratic in the field strengths (i.e. the leading terms in the weak-field expansion) which are directly related to the divergent and conformal anomaly parts of $W$ can be written in the following covariant form\footnote{Here $-D^2$ stands for a Laplacian which in general contains also curvature terms. Contributions of higher than second order in curvatures are much more complicated \footnote{Similar quadratic action was also discussed in connection with two-point and three-point correlation functions of 4d conformally invariant theories in \textit{[13]}.}.}

\[
W_2 = \frac{1}{2}\beta \int \left[ C^{mnkl} \ln\left(\frac{-D^2}{\Lambda^2}\right)C_{mnkl} 
\right.
\]

\[
+ 2F^{mn}\ln\left(\frac{-D^2}{\Lambda^2}\right)F_{mn} + 4D^2\varphi^*\ln\left(\frac{-D^2}{\Lambda^2}\right)D^2\varphi + \ldots \] .
\]

(2.10)

For $g_{mn} = e^{2\sigma}\tilde{g}_{mn}$ the term in (2.10) which is linear in $\sigma$ is indeed the same as in (2.8). As already mentioned above, the imaginary part of the quadratic $(p^{2n}\ln p^2, \ n = 4, 2)$ term in this effective action (or discontinuity of the 2-point correlation function of the corresponding operators in the SYM theory) is related to the classical D3-brane absorption \footnote{Similar quadratic action was also discussed in connection with two-point and three-point correlation functions of 4d conformally invariant theories in \textit{[13]}.} of dilatons, gravitons and other ‘minimally’ coupled CSG fields (or certain parts of the original type IIB supergravity fields).

3. $D = 4$ Super Yang-Mills – $D = 5$ gauged supergravity relation

According to the suggestion of \textit{[1,2]}, in the large $N$, $g_{YM}^2N \gg 1$ limit \footnote{Similar quadratic action was also discussed in connection with two-point and three-point correlation functions of 4d conformally invariant theories in \textit{[13]}.} of SYM theory one should have the following equality

\[
W(G) = I_{\text{GSC}}[G(G)]
\]

(3.1)
between the SYM effective action $W(G)$ in a conformal supergravity background and the action $I_{GSG}[G(G)]$ of the $D = 5$, $\mathcal{N} = 8$ gauged supergravity evaluated on the classical solution with the boundary values of the $D = 5$ fields $G$ being related to the $D = 5$ CSG fields $G$. As was pointed out in [2], the IR divergences in the GSG action on AdS$_5$ background are related to divergences and anomalies in the SYM effective. The UV cutoff $\Lambda$ in SYM theory is related to the distance to the boundary of AdS$_5$ boundary in the picture of [2] (or to the radius of AdS$_5$ in the picture of [1] which may be more natural in the context of making connection to the full D3-brane geometry).

The relation (3.1) was demonstrated at the level of the quadratic terms (2-point functions) in scalars and vectors. The scalar $\int d^5 x \sqrt{g} (\partial \mu \phi)^2$ and vector $\int d^5 x \sqrt{g} F_{\mu \nu}^2$ terms in the GSG action were shown to lead to the boundary terms $\int d^4 x \phi \partial^4 \ln \frac{-\delta^2 \phi}{\Lambda^2}$ and $\int d^4 x F_{mn} \ln \frac{-\delta^2 F_{mn}}{\Lambda^2} = -2 \int d^4 x V_m^\perp \partial^2 \ln \frac{-\delta^2 V_m^\perp}{\Lambda^2}$, in agreement with the structure of $W(2.10)$.

Below we shall consider a similar test for the term which is quadratic in the perturbation of the metric. It turns out that to ensure that only the transverse traceless part of the graviton $\tilde{h}^\perp_{mn}$ is coupled at the boundary one needs to make a special choice of the boundary term in the GSG action. We shall demonstrate in section 4 that the $D = 5$ Einstein+cosmological+boundary term expanded near AdS$_5$ background and evaluated on the solution of the Dirichlet problem to the $O(h^2)$ order reproduces the quadratic term in the Weyl tensor part of (2.10), i.e.

$$C^{m n k l} \ln \frac{-D^2}{\Lambda^2} C_{m n k l} = \frac{1}{2} \partial^2 \tilde{h}^\perp_{mn} \ln \frac{-\partial^2}{\Lambda^2} \partial^2 \tilde{h}^\perp_{mn} + O(h^3).$$  (3.2)

It remains an interesting problem to see how the full non-linear expression for the local (divergent) $D = 4$ CSG part (2.4) of the SYM effective action $W$ can emerge from the $D = 5$ GSG action.

The finite anomalous part $W_{anom}$ of the SYM action which is closely related to the local divergent part $W_\infty$ starts with terms which are cubic in the fields (2.8),(2.9).
These should thus originate from certain cubic terms on the GSG side. In particular, one should be able to reproduce the scalar term $\int d^4x \, \sigma \, \partial^2 \varphi^* \partial^2 \varphi$ in (2.8) by starting with the $\int d^5x \sqrt{g} g^{\mu\nu} \partial_{\mu} \varphi^* \partial_{\nu} \varphi$ term in the GSG action and replacing the fields by their classical expressions on the AdS$_5$ background with the boundary conditions relating them to the $D = 4$ fields. The same should be true for the vector terms $\int d^5x \, \sqrt{g} F_{\mu\nu}^2 (A) \rightarrow \int d^4x \, \sqrt{\tilde{g}} \, \sigma \, F_{mn}^2 (V)$.

As was pointed out in [2], the presence of the Chern-Simons term $\int A \wedge dA \wedge dA$ in the $D = 5$ GSG action [36] is related to the $SU(4)$ anomaly (2.7) in the SYM action (see also [25]). This implies that this CS term in the GSG action evaluated on the solution of the Dirichlet problem should reproduce the 3-point anomalous term (2.9) in $W$.

Following the same logic in the case of the conformal anomaly and considering the variation of the $D = 5$ GSG action under the Weyl transformation of the metric concentrated at the boundary we immediately reproduce the $F_{mn}^2$ term in the $D = 4$ conformal anomaly relation (2.4) (note that the Maxwell action is not conformally invariant in $D = 5$). Similar argument should apply (though in a less straightforward way) also to the scalar and graviton terms in the $D = 5$ GSG action: expressing the $D = 5$ fields in terms of their boundary values one should get (to the second order in the scalar and graviton fields) $\int d^5x \, \sigma (\partial_{\mu} \phi)^2 \rightarrow \int d^4x \, \sigma (\partial^2 \phi)^2$, $\int d^5x \, \left( \sigma R + 2 \left( D - 1 \right) \nabla^2 \sigma - \left( D - 1 \right) \left( D - 2 \right) \left( \nabla \sigma \right)^2 \right) \rightarrow \int d^4x \, \sigma (\partial^2 h_{mn}^\perp)^2$. We shall not discuss the anomalous 3-point terms any further in this paper.\(^{16}\)

The simplest non-anomalous 3-point function involving graviton is the scalar-scalar-graviton one.\(^{17}\) The term $\int d^5x \, \partial_{\mu} \phi \partial_{\nu} \phi h_{\mu\nu}$ in the GSG action should reproduce, in particular, the divergent $\phi \phi h$ term in $W$ contained in the $\varphi \Delta_4 \varphi$ part of the CSG action (2.3). We shall compute this dilaton-dilaton-graviton function in section 5.

\(^{14}\) Indeed, the transformation $\delta A = da$ gives the boundary term $\int d^4x \, \tilde{a} FF^*$, where $\tilde{a}$ (having the meaning of the $SU(4)$ gauge parameter) is the restriction of $a$ to the boundary and $V = A_{|_{\partial M}}$.

\(^{15}\) The term linear in the Weyl variation of the $D = 5$ Einstein term is determined from $(\sqrt{g} R)' = \sqrt{g} e^{(D-2)\sigma} \left[ R + 2(D-1) \nabla^2 \sigma - (D-1)(D-2)(\nabla \sigma)^2 \right]$, where $g'_{mn} = e^{2\sigma} g_{mn}$.

\(^{16}\) To reproduce them from the $D = 5$ action in a systematic way seems to require to take into account some extra boundary contributions in the evaluation of the 3-point terms in the action which result from ‘subleading’ contributions to the leading-order solution of the Dirichlet problem for the graviton discussed in the next section.

\(^{17}\) Note that there are no massless scalar-vector-vector couplings both in the $D = 4$ CSG and the $D = 5$ GSG actions. 3-point terms involving vectors only and vectors and massive KK scalars were systematically discussed in [25].
4. Metric perturbations on AdS$_{d+1}$ background and graviton 2-point function

4.1. Notation and review of the scalar 2-point function

We shall follow [2] and consider the Anti de Sitter space of dimension $D = d + 1$ with Euclidean signature and the (half-space) metric

$$ds^2 = g_{\mu \nu} dx^\mu dx^\nu = \frac{1}{x_0^2} (dx_0^2 + dx_i^2) , \quad i = 1, 2, ..., d . \tag{4.1}$$

In this hyperbolic space all conformal symmetries (e.g., scalings and inversions) are isometries. This space has two boundaries: $x_0 = \infty$, which is a single point, and $x_0 = 0$ which is $R^d$. The whole boundary is topologically $S^d$. At $x_0 = 0$ boundary we will use the flat $R^d$ metric $\tilde{g}_{Bab} = \delta_{ab}$ which is conformally related to $g_{0ij}$ (4.1). The AdS$_{d+1}$ bulk indices will be denoted by $\mu, \nu, \alpha, \beta, ...$ and will take values $0, 1, ..., d$. We shall use the notation $x = (x_0, \vec{x})$, $\vec{x} = (x_i)$, $i = 1, ..., d$. The boundary indices will be labelled by $a, b, ... = 1, ..., d$ (which, in general, should be distinguished from $i, j, ...$). The bulk indices will be raised and lowered by metric (4.1). Since the boundary metric is flat, we will not distinguish between the upper and lower boundary indices.

Let us start with a brief review (following [2]) of the calculation of the term bilinear in the scalar field. Considering the Euclidean action for a massive scalar $I(\phi) = \frac{1}{2} \int d^{d+1}\! x \sqrt{g_0} \left[ (\partial_\mu \phi)^2 + m^2 \phi^2 \right]$ we are to solve $D^2 \phi - m^2 \phi = 0$ with the Dirichlet boundary condition $\phi(x_0 = 0, \vec{x}) = \phi_0(\vec{x})$. For this we need to find a ‘propagator’ $K$ which approaches a $\delta$-function at the boundary. One may choose the $\delta$-function source to be located at $x_0 = \infty$. Since both the boundary condition and the metric do not depend on $\vec{x}$, we can take $\phi$ to be a function of $x_0$ only. Then the equation of motion simplifies and the two independent solutions are: $\phi(x_0) = x_0^{\Delta \pm}$, $\Delta \pm \equiv \frac{1}{2} (d \pm \sqrt{d^2 + 4m^2})$. The one which satisfies the boundary condition is $K(x_0) \equiv \phi(x_0) = x_0^{\Delta +}$. Next, we perform the coordinate inversion which maps the point $x_0 = \infty$ to the point $x = (x_0 = 0, \vec{x} = \vec{x}^\prime)$,

$$x^\mu \rightarrow \frac{\hat{x}^\mu}{f} , \quad f \equiv |x - x'|^2 = x_0^2 + \hat{x}^2 , \quad \hat{x}^i \equiv x^i - x'^i . \tag{4.2}$$

It is easy to see that (4.2) is an isometry of AdS space and maps the boundary to itself (at the boundary it induces a conformal transformation). After the inversion $K(x_0) \rightarrow$
K(x_0, \vec{x}; \vec{x'}) = (\frac{\Delta}{2\pi})^{\Delta+}. By superposition, we find the general solution of the Dirichlet problem
\[ \phi(x_0, \vec{x}) = c_{dm} \int d^d x' \frac{x_0^{\Delta+}}{(x_0^2 + |\vec{x} - \vec{x}'|^2)^{\Delta+}} \phi_0(\vec{x'}), \] (4.3)
where \( c_{dm} = \frac{\Gamma(\Delta_+)}{\pi^{d/2} \Gamma(\Delta_+ - \frac{d}{2})}. \) The field \( \phi_0 \) has the boundary theory interpretation of a scalar with conformal dimension \( d - \Delta_+ \). Plugging the solution (4.3) into the action \( I(\phi) \) we find
\[ I = \frac{1}{2} c_{dm} (2\Delta_+ - d) \int d^d x d^d x' \frac{\phi_0(\vec{x}) \phi_0(\vec{x'})}{(x_0^2 + |\vec{x} - \vec{x}'|^2)^{\Delta+}}. \] (4.4)
This determines the two-point function of a conformal operator \( O \) with dimension \( \Delta_+ \), which couples to \( \phi_0 \) (via \( \int d^d x \phi_0 O \)) in the boundary theory.

4.2. Quadratic term in the gravitational action

Below we shall consider the second-order term in the gravitational action expanded near AdS background. The leading-order solution of the Dirichlet problem for the gravitational perturbation we shall find allows also to find the cubic terms in the action involving gravitons (which determine the 3-point correlators of conformal field theory involving the energy-momentum operator). One particular such term (dilaton-dilaton-graviton one) will be discussed in section 5. Given that the AdS space is a solution of the same action one is perturbing, the gravitational perturbation case is more subtle than the scalar and vector cases considered in [2,24,25].

Our starting point is the Einstein action with a cosmological constant \( \lambda \) in \( d + 1 \) dimensions
\[ I = I_M + I_{\partial M}, \quad I_M = \int_M d^{d+1} x \sqrt{g} (R - 2\lambda), \] (4.5)
\[ I_{\partial M} = I_{\partial M}^{(1)} + I_{\partial M}^{(2)} , \quad I_{\partial M}^{(1)} = 2 \int_{\partial M} d^d x \sqrt{g_B} K = 2 \partial_n \int_{\partial M} d^d x \sqrt{g_B} , \] (4.6)
\[ I_{\partial M}^{(2)} = a \int_{\partial M} d^d x \sqrt{g_B} \] (4.7)
Here \( g_B \) is the metric at the boundary and \( K \) is the trace of the extrinsic curvature of the boundary, i.e. \( I_{\partial M}^{(1)} \) is the standard boundary term [37, 38]. Having the volume (cosmological) term added to the Einstein action, it is natural to introduce also the additional boundary term
\[ \frac{1}{\sqrt{g_B}} \partial_n (\sqrt{g_B} n^\alpha) = \frac{1}{\sqrt{g_B}} \partial_n \sqrt{g_B}, \]
so that \( \int_{\partial M} d^d x \sqrt{g_B} K = \int_{\partial M} d^d x \partial_n \sqrt{g_B} = \partial_n \int_{\partial M} d^d x \sqrt{g_B} \).
term $I_{\partial M}^{(2)}$ proportional to the area of the boundary. The coefficient $a$ of this term will be chosen so that to ensure the conformal invariance of the action computed on the solution of the Dirichlet problem.

We shall expand the metric near the $M=\text{AdS}_{d+1}$ background $g_{\mu\nu} \to g_{0\mu\nu} + h_{\mu\nu}$, i.e. $R_{\mu\nu} \to R_{0\mu\nu} + R_{1\mu\nu} + R_{2\mu\nu} + ...$. The background metric $g_{0\mu\nu} = x_0^{-2} \delta_{\mu\nu}$ satisfies $R_{0\mu\nu} = \frac{1}{d+1} g_{0\mu\nu} R_0$, $R_0 = R = \frac{2(d+1)}{d-1} \lambda$, where, as in [2], we have set $\lambda = -\frac{1}{2} d(d-1)$, i.e. $R = -d(d+1)$. To the first order in $h_{\mu\nu}$, the Einstein equations become

$$R_{1\mu\nu} + d h_{\mu\nu} = 0 .$$

(4.8)

Computing the value of the Einstein Lagrangian to the second order in the perturbation near the $\text{AdS}_{d+1}$ solution we get

$$\sqrt{g}(R - 2\lambda) = -2d \sqrt{g_0} + \mathcal{L}_2 + \sqrt{g_0} D_\alpha t^\alpha + O(h^3) ,$$

(4.9)

where $D_\alpha t^\alpha$ represents the total derivative terms ($D_\mu$ is the covariant derivative with respect to the background AdS metric $g_0$ and $h^\mu_{\nu} \equiv g^{\mu\rho} h_{\rho\nu}$, $h \equiv h^\mu_{\mu}$)

$$t^\alpha = h^{\nu\alpha} D_\nu h + D^\alpha h^{\mu\beta} h_{\beta\mu} - D_\mu (h^{\mu\beta} h^\alpha_{\beta}) - \frac{1}{2} hD^\alpha h + \frac{1}{2} hD^\beta h^\alpha_{\beta} - D^\alpha h + D^\beta h^\alpha_{\beta} .$$

(4.10)

$\mathcal{L}_2$ is the action for a free graviton in the AdS background,

$$\mathcal{L}_2 = \frac{1}{2} \sqrt{g_0} \left[ \frac{1}{2} D_\mu h D^\mu h - D_\mu h D^\nu h^\mu_{\nu} + D_\mu h^{\alpha\beta} D_\alpha h^\mu_{\beta} - \frac{1}{2} D_\mu h^{\alpha\beta} D^\mu h^{\alpha\beta} + d \left( \frac{1}{2} h^2 - h^\nu_{\mu} h^\mu_{\nu} \right) \right].$$

(4.11)

Using the equations of motion for $h_{\mu\nu}$, we find that $\mathcal{L}_2$ reduces to a total-derivative term

$$\mathcal{L}_2 = \sqrt{g_0} D_\alpha v^\alpha , \quad v^\alpha = \frac{1}{4} [hD^\alpha h - D_\beta (hh^{\alpha\beta}) - h^{\mu\nu} D^\alpha h_{\mu\nu} + 2h^{\mu\nu} D_\nu h^\alpha_{\mu}] .$$

(4.12)

At the boundary we shall choose the gauge in which $h^0_{\nu} = h^i_{\nu} = 0$ so that

$$\sqrt{g_B} = \sqrt{g_0 B} (1 + \frac{1}{2} h_0 - \frac{1}{4} h^i_0 h^i_j + \frac{1}{8} h^2 + ...), \quad h \equiv g^{ij}_{0} h_{ij} , \quad \sqrt{g_0 B} = x_0^{-d} ,$$

and $\frac{\partial}{\partial \theta_0} = -x_0 \frac{\partial}{\partial x_0}$. Then

$$I_{\partial M}^{(1)} = 2 \int_{\partial M} d^d x \ x_0^{1-d} \left[ \frac{d}{x_0} (1 + \frac{1}{2} h - \frac{1}{4} h^i_j h^i_j + \frac{1}{8} h^2) - \frac{1}{2} (\partial_0 h - h^i_j \partial_0 h^i_j + \frac{1}{2} h \partial_0 h) \right] ,$$

(4.13)

and

$$I_{\partial M}^{(2)} = a \int_{\partial M} d^d x \ x_0^{-d} (1 + \frac{1}{2} h - \frac{1}{4} h^i_j h^i_j + \frac{1}{8} h^2) .$$

(4.14)
Here (and below in all boundary integral expressions) \( x_0 \equiv \epsilon \rightarrow 0 \).

Rewriting (4.10), (4.12) in the explicit form, substituting the resulting expressions into the action \( I_M \) (4.5) and combining its non-constant part with \( I_{\partial M}^{(1)} \) (4.13) and \( I_{\partial M}^{(2)} \) (4.14) we find:

\[
I = I_M + I_{\partial M}^{(1)} + I_{\partial M}^{(2)} = \int_{\partial M} d^d x \, x_0^{1-d} \left( \frac{1}{4} h^j_i \partial_0 h^i_j - \frac{1}{2} h^j_i \partial_j h^i_0 \right) + [2(d-1) + a] \int_{\partial M} d^d x \, x_0^{-d} \left( 1 + \frac{1}{4} h - \frac{1}{4} h^j_i h^i_j + \frac{1}{8} h^2 \right),
\]

where we have omitted the terms proportional to \( h^0_0 \) and \( h^i_0 \) which vanish at the boundary.

Fixing the constant in the boundary area term (4.7) to be \( a = -2(d-1) \) so that the second term in (4.15) vanishes (implying, in particular, the vanishing of the volume divergence in the boundary theory), we get the following simple result for the quadratic term in the action

\[
I = \int_{\partial M} d^d x \, x_0^{1-d} \left( \frac{1}{4} h^j_i \partial_0 h^i_j - \frac{1}{4} h^j_i \partial_j h^i_0 \right).
\]

As we shall see below, (4.16) leads to the expected conformally invariant expression for the graviton 2-point function.

### 4.3. Solution of the Dirichlet problem and graviton 2-point function

We would like to solve the Einstein equations (4.8) with the Dirichlet conditions at the boundary. We shall use \( h^\mu_\nu = g^\mu_\rho h^\rho_\nu = x_0^2 h_{\mu\nu} \) as the basic variables (they are equal to the vielbein components of the metric perturbation in the present case of the conformally flat background metric \( g_0 \)). We shall assume that \( h_{00} = h_{0i} = 0 \) at the boundary, i.e. impose the following boundary conditions

\[
h^j_i(x_0 = 0, \vec{x}) = \hat{h}_{ab}(\vec{x}) , \quad h_{00}(x_0 = 0) = h_{0i}(x_0 = 0) = 0 ,
\]

where \( h^j_i \equiv g^j_0 h_{ki} \) and \( \hat{h}_{ab} \) is the fixed boundary value of the metric perturbation. One would like to find the solution which approaches a \( \delta \)-function at the boundary. As in the

Note that \( \int d^{d+1} x \left( L_2 + \sqrt{g_0} D_\alpha t^\alpha \right) = \int d^d x \sqrt{g_B} n^\alpha (v_\alpha + t_\alpha) = \int d^d x \, x_0^{1-d} (v_0 + t_0) \). The derivative terms in \( I_{\partial M} \) (the terms in the second parenthesis in (4.13)) cancel the corresponding terms in \( t_0 \) (i.e. the remaining derivative terms in \( I \) are the same as in \( v_0 \)). This is the expected effect of the boundary term \( I_{\partial M}^{(1)} \) (it should compensate those terms in the variation of the Einstein action which give rise to the normal derivatives of the metric at the boundary).

Starting with nonvanishing \( h_{00}, h_{0i} \) at the boundary we can make a gauge transformation to set them equal to zero without affecting the value of \( h^j_i \).
scalar case discussed above [2], one may first find a solution which approaches \( \delta \)-function at \( x_0 = \infty \), i.e. satisfies the boundary condition

\[
h^{ij}(x_0 \to \infty, \vec{x}) \to \infty, \quad h_{00}(x_0 \to \infty) = h_{0i}(x_0 \to \infty) = 0, \quad (4.18)
\]

(implying also \( h_{ij}(x_0 \to \infty, \vec{x}) \to \infty \)) and then use the inversion transformation. Again, it is sufficient to take \( h_{\mu\nu} \) to be a function of \( x_0 \) only. The traceless part \( \bar{h}_{ij} \) of \( h_{ij} \) then satisfies

\[
\partial_0^2 \bar{h}_{ij} - \frac{d - 5}{x_0} \partial_0 \bar{h}_{ij} - \frac{2(d - 2)}{x_0^2} \bar{h}_{ij} = 0, \quad g^0_0 \bar{h}_{ij} = 0, \quad (4.19)
\]

The solution which vanishes at \( x_0 = 0 \) and blows up at \( x_0 = \infty \) is \( h_{ij} \sim x_0^{d-2} \). The equations for the trace \( h = g^0_0 h_{ij} \) of \( h_{ij} \) and \( h_{00} \) are equivalent to the constraint

\[
h_{00} = -\frac{1}{d} (2h + x_0 \partial_0 h). \quad (4.20)
\]

It is straightforward to check that \( h_{0i} \) does not couple to \( h_{ij} \) and \( h_{00} \). In view of (4.18), we can consistently set \( h_{00} \) and \( h_{0i} \) to be zero everywhere. Then the only non-vanishing solution of (4.20) is \( h \sim x_0^{-2} \). This does not satisfy the boundary condition (4.18), so that we should set \( h = 0 \).

The solution of (4.8), (4.18) is thus

\[
h_{ij}(x) = \kappa_d x_0^{d-2} P_{ijab} \hat{h}_{ab}, \quad h_{00} = h_{0i} = 0, \quad (4.21)
\]

where \( \hat{h}_{ab} \) is an arbitrary tensor (which will be related to the perturbation of the metric at the boundary), \( \kappa_d \) is a normalization constant to be determined later, and \( P_{ijab} \) is a traceless projector (recall that \( g^0_0 = x_0^2 \delta^{ij} \); the indices \( a, b \) are contracted with flat metric)

\[
P_{ijab} = \frac{1}{2}(\delta_{ia}\delta_{jb} + \delta_{ja}\delta_{ib}) - \frac{1}{d}\delta_{ij}\delta_{ab}. \quad (4.22)
\]

Performing now the inversion (4.2) \( (x = (x_0, \vec{x}), x' = (x_0, \vec{x}')) \)

\[
h_{\mu\nu} \to f^{-2} J_{\mu\rho}(x - x') J_{\nu\lambda}(x - x') \ h_{\rho\lambda}(x), \quad f \equiv |x - x'|^2 = x_0^2 + |\vec{x} - \vec{x}'|^2,
\]

with \( J_{\mu\nu}(x) \) defined by\textsuperscript{22}

\[
J_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{|x|^2}, \quad (4.23)
\]

\textsuperscript{22} The indices of products of \( J_{\mu\nu} \) and \( P_{ijab} \) will always be contracted with flat metric.
we transform \(1.21\) into
\[
h_{\mu\nu} = \kappa_d \frac{x_0^{d-2}}{f^d} J_{\mu i}(x - x') J_{\nu j}(x - x') P_{ijab} \hat{h}_{ab}(\vec{x}') .
\]
The general solution is found by superposition,
\[
h'_\mu(x_0, \vec{x}) = \kappa_d \int d^d x' \frac{x_0^d}{f^d} J_{\mu i}(x - x') J_{\nu j}(x - x') P_{ijab} \hat{h}_{ab}(\vec{x}') .
\] (4.24)
Since
\[
\lim_{x_0 \to 0} \frac{x_0^d}{(x_0^2 + |\vec{x} - \vec{x}'|^2)^d} = \delta^{(d)}(\vec{x} - \vec{x}') , \quad c_d = \frac{\Gamma(d)}{\pi^{d/2} \Gamma(\frac{d}{2})} ,
\]
we set
\[
\kappa_d = \frac{d + 1}{d - 1} c_d ,
\] (4.26)
to ensure that \(h_i^j\) reduces to \(\hat{h}_{ab}\) at the boundary.

Alternatively, we can perform the gauge transformation,
\[
h_{\mu\nu} \rightarrow h_{\mu\nu} - \nabla_\mu \eta_\nu - \nabla_\nu \eta_\mu , \quad \eta_\mu = -\frac{x_0^{d-2}}{4(d + 1)f^{d-1}} \partial_\mu J_{ij}(x - x') P_{ijab} \hat{h}_{ab} .
\]
to get a more ‘transparent’ expression for \(h_i^j\),
\[
h_i^j(x_0, \vec{x}) = c_d \int d^d x' \frac{x_0^d}{(x_0^2 + |\vec{x} - \vec{x}'|^2)^d} P_{ijab} \hat{h}_{ab}(\vec{x}') ,
\] (4.27)
\[
h_0^i(x_0, \vec{x}) = \frac{c_d d}{d - 1} \int d^d x' \frac{x_0^{d-1}}{(x_0^2 + |\vec{x} - \vec{x}'|^2)^d} B_{iab}(x - x') \hat{h}_{ab}(\vec{x}') ,
\] (4.28)
\[
h_0^0(x_0, \vec{x}) = -\frac{c_d d}{d - 1} \int d^d x' \frac{x_0^d}{(x_0^2 + |\vec{x} - \vec{x}'|^2)^d} C_{ab}(x - x') \hat{h}_{ab}(\vec{x}') ,
\] (4.29)
where
\[
B_{iab} \equiv \frac{1}{2} \partial_i J_{jk}(x - x') P_{j kab} , \quad C_{ab} \equiv J_{ij}(x - x') P_{ijab} .
\]
Since \(h_i^j\) approaches \(\hat{h}_{ab}\) for \(x_0 \to 0\) (while \(h_0^0 \to 0\)), \(\hat{h}_{ab}\) should thus couple to the energy momentum tensor of the boundary conformal field theory.\(^23\) The consistency of this

\(^23\) That \(h_i^j\) is the relevant variable follows also from expanding the Born-Infeld action of a D3-brane probe in the AdS\(_5\) background.
interpretation can be confirmed by making a scale transformation or inversion in the AdS
space (which are isometries in the bulk, inducing conformal transformations at the boundary). It is easy to check using (4.24)–(4.29) (and following the method described in [25])
that the induced transformation on \( \hat{h}_{ab} \) indeed coincides with the conformal transformation
of the graviton field in \( R^d \).

Using (4.27), (4.28) we find

\[
\begin{align*}
\frac{\partial h_i}{\partial h^i_j} &= c_d^2 d \text{ } x_0^d \int d^d x' d^d x'' \frac{x_0^d}{|x - x'|^2d} \frac{\hat{h}_{ab}(x')P_{abcd} \hat{h}_{cd}(x'')}{|x - x''|^2d}, \\
\frac{\partial h_j}{\partial h^j_i} &= -\frac{c_d^2 d}{d-1} x_0^{d-1} \int d^d x' d^d x'' \frac{x_0^d}{|x - x'|^2d} \frac{\hat{h}_{ab}(x')X_{abcd}(x - x') \hat{h}_{cd}(x'')}{|x - x''|^2d},
\end{align*}
\]

where \(|x - x'|^2 \equiv x_0^2 + |\vec{x} - \vec{x}'|^2\) and \(X\) is defined by

\[
X_{abcd}(x) = P_{abcd} - \frac{d+1}{2|x|^2} (\delta_{bc}x_0x_d + \delta_{bd}x_0x_c + \delta_{ac}x_0x_d + \delta_{ad}x_0x_c) + \frac{2(d+1)}{|x|^4} x_0x_bx_cx_d.
\]

Substituting these expressions into (4.19), we finally obtain the following result for the
quadratic part of the action

\[
I = b_d \int d^d x' d^d x'' \frac{\hat{h}_{ab}(x')H_{abcd}(x' - x'') \hat{h}_{cd}(x'')}{|x' - x''|^2d},
\]

where \(b_d \equiv \frac{d}{4\kappa_d} = \frac{d(d+1)}{4(d-1)} c_d\) and

\[
H_{abcd}(x) \equiv \frac{1}{2}(J_{ac}J_{bd} + J_{ad}J_{bc}) - \frac{1}{d} \delta_{ab} \delta_{cd}
\]

\[
= P_{abcd} - \frac{1}{|x|^2} (\delta_{bc}x_0x_d + \delta_{bd}x_0x_c + \delta_{ac}x_0x_d + \delta_{ad}x_0x_c) + \frac{4}{|x|^4} x_0x_bx_cx_d.
\]

The kernel in (4.31) (with \(x_0 \to 0\), \(|x|^2 = x_0^2 + |\vec{x}|^2 \to |\vec{x}|^2\) has precisely the structure
expected for the correlator of the two energy-momentum tensor operators in a conformally
invariant theory (see, e.g., [38,33]).

We have obtained (4.31) by starting with the full expression for the \(D = d + 1\) gravita-
tional action (4.5) which includes the standard boundary term (4.6) and the boundary
area counter-term (4.7). Since the quadratic action (4.11) for a free graviton in AdS space
is invariant under the conformal transformations, it is natural to expect, at the same time,
that one should get a conformally invariant expression by just starting with \(\mathcal{L}_2\) alone
(4.11), (4.12). This is indeed the case: plugging (4.24) into (4.12), one finds (4.31) (though
with a different normalization factor). A non-trivial issue (which is important for the duality
[3] between the type IIB supergravity on \(AdS_5 \times S^5\) and \(\mathcal{N} = 4\) SYM) is the agreement

15
of the normalization factors. We expect that the correct normalization of anomaly-related terms is obtained only if one starts with the action that includes all relevant boundary terms (with the condition of cancellation of the boundary power divergence used to fix their relative coefficients). At the same time, the boundary terms should not be relevant for the conformally-invariant higher-point functions given by the bulk integrals.

Specifying to the case of $D = 5$ ($d = 4$) and using the momentum representation [1,23] we conclude that (4.31) coincides with the quadratic term (3.2) in the effective action (2.10) (with the $D = 5$ IR cutoff $x_0 = \epsilon \to 0$ being related to the $d = 4$ UV cutoff $\Lambda$ in (2.4) and $h_{mn} \sim \hat{h}_{ab}$).

5. Graviton-dilaton-dilaton 3-point function

Our aim here will be to compute the $h\phi\phi$ part of the term ($\phi$ is a massless scalar)

$$I = \frac{1}{2} \int d^{d+1}x \sqrt{g} g^\mu\nu \partial_\mu \phi \partial_\nu \phi$$

(5.1)

in the action of the $D = d + 1$ dimensional gauged supergravity theory on the solution of the Dirichlet problem in AdS$_{d+1}$ background and demonstrate the agreement with the expected form of the corresponding 3-point function in the boundary conformal theory.

Let us first consider a simpler example of interacting scalar theory (see also [23,24]):

$$I(\phi) = \int d^{d+1}x \sqrt{g_0} \left[ \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + V(\phi) \right].$$

The equation of motion with the Dirichlet boundary condition

$$D^2 \phi = V'(\phi), \quad \phi(x_0 = 0, \vec{x}) = \phi_0(\vec{x})$$

is solved, to leading order in perturbation theory, by

$$\phi = \phi_1 + \phi_2, \quad D^2 \phi_1 = 0, \quad D^2 \phi_2 = V'(\phi_1),$$

where

$$\phi_1 = \int d^d x' \mathcal{K}(x_0, \vec{x}; \vec{x}') \phi_0(\vec{x}'), \quad \phi_2 = \int dx_0' d^d x' G(x_0, \vec{x}; x_0', \vec{x}') V'(\phi_1(\vec{x})).$$

(5.2)

Here $\mathcal{K}(x, x') = \mathcal{K}(x_0, \vec{x}; \vec{x}')$ is the ‘propagator’ introduced in section 4.1 and $G(x, x') = G(x_0, \vec{x}; x_0', \vec{x}')$ is the standard Dirichlet Green function, i.e.

$$\mathcal{K}(x_0, \vec{x}; \vec{x}') = \sqrt{g_B} n^\alpha \partial_\alpha G(x_0, \vec{x}; x_0' \to 0, \vec{x}'), \quad G(x_0 \to 0, \vec{x}; x_0', \vec{x}'') \to 0.$$  

(5.3)

Substituting the solution (5.2) into the action $I(\phi)$, we find, to the leading order in $V$,

$$I(\phi) = \int d^{d+1}x \sqrt{g_0} \left[ \frac{1}{2} \partial_\mu \phi_1 \partial_\mu \phi_1 + \partial_\mu \phi_1 \partial_\mu \phi_2 + V(\phi_1) \right] = I_0 + I_1,$$
where \( I_0 = \frac{1}{2} \int d^{d+1}x \sqrt{g_0} \partial_\mu \phi_1 \partial_\mu \phi_1 \) was computed in section 4.1 (eq. (4.4) with \( m = 0 \)) and

\[
I_1 = \int_{\partial M} d^d x \sqrt{g_0} B \phi_2 n^\alpha \partial_\alpha \phi_1 + \int_M d^{d+1} x \sqrt{g_0} V(\phi_1) = \int_M d^{d+1} x \sqrt{g_0} V(\phi_1) ,
\]

where we have used that \( \phi_2|_{\partial M} = 0 \).\(^{24}\) Thus to find the lowest order scalar 3-point function of the we need only to plug the free-field Dirichlet problem solution \( \phi_1 \) into the potential term in the action (5.4).

Let us now return to our problem (5.1). Taking \( g_{\mu\nu} = g_{0\mu\nu} + h_{\mu\nu} \), expanding to the lowest order in \( h_{\mu\nu} \) and following the same steps as in the pure scalar case, we find

\[
I_1 = -\frac{1}{2} \int d^{d+1} x \sqrt{g_0} h_\nu^\nu T_\nu^\mu ,
\]

where \( T_\nu^\mu \) is the energy-momentum tensor of the dilaton field in the AdS\(_{d+1} \) background

\[
T_\nu^\mu = g_0^{\mu\rho} \partial_\sigma \phi \partial_\nu \phi - \frac{1}{2} \delta_\nu^\mu g_0^{\alpha\sigma} \partial_\rho \phi \partial_\sigma \phi .
\]

The solutions for \( \phi \) and \( h_\mu^\nu \) are given by (4.13) (with \( m = 0, \Delta_+ = d \)) and (4.24), i.e.

\[
\phi(x) = c_d \int d^d y \mathcal{K}(x, y) \phi_0(\vec{y}) , \quad \mathcal{K}(x, y) = \frac{x^d_0}{|x - y|^{2d}} ,
\]

\[
h_\nu^\nu(x) = \kappa_d \int d^d y \mathcal{K}(x, y) J_{\mu i}(x - y) J_{\nu j}(x - y) P_{ijab} \hat{h}_{ab}(\vec{y}) ,
\]

with \( x = (0, \vec{x}), y = (0, \vec{y}), c_d = \frac{\Gamma(d)}{\pi^{d/2} \Gamma(\frac{d}{2})}, \kappa_d = \frac{d+1}{d-1} c_d \) and \( J_{\mu\nu}(x) \) defined in (4.23). Then (5.5) takes the form

\[
I_1 = \int d^d x d^d y d^d z A_{ab}(\vec{x}, \vec{y}, \vec{z}) \hat{h}_{ab}(\vec{x}) \phi_0(\vec{y}) \phi_0(\vec{z}) ,
\]

with

\[
A_{ab} \equiv -\frac{1}{2} \kappa_d c_d^2 (A_{1ij} - \frac{1}{2} A_{2ij}) P_{ijab} ,
\]

\[
A_{1ij}(\vec{x}, \vec{y}, \vec{z}) = \int \frac{d w_0 d^d w}{w_0^{d+1}} \mathcal{K}(w, x) J_{\mu i}(w - x) J_{\nu j}(w - x) w^2_0 \partial_\mu \mathcal{K}(w, y) \partial_\nu \mathcal{K}(w, z) ,
\]

\[
A_{2ij}(\vec{x}, \vec{y}, \vec{z}) = \int \frac{d w_0 d^d w}{w_0^{d+1}} \mathcal{K}(w, x) J_{\mu i}(w - x) J_{\mu j}(w - x) w^2_0 \partial_\mu \mathcal{K}(w, y) \partial_\nu \mathcal{K}(w, z) ,
\]

\(^{24}\) Since the boundary condition is saturated by \( \phi_1 \), the correction \( \phi_2 \) should go to zero at the boundary. In the case of a compact source, this condition is satisfied automatically as a consequence of (5.3). Since the source \( V'(\phi_1) \) does not vanish at the boundary, (5.2) may have an extra boundary contribution. Here we shall ignore this problem.
where the repeated indices are contracted with flat metric and all derivatives are with respect to \( w \). Using the technique developed in [24], it is straightforward to evaluate the above integrals. We find that \( A_2 = 0 \) and

\[
A_{1ij} = -\frac{2\pi^{\frac{d}{2}}}{\Gamma(d+1)\Gamma(d+2)} \left\{ \frac{1}{|\vec{x} - \vec{y}|^{d-2}} \frac{1}{|\vec{x} - \vec{z}|^{d-2}} \frac{1}{|\vec{z} - \vec{y}|^{d+2}} \times \left[ \left( \frac{x_i - y_i}{|\vec{x} - \vec{y}|^2} - \frac{x_i - z_i}{|\vec{x} - \vec{z}|^2} \right) \left( \frac{x_j - y_j}{|\vec{x} - \vec{y}|^2} - \frac{x_j - z_j}{|\vec{x} - \vec{z}|^2} \right) - \frac{1}{d} \delta_{ij} \right] \right\} .
\]

(5.8)

If we define

\[
\lambda_{ab}^{\overrightarrow{\vec{x}}} (\vec{y}, \vec{z}) = \lambda_{a}^{\overrightarrow{\vec{x}}} (\vec{y}, \vec{z}) \lambda_{b}^{\overrightarrow{\vec{x}}} (\vec{y}, \vec{z}) - \frac{1}{d} \delta_{ab} \lambda_{c}^{\overrightarrow{\vec{x}}} (\vec{y}, \vec{z}) \lambda_{c}^{\overrightarrow{\vec{x}}} (\vec{y}, \vec{z}) , \quad \lambda_{a}^{\overrightarrow{\vec{x}}} (\vec{y}, \vec{z}) = \frac{x_a - y_a}{|\vec{x} - \vec{y}|^2} - \frac{x_a - z_a}{|\vec{x} - \vec{z}|^2} ,
\]

then the kernel in (5.7) takes the form

\[
A_{ab}(\vec{x}, \vec{y}, \vec{z}) = \frac{d^3 \Gamma(d-1)}{8\pi^d} \frac{1}{|\vec{x} - \vec{y}|^{d-2}} \frac{1}{|\vec{x} - \vec{z}|^{d-2}} \frac{1}{|\vec{z} - \vec{y}|^{d+2}} .
\]

(5.9)

\( A_{ab} \) has precisely the same structure as the conformally-invariant correlation function of the three composite conformal operators coupled to graviton and two massless scalars (see, e.g., [33], p. 104).

The conformal Ward identity requires [38]:

\[
<T_{ab}(\vec{x}) \mathcal{O}(\vec{y}) \mathcal{O}(\vec{z})> = -\frac{d \Delta_+}{d - 1} \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}} \lambda_{ab}^{\overrightarrow{\vec{x}}} (\vec{y}, \vec{z}) \left( \frac{|\vec{z} - \vec{y}|}{|\vec{x} - \vec{y}| |\vec{x} - \vec{z}|} \right)^{d-2} <\mathcal{O}(\vec{y}) \mathcal{O}(\vec{z})> ,
\]

(5.10)

where \( \Delta_+ \) is the conformal dimension of a scalar operator \( \mathcal{O} \) (see section 4).\(^{25}\) In the case of the operator coupled to the massless dilaton (\( \Delta_+ = d \)), assuming that the coupling at the boundary of AdS\(_5\) is given by \( \int d^d x (\frac{1}{2}\bar{h}_{ab} T_{ab} + \phi_0 \mathcal{O}) \), we get

\[
<T_{ab}(\vec{x}) \mathcal{O}(\vec{y}) \mathcal{O}(\vec{z})> = 4 A_{ab}(\vec{x}, \vec{y}, \vec{z}) .
\]

(5.11)

\(^{25}\) The standard differential form of the Ward identity is

\[
\partial_a < T_{ab} \mathcal{O}_1 ... \mathcal{O}_m > = - \sum_{k=1}^{m} [\delta(x - x_k) \partial^x_k + ...] < \mathcal{O}_1 ... \mathcal{O}_m >
\]

where the dots in the square brackets denote possible contact terms. For the 3-point function, the conformal invariance requires the correlator to take the form (5.10) up to an overall constant. To find the constant, we have first to regularize (5.10), and then take the derivative over \( x \). Then the Ward identity in differential form given above will determine the overall constant to be the one in (5.10). The differential form of (5.10) is

\[
\partial^x_a < T_{ab}(x) \mathcal{O}_1 (y) \mathcal{O}_2 (z) > = - \left[ \delta(x - y) \partial^x_b + \delta(x - z) \partial^x_b - \frac{\Delta_+}{d} [\partial^x_b \delta(x - y) + \partial^x_b \delta(x - z)] \right] < \mathcal{O}_1 \mathcal{O}_2 > .
\]

Note that the contact terms are also determined from the conformal symmetry.
It is easy to check that with \( A_{ab} \) given in (5.3), the relation (5.10) is indeed satisfied.

The above result for the graviton-scalar-scalar function can be generalised to the massive scalar case and is again found to be consistent with the Ward identity (5.10). For a massive scalar \( T_{\mu \nu} = g_0^{\mu \rho} \partial_\rho \phi \partial_\nu \phi - \frac{1}{2} \delta_\mu^{\nu} (g_0^{\rho \sigma} \partial_\rho \phi \partial_\sigma \phi + m^2 \phi^2) \), and \( A_{ab} \) in (5.7) is given by

\[
A_{ab} \equiv -\frac{1}{2} \kappa_d c_{dm}^2 [A_{1ij} - \frac{1}{2} (A_{2ij} + A_{3ij})] P_{ijab}, \tag{5.12}
\]

with

\[
A_{1ij}(\vec{x}, \vec{y}, \vec{z}) = \int \frac{dw_0 d^d w}{w_0^{d+1}} \mathcal{K}(w, x) J_{\mu i}(w - x) J_{\nu j}(w - x) w_0^2 \partial_\mu \mathcal{K}_{\Delta_+}(w, y) \partial_\nu \mathcal{K}_{\Delta_+}(w, z),
\]

\[
A_{2ij}(\vec{x}, \vec{y}, \vec{z}) = \int \frac{dw_0 d^d w}{w_0^{d+1}} \mathcal{K}(w, x) J_{\mu i}(w - x) J_{\mu j}(w - x) w_0^2 \partial_\rho \mathcal{K}_{\Delta_+}(w, y) \partial_\rho \mathcal{K}_{\Delta_+}(w, z),
\]

\[
A_{3ij}(\vec{x}, \vec{y}, \vec{z}) = m^2 \int \frac{dw_0 d^d w}{w_0^{d+1}} \mathcal{K}(w, x) J_{\mu i}(w - x) J_{\mu j}(w - x) \mathcal{K}_{\Delta_+}(w, y) \mathcal{K}_{\Delta_+}(w, z),
\]

where we have used (4.3) and \( \mathcal{K}_{\Delta_+}(w, z) = \left( \frac{w_z}{|w - z|^2} \right)^{\Delta_+}. \) Since \( \Delta_+ (\Delta_+ - d) = m^2 \) we find that \( A_2 + A_3 = 0 \), while

\[
A_{1ij} P_{ijab} = -\gamma \lambda_{\vec{x}}^a(\vec{y}, \vec{z}) \left( \frac{|\vec{z} - \vec{y}|}{|\vec{x} - \vec{y}|} \right)^{d-2} \frac{1}{|\vec{z} - \vec{y}|^{2\Delta_+}},
\]

\[
\gamma \equiv \frac{2\pi^{\frac{d}{2}} \Delta_+ (\Delta_+ - \frac{1}{2} d) \Gamma(\Delta_+ - \frac{1}{2} d) \Gamma(\frac{d}{2} + 1)}{\Gamma(\Delta_+) \Gamma(d + 2)} \tag{5.13}
\]

Observing that

\[
2\gamma \kappa_d c_{dm}^2 = \frac{d \Delta_+ \Gamma(\frac{d}{2})}{d - 1} \frac{\Gamma(\frac{d}{2} + 1)}{\pi^{\frac{d}{2}}} (\Delta_+ - \frac{1}{2} d) \ c_{dm},
\]

it follows from (1.4) that the Ward identity (5.11) is also satisfied in the massive case. Note that this is an independent confirmation that the normalization \( \Lambda \) in (4.4) is the consistent one.

6. Concluding remarks

We have thus established the explicit relation between the quadratic term in the \( D = 5 \) Einstein action in the AdS\(_5\) background and the quadratic term in the Weyl tensor squared part of the quantum effective action of the \( D = 4 \) SYM theory in the conformal supergravity background.

As was already mentioned above, it would be very interesting to understand, in particular, how to extend the equivalence between the logarithmically IR singular part of the
$D = 5$ gauged supergravity action evaluated on the solution of the Dirichlet problem and the $D = 4$ conformal supergravity action (2.4),(2.5) to the full non-linear level, and thus ‘derive’ the $D = 4$ conformal supergravity (Weyl +...) action from the $D = 5$ gauged supergravity (Einstein +...) action.

A simple test of this correspondence between the two gravitational actions beyond quadratic level would be to check explicitly that the divergent part of the 3-point function (5.7) is indeed in agreement with the $h\phi\phi$ term in the $\varphi^*\Delta_4\varphi$ part of the CSG action (2.3).

Acknowledgments

We acknowledge the support of PPARC, the European Commission TMR programme grant ERBFMRX-CT96-0045 and the NSF grant PHY94-07194. A.A.T. is grateful to I. Chepelev, M. Douglas, M. Green and R. Metsaev for stimulating discussions. H.L. would like to thank A. Matusis for useful correspondence.
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