An Overpartition Analogue of Bressoud’s Theorem of Rogers-Ramanujan Type

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Abstract. For $k \geq i \geq 1$, let $B_{k,i}(n)$ denote the number of partitions of $n$ such that part 1 appears at most $i - 1$ times, two consecutive integers $l$ and $l + 1$ appear at most $k - 1$ times and if $l$ and $l + 1$ appear exactly $k - 1$ times then the total sum of the parts $l$ and $l + 1$ is congruent to $i - 1$ modulo 2. Let $A_{k,i}(n)$ denote the number of partitions with parts not congruent to $i$, $2k - i$ and $2k$ modulo $2k$. Bressoud’s theorem states that $A_{k,i}(n) = B_{k,i}(n)$. Corteel, Lovejoy, and Mallet found an overpartition analogue of Bressoud’s theorem for $i = 1$, that is, for partitions not containing nonoverlined part 1. We obtain an overpartition analogue of Bressoud’s theorem in the general case. For $k \geq i \geq 1$, let $D_{k,i}(n)$ denote the number of overpartitions of $n$ such that the nonoverlined part 1 appears at most $i - 1$ times, for any integer $l$, $l$ and nonoverlined $l + 1$ appear at most $k - 1$ times and if the parts $l$ and the nonoverlined part $l + 1$ appear exactly $k - 1$ times then the total sum of the parts $l$ and nonoverlined part $l + 1$ is congruent to the number of overlined parts that are less than $l + 1$ plus $i - 1$ modulo 2. Let $C_{k,i}(n)$ denote the number of overpartitions with the nonoverlined parts not congruent to $\pm i$ and $2k - 1$ modulo $2k - 1$. We show that $C_{k,i}(n) = D_{k,i}(n)$. This relation can also be considered as a Rogers-Ramanujan-Gordon type theorem for overpartitions.

Keywords: Rogers-Ramanujan-Gordon theorem, overpartition, Bressoud’s theorem,

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1 Introduction

The Rogers-Ramanujan-Gordon theorem is a combinatorial generalization of the Rogers-Ramanujan identities [15, 16], see Gordon [10]. It establishes the equality between the number of partitions of $n$ with parts satisfying certain residue conditions and the number of partitions of $n$ with certain difference conditions. Gordon found an involution for an equivalent form of the generating function identity for this relation. An algebraic proof was given by Andrews [1] by using a recursive approach. It should be noted that the Rogers-Ramanujan-Gordon theorem is concerned only with odd moduli. Bressoud [4] succeeded in finding a theorem of
Rogers-Ramanujan-Gordon type for even moduli by using an algebraic approach in the spirit of Andrews \cite{Andrews}. The objective of this paper is to give an overpartition analogue of Bressoud’s theorem. We shall derive the equality between the number of overpartitions of \( n \) such that the nonoverlined parts belong to certain residue classes modulo some odd positive integer and the number of overpartitions of \( n \) with parts satisfying certain difference conditions. A special case of this relation has been discovered by Corteel, Lovejoy, and Mallet \cite{Corteel2014}.

An overpartition analogue of the Rogers-Ramanujan-Gordon theorem was obtained by Chen, Sang and Shi \cite{Chen}, which states that the number of overpartitions of \( n \) with nonoverlined parts belonging to certain residue classes modulo some even positive integer equals the number of overpartitions of \( n \) with parts satisfying certain difference conditions. However, as will be seen, the proof of the overpartition analogue of the Rogers-Ramanujan-Gordon theorem does not seem to be directly applicable to the case for the overpartition analogue of Bressoud’s theorem.

Let us give an overview of some definitions. A partition \( \lambda \) of a positive integer \( n \) is a non-increasing sequence of positive integers \( \lambda_1 \geq \cdots \geq \lambda_s > 0 \) such that \( n = \lambda_1 + \cdots + \lambda_s \). The partition of zero is the partition with no parts. An overpartition \( \lambda \) of a positive integer \( n \) is also a non-increasing sequence of positive integers \( \lambda_1 \geq \cdots \geq \lambda_s > 0 \) such that \( n = \lambda_1 + \cdots + \lambda_s \) and the first occurrence of each integer may be overlined. For example, \((7, 7, 6, 5, 2, 1)\) is an overpartition of 28. Many \( q \)-series identities have combinatorial interpretations in terms of overpartitions, see, for example, Corteel and Lovejoy \cite{Corteel2010}. Furthermore, overpartitions possess many analogous properties of ordinary partitions, see Lovejoy \cite{Lovejoy2004, Lovejoy2006}. For example, various overpartition theorems of the Rogers-Ramanujan-Gordon type have been obtained by Corteel and Lovejoy \cite{Corteel}, Corteel, Lovejoy and Mallet \cite{Corteel2014} and Lovejoy \cite{Lovejoy2006, Lovejoy2007, Lovejoy2008}. For a partition or an overpartition \( \lambda \) and for any integer \( l \), let \( f_l(\lambda) / f_\ell(\lambda) \) denote the number of occurrences of \( l \) non-overlined (overlined) in \( \lambda \). Let \( V_\lambda(l) \) denote the number of overlined parts in \( \lambda \) that are less than or equal to \( l \).

We shall adopt the common notation as used in Andrews \cite{Andrews}. Let

\[
(a)_{\infty} = (a; q)_{\infty} = \prod_{i=0}^{\infty} (1 - aq^i),
\]

and

\[
(a)_n = (a; q)_n = \frac{(a)_{\infty}}{(aq^n)_{\infty}}.
\]

We also write

\[
(a_1, \ldots, a_k; q)_{\infty} = (a_1; q)_{\infty} \cdots (a_k; q)_{\infty}.
\]

The Rogers-Ramanujan-Gordon theorem reads as follows.

**Theorem 1.1 (Rogers-Ramanujan-Gordon)** For \( k \geq i \geq 1 \), let \( F_{k,i}(n) \) denote the number of partitions of \( n \) of the form \( \lambda_1 + \lambda_2 + \cdots + \lambda_s \), where \( \lambda_j \geq \lambda_{j+1} \), \( \lambda_j - \lambda_{j+k-1} \geq 2 \) and part 1 appears at most \( i - 1 \) times. Let \( E_{k,i}(n) \) denote the number of partitions of \( n \) into parts \( \neq 0, \pm i \) (mod 2k + 1). Then for any \( n \geq 0 \), we have

\[
E_{k,i}(n) = F_{k,i}(n).
\]
In the algebraic proof of the above relation, Andrews \cite{1} \cite{2} introduced a hypergeometric function $J_{k,i}(a;x;q)$ as given by

$$J_{k,i}(a;x;q) = H_{k,i}(a;xq;q) - axqH_{k,i-1}(a;xq;q), \quad \text{(1.2)}$$

where

$$H_{k,i}(a;x;q) = \sum_{n=0}^{\infty} \frac{a^knq^{n^2+n-m}a^n(1-x^i q^{2ni}(axq^{n+1})/(q)_n(xq^n)_{\infty}}{(a)_n}. \quad \text{(1.3)}$$

To prove \text{(1.1)}, Andrews considered a refinement of $F_{k,i}(n)$, that is, the number of partitions enumerated by $F_{k,i}(n)$ with exactly $m$ parts, denoted by $F_{k,i}(m,n)$, and he showed that $J_{k,i}(-1/q;x;q)$ and the generating function of $F_{k,i}(m,n)$ satisfy the same recurrence relation with the same initial values. Setting $x = 1$ and using Jacobi’s triple product identity, we find that $J_{k,i}(-1/q;1;q)$ equals the generating function for $E_{k,i}(n)$. This yields that $E_{k,i}(n) = F_{k,i}(n)$.

The following Rogers-Ramanujan-Gordon type theorem for even moduli is due to Bressoud \cite{3}.

**Theorem 1.2** For $k \geq i \geq 1$, let $B_{k,i}(n)$ denote the number of partitions of $n$ of the form $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_s$ such that (i) $f_1(\lambda) \leq i - 1$, (ii) $f_l(\lambda) + f_{l+1}(\lambda) \leq k - 1$, and (iii) if $f_l(\lambda) + f_{l+1}(\lambda) = k - 1$, then $lf_l(\lambda) + (l+1)f_{l+1}(\lambda) \equiv i - 1 \pmod{2}$. Let $A_{k,i}(n)$ denote the number of partitions of $n$ with parts not congruent to $0, \pm i$ modulo $2k$. Then we have

$$A_{k,i}(n) = B_{k,i}(n). \quad \text{(1.4)}$$

The proof of Bressoud also uses the hypergeometric function $J_{k,i}(-1/q;x;q)$. But he needs a recurrence relation for $(-xq)_{\infty}J_{(k-1)/2,i/2}(a;x^2;q^2)$.

Lovejoy \cite{1} found the following overpartition analogues of Rogers-Ramanujan-Gordon theorem for the cases $i = 1$ and $i = k$.

**Theorem 1.3** For $k \geq 1$, let $\overline{B}_k(n)$ denote the number of overpartitions of $n$ of the form $\lambda_1 + \lambda_2 + \cdots + \lambda_s$ such that $\lambda_j - \lambda_{j+k-1} \geq 1$ if $\lambda_j$ is overlined and $\lambda_j - \lambda_{j+k-1} \geq 2$ otherwise. Let $\overline{A}_k(n)$ denote the number of overpartitions of $n$ into parts not divisible by $k$. Then we have

$$\overline{A}_k(n) = \overline{B}_k(n). \quad \text{(1.5)}$$

**Theorem 1.4** For $k \geq 1$, let $\overline{D}_k(n)$ denote the number of overpartitions of $n$ of the form $\lambda_1 + \lambda_2 + \cdots + \lambda_s$ such that $1$ cannot occur as a non-overlined part, and $\lambda_j - \lambda_{j+k-1} \geq 1$ if $\lambda_j$ is overlined and $\lambda_j - \lambda_{j+k-1} \geq 2$ otherwise. Let $\overline{C}_k(n)$ denote the number of overpartitions of $n$ whose non-overlined parts are not congruent to $0, \pm 1$ modulo $2k$. Then we have

$$\overline{C}_k(n) = \overline{D}_k(n). \quad \text{(1.6)}$$

Chen, Sang and Shi \cite{6} obtained an overpartition analogue of the Rogers-Ramanujan-Gordon theorem in the general case.
Theorem 1.5 For \( k \geq i \geq 1 \), let \( P_{k,i}(n) \) denote the number of overpartitions of \( n \) of the form \( \lambda_1 + \lambda_2 + \cdots + \lambda_s \) such that part 1 occurs as a non-overlined part at most \( i - 1 \) times, and \( \lambda_j - \lambda_{j+k-1} \geq 1 \) if \( \lambda_j \) is overlined and \( \lambda_j - \lambda_{j+k-1} \geq 2 \) otherwise. For \( k > i \geq 1 \), let \( Q_{k,i}(n) \) denote the number of overpartitions of \( n \) whose non-overlined parts are not congruent to 0, \( \pm i \) modulo \( 2k \) and let \( Q_{k,k}(n) \) denote the number of overpartitions of \( n \) with parts not divisible by \( k \). Then we have
\[
P_{k,i}(n) = Q_{k,i}(n). \quad (1.7)
\]

As an overpartition analogue of Bressoud’s theorem for the case \( i = 1 \), Corteel, Lovejoy, and Mallet [8] obtained the following relation.

Theorem 1.6 For \( k \geq 1 \), let \( \overline{A}_k^3(n) \) denote the number of overpartitions whose non-overlined parts are not congruent to 0, \( \pm 1 \) modulo \( 2k - 1 \). Let \( \overline{B}_k^3(n) \) denote the number of overpartitions \( \lambda \) of \( n \) such that (i) \( f_1(\lambda) = 0 \), (ii) \( f_1(\lambda) + f_7(\lambda) + f_{i+1}(\lambda) \leq k - 1 \), and (iii) if \( f_1(\lambda) + f_7(\lambda) + f_{i+1}(\lambda) = k - 1 \), then \( l f_i(\lambda) + l f_7(\lambda) + (l + 1) f_{i+1}(\lambda) \equiv V_\lambda(l) \pmod{2} \). Then we have
\[
\overline{A}_k^3(n) = \overline{B}_k^3(n). \quad (1.8)
\]

In this paper, we shall give an overpartition analogue of the Bressoud’s theorem in the general case.

2 The Main Result

The main result of this paper can be stated as follows.

Theorem 2.1 For \( k \geq i \geq 1 \), let \( D_{k,i}(n) \) denote the number of overpartitions of \( n \) of the form \( \lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_s \) such that
\[
\begin{align*}
(i) \quad & f_1(\lambda) \leq i - 1; \\
(ii) \quad & f_1(\lambda) + f_7(\lambda) + f_{i+1}(\lambda) \leq k - 1; \quad \text{and} \\
(iii) \quad & \text{if } f_1(\lambda) + f_7(\lambda) + f_{i+1}(\lambda) = k - 1, \text{ then } l f_i(\lambda) + l f_7(\lambda) + (l + 1) f_{i+1}(\lambda) \equiv V_\lambda(l) + i - 1 \pmod{2}.
\end{align*}
\]

Let \( C_{k,i}(n) \) denote the number of overpartitions of \( n \) whose nonoverlined parts are not congruent to 0, \( \pm i \) modulo \( 2k - 1 \). Then we have
\[
C_{k,i}(n) = D_{k,i}(n). \quad (2.9)
\]

In stead of using the function \( \overline{J}_{k,i}(a;x;q) \) as in the proof of Theorem 1.6 given by Corteel, Lovejoy, and Mallet [8], we find that the function \( H_{k,i}(a;x;q) \), also introduced by Corteel, Lovejoy, and Mallet [8], is related to the generating functions of the numbers \( C_{k,i}(n) \) and \( D_{k,i}(n) \). Recall that
\[
\overline{J}_{k,i}(a;x;q) = \overline{H}_{k,i}(a;xq;q) + axq \overline{H}_{k,i-1}(a;xq;q), \quad (2.10)
\]
where

\[
\widetilde{H}_{k,i}(a; x; q) = \sum_{n \geq 0} (-a)^n q^{kn^2 - \binom{n}{2} + n - in} x^{(k-1)n} (1 - x^i q^{2ni}) (-x, -1/a)_{n} (-axq^{n+1})_{\infty}.
\] (2.11)

It should be noticed that the function \(\tilde{J}_{k,i}(a; x; q)\) can be expressed as \(F_{1,k,i}(-q, -1/a; x\); q\) in the notation of Bressoud \([5]\), and the function \((-q)_{\infty} \tilde{H}_{k,i}(a; x; q)\) can be written as \(H_{k,i}(-1/a, -x; x; q)_{2}\) in the notation of Andrews \([2]\).

Let \(\tilde{B}_{k}^{3}(m, n)\) denote the number of overpartitions enumerated by \(\tilde{B}_{k}^{3}(n)\) with exactly \(m\) parts. Corteel, Lovejoy and Mallet \([8]\) have shown that the coefficients of \(x^{m}q^{n}\) in \(\tilde{J}_{k,1}(1/q; x; q)\) and \(\tilde{B}_{k}^{3}(m, n)\) satisfy the same recurrence relation with the same initial values. Moreover, they proved that the generating function of \(\tilde{B}_{k}^{3}(m, n)\) also equals \(\tilde{J}_{k,1}(1/q; x; q)\), that is,

\[
\sum_{m, n \geq 0} \tilde{B}_{k}^{3}(m, n)x^{m}q^{n} = \tilde{J}_{k,1}(-1/q; x; q).
\] (2.12)

Setting \(a = -1/q, x = 1\) and using the Jacobi’s triple product identity, the function \(\tilde{J}_{k,1}(a; x; q)\) can be expressed as an infinite product, namely,

\[
\tilde{J}_{k,1}(-1/q; 1; q) = \frac{(q, q^{2k-2}, q^{2k-1}; q^{2k-1})_{\infty} (-q)_{\infty}}{(q)_{\infty}}.
\]

Clearly, this is the generating function for \(\overline{A}_{k}^{3}(n)\). Thus we have \(\overline{A}_{k}^{3}(n) = \overline{B}_{k}^{3}(n)\).

However, the proof of Corteel, Lovejoy and Mallet does not seem to apply to the general case, since \(\tilde{J}_{k,i}(-1/q; x; q)\) cannot be expressed as an infinite product for \(i \geq 2\). Our idea goes as follows. For \(\tilde{C}_{k,i}(n)\), we shall show that the generating function for \(\tilde{C}_{k,i}(n)\) can be expressed in terms of \(\tilde{H}_{k,i}(a; x; q)\) with \(a = -1/q\) and \(x = q\). For \(\tilde{D}_{k,i}(n)\), let \(\tilde{D}_{k,i}(m, n)\) denote the number of overpartitions enumerated by \(\tilde{D}_{k,i}(n)\) with exactly \(m\) parts. We find a combinatorial interpretation of \(\tilde{D}_{k,i}(m, n) - \tilde{D}_{k,i-1}(m, n)\) from which we can derive a recurrence relation for \(\tilde{D}_{k,i}(m, n)\). Furthermore, we see that the recurrence relation and initial values of \(\tilde{D}_{k,i}(m, n)\) coincide with the recurrence relation and the initial values of the coefficients of \(x^{m}q^{n}\) in \(\tilde{H}_{k,i}(-1/q; xq; q)\). Thus we reach the conclusion that the generating function of \(\tilde{D}_{k,i}(m, n)\) equals \(\tilde{H}_{k,i}(-1/q; xq; q)\). Setting \(x = 1\), we deduce that the generating function of \(\tilde{D}_{k,i}(n)\) equals the generating function of \(\tilde{C}_{k,i}(n)\).

For convenience, we write \(W_{k,i}(x; q)\) for \(\tilde{H}_{k,i}(-1/q; xq; q)\), that is,

\[
W_{k,i}(x; q) = \sum_{n \geq 0} (-1)^{n} q^{(2k-1)(n+1)/2 - in} x^{(k-1)n} (1 - x^i q^{2n+1}) (-xq)_{\infty}.
\] (2.13)

Recall that Andrews found the following recurrence relation for \(H_{k,i}(a; x; q)\)

\[
H_{k,i}(a; x; q) - H_{k,i-1}(a; x; q) = x^{i-1} H_{k,k-i+1}(a; xq; q) - ax^i q H_{k,k-i}(a; xq; q).
\] (2.14)

A recurrence relation for \(W_{k,i}(x; q)\) is given below.
Theorem 2.2 For \( k \geq i \geq 1 \), we have

\[
W_{k,i}(x; q) - W_{k,i-1}(x; q) = (1 + qx)(xq)^{i-1}W_{k,k-i}(xq; q). 
\]

(2.15)

Proof. Since

\[
q^{-in} - x^i q^{(n+1)i} - q^{(i+1)n} + x^{i-1} q^{(n+1)(i-1)} = q^{-in}(1 - q^n) + x^{i-1} q^{(n+1)(i-1)}(1 - xq^{n+1}),
\]

it can be checked that \( W_{k,i}(x; q) - W_{k,i-1}(x; q) \) can be written as

\[
\sum_{n=1}^{\infty} q^{-in} \frac{(-1)^n x^{(k-1)n} q^{(2k-1)(\frac{n+1}{2})}(-xq)_{\infty}}{(q)_{n-1}(xq^{n+1})_{\infty}} + \sum_{n=0}^{\infty} (xq^{n+1})^{i-1} \frac{(-1)^n x^{(k-1)n} q^{(2k-1)(\frac{n+1}{2})}(-xq)_{\infty}}{(q)_{n}(xq^{n+2})_{\infty}}.
\]

(2.16)

Now, replacing \( n \) with \( n + 1 \), the first sum in (2.16) can be expressed as

\[
\sum_{n=0}^{\infty} q^{-i(n+1)} \frac{(-1)^{(n+1)} x^{(k-1)(n+1)} q^{(2k-1)(\frac{n+2}{2})}(-xq)_{\infty}}{(q)_{n}(xq^{n+2})_{\infty}}.
\]

(2.17)

Hence \( W_{k,i}(x; q) - W_{k,i-1}(x; q) \) equals

\[
- (xq)^{i-1} \sum_{n=0}^{\infty} \frac{(-1)^n (xq)^{(k-1)n} q^{(2k-1)(\frac{n+1}{2})} x^{i-k} q^{(2k-1)(n+1)} - in - 2i + 1 - (k-1)n (-xq)_{\infty}}{(q)_{n}(xq^{n+2})_{\infty}}
\]

\[
+ (xq)^{i-1} \sum_{n=0}^{\infty} \frac{(-1)^n (xq)^{(k-1)n} q^{(2k-1)(\frac{n+1}{2})} + (i-1)n - (k-1)n (-xq)_{\infty}}{(q)_{n}(xq^{n+2})_{\infty}}
\]

\[
= (1 + qx)(xq)^{i-1} \sum_{n \geq 0} \frac{(-1)^n (xq)^{(k-1)n} q^{(2k-1)(\frac{n+1}{2})} (1 - x^{k-i} q^{(2n+2)(k-i)}) (-xq^2)_{\infty}}{(q)_{n}(xq^{n+2})_{\infty}}
\]

\[
= (1 + qx)(xq)^{i-1} W_{k,k-i}(xq),
\]

as desired. \( \square \)

The following relation can be considered as a combinatorial interpretation of \( D_{k,i}(m, n) - D_{k,i-1}(m, n) \).

Theorem 2.3 For \( k \geq i \geq 1 \) and for \( m, n \geq 0 \), let \( S_{k,i}(m, n) \) denote the set of the overpartitions enumerated by \( D_{k,i}(m, n) \) with exactly one overlined part 1 and exactly \( i-1 \) nonoverlined part 1. Let \( T_{k,i}(m, n) \) denote the set of the overpartitions enumerated by \( D_{k,i}(m, n) \) with exactly one overlined part 1 and exactly \( i-1 \) nonoverlined part 1. Let \( Q_{k,i}(m, n) \) denote the number of overpartitions in \( S_{k,i}(m, n) \) and let \( R_{k,i}(m, n) \) denote the number of overpartitions in \( T_{k,i}(m, n) \). Then we have

\[
D_{k,i}(m, n) - D_{k,i-1}(m, n) = Q_{k,i}(m, n) + R_{k,i}(m, n).
\]

(2.18)

Proof. Let \( U_{k,i}(m, n) \) denote the set of overpartitions enumerated by \( D_{k,i}(n) \) with exactly \( m \) parts. By the definition of \( D_{k,i}(m, n) \) and \( D_{k,i-1}(m, n) \), it can easily seen that \( U_{k,i-1}(m, n) \) is
not contained in $U_{k,i}(m,n)$. To compute $D_{k,i}(m,n) - D_{k,i-1}(m,n)$, we wish to construct an injection $\varphi$ from overpartitions in $U_{k,i-1}(m,n)$ to overpartitions $U_{k,i}(m,n)$. From the characterization of the images of this map, we obtain the relation (2.15).

Let $\lambda$ be an overpartition in $U_{k,i-1}(m,n)$. If there exists an overlined part of $\lambda$ with the smallest underlying part, then we switch this overlined part to a nonoverlined part, otherwise we choose a smallest nonoverlined part and switch it to an overlined part. Let $\lambda'$ denote the resulting overpartition. It can be checked that this map is an injection. It is not difficult to verify that $\lambda' \in U_{k,i}(m,n)$. Hence the number $D_{k,i}(m,n) - D_{k,i-1}(m,n)$ can be interpreted as the number of overpartitions in $U_{k,i}(m,n)$ which cannot be obtained by using the above map.

By the construction of the map $\varphi$, we may generate all the overpartitions in $U_{k,i}(m,n)$ with no overlined part equal to 1 and all the overpartitions in $U_{k,i}(m,n)$ with an overlined 1 and with at most $i-3$ nonoverlined part 1. Therefore, $D_{k,i}(m,n) - D_{k,i-1}(m,n)$ is exactly the number of overpartitions in $U_{k,i}(m,n)$ with exactly one overlined part 1 such that the nonoverlined part 1 appears either $i-1$ or $i-2$ times. This completes the proof.

**Theorem 2.4** For $k \geq i \geq 1$, and $m, n \geq 0$, we have

$$Q_{k,i}(m,n) = D_{k,k-i}(m-i,n-m).$$

**Proof.** We shall define a bijection $\phi$ from $S_{k,i}(m,n)$ to $U_{k,k-i}(m-i,n-m)$ which implies (2.19). Let $\lambda$ be an overpartition in $S_{k,i}(m,n)$, the map $\phi$ is defined as follows.

Step 1. Remove all the $i-1$ parts with underlying part 1.

Step 2. Subtract 1 from each part.

Clearly, the resulting overpartition $\lambda'$ is an overpartition of $n-m$ with $m-i$ parts. Moreover, we claim that $\lambda' \in U_{k,k-i}(m-i,n-m)$.

We first show that $f_1(\lambda') \leq k-i-1$. By the construction of $\phi$, it is easy to see that $f_1(\lambda') = f_2(\lambda)$ and $f_1(\lambda) = i-1$. From the condition (ii) in the theorem, that is, $f_1(\lambda) + f_1(\lambda') + f_2(\lambda) \leq k-1$, we find that $f_2(\lambda) \leq k-1 - i$.

We still need to verify that if there is an integer $l$ such that

$$f_1(\lambda') + f_2(\lambda') + f_{l+1}(\lambda') = k-1,$$

then we have

$$lf_1(\lambda') + lf_2(\lambda') + (l+1)f_{l+1}(\lambda') \equiv V_{\lambda'}(l) + k-i-1 \pmod 2.$$  \hspace{1cm} (2.21)

By the construction of $\phi$, it is easily checked that (2.20) implies

$$f_{l+1}(\lambda) + f_{l+2}(\lambda) = k-1.$$

Since $\lambda \in S_{k,i}(m,n)$, we have

$$(l+1)f_{l+1}(\lambda) + (l+1)f_{l+1}(\lambda) + (l+2)f_{l+2}(\lambda) \equiv V_{\lambda}(l+1) + i-1 \pmod 2.$$  \hspace{1cm} (2.20)

Clearly, $f_t(\lambda') = f_{l+1}(\lambda)$ and $f_{l+2}(\lambda) = f_{l+1}(\lambda)$ for any $t \geq 1$. Thus we deduce that

$$lf_1(\lambda') + lf_2(\lambda') + (l+1)f_{l-1}(\lambda') \equiv V_{\lambda}(l+1) + i-1 - (k-1) \pmod 2.$$
Again, by the construction of \( \phi \), we find \( V_\lambda(l+1) = V_{\lambda'}(l) + 1 \). So we arrive at relation (2.21), which implies \( \lambda' \in U_{k,k-i}(m-i,n-m) \).

It is not difficult to verify that the above construction is reversible, that is, from any overpartition in \( U_{k,k-i}(m-i,n-m) \), we can recover an overpartition in \( S_{k,i}(m,n) \). This completes the proof.

**Theorem 2.5** For \( k \geq i \geq 1 \) and \( m, n \geq 0 \), we have

\[
R_{k,i}(m,n) = D_{k,k-i}(m-i+1,n-m).
\] (2.22)

**Proof.** We proceed to give a bijection \( \chi \) from \( T_{k,i}(m,n) \) to \( U_{k,k-i}(m-i+1,n-m) \). Let \( \lambda \) be an overpartition in \( T_{k,i}(m,n) \), the map \( \chi \) is defined as follows.

Step 1. Remove all \( i-1 \) parts equal to 1.

Step 2. Subtract 1 from each part.

Clearly, the resulting overpartition \( \lambda' \) is an overpartition of \( n-m \) with \( m-i+1 \) parts. We shall show that \( \lambda' \in U_{k,k-i}(m-i+1,n-m) \).

We first verify that \( f_1(\lambda') \leq k-i-1 \). It is obvious that \( f_1(\lambda') = f_2(\lambda) \). So it suffices to prove that \( f_2(\lambda) \leq k-i-1 \). Since \( \lambda \in T_{k,i}(m,n) \), we have \( f_1(\lambda) = i-2 \), \( f_1(\lambda) = 1 \) and

\[
f_1(\lambda) + f_1(\lambda) + f_2(\lambda) \leq k-1.
\] (2.23)

It follows that \( f_2(\lambda) \leq k-i \).

It remains to show that the nonoverlined part 2 cannot occur \( k-i \) times. Assume that \( f_2(\lambda) = k-i \). Then the equality in (2.23) holds, that is,

\[
f_1(\lambda) + f_1(\lambda) + f_2(\lambda) = k-1.
\]

We wish to derive a contradiction to the condition (iii) in Theorem 2.1. By the facts \( f_1(\lambda) = i-2 \), \( f_1(\lambda') = 1 \), we find

\[
1f_1(\lambda) + 1f_1(\lambda) + 2f_2(\lambda) = 2k - i - 1.
\] (2.24)

Since \( V_\lambda(1) = 1 \), from (2.24) it follows that

\[
1f_1(\lambda) + 1f_1(\lambda) + 2f_2(\lambda) \not\equiv V_\lambda(1) + i - 1 \pmod{2},
\]

a contradiction. Thus we reach the conclusion that the nonoverlined part 2 occurs at most \( k-i-1 \) times in \( \lambda \), or equivalently, the nonoverlined part 1 occurs at most \( k-i-1 \) times in \( \lambda' \).

Next, we check condition (ii) for \( \lambda' \). For any \( l \geq 1 \), we see that

\[
f_{l+1}(\lambda) = f_l(\lambda') \quad \text{and} \quad f_{l+1}(\lambda') = f_l(\lambda').
\] (2.25)

From condition (ii) for \( \lambda \), we get

\[
f_1(\lambda') + f_1(\lambda') + f_{l+1}(\lambda') \leq k-1.
\]
Finally, we proceed to verify that if there is an integer \( l \) such that
\[
f_l(\lambda') + f_{l+1}(\lambda') = k - 1, \tag{2.26}
\]
then we have
\[
lf_l(\lambda') + (l + 1)f_{l+1}(\lambda') \equiv V_\lambda(l) + k - i - 1 \pmod{2}. \tag{2.27}
\]

Notice that (2.26) implies
\[
f_{l+1}(\lambda) + f_{l+2}(\lambda) = k - 1. \tag{2.28}
\]

Since \( \lambda \in T_{k,i}(m,n) \), by condition (iii) for \( \lambda \), we have
\[
(l + 1)f_{l+1}(\lambda) + (l + 2)f_{l+2}(\lambda) \equiv V_\lambda(l + 1) + i - 1 \pmod{2}. \tag{2.29}
\]
Substituting (2.25) into (2.29), we obtain
\[
lf_l(\lambda') + f_{l+1}(\lambda') \equiv V_\lambda(l + 1) + i - 1 - (k - 1) \pmod{2}. \tag{2.30}
\]
Observing that \( V_\lambda(l + 1) = V_\lambda'(l) + 1 \), (2.30) can be rewritten as (2.27). This leads to the conclusion that \( \lambda' \in U_{k,k-i}(m-i+1,n-m) \).

It is routine to verify that the above procedure is reversible, that is, from any overpartition in \( U_{k,k-i}(m-i+1,n-m) \), one can recover an overpartition in \( T_{k,i}(m,n) \). This completes the proof.

By relations (2.18), (2.19) and (2.22), we obtain a recurrence relation of \( D_{k,i}(m,n) \).

**Theorem 2.6** For \( k \geq i \geq 1 \) and for \( m, n \geq 0 \), we have
\[
D_{k,i}(m,n) - D_{k,i-1}(m,n) = D_{k,k-i}(m-i,n-m) + D_{k,k-i}(m-i+1,n-m). \tag{2.31}
\]

By Theorem 2.2 and Theorem 2.6, we obtain a combinatorial interpretation of \( W_{k,i}(x; q) \) in terms of overpartitions.

**Theorem 2.7** For \( k \geq i \geq 1 \), we have
\[
W_{k,i}(x; q) = \sum_{m,n \geq 0} D_{k,i}(m,n)x^mq^n. \tag{2.32}
\]

**Proof.** For \( m, n \geq 0 \) and for \( k \geq i \geq 1 \), let \( W_{k,i}(m,n) \) denote the coefficient of \( x^m q^n \) in \( W_{k,i}(x; q) \), that is,
\[
W_{k,i}(x; q) = \sum_{m,n \geq 0} W_{k,i}(m,n)x^mq^n. \tag{2.33}
\]
We proceed to show that \( D_{k,i}(m,n) \) and \( W_{k,i}(m,n) \) satisfy the same recurrence relations with the same initial values.
Clearly, we have $W_{k,i}(0,0) = 1$ for $k \geq i \geq 1$ and $W_{k,0}(m,n) = 0$ for $k \geq 1, m, n \geq 0$. Moreover, we assume that $W_{k,i}(m,n) = 0$ if $m$ or $n$ is zero but not both. By Theorem 2.2, we find that

$$W_{k,i}(m,n) - W_{k,i-1}(m,n) = W_{k,k-i}(m-i,n-m) + W_{k,k-i}(m-i+1,n-m),$$

(2.34)

which is the same recurrence relation as $D_{k,i}(m,n)$ as given in Theorem 2.6.

It is clear that $D_{k,i}(0,0) = 1$ for $k \geq i \geq 1$ and $D_{k,0}(m,n) = 0$ for $k \geq 1, m, n \geq 0$. Moreover, $D_{k,i}(m,n) = 0$ if $m$ or $n$ is zero but not both. Now, we see that $D_{k,i}(m,n)$ and $W_{k,i}(m,n)$ have the same recurrence relation and the same initial values. This completes the proof.

We are now ready to finish the proof of Theorem 2.1.

Proof of Theorem 2.1. Setting $x = 1$ in (2.32), we find that the generating function for $D_{k,i}(n)$ equals $W_{k,i}(1;q)$. In other words,

$$
\sum_{n \geq 0} D_{k,i}(n) q^n = \sum_{n=0}^{\infty} (-1)^n q^{(2k-1)(n+1)/2-\text{in}} (1-q^{2n+1}i)(-q)^n (q^n+1)_{\infty}.
$$

(2.35)

The right hand side of (2.35) can be expressed as

$$
\frac{(-q)_{\infty}}{(q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n q^{(2k-1)(n+1)/2-\text{in}} + \frac{(-q)_{\infty}}{(q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n q^{(2k-1)(n+1)+i(n+1)}.
$$

(2.36)

By substituting $n$ with $-(n+1)$ in the second sum of (2.36), we get

$$
\frac{(-q)_{\infty}}{(q)_{\infty}} \sum_{n=-\infty}^{\infty} (-1)^n q^{(2k-1)(n+1)/2-\text{in}}.
$$

In view of Jacobi’s triple product identity, we obtain

$$
\sum_{n \geq 0} D_{k,i}(n) q^n = \frac{(q^i, q^{2k-1-i}, q^{2k-1}; q^{2k-1})_{\infty} (-q)_{\infty}}{(q)_{\infty}}.
$$

(2.37)

By the definition of $C_{k,i}(n)$, it is easily seen that

$$
\sum_{n=0}^{\infty} C_{k,i}(n) q^n = \frac{(q^i, q^{2k-1-i}, q^{2k-1}; q^{2k-1})_{\infty} (-q)_{\infty}}{(q)_{\infty}}.
$$

(2.38)

Comparing (2.37) and (2.38) we deduce that $C_{k,i}(n) = D_{k,i}(n)$ for $k \geq i \geq 1$ and $n \geq 0$. This completes the proof.

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