A globalization for non-complete but geodesic spaces

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Abstract

I show that if a geodesic space has curvature bounded below locally in the sense of Alexandrov then its completion has the same lower curvature bound globally.

Introduction

Let us recall few definitions.

⋄ A geodesic space is a metric space such that any two points can be connected by a minimizing geodesic.

⋄ A length space is a metric space such that any two points can be connected by a curve with length arbitrary close to the distance between the points.

⋄ A complete length space is an Alexandrov space with curvature $\geq \kappa$ if for any quadruple $(p; x^1, x^2, x^3)$ the $(1+3)$-point comparison holds; i.e., if

$$\tilde{\kappa}(p_x^{x^1}) + \tilde{\kappa}(p_x^{x^2}) + \tilde{\kappa}(p_x^{x^3}) \leq 2\pi$$

or at least one of the model angles $\tilde{\kappa}(p_x^{x^i})$ is not defined.

Theorem. Let $X$ be a geodesic space. Assume that for any point $x \in X$ there is a neighborhood $\Omega \ni x$ such that the $\kappa$-comparison holds for any quadruple of points in $\Omega$. Then the completion of $X$ is an Alexandrov space with curvature $\geq \kappa$.

The question was asked by Victor Schroeder around 2009. Later I learned from Stephanie Alexander, that this statement has an application.

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In Section 1 I give a reformulation of the above theorem, which will be proved in Section 3 after auxiliary statements proved in Section 2.

1 Observation and reformulation.

The distance between points $x$ and $y$ in a metric space will be denoted as $|x - y|$.
A geodesic from \( x \) to \( y \) will be denoted as \([xy]\), once we write \([xy]\) we implicitly make a choice of one such geodesic. Also we use the following short-cut notation:

\[
[xy] = [xy]\{x\}, \quad [xy] = [xy]\{x, y\}, \quad [xy] = [xy]\{y\}.
\]

We denote by \( \mathbb{M}^\kappa \) the model \( \kappa \)-plane and \( \pi^\kappa \overset{d_\mathbb{M}^\kappa}{=} \text{diam} \mathbb{M}^\kappa \) (so \( \pi^\kappa = \infty \) if \( \kappa \leq 0 \) and \( \pi^\kappa = \frac{\pi}{\sqrt{\kappa}} \) otherwise).

**Angles.** Assume \([px]\) and \([py]\) be two geodesics in \( X \) and \( \bar{x} \in [px] \) and \( \bar{y} \in [py] \). Since \( \kappa \)-comparison holds in a neighborhood of \( p \) the function

\[
(\|p - \bar{x}\|, \|p - \bar{y}\|) \mapsto \tilde{\angle}_\kappa(p; \bar{y})
\]

is nonincreasing in both variables for sufficiently small values of the variables. It follows that angle

\[
\angle[p; \bar{y}] \overset{d_\mathbb{M}^\kappa}{=} \lim \left\{ \tilde{\angle}_\kappa(p; \bar{y}) \mid \bar{x} \in [px], \bar{y} \in [py], \|p - \bar{x}\| \to 0, \|p - \bar{y}\| \to 0 \right\}
\]

is defined for any hinge \([p; \bar{y}] = ([px], [py])\) in \( X \).

The same way as for Alexandrov space, one can show that

\[
\angle[p; \bar{y}] + \angle[p; \bar{z}] + \angle[p; \bar{s}] \leq 2 \pi
\]

for any three hinges formed by geodesics \([px]\), \([py]\) and \([pz]\). It follows that, if \( p \in [xy]\) then

\[
\angle[p; \bar{y}] + \angle[p; \bar{z}] = \pi.
\]

**Reformulation.** To prove the theorem, it is sufficient to show that for any \( \kappa_1 < \kappa \), any point \( p \) and any geodesics \([qs]\) in \( X \) we have

\[
\tilde{\angle}_{\kappa_1}(q; \bar{s}) \leq \angle[q; \bar{s}]_{\mathbb{M}^\kappa},
\]

if \( \bar{s} \in [qs] \) and \( |q - \bar{s}| > 0 \) is small enough.

Indeed, once it is proved, it follows that the inequality \( \theta \) holds for all \( \bar{s} \in [qs] \).

Together with \( \theta \), it implies that the \((1+3)\)-point comparison for all quadruples in \( X \); this can be done exactly the same way as in Alexandrov space, see \([3, 2.8.2]\).

Hence the completion \( \bar{X} \) of \( X \) is an Alexandrov space with curvature \( \geq \kappa_1 \) for any \( \kappa_1 < \kappa \). From the standard globalization theorem (as it is stated in \([2]\)) we get the result.

## 2 Auxiliary statements

As above we denote by \( \bar{X} \) the completion of \( X \).

Note that any point \( x \in X \) admits an open neighborhood \( \Omega \) in \( \bar{X} \) such that \( \kappa \)-comparison holds for any quadruple of points in \( \Omega \). In particular the following condition holds for any point \( p \in \Omega \cap X \) and geodesic \([qs]\) which lie in \( \Omega \cap X \).

\[
\tilde{\angle}_{\kappa_1}(q; \bar{s}) \leq \angle[q; \bar{s}]_{\mathbb{M}^\kappa},
\]
where \( \bar{s} \in [q s] \).

An open domain \( \Omega \) in \( \bar{X} \) which satisfy \( \Theta \) will be called a \( \kappa \)-domain.

Note that to prove that \( \Omega \) is a \( \kappa \)-domain, it is sufficient to check that \( \Theta \) holds only if \( |q - s| \) is sufficiently small. I.e., if for any point \( p \) and geodesic \([qs]\) in \( \Omega \cap X \), the condition \( \Theta \) holds if \( |q - s| \) is small enough then \( \Theta \) holds for all \( \bar{s} \in [qs] \).

Note also that if \( B(p, 2 \cdot R) \) is a \( \kappa \)-domain in \( \bar{X} \) then \( \kappa \)-comparison holds for any quadruple of points in \( B(p, R) \). The later is proved exactly the same way as in Alexandrov space: for a quadruple \((p; x, y, z)\) we choose a geodesic \([px]\) and apply \( \Theta \) together with \( \Theta \) for \( x \in ]px[ \) such that \( x \to p \). (Everything works since geodesics with ends in \( B(p, R) \) can not leave \( B(p, 2 \cdot R) \).)

In particular, if \( \Omega \) is a \( \kappa \)-domain in \( \bar{X} \) then the curvature at each point of \( \Omega \) is \( \geq \kappa \). Therefore any local construction in Alexandrov geometry can be performed inside of \( \Omega \).

For example, we can construct so called radial curves as far as we do not get out of \( \Omega \). The radial curves are formed by trajectories which try to escape from a given point \( w \) using the greedy algorithm; these curves parametrized in a special way which makes them behave as geodesics in terms of comparisons.

The following proposition is a local version of radial monotonicity in [2] and can be proved exactly the same way.

2.1. Proposition. Let \( \Omega \subset \bar{X} \) be a \( \kappa \)-domain and \( w, a \in \Omega \). Assume that \( B[w, R] \subset \Omega \) and

\[
|a - w| = r < R < \frac{2}{\kappa}.
\]

Then there is a radial curve \( \alpha : [r, R] \to \Omega \) with respect to \( w \) such that \( \alpha(r) = a \) and the distance \( |p - \alpha(t)| \) satisfies the radial monotonicity for any point \( p \) in \( \Omega \).

I.e., if \([\bar{w} \bar{p} \alpha(t)]\) is a triangle in \( M[\kappa] \) with sides

\[
|\bar{w} - \bar{p}| = |w - p| \quad |\bar{p} - \bar{\alpha}(t)| = |p - \alpha(t)| \quad |\bar{w} - \bar{\alpha}(t)| = t
\]

then the function

\[
t \mapsto \angle[\bar{w} \bar{p} \alpha(t)]
\]

is a nonincreasing in its domain of definition.

The proposition above is used to prove the following lemma.

2.2. Key Lemma. Let \( \Omega_p \) and \( \Omega_q \) be two \( \kappa \)-domains in \( \bar{X} \). Let

\[
p \in X \cap \Omega_p, \quad q \in X \cap \Omega_q, \quad w \in X \cap \Omega_p \cap \Omega_q.
\]

Consider a triangle \([\bar{p} \bar{w} \bar{q}]\) in \( M[\kappa] \) such that

\[
|\bar{p} - \bar{w}| = |p - w| \quad |\bar{q} - \bar{w}| = |q - w| \quad \angle[\bar{w} \bar{p} \bar{q}] = \angle[w \bar{p} q]
\]

Set \( R \) to be the distance from the side \([\bar{p} \bar{q}]\) to \( \bar{w} \).
Assume \( R < \frac{2}{\kappa} \) and \( B[w, R] \subset \Omega_p \cap \Omega_q \). Then

\[
|p - q| \leq |\bar{p} - \bar{q}|.
\]

Proof of the Key Lemma. Note that if \( R = 0 \), then the lemma follows from the triangle inequality; further we assume \( R > 0 \).
Let \( \tilde{a} \in [\tilde{p} \tilde{q}] \) be a point which minimize the distance to \( \tilde{w} \); so \( R = |\tilde{w} - \tilde{a}| \).

Fix small \( \delta > 0 \); any value \( \delta < \frac{1}{10} \min \{ 1, \bar{R}/|w - p|, \bar{R}/|w - q| \} \) will do. Choose \( p_3 \in [w p] \) and \( q_3 \in [w q] \) so that
\[
|w - p_3| = \delta \cdot |w - p|, \quad |w - q_3| = \delta \cdot |w - q|.
\]

Note that geodesic \( [p_3 q_3] \) lies in \( B[w, \bar{R}] \). By Alexandrov’s lemma (stated as in [2]), one can choose a point \( a_3 \in [p_3 q_3] \) so that
\[
\tilde{Z}^\kappa(w^p_{a_3}) \leq \zeta[\tilde{w}^\kappa_{a_3}], \quad \tilde{Z}^\kappa(w^q_{a_3}) \leq \zeta[\tilde{w}^\kappa_{a_3}].
\]

Note that
\[
\tilde{Z}^\kappa(w^p_{a_3}) \leq \tilde{Z}^\kappa(w^p_{\tilde{a}}), \quad \tilde{Z}^\kappa(w^q_{a_3}) \leq \tilde{Z}^\kappa(w^q_{\tilde{a}}).
\]

Therefore
\[
\tilde{Z}^\kappa(w^p_{a_3}) \leq \zeta[\tilde{w}^\kappa_{a_3}], \quad \tilde{Z}^\kappa(w^q_{a_3}) \leq \zeta[\tilde{w}^\kappa_{a_3}].
\]

Set \( r_3 = |w - a_3| \). Consider the radial curve \( \alpha: [r_3, \bar{R}] \to \Omega \) with respect to \( w \) such that \( \alpha(r_3) = a_3 \). Set \( a = \alpha(\bar{R}) \). By Proposition 2.1, we have
\[
|p - a| \leq |\tilde{p} - \tilde{a}|, \quad |q - a| \leq |\tilde{q} - \tilde{a}|.
\]

Hence Key Lemma follows.

2.3. Lemma. Let \( \Omega_p \) and \( \Omega_q \) be two \( \kappa \)-domains in \( \bar{X} \). Let \( p \in X \cap \Omega_p, q \in X \cap \Omega_q \) and \( [pq] \subset X \cap (\Omega_p \cup \Omega_q) \).

Then for any geodesic \( [qs] \subset \Omega_q \cap X \) the condition \( \heartsuit \) holds if \( |q - s| \) is sufficiently small.

Proof. Choose \( w \in [pq] \cap \Omega_p \cap \Omega_q \). Since \( \Omega_q \) is a \( \kappa \)-domain, we have
\[
\zeta[w^s_q] \geq \tilde{Z}^\kappa(w^s_q).
\]
for \( s \in [qs] \). Therefore
\[
\zeta[w^s_q] \leq \pi - \tilde{Z}^\kappa(w^s_q).
\]

Note that for small values of \( |q - s| \), we can apply Key Lemma; hence the result.

2.4. Corollary. Let \( \Omega_1 \) and \( \Omega_2 \) be two \( \kappa \)-domains in \( \bar{X} \). Assume
\[
\Omega_3 \subset \Omega_1 \cup \Omega_2
\]
is an open set such that for any two points \( x, y \in X \cap \Omega_3 \) any geodesic \([xy]\) lies in \( \Omega_1 \cup \Omega_2 \). Then \( \Omega_3 \) is a \( \kappa \)-domain.
The following Lemma makes possible to produce triple of the domains $\Omega_1$, $\Omega_2$ and $\Omega_3$ as in the corollary above.

### 2.5. Lemma

Let $[pq]$ be a geodesic in $X$ and the points $x$, $y$ and $z$ appear on $[pq]$ in the same order. Assume that there are $\kappa$-domains $\Omega_1 \supset [xy]$ and $\Omega_2 \supset [yz]$ in $\bar{X}$. Then there is an open set $\Omega_3 \subset \bar{X}$ which contains $[xz]$ and such that for any two points $v, w \in \Omega_3 \cap X$ any geodesic $[vw]$ lies in $\Omega_1 \cup \Omega_2$.

In particular, by Corollary 2.4, $\Omega_3$ is a $\kappa$-domain.

Before the proof starts, let us discuss ultralimits of metric spaces briefly; see [2] for more details.

Fix a nonprincipal ultrafilter $\omega$ on the natural numbers. Denote by $\bar{X}^\omega$ the $\omega$-power of $\bar{X}$.

The space $\bar{X}$ will be considered as a subspace of $\bar{X}^\omega$ in the natural way. Given a point $p \in \bar{X}$, we will denote by $B(p, \varepsilon)_\bar{X}$ and $B(p, \varepsilon)_{\bar{X}^\omega}$ the $\varepsilon$-ball centered at $p$ in $\bar{X}$ and in $\bar{X}^\omega$ correspondingly.

In is straightforward to check the following

- If $B(p, \varepsilon)_\bar{X}$ is a $\kappa$-domain then so is $B(p, \varepsilon)_{\bar{X}^\omega}$.

**Proof.** Arguing by contradiction, assume that there is a sequence of geodesics $[u_n v_n]$ such that $u_n \to u \in [xz]$, $v_n \to v \in [xy]$ and $[u_n v_n] \not\subset \bar{X}_1 \cup \bar{X}_2$ for each $n$.

The $\omega$-limit of $[u_n v_n]$ is a geodesic in $\bar{X}^\omega$ from $u$ to $v$ which does not lie in $[pq]$. I.e., geodesics in $\bar{X}^\omega$ bifurcate at some point, say $w \in [xz]$.

According to $\bullet$, if $\varepsilon > 0$ is small enough, the ball $B(w, \varepsilon)_{\bar{X}^\omega}$ forms a $\kappa$-domain, a contradiction. □

### 3 The proof

**Proof of $\bullet$.** Note that one can split the geodesic $[pq]$ into segments in such a way that each segment lies in a $\kappa$-domain. More precisely, there is a sequence of points $p = p_0, p_1, \ldots, p_n = q$ on $[pq]$ such that the sequence $[p - p_0], [p - p_1], \ldots, [p - p_n]$ is increasing and each geodesic $[p_{i-1} p_i]$ lies in a $\kappa$-domain. Given $\varepsilon > 0$, the sequence above can be chosen in such a way that in addition $|p - p_i| < \varepsilon$.

Applying Lemma 2.5 few times, we get that the segment $[p_{i-1} p_i]$ belongs to one $\kappa$-domain. Applying Key Lemma (2.2) to three points $p_1, p_{n-1}$ and $\bar{s} \in [qs]$ with small enough $|q - \bar{s}|$, we get that $\bullet$ holds for $p_1$, $[qs]$ and $\kappa_1 = \kappa$.

Finally since $|p - p_1|$ can be made arbitrary small, the triangle inequality implies that $\bullet$ holds for $p$, $[qs]$ and arbitrary $\kappa_1 < \kappa$. □

### 4 Remarks

**Cat’s cradle construction.** An alternative proof of the Key Lemma can be build on the so called *Cat's cradle construction* from [1]. This way you do not have to learn who are the radial curves and what is radial monoticity.

Choose small $\varepsilon > 0$ and apply the following procedure:
Set $w_0 = w$.
- Choose $w_1 \in [pw_0]$ so that $|w_0 - w_1| = \varepsilon$.
- Choose $w_2 \in [qw_1]$ so that $|w_1 - w_2| = \varepsilon$.
- Choose $w_3 \in [pw_2]$ so that $|w_2 - w_3| = \varepsilon$.
- And so on.

Further, find a nice estimate for the value

$$\ell_n = |p - w_{2 \cdot n}| + |w_{2 \cdot n} - q|$$

in terms of $\angle [w_{\tilde{p} \tilde{q}}], |w - p|, |w - q|$ and

$$s_n = \sum_{i=1}^{n} |w_{2 \cdot (i-1)} - w_{2 \cdot i}|.$$

Note that $w_0, w_1, \ldots, w_{2 \cdot n} \in \Omega_p \cap \Omega_q$ if $s_n < R - \varepsilon$.

If you were able to do everything as suggested, you should get a weaker version of Key Lemma. Namely you prove the required estimate if $R$ is bigger than bisector of $[\tilde{p}\tilde{w}\tilde{q}]$ at $\tilde{w}$. This is good enough for the rest of the proof. Playing a bit with the construction, namely making $\varepsilon$ depend on $n$, you can actually get the Key Lemma in full generality.

(After making all this work you might appreciate the radial curves.)

**Finite dimensional case.** In case $\bar{X}$ has finite (say Hausdorff) dimension, one can build an easier proof using the formula of second variation see [4].

**References**

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