Some cycle-supermagic labelings of the calendula graphs

T R Pradipta¹ and A N M Salman²

¹Mathematics Education, Universitas Muhammadiyah Prof. DR. HAMKA, Jakarta, Indonesia
²Combinatorial Mathematics Research Group Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Indonesia

E-mail: troymath@uhamka.ac.id

Abstract. In this paper, we introduce a calendula graph, denoted by $Cl_{m,n}$. It is a graph constructed from a cycle on $m$ vertices $C_m$ and $m$ copies of $C_n$ which are $C_{n_1}, C_{n_2}, \ldots, C_{n_m}$ and grafting the $i$-th edge of $C_m$ to an edge of $C_{n_i}$ for each $i \in \{1, 2, \ldots, m\}$. A graph $G = (V, E)$ admits a $C_n$-covering, if every edge $e \in E(G)$ belongs to a subgraph of $G$ isomorphic to $C_n$. The graph $G$ is called cycle-magic, if there exists a total labeling $\varphi: V \cup E \rightarrow \{1, 2, \ldots, |V| + |E|\}$ such that for every subgraph $C_n' = (V', E')$ of $G$ isomorphic to $C_n$ has the same weight. In this case, the weight of $C_n$, denoted by $\varphi(C_n')$, is defined as $\sum_{v \in V(C_n')} \varphi(v) + \sum_{e \in E(C_n')} \varphi(e)$. Furthermore, $G$ is called cycle-supermagic, if $\varphi: V \rightarrow \{1, 2, \ldots, |V|\}$. In this paper, we provide some cycle-supermagic labelings of calendula graphs. In order to prove it, we develop a technique, to make a partition of a multiset into $m$ sub-multisets with the same cardinality such that the sum of all elements of each sub-multiset is same. The technique is called an $m$-balanced multiset.

1. Introduction

The graphs considered here are finite, undirected, and simple. The vertex set and the edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. An $H$-(super)magic labeling was first studied by Gutiérrez and Lladó in 2005 [3]. Lladó and Moragas [5] studied some cycle-(super)magic behavior of several classes of connected graphs. They gave several families of $C_r$-magic graphs for every $r \geq 3$. Maryati et al. [10] contributed to $C_n$-supermagic labelings of $c$ copies of $C_n$. Some other results on $C_n$-supermagic labelings of several classes of graphs can be found in [1, 2, 6, 7, 8, 11, 12, 13, 14].

This paper is organized as follows. In section 2, we define a new class of graph that we call a calendula graph. It is inspired by comb product graph [4, 14]. In section 3, we develop the concept of an $m$-balanced multiset [8]. It is a technique to partition a multiset to obtain $m$ sub-multisets such that each submultiset has the same cardinality and the sum of all elements in each submultiset has the same value. This result is used to prove our main result. In the last section, we study $C_n$-supermagic labelings of calendula graphs. In this paper, we use the notation $[a, b]$ to mean $\{a, b\} = \{a, a, b\}$ and $\sum A$ to mean $\sum_{a \in A} a$. We define $\{a\} \cup \{a, b\} = \{a, a, b\}$.

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

Published under licence by IOP Publishing Ltd
2. Calendula Graphs
Let \( m \geq 3 \) and \( n \geq 3 \). Let \( C_n \) be a cycle on \( m \) vertices. A calendula graph, denoted by \( Cl_{m,n} \), is a graph constructed from \( C_n \) and \( m \) copies of \( C_n \) which are \( C_n, C_n, \ldots, C_n \) and grafting the \( i \)-th edge of \( C_m \) to an edge of \( C_n \) for each \( i \in \{1, 2, \ldots, m\} \). For illustration, we can see \( Cl_{6,4} \) in figure 1.

![Figure 1. A calendula graph \( Cl_{6,4} \).](image)

We can check that the order of \( Cl_{m,n} \) is \( m(n-1) \) and the measure of \( Cl_{m,n} \) is \( mn \). It means that \( |V(\text{Cl}_{m,n})| + |E(\text{Cl}_{m,n})| = m(2n-1) \). For \( m \neq n \), \( Cl_{m,n} \) has \( m \) subgraph \( C_n \) which isomorphic to \( C_n \). As for \( m = n \), \( Cl_{m,n} \) has \( (m+1) \) subgraph \( C_n \) which isomorphic to \( C_n \). We can also check that \( Cl_{m,n} \) contains \( C_n \)-covering. Let the vertex set of \( Cl_{m,n} \) be as follows:

\[
V(\text{Cl}_{m,n}) = \{v_i \mid i \in [1, m] \text{ and } j \in [1, (n-1)]\}
\]

\[
E(\text{Cl}_{m,n}) = \{e_i \mid i \in [1, m] \text{ and } j \in [1, n]\}.
\]

3. \( m \)-Balanced Multiset
A multiset is a set which allows the same elements. Let a multiset \( V = \{a_1, a_2, \ldots, a_m\} \) and a multiset \( W = \{a_1, a_2, \ldots, a_n\} \). Define \( V \uplus W = \{a_1, a_2, \ldots, a_m, a_1, a_2, \ldots, a_n\} \). An \( m \)-balanced multiset defined as follows. Let \( m \in \mathbb{Z}^+ \) and \( Y \) is a multiset of positive integers. \( Y \) is called \( m \)-balanced, if there are \( m \) submultisets of \( Y \), that is \( Y_1, Y_2, \ldots, Y_m \), such that for each \( i \in [1, m] \), satisfies \(|Y_i| = |Y| m^{-1}\), \( \sum_{i=1}^{m} Y_i = (\sum Y) m^{-1} \in \mathbb{Z}^+ \), and \( Y_{i+1} = Y_{i} \). We need the next lemma to prove that \( Cl_{m,n} \) is \( C_n \)-supermagic.

**Lemma 1.** Let \( m \) and \( n \) are positive integers with \( m \geq 3 \) and \( n \geq 3 \). If a multiset \( X = [1, m] \cup [1, m(2n-1)] \), then \( X \) is \( m \)-balanced.

**Proof.**
For each \( i \in [1, m] \) and \( j \in [1, 2n] \), define a multiset \( X_i = \{a_{i,1}, a_{i,2}, \ldots, a_{i,2n}\} \) with

\[
a_{i,j} = \begin{cases} 
  i, & \text{for } i \in [1, m] \text{ and } j = 1; \\
  i + 1, & \text{for } i \in [1, m-1] \text{ and } j = 2; \\
  1, & \text{for } i = m \text{ and } j = 2; \\
  2m - i + 1, & \text{for } i \in [1, m] \text{ and } j = 3; \\
  3m, & \text{for } i = m \text{ and } j = 4; \\
  (j-1)m - i + 1, & \text{for } i \in [1, m] \text{ and } j \in [5, n], j \equiv 0 \text{ mod } 2; \\
  (j-2)m + i, & \text{for } i \in [1, m] \text{ and } j \in [5, n], j \equiv 1 \text{ mod } 2.
\end{cases}
\]
Next, for every $i \in [1, m - 1]$ and $j \in [1, 2n]$, we obtain
\[
\sum X_i = 1 + (i + 1) + (2m - i + 1) + (3m - i) + (3m + i) + (5m - i + 1) + (5m + 1) + (7m - i + 1) + \ldots
\]
\[
= 2m + 3m + n + \sum_{i=1}^{n} 4mt
\]
\[
= 2mn^2 - 2mn + m + n.
\]
For every $i = m$ and $j \in [1, 2n]$, we get
\[
\sum X_i = m + 1 + (m + 1) + 3m + 4m + (4m + 1) + 6m + (6m + 1) + \ldots + ((2n - 2)m) + ((2n - 2)m + 1)
\]
\[
= 2m + 3m + n + \sum_{t=2}^{n} 4mt
\]
\[
= 2mn^2 - 2mn + m + n.
\]
For every $j \in [1, 2n]$, $A_j = \{a_{ij} \mid 1 \leq i \leq m\}$, let
\[
A_j = \begin{cases} 
[1, m] & \text{for } j \in [1, 2]; \\
((j - 2)m + 1), ((j - 1)m) & \text{for } j \in [3, 2n].
\end{cases}
\]

It can be checked that $A_1 \cup A_2 \cup \ldots \cup A_{2n} = X$ and $\cup_{i=1}^{n} X_i = X$ Additionally, for each $i \in [1, m]$, we obtain $|X| = 2n$ and $\sum X_i = 2mn^2 - 2mn + m + n$. Therefore, for $m \geq 3$ and $n \geq 3$, we get that $X$ is $m$-balanced.

4. **Calendula Graphs are Some Cycle-Supermagic**

In this section we show that a calendula graph $C_{m,n}$ for any positive integers $m$ and $n$ with $m \geq 3$ and $n \geq 3$ is $C_n$-supermagic.

**Theorem 2.** Let $m$ and $n$ be two integers with $m \geq 3$ and $n \geq 3$. Let $C_{m,n}$ be a calendula graph, then $C_{m,n}$ is $C_n$-supermagic.

**Proof.**
Let $C_n'$ be a subgraph of $C_{m,n}$ which isomorphic with $C_n$. Define a total labeling $\phi: V(C_{m,n}) \cup E(C_{m,n}) \rightarrow \{1, 2, \ldots, m(2n - 1)\}$ as follows.

(i) Let $m \neq n$. Let a multiset $X = [1,m] \cup [1,m(2n - 1)]$. Partition $X$ into several submultisets, $X_i$ with $i \in [1,m]$ based on the above Lemma 1. For $i \in [1,m]$, label $v_i'$ and $e_i'$ on $C_n'$ by using elements in $X_i$ and the smallest label to label the vertices such that every subgraph $C_n'$ on $C_{m,n}$ applies
\[
\phi(C_n') = 2mn^2 - 2mn + m + n.
\]

Therefore, for $m \neq n$, we obtain $\phi$ is an $C_n$-super magic labeling on $C_{m,n}$.

(ii) Let $m = n$. Since $C_{m,n}$ has $(n + 1)$ subgraphs $C_n'$, we need a modification labeling (i) such that every subgraph $C_n'$ has the same weight. We divide into two subcases.

(ii.a) Form $n \equiv 0 \mod 2$.

First, do the labeling as in (i). Furthermore, re-do the labeling on some edge $e_i'$ by swapping a pair of edge label $e'_j$ which are at the same $C_n'$ using the following way:
• exchange the label edge $e_i^1$ with $e_i^n$, for $i = 1$;
• exchange the label edge $e_i^1$ with $e_i^{n-1}$, for $n > 4$, $i \in \left[\frac{1}{2}n - 1\right]$ and $i \in \left[\frac{1}{2}n + 1\right]$;
• exchange the label edge $e_i^1$ with $e_i^{n-2}$, for $n > 4$, $i = \frac{1}{2}n$, and for $n = 4$, $i = 2$.

This re-labeling does not change the weight of $n$-subgraph $C_n'$ which is obtained on (i). Furthermore, it is obtained labelling of a new subgraph $C_n'$ with equal weight such that there are $(n + 1)$ subgraph $C_n'$ which has same weight on $C_{lm,n}$.

(ii.b) For $m = n \equiv 1 \mod 2$.

Do labeling as in (i). Furthermore, do re-labeling on some edge $e_i^j$ by swapping a pair of label edge $e_i^j$ which are at the same $C_n'$ in the following way:

• exchange the label edge $e_i^j$ with $e_i^{j(n+1)}$, for $i = 1, n$;
• exchange the label edge $e_i^j$ with $e_i^n$, for $i \in [2, (n - 1)]$.

Similarly to (ii.a), this re-labeling does not change the weight of $n$-subgraph $C_n'$ which is obtained in (i). Furthermore, it is obtained labeling a new subgraph $C_n'$ with equal weight such that there are $(n + 1)$ subgraph $C_n'$ which has same weight on $C_{lm,n}$ for $m = n \equiv 1 \mod 2$.

From (i), (ii.a), and (ii.b), we conclude that $C_{lm,n}$ is $C_n$-supermagic for any integers $m$ and $n$ with $m \geq 3$ and $n \geq 3$. □

For illustration, in figure 2, figure 3, and figure 4 we show cycle-supermagic labelings on calendula graphs $C_{6,4}, C_{4,4},$ and $C_{5,5}$, respectively.

![Figure 2](image_url)

**Figure 2.** $C_4$—supermagic labeling on $C_{6,4}$ graph.

In figure 2, it can be checked that the number of labels of each $C_4$ is constant. We obtain the weight of 6 subgraphs $C_4$ as follows.

- $\varphi(C_4^1) = 1 + 2 + 12 + 17 + 19 + 30 + 31 + 42 = 154$
- $\varphi(C_4^2) = 2 + 3 + 11 + 16 + 20 + 29 + 32 + 41 = 154$
- $\varphi(C_4^3) = 3 + 4 + 10 + 15 + 21 + 28 + 33 + 40 = 154$
- $\varphi(C_4^1) = 4 + 5 + 9 + 14 + 22 + 27 + 34 + 39 = 154$
- $\varphi(C_4^2) = 5 + 6 + 8 + 13 + 23 + 26 + 35 + 38 = 154$
- $\varphi(C_4^6) = 6 + 1 + 7 + 18 + 24 + 25 + 36 + 37 = 154$.

Figure 3. $C_4$– supermagic labeling on $C_{4,4}$ graph.

In figure 3, it can be checked that the number of labels of each $C_4$ is constant. We obtain the weight of 5 subgraphs $C_4$ as follows.
- $\varphi(C_4^1) = 1 + 2 + 8 + 11 + 13 + 20 + 21 + 28 = 104$
- $\varphi(C_4^2) = 2 + 3 + 7 + 10 + 14 + 19 + 22 + 27 = 104$
- $\varphi(C_4^3) = 3 + 4 + 6 + 9 + 15 + 18 + 23 + 26 = 104$
- $\varphi(C_4^4) = 4 + 1 + 5 + 12 + 16 + 17 + 24 + 25 = 104$
- $\varphi(C_4^5) = 1 + 2 + 3 + 4 + 19 + 23 + 24 + 28 = 104$

Figure 4. $C_5$–supermagic labeling of $C_{5,5}$ graph
In figure 4, it can be checked that the number of labels of each $C_5$ is constant. We obtain the weight of 6 subgraphs $C_4$ as follows.

- $\varphi(C_4^1) = 1 + 2 + 10 + 14 + 16 + 25 + 26 + 35 + 36 + 45 = 210$
- $\varphi(C_4^2) = 2 + 3 + 9 + 13 + 17 + 24 + 27 + 34 + 37 + 44 = 210$
- $\varphi(C_4^3) = 3 + 4 + 8 + 12 + 18 + 23 + 28 + 33 + 38 + 43 = 210$
- $\varphi(C_4^4) = 4 + 5 + 7 + 11 + 19 + 22 + 29 + 32 + 39 + 42 = 210$
- $\varphi(C_4^5) = 5 + 1 + 6 + 15 + 20 + 21 + 30 + 31 + 40 + 41 = 210$
- $\varphi(C_4^6) = 1 + 2 + 3 + 4 + 5 + 31 + 35 + 42 + 43 + 44 = 210$

5. References

[1] Gallian J A 2015 A Dynamic survey of graph labeling The Electronic J. Comb. 6
[2] Hartsfield N and Ringel G 2003 Pearls in Graph Theory (New York: Dover Publication, Inc.)
[3] Gutiérrez A and Lladó A 2005 Magic coverings J. Comb. Math. Comb. Com. 55 43-56
[4] Jordan J 2005 Comb graphs and spectral decimation, Glasg. Math. J. 51 71 – 81
[5] Lladó A and Moragas J 2007 Cycle-magic graphs Discrete Mathematics, 307 2925-33
[6] Marbun H T and Salman A N M 2013 Wheel-supermagic labelings for a wheel $k$-multilevel corona with a cycle, AKCE International J. Graphs and Comb. 10 1-9
[7] Maryati T K, Baskoro E T and Salman A N M 2008 $P_n$ super magic labeling of some trees J. Comb. Math. Comb. Com. 65 197-204
[8] Maryati T K, Salman A N M, Baskoro E T, Ryan J and Miller M 2010 On $H$-supermagic labelings for certain shackles and amalgamations of a connected graph, Utilitas Mathematica 83 333-342
[9] Maryati T K, Salman A N M and Baskoro E T 2013 Supermagic labelings of the disjoint union graphs and amalgamations Discrete Mathematics 313 397-405
[10] Ngurah A A G, Salman A N M and Sudarsana I W 2010 On supermagic coverings of fans and ladders SUT J. Math. 46 67-78
[11] Ngurah A A G, Salman A N M, Susilowati L 2010 $H$-supermagic labelings of graphs, Discrete Mathematics 310 1293-1300
[12] Salman A N M and Purnomo A D 2010 Some cycle-supermagic labelings of the of some complete bipartite graphs. East-West Journal of Mathematics 283-291
[13] Salman A N M, Ngurah A A G and Zlatan N 2010 On (super) edge-magic total labelings of a subdivision of a star $S_n$ Utilitas Mathematica 81 275-284
[14] Saputro S W, Mardiana and Purwasih N 2013 The metric dimension of comb product graphs, Proc. of Graph Theory Conf. in honor of Egawa’s 60th birthday pp 10-14
[15] Wallis W D 2001 Magic graphs (Birkhauser Boston, Berlin)