Electrostatics in Fractal Geometry: Fractional Calculus Approach

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Abstract

The electrostatics properties of composite materials with fractal geometry are studied in the framework of fractional calculus. An electric field in a composite dielectric with a fractal charge distribution is obtained in the spherical symmetry case. The method is based on the splitting of a composite volume into a fractal volume $V_d \sim r^d$ with the fractal dimension $d$ and a complementary host volume $V_h = V - V_d$. Integrations over these fractal volumes correspond to the convolution integrals that eventually lead to the employment of the fractional integro-differentiation.

Keywords: Fractional integral, Fractional derivative, Fractal geometry, Composite materials

1. Introduction

The electrostatics of composite materials is a ubiquitous and fundamental phenomenon with a variety of applications in physical, electro-chemical, geophysical, and biological systems, see e.g. [1, 2]. The investigation of the electric properties of composite materials has been a long-lasting task, starting, e.g., from a calculation of the permittivity of a dispersed mixture [3] and continuing with the modern investigations of nanosystems, for example, of an

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enhanced scattering of electric fields in nanosystems \[4, 5, 6\]. The concept of fractals and fractal geometry is widely employed in these studies, and leads to essential progress in our understanding of the physical properties of composites. A wide class of the physical applications of fractal structures has been reviewed well, see \textit{e.g.}, \[1, 2, 7, 8\]. In its turn, the fractal concept makes it possible to involve the application of the powerful methods of fractional integro-differentiation. An illuminating example of this relation is fractional diffusion on a comb \[9, 10, 11\], where the relation between the comb geometry and non-Markovian transport is straightforwardly established in the fractional Fokker-Planck equation \[10, 11, 12\]. As obtained in \[13\] (see also \[1\] chapter 39), the mathematical formulation of the transform of the Brown line-to-line function \(B(t)\) (continuous, but not differentiable) into fractional diffusion, described by fractional line-to-line function \(B_\alpha(t)\) with \(0 < \alpha < 2\), is expressed by means of the fractional integration  

\[B_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} dB(s) . \tag{1}\]

The concept of the integration and differentiation of non-integer orders already arises from the works of Leibniz, Liouville, and Riemann, see \textit{e.g.}, \[15, 16\]. Its application is related to random processes with power law distributions. This corresponds to the absence of characteristic average values for processes that exhibit many scales \[17, 18\].

By analogy with Eq. (1), a link between fractal geometry and fractional integro-differentiation \[19\] is constituted in the procedure of averaging an extensive physical value that is expressed by means of a smooth function over a Cantor set, which leads to fractional integration. In its eventual form, it has been presented in Ref. \[20\]. The main idea of filtering or embedding a matter inside a fractal is the construction of a convolution integral  

\[F(x) = W(x) * f(x) = \int_{0}^{x} W(x-y)f(y)dy , \tag{2}\]

\footnote{Note that fractional Brown motion was originally introduced by Kolmogorov in \[14\] and was later rediscovered and explored in greater detail by Mandelbrot and Van Ness in \[13\].}
where the function $W(x)$ obeys the scaling relation $W(x) = \frac{1}{a}W(bx)$ with a solution $W(x) = x^\alpha A(x)$. Here $\alpha = \frac{\ln a}{\ln b}$, and $A(x) = A(bx)$ is a log-periodic function with a period $\ln b$. When this scaling corresponds to a Cantor set with $a = 2$, and $A(x)$ is defined explicitly, one performs averaging over this period and obtains Eq. (2) in the form of a fractional integral

$$\langle F(x) \rangle = \frac{\mathcal{V}(\alpha)}{\Gamma(\alpha)} \left( x^{\alpha-1} \ast f(x) \right) \equiv \mathcal{V}(\alpha) I_x^\alpha f(x),$$

(3)

where $\mathcal{V}(\alpha) = \frac{2^{-1+\alpha/2}}{\ln 2}$ and $\Gamma(\alpha)$ is a gamma function, while $I_x^\alpha$ designates the fractional integral of the order of $\alpha$ in the range from 0 to $x$:

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - x')^{\alpha-1} f(x') dx'.$$

(4)

Analogously one can define the fractional derivative as the inverse operation:

$$D_x^{-\alpha} = I_x^{-\alpha}.\quad \text{Extended reviews on fractional calculus can be found in } \cite{15, 16, 21}, \text{ for example. See Appendix A.}$$

This mathematical construction is relevant to the study of electrostatics of real composite structures. The main objective of the present research is to solve a standard electrostatic problem, namely, the derivation of the electric field in a composite dielectric with a fractal charge distribution. This problem was first considered in $\cite{22}$, in a slightly different mathematical formulation, where integration was carried out over the fractal with fractal dimension $\alpha$ and infinitesimal fractal volume $dr^\alpha$ inside a homogeneous dielectric.

2. Fractional Distribution of Charges

First, we demonstrate this fractal property of embedding a matter for electrostatics by implementing it in Maxwell’s integral equation for the electric field $E$. We formulate the problem in terms of a characteristic function. Let any charge distribution on a fractal be $F_\alpha(r)$, where $r = (x, y, z)$ is described by the characteristic function $\chi(r)$, such that $\chi(r) = 1$ for $r \in F_\alpha(r)$ and $\chi(r) = 0$ for $r \notin F_\alpha(r)$. Therefore, the fractal filtering of Eq. (3) now reads

$$\int_0^x \chi(x') f(x') dx' \Rightarrow \frac{1}{\Gamma(\alpha)} \int_0^x (x - x')^{\alpha-1} f(x') dx' \equiv I_x^\alpha f(x).$$

(5)
We obtain the same integration for the $y$ and $z$ coordinates.

The Gauss theorem in Cartesian’s coordinates reads

$$
\int \nabla \cdot \mathbf{E} \, dx \, dy \, dz \equiv 0
$$

where an electric charge $Q$ with a fractional distribution is embedded in a three-dimensional volume. Without restricting the generality, we consider the fractal charge in the first octant of the Euclidean space. Therefore, we have from Eq. (5)

$$
Q \equiv Q(x, y, z) = a I^\alpha_x I^\alpha_y I^\alpha_z \rho(x, y, z)
$$

where $\rho(x, y, z)$ is the charge density, and the constant multiplier $\mathcal{V}(\alpha)$ is inserted in $\rho$; for an alternate definition see also \cite{22}. The charge in Eq. (7) corresponds to a fractal structure that is the direct product of the Cantor sets defined in Eq. (3), namely $\langle F_\alpha(x) \rangle \times \langle F_\alpha(y) \rangle \times \langle F_\alpha(z) \rangle$.

To proceed, we obtain the correct dimensions of both the charge and the electric field, respectively. To this end it is convenient to work in the dimensionless space variables $x_l \to x, \ y_l \to y, \ z_l \to z$, and the properly defined fractal charge reads from Eq. (7)

$$
Q(x, y, z) = a I^\alpha_x l_0^3 I^\alpha_y I^\alpha_z \rho(x, y, z).
$$

The conventional integral in Eq. (6) yields the dimensionless integration $a I^\alpha_x I^\alpha_y I^\alpha_z \to I_0^3 I^\alpha_x I^\alpha_y I^\alpha_z$ and dimensionless differentiation $\nabla \to \frac{1}{l_0^3} \nabla$. Here $l_0$ is the characteristic size of a physical fractal, which is always finite. For example, it can be the minimal scale of self-similarity of a physical fractal. In what follows we always bear in mind that the space variables are dimensionless, and the charge density is properly scaled such that fractal charges and electric fields have correct dimensions.

When the charge density is a constant function, the fractional convolution integrals become a simple integration over a dimensionless volume $(xyz)^{\alpha}/\Gamma^3(\alpha + 1)$. Comparison of Eqs. (6) and (7) yields Maxwell’s equation in dimensionless space coordinates

$$
\nabla \cdot \mathbf{E} = \frac{\partial^3 Q}{\partial x \partial y \partial z} = \frac{(xyz)^{\alpha-1}}{\Gamma^3(\alpha)} l_0 \rho.
$$
It has a solution
\[
E = \frac{l_0 \rho}{\alpha \Gamma_\alpha(\alpha)} \left[ \hat{x} x^\alpha (yz)^{\alpha - 1} + \hat{y} y^\alpha (xz)^{\alpha - 1} + \hat{z} z^\alpha (xy)^{\alpha - 1} \right].
\] (10)

When the charge density \( \rho(x, y, z) \) is not a constant, the integrands in Eqs. (6) and (7) cannot be straightforwardly compared to each other. Therefore, we apply the fractional differential \( 0 D_x^\alpha \) to both sides of Eq. (6). Taking into account that
\[
0 D_x^\alpha 0 I_\alpha f(x) = f(x) \quad \text{and} \quad 0 D_x^\alpha 0 I_\beta f(x) = 0 D_x^{\alpha - \beta} f(x) \quad \text{(see Appendix, Eq. (A.6))},
\]
one obtains Maxwell’s equation in form either
\[
0 I_1^{1-\alpha} 0 D_y^{1-\alpha} 0 D_z^{1-\alpha} \nabla \cdot E(x, y, z) = \rho(x, y, z),
\] (11)
or
\[
\nabla \cdot E(x, y, z) = 0 D_x^{1-\alpha} 0 D_y^{1-\alpha} 0 D_z^{1-\alpha} \rho(x, y, z).
\] (12)

For a constant \( \rho \) it reduces to Eq. (9).

3. Example: Electric Field of an Infinite Fractal String

Now we consider an example where the filtering by means of the convolution integral in Eqs. (3) and (4) can be considered as a superposition of electric fields, that is the fractal filtering of the electric field. Let us find an electric field of a charged fractal string with the fractal dimension \( 0 < \nu < 1 \) and a constant linear charge density \( \rho_1 \). We use cylindrical symmetry and describe the problem in terms of the string coordinate \( z \) and the distance from the string \( r \). The \( E_r \) component of the electric field at point \((r, z)\) of an infinitesimally small charge with coordinate \( z' \) is
\[
dE_r(r, z) = \frac{\rho_1 rz}{l_0 \sqrt{r^2 + (z - z')^2}} dz',
\]
where we used that the linear charge density is scaled \( \frac{\rho_1}{l_0} \). Therefore, the complete electric field at this point is a superposition of the charges along the fractal string. This yields the fractional integration
\[
E_r(r, z) = \frac{r \rho_1}{l_0 \Gamma(\nu)} \left[ \int_{-\infty}^{z} \frac{(z - z')^{\nu - 1} dz'}{[r^2 + (z - z')^2]^{\frac{\nu}{2}}} + \int_{z}^{\infty} \frac{(z' - z)^{\nu - 1} dz'}{[r^2 + (z' - z)^2]^{\frac{\nu}{2}}} \right].
\] (13)

Performing the variable change \( y = (z - z')/r \), one obtains for the electric field
\[
E_r(r, z) \equiv E_r(r) = \frac{2 A(\nu) \rho_1}{l_0 r^{2-\nu}}.
\] (14)
where \( A(\nu) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{y^{\nu-1}dy}{[1+y^2]^{\nu/2}} \). When \( \nu = 1 \), this yields a well known result for the electric field of an infinitely long line charge. In dimension variables the electric field reads \( E_r(r) = 2A(\nu)\left(\frac{\nu}{r}\right)^{1-\nu} \), i.e., the electric field of a fractal string decreases with the distance \( r \) \( (r/l_0)^{1-\nu} \) times faster than one of the continuous counterparts.

4. A Case of Spherical Symmetry

Now we obtain an expression for the fractal charge in spherical coordinates. We define a charge on a random fractal, which has a homogeneous distribution in a three-dimensional medium with charge density \( \rho_0 \). A fractal mass inside a ball of radius \( r \) is \( \mathcal{M}(r) \sim r^d \). As shown in Sec. 2, the convolution in Eq. (7) determines the map of the charge into the fractal volume. In Cartesian’s coordinates it corresponds to the fractional Riemann-Liouville integral with elementary fractional volume \( dVd = \frac{\Gamma(3/2)}{\Gamma(d/2)} |x||y||z|^{d-3/2}dx dy dz \).

In the spherical coordinates, which correspond to the Reisz definition of the fractional integral, the elementary fractional volume is

\[
dVd = \frac{2^{3-d} \Gamma(3/2)}{\Gamma(d/2)} |r|^{d-3}r^2 d\theta d\phi .
\]

In the spherically symmetrical case, the convolution integral (7) takes a simple form. Hence, the random fractal can be considered as a direct product of the two-dimensional sphere \( S_2 \) and a Cantor set \( F_\alpha \) of the dimension \( \alpha \), such that the fractal dimension of \( S_2 \otimes F_\alpha \) is \( d = 2 + \alpha \). Therefore, the mapping of the charge inside the fractal is determined by the convolution

\[
Q(r) = \frac{1}{\Gamma(\alpha)} \int_0^r (r-r')^{\alpha-1} \rho_0(r') r' dr',
\]

where all constants are taken inside the fractional charge density \( \rho \). In what follows the charge density \( \rho_0 \) is a constant value. This choice immediately yields an expression for the charge from Eq. (15)

\[
Q(r) = \frac{2\rho_0}{\Gamma(\alpha+3)} r^d ,
\]
that correlates with the fractional mass $M(r)$. This also yields the average fractal charge density of the entire composite

$$\rho(r) \sim \frac{3Q(r)}{4\pi r^3} \sim \rho_0 r^{d-3}. \quad (17)$$

This expression yields the correct result for the electric field inside the fractal dielectric

$$E(R) = \int \frac{\rho(R)dV}{R^2} \sim \rho_0 R^{d-2} \quad (18)$$

when $R < r$. Note also that for the electric field outside the charged fractal, when $R > r$, we have for the average fractal charge density $\rho(R) \rightarrow \rho(r')\Theta(r' - r)$. This yields for the electric field $E(R) = \frac{Q(r)}{4\pi R^2}$. Here $\Theta(z)$ is the Heaviside step function.

5. Electric Field in a Fractal Dielectric Composite

The situation with respect to permittivity is different and needs special care. Let us consider the Maxwell equation inside a volume that contains a fractal dielectric. We take into account that the electric field is in a random fractal of a volume $V_d \sim r^d$ with permittivity $\varepsilon_1$ and in a free space of a complementary host volume $V_h = V_3 - V_d$ with permittivity $\varepsilon_2$. We suppose that the random fractal is a direct product of the two-dimensional sphere $S_2$ and a Cantor set $F_\alpha$ of the dimension $\alpha$, such that the fractal dimension of $S_2 \otimes F_\alpha$ is $d = 2 + \alpha$. We also suppose the spherical symmetry, $\nabla_\theta E_\theta = \nabla_\phi E_\phi = 0$. The electric displacement $D(r) = \varepsilon(r)E(r)$ is not differentiable, since the permittivity is a discontinuous function on the fractal:

$$\varepsilon(r) = \begin{cases} 
\varepsilon_1, & \text{if } r \in V_d \\
\varepsilon_2, & \text{if } r \in V_h. 
\end{cases} \quad (19)$$

The permittivity of a two-phase composite in Eq. (19) can be defined by the characteristic function $\varepsilon(r) = \varepsilon_1 \chi(r) + \varepsilon_2 [1 - \chi(r)]$. The Maxwell equation for the $\hat{r}$ component reads

$$(\varepsilon_1 - \varepsilon_2)\chi(r)\nabla_r E_r + \varepsilon_2 \nabla_r E_r + (\varepsilon_1 - \varepsilon_2)E_r \nabla_r \chi(r) = \rho_0. \quad (20)$$
The last term in the l.h.s. corresponds to a charge, which is due to the polarization of the fractal and to the discontinuity of the electric field on the fractal interface \[3\]. Now we use the Gauss theorem, which for the chosen geometry reads

\[
Q(r) = 4\pi \int_0^r \nabla_r D_r(r') r'' dr' \equiv 4\pi a I_1^\alpha r^2 \nabla_r D_r(r). \tag{21}
\]

Taking into account that \(a I_1^\alpha \chi(r) r^2 f(r) \to a I_1^\alpha r^2 f(r)\), we obtain from Eqs. \[20\] and \[21\]

\[
\frac{1}{4\pi} Q(r) = (\varepsilon_1 - \varepsilon_2) a I_1^\alpha r^2 \nabla_r E_r(r) + \varepsilon_2 a I_1^\alpha r^2 \nabla_r E_r(r) + (\varepsilon_1 - \varepsilon_2) a I_1^\alpha E_r(r) r^2 \nabla_r \chi(r). \tag{22}
\]

The last term in Eq. \[22\] can be written in the form (see Appendix B)

\(a I_1^\alpha E_r(r) r^2\). Finally, taking into account Eq. \[10\], we obtain the integral Maxwell’s equation for the electric field in the form

\[
(\varepsilon_1 - \varepsilon_2) a I_1^\alpha r^2 \nabla_r E_r(r) + \varepsilon_2 a I_1^\alpha r^2 \nabla_r E_r(r) + \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon_2} a I_1^\alpha E_r(r) r^2 = \frac{2\rho_0}{\Gamma(\alpha + 3)} r^{2+\alpha}. \tag{23}
\]

We take into account that \(r^2 \nabla_r E_r(r) = \partial_r [r^2 E_r(r)] \equiv \partial_r G(r)\), where we introduce a new function, \(G(r) = r^2 E_r(r)\). We also introduce a dimensionless parameter

\[
\epsilon = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2}. \tag{24}
\]

Applying the fractional derivative \(a D_r^\alpha\) to both sides of Eq. \[22\] and taking into account that \(a D_r^\alpha = \frac{\Gamma(p+1)}{\Gamma(p+1-q)} r^{q-d}\) one obtains

\[
\epsilon \partial_r G(r) + a I_1^{1-\alpha} \partial_r G(r) + \varepsilon_2^2 G(r) = \frac{r^2 \rho_0}{\varepsilon_2}. \tag{25}
\]

Here

\[
a I_1^{1-\alpha} \partial_r G(r) = \frac{1}{\Gamma(1-\alpha)} \int_0^r (r - r')^{-\alpha} \partial_r G(r') dr'. \tag{26}
\]

It is worth noting that for a homogeneous medium with a fractal distribution of charges, when \(\varepsilon_1 = \varepsilon_2\), Eq. \[25\] reduces to Eq. \[10\] in the spherical symmetry.

5.1. Solution by Expansion

Eq. \[25\] is the linear fractional integro-differential equation that can be solved by the Laplace transform. Defining the Laplace transform \(\mathcal{L}[G(r)] = \)
\( \tilde{G}(\lambda) \), one obtains the solution of Eq. (26) in the form

\[
\tilde{G}(\lambda) = \frac{2 \rho_0}{\varepsilon_2 \lambda^3} \cdot \frac{1}{[\lambda^\alpha + \varepsilon \lambda + \varepsilon^2]}. \tag{27}
\]

Expanding the denominator near \( \varepsilon \lambda \), one obtains

\[
\tilde{G}(\lambda) = \frac{2 \rho_0}{\varepsilon_2 \lambda} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{m=0}^{n} C_n^m \frac{\varepsilon^{2m}}{\lambda^{n+m+(n-m)\alpha}}, \tag{28}
\]

where \( C_n^m = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)} \). Carrying out the inverse Laplace transform, one obtains for the electric field

\[
E_r(r) = \frac{2 \rho_0 r}{\varepsilon_2 \lambda} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{m=0}^{n} C_n^m \frac{\varepsilon^{2m} r^{n-(n-m)\alpha}}{\Gamma[n(1-\alpha) + m\alpha + 4]}. \tag{29}
\]

When \( \varepsilon \ll 1 \), one takes only \( m = 0 \) in the second summation. This yields

\[
E_r(r) \approx \frac{2 \rho_0 r}{\varepsilon_2 \lambda} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{r^{n-(1-\alpha)/\varepsilon}}{\Gamma[n(1-\alpha) + 4]}. \tag{30}
\]

The sum in Eq. (30) is the definition of the Mittag-Leffler function (see e.g., [15, 17])

\[
\mathcal{E}_{1-\alpha,4}(\frac{-r^{1-\alpha}}{\varepsilon}) = \sum_{n=0}^{\infty} \frac{[\frac{-r^{1-\alpha}}{\varepsilon}]^n}{\Gamma[n(1-\alpha) + 4]}.
\tag{31}
\]

In the limit \( \varepsilon \to 0 \), the asymptotic behavior of the Mittag-Leffler function for \( r^{1-\alpha}/\varepsilon \gg 1 \) has a power law decay (see e.g., [15, 25])

\[
\mathcal{E}_{\alpha,\beta}(\frac{-cz^\alpha}{\varepsilon}) \sim \frac{1}{\Gamma(\beta - \alpha)cz^\alpha},
\tag{32}
\]

and we arrive at the solution (18) (see also [22])

\[
E_r(r) = \frac{\rho_0}{\varepsilon_2 \Gamma(\alpha + 3)} r^\alpha. \tag{33}
\]

Since the charge is contained inside the fractal, we also used here that \( \varepsilon \geq 0 \).

This condition also ensures the transition to a homogeneous dielectric with \( \varepsilon_1 = \varepsilon_2 \) and is valid for \( \alpha = 1 \), as well.

In the opposite case when \( \frac{r^{1-n}}{\varepsilon} \ll 1 \), the solution for the electric field reads

\[
E_r(r) \sim \frac{2r \rho_0}{\varepsilon_2 \varepsilon}. \tag{34}
\]
5.2. Approximations for $\epsilon \ll 1$ and $\epsilon \gg 1$

Note that the expansion (28) is not always valid. Therefore, it is convenient to use an approximation inside the denominator in Eq. (27). In the limiting cases $\epsilon \ll 1$ and $\epsilon \gg 1$ it can be simplified. In these cases one can look for the solutions using an expression for the Laplace transform for the Mittag-Leffler function for two parameters $E_{(\alpha,\beta)}(-az^\alpha)$. It reads

$$
\int_0^\infty e^{-\lambda z} z^{\beta-1} E_{(\nu,\beta)}(\mp az^\nu) \, dz = \frac{\lambda^{\nu-\beta}}{(\lambda^{\nu} \pm a)} , \quad (\text{Re}(\nu) > |a|^{1/\nu}) . \quad (34)
$$

We start by considering $\epsilon \ll 1$. Neglecting $\epsilon^2$, we have

$$
\tilde{G}(\lambda) \approx \frac{2\rho_0}{\epsilon^2 \lambda^{\beta+1}} \cdot \frac{1}{|\lambda^{\beta} + 1/\epsilon|} . \quad (35)
$$

Comparing Eqs. (34) and (35) one obtains

$$
E_r(r) \approx \frac{2\rho_0}{\epsilon^2} r E_{(1,4)}(\mp 1-\alpha) \left( -\frac{r^{1-\alpha}}{\epsilon} \right) , \quad (36)
$$

which yields the same expression for the electric field as in Eq. (30).

In the limit $\epsilon \gg 1$, we neglect $\lambda^\alpha$ in the denominator in Eq. (27), which yields the following solution for the electric field

$$
E_r(r) \approx \frac{2\rho_0}{\epsilon^2} r E_{(1,4)}(-\epsilon r) . \quad (37)
$$

Note also that the dimensionless radius can be both $r \gg 1/\epsilon$ and $r \ll 1/\epsilon$. Therefore, for $\epsilon r \gg 1$ we use the asymptotic behavior of the Mittag-Leffler function (32) and obtain the asymptotic behavior of the electric field $E_r(r) \approx \frac{e \rho_0}{\epsilon^2 r^{1/\epsilon}}$. In the opposite case, $\epsilon r \ll 1$, we have $E_{(1,\alpha,4)} \sim 1$, which yields $E_r(r) \approx \frac{2\rho_0}{\epsilon^2} r$.

5.3. Direct Product of Cantor Sets

In the preceding section, the random fractal was considered as a direct product of a two dimensional sphere $S_2$ and a random fractal $F_\alpha$ of the dimension $\alpha$, such that the fractal dimension of $S_2 \otimes F_\alpha$ is $d = 2 + \alpha$. It is worth noting that this is only one of many possible constructions of a random fractal [24]. Therefore, it is instructive to consider a more general example. Let us consider a
fractal dielectric as a product of the random Cantor sets $F_α(x) \times F_β(y) \times F_γ(z)$, such that the electric charge due to Eq. (7) is

$$Q(x, y, z) = \frac{\rho_0 x^α y^β z^γ}{\Gamma(1 + α)\Gamma(1 + β)\Gamma(1 + γ)}, \quad x, y, z \geq 0$$

where $α + β + γ = d$ is the fractal dimension $d < 3$. We simplify the analysis by neglecting the effect of polarization of the composite. This simplification is valid only in the case when $ε ≪ 1$. As shown, the polarization charge is of the order of $ε^2$, and it can be neglected in the Maxwell equation. Therefore, the integral Maxwell’s equation reads now

$$Q(x, y, z) = (ε_1 - ε_2) I_0 \epsilon^0 I_0^β I_2^γ \nabla \cdot E(x, y, z) + ε_2 I_0 I_0^1 I_1^γ \nabla \cdot E(x, y, z). \quad (38)$$

Applying the fractional derivatives $D_0^α, D_0^β, D_0^γ$ to both sides of the equation and denoting $F(x, y, z) = \nabla \cdot E(x, y, z)$, one obtains the Abel equation $\quad (21, 26)$

$$\frac{ρ_0}{ε_2} = εF(x, y, z) + F_0 - α x^1 - α y^1 - β z^1 - γ F(x, y, z). \quad (39)$$

Here $F(x, y, z)$ is an effective charge density. We solve this equation by a method of consequent approximations $\quad (26)$. The first step is $F_1(x, y, z) = \frac{ρ_0}{ε_2} + F_1(x, y, z)$, which yields the following equation for $F_1(x, y, z)$

$$0 = εF_1(x, y, z) + F_1(x, y, z) + F_1(x, y, z) + α x^1 - α y^1 - β z^1 - γ F_1(x, y, z) + \frac{ρ_0}{ε_2} \epsilon^0 \Gamma(2 - α)\Gamma(2 - β)\Gamma(2 - γ) \quad (40)$$

Then, defining $F_1(x, y, z) = -\frac{ρ_0}{ε_2} \epsilon^0 \frac{x^1 - α y^1 - β z^1 - γ}{\Gamma(2 - α)\Gamma(2 - β)\Gamma(2 - γ)} + F_2(x, y, z)$ and repeating the same procedure for finding $F_n(x, y, z)$, we have the solution for the effective charge density in the form of the sum

$$F(x, y, z) = \frac{ρ_0}{ε_1 - ε_2} \sum_{n=0}^{∞} F_n(x, y, z), \quad (41)$$

where

$$F_n(x, y, z) = \frac{[-β x^1 - α y^1 - β z^1 - γ/(ε)]^n}{\Gamma[1 + n(1 - α)]\Gamma[1 + n(1 - β)]\Gamma[1 + n(1 - γ)]}, \quad (42)$$

that also yields a solution for the electric field in the form of a vector combination of $F_n(x, y, z)$ functions.
There is an essential simplification of the solutions in the planes \( x = 0 \), and \( y = 0 \), and \( z = 0 \). One has for these planes

\[
\nabla \cdot \mathbf{E}(x, y, z) = \frac{\rho_0}{\varepsilon_1 - \varepsilon_2}.
\]

(42)

6. Conclusion

We demonstrated an application of fractional calculus for electrostatics of composite materials in fractal geometry. The method is based on fractional filtering of Maxwell’s equation in the framework of the Gauss theorem by means of a convolution integral in Eq. (2). One should recognize that the coarse graining procedure due to the averaging over the period \( \ln b \) is important for the application of fractional calculus. We explore the result of Ref. [20] with subsequent development by splitting the composite volume into the fractal volume \( V_d \sim r^d \) with fractal dimension \( d \) and the complementary host volume \( V_h \). Integrations over these fractal volumes correspond to the convolution integrals and, eventually, to fractional integro-differentiation. Note that this filtering procedure, in our case, is the averaging of the Maxwell equation, and not of the electric field. The latter is the solution of the obtained averaged equation. It should be noted also that the fractal dimension alone is not enough to describe a fractal completely. Therefore, although in the analysis above we discuss the Cantor sets, the results are valid for any random fractal distribution of charges, since we use the fractal dimension only.

In this connection, we should also admit that embedding the electric charge inside any random fractal, as in Eqs. (5) and (7), is based on employing the characteristic function \( \chi \) inside integrands. This integration corresponds to the convolution integral, as the result of coarse graining. The situation is the same for embedding the polarization charge inside a fractal composite. Although, instead of the characteristic function, we have its gradient \( \nabla \chi \), as shown in Appendix B, this leads to the convolution integral (B.6), as well.

One should recognize that it is also possible that the permittivity of the charged fractal dielectric is less than the permittivity of the host dielectric,
\( \epsilon < 0 \), for example, for biological macromolecules and polymers \[27\]. In this case the argument of the Mittag-Leffler function is positive, and the asymptotic behavior of the electric field is absolutely different. One finds easily that the limit \( \epsilon \to -0 \) does not correspond to the homogeneous case with \( \varepsilon_1 = \varepsilon_2 \). Obviously \( \epsilon = 0 \) is a singular point of Eq. \((23)\), since \( \epsilon \) is the parameter at the higher derivative, and solution \((36)\) is singular at \( \epsilon \to -0 \) as well. This case needs separate consideration.

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Appendix A: Fractional Integration

Extended reviews of fractional calculus can be found e.g., in \[15, 16, 21\]. Fractional integration of the order of \( \alpha \) is defined by the operator

\[
a I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(y)(x-y)^{\alpha-1}dy, \tag{A. 1}\]

where \( \alpha > 0, x > a \) and the Gamma function \( \Gamma(z) \) is defined above. Fractional derivation was developed as a generalization of integer order derivatives and is defined as the inverse operation to the fractional integral. Therefore, the fractional derivative is defined as the inverse operator to \( a I_x^\alpha \), namely \( a D_x^\alpha f(x) = a I_x^{-\alpha} f(x) \) and \( a I_x^\alpha = a D_x^{-\alpha} \). Its explicit form is

\[
a D_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_a^x f(y)(x-y)^{-1-\alpha}dy. \tag{A. 2}\]

For arbitrary \( \alpha > 0 \) this integral diverges, and as a result of this a regularization procedure is introduced with two alternative definitions of \( a D_x^\alpha \). For an integer \( n \) defined as \( n - 1 < \alpha < n \), one obtains the Riemann-Liouville fractional derivative of the form

\[
a R L D_x^\alpha f(x) \equiv a D_x^\alpha f(x) = \frac{d^n}{dx^n} a I_x^{n-\alpha} f(x), \tag{A. 3}\]

and fractional derivative in the Caputo form

\[
a C D_x^\alpha f(x) = a I_x^{n-\alpha} \frac{d^n}{dx^n} f(x). \tag{A. 4}\]
There is no constraint on the lower limit $a$. For example, when $a = 0$, one has
\[ RL_{x \rightarrow 0} x^\beta = \frac{x^{\beta - \alpha} \Gamma(\beta + 1)}{\Gamma(\beta + 1 - \alpha)}. \]
This fractional derivation with the fixed low limit is also called the left fractional derivative. However, one can introduce the right fractional derivative, where the upper limit $a$ is fixed and $a > x$. For example, the right fractional integral is
\[ \int_a^x f(y) \, dy = \frac{1}{\Gamma(\alpha)} \int_x^a (y - x)^{\alpha - 1} f(y) \, dy. \] (A. 5)

Another important property is
\[ D^\alpha D^\beta = D^{\alpha + \beta} \quad \text{and} \quad D^\alpha I^\beta = I^{\beta - \alpha}, \] (A. 6)
where other indexes are omitted for brevity. We also use here a convolution rule for the Laplace transform for $0 < \alpha < 1$
\[ \mathcal{L}[I_a^\alpha f(x)] = s^{-\alpha} \hat{f}(s). \] (A. 7)

Note that in physical applications a treatment of the Caputo fractional derivative by the Laplace transform is more convenient than the Riemann-Liouville one. In the first case, one has to define $0 \leq n < \alpha$ integer derivative at the initial point, while for the second case one defines $n$ Riemann-Liouville fractional derivative as initial conditions. Nevertheless, the use of either Caputo or Riemann-Liouville fractional derivatives is equivalent for initial value problems and leads to the same results [17, 28].

Appendix B

The last term in the l.h.s. of Eq. (22) corresponds to the induced charge, or polarization charge on the fractal, since the electric field is a discontinuous function [3]. We also note that, while the notation $\nabla_r$ is the divergence in the Gauss theorem $\nabla_r = \frac{1}{r^2} \frac{\partial}{\partial r} r^2$, it is the gradient in the last term $\nabla_r \chi(r) = \frac{\partial \chi(r)}{\partial r}$.

Let us consider the $N$th step of the fractal construction. It is a union of disjoint intervals $\Delta_N$. In the limiting case one obtains $F_\alpha = \lim_{N \to \infty} \bigcup \Delta_N$. Therefore, $\forall r_j \in F_\alpha$ the characteristic function on every closed interval $[r_j, r_j + \Delta_N]$, is
\( \chi(\Delta_N) = \Theta(r - r_j) - \Theta(r - r_j - \Delta_N) \). Differentiation of the characteristic function on the intervals yields

\[
\frac{\partial}{\partial r} \chi(\Delta_N) = \delta(r - r_j) - \delta(r - r_j - \Delta_N).
\]  

(B. 1)

Therefore, we have for any interval \( \Delta_N \) and at \( r = r_j \)

\[
r_I^1_{r+\Delta_N} E(r)r^2 \nabla_r \chi(r) = E(r)r^2 - E(r + \Delta_N)(r + \Delta_N)^2.
\]  

(B. 2)

This expression is not zero in the limit \( \Delta_N \to 0 \), since \( \varepsilon_2 E(r_j - 0) = \varepsilon_1 E(r_j + 0) \) and \( \varepsilon_1 E(r + \Delta_N - 0) = \varepsilon_2 E(r + \Delta_N + 0) \) [3]. For brevity’s sake the index \( r \) for the electric field is omitted \( E_r(r) \equiv E(r) \). We make a shift between points \( r_j - 0 \) and \( r_j + \Delta_n - 0 \). We present the result of integration in Eq. (B. 2) in the form

\[
r^2[E(r - 0) - E(r + \Delta_N - 0)] + O(\Delta_N) = \]

\[
r^2[E(r - 0) - E(r + 0) + E(r + 0) - E(r + \Delta_N - 0)] + O(\Delta_N).
\]

Since inside the interval \( \Delta_N \) the electric field is continuous, we have \( E(r + \Delta_N - 0) \approx E(r + 0) + E'(r + 0)\Delta_N \). Now the boundary conditions for the electric field at \( r_j \) can be taken into account: \( \varepsilon_2 E(r_j - 0) = \varepsilon_1 E(r_j + 0) \). This, finally, yields

\[
\lim_{\Delta_N \to 0} r_I^1_{r+\Delta_N} E(r)r^2 \nabla_r \chi(r) = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2} E(r) \quad \forall r \in F_\alpha.
\]  

(B. 3)

We have from Eq. (20) that \( \alpha I^1_r E_r(r)r^2 \nabla_r \chi(r) \) is a superposition of random values, namely

\[
\alpha I^1_r E_r(r)r^2 \nabla_r \chi(r) = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2} \sum_{r_j \in F_\alpha} r_j^2 E_r(r_j).
\]  

(B. 4)

This can be approximately considered as an “average” value \( \alpha I^1_r E_r(r)r^2 \to \alpha I^\alpha_r E_r(r)r^2 \). Indeed, let us present the sum in the form

\[
\sum_{r_j \in F_\alpha} r_j^2 E_r(r_j) = \sum_{r_j \in F_\alpha} \int_0^r r'^2 E_r(r') \delta(r' - r_j).
\]  

(B. 5)

Hence, one obtains the integration of the electric field with the fractal density \( \sum_{r_j \in F_\alpha} \delta(r' - r_j) \). This corresponds to the fractal volume \( r^\alpha \). The next step
is to consider the integration in Eq. (B. 5) as the convolution integral with the kernel \((r - r')^{\alpha - 1}\). Eventually, we obtain for the polarization charge term in Eq. (22)

\[
0 \int E_r(r) r^2 \nabla_r \chi(r) \to \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2} 0 \int r^\alpha E_r(r) r^2 .
\]

(B. 6)

Finally, we note that the shift in the intervals \(\Delta_N\) is taken between point \(r_j - 0\) and \(r_j + \Delta_N\). One can also make the shift between points \(r_j + 0\) and \(r_j + \Delta_N + 0\). The difference between electric fields at the points yields \([1 - \frac{\varepsilon_1}{\varepsilon_2}] E(r_j)\). Taking into account that this difference is opposite to the external normal, we change the sign and obtain the same expression as in Eq. (B. 3).

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