Inflation and inhomogeneities: a hybrid quantization

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Abstract. We provide a complete quantization of a homogeneous and isotropic spacetime with positive spatial curvature coupled to a massive scalar field in the framework of Loop Quantum Cosmology. The physical Hilbert space is constructed out of the space of initial data on the minimum volume section. By means of a perturbative treatment we introduce inhomogeneities and thereafter we adopt a hybrid quantum approach, in which these inhomogeneous degrees of freedom are described by a standard Fock quantization. For the considered case of compact spatial topology, the requirements of: i) invariance of the vacuum state under the spatial isometries, and ii) unitary implementation of the quantum dynamics, pick up a privileged set of canonical fields and a unique Fock representation (up to unitary equivalence).

1. Introduction
Loop Quantum Cosmology (LQC) [1] is an approach to quantize symmetry reduced systems in cosmology which has provided new insights into the early universe physics. In particular, the analysis of a homogeneous and isotropic Friedmann-Robertson-Walker (FRW) spacetime coupled to a massless scalar field has been carried out to completion, showing that the classical singularity is replaced with a quantum bounce. Besides, less symmetric models, such as anisotropic homogeneous spacetimes [2], have been successfully quantized.

To discuss the inflationary paradigm [3] that naturally explains the present state of our universe, one can add a potential for the scalar field, like in the simple case of a mass term. A careful analysis of the effective dynamics expected to correspond to this model [4] demonstrates the high probability that the inflaton produces the necessary number of \(e\)-foldings, solving the fine-tuning problem that appears in General Relativity. However, we still lack a rigorous derivation of such an effective dynamics from a complete quantization within LQC of a massive scalar field in a FRW spacetime. On the other hand, an even more realistic scenario is achieved by introducing inhomogeneities coming from the vacuum fluctuations of the inflaton. The application of perturbative methods, together with a standard Fock representation for the inhomogeneities, should satisfactorily describe the origin of the structures currently observed. Nevertheless, in general, one has available an infinite number of inequivalent Fock quantizations for fields in FRW spacetimes, resulting in a loss of prediction capability unless one can justify the selection of a preferred Fock representation.

In this work, we will consider a closed FRW spacetime coupled to a massive scalar field. We will provide a polymeric quantization of this model, in which any solution to the Hamiltonian constraint is totally determined by the data on the minimum (nonvanishing) volume section. The physical Hilbert space can be identified with such initial data space once it has been completed.
with a suitable inner product, which is uniquely determined by requiring the self-adjointness of a complete set of real observables. Moreover, we will introduce small inhomogeneities around homogeneous solutions. After linearization of the system and gauge fixing, the inhomogeneities are described by a (linear) scalar field theory. Finally, we will provide a unique $SO(4)$-invariant Fock quantization for them just by demanding the unitary implementation of the dynamics.

2. Homogeneous system

The homogeneous and isotropic system has two global degrees of freedom. One is the pair $(\phi, p_\phi)$, consisting of a massive scalar field and its canonically conjugate momentum. As for the geometry, we consider a description in terms of a densitized triad adapted to the fiducial metric of the 3-spheres, parametrized by the variable $p$, and an Ashtekar-Barbero $su(2)$-connection, determined by a variable $c$. In order to maintain the parallelism with models where the spatial slices have open, flat topology, we parametrize the phase space so that some fundamental structures, as the Poisson brackets, are independent of the fiducial volume $l_0^3 = 2\pi^2$. Therefore, $(c, p) = 8\pi G\gamma/3$. Here, $G$ and $\gamma$ are the Newton constant and the Immirzi parameter, respectively.

For the quantization of the geometry, we apply a polymeric quantization. The basic variables in LQC are fluxes of densitized triads through “square” surfaces, enclosed by four geodesic edges, which basically reduce to $p$, and holonomies of the connection along integral curves of the fiducial triads of fiducial length $\mu_0$. In this respect, we adopt the improved dynamics scheme: $\hat{\mu} = \sqrt{\Delta}/p$, with the constant $\Delta$ determined by the infrared spectrum of the area operator defined in Loop Quantum Gravity. The gravitational part of the kinematical Hilbert space will be $\mathcal{H}_{\text{kin}} = L^2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{Bohr}})$, where $\mathbb{R}_{\text{Bohr}}$ is the Bohr compactification of the real line, with Haar measure given by $d\mu_{\text{Bohr}}$. In this Hilbert space, one can find a basis of normalizable states $\{|v\rangle\}$ in which the matrix elements of the holonomies, i.e., $\hat{N}_C$, create quanta of volume, while $\hat{p}|v\rangle = \text{sign}(v)(2\pi G\gamma h\sqrt{\Delta}|v\rangle)^{2/3}|v\rangle$. For the scalar field, we will consider a standard Schrödinger representation. Now, we can promote the Hamiltonian constraint to a quantum operator (see, e.g., [5]). Adopting a suitable factor ordering, we get for the quantum Hamiltonian constraint

$$\hat{C} = \left[\frac{1}{V}\right]^{1/2} \left[\hat{C}_g + 8\pi G(\hat{p}_\phi^2 + m^2\hat{V}^2\hat{\phi}^2)\right] \left[\frac{1}{V}\right]^{1/2}, \quad \hat{C}_g = \hat{C}_{g}^0 - \frac{6}{\gamma^2} \left(1 + \gamma^2\right)\hat{V}^{4/3} - \frac{\hat{V}^2}{\Delta} \sin^2 \hat{\mu}_0,$$

$$\hat{C}_{g}^0 = -\frac{3\pi G^2\hbar^2}{2} \left[\hat{N}_{2\hat{\mu}}(\hat{f}(\hat{v}))\hat{N}_{2\hat{\mu}} + \hat{N}_{-2\hat{\mu}}(\hat{f}(\hat{v}))\hat{N}_{-2\hat{\mu}} - \hat{f}(\hat{v})\right], \quad \hat{f}(\hat{v}) = \sqrt{\hat{v}^2 - 2}\left[\hat{v}s_+(\hat{v}) + \hat{v} - 2s_-\hat{v}\right], \quad s_{\pm}(\hat{v}) = \text{sgn}(\hat{v} \pm 2) + \text{sgn}(\hat{v}).$$

The constraint is a second order difference operator, with a step of 4 units in the label $v$. Moreover, we find that $f(v) = 0 \\forall v \in [-2, 2]$. Therefore, $\hat{C}_g$ has an invariant domain composed of states with support on semilattices of the type $\mathcal{L}^+_\varepsilon = \{v = \pm(\varepsilon + 4n), n \in \mathbb{N}\}$, where $\varepsilon \in (0, 4]$ is a bounded continuous parameter labeling each sector. Besides, all the relevant operators under consideration preserve such sectors, being possible to interpret them as superselection sectors. For convenience, we restrict our considerations to semilattices with $v > 0$ from now on. Moreover, any solution to the constraint is totally determined at any $v = \varepsilon + 4n$ with $n > 0$ by its value at the minimum volume, $\Psi(\phi, v = \varepsilon)$. The physical Hilbert space can be constructed out of the functional space of initial data at $v = \varepsilon$. Its inner product can be determined by imposing reality conditions on a complete set of observables as self-adjointness relations. In this way, the physical Hilbert space of this homogeneous system can be taken as $\mathcal{H}_{\text{phy}} = L^2(\mathbb{R}, d\phi)$.

3. Inhomogeneities and gauge fixing

It is natural to assume that small initial inhomogeneities may arise from vacuum fluctuations of the inflaton. In this context, a perturbative treatment of them seems justified. We will consider
only scalar perturbations around homogeneous solutions. Thus, we parametrize the lapse, the shift and the 3-metric in the geometry sector in the form $N = \sigma(N_0 + \delta N)$, $\alpha = \lambda^2 \delta N_a$, and $h_{ab} = \sigma^2 e^{2\alpha}(\Omega_{ab} + \epsilon_{ab})$. We also introduce a massive scalar field $\Phi = \sigma^{-1}(l_0^{-3/2} \varphi + \delta \varphi)$. We have defined $\sigma^2 = 4\pi G/3\hbar^3$, and the scale factor is $e^{\alpha} [6]$. Besides, the compact spatial sections are isomorphic to $S^3$, which is endowed with a natural basis of spherical harmonics. In this basis, we can write

$$\delta N = \lambda_0^{3/2} N_0 \sum_n g_n Q_n, \quad \delta N_a = \lambda_0^{3/2} e^{\alpha} \sum_n j_n D^n_a, \quad \epsilon_{ab} = 2\lambda_0^{3/2} \sum_n (a_n Q_n \Omega_{ab} + 3b_n P^n_{ab}),$$

and $\delta \varphi = \sum_n f_n Q_n$. We will consider perturbations to second order in the Hamiltonian, which yields

$$H = H_0 \left[ C_0 + \sum_n (C^n_2 + g_n C^n_1) \right] + \sum_n j_n D^n_1,$$

where $C_0$ corresponds to the scalar constraint of the homogeneous setting ($\alpha$, $\pi_\alpha$, $\varphi$, $\pi_\varphi$) (with $\alpha$ replacing $p$ and $\pi_\alpha$ being its momentum), and each $C^n_2$ is quadratic in the scalar modes ($a_n, b_n, f_n$) and their conjugate momenta ($\pi_a, \pi_b, \pi_f$), while $C^n_1$ and $D^n_1$ are the modes of the linear perturbation of the scalar and the diffeomorphism constraints, respectively. Note that, in our formalism, the perturbations $g_n$ and $j_n$ just play the role of Lagrange multipliers.

In order to reduce the system, we introduce a partial gauge fixing. One possibility is the longitudinal gauge, what in our canonical formalism is equivalent to the conditions $b_n = 0$ and $\Pi_n = \pi_a - \pi_\alpha a_n - 3\pi_\varphi f_n = 0$. These conditions are second class with $D^n_1$ and $C^n_1$. By means of the dynamical stability conditions $b_n = 0$ and $\Pi_n = 0$, together with $D^n_1 = 0$, one can see that $j_n = 0$ (or equivalently $\pi_b = 0$) and $g_n = -a_n$. If the linear scalar constraint is imposed, we can finally solve for $a_n$ and $\pi_a$ in terms of $f_n$ and $\pi_f$. After gauge fixing, the reduced symplectic structure is nondiagonal. To remedy this, we perform a transformation in the phase space so as to arrive at a symplectic structure with canonical form. The whole transformation can be reconstructed from the redefinition $\tilde{\pi}_f_n = \pi_f_n - 3\pi_\varphi a_n$. Finally, in order to be able to apply the uniqueness results of [7, 8] to our system, we conveniently reformulate it by introducing a canonical transformation that replaces the mode functions $f_n$ and $\tilde{\pi}_f_n$, with a new set, $\tilde{f}_n = e^{\alpha} \int \tilde{f}_n$ and $\tilde{\pi}_{f_n} = e^{-\alpha}(\tilde{\pi}_f_n - \pi_\varphi f_n)$. The transformation is such that the second order equation of motion for $f_n$ approaches (in the ultraviolet limit) sufficiently fast the equation which corresponds to (the modes of) a scalar field propagating in a static spacetime, but with a time dependent mass. Besides, one can easily extend the canonical transformation to the homogeneous sector.

4. Hybrid quantization

To carry out a complete quantization of the system, we will apply the polymeric quantization prescription explained above for the homogeneous sector of the geometry, whereas, for the matter content (background + perturbations), we will consider a standard representation. Now, the problem that we have to face concerns the choice of representation for the canonical commutation relations (CCR’s) of the inhomogeneous variables. The usual procedure to pick up a unique Fock quantization is to invoke the classical symmetries (e.g., in Minkowski or pure deSitter spacetimes). In our case, the classical symmetries are not enough, and hence such a criterion is insufficient. Fortunately, significant progress on this issue has been made recently. In particular, it has been shown that any representation which is invariant under the spatial isometries and permits the unitary implementation of the dynamics belongs to a unique equivalence class of Fock quantizations [7]. Moreover, these criteria not only fix the representation of the CCR’s, but also the pair of canonical fields is uniquely determined, among all the possibilities related by a time dependent scaling of the field [8].
In our situation, the variables $\bar{f}_n$ and $\bar{\pi}_{f_n}$ satisfy those requirements. Therefore, we can choose a (unique) $SO(4)$-invariant Fock representation with unitary dynamics for them, identifying the kinematical Hilbert space $H_{\text{kin}}$ with the corresponding Fock space $\mathcal{F}$ constructed from the one-particle Hilbert space. For the homogeneous degrees of freedom, we adopt the polymeric quantization presented above. One can see then that the total kinematical Hilbert space is $H_{\text{kin}} = H_{\text{grav}} \otimes L^2(\mathbb{R}, d\phi) \otimes \mathcal{F}$.

We are now ready for the next step: studying the solutions to the Hamiltonian constraint. The strategy is based on proving that any solution is completely determined by its value on the minimum volume section. Therefore, following the lines explained above for the homogeneous scenario, one can construct the physical Hilbert space by equipping the space of initial data on the minimum volume section with a suitable inner product, which can be determined by requirements of self-adjointness on a complete set of observables (see, e.g., [9]).

Finally, in order to avoid any dependence on the gauge choice, it is natural to work with gauge invariant quantities [10]. In particular, we are interested in finding a suitable choice of gauge invariant potentials that can be identified with our fundamental variables $f_n$ and $\pi_{f_n}$ by means of a canonical transformation which is unitarily implementable in the quantum theory. All those aspects will be a matter of future research.

5. Conclusions
We have started by analyzing a homogeneous massive scalar field propagating in a closed FRW spacetime. We have quantized the system to completion, combining both Schrödinger and polymeric representations. The physical Hilbert space has been constructed out of the functional space defined on the minimum volume section and which determines the solutions to the constraint. Later on, with the purpose of studying a more realistic setting, we have introduced inhomogeneities by means of perturbation theory, under the assumption that the observed inhomogeneities come from small quantum fluctuations of the inflaton. We have considered scalar perturbations to second order in the action. We have carried out a partial gauge fixing, removing two of the three fieldlike degrees of freedom. To fulfill the uniqueness criteria of invariance and unitarity of the Fock quantization, we have introduced a canonical transformation consisting in a scaling of the remaining field which depends on the homogeneous phase space variables. Finally, we have quantized the homogeneous system combining the “loop” techniques for the homogeneous degrees of freedom with the standard ones for the inhomogeneities. Besides, we have constructed the quantum Hamiltonian constraint. Finally, this constraint must be solved by means of a characterization of its solutions, and the Hilbert space constructed out of them.

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