The Extended Thermodynamic Properties of Taub-NUT/Bolt-AdS spaces

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Abstract

We investigate the extended thermodynamic properties of higher-dimensional Taub-NUT/Bolt-AdS spaces where a cosmological constant is treated as a pressure. We find a general form for thermodynamic volumes of Taub-NUT/Bolt-AdS black holes for arbitrary dimensions. Interestingly, it is found that the Taub-NUT-AdS metric has a thermodynamically stable range when the total number of dimensions is a multiple of 4 (4, 8, 12, ...). We also explore their phase structure and find the first order phase transition holds for higher-dimensional cases.
1 Introduction

Thermodynamic properties of black holes have been studied for a long time since it was first noticed that the area of the event horizon of a black hole is proportional to its physical entropy [1]. It was found that their physical quantities are expressed in terms of the thermal energy $U$, temperature $T$, and entropy $S$ and the first law of black hole thermodynamics has the similar forms to the first law of standard thermodynamics [2].

It has been also studied in complete analogy with standard thermodynamic systems. For example, it was found that there is a phase transition in the Schwarzschild-AdS black hole [3]. Since then, the phase transitions and critical phenomena in a variety of black hole solutions have been studied [4]-[7]. In particular, they have been used for investigation to various thermodynamic issues including higher dimensional black hole with negative cosmological constant $\Lambda$ [8]-[18].

As widely known, thermodynamic quantities in black hole physics, the black hole mass $M$, surface gravity $\kappa$ and the horizon area $A$ correspond to the thermodynamic quantities of a physical system, thermal energy $U$, temperature $T$, and entropy $S$ respectively.

However, comparing the first law of standard thermodynamics with the pressure and its conjugate to the first law of black hole thermodynamics in parallel, their counterparts in black hole physics are not quite captured. Recently, there have been new and interesting developments in the question, meaning that the cosmological constant $\Lambda$ can be treated as the thermodynamic pressure $p$

$$p = -\frac{1}{8\pi} \Lambda = \frac{u(2u + 1)}{8\pi l^2},$$ (1.1)

in units where $G = c = \hbar = k_B = 1$, and the total number of dimensions $(d + 1) = 2u + 2$ is even for some integer $u$. In addition, the black hole mass $M$ is defined as the enthalpy rather than the thermal energy $U$. This framework is quite natural since when the cosmological constant is considered as pressure, its conjugate quantity becomes dimensions of volume. A lot of investigations have been performed in this direction [19]-[38].

Very recently this issue was generalized to Taub-NUT/Bolt-AdS spaces in [30, 36] and to Kerr-Bolt-AdS spaces in [35]. Interestingly, they found the thermodynamic volume in Taub-NUT-AdS metric can be negative. In the context of enthalpy, the positive thermodynamic volume may be understood as applying the work on the environment (universe) by the system (the whole black hole) considering the process of forming the black hole. In addition, the negative thermodynamic volume may be understood in that the environment (universe) applies work to the system (Taub-NUT-AdS black hole) in the process of
the Taub-NUT-AdS black hole formation \[30\]. They also found that there is the first or-
der phase transition from Taub-NUT-AdS to Taub-Bolt-AdS through exploring the phase
structure of a NUT solution and a Bolt solution \[36\].

However, their works on the thermodynamic properties of black hole of the Taub-
NUT-AdS metric have only shown the progress in the four-dimensional cases \[30, 35, 36\].
Furthermore it is well known that their thermodynamic properties are depending on odd \(u\)
(the total number of dimensions: 4, 8, 12, \ldots) or even \(u\) (the total number of dimensions:
6, 10, 14, \ldots). Here we will investigate extensively thermodynamic properties in the
higher-dimensional Taub-NUT/Bolt-AdS spaces. It will particularly show that the Taub-
NUT-AdS solution has a thermodynamically stable range as a function of the temperature
for any odd \(u\). We also will demonstrate that there is the transition from Taub-NUT-AdS
to Taub-Bolt-AdS for all odd \(u\) only.

The paper is organized as follows: In the next section we investigate thermodynamic
properties in Taub-NUT/Bolt-AdS spaces for any \(u\). We find that thermodynamic quan-
tities such as the entropy, the enthalpy, the specific heat, the temperature, and the ther-
modynamic volume. We discuss their phase structure and their instability. In the last
section we give our conclusion. In Appendix A it is shown to be explicit forms for their
thermodynamic quantities.

2 Taub-NUT/Bolt-AdS black hole

We start by considering the \((d + 1)\)-dimensional-Taub-NUT with negative cosmological
constant. The general solution in the Euclidean section, for a \(U(1)\) fibration over a series
of the space \(\mathcal{M}^2\) as the base space \(\bigotimes_{i=1}^u \mathcal{M}^2\), is given by \[39\]-\[45\] (for the generalized
versions of the issue, see e.g., \[46\]-\[47\])

\[
ds^2 = f(r) \left\{ dt + 2N \sum_{i=1}^u \cos(\theta_i) d\phi_i \right\}^2 + \frac{dr^2}{f(r)} + (r^2 - N^2) \sum_{i=1}^u \left\{ d\theta_i^2 + \sin^2(\theta_i) d\phi_i^2 \right\},
\]

with the metric function \(f(r)\)

\[
f(r) = \frac{r}{(r^2 - N^2)^u} \int_r^\infty \left\{ \frac{(a^2 - N^2)^u}{a^2} + \frac{(2u + 1)(a^2 - N^2)^{u+1}}{l^2a^2} \right\} da - \frac{2mr}{(r^2 - N^2)^u}.
\]
Here, \( N \) represents a NUT charge for the Euclidean section, \( l \) is a cosmological parameter, \( m \) is a geometric mass. The NUT solution occurs when solving \( f(r)|_{r=N}=0 \). The inverse of the temperature \( \beta \) is obtained by requiring regularity in the Euclidean time \( t_E \) and radial coordinate \( r \) \[ \beta = \frac{4\pi}{f'(r)|_{r=N}} = 4(u+1)\pi N, \] (2.4)

where \( \beta \) is the period of \( t_E \).

Using counter term subtraction method, we get the regularized action \[ I_{\text{NUT}} = \frac{(4\pi)^u N^{2u-1}(2uN^2 - l^2)}{16\pi^2 l^2} \Gamma\left(\frac{1}{2} - u\right)\Gamma(u+1)\beta, \] (2.5)

where the gamma function \( \Gamma(t) \) is defined as \[ \Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx. \] (2.6)

Solving (2.4) for the mass at \( r = N \), the NUT mass is given as \[ m_{\text{NUT}} = \frac{N^{2u-1}\{l^2 - 2(u+1)N^2\}}{\sqrt{\pi(2u-1)l^2}} \Gamma\left(\frac{3}{2} - u\right)\Gamma(u+1)N, \] (2.7)

and the entropy is found to be \[ S_{\text{NUT}} = \frac{(4\pi)^u N^{2u-1}\{2u(2u+1)N^2 - (2u-1)l^2\}}{16\pi^2 l^2} \Gamma\left(\frac{1}{2} - u\right)\Gamma(u+1)\beta, \] (2.8)

by the Gibbs-Duhem relation \( S = \beta M - I \) where \( M \) is the conserved mass \[ M = u(4\pi)^{u-1}m. \] (2.9)

As mentioned in the previous section, its conjugate variable has dimension of volume when the negative cosmological constant \( \Lambda \) is treated as thermodynamic pressure \( p \) \[ \text{(1.1)}. \]

In the context of the thermodynamic process, we should think of the energy required to create the system \( pV \) as well as the energy of the system \( U \) i.e., whole forming the system, which is quite captured by the enthalpy. Thus, the conserved mass \( M \) \[ \text{(2.9)} \] is identified with enthalpy \( H \) rather than internal energy \( U \) \[ M \equiv H = U + pV, \] (2.10)
which leads to
\[
H_{\text{NUT}} = u(4\pi)^{u-1} \left\{ \frac{N^{2u-1} \Gamma \left( \frac{3}{2} - u \right) \Gamma (u + 1)}{\sqrt{\pi}(2u - 1)} \right. \\
- \left. \frac{16\sqrt{\pi}(u + 1)N^{2u+1}\Gamma \left( \frac{3}{2} - u \right) \Gamma (u + 1)}{u(2u - 1)(2u + 1)}p \right\}. 
\] (2.11)

To check whether we perform the computation correctly, we can reproduce the temperature through
\[
T_{\text{NUT}} = \frac{\partial H}{\partial S} \bigg|_p 
\] (2.12)
\[
= u(4\pi)^{u-1} \left\{ \frac{N^{2u-2} \Gamma \left( \frac{3}{2} - u \right) \Gamma (u + 1)}{\sqrt{\pi}} \right. \\
- \left. \frac{16\sqrt{\pi}(u + 1)N^{2u}\Gamma \left( \frac{3}{2} - u \right) \Gamma (u + 1)}{u(2u - 1)}p \right\} 
\] (2.13)
\[
= \frac{1}{4(u + 1)\pi N}. 
\] (2.14)

since, from Eq. (2.8):
\[
\frac{\partial S}{\partial N} \bigg|_p = u(u + 1)(4\pi)^u N \left\{ \frac{N^{2u-2} \Gamma \left( \frac{3}{2} - u \right) \Gamma (u + 1)}{\sqrt{\pi}} \right. \\
- \left. \frac{16\sqrt{\pi}(u + 1)N^{2u}\Gamma \left( \frac{3}{2} - u \right) \Gamma (u + 1)}{u(2u - 1)}p \right\}^{-1}. 
\] (2.15)

One the other hand, employing thermal relation \( C = -\beta \partial_{\beta} S \), the specific heat is given as
\[
C_{\text{NUT}} = \frac{u(4\pi)^{u-1}N^{2u-1}\left\{ 2(u + 1)(u + 1)N^2 - (2u - 1)^2 \right\}}{2\sqrt{\pi}l^2} \Gamma(\frac{1}{2} - u)\Gamma(u + 1)\beta. 
\] (2.16)

The thermodynamic volume is also obtained as
\[
V_{\text{NUT}} = \frac{\partial H}{\partial p} \bigg|_S 
\] (2.17)
\[
= 2(-2\sqrt{\pi})^{2u-1}(u + 1)N^{2u+1}\Gamma\left(-\frac{1}{2} - u\right)\Gamma(u + 1) \\
+ \left\{ \frac{u(2u - 1)(2u + 1)(2\sqrt{\pi}N)^{2u-2}\Gamma\left(-\frac{1}{2} - u\right)\Gamma(u + 1)}{4\sqrt{\pi}} \\
- \frac{\sqrt{\pi}u(2u - 1)(2u + 1)(2\sqrt{\pi}N)^{2u-2}\Gamma\left(-\frac{1}{2} - u\right)\Gamma(u + 1)}{4}p \right\} 
\] (2.18)
\[
= -\frac{u(4\pi)^uN^{2u+1}}{2\sqrt{\pi}} \Gamma\left(-\frac{1}{2} - u\right)\Gamma(u). 
\]
From this result (2.18), we can read the thermodynamic volumes of the NUT solution are negative for all odd \( u \) since \( \Gamma(-\frac{1}{2} - u) \) produces a positive sign for all odd \( u \) only, and \( \Gamma(u) \) does a positive sign for any \( u \) due to the property of Gamma function (Table 1 in Appendix A shows their explicit volume forms for four \((u = 1)\) to ten \((u = 4)\) dimensions). Also, due to dimensional scaling arguments, the generalized Smarr formula is given as

\[
\frac{1}{2} H - \frac{u}{2u - 1} TS + \frac{1}{2u - 1} pV = 0, 
\]

which is precisely matched with that of static \( d \)-dimensional black holes with negative cosmological constant \([10, 19, 20]\).

Finally, the internal energy of Taub-NUT is obtained as

\[
U_{\text{NUT}} = \frac{u(2u + 1)}{8\pi} \left\{ (2\sqrt{\pi}N)^{2u-1} - \frac{(2u + 1)(2\sqrt{\pi}N)^{2u-1}N^2}{\ell^2} \right\} \Gamma\left(-\frac{1}{2} - u\right)\Gamma(u + 1) \tag{2.20}
\]

by an thermal relation \( U = H - pV \).

![Figure 1: Plot of the available bolt radii \( r_B \) as a function of \( N \) for \( u \) (from left to right) 1 (the dimension of spacetime \( 2u + 2 = 4 \)) to 6 (the dimension of spacetime \( 2u + 2 = 14 \)) for cosmological parameter \( \ell = 1 \).](image)

Requiring \( f(r)|_{r=r_B} > N \) and \( f'(r)|_{r=r_M} = \frac{1}{N(u+1)} \), the Bolt solution occurs. In Taub-Bolt-AdS metric, the inverse of the temperature, the action, and the mass are respectively

\[
\beta = \frac{4\pi}{f'(r)|_{r=r_B}} = \frac{4\pi\ell^2 r_B}{\ell^2 + (2u + 1)(r_B - N^2)}, \tag{2.21}
\]

\[
I_{\text{Bolt}} = \left(\frac{4\pi}{4\ell^2}\right)^{u-1} \left[ (2u + 1)(-1)^u N^{2u+2} \right]^{\frac{1}{r_B}}. \tag{2.22}
\]
\[ + \sum_{i=0}^{u} \left( \begin{array}{c} u \\ i \end{array} \right) (-1)^i N^{2i} r_B^{2u-2i} \left\{ \frac{l^2}{(2u-2i-1)r_B^2} - \frac{(2u+1)(u-2i+1)r_B}{(2u-2i+1)(u-i+1)} \right\} \beta, \]

\[ m_{\text{Bolt}} = \frac{1}{2l^2} \left\{ \sum_{i=0}^{u} \left( \begin{array}{c} u \\ i \end{array} \right) \frac{(-1)^i N^{2i} l^2 r_B^{2u-2i-1}}{(2u-2i-1)} \right. \]

\[ + \left. (2u+1) \sum_{i=0}^{u+1} \left( \begin{array}{c} u+1 \\ i \end{array} \right) \frac{(-1)^i N^{2i} r_B^{2u-2i+1}}{2u-2i+1} \right\}. \]

Here the bolt radii \( r_B \) is

\[ r_B = \frac{l^2 \pm \sqrt{l^4 + (2u+1)(2u+2)^2 N^2[(2u+1)N^2 - l^2]}}{(2u+1)(2u+2)N}, \]  

and its reality requirement implies

\[ N \leq \frac{l}{\sqrt{2(u+1)(u+1+\sqrt{u(u+2)})}} = N_{\text{max}}, \]  

where \( N_{\text{max}} \) denotes the maximum magnitude of the NUT charge. As shown in Fig.1, we plot the available bolt radius for the various dimension of spacetime. It is shown that the \( N_{\text{max}} \) grows up as the dimension of spacetime increases. Thus, the magnitude of radii of small bolts in \( r_B \) increases in the case of the higher dimension.

Using parallel way as in the case of the NUT solution, the enthalpy \( H_{\text{Bolt}} \), the entropy \( S_{\text{Bolt}} \), and thermodynamics volume \( V_{\text{Bolt}} \) for the Bolt solution yields respectively

\[ H_{\text{Bolt}} = \frac{u(4\pi)^{u-1}}{2} \left\{ \sum_{i=0}^{u} \left( \begin{array}{c} u \\ i \end{array} \right) \frac{(-1)^i N^{2i} r_B^{2u-2i-1}}{(2u-2i-1)} \right. \]

\[ + \left. \frac{8\pi}{u} \sum_{i=0}^{u+1} \left( \begin{array}{c} u+1 \\ i \end{array} \right) \frac{(-1)^i N^{2i} r_B^{2u-2i+1}}{2u-2i+1} \right\}, \]  

\[ S_{\text{Bolt}} = \frac{(4\pi)^{u-1}}{4} \left\{ \sum_{i=0}^{u} \left( \begin{array}{c} u \\ i \end{array} \right) \frac{(2u-1)(-1)^i N^{2i} r_B^{2u-2i-1}}{2u-2i-1} \right. \]

\[ + \left. \left\{ \sum_{i=0}^{u} \left( \begin{array}{c} u \\ i \end{array} \right) \frac{8\pi(2u^2 + 3u - 2i + 1)(-1)^i N^{2i} r_B^{2u-2i+1} + 8\pi(2u-1)N^{2u+2}}{u(u-i+1)(2u-2i+1)} \right\} \beta, \]  

\[ V_{\text{Bolt}} = \frac{\pi^u(2r_B)^{u-1}}{2u+1} \left\{ -2(N^2 - r_B^2) \left( 1 - \frac{N^2}{r_B^2} \right)^u \right. \]

\[ + \left. r_B^2 \left( \frac{N}{r_B} \right)^{2u+1} B \left( \frac{N^2}{r_B^2}, \frac{1}{2} - u, u + 1 \right) \right\}, \]
where the incomplete beta function \( B(x, a, b) \) is defined as
\[
B(x, a, b) = \int_0^x t^{a-1} (1 - t)^{b-1} dt.
\] (2.29)

Employing the thermal relation \( U = H - pV \), the internal energy of AdS-Taub-Bolt \( U \) is given as
\[
U_{\text{Bolt}} = \frac{(4\pi)^{u-1}}{4l^2} uN^{2u-1} \left\{ (2u + 1)N^2 - l^2 \right\} B \left( \frac{N^2}{r_B^2}, \frac{1}{2} - u, u + 1 \right). \] (2.30)

Let us start with the action difference of Taub-NUT-AdS and Taub-Bolt-AdS in order to study the phase structure of the Taub-NUT/Bolt-AdS system. Their action difference, \( I_D \) is defined as
\[
I_{\text{Bolt}} - I_{\text{NUT}} \equiv I_D = \frac{(4\pi)^u}{8r_B^2} \left[ 2N(N^2 - r_B^2)(u + 1) \left( 1 - \frac{N^2}{r_B^2} \right)^u \right]
\]
\[
\quad + (2uN^2 - l^2) \left\{ (u + 1)N^{2u} B \left( \frac{N^2}{r_B^2}, \frac{1}{2} - u, u + 1 \right) \right.
\]
\[
\quad \left. - \frac{2N^{2u}}{\sqrt{\pi}} \Gamma(\frac{1}{2} - u) \Gamma(2 + u) \right\}. \] (2.31)

Figure 2: (a) Plot of the action difference as a function of \( N \) for \( u \) (from left to right) 1 (the dimension of spacetime \( 2u + 2 = 4 \)) to 6 (the dimension of spacetime \( 2u + 2 = 14 \)) for cosmological parameter \( l = 1 \) and pressure \( p = 3 \). (b) Plot of the six-dimensional action difference \( I_6 \) as a function of \( N \) for cosmological parameter \( l \) (from left to right) 1 to 5 for pressure \( p = 3 \).

As shown in Fig. 2, taking the cosmological parameter \( l \) and the pressure \( p \) as fixed parameters, the action difference \( I_D \) becomes negative as \( N \) increases. This means that
there is the first order phase transition from Taub-NUT-AdS to Taub-Bolt-AdS at a critical NUT charge. Note that there is the transition to Taub-Bolt-AdS with $r_{B,+}$ only since the right vertical solid curves come from the larger branch of the two branches of bolts where $r_{B,+}$ represents large bolt radii in $[2,24]$. As shown in Fig 2. (a), the curves of $\mathcal{I}_D$ move more to the left as the dimension of spacetime increases. In Fig 2. (b), it is shown that the curves of $\mathcal{I}_6$ move more to the right as the pressure decreases. The pressure $p$ becomes zero as the cosmological parameter $l$ goes to infinity since $p$ is inversely proportional to $l$ i.e., $p = u(2u + 1)/(8\pi l^2)$. In such limit, curves of the action difference as a function of $N$ correspond to each dashed line in Fig. 2.

Figure 3: For the NUT solution, plot of thermodynamically stable range of $p$ as a function of $T$ for given dimension of spacetime (two red dashed curves for $u = 1$, two brown dashed curves for $u = 3$, and two blue dashed curve for $u = 5$, respectively) and for the Bolt solution, plot of $p$ as a function of $T$ for given dimension of spacetime (red solid curve for $u = 1$, brown solid curve for $u = 3$, and blue solid curve for $u = 5$, respectively).

Requiring both the entropy and the specific heat are positive, we find the following thermodynamically stable range of $p$ for the NUT solution

$$\frac{\pi u(2u - 1)(u + 1)}{T^2} < p < \frac{\pi (2u - 1)(u + 1)^2}{T^2}.$$  \tag{2.32}

Note that there is thermodynamically stable range of $p$ for all odd $u$ only, since both the entropy and specific heat at a given temperature are not positive for all even $u$ (see e.g., Fig 5. (b)). It is shown in Fig. 3 that such stable range (2.32) is given as the shaded regions between the two curves on the left.
After obtaining $N$ with a change of sign in the action difference $I_D$ \[^{(2.31)}\] through solving $I_D = 0$ for $N$, and substituting into the inverse of the temperature \[^{(2.21)}\] we may deduce an expression for the critical temperature $T_c$

$$T_c = \frac{\sqrt{p}}{\sqrt{2\pi\alpha(u+1)\sqrt{u(2u+1)}}},$$

\[^{(2.33)}\]

by employing the pressure \[^{(1.1)}\]. Here at $T_c$ a phase transition from Taub-NUT-AdS to Taub-Bolt-AdS occurs and $\alpha$ is constant. For example, for $u = 1$, the transition temperature $T_c$ reduces to the result in \[^{[36]}\]

$$\alpha = \frac{\sqrt{5} - \sqrt{2}}{6},$$

\[^{(2.34)}\]

and when $u > 1$ (beyond four-dimensional spacetime), $\alpha$ is numerically written in Table 3 in Appendix A since it is difficult to solve the equation with the analytic method for a complicated high-order polynomial equation $I_D = 0$. It is shown in Fig. 3 that the rightmost solid curves (red solid curve for $u = 1$, brown solid curve for $u = 3$, and blue solid curve for $u = 5$, respectively) denote the lines of transition between the Taub-NUT-AdS and Taub-Bolt-AdS phases for a given dimension of spacetime and this transition occurs at $T_c$ \[^{(2.33)}\].

In Fig 4 and Fig 5., dashed lines represent solutions for higher actions. As shown in Fig 4. (b) and Fig 5. (b), the entropy is zero at $T_0$

$$T_0 = \frac{\sqrt{p}}{4\sqrt{5\pi}} \quad (u = 3) \quad \text{and} \quad T_0 = \frac{\sqrt{p}}{3\sqrt{3\pi}} \quad (u = 2),$$

\[^{(2.35)}\]

and the specific heat is zero at $T_h$

$$T_h = \frac{\sqrt{p}}{2\sqrt{15\pi}} \quad (u = 3) \quad \text{and} \quad T_h = \frac{\sqrt{p}}{3\sqrt{2\pi}} \quad (u = 2),$$

\[^{(2.36)}\]

where at $T_h$ the entropy has maximum value (odd $u$) or minimum (even $u$) (for the generalized form of $T_0$ and $T_h$, see Table 3 in Appendix A). Any NUT solution in AdS space is thermodynamically stable for all odd $u$ since both the entropy $S_{u(odd)}$ and the specific heat $C_{p,u(odd)}$ become positive between $T_0$ and $T_h$ where $S_{u(odd)}$ and $C_{p,u(odd)}$ denote the $(2u-2)$-dimensional entropy for odd $u$ and the $(2u-2)$-dimensional specific heat for odd $u$ at constant pressure (see e.g., Fig. 4 (b) for $u = 3$, and \[^{[36]}\] for $u = 1$). It is also found that the thermodynamic nature of the Taub-NUT/Bolt-AdS system dramatically changes due to odd $u$ or even $u$, i.e., there is stable Bolt solution only for all even $u$. Thus, here focusing on all odd $u$, we discuss this phase transition to Taub-Bolt-AdS in detail. In
Figure 4: (a) Plot of the $(2u + 2)$-dimensional entropy $S_{u(odd)}$ as a function of temperature $T$ for odd $u$ (red solid curve for $u = 1$, brown solid curve for $u = 3$, and blue solid curve for $u = 5$, respectively) for pressure $p = 3$. (b) Plot the eight-dimensional entropy $S_8$ (brown solid curve) and specific heat $C_8$ (black solid curve) as a function of temperature $T$ for pressure $p = 3$.

Figure 5: (a) Plot of the $(2u + 2)$-dimensional entropy $S_{u(even)}$ as a function of temperature $T$ for even $u$ (orange solid curve for $u = 2$, green solid curve for $u = 4$, and purple solid curve for $u = 6$, respectively) for pressure $p = 3$. (b) Plot of the six-dimensional entropy $S_6$ (orange solid curve) and specific heat $C_6$ (black solid curve) as a function of temperature $T$ for pressure $p = 3$. 
Fig 4. (b), the brown solid curve on the left is the eight-dimensional entropy of the NUT solution, the black solid curve is the eight-dimensional specific heat of the NUT solution, and the brown solid curve on the right is the eight-dimensional entropy of the Bolt solution with the large bolt radii $r_{B, +}$. Thus possible profiles of the entropy are classified by the value of the temperature $T$: (i) when $T < T_0$ and for the NUT solution, $S_8$ is negative but $C_{p,8}$ is positive as shown in Fig 4. (b). (ii) when $T_0 < T < T_h$ and for the NUT solution, both $S_8$ and $C_{p,8}$ are positive as shown in Fig 4. (b). (iii) when $T_h < T < T_c$ and for the NUT solution, $S_8$ is positive but $C_{p,8}$ is negative as shown in Fig 4. (b). (iv) when $T > T_c$ and for the Bolt solution, $S_8$ is positive as shown in Fig 4. (b)., and $C_{p,8}$ is positive. Here we do not draw the plot of the specific heat as the function of temperature $T$ since the specific heat $C_{p,8}$ is always positive in the case of Bolt solution with the $r_{B, +}$. As pointed out in [30, 36], in the extended thermodynamics the profiles above may be interpreted as follows: For (i), the negative entropy comes from some net heat outflow during the creation of the Taub-NUT black hole. For (ii), the stable Taub-NUT black hole occurs and, in particular, since the thermodynamic volume $-\frac{1}{4\pi} \left( \frac{\sqrt{u^2-u-1}}{2u+1} \right)^{2u+1} \Gamma(-\frac{1}{2} - u)\Gamma(u+1)$ at $T_0$ and the thermodynamic volume $-\frac{1}{4\pi} \left( \frac{\sqrt{u(2u^2+u-1)}}{2(u+1)^2} \right)^{2u+1} \Gamma(-\frac{1}{2} - u)\Gamma(u+1)$ at $T_h$, the thermal stable range of the thermodynamic volume is given as

$$-\frac{1}{24\sqrt{\pi p}} < V_{\text{NUT},4} < -\frac{1}{48\sqrt{2\pi p}} \quad (\text{for } u = 1),$$

and

$$-\frac{25\sqrt{5}}{112\sqrt{\pi p}} < V_{\text{NUT},8} < -\frac{675\sqrt{15}}{14336\sqrt{\pi p}} \quad (\text{for } u = 3),$$

where it can be seen that the thermal stable range of the thermodynamic volume still increases with increasing dimensionality. For (iii), the Taub-NUT black hole develops an instability due to the negative specific heat. For (iv), the stable Taub-Bolt black hole finally occurs.

3 Conclusion

In the context of the extended thermodynamics, we considered the higher dimensional Tabu-NUT/Bolt-AdS solution and obtained their thermal quantities such as the enthalpy, the entropy, the specific heat, and the thermodynamic volume and so on.

We investigated their phase structure through their thermal quantities, and found out that the thermodynamic nature of the Taub-NUT/Bolt-AdS system dramatically changes
due to odd or even $u$. In particular, it was found out that there existed the first order phase transition from the Taub-NUT-AdS to the Taub-Bolt-AdS for all odd $u$.

Finally, we also explored a proportional behavior of their thermodynamic quantities with increasing dimensionality and found that the maximum magnitude of the NUT charge, the magnitude of radii of small bolt, and the thermal stable range of a thermodynamic volume grow up.
### Appendix

#### A Thermodynamic quantities for NUT/Bolt solution

| D | $H_{\text{NUT}}$ | $V_{\text{NUT}}$ | $U_{\text{NUT}}$ |
|---|---|---|---|
| 4 | $N(1 - \frac{32N^2\pi}{3}p)$ | $-\frac{8N^3\pi}{3}$ | $N(1 - \frac{3N^2}{l^2})$ |
| 6 | $-\frac{32}{3}N^3\pi(1 - \frac{24N^2\pi}{5}p)$ | $\frac{128N^3\pi^2}{15}$ | $-\frac{32}{3}N^3(1 - \frac{5N^2}{l^2})\pi$ |
| 8 | $\frac{384}{3}N^5\pi^2(1 - \frac{64N^2\pi}{21}p)$ | $-\frac{1024N^7\pi^3}{35}$ | $\frac{384}{3}N^5(1 - \frac{7N^2}{l^2})\pi^2$ |
| 10 | $-\frac{16384}{35}N^7\pi^3(1 - \frac{20N^2\pi}{9}p)$ | $\frac{32768N^9\pi^4}{315}$ | $-\frac{16384}{35}N^7(1 - \frac{9N^2}{l^2})\pi^3$ |

\[ u(4\pi)^{u-1}\left(\frac{N^{2u-1}}{2\sqrt{\pi}}\right)^u \frac{(\frac{3}{2}-u)\Gamma(\frac{1}{2})}{u\Gamma(u)} \left(\frac{3}{2}-u\right)^u \Gamma\left(\frac{1}{2}-u\right) \Gamma\left(\frac{1}{2}+u\right) \]

\[ u\left(\frac{2u+1}{8\pi}\right)\left\{\left(2\sqrt{\pi}\right)^{2u-1} - (2u+1)(2\sqrt{\pi})^{2u-1}\right\} \]

\[ \times\Gamma\left(\frac{1}{2}-u\right) \Gamma\left(u+1\right) \]

**Table 1:** Summary of thermodynamic quantities for Taub-NUT-AdS

| D | $H_{\text{Bolt}}$ | $V_{\text{Bolt}}$ | $U_{\text{Bolt}}$ |
|---|---|---|---|
| 4 | $\frac{r_B^2 + N^2}{2r_B^2} + 4\pi\left(r_B^2 - 6N^2r_B + 3N^4\right)p$ | $\frac{4\pi r_B}{3}(r_B^2 - 3N^2)$ | $\frac{r_B^2 + N^2}{2r_B^2}(1 - \frac{3N^2}{l^2})$ |
| 6 | $\frac{4\pi r_B}{3} - 6N^2r_B^2 - 3N^4\pi \cdot p$ | $\frac{16\pi r_B}{3}(3r_B^2 - 10N^2r_B^2 + 15N^4)$ | $\frac{4\pi r_B}{3}(6N^2r_B^2 - 3N^4\pi \cdot p)$ |
| 8 | $24\pi r_B \cdot (5r_B^2 - 35N^2r_B^2 + 5N^6) \cdot p$ | $\frac{64\pi r_B}{35}(5r_B^6 - 21N^2r_B^2 + 35N^6)$ | $\frac{24\pi r_B}{5r_B}(5r_B^6 - 5N^2r_B^2 + 5N^6)$ |
| 10 | $\frac{128\pi r_B}{35}(5r_B^6 - 28N^2r_B^2 - 35N^6) \cdot p$ | $\frac{256\pi r_B}{315}(35r_B^6 - 180N^2r_B^2 + 315N^8)$ | $\frac{128\pi r_B}{35}(5r_B^6 - 28N^2r_B^2 - 35N^6) \cdot p$ |

**Table 2:** Summary of thermodynamic quantities for Taub-Bolt-AdS
| $\mathbf{D}$ | $\mathbf{I_D}$ | $\mathbf{T_0}$ | $\mathbf{T_h}$ | $\mathbf{T_c}$ |
|---|---|---|---|---|
| 4 | $\frac{-2N\pi(r_B-N)^2}{r_B^2} \left(r_B^2 + 2N\pi + 3N^2 - l^2\right)$ | $\frac{\sqrt{p}}{2\sqrt{\pi}}$ | $\frac{\sqrt{p}}{\sqrt{2\pi}}$ | 0.84077 $\sqrt{p}$ |
| 6 | $\frac{4N\pi^2(r_B-N)^3}{r_B^2 l^2} \left\{3r_B^3 + 9N\pi r_B^2 + 13N^2 r_B + 15N^3 - (r_B + 3N)l^2\right\}$ | $\frac{\sqrt{p}}{3\sqrt{3\pi}}$ | $\frac{\sqrt{p}}{\sqrt{2\pi}}$ | 0.57253 $\sqrt{p}$ |
| 8 | $\frac{-64N\pi^4(r_B-N)^4}{5r_B^2 l^2} \left\{5r_B^4 + 20N\pi r_B^3 + 36N^2 r_B^2 + 144N^3 r_B + 81N^4 - (r_B^2 + 4N\pi + 5N^2)l^2\right\}$ | $\frac{\sqrt{p}}{4\sqrt{5\pi}}$ | $\frac{\sqrt{p}}{2\sqrt{15\pi}}$ | 0.46324 $\sqrt{p}$ |
| 10 | $\frac{64N\pi^4(r_B-N)^5}{7r_B^2 l^2} \left\{35r_B^5 + 175N\pi r_B^4 + 390N^2 r_B^3 + 780N^3 r_B^2 + 225N^4 r_B + 105N^5 - (5r_B^3 + 25N\pi r_B + 47N^2 r_B + 35N^3)l^2\right\}$ | $\frac{\sqrt{p}}{5\sqrt{7\pi}}$ | $\frac{\sqrt{p}}{2\sqrt{35\pi}}$ | 0.40000 $\sqrt{p}$ |

Table 3: Summary of thermodynamic quantities for exploring the phase structure

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