MEASUREMENT ISOMORPHISM OF GRAPHS

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Abstract. The d-measurement set of a graph is its set of possible squared edge lengths over all d-dimensional embeddings. In this note, we define a new notion of graph isomorphism called d-measurement isomorphism. Two graphs are d-measurement isomorphic if there is agreement in their d-measurement sets. A natural question to ask is “what can be said about two graphs that are d-measurement isomorphic?” In this note, we show that this property coincides with the 2-isomorphism property studied by Whitney.

1. Introduction

Given a graph Γ we can consider placing each vertex at some position in \( \mathbb{E}^d \) and then measuring the squared Euclidean length of each of the graph’s edges. This gives us the coordinates of a single “measurement point” in \( \mathbb{E}^e \), where \( e \) is the number of edges in the graph. As we alter the vertex positions, the measurement point will typically change. The d-dimensional measurement set, \( M_d(\Gamma) \), is the union of all achievable measurement points as we vary over all possible placements of the vertices in \( \mathbb{E}^d \). Suppose that, after some permutation of the \( e \) coordinate axes, we have agreement in the d-dimensional measurement sets of two graphs, Γ and ∆. Then we say that the graphs Γ and ∆ are d-measurement isomorphic.

Clearly, two isomorphic graphs must be d-measurement isomorphic. But the converse is not true. For example, the measurement set of any forest graph is the entire first octant of \( \mathbb{E}^e \) as there are no constraints on the achievable edge lengths. A natural question to ask is “what can be said about two graphs that are d-measurement isomorphic?” In this note, we relate this type of isomorphism to a graph property studied by Whitney [3] called 2-isomorphism. Our main result is that for any \( d \), two graphs are d-measurement isomorphic if and only if they are 2-isomorphic. In particular, for 3-connected graphs, this means that two graphs are d-measurement isomorphic if and only if they are isomorphic graphs.

Definition 1.1. A graph \( \Gamma \) is a set of \( v \) vertices \( V(\Gamma) \) and \( e \) edges \( E(\Gamma) \), where \( E(\Gamma) \) is a set of two-element subsets of \( V(\Gamma) \).

Definition 1.2. Two graphs \( \Gamma \) and \( \Delta \), are isomorphic if there is a bijection \( \varphi \) between \( V(\Gamma) \) and \( V(\Delta) \) such that \( \{x, y\} \in E(\Gamma) \) iff \( \{\varphi(x), \varphi(y)\} \in E(\Delta) \).

Next we define two weaker notions of graph isomorphism. These allow us to move around barely connected parts of the graph without changing the equivalence class.

Definition 1.3. A cut vertex of graph is a vertex whose removal disconnects the graph. A split operation breaks a cut vertex into two vertices to produce two disjoint subgraphs. Two graphs are 1-isomorphic if they become isomorphic under a finite sequence of split operations. See Figure 1.

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Definition 1.4. (Following [2]) If $S \subset E(\Gamma)$ then let $\Gamma[S]$ denotes the subgraph induced by $S$. A partition $\{S,T\}$ of $E(\Gamma)$ is a 2-separation of $\Gamma$ if $|S| \geq 2 \leq |T|$ and $|V(\Gamma[S]) \cap V(\Gamma[T])| = 2$. Let $\{S,T\}$ be a 2-separation of $\Gamma$ and let the cut pair $V(\Gamma[S]) \cap V(\Gamma[T])$ be $\{x,y\}$. Let $\Gamma'$ be the graph obtained from $\Gamma$ by interchanging in $\Gamma[S]$ the incidences of the edges at $x$ and $y$. Then we say that $\Gamma'$ is obtained from $\Gamma$ by a reversal operation. Two graphs are 2-isomorphic if they become 1-isomorphic after a finite sequence of reversals. See Figure 2.

Note that is not the same notion of 2-isomorphism studied in [4].

Remark 1.5. Since 3-connected graphs have no 2-separations, for such graphs 2-isomorphism coincides with graph isomorphism.

Next we define another notion of graph equivalence.

Definition 1.6. A cycle is a path of adjacent vertices that starts and ends at the same vertex, and with no vertex repeated in the path. Two graphs $\Gamma$ and $\Delta$, are cycle isomorphic if there is bijection $\sigma$ between $E(\Gamma)$ and $E(\Delta)$ such that for any $S \subset E(\Gamma)$, $\Gamma[S]$ is a cycle iff $\Delta[\sigma(S)]$ is a cycle.

In [3], Whitney proved the following theorem that will provide all of the heavy lifting that we will need in this note.

Theorem 1.7 (Whitney). Two graphs are cycle-isomorphic iff they are 2-isomorphic.

We now define some notions related to graph embeddings.

Definition 1.8. A configuration $p$ of a vertex set $V$ is a mapping from $V$ to $E^v$. Let $C^d(V)$ be the space of configurations of $V$ in $E^d$. For $p \in C(V)$ and $u \in V$, let $p(u)$ denote the image of $u$ under $p$. A framework $(p, \Gamma)$ is the pair of a graph and a configuration of its vertices. For a given graph $\Gamma$ the length-squared function $\ell_\Gamma$ is the function assigning to each edge of $\Gamma$ its squared edge length in the framework. That is, the component of $\ell_\Gamma(p)$ in the direction
of an edge \( \{u, w\} \) is \( |p(u) - p(w)|^2 \). Once we fix an (arbitrarily) identification of each edge in \( E(\Gamma) \) with an associated coordinate axis in \( \mathbb{R}^e \), we can interpret the length-squared function as being of the type: \( \ell_\Gamma : C^d(V) \to \mathbb{R}^e \).

**Definition 1.9.** The \( d \)-dimensional measurement set \( M_d(\Gamma) \) of a graph \( \Gamma \) is defined to be the image in \( \mathbb{R}^e \) of \( C^d(V) \) under the map \( \ell_\Gamma \). These are nested by \( M_d(\Gamma) \subset M_{d+1}(\Gamma) \) and eventually stabilize by \( M_{e-1}(\Gamma) \).

In our context of measurement sets of graphs, we define a new notion of isomorphism.

**Definition 1.10.** Two graphs, \( \Gamma \) and \( \Delta \), both with \( e \) edges, are \( d \)-measurement isomorphic if there is an identification of each edge in \( E(\Gamma) \) with an associated coordinate axis in \( \mathbb{R}^e \), and an identification of each edge in \( E(\Delta) \) with an associated coordinate axis in \( \mathbb{R}^e \), under which \( M_d(\Gamma) = M_d(\Delta) \).

Our main result is the following

**Theorem 1.** For any \( d \), two graphs are \( d \)-measurement isomorphic iff they are 2-isomorphic.

This also gives us the following:

**Corollary 1.11.** For any two integers \( d_1 \) and \( d_2 \), if a pair of graphs are \( d_1 \)-measurement isomorphic then they are \( d_2 \)-measurement isomorphic.

**Remark 1.12.** Testing cycle isomorphism of graphs is as computationally difficult as testing graph isomorphism \([1]\), and thus so is testing 2-isomorphism and \( d \)-measurement isomorphism.

2. **Proof**

We will prove our theorem through a cycle of implications. For these arguments we first fix \( d \) as it turns out that our arguments do not depend on it.

2.1. **2-isomorphism \( \Rightarrow \) d-measurement isomorphism.** The graphs \( \Gamma \) and \( \Delta \) are 2-isomorphic if they become isomorphic after a finite number of splits and reversals. Clearly, a split operation does not change \( M_d \).

Likewise, let \( \{S, T\} \) be a 2-separation of \( \Gamma \) with cut pair \( \{x, y\} \) and let \( \Gamma \) and \( \Gamma' \) be related by the reversal across this pair. Under reversal, there is a canonical bijection between \( E(\Gamma) \) and \( E(\Gamma') \). Let us fix our edge-axis identifications to be consistent with this bijection.

For any \( p \in C^d(V(\Gamma)) \), we can reflect, in \( \mathbb{R}^d \), the positions of the vertices of \( V(\Gamma[S]) \backslash \{x, y\} \) across the hyperplane bisecting the segment \( xy \) to obtain a new configuration \( p' \). Under this construction, we have \( \ell_\Gamma(p) = \ell_{\Gamma'}(p') \). See Figure 2. Thus \( M_d(\Gamma) = M_d(\Gamma') \) and they must be \( d \)-measurement isomorphic.

2.2. **d-measurement isomorphism \( \Rightarrow \) cycle isomorphism.** We start by showing that a cycle graph on \( k \) edges is not \( d \)-measurement isomorphic to any other graph.

**Lemma 2.1.** Let \( c \) be a cycle of \( k \) edges, and \( b \) be any other graph with \( k \) edges. Then \( c \) and \( b \) are not \( d \)-measurement isomorphic.

**Proof.** If a graph is a forest with \( k \) edges, there are no constraints on any of the achievable squared edge lengths, and thus its measurement set is the entire first octant of \( \mathbb{R}^k \).

If a graph with \( k \) edges, is not a forest then it must have a cycle as a subgraph. In this case, its measurement set cannot be the entire first octant of \( \mathbb{R}^k \) since there is no framework (in any dimension) where all but one of the edges of the cycle has zero length.
Thus, if \( c \) is a cycle and \( b \) a forest, their measurement sets cannot agree under any edge-axis identifications.

If \( c \) is a cycle and \( b \) is neither a cycle or a forest, then \( b \) must have an edge whose removal does not turn \( b \) into a forest. Meanwhile the removal of any edge turns \( c \) into a forest. In terms of measurement sets, edge removal corresponds to projecting the measurement set onto coordinates associated with the appropriate \( k - 1 \) edges. These projections for \( b \) and \( c \) cannot not agree, as one produces the measurement set of a forest and one produces the measurement set of a non-forest. Since the projected measurement sets do not agree, the original measurement set cannot have agreed as well.

Suppose that \( \Gamma \) is not cycle isomorphic to \( \Delta \). Then for any edge bijection, \( \sigma \), there must be an edge subset \( c \) of \( \Gamma \) such that \( \Gamma \setminus c \) is a cycle, while \( \Delta \setminus \sigma(c) \) is not a cycle.

Let us fix our edge-axis identifications to be consistent with \( \sigma \). Next, let us project the measurement sets \( M_d(\Gamma) \) and \( M_d(\Delta) \) down to the subspace of \( \mathbb{R}^e \) corresponding to the edge set of \( c \) and \( \sigma(c) \) respectively. In the case of \( \Gamma \) we will obtain the measurement set of a cycle, while for \( \Delta \) we will obtain the measurement set of a non-cycle. These two measurement sets cannot be the same by Lemma 2.1. Thus \( M_d(\Gamma) \) cannot be the same as \( M_d(\Delta) \) under this edge-axis identification.

This is true for all edge-axis identifications consistent with the bijection \( \sigma \). And, by assumption, this is true for all bijections. Thus it is true for all edge-axis identifications and \( \Gamma \) and \( \Delta \) cannot be d-measurement isomorphic.

2.3. cycle isomorphism ⇒ 2-isomorphism. This is simply Theorem 1.7. And we are done.

References

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