Observation of a phononic quadrupole topological insulator

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The modern theory of charge polarization in solids1,2 is based on a generalization of Berry’s phase3. The possibility of the quantization of this phase4,5 arising from parallel transport in momentum space is essential to our understanding of systems with topological band structures6–10. Although based on the concept of charge polarization, this same theory can also be used to characterize the Bloch bands of neutral bosonic systems such as photonic11 or phononic crystals12,13. The theory of this quantized polarization has recently been extended from the dipole moment to higher multipole moments14. In particular, a two-dimensional quantized quadrupole insulator is predicted to have gapped yet topological one-dimensional edge modes, which stabilize zero-dimensional in-gap corner states14. However, such a state of matter has not previously been observed experimentally. Here we report measurements of a phononic quadrupole topological insulator. We experimentally characterize the bulk, edge and corner physics of a mechanical metamaterial (a material with tailored mechanical properties) and find the predicted gapped edge and in-gap corner states. We corroborate our findings by comparing the mechanical properties of a topologically non-trivial system to samples in other phases that are predicted by the quadrupole theory. These topological corner states are an important stepping stone to the experimental realization of topologically protected wave guides12,15 in higher dimensions, and thereby open up a new path for the design of metamaterials16,17.

A non-vanishing dipole moment $\mathbf{p}$ in an insulator does not lead to any charge accumulation in the bulk. Instead, it manifests through uncompensated surface charges and hence induces potentially interesting surface physics (Fig. 1a). The dipole moment $\mathbf{p}$ is expressible through Berry’s phase1,3, which in turn can lead to the quantization of the dipole moment4,5,18–21. All observed topological insulators fit into this framework of quantized dipole moments5, or mathematical generalizations thereof22. For neutral systems, the abstract quantity $\mathbf{p}$ loses its electromagnetic content. However, it can equally well be used to predict band-structure effects such as surface modes. Whether higher-order moments, such as the quadrupole, can lead to distinctly new topological phases of matter has remained unclear.

Recently, a theory for a quantized quadrupole insulator was put forward14, based on its phenomenology: a bulk quadrupole moment in a finite two-dimensional sample gives rise to surface dipole moments on its one-dimensional edges and to uncompensated charges on the zero-dimensional corners (Fig. 1b). The former indicates gapped edge modes and the latter motivates the presence of in-gap corner excitations. This phenomenology also defines the key technological use of such a quadrupole insulator in mechanical or optical metamaterials: the localized corner modes can be used for acoustic or electromagnetic field enhancements in two dimensions22. Moreover, these states serve as a stepping stone towards topologically protected, one-dimensional channels in three dimensions: when appropriately stacked into three dimensions, the corner modes give rise to chiral one-dimensional modes along edges of the three-dimensional sample23,24–26.

The phenomenology of gapped edges and gapless corners can be formalized mathematically. Nested Wilson loops have been proposed14 as a way of obtaining a quantized quadrupole moment (see Methods for details): Wilson loop operators depend only on the bulk properties and encode the edge physics via their eigenvalues $\nu_\alpha(k_\alpha)$, $\alpha \in [x, y]$, which are known as Wannier bands27. If these Wannier bands $\nu_\alpha(k_\alpha)$ are gapped, then the eigenvectors of the Wilson loops can be used to...
The dynamical matrix (1),14 moreover, the particle–hole symmetry fixes the non-commutation of inversion symmetries \(M_x, M_y\), leading to a well-defined and quantized quadrupole phase is described by9

\[
[p_{x}^{\alpha}, p_{y}^{\alpha}] = \left[ \frac{1}{2}, \frac{1}{2} \right]
\]

(1)

Because a corner terminates two edges, equation (1) could suggest that each of them supports two in-gap states. However, it is an important hallmark of the bulk nature of the quadrupole insulator that each corner hosts only one mode14 (Fig. 1b).

A tight-binding model for a two-dimensional quantized quadrupole insulator14 is shown in Fig. 1c. The dimerized hopping with amplitudes \(\lambda\) and \(\gamma\) leads to a bandgap between two pairs of degenerate bands for \(\lambda = \gamma\) (see Methods). The black (red) lines in Fig. 1c indicate positive (negative) hoppings, effectively emulating a magnetic \(\pi\) flux per plaquette. The \(\pi\) flux requires the mirror symmetry around the horizontal axis \((M_x)\) to be accompanied by a gauge transformation, leading to the non-commutation of \(M_x\) and \(M_y\). This model also has inversion \(I\) and \(C_4\) rotational symmetry (again up to a gauge transformation), which forces \(p_{x}^{\alpha} = p_{y}^{\alpha}\). Moreover, the particle–hole symmetry fixes the corner modes to the middle of the gap. For \(\gamma < \lambda\) the topological phase satisfies equation (1), whereas for \(\gamma > \lambda\) the trivial phase \((0, 0)\) is realized14. Here, we seek a mechanical implementation of a quadrupole insulator with \(x_i = D_{ij}x_j\), where the dynamical matrix \(D_{ij}\) couples local degrees of freedom \(x_i\) according to the model in Fig. 1c.

We implement the quadrupole insulator using the concept of perturbative mechanical metamaterials28. The starting point is a single-crystal silicon plate with dimensions \(5 \, \text{mm} \times 5 \, \text{mm} \times 0.364 \, \text{mm}\), the mechanical eigenmodes of which are described by the displacement field \(u(\mathbf{r})\). We work with the first non-rigid-body mode, which is characterized by two perpendicular nodal lines in the out-of-plane component of \(u(\mathbf{r})\) (Figs 1d, 2a). By spectrally separating this mode from the modes below and above it, we can describe the dynamics in some frequency range by specifying only the amplitude \(x_i\) of the mode of interest of a given plate \(i\). The hopping elements in \(D_{ij}\) are then implemented by thin beams between neighbouring plates. The nodal structure of the mode enables us to mediate couplings of either positive or negative sign, depending on which sides of the nodal lines are connected by the beams. Moreover, the distance to the nodal line controls the coupling strength that is mediated by a given beam. A combinatorial search33 followed by a gradient optimization28 leads to the design in Fig. 1d, which is characterized by a ratio of either \(|\gamma/\lambda| = 0.28\) or \(0.28 < |\gamma/\lambda| < 0.28\) (Methods).

All measurements shown are performed using the same scheme. The plates are excited with an ultrasound air transducer. The transducer has a diameter of 5 mm and is in close proximity to the sample, such that only a single plate is excited. We measure the response of the excited plate with a laser interferometer. In this way, we measure the out-of-plane vibration amplitude \(\Delta x_i \propto \psi_i^\alpha\), where \(\psi_i\) is the eigenmode at the measured frequency (the excitation strength and the measurement both scale with \(\psi_i\)). In the insets of Fig. 2a we show the local mode of a single plate measured in this way. In all other figures where energy is shown, we display the mechanical energy \(\varepsilon_i \propto \Delta x_i^2\).

To identify the in-gap states we measure \(\varepsilon_i(\nu)\) as a function of frequency \(\nu\) on all plates \(i\). We then apply the filters \(\tilde{\varepsilon}_i(\nu) = \sum_{\nu} \varepsilon_i(\nu) F_{i,\alpha}\) (shown in Fig. 2d) to separate the response of the bulk, the edges and the corners. In Fig. 2b, e we show the resulting spectra for two different samples (Methods). In the topologically trivial case with \(\gamma > \lambda\) (Fig. 2b), we observe two frequency bands in which the system absorbs energy (the theoretically predicted location of the bands is indicated in grey). Two features characterize this trivial phase. First, no frequency range is dominated by the edge or corner response. Moreover, the relative weight of the three curves is in accordance with the respective number.
The reduced-order model $D_d$ can also be used to calculate the topological indices ($p^x, p^y$). The gapped Wannier bands $\nu_+(k_x)$ and $\nu_+(k_x)$ are shown in Fig. 4b. Note that the $M_x$ and $M_y$ symmetries imply that $\nu_+(k_x) + \nu_-(k_x) = 1/2$ and $\nu_+(k_x) + \nu_-(k_x) = 1/2$, respectively. The absence of an exact $M_y$ symmetry indeed leads to a breaking of this rule. This is also reflected in the value of the polarizations of $(p^x, p^y) = (0.5, 0.56)$. As expected from the structure of $D_d$ shown in Fig. 4a the polarizations are not precisely quantized. However, in-gap corner modes are still observed because the symmetry-breaking terms do not lead to any gap closing, on the edge or in the bulk.

The results presented here underline the power of perturbative metamaterials. We have used this technique to identify a quantized quadrupole insulator, which represents a new class of topological materials. In addition, a continuous elastic system such as a silicon wafer provides a direct route to topological applications for any theoretical idea that can be represented by a tight-binding model.

Note added in proof: Two preprints that report the observation of quadrupole insulators.
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Author Contributions S.D.H. conceived the research. M.S.-G., V.P., O.R.B., and S.D.H. designed the samples. M.S.-G., V.P., S.D.H. and O.R.B. fabricated the samples. M.S.-G., V.P., S.D.H. and R.S. conducted the experiments. L.G.V. and T.L. fabricated the samples. M.S.-G., V.P. and O.R.B. designed the samples. S.D.H. (sebastian.huber@phys.ethz.ch).

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METHODS

Topological quantum number and nested Wilson loops. Here we use the lan-
guage of fermions, where bands below a given gap can be 'filled'. For the phononic case, we have to replace 'filled bands' with 'bands below the frequency of interest'. Assuming two bands $n = 1, 2$ are filled, we can use the non-Abelian Berry phase $\mathcal{A}_n(k) = -i\langle \psi(k) | \partial_k | \psi(k) \rangle$ of the Bloch wavefunctions $\psi(k)$ to construct the Wilson-loop operators $\mathcal{W}(k) = T \exp \left[ -i \oint_{\mathcal{A}_n(k)} dk \right]$. Here, $T$ denotes the path ordering along a closed loop in the Brillouin zone. The eigenvalues $e^{\pm i \pi n}$ of $\mathcal{W}(k)$ are in one-to-one correspondence with the spectrum of an edge perpendicular to the $x$ coordinate $\gamma_x$ (or perpendicular to $y$ when $x$ and $y$ are interchanged). If the edge modes are gapped, the eigenvectors $\nu_n^\pm(k)$ of $\mathcal{W}(k)$ can be used to split the filled bands in a well-defined way:

$$w_n(k) = \sum_{\nu_n^\pm(k)} \nu_n^\pm(k) \psi_n(k)$$

The normalized polarization is then defined as

$$p_n^\pm = \frac{1}{2\pi} \int \mathcal{A}_n(k) dk$$

with $\mathcal{A}_n(k) = -i\langle \psi(k) | \partial_k | \psi(k) \rangle$. The presence of two mirror symmetries that do not commute $(M_x$ and $M_y$) is a necessary requirement for the normalized polarizations $p_x$ and $p_y$ to be quantized to $0$ or $1/2$.

Model. The model shown in Fig. 1c can be expressed using the $\Gamma$ matrices $i\gamma^4 \sigma^4 \tau^1 = -\gamma^2 \tau^1 \sigma^4$ and $i\gamma^1 \sigma^4 \tau^1 = -\gamma^2 \tau^1 \sigma^4$, where $k \in \{1, 2, 3\}$ and $\tau, \sigma$ are the standard Pauli matrices:

$$D(k) = \left[ \gamma^\pm + \lambda \sigma^\pm \cos(k) \right] \sigma^4 \tau^1 + \lambda \sin(k) \sigma^4 \tau^1$$

$$= \sum_{i=1}^4 d_i(k) \gamma^i$$

where $\gamma^i$ denotes the momentum along the $i$th direction for $i = 1, 2, 3$ and $\lambda$ is the momentum along the $4$th direction, which leads to two doubly degenerate bands. Bulk gap closings occur when $d(k) = 0$, which happens for the $\Gamma_x$-symmetry case only at $k = \pm \frac{\pi}{2}$.

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The $\Gamma_x$-symmetric version of Fig. 1c is obtained by setting $\lambda = \gamma_x$ and $\gamma_x = \gamma_x$. The mirror symmetries are represented by $D(-k_x, k_y) = m_y D(k_x, k_y) m_y^\dagger$, $D(k_x, -k_y) = m_y D(k_x, k_y) m_y^\dagger$, and $D(k_x, -k_y) = m_x D(k_x, k_y) m_x^\dagger$, where $m_y = \gamma^2 \tau^1 \sigma^4$ and $m_x = \gamma^2 \tau^1 \sigma^4$. The eigenvalues of $D(k_x, k_y)$ are given by $\pm \sqrt{1 + |d(k)|}$, which leads to two degenerate bands. Bulk gap closings occur when $d(k_x, k_y) = 0$, which happens for the $\Gamma_x$-symmetry case only at $k_x = \pm \frac{\pi}{2}$.

Signal analysis. All measurements were performed with an interferome-
ter (IDS3010 from atoacube) after exciting with an ultrasound air transducer (SMATR300H19XDA from Steiner & Martins Inc.). All measurements were subject to a systematic uncertainty of the interferometer of about 5 pm, and a statistical error determined by repeated measurements of about 10 pm, resulting in an error estimate on the displacements of around 11.2 pm. Error-propagation analysis results in error bars on all of the data presented in the figures that are smaller than the symbol size. The transducer has an essentially flat frequency response over the frequencies of interest (Extended Data Fig. 2; measured with a second air transducer). The 0.46 dB variations are negligible with respect to the variations in response of 80 dB.

To remove variations in response due to slight misalignments of the measure-
ment point, we normalize the local spectra by

$$\int \Delta \nu \psi^2(\nu) d\nu \propto \psi^2(\nu) d\nu$$

as required by the completeness of the eigenmodes. This method is valid only under the assumption that all modes suffer from the same loss or, equivalently, have the same quality factor $Q \approx 1,000$ (determined from the width of the eigenmodes). This assumption is justified for the following reason. Dissipation arises from two main sources: the viscoelasticity of the sample and the dissipation into the surrounding air. For both cases, all disconnected plates suffer from the same damping. The perturbative nature of our beams (recall the bandwidth of about 5 kHz around the centre frequency of about 74 kHz), restricts also the effects of the couplings on the dissipation. Our termination is such that all plates see identical surroundings, independent of their location in the bulk, along the edges or on the corners. Moreover, spectra based on data that are not normalized (not shown) are almost identical to those shown in this paper. In all figures where arbitrary units are indicated, we normalize to the maximal value shown in the respective figure.

Because the bulk, edge and corner modes overlap spectrally, there is no unique way to separate them in our measurements. However, because the decay length is extremely short ($\xi/a \approx 1.6$, see above), a separation using the filters shown in Fig. 2d, whereby we simply select sites in the interior, along the edge and at the corner sites, is well justified.

Greens functions. In addition to the measurement of $\psi^2(\nu)$ by moving the exciter with the measurement point, we can also measure the Greens function $\psi(\nu, \nu')$ by fixing the exciter at site $j$ and moving the measurement point $i$, while exciting at frequency $\nu$. We first measure the Greens function for the four individual corners at their respective frequencies (determined from Fig. 3d); we show the results in Extended Data Fig. 3. The density maps show the measured wave function $\psi(x, y)$ and $\psi(x, y')$ along the site index $i$. The four panels demonstrate that the four corner modes are independent and that the spread in their frequencies does not arise from their hybridization. Along the edges we show the decay of the wavefunction and compare the envelope of the edges to the theoretical prediction with a decay length of $\xi/a \approx 1.6$.

In Extended Data Fig. 4 we display the analysis of the edge physics by exciting on the bottom left corner and measuring along the lines indicated in Extended Data Fig. 4a. The goal is to show that we can determine the sign of the couplings experimentally. To this end, we model our edge states by using a simple Su–Schrieffer–Heeger model:

$$D(k) = 4\pi^2 v^2 + \sum_{i=1}^N d_i(k) \sigma_i$$

where the $\sigma_i$ matrices encode the two sublattices, $k$ is the momentum along the edge, and $d_i(k) = \xi[k] + \lambda[k] \cos(k)$ and $d_i(k) = \xi[k] \sin(k)$. Along the horizontal edge, the couplings are positive $\xi = 1$, whereas along the vertical edge we have negative matrix elements $\xi = -1$. The spectrum is given by $\omega(k) = 4\pi^2 v^2 + \sqrt{\xi [d(k)]}$, with associated eigenvectors

$$\psi_n(k) = \frac{e^{i\xi n}}{\sqrt{2}} \psi(k)$$

The highest-frequency modes below the bandgap are given by $\omega_n(\pi, \pi)$ for $\xi > 0$ and $\omega_n(\pi, \pi) < \xi < 0$. For a finite edge, we can build eigenmodes from $\psi_n(k)$ that fulfill the desired boundary conditions. Note that the $\pm 1$ in the first component of $\psi_n(k)$ determines the relative sign between the modes inside one unit cell. Without specifying the exact boundary conditions, or using knowledge of the values of $\gamma$ and $\lambda$, we cannot determine this relative sign. However, we can predict that it will be different on edges with $\xi = \pm 1$.

To find the frequency of the highest mode per edge below the gap, we show the integrated weight $w = \sum \psi^2(k)$ along the respective edge in Extended Data Fig. 4b. Fixing the excitation frequencies to the indicated values, we measure the edge wavefunction for these modes. The resulting sign change is indeed different (inside versus between unit cells) on the two edges. Finally, to further justify our filtering, we show that also the decay of the edge modes follows the expected decay with $\xi/a \approx 1.6$.

Sample design. The plate geometries that we investigated were obtained in the framework of perturbative metamaterials (Extended Data Fig. 5). We combine geometric elements (silicon plates, beams and holes) to create a mate-
rial that reproduces the discrete model of ref. 14 over a range of frequencies. A perturbative metamaterial design consists of repeating basic resonating units ($3 \times 5 \times 5 \mu m \times 0.364 \mu m$ silicon plates) that weakly interact with neighbouring resonant units. Here, this weak interaction is implemented using thin silicon beams. The weak interaction has two effects: first, the modes of isolated plates hybridize to Bloch bands of small bandwidth, preventing bands originating from unwanted modes to cross in frequency; and second, the weak interaction allows us to approximate the effect of different geometric elements by adding up individual contributions (see ref. 28 for details), resulting in a marked speedup of the calculation times. The design process starts by establishing a correspondence between the degrees of freedom in the metamaterial and those in the objective discrete model. This is
done by expressing the dynamic deformation of the basic resonant units (plates) of the metamaterial as a linear combination of free-plate eigenmodes (Extended Data Fig. 3b). For sufficiently good spectral separation and sufficiently weak interactions, a single-mode local basis is enough to capture the response of the material with high precision. Each degree of freedom in the objective model is mapped to a single plate, which is assumed to vibrate in its first non-rigid-body mode, which, for our parameters has \( \delta_{\omega} \) symmetry. Then, we evaluate individual coupling-beam geometries to identify the most suitable designs and create a database relating beam geometry and coupling strength, obtained by simulating two-beam systems (Extended Data Fig. 5c). Geometries are evaluated according to three parameters: (i) the ability to attain a broad range of couplings, (ii) a low compressional strength to prevent the in-plane acoustic bands from reaching high frequencies where they could hybridize with the topological band and (iii) the absence of beam resonances in the frequency range of interest to exclude retardation effects in the couplings.

Once the database has been assembled, we start a design by quickly constructing an approximate material geometry, and then refine it by performing a gradient optimization on a full model (Extended Data Fig. 5d) that accounts for the interactions between different geometric features.

We extract the effective theory for our design by first calculating the vibrational eigenmodes of a test system (Extended Data Fig. 5c) using the commercial finite-element method (FEM) package COMSOL Multiphysics. The displacements of the eigenmodes along the three axes \( \mathbf{R}, \mathbf{V}, \) and \( \mathbf{W} \) are then interpolated over a regularly spaced grid with a pitch of 0.05 mm. This interpolation is done for each mode \( i \) and plate \( j \), and denoted by \( \phi_{ijk} \). A similar sampling is also performed for individual free-standing plates and denoted \( \phi_{ij} \) (here, the index \( k \) labels the location and component of the displacement that is being interpolated). Once this information has been extracted from finite-element simulations, the displacements of each degree of freedom for each mode are obtained by projecting the test-system displacements onto the non-rigid-body mode of the \( j \)th plate for the \( i \)th eigenmode of the test system. The use of an interpolated grid enables us to use an individually optimized mesh for each finite-element problem while still being able to express the results of one finite-element simulation in terms of those of another.

The dynamic matrix \( \mathbf{K} \) that describes the effective theory of the test system is obtained from \( \mathbf{K}_{ij} = \alpha_{ij} \omega_{ij}^{2} \phi_{ij}^{2} \). Here, \( \omega_{ij} \) is a diagonal matrix of the elements of which contain the square angular frequencies of the modes in the frequency range of interest. \( \omega_{ij}^{2} = (2\pi f_{ij})^{2} \). The resulting matrix \( \mathbf{K} \) has the same eigenfrequencies and projected eigenmodes as the full system and therefore provides a good description of the dynamics of the system. This is highlighted in Extended Data Fig. 5g, h, which presents a comparison between the dispersion relation obtained from the effective theory and that obtained by solving a full finite-element model under Bloch boundary conditions.

**Sample fabrication.** The plate and beam geometry of Fig. 1d implements the sought-after weak and strong, positive and negative coupling-matrix elements. The definition of \( \gamma \) as the hopping strength inside a unit cell and of \( \lambda \) as that between unit cells means that \( \gamma < \lambda \) is the non-trivial phase. Connected to this identification is the notion of how we are allowed to terminate the system: surfaces have to be compatible with the unit cells; that is, they are not allowed to cut through unit cells. In turn, this means that by using the design in Fig. 1d we can realize all phases shown in this paper by starting from a \( 10 \times 10 \) sample in the \((1/2, 1/2)\) phase, then moving the cut in the \( \gamma \) direction by one row of sites to reach the \((1/2, 0)\) phase, and finally moving the termination one column and to end up in the \((0, 0)\) phase. The coupling-matrix elements are given by the ratio of the effective mass density \( \rho_{\text{eff}} \) of the mode that we use and the stiffness of the beam that connects two plates. We use a 364-\( \mu \text{m} \)-thick Si wafer in the \((100)\) orientation, where we align the \( x \) and \( y \) axes of our model with the in-plane crystalline axes. The mass density of Si is \( \rho = 2.330 \text{ kg m}^{-3} \), the Young’s moduli are \( E_x = E_y = 130 \text{ GPa} \), the Poisson ratios are \( \nu_{xx} = \nu_{yy} = \nu_{xy} = 0.28 \) and the shear moduli are \( G_{xx} = G_{yy} = 79.6 \text{ GPa} \) (ref. 33). This result in an offset frequency for our mode of \( \nu_0 = 73.895 \pm 0.037 \text{ kHz} \) and coupling-matrix elements of \( \lambda = (6.69 \pm 0.17) \times 10^{7} \text{ rad s}^{-1} \) and \( \gamma = (1.89 \pm 0.07) \times 10^{8} \text{ rad s}^{-1} \). The error estimate in the next paragraph.

Our samples are fabricated out of double-side-polished 100 mm Si wafers. We measure the thickness of each wafer individually at several points across the wafer and confirm that the overall total thickness variation within each wafer that we use is at most 1 \( \mu \text{m} \). We fabricate plate and beam geometries as illustrated in Fig. 1d using standard micro-fabrication techniques. First, 1 \( \mu \text{m} \) of \( \text{SiO}_2 \) is grown on the wafers via wet thermal oxidation (to be used as an etch mask), and a 2-\( \mu \text{m} \)-thick layer of Al (that serves to protect the structure once the whole silicon has been removed) is deposited on the back side of the wafers using e-beam evaporation. A patterned, 5-\( \mu \text{m} \)-thick photore sist is used as an etch mask when patterning the front side oxide in a reactive ion etching process. Using the remaining photore sist and the underlying oxide as etch masks, we etch through the wafer with a deep reactive ion etching following a Bosch process, alternating etching and passivation cycles. The ratio between both cycles is chosen to yield vertical side walls. This angle is characterized in several points of each wafer, confirming a variation in the angle of at most 2.5°. The Si etching terminates when the back side oxide is reached. The resulting oxide and aluminium membranes suspended between the beams and plates are removed by wet etching first the aluminium and then the oxide. This step also removes any oxide leftovers present on the front side. The main three sources of error in the targeted model that arise from the sample fabrication are as follows. The first source is the total thickness variation, which we characterize as being less than 1 \( \mu \text{m} \) and so represents less than a 0.3% variation across the wafer. The second main source of error comes from different sidewall angles between different parts of the wafer, which we measure to be less than 2.5°. Hence, variation in feature sizes, when comparing front side to back side, may be up to 10 \%. For the width of the plates used, this corresponds to an error of 0.3%. The third source arises from the misalignment of the array with the material crystalline axis (100). This error has two sources: (a) wafer specifications indicate that the flap is located within \( \pm 0.5 \); and (b) specifications of our machine state that the alignment during lithography is around \( \pm 1 \). In either case, this results in an overall error of less than 0.1% in the Young’s modulus. These errors lead to the stated uncertainties in the local plate frequencies and couplings using standard elasticity theory. Finally, the wafers are clamped between two steel plates (each of 3 mm thickness; Fig. 2c). The impedance mismatch between the steel plates and the wafer leads essentially to fixed boundary conditions \( \Delta \gamma = 0 \).

**Data availability.** The data that support the findings of this study are available from the corresponding author on reasonable request.

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Extended Data Figure 1 | Phase diagram. a, Phase diagram of the model in equation (3). The brown area marks the quantized quadrupole phase (1/2, 1/2), whereas the orange areas are the (1/2, 0) and (0, 1/2) phases with no corner modes but emergent edge physics along two parallel edges. The dashed line indicates the $C_4$-symmetric line, where the bulk gap is closing at the phase transition. The transitions away from the $C_4$-symmetric line happen through bulk-induced edge transitions, where no bulk gap is closing. b, The evolution of the Wannier bands in the $x$ and $y$ directions along the path shown in a. The transition from the quadrupole phase to the (1/2, 0) phase is marked by a gap closing at 1/2, removing any polarization in the system. The second transition is induced by a gap closing at 0.
Extended Data Figure 2 | Transducer characterization. The sound pressure level (SPL) in the frequency response of the ultrasound transducer that we used over the frequency region of interest (shaded in grey). The 0.46 dB fluctuations are negligible with respect to the 80 dB variations in the measured response.
Extended Data Figure 3 | Corner Greens functions. a–d. The corner plates at (0, 9) (a), (9, 9) (b), (0, 0) (c) and (9, 0) (d) are excited at the respective edge-mode frequency. The response recorded (amplitude and phase) enables us to reconstruct the eigenfunctions $\psi(x, y)$ of the individual corner modes. Along each edge, the measured decay of the modes (black) is shown together with the theoretical prediction (orange). Given the decay length $\xi/a \approx 1.6$, where $a$ is the lattice constant, the residual weight of at most 2% at the corners other than the one that is excited stems from spurious acoustic excitation rather than hybridization.
Extended Data Figure 4 | Edge Greens function. a, Sketch of the measured system. The bottom-left plate at \((x, y) = (0, 0)\) was excited. b, Integrated frequency response \(\tilde{\psi}^2 = \sum_i \psi_i^2(n)\), where \(i\) runs along the red and black edges indicted by the solid lines in a. The two highest-frequency peaks below the bandgap (indicated by vertical lines) are analysed. c, The decay of the two edge modes into the bulk (black) along the dashed lines in a. The orange lines show the theoretical predictions. d, The mode profiles \(\psi(x, y)\) along the two edges. The edge with positive couplings has nodes between the unit cells (top panel, black), whereas the edge with negative couplings has nodes inside the unit cell (bottom panel, red), which establishes that a direct measurement of the negative couplings gives rise to the \(\pi\) flux.
Extended Data Figure 5 | Perturbative design of a quadrupole topological insulator. 

**a**, The design approach is based on establishing a correspondence between elements of an objective model (left) and geometric features of the metamaterial (right). Each degree of freedom of the objective model is mapped into a single plate (yellow arrows) by expressing the displacement of each plate as a linear combination of free-plate modes. 

**b**, Here, only the first non-rigid-body mode, which has $d_{xy}$ symmetry, is used (top left). The other modes are the second, third and fourth non-rigid-body modes. 

**c**, Independent two-plate systems simulated to create an adequate initial guess for the geometry of the system.

**d**, Four-unit-cell design simulated during the final gradient optimization.

**e**, The refined single-plate design removes material at the maximums of nearby higher-order modes.

**f**, Small trenches at the junction between beams and plates. These trenches suppress the coupling to higher-order modes by avoiding regions where these modes have a large displacement.

**g**, Dispersion along high-symmetry lines in the Brillouin zone calculated using the finite-element method. The bands that arise from the $d_{xy}$ mode are highlighted in colour.

**h**, Detailed view of the spectrum in the frequency range of interest. The dots denote the full finite-element results whereas the lines are calculated from the extracted reduced-order model.

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