ON A THEOREM OF ARVANITAKIS

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Abstract. Arvanitakis [2] established recently a theorem which is a common generalization of Michael’s convex selection theorem [11] and Dugundji’s extension theorem [7]. In this note we provide a short proof of a more general version of Arvanitakis’ result.

1. Introduction

Arvanitakis [2] established recently the following result extending both Michael’s convex selection theorem [11] and Dugundji’s simultaneous extension theorem [7]:

Theorem 1.1. [2] Let $X$ be a space with property $c$, $Y$ a complete metric space and $\Phi: X \to 2^Y$ a lower semi-continuous set-valued map with non-empty values. Then for every locally convex complete linear space $E$ there exists a linear operator $S: C(Y, E) \to C(X, E)$ such that

\begin{equation}
S(f)(x) \in \overline{\text{conv}} f(\Phi(x)) \quad \text{for all } x \in X \text{ and } f \in C(Y, E).
\end{equation}

Furthermore, $S$ is continuous when both $C(Y, E)$ and $C(X, E)$ are equipped with the uniform topology or the topology of uniform convergence on compact sets.

Here, $C(X, E)$ is the set of all continuous maps from $X$ into $E$ (if $E$ is the real line, we write $C(X)$). We also denote by $C_b(X, E)$ the bounded functions from $C(X, E)$. Recall that a set-valued map $\Phi: X \to 2^Y$ is lower semi-continuous if the set $\{x \in X : \Phi(x) \cap U \neq \emptyset\}$ is open in $X$ for any open $U \subset Y$. A space $X$ is said to have property $c$ [2] if $X$ is paracompact and, for any space $Y$ and a map $\phi: X \to Y$, $\phi$ is continuous if and only if it is continuous on every compact subspace of $X$. It is easily seen that the last condition is equivalent to $X$ being a $k$-space (i.e., the topology of $X$ is determined by its compact subsets, see [8]).

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We provide a short proof of Theorem 1.1. Here is our slightly more general version of Theorem 1.1.

**Theorem 1.2.** Let \( X \) be a paracompact space, \( Y \) a complete metric space and \( \Phi: X \to 2^Y \) a lower semi-continuous set-valued map with non-empty values. Then:

(i) For every locally convex complete linear space \( E \) there exists a linear operator \( S_b: C_b(Y, E) \to C_b(X, E) \) satisfying condition (1) such that \( S_b \) is continuous with respect to the uniform topology and the topology of uniform convergence on compact sets;

(ii) If \( X \) is a \( k \)-space or \( E \) is a Banach space, \( S_b \) can be continuously extended (with respect to both types of topologies) to a linear operator \( S: C(Y, E) \to C(X, E) \) satisfying (1).

Our proof of Theorem 1.2 is based on the idea from a result of Repovš, P.Semenov and E.Shchepin [15] that Michael’s zero-dimensional selection theorem yields the convex-valued selection theorem.

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2. Proof of Theorem 1.2

Let \( E \) be a locally convex linear space. We denote by \( E^* \) the set of all continuous linear functionals on \( E \) with the topology of uniform convergence on the weakly bounded subsets of \( E \). The second dual \( E^{**} \) is the space of continuous functionals on \( E^* \) with the topology of uniform convergence on the equicontinuous subsets of \( E^* \). It is well known that the canonical map \( E \to E^{**} \) is an embedding, see [16].

We need Banakh’s technique [4] concerning barycenters of some probability measures. First of all, for every compact space \( X \) let \( P(X) \) be the space of all regular probability measures on \( X \) endowed with the \( w^* \)-topology. Each \( \mu \in P(X) \) can also be considered as a continuous linear positive functional on \( C(X) \) (the continuous real-valued functions on \( X \) with the uniform convergence topology) with \( \mu(1_X) = 1 \), where \( 1_X \) is the constant function on \( X \) having a value one. Recall that for any \( \mu \in P(X) \) there exists a closed nonempty set \( \text{supp}(\mu) \subset X \) such that \( \mu(g) = \mu(f) \) for any \( f, g \in C(X) \) with \( f|_{\text{supp}(\mu)} = g|_{\text{supp}(\mu)} \), and \( \text{supp}(\mu) \) is the smallest closed subset of \( X \) with this property. If \( X \) is a Tychonoff space, we consider the following subsets of \( P(\beta X) \), where \( \beta X \) is the Čech-Stone compactification of \( X \):

\[
P_\beta(X) = \{\mu \in P(\beta X) : \text{supp}(\mu) \subset X\}
\]

and

\[
\hat{P}(X) = \{\mu \in P(\beta X) : \mu_*(X) = 1\}.
\]
Here $\mu_*(X) = \sup\{\mu(B) : B \subset X\}$ is a Borel subset of $\beta X$. Every map $h: M \to E$ generates a map $P_\beta(h): P_\beta(M) \to P_\beta(E)$ defined by $P_\beta(h)(\mu) = \mu(\phi \circ h)$, where $\mu \in P_\beta(M)$ and $\phi \in C_b(E)$. In particular, if $i_M: M \hookrightarrow E$ is the inclusion of $M$ into $E$, then $P_\beta(i_M)$ is one-to-one and $P_\beta(\delta_x) = \delta_x$ for all $x \in M$ ($\delta_x$ is the Dirac measure at the point $x$). The functors $\hat{P}$ and $P_\beta$ were introduced in [13] and [6], respectively.

Banakh [4] defined barycenters of measures from $\hat{P}(M)$, where $M$ is a weakly bounded subset of some locally convex linear space $E$. For any such $M \subset E$ there exists an affine map (called a barycenter map) $b_M: \hat{P}(M) \to E^{**}$ which is continuous only when $M$ is bounded in $E$, see [4, Theorem 3.2]. A convex subset $M \subset E$ is called barycentric if $b_M(\hat{P}(M)) \subset M$. It was established in [4, Proposition 3.10] that any complete bounded convex subset of $E$ is barycentric. Since for any $M$ we have $P_\beta(M) \subset \hat{P}(M)$, we can apply the Banakh arguments with $\hat{P}(M)$ replaced by $P_\beta(M)$, and this is done in the following proposition.

**Proposition 2.1.** Let $E$ be a complete locally convex linear space. Then there exists a not necessarily continuous affine map $b_E: P_\beta(E) \to E$ such that $b_E(\mu) \in \overline{\text{conv}}(\text{supp}(\mu))$ for every $\mu \in P_\beta(E)$. Moreover, if $M \subset E$ is a bounded set then the map $b_E \circ P_\beta(i_M): P_\beta(M) \to E$ is continuous.

**Proof.** We follow the arguments from [4]. For every $\mu \in P_\beta(E)$ we consider the functional $b_E(\mu): E^* \to \mathbb{R}$, defined by $b_E(\mu)(l) = \mu(l|\text{supp}(\mu))$, $l \in E^*$.

**Claim.** $b_E(\mu)$ is continuous for all $\mu \in P_\beta(E)$.

Indeed, suppose $\{l_\alpha\} \subset E^*$ is a net in $E^*$ converging to some $l_0 \in E^*$. This means that $\{l_\alpha\}$ is uniformly convergent to $l_0$ on every weakly bounded subset of $E$. In particular, $\{l_\alpha\}$ is uniformly convergent to $l_0$ on $\text{supp}(\mu)$. Consequently, $\{\mu(l_\alpha)\}$ converges to $\mu(l_0)$.

Therefore, $b_E(\mu) \in E^{**}$ for any $\mu \in P_\beta(E)$. On the other hand, since $\text{supp}(\mu) \subset E$ is compact and $E$ is complete, $C(\mu) = \overline{\text{conv}}(\text{supp}(\mu))$ is a compact convex subset of $E$. Then, according to [4, Proposition 3.10], $C(\mu)$ is barycentric and contains $b_E(\mu)$. So, $b_E$ maps $P_\beta(E)$ into $E$. The second half of Proposition 2.1 follows from the fact that $E$ is embedded in $E^{**}$ and Theorem 3.2 from [4], which (in our situation) states that the map $b_E \circ P_\beta(i_M): P_\beta(M) \to E^{**}$ is continuous provided $M$ is bounded in $E$.

The theory of maps between compact spaces admitting averaging operators was developed by Pelczyński [13]. For noncompact spaces we use the following definition [17]: a surjective continuous map $f: X \to Y$...
admits an averaging operator with compact supports if there exists an embedding \( g: Y \to P_{\beta}(X) \) such that \( \text{supp}(g(y)) \subset f^{-1}(y) \) for all \( y \in Y \). Then the regular linear operator \( u: C_b(X) \to C_b(Y) \), defined by
\[
(2) \quad u(h)(y) = g(y)(h), \ h \in C_b(X), \ y \in Y
\]
satisfies \( u(\phi \circ f) = \phi \) for any \( \phi \in C_b(Y) \). Such an operator \( u \) is called averaging for \( f \).

**Proposition 2.2.** Let \( f: X \to Y \) be a perfect map admitting an averaging operator with compact supports and \( E \) a complete locally convex linear space. Then there exists a linear operator \( T_b: C_b(X, E) \to C_b(Y, E) \) such that:

(i) \( T_b(h)(y) \in \overline{\text{conv}}(h(f^{-1}(y))) \) for all \( y \in Y \) and \( h \in C_b(X, E) \);

(ii) \( T_b(\phi \circ f) = \phi \) for any \( \phi \in C_b(Y, E) \);

(iii) \( T_b \) is continuous when both \( C_b(X, E) \) and \( C_b(Y, E) \) are equipped with the uniform topology or the topology of uniform convergence on compact sets.

Moreover, if \( Y \) is a \( k \)-space or \( E \) is a Banach space, \( T_b \) can be extended to a linear operator \( T: C(X, E) \to C(Y, E) \) satisfying conditions (i) – (iii) with \( C_b(X, E) \) and \( C_b(Y, E) \) replaced, respectively, by \( C(X, E) \) and \( C(Y, E) \).

**Proof.** A similar statement to the first part was proved in [17] Proposition 3.1. We fix an embedding \( g: Y \to P_{\beta}(X) \) with \( \text{supp}(g(y)) \subset f^{-1}(y), y \in Y \). For every \( h \in C_b(X, E) \) consider the map
\[
(3) \quad T_b(h): Y \to E, \ T_b(h)(y) = b_E(P_{\beta}(i_{h(X)})(\nu_y)),
\]
where \( i_{h(X)}: h(X) \hookrightarrow E \) is the inclusion and \( \nu_y \in P_{\beta}(h(X)) \) is the measure \( P_{\beta}(h)(g(y)) \). According to Proposition 2.1, \( T_b(h) \) is continuous (recall that \( h(X) \subset E \) is bounded). It also follows from the definition of the map \( b_E \) that \( T_b \) is linear. Since \( \text{supp}(g(y)) \subset f^{-1}(y) \) and \( \text{supp}(P_{\beta}(i_{h(X)})(\nu_y)) \subset h(f^{-1}(y)), y \in Y \), we have \( b_E(P_{\beta}(i_{h(X)})(\nu_y)) \subset \overline{\text{conv}}(h(f^{-1}(y))) \) (see Proposition 2.1). So, \( T_b \) satisfies condition (i).

Moreover, \( T_b(h) \) belongs to \( C_b(Y, E) \) because \( T_b(h)(y) \subset \overline{\text{conv}}(h(X)) \) for all \( y \in Y \). It follows directly from (2) and (3) that \( T_b \) satisfies condition (ii). To prove (iii), assume \( K \subset Y \) is compact and let \( W_1 = \{ \phi \in C_b(Y, E) : \phi(K) \subset V_1 \} \), where \( V_1 \) is a convex neighborhood of 0 in \( E \). Obviously, \( W_1 \) is a neighborhood of the zero function in \( C_b(Y, E) \). Take a convex neighborhood \( V_2 \) of 0 in \( E \) with \( \overline{V}_2 \subset V_1 \) and let \( W_2 = \{ h \in C_b(X, E) : h(H) \subset V_2 \}, H = f^{-1}(K) \). Since \( H \) is compact (recall that \( f \) is a perfect map), \( W_2 \) is a neighborhood of 0 in \( C_b(X, E) \). Moreover, for all \( y \in Y \) and \( h \in W_2 \) we have \( T_b(h)(y) \subset \overline{\text{conv}}(h(H)) \subset \overline{V}_2 \subset V_1 \). So, \( T_b(W_2) \subset W_1 \). This provides
continuity of $T_b$ with respect to the topology of uniform convergence on compact sets. Similarly, one can show that $T_b$ is also continuous with respect to the uniform topology.

Assume that $Y$ is a $k$-space and $h \in C(X, E)$. Then formula (3) provides a map $T(h) : Y \rightarrow E$ satisfying conditions (i) and (ii). We need to show that $T(h)$ is continuous on every compact set $L \subset Y$. And this follows from Proposition 2.1 because the set $h(f^{-1}(L)) \subset E$ is compact. So, $T(h)$ is continuous and, obviously, $T(h) = T_b(h)$ for all $h \in C_b(X, E)$. Continuity of $T$ follows from the same arguments we used to prove continuity of $T_b$.

If $E$ is a Banach space, then every $T(h), h \in C(Y, E)$, is continuous without the requirement $Y$ to be a $k$-space. Indeed, we fix $y_0 \in Y$ and $h \in C(X, E)$. Let $V$ be a bounded closed neighborhood of $h(f^{-1}(y_0))$ in $E$. Then $h^{-1}(V)$ is a neighborhood of $f^{-1}(y_0)$ and, since $f$ is a perfect map, there exists a closed neighborhood $U$ of $y_0$ in $Y$ with $W = f^{-1}(U) \subset h^{-1}(V)$. Then, according to Proposition 2.1, the map $b_E \circ P_{\beta}(iv) : P_{\beta}(V) \rightarrow E$ is continuous. On the other hand $P_{\beta}(h)$ maps continuously $P_{\beta}(W)$ into $P_{\beta}(V)$ and $g(U) \subset P_{\beta}(W)$ is homeomorphic to $U$ (recall that $g$ is an embedding of $Y$ into $P_{\beta}(X)$). Hence, $T(h)$ is continuous on $U$. Because $U$ is a neighborhood of $y_0$ in $Y$, this implies continuity of $T(h)$ at $y_0$.

**Proof of Theorem 1.2.** Suppose $X, Y, \Phi$ and $E$ satisfy the hypotheses of Theorem 1.2. By [15] (see also [14]), there exists a zero-dimensional paracompact space $X_0$ and a perfect surjection $f : X_0 \rightarrow X$ admitting a regular averaging operator. By Proposition 2.2, there exists a linear operator $T_b : C_b(X_0, E) \rightarrow C_b(X, E)$ satisfying conditions (i) – (iii). The map $\Phi(x) : 2^X \rightarrow X$, $\Phi(x) = \Phi(f(x))$, is lower semi-continuous with closed non-empty values in $Y$. So, according to the Michael's 0-dimensional selection theorem [12], $\Phi$ has a continuous selection $\theta : X_0 \rightarrow Y$. Now, we define the linear operator $S_b : C_b(Y, E) \rightarrow C_b(X, E)$ by $S_b(h) = T_b(h \circ \theta), h \in C_b(Y, E)$. Obviously, $\theta(f^{-1}(x)) \subset \Phi(x)$ for every $x \in X$. Then, according to (i), for all $h \in C_b(Y, E)$ and $x \in X$ we obtain

$$S_b(h)(x) = T_b(h \circ \theta)(x) \subset \text{conv}((h \circ \theta)(f^{-1}(x))) \subset \text{conv}(h(\Phi(x))).$$

Continuity of $S_b$ follows from continuity of $T_b$ and the map $\theta$.

If $X$ is a $k$-space or $E$ is a Banach space, the operator $T_b$ can be extended to a linear operator $T : C(X_0, E) \rightarrow C(X, E)$ satisfying conditions (i) – (iii) from Proposition 2.2. Then $S : C(Y, E) \rightarrow C(X, E)$, $S(h) = T(h \circ \theta)$, is the required linear operator extending $S_b$. 

Arvanitakis' theorem

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3. Remarks

Let us show first that Theorem 1.2 implies Michael’s selection theorem. Assume $X$ is paracompact, $Y$ is a Banach space and $\Phi: X \to 2^Y$ a lower semi-continuous map with closed convex values. Then, by Theorem 1.2 there exists a linear operator $S: C(Y,Y) \to C(X,Y)$ satisfying condition (1). Since the values of $\Phi$ are convex and closed, condition (1) yields that $S(id_Y)(x) \in \Phi(x)$ for all $x \in X$, where $id_Y$ is the identity on $Y$. Hence, $S(id_Y)$ is a continuous selection for $\Phi$.

The original Dugundji theorem [7] states that if $X$ is a metric space, $A \subset X$ its closed subset and $E$ a locally convex linear space, then there exists a linear operator $S: C(A,E) \to C(X,E)$ such that $S(f)$ extends $f$ for any $f \in C(A,E)$. When both $E$ and $A$ are complete, Dugundji theorem can be derived from Theorem 1.2. Indeed, let $A$ be a completely metrizable closed subset of a paracompact $k$-space $X$ and $E$ a complete locally convex linear space. Consider the set-valued map $\Phi: X \to 2^A$, $\Phi(x) = \{x\}$ if $x \in A$ and $\Phi(x) = A$ if $x \notin A$. Let $S: C(A,E) \to C(X,E)$ be a linear operator satisfying (1). Then $S(f)(x) = f(x)$ for all $f \in C(A,E)$ and $x \in A$. So, $S$ is an extension operator. If $X$ is not necessarily a $k$-space, there exists an extension linear operator $S_b: C_b(A,E) \to C_b(X,E)$.

Heath and Lutzer [10, Example 3.3] provided an example of a paracompact $X$ and a closed set $A \subset X$ homeomorphic to the rational numbers such that there is no extension operator from $C(A)$ to $C(X)$. This space is the Michael’s line, i.e., the real line with topology consisting of all sets of the form $U \cup V$, where $U$ is an open subset of the rational numbers and $V$ is a subset of the irrational numbers. It is easily seen that this a $k$-space. So, the assumption in the above result $A$ to be completely metrizable is essential.

The original Dugundji theorem with $E$ complete can be derived from Proposition 2.2 and the well known fact that every closed subset of a zero-dimensional metric space $X$ is a retract of $X$, see for example [9, Problem 4.1.G]. Indeed, assume $X$ is a metric space and $A \subset X$ its closed subset. By [6], there exists a zero-dimensional metric space $X_0$ and a perfect surjection $f: X_0 \to X$ admitting an averaging operator. Let $A_0 = f^{-1}(A)$ and $r: X_0 \to A_0$ be a retraction. Define the linear operator $S: C(A,E) \to C(X,E)$ by $S(h) = T(h \circ f \circ r)$, where $E$ is a complete locally convex linear space, $h \in C(A,E)$ and $T: C(X_0,E) \to C(X,E)$ is the operator from Proposition 2.2. It follows from Proposition 2.2(i) that $S$ is an extension operator.

The proof of Theorem 1.2 is based on two main facts: the 0-dimensional Michael’s selection theorem and the Repovš-Semenov-Shchepin result.
that each paracompactum is a continuous image of under a perfect map admitting an averaging operator. So, the 0-dimensional Michael’s selection theorem implies not only the convex-valued section theorem, but it also implies the Dugundji extension theorem. Actually we have the following corollary from Proposition 2.2 (Sel(Φ) denotes all continuous selections for Φ).

**Corollary 3.1.** Let \( f : X \to Y \) be a perfect map admitting an averaging operator with compact supports and \( E \) a Banach space. Suppose \( \Phi : Y \to 2^E \) is a lower semi-continuous set-valued map with closed convex non-empty values. Then there exists an affine map from \( \text{Sel}(\Phi \circ f) \) to \( \text{Sel}(\Phi) \) which is continuous when both \( \text{Sel}(\Phi \circ f) \) and \( \text{Sel}(\Phi) \) are equipped with the uniform topology or the topology of uniform convergence on compact sets.

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