A Dolbeault lemma for temperate currents.

Henri Skoda *

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Dedicated to the memory of Pierre Dolbeault.

Abstract

We consider a bounded Stein open subset Ω of $\mathbb{C}^n$. We prove that if $f$ is a current on $\mathbb{C}^n$ of bidegree $(p, q + 1)$, $\overline{\partial}$-closed on $\Omega$, we can find a current $u$ on $\mathbb{C}^n$ of bidegree $(p, q)$ which is a solution of the equation $\overline{\partial}u = f$ in $\Omega$. In other words, we prove that the Dolbeault complex of temperate currents on $\Omega$ (i.e. currents on $\Omega$ which extend to currents on $\mathbb{C}^n$) is concentrated in degree 0. Moreover if $f$ is a current of order $k$, then we can find a solution $u$ on $\Omega$ which is a current of order at most $k + 2n + 1$.

Keywords: Stein open subset of $\mathbb{C}^n$, $L^2$ estimates, $\overline{\partial}$-operator, Dolbeault $\overline{\partial}$-complex, temperate distributions and currents, temperate cohomology, Sobolev spaces.

1 Introduction

We will prove the following result in the same way as the famous ” Dolbeault-Grothendieck ” lemma for $\overline{\partial}$.

Theorem 1. Let $\Omega$ be a bounded Stein open subset of $\mathbb{C}^n$ and $f$ be a given current of bidegree $(p, q + 1)$ on $\mathbb{C}^n$ (with compact support) which is $\overline{\partial}$-closed on $\Omega$. Then there exists a current $u$ of bidegree $(p, q)$ (with compact support) in $\mathbb{C}^n$ such that:

$$\overline{\partial}u = f,$$

in $\Omega$.

Moreover if $f$ is a current of order $k$ (resp. if $f \in H^{s}_{(p,q+1)}(\mathbb{C}^n)$ for some $s > 0$), we can find a solution $u$ which is a current of order at most $k + 2n + 1$ (resp. $u \in H^{-s-2n-1}_{(p,q)}(\mathbb{C}^n)$), more precisely if $k$ is the integer such that: $s \leq k < s + 1$, for every $r > k$, we can find $u \in H^{-r-2n}_{(p,q)}(\mathbb{C}^n)$.

*Corresponding author: Henri Skoda, Sorbonne University, IMJ-PRG, 4 Place Jussieu, 75005 Paris, France, E-mail: henri.skoda@imj-prg.fr
We say that a current $T$ on $\Omega \subset \mathbb{C}^n$ is temperate if and only if it can be extended to $\mathbb{C}^n$. With other words, we have:

**Corollary 2.** For a given relatively compact Stein open subset of $\mathbb{C}^n$, the Dolbeault $\bar{\partial}$-cohomology of temperate currents on $\Omega$ vanishes.

As usual, we denote by $H^s_{(p,q)}(\mathbb{C}^n)$ the space of current on $\mathbb{C}^n$ of bidegree $(p, q)$ the coefficients of which are distributions in the Sobolev space $H^s(\mathbb{C}^n)$. A distribution $T \in D'(\mathbb{R}^n)$ is of order $k \in \mathbb{N}$ if it is locally a finite linear combination of derivatives of order at most $k$ of Radon measures on $\mathbb{R}^n$ or equivalently if $T$ can be extended as a continuous linear form defined on all functions of class $C^k$ with compact support in $\mathbb{R}^n$ or equivalently if for every relatively compact open subset $\Omega \subset \mathbb{R}^n$, all functions $\phi \in D(\Omega)$ verify an inequality:

$$|< T, \phi >| \leq C(\Omega, T) \sup_{x \in \Omega} \sum_{\alpha \leq k} |D^\alpha \phi(x)|,$$

in which the constant $C(\Omega, T)$ only depends on $\Omega$ and $T$. Of course a current is of order $k$, if its coefficients are distributions of order $k$.

This result answers a question raised by Pierre Schapira in a personal discussion. He hopes it can be useful to make significant progress in the Microlocal Analysis theories highlighted for instance in the papers of M. Kashiwara and P. Schapira, [KS1996] and [Scha2017].

Even though the result is essentially a consequence of L. Hörmander’s $L^2$ estimates for $\bar{\partial}$ (corollary 4), it seems that it cannot be explicitly found in the literature on the subject (with complete proof). Let us observe the following features of the result. No assumption of smoothness is required for $\Omega$. The given current $f$ and the solution $u$ have coefficients in spaces of distributions $H^s(\mathbb{C}^n)$ with $s \leq 0$. Hence they are never supposed to be smooth but with temperate singularities as for instance derivatives of Dirac measures and the result is quite different from the most usual regularity results for $\bar{\partial}$ involving $C^k$ regularity up to the boundary of $\Omega$ ($k \geq 0$) both for $\Omega$ and for the given differential forms on $\Omega$. If $f \in H^s_{(p,q+1)}(\mathbb{C}^n)$ for some $s \geq 0$, then $f \in L^2_{(p,q+1)}(\Omega)$ and the result is an immediate consequence of Hörmander’s theorem which provides a solution $u$ in $L^2_{(p,q)}(\Omega)$. Then $u$ has a trivial extension in $L^2_{(p,q)}(\mathbb{C}^n)$ (by 0 outside $\Omega$).

The gap $2n + 1$ of regularity for the solution $u$ does not depend on $\Omega$. In the basic example $\Omega = B(0, R) \setminus H$ in which $H$ is a complex analytic hypersurface of the ball $B(0, R)$ of center 0 and radius $R$, the result does not depend at all on the complexity of the singularities of $H$ (and on the degree of $H$ when $H$ is algebraic). The gap $2n + 1$ is an automatic consequence of the method of proof. To improve this gap $2n + 1$ does not seem to immediately have a major interest for the purpose in [Scha2017].

We need four steps to prove theorem 1. At first, as P. Dolbeault in [Dol1956], by solving an appropriate Laplacian equation $\frac{1}{2} \Delta v = \bar{\partial}^* f$ on $\mathbb{C}^n$ (for the usual Laplacian on $\mathbb{C}^n$ defined on differential forms and currents) and replacing $f$ by $f - \bar{\partial} v$, we reduce the problem to the case of a current $f$
which has harmonic coefficients on $\Omega$. As $f$ is temperate, the mean value properties of harmonic functions imply that $f$ grows at the boundary of $\Omega$ like a negative power of the distance $d(z, \partial \Omega)$ to the boundary of $\Omega$ (for $z \in \Omega$). Then Hörmander’s $L^2$ estimates for $\bar{\partial}$ give a solution $u$ such that 

$$\int_{\Omega} |u|^2 |d(z, \partial \Omega)|^l d\lambda(z) < +\infty$$

with temperate growth on $\Omega$ (for some $l > 0$). Finally using an extension theorem of L. Schwartz [Schw1950] for distributions, $u$ can be extended as a current on $\mathbb{C}^n$.

Similar methods, were already used by P. Lelong [Le1964] for the Lelong-Poincaré $\partial \bar{\partial}$-equation and by H. Skoda [Sk1971] for the $\bar{\partial}$-equation to obtain solutions explicitly given on $\mathbb{C}^n$ by integral representations and with precise polynomial estimates. Y.T. Siu has already studied holomorphic functions of polynomial growth on bounded open domain of $\mathbb{C}^n$ using Hörmander’s $L^2$ estimates for $\bar{\partial}$ in Siu [1970].

We establish the preliminary results we need in Section 2 and we prove theorem 1 in Section 3.

In the case of a subanalytic bounded open Stein subset $\Omega$ in a Stein manifold $X$, Pierre Schapira in [Scha2020] gives independently a proof of the first part of Theorem 1 (i.e. of corollary 2). His proof is basically founded on cohomological methods which are particularly well adapted to the subanalytic case. It also heavily depends on Hörmander’s $L^2$ estimates for $\bar{\partial}$ in the case of a Stein open bounded subset of $\mathbb{C}^n$. He also uses Lojasiewcz inequalities and another Hörmander’s inequality for subanalytic subsets.

I thank Pierre Schapira very much for raising his insightful question which has strongly motivated this research.

2 Preliminary Definitions and Results

Before proving theorem 1 we need to remind several classical results. We have sometimes given direct proof to establish the results in the appropriate form we wish.

An open subset of $\mathbb{C}^n$ is called Stein if it is holomorphically convex: for all compact $K$ in $\Omega$ the holomorphic hull $\hat{K}_\Omega$ of $K$ is compact ($x \in \hat{K}_\Omega$ if and only if $x \in \Omega$ and for all holomorphic function $f$ on $\Omega$, $|f(x)| \leq \max_{\xi \in K} |f(\xi)|$).

Let us recall the following fundamental Hörmander’s $L^2$ existence theorem for $\bar{\partial}$ [Hör1966] or [Hör1965]. We can also use J.P. Demailly’s book [Dem2012], Chapter VIII, paragraph 6, Theorem 6.9, p. 379. We denote by $L^2_{(p,q)}(\Omega, loc)$ the vector space of current of bidegree $(p,q)$ in $\Omega$ the coefficients of which are in $L^2(\Omega, loc)$ for the usual Lebesgue measure $d\lambda$ on $\mathbb{C}^n$.

**Theorem 3.** Let $\Omega$ be an open pseudoconvex subset of $\mathbb{C}^n$ and $\phi$ a plurisubharmonic function defined on $\Omega$. For every $g \in L^2_{(p,q+1)}(\Omega, loc)$ with $\bar{\partial}g = 0$
such that: \( \int_{\Omega} |g|^2 e^{-\phi} d\lambda < +\infty \), there exists \( u \in L^2_{(p,q)}(\Omega, \text{loc}) \) such that:

\[
(2) \quad \bar{\partial} u = g
\]
in \( \Omega \) and:

\[
(3) \quad \int_{\Omega} |u|^2 e^{-\phi} (1 + |z|^2)^{-2} d\lambda \leq \frac{1}{2} \int_{\Omega} |g|^2 e^{-\phi} d\lambda.
\]

If \( \Omega \) is bounded, \( u \) verifies the \( L^2 \) estimate:

\[
(4) \quad \int_{\Omega} |u|^2 e^{-\phi} d\lambda \leq C(\Omega) \int_{\Omega} |g|^2 e^{-\phi} d\lambda
\]

with \( C(\Omega) := \frac{1}{4} (1 + \max_{z \in \Omega} |z|^2)^2 \).

The classical Oka-Norguet-Bremerman theorem (\cite{Hör1966} paragraph 2.6 and theorem 4.2.8) claims that the following assertions are equivalent:

1) \( \Omega \) is Stein,
2) \( \Omega \) is pseudoconvex: \( \text{i.e.} \) there exists a plurisubharmonic function \( \phi \) on \( \Omega \) which is exhaustive (for all \( c \in \mathbb{R} \) the subset \( \{ z \in \Omega | \phi(z) < c \} \) is relatively compact in \( \Omega \)),
3) the function \( -\log d(z, \partial \Omega) \) is plurisubharmonic in \( \Omega \).

Therefore for a given \( k \geq 0 \), we can choose \( \phi(z) = -k \log d(z, \partial \Omega) \) in the inequality (4) and we will only need to use the following special case of theorem 3 (see also \cite{Hör1965} theorem 2.2.1).

**Corollary 4.** Let \( \Omega \) be a bounded Stein open subset of \( \mathbb{C}^n \) and \( k \geq 0 \) be a given real number. Then for every \( g \in L^2_{(p,q+1)}(\Omega, \text{loc}) \) with \( \bar{\partial} g = 0 \) such that: \( \int_{\Omega} |g|^2 [d(z, \partial \Omega)]^k d\lambda < +\infty \), there exists \( u \in L^2_{(p,q)}(\Omega, \text{loc}) \) such that:

\[
(5) \quad \bar{\partial} u = g
\]
in \( \Omega \) and:

\[
(6) \quad \int_{\Omega} |u|^2 [d(z, \partial \Omega)]^k d\lambda \leq C(\Omega) \int_{\Omega} |g|^2 [d(z, \partial \Omega)]^k d\lambda
\]

If we denote by \( L^2_{(p,q)}(\Omega) \) the space of \( u \in L^2_{(p,q)}(\Omega, \text{loc}) \) such that \( \int_{\Omega} |u|^2 [d(z, \partial \Omega)]^k d\lambda < +\infty \), by:

\[
(7) \quad L^2_{0,(p,q)}(\Omega) := \{ u \in L^2_{(p,q)}(\Omega) | \bar{\partial} u \in L^2_{(p,q+1)}(\Omega) \}
\]

and by \( O^2_{(p,0)}(\Omega) := \{ u \in L^2_{(p,0)}(\Omega) | \bar{\partial} u = 0 \} \), corollary 4 means that the following Dolbeault-complex is exact:

\[
(8) \quad 0 \to O^2_{(p,0)}(\Omega) \to L^2_{0,(p,0)}(\Omega) \xrightarrow{\bar{\partial}} L^2_{0,(p,1)}(\Omega) \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} L^2_{0,(p,q)}(\Omega) \xrightarrow{\bar{\partial}} L^2_{0,(p,q+1)}(\Omega) \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} L^2_{0,(p,n)}(\Omega) \to 0.
\]

We also need two results of real analysis.
Lemma 5. Let \( w \) be a distribution on \( \mathbb{R}^n \) of order \( k \) which is harmonic (for the usual Laplacian on \( \mathbb{R}^n \)) on the bounded open subset \( \Omega \) of \( \mathbb{R}^n \). Then \( w \) is of polynomial growth on \( \Omega \): 
\[
|w(z)| \leq C(\Omega, w) \left[ d(z, \mathbb{R}^n \setminus \Omega) \right]^{-k-n}
\]
where the constant \( C(\Omega, w) \) only depends on \( \Omega \) and \( w \).
If \( w \in H^{-s}(\mathbb{R}^n) \) for \( s \geq 0 \) we have: 
\[
|w(z)| \leq C(\Omega, w) \left[ d(z, \mathbb{R}^n \setminus \Omega) \right]^{-k-\frac{n}{2}}
\]
where \( k \) is the integer such that \( s \leq k < s + 1 \).

Proof. Of course we can suppose that \( w \) has a compact support in \( \mathbb{R}^n \).
Let \( \rho \) be a non negative regularizing function in \( \mathcal{D}(\mathbb{R}^n) \) which only depends on \( |\zeta| \), has its support in the Euclidean ball of radius 1 and verifies: 
\[
\int_{\mathbb{R}^n} \rho(\zeta) d\lambda(\zeta) = 1 \quad \text{where } d\lambda \text{ is the Lebesgue measure on } \mathbb{R}^n.
\]
Let \( \rho_\epsilon(\zeta) := \frac{1}{\epsilon^n} \rho(\frac{\zeta}{\epsilon}) \) be the associate family of regularizing functions in \( \mathcal{D}(\mathbb{R}^n) \) so that \( \rho_\epsilon \) has its support in the ball of radius \( \epsilon \) and verifies too 
\[
\int_{\mathbb{R}^n} \rho_\epsilon(\zeta) d\lambda(\zeta) = 1.
\]
As \( w \) is harmonic in \( \Omega \), for every \( z \in \Omega, w(z) \) coincide with its mean-value on every Euclidean sphere of center \( z \) and radius \( r < d(z, \partial \Omega) \). Therefore using Fubini’s theorem we get for every \( \epsilon < d(z, \partial \Omega) \):
\[
(9) \quad w(z) = \int_{\mathbb{R}^n} w(z + \zeta) \rho_\epsilon(\zeta) d\lambda(\zeta) = \int_{\mathbb{R}^n} w(\zeta) \rho_\epsilon(z - \zeta) d\lambda(\zeta).
\]
\( i.e. \) \( w = w \ast \rho_\epsilon \) on \( \Omega_\epsilon := \{ z \mid d(z, \partial \Omega) < \epsilon \} \) (in which \( \ast \) represents a convolution product).
Testing \( w \) as a distribution on the test function (in the variable \( \zeta \)): \( \rho_\epsilon(z - \zeta) \) with \( \epsilon < d(z, \partial \Omega) \leq 1 \), equation (9) becomes:
\[
(10) \quad w(z) = < w(\zeta), \rho_\epsilon(z - \zeta) >_{\zeta}.
\]
As \( w \) is a distribution of order \( k \) (with compact support), we have for every function \( \phi \in \mathcal{D}(\mathbb{R}^n) \) an inequality:
\[
(11) \quad |< w, \phi >| \leq C_1(w) \sup_{\zeta \in \mathbb{C}^n} \Sigma_{|\alpha| \leq k} |D_\zeta^\alpha \phi(\zeta)|,
\]
in which \( C_1(w) > 0 \) is a constant only depending on \( w \).
Taking \( \phi(\zeta) = \rho_\epsilon(z - \zeta) \), we get:
\[
(12) \quad |w(z)| \leq C_1(w) \sup_{|k| \leq \epsilon} \Sigma_{|\alpha| \leq k} |D_\zeta^\alpha \rho_\epsilon(z - \zeta)|,
\]
and:
\[
(13) \quad |w(z)| \leq C_2(w) \epsilon^{-n-k},
\]
for some constant \( C_2(w) > 0 \).
As it is true for every \( \epsilon < d(z, \partial \Omega) \), we take the limit as \( \epsilon \to d(z, \partial \Omega) \) and we get:
\[
(14) \quad |w(z)| \leq C_2(w) \left[ d(z, \partial \Omega) \right]^{-l}.
\]
with \( l = n + k \) and then:

\[
\int_{\Omega} |w|^2 |d(z, \partial\Omega)|^2 d\lambda < \infty.
\]

If we now assume that \( w \in H^{-s}(\mathbb{R}^n) \) for a given \( s > 0 \), equation (10) becomes:

\[
|w(z)| = |< w(\zeta), \rho(\zeta - \zeta) > \zeta | \leq \|w\|_{H^{-s}(\mathbb{R}^n)} \|\rho(\zeta - \zeta)\|_{H^s(\mathbb{R}^n)}.
\]

Let \( k \) be the integer defined by \( s \leq k < s + 1 \) so that (denoting as usual by \( \hat{\phi} \) the Fourier transform of \( \phi \)):

\[
\|\phi\|_{H^k(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^k |\hat{\phi}(\xi)|^2 d\lambda(\xi) \leq \int_{\mathbb{R}^n} (1 + |\xi|^2)^k |\hat{\phi}(\xi)|^2 d\lambda(\xi) = \|\phi\|_{H^k(\mathbb{R}^n)}^2.
\]

As \( k \) is an integer, the norm \( \|\phi\|_{H^k(\mathbb{R}^n)} \) is equivalent to the sum of the \( L^2 \) norms of the derivatives of \( \phi \) of order less or equal to \( k \), we have:

\[
\|\phi\|_{H^k(\mathbb{R}^n)}^2 \leq C_2(k, n) \int_{\mathbb{R}^n} |D^\alpha \phi|^2 d\lambda
\]

We replace \( \phi \) by \( \phi_\epsilon(\zeta) := \frac{1}{\epsilon^n} \phi\left(\frac{\zeta}{\epsilon}\right) \) (with \( \epsilon \leq 1 \)) so that we get:

\[
\|\phi_\epsilon\|_{H^k(\mathbb{R}^n)}^2 \leq C_2(k, n) \epsilon^{-2k-n} \int_{\mathbb{R}^n} |D^\alpha \phi|^2 d\lambda
\]

Using (13), (17) and (19) with \( \phi(\zeta) = \rho(\zeta - \zeta) \) (for a fixed \( z \in \Omega \) with \( \epsilon < d(z, \partial\Omega) \leq 1 \)) we finally obtain:

\[
|w(z)| \leq C_3(k, n) \|w\|_{H^{-s}(\mathbb{R}^n)} \epsilon^{-k-n}\frac{\epsilon}{\epsilon}
\]

(with \( C_3(k, n) := [C_2(k, n)]^{\frac{1}{2}} \)) and when \( \epsilon \to d(z, \partial\Omega) \):

\[
|w(z)| \leq C_3(k, n) \|w\|_{H^{-s}(\mathbb{R}^n)} [d(z, \partial\Omega)]^{-k-\frac{n}{2}}.
\]

Therefore we also have:

\[
\int_{\Omega} |w(z)|^2 |d(z, \partial\Omega)|^{2k+n} d\lambda(z) < +\infty.
\]
Remark 1. Instead of using mean properties of harmonic functions, one can also use the elementar solution of $\Delta$ in $\mathbb{R}^n$ as in [KS1996] proposition 10.1., p. 53.

We also need the following theorem of L. Schwartz (in his book on distribution theory [Schw1950]). We can also directly use theory of Sobolev spaces. We say that a measure $\mu$ defined on an open bounded subset $\Omega$ of $\mathbb{R}^n$, is of polynomial growth at most $l$ in $\Omega$, if $\int_{\Omega} |d(z, \partial \Omega)|^l d\mu(z) < +\infty$.

**Theorem 6.** A measure of polynomial growth $l$ defined on an open bounded subset $\Omega$ of $\mathbb{R}^n$ can be extended as a distribution on $\mathbb{R}^n$ of order at most $l$. Moreover if $w \in L^2(\Omega, \text{loc})$ verifies the estimate: $\int_{\Omega} |w(z)|^2 [d(z, \partial \Omega)]^{2l} d\lambda(z) < +\infty$, with $l \in \mathbb{N}$, then for every $r > l$, $w$ can be extended as a distribution in $H^{-r/2}(\mathbb{R}^n)$.

**Proof.** In L. Schwartz’s book there is no proof and no references so that we give the following proof (probably well known). We consider the subspace $F \subset D(\mathbb{C}^n)$ of functions the derivatives of which vanish at the order $\leq l - 1$ in every point $\zeta \in \partial \Omega$. For a given $z \in \Omega$, we choose a point $\zeta \in \partial \Omega$ such that $|z - \zeta| = d(z, \partial \Omega)$ and we apply Taylor’s formula at the point $\zeta \in \partial \Omega$, at the order $l - 1$ (with integral remainder, cf. [Hör1983] paragraph 1.1, formula (1.1.7)) to a function $\phi \in F$ restricted to the real interval $\{tz + (1 - t)\zeta \mid t \in \mathbb{R}, 0 \leq t \leq 1\}$ linking in $\Omega$ the point $\zeta \in \partial \Omega$ to $z \in \Omega$, so that we obtain:

$$
(23) \quad \phi(z) = l \int_0^1 (1 - t)^l \left[ \sum_{|\alpha| = l} D^\alpha \phi(\zeta + t(z - \zeta)) \frac{(z - \zeta)^\alpha}{\alpha!} \right] dt,
$$

and then:

$$
(24) \quad |\phi(z)| \leq C_4(l, n) [d(z, \partial \Omega)]^l \max_{\xi \in \Omega} |\sum_{|\alpha| = l} |D^\alpha \phi(\xi)||.
$$

For all functions $\phi \in F$ and all measures $\mu$ on $\Omega$ of polynomial growth $l$, i.e $\int_{\Omega} |d(z, \partial \Omega)|^l d\mu(z) < +\infty$, (using (24)) we have:

$$
(25) \quad |\int_{\Omega} \phi d\mu| \leq C_4(l, n) \left[ \int_{\Omega} |d(z, \partial \Omega)|^l d\mu| \right] \max_{\xi \in \Omega} |\sum_{|\alpha| \leq l} |D^\alpha \phi(\xi)||.
$$

For a given measure $\mu$ of polynomial growth $l$, we consider the space $E^l(\mathbb{C}^n)$ of functions of class $C^l$ on $\mathbb{C}^n$. We apply Hahn Banach theorem to the linear form $\phi \rightarrow \int_{\Omega} \phi d\mu$ defined on the subspace $F \subset D(\mathbb{C}^n) \subset E^l(\mathbb{C}^n)$ and continuous for the seminorm $\max_{\xi \in \Omega} |\sum_{|\alpha| \leq l} |D^\alpha \phi(\xi)||$. This linear form can be extended in a continuous linear form $T$ on $E^l(\mathbb{C}^n)$, such that:

$$
(26) \quad |\langle T, \phi \rangle| \leq C_4(l, n) \left[ \int_{\Omega} |d(z, \partial \Omega)|^l d\mu| \right] \max_{\xi \in \Omega} |\sum_{|\alpha| \leq l} |D^\alpha \phi(\xi)||.
$$
for all $\phi \in \mathcal{E}^l(\mathbb{C}^n)$, i.e. a distribution of order $l$ on $\mathbb{C}^n$ (with compact support).

Let us now assume that $w \in L^2(\Omega, \text{loc})$ verifies the estimate:

$$I_l(w) := \int_{\Omega} |w(z)|^2 \left| [d(z, \partial\Omega)]^{2l} \right| d\lambda(z) < +\infty,$$

for some integer $l \geq 0$.

For every $\phi \in F$, Schwarz inequality gives:

$$| < w, \phi > |^2 = | \int_{\Omega} w\phi \ d\lambda |^2 \leq I_l(w) \int_{\Omega} |\phi(z)|^2 \left| [d(z, \partial\Omega)]^{-2l} \right| d\lambda(z).$$

Using inequality (24), (28) becomes:

$$| < w, \phi > | \leq C_5(l, n, \Omega) I_l(w) \left[ \max_{\xi \in \Omega} |D_\xi \phi(\xi)| \right]^2$$

with $C_5(l, n, \Omega) := |C_4(l, n)|^2 \int_{\Omega} d\lambda$.

For every $r > l$, we use classical Sobolev inequality:

$$\max_{\xi \in \mathbb{R}^n} \Sigma_{|\alpha| \leq l} |D_\xi^\alpha \phi(\xi)| \leq C_6(r, n) \|\phi\|_{H^{r+\frac{n}{2}}}$$

and inequality (29), so that we obtain:

$$| < w, \phi > | \leq C_7(l, n, r, \Omega) \left[ I_l(w) \right]^\frac{1}{2} \|\phi\|_{H^{r+\frac{n}{2}}}.$$

with $C_7(l, n, r, \Omega) := C_6(r, n) |C_5(l, n, \Omega)|^\frac{1}{2}$.

Using still Hahn-Banach Theorem for the linear form $\phi \rightarrow < w, \phi >$ defined on the subspace $F$ of $H^{r+\frac{n}{2}}(\mathbb{R}^n)$ and continuous for the norm of $H^{r+\frac{n}{2}}$, we extend $w$ as a distribution $T \in H^{-r-\frac{n}{2}}$ such that: $\|T\|_{H^{-r-\frac{n}{2}}} \leq C_7(l, n, r, \Omega) \left[ I_l(w) \right]^\frac{1}{2}$ (of course we can also do this extension by using orthogonal projection on the closed subspace $\bar{F}$ in the Hilbert space $H^{r+\frac{n}{2}}(\mathbb{R}^n)$).

We can now prove theorem 1.

## 3 Proof of theorem 1

We follow P. Dolbeault’s proof of the Dolbeault-Grothendieck lemma. A. Grothendieck’s proof was different, (in some sense) more elementar than P. Dolbeault’s proof but not useful for our present purpose. Of course we can suppose (w.l.o.g.) that $f$ has compact support in $\mathbb{C}^n$ (using a cutoff function in $\mathcal{D}(\mathbb{R}^n)$ equal to 1 in a neighborhood of $\bar{\Omega}$). Let us remind that $\mathbb{C}^n$ being equipped with its usual flat Hermitian metric, the Laplacian acting on differential forms and currents is defined on $\mathbb{C}^n$ by:

$$\frac{1}{2} \Delta = \frac{1}{2} (dd^* + d^* d) = \partial \bar{\partial}^* + \bar{\partial} \partial^* = \partial \bar{\partial} + \bar{\partial} \partial,$$

where $\partial$ is the exterior derivative and $\bar{\partial}$ the conjugate of the exterior derivative.
so that $\frac{1}{2}\Delta f$ is the usual Laplacian on $\mathbb{C}^n$ acting on each coefficient of the current $f$. $\bar{\partial}^*$ (resp. $\partial^*$) (resp. $d^*$) is the adjoint of $\bar{\partial}$ (resp. $\partial$) (resp. $d := \partial + \bar{\partial}$) for the same constant metric on $\mathbb{C}^n$ (there is no weight function).

At first we solve in $\mathbb{C}^n$ the Laplacian equation:

\[(33) \quad \frac{1}{2}\Delta v := (\bar{\partial}\bar{\partial}^* + \partial^*\partial) v = \bar{\partial}^* f.\]

($v$ and $\bar{\partial}^* f$ are of bidegree $(p, q)$.)

If we write: $f = \sum_{|I|=p,|J|=q+1}^{'} f_{I,J}dz_I \wedge d\bar{z}_J$ ($\Sigma'$ means that we only sum on strictly increasing multi-indices $I$ and $J$), we have (cf. [Hör1966] paragraph 4.1, p. 82 or 85 or [Dem2012] Chapter 6, paragraph 6.1):

\[(34) \quad \bar{\partial}^* f = (-1)^{p-1}\Sigma'_{|I|=p,|K|=q}^{'} \left[ \sum_{j=1}^{j=n} \frac{\partial}{\partial z_j} (f_{I,jK}) \right] dz_I \wedge d\bar{z}_K.\]

If $f$ is of bidegree $(0, 1)$, we simply have: $f = \sum_{j=1}^{j=n} f_{j} d\bar{z}_j$, $\bar{\partial}^* f = -\sum_{j=1}^{j=n} \frac{\partial f_{j}}{\partial z_j}$ and (33) is the Laplace equation in $\mathbb{C}^n$: $\sum_{j=1}^{j=n} \frac{\partial^2}{\partial z_j^2} v = \sum_{j=1}^{j=n} \frac{\partial f_{j}}{\partial z_j} v$.

$v$ is obtained by convolution of each coefficient $\frac{\partial f_{I,jK}}{\partial z_j}$ of $\bar{\partial}^* f$ in (34) with the elementar solution $E$ of the usual Laplacian in $\mathbb{C}^n$.

We set:

\[(35) \quad g := f - \bar{\partial} v\]

As $\bar{\partial}^2 = 0$, we have (by usual computation):

\[(36) \quad \frac{1}{2}\Delta (\bar{\partial} v) = (\bar{\partial}\bar{\partial}^* + \partial^*\partial) \bar{\partial} v = \bar{\partial}\bar{\partial}^* \bar{\partial} v = \bar{\partial}(\bar{\partial}\bar{\partial}^* + \partial^*\partial) v = \bar{\partial}(\frac{1}{2}\Delta v)\]

i.e. $\bar{\partial}$ commute with $\Delta$ on $\mathbb{C}^n$. Using (33), we have: $\frac{1}{2}\Delta (\bar{\partial} v) = \bar{\partial}\bar{\partial}^* f$, and:

\[(37) \quad \frac{1}{2}\Delta g = \frac{1}{2}\Delta f - \frac{1}{2}\Delta (\bar{\partial} v) = (\bar{\partial}\bar{\partial}^* + \partial^*\partial) f - \bar{\partial}\bar{\partial}^* f = \bar{\partial}^* \partial f.\]

Hence (as $\bar{\partial} f = 0$ on $\Omega$), $g$ is harmonic on $\Omega$:

\[(38) \quad \Delta g = 0.\]
with a distribution of order $k$ and compact support is still of order at most $k$. Hence $\bar{\partial}v$ is of order at most $k + 1$ and $g = f - \bar{\partial}v$ is too of order at most $k + 1$.

We write: $g = \sum_{|I|=p, |J|=q+1} g_{I,J}d\zeta_I \wedge d\bar{\zeta}_J$ with strictly increasing multi-indices $I$ and $J$. Let $g_{I,J}$ be a coefficient of $g$. As $g_{I,J}$ is harmonic in $\Omega$, we can apply lemma 5 in $\mathbb{R}^{2n}$ to $g_{I,J}$ which is a distribution of order at most $k + 1$, we get an inequality:

$$\tag{39} |g_{I,J}(z)| \leq C_1(\Omega, g_{I,J}) |d(z, \partial\Omega)|^{2n-k-1}.$$ 

Hence:

$$\tag{40} |g(z)| \leq C_2(\Omega, g) |d(z, \partial\Omega)|^{-l}$$

for some constant $C_2(\Omega, g) > 0$ and $l := 2n + k + 1$ (and $z \in \Omega$) and then:

$$\tag{41} \int_\Omega |g|^2 |d(z, \partial\Omega)|^{2l} d\lambda < \infty$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{C}^n$.

L. Hörmander’s $L^2$ estimates for $\bar{\partial}$ (corollary 4) provide a solution $u$ in $\Omega$ of the equation:

$$\tag{42} \bar{\partial}u = g$$

with the $L^2$ estimate:

$$\tag{43} \int_\Omega |u|^2 |d(z, \partial\Omega)|^{2l} d\lambda < \infty$$

As $\Omega$ is bounded, Schwarz inequality gives the following $L^1$ estimate:

$$\tag{44} \int_\Omega |u| |d(z, \partial\Omega)|^l d\lambda < \infty$$

Therefore a coefficient $u_{I,J}$ of $u$ defines a measure of polynomial growth $l$ on $\Omega$. Using L. Schwartz’s theorem 6 such a measure (of polynomial growth $l$) defined on $\Omega$ can be extended as a distribution on $\mathbb{C}^n$ (of order at most $l$) so that $u$ can be extended as a current on $\mathbb{C}^n$ of order at most $l$. Then $u + v$ is a current on $\mathbb{C}^n$ verifying:

$$\tag{45} \bar{\partial}(u + v) = f$$

on $\Omega$. Moreover $u + v$ has order at most $l = k + 2n + 1$.

We now consider the case of a given $f \in H_{(p,q+1)}^{-s}(\mathbb{C}^n)$ for some $s \geq 0$. 

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10
\[ \partial^* f \in H^{-s-1}_{(p,q)}(\mathbb{C}^n). \]

Classically we can find a solution \( v \) of the Laplace equation \( (33) \) in \( H^{-s+1}_{(p,q)}(\mathbb{C}^n) \). We apply lemma \( 5 \) to every coefficient \( g_{l,j} \) of \( g \) in \( \mathbb{C}^n = \mathbb{R}^{2n} \) so that \( |g(z)| \leq C |d(z, \mathbb{C}^n \setminus \Omega)|^{-k-n} \) where \( k \) is the integer such that \( s \leq k < s + 1 \) and therefore:

\[ \int_{\Omega} |g|^2 |d(z, \partial \Omega)|^{2k+2n} d\lambda < +\infty. \]

Corollary \( 4 \) implies we can solve \( \partial u = g = f - \partial v \) with the estimate:

\[ \int_{\Omega} |u|^2 |d(z, \partial \Omega)|^{2k+2n} d\lambda < +\infty. \]

We now apply theorem \( 9 \) in \( \mathbb{C}^n = \mathbb{R}^{2n} \) to every coefficient of \( u \) with \( l = k + n \) (\( l + n = k + 2n \)) so that for every \( r > k \), \( u \) can be extended as a current in \( \mathbb{C}^n \) in \( H^{-r-2n}_{(p,q)}(\mathbb{C}^n) \). As \( v \in H^{-s+1}_{(p,q)}(\mathbb{C}^n) \), \( u+v \) is too in \( H^{-r-2n}_{(p,q)}(\mathbb{C}^n) \) and verifies \( \partial(u+v) = f \) in \( \Omega \).

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