Coarse graining in spin foam models

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March 11, 2002

Abstract

We formulate the problem of finding the low-energy limit of spin foam models as a coarse-graining problem in the sense of statistical physics. This suggests that renormalization group methods may be used to find that limit. However, since spin foams are models of spacetime at Planck scale, novel issues arise: these microscopic models are sums over irregular, background-independent lattices. We show that all of these issues can be addressed by the recent application of the Kreimer Hopf algebra for quantum field theory renormalization to non-perturbative statistical physics. The main difference from standard renormalization group is that the Hopf algebra executes block transformations in parts of the lattice only but in a controlled manner so that the end result is a fully block-transformed lattice.

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1 Introduction

Progress on several fronts in the search for a quantum theory of gravity has resulted in the first detailed models of the microscopic structure of spacetime: spin foams. The first such models [1, 2, 3, 4] were based on the predictions for Planck scale geometry from loop quantum gravity and, in particular, the discovery that spatial areas and volumes have discrete spectra [5] (for a recent thorough review of loop quantum gravity, see [7]). Currently, there exist several such models, which also include structures from other approaches to quantum gravity, from Lorentzian path integrals [8, 9, 10] and causal sets [11, 12] to euclidean general relativity [2, 13, 14, 15], and from string networks [16] to topological quantum field theories (see [17}). For reviews see [17, 18].

In spin foam models, the microscopic degrees of freedom are representations and intertwiners of the appropriate group (originally SU(2)) and live on a branched 2-surface, or 2-complex. A specific model is given by a partition function that sums over all microscopic degrees of freedom and all (model-dependent) weights on the vertices of the 2-complex. It also sums over all 2-complexes that interpolate between the given in and out 3-geometry states, making spin foams a path-integral approach to quantum gravity.

Clearly, a spin foam model will be a good candidate for a quantum theory of gravity only if it can be shown to have a good low energy limit, which contains the known theories, namely, general relativity and quantum field theory. As in [19], in this paper we suggest that this should be treated as a problem in statistical physics. That is, for spin foams in the correct class of microscopic models, we should find the known macroscopic theory by integrating out the microscopic degrees of freedom. What is required, then, is a renormalization group approach for spin foams. This appears promising since spin foams are given by a partition function which has the same functional form as that for a spin system or lattice gauge theory (apart from the sum over all 2-complexes). Also, spin foam models are “atomic”, that is, there is a minimum length, the Planck scale.

The most intuitive way of understanding the renormalization group in statistical physics is real space renormalization, where we coarse-grain the lattice. The aim of this paper is to set up such a coarse-graining method for spin foams. We can list the basic features of the problem of spin foam coarse-graining:
1. The microscopic degrees of freedom are representations of a group or algebra. A block transformation on a spin foam involves (exact or approximate) summing over the possible values these can take. This is cumbersome, but conceptually straightforward. (Such calculations have been carried out, for example, in [15].)

2. The weights in the partition function are amplitudes rather than probabilities.

3. The lattices are the highly irregular spin foam 2-complexes. Simplifying the problem by restricting to regular 2-complexes does not appear to be an option. It would be a drastic modification of the model, as these are rare configurations in the full sum.

4. Spin foams are background-independent. This means that we cannot use a global lattice spacing as a parameter in our coarse-graining procedure. However, there is a minimum length, the Planck length $l_{\text{Pl}}$, so it is clear what a single block transformation does: it increases the scale from $l_{\text{Pl}}$ to a multiple of it. This is equivalent to increasing the lattice spacing, but the problem is that, without a background and on an irregular lattice, we cannot do this everywhere at the same time, as is required to obtain a configuration space renormalization group equation.

5. A spin foam partition function has the same functional form as a lattice gauge theory or a spin system, except for one important difference: the spin foam partition function contains a sum over all 2-complexes with the given boundary. Therefore, we need to coarse-grain sums over lattices.

1, 2 and 3 above are technically challenging, but there are existing examples of such problems in statistical physics. 4 and 5, however, are novel issues, due to the fact that these are microscopic models of spacetime itself.

In this paper, we address 4 and 5 and propose a method to deal with them. Essentially, we generalize the renormalization group transformation for a lattice gauge system to a set of local block transformations and this directly addresses 4 (by “local” we mean that we block transform parts of the lattice only). These local transformations are equivalence relations in
a Hopf algebra, thus the order in which they are performed is controlled. Furthermore, the elements of the algebra are sums over spin foams, so 5 is directly taken into account. It is not surprising that this method of coarse-graining differs from standard renormalization group. Roughly speaking, the usual renormalization group is embedded in the Hopf algebra.

The outline of this paper is as follows. In the next Section we briefly review the definition of spin foams and the sense in which they are microscopic models of spacetime. For coarse-graining purposes (configuration space renormalization group), spin foams, like any other lattice model, have to be partitioned into the appropriate subfoams that will be block-transformed. In Section 3, we define labelled partitioned spin foams and subfoams, and the corresponding (parenthesized) weights. These are the elements of the Hopf algebra whose operations are given in Section 4. Again, the labelled partitioned spin foams correspond to parenthesized weights, and the Hopf algebra operations of such weights is given in Section 5. A renormalization group transformation is an equivalence relation in this algebra (Section 6). By incorporating this equivalence relation into the antipode, in Section 7, we use this modified antipode to perform local scale transformations. This is essentially a generalization of the renormalization group equation, and we illustrate this in Section 8 by using the modified antipode to coarse-grain an ordinary regular square lattice. In Section 9, we discuss the general form of the modified antipode. We discuss our results and the possibilities for applying our method to specific spin foam models in the Conclusions.

The Hopf algebra we use is the one that Kreimer showed underlies the renormalization of quantum field theory \cite{20, 21}. In general, the elements of this algebra are rooted trees. In quantum field theory, the branches represent subdivergencies of a Feynman diagram. In our case, the branches are spin subfoams. In Kreimer’s case, the Hopf algebra takes care of the combinatorial part of the problem of renormalization in quantum field theory. However, its structure is very general and it underlies block transformations in the non-perturbative renormalization group, as we showed in \cite{19}. We can transcribe Kreimer’s algebra to spin foams even though the physics that dresses the rooted trees is very different from QFT: in spin foams we do have a UV cutoff and want the large $N$ limit.

Closing the Introduction, we would like to stress that, as in other problems in statistical physics, the renormalization group and its variation that we propose here, is more like a general set of ideas than a recipe which can be
directly applied to any system. Further progress should be made by analyzing individual models as well as by experimental input.

2 Spin foams as models of the microscopic structure of spacetime

A spin foam is a labelled 2-complex whose faces are labelled by representations of some group $G$, the edges by intertwiners in the group, and the vertices carry the evolution amplitudes. These are functions of the faces and edges that meet on that vertex that code the evolution rules for the model (figure 1).

A spin foam model is given by a partition function of the form

$$Z(s_i, s_f) = \sum_{\Gamma} N(\Gamma) \sum_{\text{labels on } \Gamma} \prod_{f \in \Gamma} \dim j_f \prod_{v \in \Gamma} A_v(j). \quad (1)$$

The first sum is over all spin foams $\Gamma$ interpolating between a given initial spin network $s_i$ and a final one $s_f$. $\dim j_f$ is the dimension of the $G$ representation

---

Footnote: A spin network is a graph whose edges are labelled by representations of the group $G$ and its nodes are labelled by the intertwiners. Here, we regard spin networks as the “spacelike slices” of a spin foam: If we take a spacelike cut through a spin foam, we obtain a graph. Its edges are cuts through the spin foam faces, and so we label them with the same representations. Its nodes are cuts of the spin foam edges, and we label them with intertwiners. **Notation:** In the literature, “spin network” often refers to a graph labelled
labelling the face \( f \) of \( \Gamma \). \( A_v \) is the amplitude on the vertex \( v \) of \( \Gamma \), a given function of the labels on the faces and edges adjacent to \( v \). A choice of the group \( G \) and the functions \( A_v \) (and possibly a restriction on the allowed 2-complexes) defines a particular spin foam model. For reviews of spin foams see \([17, 18]\).

Much of the motivation for considering spin foam models comes from the surprising result of loop quantum gravity: the quantized spatial area and volume are discrete \([6]\). \( SU(2) \) spin networks are the basis states of the loop quantization of general relativity. They diagonalize the quantum area and volume operators whose spectrum was discovered to be discrete (see \([7]\) and references therein).

In loop quantum gravity, the spin networks are embedded in the spatial manifold of the classical canonical theory. Here we are considering abstract spin foams, that is, spin foams which are combinatorial cell complexes and not embedded in some preexisting continuous manifold. Spacelike cuts through abstract \( SU(2) \) spin foams are Penrose spin networks, the combinatorial graphs Penrose used to recover angles in 3-space from the rules of angular momentum \([22]\).

As mentioned in the introduction, other degrees of freedom such as matter or supersymmetric ones can be introduced by using or adding the appropriate group representations.

Currently, there are several spin foam models, candidates for the microscopic structure of spacetime. The very first test such a model has to pass is to have a good low energy limit, where it reproduces the known theories, general relativity and quantum field theory. What that limit means is not yet clear. Our working assumption here is that spin foams are models of spacetime similar to the way that spin systems are microscopic models of matter. What is then required is that, in the low-energy limit, the spin foam observables, coarse-grained over many Planck lengths, agree with those in a classical spacetime.

In solid state physics, the correspondence between the microscopic spin system model and the macroscopic matter is obtained using renormalization group techniques. We propose that the same basic idea should also apply to spin foams. The starting point in this paper is the observation that in
equation (1), $Z_\Gamma$ for a single spin foam is a generalization of the partition function of a spin system or lattice gauge system to variables that are group representations and functions of these representations. Since $Z_\Gamma$ factorizes to a product over local labels and functions, it is possible to carry out block transformations by summing over internal variables.

In the next Section, we describe what block transformations involve for a spin foam.

3 Block transformations and partitioned spin foams

A renormalization group transformation of a lattice $\Gamma$ is a procedure that involves two steps. One is the calculation of a typical block transformation in the theory. This involves picking some small sublattice $\gamma$ of $\Gamma$ that carries only few microscopic degrees of freedom. A block transformation eliminates some of them, either by summing over all their possible values, or by some approximate recipe (decimation, truncation etc).

For a spin foam theory, an example of such a block transformation is the calculation by Perez and Rovelli on the modified Barrett-Crane model [15]. This is a block transformation that sums over all internal variables. For example, the spin foam

\[
\gamma = \quad \text{Diagram}
\]
is replaced by the coarse-grained spin foam

\[ \gamma' = \text{Diagram} \]

where a summation has been performed over all allowed values of the labels on the faces we eliminated.

Having obtained the functional form of such a block transformation, the next step is to repeatedly apply it on the entire spin foam \( \Gamma \) to obtain a coarse-grained one, \( \Gamma' \).

In lattice gauge theory, or in spin systems, the lattices are usually regular and this is an unambiguous procedure. We simply partition the lattices uniformly into sublattices of the form we have used for the calculation of the single block transformation, and perform this transformation on each such sublattice.

In the spin foam case, however, the 2-complexes are highly irregular. This makes this second step a non-trivial combinatorial problem. The algebraic method we propose in this article deals with this second part of a renormalization group transformation (and also deals with the issue of the choice of a scaling parameter in a background independent theory, as we will discuss in Section 7).

Our motivation is a rather simple observation which we will motivate diagrammatically on a 2-dimensional spin foam. Suppose I have calculated a block transformation

\[ \gamma = \text{Diagram} \quad \rightarrow \quad \gamma' = \text{Diagram} \]
(the fat node denotes a renormalized vertex), and now I wish to block-transform the larger lattice

\[ \Gamma = \]

To do so, I should first choose a partitioning of \( \Gamma \) into subfoams, such that the smallest ones are of the same form as \( \gamma \). Ideally, I should choose a partition such that, when I have block-transformed away all the smallest subfoams, I will be left with next-smallest subfoams which again have the same form as \( \gamma \), so I can use again the same block transformation. Such a partition is, for example,

\[ \Gamma = \]

(2)

Block-transforming \( \Gamma \) according to the marked partition, gives the sequence

It is this nesting structure of subfoams that we will use in this paper to construct an algebra for coarse-graining. As we saw, to begin coarse-graining a lattice, we need to first partition it into sublattices. It is not surprising,
therefore, that the elements of our algebra will be *partitioned spin foams* and their corresponding weights. In the remainder of this section we define partitioned spin foams and parenthesized spin foam weights.
3.1 Partitioned spin foam 2-complexes

A subfoam $\gamma$ of a spin foam $\Gamma$ is a subset of the faces of $\Gamma$, together with any vertices and edges that are boundaries of these faces:

$$\Gamma = \gamma \quad \rightarrow \quad \gamma = \cdot$$

The weight of a subfoam $\gamma$ is

$$\omega_\gamma = \prod_{f \in \gamma} \dim j_f \prod_{v \in \gamma} A_v. \quad (3)$$

A subfoam is connected if every face in $\gamma$ shares at least one edge with some other face in $\gamma$. A subfoam may be a set of disconnected subfoams.

A subfoam $\gamma_1$ is nested in $\gamma_2$, $\gamma_1 \subset \gamma_2$, if the set of faces of $\gamma_1$ is a proper subset of the faces of $\gamma_2$. Two subfoams are disjoint, $\gamma_1 \cap \gamma_2 = \emptyset$, if they have no faces, edges or vertices in common. When two subfoams are neither nested nor disjoint, they are overlapping.

A partitioned spin foam is a spin foam 2-complex $\Gamma$ marked with an allowed partition into subfoams $\{\gamma_i\}$. “Allowed” means that there are no overlapping subfoams, that is, any two subfoams $(\gamma_1, \gamma_2)$ in the partition are either nested: $\gamma_1 \subset \gamma_2$ or $\gamma_2 \subset \gamma_1$, or they are disjoint: $\gamma_1 \cap \gamma_2 = \emptyset$.

The remainder $\Gamma/\gamma$ of the subfoam $\gamma$ is the 2-complex we obtain by deleting $\gamma$ from $\Gamma$, that is, eliminating all faces of $\gamma$ and joining the vertices on the boundary of each of its connected components into a single vertex for

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\[\text{The edges of spin foams are labelled by intertwiners of the group } G. \text{ However, by appropriate normalizations, these can be absorbed in the vertex amplitudes. We have not, therefore, included labels on the edges in the partition function, and we will not worry about them in the definition of an allowed partition.}\]
that component. There is \textit{no} amplitude on that vertex and we will denote this in diagrams by a white vertex:

\[
\Gamma = \begin{array}{c}
\includegraphics[width=0.3\textwidth]{example1.png}
\end{array} \quad \rightarrow \quad \Gamma/\gamma = \begin{array}{c}
\includegraphics[width=0.3\textwidth]{example2.png}
\end{array}
\]

Naturally, the remainder is also a partitioned spin foam, with the partition it inherits from \( \Gamma \). \textit{In the rest of this paper we will only work with partitioned spin foams, but we will call them simply spin foams.}

While the spin foam 2-complex is a useful pictorial representation of a contribution in the partition function \( Z \) of the theory, the physical content lies in the corresponding weight. Next, we consider the analogue of the nesting structure of spin foams for their weights.

### 3.2 Parenthesized spin foam weights

Given a spin foam \( \Gamma \), the weight it contributes to \( Z \) is \( \omega_\Gamma = \prod_f \dim_j f \prod_v A_v \) in the notation defined in section 2. This is the physical content of the spin foam, and the quantity we wish to coarse-grain.

We will represent the weight of a partitioned spin foam by a \textit{parenthesized weight}. That is, we enclose in brackets the factors in \( Z \) that correspond to the subfoams in the partition. For nested subfoams, we obtain nested brackets, \((())\), and for disjoint subfoams disjoint brackets, \( ()() \). For example,
the parenthesized weight for the 2-dimensional spin foam

\[ \omega_{\Gamma} = \left( \left( (\omega_{\gamma_1}) (\omega_{\gamma_2}) \omega_{\gamma_3/\gamma_1 \cup \gamma_2} \right) \omega_{\gamma_3/\gamma_3} \right) \]

\[ = \left( ((d_i A_{v_1} A_{v_2} A_{v_3}) (d_n A_{v_4} A_{v_5} A_{v_6}) d_m A_{v_7}) d_j d_k d_p \right). \]

where we have used \( d_i \) as shorthand for \( \text{dim} i \).

An example in 3 dimensions is the spin foam

\[ \Gamma = \]

\[ \gamma_1 \]

\[ \gamma_2 \]

\[ \gamma_3 \]
This has weight
\[
\omega_\Gamma = \left( \omega_{\gamma_1} \frac{\omega_\Gamma}{\omega_{\gamma_1}} \right) = \left( (d_i d_k d_l A_1 A_2) (d_o d_p d_q d_r A_3 A_4 A_5 A_6) d_id_od_\gamma dtdw \right) .
\]  

(6)

4 The Hopf algebra of partitioned spin foams

Partitioned spin foams form a Hopf algebra whose operations we list in this section. Let \( V \) be the vector space of spin foams over the complex numbers \( \mathbb{C} \). A general element of \( V \) has the form \( \Gamma = \sum_i c_i \Gamma_i, c_i \in \mathbb{C} \). Let \( e \) be the empty spin foam. The following operations can be defined on \( V \):

- **Multiplication.** Multiplication \( m : V \otimes V \rightarrow V \) is the disjoint union of spin foams:
  \[
m(\Gamma_1 \otimes \Gamma_2) = \Gamma_1 \cdot \Gamma_2 := \Gamma_1 \cup \Gamma_2 .
\]  
  (7)

Since the order of multiplying the weights of two disjoint spin foams in \( Z \) does not matter, multiplication is commutative. Further, \( \Gamma \cdot e = e \cdot \Gamma = \Gamma \) for all \( \Gamma \in V \).

- **Unit.** The unit operation \( \epsilon : \mathbb{C} \rightarrow V \) creates spin foams:
  \[
  \epsilon(c) = ce \quad c \in \mathbb{C} .
\]  
  (8)

- **Counit.** The counit \( \bar{\epsilon} : V \rightarrow \mathbb{C} \) annihilates all non-empty spin foams:
  \[
  \bar{\epsilon}(\Gamma) = \begin{cases} 
  0 & \Gamma \neq e \\
  1 & \Gamma = e
  \end{cases} .
\]  
  (9)

- **Coproduct.** The partitioning of a spin foam lets us define an operation that unfolds the nesting structure of its subfoams. This is the coproduct \( \Delta : V \rightarrow V \otimes V \), which splits a spin foam \( \Gamma \) into all possible pairs of subfoams in the given partition paired with their remainders:
  \[
  \Delta(\Gamma) = \Gamma \otimes e + e \otimes \Gamma + \sum \gamma \otimes \Gamma/\gamma .
\]  
  (10)

The subfoams \( \gamma \) in the above sum range over all subfoams in the partition of \( \Gamma \), except the empty one and \( \Gamma \) itself which we wrote separately as the first two terms.
On the empty spin foam $e$, we get $\Delta(e) = e \otimes e$ and, for a product we have $\Delta(\Gamma_1 \cdot \Gamma_2) = \Delta(\Gamma_1) \Delta(\Gamma_2)$. Also, if $\gamma_p$ is a spin foam that has no subfoams,

$$\Delta(\gamma_p) = \gamma_p \otimes e + e \otimes \gamma_p. \quad (11)$$

We call such a spin foam *primitive*. One can check that $\Delta$ is coassociative but not cocommutative\(^3\).

The coproduct is a very important operation for our coarse-graining. Essentially, it identifies the subfoams that we should block transform (those that appear on the left of the tensor products) and so it prepares the spin foam for coarse-graining.

**Example 1.** The coproduct for the spin foam

$$\Gamma = \gamma_1 \cdot \gamma_2 \quad (12)$$

with the marked partition is

$$\Delta(\Gamma) = \Gamma \otimes e + e \otimes \Gamma + \gamma_1 \otimes \Gamma/\gamma_1 + \gamma_2 \otimes \Gamma/\gamma_2$$

$$= \quad \otimes e + e \otimes + \nabla \otimes \quad (13)$$

$$+ \nabla \otimes$$

\(^3\) Partitioned spin foams as well as their parenthesized weights are labelled rooted trees as in Kreimer [20]. Therefore, for coassociativity, cocommutativity, properties of the antipode etc, we refer the reader to [20, 21].
**Example 2.** The spin foam

\[
\Gamma = \begin{array}{c}
\gamma_1 \\
\gamma_2
\end{array}
\]

containing two disjoint spin foams \(\gamma_1\) and \(\gamma_2\) has coproduct

\[
\Delta(\Gamma) = \Gamma \otimes e + e \otimes \Gamma + \gamma_1 \otimes \Gamma/\gamma_1 + \gamma_2 \otimes \Gamma/\gamma_2 + \gamma_1\gamma_2 \otimes \Gamma/\gamma_1\gamma_2
\]

\[
= \begin{array}{c}
\gamma_1 \otimes e + e \otimes \gamma_1 \\
\gamma_2 \otimes e + e \otimes \gamma_2
\end{array}
\]

\[
+ \begin{array}{c}
\gamma_1 \otimes \gamma_2 \\
\gamma_2 \otimes \gamma_1
\end{array}
\]

\[
+ \begin{array}{c}
\gamma_1 \otimes \gamma_2 \\
\gamma_2 \otimes \gamma_1
\end{array}
\]
An operation $S: V \to V$ satisfying the above equation, and also

$$m(id \otimes S)(\Delta(\Gamma)) = \bar{\epsilon}(\Gamma),$$

(19)

is called an **antipode**.

Equations (18) and (19) are the defining property of the antipode. We still need to give an explicit expression for the action of $S$ on a partitioned spin foam. It is the iteration

$$S(\Gamma) = -\Gamma - \sum_{\gamma} S(\gamma) \cdot \Gamma/\gamma,$$

$$S(\gamma_p) = -\gamma_p,$$

(20)

which stops when a primitive lattice $\gamma_p$ is reached in the sum. As before, the subfoams in the sum range over all proper subfoams in the given partition of $\Gamma$ (i.e. excludes $\Gamma$ and $e$).

On a product of spin foams we have $S(\Gamma_1 \cdot \Gamma_2) = S(\Gamma_1) \cdot S(\Gamma_2)$, while $S(e) = e$.

It is straightforward to check, using (10), that $S$ as defined in (20) satisfies (18) and (19) for all $\Gamma \in V$ and therefore it is an antipode for $V$. One can also check that $S^2 = id$. Note that this depends on the commutativity of the product.

**Example 1.** The action of $S$ on the example spin foam (12) gives

$$S(\Gamma) = -\Gamma - S(\gamma_1) \cdot \Gamma/\gamma_1 - S(\gamma_2) \cdot \Gamma/\gamma_2.$$

(21)

The subfoam $\gamma_1$ is primitive and therefore

$$S(\gamma_1) = -\gamma_1 = -\nabla.$$

(22)

The subfoam $\gamma_2$ gives

$$S(\gamma_2) = -\gamma_2 - S(\gamma_1) \cdot \gamma_2/\gamma_1$$

$$= -\nabla + \nabla.$$

(23)
Inserting (22) and (23) in (21), we find that the antipode of $\Gamma$ is

$$S(\Gamma) = -\Gamma + \gamma_1 \cdot \Gamma / \gamma_1 + \gamma_2 \cdot \Gamma / \gamma_1 - \gamma_1 \cdot \gamma_2 / \gamma_1 \cdot \Gamma / \gamma_2$$

(24)

We can check that our result satisfies (18):

$$m(S \otimes \text{id}) \Delta(\Gamma) = m(S \otimes \text{id}) \left(\begin{array}{c}
\gamma_1 \otimes e + e \otimes \\
+ \gamma_2 \otimes \\
+ S(\gamma_1) \\
+ S(\gamma_2)
\end{array}\right)$$

$$= -\gamma_1 + \gamma_2 + \gamma_1 + \gamma_2 - \gamma_1$$

$$= 0.$$  

(25)
where in the third line we used $S(e) = e$. Similarly for (19).

**Example 2.** On the second example (14), the antipode is

$$S(\Gamma) = -\Gamma - S(\gamma_1) \cdot \Gamma / \gamma_1 - S(\gamma_2) \cdot \Gamma / \gamma_2 - S(\gamma_1 \cdot \gamma_2) \cdot \Gamma / (\gamma_1 \cdot \gamma_2). \quad (26)$$

Both $\gamma_1$ and $\gamma_2$ are primitive, while $S(\gamma_1 \cdot \gamma_2) = S(\gamma_1) \cdot S(\gamma_2)$, so

$$S(\Gamma) = -\Gamma + \gamma_1 \cdot \Gamma / \gamma_1 + \gamma_2 \cdot \Gamma / \gamma_2 - (\gamma_1 \cdot \gamma_2) \cdot \Gamma / (\gamma_1 \cdot \gamma_2)$$

$$= -\begin{array}{c}
\includegraphics[width=0.1\textwidth]{example1.pdf}
\end{array} + \begin{array}{c}
\includegraphics[width=0.1\textwidth]{example2.pdf}
\end{array} + \begin{array}{c}
\includegraphics[width=0.1\textwidth]{example3.pdf}
\end{array}$$

$$- \begin{array}{c}
\includegraphics[width=0.1\textwidth]{example4.pdf}
\end{array} - \begin{array}{c}
\includegraphics[width=0.1\textwidth]{example5.pdf}
\end{array}. \quad (27)$$

With the above five operations, partitioned spin foams are a Hopf algebra. We should emphasize that the same 2-complex with two different partitionings is two different partitioned spin foams, and therefore two different elements of the algebra.

We now wish to use the antipode as an extension of the renormalization group equation appropriate for spin foams. To do so, we will first need to consider how block transformations can be incorporated in an algebra of spin foam weights.

Recall that the spin foam 2-complexes are a convenient pictorial representation of the weights that contribute in the partition function (1). Since they carry the physical information, it is on these weights that the block transformations apply. The labelled partitioned spin foam 2-complexes are in one-to-one correspondence with parenthesized spin foam weights (of the same spin foam model). Therefore, parenthesized weights also form a Hopf algebra. We detail this in the next section.

## 5 The Hopf algebra of parenthesized spin foam weights

As we saw in section 3.2, partitioned spin foam lattices correspond to parenthesized spin foam weights. Let $W$ be the vector space of parenthesized spin foam weights over $\mathbb{C}$. The following are the Hopf algebra operations on $W$:
• **Multiplication:** Multiplication $m : W \otimes W \to W$ is simply the multiplication of the two spin foam weight functions, resulting in a new one with overall nesting structure $()()$:

$$m(\omega_{\Gamma_1} \otimes \omega_{\Gamma_2}) = \omega_{\Gamma_1} \cdot \omega_{\Gamma_2} = (\omega_{\Gamma_1})(\omega_{\Gamma_2}).$$  \hspace{1cm} (28)

We call $\omega_e$ the weight of the empty spin foam $e$. Then, $\omega_{\Gamma} \cdot \omega_e = \omega_e \cdot \omega_{\Gamma} = \omega_{\Gamma}$.

• **Unit:** The unit annihilates weights:

$$\epsilon(\omega_{\Gamma}) = \begin{cases} 
0 & \omega_{\Gamma} \neq \omega_e, \\
1 & \omega_{\Gamma} = \omega_e.
\end{cases}$$  \hspace{1cm} (29)

• **Counit:** The counit creates weights:

$$\bar{\epsilon}(c) = c\omega_e \quad c \in \mathbb{C}.$$  \hspace{1cm} (30)

• **Coproduct:** The coproduct, $\Delta : W \to W \otimes W$, splits the original weight into a sum of the weights of the subfoams paired with the weight of their remainder:

$$\Delta(\omega_{\Gamma}) = \omega_{\Gamma} \otimes \omega_e + \omega_e \otimes \omega_{\Gamma} + \sum_{\gamma} \omega_{\gamma} \otimes \omega_{\Gamma/\gamma}. \hspace{1cm} (31)$$

If $\omega_{\gamma_p}$ is the weight of a primitive spin foam $\gamma_p$, then

$$\Delta(\omega_{\gamma_p}) = \omega_{\gamma_p} \otimes e + e \otimes \omega_{\gamma_p}, \hspace{1cm} (32)$$

while $\Delta(\omega_e) = \omega_e \otimes \omega_e$.

• **Antipode:** The antipode is defined iteratively as

$$S(\omega_{\Gamma}) = -\omega_{\Gamma} - \sum_{\gamma} S(\omega_{\gamma}) \cdot \omega_{\Gamma/\gamma}, \quad S(\omega_{\gamma_p}) = -\omega_{\gamma_p}, \hspace{1cm} (33)$$

which stops at the weight of a primitive spin foam. We have $S(\omega_{\Gamma_1} \cdot \omega_{\Gamma_2}) = S(\omega_{\Gamma_1})S(\omega_{\Gamma_2})$, and $S(\omega_e) = e$.

$S$ above is an antipode in $W$ since it satisfies

$$m(S \otimes id)\Delta(\omega_{\Gamma}) = m(id \otimes S)\Delta(\omega_{\Gamma}) = \bar{\epsilon}(\omega_{\Gamma}),$$

for all $\omega_{\Gamma} \in W$. 

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Example. Again, using the spin foam example (12), which has weight
\[ \omega_\Gamma = (((A_1A_2A_3d_k) A_4A_5d_l) A_6d_id_jd_md_n), \tag{35} \]
we find
\[ \Delta(\omega_\Gamma) = \omega_\Gamma \otimes e + e \otimes \omega_\Gamma + ((A_1A_2A_3d_k) \otimes ((A_4A_5d_l) A_6d_id_jd_md_n) + ((A_1A_2A_3d_k) A_4A_5d_l) \otimes (A_6d_id_jd_md_n). \tag{36} \]
For the antipode, we get
\[ S(\omega_\Gamma) = -\omega_\Gamma - S(\omega_\gamma_1) \cdot \omega_\gamma_1 - S(\omega_\gamma_2) \cdot \omega_\gamma_2. \tag{37} \]
The subfoam \( \gamma_1 \) is primitive and therefore \( S(\omega_\gamma_1) = -\omega_\gamma_1 \), while for \( \omega_\gamma_2 \) we get
\[ S(\omega_\gamma_2) = -\omega_\gamma_2 - S(\omega_\gamma_1) \cdot \omega_\gamma_2 = -((A_1A_2A_3d_k) A_4A_5d_l) + (A_1A_2A_3d_k) (A_4A_5d_l). \tag{38} \]
Hence, (37) gives
\[ S(\omega_\Gamma) = -\omega_\Gamma + (A_1A_2A_3d_k) ((A_4A_5d_l) A_6d_id_jd_md_n) + ((A_1A_2A_3d_k) A_4A_5d_l) (A_6d_id_jd_md_n) - (A_1A_2A_3d_k) (A_4A_5d_l) (A_6d_id_jd_md_n). \tag{39} \]
Note that each term in \( S(\omega_\Gamma) \) is the weight of the same unpartitioned spin foam, but bracketed by different choice of partitions of the 2-complex. These are all different elements in \( W \). Also, note that, in the sense of (34), the weight (39) is the inverse of \( \omega_\Gamma \) in (35).

6 Block transformations as equivalence relations in the spin foam algebra

Consider the block transformation

\[ \Gamma = \quad \rightarrow \quad \Gamma' = , \]
which we obtain, for example, by summing over all possible values of the labels on the edges we eliminated. For the corresponding weights, this is a transformation \( \omega_\Gamma \to \omega_{\Gamma'} \). Now, \( \omega_\Gamma \) and \( \omega_{\Gamma'} \) are two different functions on two different sets of weights. However, we say that they are equivalent if they are related by the renormalization group equation. That is, when they both describe the same physical system but at different scales.

The renormalization group transformation is an equivalence relation

\[
(\Gamma', \omega_{\Gamma'}) := \text{RG} (\Gamma, \omega_\Gamma) \sim (\Gamma, \omega_{\Gamma'}),
\]

namely, the coarse-grained lattice and its weight is physically equivalent to the original one if the partition function of the coarse-grained lattice has a value appropriately close to the original one, that is,

\[
\text{eval} \, Z(\omega_{\Gamma'}) = \text{eval} \, Z(\omega_\Gamma) + \text{corrections}.
\]

In real-space renormalization, \((\Gamma', \omega_{\Gamma'})\) are obtained in the following way:

1. Subdivide \( \Gamma \) into nested/disjoint sublattices \( \{\gamma_i\} \) (partition \( \Gamma \)).

2. Repeatedly apply a block transformation \( R(\gamma_i) \) to the sublattices of \( \Gamma \).

Following this analogy, a block transformation \( R \) for a spin foam should have the following properties:

1. Acting on a subfoam \( \gamma \) of \( \Gamma \), it produces a new subfoam \( R(\gamma) \) such that \( \Gamma/\gamma = \Gamma/R(\gamma) \) (i.e. it applies locally to the subfoam).

2. **Exact block transformation**: \( R \) acting on the corresponding weight \( \omega_\gamma \) produces a new weight \( R(\omega_\gamma) \) on \( R(\gamma) \) which is equivalent to the old one. Equivalence has the same meaning as in eq. (41), namely, \( \text{eval} \, Z(R(\omega_\gamma)) = \text{eval} \, Z(\omega_\gamma) \).

A familiar example of an exact block transformation is summing over all values of the degrees of freedom in the interior of \( \gamma \), i.e.

\[
R(\gamma) = \partial \gamma, \\
R(\omega_\gamma) = \sum_{\text{internal labels}} \omega_\gamma.
\]

**General block transformation**: \( R \) acting on the sublattices \( \{(\gamma_i, \omega_{\gamma_i})\} \) in the partition of \( \Gamma \), produces a coarse-grained spin foam \( (\Gamma', \omega_{\Gamma'}) \)
which is equivalent to \((\Gamma, \omega_\Gamma)\) according to (41). It is not the case any-
more that \(R(\omega_\gamma)\) is equivalent to \(\omega_\gamma\). The equivalence holds only for
the final product, \((\Gamma', \omega_{\Gamma'})\).

Examples of such more general block transformations are decimation,
truncation, etc. Decimation of a sublattice produces a new one which is
not in any sense equivalent to the original sublattice, but after decima-
tion has been applied to every sublattice in the partition of \(\Gamma\), the final
product will be equivalent to \(\Gamma\) (when decimation works, naturally).

We want to incorporate the physical equivalence that the renorma-
lation group is based on as an equivalence relation in the Hopf algebra of
partitioned spin foams and, in this article, we want to do this for real-space
renormalization. The idea is the following. The renormalization group is the
equivalence

\[
\Gamma' - \Gamma \overset{\text{RG}}{\sim} 0. \tag{43}
\]

This becomes an equivalence relation in the algebra when we write it as

\[
\Gamma' - \Gamma = m (S_R \otimes \text{id}) \Delta(\Gamma) \overset{\text{RG}}{\sim} 0, \tag{44}
\]

where \(S_R\) is a modification of the antipode in which we have inserted the
transformation operation \(R\) so that it block transforms the sublattices of \(\Gamma\)
as they appear on the left of the tensor product in \(\Delta(\Gamma)\).

Of course, the antipode for a Hopf algebra is unique, so any modifica-
tion of it will give an operation which is not an antipode. The idea is that if \(R\) and
the modification of the antipode are chosen so that they contain the physical
equivalence in the renormalization group, then \(S_R\) should be an antipode in
the algebra under the equivalence relation that the RG defines:

\[
m (S_R \otimes \text{id}) \Delta(\Gamma) \overset{\text{RG}}{\sim} 0. \tag{45}
\]

All this is easier and clear to see if we use an exact block transformation
\(R\). In the next section we use the \(R\) defined in (42) and an appropriate
modified antipode on spin foam examples. The general case (general \(R\)) is
discussed in section 9.
7 Using the antipode to perform local scale transformations: exact transformations

We will obtain local scale transformations of a spin foam by inserting the equivalence relation \( R \) as defined in (42) simultaneously in the antipode (20) in \( V \) and (33) in \( W \). In effect, we will define a new antipode for \( V \) and \( W \), which performs local scale transformations on the spin foams. We call this modified antipode a shrinking antipode. It is equivalent to (20) and (33) under the equivalence relation \( R \) (and therefore it is an antipode in the algebra only under this equivalence relation).

The action of an exact block transformation is significantly different to that of an approximate one, and it turns out that different modifications of the antipode are appropriate in each of these cases. In this section, we define a shrinking antipode appropriate for exact block transformations.

A possible definition of an exact block transformation operation \( R \) on \( V \) is the following:

\[
R(\gamma) = \begin{cases} 
\text{in every connected component of } \gamma \text{ all faces are eliminated and the vertices are shrunk to a single vertex } \bullet. 
\end{cases}
\]

(46)

Then, \( R \) on \( \omega_\gamma \) sums over all possible values of the labels on the faces of \( \gamma \):

\[
R(\omega_\gamma) = \sum_{j,f \in \gamma} \omega_\gamma,
\]

(47)

In effect, \( R(\omega_\gamma) \) reduces \( \omega_\gamma \) to a product of amplitudes, one for each connected component of \( \gamma \). Note that, for \( \Gamma \) itself, \( R(\Gamma) \) eliminates all faces except those who belong to the in and out spin networks (see example below).

Given such an exact renormalization scheme \( R \), we modify \( S \) in (20) and (33) by inserting \( R \) as follows:

\[
S_{RE}(\Gamma) = -R(\Gamma) - \sum_\gamma S_{RE}(\gamma) \cdot \Gamma/\gamma,
\]

\[
S_{RE}(\gamma_p) = -R(\gamma_p)
\]

\[
S_{RE}(e) = e.
\]

(48)

On the corresponding weights we have:

\[
S_{RE}(\omega_{\Gamma}) = -R(\omega_{\Gamma}) - \sum_\gamma S_{RE}(\omega_\gamma) \cdot \omega_{\Gamma}/\gamma,
\]

\[
S_{RE}(\omega_{p}) = -R(\omega_{p})
\]

\[
S_{RE}(\omega_{e}) = \omega_{e}.
\]

(49)

We call \( S_{RE} \) the exact shrinking antipode.
Example: Consider again the spin foam (12),

\[ \Gamma = \begin{array}{c}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4 \\
\gamma_5 \\
\gamma_6 \\
n
\end{array}, \]

with weight

\[ \omega_\Gamma = ((A_1 A_2 A_3 d_k) A_4 A_5 d_l) A_6 d_i d_j d_m d_n) . \quad (50) \]

Then, from (48) and (49), the shrinking antipode of \( \Gamma \) and \( \omega_\Gamma \) is

\[ S_R^E (\Gamma) = -R(\Gamma) - S_R^E (\gamma_1) \cdot \Gamma / \gamma_1 - S_R^E (\gamma_2) \cdot \Gamma / \gamma_2, \]

\[ S_R^E (\omega_\Gamma) = -R(\omega_\Gamma) - S_R^E (\omega_\gamma_1) \omega_\Gamma / \gamma_1 - S_R^E (\omega_\gamma_2) \omega_\Gamma / \gamma_2. \quad (51) \]

For \( \gamma_1 \), we have

\[ S_R^E (\gamma_1) = \bullet \]

\[ S_R^E (\omega_\gamma_1) = \sum_k (A_1 A_2 A_3 d_k) = N A'_1, \]

where \( N = \sum d_i \) for the group \( G \) and \( A'_1 = A_1 A_2 A_3 \).

For \( \gamma_2 \) we get

\[ S_R^E (\gamma_2) = -R(\gamma_2) - S_R^E (\gamma_1) \cdot \gamma_2 / \gamma_1 \]

\[ = -\bullet + \bullet \]

\[ S_R^E (\omega_\gamma_2) = -R(\omega_\gamma_2) - S_R^E (\omega_\gamma_1) \omega_\gamma_2 / \gamma_1 \]

\[ = -\sum_{k,l} ((A_1 A_2 A_3 d_k) A_4 A_5 d_l) - R(\omega_\gamma_1) \omega_\gamma_2 / \gamma_1 \]

\[ = -N^2 A'_2 + N A'_1 (A_4 A_5 d_l), \]
where $A'_2 = A_1 A_2 A_3 A_4 A_5$. Finally, $R(\Gamma)$ is the spin foam

$$R(\Gamma) = \begin{array}{c}
\text{\includegraphics[width=1cm]{diagram.png}}
\end{array}$$

(54)

with weight

$$R(\omega_{\Gamma'}) = \sum_{k,l} \omega_{\Gamma} = N^2 A' d_i d_j d_m d_n,$$

(55)

where $A' = A_1 A_2 A_3 A_4 A_5 A_6$. Plugging the results back to equations (51), we get the answer

$$S_{RE}(\Gamma) = - \begin{array}{c}
\text{\includegraphics[width=1cm]{figure1.png}}
\end{array} + \begin{array}{c}
\text{\includegraphics[width=1cm]{figure2.png}}
\end{array} + \begin{array}{c}
\text{\includegraphics[width=1cm]{figure3.png}}
\end{array}$$

(56)

for the 2-complex, and

$$S_{RE}(\omega_{\Gamma'}) = -N^2 A' d_i d_j d_m d_n + N A'_1 (A_4 A_5 d_i) A_6 d_i d_j d_m d_n$$

$$+ N^2 A'_2 (A_6 d_i d_j d_m d_n) - N A'_1 (A_4 A_5 d_i) (A_6 d_i d_j d_m d_n)$$

(57)

for its weight.

The fully shrunk spin foam $\Gamma'$ (all faces eliminated except those who belong to the in and out spin networks) is given by

$$\Gamma' = \Gamma - m(S_{RE} \otimes \text{id})\Delta(\Gamma)$$

$$\omega_{\Gamma'} = \omega_{\Gamma} - m(S_{RE} \otimes \text{id})\Delta(\omega_{\Gamma}).$$

(58)
Note that, for the exact block transformation we defined in (46) and (47), \( \Gamma' \) is the same as \( R(\Gamma) \). However, this is only true for this particular \( R \). In general, \( R(\Gamma) \) is not the renormalized spin foam. We discuss this in detail in the next section.

Also note that the fully shrunk spin foam is a single amplitude, and the same as the evaluation of the spin foam.

**Example:** To fully coarse-grain our example (12) we use equation (58) to get:

\[
\Gamma' = \Gamma - m(S^E_R \otimes \text{id})(\Gamma \otimes e + e \otimes \Gamma + \gamma_1 \otimes \Gamma/\gamma_1 + \gamma_2 \otimes \Gamma/\gamma_2)
\]

\[
= \Gamma - S^E_R(\Gamma) - \Gamma - S^E_R(\gamma_1) \cdot \Gamma/\gamma_1 - S^E_R(\gamma_2) \cdot \Gamma/\gamma_2
\]

\[
= \begin{align*}
\includegraphics{example1.png} \\
\includegraphics{example2.png} \\
\includegraphics{example3.png} \\
\includegraphics{example4.png} \\
\includegraphics{example5.png}
\end{align*}
\]

\[
= .
\]

(59)

The shrinking antipode acting on a spin foam \( \Gamma \) produces a new foam which is a sum over all spin foams that can be obtained by local block transformations of \( \Gamma \). That is, the different terms in the equations (56) and (57) are all the original spin foam \( \Gamma \) with different parts of it coarse-grained. The coarse-graining is inhomogeneous, resulting in spin foams whose labels carry couplings at different scales (coarse-grained vertex amplitudes correspond to different multiples of \( l_{pl} \)). The sign of each term ensures that this sum over all possible local coarse-grainings is itself equivalent to the original spin foam.
This, of course, is not how coarse-graining is done in statistical physics for a usual lattice system, as there is no need and no advantage in this proliferation of terms (although it gives some insight into the renormalization group, as we will see in section 8). But recall that a main issue in developing renormalization group methods for spin foams is that the lattices are background-independent and irregular. A global block-transformation, as in standard real space renormalization, is not only difficult to implement, but it is not even clear if it is physically meaningful. Local ones are, and it is those that we use here. We will discuss this in some more detail in the next section, where we compare $S_R$ to the usual renormalization group.

8 \( S_R^E \) vs the renormalization group equation

In this section, we compare block transformations carried out using the exact shrinking antipode \( S_R^E \) and the standard renormalization group equation. This is best done by example, and we next calculate \( S_R^E \) for a familiar regular lattice.

Consider the square lattice

\[
\Gamma = \begin{array}{c}
K & K \\
K & K \\
\gamma_1 & \gamma_2
\end{array}
\]

\[ (60) \]

(e.g., a square lattice in \( Z_2 \) lattice gauge theory with spins \( \pm 1 \) on the edges and couplings \( \kappa \) on the plaquettes. It is partitioned as marked, that is, into sublattices and remainders

\[
\gamma_1 = \begin{array}{c}
K \\
K
\end{array}, \quad \Gamma/\gamma_1 = \begin{array}{c}
K \end{array},
\gamma_2 = \begin{array}{c}
K \\
K
\end{array}, \quad \Gamma/\gamma_2 = \begin{array}{c}
K \end{array}.
\]

\[ (61) \]

We have made the obvious choices of what a sublattice should be in this case so that \( \omega_\gamma \cdot \omega_{\Gamma/\gamma} = \omega_\Gamma \). Note that \( \gamma_1 \neq \gamma_2 \) since they have different labels on the edges.

With this partition, we calculate the coproduct of \( \Gamma \) to be

\[
\Delta(\Gamma) = \begin{array}{c}
K & K \\
K & K
\end{array} \otimes e + e \otimes \begin{array}{c}
K & K \\
K & K
\end{array} + \begin{array}{c}
K \\
K
\end{array} \otimes \begin{array}{c}
K \\
K
\end{array} + \begin{array}{c}
K \\
K
\end{array} \otimes \begin{array}{c}
K \\
K
\end{array}.
\]

\[ (62) \]
We can use a standard exact block transformation $R$ which removes all internal edges in a sublattice:

$$R(\gamma) = \partial \gamma,$$  \hspace{1cm} (63)

by summing over all values of the labels on these edges:

$$R(\omega_\gamma) = \sum_{\text{labels on internal edges of } \gamma} \omega_\gamma.$$  \hspace{1cm} (64)

With this $R$, we calculate the exact shrinking antipode of eq. (48) to be

$$S_{RE}(\Gamma) = -R\left(\begin{array}{c} \kappa' \\ \kappa \end{array}\right) - S_{RE}\left(\begin{array}{c} \kappa \\ \kappa \end{array}\right) \cdot \kappa + S_{RE}\left(\begin{array}{c} \kappa \\ \kappa \end{array}\right) \cdot \kappa$$

$$= - \kappa' + \kappa \cdot \kappa + \kappa \cdot \kappa + \kappa' \cdot \kappa + \kappa \cdot \kappa' + \kappa' \cdot \kappa$$

$$= - \kappa' + \kappa' \cdot \kappa + \kappa \cdot \kappa.$$

(65)

In the last line above, we have carried out the multiplications of the weights and rejoined the sublattices to match the labels on the edges. Again, in the first two lines, the last two terms are different, they have different labels on the edges.

In comparison, the standard renormalization group equation on $\Gamma$ gives

$$\text{RG}\left(\begin{array}{c} \kappa' \\ \kappa \end{array}\right) = \kappa'.$$  \hspace{1cm} (66)

We can understand the RG transformation as a special case of $S_{RE}$ in which only the homogeneous terms appear. In this example, homogeneous terms are lattices containing only square plaquettes, all carrying the same coupling. In (65) we have only one such term, the first, but clearly, if we had started with a larger lattice, all the possible homogeneous lattices that can be obtained by coarse-graining $\Gamma$ would appear.

As we saw in section 6, the correspondence between RG and $S_{RE}$ can be made precise as follows. Let us call $\Gamma'$ the result of the RG transformation:

$$\Gamma' := \text{RG} (\Gamma).$$  \hspace{1cm} (67)
\( \Gamma' \) is physically equivalent to \( \Gamma \), since they are related by the renormalization group transformation:

\[
\Gamma - \Gamma' \overset{\text{RG}}{\sim} 0.
\]  

Equation (68)

We may rewrite this as

\[
\Gamma - \Gamma' = m \left( S^E_R \otimes \text{id} \right) \Delta(\Gamma),
\]  

Equation (69)

since \( m \left( S^E_R \otimes \text{id} \right) \Delta(\Gamma) \overset{\text{RG}}{\sim} 0. \)

We can check this for our example. We have

\[
m \left( S^E_R \otimes \text{id} \right) \Delta(\Gamma) = - \kappa^* + \kappa' \kappa + \kappa \kappa' + \kappa^* \kappa - \kappa' \kappa - \kappa \kappa'
\]  

Equation (70)

which indeed gives

\[
\Gamma' = \Gamma - m \left( S^E_R \otimes \text{id} \right) \Delta(\Gamma) = - \kappa^*. \]

Equation (71)

In this sense, the standard RG equation is embedded in the shrinking antipode. The antipode is an expansion of \( \Gamma \) into all its possible coarse-grainings, with the signs in the different terms combining to respect what the RG encodes: the physical equivalence of two descriptions of the same system at different scales. It is very interesting to note that we can revisit the statement that “the renormalization group is not a group because it has no inverse” and propose that it is a Hopf algebra and it has an antipode!

One might object that on the practical, calculational side, there are no advantages in coarse-graining using \( S^E_R \) since the wanted result \( \Gamma' \) is one of the terms in \( S^E_R \), namely, \( R(\Gamma) \), and there is no need to calculate the other terms. However, this is a coincidence of our choice of block transformation \( R \). This \( R \) is itself an equivalence relation, \( \gamma' \overset{R}{\sim} \gamma \) for all sublattices \( \gamma \), and in particular

\[
\Gamma' \overset{R}{\sim} \Gamma, \quad \text{(72)}
\]
which leads to the oversimplification $\Gamma' = R(\Gamma)$ as a term in the antipode.

Other choices of $R$, for example, expanding $\Gamma$ and keeping only the lowest order terms, requires all the terms in $S_R^E$ to calculate $\Gamma'(\text{the coarse-grained system that is physically equivalent to } \Gamma)$. For such choices of approximate block transformations, alternative definitions of the shrinking antipode are also possible. We discuss this in the next section.

Closing this section, it is important to note that the RG equation can only be applied to systems with a background. In real space renormalization we always pick partitions of the lattice into identical sublattices, and block transform each one so that a RG step takes us from couplings $\{\kappa\}$ everywhere, to couplings $\{\kappa'\}$ everywhere. We need the lattice spacing as a guide everywhere on the lattice, which means we need a background and we also need to have regular lattices. Spin foams are background independent and highly irregular. Coarse-graining via the antipode does not require a global choice of lattice spacing, or regular lattices, and so it can be applied to spin foams. It can also be applied to irregular lattices with a background, where it may provide a useful calculational tool. Its strong point is that the antipode is defined as an iteration, and therefore it is suited for numerical calculations (as was shown, for example, in [23] for quantum field theory).

9 $S_R$ and general block transformations

In the previous two sections, we gave detailed examples of exact block transformations of both spin foams and lattice gauge theory, using the exact shrinking antipode $S_R^E$. The modification of $S$ to $S_R^E$ that we used is the simplest choice that satisfies $m(S_R^E \otimes \text{id})\Delta(\Gamma) = \tilde{\epsilon}(\Gamma)$ so that $S_R^E$ is still the antipode in the spin foam algebra. To check the equivalence of $S_R^E$ to $S$, we had the advantage that an exact $R$ is itself an equivalence relation.

However, it is clear that spin foams are complicated enough models that solving them using exact $R$ is unlikely to be practical. Giving specific examples of approximate coarse-graining schemes is beyond the scope of this article. In this section, we discuss the general features of such a coarse-graining using the Hopf algebra.

The basic idea is again the same. We will coarse-grain using a shrinking antipode $S_R$ which is equivalent to the original one $S$ under the physical
equivalence of the renormalization group transformation, namely, 

$$\Gamma' - \Gamma = m (S_R \otimes \text{id}) \Delta(\Gamma) \overset{\text{RG}}{\sim} 0$$

(for non-empty \(\Gamma\)) where \(\Gamma'\) is the renormalized spin foam. Clearly, further conditions on \(R\) and \(S_R\) are also required, for example \(R[R(\Gamma)] = R(\Gamma)\), and a choice of \(S_R\) that preserves the Hopf algebra structure (associativity, coassociativity etc).

We should emphasize that we have generalized the RG to apply to sums over lattices, which is necessary for spin foams. In spin foam models, we commonly use

$$\Gamma(s_1, s_2) = \sum_{\partial \Gamma_i = s_1 \cup s_2} \Gamma_i \quad (73)$$

namely, sums over all spin foams with a given boundary, spin networks \(s_1\) and \(s_2\). Since a general element of our algebra has the form \(\Gamma = \sum c_i \Gamma_i, c \in \mathbb{C}\), such \(\Gamma(s_1, s_2)\) are simply particular elements of the algebra. Note that each \(\Gamma_i\) in the sum is a partitioned spin foam. Also note that, as Kreimer showed in the original version of the Hopf algebra, the generators of the algebra are the 1-particle irreducible diagrams (here, the straightforward generalization of 1PIs to 2-complexes).

Also note that equivalence relations other than block transformations can be inserted in the algebra operations. In particular, we can use the recoupling moves as extra equivalence relations. For example, in 2-dimensional spin foams, we can use

$$R \begin{pmatrix} 
\begin{array}{c}
\circ \\
\circ \\
\circ 
\end{array}
\end{pmatrix} = \begin{pmatrix} 
\begin{array}{c}
\circ \\
\circ 
\end{array}
\end{pmatrix},$$

together with

$$M \begin{pmatrix} 
\begin{array}{c}
\circ \\
\circ \\
\circ 
\end{array}
\end{pmatrix} = \begin{pmatrix} 
\begin{array}{c}
\circ \\
\circ 
\end{array}
\end{pmatrix},$$
in \(S_R\) to keep the number of primitive spin foams small.

Concluding the general case of \(S_R\) scale transformations, we will also observe that in \(\text{[24, 25, 26]}\), spin foams are written as a field theory over a group. That case is very close in form to the original Kreimer Hopf algebra for QFT renormalization, so that is a first place where a general \(S_R\) can be
tested. Presumably, the form of $S_R$ to be used there is the same as Kreimer’s, namely,

$$S_R(\Gamma) = -R(\Gamma) - R\left(\sum_\gamma S_R(\gamma) \Gamma / \gamma \right).$$

(74)

Further work is required to set up appropriate approximate schemes or other calculations, such as linearizing near fixed points etc. We discuss this further in the Conclusions.

## 10 Conclusions

Our aim in this article was to formulate the problem of finding the low-energy limit of spin foam models as a coarse-graining problem, in the sense of statistical physics. Spin foam models, however, differ from solid state models in fundamental ways: they model spacetime itself. Therefore, standard renormalization group tools are not applicable here. In particular: 1) there is no background and therefore no useful notion of lattice spacing, 2) they are highly irregular and 3) they are sums over lattices.

We proposed that one can deal with all of these features by using a generalization of the usual renormalization group. This is a modification of the Kreimer Hopf algebra to real-space renormalization. The elements of the algebra are sums over partitioned spin foams. This method uses the nesting structure of partitioned spin foams rather than lattice spacing.

As we discussed in section 3, there are two parts to coarse-graining. The first is to find the right block transformation for the system, the second is to repeatedly apply this to the entire lattice. Here, the first part is done using an operation $R$ that applies to subfoams. The second is a combinatorial problem, trivial for regular lattices but difficult for irregular ones. We control it using a modification of the algebra antipode in which we have inserted $R$ appropriately so that scale transformations are equivalence relations in the algebra.

Coarse-graining using the Hopf algebra antipode applies to spin foams in any dimension. This is because the Kreimer Hopf algebra, which in this paper we applied to spin networks, is an algebra of rooted trees. These only encode the nesting of subfoams in the spin foam. This works in any dimension, one only needs to be careful in the definition of the subfoams.
We gave explicit expressions for $R$ and the modified antipode in the case of exact block transformations. However, one expects that it is approximate schemes that will be most relevant for spin foams and, in fact, it is here that the Hopf algebra is expected to be most powerful since the antipode is an iteration and thus suited to numerical calculations. We discussed the general features of that case in the previous section. Providing explicit expressions for specific spin foam models is an entire research program and beyond the scope of this paper.

The following are some of the obvious directions for further work using this algebra: 1) We have generalized the renormalization group equation and found that it is embedded in the antipode. One should understand the analogue of renormalization group flows for $S_R$. 2) This can be used to do linearization around fixed points. In particular, one can show that topological quantum field theories are fixed points for spin foams and linearize around them to obtain near-topological models. 3) Any local operations on spin foams act on subfoams. The Hopf algebra is then an appropriate framework for any local operations, not only scale transformations.

Closing this article, we would like to repeat that the renormalization group is a general framework and not a recipe that universally applies to any system. Further progress should be made by analyzing specific models. In this direction, one should keep in mind that experience from statistical physics teaches us that identifying the correct models for the systems we are interested in, experimental input is required.

Acknowledgments

I would like to thank Renate Loll and Thomas Thiemann for discussions and the Albert Einstein Institute, where most of this work was carried out, for its hospitality. This work was supported by the EU Network on “Discrete Random Geometry” grant HPRN-CT-1999-00161.

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