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The Einstein-Maxwell-Particle System in the York Canonical Basis of ADM Tetrad Gravity: I) The Equations of Motion in Arbitrary Schwinger Time Gauges.

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Abstract

We study the coupling of N charged scalar particles plus the electro-magnetic field to ADM tetrad gravity and its canonical formulation in asymptotically Minkowskian space-times without super-translations. To regularize the self-energies both the electric charge and the sign of the energy of the particles are Grassmann-valued. The introduction of the non-covariant radiation gauge allows to reformulate the theory in terms of transverse electro-magnetic fields and to extract the generalization of the Coulomb interaction among the particles in the Riemannian instantaneous 3-spaces of global non-inertial frames, the only ones allowed by the equivalence principle.

Then we make the canonical transformation to the York canonical basis, where there is a separation between the inertial (gauge) variables and the tidal ones inside the gravitational field and a special role of the Eulerian observers associated to the 3+1 splitting of space-time. The Dirac Hamiltonian is weakly equal to the weak ADM energy. The Hamilton equations in Schwinger time gauges are given explicitly. In the York basis they are naturally divided in four sets: a) the contracted Bianchi identities; b) the equations for the inertial gauge variables; c) the equations for the tidal ones; d) the equations for matter.

Finally we give the restriction of the Hamilton equations and of the constraints to the family of non-harmonic 3-orthogonal gauges, in which the instantaneous Riemannian 3-spaces have a non-fixed trace $^3K$ of the extrinsic curvature but a diagonal 3-metric. The inertial gauge variable $^3K$ (the general-relativistic remnant of the freedom in the clock synchronization convention) gives rise to a negative kinetic term in the weak ADM energy vanishing only in the gauges with $^3K = 0$: is it relevant for dark energy and back-reaction?

In the second paper there will be the linearization of the theory in these non-harmonic 3-orthogonal gauges to obtain Hamiltonian post-Minkowskian gravity with asymptotic Minkowski background, non-flat instantaneous 3-spaces and no post-Newtonian expansion. This will allow to explore the inertial effects induced by the York time $^3K$ in non-flat 3-spaces (they do not exist in Newtonian gravity) and to check how much dark matter can be explained as an inertial aspect of Einstein's general relativity.
I. INTRODUCTION

By re-expressing the 4-metric in terms of tetrads inside the ADM action, an ADM formulation of tetrad gravity was presented in a series of papers [1, 2, 3, 4] for globally hyperbolic space-times asymptotically Minkowskian, parallelizable (so to admit ortho-normal tetrads and a spinor structure [5]) and without super-translations (see Refs.[1, 3] for the needed boundary conditions on the 4-metric and on the tetrads). This allowed the development of canonical tetrad gravity with its fourteen Dirac constraints and the identification of the York map as a Shanmugadhasan canonical transformation adapted to ten of the constraints [6]. As a consequence we now have a York canonical basis in which the *tidal* effects of the gravitational field (the polarization of the gravitational waves in the linearized theory) are separated from the *inertial* effects, which are described by the *gauge* variables conjugated to Dirac first-class constraints. Even if this separation is only 3-covariant on the instantaneous non-flat 3-spaces corresponding to a clock synchronization convention, it allows to give a physical interpretation of all the quantities appearing in tetrad gravity and to rewrite the ADM Hamilton equations in a new form.

ADM tetrad gravity is formulated in an arbitrary admissible 3+1 splitting of the globally hyperbolic space-time, i.e. in a foliation with instantaneous space-like 3-spaces tending to a Minkowski space-like hyper-plane at spatial infinity in a direction independent way: they correspond to a clock synchronization convention and each one of them can be used as a Cauchy surface for field equations. As shown in Refs.[1, 3] the absence of super-translations implies that the SPI group of asymptotic symmetries is reduced to the asymptotic ADM Poincare’ group and the allowed 3+1 splittings must have the instantaneous 3-spaces tending to asymptotic symmetries is reduced to the asymptotic ADM Cauchy surface for field equations. As shown in Refs.[1, 3] the absence of super-translations implies that the SPI group of asymptotic symmetries is reduced to the asymptotic ADM Poincare’ group and the allowed 3+1 splittings must have the instantaneous 3-spaces tending to asymptotic special-relativistic Wigner hyper-planes orthogonal to the ADM 4-momentum in a direction-independent way. At spatial infinity there are asymptotic inertial observers, carrying a flat tetrad $e_A^\mu (4\eta_{\mu\nu} e_A^\mu e_B^\nu = 4\eta_{AB})$, whose spatial axes can be identified with the fixed stars of star catalogues.

Radar 4-coordinates $\sigma^A = (\tau; \sigma^r)$ adapted to the 3+1 splitting and centered on a time-like observer $x^\mu(\tau)$ are used instead of local 4-coordinates $x^\mu$. If $x^\mu \mapsto \sigma^A$ is the coordinate transformation to these adapted 4-coordinates, its inverse $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$ defines the embedding of the instantaneous 3-spaces $\Sigma_{\tau}$ into the space-time. The gradients $z_A^\mu(\tau, \sigma^r) = \frac{\partial z^\mu(\tau, \sigma^r)}{\partial \sigma^A}$ are the transition functions for transforming tensors: for the 4-metric we have $4g_{AB}(\tau, \sigma^r) = z_A^\mu(\tau, \sigma^r) z_B^\nu(\tau, \sigma^r) 4g_{\mu\nu}(z(\tau, \sigma^r))$, so that we get $\sqrt{|det 4g_{\mu\nu}|} = \sqrt{|det 4g_{AB}|/|det z_A^\mu|}$. Then we put the decomposition $4g_{AB}(\tau, \sigma^r) = 4E_A^{(\alpha)}(\tau, \sigma^r) 4\eta_{(\alpha)(\beta)} 4E_B^{(\beta)}(\tau, \sigma^r)$, with $4E_A^{(\alpha)}(\tau, \sigma^r)$ arbitrary cotetrads, inside the ADM action to obtain ADM tetrad gravity.

The three independent space-like 4-vectors $z_\mu(\tau, \sigma^r)$ are tangent to $\Sigma_{\tau}$ in the point $(\tau, \sigma^r)$. Instead the $\tau$-gradient of the embedding has the standard decomposition $z_\mu(\tau, \sigma^r) = (1+n) l_\mu(\tau, \sigma^r) + n^r z^r_\mu(\tau, \sigma^r)$ along the unit normal $l_\mu(\tau, \sigma^r) = \sqrt{|det 4g_{\mu\nu}(z(\tau, \sigma^r))|}/\sqrt{|det 4g_{\nu\sigma}(\tau, \sigma^r)|}$ $e_\mu_{0\beta\gamma} \left( z_1^\alpha z_2^\beta z_3^\gamma \right)(\tau, \sigma^r) = \frac{1+n(\tau, \sigma^r)}{|det z_A^\mu(\tau, \sigma^r)|} e_\mu_{0\beta\gamma} \left( z_1^\alpha z_2^\beta z_3^\gamma \right)(\tau, \sigma^r) = z_A^\mu(\tau, \sigma^r) l_A(\tau, \sigma^r)$ and the tangents to $\Sigma_{\tau}$ defining the lapse ($N = 1+n$) and shift ($N^r = n^r$) functions.

Let us remark that if we find a solution $4g_{AB}(\tau, \sigma^r)$ of the ADM Hamilton equations in some gauge, then it has an associate preferred 3+1 splitting of space-time, whose instantaneous 3-spaces are dynamically determined (dynamical clock synchronization convention)
by the given extrinsic curvature and by the given lapse and shift functions, as noted in Refs. [7]. As a consequence, there is a preferred world 4-coordinate system adapted to it. Let us take the world-line of the time-like observer as origin of the spatial coordinates, i.e. \( x^\mu(\tau) = (x^0(\tau); 0) \). Then the space-like surfaces of constant coordinate time \( x^\mu(\tau) = \text{const.} \) coincide with the dynamical instantaneous 3-spaces \( \Sigma_\tau \) with \( \tau = \text{const.} \) of the solution. Then the preferred embedding is

\[
x^\mu = z^\mu(\tau, \sigma^r) = x^\mu(\tau) + \epsilon^\mu_\nu \sigma^r = \delta^\mu_\nu \ x^\nu(\tau) + \epsilon^\mu_\nu \sigma^r.
\]

If we choose the asymptotic flat tetrads \( \epsilon^\mu_A = \delta^\mu_0 \delta^r_A + \delta^\mu_i \delta^r_A \) and \( x^\nu(\tau) = x^\nu_o + \epsilon^\nu_\sigma \tau = x^\nu_o + \tau \), we get \( z^\mu(\tau, \sigma^r) = \delta^\mu_0 x^\nu_o + \epsilon^\mu_A \tau, z^\mu_A(\tau, \sigma^r) = \epsilon^\mu_A \) and \( 4g_{\mu\nu}(x = z(\tau, \sigma^r)) = \epsilon^\mu_A \epsilon^\nu_B g_{AB}(\tau, \sigma^r) \) (\( \epsilon^A_\mu \) are the inverse flat cotetrads). Then by means of 4-diffeomorphisms we can write the solution in an arbitrary world 4-coordinate system in general not adapted to the dynamical 3+1 splitting. This gives rise to the 4-geometry containing the given solution.

In this paper we are going to study ADM tetrad gravity in presence of the following matter: \( N \) positive-energy charged scalar particles plus the electro-magnetic field.

The isolated system of \( N \) positive-energy charged scalar particles (with Grassmann-valued electric charges for a UV regularization of self-energies) with mutual Coulomb interaction plus a transverse electro-magnetic field in the radiation gauge have been studied in special relativity in Refs. [8, 9], where its \textit{inertial rest-frame instant form} was developed: in it the instantaneous 3-spaces (the Wigner hyper-planes) are orthogonal to the conserved 4-momentum in the non-inertial rest-frame instant form, was done in Ref. [10].

The starting point of these developments was the \textit{parametrized Minkowski theory} for isolated systems [11, 12] (positive-energy particles, strings, fields, fluids; see also Refs. [13, 14, 15, 16, 17, 18]) admitting a Lagrangian formulation, the precursor of the described formulation of ADM tetrad gravity. After an admissible 3+1 splitting of Minkowski space-time with the instantaneous 3-spaces \( \Sigma_\tau \) described by embedding functions \( x^\mu = z^\mu(\tau, \sigma^r) \) \([\sigma^r = (\tau; \sigma^r) \) are radar 4-coordinates centered on an arbitrary time-like observer\] one defines a Lagrangian on \( \Sigma_\tau \) depending on the given matter and on the embedding through the induced 4-metric \( g_{AB}(\tau, \sigma^r) = (z^\mu_A \eta_{\mu\nu} z^\nu_B)(\tau, \sigma^r) \). This Lagrangian is obtained by the matter Lagrangian by coupling it to an external gravitational field and by replacing the external 4-metric with \( 4g_{AB}[z(\tau, \sigma^r)] \) (a functional of the embedding) after having redefined the matter fields so that they know the clock synchronization convention (for a Klein-Gordon field \( \phi(x) \) we use \( \bar{\phi}(\tau, \sigma^r) = \phi(z(\tau, \sigma^r)) \)). Each admissible 3+1 splitting corresponds to a convention for clock synchronization and defines a global non-inertial frame centered on the observer and the components of the 4-metric \( 4g_{AB}(\tau, \sigma^r) \) play the role of the \textit{inertial potentials} in the non-inertial frame.

As said, the Lagrangian of parametrized Minkowski theories depends on the embeddings \( z^\mu(\tau, \sigma^r) \) and on matter variables adapted to the foliation. Due to the invariance of the action under frame-preserving diffeomorphisms, the embeddings \( z^\mu(\tau, \sigma^r) \) are \textit{gauge variables}. This implies the \textit{gauge equivalence} of the description of the isolated system in any non-inertial or inertial frame, namely its independence from the clock synchronization convention and from the choice of 3-coordinates on \( \Sigma_\tau \).
The transition to globally hyperbolic space-times is done in accord with the equivalence principle, according to which only a description of tetrad gravity plus matter in global non-inertial frames is possible (inertial frames exist only locally near a free falling particle). In particular the absence of super-translations identifies the non-inertial rest frames (orthogonal to the ADM 4-momentum at spatial infinity) as the only relevant ones.

The real difference with parametrized Minkowski theories is that the independent variables are now the cotetrad $\Gamma_A^{(\alpha)}(\tau, \bar{\sigma})$ (or the 4-metric $\Gamma_{AB}(\tau, \bar{\sigma})$ in metric gravity) and not the embeddings $z^\mu(\tau, \bar{\sigma})$ ($z_A^\mu$ are only transition coefficients, while in special relativity they are flat cotetrad), which are dynamically determined a posteriori instead of remaining gauge variables as it happens in special relativity.

Following Ref.[11, 13] we will describe N scalar particles of masses $m_i$, $i=1,\ldots,N$, with 3-coordinates $\eta_i^j(\tau)$ on the instantaneous 3-spaces $\Sigma_\tau$ (diffeomorphic to $R^3$) identified by the intersection with $\Sigma_\tau$ of their world-lines $x_i^\mu(\tau) = z_i^\mu(\tau, \bar{\eta}_i^\tau)$ in $M^4$. Therefore the world-lines $x_i^\mu(\tau)$ are derived quantities (as shown in Ref.[8] they are covariant non-canonical predictive coordinates) and, as shown in Ref.[6], describe particles with a definite sign of the energy (due to clock synchronization each particle is described by 3, not 4, position variables). As a consequence, the momenta $p_i^\mu(\tau)$ are not defined: as shown in Ref.[8] it is possible to define them so that the mass shell constraints $\epsilon p_i^2 - m_i^2 c^2 \approx 0$ are satisfied also in presence of interactions. However other definitions are possible.

To regularize the gravitational self-energies we will assume that the signs $\eta_i$ of the energies of the particles (a topological number with the two values $\pm 1$) are described by two complex conjugate Grassmann variables $\theta_i$, $\bar{\theta}_i$: $\eta_i = \theta_i^* \bar{\theta}_i$, $\eta_i^2 = 0$, $\eta_i \eta_j = \eta_j \eta_i \neq 0$ for $i \neq j$ (after a formal quantization they describe a 2-level system). In this way we are implementing a $i \neq j$ rule like it was done in classical electrodynamics by using Grassmann variables to describe the electric charges of charged relativistic scalar particles [14, 15]: $Q_i = \bar{\eta}_i^{(Q)} \theta_i^{(Q)}$, $Q_i^2 = 0$, $Q_i Q_j = Q_j Q_i \neq 0$ for $i \neq j$, to avoid divergencies in the electro-magnetic self-energies.

The electro-magnetic field is described by the vector potential $A_A(\tau, \sigma^u) = z_A^\mu(\tau, \sigma^u) A_\mu(\sigma^u)$ and by the field strength $F_{AB}(\tau, \sigma^u) = \partial_A A_B(\tau, \sigma^u) - \partial_B A_A(\tau, \sigma^u) = z_A^\mu(\tau, \sigma^u) z_B^\nu(\tau, \sigma^u) F_{\mu\nu}(\sigma^u)$. Following Ref.[10], we shall restrict the formulation to an electro-magnetic field in the radiation gauge in the instantaneous 3-spaces $\Sigma_\tau$. Even if this gauge is not covariant, it allows to extract the non-inertial analogue of the Coulomb potential among the particles and to work with transverse electro-magnetic fields (the non-covariant electro-magnetic Dirac observables, the only ones which can be explicitly determined).

The main aim of this paper is to write explicitly the Hamilton equations of ADM tetrad gravity coupled to this type of matter in the York canonical basis of Ref.[6] in the class of Schwinger time gauges, in which the tetrads are adapted to the 3+1 splitting of space-time (the time-like tetrad coincides with the unit normal to the 3-space $\Sigma_\tau$ and the three spatial tetrads have a conventional orientation). In these gauges the gravitational field is described by 10 configuration variables and 10 conjugate momenta like in canonical metric gravity. Since there are still 8 first-class constraints there are 8 gauge variables (inertial effects): 7 configuration variables (the lapse and shift functions and 3 variables describing the freedom in the choice of the 3-coordinates in $\Sigma_\tau$) and a momentum variable (the trace $3^K(\tau, \sigma^r)$ of the extrinsic curvature of the 3-space $\Sigma_\tau$, describing the freedom in the clock synchronization...
convention). One configuration variable (the element of 3-volume in the instantaneous 3-space \( \Sigma_\tau \), canonically conjugate to \( 3^K \)) and 7 momenta are determined by the 8 constraints. The remaining two pairs of conjugate variables describe the physical tidal effects contained in the gravitational field (they are only 3-scalars and partial Dirac observables with respect to 10 of the 14 constraints of tetrad gravity).

As we shall see, the extrinsic curvature \( 3^K_{rs}(\tau, \sigma^r) \) of the instantaneous 3-spaces \( \Sigma_\tau \), viewed as sub-manifolds of the space-time, depends upon the matter, the tidal variables, the three inertial gauge variables for the choice of the 3-coordinates and, finally, upon the inertial gauge variable \( 3^K(\tau, \sigma^r) \) (its trace). This last variable, the so-called York time, is what remains of the special-relativistic gauge freedom in the choice of the clock synchronization convention in general relativity. Once these gauge freedoms are fixed the extrinsic curvature and, as a consequence, the instantaneous 3-spaces are dynamically determined for every solution of Einstein equations with the allowed boundary conditions and with Cauchy data compatible with the constraints.

In canonical ADM tetrad gravity the Dirac Hamiltonian is equal to the weak ADM energy \( \hat{E}_{ADM} \) (its form as a volume integral over the 3-space \( \Sigma_\tau \)) plus constraints. As a consequence, there is not a frozen picture like in spatially compact without boundary space-times. We will see that the gauge momentum (the inertial York time \( 3^K(\tau, \sigma^r) \); its time nature is a reflex of Lorentz signature) gives rise to a negative kinetic term in \( \hat{E}_{ADM} \) vanishing only in the gauges \( 3^K(\tau, \sigma^r) = 0 \).

In Schwinger time gauges the ADM Hamilton equations in the York canonical basis are naturally separated in four groups. Besides the Hamilton equations for matter and for the tidal variables, there are four contracted Bianchi identities (implying the \( \tau \)-preservation of the super-Hamiltonian and super-momentum constraints) and the Hamilton equations for the four inertial gauge variables describing the 3-coordinates in \( \Sigma_\tau \) and the clock synchronization convention (if we add four gauge fixings for these variables, these Hamilton equations determine the shift and lapse functions).

Then we will define a family of non-harmonic gauges, the \( 3 \)-orthogonal ones, in which the 3-metric in the 3-spaces \( \Sigma_\tau \) is diagonal but the inertial gauge variable \( 3^K(\tau, \sigma^r) \), fixing the clock synchronization convention, is equal to an arbitrary numerical function \( F(\tau, \sigma^r) \). The restriction of the Hamilton equations to the 3-orthogonal gauges is given explicitly.

It is in this family of gauges that we will define a linearization of ADM canonical tetrad gravity plus matter in the second paper [19], to obtain a formulation of Hamiltonian post-Minkowskian gravity (without post-Newtonian expansions) with non-flat Riemannian 3-spaces and asymptotic Minkowski background. In particular this will allow to study the dynamical effects of the inertial potential \( 3^K(\tau, \sigma^r) \) and to see whether it can, at least partially, mimic the effects attributed to dark matter.

In Section II there is a review of ADM tetrad gravity, of the York canonical basis with new results beyond Ref.[6], showing the relevance of the Eulerian observers associated to a 3+1 splitting of space-time, and of the ADM Poincare’ charges. In Section III we give the action of our system in Subsection A and we evaluate the Dirac Hamiltonian and the constraints of the Hamiltonian formulation. In Subsection B
we give the Hamilton equations of the particles. In Subsection C, after having evaluated the Hamilton equations of the electro-magnetic field, we define the non-covariant electro-magnetic radiation gauge and give the restriction to it of the Hamilton equations of the particles and of the transverse electro-magnetic field. In Subsections D and E we give the Dirac Hamiltonian, the constraints and the ADM Poincare’ charges in the York canonical basis, while in Subsection F we define the Schwinger time gauges. Then in Section IV we give the Hamilton equations in the York canonical basis in these gauges.

In Section V we discuss the formulation in the York canonical basis and in the Schwinger time gauges of some of the most used gauges for canonical gravity, included the Hamiltonian formulation of the harmonic gauges for Einstein equations.

In Section VI we give the restriction of the Dirac Hamiltonian, of the constraints and of the Hamilton equations to the above defined family of 3-orthogonal Schwinger time gauges.

In the Conclusions there are some final remarks.

In Appendix A there is a discussion of the contracted Bianchi identities and a comparison with the standard ADM Hamilton equations.

In Appendix B there are the calculations needed for the Hamilton equations in the Schwinger time gauges, while in Appendix C these calculations are restricted to the 3-orthogonal gauges.
II. REVIEW OF TETRAD GRAVITY AND OF THE YORK MAP

A. Tetrads and Cotetrads

We use radar 4-coordinates $\sigma^A = (\sigma^\tau = \tau; \sigma^r)$, $A = \tau, r$, adapted to an admissible 3+1 splitting of the space-time and centered on an arbitrary time-like observer: they define a non-inertial frame centered on the observer, so that they are observer and frame-dependent. The instantaneous 3-spaces identified by this convention for clock synchronization are denoted $\Sigma_\tau$.

The 4-metric $4g_{AB}$ has signature $\epsilon(+ - - -)$ with $\epsilon = \pm$ (the particle physics, $\epsilon = +$, and general relativity, $\epsilon = -$, conventions). Flat indices $\alpha, \beta, \gamma, \delta$, are raised and lowered by the flat Minkowski metric $\eta_{\alpha\beta} = \epsilon(+ - - -)$. We define $\eta_{\alpha}(\beta) = -\epsilon \delta_{\alpha\beta}$ with a positive-definite Euclidean 3-metric. From now on we shall denote the curvilinear 3-coordinates $\sigma^r$ with the notation $\tilde{\sigma}$ for the sake of simplicity. Usually the convention of sum over repeated indices is used, except when there are too many summations.

We shall work with the tetrads $4E_A^{A}(\tau, \tilde{\sigma})$ and the cotetrads $4E_A^{(a)}(\tau, \tilde{\sigma})$. To rebuild the original tetrads $4E^B_{(a)}(\tau, \tilde{\sigma}) = \varepsilon^A_{\mu}(\tau, \tilde{\sigma}) 4E_A^{A}(\tau, \tilde{\sigma})$ we must know explicitly the embedding $\varepsilon^\mu(\tau, \tilde{\sigma})$ of the instantaneous 3-spaces, so to be able to evaluate the transformation coefficients $\varepsilon^\mu_{\tilde{\sigma}}(\tau, \tilde{\sigma})$.

General tetrads $4E_A^{A}(\tau, \tilde{\sigma})$ and cotetrads $4E_A^{(a)}(\tau, \tilde{\sigma})$ are connected to the tetrads and cotetrads adapted to the 3+1 splitting (the time-like tetrad is the unit normal $l^A$ to $\Sigma_\tau$) by a point-dependent standard Lorentz boost for time-like orbits acting on the flat indices $1$

$$4 E_A^{A}(\tau, \tilde{\sigma}) = 4 \varphi^{A}_{\beta}(\varphi(\alpha)) L^{(a)}_{(\beta)}(\varphi(\alpha)) , \quad 4 E_A^{(a)}(\tau, \tilde{\sigma}) = L^{(a)}_{(\beta)}(\varphi(\alpha)) 4 \varphi^{(\beta)}_{A} ,$$

$$4 g_{AB} = 4 E_A^{(a)}(\tau, \tilde{\sigma}) 4 \eta_{(\alpha)(\beta)} E_B^{(b)},$$

where the last line gives the resolution of the 4-metric in terms of cotetrads.

The adapted tetrads and cotetrads (in these so called Schwinger time gauges) have the expression $(N(\tau, \tilde{\sigma}) = 1 + n(\tau, \tilde{\sigma}) > 0$, with $n(\tau, \tilde{\sigma})$ vanishing at spatial infinity (absence

---

1 In each tangent plane to a point of $\Sigma_\tau$ the point-dependent standard Wigner boost for time-like Poincare’ orbits $L^{(a)}_{(\beta)}(V(z(\sigma)); \tilde{V}) = \delta^{(a)}_{(\beta)} + 2 \varepsilon V^{(a)}(z(\sigma)) V^\beta_{(\beta)} - \epsilon \frac{(V^\alpha(\varphi(\alpha); \tilde{V}))}{(1 + V^\alpha(\varphi(\alpha)) \tilde{V})} = \lim_{\tau \to \Sigma_\tau} L_{(\beta)}^{(a)}(\varphi(\alpha))$ sends the unit future-pointing time-like vector $\tilde{V}^{(a)} = (1; 0)$ into the unit time-like vector $V^{(a)} = 4 E_A^{(a)} l^A = \left(1 + \sqrt{1 + \varepsilon \phi(\varphi(\alpha)) \phi(\alpha)} \right)$, where $l^A$ is the unit future-pointing normal to $\Sigma_\tau$. We have $L^{-1}_{(\varphi(\alpha))} = 4 \eta L^T(\varphi(\alpha)) 4 \eta = L(\varphi(\alpha))$. As a consequence, the flat indices $(\alpha)$ of the adapted tetrads and cotetrads and of the triads and cotriads on $\Sigma_\tau$ transform as Wigner spin-1 indices under point-dependent SO(3) Wigner rotations $R_{(a)(b)}(V(z(\sigma)); \Lambda(z(\sigma)))$ associated with Lorentz transformations $\Lambda^{(a)}_{(\beta)}(z)$ in the tangent plane to the space-time in the given point of $\Sigma_\tau$. Instead the index $(\alpha)$ of the adapted tetrads and cotetrads is a local Lorentz scalar index.
of supertranslations), so that $N(\tau, \vec{\sigma}) \, d\tau$ is positive from $\Sigma_\tau$ to $\Sigma_{\tau + d\tau}$, is the lapse function; $N^r(\tau, \vec{\sigma}) = n^r(\tau, \vec{\sigma})$, vanishing at spatial infinity (absence of super-translations), are the shift functions)

\[
4 \delta^{(o)A}_{E} = \frac{1}{1+n} (1; -n(a) \, 3e^r_{(a)}) = l^A, \quad 4 \delta^{(o)A}_{E} = (0; 3e^r_{(a)}),
\]

\[
4 \delta^{(o)A}_{E} = (1 + n) (1; \vec{0}) = \epsilon l_A, \quad 4 \delta^{(o)A}_{E} = (n(a); 3e_{(a)r}),
\]

where $3e^r_{(a)}$ and $3e_{(a)r}$ are triads and cotriads on $\Sigma_\tau$ and $n(a) = n_r \, 3e^r_{(a)} = n^r \, 3e_{(a)r}$ 2 are adapted shift functions.

The adapted tetrads $4 \delta^{(o)A}_{E}$ are defined modulo SO(3) rotations $4 \delta^{(o)A}_{E} = R_{(a)(b)}(\alpha(c)) \, 4 \delta^{(o)A}_{E}$, $3e^r_{(a)} = R_{(a)(b)}(\alpha(c)) \, 3e^r_{(b)}$, where $\alpha(a)(\tau, \vec{\sigma})$ are three point-dependent Euler angles. After having chosen an arbitrary point-dependent origin $\alpha(a)(\tau, \vec{\sigma}) = 0$, we arrive at the following adapted tetrads and cotetrad, 

\[
4 \delta^{(o)A}_{E} = 4 \delta^{(o)A}_{E} = \frac{1}{1+n} (1; -\bar{n}(a) \, 3\bar{e}^r_{(a)}) = \bar{l}^A, \quad 4 \delta^{(o)A}_{E} = (0; 3\bar{e}^r_{(a)}),
\]

\[
4 \delta^{(o)A}_{E} = 4 \delta^{(o)A}_{E} = (1 + n) (1; \vec{0}) = \epsilon \bar{l}_A, \quad 4 \delta^{(o)A}_{E} = (\bar{n}(a); 3\bar{e}_{(a)r}),
\]

which we shall use as a reference standard.

Then Eqs.(2.1), namely

\[
4 e^A_{(a)} = 4 \delta^{(o)A}_{E} \, L^{(o)(a)}(\varphi(\epsilon)) + 4 \delta^{(o)A}_{E} \, R^{T(b)(a)}(\alpha(\epsilon)) \, L^{(a)(\epsilon)}(\varphi(\epsilon)),
\]

show that every point-dependent Lorentz transformation $\Lambda$ in the tangent planes may be parametrized with the (Wigner) boost parameters $\varphi(\epsilon)$ and the Euler angles $\alpha(\epsilon)$, being the product $\Lambda = RL$ of a rotation and a boost.

The future-oriented unit normal to $\Sigma_\tau$ and the projector on $\Sigma_\tau$ are

\[
2 \text{ Since we use the positive-definite 3-metric } \delta_{(a)(b)}, \text{ we shall use only lower flat spatial indices. Therefore for the cotriads we use the notation } 3e^r_{(a)} \text{ def } 3e_{(a)r} \text{ with } \delta_{(a)(b)} = 3e^r_{(a)} \, 3e_{(b)r}.\]
B. The 4-metric and the Canonical Variables.

The 4-metric has the following expression

\[
\begin{align*}
l_A &= \epsilon (1 + n) \begin{pmatrix} 1; 0 \end{pmatrix}, \quad 4g^{AB} l_A l_B = \epsilon, \\
l^A &= \epsilon (1 + n) 4g^{Ar} = \frac{1}{1 + n} \begin{pmatrix} 1; -n^r \end{pmatrix} = \frac{1}{1 + n} \begin{pmatrix} 1; -n(a)_r^a \end{pmatrix}, \\
3h^B_A &= \delta^B_A - \epsilon l_A l^B, \\
3h^r_\tau &= 3h^r_r = 0, \\
3h^s_\tau &= n(a)_r^a, \\
3h^r_s &= \delta^r_s,
\end{align*}
\]

\[
\begin{align*}
3h^r_\tau &= -\epsilon n(a)_r n(a), \\
3h^r_r &= -\epsilon n(a)_r e(a)_r, \\
3h^s_r &= -\epsilon e(a)_r e(a)_s,
\end{align*}
\]

(2.5)

\[
\begin{align*}
4g^{\tau \tau} &= \epsilon \left[ (1 + n)^2 - 3g^{rs} n_r n_s \right] = \epsilon \left[ (1 + n)^2 - n(a)_r n(a) \right], \\
4g^{\tau r} &= -\epsilon n_r = -\epsilon n(a)_r e(a)_r, \\
4g^{rs} &= -\epsilon^3 g_{rs}, \\
3g_{rs} &= h_{rs} = 3e(a)_r e(a)_s, \\
3g^{rs} &= h^{rs} = 3e(a)_r e(a)_s,
\end{align*}
\]

(2.6)

The 3-metric \(3g_{rs} = h_{rs}\) has signature \((+++)\), so that we may put all the flat 3-indices down. We have \(3g^{ru} 3g_{us} = \delta^r_s \left( h^{ru} h_{us} = \delta^r_s \right)\), \(\partial_A 3g^{rs} = -3g^{ru} 3g^{sv} \partial_A 3g_{uv}\).

The conditions for having an admissible 3+1 splitting of space-time are:

a) \(1 + n(\tau, \vec{\sigma}) > 0\) everywhere (the instantaneous 3-spaces never intersect each other);

b) the Møller conditions [10, 16], which imply

\[
\sqrt{-g} = \sqrt{4g} = \frac{\sqrt{3g}}{\sqrt{4g^{\tau \tau}}} = \sqrt{\gamma(1 + n)} = 3\epsilon (1 + n).
\]
i) $\epsilon^4 g_{\tau \tau} > 0$, i.e. $(1 + n)^2 > 3g^{rs} n_r n_s$ (the rotational velocity never exceeds the velocity of light $c$, so that the coordinate singularity of the rotating disk named ”horizon problem” is avoided);

ii) $\epsilon^4 g_{rr} = -3g_{rr} < 0$ (satisfied by the signature of $3 g_{rs} = h_{rs}$), $4 g_{rr}^4 g_{ss} - (4 g_{rs})^2 > 0$ and $\det \epsilon^4 g_{rs} = -\det 3 g_{rs} < 0$ (satisfied by the signature of $3 g_{rs}$) so that $\det 4 g_{AB} < 0$; these conditions imply that $3 g_{rs} = h_{rs}$ has three definite positive eigenvalues $\lambda_r = \lambda_\tau^2$ in the non-degenerate case without Killing symmetries, the only one we consider;

c) the space-time is asymptotically Minkowskian with the instantaneous 3-spaces orthogonal to the ADM 4-momentum at spatial infinity: they are non-inertial rest frames of the 3-universe (isolated system), there is an asymptotic Minkowski background 4-metric and there are asymptotic inertial observers whose spatial axes $e_\mu$ are identified by the fixed stars.

As said in Ref.[6], in ADM canonical tetrad gravity the 16 configuration variables are: the 3 boost variables $\varphi_{(a)}(\tau, \vec{\sigma})$; the lapse and shift functions $n(\tau, \vec{\sigma})$ and $n_{(a)}(\tau, \vec{\sigma})$; the cotriads $3 e_{(a)r}(\tau, \vec{\sigma})$. Their conjugate momenta are $\pi_{\varphi_{(a)}}(\tau, \vec{\sigma}), \pi_n(\tau, \vec{\sigma}), \pi_{n_{(a)}}(\tau, \vec{\sigma}), 3 \pi_{e_{(a)}}^{(r)}(\tau, \vec{\sigma})$. There are 14 first-class constraints: A) the 10 primary constraints $\pi_{\varphi_{(a)}}(\tau, \vec{\sigma}) \approx 0$, $\pi_{n}(\tau, \vec{\sigma}) \approx 0$, $\pi_{n_{(a)}}(\tau, \vec{\sigma}) \approx 0$ and the 3 rotation constraints $M_{(a)}(\tau, \vec{\sigma}) \approx 0$ implying the gauge nature of the 3 Euler angles $\alpha_{(a)}(\tau, \vec{\sigma})$; B) the 4 secondary super-Hamiltonian and super-momentum constraints $H(\tau, \vec{\sigma}) \approx 0$, $H_{(a)}(\tau, \vec{\sigma}) \approx 0$. As a consequence there are 14 gauge variables (the inertial effects) and two pairs of canonically conjugate physical degrees of freedom (the tidal effects).

At this stage the basis of canonical variables for this formulation of tetrad gravity, naturally adapted to 7 of the 14 first-class constraints, is

\[
\begin{array}{c|c|c|c}
\varphi_{(a)} & n & n_{(a)} & 3 e_{(a)r} \\
\hline
\pi_{\varphi_{(a)}} \approx 0 & n \approx 0 & n_{(a)} \approx 0 & 3 \pi_{e_{(a)}}^{(r)}
\end{array}
\]

(2.7)

From Eqs.(5.5) of Ref.[3] we assume the following (direction-independent, so to kill super-translations) boundary conditions at spatial infinity ($r = \sqrt{\sum_r (\sigma^r)^2}$; $\epsilon > 0$; $M = \text{const}$): $n(\tau, \vec{\sigma}) \rightarrow r \rightarrow \infty O(r^{-2+\epsilon})$, $\pi_n(\tau, \vec{\sigma}) \rightarrow r \rightarrow \infty O(r^{-3})$, $n_{(a)}(\tau, \vec{\sigma}) \rightarrow r \rightarrow \infty O(r^{-1+\epsilon})$, $\pi_{n_{(a)}}(\tau, \vec{\sigma}) \rightarrow r \rightarrow \infty O(r^{-5/2})$;

C. The York Canonical Basis.

In Ref.[6] we studied the following point canonical transformation (it is a Shankmugadhasan canonical transformation [20]) on the canonical variables (2.7), implementing the York map of Refs.[17, 18] and identifying a canonical basis adapted to the 10 primary first-class constraints. It is realized in two steps and leads to the following York canonical basis.
and we Abelianize the rotation constraints (with conjugate momenta \(3\pi^r\)), \(\tilde{n}_a = \sum_b \pi_{(b)(a)}(\alpha(\varepsilon))\) are the cotriads and the shift functions at \(\alpha(\varepsilon)(\tau, \bar{\sigma}) = 0\) after the extraction of the rotation matrix \(R_{(a)(b)}(\alpha(\varepsilon)(\tau, \bar{\sigma}))\), see after Eq. (2.2).

With the first canonical transformation we extract the 3 angles \(\alpha(\varepsilon)(\tau, \bar{\sigma})\) from the cotriads and we Abelianize the rotation constraints \(3M(\alpha(\varepsilon)) \approx 0\), replacing them with the momenta conjugate to the angles, \(\pi_{(a)}(\varepsilon) = -\sum_b 3M(b) A_{(b)(a)}(\alpha(\varepsilon)) \approx 0\). The O(3) Lie algebra-valued Cartan matrix \(A_{(a)(b)}(\theta^n)\) \([B = A^{-1}]\) is defined in Ref.[3].

The second canonical transformation is based on the fact that the 3-metric \(3g_{rs}\) is a real symmetric \(3 \times 3\) matrix, which may be diagonalized with an orthogonal matrix \(V(\theta^n)\), \(V^{-1} = V^T (\sum_v V_{va} V_{vb} = \delta_{ab}, \sum_v V_{va} V_{vb} = \delta_{uv}, \sum_v \epsilon_{uvw} V_{ua} V_{vb} = \sum_c \epsilon_{abc} V_{wa})\), \(\det V = 1\), depending on 3 Euler angles \(\theta^n\). The gauge Euler angles \(\theta^n\) give a description of the 3-coordinate systems on \(\Sigma_\tau\) from a local point of view, because they give the orientation of the tangents to the three 3-coordinate lines through each point (their conjugate momenta are determined by the super-momentum constraints).

In the York canonical basis we have (from now on we will use \(V_{ra}\) for \(V_{ra}(\theta^n)\) to simplify the notation)

\[ (2.8) \]

---

3 Due to the positive signature of the 3-metric, we define the matrix \(V\) with the following indices: \(V_{ua}\).

Since the choice of Shamnugadhasan canonical bases breaks manifest covariance, we will use the notation \(V_{ua} = \sum_v V_{va} \delta_{v(a)}\) instead of \(V_{v(a)}\). We use the following types of indices: \(a = 1, 2, 3\) and \(\bar{a} = 1, 2\).

4 A similar diagonalization of the 3-metric has been considered also in Ref.[21] for numerical gravity purposes. Instead of the Euler angles \(\theta^n\) one uses an exponential representation of the rotation matrices \(V_{ra}(\theta^n) = e^{\sum_a e_{ram} \zeta_m(\theta^n)} e^{\phi_a} e^{\sum_a e_{ram} \zeta_m(\theta^n)}\) with \(e^{\phi_a} = \phi^2 Q_a^2\), where \(\eta_k = \zeta_k/\zeta\) with \(\zeta = \sqrt{\zeta_1^2 + \zeta_2^2 + \zeta_3^2}\) (\(-1 < \eta_k < 1\)), one gets \(V_{ra} = \delta_{ab} \cos \zeta + \sum_c \epsilon_{abc} \eta_c \sin \zeta + \eta_a \eta_b (1 - \cos \zeta)\) and a similar expression for \(3g_{rs}\). Then \(d^3g_{rs}\) is expressed in terms of \(d\phi_b\) and \(d\zeta_a\) and, when the 3 eigenvalues are distinct, one can get \(d\phi_a\) and \(d\zeta_a\) in terms of \(d^3g_{rs}\).
\[
4g_{rr} = \epsilon \left[ (1 + n)^2 - \sum a \bar{n}_a^2 \right],
\]

\[
4g_{rs} = -\epsilon \sum a \bar{n}_a^3 \bar{e}_{(a)r} = -\epsilon \phi^{1/3} \sum a Q_a V_{ra},
\]

\[
4g_{rs} = -\epsilon^3 g_{rs} = -\epsilon \sum_{uv} V_{ru} \lambda_u \delta_{uv} V_{ev}^T = -\epsilon \sum a \left( V_{ra} \Lambda^a \right) \left( V_{sa} \Lambda^a \right) =
\]

\[
= -\epsilon \sum_a \bar{e}_{(a)r} \bar{e}_{(a)s} = -\epsilon \sum_a \bar{e}_{(a)r} \bar{e}_{(a)s} = -\epsilon \phi^{4/3} g_{rs} = -\epsilon \phi^{2/3} \sum_a Q_a^2 V_{ra} V_{sa},
\]

\[
\Lambda_a = \sum_u \delta_{au} \sqrt{\gamma} = \phi^2 Q_a = \phi^{1/3} Q_a, \quad \dot{\Lambda}_a = \epsilon \sum_a \gamma_{aa} R_a = \epsilon \Gamma_a^{(1)},
\]

\[
\dot{\phi} = \phi^6 = \sqrt{\gamma} = \sqrt{\det g} = 3 \bar{e} = \sqrt{\lambda_1 \lambda_2 \lambda_3} = \Lambda_1 \Lambda_2 \Lambda_3,
\]

\[
3 \bar{e}_{(a)r} = \sum_b R_{(a) (b)} (\alpha_{(e)}) \bar{e}_{(b)r}, \quad 3 \bar{e}_{(a)r} = \phi^{1/3} Q_a V_{ra},
\]

\[
3 \bar{e}_{(a)r} = \sum_b R_{(a) (b)} (\alpha_{(e)}) \bar{e}_{(b)r}, \quad 3 \bar{e}_{(a)r} = \phi^{-1/3} Q_a^{-1} V_{ra},
\]

\[
3 \bar{\pi}_{(a)}^{(r)} = \sum_b R_{(a) (b)} (\alpha_{(e)}) \bar{\pi}_{(b)}^{(r)},
\]

\[
3 \bar{\pi}_{(a)}^{(r)} \approx \phi^{-1/3} \left[ V_{ra} Q_a^{-1} \left( \phi \bar{\pi}_{(r)}^{(\phi)} + \sum_b \gamma_{ba} \Pi_b \right) +
\]

\[
+ \sum_{l \neq a} \sum_{twi} Q_{l}^{-1} \frac{V_{sr \ell} \epsilon_{alt} V_{ati}}{Q_{l} Q_{a}^{-1} - Q_{a} Q_{l}^{-1}} B_{lw} \pi_{(r)}^{(i)} \right],
\]

\[
\bar{\pi}_{(r)}^{(\phi)} = -\sum_{imb} A_{ml} (\theta^n) \epsilon_{mir} 3 \bar{e}_{(a)t} 3 \bar{\pi}_{(a)}^{(r)},
\]

\[
\bar{\pi}_{(r)}^{(\phi)} = \frac{c^3}{12 \pi G} 3 K \approx \frac{1}{3} \epsilon \sum_{rab} 3 \bar{\pi}_{(a)}^{(r)} R_{(a) (b)} (\alpha_{(e)}) 3 \bar{e}_{(b)r},
\]

\[
\Pi_{(r)} = \sum_{rab} \gamma_{ab} 3 \bar{\pi}_{(a)}^{(r)} R_{(a) (b)} (\alpha_{(e)}) 3 \bar{e}_{(b)r}.
\]

(2.9)

The set of numerical parameters \( \gamma_{aa} \) satisfies \([0] \sum_u \gamma_{au} = 0, \sum_u \gamma_{au} \gamma_{ba} = \delta_{ab}, \sum_a \gamma_{au} \gamma_{av} = \delta_{uv} - \frac{1}{3}\). Each solution of these equations defines a different York canonical basis.

We only consider 3-metrics with 3 distinct positive eigenvalues \( \lambda_r = \Lambda^2_r \); the degenerate cases should be treated by adding by hand the constraints \( \Lambda_a - \Lambda_b \approx 0 \) or \( \Lambda_1 \approx \Lambda_2 \approx \Lambda_3 \) and by studying the resulting constraint algebra.

The assumed boundary conditions given after Eqs. (2.2) imply \( \Lambda_a (\tau, \bar{\sigma}) = \left( \phi^{1/3} Q_a \right) (\bar{\tau}, \bar{\sigma}) \to_{\tau \to \infty} 1 + \frac{\delta}{4r} + \frac{\alpha_a}{r^{3/2}} + O(r^{-3}) \) and \( \bar{\phi} (\tau, \bar{\sigma}) \to_{\tau \to \infty} 1 + O(r^{-1}) \). Moreover
we must have \( \pi^i_0(\tau, \tilde{\sigma}) \to r^{-\infty} O(r^{-4}) \), since the requirement \( \Lambda_a(\tau, \tilde{\sigma}) \neq \Lambda_b(\tau, \tilde{\sigma}) \) for \( a \neq b \), needed to avoid singularities, implies \( a \neq a \) for \( a \neq b \) in their asymptotic behavior, so that we get \( \left( \frac{\Lambda_b}{\Lambda_a} - \frac{\Lambda_a}{\Lambda_b} \right)^{-1}(\tau, \tilde{\sigma}) = \left( Q_b Q_a^{-1} - Q_a Q_b^{-1} \right)^{-1}(\tau, \tilde{\sigma}) \to r^{-\infty} \frac{r^{3/2}}{2(a_b - a_a)} \). As a consequence, we have \( \pi^a(\tau, \tilde{\sigma}) \to r^{-\infty} O(r^{-5/2}) \). Also the angles \( \alpha_a(\tau, \tilde{\sigma}) \) and \( \theta^a(\tau, \tilde{\sigma}) \) must tend to zero in a direction-independent way at spatial infinity. We also have \( \pi^a_3(\tau, \tilde{\sigma}) \to r^{-\infty} O(r^{-5/2}) \) at spatial infinity.

In Eq.(2.9) the quantity \( 3K(\tau, \tilde{\sigma}) \) is the trace of the extrinsic curvature \( 3K_{rs}(\tau, \tilde{\sigma}) \) of the instantaneous 3-spaces \( \Sigma_r \). In the York canonical basis the extrinsic curvature \( 3K_{rs} \), the 3-spin connection \( \tilde{\omega}_{(a)} = \tilde{\omega}_{(a)}|_{\alpha = 0} \) (see Eq.(B17) of Ref.[6]) and the 3-Christoffel symbols have the following expression [6]

\[
3K_{rs} = -\frac{4\pi G}{c^3} \sum_{abu} \left[ 3\tilde{\epsilon}_{(a)(b)} \tilde{\epsilon}^{(c)} + 3\tilde{\epsilon}_{(a)s} \tilde{\epsilon}^{(b)r} \right] 3\tilde{\epsilon}_{(a)u} \tilde{\epsilon}^{(c)u} - 3\tilde{\epsilon}_{(a)r} \tilde{\epsilon}^{(b)s} \tilde{\epsilon}^{(c)u} \tilde{\epsilon}^{(b)u} \tilde{\epsilon}^{(c)u} \approx \left( \sum_a Q_a^2 V_r a V_s a + \sum_b \gamma_{ba} \Pi_b + \tilde{\phi} \pi_{\tilde{\phi}} \right) + \sum_{ab} Q_a Q_b (V_r a V_s b + V_r b V_s a) \left( \sum_{twi} \frac{\epsilon_{abt} V_{twi} B_{twi} \pi_{\tilde{\phi}}^i}{Q_b Q_a^{-1} - Q_a Q_b^{-1}} \right),
\]

\[
3\tilde{\omega}_{(a)} = \frac{1}{2} \sum_{bc} \epsilon_{(a)(b)(c)} \tilde{\omega}_{(b)(c)} =
\]

\[
= \frac{1}{2} \sum_{bcu} \epsilon_{(a)(b)(c)} \left[ \partial_r \tilde{\epsilon}_{(c)u} - \partial_u \tilde{\epsilon}_{(c)r} + \sum_{dv} \tilde{\epsilon}_{(c)v} \tilde{\epsilon}_{(d)u} \right] =
\]

\[
= \sum_{bcu} \epsilon_{(a)(b)(c)} V_{ub} \left[ -Q_b^{-1} Q_c V_r c \partial_u \left( \frac{1}{3} \ln \tilde{\phi} + \sum_{a} \gamma_{ac} R_{a} \right) \right] + \frac{1}{2} Q_b^{-1} Q_c \left( \partial_r V_{uc} - \partial_u V_{rc} \right) + \sum_{v} Q_b^{-1} Q_c^{-1} Q_d V_r d V_c \partial_r V_{uc} + \frac{1}{2} \sum_{v} Q_b^{-1} Q_c^{-1} Q_d V_r d V_c \partial_r V_{uc},
\]

\[
3\Gamma^r_{uv} = \frac{1}{3} (\delta_{ru} \tilde{\phi}^{-1} \partial_v \tilde{\phi} + \delta_{rv} \tilde{\phi}^{-1} \partial_u \tilde{\phi} - \frac{1}{3} \sum_{abs} Q_b^2 Q_a^{-2} V_r a V_s a V_v b \phi^{-1} \partial_x \phi + \sum_{a} \gamma_{aa} V_r a \left( V_u a \partial_v R_{a} + V_u a \partial_u R_{a} \right) - \sum_{bab} \gamma_{bb} Q_b^2 Q_a^{-2} V_r a V_s a V_v b \partial_s R_{b} + \frac{1}{2} \sum_{a} V_r a \left( \partial_u V_{ua} + \partial_v V_{ua} \right) + \frac{1}{2} \sum_{abs} Q_a^{-2} Q_b V_r a V_s a \left[ V_{ub} \left( \partial_u V_{sb} - \partial_s V_{vb} \right) + V_{vb} \left( \partial_u V_{sb} - \partial_s V_{ub} \right) \right],
\]

\[
\sum_{v} 3\Gamma^v_{uv} = \tilde{\phi}^{-1} \partial_u \tilde{\phi}, \quad (2.10)
\]
The previous sequence of canonical transformations realizes a York map because the gauge variable \( \pi_\phi \) (describing the freedom in the choice of the instantaneous 3-spaces \( \Sigma_r \)) is proportional to York internal extrinsic time \( ^3K \). It is the only gauge variable among the momenta: this is a reflex of the Lorentz signature of space-time, because \( \pi_\phi \) and \( \theta^a \) can be used as a set of 4-coordinates [7].

Its conjugate variable, to be determined by the super-Hamiltonian constraint, is \( \tilde{\phi} = \phi^6 = \bar{\epsilon} \), which is proportional to Misner’s internal intrinsic time; moreover \( \tilde{\phi} \) is the 3-volume density on \( \Sigma_r \): \( V_R = \int_R d^3 \sigma \phi^6 \), \( R \subset \Sigma_r \). Since we have \( ^3g_{rs} = \tilde{\phi}^{2/3} \bar{g}_{rs} \) with \( \det \bar{g}_{rs} = 1 \), \( \tilde{\phi} \) is also called the conformal factor of the 3-metric.

The two pairs of canonical variables \( R_\bar{a}, \Pi_\bar{a} \), \( \bar{a} = 1, 2 \), describe the generalized tidal effects, namely the independent degrees of freedom of the gravitational field. In particular the configuration tidal variables \( R_\bar{a} \) depend only on the eigenvalues of the 3-metric. They are Dirac observables only with respect to the gauge transformations generated by 10 of the 14 first class constraints. Let us remark that, if we fix completely the gauge and we go to Dirac brackets, then the only surviving dynamical variables \( R_\bar{a} \) and \( \Pi_\bar{a} \) become two pairs of non canonical Dirac observables for that gauge: the two pairs of canonical Dirac observables have to be found as a Darboux basis of the copy of the reduced phase space identified by the gauge and they will be (in general non-local) functionals of the \( R_\bar{a}, \Pi_\bar{a} \) variables.

Since the variables \( \tilde{\phi} \) and \( \pi_\theta^{(\theta)} \) are determined by the super-Hamiltonian and super-momentum constraints, the arbitrary gauge variables are \( \alpha(a), \varphi(a), \theta^a, \pi_\phi, n \) and \( \bar{n}_\alpha \). As shown in Refs. [5], they describe the following generalized inertial effects:

a) \( \alpha(a)(\tau, \vec{\sigma}) \) and \( \varphi(a)(\tau, \vec{\sigma}) \) are the 6 configuration variables parametrizing the \( O(3,1) \) gauge freedom in the choice of the tetrads in the tangent plane to each point of \( \Sigma_r \) and describe the arbitrariness in the choice of a tetrad to be associated to a time-like observer, whose world-line goes through the point \( (\tau, \vec{\sigma}) \). They fix the unit 4-velocity of the observer and the conventions for the orientation of gyroscopes and their transport along the world-line of the observer.

b) \( \theta^a(\tau, \vec{\sigma}) \) [depending only on the 3-metric, as shown in Eq. (2.9)] describe the arbitrariness in the choice of the 3-coordinates in the instantaneous 3-spaces \( \Sigma_r \) of the chosen non-inertial frame centered on an arbitrary time-like observer. Their choice will induce a pattern of relativistic inertial forces for the gravitational field, whose potentials are the functions \( V_{ra}(\theta^a) \) present in the weak ADM energy \( E_{ADM} \) given in Eqs. (3.14).

---

[5] If we consider the eigenvalue equation for the 3-metric \( ^3\bar{g}_{rs} \) of determinant one, we identify the following two 3-scalars depending only on the 3-metric \( ^3\bar{g}_{rs} \): i) \( \text{Tr} \ ^3\bar{g}_{rs} = \sum_r ^3\bar{g}_{rr} = \sum_a Q_a^2 \) (the sum of the eigenvalues); ii) \( ^3\bar{g}_{11} ^3\bar{g}_{22} - ^3\bar{g}_{12}^2 + ^3\bar{g}_{22}^3 \bar{g}_{33} - ^3\bar{g}_{23}^2 + ^3\bar{g}_{33} \bar{g}_{11} - ^3\bar{g}_{31}^2 = Q_1^2 Q_2^2 + Q_2^2 Q_3^2 + Q_3^2 Q_4^2 \) (the sum of the possible products of two eigenvalues). This suggest the possibility of a point canonical transformation from \( R_\bar{a}, \Pi_\bar{a} \) to 3-scalar tidal variables \( X(R_\bar{a}), \Pi_X, Y(R_\bar{a}), \Pi_Y \) with

\[
\begin{align*}
X &= e^{2 \left[ \gamma_{11} R_1 + \gamma_{21} R_2 + \gamma_{12} R_1 + \gamma_{22} R_2 + \gamma_{13} R_1 + \gamma_{23} R_2 \right]} + e^{2 \left[ \gamma_{11} R_1 + \gamma_{22} R_2 + \gamma_{12} R_1 + \gamma_{21} R_2 \right]} = \text{Tr} \ ^3\bar{g}_{rs} = \tilde{\phi}^{-2/3} \text{Tr} \ ^3\bar{g}_{rs}, \\
Y &= e^{2 \left[ \gamma_{11} + \gamma_{12} \right] R_1 + \left[ \gamma_{21} + \gamma_{22} \right] R_2} + e^{2 \left[ \gamma_{11} + \gamma_{12} \right] R_1 + \left[ \gamma_{22} + \gamma_{23} \right] R_2} + e^{2 \left[ \gamma_{11} + \gamma_{12} \right] R_1 + \left[ \gamma_{23} + \gamma_{21} \right] R_2} = \\
&= \bar{\phi}^{-4/3} \left[ ^3\bar{g}_{11} ^3\bar{g}_{22} - ^3\bar{g}_{12}^2 + ^3\bar{g}_{22}^3 \bar{g}_{33} - ^3\bar{g}_{23}^2 + ^3\bar{g}_{33} \bar{g}_{11} - ^3\bar{g}_{31}^2 \right].
\end{align*}
\]
c) \( \bar{n}_{(a)}(\tau, \vec{\sigma}) \), the shift functions appearing in the Dirac Hamiltonian, describe which points on different instantaneous 3-spaces have the same numerical value of the 3-coordinates. They are the inertial potentials describing the effects of the non-vanishing off-diagonal components \( 4_{\tau r}(\tau, \vec{\sigma}) \) of the 4-metric, namely they are the gravito-magnetic potentials \(^6\) responsible of effects like the dragging of inertial frames (Lens-Thirring effect) \(^{22}\) in the post-Newtonian approximation. The shift functions are determined by the \( \tau \)-preservation of the gauge fixings determining the gauge variables \( \theta^i(\tau, \vec{\sigma}) \).

d) \( \pi_{\tilde{\phi}}(\tau, \vec{\sigma}) \), i.e. the York time \( ^3K(\tau, \vec{\sigma}) \), describes the non-dynamical arbitrariness in the choice of the convention for the synchronization of distant clocks which remains in the transition from special to general relativity. As said in the Introduction, the choice of the shape of the instantaneous 3-space as a sub-manifold of space-time (a pure gauge choice in special relativity) is dynamically determined by the chosen solution of Einstein’s equations after the fixation of the gauge variables \( ^3K(\tau, \vec{\sigma}) \) and \( \theta^i(\tau, \vec{\sigma}) \). Since the York time is present in the Dirac Hamiltonian \(^7\), it is a new inertial potential connected to the problem of the relativistic freedom in the choice of the instantaneous 3-space, which has no non-relativistic analogue (in Galilei space-time time is absolute and there is an absolute notion of Euclidean 3-space). Its effects are completely unexplored.

e) \( n(\tau, \vec{\sigma}) \), the lapse function appearing in the Dirac Hamiltonian, describes the arbitrariness in the choice of the unit of proper time in each point of the simultaneity surfaces \( \Sigma_\tau \), namely how these surfaces are packed in the 3+1 splitting. The lapse function is determined by the \( \tau \)-preservation of the gauge fixing for the gauge variable \( ^3K(\tau, \vec{\sigma}) \).

The gauge variables \( \theta^i(\tau, \vec{\sigma}), n(\tau, \vec{\sigma}), \bar{n}_{(a)}(\tau, \vec{\sigma}) \) describe inertial effects, which are the relativistic counterpart of the non-relativistic ones (the centrifugal, Coriolis,... forces in Newton mechanics in accelerated frames) and which are present also in the non-inertial frames of Minkowski space-time \(^{10}\).

D. The Expansion and the Shear of the Eulerian Observers.

Let us now consider the geometrical interpretation of the extrinsic curvature \( ^3K_{rs} \) of the instantaneous 3-spaces \( \Sigma_\tau \) in terms of the properties of the surface-forming (i.e. irrotational) congruence of Eulerian (non geodesic) time-like observers, whose world-lines have the tangent unit 4-velocity equal to the unit normal orthogonal to the instantaneous 3-spaces \( \Sigma_\tau \). If we use radar 4-coordinates, the covariant unit normal \( \epsilon_{lA} = (1 + n) (1; 0) \) of Eqs.(2.2) has the following covariant derivative (see for instance Ref.[23])

\[^6\] In the post-Newtonian approximation in harmonic gauges they are the counterpart of the electro-magnetic vector potentials describing magnetic fields \(^{22}\): A) \( N = 1 + n, \quad n \overset{\text{def}}{=} - \frac{4\pi}{c^2} \Phi_G \) with \( \Phi_G \) the gravito-electric potential; B) \( n_r \overset{\text{def}}{=} \frac{2}{c^2} A_{Gr} \) with \( A_{Gr} \) the gravito-magnetic potential; C) \( E_{Gr} = \partial_r \Phi_G - \partial_\tau (\frac{1}{2} A_{Gr}) \) (the gravito-electric field) and \( B_{Gr} = \epsilon_{ruv} \partial_u A_{Gr} = c\Omega_{Gr} \) (the gravito-magnetic field). Let us remark that in arbitrary gauges the analogy with electromagnetism breaks down.

\[^7\] See Eqs.(3.43) and (3.44) for its presence in the super-Hamiltonian constraint and in the weak ADM energy, and Eqs.(3.41) for its presence in the super-momentum constraints.
4\nabla_A \epsilon l_B = \epsilon l_A 3 a_B + \sigma_{AB} + \frac{1}{3} \theta h_{AB} - \omega_{AB} = \epsilon l_A 3 a_B + 3 K_{AB},

3 K_{AB} = 3 K_{rs} \dot{b}_A^r \dot{b}_B^s, \quad \dot{b}_A^r = \delta_A^r + 3 e_{(a)}^r \tilde{n}_u (a) \delta_A^r, \quad h_{AB} = 4 g_{AB} - \epsilon l_A l_B.

(2.11)

The quantities appearing in Eqs.(2.11) are:

a) the acceleration of the Eulerian observers

\[ 3 a^A = l^B 4 \nabla_B l^A = 4 g^{AB} 3 a_B, \quad 3 a_A = 3 a^r _A \dot{b}_A^r, \]

\[ 3 a_r = \partial_r \ln (1 + n), \quad 3 a_A = 3 a^r _A 3 e_{(a)}^r 3 e_{(a)}^u n_u = 3 a^r _A 3 e_{(a)}^r \tilde{n}_u (a), \]

\[ 3 a^n = -\epsilon 3 \tilde{e}_{(a)}^n 3 \tilde{e}_{(a)}^s 3 a_s, \quad 3 a^r = 0; \quad (2.12) \]

b) the vorticity or twist (a measure of the rotation of the nearby world-lines infinitesimally surrounding the given one), which is vanishing because the congruence is surface-forming

\[ \omega_{AB} = -\omega_{BA} = \frac{\epsilon}{2} (l_A 3 a_B - l_B 3 a_A) - \frac{\epsilon}{2} (4 \nabla_A l_B - 4 \nabla_B l_A) = 0, \]

\[ \omega_{AB} l^B = 0, \quad \omega^A = \frac{1}{2} 4 \eta^{ABCD} \omega_{BC} l_D = 0; \]

(2.13)

c) the expansion \(^8\), which coincides with the York external time, is proportional to the Hubble parameter \( H \)^9 and to the dimensionless (cosmological) deceleration parameter \( q = 3 l^A 4 \nabla_A \frac{1}{\theta} - 1 = -3 \theta^{-2} l^A \partial_A \theta - 1, \)

\[ \theta = 4 \nabla_A l^A = -\epsilon 3 K = -\frac{4 \pi G 3 \tilde{e}_{(a)}^r 3 \tilde{n}_u (a)}{c^3} = -\epsilon \frac{12 \pi G}{c^3} \tilde{\phi}, \]

\[ H = \frac{1}{3} \theta = \frac{1}{l} l^A 4 \nabla_A l = -\epsilon \frac{4 \pi G}{c^3} \tilde{\phi}, \quad q = 3 l^A 4 \nabla_A \frac{1}{\theta} - 1; \quad (2.14) \]

\(^8\) It measures the average expansion of the infinitesimally nearby world-lines surrounding a given world-line in the congruence.

\(^9\) \( l \) is a representative length along the integral curves of \( 4 \tilde{E}_{(a)} \), describing the volume expansion (contraction) behavior of the congruence.
d) the shear $^{10}$

$$
\sigma_{AB} = \sigma_{BA} = -\frac{\epsilon}{2} (3a_A l_B + 3a_B l_A) + \frac{\epsilon}{2} \left(4 \nabla_A l_B + 4 \nabla_B l_A\right) - \frac{1}{3} \theta^3 h_{AB} = \\
= \left(3K_{rs} - \frac{1}{3} 3g_{rs} 3K\right) \tilde{b}_A \tilde{b}_B, \quad 4g^{AB} \sigma_{AB} = 0, \quad \sigma_{AB} l^B = 0. \quad (2.15)
$$

By explicit calculation we get the following components of the shear along the tetrads (2.2)

$$
\sigma_{(a)(b)} = \sigma_{(a)(b)} 4E_A 4E_B = 4g_{AC} 4g_{BD} \sigma^{CD},
$$

$$
\sigma_{\tau\tau} = \bar{n}_{(a)} \bar{n}_{(b)} \sigma_{(a)(b)} = \left(3K_{rs} - \frac{1}{3} 3g_{rs} 3K\right) \bar{n}_{(a)} 3\bar{e}_{(a)} 3\bar{e}_{(b)},
$$

$$
\bar{e}_{(a)} \bar{e}_{(b)} = 3\bar{e}_{(a)} 3\bar{e}_{(b)} 3\bar{e}_{(a)} 3\bar{e}_{(b)},
$$

$$
\sigma_{rs} = \sigma_{rs} 3\bar{e}_{(a)} 3\bar{e}_{(b)} \sigma_{(a)(b)},
$$

$$
\sigma_{(a)(b)} = \sigma_{(a)(b)} 4E_A 4E_B = \frac{1}{(1 + n)^2} \left[\sigma_{\tau\tau} - 2 \bar{n}_{(a)} \bar{e}_{(a)} \sigma_{\tau\tau} + \bar{n}_{(a)} \bar{n}_{(b)} \bar{e}_{(a)} \bar{e}_{(b)} \sigma_{rs}\right],
$$

$$
\sigma_{(a)(a)} = \sigma_{(a)(a)} 4E_A 4E_B = \frac{1}{1 + n} \left[\sigma_{rs} - \bar{n}_{(b)} \bar{e}_{(b)} \sigma_{rs}\right] 3\bar{e}_{(a)},
$$

$$
\sigma_{(a)(b)} = \sigma_{(a)(b)} 4E_A 4E_B = \sigma_{rs} 3\bar{e}_{(a)} 3\bar{e}_{(b)},
$$

$$
\downarrow
$$

$$
\sigma_{(a)(a)} = 0, \quad \sigma_{(a)(a)} = 0,
$$

$$
\sigma_{(a)(b)} = \sigma_{(b)(a)} = \left(3K_{rs} - \frac{1}{3} 3g_{rs} 3K\right) 3\bar{e}_{(a)} 3\bar{e}_{(b)}, \quad \sum_a \sigma_{(a)(a)} = 0. \quad (2.16)
$$

$^{10}$ It measures how an initial sphere in the tangent space to the given world-line, which is Lie-transported along the world-line tangent $l^\mu$ (i.e. it has zero Lie derivative with respect to $l^\mu \partial_\mu$), is distorted towards an ellipsoid with principal axes given by the eigenvectors of $\sigma^{\mu\nu}$, with rate given by the eigenvalues of $\sigma^{\mu\nu}$. 
\(\sigma_{(a)(b)}\) depends upon \(\theta^r, \tilde{\phi}, R_{\tilde{a}}, \pi_r^{(\theta)}\) and \(\Pi_{\tilde{a}}\).

As a consequence we have \(^{11}\) \((n_r|s)\) is the covariant derivative on \(\Sigma_r; \overset{\circ}{=}\) means evaluated by using the equations of motion

\[
3K_{rs} = -\frac{\epsilon}{3} g_{rs} \theta + \sigma_{(a)(b)} \frac{3}{3} \epsilon_{(a)3} \epsilon_{(b)3} = \phi^{2/3} \sum_{ab} \left( -\frac{\epsilon}{3} \theta \delta_{ab} + \sigma_{(a)(b)} \right) Q_a Q_b V_{ra} V_{sb},
\]

\[
\Rightarrow \quad \partial_r 3g_{rs} \overset{(2.9)}{=} \phi^{2/3} \sum_a Q_a^2 \left[ 2 \left( \frac{1}{3} \phi^{-1} \partial_r \tilde{\phi} + \partial_r (\Gamma_a^1) \right) V_{ra} V_{sa} + \partial_r (V_{ra} V_{sa}) \right] \overset{=} = n_{r|a} + n_{s|r} - 2 (1 + n) \phi^{2/3} \sum_{ab} Q_a Q_b V_{ra} V_{sb} \left( -\frac{\epsilon}{3} \theta \delta_{ab} + \sigma_{(a)(b)} \right).
\]

(2.17)

By using Eqs.\((2.9)\) and \((2.10)\) we get \(^{12}\)

\[
\tilde{\phi} \sigma_{(a)(a)} = -\frac{8 \pi G}{c^3} \sum_a \gamma_{\tilde{a}a} \Pi_{\tilde{a}}, \quad \Rightarrow \quad \Pi_{\tilde{a}} = -\frac{\epsilon}{8 \pi G} \tilde{\phi} \sum_a \gamma_{\tilde{a}a} \sigma_{(a)(a)},
\]

\[
\Sigma_{(a)(b)|a \neq b} \overset{\text{def}}{=} \tilde{\phi} \sigma_{(a)(b)|a \neq b} = -\frac{8 \pi G}{c^3} \sum_{tw} Q_a Q_b^{-1} - Q_b^{-1} \sum_i B_{iw} \pi_i^{(\theta)},
\]

\[
\Rightarrow \quad \pi_i^{(\theta)} = \frac{c^3}{8 \pi G} \sum_{wtab} A_{wi} V_{wt} Q_a Q_b^{-1} \epsilon_{tab} \Sigma_{(a)(b)|a \neq b},
\]

\[
3K_{rs} = \phi^{2/3} \left[ \frac{4 \pi G}{c^3} \phi \sum_a Q_a^2 V_{ra} V_{sa} + \sum_{ab} \sigma_{(a)(b)} Q_a Q_b V_{ra} V_{sb} - \right.
\]

\[
- \frac{8 \pi G}{c^3} \phi^{-1} \sum_{aa} \gamma_{\tilde{a}a} \Pi_{\tilde{a}} Q_a^2 V_{ra} V_{sa} \right],
\]

\[
\sigma^2 \overset{\text{def}}{=} \frac{1}{2} \sum_{ab} \sigma_{(a)(b)} \sigma_{(a)(b)} = \frac{1}{2} \left( \frac{8 \pi G}{c^3} \right)^2 \phi^{-2} \left[ \sum_{\tilde{a}} \Pi_{\tilde{a}}^2 + \right.
\]

\[
+ \sum_{ww'} \left( \frac{V_{w1} V_{w'1}}{(Q_2 Q_3^2 - Q_3 Q_2^2)^2} + \frac{V_{w2} V_{w'2}}{(Q_3 Q_1^2 - Q_1 Q_3^2)^2} + \frac{V_{w3} V_{w'3}}{(Q_1 Q_2^2 - Q_2 Q_1^2)^2} \right)
\]

\[
\sum_{ii'} B_{iw} B_{iw'} \pi_i^{(\theta)} \pi_{i'}^{(\theta)}. \quad (2.18)
\]

\(^{11}\) The 3-scalars associated to the symmetric matrix \(3K_{rs}\) are \(I = 3K = -\epsilon \theta, II = det^3 K_{rs}, III = 3K_{11}^1 - 3K_{12}^2 - 3K_{22}^3 + 3K_{23}^1 + 3K_{33}^2 - 3K_{31}^3\). If \(\tilde{K}_{rs} = 3K_{rs} - \frac{1}{3} g_{rs} 3K\) is the traceless extrinsic curvature, the 3-scalars \(II\) and \(III\) may be replaced by \(II' = det^3 \tilde{K}_{rs}\) and \(III' = 3K_{11}^1 \tilde{K}_{22}^2 - 3K_{12}^3 - 3K_{23}^1 - 3K_{31}^2 + 3K_{22}^3\).

\(^{12}\) The canonical tidal variables \(R_{\tilde{a}}, \Pi_{\tilde{a}}\) are 3-scalars, since they can be replaced with a canonical basis built with the 3-scalars \(X(R_{\tilde{a}}), Y(R_{\tilde{a}})\) (see footnote 5) and \(\Pi_{\tilde{a}} = -\frac{\epsilon}{8 \pi G} \tilde{\phi} \sum_a \gamma_{\tilde{a}a} \sigma_{(a)(a)}\).
Therefore the Eulerian observers associated to the 3+1 splitting of space-time allow a physical interpretation of some of the variables of the York canonical basis:

1) their expansion $\theta$ is the gauge variable $\pi_\phi$ determining the non-dynamical part of the shape of the instantaneous 3-spaces $\Sigma_\tau$;

2) the diagonal elements of their shear describe the tidal momenta $\Pi_a$, while the non-diagonal elements are connected to the variables $\pi_\phi^{(\theta)}$, determined by the super-momentum constraints.

E. The Asymptotic ADM Poincare’ Algebra, the Rest-Frame Conditions and the Center of Mass

As explained in Ref.[1], following suggestions of Dirac in Ref.[24], the limit of the embedding $z^\mu(\tau, \vec{\sigma})$ for the non-inertial rest-frame instant form of metric and tetrad gravity at spatial infinity in asymptotically Minkowskian space-times has the form

$$z^\mu(\tau, \vec{\sigma}) \rightarrow x^\mu(\infty) + b^\mu(\infty)_r \sigma^r,$$  \hspace{1cm} (2.19)

where $x^\mu(\infty)$ is an asymptotic inertial observer and $b^\mu(\infty)_r = \hat{b}^\mu(\infty)_r$ and $b^\mu(\infty)_r$ are an asymptotic tetrad (denoted $\epsilon^\mu_A$ in the Introduction).

The generators of the asymptotic ADM Poincare’ group are [1]

$$p^\mu(\infty) = b^\mu(\infty)_A(\tau) P^A_{ADM} \approx b^\mu(\infty)_r \hat{P}^r_{ADM},$$

$$j^{\mu\nu}(\infty) = x^\mu(\infty) p^\nu(\infty) - x^\nu(\infty) p^\mu(\infty) + S^{\mu\nu}(\infty),$$

$$S^{\mu\nu}(\infty) = b^\mu(\infty)_A b^\nu(\infty)_B J^{AB}_{ADM} \approx$$

$$= [l^\mu(\infty)_r b^\nu(\infty)_s - l^\nu(\infty)_r b^\mu(\infty)_s] \hat{J}^{rs}_{ADM} + [b^\mu(\infty)_r b^\nu(\infty)_s - b^\nu(\infty)_r b^\mu(\infty)_s] \hat{J}^{rs}_{ADM},$$  \hspace{1cm} (2.20)

where $p^\mu(\infty)$ is the variable canonically conjugate to $x^\mu(\infty)$. $P^A_{ADM}, J^{AB}_{ADM}$ are the strong (surface integrals at spatial infinity) ADM Poincare’ charges in adapted radar coordinates: they are weakly equal to the weak (volume integrals over $\Sigma_\tau$) ADM Poincare’ charges $\hat{P}^A_{ADM}, \hat{J}^{AB}_{ADM}$.

Since $p^\mu(\infty)$ is orthogonal to the asymptotic hyper-plane (2.19) due to the requirement of absence of super-translations [1], we can make the identifications $l^\mu(\infty) = b^\mu(\infty)_\tau = p^\mu(\infty)/\sqrt{\epsilon p^2(\infty)} = h^\mu = (\sqrt{1 + \hat{h}^2}; \vec{\hat{h}})$ and $b^\mu(\infty)_r = \epsilon^\mu_r(\vec{\hat{h}})$, where $h^\mu$ and the space-like 4-vectors $\epsilon^\mu_r(\vec{\hat{h}})$, orthogonal to $h^\mu$, are the column of the standard Wigner boost sending $P^\mu = M_c h^\mu$ to its rest-frame form $M_c (1; \vec{0})$ (see Ref.[8] for this notation).

Then for consistency the first of Eqs.(2.20) implies the rest-frame conditions

$$P^r_{ADM} \approx \hat{P}^r_{ADM} \approx 0,$$  \hspace{1cm} (2.21)

and $M_c \approx \hat{P}^r_{ADM} = \hat{E}_{ADM}/c$.  \hspace{1cm} (2.21)
From Eqs.(25) of Ref.[3] the weak or volume form of the ADM Poincaré charges appearing in Eqs.(2.20) is

\[
P_{\text{ADM}}^r = \frac{1}{c} \hat{E}_{\text{ADM}} = \int d^3\sigma \left[ -\frac{c^3}{16\pi G} \sqrt{\gamma} 3 g^{rs} \left( 3 \Gamma_{ru}^s \Gamma_{su}^r - 3 \Gamma_{rs}^u \Gamma_{ru}^s \right) + \frac{8\pi G}{c^3 \sqrt{\gamma}} 3 G_{rsuv} 3 \Pi_{rs} 3 \Pi^{uv} + \mathcal{M} \right](\tau, \bar{\sigma}),
\]

\[
P_{\text{ADM}}^s = -2 \int d^3\sigma \left[ 3 \Gamma_{su}^r(\tau, \bar{\sigma}) 3 \Pi^{su} - \frac{1}{2} 3 g^{rs} \mathcal{M}_s \right](\tau, \bar{\sigma}) \approx 0,
\]

\[
\hat{J}_{\text{ADM}}^{rr} = -\hat{J}_{\text{ADM}}^{ss} = \epsilon \int d^3\sigma \left( \sigma^r \frac{c^3}{16\pi G} \sqrt{\gamma} 3 g^{ns} \left( 3 \Gamma_{nu}^s \Gamma_{su}^n - 3 \Gamma_{ns}^u \Gamma_{nu}^s \right) - \frac{8\pi G}{c^3 \sqrt{\gamma}} 3 G_{nsuv} 3 \Pi_{ns} 3 \Pi^{uv} + \mathcal{M} \right) + \frac{c^3}{16\pi G} \delta_u^r (3 g_{vs} - \delta_{vs}) \partial_n \left( \sqrt{\gamma} 3 g^{ns} 3 g^{uv} - 3 g^{nu} 3 g^{sv} \right) \right)(\tau, \bar{\sigma}) \approx 0,
\]

\[
\hat{J}_{\text{ADM}}^{rs} = \int d^3\sigma \left[ (\sigma^r 3 \Gamma_{su}^s - \sigma^s 3 \Gamma_{ru}^s) 3 \Pi^{uv} - \frac{1}{2} (\sigma^r 3 g^{su} - \sigma^s 3 g^{ru}) \mathcal{M}_u \right](\tau, \bar{\sigma}).
\]

They are the same weak Poincaré charges of metric gravity, expressed in terms of cotriads $3 \epsilon_{(a) r}$ and their conjugate momenta $3 \pi^r_{(a)}$, by using $3 g_{rs} = 3 \epsilon_{(a) r} 3 \epsilon_{(a) s}$, $3 \Pi^{rs} = \frac{1}{4} \left[ 3 \epsilon_{(a) r} 3 \pi^s_{(a)} + 3 \epsilon_{(a) s} 3 \pi^r_{(a)} \right]$ (see Eq.(5.7) of Ref.[1]). The Christoffel symbols $3 \Gamma^{uv}_{rs}$ are built with the 3-metric $3 g_{rs}$.

In Eqs.(2.22) we use the notations A) $\mathcal{M}(\tau, \bar{\sigma})$ for the internal matter mass density (in general metric-dependent); B) $\mathcal{M}_r(\tau, \bar{\sigma})$ for the internal matter momentum density (metric-independent and universal). See Section III, Eq.(3.9), for their explicit form: there we will give the modifications of the relevant formulas of Ref.[3] due to the presence of matter, because, in absence of derivative couplings to matter, they are independent from the type of matter: only the explicit form of the mass density $\mathcal{M}$ (and also of the matter stress tensor $T^{rs}$) depends on the type of matter.\textsuperscript{13}

In Section III, Eq.(3.11), there is the form of the energy-momentum tensor $T^{AB}(\tau, \bar{\sigma})$ for the type of matter considered in this paper. Since our formulation is equivalent to the ADM

\textsuperscript{13} We have:
A) For the Klein-Gordon field the Lagrangian is $\frac{1}{2} \left[ 4 g^{AB} \partial_A \phi \partial_B \phi - m^2 \phi^2 \right](\tau, \bar{\sigma})$ and we have $\mathcal{M}(\tau, \bar{\sigma}) = \frac{1}{2} \left[ \frac{c^2}{X^2} + 3 \epsilon_{(a) r} 3 \epsilon_{(a) s} \partial_r \phi \partial_s \phi \right](\tau, \bar{\sigma})$ and $\mathcal{M}_r(\tau, \bar{\sigma}) = \pi(\tau, \bar{\sigma}) \partial_r \phi(\tau, \bar{\sigma})$, where $\pi(\tau, \bar{\sigma})$ is the KG momentum.
B) For perfect fluids [25] with Lagrangian coordinates $\alpha^i(\tau, \bar{\sigma})$ and equation of state $\rho = \rho(n, s)$ and fluid momenta $\Pi_i(\tau, \bar{\sigma})$ we have $\mathcal{M}(\tau, \bar{\sigma}) = \left( \frac{c^2}{X} \left[ \phi^2 X \rho(\phi^{-6} X) + \left( \det(\partial_r \alpha^i) \right)^2 - X^2 \right] \right)(\tau, \bar{\sigma})$ (where $X = X(3 g^{rs}, \phi, \alpha^i, \Pi_i)$ is the solution of a transcendental equation depending on the equation of state) and $\mathcal{M}_r(\tau, \bar{\sigma}) = -\Pi_i(\tau, \bar{\sigma}) \partial_r \alpha^i(\tau, \bar{\sigma})$. For a dust we have $\mathcal{M}(\tau, \bar{\sigma}) = \sqrt{\mu^2 c^2 \left[ \det(\partial_r \alpha^i) \right]^2 + \phi^{-4} \sum \sum_e^2 \sum \gamma_\mu \Pi_{\mu} \partial_{\mu} \alpha^{-n} \Pi_{\mu} \partial_{\mu} \alpha^{-n}}(\tau, \bar{\sigma})$.\textsuperscript{13}
one, and therefore to Einstein equations, we have \(4\nabla_A T^{AB}(\tau, \vec{\sigma}) \equiv 0\) as a consequence of the Bianchi identities.

In Ref.[26] it is said that the ADM Poincare’ charges coincide with those arising from the Landau-Lifshitz energy-momentum pseudo-tensor for the gravitational field \(t^{\mu\nu}_{LL} = t^{\nu\mu}_{LL}\) [27] (see Ref.[28] for a detailed analysis). As a consequence the identities \(4\nabla_A T^{AB}(\tau, \vec{\sigma}) \equiv 0\) can be rewritten in the form of conservation laws \(\partial_B \left[ (-4g) \left( T^{AB} + t^{AB}_{LL} \right) \right] = 0\).

In Ref.[10] we gave the form of the internal Poincare’ generators of an isolated system in special relativity in both the inertial and non-inertial rest frames. By restricting Eqs.(2.20) to Minkowski space-time (a special solution of Einstein’s equations when \(G = 0\)) and then to inertial rest-frames (where \(4g_{AB} = 4\eta_{AB}\) and the embedding identifies the Wigner 3-spaces), it turns out that the ADM Poincare’ generators become equal to the special relativistic ones for the given matter [8] in the inertial rest-frames. Also the non-inertial rest-frame Poincare’ generators [10] could be recovered without the restriction to inertial rest-frames (now \(4g_{AB} = 4g_{AB}[z]\)).

As a consequence, this approach leads to the following visualization of the 3-universe (with its content of gravitational field and matter) contained in the instantaneous 3-spaces \(\Sigma_{r}\): it is an isolated system described by a decoupled external canonical non-covariant 4-center of mass \(\vec{x}^\mu(\tau)\) (a non-observable decoupled pseudo-particle; it replaces the \(x^\mu(\infty)\) of Dirac proposal) carrying a pole-dipole structure with mass \(Mc \approx \tilde{E}_{ADM}/c\) and a spin \(S^r = \frac{1}{2} \epsilon^{rsv} J^{sw}_{ADM} \approx \frac{1}{2} \epsilon^{rsv} \hat{J}^{sv}_{ADM}\). It is the same structure present in special relativity [8, 10] in the inertial and non-inertial rest-frame instant forms of dynamics. There, in non-inertial rest frames the embeddings tend at spatial infinity to the Wigner hyper-planes \(z^\mu_W(\tau, \vec{\sigma}) = Y^\mu(\tau) + \epsilon^\mu(\vec{h}) \sigma^r\) (the analogue of Eq.(2.19)), where \(Y^\mu(\tau) = Y^\mu(0) + h^\mu \tau\) is the covariant non-canonical Fokker-Pryce center of inertia of the isolated system. Both \(\vec{x}^\mu(\tau)\) and \(Y^\mu(\tau)\) are well defined functions (given in Ref.[8]) of \(\tau, Mc, S^r\), and of frozen (non-evolving) Jacobi data \(\vec{z}, \vec{h}\), for the decoupled external center of mass \(\vec{x}_{NW} = \vec{z}/Mc\) is the non-covariant Newton-Wigner 3-position).

\(\vec{x}^\mu(\tau)\) carries also a universal external realization of the Poincare’ algebra with generators \(P^\mu = Mch^\mu, J^{ij} = z^i h^j - z^j h^i + \epsilon^{ij\mu} S^\mu, K^i = J^0i = -\sqrt{1 + \vec{h}^2} z^i + \frac{(\vec{S} \times \vec{h})^i}{1 + \sqrt{1 + \vec{h}^2}}\). In tetrad gravity these generators are \(p^\mu(\infty), J^{\mu\nu}(\infty)\). Let us remark that the non-observability of this decoupled pseudo-particle is in accord with the viewpoint of Ref.[29].

Inside each instantaneous 3-space there is the isolated system and an internal realization of the Poincare’ algebra, whose generators correspond to the weak ADM Poincare’ charges.

If, like in special relativity [8, 10], we eliminate the internal 3-center of mass inside the instantaneous 3-spaces \(\Sigma_{r}\) with the following gauge fixings to the constraints (2.21) (together they form three pairs of second class constraints)

\[
\hat{J}^{rr}_{ADM} \approx 0, \quad (2.23)
\]

then we can choose the world-line of the non-canonical covariant Fokker-Pryce 4-center of inertia as origin of the 3-coordinates \(\sigma^r\) in the 3-spaces \(\Sigma_{r}\) (instead of \(x^\mu(\infty)\)).
Let us add a final remark. In Eqs. (12.2) and (12.3) of Ref.[1] and in Section 7 of Ref.[3] it is shown how to make a so-called Frauendiener-Sen-Witten transport (depending on the extrinsic curvature of \( \Sigma_{\tau} \)) of the asymptotic flat tetrads \( \epsilon_{\alpha}^A(h) \) to each point of \( \Sigma_{\tau} \): this allows to define a local dynamical compass of inertia \( \tilde{E}_0^\mu(\alpha) \) to be compared in each point of \( \Sigma_{\tau} \) with the tetrad \( 4E_0^\mu(\alpha) \). These special tetrads have the following form (\( \tilde{\alpha}(a) \) are suitable Euler angles)

\[
4\tilde{E}_0^\mu(\alpha) = \frac{1}{1+n} \left( z^\tau - n^\tau z^\mu \right) \to \epsilon_{i}^\mu(h),
\]

\[
4\tilde{E}_c^\mu = z^\mu 3\tilde{e}_c^r(\alpha) \to \epsilon_c^\mu(h), \quad 3\tilde{e}_c^r(\alpha) = R_{(c)(d)}(\tilde{\alpha}(e))^{3}\tilde{e}_d^r(\alpha) \to \delta_c^a,
\]

with the special triads \( 3\tilde{e}_c^r(\alpha) \) solution of the following Frauendiener equations [30] (\( \alpha \) is a proportionality constant)

\[
3\nabla_r 3\tilde{e}_c^r(1) = 3\nabla_r 3\tilde{e}_c^r(2) = 0, \quad 3\nabla_r 3\tilde{e}_c^r(3) = -\alpha 3\tilde{K} = -\alpha \frac{12\pi G}{c^3} \pi_\phi,
\]

\[
3\tilde{e}_c^r(1) 3\tilde{e}_c^s(3) 3\nabla_r 3\tilde{e}_c^r(2) + 3\tilde{e}_c^r(3) 3\tilde{e}_c^s(2) 3\nabla_r 3\tilde{e}_c^r(1) + 3\tilde{e}_c^r(2) 3\tilde{e}_c^s(1) 3\nabla_r 3\tilde{e}_c^r(3) = 0.
\]

Therefore, these triads are formed by 3 vector fields with the properties: i) two vector fields are divergence free; ii) the third one has a non-vanishing divergence proportional to the trace of the extrinsic curvature of \( \Sigma_{\tau} \) (on a maximal slicing hyper-surface, \( 3\tilde{K} = 0 \), all three vectors would be divergence free); iii) the vectors satisfy a cyclic condition. In Ref.[30] it is shown these triads do not exist for compact \( \Sigma_{\tau} \) and in general when there is nontrivial topology for \( \Sigma_{\tau} \).

In conclusion, there are preferred ADM geometrical and dynamical Eulerian observers with unit 4-velocity \( 4\tilde{E}_0^\mu(\alpha) \) and gyroscopes along the spatial axes \( 4\tilde{E}_0^\mu(\alpha) \). The asymptotic world-lines of the congruence of these observers may replace the static concept of fixed stars in the study of the precessional effects of gravito-magnetism on gyroscopes (dragging of inertial frames) and seem to be naturally connected with the definition of post-Newtonian coordinates [31]. This congruence of time-like preferred observers is a non-Machian element of these noncompact space-times.

These preferred tetrads correspond to the non-flat preferred observers of Bergmann [32]: they are a set of privileged observers (privileged tetrads adapted to the instantaneous 3-spaces \( \Sigma_{\tau} \)) of geometrical nature and not of static nature. Since they depend on the intrinsic and extrinsic geometry of \( \Sigma_{\tau} \), on the solutions of Einstein’s equations they also acquire a dynamical nature depending on the configuration of the gravitational field itself.
III. THE ADM ACTION IN PRESENCE OF THE ELECTRO-MAGNETIC FIELD AND OF CHARGED SCALAR PARTICLES WITH GRASSMANN-VALUED ELECTRIC CHARGES AND SIGN OF THE ENERGY.

Let us now describe N charged scalar particles and the electro-magnetic field coupled to the gravitational field in ADM tetrad gravity.

A. The ADM Action and the Constraints

The tetrad ADM action 14 for tetrad gravity (see Eq. (4) of Ref. [3]) plus the electromagnetic field and N charged scalar particles with Grassmann-valued electric charges $Q_i$ and sign of the energy $\eta$, depending on the configuration variables $n(\tau, \vec{\sigma}), n_a(\tau, \vec{\sigma}), \varphi_a(\tau, \vec{\sigma}), \eta, e_{(a)r}(\tau, \vec{\sigma}), A_A(\tau, \vec{\sigma}), \bar{\eta}_i(\tau), \theta_i(\tau), \theta_i^{(Q)}(\tau)$, is 15

$$S = S_{grav} + S_{em} + S_{part} + S_{Grassmann} =$$

$$= \frac{c^3}{16\pi G} \int d\tau d^3\sigma \left\{ (1 + n)^3 e_{(a)(b)(c)}^r \epsilon_{r}^e_{(a)} \epsilon_{l}^e_{(b)} 3 \Omega_{rs(c)} + \right.$$

$$+ \frac{3e}{2(1 + n)} (3G^{-1}_{\rho})_{(a)(b)(c)(d)} 3 \epsilon_{(b)}^e (n_a)^{r} - \partial_r \epsilon_{(a)r}) 3 \epsilon_{(d)}^e (n_c)^{s} - \partial_s \epsilon_{(c)s}) \right\} (\tau, \vec{\sigma}) -$$

$$- \frac{1}{4} \int d\tau d^3\sigma \ 3 \epsilon_{(a)}^e 3 \epsilon_{(b)}^e 3 \epsilon_{(c)}^e 3 \epsilon_{(d)}^e \left\{ (1 + n) \delta_{(a)(b)} \delta_{(c)(d)} - \frac{\delta_{(a)(b)} n_{(c)} n_{(d)} + \delta_{(c)(d)} n_{(a)} n_{(b)}}{1 + n} \right\} F_{ru} F_{rs} +$$

$$+ \frac{3}{1 + n} \epsilon_{(a)}^e 3 \epsilon_{(b)}^e 3 \epsilon_{(c)}^e 3 \epsilon_{(d)}^e \left[ \frac{1}{1 + n} \right] \left[ \frac{1}{1 + n} \right] \left[ \frac{1}{1 + n} \right]$$

14 Dimensions of the quantities appearing in this Section: $[\tau] = [x^\mu] = [\bar{\sigma}] = [\bar{\eta}] = [\bar{r}] = [G] = [P^\mu] = [E/c] = [m t^{-1}], [g] = [a] = [m] = [n_a] = [e_{(a)r}^e] = [\bar{\eta}_i] = [\bar{r}_i] = [0], [G = 6.7 \times 10^{-25} cm^3 s^{-2} g^{-1}] = [m^{-1} t^3 s^{-2}], [G/c^3] = [m^{-1} t] \approx 2.5 \times 10^{-39} sec/g, [S] = [H] = [J^{AB}] = [m t^2 t^{-1}], [R] = [3 \Omega_{rs(a)}] = [H^{-2}], [\omega_{r(a)}] = [K_{rs}] = [l^{-1}], [r_{(a)}] = [3 \Pi^{rs}] = [m^{-1} t^{-1}], [T^{AB}] = [M] = [M_r] = [H] = [H_{(a)}] = [m t^{-2} t^{-1}].$

15 The use of a positive 3-metric changes the sign definition of the particle momentum and the Poisson bracket sign with respect to Ref. [13]. $G$ is the Newton constant. In $S_{Grassmann}$ we use the Planck constant $\hbar$ for dimensional reasons, since our regularization of the self-energies is considered a semi-classical approximation of a quantum theory, which exists for the electro-magnetic field (QED) but not yet for the gravitational field.
\[ - \sum_{i=1}^{N} \int d\tau d^3\sigma \delta^3(\vec{\sigma}, \vec{n}_i(\tau)) \left( m_i c \right) \]

\[ \eta_i \sqrt{\left( 1 + n(\tau, \vec{\sigma}) \right)^2} - \left( 3e(a)_{r(\tau, \vec{\sigma})} \eta_i^r(\tau) + n(a)(\tau, \vec{\sigma}) \right) \left( 3e(a)_{s(\tau, \vec{\sigma})} \eta_i^s(\tau) + n(a)(\tau, \vec{\sigma}) \right) + \frac{\eta Q_i}{c} \left[ A_r(\tau, \vec{\sigma}) + A_r(\tau, \vec{\sigma}) \eta_i^r(\tau) \right] + \]

\[ + \frac{i\hbar}{2} \int d\tau \sum_i [\dot{\theta}^r_i(\tau) \dot{\theta}_i(\tau) - \dot{\theta}^s_i(\tau) \dot{\theta}_i(\tau)] + \frac{i\hbar}{2} \int d\tau \sum_i [\theta_i^{(Q)}(\tau) \dot{\theta}^{(Q)}_i(\tau) - \theta_i^{(Q)}(\tau) \dot{\theta}^{(Q)}_i(\tau)], \]

\[ \eta_i = \theta^r_i \theta_i, \quad Q_i = \theta^{(Q)}_i \theta_i^{(Q)}. \]  

While \( Q_i \) is the Grassmann-valued electric charge of particle "\( i \)", \( \eta_i \) is its Grassmann-valued sign of the energy (particles with negative energy have the opposite electric charge: \( \eta_i Q_i \)).

In the action \( S_{grav} \) for the gravitational field \( 3\Omega_{rs(a)} = \partial_r 3\omega_{s(a)} - \partial_s 3\omega_{r(a)} - \epsilon_{a(b)(c)} 3\omega_{r(b)} 3\omega_{s(c)} \) is the field strength associated with the 3-spin connection \( 3\omega_{r(a)}(\tau, \vec{\sigma}) = \frac{1}{2} \epsilon_{a(b)(c)} \left[ \left( \partial_r 3\epsilon_{c}^{(u)} - \partial_u 3\epsilon_{c}^{(r)} \right) + \frac{1}{2} 3\epsilon_{b}^{(u)} 3\epsilon_{c}^{(v)} 3\epsilon_{d}^{(r)} \left( \partial_v 3\epsilon_{d}^{(u)} - \partial_u 3\epsilon_{d}^{(v)} \right) \right] \) and \( (3G^{-1}_o)_{(a)(b)(c)(d)} = \delta_{a(c)} \delta_{b(d)} + \delta_{a(d)} \delta_{b(c)} - 2\delta_{a(b)} \delta_{c(d)} \) is the flat (with lower indices) inverse of the flat Wheeler-DeWitt super-metric \( 3G_{o(a)(b)(c)} = \delta_{a(c)} \delta_{b(d)} + \delta_{a(d)} \delta_{b(c)} - \delta_{a(b)} \delta_{c(d)} \).

The action \( S_{em} = -\frac{1}{4} \int d\tau d^3\sigma \left[ \sqrt{-g^4 g^{AC}_{4BD} F_{AB} F_{CD}} \right](\tau, \vec{\sigma}) \) has the previous form because \( 4g^{AC}_{4BD} F_{AB} F_{CD} = 2(4g^{rs} g^{rs} - 4g^{rr} g^{rs}) F_{rr} F_{rs} + 4g^{rs} g^{su} F_{ru} F_{sr} + 4g^{rs} g^{uu} F_{ru} F_{sr} \).

The canonical momenta for the tetrad gravity variables and the electro-magnetic field are (we are in the canonical basis (2.7))

\[ \pi_{\varphi(a)}(\tau, \vec{\sigma}) = \frac{\delta S_{ADM}}{\delta \partial_\tau \varphi(a)(\tau, \vec{\sigma})} = 0, \]
\[ \pi_{n}(\tau, \vec{\sigma}) = \frac{\delta S_{ADM}}{\delta \partial_\tau n(\tau, \vec{\sigma})} = 0, \]
\[ \pi_{n(a)}(\tau, \vec{\sigma}) = \frac{\delta S_{ADM}}{\delta \partial_\tau n(a)(\tau, \vec{\sigma})} = 0, \]

\[ 3\pi^r_{(a)}(\tau, \vec{\sigma}) = \frac{\delta S_{ADM}}{\delta \partial_\tau 3\epsilon_{a(r)(\tau, \vec{\sigma})}} = \]
\[ = -\frac{c^3}{16\pi G} \left[ \frac{3e}{1 + n} (3G^{-1}_o)_{(a)(b)(c)(d)} 3\epsilon_{r(b)} 3\epsilon_{s(d)} (n_{(c)} - \partial_r 3\epsilon_{c(s)}) \right](\tau, \vec{\sigma}) = \]
\[ = -\frac{c^3}{8\pi G} [3\epsilon(3K^{rs} - 3\epsilon_{(r)} 3\epsilon_{(s)} K^3\epsilon_{(a)(s)})(\tau, \vec{\sigma}), \]

25
\[ \pi^\tau(\tau, \vec{\sigma}) = \frac{\delta S_{ADM}}{\delta \partial^\tau A_r(\tau, \vec{\sigma})} = 0, \]

\[ \pi^r(\tau, \vec{\sigma}) = \frac{\delta S_{ADM}}{\delta \partial^r A_r(\tau, \vec{\sigma})} = \left( \frac{\sqrt{\gamma}}{1 + n} \right)^3 e_{(a)}^r 3 e_{(a)}^s \left( F_{rs} - n^u F_{us} \right)(\tau, \vec{\sigma}), \]

\[ \{A_A(\tau, \vec{\sigma}), \pi^B(\tau, \vec{\sigma}')\} = c \eta_A^B \delta^3(\vec{\sigma}, \vec{\sigma}'), \]

\[ \{n(\tau, \vec{\sigma}), \pi_n(\tau, \vec{\sigma}')\} = \delta^3(\vec{\sigma}, \vec{\sigma}'), \]

\[ \{n_{(a)}(\tau, \vec{\sigma}), \pi_{n_{(b)}}(\tau, \vec{\sigma}')\} = \delta_{(a)(b)} \delta^3(\vec{\sigma}, \vec{\sigma}'), \]

\[ \{\nu(\tau, \vec{\sigma}), \pi_{\nu}(\tau, \vec{\sigma}')\} = \delta_{(a)(b)} \delta^3(\vec{\sigma}, \vec{\sigma}'), \]

\[ \{3 e_{(a)r}(\tau, \vec{\sigma}), 3 \pi_{(a)}^s(\tau, \vec{\sigma}')\} = 3 e_{(a)}^r(\tau, \vec{\sigma}) 3 e_{(a)}^s(\tau, \vec{\sigma}) \delta^3(\vec{\sigma}, \vec{\sigma}'), \]

while the particle momenta are

\[ \tilde{\kappa}_{ir}(\tau) = \frac{\partial L_{ADM}(\tau)}{\partial \tilde{\eta}_{i}^r(\tau)} \overset{def}{=} \eta_i \kappa_{iv}(\tau) = \frac{\eta_i Q_i}{c} A_r(\tau, \bar{\eta}_i(\tau)) + \]

\[ + \frac{\eta_i m_i c^3 e_{(a)r}(\tau, \bar{\eta}_i(\tau)) (3 e_{(a)}^r(\tau, \vec{\sigma}) \tilde{\eta}_{i}^s(\tau) + n_{(a)}(\tau, \vec{\sigma}))}{\sqrt{(1 + n(\tau, \vec{\sigma}))^2 - (3 e_{(a)r}(\tau, \vec{\sigma}) \tilde{\eta}_{i}^s(\tau) + n_{(a)}(\tau, \vec{\sigma}))}} \bigg|_{\vec{\sigma} = \bar{\eta}_i(\tau)}, \]

\[ \kappa_{ir}(\tau) = \int d\psi_i d\psi_i^* \tilde{\kappa}_{ir}(\tau), \quad \{\tilde{\eta}_{i}^r(\tau), \kappa_{is}(\tau)\} = \delta_{ij} \delta^r_s, \]

\[ \sqrt{m_i^2 c^2 + 3 e_{(a)}^r(\tau, \vec{\sigma}) \left( \kappa_{ir}(\tau) - \frac{Q_i}{c} A_r(\tau, \vec{\sigma}) \right) 3 e_{(a)}^s(\tau, \vec{\sigma}) \left( \kappa_{is}(\tau) - \frac{Q_i}{c} A_s(\tau, \vec{\sigma}) \right)} \bigg|_{\vec{\sigma} = \bar{\eta}_i(\tau)} = \]

\[ \frac{m_i [1 + n(\tau, \vec{\sigma})]}{\sqrt{(1 + n(\tau, \vec{\sigma}))^2 - (3 e_{(a)r}(\tau, \vec{\sigma}) \tilde{\eta}_{i}^s(\tau) + n_{(a)}(\tau, \vec{\sigma}))}} \bigg|_{\vec{\sigma} = \bar{\eta}_i(\tau)}, \]

\[ \tilde{\eta}_{i}^r(\tau) = 3 e_{(a)}^r(\tau, \bar{\eta}_i(\tau)) \left[ \frac{(1 + n)^3 e_{(a)}^r \left( \kappa_{iv}(\tau) - \frac{Q_i}{c} A_v \right)}{\sqrt{m_i^2 c^2 + 3 e_{(b)}^u 3 e_{(b)}^v \left( \kappa_{iu}(\tau) - \frac{Q_i}{c} A_u \right) \left( \kappa_{iv}(\tau) - \frac{Q_i}{c} A_v \right)}} - n_{(a)}(\tau, \bar{\eta}_i(\tau)) \right]. \]

(3.3)
To define $\kappa_{ir}$ from $\tilde{\kappa}_{ir}$ we have used the following property of Grassmann variables\textsuperscript{16}

$$
\int d\theta_i d\theta_i^* = 0, \quad \int d\theta_i d\theta_i^* \theta_i = 1.
$$

(3.4)

The momenta of the Grassmann variables, implying second-class constraints, are

$$
\pi_{\theta_i^{(Q)}}(\tau) = \frac{\partial L}{\partial \dot{\theta}_i^{(Q)}} = -i \frac{\hbar}{2} \theta_i^{(Q)*}(\tau), \quad \pi_{\theta_i^{(Q)*}}(\tau) = \frac{\partial L}{\partial \dot{\theta}_i^{(Q)*}} = -i \frac{\hbar}{2} \theta_i^{(Q)}(\tau),
$$

$$
\{\theta_i^{(Q)}(\tau), \pi_{\theta_j^{(Q)}}(\tau)\} = \{\theta_i^{(Q)}(\tau), \pi_{\theta_j^{(Q)*}}(\tau)\} = -\delta_{ij},
$$

$$
\pi_{\theta_i}(\tau) = \frac{\partial L}{\partial \dot{\theta}_i} = -i \frac{\hbar}{2} \theta_i^*(\tau), \quad \pi_{\theta_i^*}(\tau) = \frac{\partial L}{\partial \dot{\theta}_i^*} = -i \frac{\hbar}{2} \theta_i(\tau),
$$

$$
\{\theta_i(\tau), \pi_{\theta_j}(\tau)\} = \{\theta_i^*(\tau), \pi_{\theta_j^*}(\tau)\} = -\delta_{ij}.
$$

(3.5)

By going to Dirac brackets for the second-class constraints the Grassmann momenta are eliminated and we get (to simplify the notation we denote $\{..\}$ with $\{.,.\}$)

$$
\{\theta_i^{(Q)}(\tau), \theta_j^{(Q)*}(\tau)\} = -i \delta_{ij}, \quad \{\theta_i^{(Q)}(\tau), \theta_j^{(Q)}(\tau)\} = \{\theta_i^{(Q)*}(\tau), \theta_j^{(Q)*}(\tau)\} = 0,
$$

$$
\{\theta_i(\tau), \theta_j(\tau)\} = \{\theta_i^*(\tau), \theta_j^*(\tau)\} = 0.
$$

(3.6)

The primary constraints are

$$
\pi_{\varphi(a)}(\tau, \vec{\sigma}) \approx 0, \quad \pi_{\eta}(\tau, \vec{\sigma}) \approx 0, \quad \pi_{\eta(a)}(\tau, \vec{\sigma}) \approx 0,
$$

$$
3 M(a)(\tau, \vec{\sigma}) = \epsilon_{(a)(b)(c)} 3 e_{(b)r}(\tau, \vec{\sigma}) 3 \pi^r(c)(\tau, \vec{\sigma}) \approx 0,
$$

$$
\pi^r(\tau, \vec{\sigma}) \approx 0.
$$

(3.7)

By evaluating the canonical Hamiltonian by Legendre transformation (see Ref.[3]) and by asking that the primary constraints are constants of the motion under the $\tau$-evolution generated by it, we get the following secondary constraints

\textsuperscript{16}Following Ref.[33], the classical regulated theory can be obtained by taking the mean value of the solutions of the final Hamilton equations with the Berezin-Marinov distribution function. For the U(1) group behind the charge $n_i = \theta_i^* \theta_i$ (topological two-levels for the sign of energy) the positive definite distribution function is $\rho_i = a_i + \theta_i^* \theta_i$ with $a_i > 0$. Therefore the classical regulated value of a quantity $A$ is $< A > = \int A \rho_1 d\theta_1 d\theta_1^* \rho_2 d\theta_2 d\theta_2^*$ where Eqs.(3.4) have to be used.
\[ \Gamma(\tau, \vec{\sigma}) = \partial_\tau \pi^r(\tau, \vec{\sigma}) + \sum_i Q_i \eta_i \delta^3(\vec{\sigma}, \vec{\eta}_i(\tau)) \approx 0, \]

\[ \mathcal{H}(\tau, \vec{\sigma}) = \left[ \frac{c^3}{16\pi G} 3^e \varepsilon_{(a)(b)(c)} 3^e r_{(a)} 3^e s_{(b)} 3^e \Omega_{rs(c)} - \frac{2\pi G}{c^3 3^e} 3^e G_{o(a)(b)(c)(d)} 3^e r_{(a)r} 3^e r_{(b)s} 3^e \Omega_{rs(d)} \right](\tau, \vec{\sigma}) + \mathcal{M}(\tau, \vec{\sigma}) \approx 0, \]

\[ \mathcal{H}_{(a)}(\tau, \vec{\sigma}) = \left[ \partial_\tau 3^e r_{(a)} - \varepsilon_{(a)(b)(c)} 3^e r_{(b)} 3^e s_{(c)} + 3^e r_{(a)} \mathcal{M}_r \right](\tau, \vec{\sigma}) = -3^e r_{(a)}(\tau, \vec{\sigma})[3^e \Theta_r - 3^e r_{(a)}(\tau, \vec{\sigma})] \approx 0, \]

\[ 3^e \Theta_r(\tau, \vec{\sigma}) = \left[ 3^e s_{(a)} \partial_\tau 3^e r_{(a)s} - \partial_\tau \left( 3^e r_{(a)} \right) \right](\tau, \vec{\sigma}) - \mathcal{M}_r(\tau, \vec{\sigma}) \approx 0. \] (3.8)

In Eqs.(3.8) the following notation has been introduced for the matter terms (the mass density \( \mathcal{M}(\tau, \vec{\sigma}) \) and the mass current density \( \mathcal{M}_r(\tau, \vec{\sigma}) \)) to conform with the treatment of the same matter in the non-inertial rest frames of Minkowski space-time given in Ref.[10]

\[ \mathcal{M}(\tau, \vec{\sigma}) = \sum_{i=1}^{N} \delta^3(\vec{\sigma}, \vec{\eta}_i(\tau)) M_i(\tau, \vec{\sigma}) c + 3^e \mathcal{T}^{(em)}_{1\perp}(\tau, \vec{\sigma}), \]

\[ M_i(\tau, \vec{\sigma}) c = \eta_i \sqrt{m_i^2 c^2 + 3^e c_{(a)} \left( \kappa_{ir}(\tau) \right) \left( \kappa_{is}(\tau) - \frac{Q_i}{c} A_r \right) 3^e c_{(a)s}} \left( \kappa_{ir}(\tau) - \frac{Q_i}{c} A_r(\tau, \vec{\sigma}) \right) \delta^3(\vec{\sigma}, \vec{\eta}_i(\tau)) - 3^e \mathcal{T}^{(em)}_{1r}(\tau, \vec{\sigma}), \]

\[ \mathcal{T}^{(em)}_{1\perp}(\tau, \vec{\sigma}) = \frac{1}{2 c^3 3^e} \left( \frac{1}{3^e c_{(a)} 3^e r_{(a)s} \pi^r \pi^s + \frac{3^e}{2} 3^e c_{(a)} 3^e s_{(b)} 3^e u_{(b)} 3^e v_{(b)} F_{ru} F_{sv} \right)(\tau, \vec{\sigma}), \]

\[ \mathcal{T}^{(em)}_{1r}(\tau, \vec{\sigma}) = \frac{1}{c^3 3^e} F_{rs}(\tau, \vec{\sigma}) \pi^s(\tau, \vec{\sigma}). \] (3.9)

Let us remark that the mass current density \( \mathcal{M}_r(\tau, \vec{\sigma}) \) does not depend upon the 4-metric.

In Eqs.(3.8) \( \Gamma(\tau, \vec{\sigma}) \approx 0 \) is the electro-magnetic Gauss law, \( \mathcal{H}(\tau, \vec{\sigma}) \approx 0 \) and \( \mathcal{H}_{(a)}(\tau, \vec{\sigma}) \approx 0 \) are the super-Hamiltonian and super-momentum constraints respectively, while \( 3^e \Theta_r(\tau, \vec{\sigma}) \approx 0 \) are the generators of 3-diffeomorphisms on \( \Sigma_r \). In the constraint \( \mathcal{H}(\tau, \vec{\sigma}) \approx 0 \) we have \( 3^e c_{(a)(b)(c)} 3^e r_{(a)} 3^e s_{(b)} \Omega_{rs(c)} = 3^e 3^e R \) with \( 3^e R \) the scalar 3-curvature of the instantaneous 3-space \( \Sigma_r \) and \( \Omega_{rs(a)} \) the 3-field strength determined by the 3-spin connection \( 3^e \omega_{r(a)} \) given in Eq.(2.10).

One can check that the constraints are all first class with the algebra (see Ref.[34] for the terms containing the Gauss law constraint)
\[ \{^3M_{(a)}(\tau, \bar{\sigma}), ^3M_{(b)}(\tau, \bar{\sigma'})\} = \varepsilon_{(a)(b)(c)} ^3M_{(c)}(\tau, \bar{\sigma}) \delta^3(\bar{\sigma}, \bar{\sigma'}), \]
\[ \{^3M_{(a)}(\tau, \bar{\sigma}), ^3\Theta_a(\tau, \bar{\sigma'})\} = ^3M_{(a)}(\tau, \bar{\sigma'}) \frac{\partial \delta^3(\bar{\sigma}, \bar{\sigma'})}{\partial \sigma'}. \]

\[ \{^3\Theta_a(\tau, \bar{\sigma}), ^3\Theta_s(\tau, \bar{\sigma'})\} = \left[ ^3\Theta_a(\tau, \bar{\sigma'}) \frac{\partial}{\partial \sigma'} + ^3\Theta_s(\tau, \bar{\sigma}) \frac{\partial}{\partial \sigma'} \right] \frac{\partial \delta^3(\bar{\sigma}, \bar{\sigma'})}{\partial \sigma'} - \delta_{ru} \delta_{sv} \left[ ^3e_{(a)}^v \delta^3(\bar{\sigma}, \bar{\sigma'}) \frac{\partial}{\partial \sigma'} \right] (\tau, \bar{\sigma}) \Gamma(\tau, \bar{\sigma}) \frac{\partial \delta^3(\bar{\sigma}, \bar{\sigma'})}{\partial \sigma'}, \]

\[ \{^3M_{(a)}(\tau, \bar{\sigma}), ^3\Theta_s(\tau, \bar{\sigma'})\} = \left[ ^3\Theta_a(\tau, \bar{\sigma}) ^3M_s(\tau, \bar{\sigma}) + ^3\Theta_s(\tau, \bar{\sigma'}) ^3M_{(b)}(\tau, \bar{\sigma}) \right] \frac{\partial \delta^3(\bar{\sigma}, \bar{\sigma'})}{\partial \sigma'} = \left[ ^3\Theta_a(\tau, \bar{\sigma}) ^3M_s(\tau, \bar{\sigma}) + ^3\Theta_s(\tau, \bar{\sigma'}) ^3M_{(b)}(\tau, \bar{\sigma}) \right] \frac{\partial \delta^3(\bar{\sigma}, \bar{\sigma'})}{\partial \sigma'}. \]

The energy-momentum tensor of the matter, \( T^{AB}(\tau, \bar{\sigma}) = - \left[ \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{part+em}}}{\delta g_{AB}} \right] (\tau, \bar{\sigma}) \), is (we put \( M_i c = \eta_i \bar{M}_i c \))

\[ T^{rr}(\tau, \bar{\sigma}) = \frac{\mathcal{M}(\tau, \bar{\sigma})}{[^3e (1 + n)^2](\tau, \bar{\sigma})}, \]
\[ T^{rr}(\tau, \bar{\sigma}) = \frac{3^e_{(a)} [^3e_{(a)} M_s - n_{(a)} \mathcal{M}]}{[^3e (1 + n)^2]} (\tau, \bar{\sigma}), \]
\[ T^{rs}(\tau, \bar{\sigma}) = \frac{1}{[^3e (1 + n)^2](\tau, \bar{\sigma})} \sum_{i=1}^{N} \delta^3(\bar{\sigma}, \bar{\eta}_i(\tau)) \]
\[ \left[ \frac{\eta_i ^3e_{(a)}^3 e_{(b)}^r}{\bar{M}_i} \right] \left[ (1 + n)^3 e_{(a)}^m \left( \kappa_{im}(\tau) - \frac{Q_i}{c} A_m(\tau, \bar{\sigma}) \right) - n_{(a)} \bar{M}_i \right] \]
\[ \left( (1 + n)^3 e_{(b)}^n \left( \kappa_{in}(\tau) - \frac{Q_i}{c} A_n(\tau, \bar{\sigma}) \right) - n_{(b)} \bar{M}_i \right) (\tau, \bar{\eta}_i(\tau)) + T^{rs}_{(em)}(\tau, \bar{\sigma}), \]
\[ T^{rs}_{(em)}(\tau, \bar{\sigma}) = \frac{1}{c} \left[ \frac{1}{3c^2} \left( -\pi^r \pi^s + \frac{1}{2} \sum_{abcduv} 3e^r_{(a)} 3e^s_{(b)} \left( (\delta_{ab} + \frac{n_{(a)} n_{(b)}}{(1+n)^2}) \right) \delta_{cd} + \right. \right. \\
+ 4 \frac{n_{(a)} n_{(b)} n_{(c)} n_{(d)}}{(1+n)^4} \left] 3e^r_{(a)u} 3e^s_{(d)v} \pi^u \pi^v \right) + \right. \right. \\
+ \sum_{ablm} \frac{3e^r_{(a)} 3e^s_{(b)} n_{(a)} 3e^l_{(b)} + n_{(b)} 3e^l_{(a)}}{1+n} F_{lm} \pi^m + \right. \right. \\
+ \sum_{abdlmuv} 3e^r_{(a)} 3e^s_{(b)} \left[ 3e^l_{(a)} 3e^m_{(b)} - \frac{1}{4} (\delta_{ab} - \frac{n_{(a)} n_{(b)}}{(1+n)^2}) \sum_\epsilon 3e^l_{(e)} 3e^m_{(e)} \right] \delta_{cd} - \right. \right. \\
- 3e^m_3 e^l_3 \frac{n_{(c)} n_{(d)}}{(1+n)^2} \left] 3e^u_{(c)} 3e^v_{(d)} F_{ul} F_{vm} \right] (\tau, \sigma). \] (3.11)

With these conventions we have \( M = 3e T_{r r} = 3e (1+n)^2 T^r r \) and \( M_r = -3e T_{r r} = 3e (1+n) \sum_a 3e_{(a)r} \left[ \sum_s 3e_{(a)s} T^{rs} + n_{(a)} T^{rr} \right] \).

1. The Dirac Hamiltonian and the Weak ADM Energy

In the rest-frame instant form of tetrad gravity \([3]\), after making the Legendre transformation and after the addition of the DeWitt surface, the Dirac Hamiltonian has the following expression in terms of the weak ADM energy\(^{17}\) in the canonical basis (2.7)

\[ H_D = \frac{1}{c} \hat{E}_{ADM} + \int d^3\sigma \left[ n \mathcal{H} - n_{(a)} \mathcal{H}_{(a)} - \frac{1}{c} A_r \Gamma \right] (\tau, \bar{\sigma}) + \lambda_r(\tau) \hat{P}^r_{ADM} + \right. \right. \\
+ \int d^3\sigma \left[ \lambda_n \pi_n + \lambda_{n_{(a)}} \pi_{n_{(a)}} + \lambda_{\varphi_{(a)}} \pi_{\varphi_{(a)}} + \mu_{(a)} 3M_{(a)} + \mu \pi^r \right] (\tau, \bar{\sigma}), \] (3.12)

where \( \lambda_n(\tau, \bar{\sigma}), \lambda_{n_{(a)}}(\tau, \bar{\sigma}), \lambda_{\varphi_{(a)}}(\tau, \bar{\sigma}), \mu_{(a)}(\tau, \bar{\sigma}), \lambda_r(\tau) \), are the Dirac multipliers in front of the primary constraints and the mass density \( M \) is defined in Eqs.(3.9).

Since the scalar 3-curvature \( 3e^3 R = 3e e_{(a)(b)(c)} 3e_{(a)}^r 3e_{(b)}^s \Omega_{rs(c)} \) of \( \Sigma_r \) can be decomposed in the following way \([3]\)

\[ 3e(\tau, \bar{\sigma})^3 R(\tau, \bar{\sigma}) \overset{\text{def}}{=} S(\tau, \bar{\sigma}) + T(\tau, \bar{\sigma}), \]

\[ S(\tau, \bar{\sigma}) \overset{\text{def}}{=} \left[ 3e \sum_{r \mu \nu} 3e_{(a)}^r 3e_{(a)}^s \left( 3\Gamma_{\mu r}^u \Gamma_{\nu u}^v - 3\Gamma_{\nu r}^u \Gamma_{\mu u}^v \right) \right] (\tau, \bar{\sigma}), \]

\[ T(\tau, \bar{\sigma}) \overset{\text{def}}{=} \sum_r \partial_r \left[ 3e \sum_{\mu \nu} 3g^{\mu r} 3g^{\nu w} (\partial_\mu 3g_{vt} - \partial_t 3g_{vu}) \right], \] (3.13)

\(^{17}\) Since we have \( \hat{E}_{ADM} = c \hat{P}^r_{ADM}, [H_D] = [E/c] = [m l t^{-1}] \) and \( [r] = [l] \), Hamilton’s equations are written as \( \partial_r F \overset{\text{def}}{=} \{ F, H_D \} \). To agree with the standard ADM equations we need a minus sign in front of the shift function.
it follows that the weak ADM energy of Eqs.(2.22) and the super-Hamiltonian constraint of Eqs.(3.8) can be written in the following form

\begin{align*}
\hat{E}_{ADM} &= c \int d^3 \sigma \left[ M - \frac{c^3}{16 \pi G} S + \frac{2 \pi G}{c^3 \sqrt{3} c} \sum_{abcd} 3 G \delta_{(a)}(b) G_{(c)(d)} \frac{3 \pi_r}{(b)} \frac{3 \pi_s}{(c)} \frac{3 \pi_s}{(d)} \right] (\tau, \vec{\sigma}), \\
\mathcal{H}(\tau, \vec{\sigma}) &= \left[ M - \frac{c^3}{16 \pi G} (S + T) + \frac{2 \pi G}{c^3 \sqrt{3} c} \sum_{abcd} 3 G \delta_{(a)}(b) G_{(c)(d)} \frac{3 \pi_r}{(b)} \frac{3 \pi_s}{(c)} \frac{3 \pi_s}{(d)} \right] (\tau, \vec{\sigma}) \approx 0,
\end{align*}

(3.14)

B. The Hamilton Equations for the Particles

Due to the presence of the Grassmann-valued signs of the energy, which are constants of motion like the Grassmann-valued electric charges ($\{\eta_i(\tau), H_D\} \equiv 0$, $\{Q_i(\tau), H_D\} \equiv 0$), the particle Hamilton equations are defined in the following way

\begin{align*}
\frac{1}{c} \hat{E}_{ADM} + \int d^3 \sigma \left( n \partial \right) (\tau, \vec{\sigma}) &= \\
= \int d^3 \sigma \left[ (1 + n) M \right] (\tau, \vec{\sigma}) - \frac{c^3}{16 \pi G} \int d^3 \sigma \left[ (1 + n) S + n T \right] (\tau, \vec{\sigma}) + \\
+ \frac{2 \pi G}{c^3} \int d^3 \sigma \left[ \frac{1}{3} \sqrt{3} c \sum_{abcd} 3 G \delta_{(a)}(b) G_{(c)(d)} \frac{3 \pi_r}{(b)} \frac{3 \pi_s}{(c)} \frac{3 \pi_s}{(d)} \right] (\tau, \vec{\sigma}).
\end{align*}

(3.15)

The derived particle world-lines are $x^\mu_i(\tau) = z^\mu(\tau, \bar{\eta}_i(\tau))$. Looking at the expression of the momenta $\kappa_{ir}(\tau)$, $i=1,...,N$, we can define the following derived 4-momenta, corresponding to the ones of the standard manifestly covariant approach [11, 13], which satisfy the mass shell constraints for each particle also in the curved space-time $M^4$ in presence of interactions (however, as shown in Ref.[8], other definitions are possible for these derived quantities)
\[
p_i^{(\mu)} = \left(\begin{array}{c}
p_i^{(o)} = \eta_i \sqrt{m_i^2 c^2 + 3 e_{(a)}^s (\tau, \vec{\eta}_i(\tau)) \kappa_{ir}(\tau) 3 e_{(a)}^s (\tau, \vec{\eta}_i(\tau)) \kappa_{is}(\tau)}; \\
p_i^{(a)} = 3 e_{(a)}^s (\tau, \vec{\eta}_i(\tau)) \kappa_{is}(\tau)
\end{array}\right),
\]

\[
p_i^\mu = p_i^{(\mu)} (\eta_i) E_\mu;
\]

\[
p_i^A = p_i^{(\mu)} (\eta_i) E_\mu = 
\]

\[
\left(\begin{array}{c}
p_i^\tau = \frac{p_i^{(o)}}{1 + n(\tau, \vec{\eta}_i(\tau))}; \\
p_i^\eta = \left(-\left(\frac{n(a)^3 e_{(a)}^s p_i^{(a)}}{1 + n(\tau, \vec{\eta}_i(\tau))}\right) (\tau, \vec{\eta}_i(\tau)) + 3 e_{(a)}^s (\tau, \vec{\eta}_i(\tau)) p_i^{(a)}\right),
\end{array}\right)
\]

\[
p_i^\mu g_{\mu\nu} p_i^\nu = p_i^{(\mu)} (\eta_i) p_i^{(\nu)} = p_i^{(2)} g_{AB} p_i^B = m_i^2 c^2.
\]

(3.16)

The Dirac Hamiltonian (3.12) generates the following Hamilton equations for the particles (see Eq.(3.3) for the first one)

\[
\eta_i \dot{\eta}_i(\tau) = \eta_i^3 e_{(a)}^s (\tau, \vec{\eta}_i(\tau)) \left[\frac{(1 + n)^3 e_{(a)}^s}{\sqrt{m_i^2 c^2 + 3 e_{(a)}^u 3 e_{(a)}^v (\kappa_{iu}(\tau) - \frac{Q_i}{c} A_u) (\kappa_{iv}(\tau) - \frac{Q_i}{c} A_v)}}\right] - n(a) (\tau, \vec{\eta}_i(\tau));
\]

\[
\eta_i \frac{d}{d\tau} \kappa_{ir}(\tau) = \eta_i \frac{Q_i}{c} \left[\eta_i^u (\tau) \frac{\partial A_u(\tau, \vec{\eta}_i(\tau))}{\partial \eta_i^r} + \frac{\partial A_r(\tau, \vec{\eta}_i(\tau))}{\partial \eta_i^r}\right] + \eta_i F_{ir}(\tau),
\]

\[
F_{ir}(\tau) = \left(\frac{m_i c}{\sqrt{\left(1 + n\right)^2 - 3 e_{(a)}^s 3 e_{(a)}^v \left(\dot{\eta}_i^s(\tau) + 3 e_{(a)}^u n(a) b(\tau, \vec{\eta}_i(\tau))\right) (\dot{\eta}_i^s(\tau) + 3 e_{(a)}^v n(a) b(\tau, \vec{\eta}_i(\tau)) ) + \left(\frac{1}{2} \frac{\partial \left[3 e_{(a)}^s 3 e_{(a)}^t b(\tau, \vec{\eta}_i(\tau))\right]}{\partial \eta_i^t} \left(\dot{\eta}_i^s(\tau) + \left[3 e_{(a)}^u n(a) b(\tau, \vec{\eta}_i(\tau))\right] (\dot{\eta}_i^t(\tau) + \left[3 e_{(a)}^t n(a) b(\tau, \vec{\eta}_i(\tau))\right)\right) - (1 + n) \frac{\partial n(\tau, \vec{\eta}_i(\tau))}{\partial \eta_i^r} + \frac{\partial \left[3 e_{(a)}^s n(a) b(\tau, \vec{\eta}_i(\tau))\right]}{\partial \eta_i^t} \left[3 e_{(a)}^s 3 e_{(a)}^t b(\tau, \vec{\eta}_i(\tau))\right) (\dot{\eta}_i^s(\tau) + \left[3 e_{(a)}^t n(a) b(\tau, \vec{\eta}_i(\tau))\right)\right],
\]

(3.17)

where \( F_{ir}(\tau) \) denotes a set of relativistic forces, which in non-inertial frames of Minkowski space-time would be only inertial effects [10].

As a consequence, the second order form of the particle equations of motion implied by Eqs. (3.17) is
If, as in Ref. [10], we define
\[ \eta \frac{d}{d\tau} \left( \frac{3e_{(a)}^s 3e_{(a)}^r m_i c (\hat{\eta}_i^s(\tau) + 3e_{(b)}^s n_i(\tau))}{\sqrt{(1 + n)^2 - 3e_{(c)}^a 3e_{(c)}^r \left( \hat{\eta}_i^a(\tau) + 3e_{(d)}^u n_i(\tau) \right) \left( \hat{\eta}_i^c(\tau) + 3e_{(e)}^v n_i(\tau) \right)}} \right) (\tau, \hat{\eta}_i(\tau)) = \]

\[ = \eta Q_i \frac{c}{c} \left[ 3e_{(a)}^s 3e_{(a)}^r (\tau, \hat{\eta}_i(\tau)) + \left( \frac{\partial A_u(\tau, \hat{\eta}_i(\tau))}{\partial \eta_i^a} - \frac{\partial A_r(\tau, \hat{\eta}_i(\tau))}{\partial \eta_i^a} \right) + \left( \frac{\partial A_r(\tau, \hat{\eta}_i(\tau))}{\partial \tau} - \frac{\partial A_r(\tau, \hat{\eta}_i(\tau))}{\partial \eta_i^a} \right) \right] + \eta F_{ir}(\tau), \]

or

\[ \eta m_i c \frac{d}{d\tau} \left( \frac{\hat{\eta}_i^s(\tau) + 3e_{(a)}^s n_i(\tau)}{\sqrt{(1 + n)^2 - 3e_{(b)}^a 3e_{(b)}^r \left( \hat{\eta}_i^a(\tau) + 3e_{(c)}^u n_i(\tau) \right) \left( \hat{\eta}_i^c(\tau) + 3e_{(d)}^v n_i(\tau) \right)}} \right) (\tau, \hat{\eta}_i(\tau)) = \]

\[ = \eta Q_i \frac{c}{c} \left[ 3e_{(a)}^s 3e_{(a)}^r (\tau, \hat{\eta}_i(\tau)) + \left( \frac{\partial A_u(\tau, \hat{\eta}_i(\tau))}{\partial \eta_i^a} - \frac{\partial A_r(\tau, \hat{\eta}_i(\tau))}{\partial \eta_i^a} \right) + \left( \frac{\partial A_r(\tau, \hat{\eta}_i(\tau))}{\partial \tau} - \frac{\partial A_r(\tau, \hat{\eta}_i(\tau))}{\partial \eta_i^a} \right) \right] + \eta F_i^*(\tau), \]

\[ \dot{F}_i^*(\tau) = \]

\[ = \left( \frac{m_i c 3e_{(a)}^s 3e_{(a)}^r}{\sqrt{(1 + n)^2 - 3e_{(b)}^a 3e_{(b)}^r \left( \hat{\eta}_i^a(\tau) + 3e_{(c)}^u n_i(\tau) \right) \left( \hat{\eta}_i^c(\tau) + 3e_{(d)}^v n_i(\tau) \right)}} \right) (\tau, \hat{\eta}_i(\tau)) + \]

\[ \left[ \frac{1}{2} \left( \frac{\partial^3 e_{(a)}^s 3e_{(a)}^r}{\partial \eta_i^a} \right) (\tau, \hat{\eta}_i(\tau)) + \left( \frac{\partial^3 e_{(f)}^s 3e_{(f)}^r}{\partial \eta_i^a} \right) (\tau, \hat{\eta}_i(\tau)) \right] \right] \]

\[ - \left( 1 + n \right) \frac{\partial n(\tau, \hat{\eta}_i(\tau))}{\partial \eta_i^a} + \]

\[ + \left( \frac{\partial^3 e_{(a)}^s 3e_{(a)}^r}{\partial \eta_i^a} \right) \left( \frac{\partial^3 e_{(f)}^s 3e_{(f)}^r}{\partial \eta_i^a} \right) (\tau, \hat{\eta}_i(\tau)) \left( \hat{\eta}_i^a(\tau) + \left[ 3e_{(g)}^s n_i(\tau) \right](\tau, \hat{\eta}_i(\tau)) \right) \right] - \]

\[ - \left( \frac{\partial^3 e_{(a)}^s 3e_{(a)}^r}{\partial \tau} + \hat{\eta}_i^a(\tau) \right) \left( \frac{\partial^3 e_{(f)}^s 3e_{(f)}^r}{\partial \eta_i^a} \right) (\tau, \hat{\eta}_i(\tau)) \left( \hat{\eta}_i^a(\tau) + \left[ 3e_{(g)}^s n_i(\tau) \right](\tau, \hat{\eta}_i(\tau)) \right) \right]. \]  

(3.18)

Here \( \dot{F}_{ir}(\tau) \) is the new form of the relativistic forces.

If, as in Ref. [10], we define the non-inertial electric and magnetic fields in the form
\[ E_r \overset{\text{def}}{=} \left( \frac{\partial A_r}{\partial \eta^i} - \frac{\partial A_i}{\partial \tau} \right) = -F_{\tau r}, \]

\[ B_r \overset{\text{def}}{=} \frac{1}{2} \varepsilon_{rui} F_{uv} = \varepsilon_{rui} \partial_{\tau} A_{\perp v} \Rightarrow F_{uv} = \varepsilon_{uvi} B_r, \] (3.19)

the homogeneous Maxwell equations, allowing the introduction of the electro-magnetic potentials, have the standard inertial form \( \varepsilon_{rui} \partial_{\tau} B_{\perp v} = 0, \varepsilon_{rui} \partial_{\tau} E_{\perp v} + \frac{1}{c} \partial B_{\tau r} = 0. \)

Then also Eqs.(3.18) take the following form

\[ \eta_i d \left( \frac{3 e_{(a)r} 3 e_{(a)s}}{\sqrt{\left(1 + n\right)^2 - 3 e_{(c)u} 3 e_{(c)v} \left(\eta^u_i(\tau) + 3 e_{(d)u} n_{(d)}\right) \left(\eta^v_i(\tau) + 3 e_{(e)v} n_{(e)}\right)}} \right) \bigg( \tau, \vec{\eta}_i(\tau) \bigg) = \]

\[ = \eta_i Q_i \left[ E_r + \varepsilon_{rui} \eta^u_i(\tau) B_v \right] \big( \tau, \vec{\eta}_i(\tau) \big) + \eta_i F_{ir}(\tau). \] (3.20)

**C. The Hamilton Equations for the Electro-Magnetic Field**

The Hamilton-Dirac equations for the electro-magnetic field are

\[ \frac{\partial}{\partial \tau} A_r(\tau, \vec{\sigma}) \overset{\circ}{=} c \mu(\tau, \vec{\sigma}), \]

\[ \frac{\partial}{\partial \tau} A_i(\tau, \vec{\sigma}) \overset{\circ}{=} \left( \frac{\partial}{\partial \sigma^a} A_r + \frac{1}{3 e} \left[ 3 e_{(a)r} 3 e_{(a)s} \pi^a + 3 e_{(a)v} n_{(a)} F_{sr} \right] \right)(\tau, \vec{\sigma}), \]

\[ \frac{\partial}{\partial \tau} \pi^r(\tau, \vec{\sigma}) \overset{\circ}{=} \sum_i \eta_i Q_i \eta^u_i(\tau) \delta^3(\vec{\sigma}, \vec{\eta}_i(\tau)) + \]

\[ + \left( \frac{\partial}{\partial \sigma^a} \left[ (1 + n) 3 e 3 e_{(a)r} 3 e_{(a)s} 3 e_{(a)v} n_{(a)} F_{uv} - n_{(a)} \left(3 e_{(a)\pi^a} - 3 e_{(a)\pi^r} \pi^s\right) \right] \right)(\tau, \vec{\sigma}). \] (3.21)

**1. Maxwell Equations**

Eqs.(3.21) imply

\[ \pi^a(\tau, \vec{\sigma}) = - \left[ - \frac{3 e}{1 + n} 3 e_{(a)r} 3 e_{(a)s} \left( F_{rr} - 3 e_{(a)} n_{(a)} F_{rr} \right) \right](\tau, \vec{\sigma}) = \]

\[ = -\sqrt{-4 g(\tau, \vec{\sigma}) 4 g^{\pi A}(\tau, \vec{\sigma}) 4 g^{sB}(\tau, \vec{\sigma}) F_{AB}(\tau, \vec{\sigma}). \] (3.22)
If we introduce the charge density \( \bar{\rho} \), the charge current density \( \bar{J}_r \) and the total charge \( Q_{\text{tot}} = \sum_i Q_i \) on \( \Sigma_r \)

\[
\bar{\rho}(\tau, \bar{\sigma}) = \frac{1}{3e(\tau, \bar{\sigma})} \sum_{i=1}^{N} \eta_i Q_i \delta^3(\bar{\sigma}, \bar{\eta}_i(\tau)),
\]

\[
\bar{J}_r(\tau, \bar{\sigma}) = \frac{1}{3e(\tau, \bar{\sigma})} \sum_{i=1}^{N} \eta_i Q_i \dot{\eta}_i^r(\tau) \delta^3(\bar{\sigma}, \bar{\eta}_i(\tau)),
\]

\[
\Rightarrow Q_{\text{tot}} = \int d^3 \sigma 3^3e(\tau, \bar{\sigma}) \bar{\rho}(\tau, \bar{\sigma}) = \sum_{i=1}^{N} \eta_i Q_i, \tag{3.23}
\]

then the last of Eqs.(3.22) can be rewritten in form

\[
\frac{\partial}{\partial \sigma^r} \pi^r(\tau, \bar{\sigma}) \approx -3^3e(\tau, \bar{\sigma}) \bar{\rho}(\tau, \bar{\sigma}),
\]

\[
\frac{\partial}{\partial \sigma^s} \left[ \sqrt{-4g(\tau, \bar{\sigma})} g^{AB}(\tau, \bar{\sigma}) g^{CD}(\tau, \bar{\sigma}) F_{BD}(\tau, \bar{\sigma}) \right] \bigg|_{\sigma^r} = -s^C(\tau, \bar{\sigma}). \tag{3.26}
\]

Eqs.(3.26) imply the following continuity equation due to the skew-symmetry of \( F_{AB} \)

\[
\frac{1}{\sqrt{-4g(\tau, \bar{\sigma})}} \frac{\partial}{\partial \sigma^A} \left[ \sqrt{-4g(\tau, \bar{\sigma})} s^C(\tau, \bar{\sigma}) \right] = 0,
\]

or

\[
\frac{1}{\sqrt{-4g(\tau, \bar{\sigma})}} \frac{\partial}{\partial \tau} \left[ \sqrt{\gamma(\tau, \bar{\sigma})} \bar{\rho}(\tau, \bar{\sigma}) \right] + \frac{1}{3e(\tau, \bar{\sigma})} \frac{\partial}{\partial \sigma^r} \left[ 3^3e(\tau, \bar{\sigma}) \bar{J}_r(\tau, \bar{\sigma}) \right] = 0, \tag{3.27}
\]

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so that consistently we recover \( \frac{d}{d\tau} Q_{\text{tot}} = 0 \).

2. The Radiation Gauge for the Electro-Magnetic Field in Non-Inertial Frames in ADM Tetrad Gravity.

In Ref.[10] there is a general discussion about the non-covariant decomposition of the vector potential \( \vec{A}(\tau, \sigma^u) \) and its conjugate momentum \( \vec{\pi}(\tau, \sigma^u) \) (the electric field) into longitudinal and transverse parts in absence of matter. Only with this decomposition we can define a Shanmugadhasan canonical transformation adapted to the two first class constraints generating electro-magnetic gauge transformations and identify the physical degrees of freedom (Dirac observables) of the electro-magnetic field without sources. This method identifies the \textit{radiation gauge} as the natural one from the point of view of constraint theory. When there are charged particles, this method allows to find the expression of the mutual Coulomb interaction among the charges in the admissible non-inertial frames of Minkowski space-time.

As said in Appendix B of the first paper of the final version of Ref.[10] the reduction to the radiation gauge has to be done after fixing the 3+1 splitting of space-time and the associated radar 4-coordinates, i.e. after fixing the gauge variables \( \theta^i(\tau, \vec{\sigma}) \) and \( 3K(\tau, \vec{\sigma}) \). In the next Section the radiation gauge will be used in the restriction of the Hamilton equations to suitable gauges.

Here we extend the results of Refs.[10] to our class of space-times.

If \( \Delta = \sum_r \partial_r^2 \) is the non-covariant flat Laplacian, associated to the asymptotic Minkowski metric and acting in the instantaneous non-Euclidean 3-space \( \Sigma_\tau \), its inverse defines the following non-covariant distribution

\[
\frac{1}{\Delta} \delta^3(\vec{\sigma}, \vec{\sigma}') = c(\vec{\sigma}, \vec{\sigma}') = -\frac{1}{4\pi} \frac{1}{\sqrt{\sum_u (\sigma^u - \sigma'^u)^2}},
\]

with \( \delta^3(\vec{\sigma}, \vec{\sigma}') \) being the delta function for \( \Sigma_\tau \).

Then we can define the following non-covariant decomposition of the vector potential and its conjugate momentum (\( \Gamma(\tau, \vec{\sigma}) \approx 0 \) is the Gauss law of Eqs.(3.8), \( \Delta \eta_{\text{em}}(\tau, \vec{\sigma}) = \delta^{rs} \partial_r A_s(\tau, \vec{\sigma}), \eta_{\text{em}} \) describes a Coulomb cloud of longitudinal photons, see Ref.[35])

\[
A_r(\tau, \vec{\sigma}) = A_{\perp r}(\tau, \vec{\sigma}) - \partial_r \eta_{\text{em}}(\tau, \vec{\sigma}),
\]

\[
\pi^r(\tau, \vec{\sigma}) = \pi_{\perp r}(\tau, \vec{\sigma}) + \delta^{rs} \partial_s \int d^3\sigma' c(\vec{\sigma}, \vec{\sigma}') \left( \Gamma(\tau, \vec{\sigma}') - \sum_i Q_i \eta_i \delta^3(\vec{\sigma}', \vec{\eta}_i(\tau)) \right),
\]

\[
\eta_{\text{em}}(\tau, \vec{\sigma}) = -\int d^3\sigma' c(\vec{\sigma}, \vec{\sigma}') \left( \delta^{rs} \partial_r A_s(\tau, \vec{\sigma}') \right), \quad \{ \eta_{\text{em}}(\tau, \vec{\sigma}), \Gamma(\tau, \vec{\sigma}') \} = \delta^3(\vec{\sigma}, \vec{\sigma}'),
\]

\[
A_{\perp r}(\tau, \vec{\sigma}) = \delta_{ru} P_{\perp s}(\vec{\sigma}) A_s(\tau, \vec{\sigma}), \quad \pi_{\perp r}(\tau, \vec{\sigma}) = \sum_s P_{\perp s}(\vec{\sigma}) \pi^s(\tau, \vec{\sigma}),
\]

where we introduced the projector \( P_{\perp s}(\vec{\sigma}) = \delta^{rs} - \delta^{rv} \delta^{sv} \frac{\partial_r \partial_s}{\Delta} \).

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If we introduce the following new Coulomb-dressed momenta for the particles

\[
\tilde{\kappa}_{ir}(\tau) = \kappa_{ir}(\tau) + \frac{Q_i}{c} \frac{\partial}{\partial \eta_{ir}} \eta_{em}(\tau, \vec{r}_i(\tau)),
\]

\[
\Rightarrow \quad \kappa_{ir}(\tau) - \frac{Q_i}{c} A_r(\tau, \vec{r}_i(\tau)) = \tilde{\kappa}_{ir}(\tau) - \frac{Q_i}{c} A_{\perp r}(\tau, \vec{r}_i(\tau))
\]

we arrive at the following non-covariant Shanmugadhasan canonical transformation

\[
\begin{align*}
A_r(\tau, \vec{\sigma}) & \quad \eta_{ir}^*(\tau) \\
\pi^r(\tau, \vec{\sigma}) & \quad \kappa_{ir}(\tau)
\end{align*}
\rightarrow
\begin{align*}
A_{\perp r}(\tau, \vec{\sigma}) & \quad \eta_{em}(\tau, \vec{\sigma}) \\
\pi^r_\perp(\tau, \vec{\sigma}) & \quad \Gamma(\tau, \vec{\sigma}) \approx 0 \\
\tilde{\kappa}_{ir}(\tau)
\end{align*}
\]

\[
\{A_{\perp r}(\tau, \vec{\sigma}), \pi^s_\perp(\tau, \vec{\sigma}^s)\} = c \delta_{ru} P^s_\perp(\vec{\sigma}) \delta^3(\vec{\sigma}, \vec{\sigma}^s),
\]

\[
\{\eta_{ir}^*(\tau), \tilde{\kappa}_{is}(\tau)\} = \delta^r_s \delta_{ij}.
\]

The non-covariant radiation gauge is defined by adding the gauge fixing \(\eta_{em}(\tau, \vec{\sigma}) \approx 0\). As shown in Ref.[10], the \(\tau\)-constancy, \(\frac{\partial \eta_{em}(\tau, \vec{\sigma})}{\partial \tau} = \{\eta_{em}(\tau, \vec{\sigma}), H_D\} \approx 0\), of this gauge fixing implies the secondary gauge fixing for the primary constraint \(\pi^r(\tau, \vec{\sigma}) \approx 0\)

\[
A_r(\tau, \vec{\sigma}) \approx - \int d^3\sigma' c(\vec{\sigma}, \vec{\sigma}') \frac{\partial}{\partial \sigma'^r} \left[3\epsilon^s_{(a)} n_{(a)}(\tau, \vec{\sigma}') F_{sr}(\tau, \vec{\sigma}') + \left(1 + n(\tau, \vec{\sigma}')\right) [3\epsilon^s_{(a)r} 3\epsilon^s_{(a)s}](\tau, \vec{\sigma}') \left(\pi^s_\perp(\tau, \vec{\sigma}') - \delta^{sn} \sum_j Q_j \eta_j \frac{\partial c(\vec{\sigma}, \vec{r}_j(\tau))}{\partial \sigma'^m}\right)\right] + \left[\frac{\sqrt{\delta^s_{(a)}}}{1 + \epsilon_{(a)}} 3\epsilon^r_{(a)} (F_r - \epsilon_{suv} n^u B_v)(\tau, \vec{\sigma}) - \delta^{rs} \sum_i Q_i \eta_i \partial_s c(\vec{\sigma}, \vec{r}_i(\tau))\right].
\]

If we eliminate the electro-magnetic variables \(A_r, \pi^r, \eta_{em}, \Gamma\) by going to Dirac Brackets (still denoted \(\{\ldots\}\) for simplicity), we remain with only the transverse fields \(A_{\perp r}\) and \(\pi^r_\perp\) \((F_{rs} = \partial_r A_{\perp s} - \partial_s A_{\perp r})\).

Let us remark that in the radiation gauge the non-inertial magnetic field of Eqs.(3.19) is transverse: \(B_r = \epsilon_{ruv} \partial_u A_{\perp v}\). But the non-inertial electric field \(E_r = -F_{rr} = -\partial_r A_{\perp r} + \partial_r A_r\) is not transverse: it has \(E_{\perp r} = -\partial_r A_{\perp r}\) as a transverse component. Instead the transverse quantity is \(\pi^r_\perp\) (it coincides with \(\delta^{rs} E_{\perp s}\) only in the inertial frames of Minkowski space-time), whose expression in terms of the electric and magnetic fields is \(\pi^r_\perp(\tau, \vec{\sigma}) = \left[\frac{\sqrt{\delta^s_{(a)}}}{1 + \epsilon_{(a)}} 3\epsilon^r_{(a)} (F_s - \epsilon_{suv} n^u B_v)(\tau, \vec{\sigma}) - \delta^{rs} \sum_i Q_i \eta_i \partial_s c(\vec{\sigma}, \vec{r}_i(\tau))\right]\).

The electromagnetic part of the Hamiltonian (3.12) can be expressed in terms of the new canonical variables, since from Ref.[10] we have \(\tilde{\eta}^r = 3\epsilon^r_{(a)} n_{(a)}\)
\[
\int d^3\sigma \left( 3e \left[ (1 + n) T^{(em)}_{\perp \perp} + n^r T^{(em)}_{\perp r} \right] \right)(\tau, \vec{\sigma}) = \\
= \int d^3\sigma \left[ (1 + n) \left( 3e \, \hat{T}^{(em)}_{\perp \perp} + \mathcal{W}_{(n)} \right) + n^r \left( 3e \, \mathcal{\hat{T}}^{(em)}_{\perp r} + \mathcal{W}_r \right) \right](\tau, \vec{\sigma}),
\] (3.33)

where the new energy-momentum tensor has the form

\[
3e(\tau, \vec{\sigma}) \hat{T}^{(em)}_{\perp \perp}(\tau, \vec{\sigma}) = \frac{3e(a) e(a)^s}{2c} \, \pi^r_{\perp} \pi^s_{\perp} + \frac{3e^r}{4c} \, \frac{3e^s}{e^r} \, \frac{3e^u}{e^r} \, e^v \, F_{ru} F_{sv}(\tau, \vec{\sigma}),
\]

\[
3e(\tau, \vec{\sigma}) \hat{T}^{(em)}_{\perp r}(\tau, \vec{\sigma}) = \frac{1}{c} \, F_{rs}(\tau, \vec{\sigma}) \, \pi^s_{\perp}(\tau, \vec{\sigma}).
\] (3.34)

In Eq.(3.33) we rewrote the non-inertial Coulomb potential \( \mathcal{W}(\tau) \) (a function of the particle 3-coordinates \( \eta_i(\tau) \)), found in Ref.[9], in the form

\[
\mathcal{W}(\tau) = \int d^3\sigma \left[ (1 + n) \, \mathcal{W}_{(n)} + n^r \, \mathcal{W}_r \right](\tau, \vec{\sigma}),
\]

\[
\mathcal{W}_{(n)}(\tau, \vec{\sigma}) = -\frac{3e(a) e(a)^s}{2c} \left( 2 \, \pi^r_{\perp} - \delta^r m \sum_i Q_i \, \eta_i \, \frac{\partial c(\vec{\sigma}, \vec{\eta}_i(\tau))}{\partial \sigma^m} \right) \delta^s n \sum_j Q_j \, \eta_j \, \frac{\partial c(\vec{\sigma}, \vec{\eta}_j(\tau))}{\partial \sigma^n},
\]

\[
\mathcal{W}_r(\tau, \vec{\sigma}) = -\frac{1}{c} \, F_{rs}(\tau, \vec{\sigma}) \, \delta^s n \sum_i Q_i \, \eta_i \, \frac{\partial c(\vec{\sigma}, \vec{\eta}_i(\tau))}{\partial \sigma^n}.
\] (3.35)

In Minkowski space-time its limit to inertial frames is the standard Coulomb potential \( \sum_{i \neq j} \frac{Q_i Q_j \eta_i \eta_j}{4\pi|\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)|} \). Let us remark that now this potential depends not only on the particle positions, but also on the non-inertial electric and magnetic fields, on the 3-metric on \( \Sigma_{\tau} \) and on the lapse and shift functions.

The energy-momentum tensor (3.11) can be written in the radiation gauge by means of the substitutions \( A_r \rightarrow A_{\perp r} \) and \( \pi^r \rightarrow \pi^r_{\perp} - \sum_i \eta_i \, Q_i \, \frac{\partial c(\vec{\sigma}, \vec{\eta}_i(\tau))}{\partial \sigma^r} \), \( \kappa_{ir} \rightarrow \tilde{\kappa}_{ir} \).

3. The Dirac Hamiltonian and the Hamilton Equations in the Radiation Gauge

After the elimination of the variables \( \eta_{em}, \Gamma, A_r, \pi^r \) by going to Dirac brackets, the final form of the Dirac Hamiltonian (3.12) in the radiation gauge is
\[ H_D = \frac{1}{c} \dot{E}_{ADM} + \int d^3 \sigma \left[ n \mathcal{H} - n_{(a)} \mathcal{H}_{(a)} \right](\tau, \bar{\sigma}) + \lambda_r(\tau) \dot{P}_{ADM}^r + \int d^3 \sigma \left[ \lambda_n \pi_n + \lambda_{n_{(a)}} \pi_{n_{(a)}} + \lambda_{\varphi_{(a)}} \pi_{\varphi_{(a)}} + \mu_{(a)}^3 M_{(a)} \right](\tau, \bar{\sigma}), \]  

with \( \dot{E}_{ADM}, \mathcal{H}(\tau, \bar{\sigma}) \) and \( \mathcal{H}_{(a)}(\tau, \bar{\sigma}) \) given by Eqs.(3.8) and (3.14) but with \( \mathcal{M}(\tau, \bar{\sigma}) \) and \( \mathcal{M}_r(\tau, \bar{\sigma}) \) replaced with the following quantities:

\[ \mathcal{M}(\tau, \bar{\sigma}) = \sum_{i=1}^N \delta^3(\bar{\sigma}, \bar{\eta}_i(\tau)) \bar{M}_i(\tau, \bar{\sigma}) c + \left( 3 e \hat{T}_{\perp \perp}^{(em)} \right)(\tau, \bar{\sigma}) + \mathcal{W}_{(n)}(\tau, \bar{\sigma}), \]

\[ \bar{M}_i(\tau, \bar{\sigma}) c = \eta_i \sqrt{m_i^2 c^2 + \tilde{\phi}^{-2/3} \sum_{ars} Q_a^{-2} V_{ra} V_{sa} \left( \bar{\kappa}_{ir}(\tau) - \frac{Q_i}{c} A_{\perp r} \right) \left( \bar{\kappa}_{is}(\tau) - \frac{Q_i}{c} A_{\perp s} \right)}(\tau, \bar{\sigma}), \]

\[ \mathcal{M}_r(\tau, \bar{\sigma}) = \sum_{i=1}^N \eta_i \left( \bar{\kappa}_{ir}(\tau) - \frac{Q_i}{c} A_{\perp r}(\tau, \bar{\sigma}) \right) \delta^3(\bar{\sigma}, \bar{\eta}_i(\tau)) - \left( 3 e \hat{T}_{\perp \perp}^{(em)} \right)(\tau, \bar{\sigma}) - \mathcal{W}_r(\tau, \bar{\sigma}), \]

\[ \left( \tilde{\phi} \hat{T}_{\perp \perp}^{(em)} \right)(\tau, \bar{\sigma}) = \tilde{\phi}^{-1/3}(\tau, \bar{\sigma}) \left[ \frac{1}{2c} \sum_{rs} Q_a^2 V_{ra} V_{sa} \pi^r_{\perp} \pi^s_{\perp} + \frac{1}{4c} \sum_{abrsuv} Q_a^{-2} Q_b^{-2} V_{ra} V_{sa} V_{ub} V_{vb} F_{ru} F_{sv} \right](\tau, \bar{\sigma}), \]

\[ \mathcal{W}_{(n)}(\tau, \bar{\sigma}) = -\frac{1}{2c} \left[ \tilde{\phi}^{-1/3} \sum_{rs} Q_a^2 V_{ra} V_{sa} \left( 2 \pi^r_{\perp} - \delta^{rn} \sum_i Q_i \eta_i \frac{\partial c(\bar{\sigma}, \bar{\eta}_i(\tau))}{\partial \sigma^m} \right) \right] \delta^{sn} \sum_j Q_j \eta_j \frac{\partial c(\bar{\sigma}, \bar{\eta}_j(\tau))}{\partial \sigma^n} \right](\tau, \bar{\sigma}), \]

\[ \left( \tilde{\phi} \hat{T}_{\perp \perp}^{(em)} \right)(\tau, \bar{\sigma}) = \frac{1}{c} \sum_s F_{rs}(\tau, \bar{\sigma}) \pi^s_{\perp}(\tau, \bar{\sigma}), \]

\[ \mathcal{W}_r(\tau, \bar{\sigma}) = -\frac{1}{c} \sum_{sn} F_{rs}(\tau, \bar{\sigma}) \delta^{sn} \sum_i Q_i \eta_i \frac{\partial c(\bar{\sigma}, \bar{\eta}_i(\tau))}{\partial \sigma^n}. \]

Let us remark that also \( \mathcal{M}_r(\tau, \bar{\sigma}) \) does not depend upon the 4-metric like \( \mathcal{M}_r(\tau, \bar{\sigma}) \) of Eqs.(3.9). We have already given the various quantities in the York canonical basis.

In the radiation gauge the Hamilton-Dirac equations (3.17) for the particles are replaced by the following equations.
\[ \eta \ddot{\eta}^r(\tau) = \frac{\eta (1 + n) 3 e_{(a)} 3 e_{(a)} (\dot{k}_{is}(\tau) - \frac{Q_i}{c} A_{\perp s})}{\sqrt{m_i c^2 + 3 e_{(b)} 3 e_{(b)} (\dot{k}_{iu}(\tau) - \frac{Q_i}{c} A_{\perp u}) (\dot{k}_{iv}(\tau) - \frac{Q_i}{c} A_{\perp v})}}(\tau, \bar{\eta}_i(\tau)) - \\
- \eta [3 e_{(a)} n_{(a)}(\tau, \eta_i^r(\tau))]. \]

The quantity \( W(\tau) \) of Eq.(3.35) is a functional of the 3-coordinates \( \bar{\eta}_i(\tau) \) (and also of the 4-metric and the non-inertial electric and magnetic fields), which replaces the two-body Coulomb potential of the inertial frames of Minkowski space-time. In Ref.[10] it is shown that we can write \( \eta_i Q_i E_r(\tau, \bar{\eta}_i(\tau)) = -\eta_i Q_i (\frac{\partial A_{\perp l}(\tau, \bar{\eta}_i(\tau))}{\partial \tau} + \eta_i (\frac{\partial A_{\perp l}(\tau, \bar{\eta}_i(\tau))}{\partial \tau}) \mid_{\bar{\eta}} \approx -\eta_i Q_i (\frac{\partial A_{\perp l}(\tau, \bar{\eta}_i(\tau))}{\partial \tau} - \frac{\partial W(\tau)}{\partial \eta_i^r}). \)

The first of Eqs.(3.38) can be inverted to get

\[ \eta \ddot{\bar{r}}_{ir}(\tau) = \frac{\eta_i Q_i}{c} \frac{\partial A_{\perp u}(\tau, \bar{\eta}_i(\tau))}{\partial \eta_i^r} - \frac{1}{c} \frac{\partial W(\tau)}{\partial \eta_i^r}. \]

The first of Eqs.(3.39) can be inverted to get

\[ \eta \ddot{\bar{r}}_{ir}(\tau) = \eta_i m_i c \left( \frac{3 e_{(a)} 3 e_{(a)} }{\sqrt{(1 + n)^2 - 3 e_{(c)} 3 e_{(c)} (\dot{\bar{r}}_{ir}(\tau) + 3 e_{(c)} n_{(c)} (\dot{\bar{r}}_{ir}(\tau) + 3 e_{(c)} n_{(c)})}}(\tau, \bar{\eta}_i(\tau)) + \\
+ \frac{\eta_i Q_i}{c} A_{\perp r}(\tau, \bar{\eta}_i(\tau)). \right) \]

In the radiation gauge the Hamilton equations for the transverse electro-magnetic fields \( A_{\perp r} \) and \( \pi_{\perp r} \) are

\[ \partial_\tau A_{\perp r}(\tau, \bar{\sigma}) \overset{\circ}{=} \{ A_{\perp r}(\tau, \bar{\sigma}), H_D \} = \\
= \sum_{n u v a} \delta_{n u v a} P_{\perp n}^u(\bar{\sigma}) \left[ (1 + n)^2 3 e_{(a)} n_{(a)} (\dot{\bar{\sigma}}_{ir}(\tau) + 3 e_{(c)} n_{(c)} (\dot{\bar{\sigma}}_{ir}(\tau) + 3 e_{(c)} n_{(c)}) \right] (\tau, \bar{\eta}_i(\tau)) + \\
+ \eta_i (\frac{\partial A_{\perp l}(\tau, \bar{\eta}_i(\tau))}{\partial \tau} - \frac{\partial W(\tau)}{\partial \eta_i^r}). \]

\[ \partial_\tau \pi_{\perp r}(\tau, \bar{\sigma}) \overset{\circ}{=} \{ \pi_{\perp r}(\tau, \bar{\sigma}), H_D \} = \\
= \sum_{n u v a} P_{\perp n}^u(\bar{\sigma}) \delta_{n u v a} \left[ \sum_{i} \eta_i Q_i (\dot{\bar{\sigma}}_{ir}(\tau) + 3 e_{(a)} n_{(a)} (\dot{\bar{\sigma}}_{ir}(\tau) + 3 e_{(c)} n_{(c)}) \right] (\tau, \bar{\eta}_i(\tau)) + \\
\left[ \frac{1 + n}{m_i c^2} \sum_{n u v a} e_{(a)} n_{(a)} (\dot{\bar{\sigma}}_{ir}(\tau) + 3 e_{(c)} n_{(c)}) \right] (\tau, \bar{\eta}_i(\tau)) + \\
- \eta_i (\frac{\partial A_{\perp l}(\tau, \bar{\eta}_i(\tau))}{\partial \tau} - \frac{\partial W(\tau)}{\partial \eta_i^r}). \]
\[ + \left[ (1 + n) \sum_{svb} \left( 3e^a \partial_{a} 3e^b \partial_{b} \left( 3e^v \partial^m + 3e^m \partial^v \right) \right) \partial_n F_{sv} + \]
\[ + \partial_n \left[ 3e^a \partial_{a} 3e^b \partial_{b} \left( 3e^v \partial^m - 3e^m \partial^v \right) \right] F_{sv} \right) + \]
\[ + \partial_n n \sum_{svb} \left[ 3e^a \partial_{a} 3e^v \partial^m + 3e^m \partial^v \right] \left( 3e^a \partial_{a} - 3e^m \partial^m \right) F_{sv} + \]
\[ + \tilde{n}(a) \left[ 3e^a \partial_{a} \tau + \partial_n 3e^a \partial^m - \partial_n 3e^m \partial^a \right] - \sum_t \left( \partial_n 3e^a \delta^mt - \partial_n 3e^m \delta^nt \right) \sum_i \eta_i Q_i \left( \frac{\partial c(\tilde{\sigma}, \tilde{\eta}_i(t))}{\partial \sigma^t} \right) - \]
\[ - \sum_t \left( 3e^a \delta^mt - 3e^m \delta^nt \right) \sum_i \eta_i Q_i \left( \frac{\partial^2 c(\tilde{\sigma}, \tilde{\eta}_i(t))}{\partial \sigma^t \partial \sigma^u} \right) - \]
\[ - \partial_n \tilde{n}(a) \left[ 3e^a \delta^mt - 3e^m \delta^nt \right] \sum_i \eta_i Q_i \left( \frac{\partial c(\tilde{\sigma}, \tilde{\eta}_i(t))}{\partial \sigma^t} \right) (\tau, \tilde{\sigma}) \right). \] (3.40)

D. The Final Form of the Constraints in the York Canonical Basis.

1. The Super-Momentum Constraints.

If \( D_{r(a)b} = \delta_{ab} \partial_r + \bar{\omega}_{r(a)b} = \delta_{ab} \partial_r + \epsilon_{a(b)c} \bar{\omega}_{r(c)} \) is the covariant derivative defined in terms of the spin connection given in Eq. (2.10), the super-momentum constraint (3.8), with \( \mathcal{M} \) given in Eq. (3.37), has the following expression [6] in the electro-magnetic radiation gauge

\[ \mathcal{H}_{(a)} = \sum_c R_{(a)c} \mathcal{H}_{(c)} \approx 0, \]
\[ \mathcal{H}_{(a)} = \sum_r \partial_r 3\bar{n}_{(a)} + \sum_{rb} 3\bar{\omega}_{r(a)b} 3\bar{n}_{(b)} + \sum_v 3e^v \mathcal{M}_v \approx \]
\[ \approx \mathcal{H}_{(a)} = \sum_{rb} \tilde{D}_{r(a)b} 3\bar{n}_{(b)} + \sum 3e^v \mathcal{M}_v = \]
\[ = \sum_{rb} \left[ \delta_{ab} \partial_r + \sum_{u} \left( \frac{1}{3} \left[ Q_{a} Q_{b}^{-1} V_{ra} V_{ub} - Q_{b} Q_{a}^{-1} V_{rb} V_{ua} \right] \phi^{-1} \partial_u \phi + \right. \right. \]
\[ + \sum_{\tilde{a}} \left[ \gamma_{a\tilde{a}} Q_{a} Q_{b}^{-1} V_{ra} V_{ub} - \gamma_{\tilde{a}b} Q_{b} Q_{a}^{-1} V_{rb} V_{ua} \right] \partial_{u} R_{\tilde{a}} + \right. \]
\[ + \frac{1}{2} \left[ Q_{b} Q_{a}^{-1} V_{ua} \left( \partial_r V_{ub} - \partial_u V_{rb} \right) - Q_{a} Q_{b}^{-1} V_{ub} \left( \partial_r V_{ua} - \partial_u V_{ra} \right) \right] + \]
\[ + \frac{1}{2} \sum_{\nu \omega} Q_{\nu} Q_{\omega}^{-1} Q_{\nu}^{2} \left( V_{\nu a} V_{\omega b} - V_{\nu b} V_{\omega a} \right) V_{\nu \omega} \partial_{v} V_{\omega \nu} \left) \right. \]
\[
\tilde{\mathcal{H}}_{(a)}(\tau, \vec{\sigma}) \overset{\text{def}}{=} \left( \sum_{rb} D_{(r(a)b)} \phi_{(b)}^{3r} + \sum_{v} 3\bar{\epsilon}_{(a)v} \mathcal{M}_v \right)(\tau, \vec{\sigma}) = \\
= \left( \sum_{rb} [\delta_{ab} \partial_r + \sum_{u} \left( \frac{1}{3} [Q_a Q_b^{-1} V_{ra} V_{ub} - Q_b Q_a^{-1} V_{rb} V_{ua}] \right) \tilde{\phi}^{-1} \partial_u \tilde{\phi} + \\
\sum_{\tilde{a}} [\gamma_{\tilde{a}a} Q_a Q_b^{-1} V_{ra} V_{ub} - \gamma_{\tilde{a}b} Q_b Q_a^{-1} V_{rb} V_{ua}] \partial_u \tilde{R}_{\tilde{a}} + \\
\frac{1}{2} \left[ Q_a Q_b^{-1} V_{ua} (\partial_r V_{ub} - \partial_u V_{rb}) - Q_a Q_b^{-1} V_{ub} (\partial_r V_{ua} - \partial_u V_{ra}) \right] + \\
\frac{1}{2} \sum_{vw} Q_a^{-1} Q_b^{-1} Q_{vw}^2 (V_{ua} V_{vb} - V_{ub} V_{va}) V_{rw} \partial_v V_{uw} \right) \\
\right] \\
\left[ \tilde{\phi}^{-1/3} Q_b^{-1} V_{rb} \sum_{\tilde{c}} \gamma_{\tilde{c}b} \Pi_{\tilde{c}} + \tilde{\phi}^{2/3} \left( Q_b^{-1} V_{rb} \pi_{\tilde{\phi}} - \frac{c^3}{8\pi G} \sum_{c} Q_c^{-1} V_{rc} \sigma_{(b)(c)} \right) \right] + \\
+ \tilde{\phi}^{-1/3} Q_a^{-1} \sum_{v} V_{va} \mathcal{M}_v (\tau, \vec{\sigma}) \approx 0.
\]  

(3.42)

2. The Super-Hamiltonian Constraint and the Weak ADM Energy

The weak ADM energy of Eqs.(3.14) has the following form in the York canonical basis and in the electro-magnetic radiation gauge with \( \mathcal{M} \) given in Eq.(3.37)
\[
\hat{E}_{ADM} = c \int d^3 \sigma \left[ \mathcal{M} - \frac{c^3}{16\pi G} S + \frac{2\pi G}{c^3} \hat{\phi}^{-1} \left( -3 \left( \hat{\phi} \pi_\phi \right)^2 + 2 \sum_b \Pi_b^2 + \right. \right. \\
\left. + \left. \left. 2 \sum_{abtwiu} \epsilon_{abt} \epsilon_{aba} V_{wt} B_{iu} V_{uv} B_{juv} \pi_{i}^{(\theta)} \pi_{j}^{(\theta)} \right] (\tau, \bar{\sigma}) \right], \\
S = 3e^3 R - T = \hat{\phi}^{1/3} R - 8 \hat{\phi}^{1/6} \hat{\Delta} \hat{\phi}^{1/6} - T. \tag{3.43}
\]

The functions \( S(\tau, \bar{\sigma}) \) and \( T(\tau, \bar{\sigma}) \) are given in Eqs.(B8) and (B13), respectively.

In Eq.(3.43) we have also given the expression \(^{18}\) of the scalar 3-curvature \( 3R = 3R[\phi, \theta^a, R_a] \) in terms of \( \hat{3}R = \hat{3}R[\theta^a, R_a] \) and \( \hat{\Delta} \), which are the 3-curvature of \( \Sigma_\tau \) and the Laplace-Beltrami operator for the conformal 3-metric \( \hat{g}_{rs} \) (\( det \hat{g}_{rs} = 1 \)), respectively, and are given in Eqs.(B20) and (B12). This expression is needed to put \(^6\) the super-Hamiltonian constraint in the form of the Lichnerowicz equation for the conformal factor \( \phi = \hat{\phi}^{1/6} \), which has the following expression in the York canonical basis \( (3\hat{g}^{rs} \) is the inverse of the 3-metric with unit determinant)

\[
\hat{H}(\tau, \bar{\sigma}) = \frac{c^3}{16\pi G} \hat{\phi}^{1/6} (\tau, \bar{\sigma}) \left[ 8 \hat{\Delta} \hat{\phi}^{1/6} - \hat{\phi}^{1/6} 3 \hat{R} \right] (\tau, \bar{\sigma}) + \hat{M}(\tau, \bar{\sigma}) + \\
+ \frac{2\pi G}{c^3} \hat{\phi}^{-1} \left[ -3 \left( \hat{\phi} \pi_\phi \right)^2 + 2 \sum_b \Pi_b^2 + \right. \\
\left. + \left. \sum_{abtwiu} \epsilon_{abt} \epsilon_{aba} V_{wt} B_{iu} V_{uv} B_{juv} \pi_{i}^{(\theta)} \pi_{j}^{(\theta)} \right] (\tau, \bar{\sigma}) \right. \right. \\
\left. \left. \approx 0, \right. \right. \\
\hat{\Delta} = \partial_\tau \left( 3\hat{g}^{rs} \partial_s \right) = 3\hat{g}^{rs} 3\hat{\nabla}_r 3\hat{\nabla}_s = \partial_\tau \left( \sum_a Q_a^{-2} V_{ra} V_{sa} \partial_s \right) = \\
= \sum_a Q_a^{-2} \left[ V_{ra} V_{sa} \partial_\tau \partial_s + \left( 2 V_{ra} V_{sa} \sum_b \gamma_{ba} \partial_\tau R_b - \partial_\tau \left( V_{ra} V_{sa} \right) \right) \partial_s \right]. \tag{3.44}
\]

In terms of the shear the super-Hamiltonian constraint and the weak ADM energy take the following forms (again these forms cannot be used for the functional derivatives of these quantities)

\[
\hat{H}(\tau, \bar{\sigma}) = \frac{c^3}{2\pi G} \hat{\phi}^{-1} (\tau, \bar{\sigma}) \left( \hat{\phi}^{7/6} (\hat{\Delta} - \frac{1}{3} \hat{3}R) \hat{\phi}^{1/6} + \frac{2\pi G}{c^3} \hat{\phi} \hat{M} + \right. \\
\left. + \frac{8\pi^2 G^2}{c^6} \sum \Pi_a^2 + \frac{1}{8} \hat{\phi}^{2} \sum_{ab,a\neq b} \sigma_{(a)(b)} \sigma_{(a)(b)} - \frac{12\pi^2 G^2}{c^6} \hat{\phi}^2 \pi_\phi^2 \right) (\tau, \bar{\sigma}) \approx 0,
\]

\(^{18}\) See Ref.[26] and Eq.(B4) of Ref. [6].
\[ \hat{E}_{ADM} = c \int d^3\sigma \left[ \hat{\mathcal{M}} - \frac{c^3}{16\pi G} S + \frac{4\pi G}{c^3} \tilde{\phi}^{-1} \sum_b \Pi_b^2 + \tilde{\phi} \left( \frac{c^3}{16\pi G} \sum_{ab,a\neq b} \sigma_{(a)(b)} \sigma_{(a)(b)} - \frac{6\pi G}{c^3} \pi_\phi^2 \right) \right] (\tau, \vec{\sigma}). \] (3.45)

The super-momentum and super-Hamiltonian constraints are coupled equations for \( \tilde{\phi} \) and \( \pi_\phi^{(b)} \) (or \( \tilde{\phi} \sigma_{(a)(b)|a\neq b} \)). In the form given in Eqs.(3.42) and (3.45) the constraints have to be solved to get the conformal factor \( \tilde{\phi} \) and the off-diagonal terms of the shear \( \sigma_{(a)(b)|a\neq b} \).

3. The Dirac Hamiltonian in the Radiation Gauge

The Dirac Hamiltonian (3.12) takes the following form when written in the York canonical basis and restricted to the electro-magnetic radiation gauge

\[ H_D = \frac{1}{c} \hat{E}_{ADM} + \int d^3\sigma \left[ n \mathcal{H} - n_{(a)} \mathcal{H}_{(a)} \right] (\tau, \vec{\sigma}) + \lambda_r(\tau) \hat{P}_r^{ADM} + \int d^3\sigma \left[ \lambda_n \Pi_n + \lambda_{n_{(a)}} \Pi_{n_{(a)}} + \lambda_{\varphi(\varphi)} \Pi_{\varphi(\varphi)} + \lambda_{\alpha(\alpha)} \Pi_{\alpha(\alpha)} \right] (\tau, \vec{\sigma}), \] (3.46)

where \( \lambda_{\alpha(\alpha)}(\tau, \vec{\sigma}) \) are new Dirac multipliers replacing the \( \mu_{(a)}(\tau, \vec{\sigma}) \) appearing in Eq.(3.12).

E. The Asymptotic ADM Poincare’ Algebra, the Rest-Frame Conditions and the Center of Mass in the York Canonical basis

While the weak ADM energy is given in Eq.(3.43), the other weak ADM generators of Eqs.(2.22) have the following form

\[ \hat{P}_r^{ADM} = -2 \int d^3\sigma \left\{ -\tilde{\phi}^{-2/3} \sum_{a,v,b} Q^{-2}_{a} V_{ra} V_{va} \left( \frac{1}{3} \gamma_{a\alpha} \tilde{\phi}^{-1} \partial_v \tilde{\phi} + \sum_a (\gamma_{a\alpha} \gamma_{a\alpha} - (1/2) \delta_{ab}) \partial_v R_a \right) - \frac{1}{2} \gamma_{a\alpha} \partial_v(V_{ra} V_{va}) \right\} \Pi_b + \tilde{\phi}^{1/3} \sum_{a,v} Q_a^{-2} \left[ V_{ra} V_{va} \left( \frac{2}{3} \tilde{\phi}^{-1} \partial_v \tilde{\phi} + \sum_a \gamma_{a\alpha} \partial_v R_a \right) - \frac{1}{4} \partial_v(V_{ra} V_{va}) \right] \pi_\phi^+ + \tilde{\phi}^{-2/3} \sum_{a,d,v} Q_a^{-1} Q_d^{-1} \left[ V_{ra} V_{vd} \left( \frac{1}{3} \tilde{\phi}^{-1} \partial_v \tilde{\phi} + \sum_a \gamma_{a\alpha} \partial_v R_a \right) - \frac{1}{2} V_{rd} \partial_v V_{va} + \frac{1}{2} \sum_{c,u} Q_a^{-2} Q_c^{-2} V_{rc} V_{uc} V_{vd} (\partial_v V_{au} - \partial_u V_{va}) \right] \sum_{twi} \frac{\epsilon_{edt} V_{wt} B_{iw}\pi_i^q}{Q_i Q_a^{-1} - Q_a Q_i^{-1}} - \frac{1}{2} \tilde{\phi}^{-2/3} \sum_{a,s} Q_a^{-2} V_{ra} V_{sa} \mathcal{M}_s \right\} \approx 0, \]
\[
\tilde{J}^{rs}_{ADM} = \int d^3 \sigma \left\{ \sigma^r \left[ -\tilde{\phi}^{-2/3} \sum_{a,v,b} Q_a^{-2} \left[ V_{sa} V_{va} \left( \frac{1}{3} \gamma_{ba} \tilde{\phi}^{-1} \partial_v \tilde{\phi} + \sum_{\bar{a}} (\gamma_{ba} \gamma_{\bar{a}a} - (1/2)\delta_{ab}) \partial_v \bar{R}_a \right) - \right. \right. \\
- \frac{1}{2} \gamma_{ba} \partial_v (V_{sa} V_{va}) \left. \right] \right. \Pi_b + \\
\left. \left. + \tilde{\phi}^{1/3} \sum_{a,v} Q_a^{-2} \left[ V_{sa} V_{va} \left( \frac{2}{3} \tilde{\phi}^{-1} \partial_v \tilde{\phi} + \sum_{\bar{a}} \gamma_{\bar{a}a} \partial_v \bar{R}_a \right) - \frac{1}{4} \partial_v (V_{sa} V_{va}) \right] \tilde{\pi}_{\tilde{\phi}} + \\
+ \tilde{\phi}^{-2/3} \sum_{a,d,v} Q_a^{-1} Q_d^{-1} \left[ V_{sa} V_{vd} \left( \frac{1}{3} \tilde{\phi}^{-1} \partial_v \tilde{\phi} + \sum_{\bar{a}} \gamma_{\bar{a}a} \partial_v \bar{R}_a \right) - \frac{1}{2} V_{sd} \partial_v V_{va} + \\
\frac{1}{2} \sum_{c,u} Q_c^{-2} Q_v^{-1} V_{sc} V_{uc} V_{vd} (\partial_v V_{ua} - \partial_a V_{va}) \right] \left[ \sum_{tuv} \epsilon_{adt} V_{ui} B_{tw\bar{v}} \pi_{\bar{v}}^\theta - \right. \\
\left. \left. - \frac{1}{2} \tilde{\phi}^{-2/3} \sum_{a,u} Q_a^{-2} V_{sa} V_{ua} M_u \right] - \\
- \sigma^a \left[ -\tilde{\phi}^{-2/3} \sum_{a,v,b} Q_a^{-2} \left[ V_{ra} V_{va} \left( \frac{1}{3} \gamma_{ba} \tilde{\phi}^{-1} \partial_v \tilde{\phi} + \sum_{\bar{a}} (\gamma_{ba} \gamma_{\bar{a}a} - (1/2)\delta_{ab}) \partial_v \bar{R}_a \right) - \right. \right. \\
\left. \left. \right. \\
- \frac{1}{2} \gamma_{ba} \partial_v (V_{sa} V_{va}) \left. \right] \right. \Pi_b + \\
\left. \left. + \tilde{\phi}^{1/3} \sum_{a,v} Q_a^{-2} \left[ V_{ra} V_{va} \left( \frac{2}{3} \tilde{\phi}^{-1} \partial_v \tilde{\phi} + \sum_{\bar{a}} \gamma_{\bar{a}a} \partial_v \bar{R}_a \right) - \frac{1}{4} \partial_v (V_{sa} V_{va}) \right] \tilde{\pi}_{\tilde{\phi}} + \\
+ \tilde{\phi}^{-2/3} \sum_{a,d,v} Q_a^{-1} Q_d^{-1} \left[ V_{ra} V_{vd} \left( \frac{1}{3} \tilde{\phi}^{-1} \partial_v \tilde{\phi} + \sum_{\bar{a}} \gamma_{\bar{a}a} \partial_v \bar{R}_a \right) - \frac{1}{2} V_{rd} \partial_v V_{va} + \\
\frac{1}{2} \sum_{c,u} Q_c^{-2} Q_v^{-1} V_{rc} V_{uc} V_{vd} (\partial_v V_{ua} - \partial_a V_{va}) \right] \left[ \sum_{tuv} \epsilon_{adt} V_{ui} B_{tw\bar{v}} \pi_{\bar{v}}^\theta - \right. \\
\left. \left. - \frac{1}{2} \tilde{\phi}^{-2/3} \sum_{a,u} Q_a^{-2} V_{ra} V_{ua} M_u \right] \right\} ,
\]
\[
\mathcal{J}_{ADM}^{tr} = \int d^3 \sigma \left\{ \sigma^r \left[ \frac{c^3}{16 \pi G} S - \mathcal{M} - \frac{2 \pi G}{c^3} \tilde{\phi}^{-1} \left( -3(\dot{\phi} \pi)^2 + 2 \sum_b \Pi_b^2 \right) \right] + 2 \sum_{abtwiuvj} \epsilon_{abt} \epsilon_{a_{t}^{(a)}_{w}^{(a)}} V_{w} B_{i} \left[ \frac{Q_a Q_b^{-1} - Q_b Q_a^{-1}}{2} \right] \right) + \frac{c^3}{16 \pi G} \tilde{\phi}^{1/3} \sum_{a,v} \left\{ Q_a^{-2} V_r V_v \left( -\frac{2}{3} \tilde{\phi}^{-1} \partial_v \tilde{\phi} + \sum_a \gamma_a \partial_v R_a \right) + \partial_v (V_r V_v) \right) + \left[ V_r V_v \left( -\frac{1}{3} \tilde{\phi}^{-1} \partial_v \tilde{\phi} + 4 \sum_a \gamma_a Q_a^{-4} \partial_v R_a + \sum_{a,c} (\gamma_{ac} - 2 \gamma_{aa}) Q_a^{-2} Q_c^{-2} \partial_v R_a \right) + + (Q_a^{-4} - Q_a^{-2} \sum_c Q_c^{-2}) \partial_v (V_r V_v) \right] \right\} \approx 0.
\]

(3.47)

The rest-frame conditions \( \hat{P}_{ADM}^{tr} \approx 0 \) and the conditions \( \hat{J}_{ADM}^{tr} \approx 0 \) eliminate the internal 3-center of mass of the 3-universe \( \Sigma_{r} \). The gauge fixings \( \hat{J}_{ADM}^{tr} \approx 0 \) imply \( \lambda_{r}(\tau) = 0 \) in \( H_D \) of Eq.(3.36), because their time preservation implies \( \partial_{\tau} \hat{J}_{ADM}^{tr} = \{ \hat{J}_{ADM}^{tr}, H_D \} = \hat{P}_{ADM}^{tr} - \lambda_{r}(\tau) \hat{E}_{ADM} \approx -\lambda_{r}(\tau) \hat{E}_{ADM} \approx 0 \). In this way we identify the Fokker-Pryce center of inertia of the 3-universe as the origin of the 3-coordinates in the instantaneous 3-spaces \( \Sigma_{r} \).

F. The Dirac Hamiltonian in the Schwinger Time Gauges and in the Electromagnetic Radiation Gauge

In what follows we restrict ourselves to the Schwinger time gauges \( \alpha_{(a)}(\tau, \tilde{\sigma}) \approx 0, \varphi_{(a)}(\tau, \tilde{\sigma}) \approx 0, \) whose \( \tau \)-preservation implies \( \lambda_{\varphi_{(a)}}(\tau, \tilde{\sigma}) = \lambda_{\alpha_{(a)}}(\tau, \tilde{\sigma}) = 0 \) in Eq.(3.46). The following results are obtained after the elimination of the variables \( \alpha_{(a)}, \pi_{(a)}^{(a)} \) and \( \varphi_{(a)}, \pi_{\varphi_{(a)}} \) with Dirac brackets.

By using Eqs.(3.44) and (B1) for the super-Hamiltonian and super-momentum constraints with \( \mathcal{M} \) and \( \mathcal{M}_{r} \) given in Eqs.(3.37), the Dirac Hamiltonian (3.34) in Schwinger time gauges and in the electromagnetic-radiation gauge is

\[\text{The asymptotic Poincare' charges are assumed gauge invariant, i.e. } \{ \hat{J}_{ADM}^{tr}, \mathcal{H}(\tau, \tilde{\sigma}) \} \approx \{ \hat{J}_{ADM}^{tr}, \hat{H}(\tau, \tilde{\sigma}) \} \approx 0.\]
\[ H_D = \frac{1}{c} \hat{E}_{ADM} + \int d^3\sigma \left[ n \mathcal{H} - \tilde{n}_{(a)} \tilde{\mathcal{H}}_{(a)} \right](\tau, \vec{\sigma}) + \]
\[ + \int d^3\sigma \left[ \lambda_n \tilde{\pi}^n + \lambda_{(a)} \tilde{\pi}_{(a)} \right](\tau, \vec{\sigma}) = \]
\[ = \int d^3\sigma \left[ (1 + n) \mathcal{M} \right](\tau, \vec{\sigma}) - \frac{c^3}{16\pi G} \int d^3\sigma \left[ (1 + n) S + n T \right](\tau, \vec{\sigma}) + \]
\[ + \frac{2\pi G}{c^3} \int d^3\sigma \left[ (1 + n) \tilde{\phi}^{-1} \left( -3 (\tilde{\phi})_{,\vec{\sigma}}^2 + 2 \sum_b \Pi_b^2 \right) + \right. \]
\[ + 2 \sum_{a \neq b} \frac{\epsilon_{ab} \epsilon_{abu} V_{w1} V_{w2}}{Q_a Q_b^{-1} - Q_b Q_a^{-1}} \sum_i B_{iw} \tilde{\pi}_i^{(\theta)} \sum_j B_{jw} \tilde{\pi}_j^{(\theta)} \left] (\tau, \vec{\sigma}) - \right. \]
\[ - \int d^3\sigma \sum_a \tilde{n}_{(a)}(\tau, \vec{\sigma}) \tilde{\phi}^{-1/3}(\tau, \vec{\sigma}) \left( \sum_{b \neq a} \sum_{rtw} \frac{\epsilon_{ab} Q_b^{-1} V_{rb} V_{w1} B_{iw}}{Q_b Q_a^{-1} - Q_a Q_b^{-1}} \partial_r \tilde{\pi}_i^{(\theta)} + \right. \]
\[ + \sum_{rtw} \left( \sum_{b \neq a} \frac{\epsilon_{ab} Q_b Q_a^{-1} - Q_a Q_b^{-1}}{Q_b Q_a^{-1} - Q_a Q_b^{-1}} \right) \left( Q_{b}^{-1} \partial_r (V_{rb} V_{w1} B_{iw}) + \right. \]
\[ + 2 \frac{Q_a^{-1}}{Q_b Q_a^{-1} - Q_a Q_b^{-1}} \sum_c (\gamma_{ac} - \gamma_{cb}) \partial_r R_{c} V_{rb} V_{w1} B_{iw} \right) + \]
\[ + \sum_{bu} \sum_{c \neq b} \frac{\epsilon_{bca} Q_a^{-1} Q_b^{-1} Q_c^{-1}}{Q_c Q_b^{-1} - Q_b Q_c^{-1}} \left( V_{ra} V_{ua} - V_{ra} V_{uc} \right) \partial_r V_{ab} V_{w1} B_{iw} \right) \tilde{\pi}_i^{(\theta)} + \]
\[ + Q_a^{-1} \sum_r V_{ra} \left( \tilde{\phi} \partial_r \tilde{\pi}_{\vec{\sigma}} + \sum_b \gamma_{ba} \partial_r \Pi_b \right) + \]
\[ + Q_a^{-1} \sum_{rb} \left( \tilde{\gamma}_{ba} \partial_r V_{ra} - V_{ra} \partial_r R_{b} + \sum_{ab} \gamma_{bb} V_{ua} V_{rb} \partial_r V_{ab} \right) \Pi_b + \]
\[ + Q_a^{-1} \sum_r V_{ra} \mathcal{M}_r(\tau, \vec{\sigma}) + \]
\[ \left. + \int d^3\sigma \left[ \lambda_n \tilde{\pi}^n + \sum_a \lambda_{(a)} \tilde{\pi}_{(a)} \right](\tau, \vec{\sigma}), \quad (3.48) \right. \]

where we have used the last expression in Eq.(B1) for the super-momentum constraints.

\( H_D \) depends not only upon the tidal variables \( R_{a, \vec{\sigma}} \) and \( \Pi_{\vec{\sigma}} \) and the matter, but also upon the gauge variables \( n, \tilde{n}_{(a)}, \phi^\alpha, \pi^\beta \), which play the role of inertial potentials.

Let us remark that in the Hamiltonian (3.48), and in particular in the weak ADM energy (3.43), the kinetic term \(-\frac{6\pi G}{c^4} \int d^3\sigma \left[ \tilde{\phi} (1 + n) \pi^2_{\vec{\sigma}} \right](\tau, \vec{\sigma}) = -\frac{c^3}{24\pi G} \int d^3\sigma \left[ \tilde{\phi} (1 + n) \left( \frac{3}{2} \mathcal{K} \right)^2 \right](\tau, \vec{\sigma}) \right)\) connected to the momentum gauge variable \( \pi_{\vec{\sigma}}(\tau, \vec{\sigma}) = \frac{c^3}{12\pi G} \mathcal{K}(\tau, \vec{\sigma}) \), determining the instantaneous 3-space, is definite negative in every gauge. Therefore it plays
the role of a *dark energy* and it vanishes only in the CMC gauges $^3K(\tau, \vec{\sigma}) \approx 0$. Instead, the kinetic term connected to the momenta $\pi_i^{(\theta)}(\tau, \vec{\sigma})$ (or by the off-diagonal terms of the shear $\sigma_{(a)(b)}|_{a \neq b}(\tau, \vec{\sigma})$), determined by the super-momentum constraints, is always positive definite due to Eq.(3.45).
IV. THE EQUATIONS OF MOTION IN SCHWINGER TIME GAUGES.

In this Section we shall write the Hamilton equations for the gravitational and matter variables, generated by the Dirac Hamiltonian (3.48), in an arbitrary Schwinger time gauge and in the York canonical basis.

Let us remark that in the evaluation of the Hamilton equations the constraints can be used only after having done the Poisson brackets with $H_D$. When, like in Section V, some gauge fixings will be added, we shall restrict the Hamilton equations of this Section to the chosen gauge and we will not restrict the Dirac Hamiltonian to the gauge and evaluate the new Hamilton equations with the new Hamiltonian (even if the two approaches should be equivalent).

By comparison in Appendix A there are the standard ADM equations of canonical metric gravity and a discussion of the contracted Bianchi identities. The use of the York canonical basis allows to disentangle the contracted Bianchi identities from the equations for the remaining gauge variables (the inertial effects), for the tidal variables and for the matter.

See Appendix B for the explicit form in the York canonical basis of many terms appearing in the Hamilton equations.

A. The Contracted Bianchi Identities

The variables $\tilde{\phi}(\tau, \vec{\sigma})$ and $\pi^{(\theta)}_i(\tau, \vec{\sigma})$ are the quantities determined by the (non-hyperbolic) partial differential equations corresponding to the super-Hamiltonian and super-momentum constraints, respectively. The Hamilton equations for them are the contracted Bianchi identities ensuring the $\tau$-preservation of the constraint sub-manifold: they hold independently from the form of the solution of such partial differential equations.

For $\tilde{\phi}$ the Hamilton-Dirac equations obtained by using the Dirac Hamiltonian (3.48) in the York canonical basis, in the electro-magnetic radiation gauge and with the Fokker-Pryce observer as origin of the 3-coordinates are

$$\partial_\tau \tilde{\phi}(\tau, \vec{\sigma}) \overset{\delta}{=} \{\tilde{\phi}(\tau, \vec{\sigma}), H_D\} = \frac{\delta H_D}{\delta \tilde{\phi}(\tau, \vec{\sigma})} =$$

$$= \bigg[ - \frac{12\pi G}{c^3} (1 + n) \tilde{\phi} \pi_{\tilde{\phi}} + \tilde{\phi}^{2/3} \sum_{ra} Q_a^{-1} \left( \partial_r \bar{n}_{(a)} V_{ra} + \bar{n}_{(a)} \left[ V_{ra} \left( \frac{2}{3} \tilde{\phi}^{-1} \partial_\tau \tilde{\phi} - \sum_b \gamma_{ba} \partial_r R_b \right) + \partial_r V_{ra} \right] \right) \bigg](\tau, \vec{\sigma}),
$$

(4.1)

where Eq.(B2) was used.

For $\pi^{(\theta)}_i$ the Hamilton-Dirac equations obtained by using the Dirac Hamiltonian (3.48) in the York canonical basis are

\[\text{Equations continued...}\]
\[ \partial_\tau \pi_\delta^{(\theta)}(\tau, \bar{\sigma}) \overset{\circ}{=} \{ \pi_\delta^{(\theta)}(\tau, \bar{\sigma}), H_D \} = -\frac{\delta H_D}{\delta \theta^i(\tau, \bar{\sigma})} = \]

\[ = -\int d^3\sigma_1 \left[ (1 + n)(\tau, \bar{\sigma}) \left( \frac{\delta \hat{M}(\tau, \bar{\sigma}_1)}{\delta \theta^i(\tau, \bar{\sigma})} - \frac{c^3}{16\pi G} \frac{\delta S(\tau, \bar{\sigma}_1)}{\delta \theta^i(\tau, \bar{\sigma})} \right) - \frac{c^3}{16\pi G} n(\tau, \bar{\sigma}_1) \frac{\delta T(\tau, \bar{\sigma}_1)}{\delta \theta^i(\tau, \bar{\sigma})} \right] - \frac{4\pi G}{c^3} (1 + n) \bar{\phi}^{-1} \sum_{abtwuvij} \epsilon_{abt} \epsilon_{abu} \pi_\delta^{(\theta)} \pi_j^{(\theta)} (Q_a Q_b^{-1} - Q_b Q_a^{-1})^2 \frac{\partial [V_{wt} V_{vu} B_{iw} B_{jv}]}{\partial \theta^i} (\tau, \bar{\sigma}) + \int d^3\sigma_1 \bar{n}(\tau, \bar{\sigma}_1) \frac{\delta \hat{H}(\tau, \bar{\sigma}_1)}{\delta \theta^i(\tau, \bar{\sigma})}, \quad (4.2) \]

where Eq.(B21), (B11), (B19) and (B5) have to be used. With Eq.(2.18) we can replace \( \pi_\delta^{(\theta)} \) with \( \sigma_{(\theta)(\theta)}|_{a \neq b} \).

### B. The Equations of Motion for the Gauge Variables

The equation of motion for the lapse and shift functions identify their \( \tau \)-derivatives with the arbitrary Dirac multipliers

\[ \partial_\tau n(\tau, \bar{\sigma}) \overset{\circ}{=} \lambda_n(\tau, \bar{\sigma}), \quad \partial_\tau \bar{n}(\tau, \bar{\sigma}) \overset{\circ}{=} \lambda_{\bar{n}}(\tau, \bar{\sigma}). \quad (4.3) \]

For \( \pi_\delta \) the Hamilton-Dirac equations obtained by using the Dirac Hamiltonian (3.48) in the York canonical basis give the following form of the Raychaudhuri equation

\[ \partial_\tau \pi_\delta^{(\theta)}(\tau, \bar{\sigma}) \overset{\circ}{=} \{ \pi_\delta^{(\theta)}(\tau, \bar{\sigma}), H_D \} = -\frac{\delta H_D}{\delta \phi(\tau, \bar{\sigma})} = -\frac{1}{6} \phi^{-5}(\tau, \bar{\sigma}) \frac{\delta H_D}{\delta \phi(\tau, \bar{\sigma})} = \]

\[ \begin{aligned}
&= -\frac{1}{6} \bar{\phi}^{-5/6}(\tau, \bar{\sigma}) \int d^3\sigma_1 \left[ (1 + n)(\tau, \bar{\sigma}) \left( \frac{\delta \hat{M}(\tau, \bar{\sigma}_1)}{\delta \phi(\tau, \bar{\sigma})} - \frac{c^3}{16\pi G} \frac{\delta S(\tau, \bar{\sigma}_1)}{\delta \phi(\tau, \bar{\sigma})} \right) - \frac{c^3}{16\pi G} n(\tau, \bar{\sigma}_1) \frac{\delta T(\tau, \bar{\sigma}_1)}{\delta \phi(\tau, \bar{\sigma})} \right] + \\
&\quad + \frac{2\pi G}{c^3} (1 + n)(\tau, \bar{\sigma}) \left[ 3 \pi_\phi^2 + 2 \bar{\phi}^{-2} \left( \sum_b \Pi_b^2 + \right. \right. \right. \\
&\quad \left. \left. \left. + \sum_{abtwuv} \epsilon_{abt} \epsilon_{abu} V_{wt} V_{vu} \right) \frac{1}{2} \sum_i B_{iw} \pi_i^{(\theta)} \sum_j B_{jv} \pi_j^{(\theta)} \right] (\tau, \bar{\sigma}) + \\
&\quad + \left( \bar{\phi}^{-3/2} \sum_{ra} \bar{n}(\tau, \bar{\sigma}) Q_a^{-1} V_{ra} \partial_\tau \pi_\delta \right)(\tau, \bar{\sigma}), \quad (4.4) \end{aligned} \]
where Eq.(B3) has been used and Eqs.(B22), (B10) and (B17) are needed. With Eq.(2.18) we can replace \( \pi_i^{(\theta)} \) with \( \sigma_{(a)(b)}|_{a \neq b} \).

For \( \theta^i \) the Hamilton-Dirac equations obtained by using the Dirac Hamiltonian (3.48) in the York canonical basis are

\[
\partial_r \theta^i(\tau, \vec{\sigma}) = \left\{ \theta^i(\tau, \vec{\sigma}), H_D \right\} = \frac{\delta H_D}{\delta \pi_i^{(\theta)}(\tau, \vec{\sigma})} =
\]

\[
= \left[ \frac{8 \pi G}{c^3} (1 + n) \right] \tilde{\phi}^{-1} \sum_{abtwuj} \epsilon_{abt} \epsilon_{bau} V_{wt} V_{vu} B_{iu} B_{ju} \pi_j^{(\theta)} - \]

\[- \tilde{\phi}^{-1/3} (\tau, \vec{\sigma}) \sum_a \left[ \bar{n}_a(\tau, \vec{\sigma}) \left( \sum_{b \neq a} \sum_{rtw} \left[ \frac{1}{3} \tilde{\phi}^{-1} \partial_r \tilde{\phi} + \sum_c \gamma_{(a)} \partial_r R_c \right] \epsilon_{bct} \epsilon_{bau} V_{rtw} B_{iu} B_{ju} \right] + \]

\[+ \sum_{ba} \sum_{c \neq b} \epsilon_{bct} Q_a^{-1} Q_b^{-1} \left( Q_c^{-1} - Q_a^{-1} - Q_b^{-1} \right) (V_{ra} V_{uc} - V_{ra} V_{uc}) \partial_r V_{ub} V_{wt} B_{iw} \right) - \]

\[- \sum_r \partial_r \bar{n}_a(\tau, \vec{\sigma}) \sum_{b \neq a} \sum_{tuv} \epsilon_{abt} Q_a^{-1} Q_b^{-1} \left( Q_a^{-1} Q_b^{-1} - Q_a^{-1} Q_b^{-1} \right) (\tau, \vec{\sigma}), \quad (4.5)\]

where Eq.(B4) has been used.

Then one has to add the gauge fixing constraints (satisfying the orbit condition) to the super-hamiltonian and super-momentum constraints

\[
\chi(\tau, \vec{\sigma}) \approx 0,
\]

\[\text{(clock synchronization convention or determination of } \pi_{\tilde{\phi}} \text{ and of the instantaneous 3 – space)}\]

\[
\chi_r(\tau, \vec{\sigma}) \approx 0,
\]

\[\text{(determination of } \theta^i \text{ and of the 3 – coordinates).} \quad (4.6)\]

Their preservation in time, by using the Dirac Hamiltonian (3.48), generates the equations for the lapse and shift functions consistently with the clock synchronization convention and with the 3-coordinates (the partial \( \tau \)-derivatives act on the possible explicit \( \tau \)-dependence of the gauge fixings)
\[
\partial_\tau \chi(\tau, \vec{\sigma}) \overset{\circ}{=} \frac{\partial \chi(\tau, \vec{\sigma})}{\partial \tau} + \{\chi(\tau, \vec{\sigma}), H_D\} = 0, \\
\text{(determination of } n),
\]

\[
\partial_\tau \chi_r(\tau, \vec{\sigma}) \overset{\circ}{=} \frac{\partial \chi_r(\tau, \vec{\sigma})}{\partial \tau} + \{\chi_r(\tau, \vec{\sigma}), H_D\} = 0, \\
\text{(determination of } \bar{n}_{(a)}),
\]

and as a consequence the Dirac multipliers are determined: \(\lambda_n(\tau, \vec{\sigma}) \overset{\circ}{=} \partial_\tau n(\tau, \vec{\sigma})\), \(\lambda_{\bar{n}_{(a)}}(\tau, \vec{\sigma}) \overset{\circ}{=} \partial_\tau \bar{n}_{(a)}(\tau, \vec{\sigma})\).

For gauge fixings of the type \(\theta^i(\tau, \vec{\sigma}) \approx (\text{numerical function})^i\) and \(\pi_{\bar{\phi}}(\tau, \vec{\sigma}) \approx \text{numerical function},\) Eqs. (4.7) are just Eqs.(4.4) and (4.5), respectively.

**C. The Equations of Motion for the Tidal Variables and the Matter**

Let us now consider the Hamilton-Dirac equations of motion implied by the Dirac Hamiltonian (3.48) for the tidal degrees of freedom and for the particles.

**1. The Hamilton Equations for the Tidal Variables**

For the tidal variables \(R_a\) we get the following kinematical Hamilton equations

\[
\partial_\tau R_a(\tau, \vec{\sigma}) \overset{\circ}{=} \{R_a(\tau, \vec{\sigma}), H_D\} = \frac{\delta H_D}{\delta \Pi_a(\tau, \vec{\sigma})} = \\
= \left[\frac{8\pi G}{c^3} \tilde{\phi}^{-1} (1 + n) \Pi_a - \tilde{\phi}^{-1/3} \sum_{ra} Q_{a}^{-1}(\bar{n}_{(a)} \left[ \gamma_{\dot{a}a} V_{\dot{r}a} \left( \frac{1}{3} \phi^{-1} \partial_{\dot{r}} \phi + \sum_b \gamma_{ba} \partial_{\dot{r}} R_b \right) - V_{\dot{r}a} \partial_{\dot{r}} R_a + \sum_{sb} \gamma_{ab} V_{\dot{s}a} V_{\dot{r}b} \partial_{\dot{r}} V_{\dot{sb}} \right] - \gamma_{\dot{a}a} \partial_{\dot{r}} \bar{n}_{(a)} V_{\dot{r}a}\right] \right)(\tau, \vec{\sigma}),
\]

where Eq.(B6) has been used.

Eq.(4.8) can be inverted to get the momenta \(\Pi_a\) in terms of the velocities \(\partial_\tau R_a\)
\[ \Pi_\alpha(\tau, \bar{\sigma}) \overset{\circ}{=} \frac{c^3}{8\pi G} \frac{\tilde{\phi}(\tau, \bar{\sigma})}{1 + n(\tau, \bar{\sigma})} \left[ \partial_\tau R_\alpha + \right. \\
+ \tilde{\phi}^{-1/3} \sum_{ra} Q_a^{-1} \left( \tilde{n}_{(a)} V_{ra} \left( \frac{1}{3} \tilde{\phi}^{-1} \partial_r \tilde{\phi} + \sum_b \gamma_{ba} \partial_r R_b \right) - \right. \\
- V_{ra} \partial_r R_\alpha + \sum_{ab} \gamma_{ab} V_{sa} V_{rb} \partial_{r_a} V_{ab} \left. - \gamma_{a\alpha} \partial_r \tilde{n}_{(a)}(\tau, \bar{\sigma}) \right] (\tau, \bar{\sigma}). \]

(4.9)

The dynamical Hamilton equations for \( \Pi_\alpha \) are

\[ \partial_\tau \Pi_\alpha(\tau, \bar{\sigma}) = \{ \Pi_\alpha(\tau, \bar{\sigma}), H_D \} = -\frac{\delta H_D}{\delta R_\alpha(\tau, \bar{\sigma})} = \\
= - \int d^3\sigma_1 \left[ (1 + n)(\tau, \bar{\sigma}_1) \left( \frac{\delta M(\tau, \bar{\sigma}_1)}{\delta R_\alpha(\tau, \bar{\sigma})} - \frac{c^3}{16\pi G} \frac{\delta S(\tau, \bar{\sigma}_1)}{\delta R_\alpha(\tau, \bar{\sigma})} \right) - \\
- \frac{c^3}{16\pi G} n(\tau, \bar{\sigma}_1) \frac{\delta T(\tau, \bar{\sigma}_1)}{\delta R_\alpha(\tau, \bar{\sigma})} \right] - \\
- \frac{8\pi G}{c^3} \left( \tilde{\phi}^{-1} (1 + n) \sum_{ab \neq b} \gamma_{a\alpha} - \gamma_{ab} \right) \sum_{a b t \nu w u j} (\gamma_{a\alpha} - \gamma_{a\nu}) \epsilon_{abt} \epsilon_{abu} V_{wt} V_{nu} B_{iw} B_{jv} \left( Q_b Q_a^{-1} + Q_a Q_b^{-1} \right)^{-3} \pi_\alpha^{(\theta)} \pi_{a \nu}^{(\theta)}(\tau, \bar{\sigma}) + \\
+ \int d^3\sigma_1 \tilde{n}_{(a)}(\tau, \bar{\sigma}_1) \frac{\delta \tilde{H}_{(a)}(\tau, \bar{\sigma}_1)}{\delta R_\alpha(\tau, \bar{\sigma})}, \]  

(4.10)

where Eqs.(B23), (B9), (B15) and (B7) have to be used. With Eq.(2.18) we can replace \( \pi_\alpha^{(\theta)} \) with \( \sigma_{(a)(b)}|_{a \neq b} \).

If we evaluate the \( \tau \)-derivative of Eq.(4.9) and we equate it to Eq.(4.10), we get the following second order equation for \( R_\alpha \)
\[ \partial^2 R_a(\tau, \sigma) = \left[ \phi^{-1/3} \sum_{ra} Q^{-1}_{\alpha} V_{ra} \bar{n}_{(a)} \sum_b (\gamma_{ab} \gamma_{ba} - \delta_{ab}) \partial_{r} \partial_{\tau} R_b + \right. \\
+ \phi^{-1/3} \sum_{ra} Q^{-1}_{\alpha} \left[ \left( \gamma_{ab} V_{ra} \left( \frac{1}{3} \phi^{-1} \partial_{\tau} \phi + \sum_{c} \gamma_{ca} \partial_{r} R_c \right) - V_{ra} \partial_{\tau} R_a + \right. \right. \\
+ \sum_{sb} \gamma_{ab} V_{sa} V_{rb} \partial_{r} V_{sb} \bar{n}_{(a)} - \gamma_{aa} V_{ra} \partial_{r} \bar{n}_{(a)} \right] \sum_b \gamma_{ba} \partial_{r} R_b - \\
- \phi^{-1} \left[ \partial_{\tau} R_a + \frac{2}{3} \phi^{-1/3} \sum_{ra} Q^{-1}_{\alpha} \left( \left[ \gamma_{aa} V_{ra} \left( -\frac{1}{6} \phi^{-1} \partial_{\tau} \phi + \sum_{b} \gamma_{ba} \partial_{r} R_b \right) - \\
- V_{ra} \partial_{\tau} R_a + \sum_{sb} \gamma_{ab} V_{sa} V_{rb} \partial_{r} V_{sb} \bar{n}_{(a)} - \gamma_{aa} V_{ra} \partial_{r} \bar{n}_{(a)} \right) \right] \partial_{r} \phi - \\
- \frac{1}{3} \phi^{-4/3} \sum_{ra} \gamma_{aa} Q^{-1}_{\alpha} V_{ra} \bar{n}_{(a)} \partial_{r} \partial_{\tau} \phi - \\
\left. \right] \partial_{r} \phi \right) \\
- \phi^{-1/3} \sum_{ra} Q^{-1}_{\alpha} \left[ - \partial V_{ra} \frac{\partial \bar{n}_{(a)}}{\partial \theta} \left( \bar{n}_{(a)} \partial_{\tau} R_a - \gamma_{aa} \left( \frac{1}{3} \phi^{-1} \partial_{\tau} \phi + \\
+ \sum_{b} \gamma_{bb} \partial_{r} R_b \right) \right) + \gamma_{aa} \partial_{r} \bar{n}_{(a)} \right] + \sum_{sb} \gamma_{ab} \partial_{r} \bar{n}_{(a)} \right) + \sum_{sb} \gamma_{ab} \partial_{r} V_{sa} V_{rb} \partial_{r} V_{sb} \bar{n}_{(a)} \}
\sum_r \partial_{r} \theta^i + \\
+ \left[ \partial_{\tau} R_a + \phi^{-1/3} \sum_{ra} Q^{-1}_{\alpha} \left( \left[ \gamma_{aa} V_{ra} \left( \frac{1}{3} \phi^{-1} \partial_{\tau} \phi + \sum_{b} \gamma_{ba} \partial_{r} R_b \right) - \\
- V_{ra} \partial_{\tau} R_a + \sum_{sb} \gamma_{ab} V_{sa} V_{rb} \partial_{r} V_{sb} \bar{n}_{(a)} - \gamma_{aa} V_{ra} \partial_{r} \bar{n}_{(a)} \right) \right] \frac{\partial_{r} n}{1 + n} - \\
- \phi^{-1/3} \sum_{ra} Q^{-1}_{\alpha} \left( \left[ \gamma_{aa} V_{ra} \left( \frac{1}{3} \phi^{-1} \partial_{\tau} \phi + \sum_{b} \gamma_{ba} \partial_{r} R_b \right) - V_{ra} \partial_{\tau} R_a + \\
+ \sum_{sb} \gamma_{ab} V_{sa} V_{rb} \partial_{r} V_{sb} \right] \partial_{r} \bar{n}_{(a)} - \gamma_{aa} V_{ra} \partial_{r} \bar{n}_{(a)} \right) \right] (\tau, \sigma) + \\
+ \frac{1}{2} \left( \phi^{-1} (1 + n) \right) (\tau, \sigma) \int d^3 \sigma_1 \left[ (1 + n)(\tau, \sigma_1) \frac{\delta S(\tau, \sigma_1)}{\delta R_\alpha(\tau, \sigma)} + n(\tau, \sigma_1) \frac{\delta T(\tau, \sigma_1)}{\delta R_\alpha(\tau, \sigma)} \right] - \\
- \frac{8 \pi G}{c^3} \left( \phi^{-1} (1 + n) \right) (\tau, \sigma) \int d^3 \sigma_1 (1 + n)(\tau, \sigma_1) \frac{\delta M(\tau, \sigma_1)}{\delta R_\alpha(\tau, \sigma)} + \\
+ \frac{8 \pi G}{c^3} \left( \phi^{-1} (1 + n) \right) (\tau, \sigma) \int d^3 \sigma_1 \bar{n}_{(a)}(\tau, \sigma_1) \frac{\delta \mathcal{H}(\tau, \sigma_1)}{\delta R_\alpha(\tau, \sigma)} - \\
- \left( \frac{8 \pi G}{c^3} \right)^2 \left( \phi^{-2} (1 + n)^2 \right) (\tau, \sigma) \\
\left( \sum_{abtwuvij} (\gamma_{aa} - \gamma_{ab}) \frac{\epsilon_{abt} \epsilon_{uvi} V_{wu} V_{vi} B_{tw} B_{vj} (Q_{b} Q_{a}^{-1} + Q_{a} Q_{b}^{-1})}{(Q_{b} Q_{a}^{-1} - Q_{a} Q_{b}^{-1})^2} \pi_i^{(\theta)} \pi_j^{(\theta)} \right) (\tau, \sigma). \tag{4.11} \right] \]
This equation depends:
a) on the $\tau$-derivatives $\partial_{\tau} \phi(\tau, \vec{\sigma})$ and $\partial_{\tau} \theta^i(\tau, \vec{\sigma})$, given in Eqs.(4.1) and (4.5), respectively;
b) on the $\tau$-derivatives of the lapse and shift functions, namely on the arbitrary Dirac multipliers appearing in Eqs.(4.3).

2. The Hamilton Equations for the Particles

By using Eqs.(3.15), the Hamilton equations (3.38) for the particles in the electromagnetic radiation gauge take the following form in the York canonical basis (Eq.(3.35) and (3.37) are used for $W$)

$$
\begin{align*}
\eta_i \dot{\eta}_i^r(\tau) & \overset{\circ}{=} \{\eta_i^r(\tau), H_D\} = \\
& = \int d^3\sigma \left[ 1 + n(\tau, \vec{\sigma}) \right] \frac{\partial \mathcal{M}(\tau, \vec{\sigma})}{\partial \kappa_{ir}} - \\
& - \sum_a \tilde{n}_a(\tau, \vec{\sigma}) \left[ \phi^{-2} \sum_v Q_v^{-1} V_v a \right] (\tau, \vec{\sigma}) \frac{\partial \mathcal{M}_v(\tau, \vec{\sigma})}{\partial \kappa_{ir}} = \\
& = \eta_i \left( \frac{\phi^{-4} (1 + n) \sum_a Q_a^{-2} V_r V_a \left( \tilde{k}_i s(\tau) - \frac{Q_i}{c} A_{\perp s} \right)}{\sqrt{m_i c^2 + \phi^{-4} \sum_{cv} Q_c^{-2} V_c V_v \left( \tilde{k}_i u(\tau) - \frac{Q_i}{c} A_{\perp u} \right) \left( \tilde{k}_i u(\tau) - \frac{Q_i}{c} A_{\perp v} \right)}} - \\
& - \phi^{-2} \sum_a Q_a^{-1} V_r \tilde{n}_a(\tau, \vec{\sigma}) \right) (\tau, \vec{\eta}_i(\tau)),
\end{align*}
$$

$$
\begin{align*}
\eta_i \frac{d}{d\tau} \tilde{\kappa}_{ir}(\tau) & \overset{\circ}{=} \{\tilde{\kappa}_{ir}(\tau), H_D\} = \\
& = - \int d^3\sigma \left[ 1 + n(\tau, \vec{\sigma}) \right] \frac{\partial \mathcal{M}(\tau, \vec{\sigma})}{\partial \eta_i^r} - \\
& - \sum_a \tilde{n}_a(\tau, \vec{\sigma}) \left[ \phi^{-2} \sum_v Q_v^{-1} V_v a \right] (\tau, \vec{\sigma}) \frac{\partial \mathcal{M}_v(\tau, \vec{\sigma})}{\partial \eta_i^r} = \\
& = - \frac{\partial}{\partial \eta_i^r} \int d^3\sigma \left[ (1 + n) \frac{\partial W_{(n)}}{\partial \eta_i^r} + \\
& + \phi^{-2} \sum_a Q_a^{-1} \tilde{n}_a(\tau, \vec{\sigma}) \sum_v V_v a \frac{\partial W_v}{\partial \eta_i^r} \right] (\tau, \vec{\sigma}) + \\
& + \eta \sum_{asv} \left[ \frac{\phi^{-4} (1 + n) Q_a^{-2} V_s V_a \left( \tilde{k}_i s(\tau) - \frac{Q_i}{c} A_{\perp s} \right)}{\sqrt{m_i c^2 + \phi^{-4} \sum_{cv} Q_c^{-2} V_c V_v \left( \tilde{k}_i u(\tau) - \frac{Q_i}{c} A_{\perp u} \right) \left( \tilde{k}_i u(\tau) - \frac{Q_i}{c} A_{\perp v} \right)}} - \\
& - \phi^{-2} Q_a^{-2} V_s \tilde{n}_a(\tau, \vec{\sigma}) \right] \left( \frac{Q_i}{c} \frac{\partial A_{\perp s}}{\partial \eta_i^r} \right) (\tau, \vec{\eta}_i(\tau)) + \\
& + \eta_i \tilde{F}_{ir}(\tau, \vec{\eta}_i(\tau)),
\end{align*}
$$

55
\[ F_{ir}(\tau, \tilde{\eta}_i(\tau)) = \left( \sqrt{m_i^2 c^2 + \phi^{-4} \sum_{cmm} Q_c^{-2} V_{mc} V_{nb} \left( \bar{k}_{im}(\tau) - \frac{Q_i}{c} A_{\perp m} \right) \left( \bar{k}_{in}(\tau) - \frac{Q_i}{c} A_{\perp n} \right)} \right. \]

\[ \left. \left[ - \frac{\partial n}{\partial \eta_i} + \phi^{-2} \sum_a Q_a^{-1} V_{sa} \frac{\partial \tilde{n}(a)}{\partial \eta_i} \times \left( \bar{k}_{is}(\tau) - \frac{Q_i}{c} A_{\perp s} \right) \right. \right. \]

\[ \left. \left. \left( \sqrt{m_i^2 c^2 + \phi^{-4} \sum_{cmm} Q_c^{-2} V_{mc} V_{nb} \left( \bar{k}_{im}(\tau) - \frac{Q_i}{c} A_{\perp m} \right) \left( \bar{k}_{in}(\tau) - \frac{Q_i}{c} A_{\perp n} \right)} \right) \right)^2 \right] - \phi^{-2} \sum_{as} Q_a^{-1} \tilde{n}(a) \left( V_{sa} \left( \frac{1}{3} \bar{\phi}^{-1} \partial_r \tilde{\phi} + \sum_b \gamma_{bb} \partial_r R_b \right) - \partial_r V_{sa} \right) \]

\[ \left. \left. \left( \bar{k}_{is}(\tau) - \frac{Q_i}{c} A_{\perp s} \right) \right. \right. \]

\[ \left. \left. \left. \left( \sqrt{m_i^2 c^2 + \phi^{-4} \sum_{cmm} Q_c^{-2} V_{mc} V_{nb} \left( \bar{k}_{im}(\tau) - \frac{Q_i}{c} A_{\perp m} \right) \left( \bar{k}_{in}(\tau) - \frac{Q_i}{c} A_{\perp n} \right)} \right) \right)^2 \right] \]

\[ \left. \left. \left. \left( \bar{k}_{is}(\tau) - \frac{Q_i}{c} A_{\perp s} \right) \right. \right. \]

\[ \left. \left. \left. \left( \sqrt{m_i^2 c^2 + \phi^{-4} \sum_{cmm} Q_c^{-2} V_{mc} V_{nb} \left( \bar{k}_{im}(\tau) - \frac{Q_i}{c} A_{\perp m} \right) \left( \bar{k}_{in}(\tau) - \frac{Q_i}{c} A_{\perp n} \right)} \right) \right)^2 \right] \]

\[ \left( \tau, \tilde{\eta}_i(\tau) \right). \]

(4.12)

By inverting the first of Eqs.(4.12) we get the following form of Eq.(3.39)

\[ \bar{k}_{ir}(\tau) = \frac{Q_i}{c} A_{\perp r}(\tau, \tilde{\eta}_i(\tau)) + \]

\[ + m_i c \left( \tilde{\phi}^{2/3} \sum_{sa} Q_s^a V_{sa} \left( \tilde{\eta}_i^s(\tau) + \tilde{\phi}^{-1/3} \sum_b Q_b^{-1} V_{sb} \tilde{n}(b) \right) \right] \left[ \left( 1 + n \right)^2 - \right. \]

\[ - \tilde{\phi}^{2/3} \sum_{uv} Q_u^c V_{uc} V_{ve} \left( \tilde{\eta}_i^u(\tau) + \tilde{\phi}^{-1/3} \sum_d Q_d^{-1} V_{ud} \tilde{n}(d) \right) \left( \tilde{\eta}_i^v(\tau) + \tilde{\phi}^{-1/3} \sum_e Q_e^{-1} V_{ve} \tilde{n}(e) \right) \right]^{-1/2} \]

\[ (\tau, \tilde{\eta}_i(\tau)). \]

(4.13)

If we put Eq.(4.13) into the second of Eqs.(4.12), we get the second order equation for \( \eta_i^s(\tau) \) (corresponding to Eq.(3.17))
\( \eta_i \frac{d \kappa_{iir}}{d \tau} = \left( -\frac{\partial}{\partial \eta_i^r} W + \frac{\eta_i Q_i}{c} \dot{\eta}_i^r(\tau) \frac{\partial A_{iis}}{\partial \eta_i^r} + \eta_i \ddot{F}_{iir} \right)(\tau, \bar{\eta}_i(\tau)), \)

\( W(\tau) = \int d^3 \sigma \left[ (1 + n) W_{(n)} + \phi^{-2} \sum_a Q_a^{-1} \bar{\eta}_a(\tau) \sum_v V_v W_v \right](\tau, \bar{\sigma}), \)

\( W_{(n)}(\tau, \bar{\sigma}) = -\frac{1}{2c} \left[ \phi^{-2} \sum_{ar} Q_a^2 V_{ra} V_{sa} \left( 2 \pi_{\perp}^r - \delta^r_m \sum_i Q_i \eta_i \frac{\partial c(\bar{\sigma}, \bar{\eta}_i(\tau))}{\partial \sigma^m} \right) + \delta^m_n \sum_j Q_j \eta_j \frac{\partial c(\bar{\sigma}, \bar{\eta}_j(\tau))}{\partial \sigma^m} \right](\tau, \bar{\sigma}), \)

\( W_r(\tau, \bar{\sigma}) = -\frac{1}{c} F_{rs}(\tau, \bar{\sigma}) \delta^m_n \sum_i Q_i \eta_i \frac{\partial c(\bar{\sigma}, \bar{\eta}_i(\tau))}{\partial \sigma^m}, \)

\( \ddot{F}_{iir} = m_i c \left[ \left( 1 + n \right)^2 - \bar{\phi}^{2/3} \sum_{uc} Q_c^2 V_{uc} V_{vc} \left( \dot{\bar{\eta}}_i^u(\tau) + \bar{\phi}^{-1/3} \sum_d Q_d^{-1} V_{ud} \bar{n}(d) \right) \right] \left( \bar{\dot{\eta}}_i^u(\tau) + \bar{\phi}^{-1/3} \sum_e Q_e^{-1} V_{ve} \bar{n}(e) \right)^{-1/2} \)

\( - \left[ (1 + n) \frac{\partial n}{\partial \eta_i^r} + \phi^2 \sum_{acsu} Q_a^{-1} Q_c \left( V_{sa} \frac{\partial \bar{n}(a)}{\partial \eta_i^r} - \partial_r V_{sa} \partial_r \bar{n}(a) \right) \right] V_{ac} V_{sc} \left( \dot{\bar{\eta}}_i^u(\tau) + \bar{\phi}^{-2} \sum_b Q_b^{-1} V_{ub} \bar{n}(b) \right) + \)

\( \phi^4 \sum_{auv} Q_a^2 V_{ua} \left( (1 + n) \frac{\partial n}{\partial \eta_i^r} + \bar{\phi}^{-1} \partial_r \bar{\phi} + \sum_a \gamma_{aa} \partial_r R_{a} V_{sa} - \partial_r V_{sa} \right) \bar{n}(a) \right) V_{ac} V_{sc} \left( \dot{\bar{\eta}}_i^u(\tau) + \bar{\phi}^{-2} \sum_b Q_b^{-1} V_{ub} \bar{n}(b) \right) + \)

\( \phi^4 \sum_{auv} Q_a^2 V_{ua} \left( (1 + n) \frac{\partial n}{\partial \eta_i^r} + \bar{\phi}^{-1} \partial_r \bar{\phi} + \sum_a \gamma_{aa} \partial_r R_{a} V_{sa} - \partial_r V_{sa} \right) \bar{n}(a) \right) V_{ac} V_{sc} \left( \dot{\bar{\eta}}_i^u(\tau) + \bar{\phi}^{-2} \sum_b Q_b^{-1} V_{ub} \bar{n}(b) \right) \right]. \)

(4.14)

3. The Hamilton Equations for the Transverse Electro-Magnetic Field

Eqs.(3.40) for the transverse electro-magnetic fields \( A_{\perp r}(\tau, \bar{\sigma}) \) and \( \pi^r_{\perp}(\tau, \bar{\sigma}) \) in the radiation gauge take the following form in the York canonical basis
\[ \partial_r A_{\perp}(\tau, \bar{\sigma}) \triangleq \{ A_{\perp}(\tau, \bar{\sigma}), H_D \} = \]
\[ = \sum_{n_{\perp}a} \delta_{rn} P_{\perp}^{nu}(\bar{\sigma}) \left[ \phi^{-1/3} (1 + n) Q_a^2 V_{u\nu} \left( \pi^v - \sum_m \delta_{vm} \sum_i Q_i \eta_i \frac{\partial c(\bar{\sigma}, \bar{\eta}_i(\tau))}{\partial \sigma_m} \right) + \right. \\
\[ + \left. \phi^{-1/3} Q_a^{-1} V_{ua} \bar{r}_{(a)} F_vu \right] (\tau, \bar{\sigma}), \]
\[ \partial_r \pi^r(\tau, \bar{\sigma}) \triangleq \{ \pi^r(\tau, \bar{\sigma}), H_D \} = \]
\[ = \sum_{w_{\perp}a} P_{\perp}^{uw}(\bar{\sigma}) \delta_{wm} \left( \sum_i \eta_i Q_i \delta^3(\bar{\sigma}, \bar{\eta}_i(\tau)) \right) \]
\[ \left[ \frac{\phi^{-2/3} (1 + n) Q_a^{-2} V_{ma} \sum_i V_{sa} \bar{\kappa}_{iu}(\tau)}{\sqrt{m_i^2 c^2 + |\phi^{-2/3} \sum_{u_{\perp}b} Q_b^{-2} V_{ub} V_{vb} (\bar{\kappa}_{iu}(\tau) - Q_i A_{\perp u}) (\bar{\kappa}_{iu}(\tau) - Q_i A_{\perp v})|}} - \right. \\
\[ - \left. \phi^{-1/3} Q_a^{-1} V_{ma} \bar{n}_{(a)}(\tau), \bar{\eta}_i(\tau) \right] + \]
\[ + \left[ (1 + n) \sum_{sv_{\perp}b} \left( \phi^{-1/3} Q_a^{-2} Q_b^{-2} V_{sa} V_{vb} (V_{na} V_{mb} - V_{nb} V_{ma}) \partial_n F_{sv} + \right. \\ \[ + \left. \phi^{-1/3} Q_a^{-2} Q_b^{-2} \left[ \partial_n \left( V_{sa} V_{vb} (V_{na} V_{mb} - V_{nb} V_{ma}) \right) - \right. \\
\[ - V_{sa} V_{vb} (V_{na} V_{mb} - V_{nb} V_{ma}) \left( \frac{1}{3} \phi^{-1} \partial_n \phi + 2 \sum_b (\gamma_{ba} + \gamma_{bb}) \partial_n R_b \right) \right] F_{sv} \right) + \right. \\
\[ + \left. \phi^{-1/3} \sum_{sv_{\perp}b} \partial_n n Q_a^{-2} Q_b^{-2} V_{sa} V_{vb} (V_{na} V_{mb} - V_{nb} V_{ma}) F_{sv} + \right. \\
\[ + \left. \phi^{-1/3} n_{(a)} Q_a^{-1} \sum_n \left( V_{na} \partial_n \pi^m_{\perp} + \right. \\
\[ + \left[ \partial_n V_{na} - V_{na} \left( \frac{1}{3} \phi^{-1} \partial_n \phi + \sum_b \gamma_{ba} \partial_n R_b \right) \right] \pi^m_{\perp} - \right. \\
\[ - \left[ \partial_n V_{ma} - V_{ma} \left( \frac{1}{3} \phi^{-1} \partial_n \phi + \sum_b \gamma_{ba} \partial_n R_b \right) \right] \pi^m_{\perp} - \right. \\
\[ - \sum_t \left( \left[ \partial_n V_{na} - V_{na} \left( \frac{1}{3} \phi^{-1} \partial_n \phi + \sum_b \gamma_{ba} \partial_n R_b \right) \right] \delta^{mt} - \right. \\
\[ - \left[ \partial_n V_{ma} - V_{ma} \left( \frac{1}{3} \phi^{-1} \partial_n \phi + \sum_b \gamma_{ba} \partial_n R_b \right) \right] \delta^{nt} \right) \sum_i Q_i \frac{\partial c(\bar{\sigma}, \bar{\eta}_i(\tau))}{\partial \sigma^t} - \right. \\
\[ - \sum_t (V_{na} \delta^{mt} - V_{ma} \delta^{nt}) \sum_i Q_i \frac{\partial^2 c(\bar{\sigma}, \bar{\eta}_i(\tau))}{\partial \sigma^t \partial \sigma^n} - \right. \\
\[ - \left. \phi^{-1/3} \sum_n \partial_n n_{(a)} Q_a^{-1} \sum_t (V_{na} \delta^{mt} - V_{ma} \delta^{nt}) \right) \sum_i Q_i \frac{\partial c(\bar{\sigma}, \bar{\eta}_i(\tau))}{\partial \sigma^t} \right) (\tau, \bar{\sigma}). \]

(4.15)
4. The Constraints to be added to the Hamilton Equations

To the previous Hamilton equations we have to add the super-Hamiltonian constraint (3.44) [or (3.45)] for the determination of \( \tilde{\phi}(\tau, \vec{\sigma}) \) (the Lichnerowicz equation) and the super-momentum constraints (3.41) [or (3.42)] for the determination of the momenta \( \pi_{i}(\theta)(\tau, \vec{\sigma}) \) [or \( \tilde{\phi}(\tau, \vec{\sigma}) \sigma_{(a)(b)}(\tau, \vec{\sigma}) \) of Eq.(2.18)] or their explicit expression given in Ref.[6].

Also the six rest-frame conditions contained in Eqs.(3.47) have to be added.
V. GAUGES IN ADM CANONICAL GRAVITY IN THE YORK CANONICAL BASIS

Let us consider some gauge fixings for the secondary super-hamiltonian and super-momentum first class constraints determining the gauge variables $\pi_{\tilde{\phi}}(\tau, \vec{\sigma})$ and $\theta^n(\tau, \vec{\sigma})$ inside the family of Schwinger time gauges.

As shown in Eqs.(4.7), their preservation in time by using $H_D$ will generate four secondary gauge fixings determining the lapse and shift functions consistently with the chosen definition of instantaneous 3-space (clock synchronization convention) and of 3-coordinates in it.

A. ADM 4-Coordinate Gauges

These are CMC (constant mean curvature) gauges with $3K(\tau, \vec{\sigma}) \approx 0$ and with the following gauge fixings for the 3-coordinates

$$\pi_{\tilde{\phi}}(\tau, \vec{\sigma}) \approx 0,$$

$$\chi_r(\tau, \vec{\sigma}) = \left[ \sum_s \partial_s 3g_{rs} - \frac{1}{3} \partial_r \sum_s 3g_{ss} \right](\tau, \vec{\sigma}) =$$

$$= \left[ \tilde{\phi}^{2/3} \sum_a Q_a^2 \left( \sum_s \partial_s (V_{ra} V_{sa}) + 2 \sum_s \left( \frac{1}{3} \tilde{\phi}^{-1} \partial_s \tilde{\phi} + \sum_b \gamma_{ba} \partial_s R_b \right) V_{ra} V_{sa} - \frac{2}{3} \left( \frac{1}{3} \tilde{\phi}^{-1} \partial_r \tilde{\phi} + \sum_b \gamma_{ba} \partial_r R_b \right) \right) \right](\tau, \vec{\sigma}) \approx 0. \quad (5.1)$$

These are 3 equations of the form $\sum_{as} F_{1a}(\tau, \vec{\sigma}) \partial_s T_{(rs)a}(\theta^n(\tau, \vec{\sigma})) + \sum_{as} F_{2as}(\tau, \vec{\sigma}) T_{(rs)a}(\theta^n(\tau, \vec{\sigma})) + F_3(\tau, \vec{\sigma}) = 0$ with $T_{(rs)a} = V_{ra}(\theta^n) V_{sa}(\theta^n) = T_{(sr)a}$

Once $T_{(rs)a}(\tau, \vec{\sigma})$ has been found, one has to find the compatible Euler angles $\theta^n(\tau, \vec{\sigma})$ of the rotation matrix $V(\theta^n)$.

A similar gauge with $3K(\tau, \vec{\sigma}) \approx 0$ and with $\chi_r(\tau, \vec{\sigma}) \approx 0$ replaced by $3g_{rs}(\tau, \vec{\sigma}) \approx [(1 + \frac{1}{8} \varphi)^4 \delta_{rs} + h_{rs}^{TT}](\tau, \vec{\sigma})$ ($h_{rs}^{TT}$ is a transverse-traceless tensor) is used in Ref.[36].

B. The 3-Harmonic Gauges

These are gauges in which only the 3-coordinates are fixed but not the instantaneous 3-space
\[
\pi_{\tilde{\phi}}(\tau, \vec{\sigma}) \approx \text{not specified},
\]
\[
\sum_s \partial_s \left( 3 e^3 g^{rs} \right)(\tau, \vec{\sigma}) = \sum_s \partial_s \left( \tilde{\phi}^{1/3} \sum_a Q_a^{-2} V_{ra} V_{sa} \right)(\tau, \vec{\sigma}) =
\]
\[
= \left[ \tilde{\phi}^{1/3} \sum_{as} Q_a^{-2} \left( \partial_s (V_{ra} V_{sa}) + \frac{1}{3} \tilde{\phi}^{-1} \partial_s \tilde{\phi} - 2 \sum_b \gamma_{ba} \partial_s R_b V_{ra} V_{sa} \right) \right](\tau, \vec{\sigma}) \approx 0. \quad (5.2)
\]

Again we have a linear partial differential equation for \(T_{(rs)a} = V_{ra}(\theta^n) V_{sa}(\theta^n) = T_{(sr)a} \).

C. The 4-Harmonic Gauges

The 4-harmonic gauge, see Eq.(6.4) of Ref.[6], is defined by the four gauge fixings
\[
\chi^A(\tau, \vec{\sigma}) = \sum_B \partial_B \left( \sqrt{\left| g^{-1} g_{AB} \right|} \right)(\tau, \vec{\sigma}) \approx 0. \quad \text{By using Eqs.(2.6), (2.10), (4.1) (the kinematical Hamilton equation for } \tilde{\phi} \text{) and Eqs.(4.3) (the } \tau \text{-derivatives of the lapse and shift functions are Dirac multipliers) we get the following expression for } \chi^\tau(\tau, \vec{\sigma}) \approx 0 \text{ in the York canonical basis}
\]
\[
\chi^\tau = \partial_\tau \left( (1 + n)^3 e^4 g^{\tau\tau} \right) + \sum_s \partial_s \left( (1 + n)^3 e^4 g^{\tau s} \right) = \epsilon \left[ \partial_\tau \frac{3 e}{N} - \sum_s \partial_s \frac{3 e n^s}{N} \right] \approx 0,
\]
\[
\Downarrow
\]
\[
\frac{12 \pi G}{c^3} \pi_{\tilde{\phi}}(\tau, \vec{\sigma}) = 3 K(\tau, \vec{\sigma}) \approx \frac{1}{\left( 1 + n(\tau, \vec{\sigma}) \right)^2} \left( \lambda_n + \tilde{\phi}^{-1/3} \sum_{ra} Q_a^{-1} V_{ra} \partial_r n \right)(\tau, \vec{\sigma}), \quad (5.3)
\]

The other three gauge fixings are
\[\chi^r = \partial_r \left( (1 + n)^3 e^4 g^{rr} \right) + \sum_s \partial_s \left( (1 + n)^3 e^4 g^{rs} \right) = \]

\[= \epsilon \left[ - \partial_r \frac{3e n^r}{1+n} - \sum_s \partial_s \left( (1 + n)^3 e \left( g^{rs} - \frac{n^r n^s}{(1+n)^2} \right) \right) \right] = \]

\[= -\epsilon \left[ \partial_r \frac{\phi^6}{1+n} \sum_a 3 \tilde{e}_r^{(a)} \tilde{n}_a - \sum_s \partial_s \left( (1 + n) \phi^6 \sum_{ab} 3 \tilde{e}_r^{(a)} \tilde{e}_r^{(b)} \right) \right] \]

\[\left[ \delta_{ab} - \frac{\tilde{n}_a \tilde{n}_b}{(1+n)^2} \right] = \]

\[= \left( -\tilde{\phi}^{2/3} \sum_a Q^{-1}_a \left( V_{ra} \lambda_{\tilde{n}_a} + \tilde{n}_a \left[ \frac{\partial V_{ra}}{\partial \tilde{\theta}} \partial_r \tilde{\theta}^i + V_{ra} \frac{\lambda_n}{1+n} \right] \right) \right) - \]

\[- \frac{\tilde{\phi}^{1/3}}{(1+n)^2} \sum_{sab} Q^{-1}_a Q^{-1}_b \left( V_{ra} V_{sa} \left[ \left( 1+n \right)^2 \delta_{ab} + \tilde{n}_a \tilde{n}_b \right] \partial_s n - \right. \]

\[- \left. (1+n) \left( \tilde{n}_a (\partial_s \tilde{n}_b) + \tilde{n}_b (\partial_s \tilde{n}_a) \right) \right] + \]

\[- \left. (1+n) \left( (1+n)^2 \delta_{ab} - \tilde{n}_a \tilde{n}_b \right) \left[ \partial_s (V_{ra} V_{sb}) + V_{ra} V_{sb} \left( \frac{1}{3} \tilde{\phi}^{-1} \partial_s \tilde{\phi} - \sum \left( \gamma_{ba} + \gamma_{ab} \right) \partial_s R_b \right) \right] \right)(\tau, \bar{\sigma}) \approx 0. \]

(5.4)

The kinematical Hamilton equations (4.1), (4.5) and (4.8) are needed to get this final form.

These unconventional Hamiltonian constraints \(\chi^r \approx 0\) does not define a CMC gauge \(3K(\tau, \bar{\sigma}) = \text{const.}\) are four coupled equations for \(\pi_\phi\) and \(\theta^i\) in terms of \(\phi, R_a, \Pi_a, n, \lambda_n = \partial_r n, \tilde{n}_a, \lambda_{\tilde{n}_a} = \partial_r \tilde{n}_a\).

The stability of these gauge fixings requires to impose \(\partial_r \tilde{\chi}_a(\tau, \bar{\sigma}) \approx 0\) and \(\partial_r \tilde{\chi}_r(\tau, \bar{\sigma}) \approx 0\). In this way we get four equations for the determination of \(n\) and \(\tilde{n}_a\). But these are not equations of the "elliptic" type like with ordinary gauge fixings. They are coupled equations depending upon \(n, \partial_r n, \partial_s n, \partial_r^2 n\ and \tilde{n}_a, \partial_r \tilde{n}_a, \partial_r \tilde{n}_a, \partial_r^2 \tilde{n}_a\), namely hyperbolic equations. As a consequence there is a problem of initial conditions not only for \(R_a\) but also for the lapse and shift functions of the harmonic gauge. Each possible set of initial values should correspond to a different completely fixed harmonic gauge, since once we have a solution for \(n\) and \(\tilde{n}_a\) the corresponding Dirac multipliers are determined by taking their \(\tau\)-derivative.

Instead of reading these constraints as gauge fixings determining \(\pi_\phi(\tau, \bar{\sigma})\), Eq.(5.3), and \(\theta^i(\tau, \bar{\sigma})\), Eq.(5.4), let us solve them for the Dirac multipliers \(\lambda_n\) and \(\lambda_{\tilde{n}_a}\). Then it is more natural the following interpretation: the family of 4-harmonic gauges is determined by all the Hamiltonian gauge fixings of the type (4.6) for \(\pi_\phi(\tau, \bar{\sigma})\) and \(\theta^i(\tau, \bar{\sigma})\), with induced secondary gauge fixings (4.7) for the lapse and shift functions, such that the Dirac multipliers
\( \lambda_n(\tau, \vec{\sigma}) \equiv \partial_\tau n(\tau, \vec{\sigma}), \lambda_{\bar{n}(a)}(\tau, \vec{\sigma}) \equiv \partial_\tau \bar{n}(a)(\tau, \vec{\sigma}) \), have the form implied by Eqs. (5.3) and (5.4).

As a consequence, let us remark that in the family of harmonic gauges there is no natural way to visualize the instantaneous 3-spaces (they cannot be Euclidean due to the equivalence principle) and to check which properties have an inertial origin due to the freedom in the clock synchronization convention (the gauge freedom in the York time). Most of the applications (for instance the IAU conventions for the solar system [37]) are based on post-Newtonian expansions inside a fixed Euclidean 3-space of an inertial frame in Minkowski space-time containing Newton gravity as the zero order!

D. The Synchronous gauges

The synchronous gauges are defined by the gauge fixings

\[ \bar{n}(a)(\tau, \vec{\sigma}) \approx 0, \]

(5.5)

implying \( 4g_{\tau\tau}(\tau, \vec{\sigma}) \approx 0 \) (no gravito-magnetism) and \( 4g_{\tau\tau}(\tau, \vec{\sigma}) \approx \epsilon \left(1 + n(\tau, \vec{\sigma})\right)^2 \), \( \lambda_{\bar{n}(a)}(\tau, \vec{\sigma}) \approx 0 \).

These unconventional gauges have a residual gauge freedom in any fixation of the 3-coordinates implying \( \lambda_{\bar{n}(a)}(\tau, \vec{\sigma}) \approx 0 \). If we add the ”comoving” condition \( n(\tau, \vec{\sigma}) \approx 0 \), we are also restricting the freedom in the fixation of the York time.

These gauges are used in cosmology in presence of Killing vectors implying that the solution of Einstein’s equations are homogeneous and isotropic. However their use in absence of Killing symmetries is questionable, since gravito-magnetism cannot be eliminated in general.
VI. THE 3-ORTHOGONAL SCHWINGER TIME GAUGES.

Let us now consider the most natural family of Schwinger time gauges in the framework of the York canonical basis, i.e. the 3-orthogonal gauges

\[ \theta^i(\tau, \vec{\sigma}) \approx 0, \]  

(6.1)

in which the 3-metric is diagonal in each point of the instantaneous 3-space \( \Sigma_\tau \) and \( V_{ra}(\theta^n) = \delta_{ar} \). The residual gauge freedom is the selection of the instantaneous 3-spaces by fixing the gauge variable \( \pi_{\tilde{\phi}}(\tau, \vec{\sigma}) \).

Till now we have not make any hypothesis on the type of angles \( \theta^i(\tau, \vec{\sigma}) \) to be used to parametrize the matrices \( V_{ra}(\theta^i(\tau, \vec{\sigma})) \) and its derivatives

\[ V_{(i)ra} = \left. \frac{\partial V_{ra}(\theta^n)}{\partial \theta^i} \right|_{\theta^i=0}, \]

\[ B_{(i)jw} = \left. \frac{\partial B_{jw}(\theta^n)}{\partial \theta^i} \right|_{\theta^i=0}, \]

needed in the 3-orthogonal gauge.

It is convenient to parametrize the group manifold of the Lie group O(3) in terms of canonical coordinates of first kind (see Appendix A of Ref.[38] and references therein), because then we have:

A) \( \theta^n \) are first kind coordinates (not the standard Euler angles);
B) \( V(\theta^n) = e^{\sum_i \theta^n T_i} \), where \( (T_i)_{ab} = \epsilon_{iab} \) (\( T_i \) are the generators of the O(3) Lie algebra);
C) \( V_{(i)ab} = \epsilon_{iab}, \quad B_{(i)ab} = \frac{1}{2} \epsilon_{iab} \) with \( B = A^{-1}, \quad A_{(i)ab} = \frac{1}{2} \epsilon_{iab}. \)

In Appendix C there the restriction at the 3-orthogonal gauges of the quantities evaluated in Appendix B.

In the 3-orthogonal gauges Eqs.(2.18) become

\[ \Pi_{\tilde{a}} = \left. -\frac{c^3}{8\pi G} \tilde{\phi} \sum_a \gamma_{\tilde{a}a} \sigma_{(a)(a)} \right|_{\theta^i=0}, \]

\[ \pi_{i}^{(\theta)} |_{\theta^i=0} = \left. \frac{c^3}{8\pi G} \sum_{ab} \epsilon_{iab} \tilde{\phi} \sigma_{(a)(b)} \right|_{a \neq b} Q_a Q_b^{-1}, \]

\[ \tilde{\phi} \sigma_{(a)(b)} |_{a \neq b} = \left. -\frac{8\pi G}{c^3} \sum_i Q_b Q_a^{-1} - Q_a Q_b^{-1} \right| \epsilon_{abi} \pi_{i}^{(\theta)} \]

\[ 3K_{rs} |_{\theta^i=0} = \tilde{\phi}^2/3 \left[ Q_r Q_s \sigma_{(r)(s)} |_{r \neq s} - \frac{4\pi G}{c^3} \delta_{rs} Q_r^2 \left( 2 \tilde{\phi}^{-1} \sum_{\tilde{a}} \gamma_{\tilde{a}r} \Pi_{\tilde{a}} - \pi_{\tilde{\phi}} \right) \right]. \]  

(6.2)

A. The Weak ADM Energy

The weak ADM energy of Eq.(3.45) becomes
\[
\dot{E}_{ADM}|_{\theta^i=0} = c \int d^3\sigma \left[ \mathcal{M}|_{\theta^i=0} - \frac{c^3}{16\pi G} S|_{\theta^i=0} + \right.
\]
\[
\left. + \frac{2\pi G}{c^3} \phi^{-1} \left( -3 (\dot{\phi} \pi_{\phi})^2 + 2 \sum_b \Pi_{b}^2 + 2 \sum_{abj} \frac{\epsilon_{abc} \epsilon_{abj} \pi_i^{(\theta)} \pi_j^{(\theta)}}{[Q_a Q_b^{-1} - Q_b Q_a^{-1}]^2} \right) \right](\tau, \vec{\sigma}) =
\]
\[
= c \int d^3\sigma \left[ \mathcal{M}|_{\theta^i=0} - \frac{c^3}{16\pi G} S|_{\theta^i=0} + \right.
\]
\[
\left. + \frac{4\pi G}{c^3} \phi^{-1} \sum_{\bar{a}} \Pi_{\bar{a}}^2 + \frac{c^4}{16\pi G} \tilde{\phi} \sum_{a \neq b} \sigma_{(a)(b)}^2 - \frac{6\pi G}{c^3} \phi \tilde{\phi}^2 \right](\tau, \vec{\sigma}),
\]
(6.3)

where Eq.(6.2) has been used and Eq. (C8) has to be used.

The mass density appearing in Eq.(6.3) has the following expression implied by Eqs.(3.37)
\[
\mathcal{M}(\tau, \vec{\sigma})|_{\theta^i=0} = \sum_i \delta^3(\vec{\sigma}, \vec{\eta}_i(\tau)) \eta_i \sqrt{m_i^2 c^2 + \phi^{-4} \sum_a Q_{a}^{-2} \left( \tilde{\kappa}_{ia}(\tau) - \frac{Q_i}{c} A_{\perp a} \right)^2 (\tau, \vec{\sigma}) +}
\]
\[
+ \tilde{\phi}^2 \tilde{T}_{\perp \perp}^{(em)}(\tau, \vec{\sigma}) + W_{(n)}(\tau, \vec{\sigma}),
\]

\[
\tilde{T}_{\perp \perp}^{(em)}(\tau, \vec{\sigma}) = \tilde{\phi}^{-4/3}(\tau, \vec{\sigma}) \left[ \frac{1}{2c} \sum_{r} Q_{a}^2 \delta_{ra} \delta_{sa} \pi_{s}^r \pi_{s}^r + \frac{1}{4c} \sum_{ab} Q_{a}^{-2} Q_{b}^{-2} F_{ab} F_{ab} \right](\tau, \vec{\sigma}),
\]
\[
W_{(n)}(\tau, \vec{\sigma}) = -\frac{1}{2c} \left[ \tilde{\phi}^{-1/3} \sum_{r} Q_{a}^2 \delta_{ra} \delta_{sa} \left( 2 \pi_{\perp}^r - \sum_{m} \delta^{rm} \sum_{i} Q_i \eta_i \frac{\partial c(\vec{\sigma}, \vec{\eta}_i(\tau))}{\partial \sigma^m} \right) \right. \delta^{sn} \sum_{j} Q_{j} \eta_j \frac{\partial c(\vec{\sigma}, \vec{\eta}_j(\tau))}{\partial \sigma^n} \left(\tau, \vec{\sigma}, \right)
\]
(6.4)

B. The Super-Hamiltonian and Super-Momentum Constraints

From Eqs.(3.44) (or (3.45)) and (3.42) the constraints determining \(\tilde{\phi}\) and \(\pi_i^{(\theta)}\) are (Eqs.(C12), (C20) and (6.3) are also needed)
\[ \mathcal{H}(\tau, \vec{\sigma})|_{\theta^i=0} = \frac{c^3}{16\pi G} \tilde{\phi}^{1/6}(\tau, \vec{\sigma}) \left[ 8 \Delta \tilde{\phi}^{1/6} - 3 \tilde{\dot{R}}|_{\theta^i=0} \tilde{\phi}^{1/6} \right] \left( \tau, \vec{\sigma} \right) + \tilde{\mathcal{M}}|_{\theta^i=0}(\tau, \vec{\sigma}) + \\
+ \frac{2\pi G}{c^3} \tilde{\phi}^{-1} \left[ - 3 (\tilde{\phi} \tilde{\pi}_i)^2 + 2 \sum_b \Pi_b^2 + 2 \sum_{abij} \frac{\epsilon_{abi} \epsilon_{ajb} \pi_i^{(\theta) \pi_j^{(\theta)}}}{[Q_a Q_b^{-1} - Q_b Q_a^{-1}]^2} \right] (\tau, \vec{\sigma}) = \\
= \frac{c^3}{2\pi G} \phi^{-6}(\tau, \vec{\sigma}) \left( \phi^7 (\Delta|_{\theta^i=0} - \frac{1}{8} 3 \tilde{\dot{R}}|_{\theta^i=0}) \phi + \frac{2\pi G}{c^3} \phi^6 \tilde{\mathcal{M}}|_{\theta^i=0} + \\
+ \frac{8\pi^2 G^2}{c^3} \sum_a \Pi_a^2 + \frac{1}{8} \phi^{12} \sum_{ab,a\neq b} \sigma_{(a)(b)} \sigma_{(a)(b)} - \frac{12\pi^2 G^2}{c^3} \phi^{12} \tilde{\phi}^2 \right) (\tau, \vec{\sigma}) \approx 0,
\] 

(6.5)

\[ \tilde{\mathcal{H}}_{(a)}|_{\theta^i=0}(\tau, \vec{\sigma}) = \phi^{-2}(\tau, \vec{\sigma}) \left[ \sum_{b \neq a} \sum_i \frac{\epsilon_{abi} Q_b^{-1}}{Q_b Q_a^{-1} - Q_a Q_b^{-1}} \partial_b \pi_i^{(\theta)} + \\
+ 2 \sum_{b \neq a} \sum_i \frac{\epsilon_{abi} Q_a^{-1}}{Q_b Q_a^{-1} - Q_a Q_b^{-1}} \frac{1}{2} \left( \tilde{\gamma}_{ab} - \gamma_{ab} \right) \partial_b R_{\tilde{\phi} \pi_i^{(\theta)}} + \\
+ Q_a^{-1} \left( \phi^6 \partial_a \pi_{\tilde{\phi}} + \sum_b \left( \gamma_{ba} \partial_a \Pi_b - \partial_a R_{\tilde{\phi} \Pi_b} \right) + \tilde{\mathcal{M}}_{(a)} \right) \right] (\tau, \vec{\sigma}) = \\
= -\frac{c^3}{8\pi G} \tilde{\phi}^{2/3}(\tau, \vec{\sigma}) \left[ \sum_{b \neq a} \frac{Q_b^{-1}}{Q_b Q_a^{-1}} \partial_b \sigma_{(a)(b)} + \\
+ \left( \partial_{\tilde{\phi}} \tilde{\phi} + \sum_b \left( \gamma_{ba} - \gamma_{bb} \right) \partial_b R_{\tilde{\phi}} \right) \sigma_{(a)(b)} \right] - \\
- \frac{8\pi G}{c^3} \tilde{\phi}^{-1} Q_a^{-1} \left( \partial_a \pi_{\tilde{\phi}} + \sum_b \left( \gamma_{ba} \partial_a \Pi_b - \partial_a R_{\tilde{\phi} \Pi_b} \right) + \tilde{\mathcal{M}}_{(a)} \right) (\tau, \vec{\sigma}) \approx 0.
\] 

(6.6)

To get the second form of the super-momentum constraints we used Eq.(6.2), which implies \( \partial_r \pi_i^{(\theta)}|_{\theta^i=0} = \frac{c^3}{8\pi G} \tilde{\phi} \sum_{a\neq b} \epsilon_{iab} Q_a^{-1} \left[ \partial_r \sigma_{(a)(b)} + \left( \tilde{\phi}^{-1} \partial_r \tilde{\phi} + \sum_b \left( \gamma_{ba} - \gamma_{bb} \right) \partial_r R_b \right) \sigma_{(a)(b)} \right]. \)

For the solution of Eq.(6.6) the results of Ref.[6] are needed.

**C. The Contracted Bianchi Identities**

From Eqs. (4.1) and (4.2) the Hamilton equations for the unknowns in the constraints are (the first is the Raychaudhuri equation)
\[
\partial_{\tau} \bar{\phi}(\tau, \bar{\sigma})|_{\theta^i=0} = \left[- \frac{12\pi G}{c^3} (1+n) \bar{\phi} \pi_{\bar{\phi}} + \bar{\phi}^{2/3} \sum_a Q_a^{-1} \left( \partial_{a} \bar{n}_a + \right. \right. \\
+ \left. \left. \bar{n}_a \left( \frac{2}{3} \bar{\phi}^{-1} \partial_{a} \bar{\phi} \right. - \sum_b \gamma_{ba} \partial_{a} R_b \left. \right) \right) \right](\tau, \bar{\sigma}),
\]

\[\text{(6.7)}\]

\[
\partial_{\tau} \pi_i^{(\theta)}(\tau, \bar{\sigma})|_{\theta^i=0} = - \int d^3 \bar{\sigma}_1 \left[(1+n)(\tau, \bar{\sigma}_1) \left( \frac{\delta \hat{M}(\tau, \bar{\sigma}_1)}{\delta \theta^i(\tau, \bar{\sigma})} \right)|_{\theta^i=0} - \frac{c^3}{16\pi G} \frac{\delta S(\tau, \bar{\sigma}_1)}{\delta \theta^i(\tau, \bar{\sigma})}|_{\theta^i=0} \right. \\
- \frac{c^3}{16\pi G} n(\tau, \bar{\sigma}_1) \frac{\delta T(\tau, \bar{\sigma}_1)}{\delta \theta^i(\tau, \bar{\sigma})}|_{\theta^i=0} \right. \\
- \frac{8\pi G}{c^3} \left[(1+n) \bar{\phi}^{-1} \sum_{ab klj} \epsilon_{ab} \epsilon_{lj} \left( V_{i(l)} + B_{i(l)j} \right) \pi_k(\theta) \pi_j(\theta) \right. \\
\left. \left. \left( Q_a Q_b - Q_b Q_a \right)^{-1} \right) (\tau, \bar{\sigma}) \right. + \\
\left. \int d^3 \bar{\sigma}_1 \bar{n}_a(\tau, \bar{\sigma}_1) \frac{\delta \hat{H}_a(\tau, \bar{\sigma}_1)}{\delta \theta^i(\tau, \bar{\sigma})}|_{\theta^i=0}. \right.
\]

\[\text{(6.8)}\]

Eqs.(C21), (C11), (C19) and (C5) are needed.

D. The Shift Functions

By using Eq.(4.5) for the \(\tau\)-preservation of the gauge fixings (6.1), we get the following equations for the shift functions (the second expression uses Eqs.(6.2))
$$\left[ \sum_{ab} \frac{\epsilon_{ab}}{Q_a Q_b^{-1} - Q_b Q_a^{-1}} \left( - \tilde{\phi}^{-1/3} Q_b^{-1} \left[ \frac{1}{3} \tilde{\phi}^{-1} \partial_b \tilde{\phi} + \sum_b \gamma_{bb} \partial_b R_b \right] \tilde{n}_{(a)} - \partial_b \tilde{n}_{(a)} \right) - \\
- \frac{8\pi G}{c^3} (1 + n) \tilde{\phi}^{-1} \sum_j \frac{\epsilon_{abj}}{Q_a Q_b^{-1} - Q_b Q_a^{-1}} \right] (\tau, \vec{\sigma}) =$$

$$= \left[ \sum_{ab} \frac{\epsilon_{ab}}{Q_a Q_b^{-1} - Q_b Q_a^{-1}} \left( - \tilde{\phi}^{-1/3} Q_b^{-1} \left[ \frac{1}{3} \tilde{\phi}^{-1} \partial_b \tilde{\phi} + \sum_b \gamma_{bb} \partial_b R_b \right] \tilde{n}_{(a)} - \partial_b \tilde{n}_{(a)} \right) - \\
- (1 + n) \sigma_{(a)(b)} \right] (\tau, \vec{\sigma}) \approx 0. \quad (6.9)$$

By saturating with $\epsilon_{cdi}$ we get for $a \neq b$

$$\left( Q_b^{-1} \partial_b \tilde{n}_{(a)} + Q_a^{-1} \partial_a \tilde{n}_{(b)} - \left[ Q_b^{-1} \left( \frac{1}{3} \tilde{\phi}^{-1} \partial_b \tilde{\phi} + \sum_a \gamma_{aa} \partial_a R_a \right) \tilde{n}_{(a)} + \\
+ Q_a^{-1} \left( \frac{1}{3} \tilde{\phi}^{-1} \partial_a \tilde{\phi} + \sum_a \gamma_{ab} \partial_a R_a \right) \tilde{n}_{(b)} \right] \right) (\tau, \vec{\sigma}) \approx$$

$$\approx 2 \left[ \tilde{\phi}^{1/3} (1 + n) \sigma_{(a)(b)} \big|_{a \neq b} \right] (\tau, \vec{\sigma}). \quad (6.10)$$

This is the final form of the equations for the shift functions.

**E. The Instantaneous 3-Space and the Lapse Functions**

If we recall from Eq.(2.9) the definition $\pi_{\tilde{\phi}} = \frac{c^3}{12\pi G} 3\tilde{K}$, we can restrict ourselves to the family of gauges

$$\pi_{\tilde{\phi}}(\tau, \vec{\sigma}) - \frac{c^3}{12\pi G} F(\tau, \vec{\sigma}) \approx 0, \quad \Rightarrow 3K(\tau, \vec{\sigma}) \approx F(\tau, \vec{\sigma}), \quad (6.11)$$

where $F(\tau, \vec{\sigma})$ is a numerical function independent from the canonical variables, which describes the relativistic inertial effects associated with the freedom in the clock synchronization (they do not exist in Newton theory in Galilei space-times).

For $F(\tau, \vec{\sigma}) = const.$ we have the CMC gauges.

By using Eqs.(4.4) and (6.2) restricted to the 3-orthogonal gauges, the preservation in $\tau$ of the gauge fixing constraint (6.11) gives the following equation for the determination of the lapse function
where Eqs. (C22), (C10) and (C17) have to be used.

Note that this equation depends also upon the shift functions.

Eqs.(6.10) and (6.12) are 4 coupled equations for the lapse and shift functions.

F. The Equations of Motion

1. The Tidal Variables

The second order equations of motion(4.11) for the tidal variables $R_a$ and Eq.(4.9) for the momentum $\Pi_{ab}$ become respectively

\[ \frac{1}{6} \phi^{-5/6}(\tau, \bar{\sigma}) \int d^3 \sigma \left[ \left(1 + n\right)(\tau, \bar{\sigma}) \left( \frac{\delta \tilde{M}(\tau, \bar{\sigma})}{\delta \phi(\tau, \bar{\sigma})} \right)_{\theta^i=0} - \frac{c^3}{16 \pi G} \frac{\delta S(\tau, \bar{\sigma})}{\delta \phi(\tau, \bar{\sigma})} \right]_{\theta^i=0} - \]

\[ - \frac{2 \pi G}{c^3} (1 + n)(\tau, \bar{\sigma}) \left[ 3 \pi^2 \phi + 2 \phi^{-2} \left( \sum_b \Pi_b^2 + \sum_{a \neq b} \epsilon_{abij} \frac{\pi_i^{(a)} \pi_j^{(b)}}{\sigma_{(a)(b)}} \right) \right] (\tau, \bar{\sigma}) - \]

\[ - \left( \tilde{\phi}^{-1/3} \sum_a \bar{n}_a Q_a^{-1} \partial_a \pi_{\phi} \right)(\tau, \bar{\sigma}) = \]

\[ = \frac{1}{6} \phi^{-5/6}(\tau, \bar{\sigma}) \int d^3 \sigma \left[ \left(1 + n\right)(\tau, \bar{\sigma}) \left( \frac{\delta \tilde{M}(\tau, \bar{\sigma})}{\delta \phi(\tau, \bar{\sigma})} \right)_{\theta^i=0} - \frac{c^3}{16 \pi G} \frac{\delta S(\tau, \bar{\sigma})}{\delta \phi(\tau, \bar{\sigma})} \right]_{\theta^i=0} - \]

\[ - \frac{2 \pi G}{c^3} (1 + n)(\tau, \bar{\sigma}) \left[ 3 \pi^2 \phi + 2 \phi^{-2} \left( \sum_b \Pi_b^2 + \frac{c^6}{64 \pi^2 G^2} \sum_{a \neq b} \sigma_{(a)(b)} \right) \right] (\tau, \bar{\sigma}) - \]

\[ - \left( \tilde{\phi}^{-1/3} \sum_a \bar{n}_a Q_a^{-1} \partial_a \pi_{\phi} \right)(\tau, \bar{\sigma}) \approx \]

\[ \approx - \frac{c^3}{12 \pi G} \frac{\partial F(\tau, \bar{\sigma})}{\partial \tau}. \]
\[
\partial^2 R_a (\tau, \vec{\sigma}) \overset{\circ}{=} \left[ \tilde{\phi}^{-1/3} \sum_a Q^{-1}_a \tilde{n}(a) \sum_b (\gamma_{aa} \gamma_{ba} - \delta_{ab}) \partial_a \partial_\tau R_b + \\
+ \hat{\phi}^{-1/3} \sum_a Q^{-1}_a \left( \left( \gamma_{aa} \left( \frac{1}{3} \hat{\phi}^{-1} \partial_a \tilde{\phi} + \sum_c \gamma_{ca} \partial_a R_c \right) - \partial_a R_a \right) \right) \tilde{n}(a) - \\
- \gamma_{aa} \partial_a \tilde{n}(a) \right] \sum_b \gamma_{ba} \partial_\tau R_b - \\
- \tilde{\phi}^{-1} \left[ \partial_\tau R_a + \frac{2}{3} \tilde{\phi}^{-1/3} \sum_a Q^{-1}_a \left( \left( \gamma_{aa} \left( \frac{1}{3} \hat{\phi}^{-1} \partial_a \tilde{\phi} + \sum_b \gamma_{ba} \partial_a R_b \right) - \\
- \partial_a R_a \right) \tilde{n}(a) - \gamma_{aa} \partial_a \tilde{n}(a) \right) \right] \partial_\tau \tilde{\phi} - \\
- \frac{1}{3} \tilde{\phi}^{-1/3} \sum_a \gamma_{aa} Q^{-1}_a \tilde{n}(a) \partial_a \partial_\tau \tilde{\phi} + \\
+ \left[ \partial_\tau R_a + \tilde{\phi}^{-1/3} \sum_a Q^{-1}_a \left( \left( \gamma_{aa} \left( \frac{1}{3} \hat{\phi}^{-1} \partial_a \tilde{\phi} + \sum_b \gamma_{ba} \partial_a R_b \right) - \\
- \partial_a R_a \right) \tilde{n}(a) - \gamma_{aa} \partial_a \tilde{n}(a) \right) \right] \frac{\partial_\tau n}{1 + n} - \\
- \tilde{\phi}^{-1/3} \sum_a Q^{-1}_a \left( \left( \gamma_{aa} \left( \frac{1}{3} \hat{\phi}^{-1} \partial_a \tilde{\phi} + \sum_b \gamma_{ba} \partial_a R_b \right) - \partial_a R_a \right) \right) \partial_\tau \tilde{n}(a) - \\
- \gamma_{aa} \partial_a \partial_\tau \tilde{n}(a) \right] (\tau, \vec{\sigma}) + \\
+ \frac{1}{2} \left( \tilde{\phi}^{-1} (1 + n) \right) (\tau, \vec{\sigma}) \int d^3 \sigma_1 \left[ (1 + n) (\tau, \vec{\sigma}_1) \frac{\delta S(\tau, \vec{\sigma}_1)}{\delta R_a(\tau, \vec{\sigma})} |_{\theta^i = 0} + n(\tau, \vec{\sigma}_1) \frac{\delta T(\tau, \vec{\sigma}_1)}{\delta R_a(\tau, \vec{\sigma})} |_{\theta^i = 0} \right] - \\
- \frac{8\pi G}{c^3} \left( \tilde{\phi}^{-1} (1 + n) \right) (\tau, \vec{\sigma}) \int d^3 \sigma_1 (1 + n) (\tau, \vec{\sigma}_1) \frac{\delta \mathcal{M}(\tau, \vec{\sigma}_1)}{\delta R_a(\tau, \vec{\sigma})} |_{\theta^i = 0} + \\
+ \frac{8\pi G}{c^3} \left( \tilde{\phi}^{-1} (1 + n) \right) (\tau, \vec{\sigma}) \int d^3 \sigma_1 \tilde{n}(a) (\tau, \vec{\sigma}_1) \frac{\delta \mathcal{H}_a(\tau, \vec{\sigma}_1)}{\delta R_a(\tau, \vec{\sigma})} |_{\theta^i = 0} - \\
- \left( \frac{8\pi G}{c^3} \right)^2 \left( \tilde{\phi}^{-2} (1 + n)^2 \right) (\tau, \vec{\sigma}) \left( \sum_{a \neq b} (\gamma_{aa} - \gamma_{ab}) \epsilon_{abi} \epsilon_{abj} (Q_b Q^{-1}_a + Q_a Q^{-1}_b) \pi^{(a)} \pi^{(b)} \right) (\tau, \vec{\sigma}),
\]

\[\Pi_a(\tau, \vec{\sigma}) = - \frac{c^3}{8\pi G} \left[ \hat{\phi} \sum_a \gamma_{aa} \sigma_{(a)(a)} \right] (\tau, \vec{\sigma}) \overset{\circ}{=} \]

\[= \frac{c^3}{8\pi G \frac{\hat{\phi}(\tau, \vec{\sigma})}{1 + n(\tau, \vec{\sigma})}} \left[ \partial_\tau R_a + \tilde{\phi}^{-1/3} \sum_a Q^{-1}_a \left( \left( \gamma_{aa} \left( \frac{1}{3} \hat{\phi}^{-1} \partial_a \tilde{\phi} + \sum_b \gamma_{ba} \partial_a R_b \right) - \partial_a R_a \right) \tilde{n}(a) - \gamma_{aa} \partial_a \tilde{n}(a) \right) \right] (\tau, \vec{\sigma}),\]

(6.13)
where Eqs. (4.3), (6.7), (C9), (C15), (C23) and (C7) have to be used.

The last term in the equation for \( \partial^2_{\tau} R_a \) can be written as

\[
-\left(1 + n\right) \sum_{ab,a \neq b} (\gamma_{aa} - \gamma_{ab}) \frac{Q_a Q_b^{-1} + Q_b Q_a^{-1}}{Q_a Q_b^{-1} - Q_a Q_b} \sigma^2_{(a)(b)}(\tau, \bar{\sigma}).
\]

2. The Particles

The Hamilton equations (4.12) for the particles become

\[
\eta_i \dot{\eta}_i^r(\tau) = \eta_i \left( -\frac{\phi^{-4} \left(1 + n\right) Q^{-2}_r \left(\bar{\kappa}_{ir}(\tau) - \frac{Q_i}{c} A_{\perp r}\right)}{\sqrt{m_i^2 c^2 + \phi^{-4} \sum_c Q^{-2}_c \left(\bar{\kappa}_{ic}(\tau) - \frac{Q_i}{c} A_{\perp c}\right)^2}} - \right.
\]

\[
-\phi^{-2} Q^{-1}_r \bar{n}(\tau) \right)(\tau, \bar{\eta}_i(\tau)),
\]

\[
\eta_i \frac{d}{d \tau} \left( \frac{Q_i}{c} A_{\perp r} + m_i c \left(\bar{\phi}^{2/3} Q^2_r \left(\bar{\eta}_i^r(\tau) + \bar{\phi}^{-1/3} Q^{-1}_r \bar{n}(\tau)\right) \right) \right] \left(1 + n\right)^2 - \phi^{-2} \sum_c Q^2_c \left(\bar{\eta}_i^c(\tau) + \bar{\phi}^{-1/3} Q^{-1}_c \bar{n}(\tau)\right) \right)^{-1/2} \right)(\tau, \bar{\eta}_i(\tau)) =
\]

\[
\left(-\frac{\partial}{\partial \eta_i} \mathcal{W} + \frac{\eta_i Q_i}{c} \bar{\eta}_i^c(\tau) \frac{\partial A_{\perp s}}{\partial \eta_i^c} + \eta_i \bar{F}_{ir}\right)(\tau, \bar{\eta}_i(\tau)),
\]

\[
\mathcal{W}(\tau) = \int d^3 \sigma \left[ (1 + n) \mathcal{W}_n + \phi^{-2} \sum_a Q^{-1}_a \bar{n}(a) \mathcal{W}_a \right](\tau, \bar{\sigma}),
\]

\[
\mathcal{W}_n(\tau, \bar{\sigma}) = -\frac{1}{2c} \left[ \phi^{-2} \sum_a Q^2_a \left(2 \pi^a_\perp - \delta^{am} \sum_i Q_i \eta_i \frac{\partial c(\bar{\sigma}, \bar{\eta}_i(\tau))}{\partial \sigma^m} \right) \right.
\]

\[
\left. \delta^{an} \sum_i Q_i \eta_i \frac{\partial c(\bar{\sigma}, \bar{\eta}_i(\tau))}{\partial \sigma^n} \right](\tau, \bar{\sigma}),
\]

\[
\mathcal{W}_r(\tau, \bar{\sigma}) = -\frac{1}{c} \bar{F}_{rs}(\tau, \bar{\sigma}) \delta^{sn} \sum_i Q_i \eta_i \frac{\partial c(\bar{\sigma}, \bar{\eta}_i(\tau))}{\partial \sigma^n},
\]
\[
\tilde{F}_{ir} = m_i c \left[ \left( 1 + n \right)^2 - \tilde{\phi}^{2/3} \sum_c Q_c^2 \left( \tilde{\eta}_i^c(\tau) + \phi^{-1/3} Q_c^{-1} \tilde{n}_c(\tau) \right)^2 \right]^{-1/2} \\
\left[ - (1 + n) \frac{\partial n}{\partial \eta_i^c} + \phi^2 \sum_a Q_a \left( \frac{\partial \tilde{n}_a(\tau)}{\partial \eta_i^c} - \frac{1}{3} \phi^{-1} \partial_r \phi + \sum a \gamma_{aa} \partial_r R_a \right) \tilde{n}_a(\tau) \right) \left( \tilde{\eta}_i^c(\tau) + \phi^{-2} Q_a^{-1} \tilde{n}_a(\tau) \right) + \\+ \phi^4 \sum_a Q_a^2 \left( \frac{1}{3} \tilde{\phi}^{-1} \partial_r \phi + \sum a \gamma_{aa} \partial_r R_a \right) \left( \tilde{\eta}_i^a(\tau) + \phi^{-2} Q_a^{-1} \tilde{n}_a(\tau) \right)^2 \right].
\]

(6.14)

The inversion of the first of Eqs.(6.11) gives

\[
\tilde{\kappa}_{ir}(\tau) = \frac{Q_i}{c} A_{ir}(\tau, \tilde{\eta}_i(\tau)) + m_i c \left[ \tilde{\phi}^{2/3} Q_r^2 \left( \tilde{\eta}_i^r(\tau) + \phi^{-1/3} Q_r^{-1} \tilde{n}_r(\tau) \right) \left( (1 + n)^2 - \tilde{\phi}^{2/3} \sum c Q_c^2 \left( \tilde{\eta}_i^c(\tau) + \phi^{-1/3} Q_c^{-1} \tilde{n}_c(\tau) \right) \right) \right]^{-1/2}(\tau, \tilde{\eta}_i(\tau)).
\]

(6.15)

3. The Transverse Electro-Magnetic Field

Finally the Hamilton equations (4.15) for the transverse electro-magnetic fields in the radiation gauge become

\[
\partial_{\tau} A_{\perp \tau}(\tau, \tilde{\sigma}) = \sum_{nu} \delta_{nu} P_{\perp}(\tilde{\sigma}) \left[ \tilde{\phi}^{-1/3} (1 + n) Q_a^2 \delta_{ua} \left( \frac{\tilde{\phi}}{2} - \sum m \delta_{am} \sum_i Q_i \eta_i \frac{\partial c(\tilde{\sigma}, \tilde{\eta}_i(\tau))}{\partial \sigma_m} \right) + \tilde{\phi}^{-1/3} Q_a^{-1} \tilde{n}_a(\tau) F_{au} \right] \left( \tau, \tilde{\eta}_i(\tau) \right),
\]

\[
\partial_{\tau} \pi_{\perp}(\tau, \tilde{\sigma}) = \sum_m P_{\perp}(\tilde{\sigma}) \left( \sum_n \delta_{na} \sum_i Q_i \delta^2(\tilde{\sigma}, \tilde{\eta}_i(\tau)) \right) \left[ \frac{\tilde{\phi}^{-2/3} (1 + n) Q_a^{-2} \tilde{\kappa}_{ia}(\tau)}{\sqrt{m_i^2 c^2 + \tilde{\phi}^{-2/3} \sum b Q_b^{-2} \left( \tilde{\kappa}_{ib}(\tau) - \frac{Q_b}{c} A_{\perp b} \right)^2} - \tilde{\phi}^{-1/3} Q_a^{-1} \tilde{n}_a(\tau) \right] \left( \tau, \tilde{\eta}_i(\tau) \right).
\]

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\[-2 \tilde{\phi}^{-1/3} (1 + n) \sum_{ab} Q_a^{-2} Q_b^{-2} \delta_{ma} \left( \partial_b F_{ab} - \left[ \frac{1}{3} \tilde{\phi}^{-1} \partial_b \tilde{\phi} + 2 \sum_b (\gamma_{ba} + \gamma_{bb}) \partial_b R_b \right] F_{ab} \right) +
\]
\[+ 2 \tilde{\phi}^{-1/3} \sum_{ab} Q_a^{-2} Q_b^{-2} \delta_{ma} \partial_b n F_{ab} -
\]
\[-\tilde{\phi}^{-1/3} \sum_a \bar{n}_{(a)} Q_a^{-1} \left( \partial_a \pi^m_{\perp} - \left[ \frac{1}{3} \tilde{\phi}^{-1} \partial_a \tilde{\phi} + \sum_b \gamma_{ba} \partial_a R_b \right] \pi^m_{\perp} +
\]
\[+ \delta_{ma} \sum_n \left[ \frac{1}{3} \tilde{\phi}^{-1} \partial_n \tilde{\phi} + \sum_b \gamma_{ba} \partial_n R_b \right] \pi^n_{\perp} +
\]
\[+ \sum_i \eta_i Q_i \left[ \left( \frac{1}{3} \tilde{\phi}^{-1} \partial_a \tilde{\phi} + \sum_b \gamma_{ba} \partial_a R_b \right) \frac{\partial c(\tilde{\sigma}, \tilde{\eta}_i(\tau))}{\partial \sigma^m} - \frac{\partial^2 c(\tilde{\sigma}, \tilde{\eta}_i(\tau))}{\partial \sigma^m \partial \sigma^a} -
\]
\[-\delta_{ma} \sum_n \left( \left[ \frac{1}{3} \tilde{\phi}^{-1} \partial_n \tilde{\phi} + \sum_b \gamma_{ba} \partial_n R_b \right] \frac{\partial c(\tilde{\sigma}, \tilde{\eta}_n(\tau))}{\partial \sigma^n} - \frac{\partial^2 c(\tilde{\sigma}, \tilde{\eta}_n(\tau))}{\partial \sigma^n \partial \sigma^n} \right) \right) +
\]
\[+ \tilde{\phi}^{-1/3} \sum_{a} Q_a^{-1} \sum_i \eta_i Q_i \left( \partial_a \bar{n}_{(a)} \frac{\partial c(\tilde{\sigma}, \tilde{\eta}_i(\tau))}{\partial \sigma^m} -
\]
\[-\delta_{ma} \sum_n \partial_n \bar{n}_{(a)} \frac{\partial c(\tilde{\sigma}, \tilde{\eta}_n(\tau))}{\partial \sigma^n} \right) (\tau, \tilde{\sigma}) \right). \]

(6.16)
VII. CONCLUSIONS

In this paper we gave the Hamilton equations of ADM tetrad gravity coupled to charged scalar particles and to the electro-magnetic field in the York canonical basis first in an arbitrary Schwinger time gauge and then in a sub-family of 3-orthogonal gauges. The electro-magnetic field has been specialized to the non-covariant radiation gauge, in which we can eliminate the electro-magnetic gauge variables and visualize the non-inertial Coulomb potential among the particles.

We gave some refinements of the York canonical basis of Ref.[6], connected with the congruence of the Eulerian observers associated to the 3+1 splitting of space-time.

We emphasized the role of the inertial gauge variable $^3K(\tau, \vec{\sigma})$, the York time, connected to the freedom in the choice of the non-dynamical part of the clock synchronization convention defining the instantaneous 3-spaces: in the York canonical basis it is the only gauge variable described by a momentum (a reflex of the Lorentz signature of space-time) and it gives rise to a negative kinetic term in the weak ADM energy vanishing only in the gauges $^3K(\tau, \vec{\sigma}) = 0$.

Moreover, in connection with the weak ADM Poincare’ charges, we showed that in these asymptotically Minkowskian space-times one can introduce an interpretation of the isolated system gravitational field plus matter like in the inertial and non-inertial rest frames of Minkowski space-time [8, 10]. The 3-universe contained in an instantaneous 3-space can be described as a decoupled non-covariant non-observable pseudo-particle carrying a pole-dipole structure, whose mass and spin are those identifying the configuration of the ”gravitational field plus matter” isolated system present in the 3-universe.

In the next paper [19], starting from the results obtained in the family of non-harmonic 3-orthogonal Schwinger gauges with an arbitrary numerical function describing the inertial York time, $^3K(\tau, \vec{\sigma}) \approx F(\tau, \vec{\sigma})$, we will define a linearization of ADM canonical tetrad gravity plus matter, to obtain a formulation of Hamiltonian post-Minkowskian gravity (without post-Newtonian expansions) with non-flat Riemannian 3-spaces and asymptotic Minkowski background: i.e. with a decomposition of the 4-metric tending to the asymptotic Minkowski metric at spatial infinity, $^4g_{AB} \rightarrow ^4\eta_{AB}$. We can write $^4g_{AB} = ^4\eta_{AB} + ^4h_{AB}$, with the small perturbation $^4h_{AB}$ vanishing at spatial infinity, but this decomposition has no intrinsic meaning in the bulk differently from what happens in harmonic gauges with a fixed background (in them one has Euclidean 3-spaces with a violation of the equivalence principle). We will show that a consequence of this approach is the possibility of describing part (or maybe all) dark matter as a relativistic inertial effect determined by the gauge variable $^3K(\tau, \vec{\sigma}^*)$ (not existing in Newtonian gravity, where the Euclidean 3-space is an absolute notion): the rotation curves of galaxies would then experimentally determine a preferred choice of the instantaneous 3-spaces.

Finally in a future paper we will replace the matter with a perfect fluid (for instance dust [25]) and we will try to see whether the York canonical basis can help in developing the back-reaction [39] approach to dark energy.
APPENDIX A: THE COMPARISON WITH THE STANDARD ADM EQUATIONS OF MOTION, THE RAYCHAUDHURI EQUATION AND THE CONTRACTED BIANCHI IDENTITIES.

In absence of matter, after the 3+1 splitting of space-time and the introduction of the radar 4-coordinates, the standard ADM equations of motion of canonical metric gravity are (see Eqs.(4.11) of Ref.[2])

\[
\partial_\tau \, g_{rs} = n_r |s + n_s |r - 2 (1 + n) \, K_{rs}, \\
\partial_\tau \, K_{rs} = (1 + n) \left[ 3 R_{rs} + 3 K_{sr} - 2 \, K_{ru} \, K_{us} \right] - n_r |s + n_u |s \, K_{ur} + n_r |s \, K_{us} + n_r |s \, K_{rs} |u. 
\]

(A1)

Since from Eqs.(2.6) and (2.14) we have \( \gamma = \det g_{rs} = (3e)^2 = \phi^{12} = \tilde{\phi}^2 \) and \( 3K = -\epsilon \theta = \frac{12 \pi G}{c^3} \tau_{\tilde{\phi}} \) (\( \theta \) is the expansion of the non-geodetic Eulerian observers), we can deduce from Eqs.(A1) the following equations

\[
\partial_\tau \, \tilde{\phi} = \left[ - \frac{12 \pi G}{c^3} (1 + n) \, \pi_{\tilde{\phi}} + \left( \frac{3}{c^3} e_{(a)} \, \bar{n}(a) \right) \right] \tilde{\phi}, \\
\partial_\tau \, K = \frac{12 \pi G}{c^3} \partial_\tau \, \pi_{\tilde{\phi}} = -\epsilon \, \partial_\tau \, \theta = \\
= (1 + n) \left[ 3 R + (3K)^2 \right] - n |u + n \, K |u = \\
= (1 + n) \left[ 3 R + \left( \frac{c^3}{12 \pi G} \pi_{\tilde{\phi}} \right)^2 \right] - n |u + \frac{12 \pi G}{c^3} \pi_{\tilde{\phi}} |u \, \bar{e}_{(a)} \, \bar{n}(a). 
\]

(A2)

The second of Eqs.(A2) is an equation for the time evolution \( \partial_\tau \, \pi_{\tilde{\phi}} \) of the gauge variable \( \pi_{\tilde{\phi}} \), named Raychaudhuri equation. For geodetic congruences of time-like curves it is connected with the geodesic deviation equation and is a fundamental ingredient for the singularity theorems, see Ref.[40], where it is shown that in general (also in Minkowski space-time) in a congruence of ”time-like geodesics” caustics will develop if convergence (\( \theta < 0 \)) occurs anywhere. In certain space-times there will be real space-time singularities if other global assumptions hold [40]. As shown in Ref.[41] the singularity theorems imply that the gauge fixing identifying the instantaneous 3-spaces (clock synchronization convention) have to be divided into inequivalent classes. Those who satisfy the strong energy conditions and have non-negative 3-curvature imply a singularity (geodesic incompleteness) in the past if the spatial average of the expansion does not vanish. In our class of space-times, including the Christodoulou-Klainermann ones [42] in absence of matter, the assumed boundary conditions should eliminate the singularities (at least for finite intervals of time; but in presence of matter the problem is open). Moreover the congruence of Eulerian observers, relevant for the York canonical basis, is not geodesic.

Instead the first of Eqs.(A2) is a contracted Bianchi identity. Let us clarify this point.

Due to the Bianchi identities \( G_{\mu\nu} = 0 \), the 10 Einstein’s equations \( G_{\mu\nu} = 4R_{\mu\nu} - \frac{1}{2} \, g_{\mu\nu} \, 4R = \frac{8 \pi G}{c^4} T_{\mu\nu} \) imply the equations of motion \( T_{\mu\nu} = 0 \) for matter and 4 contracted
Bianchi identities, which say that 4 Einstein’s equations are not independent from the others and their time-derivatives. Therefore there are only 6 Einstein’s equations functionally independent. But, since 4 Einstein’s equations are the super-Hamiltonian and super-momentum constraints (restrictions on the Cauchy data), it turns out that only 2 combinations of the 10 Einstein equations are dynamical, i.e. depending on the accelerations (they can be put in normal form; see the Hamilton-Dirac equations for \(R_\alpha, \Pi_\alpha\) in Section III). The four contracted Bianchi identities, \(4G^{\mu\nu};_\rho \equiv 0\), imply [40] that, if the restrictions of Cauchy data are satisfied initially and the spatial equations \(4G_{ij} \equiv 0\) are satisfied everywhere, then the secondary constraints are satisfied also at later times. Behind these identities there is the gauge invariance of the Einstein-Hilbert action under 4-diffeomorphisms. However to get a canonical formulation we need a 3+1 splitting of space-time and the replacement of the Einstein-Hilbert action with the ADM action.

At the Hamiltonian level, while the first 6 equations (A1) are nothing else that the definition of the extrinsic curvature of the instantaneous 3-space \(\Sigma_\tau\), only two of the other 6 equations (A1) are independent due to the ADM version of the 4 contracted Bianchi identities, now implied by the gauge invariances of the ADM action which are no more the 4-diffeomorphisms due to the presence of the 3+1 splitting of space-time. At the Hamiltonian level these gauge invariances are generated by the 8 (14 in tetrad gravity \(^{20}\)) first class constraints of ADM canonical metric gravity defining a pre-symplectic sub-manifold of the phase space symplectic manifold. However at each fixed value of the radar time \(\tau\) the gauge group is not a Lie group \(^{21}\) and its group manifold, as a transformation group on the symplectic manifold of ADM canonical metric gravity, does not have 8 functional gauge parameters but only 4, the 4 Dirac multipliers \(^{22}\) in front of the 4 primary constrains (the vanishing of the lapse and shift momenta). The secondary constraints (the super-Hamiltonian and super-momentum constraints) have as gauge parameters the gauge variables corresponding to the lapse and shift functions, i.e. 4 elements of a Darboux basis of the symplectic manifold. As a consequence, it is not clear how to define an abstract group manifold for this type of symplectic gauge group \(^{23}\). Moreover, the group manifold is \(\tau\)-dependent: we have (a-priori) different group manifolds for the gauge group on each instantaneous 3-space \(\Sigma_\tau\). Therefore Eqs.(A1) contain an information on the \(\tau\)-dependence of the effective group manifold of the gauge transformation group of canonical gravity describing its modification.

\(^{20}\) In canonical tetrad gravity there are 6 extra primary constraints with the associated Dirac multipliers in the Dirac Hamiltonian and no extra contracted Bianchi identity.

\(^{21}\) It contains spatial diffeomorphisms acting on \(\Sigma_\tau\) and the gauge transformations, replacing the time diffeomorphisms, generated by the super-Hamiltonian constraint, which modify the shape of the instantaneous 3-space \(\Sigma_\tau\) with deformations along the normal to \(\Sigma_\tau\) in every point, proportional to the conjugate gauge variable \(\pi_{\delta} = \frac{\alpha^3}{12\gamma\sigma} \gamma \dot{3} K\), namely to the value of the York time in that point.

\(^{22}\) They correspond to a special choice of the arbitrary generalized velocities existing at the Lagrangian level due to the gauge invariances. The kinematical set of the Hamilton-Dirac equations identify them as the \(\tau\)-derivatives of the lapse and shift functions.

\(^{23}\) From what has been said the natural gauge parameters of the symplectic gauge group are the lapse and shift gauge functions and their \(\tau\)-derivatives, so that there are only 4 arbitrary functions replacing the space-time 4-diffeomorphisms (whose abstract group manifold, depending on 4 arbitrary space-time functions as gauge parameters, is not under mathematical control in the large), the gauge group of the Einstein-Hilbert action.
from an instantaneous 3-space $\Sigma_\tau$ to a modified one $\Sigma_{\tau+\delta\tau}$.

Let us now look at the Hamiltonian interpretation of the ADM equations (A1) in the York canonical basis of tetrad gravity. Since in the canonical York basis there is an identification both of the gauge variables $(\alpha(a), \phi(a), n, n_r(a), \theta^r, \pi^\alpha_r)$ and of the variables $(\bar{\phi}, \bar{\pi}^\alpha_r)$ determined by the super-Hamiltonian and super-momentum constraints, the content of Eqs.(A1) rewritten in the York canonical basis will be:

i) 4 physical (hyperbolic) equations for the tidal variables $R_a$, $\Pi_a$, see Eqs.(4.8) and (4.10).

ii) 4 equations for the variables $\bar{\phi}$, $\pi^\alpha_r$, see Eqs.(4.1) and (4.2), which hold independently from the form of the solution of the super-Hamiltonian and super-momentum constraints determining them. They are the Hamiltonian version of the 4 contracted Bianchi identities implying the time-preservation of these secondary constraints. Let us remark that the super-Hamiltonian constraint (3.44) and the super-momentum constraints (3.41) are coupled elliptic partial differential equations for the unknowns $\bar{\phi}$ and $\pi^\alpha_r$ (not upon the lapse and shift functions) and upon the tidal variables $R_a$, $\Pi_a$. Therefore their general solution (depending also on some arbitrary functions) will be of the form $\bar{\phi} \approx F[R_a, \Pi_a, \theta^i, \pi^\alpha_r]$, $\pi^\alpha_r \approx F_i[R_a, \Pi_a, \theta^i, \pi^\alpha_r]$. iii) 4 equations (4.4) and (4.5) for the gauge variables $\theta^r$, $\pi^\alpha_r$: therefore, after the addition of gauge fixings (4.6) these equations allow to evaluate the terms $\partial_r \theta^r$, $\partial_r \pi^\alpha_r$ appearing in Eqs.(4.7) and generate the equations for the determination of the lapse and shift functions.

By comparison let us consider electro-magnetism (with its Abelian gauge group) in the Shanmugadhasan canonical basis (3.31). In the rest-frame instant form of Ref.[8] the canonical variables and the constraint are $A_r$, $\pi_r \approx 0$, $\eta_{em} = -\frac{1}{\Delta} \vec{\Delta} \cdot \vec{A}$, $\pi_{\eta_{em}} = \Gamma = \vec{\Delta} \cdot \vec{\pi} \approx 0$, $\vec{A}_\perp$, $\vec{\pi}_\perp$. The Dirac Hamiltonian is $H_D = \int d^3\sigma \left[ \vec{\pi}_\perp^2 + B^2 - A_r \pi_{\eta_{em}} + \lambda \pi_r \right] (r, \vec{\sigma})$ and the gauge variable $A_r$ plays the role of the lapse and shift functions $(\lambda \overset{0}{=} \partial_r A_r)$. The analogue of the equations (4.4), (4.5), for the gauge variables is $\partial_r \eta_{em} \overset{0}{=} -A_r$: this equation determines $A_r$ once the gauge fixing for $\eta_{em}$ has been given (the analogue of Eqs.(4.7)): $\eta - f[.., A_r] \approx 0$. The analogue of the Hamiltonian contracted Bianchi identities (4.1), (4.2), is $\partial_r \pi_{\eta_{em}} \overset{0}{=} 0$.  

In conclusion in the York canonical basis we have the following interpretation of Eqs.(A1):

1) The first six equations (A1) involve the 3-metric $g^\alpha_r = \bar{\phi}^{2/3} g^\alpha_r$ with $det g^\alpha_r = 1$ and $g^\alpha_r = \sum_a \epsilon a V_a(\theta^r) V_{sa}(\theta^r)$. While $\gamma = det g^\alpha_r = \bar{\phi}^2$ is determined by the super-Hamiltonian constraint, $g^\alpha_r = \bar{\phi}^{2/3} g^\alpha_r$ depends upon the three gauge variables $\theta^r$ (the freedom in the 3-coordinates on $\Sigma_\tau$) and on the tidal variables $R_a$ through $Q_a = \epsilon \sum_a \gamma_{aa} R_a$. Therefore the first six equations (A1) imply Eq.(4.1) for $\bar{\phi}$ and $\pi^\alpha_r$. 

\[ \text{The final equations for } \bar{\phi} \text{ and } \pi^\alpha_r \text{ in tetrad gravity are so complicated because the gauge algebra of metric gravity is non-Abelian and not a Lie algebra.} \]
\[ \partial_{r}^{3}g_{rs} = \sum_{a} Q_{a}^{2} \left[ 2V_{ra}(\theta^{i}) V_{sa}(\theta^{i}) \sum_{\bar{a}} \gamma_{\bar{a}a} \partial_{r} R_{\bar{a}} + \partial_{r} \left( V_{ra}(\theta^{i}) V_{sa}(\theta^{i}) \right) \right] + \\
= -\frac{2}{3} \left[ -\frac{12 \pi G}{c^{3}} (1 + n) \pi_{a}^{r} + \left( \bar{\phi}^{-2/3} Q_{a}^{-1} V_{aa}(\theta^{i}) \bar{n}_{(a)} \right) \right] + \\
+ \bar{\phi}^{-2/3} \left[ \left( \bar{\phi}^{2/3} \sum_{a} Q_{a} V_{ra}(\theta^{i}) \bar{n}_{(a)} \right) |_{s} + \left( \bar{\phi}^{2/3} \sum_{a} Q_{a} V_{sa}(\theta^{i}) \bar{n}_{(a)} \right) |_{r} - 2 (1 + n)^{3} K_{rs} \right]. \]

(A3)

The five equations (A3) are a mixture of the two Hamilton-Dirac equations (4.8) for \( R_{\bar{a}} \) plus the three equations (4.5) for \( \partial_{r} \theta^{i} \): in the York canonical basis it is possible to separate the two Eqs.(4.8) from the three Eqs.(4.5).

2) The second group of six equations (A1) involve the extrinsic curvature tensor given in Eq.(2.10). These six equations split into the second one of Eqs.(A2), i.e. Eq.(4.4), for the trace of the extrinsic curvature plus five equations for the traceless extrinsic curvature \( \bar{\phi}^{3} K_{rs} = 3 K_{rs} - \frac{4}{3} g_{rs} \bar{\phi}^{3} K \), which, in the York canonical basis may be decoupled to give the two Eqs.(4.10) for \( \Pi_{\bar{a}} \) and the three equations (4.2) for \( \pi_{\bar{a}}^{(\theta)} \).
APPENDIX B: CALCULATIONS.

In this Appendix there are the calculations needed for the form in the York canonical basis of the Hamilton equations in arbitrary Schwinger time gauges, generated by the Dirac Hamiltonian (3.48), of Section IV.

1. The Functional Derivatives of the Super-Momentum Constraints.

a. The Super-Momentum Constraints

For the evaluation of the functional derivatives it is more convenient to use the following form of the super-momentum constraints obtained from Eq.(3.41)

\[
\tilde{H}_{(a)}(\tau, \bar{\sigma}) = \phi^{-2}(\tau, \bar{\sigma}) \left( \sum_{b \neq a} \sum_{rtiw} \frac{\epsilon_{a}^{\tilde{b}}}{Q_{b} Q_{a}^{-1} - Q_{b} Q_{a}^{-1}} Q_{b}^{-1} V_{rb} V_{wt} B_{iw} \partial_{r} \pi_{i}^{(\theta)} + \right.
\]

\[
+ \sum_{rtiw} \left( \sum_{b \neq a} \frac{\epsilon_{a}^{\tilde{b}}}{Q_{b} Q_{a}^{-1} - Q_{b} Q_{a}^{-1}} \left[ Q_{b}^{-1} \partial_{r} (V_{rb} V_{wt} B_{iw}) + \right. \right.
\]

\[
+ \sum_{ba} \frac{\epsilon_{ba}^{c}}{Q_{c} Q_{b}^{-1} - Q_{b} Q_{c}^{-1}} (V_{rc} V_{ua} - V_{ra} V_{uc}) \partial_{r} V_{ub} V_{wt} B_{iw} \right) \pi_{i}^{(\theta)} + \]

\[
+ Q_{a}^{-1} \sum_{r} V_{ra} \left( \phi^{-1} \partial_{r} \pi_{\tilde{\phi}} + \sum_{b} \gamma_{ba} \partial_{r} \Pi_{b} \right) + \]

\[
+ Q_{a}^{-1} \sum_{rb} \left( \gamma_{ba} \partial_{r} V_{ra} - V_{ra} \partial_{r} R_{b} + \sum_{ub} \gamma_{ub} V_{ua} V_{rb} \partial_{r} V_{ub} \right) \Pi_{b} + \]

\[
+ Q_{a}^{-1} \sum_{r} V_{ra} \tilde{M}_{r}(\tau, \bar{\sigma}) \approx 0, \quad (B1)
\]

This form of the super-momentum constraints will be used to evaluate their functional derivatives with respect to the arguments \( \tilde{\phi} = \phi^{6}, \pi_{\tilde{\phi}}, \theta^{i}, \pi_{i}^{(\theta)}, R_{a}, \Pi_{a} \).

b. The Functional Derivative with Respect to \( \pi_{\tilde{\phi}} \)

\[
\int d^{3}\sigma_{1} \sum_{a} \bar{n}(a)(\tau, \bar{\sigma}_{1}) \frac{\delta \tilde{H}_{(a)}(\tau, \bar{\sigma}_{1})}{\delta \pi_{\tilde{\phi}}(\tau, \bar{\sigma})} =
\]

\[
= - \left[ \phi^{4} \sum_{ra} Q_{a}^{-1} \left( \partial_{r} \bar{n}(a) V_{ra} + \right. \right.
\]

\[
+ \bar{n}(a) \left[ V_{ra} \left( 4 \phi^{-1} \partial_{r} \phi - \sum_{b} \gamma_{ba} \partial_{r} R_{b} \right) + \partial_{r} V_{ra} \right] \right] \right)(\tau, \bar{\sigma}). \quad (B2)
\]
c. The Functional Derivative with Respect to $\tilde{\phi} = \phi^6$

Since we have $\frac{\delta}{\delta \phi(\tau, \sigma)} = \frac{1}{6} \phi^{-5}(\tau, \sigma) \frac{\delta}{\delta \phi(\tau, \sigma)}$, we get

$$
\int d^3\sigma_1 \sum_a \bar{n}_{(a)}(\tau, \bar{\sigma}_1) \frac{\delta \tilde{H}_{(a)}(\tau, \bar{\sigma}_1)}{\delta \phi(\tau, \bar{\sigma})} \approx 6 \left[ \phi^3 \sum_{ra} \bar{n}_{(a)} Q_a^{-1} V_{ra} \partial_r \pi_{\tilde{\phi}} \right](\tau, \bar{\sigma}). \quad (B3)
$$

d. The Functional Derivative with Respect to $\pi_i^{(\theta)}$

By using the results

$$
\frac{\partial}{\partial \pi_i} \frac{Q_b^{-1}}{Q_a Q_a^{-1} - Q_b Q_b^{-1}} = - \frac{Q_b^{-1}}{Q_a Q_a^{-1} - Q_b Q_b^{-1}} \sum_{b'} \left[ \gamma_{bb} - \gamma_{bb} - \gamma_{bc} - (\gamma_{bb} - \gamma_{bc}) \frac{Q_a^{-1} Q_b Q_b^{-1}}{Q_c Q_c^{-1} - Q_b Q_b^{-1}} \right] \partial_{\pi_i} \bar{V}_{rb} \bar{V}_{tw} \bar{B}_{iw}
$$

and

$$
\frac{\partial}{\partial \pi_i} \frac{Q_b^{-1} Q_b Q_c^{-1}}{Q_c Q_c^{-1} - Q_b Q_b^{-1}} = - \frac{Q_b^{-1} Q_b Q_c^{-1}}{Q_c Q_c^{-1} - Q_b Q_b^{-1}} \sum_{b'} \left[ \gamma_{bb} - \gamma_{bb} - \gamma_{bc} - (\gamma_{bb} - \gamma_{bc}) \frac{Q_a^{-1} Q_b Q_b^{-1}}{Q_c Q_c^{-1} - Q_b Q_b^{-1}} \right] \partial_{\pi_i} \bar{V}_{rb} \bar{V}_{tw} \bar{B}_{iw},
$$

we get

$$
\int d^3\sigma_1 \sum_a \bar{n}_{(a)}(\tau, \bar{\sigma}_1) \frac{\delta \tilde{H}_{(a)}(\tau, \bar{\sigma}_1)}{\delta \pi_i^{(\theta)}(\tau, \bar{\sigma})} =
$$

$$
= \phi^{-2}(\tau, \bar{\sigma}) \sum_a \left[ \bar{n}_{(a)} \times \left( \sum_{b' \neq a} \sum_{rtw} \left[ 2 \phi^{-1} \partial_r \phi + \sum_{c \neq a} \gamma_{ca} \partial_r \bar{R}_c \right] \epsilon_{abc} Q_b^{-1} V_{rb} V_{tw} B_{iw} + \sum_{b' \neq a} \sum_{c' \neq b} \epsilon_{bac} \frac{Q_a^{-1} Q_b Q_c^{-1}}{Q_c Q_c^{-1} - Q_b Q_b^{-1}} \left( V_{rc} V_{ua} - V_{ra} V_{uc} \right) \partial_r V_{ab} V_{tw} B_{iw} \right) - \sum_{r} \partial_r \bar{n}_{(a)} \sum_{b' \neq a} \sum_{tw} \epsilon_{abc} \frac{Q_b^{-1} V_{rb} V_{tw} B_{iw}}{Q_b Q_b^{-1} - Q_a Q_a^{-1}} \right](\tau, \bar{\sigma}). \quad (B4)
$$

e. The Functional Derivative with Respect to $\theta_i$

By using the result $\frac{\delta V_{ra}(\theta_i(\bar{\sigma}_1))}{\delta \theta_i(\bar{\sigma}_1)} = \frac{\partial V_{ra}(\theta_i(\bar{\sigma}_1))}{\partial \theta_i(\bar{\sigma}_1)} \delta^3(\bar{\sigma}, \bar{\sigma}_1)$, we get

$$
\sum_r \frac{\partial V_{ra}(\theta_i(\bar{\sigma}_1))}{\delta \theta_i(\bar{\sigma})} \partial_r \delta^3(\bar{\sigma}, \bar{\sigma}_1), \quad \sum_r \partial_r \frac{\delta V_{ra}(\theta_i(\bar{\sigma}_1))}{\delta \theta_i(\bar{\sigma})} =
$$
\[
\int d^3 \sigma_1 \sum_a \bar{\eta}_a(\tau, \bar{\sigma}_1) \frac{\delta \tilde{H}_a(\tau, \bar{\sigma}_1)}{\delta \theta^i(\tau, \bar{\sigma})} = -\phi^{-2}(\tau, \bar{\sigma}) \sum_r \partial_r \bar{\eta}_a(\tau, \bar{\sigma}) \\
\left[ Q_a^{-1} \sum_b \left( \gamma_{ba} \frac{\partial V_{ra}}{\partial \theta^i} + \sum_{sb} \gamma_{bb} V_{sa} \frac{\partial V_{sb}}{\partial \theta^i} \right) \Pi_b + \\
\sum_{jtw} \left( \sum_{b \neq a} \frac{\epsilon_{abt} Q_b^{-1}}{Q_b Q_a^{-1} - Q_c Q_b^{-1}} \frac{\partial V_{rb} V_{wt}}{\partial \theta^i} B_{jw} \right) + \\
\sum_{sb} \sum_{c \neq b} \frac{\epsilon_{bc} Q_a^{-1} Q_b^{-1} Q_c^{-1}}{Q_c Q_b^{-1} - Q_b Q_c^{-1}} (V_{rc} V_{sa} - V_{ra} V_{sc}) \frac{\partial V_{sb}}{\partial \theta^i} V_{wt} B_{jw} \right) \right](\tau, \bar{\sigma}) + \\
+ \phi^{-2}(\tau, \bar{\sigma}) \sum_a \bar{\eta}_a(\tau, \bar{\sigma}) \\
\left[ Q_a^{-1} \sum_r \left( \frac{\partial V_{ra}}{\partial \theta^i} \left[ \phi \partial_r \bar{\eta} + \sum_b \gamma_{ba} \partial_r \Pi_b + \\
\sum_b \left( \gamma_{ba} (2 \phi^{-1} \partial_r \phi + \sum_c \gamma_{ca} \partial_r \Pi_b - \partial_r R_b \Pi_b) \right) \Pi_b \right] - \\
\sum_{sbb} \frac{\partial V_{sb}}{\partial \theta^i} \gamma_{bb} \left[ V_{sa} V_{rb} \left( \partial_r \Pi_b - (2 \phi^{-1} \partial_r \phi + \sum_c \gamma_{ca} \partial_r \Pi_b) \right) + \\
\partial_r (V_{sa} V_{rb}) \Pi_b \right] + \sum_{sbb} \gamma_{bb} \frac{\partial V_{sa} V_{rb}}{\partial \theta^i} \partial_r V_{sb} \Pi_b \right) - \\
\sum_{rjtw} \sum_{c \neq b} \frac{\epsilon_{bc} Q_a^{-1} Q_b^{-1} Q_c^{-1}}{Q_c Q_b^{-1} - Q_b Q_c^{-1}} (V_{rc} V_{sa} - V_{ra} V_{sc}) \frac{\partial V_{sb}}{\partial \theta^i} V_{wt} B_{jw} \partial_r \bar{\eta}^{(\theta)} + \\
\sum_{jrtw} \left( \sum_{b \neq a} \frac{\epsilon_{abt} Q_b^{-1}}{Q_b Q_a^{-1} - Q_c Q_b^{-1}} (2 \phi^{-1} \partial_r \phi + \sum_c \gamma_{ca} \partial_r \Pi_b) \frac{\partial V_{rb} V_{wt}}{\partial \theta^i} B_{jw} \right) + \\
\sum_{sb} \sum_{c \neq b} \frac{\epsilon_{bc} Q_a^{-1} Q_b^{-1} Q_c^{-1}}{Q_c Q_b^{-1} - Q_b Q_c^{-1}} \left[ \frac{\partial V_{sb}}{\partial \theta^i} \left( [2 \phi^{-1} \partial_r \phi + \sum_b \left( \gamma_{ba} - \gamma_{bb} + \gamma_{bc} - \\
(\gamma_{bb} - \gamma_{bc}) \frac{Q_c Q_b^{-1} + Q_b Q_c^{-1}}{Q_c Q_b^{-1} - Q_b Q_c^{-1}} \partial_r R_b \right] (V_{rc} V_{sa} - V_{ra} V_{sc}) V_{wt} B_{jw} - \\
\partial_r \left[ (V_{rc} V_{sa} - V_{ra} V_{sc}) V_{wt} B_{jw} \right] + \\
\partial_r \frac{\partial (V_{rc} V_{sa} - V_{ra} V_{sc}) V_{wt} B_{jw}}{\partial \theta^i} \right] \bar{\eta}^{(\theta)} \right](\tau, \bar{\sigma}). \right]
\right]
\text{(B5)}
f. The Functional Derivative with Respect to $\Pi_{\bar{a}}$

$$\int d^3\sigma_1 \sum_a \tilde{n}_{(a)}(\tau, \bar{\sigma}_1) \frac{\delta \tilde{H}_{(a)}(\tau, \bar{\sigma}_1)}{\delta \Pi_{\bar{a}}(\tau, \bar{\sigma})} =$$

$$= \left[ \phi^{-2} \sum_{ra} Q^{-1}_a \left( \tilde{n}_{(a)} \left[ \gamma_{\bar{a}a} V_{ra} \left( 2 \phi^{-1} \partial_r \phi + \sum_b \gamma_{ba} \partial_r R_b \right) - V_{ra} \partial_r R_{\bar{a}} + \sum_{sb} \gamma_{\bar{a}b} V_{sa} V_{rb} \partial_r V_{sb} \right] - \gamma_{\bar{a}a} \partial_r \tilde{n}_{(a)} V_{ra} \right) \right](\tau, \bar{\sigma}).$$  \hspace{1cm} (B6)

g. The Functional Derivative with Respect to $R_{\bar{a}}$

By using the results

$$\frac{\delta Q_{a}^{n}(\tau, \bar{\sigma}_1)}{\delta R_{a}(\tau, \bar{\sigma})} = n \gamma_{\bar{a}a} Q_{a}^{n}(\tau, \bar{\sigma}) \delta^{3}(\bar{\sigma}, \bar{\sigma}_1),$$

$$\frac{\delta}{\delta R_{a}(\tau, \bar{\sigma})} \frac{Q_{b}^{-1}}{Q_{a}^{-1} - Q_{b}^{-1}} \left( \gamma_{\bar{a}b} - (\gamma_{\bar{a}a} - \gamma_{\bar{a}b}) \left( \frac{Q_{a}^{-1} + Q_{b}^{-1}}{Q_{a}^{-1} - Q_{b}^{-1}} \right) \right) \left( \gamma_{\bar{a}b} - \left[ \gamma_{\bar{a}a} - \gamma_{\bar{a}b} \left( \frac{Q_{a}^{-1} + Q_{b}^{-1}}{Q_{a}^{-1} - Q_{b}^{-1}} \right) \right] \left( \gamma_{\bar{a}b} - \gamma_{\bar{a}c} \left( \frac{2 Q_{b}^{-1}}{Q_{a}^{-1} - Q_{b}^{-1}} \right) \right) \right)(\tau, \bar{\sigma})$$

we get

$$\int d^3\sigma_1 \sum_a \tilde{n}_{(a)}(\tau, \bar{\sigma}_1) \frac{\delta \tilde{H}_{(a)}(\tau, \bar{\sigma}_1)}{\delta R_{a}(\tau, \bar{\sigma})} =$$

$$= \phi^{-2}(\tau, \bar{\sigma}) \sum_{ra} \partial_r \tilde{n}_{(a)}(\tau, \bar{\sigma}) \left[ Q^{-1}_a V_{ra} \Pi_{\bar{a}} - \sum_{b \neq a} \gamma_{\bar{a}b} \sum_{tw} \frac{\epsilon_{abt} Q^{-1}_a}{(Q_{b}^{-1} - Q_{a}^{-1})^2} V_{rb} B_{twv} \pi^{(\theta)}_v \right](\tau, \bar{\sigma}) +$$

$$+ \phi^{-2}(\tau, \bar{\sigma}) \sum_a \tilde{n}_{(a)}(\tau, \bar{\sigma})$$

$$\left[ Q^{-1}_a \sum_{r} \left( \partial_r V_{ra} \Pi_{\bar{a}} + V_{ra} \left[ \partial_r \Pi_{\bar{a}} - (2 \phi^{-1} \partial_r \phi + \sum \gamma_{ba} \partial_r R_b) \Pi_{\bar{a}} \right) - \gamma_{\bar{a}a} \left( \sum_{b} \left( \gamma_{ba} \partial_r V_{ra} + \sum_{sb} \gamma_{\bar{a}b} V_{sa} V_{rb} \partial_r V_{sb} \right) \Pi_{\bar{b}} + \right. \right.$$
- \gamma_{aa} \sum_{b \neq a} \sum_{rtwi} \frac{\epsilon_{abt} Q_a^{-1}}{Q_b Q_a^{-1} - Q_a Q_b^{-1}} V_{rb} V_{wt} B_{iw} \partial_r \pi_{i}^{(0)} + \\
+ \sum_i \sum_{rtw} \left( 2 \sum_{b \neq a} \frac{\epsilon_{abt} Q_a^{-1}}{(Q_b Q_a^{-1} - Q_a Q_b^{-1})^2} \left[ 2 \left( \gamma_{aa} - \gamma_{ab} \right) \phi^{-1} \partial_r \phi + \right.ight. \\
+ \sum_b \left( \gamma_{ab} \gamma_{bb} - \gamma_{ab} \gamma_{ba} \right) \partial_r R_b \right] V_{rb} V_{wt} B_{iw} - \\
- \gamma_{aa} \sum_{b \neq a} \frac{\epsilon_{abt} Q_a^{-1}}{Q_b Q_a^{-1} - Q_a Q_b^{-1}} \partial_r \left( V_{rb} V_{wt} B_{iw} \right) - \\
- \sum_b \sum_{c \neq b} \frac{\epsilon_{bct} Q_c^{-1} Q_b Q_c^{-1}}{Q_b Q_c^{-1} - Q_c Q_b^{-1}} \left[ \gamma_{aa} - \left( \gamma_{ab} - \gamma_{ac} \right) \frac{2 Q_a Q_b^{-1}}{Q_c Q_b^{-1} - Q_b Q_c^{-1}} \right] \\
\sum_s \left( V_{rc} V_{sa} - V_{ra} V_{sc} \right) \partial_r V_{sb} V_{wt} B_{iw} \right) \pi_{i}^{(0)} \right) (\tau, \vec{\sigma}). \quad (B7)

2. The Function \(S\) and its Functional Derivatives.

By using Eqs. (B1) and (B2) of Ref.[4] we have the following expression for the function \(S(\tau, \vec{\sigma})\) defined in Eq.(3.13)

\[ S \overset{def}{=} 3 e \sum_{rs} 3 e_{rs} 3 e_{rs} \sum_{uv} \left( 3 \Gamma_{ru} 3 \Gamma_{sv} - 3 \Gamma_{rs} 3 \Gamma_{uv} \right) = \]

\[ = \phi^2 \sum_a Q_a^{-2} \left( \sum_r \left[ 2 \left( V_{ra} \partial_v \ln \phi + \delta_{rv} \sum_u V_{ua} \partial_u \ln \phi \right) - \right. \right. \\
- 2 \sum_{bu} V_{rb} V_{ab} V_{va} Q_a^{-2} Q_b^{-2} \partial_u \ln \phi + \right. \\
+ \sum_{bbu} \gamma_{bb} V_{rb} \left( \delta_{ab} \partial_v R_b + V_{cb} \sum_u V_{ua} \partial_u R_b \right) - \right. \\
- \sum_{bbu} \gamma_{ba} V_{rb} V_{ab} V_{va} Q_a^{-2} Q_b^{-2} \partial_u R_b + \right. \\
+ \frac{1}{2} \sum_{bu} V_{rb} V_{ua} \left( \partial_u V_{vb} + \partial_v V_{ub} \right) + \right. \\
+ \frac{1}{2} \sum_{bucw} V_{rb} V_{wbcw} Q_c^{-2} \left( \delta_{ac} \left[ \partial_v V_{wc} - \partial_w V_{vc} \right] + \right. \\
+ V_{vc} \sum_u V_{ua} \left[ \partial_u V_{wc} - \partial_w V_{uc} \right] \right) \times \]

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\[
2 \left( V_{va} \partial_r \ln \phi + \delta_{rv} \sum_s V_{sa} \partial_s \ln \phi \right) - \\
- \sum_{ds} V_{vd} V_{sd} V_{ra} Q_a^2 Q_d^{-2} \partial_s \ln \phi \\
+ \sum_{cd} \gamma_{cd} V_{vd} \left( \delta_{ad} \partial_r R_c + V_{rd} \sum_s V_{sa} \partial_s R_c \right) - \\
- \sum_{c\bar{d}s} \gamma_{\bar{c}a} V_{vd} V_{sd} V_{ra} Q_a^2 Q_d^{-2} \partial_s R_c + \\
+ \frac{1}{2} \sum_{ds} V_{vd} V_{sa} \left( \partial_s V_{rd} + \partial_r V_{sd} \right) + \\
+ \frac{1}{2} \sum_{des} V_{vd} V_{sd} Q_e^2 Q_d^{-2} \left( \delta_{ae} \left[ \partial_r V_{se} - \partial_s V_{re} \right] + \\
+ V_{re} \sum_t V_{ta} \left[ \partial_t V_{se} - \partial_s V_{te} \right] \right) + \\
+ \frac{1}{4} \sum_{uvbc} \left[ 2 \partial_r V_{sa} \partial_v V_{sb} \left( V_{ra} V_{rb} V_{bd} \partial_b V_{sa} \right) - \\
2 \sum_{b} \left[ \gamma_{ba} V_{sa} \partial_s V_{ra} + \sum_{vb} \gamma_{bb} V_{ra} V_{vb} V_{sb} \partial_v V_{sa} \right] \partial_r R_b + \\
+ Q_b^2 Q_c^{-2} \partial_r V_{sa} \partial_v V_{rb} \left( (V_{ra} V_{sc} + V_{rc} V_{sa}) (V_{ua} V_{uc} + V_{uc} V_{va}) - 4 V_{rc} V_{sa} V_{uc} V_{va} \right) \right]. \tag{B8}
\]

\( S(\tau, \bar{\sigma}) \) is a function of \( \tilde{\phi} = \phi^6, \theta^i \) and \( R_a \), which, being 3-coordinate-dependent, plays the role of an inertial potential for the gravitational field.

Due to Eq.(3.14), the function \( S(\tau, \bar{\sigma}) \) appears in the Dirac Hamiltonian in the form
\[
\int d^3\sigma \left[ (1 + n) S \right](\tau, \bar{\sigma}),
\]
so that we need the functional derivatives of this quantity.
a. The Functional Derivative with Respect to $R_a$

\[
\int d^3 \sigma_1 \left[ 1 + n(\tau, \bar{\sigma}_1) \right] \frac{\delta S(\tau, \bar{\sigma}_1)}{\delta R_a(\tau, \bar{\sigma})} =
\]

\[
= 2 \left( \phi^2 \sum \limits_{rsa} Q_a^{-2} \right) \\
\left[ \partial_r n \left( V_{ra} V_{sa} \left[ 2 \gamma_{aa} \phi^{-1} \partial_s \phi - \sum \limits_b (2 \gamma_{aa} \gamma_{ba} - \delta_{a \bar{b}}) \partial_s R_b \right] + \right. \right.
\]

\[
+ \gamma_{aa} V_{sa} \partial_s V_{ra} + \sum \limits_{vb} \gamma_{ba} V_{ra} V_{vb} V_{sb} \partial_v V_{sa} \right) -
\]

\[
- (1 + n) \left( V_{ra} V_{sa} \left[ 2 \gamma_{aa} (-\phi^{-1} \partial_r \partial_s \phi + 3 \phi^{-1} \partial_r \phi \phi^{-1} \partial_s \phi) + \right. \right.
\]

\[
+ 2 \phi^{-1} \partial_r \phi \sum \limits_b (2 \gamma_{aa} \gamma_{ba} - \delta_{a \bar{b}}) \partial_s R_b +
\]

\[
+ \sum \limits_b \left( 2 \gamma_{aa} \gamma_{ba} - \delta_{a \bar{b}} \right) \partial_r \partial_s R_b + \sum \limits_{b \bar{c}} \left[ 2 \gamma_{ba} (\delta_{a \bar{c}} - \gamma_{aa} \gamma_{ca} - \gamma_{aa} \delta_{b \bar{c}}) \partial_r R_b \partial_s R_c \right] -
\]

\[
- 2 \phi^{-1} \partial_r \phi \left[ \gamma_{aa} V_{sa} \partial_s V_{ra} + \sum \limits_{vb} \gamma_{ba} V_{ra} V_{vb} V_{sb} \partial_v V_{sa} \right] +
\]

\[
+ \sum \limits_b \left[ (2 \gamma_{aa} \gamma_{ba} - \delta_{a \bar{b}}) \partial_s \left( V_{ra} V_{sa} \right) + 2 \sum \limits_{vb} \left( \gamma_{ba} \gamma_{a \bar{b}} - \gamma_{aa} \gamma_{b \bar{b}} \right) V_{ra} V_{vb} V_{sb} \partial_v V_{sa} \right] \partial_r R_b -
\]

\[
- \partial_r \left( \gamma_{aa} V_{sa} \partial_s V_{ra} + \sum \limits_{vb} \gamma_{a \bar{b}} V_{ra} V_{vb} V_{sb} \partial_v V_{sa} \right) + \right.
\]

\[
+ \frac{1}{4} \sum \limits_{uvbc} \left[ 2 \gamma_{aa} \partial_r V_{sc} \partial_u V_{vb} \left( V_{ra} V_{sb} (V_{uc} V_{va} - V_{ua} V_{vc}) + V_{rb} V_{sa} (V_{ua} V_{vc} + V_{uc} V_{va}) \right) + \right.
\]

\[
+ \left( \gamma_{aa} - \gamma_{a \bar{b}} + \gamma_{a \bar{c}} \right) Q_2^{-2} Q_c^{-2} \partial_r V_{sb} \partial_u V_{vb} \left( (V_{ra} V_{sc} + V_{rb} V_{sa}) (V_{ua} V_{vc} + V_{uc} V_{va}) - \right.
\]

\[
- 4 V_{rc} V_{sa} V_{uc} V_{va} \right] \right) \right) (\tau, \bar{\sigma}). \quad (B9)
\]
b. The Functional Derivative with Respect to $\bar{\phi} = \phi^6$

$$\int d^3\sigma_1 [1 + n(\tau, \vec{\sigma}_1)] \frac{\delta S(\tau, \vec{\sigma}_1)}{\delta \phi(\tau, \vec{\sigma})} =$$

$$= 2 \left( \phi^6 (1 + n) \sum_{r,s} Q_a^{-2} \right) - 8 \left( \phi^{-1} \partial_r \phi \partial_s (V_{ra} V_{sa}) + V_{ra} V_{sa} \phi^{-1} \partial_r \phi \sum_b \gamma_{ba} \partial_s \phi \right) +$$

$$+ V_{ra} V_{sa} \left( \phi^{-1} \partial_r \phi \sum_b \gamma_{ba} \partial_s \phi - 2 \phi^{-1} \partial_r \phi \sum_b \gamma_{ba} \partial_s \phi \right) +$$

$$+ V_{ra} V_{sa} \left( 2 \sum_b \gamma_{ba} \partial_r \phi \sum_{b,c} (2 \gamma_{ba} \gamma_{ca} + \delta_{bc}) \partial_r \phi \partial_s \phi \right) -$$

$$- 2 \sum_b \left( \gamma_{bb} V_{ra} V_{rb} \partial_s \phi \right) \partial_r \phi \partial_s \phi +$$

$$+ \sum_{r,s} \gamma_{bb} V_{ra} V_{rb} \partial_s \phi \partial_r \phi \partial_s \phi - \partial_r \phi \partial_s \phi (V_{ra} V_{sa}) +$$

$$+ \frac{1}{4} \sum_{svbc} \left[ 2 \partial_r \phi \partial_s \phi \partial_t \phi V_{ra} V_{rb} \partial_s \phi \partial_r \phi \partial_s \phi \right] +$$

$$+ Q_b^2 Q_c^{-2} \partial_r \phi \partial_s \phi \partial_t \phi \left[ (V_{ra} V_{sc} + V_{rc} V_{sa}) (V_{sa} V_{rc} + V_{uc} V_{va}) - 4 V_{ra} V_{sa} V_{rc} V_{va} \right] \right] -$$

$$- \phi \sum_{r,s} \partial_r \phi \partial_s \phi \left( 2 V_{ra} V_{sa} \left( 4 \phi^{-1} \partial_r \phi \partial_s \phi - \sum_b \gamma_{ba} \partial_s \phi \partial_r \phi \right) \right) (\tau, \vec{\sigma}).$$

(B10)

c. The Functional Derivative with Respect to $\theta^i$

$$\int d^3\sigma_1 [1 + n(\tau, \vec{\sigma}_1)] \frac{\delta S(\tau, \vec{\sigma}_1)}{\delta \theta^i(\tau, \vec{\sigma})} =$$

$$= - \left[ \phi^2 \sum_{r,s} Q_a^{-2} \partial_r \phi \sum_{v,b} \left( \gamma_{ba} V_{ra} \delta_{sv} \partial_v \phi \right) -$$

$$- 2 \partial_r \phi \partial_s \phi \sum_{v,b} \left[ \gamma_{ba} V_{ra} \delta_{sv} \partial_v \phi \right] \partial_v \phi \partial_s \phi +$$

$$+ \sum_{u,v,b} \partial_r \phi \partial_s \phi \left[ V_{ra} V_{sb} (V_{uc} V_{va} - V_{ua} V_{vc}) + V_{ra} V_{sb} (V_{ua} V_{vc} + V_{uc} V_{va}) \right] +$$

$$+ \frac{1}{2} \sum_{u,v,b} Q_b^2 Q_c^{-2} \partial_r \phi \partial_s \phi \partial_t \phi \partial_v \phi \left[ (V_{ra} V_{sc} + V_{rc} V_{sa}) (V_{sa} V_{rc} + V_{uc} V_{va}) - 4 V_{ra} V_{sa} V_{rc} V_{va} \right] \right] (\tau, \vec{\sigma}) +$$

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\[
+ \left[ \phi^2 (1 + n) \sum_{rsab} Q_a^{-2} \left( \frac{\partial V_{rb} V_{sa}}{\partial \theta_i} \right) - 2 \phi^{-1} \partial_r \partial_s \phi + 6 \phi^{-1} \partial_r \phi \phi^{-1} \partial_s \phi + \\
+ \sum_{\delta_c} (2 \gamma_{\delta a} \gamma_{\delta} - \delta_{\delta c}) \partial_r R_{\delta b} \partial_s R_c \right] - \\
- 2 \sum_{\delta_{bb}} \gamma_{\delta b} \frac{\partial V_{rb} V_{sb}}{\partial \theta_i} \partial_{\tau} V_{sa} \partial_r R_b + \\
+ 2 \partial V_{sa} \sum_{\delta_{bb}} \left[ \gamma_{\delta a} \left( V_{rb} \partial_r \partial_s R_b + \left[ \partial_r V_{rb} + 2 V_{rb} \left( \phi^{-1} \partial_r \phi - \sum_{\delta} \gamma_{\delta a} \partial_r R_c \right) \partial_s R_b - \partial_s V_{rb} \partial_r R_b \right) \right] + \\
+ \sum_{\delta_{bb}} \gamma_{\delta b} \left( V_{rb} V_{rb} V_{sb} \left[ \partial_r \partial_s R_b \right] + \partial_r \left( V_{rb} V_{rb} V_{sb} \partial_{\tau} R_b \right) \right] + \\
+ 2 \sum_{\delta_{bb}} \left[ \frac{1}{2} \partial_r V_{sc} \partial_u V_{vb} \right.
\]
3. The Laplace-Beltrami operator $\hat{\triangle}$, the Function $T$ and their Functional Derivatives

The Laplace-Beltrami operator associated with the 3-metric $\hat{g}_{rs}$ ($\det \hat{g}_{rs} = 1$) appearing in Eq.(3.43) is

$$\hat{\triangle} = \partial_r (\hat{g}_{rs} \partial_s) = \hat{g}_{rs} \hat{\nabla}_r \hat{\nabla}_s = \partial_r \left( \sum_a Q_a^{-2} V_{ra} V_{sa} \partial_s \right) = \sum_a Q_a^{-2} \left[ V_{ra} V_{sa} \partial_r \partial_s - 2 V_{ra} V_{sa} \sum_b \gamma_{ba} \partial_r R_b - \partial_r (V_{ra} V_{sa}) \partial_s \right]. \quad (B12)$$

From Eqs.(3.13) and (2.10) we have

$$T(\tau, \vec{\sigma}) = \sum_r \partial_r \left( 3 \hat{c} \sum_{suv} 3 \hat{g}_{rs} \hat{g}_{uv} (\partial_u \hat{g}_{us} - \hat{h}_s \hat{g}_{uv}) \right)(\tau, \vec{\sigma}) =$$

$$= - \sum_{r,s,a} \partial_r \left( \phi^2 Q_a^{-2} \left[ 2 V_{ra} V_{sa} (4 \phi^{-1} \partial_s \phi - \sum_b \gamma_{ba} \partial_s R_b) + \partial_s (V_{ra} V_{sa}) \right] \right)(\tau, \vec{\sigma}) =$$

$$= - \phi^2 \sum_{r,s,a} Q_a^{-2} \left[ 2 V_{ra} V_{sa} \left[ 4 (\phi^{-1} \partial_r \partial_s \phi + \phi^{-1} \partial_r \phi \partial_s \phi) - \sum_b \gamma_{ba} \partial_r \partial_s R_b + 2 (5 \phi^{-1} \partial_s \phi - \sum_c \gamma_{ca} \partial_s \partial_c \phi) \partial_r R_b \right] \right] +$$

$$+ 2 \partial_r (V_{ra} V_{sa}) \left( 5 \phi^{-1} \partial_s \phi - 2 \sum_b \gamma_{ba} \partial_s R_b \right) + \partial_r \partial_s (V_{ra} V_{sa}) \right)(\tau, \vec{\sigma}) =$$

$$\overset{\text{def}}{=} \left( - 8 \phi \hat{\triangle} \phi + T_1 \right)(\tau, \vec{\sigma}),$$

$$T_1(\tau, \vec{\sigma}) = - \phi^2 \sum_{r,s,a} Q_a^{-2} \left[ 2 V_{ra} V_{sa} \left[ 4 \phi^{-1} \partial_r \phi \partial_s \phi - \sum_b \gamma_{ba} \partial_r \partial_s R_b \right] + 2 (\phi^{-1} \partial_s \phi - \sum_c \gamma_{ca} \partial_s \partial_c \phi) \partial_r R_b \right] +$$

$$+ 2 \partial_r (V_{ra} V_{sa}) \left( \phi^{-1} \partial_s \phi - 2 \sum_b \gamma_{ba} \partial_s R_b \right) + \partial_r \partial_s (V_{ra} V_{sa}) \right)(\tau, \vec{\sigma}). \quad (B13)$$

In the Dirac Hamiltonian (3.48) there is the term $-\frac{e^3}{16\pi G} \int d^3 \sigma \left[ n T \right](\tau, \vec{\sigma})$. To evaluate its functional derivatives we use the form $T = -8 \phi \hat{\triangle} \phi + T_1$. First we evaluate the functional derivatives of the term $\frac{e^3}{2\pi G} \int d^3 \sigma \left[ n \phi \hat{\triangle} \phi \right](\tau, \vec{\sigma})$ and then of the quantity $-\frac{e^3}{16\pi G} \int d^3 \sigma \left[ n T \right](\tau, \vec{\sigma}) = -\frac{e^3}{16\pi G} \int d^3 \sigma \left[ n \left( -8 \phi \hat{\triangle} \phi + T_1 \right) \right](\tau, \vec{\sigma})$. 

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a. The Functional Derivative with Respect to $R_a$

\[
\int d^3 \sigma_1 \frac{\delta}{\delta R_a(\tau, \vec{\sigma})} \left[ n(\phi \hat{\Delta} \phi)(\tau, \vec{\sigma}_1) \right] = 2 \left[ \phi^2 \gamma_{aa} Q_a^{-2} V_{ra} V_{sa} \phi^{-1} \partial_s \phi \left( n \phi^{-1} \partial_r \phi + \partial_r n \right) \right](\tau, \vec{\sigma}),
\]

(B14)

\[
\int d^3 \sigma_1 n(\tau, \vec{\sigma}_1) \frac{\delta T(\tau, \vec{\sigma}_1)}{\delta R_a(\tau, \vec{\sigma})} = -8 \int d^3 \sigma_1 \frac{\delta}{\delta R_a(\tau, \vec{\sigma})} \left[ n(\phi \hat{\Delta} \phi)(\tau, \vec{\sigma}_1) \right] + 2 \left[ \phi^2 \gamma_{aa} Q_a^{-2} V_{ra} V_{sa} \left( \partial_r \partial_s n + 2 \partial_r n \phi^{-1} \partial_s \phi + 8 n \phi^{-1} \partial_r \phi \phi^{-1} \partial_s \phi \right) \right](\tau, \vec{\sigma}) =
\]

\[
= 2 \left[ \phi^2 \gamma_{aa} Q_a^{-2} V_{ra} V_{sa} \left( \partial_r \partial_s n - 6 \phi^{-1} \partial_s \phi \partial_r n \right) \right](\tau, \vec{\sigma}).
\]

(B15)

b. The Functional Derivative with Respect to $\tilde{\phi} = \phi^6$

\[
\int d^3 \sigma_1 \frac{\delta}{\delta \tilde{\phi}(\tau, \vec{\sigma})} \left[ n(\phi \hat{\Delta} \phi)(\tau, \vec{\sigma}_1) \right] =
\]

\[
= \left[ 2 n \hat{\Delta} \phi + 2 \sum_a Q_a^{-2} V_{ra} V_{sa} \partial_r n \partial_s \phi + \phi \hat{\Delta} n \right](\tau, \vec{\sigma}).
\]

(B16)

\[
\int d^3 \sigma_1 n(\tau, \vec{\sigma}_1) \frac{\delta T(\tau, \vec{\sigma}_1)}{\delta \tilde{\phi}(\tau, \vec{\sigma})} = -8 \int d^3 \sigma_1 \frac{\delta}{\delta \tilde{\phi}(\tau, \vec{\sigma})} \left[ n(\phi \hat{\Delta} \phi)(\tau, \vec{\sigma}_1) \right] + 2 \left[ 8 n \hat{\Delta} \phi + \phi Q_a^{-2} \partial_r n \left[ \partial_s (V_{ra} V_{sa}) + 2 V_{ra} V_{sa} \left( 4 \phi^{-1} \partial_s \phi - \sum_b \gamma_{ba} \partial_s R_b \right) \right] \right](\tau, \vec{\sigma}) =
\]

\[
= -2 \left[ \phi \sum Q_a^{-2} \left[ 4 V_{ra} V_{sa} \partial_r \partial_s n - 3 \left( 2 V_{ra} V_{sa} \gamma_{ba} \partial_r R_b - \partial_r (V_{ra} V_{sa}) \partial_s n \right) \right] \right](\tau, \vec{\sigma}).
\]

(B17)
c. The Functional Derivative with Respect to $\theta^i$

$$
\int d^3\sigma_1 \frac{\delta \left[ n \phi \hat{\Delta} \phi \right](\tau, \vec{\sigma}_1)}{\delta \theta^i(\tau, \vec{\sigma})} = -\left( \phi^2 \sum_a Q_{a}^{-2} \frac{\partial V_{ra}}{\partial \theta^i} \left[ n \phi^{-1} \partial_r \phi + \partial_r n \right] \phi^{-1} \partial_s \phi \right)(\tau, \vec{\sigma}),
$$

(B18)

$$
\int d^3\sigma_1 n(\tau, \vec{\sigma}_1) \frac{\delta T(\tau, \vec{\sigma}_1)}{\delta \theta^i(\tau, \vec{\sigma})} = -8 \int d^3\sigma_1 \frac{\delta \left[ n \phi \hat{\Delta} \phi \right](\tau, \vec{\sigma}_1)}{\delta \theta^i(\tau, \vec{\sigma})} -

- \left[ \phi^2 \sum_{rsa} Q_{a}^{-2} \frac{\partial V_{ra}}{\partial \theta^i} \left( \partial_r \partial_s n + 2 \partial_r n \phi^{-1} \partial_s \phi - 8 n \phi^{-1} \partial_r \phi \partial_s \phi \right) \right](\tau, \vec{\sigma}) =

- \left( \phi^2 \sum_{rsa} Q_{a}^{-2} \frac{\partial V_{ra}}{\partial \theta^i} \left[ \partial_r \partial_s n - 6 \partial_r n \phi^{-1} \partial_s \phi \right] \right)(\tau, \vec{\sigma}).
$$

(B19)

4. The Function $^3\hat{R}$.

As shown in Ref.[6], for the evaluation of $^3\hat{R}$ (a function of $\theta^i$ and $R_{a}$), appearing in Eqs. (3.43) and (3.44), we need the following results (see Eqs.(A25) of Ref.[2] and Eq.(2.10) for the spin connection $^3\omega_{r(a)}$)

$$
^3R[\theta^n, \phi, R_a] = \sum_{abrs} 3e \epsilon_{(a)(b)(c)} \epsilon^r_{(a)} \epsilon^s_{(b)} \hat{\Omega}_{rs(c)} = 3e \left[ \partial_r ^3\omega_{s(a)(b)} - \partial_s ^3\omega_{r(a)(b)} + \right.

+ \left. \sum_e \left( ^3\omega_{r(a)(e)} ^3\omega_{s(e)(b)} - ^3\omega_{s(a)(e)} ^3\omega_{r(e)(b)} \right) \right] =

= \phi^{-5} \left( -8 \hat{\Delta} \phi + ^3\hat{R} \phi \right) = \phi^{-6} \left( S + T \right),

\downarrow

^3\hat{R}[\theta^n, R_a] = \phi^{-2} \left( S + T_1 \right) =
$$
\[
\begin{align*}
&= \sum_{rsa} Q_a^{-2} \left( - \partial_r \partial_s (V_{ra} V_{sa}) + \\
&+ 2 \sum_{b} \left[ \gamma_{ba} \left( 2 \partial_s (V_{ra} V_{sa}) - V_{sa} \partial_s V_{ra} \right) - \sum_{vb} \gamma_{\bar{b}a} V_{ra} V_{vb} V_{sb} \partial_v V_{sa} \right] \partial_r R_b - \\
&- V_{ra} V_{sa} \sum_{b} \left[ \partial_r R_b \partial_s R_b + 2 \gamma_{ba} \left( \sum_{c} \gamma_{\bar{c}a} \partial_s R_c \partial_r R_b - \partial_r \partial_s R_b \right) \right] + \\
&+ \frac{1}{4} \sum_{uvbc} \left[ 2 \partial_r V_{sc} \partial_a V_{vb} \left( V_{ra} V_{sb} (V_{uc} V_{va} - V_{ua} V_{vc}) + V_{rb} V_{sa} (V_{uc} V_{vc} + V_{uc} V_{va}) + \\
&+ Q_b Q_c^{-2} \partial_r V_{sb} \partial_a V_{vb} \left( (V_{ra} V_{sc} + V_{rc} V_{sa}) (V_{ua} V_{vc} + V_{uc} V_{va}) - 4 V_{rc} V_{sa} V_{uc} V_{va} \right) \right] \right). \\
\end{align*}
\]

(B20)

5. The Functional Derivatives of $\dot{\mathcal{M}}$

We need the derivatives with respect $\theta^i$, $\phi$ and $R_a$ of the mass density $\dot{\mathcal{M}}(\tau, \vec{\sigma})$, appearing in the combination $\int d^3\sigma \left[ (1 + n) \dot{\mathcal{M}}(\tau, \vec{\sigma}) \right]$ in the Dirac Hamiltonian (3.48), given in Eqs.(3.37). They are

\[
\int d^3\sigma \left[ (1 + n(\tau, \vec{\sigma})) \right] \frac{\delta \dot{\mathcal{M}}(\tau, \vec{\sigma})}{\delta \theta^i(\tau, \vec{\sigma})} =
\]

\[
= \frac{1}{2} \sum_i \delta^3(\vec{\sigma}, \vec{\eta}_i(\tau)) \eta_i \left( 1 + n \right)
\]

\[
\frac{\phi^{-4} \sum_{rsa} Q_a^{-2} \frac{\partial V_{ra} V_{sa}}{\partial \theta^i} \left( \kappa_{ir}(\tau) - \frac{Q_c}{c} A_{\perp r} \right) \left( \kappa_{is}(\tau) - \frac{Q_c}{c} A_{\perp s} \right)}{\sqrt{m_i^2 c^2 + \phi^{-4} \sum_{rsa} Q_a^{-2} V_{ra} V_{sa} \left( \kappa_{ir}(\tau) - \frac{Q_c}{c} A_{\perp r} \right) \left( \kappa_{is}(\tau) - \frac{Q_c}{c} A_{\perp s} \right)}} (\tau, \vec{\sigma}) +
\]

\[
+ \left( 1 + n(\tau, \vec{\sigma}) \right) \left( \phi^{-1/3} \right)
\]

\[
\left[ \frac{1}{2c} \sum_{rsa} Q_a^2 \frac{\partial V_{ra} V_{sa}}{\partial \theta^i} \pi^s_{\perp} \pi^s_{\perp} + \frac{1}{4c} \sum_{abrsuv} Q_a^{-2} Q_b^{-2} \frac{\partial V_{ra} V_{sa} V_{ub} V_{vb}}{\partial \theta^i} F_{ru} F_{sv} - \\
- \frac{1}{2c} \sum_{rsan} Q_a^2 \frac{\partial V_{ra} V_{sa}}{\partial \theta^i} \left( 2 \pi^r_{\perp} - \sum_m \delta^{rm} \sum_i Q_i \eta_i \partial_m c(\vec{\sigma}, \vec{\eta}_i(\tau)) \right) \right] (\tau, \vec{\sigma}),
\]

(B21)
\[
\int d^3\sigma_1 \left(1 + n(\tau, \bar{\sigma}_1)\right) \frac{\delta \tilde{M}(\tau, \bar{\sigma}_1)}{\delta \phi(\tau, \bar{\sigma})} = \\
= -2 \sum_i \delta^3(\bar{\sigma}, \bar{\eta}_i(\tau)) \eta_i \left(1 + n\right) \\
\frac{\phi^{-5} \sum_{rs} Q_a^2 V_r V_s \left(\bar{\kappa}_{ir}(\tau) - \frac{\Omega_4}{c} A_{ir}\right) \left(\bar{\kappa}_{is}(\tau) - \frac{\Omega_4}{c} A_{is}\right)}{\sqrt{m_i^2 c^2 + \phi^{-4} \sum_{rs} Q_a^2 V_r V_s \left(\bar{\kappa}_{ir}(\tau) - \frac{\Omega_4}{c} A_{ir}\right) \left(\bar{\kappa}_{is}(\tau) - \frac{\Omega_4}{c} A_{is}\right)}}(\tau, \bar{\sigma}) - \\
\left(1 + n(\tau, \bar{\sigma})\right) \left(\bar{\phi}^{-1/3}\right) \\
\left[\frac{1}{c} \sum_{rs} Q_a^2 V_r V_s \pi_{\perp}^r \pi_{\perp}^s + \frac{1}{2c} \sum_{a brs uv} \frac{Q_a^2}{Q_b^2} V_r V_s V_{ub} V_{vb} F_{ru} F_{sv} - \\
\frac{1}{c} \sum_{rs} Q_a^2 V_r V_s \left(2 \pi_{\perp}^r - \sum_m \delta_{rm} \sum_i Q_i \eta_i \partial_m c(\bar{\sigma}, \bar{\eta}_j(\tau))\right) \\
\delta^{sn} \sum_j Q_j \eta_j \partial_n c(\bar{\sigma}, \bar{\eta}_j(\tau))\right](\tau, \bar{\sigma}),
\]

(B22)

\[
\int d^3\sigma_1 \left(1 + n(\tau, \bar{\sigma}_1)\right) \frac{\delta \tilde{M}(\tau, \bar{\sigma}_1)}{\delta \tilde{R}_a(\tau, \bar{\sigma})} = \\
= -\sum_i \delta^3(\bar{\sigma}, \bar{\eta}_i(\tau)) \eta_i \left(1 + n\right) \\
\frac{\phi^{-4} \sum_{rs} \gamma_{\bar{a}} Q_a^2 V_r V_s \left(\bar{\kappa}_{ir}(\tau) - \frac{\Omega_4}{c} A_{ir}\right) \left(\bar{\kappa}_{is}(\tau) - \frac{\Omega_4}{c} A_{is}\right)}{\sqrt{m_i^2 c^2 + \phi^{-4} \sum_{rs} Q_a^2 V_r V_s \left(\bar{\kappa}_{ir}(\tau) - \frac{\Omega_4}{c} A_{ir}\right) \left(\bar{\kappa}_{is}(\tau) - \frac{\Omega_4}{c} A_{is}\right)}}(\tau, \bar{\sigma}) + \\
\left(1 + n(\tau, \bar{\sigma})\right) \left(\bar{\phi}^{-1/3}\right) \\
\left[\frac{1}{c} \sum_{rs} \gamma_{\bar{a}} Q_a^2 V_r V_s \pi_{\perp}^r \pi_{\perp}^s - \frac{1}{2c} \sum_{a brs uv} \frac{\gamma_{\bar{a}}}{\gamma_{\bar{b}}} Q_a^2 Q_b^{-2} V_r V_s V_{ub} V_{vb} F_{ru} F_{sv} - \\
\frac{1}{c} \sum_{rs} \gamma_{\bar{a}} Q_a^2 V_r V_s \left(2 \pi_{\perp}^r - \sum_m \delta_{rm} \sum_i Q_i \eta_i \partial_m c(\bar{\sigma}, \bar{\eta}_j(\tau))\right) \\
\delta^{sn} \sum_j Q_j \eta_j \partial_n c(\bar{\sigma}, \bar{\eta}_j(\tau))\right](\tau, \bar{\sigma}),
\]

(B23)
APPENDIX C: THE QUANTITIES OF APPENDIX B IN THE 3-ORTHOGONAL GAUGES.

In this Appendix we give the restriction of the quantities evaluated in Appendix B to the 3-orthogonal gauges.

We shall use the notation introduced after Eq.(6.1).

The functional derivatives of Eqs.(B2) - (B7) become

\[ \tilde{\mathcal{H}}_{(a)}|_{\theta^i=0}(\tau, \bar{\sigma}) = \phi^{-2}(\tau, \bar{\sigma}) \left[ \sum_{b \neq a} \sum_i \frac{\epsilon_{aba} Q_b^{-1}}{Q_b Q_a^{-1} - Q_a Q_b^{-1}} \partial_b \pi_i^{(\theta)} + \right. \\
+ 2 \sum_{b \neq a} \sum_i \frac{\epsilon_{aba} Q_b^{-1}}{(Q_b Q_a^{-1} - Q_a Q_b^{-1})^2} \sum_c (\gamma_{ca} - \gamma_{cb}) \partial_b R_c \pi_i^{(\theta)} + \\
+ Q_a^{-1} \left( \phi^6 \partial_a \pi_\phi + \sum_b (\gamma_{ba} \partial_a \Pi_b - \partial_a R_b \Pi_b) + \mathcal{N}_{(a)} \right) \right] (\tau, \bar{\sigma}) \approx 0. \]  

(C1)

\[
\int d^3 \sigma_1 \sum_a \tilde{n}_{(a)}(\tau, \bar{\sigma}_1) \frac{\delta \tilde{\mathcal{H}}_{(a)}(\tau, \bar{\sigma}_1)}{\delta \pi_\phi(\tau, \bar{\sigma})} \big|_{\theta^i=0} = \\
= -\left[ \phi^4 \sum_a Q_a^{-1} \left( \partial_a \tilde{n}_{(a)} + \tilde{n}_{(a)} \left( 4 \phi^{-1} \partial_a \phi - \sum_b \gamma_{ba} \partial_a R_b \right) \right) \right] (\tau, \bar{\sigma}). \]  

(C2)

\[
\int d^3 \sigma_1 \sum_a \tilde{n}_{(a)}(\tau, \bar{\sigma}_1) \frac{\delta \tilde{\mathcal{H}}_{(a)}(\tau, \bar{\sigma}_1)}{\delta \partial_\theta \pi_i^{(\theta)}(\tau, \bar{\sigma})} \big|_{\theta^i=0} \approx 6 \left[ \phi^3 \sum_a \tilde{n}_{(a)} Q_a^{-1} \partial_a \pi_\phi \right] (\tau, \bar{\sigma}). \]  

(C3)

\[
\int d^3 \sigma_1 \sum_a \tilde{n}_{(a)}(\tau, \bar{\sigma}_1) \frac{\delta \tilde{\mathcal{H}}_{(a)}(\tau, \bar{\sigma}_1)}{\delta \partial_\theta \pi_i^{(\theta)}(\tau, \bar{\sigma})} \big|_{\theta^i=0} = \\
= \left( \phi^{-2} \sum_{a \neq b} \frac{\epsilon_{iab} Q_b^{-1}}{Q_b Q_a^{-1} - Q_a Q_b^{-1}} \left[ \left( \frac{1}{3} \phi^{-1} \partial_b \phi + \sum_b \gamma_{ba} \partial_b R_b \right) \tilde{n}_{(a)} - \partial_b \tilde{n}_{(a)} \right] \right) (\tau, \bar{\sigma}). \]  

(C4)
\[
\int d^3 \sigma_1 \sum_a n(a) (\tau, \vec{\sigma}_1) \left. \frac{\delta \tilde{H}(a)(\tau, \vec{\sigma}_1)}{\delta \theta^i(\tau, \vec{\sigma})} \right|_{\theta^i=0} = \\
= -\phi^{-2}(\tau, \vec{\sigma}) \sum_{ra} \partial_r n(a)(\tau, \vec{\sigma}) \\
\left[ Q_a^{-1} \sum_b \left( \gamma_{ba} V_{(i)ra} + \gamma_{br} V_{(i)ar} \right) \Pi_b + \\
\sum_j \left( \sum_t \sum_{b \neq a} \frac{\epsilon_{abt} Q^{-1}_b}{Q_b Q^{-1}_a - Q_b Q^{-1}_c} \left[ \delta_{tj} V_{(i)rb} + \delta_{rb} (V_{(i)jt} + B_{(i)jt}) \right] + \\
\sum_c \sum_{c \neq b} \frac{\epsilon_{bcj} (\delta_{rc} V_{(i)ab} - \delta_{ra} V_{(i)cb}) (Q^{-1}_a Q^{-1}_b - Q^{-1}_c)}{Q_c Q^{-1}_b - Q_b Q^{-1}_c} \right) \pi_j^{(\theta)}(\tau, \vec{\sigma}) + \\
\phi^{-2}(\tau, \vec{\sigma}) \sum_a n(a)(\tau, \vec{\sigma}) \\
\left[ Q_a^{-1} \left( \sum_r V_{(i)ra} \left[ \phi^6 \partial_r \Pi_c + \sum_b \left( \gamma_{ba} (2 \phi^{-1} \partial_r \phi + \sum_c \gamma_{ca} \partial_r R_c) \Pi_b - \\
\partial_r R_b \Pi_b \right) \right] + \sum_b \gamma_{ba} \partial_r \Pi_b + \mathcal{M}_r \right] - \\
\sum_{bb} V_{(i)ab} \gamma_{bb} \left[ \partial_b \Pi_b - \left( 2 \phi^{-1} \partial_{rb} \phi + \sum_c \gamma_{ca} \partial_b R_c \right) \Pi_b \right] \right] - \\
\sum_{rjb} \sum_{c \neq b} \frac{\epsilon_{bcj} Q^{-1}_a Q^{-1}_c Q^{-1}_b}{Q_c Q^{-1}_b - Q_b Q^{-1}_c} \left[ \delta_{rc} V_{(i)ab} - \delta_{ra} V_{(i)cb} \right] \partial_r \pi_j^{(\theta)} + \\
\sum_{jr} \left( \sum_t \sum_{b \neq a} \frac{\epsilon_{abt} Q^{-1}_b}{Q_b Q^{-1}_a - Q_b Q^{-1}_c} \left( 2 \phi^{-1} \partial_r \phi + \sum_c \gamma_{ca} \partial_r R_c \right) \right] \left[ \delta_{tj} V_{(i)rb} + \delta_{rb} (V_{(i)jt} + B_{(i)jt}) \right] + \\
\sum_b \sum_{c \neq b} \frac{\epsilon_{bcj} Q^{-1}_a Q^{-1}_c Q^{-1}_b}{Q_c Q^{-1}_b - Q_b Q^{-1}_c} \left( \delta_{rc} V_{(i)ab} - \delta_{ra} V_{(i)cb} \right) \left[ 2 \phi^{-1} \partial_r \phi + \\
\sum_b \left( \gamma_{ba} - \gamma_{bb} + \gamma_{bc} - (\gamma_{bb} - \gamma_{bc}) \frac{Q_c Q^{-1}_b + Q_b Q^{-1}_c}{Q_c Q^{-1}_b - Q_b Q^{-1}_c} \partial_r R_b \right) \right] \right] \\
\pi_j^{(\theta)}(\tau, \vec{\sigma}) =
\right]
\]
\[= -\left[\tilde{\phi}^{-1/3} \sum_a \sum_r \partial_r \tilde{n}_a \left[ Q_a^{-1} \sum_b (\gamma_{ba} V_{(i)rb} + \gamma_{br} V_{(i)ar}) \Pi_b \right] - \frac{c^3}{8\pi G} \tilde{\phi} \left( \sum_{b \neq a} Q_b^{-1} V_{(i)rb} \sigma_{(a)(b)} + Q_a^{-1} \sum_{b \neq r} Q_b Q_r^{-1} V_{(i)ab} \sigma_{(r)(b)} \right) - \sum_{tj} \sum_{c \neq d} \epsilon_{art} Q_r^{-1} \left( V_{(i)jt} + B_{(i)jt} \right) \epsilon_{jcd} Q_c Q_d^{-1} \sigma_{(c)(d)} \right]\]

\[+ \frac{c^3}{8\pi G} \tilde{\phi} \partial_a \tilde{n}_a Q_a^{-1} \sum_{b \neq c} Q_b Q_c^{-1} V_{(i)cb} \sigma_{(b)(c)} \right] (\tau, \tilde{\sigma}) + \left[\tilde{\phi}^{-1/3} \sum_a \tilde{n}_a \left( Q_a^{-1} \sum_b \left[ V_{(i)ba} \left( \tilde{\phi} \bar{\partial}_b \bar{\phi} - \sum_b \partial_b \Pi_b \right) + \right. \right. \]

\[+ \sum_b (\gamma_{ba} V_{(i)ba} - \gamma_{bb} V_{(i)ab}) \partial_b \Pi_b + \sum_b (\gamma_{ba} V_{(i)ba} + \gamma_{bb} V_{(i)ab}) \left( \frac{1}{3} \tilde{\phi}^{-1} \partial_b \tilde{\phi} + \sum_b \gamma_{cb} \partial_b R_c \right) \Pi_b + V_{(i)ba} \tilde{\mathcal{M}}_b \right) + \frac{c^3}{8\pi G} \tilde{\phi} \left[ Q_a^{-1} \sum_{b,c \neq b} Q_b Q_c^{-1} \left( V_{(i)ab} \bar{\partial}_c \sigma_{(b)(c)} - V_{(i)cb} \partial_a \sigma_{(b)(c)} \right) + \frac{2}{3} (V_{(i)ab} \tilde{\phi}^{-1} \partial_c \tilde{\phi} - V_{(i)cb} \tilde{\phi}^{-1} \partial_a \tilde{\phi}) \right] \sigma_{(b)(c)} - \sum_b (\gamma_{ba} - \gamma_{bb} + \gamma_{bc}) (V_{(i)ab} \partial_c R_b - V_{(i)cb} \partial_a R_b) \sigma_{(b)(c)} \right) + \sum_{tj} \sum_{b \neq a, c \neq d} \epsilon_{abb} Q_b^{-1} (V_{(i)jt} + B_{(i)jt}) \epsilon_{jcd} Q_c Q_d^{-1} \left( \frac{1}{3} \tilde{\phi}^{-1} \partial_b \tilde{\phi} + \sum_b \gamma_{ba} \partial_b R_b \right) \sigma_{(c)(d)} \right] - \sum_{r b \neq a} Q_b^{-1} \sigma_{(a)(b)} V_{(i)rb} \left( \frac{1}{3} \tilde{\phi}^{-1} \partial_r \tilde{\phi} + \sum_b \gamma_{ca} \partial_r R_c \right) \right] (\tau, \tilde{\sigma}).

\tag{C5}

\int d^3 \sigma_1 \sum_a \tilde{n}_a (\tau, \tilde{\sigma}_1) \frac{\delta \tilde{\mathcal{H}}_a(\tau, \tilde{\sigma}_1)}{\delta \Pi_a (\tau, \tilde{\sigma})} \bigg|_{\tilde{\phi} = 0} = \left[ \phi^{-2} \sum_a Q_a^{-1} \left( \tilde{n}_a \left[ \gamma_{aa} \left( 2 \phi^{-1} \partial_a \phi + \sum_b \gamma_{ba} \partial_a R_b \right) - \partial_a R_a \right] - \gamma_{aa} \partial_a \tilde{n}_a \right) \right] (\tau, \tilde{\sigma}).

\tag{C6}
\begin{equation}
\int d^3 \sigma_1 \sum_a \tilde{n}_{(a)}(\tau, \bar{\sigma}_1) \frac{\delta \tilde{H}_{(a)}(\tau, \bar{\sigma}_1)}{\delta R_a(\tau, \bar{\sigma})} |_{\theta^i = 0} =
\end{equation}

\begin{align*}
&= \phi^{-2}(\tau, \bar{\sigma}) \sum_{ra} \partial_r \tilde{n}_{(a)}(\tau, \bar{\sigma}) \left[ Q_a^{-1} \delta_{ra} \Pi_a - 
\right.
&- 2 \sum_{b \neq a} (\gamma_{\bar{a}a} - \gamma_{\bar{a}b}) \sum_i \epsilon_{abi} \frac{Q_a^{-1}}{(Q_b Q_a^{-1} - Q_a Q_b^{-1})^2} \delta_{rb} \pi_i(\theta) \right] (\tau, \bar{\sigma}) + 
\end{align*}

\begin{align*}
&+ \phi^{-2}(\tau, \bar{\sigma}) \sum_a \tilde{n}_{(a)}(\tau, \bar{\sigma}) \left[ Q_a^{-1} \left( \partial_a \Pi_{\bar{a}} - (2 \phi^{-1} \partial_a \phi + \sum_b \gamma_{\bar{b}a} \partial_a R_b) \Pi_{\bar{a}} - 
\right.
&- \gamma_{\bar{a}a} \left[ \phi^3 \partial_a \pi_{\bar{a}} + \sum_b (\gamma_{\bar{b}a} \partial_a \Pi_{\bar{b}} - \partial_a R_b \Pi_{\bar{b}}) \right] \right) =
\end{align*}

\begin{align*}
&= \left[ \phi^{-1/3} \sum_a Q_a^{-1} \left( \partial_a \tilde{n}_{(a)} \Pi_{\bar{a}} + \frac{\epsilon^3}{4 \pi G} \phi \sum_{b \neq a} (\gamma_{\bar{a}a} - \gamma_{\bar{a}b}) \frac{\partial_b \tilde{n}_{(a)} \sigma_{(a)(b)}}{Q_b Q_a^{-1} - Q_a Q_b^{-1}} \right) \right] (\tau, \bar{\sigma}) + 
\end{align*}

\begin{align*}
&+ \left[ \phi^{-1/3} \sum_a \tilde{n}_{(a)} \left( Q_a^{-1} \left[ \partial_a \Pi_{\bar{a}} - \left( \frac{1}{3} \phi^{-1} \partial_a \phi + \sum_b \gamma_{\bar{b}a} \partial_a R_b \right) \Pi_{\bar{a}} - 
\right.
&- \gamma_{\bar{a}a} \left( \phi \partial_a \pi_{\bar{a}} + \sum_b (\gamma_{\bar{b}a} \partial_a \Pi_{\bar{b}} - \partial_a R_b \Pi_{\bar{b}}) \right) \right] \right) + 
\end{align*}

\begin{align*}
&+ \frac{\epsilon^3}{8 \pi G} \phi \sum_{b \neq a} Q_b^{-1} \left[ \gamma_{\bar{a}a} \partial_b \sigma_{(a)(b)} + 
\right.
&+ \left( \gamma_{\bar{a}a} - \frac{2}{3} (\gamma_{\bar{a}a} - \gamma_{\bar{a}b}) \frac{Q_b Q_a^{-1}}{Q_b Q_a^{-1} - Q_a Q_b^{-1}} \right) \phi^{-1} \partial_b \phi - 
\right.
\end{align*}

\begin{align*}
&- \sum_b \gamma_{\bar{a}a} (\gamma_{\bar{b}a} - \gamma_{\bar{b}b}) [Q_b Q_a^{-1} + Q_a Q_b^{-1}] + 2 (\gamma_{\bar{a}a} \gamma_{\bar{b}b} - \gamma_{\bar{a}b} \gamma_{\bar{b}a}) Q_b Q_a^{-1} \partial_b R_b \sigma_{(a)(b)} \right] \right] (\tau, \bar{\sigma}) .
\end{align*}

(C7)

The function \( S(\tau, \bar{\sigma}) \) of Eqs.(B8) becomes

\[96\]
\[S|_{\theta^i=0} = \phi^2 \sum_a Q_a^{-2} \left( \sum_b \left[ \sum_c \left( 2 \gamma_{ba} \gamma_{ca} - \delta_{bc} \right) \partial_a R_b \partial_a R_c - 4 \gamma_{ba} \phi^{-1} \partial_a \phi \partial_a R_b \right] + 8 (\phi^{-1} \partial_a \phi)^2 \right), \]

(C8)

and its functional derivatives (B9), (B10) and (B11) become

\[\int d^3 \sigma_1 \left[ 1 + n(\tau, \vec{\sigma}_1) \right] \frac{\delta S(\tau, \vec{\sigma}_1)}{\delta R_a(\tau, \vec{\sigma})} \bigg|_{\theta^i=0} = \]

\[= 2 \left( \phi^2 \sum_a Q_a^{-2} \left[ \partial_a n \left( 2 \gamma_{aa} \phi^{-1} \partial_a \phi - \sum_b \left( 2 \gamma_{aa} \gamma_{ba} - \delta_{ab} \right) \partial_a R_b \right) - (1 + n) \left( 2 \gamma_{aa} \left( - \phi^{-1} \partial_a^2 \phi + 3 (\phi^{-1} \partial_a \phi)^2 \right) + \sum_b (2 \gamma_{aa} \gamma_{ba} - \delta_{ab}) \left( \partial_a^2 R_b + 2 \phi^{-1} \partial_a \phi \partial_a R_b \right) + \sum_{b\epsilon} \left( 2 \gamma_{ba} (\delta_{a\epsilon} - \gamma_{aa} \gamma_{ca}) - \gamma_{ca} \delta_{b\epsilon} \right) \partial_a R_b \partial_a R_c \right) \right) \right](\tau, \vec{\sigma}). \]

(C9)

\[\int d^3 \sigma_1 \left[ 1 + n(\tau, \vec{\sigma}_1) \right] \frac{\delta S(\tau, \vec{\sigma}_1)}{\delta \phi(\tau, \vec{\sigma})} \bigg|_{\theta^i=0} = \]

\[= -2 \left( \phi \sum_a Q_a^{-2} \left[ \partial_a n \left( 4 \phi^{-1} \partial_a \phi - \sum_b \gamma_{ba} \partial_a R_b \right) + (1 + n) \left( 8 (\phi^{-1} \partial_a^2 \phi - 2 \phi^{-1} \partial_a \phi \sum_b \gamma_{ba} \partial_a R_b) - 2 \sum_b \gamma_{ba} \partial_a^2 R_b + \sum_{b\epsilon} \left( 2 \gamma_{ba} \gamma_{ca} + \delta_{b\epsilon} \right) \partial_a R_b \partial_a R_c \right) \right) \right](\tau, \vec{\sigma}). \]

(C10)

\[\int d^3 \sigma_1 \left[ 1 + n(\tau, \vec{\sigma}_1) \right] \frac{\delta S(\tau, \vec{\sigma}_1)}{\delta \theta^i(\tau, \vec{\sigma})} \bigg|_{\theta^i=0} = \]

\[= -2 \left[ \phi^2 \sum_{ra} Q_a^{-2} V(i)_{ra} \left( \partial_a n \left( \phi^{-1} \partial_r \phi - \sum_b \gamma_{ba} \partial_r R_b \right) + \partial_r n \left( \phi^{-1} \partial_a \phi - \sum_b \gamma_{br} \partial_a R_b \right) - (1 + n) \left[ 2 (3 \phi^{-1} \partial_a \phi \phi^{-1} \partial_r \phi - \phi^{-1} \partial_a \partial_r \phi) + \sum_b (\gamma_{ba} + \gamma_{br}) \partial_a \partial_r R_b \right. \right. \right. \]

\[+ \left. \left. \sum b \left( 2 \gamma_{br} \partial_a \phi \partial_r R_b + \gamma_{br} \phi^{-1} \partial_r \phi \partial_a R_b \right) - \sum_{b\epsilon} \left( 2 \gamma_{br} \gamma_{ca} + \delta_{b\epsilon} \right) \partial_a R_b \partial_r R_c \right) \right](\tau, \vec{\sigma}). \]

(C11)
The Laplace-Beltrami operator of Eq.(B12) becomes

\[ \hat{\Delta}_{\theta_i=0} = \sum_a Q_a^{-2} \left[ \partial_a^2 - 2 \sum_b \gamma_{ba} \partial_a R_b \partial_a \right]. \]  

(C12)

The functions \( T(\tau, \vec{\sigma}) \) and \( T_1(\tau, \vec{\sigma}) \) of Eq.(B13) become

\[ T(\tau, \vec{\sigma})|_{\theta_i=0} = -2 \left[ \phi^2 \sum_a Q_a^{-2} \left( 4 \left( \phi^{-1} \partial_a^2 \phi + (\phi^{-1} \partial_a \phi) \right) \right) - \sum_b \gamma_{ba} \left( \partial_a^2 R_b + 2 \left( 5 \phi^{-1} \partial_a \phi - \sum_c \gamma_{ca} \partial_a R_c \partial_a R_b \right) \right) \right](\tau, \vec{\sigma}), \]

\[ T_1(\tau, \vec{\sigma})|_{\theta_i=0} = -2 \left[ \phi^2 \sum_a Q_a^{-2} \left( 4 \left( \phi^{-1} \partial_a \phi \right)^2 - \sum_b \gamma_{ba} \left( \partial_a^2 R_b + 2 \left( \phi^{-1} \partial_a \phi - \sum_c \gamma_{ca} \partial_a R_c \partial_a R_b \right) \right) \right) \right](\tau, \vec{\sigma}). \]  

(C13)

The functional derivatives (B14)-(B19) become

\[ \int d^3 \sigma_1 \frac{\delta}{\delta R_a(\tau, \vec{\sigma})} \left[ n \phi \hat{\Delta} \phi \right](\tau, \vec{\sigma}_1)|_{\theta_i=0} =
\]

\[ = 2 \phi^2 \left( \sum_a Q_a^{-2} \gamma_{aa} \phi^{-1} \partial_a \phi \left( n \phi^{-1} \partial_a \phi + \partial_a n \right) \right)(\tau, \vec{\sigma}), \]  

(C14)

\[ \int d^3 \sigma_1 n(\tau, \vec{\sigma}_1) \frac{\delta}{\delta R_a(\tau, \vec{\sigma})} \left[ \frac{\delta T(\tau, \vec{\sigma}_1)}{\delta \phi(\tau, \vec{\sigma})} \right]|_{\theta_i=0} =
\]

\[ = 2 \left[ \phi^2 \sum_a \gamma_{aa} Q_a^{-2} \left( \partial_a^2 n - 6 \phi^{-1} \partial_a \phi \partial_a n \right) \right](\tau, \vec{\sigma}). \]  

(C15)

\[ \int d^3 \sigma_1 \frac{\delta}{\delta \phi(\tau, \vec{\sigma})} \left[ n \phi \hat{\Delta} \phi \right](\tau, \vec{\sigma}_1)|_{\theta_i=0} =
\]

\[ = \left[ 2 n \hat{\Delta}|_{\theta_i=0} \phi + 2 \sum_a Q_a^{-2} \partial_a n \partial_a \phi + \phi \hat{\Delta}|_{\theta_i=0} n \right](\tau, \vec{\sigma}). \]  

(C16)

\[ \int d^3 \sigma_1 n(\tau, \vec{\sigma}_1) \frac{\delta}{\delta \phi(\tau, \vec{\sigma})} \left[ \frac{\delta T(\tau, \vec{\sigma}_1)}{\delta \phi(\tau, \vec{\sigma})} \right]|_{\theta_i=0} =
\]

\[ = -4 \left[ \phi \sum_a Q_a^{-2} \left( 2 \partial_a^2 n - 3 \sum_b \gamma_{ba} \partial_a R_b \partial_a n \right) \right](\tau, \vec{\sigma}). \]  

(C17)
\[
\int d^3 \sigma_1 n(\tau, \bar{\sigma}_1) \frac{\delta T(\tau, \bar{\sigma}_1)}{\delta \theta^i(\tau, \bar{\sigma})} |_{\theta^i=0} = \\
= -2 \left[ \phi^2 \sum_{ra} Q_a^{-2} V(i)ra \left( \partial_r n - 3 \left( \partial_r n \phi^{-1} \partial_a \phi + \partial_a n \phi^{-1} \partial_r \phi \right) \right) \right](\tau, \bar{\sigma}).
\]  
(C19)

The scalar 3-curvature of Eq.(B20) becomes

\[
3 \hat{R}(\tau, \bar{\sigma})|_{\theta^i=0} = \sum_a \left( Q_a^{-2} \sum_b \left[ 2 \gamma_{ba} \partial_a^2 R_b - (\partial_a R_b)^2 - 2 \gamma_{ba} \sum_c \gamma_{ca} \partial_c \partial_a R_c \right] \right)(\tau, \bar{\sigma}).
\]  
(C20)

Finally the functional derivatives (B21), (B22) and (B23) of the mass density become

\[
\int d^3 \sigma_1 \left( 1 + n(\tau, \bar{\sigma}_1) \right) \frac{\delta \dot{M}(\tau, \bar{\sigma}_1)}{\delta \theta^i(\tau, \bar{\sigma})} |_{\theta^i=0} = \\
= \frac{1}{2} \sum_i \delta^3(\bar{\sigma}, \bar{\eta}_i(\tau)) \eta_i \left( 1 + n \right) \\
\phi^{-4} \sum_{r,s} Q_a^{-2} V(i)ra \delta_{sa} + \delta_{ra} V(i)sa \left( \bar{\kappa}_{iv}(\tau) - \frac{Q_i}{c} A_{\perp r} \right) \left( \bar{\kappa}_{is}(\tau) - \frac{Q_i}{c} A_{\perp s} \right) \right)(\tau, \bar{\sigma}) + \\
\sqrt{m_i^2 c^2 + \phi^{-4} \sum_a Q_a^{-2} \left( \bar{\kappa}_{ia}(\tau) - \frac{Q_i}{c} A_{\perp a} \right)^2} \\
+ \left( 1 + n(\tau, \bar{\sigma}) \right) \left( \phi^{-1/3} \frac{1}{c} \left[ \sum_{ar} Q_a^2 V(i)ra \delta_{sa} \pi_1^r \pi_1^s + \sum_{ab} Q_a^{-2} Q_b^{-2} V(i)ra F_{rb} F_{ab} - \right. \right. \\
\left. \left. - \frac{1}{2} \sum_{ar} Q_a^2 V(i)ra \delta_{ra} \right] \left( 2 \pi_1^r - \sum_{m} \delta_{rm} \sum_i Q_i \eta_i \partial_m c(\bar{\sigma}, \bar{\eta}_i(\tau)) \right) \\
\delta^{sn} \sum_f Q_j \eta_j \partial_n c(\bar{\sigma}, \bar{\eta}_j(\tau)) \right)(\tau, \bar{\sigma}),
\]  
(C21)
\[ \int d^3 \sigma_1 \left( 1 + n(\tau, \vec{\sigma}_1) \right) \frac{\delta \hat{M}(\tau, \vec{\sigma}_1)}{\delta \phi(\tau, \vec{\sigma})} = \]

\[ = -2 \sum_i \delta^3(\vec{\sigma}, \vec{\eta}_i(\tau)) \eta_i \left( 1 + n \right) \]

\[ \frac{\phi^{-5} \sum_a Q_a^{-2} \left( \hat{\kappa}_{ia}(\tau) - \frac{Q_a}{c} A_{\perp a} \right)^2}{\sqrt{m_i^2 c^2 + \phi^{-4} \sum_a Q_a^{-2} \left( \hat{\kappa}_{ia}(\tau) - \frac{Q_a}{c} A_{\perp a} \right)^2}}(\tau, \vec{\sigma}) - \]

\[ - \left( 1 + n(\tau, \vec{\sigma}) \right) \left( \frac{\phi^{-1/3}}{c} \sum_{ars} \gamma_{aa} Q_a^{-2} \left( \hat{\kappa}_{ia}(\tau) - \frac{Q_a}{c} A_{\perp a} \right)^2 + \frac{1}{2c} \sum_{ab} Q_a^{-2} Q_b^{-2} F_{ab} F_{ab} - \right. \]

\[ - \frac{1}{c} \sum_{ar sn} Q_a^2 \delta_{ra} \delta_{sa} \left( 2 \pi^r - \sum_m \delta^{rm} \sum_i Q_i \eta_i \partial_m c(\vec{\sigma}, \vec{\eta}_i(\tau)) \right) \]

\[ \delta^{sn} \sum_j Q_j \eta_j \partial_n c(\vec{\sigma}, \vec{\eta}_j(\tau)) \right) \right)(\tau, \vec{\sigma}), \]

\( (C22) \)

\[ \int d^3 \sigma_1 \left( 1 + n(\tau, \vec{\sigma}_1) \right) \frac{\delta \hat{M}(\tau, \vec{\sigma}_1)}{\delta R_a(\tau, \vec{\sigma})} = \]

\[ = - \sum_i \delta^3(\vec{\sigma}, \vec{\eta}_i(\tau)) \eta_i \left( 1 + n \right) \]

\[ \frac{\phi^{-4} \sum_a \gamma_{aa} Q_a^{-2} \left( \hat{\kappa}_{ia}(\tau) - \frac{Q_a}{c} A_{\perp a} \right)^2}{\sqrt{m_i^2 c^2 + \phi^{-4} \sum_a Q_a^{-2} \left( \hat{\kappa}_{ia}(\tau) - \frac{Q_a}{c} A_{\perp a} \right)^2}}(\tau, \vec{\sigma}) + \]

\[ + \left( 1 + n(\tau, \vec{\sigma}) \right) \left( \frac{\phi^{-1/3}}{c} \sum_{ars} \gamma_{aa} Q_a^{-2} \left( \hat{\kappa}_{ia}(\tau) - \frac{Q_a}{c} A_{\perp a} \right)^2 + \frac{1}{2c} \sum_{ab} (\gamma_{aa} + \gamma_{ab}) Q_a^{-2} Q_b^{-2} F_{ab} F_{ab} - \right. \]

\[ - \frac{1}{c} \sum_{ar sn} \gamma_{aa} Q_a^2 \delta_{ra} \delta_{sa} \left( 2 \pi^r - \sum_m \delta^{rm} \sum_i Q_i \eta_i \partial_m c(\vec{\sigma}, \vec{\eta}_i(\tau)) \right) \]

\[ \delta^{sn} \sum_j Q_j \eta_j \partial_n c(\vec{\sigma}, \vec{\eta}_j(\tau)) \right) \right)(\tau, \vec{\sigma}). \]

\( (C23) \)
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