SHORT

COMMUNICATIONS

On the Convergence of Mappings with $k$-Finite Distortion

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It is well known that the limit of a uniformly converging sequence of analytic functions is an analytic function. Reshetnyak generalized this result to mappings with bounded distortion: the limit of a locally uniformly converging sequence of mappings with bounded distortion is a mapping with bounded distortion.

Reshetnyak used the weak convergence of Jacobians to prove the following theorem on the limit of a sequence of mappings with bounded distortion.

Theorem 1 ([1]). Let $f_m : \Omega \to \mathbb{R}^n$, $m = 1, 2, \ldots$, be an arbitrary sequence of mappings with bounded distortion locally converging in $L^p(\Omega)$ to a mapping $f_0 : \Omega \to \mathbb{R}^n$. Assume that the sequence of distortion coefficients $K(f_m)$, $m = 1, 2, \ldots$, is bounded. Then the limit mapping $f_0$ is a mapping with bounded distortion and the following inequality holds:

$$K(f_0) \leq \lim_{k \to \infty} K(f_m).$$

(1)

The weighted mappings with bounded $(p, q)$-distortion, which generalize the mappings with bounded distortion, were defined and studied in [2]. It was shown in [3] that the locally uniform limit of mappings with bounded $(\theta, 1)$-weighted $(p, q)$-distortion is also a mapping with bounded $(\theta, 1)$-weighted $(p, q)$-distortion, and an estimate similar to (1) was obtained. The proofs of theorems in both [3] and the present paper are based on a new method developed in [4] for generalizing Reshetnyak’s results to the Carnot groups.

In the present paper, we generalize these assertions to classes of mappings with $k$-finite distortion which naturally arise in the problem of pull-backs of differential forms of degree $k$ (see [5]).

1. PRELIMINARIES

Let $U$ be a domain in $\mathbb{R}^n$. We consider the Banach space $\mathcal{L}_p(U, \Lambda^k)$ of differential forms $\omega$ of degree $k$ with measurable coefficients which have the following finite norm:

$$\|\omega\|_p = \left( \int_U |\omega|^p \, dx \right)^{1/p}.$$
A mapping \( f: U \to \mathbb{R}^n \) is said to be \emph{approximate differentiable} at a point \( x \in U \) [6] if there exists a linear mapping \( L: \mathbb{R}^n \to \mathbb{R}^n \) such that
\[
\lim_{r \to 0} \left| \frac{\left( \{ y \in B(x, r) : |f(y) - f(x) - L(y-x)| > \varepsilon \} \right)}{r^n} \right| = 0
\]
for any \( \varepsilon > 0 \). It is well known that the approximate differential is unique [6] if \( x \) is a density point. In what follows, it is denoted by the symbol \( \text{ap} \, Df(x) \).

Let a mapping \( f = (f_1, \ldots, f_n): U \to W \) of Euclidean domains \( U, W \subset \mathbb{R}^n \) be approximate differentiable almost everywhere in \( U \). The differential
\[
\text{ap} \, Df(x): T_x U \to T_{f(x)} W
\]
canonicaly generates a linear mapping
\[
\Lambda_k f(x): \Lambda_k T_x U \to \Lambda_k T_{f(x)} W
\]
of spaces of \( k \)-vectors and the following operation \( f^* \) of pull-back of \( k \)-forms. Namely, any \( k \)-form
\[
\omega = \sum \omega_\beta \, dy^\beta
\]
in \( W \) with continuous coefficients \( \omega_\beta: W \to \mathbb{R} \), where the summation is performed over all \( k \)-dimensional ordered multi-indices \( \beta = (\beta_1, \ldots, \beta_k), \, 1 \leq \beta_1 < \cdots < \beta_k \leq n \), and where
\[
dy^\beta = dy_{\beta_1} \wedge dy_{\beta_2} \wedge \cdots \wedge dy_{\beta_k},
\]
can be pulled back to the domain \( U \) so that one obtains a \( k \)-form
\[
f^* \omega(x) = \sum_\beta \omega_\beta(f(x)) \, df_{\beta_1} \wedge df_{\beta_2} \wedge \cdots \wedge df_{\beta_k} = \sum_\alpha \sum_\beta \omega_\beta(f(x)) M_{\alpha}^{\beta}(x) \, dx^\alpha
\]
with measurable coefficients defined for almost all \( x \in U \). Here
\[
df_{\beta_k} = \sum_{i=1}^n \left( \frac{\partial f_\beta}{\partial x_i} \right) \, dx_i,
\]
the partial derivatives are understood in an approximate sense, and \( M_{\alpha}^{\beta}(x) \) are \( k \times k \) minors of the matrix
\[
\text{ap} \, Df(x) = \left( \frac{\partial f_i}{\partial x_j} \right), \quad i, j = 1, \ldots, n,
\]
with ordered rows and columns. We denote the norm of this linear mapping by the symbol \( |\Lambda^k f(x)| \).

The minimal analytic and geometric properties of the mapping \( f \) generating a bounded operator
\[
f^*: \mathcal{L}_p(W, \Lambda^k) \cap \mathcal{C}(W, \Lambda^k) \to \mathcal{L}_q(U, \Lambda^k), \quad 1 \leq q \leq p \leq \infty,
\]
of pull-back of differential forms of degree \( k \) were obtained in [5].

Further, let \( Z = \{ x \in U : \text{det} \, \text{ap} \, Df(x) = 0 \} \). We shall say that an approximate differentiable mapping \( f: U \to W \) has a \( k \)-finite distortion\(^1\) (briefly, \( f \in \mathcal{C} \mathcal{D}^k(U; W) \)) if \( |Z| = 0 \) for \( k = 0 \) and \( \text{rank} \, \text{ap} \, Df(x) < k \) for a.a. \( x \in Z \) for \( 1 \leq k \leq n \) [5].

In addition to the property of \( k \)-finite distortion, the mapping in (3) must also exhibit a certain behavior of a distortion characteristic containing the ratios \( |\Lambda^k f(x)|^q / |J(x, f)| \), where \( J(x, f) = \text{det} \, \text{ap} \, Df(x) \) (for details, see [5]). Here we formulate a simpler version of this result for homeomorphic mappings.

\textbf{Proposition ([5])}. \textit{Let} \( f: U \to W \) \textit{be an approximate differentiable homeomorphism. An operator}
\[
f^*: \mathcal{L}_p(W, \Lambda^k) \cap \mathcal{C}(W, \Lambda^k) \to \mathcal{L}_q(U, \Lambda^k), \quad 1 \leq q \leq p \leq \infty,
\]
\textit{is bounded if and only if the following conditions are satisfied:}

\[^1\text{This notion generalizes the notion (which is well known in the literature for } k = 1 \text{) of family of mappings with finite distortion of the Sobolev class } W^1_{1, \text{loc}}(U), \text{ which are characterized by the following property: } Df(x) = 0 \text{ almost everywhere on the set } Z.\]
Theorem 2. \( f : U \to W \) has a \( k \)-finite distortion;

(2) the function

\[
K_{k,p}(x) = \begin{cases} 
\frac{|A^k f(x)|}{|J(x,f)|^{1/p}} & \text{if } x \notin \mathbb{Z}, \\
0 & \text{otherwise},
\end{cases}
\]

belongs to \( L_\infty(U) \), where \( 1/\infty_0 = 1/q - 1/p. \)

In this case, the norm of the operator \( f^* \) is comparable with \( \|K_{k,p}(\cdot) \|_{L_\infty(U)} \).

In our paper, we consider the mappings differentiable in the sense of Sobolev. Such mappings are unconditionally approximate differentiable.

We say that a homeomorphism \( f \in CD^k(U;W) \) has a \( (q,p) \)-bounded distortion \( f \in CD^k_{q,p}(U;W) \) if it satisfies condition (2) of the proposition stated above.

2. MAIN RESULT

Theorem 2. Let \( f_m : U \to W \) be homeomorphisms of Sobolev class \( W^1_{1,\text{loc}}(U) \) with \( k \)-finite distortions which are locally bounded in \( W^1(U) \), have a \( (q,p) \)-bounded distortion \( f_m \in CD^k_{q,p}(U;W) \).

\( k < q, k < l, 1 < q \leq p \leq \infty, \) and locally uniformly converge to a homeomorphism \( f : U \to W. \) Assume also that there exists a function \( M(x) \in L_\infty(U), 1/\infty_0 = 1/q - 1/p, \) such that

\[
K_{k,p}^m(x) = K_{k,p}(f_m)(x) \leq M(x) \quad \text{for all } m \in \mathbb{N}
\]

almost everywhere in \( U. \) Then the limit mapping \( f \) is also a mapping with \( k \)-finite distortion and has a \( (q,p) \)-bounded distortion, and the inequality \( K_{k,p}(f)(x) \leq M(x) \) holds for its distortion coefficient.

Proof. It follows from the conditions of the theorem that \( f \in W^1_{1,\text{loc}}(U). \) First, we show that the limit mapping \( f \) is also a mapping with \( k \)-finite distortion and has a \( (q,p) \)-bounded distortion. For this, we first show that the mapping \( f \) induces a bounded operator

\[
f^* : \mathcal{L}_p(W, \Lambda^k) \cap \mathcal{E}(W, \Lambda^k) \to \mathcal{L}_q(U, \Lambda^k), \quad 1 < q \leq p \leq \infty.
\]

Since each mapping \( f_m \) is a mapping with \( k \)-finite and \( (q,p) \)-bounded distortion, it follows from the assumption that the homeomorphism \( f_m : U \to W \) induces a bounded operator

\[
f^*_m : \mathcal{L}_p(W, \Lambda^k) \cap \mathcal{E}(W, \Lambda^k) \to \mathcal{L}_q(U, \Lambda^k), \quad 1 < q \leq p \leq \infty.
\]

Moreover, the norms of the operators \( f^*_m \) are totally bounded

\[
\|f^*_m\| \leq \|K_{f_m,p}(\cdot) \|_{L_\infty(U)} \leq \|M(x)(\cdot) \|_{L_\infty(U)} \leq \tilde{M} < \infty, \quad m \in \mathbb{N}.
\]

Take a \( k \)-form \( \omega \in \mathcal{L}_p(W, \Lambda^k) \cap \mathcal{E}(W, \Lambda^k) \) and set \( u_m = f^*_m(\omega). \) Since \( \|f^*_m\| \leq \tilde{M} \), the sequence of forms \( u_m \) is bounded in \( \mathcal{L}_q(U, \Lambda^k). \) Therefore, we can extract a weakly converging subsequence.

We assume that the sequence \( u_m \) weakly converges in \( \mathcal{L}_q(U, \Lambda^k) \) to a form \( u_0. \) The weak convergence of forms means that the coefficients of the forms \( u_m \) weakly converge in \( L_q(U) \) to the corresponding coefficients of the form \( u_0. \)

Since the sequence \( u_m \) weakly converges to \( u_0 \) in \( \mathcal{L}_q(U, \Lambda^k), \) we have

\[
\|u_0 \|_{L_q(U)} \leq \lim_{m \to \infty} \|u_m \|_{L_q(U)} = \lim_{m \to \infty} \|f^*_m \omega \|_{L_q(U)} \leq \lim_{k \to \infty} \|f^*_m \| \|\omega \|_{L_p(U)} \leq \|M(x)(\cdot) \|_{L_\infty(U)} \|\omega \|_{L_p(U)}.
\]

The following lemma is proved in the book [1, Chap. 2, Sec. 4].
Lemma. Suppose $U$ is an open subset in $\mathbb{R}^n$, and suppose that $\varphi_m = (\varphi_{m1}, \varphi_{m2}, \ldots, \varphi_{mk})$, $1 \leq k \leq n$, $m = 1, 2, \ldots$, be a sequence of vector functions of class $W^1_{p, \text{loc}}(U)$, $p \geq k$, locally bounded in $W^1_p(U)$. Assume that, as $m \to \infty$, the functions $\varphi_m$ converge in $L^1_{k, \text{loc}}$ to a vector function $\varphi_0 = (\varphi_{01}, \varphi_{02}, \ldots, \varphi_{0k})$ and set $\omega_m = d\varphi_{m1} \wedge d\varphi_{m2} \wedge \cdots \wedge d\varphi_{mk}$. Then the sequence of forms $\omega_m$ weakly converges in $L^{p/k, \text{loc}}$ to a form $\omega$.

Since the homeomorphisms $f_m$ locally uniformly converge to $f$ and the form $\omega$ has continuous coefficients, the functions $\omega_m(f_m(x))$ converge to $\omega(f(x))$ locally uniformly. The lemma implies that the minors of the matrices $Df_m$ weakly converge in $L^1_{k, \text{loc}}$ to minors of the matrix $Df$. Therefore, the forms $u_m$ weakly converge to $f^*(\omega)$ and hence the mapping $f$ induces a bounded operator

$$f^*: \mathcal{L}_p(W, \Lambda^k) \to \mathcal{L}_q(U, \Lambda^k), \quad 1 < q \leq p \leq \infty.$$ 

Now we estimate the distortion coefficient of the limit mapping $f$. Let $\theta \in C^\infty_0(U)$ be a test function. Let $Z_m$ be the set of zeros of the Jacobian of the mapping $f_m$. Since the rank of the matrix $Df_m$ on the set $Z_m$ is less than $k$, it follows that all $k$th-order minors are equal to zero on the set $Z_m$. First, we consider the case $q < p$. We apply Hölder’s inequality to derive

$$\int_U |\Lambda^k f_m(x)|^q \theta(x) \, dx = \int_{U \setminus Z_m} \frac{|\Lambda^k f_m(x)|^q}{|J(x, f_m)|^{q/p}} |J(x, f_m)|^{q/p} \theta(x) \, dx$$

$$\leq \left( \int_U (K_{k,p}(f_m) \theta(x) \, dx \right)^{q/\infty} \left( \int_U |J(x, f_m)| \theta(x) \, dx \right)^{q/p}$$

$$\leq \left( \int U M^\infty(x) \theta(x) \, dx \right)^{q/\infty} \left( \int_U |J(x, f_m)| \theta(x) \, dx \right)^{q/p}.$$

Hölder’s inequality can be used, because $q/\infty + q/p = 1$.

It follows from the lemma that the elements of the matrix $\Lambda^k(f_m)(x)$, i.e., the $k$th-order minors of the mapping $f_m$, weakly converge in $L^1_{k, \text{loc}}$ to the elements of the matrix $\Lambda^k(f)(x)$. Since the norm is semicontinuous in the Banach space $L^1_{k, \text{loc}}$, the left-hand side of the inequality can be estimated as

$$\int_U |\Lambda^k f(x)|^q \theta(x) \, dx \leq \lim_{j \to \infty} \int_U |\Lambda^k f_m(x)|^q \theta(x) \, dx.$$

Let $x_0 \in U$, and let $r \leq R$ be such that the ball $B(x_0, R)$ lies in the domain $U$. We consider the following family of functions $\theta_{r, \varepsilon, y} \in C^\infty_0(U)$:

$$\theta_{r, \varepsilon, x_0}(x) = \begin{cases} 1 & \text{if } x \in B(x_0, r - \varepsilon), \\ 0 & \text{if } x \notin B(x_0, r), \\ 0 < \theta_{r, \varepsilon, x_0}(x) < 1 & \text{otherwise}. \end{cases}$$

As the test function $\theta(x)$, we take the function $\theta_{r, \varepsilon, x_0}(x)$. Then we have the inequality

$$\int_U |\Lambda^k f(x)|^q \theta_{r, \varepsilon, x_0}(x) \, dx \leq \left( \int_{B(x_0, r)} M^\infty(x) \, dx \right)^{q/\infty} \left( \int_{B(x_0, r)} |J(x, f_m)| \, dx \right)^{q/p}. \quad (4)$$

Since $|\theta_{r, \varepsilon, x_0}(x)| \leq 1$, it follows from Lebesgue’s dominated convergence theorem that an arbitrary function $g \in L^1_{k, \text{loc}}(\Omega)$ satisfies the relation

$$\int_{B(x_0, r)} g(x) \, dx = \lim_{\varepsilon \to 0} \int_{\Omega} g(x) \theta_{r, \varepsilon, x_0}(x) \, dx.$$

Since the mapping $f_m$ is differentiable, there is a set $\Sigma_m$ of measure zero [6], [7] such that

$$\int_{B(x_0, r)} |J(x, f_m)| \, dx = \int_{f_m(B(x_0, r) \setminus \Sigma_m)} dy = |f_m(B(x_0, r) \setminus \Sigma_m)|.$$
Therefore,

\[ \int_{B(x_0, r)} |J(x, f_m)| \, dx \leq |f_m(B(x_0, r)|. \]

Let us show that

\[ |f_m(B(x_0, r)| \to |f(B(x_0, r)| \quad \text{as} \quad m \to \infty \quad \text{for almost all} \quad r. \]

Since \( |f_m(B(x_0, r)| < \infty \), the mappings \( f_m \) are homeomorphisms, so that the images of the spheres \( f_m(S(x_0, r)) \) do not intersect, it follows that the measure of the image of any sphere is zero for almost all \( r \),

\[ |f_m(S(x_0, r))| = 0. \]

We fix \( r \) so that

\[ |f(S(x_0, r))| = 0 \quad \text{and} \quad |f_m(S(x_0, r))| = 0 \]

and surround the image of the sphere \( f(S(x_0, r)) \) by an \( \varepsilon \)-neighborhood \( U_\varepsilon \). Since the mappings \( f_m \) locally uniformly converge to the mapping \( f \), it follows that, starting from a number \( M \), the images of the spheres \( f_m(S(x_0, r)), m \geq M \), are contained in this \( \varepsilon \)-neighborhood. It is clear that \( |U_\varepsilon| \to 0 \) as \( \varepsilon \to 0 \), and hence

\[ |f_m(B(x_0, r)| \to |f(B(x_0, r)| \quad \text{as} \quad m \to \infty. \]

Thus, passing to the limit as \( m \to \infty \) and then as \( \varepsilon \to 0 \) in inequality (4), we obtain

\[ \int_{B(x_0,r)} |\Lambda^k f(x)|^q \, dx \leq \left( \int_{B(x_0,r)} M^\varepsilon(x) \, dx \right)^{q/p} |f(B(x_0,r)|^q/p. \]

Dividing the result by the measure of the ball \( B(x_0, r) \), we obtain the inequality

\[ \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} |\Lambda^k f(x)|^q \, dx \leq \left( \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} M^\varepsilon(x) \, dx \right)^{q/p} \left( \frac{|f(B(x_0,r)|}{|B(x_0, r)|} \right)^{q/p}. \]

Since the homeomorphism \( f \) is differentiable in the Sobolev sense, by formula (2.5) in [8, Sec. 2.3], we can write

\[ \frac{|f(B(x_0,r)|}{|B(x_0, r)|} \to |J(x_0, f)| \quad \text{as} \quad r \to 0 \quad \text{a.e. in} \quad U. \]

For a function \( g \in L_{1,\text{loc}}(U) \), we use the Lebesgue fundamental theorem to derive (for details, see [9])

\[ g(x_0) = \lim_{r \to 0} \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} f(x) \, dx \quad \text{for a.a.} \quad x_0 \in U. \]

Let us use this property, and let \( r \) tend to 0. Almost everywhere in \( U \), we obtain the inequality

\[ |\Lambda^k f(x)|^q \leq M^q(x)|J(x, f)|^{q/p}. \]

Therefore, the distortion coefficient of the limit mapping \( f \) satisfies the estimate

\[ K_{k,p}(f)(x) \leq M(x) \quad \text{for almost all} \quad x \in U. \]

In the case \( q = p \), we have

\[ \int_U |\Lambda^k f_m(x)|^q \theta(x) \, dx \leq M \left( \int_U |J(x, f_m)| \theta(x) \, dx \right). \]

Further, proceeding as in the case \( q < p \), we obtain the desired estimate

\[ K_{k,p}(f)(x) \leq M \quad \text{for almost all} \quad x \in U. \]

Therefore, applying Theorem 2, we use the Lebesgue fundamental theorem to derive (for details, see [9])

\[ g(x_0) = \lim_{r \to 0} \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} f(x) \, dx \quad \text{for a.a.} \quad x_0 \in U. \]

Let us use this property, and let \( r \) tend to 0. Almost everywhere in \( U \), we obtain the inequality

\[ |\Lambda^k f(x)|^q \leq M^q(x)|J(x, f)|^{q/p}. \]

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Further, proceeding as in the case \( q < p \), we obtain the desired estimate

\[ K_{k,p}(f)(x) \leq M \quad \text{for almost all} \quad x \in U. \]

For applications of Theorem 2, see [10].
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