Abstract. For a given infinite connected graph $G = (V, E)$ and an arbitrary but
fixed conductance function $c$, we study an associated graph Laplacian $\Delta_c$; it is a
generalized difference operator where the differences are measured across the edges
$E$ in $G$; and the conductance function $c$ represents the corresponding coefficients.
The graph Laplacian (a key tool in the study of infinite networks) acts in an energy
Hilbert space $\mathcal{H}_E$ computed from $c$. Using a certain Parseval frame, we study the
spectral theoretic properties of graph Laplacians. In fact, for fixed $c$, there are two
versions of the graph Laplacian, one defined naturally in the $l^2$ space of $V$, and the
other in $\mathcal{H}_E$. The first is automatically selfadjoint, but the second involves a Krein
extension. We prove that, as sets, the two spectra are the same, aside from the point
0. The point zero may be in the spectrum of the second, but not the first.

We further study the fine structure of the respective spectra as the conductance
function varies; showing now how the spectrum changes subject to variations in
the function $c$. Specifically, we study an order on the spectra of the family of
operators $\Delta_c$, and we compare it to the ordering of pairs of conductance functions.
We show how point-wise estimates for two conductance functions translate into
spectral comparisons for the two corresponding graph Laplacians; involving a certain
similarity: We prove that point-wise ordering of two conductance functions $c$ on $E$,
induces a certain similarity of the corresponding (Krein extensions computed from
the) two graph Laplacians $\Delta_c$. The spectra are typically continuous, and precise
notions of fine-structure of spectrum must be defined in terms of equivalence classes
of positive Borel measures (on the real line.) Our detailed comparison of spectra is
analyzed this way.

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1. **Introduction**

By an electrical network we mean a graph $G$ of vertices and edges satisfying suitable conditions which allow for computation of voltage distribution from a network of prescribed resistors assigned to the edges in $G$. The mathematical axioms are prescribed in a way that facilitates the use of the laws of Kirchhoff and Ohm in computing voltage distributions and resistance distances in $G$. It will be more convenient to work with prescribed conductance functions $c$ on $G$. Indeed with a choice of conductance function $c$ specified we define two crucial tools for our analysis, a graph Laplacian $\Delta (= \Delta_c)$, a discrete version of more classical notions of Laplacians, and an energy Hilbert space $\mathcal{H}_E$.

To make it more clear what are the new contributions in the present paper relative to previous work (e.g., [JP10, JP11a, JP11b, JT14]), we note that they are two-fold: our spectral theoretic conclusions in Section 4 below, and our applications in Section 5. The key to our construction in sect 4 is our identification of a canonical closable operator $L_0$ (with dense domain) from $l^2$ into $\mathcal{H}_E$. Once closability is established, from the graph-closure $L$, we then get the following two selfadjoint operators $LL^*$ in $\mathcal{H}_E$, and $L^*L$ in $l^2$. We show that the first is a Krein extension (referring to $\mathcal{H}_E$), and that the second is the selfadjoint $l^2$-Laplacian $\Delta_2$. Of course $l^2$ does not depend on choice of conductance function, but the graph Laplacian $\Delta$ does. A number of issues are resolved, and care is exercised: Since $L_0$ maps between different Hilbert spaces, i.e., from a dense domain in $l^2$ to $\mathcal{H}_E$, its adjoint $L_0^*$ acts in the reverse direction. We show that the domain of $L_0^*$ is dense in $\mathcal{H}_E$, and $L_0$ is closable. From this we proceed to establish unitary equivalence between an associated pair of selfadjoint operators. Specifically we show that two selfadjoint operators $LL^*$ in $\mathcal{H}_E$, and $L^*L$ are unitarily equivalent; i.e., that the corresponding spectral measures are unitarily equivalent, and we specify an explicit intertwining operator.

Note that unitary equivalence is a global conclusion, and that it is a much stronger conclusion than related local notions of “same spectrum,” such as comparison of local Radon-Nikodym derivatives for the respective spectral measures computed in cyclic subspaces.

Because of statistical consideration, and our use of random walk models, we focus our study on infinite electrical networks, i.e., the case when the graph $G$ is countable.
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infinite. In this case, for realistic models the graph Laplacian $\Delta_c$ will then be an unbounded operator with dense domain in $\mathcal{H}_E$, Hermitian and semibounded. Hence it has a unique Krein extension. Our main theorem gives an answer to how the Krein extensions depend on assignment of conductance function. Our theorem offers a direct comparison the Krein extensions $\Delta_c$ associated to pairs conductance functions of say $c_1$, and $c_2$ both defined on the same $G$.

Large networks arise in both pure and applied mathematics, e.g., in graph theory (the mathematical theory of networks), and more recently, they have become a current and fast developing research area; with applications including a host of problems coming from for example internet search, and social networks. Hence, of the recent applications, there is a change in outlook from finite to infinite.

More precisely, in traditional graph theoretical problems, the whole graph is given exactly, and we are then looking for relationships between its parameters, variables and functions; or for efficient algorithms for computing them. By contrast, for very large networks (like the Internet), variables are typically not known completely; – in most cases they may not even be well defined. In such applications, data about them can only be collected by indirect means; hence random variables and local sampling must be used as opposed to global processes.

Although such modern applications go far beyond the setting of large electrical networks (even the case of infinite sets of vertices and edges), it is nonetheless true that the framework of large electrical networks is helpful as a basis for the analysis we develop below; and so our results will be phrased in the setting of large electrical networks, even though the framework is much more general.

The applications of “large” or infinite graphs are extensive, counting just physics; see for example [BCD06, RAKK05, KMRS05, BC05, TD03, VZ92].

2. Preliminaries

Starting with a given network $(V, E, c)$, we introduce functions on the vertices $V$, voltage, dipoles, and point-masses; and on the edges $E$, conductance, and current. We introduce the graph-Laplacian $\Delta$ (see Definition 2.4 below.) There are two Hilbert spaces serving different purposes, $l^2(V)$, and the energy Hilbert space $\mathcal{H}_E$; the latter depending on choice of conductance function $c$.

The graph Laplacian $\Delta$ (Definition 2.4) has an easy representation as a densely defined semibounded operator in $l^2(V)$ via its matrix representation, see Remark 2.5. To do this we use implicitly the standard orthonormal (ONB) basis $\{\delta_x\}$ in $l^2(V)$. But in network problems, and in metric geometry, $l^2(V)$ is not useful; rather we need the energy Hilbert space $\mathcal{H}_E$ (see Section 2.2.)

We are motivated in part by earlier papers on analysis on discrete networks, see e.g., [Ana11, Gri10, Zem96, Bar93, Tet91].

Properties of $\mathcal{H}_E$: The case when $V$ is countably infinite, i.e., $\#V = \aleph_0$.

(i) The equation $\Delta h = 0$ typically has non-zero solutions in $\mathcal{H}_E$, but not in $l^2(V)$. 
(ii) Let $x$ and $y$ be two distinct vertices; then the equation
\[ \Delta v_{xy} = \delta_x - \delta_y \] (2.1)
has a solution in $\mathcal{H}_E$, but $v_{xy} \notin L^2(V)$.

(iii) Given $x \neq y$ in $V$, then there is a unique $v_{xy} \in \mathcal{H}_E$ (dipole) such
\[ \langle v_{xy}, u \rangle_{\mathcal{H}_E} = u(x) - u(y), \] (2.2)
and $v_{xy}$ satisfies (2.1); i.e., (2.2) implies (2.1).

(iv) Fix a base-point $o \in V$, and for $x \in V' := V \setminus \{o\}$, set $v_x := v_{xo}$; and
\[ G(x,y) := \langle v_x, v_y \rangle_{\mathcal{H}_E}, \] (2.3)
then
\[ \Delta_y G(x,y) = \delta_{x,y}. \] (2.4)

The function $G$ in (2.3) is called the Gramian.

(v) The dipole vectors $v_{xy}$ from (iii) satisfy the following:
\[ \|v_{xy}\|_{\mathcal{H}_E}^2 = \sup \left\{ \frac{1}{\|u\|_{\mathcal{H}_E}^2} \mid u \in \mathcal{H}_E, u(x) = 1, u(y) = 0 \right\} \]
and
\[ \text{dist}_c(x,y) = \|v_{xy}\|_{\mathcal{H}_E}^2 = v_{xy}(x) - v_{xy}(y) \] (2.5)
is a metric on $V$.

**Remark 2.1.** In the notation of electrical networks, (2.5) states that the distance between $x$ and $y$ is the voltage drop measured in the dipole.

**Remark 2.2 (Properties of the metric $d_c$ in (2.5)).** Combining (2.5) and (2.2) we immediately obtain the following estimate
\[ |u(x) - u(y)|^2 \leq \text{dist}_c(x,y) \|u\|_{\mathcal{H}_E}^2 \]
valid for all $u \in \mathcal{H}_E$ and all pairs of vertices $x$ and $y$ in $V$.

**Consequence (i):** If $\tilde{V}_c$ denotes the completion of the metric space $(V, \text{dist}_c)$, then every $u \in \mathcal{H}_E$ extends by completion to be a continuous Lipschitz-function (denoted $\tilde{u}$) on $\tilde{V}_c$, and we have
\[ |\tilde{u}(x^*) - \tilde{u}(y^*)|^2 \leq \text{dist}_c(x^*, y^*) \|u\|_{\mathcal{H}_E}^2 \]
valid for all points $x^*, y^*$ in $\tilde{V}_c$.

**Consequence (ii):** With notations as above; we conclude that the space of continuous functions
\[ \left\{ \tilde{u} \text{ on } \tilde{V}_c \mid \|u\|_{\mathcal{H}_E} \leq 1 \right\} \]
is relatively compact in $C(\tilde{V}_c)$.

**Proof of (ii).** Follows from (i), and the Arzelà-Ascoli theorem. \qed
2.1. Basic Setting

Here we define the graph Laplacian $\Delta (= \Delta_c)$, and the energy Hilbert space $\mathcal{H}_E$, and we outline some of their properties.

Let $V$ be a countable discrete set, and let $E \subset V \times V$ be a subset such that:

1. $(xy) \in E \iff (yx) \in E$; $x, y \in V$;
2. $\# \{ y \in V \mid (xy) \in E \}$ is finite, and $> 0$ for all $x \in V$;
3. $(xx) \notin E$; and
4. $\exists o \in V$ s.t. for all $y \in V \exists x_0, x_1, \ldots, x_n \in V$ with $x_0 = o, x_n = y, (x_{i-1}x_i) \in E, \forall i = 1, \ldots, n$. (This property is called connectedness.)
5. If a conductance function $c$ is given we require $c_{xy} > 0$. See Definition 2.3 below.

Definition 2.3. A function $c : E \to \mathbb{R}_+$ is called conductance function if $c_{xy} > 0, c_{xy} = c_{yx}$, for all $(xy) \in E$. Given $x \in V$, we set

$$c(x) := \sum_{(xy) \in E} c_{xy}. \tag{2.6}$$

The summation in (2.6) is denoted $x \sim y$; i.e., $x \sim y$ if $(xy) \in E$.

Definition 2.4. When $c$ is a conductance function (see Def. 2.3) we set $\Delta = \Delta_c$ (the corresponding graph Laplacian)

$$\left( \Delta u \right)(x) = \sum_{y \sim x} c_{xy} \left( u(x) - u(y) \right) = c(x)(u(x) - \sum_{y \sim x} c_{xy} u(y)) \tag{2.7}$$

Remark 2.5. Given $G = (V, E, c)$ as above, and let $\Delta = \Delta_c$ be the corresponding graph Laplacian. With a suitable ordering on $V$, we obtain the following banded $\infty \times \infty$ matrix-representation for $\Delta$ (eq. (2.8)). We refer to [GLS12] for a number of applications of infinite banded matrices.

\[
\begin{bmatrix}
c(x_1) & -c_{x_1x_2} & 0 & \cdots & \cdots & \cdots & 0 & \cdots \\
-c_{x_2x_1} & c(x_2) & -c_{x_2x_3} & 0 & \cdots & \cdots & \vdots & \vdots \\
0 & -c_{x_3x_2} & c(x_3) & -c_{x_3x_4} & 0 & \cdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\
0 & \cdots & \cdots & -c_{x_nx_{n-1}} & c(x_n) & -c_{x_nx_{n+1}} & 0 & \cdots \\
\vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{bmatrix} \tag{2.8}
\]

(The above planar matrix representation for $\Delta$ in eq (2.7) is an oversimplification in two ways: First, in the general case, the width of the band down the diagonal in (2.8) is typically more than 3; i.e., the matrix representation for the general graph Laplacian is typically banded with much wider bands. Secondly, the width is not in general constant, i.e., for each row in the infinite by infinite matrix, the number of...
non-zero entries may vary and even be unbounded. Furthermore the row and column size in the matrix is typically double infinite, and if a base point is chosen in \(V\), it may occur inside the matrix, in a diagonal position.)

**Overview of main results in the paper.** Our main results from sections 4 and 5 are as follows: In section 4, we prove the following result (Theorem 4.1): Starting with a fixed graph \((V, E)\) and an arbitrary but fixed conductance function \(c\), as noted, we then arrive at two versions of a selfadjoint graph Laplacian, one defined naturally in the \(l^2\) space of \(V\), and the other in the energy Hilbert space \(\mathcal{H}_E\) defined from \(c\). The first is automatically selfadjoint, but the second involves a Krein extension (see Definition 3.3). We prove that, as sets, the two spectra are the same, aside from the point 0. The point zero may be in the spectrum of the second, but not the first. In addition to this theorem, we isolate other spectral similarities.

In section 5 we turn to fine structure of the respective spectra. We study how the spectrum changes subject to change of conductance function \(c\). Specifically, we study how the spectrum of the graph Laplacian \(\Delta_c\) changes subject to variations in choice of conductance function \(c\), how estimates for two conductance functions translate into spectra comparisons for the two corresponding graph Laplacians. We prove (Theorems 5.8 and 5.16) that the natural order of conductance functions, i.e., pointwise as functions on \(E\), induces a certain similarity of the corresponding (Krein extensions computed from the) two graph Laplacians. Since the spectra are typically continuous, precise notions of fine-structure of spectrum must be defined in terms of equivalence classes of positive Borel measures (on the real line.) Hence our detailed comparison of spectra must be phrased involving these; see Definition 5.14.

2.2. The Energy Hilbert Spaces \(\mathcal{H}_E\)

Here we prove some technical lemmas for the energy Hilbert space \(\mathcal{H}_E\), and operators which will be needed later.

Let \(G = (V, E, c)\) be an infinite connected network introduced in section 2.1. Set 
\[ \mathcal{H}_E := \text{completion of all compactly supported functions } u : V \to \mathbb{C} \text{ with respect to} \]
\[\langle u, v \rangle_{\mathcal{H}_E} := \frac{1}{2} \sum_{(x,y) \in E} c_{xy} (u(x) - u(y))(v(x) - v(y)) \tag{2.9}\]
\[\|u\|_{\mathcal{H}_E}^2 := \frac{1}{2} \sum_{(x,y) \in E} c_{xy} |u(x) - u(y)|^2 \tag{2.10}\]
then \(\mathcal{H}_E\) is a Hilbert space [JP10].

**Lemma 2.6.** For all \(x, y \in V\), there is a unique real-valued dipole vector \(v_{xy} \in \mathcal{H}_E\) s.t.
\[\langle v_{xy}, u \rangle_{\mathcal{H}_E} = u(x) - u(y), \forall u \in \mathcal{H}_E. \tag{2.11}\]

**Proof.** By the connectedness assumption (see (4), Section 2.1), one checks that 
\[\mathcal{H}_E \ni u \mapsto u(x) - u(y) \in \mathbb{C}\]
is a bounded linear functional on $\mathcal{H}_E$; so by Riesz’s theorem there exists a unique $v_{xy} \in \mathcal{H}_E$ s.t. (2.11) holds. For details, see e.g., [JP10, JP11a].

A difficulty with $\Delta$ is that there is not an independent characterization of the domain $\text{dom} (\Delta, \mathcal{H}_E)$ when $\Delta$ is viewed as an operator in $\mathcal{H}_E$ (as opposed to in $l^2(V)$); other than what we do in Definition 2.11, i.e., we take for its domain $D_{\mathcal{E}} = \text{finite span of dipoles.}$ This creates an ambiguity with functions on $V$ versus vectors in $\mathcal{H}_E$. Note, vectors in $\mathcal{H}_E$ are equivalence classes of functions on $V$. In fact we will see that it is not feasible to aim to prove properties about $\Delta$ in $\mathcal{H}_E$ without first introducing dipoles; see Lemma 2.6. Also the delta-functions $\{\delta_x\}$ from the $l^2(V)$-ONB will typically not be total in $\mathcal{H}_E$. In fact, the $H_E$ ortho-complement of $\{\delta_x\}$ in $\mathcal{H}_E$ consists of the harmonic functions in $\mathcal{H}_E$, see Lemma 2.12 (5).

**Definition 2.7.** Let $\mathcal{H}$ be a Hilbert space with inner product denoted $\langle \cdot, \cdot \rangle$, or $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ when there is more than one possibility to consider. Let $J$ be a countable index set, and let $\{w_j\}_{j \in J}$ be an indexed family of non-zero vectors in $\mathcal{H}$. We say that $\{w_j\}_{j \in J}$ is a frame for $\mathcal{H}$ iff (Def.) there are two finite positive constants $b_1$ and $b_2$ such that

$$b_1 \|u\|_{\mathcal{H}}^2 \leq \sum_{j \in J} |\langle w_j, u \rangle_{\mathcal{H}}|^2 \leq b_2 \|u\|_{\mathcal{H}}^2$$

(2.12)

holds for all $u \in \mathcal{H}$. We say that it is a Parseval frame if $b_1 = b_2 = 1$.

For references to the theory and application of frames, see e.g., [HJL+13, KLZ09, CM13, SD13, KOPT13, EO13].

**Lemma 2.8.** If $\{w_j\}_{j \in J}$ is a Parseval frame in $\mathcal{H}$, then the (analysis) operator $A = A_{\mathcal{H}} : \mathcal{H} \rightarrow l^2 (J)$,

$$Au = (\langle w_j, u \rangle_{\mathcal{H}})_{j \in J}$$

(2.13)

is well-defined and isometric. Its adjoint $A^* : l^2 (J) \rightarrow \mathcal{H}$ is given by

$$A^* \left( (\gamma_j)_{j \in J} \right) := \sum_{j \in J} \gamma_j w_j$$

(2.14)

and the following hold:

1. The sum on the RHS in (2.14) is norm-convergent;
2. $A^* : l^2 (J) \rightarrow \mathcal{H}$ is co-isometric; and for all $u \in \mathcal{H}$, we have

$$u = A^*Au = \sum_{j \in J} \langle w_j, u \rangle w_j$$

(2.15)

where the RHS in (2.15) is norm-convergent.

**Proof.** The details are standard in the theory of frames; see the cited papers above. Note that (2.12) for $b_1 = b_2 = 1$ simply states that $A$ in (2.13) is isometric, and so $A^*A = I_{\mathcal{H}} = \text{the identity operator in } \mathcal{H}$, and $AA^* = \text{the projection onto the range of } A$. □
Theorem 2.9. Let $G = (V, E, c)$ be an infinite network. Choose an orientation on the edges, denoted by $E^{(ori)}$. Then the system of vectors
\[
\left\{ w_{xy} := \sqrt{c_{xy}} v_{xy}, \ (xy) \in E^{(ori)} \right\}
\] is a Parseval frame for the energy Hilbert space $\mathcal{H}_E$. For all $u \in \mathcal{H}_E$, we have the following representation
\[
u = \sum_{(xy) \in E^{(ori)}} c_{xy} \langle v_{xy}, u \rangle v_{xy}, \quad \text{and}
\] \[
\|u\|_{\mathcal{H}_E}^2 = \sum_{(xy) \in E^{(ori)}} c_{xy} |\langle v_{xy}, u \rangle|^2
\] (2.17) (2.18)

Proof. See [JT14, CH08].

Remark 2.10. While the vectors $w_{xy} := \sqrt{c_{xy}} v_{xy}, (xy) \in E^{(ori)}$, form a Parseval frame in $\mathcal{H}_E$ in the general case, typically this frame is not an orthogonal basis (ONB) in $\mathcal{H}_E$.

2.3. The Graph-Laplacian

Here we prove some technical lemmas for graph Laplacian in the energy Hilbert space $\mathcal{H}_E$.

Let $G = (V, E, c)$ be as above; assume $G$ is connected; i.e., there is a base point $o$ in $V$ such that every $x \in V$ is connected to $o$ via a finite path of edges.

If $x \in V$, we set
\[
\delta_x (y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}
\] (2.19)

Definition 2.11. Let $(V, E, c, o, \Delta)$ be as above. Let $V' := V \setminus \{o\}$, and set
\[
v_x := v_{x,o}, \ \forall x \in V'.
\]

Further, let
\[
\mathcal{D}_2 := \text{span} \left\{ \delta_x \mid x \in V \right\}, \quad \text{and}
\]
\[
\mathcal{D}_E := \left\{ \sum_{\text{finite}} \xi_x v_x \mid \xi_x \in \mathbb{C}, \ x \in V' \right\}
\] (2.20) (2.21)

where by “span” we mean the set of all finite linear combinations.

Lemma 2.12 below summarizes the key properties of $\Delta$ as an operator, both in $l^2(V)$ and in $\mathcal{H}_E$.

Lemma 2.12. The following hold:
(1) $\langle \Delta u, v \rangle_{l^2} = \langle u, \Delta v \rangle_{l^2}, \ \forall u, v \in \mathcal{D}_2$;
(2) $\langle \Delta u, v \rangle_{\mathcal{H}_E} = \langle u, \Delta v \rangle_{\mathcal{H}_E}, \ \forall u, v \in \mathcal{D}_E$;
(3) $\langle u, \Delta u \rangle_{l^2} \geq 0, \ \forall u \in \mathcal{D}_2$, and
(4) $\langle u, \Delta u \rangle_{\mathcal{H}_E} \geq 0, \ \forall u \in \mathcal{D}_E$. 
Moreover, we have
\[ (5) \langle \delta_x, u \rangle_{\mathcal{H}_E} = (\Delta u)(x), \forall x \in V, \forall u \in \mathcal{H}_E. \]
\[ (6) \Delta v_{xy} = \delta_x - \delta_y, \forall v_{xy} \in \mathcal{H}_E. \text{ In particular, } \Delta v_x = \delta_x - \delta_o, x \in V' \setminus \{o\}. \]
\[ (7) \delta_x(\cdot) = c(x)v_x(\cdot) - \sum_{y \sim x} c_{xy}v_y(\cdot), \forall x \in V'. \]
\[ (8) \langle \delta_x, \delta_y \rangle_{\mathcal{H}_E} = \begin{cases} c(x) = \sum_{t \sim x} c_{tx} & \text{if } y = x \\ -c_{xy} & \text{if } (xy) \in E \\ 0 & \text{if } (xy) \notin E, \ x \neq y \end{cases} \]

**Proof.** See [JP10, JP11a, JT14]. Note that the numbers in (8) are precisely the entries in the $\infty \times \infty$ banded matrix (2.8).

\[ \square \]

### 3. The Krein Extension

Fix a conductance function $c$. In this section we turn to some technical lemmas we will need for the Krein extension of $\Delta (= \Delta_c)$; see Definition 3.3 below.

It is known the graph-Laplacian $\Delta$ is automatically essentially selfadjoint as a densely defined operator in $l^2(V)$, but not as a $\mathcal{H}_E$ operator [Jor08, JP11b]. Since $\Delta$ defined on $\mathcal{D}_E$ is semibounded, it has the Krein extension $\Delta_{Kre}$ (in $\mathcal{H}_E$).

**Lemma 3.1.** Consider $\Delta$ with $\text{dom}(\Delta) := \text{span}\{v_{xy} : x, y \in V\}$, then
\[ \langle \varphi, \Delta \varphi \rangle_{\mathcal{H}_E} = \sum_{xy \in E} c_{xy}^2 \left| \langle v_{xy}, \varphi \rangle_{\mathcal{H}_E} \right|^2. \]

**Proof.** Suppose $\varphi = \sum \varphi_{xy}v_{xy} \in \text{dom}(\Delta)$. Note the edges are not oriented, and a direct computation shows that
\[ \langle \varphi, \Delta \varphi \rangle_{\mathcal{H}_E} = 4 \sum_{x,y} |\varphi_{xy}|^2. \]

Using the Parseval frames in Theorem 2.9, we have the following representation
\[ \varphi = \sum_{(xy) \in E} \frac{1}{2} c_{xy} \overbrace{\langle v_{xy}, \varphi \rangle_{\mathcal{H}_E}}^{= \varphi_{xy}} v_{xy} \]

Note $\varphi \in \text{span}\{v_{xy} : x, y \in V\}$, so the above equation contains a finite sum.

It follows that
\[ \langle \varphi, \Delta \varphi \rangle_{\mathcal{H}_E} = 4 \sum_{(xy) \in E} |\varphi_{xy}|^2 = \sum_{(xy) \in E} c_{xy}^2 \left| \langle v_{xy}, \varphi \rangle_{\mathcal{H}_E} \right|^2 \]
which is the assertion. \[ \square \]

Throughout below, we shall assume that $V$ is infinite and countable.
Theorem 3.2. Let $G = (V, E, c)$ be an infinite network. If the deficiency indices of $\Delta (= \Delta_c)$ are $(k, k)$, $k > 0$, where $\text{dom}(\Delta) = \text{span}\{v_{xy}\}$, then the Krein extension $\Delta_{Kre} \supset \Delta$ is the restriction of $\Delta^*$ to

$$\text{dom}(\Delta_{Kre}) := \left\{ u \in \mathcal{H}_E \mid \sum_{(xy) \in E} c_{xy}^2 |\langle v_{xy}, u \rangle_E|^2 < \infty \right\}. \quad (3.1)$$

Proof. Follows from Lemma 3.1, and the characterization of Krein extensions of semi-bounded Hermitian operators; see, e.g., [DS88, AG93, RS75]. □

Definition 3.3. The Krein extension $\Delta_{Kre}$ is specified by a selfadjoint and contractive operator $B$ in $\mathcal{H}_E$ satisfying $B(\varphi + \Delta \varphi) = \varphi$, $\forall \varphi \in \mathcal{D}_E$; and then $\Delta_{Kre} = B^{-1} - I_{\mathcal{H}_E}$; noting that $B^{-1}$ is well defined, selfadjoint, but unbounded.

Remark 3.4. We shall return to the Krein extension after Corollary 3.6, in Theorem 4.1, and in Corollary 4.3 where we give an explicit formula for the s.a. contraction $B = B_{Kr}$.

3.1. A Factorization via $l^2(V')$

We begin with some preliminary lemmas about closable operators in Hilbert space.

Lemma 3.5. Let $\mathcal{H}_i$ and $\mathcal{H}_2$ be two Hilbert spaces with respective inner products $\langle \cdot, \cdot \rangle_i$, $i = 1, 2$; let $\mathcal{D}_i \subset \mathcal{H}_i$, $i = 1, 2$, be two dense linear subspaces; and let $L_0 : \mathcal{D}_1 \to \mathcal{H}_2$, and $M_0 : \mathcal{D}_2 \to \mathcal{H}_1$ be linear operators such that

$$\langle L_0 u, v \rangle_2 = \langle u, M_0 v \rangle_1, \forall u \in \mathcal{D}_1, \forall v \in \mathcal{D}_2. \quad (3.2)$$

Then both operators $L_0$ and $M_0$ are closable. The closures $L = \overline{L_0}$, and $M = \overline{M_0}$ satisfy

$$L^* \subseteq M \text{ and } M \subseteq L^*. \quad (3.3)$$

Moreover, equality in (3.3) holds if and only if

$$\mathcal{N}(I_{\mathcal{H}_2} + M^* L^*) = \mathcal{N}(I_{\mathcal{H}_1} + L^* M^*) = 0. \quad (3.4)$$

Proof. Note (3.2) is equivalent to

$$L_0 \subseteq M_0^*, \text{ and } M_0 \subseteq L_0^*.$$

Since the adjoints $M_0^*, L_0^*$ are closed, it follows that both $L_0$ and $M_0$ are closable. Below we give a direct argument:

Suppose $w \in \mathcal{H}_2$, $u_n \in \mathcal{D}_1$, satisfying $\|L_0 u_n - w\|_2 \to 0$, $\|u_n\|_1 \to 0$, then, for all $v \in \mathcal{D}_2$, we have

$$\langle L_0 u_n, v \rangle_2 = \langle u_n, M_0 v \rangle_1$$

and as $n \to \infty$, we get

$$\langle w, v \rangle_2 = \langle 0, M_0 v \rangle_1 = 0.$$

Since $\mathcal{D}_2$ is dense in $\mathcal{H}_2$, it follows that $w = 0$. Thus, $L_0$ is closable. Similarly, $M_0$ is closable as well. See also Lemma 3.11 below.
From (3.2) we conclude that
\[ M \subset L^* \quad (= L_0^*), \quad \text{and} \quad L \subset M^* \quad (= M_0^*). \]  
(3.5)

We give conditions for the inclusions in (3.5) to be equal. To verify \( M = L^* \), we must prove that the graph of \( M \) is dense in that of \( L^* \), i.e., if \( w \in \text{dom} \ (L^*) \), satisfying
\[
\left\langle \left( \begin{array}{c} v \\ Mv \end{array} \right), \left( \begin{array}{c} w \\ L^*w \end{array} \right) \right\rangle_{\mathcal{H}_2 \oplus \mathcal{H}_1} = 0, \quad \forall v \in \mathcal{D}_2
\]
(3.6) then \( w = 0 \).

Now (3.6) reads
\[
\langle v, w \rangle_2 + \langle Mv, L^*w \rangle_1 = 0, \quad \forall v \in \mathcal{D}_2.
\]
Hence \( L^*w \in \text{dom} \ (M^*) \), and \( M^*L^*w = -w \); i.e., we arrive at the implication
\[
(I_{\mathcal{H}_2} + M^*L^*)w = 0 \implies w = 0.
\]
(3.4) follows from this. \( \square \)

For the result to the effect that \( \Delta \) as an operator in \( l^2(V) \), with domain consisting of finitely supported functions, is essentially selfadjoint we cite [KL12, Woj07, Jor08, JP10].

**Corollary 3.6.** Let \( L \) and \( M \) be as above, including (3.4), then
1. \( ML \) is selfadjoint in \( \mathcal{H}_1 \), and
2. \( LM \) is selfadjoint in \( \mathcal{H}_2 \).

**Proof.** This follows from von Neumann’s theorem which states that if \( T \) is any closed operator with dense domain, then \( T^*T \) is selfadjoint. See [DS88, RS75]. \( \square \)

Let \( \Delta_{Kre} \) denote the Krein-extension of the operator \( \Delta \big|_{\mathcal{H}_E} \) defined in Definition 2.11. We have the following:

**Remark 3.7.** As an application, we consider the factorization \( \Delta_{Kre} = LL^* \), where \( \Delta_{Kre} \) is the Krein extension of the graph-Laplacian in \( \mathcal{H}_E \). See Theorem 3.9 below.

Let \( G = (V, E, c, \Delta (= \Delta_c), \mathcal{H}_E, \Delta_{Kre}) \) be as above.

**Definition 3.8.** Let \( \mathcal{D}'_{l^2} \) be the dense subspace in \( l^2(V') \), given by
\[
\mathcal{D}'_{l^2} = \left\{ (\xi_x) \in l^2(V') \mid \text{finite support, and} \sum_{x \in V'} \xi_x = 0 \right\}.
\]
(3.7)

**Theorem 3.9.** Let \( (V, E, c, \Delta (= \Delta_c), \mathcal{H}_E, \Delta_{Kre}) \) be as above.
1. Set
\[
L \left( (\xi_x) \right) := \sum_x \xi_x \delta_x \in \mathcal{H}_E; \quad \text{(3.8)}
\]
then \( L : l^2(V') \rightarrow \mathcal{H}_E \) is a closable operator with dense domain \( \mathcal{D}'_{l^2} \); and the corresponding adjoint operator \( L^* : \mathcal{H}_E \rightarrow l^2(V') \) satisfies
\[
L^* \left( \sum_{x \in V'} \xi_x v_x \right) = \xi \left( (= (\xi_x)) \right). \quad \text{(3.9)}
\]
(2) $LL^*$ is selfadjoint,
(3) $LL^* = \Delta_{Kre}$; and
(4) Using Lemma 2.12 \((7)\) we note that $L$ in \((3.8)\) may also be written in the following form:

$$L(\xi) = \sum_x \xi_x c(x) v_x - \sum_y \left( \sum_{x \sim y} \xi_x c_{xy} \right) v_y.$$  \((3.10)\)

**Remark 3.10.** In the proof of Theorem 3.9, we will use Lemma 3.5 and Corollary 3.6. (See also [JT14] for more details about graph Laplacians.)

The following are immediate from our constructions:

\[
\begin{array}{c}
l^2(V') \xrightarrow{L} \mathcal{H}_E \xrightarrow{L^*} l^2(V') \xrightarrow{M^*} \mathcal{H}_E \xrightarrow{M} l^2(V') \\
\end{array}
\]  \((3.11)\)

- $L : l^2(V') \rightarrow \mathcal{H}_E$, with dense domain

$$L((\xi_x)) := \sum_x \xi_x \delta_x, \forall (\xi_x) \in \mathcal{D}_l \subset l^2(V'),$$  \((3.12)\)

where $(\xi_x) \in \mathcal{D}_l$ satisfies $\sum_x \xi_x = 0$; see Def. 3.8.

- $M : \mathcal{H}_E \rightarrow l^2(V')$,

$$M \left( \sum_x \xi_x v_x \right) := (\xi_x), \forall \sum_x \xi_x v_x \in \mathcal{D}_E \subset \mathcal{H}_E;$$  \((3.13)\)

where $\mathcal{D}_E := \{ \sum_{\text{finite}} \xi_x v_x \mid \xi_x \in \mathcal{C}, x \in V' \}$, see Def. 2.11.

- $\mathcal{D}_l$ is dense in $l^2(V')$, and $\mathcal{D}_E$ is dense in $\mathcal{H}_E$.

**Proof of Theorem 3.9.** Step 1.

$$\langle L(\xi), u \rangle_{\mathcal{H}_E} = \langle \xi, Mu \rangle_{l^2(V')}$$  \((3.15)\)

holds $\forall \xi \in \mathcal{D}_l \subset l^2(V'), \forall u \in \mathcal{D}_E \subset \mathcal{H}_E$. It follows that

$$L \subset M^*, \text{ and } M \subset L^*.$$  \((3.16)\)

(We define containment of operators as containment of the respective graphs. We will show that “equality” holds in \((3.16)\).)

The proof will resume after the following digression: \(\square\)
**Lemma 3.11.** Let $\mathcal{H}_1 \supset \mathcal{D} \xrightarrow{T} \mathcal{H}_2$ be densely defined, with $\text{dom}(T) = \mathcal{D}$, and let $\mathcal{G}(T) = \{(u, Tu) : u \in \mathcal{D}\} \subset \mathcal{H}_1 \oplus \mathcal{H}_2$ be the graph of $T$. Set
$$\chi(v) = \begin{pmatrix} -v \\ v \end{pmatrix}$$
for all $(u, v) \in \mathcal{H}_1 \oplus \mathcal{H}_2$. Then
$$\chi(\mathcal{G}(T)) = \mathcal{G}(T^*) .$$
See the diagram below.

**Proof.** Note the following statements are all equivalent:
\[
\begin{align*}
\left\langle \begin{pmatrix} -Tu \\ u \end{pmatrix}, \begin{pmatrix} w_2 \\ w_1 \end{pmatrix} \right\rangle &= 0, \quad \forall u \in \mathcal{D} \\
\uparrow \\
(Tu, w_2) &= \langle u, w_1 \rangle, \quad \forall u \in \mathcal{D} \\
\downarrow \\
w_2 &\in \text{dom}(T^*) \text{ and } T^*w_2 = w_1 \\
\downarrow \\
\begin{pmatrix} w_2 \\ w_1 \end{pmatrix} &\in \mathcal{G}(T^*) .
\end{align*}
\]

**Corollary 3.12.** $T$ is closable $\iff$ $\text{dom}(T^*)$ is dense in $\mathcal{H}_2$.

**Corollary 3.13.** Let $M, M^*, L, L^*$ be as before. Then
\begin{enumerate}
\item $\xi \in \text{dom}(M^*) \iff \xi \in l^2(V')$ and $\exists C = C_\xi < \infty$ s.t.
\[
\left| \langle \xi, Mu \rangle_{l^2(V')} \right| \leq C \|u\|_{\mathcal{H}_E}, \quad \forall u \in \mathcal{D}_E;
\] (3.17)
\item $u \in \text{dom}(L^*) \iff u \in \mathcal{H}_E$ and $\exists C = C_u < \infty$ s.t.
\[
\left| \langle L(\xi), u \rangle_{\mathcal{H}_E} \right| \leq C \|\xi\|_{l^2(V')}, \quad \forall \xi \in \mathcal{D}_E' .
\] (3.18)
\end{enumerate}

Comments on (3.17), i.e., $\text{dom}(M^*)$

Let $u = \sum_x \eta_x v_x$, $\sum \eta_x = 0$, finite support; then
\[
\langle \xi, Mu \rangle_{l^2(V')} = \sum_x \xi_x \eta_x
\] (3.19)
(It suffices to specialize to real valued functions.) Note (3.19) \(\iff\) 
\[
|\sum_x \xi_x \eta_x| \leq C_\xi \|u\|_{\mathcal{H}_E}, \forall u \in \mathcal{D}_E;
\]
i.e., \(\xi \in \text{dom}(M^*) \subseteq l^2(V')\).

Comments on (3.18), i.e., \(\text{dom}(L^*)\)
Using the same notations as above, we get (3.18) \(\iff\)
\[
|\sum_x \xi_x \eta_x| \leq C_\eta \|\xi\|_{l^2}, \forall \xi \in \mathcal{D}_{l^2}(V');
\]
i.e., \(u \in \text{dom}(L^*) \subseteq \mathcal{H}_E\).

Corollary 3.14. We have 
\[
\begin{align*}
\text{dom}(L^*) & \iff 
\left\{ u = \sum_x \eta_x v_x \in \mathcal{H}_E \text{ with } 
\eta_x \in l^2(V'), \text{ i.e., } \sum_x |\eta_x|^2 < \infty \right\}.
\end{align*}
\]

Proof of Theorem 3.9, continued. Step 2. We have \(M \subseteq L^*\) so \(\text{dom}(L^*)\) is dense. But we need \(M = L^*\) where \(M\) is the closure. Turn to ortho-complement:

Fix \(\left( \frac{w}{L^*w} \right) \in \mathcal{G}(L^*) = \text{graph of } L^* \) s.t.
\[
\left\langle \mathcal{H}_E \ni w := \sum_x \xi_x v_x, \left( l^2 \ni \zeta := L^* \left( \sum_x \xi_x v_x \right) \right) \right\rangle = 0, \forall \left( \frac{u}{Mu} \right) \in \mathcal{G}(M)
\]
\[
\downarrow
\]
\[
\langle w, u \rangle_{\mathcal{H}_E} + \sum_x \zeta_x \eta_x = 0, \forall u = \sum_x \eta_x v_x \in \mathcal{D}_E
\]
\[
\downarrow
\]
\[
\sum_x (w(x) - w(o)) \eta_x + \sum_x \zeta_x \eta_x = 0, \forall u = \sum_x \eta_x v_x \in \mathcal{D}_E
\]
\[
\downarrow
\]
\[
\zeta_x = w(o) - w(x) \in l^2(V')
\]
\[
\downarrow
\]
\[
L^*w = \xi \text{ (Recall } w = \sum_x \xi_x v_x) \Rightarrow L^* = M
\]
where we identify \(M\) with its closure.

The second equality, \(M = L^*\), is Corollary 3.23 below.

Details for the rest of the proof are in Lemma 3.20-3.22, and the corollaries below. \(\Box\)

Remark 3.15. For a general symmetric pair of unbounded operators with dense domains, \(L_0\) and \(M_0\) (as above), we have that each will be contained in the adjoint of the other. So this means that each of the two adjoints must have dense domain; and so be closable. If the closures are denoted \(L\) and \(M\), respectively, then this new pair will still have each is contained in the adjoint of the other, but in general these inclusions might be strict. Nonetheless, if one is an equality, then so is the other; i.e, the conclusion \(L^* = M\) implies \(M^* = L\). (See also Corollary 3.21 below.)
Example 3.16. Let \( G = (V, E, c) \), with \( V = \mathbb{Z} \cup \{0\} \); i.e., a nearest neighbor graph \((x, y \in V \text{ are connected by an edge whenever } x \text{ is a nearest neighbor of } y)\).

Fix \( A > 1 \). Set
\[
c_{n,n+1} \equiv 1, \quad c^{(A)}_{n,n+1} = A^n;
\]
then \( c < c^{(A)} \) point-wise on \( E \).

If we introduce \( H_C \) and \( H_A \), where
\[
H_C = \left\{ u : \sum_n |u(n) - u(n+1)|^2 < \infty \right\}
\]
\[
H_A = \left\{ u : \sum_n A^n |u(n) - u(n+1)|^2 < \infty \right\}
\]
then
\[
v_{xy}(\cdot) = \begin{cases} v_{xy}(x) & \text{if } n < x \\ v_{xy}(y) & \text{if } n > y \\ v_{xy}(x) - n & \text{if } x < n \leq y \end{cases}
\]
and
\[
v^{(A)}_{xy}(n) = \begin{cases} v^{(A)}_{xy}(x) & \text{if } n < x \\ v^{(A)}_{xy}(y) & \text{if } n > y \\ v_{xy}(x) - \sum_{x<k \leq n} \frac{1}{A^k} & \text{if } x < n \leq y \end{cases}
\]
See Fig 3.1 below.

Moreover, if \( j_A : H_A \longrightarrow H_C \) is the natural inclusion then
\[
j^{*}_A v_{xy} = v^{(A)}_{xy}.
\]

**Proof.** See Lemma 5.2. \(\square\)

**Remark 3.17.** In \( H_A \), we consider the associated Laplacian
\[
(\Delta_A f)(n) = A^n (f(n) - f(n+1)) + A^{n-1} (f(n) - f(n-1))
\]
defined initially on the dense domain \( D_A = \text{span} \{v_{(n,n+1)}^{(A)} \} \). This operator has deficiency indices \((1, 1)\) as one checks using domination with the geometric series \( \sum_{k=1}^{\infty} A^{-n} \).

**Remark 3.18 ([JT14]).** In the example, \( c_{n,n+1} \equiv 1 \), the Laplacian \( \Delta = \Delta_c \) is bounded.

In the example \( c^{(A)}_{n,n+1} = A^n, A > 1 \), the Laplacian \( \Delta_A \) is unbounded, and its von Neumann deficiency indices are \((1, 1)\). Note since \( A > 1 \), so \( \{A^{-n}\}_{n=1}^{\infty} \) is monotone decreasing, and \( \sum_{n \geq k} A^{-n} = \frac{A^{-k+1}}{A-1} \) is convergent.
Remark 3.19. The metric space \((\mathbb{Z}_+, d_c)\) is unbounded when \(c \equiv \text{unit conductance}\), while \((\mathbb{Z}_+, d_A)\) is a bounded metric space when \(c^{(A)}_{n,n+1} = A^n, A > 1\).

3.2. Further Comments

Fix \((V, E, c, o)\) be as above. Recall

- \(V\) - vertices, \#\(V = \aleph_0\), so infinite;
- \(E\) - edges, see section 2.1 above;
- \(c\) - conductance, defined on \(E\);
- \(o\) - a fixed base-point in \(V\);
- \(\mathcal{H}_E\) - energy space;
- \(l^2 := l^2(V'), V' := V \setminus \{o\}\), and set \(v_x := v_{xo}, x \in V'\).

Set

\[
\mathcal{D}_E = \left\{ \sum_{x} \xi_x v_x \mid (\xi_x) \text{ finite, } \sum_{x} \xi_x = 0 \right\} \quad (3.20)
\]

\[
l_0^2 := \mathcal{D}_E^l = \left\{ (\xi_x) \mid (\xi_x) \text{ finite support, } \sum_{x} \xi_x = 0 \right\} \quad (3.21)
\]

\(\mathcal{D}_E\) is dense in \(\mathcal{H}_E\), and \(l_0^2\) dense in \(l^2\).

Let \(L_0\) and \(M_0\) be as follows, see Remark 3.10:

\[
\begin{array}{c}
l^2 \quad L_0 \\
\searrow \quad \nearrow \quad \mathcal{H}_E \\
\searrow \quad \nearrow \quad M_0
\end{array}
\]

Set \(\text{dom} \ (L_0) := l_0^2\), and

\[
L_0 \xi := \sum_{x} \xi_x \delta_x, \quad \forall \xi \in l_0^2; \quad (3.22)
\]
Set $\text{dom} \,(M_0) := \mathcal{D}_E$, and

$$M_0 u = \xi, \ \forall u := \sum_x \xi_x v_x \in \mathcal{H}_E. \quad (3.23)$$

**Lemma 3.20.** We have

$$\langle L_0 \eta, u \rangle_{\mathcal{H}_E} = \langle \eta, M_0 u \rangle_{l^2}$$

for all $\eta \in l^2_0$, and all $u \in \mathcal{D}_E$.

**Corollary 3.21.** $M_0 \subset L_0^*$, and $L_0 \subset M_0^*$. Identify $M$ with the closure of the operator defined initially only on $\mathcal{D}_E$; and identify $L$ with the closure of the operator defined initially only on $l^2_0$, i.e.,

$$M := \overline{M_0}, \quad L = \overline{L_0}.$$

**Lemma 3.22.** $L_0^* = L^*$ is as follows: A vector $u \in \mathcal{H}_E$ is in $\text{dom}(L^*) \iff \Delta u \in l^2(V')$; see Definition 2.4.

**Proof.** By definition of the adjoint $L^*$, $u \in \text{dom}(L^*) \iff \exists C < \infty \text{ s.t.}$

$$\langle L \eta, u \rangle_{\mathcal{H}_E} \leq C \|\eta\|_{l^2}, \ \forall \eta \in l^2_0; \quad (3.24)$$

see the diagram below.

\[ \begin{array}{c}
 l^2 \overset{L^*}{\longrightarrow} \mathcal{D}_E \\
 \downarrow L \downarrow
\end{array} \]

Compute (3.24) as follows: Let $\eta \in l^2_0$, then

$$(\text{LHS})_{(3.24)} = \langle L \eta, u \rangle_{\mathcal{H}_E}$$

(by (3.22))

$$= \sum_x \eta_x \langle \delta_x, u \rangle_{\mathcal{H}_E}$$

$$= \sum_x \eta_x (\Delta u) (x),$$

where the last equality follows from Lemma 2.12 (5). So if (3.24) holds for some $C < \infty$, then $(\Delta u) (\cdot) \in l^2$, and $(L^* u) (x) = (\Delta u) (x)$, so

$$\langle L \eta, u \rangle_{\mathcal{H}_E} = \sum_x \eta_x (\Delta u) (\cdot) = \langle \eta, L^* u \rangle_{l^2}, \quad (3.25)$$

i.e., $L^* u = (\Delta u) (\cdot)$. \hfill \square

**Corollary 3.23.**

$$M = L^*. \quad (3.26)$$

**Proof.** We have (3.26) $M \subset L^*$ by Cor. 3.21. So $\mathcal{G}(M) \subset \mathcal{G}(L^*)$, i.e., containment of operator graphs. Now compute $\mathcal{G}(L^*) \cap \mathcal{G}(M)$: (see (3.22)-(3.23))
Variation of Conductance Functions in Discrete Laplacians

We have the graph inner product in $\mathcal{H}_E \oplus l^2$

\[
\left\langle \left( \sum_{x} \xi_x v_x \in \mathcal{H}_E \right), \left( \begin{array}{c} u \in \mathcal{H}_E \\ (\Delta u)(x) \in l^2 \end{array} \right) \right\rangle \in \mathcal{G}(M) \cup \mathcal{G}(L^*)
\]

(3.27)

Assume $\left( \begin{array}{c} u \\ L^* u \end{array} \right) \in \mathcal{G}(M)$. Claim $u = 0$.

Proof of the claim: It is equivalent to prove

\[
\langle \sum_{x} \xi_x v_x, u \rangle_{\mathcal{H}_E} + \langle (\xi_x), L^* u \rangle_{l^2} = 0
\]

(3.28)

for all $(\xi_x)$ finite supported, s.t. $\sum_{x} \xi_x = 0$. See (3.23). Now rewrite (3.28) as follows

\[
0 = \sum_{x} \xi_x \langle v_x, u \rangle_{\mathcal{H}_E} + \sum_{x} \xi_x (\Delta u)(x)
\]

(3.29)

for all $(\xi_x)$ finite support, $\sum_{x} \xi_x = 0$; and use

\[
\langle v_x, u \rangle_{\mathcal{H}_E} = u(x) - u(o)
\]

\[
\sum_{x} \xi_x (u(x) - u(o)) = \sum_{x} \xi_x u(x), \text{ since } \sum_{x} \xi_x = 0;
\]

then (3.29) $\iff$

\[
\sum_{x} \xi_x \{ u(x) + (\Delta u)(x) \} = 0
\]

for all finite support $(\xi_x)$ s.t. $\sum_{x} \xi_x = 0$.

But then we get

$\mathcal{H}_E \ni u = -\Delta u \in l^2$

and therefore $u \in \mathcal{H}_E \cap l^2$, and $\langle u, \Delta u \rangle_{\mathcal{H}_E} \geq 0$ by Lemma 2.12. So $\|u\|^2_{\mathcal{H}_E} \geq 0$.

$\implies u = 0$. □

4. The Spectrum of the Graph Laplacians

In this section we show the following result (Theorem 4.1): Starting with a fixed graph $(V, E)$ and a fixed conductance function $c$, we arrive at two versions of a self-adjoint graph Laplacian, one defined naturally in the $l^2$ space of $V$, and the other in the energy Hilbert space defined from $c$. We prove that, as sets, the two spectra are the same, aside from the point $0$. The point zero may be in the spectrum of the second, but not the first. In addition to this theorem, we isolate other spectral similarities. In section 5 below we then study how the spectrum changes subject to variations in choice of conductance function $c$.

Let $(V, E, c, o, \Delta (= \Delta_c), \mathcal{H}_E)$ be as above. As in Definition 3.8, we introduce the operator $L : l^2 \longrightarrow \mathcal{H}_E$ where $l^2 := l^2(V')$, $V' := V \setminus \{o\}$. Recall $L$ is defined initially as $L_0$; and

\[
dom(L_0) = \left\{ (\xi_x) \mid \text{finitely supported, } \sum_{x} \xi_x = 0 \right\};
\]

(4.1)
and we let
\[ L := L_0 \]  \hfill (4.2)
i.e., the operator arises as the closure of \( L_0 \)
\[ \mathcal{G} (L) = \mathcal{G} (L_0) \]  \hfill (norm closure)
where
\[ \mathcal{G} (L_0) = \{ (\xi, L_0 \xi) \mid \xi \in \text{dom} (L_0) \} \subset l^2 \oplus \mathcal{H}_E. \]  \hfill (4.4)
(For the discussion of closed operators, see [DS88, RS75].)

We assume throughout the conductance \( c \) is fixed, and that \((V,E,c)\) is connected. Also, we fix a base-point \( o \), and set \( \mathbf{v}_x := v_{xo} \) (the \((xo)\) dipole for \( x \in V' \)), and we recall that
\[ L_0 (\xi) := \sum_x \xi_x \delta_x \in \mathcal{H}_E, \]  \hfill (4.5)
\[ L_0^* \left( \sum_x \xi_x v_x \right) := (\xi_x) \in l^2 (V'). \]  \hfill (4.6)

We shall compute the spectrum of the following two versions of the graph Laplacian:
\[ (\Delta u) (x) = \sum_{y \sim x} c_{xy} (u(x) - u(y)). \]  \hfill (4.7)

**V1.** \( \Delta \) is viewed as a selfadjoint operator in the Hilbert space \( l^2 (V') \)
**V2.** Refers to \( \Delta \) as the selfadjoint Krein extension \( \Delta_{Kr} \) in \( \mathcal{H}_E \).

For the operator in version 1, we shall write \( \Delta_{l^2} \) for identification; and
\[ \text{dom} (\Delta_{l^2}) = \{ \xi \in l^2 \mid \Delta \xi \in l^2 \} \]  \hfill where
\[ (\Delta \xi) (x) := \sum_{y \sim x} c_{xy} (\xi_x - \xi_y). \]  \hfill (4.8)

It is known [Jor08, JP11b] that the operator \( \Delta_{l^2} \) is selfadjoint in \( l^2 \); generally unbounded; i.e., its deficiency indices are \((0,0)\) referring to \( l^2 \).

**Theorem 4.1.** Let \( \Delta_{l^2} \) and \( \Delta_{Kr} \) be the selfadjoint graph-Laplacians in \( l^2 (V') \) and \( \mathcal{H}_E \) respectively. Then,
\[ \text{spectrum} (\Delta_{l^2}) = \text{spectrum} (\Delta_{Kr}) \setminus \{0\}; \]  \hfill (4.9)
i.e., the spectrum of the two operators agree except for the point 0. In particular, one of the two operators is bounded iff the other is; and then the bounds agree.

We shall need the following lemma.

**Lemma 4.2.** \( \Delta_{l^2} = L^* L \).

**Proof.** Using (4.5), we note that it suffices to prove that for all \( \xi \in \text{dom}(L_0) \) (eq. (4.1)) we have
\[ \langle \xi, \Delta \xi \rangle_{l^2} = \| L \xi \|_{\mathcal{H}_E}^2 = \langle \xi, L^* L \xi \rangle_{l^2}. \]  \hfill (4.10)
But a direct computation yields (4.10); indeed
\[ (\text{LHS})_{(4.10)} = \sum_x \xi_x \sum_{y \sim x} c_{xy} (\xi_x - \xi_y) \]
\[
= \sum_x \xi_x^2 c(x) - \sum \sum_{(xy) \in E_{\text{dir}}} c_{xy} \xi_x \xi_y,
\]
where \(c(x) = \sum_{y \sim x} c_{xy}\).

Now using
\[
\langle \delta_x, \delta_y \rangle_{\mathcal{H}} = \begin{cases} 
  c(x) & \text{if } x = y \\
  -c_{xy} & \text{if } (xy) \in E \\
  0 & \text{otherwise}
\end{cases};
\]
we get
\[
\text{(LHS)}_{(4.10)} \quad \begin{aligned}
  &= \sum_x \sum_y \xi_x \xi_y \langle \delta_x, \delta_y \rangle_{\mathcal{H}} \\
  &= \left\| \sum_x \xi_x \delta_x \right\|^2_{\mathcal{H}} \\
  &= \left\| L\xi \right\|^2_{\mathcal{H}} = \text{(RHS)}_{(4.10)}
\end{aligned}
\]
which is the desired conclusion. \(\square\)

**Proof of Theorem 4.1.** From Theorem 3.9, we have \(\Delta_{K_F} = LL^*\). Using the polar decomposition [DS88, RS75] (polar factorization) for the closed operator \(L : l^2 \to \mathcal{H}_{l^2}\), we get a unique isometry \(U : l^2 \to \mathcal{H}_{l^2}\) such that
\[
L = U (L^* L)^{1/2} = (LL^*)^{1/2} U;
\]
and therefore by Theorem 3.9 and Lemma 4.2,
\[
L = U (\Delta_{l^2})^{1/2} = (\Delta_{K_F})^{1/2} U.
\]
Note that the two operators in (4.13) under the square root are selfadjoint (and semibounded).
\(\Delta_{l^2} \geq 0\) is selfadjoint in \(l^2\); and \(\Delta_{K_F} \geq 0\) is selfadjoint in \(\mathcal{H}_{F}\). Hence the conclusion in Theorem 4.1 is a consequence of the polar decomposition in (4.12), i.e., that \(U\) is isometric in the ortho-complement of the kernel of \(\Delta_{l^2}\). The final space \(UU^*\) is \(\mathcal{H}_{l^2} \ominus \text{Ker} (\Delta_{l^2})\). \(\square\)

**Corollary 4.3.** The contractive selfadjoint operator \(B_{K_F} : \mathcal{H}_E \to \mathcal{H}_F\) from Definition 3.3 giving the Krein extension \(\Delta_{K_F}\) is as follows:
\[
B_{K_F} = U (\Delta_{l^2} + I_{l^2})^{-1} U^* \quad (4.14)
\]
where \(U\) is the isometry in (4.12).

**Proof.** Follows from the unitary equivalence assertion in Theorem 4.1. \(\square\)

We reference here Krein’s theory of semibounded operators and their semibounded selfadjoint extensions. In brief outline, Krein developed a theory (see e.g., [HMDs04, AD98, Har56]) for all the selfadjoint extensions of semibounded Hermitian operators such that the selfadjoint extensions preserve the same lower bound. (There are other selfadjoint extensions, not with the same lower bound, but we shall not be concerned
Krein showed that the family of selfadjoint extensions (with the same lower bound) is in bijective correspondence with a certain family of contractive selfadjoint operators, which in turn has a natural order such that the two cases, the Krein extension, and the Friedrichs extension, are the extreme ends in the associated “order-interval.” From our construction, we note that our particular s.a. extension in the Hilbert space $\mathcal{H}_E$ is in fact the Krein-extension.

From general theory we have that $U$ in \((4.12)\) is a partial isometry with initial space $U^*U = (\ker (L^*L))^\perp = (\ker (\Delta_2))^\perp$, but $\ker (\Delta_2) = 0$ on account of assumption \((4)\) in Section 2.1, i.e., connectedness. This is an application of a maximum principle for $\Delta_2$. Recall $\Delta = c(I - P)$ where

$$(P\xi)(x) = \sum_{y \sim x} p_{xy}\xi(y), \quad p_{xy} = \frac{c_{xy}}{c(x)}; \quad \Delta \xi = 0 \iff P\xi = \xi;$$

and $P$ is a Markov-operator.

The fact that $LL^*$ (as a selfadjoint operator in $\mathcal{H}_E$) is the Krein extension of $\Delta_E$ from Lemma 2.12 \((4)\) follows from the following:

If $f \in \mathcal{H}_E$, and

$$\langle f, L\delta_x \rangle_{\mathcal{H}_E} = 0, \quad \text{for } \forall x \in V, \quad (4.15)$$

then $f \in \text{dom} (L^*)$, and $L^*f = 0$. Indeed, from \((4.15)\), we have

$$0 = \langle L\delta_x, f \rangle_{\mathcal{H}_E} = \langle \delta_x, L^*f \rangle_{L^2} = (L^*f)(x).$$

### 5. Variation of Conductance

In this section we study how the spectrum of the graph Laplacian $\Delta_c$ changes subject to variations in choice of conductance function $c$. We prove (Theorems 5.8 and 5.16) that the natural order of conductance functions, i.e., point-wise as functions on $E$, induces a certain similarity of the corresponding (Krein extensions of the) two graph Laplacians. Since the spectra are typically continuous, fine-structure of spectrum must be defined in terms of equivalence classes of positive Borel measures on the real line. Hence our detailed comparison of spectra must be phrased involving these; see Definition 5.14.

In this section we turn to the details on the comparison the Krein extensions of $\Delta_c$ as the conductance function varies.

Let $G = (V, E)$ be a network with vertices $V$ and edges $E$. We assume $V$ is countable infinite, and $G$ is connected, i.e., for all $x, y \in V$, $\exists$ a finite path $\{e_i = (x_i, x_{i+1})\}$ in $E$ s.t. $x_0 = x, x_n = y$. See Fig 5.1.
Variation of Conductance Functions in Discrete Laplacians

Figure 5.1. A finite path connecting $x, y$, $(x_i, x_{i+1}) \in E$, $c_{x_i, x_{i+1}} > 0$, $i = 1, \ldots, n - 1$.

Given a conductance function $c$, let $\mathcal{H}_C$ be the energy Hilbert space introduced in section 2.2; where

$$\langle u, v \rangle_C := \frac{1}{2} \sum_{(xy) \in E} c_{xy} (u(x) - u(y)) (v(x) - v(y))$$

(5.1)

$$\| u \|_C^2 := \frac{1}{2} \sum_{(xy) \in E} c_{xy} |u(x) - u(y)|^2 .$$

(5.2)

Let $c^A$ be another conductance function on $E$, and $\mathcal{H}_A$ be the corresponding energy Hilbert space. If $c \leq c^A$, i.e., $c_{xy} \leq c^A_{xy}$ for all $(xy) \in E$, then by (5.2), we have

$$\| u \|_C^2 \leq \| u \|_A^2 , \quad \forall u \in \mathcal{H}_A ;$$

(5.3)

and so $\mathcal{H}_A \subset \mathcal{H}_C$.

Figure 5.2. $c^A \geq c$ point-wise on $E$.

Definition 5.1. Let

$$j_A : \mathcal{H}_A \rightarrow \mathcal{H}_C , \quad j_A u = u , \quad \forall u \in \mathcal{H}_A$$

be the natural inclusion mapping. It follows from (5.3) that $j_A$ is a continuous (contracutive) inclusion.

Lemma 5.2. Let $c, c^A, \mathcal{H}_C, \mathcal{H}_A$ and $j_A$ be as before.

(1) Then,

$$v_{xy}^A = j_A v_{xy}$$

(5.4)
where \( v_{xy} \) and \( v_{xy}^{(A)} \) are the dipoles in the respective energy spaces satisfying

\[
\begin{align*}
\langle v_{xy}, u \rangle_C &= u(x) - u(y) \quad \forall u \in \mathcal{H}_C \\
\langle v_{xy}^{(A)}, u \rangle_A &= u(x) - u(y) \quad \forall u \in \mathcal{H}_A \subset \mathcal{H}_C
\end{align*}
\] (5.5)

(See Lemma 2.6, and Fig 3.1 for an illustration.)

(2) \( j_A \left( \delta_x^{(A)} \right) = \delta_x, \forall x \in V. \) (5.6)

(3) \( j_A^* (\delta_x) = \sum_{y \sim x} c_{xy} v_{xy}^{(A)}, \forall x \in V. \) (5.7)

(4) The following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{H}_A & \xleftarrow{j_A^*} & \mathcal{H}_C \\
\Delta_A & \downarrow & \Delta(=\Delta_C) \\
\mathcal{H}_A & \xrightarrow{j_A} & \mathcal{H}_C
\end{array}
\]

i.e., \( j_A \Delta_A j_A^* = \Delta \) (5.8)

valid on \( \text{span} \{ v_{xy} : x, y \in V \} \).

Proof. Below we consider pairs of vertices \( x, y \), and \( s, t \) as follows:

1. Let \( u \in \mathcal{H}_A \), then

\[
\langle v_{xy}, j_A u \rangle_C = \langle j_A^* v_{xy}, u \rangle_A ;
\]

which implies \( j_A^* v_{xy} = v_{xy}^{(A)}. \)

2. For all \( v_{st} \in \mathcal{H}_C, \)

\[
\langle j_A \delta_x^{(A)}, v_{st} \rangle_C = \langle \delta_x^{(A)}, j_A^* v_{st} \rangle_A = \langle \delta_x^{(A)}, v_{st}^{(A)} \rangle_A = \delta_{s,x}^{(A)} - \delta_{t,x}^{(A)} = \delta_{s,x} - \delta_{t,x} = \langle \delta_x, v_{st} \rangle_C
\]

and (5.6) follows.

3. For all \( u \in \mathcal{H}_A \), we have

\[
\langle j_A^* \delta_x, u \rangle_A = \langle \delta_x, u \rangle_C = (\Delta_C u)(x)
\]

\[
= \sum_{y \sim x} c_{xy} (u(x) - u(y))
\]

\[
= \sum_{y \sim x} c_{xy} v_{xy}^{(A)} ;
\]

eq. (5.7) follows from this.
(4) Let \( v_{xy} \in \mathcal{H}_C \), then
\[
\begin{align*}
 j_A \Delta A^* v_{xy} &= \text{(by (5.4))} j_A \Delta(A) v_{xy} \\
 &\text{lem 2.12} j_A \left( \delta_x(A) - \delta_y(A) \right) \\
 &\text{(by (5.6))} \delta_x - \delta_y \\
 &\text{lem 2.12} \Delta_C v_{xy}
\end{align*}
\]
which gives (5.8).

Let \((V, E)\) be as above, and \(\mathcal{H}_C\) and \(\mathcal{H}_A\) be the energy Hilbert spaces. Fix a base point \(o \in V\), and set \(v_x := v_{xo}, v^A_x := v^A_{xo}\), for all \(x \in V' := V \setminus \{o\}\).

Let
\[ G(x, y) = \langle v_x, v_y \rangle_{\mathcal{H}} \]
\[ G^A(x, y) = \langle v_x, v_y \rangle_{\mathcal{H}_A} \]
be the respective Gramians.

**Lemma 5.3.** If \(c \leq c^A\) on \(E\), let \(j_A : \mathcal{H}_A \to \mathcal{H}\) be the natural inclusion as before.

(i) Then
\[ \langle v_x, (j_Aj_A^*) v_y \rangle_{\mathcal{H}_C} = G^A(x, y) \]
for all \(x, y\) in \(V'\); and

(ii) We have \(\Delta_y G_{xy} = \delta_{xy}\).

**Proof.** For (i), we have
\[
\langle v_x, j_Aj_A^* v_y \rangle_{\mathcal{H}_C} = \langle j_A^* v_x, j_A^* v_y \rangle_{\mathcal{H}_A}
\]
\[
= \langle v^A_x, v^A_y \rangle_{\mathcal{H}_A} = G^A(x, y).
\]

See the diagram below.

\[
\begin{array}{c}
\text{H}_A \\
\text{j}_A \\
\text{H}\end{array} \xrightarrow{\text{j}_A^*} \begin{array}{c}
\text{H}_A \\
\text{j}_A \\
\text{H}\end{array} \xrightarrow{\text{j}_A^*} \begin{array}{c}
\text{H}_A \\
\text{j}_A \\
\text{H}\end{array}
\]

Proof of (ii). We have the following facts for the Gramian \(G(x, y) = \langle v_x, v_y \rangle_{\mathcal{H}}\):

1. \(G(x, y) = v_y(x) - v_y(o)\), where \(x, y \in V' = V \setminus \{o\}\), and \(o\) is a fixed chosen base point;
2. \(\Delta_y G_{xy} = \delta_{xy}\).

**Corollary 5.4.** The following holds:
\[
v^A_x(y) - v^A_x(o) = (j_Aj_A^*) (v_x) (y) - (j_Aj_A^*) (v_x) (o).
\]
Corollary 5.5. Let \((V, E, c, o, \Delta (= \Delta_c), \mathcal{H}_E)\) be as above. Let \(c_A\) be a second conductance function defined on \(E\) and satisfying \(c_A \geq c\) point-wise on \(E\). (Note \(c_A\) may be unbounded, even though \(c\) might be bounded.) Let the operators \(j_A\) and \(j_A^*\) be as above, i.e.,

\[
\begin{align*}
\mathcal{H}_A & \xrightarrow{j_A} \mathcal{H}_E \\
\mathcal{H}_E & \xleftarrow{j_A^*}
\end{align*}
\] (5.9)

then

\[
\text{Ker}(j_A) = \{0\}.
\] (5.10)

Proof. For operators in Hilbert space, we denote by Ker, and Ran, kernel and range, respectively. We have

\[
\text{Ker}(j_A) = \overline{\text{Ran}(j_A^*)}
\] (5.11)

with “⊥” on the RHS in (5.11) denoting ortho-complement. But by Lemma 5.2 (eq. (5.4)), we note that all dipole vectors \(v_{xy}^{(A)}\), for \(x\) and \(y\) in \(V\), are in \(\text{Ran}(j_A^*)\). Since their span is dense in \(\mathcal{H}_A\), by Lemma 2.8, the desired conclusion (5.10) follows, i.e., \(\text{Ker}(j_A) = \{0\}\). □

Corollary 5.6. Let \(c\) and \(c_A\) be as above, \(c \leq c_A\) assumed to hold on \(E\). Then the partial isometry \(W : \mathcal{H}_A \hookrightarrow \mathcal{H}_E\), in

\[
j_A = W (j_A^* j_A)^{\frac{1}{2}} = (j_A^* j_A)^{\frac{1}{2}} W
\] (5.12)

is isometric, i.e.,

\[
W^* W = I_{\mathcal{H}_A}.
\] (5.13)

Proof. This is immediate from the previous corollary, and an application of the polar-decomposition theorem for bounded operators, see e.g., [DS88, RS75]. □

Corollary 5.7. Let \(V, E, c\) and \(c_A\) be as above, with \(c_A \geq c\), and let the operators \(j_A = W (j_A^* j_A)^{\frac{1}{2}},\ \Delta_F (= (\Delta_c)_F)\) and \(\Delta^{(A)}_F\) be as described, i.e., the respective Krein extensions; then

\[
W \Delta^{(A)}_F W^* = \Delta_F, \quad \text{and}\n\]

\[
W \Delta^{(A)}_F = \Delta_F W.
\] (5.14) (5.15)

Proof. Immediate from the previous two corollaries.

Note in particular that (5.14)\(\iff\)(5.15) in view of the isometry conclusion (5.13) in the last corollary. □

Theorem 5.8. Fix \(G = (V, E)\). Let \(o\) be a base point in \(V\), and set \(V' := V \setminus \{o\}\). Let \(c\) and \(c_A\) be the conductance functions s.t. \(c \leq c_A\) holds point-wise on \(E\). Let \(\Delta, \Delta^{(A)}\) be the graph-Laplacians, \(v_x := v_{xo}, v^{(A)}_x := v^{(A)}_{xo}, x \in V'\), be the dipoles. Then,

\[
j_A \Delta^{(A)}_{Kre} j_A^* = \Delta_{Kre}.
\] (5.16)

The key step of the proof are the lemmas below.
Lemma 5.9. Let \((V,E), c, c^A\) and \(L, L_A\) be as before, assume \(c \leq c^A\) point-wise on \(E\); and let \(j_A : \mathcal{H}_A \rightarrow \mathcal{H}\) be the natural inclusion.

(i) Then
\[
L^*_A j^*_A \subseteq L^*; \quad (5.17)
\]
i.e., the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{j^*_A} & \mathcal{H}_A \\
\downarrow & & \downarrow \\
L^* & \xrightarrow{L^*_A} & L^*_A \hookrightarrow \mathcal{H}
\end{array}
\]

(ii) We have \(L \supseteq j_A L_A\).

Proof. Let \(u = \sum \xi_x v_x, \sum \xi_x = 0\), be any vector in \(\mathcal{D}_E\) (eq. (2.21)); see part (1) of Theorem 3.9. Then,
\[
L^*_A (j^*_A u) = L^*_A \left( \sum \xi_x j^*_A (v_x) \right) = L^*_A \left( \sum \xi_x v_x^A \right) = (\xi_x) = L^* u
\]
and the assertion follows.

Proof of part (ii): Taking adjoint of eq. (5.17).

Lemma 5.10. Every selfadjoint \(H\) operator with dense domain in a Hilbert space is maximal Hermitian.

Proof. Suppose \(H\) is selfadjoint, and that \(T\) is a Hermitian extension; then \(H \subset T\), and by taking adjoints, we get \(T^* \subseteq H^*\). Hence
\[
T \subseteq T^* \subset H^* = H^* \subseteq T
\]
so \(T = H\).

Proof of Theorem 5.8. We apply Theorem 3.9 (also see [JT14]) to both \((\Delta, \mathcal{H}_C)\) and \((\Delta^A, \mathcal{H}_A)\), and arrive at the following factorizations:
\[
LL^* = \Delta_{Kre} \quad \text{in} \quad \mathcal{H} \quad (5.18)
\]
\[
L_A L_A^* = \Delta^A_{Kre} \quad \text{in} \quad \mathcal{H}_A \quad (5.19)
\]

Then, by (5.18), we have
\[
\Delta_{Kre} \underset{(5.18)}{=} LL^* \subseteq \underset{(5.17)}{(j_A L_A)} (L^*_A j^*_A) = j_A (L_A L^*_A) j^*_A = j_A \Delta^A_{Kre} j_A.
\]
To finish the proof we apply Lemma 5.10 to the two operators in $H_E, H = \Delta_{Kre}$ and $T_A = j_A \Delta_{Kre}^{(A)} j_A^*$. Here $H$ is selfadjoint on $H_E$, and $T_A$ is a Hermitian extension; hence by the Lemma 5.10, $T_A = \Delta_{Kre}$. \hfill $\Box$

Remark 5.11. Fix two conductance functions $c$ and $c_A$, and let $j_A$ be as before. See (5.9) and the diagram below:

\begin{center}
\begin{tikzpicture}

\node (HA) at (0,0) {$H_A$};
\node (HC) at (0,-2) {$H_C$};
\node (HE) at (2,0) {$H_E$};
\node (HF) at (2,-2) {$H_F$};
\node (jA) at (1,-1) {$j_A$};
\node (jA*) at (1,-0.5) {$j_A^*$};
\node (Delta) at (1,1) {$\Delta_{F}^{(A)}$};

\draw[->] (HA) -- (HE);
\draw[->] (HE) -- (HC);
\draw[->] (HC) -- (HF);
\draw[->] (HA) -- (HC);
\draw[->] (HC) -- (HA);
\end{tikzpicture}
\end{center}

Since for $u \in H_A$, we have
\[
\sum \sum c_{x,y} |u(x) - u(y)|^2 \leq \sum \sum c_{x,y}^{(A)} |u(x) - u(y)|^2 \quad (5.20)
\]
it follows that $j_A : H_A \xrightarrow{j_A} H_C$ is contractive.

Since $j_A, j_A^*$ are bounded operators, we have
\[
\|j_A^*\|_{H_A} = \|j_A\|_{H_C} \leq 1. \quad (5.21)
\]

Example 5.12 (Example 3.16 revisited). Let $V = \{0\} \cup \mathbb{Z}_+$, i.e., nearest neighbors. Let $c, c_A$ be two conductance functions given by
\[
c_{n,n+1} := n \\
c_{n,n+1}^{(A)} := A^n, \ A > 1, \ n \in V.
\]
and we have
\[
H_C = \left\{ u : V \rightarrow \mathbb{C} \mid \sum_n n |u(n) - u(n+1)|^2 < \infty \right\}
\]
\[
H_A = \left\{ u : V \rightarrow \mathbb{C} \mid \sum_n A^n |u(n) - u(n+1)|^2 < \infty \right\}.
\]

Let $v_{n,n+1}$ and $v_{n,n+1}^{(A)}$ be the respective dipoles, as illustrated in Fig 3.1.

Then, $j_A^* (v_{n,n+1}) = v_{n,n+1}^{(A)}$, for all $n \in \mathbb{Z}_+$ (by (5.4)), and
\[
\|v_{n,n+1}\|_{H_C}^2 = 1 \\
\|v_{n,n+1}^{(A)}\|_{H_A}^2 = \frac{1}{A^n} \rightarrow \infty \\
\|v_{n,n+1}\|_{H_A}^2 = A^n \rightarrow 0.
\]
Lemma 5.13. In Example 5.12, the operator $j_{AJ_A}^* : \mathcal{H}_C \rightarrow \mathcal{H}_C$ is trace class, and

$$\text{trace} (j_{AJ_A}^*) = \sum_{n=0}^{\infty} \frac{1}{A^n}.$$ 

Proof. Note the dipoles $\{v_{n,n+1} : n \in V\}$ forms an ONB in $\mathcal{H}_C$, and

$$\text{trace} (j_{AJ_A}^*) = \sum_{n=0}^{\infty} \langle v_{n,n+1}, j_A j_A^* v_{n,n+1} \rangle_{\mathcal{H}_C}$$

$$= \sum_{n=0}^{\infty} \langle j_A^* v_{n,n+1}, j_A^* v_{n,n+1} \rangle_{\mathcal{H}_A}$$

$$= \sum_{n=0}^{\infty} \langle v_{n,n+1}^{(A)}, v_{n,n+1}^{(A)} \rangle_{\mathcal{H}_A}$$

$$= \sum_{n=0}^{\infty} \frac{1}{A^n} < \infty.$$

□

5.1. Comparing Spectra

Let $(V, E, c, o, \Delta, \mathcal{H}_E)$ be a network, with a fixed conductance function $c$ on $E$. We assume, as above, that $(V, E, c)$ is connected, and that $#V = \aleph_0$.

We fix a second conductance function $c_A$ on $E$, and assume the following estimate:

$$c_A \geq c \quad \text{holds on } E. \quad (5.22)$$

Hence we get two energy Hilbert spaces $\mathcal{H}_E (= \mathcal{H}_C$, see (5.1)-(5.2)) and $\mathcal{H}_A$ with the natural inclusion mapping $j_A : \mathcal{H}_A \hookrightarrow \mathcal{H}_E$ as follows

$$\begin{array}{c}
\text{\mathcal{H}_A} \\
\downarrow j_A \\
\text{\mathcal{H}_E} \\
\uparrow j_A^*
\end{array}$$

and the respective graph Laplacians:

- $\Delta_F$ with its dense domain in $\mathcal{H}_E$; and
- $\Delta_F^{(A)}$ with its dense domain in $\mathcal{H}_A$.

We shall study the following notions from spectral theory of selfadjoint operators:

Definition 5.14. Let $T$ be a selfadjoint operator with dense domain in a Hilbert space $\mathcal{H}$; and let $P_T (\cdot)$ be the associate projection valued measure; see e.g., [DS88]. For all vectors $u \in \mathcal{H}$, set

$$d\mu_u (\cdot) = \|P_T (\cdot) u\|^2.$$
$\mu_u$ is a finite Borel measure on $\mathbb{R}$, i.e., defined on the Borel sigma algebra $\mathcal{B}$. We have:

$$u \in \text{dom}(T) \iff \int_{\mathbb{R}} \lambda^2 d\mu_u(\lambda) < \infty; \quad (5.24)$$

and then

$$\langle u, Tu \rangle = \int_{\mathbb{R}} \lambda d\mu_u(\lambda) \quad (5.25)$$

where the integral is absolutely convergent.

If $\psi$ is a Borel function, then a vector $u \in \mathcal{H}$ is in the domain of the operator $\psi(T)$ iff $\psi \in L^2(\mu_u)$; and then

$$\|\psi(T)u\|^2 = \int |\psi|^2 d\mu_u < \infty. \quad (5.26)$$

**Definition 5.15.** A finite positive Borel measure $\mu$ is said to be in $\mathcal{M}(T)$ iff (Def.) $\exists u \in \mathcal{H} \setminus \{0\}$ such that

$$d\mu = d\mu_u = d\|P^T(\cdot)u\|^2. \quad (5.27)$$

We say that $\mathcal{M}(T)$ is the spectral contents of the operator $T$.

If $\mu$ and $\nu$ are positive finite measures, we say that $\mu \ll \nu$ if the following implication holds:

$$[S \in \mathcal{B}, \nu(S) = 0] \implies \mu(S) = 0;$$

and we then introduce the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$ i.e., $\frac{d\mu}{d\nu} \in L^2_+(\mathbb{R}, \nu)$, and

$$\mu(S) = \int_S \frac{d\mu}{d\nu}(\lambda) d\nu(\lambda), \forall S \in \mathcal{B}. \quad (5.28)$$

**Theorem 5.16.** Let $\mathcal{H}_E$ and $\mathcal{H}_A$ be as above, consider $u \in \mathcal{H}_E \setminus \{0\}$, and $\mu = \mu_u \in \mathcal{M}(\Delta_{Kr})$; then $u_A := j_A^*u$ yields $\mu_u(A) \in \mathcal{M}(\Delta_{Kr}^A)$ if and only if $u_A \neq 0$. In this case, we have

$$\mu_u \ll \mu_u(A). \quad (5.29)$$

**Proof.** By Theorem 5.8 (eq. (5.16)) we have

$$j_A \Delta_{Kr}^A j_A^* = \Delta_{Kr},$$

and $j_A \Delta_{Kr}^A j_A^* j_A \Delta_{Kr}^A j_A^* = \Delta_{Kr}^2$; and so in the natural order of Hermitian operators, we have

$$\Delta_{Kr}^2 \leq j_A \left(\Delta_{Kr}^A\right)^2 j_A^*$$

since $j_A j_A^* \leq I_{\mathcal{H}_E}$ by Lemma 5.2. By induction, we get

$$\Delta_{Kr}^n \leq j_A \left(\Delta_{Kr}^A\right)^n j_A^*, \quad (5.30)$$

for all $n \in \mathbb{Z}_+$, where again we use the order “$\leq$” on Hermitian operators in the Hilbert space $\mathcal{H}_E$. 
By Stone-Weierstrass and measurable approximation, we get the following estimate:
\[
\|P_{\Delta_{KR}}(S)u\|_{\mathcal{H}_E} \leq \left\| P_{\Delta_{(A)}}(S)j_A^*u \right\|_{\mathcal{H}_A}^2
\]  
for all $S \in \mathcal{B}(\mathbb{R})$ (= all Borel subsets of $\mathbb{R}$). Now, introduce the measures from (5.26)-(5.27), and we get:
\[
\mu_{(\Delta_{KR})}^u(S) \leq \mu_{(\Delta_{(A)})}^{j_A^*u}(S)
\]  
for all $S \in \mathcal{B}(\mathbb{R})$; which is the desired conclusion in the theorem, see (5.29). From (5.32), we are then able to verify relative absolute continuity for the respective measures, and to compute the Radon-Nikodym derivatives:
\[
d\mu_{(\Delta_{KR})}^u/d\mu_{(\Delta_{(A)})}^{j_A^*u} \in L^1_{+}\left(d\mu_{(\Delta_{KR})}^{j_A^*u}\right).
\]

Remark 5.17. Eq (5.31) refers to extending information for a given intertwining operator, intertwining a given pair of selfadjoint operators. We pass from intertwining of the operators to intertwining of the respective projection valued measures as follows. There are in four steps; first from a given intertwining of the operators themselves, pass to the intertwining property for polynomials in the respective selfadjoint operators; then passing to the functional calculus for continuous functions of compact support applied to the operators (Stone-Weierstrass), and finally to the measurable functional calculus. When the latter is applied to indicator functions for Borel subsets, we get the respective projection valued measures corresponding to the given pair of selfadjoint operators.

5.2. Comparing Harmonic Functions, and Dirac Masses

Let $(V,E,c,\Delta,\mathcal{H}_E)$ be as before. Set
\[
\text{Fin}(\mathcal{H}_E) = \mathcal{H}_E\text{-closed span of } \{ \delta_x \mid x \in V \}
\]
\[
\text{Harm}(\mathcal{H}_E) = \{ h \in \mathcal{H}_E \mid \Delta h = 0 \}
\]

Lemma 5.18.
\[
\mathcal{H}_E = \text{Fin}(\mathcal{H}_E) \oplus \text{Harm}(\mathcal{H}_E).
\]  
Proof. Immediate from the fact
\[
\langle \delta_x, f \rangle_{\mathcal{H}_E} = (\Delta f)(x) = \sum_{y \sim x} c_{xy}(f(x) - f(y)),
\]  
see Lemma 2.12 (5).
Corollary 5.19.

\[ j^*_A (\text{Harm}(\mathcal{H}_C)) \subset \text{Harm}(\mathcal{H}_A). \]  (5.35)

Proof. Let \( h \in \text{Harm}(\mathcal{H}_C) \), and let \( x \in V \), then

\[
\Delta_A (j^*_A (h)) (x) = \langle j^*_A (h), \delta^A_x \rangle_{\mathcal{H}_A}
= \langle h, j_A (\delta^A_x) \rangle_{\mathcal{H}_C}
= \langle h, \delta_x \rangle_{\mathcal{H}_C} = (\Delta c h) (x) = 0;
\]

so \( j^*_A (h) \in \text{Harm}(\mathcal{H}_A) \) as claimed. \( \square \)

For the action of \( j^*_A \) on the point masses \( \delta_x \), we then have the following:

Theorem 5.20.

\[ j^*_A (\delta_x) - \delta^A_x = (c(x) - c^{(A)} (x)) v_x^{(A)} - \sum_{y \sim x} (c_{xy} - c^{(A)}_{xy}) v_{xy}^{(A)}, \]

where \( v_x \) and \( v_x^{(A)} \) are the respective dipoles, i.e., with a fixed base point \( o \in V \), \( v_x := v_{xo} \), \( v_x^{(A)} := v_{xo}^{(A)} \), for \( x \in V' := V \setminus \{o\} \).

Proof. We have \( \delta_x = c(x) v_x - \sum_{y \sim x} c_{xy} v_y \) and therefore

\[
j^*_A (\delta_x) = c(x) j^*_A (v_x) - \sum_{y \sim x} c_{xy} j^*_A (v_y)
= c(x) v_x^{(A)} - \sum_{y \sim x} c_{xy} v_y^{(A)};
\]

and the desired formula now follows by a subtract, and

\[ \delta^A_x = c^{(A)} (x) v_x^{(A)} - \sum_{y \sim x} c_{xy}^{(A)} v_{xy}^{(A)}. \]

\( \square \)

Below, we contrast our use of the Krein extensions (above) with the corresponding Friedrichs extensions and their respective quadratic forms.

Let \( G = (V,E) \) be an infinite connected network as before. Fix two conductance functions \( c, c^{(A)} : E \to \mathbb{R}_+ \), and assume \( c^{(A)}_{xy} \geq c_{xy} \) for all \((xy) \in E\).

Settings:

- \( \mathcal{H}_A, \mathcal{H}_C \) - the energy spaces with respect to \( c^{(A)} \) and \( c \); i.e., the completion of span of dipoles w.r.t. the corresponding energy inner products.

- \( j_A : \mathcal{H}_A \to \mathcal{H}_C \) the natural inclusion in Definition 5.1.

- \( v_{xy}, v_{xy}^{(A)} \) - dipoles in the respective energy spaces.

- \( \delta_x, \delta^A_x \) - as defined in (2.19).
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- $\Delta (= \Delta_c), \Delta_A$ - graph Laplacians in the respective energy spaces.

Recall that $j_A : \mathcal{H}_A \rightarrow \mathcal{H}_C$ is a continuous injection, and

$$\|u\|_C \leq \|u\|_A, \forall u \in \mathcal{H}_A.$$ 

The operator $j_A j_A^* : \mathcal{H}_C \rightarrow \mathcal{H}_C$ is positive, selfadjoint, $\|j_A j_A^*\|_C \leq 1$.

Let $\Delta_A, \Delta$ be the graph-Laplacians, where

$$\text{dom}(\Delta_A) = \text{span}\{v_{xy}(A)\},$$

and

$$\text{dom}(\Delta) = \text{span}\{v_{xy}\}.$$ 

Set

$$M_A := I_{\mathcal{H}_A} + \Delta_A, \quad M := I_{\mathcal{H}_C} + \Delta$$

and let

$$\mathcal{H}_{M_A} = \text{completion of dom}(\Delta_A) \text{ w.r.t } \|\varphi\|_{M_A}^2 := \langle \varphi, M_A \varphi \rangle_A$$

$$\mathcal{H}_M = \text{completion of dom}(\Delta) \text{ w.r.t } \|\varphi\|_{M}^2 := \langle \varphi, M \varphi \rangle_C.$$ 

By Lemma 3.1, and the assumption $c(A) \geq c$, we have

$$\|u\|_{M_A} \geq \|u\|_M \geq \|u\|_C$$

and

$$\|u\|_{M_A} \geq \|u\|_A \geq \|u\|_C, \forall u \in \mathcal{H}_{M_A}.$$ 

Consequently, the inclusions in the diagram below are all contractive:

$$\mathcal{H}_A \xrightarrow{j_A} \mathcal{H}_C$$

$$\mathcal{H}_{M_A} \xrightarrow{j_1} \mathcal{H}_M \xrightarrow{j_2} \mathcal{H}_C$$

Let $\widetilde{M}_A \supset M_A, \widetilde{M} \supset M$ be the respective Krein extensions; i.e.,

$$\widetilde{M}_A = I_{\mathcal{H}_A} + \Delta_{Kre}^{(A)}$$

$$\widetilde{M} = I_{\mathcal{H}_C} + \Delta_{Kre}.$$ 

**Lemma 5.21.** The following hold.

1. $\text{dom}(\widetilde{M}) = j_2^*(\mathcal{H}_C), \mathcal{H}_M = \text{dom}(\widetilde{M}^{1/2});$

2. For all $u \in \text{dom}(\widetilde{M}), v \in \mathcal{H}_M$,

$$\langle u, v \rangle_M = \langle \widetilde{M} u, v \rangle_C; \quad (5.36)$$

and

3. $\text{dom}(\widetilde{M}_A) = j_1^*(\mathcal{H}_A), \mathcal{H}_{M_A} = \text{dom}(\widetilde{M}_A^{1/2});$
(1)

\[ j_A \widetilde{M} j_A^* = j_A j_A^* + \Delta_{F_{ri}} = \widetilde{M} + (j_A j_A^* - I_{W_G}). \]

Proof. See e.g., [RS75, DS88]. \qed

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