Normality of the dual nilcone in bad characteristic

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Abstract

Let $G$ be a simple simply connected algebraic group defined over an algebraically closed field $K$ of positive characteristic. We demonstrate that the dual nilcone $\mathcal{N}^* \subseteq \mathfrak{g}^*$ is a normal variety in adequate characteristics, which are a subset of the bad characteristics, dependent on the root system of $G$. These characteristics are precisely those where the Springer map $\mu : T^*B \rightarrow \mathcal{N}$ is a resolution of singularities.

As an application, we extend the results of Ardakov and Wadsley on representations of $p$-adic Lie groups. Under the same restrictions on the characteristic, we show that the canonical dimension of a coadmissible representation of a semisimple $p$-adic Lie group in a $p$-adic Banach space is either zero or at least half the dimension of a nonzero coadjoint orbit.

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1 Introduction

Let $G$ be a simple simply connected algebraic group defined over an algebraically closed field $K$ of positive characteristic. In case the characteristic of $K$, $p$, is very good for $G$, which, broadly speaking, implies that $G$ is not of type $A$ and $p > 5$, it is known that the dual nilpotent cone $N^* \subseteq g^*$ is a normal variety, and it admits a desingularisation $\mu : T^*B \to N$ from the cotangent bundle of the flag variety $B$ of $G$; the so-called Springer resolution of $N$.

When $p$ is small, the classical proofs of these results break down. The goal of this paper is to investigate in which bad characteristics the dual nilcone $N^*$ remains a normal variety and the Springer map is a resolution of singularities. We define the notion of an adequate prime, which is a prime $p$ for $G$ contained in the following list (Dynkin diagram of $G$, $p$):

(i) $(E_6, 2)$,
(ii) $(E_7, 3)$,
(iii) $(E_8, 2, 3, 5)$,
(iv) $(F_4, 3)$,
(v) $(G_2, 2)$.

This is a subset of the bad primes. We then have the following main theorem:

**Theorem A.** Let $G$ be a simple simply connected algebraic group, defined over an algebraically closed field $K$ of adequate characteristic. Then the dual nilpotent cone $N^* \subseteq g^*$ is a normal variety.

As an application, let $p$ be a prime, $G$ be a semisimple compact $p$-adic Lie group and let $K$ be a finite extension of $\mathbb{Q}_p$. Ardakov and Wadsley studied the coadmissible representations of $G$, which are finitely generated modules over the completed group ring $KG$ with coefficients in $K$, in [2]. These completed group rings may be realised as Iwasawa algebras, which are important objects in noncommutative Iwasawa theory.

One of the central results in [2] is an estimate for the canonical dimension of a coadmissible representation of a semisimple $p$-adic Lie group in a $p$-adic Banach space. When $p$ is very good for $G$, Ardakov and Wadsley showed that this canonical dimension is either zero or at least half the dimension of a nonzero coadjoint orbit. We extend their results to the case where $p$ is adequate for $G$. The main result of this section is as follows:

**Theorem B.** Let $G$ be a compact $p$-adic analytic group whose Lie algebra is
semisimple. Let \( p \) be an adequate prime for \( G \), and let \( G_C \) be a complex semisimple algebraic group with the same root system as \( G \). Let \( r \) be half the smallest possible dimension of a nonzero coadjoint \( G_C \)-orbit. Then any coadmissible \( KG \)-module \( M \) that is infinite-dimensional over \( K \) satisfies \( d(M) \geq r \).

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2 The nilpotent cone and the Springer resolution

2.1 Characteristic

In this section, we study the geometric structure of the nilpotent cone \( \mathcal{N} \) of a semisimple Lie algebra \( \mathfrak{g} \) in arbitrary characteristic. We begin with a discussion of the ordinary nilpotent cone, defined as a subvariety of \( \mathfrak{g} \), and then give a characterisation of the dual nilpotent cone \( \mathcal{N}^* \).

Our treatment of the material on \( \mathcal{N} \) is based on that of Jantzen in [10]. We generalise some of his arguments which are dependent on certain restrictions on the characteristic.

Let \( G \) be a split reductive algebraic group scheme, defined over \( \mathbb{Z} \), and \( K \) an algebraically closed field of characteristic \( p > 0 \). Let \( G := G(K) \). Let \( \mathfrak{g} \) denote the Lie algebra of \( G \) and \( W(G) \) the Weyl group of \( G \). When \( G \) is clear from context, we will abbreviate \( W(G) \) to \( W \). Since \( G \) is a linear algebraic group, identify \( G \subseteq GL(V) \) for some \( n \)-dimensional \( K \)-vector space \( V \).

Definition 2.1.1. Let \( \alpha_i \) be the simple roots of the root system \( R \) of \( G \), and let \( \beta \) be the highest-weight root. Writing \( \beta = \sum_i m_i \alpha_i \), \( p \) is bad for \( G \) if \( p = m_i \) for some \( i \). \( p \) is good if \( p \) is not bad.

If \( G \) is not of type \( A \) and \( p \) is good, \( p \) is furthermore very good. If \( G \) is of type \( A_n \), \( p \) is very good if and only if \( p \) does not divide \( n + 1 \).

In practice, we have the following classification. In types \( B, C \) and \( D \), the only bad prime is 2. For the exceptional Lie algebras, the bad primes are 2 and 3 for types \( E_6, E_7, F_4 \) and \( G_2 \), and 2, 3 and 5 for type \( E_8 \). In type \( A \), there are no bad primes. For more details of this classification, see [27, I.4.3].
Definition 2.1.2. A prime $p$ is **special** for $G$ if the pair (Dynkin diagram of $G$, $p$) lies in the following list:

(i) $(B, 2)$,
(ii) $(C, 2)$,
(iii) $(F_4, 2)$,
(iv) $(G_2, 3)$.

This definition, and material on the importance of nonspecial primes, can be found in [24, Section 5.6].

Definition 2.1.3. Suppose $G$ is a split reductive algebraic group scheme and let $p$ be a nonspecial prime for $G$. $p$ is **adequate** for $G$ if the pair (Dynkin diagram of $G$, $p$) lies in the following list:

(i) $(E_6, 2)$,
(ii) $(E_7, 3)$,
(iii) $(E_8, 2, 3, 5)$,
(iv) $(F_4, 3)$,
(v) $(G_2, 2)$.

### 2.2 The $W$-invariants of $S(\mathfrak{h})$

Let $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. In case $G$ is a simply connected algebraic group, and the prime $p$ is good for $G$, there is a $G$-equivariant isomorphism $\mathfrak{g} \to \mathfrak{g}^*$, by [3, Corollary 9.3.4]. This isomorphism does not exist when $p$ fails to be very good for $G$, which motivates our separate treatment of the varieties $\mathcal{N}$ and $\mathcal{N}^*$. Since $\mathfrak{g}$ is a finite-dimensional vector space, we naturally identify the symmetric algebra $S(\mathfrak{g})$ and the algebra of polynomial functions $K[\mathfrak{g}^*]$.

Let $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$. The Weyl group $W$ has a natural action on $\mathfrak{h}$, which can be extended linearly to an action of $W$ on the symmetric algebra $S(\mathfrak{h})$. The identification $S(\mathfrak{h}) \cong K[\mathfrak{h}^*]$ is compatible with the $W$-action. We begin this section by studying the $W$-invariants under this action.

**Theorem 2.2.1.** Suppose $G$ is simply connected and $p$ is adequate for $G$. Then $S(\mathfrak{h})^W$ is a polynomial ring.

**Proof.** Suppose $p$ is adequate for $G$. Let $R$ be the root system of $G$. Since $G$ is simply connected, the cocharacter group $Y(T) = \mathbb{Z}R^\vee$, the coroot lattice. Dualising, the character group $X(T)$ may be identified with the weight lattice $\mathbb{Z}Q$. 


Let $h_Z$ denote the $\mathbb{Z}$-span of the basis elements of the Cartan subalgebra $h$. Since $G$ is simply connected, it follows that $h_Z = Y(T)$, and hence $\mathfrak{h} = h_Z \otimes_{\mathbb{Z}} K$. We also identify $h^*_{\mathbb{Z}} \cong X(T)$. Then:

$$S(h_Z)^W = S(Y(T))^W = Z(X(T))^W.$$ 

Now $Z(X(T))^W$ is isomorphic to a polynomial algebra by [6, Theorem VI.3.1]. We next prove the following lemma.

**Lemma 2.2.2.** There is a $K$-algebra isomorphism:

$$Z(X(T))^W \otimes_{\mathbb{Z}} K \cong (Z(X(T)) \otimes_{\mathbb{Z}} K)^W.$$ 

**Proof.** Let $N$ be a $\mathbb{Z}W$-module. Then we have the following chain of natural $\mathbb{Z}$-module isomorphisms:

$$N^W \cong \text{Hom}_{\mathbb{Z}}(Z, N)^W \cong \text{Hom}_{\mathbb{Z}^N}(Z, N) = \text{Ext}^0_{\mathbb{Z} W}(Z, N) =: H^0(W; N).$$

Since $Z(X(T))$ is $\mathbb{Z}$-torsion free, and $W$ acts trivially on $\mathbb{Z}$, one may apply the Universal Coefficient Theorem, [13, Theorem 3.2], to obtain the short exact sequence of $\mathbb{Z}$-modules:

$$0 \to H^0(W; Z(X(T)) \otimes_{\mathbb{Z}} K) \to H^0(W; Z(X(T) \otimes_{\mathbb{Z}} K) \to \text{Tor}^1_{\mathbb{Z}}(H^1(W; Z(X(T)), K) \to 0.$$

Hence the statement in the lemma is equivalent to $\text{Tor}^1_{\mathbb{Z}}(H^1(W; Z(X(T)), K) = 0.$

Let $\mathfrak{g}_C$ denote the complex semisimple Lie algebra with the same root system as $\mathfrak{g}$, and $G_C$ the simply connected complex algebraic group associated with $\mathfrak{g}_C$, with maximal torus $T_C$. Since $p$ is an adequate prime for $G$, the cohomology group $H^1(W; T_C) = 0$, by [13, Main Theorem], except in the case $(E_7, 3)$. As abelian groups, we have an isomorphism $T_Z \cong \mathbb{Z}[X(T)]$, since they are both free abelian groups of rank $l$. Since $T_C$ may be viewed as a $\mathbb{Z}$-module and $\mathbb{Z}$ is a principal ideal domain, we may apply the Universal Coefficient Theorem to obtain the short exact sequence:
\[ 0 \to \text{Ext}_Z^1(H_0(W;Z), T_C) \to H^1(W;T_C) \to \text{Hom}_Z(H_0(W,Z), T_C) \to 0. \]

The middle term is zero, and hence all terms in this short exact sequence vanish. Since the \( W \)-action on \( Z \) is trivial, \( H_0(W;Z) \cong Z \) is a projective \( Z \)-module. It follows that both \( \text{Ext}_Z^1(H_0(W;Z), -) \) and \( \text{Hom}_Z(H_0(W,Z), -) \) are exact functors \( \text{Ab} \to \text{Ab} \). Since there is a natural injection \( T_Z \to T_C \), it follows that \( \text{Ext}_Z^1(H_0(W;Z), T_Z) \cong \text{Hom}_Z(H_0(W,Z), T_Z) = 0 \). Applying the Universal Coefficient Theorem again yields the short exact sequence:

\[ 0 \to \text{Ext}_Z^1(H_0(W;Z), T_Z) \to H^1(W;T_Z) \to \text{Hom}_Z(H_0(W,Z), T_Z) \to 0 \]

and so \( H^1(W;T_Z) = 0 \). It follows that \( \text{Tor}_1^Z(H^1(W;Z(X(T)), K) = 0 \). This proves the lemma in all cases.

It follows that:

\[ S(\mathfrak{h})^W \cong S(\mathfrak{h}_Z \otimes_Z K)^W \cong (\mathfrak{h}_Z^* \otimes_Z K)^W \cong Z[\mathfrak{h}_Z^*]^W \otimes_Z K. \]

where the last isomorphism follows from Lemma 2.2.2. The last algebra is a polynomial \( K \)-algebra, and it follows that \( S(\mathfrak{h})^W \) is a polynomial \( K \)-algebra.

**Lemma 2.2.3.** \( S(\mathfrak{h}) \) is a free \( S(\mathfrak{h})^W \)-module if and only if \( S(\mathfrak{h})^W \) is a polynomial ring.

**Proof.** See [25] Corollary 6.7.13. \( \square \)

### 2.3 Properties of the nilpotent cone

We now outline some general preliminaries on the structure theory of groups acting on varieties. At first, we do not impose any restriction on the characteristic.

Let \( M \) be a variety which admits a group action by \( G \), and let \( x \in M \). The closure \( \overline{Gx} \) of the orbit \( Gx \) of \( x \) is a closed subvariety of \( M \). By [17] Proposition 8.3, \( Gx \) is
open in $\overline{Gx}$ and so $Gx$ has the structure of an algebraic variety.

The orbit map $\pi_x : G \to Gx$, $\pi_x(g) = gx$, is a surjective morphism of varieties. The stabiliser $G_x := \{g \in G \mid gx = x\}$ is a closed subgroup of $G$, and $\pi_x$ induces a bijective morphism:

$$\pi_x : G/G_x \to Gx$$

by [17, Section 12].

We now specialise to the case where $M = g$ and $G$ acts on $g$ via the adjoint action. Let $X \in g$ and let $GX$ denote the $G$-orbit of $X$ under the adjoint action $Ad : G \to \text{Ad}(G)$.

Recall that an element $g \in g$ is nilpotent if the operator $\text{ad}_g(y)$ is nilpotent for each $y \in g$. The set of nilpotent elements is denoted $\mathcal{N}$.

Since $G$ is a linear algebraic group, identify $G \subseteq GL(V)$ for some $n$-dimensional $K$-vector space $V$. Then $\mathcal{N} = g \cap \mathcal{N}(\mathfrak{gl}(V))$, where $\mathcal{N}(\mathfrak{gl}(V))$ denotes the set of nilpotent elements of the Lie algebra of $GL(V)$. It follows that $\mathcal{N}$ is closed in $g$, and hence $\mathcal{N}$ has the structure of a subvariety of the algebraic variety $g$.

Let:

$$P_X(t) := \det(tI - X)$$

denote the characteristic polynomial of $X$ in the variable $t$. Then:

$$P_X(t) := t^n + \sum_{i=1}^{n} (-1)^i s_i(X) t^{n-i}$$

where each $s_i$ is a homogeneous polynomial of degree $i$ in the entries of $X$. If $a_1, \cdots, a_n$ are the eigenvalues of $X$, counted with algebraic multiplicity, then, since $K$ is algebraically closed, $P_X(t) = \prod_{i=1}^{n} (t - a_i)$, and so $s_i(X)$ can be identified with the $i$th elementary symmetric function in the $a_j$. It follows that $X$ is nilpotent if and only if $P_X(t) = t^n$ if and only if $s_i(X) = 0$ for each $i$: 7
\[ \mathcal{N}(\mathfrak{gl}(V)) = \{ X \in \mathfrak{gl}(V) \mid s_i(X) = 0 \text{ for all } i \}. \]

Let \( S(V) \) denote the algebra of polynomial functions on \( V \). This has a natural grading by degree, with \( S(V) = \bigoplus_{i \geq 0} S^i(V) \). Set \( S^+(V) := \bigoplus_{i \geq 1} S^i(V) \).

Now the restrictions of the \( s_i \) to \( \mathfrak{g} \) are \( G \)-invariant polynomial functions on \( \mathfrak{g} \), and so \( s_{i|\mathfrak{g}} \in S^i(\mathfrak{g}^*)^G \). It follows that there exist \( f_1, \cdots, f_n \in S^+(\mathfrak{g}^*)^G \) such that:

\[ \mathcal{N} = \{ X \in \mathfrak{g} \mid f_i(X) = 0 \text{ for all } i \}. \]

**Proposition 2.3.1.** The nilpotent cone \( \mathcal{N} \) may be realised as:

\[ \mathcal{N} = \{ X \in \mathfrak{g} \mid f(X) = 0 \text{ for all } f \in S^+(\mathfrak{g}^*)^G \}. \]

Hence \( \mathcal{N} = V(S^+(\mathfrak{g}^*))^G \) is an affine variety.

**Proof.** It is clear that \( \{ X \in \mathfrak{g} \mid f(X) = 0 \text{ for all } f \in S^+(\mathfrak{g}^*)^G \} \subseteq \mathcal{N} \) by the above discussion. Conversely, given \( f \in S^+(\mathfrak{g}^*)^G \), \( f(0) = 0 \), and \( f \) is constant on the closure of the orbits under the adjoint action. Then \( f \) is constant on \( G\mathfrak{X} \), the closure of the regular orbit under the adjoint action, and \( 0 \in G\mathfrak{X} \) by [19, Proposition 2.11(1)]. \( \square \)

**Lemma 2.3.2.** Let \( \mathcal{B} \) be the set of all Borel subalgebras of \( \mathfrak{g} \). Then there is a bijection \( G/B \leftrightarrow \mathcal{B} \).

**Proof.** By definition, \( \mathcal{B} \) is the closed subvariety of the Grassmannian of \( \mathfrak{b} \)-dimensional subspaces in \( \mathfrak{g} \) formed by solvable Lie algebras. Hence \( \mathcal{B} \) is a projective variety. All Borel subalgebras are conjugate under the adjoint action of \( G \), and the stabiliser subgroup \( G_b \) of \( \mathfrak{b} \) in \( G \) is equal to \( B \) by [3, Theorem 11.16]. Hence the claimed bijection follows via the assignment \( g \mapsto g \cdot \mathfrak{b} \cdot g^{-1} \). \( \square \)

**Definition 2.3.3.** Let \( \mu : \mathcal{N} \times \mathcal{B} \to \mathcal{N} \) be the projection onto the first coordinate. The **enhanced nilpotent cone** is the preimage:

\[ \mathcal{N} := \mu^{-1}(\mathcal{N}) = \{ (x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b} \}. \]

**Lemma 2.3.4.** \( \mathcal{N} \) is a smooth irreducible variety.
Proof. Let $b \in B$ be a fixed Borel subalgebra. The fibre over $b$ of the second projection $\pi : \widetilde{N} \to B$ is the set of nilpotent elements of $b$. Decomposing $b = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n} := [b, b]$ is the nilradical of $b$, an element $x \in b$ is nilpotent if and only if it is has no component in the Cartan subalgebra $\mathfrak{h}$. Hence $\pi$ makes $\widetilde{N}$ a vector bundle over $B$ with fibre $\mathfrak{n}$.

The canonical map $G \to G/B$ is locally trivial by [18 II.1.10(2)], so the set of $B$-orbits on $G \times \mathfrak{n}$ has a natural structure of a variety, denoted $G \times_B \mathfrak{n}$. The above construction yields a $G$-equivariant vector bundle isomorphism:

$$\widetilde{N} = \mu^{-1}(\mathcal{N}) \cong G \times_B \mathfrak{n},$$

where $B$ is the Borel subgroup of $G$ corresponding to $b$. Hence $\widetilde{N}$ is a smooth variety.

Consider the morphism $f : \mathfrak{g} \times G \to \mathfrak{g} \times B$ defined by $f(x, g) = (x, gB)$. The inverse image:

$$f^{-1}(\widetilde{N}) = \{(x, g) \in \mathcal{N} \times G | x \in b\}$$

is closed in $\mathfrak{g} \times G$ since it is the inverse image of $\mathfrak{n}$ under the projection map $\mu : \widetilde{\mathfrak{g}} \to \mathfrak{g}$. Since $f$ is an open map and $f^{-1}(\widetilde{N})$ is closed, $\widetilde{N}$ is a closed subvariety of $N \times B$.

The morphism $\mathfrak{n} \times G \to \widetilde{N}$, $(x, g) \mapsto (\text{ad}(g)(x), gB)$ is surjective by definition. Hence $\widetilde{N}$ is irreducible.

By [19 Theorem 2.8(1)]], there are only finitely many orbits for the $G$-action in the nilpotent cone $\mathcal{N}$. Let $X_1, \cdots, X_r$ be representatives for these orbits. Then:

$$\mathcal{N} = \bigcup_{i=1}^{r} \overline{O_{X_i}}$$

Since $\mathcal{N}$ is irreducible by Lemma 2.5.4, one of these closed subsets must be all of $\mathcal{N}$: let $\overline{GZ} = \mathcal{N}$. Then, by [28 1.13, Corollary 1], this orbit is open in $\mathcal{N}$ and $\dim(GZ) = \dim(\mathcal{N})$, while $\dim(GY) < \dim(GZ)$ for any $GY \neq GZ$. Hence $GZ$ is unique with respect to this property.
Definition 2.3.5. An element $X \in \mathfrak{g}$ is regular if it lies in $GZ$, the unique open dense $G$-orbit of $\mathcal{N}$.

For the remainder of this section, we assume that $p$ is adequate for $G$.

**Proposition 2.3.6.** Let $X$ be a regular nilpotent element of $\mathfrak{g}$. Then the following hold and are equivalent:

(i) the bijection $\pi_X$ is an isomorphism of varieties $G/G_X \rightarrow GX$.

(ii) $\text{Lie } (G_X) = \mathfrak{g}_X$.

**Proof.** The equivalence of these two criteria follows from [19, Section 2.1]. Since $p$ is adequate for $G$, [17, Section 0.13] demonstrates that the Lie algebra $\mathfrak{g}$ of $G$ is simple. It follows that the induced map $\text{ad} : \mathfrak{g} \rightarrow \text{dAd}(G)$ is injective, and so, by applying [19, Proposition 2.7(b)], we may assume $G$ is of adjoint type.

The condition (ii) is equivalent to the statement $\dim \mathfrak{g}_X = \dim G_X$. Now $\dim \mathfrak{g}_X$ is equal to the number of Jordan blocks in $X$. Via the tables in [22], we see that, whenever $X$ is regular and $p$ is adequate for $G$, the number of Jordan blocks is equal to the number of Jordan blocks when $G$ is considered over a field of characteristic 0. Since 0 is good for any $G$, and $\dim G_X$ is independent of $p$ by the tables in [23, Section 22.1], the result follows from the argument in [19, Section 2.9].

Let $\mathfrak{g}_{ad}$ (resp. $\mathfrak{g}_{sc}$) denote the Lie algebra of the adjoint (resp. simply connected) group $G_{ad}$ (resp. $G_{sc}$) with a given Dynkin diagram.

**Proposition 2.3.7.** There is a $G$-equivariant bijection $\mathfrak{g} = \mathfrak{g}^*$, provided $(G, p) \neq (E_6, 2), (E_7, 2)$. In these cases, there is a $G$-equivariant bijection $\mathfrak{g}_{ad} \rightarrow \mathfrak{g}_{sc}$.

**Proof.** This follows from [29, Section 4.1].

**Lemma 2.3.8.** There is a natural $G$-equivariant vector bundle isomorphism:

$$\tilde{\mathcal{N}} \cong T^* \mathcal{B}.$$ 

**Proof.** By the proof of Lemma 2.3.4 there is a $G$-equivariant vector bundle isomorphism:

$$\tilde{\mathcal{N}} \cong G \times_B \mathfrak{n}.$$
The map \( \pi : G \to G/B \) induces an isomorphism \( \mathfrak{g}/b \to T_1B(G/B) \), the tangent space of the identity of \( G/B \). Since \( p \) is nonspecial for \( G \), the dual of this tangent space identifies with \( (\mathfrak{g}/b)^* \) and the cotangent bundle on \( G/B \) identifies with \( G \times_B (\mathfrak{g}/b)^* \). But then \( \mathfrak{n} \) identifies with \( (\mathfrak{g}/b)^* \), by Proposition 2.3.7 and the result follows. \( \square \)

**Definition 2.3.9.** The map \( \mu : T^*B \to \mathcal{N} \) is the Springer resolution for the nilpotent cone \( \mathcal{N} \).

**Lemma 2.3.10.** Let \( \mathcal{N}_s \) denote the set of smooth points of \( \mathcal{N} \). Then \( \mu^{-1}(\mathcal{N}_s) \) is dense in \( \mathcal{N} \).

**Proof.** \( \mathcal{N}_s \) is an open and non-empty subset of \( \mathcal{N} \). Hence it is dense, and its preimage is open and non-empty in \( \mathcal{N} \). By Lemma 2.3.11, \( \mathcal{N} \) is irreducible and so \( \mu^{-1}(\mathcal{N}_s) \) is dense. \( \square \)

**Lemma 2.3.11.** Let \( G/Z \) denote the orbit of all regular nilpotent elements. The morphism \( \mu^{-1}(G/Z) \to G/Z \) is an isomorphism of varieties.

**Proof.** By [19, Corollary 6.8], \( G/Z \) is an open subset of \( \mathcal{N} \), and \( |\mu^{-1}(X)| = 1 \) for \( X \in G/Z \). Hence \( \mu \) induces a bijection \( \mu^{-1}(G/Z) \to G/Z \). Pick \( X \in G/Z \) and let \( \tilde{X} \) denote its inverse image in \( \mathcal{N} \). The maps \( g \mapsto \text{Ad}(g)(X) \) and \( g \mapsto g \cdot \tilde{X} \) induce bijective morphisms \( \pi_X : G/G_X \to G/Z \) and \( \pi_{\tilde{X}} : G/G_X \to \mu^{-1}(G/Z) \), and \( \pi_X = \mu \circ \pi_{\tilde{X}} \).

By Proposition 2.3.6, \( \mathfrak{g}_X = \text{Lie}(G_X) \) for any \( X \in G/Z \) and so \( \pi_X \) is an isomorphism. Hence \( d\pi_X \) is an isomorphism at each point of \( G/G_X \), so \( d\pi_{\tilde{X}} \) is injective at each point of \( G/G_X \). By [5, Proposition 6.7], \( \tilde{\pi}_X \) is an isomorphism and so the result follows. \( \square \)

Recall from Theorem 2.2.1 that \( S(\mathfrak{h})^W \) is a polynomial ring, with algebraically independent generators \( f_1, \ldots, f_n \).

**Definition 2.3.12.** The Steinberg quotient is the map \( \chi : \mathfrak{g} \to K^n \) defined by \( \chi(Z) = (f_1(Z), \ldots, f_n(Z)) \). Note that the nilpotent cone \( \mathcal{N} = \chi^{-1}(0) \).

**Lemma 2.3.13.** The smooth points of \( \mathcal{N} \) are precisely the regular nilpotent elements.

**Proof.** By the assumptions on the prime \( p \), applying Theorem 2.2.1 and Lemma 2.2.8 shows that \( S(\mathfrak{h}) \) is a free \( S(\mathfrak{h})^W \)-module and \( S(\mathfrak{h})^W \) is a polynomial ring, with generators \( f_1, \ldots, f_n \). Hence the argument for [7, Claim 6.7.10] applies and the Steinberg quotient \( \chi \) satisfies, for \( Z \in \mathfrak{g} \), the condition that \( (d\chi)_Z \) is surjective if and only if \( Z \) is regular. By [19, Proposition 7.11], for each \( b = (b_1, \ldots, b_n) \in K^n \),
the ideal of $\chi^{-1}(b)$ is generated by all $f_i - b_i$.

By [12, I.5], $Z \in \chi^{-1}(b)$ is a smooth point if and only if the $d(f_i - b_i)$ are linearly independent at $Z$ if and only if the map $(d\chi)_Z$ is surjective. Let $b = 0$. Then the smooth points in $\chi^{-1}(0)$ are the regular elements contained in $\chi^{-1}(0)$, and so the smooth points of $\mathcal{N}$ are precisely the regular nilpotent elements. \qed

**Theorem 2.3.14.** $\mu : T^*B \to \mathcal{N}$ is a resolution of singularities for $\mathcal{N}$.

**Proof.** By Lemma 2.3.4 and Lemma 2.3.8 $\tilde{\mathcal{N}}$ is a smooth irreducible variety. Furthermore, $\mu$ is proper by [19, Lemma 6.10(1)]. By Lemma 2.3.10 $\mu^{-1}(\mathcal{N}_s)$ is dense in $\tilde{\mathcal{N}}$, and by Lemma 2.3.11 $\mu$ is a birational morphism between $\mu^{-1}(\mathcal{N}_s)$ and $\mathcal{N}_s$. Hence $\mu$ is a resolution of singularities. \qed

2.4 The dual nilpotent cone

In this section, we give an intrinsic definition of the dual nilpotent cone in arbitrary characteristic and discuss some basic properties. We make no overall assumption on the characteristic in this subsection.

**Definition 2.4.1.** Let $g^*$ denote the dual vector space of $g$. The coadjoint action of $G$ on $g^*$ is:

$$g \cdot \zeta(x) = \zeta(\text{ad}_{g^{-1}}(x))$$

for $g \in G, \zeta \in g^*, x \in g$.

**Lemma 2.4.2.** Let $K$ be a field of very good characteristic. Then $\mathcal{N}^*$ coincides with the set:

$$\{\zeta \in g^* \mid G \cdot \zeta \cap b^\perp \neq \emptyset\},$$

where $b^\perp := \{\zeta \in g^* \mid \zeta(b) = 0\}$, and $b$ is a Borel subalgebra of $g$.

**Proof.** Fix a Borel subgroup $B$ of $G$, with Levi decomposition $B = TU$, where $T$ is a maximal torus of $G$ and $U$ is unipotent. Given $x \in U$, $x$ lies in some conjugate of $U$ by [5, Theorem 11.10]. Let $f : U \to \mathcal{N}$ be a $G$-equivariant isomorphism, which exists since $p$ is very good for $G$, by [3, Corollary 9.3.4]. Then $f(x) \in f(U)$, which is a conjugate of the nilpotent subalgebra $\mathfrak{n}$: we may write $f(U) = g^{-1} \cdot \mathfrak{n} \cdot g$ for
some $g \in G$. It follows that $\mathcal{N}$ has a cover by sets of the form $g^{-1} \cdot n \cdot g$ for some $g \in G$, so it may be covered by conjugates of $n$ under the adjoint action.

Let $\kappa : g \to g^*$ be a $G$-equivariant map. We can rewrite the above condition as:

$$\mathcal{N} = \{x \in g \mid G \cdot x \cap n \neq \emptyset\}.$$  

Put another way, there exists some $g \in G$ with $g \cdot x \in n$. Applying $\kappa$ to both sides, we have:

$$\kappa(g \cdot x) = g \cdot \kappa(x) = g \cdot \zeta \in \kappa(n).$$

But $\kappa(n) = b^\perp$ by [21, VI.5.3]. Also, $\kappa(\mathcal{N}) = \mathcal{N}^*$ by definition. Hence we have:

$$\mathcal{N}^* = \{\zeta \in g^* \mid G \cdot \zeta \cap b^\perp \neq \emptyset\},$$

as required.

Lemma 2.4.2 motivates the following definition.

**Definition 2.4.3.** For $p$ arbitrary, the *dual nilcone* $\mathcal{N}^*$ is the subset of $g^*$ defined by:

$$\mathcal{N}^* = \{\zeta \in g^* \mid G \cdot \zeta \cap b^\perp \neq \emptyset\}.$$  

We now demonstrate that the dual nilpotent cone $\mathcal{N}^*$ is an affine algebraic variety.

**Theorem 2.4.4.** Let $G$ be a simple algebraic group, and suppose $(G,p) \neq (B,2)$. There is a projection map $g = n^- \oplus h \oplus n \to h$, which induces a map $S(g) \to S(h)$.

This map induces a map $\eta : S(g)^G \to S(h)^W$, which is an isomorphism.

**Proof.** This is [20, Theorem 4].

In case $p$ is nonspecial for $G$, the hypotheses in Theorem 2.4.4 will always be satisfied.
Proposition 2.4.5. Suppose $K$ has nonspecial characteristic. Then we can identify the dual nilpotent cone as:

$$N^* = \{ \zeta \in g^* \mid f(\zeta) = 0 \text{ for all } f \in S^+(g)^G \}.$$ 

Hence $N^* = V(S^+(g)^G)$ is an affine variety.

Proof. Since $K$ has nonspecial characteristic, we have an isomorphism $S(h)^W \to S(g)^G$ by Theorem 2.4.4, and $S(h)^W$ is a polynomial ring by Theorem 2.2.1. Let $\{f_i\}$ be a set of algebraically independent homogeneous generators for $S(h)^W$: then the set of preimages $\{\eta^{-1}(f_i)\}$ is a set of algebraically independent homogeneous generators for $S(g)^G$. By the same argument as in the discussion above Proposition 2.3.1, $N^* = \{ \zeta \in g^* \mid \eta^{-1}(f_i)(\zeta) = 0 \text{ for all } i \}$. Hence $\{ \zeta \in g^* \mid \eta^{-1}(f)(\zeta) = 0 \text{ for all } f \in S^+(g)^G \} \subseteq N^*$.

Conversely, given $f \in S^+(g)^G, f(0) = 0$, and $f$ is constant on the closure of the orbits under the coadjoint action. Given a regular element $X \in g$, by Proposition 2.3.1 and Proposition 2.3.7, $f$ is constant on $GX$, the closure of the regular orbit under the coadjoint action, and by the proof of part (ii) of loc. cit, $0 \in GX$. The claim follows.

2.5 The dual nilpotent cone is a normal variety

In this section, we demonstrate that the dual nilcone $N^*$ is a normal variety when the characteristic of $K$ is adequate for $G$. We begin with some basic properties of normal rings and varieties. In this section, we always assume $p$ is adequate for $G$.

Definition 2.5.1. [7, Definition 2.2.12] A finitely generated commutative $K$-algebra $A$ is Cohen-Macaulay if it contains a subalgebra of the form $O(V)$ such that $A$ is a free $O(V)$-module of finite rank, and $V$ is a smooth affine scheme.

A scheme $X$ defined over $K$ is Cohen-Macaulay if the ring of regular functions $O(X)$ is a Cohen-Macaulay ring.

Definition 2.5.2. A commutative ring $A$ is normal if the localization $A_p$ for each prime ideal $p$ is an integrally closed domain.
A variety $V$ is normal if $\mathcal{O}(V)$ is a normal ring. This definition is equivalent to the condition that, for any $x \in V$, the local ring $\mathcal{O}_{V,x}$ is a normal ring.

We now begin the proof of the normality of the dual nilpotent cone $\mathcal{N}^\ast$. We adapt the arguments in [4] to our situation.

**Theorem 2.5.3.** Let $X$ be an irreducible affine Cohen-Macaulay scheme defined over $K$ and $U \subseteq X$ an open subscheme. Suppose $\dim (X/U) \leq \dim X - 2$ and that the scheme $U$ is normal. Then the scheme $X$ is normal.

This theorem is proved in stages. We begin with the following lemma, a variant on Hartogs’ lemma.

**Lemma 2.5.4.** Let $Y$ be a normal variety and $Z \subseteq Y$ be a subvariety of codimension at least 2. Then any rational function on $Y$ which is regular on $Y \setminus Z$ can be extended to a regular function on $Y$.

**Proof.** Without loss of generality we may suppose $Y$ is affine, so write $Y = \text{Spec } B$, where $B$ is a normal domain. Set $Z := V(I)$ for some ideal $I$, and write $U := Y \setminus Z$. Then $U = \bigcup_{f \in I} D(f)$, where $D(f)$ denotes the basic open sets in the Zariski topology.

Let $p$ be a prime ideal of height 1. By assumption, $\text{ht } I \geq 2$, and so there exists some $f \in I$ with $f \not\in p$. It follows that $B_f \subseteq B_p$.

Let $a/b$ be a regular function on $U$, with $a/b \in \text{Frac } B$, the field of fractions of $B$. Since $p$ has height 1, we can find $f \in I \setminus p$. Then $a/b$ is regular on $D(f)$, and so $a/b \in \mathcal{O}(D(f)) = B_f \subseteq B_p$. As $p$ was arbitrary, $a/b \in \bigcap_{\text{ht } p = 1} B_p = B$. Hence $a/b$ can be extended to a regular function on $Y$. \qed

**Lemma 2.5.5.** Let $X$ be an affine Cohen-Macaulay scheme with an open subscheme $U$. Let $r : \mathcal{O}(X) \to \mathcal{O}(U)$ be the restriction morphism. Then:

(i) if $\dim (X \setminus U) < \dim X$, then $r$ is injective,

(ii) if $\dim (X \setminus U) \leq \dim X - 2$, then $r$ is an isomorphism.

**Proof.** For ease of notation, we suppose $\mathcal{O}(X)$ is a finitely generated $\mathcal{O}(Y)$-module for some smooth affine scheme $Y$. Now the projection map $p : X \to Y$ is a finite morphism and hence is closed. Without loss of generality, we can shrink $U$, replacing it by a smaller open subset $p^{-1}(W)$, where $W = Y \setminus p(X \setminus U)$ is an open subset of $Y$. 

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Let $F := p_*(\mathcal{O}_X)$. This is a free $\mathcal{O}_Y$-module and we clearly have $\Gamma(Y, F) = p_*(\mathcal{O}_X)(Y) = \mathcal{O}_X(p^{-1}(Y)) = \mathcal{O}_X(X)$, and similarly $\Gamma(W, F) = p_*(\mathcal{O}_X)(W) = \mathcal{O}_X(p^{-1}(W)) = \mathcal{O}_X(U)$. Hence the restriction morphism $r$ agrees with the natural restriction map $r: \Gamma(Y, F) \to \Gamma(W, F)$.

If $\dim (X \setminus U) < \dim X$, then $\dim (p(U \setminus X)) < \dim (p(X))$, so $\dim (Y \setminus W) < \dim Y$, and so $r$ is injective.

Similarly, if $\dim (X \setminus U) \leq \dim X - 2$, then $\dim (Y \setminus W) \leq \dim Y - 2$. Hence, by Lemma [2.5.4] any regular function on $W$ can be extended to a regular function on $Y$. Furthermore, $F$ is a free $\mathcal{O}_Y$-module; it follows that $r$ is surjective.

As an immediate consequence, we see that if the scheme $U$ is reduced and normal, then so is $X$.

We now demonstrate that the hypotheses in Theorem [2.5.3] are satisfied in our situation. By Proposition [2.4.5], $\mathcal{N}^*$ is an affine variety. It suffices to show that $\mathcal{N}^*$ is irreducible and Cohen-Macaulay.

Definition 2.5.6. $\lambda \in \mathfrak{h}^*$ is regular if its centraliser in $\mathfrak{g}$ under the natural $\mathfrak{g}$-action on $\mathfrak{g}^*$ coincides with the Cartan subalgebra $\mathfrak{h}$. A general $\lambda \in \mathfrak{g}^*$ is regular if its coadjoint orbit contains a regular element of $\mathfrak{h}^*$.

The subvariety $U$ in Lemma [2.5.5] will be taken to be the subset of regular nilpotent elements.

Proposition 2.5.7. Suppose $p$ is nonspecial for $G$. Then:

(i) the dual nilcone $\mathcal{N}^* \subseteq \mathfrak{g}^*$ is a closed irreducible subvariety of $\mathfrak{g}^*$, and it has codimension $r$ in $G$, where $r$ is the rank of $G$.

(ii) Let $U$ denote the set of regular elements of $\mathcal{N}^*$. Then $U$ is a single coadjoint orbit, which is open in $\mathcal{N}^*$, and its complement has codimension $\geq 2$.

Proof. (i) We define an auxiliary variety $S$ via:

$S := \{(gB, \zeta) \in G/B \times \mathfrak{g}^* | g \cdot \zeta \in \mathfrak{b}^\perp\}$.

This subset of $G/B \times \mathfrak{g}^*$ is closed. Define a map $\phi : G \times \mathfrak{b}^\perp \to G/B \times \mathfrak{g}^*$ by $\phi(g, \zeta) = (gB, g^{-1} \cdot \zeta)$. Now the image of $\phi$ is contained in $S$, and we can also see
that $\text{im}(\phi) \cong S$ since we have a linear isomorphism $b^\perp \to g^{-1} \cdot b^\perp$. Hence the image of $\phi$ coincides with $S$. It follows that $S$ is a morphic image of an irreducible variety, and hence $S$ is itself an irreducible subvariety.

Let $p_1 : G/B \times g^* \to G/B$ and $p_2 : G/B \times g^* \to g^*$ be the obvious projection maps. Clearly $p_1(S) = G/B$. The fiber of $gB$ under the map $p_1$ is $g^{-1} \cdot n$, which is isomorphic to $n$. Hence the fibers are equidimensional, and we have:

\begin{align*}
\dim S &= \dim G/B + \dim n, \\
\dim S &= \dim G/B + \dim U \\
&= \dim G - r.
\end{align*}

Using the second projection, $\dim S \leq \dim G - r$, with equality if some fibre is finite (as a set). First notice that:

\[ p_2(S) = \{ \zeta \in g^* \mid \exists g \in G \text{ s.t. } g \cdot \zeta \in b^\perp \} = N^*. \]

Hence $N^*$ is irreducible, and, since the flag variety $G/B$ is complete by [5], $N^*$ is closed. We show that there exists some $\zeta \in g^*$ with:

\[ |\{ gB \mid g \cdot \zeta \in b^\perp \}| < \infty, \]

\[ |\{ gB \mid \zeta(\text{Ad}_{g}^{-1}(b)) = 0 \}| < \infty. \]

By [15] Proposition 2, and Corollary 2.3.7, we have the following dimension formula:

\[ \dim p_1(p_2^{-1}(\zeta)) = \frac{\dim Z_G(\zeta) - r}{2}. \]

Since $p$ is nonspecial for $G$, the set of regular nilpotent elements $U$ in $N^*$ is non-empty, by [10] Section 6.4, and thus we can always pick some $\zeta \in g^*$ such that $\dim Z_G(\zeta) - r = 0$. Thus there exists $\zeta$ with $|\{p_1(p_2^{-1}(\zeta))\}| < \infty$.

Now consider two points $(gB, \zeta), (hB, \zeta) \in S$. By definition, $g \cdot \zeta \in b^\perp$ and $h \cdot \zeta \in b^\perp$. The coadjoint action then gives $\zeta(\text{ad}_{g}^{-1}(b)) = \zeta(\text{ad}_{h}^{-1}(b)) = 0$. It follows that $gB = hB$, and so $p_1$ is injective when restricted to the fibre $p_2^{-1}(\zeta)$. It follows that
there is a fibre of $p_2$ which is finite as a set.

Given the existence of a finite fibre of $p_2$, we have $\dim S = \dim p_2(S) = \dim \mathcal{N}^* = \dim G - r$.

(ii) Now $\mathcal{N}^*$ has only finitely many $G$-orbits by [31] and [30, Proposition 7.1], so the dimension of $\mathcal{N}^*$ is equal to the dimension of at least one of these orbits. Since $\dim \mathcal{N}^* = \dim G - r$, some orbit in $\mathcal{N}^*$ also has dimension equal to $\dim G - r$. This orbit is regular and its closure is all of $\mathcal{N}^*$, since the dimensions are equal and $\mathcal{N}^*$ is irreducible. Since any $G$-orbit is open in its closure, by [28, 1.13, Corollary 1], this class is open in $\mathcal{N}^*$ and thus is dense.

Let $R$ be the root system of $G$ and fix a subset of positive roots $R^+ \subseteq R$. Let $\alpha_i$ be a simple root, $X_\alpha$ the corresponding root subgroup, and set $U_i := \prod_{\alpha \in R^+, \alpha \neq \alpha_i} X_\alpha$. Let $T$ be the maximal torus of $G$ defined by this root system and let $P_i := T \cdot \langle X_{\alpha_i}, X_{-\alpha_i} \rangle \cdot U_i$. Since both $T$ and $\langle X_{\alpha_i}, X_{-\alpha_i} \rangle$ normalise $U_i$ by the commutation formulae in [28, 3.7], we see that $P_i$ is a rank 1 parabolic subgroup of $G$, $U_i$ is its unipotent radical and $T \cdot \langle X_{\alpha_i}, X_{-\alpha_i} \rangle$ is a Levi subgroup of $P_i$.

Note that $\dim T \cdot \langle X_{\alpha_i}, X_{-\alpha_i} \rangle = r + 2$ and so $\dim P_i - \dim U_i = r + 2$.

Parallel to the definition of the variety $S$, we set:

$$S_i := \{(gP_i, \zeta) \in G/P_i \times g^* \mid g \cdot \zeta \in b_i^+\}$$

where $b_i^+ = \{\zeta \in g^* \mid \zeta(\text{Lie}(U_i T)) = 0\}$. Then $S_i$ is a closed and irreducible variety and, by the same argument as in part (i) of the proposition:

$$\dim S_i = \dim G/P_i + \dim U_i = \dim G - (r + 2).$$

Projecting onto the second factor, we see that:

$$\dim p_2(S_i) \leq \dim S_i = \dim G - (r + 2).$$
But an element $\zeta \in \mathcal{N}^*$ fails to be regular if and only if $G \cdot \zeta \cap \mathfrak{h}^*_{\text{reg}} = \emptyset$. By the decomposition in [10, Section 6.4], this occurs precisely when the centraliser of each $\xi \in G \cdot \zeta \cap \mathfrak{h}^*$ contains some non-zero root $\alpha$ such that $\xi(\alpha^\vee(1)) = 0$, where $\alpha^\vee$ is the coroot corresponding to $\alpha$. It follows that $\zeta \in \mathcal{N}^*$ fails to be regular if and only if it lies in $p_2(S_i)$ for some $i$. Then:

$$\dim (\mathcal{N}^* \setminus U) = \sup_i \dim p_2(S_i) \leq \dim G - (r + 2).$$

\[\square\]

**Lemma 2.5.8.** Let $r : S(\mathfrak{g}) \to S(\mathfrak{h})$ be the natural map, and $r'$ its restriction to the graded subalgebra $S(\mathfrak{g})^G$. Suppose that $r'$ is an isomorphism onto its image $S(\mathfrak{h})^W$ and $S(\mathfrak{h})$ is a free $S(\mathfrak{h})^W$-module. Then $S(\mathfrak{g})$ is a free $R$-module, where $R := S(\mathfrak{g}/\mathfrak{h}) \otimes S(\mathfrak{g})^G$, and hence is a free $S(\mathfrak{g})^G$-module.

**Proof.** The argument is similar to that which is set out in [7, 2.2.12] and the following discussion. Consider the projection map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. This makes $\mathfrak{g}$ a vector bundle over $\mathfrak{g}/\mathfrak{h}$, and defines a natural increasing filtration on $S(\mathfrak{g})$ via:

$$F_pS(\mathfrak{g}) = \{ P \in S(\mathfrak{g}) \mid P \text{ has degree } \leq p \text{ along the fibers} \}.$$ 

Let $\text{gr}_F(S(\mathfrak{g}))$ denote the associated graded ring corresponding to this filtration, and set $S(\mathfrak{g})(p)$ to denote the $p$-th graded component. Clearly $S(\mathfrak{g})(0) = S(\mathfrak{g}/\mathfrak{h})$, and each graded component is an infinite-dimensional free $S(\mathfrak{g}/\mathfrak{h})$-module. There is a $K$-algebra isomorphism:

$$S(\mathfrak{g})(p) \cong S(\mathfrak{g}/\mathfrak{h}) \otimes_K S^p(\mathfrak{h}),$$

where $S^p(\mathfrak{h})$ denotes the space of degree $p$ homogeneous polynomials on $\mathfrak{h}$.

Let $\sigma_p : F_pS(\mathfrak{g}) \to S(\mathfrak{g})(p)$ be the principal symbol map. Suppose $f \in F_pS(\mathfrak{g})$ is a homogeneous degree $p$ polynomial whose restriction $r(f)$ to $\mathfrak{h}$ is non-zero. Then $\sigma_p(f)$ equals the image of the element $1 \otimes_k r(f)$ under the above isomorphism, and so is non-zero in $S(\mathfrak{g})(p)$.

To see this, choose a vector subspace $\mathfrak{j}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{j}$. This yields a graded algebra isomorphism $S(\mathfrak{g}) = S(\mathfrak{h}) \otimes S(\mathfrak{j})$, and so one writes $F_pS(\mathfrak{g}) = \sum_{i \leq p} S^i(\mathfrak{h}) \otimes S^{p-i}(\mathfrak{j})$. Hence $f \in F_pS(\mathfrak{g})$ has the form:

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\[ f = e_p \otimes 1 + \sum_{i \leq p} e_i \otimes w_{p-i}, \]

where \( e_i \in S^i(\mathfrak{h}) \) and \( w_{p-i} \in S^{p-i}(\mathfrak{j}) \). Hence \( r(f) = e_p \) and \( \sigma_p(f) = e_p \otimes 1 \), as required.

Given this claim, consider the filtration \( F^p S(\mathfrak{g})^G \) in \( S(\mathfrak{g}) \). For any homogeneous element \( f \in S(\mathfrak{g})^G \), its symbol \( \sigma_p(f) \) coincides with \( r(f) \in S(\mathfrak{h}) \subseteq \text{gr}_p(S(\mathfrak{g})) \). Hence the subalgebra \( \sigma_p(f) \subseteq \text{gr}_p(S(\mathfrak{g})) \) coincides with \( r(S(\mathfrak{g})^G) = S(\mathfrak{h})^W \).

Let \( \{a_k\} \) be a free basis for the \( S(\mathfrak{h})^W \)-module \( S(\mathfrak{g}) \), and fix \( b_k \in S(\mathfrak{g}) \) with \( r(b_k) = a_k \). Then \( \sigma_p(b_k) = a_k \). The \( a_k \) form a free basis of the \( \text{gr}_p R \)-module \( \text{gr}_p(S(\mathfrak{g})) = S(\mathfrak{h}) \otimes S(\mathfrak{g}/\mathfrak{h}) \), via tensoring on the right and applying the second part of the claim. It follows that the \( \{b_k\} \) form a free basis of the \( R \)-module \( S(\mathfrak{g}) \).

**Theorem 2.5.9.** Let \( G \) be a split semisimple simply connected algebraic group, defined over a field \( K \) of adequate characteristic. Then the dual nilpotent cone \( N^* \) is a normal variety.

**Proof.** Recall from Proposition 2.4.5 that \( N^* \) is an affine variety with defining ideal \( J := V(S^+(\mathfrak{g})^G)) \). It follows that its algebra of global functions \( \mathcal{O}(N^*) = S(\mathfrak{g})/J \). Consider \( Y := \mathfrak{g}/\mathfrak{h} \) as an affine variety. Then Lemma 2.5.8 implies that \( \mathcal{O}(N^*) \) is a free finitely generated module over the polynomial algebra \( S(Y) \). Hence \( N^* \) is a Cohen-Macaulay variety.

By Proposition 2.5.7, \( N^* \) is a closed irreducible subvariety of \( \mathfrak{g}^* \), and the complement of the set of regular elements \( U \) in \( N^* \) has codimension \( \geq 2 \). Hence all conditions in the statement of Theorem 2.5.3 are satisfied, and so \( N^* \) is normal.  

**Proof of Theorem A:** This is immediate from Theorem 2.5.9.

We conclude this section with an application of this result, which will be used in later sections.

**Corollary 2.5.10.** We have an isomorphism \( \mu^* : \mathcal{O}(N^*) \to \mathcal{O}(T^*\mathcal{B}) \).

**Proof.** The map \( T^*\mathcal{B} \to N^* \) is a resolution of singularities by Theorem 2.3.14, and so induces an isomorphism \( \mu^* : \mu^{-1}(N^*)^s \to (N^*)^s \) on the smooth points. These are non-empty open subsets of \( T^*\mathcal{B} \) and \( N^* \) respectively, and so \( T^*\mathcal{B} \) and \( N^* \) are
birationally equivalent.

Let \( Q(A) \) denote the field of fractions of an integral domain \( A \). By \([12, \text{I.4.5}]\), \( \mu \) induces an isomorphism \( Q(O(N^*)) \to Q(O(T^*B)) \), and so \( O(T^*B) \) can be considered as a subring of \( Q(O(N^*)) \).

Since the map \( T^*B \to N^* \) is surjective, and \( O(T^*B), O(N^*) \) are integral domains, there is an inclusion \( O(N^*) \to O(T^*B) \). The map \( \mu \) is proper, and so the direct image sheaf \( \mu_*O_{T^*B} \) is a coherent \( O_{N^*} \)-module. In particular, taking global sections, we have that \( \Gamma(N^*, \mu_*O_{T^*B}) \) is a finitely generated \( O(N^*) \)-module. By definition, \( \Gamma(N^*, \mu_*O_{T^*B}) = O(T^*B) \), so \( O(T^*B) \) is a finitely generated \( O(N^*) \)-module.

The variety \( N^* \) is normal, and so \( O(N^*) \) is an integrally closed domain. Let \( b \in O(T^*B) \). Then clearly \( O(T^*B)b \subseteq O(T^*B) \), and hence \( b \) is integral over \( O(N^*) \). Hence, by integral closure, \( b \in N^* \) and there is an isomorphism \( O(N^*) \to O(T^*B) \).

\[ \square \]

3 Applications to representations of \( p \)-adic Lie groups

3.1 Generalising the Beilinson-Bernstein theorem for \( D_{n,K}^{\lambda} \)

In this chapter, we apply the results of Section 2 to the constructions given in [2]. This allows us to weaken the restrictions on the characteristic of the base field given in [2, Section 6.8], thereby providing us with generalisations of their results.

Throughout Section 3 we suppose \( R \) is a fixed complete discrete valuation ring with uniformiser \( \pi \), residue field \( k \) and field of fractions \( K \). Assume that \( G \) is a semisimple simply connected algebraic group, \( K \) is a field of characteristic 0, and \( k \) is an algebraically closed field of adequate characteristic \( p \).

We recall some of the arguments from [2, Section 4], to define the sheaf of enhanced vector fields \( \widehat{\mathcal{V}} \) on a smooth scheme \( X \), and the relative enveloping algebra \( \widehat{\mathcal{D}} \) of an \( H \)-torsor \( \xi : \widetilde{X} \to X \).

Let \( X \) be a smooth separated \( R \)-scheme that is locally of finite type. Let \( H \) be a flat affine algebraic group over \( R \) of finite type, and let \( \widetilde{X} \) be a scheme equipped with an \( H \)-action.
Definition 3.1.1. A morphism $\xi : \widetilde{X} \to X$ is an $H$-torsor if:

(i) $\xi$ is faithfully flat and locally of finite type,
(ii) the action of $H$ respects $\xi$,
(iii) the map $\widetilde{X} \times H \to \widetilde{X} \times_X \widetilde{X}$, $(x, h) \to (x, hx)$ is an isomorphism.

An open subscheme $U$ of $X$ trivialises the torsor $\xi$ if there is an $H$-invariant isomorphism:

$$U \times H \to \xi^{-1}(U)$$

where $H$ acts on $U \times H$ by left translation on the second factor.

Definition 3.1.2. Let $S_X$ denote the set of open subschemes $U$ of $X$ such that:

(i) $U$ is affine,
(ii) $U$ trivialises $\xi$,
(iii) $\mathcal{O}(U)$ is a finitely generated $R$-algebra.

$\xi$ is locally trivial for the Zariski topology if $X$ can be covered by open sets in $S_X$.

Lemma 3.1.3. If $\xi$ is locally trivial, then $S_X$ is a base for $X$.

Proof. Since $X$ is separated, $S_X$ is stable under intersections. If $U \in S_X$ and $W$ is an open affine subscheme of $U$, then $W \in S_X$. Hence $S_X$ is a base for $X$.  

The action of $H$ on $\widetilde{X}$ induces a rational action of $H$ on $\mathcal{O}(V)$ for any $H$-stable open subscheme $V \subseteq \widetilde{X}$, and therefore induces an action of $H$ on $T_{\widetilde{X}}$ via:

$$(h \cdot \partial)(f) = h \cdot \partial(h^{-1} \cdot f)$$

for $\partial \in T_{\widetilde{X}}$, $f \in \mathcal{O}(\widetilde{X})$ and $h \in H$. The sheaf of enhanced vector fields on $X$ is:

$$\widetilde{T} := (\xi_* T_{\widetilde{X}})^H.$$

Differentiating the $H$-action on $\widetilde{X}$ gives an $R$-linear Lie algebra homomorphism:  

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$j : \mathfrak{h} \to \mathcal{T}_\tilde{X}$

where $\mathfrak{h}$ is the Lie algebra of $H$.

**Definition 3.1.4.** Let $\xi : \tilde{X} \to X$ be an $H$-torsor. Then $\xi_* \mathcal{D}_{\tilde{X}}$ is a sheaf of $R$-algebras with an $H$-action. The *relative enveloping algebra* of the torsor is the sheaf of $H$-invariants of $\xi_* \mathcal{D}_{\tilde{X}}$:

$$\tilde{\mathcal{D}} := (\xi_* \mathcal{D}_{\tilde{X}})^H.$$  

This sheaf has a natural filtration:

$$F_m \tilde{\mathcal{D}} := (\xi_* F_m \mathcal{D}_{\tilde{X}})^H$$

induced by the filtration on $\mathcal{D}_{\tilde{X}}$ by order of differential operator.

Let $B$ be a Borel subgroup of $G$. Let $N$ be the unipotent radical of $B$, and $H := B/N$ the abstract Cartan group. Let $\tilde{B}$ denote the homogeneous space $G/N$. There is an $H$-action on $\tilde{B}$ defined by:

$$bN \cdot gN := gbN$$

which is well-defined since $[B, B]$ is contained in $N$. $B := G/B$ is the *flag variety* of $G$. $\tilde{B}$ is the *basic affine space* of $G$.

By the splitting assumption of $G$, we can find a Cartan subgroup $T$ of $G$ complementary to $N$ in $B$. This is naturally isomorphic to $H$, and induces an isomorphism of the corresponding Lie algebras $t \to \mathfrak{h}$.

We let $W$ denote the Weyl group of $G$, and let $W_k$ denote the Weyl group of $G_k$, the $k$-points of the algebraic group $G$.

We may differentiate the natural $G$-action on $\tilde{B}$ to obtain an $R$-linear Lie homomorphism:
\[ \varphi : \mathfrak{g} \to \mathcal{T}_B. \]

Since the \( \mathbf{G} \)-action commutes with the \( \mathbf{H} \)-action on \( \mathcal{B} \), this map descends to an \( R \)-linear Lie homomorphism \( \varphi : \mathfrak{g} \to \mathcal{T}_B \) and an \( \mathcal{O}_B \)-linear morphism:

\[ \varphi : \mathcal{O}_B \otimes \mathfrak{g} \to \mathcal{T}_B \]

of locally free sheaves on \( B \). Dualising, we obtain a morphism of vector bundles over \( B \):

\[ \varphi^* : \mathcal{T}^* \mathcal{B} \to B \times \mathfrak{g}^* \]

from the enhanced cotangent bundle to the trivial vector bundle of rank \( \dim \mathfrak{g} \).

**Definition 3.1.5.** The *enhanced moment map* is the composition of \( \varphi^* \) with the projection onto the second coordinate:

\[ \beta : \mathcal{T}^* \mathcal{B} \to \mathfrak{g}^*. \]

We may apply the deformation functor ( [2, Section 3.5]) to the map \( j : U(\mathfrak{h}) \to \tilde{\mathcal{D}} \), defined above Definition 3.1.4, to obtain a central embedding of the constant sheaf \( U(\mathfrak{h})_n \) into \( \tilde{\mathcal{D}}_n \). This gives \( \mathcal{D}_n \) the structure of a \( U(\mathfrak{h})_n \)-module.

Let \( \lambda \in \text{Hom}_R(\pi^n \mathfrak{h}, R) \) be a linear functional. This extends to an \( R \)-algebra homomorphism \( U(\mathfrak{h})_n \to R \), which gives \( R \) the structure of a \( U(\mathfrak{h})_n \)-module, denoted \( R_\lambda \).

**Definition 3.1.6.** The *sheaf of deformed twisted differential operators* \( \mathcal{D}_n^\lambda \) on \( B \) is the sheaf:

\[ \mathcal{D}_n^\lambda := \mathcal{D}_n \otimes_{U(\mathfrak{h})_n} R_\lambda \]

By [2, Lemma 6.4(b)], this is a sheaf of deformable \( R \)-algebras.
Definition 3.1.7. The $\pi$-adic completion of $D_\lambda^n$ is 
$$\hat{D}_\lambda^n := \lim_{\leftarrow} D_\lambda^n / \pi^n D_\lambda^n.$$ 
Furthermore, set $\hat{D}_\lambda^n,\mathbb{K} := D_\lambda^n \otimes_R \mathbb{K}$.

Proposition 3.1.8. The rows of the diagram:

\begin{center}
\begin{tikzcd}
0 \rightarrow \text{gr}(U(g)^G) \ar[r, shift left=2pt, \pi] \ar[d, \iota] & \text{gr}(U(g)^G) \ar[r, shift left=2pt, \pi] \ar[d, \iota] & \text{gr}(U(g_k)^G_k) \ar[r, \pi] \ar[d, \iota_k] & 0 \\
0 \rightarrow S(g)^G \ar[r, \pi] \ar[d, \psi] & S(g)^G \ar[r, \pi] \ar[d, \psi] & S(g_k)^G_k \ar[r, \psi_k] \ar[d] & 0 \\
0 \rightarrow S(t)^W \ar[r, \pi] & S(t)^W \ar[r] & S(t_k)^W_k \ar[r] & 0
\end{tikzcd}
\end{center}

are exact, and each vertical map is an isomorphism.

Proof. View the diagram as a sequence of complexes $C^* \rightarrow D^* \rightarrow E^*$. Since $\pi$ generates the maximal ideal $m$ of $R$ by definition, and $R/m = k$, it is clear that each complex is exact in the left and in the middle. The exactness of $E^*$ follows from the fact that $S(t_k)^W_k$ is a polynomial ring by Theorem 2.2.1: fix homogeneous generators $s_1, \cdots, s_l$ and lift these generators to homogeneous generators $S_1, \cdots, S_l$ of the ring $S(t)^W$ with $s_i = S_i(\text{mod } m)$. Hence the map $S(t)^W \rightarrow S(t_k)^W_k$ is surjective, and the complex $E^*$ is exact.

By [9, Theorem 7.3.7], $\psi$ is injective, and since $p$ is nonspecial, $\psi_k$ is an isomorphism by Theorem 2.4.4. Thus the composite map of complexes $\psi^* \circ \iota^*$ is injective. Set $F^* := \text{coker}(\psi^* \circ \iota^*)$: by definition, the sequence of complexes $0 \rightarrow C^* \rightarrow E^* \rightarrow F^* \rightarrow 0$ is exact.

Since $C^*$ is exact in the left and in the middle, $H^0(C^*) = H^1(C^*) = 0$. As $E^*$ is exact, taking the long exact sequence of cohomology shows that $H^0(F^*) = H^2(F^*) = 0$ and yields an isomorphism $H^1(F^*) \cong H^2(C^*)$.

Since $K$ is a field of characteristic zero, the map $\psi_K \circ \iota_K : \text{gr}(U(g_K)^G_K) \rightarrow S(t_K)^W_k$ is an isomorphism by [9, Theorem 7.3.7]. Hence $F^0 = F^1 = \text{coker}(\psi \circ \iota)$ is $\pi$-torsion. Now $H^0(F^*) = 0$, and so we have an exact sequence $0 \rightarrow F^0 \rightarrow F^1$. So $F^0 = F^1 = 0$, and hence $H^1(F^*) = H^2(C^*) = 0$. It follows that the top row $C^*$ is exact.

Hence $\psi^* \circ \iota^* : C^* \rightarrow E^*$ is an isomorphism in all degrees except possibly 2, and so
is an isomorphism via the Five Lemma. The result follows from the fact that $\psi^\bullet$ and $\iota^\bullet$ are both injections.

It follows that, since $\psi \circ \iota$ is a graded isomorphism and $p$ is nonspecial, $\text{gr}(U(g)^G)$ is isomorphic to a commutative polynomial algebra over $R$ in $l$ variables by Theorem 2.2.1. The commutative polynomial algebra $R[x_1, \cdots, x_l]$ is a free $R$-module and hence is flat, and so $(U(g)^G)$ is a deformable $R$-algebra by [2] Definition 3.5. Furthermore, $\widehat{U(g)^G}_{n,K}$ is also a commutative polynomial algebra over $R$ in $l$ variables, so the $\pi$-adic completion $\widehat{U(g)^G}_{n,K}$ is a commutative Tate algebra.

By [2] Proposition 4.10, we have a commutative square consisting of deformable $R$-algebras:

$$
\begin{array}{ccc}
U((g)^G)_n & \xrightarrow{\phi_n} & U(t)_n \\
\downarrow^{i_n} & & \downarrow^{(j \circ i)_n} \\
U(g)_n & \xrightarrow{U(\phi)_n} & \widehat{D}_n,
\end{array}
$$

We set:

$$
U^\lambda := U(g) \otimes_{(U(g)^G)_n} R^\lambda,
$$

$$
\widehat{U}^\lambda := \lim_{\leftarrow} \pi^a U^\lambda_n,
$$

$$
\widehat{U}^\lambda_{n,K} := \widehat{U}^\lambda_n \otimes_R K.
$$

By commutativity of the diagram, the map:

$$
U(\phi)_n \otimes (j \circ i)_n : U(g)_n \otimes U(t)_n \to \widehat{D}_n
$$

factors through $U((g)^G)_n$, and we obtain the algebra homomorphisms:

$$
\phi^\lambda_n : U^\lambda_n \to \widehat{D}^\lambda_n,
$$

$$
\widehat{\phi}^\lambda_n : \widehat{U}^\lambda_n \to \widehat{D}^\lambda_n,
$$

$$
\widehat{\phi}^\lambda_{n,K} : \widehat{U}^\lambda_{n,K} \to \widehat{D}^\lambda_{n,K}.
$$
Theorem 3.1.9. (a) $\hat{U}_{n,K}^\lambda \cong U(\mathfrak{g})_{n,K} \otimes_{U(\mathfrak{g})_{n,K}} ^\mathbb{G} K$ is an almost commutative affinoid $K$-algebra.

(b) The map $\hat{\phi}_{n,K}^\lambda : \hat{U}_{n,K}^\lambda \to \Gamma(B, \mathcal{D}_{n,K}^\lambda)$ is an isomorphism of complete doubly filtered $K$-algebras.

(c) There is an isomorphism $S(\mathfrak{g}_k) \otimes_{S(\mathfrak{g}_k)} \mathcal{C}_k \cong Gr(\hat{U}_{n,K}^\lambda)$.

Proof. (a): This is identical to the proof given in [2, Theorem 6.10(a)].

(b): Let $\{U_1, \cdots, U_m\}$ be an open cover of $B$ by open affines that trivialise the torsor $\xi$, which exists by [2, Lemma 4.7(c)]. The special fibre $B_k$ is covered by the special fibres $U_{i,k}$. It suffices to show that the complex:

$$C^\bullet : 0 \to \hat{U}_{n,K}^\lambda \to \bigoplus_{i=1}^m \mathcal{D}_{n,K}^\lambda(U_i) \to \bigoplus_{i<j} \mathcal{D}_{n,K}^\lambda(U_i \cap U_j)$$

is exact.

Clearly, $C^\bullet$ is a complex in the category of complete doubly-filtered $K$-algebras, and so it suffices to show that the associated graded complex $Gr(C^\bullet)$ is exact. By [2, Corollary 3.7], there is a commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
0 & \rightarrow & \text{gr}(U(\mathfrak{g})) & \rightarrow & \text{gr}(U(\mathfrak{g})) & \rightarrow & \text{Gr}(\hat{U}_{n,K}^\lambda) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{gr}(U(\mathfrak{g})) & \rightarrow & \text{gr}(U(\mathfrak{g})) & \rightarrow & \text{Gr}(U(\mathfrak{g})) & \rightarrow & 0.
\end{array}
$$

Via the identification $\text{gr}(U(\mathfrak{g})) = S(\mathfrak{g})$, Proposition 3.1.8 induces a commutative square:

$$
\begin{array}{ccc}
\text{Gr}(U(\mathfrak{g}))_{n,K} & \rightarrow & S(\mathfrak{g}_k) \\
\downarrow & & \downarrow \\
\text{Gr}(U(\mathfrak{g}))_{n,K} & \rightarrow & S(\mathfrak{g}_k)
\end{array}
$$
where the horizontal maps are isomorphisms and the vertical maps are inclusions. Since \( \text{Gr}(K_\lambda) \) is the trivial \( \text{Gr}(\hat{U}_{n,K}) \)-module, we have a natural surjection:

\[
S(\mathfrak{g}_k) \otimes_{S(\mathfrak{g}_k)} \mathfrak{g}_k k \cong \text{Gr}(U(\mathfrak{g})_{n,K} \otimes_{\text{Gr}(U(\mathfrak{g})_{n,K})} \text{Gr}(K_\lambda)) \to \text{Gr}(\hat{U}_{n,K}).
\]

This surjection fits into the commutative diagram:

\[
\begin{array}{c}
0 \\
\downarrow
\end{array} \quad \begin{array}{c}
S(\mathfrak{g}_k) \otimes_{S(\mathfrak{g}_k)} \mathfrak{g}_k k \\
\oplus_{i=1}^m O(T^*U_{i,k}) \\
\oplus_{i<j} O(T^*(U_{i,k} \cap U_{j,k}))
\end{array} \quad \begin{array}{c}
\oplus_{i=1}^m \text{Gr}(\hat{U}_{n,K}) \\
\oplus_{i<j} \text{Gr}(\hat{D}_{n,K}(U_i))
\end{array} \quad \begin{array}{c}
0 \\
\downarrow
\end{array}
\]

The bottom row is \( \text{Gr}(C^\bullet) \) by definition, and the top row is induced by the moment map \( T^*B \to \mathfrak{g}_k^* \). To see this, note that by Lemma 2.3.8, we have an identification \( \tilde{N}^* \to T^*B \) under our assumptions on \( p \), and so exactness of the top row is equivalent to the existence of an isomorphism:

\[
S(\mathfrak{g}) \otimes_{S(\mathfrak{g})} \mathfrak{g} k \cong \Gamma(\tilde{N}^*, O_{\tilde{N}^*}).
\]

By Theorem 2.5.9, \( N^* \) is a normal variety and, by Theorem 2.3.14, the map \( \gamma : T^*B \to N^* \) is a resolution of singularities. It follows, by Corollary 2.5.10, that there is an isomorphism of global sections:

\[
\gamma^* : \Gamma(N^*, O_{N^*}) \to \Gamma(T^*B, O_{T^*B}).
\]

By Proposition 2.4.5, \( O(N^*) = S(\mathfrak{g}) \otimes_{S(\mathfrak{g})} \mathfrak{g} k \). Putting these isomorphisms together, \( S(\mathfrak{g}) \otimes_{S(\mathfrak{g})} \mathfrak{g} k \cong \Gamma(\tilde{N}^*, O_{\tilde{N}^*}) \).

Now the second and third vertical arrows are isomorphisms by [2, Proposition 6.5(a)], which shows that \( \text{Gr}(C^\bullet) \) is exact.

(c) This is immediate, since one can also show that the first vertical arrow in the above diagram is an isomorphism via the Five Lemma. \( \square \)
Definition 3.1.10. For each $\lambda \in \text{Hom}_R(\pi^n \mathfrak{h}, R)$, there is a functor:

$$\text{Loc}^\lambda : \widehat{U(\mathfrak{g})}_{n,K} - \text{mod} \rightarrow \widehat{D}_{n,K}^\lambda - \text{mod}$$

given by $M \mapsto \widehat{D}_{n,K}^\lambda \otimes \widehat{U(\mathfrak{g})}_{n,K} M$.

3.2 Modules over completed enveloping algebras

The adjoint action of $G$ on $\mathfrak{g}$ induces an action of $G$ on $U(\mathfrak{g})$ by algebra automorphisms. Composing the inclusion $U(\mathfrak{g})^G \rightarrow U(\mathfrak{g})$ with the projection $U(\mathfrak{g}) \rightarrow U(\mathfrak{t})$ defined by the direct sum decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}^+$ yields the Harish-Chandra homomorphism:

$$\phi : U(\mathfrak{g})^G \rightarrow U(\mathfrak{t})$$

This is a morphism of deformable $R$-algebras, so by applying the deformation and $\pi$-adic completion functors, one obtains the deformed Harish-Chandra homomorphism:

$$\widehat{\phi}_{n,K} : \widehat{U(\mathfrak{g})}_{n,K}^G \rightarrow \widehat{U(\mathfrak{t})}_{n,K}$$

which we will denote via the shorthand $\widehat{\phi} : Z \rightarrow \widehat{Z}$. We have an action of the Weyl group $W$ on the dual Cartan subalgebra $\mathfrak{t}_K^*$ via the shifted dot-action:

$$w \bullet \lambda = w(\lambda + \rho') - \rho'$$

where $\rho'$ is equal the half-sum of the T-roots on $\mathfrak{n}^+$. Viewing $U(\mathfrak{t})_K$ as an algebra of polynomial functions on $\mathfrak{t}_K^*$, we obtain a dot-action of $W$ on $U(\mathfrak{t})_K$. This action preserves the $R$-subalgebra $U(\mathfrak{t})_n$ of $U(\mathfrak{t})_K$ and so extends naturally to an action of $W$ on $\widehat{Z}$.

Theorem 3.2.1. Let $p$ be an adequate prime for $G$. Then:

(a) set $A := \widehat{U(\mathfrak{g})}_{n,K}$. The algebra $Z$ is contained in the centre of $A$. 

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(b) the map $\hat{\phi}$ is injective, and its image is the ring of invariants $\hat{\mathbb{Z}}^W$.

(c) the algebra $\mathbb{Z}$ is a free rank $|W|$-module over $\hat{\mathbb{Z}}^W$.

(d) $\hat{\mathbb{Z}}^W$ is isomorphic to a Tate algebra $K\langle S_1, \cdots, S_l \rangle$ as a complete doubly filtered $K$-algebra.

Proof. (a): The algebra $U(g)^G_K$ is central in $U(g^K)$ via [16, Lemma 23.2]. Since $U(g)^G_K$ is dense in $A$, it is also contained in the centre of $A$. But $U(g)^G_K$ is also dense in $Z$, and so $Z$ is central in $A$.

(b): By the Harish-Chandra homomorphism (see [9, Theorem 7.4.5]), $\phi$ sends $U(g)^G_K$ onto $U(t)^W_K$, and so $\hat{\phi}(Z)$ is contained in $\hat{\mathbb{Z}}^W$. This is a complete doubly filtered algebra whose associated graded ring $\text{Gr}(\hat{\mathbb{Z}}^W)$ can be identified with $S(t_k)^W_k$. This induces a morphism of complete doubly filtered $K$-algebras $\alpha: Z \to \hat{\mathbb{Z}}^W$. Its associated graded map $\text{Gr}(\alpha): \text{Gr}(Z) \to \text{Gr}(\hat{\mathbb{Z}}^W)$ can be identified with the isomorphism $\psi_k: S(g_k)^G_k \to S(t_k)^W_k$ by Proposition 3.1.8. Hence $\text{Gr}(\alpha)$ is an isomorphism, and so $\alpha$ is an isomorphism by completeness.

(c): By Lemma 2.2.3 and Theorem 2.2.1, $S(t_k)$ is a free graded $S(t_k)^W_k$-module of rank $|W|$. Hence, by [2, Lemma 3.2(a)], $\hat{\mathbb{Z}}$ is finitely generated over $Z$, and in fact is free of rank $|W|$.

(d): By Theorem 2.2.1, $S(t_k)^W_k$ is a polynomial algebra in $l$ variables. Fix double lifts $s_1, \cdots, s_l \in U(t)^W_K$ of these generators. Define an $R$-algebra homomorphism $R[S_1, \cdots, S_l] \to \hat{\mathbb{Z}}^W$ which sends $S_i$ to $s_i$. This extends to an isomorphism $K\langle S_1, \cdots, S_l \rangle \to \hat{\mathbb{Z}}^W$ of complete doubly filtered $K$-algebras. □

We identify the $k$-points of the scheme $g^* := \text{Spec}(\text{Sym}_R g)$ with the dual of the $k$-vector space $g^*$, so $g^*(k) = g^*_k$. Let $G$ denote the $k$-points of the algebraic group scheme $G$. $G$ acts on $g_k$ and $g_k^*$ via the adjoint and coadjoint action respectively.

Recall the definition of the enhanced moment map $\beta: T^*B(k) \to g^*_k$ from Definition 3.1.5. Given $y \in g^*_k$, write $G.y$ to denote the $G$-orbit of $y$ under the coadjoint action. We write $N^*$ (resp. $N^*$) to denote the nilpotent cone (resp. dual nilpotent cone) of the $k$-vector spaces $g_k$ and $g_k^*$.

**Proposition 3.2.2.** Suppose $p$ is nonspecial for $G$. For any $y \in N^*$, we have $\dim \beta^{-1}(y) = \dim B - \frac{1}{2} \dim G.y$. 30
Proof. This is stated for $\mathcal{N}$ as [19, Theorem 10.11]. The result follows by applying Proposition 2.3.7.

We now let $\mathfrak{g}_C$ denote the complex semisimple Lie algebra with the same root system as $G$, and let $G_C$ be the corresponding adjoint algebraic group. By [8, Remark 4.3.4], there is a unique non-zero nilpotent $G_C$-orbit in $\mathfrak{g}_C^*$, under the coadjoint action, of minimal dimension. Since each coadjoint $G_C$-orbit is a symplectic manifold, it follows that each of these dimensions is an even integer. We set:

$$r := \frac{1}{2} \min \{ \dim G_C \cdot y \mid 0 \neq y \in \mathfrak{g}_C \}$$

Proposition 3.2.3. For any non-zero $y \in \mathcal{N}^*$, $\frac{1}{2} \dim G \cdot y \geq r$, with no restrictions on $(G, p)$.

Proof. We will demonstrate that this inequality holds for all split semisimple algebraic groups $G$ defined over an algebraically closed field $k$ of positive characteristic. When the characteristic $p$ is small, we will proceed via a case-by-case calculation of the maximal dimension of the centraliser $Z_G(y)$ of $y \in \mathcal{N}^*$.

By Proposition 3.2.2, $\dim \beta^{-1}(y) = \dim \mathcal{B} - \frac{1}{2} \dim G \cdot y$. Suppose $(G, p) \neq (E_6, 2), (E_7, 2)$. We may assume $y \in \mathcal{N}$ and $G$ acts on $\mathfrak{g}$ via the adjoint action by Proposition 2.3.7. By [15, Theorem 2], we see that:

$$\dim \beta^{-1}(y) = \frac{1}{2}(\dim Z_G(y) - \text{rk } (G))$$

where $Z_G(y)$ denotes the centraliser of $y$ in $G$. Hence it suffices to demonstrate that the following inequality:

$$\dim \mathcal{B} - \frac{1}{2}(\dim Z_G(y) - \text{rk } (G)) \geq r$$

holds in all types. We evaluate on a case-by-case basis, aiming to find the maximal dimension of the centraliser. We first note that, using the work of [26, 1.6], we have the following table:
By [23, Theorem 2.33], when $p$ is nonspecial, the dimension of the centraliser is independent of the isogeny type of $G$.

Since $p$ is always nonspecial for a group of type $A$, it therefore suffices to consider $Z_{\text{sl}_n}(y)$. Since $p$ is good and $\text{SL}_n$ is a simply connected algebraic group, by [23, Lemma 2.15], it suffices to consider the centraliser of a non-identity unipotent element in $\text{SL}_n$. Via the identification $GL_n(k) = \text{SL}_n(k)Z(GL_n(k))$, it is sufficient to compute $Z_{GL_n(k)}(u)$, for some unipotent matrix $u$. This dimension is bounded above by $n^2 - 1$, the dimension of $GL_n(k)$ as an algebraic group, and so we have the expression:

$$
\dim B - \frac{1}{2}(\dim Z_G(y) - \text{rk } (G)) \\
\geq \frac{1}{2}n(n + 1) - \frac{1}{2}(n^2 - 1 - n) \geq n.
$$

Hence the inequality is verified in type $A$.

For the remaining classical groups, view $y \in \mathcal{N}$ as a nilpotent matrix, which without loss of generality may be taken to be in Jordan normal form. Let $m_1 \geq \cdots \geq m_r$ be the sizes of the Jordan blocks, with $\sum_{i=1}^r m_i = n$, the rank of the group. By [14, Theorem 4.4], we have:

$$
\dim Z_G(y) = \sum_{i=1}^r (im_i - \chi_V(m_i))
$$

where $\chi_V$ is a function $\chi_V : \mathbb{N} \rightarrow \mathbb{N}$. It follows that:
\[
\dim Z_G(y) \leq \sum_{i=1}^{r} i m_i = \sum_{j=1}^{n} \sum_{i=j}^{r} m_i.
\]

Since \( m_1 \geq \cdots \geq m_r \) by construction, the maximum value of this sum is attained when \( m_k = 1 \) for all \( k \). Hence we obtain the inequality \( \dim Z_G(y) \leq \frac{1}{2} n(n + 1) \).

Using this, it is easy to see that the required inequality holds except possibly in the cases \( B_2, B_3, D_4 \) and \( D_5 \).

For these cases, along with all exceptional cases, we directly verify that the inequality holds using the calculations on dimensions of centralisers in [23, Chapter 8 and Chapter 22].

Now suppose \((G, p) = (E_6, 2) \) or \((E_7, 2) \). By Proposition 2.3.7, there is a \( G \)-equivariant bijection \( \mathcal{N}_{ad} \rightarrow \mathcal{N}_{sc} \). By the calculations in [23, Chapter 22], we may apply the same argument as above, replacing \( \mathcal{N} \) with \( \mathcal{N}_{ad} \), to deduce the result. \( \square \)

This allows us to prove our generalisation of [2, Theorem 9.10]; a result on the minimal dimension of finitely generated modules over \( \pi \)-adically completed enveloping algebras.

**Definition 3.2.4.** Let \( A \) be a Noetherian ring. \( A \) is *Auslander-Gorenstein* if the left and right self-injective dimension of \( A \) is finite and every finitely generated left or right \( A \)-module \( M \) satisfies, for \( i \geq 0 \) and every submodule \( N \) of \( \text{Ext}^i_A(M, A) \), \( \text{Ext}^j_A(N, A) = 0 \) for \( j < i \).

In this case, the *grade* of \( M \) is given by:

\[
j_A(M) := \inf \{ j \mid \text{Ext}^j_A(M, A) \neq 0 \}
\]

and the *canonical dimension* of \( M \) is given by:

\[
d_A(M) := \text{inj.dim}_A(A) - j_A(M).
\]

By the discussion in [2, Section 9.1], the ring \( \hat{U}(\mathfrak{g})_{n, K} \) is Auslander-Gorenstein and so it makes sense to define the canonical dimension function:

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Theorem 3.2.5. Suppose \( n > 0 \) and let \( M \) be a finitely generated \( \widehat{U(g)}_{n,K} \)-module with \( d(M) \geq 1 \). Then \( d(M) \geq r \).

Proof. By [2, Proposition 9.4], we may assume that \( M \) is \( \mathbb{Z} \)-locally finite. We may also assume that \( M \) is a \( \widehat{U}_{n,K}^{\lambda} \)-module for some \( \lambda \in h^*_K \), by passing to a finite field extension if necessary and applying [2, Theorem 9.5].

By Proposition 3.2.1(b), \( \lambda \circ (i \circ \hat{\phi}) = (w \bullet \lambda) \circ (i \circ \hat{\phi}) \) for any \( w \in W \). Hence we may assume \( \lambda \) is \( \rho \)-dominant by [2, Lemma 9.6]. Hence \( \text{Gr}(M) \) is a \( \text{Gr}(\widehat{U}_{n,K}^{\lambda}) \cong S(g_k) \otimes S(g_k)_{\mathbb{C}_n} \) \( k \)-module by Theorem 3.1.9 If \( \mathcal{M} := \text{Loc}^{\lambda}(M) \) is the corresponding coherent \( \mathcal{D}_{n,K}^{\lambda} \)-module in the sense of Definition 3.1.10 then \( \beta(\text{Ch}(\mathcal{M})) = \text{Ch}(M) \) via [2, Corollary 6.12].

Let \( X \) and \( Y \) denote the \( k \)-points of the characteristic varieties \( \text{Ch}(\mathcal{M}) \) and \( \text{Ch}(M) \) respectively. Now \( \text{Gr}(M) \) is annihilated by \( S^+(g_k)^{G_k} \), and so \( Y \subseteq N^* \). We see that the map \( \beta : T^*B \to g \) maps \( X \) onto \( Y \).

Let \( f : X \to Y \) be the restriction of \( \beta \) to \( X \). By [2, Corollary 9.1], since \( \dim Y = d(M) \geq 1 \) we can find a non-zero smooth point \( y \in Y \). By surjectivity, we have a smooth point \( x \in f^{-1}(y) \). The induced differential \( df_x : T_{X,x} \to T_{Y,y} \) on Zariski tangent spaces yields the inequality:

\[
\dim Y + \dim f^{-1}(y) \geq \dim T_{X,x}
\]

By [2, Theorem 7.5], \( \dim T_{X,x} \geq \dim B \). Hence:

\[
d(M) = \dim Y \geq \dim B - \dim \beta^{-1}(y)
\]

By Proposition 3.2.2 and Proposition 3.2.3 the RHS equals \( r \). □

Proof of Theorem B: This follows from Theorem 3.2.5 and [2, Section 10] in the split semisimple case. We may then apply the same argument as in [1] to remove the split hypothesis on the Lie algebra.
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