Solvable reaction-diffusion processes without exclusion

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Abstract
For reaction-diffusion processes without exclusion, in which the particles can exist in the same site of a one-dimensional lattice, we study all the integrable models which can be obtained by imposing a boundary condition on the master equation of the asymmetric diffusion process. The annihilation process is also added. The Bethe ansatz solution and the exact \(N\)-particle conditional probabilities are obtained.

1 Introduction
There are a variety of phenomena that can be explained by stochastic models, and their non-equilibrium behaviors can be understood by rather simple rules [1–3]. One of the important examples of these models are the reaction-diffusion processes on a one-dimensional lattice for which their dynamics are fully specified by their master equation [4, 5]. A simple example of reaction-diffusion process is asymmetric simple exclusion process (ASEP) [2, 6, 7], which is known to be relevant to various fields of science like the kinetics of biopolymerization [8, 9], traffic models [10], polymers in random media, dynamical models of interface growth [11, 12], noisy Burgers equation [13], study of shocks [14, 15], sequence alignment [16], and molecular motors [17]. For recent reviews, see for instance [18–20].

The ASEP is a lattice model in which each particle hops to its right (left) nearest-neighboring site with a probability \(D_R dt(D_L dt)\) in an infinitesimal interval \(dt\). In addition, particles are subject to hard-core exclusion: each site is either occupied only by one particle or empty. ASEP has been studied in [21] by introducing a master equation which describes the time evolution of probabilities \(P(x_1, \cdots, x_N; t)\), when the particles are not in neighboring sites, and a so-called boundary condition, which specifies the situations in which the probabilities go outside the physical region \(x_1 < x_2 < \cdots < x_N\). These happen when some of the particles are in adjacent sites and the master equation can not be applied to them. It has been shown that the model is integrable in the sense that the \(N\)-particle \(S\)-matrix is factorized into a product of two-particle

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$S$-matrices. The coordinate Bethe ansatz has been used in this proof. Note that the $S$-matrix can completely determine the dynamics of a Markovian process, i.e. the $N$-particle probabilities of a model.

By choosing other suitable boundary conditions, without changing the master equation, one may study the more complicated reaction-diffusion processes, even with long-range interaction. In ref. [22], the so-called drop-push model has been studied by this method. In this model the particle hops to the next right site even it is occupied. The particle hops to this site by pushing all the neighboring particles to their next right sites, with a rate depending on the number of right neighboring particles. The generalization of this model, by considering both the right and left hopping, has been done in ref. [23]. This method has been also applied to more-than-one-species situations, which become more complicated. The complexity arises from the above mentioned factorization of $N$-particle scattering matrix. In these cases, the factorization demands the two-particle $S$-matrices to satisfy the spectral Yang-Baxter equation. Various solvable multi-species models have been studied in this way, of which the most recent general cases have been discussed in [24] and [25].

All of the above studies have been restricted to interactions which include the hard-core exclusion. This made some simplification. In ref. [26, 27], an asymmetric diffusion model without exclusion has been shown to be integrable and to have the same $R$-matrix as that of the ASEP. In [28], it has been shown that the processes:

\[
\begin{aligned}
  mn &\rightarrow m - l, n + l \quad \text{with rate } \frac{D_R}{[l]}, \\
mn &\rightarrow m + l, n - l \quad \text{with rate } \frac{D_R \gamma}{[l]},
\end{aligned}
\]

is integrable, in the sense of the above mentioned two-particle factorization. The numbers "$m", "n", \cdots" indicate the particle numbers on a site, $\gamma = D_L / D_R$, and

\[
[l] = \frac{1 - \gamma}{1 - \gamma}.
\]

Note that in [28], the reaction rates of eq. (1) are scaled by $D_R$. These processes are obtained by imposing the boundary condition

\[
P_N(\cdots, x_j, x_j - 1, \cdots; t) = D_R P_N(\cdots, x_j - 1, x_j - 1, \cdots; t) + D_L P_N(\cdots, x_j, x_j, \cdots; t) \quad (j = 1, \cdots, N - 1),
\]

on master equation of ASEP:

\[
\begin{aligned}
\frac{\partial}{\partial t} P(x_1, \cdots, x_j, \cdots, x_N; t) &= \sum_{j=1}^{N} D_R P(x_1, \cdots, x_{j-1}, x_j - 1, x_{j+1} \cdots, x_N; t) \\
&\quad + \sum_{j=1}^{N} D_L P(x_1, \cdots, x_{j-1}, x_j + 1, x_{j+1} \cdots, x_N; t) \\
&\quad - NP(x_1, \cdots, x_j, \cdots, x_N; t),
\end{aligned}
\]

in which we have used a time scale so that $D_R + D_L = 1$. In above equation, $P(x_1, \cdots, x_N; t)$ is the probability for finding at time $t$ the particles at sites $x_1, \cdots, x_N$. We take these functions to define probabilities only in the physical
region $x_1 \leq x_2 \leq \cdots \leq x_N$. In fact, in the domain $\Omega_N = x_1 \leq x_2 \leq \cdots \leq x_N \subset \mathbb{Z}^N$, the function $P$ is the probability defined above, whereas in $\mathbb{Z}^N/\Omega_N$ it is defined by the master equation (4), but it is not a probability. The master equation (4) is only valid for $x_i < x_{i+1}$, since for $x_i = x_{i+1}$, there will be terms with $x_i = x_{i+1} + 1$ in the right-hand side of eq. (4), which is out of the physical region. One can, however, assume that (4) is valid for all physical region $x_i \leq x_{i+1}$, and impose certain boundary condition for $x_i = x_{i+1}$. The boundary condition (5) leads to interactions (1).

In this paper we want to study all possible boundary conditions for single-species systems and derive all the integrable one-dimensional reaction-diffusion processes without exclusion which can be obtained by this method. The scheme of the paper is the following. In section 2, we show that there are two types of boundary conditions when $D_R \neq 0$ and $D_L \neq 0$. The first is one considered in [28], i.e. eq. (3), which we call it the type 1 model, and the second one is:

$$D_R P(x, x-1) + D_L P(x+1, x) = P(x, x).$$

Here we have suppressed all the other coordinates for simplicity. We show that the interactions of this type 2 model are

$$mn \rightarrow m-1, n+1 \quad \text{with rate } D_R,$$

$$mn \rightarrow m+1, n-1 \quad \text{with rate } D_L. \quad (7)$$

This is the reactions (1), restricted to $l = 1$. In other words, to have an integrable model, it is not necessary to have the simultaneous hoppings of any number of particles from a common site to the neighboring site (as indicated in (1)), but the one particle hoppings can also lead to integrable models. Note that the reactions (7) is not a subset of processes (1).

In the totally asymmetric case with $D_L = 0$, we show in section 3 that there is a new boundary condition which is a linear combination of boundary conditions (3) and (6), i.e.

$$P(x, x-1) = lP(x, x) + \mu P(x-1, x-1). \quad (8)$$

It is shown that this type 3 model contains the reactions

$$mn \rightarrow m-l, n+l, \quad (9)$$

with rates

$$r_l = \frac{1}{1 + \frac{l}{\mu} + \cdots + \left(\frac{l}{\mu}\right)^{l-1}}. \quad (10)$$

This is an interesting one-parameter family of interactions.

In section 4, we generalize the boundary conditions (3), (6) and (8) to include annihilation to the processes (1), (7) and (9), respectively. The resulting models are rather involved. To be specific, we consider $D_L = 0$ case of (3), and show that the boundary condition

$$P(x, x-1) = \mu P(x-1, x-1) \quad (\mu < 1) \quad (11)$$

3
describes the reactions

\[
\begin{align*}
    mn & \rightarrow m - l, n + l \quad \text{with rate } \mu^{l-1}, \\
    n & \rightarrow \begin{cases} 
        n - 1 \\
        n - 2 \\
        \vdots 
    \end{cases} \quad \text{with total rates } n - \sum_{l=0}^{n-1} \mu^l. 
\end{align*}
\]  

(12)

The second reactions are the annihilation processes. We call this model as the type 4 model. It must be mentioned that we cannot extend our investigation to include the creation processes. The main reason is that if we do so, the evolution equation of n-particle sector will not become closed and will depend on the more-than-n-particle configurations.

The Bethe ansatz solution for different models is discussed in section 5 and the exact \(N\)-point conditional probabilities of type 4 model is obtained in section 6. Some interesting physical quantities are also obtained. Finally we discuss the multi-species extension of these processes in the last section and show that this generalization is not possible for the reactions without exclusion.

\section{Reactions with \(D_L \neq 0\) and \(D_R \neq 0\)}

Consider the master equation (11) for two particle sector, when the particles are at site \(x_1 = x_2 = x\):

\[
\dot{P}(x, x) = D_R P(x-1, x) + D_R P(x, x-1) + D_L P(x+1, x) + D_L P(x, x+1) - 2P(x, x).
\]  

(13)

The second and third probabilities in the right-hand side of the above equation are out of the physical region, and must be defined through some boundary conditions. There are only two possibilities which are consistent in more-than-two-particle sectors. The first one is

\[
\mu' P(x, x - 1) = D_R P(x - 1, x - 1) + D_L P(x, x),
\]  

(14)

and the second one is

\[
D_R P(x, x - 1) + D_L P(x + 1, x) = l' P(x, x).
\]  

(15)

In the first choice, we take any unphysical terms of eq. (13) as a linear combination of physical functions, and in the second choice, we take the whole unphysical terms as a linear combination of the physical probabilities. The right-hand sides of eqs. (14) and (15) are the only allowed combinations which one can write. This was discussed in [25] for ASEP cases. In fact, one can use any other parameters \(\alpha\) and \(\beta\) instead of \(D_R\) and \(D_L\) in eq. (14), with condition \(\alpha + \beta = 1\). Also one can add two other terms \(\mu P(x - 1, x - 1)\) and \(\nu P(x + 1, x + 1)\) to the right-hand side of eq. (14), but it can be shown that for obtaining a consistent description in more-than-two-particle sectors, one must take \(\mu = \nu = 0\) (see section 3 of [25] for more details in ASEP case).

To obtain the range of parameters \(\mu'\) and \(l'\), one can use eq. (11) to show that

\[
\partial_t \sum_{x_2} \sum_{x_1 \leq x_2} P(x_1, x_2; t) = - \sum_x P(x, x; t) + D_R \sum_x P(x, x - 1; t) + D_L \sum_x P(x + 1, x; t)
\]
\[ \sum_{x} P(x, x; t) + \sum_{x} P(x, x - 1; t). \] (16)

For boundary condition (14), eq. (16) results
\[ \partial \phi \partial t \sum_{x_2} \sum_{x_1 \leq x_2} P(x_1, x_2; t) = (10\mu' - 1) \sum_{x} P(x, x; t), \] (17)

and for boundary condition (15), it results
\[ \partial \phi \partial t \sum_{x_2} \sum_{x_1 \leq x_2} P(x_1, x_2; t) = (l' - 1) \sum_{x} P(x, x; t). \] (18)

In the first step, let us exclude the annihilation processes and therefore it is assumed that the number of particles is constant in time. In this way, the eqs. (17) and (18) lead to
\[ \mu' = 1 \quad \text{(for boundary condition (14))}, \]
\[ l' = 1 \quad \text{(for boundary condition (15))}. \] (19)

Eq. (14) exactly becomes the one studied in [28], i.e. eq. (3), and it induces the reactions (1), but the second one, eq. (15), is new. To obtain the reaction introduced by this boundary condition, with \( l' = 1 \), we first consider \( \dot{P}(x, x) \) in eq. (13). Using eqs. (15) and (19), one finds
\[ \dot{P}(x, x) = D_R P(x - 1, x) + D_L P(x, x + 1) - P(x, x), \] (20)

which is the evolution equation of the following two-particle reactions
\[ 10 \rightarrow 01 \text{ with rate } D_R, \]
\[ 01 \rightarrow 10 \text{ with rate } D_L. \] (21)

Remember \( D_R + D_L = 1 \). Generally, for \( n \) particles existing at a common site \( x \), one finds from (1):
\[
\frac{\partial}{\partial t} P(x, \ldots, x) = D_R P(x - 1, x, \ldots, x) + D_L P(x, x, \ldots, x + 1) \\
+ \sum_{j=2}^{n} D_R P(x_1 = x, \ldots, x_{j-1} = x, x_j = x - 1, x_{j+1} = x \ldots, x_n = x) \\
+ \sum_{j=2}^{n} D_L P(x_1 = x, \ldots, x_{j-2} = x, x_{j-1} = x + 1, x_j = x, \ldots, x_n = x) \\
- nP(x, \ldots, x) \\
= D_R P(x - 1, \ldots, \ldots, x) + D_L P(x, \ldots, x + 1) - P(x, \ldots, x), \] (22)

which is obviously the evolution equation of reactions (7).

### 3 Totally asymmetric diffusion with \( D_L = 0 \)

In \( D_L = 0 \), the boundary condition (8) and (9) becomes
\[ P(x, x - 1) = P(x - 1, x - 1), \]
\[ P(x, x - 1) = P(x, x), \] (23)

respectively, which describe the interactions

\[ mn \rightarrow m - l, n + l \quad \text{with rate 1} \] (24)

and

\[ mn \rightarrow m - 1, n + 1 \quad \text{with rate 1}, \] (25)

respectively. But as was first noted in [22] for ASEP, the linear combination of these two boundary conditions may result in a new integrable model. So we consider

\[ P(x, x - 1) = lP(x, x) + \mu P(x - 1, x - 1), \] (26)

as the boundary condition for the following master equation

\[
\frac{\partial}{\partial t} P(x_1, \ldots, x_N; t) = \sum_{j=1}^{N} \left[ P(x_1, \ldots, x_{j-1}, x_j - 1, x_{j+1} \ldots, x_N; t) - P(x_1, \ldots, x_N; t) \right].
\] (27)

First we note that

\[
\frac{\partial}{\partial t} \sum_{x_2} \sum_{x_1 \leq x_2} P(x_1, x_2; t) = (l + \mu - 1) \sum_x P(x, t),
\] (28)

from which the conservation of number of particles results

\[ l + \mu = 1. \] (29)

To obtain the resulting reactions, besides the diffusion 10 \rightarrow 01 with rate 1, we first consider \( \dot{P}(x, x) \). Using (26) and (27), it is found

\[
\dot{P}(x, x) = P(x - 1, x) + \mu P(x - 1, x - 1) - (2 - l)P(x, x),
\] (30)

which represents the reaction 20 \rightarrow 02, with rate \( \mu = 1 - l \), as the source and sink of this state. To find the reactions in general case, we first prove a lemma.

**Lemma:** Equation (26) implies, for arbitrary \( n \), the following

\[
P(x, \ldots, x, x - 1) = r_{n+1} P(x - 1, \ldots, x - 1) + (1 - r_{n+1}) P(x, \ldots, x),
\] (31)

with \( r_n \) defined through (10).

**Proof:** We proceed by induction. For \( n = 1 \), (31) reduces to (26). Assuming (31) is correct for \( n - 1 \), then using (26), we have

\[
P(x, \ldots, x, x - 1) = lP(x, \ldots, x) + \mu P(x, \ldots, x, x - 1, x - 1)
\]

\[
= lP(x, \ldots, x) + \mu [r_n P(x - 1, \ldots, x - 1) + (1 - r_n) P(x, \ldots, x, x - 1)],
\] (32)

so

\[
P(x, \ldots, x, x - 1) = r_{n+1} P(x - 1, \ldots, x - 1) + s_{n+1} P(x, \ldots, x),
\] (33)
where
\[ r_{n+1} = \mu r_n \delta 1 - \mu (1 - r_n), \quad s_{n+1} = \lambda \delta 1 - \mu (1 - r_n). \] (34)

It is seen that \( r_{n+1} + s_{n+1} = 1 \). The value of \( r_n \) can be found by solving the first equation of (34), which can be written as
\[ r_n^{-1} = 1 + \lambda \delta r_n^{-1}. \] (35)

Using \( r_n^{-1} = 1 + \lambda \delta \mu \), (34) leads to eq. (10) for \( r_n \)'s. This proves the lemma. ■

We now consider the evolution of \( P_{\cdots}(x) \). Using (27) and (31), we find
\[ \frac{\partial}{\partial t} P_{\cdots}(x) = \sum_{j=1}^{n} P(x_1 = x, \cdots, x_{j-1} = x, x_j = x - 1, x_{j+1} = x \cdots, x_n = x) - n P_{\cdots}(x) \]
\[ = \sum_{j=1}^{n} [r_j P(x_1 = x - 1, \cdots, x_j = x - 1, x_{j+1} = x \cdots, x_n = x) + (1 - r_j) P_{\cdots}(x)] 
- n P_{\cdots}(x) \]
\[ = \sum_{j=1}^{n} r_j P(x_1 = x - 1, \cdots, x_j = x - 1, x_{j+1} = x \cdots, x_n = x) - \sum_{j=1}^{n} r_j P_{\cdots}(x). \] (36)

This equation shows that a collection of \( j \) particles hops from site \( x - 1 \) to site \( x \) with rate \( r_j \), and this proves that the induced reactions of the boundary condition (8) are those indicated in (9).

At \( l = 1 \) (\( \mu = 0 \)), eq.(10) results \( r_l = \delta \delta_1 \). So the reactions are
\[ mn \rightarrow m - 1, n + 1, \quad \text{with rate } 1. \] (37)

This is nothing but eq.(25). At \( l = 0 \) (\( \mu = 1 \)), eq.(10) gives \( r_l = 1 \) for all \( ls \). So the reactions are those shown in (24). In fact, at \( l = 0 \), all the multi-particle hoppings occur with equal rate 1. By increasing \( l \), the greater number of simultaneous hoppings happen with lower rates, until at \( l = 1 \), in which only the one-particle hopping is allowed.

### 4 Annihilation-diffusion processes

Adding annihilation to the previous reactions, results in the decreasing of the number of particles with time. Therefore the parameters \( \mu' \), \( l' \) and \( (l, \mu) \) in eqs.(14), (15) and (26) become
\[ \mu' > 1 \quad \text{(for boundary condition (15))}, \]
\[ l' < 1 \quad \text{(for boundary condition (15))}, \]
\[ \mu + l < 1 \quad \text{(for boundary condition (26))}. \] (38)

These are obtained from eqs.(17), (18) and (28), respectively. Note that the annihilation interactions only appear in sink terms of the master equation, since
In this case, we have a two-parameter family of interactions.

To obtain the corresponding interactions of either of boundary conditions, we begin with (15) with \( l' < 1 \). Considering \( \dot{P}(x, x) \), one finds

\[
\dot{P}(x, x) = D_R P(x - 1, x) + D_L P(x, x + 1) - (2 - l') P(x, x),
\]

(39)

which shows the annihilation rate \( 1 - l' \) for the state of two particle at a common site. In general case, one finds, similar to eq.(22):

\[
\frac{\partial}{\partial t} \underbrace{P(x, \ldots, x)}_n = D_R \underbrace{P(x-1, x, \ldots, x)}_{n-1} + D_L \underbrace{P(x, \ldots, x, x+1)}_{n-1} - \left[ n-(n-1)l' \right] \underbrace{P(x, \ldots, x)}_n.
\]

(40)

which is the evolution equation of the following reactions:

\[
m n \rightarrow m - 1, n + 1 \quad \text{with rate} \quad D_R,
\]

\[
m n \rightarrow m + 1, n - 1 \quad \text{with rate} \quad D_L,
\]

\[
n \rightarrow \begin{cases} 
  n - 1 \\
  n - 2 \\
  \vdots 
\end{cases} \quad \text{with total rates} \quad (n - 1)(1 - l').
\]

(41)

In the case of boundary condition (26) with \( l + \mu < 1 \), it is not easy to obtain a compact form for interaction rates in general case. For example in two- and three-particle sectors, considering \( \dot{P}(x, x) \) and \( \dot{P}(x, x, x) \), one finds

\[
m n \rightarrow m - 1, n + 1 \quad \text{with rate} \quad 1,
\]

\[
m n \rightarrow m - 2, n + 2 \quad \text{with rate} \quad \mu,
\]

\[
m n \rightarrow m - 3, n + 3 \quad \text{with rate} \quad \mu^2 \alpha 1 - l \mu,
\]

\[
2 \rightarrow 1 \quad \text{with rate} \quad \alpha = 1 - (l + \mu),
\]

\[
3 \rightarrow \begin{cases} 
  2 \\
  1 
\end{cases} \quad \text{with total rates} \quad \alpha(\mu^2 + \alpha \mu + 2) \alpha 1 - l \mu.
\]

(42)

In this case, we have a two-parameter family of interactions.

When we consider the boundary condition (14) with \( \mu' > 1 \), again we can not find the compact relations and it is better to restrict ourselves to the subset of totally asymmetric reactions with \( D_L = 0 \). This means that we take

\[
P(x, x - 1) = \mu P(x - 1, x - 1) \quad (\mu < 1)
\]

(43)

as the boundary condition (\( \mu = 1/\mu' \)). Considering the general case \( P(x, \cdots, x) \), results

\[
\frac{\partial}{\partial t} \underbrace{P(x, \ldots, x)}_n = \sum_{j=1}^n P(x_1 = x, \cdots, x_{j-1} = x, x_j = x - 1, x_{j+1} = x \cdots, x_n = x) - nP(x, \cdots, x)
\]

\[
= \sum_{j=1}^n \mu^{j-1} P(x_1 = x - 1, \cdots, x_j = x - 1, x_{j+1} = x \cdots, x_n = x) - nP(x, \cdots, x).
\]

(44)

This equation shows that the sources are the simultaneous hoppings of \( j \) particles \((j = 1, \cdots, n)\) from the common site \( x - 1 \) to \( x \), with rates \( \mu^{j-1} \). So the sinks are also these hoppings and the remaining rate of the sink terms in eq.(44), i.e. \( n - \sum_{j=1}^{n-1} \mu^j \), is due to the annihilations of these \( n \) particles at site \( x \). So, it is proved that the boundary condition (43) induces the reactions (12).
5  Bethe ansatz solution

Now we try to solve the resulting evolution equations, in all the discussed cases, by the following Bethe ansatz

\[ P(x; t) = e^{-ENt}\psi(x), \quad (45) \]

where \( x = (x_1, \cdots, x_N) \), and

\[ \psi(x) = \sum_\sigma A_\sigma e^{i\sigma(p).x}. \quad (46) \]

The summation runs over the elements of permutation group of \( N \) object. Inserting (46) in master equation (4), results

\[ E_N = \sum_{k=1}^{N} \left( 1 - D_RE^{-ip_k} - D_L e^{ip_k} \right). \quad (47) \]

To determine \( A_\sigma \), we must insert the expression (46) in the boundary conditions. For type 2 model, for instance, we must use eq.(6), which results

\[ D_R \psi(\cdots, x_i = x, x_{i+1} = x-1, \cdots) + D_L \psi(\cdots, x_i = x+1, x_{i+1} = x, \cdots) = \psi(\cdots, x_i = x, x_{i+1} = x, \cdots), \quad (48) \]

and using (46), gives

\[ \left[ D_RE^{-i\sigma(p_{k+1})} + D_L e^{i\sigma(p_k)} - 1 \right] A_\sigma + \left[ D_RE^{-i\sigma(p_k)} + D_L e^{i\sigma(p_{k+1})} - 1 \right] A_\sigma_{\sigma_k} = 0. \quad (49) \]

\( \sigma_k \) is an element of permutation group which only interchanges \( p_k \) and \( p_{k+1} \).

\[ \sigma_k : (p_1, \cdots, p_k, p_{k+1}, \cdots, p_N) \rightarrow (p_1, \cdots, p_{k+1}, p_k, \cdots, p_N). \quad (50) \]

Eq.(49) gives \( A_{\sigma_\sigma_k} \) in terms of \( A_\sigma \) as following

\[ A_{\sigma_\sigma_k} = S^{(2)}(\sigma(p_k), \sigma(p_{k+1}))A_\sigma \quad (51) \]

where

\[ S^{(2)}(z_1, z_2) = -D_R z_2 + D_L z_1^{-1} - l_0 D_R z_1 + D_L z_2^{-1} - 1, \quad (52) \]

where \( z_k = e^{-ip_k} \). Eq.(51) allows one to compute all \( A_\sigma \)'s in terms of \( A_1 \), which is set to unity.

The same procedure can be applied to other boundary conditions. For example for type 3 model, boundary condition (8), one finds

\[ S^{(3)}(z_1, z_2) = -l + \mu(z_1 + z_2) - z_2 d + \mu(z_1 + z_2) - z_1, \quad (53) \]

and for boundary condition (15), with \( l' < 1 \),

\[ S(z_1, z_2) = -D_R z_2 + D_L z_1^{-1} - l' D_R z_1 + D_L z_2^{-1} - l'. \quad (54) \]

These solutions can be used, in principle, to calculate the conditional probabilities \( P(x_1, \cdots, x_N; t; y_1, \cdots, y_N; 0) \). This is the probability of finding the particles...
at time $t$ at sites $x_1, \ldots, x_N$ if at $t = 0$, they were at sites $y_1, \ldots, y_N$, respectively. But unfortunately, the standard method, used for example in [21–25], can not be used here. This is because the initial condition
\begin{equation}
P(x; 0 | y; 0) = \delta_{x,y},
\end{equation}
is satisfied by standard expression
\begin{equation}
P(x; t | y; 0) = \int d^N p \phi(2\pi)^N e^{-E_N t} e^{-i p \cdot y} \psi(x),
\end{equation}
only when $y_i < y_{i+1}$ and $x_i < x_{i+1}$. This condition only satisfied by exclusion processes and for processes without exclusion, we must look for other methods.

6 \hspace{1em} N\text{-particle conditional probabilities}

In some special cases, it is possible to calculate the conditional probabilities in terms of a specific determinant. This was first proved in [21], and then used for other cases in [22, 23]. Now we want to check that this method does work here, and as a specific example, we consider the type 4 model with the boundary condition (11).

We set the following ansatz for $N$-particle conditional probabilities
\begin{equation}
P(x; t | y; 0) = e^{-N t} \det [G(x; t | y; 0)],
\end{equation}
where $G$ is a $N \times N$ matrix with elements
\begin{equation}
G_{ij}(x; t | y; 0) = g_{i-j}(x_i - y_j; t).
\end{equation}
Inserting (57) in eqs. (27) and (11), results
\begin{equation}
\partial \phi \partial t G_i(x; t) = G_i(x-1; t),
\end{equation}
\begin{equation}
G_{i-1}(x; t) = \mu G_{i-1}(x-1; t) + \beta G_i(x-1; t),
\end{equation}
where $G_i$ denotes the $i$-th row of matrix $G$, and $\beta$ is an arbitrary constant. In terms of functions $g_p(n; t)$, (59) becomes
\begin{equation}
\partial \phi \partial t g_p(n; t) = g_p(n-1; t),
\end{equation}
\begin{equation}
g_p(n; t) = \mu g_p(n-1; t) + \beta g_{p+1}(n-1; t).
\end{equation}
Introducing the $z$-transform
\begin{equation}
\tilde{g}_p(z, t) = \sum_{n=-\infty}^{\infty} z^n g_p(n; t),
\end{equation}
we find
\begin{equation}
\partial \phi \partial t \tilde{g}_p(z, t) = z \tilde{g}_p(z, t) \Rightarrow \tilde{g}_p(z, t) = e^{zt} \tilde{g}_p(z, 0),
\end{equation}
\begin{equation}
\tilde{g}_{p+1}(z, t) = 1 - \mu z \beta z \tilde{g}_p(z, t).
\end{equation}
The second equation yields
\begin{equation}
\tilde{g}_p(z, 0) = (1 - \mu z \beta z)^p \tilde{g}_0(z, 0).
\end{equation}
Using $P(x; 0|y; 0) = g_0(x - y; 0) = \delta_{x,y}$, one finds $\tilde{g}_0(z, 0) = 1$. Finally

$$\tilde{g}_p(z, t) = e^{zt} (1 - \mu z \phi(z))^p,$$  \hspace{1cm} (64)

in which we choose $\beta = 1$. The functions $g_p(n; t)$ can be obtained by expanding the generating functions $\tilde{g}_p(z, t)$. For $p \geq 0$, the expansion yields

$$g_p(n; t) = \sum_{m=0}^{p} \binom{p}{m} (-\mu)^m t^{n+p-m} \phi(n+p-m)!,$$  \hspace{1cm} (65)

and for negative $p$, it yields

$$g_{-|p|}(n; t) = \sum_{m=0}^{-|p|} \binom{|p|+m-1}{m} \mu^m t^{n-|p|-m} \phi(n-|p|-m)!.$$  \hspace{1cm} (66)

We have thus obtained the explicit relation for conditional probabilities.

It can be checked that the resulting function satisfies the desired initial condition. At $t = 0$, where $x_i = y_i$, we have $g_{-|p|}(0; t) = 0$ and $g_0(n; 0) = \delta_{n,0}$, which result $P(x; 0|x; 0) = 1$. Also it may be interesting to obtain the rate of decay of a delta function distribution. Suppose at $t = 0$, there are $N$ particles at the same site $y$. We want to obtain the probability of finding all the particles at their initial positions at later time $t$, i.e. $P(y; t|y; 0)$, where $y = (y, \cdots, y)$. Using $g_{-|p|}(0; t) = 0$ and $g_0(0; t) = 1$, we obtain

$$P(y; t|y; 0) = e^{-\mu t} \det[G(y; t|y; 0)] = e^{-\lambda t},$$  \hspace{1cm} (67)

which is independent of $\mu$! It is an interesting result. At $\mu = 1$, where there is no annihilation, the rate of simultaneous hoppings of particles are all 1, irrespective of the number of particles. By decreasing $\mu$, the rates of simultaneous hoppings decrease with increasing the number of particles (see eq.(12)). At the same time, the rate of annihilation increases in such a way that the total decaying rate remains constant.

The above determinant method can be applied to any model which its boundary condition equation contains only two terms. Otherwise it does not lead to the consistent relations for matrix elements $g_p(n; t)$’s. So for $D_R = 0$ or $D_L = 0$ cases of boundary conditions (3) and (6), and $\lambda = 0$ or $\mu = 0$ cases of boundary condition (5), this method leads to explicit expressions for $N$-particle conditional probabilities.

7 Conclusion

In the previous sections, we use all the allowed generalization of boundary condition in the asymmetric reaction-diffusion processes without exclusion, to obtain several new integrable models for one-species cases. The further natural generalization, which has been discussed in many papers for the ASEP case, is the multi-species extensions of these models. In multi-species studies, one considers a $p$-species system with particles $A_1, \cdots, A_p$. The basic objects are the probabilities $P_{\alpha_1, \cdots, \alpha_N}(x_1, \cdots, x_N; t)$ for finding at time $t$ the particles of type $\alpha_1$ at
The master equation, instead of (4), is
\[
\frac{\partial}{\partial t} P_{\alpha_1\cdots\alpha_N}(x_1,\cdots,x_j,\cdots,x_N; t) = \sum_{j=1}^{N} D_R P_{\alpha_1\cdots\alpha_N}(x_1,\cdots,x_{j-1},x_j-1,x_{j+1}\cdots,x_N; t) \\
+ \sum_{j=1}^{N} D_L P_{\alpha_1\cdots\alpha_N}(x_1,\cdots,x_{j-1},x_j+1,x_{j+1}\cdots,x_N; t) \\
- NP_{\alpha_1\cdots\alpha_N}(x_1,\cdots,x_j,\cdots,x_N; t).
\]
(68)

Now if we want to use this equation for reactions without exclusion, the problem will arise when some of the particles are in the same position. Consider, for example, the two-species case \( P_{\alpha_1\alpha_2}(x, x) \). It is seen that eq.(68) has the term \( P_{\alpha_1\alpha_2}(x-1, x) \) as the source term of the desired state, but does not contain \( P_{\alpha_2\alpha_1}(x-1, x) \), which is as important as the first term. In fact, this is the source of many difficulties arise in multi-species extension of reaction-diffusion processes which have not exclusion. So it seems that the integrable models discussed in the previous sections are all that one can obtain in this context.

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