Semiclassical approach to $S$ matrix energy correlations and time delay in chaotic systems

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(Dated: February 18, 2022)

The $M$-dimensional scattering matrix $S(E)$ which connects incoming to outgoing waves in a chaotic system is always unitary, but shows complicated dependence on the energy. This is partly encoded in correlators constructed from traces of powers of $S(E + \epsilon)S^\dagger(E - \epsilon)$, averaged over $E$, and by the statistical properties of the time delay operator, $Q(E) = -i\hbar S^\dagger dS/dE$. Using a semiclassical approach for systems with broken time reversal symmetry, we derive two kind of expressions for the energy correlators: one as a power series in $1/M$ whose coefficients are rational functions of $\epsilon$, and another as a power series in $\epsilon$ whose coefficients are rational functions of $M$. From the latter we extract an explicit formula for $\text{Tr}(Q^n)$ which is valid for all $n$ and is in agreement with random matrix theory predictions.

I. INTRODUCTION

Scattering of waves of energy $E$ can be described by the $S(E)$ matrix, which connects incoming to outgoing amplitudes. We consider a finite region with chaotic classical dynamics, characterized by a single time scale $\tau_D$, the dwell time, the average amount of time spent inside the region by a classical particle injected at random. This chaotic region is connected to the outside world by means of $M$ channels, so that $S$ is $M$-dimensional and always unitary as a consequence of the energy conservation.

If time-reversal symmetry is broken, one statistical approach, random matrix theory (RMT), assumes $S(E)$ to be uniformly distributed in the unitary group $U[M]$, according to the invariant Haar measure, for every $E$. To understand the correlations between $S$ matrices at different energies has always been a challenge. One way to quantify this is to compute $S$ at one energy and $S^\dagger$ at another, and take the trace of their product, \[ \text{Tr} \left[ S \left( E + \frac{i\hbar}{2\tau_D} \right) S^\dagger \left( E - \frac{i\hbar}{2\tau_D} \right) \right]. \] This will be equal to $M$ for $\epsilon = 0$, but in general a widely fluctuating function of $E$. Averaging within a local energy window produces a well behaved function of $\epsilon$. Such energy correlations have traditionally been studied by modelling the Hamiltonian of the system as a random hermitian matrix coupled to scattering channels.

A more detailed characterization of energy correlations is the calculation of

$$ C_n(M, \epsilon) = \left\langle \text{Tr} \left[ S \left( E + \epsilon' \right) S^\dagger \left( E - \epsilon' \right) \right]^n \right}\rangle $$

for integer $n$, where $\epsilon' = \frac{i\hbar}{2\tau_D}$. The above quantity is expected to be universal, i.e. independent of the system’s details as long as it is chaotic. Besides $M$ and $\epsilon$, it should depend only on whether time-reversal symmetry is present or not. In this work we focus our attention on systems where this symmetry is broken.

Related to energy dependence of the $S$ matrix is the time delay matrix $Q(E)$

$$ Q(E) = -i\hbar S^\dagger \frac{dS}{dE}. \quad (2) $$

Its real eigenvalues $\{\tau_1, ..., \tau_M\}$ are commonly referred to as proper time delays and provide the lifetimes of metastable states. Its normalized trace $\tau_W = \frac{1}{M} \text{Tr}(Q)$ is known as the Wigner time delay, which provides a measure of the density of states of the open system. Its average value equals the classical dwell time, $\langle \tau_W \rangle = \tau_D$. More detailed information is encoded in higher spectral moments such as

$$ Q_n = \langle \text{Tr}(Q^n) \rangle. \quad (3) $$

The statistical properties of time delay have been much studied. Within RMT, perhaps the main point of departure is the distribution of the inverse matrix $Q^{-1}$, which is known to conform to the Laguerre ensemble. This lead to the calculation of the distribution function of $\tau_W$ in different regimes and to expressions for the above spectral moments (see the review).

In this work we do not rely on random matrices, but instead employ a semiclassical approach, in which the elements of $S$ are approximated, in the short-wavelength regime, as infinite sums over scattering rays. It has been very successful in treating transport properties at fixed energy. It was adapted by Kuipers and Sieber in order to take into account the variable $\epsilon$ and handle correlators like $\langle \tau_W \rangle$. It has grown into an independent line of attack to this kind of problems.

We follow recent advances in the semiclassical theory and formulate correlation functions in terms of auxiliary matrix integrals. These integrals are then computed using Schur polynomials. This leads to two explicit formulas for $C_n(M, \epsilon)$: one as a power series in $1/M$ whose coefficients are rational functions of $\epsilon$, and another as a power series in $\epsilon$ whose coefficients are rational functions of $M$. From the latter we extract an explicit formula for $\text{Tr}(Q^n)$ which is valid for arbitrary values of $n$ and $M$ and which is in agreement with random matrix theory predictions.

In Section 2 we present the semiclassical matrix integral which is the crux of the theory. In Sections 3 and 4 we use it to compute $C_n(M, \epsilon)$ in two different ways. In Section 5 we make the connection with $Q_n$. We conclude in Section 6.
II. SEMICLASSICAL MATRIX INTEGRALS

The semiclassical approximation to quantum scattering has been extensively discussed in previous works [27, 28, 31, 43]. When correlations among scattering trajectories are taken into account, and the required integrations over phase space have been performed, the theory has a diagrammatic formulation which is a perturbative theory in the parameter $M^{-1}$. Kuipers and Sieber obtained the diagrammatic rules governing this theory [32, 36] when applied to [11]. The contribution of any given diagram factorizes into the contributions of individual vertices and edges: a vertex of valence 2 gives rise to $-M(1-i\epsilon)$; channels of any valence give rise to $M$; each edge gives rise to $[M(1-i\epsilon)]^{-1}$.

Recently, the semiclassical approach has been developed in terms of appropriate matrix integrals [34, 43–45] into which the diagrammatic rules are built by design. For systems with broken time-reversal symmetry, which are our focus, the result is that

$$C_n = \lim_{N \to 0} \int e^{-\sum_{q=1}^{\infty} \frac{1}{4q}(1-i\epsilon)Tr(Z^q)^n} \text{Tr}[ZPZP]^n \frac{dZ}{Z},$$

(4)

where $Z$ is an $N$-dimensional complex matrix, $P$ is an orthogonal projector from $\mathbb{R}^N$ to $\mathbb{R}^M$ and

$$Z = \int e^{-M(1-i\epsilon)Tr(Z^2)}dZ$$

(5)
is a normalization.

The way this matrix model works is that the factor $e^{-M(1-i\epsilon)\text{Tr}(Z^2)}$ is kept as a Gaussian measure while the rest of the exponential is Taylor expanded. Each trace then becomes a vertex in a diagram, along with the correct factor $-M(1-i\epsilon)$. Then the integration is performed by invoking Wick’s rule, and edges are produced along with the correct factor $[M(1-i\epsilon)]^{-1}$. The term $\text{Tr}[ZPZP]^n$ mimicks the correlator we want to compute. Finally, the limit $N \to 0$ is necessary to remove spurious contributions coming from unwanted periodic orbits [32].

The traditional singular value decomposition $Z = UDV^\dagger$, where $U$ and $V$ are unitary matrices, leads to $Z = \mathcal{G} \int e^{-M(1-i\epsilon)\text{Tr}(X)^n} |\Delta(X)|^2dX$, where the Vandermonde

$$\Delta(X) = \prod_{1 \leq i < j \leq N} (x_j - x_i)$$

(6)
is the jacobian of the change of variables and $\mathcal{G} = \int dUdV$ is the result of a double integration over the unitary group. This integral gives

$$Z = \mathcal{G}(M(1-i\epsilon))^{-N^2} \prod_{j=1}^{N} j!(j-1)!.$$  

(7)

Let $\chi_\lambda(\mu)$ be the characters of the irreducible representations of the permutation group $S_n$ (these representations are labelled by integer partitions, denoted by $\lambda \vdash n$ or $|\lambda| = n$). They are useful in expressing the trace in terms of Schur polynomials,

$$\text{Tr}(A^n) = \sum_{\lambda \vdash n} \chi_\lambda(n)s_\lambda(A).$$

(8)

Using this fact and the identity

$$\frac{1}{\mathcal{Z}} \int dUdV s_\lambda(UAU^\dagger B) = \frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(1^n)}$$

(9)

we get

$$C_n(M, \epsilon) = \lim_{N \to 0} \sum_{\lambda \vdash n} \left( \frac{s_\lambda(1^M)}{s_\lambda(1^n)} \right)^2 \chi_\lambda(n) \mathcal{I}_\lambda,$$

(10)

The value of the Schur polynomial at an identity matrix is

$$s_\lambda(1^n) = \frac{d_\lambda}{m!}[N]^\lambda,$$

(11)

where $d_\lambda$ is the dimension of the associated irreducible representation, given by

$$\chi_\lambda(1^n) = n! \prod_{i=1}^{\ell(\lambda)} (\lambda_i - i + \ell)! \prod_{j=1}^{\ell(\lambda)} (\lambda_j - j - \lambda + i),$$

(12)

and $[N]^\lambda$ is a monic polynomial in $N$,

$$[N]^\lambda = \prod_{j=1}^{\ell(\lambda)} \frac{(N + \lambda_j - j)!}{(N - j)!},$$

(13)

which is a generalization of the rising factorial. For future reference, let us also define a corresponding generalization of the falling factorial,

$$[N]_{\lambda} = \prod_{j=1}^{\ell(\lambda)} \frac{(N - \lambda_j + j)!}{(N + j)!}.$$  

(14)

Therefore,

$$C_n(M, \epsilon) = \lim_{N \to 0} \sum_{\lambda \vdash n} \left( \frac{[N]^\lambda}{[N]_{\lambda}} \right)^2 \chi_\lambda(n)$$

(15)

with

$$\mathcal{I}_\lambda = \frac{\mathcal{G}}{\mathcal{Z}} \int e^{-M \sum_{q=1}^{\infty} \frac{1}{4q}(1-i\epsilon)\text{Tr}(X)^n} s_\lambda(X)dX.$$  

(16)

It is known that $\chi_\lambda(n$) is different from zero only if $\lambda = (n-k, k^s)$, a so-called hook partition. In that case, $\chi_\lambda(n) = (-1)^s$ and $d_\lambda = (n-k)!$. We also define the quantity

$$t_\lambda = (n-k)!k!.$$  

(17)

We denote by $H_n$ the set of all hook partitions of $n$. For example, $H_4 = \{(4), (3, 1), (2, 1, 1), (1^4)\}$. 


III. CORRELATOR AS POWER SERIES IN 1/M

Let $b_\beta$ be the size of the conjugacy class of the permutation group containing permutations of cycle type $\beta$ and let us define the function
\[
g_\beta(e) = \prod_{\alpha \in \beta} (1 - i\alpha e).
\] (18)

Then we can expand $e^{-\sum_{\gamma} \frac{1}{2\gamma^2}(1-\gamma e)\text{Tr}(X^\gamma)}$ as
\[
\sum_{m} \sum_{\beta \vdash m} \frac{1}{m!} b_\beta(-M)^{\ell(\beta)} g_\beta(e) p_\beta(X),
\] (19)

where
\[
p_\beta(X) = \prod_{j=1}^{N} \sum_{i=1}^{\beta_j} x_i^{\beta_j}
\] (20)
is a power sum symmetric polynomial. In the sum the term $m = 1$ is excluded, and the partition $\beta$ has no parts equal to 1.

Next, we write $p_\beta(X) = \sum_\nu \chi_\nu(\beta)s_\nu(X)$ and then join this Schur polynomial with the one already in the integrand, according to
\[
s_\nu(X)s_\lambda(X) = \sum_\nu \epsilon_{\lambda\nu} c^\nu_{\lambda\nu} s_\nu,
\] (21)

where $c_{\lambda\nu}^\nu$ are the Littlewood-Richardson coefficients. The integral to be done is then
\[
\frac{G}{Z} \int e^{-M(1-i\epsilon)\text{Tr}(X)}|\Delta(X)|^2 s_\nu(X) dX.
\] (22)

But this is an integral of Selberg type, and is given by
\[
d_\nu|N\nu|^2 |M(1-i\epsilon)|^{N^2-|\nu|}.
\] (23)

The limit $N \to 0$ can be taken by noticing that, since $\lambda$ is a hook, we have
\[
|N|^\lambda = Nt_\lambda + O(N^2).
\] (24)

This means only partitions $\nu$ that are also hooks will contribute and we get
\[
C_n = \sum_{\lambda \in H_n} \frac{\chi_\lambda(n)}{t_\lambda^\lambda} (|M|^\lambda)^2 B_\lambda,
\] (25)

where
\[
B_\lambda = \sum_{m \beta \vdash m} \frac{b_\beta(-M)^{\ell(\beta)} g_\beta(e)(n + m - 1)!}{m!(n + m)! |M(1-i\epsilon)|^{n+m}} D_{\lambda\beta},
\] (26)

with
\[
D_{\lambda\beta} = \sum_{\rho \nu} \chi_\nu(\beta) c_{\lambda\nu}^\nu d_\nu.
\] (27)

For a given pair of hooks, $\lambda, \nu$, there are two different $\rho$ for which $c_{\lambda\rho}^\nu$ is not zero. If $\lambda = (n-k, 1^k)$ and $\nu = (n+m-r, 1^r)$, then $\rho_1 = (m+k-r, 1^{r-k})$ and $\rho_2 = (m+k-r+1, 1^{r-k-1})$. We thus have the sum
\[
\sum_{\rho \in H_m} \chi_\rho(\beta) c_{\lambda\rho}^\nu = \chi_{\rho_1}(\beta) + \chi_{\rho_2}(\beta).
\] (28)

It is a standard fact from representation theory that the restriction from $S_{n+1}$ to $S_n$ of $\lambda_{1^n}$ is the sum of $\chi_\rho$ over all partitions $\rho$ that result from the Young diagram of $\lambda$ by removing a box. Hence, the above sum equals $\chi_{\omega}(\beta, 1)$ with $\omega = (m+k-r+1, 1^{r-k})$.

We now have to compute
\[
\sum_{\nu \in H_{n+m}} \chi_{\nu}(\beta, 1) x^r = \sum_{r=0}^{n+m-1} \chi_{\omega}(\beta, 1)x^r,
\] (31)

where $x = u/(1-u)$. But the characters $\chi_\omega$ have been studied and it turns out that
\[
\sum_{r=0}^{n+m-1} \chi_{\omega}(\beta, 1)x^r = xf_\beta(x),
\] (32)

where
\[
f_\beta(x) = \prod_{\gamma \in \beta} [1 - (-x)^\gamma].
\] (33)

Hence,
\[
\sum_{\nu} \chi_{\nu}(n+m) = \int_0^1 u^k (1-u)^{n+m-k-1} f_\beta \left( \frac{u}{1-u} \right) du.
\] (34)

Finally,
\[
B_\lambda = \sum_{m \beta \vdash m} \frac{b_\beta(-M)^{\ell(\beta)} g_\beta(e)(n + m - 1)!}{m!(n + m)! |M(1-i\epsilon)|^{n+m}} F_{n,m,k}(\beta)
\] (35)

where
\[
F_{n,m,k}(\beta) = \int_0^1 u^k (1-u)^{n+m-k-1} f_\beta \left( \frac{u}{1-u} \right) du.
\] (36)

Since $|M|^\lambda = (M-k)^n$ we get
\[ C_n = \sum_{k=0}^{n-1} \frac{(-1)^k}{t^k_\lambda} ((M - k)^{(n)})^2 \sum_{\beta+m} b_\beta(-M)^{(\beta)} q_\beta(\epsilon)(n + m - 1)! \frac{m!}{M(1 - i\epsilon)^{n+m}} F_{n,m,k}(\beta). \]  

which gives
\[ I_{\lambda\mu} = N! \frac{G}{Z} \det \left[ \int_0^1 (1 - x)^{M-\mu_j+j-1} x^{2N+\mu_j-j+\lambda_i-1} dx \right], \]

or
\[ I_{\lambda\mu} = N! \frac{G}{Z} \prod_{j=1}^N \frac{(M - \mu_j + j - 1)!}{(M + 2N + \lambda_j - j)!} \times \det [(2N + \mu_j - j + \lambda_i - i)!]. \]

The above determinant can be computed resorting again to the Andrieu identity,
\[ \det [(2N + \mu_j - j + \lambda_i - i)!] \]
\[ = \det \left[ \int_0^\infty x^{2N+\mu_j-j+\lambda_i-i} e^{-x} dx \right] \]
\[ = \frac{1}{N!} \int_0^\infty \det [x^{N+\lambda_i-i}] \det [x_i^{N+\mu_j-j}] e^{-\text{Tr}(X)} dX \]
\[ = \frac{1}{N!} \int_0^\infty s_\lambda(X) s_\mu(X) |\Delta(X)|^2 e^{-\text{Tr}(X)} dX \]
\[ = \frac{1}{N!} \sum_{\nu} c^{\nu}_{\lambda\mu} \frac{d_\nu}{\mu!} (|N|\nu)^2. \]

When we take \( N \to 0 \), we get that \( \nu \) must be a hook, as well as \( \mu \). We also recognize in \( \text{(46)} \) an expression for the generalized falling factorial, so that
\[ C_n = \sum_{\lambda \in H_n} \chi_\lambda(n) \frac{[M]^\lambda}{[\lambda]_1^t} B_\lambda, \]

with
\[ B_\lambda = \sum_{m=0}^\infty \frac{(iMe)^m}{m!} \sum_{\mu \in H_m} \frac{d_\mu}{\mu!} \frac{(n + m - 1)!}{(n + m)!} \sum_{\nu} c^{\nu}_{\lambda\mu} \frac{d_\nu}{\nu!}. \]

When \( \lambda = (n - k, 1^k) \) and \( \mu = (m - l, 1^l) \), the coefficient \( c^{\nu}_{\lambda\mu} = 1 \) if and only if \( \nu = (n + m - k - l, 1^{k+l}) \) or \( \nu = (n + m - k - l - 1, 1^{k+l+1}) \). Hence,
\[ \sum_{\nu \in H_{n+m}} c^{\nu}_{\lambda\mu} \frac{d_\nu}{\nu!} = \frac{(n + m - k - l - 2)! (k + l)! (n + m)}{(n + m - 1)!} = \frac{(n + m)}{(n + m - 1)!} t_{\lambda\circ\mu}, \]

where \( \lambda \circ \mu = (n + m - k - l - 1, 1^{k+l}). \)

Finally,
\[ C_n = \sum_{m=0}^\infty \frac{(iMe)^m}{m!} \sum_{\lambda \in H_n} \chi_\lambda(n) d_\mu \frac{t_{\lambda\circ\mu}}{\lambda_1^t} [M]^\lambda_{\mu}. \]
This expression is of a different nature than the one obtained in the previous Section, but it is also very explicit and easy to implement.

V. STATISTICS OF TIME DELAY

As discussed by Berkolaiko and Kuipers [57], the time delay moments $Q_m = \langle \text{Tr}(Q^m) \rangle$ can be obtained from appropriate derivatives of the energy correlators,

$$Q_m = \frac{M \tau_D^m}{i^m m!} \frac{d^m}{d\epsilon^m} \sum_{\epsilon=0}\epsilon^{m-n} \binom{m}{n} C_n(\epsilon). \quad (56)$$

Using the expression we have just derived for $C_n$ as a power series in $\epsilon$, Eq. (55), it is easy to see that

$$\frac{M}{i^m} \frac{d^m C_n}{d\epsilon^m} = M^m \sum_{\lambda \in H_n} \sum_{\mu \in H_m} \chi_{\lambda}(n) d_{\mu} \frac{[M]^{\lambda} t_{\lambda\mu}}{|M|^m} \quad (57)$$

$$\sum_{n=1}^{m-1} \sum_{k=0}^{m-n-1} \binom{m}{n} (-1)^{n-k} \binom{m-k-1}{l} (k+l)! (m-k)^{n} = (-1)^{m-l} (M-l)^{(m)}, \quad (61)$$

with $\mu = (m-l, 1^l)$. Therefore, we arrive at a very simple expression,

$$Q_m = \frac{(M \tau_D)^m}{m} \sum_{\mu \in H_m} \chi_{\mu}(m) d_{\mu} \frac{[M]^{\mu}}{|M|^m}. \quad (62)$$

This coincides exactly for any $n$, with the result derived from random matrix theory [22].

VI. CONCLUSION

Using a powerful semiclassical approach, based on matrix integrals, we investigated energy correlations in the scattering matrices of chaotic systems with broken time-reversal symmetry. We expressed the basic correlator $C_n(M, \epsilon)$, Eq. (41), in two different ways: as a power series in $1/M$ and as a power series in $\epsilon$. From the latter we were then able to extract average spectral moments of the time delay operator. We found complete agreement with RMT predictions, thereby microscopically justifying that approach.

A natural extension of this work would be to perform analogous calculations for systems with intact time-reversal symmetry. That remains a challenge. Moreover, nonlinear statistics of time delay, like $\langle \text{Tr}(Q)^m \rangle$, have been computed within RMT, but are not accessible to the present approach. We believe the alternative semiclassical treatment introduced in [50] is promising in that respect.

ACKNOWLEDGMENTS

Financial support from CNPq, grant 306765/2018-7, is gratefully acknowledged. I would like to thank Marko Riedel for providing a proof of equation (61).

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