SOME HOMOLOGICAL PROPERTIES OF IDEALS WITH COHOMOLOGICAL DIMENSION ONE

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ABSTRACT. Let $R$ denote a commutative Noetherian ring and let $I$ be an ideal of $R$ such that $H^i_I(R) = 0$, for all integers $i \geq 2$. In this paper we shall prove some results concerning the homological properties of $I$.

1. Introduction

Let $R$ denote a commutative Noetherian ring and let $I$ be an ideal of $R$. In [11], Hartshorne defined an $R$-module $L$ to be $I$-cofinite, if $\text{Supp } L \subseteq V(I)$ and $\text{Ext}_{R}^{i}(R/I, L)$ is a finitely generated module for all $i$. He posed the following question:

Whether the category $\mathcal{C}(R, I)_{\text{cof}}$ of $I$-cofinite modules is an Abelian subcategory of the category of all $R$-modules? That is, if $f : M \rightarrow N$ is an $R$-homomorphism of $I$-cofinite modules, are $\text{ker } f$ and $\text{coker } f$ $I$-cofinite?

Hartshorne gave a counterexample to show that this question has not an affirmative answer in general, (see [11, §3]). On the positive side, Hartshorne proved that if $I$ is a prime ideal of dimension one in a complete regular local ring $R$, then the answer to his question is yes. On the other hand, in [7], Delfino and Marley extended this result to arbitrary complete local rings. Kawasaki in [18] generalized the Delfino and Marley’s result for an arbitrary ideal $I$ of dimension one in a local ring $R$. Finally, Melkersson in [23] generalized the Kawasaki’s result for all ideals of dimension one of any arbitrary Noetherian ring $R$. More recently, in [5] as a generalization of Melkersson’s result it is shown that for any ideals $I$ in a commutative Noetherian ring $R$, the category of all $I$-cofinite $R$-modules $M$ with $\dim M \leq 1$ is an Abelian subcategory of the category of all $R$-modules.

Recall that, for an $R$-module $M$, the cohomological dimension of $M$ with respect to $I$, denoted by $\text{cd}(I, M)$, is the smallest integer $n \geq 0$ such that $H^i_I(M) = 0$ for all $i > n$.

The cohomological dimension have been studied by several authors; see, for example, Faltings [9], Hartshorne [12], Huneke-Lyubeznik [16], Divaani-Aazar et al [8], Hellus [13].

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Recall that, for any proper ideal \( I \) of \( R \), the arithmetic rank of \( I \), denoted by \( \text{ara}(I) \), is the least number of elements of \( I \) required to generate an ideal which has the same radical as \( I \).

Now, let \( I \) be an ideal of an arbitrary Noetherian ring \( R \). Kawasaki in [17, Theorem 2.1] proved that if \( \text{ara}(I) = 1 \) then the category \( \mathcal{C}(R, I)_{\text{cof}} \) of \( I \)-cofinite modules is an Abelian subcategory of the category of all \( R \)-modules.

It is well known that, for any proper ideal \( I \) of a Noetherian ring \( R \) we have \( \text{cd}(I, R) \leq \text{ara}(I) \). (See [6, Corollary 3.3.3]). In particular, for any ideal \( I \) with \( \text{ara}(I) = 1 \) we have \( \text{cd}(I, R) \leq 1 \). So, as a generalization of Kawasaki’s interesting result, it is more natural that we ask the following question:

**Question 1:** Let \( R \) be a Noetherian ring and \( I \) be an ideal of \( R \) with \( \text{cd}(I, R) \leq 1 \). Whether the category \( \mathcal{C}(R, I)_{\text{cof}} \) of \( I \)-cofinite modules is an Abelian subcategory of the category of all \( R \)-modules?

In section 2 of this paper we present an affirmative answer to the Question 1, whenever \( R \) is a local Noetherian ring. In section 3 we present some equivalent conditions for the exactness of ideal transform functors. Using these results, for any ideal \( I \) generated by two elements, we provide some necessary and sufficient conditions for the non-vanishing of the local cohomology module \( H^2_I(R) \). Finally, in section 4 we prepare some vanishing conditions for the Bass and Betti numbers of some special local cohomology modules. Also, we give a formula for the cohomological dimension of some special ideals in Noetherian domains.

Throughout this paper, \( R \) will always be a commutative Noetherian ring and \( I \) will be an ideal of \( R \). Also, for an \( R \)-module \( M \), \( \Gamma_i(M) \) denotes the submodule of \( M \) consisting of all elements annihilated by some power of \( I \), i.e., \( \bigcup_{n=1}^{\infty} (0 :_M I^n) \). For every \( R \)-module \( L \), we denote by \( \text{mAss}_R L \) the set of minimal elements of \( \text{Ass}_R L \) with respect to inclusion. Also, for any ideal \( a \) of \( R \), we denote \( \{ p \in \text{Spec } R : p \supseteq a \} \) by \( V(a) \). Finally, for any ideal \( b \) of \( R \), the radical of \( b \), denoted by \( \text{Rad}(b) \), is defined to be the set \( \{ x \in R : x^n \in b \text{ for some } n \in \mathbb{N} \} \). For any unexplained notation and terminology we refer the reader to [6] and [21].

2. A CATEGORY OF MODULES WHICH IS ABELIAN

The following well known lemma plays a key role in the proof of Proposition 2.2.

**Lemma 2.1.** Let \( R \) be a Noetherian ring, \( I \) be an ideal of \( R \) and \( T \) be an \( R \)-module. Then the following statements are equivalent:

(i) The \( R \)-module \( H^i_I(T) \) is Artinian, for \( i \geq 0 \).
(ii) The $R$-module $\text{Ext}^i_R(R/I, T)$ is Artinian, for $i \geq 0$.
(iii) The $R$-module $\text{Ext}^i_R(R/I, T)$ is Artinian, for $0 \leq i \leq \text{cd}(I, R)$.

Proof. See [24, Theorem 5.5] and [2, Theorem 2.9]. \hfill \Box

**Proposition 2.2.** Let $(R, m)$ be a Noetherian local ring, $I$ be an ideal of $R$ and $M$ be an $R$-module. Then the following conditions are equivalent:

(i) The $R$-modules $\text{Tor}_i^R(R/I, M)$ are finitely generated for all $i \geq 0$.
(ii) The $R$-modules $\text{Tor}_i^R(R/I, M)$ are finitely generated for all $0 \leq i \leq \text{cd}(I, R)$.

Proof. (i)$\Rightarrow$(ii) Is clear.

(ii)$\Rightarrow$(i) Let $T := D(M)$, where $D(-) := \text{Hom}_R(-, E)$ denotes the Matlis dual functor and $E := E_R(R/m)$ is the injective hull of the residue field $R/m$. Then by the adjointness we have $\text{Ext}^i_R(R/I, T) \simeq D(\text{Tor}_i^R(R/I, M))$, for all $0 \leq i \leq \text{cd}(I, R)$. In particular, the $R$-modules $\text{Ext}^i_R(R/I, T)$ are Artinian for all $0 \leq i \leq \text{cd}(I, R)$. So, it follows from Lemma 2.1 and adjointness, that the $R$-modules $\text{Ext}^i_R(R/I, T) \simeq D(\text{Tor}_i^R(R/I, M))$ are Artinian for all $i \geq 0$. Now, it follows from [19, Lemma 1.15(a)] that, the $R$-modules $\text{Tor}_i^R(R/I, M)$ are finitely generated for all $i \geq 0$, as required. \hfill \Box

The following easy consequence of Proposition 2.2 plays a key role in the proof of Theorem 2.4.

**Corollary 2.3.** Let $(R, m)$ be a Noetherian local ring, $I$ be an ideal of $R$ with $\text{cd}(I, R) = n$ and let $M$ be an $R$-module with $\text{Supp} M \subseteq V(I)$. Then the $R$-module $M$ is $I$-cofinite, if and only if, the $R$-modules $\text{Tor}_i^R(R/I, M)$ are finitely generated for all $0 \leq i \leq n$.

Proof. The assertion follows from Proposition 2.2 and [24, Theorem 2.1]. \hfill \Box

The following result gives an affirmative answer to the Question 1, for the local case.

**Theorem 2.4.** Let $I$ be an ideal of a Noetherian local ring $(R, m)$ such that $\text{cd}(I, R) \leq 1$. Let $\mathcal{C}(R, I)_{cof}$ denote the category of $I$-cofinite $R$-modules. Then $\mathcal{C}(R, I)_{cof}$ is an Abelian subcategory of the category of all $R$-modules.

Proof. Let $M, N \in \mathcal{C}(R, I)_{cof}$ and let $f : M \rightarrow N$ be an $R$-homomorphism. It is enough to show that the $R$-modules $\ker f$ and $\text{coker} f$ are $I$-cofinite.

To this end, the exact sequence

$$0 \rightarrow \ker f \rightarrow M \rightarrow \text{im} f \rightarrow 0,$$

induces an exact sequence

$$\text{Tor}_0^R(R/I, M) \rightarrow \text{Tor}_0^R(R/I, \text{im} f) \rightarrow 0,$$

which using [24, Theorem 2.1], implies that the $R$-module $\text{Tor}_0^R(R/I, \text{im} f)$ is finitely generated. Now, the exact sequence

$$0 \rightarrow \text{im} f \rightarrow N \rightarrow \text{coker} f \rightarrow 0.$$
induces an exact sequence
\[
\text{Tor}_1^R(R/I, N) \to \text{Tor}_1^R(R/I, \text{coker} f) \to \text{Tor}_0^R(R/I, \text{im} f) \\
\to \text{Tor}_0^R(R/I, N) \to \text{Tor}_1^R(R/I, \text{coker} f) \to 0.
\]

By [24, Theorem 2.1] the modules \( \text{Tor}_1^R(R/I, N) \) and \( \text{Tor}_0^R(R/I, N) \) are finitely generated \( R \)-modules, which implies that the \( R \)-modules \( \text{Tor}_1^R(R/I, \text{coker} f) \) and \( \text{Tor}_1^R(R/I, \text{coker} f) \) are finitely generated. Therefore, it follows from Corollary 2.3, that the \( R \)-module \( \text{coker} f \) is \( I \)-cofinite. Now, the assertion follows from the following exact sequences
\[
0 \to \text{im} f \to N \to \text{coker} f \to 0,
\]
and
\[
0 \to \ker f \to M \to \text{im} f \to 0.
\]

\[\square\]

**Corollary 2.5.** Let \( I \) be an ideal of a Noetherian local ring \((R, m)\) such that \( \text{cd}(I, R) \leq 1 \). Let \( \mathcal{C}(R, I)_{\text{cof}} \) denote the category of \( I \)-cofinite modules over \( R \). Let
\[
X^\bullet : \cdots \to X^i \xrightarrow{f_i} X^{i+1} \xrightarrow{f_{i+1}} X^{i+2} \to \cdots,
\]
be a complex such that \( X^i \in \mathcal{C}(R, I)_{\text{cof}} \) for all \( i \in \mathbb{Z} \). Then for each \( i \in \mathbb{Z} \) the \( i \)-th cohomology module \( H^i(X^\bullet) \) is in \( \mathcal{C}(R, I)_{\text{cof}} \).

**Proof.** The assertion follows from Theorem 2.4. \[\square\]

**Corollary 2.6.** Let \((R, m)\) be a Noetherian local ring, \( I \) be an ideal of \( R \) with \( \text{cd}(I, R) \leq 1 \) and let \( M \) be an \( I \)-cofinite \( R \)-module. Then, the \( R \)-modules \( \text{Tor}_i^R(N, M) \) and \( \text{Ext}^i_R(N, M) \) are \( I \)-cofinite, for all finitely generated \( R \)-modules \( N \) and all integers \( i \geq 0 \).

**Proof.** Since \( N \) is finitely generated it follows that, \( N \) has a free resolution with finitely generated free \( R \)-modules. Now the assertion follows using Corollary 2.5 and computing the \( R \)-modules \( \text{Tor}_i^R(N, M) \) and \( \text{Ext}^i_R(N, M) \), using this free resolution. \[\square\]

### 3. Vanishing of the Extension and Torsion Functors

In this section we present some equivalent conditions for the exactness of ideal transform functors. Using these results, for any ideal \( I \) generated by two elements, we provide some necessary and sufficient conditions for the non-vanishing of the local cohomology module \( H^2_I(R) \).

The following lemma is needed in the proof of Proposition 3.2.

**Lemma 3.1.** Let \( R \) be a Noetherian ring and \( I \) be an ideal of \( R \). Let \( M \) be an \( R \)-module such that \( \text{Tor}_i^R(R/I, M) = 0 \), for all integers \( i \geq 0 \). Then \( \text{Hom}_R(R/I, M) = 0 \).
Proof. Let $I = (x_1, \ldots, x_n)$ and let

$$K_*(\underline{x}; M) : 0 \rightarrow M \xrightarrow{f_0} \bigoplus_{k=1}^{C_1^n} M \xrightarrow{f_1} \bigoplus_{k=1}^{C_2^n} M \rightarrow \cdots \rightarrow \bigoplus_{k=1}^{C_n^n} M \xrightarrow{f_{n-1}} M \rightarrow 0,$$

be the Koszul complex of $M$ with respect to $\underline{x} = x_1, \ldots, x_n$.

We prove that $H_i(\underline{x}; M) = 0$ for all $0 \leq i \leq n$. By the definition we have

$$H_0(\underline{x}; M) = \text{coker } f_{n-1} = M/IM \simeq R/I \otimes_R M = 0.$$ 

So, we have $\text{im } f_{n-1} = M$. Therefore, using the hypothesis it follows that

$$\text{Tor}_i^R(R/I, \text{im } f_{n-1}) = 0, \text{ for each } i \geq 0.$$ 

Hence, the exact sequence

$$0 \rightarrow \text{ker } f_{n-1} \rightarrow \bigoplus_{k=1}^{C_{n-1}^n} M \rightarrow \text{im } f_{n-1} \rightarrow 0,$$

implies that $\text{Tor}_i^R(R/I, \text{ker } f_{n-1}) = 0$, for each $i \geq 0$. The exact sequence

$$0 \rightarrow \text{im } f_{n-2} \rightarrow \text{ker } f_{n-1} \rightarrow H_1(\underline{x}; M) \rightarrow 0, \quad (3.1.1)$$

induces the exact sequence

$$R/I \otimes_R \text{ker } f_{n-1} \rightarrow R/I \otimes_R H_1(\underline{x}; M) \rightarrow 0. \quad (3.1.2)$$

Now as $R/I \otimes_R \text{ker } f_{n-1} = 0$, it follows from (3.1.2) that

$$R/I \otimes_R H_1(\underline{x}; M) = 0.$$ 

By the definition of the Koszul complex we have $\text{I } H_1(\underline{x}; M) = 0$. Therefore, we have

$$H_1(\underline{x}; M) \simeq R/I \otimes_R H_1(\underline{x}; M) = 0.$$ 

Now it follows from the exact sequence (3.1.1) that

$$\text{Tor}_i^R(R/I, \text{im } f_{n-2}) = 0, \text{ for each } i \geq 0.$$ 

Moreover, the exact sequence

$$0 \rightarrow \text{ker } f_{n-2} \rightarrow \bigoplus_{k=1}^{C_{n-2}^n} M \rightarrow \text{im } f_{n-2} \rightarrow 0,$$

implies that $\text{Tor}_i^R(R/I, \text{ker } f_{n-2}) = 0$, for each $i \geq 0$. Proceeding in the same way we can see $H_i(\underline{x}; M) = 0$, for all $0 \leq i \leq n$.

Now, since $H_i(\underline{x}; M) \simeq H^{n-i}(\underline{x}; M)$ for all $0 \leq i \leq n$, it follows that

$$\text{Hom}_R(R/I, M) \simeq (0 :_M I) \simeq H^0(\underline{x}; M) \simeq H_n(\underline{x}; M) = 0.$$ 

The following proposition plays a key role in the proof of our main results.

**Proposition 3.2.** Let $R$ be a Noetherian ring, $I$ be an ideal of $R$ and $M$ be an $R$-module. Then the following conditions are equivalent:

(i) $\text{Tor}_n^R(R/I, M) = 0$, for all integers $n \geq 0$.
(ii) $\text{Ext}_n^R(R/I, M) = 0$, for all integers $n \geq 0$. 

□
Proof. (i)⇒(ii) We argue by induction on \(n\). For \(n = 0\), the assertion holds by Lemma 3.1. We therefore assume, inductively, that \(n > 0\) and the result has been proved for smaller values of \(n\). Then there is an exact sequence

\[
0 \rightarrow M \rightarrow E_R(M) \rightarrow E_R(M)/M \rightarrow 0. \tag{3.2.1}
\]

Since, by the hypothesis we have \((0 :_M I) = 0\) it follows that \((0 :_{E_R(M)} I) = 0\) and hence \(\Gamma_I(E_R(M)) = 0\). By [21] Theorem 18.5 the \(R\)-module \(E_R(M)\) is isomorph with a direct sum of a family of indecomposable injective \(R\)-modules. Let \(p\) be a prime ideal of \(R\) such that \(E_R(R/p)\) is a direct summand of \(E_R(M)\). Then we have \(\Gamma_I(E_R(R/p)) = 0\). Therefore, form the fact that \(\text{Ass}_R E_R(R/p) = \{p\}\) it follows that \(I \not\subseteq p\). So there exists an element \(a \in I\) such that \(a \not\in p\). Then by [21] Theorem 18.4(iii), multiplication by \(a\) is an automorphism on \(E_R(R/p)\). Therefore, multiplication by \(a\) is an automorphism on \(\text{Tor}^R_i(R/I, E_R(R/p))\), for all \(i \geq 0\). But, since \(a \in I\) it follows that, multiplication by \(a\) on \(\text{Tor}^R_i(R/I, E_R(R/p))\) is the zero map, for all \(i \geq 0\). Thus, \(\text{Tor}^R_i(R/I, E_R(R/p)) = 0\) for all \(i \geq 0\). Since for each \(i \geq 0\), the torsion functor \(\text{Tor}^R_i(R/I, -)\) commutes with the direct sums it follows that \(\text{Tor}^R_i(R/I, E_R(M)) = 0\), for all \(i \geq 0\). So, from the exact sequence (3.2.1) it follows that \(\text{Tor}^R_i(R/I, E_R(M)/M) = 0\), for all \(i \geq 0\). Hence, by applying the inductive hypothesis to the \(R\)-module \(E_R(M)/M\) we have

\[
\text{Ext}^{i+1}_R(R/I, M) \simeq \text{Ext}^i_R(R/I, E_R(M)/M) = 0 \quad \text{for} \; 0 \leq i \leq n - 1.
\]

So, we have \(\text{Ext}^n_R(R/I, M) = 0\). This completes the inductive step. \(\Box\)

(ii)⇒(i) Assume the opposite. Then there is an integer \(j \geq 0\) such that \(\text{Tor}^R_j(R/I, M) \neq 0\). Let \(p \in \text{Supp} \text{Tor}^R_j(R/I, M)\). Then localizing at \(p\), without loss of generality, we may assume that \((R, m)\) is a Noetherian local ring. Let \(T := D(M)\), where \(D(-)\) denotes the Matlis dual functor. Then by the adjointness we have

\[
\text{Tor}^R_i(R/I, T) \simeq D(\text{Ext}_R^i(R/I, M)) = 0,
\]

for all \(i \geq 0\). Therefore, by the previous part of the proof we have \(\text{Ext}_R^i(R/I, T) = 0\), for each \(i \geq 0\). So, by the adjointness we have

\[
D(\text{Tor}^R_j(R/I, M)) \simeq \text{Ext}_R^j(R/I, T) = 0,
\]

which implies that \(\text{Tor}^R_j(R/I, M) = 0\). This is the desired contradiction. \(\Box\)

Lemma 3.3. Let \(R\) be a Noetherian ring and \(I\) be an ideal of \(R\). Let \(E\) be an injective \(R\)-module and \(K\) be a submodule of \(E\). Then

\[
\frac{L}{\Gamma_I(L)} \otimes_R R \frac{I}{I} = 0, \quad \text{where} \quad L := \frac{E}{K}.
\]

Proof. In view of [3] Proposition 2.1.4] the \(R\)-module \(E_1 := \Gamma_I(E)\) is injective. Therefore, there exists an injective submodule \(E_2\) of \(E\) such that \(E_1 + E_2 = E\) and \(E_1 \cap E_2 = 0\). Since \(\frac{E_1 + K}{K}\) is a submodule of \(\Gamma_I(L)\) it follows that the \(R\)-module \(\frac{L}{\Gamma_I(L)}\) is a homomorphic image of the \(R\)-module \(E_2\). So, in order to prove the assertion it is enough to prove that
On the other hand, by [6, Lemma 2.1.1] there is an exact sequence

$$E_2 = (0 :_{E_2} \Gamma_1(R)) \cong \text{Hom}_R(R/\Gamma_1(R), E_2).$$

On the other hand, by [6, Lemma 2.1.1] there is an exact sequence

$$0 \longrightarrow R/\Gamma_1(R) \overset{a}{\longrightarrow} R/\Gamma_1(R),$$

for some element $a \in I$, which effecting the $R$-linear exact functor $\text{Hom}_R(-, E_2)$ induces the exact sequence

$$\text{Hom}_R(R/\Gamma_1(R), E_2) \overset{a}{\longrightarrow} \text{Hom}_R(R/\Gamma_1(R), E_2) \longrightarrow 0.$$

Therefore, we have $E_2/aE_2 \cong E_2 \otimes_R R/Ra \cong \text{Hom}_R(R/\Gamma_1(R), E_2) \otimes_R R/Ra 
 \cong \text{Hom}_R(R/\Gamma_1(R), E_2)/a \text{Hom}_R(R/\Gamma_1(R), E_2) 
 \cong 0.$

So, we have $E_2 = aE_2$ and hence $E_2 = IE_2$. \hfill \Box

**Lemma 3.4.** Let $R$ be a Noetherian ring and let $I \subseteq J$ be two ideals of $R$. Let $M$ be an $R$-module and $t \geq 2$ be an integer such that $\text{Tor}_j^R(R/J, H_i^j(M)) = 0$ for all $i > t$ and all $j \geq 0$. Then we have $\text{Tor}_j^R(R/J, H_i^j(M)) = 0$ for $j = 0, 1$.

**Proof.** Let

$$0 \longrightarrow M \overset{\varepsilon}{\longrightarrow} E_0 \overset{f_0}{\longrightarrow} E_1 \overset{f_1}{\longrightarrow} E_2 \overset{f_2}{\longrightarrow} \cdots$$

be a minimal injective resolution for $M$. Set $K_i := \ker f_i$ for $i \geq 0$. By splitting this minimal injective resolution to some short exact sequences we get the isomorphisms

$$H_i^j(M) \cong H_i^{j-1}(K_1) \cong H_i^{j-2}(K_2) \cong \cdots \cong H_i^2(K_{t-2}) \cong H_i^1(K_{t-1}) \cong H_i^1(K_{t-1}/\Gamma_1(K_{t-1})).$$

Set $N := K_{t-1}/\Gamma_1(K_{t-1})$. Then we have $\Gamma_1(N) = 0$ and

$$K_{t-1} = \ker f_{t-1} = \text{im} f_{t-2} \cong E_{t-2}/\ker f_{t-2} = E_{t-2}/K_{t-2}.$$

So, there exists a submodule $K$ of $E := E_{t-2}$ such that $N \cong E/K$. By [6, Remark 2.2.7] there is an exact sequence

$$0 \longrightarrow N \longrightarrow D_1(N) \longrightarrow H_1^1(N) \longrightarrow 0. \quad (3.4.1)$$

For each $i \geq 2$ we have

$$H_i^j(N) \cong H_i^j(K_{t-1}) \cong H_i^{j+i-1}(M).$$

Therefore, by the hypothesis we have

$$\text{Tor}_j^R(R/J, H_i^j(N)) \cong \text{Tor}_j^R(R/J, H_i^{j+i-1}(M)) = 0.$$
for all $i \geq 2$ and all $j \geq 0$. Thus, by Proposition 3.2 we have $\text{Ext}^i_R(R/J, H^i_1(N)) = 0$ for all $i \geq 2$ and all $j \geq 0$. The exact sequence (3.4.1) yields the isomorphisms

$$H^i_1(D_1(N)) \simeq H^i_1(N),$$

for $i \geq 2$. So, we have

$$\text{Ext}^i_R(R/J, H^i_1(D_1(N))) \simeq \text{Ext}^i_R(R/J, H^i_1(N)) = 0$$

for all $i \geq 2$ and all $j \geq 0$. Also, by [3, Corollary 2.2.8] we have $H^i_1(D_1(N)) = 0$, for $i = 0, 1$. So $\text{Ext}^i_R(R/J, H^i_1(D_1(N))) = 0$ for all $i \geq 0$ and all $j \geq 0$. Therefore, by [10, Lemma 2.1] we have $\text{Ext}^j_R(R/J, D_1(N)) = 0$, for all integers $j \geq 0$. Thus by Proposition 3.2 we have $\text{Tor}^j_i(R/J, D_1(N)) = 0$, for all integers $i \geq 0$. On the other hand, in view of Lemma 3.3 we have $N \otimes_R R/I = 0$ and hence it follows from the hypothesis $I \subseteq J$ that $N \otimes_R R/J = 0$. Hence, using the long exact sequence induced by the exact sequence (3.4.1) it follows that $\text{Tor}^j_i(R/J, H^1_1(N)) = 0$, for $j = 0, 1$. Therefore, $\text{Tor}^j_i(R/J, H^1_1(M)) \simeq \text{Tor}^j_i(R/J, H^1_1(N)) = 0$, for $j = 0, 1$.

**Corollary 3.5.** Let $R$ be a Noetherian ring and $I \subseteq J$ be two ideals of $R$. Let $K$ be an $R$-module such that $2 \leq \text{cd}(I, K) = t$. Then $\text{Tor}^i_R(R/J, H^1_1(K)) = 0$, for $i = 0, 1$.

**Proof.** The assertion follows from Lemma 3.4. \hfill $\square$

**Proposition 3.6.** Let $R$ be a Noetherian ring, $I$ be an ideal of $R$ and $M$ be an $R$-module. Then the following conditions are equivalent:

(i) $\text{Tor}^i_R(R/I, M) = 0$, for all $i \geq 0$.

(ii) $\text{Tor}^i_R(R/I, M) = 0$, for all $0 \leq i \leq \text{cd}(I, R)$.

**Proof.** (i)$\Rightarrow$(ii) is clear.

(ii)$\Rightarrow$(i) Assume the opposite. Then there is an integer $j > \text{cd}(I, R)$ such that $\text{Tor}^j_i(R/I, M) \neq 0$. Let $p \in \text{Supp} \text{Tor}^j_i(R/I, M)$. Then localizing at $p$, without loss of generality, we may assume that $(R, \mathfrak{m})$ is a Noetherian local ring. Let $T := D(M)$, where $D(\cdot)$ denotes the Matlis dual functor. Then by the adjointness we have

$$\text{Ext}^i_R(R/I, T) \simeq D(\text{Tor}^i_R(R/I, M)) = 0,$$

for all $0 \leq i \leq \text{cd}(I, R)$. Now, by [2, Theorem 2.9] we have $H^i_1(T) = 0$, for all $0 \leq i \leq \text{cd}(I, R)$. Hence, $H^i_1(T) = 0$, for all integers $i \geq 0$. So, in view of [2, Theorem 2.9] we have $\text{Ext}^i_R(R/I, T) = 0$, for all integers $i \geq 0$. Consequently, by the adjointness, we have

$$D(\text{Tor}^i_R(R/I, M)) \simeq \text{Ext}^i_R(R/I, T) = 0,$$

for all $i \geq 0$. Thus, $\text{Tor}^i_R(R/I, M) = 0$ for all $i \geq 0$, which is a contradiction. \hfill $\square$

**Theorem 3.7.** Let $R$ be a Noetherian ring and $I$ be an ideal of $R$. Then the following conditions are equivalent:
(i) $\text{cd}(I, R) \leq 1$.
(ii) The ideal transform functor $D_I(-)$ is exact.
(iii) For every $R$-module $M$, if $\text{Tor}_i^R(R/I, M) = 0$ for $i = 0, 1$, then $\text{Tor}_i^R(R/I, M) = 0$, for all integers $i \geq 0$.
(iv) For every $R$-module $M$, if $\text{Ext}_i^R(R/I, M) = 0$ for $i = 0, 1$, then $\text{Ext}_i^R(R/I, M) = 0$, for all integers $i \geq 0$.

Proof. (i)$\iff$(ii) See [6, Lemma 6.3.1].

(i)$\implies$(iii) Follows from Proposition 3.6.

(iii)$\implies$(i) Assume the opposite. Then we have $\text{cd}(I, R) \geq 2$. Let $t = \text{cd}(I, R)$. Then by Corollary 3.5 for the $R$-module $M := H_I^1(R)$ we have $\text{Tor}_i^R(R/I, M) = 0$ for $i = 0, 1$. So, by the hypothesis we have $\text{Tor}_i^R(R/I, M) = 0$, for all integers $i \geq 0$. Therefore, by Proposition 3.2 we have $\text{Ext}_i^R(R/I, M) = 0$, for all integers $i \geq 0$. Therefore $(0 : H_I^1(R)) \simeq \text{Hom}_R(R/I, H_I^1(R)) = 0$. Hence $H_I^1(R) = 0$, which is a contradiction.

(i)$\implies$(iv) Let $M$ be an $R$-module such that $\text{Ext}_i^R(R/I, M) = 0$ for $i = 0, 1$. Then, by [2, Theorem 2.9] we have $H_I^1(M) = 0$ for $i = 0, 1$. On the other hand by [6, Lemma 6.3.1] we have $H_I^1(M) = 0$, for all integers $i \geq 2$. Therefore, $H_I^1(M) = 0$, for all integers $i \geq 0$. Hence, it follows from [2, Theorem 2.9] that $\text{Ext}_i^R(R/I, M) = 0$ for all integers $i \geq 0$.

(iv)$\implies$(i) By [6, Corollary 2.2.8] we have $H_I^1(D_I(R)) = 0$, for $i = 0, 1$. Hence, $\text{Ext}_i^R(R/I, D_I(R)) = 0$ for $i = 0, 1$ by [2, Theorem 2.9]. So, by the hypothesis we have $\text{Ext}_i^R(R/I, D_I(R)) = 0$, for all integers $i \geq 0$. Hence, by Proposition 3.2 it follows that $R/I \otimes_R D_I(R) = 0$ and so $D_I(R) = ID_I(R)$. Now the assertion follows from [6, Lemma 6.3.1 and Proposition 6.3.5].

Theorem 3.8. Let $R$ be a Noetherian ring and let $I = Ra_1 + Ra_2$ be an ideal of $R$. Then the following conditions are equivalent:

(i) $\text{cd}(I, R) = 2$.
(ii) There exists an $R$-module $M$, such that $\text{Tor}_i^R(R/I, M) = 0$ for $i = 0, 1$ and $\text{Tor}_2^R(R/I, M) \neq 0$.
(iii) There exists an $R$-module $M$, such that $\text{Ext}_i^R(R/I, M) = 0$ for $i = 0, 1$ and $\text{Ext}_2^R(R/I, M) \neq 0$.

Proof. (i)$\implies$(ii) If $\text{cd}(I, R) = 2$, then for the $R$-module $M := H_I^2(R)$ by Corollary 3.5 we have $\text{Tor}_i^R(R/I, M) = 0$ for $i = 0, 1$. Now, we claim that $\text{Tor}_2^R(R/I, M) \neq 0$. Assume the opposite. Then, it follows from Proposition 3.6 that $\text{Tor}_i^R(R/I, M) = 0$, for all integers $i \geq 0$. Then it follows from Proposition 3.2 that $\text{Ext}_i^R(R/I, M) = 0$, for all integers $i \geq 0$. Therefore, $\text{Hom}_R(R/I, H_I^2(R)) = 0$, which implies that $H_I^2(R) = 0$. This is a contradiction.
(ii)⇒(i) Under the given hypothesis it follows from Theorem 3.7 that cd(I, R) ≥ 2. On the other hand, by [6, Theorem 3.3.1] we have cd(I, R) ≤ 2. So, we have cd(I, R) = 2.

(i)⇒(iii) If cd(I, R) = 2, then for the R-module M := D_I(R), by [6, Corollary 2.2.8] we have \( H^i_J(M) = 0 \), for \( i = 0, 1 \). Hence, by [2, Theorem 2.9] we have Ext^i_R(R/I, M) = 0 for \( i = 0, 1 \). Moreover, by [6, Remark 2.2.7] there is an exact sequence

\[ 0 \rightarrow R/\Gamma_I(R) \rightarrow D_I(R) \rightarrow H^1_I(R) \rightarrow 0, \]

which induces the isomorphisms

\[ H^2_I(M) \cong H^2_I(R/\Gamma_I(R)) \cong H^2_I(R) \neq 0. \]

Now, if Ext^2_R(R/I, M) = 0, then Ext^i_R(R/I, M) = 0, for \( i = 0, 1, 2 \). So, it follows from [2, Theorem 2.9] that \( H^i_I(M) = 0 \) for \( i = 0, 1, 2 \), which is a contradiction, because \( H^2_I(M) \neq 0 \). So, for the R-module \( M := D_I(R) \) we have Ext^i_R(R/I, M) = 0 for \( i = 0, 1 \) and Ext^2_R(R/I, M) ≠ 0.

(iii)⇒(i) Under the given hypothesis it follows from Theorem 3.7 that cd(I, R) ≥ 2. On the other hand, by [6, Theorem 3.3.1] we have cd(I, R) ≤ 2. So, we have cd(I, R) = 2. □

**Remark 3.9.** For a given proper ideal \( I \) of a Noetherian ring \( R \), there are some other well known equivalent conditions for cd(I, R) ≤ 1. For instance, see [6] Lemma 6.3.1 and Proposition 6.3.5] and for more properties of such ideals see [4].

### 4. Vanishing of the Bass and Betti numbers of local cohomology modules

In this section we prepare some vanishing conditions for some of the Bass and Betti numbers of special local cohomology modules. Also, we give a formula for the cohomological dimension of special ideals over Noetherian domains.

**Theorem 4.1.** Let \( R \) be a Noetherian ring and let \( I \subseteq J \) be two ideals of \( R \) such that cd(\( J, R \)) = 1. Then the following statements hold:

(i) For every \( R \)-module \( M \) we have \( \text{Ext}^1_R(R/J, H^i_I(M)) = 0 \), for all integers \( i \geq 2 \) and \( j \geq 0 \).

(ii) For every \( R \)-module \( M \) and every finitely generated \( R \)-module \( N \) with \( \text{Supp} \ N \subseteq \ V(J) \) we have \( \text{Ext}^i_R(N, H^1_I(M)) = 0 \), for all integers \( i \geq 2 \) and \( j \geq 0 \).

(iii) For every \( R \)-module \( M \) and each prime ideal \( \mathfrak{p} \in V(J) \) we have \( \mu^j(\mathfrak{p}, H^1_I(M)) = 0 \), for all integers \( i \geq 2 \) and \( j \geq 0 \). (Here \( \mu^j(\mathfrak{p}, H^1_I(M)) \) denotes the \( j \)-th Bass number of the \( R \)-module \( H^1_I(M) \) with respect to \( \mathfrak{p} \)).

(iv) For every \( R \)-module \( M \) and each prime ideal \( \mathfrak{p} \in V(J) \) we have \( \beta_j(\mathfrak{p}, H^1_I(M)) = 0 \), for all integers \( i \geq 2 \) and \( j \geq 0 \). (Here \( \beta_j(\mathfrak{p}, H^1_I(M)) \) denotes the \( j \)-th Betti number of the \( R \)-module \( H^1_I(M) \) with respect to \( \mathfrak{p} \)).

**Proof.** (i) Let \( M \) be an arbitrary \( R \)-module. In order to prove the assertion we may assume cd(\( I, M \)) = \( t \geq 2 \). By Corollary 3.5 we have Tor^1_R(R/J, H^1_I(M)) = 0 for \( j = 0, 1 \).
Now, it follows from Proposition 3.6 that $\text{Tor}_i^R(R/J, H^1_J(M)) = 0$, for all integers $i \geq 0$. Therefore, by Proposition 3.2 we have $\text{Ext}_i^R(R/J, H^1_J(M)) = 0$, for all integers $i \geq 0$. Now, if $t \geq 3$ then by Lemma 3.4 we have $\text{Tor}_i^R(R/J, H^{t-1}_J(M)) = 0$ for $j = 0, 1$. So, it follows from Proposition 3.6 that $\text{Tor}_i^R(R/J, H^{t-1}_J(M)) = 0$, for all integers $i \geq 0$. Therefore, by Proposition 3.2 we have $\text{Ext}_i^R(R/J, H^{t-1}_J(M)) = 0$, for all integers $i \geq 0$. Proceeding in the same way we see that $\text{Ext}_i^R(R/J, H^1_J(M)) = 0$, for all integers $i \geq 2$ and $j \geq 0$.

(iii) Follows from (ii).

(iv) Follows from (ii) using Proposition 3.2. □

Theorem 4.2. Let $R$ be a Noetherian domain and let $I$ and $J$ be two non-zero proper ideals of $R$ such that $\text{cd}(J, R) = 1$ and $J \not\subseteq \text{Rad}(I)$. Then we have

$$\text{cd}(I \cap J, R) = \max\{i \in \mathbb{Z} : \text{Supp} H^1_J(R) \not\subseteq V(J)\}. $$

Proof. Since by the hypothesis we have $J \not\subseteq \text{Rad}(I)$ it follows that

$$J \not\subseteq \bigcap_{p \in \text{mAss}_R R/I} p,$$

which implies that $J \not\subseteq q$ for some $q \in \text{mAss}_R R/I$. Assume that $k := \text{height} q$. Then by Grothendieck’s Non-vanishing Theorem we have

$$0 \neq H^k_{q, R} (R_q) = H^k_{I, R} (R_q) \simeq (H^k_I(R))_q,$$

which implies that $q \in \text{Supp} H^k_I(R)$. Hence we have $\text{Supp} H^k_I(R) \not\subseteq V(J)$ and so we have

$$k \in \{i \in \mathbb{Z} : \text{Supp} H^1_J(R) \not\subseteq V(J)\}. $$

In particular, we have $\{i \in \mathbb{Z} : \text{Supp} H^1_J(R) \not\subseteq V(J)\} \neq \emptyset$.

Now, set $\ell := \max\{i \in \mathbb{Z} : \text{Supp} H^i_J(R) \not\subseteq V(J)\}$ and $t := \text{cd}(I \cap J, R)$. Then as by [R Corollary 3.3.3] we have $\ell \leq \text{ara}(I)$ and $\ell \leq \text{ara}(I \cap J)$ it follows that $0 \leq \ell < \infty$ and $0 \leq t < \infty$. By the Mayer-Vietoris exact sequence for each integer $i > t$ we have the exact sequence

$$H^i_{I + J}(R) \rightarrow H^i_J(R) \oplus H^i_J(R) \rightarrow H^i_{I \cap J}(R),$$

which gives the exact sequence

$$H^i_{I + J}(R) \rightarrow H^i_J(R) \oplus H^i_J(R) \rightarrow 0$$

and hence we have

$$\text{Supp} H^i_J(R) \subseteq \text{Supp} H^i_{I + J}(R) \subseteq V(I + J) \subseteq V(J).$$

So, it is clear that $\ell \leq t$. On the other hand, since by the hypothesis $R$ is a domain and $I \cap J \neq 0$, it follows that $\ell \geq 1$ and $t \geq 1$. Hence, if $t = 1$, then it is clear that $\ell = 1 = t$. 

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Now, assume that \( t \geq 2 \) and \( \ell < t \). Then, we have \( \text{Supp} H^t_I(R) \subseteq V(J) \). Also, by the Mayer-Vietoris exact sequence we have the exact sequence

\[
H^t_I(R) \oplus H^t_J(R) \longrightarrow H^{t+1}_{I \cap J}(R),
\]

which implies that

\[
\text{Supp} H^{t}_{I \cap J}(R) \subseteq [\text{Supp} H^t_I(R) \cup \text{Supp} H^t_J(R) \cup \text{Supp} H^{t+1}_{I \cap J}(R)] \subseteq V(J).
\]

So, the non-zero \( R \)-module \( H^{t}_{I \cap J}(R) \) is \( J \)-torsion and hence we have

\[
\text{Hom}_R(R/J, H^{t}_{I \cap J}(R)) \neq 0.
\]

But, by Theorem 4.1 we have

\[
\text{Hom}_R(R/J, H^{t}_{I \cap J}(R)) = 0,
\]

which is a contradiction. So, we have \( \ell = t \), whenever \( t \geq 2 \).

**Proposition 4.3.** Let \( (R, m) \) be a Noetherian local ring and let \( I \) and \( J \) be two proper ideals of \( R \) such that \( \text{cd}(J, R) = 1 \). Let \( k \geq 2 \) be an integer such that \( \text{Supp} H^k_I(R) \not\subseteq V(J) \).

Then, the \( R \)-module \( H^k_{I \cap J}(R) \) is not \( I \cap J \)-cofinite. In particular, \( H^k_{I \cap J}(R) \neq 0 \).

**Proof.** By the Mayer-Vietoris exact sequence we have the exact sequence

\[
H^k_{I \cap J}(R) \longrightarrow H^k_I(R) \oplus H^k_J(R) \longrightarrow H^k_{I \cap J}(R),
\]

which considering that fact that \( \text{Supp} H^k_{I \cap J}(R) \subseteq V(I + J) \subseteq V(J) \) implies that \( \text{Supp} H^k_{I \cap J}(R) \not\subseteq V(J) \). (Note that by the hypothesis we have \( \text{Supp} H^k_I(R) \not\subseteq V(J) \). In particular, we have \( H^k_{I \cap J}(R) \neq 0 \). In order to prove the assertion, assume the opposite and assume that \( \dim \text{Supp} H^k_{I \cap J}(R) = d \). Then, in view of [20] Theorem 2.9 we have \( H^d_m(H^k_{I \cap J}(R)) \neq 0 \). On the other hand, by Theorem 4.1 we have \( \text{Ext}^j_{R}(R/m, H^k_{I \cap J}(R)) = 0 \), for all integers \( j \geq 0 \). Hence it follows from [2] Theorem 2.9 that \( H^d_m(H^k_{I \cap J}(R)) = 0 \), for all integers \( j \geq 0 \). Therefore, we have \( H^d_m(H^k_{I \cap J}(R)) = 0 \), which is a contradiction.

**Remark 4.4.** There are examples of Noetherian local rings \( (R, m) \) with proper ideals \( I \), for which \( \text{cd}(I, R) = 1 \) and \( \text{ara}(I) \geq 2 \). For instance, the following example is given by Hellus and Stöckrad in [15].

**Example 4.5.** Let \( k \) be a field and let \( S = k[[x, y, z, w]] \), where \( x, y, z, w \) are independent indeterminacies over \( k \). Let \( f = xw - yz, g = y^3 - x^2z \) and \( h = z^3 - w^2y \). Let \( R = S/fS \) and \( I = (f, g, h)S/fS \). Then \( R \) is a Noetherian local ring of dimension 3 with maximal ideal \( m = (x, y, z, w)S/fS \). Also, for the ideal \( I \) of \( R \), we have \( \text{cd}(I, R) = 1 \) and \( \text{ara}(I) = 2 \). (See [15] Remark 2.1(ii)).

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