The approximate calculation of 2D field strength with a Coulomb potential by using multipole expansion

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Abstract. The necessary formulae are derived for approximate computation of the Coulomb-type potential and the field strength, generated by system of particles in 2D case. On the one hand, these formulae represent the multipole expansion in which monopole, dipole, quadrupole and octupole terms are taken into account that allows for computing the potential and the field strength, generated by set of the charges with rather high accuracy. On the other hand, polynomial expansions of the obtained formulae are derived up to the cubic terms with respect to the observation point coordinates, which make it possible to compute the field strength at closely located points approximately. The expressions are derived that permit one to compute the multipole moments for the system of charges, which consists of smaller subsystems. Such formulae are required in tree-codes when the hierarchical (tree-type) structure of geometrical regions is introduced in the region of the initial system of the charges. In particular case these expressions allows for changing of the reference point of multipole moments computation (multipole moments shifting). The test problem is considered for rather small number of the charges, which shows the dependency of the relative error of the field strength reconstruction on the type of multipole expansion and number of considered terms in the Taylor expansion.

1. Introduction
At numerical simulation of the interaction of the point charges, or some other problems of similar type, where the particles interact with the Coulomb potential, it is necessary to take into account their mutual influences, so the computational complexity of the trivial algorithm is proportional to $N^2$, where $N$ is number of charges. In order to reduce time of computations, approximate fast methods can be applied, which review can be found in [1, 2]. The Barnes – Hut method [3] and multipole expansions method [4] are rather efficient, as well as their hybrid modifications, e.g., described in [5], which is actually multipole method with monopole and dipole terms only and their linear expansion with respect to the observation point. These methods have computational complexity proportional to $N \log N$, however, numerical experiments show that providing the same error level (about 0.1 %), the method [5] yields to an other approximate fast method [6], based on the FFT technique. The results of their comparison are described in [7].

The aim of the present paper is to derive main computational formulae for the fast method, based on the multipole expansions method for the particles (charges) interaction with a Coulomb potential. The idea of their derivation is rather clear, however it is important not only to write down the formulae, which are ‘formally correct’, but to develop also the efficient approach to implementation of...
the numerical algorithm, taking into account the symmetry of the corresponding matrices, tensors, etc.
In those formulae the following features should be considered:

- monopole, dipole, quadrupole and octupole terms, i.e., 4 terms in the Laurent expansion at computation of the influence from the group of closely placed particles;
- constant, linear, quadratic and cubic terms, i.e., 4 terms in the Taylor expansion for the field strength approximate computation at closely placed points.

2. Problem statement
We consider system of the charges, each of them generates the Coulomb-type potential, and the total potential in 2D space is their superposition:

\[ \Phi(\vec{r}) = \sum_{p=1}^{N} g_p \ln \frac{1}{|\vec{r} - \vec{r}_p|}, \]

where \( g_p \) is the charge of the \( p \)-th particle; \( \vec{r}_p \) is its position vector.

Then the field strength at the point \( \vec{r} \) is calculated as

\[ \vec{E}(\vec{r}) = -\nabla \Phi = \sum_{p=1}^{N} \frac{g_p (\vec{r} - \vec{r}_p)}{|\vec{r} - \vec{r}_p|^3}. \]  

(1)

It is required to derive approximate formulae for the field strength computation, under the assumption that point \( r \) is placed rather far from the charges; and also to write down the approximate expression for the field strength at point \( (\vec{r} + \vec{h}) \), placed rather close to the point \( r \).

3. Monopole, dipole, quadrupole and octupole terms in potential expansion
The potential of the system of charges, being represented as a multipole expansion, can be reconstructed approximately with different accuracy:

\[ \Phi(\vec{r}) = \sum_{m=0}^{N} \Phi^m(\vec{r}) + \Phi^0(\vec{r}), \]

\[ = \Phi^0(\vec{r}) + \Phi^0(\vec{r}) + \Phi^0(\vec{r}) + \Phi^0(\vec{r}) + O(|\vec{\rho}|^4), \]

where \( \vec{\rho} = \vec{r} - \vec{r}_0 \) is vector, which connects the origin of the potential expansion (the reference point) with the point of its approximate computation; \( \Phi^0(\vec{r}) \), \( \Phi^0(\vec{r}) \), \( \Phi^0(\vec{r}) \) and \( \Phi^0(\vec{r}) \) are the potentials of a monopole, dipole, quadrupole and octupole, respectively:

\[ \Phi^0(\vec{r}) = \ln \frac{1}{|\vec{\rho}|} \sum_{p=1}^{N} g_p, \]

\[ \Phi^0(\vec{r}) = \frac{1}{2} \sum_{p=1}^{N} \left( g_p \nabla \cdot \left( \ln \frac{1}{|\vec{\rho}|} \right) \right) \left( \Delta \vec{r}_p \otimes \Delta \vec{r}_p \right), \]

\[ \Phi^0(\vec{r}) = \frac{1}{6} \sum_{p=1}^{N} \left( g_p \nabla \cdot \left( \ln \frac{1}{|\vec{\rho}|} \right) \right) \left( \Delta \vec{r}_p \otimes \Delta \vec{r}_p \otimes \Delta \vec{r}_p \right). \]  

(3)

We use traditional denotations for the gradient and the Hessian of the fundamental solution of the Laplacian equation in 2D:

\[ \nabla \ln \frac{1}{|\vec{\rho}|} = -\frac{\vec{\rho}}{|\vec{\rho}|^2} \quad \text{and} \quad \nabla \nabla \ln \frac{1}{|\vec{\rho}|} = 2\frac{\vec{\rho} \otimes \vec{\rho} - |\vec{\rho}|^2 \hat{I}}{|\vec{\rho}|^4}, \]

as well as for the tensor consists of its third partial derivatives:

\[ \nabla \nabla \nabla \ln \frac{1}{|\vec{\rho}|} = \frac{8 \vec{\rho} \otimes \vec{\rho} \otimes \vec{\rho} - 6 |\vec{\rho}|^2 \text{Sym} (\hat{I} \otimes \vec{\rho})}{|\vec{\rho}|^6}, \]

and we denote the monopole moment, the dipole moment vector and the quadrupole and octupole moment tensors as follows:
\[
m^m = \sum_{p=1}^{N} g_p, \quad \tilde{m}^d = \sum_{p=1}^{N} (g_p \Delta \tilde{r}_p), \quad \tilde{m}^q = \sum_{p=1}^{N} g_p \left( \Delta \tilde{r}_p \otimes \Delta \tilde{r}_p - \frac{1}{2} |\Delta \tilde{r}_p|^2 \hat{I} \right),
\]

\[
m^e = \sum_{p=1}^{N} g_p \left( \frac{4}{3} \Delta \tilde{r}_p \otimes \Delta \tilde{r}_p \otimes \Delta \tilde{r}_p - \frac{3}{2} |\Delta \tilde{r}_p|^2 \text{Sym} \left( \hat{I} \otimes \Delta \tilde{r}_p \right) \right),
\]

where \( \hat{I} \) is unit 2-nd rank tensor, which corresponds to the identity matrix. Now the monopole, dipole, quadrupole and octupole potentials (3) have the following form:

\[
\Phi^w(\bar{r}) = m^w \ln \frac{1}{|\bar{\rho}|}, \quad \Phi^d(\bar{r}) = \frac{\tilde{m}^d \cdot \bar{\rho}}{|\bar{\rho}|^2}, \quad \Phi^q(\bar{r}) = \frac{\tilde{m}^q : (\bar{\rho} \otimes \bar{\rho})}{|\bar{\rho}|^3}, \quad \Phi^o(\bar{r}) = \frac{\tilde{m}^e \cdot (\bar{\rho} \otimes \bar{\rho} \otimes \bar{\rho})}{|\bar{\rho}|^5}, \quad \bar{\rho} = \bar{r} - \bar{r}^0.
\]

Hereinafter in addition to standard operations of the gradient \( \nabla, \nabla \Psi(\bar{r}), \) the Hessian \( \nabla, \nabla, \nabla \Psi(\bar{r}), \) and the third derivatives tensor \( \nabla, \nabla, \nabla, \Psi(\bar{r}) \) for the scalar field, which correspond to vector and the 2-nd and 3-rd rank tensors with components

\[
(\nabla, \nabla \Psi(\bar{r}))_{ij} = \frac{\partial^2 \Psi(\bar{r})}{\partial x_i \partial x_j}, \quad (\nabla, \nabla, \nabla \Psi(\bar{r}))_{ijkl} = \frac{\partial^3 \Psi(\bar{r})}{\partial x_i \partial x_j \partial x_k},
\]

and the scalar product operation (\( \bar{u} \cdot \bar{v} = \sum_i u_i v_i \)), the outer (tensor) product operation \( \bar{u} \otimes \bar{v} \) is introduced, which result gives the 2-nd rank tensor with the components \((\bar{u} \otimes \bar{v})_{ij} = u_i v_j \), and the operation of matrix-vector outer product, which leads to the 3-rd rank tensor with components \((\bar{p} \otimes \bar{u})_{ijk} = p_{ij} u_k \). We also use the notations for double and triple internal (scalar) product for the 2-nd and 3-rd rank tensors:

\[
\hat{p} : \hat{q} = \sum_{ij} p_{ij} q_{ij}, \quad m : s = \sum_{ij} \sum_k m_{ijk} s_{ijk}.
\]

In the above presented formulae \( \Psi(\bar{r}) \) is arbitrary scalar field; \( \bar{u} \) and \( \bar{v} \) are arbitrary vectors; \( \hat{p} \) and \( \hat{q} \) are the 2-nd rank tensors; \( m \) and \( s \) are the 3-rd rank tensors.

The operations \( \text{Sym}[\hat{p}], \text{Sym}[m] \) and \( \text{Sym}[z] \), being applied to the tensors of the 2-nd, 3-rd and 4-th rank, mean their symmetric parts:

\[
(\text{Sym}[\hat{p}])_{ij} = p_{ij}, \quad (\text{Sym}[m])_{ijk} = m_{ijk}, \quad (\text{Sym}[z])_{ijkl} = z_{ijkl},
\]

where the summation is performed over all possible index permutations.

The dipole moment of the system of charges is the vector \( \tilde{m}^d = (m^d, m^q) \), the quadrupole moment tensor \( \tilde{m}^q \) is symmetric and has zero trace, so it is defined by two numbers \( m^q_{ij} \) and \( m^q_{ij} \), and its matrix has the following form:

\[
\tilde{m}^q = \begin{pmatrix} m^q_{ij} & m^q_{ij} \\ m^q_{ij} & -m^q_{ij} \end{pmatrix};
\]

the octupole moment tensor is also fully symmetric (it remains the same after arbitrary index permutation) and its convolution over arbitrary indices pair is equal to zero, so it is also defined by only two numbers \( m^o_{ij} \) and \( m^o_{ij} \), and its components are

\[
m^o_{1,1,1} = -m^o_{1,2,2} = -m^o_{2,1,2} = m^o_{1,2,1} = m^o_{2,1,1} = m^o_{1,1,2} = m^o_{1,1,2} = m^o_{2,2,2} = -m^o_{2,1,1} = -m^o_{1,2,1} = -m^o_{1,1,2} = -m^o_{1,2,2},
\]

where \( g_{ij} \) are the 2-nd rank, mean their symmetric parts:

\[
\text{Sym}[p]_{ij} = p_{ij}, \quad \text{Sym}[m]_{ijk} = m_{ijk}, \quad \text{Sym}[z]_{ijkl} = z_{ijkl},
\]

where the summation is performed over all possible index permutations.
This property of the quadrupole and octupole moments makes their computation procedure rather easy; it is also taken into account in the further formulæ.

Note, that if we choose the reference point \( \mathbf{r}^0 \) (the origin of the multipole expansion and the multipole moments computation) as the ‘center of charges’

\[
\mathbf{r}^0 = \frac{\sum_{p=1}^{N} g_p \mathbf{r}_p}{\sum_{p=1}^{N} g_p},
\]

then the dipole moment \( \mathbf{m}^d \) is equal to zero. If all the charges have the same sign, then their center of charges is always placed inside the convex polygon, which contains all the charges. Otherwise its computation seems to be not useful; the better choice is to set the reference point \( \mathbf{r}^0 \) as close as possible to the geometric center of the region with charges. The other possible approach consists in the splitting of the initial charges system into subsystems which contain only positive and only negative charges and considering these subsystems separately.

If the system of charges is considered as an aggregate of \( M \) other (smaller) subsystems of charges, for each of them monopole moments \( m^m_{s(i)} \) are known, as well as dipole, quadrupole and octupole moments \( \mathbf{m}^d_{s(i)}, \mathbf{m}^q_{s(i)} \) and \( m^m_{s(i)} \), calculated with respect to some reference points \( \mathbf{r}^0_{s(i)}, s = 1, \ldots, M \), then the monopole moment of total system is calculated simply as a sum of the monopole moments of the subsystems,

\[
m^m_k = \sum_{s=1}^{M} m^m_{s(i)},
\]

the dipole, quadrupole and octupole moments with respect to arbitrary chosen reference point \( \mathbf{r}^0_k \) are expressed as the following:

\[
m^d_k = \sum_{s=1}^{M} \left( \mathbf{m}^d_{s(i)} + \mathbf{h}_{s(i)} m^m_{s(i)} \right),
\]

\[
m^q_k = \sum_{s=1}^{M} \left( \mathbf{m}^q_{s(i)} + 2 \text{Sym} \left[ \left( \mathbf{m}^d_{s(i)} + \frac{1}{2} \mathbf{h}_{s(i)} m^m_{s(i)} \right) \otimes \mathbf{h}_{s(i)} \right] - \left( \mathbf{m}^d_{s(i)} + \frac{1}{2} \mathbf{h}_{s(i)} m^m_{s(i)} \right) \cdot \mathbf{h}_{s(i)} \right) \hat{I},
\]

\[
m^o_k = \sum_{s=1}^{M} \left( m^m_{s(i)} + 4 \text{Sym} \left[ \left( \mathbf{m}^d_{s(i)} + \mathbf{h}_{s(i)} \otimes \mathbf{m}^d_{s(i)} \right) \otimes \mathbf{h}_{s(i)} \right] - 2 \text{Sym} \left[ \left( \mathbf{m}^d_{s(i)} + \mathbf{h}_{s(i)} \otimes \mathbf{m}^d_{s(i)} \right) \cdot \mathbf{h}_{s(i)} + \frac{1}{2} \left( \mathbf{h}_{s(i)} \cdot \mathbf{h}_{s(i)} \right) \left( \mathbf{m}^d_{s(i)} + \mathbf{h}_{s(i)} m^m_{s(i)} \right) \right] \otimes \hat{I} + \frac{4}{3} m^m_{s(i)} \left( \mathbf{h}_{s(i)} \otimes \mathbf{h}_{s(i)} \otimes \mathbf{h}_{s(i)} \right) \right),
\]

where \( \mathbf{h}_{s(i)} = \mathbf{r}^0_{s(i)} - \mathbf{r}^0_k \).

The above written formulæ remain true at \( M = 1 \), for this particular case they express the rule of recalculation (shifting) of the dipole, quadrupole and octupole moments at varying the reference point.

If the dipole moments of all the subsystems are equal to zero, \( \mathbf{m}^d_{s(i)} = 0, \; s = 1, \ldots, M \), i.e., \( \mathbf{r}^0_{s(i)} \) are the centers of charges for the subsystems, then the point

\[
\mathbf{r}^0_k = \frac{\sum_{s=1}^{M} m^m_{s(i)} \mathbf{r}^0_{s(i)}}{\sum_{s=1}^{M} m^m_{s(i)}},
\]

is the center of charges of total system, consequently its dipole moment with respect to this point is equal to zero (\( m^d_k = 0 \)), while the expressions for the quadrupole and octupole moments become easier:

\[
m^d_k = \sum_{s=1}^{M} \left( \mathbf{m}^d_{s(i)} + \left( \mathbf{h}_{s(i)} \otimes \mathbf{h}_{s(i)} \right) m^m_{s(i)} - \frac{m^m_{s(i)}}{2} \left( \mathbf{h}_{s(i)} \cdot \mathbf{h}_{s(i)} \right) \hat{I} \right),
\]
4. Multipole expansion for the field strength

The expression (1) for the field strength, generated by system of charges at the point \( \mathbf{r} \), can be replaced approximately by its multipole expansion, obtained by calculating of the gradient from the potential expansion (2), taking into account (4):

\[
\hat{E}(\mathbf{r}) = -\nabla \Phi \approx \hat{\mathbf{E}}^m(\mathbf{r}) + \hat{E}^d(\mathbf{r}) + \hat{E}^s(\mathbf{r}) = \frac{m^m}{|\mathbf{\hat{p}}|^2} \hat{\mathbf{p}} + \left( \frac{-\hat{m}^d}{|\mathbf{\hat{p}}|^2} + \frac{2\hat{m}^d \cdot \mathbf{\hat{p}}}{|\mathbf{\hat{p}}|^3} \right) + \left( \frac{-3\hat{m}^s}{|\mathbf{\hat{p}}|^3} + \frac{6m^s (\mathbf{\hat{p}} \cdot \mathbf{\hat{p}}) \cdot \mathbf{\hat{p}}}{|\mathbf{\hat{p}}|^5} \right),
\]

where vector \( \mathbf{p} = \mathbf{r} - \mathbf{r}^0 \), as earlier, connects the reference point (the origin of the multipole moments computation) with the point of the field strength computation.

Let us consider an example. The system consists of 40 point charges is shown in Figure 1; positive and negative charges are shown as blue and red circles, respectively. The position points are chosen in a random way inside the circle of unit radius, their charges are also chosen randomly in range from \(-1\) to \(2\). Now we calculate approximately the potential and the field strength, generated by this system of charges, according to the formulae (2)-(3) and (5), respectively, in two ways: as the influence of the total system of charges, by computing their multipole moments, and as the sum of influences of 4 small subsystems of charges, placed in the sectors of the circle, which are denoted in Figure 1 by numerals 1...4. Reference points for the multipole moments computation are chosen as mean coordinates of all the charges in the corresponding subsystems.

![Figure 1](image-url)

**Figure 1.** Positions of the charges in the model problem (blue and red circles), origins of the multipole moments computation for the small subsystems of charges 1...4 (green crossings in the corresponding sectors) and for the total system of charges (brown crossing)
In Figure 2 relative error values are shown for the potential and the field strength at the point placed at the distance \( r \) from the origin of coordinates (which coincides with the center of the circle containing all the charges) for monopole, dipole, quadrupole and octupole approximations. Computations have been performed for 360 radial directions, and then the results were averaged. Dots in the Figure 2 correspond to the influence of the total system of charges, solid lines – to the sums of influences of subsystems 1...4.

**Figure 2.** Relative errors for the potential (a) and the field strength (b) computation for different distance from the system of charges. Dots corresponds to the influence of the total system of charges, solid lines – to the sums of influences of subsystems 1...4

5. **Field strength expansion with respect to observation point coordinates**

In order to compute the field strength at some points, placed at rather small distance one from the others, the expression (5), calculated at point \( (\vec{r} + \vec{h}) \), can be expanded into the Taylor series at the point \( \vec{r} \); we take into account 4 terms, which correspond to constant, linear, quadratic and cubic approximations:

\[
E(\vec{r} + \vec{h}) = E(\vec{r}) + \left( \nabla, E(\vec{r}) \right) \cdot \vec{h} + \frac{1}{2} \left( \nabla \cdot \nabla, E(\vec{r}) \right) : \left( \vec{h} \otimes \vec{h} \right) + \frac{1}{6} \left( \nabla \cdot \nabla \cdot \nabla, E(\vec{r}) \right) : \left( \vec{h} \otimes \vec{h} \otimes \vec{h} \right) + O\left( \| \vec{h} \|^4 \right).
\]

The gradient of the vector field \( E(\vec{r}) \), denoted as \( \nabla, E(\vec{r}) \), is a tensor of the second rank; \( \nabla \cdot \nabla, E(\vec{r}) \) – tensor of the third rank; \( \nabla \cdot \nabla \cdot \nabla, E(\vec{r}) \) – tensor of the fourth rank; their components are equal to the corresponding partial derivations:

\[
\left( \nabla, E(\vec{r}) \right)_q = \frac{\partial E_i}{\partial x_j}, \quad \left( \nabla \cdot \nabla, E(\vec{r}) \right)_{ij} = \frac{\partial^2 E_i}{\partial x_k \partial x_j}, \quad \left( \nabla \cdot \nabla \cdot \nabla, E(\vec{r}) \right)_{ijkl} = \frac{\partial^3 E_i}{\partial x_l \partial x_k \partial x_j}.
\]

Calculating all the derivatives in (6) and denoting the corresponding components as the following

\[
\bar{E}(\vec{r}) = \sum_p \bar{E}_p(\vec{r}), \quad \nabla, \bar{E}(\vec{r}) = \sum_p \nabla, \bar{E}_p(\vec{r}), \quad \frac{1}{2} \nabla \cdot \nabla, \bar{E}(\vec{r}) = \sum_p \frac{1}{2} \nabla \cdot \nabla, \bar{E}_p(\vec{r}), \quad \frac{1}{6} \nabla \cdot \nabla \cdot \nabla, \bar{E}(\vec{r}) = \sum_p \frac{1}{6} \nabla \cdot \nabla \cdot \nabla, \bar{E}_p(\vec{r}),
\]
where summation with respect to index $\rho$ is performed over all multipoles terms, which are taken into account, it is possible to write down the following formulae for the components of the field strength, which correspond to the monopole, dipole, quadrupole and octupole approximations.

For the influence of the monopole we obtain:

$$E^m_0 = \frac{m^m}{|\rho|^4} \rho,$$
$$E^m_1 = \frac{m^m}{|\rho|^5} \hat{i} - 2 \frac{m^m}{|\rho|^5} \hat{\rho} \otimes \hat{\rho},$$
$$E^m_2 = \text{Sym} \left[ -3 \frac{m^m}{|\rho|^5} \hat{\rho} \otimes \hat{i} + 4 \frac{m^m}{|\rho|^5} \hat{\rho} \otimes \hat{\rho} \otimes \hat{\rho} \right],$$
$$E^m_3 = \text{Sym} \left[ \frac{m^m}{|\rho|^5} \hat{i} \otimes \hat{i} + 8 \frac{m^m}{|\rho|^5} \hat{\rho} \otimes \hat{\rho} \otimes \hat{i} - 8 \frac{m^m}{|\rho|^5} \hat{\rho} \otimes \hat{\rho} \otimes \hat{\rho} \otimes \hat{\rho} \right].$$

For the dipole approximation Taylor’s coefficients have the following form:

$$E^d_0 = -\frac{m^d}{|\rho|^4} + \frac{2m^d \cdot \hat{\rho}}{|\rho|^5} \hat{\rho},$$
$$E^d_1 = \text{Sym} \left[ 4 \frac{m^d}{|\rho|^4} \otimes \hat{\rho} + 2 \frac{m^d \cdot \hat{\rho}}{|\rho|^5} \hat{i} - 8 \frac{m^d \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{\rho} \right],$$
$$E^d_2 = \text{Sym} \left[ 3 \frac{m^d}{|\rho|^4} \otimes \hat{i} - 12 \frac{m^d}{|\rho|^5} \otimes \hat{\rho} \otimes \hat{\rho} - 12 \frac{m^d \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{i} + 16 \frac{m^d \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{\rho} \otimes \hat{\rho} \right],$$
$$E^d_3 = \text{Sym} \left[ -16 \frac{m^d}{|\rho|^4} \otimes \hat{\rho} \otimes \hat{i} - 4 \frac{m^d \cdot \hat{\rho}}{|\rho|^5} \hat{i} \otimes \hat{i} + 32 \frac{m^d \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{\rho} \otimes \hat{\rho} + 48 \frac{m^d \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{\rho} \otimes \hat{i} - 64 \frac{m^d \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{\rho} \otimes \hat{\rho} \otimes \hat{\rho} \right].$$

For the expansion of the quadrupole one can obtain

$$E^q_0 = -\frac{2m^q \cdot \hat{\rho}}{|\rho|^4} + \frac{4m^q \cdot \hat{\rho}}{|\rho|^5} \hat{\rho},$$
$$E^q_1 = \text{Sym} \left[ -2 \frac{m^q}{|\rho|^4} + 16 \frac{m^q \cdot \hat{\rho}}{|\rho|^5} \otimes \hat{\rho} + 4 \frac{m^q \cdot \hat{\rho}}{|\rho|^5} \hat{i} - 24 \frac{m^q \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{\rho} \right],$$
$$E^q_2 = \text{Sym} \left[ 12 \frac{m^q}{|\rho|^4} \otimes \hat{i} + 12 \frac{m^q \cdot \hat{\rho}}{|\rho|^5} \otimes \hat{\rho} \otimes \hat{i} - 72 \frac{m^q \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{i} \otimes \hat{i} - 36 \frac{m^q \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{i} \otimes \hat{i} + 96 \frac{m^q \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{i} \otimes \hat{i} \otimes \hat{\rho} \right],$$
$$E^q_3 = \text{Sym} \left[ 8 \frac{m^q}{|\rho|^4} \otimes \hat{\rho} \otimes \hat{i} - 8 \frac{m^q \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{i} + 96 \frac{m^q \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{i} - 96 \frac{m^q \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{i} \otimes \hat{\rho} + 256 \frac{m^q \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{i} \otimes \hat{i} \otimes \hat{\rho} + 192 \frac{m^q \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{i} \otimes \hat{i} \otimes \hat{\rho} \otimes \hat{\rho} \right].$$

The coefficients of the octupole term expansion have the most complicated form:

$$E^o_0 = -\frac{3m^o \cdot \hat{\rho}}{|\rho|^4} + \frac{6m^o \cdot \hat{\rho}}{|\rho|^5} \hat{\rho},$$

$$E^o_1 = \text{Sym} \left[ -6 \frac{m^o}{|\rho|^4} + 24 \frac{m^o \cdot \hat{\rho}}{|\rho|^5} \otimes \hat{\rho} + 12 \frac{m^o \cdot \hat{\rho}}{|\rho|^5} \hat{i} - 48 \frac{m^o \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{\rho} \otimes \hat{\rho} + 384 \frac{m^o \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{\rho} \otimes \hat{i} \otimes \hat{\rho} \right],$$

$$E^o_2 = \text{Sym} \left[ 12 \frac{m^o}{|\rho|^4} \otimes \hat{i} + 12 \frac{m^o \cdot \hat{\rho}}{|\rho|^5} \otimes \hat{\rho} \otimes \hat{i} - 72 \frac{m^o \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{i} \otimes \hat{i} - 36 \frac{m^o \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{i} \otimes \hat{i} + 96 \frac{m^o \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{i} \otimes \hat{i} \otimes \hat{\rho} + 256 \frac{m^o \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{i} \otimes \hat{i} \otimes \hat{\rho} + 192 \frac{m^o \cdot \hat{\rho}}{|\rho|^5} \hat{\rho} \otimes \hat{i} \otimes \hat{i} \otimes \hat{\rho} \otimes \hat{\rho} \right].$$
In the Figure 3 the scheme of the numerical experiment is shown, which makes it possible to estimate the accuracy of the derived approximate formulae. The field strength has been computed at 144 point, placed at the distance $h$ from the center of expansion, which, in turn, is placed at the distance $r$ from the circle with the initial system of charges. Then the relative errors were averaged over all these point and also over 36 positions of the point $\tilde{r}$ (i.e., for 36 different values of the angle $\theta$).

The results of computations, obtained for $r = 5$, are shown in Figure 4 for monopole (a), dipole (b), quadrupole (c) and octupole (d) approximations. As earlier in Figure 2, dots in Figure 4 correspond to the influence of the total system of charges, solid lines – to the sums of influences of subsystems $I...4$.

6. Conclusion
It is seen from Figure 4, that the minimal error for total charges system is about 2 %, 0.5 %, 0.04 % and 0.007 % for 4 considered expansions. The corresponding values for the influence of subsystems $I...4$ are about 0.5 %, 0.2 %, 0.03 % and less than 0.002 %, respectively. It is also seen, that nearly the same levels of error are typical for the points in the circle with center at the Taylor expansion origin, which radius $h$ depends significantly on number of terms in the considered Taylor expansion. For 3-terms (quadratic) approximation it is approximately 2-3 times larger than for 2-terms (linear) approximation, for octupole expansion this ratio is even higher.
However, cubic terms improve the accuracy not so much, while their computation is connected with rather high computation cost. Then, if the acceptable level of the error has order of 0.1 % or even higher, it seems to be not efficiently to take into account the octupole expansion. The cost of its computation is also rather high, while the accuracy improvement for the considered case is not significant.

**Figure 4.** Relative errors of the field strength calculation for monopole (a), dipole (b), quadrupole (c) and octupole (d) approximations for different values of the $|\vec{h}|/|\vec{r}|$ ratio, taking into account constant, linear, quadratic and cubic terms in Taylor expansion with respect to observation point coordinates. Dots corresponds to the influence of the total system of charges, solid lines – to the sums of influences of subsystems 1...4.

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