The nonlinear directional coupler. An analytic solution

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Abstract

Linear and nonlinear directional couplers are currently used in fiber optics communications. They may also play a role in multiphoton approaches to quantum information processing if accurate control is obtained over the phases and polarizations of the signals at the output of the coupler. With this motivation, the constants of motion of the coupler equation are used to obtain an explicit analytical solution for the nonlinear coupler.

1 Introduction

Directional couplers are useful devices currently used in fiber optics communications. Because of the interaction between the signals in the input fibers, power fed into one fiber is transferred to the other. The amount of power transfer can be controlled by the coupling constant, the interaction length or the phase mismatch between the inputs. If, in addition, the material in the coupler region has nonlinearity properties, the power transfer will also depend on the intensities of the signals [1] [2]. A large number of interesting effects take place in nonlinear directional couplers [3] [4] [5] [6] with, in particular, the possibility of performing all classical logic operations by purely optical means [7]. They may also play a role in quantum information processing.

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The use of the intensity-dependent phase shifts associated to the Kerr nonlinearity was, in the past, proposed for the construction of quantum gates [8] [9] [10]. However they rely on one-photon processes and therefore would require very strong nonlinearities, not available in the low loss optical materials. On the other hand, the quantum computation scheme based on linear optics of Knill, Laflamme and Milburn is probabilistic and relies on a delicate sensitivity of one-photon detectors. For this reason multiphoton approaches have been explored based either on the quantumlike behavior [13] [14] [15] of optical modes on a fiber [12] or on coherent states [16]. For light beams on a fiber, sizable nonlinear effects are easy to achieve with available materials. In particular the directional coupler might provide an already available tool for the implementation of linear or nonlinear gates.

For quantum information purposes one would require accurate information on the phases and polarizations of the signals at the output of the coupler. Analytic solutions are ideal for this purpose although, in general, difficult to obtain for nonlinear systems. Here, by exploring the constants of motion of the coupler equation, an explicit analytical solution is obtained for the nonlinear coupler.

2 An analytic solution

Consider two linear optical fibers coming together in a coupler of nonlinear material. The equation for the electric field is

$$\Delta E - \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_L}{\partial t^2} + \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$$

(1)

$P_L$ and $P_{NL}$ being the linear and nonlinear components of the medium polarization.

$$P_L (r, t) = \varepsilon_0 \chi^{(1)} E (r, t)$$

(2)

For symmetric molecules (like SiO$_2$) the leading nonlinear term is

$$P_{NL} (r, t) = \varepsilon_0 \chi^{(3)} |E (r, t)|^2 E (r, t)$$

(3)

where an instantaneous nonlinear response may be assumed (except for extremely short pulses) because in current fibers the electronic contribution to $\chi^{(3)}$ occurs on a time scale of $1 - 10$ fs.

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There have been some speculations [17] that nonlinear quantum(like) effects might endow quantum computation with yet additional power.
Separating fast and slow (time) variations

\[ E(r, t) = \frac{1}{2} \{ E(r, t) e^{-i\omega_0 t} + c.c. \} \]
\[ P_{NL}(r, t) = \frac{1}{2} \{ P_{NL}(r, t) e^{-i\omega_0 t} + c.c. \} \] (4)

and using Eqs. (3) and (4) one obtains for the \( e^{-i\omega_0 t} \) part of a transversal mode

\[ P_{NL,1,2}(r, t) = \frac{3\varepsilon_0}{8} \chi^{(3)} \left\{ e^{-i\omega_0 t} \left[ \left( |E_{1,2}|^2 + \frac{2}{3} |E_{2,1}|^2 \right) E_{1,2} + \frac{1}{3} E_{2,1} E_{2,1}^* \right] + c.c. \right\} \] (5)

The labels 1 and 2 denote two orthogonal polarizations.

The dependence on transversal coordinates \((x, y)\) is separated by considering

\[ E^{(i)}(r, t) = \Psi^{(i)}(x, y, z) e^{i\beta z} e^{-i\omega_0 t} \] (6)

\( \Psi^{(i)}(x, y, z) \) being an eigenmode of the coupler with slow variation along \( z \)

\[ \Delta_2 \Psi^{(i)} + \left( \frac{\omega_0^2}{c^2} \left( 1 + \chi^{(1)} \right) - \beta^{(i)2} \right) \Psi^{(i)} = 0 \] (7)

\((i)\) denotes the mode number, \( k \) the polarization and \( \Delta_2 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \).

Neglecting \( \frac{\partial^2 \Psi^{(i)}}{\partial z^2} \) one obtains

\[ 2i\beta^{(i)} \frac{\partial \Psi^{(i)}_{1,2}}{\partial z} = - \frac{3\omega_0^2}{4c^2} \chi^{(3)} \left\{ \left( |\Psi^{(i)}_{1,2}|^2 + \frac{2}{3} |\Psi^{(i)}_{2,1}|^2 \right) \Psi^{(i)}_{1,2} + \frac{1}{3} \Psi^{(i)}_{2,1} \Psi^{(i)}_{2,1} \Psi^{(i)\ast}_{1,2} \right\} \] (8)

In directional couplers the propagating beams are made to overlap along one of the transversal coordinates \((x)\). Typically, in the nonlinear region of the directional coupler, the eigenmodes are symmetric \((+\) and antisymmetric \((-\) functions of \( x \), the amplitudes in each fiber at the input and output of the coupler being recovered by

\[ \Psi^{(1)}_k = \frac{1}{2} \left( \Psi^{(+)}_k + \Psi^{(-)}_k \right) \]
\[ \Psi^{(2)}_k = \frac{1}{2} \left( \Psi^{(+)}_k - \Psi^{(-)}_k \right) \] (9)

\(^2\)Justified for slow variations of the refractive index along the beam axis over distances of the order of one wavelength

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An explicit analytic solution for the nonlinear coupler equation (8) is now obtained by noticing that it has two constants of motion, namely
\[
\frac{\partial}{\partial z} \left\{ \left| \psi_1^{(i)} \right|^2 + \left| \psi_2^{(i)} \right|^2 \right\} = 0
\]
\[
\frac{\partial}{\partial z} \left\{ \psi_1^{(i)} \psi_2^{(i)} - \psi_1^{(i)} \psi_2^{(i)*} \right\} = 0
\]

(10)

Therefore, defining
\[
\left| \psi_1^{(i)} \right|^2 + \left| \psi_2^{(i)} \right|^2 = \alpha^{(i)}
\]
\[
\psi_1^{(i)} \psi_2^{(i)} - \psi_1^{(i)} \psi_2^{(i)*} = i \gamma^{(i)}
\]

(11)

one obtains for the electrical field of the eigenmodes
\[
i \frac{\partial E_1^{(i)}}{\partial z} = - \beta^{(i)} E_1^{(i)} - k E_2^{(i)}
\]
\[
i \frac{\partial E_2^{(i)}}{\partial z} = - \beta^{(i)} E_2^{(i)} + k E_1^{(i)}
\]

(12)

with
\[
\beta^{(i)} = \beta^{(i)} + \frac{3 \omega_0^2}{8 c^2 \chi^{(3)}} \alpha^{(i)}
\]
\[
k^{(i)} = \frac{\omega_0^2}{8 c^2 \chi^{(3)}} \gamma^{(i)}
\]

(13)

Notice that, through \(\alpha^{(i)}\) and \(\gamma^{(i)}\), \(\beta^{(i)}\) and \(k^{(i)}\) depend on the material properties, on the geometry of the mode and also on its intensity. One may now obtain, for each eigenmode, the input-output relation of the nonlinear coupler
\[
E_1^{(i)} (z) = e^{i \beta^{(i)} z} \left\{ E_1^{(i)} (0) \cos \left( k^{(i)} z \right) - E_2^{(i)} (0) \sin \left( k^{(i)} z \right) \right\}
\]
\[
E_2^{(i)} (z) = e^{i \beta^{(i)} z} \left\{ E_1^{(i)} (0) \sin \left( k^{(i)} z \right) + E_2^{(i)} (0) \cos \left( k^{(i)} z \right) \right\}
\]

(14)

\(((i) = (+) or (-)), the nonlinearity being embedded into \(\beta^{(i)}\) and \(k^{(i)}\)
\[
\beta^{(i)} = \beta^{(i)} + \frac{3 \omega_0^2}{8 c^2 \chi^{(3)}} \left( \left| E_1^{(i)} (0) \right|^2 + \left| E_2^{(i)} (0) \right|^2 \right)
\]
\[
k^{(i)} = \frac{\omega_0^2}{4 c^2 \chi^{(3)}} \text{Im} \left( E_1^{(i)*} (0) E_2^{(i)} (0) \right)
\]

(15)
To obtain the corresponding input-output relations in the two fibers (1) and (2) one uses Eqs.(9), namely

\[ E_k^{(1)}(z) = \frac{1}{2} \left( E_k^{(+)}(z) + E_k^{(-)}(z) \right) \]
\[ E_k^{(2)}(z) = \frac{1}{2} \left( E_k^{(+)}(z) - E_k^{(-)}(z) \right) \] (16)

In conclusion: Eqs.(14)-(16) provide an analytic solution for the nonlinear directional coupler, from which phases and polarizations may be obtained explicitly.

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