Improved Prediction and Network Estimation Using the Monotone Single Index Multi-variate Autoregressive Model

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Abstract

Network estimation from multi-variate point process or time series data is a problem of fundamental importance. Prior work has focused on parametric approaches that require a known parametric model, which makes estimation procedures less robust to model mis-specification, non-linearities and heterogeneities. In this paper, we develop a semi-parametric approach based on the monotone single-index multi-variate autoregressive model (SIMAM) which addresses these challenges. We provide theoretical guarantees for dependent data, and an alternating projected gradient descent algorithm (based on Dai et al. (2021)). Significantly we do not explicitly assume mixing conditions on the process (although we do require conditions analogous to restricted strong convexity) and we achieve rates of the form $O(T^{-\frac{1}{3}} s \log(TM))$ (optimal in the independent design case) where $s$ is the threshold for the maximum in-degree of the network that indicates the sparsity level, $M$ is the number of actors and $T$ is the number of time points. In addition, we demonstrate the superior performance both on simulated data and two real data examples where our SIMAM approach out-performs state-of-the-art parametric methods both in terms of prediction and network estimation.

1. Introduction

Multi-variate time series or point process data arises in a number of settings such as social networks Zhou et al. (2013); Richey (2008); Mark et al. (2019b); Raginsky et al. (2012); crime networks Stomakhin et al. (2011b); Mark et al. (2019a); Egesdal et al. (2010); electrical systems Ertekin et al. (2015); neuroscience Brown et al. (2004); Hall and Willett (2015); Smith and Brown (2003); Fujita et al. (2007) and many others. One of the questions of interest in multi-variate time series/point process data is estimating an influence network which captures the temporal influence amongst different nodes. For instance, in a social network, different nodes represent different individuals or media sources, whose behaviours, such as posting articles or reporting hot events, can be observed through time. By investigating such time-stamped data, we seek to discover the flow of information or potential communities through the inference of the underlying influence network.

There is a large body of recent work on parametric models and estimators for learning influence networks (see e.g. Hall et al. (2016); Mark et al. (2019a)). In many scenarios,
underlying non-linearities and heterogeneities make it difficult to posit a parametric model. Furthermore, parametric models often do not yield good prediction performance due to their lack of flexibility and inability to model non-linearities. In this work, we use a semi-parametric network estimation approach that addresses these challenges. In particular, rather than using standard parametric approaches, we use the monotone single index model (SIM) for network estimation.

The monotone single index model (MSIM) has been widely used in many settings [Foster et al. 2013; Balabdaoui et al. 2019; Groeneboom and Hendrickx 2019]. The semi-parametric construction allows the interpretation using the “parametric” part while the “non-parametric” part allows the flexibility to model non-linearities and misspecified link functions. Typically, the non-parametric function is assumed to be monotone, which covers a number of interesting examples, including all generalized linear models and many other examples and this function does not need to be pre-specified. From a statistical and algorithmic perspective, MSIM in high dimensions presents a number of technical challenges (see e.g. Chen and Samworth 2014; Foster et al. 2013). Many of these have been addressed in settings where we have independent samples Foster et al. 2013; Balabdaoui et al. 2019).

In the context of multi-variate time series and point process models, the MSIM provides a natural semi-parametric framework by modelling each time series as a separate MSIM, where the features/co-variates are the data from previous time points. One of the major theoretical challenges with applying the MSIM to autoregressive point processes is providing theoretical guarantees while accounting for the complex nonlinear dependence.

This paper addresses this challenge by providing an alternating PGD algorithm and its theoretical guarantees for the monotone single index multi-variate autoregressive model (SIMAM). Significantly, we do not explicitly assume any mixing condition on the multi-variate time series as is done in Zhou and Raskutti 2018, although we do require conditions analogous to restricted strong convexity. We also support our theoretical findings by giving empirical evidence through simulations and real data examples, illustrating the superior performance for monotone SIMAM in terms of prediction and variable selection compared to existing state-of-the-art approaches.

1.1 Related works

Various parametric approaches have been widely explored in a large body of prior work to learn the influence network from multi-variate time series or point process data. One standard approach is the vector autoregressive (VAR) model Lütkepohl 2013; Canova 1995; Hsu et al. 2008. To avoid the limitations of VAR in non-Gaussian or non-linear autoregressive processes, vector generalized linear autoregressive (GLAR) model Hall et al. 2016; Dunsmuir 2015; Hall et al. 2018; Shephard et al. 1995 is proposed and widely used as an extension of VAR, in which non-linear structure is introduced by a known link function according to prior knowledge. By specifying the link function and the conditional distribution within the exponential family, GLAR can be adjusted to many specific models, such as the Bernoulli autoregressive model Pandit et al. 2019 and log-linear Poisson autoregressive (PAR) model Fokianos et al. 2009; Zhu and Wang 2011. Although these
models are more flexible, they are still restricted to non-linear models with known and fixed parametrization, which may not be applicable to real-world settings.

In order to improve the robustness and flexibility of parametric autoregressive models in multi-variate time series, non-parametric approaches have been explored and developed (see e.g. Scaillet (2004); Härdle and Vieu (1992)). For instance, in the recent work Zhou and Raskutti (2018), a non-parametric additive autoregressive network model is developed, involves replacing linear terms with additive functions belonging to a reproducing kernel Hilbert space (RKHS). The estimators are obtained through a penalized maximum likelihood procedure.

In this work, we don’t directly impose any smoothness assumptions as we formulate the conditional expectations directly through an isotonic single index autoregressive model for a $M$-dimensional multi-variate time series, i.e.,

$$\mathbb{E}(X_{t,j}|X_{t-1}) = f_j^*(X_{t-1}^T u_j^*), \ j \in [M],$$

where $f_j^*$ is an unknown link function and $u_j^*$ is the direction or the index to be estimated as one column of the influence network. We approximate the network parameters by minimizing the mean squared loss rather than maximizing the unknown likelihood function.

In a non-parametric regression setting, the single index model has been well developed over the past decades. Classical approaches in estimating the single index model include profile likelihoods and smooth kernels Carroll et al. (1997); Xue and Zhu (2006); Hristache et al. (2001); Wang et al. (2010); Naik and Tsai (2001). For multi-variate time series data, Wu et al. (2011); Guo et al. (2017) respectively constructed a single index coefficient model and a partial linear model to deal with the non-linearity. In Li and Genton (2009), the authors proposed a single index additive autoregressive model for a multi-variate time series. All of the above literature estimates the single index model in the time series data by penalized splines, which involved the selection for smoothing parameters. Large sample results were derived based on mixing conditions, yet non-asymptotic results are not provided, making them not applicable for high dimensional (large $M$) settings.

A multitude of advances on isotonic regression analysis (Durot (2002); Zhang et al. (2002); Chatterjee et al. (2014, 2015); Bellec et al. (2018)) substitutes the smoothness assumption by monotonicity of the link function, which leads to the isotonic single index model. To estimate this semi-parametric model, the Isotron algorithm and estimators were proposed and studied in Kakade et al. (2011); Kalai and Sastry (2009). To further address the high-dimensional challenges, a variable selection procedure using LASSO was combined with the isotonic single index model in Neykov (2019); Foster et al. (2013), both of which considered independent Gaussian data. In Balabdaoui et al. (2019), the authors showed that the rate of the least squared estimator of the bundled isotonic single index function in the $\ell_2$ norm with respect to the sample size $n$ is $n^{-1/3}$ under appropriate conditions. In Dai et al. (2021), the “Sparse Orthogonal Descent Single-Index Model” (SOD-SIM) is developed with the isotonic regression and a projection-based iterative approach, where a $n^{-1/3}$ convergence guarantee in the high dimensional setting is given. Both of these papers (Balabdaoui et al. (2019); Dai et al. (2021)) are focusing on the regression setting (i.e., $\mathbb{E}(Y|X) = f^*(X^T u^*)$) with independent data.

Perhaps the most closely related prior work to our setting is Wang et al. (2016), which proposed an Isotonic-Hawkes process whose intensity function was formulated in the form
of an isotonic single index model. They used an alternating minimization procedure in the algorithm, which shares the same framework as ours and showed the efficiency of the estimated near optimal indices and link functions. On the other hand, there are significant differences between this work and ours. Our results can not only be applied to the counting process as the Isotonic Hawkes process dealt with, but can also be applied to more general multi-variate time series with continuous-value co-variates. Besides, we take the influence network’s sparsity into account and introduce a hard thresholding operator to enforce the sparsity, which is particularly helpful in the high dimensional setting.

1.2 Contributions

Our major contributions in this paper are as follows:

- We formulate the monotone single index multi-variate autoregressive model (monotone SIMAM) in the high-dimensional settings to learn the influence network from a multi-variate time series or point process data, which is more flexible and robust compared to existing parametric models, while entails more interpretability than other non-parametric ones. Based on this model, a feasible algorithm in an alternating framework combining the iterative hard-thresholding (IHT) method and suitable initialization is provided to solve the non-convex problem.

- In terms of theoretical analysis, we provide the convergence rate for our network estimator from the proposed algorithm in a non-asymptotic manner that applies to the high-dimensional setting using martingale concentration inequalities. The result indicates that after applying sufficiently many iterations, given a multi-variate time series with $T$ observations, the Frobenius norm of the influence network estimation error converges in the order of $O(T^{-\frac{1}{3}})$ up to some poly-log terms. Specifically, our rate depends on the sparsity of the network, the noise level of the data, the Lipschitz continuity of the monotone function, and the dimension of the network. In addition, the empirical one-step prediction error also has the rate of $O(T^{-\frac{1}{3}})$. We also prove that the angle between our initialization and the true parameters is acute with high probability which is sufficient to guarantee our $O(T^{-\frac{1}{4}})$ rate.

- Simulation results are given to support the $O(T^{-\frac{1}{3}})$ convergence rate of the estimator derived from our algorithm with the nonlinear link functions unknown, after sufficiently many iterations. It is also illustrated that our method has a better performance in terms of both in-sample and out-of-sample prediction errors, compared to VAR with $\ell_1$ penalty.

- Two real data examples, the Chicago crime data and Memetracker data, are analyzed, which indicate that there exist highly nonlinear and non-smooth structures in point process data. In terms of prediction and estimation, we observe a significant advantage of our proposed monotone SIMAM over other popular parametric network estimating models, such as vector autoregressive (VAR) model, VAR with LASSO type of penalty and Poisson autoregressive (PAR) with the $\ell_1$ penalty.
2. Preliminaries

2.1 Notations

Let \{X_0, X_1, \ldots, X_T\} \subset \mathbb{R}^M be a \(M\)-dimensional time series with \(T+1\) observed time points. Combining them together as rows of a matrix gives \(x \in \mathbb{R}^{(T+1) \times M}\), whose entries are \(X_{i,j} \in \mathbb{R}\), where \(i \in [T + 1] - 1 = \{0, \ldots, T\}, j \in [M] = \{1, \ldots, M\}\).

We write \(X_{-i} \in \mathbb{R}^{T \times M}\) as the matrix with \(i\)-th row deleted from \(X\); \(X_{-i,j} \in \mathbb{R}^T\), in this manner, denotes the \(j\)-th column of the matrix \(X_{-i}\). Specifically for example, as will appear repeatedly in the paper, \(X_{-T}\) denotes all the data collected in time points \(t = 0, 1, \ldots, T - 1\); \(X_{-0,j}\) denotes the \(j\)-th co-variates observations in time points \(t = 1, \ldots, T\), with the first observation deleted.

For an index set \(I = \{i_1, \ldots, i_k\} \subseteq \mathbb{N}\) and any vector \(v = (v_1, \ldots, v_n)^T \in \mathbb{R}^n\) with length \(n \geq \max\{I\}\), we denote \(v_I = (v_{i_1}, \ldots, v_{i_k})^T\) as the extraction of all elements in \(v\) whose indices are included in \(I\); \(|I| = k\) represents the cardinality of set \(I\). For any \(l \in \mathbb{N}\), \(I + l\) is short for the set with all elements being added \(l\): \(I + l = \{i_1 + l, \ldots, i_k + l\}\).

Let \(\Phi_s : \mathbb{R}^M \rightarrow \mathbb{R}^M\) be the hard thresholding operator whose image is always a subset of all \(s\)-sparse vectors in \(\mathbb{R}^M\), i.e.

\[
\Phi_s(x) = \operatorname{arg\,min}_{y \in \mathbb{R}^M} \{\|y - x\|_2 : \|y\|_0 = s\}, \quad \forall x \in \mathbb{R}^M.
\] (2)

Another important projection operator we would use is the orthogonal projection operator \(P_u^\perp(\cdot)\) for any \(u \in \mathbb{R}^M\), which projects any vector in \(\mathbb{R}^M\) onto the subspace of \(\mathbb{R}^M\) orthogonal to \(u\):

\[
P_u^\perp(x) = \operatorname{arg\,min}_{y \in \mathbb{R}^M} \{\|y - x\|_2^2 : \langle u, y \rangle = 0\}, \quad \forall x \in \mathbb{R}^M.
\] (3)

2.2 Partial ordering and Isotonic Regression

The isotonic regression problem is:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{i=1}^{T} (v_i - x_i)^2 \\
\text{Subject to} & \quad x_i \leq x_j \text{ when } i \preceq j,
\end{align*}
\] (4)

where \(\preceq\) is a specified partial ordering on \(\Omega = \{1, 2, \ldots, T\}\). The solution to this problem is referred to as isotonic regression. The vector \(x = (x_1, \ldots, x_T)\) is said to be isotonic or order preserving if \(i \preceq j\) implies \(x_j \leq x_j\). Note that the set of isotonic vectors \(x \in \mathbb{R}^T\) is a closed convex cone, which guarantees that there is a unique solution \(x \in \mathbb{R}^T\) that solves the above isotonic regression problem.

To specify the partial ordering \(\preceq\), we could introduce a collection of real numbers \(z = (z_1, \ldots, z_T)\):

\[
i \preceq j \text{ if } z_i \leq z_j, \quad \forall i, j \in \Omega.
\] (5)

Hence, for any collections \(z = (z_1, \ldots, z_T)\) and \(v = (v_1, \ldots, v_T)\), define \(\operatorname{iso}_z(v) \in \mathbb{R}^T\) as the isotonic vector of \(v\) with respect to \(z\), i.e. the solution for the following constrained
minimization:

\[ \text{iso}_z(v) = \arg \min_{x \in \mathbb{R}^T} \{ \| v - x \|_2^2 : x_i \leq x_j \text{ whenever } z_i \leq z_j \text{ for } \forall i, j \in [T] \}. \]  

(6)

Essentially, \( \text{iso}_z(v) \) preserves the ordering of \( z \) while fitting \( v \). Important properties of isotonic regression include its contractiveness with respect to some seminorms, as is given in Lemma 16 and Corollary 17. Computationally, using pool-adjacent-violators algorithm (PAVA) [Mair et al. (2009)] with \( z, v \in \mathbb{R}^T \) being the algorithm inputs, \( \text{iso}_z(v) \) can be solved with a computational complexity \( O(T) \).

Since the above partial ordering in Eq. (5) is only defined by collections of scalars, when it turns to high dimensional vectors, such ordering has to be induced by some projection. Consider a collection of \( T \) points \( X_1, \ldots, X_T \) in \( M \)-dimensional Euclidean space and a reference vector \( u \in \mathbb{R}^M \), for a permutation \( \pi \) of the set \( \{1, \ldots, T\} \), if

\[ \langle X_{\pi(1)}, u \rangle \leq \cdots \leq \langle X_{\pi(T)}, u \rangle, \]

(7)

we say that \( u \in \mathbb{R}^M \) induces the ordering \( \pi \).

3. Model and Algorithm

3.1 Monotone Single Index Multivariate Autoregressive Model

We assume that the time series \( \{X_t\}_{t=0}^T \subset \mathbb{R}^M \) follows the monotone single index multivariate autoregressive model (SIMAM) and is conditionally independent across \( j \in [M] \), i.e.,

\[ \mathbb{E}(X_{t,j} | X_{t-1}) = f_j^*(X_{t-1}^T u_j^*) \]

(8)

for all \( t \in [T] \) almost surely with an unknown index \( u_j^* \in \mathbb{R}^M \setminus \{0\} \) and a monotone function \( f_j^* \), which is also unknown. According to the conditional independence across \( j \in [M] \), conditioned on the previous data, the elements \( X_{t,1}, \ldots, X_{t,M} \) of the \( t \)-th observation are independent of one another. Specifically, the index \( u_j^* = (u_{j1}^*, \ldots, u_{jM}^*)^T \), also known as the direction vector, is assumed to lie on a unit sphere \( S^{M-1} \subset \mathbb{R}^M \) with \( s_j^* \) nonzero elements, where \( s_j^* \) refers to the sparsity parameter. Let

\[ s^* := \max\{s_1^*, \ldots, s_M^*\}, \]

then \( s^* \) is the maximum in-degree of the directed graph induced by the network, in which the \( i \)-th node represents the \( i \)-th co-variate, and the edge from node \( i \) to node \( j \) exists when \( u_{ji}^* \neq 0 \).

The univariate function \( f_j^* : \mathbb{R} \to \mathbb{R} \), which contributes to the non-parametric flexibility of the model, is assumed to be \( L_j \)-Lipschitz continuous and non-decreasing on its domain that contains the range of the linear predictors \( \{X_{t-1}^T u_j^*\}_{t=1}^T \). Let \( \mathcal{M} \) denote the function class that contains all monotonically non-decreasing functions, then we have \( f_j^* \in \mathcal{M} \) for any \( j \in [M] \). For technical reasons, we extend all functions outside their actual support by taking the extension to be constant to the left and right of the original support’s endpoints.

The noise terms in the \( j \)-th co-variate, denoted as \( Z_{t,j} = (Z_{1,j}, \ldots, Z_{t,j}) \) where

\[ Z_{t,j} = X_{t,j} - f_j^*(X_{t-1}^T u_j^*), \]

(9)
are generally assumed to be martingale differences with conditional sub-Gaussian tails.

Due to the monotonicity of \( f^*_j \) for each \( j \in [M] \), the nonlinear function \( f^*_j \) is order-preserving, while the index \( u^*_j \) essentially captures the direction that finds the best ordering for the variables to project on, i.e., \( u^*_j \) is the linear projector that induces the variable ordering.

Based on this model, we can make inference on the influence network by the direction vectors \( \{ u^*_1, \ldots, u^*_M \} \). Let \( A^* = (u^*_1, \ldots, u^*_M) \) denote the coefficient matrix with the direction vectors being its columns, then its element \( A_{ij} \) with \( i, j \in [M] \) represents the temporal influence of \( X_{t-1,i} \) on \( X_{t,j} \) for every time series observation \( t \in [T] \). From the perspective of graphs/networks, \( A^* \) is an adjacency matrix of the weighted directed graph that indicates the influence network.

### 3.2 Connection with Generalized Linear autoregressive Models

In the generalized linear autoregressive models (GLAR) [Hall et al. 2016], for any \( j \in [M] \), the density of \( X_{t+1,j} \) given \( X_t = x_t \) with respect to a given base measure is an exponential family of the form

\[
p(x_{t+1,j}|X_t = x_t) = h_j(x_{t+1,j}, \phi_j) \exp \left\{ \frac{x_{t+1,j}(x^T_t u^*_j + \nu_j) - Z_j(x^T_t u^*_j + \nu_j)}{\phi_j} \right\},
\]

where \( h_j \) is the base measure of the conditional distribution, and \( \phi_j > 0 \) is the dispersion parameter. \( u^*_j \) is the unknown vector containing network parameters of our interest, while \( Z_j(\cdot) \) is referred to as the log partition function, whose second-order derivative satisfies \( Z''_j(\cdot) > 0 \) for all elements in its domain. Further since \( p(\cdot) \) belongs to an exponential family, we have

\[
\mathbb{E}(X_{t+1,j}|X_t) = Z'_j(X^T_t u^*_j + \nu_j).
\]

By the convexity of the log partition function \( Z_j(\cdot) \), we know that its first-order derivative \( Z'_j(\cdot) \) is monotonically non-decreasing. Thus there exists a monotone function \( f^*_j \) for each \( j \in [M] \), such that the monotone single index multi-variate autoregressive model Eq. (8) holds true.

Hence, the generalized linear autoregressive model is a special case of the monotone SIMAM. A key difference between these two models is that in GLAR, the inverse link function (or transfer function) \( f^*_j(\cdot) = Z'_j(\cdot) + c \), where \( c \) is a constant thanks to the monotonicity of \( Z'_j(\cdot) \), is assumed known. In the monotone SIMAM, however, the function \( f^*_j \) is allowed to be unknown and only assumed to be isotonic. Further, the conditional distribution of \( X_{t+1,j} \) given \( X_t \) is no longer assumed to take the form Eq. (10), making the model more flexible.

### 3.3 Non-convex optimization and algorithm

Based on the monotone SIMAM, we want to find the solutions to the following non-convex optimization problem over the unknown functions \( \{ f_1, \ldots, f_M \} \) and direction vectors \( \{ u_1, \ldots, u_M \} \): given a multi-variate time series or point process \( \{ X_0, \ldots, X_T \} \) and sparsity
levels \( \{s_1, \ldots, s_M\} \),

\[
\text{Minimize } \frac{1}{T} \sum_{j=1}^{M} \sum_{t=0}^{T-1} [X_{t+1,j} - f_j(X_t^T u_j)]^2
\]

Subject to \( \|u_j\|_2 = 1; \|u_j\|_0 = s_j \) and \( f_j \) is non-decreasing, \( \forall j \in [M] \).

Since \( \{X_{t+1,1}, \ldots, X_{t+1,M}\} \) is conditionally independent given the prior observation \( X_t \), the loss function is separable with respect to the sum over \( j \in [M] \). Therefore, we can estimate the pairs \( \{(f_j^*, u_j^*)\}_{j=1}^{M} \) separately. For any \( j \in [M] \), to simultaneously estimate \( f_j^* \) and \( u_j^* \), an alternating procedure is proposed. Note that once the direction \( u_j \in \mathbb{R}^M \) is given, \( f_j \) can be estimated by minimizing the profile loss function \( \mathcal{L}_j(f; u_j) \) where

\[
\mathcal{L}_j(f; u) = \frac{1}{T} \sum_{t=0}^{T-1} [X_{t+1,j} - f(X_t^T u)]^2.
\]

By the definition of isotonic regression in Eq. (6), we know that

\[
\text{iso}_{X_{-T} u_j}(X_{-0,j}) = (\hat{f}_j(X_0^T u_j), \ldots, \hat{f}_j(X_{T-1}^T u_j)) \text{ for any } \hat{f}_j \in \arg \min_{f \in \mathcal{M}} \mathcal{L}_j(f; u_j),
\]

where \( X_{-T} u_j = (X_0^T u_j, \ldots, X_{T-1}^T u_j)^T \) and \( X_{-0,j} = (X_{1,j}, \ldots, X_{T,j})^T \).

On the other hand, even if the non-decreasing function \( f_j \) is given, minimizing \( \mathcal{L}_j(f_j; u) \) over \( \{u \in \mathbb{R}^M : \|u\|_2 = 1\} \) with the non-convex constraint \( \|u\|_0 = s_j \) is still challenging. In linear settings without the non-linear transformation \( f_j(\cdot) \), projected gradient descent (PGD) (also known as iterative hard-thresholding (IHT)) algorithms are used to solve the \( \ell_0 \)-norm constrained problem (see e.g., [Blumensath and Davies (2009); Jain et al. (2014)]). To implement PGD, we first need to find the gradient with respect to \( u \) in \( \mathcal{L}_j(f_j; u) \): (for heuristic purpose, we assume \( f_j \) has the first-order derivative \( f'_j \) here)

\[
\nabla_u \mathcal{L}_j(f_j; u) = \frac{1}{T} \sum_{t=0}^{T-1} [X_{t+1,j} - f_j(X_t^T u)] \cdot f'_j(X_t^T u) \cdot X_t.
\]

Although \( f'_j(\cdot) \) is unknown, due to the monotonicity we know that \( f'_j(X_t^T u) \) is a non-negative scalar for all \( t \in [T] - 1 \), thus in the gradient descent step, we remove this term as an approximation to the gradient direction. Therefore for any \( u_j \), if we have obtained an estimated \( \hat{f}_j \) satisfying Eq. (14), to update \( u_j \) using PGD, the pseudo gradient we use is

\[
\frac{1}{T} \sum_{t=0}^{T-1} [X_{t+1,j} - \hat{f}_j(X_t^T u_j)] \cdot X_t = \frac{1}{T} X_{-T}^T \left[ X_{-0,j} - \text{iso}_{X_{-T} u_j}(X_{-0,j}) \right],
\]

the specific usage of which would be further indicated in Eq. (19).

With the above alternating framework, we are now in position to introduce the procedure to estimate the direction vector \( u_j^* \) and the corresponding monotone link function \( f_j^* \). The pseudo-code of the overall procedure is given in alg. [1].

For any \( j \in [M] \), fix a step-size \( \eta_j = \frac{1}{\beta} \) and a maximum iteration count \( K_j \), where \( \beta \) is the largest eigenvalue of \( X_{-T} \) defined in Eq. (27). We use \( s_j \) as an estimated sparsity level for the \( j \)-th direction vector. For theoretical convenience, we require \( s_j \) to be larger than the true sparsity \( s^*_j \) which is often standard. The algorithm is:
1. Initialization:

\[
\tilde{u}_j^{(0)} = \frac{1}{T} X_{-T}^T (X_{-0,j} - \overline{X}_{-0,j} \cdot 1_T);
\]  

(17)

where \(\overline{X}_{-0,j}\) denotes the mean of \(X_{-0,j} \in \mathbb{R}^T\), and \(1_T\) is the all-one vector in \(\mathbb{R}^T\).

Taking hard-thresholding and normalization to enforce the sparsity and unit norm, we have

\[
u_j^{(0)} = \frac{\Phi_{s_j}(\tilde{u}_j^{(0)})}{\| \Phi_{s_j}(\tilde{u}_j^{(0)}) \|_2}.
\]  

(18)

2. In each iteration \(k = 1, \cdots, K_j\),

(a) Compute iso \(X_{-T} u_j^{(k-1)}(X_{-0,j})\);

(b) Take an orthogonal pseudo gradient step,

\[
\tilde{u}_j^{(k)} = u_j^{(k-1)} + \eta_j \cdot \mathcal{P}_{u_j^{(k-1)}} \left( \frac{1}{T} X_{-T}^T \left[ X_{-0,j} - \text{iso}_{X_{-T} u_j^{(k-1)}(X_{-0,j})} \right] \right);
\]  

(19)

(c) Enforce sparsity and unit norm,

\[
u_j^{(k)} = \frac{\Phi_{s_j}(\tilde{u}_j^{(k)})}{\| \Phi_{s_j}(\tilde{u}_j^{(k)}) \|_2}.
\]  

(20)

(d) Stop when \(k = K_j\).

Note that for any given \(u \in \mathbb{R}^M\), the minimum of \(f \to L_j(f;u)\) over the monotone function class \(M\) can always be achieved, yet the minimizer is not unique over \(M\). In fact, it is only uniquely defined at the points \(\{X_T^T u\}_{T=0}^{T-1}\) (see theorem 2.1 in [Balabdaoui et al., 2019]). In other words, the best isotonic function that regresses \(X_{-0,j}\) on \(X_{-T} u_j^{(k-1)}\) is uniquely determined only at all the observed linear predictors \(X_{-T} u_j^{(k-1)}\), which take the value \(\text{iso}_{X_{-T} u_j^{(k-1)}(X_{-0,j})}\).

Such uniqueness makes the above estimation procedure work well, yet when it comes to prediction, new data outside of the support of previous observations may not give well-defined predictions. Thus we consider below the estimated monotone function \(f_j^{(k)}\) to be left continuous and piece-wise constant, with jumps only possible at \(T\) linear predictors \(\langle X_0, u_j^{(k-1)} \rangle, \cdots, \langle X_{T-1}, u_j^{(k-1)} \rangle\), for the sake of convenience. In practice, we would use the classical algorithm PAVA (Pool Adjacent Violators Algorithm) [Mair et al., 2009] to compute this least squares estimator \(\text{iso}_{X_{-T} u_j^{(k)}(X_{-0,j})}\) in each iteration step.

4. Main results

4.1 Assumptions

4.1.1 Assumptions for model identifiability

To ensure identifiability, we assume \(u_j^*\) lies on the unit sphere, i.e. \(\| u_j^* \|_2 = 1\); \(f_j^*\) is a monotonically non-decreasing function with \(L_j\)-Lipschitz continuity: for any \(x \in \mathbb{R}\) and
\[ \Delta_x > 0, \]
\[ 0 \leq f_j^*(x + \Delta_x) - f_j^*(x) \leq L_j \cdot \Delta_x. \]  
\[ (21) \]

Such non-parametric function class, due to its large complexity, still suffers from identifiability issues, thus we use the following condition to ensure that the model is identifiable: for \( j = 1, \ldots, M \), there exists a small relaxation term \( \epsilon_j \geq 0 \) and a positive value \( \alpha_j > 0 \), such that
\[
\frac{1}{N} \| f_j(X_{-T} \cdot u_j) - f_j^*(X_{-T} \cdot u_j^*) \|_2^2 \geq \alpha_j \| u_j - u_j^* \|_2^2 - \epsilon_j^2,
\]
for any monotonically non-decreasing \( f_j \) and \( s_j \)-sparse unit vector \( u_j \). Note that in this paper, for any univariate function \( f \) and any vector \( v \), \( f(v) \) denotes the vector of the same length as \( v \) with \( [f(v)]_i = f(v_i) \).

**Remark 1.** Suppose we further assume the function class containing \( f_j \) and \( f_j^* \) of interest could be uniformly lower bounded by a linear function. In that case, we know that assumption Eq. (22) is equivalent to the restricted eigenvalue condition (REC). Indeed, this assumption implicitly involves conditions analogous to restricted strong convexity to capture the dependence structure. In much of the literature, this condition can be verified in cases of independent design. While the dependence structure introduced in our autoregressive framework makes it a more complex condition to verify, such REC type of assumption is also included in Mark et al. (2019a) when dealing with dependent data.

In fact, the identifiability of the class of non-decreasing functions \( M_j \) in which \( f_j \) lies in, depends on the data structure of \( X_{-T} \) and properties of the true monotone function \( f_j^* \). For example, in order to include all constant functions in the identifiable function class, we need
\[
\inf_{c \in \mathbb{R}} \frac{1}{T} \| f_j^*(X_{-T} \cdot u_j^*) - c1_T \|_2^2 = \| f_j^*(X_{-T} \cdot u_j^*) - f_j^*(X_{-T} \cdot u_j^*)1_T \|_2^2 \geq 4\alpha_j - \epsilon_j^2,
\]
\[ (23) \]

---

**Algorithm 1:** AlternatingProjected Gradient Descent for monotone SIMAM

**Input:** \( M \)-dimensional time series \( \{X_0, X_1, \ldots, X_T\} \subset \mathbb{R}^M \);

**Parameters:** sparsity level \( s_j \), step size \( \eta_j \), iteration number \( K_j \), for any \( j \in [M] \);

**foreach** \( j = 1, \ldots, M \), **do**

Initialize with \( u_j^{(0)} = \frac{\Phi_{s_j}(\hat{u}_j^{(0)})}{\|\Phi_{s_j}(\hat{u}_j^{(0)})\|_2} \) where \( \hat{u}_j^{(0)} = \frac{1}{T}X_{-T}^T(X_{-0,j} - \bar{X}_{-0,j} \cdot 1_T) \);

**foreach** iteration \( k = 1, \ldots, K_j \), **do**

- Compute iso \( X_{-T}u_j^{(k-1)}(X_{-0,j}) \) with PAVA;
- \( \hat{u}_j^{(k)} = u_j^{(k-1)} + \eta_j \cdot P_{u_j^{(k-1)}}(\frac{1}{T}X_{-T}^T(X_{-0,j} - \text{iso}_{X_{-T}u_j^{(k-1)}}(X_{-0,j}))) \);
- \( u_j^{(k)} = \frac{\Phi_{s_j}(\hat{u}_j^{(k)})}{\|\Phi_{s_j}(\hat{u}_j^{(k)})\|_2} \);

**end**

**Output:** \( u_j^{(K_1)}, \ldots, u_j^{(K_M)} \).
i.e., the underlying signals \( \left\{ f_j^*(\langle X_0, u_j^* \rangle), f_j^*(\langle X_1, u_j^* \rangle), \ldots, f_j^*(\langle X_{T-1}, u_j^* \rangle) \right\} \) should have a variance no less than \( 4\alpha_j - \epsilon_j^2 \).

4.1.2 Noise distribution assumptions

For any \( j \in [M] \), the noise sequence \( \{Z_{t,j}\}_{t=1}^T \) are assumed to be a martingale difference sequence satisfying the \( \sigma_j \)-sub-Gaussian tail condition:

\[
E[|Z_{t,j}|] < \infty; \quad E[Z_{t,j}|\mathcal{F}_{t-1}] = 0; \quad E[e^{\lambda Z_{t,j}|\mathcal{F}_{t-1}}] \leq e^{\lambda^2 \sigma_j^2/2}, \quad \forall t = 1, \ldots, T. \quad (24)
\]

**Remark 2.** Being a martingale difference sequence, the noise sequence \( \{Z_{t,j}\} \) are allowed to be signal-dependent. Combined with the sub-Gaussian tail condition, such an assumption is weak enough to include many popular distribution assumptions for a time series. For example, \( \{Z_{t,j}\} \) could be independent mean-zero Gaussian noise with variance \( \sigma_j^2 \), which is a common assumption in analyzing continuous data. Another popular situation is to deal with the count data, when \( Z_{t,j}, t = 1, \ldots, T \) are typically signal-dependent noises from the Poisson Auto-Regressive (PAR) model:

\[
X_{t,j} | \mathcal{F}_{t-1} \sim \text{Poisson}(f_j^*(X_t^T u_j^*)). \quad (25)
\]

In this case, \( \{Z_{t,j}\}_{t=1}^T \) is still a martingale difference sequence. Although the Poisson tail is heavier than the sub-Gaussian tail, we can always conduct a truncation on the tails to make it sub-Gaussian. In practice, as long as the noise is bounded by a constant \( \sigma_j \), i.e., \( |Z_{t,j}| \leq \sigma_j \), the \( \sigma_j \) sub-Gaussian condition can be met.

4.1.3 Assumptions for data boundedness

We assume the observed data \( X_{-T} = (X_0^T, \ldots, X_{T-1}^T)^T \in \mathbb{R}^{T \times M} \) is entry-wise bounded:

\[
\max\{|X_{t,j}| : t \in \{T-1\} \text{ and } j \in [M]\} \leq M_x < \infty. \quad (26)
\]

The eigenvalues of \( X_{-T} \) are also upper bounded in the sparse setting:

\[
\frac{1}{T} \| X_{-T} \cdot u \|_2^2 \leq \beta \| u \|_2^2, \quad \forall u \in \mathcal{S}^{p-1} \text{ with sparsity at most } \max_j (2s_j + s_j^*). \quad (27)
\]

4.2 Convergence Guarantee

The following main theorem (Theorem 3) gives the non-asymptotic result for our network estimator from alg. 1 which indicates that our algorithm alg. converges at a geometric rate, and after sufficiently many iterations, it converges to the statistical error with the rate \( O(T^{-\frac{1}{4}} s \log(TM)) \) up to poly-log terms.

**Theorem 3.** Suppose the \( M \)-dimensional multi-variate time series data follows the monotone SIMAM in Eq. (8), and the assumptions for Lipschitz continuity (Eq. (21)), model identifiability (Eq. (22)), sub-Gaussian martingale noise (Eq. (24)) and boundedness (Eq. (26), Eq. (27)) are satisfied. Denote \( \Delta_j \) as the quantity listed in Eq. (43). For any \( j \in [M] \), after
running alg. 1 for $K$ times, with the step size being $\eta_j = \frac{1}{L_j \beta}$ and hard-thresholding sparsity level $s_j$ satisfying
\begin{equation}
\tag{28}
s_j^* < s_j \cdot \min \left\{ 1 - \frac{\delta_j \alpha_j}{L_j \beta}, \frac{\epsilon_j^2}{2 L_j^2 \beta} - \frac{\Delta_j^2}{2 \delta_j \alpha_j \beta} \right\}^2
\end{equation}

for some $0 < \delta_j < 1$, the following bound holds with probability at least $1 - 4\gamma^{2s_j}$:
\begin{equation}
\| u_j^{(K)} - u_j^* \|^2 \leq 2\theta_j^2 + R_j^2,
\end{equation}

where $\theta_j = \frac{1 - \frac{\alpha_j}{L_j \beta}}{1 - \frac{\delta_j \alpha_j}{L_j \beta} - \sqrt{\frac{s_j^*}{s_j}}} < 1$,
\begin{equation}
\text{and } R_j^2 = \frac{\epsilon_j^2}{L_j^2 \beta} + \frac{\Delta_j^2}{\delta_j \alpha_j \beta} = O \left( T^{-\frac{2}{3}} s_j \log \left( \frac{TM}{\gamma} \right) \right),
\end{equation}

as long as we have a warm initialization satisfying $\langle u_j^{(0)}, u_j^* \rangle \geq 0$.

Therefore, for any tolerance $\tau_j > R_j$, running alg. 1 for $K_j \geq \frac{\log(\epsilon_j^2 - R_j^2)}{\log \theta_j}$ many iterations will guarantee that $\| u_j^{(K)} - u_j^* \|_2 \leq \tau_j$. With the conditions in Theorem 3 satisfied, let $s = \max_{j=1,...,M} s_j$ be the thresholded maximum in-degree of the network, after running alg. 1 for sufficiently many iterations in the $j$-th co-variate for $j \in [M]$, the influence network estimator $\hat{A} = (u_1^{(K_1)}, \ldots, u_M^{(K_M)})$ approaches the truth $A^*$ with the rate
\begin{equation}
\| \hat{A} - A^* \|_F \approx O_p \left( T^{-\frac{1}{3}} s \log(TM) \right).
\end{equation}

Remark 4. The remaining term $R_j$ gives the statistical error bound in Theorem 3. To assure the convergence, the sparsity level $s_j$ used in the hard-thresholding projection should be larger than its truth $s_j^*$, which is commonly needed in the projected gradient descent literature (see e.g., Jain et al. (2014)), due to the greedy nature of IHT algorithm. The trade-off is that a larger sparsity threshold $s_j$ speeds up the algorithm convergence, while sacrificing the statistical error rate in $R_j$, as is indicated in Theorem 3.

In Theorem 3 a good starting point for the algorithm that has an acute angle with the truth is needed to ensure the convergence. The following lemma shows that our proposed initialization in alg. 1 satisfies this condition with high probability.

Lemma 5. (Initialization guarantee) The angle between the initialization $u_j^{(0)}$ in alg. 1 and the truth $u_j^*$ is acute for any $j \in [M]$:
\begin{equation}
\langle u_j^{(0)}, u_j^* \rangle > 0, \quad j \in [M],
\end{equation}
as long as \( s_j > s_j^* \cdot \max\{1, \left( \frac{L_j}{\sqrt{2(4\alpha_j - \epsilon_j^2)}} + O\left(\frac{\beta L_j^3}{T}\right)\right)^2 \}, \) with high probability \( 1 - 2(M + 1) \exp\left(-\frac{T U_j^+}{8s_j^* M^2}\right), \) where

\[
U_j^+ = \frac{1}{2} \left( \frac{1}{s_j} + 4 \left( \frac{1}{L_j} \sqrt{s_j s_j^*} - \frac{2\beta}{T} (4\alpha_j - \epsilon_j^2) - \frac{1}{2s_j} \right) - \sqrt{1} \right). \tag{32}
\]

**Proposition 6.** For each \( j \in [M], \) after running alg. \( \mathcal{I} \) with a sufficient number of iterations such that the error \( \|\hat{u}_j - u_j^*\|_2 \) of the \( j \)-th variable estimate \( \hat{u}_j \) is dominated by its statistical error term \( R_j = O_p(T^{-1/3}) \) up to some poly-log terms in Theorem \( 3, \) with high probability we have

\[
\frac{1}{\sqrt{T}} \| \text{iso}_X_{-\hat{u}_j}(X_{-0,j}) - f_j^*(X_{-Tu_j}) \|_2 \leq O_p \left( T^{-\frac{1}{3}} \log(T) \right). \tag{33}
\]

Let \( \hat{f}_j \) denote the monotone function estimated based on \( \hat{u}_j \) (see Eq. (14)), then the result in Proposition 6 can also be expressed as:

\[
\sqrt{\frac{1}{T} \sum_{t=0}^{T-1} \left[ \hat{f}_j(X_t^\top \hat{u}_j) - f_j^*(X_t^\top u_j^*) \right]^2} \leq O_p \left( T^{-\frac{1}{3}} \log(T) \right), \tag{34}
\]

which demonstrates that Proposition 6 essentially gives the in-sample prediction error bound.

### 5. Simulation Study

In this section, we validate our theoretical results and explore the properties of monotone SIMAM on synthetic data. For a given number of time points \( T \) and node size \( M, \) we construct a large sparse \( M \times M \) coefficient matrix whose \( j \)-th column \( u_j^* \) lies on a unit sphere and has \( s_j^* \) non-zero elements for any \( j \in [M]. \) We define a sequence of monotone nonlinear functions \( \{f_j^*\}_{j=1}^M \) in the form of

\[
f_j^*(x) = \frac{\exp(j \cdot x)}{\exp(j \cdot x) + 1}, \quad j \in [M]. \tag{35}
\]

These functions are isotonic, Lipschitz continuous and well-bounded. When \( j = 1, \) \( f_j^* \) is the logistic link function. As the value of \( j \) increases, the nonlinearity of \( f_j^*(x) \) increases accordingly. First we consider the noise variables to be drawn from a Gaussian distribution. Using the equation \( X_{t+1,j} = f_j^*(X_t^\top \beta_j) + Z_{t+1,j}, \) where \( \{Z_{t,j}\} \) for \( j \in [M], t \in [T] \) is the sequence of noise generated independently from \( N(0, \sigma^2), \) we define a multi-variate time series \( \{X_t\}_{t=0}^T \) whose initial vector \( X_0 \) is randomly generated from a normal distribution \( N(0, I_M). \) Specifically, we take the dimension \( M = 9, \) the sparsity \( s_j^* = 3, \forall j \in [M] \) and noise level \( \sigma = 0.05. \) The sample size \( T, \) also referred to as the length of the time series, takes values from an integer sequence \( \{50 \times (i + 1)\}_{i=1}^{20}. \)
Given the above generated time series \( \{X_0, X_1, \ldots, X_T\} \), we estimate the underlying network structure by using alg. 1. We take sparsity thresholds in alg. 1 as \( s_j = 4 \) for \( j = 1, \ldots, 9 \), which are slightly larger than the true sparsity levels \( (s^*_j = 3) \). We set constant step-sizes \( \eta_j \) for \( j = 1, \ldots, 9 \) as 0.1, and the maximum steps for iteration for all dimensions are set to be sufficiently large enough as \( K_j = 2000, j = 1, \ldots, 9 \). Once the direction vectors \( \{\hat{u}_1, \ldots, \hat{u}_9\} \) are estimated, we calculate the RMSE (square root of mean squared error) in the form of \( \sqrt{\frac{1}{T} \sum_{j=1}^{9} \| \hat{u}_j - u^*_j \|_2^2} \), which is in fact the Frobenius norm of the influence network estimation error: \( \| \hat{A} - A^* \|_F \).

For each given sample size \( T \in \{100, 150, \ldots, 1050\} \), we independently generate 100 multi-variate time series of length \( T \) with different random seeds and perform the above estimation procedure repeatedly for each of the 100 realizations. At each sample size, we take the average over the 100 network estimation errors. Plotting the network estimation against the sample size \( T \) and in particular against \( T^{-1/3} \), as is shown in Fig. 1, we find that the estimation error converges in the rate \( O(T^{-1/3}) \) which supports the theoretical analysis.

![Convergence of network estimation error](image.png)

**Figure 1:** The convergence of the network (coefficient matrix) estimation in RMSE with respect to the sample size

Another thing we are interested in is whether the isotonic SIMAM framework works better when there is unknown non-linear structure in the data compared with other popular autoregressive methods in terms of prediction. We start with multivariate time series of node size \( M = 9 \) with monotone functions \( f_2, \ldots, f_{10} \) in Eq. (35). (In the later example, we will increase the node size to \( M = 36 \) to see the prediction performance in a higher-
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dimension setting.) The true coefficient matrix $A^*$ is randomly generated satisfying

$$\|A^*_j\|_2 = 1; \|A^*_j\|_0 = s^*_j = 3, \text{ for any } j \in [9],$$

where $A^*_j$ is the $j$-th column of $A^*$. Two types of noise distributions are considered: (a) Gaussian noise from $N(0, 0.05^2)$; (b) Bounded Uniform noise from $\text{Uniform}(-0.1, 0.1)$. The total number of observations is 1000, split into a training set (the first 9/10 samples) and a testing set (the remaining 1/10 samples).

As indicated in the main result (see Theorem 3), as long as the initialization satisfies $\langle u_j^{(0)}, u^*_j \rangle \geq 0$, the convergence of alg. 1 is guaranteed. Therefore alg. 1 can be adapted by replacing the initialization in Eq. (17) with other initializations satisfying the above criteria. Here, we use the solutions from the LASSO method as a warm start for alg. 1 to approximate SIMAM model. The step-size for alg. 1 is taken as 0.01, and the hard-threshold levels $s_j, j \in [M]$ are still taken as 4, not accurate but slightly larger than the true parameters $s^*_j = 3$. The prediction results in MSE are evaluated both on the training (in-sample) and test (out-of-sample) data. As is shown in Fig. 2, for both noise types of data, SIMAM performs better not only on the in-sample data, but also on the out-of-sample data with a better generalization performance. The convergence is achieved within 100 steps, yet the generalization error increases slightly after the convergence in Fig. 2b, suggesting that an early stopping after the convergence is recommended.

In fact, as the dimension (node size) increases, such superior performance of SIMAM compared with LASSO still exists. In Fig. 3, we increase the dimension from $M = 9$ to $M = 36$, with the number of time points unchanged. We use the same set of monotone functions to introduce the nonlinear structure: for the $j$-th node, we generate the data with monotone function $f_l(x)$ in Eq. (35) where $l = [j \mod 9] + 1, \forall j \in [36]$. The true sparsity level in this case changes to $s^*_j = 6$. We still consider two types of noises: (a) Gaussian noise from $N(0, 0.05^2)$; (b) Bounded Uniform noise from $\text{Uniform}(-0.1, 0.1)$. To solve SIMAM in this case, the hard-threshold levels increases to $s_j = 8$ accordingly. The step-size is still 0.01, and we start alg. 1 with the corresponding LASSO solutions. As shown in Fig. 3 when the node size has been enlarged, SIMAM still has a better prediction performance than LASSO in terms of both training and test prediction error.

6. Real Data Examples

We validate our methodology and the main hypothesis on the Chicago crime data set and the MemeTracker data set. One challenge of real-data network estimation is the validation since there is no obvious ground truth. For both applications, we provide two types of validations:

1. Out-of-sample prediction performance showing that SIMAM fits the real data well without underfitting or overfitting compared to other popular learning methods;

2. External knowledge of the influence network that are not included in the data set when training the model. For the Chicago crime example, we use the geographical information of communities to validate our learned community clusters. For the MemeTracker example, we use ‘time lag’ and ‘pct of top quotes’ indices for media influence to evaluate the top influential media sites we learned.
Figure 2: Prediction Mean Squared Error (MSE) by LASSO and SIMAM for multivariate time series with $M = 9$ nodes, 900 samples for training (in-sample) and 100 samples for testing (out-of-sample). The left and right panels are from data generated by a SIMAM model respectively with Gaussian $N(0, 0.05^2)$ noise and Uniform($-0.1, 0.1$) noise. For the SIMAM model, we use alg. 1 with the LASSO solution as a warm start. Note that in all the panels, the $x$-axis denotes the iteration step for SIMAM in alg. 1, the prediction MSE by LASSO does not change along these iterations, making it a horizontal line. The experiment is repeated on 50 independently generated time series data with different random seeds.

6.1 Chicago Crime Data

We begin by seeing how our method performs on inferring a crime network based on records of Chicago crimes. Inferring the patterns of the crime locations over time and predicting the number of crime events in different areas could help the police forces work better and provide on-time security information for residents [Daniel Rivera Ruiz (2019)], which has been investigated by a number of studies, including Stomakhin et al. (2011a); Mark et al. (2019a); Zhou and Raskutti (2018). In the following, we use the monotone SIMAM method to conduct inference and prediction for the Chicago crime data, comparing with other popular methods.

Specifically, the crime data in 77 pre-defined community areas of Chicago from Jan 2004 to Dec 2018 are collected, focusing on severe types of crimes in each area (including homicide and battery) with a 2-day discretization. The first 90% of the data (before June 30, 2017) are regarded as the observed training set, while the remaining 10% (final 18-month) data from July 2017 to Dec 2018 would serve as an out-of-sample test set to evaluate the
prediction performance. To learn the SIMAM model, among the training set with 2465 time points, the last $1/10$ part would be held out for tuning a good time $K_j$ ($K_j \leq 500$) for alg. 1 to stop the iteration in the $j$-th area, where $j \in \{M\}$, $p = 77$. Besides, we use 0.02 as the step size, and cross validated sparsity from LASSO as the sparsity levels for hard-threshold in alg. 1.

We then obtain the estimated coefficient matrix $\hat{A} = (\hat{u}_1, \ldots, \hat{u}_M)$ that contains the information of the influence network, and the nonlinear monotone functions $\hat{f}_j, j = 1, \ldots, 77$. In Fig. 4 some example monotone functions extracted from the Chicago crime data are given, indicating that there exist highly nonlinear and heterogeneous structure.

In terms of predicting the number of the crime events in different areas, we observe a significant improvement by using monotone SIMAM, compared with the popular high dimensional multi-variate autoregressive methods including VAR (vector autoregressive model), LASSO (VAR with $\ell_1$ penalty), and PAR LASSO (generalized Poisson autoregressive model with $\ell_1$-regularization). Specifically, after constructing the above models based on the training set, we predict the crime counts in the test set, and then calculate the out-of-sample prediction RMSEs for each of the 77 pre-defined Chicago areas.

To see whether SIMAM improves the prediction performance, since the crime counts from different Chicago areas are highly heteroscedastic, a paired t-test among the prediction RMSEs over the 77 areas is recommended to compare the out-of-sample prediction performances. As shown in Fig. 5, the differences of RMSEs in all 77 areas between any baseline method and SIMAM tend to be larger than zero. More rigorously, 3 sets of paired
Figure 4: Example monotone functions extracted from Chicago crime data using alg. 1

50 100 150
0 2 4 6
x
y

y = ˆf6(x)

y = ˆf9(x)

y = ˆf10(x)

Figure 5: The prediction RMSE differences between baseline methods and SIMAM in out-of-sample Chicago crime data. For each of the 77 pre-defined Chicago areas, by using VAR, PAR LASSO, LASSO and SIMAM method, the crime counts are predicted and the prediction errors in the test set are calculated in terms of RMSE. We take the difference between errors from the baseline methods and SIMAM for each area. In the figure, all these 77 prediction RMSE differences for each baseline method are given.

To infer the patterns of crime locations over time, we conduct a spectral clustering based on the coefficient matrix ˆA derived from the above SIMAM procedure. Specifically, we first transform the coefficient matrix to an undirected adjacency matrix ˜A, by replacing all positive entries in the coefficient matrix with 1 and otherwise with 0, followed by a
Table 1: Paired t-tests for prediction RMSEs in the out-of-sample Chicago crime data. As shown in the table, the p-values for the null hypotheses that SIMAM doesn’t have a lower prediction RMSE than PAR LASSO, LASSO and VAR are all smaller than 0.05, indicating that SIMAM has a significantly lower prediction error in the test set than the others.

| Chicago Crime | Method 1  | Method 2   | p-value | p.signif | alternative |
|---------------|-----------|------------|---------|----------|-------------|
| RMSE          | SIMAM     | PAR LASSO  | 0.0317  | *        | <           |
| RMSE          | SIMAM     | LASSO      | 0.00709 | **       | <           |
| RMSE          | SIMAM     | VAR        | 8e-9    | ****     | <           |

symmetrization with the or-operator:

\[
\tilde{A}_{ij} = \begin{cases} 
1, & \text{if } \hat{A}_{ij} > 0 \text{ or } \hat{A}_{ji} > 0; \\
0, & \text{else.}
\end{cases}
\] (37)

Therefore \(\tilde{A}_{ij} = 1\) means that the crime events in the \(i\)-th area influence or is influenced by the \(j\)-th area. Having acquired the adjacency matrix \(\tilde{A}\), we apply the standard spectral clustering algorithm with cluster number \(K_{\text{cluster}} = 4\) (see e.g., [Rohe et al. (2011); Zhou and Raskutti (2018)]). Hence, we obtain a block clustering for the patterns of crime locations in Chicago only according to our estimated coefficient matrix. The result is shown in Fig. 6 with colors indicating cluster membership. Note that without any knowledge of the geographical information such as latitudes and longitudes in the data, the clustering results based on the SIMAM estimated coefficients have clear patterns that conform with actual geographical locations, providing some validation to the estimated influences of crime events among these areas.

6.2 MemeTracker: social media data

An interest in social networks is to infer the influence network among media sources, based on which one can have a better sight of the information flow. We collect news event data for 197 sources of media in a period from August 2008 to December 2009, with one-hour discretization. Thus the total sample size of such time series is 3602 ([Mark et al., 2019a]).

We use popular methods including LASSO, PAR LASSO and VAR to analyze this MemeTracker data set as baseline methods to compare with our proposed monotone SIMAM. When we implement alg. 1 for SIMAM, the maximum number of iterations is set to be 100, up to which we have observed a good performance in convergence and accuracy. Step sizes \(\eta_j, j \in [197]\) are equally fixed as 0.05, according to pre-experiment. To avoid underfitting or overfitting, we split the data into three parts with the proportions 8 : 1 : 1 (training : validation: testing), through which the iteration stop time is tuned separately for each of the 197 media source based on the prediction performance on the validation set. The sparsity levels for hard-thresholding are estimated by the cross validation procedure in LASSO, i.e., for each \(j \in [197]\), we use the \(\ell_0\) norm of the LASSO solution under the sparsity parameter chosen by cross validation as the sparsity threshold \(s_j\).

Once we have constructed the all the abovementioned models within the training set, we calculate their prediction RMSEs on the out-of-sample data for each of the 197 media
Chicago Crime Clustered with SIM

Figure 6: Clusters learned from crime data using SIMAM. The clusters are overlaid on a map of community areas in Chicago. No geospatial information is provided in the data, but clusters show geographical patterns.

sources. Since the post numbers from different media sources might be highly heteroscedastic, a paired t-test is recommended to test the differences of prediction RMSEs between any baseline method and SIMAM. The out-of-sample prediction RMSE differences are visualized in Fig. 7. Following that, three paired t-tests are conducted with null hypotheses that SIMAM has no smaller prediction RMSEs than ‘Method 2’ in Table 2. The p-values as shown in Table 2 are all smaller than $10^{-4}$, indicating a very strong evidence that SIMAM has smaller prediction RMSEs than other methods.

| MemeTracker Method 1 Method 2 p-value p.signif alternative |
|-----------------|------------------------------|-----------------|-----------------|-----------------|
| RMSE SIMAM LASSO 5.06e-04 *** < |
| RMSE SIMAM PAR LASSO 3.56e-06 **** < |
| RMSE SIMAM VAR 7.72e-20 **** < |

Table 2: Paired t-tests for prediction RMSEs in the out-of-sample MemeTracker data. As shown in the table, the p-values for the null hypotheses that SIMAM doesn’t have a lower prediction RMSE than PAR LASSO, LASSO and VAR are all smaller than 0.05, indicating that SIMAM has a significantly lower prediction error in the test set than the others.
Figure 7: The prediction RMSE difference between baseline methods and SIMAM in out-of-sample MemeTracker data. For each of the 197 media sources, by using VAR, PAR LASSO, LASSO and SIMAM method, the crime counts are predicted and the prediction errors in the test set are calculated in terms of RMSE. In the figure, all these 197 prediction RMSE differences for each baseline method are given.

Since the element of the coefficient matrix $A_{ij}$ indicates the influence of the $i$-th media source on the $j$-th media source for any $i, j \in [197]$, the $i$-th row sum of the coefficient matrix $A$ can measure the overall influence of the $i$-th media source upon the entire investigated social media network. Therefore, to examine the structure of our learned coefficient matrix $\hat{A}_{SIMAM}$ using SIMAM compared with $\hat{A}_{LASSO}$ using LASSO, we add up each row of $\hat{A}_{SIMAM}$ and $\hat{A}_{LASSO}$. We then rank the influences of the $i$-th media source respectively by $\sum_{j=1}^{197} \hat{A}_{ij}^{SIMAM}$ and $\sum_{j=1}^{197} \hat{A}_{ij}^{LASSO}$.

We therefore extract the top 20 media sources with the first 20 largest row sums among all 197 sources, using both Lasso and SIMAM. To see whether these learned top influential media sources are indeed influential, we use some external knowledge of these media sources provided in [http://snap.stanford.edu/memetracker/](http://snap.stanford.edu/memetracker/). The measures of influence for media sources include:

1. **Time lag**: number of hours between the time a media site first reported a story and when the story (quote) reached its peak. Negative times mean that site reported the news before it reached its peak, and positive numbers mean that the site was lagging and only reported the news after it reached its peak.

2. **Pct of top quotes**: Fraction of top stories (quotes) the site covered. The higher the number, the more important news was covered by the site.

Hence, to compare the actual influence of the top 20 media sources learned respectively from SIMAM and LASSO, we look up their ‘time lag’ and ‘pct of top quotes’ indices in the MemeTracker website. As shown in Table 3, the top 20 media sources extracted by SIMAM have significantly lower negative time lags, and higher fractions of top quotes on average than LASSO. Therefore the media sources learned by SIMAM are more ‘influential’, in the sense that they tend to report hot stories earlier and have more top stories (quotes) covered than the ones learned from LASSO. This also gives some validation that SIMAM can help us to have a better sight of the information flow in the social media network.
Table 3: Top 20 influential media sources learned by SIMAM and LASSO. In the table, the column ‘media source’ contains the top 20 media sources learned respectively by SIMAM and LASSO; we evaluate the quality and influence of these learned media sources by two external indices: ‘time lag’ contains the corresponding time lag indices for each media source, the smaller the number, the earlier the media source report the hot news; ‘pct of top’ represents the percentage of top stories (quotes) a media source covered, the larger the number, the more important news was covered. Both ‘time lag’ and ‘pct of top’ are not provided in the data.

7. Proof Overview

One of the major challenges to prove the theoretical results comes from the unknown non-linear structures in the monotone SIMAM. Compared to prior methods such as the AR model and GLAR model, whose proofs rely heavily on the parametric assumptions and the constraints of the parameters, SIMAM introduces a set of highly non-linear functions that are unknown, making it hard to find analytical solutions and optimize simultaneously over the direction vectors and the functions. The theoretical guarantees for the estimator from alg. 1 are more challenging to derive than the previous MLE type of estimators, since the optimization of the unknown monotone function changes each time after the direction vector is updated, and vice versa for estimating the direction vector. To tackle this challenge, we utilize the contractiveness property of isotonic functions (Lemma 16) and the bounded complexity of the reference space (Lemma 19), so as to give the uniform bound over all possible isotonic reference vectors to control the deviation regardless of the changes in the alternating procedure (see Lemma 9).
Another technical challenge comes from the dependence structure of the data, which makes the proofs for single index models in the i.i.d case no longer suitable. Based on the conditions analogous to the restricted strong convexity implicitly included in the identifiability assumption (Eq. (22)), and the mild assumption for the noise (Eq. (24)), we use the martingale concentration inequalities to prove the non-asymptotic results for the dependent data.

In the following, we provide the major steps to prove Theorem 3, while the mathematical details and complete proofs for other results would be deferred to the appendix (Section 9).

7.1 Iterative Guarantee

To prove Theorem 3, we first show that after each iteration step, the updated estimator $u_j^{(k)}$ get closer to the true parameters $u_j^*$ compared to $u_j^{(k-1)}$ in a linear convergence manner with a remainder term, for any $j \in [M], k \geq 1$.

Lemma 7. For any $j \in [M]$ and iteration step $k \in \mathbb{N}^+$, suppose the assumptions Eq. (21), Eq. (22) and Eq. (27) are satisfied, as long as $\langle u_j^{(k-1)}, u_j^* \rangle \geq 0$, we have

$$
(1 - \sqrt{s_j^* / s_j}) \| u_j^{(k)} - u_j^* \|^2 \leq (1 - \frac{\alpha_j}{L_j^* \beta}) \| u_j^{(k-1)} - u_j^* \|^2 + \frac{c_j^2}{L_j^* \beta} + \frac{C^2(Z, u_j^{(k-1)})}{\delta_j \alpha_j \beta}
$$

for any positive scalar $\delta_j > 0$, with the numerator of the last term formulated as

$$
C(Z, u_j^{(k-1)}) = \sqrt{\frac{2s_j + s_j^*}{N}} \| X_{-N}^T (X_{-0,j} - f_j^* (X_{-N} \cdot u_j^*)) \|_\infty
$$

(39)

where $v_{k-1} = X_{-N} \cdot u_j^{(k-1)}$.

7.2 Bounding the remainder term

To further control $C(Z, u_j^{(k-1)})$ in the remaining term for every iteration step $k = 1, 2, \ldots$ which determines the statistical error bound, the following two lemmas are provided that can reveal some rationales behind alg. 1.

Lemma 8. With the martingale difference assumption Eq. (24) and the data boundedness assumption Eq. (26) satisfied, for any $j \in [M]$, we have

$$
\frac{1}{M} \| X_{-M}^T (X_{-0,j} - f_j^* (X_{-M} \cdot u_j^*)) \|_\infty \leq \frac{\sigma_j M x}{\sqrt{M}} \sqrt{2 \log \left( \frac{2M}{\gamma} \right)}
$$

(40)

with probability at least $1 - \gamma$.

Lemma 8 essentially controls the magnitude of the martingale difference $\{X_{t,j} - f_j^* (X_{t-1} u_j^*)^T \}_{t=1}$ after projecting them on the space spanned by $\{X_{t-1}\}$. The following lemma, which is more complicated than the former, controls the distance between the node observations and their
corresponding conditional expectations after both are isotonically projected with respect to a reference vector $v$. Since we desire to bound this term no matter how the reference vector $v$ varies in the iteration procedure, we need to derive a uniform bound for all possible reference vectors.

**Lemma 9.** For any $j \in [M]$, if the assumptions Eq. (21), Eq. (22), Eq. (24) and Eq. (26) are satisfied, the following 2-norm distance between isotonically projected observations $X_{0,j}$ and their corresponding conditional means $f_{j}^{*}(X_{-T}u_{j}^{*})$ can be uniformly bounded over the set $V = \{v = X_{-T}u \in \mathbb{R}^{T}: u \in S^{M-1} \text{ with sparsity } s_{j}\}$ with probability at least $1 - 2\gamma$,

\[
\sup_{v \in V} \frac{1}{\sqrt{v}} \| iso_{v}(f_{j}^{*}(X_{-T}u_{j}^{*})) - iso_{v}(X_{-0,j}) \|_{2} \leq 4T^{-\frac{1}{3}} \left[ \left( 2 \frac{2\sigma_{j}^{2} \log(2T^{2}s_{j}M_{s_{j}})}{\gamma} + 2L_{j}M_{x}\sqrt{s_{j}} \right) \sigma_{j}^{2} \log\left( \frac{T^{2}s_{j}(T+1)M_{s_{j}}}{\gamma} \right) \right]^{\frac{1}{3}}
\]

With Lemma 8 and Lemma 9, we could then get the non-asymptotic uniform bound for the remaining term $C(Z_{j}, u_{j})$ over all possible $u_{j}$, the rate of which would determine the statistical error bound in the main result.

**Corollary 10.** Denote $B_{0}(s_{j}) = \{u \in \mathbb{R}^{M}: \|u\|_{2} = 1, \|u\|_{0} = s_{j}\}$ as the set of all direction vectors with sparsity level $s_{j}$. Suppose assumptions listed in Lemma 8 and Lemma 9 are all satisfied, then with probability at least $1 - 3\gamma$, we have

\[
\sup_{u \in B_{0}(s_{j})} C(Z_{j}, u) \leq \Delta_{j} = T^{-\frac{1}{3}} \sigma_{j} \left[ M_{x} \sqrt{2(2s_{j} + s_{j}^{*}) \log\left( \frac{2M}{\gamma} \right)} + 2\sqrt{\beta \log\left( \frac{T^{2}s_{j}(T+1)M_{s_{j}}}{\gamma} \right)} \right]^{\frac{1}{2}} + 4T^{-\frac{1}{3}} \sqrt{\beta} \left[ 2^{2s_{j}^{*}} \log\left( \frac{T^{2}s_{j}(T+1)M_{s_{j}}}{\gamma} \right) + 2L_{j}M_{x}\sqrt{s_{j}} \right] \sigma_{j}^{2} \log\left( \frac{T^{2}s_{j}(T+1)M_{s_{j}}}{\gamma} \right) \right]^{\frac{1}{3}}
\]

\[
= O\left( T^{-\frac{1}{3}} \left[ \log\left( \frac{T^{2}s_{j}+1M_{s_{j}}}{\gamma} \right) \right]^{\frac{1}{2}} \right).
\]

### 7.3 Mathematical Induction

Based on the above lemmas, we use the mathematical induction to complete the proof of Theorem 3.

**Proof** Given a good initialization $u_{j}^{(0)}$ such that $\langle u_{j}^{(0)}, u_{j}^{*} \rangle \geq 0$, we know that

\[
\|u_{j}^{(0)} - u_{j}^{*}\|_{2}^{2} = \|u_{j}^{(0)}\|_{2}^{2} + \|u_{j}^{*}\|_{2}^{2} - 2\langle u_{j}^{(0)}, u_{j}^{*} \rangle \leq 2.
\]

\[
(44)
\]
Recall the definition
\[ \theta_j = \frac{1 - \frac{\alpha_j}{L_j^2 \beta}}{1 - \frac{\delta_j \alpha_j}{L_j^2 \beta} - \sqrt{\frac{s_j^*}{s_j}}} \]
for some \( \delta_j > 0 \), and \( R_j = \frac{c_j^2}{L_j^2 \beta} + \frac{\Delta_j^2}{\delta_j \alpha_j \beta} \) \( \sqrt{\frac{s_j^*}{s_j}} \).

Now we use mathematical induction to prove the result for any \( j \in [M] \). By Eq. (44), when \( k = 0 \), \( \| u_j^{(k)} - u_j^* \|_2^2 \leq 2 \theta_j^k + (1 - \theta_j^k) R_j \) is true. For \( K \geq 1 \), assume
\[ \| u_j^{(k)} - u_j^* \|_2^2 \leq 2 \theta_j^k + (1 - \theta_j^k) R_j \] \( \text{(46)} \)
holds for all \( k \leq K - 1 \). First, we want to show that for these \( k \leq K - 1 \), once the induction assumption (Eq. (46)) is true, the angle between these estimators and the truth is acute. With the sparsity condition (Eq. (28)) satisfied, we have that \( R_j \leq 2 \). Therefore,
\[ \langle u_j^{(k)}, u_j^* \rangle = \frac{1}{2} \left( \| u_j^{(k)} \|_2^2 + \| u_j^* \|_2^2 - \| u_j^{(k)} - u_j^* \|_2^2 \right) \]
\[ \geq 1 - \frac{1}{2} \left[ 2 \theta_j^k + (1 - \theta_j^k) R_j \right] \]
\[ \geq 1 - \frac{1}{2} \left[ 2 \theta_j^k + 2(1 - \theta_j^k) \right] = 0. \] \( \text{(47)} \)

With this condition satisfied for all \( k \leq K - 1 \), by Lemma [7] and Corollary [10] we have
\[ \| u_j^{(K)} - u_j^* \|_2^2 \leq \theta_j \| u_j^{(K-1)} - u_j^* \|_2^2 + \frac{c_j^2}{L_j^2 \beta} + \frac{\Delta_j^2}{\delta_j \alpha_j \beta} \]
\[ \cdot \left( 1 - \frac{\delta_j \alpha_j}{L_j^2 \beta} - \sqrt{\frac{s_j^*}{s_j}} \right) \]
\[ = \theta_j \| u_j^{(K-1)} - u_j^* \|_2^2 + (1 - \theta_j) R_j \]
\[ \leq \theta_j \left[ 2 \cdot \theta_j^{K-1} + (1 - \theta_j^{K-1}) \cdot R_j \right] + (1 - \theta_j) R_j \]
\[ = 2 \theta_j^K + (1 - \theta_j^K) R_j. \] \( \text{(48)} \)

Hence, the induction reveals that
\[ \| u_j^{(K)} - u_j^* \|_2^2 \leq 2 \theta_j^K + (1 - \theta_j^K) R_j \leq 2 \theta_j^K + R_j \] \( \text{(49)} \)
is true for any \( K \in \mathbb{N}^+ \).

8. Conclusion

In this paper, we construct the monotone SIMAM to improve the prediction and influence network estimation for high-dimensional multi-variate time series or point process data. Such semi-parametric formulation gains more flexibility and robustness against model
mis-specification due to its enlarged model space while retaining parametric models’ desirable interpretation. Based on this model, we developed an alternating PGD algorithm for SIMAM (alg. 1) to estimate the underlying network structure, which takes the non-convex sparsity constraint in a high dimensional setting into account. Theoretically, using martingale concentration inequalities, we show that our algorithm converges in a geometric rate, and after sufficiently many iterations, the statistical error for the influence network estimation achieves the rate \( O(T^{-\frac{3}{s}}\sqrt{s\log(TM)}) \). We also demonstrated that our algorithm for monotone SIMAM has a superior performance both on simulated data and two popular real data examples (Chicago crime data and MemeTracker social media data) compared to state-of-the-art parametric methods.
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9. Appendix

9.1 Proof of Lemma \[7\]

Proof Using three point identity, we have

\[
\| \Phi_{s_j}(u_j^{(k)}) - u_j^* \|_2^2 = \| u_j^{(k-1)} - u_j^* \|_2^2 - \| u_j^{(k-1)} - \Phi_{s_j}(u_j^{(k)}) \|_2^2 + 2\langle u_j^{(k-1)} - \Phi_{s_j}(u_j^{(k)}), u_j^* - \Phi_{s_j}(u_j^{(k)}) \rangle;
\]

(50)

First we bound the third term in Eq. (50), through

\[
\langle u_j^{(k-1)} - \Phi_{s_j}(u_j^{(k)}), u_j^* - \Phi_{s_j}(u_j^{(k)}) \rangle = \langle u_j^{(k-1)} - \tilde{u}_j^{(k)}, u_j^* - \Phi_{s_j}(\tilde{u}_j^{(k)}) \rangle + \langle \tilde{u}_j^{(k)} - \Phi_{s_j}(\tilde{u}_j^{(k)}), u_j^* - \Phi_{s_j}(\tilde{u}_j^{(k)}) \rangle
\]

\[
\leq \sqrt{s_{j}} \| u_j^* - \Phi_{s_j}(\tilde{u}_j^{(k)}) \|_2, \quad \leq \frac{\sqrt{s_{j}}}{2} \| u_j^* - \Phi_{s_j}(\tilde{u}_j^{(k)}) \|_2,
\]

(51)

where the last step comes from the Lemma \[18\]

Hence we have

\[
(1 - \frac{\sqrt{s_{j}}}{\sqrt{s_{j}}}) \| u_j^* - \Phi_{s_j}(\tilde{u}_j^{(k)}) \|_2^2 \leq \| u_j^{(k-1)} - u_j^* \|_2^2 - \| u_j^{(k-1)} - \Phi_{s_j}(\tilde{u}_j^{(k)}) \|_2^2
\]

\[
+ 2 \frac{\eta_j}{T} (\mathcal{P}_{u_j^{(k-1)}}[X_T(X_{-0,j} - \text{iso}_{v_{k-1}}(X_{-0,j}))], \Phi_{s_j}(\tilde{u}_j^{(k)}) - u_j^*).
\]

(52)

Splitting the inner product term in the way that

\[
X_{-0,j} - \text{iso}_{v_{k-1}}(X_{-0,j}) = X_{-0,j} - f_j^*(X_T u_j^*) + f_j^*(X_T u_j^*) - \text{iso}_{v_{k-1}}(f_j^*(X_T u_j^*))
\]

\[
+ \text{iso}_{v_{k-1}}(f_j^*(X_T u_j^*)) - \text{iso}_{v_{k-1}}(X_{-0,j});
\]

(53)

So as to ease notation, since we are dealing with the \(j\)-th variate, which shares exactly the same method and theoretical technique across all \(j \in \{\mathcal{M}\}\), we drop the subscript \(j\) in the following of this proof. Specifically, we denote the projection operator \(\mathcal{P}_{u_j^{(k-1)}}\) as \(\mathcal{P}_{k-1}\); \(u_j^*\) as \(u^*\) and \(u_j^{(k)}\) as \(u^{(k)}\); \(X_{-0,j}\) as \(X_{-0}\); \(s_j^*\) and \(s_j\) as \(s^*\) and \(s\) respectively, \(f_j^*\) as \(f^*\).

By the property of inner product, we have

\[
\langle \mathcal{P}_{k-1}(X_T(X_{-0} - f^*(X_T u^*))), \Phi_{s}(\tilde{u}^{(k)}) - u^* \rangle \leq \| X_T(X_{-0} - f^*(X_T u^*)) \|_{\infty} \cdot \| \mathcal{P}_{k-1}(\Phi_{s}(\tilde{u}^{(k)}) - u^*) \|_1
\]

\[
\leq \sqrt{2s + s^*} \| X_T(X_{-0} - f^*(X_T u^*)) \|_{\infty} \cdot \| \Phi_{s}(\tilde{u}^{(k)}) - u^* \|_2,
\]

(54)
where the last step comes from the inequality of norms that \( \| x \|_1 \leq \sqrt{\| x \|_0 \cdot \| x \|_2} \) for any vector \( x \), and the fact that \( \mathcal{P}_{k-1}^\perp(\Phi_s(\tilde{u}^{(k)}) - u^*) \) has at most \( 2s + s* \) non-zero elements since \( \| u^{(k-1)} \|_0 = s, \| u^* \|_0 = s* \) and \( \| \Phi_s(\tilde{u}^{(k)}) \|_0 = s \).

By the assumption of data boundedness in Eq. (27) and Cauchy inequality, we have

\[
\left\langle \mathcal{P}_{k-1}^\perp(X_T^T(\text{iso}_{k-1}(f^*(X_Tu^*)) - \text{iso}_{k-1}(X_0))), \Phi_s(\tilde{u}^{(k)}) - u^* \right\rangle \\
\leq \| \text{iso}_{k-1}(f^*(X_Tu^*)) - \text{iso}_{k-1}(X_0) \|_2 \cdot \| X_T \mathcal{P}_{k-1}^\perp(\Phi_s(\tilde{u}^{(k)}) - u^*) \|_2 \\
\leq \| \text{iso}_{k-1}(f^*(X_Tu^*)) - \text{iso}_{k-1}(X_0) \|_2 \cdot \sqrt{\beta T} \| \Phi_s(\tilde{u}^{(k)}) - u^* \|_2.
\]

To deal with the remaining part of the inner product generated from the last splitting term, first we use the relationship between inner product and dual norms for 2-norm again to have

\[
\left\langle f^*(X_Tu^*) - \text{iso}_{k-1}f^*(X_Tu^*), X_T \mathcal{P}_{k-1}^\perp \Phi_s(\tilde{u}^{(k)}) \right\rangle \\
\leq \| f^*(X_Tu^*) - \text{iso}_{k-1}f^*(X_Tu^*) \|_2 \| X_T \mathcal{P}_{k-1}^\perp \Phi_s(\tilde{u}^{(k)}) \|_2 \\
\leq \| f^*(X_Tu^*) - \text{iso}_{k-1}f^*(X_Tu^*) \|_2 \cdot \sqrt{\beta T} \| \mathcal{P}_{k-1}^\perp \Phi_s(\tilde{u}^{(k)}) \|_2 \\
\leq \| f^*(X_Tu^*) - \text{iso}_{k-1}f^*(X_Tu^*) \|_2 \cdot \sqrt{\beta T} \| \Phi_s(\tilde{u}^{(k)}) - u^{(k-1)} \|_2 \\
\leq \frac{1}{2L_j^2} \| f^*(X_Tu^*) - \text{iso}_{k-1}f^*(X_Tu^*) \|_2^2 + \frac{L_j^2 \beta T}{2} \| \Phi_s(\tilde{u}^{(k)}) - u^{(k-1)} \|_2^2,
\]

where the third line is true again due to Eq. (27); the last line comes from the fact that for any \( a, b \in \mathbb{R} \) and any \( L > 0 \), we have \( ab \leq \frac{a^2}{2L} + \frac{b^2 L}{2} \).

On the other hand,

\[
\left\langle f^*(X_Tu^*) - \text{iso}_{k-1}f^*(X_Tu^*), X_T \mathcal{P}_{k-1}^\perp(u^*) \right\rangle \\
= \left\langle f^*(X_Tu^*) - \text{iso}_{k-1}f^*(X_Tu^*), X_T \left( u^* - \langle u^*, u^{(k-1)} \rangle u^{(k-1)} \right) \right\rangle \\
= \left\langle f^*(X_Tu^*) - \text{iso}_{k-1}f^*(X_Tu^*), X_T u^* \right\rangle \\
- \langle u^*, u^{(k-1)} \rangle \left\langle f^*(X_Tu^*) - \text{iso}_{k-1}f^*(X_Tu^*), X_T u^{(k-1)} \right\rangle \\
\geq \frac{1}{L_j} \| f^*(X_Tu^*) - \text{iso}_{k-1}f^*(X_Tu^*) \|_2^2 \\
- \langle u^*, u^{(k-1)} \rangle \left\langle f^*(X_Tu^*) - \text{iso}_{k-1}f^*(X_Tu^*), \nu_{k-1} \right\rangle \\
\geq \frac{1}{L_j} \| f^*(X_Tu^*) - \text{iso}_{k-1}f^*(X_Tu^*) \|_2^2,
\]

where the forth line is derived from Lemma 12; the last step comes from the acuteness of the angle between \( u^* \) and \( u^{(k-1)} \) as assumed, and the fact that

\[
\langle \omega - \text{iso}_v(\omega), v \rangle \leq 0
\]

for any vectors \( v, \omega \in \mathbb{R}^T \), since \( \text{iso}_v(\omega) \) is the projection of \( \omega \) onto the convex cone satisfying the isotonic constraints, while \( v \) itself is a vector contained in this convex cone.
Combining Eq. (56) and Eq. (57), we have

\[
\left\langle \frac{1}{\lambda_{k-1}} \left[ X_{-T} f^* \left( X_{-T} u^* \right) - \text{Iso}_{\lambda_{k-1}} f^* \left( X_{-T} u^* \right) \right], \Phi_s(\tilde{u}^{(k)}) - u^* \right\rangle \\
= \left\langle f^* \left( X_{-T} u^* \right) - \text{Iso}_{\lambda_{k-1}} f^* \left( X_{-T} u^* \right), X_{-T} \mathcal{P}_{k-1}^\perp(\Phi_s(\tilde{u}^{(k)})) \right\rangle \\
- \left\langle f^* \left( X_{-T} u^* \right) - \text{Iso}_{\lambda_{k-1}} f^* \left( X_{-T} u^* \right), X_{-T} \mathcal{P}_{k-1}^\perp(\Phi_s(\tilde{u}^{(k)})) \right\rangle \\
\leq \frac{L_j \beta T}{2} \| \Phi_s(\tilde{u}^{(k)}) - u^{(k-1)} \|_2^2 - \frac{1}{2L_j} \| f^* \left( X_{-T} u^* \right) - \text{Iso}_{\lambda_{k-1}} f^* \left( X_{-T} u^* \right) \|_2^2 .
\]

Plugging Eq. (54), Eq. (55) and Eq. (59) into Eq. (52), we have

\[
(1 - \sqrt{s_k \beta}) \| u^* - \Phi_s(\tilde{u}^{(k)}) \|_2^2 \\
\leq (\eta_j L_j \beta - 1) \|\Phi_s(\tilde{u}^{(k)}) - u^{(k-1)} \|_2^2 + 2\eta_j C(Z, u_j^{(k-1)}) \| u^* - \Phi_s(\tilde{u}^{(k)}) \|_2 \\
+ \| u^* - u^{(k-1)} \|_2^2 - \frac{\eta_j}{T L_j} \| f^* \left( X_{-T} u^* \right) - \text{Iso}_{\lambda_{k-1}} f^* \left( X_{-T} u^* \right) \|_2^2.
\]

Let the step size be chosen as \( \eta_j = \frac{1}{L_j \beta} \), by Cauchy inequality, for any \( \delta_j > 0 \),

\[
\frac{2}{L_j \beta} C(Z, u_j^{(k-1)}) \| u^* - \Phi_s(\tilde{u}^{(k)}) \|_2 \\
\leq \frac{\delta_j \alpha_j}{L_j^2 \beta} \| u^* - \Phi_s(\tilde{u}^{(k)}) \|_2^2 + \frac{1}{\beta \alpha_j \delta_j} C^2(Z, u_j^{(k-1)}).
\]

Hence, putting back the subscript \( j \) we have that

\[
(1 - \sqrt{s_k \beta}) \| u_j^* - \Phi_{s_j}(\tilde{u}_j^{(k)}) \|_2^2 \\
\leq \| u_j^* - u_j^{(k-1)} \|_2^2 + \frac{1}{\beta \alpha_j \delta_j} C^2(Z, u_j^{(k-1)}) - \frac{\eta_j}{L_j} \| u_j^* - u_j^{(k-1)} \|_2^2 - \epsilon_j^2 \\
(1 - \frac{\alpha_j}{L_j^2 \beta}) \| u_j^* - u_j^{(k-1)} \|_2^2 + \frac{C^2(Z, u_j^{(k-1)})}{\beta \alpha_j \delta_j} + \frac{\epsilon_j^2}{L_j^2 \beta}.
\]

Next we only need to show that \( \| u_j^* - u_j^{(k)} \|_2^2 \leq \| u_j^* - \Phi_{s_j}(\tilde{u}_j^{(k)}) \|_2^2 \) to complete the proof. Let \( S_j^{(k-1)} \) be the support of \( u_j^{(k-1)} \), then we know \( |S_j^{(k-1)}| = s_j \). Since \( \Phi_{s_j}(\cdot) \) is the hard-thresholding operator that finds the largest \( s_j \) elements in absolute values, we have

\[
\| \Phi_{s_j}(\tilde{u}_j^{(k)}) \|_2^2 \geq \left\| \left( \tilde{u}_j^{(k)} \right)_{S_j^{(k-1)}} \right\|_2^2 \\
= \left\| \left( u_j^{(k-1)} + \eta_j \mathcal{P}_{u_j^{(k-1)}} \left[ \frac{1}{T} X_{-T} \left( X_{-0,j} - \text{Iso}_{\lambda_{k-1}}(X_{-0,j}) \right) \right] \right)_{S_j^{(k-1)}} \right\|_2^2 \]

\[
\geq \left\| u_j^{(k-1)} \right\|_2^2 = 1.
\]
Therefore, by the fact that \( u_j^{(k)} = \frac{\Phi_{sj}(\tilde{u}_j^{(k)})}{\|\Phi_{sj}(\tilde{u}_j^{(k)})\|_2} \), we know that \( u_j^{(k)} \) is the projection of \( \Phi_{sj}(\tilde{u}_j^{(k)}) \) not only onto the unit-sphere, but also onto the unit-ball. Since a unit ball \( \{ u \in \mathbb{R}^M : \| u \|_2 \leq 1 \} \) is a convex set containing \( u_j^* \), we have

\[
\langle \Phi_{sj}(\tilde{u}_j^{(k)}) - u_j^{(k)} , u_j^* - u_j^k \rangle \leq 0; \tag{64}
\]

Equipped with the three-point identity, we have

\[
\| u_j^* - \Phi_{sj}(\tilde{u}_j^{(k)}) \|^2_2 = \| u_j^* - u_j^{(k)} \|^2_2 + \| \Phi_{sj}(\tilde{u}_j^{(k)}) - u_j^{(k)} \|^2_2 + 2 \langle u_j^* - u_j^{(k)} , \Phi_{sj}(\tilde{u}_j^{(k)}) - u_j^{(k)} \rangle \geq \| u_j^* - u_j^{(k)} \|^2_2. \tag{65}
\]

Combining Eq. (62) and Eq. (65), we finish the proof of Lemma 7.

9.2 Proof of Lemma 8

**Proof** For any \( j \in [M] \) of interest, the following inequalities hold by union bound for any \( y > 0 \):

\[
P\left( \frac{1}{T} \| X_{-T}^T (X_{-0,j} - f_j^*(X_{-T} \cdot u_j^*)) \|_\infty \geq y \right) \leq \sum_{i=1}^{M} P\left( \frac{1}{T} \sum_{t=0}^{T-1} X_{t,i} (X_{t+1,j} - f_j^*(X_t^T \cdot u_j^*)) \geq y \right) \tag{66}
\]

By the assumption that for any \( \lambda \in \mathbb{R} \), the noises \( Z_{t,j}, t = 1, \ldots, T \) are conditionally sub-Gaussian in the way that \( \mathbb{E}[\exp(\lambda Z_{t,j}) | F_{t-1}] \leq e^{\sigma_j^2 \lambda^2/2} \), and the fact that \( X \) is entrywise bounded by \( M_x \), we have

\[
\mathbb{E}\left( e^{\lambda X_{t-1,i} Z_{t,j}} | F_{t-1} \right) \leq \mathbb{E}\left[ \exp\left( \frac{\lambda^2 \sigma_j^2 X_{t-1,i}^2}{2} \right) | F_{t-1} \right] \leq \exp\left( \frac{\lambda^2 \sigma_j^2 M_x^2}{2} \right), \quad t = 1, \ldots, T.
\tag{67}
\]

We could then bound the moment generating function of the sum \( \sum_{t=1}^{T} X_{t-1,i} Z_{t,j} \) in the following recursive manner using conditional expectations. Define \( S_{n,i,j} = \sum_{t=1}^{n} X_{t-1,i} Z_{t,j}, \ n \in \mathbb{N} \).
[T]. For any $\lambda \in \mathbb{R}$ and any $i = 1, \ldots, M$,

\[
\mathbb{E} \left[ \exp(\lambda S_{n,i,j}) \right] = \mathbb{E} \left[ \exp \left( \lambda \sum_{t=1}^{n} X_{t-1,i}Z_{t,j} \right) \right] = \mathbb{E} \left[ \mathbb{E} \left( e^{\lambda \sum_{t=1}^{n} X_{t-1,i}Z_{t,j}} | \mathcal{F}_{n-1} \right) \right] = \mathbb{E} \left[ \exp \left( \lambda \sum_{t=1}^{n-1} X_{t-1,i}Z_{t,j} \right) \right].
\]

(68)

Therefore we have

\[
\mathbb{E} \left[ \exp(\lambda S_{T,i,j}) \right] \leq \exp \left( \frac{(T-1)\lambda^2 \sigma_j^2 M_x^2}{2} \right) \cdot \mathbb{E} \left[ \exp(\lambda S_{1,i,j}) \right] = \exp \left( \frac{(T-1)\lambda^2 \sigma_j^2 M_x^2}{2} \right) \cdot \mathbb{E} \left[ \exp(\lambda S_{1,i,j}) \right].
\]

(69)

Hence, by definition, $S_{T,i,j}$ is sub-Gaussian with parameter $\sqrt{T} \sigma_j M_x^2$ and mean zero.

Using the concentration inequality and Eq. (66), we have

\[
\mathbb{E} \left[ S_{T,i,j} \right] = \sum_{t=1}^{T} \mathbb{E} \left[ X_{t-1,i}Z_{t,j} \right] = \sum_{t=1}^{T} \mathbb{E} \left[ \mathbb{E} \left( X_{t-1,i}Z_{t,j} | \mathcal{F}_{t-1} \right) \right] = \sum_{t=1}^{T} \mathbb{E} \left[ X_{t-1,i} \mathbb{E} \left( Z_{t,j} | \mathcal{F}_{t-1} \right) \right] = 0.
\]

(70)

By the fact that $\mathbb{E} (Z_{t,j} | \mathcal{F}_{t-1}) = 0$, we have

\[
\mathbb{E} [S_{T,i,j}] = \sum_{t=1}^{T} \mathbb{E} [X_{t-1,i}Z_{t,j}] = \sum_{t=1}^{T} \mathbb{E} [\mathbb{E} (X_{t-1,i}Z_{t,j} | \mathcal{F}_{t-1})] = \sum_{t=1}^{T} \mathbb{E} [X_{t-1,i} \mathbb{E} (Z_{t,j} | \mathcal{F}_{t-1})] = 0.
\]

Hence, by definition, $S_{T,i,j}$ is sub-Gaussian with parameter $\sqrt{T} \sigma_j M_x^2$ and mean zero. Using the concentration inequality and Eq. (66), we have

\[
P \left( \frac{1}{T} \| X^T_{-T} (X_{-0,j} - f_j^* (X_{-T} \cdot u_j^*)) \|_{\infty} \geq y \right) \leq \sum_{i=1}^{M} P \left( \frac{1}{T} |S_{T,i,j}| \geq y \right) \leq M \exp \left( - \frac{T y^2}{2 \sigma_j^2 M_x^2} \right).
\]

(71)
9.3 Proof of Lemma 9

Before showing the uniform $\ell_2$ norm bound in Lemma 9 over $V = \{v = X_Tu : u \in S^{M-1} \text{ with sparsity } s_j \}$, we first need the following lemma for the $\ell_2$ norm bound for arbitrary vector $v \in V$.

**Lemma 11.** For any $v \in V = \{v = X_Tu : u \in S^{M-1} \text{ with sparsity } s_j \}$, with probability at least $1 - 2\gamma$,

$$\|\text{iso}_v(f_j^*(X_Tu_j^*)) - \text{iso}_v(X_{0,j})\|_2^2 \leq 16T^{3/2} \left[ \left( 2 + 2T\sqrt{T+1} \right) \sigma^2 \log \left( \frac{T(T+1)}{\gamma} \right) \right] + 4\sigma^2 \log \left( \frac{T(T+1)}{\gamma} \right).$$

**(Proof (Lemma 11))**

For any $v \in V$, we first bound the range of $\text{iso}_v(f_j^*(X_Tu_j^*))$ by the nature of isotonic regression as well as the monotonicity and the $L_j$-Lipschitz continuity of the function $f_j^*$:

$$\|\text{iso}_v(f_j^*(X_Tu_j^*)) - \text{iso}_v(f_j^*(X_Tu_j^*)))_n(1)\| \leq \left| \text{iso}_v(f_j^*(X_Tu_j^*))_{max} - \text{iso}_v(f_j^*(X_Tu_j^*))_{min}\right|$$

$$\leq f_j^*(M_x \sqrt{s_j^*}) - f_j^*(-M_x \sqrt{s_j^*}) \leq 2L_jM_x \sqrt{s_j^*},$$

where we made use of the boundedness of $X_T$ and the boundedness of $u_j^*$ in the way that

$$\|X_Tu_j^*\|_\infty = \max_{t \in [T]-1}(\|X_t, u_j^*\|)$$

$$\leq \max_{t \in [T]-1} \|X_t\|_\infty \|u_j^*\|_1 \leq M_x \sqrt{\|u_j^*\|_0} \|u_j^*\|_2 \leq M_x \sqrt{s_j^*}. \quad (74)$$

Following that, we further bound the range of $\text{iso}_v(X_{0,j})$:

$$\left| \text{iso}_v(X_{0,j}))_{n(1)} - \text{iso}_v(X_{0,j}))_{(1)} \right| \leq \left| \text{iso}_v(f_j^*(X_Tu_j^*)))_{n(1)} - \text{iso}_v(f_j^*(X_Tu_j^*))_{(1)} \right|$$

$$+ \left| \text{iso}_v(f_j^*(X_Tu_j^*)))_{(1)} - \text{iso}_v(f_j^*(X_Tu_j^*))_{(1)} \right|$$

$$\leq 2 \|\text{iso}_v(X_{0,j}) - \text{iso}_v(f_j^*(X_Tu_j^*))\|_\infty$$

$$+ \left| \text{iso}_v(f_j^*(X_Tu_j^*)))_{n(1)} - \text{iso}_v(f_j^*(X_Tu_j^*))_{(1)} \right|$$

$$\leq 2 \|X_{0,j} - f_j^*(X_Tu_j^*)\|_\infty + \left| \text{iso}_v(f_j^*(X_Tu_j^*))_{n(1)} - \text{iso}_v(f_j^*(X_Tu_j^*))_{(1)} \right|$$

$$\leq 2 \|Z_j\|_\infty + 2L_jM_x \sqrt{s_j^*}. \quad (75)$$

The first inequality comes from triangle inequality, the second holds directly from the definition of infinity norm, while the third inequality holds because of the contractive property of isotonic regression in Corollary 17.
Let \( \mathcal{A} = \{ \| Z_j \|_\infty \leq \sqrt{2\sigma^2_j \log \frac{2T}{\gamma}} \} \). By the sub-Gaussianity of \( Z_j = (Z_{t,j})_{t=1, \ldots, T} \) with parameter \( \sigma_j \) and the union bound inequality, we know that \( P(\mathcal{A}) \geq 1 - \gamma \). Hence, based on Eq. (75), with probability at least \( 1 - \gamma \) on event \( \mathcal{A} \),

\[
|\text{iso}_Y(X_{0:j})| - |\text{iso}_Y(X_{0:j})|_{(1)}| \leq \tilde{H}_j := 2\sqrt{2\sigma^2_j \log \frac{2T}{\gamma}} + 2L_j M_x \sqrt{s_j^*}. \tag{76}
\]

Next, we apply a similar proof technique used in \((\text{Dai et al.} (2021))\), cutting the range of \( \text{iso}_Y(X_{0:j}) \) and \( \text{iso}_Y(f_j^* (X_{-T} u_j^*)) \) into arbitrarily short segments with \( Q \) cutting points \( \{M_0 = 1, M_1, \ldots, M_Q = T\} \): for any \( n \geq 1 \) and fixed \( j \), on event \( \mathcal{A} \) we have

\[
\begin{align*}
|\text{iso}_Y(f_j^* (X_{-T} u_j^*))|_{(M_i)} - |\text{iso}_Y(f_j^* (X_{-T} u_j^*))|_{(M_i+1)} & \leq \tilde{H}_j/n; \\
|\text{iso}_Y(X_{0:j})|_{(M_i)} - |\text{iso}_Y(X_{0:j})|_{(M_i+1)} & \leq \tilde{H}_j/n, \forall i = 1, \ldots, Q.
\end{align*}
\tag{77}
\]

Let the index set of \( j \)th segment be \( \mathcal{I}_l = \{(M_{l-1} + 1), (M_{l-1} + 2), \ldots, (M_l)\} \), then we know that Eq. (77) is equivalent to:

\[
\begin{align*}
\text{range}(\text{iso}_Y(f_j^* (X_{-T} u_j^*))|_{\mathcal{I}_l}) & \leq \tilde{H}_j/n; \\
\text{range}(\text{iso}_Y(X_{0:j})|_{\mathcal{I}_l}) & \leq \tilde{H}_j/n, \forall l = 1, \ldots, Q.
\end{align*}
\tag{78}
\]

Thanks to the above segmentation, we could first bound the distances of points in each segment to its grouped mean:

\[
\| \text{iso}_Y(x) \|_{\mathcal{I}_l} - |\text{iso}_Y(x)|_{\mathcal{I}_l} 1_{|\mathcal{I}_l|} \|_2 \leq \sqrt{\|\mathcal{I}_l\| \cdot \tilde{H}_j/n}, \forall l \in [Q],
\tag{79}
\]

which holds for both \( x = X_{0:j} \) and \( x = f_j^* (X_{-T} u_j^*) \).

On the other hand, using union bounds and the fact that \( \{Z_{t,j}\}_{t=1}^T \) is a martingale difference sequence with conditional sub-Gaussian tail, we could bound the distance of the centers by

\[
P \left( \max_{l \in [Q]} \left\{ \sqrt{|\mathcal{I}_l|} \cdot \left| (\text{iso}_Y(X_{0:j})|_{\mathcal{I}_l}) - (\text{iso}_Y(f_j^* (X_{-T} u_j^*))|_{\mathcal{I}_l}) \right| \right\} > t \right)
\leq P \left( \max_{l \in [Q]} \sqrt{|\mathcal{I}_l|} \cdot \left| X_{0:j} - (f_j^* (X_{-T} u_j^*))|_{\mathcal{I}_l} \right| > t \right)
\leq \left( \frac{T+1}{2} \right) P \left( \sqrt{|\mathcal{I}_l|} \cdot |Z_{j}| > t \right)
\leq T(T+1) \exp \left( - \frac{t^2}{2\sigma^2_j} \right). \tag{80}
\]

The second line holds by using contractive property again in Corollary 17 and the third line comes from union bound relaxation. Thus the following event \( \mathcal{B} \) takes place with probability at least \( 1 - \gamma \), where

\[
\mathcal{B} = \left\{ \max_{l \in [Q]} \left\{ \sqrt{|\mathcal{I}_l|} \cdot \left| (\text{iso}_Y(X_{0:j})|_{\mathcal{I}_l}) - (\text{iso}_Y(f_j^* (X_{-T} u_j^*))|_{\mathcal{I}_l}) \right| \right\} \leq \sqrt{2\sigma^2_j \log \frac{T(T+1)}{\gamma}} \right\}.
\tag{81}
\]
Hence, by triangle inequality, in each segment, we have
\[
\| (\text{iso}_v(X_{-0,j}))_{I_l} - (\text{iso}_v(f_j^*(X_{-Tu_j}^*)))_{I_l} \|_2 \\
\leq 2\sqrt{|I_l|} \tilde{H}_j/n + \sqrt{2\sigma_j^2 \log(T(T+1)/\gamma)}.
\]  
(82)

Putting all the segments together, we have on the event \( \mathcal{A} \cap \mathcal{B} \),
\[
\| \text{iso}_v(X_{-0,j}) - \text{iso}_v(f_j^*(X_{-Tu_j}^*))) \|_2^2 \\
\leq 8\sum_{l=1}^{Q} |I_l| \tilde{H}_j^2/n^2 + 4Q\sigma_j^2 \log(T(T+1)/\gamma) \\
\leq 8T\tilde{H}_j^2/n^2 + 4(2n - 1)\sigma_j^2 \log(T(T+1)/\gamma),
\]  
(83)

where the last line comes from the trivial results in the segmentation step that \( Q \leq 2n - 1 \).

Take the segmentation parameter \( n \) that could roughly minimize the bound in Eq. (83):
\[
n = \left\lceil \left( \frac{T\tilde{H}_j^2}{\sigma_j^2 \log(T(T+1)/\gamma)} \right)^{\frac{1}{3}} \right\rceil,
\]  
(84)

we have on the event \( \mathcal{A} \cap \mathcal{B} \), with probability at least \( 1 - 2\gamma \),
\[
\| \text{iso}_v(X_{-0,j}) - \text{iso}_v(f_j^*(X_{-Tu_j}^*))) \|_2^2 \\
\leq 16T^{\frac{1}{3}}\tilde{H}_j^2 \left[ \sigma_j^2 \log(T(T+1)/\gamma) \right]^{\frac{1}{3}} + 4\sigma_j^2 \log(T(T+1)/\gamma) \\
= 16T^{\frac{1}{3}} \left[ \left( 2\sqrt{2\sigma_j^2 \log(T(T+1)/\gamma) + 2L_jM_x\sqrt{s_j^*} \sigma_j^2 \log(T(T+1)/\gamma)} \right)^{\frac{1}{3}} \right] \\
+ 4\sigma_j^2 \log(T(T+1)/\gamma),
\]  
(85)

where the last line is derived by plugging in the definition of the range bound \( \tilde{H}_j \) in Eq. (76).

With the help of Lemma 11, now we could derive the uniform bound in Lemma 9.

**Proof (Lemma 9)** In Lemma 9 we intend to uniformly bound
\[
\| \text{iso}_v(X_{-0,j}) - \text{iso}_v(f_j^*(X_{-Tu_j}^*))) \|_2
\]  
over the set \( V = \{ v = X_{-Tu} : u \in S^{M-1} \text{ with sparsity } s_j \} \). To this end, we first notice that for the isotonic regression \( \text{iso}_v(\cdot) \), it is the ordering of \( v \) instead of \( v \) itself that makes
by the fact that 

\[\| \text{iso}_v(X_{-0,j}) - \text{iso}_v(f_j^*(X_{-Tu_j}^*)) \|_2 = \| \text{iso}_u(X_{-0,j}) - \text{iso}_u(f_j^*(X_{-Tu_j}^*)) \|_2. \] (86)

For any \(n\) points \(x_1, \ldots, x_n \in \mathbb{R}^M\) and a given sparsity level \(s\), define the set

\[S_{n,M}^{s,-\text{sparse}}(x_1, \ldots, x_n) = \left\{ \pi \in S_n : x_{\pi(1)}^T u \leq x_{\pi(2)}^T u \leq \cdots \leq x_{\pi(n)}^T u \text{ for some } s\text{-sparse } u \in \mathbb{R}^M \right\}, \] (87)

where \(S_n\) contains all permutations for \(n\) objects. By union bound and the preserving order property for isotonic regression, we have for any \(j \in [M]\),

\[P \left( \sup_{v \in V} \| \text{iso}_v(X_{-0,j}) - \text{iso}_v(f_j^*(X_{-Tu_j}^*)) \|_2^2 > t \right) \leq \left| S_{n,M}^{s,-\text{sparse}}(X_0, \ldots, X_{T-1}) \right| \cdot P \left( \| \text{iso}_v(X_{-0,j}) - \text{iso}_v(f_j^*(X_{-Tu_j}^*)) \|_2^2 > t \right) \leq 2^{s_j - 1} M^{s_j - 1} \cdot P \left( \| \text{iso}_v(X_{-0,j}) - \text{iso}_v(f_j^*(X_{-Tu_j}^*)) \|_2^2 > t \right), \] (88)

where the last line comes from Lemma 19. Therefore, combining Lemma 11, we know that with probability \(1 - 2\gamma\),

\[\sup_{v \in V} \| \text{iso}_v(X_{-0,j}) - \text{iso}_v(f_j^*(X_{-Tu_j}^*)) \|_2^2 \leq 16T^3 \left[ \left( 2 \sqrt{2\sigma_j^2 \log \frac{2T}{\gamma/(T^{2s_j - 1} M^{s_j})} + 2L_j M_x \sigma_j \sqrt{s_j}} \right)^2 \sigma_j^2 \log \left( \frac{T(T + 1)}{\gamma/(T^{2s_j - 1} M^{s_j})} \right) \right]\] (89)

by the fact that \(\sqrt{a^2 + b^2} \leq a + b\) for any \(a, b > 0\), we could conclude that

\[\sup_{v \in V} \| \text{iso}_v(f_j^*(X_{-Tu_j}^*)) - \text{iso}_v(X_{-0,j}) \|_2 \leq 4T^{1/6} \left[ \left( 2 \sqrt{2\sigma_j^2 \log \frac{2T^{2s_j} \sigma_j^2}{\gamma} + 2L_j M_x \sigma_j \sqrt{s_j}} \right)^2 \sigma_j \log \left( \frac{T^{2s_j} (T + 1) M^{s_j}}{\gamma} \right) \right]^1/3\] (90)

with probability at least \(1 - 2\gamma\).
9.4 Proof of Lemma 5

Proof As is defined, \( u_j^{(0)} = \frac{\Phi_{s_j}(\tilde{u}_j^{(0)})}{\|\Phi_{s_j}(\tilde{u}_j^{(0)})\|_2} \). Hence Eq. (31) is equivalent to \( \langle \Phi_{s_j}(\tilde{u}_j^{(0)}), u_j^* \rangle > 0 \).

We conquer it by splitting it into two parts:

\[
\langle \Phi_{s_j}(\tilde{u}_j^{(0)}), u_j^* \rangle = \langle \tilde{u}_j^{(0)}, u_j^* \rangle + \langle \Phi_{s_j}(\tilde{u}_j^{(0)}) - \tilde{u}_j^{(0)}, u_j^* \rangle. \tag{91}
\]

For the first term in the right-hand side, we have

\[
\langle \tilde{u}_j^{(0)}, u_j^* \rangle = \langle \frac{1}{T}X^T(X_{-j} - \overline{X}_{-j}1_T), u_j^* \rangle
\]

\[
= \frac{1}{T} \langle f_j^*(X_{-j}u_j^*) - f_j^*(X_{-j}u_j^*) \cdot 1_T, X_{-j}u_j^* \rangle + \langle \frac{1}{T}X^T(Z_j - Z_j \cdot 1_T), u_j^* \rangle
\]

\[
\geq \frac{1}{T} L_{X,j}^{-1} \| f_j^*(X_{-j}u_j^*) - f_j^*(X_{-j}u_j^*) \cdot 1_T \|_2^2 - \sqrt{s_j^*} \| \frac{1}{T}X^T(Z_j - Z_j \cdot 1_T) \|_{\infty}. \tag{92}
\]

Note that in the above last line, the first term came from Lemma 12 with \( u \) taken as \( -u_j^* \); the second term is derived from the duality equality and \( \| u_j^* \|_1 \leq \sqrt{s_j^*} \).

As for the second term of Eq. (91), with the fact that \( \langle \tilde{u}_j^{(0)} - \Phi_{s_j}(\tilde{u}_j^{(0)}), \Phi_{s_j}(\tilde{u}_j^{(0)}) \rangle = 0 \), we have

\[
\langle \Phi_{s_j}(\tilde{u}_j^{(0)}) - \tilde{u}_j^{(0)}, u_j^* \rangle
\]

\[
= -\langle \tilde{u}_j^{(0)} - \Phi_{s_j}(\tilde{u}_j^{(0)}), u_j^* - \Phi_{s_j}(\tilde{u}_j^{(0)}) \rangle
\]

\[
\geq -\frac{\sqrt{s_j^*}}{2\sqrt{s_j}} \| u_j^* - \Phi_{s_j}(\tilde{u}_j^{(0)}) \|_2^2 \tag{93}
\]

\[
= -\frac{\sqrt{s_j^*}}{2\sqrt{s_j}} \left( 1 + \| \Phi_{s_j}(\tilde{u}_j^{(0)}) \|_2^2 \right) + \frac{s_j^*}{s_j} \langle u_j^*, \Phi_{s_j}(\tilde{u}_j^{(0)}) \rangle.
\]

Specifically, the norm of \( s_j \)-sparsity enforced vector \( \Phi_{s_j}(\tilde{u}_j^{(0)}) \) can be bounded by:

\[
\| \Phi_{s_j}(\tilde{u}_j^{(0)}) \|_2^2
\]

\[
\leq 2 \left\{ \left\| \Phi_{s_j} \left( \frac{1}{T}X^T(f_j^*(X_{-j}u_j^*) - f_j^*(X_{-j}u_j^*) \cdot 1_T) \right) \right\|_2^2 + \left\| \frac{1}{T} \Phi_{s_j} (X^T(Z_j - Z_j \cdot 1_T)) \right\|_2^2 \right\}
\]

\[
\leq 2 \frac{2}{T^2} \| f_j^*(X_{-j}u_j^*) - f_j^*(X_{-j}u_j^*) 1_T \|_2^2 + \| X^T 1_A_{s_j} \|_2^2 + 2s_j \| \frac{1}{T} X^T (Z_j - Z_j \cdot 1_T) \|_{\infty}^2
\]

\[
\leq \frac{4s_j}{T^2} \| f_j^*(X_{-j}u_j^*) - f_j^*(X_{-j}u_j^*) 1_T \|_2^2 + 2s_j \| \frac{1}{T} X^T (Z_j - Z_j \cdot 1_T) \|_{\infty}^2. \tag{94}
\]
Putting things together, we have

\[
(1 - \sqrt{s_j^* s_j}) \langle \Phi_{s_j}(\tilde{u}_j^{(0)}), u_j^* \rangle \\
\geq (1 - \frac{2\beta}{L_j} \sqrt{s_j s_j^*}) : \frac{1}{T} \| f_j^*(X_{-T}^T u_j^*) - f_j^*(X_{-T}^T u_j^*) \|_2^2 \\
- \sqrt{s_j s_j^*} \| \frac{1}{T} X_{-T}^T (Z_j - Z_j 1_T) \|_2^2 - \sqrt{s_j^*} \| \frac{1}{T} X_{-T}^T (Z_j - Z_j 1_T) \|_\infty - \sqrt{s_j^*} \|
\]

Note that for any \( u \in \mathbb{R}^p \) with norm 1, by the optimality of iso\( X_{-T}^T u(\cdot) \) and the identifiability condition Eq. (22), we have

\[
\frac{1}{T} \| f_j^*(X_{-T}^T u_j^*) - f_j^*(X_{-T}^T u_j^*) \|_2^2 \\
\geq \frac{1}{T} \sup_{u \in \mathbb{R}^p, \|u\|_2 = 1} \| f_j^*(X_{-T}^T u_j^*) - \text{iso}_{X_{-T}} u(f_j^*(X_{-T}^T u_j^*)) \|_2^2 \\
\geq \alpha_j \sup_{u \in \mathbb{R}^p, \|u\|_2 = 1} \| u - u_j^* \|_2 - \epsilon_j^2 \geq 4\alpha_j - \epsilon_j^2,
\]

where the last inequality can be reached when we take a \( u = -u_j^* \).

Plug Eq. (96) into Eq. (95), we have that \( (1 - \sqrt{s_j^* s_j}) \langle \Phi_{s_j}(\tilde{u}_j^{(0)}), u_j^* \rangle > 0 \) as long as

\[
\| \frac{1}{T} X_{-T}^T (Z_j - Z_j 1_T) \|_\infty < U_j^+ := \frac{1}{2} \left( \sqrt{\frac{1}{s_j} + 4c_j^+} - \sqrt{\frac{1}{s_j}} \right),
\]

where

\[
c_j^+ := \left( \frac{1}{L_j \sqrt{s_j s_j^*}} - \frac{2\beta}{T} \right) (4\alpha_j - \epsilon_j^2) - \frac{1}{2s_j}.
\]

To assure \( U_j^+ > 0 \), we need \( c_j^+ > 0 \). \( 4\alpha_j - \epsilon_j > 0 \), we have

\[
s_j > s_j^* \left( \frac{L_j}{2(4\alpha_j - \epsilon_j^2)} + O\left( \frac{\beta L_j^3}{T} \right) \right)^2.
\]

By union bound, data boundedness and sub-Gaussian tail, we have

\[
P \left( \left\| \frac{1}{T} X_{-T}^T (Z_j - Z_j 1_T) \right\|_\infty \geq t \right) \\
\leq P \left( \frac{1}{T} \left\| X_{-T}^T Z_j \right\|_\infty \geq \frac{t}{2} \right) + P \left( \frac{1}{T} \left\| X_{-T}^T Z_j 1_T \right\|_\infty \geq \frac{t}{2} \right) \leq 2(M + 1) \exp\left( -\frac{T_1^2}{8\sigma_j^2 M_x} \right).
\]


Hence, with probability
\[ 1 - 2(M + 1) \exp\left(-\frac{T U_j^2}{8 \sigma_j^2 M_x}\right), \tag{101} \]
we have \( \langle u_j^{(0)}, u_j^* \rangle > 0 \), when \( s_j > s_j^* \cdot \max\left\{ 1, \left( \frac{L_j}{2(4\sigma_j - \epsilon_j^2)} + O(\frac{L_j^3}{T}) \right)^2 \right\} \).

9.5 Proof of Proposition 6

Proof By Cauchy inequality, the definition of isotonic regression and its contractive property, for any \( j \in [M] \), let \( \hat{u}_j = u_j^{(K)} \), where \( K \geq \frac{\log(R_j^2)}{\log(\theta_j)} \), where \( R_j \) and \( \theta_j \) are defined in Theorem 3. Then we have
\[
\frac{1}{\sqrt{T}} \| \text{iso}_x T \hat{u}_j (X_{-j,0}) - f_j^* (X_{-j} u_j^*) \|_2 \\
\leq \frac{1}{\sqrt{T}} \| \text{iso}_x T \hat{u}_j (X_{-j,0}) - \text{iso}_x T \hat{u}_j (f_j^* (X_{-j} u_j^*)) \|_2 \\
+ \frac{1}{\sqrt{T}} \| \text{iso}_x T \hat{u}_j (f_j^* (X_{-j} u_j^*)) - f_j^* (X_{-j} u_j^*) \|_2 \\
\leq \frac{1}{\sqrt{T}} \| \text{iso}_x T \hat{u}_j (X_{-j,0}) - \text{iso}_x T \hat{u}_j (f_j^* (X_{-j} u_j^*)) \|_2 \\
+ \frac{1}{\sqrt{T}} \| f_j^* (X_{-j} \hat{u}_j) - f_j^* (X_{-j} u_j^*) \|_2 \\
\leq \frac{1}{\sqrt{T}} \| \text{iso}_x T \hat{u}_j (X_{-j,0}) - \text{iso}_x T \hat{u}_j (f_j^* (X_{-j} u_j^*)) \|_2 + L_j \sqrt{\beta} \| \hat{u}_j - u_j^* \|_2 \\
\leq O_p(T^{-\frac{1}{2}} \log T),
\]
where the last line comes directly from Lemma 11 and Theorem 3.

9.6 Other Supporting Lemmas

Lemma 12. (Lemma 6 in Dai et al. (2021)) For any vector \( u \in \mathbb{R}^M \),
\[
(X_{-j} u^*, f^* (X_{-j} u^*)) - \text{iso}_x u (f^* (X_{-j} u^*)) \geq L^{-1} \| f^* (X_{-j} u^*) - \text{iso}_x u (f^* (X_{-j} u^*)) \|_2,
\]
where \( f^* \) is an L-Lipschitz monotone non-decreasing function.

Lemma 13. For any \( j \in [M] \), and an index set \( I \subset [T] \), we have
\[
P \left( \sqrt{|I|} \cdot |(Z_j)_I| > t \right) \leq 2 \exp\left(-\frac{t^2}{2 \sigma_j^2}\right), \tag{104}
\]
where \( |I| \) denotes the cardinality of \( I \).
Proof. To show the tail bound of the martingale difference mean, we first bound the moment generating function. For any $\lambda > 0$, we have

$$
\mathbb{E} \left[ \exp \left( \lambda (|I| \cdot (Z_j)_I) \right) \right] = \mathbb{E} \left[ \exp \left( \lambda \sum_{t \in I} Z_{t,j} \right) \right] = \mathbb{E} \left\{ \mathbb{E} \left( \exp(\lambda \sum_{t \in I} Z_{t,j}) | \mathcal{F}_{T-1} \right) | \mathcal{F}_{T-2} \right\} \cdots | \mathcal{F}_0 \right\} \left(105\right)
$$

$$
\leq \exp \left( \frac{\lambda^2 |I| \sigma_j^2}{2} \right).
$$

Then by the tail bound of a sub-Gaussian variable, we finish the proof.

**Definition 14.** (Seminorm) Let $\mathcal{V}$ be a vector space over real numbers in $\mathbb{R}$. A map $\| \cdot \| : \mathcal{V} \mapsto \mathbb{R}$ is called a seminorm if it satisfies the following two conditions:

1. **Subadditivity (Triangle inequality):** $\| x + y \| \leq \| x \| + \| y \|$, $\forall x, y \in \mathcal{V}$;

2. **Homogeneity:** $\| cx \| = |c| \cdot \| x \|$, $\forall c \in \mathbb{R}, x \in \mathcal{V}$.

**Remark 15.** If a seminorm $\| \cdot \|$ separates points, i.e., $\| x \| = 0$ implies $x = 0$ for any $x \in \mathcal{V}$, it is also a norm. Define a mapping $\| \cdot \|_{\text{mean}} : \mathcal{V} \mapsto \mathbb{R}$ such that $\| x \|_{\text{mean}} = |\bar{x}|$, we could find that $\| \cdot \|_{\text{mean}}$ is not a norm, since for $x = [1, -1]^T$, by definition $\| x \|_{\text{mean}} = 0$. However, it's a seminorm, by the fact that for any vector $x, y \in \mathcal{V}$ and any $c \in \mathbb{R}$, we have

$$
\begin{align*}
1. & \quad \| x + y \|_{\text{mean}} = |\bar{x} + \bar{y}| = |\bar{x}| + |\bar{y}| = \| x \|_{\text{mean}} + \| y \|_{\text{mean}}; \\
2. & \quad \| cx \|_{\text{mean}} = |c\bar{x}| = |c| \cdot \| x \|_{\text{mean}}.
\end{align*}
$$

**Lemma 16.** (Lemma 1 in (Yang and Barber, 2017)) For any $n \in \mathbb{T}^+$ and seminorm $\| \cdot \|$ on the vector space $\mathbb{R}^n$, the isotonic projection is contractive with respect to $\| \cdot \|$, i.e.,

$$
\| \text{iso}(x) - \text{iso}(y) \| \leq \| x - y \| \quad \text{for all} \ x, y \in \mathbb{R}^n, \quad \left(106\right)
$$

as long as the seminorm $\| \cdot \|$ is invariant to permutations of the entries of the vector, that is, for any vector $x \in \mathbb{R}^n$ and permutation $\pi$ on $\{1, \cdots, n\}$,

$$
\| x \| = \| x_\pi \|. \quad \left(107\right)
$$

**Corollary 17.** With regard to the infinity norm $\| \cdot \|_{\infty}$, one-norm $\| \cdot \|_1$, 2-norm $\| \cdot \|_2$, as well as the mean seminorm $\| \cdot \|_{\text{mean}}$, the isotonic projection is contractive.

**Lemma 18.** (Lemma 1 in (Liu and Barber, 2018)) For any $v \in \mathbb{R}^M$ and $s^*$-sparse $\omega \in \mathbb{R}^M$, it is true for the $s$-sparse hard-threshold operator $\Phi_s(\cdot)$ that

$$
\frac{\langle v - \Phi_s(v), \omega - \Phi_s(v) \rangle}{\| \omega - \Phi_s(v) \|_2^2} \leq \frac{s^2}{2s^2}. \quad \left(108\right)
$$
Lemma 19. [Cover 1967] Given a set of points \( \{x_1, \ldots, x_n\} \subset \mathbb{R}^M \) with sample size \( n \) and dimension \( M \), for any sparsity level \( s \leq M \), define the set

\[
S_{n,M}^{s\text{-sparse}}(x_1, \ldots, x_n) = \left\{ \pi \in S_n : x_{\pi(1)}^T u \leq x_{\pi(2)}^T u \leq \cdots \leq x_{\pi(n)}^T u \text{ for some } s\text{-sparse } u \in \mathbb{R}^M \right\},
\]

where \( S_n \) contains all permutations for \( n \) objects, i.e., all bijections from the set \( \{1, \ldots, n\} \) onto itself. Then the cardinality of the set \( S_{n,M}^{s\text{-sparse}} \) can be bounded as

\[
\left| S_{n,M}^{s\text{-sparse}}(x_1, \ldots, x_n) \right| \leq n^{2s-1} M^s.
\]