BOUNDEDLY SIMPLE GROUPS OF AUTOMORPHISMS OF TREES

JAKUB GISMATULLIN

Abstract. A group is boundedly simple if for some constant $N$, every nontrivial conjugacy class generates the whole group in $N$ steps (bounded simplicity implies simplicity). For a large class of trees, Tits proved simplicity of automorphism groups generated by stabilizers of edges. We prove that for only subdivisions of bi-regular trees such groups are boundedly simple (in fact $32$-boundedly simple). As a consequence, we show that if a boundedly simple group $G$ acts by automorphisms on a tree and $G$ contains a nontrivial stabilizer of some edge, then there is $G$-invariant subtree which is a subdivision of a bi-regular tree.

1. Introduction

A group $G$ is simple (in the algebraic sense) if and only if $G$ is generated by every nontrivial conjugacy class. A finer notion is that of bounded simplicity. A group $G$ is called $N$-boundedly simple if for every two nontrivial elements $g, h \in G$, the element $h$ is the product of $N$ or fewer conjugates of $g^{\pm 1}$, i.e.

$$G = (g^n \cup h^{-1}g^n)^{\leq N}.$$

$G$ is boundedly simple if it is $N$-boundedly simple, for some natural $N$. The simplest example of a boundedly simple group is a finite simple group. Also many groups of Lie type e.g. $\text{PSL}_n(K), \text{PSp}_n(K)$ for an arbitrary field $K$ with $|K| > 4$, are boundedly simple (see Lemma below).

In this paper we are interested in actions of boundedly simple groups on trees. Our results were inspired by the following theorem due to Tits.

Theorem. \[6, \text{Theorem 4.5}\] Suppose that $A$ is a tree and $G$ is a group acting by automorphisms on $A$ without stabilizing any nonempty proper subtree of $A$ nor any end of $A$. Assume that $G$ has Tits’ independence property $(P)$ (see Definition \[2.2\]). Let $G^+$ be a subgroup of $G$ generated by stabilizers of edges of $A$. Then $G^+$ is a simple group (or trivial). Even more: every subgroup normalizing by $G^+$ is trivial or contains $G^+$.

The full group of automorphisms $\text{Aut}(A)$ has property $(P)$ and usually does not stabilize subtrees nor ends of $A$. In this case $\text{Aut}^+(A)$ is simple. We determine trees such that $\text{Aut}^+(A)$ is boundedly simple. In fact, we consider a more general situation of a tree with a coloring $f$ of vertices and group $\text{Aut}^+_f(A)$ of color-preserving automorphisms.

Theorem. \[3.5\] The full automorphism group $\text{Aut}(A_{n,m})$ of a bi-regular tree $A_{n,m}$ has $32$-boundedly simple subgroup $\text{Aut}^+(A_{n,m})$ of index $\leq 2$. 

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Theorem. 3.12 Suppose $(A, f: S(A) \to I)$ is a colored tree and $\text{Aut}_{f}^{+}(A)$ leaves no nonempty proper subtree of $A$ invariant. If $\text{Aut}_{f}^{+}(A)$ is boundedly simple (and non-trivial), then for some $n, m \geq 3$, $A$ is almost $(n, m)$-regular tree (a subdivision of a bi-regular tree), so $\text{Aut}_{f}^{+}(A)$ is 32-boundedly simple.

Therefore, the bounded simplicity of automorphism groups characterizes the bi-regular trees. We do not expect that the bound 32 is sharp.

In the last section we consider a more general set-up of an action of a boundedly simple group on a tree. Our motivation for study such actions comes from Bruhat-Tits buildings for $\text{PSL}_2(K)$, where $K$ is a field with discrete valuation (see [4, Chapter II]). That is, $\text{PSL}_2(K)$ acts by automorphisms on a $n$-regular tree (its Bruhat-Tits building), where $n$ is the cardinality of the residue field. In fact, $\text{PSL}_2(K)$ is a subgroup of an automorphism group of a $n$-regular tree generated by stabilizers of edges. On the other hand it is well known that for an arbitrary field $K$, the group $\text{PSL}_2(K)$ is boundedly simple (by [7], $\text{PSL}_2(K)$ is 5-boundedly simple). We prove the following generalization of Theorem 3.12.

Theorem. 4.1 Suppose that $A$ is a tree and $G < \text{Aut}(A)$ leaves no nonempty proper subtree of $A$ invariant nor stabilizing any end of $A$. Assume that $G^{+}$ is boundedly simple and nontrivial. Then $A$ is an almost $(n, m)$-regular tree (a subdivision of a bi-regular tree $A_{n,m}$) for some $n, m \geq 3$.

Corollary. 4.2 Given a boundedly simple group $G$ acting by automorphisms on a tree $A$ without stabilizing any end. If $G^{+}$ is nontrivial (i.e some nontrivial element from $G$ stabilizes some edge), then there is a $G$-invariant subtree $A' \subseteq A$ which is a subdivision of a bi-regular tree.

There are many examples and results related to boundedly simple groups (see e.g. [1, 3]). Bounded simplicity is a stronger property than usual simplicity (the infinite alternating group is simple but not boundedly simple). This property arises naturally in the study of the logic (model theoretic) nature of simple groups. For fixed $N$, $N$-bounded simplicity is a first order logic property, i.e. can be written as a sentence in the first order logic. Therefore, for each natural $N$, the class of $N$-boundedly simple groups is an elementary class (or an axiomatizable class) of structures. Every elementary class of structures is closed under taking ultraproducts (and elementary extensions). On the other hand the following result characterizes bounded simplicity.

Lemma. The following conditions are equivalent for any group $G$.

1. $G$ is boundedly simple.
2. Some (equivalently every) ultrapower $G^{N}/U$ of $G$ over a non-principal ultrafilter $U$ is a simple group.

Proof. (1) $\Rightarrow$ (2) Bounded simplicity is a first order property and ultrapowers preserve first-order structure. Hence every ultrapower is boundedly simple, and thus simple.

(2) $\Rightarrow$ (1) Let an ultrapower $G^{N}/U$ be simple. Assume contrary to (1), that for every natural $N$, there is $e \neq g_{N} \in G$ and

$$h_{N} \in G \setminus \left(g_{N}^{G} \cup g_{N}^{-1}G \right)^{\leq N}.$$
Consider \( g = (g_N)_{N \in \mathbb{N}}/\mathcal{U} \) and \( h = (h_N)_{N \in \mathbb{N}}/\mathcal{U} \) from \( G^N/\mathcal{U} \). Then the normal closure

\[
H = \bigcup_{n \in \mathbb{N}} \left( g^{g_N/\mathcal{U}} \cup g^{-1}g^{g_N/\mathcal{U}} \right)^n
\]

of \( g \) in \( G^N/\mathcal{U} \) is a nontrivial proper (\( h \notin H \)) subgroup of \( G^N/\mathcal{U} \), which is impossible. \( \square \)

Using above Lemma one can give an easy proof of bounded simplicity of e.g. projective special linear groups \( \text{PSL}_n(K) \) or projective symplectic groups \( \text{PSp}_n(K) \) \((n \geq 2)\). Namely, let \( K \) be an arbitrary infinite field, then

\[
\text{PSL}_n(K)^N/\mathcal{U} \cong \text{PSL}_n\left( K^N/\mathcal{U} \right), \quad \text{PSp}_n(K)^N/\mathcal{U} \cong \text{PSp}_n\left( K^N/\mathcal{U} \right).
\]

By the Lemma, simplicity of \( \text{PSL}_n\left( K^N/\mathcal{U} \right) \) and \( \text{PSp}_n\left( K^N/\mathcal{U} \right) \) implies bounded simplicity of \( \text{PSL}_n(K) \) and \( \text{PSp}_n(K) \) (for an arbitrary infinite field \( F \), \( \text{PSL}_n(F) \) and \( \text{PSp}_n(F) \) are simple groups).

In the model theoretic language we can translate above considerations as: if \( G \) is saturated group, then \( G \) is simple if and only if \( G \) is boundedly simple. For instance, if \( K \) is a saturated field and \( G(K) \) is simple group definable in \( K \), then \( G(K) \) is also saturated (with induced structure), hence is boundedly simple.

Using results from \([1]\), we have a more general picture. Namely, we say that a covering number of a group \( G \) is \( \leq N \), if

\[
C^N = G,
\]

for an arbitrary nontrivial conjugacy class \( C \) of \( G \). Every group with a finite covering number is obviously boundedly simple. By \([1] \text{ Theorem M}\), all simple Chevalley groups (i.e. groups generated by root subgroups corresponding to an irreducible root system, see \([3]\)) have finite covering number. Thus they are boundedly simple (note that \( \text{PSL}_n(F) \) and \( \text{PSp}_n(F) \) are simple Chevalley groups).

2. Basic Notation and Prerequisites

We use the notation and basic facts from \([3]\). A tree is a connected graph without cycles. In this paper \( A \) always denotes a tree. By \( S(A) \) we denote the set of vertices of \( A \). The set of edges \( \text{Ar}(A) \) is a collection of some 2-element subsets of \( S(A) \). Let \( \text{Ch}(A) \) be the set of all infinite paths starting at some vertex of \( A \). \( \text{Ends} \) are equivalence classes of the following relation defined on \( \text{Ch}(A) \): \( C \sim C' \Leftrightarrow C \cap C' \in \text{Ch}(A) \). The set of ends is denoted by \( \text{Bout}(A) \).

By \( \text{Aut}(A) \) we denote the group of all automorphisms of \( A \), i.e. bijections of \( S(A) \) preserving edges. An automorphism \( \alpha \in \text{Aut}(A) \) is called a rotation if it stabilizes some vertex \( s \in S(A) \), i.e. \( \alpha(s) = s \). \( \alpha \) is a symmetry if for some edge \( \{s, s'\} \in \text{Ar}(A) \), \( \alpha(s) = s' \) and \( \alpha(s') = s \). If for some doubly infinite path \( C \) in \( A \), an automorphism \( \alpha \) fixes \( C \) setwise and is not a rotation nor a symmetry, then we call \( \alpha \) a translation (in this case \( C \) is the unique doubly infinite path with above properties and \( \alpha \) restricted to \( C \) is a nontrivial translation). The translation length of a translation \( \alpha \) is the infimum of distances between \( s \) and \( \alpha(s) \), for all \( s \in S(A) \). Note that, the translation length of an arbitrary translation is always positive. By \([6] \text{ Proposition 3.2}\) the group \( \text{Aut}(A) \) is a disjoint union of rotations, symmetries and translations. The subtree of \( A \) consisting of vertices fixed pointwise by \( \alpha \) is called a fixed tree of \( \alpha \) and is denoted by \( \text{Fix}(\alpha) \). The subgroup of \( \text{Aut}(A) \) stabilizing pointwise a given subtree \( A' \) of \( A \) is denoted by \( \text{Stab}(A') \).
For $G < \text{Aut}(A)$, by $\text{Stab}^G(A')$ we denote $\text{Stab}(A') \cap G$. Aut($A$) acts naturally on ends $\text{Bout}(A)$.

**Definition 2.1.** [6, 2.5] Let $\alpha \in \text{Aut}(A)$ and $b \in \text{Bout}(A)$.

1. $\alpha$ stabilizes $b$, if $\alpha(b) = b$.
2. $\alpha$ centralizes $b$, if $\alpha$ fixes pointwise some infinite path $C$ from $b$. The set of all elements that centralize $b$ form a group, called centralizer of $b$.

Clearly, if $\alpha$ stabilizes $b$, then $\alpha$ also centralizes $b$. If $\alpha$ is not a nontrivial translation, then the converse is also true, i.e. if $\alpha(b) = b$, then for some $C \in b$, $\alpha|_C = id_C$. To see this, take an arbitrary $D \in b$. Then $C = D \cap \alpha(D)$ is an infinite path and $C \subseteq \alpha(C)$ or $\alpha(C) \subseteq C$. We claim that $\alpha|_C = id_C$. If not, then e.g. $C \subsetneq \alpha(C)$. Hence $C' = \bigcup_{n \in \mathbb{N}} \alpha^n(C)$ is a double infinite path and $\alpha$ is a translation along $C'$.

**Lemma 2.2.** Let $\alpha, \beta \in \text{Aut}(A)$, and for some $x, y \in S(A)$

$$\alpha(x) = x, \ \beta(y) = y, \ \alpha(y) \neq y \text{ and } \beta(x) \neq x.$$ 

Then $\alpha \circ \beta$ is a translation with an even translation length.

**Proof.** Let $\gamma = \alpha \circ \beta$. We use the following criterion ([6, Lemma 3.1]) for an automorphism $\gamma \in \text{Aut}(A)$ to be a translation:

- (♠) if for some edge $\{x, y\} \in \text{Ar}(A)$, $x$ is on the shortest path from $y$ to $\gamma(y)$ and $\gamma(y)$ is on the shortest path from $x$ to $\gamma(x)$, then $\gamma$ is a translation along a doubly infinite path containing $y, x, \gamma(y)$ and $\gamma(x)$, with the translation length $\text{dist}(x, \gamma(x)) = \text{dist}(y, \gamma(y))$.

We may assume that on the shortest path from $x$ to $y$ there is no vertex fixing by $\alpha [\beta]$ other than $x [y$ respectively]. Since $\alpha(x) = x$, $\alpha(y) = \gamma(y) \neq y$ and $\gamma(x) = \alpha(\beta(x)) \neq \alpha(x) = x$, the shortest path from $y$ to $\gamma(x)$ first goes through $x$ and then through $\gamma(y)$ (see Figure 2.1). Therefore by (♠), the translation length of $\gamma$ is $\text{dist}(y, \gamma(y)) = 2 \text{dist}(y, x)$. □

It is proved in [6, Proposition 3.4] that if a subgroup $G < \text{Aut}(A)$ does not contain translations, then $G$ stabilizes some vertex or edge of $A$, or centralizes some end of $A$. The proof of this uses the assumption that $G$ is a group in a very limited way, so a slightly general fact is true (Lemma 2.3). We use this generalization in the proof of Theorem 2.7.

**Lemma 2.3.** If $T \subseteq \text{Aut}(A)$ and $T \cup TT$ does not contain translations, then the group generated by $T$ also does not contain translations. Hence $T$ stabilizes some vertex or edge of $A$ or centralizes some end of $A$. 

![Figure 2.1. Composition of two rotations](image-url)
Proof. It is enough to prove that $G = \langle T \rangle$ does not contain translations (the rest follows from [6 Proposition 3.4]).

Upon replacing the tree $A$ by its first barycentric subdivision, there is no loss of generality in assuming that $T$ contains no symmetries. Hence, every element of $T$ is a rotation. The family of fixed trees of automorphisms from $T$

$$\{\text{Fix}(\alpha) : \alpha \in T\}$$

is linearly ordered by the inclusion (because otherwise by Lemma 2.2, $TT$ contains a translation). Therefore for $\alpha_1, \ldots, \alpha_n \in T \cup T^{-1}$,

$$\text{Fix}(\alpha_1 \circ \cdots \circ \alpha_n) \supseteq \text{Fix}(\alpha_1) \cap \cdots \cap \text{Fix}(\alpha_n) \neq \emptyset$$

(note that $\text{Fix}(\alpha) = \text{Fix}(\alpha^{-1})$). Hence $\alpha_1 \circ \cdots \circ \alpha_n$ is not a translation, so $G = \langle T \rangle$ does not contain translations.

□

Some groups of automorphisms of trees we are going to deal with satisfy Tits’ independence property $(P)$ ([6 4.2]). Let $G < \text{Aut}(A)$ and $C$ be an arbitrary (finite or infinite) path in $A$. Consider a natural projection

$$\pi : S(A) \to S(C)$$

($\pi(x) \in S(C)$ is the closest vertex to $x$) and for every $s \in S(C)$ an induced projection of stabilizer

$$\rho_s : \text{Stab}^G(C) \longrightarrow \text{Aut}(\pi^{-1}[s]).$$

Definition 2.4. We say that $G < \text{Aut}(A)$ has the property $(P)$ if for an arbitrary path $C$ in $A$, the mapping

$$\rho = (\rho_s)_{s \in S(C)} : \text{Stab}^G(C) \longrightarrow \prod_{s \in S(C)} \text{Im}(\rho_s)$$

is an isomorphism.

For example, the full group of automorphisms $\text{Aut}(A)$ has property $(P)$.

Definition 2.5. Let $A$ be a tree and $G < \text{Aut}(A)$.

1. A vertex having at least three edges adjacent to it is called a ramification point ([6 2.1]).

2. $G^+$ is a subgroup of $\text{Aut}(A)$ generated by the stabilizers of edges: ([6 4.5])

$$G^+ = \langle \text{Stab}^G(x, y) : \{x, y\} \in \text{Ar}(A) \rangle.$$ 

Lemma 2.6. Every element of $G^+$ is either a rotation or a translation with an even translation length.

Proof. Consider an equivalence relation $E$ on $S(A)$:

$$E(x, y) \iff \text{distance from } x \text{ to } y \text{ is even.}$$

Every stabilizer of an edge preserves $E$, so $G^+$ preserves $E$. On the other hand, only rotations and translations with even translation lengths preserve $E$. □

Assume that $G < \text{Aut}(A)$ has property $(P)$ and does not preserve any proper subtree nor stabilizes any end of $A$. [6, Theorem 4.5] implies that every subgroup of $G$ normalizing by $G^+$ is trivial or contains $G^+$. In particular $G^+$ is a simple group. Modifying one step in the proof of this theorem (using Lemma 2.3), we can show a more precise result regarding conjugacy classes in $G^+$. 
By $h^H = \{ x^{-1}hx : x \in H \}$ we mean the conjugacy class of element $h$ of the group $H$.

**Theorem 2.7.** Let $A$ be a tree and $G < \text{Aut}(A)$. Assume that $G$ has property (P) and $G$ leaves no nonempty proper subtree of $A$ invariant nor stabilizing any end of $A$. Then for every $g \in G^+$ and edge $\{x, y\} \in \text{Ar}(A)$

$$\text{Stab}^G(x, y) \subseteq \left( g^{G^+} \cup g^{G^+} \cdot g^{G^+} \right)^2 \cdot \left( g^{-1G^+} \cup g^{-1G^+} \cdot g^{-1G^+} \right)^2.$$ 

**Proof.** The proof is a repetition of a proof of [6, Theorem 4.5], so we will be brief. By removing the edge $\{x, y\}$ from $A$ we obtain two subtrees $A'$ and $A''$ of $A$. Using property (P) we have

$$\text{Stab}^G(x, y) = \text{Stab}^G(A') \cdot \text{Stab}^G(A'').$$

Hence, it is enough to show that

$$\text{Stab}^G(A') \cup \text{Stab}^G(A'') \subseteq \left( g^{G^+} \cup g^{G^+} \cdot g^{G^+} \right) \cdot \left( g^{-1G^+} \cup g^{-1G^+} \cdot g^{-1G^+} \right).$$

**Claim.** There is a translation $h$ from $g^{G^+} \cup g^{G^+} \cdot g^{G^+}$ along a doubly infinite path $D$ from $A'$.

**Proof of the claim.** By [5, Lemma 4.4] applied to $G^+$, we have that $g^{G^+}$ does not preserve any proper subtree nor stabilizes any end of $A$. Therefore by Lemma 2.3, $g^{G^+} \cup g^{G^+} \cdot g^{G^+}$ contains a translation $h$ along some doubly infinite path $D$. Without loss of generality we may assume that

$$D \cap A' \neq \emptyset.$$

Namely, take an arbitrary vertex $s \in S(D)$. By [5, Lemma 4.1], there is $g'' \in G^+$ with $g''^{-1}(s) \in S(A')$. If $h' = h^{g''}$ and $D' = g''^{-1}[D]$, then $h' \in g^{G^+} \cup g^{G^+} \cdot g^{G^+}$ is a translation along $D'$ and $D' \cap A' \neq \emptyset$.

We may also assume that $D \not\subseteq A'$.

Let $b'$ and $b''$ be two ends induced by $D$. There is $g' \in G^+$ with $g'(b'') \not\in \{b', b''\}$ (otherwise $G^+$ leaves invariant $D$). Denote by $\pi : S(A) \to S(D)$ a projection from $A$ to $D$ (i.e. $\text{dist}(\{x\}, D) = \text{dist}(x, \pi(x))$). Then $\pi[S(A'')] \subseteq S(A'' \cap D'')$. Since $b'' \not\in \{g''^{-1}(b'), g''^{-1}(b'')\}$ we can find $n \in \mathbb{Z}$ such that

$$D' = h^n (g^{-1}[D]) \subseteq A'.$$

Thus $h' = h^{g' h^{-n}}$ is a translation from $g^{G^+} \cup g^{G^+} \cdot g^{G^+}$ along a doubly infinite path $D'$ from $A'. \hfill \square$

(♠) follows directly from the claim: by [5, Lemma 4.3],

$$\text{Stab}^G(D) = \{ hfh^{-1}f^{-1} : f \in \text{Stab}^G(D) \},$$

so $\text{Stab}^G(A') \subset \text{Stab}^G(D) = h \cdot h^{-1} \text{Stab}^G(D) \subseteq h^{G^+} \cdot h^{-1G^+}$ is included in $\left( g^{G^+} \cup g^{G^+} \cdot g^{G^+} \right) \cdot \left( g^{-1G^+} \cup g^{-1G^+} \cdot g^{-1G^+} \right). \hfill \square$

We recall from [6, Section 5] a convenient way to describe trees. Let $I$ be a set of “colors” and

$$f : S(A) \to I$$

a coloring function. Define a group of automorphisms preserving $f$ as

$$\text{Aut}_f(A) = \{ \alpha \in \text{Aut}(A) : f \circ \alpha = f \}.$$
We say that \( f \) is normal if \( f \) is onto and for every \( i \in I \), \( \operatorname{Aut}_f(A) \) is transitive on \( f^{-1}[i] \).
Clearly, for every coloring function \( f \) there is a normal coloring function \( f' \) such that \( \operatorname{Aut}_f(A) = \operatorname{Aut}_{f'}(A) \). Hence we may always assume that \( f \) is normal.

It is easy to see that \( \operatorname{Aut}_f(A) \) has the property \((P)\).

Let \((A, f: S(A) \to I)\) be an arbitrary colored tree (\( f \) is normal). Define a function \( a: I \times I \to \operatorname{Card} \) as follows: take an arbitrary \( x \in f^{-1}[i] \) and set

\[
a(i, j) = |\{y \in f^{-1}[j] : \{x, y\} \in \operatorname{Ar}(A)\}|.
\]

Since \( f \) is normal, the value \( a(i, j) \) does not depend on the choice of \( x \) from \( f^{-1}[i] \). Functions \( a \) arising this way can be characterized by two conditions \cite[Proposition 5.3]{6}:

1. if \( a(i, j) = 0 \), then \( a(j, i) = 0 \)
2. the directed graph \( G(\alpha) = (I, E) \), where \( E = \{\{i, j\} \subseteq I : a(i, j) \neq 0\} \), is connected.

If a function \( a: I \times I \to \operatorname{Card} \) has properties (1) and (2), then there is a colored tree \( A \) with a normal coloring function \( f \) such that for every \( x \in f^{-1}[i] \),

\[
a(i, j) = |\{y \in f^{-1}[j] : \{x, y\} \in \operatorname{Ar}(A)\}|.
\]

We say then, that \( a \) is a code of colored tree \( A \). We note also \cite[5.7]{6} that if \( 1 \not\in a[I \times I] \), then \( \operatorname{Aut}_f(A) \) leaves no nonempty proper subtree invariant nor stabilizes any end of the tree (hence by \cite[Theorem 4.5]{6} or our Theorem 2.7, \( \operatorname{Aut}_f(A) \) is a simple group).

An element \( i \in I \) is a ramification color, if \( i = f(x) \), for some ramification point \( x \in S(A) \). The set of all ramification colors we denote by \( I_{\text{ram}} \).

By the color of (possibly infinite) path we mean the sequence of colors of its vertices.
We will use the next fact from \cite{6}.

**Proposition 2.8.** \cite[6.1]{6} Let \( A \) be a colored tree. Then \( \operatorname{Aut}_f(A) \) is generated by stabilizers of ramification points in \( \operatorname{Aut}_f(A) \):

\[
\operatorname{Aut}_f(A) = \left\langle \operatorname{Stab}^{\operatorname{Aut}_f(A)}(r) : r \in S(A) \text{ is a ramification point} \right\rangle.
\]

Since \( \operatorname{Aut}_f(A) \) is the subgroup of \( \operatorname{Aut}_f(A) \), we may consider the following coloring function

\[
f^+: S(A) \to I^+ = \{\text{orbits of } \operatorname{Aut}_f(A) \text{ on } S(A)\},
\]

which is just the quotient map.

**Proposition 2.9.**

1. \( f^+ \) is normal and \( f^+ \) refines \( f \), i.e. if \( f^+(x) = f^+(y) \), then \( f(x) = f(y) \).
2. \( \operatorname{Aut}_{f^+}(A) = \operatorname{Aut}_{f^+}(A) \)

**Proof.** (1) and the inclusion \( \subseteq \) in (2) is obvious. If \( \alpha \in \operatorname{Aut}_{f^+}(A) \) and \( r \in S(A) \) is a ramification point, then \( \alpha(r) \in \operatorname{Aut}_{f^+}(A) \cdot r \). Thus, by Proposition 2.8 \( \alpha \in \operatorname{Stab}^{\operatorname{Aut}_f(A)}(r) \cdot \operatorname{Aut}_{f^+}(A) = \operatorname{Aut}_{f^+}(A) \). \( \Box \)
3. Bounded simplicity of $\text{Aut}_f^+(A)$

We begin with the criterion for bounded simplicity of $G^+$. 

**Proposition 3.1.** Let $A$ be a tree. Assume that $G < \text{Aut}(A)$ has property (P) and leaves no nonempty proper subtree of $A$ nor stabilizes any end of $A$ (so $G^+$ is simple).

Then $G^+$ is boundedly simple if and only if there is a natural number $K$ such that

1. every translation from $G^+$ is a product of $K$ rotations from $G^+$,
2. every rotation from $G^+$ is a product of $K$ elements from $\bigcup_{(x,y) \in \text{Ar}(A)} \text{Stab}^G(x,y)$.

**Proof.** $\Rightarrow$ is clear. $\Leftarrow$ Let $g \in G^+$ be nontrivial. By Theorem 2.7 for an arbitrary edge $(x,y) \in \text{Ar}(A)$

$$\text{Stab}^G(x,y) \subseteq \left(g^{G^+} \cup g^{-1G^+}\right)^{\leq 8}.$$ 

Lemma 2.6 with the assumption lead to $G^+ = \left(g^{G^+} \cup g^{-1G^+}\right)^{\leq 8K^2}$. \Halmos

Next lemma states that the condition (2) from Proposition 3.1 is always satisfied in $\text{Aut}_f^+(A)$, with $K = 2$.

**Lemma 3.2.** Assume that $A$ is a colored tree and the group $\text{Aut}_f^+(A)$ is nontrivial. Then every nontrivial rotation from $\text{Aut}_f^+(A)$ fixes some ramification point and is a composition of two elements from $\bigcup_{(x,y) \in \text{Ar}(A)} \text{Stab}^{\text{Aut}_f(A)}(x,y)$.

**Proof.** By [6, 6.1], if $\alpha \in \text{Aut}_f^+(A)$ stabilizes a ramification point, then $\alpha$ is a product of two elements from $\bigcup_{(x,y) \in \text{Ar}(A)} \text{Stab}^{\text{Aut}_f(A)}(x,y)$.

We prove, that every rotation $\alpha \in \text{Aut}_f^+(A)$ fixes some ramification point. Let $x$ be a non-ramification point and $\alpha(x) = x$. We may assume that $x$ has two adjacent vertices $y$ and $z$ of the same color. It is enough to show, that $\alpha(y) = y$ and $\alpha(z) = z$. If $f(x) = f(y)$, then $A$ is just a doubly infinite path, so let $i = f(x) \neq f(y)$. Consider on $S(A)$ the following equivalence relation: $E(r,s)$ if and only if on the shortest path from $r$ to $s$ there is even number of vertices of color $i$. Clearly $\neg E(y,z)$. It suffices to show that for every $\beta \in \text{Aut}_f^+(A)$ and $r \in S(A)$

$$E(r, \beta(r)).$$

Let $\beta \in \text{Stab}^{\text{Aut}_f(A)}(x',y')$ (where $\{x',y'\} \in \text{Ar}(A)$) and consider the shortest path $C$ from $r$ to $\beta(r)$. $\beta$ fixes some ramification point $t$ from $C$. Since $x$ is not a ramification point, $f(t) \neq i$. Therefore $E(r, \beta(r))$. \Halmos

The main ingredient in proofs of results of this paper is (defined in the next definition) the notion of *the type a translation*. We associate with each translation, a finite sequence of colors.

**Definition 3.3.** Let $A$ be an arbitrary colored tree. The *type* of a translation $\alpha \in \text{Aut}_f(A)$ along a doubly infinite path $C$ is a set of all cyclic shifts of a particular finite sequence from $I$:

$$t = [i_1, \ldots, i_n] = \{(i_1, \ldots, i_n), (i_2, \ldots, i_n, i_1), \ldots, (i_n, i_1, \ldots, i_{n-1})\}$$

such that if for some $x \in C$, $x = x_1, \ldots, x_n, x_{n+1} = \alpha(x)$ is a subpath of $C$, then

$$f(x_1) = i_1, \ldots, f(x_n) = i_n$$

(note that $f(x_{n+1}) = i_1$).
Any two translations which are conjugate have the same type. We calculate types of some translations: a composition of two rotations and a composition of a rotation and a translation.

**Lemma 3.4.** Let $\alpha, \beta \in \text{Aut}(A)$ be rotations and $\gamma \in \text{Aut}(A) — a$ translation.

1. Assume that $\alpha(x) = x, \beta(y) = y$ for some $x, y \in S(A)$, and on the shortest path $D$ from $y$ to $x$ there is no vertex except $x, y$ fixing $\alpha, \beta$, respectively. If the color of $D$ is

   \[ \Box = (i_1, \ldots, i_n), ~ f(y) = i_1, ~ f(x) = i_n, ~ n \geq 2, \]

   then the type of $\alpha \circ \beta$ is

   \[ \lozenge = [i_1, i_2, \ldots, i_{n-1}, i_n, i_{n-1}, \ldots, i_2]. \]

   On the other hand, every translation of the type $\lozenge$ is a composition of two rotations. As a consequence we see, that the type of $\alpha \circ \beta$ depends only on the type of the shortest path between $\text{Fix}(\beta)$ and $\text{Fix}(\alpha)$.

2. Assume that $\gamma$ is a translation of the type

   \[ \triangle = [i_1, j_2, \ldots, j_m], ~ m \geq 2, \]

   along a doubly infinite path $C$ and $x$ is the vertex fixed by $\alpha$ which is closest to $C$.

   2.1 Assume that $x$ lies outside $C$ (see Figure 3.1). Take a vertex $y \in C$ — the closest vertex to $x$ in $C$ (a projection of $x$ on $C$). Let $D$ be the shortest path from $y$ to $x$ (where $x$ is the only vertex in $D$ fixed by $\alpha$). Let the color of $D$ be $\Box$ from the previous point (with $f(y) = i_1, f(x) = i_n$). Then $\alpha \circ \gamma$ and $\gamma \circ \alpha$ are translations of the type

   \[ \lozenge = \lozenge \triangle = [i_1, i_2, \ldots, i_{n-1}, i_n, i_{n-1}, \ldots, i_2, i_1, j_2, \ldots, j_m]. \]

   Also every translation of the type $\lozenge$ is a composition of a rotation, and translation of the type $\triangle$. Hence, the type of $\alpha \circ \gamma$ only depends on the type of $\gamma$ and the type of the shortest from $C$ to $\text{Fix}(\alpha)$.

   2.2 Assume that $x$ lies on $C$. Let $D$ be the shortest path from $x$ to $\gamma(x)$ and assume that $f(x) = i_1$. Let $y$ be a vertex from $D$ next to $x$ (so $f(y) = j_2$).

   2.2.1 If $\gamma(\alpha(y))$ lies outside $D$, then $\gamma \circ \alpha$ is a translation of the same type as $\gamma$, i.e.

   \[ \triangle = [i_1, j_2, \ldots, j_m]. \]

   2.2.2 Assume that $\gamma(\alpha(y))$ lies on $D$ (so $j_2 = j_m$). Take $y' \neq x$ — a vertex from $D$, next to $y$ (so $f(y') = j_3$). Again, if $\gamma(\alpha(y'))$ is outside $D$, then $\gamma \circ \alpha$ is a translation of the type

   \[ [j_2, \ldots, j_{m-1}]. \]

   Continuing this way we see, that either $\gamma \circ \alpha$ is a translation of the type being the subtype of $\triangle$ or $\gamma \circ \alpha$ is a rotation. In the last case $m$ is even and

   \[ j_2 = j_m, ~ j_3 = j_{m-1}, ~ \ldots, j_{\frac{m}{2}-1} = j_{\frac{m}{2}+2}. \]

   Thus, $\gamma \circ \alpha$ stabilises vertex of type $j_{\frac{m}{2}+1}$. Since $\alpha \circ \gamma = (\gamma \circ \alpha)^{-1}$, the same applies to $\alpha \circ \gamma$.

**Proof.** It is enough to apply (♠) from Lemma 2.2 to:
Figure 3.1. Composition of translation and rotation

- $y, x, \alpha(\beta(y)), \alpha(\beta(x))$, in (1),
- $y, x, \alpha(\gamma(y)), \alpha(\gamma(x))$ (see Figure 3.1), in (2.1),
- $x, y, \gamma(x) = \gamma(\alpha(x))$ and $\gamma(\alpha(y))$, in (2.2.1),
- $y, y', \gamma(\alpha(y))$ and $\gamma(\alpha(y'))$ (see Figure 3.1), in (2.2.2).

A $(n, m)$-regular (bi-regular) tree $A_{n,m}$, is a 2-colored tree with the following code:

$$a(0,0) = a(1,1) = 0, \ a(0,1) = n, \ a(1,0) = m,$$
where $I = \{0,1\}$ and $n, m$ are some cardinal numbers $\geq 3$. Intuitively, in a bi-regular tree every vertex is black or white, every white vertex is connected with $n$ black vertices and every black vertex is connected with $m$ white vertices (if $n = 2$ and $m \geq 3$, then after removing vertices of color 0 we get the $m$-regular tree).

**Theorem 3.5.** The full automorphism group $\text{Aut}(A_{n,m})$ of a bi-regular tree $A_{n,m}$ has 32-boundedly simple subgroup $\text{Aut}^+(A_{n,m})$ of index $\leq 2$.

**Proof.** Clearly, $\text{Aut}^+(A_{n,m})$ has property (P) and $1 \not\in a[I \times I]$ (so $\text{Aut}^+(A_{n,m})$ leaves no nonempty proper subtree of $A_{n,m}$ invariant nor stabilizes any end of $A_{n,m}$). $\text{Aut}^+(A_{n,m})$ consists on translations with even translation lengths and rotations. The type of a translation from $\text{Aut}^+(A_{n,m})$ has to be of the form $[ijij \ldots ij]$. Therefore, it follows from Lemma 3.4(1), that every translation is a product of two rotations. It is enough to apply Proposition 3.1 with $K = 2$. □

**Definition 3.6.** An almost $(n, m)$-regular tree (almost bi-regular tree) is the $(n, m)$-regular tree subdivided (in an equivariant way) by non-ramification points. Namely, it is the tree with the set of colors $I = \{0, \ldots , k\}$ and the following code: $a(0,1) = n, \ a(k,k-1) = m$ and $a(i, i+1) = a(i+1, i) = 1$ for $i \in I \setminus \{0,k\}$. For all other pairs $(p,q)$ from $I^2$, $a$ has value 0.

If $A$ is an almost $(n,m)$-regular tree, then $A$ is the subdivision of a bi-regular tree $A_{n,m}$ and $\text{Aut}^+(A_{n,m})$. Hence $\text{Aut}^+(A_{n,m})$ is 32-boundedly simple.

Except for $A_{n,m}$ there are no other colored trees $A$ with boundedly simple groups $\text{Aut}^+(A)$ and with the property that $\text{Aut}^+(A)$ leaves no nonempty proper subtree of $A$ invariant (Theorem 3.12). The next proposition is the main technical step in the proof of this fact. We prove that, if $\text{Aut}^+(A)$ is boundedly simple, then some particular configuration in the code of $A$ is forbidden.

**Proposition 3.7.** Assume that $A$ is a colored tree and $\text{Aut}^+(A)$ is nontrivial and boundedly simple. Take two rotations $\alpha, \beta$ from $\text{Aut}^+(A)$. Suppose that for three different ramification points $x, y, z \in S(A)$,
BOUNDEDLY SIMPLE GROUPS OF AUTOMORPHISMS OF TREES

Figure 3.2. Composition of three rotations

- \( \alpha(x) = x, \beta(y) = y \) and \( \beta \circ \alpha \) is a translation along a doubly infinite path \( C \),
- \( t \) is the projection of \( z \) onto \( C \) (see Figure 3.2) and \( s \) is a vertex next to \( z \) lying on the shortest path from \( z \) to \( t \),
- on the shortest paths: from \( x \) to \( y \) and from \( s \) to \( t \), there are no vertices of color \( f(z) \) (so also on \( C \) there are no such vertices).

If \( \gamma \) is an arbitrary element from \( \text{Aut}_f^+(A) \) fixing \( z \), then \( \gamma \) fixes also \( s \).

\[ \gamma(s) = s. \]

**Proof.** By Proposition 2.9, there is a normal function \( f^+: S(A) \to I^+ \) with \( \text{Aut}_{f^+}(A) = \text{Aut}_f^+(A) \). We may assume further that \( f = f^+ \) and \( I = I^+ \).

Suppose, contrary to our claim, that \( \gamma(s) \neq s \). For each natural \( K \) we construct a composition of some rotations which cannot be written as a composition of \( K \) rotations.

Then, Proposition 3.1 implies that \( \text{Aut}_{f^+}(A) \) is not boundedly simple.

Our situation is described by Figure 3.2. We may assume that \( t \) belongs to the path in \( C \) from \( x \) to \( y \) (if \( t \) belongs to the path from \( (\beta \circ \alpha)^n(x) \) to \( (\beta \circ \alpha)^n(y) \), for some integer \( n \), then just take \( z := (\beta \circ \alpha)^{-n}(z) \) and conjugate \( \gamma := \gamma^{(\beta \circ \alpha)^n} \).

Denote by 0, 1 and 2 sequences of colors, corresponding to paths in Figure 3.3. Namely, let

- 0 corresponds to the shortest path from \( t \) to \( \gamma(t) \) (through \( z \)) without the last term of color \( f(t) \),
- 1 — from \( \alpha^{-1}(t) \) to \( t \) (through \( x \)) without the last term of color \( f(t) \),
- 2 — from \( t \) to \( \beta(t) \) (through \( y \)) without the last term of color \( f(t) \).

Note that 1 or 2 might be empty, but 12 and 0 are always nonempty.

For example, by Lemma 3.4(2.1), translations \( \alpha \circ \beta \), \( \beta \circ \alpha \) have type \([1, 2]\), translations \( \delta = \gamma \circ \beta \circ \alpha \), \( \beta \circ \alpha \circ \gamma \), \( \gamma \circ \alpha \circ \beta \), \( \alpha \circ \beta \circ \gamma \) have type \([1, 0, 2]\) and the path \( C \) has color \((\ldots 1212 \ldots)\) (see Figure 3.3).

Define by induction the following sequences

- \( t_2 = (1, 2, 1, 0, (1, 2)^2, 1, 0), \)
- \( t_{n+1} = (1, 2, 1, 0, (1, 2)^{2n-1}, t_n, (1, 2)^{2n-1}, 1, 0), \) for \( n \geq 2 \).

Let \( \alpha_n \) be a translation of type \([t_n]\) \( (\alpha_n \) exists, because \( t_n \) induces a double infinite path in \( A \) of the color \((\ldots t_n t_n \ldots) \), and \( \alpha_n \) is the translation along this path). \( \alpha_n \) is the composition of \( n \) rotations from \( \text{Aut}_{f^+}(A) \). Particularly, by Lemma 3.4(1),

\[ [t_2] = [2, 1, 0, (1, 2)^2, 1, 0, 1] \]
is a type of composition of two rotations from Aut$_{f^+}(A)$ (because $x$ and $y$ are ramification points). $\alpha_{n+1}$ has the type

$$[1, 2, 1, 0, (1, 2)^{2n-1}, t_n, (1, 2)^{2n-1}, 1, 0] = [(1, 2)^{2n-1}, 1, 0, 1, 2, 1, 0, (1, 2)^{2n-1}, t_n],$$

being (by Lemma 3.4(2.1)) the type of the composition of a translation of type $t_n$ and a rotation. Hence, $\alpha_{n+1}$ is a composition of $n+1$ rotations.

The proof will be completed by showing that $\alpha_n$ cannot be written as a composition of less than $\frac{n+3}{2}$ rotations. In order to do this, we introduce notions describing the complexity of distances of colors in types.

**Definition 3.8.** For $i \in I$ and type $t = [i_1, \ldots, i_n]$ define the $i$-sequence of $t$ in the following way.

- If there is no occurrence of $i$ in $t$, then the $i$-sequence of $t$ is empty.
- Let $i_k$ be the first occurrence of $i$ in $(i_1, \ldots, i_n)$. The $i$-sequence of $t$ is a sequence (modulo all cyclic shifts) of distances between consecutive occurrences of $i$ in the sequence $(i_k, i_{k+1}, \ldots, i_{n-1}, i_n, i_1, \ldots, i_k)$.

Note that if $t$ has $N$ occurrences of $i$, then its $i$-sequence is of length $N$.

We compute the $f(z)$-sequence of $t_n$ (note that, by the assumption $f(z)$ appears once only in the path 0). However, first we compute the 0-sequence for $t_n$ (regarding 0 as an additional color). The 0-sequence for $[t_2] = [1, 2, 1, 0, 1, 2, 1, 2, 1, 0]$ is $[6, 4]$ and for $[t_3] = [1, 2, 1, 0, (1, 2)^4, 1, 0, (1, 2)^2, 1, 0, (1, 2)^3, 1, 0]$ is $[10, 6, 8, 4]$. It can be proved by induction that the 0-sequence for $[t_{n+1}]$ is

$$(\star) \quad (4n + 2, 4n - 2, \ldots, 14, 10, 6, 8, 12, \ldots, 4n - 4, 4n, 4).$$

Let $p$ be the length of the path 01 and $q$ — of the path 12 ($p, q$ are even and $\geq 2$). Note that the 0-sequence for $t' = [0, (1, 2)^n, 1]$ is $(2n + 2)$ and $f(z)$-sequence for $t'$ is $(p + nq)$. Therefore by $(\star)$, the $f(z)$-sequence for $[t_{n+1}]$ is

$$[p + (2n)q, p + (2n - 2)q, \ldots, p + 4q, p + 2q, p + 3q, \ldots, p + (2n - 3)q, p + (2n - 1)q, p + q].$$

**Definition 3.9.** Suppose $i \in I$, $t = [i_1, \ldots, i_n]$ is a type and $m \in \mathbb{N}$. Define

- $L_m(t, i)$ as the maximal even number, less or equal that the number of occurrences of $m$ in the $i$-sequence of $t$,
- $L_\infty(t, i) = \sum_{m \in \mathbb{N}} L_m(t, i)$. 

![Figure 3.3. Types of paths](image-url)
Table 3.1. *i*-sequence of the composition of two rotations

| Case                        | *i*-sequence                                                                 |
|-----------------------------|-----------------------------------------------------------------------------|
| \(i_1 = i \) and \(i_n = i\) | \(m_1, m_2, \ldots, m_{\frac{N - 1}{2}}, m_{\frac{N + 1}{2}}, m_{\frac{N - 1}{2}}, \ldots, m_2, m_1\) |
| \(i_1 = i \) and \(i_n \neq i\) | \(m_1, m_2, \ldots, m_{\frac{N - 1}{2}}, m_{\frac{N + 1}{2}}, m_{\frac{N - 1}{2}}, \ldots, m_2, m_1\) |
| \(i_1 \neq i \) and \(i_n = i\) | \(m_1, m_2, \ldots, m_{\frac{N - 1}{2}}, m_{\frac{N + 1}{2}}, m_{\frac{N - 1}{2}}, \ldots, m_2, m_1, m_0\) |
| \(i_1 \neq i \) and \(i_n \neq i\) | \(m_1, m_2, \ldots, m_{\frac{N - 1}{2}}, m_{\frac{N + 1}{2}}, m_{\frac{N - 1}{2}}, \ldots, m_2, m_1, m_0\) |

E.g. for an arbitrary \(m\), \(L_m(t_n, f(z)) = 0\), i.e. \(t_n\) has no multiple occurrences of any value.

To prove the lower bound for the number of rotations needed to generate \(\alpha_n\), we will use the next lemma.

**Lemma 3.10.** Let \(t\) be a type of a translation which is a composition of \(K\) rotations. Suppose \(t\) has \(N\) occurrences of \(i\). Then

\[
L_\infty(t, i) = \sum_{m \in \mathbb{N}} L_m(t, i) \geq N - 4K + 6.
\]

**Proof.** We prove the lemma by induction on \(K\).

Let \(K = 2\). By Lemma 3.4(1), \(t\) is of the form \([i_1, i_2, \ldots, i_{n-1}, i_n, i_{n-1}, \ldots, i_2]\). In the Table 3.1 we describe all possibilities for the shape of \(i\)-sequence of \(t\). In all cases \(L_\infty(t, i) \geq N - 2\).

Let \(t\) be the type of the composition \(\tau = \tau_1 \circ \ldots \circ \tau_{K+1}\) of \(K + 1\) rotations. Denote by \(\rho = \tau_1 \circ \ldots \circ \tau_K\). If \(\rho\) is a rotation, then \(\tau = \rho \circ \tau_{K+1}\) is the composition of two rotations and we may use the induction hypothesis. Otherwise, \(\rho\) is the translation along some double infinite path \(C'\). Let

\[
s = [i_1, j_2, \ldots, j_m], \quad m \geq 2
\]

be the type of \(\rho\) and \(x_{K+1}\) be a vertex fixed by \(\tau_{K+1}\), which is the nearest to \(C'\). We have two main cases: \(x_{K+1}\) is in \(C'\) or not.

Assume first that \(x_{K+1} \not\in C'\), i.e. the case (2.1) from Lemma 3.4 holds. Then

\[
t = [i_1, i_2, \ldots, i_{n-1}, i_n, i_{n-1}, \ldots, i_2, i_1, j_2, \ldots, j_m].
\]

Let \(N_1\) and \(N_2\) be numbers of occurrences of \(i\) in \((i_1, \ldots, i_n, \ldots, i_2)\) and \((i_1, j_2, \ldots, j_m)\) respectively. \(t\) has \(N = N_1 + N_2\) occurrences of \(i\). Denote by

\[
[n_1, n_2, \ldots, n_{N_2}]
\]

the \(i\)-sequence of \(s\). Again, there are four possibilities for the shape of \(i\)-sequence of \(t\) (presented in the Table 3.2). By induction hypothesis \(L_\infty(s, i) \geq N_2 - 4K + 6\). Therefore, by the definition of \(L_m(t, i)\), in the worst (i.e. fourth) case we have

\[
L_\infty(t, i) \geq (L_\infty(s, i) - 2) + (N_1 - 2) = N - 4K + 2 = N - 4(K + 1) + 6.
\]

Assume that \(x_{K+1} \in C'\), i.e. the case (2.2) from Lemma 3.4 holds. Then

\[
t = [i_1, j_2, \ldots, j_m]
\]

and \(\rho\) has type \(s = [i_1, i_2, \ldots, i_{n-1}, i_n, i_{n-1}, \ldots, i_2, i_1, j_2, \ldots, j_m]\). Let \(N_1\) and \(N_2\) be numbers of occurrences of \(i\) in \((i_1, i_2, \ldots, i_{n-1}, i_n, i_{n-1}, \ldots, i_2)\) and \(t\) respectively. We
may assume that the \( i \)-sequence of \( s \) is given by the Table 3.2 (where \([n_1, n_2, \ldots, n_{N_2}]\) is the \( i \)-sequence of \( t \)). By induction hypothesis \( L_\infty(s, i) \geq N_1 + N_2 - 4K + 6 \). We have to show that \( L_\infty(t, i) \geq N_2 - 4(K + 1) + 6 \). In the first case (i.e. \( i_1 = i \) and \( i_n = i \))

\[
L_\infty(t, i) = L_\infty(s, i) - N_1.
\]

In the second case

\[
L_\infty(t, i) \geq L_\infty(s, i) - (N_1 - 1) - 2.
\]

In the third case

\[
L_\infty(t, i) \geq L_\infty(s, i) - (N_1 - 1) - 4.
\]

In the fourth case

\[
L_\infty(t, i) \geq L_\infty(s, i) - (N_1 - 2) - 6 = N_2 - 4(K + 1) + 6.
\]

The \( f(z) \)-sequence for \( t_n \) has no multiple occurrences of any value and \( t_n \) has \( 2n \) occurrences of \( f(z) \). If \( t_n \) is the type of the composition of \( K \) rotations, then by the Lemma 3.10, \( 0 \geq 2n - 4K + 6 \), so \( K \geq \frac{2n+3}{2} \). This finishes the proof of Proposition 3.7.

Proposition 3.7 implies that for many trees \( A \), groups \( \text{Aut}_{f^+}(A) \) are not boundedly simple. That is, after adding to “an almost arbitrary” tree \( A \) one new color \( k \), such that for some old color \( j \), \( a(k,j) \geq 2 \), we obtain a tree \( A' \) where \( \text{Aut}_{f^+}(A') \) is not boundedly simple.

**Corollary 3.11.** Assume that \( A \) is a colored tree and \( \text{Aut}_{f^+}(A) \) does not stabilize any vertex. Extend the code \( a \) of \( A \) by adding one new color \( I' = I \cup \{k\} \) (\( k \not\in I \)) to get a code \( a' \supset a \) such that for every \( i \in I \), \( a'(i,k) = 0 \) if and only if \( a'(k,i) = 0 \), and for some \( j \in I \)

\[
a'(k,j) \geq 2.
\]

If \( (A', f'): S(A') \to I' \) is a tree corresponding to \( a' \), then \( \text{Aut}_{f^+}(A') \) is not boundedly simple.

**Proof.** \( A' \) contains a subtree \( A \) corresponding to \( a \). Let \( z \) be a vertex in \( A' \) of color \( k \) and let \( s \) be a vertex in \( A \) of color \( j \) adjacent to \( z \). Since \( \text{Aut}_{f^+}(A) \) does not stabilize any vertex, there is a translation in \( \text{Aut}_{f^+}(A) \) along a doubly infinite path \( C \) in \( A \). Let \( t \) be the projection of \( s \) onto \( C \) in the tree \( A \). Applying Proposition 3.7 to \( z \), \( s \), \( t \) and \( C \), we conclude that \( \text{Aut}_{f^+}(A') \) is not boundedly simple (because there is \( \gamma \in \text{Aut}_{f^+}(A') \), such that \( \gamma(z) = z \) and \( \gamma(s) \neq s \)).

| Case | \( i \)-sequence |
|------|-----------------|
| \( i_1 = i \) and \( i_n = i \) | \( m_1, \ldots, m_{N_1}, \frac{m_{N_1}}{2}, m_{N_1+1}, \frac{m_{N_1+1}}{2}, \ldots, m_1, n_1, n_2, \ldots, n_{N_2} \) |
| \( i_1 = i \) and \( i_n \neq i \) | \( m_1, \ldots, m_{N_1-1}, \frac{m_{N_1-1}}{2}, m_{N_1}, \frac{m_{N_1}}{2}, \ldots, m_1, n_1, n_2, \ldots, n_{N_2} \) |
| \( i_1 \neq i \) and \( i_n = i \) | \( m_1, \ldots, m_{N_1-1}, \frac{m_{N_1-1}}{2}, m_{N_1}, \frac{m_{N_1}}{2}, \ldots, m_1, m_0, n_1, n_2, \ldots, n_{N_2-1}, n_{N_2}' \) |
| \( i_1 \neq i \) and \( i_n \neq i \) | \( m_1, \ldots, m_{N_1-2}, \frac{m_{N_1-2}}{2}, m_{N_1-1}, \frac{m_{N_1-1}}{2}, \ldots, m_1, m_0, n_1, n_2, \ldots, n_{N_2-1}, n_{N_2}' \) |
We characterize all colored trees $A$ with boundedly simple group $\text{Aut}_f^+(A)$ (assuming that $\text{Aut}_f^+(A)$ leaves no nonempty proper subtree invariant).

**Theorem 3.12.** Suppose $(A, f): S(A) \to I$ is a colored tree and $\text{Aut}_f^+(A)$ leaves no nonempty proper subtree of $A$ invariant. If $\text{Aut}_f^+(A)$ is boundedly simple (and nontrivial), then for some $n, m \geq 3$, $A$ is almost $(n, m)$-regular tree (a subdivision of a bi-regular tree), so $\text{Aut}_f^+(A)$ is $32$-boundedly simple.

**Proof.** By Proposition 2.3 there is a normal function $f^+: S(A) \to I^+$ such that $\text{Aut}_f^+(A) = \text{Aut}_{f^+}(A)$. Let $a^+$ be the code for $(A, f^+: S(A) \to I^+)$. By the assumption, $\text{Aut}_f^+(A)$ contains some translation which is a composition of two rotations. Let $\alpha \in \text{Aut}_f^+(A)$ be a translation of minimal possible translation length amongst all translations which are products of two rotations. Let $\gamma$ be the type of $\alpha$ according to the coloring $f^+$. We may assume that $\alpha = \beta \circ \gamma$, $\beta(x) = x$, $\gamma(y) = y$, colors of $x$ and $y$ are $j_0$ and $j_k$ respectively and

$$(x = x_0, x_1, \ldots, x_{k-1}, x_k = y)$$

is the shortest path in $A$ from $x$ to $y$. Then

$$n = a^+(j_0, j_1), \ m = a^+(j_k, j_{k-1}) \geq 2.$$ 

The minimality of translation length of $\alpha$ implies that if $k > 1$, then

$$a^+(j_1, j_2) = \ldots = a^+(j_{k-1}, j_k) = 1 \text{ and } a^+(j_{k-1}, j_{k-2}) = \ldots = a^+(j_1, j_0) = 1,$$

(because e.g. if $a^+(j_1, j_0) > 1$, then $j_1 = j_k$, and if $a^+(j_1, j_2) > 1$, then instead of $x_0$ we may consider $x_1$).

We claim that $k = 1$ or for $s, t \in \{1, \ldots, k - 1\}$

(0) $j_s \neq j_t$,

(1) if $|s - t| \neq 1$, then $a^+(j_s, j_t) = 0$,

(2) if $s \neq 1$ and $t \neq k - 1$, then $a^+(j_0, j_s) = a^+(j_k, j_t) = a^+(j_0, j_k) = 0$.

(0) and (1) follows from the minimality of translation length of $\alpha$ (otherwise we can find a translation with shorter translation length). For (2) suppose, contrary to our claim, that $a^+(j_0, j_s) > 0$ (now $s \in \{2, \ldots, k\}$). Then (again by the minimality of translation length),

$$a^+(j_0, j_s) = 1.$$

Therefore, there is in $G(a^+)$ the path $(j_0, j_s, j_{s-1}, \ldots, j_1, j_0)$. We may assume that $j_0 \notin \{j_1, \ldots, j_s\}$. There is also a corresponding path

$$Q = (x_0, x'_s, x'_{s-1}, \ldots, x'_1, x'_0)$$

in $A$, i.e. $f(x'_i) = j_i$. Since $j_0, j_1, \ldots, j_s$ are pairwise distinct, $Q$ is the shortest path between $x_0$ and $x'_0$. Vertices $x_0$ and $x'_0$ have the same color, so there is $\alpha \in \text{Aut}_f^+(A)$, with $\alpha(x_0) = x_0$. $\alpha$ cannot be a rotation ($s \geq 2$ and $j_0, j_1, \ldots, j_s$ are pairwise distinct), so $\alpha$ is a translation. The translation length of $\alpha$ is $\geq 4$ (because $s \geq 2$). On the other hand, the type of every translation from $\text{Aut}_f^+(A)$ with translation length $\geq 4$, contains a multiple occurrence of some color. However, this observation is not true for the type of $\alpha$. Hence, (2) is proved.

We claim that

(3) $I^+ = \{j_0, j_1, \ldots, j_{k-1}, j_k\}$. 


This follows from Proposition \ref{prop:37} and our assumption that \( \text{Aut}_{f^+}(A) \) leaves no nonempty proper subtree of \( A \) invariant. That is, take \( * \in I^+ \setminus \{j_0, j_1, \ldots, j_k\} \), such that \( * \) is adjacent in \( G(a^+) \) to some \( j_s \), \( s \in \{0, \ldots, k\} \). Then by Proposition \ref{prop:37}, \( a^+(*) = 1 \). Therefore (e.g. by \cite[Lemma 4.1]{Gismatullin2011}) the subtree \( A' \) of \( A \), corresponding to the code \( a^+_{\{j_0, \ldots, j_k\}} \), is \( \text{Aut}_{f^+}(A) \)-invariant, so \( A' = A \).

Recall that \( I^{+\text{ram}} \) is the set of ramification colors from \( I^+ \). Clearly \( I^{+\text{ram}} \neq \emptyset \). It cannot happen that \( |I^{+\text{ram}}| = 1 \). Otherwise, if e.g. \( I^{+\text{ram}} = \{j_0\} \), then consider on the set of vertices of color \( j_0 \), the following equivalence relation: \( E(r, s) \) if and only if on the shortest path from \( r \) to \( s \) there is odd number of vertices of color \( j_0 \). One can easily show that for every rotation \( \alpha \in \text{Aut}_{f^+}(A) \) and \( r \in S(A) \) with \( f^+(r) = j_0 \), \( E(\alpha(r), r) \).

Hence \( E \) is \( \text{Aut}_{f^+}(A) \)-invariant. There is at least two vertices of color \( j_0 \), thus for some such \( r \) and some \( \beta \in \text{Aut}_{f^+}(A) \), \( \neg E(r, \beta(r)) \), which is impossible.

Therefore \( |I^{+\text{ram}}| = 2 \) and \( j_0, j_k \) are ramification colors. Hence \( n, m \geq 3 \). By (0), (1), (2) and (3), \( A \) is almost \((n, m)\)-regular tree.

4. Boundedly simple action on trees

In this section we extend our results to boundedly simple groups acting on trees.

For a group \( G \) acting on a tree \( A \) we may consider the following coloring function

\[ f^G : S(A) \to \{\text{orbits of } G \text{ on } S(A)\}. \]

\( f^G \) is normal and \( G < \text{Aut}_{f^c}(A) \).

If \( G \) leaves no nonempty proper subtree of \( A \) invariant nor stabilizing any end of \( A \) and \( G^+ \) is boundedly simple, then we show that \( \text{Aut}_{f^c+}(A) \) is also boundedly simple. Hence, applying Theorem \ref{theo:312} we get that \( A \) is an almost bi-regular tree.

**Theorem 4.1.** Suppose that \( A \) is a tree and \( G < \text{Aut}(A) \) leaves no nonempty proper subtree of \( A \) invariant nor stabilizing any end of \( A \). Assume that \( G^+ \) is boundedly simple and nontrivial. Then \( A \) is an almost \((n, m)\)-regular tree (a subdivision of a bi-regular tree \( A_{n,m} \)) for some \( n, m \geq 3 \).

**Proof.** Since \( G^+ \) is nontrivial, \( G^+ \) contains some nontrivial rotation (by Definition \ref{def:25}(2)). \( G^+ \) is the subgroup of \( \text{Aut}_{f^c+}(A) \), so \( \text{Aut}_{f^c+}(A) \) also contains some nontrivial rotation. By Lemma \ref{lem:32} there is a ramification point \( r \in S(A) \).

Take an arbitrary \( \alpha \in \text{Aut}_{f^c+}(A) \). There is \( h \in G^+ \) with

\[ \alpha(r) = h(r). \]

Hence (by Proposition \ref{prop:28}), for some rotation \( \beta \in \text{Aut}_{f^c+}(A) \) fixing \( r \),

\[ \alpha = \beta \circ h. \]

Assuming that \( G^+ \) is \( N \)-boundedly simple, \( \alpha \) is a composition of \((N + 1)\) rotations from \( \text{Aut}_{f^c+}(A) \) (\( h \) is a composition of \( N \) rotations from \( G^+ \)).

\( G \) leaves invariant no proper subtree nor stabilizes any end, so by \cite[Lemma 4.4]{Gismatullin2011} the same is true for \( G^+ \) and for \( \text{Aut}_{f^c+}(A) \) (since \( G^+ < \text{Aut}_{f^c+}(A) \)). Thus, by Proposition \ref{prop:33} and Lemma \ref{lem:32} \( \text{Aut}_{f^c}(A) \) is \( 16(N + 1) \)-boundedly simple. It is enough to apply Theorem \ref{theo:312}.

\[
\square
\]
The proof of the previous theorem works under the slightly weaker assumption than “$G^+$ is boundedly simple and nontrivial” i.e. we may assume that for some nontrivial rotation $g \in G^+$ and natural $N$, $G^+ = (g^{G^+} \cup g^{-1}G^+) \leq N$.

For boundedly simple groups acting on a tree without stabilizing any end we may derive the following corollary from Theorem 4.1.

**Corollary 4.2.** Given a boundedly simple group $G$ acting by automorphisms on a tree $A$ without stabilizing any end. If $G^+$ is nontrivial (i.e. some nontrivial element from $G$ stabilizes some edge), then there is a $G$-invariant subtree $A' \subseteq A$ which is a subdivision of a bi-regular tree.

**Proof.** Since $G^+ \trianglelefteq G$ and $G^+$ is nontrivial, $G = G^+$. We may assume that $G$ has no fixed vertex. Then by [6, Corollary 3.5], there is a nonempty minimal $G$-invariant subtree $A'$ of $A$. Since $G$ is simple, $G < Aut(A')$. $G$ leaves no nonempty proper subtree of $A'$ invariant nor stabilizes any end of $A'$. By Theorem 4.1, $A'$ is a subdivision of a bi-regular tree. □

**Question 4.3.** Can we weaken the assumption “$G^+$ is nontrivial” in Corollary 4.2?

If $G$ is a boundedly simple group acting on a tree without inversion (i.e. no element of $G$ is a symmetry), then $G$ contains a nontrivial rotation (because otherwise $G$ acts freely on the tree, and hence must be a free group, as follows from Bass-Serre theory [4, Section I.3, Theorem 4]). We do not know if this observation can be generalized to stabilizers of edges. However, when $G$ is big enough, then the description of $G^+$ from Proposition 2.8 ([6, 6.1]) should be true. Hence, if $A$ contains only ramification vertices, then every rotation is a composition of stabilizers of edges, so $G^+$ is nontrivial.

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**Instytut Matematyczny Uniwersytetu Wrocławskiego, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland**

**And**

**Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warsaw, Poland**

E-mail address: gismat@math.uni.wroc.pl, www.math.uni.wroc.pl/~gismat