ON NONHOMEOMORPHIC MAPPINGS BETWEEN RIEMANNIAN MANIFOLDS

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Abstract

We consider mappings of domains of Riemannian manifolds that admit branch points and satisfy a certain condition regarding the distortion of the modulus of families of paths. We have established logarithmic estimates of distance distortion under such mappings. A separate study relates to the situation when the mappings are defined in metric spaces, and one of them is the Lowner space. We also studied the question of equicontinuity of the families of the indicated mappings in the closure of the domain. In addition, we have established the possibility of continuous extension of these mappings to an isolated point of the boundary.

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1 Introduction

This article is devoted to the study of the local and boundary behavior of mappings of Riemannian manifolds satisfying some conditions on the distortion of the modulus of families of paths. Note that homeomorphisms with a similar condition were partially investigated in our previous article [IS] (see also [SevSkv1–SevSkv3]). Therefore, this manuscript is primarily devoted to mappings with branching. The estimates of the distortion of the distance under mappings are especially important for us. In particular, the next two sections are devoted to this problem. It is worth noting that the distortion estimates under mappings may be applied to the problem of the existence of homeomorphic solutions of the Beltrami equations (see, for example, [GRSY], [RSY]). We note numerous results related to the local behavior of quasiconformal mappings and their generalizations, including estimates of the distortion under them (see, for example, [Ci Theorem 5], [LV Theorem 3.2.II], [MRV2 Theorem 3.2], [MRSY Theorem 7.3] and [Re Theorem 1.1.II]). Other goals pursued in the article are the boundary behavior of mappings, the behavior of mappings in the closure of a domain, the problem of removability of isolated singularities of mappings. Let us designate the structure of the article:

1. Mappings satisfying generalized Hölder-type estimates ("logarithmic Hölder property").
2. Equicontinuity of families of mappings in terms of \( \varepsilon-\delta \).

3. Boundary behavior of mappings.

4. Global behavior of mappings (equicontinuity of families inside and on the boundary of the domain).

5. Continuous extension of mappings to isolated points of the boundary of a domain.

6. Examples.

Recall that quasiconformal mappings, as well as mappings with bounded distortion, satisfy

\[
M(\Gamma) \leq N(f, A) \cdot K \cdot M(f(\Gamma)), \tag{1.1}
\]

where \( M \) denotes a modulus of families of paths \( \Gamma \) in \( D \),

\[
N(y, f, A) = \text{card} \{ x \in A : f(x) = y \}, \quad N(f, A) = \sup_{y \in \mathbb{R}^n} N(y, f, A), \tag{1.2}
\]

\( A \) is an arbitrary Borel set in \( D \), and \( K \geq 1 \) is some constant that can be calculated as

\[
K = \text{ess sup} K_O(x, f),
\]

where \( K_O(x, f) = \|f'(x)\|^n / J(x, f) \) for \( J(x, f) \neq 0 \); \( K_O(x, f) = 1 \) for \( f'(x) = 0 \), and \( K_O(x, f) = \infty \) for \( f'(x) \neq 0 \), but \( J(x, f) = 0 \) (see, e.g., [MRV] Theorem 3.2 or [Ri] Theorem 6.7.II]). In this article, the main object of research are mappings that satisfy even some more general condition than (1.1). Let us introduce this condition into consideration.

We will assume that the main objects related to Riemannian manifolds are known: the concept of length and volume, a normal neighborhood of a point, etc. (see, for example, [IS]). We also consider known the definition of the modulus \( M(\Gamma) \) of families of paths \( \Gamma \), including the concept of an admissible function \( \rho \in \text{adm } \Gamma \). Let \( \mathbb{M}^n \) and \( \mathbb{M}^n_* \) are Riemannian manifolds of dimension \( n \) with geodesic distances \( d \) and \( d_* \), respectively,

\[
B(x_0, r) = \{ x \in \mathbb{M}^n : d(x, x_0) < r \}, \quad S(x_0, r) = \{ x \in \mathbb{M}^n : d(x, x_0) = r \}, \tag{1.3}
\]

\[
A = A(y_0, r_1, r_2) = \{ y \in \mathbb{M}^n_* : r_1 < d(y, y_0) < r_2 \}, \quad 0 < r_1 < r_2 < r_0, \tag{1.4}
\]

d\(v\) and \(dv_*\) are volume measures on \( \mathbb{M}^n \) and \( \mathbb{M}^n_* \), respectively (see [IS]). For the sets \( A, B \subset \mathbb{M}^n \) we use the notation

\[
\text{dist} (A, B) = \inf_{x \in A, y \in B} d(x, y), \quad d(A) = \sup_{x, y \in A} d(x, y).
\]

Sometimes instead of \( \text{dist} (A, B) \), we write \( d(A, B) \), if a misunderstanding is impossible.

Let \( x_0 \in D \), and the number \( r_0 > 0 \) be such that the ball \( B(x_0, r_0) \) lies with its closure in some normal neighborhood \( U \) of the point \( x_0 \). Denote by \( S_i = S(x_0, r_i) \), \( i = 1, 2 \), geodesic spheres centered at the point \( x_0 \) and radii \( r_1 \) and \( r_2 \). Given sets \( E, F \) and \( G \) in \( \mathbb{M}^n \), we denote by \( \Gamma(E, F, G) \) the family of all paths \( \gamma : [a, b] \to \mathbb{M}^n \), joining \( E \) and \( F \) in \( G \), in other words, \( \gamma(a) \in E, \gamma(b) \in F \) and \( \gamma(t) \in G \) for \( t \in (a, b) \). If \( D \) is a domain of a Riemannian manifold
\( \mathbb{M}^n, f : D \to \mathbb{M}^n \) is some mapping, \( y_0 \in f(D) \) and \( 0 < r_1 < r_2 < d_0 = \sup_{y \in f(D)} d_*(y, y_0) \), then by \( \Gamma_f(y_0, r_1, r_2) \) we denote the family of all paths \( \gamma \) in the domain \( D \) such that \( f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2)) \). Let \( Q : \mathbb{M}^n \to [0, \infty) \) be a measurable function with respect to the volume measure \( v_* \). We will say that \( f \) satisfies the inverse Poletskii inequality at the point \( y_0 \in f(D) \), if the relation

\[
M(\Gamma_f(y_0, r_1, r_2)) \leq \int_{A(y_0, r_1, r_2) \cap f(D)} Q(y) \cdot \eta^n(d_*(y, y_0)) \, dm(y)
\]

(1.5)

holds for any Lebesgue measurable function \( \eta : (r_1, r_2) \to [0, \infty] \) such that

\[
\int_{r_1}^{r_2} \eta(r) \, dr \geq 1.
\]

(1.6)

It is easy to verify that inequalities of the form (1.5) transform into relations of the form

\[
M(\Gamma_f(y_0, r_1, r_2)) \leq K \cdot M(\Gamma_f(y_0, r_1, r_2)),
\]

as soon as the function \( Q \) is bounded by the number \( K \geq 1 \). Moreover, if the mapping \( f \) with a bounded distortion has a bounded multiplicity function \( N(f, D) \), then we also have the relation (1.1), and therefore the inequality (1.5). In fairness it is worth note that such inequalities are satisfied not for all families of paths \( \Gamma \), but only for ”special” families \( \Gamma := \Gamma_f(y_0, r_1, r_2) \) and only at a fixed point \( y_0 \). However, for mappings whose characteristic is unbounded, inequalities (1.5) are also established, and the families of paths \( \Gamma \) in this case may be arbitrary (see \textbf{[MRSY, Theorem 8.5]}).

Let us now formulate the main results of this article. Let \( X \) and \( Y \) be two topological spaces. A mapping \( f : X \to Y \) is called an open if \( f(A) \) is open in \( Y \) for any open \( A \subset X \), and a discrete if for each \( y \in Y \) any two different points of the set \( f^{-1}(y) \) have pairwise disjoint neighborhoods. Let be \( D \subset X \) and \( D_\ast \subset Y \). A mapping \( f : D \to D_\ast \) is called a closed, if \( f \) takes any set \( A \) closed with respect to \( D \), onto a set \( f(D) \) closed with respect to \( D_\ast \). Everywhere below, the closure \( \overline{A} \) and the boundary \( \partial A \) of the set \( A \subset \mathbb{M}^n \) should be understood in the sense of the geodesic distance \( d \) in \( \mathbb{M}^n \). Let \( I \) be an open, half-open or closed interval of the real axis. Given a path \( \alpha : I \to X \), a locus of \( \alpha \) is called the set

\[
|\alpha| = \{ x \in X : \exists t \in I : \alpha(t) = x \}.
\]

We say that in the domain \( D' \) of the metric space \( X' \) the condition of the complete divergence of paths is satisfied, if for any different points \( y_1 \) and \( y_2 \in D' \) there are some \( w_1, w_2 \in \partial D' \) and paths \( \alpha_2 : (-2, -1] \to D', \alpha_1 : [1, 2) \to D' \) such that \( 1) \alpha_1 \) and \( \alpha_2 \) are subpaths of some geodesic path \( \alpha : [-2, 2] \to X' \), that is, \( \alpha_2 := \alpha|_{(-2, -1]} \) and \( \alpha_1 := \alpha|_{[1, 2)} \); 2) the geodesic path \( \alpha \) joins the points \( w_2, y_2, y_1 \) and \( w_1 \) such that \( \alpha(-2) = w_2, \alpha(-1) = y_2, \alpha(1) = y_1, \alpha(2) = w_2 \).

Note that the condition of the complete divergence of the paths is satisfied for an arbitrary bounded domain \( D' \) of the Euclidean space \( \mathbb{R}^n \), since as paths \( \alpha_1 \) and \( \alpha_2 \) we may take line segments starting at points \( y_1 \) and \( y_2 \) and directed to opposite sides of each other. In this case, the points \( w_1 \) and \( w_2 \) are automatically detected due to the boundedness of \( D' \) (see, e.g., \textbf{[SevSkv2, Proof of Theorem 1.5]}; see also Figure \textbf{[1]} on this matter).
We also note that this condition does not hold for every manifold and domain on it. For example, on a Riemannian sphere with a cut out (sufficiently small) disk, the geodesic distance between the taken points may be greater than the distance between any points of the "small disk". This means that the indicated curves $\alpha_1$ and $\alpha_2$ does not exist in this case (see Figure 2).

Given domains $D \subset M^n$ and $D_\ast \subset M^n_\ast$, $n \geq 2$, and a function $Q : M^n_\ast \to [0, \infty]$ that is measurable with respect to the volume measure $v_\ast$ denote by $\mathcal{F}_Q(D, D_\ast)$ the family of all open discrete mappings $f : D \to D_\ast$ such that the relation (1.5) holds for every $y_0 \in f(D)$. An analogue of the following theorem is proved in [SSD, Theorem 1.1] in the case of an ordinary Euclidean space.

**Theorem 1.1.** Suppose that $D_\ast$ satisfies the condition of complete divergence of paths and, in addition, $Q \in L^1(D_\ast)$. Let $x_0 \in D$. Then there exist $r_0 = r_0(x_0) > 0$, $R_0 = R_0(x_0) > 0$, and a constant $C_n > 0$, depending only on $n$, such that the inequality

$$d_\ast(f(x), f(x_0)) \leq \frac{C_n \cdot (\|Q\|_1)^{1/n}}{\log^{1/n} \left( \frac{R_0}{d(x, x_0)} \right)}$$

(1.7)
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for any \( x \in B(x_0, r_0) \) and all \( f \in \mathcal{F}_Q(D, D_*) \), where \( \|Q\|_1 \) denotes \( L^1 \)-norm of \( Q \) in \( D_* \). In particular, \( \mathcal{F}_Q(D, D_*) \) is equicontinuous in \( D \).

Note that the theorem is a special case of some more general statement that holds for metric spaces of a wide spectrum, including Riemannian manifolds. Here is the formulation of this statement. First of all, let us denote by \((X, d, \mu)\) and \((X', d', \mu')\) metric spaces \( X \) and \( X' \) with metrics \( d \) and \( d' \) and Borel measures \( \mu \) and \( \mu' \), respectively. Let \((X, d, \mu)\) be a metric space with measure \( \mu \). Define the Loewner function \( \phi_n : (0, \infty) \to [0, \infty) \) on \( X \) according to the following rule:

\[
\phi_n(t) = \inf \{ M_n(\Gamma(E, F, X)) : \Delta(E, F) \leq t \},
\]

where \( \inf \) is taken over all disjoint nondegenerate continua \( E, F \) in \( X \), and \( \Delta(E, F) \) is defined as

\[
\Delta(E, F) := \frac{\text{dist}(E, F)}{\min\{\text{diam } E, \text{diam } F\}}.
\]

A space \( X \) is called a Loewner space if the function \( \phi_n(t) \) is positive for all positive \( t \) (see [MRSY, section 2.5] or [He, Chap. 8]). Notice, that the definition of mappings of the form (1.5) is easily carried over to the case of arbitrary metric spaces. Indeed, suppose that \( X \) and \( X' \) are metric spaces with Hausdorff dimensions \( n \) and \( n' \), respectively. We define the modulus of the family of paths \( \Gamma \) in the space \( X \) by the relation

\[
M_n(\Gamma) = \inf_{\rho \in \text{adm } \Gamma} \int_X \rho^n(x) \, d\mu(x),
\]

where the notation \( \rho \in \text{adm } \Gamma \) means that \( \rho : X \to [0, \infty] \) is a Borel function on \( X \) satisfying the condition \( \int_{\gamma} \rho(x) \, |dx| \geq 1 \) for any locally rectifiable \( \gamma \in \Gamma \). Similarly, we may define the modulus of the family of paths \( \Gamma \) in \( X' \). The notations used above for Riemannian manifolds, including in the relations (1.1)–(1.3) we also use for metric spaces without additional explanations. In particular, given a domain \( D \subset X \), we say that \( f : D \to X' \) satisfies the inverse Poletskii inequality at the point \( y_0 \in f(D) \), if inequality

\[
M_n(\Gamma_f(y_0, r_1, r_2)) \leq \int_{A(y_0, r_1, r_2) \cap f(D)} Q(y) \cdot \eta^{n'}(d'(y, y_0)) \, d\mu'(y)
\]

holds for any Lebesgue measurable function \( \eta : (r_1, r_2) \to [0, \infty) \) for which the condition (1.10) holds. A metric space \((X, d, \mu)\) is called \( \tilde{Q} \)-Ahlfors regular if there exist \( \tilde{Q} \geq 1 \) and \( C \geq 1 \) such that the relation

\[
\frac{1}{C} R^{\tilde{Q}} \leq \mu(B(x_0, R)) \leq CR^{\tilde{Q}}
\]

holds for any \( x_0 \in X \) and any \( 0 < R < \text{diam } X \) (in particular, \( C \) does not depend on \( x_0 \)). Observe that Riemannian manifolds are locally Ahlfors \( n \)-regular (see, e.g. [ARS, Lemma 5.1]), where \( \mu \) is the volume measure \( v \) on the manifold.
Given domains $D \subset X$ and $D' \subset X'$ and a $\mu'$-measurable function $Q : X' \to [0, \infty]$ we denote by $\mathfrak{F}_Q(D, D')$ the family of all open discrete mappings $f : D \to D'$ such that the relation $(1.10)$ holds for any $y_0 \in f(D)$. The following statement holds.

**Theorem 1.2.** Suppose the space $X$ is locally compact and locally connected; moreover, assume that the condition of complete divergence of paths is satisfied in $D'$. Let $x_0 \in D$ and let $Q \in L^1(D')$. Suppose that for any neighborhood $W$ of the point $x_0$ there exists a neighborhood $U \subset W$ which is the Loewner space, and which is also Ahlfors regular. Then there are $r_0 = r_0(x_0) > 0$, $R_0 = R_0(x_0) > 0$ and a constant $\bar{C} > 0$, depending only on $X$ and $X'$, such that the inequality

$$d'(f(x), f(x_0)) \leq \frac{\bar{C} \cdot (\|Q\|_1)^{1/n'}}{\log^{1/n'} \left( \frac{R_0}{d(x, x_0)} \right)},$$

holds for any $x \in B(x_0, r_0)$ and $f \in \mathfrak{F}_Q(D, D')$, where $\|Q\|_1$ denotes $L^1$-norm of the function $Q$ in $D'$. In particular, the family $\mathfrak{F}_Q(D, D')$ is equicontinuous in $D$.

Note that the additional condition of complete divergence of paths, which is present in Theorems 1.1 and 1.2, is compensated by the presence of explicit estimates of distance distortion (1.7) and (1.12). At the same time, for the equicontinuity of similar families in more abstract terms "$\varepsilon$-$\delta$" there is no need for such conditions. Moreover, in this situation, the conditions on the function $Q$ have sufficient general form, which does not even require its integrability. Let us formulate a corresponding statement related to this case. In what follows,

$$q_{x_0}(r) = \frac{1}{\nu_{n-1}} \int_{S(x_0, r)} Q(x) \, d\mathcal{A}, \quad (1.13)$$

where $d\mathcal{A}$ is the area element of $S(x_0, r)$. For more details on the definition of the area element and integrals over a surface on Riemannian manifolds, see, for example, [IS$_3$, §4].

For domains $D \subset M^n$, $D_\ast \subset M^n_*$, $n \geq 2$, and a function $Q : M^n_* \to [0, \infty]$, $Q(x) \equiv 0$ for $x \notin D_\ast$, denote by $\mathfrak{M}_Q(D, D_\ast)$ the family of all open discrete mappings $f : D \to M^n_*$, $f(D) = D_\ast$, for which $f$ satisfies the condition (1.5) at each point $y_0 \in D_\ast$. The following result holds.

**Theorem 1.3.** Assume that, $\overline{D}$ and $\overline{D_\ast}$ a compact sets in $M^n$ and $M^n_*$, respectively, $\overline{D_\ast} \neq M^n_*$ and, in addition, $M^n_*$ is connected. Suppose also that the following condition is satisfied: for each point $y_0 \in \overline{D_\ast}$ there is $r_0 = r_0(y_0) > 0$ such that $q_{y_0}(r) < \infty$ for each $r \in (0, r_0)$. Then the family $\mathfrak{M}_Q(D, D_\ast)$ is equicontinuous in $D$.

The following results are related to the possibility of continuous extension of mappings to the boundary, as well as to the equicontinuity of families of mappings not only at the inner, but also at the boundary points of the domain. We emphasize that they are all true not only for integrable functions $Q$, but also in some more general case, which is also will be considered by us. Note that, for a Euclidean $n$-dimensional space, such results are established in [SSD], [SevSkv$_3$] and [Sev] under various conditions on the mappings under study.
Recall that a domain \( D \subset \mathbb{M}^n \) is called \textit{locally connected at the point} \( x_0 \in \partial D \), if for any neighborhood \( U \) of \( x_0 \) there is a neighborhood \( V \subset U \) of this point such that \( V \cap D \) is connected. The domain \( D \) is \textit{locally connected on} \( \partial D \) if \( D \) is locally connected at each point \( x_0 \in \partial D \). The boundary of the domain \( D \) is called \textit{weakly flat} at the point \( x_0 \in \partial D \), if for each \( P > 0 \) and for any neighborhood \( U \) of the point \( x_0 \) there is such a neighborhood \( V \subset U \) of this point such that \( M(\Gamma(E, F, D)) > P \) for any continua \( E, F \subset D \), intersecting \( \partial U \) and \( \partial V \). The boundary of the domain \( D \) is called \textit{weakly flat} if the corresponding property holds at any point of \( \partial D \).

Given a set \( E \subset \overline{D} \), we put

\[
C(f, E) = \{ y \in \mathbb{M}^n_* : \exists x_0 \in E, x_k \in D : x_k \xrightarrow{d} x_0, f(x_k) \xrightarrow{d} y, k \to \infty \}.
\]

The following theorem holds.

**Theorem 1.4.** Let \( D \subset \mathbb{M}^n, D_* \subset \mathbb{M}^n_* \), \( n \geq 2 \), \( x_0 \in \partial D \), and let \( f \) be an open, discrete and closed mapping of \( D \) onto \( D_* \). Assume that, \( \overline{D}_* \) is a compact in \( \mathbb{M}^n_* \), and \( \partial D \) is weakly flat, and \( D_* \) is locally connected on the boundary. Let, in addition, the following condition be satisfied: there are \( z_0 \in C(f, x_0) \) and \( r_0 = r_0(z_0) > 0 \) such that \( q_{z_0}(r) < \infty \) for any \( r \in (0, r_0) \). Then the mapping \( f \) has a continuous extension to \( x_0 \). If the specified condition holds at each point \( x_0 \in \partial D \), then \( f \) has a continuous extension \( \overline{f} : \overline{D} \to \overline{D}_* \) such that \( \overline{f}(\overline{D}) = \overline{D}_* \).

Given a number \( \delta > 0 \), domains \( D \subset \mathbb{M}^n, D_* \subset \mathbb{M}^n_* \), \( n \geq 2 \), a continuum \( A \subset D_* \) and an arbitrary function \( Q : D_* \to [0, \infty] \) which is measurable with respect to the volume measure \( v_* \), denote by \( \mathfrak{S}_{\delta, A, Q}(D, D_*) \) the family if all open, discrete and closed mappings \( f \) of \( D \) onto \( D_* \), satisfying (1.5) for any \( y_0 \in D_* \) and such that \( d(f^{-1}(A), \partial D) \geq \delta \). The following assertion holds.

**Theorem 1.5.** Let \( \overline{D} \) and \( \overline{D}_* \) be compact sets in \( \mathbb{M}^n \) and \( \mathbb{M}^n_* \), respectively, let \( \overline{D}_* \neq \mathbb{M}^n_* \) and let \( \mathbb{M}^n_* \) be connected. Assume that \( \partial D \) is weakly flat, and \( D_* \) is locally connected on the boundary. Assume also that, for any \( y_0 \in \overline{D}_* \) there is \( r_0 = r_0(y_0) > 0 \) such that \( q_{y_0}(r) < \infty \) for any \( r \in (0, r_0) \). Now, any \( f \in \mathfrak{S}_{\delta, A, Q}(D, D_*) \) has a continuous extension \( \overline{f} : \overline{D} \to \overline{D}_* \) for which \( \overline{f}(\overline{D}) = \overline{D}_* \) and, in addition, the family \( \mathfrak{S}_{\delta, A, Q}(\overline{D}, \overline{D}_*) \), consisting of all extended mappings \( \overline{f} : \overline{D} \to \overline{D}_* \) is equicontinuous in \( \overline{D} \).

Finally, we formulate a result related to the removability of isolated singularities of mappings.

**Theorem 1.6.** Let \( D \subset \mathbb{M}^n, D_* \subset \mathbb{M}^n_* \), \( n \geq 2 \), be domains which have compact closures, \( x_0 \in D \), and let \( f \) be an open discrete mapping of \( D \setminus \{x_0\} \) onto \( D_* \) for which the condition (1.5) holds at least for one \( y_0 \in C(f, x_0) \). Let \( C(f, x_0) \subset \partial D_* \). Assume that, for any \( y_0 \in \overline{D}_* \) there is \( r_0 = r_0(y_0) > 0 \) such that \( q_{y_0}(r) < \infty \) for any \( r \in (0, r_0) \). Then \( f \) has a continuous extension \( f : D \to \overline{D}_* \).
2 Logarithmic Hölder continuity of mappings

In this section, $X$ and $X'$ are metric spaces, and $D$ and $D'$ are domains in them. Before proceeding to the proof of the main results, we give the minimum necessary information about lifting of paths.

Let $D \subset X$, let $f : D \to X'$ be an open discrete mapping, let $\beta : [a, b) \to X'$ be a path, and let $x \in f^{-1}(\beta(a))$. A path $\alpha : [a, c) \to D$ is called a maximal $f$-lifting of $\beta$ starting at $x$, if (1) $\alpha(a) = x$; (2) $f \circ \alpha = \beta|_{[a, c)}$; (3) for any $c < c' \leq b$, there is no a path $\alpha' : [a, c') \to D$, for which $\alpha = \alpha'|_{[a, c)}$ and $f \circ \alpha' = \beta|_{[a, c)}$. If $X = X' = \mathbb{R}^n$, the above assumption on the mapping $f$ implies the existence of the maximal $f$-lifting of $\beta$ starting at $x$ for any $x \in f^{-1}(\beta(a))$ (see [Ri, Corollary II.3.3]). The maximal lifting $\alpha$ will be called complete, if $a = b$. The metric space $X$ is called locally connected if for each point $x_0 \in X$ and any neighborhoods $U$ of $x_0$ there exists a neighborhood $V$ of $x_0$, $V \subset U$ such that $V$ is connected. A space $X$ is called locally compact if for each point $x_0 \in X$ there is a neighborhood $U$ of this point such that $U$ is a compact set in $X$. The following statement is established in [SM] Lemma 2.1.

**Lemma 2.1.** Let $X$ and $X'$ are locally compact metric spaces, let $X$ is locally connected, let $D$ be a domain in $X$, and let $f : D \to X'$ be an open discrete mapping. If $x \in f^{-1}(\beta(a))$, then any path $\beta : [a, b) \to X'$ has a maximal $f$-lifting starting at $x$.

**Proof of Theorem 2.2.** Fix $x_0 \in D$ and $f \in \mathcal{F}_Q(D, D')$. By hypothesis, there is a neighborhood $U$ of $x_0$ which is Ahlfors regular Loewner space. Since such a neighborhood may be chosen arbitrarily small, and the space $X$ is locally compact, we may assume that $U$ is a compact set in $D$.

Let $\Phi_n(t)$ be a Loewner function in (1.8), corresponding to the space $U$. Then, by virtue of [He] Theorem 8.23, there is $\delta_0 > 0$ and some constant $C > 0$ such that

$$\Phi_n(t) \geq C \log \frac{1}{t} \quad \forall \ t > 0 : |t| < \delta_0.$$  \hfill(2.1)

We may consider that $\delta_0 < 1$. Let

$$R_0 := (1/2) \cdot d(x_0, \partial D),$$

and let

$$r_0 < R_0 \cdot \delta_0.$$

Let $0 < r < r_0$ and $x \in B(x_0, r)$. We put

$$d'(f(x), f(x_0)) := \varepsilon_0.$$  \hfill(2.2)

If $\varepsilon_0 = 0$, there is nothing to prove. Now let $\varepsilon_0 > 0$.

By hypothesis, for the points $f(x)$ and $f(x_0) \in D'$ there are $w_1, w_2 \in \partial D'$ and paths $\alpha_2 : (-2, -1] \to D'$, $\alpha_1 : [1, 2) \to D'$, such that

1) $\alpha_1$ and $\alpha_2$ are subpaths of some geodesic path $\alpha : [-2, 2] \to X'$, that is, $\alpha_2 := \alpha|_{(-2, -1]}$ and $\alpha_1 := \alpha|_{[1, 2]}$;
Let’s use the method by contradiction: suppose that (2.3) does not hold. Then \( \bar{\gamma_1} \) is compact in \( D \). Note that \( c \neq t_1 \), since otherwise \( \bar{\alpha_1} \) is compact in \( f(\bar{\gamma_1}) \subset f(D) \), which contradicts the condition \( \bar{\alpha_1}(t) \to \bar{\alpha}(t_1) \in \partial f(D) \) as \( t \to t_1 \). Consider the cluster set \( G \) of \( \gamma_1 = \gamma_1(t) \) as \( t \to c - 0 \),

\[
G = \left\{ x \in X : x = \lim_{k \to \infty} \gamma_1(t_k) \right\}, \quad t_k \in [1, c), \quad \lim_{k \to \infty} t_k = c.
\]

Note that passing to subsequences, we may restrict ourselves to monotone sequences \( t_k \). For \( z \in G \), since \( f \) is continuous, we have \( f(\gamma_1(t_k)) \to f(z) \) as \( k \to \infty \), where \( t_k \in [1, c) \), \( t_k \to c \) as \( k \to \infty \). However, \( f(\gamma_1(t_k)) = \beta(t_k) \to \beta(c) \) as \( k \to \infty \). Hence we conclude that \( f \) is constant on \( G \subset D \). By Cantor’s condition on the compact set \( \bar{\gamma_1} \), due to the monotonicity
of the sequence of connected sets $\gamma_1([t_k, c])$,

$$G = \bigcap_{k=1}^{\infty} \gamma_1([t_k, c]) \neq \emptyset,$$

see [Ku, 1.II.4, §41]. By [Ku, Theorem 5.II.5, §47] the set $G$ is connected. Since $f$ is discrete, $G$ is one-point. Thus, the path $\gamma_1: [1, c) \to D$ may be extended to the closed path $\gamma_1: [1, c] \to D$, moreover $f(\gamma_1(c)) = \tilde{\alpha}_1(c)$. Again, by Lemma [2.1], there is a maximal $f$-lifting $\gamma'_1$ of $\alpha|_{[c, t_1]}$ starting at a point $\gamma_1(c)$. Combining lifting $\gamma_1$ and $\gamma'_1$, we obtain a new $f$-lifting $\gamma''_1$ of the path $\tilde{\alpha}_1$, defined on some semi-interval $[1, c')$, $c' \in (c, t_1)$, which contradicts the "maximality" of the lifting $\gamma_1$. The resulting contradiction indicates the validity of the relation (2.3). Similarly, one can show that for some sequence the condition

$$d(\gamma_2(t'_k), \partial D) \to 0 \quad \text{при} \quad t'_k \to d + 0, \quad (2.4)$$

is satisfied for some sequence $t'_k \in (d, -1)$, $k = 1, 2, \ldots$. By hypothesis, $B(x_0, 2R_0) \subset U$, so using [Ku, Theorem 1.I.5.46] and taking into account the above relations (2.3) and (2.4), we obtain that $\gamma_1(q_1) \in S(x_0, R_0)$, $\gamma_1(p_1) \in S(x_0, 2R_0)$, $\gamma_2(q_2) \in S(x_0, R_0)$, $\gamma_2(p_2) \in S(x_0, 2R_0)$ for some $1 < q_1 < p_1 < t_1$ and $t_2 < p_2 < q_2 < -1$. Without loss of generality, we may assume that $\gamma_1(t) \in A(x_0, R_0, 2R_0)$ for $q_1 < t < p_1$ and $\gamma_2(t) \in A(x_0, R_0, 2R_0)$ $p_2 < t < q_2$. Denote

$$\beta_1 := \gamma_1|_{[1, p_1]}, \quad \beta_2 := \gamma_2|_{[p_2, -1]}, \quad \beta_i := \gamma_i(p_i), \quad \beta_i^* := \gamma_i(q_i).$$

It follows from the triangle inequality that

$$\text{diam } (|\beta_1|) \geq d(x_i, x_i^*) \geq R_0. \quad (2.5)$$

Note that $|\beta_1|$ and $|\beta_2|$ are two continua in $U$, whose diameters are not less than $R_0$ due to (2.5). Then

$$\Delta(|\beta_1|, |\beta_2|) := \text{dist } (|\beta_1|, |\beta_2|) \leq \frac{d(x, x_0)}{R_0}, \quad (2.6)$$

because $\text{dist } (|\beta_1|, |\beta_2|) \leq d(x, x_0)$. Then, by the definition of the Loewner function $\Phi_n(t)$ in (1.8) we will have that

$$\Phi_n \left( \frac{d(x, x_0)}{R_0} \right) \leq M_n(\Gamma(|\beta_1|, |\beta_2|, U)) \leq M_n(\Gamma(|\beta_1|, |\beta_2|, D)). \quad (2.7)$$

Observe that $d(x, x_0) < r_0$, so that

$$\frac{d(x, x_0)}{R_0} \leq \frac{r_0}{R_0} < \frac{\delta_0 R_0}{R_0} = \delta_0.$$

Therefore, by (2.1)

$$C \cdot \log \frac{R_0}{d(x, x_0)} \leq \Phi_n \left( \frac{d(x, x_0)}{R_0} \right). \quad (2.8)$$

Now, by (2.7) and (2.8) we obtain that

$$C \cdot \log \frac{R_0}{d(x, x_0)} \leq M_n(\Gamma(|\beta_1|, |\beta_2|, D)). \quad (2.9)$$
We now obtain an upper bound for $M_n(\Gamma(|\beta_1|, |\beta_2|, D))$. Denote $z_1 := f(x_1)$. Since $\alpha_1$ and $\alpha_2$ are part of one geodesic $\alpha : [-2, 2] \to X'$, moreover, $\alpha_2 := \alpha|_{[-2, -1]}$, $\alpha_1 := \alpha|_{[1, 2]}$ and $\alpha_1(p_1) = x_1$, $1 < q_1 < p_1 < t_1$, then
\[
d'(z_1, \alpha(\kappa_1)) = d'(z_1, \alpha(\kappa_2)) + d'(\alpha(\kappa_2), \alpha(\kappa_1)), \quad -2 \leq \kappa_1 < \kappa_2 \leq p_1.
\]
Hence it follows that
\[
|f(\beta_1)| \subset |\tilde{\alpha}_1| \subset B(z_1, r_1)
\] (2.10)
and
\[
|f(\beta_2)| \subset |\tilde{\alpha}_2| \subset X' \setminus B(z_1, r_2),
\] (2.11)
where $r_1 := d'(z_1, f(x))$ and $r_2 := d'(z_1, f(x_0))$. Let $\gamma \in \Gamma(|\beta_1|, |\beta_2|, U)$. Then $f(\gamma) \in \Gamma(f(|\beta_1|), f(|\beta_2|), D'))$. By [Ku, Theorem 1.1.5.46] and by (2.10)–(2.11) we obtain that
\[
\Gamma(f(|\beta_1|), f(|\beta_2|), D')) > \Gamma(S(z_1, r_1), S(z_1, r_2), A(z_1, r_1, r_2)),
\]
where $r_1 := d'(z_1, f(x)) \cup r_2 := d'(z_1, f(x_0))$. Hence it follows that
\[
\Gamma(|\beta_1|, |\beta_2|, D) > \Gamma_f(z, r_1, r_2)
\]
and by the minorization of the modulus of families of paths
\[
M_n(\Gamma(|\beta_1|, |\beta_2|, D)) \leq M_n(\Gamma_f(z_1, r_1, r_2)). \tag{2.12}
\]
Now let us use the definition of the mapping from $f$ in (1.10). According to this definition
\[
M_n(\Gamma_f(z_1, r_1, r_2)) \leq \int_{A(z_1, r_1, r_2) \cap D'} Q(y) \cdot \eta^\alpha(d'(z_1, y)) \, d\mu'(y). \tag{2.13}
\]
Consider the function
\[
\eta(t) = \begin{cases} \frac{1}{\varepsilon_0}, & t \in [r_1, r_2], \\ 0, & t \notin [r_1, r_2], \end{cases}
\]
where, as above, $\varepsilon_0 = d'(f(x), f(x_0))$. We obtain that
\[
\int_{r_1}^{r_2} \eta(t) \, dt = \frac{r_2 - r_1}{\varepsilon_0} = \frac{d'(z_1, f(x_0)) - d'(z_1, f(x))}{\varepsilon_0} = 1,
\]
since all three points $f(x_0)$, $f(x)$ and $z_1$ are sequentially located on one geodesic, which means that
\[
r_2 = d'(z_1, f(x_0)) = d'(z_1, f(x)) + d'(f(x_0), f(x)) = r_1 + \varepsilon_0.
\]
Now, it follows from (2.13) that
\[
M_n(\Gamma_f(z_1, r_1, r_2)) \leq \frac{1}{(d'(f(x), f(x_0)))^{\alpha'}} \int_{D'} Q(y) \, d\mu'(y) = \frac{\|Q\|_{L_1(D')}}{(d'(f(x), f(x_0)))^{\alpha'}}. \tag{2.14}
\]
Finally, combining (2.9), (2.12) and (2.14), we will have that

\[ C \cdot \log \frac{R_0}{d(x,x_0)} \leq \frac{\|Q\|_{L^1(D')}^{1/n'}}{\left( d'(f(x), f(x_0)) \right)^{n'}} , \]

whence it follows that

\[ d'(f(x), f(x_0)) \leq \left( \frac{\|Q\|_{L^1(D')}^{1/n'}}{C^{1/n'} \cdot \log \frac{R_0}{d(x,x_0)}} \right)^{1/n'} \cdot C_1^{1/n'} \cdot \log \frac{1}{n'} R_0 d(x,x_0) . \]

To complete the proof, it remains to put \( \tilde{C} := C_1^{1/n'} \cdot \log \frac{1}{n'} R_0 d(x,x_0) \).

\[ \square \]

The proof of Theorem 1.1 is reduced to the statement of Theorem 1.2. To verify this, we show that all the conditions of this theorem are satisfied. Indeed, local compactness and the local connectedness of the space \( M^n \) is obvious, since they are a consequence of the definition of a smooth manifold. It remains to verify that for any neighborhood \( W \) of the point \( x_0 \in D \subset M^n \) there exists a neighborhood \( U \subset W \), which is a Loewner space as a metric space, and which is also Ahlfors regular. Put \( U := B(x_0, r_0) \subset W, r_0 < d(x_0, \partial U) \). Let us prove, first of all, that \( U \) is a Loewner space. We may assume that \( r_0 > 0 \) is so small that an arbitrary ball \( B(x_0, r) \), \( 0 < r < r_0 \), is transformed by the corresponding coordinate mapping \( \varphi : U \rightarrow \mathbb{R}^n \) into the Euclidean ball \( B(0, r) \), and the metric tensor \( g_{ij}(x) \) is arbitrarily close to the identity matrix in \( U \) and coincides with it at the origin (see [Lee, Lemma 5.10, Proposition 5.11 and Corollary 6.11], see also [ARS, Proposition 1.1, Remark 1.1]). Then the volume element \( dv(p) = \sqrt{\det g_{ij}} dx^1 \ldots dx^n, x = \varphi(p), \) is arbitrarily close to \( dx^1 \ldots dx^n \). In particular, for \( A \subset U \)

\[ C_1 m(\varphi(A)) \leq v(A) \leq C_2 m(\varphi(A)) , \]

where \( C_1 \) and \( C_2 \) are some positive constants depending only on \( U \), and \( v \) is a volume on \( M^n \), and \( m \) is the Lebesgue measure in \( \mathbb{R}^n \). Observe that, the neighborhood of \( U \), we also have a two-sided estimate of the geodesic distance through the Euclidean, namely,

\[ m \cdot |\varphi(p) - \varphi(q)| \leq d(p, q) \leq M \cdot |\varphi(p) - \varphi(q)| \]

for some \( m, M > 0 \) and any \( p, q \in U \) (see [ARS] proof of Lemma 5.1). Let \( E, F \) are continua in \( U \). As above, we put

\[ \phi(t) = \inf \{ M(\Gamma(E,F,U)) : \Delta(E,F) \leq t \} , \]

where inf is taken over all disjoint nondegenerate continua \( E, F \) in \( U \), and \( \Delta(E,F) \) is defined as

\[ \Delta(E,F) := \frac{\text{dist}(E,F)}{\min\{\text{diam } E, \text{diam } F\}} . \]

By (2.15), we obtain that

\[ C_1 M(\Gamma(\varphi(E), \varphi(F), B(0, r_0))) \leq \]

\[ \leq M(\Gamma(E,F,U)) \leq C_2 M(\Gamma(\varphi(E), \varphi(F), B(0, r_0))) . \]
Similarly,
\[ m \cdot \text{dist} (\varphi(E), \varphi(F)) \leq \text{dist} (E, F) \leq M \cdot \text{dist} (\varphi(E), \varphi(F)) \]
and
\[ m \cdot \min \{ \text{diam} \varphi(E), \text{diam} \varphi(F) \} \leq \min \{ \text{diam} E, \text{diam} F \} \leq M \min \{ \text{diam} \varphi(E), \text{diam} \varphi(F) \} . \]

Hence, taking into account (2.18), it follows that
\[ \frac{m}{M} \Delta (\varphi(E), \varphi(F)) \leq \Delta (E, F) \leq \frac{M}{m} \Delta (\varphi(E), \varphi(F)) . \] (2.20)

Now we fix \( t > 0 \), and let \( \Delta (E, F) \leq t \). Then it follows from (2.20) that \( \Delta (\varphi(E), \varphi(F)) \leq \frac{M}{m} t \). Denote by \( \tilde{\varphi}(t) \) the Loewner function \( \tilde{\varphi}(t) \), similar to (2.17), but for \( B(0, r_0) \subset \mathbb{R}^n \). Now, by (2.19) we obtain that
\[ M(\Gamma(E, F, U)) \geq C_1 \cdot M(\Gamma(\varphi(E), \varphi(F), B(0, r_0))) \geq C_1 \cdot \tilde{\varphi} \left( \frac{M}{m} t \right) , \] (2.21)
where we also used the fact that \( \Delta (\varphi(E), \varphi(F)) \leq \frac{M}{m} t \). Passing in (2.21) to inf over all continua \( E, F \subset U \) such that \( \Delta (E, F) \leq t \), we will have:
\[ \phi(t) \geq C_1 \cdot \tilde{\varphi} \left( \frac{M}{m} t \right) , \quad t > 0 . \] (2.22)

It remains to note that the usual Euclidean ball \( B(0, r_0) \) is a Loewner space: on the one hand, such is the entire Euclidean space \( \mathbb{R}^n \) (see [He, Theorem 8.2]), and on the other hand, the modulus of families of paths joining a pair of continua inside the ball is at least half the modulus of a family of paths joining the same continua in the entire \( n \)-dimensional Euclidean space (see [Vu, Lemma 4.3]). Thus, \( \tilde{\varphi} \left( \frac{M}{m} t \right) > 0 \) and, therefore, also \( \phi(t) > 0 \) due to (2.22), which should be installed. We proved that \( U \) is a Loewner space.

Finally, by virtue of [He, Proposition 8.19], the left inequality in (1.11) holds for some constant \( C > 0 \) and all \( 0 < R < \text{diam} X \), where \( X := U \) and \( \mu := v \). The right inequality in (1.11) is obvious by the definition of volume on the manifold and the right inequality in (2.19). Thus, \( U \) is Ahlfors regular. Theorem 1.1 is completely proved. \( \square \)

3 Equicontinuity of families of mappings with finite mean over spheres

The proof of Theorem 1.3 is largely based on the approach we used to prove a similar statement for homeomorphisms (see Theorem 1.1 in [IS4]).

Let us prove Theorem 1.3 by contradiction. Suppose that the family \( \mathcal{R}_Q(D, D_*) \) is not equicontinuous at some point \( x_0 \in D \). Then there is \( \varepsilon_0 > 0 \), for which the following condition is true: for any \( m \in \mathbb{N} \) there is an element \( x_m \in D \) with \( d(x_m, x_0) < 1/m \), and a mapping \( f_m \in \mathcal{R}_Q(D, D_*) \), such that
\[ d_*(f_m(x_m), f_m(x_0)) \geq \varepsilon_0 . \] (3.1)
Let $M$ points, since a finite (or even countable) set does not split $\partial D$ by a path in $D$ and to the boundary. Indeed, by the condition $M$ by disjoint paths $\gamma$ by subpaths of these paths with ends in others, generally speaking, boundary points $\gamma$ by disjoint paths $\gamma_1: [1/2, 1] \rightarrow M^n$ and $\gamma_2: [1/2, 1] \rightarrow M^n$ respectively. Without loss of generality, we may assume that both paths lie in the domain $D_*$, with the exception of their endpoints $\overline{x_1}$ and $\overline{x_2}$ (otherwise, by virtue of [Ku Theorem 1.1.5.46], there would be subpaths of these paths with ends in others, generally speaking, boundary points $\overline{x_1}$ and $\overline{x_2}$). Let $R_1 > 0$ be such that $B(\overline{x_1}, R_1) \cap |\gamma_1| = \emptyset$, and let $R_2 > 0$ be such that $B(\overline{x_1}, R_1) \cup |\gamma_1| \cap B(\overline{x_2}, R_2) = \emptyset$.

Since the infinitesimal balls on the manifold are connected, we may assume that $B(\overline{x_1}, r)$ and $B(\overline{x_2}, r)$ are path-connected sets for every $r \in [0, \max\{R_1, R_2\}]$. We may also assume that $f_m(x_m) \in B(\overline{x_1}, R_1)$ and $f_m(x_0) \in B(\overline{x_2}, R_2)$ for any $m \geq 1$. Join the points $f_m(x_m)$ and $\overline{x_1}$ by a path $\alpha_m^*: [0, 1/2] \rightarrow B(\overline{x_1}, R_1)$, and join the point $f_m(x_0)$ with the point $\overline{x_2}$ by a path $\beta_m^*: [0, 1/2] \rightarrow B(\overline{x_2}, R_2)$ (see Figure 4).

Set $\alpha_m(t) = \begin{cases} \alpha_m^*(t), & t \in [0, 1/2], \\ \gamma_1(t), & t \in [1/2, 1] \end{cases}$ and $\beta_m(t) = \begin{cases} \beta_m^*(t), & t \in [0, 1/2], \\ \gamma_2(t), & t \in [1/2, 1]. \end{cases}$

![Figure 4: To the proof of Theorem 1.1](image_url)
By the construction, the sets

$$A_1 := |\gamma_1| \cup \overline{B(x_1, R_1)}, \quad A_2 := |\gamma_2| \cup \overline{B(x_2, R_2)}$$

do not intersect, in particular, there is $\varepsilon_1 > 0$ such that

$$d(A_1, A_2) \geq \varepsilon_1 > 0. \quad (3.2)$$

Let $r_0 = r_0(y) > 0$ be the number from the conditions of the theorem, defined for each $y_0 \in \overline{D}$. According to [IS, Remark 2.1], for any $y_0 \in \mathbb{M}_n^*$ there is $\delta(y_0) > 0$ and a constant $C = C(y_0) > 0$ such that

$$\int_{\varepsilon_1 < d_s(y, y_0) < \varepsilon_2} Q(y) \, dv_s(y) \leq C \cdot \int_{\varepsilon_1}^{\varepsilon_2} \int_{S(y_0, r)} Q(y) \, dA \, dr \quad (3.3)$$

for any $0 \leq \varepsilon_1 < \varepsilon_2 \leq \delta(y_0)$. Set

$$r_*(y) := \min\{\varepsilon_1, r_0(y), \delta(y)\}.$$ 

Cover the set $A_1$ with balls $B(y, r_*/4), \ y \in A_1$. Note that $|\gamma_1|$ is a compact set in $\mathbb{M}_n^*$ as a continuous image of the compact set $[1/2, 1]$ under the mapping $\gamma_1$. Then, by the Heine-Borel-Lebesgue lemma, there is a finite subcover $\bigcup_{i=1}^{p} B(y_i, r_*/4)$ of the set $A_1$. In other words,

$$A_1 \subset \bigcup_{i=1}^{p} B(y_i, r_i/4), \quad 1 \leq p < \infty, \quad (3.4)$$

where $r_i$ denotes $r_*(y_i)$ for any $y_i \in \overline{D}$. 

Let $\alpha^0_m : [0, c_1) \to \mathbb{M}^n$ and $\beta^0_m : [0, c_2) \to \mathbb{M}^n$ be maximal $f_m$-liftings of paths $\alpha_m$ and $\beta_m$ starting at points $x_m$ and $x_0$, respectively. Such maximal liftings exist by Lemma 2.1. Arguing in the same way as in the proof of Theorem 1.2 one can show that $\alpha^0_m(t_k) \to \partial D$ and $\beta^0_m(t'_k) \to \partial D$ for some sequences $t_k \to c_1 - 0$ and $t'_k \to c_2 - 0, \ k \to \infty$. Then there are sequences of points $z^1_m \in [\alpha^0_m]$ and $z^2_m \in [\beta^0_m]$ such that $d(z^1_m, \partial D) < 1/m$ and $d(z^2_m, \partial D) < 1/m$. Since $\overline{D}$ is a compact set, we may consider that $z^1_m \to p_1 \in \partial D$ and $z^2_m \to p_2 \in \partial D$ as $m \to \infty$. Let $P_m$ be the part of the support of the path $\alpha^0_m$ in $\mathbb{M}^n$, located between the points $x_m$ and $z^1_m$, and $Q_m$ the part of the support of the path $\alpha^0_m$ in $\mathbb{M}^n$, located between the points $x_0$ and $z^2_m$. By the construction, $f_m(P_m) \subset A_1$ and $f_m(Q_m) \subset A_2$. Put $\Gamma_m := \Gamma(P_m, Q_m, D)$. Recall that we write $\Gamma_1 > \Gamma_2$ if and only if each path $\gamma_1 \in \Gamma_1$ has a subpath $\gamma_2 \in \Gamma_2$. (In other words, if $\gamma_1 : I \to \mathbb{M}^n$, then $\gamma_2 : J \to \mathbb{M}^n$, where $J \subset I$ and $\gamma_2(t) = \gamma_1(t)$ for $t \in J$, and $I, J$ are segments, intervals, or half-intervals). Then, by (3.2) and (3.4), and by [Kn, Theorem 1.1.5.46], we obtain that

$$\Gamma_m > \bigcup_{i=1}^{p} \Gamma_{im}, \quad (3.5)$$

where $\Gamma_{im} := \Gamma_{f_m(y_i, r_i/4, r_i/2)}$. 

ON NONHOMEOMORPHIC MAPPINGS...

Set \( \tilde{Q}(y) = \max\{Q(y), 1\} \) and

\[
\tilde{q}_{y_i}(r) = \int_{S(y, r)} \tilde{Q}(y) \, dA.
\]

Now, we have also that \( \tilde{q}_{y_i}(r) \neq \infty \) for any \( r \in [r_i/4, r_i/2] \). Set

\[
I_i = I_i(y_0, r_i/4, r_i/2) = \int_{r_i/4}^{r_i/2} \frac{dr}{r \tilde{q}_{y_i}(r)}.
\]

Observe that \( I \neq 0 \), because \( \tilde{q}_{y_i}(r) \neq \infty \) for any \( r \in [r_i/4, r_i/2] \). Besides that, note that \( I \neq \infty \), since

\[
I_i \leq \log \frac{r_2}{r_1} < \infty, \quad i = 1, 2, \ldots, p.
\]

Now, we put

\[
\eta_i(r) = \begin{cases} 
\frac{1}{I_i r \tilde{q}_{y_i}(r)}, & r \in [r_i/4, r_i/2], \\
0, & r \notin [r_i/4, r_i/2].
\end{cases}
\]

Observe that, a function \( \eta_i \) satisfies the condition \( \int_{r_i/4}^{r_i/2} \eta_i(r) = 1 \), therefore it can be substituted into the right side of the inequality (1.5) with the corresponding values \( f, r_1 \) and \( r_2 \).

We will have that

\[
M(\Gamma_{im}) \leq \int_{A(y_i, r_i/4, r_i/2)} \tilde{Q}(y) \eta_i^n(d^*(y, y_i)) \, dv^*(y). \tag{3.6}
\]

Let us use the estimate (3.3) on the right-hand side (3.6). We obtain that

\[
\int_{A(y_i, r_i/4, r_i/2)} \tilde{Q}(y) \eta_i^n(d^*(y, y_i)) \, dv^*(y) \leq
\]

\[
\leq C_i \int_{r_i/4}^{r_i/2} Q(y) \eta_i^n(d^*(y, y_i)) \, dA \, dr = \tag{3.7}
\]

\[
= C_i \frac{r_i^n}{T_i} \int_{r_i/4}^{r_i/2} r^{-n-1} \tilde{q}_{y_i} \, dr = \frac{C_i}{T_i} \int_{r_i/4}^{r_i/2} \frac{dr}{r \tilde{q}_{y_i}(r)} = C_i \frac{1}{T_i},
\]

where \( C_i \) is a constant corresponding to \( y_i \) in (3.3). Now, by (3.6) and (3.7) we obtain that

\[
M(\Gamma_{im}) \leq \frac{C_i}{T_i},
\]

whence from (3.5)

\[
M(\Gamma_m) \leq \sum_{i=1}^p M(\Gamma_{im}) \leq \sum_{i=1}^p \frac{C_i}{r_i^{p-1}} := C_0, \quad m = 1, 2, \ldots. \tag{3.8}
\]
Further reasoning is related to the "weak flatness" of the inner points of the domain $D$, see [IS] Lemma 2.1. Notice, that $d(P_m) \geq d(x_m, z_m^1) \geq (1/2) \cdot d(x_0, p_1) > 0$ and $d(Q_m) \geq d(x_0, z_m^2) \geq (1/2) \cdot d(x_0, p_2) > 0$, in addition, 

$$d(P_m, Q_m) \leq d(x_m, x_0) \to 0, \quad m \to \infty.$$ 

Now, by [IS] Lemma 2.1 

$$M(\Gamma_m) = M(\Gamma(P_m, Q_m, D)) \to \infty, \quad m \to \infty,$$ 

which contradicts the relation (3.8). The resulting contradiction indicates that the assumption in (3.1) is wrong, which completes the proof of the theorem. \hfill \Box 

**Remark 3.1.** In particular, the assertion of Theorem 1.3 holds if $Q \in L^1_{loc}(\mathbb{M}_n^*)$. Indeed, by [IS], Remark 2.1, for any point $\gamma_0 \in \mathbb{M}_n^*$ there is a number $\delta(\gamma_0) > 0$ and a constant $\tilde{C} = \tilde{C}(\gamma_0) > 0$ such that 

$$\int_{\varepsilon_1}^{\varepsilon_2} \int \frac{1}{S(\gamma_0,r)} Q(y) \cdot dA \cdot dr \leq \tilde{C} \cdot \int_{\varepsilon_1}^{\varepsilon_2} Q(y) \cdot dv_*(y) \quad (3.9)$$ 

for any $0 \leq \varepsilon_1 < \varepsilon_2 < \delta(\gamma_0)$. By (3.9), it follows that $q_{\gamma_0}(r) < \infty$ for $\varepsilon_1 < r < \varepsilon_2$. 

In conclusion of this section, we formulate one more important statement. 

**Corollary 3.1.** *The conclusion of Theorem 1.3 holds if, instead of the condition $q_{\gamma_0}(r) < \infty$, we require a simpler condition: $Q \in L^1(D_*)$.* 

**Proof.** We may assume that $Q(y) \equiv 0$ for $y \in \mathbb{M}_n^* \setminus D_*$ (if this is not the case, then we may consider a new function 

$$Q'(y) = \begin{cases} Q(y), & y \in D_*, \\ 0, & y \in \mathbb{M}_n^* \setminus D_* \end{cases}.$$ 

Then, if the condition (1.10) was satisfied for $Q$, then it will also hold for $Q')$. Then $Q \in L^1_{loc}(\mathbb{M}_n^*)$, so that the required conclusion follows from Remark 3.1. \hfill \Box 

4 **Boundary behavior of mappings** 

The proof of Theorem 1.4 is largely based on the approach used in the proof of Theorem 3.1 in [SSD]. 

Suppose the opposite, namely, that the mapping $f$ has no a continuous extension to some point $x_0 \in \partial D$. In this case, there are at least two sequences $x_i, y_i \in D$, $i = 1, 2, \ldots$, such that $x_i, y_i \to x_0$ as $i \to \infty$, and 

$$d_*(f(x_i), f(y_i)) \geq a > 0 \quad (4.1)$$
for some $a > 0$ and any $i \in \mathbb{N}$. Since $\overline{D_\ast}$ is compact, we may also assume that the sequences $f(x_i)$ and $f(y_i)$ converge as $i \to \infty$ to some elements $z_1$ and $z_2$. Since the mapping $f$ is closed, it preserves the boundary (see, for example, [IS3, Proposition 4.1]). Then $z_1, z_2 \in \partial D_\ast$. Since the domain $D_\ast$ is locally connected on its boundary, there are disjoint neighborhoods $U_1$ and $U_2$ of points $z_1$ and $z_2$ such that $W_1 = D_\ast \cap U_1$ and $W_2 = D_\ast \cap U_2$ are connected. We may assume that $W_1$ and $W_2$ are path-connected, since $U_1$ and $U_2$ may be chosen open (see, for example, [MRSY, Proposition 13.2]; see Figure 5). We may also consider that $z_1$ is a boundary point from the condition of the theorem such that

$$U_1 \subset B(z_1, R_0), \quad \overline{B(z_1, 2R_0)} \cap \overline{U_2} = \emptyset, \quad R_0 > 0,$$

and, in addition, $f(x_i) \in W_1$ and $f(y_i) \in W_2$ for any $i = 1, 2, \ldots$. In addition, reducing the neighborhood $U_1$, if necessary, we may choose a number $R_0$ so small that $q_{z_i}(r) < \infty$ for almost all $r \in (0, 2R_0)$. Join the points $f(x_i)$ and $f(x_1)$ by a path $\alpha_i : [0, 1] \to D_\ast$, and points $f(y_i)$ and $f(y_1)$ by a path $\beta_i : [0, 1] \to D_\ast$ such that $|\alpha_i| \subset W_1$ and $|\beta_i| \subset W_2$ as $i = 1, 2, \ldots$. Let $\widetilde{\alpha}_i : [0, 1] \to D_\ast$ and $\widetilde{\beta}_i : [0, 1] \to D_\ast$ be total $f$-liftings of paths $\alpha_i$ and $\beta_i$ starting at the points $x_i$ and $y_i$, respectively (these liftings exist by [IS3, Proposition 4.2]). Note that the points $f(x_1)$ and $f(y_1)$ can have at most a finite number of pre-images in $D$ under mapping $f$, because $f$ is open, discrete and closed (see [MS, Theorem 2.8]). Then there is $r_0 > 0$ such that $\widetilde{\alpha}_i(1), \widetilde{\beta}_i(1) \in D \setminus B(x_0, r_0)$ for any $i = 1, 2, \ldots$. Since the boundary of $D$ is weakly flat, for any $P > 0$ there exists $i = i_P \geq 1$ such that

$$M(\Gamma(|\widetilde{\alpha}_i|, |\widetilde{\beta}_i|, D)) > P \quad \forall \ i \geq i_P.$$  

Let us show that the condition (4.3) contradicts the definition of the mapping from $f$ in (1.5). Indeed, by (4.2) and [Kul, Theorem 1.1.5.46]

$$f(\Gamma(|\widetilde{\alpha}_i|, |\widetilde{\beta}_i|, D)) > \Gamma(S(z_1, R_0), S(z_1, 2R_0), A(z_1, R_0, 2R_0)).$$

By (4.4)

$$\Gamma(|\widetilde{\alpha}_i|, |\widetilde{\beta}_i|, D) > \Gamma_f(z_1, R_0, 2R_0).$$

Figure 5: To the proof of Theorem 1.4

\[\begin{array}{c}
\begin{array}{c}
D \\
\Gamma(x, x, D)
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
f
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
U_1
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
f(x_i)
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
f(y_i)
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
f(D) = D_\ast
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\Gamma(\xi, \xi, D)
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\Gamma(\xi, \xi, D)
\end{array}
\end{array}\]
In turn, it follows from (4.5) that
\[
M(\Gamma(|\tilde{\alpha}_1|, |\tilde{\beta}_1|, D)) \leq M(\Gamma_f(z_1, R_0, 2R_0)) \leq \int_A Q(y) \cdot \eta^n(d_*(y, z_1)) \, dv_*(y),
\] (4.6)
where \( A = A(z_1, R_0, 2R_0) \) is defined in (1.3) and \( \eta \) is any nonnegative Lebesgue measurable function satisfying the relation (1.6) with \( r_1 := R_0 \) and \( r_2 := 2R_0 \). Set \( \tilde{Q}(y) = \max\{Q(y), 1\} \) and
\[
\tilde{q}_{z_1}(r) = \int_{S(z_1, r)} \tilde{Q}(y) \, dA.
\]
Then \( \tilde{q}_{z_1}(r) \neq \infty \) for a.a. \( r \in [R_0, 2R_0] \). Set
\[
I = \int_{R_0}^{2R_0} \frac{dt}{t^{q_{z_1}(t)}}.
\] (4.7)
Observe that \( 0 \neq I \neq \infty \). Now, the function \( \eta_0(t) = \frac{1}{t^{q_{z_1}(t)}} \) satisfies the relation (1.6) with \( r_1 := R_1 \) and \( r_2 := 2R_0 \). Substituting the function \( \eta_0 \) into the defining relation (1.5), and also taking into account the condition (3.3), we obtain that
\[
M(\Gamma) \leq \frac{C}{t^{n-1}} < \infty,
\] (4.8)
where \( C > 0 \) is some constant. The relation (4.8) contradicts (4.3), which indicates that the assumption made in (4.1) is wrong.

The equality \( \overline{f(D)} = \overline{D}_* \) is proved similarly to the last part of the proof of Theorem 3.1 in [SSD]. \( \square \)

**Corollary 4.1.** The conclusion of Theorem 1.4 holds if in this theorem, instead of the condition \( q_{z_1}(r) < \infty \), we require a stronger condition: \( Q \in L^1(D_*) \).

**Proof of Corollary 4.1** follows immediately from Theorem 1.4 and Remark 3.1. \( \square \)

### 5 Equicontinuity of families in the closure of a domain

**Proof of Theorem 1.5** Let \( f \in \mathcal{G}_{\delta, A, Q}(D, D_*) \). By Theorem 1.4, \( f \) has a continuous extension \( \overline{f} : \overline{D} \to \overline{D}_* \), at the same time, \( \overline{f(D)} = \overline{D}_* \). Equicontinuity of \( \mathcal{G}_{\delta, A, Q}(D, D_*) \) in \( D \) is the assertion of Theorem 1.3. It remains to establish the equicontinuity of this family in \( \partial D \).

Let us prove the theorem by contradiction. Suppose there is \( x_0 \in \partial D \), a number \( \varepsilon_0 > 0 \), a sequence \( x_m \in \overline{D} \), converging to \( x_0 \), as well as the corresponding maps \( \overline{f}_m \in \mathcal{G}_{\delta, A, Q}(\overline{D}, \overline{D}) \) such that
\[
d''(\overline{f}_m(x_m), \overline{f}_m(x_0)) \geq \varepsilon_0, \quad m = 1, 2, \ldots
\] (5.1)
We set \( f_m := |\overline{f}_m|_{D} \). Since \( f_m \) has a continuous extension to \( \partial D \), we may assume that \( x_m \in D \). Therefore, \( \overline{f}_m(x_m) = f_m(x_m) \). In addition, there is a sequence \( x'_m \in D \) such that \( x'_m \to x_0 \) as \( m \to \infty \) and \( d'(f_m(x'_m), \overline{f}_m(x_0)) \to 0 \) as \( m \to \infty \). Since \( \overline{D}_* \) is compact, we
may also assume that the sequences \( f_m(x_m) \) and \( \overline{f_m(x_0)} \) are convergent as \( m \to \infty \). Let \( f_m(x_m) \to \overline{x}_1 \) and \( \overline{f_m(x_0)} \to \overline{x}_2 \) as \( m \to \infty \). By the continuity of the metric in (5.1), \( \overline{x}_1 \neq \overline{x}_2 \). Since the mappings \( f_m \) are closed, they preserve the boundary (see, for example, [IS₃, Proposition 4.1]), therefore \( \overline{x}_2 \in \partial D \). Let \( \overline{x}_1 \) and \( \overline{x}_2 \) be different points of the continuum \( A \), none of which coincides with \( \overline{x}_1 \). By [IS₄, Lemma 3.3] two pairs of points \( \overline{x}_1, \overline{x}_1 \) and \( \overline{x}_2, \overline{x}_2 \) can be joined by paths \( \gamma_1 : [0,1] \to \overline{D} \) and \( \gamma_2 : [0,1] \to \overline{D} \) such that \( |\gamma_1(0) - \gamma_2(0)| = 0 \), \( \gamma_1(t), \gamma_2(t) \in D \) for \( t \in (0,1) \), \( \gamma_1(0) = \overline{x}_1, \gamma_1(1) = \overline{x}_1, \gamma_2(0) = \overline{x}_2 \) and \( \gamma_2(1) = \overline{x}_2 \). Also, since \( D_* \) is locally connected on \( \partial D_* \), there are neighborhoods \( U_1 \) and \( U_2 \) of points \( \overline{x}_1 \) and \( \overline{x}_2 \), whose closures do not intersect; moreover, the sets \( W_t := D_* \cap U_1 \) are path-connected. Without loss of generality, we may assume that \( \overline{U}_1 \subset B(\overline{x}_1, \tilde{\delta}_0) \) and

\[
B(\overline{x}_1, \tilde{\delta}_0) \cap |\gamma_2| = \emptyset = U_2 \cap |\gamma_1|, \quad B(\overline{x}_1, \tilde{\delta}_0) \cap U_2 = \emptyset, \tag{5.2}
\]

where

\[
0 < \tilde{\delta}_0 < (1/2) \cdot \min\{\delta_0(\overline{x}_1), r_0(\overline{x}_1)\},
\]

\( r_0(\overline{x}_1) \) corresponds to the condition of the theorem, and \( \delta_0(\overline{x}_1) \) corresponds to [IS₃]. We may also assume that \( f_m(x_m) \in W_1 \) and \( f_m(x_m') \in W_2 \) for any \( m \in \mathbb{N} \). Let \( a_1 \) and \( a_2 \) be two different points belonging to \( |\gamma_1| \cap W_1 \) and \( |\gamma_2| \cap W_2 \); in addition, let \( 0 < t_1, t_2 < 1 \) such that \( \gamma_1(t_1) = a_1 \) and \( \gamma_2(t_2) = a_2 \). Join the points \( a_1 \) and \( f_m(x_m) \) by a path \( \alpha_m : [t_1,1] \to W_1 \) such that \( \alpha_m(t_1) = a_1 \) and \( \alpha_m(1) = f_m(x_m) \). Similarly, join \( a_2 \) and \( f_m(x_m') \) by a path \( \alpha_m : [t_1,1] \to W_1 \beta_m : [t_2,1] \to W_2 \) such that \( \beta_m(t_2) = a_2 \) and \( \beta_m(1) = f_m(x_m') \) (see Figure 6).

![Figure 6: To the proof of Theorem 15](image_url)

\[
C_m^1(t) = \left\{ \begin{array}{ll} \gamma_1(t), & t \in [0,t_1], \\ \alpha_m(t), & t \in [t_1,1] \end{array} \right., \quad C_m^2(t) = \left\{ \begin{array}{ll} \gamma_2(t), & t \in [0,t_2], \\ \beta_m(t), & t \in [t_2,1] \end{array} \right.. \]

Let \( D_m^1 \) and \( D_m^2 \) be total \( f_m \)-liftings of paths \( C_m^1 \) and \( C_m^2 \) starting at \( x_m \) and \( x_m' \), respectively (these liftings exist by [IS₃, Proposition 4.2]). In particular, since \( d(f_m^{-1}(A), \partial D) \geq \delta > 0 \) by the condition of the theorem, the end points \( b_m^1 \) and \( b_m^2 \) of \( D_m^1 \) and \( D_m^2 \) are at a distance from the boundary of the domain \( D \) not less than \( \delta \). Let \( |C_m^1| \) and \( |C_m^2| \) be loci of \( C_m^1 \) and
Let $\delta_0 = \delta_0(y) > 0$ be a number corresponding to the relation (3.3), and let $r_0 = r_0(y) > 0$ be a number form the condition of the theorem. Set

$$l_0 = l_0(y) := \min\{\text{dist}(|\gamma_1|, |\gamma_2|), \text{dist}(|\gamma_1|, U_2 \setminus \{\infty\}), \delta_0, r_0\}$$

and consider the coverage $A_0 := \bigcup_{x \in |\gamma_1|} B(x, l_0/4)$ of the path $|\gamma_1|$ by balls. Since $|\gamma_1|$ is a compactum, there is a finite set of indices $1 \leq N_0 < \infty$ and points $z_1, \ldots, z_{N_0} \in |\gamma_1|$ such that $|\gamma_1| \subset B_0 := \bigcup_{i=1}^{N_0} B(z_i, l_0/4)$. In this case,

$$|C^1_m| \subset U_1 \cup |\gamma_1| \subset B(\overline{x_1}, \delta_0) \cup \bigcup_{i=1}^{N_0} B(z_i, l_0/4).$$

Let $\Gamma_m$ be a family of paths joining $|C^1_m|$ and $|C^2_m|$ in $D_*$. Now, we obtain that

$$\Gamma_m = \bigcup_{i=0}^{N_0} \Gamma_{mi},$$

where $\Gamma_{mi}$ is a family of paths $\gamma : [0, 1] \to D_*$ such that $\gamma(0) \in B(z_i, l_0/4) \cap |C^1_m|$ and $\gamma(1) \in |C^2_m|$ for $1 \leq i \leq N_0$. Similarly, let $\Gamma_{m0}$ be a family of paths $\gamma : [0, 1] \to D_*$ such that $\gamma(0) \in B(\overline{x_1}, \delta_0) \cap |C^1_m|$ and $\gamma(1) \in |C^2_m|$. By (3.2) there is $\sigma_0 > \delta_0 > 0$ such that

$$\overline{B(\overline{x_1}, \sigma_0) \cap |\gamma_2|} = \emptyset = \overline{U_2 \cap |\gamma_1|}, \quad \overline{B(\overline{x_1}, \sigma_0) \cap U_2} = \emptyset,$$

in addition,

$$0 < \sigma_0 < \min\{\delta_0(\overline{x_1}), r_0(\overline{x_1})\}.$$

By [Ku, Theorem 1.15.46],

$$\Gamma_{m0} > \Gamma(S(\overline{x_1}, \delta_0), S(\overline{x_1}, \sigma_0), A(\overline{x_1}, \delta_0, \sigma_0)),$$

$$\Gamma_{mi} > \Gamma(S(z_i, l_0/4), S(z_i, l_0/2), A(z_i, l_0/4, l_0/2)).$$

Set $\tilde{Q}(y) = \max\{Q(y), 1\}$ and

$$\tilde{q}_{z_i}(r) = \int_{S(z_i, r)} \tilde{Q}(y) \, dA, \quad \tilde{q}_{\overline{x_1}}(r) = \int_{S(\overline{x_1}, r)} \tilde{Q}(y) \, dA.$$

Then also $\tilde{q}_{z_i}(r) \neq \infty$ for a.a. $r \in [0, l_0(z_i)]$ and $\tilde{q}_{\overline{x_1}}(r) \neq \infty$ for a.a. $r \in [0, l_0(\overline{x_1})]$. Set

$$I_i = \int_{l_0(z_i)/4}^{l_0(z_i)/2} \frac{dt}{t q_{z_i}^{1/(n-1)}(t)}, \quad I_0 = \int_{\delta_0}^{\sigma_0} \frac{dt}{t q_{\overline{x_1}}^{1/(n-1)}(t)}.$$

Observe that $0 \neq I_i \neq \infty$ и $0 \neq I_0 \neq \infty$. Now, the functions

$$\eta_i(t) = \frac{1}{I_i t q_{z_i}^{1/(n-1)}(t)}, \quad \eta_0(t) = \frac{1}{I_0 t q_{\overline{x_1}}^{1/(n-1)}(t)}.$$
satisfy the relation (1.6) for corresponding $r_1$ and $r_2$. Set $\Gamma_m^* := \Gamma(\{|D^1_m|, |D^2_m|, D\})$. Observe that $f_m(\Gamma_m^*) \subset \Gamma_m$. Now, by (5.3) and (5.4)

$$\Gamma_m^* > \left( \bigcup_{i=1}^{N_0} \Gamma_{f_m}(z_i, l_0/4, l_0/2) \right) \cup \Gamma_{f_m}(\overline{x_1}, \delta_0, \sigma_0). \quad (5.5)$$

Since mappings $f_m$ satisfy the relation (1.5) in $D$, by (5.5) and by (3.3) we obtain that

$$M(\Gamma_m^*) \leq \sum_{i=1}^{N_0} \frac{C_i}{I_{n-1}} + \frac{C}{I_{n-1}} = c < \infty, \quad (5.6)$$

where $C_i$ is some constant corresponding to $z_i$ in (3.3), and $C$ is a constant corresponding to a point $\overline{x_1}$ here. Let us show that the relation (5.6) contradicts the condition of the weak flatness of the boundary of the original domain $D$. Indeed, by construction

$$d(|D^1_m|) \geq d(x_m, b^1_m) \geq (1/2) \cdot d(f_m^{-1}(A), \partial D) > \delta/2,$$

$$d(|D^2_m|) \geq d(x'_m, b^2_m) \geq (1/2) \cdot d(f_m^{-1}(A), \partial D) > \delta/2 \quad (5.7)$$

for any $m \geq M_0$ and some $M_0 \in \mathbb{N}$. Set $U := d(x_0, r_0^*)$, where $0 < r_0^* < \delta/4$ and a number $\delta$ refers to the condition (5.7). Note that $|D^1_m| \cap U \neq \emptyset \neq |D^2_m| \cap (D \setminus U)$ for any $m \in \mathbb{N}$, since $d(|D^1_m|) \geq \delta/2$ and $x_m \in |D^1_m|, x_m \to x_0$ as $m \to \infty$. Similarly, $|D^2_m| \cap U \neq \emptyset \neq |D^2_m| \cap (D \setminus U)$.

Since $|D^1_m|$ and $|D^2_m|$ are continua, we obtain that

$$|D^1_m| \cap \partial U \neq \emptyset, \quad |D^2_m| \cap \partial U \neq \emptyset, \quad (5.8)$$

see e.g. [Kn, Theorem 1.1.5.46]. Since $\partial D$ is weakly flat, for $P := c > 0$ (where $c$ is a number from (5.6)) there exists a neighborhood $V \subset U$ of the point $x_0$ such that

$$M(\Gamma(E, F, D)) > c \quad (5.9)$$

for any continua $E, F \subset D$ such that $E \cap \partial U \neq \emptyset \neq E \cap \partial V$ and $F \cap \partial U \neq \emptyset \neq F \cap \partial V$. Let us show that for sufficiently large $m \in \mathbb{N}$

$$|D^1_m| \cap \partial V \neq \emptyset, \quad |D^2_m| \cap \partial V \neq \emptyset. \quad (5.10)$$

Indeed, $x_m \in |D^1_m|$ and $x'_m \in |D^2_m|$, where $x_m, x'_m \to x_0 \in V$ as $m \to \infty$. In this case, $|D^1_m| \cap V \neq \emptyset \neq |D^2_m| \cap V$ for sufficiently large $m \in \mathbb{N}$. Note that $d(V) \leq d(U) \leq 2r_0^* < \delta/2$. Due to (5.7) $d(|D^1_m|) > \delta/2$. Therefore, $|D^1_m| \cap (D \setminus V) \neq \emptyset$ and thus $|D^1_m| \cap \partial V \neq \emptyset$ (see, e.g., [Kn, Theorem 1.1.5.46]). Similarly, $d(V) \leq d(U) \leq 2r_0^* < \delta/2$. By (5.7) it follows that $d(|D^2_m|) > \delta/2$. Then $|D^2_m| \cap (D \setminus V) \neq \emptyset$. By [Kn, Theorem 1.1.5.46] we obtain that $|D^2_m| \cap \partial V \neq \emptyset$. Thus, the relation (5.10) is established. Combining (5.8), (5.9) and (5.10), we obtain that $M(\Gamma_m^*) = M(\Gamma(|D^1_m|, |D^2_m|, D)) > c$. The latter contradicts (5.6), which completes the proof of the theorem. \(\Box\)
6 Removability of isolated singularities

Proof of Theorem 1.6. Due to the discreteness of the mapping $f$ there is $0 < \varepsilon_0 < \text{dist}(x_0, \partial D)$ such that $\infty \not\in f(S(x_0, \varepsilon))$ (if $\partial D = \emptyset$, we fix an arbitrary $\varepsilon_0 > 0$ with the specified condition). We denote

$$g := f|_{B(x_0, \varepsilon_0) \setminus \{x_0\}}.$$ 

Suppose that the assertion of the theorem is not true, namely, that $f$ has no a continuous extension to $x_0$. Then $g$ has no a continuous extension to the same point, as well. By virtue of compactness of $\overline{D}_s$, $C(f, x_0) = C(g, x_0) \neq \emptyset$. Then there are $y_1, y_2 \in C(f, x_0)$, $y_1 \neq y_2$, and corresponding sequences $x_m, x'_m \in B(x_0, \varepsilon_0) \setminus \{x_0\}$ such that $x_m, x'_m \to x_0$ as $m \to \infty$, wherein, $z_m := g(x_m) \to y_1$, $z'_m = g(x'_m) \to y_2$ as $m \to \infty$.

Let

$$D_{**} := f(B(x_0, \varepsilon_0) \setminus \{x_0\}).$$

Let us show that there exists some number $\varepsilon_1 > 0$ such that

$$B(y_1, \varepsilon_1) \cap f(S(x_0, \varepsilon_0)) = \emptyset. \quad (6.1)$$

Observe that $y_1 \in \partial D_{**}$. Indeed, if $y_1$ is an inner point of $D_{**}$, then $y_1$ is also an inner point of $D_*$, because $D_{**} \subset D_*$. The latter contradicts the condition $C(f, x_0) \subset \partial D_*$. Further, since $S(x_0, \varepsilon_0)$ is a compactum in $D$, then $f(S(x_0, \varepsilon_0))$ is a compactum in $D_*$, so that

$$d_*(f(S(x_0, \varepsilon_0)), y_1) > \delta_1 > 0. \quad (6.2)$$

By (6.2), the relation (6.1) holds for $\varepsilon_1 := \delta_1$.

Arguing similarly to the proof of the relation (6.1), one can show the existence of $\varepsilon_2 > 0$, such that

$$B(y_2, \varepsilon_2) \cap f(S(x_0, \varepsilon_0)) = \emptyset. \quad (6.3)$$

Without loss of generality, we may assume that $\overline{B(y_1, \varepsilon_1)} \cap \overline{B(y_2, \varepsilon_2)} = \emptyset$, in addition, $z_m \in B(y_1, \varepsilon_1)$ and $z'_m \in B(y_2, \varepsilon_2)$ for any $m = 1, 2, \ldots$ (see Figure 7). We may also assume that $B(y_1, \varepsilon_1)$ and $B(y_2, \varepsilon_2)$ are path-connected sets, since sufficiently small balls on a manifold are homeomorphic to Euclidean balls of the same radius (see [Lee, Proposition 5.11]). In this case, the points $z_1$ and $y_1$ may be joined by some path $I = I(t)$, $t \in (0, 1)$, completely lying in $B(y_1, \varepsilon_1)$. Similarly, the points $z'_1$ and $y_2$ may also be connected by a path $J = J(t)$, $t \in [0, 1]$, lying in $B(y_2, \varepsilon_2)$. Finally, we may consider that $\varepsilon_1 < \min\{\delta_0(y_1), r_0(y_1)\}$, where $r_0$ is the number from the condition of the theorem, and $\delta_0$ is the number corresponding to the condition (3.3). Observe that, by the construction, $|I| \cap \partial D_* \neq \emptyset \neq |J| \cap \partial D_*$. Denote

$$t_* := \sup_{t \in [0, 1]: I(t) \in D_*} t, \quad p_* := \sup_{t \in [0, 1]: J(t) \in D_*} t.$$

Put

$$C_1 := I_{[0, t_*)}, \quad C_2 := J_{[0, p_*)}.$$
By Lemma 2.1 the paths $C_1$ and $C_2$ have maximal $g$-liftings $C_1^* : [0, c_1) \rightarrow B(x_0, \varepsilon_0) \setminus \{x_0\}$ and $C_2^* : [0, c_2) \rightarrow B(x_0, \varepsilon_0) \setminus \{x_0\}$ starting at points $x$ and $x'$, respectively. Note, that $C_i^*(t_k) \rightarrow \partial D_{n^*}$ and $C_i^*(t_k') \rightarrow \partial D_{n^*}$ for some sequences $t_k \rightarrow c_1$ and $t_k' \rightarrow c_2$ (this can be proved in the same way as in the proof of the relation \((2.3)\)). Let us show that the situation when $d(C_1^*(t_k), S(x_0, \varepsilon_0)) \rightarrow 0$ as $k \rightarrow \infty$ and some sequence $t_k \rightarrow c_0$ is also not possible. Indeed, due to the compactness of the sphere $S(x_0, \varepsilon_0)$ there is a sequence $w_k \in S(x_0, \varepsilon_0)$ such that $d(C_1^*(t_k), S(x_0, \varepsilon_0)) = d(C_1^*(t_k), w_k)$. Further, since the sphere $S(x_0, \varepsilon_0)$ is compact, we may assume that $w_k \rightarrow w_0$ as $k \rightarrow \infty$. Then $C_1^*(t_k) \rightarrow w_0$ as $k \rightarrow \infty$, whence by the continuity of $f$ in $D$, we obtain that

$$f(C_1^*(t_k)) = C_1(t_k) \rightarrow f(w_0) \in f(S(x_0, \varepsilon_0))$$  \hspace{1cm} (6.4)

as $k \rightarrow \infty$. The latter contradicts the condition \((6.1)\), since at the same time we have that $f(w_0) \in f(S(x_0, \varepsilon_0))$ and $f(w_0) \in \{1\} \subset B(y_1, \varepsilon_1)$. Then we have that

$$d(C_1^*(t_k), x_0) \rightarrow 0, \hspace{0.5cm} t_k \rightarrow c_1 - 0. \hspace{1cm} (6.5)$$

Applying similar statements to the path $C_2^*(t)$, one can show that

$$d(C_2^*(t_k'), x_0) \rightarrow 0, \hspace{0.5cm} t_k' \rightarrow c_2 - 0. \hspace{1cm} (6.6)$$

From conditions \((6.5)\) and \((6.6)\), it follows that

$$M(\Gamma(|C_1^*(t)|, |C_2^*(t)|, B(x_0, \varepsilon_0) \setminus \{x_0\})) = \infty, \hspace{1cm} (6.7)$$

since both inner and isolated points of the domains of Riemannian manifolds are weakly flat (see [18], Lemma 2.1). Let us show that \((6.7)\) contradicts the condition \((1.5)\) at the point $y_1$. Since $B(y_1, \varepsilon_1) \cap B(y_2, \varepsilon_2) = \varnothing$, there exists $\varepsilon_1^* > \varepsilon_1$, for which we still have $B(y_1, \varepsilon_1^* \cap B(y_2, \varepsilon_2) = \varnothing$, where $r_0$ is the number from the conditions of the theorem, and $\delta_0$ is the number corresponding to the condition \((3.3)\). Let $\Gamma_* = \Gamma(|C_1|, |C_2|, D_*)$. Observe that

$$\Gamma_* > \Gamma(S(y_1, \varepsilon_1^*), S(y_1, \varepsilon_1), A(y_1, \varepsilon_1, \varepsilon_1^*)) \hspace{1cm} (6.8)$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7}
\caption{To the proof of Theorem 1.6}
\end{figure}
Indeed, let \( \gamma \in \Gamma_* \), \( \gamma : [a, b] \to \mathbb{R}^n \). Since \( \gamma(a) \in |C_1| \subset B(y_1, \varepsilon_1) \) and \( \gamma(b) \in |C_2| \subset \mathbb{R}^n \setminus B(y_1, \varepsilon_1) \), by [Ku. Theorem 1.1.5.46] there is \( t_1 \in (a, b) \) such that \( \gamma(t_1) \in S(y_1, \varepsilon_1) \). Without loss of generality, we may consider that \( d_*(\gamma(t), y_1) > \varepsilon_1 \) for \( t > t_1 \). Further, since \( \gamma(t_1) \in B(y_1, \varepsilon_1^*) \) and \( \gamma(b) \in |C_2| \subset \mathbb{R}^n \setminus B(y_1, \varepsilon_1^*) \), by [Ku. Theorem 1.1.5.46] there is \( t_2 \in (t_1, b) \) such that \( \gamma(t_2) \in S(y_1, \varepsilon_1^*) \). Without loss of generality, we may consider that \( d_*(\gamma(t), y_1) < \varepsilon_1^* \) as \( t_1 < t < t_2 \). Thus, \( \gamma|_{[t_1, t_2]} \) is a subpath of \( \gamma \), belonging to \( \Gamma(S(y_1, \varepsilon_1^*), S(y_1, \varepsilon_1), A(y_1, \varepsilon_1, \varepsilon_1^*)) \). Thus, the relation (6.8) is proved.

Now, let us prove that

\[
\Gamma(|C_1^*(t)|, |C_2^*(t)|, B(x_0, \varepsilon_0) \setminus \{x_0\}) > \Gamma_f(y_1, \varepsilon_1, \varepsilon_1^*). \tag{6.9}
\]

Indeed, if \( f : [a, b] \to B(x_0, \varepsilon_0) \setminus \{x_0\} \) belongs to

\[
\Gamma(|C_1^*(t)|, |C_2^*(t)|, B(x_0, \varepsilon_0) \setminus \{x_0\}),
\]

then \( f(\gamma) \) belongs to \( D_* \). Now, \( f(\gamma(a)) \in |C_1| \) and \( f(\gamma(b)) \in |C_2| \), that is, \( f(\gamma) \in \Gamma_* \). Then, by what was proved above and by (6.8), the path \( f(\gamma) \) has a subpath \( f(\gamma)^* : = f(\gamma)|_{[t_1, t_2]} \), \( a \leq t_1 < t_2 \leq b \), belongs to \( \Gamma(S(y_1, \varepsilon_1^*), S(y_1, \varepsilon_1), A(y_1, \varepsilon_1, \varepsilon_1^*)) \). Then \( \gamma^* : = \gamma|_{[t_1, t_2]} \) is a subpath of \( \gamma \), belonging to \( \Gamma_f(y_1, \varepsilon_1, \varepsilon_1^*) \), which should be proved.

In turn, by (6.9) we have the following:

\[
M(\Gamma(|C_1^*(t)|, |C_2^*(t)|, B(x_0, \varepsilon_0) \setminus \{x_0\})) \leq
\leq M(\Gamma_f(y_1, \varepsilon_1, \varepsilon_1^*)) \leq \int_A Q(y) \cdot \eta^*(d_*(y, y_1)) \, dv_*(y), \tag{6.10}
\]

where \( A = A(y_1, \varepsilon_1, \varepsilon_1^*) \) and \( \eta \) is a Lebesgue measurable nonnegative function satisfying the condition (1.6) for \( r_1 := \varepsilon_1 \) and \( r_2 := \varepsilon_1^* \).

Set \( \widetilde{Q}(y) = \max\{Q(y), 1\} \) and

\[
\widetilde{q}_{y_1}(r) = \int_{S(y_1, r)} \widetilde{Q}(y) \, dA.
\]

Then also \( \widetilde{q}_{y_1}(r) \neq \infty \) for almost all \( r \in [0, r_0(y_1)] \). Put

\[
I = \int_{\varepsilon_1}^{\varepsilon_1^*} \frac{dt}{td_{q_{y_1}}^{1/(n-1)}(t)}. \tag{6.11}
\]

Observe that \( 0 \neq I \neq \infty \). Now, the function \( \eta_0(t) = \frac{1}{td_{q_{y_1}}^{1/(n-1)}(t)} \) satisfies the relation (1.6) for \( r_1 := \varepsilon_1 \) and \( r_2 := \varepsilon_1^* \). Substituting this function into the right-hand side of the inequality (6.10) and applying the analog Fubini theorem (3.3), we obtain that

\[
M(\Gamma(|C_1^*(t)|, |C_2^*(t)|, B(x_0, \varepsilon_0) \setminus \{x_0\})) \leq \frac{C}{I^{n-1}} < \infty. \tag{6.12}
\]

Relationships (6.12) and (6.7) contradict each other. The resulting contradiction completes the proof of the theorem. \( \square \)
7 Examples

To illustrate some of the assertions of the article, we slightly modify the examples given in Section 4 of [Skv].

**Example 1.** Let \( n \geq 2 \), and let \( p \geq 1 \) be such that \( n/p(n-1) < 1 \). Let also \( \alpha \in (0, n/p(n-1)) \). Define a sequence of mappings \( f_m \) of the unit ball \( B^n = \{ x \in \mathbb{R}^n : |x| < 1 \} \) as follows:

\[
f_m(x) = \begin{cases} \frac{1+|x|^n}{|x|^n} \cdot x, & 1/m \leq |x| \leq 1, \\ \frac{1+(1/m)^n}{(1/m)^n} \cdot x, & 0 < |x| < 1/m. \end{cases}
\]

Note that, the mappings \( f_m \) satisfy the condition

\[
M(f_m(\Gamma(S(x_0, r_1), S(x_0, r_2), B^n))) \leq \int_{A(x_0, r_1, r_2) \cap \overline{B^n}} Q(x) \cdot \eta^n(|x-x_0|) \, dm(x) \tag{7.1}
\]

for any \( m = 1, 2, \ldots \) and all \( x_0 \in B^n \), any \( 0 < r_1 < r_2 < d_0 := \sup_{x \in B^n} |x-x_0| \) and all Lebesgue measurable functions \( \eta : (r_1, r_2) \to [0, \infty] \) satisfying the condition

\[
\int_{r_1}^{r_2} \eta(r) \, dr \geq 1,
\]

where \( Q(x) = \frac{1+|x|^\alpha}{|x|^\alpha} \); moreover, \( Q \in L^p(B^n) \) (see the reasoning used for considering [MRSY, Proposition 6.3]). Using direct calculations, we may verify that the inverse mappings \( g_m = f_m^{-1}(x) \) have the following form:

\[
g_m(x) = \begin{cases} \frac{(|x|-1)^{1/n}}{|x|^{1/n}} \cdot x, & 1 + 1/(m^n) \leq |x| \leq 2, \\ \frac{|x|}{1+(1/m)^n} \cdot x, & 0 < |x| < 1 + 1/(m^n). \end{cases}
\]

Moreover, the relation (7.1) may be written in another form:

\[
M(\Gamma_{g_m}(y_0, r_1, r_2)) \leq \int_{A(y_0, r_1, r_2) \cap \overline{B^n}} Q(y) \cdot \eta^n(|y-y_0|) \, dm(y). \tag{7.2}
\]

Note that the mappings \( g_m \) satisfy all the conditions Theorems [1.1] and [1.3] (the condition of complete divergence of paths in \( \mathbb{R}^n \) is always satisfied, since as such paths segments of some straight line, diverging in different directions from each other, may be taken. In addition, the condition \( \eta_0(r) < \infty \) is a consequence of the usual Fubini theorem in \( \mathbb{R}^n \), see also the inequality [1.3]).

**Example 2.** It is not difficult to point out a similar example of mappings with branching. To construct it, we will use twisting around the axis, or winding map, see [Re, example 3, item 4.3.1]. Let Пусть \( m \in \mathbb{N}, x = (x_1, x_2, x_3, \ldots, x_n) \in B^n, x_1 = r \cos \varphi, x_2 = r \sin \varphi, r \geq 0, \varphi \in [0, 2\pi) \). Put \( l(x) = (r \cos m\varphi, r \sin m\varphi, x_3, \ldots, x_n) \). It is clear that \( N(f, B^n) = m \), where the multiplicity function \( N \) is defined by the relation (1.2), in this case, \( K_0(x, f) = m^{n-1} \)
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(see result of consideration [Re, example 3, item 4.3.I]). Then the mapping \( f \) satisfies the relation (1.1) to \( B \) (see [MRV₁] Theorem 3.2 or [Ri] Theorem 6.7.II), in other words,

\[
M(\Gamma) \leq m^{n-1} \cdot M(l(\Gamma)).
\] (7.3)

Set \( h_m = (g_m \circ l)(x) \). Observe that \( l(\Gamma_{h_m}(y_0, r_1, r_2)) \subset \Gamma_{g_m}(y_0, r_1, r_2) \). Now, by (7.2) and (7.3) it follows that

\[
M(\Gamma_{h_m}(y_0, r_1, r_2)) \leq m^{n-1} \int_{A(y_0, r_1, r_2) \cap B^n} Q(y) \cdot \eta^n(|y - y_0|) \, dm(y).
\]

The family \( h_m, m = 1, 2, \ldots \), also satisfies all conditions of Theorems [1.1] and [1.3].

**Example 3.** We now construct similar examples of mappings on Riemannian manifolds. Let \( M^n \) be a Riemannian manifold, \( D \) a domain in \( M^n \), and let \( p_0 \in D \). Suppose that \( \varphi : U \to B(0, r_0) \) is a mapping of the normal neighborhood \( U := B(p_0, r_0) \) of the point \( p_0 \) onto the ball \( B(0, r_0) \) in \( \mathbb{R}^n \). Put

\[
\tilde{g}_m(x) = \begin{cases} 
\frac{r_0}{|x|} \left( \frac{1}{m^\alpha} - 1 \right)^{1/\alpha} \cdot x, & \frac{r_0}{2} (1 + 1/(m^\alpha)) \leq |x| \leq r_0, \\
\frac{r_0}{2m} \left( \frac{1}{1/(m^\alpha)} - 1 \right)^{1/\alpha} \cdot x, & 0 < |x| < \frac{r_0}{2} (1 + 1/(m^\alpha)).
\end{cases}
\]

Observe that \( \tilde{g}_m(x) = (f_1 \circ g_m \circ f_2)(x) \), where \( f_2(x) = \frac{2}{r_0} \cdot x \) and \( f_1(x) = r_0 x \). We set

\[
G_m(p) = \begin{cases} 
(\varphi^{-1} \circ \tilde{g}_m \circ \varphi)(p), & p \in U, \\
\varphi(p), & p \in M^n \setminus U,
\end{cases}
\]

see Figure [8]. Applying arguments similar to those used in considering [MRSY] Proposition 6.3, we may show that the mappings \( \tilde{g}_m, m = 1, 2, \ldots \), satisfy the inequality

\[
M(\Gamma_{\tilde{g}_m}(y_0, r_1, r_2)) \leq \int_{A(y_0, r_1, r_2) \cap B(0, r_0)} Q(y) \cdot \eta^n(|y - y_0|) \, dm(y)
\]
for
\[ Q(x) = \frac{C}{|x|^\alpha(n-1)}, \]
where \( C > 0 \) is some constant depending only on \( \alpha \) and \( r_0 \). It can also be established that the function \( Q \) is integrable to the power \( p \) in \( B(0, r_0) \). Let us show that for the mappings \( G_m, m = 1, 2, \ldots, \) and any point \( q_0 \in U \) the inequality
\[ M(\Gamma_{G_m}(q_0, r_1, r_2)) \leq \int_{A(q_0, r_1, r_2) \cap B(p_0, r_0)} \tilde{Q}(p) \cdot \eta^n(d(p, q_0)) \, dv(p), \quad (7.4) \]
holds for some function
\[ \tilde{Q} = \tilde{Q}(p) \]
defined in \( B(p_0, r_0) \). For this purpose, we may use Theorem 5.4 in [IS2], which we apply to the inverse mappings \( F_m = G_m^{-1} \). It is easy to understand that these inverse mappings are calculated by the formula:
\[ F_m(p) = \begin{cases} \varphi^{-1} \circ \tilde{g}_m^{-1} \circ \varphi)(p), & p \in U, \\ p, & p \in \mathbb{M}^n \setminus U, \end{cases} \]
where
\[ \tilde{f}_m(x) = \tilde{g}_m^{-1}(x) = \begin{cases} \frac{\alpha}{2} \cdot \frac{1+|x|}{|x|} \cdot x, & \frac{\alpha}{m} \leq |x| < r_0, \\ \frac{1+(1/m)^n}{2/m} \cdot x, & |x| < \frac{\alpha}{m}. \end{cases} \]
Obviously, the mappings \( G_m \) are differentiable almost everywhere and possess the \( N \) and \( N^{-1} \)-Luzin properties. It can be shown that the mappings \( G_m^{-1} \) are locally Lipschitz and, therefore, belong to the class \( W_{1,n}^{1,\text{loc}} \). Then by [IS2] Theorem 5.4
\[ M(F_m(\Gamma)) \leq \int_D K_I(p, F_m) \cdot \rho^n(p) \, dv(p) \quad (7.5) \]
for any function \( \rho \in \text{adm} \Gamma \) and family of paths \( \Gamma \) in \( D \), where
\[ K_I(p, F_m) = \frac{|J(p, F_m)|}{l^n(p, F_m)}, \]
\[ l(p, F_m) = \liminf_{y \to p} \frac{d(F_m(p), F_m(y))}{d(p, y)}, \quad J(p, F_m) = \limsup_{r \to 0} \frac{v(F_m(B(p, r))))}{v(B(p, r))}. \]
It is seen from the definition that
\[ K_I(p_0, F_m) = K_I(0, \tilde{g}_m^{-1}), \quad (7.6) \]
where
\[ K_I(x, \tilde{g}_m^{-1}) = \frac{|J(x, \tilde{g}_m^{-1})|}{l^n(\tilde{g}_m^{-1}(x))}, \]
\[ J(x, \tilde{g}_m^{-1}) = \det \tilde{g}_m^{-1}(x), \quad l(\tilde{g}_m^{-1}(x)) = \min_{|h|=1} |\tilde{g}_m^{-1}(x)h|. \]
Moreover, since
\[ C_1 \cdot |\varphi(x) - \varphi(y)| \leq d(x, y) \leq C_2 \cdot |\varphi(x) - \varphi(y)|, \]
in the normal neighborhood of \( U \), for any \( x, y \in U \) and some constants \( C_1, C_2 > 0 \), depending only on \( U \), by (7.6) we obtain that
\[ K_I(p, F_m) \leq C \cdot K_I(x, \tilde{g}^{-1}_m), \quad x = \varphi(p), \quad (7.7) \]
where \( C > 0 \) is some constant depending only on \( U \). By reasoning similar to those used in considering [MRSY, Proposition 6.3], we can show that \( K_I(x, \tilde{g}^{-1}_m) \leq \frac{C_3}{|x|^\alpha} \), where \( C_3 > 0 \) is some constant depending only on \( r_0 \) and \( \alpha \). Then, by (7.7) we will have that
\[
\int_U K_I(p, F_m) \, dv(p) \leq C \int_{B(0,r_0)} K_I(x, \tilde{g}^{-1}_m) \, dm(x) \leq C \cdot C_3 \cdot \frac{dm(x)}{|x|^{\alpha(n-1)}} = \omega_{n-1} C C_3 \int_0^{r_0} \frac{dr}{p^{(\alpha-1)(n-1)}} < \infty,
\]
because \((\alpha - 1)(n - 1) < 1\) by the choice of \( \alpha < \frac{1}{n-1} \). It remains to establish that the inequality (7.5) leads to (7.4) for \( \bar{Q}(p) := \frac{C_3}{|x|^\alpha} \), where \( x := \varphi(p) \). Let \( A = A(q_0, r_1, r_2) \) be a ring in \( \mathbb{M}^n \) centered at the point \( q_0 \in U \) and \( \Gamma = \Gamma(S(q_0, r_1), S(q_0, r_2), A(q_0, r_1, r_2)) \), then consider the function \( \rho(p) := \eta(d(p, q_0)) \), where \( \eta \) is an arbitrary Lebesgue measurable function, satisfying the condition \( \int_{r_1}^{r_2} \eta(t) \, dt \geq 1 \), \( \eta(t) = 0 \) for \( t \not\in [r_1, r_2] \). Fix \( \gamma \in \Gamma_* \). Then, by [MRSY, Proposition 13.4]
\[
\int_{r_1}^{r_2} \rho \, ds \geq \int_{r_1}^{r_2} \eta(t) \, dt \geq 1.
\]
By (7.5) it follows that
\[
M(F_m(\Gamma(S(q_0, r_1), S(q_0, r_2), A(q_0, r_1, r_2)))) \leq \int_{D \cap A(q_0, r_1, r_2)} K_I(p, F_m) \cdot \eta^n(d(p, q_0)) \, dv(p) \leq \int_{D \cap A(q_0, r_1, r_2)} \bar{Q}(p) \cdot \eta^n(d(p, q_0)) \, dv(p). \quad (7.8)
\]
But the inequality (7.8) is the relation (7.4), since
\[
F_m(\Gamma(S(q_0, r_1), S(q_0, r_2), A(q_0, r_1, r_2))) = \Gamma_m(q_0, r_1, r_2).
\]
All conditions of the theorems 1.1 or 1.3 are satisfied, and the family of mappings \( G_m, m = 1, 2, \ldots \), satisfy the conclusions of these theorems.

**Example 4.** Finally, let us point out an example of a similar family of mappings with branching acting between Riemannian manifolds. We put
\[
H_m(p) = \begin{cases} (\varphi^{-1} \circ \tilde{g}_m \circ l \circ \varphi)(p), & p \in U, \\ p, & p \in \mathbb{M}^n \setminus U, \end{cases}
\]
where \( l(x) = (r \cos m\varphi, r \sin m\varphi, x_3, \ldots, x_n) \). Notice that

\[
H_m = G_m \circ L,
\]

where \( L = \varphi^{-1} \circ l \circ \varphi \). It can be shown that

\[
L(\Gamma_{H_m}(q_0, r_1, r_2)) \subset \Gamma_{G_m}(q_0, r_1, r_2).
\] (7.9)

Let \( \Gamma \) be a family of paths in \( U \). Then, taking into account that the value of the module \( M(\Gamma) \) of the family of paths \( \Gamma \) in a normal neighborhood of \( U \) is sufficiently close to \( M(\varphi(\Gamma)) \), as well as taking into account (7.3), we will have that

\[
M(\Gamma) \leq K_0 M(L(\Gamma))
\] (7.10)

for some constant \( K_0 > 0 \). Combining (7.9) and (7.10), by (7.4) we obtain that

\[
M(\Gamma_{H_m}(q_0, r_1, r_2)) \leq K_0 \cdot M(L(\Gamma_{H_m}(q_0, r_1, r_2))) \leq K_0 M(\Gamma_{G_m}(q_0, r_1, r_2)) \leq \int_{A(q_0, r_1, r_2) \cap B(p_0, r_0)} \tilde{Q}(p) \cdot \eta^n(d(p, q_0)) \, dv(p).
\]

Hence, the theorems 1.1 and 1.3 may also be applied to the mappings \( H_m, m = 1, 2, \ldots. \). Note that each of the mappings \( H_m \) is a mapping with branching.

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