Existence and uniqueness of the global solution to the Navier-Stokes boundary problem

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Abstract

A proof is given of the global existence and uniqueness of a weak solution to Navier-Stokes boundary problem. The proof is short and essentially self-contained.

1 Introduction

Let $D \subset \mathbb{R}^3$ be a bounded domain with a connected $C^2$--smooth boundary $S$, and $N$ be the unit normal to $S$ pointing out of $D$.

Consider the Navier-Stokes boundary problem:

$$u_t + (u, \nabla)u = -\nabla p + \nu \Delta u + f, \quad x \in D, t \geq 0, \quad (1)$$

$$\nabla \cdot u = 0, \quad (2)$$

$$u|_S = 0, \quad u|_{t=0} = u_0(x). \quad (3)$$

Here $f$ is a given vector-function, $p$ is the pressure, $u = u(x, t)$ is a velocity vector-function, $\nu = \text{const} > 0$ is the viscosity coefficient, $u_0$ is the given initial velocity, $u_t = \partial_t u$, and $\nabla \cdot u_0 = 0$. We assume that $u \in W$, where

$$W := \{u|L^2(0, T; H^1_0(D)) \cap L^\infty(0, T; L^2(D)) \cap u_t \in L^2(D \times [0, T]); \nabla \cdot u = 0\},$$

where $T > 0$ is arbitrary. All functions are assumed real-valued. Let $(u, v) := \int_D u_j v_j dx$ denote the inner product in $L^2(D)$, $\|u\| := (u, u)^{1/2}$. Over the repeated indices summation is understood, $1 \leq j \leq 3$. By $u_{j,i}$ the derivative $\frac{\partial u_j}{\partial x_i}$ is denoted. Equation (2) can be written as $u_{i,i} = 0$ in these notations.

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Let us define a weak solution to problem (1)-(3) as an element of \(W\) which satisfies the identity:

\[
(u_t, v) + (u, v_{x,i}) + \nu(\nabla u, \nabla v) = (f, v), \quad \forall v \in W.
\]

(4)

Here we took into account that \(- (\Delta u, v) = (\nabla u, \nabla v)\) and \((\nabla p, v) = -(p, v_{x,i}) = 0\) if \(v \in H_0^1(D)\). Equation (4) is equivalent to the integrated equation:

\[
\int_0^t [(u_s, v) + (u_x, v_{x,i}) + \nu(\nabla u, \nabla v)] ds = \int_0^t (f, v) ds, \quad \forall v \in W \quad (*).
\]

Equation (4) implies the above equation, and differentiating the above equation with respect to \(t\) one gets equation (4) for almost all \(t \geq 0\).

The aim of this paper is to prove the global existence and uniqueness of the weak solution to the Navier-Stokes boundary problem, that is, solution in \(W\) existing for all \(t \geq 0\). Let us assume that

\[
\sup_{t \geq 0} \int_0^t ||f||^2 ds \leq c, \quad (u_0, u_0) \leq c. \quad (A)
\]

**Theorem 1.** If assumptions (A) hold and \(u_0 \in H_0^1(D)\) satisfies equation (2), then there exists for all \(t > 0\) a solution \(u \in W\) to (4) and this solution is unique in \(W\).

In Section 2 we prove Theorem 1. There is a large literature on Navier-Stokes equations, of which we mention only [1] and [2]. The global existence and uniqueness of the solution to Navier-Stokes boundary problems has not been proved for a long time. The history of this problem see, for example, in [1]. In Section 3 we prove exponential decay of the solution as \(t \to \infty\), see Theorem 2.

## 2 Proof of Theorem 1

**Proof of Theorem 1.** The steps of the proof are: a) derivation of a priori estimates; b) proof of the existence of the solution in \(W\); c) proof of the uniqueness of the solution in \(W\).

a) **Derivation of a priori estimates.**

Take \(v = u\) in (4). Then

\[
(u_x, v_{x,i}) = -(u_x, u_{x,i}) = -\frac{1}{2} (u_x, u^2_x) = -\frac{1}{2} (u_{x,i}, u^2_x) = 0,
\]

and

\[
\frac{1}{2} \partial_t(u, u) + \nu(\nabla u, \nabla u) = (f, u) \leq ||f|| ||u||.
\]

(5)

Let us use the known inequality \(||u|| ||f|| \leq \epsilon ||u||^2 + \frac{1}{2\epsilon} ||f||^2\) with a small \(\epsilon > 0\), and the Poincare inequality \(||\nabla u|| \geq c_p ||u||\), valid for \(u \in H_0^1(D)\), where \(D\) is a bounded domain and \(c_p > 0\) depends on \(D\). Below by \(c > 0\) we denote various estimation constants. One has \(\nu||\nabla u||^2 \geq \nu c_p^2 ||u||^2\), and

\[
\partial_t ||u||^2 + \nu ||\nabla u||^2 \leq \frac{1}{2\epsilon} ||f||^2 + 2\epsilon ||u||^2.
\]
Thus, taking $2\epsilon = \nu c^2_p$ and using the Poincare inequality, one gets

$$\partial_t(u, u) + \nu(\nabla u, \nabla u) \leq \frac{1}{2\epsilon} \|f\|^2.$$  \hfill (6)

Recall that assumptions (A) hold. Integrating (6) over $[0, t]$, $t \in [0, T]$, one obtains:

$$(u(t), u(t)) + \nu \int_0^t (\nabla u(s), \nabla u(s)) ds \leq \frac{1}{2\epsilon} \int_0^t \|f\|^2 ds + (u_0, u_0) := c. \hfill (7)$$

A priori estimates (5)-(7) imply

$$u \in L^\infty(0, T; L^2(D)), \quad u \in L^2(0, T; H^1_0(D)).$$

This and equation (4) imply that $u_t \in L^2(D \times [0, T])$.

b) Proof of the existence of the solution $u \in W$ to (4) and (*)

The idea of the proof is to reduce the problem to the existence of the solution to a Cauchy problem for ordinary differential equations (ODE) of finite order, and then to use a priori estimates to establish convergence of these solutions of ODE to a solution of equations (4) and (*). This idea is not new, see, for example, [1], [2]. Our argument differs from the arguments in the literature, for example, in treating the limit of the term $\int_0^t (u_t^n, v) ds$.

Let us look for a solution to equation (4) of the form $u^n := \sum_{j=1}^n c^{n}_j(t) \phi_j(x)$, where $\{\phi_j\}_{j=1}^\infty$ is an orthonormal basis of the space $L^2(D)$ of divergence-free vector functions belonging to $H^1_0(D)$ and in the expression $u^n$ the upper index $n$ is not a power. If one substitutes $u^n$ into equation (4), takes $v = \phi_m$, and uses the orthonormality of the system $\{\phi_j\}_{j=1}^\infty$ and the relation $(\nabla \phi_j, \nabla \phi_m) = \lambda_m \delta_{jm}$, where $\lambda_m$ are the eigenvalues of the vector Dirichlet Laplacian in $D$ on the divergence-free vector fields, then one gets a system of ODE for the unknown coefficients $c^{n}_j$.

$$\partial_t c^{n}_m + \nu \lambda_m c^{n}_m + \sum_{i,j=1}^n (\phi_i \phi_{j,i}, \phi_m) c^{n}_i c^{n}_j = f_m, \quad c^{n}_m(0) = (u_0, \phi_m).$$

This problem has a unique global solution because of the a priori estimate

$$\sup_{t \geq 0} (u^n, u^n) = \sup_{t \geq 0} \sum_{j=1}^n |c^{n}_j(t)|^2 \leq c.$$
Since equation (*) holds, and the limits of all its terms, except $\int_0^t(u^n_v, v(s))ds$, do exist, then there exists the limit $\int_0^t(u^n_v, v(s))ds \rightarrow \int_0^t(u, v(s))ds$ for all $v \in W$. By passing to the limit $n \rightarrow \infty$ one proves that the limit $u$ satisfies equation (*). Differentiating equation (*) with respect to $t$ gives equation (4) almost everywhere.

c) Proof of the uniqueness of the solution $u \in W$.

Suppose there are two solutions to equation (4), $u$ and $w$, belonging to $W$, and let $z := u - w$. Then

$$\langle z_t, v \rangle + \nu(\nabla z, \nabla v) + \langle u_i u_{j,i} - w_i w_{j,i}, v_j \rangle = 0. \quad (8)$$

Since $z = u - w \in W$, one may set $v = z$ in (8) and get

$$\langle z_t, z \rangle + \nu(\nabla z, \nabla z) + \langle z_i u_{j,i}, z_j \rangle = 0. \quad (9)$$

Note that $\langle z_i u_{j,i}, z_j \rangle = \langle z_i u_{j,i}, z_j \rangle + \langle w_i z_{j,i}, z_j \rangle$, and $\langle w_i z_{j,i}, z_j \rangle = 0$ due to the equation $w_{i,i} = 0$. Thus, equation (9) implies

$$\langle z_t, z \rangle + \nu(\nabla z, \nabla z) + \langle z_i u_{j,i}, z_j \rangle = 0. \quad (10)$$

One has

$$|(z_i u_{j,i}, z_j)| \leq \int_D |z|^2 |\nabla u| dx \leq \|z\|^2_{L^4(D)} \|\nabla u\|. \quad (11)$$

The compactness of the imbedding $i : H_0^1(D) \rightarrow L^4(D)$, where $D$ is a $C^2$-smooth bounded domain in $\mathbb{R}^3$, implies:

$$\|z\|_{L^4(D)} \leq \epsilon \|\nabla z\| + c(\epsilon)\|z\|_{L^2(D)}, \quad (12)$$

and $\epsilon > 0$ is an arbitrary small constant. Thus,

$$\|z\|^2_{L^4(D)} \leq 2\epsilon^2 \|\nabla z\|^2 + 2c^2(\epsilon)\|z\|^2_{L^2(D)}. \quad (13)$$

Choose $2\epsilon^2 = \nu/2$, denote $\phi := \langle z, z \rangle$, take into account that $\|\nabla u\| \leq c$ because $u \in W$, and get from (10)-(13) the inequality

$$\phi' + \frac{1}{2} \nu(\nabla z, \nabla z) \leq c \phi, \quad \phi|_{t=0} = 0, \quad (14)$$

with $c = 2c^2(\epsilon)$. It follows from (14) by a standard argument that $\phi = 0$ for all $t \geq 0$.

Theorem 1 is proved. \hfill \blacksquare

In [1] it is shown that he smoothness properties of the solution $u$ are improved when the smoothness properties of $f$, $u_0$ and $S$ are improved.

Under suitable assumptions one can prove exponential decay of $u$ as $t \rightarrow \infty$, see the next Section.
3 Time decay of the solution

Let us give the rate of the time decay of the solution using the following result.

**Lemma 1.** Suppose that \( g(t) \geq 0, \ g' + 2ag \leq 2b(t)g^{1/2}, \ a = \text{const} > 0, \ b(t) \geq 0 \) is a locally integrable function of \( t \in [0, \infty) \). Then the following inequality holds:

\[
g(t) \leq 2e^{-2at}g(0) + 2e^{-2at}\left(\int_0^t e^{as}b(s)ds\right)^2.
\]

(15)

**Proof.** Let \( h := e^{2at}g \). Then \( h(0) = g(0) \) and \( h' \leq 2b(t)e^{at}h^{1/2} \). Thus,

\[
h^{1/2}(t) - h^{1/2}(0) \leq \int_0^t b(s)e^{as}ds.
\]

Since \( g(0) = h(0) \), one gets

\[
h(t) \leq 2g(0) + 2\left(\int_0^t b(s)e^{as}ds\right)^2.
\]

This and the relation \( g := e^{-2at}h \) imply that inequality (15) holds. \( \square \)

To use Lemma 1, one starts with inequality (5), denotes \( g := (u, u) \), uses the estimate \( (\nabla u, \nabla u) \geq c_p^2g \), denotes \( a = \nu c_p^2 \), assumes that \( b(t) := \|f\| \), and gets inequality (15) with \( g = (u, u) \) and \( b(t) := \|f\| \).

Let us formulate the result.

**Theorem 2.** Assume that \( \int_0^t e^{as}\|f\|(s)ds \leq ce^{kt}, \) where \( k < a := \nu c_p^2 \). Then the solution \( u \) to problem (1)-(3) decays exponentially fast as \( t \to 0 \).

References

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[2] R. Temam, Navier-Stokes equations. Theory and numerical analysis, North Holland, Amsterdam, 1984.