Barrett-Crane spin foam model
from generalized BF-type action for gravity

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We study a generalized action for gravity as a constrained BF theory, and its relationship with the
Plebanski action. We analyse the discretization of the constraints and the spin foam quantization
of the theory, showing that it leads naturally to the Barrett-Crane spin foam model for quantum
gravity. Our analysis holds true in both the Euclidean and Lorentzian formulation.

I. INTRODUCTION

Spin foam models [1] [2] are an attempt to formulate a non-perturbative and background-independent theory of
quantum gravity, and can be seen as a new way to construct a sum-over-geometries out of algebra and combinatorics,
with the geometrical quantities emerging in the semiclassical limit only. With a striking convergence of results and
ideas, spin foam models emerged [3] also in contexts as different as canonical loop quantum gravity [4] and discrete
topological field theory [5]. The most studied and developed of these is the Barrett-Crane model for both Euclidean [6]
and Lorentzian [7] quantum gravity. It was first obtained as a quantization of simplicial geometry, using the methods
of category theory and geometric quantization. It was then re-derived using a field theory over a group manifold
[8] [9] [10] [11] [12] [13], or generalized matrix model. And finally, it was shown to arise from a discretization and
quantization of BF theory [14].

Recently, a new action for gravity as a constrained BF theory was proposed [15] and it seemed that a discretization
and spin foam quantization of it would lead to a model necessarily different from the Barrett-Crane one. Also, a
natural outcome would be a one-parameter ambiguity in the corresponding spin foam model, related to the Immirzi
parameter of loop quantum gravity [16] [17], and this gives an additional reason to study the spin foam quantization
of the generalized action since it could help understand the link between the current spin foam models and loop
quantum gravity.

We show here that a careful discretization of the new form of the constraints, an analysis of the field content of
the theory, at the classical level, and a spin foam quantization taking all this into account, lead naturally to the
Barrett-Crane model as a quantum theory corresponding to that action. No one-parameter ambiguity arises in the
spin foam model and in the quantum geometry described by it. This suggests that the Barrett-Crane model is more
universal than at first thought and that its continuum limit may be described by several different lagrangians. Finally,
in section VII, we also briefly discuss the issue of the area spectrum in the spin foam framework.

II. THE BARRETT-CRANE MODEL

First of all, we recall the basic elements of the Barrett-Crane model [1] [4]. Consider a geometric 4-simplex, i.e. the
convex hull of 5 points in $\mathbb{R}^4$. It determines and is uniquely characterized (up to parallel translation and inversion
through the origin) by a set of 10 bivectors $b_i \in \mathbb{R}^4 \wedge \mathbb{R}^4$, assigned one to each of its triangles, satisfying the following properties:

- A different orientation of the triangle corresponds to a bivector with opposite sign;
the bivectors assigned to the triangles are simple, i.e. given by a wedge product of two vectors;
• two bivectors corresponding to two triangles sharing an edge add to a simple bivector;
• the (oriented) sum of the 4 bivectors assigned to the 4 triangles of a tetrahedron is zero;
• the six bivectors corresponding to six triangles sharing a vertex are linearly independent;
• given 3 triangles meeting at a vertex of a tetrahedron, the volume spanned by the 3 corresponding bivectors
must be positive i.e the bivectors (considered as operators by means of the metric) satisfy: $Tr b_1[b_2, b_3] > 0$.

For quantum 4-simplices, we deal with possibly degenerate 4-simplices, so we drop the linear independence (condition 5) and allow for zero volume in the last condition ($\geq 0$ instead of $> 0$). Now we can proceed to the quantization. We associate to each triangle an element of the Lie algebra of the local gauge group (so(4) in the Euclidean case, and so(3,1) in the Lorentzian) using the isomorphism between bivectors and Lie algebra elements, and then turn them into operators choosing a representation, so that we obtain bivector operators acting on the Hilbert space given by the representation space chosen. To each tetrahedron is then associated an element of the tensor product of the 4 Hilbert spaces associated to its faces. To characterize completely the quantum geometry of the 4-simplex the chosen representations have to satisfy a quantum analog of the conditions above:

• the representations corresponding to different orientations of the same triangle are dual to each other;
• the representations used are simple representations, i.e. characterized by a vanishing second casimir of the
algebra;
• given two faces of a tetrahedron, we can decompose the corresponding pair of representations into its Clebsch-
Gordon series; then the tensor for the tetrahedron have to decompose into summands given by simple representa-
tions only;
• the tensor for the tetrahedron is invariant under the local gauge group.

In the Euclidean case a bivector is simple when its selfdual and anti-selfdual parts have the same magnitude. The splitting into selfdual and anti-selfdual parts corresponds to the splitting of the $Spin(4)$ algebra (universal covering of so(4)) into a sum of two su(2) algebra, so that the irreducible representations are given by pairs of spins $(j^+, j^-)$. The simplicity of these representations corresponds to the vanishing of the second casimir, $\epsilon_{IJKL}^{+1} J^{IJKL}$ given in the canonical basis, $C_2 = j^+(j^+ + 1) - j^-(j^+ + 1)$. This implies that $j^+ = j^-$ so that the representations to be used are of the type $(j, j)$. In the Lorentzian case, the same splitting is possible only through complexification of the fields. Nevertheless, we can work with the principal series of irreducible unitary representations of so(3,1) [12] [13]. They are labelled by a half-integer number $j$ and a positive real number $\rho$. The second casimir of so(3,1) is given by $C_2 = \frac{1}{2} j \rho$ so that simplicity implies that we restrict ourselves to using representations corresponding to the two series $(j, 0)$ and $(0, \rho)$.

With these conditions, an amplitude for a 4-simplex can be obtained (as the evaluation of a spin network), and one can characterize completely the 4-geometry at the quantum level [14] [15]. This model, both in its Euclidean version and in its Lorentzian formulation, is shown to be finite given a fixed triangulation (we sum over all possible colorings of the faces) and therefore the spin foam model is well-defined [16] [17] [18]. We don’t give here the details of this construction for which we refer to the literature. Instead we think is useful for our purposes to describe how the association of bivectors to the triangles of a 4-simplex can be naturally made, in a gravitational context. Suppose we have a tetrad field $e: TM_p \rightarrow \mathbb{R}^4$, where $M$ is our spacetime manifold, so that $e \wedge e: \wedge^2(TM_p) \rightarrow \wedge^2(\mathbb{R}^4)$, mapping any wedge product of vectors $u_1 \wedge u_2$ into a bivector $e(u_1) \wedge e(u_2)$. Then the bivector $b_T$ associated to the triangle $T$ is naturally given by: $b_T = \int_T e \wedge e = e(u) \wedge e(v)$, where $u$ and $v$ are the vectors corresponding to two edges of the triangle.

The Barrett-Crane model was argued to be a quantization of the Plebanski action [19] for gravity in [20], and this
conclusion is also supported by the results of [21]. The Plebanski action is a a BF-type action, in the sense that it
gives gravity as a constrained BF theory, with quadratic constraints on the B field (we note that such a formulation
of general relativity has been generalised to any dimension [22]). More precisely the action is given by:

$$S = S(\omega, B, \phi) = \int_M \left[ B^{IJ} \wedge F_{IJ}(\omega) - \frac{1}{2} \phi_{IJKL} B^{KL} \wedge B^{IJ} \right]$$  

(1)
where \( \omega \) is a connection 1-form valued in \( so(4) \) (\( so(3,1) \)), \( \omega = \sum_{I,J} J_{IJ} dx^a \), \( J_{IJ} \) are the generators of \( so(4) \) (\( so(3,1) \), \( F = dw \) is the corresponding two-form curvature, \( B \) is a 2-form also valued in \( so(4) \) (\( so(3,1) \)), \( B = B_{abcd} J_{IJ} dx^a \wedge dx^b \), and \( \phi_{IJKL} \) is a Lagrange multiplier satisfying \( \phi_{IJKL} \epsilon^{IJKL} = 0 \). Here and in the following \( a, b, .. \) are spacetime indices and \( I, J, K, ... \) are internal indices. The equations of motion are:

\[
dB + [\omega, B] = 0 \quad F^{I\bar{J}}(\omega) = \phi^{IJKL} B_{KL} \quad B^{I\bar{J}} \wedge B^{KL} = e^{IJKL}
\]

where \( e = \frac{1}{4} \epsilon^{IJKL} B^{I\bar{J}} \wedge B^{KL} \). When \( e \neq 0 \), the constraint (2) is equivalent to \( \epsilon^{IJKL} B_{ab} B^{KL} = \epsilon_{abcd}e \) [20], which implies that \( \epsilon^{IJKL} B_{ab} B^{KL} = 0 \). \( B_{ab} \) is a simple bivector. Moreover, (2) is satisfied if and only if there exists a real tetrad field \( e^I = e_0^I dx^a \) so that one of the following equations holds:

\[
I \quad B^{I\bar{J}} = \pm e^I \wedge e^J \\
II \quad B^{I\bar{J}} = \pm \frac{1}{2} \epsilon^{I\bar{J}KL} e^K \wedge e^L.
\]

Restricting the field \( B \) to be always in the sector \( II_+ \) (which is always possible classically), the action becomes:

\[
S = \int_M \epsilon^{IJKL} e^I \wedge e^J \wedge F^{KL}
\]

which is the action for general relativity in the first order Palatini formalism. Then a discretization of the constraints [2] giving a bivector field from the \( B \) field for each triangle, shows [20] that they correspond exactly to the conditions given above for the bivectors characterizing a 4-simplex, and this leads to the conclusion that the Barrett-Crane model gives the quantization of the Plebanski action.

More precisely, going from the field theory to the Barrett-Crane model is achieved in 3 main steps. First, it is the \textit{discretization} of the two-form into bivectors associated to each face to the triangulation. Then, we translate bivectors as elements of \( so(4)^* \) or \( so(3,1)^* \), as described in [18], using the function

\[
\theta : \Lambda^2 \mathbb{R}^4 \to so(4)^* \text{ or } so(3,1)^* \\
e \wedge f \to \theta(e \wedge f)(l) \to \eta(e, f)
\]

where \( \eta \) is the Euclidean or Lorentzian metric. Less formally, it is the step we call \textit{correspondence} between bivectors \( B \) and elements of the Lie algebra \( J \). The last step is the \textit{quantization} itself, using techniques from geometric quantization. This gives representation labels to the faces of the triangulation and gives the Barrett-Crane model (to some normalisation factors).

In fact, there is an ambiguity at the level of the correspondence. We can also choose to use the isomorphism \( \theta o^* \) where \( ^* \) is the Hodge operator. This leads to the so-called flipped Poisson bracket, and it is indeed the right thing to do. In the Euclidean case, this leads us to only “real” tetrahedra, whose faces are given by the bivectors and not the Hodge dual of the bivectors. This amounts to selecting the sector \( II \) of the theory which is the sector we want [18]. In the Lorentzian case, such a check on the tetrahedra hasn’t been done yet, however the use of the flipped correspondence has a nice consequence: it changes the sign of the area to being given by \(-C_1\) instead of \(C_1\), so that the discrete series of representations \((n,0)\) truly correspond to time-like faces and the continuous series \((0,\rho)\) to space-like faces, as implied by the algebraic properties of these representations.

Nevertheless, we can generalize this correspondence. We have a family of such isomorphisms given by \( \theta o(\alpha + \beta) \). For \( \alpha \) and \( \beta \) different from 0, it wouldn’t give anything interesting when dealing with the Barrett-Crane conditions. However, it is this generalized correspondence we are going to use to deal with the generalized BF-type action. And, at the end, we will find again the same Barrett-Crane model.

### III. GENERALIZED BF-TYPE ACTION FOR GRAVITY

In [14], the following BF-type action was proposed for general relativity:

\[
S = \int B^{I\bar{J}} \wedge F_{IJ} - \frac{1}{2} \phi_{IJKL} B^{IJ} \wedge B^{KL} + \mu H
\]

where \( H = a_1 \phi_{IJKL}^{IJ} + a_2 \phi_{IJKL} \epsilon^{IJKL} \), where \( a_1 \) and \( a_2 \) are arbitrary constants. \( B \) is a 2-form and \( F \) is the curvature associated to the connection \( \omega \). \( \phi \) (spacetime scalar) and \( \mu \) (spacetime 4-form) are Lagrange multipliers, with \( \phi \) having
The equations of motion for $\omega$ now are:

\[ B^{IJ} \wedge B^{KL} = \frac{1}{6}(B^{MN} \wedge B^{MN})\eta^{IJKL} + \frac{\epsilon}{12}(B^{MN} \wedge *B_{MN})e^{IJKL} \tag{8} \]

\[ 2a_2 B^{IJ} \wedge B_{IJ} - e a_1 B^{IJ} \wedge *B_{IJ} = 0 \tag{9} \]

The solution of these constraints \[22\], for non-degenerate $B$ ($B^{IJ} \wedge *B_{IJ} \neq 0$), is:

\[ B^{IJ} = \alpha* (e^I \wedge e^J) + \beta e^I \wedge e^J \tag{10} \]

with:

\[ \frac{a_2}{a_1} = \frac{\alpha^2 + \epsilon \beta^2}{4\alpha \beta} \tag{11} \]

Inserting this solution into \[1\], we get:

\[ S = \alpha \int * (e^I \wedge e^J) \wedge F_{IJ} + \beta \int e^I \wedge e^J \wedge F_{IJ} \tag{12} \]

so that there is a coupling between the geometric sector given by $*(e \wedge e)$ (general relativity) and the non-geometric one given by $e \wedge e$. Nevertheless, we note that the second term vanishes on shell so that the equations of motion ignore the non-geometric part and are still given by the Einstein equations.

In the usually studied case $a_1 = 0$, \[24\] \[21\], we are back to the Plebanski action, the sectors of solutions being given by $\alpha = 0$ and $\beta = 0$ so that we have either the general relativity sector or the non-geometric sector $e \wedge e$. In the particular case $a_2 = 0$, in the Euclidean case, the only solution to \[11\] is $\alpha = \beta = 0$ so that only degenerate tetrads are going to contribute. On the other hand, in the Lorentzian case we have instead $\alpha = \pm \beta$.

Looking at \[11\], once we have chosen a couple $(\alpha, \beta)$, we see that we have four possible sectors as with the Plebanski action \[21\]. In the Euclidean case, we can exchange $\alpha$ and $\beta$. Under this transformation, the $B$ field gets changed into its Hodge dual, so we can trace back this symmetry to the fact that we can use both $B$ and $*B$ as field variables in our original action, without any change in the physical content of the theory. We can also change $B \rightarrow -B$ without affecting the physics of our model. This gives us the following four sectors:

\[ (\alpha, \beta) \ (\alpha, -\beta) \ (\beta, \alpha) \ (-\beta, -\alpha) \tag{13} \]

In the Lorentzian case, the same $*$-symmetry brings us the following four sectors:

\[ (\alpha, \beta) \ (\beta, -\alpha) \ (-\alpha, -\beta) \ (-\beta, \alpha) \tag{14} \]

The canonical analysis of the action \[12\] was performed in \[23\], leading to the presence of the Immirzi parameter of loop quantum gravity given by $\gamma = \alpha/\beta$ and related to $a_1$ and $a_2$ by:
\[ \frac{a_2}{a_1} = \frac{1}{4} \left( \gamma + \frac{\epsilon}{\gamma} \right) \]  

(15)

We can notice that we have two sectors in our theory with different Immirzi parameters: \( \gamma \) and \( \epsilon/\gamma \), corresponding to a symmetry exchanging \( \alpha \) and \( \epsilon \beta \).

The full symmetry group of the theory is then \( Diff(M) \times SO(4) \times Z_2 \times Z_2 \), with \( SO(4) \) replaced by \( SO(3,1) \) in the Lorentzian case. The \( Z_2 \times Z_2 \) comes from the existence of the four sectors of solutions and is responsible for their interferences at the quantum level.

We want to study the relationship between this new action and the usual Plebanski action, and its spin foam quantization, in order to understand if and in which cases the corresponding spin foam model at the quantum level is still given by the Barrett-Crane one. To this aim it is important to note that the constraints (8) and (9) can be recast in an equivalent form (see in appendix for more details) leading to the same set of solutions, for \( a_2 \neq 0 \), \( B \) non-degenerate, and \( \left( \frac{a_1}{2a_2} \right)^2 \neq \epsilon \) (this excludes the purely selfdual and the purely anti-selfdual cases). The situation is analogous to that analyzed in [20] for the Plebanski action. The new constraint is

\[ \left( \epsilon_{IJKM} - \frac{a_1}{a_2} \eta_{I|M} \eta_{J|N} \right) B^{MN}_{cd} B^{IJ}_{ab} = e \epsilon_{abcd} \left( 1 - \epsilon \left( \frac{a_1}{2a_2} \right)^2 \right) \]  

(16)

where

\[ e = \frac{1}{4! \epsilon_{IJKL}} B^{IJ} \wedge B^{KL} \]  

(17)

This constraint can then be discretised to give the simplicity constraint and the intersection constraint leading to the Barrett-Crane model, as we will see in section (IV). But we can already notice that in the case \( (ab) = (cd) \), (18) gives an equivalent to (8) for the Plebanski action.

In the following, we mainly use this second form of the constraints, discussing the discretization and possible spin foam quantization of the first one in section (VI).

Looking at the constraints on \( B \) it is apparent that they are not anymore just simplicity constraints like in the Plebanski case, so that a direct discretization of them for \( B \) would not give the Barrett-Crane constraints. We can see this by translating the constraints into a condition on the Casimirs of \( so(4) \) (or \( so(3,1) \)), using the isomorphism between bivectors and Lie algebra. We can naively replace \( B^{IJ}_{ab} \) with the canonical generators \( J^{IJ} \) of \( so(4) \) (or \( so(3,1) \)), giving the correspondence:

\[ B^{IJ}_{ab} B^{Ib}_{J} = 2 C_1 \]  

(19)

\[ \frac{1}{2} \epsilon_{IJKL} B^{IJ}_{ab} B^{KL}_{ab} \rightarrow \frac{1}{2} \epsilon_{IJKL} J^{IJ} J^{KL} = 2 C_2 \]  

(20)

Using this, (18) gets transformed into:

\[ 2a_2 C_2 - a_1 C_1 = 0 \]  

(21)

or equivalently:

\[ 2\alpha \beta C_1 = (\alpha^2 + \beta^2) C_2 \]  

(22)

Then we would conclude as suggested in [15] that we should use non-simple representations in our spin foam model. Actually the situation would be even worse than this, since it happens that in general (for arbitrary values of \( a_1 \) and \( a_2 \)) no spin foam model can be constructed using only representations satisfying (21) in the Euclidean case. More precisely, using the splitting of the algebra \( so(4) \simeq su(2)_+ \oplus su(2)_- \), the two Casimirs are:

\[ C_1 = j^+(j^++1) + j^-(j^-+1) \]

\[ C_2 = j^+(j^++1) - j^-(j^-+1) \]  

(23)
Apart from the case $a_1 = 0$ which gives us the Barrett-Crane simplicity constraint $C_2 = 0$, the equation (21) have an infinite number of solutions only when $2a_2 = \pm a_1$, in which cases we get representations of the form $(j^+, 0)$ (or $(0, j^-)$).

This could be expected since the constraint (9) with this particular value of the parameters implies that the $B$ field is selfdual or anti-selfdual when non-degenerate. In the other cases, (21) can be written as $j^+(j^+ + 1) = \lambda j^-(j^- + 1)$, with $2a_2 = (1 + \lambda)/(1 - \lambda)a_1$ and in general has no solutions. For particular values of $\lambda$, it can have one and only one solution. This would lead us to an ill-defined spin foam model using only one representation $(j^+, j^-)$. However, using the framework set up in [29], it can be easily proven that it is not possible to construct an intertwiner for a spin network built out of a single representation (a single representation is not stable under change of tree expansion for the vertex) so that no spin foam model can be created.

In the Lorentzian case, the situation is more complex. The representations are labelled by couples $(j \in \mathbb{N}/2, \rho \geq 0)$ and the two Casimirs are [12] [28]:

$$C_1 = j^2 - \rho^2 - 1$$
$$C_2 = \frac{1}{2}j\rho$$

The equation (21) now reads:

$$\rho^2 + \frac{a_2}{a_1}j\rho - j^2 + 1 = 0$$

and admits the following solutions if $a_1 \neq 0$:

$$\rho = \frac{1}{2} \left[ -\frac{a_2}{a_1}j \pm \sqrt{\left(\frac{a_2}{a_1}\right)^2 j^2 + 4(j^2 - 1)} \right]$$

So we always have some solutions to the mixed simplicity condition (21). However, instead of having a discrete series of representations $(n, 0)$ and a continuous series $(0, \rho)$, which are said to correspond to space-like and time-like degrees of freedom, as in the simple case $a_1 = 0$, we end up with a (couple of) discrete series representations with no direct interpretation. We think it is not possible to construct a consistent vertex using them (it is also hard to imagine how to construct a field theory over a group manifold formulation of such a theory whereas, in the case of simple representations, the construction was rather straightforward [12] [13] adapting the Euclidean case to the Lorentzian case). But this should be investigated.

As we will show, on the contrary, a careful analysis of the field content of the theory, and of the correspondence between Lie algebra elements and bivectors shows that not only a spin foam quantization is possible, but that the resulting spin foam model should again be based on the Barrett-Crane quantum constraints on the representations.

**IV. FIELD CONTENT AND RELATIONSHIP WITH THE PLEBANSKI ACTION**

We have seen that the $B$ field, even if subject to constraints which are more complicated than the Plebanski constraints, is forced by them to be in 1-1 correspondence with the 2-form built out of the tetrad field, $*(e \wedge e)$, which in turn is to be considered the truly physical field of interest, since it gives the geometry of the manifold through the Einstein equations. To put it in another way, we can argue that the physical content of the theory, expressed by the Einstein equations, is independent of the fields we use to derive it. In a discretized context, in particular, we know that the geometry of the manifold is captured by bivectors associated to 2-dimensional simplices, and constrained to be simple. Consequently we would expect that in this context we should be able to put the $B$ field in 1-1 correspondence with another 2-form field, say $E$, then discretized to give a bivector for each triangle, in such a way that the mixed constraints (18) would imply the simplicity of this new bivector and the other Barrett-Crane constraints. If this happens, then it would also mean that the action is equivalent to the Plebanski action in terms of this new 2-form field, at least for what concerns the constraints. This is exactly the case, as we are going to prove.

The correspondence between $B$ and $E$ is actually suggested by the form of the solution [10]. We take

$$B^{IJ}_{ab} = (\alpha I + \epsilon \beta \ast) E^{IJ}_{ab}$$

This is an invertible transformation, so that it really gives a 1-1 correspondence, if and only if $\alpha^2 - \epsilon \beta^2 \neq 0$, which we will assume to be the case in the following.
Formally, we can do such a change of variable directly on the action and express the action itself in terms of the field $E$, making apparent that the constraints are just the Plebanski constraints:

$$
\begin{align*}
B^{IJ} &= \alpha E^{IJ} + \epsilon \beta \ast E^{IJ} \\
\hat{\phi}_{IJKL} &= (\alpha + \epsilon \beta \frac{1}{2} \epsilon_{IJ}^{AB} \phi_{ABCD}(\alpha + \epsilon \beta \frac{1}{2} \epsilon_{KL}^{CD})
\end{align*}
$$

After this change, the action (2) becomes:

$$
S = \frac{1}{|\alpha^2 - \epsilon \beta|^2} \int (\alpha E^{IJ} + \epsilon \beta \ast E^{IJ}) \wedge F_{IJ} - \frac{1}{2} \hat{\phi}_{IJKL} E^{IJ} \wedge E^{KL} + \mu \epsilon^{IJKL} \hat{\phi}_{IJKL}
$$

so that we have the Plebanski constraints on the $E$ field and we can derive directly from this expression the Holst action (12). In the Euclidean case, decomposing into selfdual and anti-selfdual components can be quite useful to understand the structure of the theory. This is done in the appendix and shows that the previous change of variable is simply a rescaling of the selfdual and anti-selfdual parts of the $B$ field.

Let’s now look at the discretization of the constraints. We will follow the same procedure as in [20]. Using the 2-form $B$, a bivector can be associated to each triangle of the triangulation using the procedure above. This is equivalent to supposing $E$ constant inside each 4-simplex, i.e. $dB = 0$, and we associate a bivector to each triangle of the triangulation using the procedure above. This is equivalent to supposing $E$ constant in the 4-simplex since the map (27) is invertible. Then we can use Stokes’ theorem to prove that the sum of all the bivectors $E$ associated to the 4 faces $t$ of a tetrahedron $T$ is:

$$
0 = \int_T dE = \int_{\partial T} E = \int_{t_1} E + \int_{t_2} E + \int_{t_3} E + \int_{t_4} E = E(t_1) + E(t_2) + E(t_3) + E(t_4)
$$

meaning that the bivectors $E$ satisfy the fourth Barrett-Crane constraints (closure constraint).

Let’s consider the constraints in the form (16). Using (22), we obtain a much simpler constraint on the $E$ field:

$$
(\alpha^2 + \epsilon \beta^2) \epsilon^{IJKL} E^{IJ} E^{KL} = \epsilon_{abcd}
$$

Now, $e = \frac{1}{4!} \epsilon^{IJKL} B^{IJ} \wedge B^{KL} = \frac{1}{4!} (\alpha^2 + \epsilon \beta^2) \epsilon^{IJKL} E^{IJ} \wedge E^{KL}$ is a sensible volume element, since we assumed that $B$ is non-degenerate.

After imposing the equation of motion, it appears that it is also the “right” geometric volume element, i.e. the one constructed out of the tetrad field. More precisely, equation (32) implies the simplicity of the field $E$. Thus there exist a tetrad field such that $E^{IJ} = \pm e^{IJ}$ or $E^{IJ} = \pm \ast (e^{IJ} \wedge e^{KL})$ and this tetrad field is the one defining the metric after imposing the Einstein equation. The scalar $e$ is then proportional to $\epsilon^{IJKL} e^{IJ} \wedge e^{KL} = det(e)$. Consequently, up to a factor, the 4-volume spanned by two faces $t$ and $t'$ of a 4-simplex is given by:

$$
V(t, t') = \int_{x \in t; y \in t'} e \epsilon_{abcd} dx^a \wedge dx^b \wedge dy^c \wedge dy^d
$$

Then, integrating equation (24) gives directly:

$$
\epsilon^{IJKL} E^{IJ}(t) E^{KL}(t') = \frac{1}{(\alpha^2 + \epsilon \beta^2)} V(t, t')
$$

Considering only one triangle $t$:

$$
\epsilon^{IJKL} E^{IJ}(t) E^{KL}(t) = 0
$$

so that for any of the bivectors $E$ the selfdual part has the same magnitude of the anti-selfdual part, So that the $E(t)$ are simple bivectors. This corresponds to the second of the Barrett-Crane constraints (simplicity constraint). For two triangles sharing an edge, we similarly have:
\[ \epsilon_{IJKL} E^{IJ}(t) E^{KL}(t') = 0 \]  
\[ \epsilon_{IJKL}(E^{IJ}(t) + E^{IJ}(t'))(E^{KL}(t) + E^{KL}(t')) - \epsilon_{IJKL} E^{IJ}(t) E^{KL}(t) - \epsilon_{IJKL} E^{IJ}(t') E^{KL}(t') = 0 \]

This can be rewritten as:

\[ \epsilon_{IJKL} E^{IJ}(t) E^{KL}(t') = 0 \]

and this, together with the simplicity constraint, implies that the sum of the two bivectors associated to the two triangles is again a simple bivector. This implements the third of the Barrett-Crane constraints (intersection constraint).

Let’s note that in the Euclidean case, we can use the decomposition into selfdual and anti-selfdual components to write the previous constraints (for \( t = t' \) or \( t \) and \( t' \) sharing an edge) as:

\[ \delta_{IJ} \left[ E^{(+)}(t)E^{(+)}(t') - E^{(-)}(t)E^{(-)}(t') \right] = 0 \]

thus showing that the simplicity for the bivectors \( E(t) \) and \( E(t) + E(t') \) is that their selfdual part and anti-self part have the same magnitude. We note that in the Lorentzian case this decomposition implies a complexification of the fields, so the physical interpretation is somehow more problematic. However it presents no problems formally, and corresponds to the splitting of the Lie algebra of \( so(3,1) \), to which the bivectors in Minkowski space are isomorphic, into \( su(2)_C \oplus su(2)_C \).

Thus, it is clear that the complicated constraints \( E \) and \( E' \) for the field \( B \) are just the Plebanski constraint for the field \( E \), associated to \( B \) by means of the transformation \( (27) \), and, when discretized, are exactly the Barrett-Crane constraints. The fields characterizing the 4-geometry of the triangulated manifold are then the bivectors \( E(t) \).

This result, by itself, does not imply necessarily that a spin foam quantization of the generalized BF-type action gives the Barrett-Crane model, but it means anyway that the geometry is still captured by a field which, when discretized, gives a set of bivectors satisfying the Barrett-Crane constraints. This in turn suggests strongly that the Barrett-Crane constraints characterize the quantum geometry also in this case, even if a first look at the constraints seemed to contradict this, and consequently the simple representations are the right representations of \( so(4) \) and \( so(3,1) \) that have to be used in constructing the spin foam.

V. SPIN FOAM QUANTIZATION AND CONSTRAINTS ON THE REPRESENTATIONS

We have already proven that the spin foam quantization cannot be performed using a naive association between the \( B \) field and the canonical generators of the Lie algebra. On the other hand, we have seen that the \( B \) field can be put in correspondence with a field \( E \) such that the constraints on this are the Barrett-Crane constraints. This suggest that a similar transformation between Lie algebra elements would make everything work again, giving again the simplicity conditions for the representations, as in the Barrett-Crane model.

In light of the discussion in section \( (10) \) on the natural way to associate a bivector to a triangle in a gravitational context, using the frame field, and also because the 2-forms \(* (e^I \wedge e^J)\) are a basis for the space of 2-forms, we could argue that it is the bivector coming from \(* (e \wedge e)\) that has to be associated to the canonical generators of the Lie algebra.

That this is the right choice can be proven easily. In fact the isomorphism between bivectors and Lie algebra elements is realized choosing a basis for the bivectors such that they are represented by 4 by 4 antisymmetric matrices, and interpreting these matrices as being the 4-dimensional representation of Lie algebra elements. If this is done for the basis 2-forms \(* (e \wedge e)\), the resulting matrices give exactly the canonical generators of \( so(4) \) or \( so(3,1) \), so that \(* (e \wedge e) \leftrightarrow J\).

Then equation \( (10) \) suggests us that the field \( B \) has to be associated to elements \( \tilde{J} \) of the Lie algebra such that:

\[ \tilde{B} \leftrightarrow \tilde{J}^{IJ} = \alpha J^{IJ} + \epsilon \beta * J^{IJ} \]

This is simply a change of basis (but not a Lorentz rotation), since the transformation is invertible, provided that \( \alpha^2 - \epsilon \beta^2 \neq 0 \). In some sense, we can say that working with the \( B \) in the action is like working with a non-canonical basis in the Lie algebra, the canonical basis being instead associated to \(* (e \wedge e)\).

Now we consider the constraint \( (18) \) on the \( B \) field, and use the correspondence above to translate it into a constraint on the representations of the Lie algebra. The Casimir corresponding to \( \frac{1}{2} \epsilon_{IJKL} B^{IJ}_{ab} B^{abKL} \) is

\[ \tilde{C}_2 = \frac{1}{2} \epsilon_{IJKL} \tilde{J}^{IJ} \tilde{J}^{KL} = 2 \alpha \beta C_1 + (\alpha^2 + \epsilon \beta^2) C_2 \]
where $C_1$ and $C_2$ are the usual Casimirs associated to the canonical generators $J^{IJ}$ (being $C_1 = j^+ (j^+ + 1) + j^- (j^- + 1)$ and $C_2 = j^+ (j^+ + 1) - j^- (j^- + 1)$) in the Euclidean case, and $C_1 = j^2 - \rho^2 - 1$ and $C_2 = \frac{1}{2} j \rho$ in the Lorentzian case), while the Casimir associated to $B^{ab}_{IJ} B^{ab}_{IJ}$ is:

$$\tilde{C}_1 = J^{IJ} \tilde{J}_{IJ} = (\alpha^2 + \epsilon \beta^2) C_1 + 2 \epsilon \alpha \beta C_2$$

Substituting these expressions in (18), we get:

$$2 \alpha \beta \tilde{C}_1 = (\alpha^2 + \epsilon \beta^2) \tilde{C}_2 \Rightarrow (\alpha^2 - \epsilon \beta^2)^2 C_2 = 0$$

In the assumed case $\alpha^2 \neq \epsilon \beta^2$, we find the usual Barrett-Crane simplicity condition $C_2 = 0$ with a restriction to the simple representations of so(4) $(j^+ = j^-)$ or so(3,1) ($n = 0$ or $\rho = 0$). We see that, at least for what concerns the representations to be used in the spin foam model, the whole modification of the initial action is absorbed by a suitable redefinition of the correspondence between the field $B$ and the generators of the Lie algebra.

Now we want to discuss briefly how general is the association we used between the $B$ field and Lie algebra elements, i.e. how many other choices would give still the Barrett-Crane simplicity constraint on the representations starting from the constraint (15) on the $B$ field. Suppose we associate to $B$ a generic element $\tilde{J}$ of the Lie algebra, related to the canonical basis by a generic invertible transformation $\tilde{J} = \Omega J$. Any such transformation can be split into $\tilde{J}^{IJ} = \Omega^{IJ}_{KL} J^{KL} = (\alpha I + \epsilon \beta *)_{MN} U^{IJ}_{KL} J^{KL} = (\alpha I + \epsilon \beta *)_{MN} J^{IJ}$, for $\alpha^2 \neq \epsilon \beta^2$, so shifting all the ambiguity into $U$, inserting this into the constraints we obtain the condition $C'_2 = 0$, where $C'_2 = J^* J'$. If we now require that this transformation should still give the Barrett-Crane simplicity constraint $C_2 = 0$, then this amount to require $C'_2 = \lambda C_2$ for a generic $\lambda$. But if $C_2 = 0$ and the transformation preserves the second Casimir modulo rescaling, then it should preserve, modulo rescaling, also the first one. This means that the transformation $U$ preserves, modulo rescaling, the two bilinear forms in the Lie algebra which the two Casimirs are constructed with, i.e. the “identity” and the completely antisymmetric 4-tensor in the 6-dimensional space of generators. Consider the Euclidean case. The set of transformations preserving the first is given by $O(6) \times Z_2 \simeq SO(6) \times Z_2 \times Z_2$, while the set of transformations preserving the second is given by $O(3,3) \times Z_2 \simeq SO(3,3) \times Z_2 \times Z_2$, so that the $U$ preserving both are given by the intersection of the two groups, i.e. by the transformations belonging to a common subgroup of them. Certainly a common subgroup is given by $SO(3) \times SO(3) \times Z_2 \times Z_2 \simeq SO(4) \times Z_2 \times Z_2$, and we can conjecture that this is the largest one, since it covers all the symetries of the original action, so that any solution of the theory, like (10), should be defined up to such transformation, and we expect this to be true also in the association between fields and Lie algebra elements. Of course we can in addition rescale the $\tilde{J}$ with an arbitrary real number. The argument in the Lorentzian case goes similarly.

Coming back to the spin foam model corresponding to the new generalized action, our results prove that it should still be based on the simple representations of the Lie algebra, and that no ambiguity in the choice of the labelling of the spin foam faces results from the more general form of the constraints, since this can be naturally and unambiguously re-absorbed in the correspondence between the $B$ field and the Lie algebra elements. This suggests strongly that the resulting spin foam model corresponding to the classical action here considered is still the Barrett-Crane model, as for the Plebanski action, but it is not completely straightforward to prove it due to the more complicated form of the action (29). Anyway a motivation for this is provided by the fact that our analysis shows that the physics is still given by a set of bivectors $E$, in 1-1 correspondence with the field $B$ on which the generalized action is based, and that these bivectors satisfy the Barrett-Crane constraints at the classical level, with the translation of them at the quantum level being straightforward. The action is in fact that of BF theory plus constraints, which, as shown in this section, at the quantum level are just the Barrett-Crane constraints on the representations used as labelling in the spin foam, so that a spin foam quantization procedure of the type performed in [3] seems viable.

VI. ALTERNATIVE: THE REISENBERGER MODEL

We have seen that a natural discretization and spin foam quantization of the constraints in this generalized BF-type action leads to the Barrett-Crane model. In this section we want to discuss and explore a bit the alternatives to our procedure, and the cases not covered in our previous analysis.

In order to prove the equivalence of the two form of the constraints (15) and (18), we assumed that $a_2 \neq 0$ and that $\frac{2 a_2}{a_1} \neq \epsilon$, which, in terms of $\alpha$ and $\beta$ is requiring that $\alpha^2 \neq \epsilon \beta^2$ ($\alpha = \pm \beta$ in the Euclidean case, and $\alpha \neq \pm i \beta$ in the Lorentzian one). Later, the transformations we used both at the classical level ($B \rightarrow E$) and at the quantum (better, Lie algebra) level ($J \rightarrow J$) were well-defined, i.e. invertible, provided that $\alpha^2 \neq \epsilon \beta^2$, again. Actually, the only
interesting condition is the last one, since we already mentioned at the beginning (section (7)) that the case \(a_2 = 0\) leads to considering only degenerate \(B\) fields and degenerate tetrads (in the Euclidean case).

The case \(a_2 \neq 0\) corresponds, at the canonical level, to the Barbero’s choice of the connection variable (with Immirzi parameter \(\gamma = \pm 1\) in the Euclidean case, and to Ashtekar variables (\(\gamma = \pm i\) in the Lorentzian case. It amounts to formulate the theory using a self-dual (or anti-selfdual) 2-form field \(\epsilon\) where \(J\) with \((18)\) cannot be proved and the transformations we used are not invertible. The constraints on the field \(B\) lead to considering only degenerate \(B\) state that the selfduality (or anti-selfduality) of the field \(B\). There is no rigorous way to relate the Barrett-Crane conditions and spin foam model to the classical action when this happens, at least using our procedure, since the equivalence of \((8)\) and \((18)\) cannot be proved and the transformations we used are not invertible. The constraints on the field \(B\) just state that \(B\) is the (anti-)selfdual part of a field \(E\) (and in this case, the action \((8)\) corresponds to the (anti-)selfdual Plebanski action for \(E\)). We note however that if we use still the constraints in the form \((18)\) in the Euclidean case, in spite of the fact that we are not able to prove their equivalence with the original ones, and translate them into a constraint on the representations of \(so(4)\) using the naive correspondence \(B \rightarrow J\), where \(J\) is the canonical basis of the algebra, we get: \(C_1 = \pm C_2\) for \(2a_2 = \pm a_1\). The first case leads to \(j^- = 0\) and the second to \(j^+ = 0\), so we are reduced from \(so(4)\) to \(su(2)_L\) or to \(su(2)_R\), with a precise correspondence between the (anti-)selfduality of the variables used and the (anti-)selfduality of the representations labelling the spin foam. There exist a spin foam model in these “degenerate” cases. It is the Reisenberger model for left-handed (or right-handed) Euclidean gravity [24] [25], whose relationship with the Barrett-Crane model is unfortunately not yet clear. This model [24] [25] can be associated to a different discretization of the generalized action \((7)\) using the original form \((8)\) of the constraints, as it was analyzed in [21] for the Plebanski action. This is indeed the only other (known) alternative to our procedure, at least in the Euclidean case.

As before we define the volume spanned by the two triangles \(S\) and \(S'\):

\[
V(S, S') = \int_{x \in S \cap y \in S'} \epsilon e_{abcd} dx^a \wedge dx^b \wedge dy^c \wedge dy^d
\]  

(43)

To use \((8)\), we decompose the 2-form \(B\) inside the 4-simplex into a sum of 2-forms associated to the faces (triangles) of the 4-simplex [20] [24] [27]:

\[
B^{IJ}(x) = \sum_S B^{IJ}_S(x)
\]  

(44)

where \(B^{IJ}_S(x)\) is such that

\[
\int B^{IJ}_S \wedge J = B^{IJ}[S] \int_{S'} J
\]  

(45)

with \(J\) is any 2-form and \(S'\) the dual face of \(S\) (more precisely the wedge dual to \(S\), i.e. the part of the dual face to \(S\) lying inside the considered 4-simplex). Then, it is clear that:

\[
\int_S B^{IJ}_S = \delta_{S,S'} B^{IJ}[S]
\]

\[
\int B^{IJ}_S \wedge B^{KL}_S = B^{IJ}[S] B^{KL}[S'] \epsilon(S, S')
\]

where \(\epsilon(S, S')\) is the sign of the oriented volume \(V(S, S')\). More precisely \(\epsilon(S, S') = \pm 1\) if \(S, S'\) don’t share any edge, and \(\epsilon(S, S') = 0\) if they do.

Using that, we can translate \((8)\) and \((18)\) into:

\[
\hat{\Omega}^{IJKL} = \Omega^{IJKL} - \frac{1}{6} \eta^{[IK} \eta^{JL]} \Omega^{AB} - \frac{1}{24} \epsilon^{IJKL} \epsilon_{ABCD} \Omega^{ABCD} = 0
\]  

(46)

and

\[
4a_2 \Omega^{AB} = a_1 \epsilon_{ABCD} \Omega^{ABCD}
\]  

(47)

where

\[
\Omega^{IJKL} = \sum_{S,S'} B^{IJ}[S] B^{KL}[S'] \epsilon(S, S')
\]  

(48)
These are the \( so(4) \) analogs of the Reisenberger constraints. Let’s note that the constraints involve associations triangles not sharing any edge whereas the Barrett-Crane procedure was to precisely study triangles sharing an edge i.e being in the same tetrahedron. This is one of the reasons why it is hard to link these two models. Then following [23][2], it is possible to calculate the amplitude associated to the 4-simplex and the corresponding spin foam model. For this purpose, it is useful to project these constraint on the selfdual and anti-selfdual sectors as in [20]:

\[
\tilde{\Omega}^{ij}_{++} = \Omega^{ij}_{++} - \delta^{ij} \frac{1}{3} \text{tr}(\Omega_{++})
\]

(49)

\[
\tilde{\Omega}^{ij}_{--} = \Omega^{ij}_{--} - \delta^{ij} \frac{1}{3} \text{tr}(\Omega_{--})
\]

(50)

\[
\Omega^{ij}_{+-} = 0
\]

(51)

\[
\Omega_0 = (\alpha - \beta)^2 \text{tr}(\Omega_{++}) + (\alpha + \beta)^2 \text{tr}(\Omega_{--}) = 0
\]

(52)

The two first constraints are the same as in the Reisenberger model for \( su(2) \) variables. The two last constraints link the two sectors (+ and -) of the theory, mainly stating that they is no correlation between them except for the constraint \( \Omega = 0 \). Indeed, only that last constraint is modified by the introduction the Immirzi parameter. Following the notations of [25], we replace the field \( B^{IJ} \) by the generator \( J^{IJ} \) of \( so(4) \) (this is a formal quantization of the discretised BF action, see [24] for more details) and we define the projectors \( P_{1,2,3,4} \) (or some gaussian-regularised projectors) on the kernel of the operators corresponding to the four above constraints. Then, the amplitude for the vertex \( \nu \) is a function of the holonomies on the 1-dual skeleton of the three-dimensional frontier \( \partial \nu \) of the vertex. It is given by projecting an universal state (or topological state since that without the projectors, the amplitude gives the \( so(4) \) Ooguri’s topological model) and then integrating it over the holonomies around the ten wedges \( \{h_i\}_{i=1\ldots10} \):

\[
a(g_{\partial \nu}) = \int dh_1 \prod_{s \text{ wedge}} \sum_{s_j} \text{tr}^{g_1\ldots g_{j=10}} \left[ P_1 P_2 P_3 P_4 \bigotimes_s (2j_s + 1) U^{(j_s)}(g_{\partial \nu}) \right]
\]

(53)

It seems that a vertex including the Immirzi parameter is perfectly well-defined in this case. Let’s analyse this more closely. As only one constraint involves the Immirzi parameter, in a first time, we will limit ourselves to studying its amplitude.

As calculating the action of \( \Omega_0 \) might involve some change of basis and thus some Clebsch-Gordon coefficient. However, those are still rational. So we conjecture that the amplitude of the vertices will be zero (no state satisfying \( \Omega_0 = 0 \)) except if \( \alpha = \beta \), \( \alpha = 0 \) or \( \beta = 0 \) as in our first approach to quantizing the \( B \)-field constraints in the Barrett-Crane framework. If this conjecture is verified, we will have two possibilities: either the discretization procedure is correct and we are restricted to a few consistent cases, or we need to modify the discretization or quantization procedures.

Resuming, the situation looks as follows. We have the most general BF-type action for gravity, depending on two arbitrary parameters, both in the Euclidean and Lorentzian signature. In both signatures, and for all the values of the parameters except one (corresponding to the Ashtekar-Barbero choice of canonical variables), the constraints which give gravity from BF theory can be expressed in such a form that a spin foam quantization of the theory leads to the Barrett-Crane spin foam model. In the Euclidean case, for all the values of the parameters, a different discretization, and quantization procedure, leads to the Reisenberger model, but it also seems that the last spin foam model is non-trivial (i.e. non-zero vertex amplitude) only for some particular values of the parameters \( \alpha \) and \( \beta \). These two spin foam models may well turn out to be equivalent, but they are a priori different, and their relationship is not known at this stage. There is no need to stress that an analysis of this relationship would be of paramount importance.
VII. THE ROLE OF THE IMMIRZI PARAMETER IN THE SPIN FOAM MODELS

We would like to discuss briefly what our results suggest regarding the role of the Immirzi parameter in the spin foam models, stressing that this suggestion can at present neither be well supported nor disproven by precise calculations. As we said in section (III), the BF-type action (6), after the imposition of the constraints on the field $B$, reduces to a generalized Hilbert-Palatini action for gravity, in a form studied within the canonical approach in [23]. The canonical analysis performed in that work showed that this action is the lagrangian counterpart of the Barbero’s hamiltonian formulation [29] introducing the Immirzi parameter in the definition of the connection variable and then in the area spectrum. This led to the suggestion [16] that the spin foam model corresponding to the new action (6) would present non-simple representations and an arbitrary (Immirzi) parameter as well.

On the contrary we have shown that the spin foam model corresponding to the new action is given again by the Barrett-Crane model, with the representations labelling the faces of the 2-complex being still the simple representations of $so(4)$ or $so(3,1)$. The value of the area of the triangles in this model is naturally given by the (square root of the) first Casimir of the gauge group in the representation assigned to the triangle, with no additional (Immirzi) parameter. From this point of view it can be said that the prediction about the area spectrum of spin foam models and canonical (loop) quantum gravity do not coincide.

However both the construction of the area operator and its diagonalization imply working with an Hilbert space of states, and not with their histories as in the spin foam context, and the canonical structure of the spin foam models, like the Barrett-Crane one, is not fully understood yet. Consequently, the comparison with the loop quantum gravity approach and results is not straightforward. In fact, considering for example the Barrett-Crane model, it assigns an Hilbert space to boundaries of spacetime, and these correspond, in turn, to boundaries of the spin foam, i.e. spin networks, so that again a spin network basis spans the Hilbert space of the theory, like in loop quantum gravity. The crucial difference, however, is that the spin networks used in the Barrett-Crane model are constructed out of (simple) representation of $so(4)$ or $so(3,1)$, i.e. the full local gauge group of the theory, while in loop quantum gravity (for a nice introduction see [30]) the connection variable used is an $su(2)$-valued connection resulting from a breaking of the gauge group from $so(4)$ or $so(3,1)$ to that subgroup, in the process of the canonical $3 + 1$ decomposition, so that the spin network basis uses only $su(2)$ representations. Consequently, a comparison of the results in spin foam models could possibly be made more easily with a covariant (with respect to the gauge group used) version of loop quantum gravity, i.e. one in which the group used is the full $so(4)$ or $so(3,1)$.

The only results in this sense we are aware of were presented recently in [31] [32]. In these two papers, a Lorentz covariant version of loop quantum gravity is sketched at the algebraic level. However, the quantization was not yet achieved and the Hilbert space of the theory (“spin networks”) not constructed because of problems arising from the non-commutativity of the connection variable used. Nevertheless, two results were derived from the formalism. The first one is that the path integral of the theory formulated in the covariant variables is independent from the Immirzi parameter [31], which becomes an unphysical parameter whose role is to regularize the theory. The second result was the construction of an area operator acting on the hypothetic “spin network” states of the theory [32]:

$$A \approx l^2 \sqrt{-C(so(3,1)) + C(su(2))}$$  \hspace{1cm} (55)

where $C(so(3,1))$ is a quadratic Casimir of the Lie algebra $so(3,1)$ (the Casimir $C_1$ to a factor) and $C(su(2))$ the Casimir of the spatial pull-back $so(3)$. So a “spin network” state would be labelled by both a representation of $so(3,1)$ and a representation of $su(2)$. As we see, the area, whose spectrum differ from both the spin foam and the loop quantum gravity one, is independent on the Immirzi parameter. However, we do not know yet if such an area spectrum has any physical meaning, since no Hilbert space has been constructed for the theory. Nevertheless, it suggests that a “canonical” interpretation of a spin foam might not be as straightforward as it is believed. Instead of taking as spatial slice a $SO(3,1)$ spin networks by cutting a spin foam [33], we might have to also project the $SO(3,1)$ structure onto a $SU(2)$ one; the resulting $SU(2)$ spin network been our space and the background $SO(3,1)$ structure describing its space-time embedding.

It is clear that this issue has not found at present any definite solution, and it remains rather intricate. Thus all we can say is that our results (that do not regard directly the issue of the area spectrum) and those in [32] suggest that the appearence of the Immirzi parameter in loop quantum gravity is an indication of the presence of a quantum anomaly, as discussed in [14], but not of a fundamental one, i.e. not one originating from the breaking up of a classical symmetry at the quantum level, and indicating that some new physics takes place. Instead what seems to happen is that the symmetry is broken by a particular choice of quantization procedure, and that a fully covariant quantization, like the spin foam quantization or the manifestly Lorentzian canonical one, does not originate any one-parameter ambiguity in the physical quantities to be measured, i.e. no Immirzi parameter. Of course, much more work is needed.
to understand better this issue, in particular the whole topic of the relationship between the canonical loop quantum gravity approach and the covariant spin foam one is to be explored in details, and to support or disprove this idea.

VIII. CONCLUSIONS

To conclude, we have shown that starting from the most general BF-type action for gravity we are naturally lead to the Barrett-Crane spin foam model as its corresponding quantization. More precisely, the B field in that action can be put in 1-1 correspondence with a set of bivectors such that the constraints on them are exactly the Barrett-Crane ones, at the classical level; moreover, a translation of these at the quantum level, based on the association between B field and Lie algebra elements, gives exactly the simplicity constraint on the representations of so(4) or so(3, 1) to be used in the spin foam model. We also explored the possible alternatives, all leading to the Reisenberger model in the Euclidean case, based on (anti-)selfdual fields at the classical level and on the $su(2)_L$ ($su(2)_R$) subset of representations of so(4) at the quantum one. Regarding the possible role of the Immirzi parameter in the spin foam models, our results suggest that it does not appear in the quantization of the generalized action, in agreement with the idea that its appearance is a result of the breaking of Lorentz covariance in the usual canonical approach. This issue, however, remains to be understood. Further work is also needed to investigate the relationship between the Reisenberger model and the Barrett-Crane one, in order to understand whether they represent two formulation of the same quantum theory or two different and inequivalent quantum models. In light of the results presented here, we believe this is really a central issue.

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APPENDIX A: EQUIVALENCE OF THE TWO FORMS OF CONSTRAINTS

Here, we follow the same procedure as in [20] to show the equivalence of the two forms of the constraints for the B field, the one leading to the discrete constraints as needed in the Reisenberger model and the one leading to the Barrett-Crane discrete geometry picture. First, the constraints (3) and (5) can be condensed into a single equation:

$$
\epsilon^{abcd} B_{ab}^{ij} B_{cd}^{kl} = \epsilon \epsilon^{ijkl} + \frac{a_1}{a_2} \eta^{[i} [k] \eta^{j]} [l]} = \epsilon \Omega^{ijkl}
$$

(A1)

Considering all the variables as 6 $\times$ 6 matrices (on the antisymmetric couples $[IJ]$ and $[ab]$), we write the previous equation as:

$$
B_{ij}^{ab} \epsilon_{cd} B_{cd}^{kl} = \epsilon \Omega_{IJKL}^{cd}
$$

(A2)

Then $\Omega_{IJKL}^{cd}$ is invertible when

$$
\left( \frac{a_1}{2a_2} \right)^2 \neq \epsilon
$$

(A3)

Assuming that this is the case, and that $\epsilon \neq 0$ (B field non-degenerate), we can define

$$
\Sigma_{ij}^{ab} = \frac{1}{\epsilon} \Omega_{IJKL} B_{cd}^{KL} \epsilon_{cd} = \frac{1}{4\epsilon} \epsilon^{abcd} \left( \frac{a_1}{2a_2} \right)^2 - \epsilon \epsilon^{ijkl} + \frac{a_1}{a_2} \eta^{[i} [k] \eta^{j]} [l]} B_{cd}^{KL}
$$

(A4)

and rewrite (A3) as:

$$
\Sigma_{ij}^{cd} B_{cd}^{KL} = \delta_{ij}^{KL}
$$

(A5)

(A3) means that $\Sigma_{ij}^{cd}$ and $B_{cd}^{KL}$ are invertible and inverse of each other, so that it is equivalent to:

$$
B_{cd}^{KL} \Sigma_{ij}^{ab} = \delta_{cd}^{ab}
$$

(A6)
Expanding this last equation and inverting $\epsilon^{abcd}$, we get:

$$
(-\epsilon_{IJMN} + \frac{a_1}{a_2} \eta_{[IJ} \eta_{N]}) B^{MN}_{cd} B_{ab}^{IJ} = \epsilon \epsilon^{abcd} \left( \left( \frac{a_1}{2a_2} \right)^2 - \epsilon \right)
$$

(A7)

We can check that in the case $a_1 = 0$ (Plebanski action), we find the same constraint as in [20] which leads to the Barrett-Crane simplicity constraint after discretization.

Let’s note that in the Lorentzian case ($\epsilon = -1$), the condition on $a_1, a_2$ is automatically satisfied if we keep real variables.

**APPENDIX B: SELFDUAL AND ANTI-SELFDUAL COMPONENTS OF THE ACTION**

In the Euclidean case, decomposing the action into selfdual and anti-self dual components gives another way of understanding its structure. Following [20], we decompose $B = B^+ + B^-$, $\omega = \omega^+ + \omega^-$ and $\phi = \phi^+ + \psi + \phi^- + \phi_0$. $\phi_{ij}^{(+)}$ and $\phi_{ij}^{(-)}$ are two symmetric traceless matrices (left and right parts of the Weyl), $\psi_{ij}$ is an antisymmetric matrix (traceless part of the Ricci) and $\phi_0$ is a scalar (the only remaining one after imposing the constraint $H(\phi) = 0$ on the field $\phi$. The action (7) then writes:

$$
S = S^+ + S^- + S^\psi + S^0
$$

(B1)

$$
S^\pm = \int \left[ \delta_{ij} B^{(\pm)i} \wedge F^{(\pm)j} - \frac{1}{2} \phi_{ij}^{(\pm)} B^{(\pm)i} \wedge B^{(\pm)j} \right]
$$

(B2)

$$
S^\psi = \int \left[ -\psi_{ij} B^{(+i)} \wedge B^{(-j)} \right]
$$

(B3)

$$
S^0 = \int \left[ -\frac{\phi_0}{2} \delta_{ij} \left( 2a_2 (B^{(+i)} \wedge B^{(+j)} + B^{(-i)} \wedge B^{(-j)}) - a_1 (B^{(+i)} \wedge B^{(+j)} - B^{(-i)} \wedge B^{(-j)}) \right) \right]
$$

(B4)

In this last equation (B4), we can also replace $2a_2$ by $\alpha^2 + \beta^2$ and $a_1$ by $2\alpha \beta$ using (11) and rewrite it as

$$
S^0 = \int \left[ -\frac{\phi_0}{2} \delta_{ij} \left( (\alpha - \beta)^2 B^{(+i)} \wedge B^{(+j)} + (\alpha + \beta)^2 B^{(-i)} \wedge B^{(-j)} \right) \right]
$$

(B5)

This motivates to renormalise the selfdual and anti-selfdual parts of $B$:

$$
B^{(\pm)} = (\alpha + \beta) E^{(\pm)}
$$

(B6)

And renormalising also the components of $\phi$ to absorb the changes, we find that only the dynamical terms $B^{(\pm)} \wedge F^{(\pm)}$ get modified and also the scalar constraint:

$$
B^{(\pm)} \wedge F^{(\pm)} = (\alpha \pm \beta) E^{(\pm)} \wedge F^{(\pm)}
$$

(B7)

$$
S^0 = \int \left[ -\frac{\phi_0}{2} \delta_{ij} \left( E^{(+i)} \wedge E^{(+j)} + E^{(-i)} \wedge E^{(-j)} \right) \right]
$$

(B8)

Thus we recover exactly the action [20].

[1] John C Baez, Spin foam models, Class.Quant.Grav. 15 (1998) 1827-1858, gr-qc/9709052

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