Continuous and discrete inf-sup conditions for surface incompressibility of a deformable continuum

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Abstract

Surface incompressibility, also called inextensibility, imposes a zero-surface-divergence constraint on the velocity of a closed deformable material surface. The well-posedness of the mechanical problem under such constraint depends on an inf-sup or stability condition for which an elementary proof is provided. The result is also shown to hold in combination with the additional constraint of preserving the enclosed volume, or isochoricity. These continuous results are then applied to prove a modified discrete inf-sup condition that is crucial for the convergence of stabilized finite element methods.

Keywords: Inextensibility, Surface incompressibility, Volume preservation, Inf-sup condition, Surface finite elements, Stabilization.

1. Introduction

The rate of change of the volume $|\omega|$ of an infinitesimal piece of continuum $\omega$ located at point $x$ at time $t$ is given by

$$\frac{d|\omega|}{dt} = |\omega| \nabla \cdot \mathbf{u}(x, t). \quad (1)$$

There exist materials that preserve volume exactly, or within experimental accuracy, and are modeled as incompressible. The incompressibility constraint ($\nabla \cdot \mathbf{u} = 0$) materializes in the equations of motion as a reaction force, which is the gradient of an unknown pressure field.

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p. The pressure field is a uniquely determined element of \( Q = L^2(\Omega) \) because the divergence operator \( \nabla \cdot \) is surjective onto \( Q \). The surjectivity is equivalent to the inf-sup condition
\[
\inf_{q \in Q} \sup_{v \in H^1(\Omega)} \frac{\int_{\Omega} q \nabla \cdot v}{\|v\|_1 \|q\|_0} > 0 .
\] (2)

If the continuum is restricted at the boundary, more precisely, if the velocity field normal to the boundary is constrained all over \( \partial \Omega \), then the mean value of the pressure is undetermined. This reflects mathematically in that to satisfy (2) the pressure space \( Q \) must be chosen as a strict subspace of \( L^2(\Omega) \), for example \( L^2_0(\Omega) \) which consists of functions with zero mean.

If we now consider a smooth, closed, orientable surface \( \Gamma \) evolving in \( \mathbb{R}^3 \), the analogous property to volume preservation is area preservation, also called inextensibility or surface incompressibility, which is indeed exhibited by many relevant materials. Our main interest is in lipid membranes \([4, 5, 6, 7]\), which are area-preserving surface fluids \([8]\), but the inextensibility constraint can also hold in other, fluid or solid, material surfaces.

The rate of change of the area \(|\varpi|\) of an infinitesimal piece of surface \( \varpi \) located at \( x \) at time \( t \) and moving along a velocity field \( u \) is given by
\[
\frac{d|\varpi|}{dt} = |\varpi| \nabla_{\Gamma} \cdot u(x, t)
\] (3)
where we have introduced the surface divergence operator \( \nabla_{\Gamma} \cdot \). The velocity field of an inextensible surface \( \Gamma \) must thus satisfy
\[
\nabla_{\Gamma} \cdot u(x, t) = 0 \quad \text{a.e. in } \Gamma .
\] (4)

**Remark 1.** We adopt here the operators of tangential calculus as presented by Delfour and Zolésio \([9]\), with the specific notation of Buscaglia and Ausas \([10]\). In particular, the surface gradient \( \nabla_{\Gamma} f \) of a function \( f : \Gamma \to \mathbb{R} \) at the point \( x \in \Gamma \subset \mathbb{R}^3 \) is the three-dimensional vector
\[
\nabla_{\Gamma} f(x) \equiv \nabla \hat{f}(x) ,
\] (5)
where
\[
\hat{f}(x) \equiv f(\Pi_{\Gamma} x) ,
\] (6)
ΠΓx being the normal projection of x ∈ R³ onto Γ. Surface gradients of vector fields are computed one Cartesian component at a time ((∇Γu)ij = (∇ũi)j).

If the velocity of the surface u ∈ V satisfies a variational formulation

\[ a(u, v) = f(v) \quad \forall v \in V , \]  

for some continuous bilinear and linear forms \( a(\cdot, \cdot) \) and \( f(\cdot) \), then the mixed formulation that enforces the inextensibility constraint is:

Mixed inextensible formulation: Find \((u, \sigma) \in V \times \Sigma\) such that

\[ a(u, v) + b(v, \sigma) = f(v) \quad \forall v \in V \]

\[ b(u, \xi) = 0 \quad \forall \xi \in \Sigma , \]

where

\[ b(v, \xi) = \int_{\Gamma} \xi \nabla_{\Gamma} \cdot v . \]

The Lagrange multiplier \( \sigma \) is the reaction to the inextensibility constraint, a scalar field that is known as surface tension. It is well known [3, 11] that a necessary condition for (8)-(9) to be well posed, and in particular for \( \sigma \) to exist and be unique, is the inf-sup condition

\[ \inf_{0 \neq \xi \in \Sigma} \sup_{0 \neq v \in V} \frac{b(v, \xi)}{\|v\|_X \|\xi\|_{\Sigma}} \geq \alpha > 0 . \]  

We assume that the velocity space \( V \) is a subspace of

\[ X = \{ v \in H^1(\Gamma) \mid v_n = v \cdot \mathbf{n} \in H^m(\Gamma) \} , \]

\( \mathbf{n} \) being the unit normal, with two possibilities:

• If \( m = 1 \), we have \( X = H^1(\Gamma) \) and we set

\[ \|v\|_X = \|v\|_0 + \ell \|v\|_1 . \]
If \( m > 1 \) the space \( X \) has extra regularity in the normal component, and the norm is chosen as
\[
\| \mathbf{v} \|_X = \| \mathbf{v} \|_0 + \ell \| \mathbf{v}_1 + \ell^m |v_n|_m .
\] (14)

Above, \( \| \cdot \|_m \) (respectively, \( | \cdot |_m \)) denotes the usual norm (respectively, seminorm) on \( H^m(\Gamma) \). The same notation is used, without risk of confusion, for the norm and seminorm of \( H^m(\Gamma) \) (space of vector fields with components in \( H^m(\Gamma) \)). The length scale \( \ell \) is a constant introduced to make the units consistent. The added regularity in the normal component results, in the case of lipid membrane models, from a curvature dependent energy \([12, 13, 14, 8, 15]\).

Our interest in this article lies in identifying appropriate combinations of spaces \( V \subset X \) and \( \Sigma \) so that (11) is satisfied.

Notice that \( b(\mathbf{v}, \sigma) \) in (8) is the dynamical action of \( \sigma \). Using the integration by parts formula for closed surfaces \([9, 10, 16]\)
\[
\int_{\Gamma} \sigma \nabla \cdot \mathbf{v} = \int_{\Gamma} \mathbf{v} \cdot (-\nabla_{\Gamma} \sigma + H \sigma \mathbf{\tilde{n}}) \quad \forall \mathbf{v} \in X, \quad \forall \sigma \in H^1(\Gamma) ,
\] (15)
where \( H = \nabla_{\Gamma} \cdot \mathbf{\tilde{n}} \) is the mean curvature, one recovers the classical expression of the surface tension force
\[
f_\sigma = \nabla_{\Gamma} \sigma - H \sigma \mathbf{\tilde{n}} .
\] (16)

The inextensibility constraint is frequently imposed together with the constraint of isochoricity, that is, of preserving the enclosed volume. Many physical situations admit such an idealization, as for example the situation in which an impermeable material surface encloses an incompressible medium. In the case of lipid membranes, isochoricity is a consequence of osmotic equilibrium \([8]\).

The rate of change of the enclosed volume \( V \) when \( \Gamma \) moves along the velocity field \( \mathbf{u} \) is
\[
\frac{dV}{dt} = \int_{\Gamma} \mathbf{u} \cdot \mathbf{\tilde{n}} .
\] (17)

The Lagrange multiplier that enforces this constraint turns out to be an internal uniform pressure \( p \in \mathbb{R} \) acting as a uniform normal force \( p \mathbf{\tilde{n}}(\mathbf{x}) \) at each \( \mathbf{x} \in \Gamma \).
If just the isochoricity constraint is imposed, the corresponding inf-sup condition is

$$\inf_{r \in \mathbb{R}} \sup_{v \in \mathbf{V}} \frac{r \int_{\Gamma} v \cdot \mathbf{n}}{\|r\|\|v\|_{\Sigma}} > 0.$$ (18)

If both the inextensibility and isochoricity constraints hold simultaneously, the mixed formulation becomes:

**Mixed inextensible-isochoric formulation:** Find \((u, \sigma, p) \in \mathbf{V} \times \Sigma \times \mathbb{R}\) such that

$$a(u, v) + c(v, (\sigma, p)) = f(v) \quad \forall v \in \mathbf{V}$$

$$c(u, (\xi, q)) = 0 \quad \forall (\xi, q) \in \Sigma \times \mathbb{R},$$

where

$$c(v, (\xi, q)) \doteq \int_{\Gamma} (\xi \nabla_{\Gamma} \cdot v + q v \cdot \mathbf{n}) \, .$$ (21)

In this case both its surface tension \(\sigma\) (as a function of \(x \in \Gamma\)) and its internal pressure \(p\) are uniquely defined in \(\Sigma \times \mathbb{R}\), under the following condition on \(\mathbf{V}-\Sigma\),

$$\inf_{(\xi, q) \in \Sigma \times \mathbb{R}} \sup_{v \in \mathbf{V}} \frac{c(v, (\xi, q))}{\|v\|_{\Sigma} \|(\xi, q)\|_{\Sigma \times \mathbb{R}}} = \inf_{(\xi, q) \in \Sigma \times \mathbb{R}} \sup_{v \in \mathbf{V}} \frac{\int_{\Gamma} (\xi \nabla_{\Gamma} \cdot v + q v \cdot \mathbf{n})}{\|v\|_{\Sigma} \|(\xi, q)\|_{\Sigma \times \mathbb{R}}} \doteq \beta > 0.$$ (22)

In what follows, we will select appropriate spaces \(\Sigma\) and prove the stability inequalities (11) and (22) considering two possibilities for \(\mathbf{V}\): (a) the shape \(\Gamma\) is fixed, implying that \(\mathbf{V} \subseteq \mathbf{X}\) consists solely of tangential fields; and (b) the space \(\mathbf{V}\) is unconstrained, i.e., \(\mathbf{V} = \mathbf{X}\).

2. Inf-sup conditions for purely tangential motions

If a vector field \(v \in \mathbf{X}\) is decomposed into its tangential component \(v_{\tau}\) and its normal part \(v_{n} \mathbf{n}\), i.e.,

$$v = v_{\tau} + v_{n} \mathbf{n},$$ (23)

its gradient takes the form

$$\nabla_{\Gamma} v = \nabla_{\Gamma} v_{\tau} + \mathbf{n} \otimes \nabla_{\Gamma} v_{n} + v_{n} \nabla_{\Gamma} \mathbf{n}.$$ (24)
and its surface divergence is given by
\[ \nabla \Gamma \cdot \mathbf{v} = \nabla \Gamma \cdot \mathbf{v} + v_n \nabla \Gamma \cdot \mathbf{n}. \]  
(25)

The last term in (24) contains the \textit{curvature tensor}
\[ \mathbf{H} = \nabla \Gamma \mathbf{n}. \]  
(26)

If \( \mathbf{V} \) only consists of tangential motions, then \( v_n = 0 \) for all \( \mathbf{v} \in \mathbf{V} \), the numerator inside the inf-sup in (18) vanishes identically and thus the condition is not satisfied. The internal pressure is not uniquely defined, and modifying it has no effect on the motion of the surface continuum.

The surface tension \( \sigma \), on the other hand, is well defined in \( L^2(\Gamma) \) \textit{up to an arbitrary additive constant}. It is uniquely defined, for example, in
\[ L_0^2(\Gamma) = \{ q \in L^2(\Gamma) \mid \int_{\Gamma} q = 0 \}. \]  
(27)

Prop. 2. \textit{If} \( \mathbf{V} \) \textit{is the closed subspace of} \( \mathbf{X} \) \textit{consisting of purely tangential motions, i.e.,}
\[ \mathbf{V} = \{ \mathbf{v} \in \mathbf{X} \mid \mathbf{v} \cdot \mathbf{n} = 0 \ \text{a.e. in} \ \Gamma \} \]  
(28)

\textit{and} \( \Sigma = L_0^2(\Gamma) \), \textit{then the inf-sup condition} (17) \textit{holds with}
\[ \alpha = \frac{1}{c_r \ell} \]  
(29)

where \( c_r \) \textit{is the elliptic regularity constant} (see (32) below).

\textit{Proof.} This is proved in the same way as (2) is proved. Given \( \xi \) arbitrary in \( L_0^2(\Gamma) \), let \( \varphi \in H^1(\Gamma) \cap L_0^2(\Gamma) \) be the solution of
\[ \Delta_\Gamma \varphi = \xi. \]  
(30)

Taking \( \mathbf{v} = \nabla_\Gamma \varphi \), one has
\[ b(\mathbf{v}, \xi) = \int_{\Gamma} \xi \nabla \mathbf{v} \cdot \mathbf{v} = \| \xi \|_0^2. \]  
(31)

On the other hand, from the regularity estimate (16) \( \| \varphi \|_0 + \ell \| \varphi \|_1 + \ell^2 \| \varphi \|_2 \leq c_r \ell^2 \| \xi \|_0 \) \( (32) \)

(where again \( \ell \) is introduced to render the units consistent) one has that \( \| \mathbf{v} \|_{\mathbf{X}} = \| \mathbf{v} \|_0 + \ell \| \mathbf{v} \|_1 = \| \varphi \|_1 + \ell \| \varphi \|_2 \leq c_r \ell \| \xi \|_0 \), and thus the claim (17) is proved. \( \square \)
3. The inf-sup condition for arbitrary motions

If a surface can move along its normal direction then condition (18) is seen to hold simply taking \( v = \mathbf{n} \). The preservation of the enclosed volume uniquely defines an internal pressure.

Turning to the inextensibility condition, one has the following:

**Prop. 3.** If \( V = X \) and \( \Sigma = L^2(\Gamma) \), then \( \text{(17)} \) holds.

This means that the surface tension is completely defined on an inextensible surface that can move freely in space. We will prove this proposition after proving Prop. 4 since by then it will become straightforward.

Consider now the case of a surface that both is inextensible and preserves the enclosed volume. Rewriting the corresponding inf-sup condition (22) as

\[
\inf_{(\xi,q) \in \Sigma \times \mathbb{R}} \sup_{v \in V} \left\{ \int_{\Gamma} \left[ \nabla \cdot (H\xi + q) n + (H\xi + q) v_n \right] \right\} = \beta > 0
\]

one immediately sees that these two conditions are linearly dependent if the surface has constant mean curvature (a sphere). Taking \( q = H \) (constant) and \( \xi = -1 \) (constant), and remembering that the integral of the surface divergence of a purely tangential field over a closed surface vanishes (a consequence of (15) taking \( \sigma = 1 \)), one has the numerator inside the inf-sup equal to zero for all \( v \). For a sphere, thus, inextensibility implies the preservation of the enclosed volume. The internal pressure and the mean surface tension are not uniquely defined. The gradient of the surface tension, on the other hand, is well determined.

For all surfaces other than spheres, however, one has:

**Prop. 4.** If the surface \( \Gamma \) is not a sphere, then \( \text{(22)} \) holds with \( V = X \) and \( \Sigma = L^2(\Gamma) \).

The proof is quite straightforward, but let us for clarity first state a couple of preliminary facts. Let \( \overline{g} \) denote the mean value of \( g \in L^1(\Gamma) \).

**Lemma 5.** The expression

\[
\| (\xi,q) \|_{L^2(\Gamma) \times \mathbb{R}} = \left[ \| \xi - \overline{\xi} \|^2_0 + \ell^4 \| H \overline{\xi} + q \|^2 + \ell^2 \overline{\xi}^2 \right]^{\frac{1}{2}}
\]

is indeed a norm on \( L^2(\Gamma) \times \mathbb{R} \). □
As a consequence of (24) one also has:

Lemma 6.

\[ \|v\|_0 + \ell |v|_1 \leq \|v_r\|_0 + \ell |v_r|_1 + (1 + \ell \|H\|_{L^\infty(\Gamma)}) \|v_n\|_0 + \ell |v_n|_1 \quad (35) \]

for all \( v \in H^1(\Gamma) \).

Now we proceed to prove Prop. 4.

Proof. Let \((\xi, q)\) be an arbitrary element of \( \Sigma \times \mathbb{R} \). Taking \( \varphi \in H^1(\Gamma) \cap L^2(\Gamma) \) as the unique solution of \( \Delta_\Gamma \varphi = \xi - \zeta \), we select \( v \in V \) as

\[ v = \nabla_\Gamma \varphi + v_n \tilde{n} , \quad \text{with} \quad v_n = k_1(\tilde{H} \tilde{\xi} + q) + k_2(\tilde{H} - \tilde{H}) \tilde{\xi} . \quad (36) \]

The positive constants \( k_1 \) and \( k_2 \) are left unspecified for now. Rewriting

\[ c(v, (\xi, q)) = \int_\Gamma [\xi \nabla_\Gamma \cdot v_r + (H\xi + q) v_n] \] (37)

and using (36) one has

\[ c(v, (\xi, q)) = \|\xi - \zeta\|_0^2 + k_1 \left[ |\Gamma| (\tilde{H} \tilde{\xi} + q)^2 + (\tilde{H} \tilde{\xi} + q) \int_\Gamma (H, \xi - \tilde{H} \tilde{\xi}) \right] + \\
+ k_2 \int_\Gamma [(H - \tilde{H})^2 \tilde{\xi}^2 + (H - \tilde{H}) H \tilde{\xi} (\xi - \tilde{\xi}) + (H - \tilde{H}) (H \tilde{\xi} + q) \tilde{\xi}] \quad (38) \]

Noticing that the last term in the second integral cancels out and using that \( \int_\Gamma (H\xi - \tilde{H} \tilde{\xi}) \leq \|H - \tilde{H}\|_0 \|\xi - \tilde{\xi}\|_0 \), one arrives at

\[ c(v, (\xi, q)) \geq \|\xi - \zeta\|_0^2 + k_1 |\Gamma| (\tilde{H} \tilde{\xi} + q)^2 + k_2 \|H - \tilde{H}\|_0^2 \tilde{\xi}^2 - \\
k_1 \|H - \tilde{H}\|_0 |\tilde{H} \tilde{\xi} + q| \|\xi - \tilde{\xi}\|_0 - k_2 \|H - \tilde{H}\|_0^2 |\tilde{\xi}| \|\xi - \tilde{\xi}\|_0 \quad (39) \]

which using Young’s inequality twice yields

\[ c(v, (\xi, q)) \geq \left( 1 - k_1 \frac{\|H - \tilde{H}\|_0^2}{2 |\Gamma|} - k_2 \frac{(H - \tilde{H})^2 \|\tilde{\xi}\|_0^2}{2} \right) \|\xi - \tilde{\xi}\|_0^2 + \\
+ k_1 \frac{|\Gamma|}{2} (\tilde{H} \tilde{\xi} + q)^2 + k_2 \frac{\|H - \tilde{H}\|_0^2}{2} \tilde{\xi}^2 . \quad (40) \]
Let us now choose
\[
k_1 = \min \left\{ \frac{|\Gamma|}{2\|H - \overline{H}\|_0^2}, \ell^2 \right\}
\]
\[
k_2 = \min \left\{ \frac{\|H - \overline{H}\|_0^2}{2\|H - \overline{H}\|^2_0}, \ell^2 \right\}
\]
which gives
\[
c(v, (\xi, q)) \geq \frac{1}{2} \|\xi - \overline{\xi}\|^2_0 + k_1 \frac{|\Gamma|}{2} (\overline{H} \overline{\xi} + q)^2 + k_2 \frac{\|H - \overline{H}\|^2_0}{2} \xi^2
\]
\[
\geq A \|\xi, q\|^2_{L^2(\Gamma) \times \mathbb{R}},
\]
with
\[
A = \min \left\{ \frac{1}{2}, \frac{|\Gamma|^2}{4 \ell^4 \|H - \overline{H}\|_0^2}, \frac{|\Gamma|}{2\ell^2}, \frac{\|H - \overline{H}\|_0^4}{4 \ell^2 \|H - \overline{H}\|^2_0}, \frac{\|H - \overline{H}\|^2_0}{2} \right\}.
\]
At the same time, from (35) and the estimates
\[
\|v_r\|_0 + \ell \|v_r\|_1 \leq c_r \ell \|\xi - \overline{\xi}\|_0
\]
\[
\|v_n\|_0 \leq \ell^2 |\Gamma|^\frac{1}{2} |\overline{H} \overline{\xi} + q| + \ell^2 \|H - \overline{H}\|_0 \|\overline{\xi}\|
\]
\[
\|v_n\|_k \leq \ell^2 |H|_k \|\overline{\xi}\| \quad \forall k \geq 1
\]
it follows that
\[
\|v\|_X \leq c_r \ell \|\xi - \overline{\xi}\|_0 + (1 + \ell \|H\|_{L^\infty(\Gamma)}) \ell^2 |\Gamma|^\frac{1}{2} |\overline{H} \overline{\xi} + q| +
\]
\[
+ \left[ (1 + \ell \|H\|_{L^\infty(\Gamma)}) \ell^2 \|H - \overline{H}\|_0 + \ell^3 |H|_1 + \ell^{m+2} |H|_m \right] \|\overline{\xi}\|
\]
\[
\leq B \|\xi, q\|_{L^2(\Gamma) \times \mathbb{R}}
\]
with
\[
B^2 = c_r^2 \ell^2 + (1 + \ell \|H\|_{L^\infty(\Gamma)})^2 |\Gamma| + \left[ (1 + \ell \|H\|_{L^\infty(\Gamma)}) \ell \|H - \overline{H}\|_0 + \ell^2 |H|_1 + \ell^{m+1} |H|_m \right]^2
\]
The claim is proved with $\beta = A/B$. \qed
Notice that \( A \) is equal to zero for a sphere, and thus \( \beta = 0 \). However, Prop. 3 is true irrespective of \( \Gamma \) being a sphere or not. Let us modify the previous proof to prove it.

**Proof.** (of Prop. 3) Taking the same \( v \) as before, and particularizing (43) to \( q = 0 \) one gets

\[
b(v, \xi) \geq \frac{1}{2} \| \xi - \xi_0 \|^2 + \left( k_1 \frac{\| \Gamma \|}{2} \bar{H}^2 + k_2 \frac{\| H - \bar{H} \|^2}{2} \right) \xi^2 \geq C \| \xi \|^2 . \tag{51}
\]

with

\[
C = \min \left\{ \frac{1}{2}, \frac{k_1 \bar{H}^2}{2} + \frac{k_2 \| H - \bar{H} \|^2}{2 \| \Gamma \|} \right\} . \tag{52}
\]

and now \( C > 0 \) even if \( H = \bar{H} \). Proposition 3 is thus proved with \( \Sigma = L^2(\Gamma) \) and \( \alpha = C/B \).

4. Discrete inf-sup condition

The continuous inf-sup conditions proved above also allow for the extension to surface finite elements of the discrete counterpart known as Verfürth’s lemma \[17\]. It is central in the numerical analysis of stabilized finite element methods such as the Galerkin-Least-Squares method \[18, 19\]. A variant of one such method, the pressure gradient projection method \[20, 21, 22\], has recently been successfully implemented for lipid membrane models \[5, 6\].

Let \( V_h \subset V \) and \( \Sigma_h \subset \Sigma \) be surface finite element spaces, as defined in Dziuk and Elliott \[16\]. Notice that these are lifted spaces, which are defined with the aid of a faceted surface but consist of scalar functions (or vector fields) defined on the “exact” surface \( \Gamma \).

**Prop. 7.** If the space \( \Sigma_h \) consists of continuous functions, then there exist \( \gamma > 0 \) and \( \delta > 0 \), independent of the mesh size \( h = \max_K h_K \), such that

\[
\sup_{v_h \in V_h} \frac{\int_{\Gamma}(\xi_h \nabla_{\Gamma} \cdot v_h + q \nabla_{\Gamma} \cdot \bar{n})}{\| v_h \|_X} \geq \gamma \| (\xi_h, q) \|_{\Sigma \times \mathbb{R}} - \delta \left( \sum_{K \in T_h} h_K^2 \| \nabla_{\Gamma} \xi_h \|_{L^2(K)}^2 \right)^{\frac{1}{2}} \tag{53}
\]

for all \( (\xi_h, q) \in \Sigma_h \times \mathbb{R} \).
The proof assumes the existence of a Clément-type quasi-interpolatory operator $\mathcal{I}_h : $ $V \rightarrow V_h$, satisfying
\begin{align}
\| \mathcal{I}_h v \|_X & \leq c_1 \| v \|_X \tag{54} \\
\| v - \mathcal{I}_h v \|_{L^2(K)} & \leq c_2 h_K \| \nabla v \|_{L^2(\omega_K)} \tag{55}
\end{align}
where $\omega_K$ is the union of all elements that share at least one node with element $K$.

**Proof.** For arbitrary $\xi_h$ and $q$, from Prop. 4 we have
\begin{equation}
\beta \| (\xi_h, q) \|_{\Sigma \times \mathbb{R}} \leq \sup_{v \in V} \frac{c(\mathcal{I}_h v, (\xi_h, q))}{\| v \|_X} + \sup_{v \in V} \frac{c(v - \mathcal{I}_h v, (\xi_h, q))}{\| v \|_X} . \tag{56}
\end{equation}

It is clear that
\begin{equation}
\sup_{v \in V} \frac{c(\mathcal{I}_h v, (\xi_h, q))}{\| v \|_X} \leq c_1 \sup_{v \in V} \frac{c(\mathcal{I}_h v, (\xi_h, q))}{\| \mathcal{I}_h v \|_X} \leq c_1 \sup_{v_h \in V_h} \frac{c(v_h, (\xi_h, q))}{\| v_h \|_X} . \tag{57}
\end{equation}

Concerning the second term in the right-hand side of (56), and denoting $w = v - \mathcal{I}_h v$, we have
\begin{align}
c(v - \mathcal{I}_h v, (\xi_h, q)) & = \sum_{K \in T_h} \int_K [\xi_h \nabla v \cdot \mathbf{w} + q H w \cdot \mathbf{n}] \\
& = \sum_{K \in T_h} \int_{\partial K} \xi_h w \cdot \mathbf{n} + \sum_{K \in T_h} \int_K (-\nabla v \cdot \mathbf{n} + \xi_h H w \cdot \mathbf{n}) . \tag{58}
\end{align}
The first sum vanishes if $\xi_h$ is continuous across element boundaries and $\Gamma$ is $C^1$. As a consequence, since $\| w \|_{L^2(K)} \leq c_2 h_K \| \nabla v \|_{L^2(\omega_K)}$,
\begin{equation}
c(v - \mathcal{I}_h v, (\xi_h, q)) \leq c_3 \ell \left[ \left( \sum_{K \in T_h} h_K^2 \| \nabla v \|_{L^2(K)}^2 \right)^{\frac{1}{2}} + h \| H \|_{L^\infty(\Gamma)} \| \xi_h \|_0 \right] \| v \|_X . \tag{59}
\end{equation}
where
\begin{equation}
c_3 = c_2 \ell \sup_{v \in V} \frac{\left( \sum_{K \in T_h} \| \nabla v \|_{L^2(\omega_K)}^2 \right)^{\frac{1}{2}}}{\| v \|_X} . \tag{60}
\end{equation}
Replacing into (56) one gets
\begin{align}
\beta \| (\xi_h, q) \|_{\Sigma \times \mathbb{R}} & \leq c_1 \sup_{v_h \in V_h} \frac{c(v_h, (\xi_h, q))}{\| v_h \|_X} + c_3 \left( \sum_{K \in T_h} h_K^2 \| \nabla v \|_{L^2(K)}^2 \right)^{\frac{1}{2}} + \frac{c_3 h \| H \|_{L^\infty(\Gamma)} \| \xi_h \|_0}{\ell} . \tag{61}
\end{align}
which proves (53) taking $\gamma = \beta/(2 c_1)$, $\delta = c_3/(\ell c_1)$ and $h$ small enough. \qed
5. Concluding remarks

It has been shown that the inextensibility of a surface continuum, analogous to the
incompressibility of a volumetric medium, is a well-posed constraint for closed surfaces
evolving in $\mathbb{R}^3$. It gives rise to a surface tension field $\sigma$ that is uniquely defined in $L^2(\Gamma)$.

The simultaneous imposition of both the inextensibility constraint and the isochoricity
constraint (preservation of the enclosed volume) is also well posed, the only exception being
that of a spherical configuration of the surface continuum. In all other cases, both the
surface tension field $\sigma \in L^2(\Gamma)$ and the internal pressure $p \in \mathbb{R}$ are uniquely determined by
the constrained problem.

The estimates in the proofs require the surface $\Gamma$ to be of class $C^2$ and have its mean
curvature $H$ in $H^m(\Gamma)$, $m \geq 1$.

On the basis of the exact well-posedness, a discrete stability result was established for
discretizations of the surface tension field that consist of continuous interpolants. It consists
of a modified inf-sup condition (sometimes called Verfürth’s lemma) which plays a central
role in the justification of stabilized methods for incompressible flow. The extension of this
inf-sup condition to deformable surfaces thus justifies the stabilized treatment of the surface
tension proposed in recent work on lipid membranes \[5, 6\].

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References

[1] M. Gurtin. An Introduction to Continuum Mechanics. Academic Press Inc., London, 1981.
[2] V. Girault and P.-A. Raviart. Finite Element Method for Navier-Stokes Equations: Theory and Algo-
rithms. Springer Verlag, Berlin, 1986.
[3] F. Brezzi and M. Fortin. Mixed and Hybrid Finite Element Methods. Springer-Verlag, 1991.
[4] M. Arroyo and A. DeSimone. Relaxation dynamics of fluid membranes. Phys. Rev. E, 79:031925, 2009.
[5] I. Tasso and G. Buscaglia. A finite element method for viscous membranes. *Comput. Methods Appl. Mech. Engrg.*, 255:226–237, 2013.

[6] D. Rodrigues, R. Ausas, F. Mut, and G. Buscaglia. A semi-implicit finite element method for viscous lipid membranes. *J. Comput. Phys.*, 298:565–584, 2015.

[7] J. Barrett, H. Garcke, and R. Nürnberg. A stable numerical method for the dynamics of fluidic membranes. *Univ. Regensburg Preprint*, 18, 2014.

[8] U. Seifert. Configurations of fluid membranes and vesicles. *Adv. Phys.*, 46:13–137, 1997.

[9] M. Delfour and J.-P. Zolésio. *Shapes and Geometries. Metrics, Analysis, Differential Calculus, and Optimization*. SIAM, 2nd edition, 2011.

[10] G. Buscaglia and R. Ausas. Variational formulations for surface tension, capillarity and wetting. *Comput. Methods Appl. Mech. Engrg.*, 200(45-46):3011–3025, 2011.

[11] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*. Springer, 2004.

[12] P. Canham. The minimum energy of bending as a possible explanation of the biconcave shape of the human red blood cell. *J. Theor. Biol.*, 26:61–81, 1970.

[13] W. Helfrich. Elastic properties of lipid bilayers – theory and possible experiments. *Z. Naturforsch. C*, 28:693–703, 1973.

[14] H. Deuling and W. Helfrich. Red blood cell shapes as explained on the basis of curvature elasticity. *Biophys. J.*, 16(8):861–868, 1976. PMCID: PMC1334911.

[15] B. Seguin and E. Fried. Microphysical derivation of the Canham-Helfrich free-energy density. *J. Math. Biol.*, 68(3):647–665, 2014.

[16] G. Dziuk and C. Elliott. Finite element methods for surface PDEs. *Acta Numerica*, 22:289–396, 2013.

[17] L. Franca, T. Hughes, and R. Stenberg. Stabilized finite element methods. In M. Gunzburger and R. Nicolaides, editors, *Incompressible Computational Fluid Dynamics*. Cambridge Univ. Press, 1993.

[18] L. Franca and T. Hughes. Two classes of mixed finite element methods. *Comp. Meth. Applied Mech. and Engrg.*, 69:89–129, 1987.

[19] L. Franca and S. Frey. Stabilized finite element methods: II. The incompressible navier-stokes equations. *Comp. Meth. Appl. Mech. Engrg.*, 99:209–233, 1992.

[20] R. Codina and J. Blasco. A finite element formulation for the Stokes problem allowing equal velocity-pressure interpolation. *Comput. Meth. Appl. Mech. Engrg.*, 143:373–391, 1997.

[21] G. Buscaglia, F. Basombrio, and R. Codina. Fourier analysis of an equal-order incompressible flow solver stabilized by pressure-gradient projection. *Int. J. Numer. Methods Fluids*, 34:65–92, 2000.

[22] R. Codina, J. Blasco, G. C. Buscaglia, and A. Huerta. Implementation of a stabilized finite element formulation for the incompressible navier-stokes equations based on a pressure gradient projection. *Int. J. Num. Meth. Fluids*, 37(4):419–444, 2001.
[23] P. Clément. Approximation by finite element functions using local regularization. *RAIRO Anal. Numér.*, 9:77–84, 1975.