The Zamolodchikov $C$-Function, Classical Closed String Field Theory, The Duistermaat-Heckman Theorem, The Renormalization Group, and all that ...

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Abstract

In this article we formulate a ‘topological’ field theory by employing a generalization of the Duistermaat-Heckman Theorem to localize the path-integral of the ‘topological action’ $C^2$, where $C$ is a slight modification of the Zamolodchikov $C$-Function, over the space of all two-dimensional field theories to the fixed points of the renormalization group’s identity component. Also, we propose an interpretation of the background independent classical closed string field theory action $S$ in terms of the Zamolodchikov $C$-Function’s modification.

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1. Introduction

In this article we will formulate a ‘topological’ field theory which has the action $C^2$, $C$ being a slight modification of the Zamolodchikov $C$-Function, by employing a generalization of the Duistermaat-Heckman Theorem to localize the path-integral over the space of two-dimensional field theories to the fixed points of the renormalization group. Furthermore, we will express the background independent classical closed string field theory action $S$ in terms of the Zamolodchikov $C$-Function’s modification. As far as the author knows, this is the first example of an explicitly background independent classical closed string field theory.

Previous formulations of closed string field theory have run up against several seemingly impenetrable barriers. The first, and most significant, barrier which these formulations have faced is that of background independence. Basically, all such string field theories start by postulating that space-time interactions are governed by a string theory which is conformally invariant \[1\]. However, requiring such a theory to be conformally invariant is equivalent to requiring that the background space-time fields are solutions to string theory’s classical equations of motion. Thus, such string field theories are said to be background dependent, as they depend upon a classical background. String field theory then makes its appearance only to describe the perturbations of these fields about their classical backgrounds. This state of affairs is unsatisfactory on at least two fronts. First of all, the string theorist, not the string theory, chooses the particular classical background. Thus, any relevant physics which string theory may have graced us with is lost. Secondly, the interactions are only formulated perturbatively. Any non-perturbative interactions, which, for instance, would act to choose the vacuum, are ignored. Therefore, this state of affairs seems to desperately call for a background independent notion of string field theory. In this paper we will attempt to take the first steps towards this goal by formulating a background independent classical closed string field theory.

The second problem which string field theory faces is that of obtaining a well defined configuration space. Conventionally, one defines the configuration space of a theory to be the set of all possible space-time fields modded by an equivalence relation derived from the theory’s gauge group. However, in contrast to most ‘conventional’ field theories, the space-time fields in closed string theory appear in a very unusual setting. In the case of closed string field theory, the set of space-time fields do not appear as dynamical fields in the two-dimensional field theory, they appear as coupling constants. Thus, if we consider
a two-dimensional field theory as being completely defined by a set of coupling constants, the configuration space of string field theory can be taken to be the space of ‘all’ two-dimensional field theories modded by some appropriate gauge group. However, there are various problems which arise with such a simplistic outlook. Ideally, to properly formulate a closed string field theory one would like to impose various structures on the space of ‘all’ two-dimensional field theories such as a metric, a symplectic form, a volume form . . . But, to impose such structures on the space of ‘all’ two-dimensional field theories is a difficult task. All such attempts have settled with rather badly defined structures over this space [2]. In this article we will avoid the most difficult of these problems by simply assuming there exists a symplectic form over the space of ‘all’ two-dimensional field theories.

A third hurdle which string field theory has faced is the search for a formulation of a full, background independent classical closed string field theory action. The major obstacle to this goal is our lack of knowledge in two areas: formulating string field theories around backgrounds which are not solutions to string theory’s classical equations of motion and formulating string field theories around backgrounds which correspond to non-renormalizable two-dimensional field theories [1][2]. However, again, in this paper we side-step this problem by only considering very general properties of the background independent classical closed string field theory action which do not depend upon the backgrounds being solutions to closed string field theory’s classical equations of motion or corresponding to renormalizable two-dimensional fields theories.

The outlay of this paper is as follows. First, we will review a non-abelian generalization of the Duistermaat-Heckman Theorem, this will occupy the second section. Second, we will define the configuration space of our theory and some of its geometrical properties, this will occupy section three. After this, we will set out to define the renormalization group of our theory and some of its topological properties. Finally, in the fifth section, we will tie this all together and compute some path-integrals in the theory $C^2$ and derive the relation between the background independent classical closed string field theory action $S$ and the Zamolodchikov $C$-Function’s modification. The sixth section will be occupied with conclusions and wild speculations.

2. The Non-Abelian Duistermaat-Heckman Theorem

In this section we will explain the non-abelian version of the Duistermaat-Heckman Theorem, following the work of Witten [3]. We will explain the theorem in two steps.
The first step will occupy the first subsection, and the second step will occupy the second subsection. In the first subsection we will explain the notion of equivariant integration, and in the second subsection we will explain the localization principle itself.

2.1. Equivariant Integration

In this subsection we will explain equivariant integration. This explanation will consist of two portions. In the first portion we will introduce equivariant cohomology, and in the second portion we will introduce equivariant integration.

Let us start by considering a manifold $X$ which is acted upon by a compact, connected group $G$ with Lie algebra $\mathcal{G}$. Furthermore, let us assume that the manifold $X$ is symplectic and of dimension $2n$. Now, consider the deRham complex of $X$ with complex coefficients, $\Omega^*(X)$, and the space of zero-forms on $\mathcal{G}$, $\Lambda^0(\mathcal{G})$. Also, let us grade $\Lambda^0(\mathcal{G})$ such that a $n^{th}$ order homogenous polynomial is of degree $2n$. Now, we define the equivariant forms on $X$ to be the elements of $\Omega^*(X) \otimes \Lambda^0(\mathcal{G})$ which are invariant under the action of $G$. Let us denote the equivariant differential forms over $X$ by $\Omega^*_G(X)$.

As we have a notion of an equivariant differential form, let us now consider defining a notion of equivariant cohomology. With this goal in mind, we must first define an equivariant deRham $d$ operator. This is done by considering the $G$ action on $X$. The $G$ action on $X$ is given by a homomorphism from an element in $\mathcal{G}$ to a vector field on $X$, and the flow along this field is $G$’s action. Thus, if $v$ is an element of $\mathcal{G}$, then there is a corresponding vector field $V(v)$ on $X$. Then, we define the equivariant deRham $d$ operator $d_G$ as,

$$d_G \equiv d - i_{V(v)}, \quad (2.1.1)$$

where $i_{V(v)}$ denotes contraction with the vector field $V(v)$. However, a short computation yields the following result,

$$d_G^2 = -iL_{V(v)}, \quad (2.1.2)$$

where $L_{V(v)}$ denotes Lie differentiation with respect to the vector field $V(v)$. Thus, the operator is nil-potent on precisely the equivariant differential forms. Therefore, we have
a natural notion of a $G$-equivariant cohomology precisely on the equivariant differential forms. Let us denote this $G$-equivariant cohomology by $H^*_G(X)$.

As we have now defined the notion of a $G$-equivariant cohomology, let us define the notion of equivariant integration. Among the things on our wish list for the properties of equivariant integration, we should request that the integration only depend upon the equivariant cohomology class of the integrand, and, of course, the integral should not diverge. These two points will guide our definition.

The vector space $G$ has a natural invariant measure, unique up to a constant factor. To fix this factor let us consider $G$ in a different setting; $G$ is, by definition, in isomorphism with $TG|_{id}$. A choice of Harr measure on $G$ then defines a measure on $G$. As $G$ is compact the Harr measure on $G$ yields a finite volume $Vol(G)$ for $G$. Now, choose coordinates $v_m$ on $G$ such that the measure $dv_1 dv_2 \ldots dv_m$ on $G$ coincides with the Harr measure at $id$ of $G$. Thus, we now have a natural measure,

$$\frac{1}{Vol(G)} dv_1 dv_2 \ldots dv_m,$$

(2.1.3)
on $G$ which is independent of the chosen Harr measure. The equivariant integration we wish to define on $X$ is now taken to be the map from $H^*_G(X) \to \mathbb{R}$ given by,

$$\alpha \to \frac{1}{Vol(G)} \int_{G \times X} \frac{dv_1 dv_2 \ldots dv_m}{(2\pi)^m} \alpha.$$ 

(2.1.4)

However, this definition is not quite up to snuff as the integral does not generically converge. But, we may fix this by putting in a convergence factor. If we take $s$ as a positive, real number and $(\ ,\ )$ as a positive definite invariant quadratic form on $G$, then we define equivariant integration of an element $\alpha$ in $H^*_G(X)$ as follows,

$$\int_X \alpha \equiv \frac{1}{Vol(G)} \int_{G \times X} \frac{dv_1 dv_2 \ldots dv_m}{(2\pi)^m} \alpha \exp \left( -\frac{1}{4s}(v,v) \right).$$

(2.1.5)

With this added exponential convergence factor, we can integrate equivariant differential forms with arbitrary polynomial dependence upon $v$ and all such integrals will converge as a result of our exponential factor. We will call the above map equivariant integration.
2.2. Localization Principle

In this subsection we will explain the non-abelian localization principle which is a generalization of the localization principle of Duistermaat and Heckman.

Consider an equivariantly closed form \( \alpha \) on \( X \); then, for any real number \( t \) and any ‘nice’ \( \lambda \in \Omega^*_G(X) \), one has,

\[
\oint_X \alpha = \oint_X \alpha \exp(t d_G \lambda).
\]  \hspace{1cm} (2.2.1)

This is a result of the fact that the form \( \alpha(1 - \exp(t d_G \lambda)) \) is equivariantly exact and thus integrates to zero, by construction. Thus, if one writes the integral of \( \alpha(1 - \exp(t d_G \lambda)) \) and takes the \( \alpha \) term to one side and the exp term to the other side, we have the result.

Now, let us consider a specific case of the above formula. For our purposes, we will consider \( \alpha \) to be independent of \( v \) and we shall assume \( \lambda \) is independent of \( v \) also. Furthermore, we shall assume that \( \lambda \) is an equivariant one-form. Thus, if we choose an orthonormal basis \( T_a \) of \( G \) and write \( V(v) \) as \( V_a v^a \), where \( V_a \) is a vector field on \( X \) corresponding to \( T_a \) and \( v^a \) are linear functions on \( G \). Then, equation (2.2.1) takes the form,

\[
\oint_X \alpha = \frac{1}{\text{Vol}(G)} \int_{G \times X} \frac{dv_1 dv_2 \ldots dv_m}{(2\pi)^m} \alpha \exp \left( t d\lambda - it \lambda(V_a)v^a - \frac{1}{4s}(v^a, v^a) \right),
\]  \hspace{1cm} (2.2.2)

where repeated indices are summed over. As the only \( v^a \) dependence resides in the exponential, we may complete the square and integrate out the \( v_a \) dependence. Thus, one obtains,

\[
\oint_X \alpha = \frac{1}{\text{Vol}(G)(\pi/s)^{m/2}} \int_X \alpha \exp \left( t d\lambda - t^2 s(\lambda(V_a), \lambda(V_a)) \right).
\]  \hspace{1cm} (2.2.3)

Now, as the integral is formally independent of the value of \( t \), we may take the limit as \( t \to \infty \). Thus, upon taking this limit, one sees that the points on \( X \) at which \( \lambda(V_a) \neq 0 \) do not contribute to the above integral. Thus, the integral is localized to the set of points on \( X \) at which \( \lambda(V_a) = 0 \). Let us enumerate the connected components of this set by the index \( \sigma \in U \), where \( U \) is some indexing set. Thus, the integral above takes the form,
\[ \int_X \alpha = \sum_{\sigma \in \mathcal{U}} Z_\sigma, \tag{2.2.4} \]

where the summand \( Z_\sigma \) corresponds to the contribution given by the connected component \( \sigma \). Now, let us consider a particular example.

If the action of \( G \) on \( X \) is Hamiltonian, as we assume it is, then corresponding to \( V_\alpha \) there exists a function on \( X \), \( \mu_\alpha \) say, such that \( -iV_\alpha \omega = d\mu_\alpha \), where \( \omega \) is the symplectic form on \( X \). Now, let us consider an \( \alpha \) for our specific example given by,

\[ \alpha = \exp(\omega - i\mu_\alpha v^\alpha). \tag{2.2.5} \]

Thus, with this particular \( \alpha \) the equivariant integral takes the form,

\[ \int_X \alpha = \frac{1}{\text{Vol}(G)} \int_{G \times X} \frac{dv_1dv_2\ldots dv_m}{(2\pi)^m} \exp \left( \omega - i\mu_\alpha v^\alpha - \frac{1}{4s}(v^\alpha, v^\alpha) \right). \tag{2.2.6} \]

Now, if we perform the \( v^\alpha \) integral, then the above integral takes the form,

\[ \int_X \alpha = \frac{1}{\text{Vol}(G)(\pi/s)^{2m}} \int_X \frac{\omega^n}{n!} \exp \left( -s(\mu, \mu) \right). \tag{2.2.7} \]

Thus, in our particular case, the integral is localized over the elements of \( X \) at which \((\mu, \mu) = 0\). This result will come in handy later on when we apply all this stuff to background independent classical closed string field theory and the Zamolodchikov C-Function. In fact, this will be the exact integral that we employ.

3. The Space of All Two-Dimensional Field Theories

In this section we will derive the basic properties of the space of all two-dimensional field theories. First, let us consider what we will mean when we say the space of ‘all’ two-dimensional field theories. Essentially, what we will mean is the space of all two-dimensional field theories which have an interpretation as string theories. Therefore, if we wish to consider all such field theories, we must have a natural notion of what fields are
space-time fields and what fields are world-sheet fields. This notion is easily obtained by
limiting ourselves in considering only the common world-sheet fields $X^\rho$ and $h_{ab}$. Given
these world-sheet fields, we may now easily construct the space-time fields of our the-
ory. This is done by first considering all possible combinations of the world-sheet fields
$\{X^\rho, h_{ab}, \epsilon_{ab}, \nabla_{a_1} X^\rho, \nabla_{a_2} \nabla_{a_1} X^\rho, \ldots \}$ into operators with only space-time indices. For ex-
ample, one could have an operator of the form $h^{ab} \nabla_a X^\rho \nabla_b X^\xi \sqrt{h}$. Now, let us denote an
arbitrary such operator as $\hat{O}^{\rho_1 \rho_2 \ldots \rho_n}(r)$, where $r$ is a world-sheet point. Given the set of all
such operators, let us introduce one space-time field for each such operator. For instance,
$\hat{O}^{\rho_1 \rho_2 \ldots \rho_n}(r)$ would correspond to a space-time field $\bar{F}_{\rho_1 \rho_2 \ldots \rho_n}(x)$, where $x$ is a space-time
point. Now, with this information we may define a two-dimensional field theory.

We can define a two-dimensional field theory with this data by simply defining the La-
grangian to be the summation of all possible operators contracted with their corresponding
space-time fields. More specifically, we define the action of this two-dimensional theory as
follows,

$$ S_{2DFT} = \int_\Sigma \sum_{\text{All Possible } \hat{O}^{\rho_1 \rho_2 \ldots \rho_n}(r)} \hat{O}^{\rho_1 \rho_2 \ldots \rho_n}(r) \bar{F}_{\rho_1 \rho_2 \ldots \rho_n}(X^\rho(r)) \ d^2 r, \quad (3.1) $$

where $\Sigma$ is a Riemann surface. However, note that this is a two-dimensional field theory
and the dynamical fields are $X^\rho$ and $h_{ab}$, not the space-time fields. The space-time fields
only appear as coupling constants. Thus, any given set of space-time fields which saturates
all possible $\hat{O}^{\rho_1 \rho_2 \ldots \rho_n}(r)$’s defines a two-dimensional field theory for $X^\rho$ and $h_{ab}$. Therefore,
the space of all two-dimensional field theories is equivalent to the space of all possible sets
of space-time fields which saturate the $\hat{O}^{\rho_1 \rho_2 \ldots \rho_n}(r)$’s.

However, in our considerations we will only be concerned with the space of all renor-
malizable two-dimensional interactions. Therefore, we will only have need of the renor-
malizable interactions in equation (3.1). Let us denote a generic renormalizable operator by $O^{\rho_1 \rho_2 \ldots \rho_n}(r)$. Similarly, let us denote a space-time field which corresponds to a two-
dimensional renormalizable operator by $F_{\rho_1 \rho_2 \ldots \rho_n}(x)$. Thus, as before, the space of all
renormalizable two-dimensional field theories is equivalent to the space of all possible sets
of space-time fields which saturate the $O^{\rho_1 \rho_2 \ldots \rho_n}(r)$’s. Let us denote the space of all renor-
malizable two-dimensional field theories by $\mathcal{M}$. A point in this space is a set of space-time
fields $\{\ldots, F_{\rho_1 \rho_2 \ldots \rho_n}(x), F_{\rho_1 \rho_2 \ldots \rho_n}(x), \ldots\}$ which saturates all possible renormalizable combi-
nations of $\{X^\rho, h_{ab}, \epsilon_{ab}, \nabla_{a_1} X^\rho, \nabla_{a_2} \nabla_{a_1} X^\rho, \ldots \}$. 

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Now, as in most cases of ‘theories with symmetries,’ we must mod the naïve configuration space of the theory, $\mathcal{M}$ in this case, by the gauge symmetries of the theory. However, we run into a bit of trouble here as the fields in our naïve configuration space $\mathcal{M}$ do not appear dynamically in the two-dimensional action, they appear as coupling constants. But, we have no ‘natural’ notion of a gauge symmetry for coupling constants. Thus, we must improvise. In general, a gauge symmetry is a canonical transformation of the fields in the theory. A canonical transformation on the theory’s fields is defined in such a manner that it leaves the Lagrangian invariant up to a total derivative. Thus, if we extend this notion to the case at hand, we can define a gauge transformation of $p \in \mathcal{M}$ to be any transformation of $p$’s fields which leaves the two-dimensional theory $p$ corresponds to invariant up to a total derivative. As one may check explicitly in the case of the space-time metric, which corresponds to the operator $h^{ab}\nabla_a X^\rho \nabla_b X^\xi \sqrt{h}$, this is the correct prescription. Thus, as we now have a natural notion of a gauge symmetry for all the fields in $\mathcal{M}$, we are ready to pass to the true configuration space. This is simply done by modding $\mathcal{M}$ by the action of the gauge symmetries above. Let us denote the resulting space by $\mathcal{M}$, the true configuration space.

As we now have some knowledge as to what the true configuration space of closed string field theory is as a set, we may further explore its properties by imposing various structures on $\mathcal{M}$, a metric and a symplectic form.

First, we will impose a metric upon the space $\mathcal{M}$. Before we set off demanding a metric on $\mathcal{M}$, let us remind ourselves of what a metric is. A metric at a point $p \in \mathcal{M}$, by definition, is a multi-linear symmetric non-degenerate map from $T_p \mathcal{M} \otimes T_p \mathcal{M}$ to the real numbers. Thus, if we are to define a metric on $\mathcal{M}$, we must exhibit such a map for all $p \in \mathcal{M}$. Let us choose some $p \in \mathcal{M}$ and also two elements of $T_p \mathcal{M}$, $V_{1A}$ and $V_{2B}$ say, where the indices $A$ and $B$ run over all of the ‘field directions’ of $\mathcal{M}$. Now, we must exhibit a scalar corresponding to these two vectors. Let us first note that we have a natural map from the space of vectors over $p$ to the space of operator valued vectors over $p$. This is simply the map $V_A \rightarrow O^A V_A$. Now, we have a natural notion of a scalar which is associated to two operator valued vectors, their space-time scattering amplitude. Thus, we define the metric $g^{AB}$ on $\mathcal{M}$ at $p$ as follows,

$$
g^{AB} V_{1A} V_{2B} \equiv \sum_{\text{All Riemann Surfaces} \Sigma \text{ Mod Symmetries}} \int_{\Sigma \times \Sigma} < O^A V_{1A}(\sigma_1) O^B V_{2B}(\sigma_2) > d^2 \sigma_1 d^2 \sigma_2, \quad (3.2)$$
where the summation is taken over all $g = 0$ Riemann surfaces mod the symmetries of the two-dimensional theory defined by the point $p$. As one may note, this is a rather poorly defined metric, some of its entries may yield infinities. However, to get rid of these infinities, we will introduce a world-sheet cut-off $\lambda$. With this cut-off in place, the metric above is well defined, but dependent upon the cut-off. However, in the next section we will implement a renormalization program which will allow us to remove the dependence upon the cut-off. Also, note that the metric is trivially multi-linear and symmetric. Its non-degeneracy follows from its relation to Zamolodchikov’s metric or from space-time unitary.

Now, we will simply assume the existence of a symplectic form $\omega_{AB}$ over $\mathcal{M}$. As seen in Witten’s paper, the construction of such a symplectic form is fiendishly difficult in the case of open strings, and, as a rule, closed strings present much more of a problem. Therefore, we will simply assume one exists and see what we may derive from its existence. With these two structures over $\mathcal{M}$, we are ready to use some of $\mathcal{M}$’s geometrical properties to define the renormalization group and explore the renormalization group’s topological properties.

4. The Renormalization Group

In this section we will derive various properties of the renormalization group which acts on $\mathcal{M}$, the true configuration space. This section will be divided into two subsections. In the first subsection we will loosely recount the article of Dole on the renormalization ‘group.’ In the second subsection we will define a more general notion of the renormalization group, the actual group as opposed to the renormalization Lie algebra which is commonly referred to as the renormalization group. Then, we will derive various topological properties of the renormalization group which we will use in our computations.

4.1. Renormalization Lie Algebra and Lie Transport

In this subsection we will derive various properties of the renormalization Lie algebra, which is commonly called the renormalization group. These geometrical properties of the renormalization Lie algebra will serve as motivation for the definition of the full renormalization group in the next subsection.

Let us start this subsection by noting that, as mentioned previously, the space-time fields, which define coordinates on the space $\mathcal{M}$, appear as coupling constants in the
two-dimensional world-sheet field theories. However, as of yet, we have not renormalized these parameters. Thus, they are bare parameters and, as of yet, unphysical. Thus, as in conventional quantum field theory, we must implement some renormalization program so as to yield physical values for the coordinates of $\mathcal{M}$.

Let us start in a rather conventional manner by assuming that the bare fields which describe points in $\mathcal{M}$ are functions of the renormalized fields. Also, let us introduce a regularization parameter or parameters $\epsilon$; for a cut-off $\lambda$ one has $\epsilon = \kappa/\lambda$, where $\kappa$ is a renormalization point. Furthermore, let us assume the bare fields are functions of the regularization parameter(s) $\epsilon$. Now, as the bare couplings previously provided a coordinate system on the space $\mathcal{M}$, the new renormalized couplings will also provide a coordinate system on $\mathcal{M}$, provided that the transformation between the bare and renormalized couplings is not singular. Similarly, if we choose some other renormalization scheme, this will lead to a new set of renormalized space-time fields and thus a new set of coordinates on $\mathcal{M}$. In other words, various schemes of renormalization correspond to different coordinate systems on $\mathcal{M}$.

Now, let us consider how correlation functions are affected by the choice of renormalization scheme. Let us consider computing a correlation function, in some renormalization scheme, which is of the following form $< \ldots O_N(r_n)O_M(r_m) \ldots >$, where $r_n$ and $r_m$ are points on a Riemann surface such that $r_n \neq r_m$ for all $n \neq m$ and the indices $N$ and $M$ refer to specific operators in the space of renormalizable two-dimensional operators. If we choose some second renormalization scheme in which to compute this correlation function, we find a new correlation function. As a new renormalization scheme is equivalent to a change of coordinates on $\mathcal{M}$, the new coordinates of $\mathcal{M}$ will be related to the old via a simple functional relationship. In other words, if the old renormalization scheme had coordinates $F^A$, where $A$ runs over all ‘field directions’ of the theory, the new scheme has coordinates $\hat{F}^A(F^A)$. The correlation function, as computed in the new system, is then related to the correlation function computed in the old system via a tensor transformation $\mathbb{F}$. In other words, such correlation functions are tensors over $\mathcal{M}$ and transform accordingly under a change of coordinates on $\mathcal{M}$, which, of course, corresponds to a change in the renormalization scheme.

However, as we wish physically measurable correlation functions to be invariant under choice of renormalization scheme, such correlation functions are not to be considered ‘physical.’ We need to define some other quantity which holds all of the information of these correlation functions, but is invariant under a change of coordinates on $\mathcal{M}$, equivalently,
a change in the renormalization scheme. Thus, what we really desire is to describe the correlation functions as scalars on \( \mathcal{M} \). The simplest manner in which to get a scalar from the tensor correlation functions above is to contract the tensors above with the appropriate number of one-forms so as to obtain a scalar. This is what we shall do.

If we consider a point \( p \) which is in \( \mathcal{M} \), then it is defined by a set of space-time fields \{\ldots, F^N(x), F^M(x), \ldots\} which saturates all possible renormalizable combinations of \{\( X^\rho, h_{ab}, \epsilon_{ab}, \nabla_a X^\rho, \nabla_a \nabla_b X^\rho, \ldots \)\}. Thus, a one-form on \( \mathcal{M} \) at \( p \) corresponds to an arbitrary variation of the fields which define \( p \) modded by the gauge symmetries of the theory. In other words, \( dF^N = \delta F^N \) mod symmetries. Thus, if we have a correlation function of the form \(< \ldots O_N(r_n)O_M(r_m) \ldots >\), then we must produce a set of one-forms \{\ldots dF^N, dF^M \ldots\}. With these one-forms we can create a scalar over \( \mathcal{M} \) by contracting, \(< \ldots O_N dF^N(r_n)O_M dF^M(r_m) \ldots >\). This quantity is a scalar over \( \mathcal{M} \) and is therefore invariant under changes in the coordinates on \( \mathcal{M} \), which, of course, correspond to changes in the renormalization scheme. Therefore, we will take these scalars as our physical observables.

However, the thrust of renormalization theory is the idea that the physical observables calculated at some point \( p \in \mathcal{M} \) should be the same as those calculated at a point \( p + \delta \beta \), where \( \delta \) is an infinitesimal parameter and \( \beta \) is the theory’s beta-function. In other words, the beta-function \( \beta_A \) is a vector field on \( \mathcal{M} \) and it produces a one-parameter flow on \( \mathcal{M} \) such that the physical correlation functions are invariant under this flow. In terms of equations, we may write this as follows,

\[
< \ldots O_N dF^N(r_n)O_M dF^M(r_m) \ldots > (p) - < \ldots O_N dF^N(r_n)O_M dF^M(r_m) \ldots > (p + \delta \beta) = 0. \tag{4.1.1}
\]

Now, with this statement of the renormalization group equation, we can, with a bit of mathematics, interpret the renormalization Lie algebra in a more geometric setting. The first step in doing so is dividing both sides of the above equation by \( \delta \). Then, we take the limit as \( \delta \to 0 \). Then, after noting the definition of the Lie derivative, one obtains,

\[
\mathcal{L}_B < \ldots O_N dF^N(r_n)O_M dF^M(r_m) \ldots >= 0 \tag{4.1.2}
\]

In other words, the renormalization group equation expresses the fact that the physical correlation functions of the theory are invariant with respect to Lie transport along the vector fields defined by the beta-functions.
4.2. The Renormalization Group

Before we plunge straight into a definition of the renormalization group, let us consider a simple example from general relativity which will act to guide us through this definition. In general relativity when we consider a space-time manifold $\mathcal{M}$ and the space of diffeomorphisms which act on this manifold $\text{Diff}(\mathcal{M})$, the tangent space at the identity of $\text{Diff}(\mathcal{M})$ is the Lie algebra $\text{diff}(\mathcal{M})$. The Lie algebra $\text{diff}(\mathcal{M})$ acts on $\mathcal{M}$ in a simple manner. If $v$ is an element of this Lie algebra, then its action on $\mathcal{M}$ corresponds to a diffeomorphism of $\mathcal{M}$ generated by a vector field $V(v)$ on $\mathcal{M}$. This is simply the normal notion of an infinitesimal diffeomorphism, common in general relativity. Let us now consider how this applies to our present case of the renormalization group.

As we saw in the last section, the beta-functions act on $\mathcal{M}$ as infinitesimal transformations. This is exactly analogous to the action of the vector fields $V(v)$ on $\mathcal{M}$ in the case of general relativity. Thus, we are led to consider the beta-functions as the vectors on $\mathcal{M}$ which correspond to the Lie algebra of the renormalization group. However, we are now left with the task of defining exactly what the renormalization group is.

If we consider the physical motivation behind the renormalization program, then we see that its core idea is that the physical correlation functions calculated at some point $p \in \mathcal{M}$ should be the same as those calculated at a point $p + \delta \beta$. However, in the last section we proved that this is equivalent to requiring that the transformations of the renormalization Lie algebra on $\mathcal{M}$ be such that they are Killing vectors of all physical correlation functions over $\mathcal{M}$. Thus, with this as a motivation, we should define the renormalization group as the set of all homeomorphisms of $\mathcal{M}$ to itself which leave the physical correlation functions invariant. In other words, if $\phi$ is a member of the renormalization group $\mathcal{RG}$, then for any physical correlation function $\left< \ldots O_N dF^N(r_n) O_M dF^M(r_m) \ldots \right> \!(p)$ at $p \in \mathcal{M}$ one has $\left< \ldots O_N dF^N(r_n) O_M dF^M(r_m) \ldots \right> \!(p) = \left< \ldots O_N dF^N(r_n) O_M dF^M(r_m) \ldots \right> \!(\phi(p))$. We will take this as our definition of the renormalization group. It seems the only logical definition which we may make, and, of course, it has the correct tangent space, the beta-functions.

Now, as we’ve defined the renormalization group, let us consider some of its topological properties. First of all, we may introduce a ‘natural’ metric $G(\ , \ )$ on $\mathcal{RG}$. To do this let us first consider what a metric on $\mathcal{RG}$ would be. By definition, a metric at $q \in \mathcal{RG}$ is a multi-linear symmetric non-degenerate map from $\mathcal{TRG} \otimes \mathcal{TRG}|_q$ to the real numbers. Now, as we have previously said in the case that $q$ is the identity, the tangent space of $\mathcal{RG}$
at a point $q$ can be identified with a set of vectors on $q(M)$. Thus, a metric at $q$ on $\mathcal{RG}$ is equivalent to a metric on the space of vectors over $q(M)$. To wit, if we consider two elements of $T\mathcal{RG}|_q$, $b_1$ and $b_2$ say, then they correspond to two vector fields $\beta_{1A}$ and $\beta_{2B}$ over $q(M)$. Now, we may introduce a scalar which corresponds to the inner product of $b_1$ and $b_2$,

$$
G(b_1, b_2) \equiv \int_{q(M)} g^{AB} \beta_{1A} \beta_{2B} \, DF.
$$

(4.2.1)

However, as $\mathcal{RG}$ is now a metric space, we can appeal to a theorem of topology to draw some conclusions about the behavior of $\mathcal{RG}$. A standard theorem of topology states that if a space is a metric space, then limit point compactness is equivalent to normal notion of compactness. What this means is that to prove a metric space is compact one only needs to prove that every infinite sequence of points in the space has a limit point which is also in the space. We will use this to explicitly compactify $\mathcal{RG}$.

Consider the set of all infinite sequences in $\mathcal{RG}$. Let us denote this set by $\{\{q_n\}\}$. Furthermore, let us consider the set of all limit points of these sequences. We will notate this set by $\{q_\infty\}$. Now, we will, rather explicitly, compactify $\mathcal{RG}$ by defining all the points of $\{q_\infty\}$ to be elements of $\mathcal{RG}$. We will refer to this compactified renormalization group as $\overline{\mathcal{RG}}$. For future reference, we will also need to consider the component of $\overline{\mathcal{RG}}$ which is homotopic to the identity. We will call this space $\overline{\mathcal{RG}}_0$. Now that we’ve explored some of the topological properties of the renormalization group, we are ready to put them to use in computing some exact path-integrals.

5. String Field Theory and the Zamolodchikov $C$-Function

In this section we will employ a non-abelian generalization of the Duistermaat-Heckman Theorem to localize the path-integral of the action $C^2$ over $\mathcal{M}$ to the fixed points of the group $\overline{\mathcal{RG}}_0$. Thus, as we will use the non-abelian generalization of the Duistermaat-Heckman Theorem, let us first remind ourselves of the arena in which it applies.

Consider the action of a compact, connected group $G$, with Lie algebra $\mathcal{G}$, on a manifold $X$. Let us assume that $X$ is a symplectic manifold of dimension $2n$ with a symplectic form $\omega$. The action of $G$ is said to be Hamiltonian, as we assume it is, if it is induced from a map $\tilde{\mu} : \mathcal{G} \to \Lambda^0(X)$. In other words, for each element $t$ of $\mathcal{G}$ there exists a vector
field \( T(t) \) on \( X \), which represents the action of \( t \) on \( X \), and function \( \mu(t) \) on \( X \) such that 
\[-i_{T(t)} \omega = d\mu(t) \]. As \( G \) is taken to be multi-dimensional, we can choose various such \( t \)'s. Let us denote a set of them by \( t_n \). As the \( t_n \) are members of \( G \), a vector space, we may take their vector sum \( t_\Sigma \) and introduce a function \( \mu(t_\Sigma) \) which corresponds to this vector sum. Also, as the elements of \( G \) naturally correspond to vector fields on \( X \), the dual of \( G \), \( G^* \), corresponds naturally to some set of one-forms on \( X \). Thus, we may associate to the function \( \mu(t_\Sigma) \) a natural element of \( G^* \) given by \( d\mu(t_\Sigma) \). Now, if we introduce a positive definite invariant quadratic form \( (\ , \ ) \) on \( G^* \), then we may consider integrals of the form,

\[
Z = \int_X \frac{\omega^n}{n!} \exp \left( -s(d\mu(t_\Sigma), d\mu(t_\Sigma)) \right),
\]

(5.1)

where \( s \) is a positive constant. Such integrals can be exactly evaluated. As we proved previously, they are localized about the minima of \( (d\mu(t_\Sigma), d\mu(t_\Sigma)) \). As \( (\ , \ ) \) is a positive definite invariant quadratic form, the minima of \( (d\mu(t_\Sigma), d\mu(t_\Sigma)) \) occur at the points \( d\mu(t_\Sigma) = 0 \). Thus, the integral is equivalent to a summation of various terms, each term in the sum corresponds to a connected component of the set of points at which \( d\mu(t_\Sigma) = 0 \). Thus, the integral takes the following form,

\[
Z = \sum_{\text{Components of } d\mu(t_\Sigma) = 0} Z_\sigma.
\]

(5.2)

Now, let us consider how to apply this to the case at hand. In the case at hand, we will take the manifold \( X \) as \( \mathcal{M} \). However, we will not take the group \( G \) as \( \overline{G}_0 \); we will use a group closely related to \( \overline{G}_0 \). So, let us set about defining this auxiliary group. We will call this auxiliary group \( \overline{G}_0 \perp \). First, consider an element \( q \) of \( \overline{G}_0 \). As all the elements of \( \overline{G}_0 \) are, by definition, homotopic to the identity, there exists a path \( \Gamma \) in \( \overline{G}_0 \) which connects \( q \) with the identity. Consider starting at the identity end of this path. As the exponential map from \( \overline{T \mathcal{G}_0}_{id} \) to a small neighborhood of \( id \) is an isomorphism, we may parameterize \( \Gamma \) in some small neighborhood about \( id \) with a scalar \( \bar{\epsilon}_1 \) and an element of \( \overline{T \mathcal{G}_0}_{id} \), \( \beta_{1A} \) say. Furthermore, let us assume that \( \epsilon_1 \) corresponds to the point on \( \Gamma \) which is the farthest from \( id \), but yet still within the range of the exponential map’s isomorphism status. Let us denote the element of \( \Gamma \) corresponding to \( \epsilon_1 \) by \( q_1 \). Again, we may repeat this construction at \( q_1 \) and obtain a point \( q_2 \) on \( \Gamma \) and an element \( \beta_{2A} \) of \( \overline{T \mathcal{G}_0}_{q_1} \) and a scalar
\( \epsilon_2 \). We can proceed to parameterize all of \( \Gamma \) in this manner. (Note that this will require a finite number of steps, if we choose a ‘reasonable’ path, as \( \mathcal{RG}_0 \) is compact). So, we end up with a set \( \{ (\epsilon_n, \beta_{nA}) \} \) which parameterizes the path \( \Gamma \). However, we can also obtain the original transformation \( q \) by applying the sequence of infinitesimal transformations given by \( \{ (\epsilon_n, \beta_{nA}) \} \) to \( M \), where at the \( n^{th} \) step for \( p \in q_{n-1}(M) \), \( p \rightarrow p + \epsilon_n \beta_n \). Now, let us relate all this to the auxiliary group \( \mathcal{RG}^\perp_0 \).

For each element \( q \) in \( \mathcal{RG}^\perp_0 \), we will define a corresponding element \( q^\perp \) in \( \mathcal{RG}^\perp_0 \) as follows. Choose a path \( \Gamma \) in \( \mathcal{RG}^\perp_0 \) as above and parameterize such a path by the set \( \{ (\epsilon_n, \beta_{nA}) \} \). Now, we define \( q^\perp \) as the transformation on \( M \) which results from the sequence of transformations \( \{ (\epsilon_n, \beta_{nA}) \omega^{AB} g_{BC} \} \). Thus, as one may trivially see, \( \mathcal{RG}^\perp_0 \) is compact as a result of \( \mathcal{RG}_0 \)’s compactness; also, \( \mathcal{RG}^\perp_0 \) is connected by construction. Thus, we will take our group to be \( \mathcal{RG}^\perp_0 \), as it satisfies the hypothesis of the localization theorem and, as we shall see, its Hamiltonian has a nice interpretation.

So, now let us consider what the function \( \mu(t_\Sigma) \) will be in the case at hand. As above, we may consider an element in \( T\mathcal{RG}^\perp_0 |_{id} \) and we have a natural notion of which element to choose, that corresponding to the beta-function on \( M \). Thus, the vector field on \( M \) which corresponds to this vector in \( T\mathcal{RG}^\perp_0 |_{id} \) is simply the vector field \( \beta_A \omega^{AB} g_{BC} \). This vector field is easily shown to have the same fixed point set as the original beta-function vector field \( \beta_A \), as \( \omega^{AB} \) and \( g_{AB} \) are non-degenerate \( \beta_A \) and \( \beta_A \omega^{AB} g_{BC} \) both vanish at the same points. The function on \( M \) which generates this vector field via its Hamiltonian flow is \( \Psi \), where,

\[
(\beta_C \omega^{CB} g_{BA}) \omega^{AD} = d\Psi^D. \tag{5.3}
\]

Thus, if we note that \( \omega_{AB} \omega^{BC} = \delta_A^C \), then the above equation becomes,

\[
\beta_A g^{AB} = d\Psi^B. \tag{5.4}
\]

However, this is essentially the definition of the Zamolodchikov \( C \)-Function. For the Zamolodchikov \( C \)-Function \( C \) at any point \( p \in M \) one has \( \beta_B dc^B = \Omega^{-1} \beta_A g^{AB} \beta_B \), where \( \Omega \) is some normalization constant. Also, Zamolodchikov proved \( C \) that near the fixed point set of \( T\mathcal{RG}^\perp_0 \) one has \( dc^B = \Omega^{-1} \beta_A g^{AB} \). Thus, we will introduce a generalization of the Zamolodchikov \( C \)-Function, \( C \), such that \( dc^B = \Omega^{-1} \beta_A g^{AB} \) at all \( p \in M \). As one
easily sees, this reduces to the original Zamolodchikov $C$-Function in the two cases above. Thus, $\Psi$ is equivalent to the new Zamolodchikov $C$-Function up to an additive and a multiplicative constant. However, this has some interesting consequences. If we consider the integral of equation (5.1) in this case, then we find it takes the following form,

$$Z = \int_{\mathcal{M}} DF \exp \left( -s(\Omega dC, \Omega dC) \right),$$  \tag{5.5}$$

where $(\cdot, \cdot)$ is a positive definite invariant quadratic form on $T^*\mathcal{R}G_0|_{id}$ and $DF$ is the symplectic measure on $\mathcal{M}$. However, this path-integral, as stated previously, is localized over the set of points at which $dC = 0$, just the set of points at which $c$ and $C$ agree. But, as is commonly known \cite{7}, see chapter three, requiring the beta-functions to vanish is equivalent to requiring that the space-time field’s classical stringy equations of motion are satisfied. Thus, if $\beta_A = 0$ at some point $p$, then, for the space-time classical closed string field theory action $S$, $dS^A = 0$ at $p$. From the definition of the new Zamolodchikov $C$-Function, one sees that if $\beta_A = 0$, then $dC = 0$, as $g_{AB}$ is non-degenerate. Thus, the points at which $dC = 0$ are exactly the classical solutions of classical closed string field theory. Furthermore, we may choose the positive definite invariant quadratic form $(\cdot, \cdot)$ such that $(\Omega dC, \Omega dC) = \Omega^2 C^2$. Thus, with this positive definite invariant quadratic form, we can exactly compute the path-integral of the field theory defined by $\Omega^2 C^2$. In other words, we may evaluate,

$$Z = \int_{\mathcal{M}} DF \exp \left( -s\Omega^2 C^2 \right),$$  \tag{5.6}$$

exactly as it reduces to a sum over the points on $\mathcal{M}$ which satisfy $dS = 0$, the classical solutions to string field theory. Thus, it seems from the above argument that the field theory with action $\Omega^2 C^2$ is a ‘topological field theory!’ I am not really sure of the deeper meaning of this fact; however, it is rather interesting and deserves further investigation.

An additional interpretation of the background independent classical closed string field theory action $S$ is also afforded by the above exposition. But, we must make some regularity assumptions on the form of the background independent classical closed string field theory action, see \cite{8}. We assume that we can split the derivatives $\delta S/\delta F^A$ into $k$ ‘independent functions’ $I_A$ and some ‘dependent functions’ $D_A$ in such a way that the field equations are $I_A = 0$, and the form $dI_1 \wedge dI_2 \ldots \wedge dI_k$ does not vanish on the set of points.
at which $dS = 0$. Thus, if we make these regularity assumptions on the action $S$, then we have the following result. Any function $\Phi(F)$ on $\mathcal{M}$ which vanishes only on the set $dS = 0$ is of the following form,

$$
\Phi(F) = Q^A \frac{\delta S}{\delta F^A},
$$

(5.7)

where $Q^A$ is a one-form on $\mathcal{M}$. This result is rather easy to prove. The regularity conditions that we imposed imply that the set of ‘functions’ $I_A$ can be used as the first $k$ coordinates of a new coordinate system on $\mathcal{M}$. Thus, if we define the remaining coordinates to be given by $J_A$, then we may write $\Phi(F)$ in the following form,

$$
\Phi(I, J) = \Phi(I = 0, J) + \int_0^1 d\tau \frac{d\Phi(\tau I, J)}{d\tau}.
$$

(5.8)

Thus, we can simplify the above equation by making a change of variables in the integral and noting $\Phi$ vanishes at $I = 0$, to obtain,

$$
\Phi(I, J) = I_A \int_0^1 d\tau \frac{d\Phi(\tau I, J)}{dI_A}.
$$

(5.9)

Now, if we remember the fact that $I_A$ are the field equations, one has our result,

$$
\Phi(I, J) = \frac{\delta S}{\delta F^A} \int_0^1 d\tau \frac{\delta \Phi(\tau I, J)}{\delta I_A},
$$

(5.10)

were we have called the integral above $Q^A$. Thus, applying this to the case at hand, we find that, as the ‘functions’ $dC^A$ vanish only on $\mathcal{M}$ at the points $dS = 0$, $dC^A$ must be of the following form,

$$
dC^A = Q^A_B dS^B,
$$

(5.11)

where $Q^A_B$ is a vector valued one-form over $\mathcal{M}$ given by,

$$
Q^A_B \equiv \int_0^1 d\tau \frac{\delta^2 C(\tau I, J)}{\delta I_A \delta F^B},
$$

(5.12)
where $F$ runs over both $I$ and $J$. Thus, if we assume all of the equations of motion of string field theory are independent, we have a natural relation between the new Zamolodchikov $C$-Function and the background independent classical closed string field theory action $S$,

$$dC^A = dS^B \int_0^1 d\tau \frac{\delta^2 C(\tau F)}{\delta F_A \delta F^B}$$  \hspace{1cm} (5.13)

Again, what this means? However, it does lead to a new and novel interpretation of the action $S$. Indeed, if $Q^A_B$ is invertible, then one has, up to a physically irrelevant constant,

$$S = \int Q^{-1}_B A dC^A \ DF_B.$$  \hspace{1cm} (5.14)

Which expresses the background independent classical closed string field theory action $S$ completely in terms of the new Zamolodchikov $C$-Function!

\section{6. Conclusions}

In this article we’ve proved that the field theory action $C^2$ yields a ‘topological field theory’ over $\mathcal{M}$, the space of all two-dimensional field theories. Also, we’ve proved that the background independent classical closed string field theory action $S$ is intimately related to the new Zamolodchikov $C$-function. (However, with both we’ve assumed $\omega_{AB}$’s existence). Thus, with these new results we should be able to gain some insights into background independent closed string field theory. As the theory with the action $C$ is the ‘square-root’ of a ‘topological field’ theory and $C$ is so intimately related to $S$, we should probably think of $S$ as the ‘square-root’ of the ‘topological field theory’ $C^2$. What this means?

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