NONCOMPACT COMPLETE RIEMANNIAN MANIFOLDS WITH DENSE EIGENVALUES EMBEDDED IN THE ESSENTIAL SPECTRUM OF THE LAPLACIAN

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Abstract. We prove sharp criteria on the behavior of radial curvature for the existence of asymptotically flat or hyperbolic Riemannian manifolds with prescribed sets of eigenvalues embedded in the spectrum of the Laplacian. In particular, we construct such manifolds with dense embedded point spectrum and sharp curvature bounds.

1. Introduction and main results

Let \((M_n, g)\) be an \(n\)-dimensional noncompact complete Riemannian manifold. The Laplace-Beltrami operator \(\Delta = \Delta_g\) on \((M_n, g)\) is essentially self-adjoint on \(C_0^\infty(M_n)\). We also denote by \(\Delta\) its unique self-adjoint extension to \(L^2(M_n, dv_g)\).

For compact \(M_n\), there is a wealth of results on the relations between the geometry of the manifold and spectral properties of the Laplacian (which in this case only has discrete eigenvalues). These relations are not so well studied for the noncompact case, which is the subject of this paper. We will mention some results later in this article but refer the readers to [6] for a more complete review. The past work has been mostly focused on proofs of the purity of absolutely continuous spectrum and absence of embedded eigenvalues. Here we study the opposite question.

If \(M\) has constant radial curvature \(-K_0\), then, for negative \(K_0\), \(-\Delta\) has a complete set of eigenvalues \(\{\lambda_n \geq 0\}\) with \(\lambda_n \to \infty\) \([3]\). For \(K_0 \geq 0\), \(\sigma(-\Delta) = \sigma_{\text{ess}}(-\Delta) = \left[\frac{K_0}{4} (n-1)^2, \infty\right)\) and there are no eigenvalues \([3]\). For perturbations of the latter case it is natural to expect that whether there are eigenvalues will depend on the size of the perturbation. Perturbations on a compact set can only create eigenvalues below the essential spectrum. Thus the question whether one can embed eigenvalues in the essential spectrum will depend on the rate of approach of \(-K_0\) by the radial curvature \(K(r)\) of the perturbation, at infinity. Manifolds with \(K(r) \to -K_0\) as \(r \to \infty\) are called asymptotically flat if \(K_0 = 0\) and asymptotically hyperbolic if \(K_0 = 1\). The case \(K_0 > 0\) can be rescaled to \(K_0 = 1\) but we find it more useful to keep \(K_0\) and will call all manifolds with \(K(r) \to -K_0, K_0 > 0\), asymptotically hyperbolic.

In this paper we answer the following question. Given any finite or countable (possibly dense) set \(A \subset \sigma(-\Delta) = \left[\frac{K_0}{4} (n-1)^2, \infty\right)\), can we construct an asymptotically flat or hyperbolic \(n\)-dimensional manifold with an embedded (in the absolutely continuous spectrum) eigenvalue at each \(\lambda \in A\)? How is it influenced by the asymptotical behavior of the radial curvature?

We prove

**Theorem 1.1.** For any countable \(A \subset \left[\frac{K_0}{4} (n-1)^2, \infty\right)\) there exist asymptotically flat and asymptotically hyperbolic \(n\)-dimensional Riemannian manifolds with \(\sigma_{\text{ac}}(-\Delta) = \left[\frac{K_0}{4} (n-1)^2, \infty\right)\) and an embedded eigenvalue at each \(\lambda \in A\).

In particular, this of course implies
Corollary 1.2. There exist asymptotically flat and asymptotically hyperbolic $n$-dimensional Riemannian manifolds with dense point spectrum embedded in the absolutely continuous spectrum.

An interesting question is to study the curvature conditions for possibility to embed an arbitrary countable set in the absolutely continuous spectrum. It turns out, for finite sets it can be done with $r|K(r) + K_0| = O(1)$, while for countable sets it is enough to require that $r|K(r) + K_0| \to \infty$, no matter how slowly. We have

Theorem 1.3.  
(1) For any finite $A \subset [\frac{K_0(n-1)^2}{4}, \infty)$, there exists an $n$-dimensional manifold with $r|K(r) + K_0| < C$, $\sigma_{ac}(\Delta) = [\frac{K_0(n-1)^2}{4}, \infty)$, and an embedded eigenvalue at each $\lambda \in A$.

(2) For any countable $A \subset [\frac{K_0(n-1)^2}{4}, \infty)$ and any $C(r) > 0$ with $\lim_{r \to \infty} rC(r) = \infty$, there exists an $n$-dimensional manifold with $|K(r) + K_0| < C(r)$, $\sigma_{ac}(\Delta) = [\frac{K_0(n-1)^2}{4}, \infty)$, and an embedded eigenvalue at each $\lambda \in A$.

Remark.  
(1) Theorem 1.3 is sharp in the asymptotically hyperbolic case in the following sense. If $r|K(r) + K_0| \to 0$ (or even is bounded by a sufficiently small constant), there can be no embedded eigenvalues [13]. We conjecture it is also sharp in a similar sense in the asymptotically flat case.

(2) We conjecture that, at least in the hyperbolic case, (2) of Theorem 1.3 is sharp in an even stronger sense. Namely, that given a monotone $C(r) > 0$, for any countable $A \subset [0, \infty)$ there exists an $n$-dimensional manifold with $|K(r) + K_0| < C(r)$, $\sigma_{ac}(\Delta) = [\frac{K_0}{4}(n-1)^2, \infty)$, and an embedded eigenvalue at each $\lambda \in A$ if and only if $\lim_{r \to \infty} rC(r) = \infty$. Given Theorem 1.3, this statement would only require proving that if $C(r)$ is bounded, only eigenvalues below a certain threshold can be embedded, as is the case for $n = 1$ [15].

Let us present more detail on the history. For the asymptotically hyperbolic case, the sharp transition on a possibility to embed one eigenvalue was given by Kumura [15] based on the arguments of Kato [11]. He excluded eigenvalues greater than $\frac{K_0(n-1)^2}{4}$ under the assumption that $K_{rad} + K_0 = o(r^{-1})$, and also constructed a manifold with the radial curvature $K_{rad} + K_0 = O(r^{-1})$ and with an eigenvalue $\frac{K_0(n-1)^2}{4} + 1$ embedded into the essential spectrum $[\frac{K_0(n-1)^2}{4}, \infty)$.

Previous results on absence of embedded eigenvalues under the radial curvature conditions are reviewed in [6]. They go back to Pinsky [25], with a later milestone by Donnelly [4]. Some recent results on the absence of eigenvalues can be found in [10, 21].

For the asymptotically flat case, the absence of embedded eigenvalues results go back to [10]. Kumura, Donnelly, and Garofalo [5, 7, 16] showed the absence of positive eigenvalues of the Laplacian if the curvature $K_{rad} = o(r^{-2})$. As mentioned, we conjecture here that Laplacian has no positive eigenvalues if $K_{rad} = o(r^{-1})$.

Under certain stronger curvature decay assumption on the perturbation, limiting absorption principle, originally from Agmon’s theory [1], holds for the Laplacian. See [17, 26] and references therein. In this case, Laplacian has purely absolutely continuous spectrum.

This paper is the first one in the series where we construct manifolds with unusual spectral properties of the Laplacian and certain sharp curvature bounds. For example, in the upcoming [8], we obtain Riemannian manifolds with singular continuous spectrum embedded in the spectrum of the Laplacian.
The Riemannian manifolds \((M, g)\) we construct are rotationally symmetric, and we construct rotationally symmetric eigenfunctions, thus reducing the problem to a one-dimensional Schrödinger operator. Fix some \(O \in M\) as the origin. Using the radial coordinates (from \(O\)) we construct Riemannian manifold with the structure of the form \((M, g) = (\mathbb{R}^n, dr^2 + f_1^2(r)g_{S^{n-1}(1)})\) where \(g_{S^{n-1}(1)}\) is the standard Riemannian metric on the unit sphere, and we need to construct \(f_1\) so that the Laplacian has the desired properties. Suppose \(h(r)\) is a function on \(M\) only depending on the radius \(r\). Then the Laplacian is equivalent to the following one-dimensional Schrödinger operator,

\[
-\Delta_g(h(r)) = -\left\{ \frac{\partial^2}{\partial r^2} + (n - 1)S(r)\frac{\partial}{\partial r} \right\} h(r),
\]

where

\[
S(r) = \frac{f_1'(r)}{f_1(r)}.
\]

In order to make the manifold smooth in the neighborhood of \(O\), \(f_1^{\text{even}}(0)\) must vanish at 0 and \(f_1'(0) \neq 0\). This implies \(S(r)\) is singular at 0. Thus we need to deal with one-dimensional Schrödinger operator (1) with singularities at both 0 and \(\infty\).

In the neighborhood of \(\infty\) by the Liouville transformation, Laplacian (1) can be normalized to a Schrödinger operator of the form

\[
-\frac{d^2}{dx^2} + q(x).
\]

Constructing operators (3) with given sets embedded as eigenvalues in the essential spectrum is an old question, going back to the celebrated work of Wigner-von Neuman \[28\] who constructed an explicit potential with an embedded (given) eigenvalue.

Simon \[27\] showed that for any rate of decay \(h(x)\) that is slower than Coulomb and any countable subset \(A \subset \mathbb{R}^+\), there exists \(q(x)\) bounded by \(h(x)\), so that operator (3) (whole-line or with the desired boundary conditions) has an embedded eigenvalue at each \(\lambda \in A\). However, the potential in Simon’s construction is not continuous and thus cannot be used for our purposes. Previously, Naboko \[24\] constructed smooth potentials with the same property but only if elements of \(A\) are rationally independent.

Naboko’s construction starts from the origin. Then he first constructs piecewise-constant potentials with desired properties, which are then smoothed out. Simon uses a different method. He uses the Wigner-von Neumann type to construct the desired potential and the method of \(L^2\) perturbations to guarantee boundary conditions at the origin 0 for the eigenfunctions. His construction starts at \(\infty\), thus it is nontrivial to make it smooth.

In this paper we develop a new construction, based on piecewise Wigner-von Neumann potentials, different from both \[21\] and \[27\]. In fact, we view the construction itself as one of the important achievements of this paper. It is robust and fundamental in that it can be applied in a variety of contexts to construct embedded eigenvalues. In the forthcoming work it is adapted by one of the authors and Ong to construct eigenvalues embedded into the spectral band for perturbed periodic operator, in both continuous and discrete settings \[20, 23\], and also to construct eigenvalues embedded into the absolutely continuous spectrum for perturbed Stark type operators \[18, 22\].

First, in order to deal with singularities at both 0 and infinity, it is natural to construct Riemannian metric around 0 and \(\infty\) separately so that the two operators \(-\frac{\partial^2}{\partial r^2} - (n - 1)S(r)\frac{\partial}{\partial r}\) (one around 0, another around \(\infty\)) have the given eigenvalues.

We start at \(O\) with the standard Euclidean metric for \(r \leq \frac{1}{2}\). Thus the eigenfunctions of the Laplacian (1) are given by the Bessel functions. In the neighborhood of \(\infty\) \((r \geq 3)\) we use
piecewise Wigner-von Neumann type functions to construct the $f_1(r)$, adding the eigenvalues one (or fewer) at a time. We allow Wigner-von Neumann type potentials to be adapted in the next segment to balance the boundary conditions of the associated eigenfunctions.

When further eigenvalues are taken into consideration, we need to adapt the next segment of Wigner-von Neumann type potential to balance the new boundary conditions. However, Wigner-von Neumann type potential (associated to a fixed eigenvalue) may significantly change other eigenfunctions. As we add new eigenvalues, the change will accumulate. To overcome this difficulty, we use the quantitative analysis to study the relationships for all the Wigner-von Neumann type potentials, corresponding eigenfunctions and the other eigenfunctions. In particular, an important building block is a Theorem that allows to construct a Wigner-von Neumann type potential (associated to a fixed eigenvalue) may significantly change the decay for each previously treated energy, to ensure the overall decay.

After the separate construction, we need to connect the Riemannian metric at $r < \frac{1}{2}$ and $r > 3$ smoothly so that the eigenfunctions of the two separate operators on $r < \frac{1}{2}$ and $r > 3$ connect smoothly. This can be done if the boundary conditions of eigenfunctions match at some fixed point $r \in [\frac{1}{2}, 3]$.

As we work with the one-dimensional construction, the proof of Theorems 3.1 and 3.2 establishes also the following Theorems.

**Theorem 1.4.** Let $\{\lambda_j\}$ be an arbitrary set of distinct positive numbers. Let $\{\theta_j\}$ be a sequence of angles in $[0, \pi]$. If the set $\{\lambda_j\}$ is finite, then there exist potentials $q(x) \in C^\infty[0, \infty)$ such that

1. for each $j$, $(-D^2 + q)u = \lambda_j u$ has an $L^2(\mathbb{R}^+) \quad \Rightarrow \quad \frac{u'(0)}{u(0)} = \tan \theta_j,$

2. $|q(x)| = \frac{C(1)}{|x|^\frac{1}{2}}.$

If the set $\{\lambda_j\}$ is countable, then for any function $C(x) > 0$ on $(0, \infty)$ with $\lim_{x \to \infty} C(x) = \infty$, there exist potentials $q(x) \in C^\infty[0, \infty)$ such that

1. for each $j$, $(-D^2 + q)u = \lambda_j u$ has an $L^2(\mathbb{R}^+) \quad \Rightarrow \quad \frac{u'(0)}{u(0)} = \tan \theta_j,$

2. $|q(x)| \leq \frac{C(x)}{|x|^\frac{1}{2}}.$

**Theorem 1.5.** Let $\{\lambda_j\}$ be an arbitrary set of distinct positive numbers. If the set $\{\lambda_j\}$ is finite, then there exist potentials $q(x) \in C^\infty(-\infty, \infty)$ such that

1. for each $j$, $(-D^2 + q)u = \lambda_j u$ has an $L^2(\mathbb{R}) \quad \Rightarrow \quad \frac{u'(0)}{u(0)} = \tan \theta_j,$

2. $|q(x)| = \frac{C(1)}{|x|^\frac{1}{2}}.$

If the set $\{\lambda_j\}$ is countable, then for any function $C(x) > 0$ on $(0, \infty)$ with $\lim_{x \to \infty} C(x) = \infty$, there exist potentials $q(x) \in C^\infty(-\infty, \infty)$ such that

1. for each $j$, $(-D^2 + q)u = \lambda_j u$ has an $L^2(\mathbb{R}) \quad \Rightarrow \quad \frac{u'(0)}{u(0)} = \tan \theta_j,$

2. $|q(x)| \leq \frac{C(x)}{|x|^\frac{1}{2}}$ for $x \in \mathbb{R}.$
2. Preparation

The following result is well known. See [13] or page 93 in [8].

**Theorem 2.1.** Let \( V(x) \) be a continuous function on \((R_0, \infty)\) of the form \( V(x) = 4 \arcsin(2x \phi) / x + V_1(x) \) with \( x > R_0 \) and \( a \neq 0 \), where \( |V_1(x)| \leq \frac{C}{x^2} \). Consider the differential equation \(-y'' + V y = \lambda y\) with \( \lambda > 0 \). Then the following asymptotics holds (uniform with respect to \( \phi \)) as \( x \) goes to infinity

1. if \( \kappa \neq \pm \sqrt{\lambda} \), then there exists a fundamental system of solutions \( \{y_1(x), y_2(x)\}\) such that \( y_1(x) = \cos \sqrt{\lambda} x + O(\frac{1}{x}) \), \( y_1'(x) = -\sqrt{\lambda} \sin \sqrt{\lambda} x + O(\frac{1}{x}) \), and \( y_2(x) = \sin \sqrt{\lambda} x + O(\frac{1}{x}) \);

2. if \( \kappa = \pm \sqrt{\lambda} \), then there exists a fundamental system of solutions \( \{y_1(x), y_2(x)\}\) such that \( y_1(x) = x^{-\alpha} (\cos(\sqrt{\lambda} x + \frac{\theta}{2}) + O(\frac{1}{x})) \), \( y_1'(x) = -\sqrt{\lambda} x^{-\alpha} (\sin(\sqrt{\lambda} x + \frac{\theta}{2}) + O(\frac{1}{x})) \), and \( y_2(x) = x^{\alpha} (\cos(\sqrt{\lambda} x + \frac{\theta}{2}) + O(\frac{1}{x})) \). Moreover, \( y_1(x), y_1'(x), y_2(x), y_2'(x) \) are jointly continuous with respect to \( x, \phi \).

The following theorem is an important building block for our inductive construction. It allows to construct a potential with desired bounds that ensures decay of the solution for a given energy/boundary condition and stabilization (not much growth) of solutions for energies from a given finite set with arbitrary boundary conditions.

**Theorem 2.2.** Suppose \( \lambda > 0 \) and \( A = \{\hat{\lambda}_j > 0\}_{j=1}^k \) with \( \lambda \notin A \). Suppose \( \theta_0 \in [0, \pi] \). Let \( x_1 > x_0 > b \). For any function \( \tilde{V} \), we define

\[
\tilde{q}(x) = \frac{(n - 1)^2}{4} (\sqrt{K_0} + \tilde{V}(x))^2 + \frac{n - 1}{2} \tilde{V}'(x).
\]

Then there exist constants \( K(E, A, K_0), C(E, A, K_0) \) (independent of \( b, x_0 \) and \( x_1 \)) and potential \( \tilde{V}(x, E, A, x_0, x_1, b, \theta_0) \) such that for \( x_0 - b > K(E, A, K_0) \) the following holds:

**Curvature:** for \( x_0 \leq x \leq x_1 \), supp(\( \tilde{V} \)) \( \subset (x_0, x_1) \) and \( \tilde{V} \in C^\infty(x_0, x_1) \), and

\[
|\tilde{V}(x, E, A, x_0, x_1, b, \theta_0)| \leq \frac{C(E, A, K_0)}{x - b},
\]

and

\[
|\tilde{V}'(x, E, A, x_0, x_1, b, \theta_0)| \leq \frac{C(E, A, K_0)}{x - b}.
\]

**Solution for \( \lambda \):** the solution of \((-D^2 + \tilde{q}) y_\lambda = (\lambda + \frac{(n - 1)^2}{4} K_0) y_\lambda\) with boundary condition \( y_\lambda(x_0) = \tan \theta_0 \) satisfies

\[
\left\| \begin{pmatrix} y_\lambda(x_0) \\ \sqrt{\lambda} y_\lambda'(x_0) \end{pmatrix} \right\| \leq 2 \left( \frac{x_1 - b}{x_0 - b} \right)^{-100} \left\| \begin{pmatrix} y_\lambda(x_0) \\ \sqrt{\lambda} y_\lambda'(x_0) \end{pmatrix} \right\|,
\]

and for \( x \in [x_0, x_1] \),

\[
\left\| \begin{pmatrix} y_\lambda(x) \\ \sqrt{\lambda} y_\lambda'(x) \end{pmatrix} \right\| \leq 2 \left\| \begin{pmatrix} y_\lambda(x_0) \\ \sqrt{\lambda} y_\lambda'(x_0) \end{pmatrix} \right\|.
\]

**Solution for \( \lambda_j \):** for any solution of \((-D^2 + \tilde{q}) y_{\lambda_j} = (\lambda_j + \frac{(n - 1)^2}{4} K_0) y_{\lambda_j}\), we have

\[
\left\| \begin{pmatrix} y_{\lambda_j}(x_0) \\ \sqrt{\lambda_j} y_{\lambda_j}'(x_0) \end{pmatrix} \right\| \leq 2 \left\| \begin{pmatrix} y_{\lambda_j}(x_0) \\ \sqrt{\lambda_j} y_{\lambda_j}'(x_0) \end{pmatrix} \right\|,
\]
for all \( x_0 \leq x \leq x_1 \).

**Proof.** In this proof \( C(E, A, K_0) \), \( K(E, A, K_0) \) will denote constants (possibly different in different equations) that depend on \( E, A, K_0 \) only. In the future, however, \( C(E, A, K_0), K(E, A, K_0) \) will refer to the specific constants as given in the statement of Theorem 2.2.

By shifting the Schrödinger equation, we can assume \( b = 0 \). Let

\[
V(x) = C(E, A, K_0)\chi_{[x_0+1, x_1-1]}(x)\frac{\sin(2\sqrt{\lambda}x + \phi)}{x},
\]

and define

\[
q(x) = \frac{(n-1)^2}{4}(\sqrt{K_0} + V(x))^2 + \frac{n-1}{2}V'(x).
\]

Direct computation implies, for \( x_0 + 1 \leq x \leq x_1 - 1 \),

\[
q(x) = \frac{(n-1)^2}{4}K_0 + C(E, A, K_0)\frac{\sin(2\sqrt{\lambda}x + \phi + \phi')}{x} + V_1(x),
\]

where \( \phi' \in \mathbb{R} \) depends on \( K_0, n, \lambda \) explicitly, and \( \text{supp}(V_1) \subset [x_0 + 1, x_1 - 1] \)

\[
|V_1(x)| \leq \frac{C(E, A, K_0)}{x^2}.
\]

We extend \( V(x) \) smoothly to \( \tilde{V} \) for \( x_0 < x < x_1 \), with \( \text{supp}(\tilde{V}) \subset (x_0, x_1) \) and \( \tilde{V} \in C^\infty(x_0, x_1) \), so that (11) and (12) hold.

Let

\[
\tilde{q}(x) = \frac{(n-1)^2}{4}(\sqrt{K_0} + \tilde{V}(x))^2 + \frac{n-1}{2}\tilde{V}'(x).
\]

By (10), we have

\[
\tilde{q}(x) = \frac{(n-1)^2}{4}K_0 + C(E, A, K_0)\chi_{[x_0+1, x_1-1]}(x)\frac{\sin(2\sqrt{\lambda}x + \phi + \phi')}{x} + V_1(x) + V_2(x),
\]

where \( \text{supp}(V_2) \subset (x_0, x_0 + 1) \cup [x_1 - 1, x_1) \) and

\[
|V_2(x)| \leq \frac{C(E, A, K_0)}{x}.
\]

We first prove the property of the solution for \( \lambda \). By (2) of Theorem 2.1 for \( x_0 > K(\lambda, A, K_0) \), there is a solution \( y \) of \((-D^2 + \tilde{q})y = (\lambda + \frac{(n-1)^2}{4}K_0)y \) (we only consider \( x_0 + 1 \leq x \leq x_1 - 1 \) so that \( q = \tilde{q} \)) such that

\[
|y_\lambda(x) - \frac{\cos(\sqrt{\lambda}x + \frac{\phi}{2})}{x^{100}}| \leq \frac{C(E, A, K_0)}{x^{101}},
\]

\[
|y_\lambda'(x) + \sqrt{\lambda}\frac{\sin(\sqrt{\lambda}x + \frac{\phi}{2})}{x^{100}}| \leq \frac{C(E, A, K_0)}{x^{101}},
\]

for \( x_0 + 1 \leq x \leq x_1 - 1 \). In particular,

\[
|y_\lambda(x_0 + 1) - \frac{\cos(\sqrt{\lambda}(x_0 + 1) + \frac{\phi}{2})}{(x_0 + 1)^{100}}| \leq \frac{C(E, A, K_0)}{(x_0 + 1)^{101}},
\]

\[
|y_\lambda'(x_0 + 1) + \sqrt{\lambda}\frac{\sin(\sqrt{\lambda}(x_0 + 1) + \frac{\phi}{2})}{(x_0 + 1)^{100}}| \leq \frac{C(E, A, K_0)}{(x_0 + 1)^{101}}.
\]

Now let us consider \((-D^2 + \tilde{q})y = (\lambda + \frac{(n-1)^2}{4}K_0)y \) for \( x_0 < x \leq x_0 + 1 \).

For \( x_0 < x \leq x_0 + 1 \),

\[
\tilde{q} = \chi_{(x_0, x_0+1)}(x)V_2(x).
\]

By (10), one has for \( x_0 \leq x \leq x_0 + 1 \)

\[
\| \begin{pmatrix} y_\lambda(x) \\ y_\lambda'(x) \end{pmatrix} - \begin{pmatrix} y_\lambda(x_0 + 1) \\ y_\lambda'(x_0 + 1) \end{pmatrix} \| \leq \frac{C(E, A, K_0)}{x_0} \| \begin{pmatrix} y_\lambda(x_0 + 1) \\ y_\lambda'(x_0 + 1) \end{pmatrix} \|.
\]
By (12) and (13), we have

\[ |y_\lambda(x_0) - \cos(\sqrt{\lambda}(x_0 + 1) + \frac{\pi}{2})| \leq \frac{C(E, A, K_0)}{x_0^{101}} \left| y'_\lambda(x_0) + \sqrt{\lambda} \sin(\sqrt{\lambda}(x_0 + 1) + \frac{\pi}{2}) \right| \leq \frac{C(E, A, K_0)}{x_0^{101}}. \]

If \( x_0 \) is large enough, the range of \( \frac{y'_\lambda(x_0)}{y_\lambda(x_0)} \) is \( \mathbb{R} \) when \( \phi \) is varied. Choose a \( \phi \) such that

\[ \frac{y'_\lambda(x_0)}{y_\lambda(x_0)} = \tan \theta_0. \]

Arguing as in the proof of (13), one has for \( x_1 - 1 \leq x \leq x_1 \)

\[ \| \begin{pmatrix} y_\lambda(x) \\ y'_\lambda(x) \end{pmatrix} - \begin{pmatrix} y_\lambda(x_1 - 1) \\ y'_\lambda(x_1 - 1) \end{pmatrix} \| \leq \frac{C(E, A, K_0)}{x_1} \| \begin{pmatrix} y_\lambda(x_1 - 1) \\ y'_\lambda(x_1 - 1) \end{pmatrix} \|. \]

By (12), (13) and (15), we get that the solution (up to a constant) of \((-D^2 + \tilde{q})y_\lambda = (\lambda + \frac{(n-1)^2}{4} K_0)y_\lambda\) with boundary condition \( \frac{y'_\lambda(x_0)}{y_\lambda(x_0)} = \tan \theta_0 \) satisfies

\[ \| \begin{pmatrix} y_\lambda(x_1) \\ y'_\lambda(x_1) \end{pmatrix} \| \leq \frac{2}{x_0^{100}} \| \begin{pmatrix} y_\lambda(x_0) \\ y'_\lambda(x_0) \end{pmatrix} \|, \]

and for \( x \in [x_0, x_1] \),

\[ \| \begin{pmatrix} y_\lambda(x) \\ y'_\lambda(x) \end{pmatrix} \| \leq 2 \| \begin{pmatrix} y_\lambda(x_0) \\ y'_\lambda(x_0) \end{pmatrix} \|. \]

Suppose \( x_0 \geq K(\lambda, A, K_0) \). By Theorem 2.1 again and following the proof of (16), (17), for any solution of \((-D^2 + \tilde{q})y_{\lambda_j} = (\lambda_j + \frac{(n-1)^2}{4} K_0)y_{\lambda_j} \), we have

\[ \| \begin{pmatrix} y_{\lambda_j}(x) \\ y'_{\lambda_j}(x) \end{pmatrix} \| \leq 2 \| \begin{pmatrix} y_{\lambda_j}(x_0) \\ y'_{\lambda_j}(x_0) \end{pmatrix} \|, \]

for all \( x_0 \leq x \leq x_1 \).

\[ \square \]

Now consider the Riemannian manifold with rotationally symmetric structure

\[ (M, g) = (\mathbb{R}^+ \times S^{n-1}(1), dr^2 + f_1^2(r) g_{S^{n-1}(1)}). \]

Let \( d\theta_0 \) be the standard measure on the unit sphere \((S^{n-1}(1), g_0)\). Assume that in the neighborhood of the origin \((M, g)\) is the usual Euclidean space with its standard metric \( g_0 \), i.e., \( f_1(r) = r \) and \( S(r) = \frac{1}{r} \) for \( 0 < r < \frac{1}{2} \). Then we have

**Theorem 2.3.** For any positive number \( \lambda > 0 \), there exists a rotationally invariant function \( h_{1,\lambda}(r) \) such that

\[ -\Delta_g(h_{1,\lambda}) = -h''_{1,\lambda} + \frac{n-1}{r} h'_{1,\lambda} = (K_0(n-1)^2/4 + \lambda) h_{1,\lambda} \]

and \( h_{1,\lambda} \in L^2(M_1, dv_g) \), where \((M_1, g) = \left( (0, \frac{1}{2}] \times S^{n-1}(1), dr^2 + r^2 g_{S^{n-1}(1)} \right)\).

**Proof.** Let us consider the ODE,

\[ u'' + \frac{n-1}{r} u' + (K_0(n-1)^2/4 + \lambda) u = 0. \]

By the Frobenius method, (20) has a power series solution of the form,

\[ u(r) = \sum_{j=0}^{\infty} c_j r^{j+s}, \]
and \( s \) satisfies
\[
s(s - 1) + (n - 1)s = 0.
\]
It implies we can let \( s = 0 \) in (20). Thus (20) has a solution of the form
\[
u(r) = \sum_{j=0}^{\infty} c_j r^j.
\]
By the definition \( dv_g = r^{n-1}drdg_0 \) for \( 0 \leq r < \frac{1}{2} \). This implies the solution given by (21) is in \( L^2(M_1, dv_g) \).

3. Inductive Construction

We reformulate Theorem 1.3 in a more convenient way.

**Theorem 3.1.** Suppose \( K_0 \geq 0 \). Let \( \{\lambda_j > 0\} \) be a finite set of distinct numbers. There exists a rotationally symmetric Riemannian manifold \((M_n, g) = (\mathbb{R}^n, dr^2 + f_1^2(r)g_{S^{n-1}(1)})\) such that the following holds,
\[
\begin{align*}
(1) \quad &\sigma_{\text{ess}}(-\Delta_g) = \sigma_{\text{ac}}(-\Delta_g) = \left[\frac{K}{4} (n-1)^2, \infty\right), \\
(2) \quad &\left\{\frac{K_0}{4} (n-1)^2 + \lambda_j\right\} \subset \sigma_p(-\Delta_g) \cap \left(\frac{K}{4} (n-1)^2, \infty\right), \\
(3) \quad &K_{\text{rad}}(r) + K_0 = O(r^{-1}) \quad \text{as } r \to \infty.
\end{align*}
\]

**Theorem 3.2.** Suppose \( K_0 \geq 0 \). Let \( \{\lambda_j > 0\} \) be a countable set of distinct numbers. Let \( C(r) > 0 \) be any function on \((0, \infty)\) with \( \lim_{r \to \infty} C(r) = \infty \). Then there exists a rotationally symmetric Riemannian manifold \((M_n, g) = (\mathbb{R}^n, dr^2 + f_1^2(r)g_{S^{n-1}(1)})\) such that the following holds,
\[
\begin{align*}
(1) \quad &\sigma_{\text{ess}}(-\Delta_g) = \sigma_{\text{ac}}(-\Delta_g) = \left[\frac{K}{4} (n-1)^2, \infty\right), \\
(2) \quad &\left\{\frac{K_0}{4} (n-1)^2 + \lambda_j\right\} \subset \sigma_p(-\Delta_g) \cap \left(\frac{K}{4} (n-1)^2, \infty\right), \\
(3) \quad &K_{\text{rad}}(r) + K_0 \leq C(r) \quad \text{as } r \to \infty.
\end{align*}
\]

We will first construct \( f_1 \) near the origin, and then extend it inductively, adding one segment at a time.

For \( r \leq \frac{1}{2} \), let
\[
f_1(r) = r.
\]
For \( r \in [1, 3] \), let
\[
f_1(r) = e^{\sqrt{K_0}(r-1)}.
\]
We extend \( f_1(r) \) to \((0, 3)\) so that \( f_1(r) > 0 \) and \( f_1 \in C^\infty(0, 3) \). Suppose \( K_0 \geq 0 \).

For any function \( f(x) \) on \([0, \infty)\) such that \( \text{supp} f \subset (3, \infty) \) and \( f \in C^\infty(3, \infty) \), if for \( r \geq 1 \), we let
\[
f_1(r) = \exp\left(\int_1^r \sqrt{K_0} + f(x)dx\right),
\]
then \( f_1 \in C^\infty(0, \infty) \) and for \( r \in [1, 3] \) (23) holds.

Our objective is to construct \( f(x) \) so that Riemannian manifold \((M, g) = (\mathbb{R}^+ \times S^{n-1}(1) \cup \{O\}, dr^2 + f_1^2(r)g_{S^{n-1}(1)})\) satisfies Theorem 3.1 or Theorem 3.2.
Set for \( r \geq 1 \),

\[
S(r) := \frac{f_1(r)}{f_1(r)} = \sqrt{K_0} + f(r),
\]

\[
K(r) := -\frac{f_1''(r)}{f_1(r)} = -((\sqrt{K_0} + f(r))^2 - f'(r)) = -K_0 - 2\sqrt{K_0}f(r) - f^2(r) - f'(r),
\]

\[
q(r) := \frac{(n-1)(n-3)}{4}S(r)^2 - \frac{(n-1)}{2}K(r) = \frac{(n-1)^2}{4}(\sqrt{K_0} + f(r))^2 + \frac{n-1}{2}f'(r).
\]

Thus we have

\[
S(r) - \sqrt{K_0} = f(r),
\]

\[
K(r) - K_0 = -2\sqrt{K_0}f(r) - f^2(r) - f'(r),
\]

\[
q(r) - \frac{(n-1)^2}{4}K_0 = \frac{(n-1)^2}{2}\sqrt{K_0}f(r) + \frac{(n-1)^2}{4}f^2(r) + \frac{n-1}{2}f'(r).
\]

**Remark 3.3.** Actually \( K(r) \) is the radial curvature and \( \Delta r = (n - 1)S(r) \).

Let \( h_{1,\lambda} \) be given by Theorem 2.3 on \((0, \frac{1}{2})\) and we extend it to \([0, 1]\) by solving

\[-\Delta_g h_{1,\lambda}(r) = -\left\{ \frac{\partial^2}{\partial r^2} + (n - 1)\frac{f'_1(r)}{f_1(r)} \frac{\partial}{\partial r} \right\} h_{1,\lambda}(r) = \left( \frac{(n-1)^2}{4}K_0 + \lambda \right) h_{1,\lambda}(r).
\]

Suppose there exists a nontrivial solution \( w_{\lambda}(x) \in L^2([1, \infty), dx) \) to the equation

\[
\left( -\frac{d^2}{dx^2} + q(x) - \frac{(n-1)^2}{4}K_0 \right) w_{\lambda}(x) = \lambda w_{\lambda}(x),
\]

with boundary condition

\[
\frac{w'_0(1)}{w_{\lambda}(1)} = \frac{h'_1(1)}{h_{1,\lambda}(1)} + \frac{n-1}{2} \sqrt{K_0}.
\]

Using this function \( w_{\lambda} \), we define a function \( h_{2,\lambda} \) by

\[
h_{2,\lambda} := f_1^{-\frac{n-1}{2}} w_{\lambda}.
\]

It is easy to verify that

\[
h_{2,\lambda}(1) = \frac{h'_{2,\lambda}(1)}{h_{2,\lambda}(1)} = \frac{h'_{1,\lambda}(1)}{h_{1,\lambda}(1)}.
\]

A direct computation shows that the function \( h_{2,\alpha}(r) \) for \( r \geq 1 \) satisfies the eigenvalue equation on \((M, g)\):

\[-\Delta_g h_{2,\lambda}(r) = -\left\{ \frac{\partial^2}{\partial r^2} + (n - 1)S(r) \frac{\partial}{\partial r} \right\} h_{2,\lambda}(r) = \left( \frac{(n-1)^2}{4}K_0 + \lambda \right) h_{2,\lambda}(r)
\]

and \( h_{2,\lambda}(r) \in L^2(M, dv_g) \), where \( dv_g = f_1^{n-1}(r) dr dg_0 \) for \( r \geq 1 \).

Define \( h_{\lambda}(r) = h_{1,\lambda}(r) \) for \( r \leq 1 \) and \( h_{\lambda}(r) = \frac{h_{1,\lambda}(1)}{h_{2,\lambda}(1)} h_{2,\lambda}(r) \) for \( r \geq 1 \). Combining with (34), we have for all \( r > 0 \)

\[
-\Delta_g (h_{\lambda}(r)) = \left( \frac{(n-1)^2}{4}K_0 + \lambda \right) h_{\lambda}(r).
\]

Thus \( \frac{(n-1)^2}{4}K_0 + \lambda \) is an eigenvalue and \( h_{\lambda} \) is the corresponding eigenfunction.
Now given a set \( \{ \lambda_j > 0 \} \), we will construct \( f(x) \) piecewise of the form as in Theorem 2.2 such that for any \( j \), there exists eigenfunction \( w_{\lambda_j}(x) \in L^2([1, \infty), dx) \) that solves equation (31) with \( \lambda = \lambda_j \) and satisfies the boundary condition (32). By (33), \( \{ (n-1)^2 K_0 + \lambda_j \} \) are the eigenvalues of Laplacian \( \Delta_\gamma \).

Let \( N(k) \in \mathbb{Z}^+ \) be a non-decreasing function on \( \mathbb{Z}^+ \), \( N(1) = 1 \) (\( N(k) \) that we choose will be growing very slowly). Let \( A_k = \{ \lambda_1, \lambda_2, \cdots, \lambda_{N(k)} \} \). For \( k \geq 1 \), let

\[
K_k = 10 + \max_{1 \leq j \leq N(k)} K(\lambda_j, A_k \{ \lambda_j \}, K_0),
\]

and

\[
C_k = 10 + 4^{N(k)} + N(k)^{100} + K_k + \max_{1 \leq j \leq N(k)} C(\lambda_j, A_k \{ \lambda_j \}, K_0),
\]

where the \( C \) in (37) and \( K \) in (36) are given by Theorem 2.2. Define \( T_0 = 1 \) and \( T_k = T_{k-1} C_k \). Let \( J_k = \sum_{i=1}^{k} N(i) T_i \). Then we have

\[
C_k \geq 4^{N(k)}, C_k \geq N(k)^{100},
\]

and

\[
T_k \geq 10^k, T_k \geq K_k.
\]

We can assure that \( C_k \) goes to infinity arbitrarily slowly if we choose appropriately slowly growing \( N(k) \). We choose \( N(k) \) to be the largest integer such that

\[
C_k \leq C \ln k,
\]

and

\[
2C_{k+1}^4 \leq C \min_{x \in [J_k, J_{k+1}]} \min \{ C(x), \ln x \},
\]

where \( C(x) \) is given by Theorem 3.2 and \( C = C(\lambda_1) \). We then have \( N(k) = N \) for sufficiently large \( k \) in the construction of Theorem 3.1 and \( \lim_k N(k) = \infty \) in the construction of Theorem 3.2.

We will also define function \( f(x) \) (\( \sup f \subset (3, \infty) \)) and \( w_{\lambda_j}(x), j = 1, 2, \cdots \) on \( (1, J_k) \) by induction, so that

1. \( w_{\lambda_j}(x) \) solves

\[
\left( -\frac{d^2}{dx^2} + q(x) - \frac{(n-1)^2}{4} \right) w_{\lambda_j}(x) = \lambda w_{\lambda_j}(x),
\]

for \( x \in (1, J_k) \) where \( q(x) \) on \( (1, J_k) \) is given by (27), and satisfies boundary condition

\[
\frac{w'_{\lambda_j}(1)}{w_{\lambda_j}(1)} = \frac{h_{1, \lambda_j}(1)}{h_{1, \lambda_j}(1)} + \frac{n-1}{2} \sqrt{K_0},
\]

where \( h_{1, \lambda_j} \) is given by (19).

2. \( w_{\lambda_i}(x) \) for \( i = 1, 2, \cdots, N(k) \) and \( k \geq 2 \), satisfies

\[
\| \left( \frac{1}{\sqrt{n}} w_{\lambda_j}(J_k) \right) \| \leq 2^{N(k)+1} N(k)^{100} C_k^{-99} \left( \frac{1}{\sqrt{n}} w_{\lambda_i}(J_{k-1}) \right) \|.
\]

3. \( \sup f \subset (J_{k-1}, J_k) \) and \( f \in C^\infty(J_{k-1}, J_k) \), and

\[
|S(x) - \sqrt{K_0}| \leq \frac{N(k) C_k^2}{x},
\]

\[
|K(x) + K_0| \leq 4\sqrt{K_0} + 8 \frac{N(k) C_k^2}{x},
\]
for $x \in (3, J_k)$, where $S(x)$ and $K(x)$ are given by (25), (26) respectively.

By our construction, one has

$$
\frac{J_k}{T_{k+1}} = \sum_{i=1}^{k} \frac{N(i)T_i}{T_{k+1}}
$$

(45)

$$
\leq \frac{N(k)}{C_{k+1}} \sum_{i=1}^{k} \frac{T_i}{T_k}
$$

(46)

$$
\leq 2 \frac{N(k)}{C_{k+1}}.
$$

(47)

Step 1: Let $f(x) = 0$ on $(2, J_1)$. Then $q(x)$ given by (27) is well defined on $(2, J_1)$.

Let $w_{\lambda_j}(x), j = 1, 2, \cdots$ for $x \in (1, J_1)$ be solutions of the equation

$$
\left(-\frac{d^2}{dx^2} + q(x) - \frac{(n-1)^2}{4} K_0\right) w_{\lambda_j}(x) = \lambda_j w_{\lambda_j}(x),
$$

(48)

with boundary condition

$$
\frac{w_{\lambda_j}'(1)}{w_{\lambda_j}(1)} = \frac{h_{1,\lambda_j}(1)}{h_{1,\lambda_j}(1)} + \frac{n-1}{2} \sqrt{K_0},
$$

(49)

where $h_{1,\lambda_j}$ is given by (19).

Step $k + 1$, for $k \geq 1$:

Suppose we completed the construction of $f(x)$ for step $k$. That is we have defined $f(x), w_{\lambda_j}(x), q(x)$ on $(1, J_k)$.

Denote $B_{k+1} = \{\lambda_i\}_{i=1}^{N(k+1)}$. Applying Theorem 2.2 to $x_0 = J_k, x_1 = J_k + T_{k+1}, b = 0, \lambda = \lambda_1, \tan \theta_0 = \frac{w_{\lambda_1}'(J_k)}{w_{\lambda_1}(J_k)}$ and $A = B_{k+1} \setminus \{\lambda_1\}$, we can define $f(x) = \tilde{V}(x, \lambda_1, B_{k+1} \setminus \{\lambda_1\}, J_k, J_k + T_{k+1}, 0, \theta_0)$ for $x \in (J_k, J_k + T_{k+1}]$. Thus we can define $w_{\lambda_j}(x)$ on $(0, J_k + T_{k+1})$ for all possible $j$. Since the boundary condition of $w_{\lambda_j}(x)$ matches at the point $J_k$ (guaranteed by $\tan \theta_0 = \frac{w_{\lambda_j}'(J_k)}{w_{\lambda_j}(J_k)}$) and by Theorem 2.2 one has

1. $w_{\lambda_j}(x)$ solves

$$
\left(-\frac{d^2}{dx^2} + q(x) - \frac{(n-1)^2}{4} K_0\right) w_{\lambda_j}(x) = \lambda_j w_{\lambda_j}(x),
$$

(50)

for $x \in (1, J_k + T_{k+1})$, and satisfies the boundary condition $\frac{w_{\lambda_j}'(1)}{w_{\lambda_j}(1)} = \frac{h_{1,\lambda_j}(1)}{h_{1,\lambda_j}(1)} + \frac{n-1}{2} \sqrt{K_0}$.

(2) $w_{\lambda_j}(x)$ satisfies

$$
\| \left( \frac{w_{\lambda_j}(J_k + T_{k+1})}{w_{\lambda_j}(J_k + T_{k+1})} \right) \| \leq 2^{100} \left( \frac{J_k + T_{k+1}}{J_k} \right)^{100} \| \left( \frac{w_{\lambda_j}(J_k)}{\sqrt{\lambda_j} w_{\lambda_j}'(J_k)} \right) \|
$$

$$
\leq 2^{101} N(k)^{100} C_{k+1}^{100} \| \left( \frac{w_{\lambda_j}(J_k)}{\sqrt{\lambda_j} w_{\lambda_j}'(J_k)} \right) \|
$$

$$
\leq N(k)^{100} C_{k+1}^{99} \| \left( \frac{w_{\lambda_j}(J_k)}{\sqrt{\lambda_j} w_{\lambda_j}'(J_k)} \right) \|
$$

(51)

where the second inequality holds by (17). At the same time, by (8), the solutions for

\[\left(-\frac{d^2}{dx^2} + q(x) - \frac{(n-1)^2}{4} K_0\right) w_{\lambda_j}(x) = \lambda_j w_{\lambda_j}(x), j = 2, 3, \cdots, N(k+1),\]

(52) satisfy

$$
\| \left( \frac{w_{\lambda_j}(J_k + T_{k+1})}{\sqrt{\lambda_j} w_{\lambda_j}'(J_k + T_{k+1})} \right) \| \leq 2 \| \left( \frac{w_{\lambda_j}(J_k)}{\sqrt{\lambda_j} w_{\lambda_j}'(J_k)} \right) \|.
$$
Suppose we have defined $f(x)$ on $(0, J_k + tT_{k+1})$ for $t \leq N(k + 1) - 1$. Let us give the definition on $(0, J_k + (t + 1)T_{k+1})$.

Applying Theorem 2.2 to $x_0 = J_k + tT_{k+1}$, $x_1 = J_k + (t + 1)T_{k+1}$, $b = tT_{k+1}$, $\lambda = \lambda_{t+1}$, $A = B_{k+1} \setminus \lambda_{t+1}$ and $\tan \theta_0 = \frac{w_{\lambda_{t+1}}(J_k + tT_{k+1})}{w_{\lambda_{t+1}}'(J_k + tT_{k+1})}$, we can define $f(x) = V(x, \lambda_{t+1}, B_{k+1} \setminus \lambda_{t+1}, J_k + tT_{k+1}, J_k + (t + 1)T_{k+1}, \theta_0)$ on $x \in (J_k + tT_{k+1}, J_k + (t + 1)T_{k+1})$. Thus we can define $w_{\lambda_j}(x)$ on $(1, J_k + (t + 1)T_{k+1})$ for all possible $j$.

Since the boundary condition of $w_{\lambda_{t+1}}(x)$ matches at the point $J_k + tT_{k+1}$ (guaranteed by \( \tan \theta_0 = \frac{w_{\lambda_{t+1}}(J_k + tT_{k+1})}{w_{\lambda_{t+1}}'(J_k + tT_{k+1})} \)) and by Theorem 2.2 one has

1. $w_{\lambda_{t+1}}(x)$ solves

$$
\left(-\frac{d^2}{dx^2} + q(x) - \frac{(n-1)^2}{4}K_0\right) w_{\lambda_{t+1}}(x) = \lambda_{t+1} w_{\lambda_{t+1}}(x),
$$

for $x \in (1, J_k + (t+1)T_{k+1})$, and satisfies the boundary condition $\frac{w_{\lambda_{t+1}}(1)}{w_{\lambda_{t+1}}'(1)} = \frac{h_1^{\lambda_{t+1}}(1)}{n_1^{\lambda_{t+1}}(1)} + \frac{n_1^{\lambda_{t+1}}(1)}{\sqrt{K_0}}$.

2. $w_{\lambda_{t+1}}(x)$ satisfies \( (w_{\lambda_{t+1}}(J_k + (t + 1)T_{k+1})/\sqrt[\lambda_{t+1}]{w_{\lambda_{t+1}}'(J_k + (t + 1)T_{k+1})}) \) \( \leq \frac{2(J_k + T_{k+1})}{J_k} \left(\frac{w_{\lambda_{t+1}}(J_k + T_{k+1})}{\sqrt[\lambda_{t+1}]{w_{\lambda_{t+1}}'(J_k + T_{k+1})}}\right) \)

\( \leq N(k)100C_{k+1}^{-99}\left(\frac{w_{\lambda_{t+1}}(J_k + tT_{k+1})}{\sqrt[\lambda_{t+1}]{w_{\lambda_{t+1}}'(J_k + tT_{k+1})}}\right) \).

At the same time, by (53), the solutions for \( \left(-\frac{d^2}{dx^2} + q(x) - \frac{(n-1)^2}{4}K_0\right) w_{\lambda_j}(x) = \lambda_j w_{\lambda_j}(x), \) $j = 1, 2, 3, \cdots, N(k+1)$ and $j \neq t+1$, satisfy,

\( \left(\frac{w_{\lambda_j}(J_k + (t + 1)T_{k+1})}{\sqrt[\lambda_j]{w_{\lambda_j}'(J_k + (t + 1)T_{k+1})}}\right) \leq 2\left(\frac{w_{\lambda_j}(tJ_k)}{\sqrt[\lambda_j]{w_{\lambda_j}'(tJ_k)}}\right) \).

Thus we have defined $f(x)$ by induction in $t$ on each $(1, J_k + tT_{k+1})$ and therefore on $(1, J_k + N(k+1)T_{k+1}) = (1, J_{k+1})$.

Let us mention that for $x \in [J_k + tT_{k+1}, J_k + (t + 1)T_{k+1}]$ and $0 \leq t \leq N(k + 1) - 1,

\( f(x) = V\left(x, \lambda_{t+1}, B_{k+1} \setminus \lambda_{t+1}, J_k + tT_{k+1}, J_k + (t + 1)T_{k+1}, \frac{w_{\lambda_{t+1}}(J_k + tT_{k+1})}{w_{\lambda_{t+1}}'(J_k + tT_{k+1})}\right), \)

where $V$ is taken from Theorem 2.2.

Now we should show that our definition satisfies the $k + 1$ step conditions (40)-(44).

Let us consider $\left(\frac{w_{\lambda_i}(x)}{\sqrt[\lambda_i]{w_{\lambda_i}'(x)}}\right)$ for $i = 1, 2, \cdots, N(k+1)$. $\left(\frac{w_{\lambda_i}(x)}{\sqrt[\lambda_i]{w_{\lambda_i}'(x)}}\right)$ decreases from point $J_k + (i - 1)T_{k+1}$ to $J_k + iT_{k+1}$, $i = 1, 2, \cdots, N(k + 1)$, and may increase from any point $J_k + (m - 1)T_{k+1}$ to $J_k + mT_{k+1}$, $m = 1, 2, \cdots, N(k + 1)$ and $m \neq i$, but no more than by a factor of 2. That is

\( \left(\frac{w_{\lambda_i}(J_k + iT_{k+1})}{\sqrt[\lambda_i]{w_{\lambda_i}'(J_k + iT_{k+1})}}\right) \leq N(k)100C_{k+1}^{-99}\left(\frac{w_{\lambda_i}(J_k + (i - 1)T_{k+1})}{\sqrt[\lambda_i]{w_{\lambda_i}'(J_k + (i - 1)T_{k+1})}}\right) \).
and for $m \neq i$,

$$\| \left( \frac{w_{\lambda_i}(J_k + m T k)}{\sqrt{\lambda_i}} w'_{\lambda_i}(J_k + m T k) \right) \| \leq 2 \| \left( \frac{w_{\lambda_i}(J_k + (m-1) T k)}{\sqrt{\lambda_i}} w'_{\lambda_i}(J_k + (m-1) T k) \right) \|$$

by Theorem 2.2.

Thus for $i = 1, 2, \cdots, N(k+1)$,

$$\| \left( \frac{w_{\lambda_i}(J_k + 1)}{\sqrt{\lambda_i}} w'_{\lambda_i}(J_k + 1) \right) \| \leq 2^{N(k+1) + 1} N(k)^{100} C_{k+1} \left( \frac{w_{\lambda_i}(J_k)}{\sqrt{\lambda_i}} w'_{\lambda_i}(J_k) \right).$$

This implies (12) for $k + 1$. By the construction of $f(x)$ (50), (4), and (5) we have for $x \in [J_k + t T k + 1, J_k + (t+1) T k + 1]$ and $0 \leq t \leq N(k+1) - 1$,

$$|f'(x)|, |f(x)| \leq \frac{C_{k+1}}{x - t T k + 1} \leq 2 \frac{N(k+1) C_{k+1}^2}{x + 1}.$$

(57)

This implies (13) and (14) by (28) and (29).

4. PROOF OF THEOREMS 3.1 AND 3.2

Proof. Fix a set $\{ \lambda_i > 0 \}$. We construct $f(x)$ such that (40), (41), (42), (43) and (44) hold for any $k$. It is known that

$$\Delta r = (n-1) S(r),$$

$$K_{rad} = K(r).$$

By our construction, one has

$$\lim_{r \to \infty} \Delta r = \lim_{r \to \infty} (n-1) S(r) = (n-1) \sqrt{K_0}.$$

Thus by Theorem 1.2 in [14], we have

$$\sigma_{ess}(-\Delta_g) = \left( \frac{(n-1)^2}{4} K_0, \infty \right).$$

Thus

$$\sigma_{ac}(-\Delta_g) \subset \left( \frac{(n-1)^2}{4} K_0, \infty \right).$$

By (30) and (57), one has for $r \in [J_{k-1}, J_k]$,

$$|q(r) - \frac{(n-1)^2}{4} K_0| \leq O(1) \frac{N(k) C^2(k)}{r}.$$

By (39), one has for $r \in [J_{k-1}, J_k]$,

$$\frac{N(k) C^2(k)}{r} = O(1) \frac{C^3}{r} = O(1) \frac{\ln r}{r}.$$

Thus

$$|q(r) - \frac{(n-1)^2}{4} K_0| = O(1) \frac{\ln r}{r}.$$

It implies (e.g. [2, 12]), for the Schrödinger operator, $\sigma_{ac}(-D^2 + q) = \left( \frac{(n-1)^2}{4} K_0, \infty \right)$, and then

$$\left( \frac{(n-1)^2}{4} K_0, \infty \right) \subset \sigma_{ac}(-\Delta_g).$$
Thus
\[ \sigma_{ac}(-\Delta_g) = \frac{(n-1)^2}{4}K_0, \infty. \]

It yields (1) of Theorems 3.1 and 3.2.

Also, by (44) and (39), we have (3) of Theorems 3.1, 3.2 hold.

By (44) and (35), it suffices to show that for any \( j \), \( w_{\lambda_j}(x) \in L^2([1, \infty), dx) \).
Below we give the details.

For any \( N(k_0 - 1) < j \leq N(k_0) \), by the construction (see (42)), we have for \( k \geq k_0 \)
\[
\left\| \left( \frac{w_{\lambda_j}(J_{k+1})}{\sqrt{\lambda_j}} \right) \right\| \leq 2^{N(k+1)+1}N(k+1)^{100}C_{k+1}^{-99} \left\| \left( \frac{w_{\lambda_j}(J_k)}{\sqrt{\lambda_j}} \right) \right\|
\]
\[
\leq C_{k+1}^{-50} \left\| \left( \frac{w_{\lambda_j}(J_k)}{\sqrt{\lambda_j}} \right) \right\|
\]
where the second inequality holds by (38).

This implies for \( k \geq k_0 \)
\[
(60) \quad \left\| \left( \frac{w_{\lambda_j}(J_{k+1})}{\sqrt{\lambda_j}} \right) \right\| \leq T_{k_0}^{50}T_{k+1}^{-50} \left\| \left( \frac{w_{\lambda_j}(J_{k_0})}{\sqrt{\lambda_j}} \right) \right\|
\]
By (38) and (37), for all \( x \in [J_{k+1}, J_{k+2}] \),
\[
\left\| \left( \frac{w_{\lambda_j}(x)}{\sqrt{\lambda_j}} \right) \right\| \leq 2^{N(k+2)} \left\| \left( \frac{w_{\lambda_j}(J_{k+1})}{\sqrt{\lambda_j}} \right) \right\|
\]
\[
\leq 2^{N(k+2)}T_{k_0}^{50}T_{k+1}^{-50} \left\| \left( \frac{w_{\lambda_j}(J_{k_0})}{\sqrt{\lambda_j}} \right) \right\|
\]
\leq T_{k_0}^{50}T_{k+1}^{-49} \left\| \left( \frac{w_{\lambda_j}(J_{k_0})}{\sqrt{\lambda_j}} \right) \right\|
\]
(61)
where the third inequality holds by (38).

Then by (61), we have
\[
\int_{J_{k_0+1}}^{\infty} \left( \left\| \left( \frac{w_{\lambda_j}(x)}{\sqrt{\lambda_j}} \right) \right\| \right|^2 dx = \sum_{k \geq k_0+1} \int_{J_k}^{J_{k+1}} \left( \left\| \left( \frac{w_{\lambda_j}(x)}{\sqrt{\lambda_j}} \right) \right\| \right|^2 dx
\]
\[
\leq T_{k_0}^{100} \left\| \left( \frac{w_{\lambda_j}(J_{k_0})}{\sqrt{\lambda_j}} \right) \right\|^2 \sum_{k \geq k_0+1} \int_{J_k}^{J_{k+1}} T_k^{-98} dx
\]
\[
\leq T_{k_0}^{100} \left\| \left( \frac{w_{\lambda_j}(J_{k_0})}{\sqrt{\lambda_j}} \right) \right\|^2 \sum_{k \geq k_0+1} N(k+1)T_{k+1}^{-98}
\]
\[
\leq T_{k_0}^{100} \left\| \left( \frac{w_{\lambda_j}(J_{k_0})}{\sqrt{\lambda_j}} \right) \right\|^2 \sum_{k \geq k_0+1}(k+1)C_{k+1}T_k^{-96}
\]
\[
\leq T_{k_0}^{100} \left\| \left( \frac{w_{\lambda_j}(J_{k_0})}{\sqrt{\lambda_j}} \right) \right\|^2 \sum_{k \geq k_0+1} T_k^{-90} < \infty.
\]
This completes the proof. \( \square \)
Acknowledgments

We are grateful to A. Mramor for sharing with us the review [6], which led to the idea of this project. W.L. would like to thank M. Lukic and H. Xu for some useful discussions. W.L. was supported by the AMS-Simons Travel Grant 2016-2018. This research was supported by NSF DMS-1401204 and NSF DMS-1700314.

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