A VARIATION ON THE THEME OF NICOMACHUS

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Abstract

In this paper, we prove some conjectures of K. Stolarsky concerning the first and third moments of the Beatty sequences with the golden section and its square.

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1. Introduction

Nicomachus’ theorem asserts that the sum of the first \( m \) cubes is the square of the \( m \)th triangular number,

\[
1^3 + 2^3 + \cdots + m^3 = (1 + 2 + \cdots + m)^2. \tag{1.1}
\]

(See [2].) With the notation

\[
Q(\alpha, m) := \frac{\sum_{n=1}^{m} \lfloor \alpha n \rfloor^3}{\left(\sum_{n=1}^{m} \lfloor \alpha n \rfloor\right)^2}, \tag{1.2}
\]

where \( \alpha \in \mathbb{R} \setminus \{0\} \), it implies that

\[
\lim_{m \to \infty} Q(\alpha, m) = \alpha. \tag{1.3}
\]

Here, \( \lfloor x \rfloor \) is the integer part of the real number \( x \). The limit in (1.3) follows from \( \lfloor \alpha n \rfloor = \alpha n + O(1) \) and Nicomachus’ theorem (1.1).

Recall that the Fibonacci and Lucas sequences, \( \{F_n\}_{n \geq 0} \) and \( \{L_n\}_{n \geq 0} \), are given by \( F_0 = 0, F_1 = 1 \) and \( L_0 = 2, L_1 = 1 \) and the recurrence relations

\[
F_{n+2} = F_{n+1} + F_n, \quad L_{n+2} = L_{n+1} + L_n
\]

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for \( n \geq 0 \). In a personal communication to the second author, K. Stolarsky observed that the limit relation (1.3) can be ‘quantified’ for \( \alpha = \phi \) and \( \phi^2 \), where \( \phi := \frac{1}{2}(1 + \sqrt{5}) \) is the golden mean, and a specific choice of \( m \) along the Fibonacci sequence. The corresponding result is Theorem 2.1 below. We complement it by a general analysis of moments of the Beatty sequences and give a solution to a related arithmetic question in Theorem 2.5.

2. Principal results

**Theorem 2.1.** For \( k \geq 1 \) an integer, define \( m_k := F_k - 1 \). Then

\[
Q(\phi^2, m_{2k}) - Q(\phi, m_{2k}) = \begin{cases} 
1 - \frac{1}{(F_{k+1})^2L_{k+2}L_{k-1}} & \text{if } k \text{ is even,} \\
1 - \frac{1}{(L_{k+1})^2F_{k+2}F_{k-1}} & \text{if } k \text{ is odd}
\end{cases}
\]

and

\[
Q(\phi^2, m_{2k-1}) - Q(\phi, m_{2k-1}) = \begin{cases} 
1 - \frac{F_{k-2}}{F_{k+1}(F_k)^2(L_{k-1})^2} & \text{if } k \text{ is even,} \\
1 - \frac{L_{k-2}}{L_{k+1}(L_k)^2(F_k)^2} & \text{if } k \text{ is odd.}
\end{cases}
\]

The theorem motivates our interest in the numerators and denominators of \( Q(\phi, F_k - 1) \) and \( Q(\phi^2, F_k - 1) \), which can be thought of as expressions of the form

\[
A(k, s) := \sum_{n=1}^{F_{j-1}} \lfloor \phi n \rfloor^s \quad \text{and} \quad A'(k, s) := \sum_{n=1}^{F_{j-1}} \lfloor \phi^2 n \rfloor^s
\]

for \( k, s = 1, 2, \ldots \) and \( s = 1, 3 \). More generally, our analysis in Section 3 covers the sums

\[
A(k, s, j) := \sum_{n=1}^{F_{j-1}} n^j \lfloor \phi n \rfloor^s \quad \text{where } k, s, j = 1, 2, \ldots\quad \text{and } s, j = 0, 1, 2, \ldots \quad (2.1)
\]

Namely, we find a recurrence relation for \( A(k, j, s) \) and deduce recursions from it for \( A(k, s) = A(k, s, 0) \) and \( A'(k, s) \). The strategy leads to the following expressions for the numerators and denominators in Theorem 2.1, which are given in Lemmas 2.2–2.4.

**Lemma 2.2.** Let \( k \geq 1 \) be an integer. Then

\[
A(k, 1) = \frac{1}{2}(F_{k+1} - 1)(F_k - 1), \quad A'(k, 1) = \frac{1}{2}(F_{k+2} - 1)(F_k - 1). \quad (2.2)
\]

**Lemma 2.3.** Let \( k \geq 1 \) be an integer. Then

\[
A(2k, 3) = \frac{1}{4}(F_{2k-1} - 1)^2(F_{2k+1} - 1)(F_{2k+2} - 1), \quad A(2k - 1, 3) = \frac{1}{4}(F_{2k-1} - 1)(F_{2k} - 1) \times \frac{1}{5}(L_{4k} - 3L_{2k+1} - L_{2k} + 3).
\]
Lemma 2.4. Let \( k \geq 1 \) be an integer. Then
\[
A'(2k, 3) = \frac{1}{3}(F_{2k} - 1)(F_{2k+2} - 1) \times \frac{1}{3}(L_{4k+4} - 5L_{2k+3} + 13),
\]
\[
A'(2k - 1, 3) = \frac{1}{3}(F_{2k-1} - 1)(F_{2k+1} - 1) \times \frac{1}{3}(L_{4k+2} - 5L_{2k+2} + 7).
\]

Finally, we present an arithmetic formula inspired by Stolarsky’s original question.

Theorem 2.5. For \( k \geq 1 \),
\[
\text{LCM}(A(2k, 1), A'(2k, 1)) = \begin{cases} 
\frac{1}{2}F_{k+1}F_kL_{k+2}L_{k+1}L_{k-1} & \text{if } 2 \mid k, \\
\frac{1}{2}F_{k+2}F_{k+1}F_{k-1}L_{k+1}L_k & \text{if } 2 \nmid k.
\end{cases}
\]

Remark 2.6. Lemmas 2.2–2.4 indicate that the expression
\[
Q(\phi^2, F_k - 1) - Q(\phi, F_k - 1)
\]
is expressible as a fraction whose numerator and denominator are polynomials in Fibonacci and Lucas numbers with indices depending linearly on \( k \) according to the parity of \( k \), yet the statement of Theorem 2.1 presents formulas for these quantities according to the congruence class of \( k \) modulo 4 rather than modulo 2. The discrepancy is related to different factorisations of the factors \( F_n - 1 \) that occur in the formulas for \( A(k, j) \) and \( A'(k, j) \) for \( j \in \{1, 3\} \), since each of the factors \( F_n - 1 \) happens to be a product of a Fibonacci and a Lucas number according to the congruence class of \( n \) modulo 4 (see formulas (6.1)).

3. Recurrence relations for auxiliary sums

Here, we show how to compute the integer-part sums (2.1). This clearly covers the cases \( A(k, s) = A(k, s, 0) \). On using
\[
\phi^2 = 1 + \phi,
\]
which upon multiplication by the integer \( n \) and taking integer parts becomes
\[
[\phi^2 n] = n + [\phi n],
\]
one also gets the explicit formulas
\[
A'(k, s) = \sum_{i=0}^{s} \binom{s}{i} A(k, s - i, i).
\]

Using the Binet formula
\[
F_k = \frac{\phi^k - (-\phi^{-1})^k}{\sqrt{5}} \quad \text{for all } k \geq 0,
\]
one easily proves that
\[
[\phi F_k] = F_{k+1} - \epsilon_k \quad \text{where } \epsilon_k = \frac{1 + (-1)^k}{2}
\]
and 
\[ [\phi(F_k + n)] = F_{k+1} + \lfloor \phi n \rfloor \quad \text{for } 1 \leq n \leq F_{k-1} - 1 \]
(see, for example, [1]). Thus, 
\[
A(k + 1, s, j) = \sum_{n=1}^{F_k-1} n^j[\phi n]^s + F_k^j[\phi F_k]^s + \sum_{n=F_k+1}^{F_{k+1}-1} n^j[\phi n]^s
\]
\[
= A(k, s, j) + F_k^j(F_k+1 - \epsilon_k)^s + \sum_{n=1}^{F_{k+1}-F_k-1} (F_k + n)^j[\phi(F_k + n)]^s
\]
\[
= A(k, s, j) + F_k^j(F_k+1 - \epsilon_k)^s + \sum_{n=1}^{F_{k+1}-1} (F_k + n)^j(F_k+1 + [\phi n])^s
\]
\[
= A(k, s, j) + F_k^j \sum_{i=0}^{s} \binom{s}{i} F_{k+1}^i(-\epsilon_k)^{s-i}
\]
\[
+ \sum_{n=1}^{F_{k+1}-1} \sum_{\ell=0}^{j} \binom{j}{\ell} F_k^n j^{i-\ell} \sum_{i=0}^{s} \binom{s}{i} F_{k+1}^i[\phi n]^{s-i}
\]
\[
= A(k, s, j) + \sum_{i=0}^{s} \binom{s}{i} (-\epsilon_k)^{s-i} F_k^i F_{k+1}^i
\]
\[
+ \sum_{\ell=0}^{j} \sum_{i=0}^{s} \binom{j}{\ell} \binom{s}{i} F_k^n F_{k+1}^i A(k-1, s-i, j-\ell).
\]
The above reduction, the identity \( A(k, 0, 0) = F_k - 1 \) and induction on \( k + j + s \) imply that
\[
A(k, s, j) \in \text{span}([\phi^i]^k, (-\phi^i)^k : |i| \leq j + s + 1];
\]
in particular, for a fixed choice of \( s \) and \( j \), the sequence \( \{A(k, s, j)\}_{k \geq 1} \) is linearly recurrent of order at most \( 4(s + j) + 6 \). In the following section, we will use this observation about linear recurrence together with the following facts.

- If \( u = \{u_n\}_{n \geq 0} \) is a linearly recurrent sequence whose roots are all simple in some set \( U \), then, for fixed integers \( p \) and \( q \), the sequence \( \{u_{pn+q}\}_{n \geq 0} \) is linearly recurrent with simple roots in \( \{\alpha^p : \alpha \in U\} \).
- If \( u = \{u_n\}_{n \geq 0} \) and \( v = \{v_n\}_{n \geq 0} \) are linearly recurrent and their roots are all simple in some sets \( U \) and \( V \), respectively, then \( uv = \{u_n v_n\}_{n \geq 0} \) is linearly recurrent and its roots are all simple in \( UV = \{\alpha\beta : \alpha \in U, \beta \in V\} \).

In this context, the roots of a linearly recurrent sequence are defined as the zeros of its characteristic polynomial, counted with their multiplicities. It then follows that, for a fixed \( s \), each of the sequences \( \{A(k, s)\}_{k \geq 1} \) and \( \{A'(k, s)\}_{k \geq 1} \) is linearly recurrent of order at most \( 4s + 6 \).
4. The proofs of the lemmas

We first establish Lemma 2.2. By the argument in Section 3, both $A(k, 1)$ and $A'(k, 1)$ are linearly recurrent with simple roots in the set $\{\pm \phi^j : |l| \leq 2\}$. The same is true for the right-hand sides in (2.2). Since the set of roots is contained in a set with 10 elements, it follows that the validity of (2.2) for $k = 1, \ldots, 10$ implies that the relations hold for all $k \geq 1$.

Lemmas 2.3 and 2.4 are similar. The argument in Section 3 shows that the left-hand sides, $\{A(k, 3)\}_{k \geq 1}$ and $\{A'(k, 3)\}_{k \geq 1}$, are linearly recurrent with simple roots in $\{\pm \phi^j : |l| \leq 4\}$. Splitting according to the parity of $k$, we deduce that $\{A(2k, 3)\}_{k \geq 1}$, $\{A(2k - 1, 3)\}_{k \geq 1}$, $\{A'(2k, 3)\}_{k \geq 1}$ and $\{A'(2k - 1, 3)\}_{k \geq 1}$ are linearly recurrent with simple roots in $\{\phi^j : |l| \leq 4\}$, a set with nine elements. The same is true about the right-hand sides in the lemmas. Thus, if the relations hold for $k = 1, \ldots, 9$, then they hold for all $k \geq 1$.

A few words about the computation. For the identities presented in Lemmas 2.2–2.4, one can use brute force to compute $A(k, s)$ and $A'(k, s)$ for $s = 1, 3$ and $k = 2\ell + i$, where $i \in \{0, 1\}$ and $\ell$ is reasonably small (we dealt with $\ell \leq 9$), with any computer algebra system. To check them up to larger values of $k$ (around 100, say), the brute force strategy no longer works since the summation range up to $F_k - 1$ becomes too large. Instead one can use the recursion from Section 3 together with $A(k, 0, 0) = F_k - 1$ to find $A(k, 1, 0), A(k, 2, 0)$ and $A(k, 3, 0)$ for all desired $k$ and, similarly, $A(k, s, j)$ for small $j$, to evaluate $A'(k, s)$.

5. The proof of Theorem 2.1

Let us now address Theorem 2.1. When $k = 4\ell$, this can be rewritten as

$$F_{2\ell+1}^2 L_{2\ell+2} L_{2\ell-1} (A'(4\ell, 3)A(4\ell, 1)^2 - A(4\ell, 3)A'(4\ell, 1)^2) = A(4\ell, 1)^2 A'(4\ell, 1)^2 (F_{2\ell+1}^2 L_{2\ell+2} L_{2\ell-1} - 1).$$

(5.1)

Since $A(4\ell, s)$ and $A'(4\ell, s)$ are linearly recurrent (in $\ell$) with roots contained in $\{\phi^{4l} : |l| \leq s + 1\}$, and both the left-most factor in the left-hand side and the right-most factor in the right-hand side each have simple roots in $\{\phi^{4l} : |l| \leq 2\}$, it follows that both the left-hand side and the right-hand side are linearly recurrent with simple roots contained in $\{\phi^{4l} : |l| \leq 10\}$, a set with 21 elements. Thus, if the above formula holds for $\ell = 1, \ldots, 21$, then it holds for all $\ell \geq 1$. A similar argument applies to the case when $k = 4\ell + i$ for $i \in \{1, 2, 3\}$. Hence, all claimed formulas hold provided they hold for all $k \leq 100$, say.

Now we use the lemmas. For $k = 4\ell$, Lemmas 2.2–2.4 tell us that (5.1), after eliminating the common factor $(F_{4\ell} - 1)^2 (F_{4\ell+1} - 1)^2 (F_{4\ell+2} - 1)/16$, is equivalent to

$$F_{2\ell+1}^2 L_{2\ell+2} L_{2\ell-1} \times \left( \frac{1}{5} (F_{4\ell} - 1)(L_{8\ell+4} - 5L_{4\ell+3} + 13) - (F_{4\ell+2} - 1)(F_{4\ell+1} - 1)(F_{4\ell+2} - 1) \right)$$

$$= (F_{4\ell} - 1)^2 (F_{2\ell+1}^2 L_{2\ell+2} L_{2\ell-1} - 1)$$
(and one can perform further reduction using (6.1)). It is sufficient to verify the resulting equality for \( \ell = 1, \ldots, 15 \) and we have checked it for all \( \ell = 1, \ldots, 100 \). The remaining cases for \( k \) modulo 4 are similar. We do not give further details here.

6. The proof of Theorem 2.5

This follows from Lemma 2.2, the classical formulas
\[
F_{4\ell} - 1 = F_{2\ell+1}L_{2\ell-1}, \quad F_{4\ell+1} - 1 = F_{2\ell}L_{2\ell+1}, \\
F_{4\ell+2} - 1 = F_{2\ell}L_{2\ell+2}, \quad F_{4\ell+3} - 1 = F_{2\ell+2}L_{2\ell+1}
\]
(6.1)
as well as known facts about the greatest common divisor of Fibonacci and Lucas numbers with close arguments. For example, for \( k = 2\ell \),
\[
\text{LCM}(2\alpha(4\ell, 1), 2\alpha'(4\ell, 1)) = \text{LCM}((F_{4\ell+1} - 1)(F_{4\ell} - 1), (F_{4\ell+2} - 1)(F_{4\ell} - 1))
\]
\[
= \text{LCM}(F_{2\ell}L_{2\ell+1}, F_{2\ell}L_{2\ell+2})F_{2\ell+1}L_{2\ell-1}
\]
\[
= F_{2\ell}L_{2\ell+1}L_{2\ell+2}F_{2\ell+1}L_{2\ell-1}
\]
\[
= F_{k+1}F_kL_{k+2}L_{k+1}L_{k-1},
\]
where we used the fact that \( \text{gcd}(L_{2\ell+1}, L_{2\ell+2}) = 1 \). The case \( k = 2\ell + 1 \) is similar.

7. Further variations

First, we give an informal account of a more general result lurking, perhaps, behind the formulas in Theorem 2.1. Consider a homogeneous (rational) function \( r(x) = r(x_1, \ldots, x_m) \) of degree 1, that is, satisfying
\[
r(tx) = tr(x) \quad \text{for} \quad t \in \mathbb{Q},
\]
and an algebraic number \( \alpha \) solving the equation
\[
\sum_{k=0}^{m} c_k \alpha^k = 0, \quad (7.1)
\]
where the \( c_k \) are integers. If \( r(x) \) vanishes at a vector \( x^* = (x_1^*, \ldots, x_m^*) \), then automatically
\[
\sum_{k=0}^{m} c_k r(\alpha^k x^*) = 0 \quad (7.2)
\]
in view of the homogeneity of the function. We can then enquire whether equation (7.2) is ‘approximately’ true if \( r(x^*) = 0 \) is ‘approximately’ true. In this note, we merely examined the golden ratio case in which (7.1) is \( \alpha^2 - \alpha - 1 = 0 \), while the choice
\[
r(x_1, \ldots, x_m) = \frac{\sum_{n=1}^{m} x_n^3}{(\sum_{n=1}^{m} x_n)^2}
\]
for the rational function and \( x^* = (1, 2, \ldots, m) \) for its exact solution originated from the Nicomachus identity.
Notice that Nicomachus’ theorem (1.1) is the first entry in the chain of identities
\[ 1^{2r-1} + 2^{2r-1} + \cdots + m^{2r-1} = P_r(1 + 2 + \cdots + m) \quad \text{for } r = 2, 3, \ldots, \]
where \( P_r(x) \) are known as the Faulhaber polynomials. Our approach in this note gives a clear strategy to deal with the quantities that replace (1.2) in these settings.

Some further variations on the topic can be investigated in the \( q \)-direction, based on \( q \)-analogues of (1.1) (see [3]).

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