Boundary values, random walks and $\ell^p$-cohomology in degree one

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Abstract

The vanishing of reduced $\ell^2$-cohomology for amenable groups can be traced to the work of Cheeger & Gromov in [8]. The subject matter here is reduced $\ell^p$-cohomology for $p \in [1, \infty[$, particularly its vanishing. Results for the triviality of $\ell^p H^1(G)$ are obtained, for example: when $p \in [1, 2]$ and $G$ is amenable; when $p \in ]1, \infty[$ and $G$ is Liouville (e.g. of intermediate growth).

This is done by answering a question of Pansu in [36, §1.9] for graphs satisfying isoperimetric profile. Namely, the triviality of the reduced $\ell^p$-cohomology is equivalent to the absence of non-constant harmonic functions with gradient in $\ell^q$ ($q$ depends on the profile). In particular, one reduces questions of non-linear analysis ($p$-harmonic functions) to linear ones (harmonic functions with a very restrictive growth condition).

1 Introduction

A graph $\Gamma = (X, E)$ is defined by $X$, its set of vertices, and $E$, its set of edges. All graphs will be assumed to be of bounded valency. The set of edges will be thought of as a subset of $X \times X$. The subject matter is the reduced $\ell^p$-cohomology in degree one of the graph $\Gamma$. This is the quotient

$$\ell^p H^1(\Gamma) := \mathcal{D}^p(\Gamma)/\mathcal{P}(X) + \mathbb{K}\mathcal{D}^p.$$ 

See subsection §2.1 for more details. The main goal of this paper is to give partial answers to a question (dating back at least to Gromov [20, §8.A1.(A2), p.226]):

Question 1.1. Let $G$ be an amenable group, is it true that for one (and hence all) Cayley graph $\Gamma$ and all $1 < p < \infty$, $\ell^p H^1(\Gamma) = 0$?

The case $p = 1$ is slightly singular and the case $p = \infty$ is trivially false (see appendix A). For $p = 2$, the positive answer is a famous result of Cheeger & Gromov [8] (see also Lück’s book [30]). The results presented here will give a positive answer for all $p \in ]1, 2]$ and covers (for all $p$) many cases.

The basic idea relies on a standard argument which shows that a function which (essentially) only takes one value at infinity has trivial cohomology class. With a few more efforts, this determines $H^1$ via boundary values on the end (see appendix A). The idea is

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to define a “boundary value” of \( g \in \mathbb{D}^p(\Gamma) \) on another ideal boundary, namely the Poisson boundary. This boundary is made up by harmonic functions, so a natural candidate for this boundary value is \( \lim_{n \to \infty} P(n)g \) where \( P \) is the random walk operator.

The convergence of this limit can be obtained as a consequence of return probability (or heat kernels) estimates. To see that the behaviour of the boundary value still says something about the behaviour of \( g \) at infinity, a transport problem (between the a Dirac measure and the time \( n \) distribution of a simple random walk) has to be studied. This gives the following result.

**Theorem 1.2.** Let \( \Gamma \) be a graph satisfying IS\(_d\) and let \( 1 \leq q \leq p < d/2 \). Then

- the natural quotient \( \ell^q\mathcal{H}^1(\Gamma) \to \ell^p\mathcal{H}^1(\Gamma) \) is an injection;
- if \( \Gamma \) has no non-constant bounded harmonic functions whose gradient is in \( \ell^p(E) \) then \( \forall q < \frac{pd}{p+2}, \ell^q\mathcal{H}^1(\Gamma) = \{0\} \);
- if \( \Gamma \) has a non-constant (bounded or not) harmonic functions whose gradient is in \( \ell^p(E) \) then \( \ell^p\mathcal{H}^1(\Gamma) \neq \{0\} \).

More precisely, a map from \( \mathbb{D}^p(\Gamma) \) to harmonic functions modulo constants on \( \Gamma \) is exhibited, and it is shown does not depends on the representative of the reduced \( \ell^p \)-cohomology class \((1 \leq p < \infty)\). This maps sends bounded functions to bounded functions. To establish vanishing of reduced \( \ell^p \)-cohomology in degree 1, it is sufficient to consider only bounded functions (for \( 1 < p < \infty \)), thanks to a lemma of Holopainen & Soardi [22, Lemma 4.4] (see the end of subsection 2.2 for a simple proof).

Theorem 1.2 almost answers a question of Pansu [36, Question 6 in §1.9]: if \( \Gamma \) has IS\(_d\) for all \( d \), is the existence of a non-constant harmonic form whose gradient is in \( \ell^p(E) \) equivalent to non-trivial reduced \( \ell^p \)-cohomology in degree 1? Theorem 1.2 shows this holds if one allows to lose some regularity (\( q \) is bigger than \( p \)). This theorem is the compilation of Corollaries 3.3.1, 3.3.2 and 3.3.5.

Recall that all groups of subexponential growth are Liouville, i.e. the Poisson boundary associated to the simple random walk on the Cayley graph is trivial (in this article, by “trivial Poisson boundary”, one should always read trivial Poisson boundary for simple random walk in the Cayley graph).

**Corollary.** If \( G \) is a group of growth at least polynomial of degree \( d \) and its Cayley graph has trivial Poisson boundary (for the simple random walk, i.e. \( G \) is Liouville), then \( \ell^p\mathcal{H}^1(\Gamma) = \{0\} \) for any \( 1 \leq p < d/2 \).

In particular, groups of intermediate growth and \( \mathbb{Z}_2 \wr \mathbb{Z}^d \) has trivial reduced \( \ell^p \)-cohomology in degree 1, for any \( p \in [1, \infty[ \).

Note it is unknown whether being Liouville is an invariant of quasi-isometry for Cayley graphs (it is not even known whether it is invariant under changing the generating set). It actually suffices that \( G \) has a Cayley graph quasi-isometric to a Liouville graph in order to have that its \( \ell^p \)-cohomology vanish for all \( p \in [1, \infty[ \).

Using [17, Theorem 1.3] and 1.2 one can show that [amenable] lamplighters on \( \mathbb{Z}^d \) (e.g. \( \mathbb{Z}_2 \wr \mathbb{Z}^d \)) have harmonic functions with gradient in \( \ell^p \).
The reduced $\ell^2$-cohomology is trivial by Cheeger & Gromov [8], using Theorem 1.2 one gets a positive answer to question 1.1:

**Corollary.** Any finitely generated amenable groups has trivial reduced $\ell^p$-cohomology (in degree one) for all $p \in ]1,2]$. Virtually-$\mathbb{Z}$ groups are the only amenable groups with non-trivial reduced $\ell^1$-cohomology in degree 1.

The Poisson boundary is not an invariant of quasi-isometry (see, for example, T. Lyons’ examples [32]). However, the following corollary, which was known for $p = 2$ (trivially), may now be extended:

**Corollary.** If $\Gamma$ is a graph satisfying $\text{IS}_d$ and $\ell^pH^1(\Gamma) \neq \{0\}$ for some $1 \leq p < d/2$, then any graph quasi-isometric to $\Gamma$ has non-trivial Poisson boundary.

Indeed, if $\Gamma$ satisfies $\text{IS}_d$ and given that $p < d/2$, then inside the Poisson boundary lives the image (by boundary values) of the non-trivial reduced $\ell^p$-cohomology class representable by bounded functions; denote this image $\mathcal{P}_p$. Furthermore, if $D = \sup_{\Gamma\text{ has }\text{IS}_d} d \in [1,\infty]$, then $\mathcal{P} = \bigcup_{p < 2D} \mathcal{P}_p$ is a part of the Poisson boundary which will persist under quasi-isometry. One could also try to thicken $\mathcal{P}$ by considering more generic Banach spaces (e.g. Orlicz spaces).

The second result is but a consequence of the first.

**Theorem 1.3.** If $\Gamma$ has $\text{IS}_d$ with constant $\kappa$, then the cohomology is always reduced (i.e. $\ell^pH^1(\Gamma) = \ell^pH^1(\Gamma')$). Furthermore, if $n = \lceil \kappa^{-1} \rceil$ and $\Gamma^{[n]}$ is the $n$-fuzz of $\Gamma$ (the graph obtained from $\Gamma$ by adding edges between all points at distance $n$ in $\Gamma$), then there is a spanning tree $T$ in $\Gamma^{[n]}$ so that the non-trivial cohomology classes are exactly those non-trivial cohomology class in $T$ which also belong to $\mathcal{D}^p(\Gamma^{[n]})$.

The equality between reduced and unreduced cohomology was already known. For Cayley graphs, see Guichardet [21, Corollaire 1] or Martin & Valette [33, Corollary 2.4]): a group is amenable if and only if $\ell^pH^1(G) \neq \ell^pH^1(G)$, for some, and hence any, $p \in ]1,\infty]$. For more general graphs, it is implicit at least in Lohoué [29].

This result is interesting in that to compute the $\ell^p$ cohomology of a graph with positive isoperimetric constant, one needs only to run through the list of boundary values for a spanning tree, and look which one are in $\mathcal{D}^p$ of the initial graph. Of course, even if boundary values of the tree are somehow much more reasonable to compute (either by the methods of Bourdon & Pajot in [7] or as the harmonic functions associated to the random walk), this is probably not directly usable unless the spanning tree produced by Benjamini & Schramm in [5] may be made explicit.

Some results on $\ell^{p,q}$-cohomology are presented in §4.4.

Using Theorem 1.2, it is possible to get a vanishing result for groups with normal subgroups:

**Theorem 1.4.** Let $p \in ]1,\infty]$. Assume $G$ is a finitely generated group, $N < G$ is finitely generated as a group and the growth of $N$ is at least polynomial of degree $> 2p$. Assume further that $G/N$ is infinite and $\ell^pH^1(\Gamma_N) = \{0\}$, where $\Gamma_N$ is some Cayley graph of $N$. Then for all Cayley graphs $\Gamma_G$ of $G$, $\ell^pH^1(\Gamma_G) = 0$. If further $N$ is non-amenable then the statement is true in unreduced cohomology.
It is also possible to show that certain semi-direct products (where $N$ is not finitely generated) have trivial reduced $\ell^p$ cohomology. More precisely, if $H$ satisfies IS$_d$, $H$ has trivial reduced $\ell^p$ cohomology and $N$ is not finitely generated as a group, then $G = N \times H$ has also trivial reduced $\ell^p$ cohomology for $p < d/2$. See [18] for more details. This seems to support a positive answer to 1.1. Here is a probably easier question which should shed more light on this topic:

**Question 1.5.** Is there an amenable group $G$ so that its Cayley graph has non-constant harmonic functions with gradient in $c_0$?

Note that this condition is much stronger than asking for harmonic functions of, say, sublinear growth. In fact, an answer to this question for $G$ solvable would already be interesting.

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## 2 Definitions and further discussions

### 2.1 Preliminaries

A graph $\Gamma = (X, E)$ is defined by $X$, its set of vertices, and $E$, its set of edges. All graphs will be assumed to be of bounded valency. The set of edges will be thought of as a subset of $X \times X$. The set of edges will be assumed symmetric (i.e. $(x, y) \in E \Rightarrow (y, x) \in E$). Functions will take value in $K = \mathbb{R}$ or $\mathbb{C}$. Furthermore, functions on $E$ will always be assumed to be anti-symmetric (i.e. $f(x, y) = -f(y, x)$). This said $\ell^p(X)$ is the Banach space of functions on the vertices which are $p$-summable, while $\ell^p(E)$ will be the subspace of (anti-symmetric) functions on the edges which are $p$-summable.

The gradient $\nabla : K^X \to K^E$ is defined by $\nabla g(\gamma, \gamma') = g(\gamma') - g(\gamma)$. Given a finitely generated group $G$ and a finite set $S$, the Cayley graph Cay$(G, S)$ is the graph whose vertices are the element of $G$ and $(\gamma, \gamma') \in E$ if $\exists s \in S$ such that $s^{-1}\gamma = \gamma'$. In order for the resulting graph to have a symmetric edge set, $S$ is always going to be symmetric (i.e. $s \in S \Rightarrow s^{-1} \in S$). Also, Cayley graphs are always going to be connected (i.e. $S$ is generating). This said, it is worthwhile to observe that the gradient is made of $\{\lambda_{s - \text{Id}}g\}_{s \in S}$ where $\lambda$ is the left-regular representation. As for the right-regular representation, it is a (injective) homomorphism from $G$ into Aut(Cay$(G, S)$), the automorphism group of the Cayley graph.

The Banach space of $p$-Dirichlet functions is the space of functions $f$ on $X$ such that $\nabla f \in \ell^p(E)$. It will be denoted $\mathbb{D}^p(\Gamma)$. In order to introduce the $\mathbb{D}^p(\Gamma)$-norm on $K^X$, it is necessary to choose a vertex, denoted $e_\Gamma$ (in a Cayley graph, it is convenient to choose the neutral element). This said $|f|_{\mathbb{D}^p(\Gamma)}^p = \|\nabla f\|_{\ell^p(E)}^p + |f(e_\Gamma)|^p$. Lastly, $p'$ will denote the Hölder conjugate exponent of $p$, i.e. $p' = p/(p - 1)$ (with the usual convention that 1 and $\infty$ are conjugate).
The subject matter is the \( \ell^p \)-cohomology in degree one of the graph \( \Gamma \). This is the quotient
\[
\ell^p H^1(\Gamma) := (\ell^p(E) \cap \nabla K^X)/\nabla \ell^p(X).
\]
This space is not always separated, and it is sometimes more convenient to look at the largest separated quotient, the reduced \( \ell^p \)-cohomology,
\[
\ell^p H^1(\Gamma) := (\ell^p(E) \cap \nabla K^X)/\nabla \ell^p(X)^{\ell^p(E)}.
\]
By taking the primitive of these gradients, one may also prefer to define this by
\[
\ell^p H^1(\Gamma) := D^p(\Gamma)/\ell^p(X) + K\ell^p.
\]
A common abuse of language/notation will happen when we say the reduced cohomology is equal to the non-reduced one: this means that the “natural” quotient map \( \ell^p H^1(\Gamma) \to \ell^p H^1(\Gamma) \) is injective.

When \( G \) is a finitely generated group, this is isomorphic to the cohomology of the left-regular representation on \( \ell^p(G) \), see Puls’ paper [38] or Martin & Valette [33]. Another important result is that \( \ell^p \)-cohomology is an invariant of quasi-isometry:

**Theorem.** (see Élek [11, §3] or Pansu [37]) If two graphs of bounded valency \( \Gamma \) and \( \Gamma' \) are quasi-isometric, then they have the same \( \ell^p \)-cohomology (in all degrees, reduced or not).

The result is actually much more powerful, in the sense that it holds for a large category of measure metric spaces (see above mentioned references). For shorter proofs in more specific situations see Puls [40, Lemma 6.1] or Bourdon & Pajot [7, Théorème 1.1]. A first useful consequence is that it is possible to work on graphs and obtain results about manifolds (or *vice-versa*, when it is more convenient). A second corollary is that if \( G \) is a finitely generated group, the \( \ell^p \)-cohomology of any two Cayley graphs are isomorphic. Consequently, one may speak of the \( \ell^p \)-cohomology of a group without making reference to a Cayley graph.

Kanai has shown [26] that any Riemannian manifold with Ricci curvature and injectivity radius bounded from below is quasi-isometric to a graph (of bounded valency). Also, if \( M \) is a compact Riemannian manifold and \( \tilde{M} \) its universal covering, then \( \tilde{M} \) is quasi-isometric to the fundamental groups \( \pi_1(M) \). Even if the language is that of graphs, there is always a corresponding result for Riemannian manifolds of bounded geometry (see Corollary for a summary of the results expressed on Riemannian manifolds).

A last important quantity to introduce, before moving on to the results, are isoperimetric profiles (see [49, (4.1) Definition]). For a set of vertices \( A \) let \( \partial A \) be the edges between \( A \) and \( A^c \). Let \( \mathfrak{g} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) be a function. Then \( \Gamma \) (of bounded valency) satisfies the isoperimetric profile \( IS_{\mathfrak{g}} \) if there is a \( \kappa > 0 \) such that, for any non-empty finite set of vertices \( A \)
\[
\mathfrak{g}(|A|) \leq \kappa |\partial A|
\]
If \( \mathfrak{g}(t) = t^{1-1/d} \) then this is called a \( d \)-dimensional isoperimetric profile; the short notation is \( IS_d \). If \( \mathfrak{g}(t) = t \) this is called a strong isoperimetric profile (or inequality); the short...
notation used here will be $\text{IS}_\omega$. In the latter case, the constant $\kappa$ is sometimes referred to as the isoperimetric constant. It is straightforward to see that $\text{IS}_\omega \Rightarrow \text{IS}_d$ for all $d$. The converse is false (hence the choice of $\omega$ rather than $\infty$). Obviously $d' \leq d$, then $\text{IS}_d \Rightarrow \text{IS}_{d'}$.

The constant $\kappa$ in the various isoperimetric profiles is not an invariant of quasi-isometry. However (see [49, (4.7) Theorem]), satisfying a $d$-dimensional or a strong isoperimetric profile is an invariant of quasi-isometry.

Under the convention that Cayley graphs are always connected, recall that $G$ is a non-amenable group precisely when one (hence all) of its Cayley graph have a strong isoperimetric profile (Følner’s classical result, [15]).

2.2 Discussion

Theorem 1.2 extends the result of Bourdon & Pajot [7, Théorème 1.1] from hyperbolic groups or spaces to those satisfying a $d$-dimensional isoperimetric profile, and extends the result of Lohoué [29] from graphs with $\text{IS}_\omega$ to graphs with $\text{IS}_d$ for all $d$.

To position this theorem with respect to Tessera’s [45, Theorem 2.2], note that many groups with CF were already known to have trivial Poisson boundary, see Kaimanovich’s [25, Theorem 3.3] for $F \wr \mathbb{Z}$ and [25, Corollary on p.23] for polycyclic groups. R. Tessera pointed out to the author that this is indeed the case for all groups with CF. Indeed, one can construct “controlled” almost-invariant vectors (in the $\ell^2$-left-regular representation) and, using an argument of Austin, Naor & Peres from [1], conclude that their Poisson boundary must be trivial (see Tessera’s article [46]).

However, $\mathbb{Z}_2 \wr \mathbb{Z}^2$ has a trivial Poisson boundary, but is not CF. This is a consequence of estimates of Erschler on the isoperimetric profile [13] of wreath products. These estimates is not compatible with the isoperimetric profile of CF groups computed in [46]. Bartholdi & Virág proved the Basilica group also has trivial Poisson boundary [3, Theorem 1], but is not CF: the return probability of CF groups of exponential growth is $\approx c e^{n-1/3}$ (see [46]) but that of the Basilica is $\geq c' e^{n-2/3}$ (see [3, Theorem 2]). For more examples, see the results of Revelle in [41] and the references therein.

Let us also mention that groups of intermediate growth are most likely not CF (though the author ignores the existence of a direct argument). Note further that any semi-direct product $N \rtimes H$ where $N$ is finitely generated nilpotent and $H$ is finitely generated and Liouville has trivial Poisson boundary by Kaimanovich [25, Theorem 3.2]. Note that the result is stated there in terms of rate of escape (also called drift speed, i.e. the expected length of a element after $n$ steps of the random walk starting at the identity), but a result of Varopoulos [48] implies this coincides with triviality of the Poisson boundary (the group being finitely generated).

Thus, any semi-direct product $N \rtimes H$ of finitely generated groups with $N$ nilpotent and $H$ Liouville is Liouville again; consequently, 1.2 implies its reduced $\ell^p$ cohomology is also trivial (for all $p \in [1, \infty]$), unless the resulting group has polynomial growth (in which case the vanishing is well-known).

Indeed, Theorem 1.2 does not cover (for all $p$) groups of polynomial growth. But for these groups, many (quite different) proofs of the vanishing of reduced $\ell^p$-cohomology are available to the reader: groups of polynomial growth have infinitely many finite conjugacy
class and see [24, Theorem 6.4] or [17, Theorem 3.2]; they are also polycyclic, hence CF, and see [45, Theorem 2.2] or [17, Theorem 1.3]; lastly they satisfy certain Poincaré inequalities and see [23, Corollary 1.10]. The first assertion requires to use that groups of polynomial growth are virtually nilpotent by Gromov’s famous result [19].

Theorems 1.3 and/or 1.2 essentially unify many preceding notions of an ideal boundary which allows to compute the reduced $\ell^p$-cohomology (in degree one). These boundaries are $\ell^p$-corona (see Gromov [20, §8.C] and Élek [10]), the Bourdon & Pajot boundary for hyperbolic spaces (see [7]), the Floyd boundary (Puls, see [39]) and the $p$-harmonic boundary (Puls, see [40]). The advantage of the Poisson boundary is that it is better understood than most of the above (e.g. it possesses a linear structure). It is worthwhile to underline that, in non-amenable groups, a result of Karlsson [27] exhibits a strong link between Floyd and Poisson boundaries.

[17, Theorem 1.3] shows some wreath products also have trivial reduced $\ell^p$-cohomology. This means that groups such as $H \wr \mathbb{Z}^k$ (where $H$, the “lamp state” group, is amenable and $k > 0$) have no [bounded or not] harmonic functions with gradient in $\ell^p$ (though they have bounded harmonic functions if $k > 2$). Actually, the Poisson boundary of these groups is fully described in [14, Theorem 1] (under the further assumption that $k > 4$), and one may directly check that these do not have harmonic functions in $\ell^p$.

Also, using [14, Theorem 2], one may check that the free metabelian groups of rank $\geq 5$ also do not have harmonic functions with $\ell^p$ gradient, and hence trivial reduced $\ell^p$ cohomology in degree 1. As a last note on this topic, Martin & Valette [33, Theorem (iv)] shows that wreath products $H' \wr H$ where $H'$ is non-amenable have trivial reduced $\ell^p$-cohomology. See [18] for even more wreath products with trivial reduced $\ell^p$-cohomology in degree 1.

In higher degree, there is no hope to extend Theorem 1.2 or 1.3. Pansu computed in [36, Théorème B] that, already in degree 2, some groups have non-trivial cohomology exactly in an interval. Theorem 1.4 extends Bourdon, Martin & Valette [6, Theorem 1,1)].

Finally, let us sum up the results in the language of $p$-harmonic functions and on Riemannian manifolds. When $1 < p < \infty$, it is known (see Puls [38, §3] or Martin & Valette [33, §3]) that the existence of non-constant $p$-harmonic function (i.e. $h \in D^p(\Gamma)$ with $\nabla^* \mu_{p,p'} \nabla h = 0$, where $\mu_{p,p'}$ is the Mazur map defined by $(\mu_{p,p'} f)(\gamma) = |f(\gamma)|^{p-2} f(\gamma)$) is equivalent to the non-vanishing of reduced cohomology in degree 1. In this light, Theorem 1.2 is even more surprising, as one may replace solutions of a non-linear equation (the $p$-Laplacian) by solutions to a linear one (the Laplacian).

Using the fact that the reduced $\ell^p$-cohomology of groups of polynomial growth is trivial, let’s sum up the results for groups in terms of $p$-harmonic functions:

**Corollary.** Let $p \in [1, \infty]$ and $\Gamma$ be the Cayley graph of a finitely generated group $G$. Assume one of the following holds

- $G$ is amenable and $p \in [1, 2]$;
- $\exists q \in [p, \infty]$ such that $\Gamma$ has no non-constant bounded harmonic function whose gradient is in $\ell^q(E)$ (e.g. $\Gamma$ is Liouville, e.g. $G$ is of intermediate growth, e.g. $G$ is...
polycyclic, e.g. \( G = N \rtimes H \) where \( N \) is finitely generated nilpotent and \( H \) is finitely generated Liouville);

- \( \exists q \in [p, d/2] \) such that there are no non-constant bounded \( q \)-harmonic functions on \( \Gamma \).
- there is an infinite finitely generated normal subgroup \( N \triangleleft G \) with \( G/N \) infinite and the growth of \( N \) is at least a polynomial of degree \( > 2p \);

Then there are no non-constant \( p \)-harmonic functions on \( \Gamma \).

Note that the corollary also holds

- when \( G \) has infinitely many finite conjugacy classes (e.g. \( G \) has infinite centre, e.g. \( G \) is of polynomial growth);
- when \( G \) is transport amenable (e.g. \( G \) is of the form \( H \wr \mathbb{Z}^k \) where \( H \) is amenable and \( k > 0 \));

thanks to [17, Theorem 1.3] and [17, Theorem 3.2] (see also [24], [33] and [45]).

Using quasi-isometry to re-express the results on Riemannian manifolds, one obtains:

**Corollary.** Let \( M \) be a Riemannian manifold (with Ricci curvature and injectivity radius bounded below). The degree one reduced \( \ell^p \)-cohomology of \( M \) vanishes and (equivalently) there are no non-constant continuous \( p \)-harmonic functions on \( M \)

- if \( M \) is the universal cover of a compact Riemannian manifold \( M' \) with \( \pi_1(M') =: G \) and \( p \) satisfying one of the conditions of Corollary;
- if \( M \) satisfies a \( d \)-dimensional isoperimetric profile with \( d > 2p \) and \( M \) is Liouville.
- if \( M \) satisfies a \( d \)-dimensional isoperimetric profile, \( \ell^q H^1(M) = 0 \) for some \( q \in [p, \infty] \) and \( d > 2q \).

Furthermore, \( \ell^1 H^1(M) = \{0\} \) if and only if \( M \) has one end.

In particular, if \( M \) is the universal covering of \( M' \) with \( \pi_1(M') \) amenable, then \( \ell^p H^1(M) = \{0\} \) for \( p \in ]1, 2] \).

Other known consequences of the triviality of the reduced \( \ell^p \)-cohomology include the triviality of the \( p \)-capacity between finite sets and \( \infty \) (see [50] and [40, Corollary 2.3]) and existence of continuous translation invariant linear functionals on \( D^p(\Gamma)/\mathbb{K} \) (see [40, §8]).

Since the lemma 4.4 from Holopainen & Soardi [22] will often be used and its proof relies on \( p \)-harmonic functions, the author feels he owes the reader a proof which does not require the use of \( p \)-harmonic functions.

**Lemma 2.2.1** (Holopainen & Soardi [22], 1994). Let \( g \in D^p(\Gamma) \) be such that \( g \notin [0] \in \ell^p H^1(\Gamma) \). For \( t \in \mathbb{R}_{>0} \), let \( g_t \) be defined as

\[
g_t(x) = \begin{cases} 
  g(x) & \text{if } |g(x)| < t, \\
  \frac{g(x)}{|g(x)|} & \text{if } |g(x)| \geq t.
\end{cases}
\]
Then there exists \( t_0 \) such that \( g_t \notin [0] \), for any \( t > t_0 \). In particular, the reduced \( \ell^p \) cohomology is trivial if and only if all bounded functions in \( D^p(\Gamma) \) have trivial class.

**Proof.** The proof goes essentially as in Proposition A.1. Assume, without loss of generality that \( g(o) = 0 \) for some preferred vertex (i.e. root) \( o \in X \). Since \( \| \nabla g \|_{\ell^\infty(E)} \leq \| \nabla g \|_{\ell^p(E)} =: K \), given \( x \in X \) and \( P \) a path from \( o \) to \( x \),

\[
|g(x)| = |g(x) - g(o)| = \sum_{e \in P: o \rightarrow x} \nabla g(e) \leq d(o, x) \| \nabla g \|_{\ell^p(E)}.
\]

In particular, \( g_t \) is identical to \( g \) on \( B_{t/K} \). Hence \( \| g - g_t \|_{D^p(\Gamma)} \leq \| \nabla g \|_{\ell^p(B_{t/K})} \), where \( \ell^p(B_{t/K}) \) denotes the \( \ell^p \)-norm restricted to edges which are not inside \( B_{t/K} \). Because \( \nabla g \in \ell^p(E) \), \( \| \nabla g \|_{\ell^p(B_{t/K})} \) tends to 0, as \( t \) tends to \( \infty \).

Now if there is an infinite sequence \( t_n \) such that \( g_{t_n} \) are in \( [0] \) and \( t_n \rightarrow \infty \), then \( g_{t_n} \) is a sequence of functions in \( [0] \) which tends (in \( D^p \)-norm) to \( g \). This implies \( g \in [0] \), a contradiction. Hence, for some \( t_0 \), \( g_t \notin [0] \) given that \( t > t_0 \). \[ \square \]

It seems worthwhile to note that this proof works for \( c_0 \) (though it is clearly false for \( \ell^\infty \), see Proposition A.3), whereas the original proof requires \( p \)-harmonic functions (which are not defined in this extremal cases). Of course, one can check directly that \( D^1(\Gamma) \subset \ell^\infty(X) \) (see §A), so that even if the proof still works for \( p = 1 \), this is not much of a surprise.

### 3 Boundary values and simple random walks

Let \( \Gamma = (X, E) \) be a graph. Let \( \xi_x \in \ell^1(X) \) be a family of finitely supported, positive elements of \( \ell^1 \)-norm one. Define

\[
\xi * g(x) = \int_X g(y) \xi_x(y) = \sum_{y \in X} g(y) \xi_x(y).
\]

If there exists a \( R \in \mathbb{Z}_{\geq 0} \) so that the support of \( \xi_x \) is contained in a ball of radius \( R \) at \( x \), then \( \xi * g \in [g] \). Assume given a sequence \( \xi_x^{(n)} \) of such families and suppose that \( \xi_x^{(n)} * g \) converges pointwise. The main idea of this section is that it might prove more useful to try to get directly information about \( g \) instead of trying to show convergence of \( \xi_x^{(n)} * g \) in \( D^p(\Gamma) \) (this second option is investigated in [17]).

The first subsection treats a fairly generic situation. Existence of boundary values when \( \xi_x^{(n)} \) is the time \( n \) distribution of a simple random walker starting at \( x \) can be significantly simplified. This may be achieved by interested reader should read Remark 3.3.3.

#### 3.1 Boundary values

Here is the avatar of this viewpoint: if \( g \) does not take “enough distinct values” then its class is trivial. This motivates the introduction of boundary values. The following argument is well-known, it may probably be traced back to Strichartz [44], if not earlier.
Lemma 3.1.1. Assume $h$ is a $p$-harmonic function (i.e. an element of minimal $\mathbb{D}^p$-norm in its cohomology class). If there exists a $c \in \mathbb{K}$ such that $\forall \varepsilon > 0$ the set $X \setminus h^{-1}(B_\varepsilon(c))$ is finite (where $B_\varepsilon(c) = \{k \in \mathbb{K} \mid |c - k| < \varepsilon\}$). Then $h$ is constant (so $[h] = 0$).

Proof. First, one may assume $c = 0$ by changing $h$ up to a constant. Then $\forall t > 0$, the truncated function $h_t$ defined by

$$h_t(\gamma) = \begin{cases} th(\gamma)/|h(\gamma)| & \text{if } |h(\gamma)| > t \\ h(\gamma) & \text{otherwise} \end{cases}$$

is distinct from $h$ only on a finite set, and so $[h_t] = [h]$. However, its $\mathbb{D}^p$ norm is smaller, which contradicts minimality, unless $h_t = h = 0$. \qed

In fact, with a little more effort, one could check the following. Let $h \in \mathbb{D}^p(\Gamma)$ be as in the lemma but not necessarily $p$-harmonic. Then the functions $h - h_t$ are finitely supported (thus in the trivial class) and $h - h_t \mathbb{D}^p \to h$ (see the proof of Proposition A.1). Thus $h \in [0]$.

The aim here is to define a “boundary value” for functions so that the “value” does not depend on $p$ or the representative in the reduced cohomology class and it is constant exactly when the hypothesis of Lemma 3.1.1 apply. In order to do so, one must show some continuity in $\mathbb{D}^p$-norm.

Let $\xi$ and $\phi$ be two (finitely supported probability) measures. Let $m_{\xi,\phi}$ be a (finitely supported probability) measure on $\phi(\Gamma)$ the set of (oriented) paths in $\Gamma$. The marginals of this measure at the beginning and the end of the paths yield respectively $\xi$ and $\phi$. Denote by $E(p)$ the edges of the path $p$. Then for any such measure define

$$\forall a \in E, m_{\xi,\phi}^\flat(a) = \sum_{p \ni a} m(p).$$

The (weighted) number of times an edge is used during the transport from $\xi$ to $\phi$ is encoded in the (finitely supported positive, but not probability) measure $m_{\xi,\phi}^\flat$. There are many measures satisfying the condition on the marginals so $m_{\xi,\phi}^\flat$ is never unique.

Here is a more convenient formulation.

Definition 3.1.2. A transport pattern from $\xi$ to $\phi$ (two finitely supported probability measures) is a finitely supported function on the edges $\tau_{\xi,\phi}$ such that $\nabla^* \tau_{\xi,\phi} = \xi - \phi$.

Lemma 3.1.3. Let $\xi, \phi$ be as above, $g \in \mathbb{D}^p(\Gamma)$ and $\tau_{\xi,\phi} \in \ell^p'(E)$ be a transport pattern. Then

$$\left| \int g d\xi - \int g d\phi \right| \leq \|\nabla g\|_{\ell^p(E)} \|\tau_{\xi,\phi}\|_{\ell^p'(E)}.$$

Proof. Simply note that

$$\int g d\xi - \int g d\phi = \langle g \mid \xi - \phi \rangle = \langle g \mid \nabla^* \tau_{\xi,\phi} \rangle = \langle \nabla g \mid \tau_{\xi,\phi} \rangle,$$

then conclude using Hölder’s inequality. \qed

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As a consequence obtaining the following bound will become important:

\[ \lim_{n \to \infty} \sup_{k \geq 0} \| \tau_{\xi_x^{(n)}}(\xi_x^{(n+k)}) \|_{\ell^r(E)} = 0. \]  

where \( \xi_x^{(n)} \) is a sequence of (finitely supported probability) measures.

**Definition 3.1.5.** Let \( \{\xi_x^{(n)}\}_{x \in X, n \in \mathbb{Z}_{\geq 0}} \) be (finitely supported probability) measures such that (3.1.4) holds for \( r = p' \) and for all \( x \in X \). If \( g \in D^q(\Gamma) \), then \( \xi^{(n)} \ast g \) converges pointwise and the resulting pointwise limit is called the boundary value of \( g \) for \( \xi^{(n)} \).

The above definition of boundary value is closer to that of Pansu [35] than Bourdon & Pajot [7]. With the choice of \( \xi_x^{(n)} \) which will considered in a few paragraphs, these boundary values are harmonic functions.

An important point in the preceding definition is that, if the condition (3.1.4) holds for \( r = p' \) it holds for \( r = q' > p' \), and if \( g \in D^q(\Gamma) \) then \( g \in D^p(\Gamma) \). In other words, if this boundary value exists for \( p \), it does not depend on \( p \) and exists for all \( q < p \).

The choice of \( \tau \) is very important in the computation of the norm. Even in \( \mathbb{Z}^2 \) and for \( \xi_x^{(n)} \) the normalised characteristic function of the balls around \( x \), some (too simple) choices of \( \tau \) will not satisfy the required condition whereas others will.

### 3.2 Simple random walks

In order to start with the easiest setting, the measure \( \xi_x^{(n)} \) will, in this subsection and subsection 3.3, be \( P_x^{(n)} \) where \( P_x^{(n)}(y) \) is the probability a simple random walker starting at \( x \) reaches \( y \) in \( n \) steps. With this vocabulary, \( P^{(n)} \ast g(x) \) is the expected value of \( g \) after \( n \) steps of a simple random walk starting at \( x \). When the boundary value exists, it is a harmonic functions (for the simple random walk); see also Remark 3.3.3 below.

The advantage of these function is that there is a canonical choice for \( \tau_{P_x^{(n)}_x, P_x^{(n+k)}} \). Namely, \( \tau_{P_x^{(n)}_x, P_x^{(n+k)}} \) is given by continuing the random walk \( k \) steps. This understood, let \( P_x^{(i)} \) be the measure on the edges defined by

\[ \forall y \in X, \forall s \in S, P_x^{(i)}(y, sy) = \frac{1}{|S|} P_x^{(i)}(y). \]

Then \( \tau_{P_x^{(n)}_x, P_x^{(n+k)}} = \sum_{i=n}^{n+k-1} P_x^{(i)} \). This formula is particularly useful as one can give an upper bound in terms of more well-studied quantities:

\[ \| \tau_{P_x^{(n)}_x, P_x^{(n+k)}} \|_{\ell^r(E)} \leq \frac{1}{|S|^{n+k}} \sum_{i=n}^{n+k-1} \| P_x^{(i)} \|_{\ell^r(X)}. \]

This proves:

**Lemma 3.2.1.** If \( \xi_x^{(n)} = P_x^{(n)} \) as above, and, for all \( x \in X \), \( \sum_{i=0}^{\infty} \| P_x^{(i)} \|_{\ell^r(X)} < +\infty \), then any \( g \in D^p(\Gamma) \) admits a boundary value for \( \xi^{(n)} \).
In particular, the simple random walk is transient exactly when the condition of Lemma 3.2.1 holds for \( p' = \infty \). (It never holds in the case \( p' = 1 \), but \( p = \infty \) is also not of interest.)

Fortunately, there are very good estimates at hand for \( \| P_x^{(n)} \|_{\ell^p}(X) \), and some of them rely only on isoperimetric profiles (e.g., in the case of a Cayley graph, the growth of the group).

The appropriate assumption is that the graph \( \Gamma \) must satisfies some isoperimetric profile (typically \( IS_d^{\delta_d} \)), see [49, (14.5) Corollary]. Recall that quasi-homogeneous graphs with a certain (uniformly bounded below) volume growth in \( n_d \) will satisfy these isoperimetric profiles, see [49, (4.18) Theorem]. If \( \Gamma \) has \( IS_d^{\delta_d} \), then

\[
\exists K > 0, \forall x, y \in X, P_x^{(n)}(y) \leq Kn^{-d/2}
\]

**Corollary 3.2.2.** If \( \Gamma \) has \( IS_d^{\delta_d} \), then boundary values of \( g \in D^p(\Gamma) \) exists for \( \xi^{(n)} = P_x^{(n)} \) if \( p < d/2 \).

**Proof.** Obviously \( \| P_x^{(n)} \|_{\ell^1(X)} = 1 \). By Hölder’s inequality,

\[
\| P_x^{(n)} \|_{\ell^{p'}(X)} \leq \| P_x^{(n)} \|_{\ell^1(X)}^{1/p'} \| P_x^{(n)} \|_{\ell^\infty(X)}^{1/p}.
\]

Hence \( \| P_x^{(n)} \|_{\ell^{p'}(X)} \leq K' n^{-d/2p} \) uniformly in \( x \), for some \( K' > 0 \).

This shows that there are plenty of graphs where boundary values may be defined. It remains to prove these boundary values have the desired properties.

**Lemma 3.2.3.** Assume that the boundary values for \( P_x^{(n)} \) may be defined in \( D^p(\Gamma) \) (and \( X \) is not finite). If \( g \in D^p(\Gamma) \) then its boundary value is trivial.

**Proof.** If \( g \in \ell^p(G) \) then \( P_x^{(n)} g \) tends to 0 (as \( \ell^p \subset \ell^0 \) and the mass of \( P^{(n)} \) tends to 0 on finite sets). It remains to be checked that convergence in \( D^p(\Gamma) \) does not alter the boundary value. However, by Lemma 3.1.3, if \( g - g_n \) tends to 0 in \( D^p(\Gamma) \) norm so does the difference of their boundary values.

In fact, since the boundary value of a sum is the sum of boundary values, this shows the boundary value does not depend on the representative of the reduced cohomology class, up to a constant function.

**Lemma 3.2.4.** Assume \( g \in D^p(\Gamma) \) and \( \sum_{i \geq 0} \| P^{(i)} \|_{\ell^{p'}(G)} < +\infty \). If the boundary value of \( g \) for \( P^{(n)} \) is a constant function, then \([g] = 0 \in \ell^p H^1(\Gamma)\).

**Proof.** By changing the representative of the reduced class one may assume that \( g \) is \( p \)-harmonic and by adding by a constant, that the boundary value is trivial. Fix a root \( o \in X \) of the graph. For any \( \varepsilon > 0 \), let \( n_\varepsilon \) be such that \( \| \nabla g \|_{\ell^p(E \setminus B_{n_\varepsilon}(o))} < \varepsilon \) and, uniformly in \( x \),

\[
\sum_{i \geq 0} \| P_x^{(i)} \|_{\ell^{p'}(E)} < \varepsilon.
\]
Let $E[Y]$ denote the edges incident with $Y \subset X$ and $\langle \cdot \mid \cdot \rangle_{E'}$ restriction of the pairing to the set $E'$. If $x \notin B_{3\nu}(c)$, then

$$|P^{(k)} * g(x) - g(x)| = |\langle g \mid P^{(k)}_x \delta_x \rangle|$$

$$= |\langle g \mid \nabla^* \tau_{P^{(k)}_x} \delta_x \rangle|$$

$$\leq \|\nabla g\|_{\ell^p(E' \setminus B_{3\nu}(x))}\|\tau_{P^{(k)}_x} \delta_x\|_{L^q(E)} + \|\nabla g\|_{\ell^p(E')}\|\tau_{P^{(k)}_x} \delta_x\|_{L^q(E\setminus B_{3\nu}(x))}$$

$$\leq \varepsilon \sum_{i \geq 0} \|P^{(i)}_x\|_{L^q(E')} + \|\nabla g\|_{\ell^p(E')}\sum_{i \geq 0} \|P^{(i)}_x\|_{L^q(E')}$$

$$\leq \varepsilon \sum_{i \geq 0} \|P^{(i)}_x\|_{L^q(E')} + \|\nabla g\|_{\ell^p(E')}\varepsilon$$

where $c$ is a constant depending only on the constant in $\text{IS}_d$ and the $D_p$-norm of $g$. Thus, letting $k \to \infty$, for all $x \notin B_{3\nu}(c)$, $|g(x)| \leq \varepsilon$. Thus, Lemma 3.1.1 may be applied to yield that $|g| = 0$. 

A trivial, but useful, remark, is that if $g$ is bounded (i.e. in $\ell^{\infty}(X)$) then its boundary value is also bounded.

3.3 Inclusion and vanishing

**Corollary 3.3.1.** Let $\Gamma$ be a graph with $\text{IS}_d$ and $1 \leq q \leq p < d/2$. Then the natural quotient $\ell^q H^1(\Gamma) \to \ell^p H^1(\Gamma)$ is an injection.

**Proof.** Assume $[g] \neq [0] \in \ell^q H^1(\Gamma)$. Then, by Lemma 3.2.4, its boundary value is not trivial. However, this boundary value does not depend on $p$ and so, by Lemma 3.2.3, $g$ is not trivial in $\ell^p H^1(\Gamma)$. 

Finally, let us consider the case where the graph has is Liouville, i.e. there are no non-constant bounded harmonic functions. This means that, if $g \in \ell^{\infty}(X)$ and $P^{(n)} * g$ converges pointwise, then the limit is a constant function.

**Corollary 3.3.2.** Let $\Gamma$ be a Liouville graph with $\text{IS}_d$ and $1 \leq q \leq p < d/2$. Then $\ell^p H^1(\Gamma) = \{0\}$ for all $p \in [1, \infty]$.

**Proof.** By a lemma of Holopainen & Soardi [22, Lemma 4.4] (see Lemma 2.2.1), it suffices to show all bounded functions in $D^p(\Gamma)$ are trivial. But if $g$ is in $\ell^{\infty}(X)$ and the graph is Liouville, the boundary value of $g$ for $P^{(n)}_x$ is constant. By Lemma 3.2.4, the conclusion follows.

**Remark 3.3.3.** As P. Pansu pointed out to the author, a result of Lohoué [29] shows that in graphs satisfying $\text{IS}_\omega$, there is a harmonic element in each (unreduced) class. This element is the boundary value above. To see this, first recall that the representative in the (reduced or unreduced) class of $g$ exhibited by N. Lohoué is defined by $g + u$ where $u = \Delta^{-1}(-\Delta g)$, where $\Delta = \text{Id} - P$ and $P$ is the common notation for $Pg = P^{(1)} * g$. 

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Thanks to H. Kesten, graphs satisfying $\text{IS}_d$ are exactly those where $\|P\|_{\ell^2 \to \ell^2} < 1$. On the other hand, $\|P\|_{\ell^p \to \ell^p} \leq 1$ for $p = 1$ or $\infty$. Thus $\|P\|_{\ell^p \to \ell^p} < 1$ for any $p \in ]1, \infty[$ by Riesz-Thorin interpolation, and $\Delta^{-1} = \sum_{i \geq 0} P^{(i)}$ is bounded from $\ell^p(X)$ to itself. Next notice
\[
\tilde{g} - g = \lim_{n \to \infty} P^{(n)} * g - g = \sum_{i \geq 0} (P^{(i+1)} * g - P^{(i)} * g) = (\sum_{i \geq 0} P^{(i)}) * (\Delta g),
\]
to conclude that $\tilde{g} = g + u$.

This argument may not apply in (many, if not all) Cayley graphs of amenable groups. Indeed, this would mean the harmonic function belongs to the same (unreduced!) cohomology class. But, in amenable groups, the reduced and unreduced $\ell^p$-cohomologies are never equal (see [21, Corollaire 1]). Thus, if the reduced $\ell^p$-cohomology is trivial (which is already known for many amenable groups), this would give a contradiction.

However, it is still possible answer positively a weaker form of P. Pansu’s question [36, Question 6 in §1.9] (i.e. is the absence of non-constant harmonic function whose gradient has finite $\ell^p$-norm is equivalent to $H^1(\Gamma) = \{0\}$). Lemma 3.2.4 shows that if there is such a harmonic function and $\Gamma$ has $\text{IS}_d$ for $d > 2p$, then $H^1(\Gamma)$ is not trivial. Indeed, such a function would be its own boundary value, and being non-constant, it would be non-trivial in cohomology.

Let us now address the reverse implication.

Lemma 3.3.4. Let $\Gamma$ be a graph with $\text{IS}_d$ and $1 \leq p < d/2$. For any $g \in D^p(\Gamma)$, let $\tilde{g}$ be its boundary value for $P_\infty^{(n)}$. Then $\tilde{g}$ is in the same $\ell^p H^1(\Gamma)$ class as $g$ for all $q > \frac{dp}{d-2p}$.

Proof. As before, write:
\[
\tilde{g} - g = \lim_{n \to \infty} P^{(n)} * g - g = \sum_{i \geq 0} (P^{(i+1)} * g - P^{(i)} * g) = \sum_{i \geq 0} P^{(i)} * (P^{(i)} - \text{Id}) g
\]
Let $h = (P^{(i)} - \text{Id}) * g = \Delta g \in \ell^p(X)$ and remember $P^{(i)}$ are operators defined by a kernel. Using Young’s inequality (see [43, Theorem 0.3.1]) for $r > p$ and $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$,
\[
\|\tilde{g} - g\|_{\ell^r(X)} = \left\| \left( \sum_{i \geq 0} P^{(i)} \right) * h \right\|_{\ell^r(X)} \leq \left\| \sum_{i \geq 0} P^{(i)} \right\|_{\ell^q(X)} \|h\|_{\ell^p(E)} \leq S \|\nabla g\|_{\ell^p(E)} \sum_{i \geq 0} P^{(i)} \|h\|_{\ell^r(X)}
\]
Since $\sum_{i \geq 0} P^{(i)} \|h\|_{\ell^q(X)} < +\infty$ for all $q < d/2$, this means $\tilde{g} - g \in \ell^r(X)$ (for all $r > \frac{dp}{d-2p}$) and consequently that $\tilde{g}$ and $g$ belong in the same (unreduced) $\ell^r$ cohomology class.

To address the question of whether $\tilde{g}$ and $g$ are in the same reduced $\ell^p$-cohomology class, the author believes one would need to consider question similar to those of the transport problem from [17].

Corollary 3.3.5. Let $\Gamma$ be a graph with $\text{IS}_d$ and $1 \leq p < d/2$. If there are no non-constant bounded harmonic functions whose gradient has finite $\ell^p(E)$ norm, then $H^1(\Gamma) = \{0\}$ for all $q < \frac{dp}{d-2p}$. Conversely, if there is a non-constant harmonic functions whose gradient has finite $\ell^p(E)$ norm, then $H^1(\Gamma) \neq \{0\}$ and there is also a bounded non-constant harmonic functions whose gradient has finite $\ell^p(E)$ norm.
Proof. Under this new hypothesis, one has that \( \tilde{g} \) is constant (by Lemma 3.3.4), so Lemma 3.2.4 implies that \([g] = 0 \in \ell^p H^1(\Gamma)\). As usual, boundedness may be added thanks to Holopainen & Soardi [22, Lemma 4.4] (see also Lemma 2.2.1).

As mentioned above, the converse is a consequence of Lemma 3.2.4.

Corollaries 3.3.1, 3.3.2 and 3.3.5 are particularly interesting for graphs satisfying IS\(_d\) for every \( d \in \mathbb{Z}_{\geq 1} \). This will be used in subsection 4.2.

4 Consequences

This section finishes the proof of theorems announced in the introduction, namely Theorems 1.3 and 1.4.

4.1 Graphs having IS\(_\omega\)

Proof of Theorem 1.3. When the graph has IS\(_\omega\), the fact that the quotient is always separated (i.e. \( \ell^q H^1(\Gamma) = \ell^q H^1(\Gamma) \)) is a consequence Remark 3.3.3.

Let \( \Gamma := \Gamma_1 \). For any finite subset \( F \) of \( \Gamma_1 \), \( |\partial^{\Gamma_1} F| \geq c|F| \) for some \( c > 0 \). So take \( n \in \mathbb{N} \) such that \( nc \geq 1 \). Let \( \Gamma_k \) be graph obtained by adding, for all \( \gamma \in \Gamma \), an edge between \( \gamma \) and all vertices at distance \( \leq n \) from \( \gamma \) (i.e. the \( k \)-fuzz of \( \Gamma \)). Denote \( \partial^{\Gamma_k} F \) to be the boundary of \( F \) in \( \Gamma_k \) and \( B_k(F) \) to be the sets of vertices at distance \( \leq k \) from \( F \) (\( F \) included). Then, for any finite subset \( F \) of \( \Gamma_n \),

\[
|\partial^{\Gamma_n} F| \geq |\partial^{\Gamma_1} B_n(F)| + |\partial^{\Gamma_n-1} F| \geq c|F| + |\partial^{\Gamma_n-1} F| \geq \ldots \geq nc|F| \geq |F|.
\]

In other words, there is a graph \( \Gamma_n \), quasi-isometric to \( \Gamma_1 \), satisfying a strong isoperimetric profile with constant \( \geq 1 \). But by a result of Benjamini and Schramm [5, Theorem 1.2], there is then a spanning tree in \( \Gamma_n \) which is obtained by adding edges between disjoint copies of binary trees. Let \( T \) be this tree, and look at the simple random walk on this tree. The boundary value of \( g \) for this tree will be in the same class for \( \ell^p H^1(\Gamma) \) as \( g \) (the difference being in \( \ell^p(X) \), the class is preserved for any set of edges as long as the valency remains bounded).

Furthermore, the fact that the boundary value is constant depends only on the boundary value for the tree. Boundary values actually shows the existence of a commuting diagram of “natural” injections (where \( T \) is the spanning tree inside \( \Gamma_n \)):

\[
\begin{array}{ccc}
\ell^q H^1(\Gamma_n) & \to & \ell^q H^1(T) \\
\downarrow & & \downarrow \\
\ell^p H^1(\Gamma_n) & \to & \ell^p H^1(T)
\end{array}
\]

As a consequence, the only boundary values possible (in a graph \( \Gamma \) having IS\(_\omega\)) are those of \( T \). To know if a boundary value of the tree is an actual cohomology class in \( \Gamma \), it remains to check that it belongs to \( \mathbb{D}^p(\Gamma) \).
4.2 Groups and growth

N. Varopoulos showed that superpolynomial growth implies superpolynomial decay of \( \|P^{(i)}\|_{L^\infty(G)} \) (e.g. see [47]). Let us also mention Woess’ book [49][14.5] Corollary, p.148:\n
\[ \text{if } |S^n| \geq n^d \text{ (for } d \geq 1 \text{) then } \|P^{(n)}\|_{L^\infty(G)} \leq n^{-d/2} \text{ and if } |S^n| \geq e^{\alpha n} \text{ (for } 0 < \alpha \leq 1 \text{) then } \|P^{(n)}\|_{L^\infty(G)} \leq e^{-n^{\alpha/(\alpha+2)}}. \]

In short, Cayley graphs of groups of polynomial growth of degree \( d \) satisfy \( IS_d \) and Cayley graphs of groups of superpolynomial growth will satisfy \( IS_d \) for any \( d \in \mathbb{Z}_{\geq 1} \). In the latter case, Corollaries 3.3.1, 3.3.2 and 3.3.5 yield

**Corollary 4.2.1.** Let \( G \) be a group of superpolynomial growth and \( \Gamma \) a Cayley graph. Let \( 1 \leq p < \infty \). Then there exists a map \( \pi \) from \( L^p(\Gamma) \) to the space of harmonic functions modulo constants such that

1. \( \pi(g) = \pi(h) \iff [g] = [h] \in L^pH^1(\Gamma); \)
2. \( g \in L^\infty(G) \implies \pi(g) \in L^\infty; \)
3. \( \pi(g) \in L^q(\Gamma) \) for any \( q > p. \)

Using Theorem 1.2, it also easy to find graphs of polynomial growth which have \( L^pH^1(\Gamma) \neq \{0\}. \)

**Example 4.2.2.** Take \( \Gamma \) to be two copies of the usual Cayley graph of \( \mathbb{Z}^d \) and join them by an edge (say between their respective identity elements). So a vertex \( x \in \Gamma \) can be written as \( x = (z, i) \) with \( i \in \{1, 2\} \) and \( z \in \mathbb{Z}^d \). Consider \( f = \sum_{i \geq 0} P_i \) on \( \mathbb{Z}^d \). Define

\[ g(z, i) = \begin{cases} f(z) & \text{if } i = 1, \\ K + 2f(0) - f(z) & \text{if } i = 2, \end{cases} \]

where \( K = \nabla^d f(0) \). Then \( g \) is harmonic and \( \nabla g \in L^p(E) \) for \( p < 2d \) (since \( f \in L^p(\mathbb{Z}^d) \) for \( p < 2d \)). Since \( g \) is non-constant, \( \ell^pH^1(\Gamma) \neq \{0\} \) for any \( p < d/2. \)

Actually, since \( \Gamma \) has two ends, using Proposition A.1 one has that \( \ell^pH^1(\Gamma) \neq \{0\}. \) Since a corollary of Theorem 1.2 is that \( \ell^pH^1(\Gamma) \to \ell^qH^1(\Gamma) \) for \( 1 \leq q \leq p < d/2 \), one also sees that \( \ell^pH^1(\Gamma) \neq \{0\} \) for \( p \in [1, d/2]. \)

4.3 Normal subgroups

The existence of boundary values is a convenient tool. Let \( N \triangleleft G \) be an infinite normal subgroup of \( G \). If \( N \) is finitely generated as a group, it is possible to pick a Cayley graph \( \Gamma = \text{Cay}(G, S) \) so that \( S \) generates \( N \) and \( G \). Denote by \( \Gamma_N = \text{Cay}(N, S \cap N) \). Then \( \Gamma \) admits \( |G : N| \) copies of \( \Gamma_N \) as a subgraph. A function on \( \Gamma \) may be restricted to any of these copies and then be identified with a function on \( \Gamma_N \).

A preliminary result on graphs “stitched” together is necessary.

**Corollary 4.3.1.** Let \( \Gamma_i \) be a family of graphs all satisfying \( IS_d \) with the same constant. Let \( \Gamma^{\Pi} \) be the disjoint union of the \( \Gamma_i \). Fix some \( k \in \mathbb{Z}_{\geq 2} \) and let \( \Gamma' \) be obtained by adding \( k \) edges to each \( \Gamma_i \) inside \( \Gamma^{\Pi} \) so that the resulting graph is connected. Then \( [g] \notin 0 \in \ell^pH^1(\Gamma') \) if and only one of the following holds:
• there exists \( i \) such that \( [g_{|\Gamma_i}] \neq 0 \in \ell^pH^1(\Gamma_i) \);

• there exists \( i \neq j \) such that \( g_{|\Gamma_i} \) and \( g_{|\Gamma_j} \) are constant at \( \infty \) (in the sense of Lemma 3.1.1) but not the same constant.

If the \( \Gamma_i \) have IS\( \omega \), the statement holds for unreduced cohomology.

**Proof.** If all \( \Gamma_i \) satisfy IS\( d \) with the same constant then so do \( \Gamma' \) and \( \Gamma'' \) (a consequence of the concavity of \( t \mapsto t^{1-1/d} \)).

Assume \( g \) is non-trivial in cohomology, then, by Lemma 3.2.4, the boundary value of \( g \) in \( \Gamma' \) is non-constant. On the other hand, since the graphs \( \Gamma_i \) are transient, the boundary value of \( g \) restricted to each \( \Gamma_i \) will be arbitrarily close (outside finite sets) to that of \( g \) in \( \Gamma'' \). Thus, a first possibility is that the boundary of \( g \) restricted to one of the \( \Gamma_i \) will be non-trivial. If this is not the case, since the boundary value of \( g \) in \( \Gamma'' \) is non-trivial, the constants must be different.

Similarly, if all \( [g_{|\Gamma_i}] = 0 \), then, by Lemma 3.2.3, all boundary values of the copies of \( \Gamma_i \) are constant. By Lemma 3.2.4, \( g \) is trivial exactly when these constant are the same.

Lastly, if the \( \Gamma_i \) have IS\( \omega \), boundary values of \( g \) are in the same unreduced class as \( g \) (see Remark 3.3.3), so unreduced cohomology may be considered.

**Proof of Theorem 1.4.** Note that Corollary 4.3.1 holds in the following context. Let \( G \) be a finitely generated group and assume \( N \triangleleft G \) is finitely generated as a group and has polynomial growth at least \( d \) (where \( d \geq 3 \)). Let \( p \in [1, d/2] \). Let \( \Gamma \) and \( \Gamma_N \) be as above. Pick the \( \Gamma_i \) to be the various copies of \( \Gamma_N \) in \( \Gamma \). Since they are all isomorphic, they have IS\( d \) with the same constant. Consider two graphs: the graph of the disjoint copies of \( \Gamma_N \), say \( \Gamma_N' \) as above, and this same graph with edges added between the different copies so that the result is connected, call it \( \Gamma_N'' \). Note that only finitely many edges need to be added to each copy of \( \Gamma_N \) to construct \( \Gamma_N'' \) as \( G \) is finitely generated.

Assume there is a non-trivial element \( g \) in \( D^p(\Gamma) \). By Corollary 4.3.1, there are two possibilities.

First possibility: one of its restriction to some \( N\gamma_0 \) (one may assume \( \gamma_0 = e_G \)) possesses a non-constant boundary value. But after this restriction, the quotient is \( \ell^pH^1(\Gamma_N) \), which is trivial by hypothesis.

Second possibility: the restrictions all have constant boundary values, but not the same constant. This would imply that, outside large enough finite sets, these restrictions are arbitrarily close to their constant value. But the gradient of \( g \) on the initial graph \( \Gamma \) would then not be in \( \ell^p(E) \) (in fact, not even in \( c_0(E) \)) but in \( \ell^\infty(E) \). This contradicts \( g \in D^p(\Gamma) \).

As before, in the case when \( N \) is non-amenable, boundary values belong to the same unreduced class (see Remark 3.3.3), so the result will extend to unreduced cohomology in this case.

**4.4 \( \ell^{p,q} \)-cohomology**

Another quotient which is sometimes studied is the \( \ell^{p,q} \)-cohomology. This is the quotient,

\[
\ell^{p,q}H^1(\Gamma) := \frac{D^p(\Gamma)}{\ell^q(X) + K}.
\]
Recall that in Lemma 3.3.4, if $g \in D^p(\Gamma)$ and $\Gamma$ has IS$_d$, then $\tilde{g} - g \in \ell^q(X)$ for $q > \frac{dp}{d-2p}$.

As a corollary

**Corollary 4.4.1.** If $\Gamma$ is Liouville and has IS$_d$, then $\ell^q \mathcal{H}^1(\Gamma)$ is trivial for all $q > \frac{dp}{d-2p}$.

There is also an analogue of Theorem 1.3:

**Proposition 4.4.2.** Assume $\Gamma'$ has IS$_d$ and $\Gamma'$ is quasi-isometric to a spanning subgraph of $\Gamma$. Let $p < \infty$ and $q > \frac{dp}{d-2p}$. Then non-trivial $\ell^p,q$-classes of $\Gamma$ are given by functions $g \in D^p(\Gamma)$ whose class is non-trivial in $\ell^p,q \mathcal{H}^1(\Gamma')$.

### A Ends and degree one reduced $\ell^1$-cohomology

This section is devoted to the reduced degree one $\ell^1$-cohomology. This result is known (even well-known, if one adheres to the rule that at least 3 persons were aware of it): P. Pansu was aware of this, the main argument of non-vanishing is present in Martin & Valette [33, Example 3 in §4] who mention hearing it from M. Bourdon. It is included here for the sake of completion.

The ends of a graph are the infinite components of a group which cannot be separated by a finite (i.e. compact) set. More precisely, an end $\xi$ is a function from finite sets to infinite connected components of their complement so that $\xi(F) \cap \xi(F') \neq \emptyset$ (for any $F$ and $F'$). It may also be seen as an equivalence class of (infinite) rays who eventually leave any finite set. Two rays $r$ and $r'$ are equivalent if, for any finite set $F$, the infinite part of $r$ and $r'$ lie in the same (infinite) connected component.

Thanks to Stallings’ theorem, groups with infinitely many ends contain an (non-trivial) amalgamated product or a (non-trivial) HNN extension (and in particular are not amenable). Being without ends is equivalent to being finite, and amenable groups may not have infinitely many ends. This may be seen using Stallings’ theorem, see also Moon & Valette [34] for a direct proof (of a more general statement). An intuitive idea is that a Cayley graph with infinitely many ends contains has a quasi-isometry to a tree $T$ with strong isoperimetric constant, and hence cannot be amenable. Groups with two ends admit $\mathbb{Z}$ as a finite index subgroup. These groups are peculiar, as they have non-trivial reduced $\ell^1$-cohomology in degree 1, even if their reduced $\ell^p$-cohomology (in all degrees) vanishes for $1 < p < \infty$.

So outside virtually-$\mathbb{Z}$ groups, all infinite amenable groups have one end.

**Proposition A.1.** Let $\Gamma$ be a connected graph, then $\ell^1 \mathcal{H}^1(\Gamma) = 0$ if and only if the number of ends of $\Gamma$ is $\leq 1$. More precisely, let $\mathcal{N} = \mathbb{K}^{\text{ends}(\Gamma)}/\mathbb{K}$ be the vector space of functions on ends modulo constants. There is a boundary value map $\beta : D^1(\Gamma) \rightarrow \mathcal{N}$ such that $\beta(g) = \beta(h) \iff [g] = [h] \in \ell^1 \mathcal{H}^1(\Gamma)$.

Note that the isomorphism is in the category of vector spaces, not of normed vector spaces. In a few cases, the norm on $\mathcal{N}$ resembles the norm of the quotient $\ell^\infty(|\text{ends}|)/\mathbb{K}$.

**Proof.** In [33, Example 3 in §4], Martin & Valette exhibit functionals $T : D^1(\Gamma) \rightarrow \mathbb{K}$ (one for each pair of ends) on functions with a $\ell^1$-gradient which vanishes on $\nabla_1 \ell^1(X)^1(E)$. 

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A different method may be used. Note that $D^1(\Gamma) \subset \ell^\infty(X)$: if $g \in D^1(\Gamma)$, then, for $P$ a path from $x$ to $y$,

$$|g(y)| = |g(x) + \sum_{e \in P} g(e)| \leq |g(x)| + \|\nabla g\|_{\ell^1(E)}.$$

In fact, $\|g\|_{\ell^\infty(X)} \leq \|g\|_{D^1(\Gamma)} + \inf_{x \in X} |g(x)|$. Since functions in $\ell^1$ decrease at $\infty$, if one removes a large enough finite set, the function $g$ on the resulting graph is almost constant. In particular, it is possible to define a value of $g$ on each end:

Alternatively, if $r : \mathbb{Z}_{\geq 0} \to X$ is a ray representing the end $\xi$, then the value at $\xi$ is $\lim_{n \to \infty} g(r(n))$.

Fix an end $\xi_0$. Then, define $\beta : D^1(\Gamma) \to \mathcal{N}$ by changing with a constant the value of $g$ to be 0 at $\xi_0$ and then looking at the values at the ends. This map is continuous and trivial on $\ell^1(X) + \mathbb{K}$ (since functions in $\ell^1(X)$ have trivial value at the ends). By continuity, $\ell^1(X) + \mathbb{K}D^1(\Gamma) \subset \ker \beta$.

Assume, $\beta(f) = 0$, this means that, $\forall \varepsilon > 0, \exists X_\varepsilon \subset X$ a finite set such that $|f(X_\varepsilon^\xi)| < \varepsilon$. As in Lemma 3.1.1, set

$$f_\varepsilon ps(\gamma) = \begin{cases} \varepsilon f(\gamma)/|f(\gamma)| & \text{if } |f(\gamma)| > \varepsilon, \\ f(\gamma) & \text{otherwise,} \end{cases}$$

Then $g_\varepsilon := f - f_\varepsilon$ is finitely supported, so in $\ell^1(X)$. Furthermore, $\|f - g_\varepsilon\|_{D^1(\Gamma)} = \|f_\varepsilon\|_{D^1(\Gamma)}$.

Let $X_\varepsilon$ be as before, then

$$\nabla f_\varepsilon \begin{cases} \text{equal to } \nabla f & \text{on } E \cap (X_\varepsilon^\xi \times X_\varepsilon^\xi), \\ \text{smaller in } | \cdot | \text{ than } \nabla f & \text{on } \partial X_\varepsilon, \\ 0 & \text{on } E \cap (X_\varepsilon \times X_\varepsilon). \end{cases}$$

But $E \cap (X_\varepsilon \times X_\varepsilon)$ increases, as $\varepsilon \to 0$, to the whole of $E$. More importantly, the $\ell^1$-norm of $\nabla f$ outside this set tends to 0. Thus $\|f_\varepsilon\|_{D^1(\Gamma)} \to 0$ as $\varepsilon \to 0$, and consequently $f \in \ell^1(X)D^1(\Gamma)$.

Amenable groups with two ends step strangely out of the crowd: although their $\ell^p$-cohomology is always trivial if $p > 1$, it is non-trivial for $p = 1$ (actually isomorphic to the base field). An amusing corollary is

**Corollary A.2.** Let $G$ be a finitely generated group. $G$ has infinitely many ends if and only if for some (and hence all) Cayley graph $\Gamma$, $\forall p \in [1, \infty[, \ell^pH^1(\Gamma) \neq 0$. $G$ has two ends if and only if for some (and hence all) Cayley graph $\Gamma$, $\forall p \in ]1, \infty[, \ell^pH^1(\Gamma) = 0$ but $\ell^1H^1(\Gamma) = \mathbb{K}$.

**Proof.** Use Proposition A.1 for reduced $\ell^1$-cohomology, use any vanishing theorem on groups of polynomial growth ([24], [17, Theorem 3.2], or [45]) to get the remaining values of $p$ for groups with two ends, and finally use Theorem 1.3 on groups with infinitely many ends (which are in particular non-amenable).
It is worth noting that Bekka & Valette showed in [4, Lemma 2, p.316] that (for $G$ discrete) the cohomology $H^1(G, \mathbb{C}G)$ is also isomorphic as a vector space to $\mathcal{N}$. Furthermore, by [4, Proposition 1], there is an embedding $H^1(G, \mathbb{C}G) \hookrightarrow \ell^1 H^1(G)$. A careful reading would probably reveal this remains injective in reduced cohomology (the only case to check is when $G$ has two ends).

For completion, let us also mention an other extremal case:

**Proposition A.3.** Let $\Gamma$ be an infinite graph, then $\ell^\infty H^1(\Gamma) \neq \{0\}$.

*Proof.* $\ell^\infty H^1(\Gamma)$ is the quotient of Lipschitz functions by bounded Lipschitz functions so is manifestly never trivial. Further, if one takes $g$ to be the distance to a fixed vertex $r$, i.e. $g(\gamma) = d(\gamma, r)$, then $[g]$ is not trivial in the reduced cohomology. Indeed, a function with $\|g - h\| < 1/2$ has positive gradient on all edges between the spheres around $r$. As a consequence $h$ may not be bounded, and no element of $\ell^\infty(X)$ may be close to $g$. 

It seems quite plausible that $c_0 H^1(\Gamma) = \{0\}$ for any graph. Indeed, let $\Omega_\varepsilon$ a big ball such that $\|f\|_{D^\infty(\Gamma \setminus \Omega_\varepsilon)} \leq \varepsilon$. Start constructing $g_\varepsilon$ by making it equal to $f$ on $\Omega_\varepsilon$. If these functions can be extended so that its gradient is always $< \varepsilon$ outside $\Omega_\varepsilon$ and that it is finitely supported, then $\|f - g_\varepsilon\|_{D^\infty} < 2\varepsilon$. Hence the class of $f$ would be trivial. It might be useful to use Lemma 2.2.1 (i.e. $f$ may be assumed bounded) to conclude.

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