Approximate Dynamic Programming based on Projection onto the \((\min, +)\) subsemimodule

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Abstract

We develop a new Approximate Dynamic Programming (ADP) method for infinite horizon discounted reward Markov Decision Processes (MDP) based on projection onto a subsemimodule. We approximate the value function in terms of a \((\min, +)\) linear combination of a set of basis functions whose \((\min, +)\) linear span constitutes a subsemimodule. The projection operator is closely related to the Fenchel transform. Our approximate solution obeys the \((\min, +)\) Projected Bellman Equation (MPPBE) which is different from the conventional Projected Bellman Equation (PBE). We show that the approximation error is bounded in its \(L_\infty\)-norm. We develop a Min-Plus Approximate Dynamic Programming (MPADP) algorithm to compute the solution to the MPPBE. We also present the proof of convergence of the MPADP algorithm and apply it to two problems, a grid-world problem in the discrete domain and mountain car in the continuous domain.

1 Introduction

Markov Decision Process (MDP) is a useful mathematical framework for posing, analyzing and solving stochastic optimal sequential decision making problems. An MDP is characterized by its state space, action space, the model parameters namely reward structure, and the probability of transition from one state to another under any given action. We consider an MDP with \(n\) states and \(d\) actions. A policy \(u\) specifies the manner in which states are mapped to actions. The value of a state under a policy is the discounted sum of rewards starting in that state and performing actions according to that policy. Thus a given policy \(u\) induces a map from the state space to reals. This map is called the value-function, denoted by \(J_u \in \mathbb{R}^n\). Solving an MDP means computing the optimal value function \(J^* = \max_u J_u\) and the optimal policy \(u^* = \arg \max_u J_u\). The Bellman operator \(T\) (\([2]\)) is defined using the model parameters of an MDP, and is a map \(T : \mathbb{R}^n \rightarrow \mathbb{R}^n\). The Bellman Equation (BE) states that \(J^* = T J^*\) (\([2]\)), i.e., the optimal value function \(J^*\), is a fixed point of \(T\). Most methods to solve MDP such as value/policy iteration (\([2]\)) are based on solving the BE.

The phenomenon called Curse of Dimensionality (or simply curse) refers to the fact that the size of the state space grows exponentially in the number of the state variables. Most problems of practical interest suffer from the curse, i.e., have large number of states. In such situations it is expensive to compute

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the optimal policy/value-function and we need to resort to the use of approximate methods. Approximate Dynamic Programming (ADP) refers to an entire spectrum of methods that aim to obtain sub-optimal policies and approximate value-functions. Value-function based ADP methods consider a family of functions and pick a function that approximates the value function well. Typically, the family of functions considered is the linear span of a set of basis functions. This is known as linear function approximation (LFA) wherein the value function is approximated as \( J^* \approx \tilde{J} = \Phi r^* \). Here \( \Phi \) is an \( n \times k \) feature matrix and \( r^* \in \mathbb{R}^k \) is the weight vector with \( k << n \).

Given a \( \Phi \) matrix, ADP methods vary in the way they learn the weight vector and hence the approximate solution varies across the various ADP methods. In a class of ADP methods ([7]) \( r^* \) satisfies the below relation known as the Projected Bellman Equation (PBE).

\[ \Phi r^* = \Pi T \Phi r^* \tag{1} \]

where the projection matrix, \( \Pi = \Phi (\Phi^T D \Phi)^{-1} \Phi^T \) and \( D \) is any positive definite matrix. The approximation error can be bounded as below ([7]):

\[ ||\Phi r^* - J^*|| \propto ||\Pi J^* - J^*||_D. \tag{2} \]

Alternatively, there are ADP methods such as the Approximate Linear Program (ALP), wherein \( r^* \) does not obey a PBE, and is the solution to the below linear program.

\[ \min c^T \Phi r \]
\[ s.t \quad \Phi r \geq T \Phi r, \tag{3} \]

where \( c \in \mathbb{R}^n \) is such that \( c(i) \geq 0, i = 1, \ldots, n \) and \( \sum_{i=1}^{n} c(i) = 1 \). The approximation error is bounded as below ([4]):

\[ ||\Phi r^* - J^*||_{1,c} \propto ||\Pi J^* - J^*||_\infty, \tag{4} \]

where \( ||v||_{1,c} = \sum_{i=1}^{n} |v(i)|c(i) \). It is evident from (2) and (4) that the choice of ADP method is dictated by the kind of approximation guarantees required in the application at hand.

In this paper, we develop a ADP method based on LFA in \((\min, +)\) algebra called \((\min, +)\) approximate dynamic programming (MPADP). The \((\min, +)\) algebra differs from conventional algebra, in that + and \( \times \) operators are replaced by \( \min \) and \( + \) respectively. \( \mathbb{R}_{\min}^n = (\mathbb{R} \cup +\infty, \min, +) \) is a semiring and semimodule \( \mathbb{R}_{\min}^n \) can be defined over \( \mathbb{R}_{\min} \) in a manner similar to the vector space \( \mathbb{R}^n \) over \( \mathbb{R} \). Naturally, \( J^* \in \mathbb{R}_{\min}^n \), and given an \( n \times k \) feature matrix \( \Phi \), with columns \( \{\phi_j, j = 1, \ldots, k\} \), we consider the set \( \mathcal{V} = \{v|v = \Phi \otimes r \overset{\Delta}{=} \min(\phi_1 + r(1), \phi_2 + r(2), \ldots, \phi_k + r(k), r \in \mathbb{R}^k)\} \), where \( \otimes \) in \( \Phi \otimes r \) emphasizes the fact that the approximation is linear in \((\min, +)\). Our function class \( \mathcal{V} \) is a subsemimodule as opposed to the subspace in the conventional LFA. Akin to the PBE (1), in order to obtain the approximate value function \( \tilde{J} = \Phi \otimes r^* \) we project onto the subsemimodule \( \mathcal{V} \), i.e., \( r^* \) obeys the following \((\min, +)\) Projected Bellman Equations (MPPBE).

\[ \Phi \otimes r^* = \Pi M T \Phi \otimes r^*, \Phi \otimes r^* \in \mathcal{V} \tag{5} \]
where \( \Pi_M : R^n \to V \), is the \((\min, +)\) projection operator (defined in section 3).

Approximate Dynamic Programs based on the \((\min, +)\) semiring have been developed for deterministic control problems [1, 5] using the fact that the Bellman operator \( T \) is \((\min, +) \) - linear. However, in the case of infinite horizon discounted reward MDP, the presence of probability transition matrix, and discount factor destroys the linearity of the Bellman operator. This makes our MPADP algorithm significantly different from [1, 5]. Also the projection operator \( \Pi_M \) onto subsemimodules have been studied before in the literature [3]. Nevertheless, we use them in the context of finding approximate solution to MDPs. Our specific contributions in this paper are as given below.

1. We develop for the first time an ADP method that makes use of \((\min, +)\) LFA. Another novel aspect of our approach is the \((\min, +)\) PBE.

2. We characterize the approximation error of \( \tilde{J} = \Phi \otimes r^* \), the solution to MPPBE in (5). In particular, we show that the error bound of the form \( ||J^* - \tilde{J}||_\infty \propto \min_r ||J^* - \Phi \otimes r||_\infty \).

3. We show that \( \Pi_M \) is similar to the Fenchel transform and the MPPBE equation is similar to the ALP formulation.

4. We present the MPADP algorithm to solve (5). We also provide the proof of convergence for our algorithm.

5. We demonstrate our method on two benchmark planning problems namely the grid world and mountain car.

The rest of the paper is organized as follows. In section 2, we provide a brief introduction to discounted reward infinite horizon MDPs. In section 3, we define the \( R_{\min} \) semiring, and semimodules, and the \((\min, +)\) projection operator \( \Pi_M \) onto subsemimodules. In section 4, we discuss the similarities of the \((\min, +)\) projection operator \( \Pi_M \) and the Fenchel-Legendre transform. In section 5, we introduce the MPPBE equation and derive the approximation guarantees. Section 6 contains the MPADP algorithm with a proof of convergence. Section 7 contains experiments conducted on the “grid world” and “mountain car” problems. In section 8, we present the conclusions and also discuss future work.

## 2 Discounted Reward Markov Decision Processes

The ADP methods that we develop in this paper are for infinite horizon discounted reward Markov decision processes. Here, we provide a brief overview of MDPs (please refer to [2, 6] for a more detailed presentation). We consider an MDP with state space, \( S = \{1, 2, \ldots, n\} \) and action set, \( A = \{1, 2, \ldots, d\} \). We denote by \( p_a(i, j) \) the probability of transitioning from state \( i \) to \( j \) (\( i, j \in S \)) under action \( a \in A \). For simplicity, we assume that all actions \( a \in A \) are feasible in every state \( s \in S \). The reward is given by the map \( g : S \to R \) and the discount factor is \( \alpha, 0 < \alpha < 1 \).

By policy we mean a sequence \( \mu = \{\mu_0, \mu_1, \ldots\} \) of functions \( \mu_i \) that map states to actions at time \( i \). When \( \mu_i = \mu, \forall i = 1, 2, \ldots \), the policy is said to be stationary. Stationary policies are of two types:

1. Deterministic, wherein \( \mu = \{u, u, \ldots, u, \ldots\} \), where \( u : S \to A \). We denote the class of stationary deterministic policies (SDP) by \( U \), and a given SDP by \( u \).

2. Randomized, wherein \( \mu = \{\pi, \pi, \ldots, \pi, \ldots\} \), where given any \( s \in S \), \( \pi(s, \cdot) \) is a distribution among actions. Thus in state \( s \) action \( a \) is performed with probability \( \pi(s, a) \). We denote the class of stationary randomized policies (SRP) by \( \Pi \), and a given SRP by \( \pi \).
Under a stationary policy \( u \) (or \( \pi \)) the MDP is a Markov chain and we denote its probability transition kernel by \( P_u = (p_{u(i)}(i, j), i = 1 \text{ to } n, j = 1 \text{ to } n) \) (or \( P_\pi \)). The discounted reward starting from state \( s \) following policy \( u \) is denoted by \( J_u(s) \), where

\[
J_u(s) = \mathbb{E}\left[ \sum_{t=0}^{\infty} \alpha^t g(s_t) | s_0 = s, u \right]. \tag{6}
\]

Here \( \{s_t\} \) is the trajectory of the Markov chain under \( u \). We call \( J_u(s) \) the value function for policy \( u \).

We denote the optimal policy by \( u^* \) where

\[
u^* = \arg \max_{u \in \mathcal{U}} J_u(s), \forall s \in S. \tag{7}\]

The optimal value function is given by \( J^*(s) = J_{u^*}(s), \forall s \in S \). The optimal value function and optimal policy are related by the Bellman equation below:

\[
J^*(s) = \max_{a \in A} (g(s) + \alpha \sum_{s' = 1}^{n} p_{a}(s, s') J^*(s')), \tag{8}
\]

\[
u^*(s) = \arg \max_{a \in A} (g(s) + \alpha \sum_{s' = 1}^{n} p_{a}(s, s') J^*(s')). \tag{9}\]

Once an MDP is posed, our aim is to find \( u^* \). Again, once \( J^* \) is known, \( u^* \) can always be found by plugging \( J^* \) in (9). Thus, in most cases, we are interested in computing \( J^* \). Taking cue from (8) we define the Bellman operator \( T: \mathbb{R}^n \rightarrow \mathbb{R}^n \) as

\[
(TJ)(s) = \max_{a \in A} (g(s) + \alpha \sum_{s' = 1}^{n} p_{a}(s, s') J(s')), J \in \mathbb{R}^n. \tag{10}\]

Given \( J \in \mathbb{R}^n, TJ \) is the one-step, greedy value function. Also \( J^* \) is a fixed point of \( T \) i.e., \( J^* = TJ^* \), and from Lemma 1, Corollary 1, it follows that it is also unique (for proofs, please see [2]).

**Lemma 1** \( T \) is a max-norm contraction operator, i.e., given \( J_1, J_2 \in \mathbb{R}^n \)

\[
||TJ_1 - TJ_2||_\infty \leq \alpha ||J_1 - J_2||_\infty \tag{11}\]

**Corollary 1** \( J^* \) is a unique fixed point of \( T \).

Further, Bellman operator \( T \) exhibits two more important properties presented in the following Lemmas (see [2] for proofs)

**Lemma 2** \( T \) is a monotone map, i.e., given \( J_1, J_2 \in \mathbb{R}^n \) such that \( J_2 \geq J_1 \) then \( TJ_2 \geq TJ_1 \). Further if \( J \in \mathbb{R}^n \) is such that \( J \geq TJ \), it follows that \( J \geq J^* \).

**Lemma 3** Given \( J \in \mathbb{R}^n \), and \( k \in \mathbb{R} \) and \( \mathbf{1} \in \mathbb{R}^n \) a vector with all entries 1, then

\[
T(J + k\mathbf{1}) = TJ + \alpha k\mathbf{1}. \tag{12}\]
$J^*$ can also be seen to be the solution to the following linear program

$$\min \ c^T J$$
$$s.t \quad J \geq T J,$$

where $c \in \mathbb{R}^n$, $c \geq 0$.

Similarly one can define the Bellman operator restricted to a policy $u$ as

$$(T_u J)(s) = g(s) + \alpha \sum_{s'} p_u(s)(s, s') J(s'),$$

and it is straightforward to show that the value function of policy $u$ obeys the Bellman equation $J_u = T_u J_u$.

Due to the curse, as the number of variables increase, it is hard to compute exact values of $J^*$ and $u^*$. Approximate Dynamic Programming (ADP) methods make use of (8) and dimensionality reduction techniques to compute suboptimal policies $\tilde{u}$ instead of $u^*$. ADP methods approximate $J^*$ by means of lower dimensional quantities, i.e. $J^* \approx \tilde{J}$, where $J \in V \subset \mathbb{R}^n$. Typically $V$ is the subspace spanned by a set of preselected basis functions $\{\phi_i, i = 1, \ldots, k\}, \phi_i \in \mathbb{R}^n$. Let $\Phi$ be the $n \times k$ matrix with columns $\phi_i, i = 1, \ldots, k$, and $V = \{\Phi r | r \in \mathbb{R}^k\}$, then the approximate value function $\tilde{J}$ is of the form $\Phi r^*$ for some $r^* \in \mathbb{R}^k$, i.e., $J^* \approx \tilde{J} = \Phi r^*$. Computing $r^* \in \mathbb{R}^k (k << n)$ is easier than computing $J^* \in \mathbb{R}^n$. Since $J^*$ is not known, one cannot obtain its projection onto $V$. Hence one obtains $r^*$ either as a solution to the PBE in (1) or solution to the ALP (3). It is important to note that whilst PBE methods are based on value iteration[2], the ALP method is based on the LP formulation (13). Once the approximate value function $\tilde{J}$ is obtained, the suboptimal/greedy policy $\tilde{u}$ is obtained as below:

$$\tilde{u}(s) = \arg \max_{a \in A} (g(s) + \alpha \sum_{s'=1}^{n} p_u(s, s') \tilde{J}(s')).$$

The following lemma characterizes the degree of sub-optimality of the greedy policy $\tilde{u}$.

**Lemma 4** Let $\tilde{J} = \Phi r^*$ be the approximate value function and $\tilde{u}$ be as in (15), then

$$||J_{\tilde{u}} - J^*||_{\infty} \leq \frac{2}{1 - \alpha} ||J^* - \tilde{J}||_{\infty}$$

**Proof:** We know that

$$(T_{\tilde{u}} \tilde{J})(s) = g(s) + \alpha \sum_{s'} p_{\tilde{u}}(s)(s, s') \tilde{J}(s'),$$

$$J_{\tilde{u}}(s) = g(s) + \alpha \sum_{s'} p_{\tilde{u}}(s)(s, s') J_{\tilde{u}}(s').$$

Hence we can write by subtracting (17) from (18)

$$J_{\tilde{u}} - \tilde{J} = T_{\tilde{u}} \tilde{J} - \tilde{J} + \alpha P_{\tilde{u}} (J_{\tilde{u}} - \tilde{J})$$

$$J_{\tilde{u}} - \tilde{J} = (I - \alpha P_{\tilde{u}})^{-1} (T_{\tilde{u}} \tilde{J} - \tilde{J})$$

$$||J_{\tilde{u}} - \tilde{J}||_{\infty} \leq \frac{1}{1 - \alpha} ||T_{\tilde{u}} \tilde{J} - \tilde{J}||_{\infty}.$$
We know from (15) that $T_u\tilde{J} = T\tilde{J}$. Also from the fact that $J^* = TJ^*$ and the contraction property of $T$, we know $||TJ - J^*||_\infty \leq \alpha ||\tilde{J} - J^*||_\infty$ and $||T_u\tilde{J} - J||_\infty \leq (1 + \alpha) ||\tilde{J} - J^*||_\infty$. Hence we have

\[
\begin{align*}
||J_u - J^*||_\infty &= ||J_u - J^* + \tilde{J} - \tilde{J}||_\infty \\
&\leq ||J_u - \tilde{J}||_\infty + ||\tilde{J} - J^*||_\infty \\
&\leq \frac{1}{1 - \alpha} ||T_u\tilde{J} - \tilde{J}||_\infty + ||J^* - \tilde{J}||_\infty \\
&\leq \frac{1 + \alpha}{1 - \alpha} ||J^* - \tilde{J}||_\infty + ||J^* - \tilde{J}||_\infty \\
&\leq \frac{2}{1 - \alpha} ||J^* - \tilde{J}||_\infty 
\end{align*}
\]

Irrespective of the formulation (PBE or ALP), it is important to choose the basis such that $||J^* - \tilde{J}||_\infty$ is as small as possible. Error bounds for the PBE based methods are in the $L_2$-norm ([7]) and hence the sub-optimality of the greedy policy cannot be ascertained. However, in the case of ALP the sub-optimality of the greedy policy is characterized by error bounds in a modified $L_1$-norm. In this paper, we look at a novel method of approximating $J^*$ using linear function approximators (LFA), which are linear in $(\text{min}, +)$. As we shall see in section 5, our approximate solution has error bounds in the $L_\infty$ norm and hence the sub-optimality of the greedy policy can be ascertained via Lemma 4. In the next section we describe the $(\text{min}, +)$ LFAs.

3 Semiring, Semimodules and Projections

We define the semiring as $R_{\text{min}} = (R \cup \{+\infty\}, \text{min}, +)$. In $R_{\text{min}}$, the usual multiplication is replaced with $+$, and addition is replaced by $\text{min}$ given as below.

**Definition 5**

\[
\begin{align*}
\text{Addition:} & \quad x \oplus y = \min\{x, y\} \\
\text{Multiplication:} & \quad x \otimes y = x + y
\end{align*}
\]

Henceforth we use, $(+, -)$ and $(\oplus, \otimes)$ to respectively denote the conventional and $R_{\text{min}}$ addition and multiplication respectively. In $R_{\text{min}}$, the multiplicative identity is denoted by $e$ with $e = 0 \in R$ and the additive identity is denoted by $1$ and is $+\infty$. The $R_{\text{min}}$ is an idempotent semiring, i.e., $\alpha \oplus a = a, \forall a \in R_{\text{min}}$. We can define a semimodule $M$ over this semiring, in a similar manner as vector spaces are defined over fields. In particular we are interested in the semimodule $M = R_{\text{min}}^n$. Given $u, v \in R_{\text{min}}^n$, and $\lambda \in R_{\text{min}}$, we define addition and scalar multiplication as follows:

**Definition 6**

\[
\begin{align*}
(u \oplus v)(i) &= \min\{u(i), v(i)\} = u(i) \oplus v(i), \forall i = 1, 2, \ldots, n. \\
(u \otimes \lambda)(i) &= u(i) \otimes \lambda = u(i) + \lambda, \forall i = 1, 2, \ldots, n.
\end{align*}
\]
Subsemimodule of semimodules are similar to subspaces of a given vector space. The \((\min, +)\) projection operator \(\Pi_M\) is given by \([1, 3, 5]\)

\[
\Pi_M u = \min \{v | v \in \mathcal{V}, v \geq u\}, \forall u \in \mathcal{M}.
\] (22)

In this paper, we consider semimodule \(\mathcal{M} = \mathbb{R}^n_{\min}\) and \(k\)-dimensional subsemimodule \(\mathcal{V}\) which is a linear span of a given basis, i.e., \(\mathcal{V} = \text{Span}\{\phi_i | \phi_i \in \mathbb{R}^n_{\min}, i = 1, \ldots, k\} = \{v | v = \Phi \otimes r, r = \phi_1 \otimes r(1) + \phi_2 \otimes r(2) + \cdots + \phi_k \otimes r(k), r(i) \in \mathbb{R}^n_{\min}, i = 1, \ldots, k\}.\) We now show that \(\Pi_M\) in (22) is closely related to the Fenchel transform, or the \(\sup\)-transform. (For a detailed discussion on projection onto subsemimodules, see [1]).

4 Fenchel Dual and Projection on Subsemimodules

In this section, we demonstrate the connections between the Fenchel-Legendre transform (FLT) and the \((\min, +)\) projection defined in (22). Given a function \(f : \mathbb{R}^n \rightarrow \mathbb{R}\), its FLT is defined by \(f^* : \mathbb{R}^n \rightarrow \mathbb{R}\), with

\[
f^*(y) = \sup_{x \in \mathbb{R}^n} (y^\top x - f(x)), y \in \mathbb{R}^n.
\] (23)

If \(f\) is convex, then it can be recovered as \(f = f^{**}\), i.e.,

\[
f(x) = f^{**}(x) = \sup_{y \in \mathbb{R}^n} (x^\top y - f^*(y)), x \in \mathbb{R}^n.
\] (24)

We can rewrite (23) as below

\[
f^*(y) = \sup_{x \in \mathbb{R}^n} (f_y(x) - f(x)), y \in \mathbb{R}^n, \text{ where } f_y(x) = y^\top x.
\] (25)

Now instead of considering functions \(f_y(x)\) indexed by \(y \in \mathbb{R}^n\), we consider the sequence \(\{\phi_j\}, j \in \mathcal{J} = \{1, 2, \ldots, k\}, \phi_j : \mathbb{R}^n \rightarrow \mathbb{R}\). Then (25) can be modified as below:

\[
f^*(j) = \sup_{x \in \mathbb{R}^n} (\phi_j(x) - f(x)), j \in \mathcal{J}.
\] (26)

We call (26), the \(\sup\)-Transform or the max-Transform. It is easy to check that \(\phi_j(x) - f^*(j) < f(x), \forall x \in \mathbb{R}^n, j \in \mathcal{J}\). Since our index set in (26) is finite (as opposed to \(\mathbb{R}^n\) as in (23)), it is not necessary that the original function \(f\) can be reconstructed from \(f^*(j), j \in \mathcal{J}\). However, we can get an approximation \(\tilde{f}\) as below:

\[
f(x) \approx \tilde{f}(x) = \sup_{j \in \mathcal{J}} (\phi_j(x) - f^*(j)).
\] (27)

In the light of (26) and (27), the projection in (22) is nothing but the \(\min\)-Transform (as opposed to the max-Transform (26)). It is more clear if we rewrite (22) for the case when \(\mathcal{V} = \text{Span}\{\phi_j | \phi_j \in \mathbb{R}^n_{\min}, j = 1, \ldots, k\}\). Let \(\Pi_M u = \Phi \otimes r^u\), then one can see that

\[
\Pi_M u = \{\min \Phi \otimes r | \Phi \otimes r \geq u, r \in \mathbb{R}^n_{\min}\}.
\] (28)

\[
r^u(j) = \min_{i=1,2,\ldots,n} (\phi_j(i) - u(i)), \forall j = 1, 2, \ldots, k.
\] (29)
Note the similarity between \( r^u(j) \) in (29) and \( f^*(j) \) in (26). Then the approximation/projection of \( u \) onto \( V \) is given by

\[
\tilde{u} = \Pi_{M} u = \Phi \otimes r^u \text{ with }

\Pi_{M} u(i) = \min_{j=1,\ldots,k} (\phi_j(i) + r^u(j))

= \phi_1(i) \otimes r^u(1) \oplus \ldots \oplus \phi_k(i) \otimes r^u(k).
\]

(30)

Also, it is important to note that (26) deals with projecting a function, while (22) deals with projecting the elements of \( n \)-dimensional semimodule. Nevertheless, the spirit of the projection is similar in both cases. Also, \( \phi_j(i) + r^u_j - u(i) > 0 \), i.e., the min-Transform approximates the given element \( u \) by point-wise minimum of functions that upper bound \( u \). We end this section with the following illustration.

**Example 1** Let \( f(x) = x^2 \), and let \( a = (a(j), j = 1,\ldots,5) = (-0.8, -0.4, 0, 0.4, 0.8) \), and \( \phi_j(x) = 2|x - a(j)| \). Then (min,+) LFA of \( f(x) \) via the min-Transform using the \( \{\phi_j(x), j = 1,\ldots,5\} \) as the (min,+) basis, is given in the Figure 1.

5 \ (\text{min, +}) \textbf{Projected Bellman Equation}

Given a \( n \times k \) feature matrix \( \Phi \), since we do not know \( J^* \in \mathbb{R}^n_{\min} \), \( \Pi_M J^* \) cannot be obtained. Thus taking a cue from (1), we have the approximate value function \( \tilde{J} = \Phi \otimes r^* \) to obey the \( \text{(min, +)} \) Projected Bellman Equation (MPPBE) given below:

\[
\Phi \otimes r^* = \Pi_{M} T\Phi \otimes r^*.
\]

(31)

We can expand (31) based on (22), as follows:

\[
\min \{ \Phi \otimes r | \Phi \otimes r \geq T\Phi \otimes r, r \in \mathbb{R}^k_{\min} \}.
\]

(32)
The above (32) is similar to another class of ADP methods called Approximate Linear Program (ALP) in (3). However, despite the apparent similarity in structure between the ALP (3) and the PBE in the \((\text{min}, +)\) basis (32), the key difference is in the type of basis representation. We assume that (32) is feasible, until we establish that fact in Corollary 3. We also make the following definition and assumption:

**Definition 7** We call the set of column vectors \(\{\phi_i\}, i = 1, \ldots, k, \phi_i \in \mathbb{R}^n\) of the \(n \times k\) matrix \(\Phi\) to be linearly independent if \(\Phi \otimes x = \Phi \otimes y \iff x = y\).

**Assumption 1** The columns of the feature matrix \(\Phi\) are independent.

**Lemma 8** Let \(r_1, r_2 \in \mathbb{R}^k\) be such that \(\Phi \otimes r_1 \geq T\Phi \otimes r_1\), and \(\Phi \otimes r_2 \geq T\Phi \otimes r_2\) and let \(r_{\text{new}} = r_1 \oplus r_2\), then
\[
\Phi \otimes r_{\text{new}} \geq T\Phi \otimes r_{\text{new}}
\]

**Proof:** From Lemma 2, it follows that
\[
\begin{align*}
\Phi \otimes r_1 & \geq T(\Phi \otimes r_1 \oplus \Phi \otimes r_2), \\
\Phi \otimes r_2 & \geq T(\Phi \otimes r_1 \oplus \Phi \otimes r_2).
\end{align*}
\]
From (33) and (34) we have
\[
\begin{align*}
(\Phi \otimes r_1) \oplus (\Phi \otimes r_2) & \geq T(\Phi \otimes r_1 \oplus \Phi \otimes r_2), \\
\Phi \otimes (r_1 \oplus r_2) & \geq T\Phi \otimes (r_1 \oplus r_2), \\
\Phi \otimes (r_{\text{new}}) & \geq T\Phi \otimes (r_{\text{new}}).
\end{align*}
\]

### 5.1 Approximation Guarantees of the \((\text{min}, +)\) PBE

The minimization in \((\text{min}, +)\) PBE in (32) is component-wise. It is desirable to identify an equivalent optimization problem wherein the objective function is not multivalued. To this end, we consider the following program:

\[
\begin{align*}
\min & \quad c^T \Phi \otimes r \\
\text{s.t.} & \quad \Phi \otimes r \geq T\Phi \otimes r, \\
& \quad \text{where } c^T \Phi \otimes r = \sum_{i=1}^n c(i)(\Phi \otimes r)(i).
\end{align*}
\]

**Lemma 9** (38) has a unique solution.

**Proof:** Let \(r_1^*\) and \(r_2^*\) be two distinct solutions of (38). Then let \(r_{\text{new}} = r_1^* \oplus r_2^*\), and \(r_{\text{new}}\) is feasible from Lemma 8. Since \(r_1^*\) and \(r_2^*\) are distinct, there exists a \(j\) such that \(r_{\text{new}}(j) < r_1^*(j)\) or \(r_{\text{new}}(j) < r_2^*(j)\), and hence from Assumption 1, \(c^T \Phi \otimes r_{\text{new}} < c^T \Phi \otimes r_1^* = c^T \Phi \otimes r_2^*\). This contradicts that fact that \(r_1^*\) and \(r_2^*\) are optimizers. Thus \(r_1^* = r_2^* = r_{\text{new}}\).
Corollary 2 Let \( r_f \) be any feasible solution and \( r^* \) the optimal solution for (38). Then \( r_f \geq r^* \) (\( r_f(i) \geq r^*(i), i = 1, \ldots, k \)).

Proof: Let \( r_1 = r_f + r^* \). From Lemma 8 we know that \( r_1 \) is feasible, and from Lemma 9 that \( r_1 = r^* \). The following Lemma 10, shows that (38) and (32) are equivalent.

Lemma 10 For any \( c \in \mathbb{R}^n, c > 0 \) (all components are positive), program (38) and the \((\min, +)\) PBE in (32) are equivalent. i.e., \( r^* \in \mathbb{R}^k_{\min} \) is a solution to (32) \( \iff \) \( r^* \in \mathbb{R}^k_{\min} \) is a solution to (38).

Proof: Let \( r_1^* \) and \( r_2^* \) be the solutions to (32) and (38) respectively.

\[ \Rightarrow \]
It clearly follows that \( r_1^* \) is feasible for (38). Now \( r_2^* \leq r_1^* \). Suppose not, then define \( r_{new}^* = r_1^* \oplus r_2^* \). From Lemma 8 we know that \( r_{new} \) is feasible. It then follows that for \( c > 0 \), \( c^T \Phi \otimes r_{new}^* \leq c^T \Phi \otimes r_2^* \). But since \( c > 0 \) and \( r_2^* \) is the solution to (38), which implies \( r_{new}^* = r_2^* \), hence \( r_2^* \leq r_1^* \).

\[ \Leftarrow \]
It is easy to check that \( r_2^* \) is feasible for (32). Then \( r_1^* \leq r_2^* \). Suppose not, and let \( r_{new}^* = r_1^* \oplus r_2^* \). From Lemma 8 we know that \( r_{new} \) is feasible. Then we know that \( \Phi \otimes r_{new}^* \leq \Phi \otimes r_1^* \). But \( r_1^* \) is the solution to (38), so \( r_1^* = r_{new}^* \), hence \( r_1^* \leq r_2^* \).

Lemma 11 \( r^* \) is the optimal solution of (38) if and only if

\[ r^* = \arg \min_r \| J^* - \Phi \otimes r \|_{\infty} \quad \text{s.t.} \quad \Phi \otimes r \geq T \Phi \otimes r. \]  

(39)

Proof: \( \Rightarrow \)
Suppose not. Let \( r_1^* \) be the solution to (38) and \( r_2^* \) be the solution to (39). Then \( \tilde{r} = r_1^* \oplus r_2^* \) is feasible for (38). We also know from Lemma 2 that \( \Phi \otimes r_2^* \geq \Phi \otimes r_1^* \geq J^* \), but we know that \( r_2^* \) is solution of (39), which implies \( r_1^* = r_2^* \).

\[ \Leftarrow \]
Suppose not. Let \( r_1^* \) be the solution to (38) and \( r_2^* \) be the solution to (39). Then \( \tilde{r} = r_1^* \oplus r_2^* \) is feasible for (38). But we from Corollary 2 know that \( r_1^* \leq \tilde{r} \) which is a contradiction. Thus \( r_1^* \) and \( r_2^* \) must be identical.

Lemma 12 There exists \( \tilde{r} \in \mathbb{R}^k_{\min} \) such that \( \Phi \otimes \tilde{r} \geq T(\Phi \otimes \tilde{r}) \) and \( \| J^* - \Phi \otimes \tilde{r} \|_{\infty} \leq \frac{2}{1 - \alpha} \| J^* - \Phi \otimes \tilde{r} \|_{\infty} \), where \( \| V \|_{\infty} = \max_i |V(i)| \), \( \tilde{r} = \arg \min_{r \in \mathbb{R}^k_{\min}} \| J^* - \Phi \otimes r \|_{\infty} \).

Proof: Let \( \epsilon = \| J^* - \Phi \otimes \tilde{r} \|_{\infty} \). Now due to the max-norm contraction property of \( T \) (Lemma 1), we have \( \| T J^* - T \Phi \otimes \tilde{r} \| \leq \alpha \epsilon \). So we know that

\[ \Phi \otimes \tilde{r} \geq T \Phi \otimes \tilde{r} - (1 + \alpha) \epsilon 1. \]  

(40)

Now for any \( p \in \mathbb{R} \), let \( \tilde{r} = (\tilde{r}(1) + p, \tilde{r}(2) + p, \ldots, \tilde{r}(k) + p) \), then

\[ \Phi \otimes \tilde{r} = \Phi \otimes \tilde{r} + p 1. \]

\[ T \Phi \otimes \tilde{r} = T \Phi \otimes \tilde{r} + \alpha p 1. \]  

(41)
For \( p = \frac{1 + \alpha}{1 - \alpha} \), from (41) and (40), we have

\[
\Phi \otimes \tilde{r} - T\Phi \otimes \tilde{r} = \Phi \otimes \tilde{r} - T\Phi \otimes \tilde{r} + (1 - \alpha)\frac{1 + \alpha}{1 - \alpha} \epsilon 1
\]

\[
\geq 0.
\]

(42)

Now

\[
||J^* - \Phi \otimes \vec{r}||_{\infty} \leq ||J^* - \Phi \otimes \vec{r}||_{\infty} + ||\Phi \otimes \vec{r} - \Phi \otimes \vec{r}||_{\infty}
\]

\[
= (1 + \frac{1 + \alpha}{1 - \alpha})||J^* - \Phi \otimes \vec{r}||_{\infty}
\]

\[
= \frac{2}{1 - \alpha} ||J^* - \Phi \otimes \vec{r}||_{\infty}.
\]

Corollary 3 (38) is feasible.

We now state the approximation bound

**Theorem 13** Let \( r^* \) be the solution of (38), and \( \hat{r} = \arg \min \|J^* - \Phi \otimes r\|_{\infty} \). Then we have

\[
||J^* - \Phi \otimes r^*||_{\infty} \leq \frac{2}{1 - \alpha} ||J^* - \Phi \otimes \hat{r}||_{\infty}.
\]

**Proof:** We have shown in Lemma 12 that there exists \( \vec{r} \) feasible such that \( ||J^* - \Phi \otimes \vec{r}||_{\infty} \leq \frac{2}{1 - \alpha} ||J^* - \Phi \otimes \vec{r}||_{\infty} \). Now we know from Lemma 11 that \( ||J^* - \Phi \otimes r^*||_{\infty} \leq ||J^* - \Phi \otimes \vec{r}||_{\infty} \). Thus irrespective of the choice of \( \epsilon \) the \( L_{\infty} \)-norm bound on the approximation error always holds, which is not the case of conventional ALP. Going forward we would want to further understand (38) and develop an algorithm to solve it.

**Definition 14** At a given \( r \in \mathbb{R}^k \):

1. We say that column vector \( \phi_j \) participates in row \( i \), if \( (\Phi \otimes r)(i) = \phi_j(i) + r(j) \).
2. We call row \( i \) to be active if \( (\Phi \otimes r)(i) = (T\Phi \otimes r)(i) \)

**Definition 15** We call a point \( \vec{r} \) to be an active-point if the following hold:

1. Each column of \( \Phi \) participates in at least one row of \( \Phi \).
2. Atleast one of the rows is active, i.e., \( \exists i \) such that \( \Phi \otimes \vec{r}(i) = (T\Phi \otimes \vec{r})(i) \).
3. Each column of \( \Phi \) participates in one or more active rows.
4. It is feasible i.e., \( \Phi \otimes \vec{r} \geq T(\Phi \otimes \vec{r}) \).
Lemma 16 Let \( r \in \mathbb{R}^k \) be any point feasible point, i.e., \( \Phi \otimes r \geq T(\Phi \otimes r) \). Let \( g \in \mathbb{R}^k \) be defined as \( g(j) = \min_i (\phi_j(i) + r(j) - T(\Phi \otimes r)(i)) \) and \( r_{\text{new}} \) be defined as \( r_{\text{new}} = r - g \). Then \( r_{\text{new}} \) is feasible.

Proof: Since \( r_{\text{new}} \leq r \), we have
\[
T(\Phi \otimes r_{\text{new}}) \leq T(\Phi \otimes r).
\]
Pick any column \( j \), and let \( i \) be any row in which column \( j \) participates at \( r_{\text{new}} \). Then we have
\[
(\Phi \otimes r_{\text{new}})(i) = \phi_j(i) + r_{\text{new}}(j) = \phi_j(i) + r(j) - g(j).
\]
Now
\[
(\Phi \otimes r_{\text{new}})(i) - (T\Phi \otimes r_{\text{new}})(i) = \phi_j(i) + r(j) - g(j) - (T\Phi \otimes r_{\text{new}})(i) \geq 0.
\]

Corollary 4 \( r_{\text{new}} = r - g' \) is feasible for any \( g' \leq g \).

Lemma 17 Let \( \tilde{r} \) be an active point and \( v > 0 \) be any positive vector in \( \mathbb{R}^k \). Then any \( r_{\text{new}} \) such that \( r_{\text{new}} = \tilde{r} - v \) is not feasible.

Proof: Let \( j = \arg \max_{p=1}^k v(p) \). By part 3 of Definition 15 column \( j \) should participate in any one or more active rows. So w.l.o.g, we assume that column \( j \) participates in the active row \( i \) at \( \tilde{r} \). Then it follows from definition of \( j \) that column \( j \) participates in row \( i \) at \( r_{\text{new}} \). Now
\[
(\Phi \otimes r_{\text{new}})(i) - (T\Phi \otimes r_{\text{new}})(i) \leq (\Phi \otimes \tilde{r})(i) - (T\Phi \otimes \tilde{r})(i) - v(j) + \alpha v(j),
\]
where \( \alpha \) is a constant. Then
\[
(\Phi \otimes r_{\text{new}})(i) - (T\Phi \otimes r_{\text{new}})(i) \leq 0.
\]

(50) follows from (49) from Lemma 3, and due to the fact that \( v \leq v(j)1 \), where \( 1 \in \mathbb{R}^k \) is vector with all entries equal to 1.

The following Lemma characterizes the optimal solution of (38)

Theorem 18 \( r^* \) is an optimal solution of (38) iff \( r^* \) is an active-point.

Proof: Let us assume on the contrary that part 1 of Definition 15 is not true for \( r^* \). Then \( \exists \) some \( j \) such that \( \phi_j \) does not participate in any of the rows. Define \( a = \min_i [\phi_j(i) + r^*(j) - (\Phi \otimes r^*)(i)] \). Now define
\( r_{new} \triangleq r^* - de_j \) (where \( e_j \) is the standard basis with 1 in the \( j^{th} \) coordinate and all other entries set to 0). From Corollary 4 it follows that \( r_{new} \) is feasible for (38) and \( r_{new} \leq r^* \), which is a contradiction by Lemma 9. So part 1 of Definition 15 has to be true for \( r^* \).

Suppose part 2 of Definition 15 is not true for \( r^* \). Define \( V = \Phi \otimes r^* - T\Phi \otimes r^* \). Since \( r^* \) is feasible and none of the rows are active we know that \( V > 0 \). Also, none of the columns participate in any of the active rows (since no row is active). Pick any column \( j \), and let \( d = \min_i (\phi_j(i) + r^*(j) - (T\Phi \otimes r^*)(i)) \), and \( r_{new} = r^* - de_j \). Then from Corollary 4, \( r_{new} \) is also feasible, but \( r_{new} \leq r^* \), which is not possible by Lemma 9. So part 2 of Definition 15 has to be true for \( r^* \).

Finally let us assume on the contrary that part 3 of Definition 15 is not true for \( r^* \). Then \( \exists \) some \( j \) such that \( \phi_j \) does not participate in any of the active rows. Let \( \mathcal{I} \) denote the set of active rows, and define \( d_1 \trianglerighteq \min_{i \in \mathcal{I}} (\phi_j(i) + r(j) - (T\Phi \otimes r^*)(i)) \), \( d_2 \trianglerighteq \min_{i \in \mathcal{I}} (\phi_j(i) + r(j) - (T\Phi \otimes r^*)(i)) \), and \( d \trianglerighteq \min \{d_1, d_2\} \).

Define \( r_{new} \triangleq r^* - de_j \). Now we have

1. \( i \notin \mathcal{I} \)
   \[
   \Phi \otimes r_{new}(i) - (T\Phi \otimes r_{new})(i) \\
   \geq (\Phi \otimes r^*)(i) - (T\Phi \otimes r_{new})(i) - d \\
   \geq (\Phi \otimes r^*)(i) - (T\Phi \otimes r^*)(i) - d \\
   \geq 0
   \]

2. \( i \in \mathcal{I} \)
   \[
   \Phi \otimes r_{new}(i) - (T\Phi \otimes r_{new})(i) \\
   = (\Phi \otimes r^*)(i) - (T\Phi \otimes r_{new})(i) \\
   \geq (\Phi \otimes r^*)(i) - (T\Phi \otimes r^*)(i) \\
   \geq 0
   \]

Thus \( r_{new} \) is a feasible solution for (38) and \( r_{new} \leq r^* \), which is a contradiction from Lemma 9. So part 3 of Definition 15 has to be true for \( r^* \). It is easy to check that part 4 holds trivially.

\( \Leftarrow \)

Let \( \tilde{r} \) be an active-point. Let the optimal point \( r^* \) be different from \( \tilde{r} \). We know from part 4 of Definition 15 that \( \tilde{r} \) is feasible for (38). We know from that Corollary 2 that \( \tilde{r} \leq r^* \), which is a contradiction according to Lemma 17. So \( \tilde{r} = r^* \).

### 5.2 Finding a feasible point

We now split the program (38) in \( k \)-variables into \( k \) programs in one variable each. We call these programs as Sub \((\min, +)\) Projected Bellman Equation (SMPPBE). The \( i^{th} \) SMPPBE is given by

\[
\begin{align*}
\min \ c^T \phi_i \otimes r(i) \\
\text{s.t.} \ \phi_i \otimes r(i) \geq T\phi_i \otimes r(i).
\end{align*}
\]
The objective in (46) can be simplified further.

\[ c^T \phi_i \otimes r(i) = \sum_{j=1}^{k} c(i)(\phi_i(j) + r(i)) \]
\[ = \sum_{j=1}^{k} c(i)\phi_i(j) + \sum_{j=1}^{k} c(i)r(i) \]
\[ = \sum_{j=1}^{k} c(i)\phi_i(j) + r(i) \sum_{j=1}^{k} c(i) \]

(47)

The first term on the right hand side of (47) is a constant and since \( \sum_{j=1}^{k} c(i) > 0 \), the \( i^{th} \) SMPPBE can be equivalently written as below:

\[ \min r(i) \]
\[ \text{s.t.} \quad \phi_i \otimes r(i) \geq T \phi_i \otimes r(i). \]

Let \( r^*_i(i) \) be the optimal value of the \( i^{th} \) SMPPBE. We define \( r^*_s \in \mathbb{R}^{k} \) as \( r^*_s = (r^*_s(1), r^*_s(2), \ldots, r^*_s(k)) \).

**Theorem 19** \( r^*_s \) is feasible for (38).

**Proof:** Since \( r^*_s(i) \) is the solution for the \( i^{th} \) SMPPBE, we know that

\[ \phi_i \otimes r^*_s(i) \geq T \phi_i \otimes r^*_s(i). \]  

(49)

Hence,

\[ \phi_i \otimes r^*_s(i) \geq T \min\{\phi_1 + r^*_s(1), \ldots, \phi_k + r^*_s(k)\}, \]  

(50)

or,

\[ \phi_i \otimes r^*_s(i) \geq T \Phi \otimes r^*_s, \]  

(51)

where (50) follows from (49) due to the monotonicity property of \( T \), and (51) follows from (50) due to the definition of \( \Phi \otimes r \). Now since (51) is true for every \( i \), we have

\[ \min\{\phi_1 + r^*_s(1), \ldots, \phi_k + r^*_s(k)\} \geq T \Phi \otimes r^*_s, \]

or

\[ \Phi \otimes r^*_s \geq T \Phi \otimes r^*_s. \]

**6 (min, +) Approximate Dynamic Programming Algorithm (MPADP)**

From Lemma 16, we know that \( r_n \) in Algorithm 1 is feasible for all \( n \).

**Theorem 20** The Algorithm 1 converges in a finite number of iterations for \( \epsilon > 0 \).
Algorithm 1 (min, +) Approximate Dynamic Programming Algorithm

1: Start with any feasible point \( r_0 \), a small number \( \epsilon > 0 \) a small number and \( n = 0 \).
2: while \( \|g_n\|_\infty > \epsilon \) do
3: Compute the gradient \( g_n(j) = \min_{s \in S} (\phi_j(s) + r_n(j) - (T\Phi \otimes r_n)(s)) \).
4: \( r_{n+1} = r_n - g_n \).
5: \( n = n + 1 \).
6: end while
7: return \( r_{opt} = r_n \), and approximate value function \( \tilde{J} = \Phi \otimes r_{opt} \).

Proof: Suppose not, then at each step, the value function decreases by at least \( \min_i c(i)e \). However the objective function is lower bounded. The claim follows. It is important to note that when \( \|g\|_\infty = 0 \), \( r_{opt} \) is an active point, (Definition 15), i.e., the optimal solution. For any other \( \epsilon > 0 \), \( r_{opt} \) is in the \( \epsilon \)-neighbourhood of the active point, as characterized by the following Lemmas.

Lemma 21 Let \( v \in \mathbb{R}^k \) be any positive vector with \( \|v\|_\infty > \frac{\epsilon}{1 - \alpha} \), and \( r_{new} \) defined as \( r_{new} = r_{opt} - v \). Then \( r_{new} \) is not feasible.

Proof: Let \( j = \arg \max_{p=1}^k v(p) \). Now from line 3 of Algorithm 1, there is an \( i \ni (\Phi \otimes r_{opt})(i) - (T\Phi \otimes r_{opt})(i) < \epsilon \). Now

\[
\begin{align*}
(\Phi \otimes r_{new})(i) - (T\Phi \otimes r_{new})(i) &\leq (\Phi \otimes r_{opt})(i) - (T\Phi \otimes r_{opt})(i) - \frac{\epsilon}{1 - \alpha} \\
&\leq (\Phi \otimes r_{opt})(i) - (T\Phi \otimes r_{opt})(i) - \frac{\epsilon}{1 - \alpha} + \alpha \frac{\epsilon}{1 - \alpha} \\
&\leq 0.
\end{align*}
\]

Corollary 5 \( r_{opt} - r^* < \frac{\epsilon}{1 - \alpha} \), where \( r^* \) is the optimal solution to (38) and \( r_{opt} \) is the solution returned by Algorithm 1.

Proof: We know that \( r^* \leq r_{opt} \). Let \( v = r_{opt} - r^* \). Now \( \|v\|_\infty < \frac{\epsilon}{1 - \alpha} \).

7 Experiments

We test our MPADP algorithm (Algorithm 1) on a 10 \times 10 grid world problem. There are a total of 100 states, i.e., \( S = \{1, 2, \ldots, 100\} \), the co-ordinate \((x_i, y_j)\) is encoded as the state \( s = (i - 1) \times 10 + j \). The reward matrix is as given in Table 1, where each entry is an integer between 1 and 10. The grid world problem is used to model terrain exploration by autonomous decision making agents (robots). In each grid position, the agent has 8 actions corresponding to the 8 possible directions. In the corners, fewer directions are feasible, and the rest of the directions lead to the current grid position. So \( A = \{1, 2, \ldots, 8\} \). Actions fail with probability of 0.1 and no movement is made and the same grid position
Table 1: Grid world with rewards

|   | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $y_1$ | 2 | 5 | 9 | 5 | 8 | 3 | 6 | 10 | 7 | 3 |
| $y_2$ | 10 | 10 | 7 | 1 | 4 | 4 | 3 | 8 | 4 | 4 |
| $y_3$ | 1 | 2 | 4 | 10 | 3 | 9 | 8 | 5 | 9 | 5 |
| $y_4$ | 8 | 3 | 6 | 10 | 5 | 1 | 2 | 5 | 6 | 3 |
| $y_5$ | 9 | 2 | 5 | 5 | 1 | 1 | 7 | 5 | 4 | 9 |
| $y_6$ | 9 | 2 | 1 | 5 | 2 | 2 | 2 | 4 | 10 | 2 |
| $y_7$ | 1 | 9 | 3 | 4 | 10 | 7 | 4 | 6 | 9 | 3 |
| $y_8$ | 4 | 6 | 2 | 10 | 10 | 8 | 7 | 6 | 6 | 2 |
| $y_9$ | 3 | 6 | 2 | 4 | 6 | 7 | 8 | 9 | 7 | 3 |
| $y_{10}$ | 9 | 2 | 3 | 2 | 1 | 5 | 1 | 8 | 6 | 5 |

is retained, i.e., $p_a(s, s) = 0.1$, $a \in A, s \in S$, and with probability 0.9 the agent reaches the intended grid position.

Let $\{\phi_j, j = 1, \ldots, k\}, \phi_j \in \mathbb{R}_{\min}^n$ and $\{\phi^i, i = 1, \ldots, n\}, \phi^i \in \mathbb{R}_{\min}^k$ be the columns and rows respectively of the feature matrix $\Phi$. Under the feature representation $\Phi$ the similarity of states $s, s' \in S$ is given by the dot product below:

$$< \phi^s, \phi^{s'} > = \phi^s(1) \otimes \phi^{s'}(1) \oplus \ldots \oplus \phi^s(k) \otimes \phi^{s'}(k).$$

We desire the following in the feature matrix $\Phi$.

1. Features $\phi^i$ should have unit norm, i.e., $||\phi^i|| = < \phi^i, \phi^i > = 0$, since 0 is the multiplicative identity in the $(\min, +)$ algebra.

2. For dissimilar states $s, s' \in S$, we prefer $< \phi^s, \phi^{s'} > = +\infty$, since $+\infty$ is the additive identity in $(\min, +)$ algebra.

Keeping these in mind, we design the feature matrix $\Phi$ for the grid world problem. Since the state space is similar in the connectivity, we aggregate the states based on the reward forming $k$ partitions. Let $g_{\min} = \min_s g(s), s \in S$, $g_{\max} = \max_s g(s), s \in S$ and $L = g_{\max} - g_{\min}$, then we select the features as follows:

$$\phi^s(i) = \begin{cases} 0 & : g(s) \in [g_{\min} + \frac{(i-1)L}{k}, g_{\min} + \frac{(i)L}{k}], \\ 1000 & : g(s) \notin [g_{\min} + \frac{(i-1)L}{k}, g_{\min} + \frac{(i)L}{k}], \\ & \forall i = 1, \ldots, k. \end{cases}$$

We use 1000 in place of $+\infty$, and set $\epsilon = 0$ (see Algorithm 1). It is easy to verify that $\Phi$ in (54) has the enumerated properties. The errors are given in Table 2 for discount factors 0.9 and 0.99, where $r_{opt}$ is the result returned by the MPADP in Algorithm 1, and $\tilde{u}$ is the greedy policy given by

$$\tilde{u} = \arg\max_{a \in A} \left( g(s) + \alpha \sum_{a'} p_a(s, s') \tilde{J}(s') \right),$$

where $\tilde{J} = \Phi \otimes r_{opt}$. 
We also noted that the algorithm finds the optimal actions for about 75 states. The number of steps taken for the mountain car to reach the goal in each of these settings is presented in Table 2.

Table 2: Error Table

| Error Term | Error for $\alpha = 0.9$ | Error for $\alpha = 0.99$ |
|------------|--------------------------|--------------------------|
| $||J^* - \Phi \otimes \pi_{opt}||_\infty$ | 9.2768 | 18.657 |
| $||J^* - J_\lambda||_\infty$ | 9.3248 | 99.149 |

The results are plotted in Figure 2. Note that $\tilde{J} \geq J^*$. Also the errors in the table obey the error bounds. We also noted that the algorithm finds the optimal actions for about 75 states.

Next we apply the MPADP algorithm to solve the mountain car problem described in the next subsection.

7.1 Mountain Car

The problem is to make an underpowered car climb a one-dimensional hill (Figure 3), whose position $x$ lies in the interval $[-1.2, 0.5]$. There are 3 actions available to the car, i.e., $A = \{0, 1, 2\}$. $a = 0$, $a = 2$ correspond to accelerating to left and right respectively, $a = 1$ corresponds to no acceleration. The velocity $y$ is limited between $[-0.07, 0.07]$. The goal is reached once the car crosses the position $x \geq 0.5$ with a reward of 100 and everywhere else, the reward is 0. The dynamics is given by

\[
y_{t+1} = y_t + 0.001(a_t - 1) - 0.0025cos(3x_t),
\]

\[
x_{t+1} = x_t + y_t.
\]

The state space is continuous with $S = [-1.2, 0.5] \times [-0.07, 0.07]$ and the state is given by $s = (x, y), x \in [-1.2, 0.5], y \in [-0.07, 0.07]$. The feature vector for state $s$ is

\[
\phi^*(i) = |\beta(x + 1.2 - x_i)|^\gamma + |\beta(y + 0.07 - y_i)|^\gamma, i = 1, \ldots, k,
\]

where $\beta > 0$ is a scaling factor and $\gamma > 1$ is the order. $(x_i, y_j), i = 1, \ldots, k, j = 1, \ldots, k$ are the $k \times k$ centers, with $s_{ij} = (x_i, y_j) \in S$. We note that, it is difficult to perform the minimization in line-3 of Algorithm 1 over all $s \in S$ and hence we discretize $S$ by means of $k_1 \times k_1$ grid points. These grid points were generated by choosing $x_i^j, i = 1, \ldots, k_1$ and $y_j^i, j = 1, \ldots, k_1$, with $s_{ij} = (x_i^j, y_j^i)$.

In our experiments we fixed $\beta = 100$ and $\gamma = 2$, and varied $k = 5, 7, 9, 11$ and $k_1 = 30, 40, 50$, and the discount factor was set to $\alpha = 0.95$, and $\epsilon = 1e^{-5}$. The number of steps taken for the mountain car to reach the goal in each of these settings is presented in Table 3. The value function learnt in the various cases is presented in Table 4. The actual value function is shown in Figure 4. The brighter regions denote higher values and darker regions denote lower values.

Near optimal policy for the mountain car problem is known to achieve the goal within 150 steps.

8 Conclusion

We introduced a novel ADP method to approximate the value function of infinite horizon discounted reward MDP. The novelty was in the use of $(\min, +)$ linear basis as opposed to the conventional linear basis. Our approximate value function belonged to the subsemimodule formed by the $(\min, +)$ linear
**Figure 2:** Optimal, Approximate and Greedy Policy Value Function

**Figure 3:** Mountain Car
Table 3: Number of steps taken by the *Greedy* policy

| $k$ | $k_1$ | Steps to reach the goal |
|-----|-------|-------------------------|
| 5   | 30    | 285                     |
| 5   | 40    | 285                     |
| 5   | 50    | 285                     |
| 7   | 30    | 322                     |
| 7   | 40    | 322                     |
| 7   | 50    | 327                     |
| 9   | 30    | 218                     |
| 9   | 40    | 317                     |
| 9   | 50    | 324                     |
| 11  | 30    | 267                     |
| 11  | 40    | 260                     |
| 11  | 50    | 257                     |

$k = 5$, $k_1 = 30$, $V_{\text{max}} = 2.83 \cdot 10^{-3}$, $V_{\text{min}} = 0.73 \cdot 10^{-3}$

$k = 5$, $k_1 = 40$, $V_{\text{max}} = 3.29 \cdot 10^{-3}$, $V_{\text{min}} = 0.74 \cdot 10^{-3}$

$k = 5$, $k_1 = 50$, $V_{\text{max}} = 3.28 \cdot 10^{-3}$, $V_{\text{min}} = 0.77 \cdot 10^{-3}$

$k = 7$, $k_1 = 30$, $V_{\text{max}} = 2.60 \cdot 10^{-3}$, $V_{\text{min}} = 0.45 \cdot 10^{-3}$

$k = 7$, $k_1 = 40$, $V_{\text{max}} = 2.83 \cdot 10^{-3}$, $V_{\text{min}} = 0.47 \cdot 10^{-3}$

$k = 7$, $k_1 = 50$, $V_{\text{max}} = 2.88 \cdot 10^{-3}$, $V_{\text{min}} = 0.48 \cdot 10^{-3}$

$k = 9$, $k_1 = 30$, $V_{\text{max}} = 2.42 \cdot 10^{-3}$, $V_{\text{min}} = 0.25 \cdot 10^{-3}$

$k = 9$, $k_1 = 40$, $V_{\text{max}} = 2.58 \cdot 10^{-3}$, $V_{\text{min}} = 0.27 \cdot 10^{-3}$

$k = 9$, $k_1 = 50$, $V_{\text{max}} = 2.57 \cdot 10^{-3}$, $V_{\text{min}} = 0.26 \cdot 10^{-3}$

$k = 11$, $k_1 = 30$, $V_{\text{max}} = 2.32 \cdot 10^{-3}$, $V_{\text{min}} = 0.18 \cdot 10^{-3}$

$k = 11$, $k_1 = 40$, $V_{\text{max}} = 2.42 \cdot 10^{-3}$, $V_{\text{min}} = 0.20 \cdot 10^{-3}$

$k = 11$, $k_1 = 50$, $V_{\text{max}} = 2.42 \cdot 10^{-3}$, $V_{\text{min}} = 0.20 \cdot 10^{-3}$

Table 4: Approximate Value Function for various values of $k$ and $k_1$
span of the basis and obeyed the \((\text{min}, +)\) Projected Bellman Equation (MPPBE). The salient feature of the approximate value function was that the error was bounded in the \(L_\infty\) norm. We also presented the MPADP algorithm (Algorithm 1) to solve the MPPBE and showed that the algorithm converges to the desired solution. We also applied our method on two example problems.

The use of \((\text{min}, +)\) LFAs in ADP methods is quite new and there are several interesting directions that can be furthered. A question of immediate interest is to find the possibilities of a reinforcement learning (RL) algorithm based on \((\text{min}, +)\) LFA, that solve MDP in the absence of model information. It will be interesting to investigate whether it is possible to develop \(Q\)-learning algorithm using \((\text{min}, +)\) LFA. Also, further research is required to find the right choice of basis functions in the new algebra. These might together throw light on the right kind of LFA architecture to be chosen for any given problem.
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