Research article

On a nonlinear mixed-order coupled fractional differential system with new integral boundary conditions

Bashir Ahmad$^{1, *}$, Soha Hamdan$^1$, Ahmed Alsaedi$^1$ and Sotiris K. Ntouyas$^{1, 2}$

$^1$ Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia

$^2$ Department of Mathematics, University of Ioannina, 45110, Ioannina, Greece

* Correspondence: Email: bashirahmad_qau@yahoo.com.

Abstract: We present the criteria for the existence of solutions for a nonlinear mixed-order coupled fractional differential system equipped with a new set of integral boundary conditions on an arbitrary domain. The modern tools of the fixed point theory are employed to obtain the desired results, which are well-illustrated by numerical examples. A variant problem dealing with the case of nonlinearities depending on the cross-variables (unknown functions) is also briefly described.

Keywords: fractional differential equations; Caputo fractional derivative; systems; existence; fixed point theorems

Mathematics Subject Classification: 34A08, 34B15

1. Introduction

It is well known that the classical boundary conditions cannot describe certain peculiarities of physical, chemical, or other processes occurring within the domain. In order to overcome this situation, the concept of nonlocal conditions was introduced by Bicadze and Samarskiĭ [1]. These conditions are successfully employed to relate the changes happening at nonlocal positions or segments within the given domain to the values of the unknown function at end points or boundary of the domain. For a detailed account of nonlocal boundary value problems, for example, we refer the reader to the articles [2–6] and the references cited therein.

Computational fluid dynamics (CFD) technique directly deals with the boundary data [7]. In case of fluid flow problems, the assumption of circular cross-section is not justifiable for curved structures. The idea of integral boundary conditions serves as an effective tool to describe the boundary data on arbitrary shaped structures. One can find application of integral boundary conditions in the study of
thermal conduction, semiconductor, and hydrodynamic problems [8–10]. In fact, there are numerous applications of integral boundary conditions in different disciplines such as chemical engineering, thermoelasticity, underground water flow, population dynamics, etc. [11–13]. Also, integral boundary conditions facilitate to regularize ill-posed parabolic backward problems, for example, mathematical models for bacterial self-regularization [14]. Some recent results on boundary value problems with integral boundary conditions can be found in the articles [15–19] and the references cited therein.

The non-uniformities in form of points or sub-segments on the heat sources can be relaxed by using the integro multi-point boundary conditions, which relate the sum of the values of the unknown function (e.g., temperature) at the nonlocal positions (points and sub-segments) and the value of the unknown function over the given domain. Such conditions also find their utility in the diffraction problems when scattering boundary consists of finitely many sub-strips (finitely many edge-scattering problems). For details and applications in engineering problems, for instance, see [20–23].

The subject of fractional calculus has emerged as an important area of research in view of extensive applications of its tools in scientific and technical disciplines. Examples include neural networks [24, 25], immune systems [26], chaotic synchronization [27, 28], Quasi-synchronization [29, 30], fractional diffusion [31–33], financial economics [34], ecology [35], etc. Inspired by the popularity of this branch of mathematical analysis, many researchers turned to it and contributed to its different aspects. In particular, fractional order boundary value problems received considerable attention. For some recent results on fractional differential equations with multi-point and integral boundary conditions, see [36, 37]. More recently, in [38, 39], the authors analyzed boundary value problems involving Riemann-Liouville and Caputo fractional derivatives respectively. A boundary value problem involving a nonlocal boundary condition characterized by a linear functional was studied in [40]. In a recent paper [41], the existence results for a dual anti-periodic boundary value problem involving nonlinear fractional integro-differential equations were obtained.

On the other hand, fractional differential systems also received considerable attention as such systems appear in the mathematical models associated with physical and engineering processes [42–46]. For theoretical development of such systems, for instance, see the articles [47–52].

Motivated by aforementioned applications of nonlocal integral boundary conditions and fractional differential systems, in this paper, we study a nonlinear mixed-order coupled fractional differential system equipped with a new set of nonlocal multi-point integral boundary conditions on an arbitrary domain given by

\[ \begin{align*} 
\alpha^\chi D^{\chi}_{a^+} x(t) &= \varphi(t, x(t), y(t)), \quad 0 < \chi \leq 1, \quad t \in [a, b], \\
\alpha^\chi D^{\chi}_{a^+} y(t) &= \psi(t, x(t), y(t)), \quad 1 < \chi \leq 2, \quad t \in [a, b], \\
p x(a) + q y(b) &= y_0 + x_0 \int_a^b (x(s) + y(s)) ds, \\
y(a) &= 0, \quad y'(b) = \sum_{i=1}^m \delta_i x(\sigma_i) + \lambda \int_a^\tau x(s) ds, \\
0 < \sigma_1 < \sigma_2 < \ldots < \sigma_m < \tau < b, 
\end{align*} \]

where \( \alpha^\chi D^{\chi} \) is Caputo fractional derivative of order \( \chi \in [\xi, \zeta], \varphi, \psi : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are given functions, \( p, q, \delta_i, x_0, y_0 \in \mathbb{R}, i = 1, 2, \ldots, m \).

Here we emphasize that the novelty of the present work lies in the fact that we introduce a coupled
system of fractional differential equations of different orders on an arbitrary domain equipped with coupled nonlocal multi-point integral boundary conditions. It is imperative to notice that much of the work related to the coupled systems of fractional differential equations deals with the fixed domain. Thus our results are more general and contribute significantly to the existing literature on the topic. Moreover, several new results appear as special cases of the work obtained in this paper.

We organize the rest of the paper as follows. In Section 2, we present some basic concepts of fractional calculus and solve the linear version of the problem (1.1). Section 3 contains the main results. Examples illustrating the obtained results are presented in Section 4. Section 5 contains the details of a variant problem. The paper concludes with some interesting observations.

2. Preliminaries

Let us recall some definitions from fractional calculus related to our study [53].

**Definition 2.1.** The Riemann–Liouville fractional integral of order $\alpha \in \mathbb{R}$ ($\alpha > 0$) for a locally integrable real-valued function $\varrho$ of order $\alpha \in \mathbb{R}$, denoted by $I_{a}^{\alpha} \varrho$, is defined as

$$I_{a}^{\alpha} \varrho (t) = \left( \varrho \ast \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) (t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-s)^{\alpha-1} \varrho (s) ds, \quad -\infty \leq a < t < b \leq +\infty,$$

where $\Gamma$ denotes the Euler gamma function.

**Definition 2.2.** The Riemann–Liouville fractional derivative $D_{a}^{\alpha} \varrho$ of order $\alpha \in ]m-1, m]$, $m \in \mathbb{N}$ is defined as

$$D_{a}^{\alpha} \varrho (t) = \frac{d^{m}}{dt^{m}} I_{a}^{1-\alpha} \varrho (t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^{m}}{dt^{m}} \int_{a}^{t} (t-s)^{m-1-\alpha} \varrho (s) ds, \quad -\infty \leq a < t < b \leq +\infty,$$

while the Caputo fractional derivative $^{c}D_{a}^{\alpha} \varrho$ is defined as

$$^{c}D_{a}^{\alpha} \varrho (t) = D_{a}^{\alpha} \left[ \varrho (t) - \varrho (a) - \varrho' (a) \frac{(t-a)}{1!} - \ldots - \varrho^{(m-1)} (a) \frac{(t-a)^{m-1}}{(m-1)!} \right],$$

for $\varrho, \varrho^{(m)} \in L^{1}[a, b]$.

**Remark 2.1.** The Caputo fractional derivative $^{c}D_{a}^{\alpha} \varrho$ is also defined as

$$^{c}D_{a}^{\alpha} \varrho(t) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{t} (t-s)^{m-1-\alpha} \varrho^{(m)} (s) ds.$$
Lemma 2.1. Let $\Phi, \Psi \in C([a, b], \mathbb{R})$. Then the unique solution of the system

\[
\begin{align*}
\begin{cases}
^cD_{a^+}^\xi x(t) &= \Phi(t), \quad 0 < \xi \leq 1, \quad t \in [a, b], \\
^cD_{a^+}^\zeta y(t) &= \Psi(t), \quad 1 < \zeta \leq 2, \quad t \in [a, b], \\
p x(a) + q y(b) &= y_0 + x_0 \int_a^b (x(s) + y(s))ds,
\end{cases}
\end{align*}
\]  

(2.1)

is given by a pair of integral equations

\[
x(t) = I_{a^+}^\xi \Phi(t) + \frac{1}{\Delta} \left\{ y_0 + x_0 \int_a^b \frac{(b - s)^\xi}{\Gamma(\xi + 1)} \Phi(s)ds \\
+ \int_a^b \frac{x_0 (b - s)^\xi + \epsilon_1 (b - s)^{\xi - 2} - q (b - s)^{\xi - 1}}{\Gamma(\xi)} \Psi(s)ds \\
- \epsilon_1 \sum_{i=1}^m \delta_i \int_a^{\sigma_i} \frac{(\sigma_i - s)^{\xi - 1}}{\Gamma(\xi)} \Phi(s)ds - \epsilon_1 \lambda \int_\tau^b \int_a^s \frac{(s - u)^{\xi - 1}}{\Gamma(\xi)} \Phi(u)du ds \right\},
\]

(2.2)

\[
y(t) = I_{a^+}^\zeta \Psi(t) + \frac{(t - a)}{\Delta} \left\{ e_2 y_0 + e_2 x_0 \int_a^b \frac{(b - s)^\zeta}{\Gamma(\zeta + 1)} \Phi(s)ds \\
+ \int_a^b \frac{e_2 x_0 (b - s)^\zeta - e_2 q (b - s)^{\zeta - 1} + \epsilon_3 (b - s)^{\xi - 2} - \epsilon_3 \epsilon_1 (b - s)^{\xi - 1}}{\Gamma(\zeta)} \Psi(s)ds \\
+ \epsilon_3 \sum_{i=1}^m \delta_i \int_a^{\sigma_i} \frac{(\sigma_i - s)^{\xi - 1}}{\Gamma(\xi)} \Phi(s)ds + \epsilon_3 \lambda \int_\tau^b \int_a^s \frac{(s - u)^{\xi - 1}}{\Gamma(\xi)} \Phi(u)du ds \right\},
\]

(2.3)

where

\[
\epsilon_1 = q(b - a) - x_0 \frac{(b - a)^2}{2}, \quad \epsilon_2 = \sum_{i=1}^m \delta_i + \lambda(b - \tau), \quad \epsilon_3 = p - (b - a)x_0,
\]

(2.4)

and it is assumed that

\[
\Delta = \epsilon_3 + \epsilon_2 \epsilon_1 \neq 0.
\]

(2.5)

Proof. Applying the integral operators $I_{a^+}^\xi$ and $I_{a^+}^\zeta$ respectively on the first and second fractional differential equations in (2.1), we obtain

\[
x(t) = I_{a^+}^\xi \Phi(t) + c_1 \quad \text{and} \quad y(t) = I_{a^+}^\zeta \Psi(t) + c_2 + c_3 (t - a),
\]

(2.6)

where $c_i \in \mathbb{R}, i = 1, 2, 3$ are arbitrary constants. Using the condition $y(a) = 0$ in (2.6), we get $c_2 = 0$. Making use of the conditions $px(a) + qy(b) = y_0 + x_0 \int_a^b (x(s) + y(s))ds$ and $y'(b) = \sum_{i=1}^m \delta_i x(\sigma_i) + \lambda \int_\tau^b x(s)ds$ in (2.6) after inserting $c_2 = 0$ in it leads to the following system of equations in the unknown constants $c_1$ and $c_3$:

\[
\left( p - (b - a)x_0 \right) c_1 + \left( q(b - a) - x_0 \frac{(b - a)^2}{2} \right) c_3 = y_0 + x_0 \int_a^b \frac{(b - r)^\xi}{\Gamma(\xi + 1)} \Phi(r)dr
\]
where (2.7) and (2.8) for $c_1$ and $c_3$ and using the notation (2.5), we find that

$$c_1 = \frac{1}{\Delta} \left\{ \varepsilon_1 (I^{\varepsilon-1}_{a^+} \Psi(b) - \sum_{i=1}^{m} \delta_i I^{\varepsilon}_{a^+} \Phi(\sigma_i)) - \lambda \int_{a}^{b} I^{\varepsilon}_{a^+} \Phi(s)ds \right\} + x_0 \int_{a}^{b} \frac{(b - r)^{\varepsilon}}{\Gamma(\xi + 1)} \Psi(r)dr - q I^{\varepsilon}_{a^+} \Psi(b) \right\},$$

$$c_3 = \frac{1}{\Delta} \left\{ \varepsilon_2 (y_0 + \int_{a}^{b} \frac{(b - r)^{\varepsilon}}{\Gamma(\xi + 1)} \Phi(r)dr + x_0 \int_{a}^{b} \frac{(b - r)^{\varepsilon}}{\Gamma(\xi + 1)} \Psi(r)dr - q I^{\varepsilon}_{a^+} \Psi(b)) - \varepsilon_3 (I^{\varepsilon-1}_{a^+} \Psi(b) - \sum_{i=1}^{m} \delta_i I^{\varepsilon}_{a^+} \Phi(\sigma_i)) - \lambda \int_{a}^{b} I^{\varepsilon}_{a^+} \Phi(s)ds \right\}.$$

Inserting the values of $c_1, c_2,$ and $c_3$ in (2.6) leads to the solution (2.2) and (2.3). One can obtain the converse of the lemma by direct computation. This completes the proof. \(\square\)

### 3. Main results

Let $X = C([a, b], \mathbb{R})$ be a Banach space endowed with the norm $||x|| = \sup ||x(t)||, t \in [a, b]$. In view of Lemma 2.1, we define an operator $T : X \times X \rightarrow X$ by:

$$T(x(t), y(t)) = (T_1(x(t), y(t)), T_2(x(t), y(t))),$$

where $(X \times X, ||(x, y)||)$ is a Banach space equipped with norm $||(x, y)|| = ||x|| + ||y||, x, y \in X,

$$T_1(x, y)(t) = I^{\varepsilon}_{a^+} \psi(t, x(t), y(t)) + \frac{1}{\Delta} \left( y_0 + x_0 \int_{a}^{b} \frac{(b - s)^{\varepsilon}}{\Gamma(\xi + 1)} \phi(s, x(s), y(s))ds \right. + \int_{a}^{b} \rho_1(s) \psi(s, x(s), y(s))ds - \varepsilon_1 \sum_{i=1}^{m} \delta_i \int_{a}^{\sigma_i \varepsilon} \frac{(\sigma_i - s)^{\varepsilon}}{\Gamma(\xi)} \phi(s, x(s), y(s))ds \left. - \varepsilon_1 \lambda \int_{a}^{b} \int_{a}^{s} \frac{(s - u)^{\varepsilon-1}}{\Gamma(\xi)} \phi(u, x(u), y(u))du ds \right),$$

$$T_2(x, y)(t) = I^{\varepsilon}_{a^+} \psi(t, x(t), y(t)) + \frac{(t - a)}{\Delta} \left( \varepsilon_2 y_0 + x_0 \int_{a}^{b} \frac{(b - s)^{\varepsilon}}{\Gamma(\xi + 1)} \phi(s, x(s), y(s))ds \right. + \int_{a}^{b} \rho_2(s) \psi(s, x(s), y(s))ds + \varepsilon_3 \sum_{i=1}^{m} \delta_i \int_{a}^{\sigma_i \varepsilon} \frac{(\sigma_i - s)^{\varepsilon-1}}{\Gamma(\xi)} \phi(s, x(s), y(s))ds \left. + \varepsilon_3 \lambda \int_{a}^{b} \int_{a}^{s} \frac{(s - u)^{\varepsilon-1}}{\Gamma(\xi)} \phi(u, x(u), y(u))du ds \right).$$
\[
\begin{align*}
\rho_1(s) &= x_0 \frac{(b-s)^c}{\Gamma(\zeta+1)} + \varepsilon_1 \frac{(b-s)^{c-2}}{\Gamma(\zeta-1)} - q \frac{(b-s)^{c-1}}{\Gamma(\xi)}, \\
\rho_2(s) &= \varepsilon_2 x_0 \frac{(b-s)^c}{\Gamma(\zeta+1)} - \varepsilon_2 q \frac{(b-s)^{c-1}}{\Gamma(\xi)} - \varepsilon_3 \frac{(b-s)^{c-2}}{\Gamma(\zeta-1)}.
\end{align*}
\]

For computational convenience we put:
\[
\begin{align*}
L_1 &= \frac{(b-a)^c}{\Gamma(\xi + 1)} + \frac{1}{|\Delta|} \left( |x_0| \frac{(b-a)^{c+1}}{\Gamma(\xi + 2)} + |\varepsilon_1| \sum_{i=1}^m |\sigma_i - a| \frac{(b-a)^{c+1}}{\Gamma(\xi + 1)} \right) \\
+ |\varepsilon_2| \frac{(b-a)^{c+1} - (\tau - a)^{c+1}}{\Gamma(\xi + 2)}, \\
M_1 &= \frac{1}{|\Delta|} \left( |x_0| \frac{(b-a)^{c+1}}{\Gamma(\xi + 2)} + |\varepsilon_1| \frac{(b-a)^{c+1} - (\tau - a)^{c+1}}{\Gamma(\xi + 1)} + |q| \frac{(b-a)^c}{\Gamma(\xi + 1)} \right), \\
L_2 &= \frac{(b-a)^c}{|\Delta|} \left( |x_0| \frac{(b-a)^{c+1}}{\Gamma(\xi + 2)} + |\varepsilon_1| \sum_{i=1}^m |\sigma_i - a| \frac{(b-a)^{c+1}}{\Gamma(\xi + 1)} \right) \\
+ |\varepsilon_3| \frac{(b-a)^{c+1} - (\tau - a)^{c+1}}{\Gamma(\xi + 2)}, \\
M_2 &= \frac{(b-a)^c}{\Gamma(\xi + 1)} + \frac{b-a}{|\Delta|} \left( |x_0| \frac{(b-a)^{c+1}}{\Gamma(\xi + 2)} + |\varepsilon_2| \frac{(b-a)^c}{\Gamma(\xi + 1)} + |\varepsilon_3| \frac{(b-a)^{c-1}}{\Gamma(\xi)} \right).
\end{align*}
\]

Our first existence result for the system (1.1) relies on Leray-Schauder alternative [54].

**Theorem 3.1.** Assume that:

(H₁) \( \varphi, \psi : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are continuous functions and there exist real constants \( k_i, \gamma_i \geq 0 \), \( i = 1, 2 \) and \( k_0 > 0, \gamma_0 > 0 \) such that \( \forall x, y \in \mathbb{R}, \)

\[
|\varphi(t, x, y)| \leq k_0 + k_1|x| + k_2|y|, \\
|\psi(t, x, y)| \leq \gamma_0 + \gamma_1|x| + \gamma_2|y|.
\]

Then there exists at least one solution for the system (1.1) on \([a, b]\) if

\[
(L_1 + L_2)k_1 + (M_1 + M_2)\gamma_1 < 1 \quad \text{and} \quad (L_1 + L_2)k_2 + (M_1 + M_2)\gamma_2 < 1,
\]

where \( L_i, M_i, i = 1, 2 \) are given by (3.1).

**Proof.** Let us note that continuity of the functions \( \varphi \) and \( \psi \) implies that of the operator \( T : X \times X \to X \times X \). Next, let \( \Omega \subset X \times X \) be bounded such that

\[|\varphi(t, x(t), y(t))| \leq K_1, \quad |\psi(t, x(t), y(t))| \leq K_2, \quad \forall (x, y) \in \Omega,\]

for positive constants \( K_1 \) and \( K_2 \). Then for any \((x, y) \in \Omega\), we have

\[
|T_1(x, y)(t)| \leq \int_a^b |\varphi(t, x(t), y(t))| + \frac{1}{|\Delta|} \left( |y_0| + |x_0| \int_a^b \frac{(b-s)^c}{\Gamma(\xi + 1)}|\varphi(s, x(s), y(s))|ds \right)
\]

\AIMS Mathematics
Volume 6, Issue 6, 5801–5816.
\[ \begin{align*}
+ & \int_a^b |\rho_1(s)||\psi(s, x(s), y(s))|ds \\
+ & |\varepsilon_1| \sum_{i=1}^m |\delta_i| \int_a^{c_i} \frac{(s_i - s)^{\varepsilon-1}}{\Gamma(\varepsilon)} |\varphi(s, x(s), y(s))|ds \\
+ & |\varepsilon_1| \lambda \int_a^b \int_{t_1}^t \left( \frac{(s-u)^{\varepsilon-1}}{\Gamma(\varepsilon)} |\varphi(u, x(u), y(u))|duds \right)
\end{align*} \]

\[ \leq \frac{|y_0|}{|\Delta|} + \left\{ \frac{(b-a)^{\varepsilon}}{\Gamma(\varepsilon + 1)} + \frac{1}{|\Delta|} \left( |x_0| \frac{(b-a)^{\varepsilon+1}}{\Gamma(\varepsilon + 2)} + |\varepsilon_1| \sum_{i=1}^m |\delta_i| \frac{(s_i - a)^{\varepsilon}}{\Gamma(\varepsilon + 1)} \right) \right\} K_1 \\
+ & \frac{1}{|\Delta|} \left( |x_0| \frac{(b-a)^{\varepsilon+1}}{\Gamma(\varepsilon + 2)} + |\varepsilon_1| \frac{(b-a)^{\varepsilon-1}}{\Gamma(\varepsilon)} + |\varepsilon_0| \frac{(b-a)^{\varepsilon}}{\Gamma(\varepsilon + 1)} \right) \right\} K_2 \\
= & \frac{|y_0|}{|\Delta|} + L_1 K_1 + M_1 K_2, \quad (3.3)
\]

which implies that

\[ \|T_1(x, y)\| \leq \frac{|y_0|}{|\Delta|} + L_1 K_1 + M_1 K_2. \]

In a similar manner, one can obtain that

\[ \|T_2(x, y)\| \leq \frac{|\varepsilon_2 y_0| (b-a)}{|\Delta|} + L_2 K_1 + M_2 K_2. \]

In consequence, the operator \( T \) is uniformly bounded as

\[ \|T(x, y)\| \leq \frac{|y_0|}{|\Delta|} + \frac{|\varepsilon_2 y_0| (b-a)}{|\Delta|} + (L_1 + L_2) K_1 + (M_1 + M_2) K_2. \]

Now we show that \( T \) is equicontinuous. Let \( t_1, t_2 \in [a, b] \) with \( t_1 < t_2 \). Then we have

\[ \begin{align*}
|T_1(x(t_2), y(t_2)) - T_1(x(t_1), y(t_1))| & \leq K_1 \left| \frac{1}{\Gamma(\varepsilon)} \int_a^{c_2} (t_2 - s)^{\varepsilon-1} ds - \frac{1}{\Gamma(\varepsilon)} \int_a^{c_1} (t_1 - s)^{\varepsilon-1} ds \right| \\
& \leq K_1 \left\{ \frac{1}{\Gamma(\varepsilon)} \int_a^{c_2} [(t_2 - s)^{\varepsilon-1} - (t_1 - s)^{\varepsilon-1}] ds + \frac{1}{\Gamma(\varepsilon)} \int_{t_1}^{c_2} (t_2 - s)^{\varepsilon-1} ds \right\} \\
& \leq \frac{K_1}{\Gamma(\varepsilon + 1)} |2(t_2 - t_1)^{\varepsilon} + |t_2|^\varepsilon - |t_2|^\varepsilon|.|. \quad (3.4)
\end{align*} \]

Analogously, we can obtain

\[ \begin{align*}
|T_2(x(t_2), y(t_2)) - T_2(x(t_1), y(t_1))| & \leq \frac{K_2}{\Gamma(\varepsilon + 1)} |2(t_2 - t_1)^{\varepsilon} + |t_2|^\varepsilon - |t_2|^\varepsilon|.| + \frac{|t_2 - t_1|}{|\Delta|} \left\{ |\varepsilon_2 y_0| \frac{(b-a)^{\varepsilon+1}}{\Gamma(\varepsilon + 2)} K_1 \right\}
\end{align*} \]
\[ + \left\{ \varepsilon_2 x_0 \frac{(b-a)\xi}{\Gamma(\xi+2)} + \varepsilon_2 q \frac{(b-a)\xi}{\Gamma(\xi+1)} + \varepsilon_3 \frac{(b-a)^{\xi-1}}{\Gamma(\xi)} \right\} K_2 + \varepsilon_3 \sum_{i=1}^{m} [\sigma_i - b] \frac{\xi}{\Gamma(\xi+1)} K_1 + \varepsilon_3 \frac{(b-a)^{\xi-1} - (\tau-a)^{\xi-1}}{\Gamma(\xi+2)} K_1 \right\}. \]

From the preceding inequalities, it follows that the operator \( T(x,y) \) is equicontinuous. Thus the operator \( T(x,y) \) is completely continuous.

Finally, we consider the set \( \mathcal{P} = \{(x,y) \in X \times X : (x,y) = vT(x,y), 0 \leq v \leq 1 \} \) and show that it is bounded.

Let \( (x,y) \in \mathcal{P} \) with \( (x,y) = vT(x,y) \). For any \( t \in [a,b] \), we have
\[
|x(t)| \leq \frac{|y_0|}{|\Delta|} + L_1 k_0 + M_1 |x| + M_1 |y_0| + M_1 |y_1| |x| + (L_1 k_2 + M_1 y_2) |y|,
\]
and
\[
|y(t)| \leq \frac{|x_0| (b-a)}{|\Delta|} + L_2 k_0 + M_2 |y_0| + M_2 |y_1| |x| + (L_2 k_2 + M_2 y_2) |y|.
\]

In consequence of the above inequalities, we deduce that
\[
||x|| \leq \frac{|y_0|}{|\Delta|} + L_1 k_0 + M_1 |y_0| + (L_1 k_1 + M_1 |y_1|) ||x|| + (L_1 k_2 + M_1 y_2) ||y||,
\]
and
\[
||y|| \leq \frac{|x_0| (b-a)}{|\Delta|} + L_2 k_0 + M_2 |y_0| + (L_2 k_1 + M_2 |y_1|) ||x|| + (L_2 k_2 + M_2 y_2) ||y||,
\]
which imply that
\[
||x|| + ||y|| \leq \frac{|y_0|}{|\Delta|} + \frac{|x_0| (b-a)}{|\Delta|} + (L_1 + L_2) k_0 + (M_1 + M_2) |y_0| + (L_1 + L_2) k_1 + (M_1 + M_2) |y_1| ||x|| + [(L_1 + L_2) k_2 + (M_1 + M_2) y_2] ||y||.
\]

Thus
\[
||(x,y)|| \leq \frac{1}{M_0} \left\{ \frac{|y_0|}{|\Delta|} + \frac{|x_0| (b-a)}{|\Delta|} + (L_1 + L_2) k_0 + (M_1 + M_2) |y_0| \right\},
\]
where \( M_0 = \min \{1 - [(L_1 + L_2) k_1 + (M_1 + M_2) |y_1|], 1 - [(L_1 + L_2) k_2 + (M_1 + M_2) y_2]\} \). Hence the set \( \mathcal{P} \) is bounded. As the hypothesis of Leray-Schauder alternative [54] is satisfied, we conclude that the operator \( T \) has at least one fixed point. Thus the problem (1.1) has at least one solution on \( [a,b] \). \( \square \)

By using Banach’s contraction mapping principle we prove in the next theorem the existence of a unique solution of the system (1.1).

AIMS Mathematics

Volume 6, Issue 6, 5801–5816.
Theorem 3.2. Assume that:

\[(H_2) \quad \varphi, \psi : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \text{ are continuous functions and there exist positive constants } l_1 \text{ and } l_2 \text{ such that for all } t \in [a, b] \text{ and } x_i, y_i \in \mathbb{R}, \ i = 1, 2, \text{ we have}
\]
\[
|\varphi(t, x_1, x_2) - \varphi(t, y_1, y_2)| \leq l_1(|x_1 - y_1| + |x_2 - y_2|),
\]
\[
|\psi(t, x_1, x_2) - \psi(t, y_1, y_2)| \leq l_2(|x_1 - y_1| + |x_2 - y_2|).
\]

If
\[
(L_1 + L_2)l_1 + (M_1 + M_2)l_2 < 1,
\]
where \(L_i, M_i, i = 1, 2\) are given by (3.1) then the system (1.1) has a unique solution on [a, b].

Proof. Define \(\sup_{t \in [a, b]} \varphi(t, 0, 0) = N_1 < \infty, \sup_{t \in [a, b]} \psi(t, 0, 0) = N_2 < \infty\) and \(r > 0\) such that
\[
r > \frac{(y_0/(|\Delta|))(1 + (b - a)|x_2|) + (L_1 + L_2)N_1 + (M_1 + M_2)N_2}{1 - (L_1 + L_2)l_1 - (M_1 + M_2)l_2}.
\]

Let us first show that \(TB_r \subset B_r\), where \(B_r = \{(x, y) \in X \times X : \| (x, y) \| \leq r \}\). By the assumption \((H_2)\), for \((x, y) \in B_r, \ t \in [a, b]\), we have
\[
|\varphi(t, x(t), y(t))| \leq |\varphi(t, x(t), y(t)) - \varphi(t, 0, 0)| + |\varphi(t, 0, 0)|
\]
\[
\leq l_1(|x(t)| + |y(t)|) + N_1
\]
\[
\leq l_1(\|x\| + \|y\|) + N_1 \leq l_1r + N_1.
\]

Similarly, we can get
\[
|\psi(t, x(t), y(t))| \leq l_2(\|x\| + \|y\|) + N_2 \leq l_2r + N_2.
\]

Using (3.6) and (3.7), we obtain
\[
|T_1(x, y)(t)| \leq \int_a^b |\varphi(t, x(t), y(t))| + \frac{1}{|\Delta|} \left( |y_0| + |x_0| \right) \int_a^b \frac{(b - s)^\xi}{\Gamma(\xi + 1)} |\varphi(s, x(s), y(s))| ds
\]
\[
+ \int_a^b |\varphi(s)| |\psi(s, x(s), y(s))| ds
\]
\[
+ |\varepsilon_1| \sum_{i=1}^m |\delta_i| \int_a^\sigma \frac{(\sigma_i - s)^\xi-1}{\Gamma(\xi)} |\varphi(s, x(s), y(s))| ds
\]
\[
+ |\varepsilon_1| \lambda \int_a^b \int_a^s \frac{(s - u)^\xi-1}{\Gamma(\xi)} |\varphi(u, x(u), y(u))| duds
\]
\[
\leq \frac{|y_0|}{|\Delta|} + \left( \frac{(b - a)^\xi}{\Gamma(\xi + 1)} + \frac{1}{|\Delta|} \left( |x_0| \frac{(b - a)^{\xi+1}}{\Gamma(\xi + 2)} + |\varepsilon_1| \sum_{i=1}^m |\delta_i| \frac{((\sigma_i - a)^\xi}{\Gamma(\xi + 1)} \right)
\]
\[
+ \left( \frac{1}{|\Delta|} \left( |x_0| \frac{(b - a)^{\xi+1}}{\Gamma(\xi + 2)} + |\varepsilon_1| \frac{((\sigma_i - a)^{\xi+1}}{\Gamma(\xi + 1)} \right)ight)
\]
\[
\leq (l_1r + N_1)
\]
\[
\begin{align*}
\|T_1(x, y)\| &\leq \frac{|y_0|}{|\Delta|} + (L_1 l_1 + M_1 l_2) r + L_1 N_1 + M_1 N_2. \\
\|T_2(x, y)\| &\leq \frac{|\varepsilon_2 y_0| (b - a)}{|\Delta|} + (L_2 l_1 + M_2 l_2) r + L_2 N_1 + M_2 N_2.
\end{align*}
\]

Consequently,
\[
\|T(x, y)\| \leq \frac{|y_0|}{|\Delta|} + \frac{|\varepsilon_2 y_0| (b - a)}{|\Delta|} + \left( (L_1 + L_2) l_1 + (M_1 + M_2) l_2 \right) r + (L_1 + L_2) N_1 + (M_1 + M_2) N_2 \\
\leq r.
\]

Now, for \((x_1, y_1), (x_2, y_2) \in X \times X\) and for any \(t \in [a, b]\), we get
\[
|T_1(x_2, y_2)(t) - T_1(x_1, y_1)(t)| \\
\leq \left\{ \frac{(b - a)^{\xi}}{\Gamma(\xi + 1)} + \frac{1}{|\Delta|} \left( |x_0| \frac{(b - a)^{\xi+1}}{\Gamma(\xi + 2)} + |\varepsilon_1| \sum_{i=1}^{m} |\delta_i| \frac{(\sigma_i - a)^{\xi}}{\Gamma(\xi + 1)} \right) l_1(||x_2 - x_1|| + ||y_2 - y_1||) \right. \\
+ \left. \frac{1}{|\Delta|} \left( |x_0| \frac{(b - a)^{\xi+1} - (\tau - a)^{\xi+1}}{\Gamma(\xi + 2)} + |\varepsilon_1| \frac{(b - a)^{\xi+1}}{\Gamma(\xi + 1)} \right) l_2(||x_2 - x_1|| + ||y_2 - y_1||) \right\} \\
= (L_1 l_1 + M_1 l_2)(||x_2 - x_1|| + ||y_2 - y_1||),
\]
which implies that
\[
\|T_1(x_2, y_2) - T_1(x_1, y_1)\| \leq (L_1 l_1 + M_1 l_2)(||x_2 - x_1|| + ||y_2 - y_1||). \tag{3.9}
\]

Similarly, we find that
\[
\|T_2(x_2, y_2) - T_2(x_1, y_1)\| \leq (L_2 l_1 + M_2 l_2)(||x_2 - x_1|| + ||y_2 - y_1||). \tag{3.10}
\]

It follows from (3.9) and (3.10) that
\[
\|T(x_2, y_2) - T(x_1, y_1)\| \leq \left( (L_1 + L_2) l_1 + (M_1 + M_2) l_2 \right)(||x_2 - x_1|| + ||y_2 - y_1||).
\]

From the above inequality, we deduce that \(T\) is a contraction. Hence it follows by Banach’s fixed point theorem that there exists a unique fixed point for the operator \(T\), which corresponds to a unique solution of problem (1.1) on \([a, b]\). This completes the proof. \(\square\)
3.1. Example

Consider the following mixed-type coupled fractional differential system

\[
\begin{align*}
D_{\alpha^+}^{\frac{3}{2}} x(t) &= \varphi(t, x(t), y(t)), \ t \in [1, 2], \\
D_{\alpha^+}^{\frac{1}{2}} y(t) &= \psi(t, x(t), y(t)), \ t \in [1, 2] \\
\frac{1}{5} x(1) + \frac{1}{10} y(2) &= \frac{1}{1000} \int_{1}^{2} (x(s) + y(s)) ds, \\
y(1) &= 0, \ y'(2) = \sum_{i=1}^{2} \delta_{i} x(\sigma_{i}) + \frac{1}{10} \int_{\tau}^{2} x(s) ds,
\end{align*}
\]  

(3.11)

where \( \xi = 3/4, \zeta = 7/4, p = 1/5, q = 1/10, x_{0} = 1/1000, y_{0} = 0, \delta_{1} = 1/10, \delta_{2} = 1/100, \sigma_{1} = 5/4, \sigma_{2} = 3/2, \tau = 7/4, \lambda = 1/10. \) With the given data, it is found that \( L_{1} \approx 3.5495 \times 10^{-2}, L_{2} = 6.5531 \times 10^{-2}, M_{1} = 1.0229, M_{2} = 0.9074. \)

1. In order to illustrate Theorem 3.1, we take

\[
\varphi(t, x, y) = e^{-t} + \frac{1}{8} x \cos y + \frac{e^{-t}}{3} \sin y, \\
\psi(t, x, y) = t \sqrt{t^{2} + 2} + \frac{e^{-t}}{3\pi} x \tan^{-1} y + \frac{1}{\sqrt{48 + t^{2}}} y.
\]  

(3.12)

It is easy to check that the condition \((H_{1})\) is satisfied with \( k_{0} = 1/e^{2}, k_{1} = 1/8, k_{2} = 1/(3e), \gamma_{0} = 2 \sqrt{7}, \gamma_{1} = 1/(6e), \gamma_{2} = 1/7. \) Furthermore, \((L_{1} + L_{2})k_{1} + (M_{1} + M_{2})\gamma_{1} \approx 0.13098 < 1, \) and \((L_{1} + L_{2})k_{2} + (M_{1} + M_{2})\gamma_{2} \approx 0.28815 < 1. \) Clearly the hypotheses of Theorem 3.1 are satisfied and hence the conclusion of Theorem 3.1 applies to problem (3.11) with \( \varphi \) and \( \psi \) given by (3.12).

2. In order to illustrate Theorem 3.2, we take

\[
\varphi(t, x, y) = \frac{e^{-t}}{\sqrt{3 + t^{2}}} \cos x + \cos t, \quad \psi(t, x, y) = \frac{1}{5 + t} (\sin x + |y|) + e^{-t},
\]  

(3.13)

which clearly satisfy the condition \((H_{2})\) with \( l_{1} = 1/(2e) \) and \( l_{2} = 1/6. \) Moreover \((L_{1} + L_{2})l_{1} + (M_{1} + M_{2})l_{2} \approx 0.3403 < 1. \) Thus the hypothesis of Theorem 3.2 holds true and consequently there exists a unique solution of the problem (3.11) with \( \varphi \) and \( \psi \) given by (3.13) on \([1, 2].\)

4. A variant problem

In this section, we consider a variant of the problem (1.1) in which the nonlinearities \( \varphi \) and \( \psi \) do not depend on \( x \) and \( y \) respectively. In precise terms, we consider the following problem:

\[
\begin{align*}
D_{\alpha^+}^{\xi} x(t) &= \bar{\varphi}(t, y(t)), \ 0 < \xi \leq 1, \ t \in [a, b], \\
D_{\alpha^+}^{\zeta} y(t) &= \bar{\psi}(t, x(t)), \ 1 < \zeta \leq 2, \ t \in [a, b], \\
x(a) + qy(b) &= y_{0} + x_{0} \int_{a}^{b} (x(s) + y(s)) ds, \\
y(a) &= 0, \ y'(b) = \sum_{i=1}^{m} \delta_{i} x(\sigma_{i}) + \lambda \int_{\tau}^{b} x(s) ds,
\end{align*}
\]  

(4.1)

\( a < \sigma_{1} < \sigma_{2} < \ldots < \sigma_{m} < \tau < \ldots < b, \)
where \( \varphi, \psi : [a, b] \times \mathbb{R} \rightarrow \mathbb{R} \) are given functions. Now we present the existence and uniqueness results for the problem (4.1). We do not provide the proofs as they are similar to the ones for the problem (1.1).

**Theorem 4.1.** Assume that \( \tilde{\varphi}, \tilde{\psi} : [a, b] \times \mathbb{R} \rightarrow \mathbb{R} \) are continuous functions and there exist real constants \( \tilde{k}_i, \tilde{\gamma}_i \geq 0, \ (i = 0, 1) \) and \( \tilde{k}_0 > 0, \tilde{\gamma}_0 > 0 \) such that, \( \forall x, y \in \mathbb{R}, \)

\[
|\tilde{\varphi}(t, y)| \leq \tilde{k}_0 + \tilde{k}_1|y|, \ |\tilde{\psi}(t, x)| \leq \tilde{\gamma}_0 + \tilde{\gamma}_1|x|.
\]

Then the system (4.1) has at least one solution on \([a, b]\) provided that \((M_1 + M_2)\tilde{\gamma}_1 < 1\) and \((L_1 + L_2)\tilde{k}_1 < 1\), where \(L_1, M_1\) and \(L_2, M_2\) are given by (3.1).

**Theorem 4.2.** Let \( \tilde{\varphi}, \tilde{\psi} : [a, b] \times \mathbb{R} \rightarrow \mathbb{R} \) be continuous functions and there exist positive constants \( \tilde{l}_1 \) and \( \tilde{l}_2 \) such that, for all \( t \in [a, b] \) and \( x_i, y_i \in \mathbb{R}, \ i = 1, 2, \)

\[
|\tilde{\varphi}(t, x_1) - \tilde{\varphi}(t, y_1)| \leq \tilde{l}_1|x_1 - y_1|, \quad |\tilde{\psi}(t, x_1) - \tilde{\psi}(t, y_1)| \leq \tilde{l}_2|x_1 - y_1|.
\]

If \((L_1 + L_2)\tilde{l}_1 + (M_1 + M_2)\tilde{l}_2 < 1\), where \(L_1, M_1\) and \(L_2, M_2\) are given by (3.1) then the system (4.1) has a unique solution on \([a, b]\).

5. Conclusions

We studied the solvability of a coupled system of nonlinear fractional differential equations of different orders supplemented with a new set of nonlocal multi-point integral boundary conditions on an arbitrary domain by applying the tools of modern functional analysis. We also presented the existence results for a variant of the given problem containing the nonlinearities depending on the cross-variables (unknown functions). Our results are new not only in the given configuration but also yield some new results by specializing the parameters involved in the problems at hand. For example, by taking \( \delta_i = 0, \ i = 1, 2, \ldots, m \) in the obtained results, we obtain the ones associated with the coupled systems of fractional differential equations in (1.1) and (4.1) subject to the boundary conditions:

\[
p x(a) + q y(b) = y_0 + \int_a^b (x(s) + y(s))ds, \ y(a) = 0, \ y'(b) = \lambda \int_t^b x(s)ds.
\]

For \( \lambda = 0 \), our results correspond to the boundary conditions of the form:

\[
p x(a) + q y(b) = y_0 + \int_a^b (x(s) + y(s))ds, \ y(a) = 0, \ y'(b) = \sum_{i=1}^m \delta_i x(\sigma_i). \quad (5.1)
\]

Furthermore, the methods employed in this paper can be used to solve the systems involving fractional integro-differential equations and multi-term fractional differential equations complemented with the boundary conditions considered in the problem (1.1).

**Conflict of interest**

All authors declare no conflicts of interest in this paper.
Acknowledgements

This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (KEP-PhD-41-130-41). The authors, therefore, acknowledge with thanks DSR technical and financial support. The authors also thank the reviewers for their useful suggestions on our work.

References

1. A. V. Bicadze, A. A. Samarskiĭ, Some elementary generalizations of linear elliptic boundary value problems, *Dokl. Akad. Nauk SSSR*, 185 (1969), 739–740.
2. J. Andres, A four-point boundary value problem for the second-order ordinary differential equations, *Arch. Math.*, 53 (1989), 384–389.
3. P. W. Eloe, B. Ahmad, Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions, *Appl. Math. Lett.*, 18 (2005), 521–527.
4. J. R. L. Webb, G. Infante, Positive solutions of nonlocal boundary value problems: A unified approach, *J. Lond. Math. Soc.*, 74 (2006), 673–693.
5. J. R. Graef, J. R. L. Webb, Third order boundary value problems with nonlocal boundary conditions, *Nonlinear Anal. Theor.*, 71 (2009), 1542–1551.
6. M. Feng, X. Zhang, W. Ge, Existence theorems for a second order nonlinear differential equation with nonlocal boundary conditions and their applications, *J. Appl. Math. Comput.*, 33 (2010), 137–153.
7. L. Zheng, X. Zhang, *Modeling and analysis of modern fluid problems*, Academic Press, 2017.
8. J. R. Cannon, The solution of the heat equation subject to the specification of energy, *Q. Appl. Math.*, 21 (1963), 155–160.
9. N. I. Ionkin, The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition, *Differentsial’nye Uravnienia*, 13 (1977), 294–304.
10. R. Yu. Chegis, Numerical solution of a heat conduction problem with an integral condition, *Liet. Mat. Rink.*, 24 (1984), 209–215.
11. C. Taylor, T. Hughes, C. Zarins, Finite element modeling of blood flow in arteries, *Comput. Method. Appl. M.*, 158 (1998), 155–196.
12. J. R. Womersley, Method for the calculation of velocity, rate of flow and viscous drag in arteries when the pressure gradient is known, *J. Physiol.*, 127 (1955), 553–563.
13. F. Nicoud, T. Schfonfeld, Integral boundary conditions for unsteady biomedical CFD applications, *Int. J. Numer. Meth. Fl.*, 40 (2002), 457–465.
14. R. Ćiegis, A. Bugajew, Numerical approximation of one model of the bacterial self-organization, *Nonlinear Anal. Model.*, 17 (2012), 253–270.
15. B. Ahmad, A. Alsaedi, Existence of approximate solutions of the forced Duffing equation with discontinuous type integral boundary conditions, *Nonlinear Anal. Real*, 10 (2009), 358–367.
16. A. Boucherif, Second-order boundary value problems with integral boundary conditions, *Nonlinear Anal. Theor.*, 70 (2009), 364–371.

17. M. Boukrouche, D. A. Tarzia, A family of singular ordinary differential equations of the third order with an integral boundary condition, *Bound. Value Probl.*, 2018 (2018), 1–11.

18. J. Henderson, Smoothness of solutions with respect to multi-strip integral boundary conditions for nth order ordinary differential equations, *Nonlinear Anal. Model.*, 19 (2014), 396–412.

19. J. R. L. Webb, G. Infante, Positive solutions of nonlocal boundary value problems involving integral conditions, *NoDEA, Nonlinear Diff.*, 15 (2008), 45–67.

20. G. S. Wang, A. F. Blom, A strip model for fatigue crack growth predictions under general load conditions, *Eng. Fract. Mech.*, 40 (1991), 507–533.

21. B. Ahmad, T. Hayat, Diffraction of a plane wave by an elastic knife-edge adjacent to a rigid strip, *Canad. Appl. Math. Quart.*, 9 (2001), 303–316.

22. T. V. Renterghem, D. Botteldooren, K. Verheyen, Road traffic noise shielding by vegetation belts of limited depth, *J. Sound Vib.*, 331 (2012), 2404–2425.

23. E. Yusufoglu, I. Turhan, A mixed boundary value problem in orthotropic strip containing a crack, *J. Franklin I.*, 349 (2012), 2750–2769.

24. M. S. Ali, G. Narayanan, V. Shekher, A. Alsaeedi, B. Ahmad, Global Mittag-Leffler stability analysis of impulsive fractional-order complex-valued BAM neural networks with time varying delays, *Commun. Nonlinear Sci.*, 83 (2020), 105088.

25. Y. Xu, Y. Li, W. Li, Adaptive finite-time synchronization control for fractional-order complex-valued dynamical networks with multiple weights, *Commun. Nonlinear Sci.*, 85 (2020), 105239.

26. Y. Ding, Z. Wang, H. Ye, Optimal control of a fractional-order HIV-immune system with memory, *IEEE T. Contr. Syst. T.*, 20 (2012), 763–769.

27. Y. Xu, W. Li, Finite-time synchronization of fractional-order complex-valued coupled systems, *Physica A*, 549 (2020), 123903.

28. F. Zhang, G. Chen C. Li, J. Kurths, Chaos synchronization in fractional differential systems, *Philos. T. R. Soc. A*, 371 (2013), 201201553.

29. L. Xu, X. Chu, H. Hu, Quasi-synchronization analysis for fractional-order delayed complex dynamical networks, *Math. Comput. Simulat.*, 185 (2021), 594–613.

30. X. Chu, L. Xu, H. Hu, Exponential quasi-synchronization of conformable fractional-order complex dynamical networks, *Chaos Soliton. Fract.*, 140 (2020), 110268.

31. V. J. Ervin, N. Heuer, J. P. Roop, Regularity of the solution to 1–D fractional order diffusion equations, *Math. Comput.*, 87 (2018), 2273–2294.

32. H. Wang, X. Zheng, Well posedness and regularity of the variable-order time-fractional diffusion equations, *J. Math. Anal. Appl.*, 475 (2019), 1778–1802.

33. X. Zheng, H. Wang, An error estimate of a numerical approximation to a hidden-memory variable-order space-time fractional diffusion equation, *SIAM J. Numer. Anal.*, 58 (2020), 2492–2514.

34. H. A. Fallahgoul, S. M. Focardi, F. J. Fabozzi, *Fractional Calculus and Fractional Processes with Applications to Financial Economics, Theory and Application*, Academic Press, 2017.
35. M. Javidi, B. Ahmad, Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton-zooplankton system, *Ecol. Model.*, **318** (2015), 8–18.

36. Z. Ming, G. Zhang, H. Li, Positive solutions of a derivative dependent second-order problem subject to Stieltjes integral boundary conditions, *Electron. J. Qual. Theo.*, **2019** (2019), 1–15.

37. Y. Wang, S. Liang, Q. Wang, Existence results for fractional differential equations with integral and multi-point boundary conditions, *Bound. Value Probl.*, **2018** (2018), 1–11.

38. Z. Cen, L. B. Liu, J. Huang, A posteriori error estimation in maximum norm for a two-point boundary value problem with a Riemann-Liouville fractional derivative, *Appl. Math. Lett.*, **102** (2020), 106086.

39. G. Iskenderoglu, D. Kaya, Symmetry analysis of initial and boundary value problems for fractional differential equations in Caputo sense, *Chaos Soliton. Fract.*, **134** (2020), 109684.

40. C. S. Goodrich, Coercive nonlocal elements in fractional differential equations, *Positivity*, **21** (2017), 377–394.

41. B. Ahmad, Y. Alruwaily, A. Alsaedi, J. J. Nieto, Fractional integro-differential equations with dual anti-periodic boundary conditions, *Differ. Integral Equ.*, **33** (2020), 181–206.

42. D. Ben-Avraham, S. Havlin, *Diffusion and Reactions in Fractals and Disordered Systems*, Cambridge University Press, 2000.

43. A. S. Deshpande, V. Daftardar-Gejji, On disappearance of chaos in fractional systems, *Chaos Soliton. Fract.*, **102** (2017), 119–126.

44. S. Wang, M. Xu, Axial Couette flow of two kinds of fractional viscoelastic fluids in an annulus, *Nonlinear Anal. Real*, **10** (2009), 1087–1096.

45. L. Xu, X. Chu, H. Hu, Exponential ultimate boundedness of non-autonomous fractional differential systems with time delay and impulses, *Appl. Math. Lett.*, **99** (2020), 106000.

46. D. He, L. Xu, Exponential stability of impulsive fractional switched systems with time delays, *IEEE T. Circuits-II*, 2020, doi: 10.1109/TCSII.2020.3037654.

47. B. Ahmad, N. Alghamdi, A. Alsaedi, S. K. Ntouyas, A system of coupled multi-term fractional differential equations with three-point coupled boundary conditions, *Fract. Calc. Appl. Anal.*, **22** (2019), 601–618.

48. B. Ahmad, S. K. Ntouyas, A. Alsaedi, Fractional order differential systems involving right Caputo and left Riemann-Liouville fractional derivatives with nonlocal coupled conditions, *Bound. Value Probl.*, **2019** (2019), 1–12.

49. B. Ahmad, A. Alsaedi, S. K. Ntouyas, Fractional order nonlinear mixed coupled systems with coupled integro-differential boundary conditions, *J. Appl. Anal. Comput.*, **10** (2020), 892–903.

50. B. Ahmad, A. Alsaedi, Y. Alruwaily, S. K. Ntouyas, Nonlinear multi-term fractional differential equations with Riemann-Stieltjes integro-multipoint boundary conditions, *AIMS Mathematics*, **5** (2020), 1446–1461.

51. J. Henderson, R. Luca, A. Tudorache, On a system of fractional differential equations with coupled integral boundary conditions, *Fract. Calc. Appl. Anal.*, **18** (2015), 361–386.
52. S. K. Ntouyas, H. H. Al-Sulami, A study of coupled systems of mixed order fractional differential equations and inclusions with coupled integral fractional boundary conditions, *Adv. Differ. Equ.*, 2020 (2020), 1–21.

53. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.

54. A. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.