The Binary $\text{aff}(n|1)$-Invariant Differential Operators On Weighted Densities On The Superspace $\mathbb{R}^{1|n}$ And $\text{aff}(n|1)$-Relative Cohomology

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Abstract

We consider the $\text{aff}(n|1)-$module structure on the spaces of differential bilinear operators acting on the superspaces of weighted densities. We classify $\text{aff}(n|1)-$invariant binary differential operators acting on the spaces of weighted densities. This result allows us to compute the first $\text{aff}(n|1)-$relative differential cohomology of $\mathcal{K}(n)$ with coefficients in the superspace of linear differential operators acting on the superspaces of weighted densities.

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1 Introduction

Let $\text{vect}(1)$ be the Lie algebra of polynomial vector fields on $\mathbb{R}$. Consider the 1–parameter deformation of the $\text{vect}(1)-$action on $\mathbb{R}[x]$: $L^\lambda_{\frac{d}{dx}}(f) = Xf' + \lambda Xf$, where $X, f \in \mathbb{R}[x]$ and $X' := \frac{dX}{dx}$. Denote by $\mathcal{F}_\lambda$ the $\text{vect}(1)-$module structure on $\mathbb{R}[x]$ defined by $L^\lambda$ for a fixed $\lambda$. Geometrically, $\mathcal{F}_\lambda = \{f dx^\lambda \mid f \in \mathbb{R}[x]\}$ is the space of polynomial weighted densities of weight $\lambda \in \mathbb{R}$. The space $\mathcal{F}_\lambda$ coincides with the space of vector fields, functions and differential 1–forms for $\lambda = -1, 0$ and 1, respectively.

Denote by $D_{\lambda,\mu} := \text{Hom}_{\text{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$ the $\text{vect}(1)-$module of linear differential operators with the natural $\text{vect}(1)-$action. Feigin and Fuchs [6] computed $H_{\text{diff}}^1(\text{vect}(1); D_{\lambda,\mu})$, where $H_{\text{diff}}^*$ denotes the differential cohomology; that is, only cochains given by differential operators are considered. They showed that non-zero cohomology $H_{\text{diff}}^1(\text{vect}(1); D_{\lambda,\mu})$ only appear for particular values of weights that we call resonant and which satisfy $\mu - \lambda \in \mathbb{N}$.

If we restrict ourselves to the Lie subalgebra of $\text{vect}(1)$ generated by $\{\frac{d}{dx}, x \frac{d}{dx}\}$, isomorphic to $\text{aff}(1)$, we get a family of infinite dimensional $\text{aff}(1)-$modules, still denoted $\mathcal{F}_\lambda$ and $D_{\lambda,\mu}$.

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In [3], Basdouri and Nasri classified all \(\text{aff}(1)\)--invariant binary differential operators on \(\mathbb{R}\) acting in the spaces \(F_\lambda\). This allows us to show, in this paper, that nonzero cohomology \(H^1_{\text{diff}}(\text{vect}(1), \text{aff}(1); D_{\lambda,\mu})\) only appear for particular values of weights which satisfy \(\mu - \lambda \in \mathbb{N}\) and give explicit basis of this cohomology space. This space arises in the classification of \(\text{aff}(1)\)--trivial infinitesimal deformations of the \(\text{vect}(1)\)--module \(S_{\mu-\lambda} = \bigoplus_{k=0}^{\infty} F_{\mu-\lambda-k}\), the space of symbols of \(D_{\lambda,\mu}\).

In this paper we study also the simplest super analog of this problem. Namely, we consider the superspace \(\mathbb{R}^{1|n}\) equipped with the contact structure determined by a 1-form \(\alpha\), and the Lie superalgebra \(K(n)\) of contact polynomial vector fields on \(\mathbb{R}^{1|n}\). We introduce the \(K(n)\)--module \(\mathbb{F}_\lambda^n\) of \(\lambda\)-densities on \(\mathbb{R}^{1|n}\) and the \(K(n)\)--module of linear differential operators, \(D_{\lambda,\mu}^n := \text{Hom}_{\text{diff}}(\mathbb{F}_\lambda^n, \mathbb{F}_\mu^n)\), which are super analogues of the spaces \(F_\lambda\) and \(D_{\lambda,\mu}\), respectively. The Lie superalgebra \(\text{aff}(n|1)\), a super analogue of \(\text{aff}(1)\), can be realized as a subalgebra of \(K(n)\). We classify all \(\text{aff}(n|1)\)--invariant binary differential operators on \(\mathbb{R}^{1|n}\) acting in the spaces \(\mathbb{F}_\lambda^n\) for \(n = 1\) and \(2\). We use this result to compute \(H^1_{\text{diff}}(K(n), \text{aff}(n|1); D_{\lambda,\mu}^n)\) for \(n = 1\) and \(2\). We show that nonzero cohomology \(H^1_{\text{diff}}(K(n), \text{aff}(n|1); D_{\lambda,\mu}^n)\) only appear for resonant values of weights that satisfy \(\mu - \lambda \in \frac{1}{2}\mathbb{N}\). Moreover, we give explicit basis of these cohomology spaces.

## 2 Definitions and Notations

### 2.1 The Lie superalgebra of contact vector fields on \(\mathbb{R}^{1|n}\)

Let \(\mathbb{R}^{1|n}\) be the superspace with coordinates \((x, \theta_1, \ldots, \theta_n)\), where \(\theta_1, \ldots, \theta_n\) are the odd variables, equipped with the standard contact structure given by the following 1-form:

\[
\alpha_n = dx + \sum_{i=1}^{n} \theta_i d\theta_i. \quad (2.1)
\]

On the space \(\mathbb{R}[x, \theta] := \mathbb{R}[x, \theta_1, \ldots, \theta_n]\), we consider the contact bracket

\[
\{F, G\} = FG' - F'G - \frac{1}{2}(-1)^{|F|} \sum_{i=1}^{n} \eta_i(F) \cdot \eta_i(G), \quad (2.2)
\]

where \(\eta_i = \frac{\partial}{\partial \theta_i} - \theta_i \frac{\partial}{\partial x}\) and \(|F|\) is the parity of \(F\). Note that the derivations \(\eta_i\) are the generators of \(n\)-extended supersymmetry and generate the kernel of the form (2.1) as a module over the ring of polynomial functions. Let \(\text{Vect}_{\text{Pol}}(\mathbb{R}^{1|n})\) be the superspace of polynomial vector fields on \(\mathbb{R}^{1|n}\):

\[
\text{Vect}_{\text{Pol}}(\mathbb{R}^{1|n}) = \left\{ F_0 \partial_x + \sum_{i=1}^{n} F_i \partial_{\theta_i} \mid F_i \in \mathbb{R}[x, \theta] \text{ for all } i \right\},
\]

where \(\partial_i = \frac{\partial}{\partial \theta_i}\) and \(\partial_x = \frac{\partial}{\partial x}\), and consider the superspace \(K(n)\) of contact polynomial vector fields on \(\mathbb{R}^{1|n}\). That is, \(K(n)\) is the superspace of vector fields on \(\mathbb{R}^{1|n}\) preserving the distribution singled out by the 1-form \(\alpha_n\):

\[
K(n) = \left\{ X \in \text{Vect}_{\text{Pol}}(\mathbb{R}^{1|n}) \mid \text{there exists } F \in \mathbb{R}[x, \theta] \text{ such that } L_X(\alpha_n) = F \alpha_n \right\}.
\]
The Lie superalgebra $K(n)$ is spanned by the fields of the form:

$$X_F = F\partial_x - \frac{1}{2} \sum_{i=1}^{n} (-1)^{|F|}\eta_i(F)\eta_i,$$

where $F \in \mathbb{R}[x, \theta]$.

In particular, we have $K(0) = \text{vect}(1)$. Observe that $L_{X_F}(\alpha_n) = X_1(F)\alpha_n$. The bracket in $K(n)$ can be written as:

$$[X_F, X_G] = X_{\{F, G\}}.$$

2.2 The superalgebra $\text{aff}(1|1)$

The Lie algebra $\text{aff}(1)$ is isomorphic to the Lie subalgebra of $\text{vect}(1)$ generated by

$$\left\{ \frac{d}{dx}, x \frac{d}{dx} \right\}.$$

Similarly, we consider the affine Lie superalgebra ([2])

$$\text{aff}(1|1) = \text{Span}(X_1, X_x, X_\theta),$$

where

$$\text{(aff}(1|1)_0 = \text{Span}(X_1, X_x) \text{ and } (\text{aff}(1|1))_1 = \text{Span}(X_\theta).$$

The new commutation relations are

$$[X_1, X_x] = X_1, \quad [X_x, X_\theta] = -\frac{1}{2}X_\theta, \quad [X_1, X_\theta] = 0, \quad [X_\theta, X_\theta] = \frac{1}{2}X_1.$$

More generally, the affine Lie superalgebra $\text{aff}(n|1)$ can be realized as a subalgebra of $K(n)$:

$$\text{aff}(n|1) = \text{Span}(X_1, X_x, X_{\theta_1}, \ldots, X_{\theta_i}, \ldots) \quad 1 \leq i, j \leq n.$$ 

The Lie superalgebra $\text{aff}(n-1|1)$ can be realized as a subalgebra of $\text{aff}(n|1)$:

$$\text{aff}(n-1|1) = \left\{ X_F \in \text{aff}(n|1) \mid \partial_n F = 0 \right\}.$$

Note that, for any $i$ in $\{1, 2, \ldots, n-1\}$, $\text{aff}(n-1|1)$ is isomorphic to

$$\text{aff}(n-1|1)_i = \left\{ X_F \in \text{aff}(n|1) \mid \partial_i F = 0 \right\}.$$

2.3 Modules of weighted densities

We introduce a one-parameter family of modules over the Lie superalgebra $K(n)$. As vector spaces all these modules are isomorphic to $\mathbb{R}[x, \theta]$, but not as $K(n)-$modules.

For every contact polynomial vector field $X_F$, define a one-parameter family of first-order differential operators on $\mathbb{R}[x, \theta]$:

$$L^\lambda_{X_F} = X_F + \lambda F', \quad \lambda \in \mathbb{R}. \quad (2.3)$$
We easily check that
\[ [L^\lambda_{X_F}, L^\lambda_{X_G}] = L^\lambda_{[X_F,X_G]} \]  \quad (2.4)
We thus obtain a one-parameter family of \( K(n) \)-modules on \( \mathbb{R}[x,\theta] \) that we denote \( F^n_\lambda \), the space of all polynomial weighted densities on \( \mathbb{R}^{1|n} \) of weight \( \lambda \) with respect to \( \alpha_n \):
\[
F^n_\lambda = \left\{ F\alpha^n_\lambda \mid F \in \mathbb{R}[x,\theta] \right\}.
\]  \quad (2.5)
In particular, we have \( F^0_\lambda = \mathcal{F}_\lambda \). Obviously the adjoint \( K(n) \)-module is isomorphic to the space of weighted densities on \( \mathbb{R}^{1|n} \) of weight \(-1\).

2.4 Differential operators on weighted densities
A differential operator on \( \mathbb{R}^{1|n} \) is an operator on \( \mathbb{R}[x,\theta] \) of the form:
\[
A = \sum_{k=0}^{M} \sum_{\varepsilon=\varepsilon_1,\ldots,\varepsilon_n} a_{k,\varepsilon}(x,\theta) \partial_{\varepsilon}^k \cdot \partial_{\varepsilon_{n}}^1 \cdots \partial_{\varepsilon_{n}}^{\varepsilon_n}; \quad \varepsilon_i = 0, 1; \quad M \in \mathbb{N}.
\]  \quad (2.6)
Of course any differential operator defines a linear mapping \( F\alpha^n_\lambda \mapsto (AF)\alpha^n_\mu \) from \( F^n_\lambda \) to \( F^n_\mu \) for any \( \lambda, \mu \in \mathbb{R} \), thus the space of differential operators becomes a family of \( K(n) \)-modules \( \mathbb{D}^n_{\lambda,\mu} \) for the natural action:
\[
X_F \cdot A = L^n_{X_F} \circ A - (-1)^{|A||F|} A \circ L^n_{X_F},
\]  \quad (2.7)
Similarly, we consider a multi-parameter family of \( K(n) \)-modules on the space \( \mathbb{D}^n_{\lambda_1,\ldots,\lambda_m;\mu} \) of multi-linear differential operators: \( A : F^n_{\lambda_1} \otimes \cdots \otimes F^n_{\lambda_m} \rightarrow F^n_\mu \) with the natural \( K(n) \)-action:
\[
X_F \cdot A = L^n_{X_F} \circ A - (-1)^{|A||F|} A \circ L^n_{X_F} \otimes \cdots \otimes L^n_{X_F},
\]  \quad (2.8)
where \( L^n_{X_F} \otimes \cdots \otimes L^n_{X_F} \) is defined by the Leibnitz rule. We also consider the \( K(n) \)-module \( \Pi \left( \mathbb{D}^n_{\lambda_1,\ldots,\lambda_m;\mu} \right) \) with the \( K(n) \)-action (\( \Pi \) is the change of parity operator):
\[
X_F \cdot \Pi(A) = \Pi \left( L^n_{X_F} \circ A - (-1)^{|A||F|} A \circ L^n_{X_F} \otimes \cdots \otimes L^n_{X_F} \right).
\]  \quad (2.9)
Since \( -\eta^2_0 = \partial_x \) and \( \partial_i = \eta_i - \partial_{\eta^2_i} \), every differential operator \( A \in \mathbb{D}^n_{\lambda,\mu} \) can be expressed in the form
\[
A(F\alpha^n_\lambda) = \sum_{\ell=(\ell_1,\ldots,\ell_n)} a_\ell(x,\theta)\eta^\ell_1 \cdots \eta^\ell_n(F)\alpha^n_\lambda,
\]  \quad (2.10)
where the coefficients \( a_\ell(x,\theta) \) are arbitrary polynomial functions.

The Lie superalgebra \( K(n-1) \) can be realized as a subalgebra of \( K(n) \):
\[
K(n-1) = \left\{ X_F \in K(n) \mid \partial_n F = 0 \right\}.
\]
Therefore, \( \mathbb{D}^n_{\lambda_1,\ldots,\lambda_m;\mu} \) and \( F^n_\lambda \) are \( K(n-1) \)-modules. Note that, for any \( i \) in \( \{1, 2, \ldots, n-1\} \), \( K(n-1) \) is isomorphic to
\[
K(n-1)^i = \left\{ X_F \in K(n) \mid \partial_i F = 0 \right\}.
\]
Proposition 2.1. As a $\text{aff}(n-1|1)$–module, we have

$$D_{\lambda,\mu,\nu}^n \cong \tilde{D}_{\lambda,\mu,\nu}^{n-1} := D_{\lambda,\mu,\nu}^{n-1} + D_{\lambda,\mu+\frac{1}{2},\nu}^{n-1} + D_{\lambda,\mu+\frac{1}{2},\nu+\frac{1}{2}}^{n-1} + D_{\lambda+\frac{1}{2},\mu,\nu+\frac{1}{2}}^{n-1} + D_{\lambda+\frac{1}{2},\mu+\frac{1}{2},\nu+\frac{1}{2}}^{n-1}.$$ \hfill (2.9)

Proof. For any $F \in \mathbb{R}[x, \theta]$, we write

$$F = F_1 + F_2 \theta_n \quad \text{where} \quad \partial_n F_1 = \partial_n F_2 = 0$$

and we prove that

$$L^{\lambda \mu}_{X\mu}F = L^{\lambda \mu}_{X\mu}F_1 + (L^{\lambda+\frac{1}{2} \mu}_{X\mu}F_2)\theta_n.$$ 

Thus, it is clear that the map

$$\varphi_{\lambda} : \mathbb{F}_{\lambda}^n \rightarrow \mathbb{F}_{\lambda}^{n-1} \oplus \Pi(\mathbb{F}_{\lambda+\frac{1}{2}}^{n-1}),$$ \hfill (2.10)

$$F \alpha_{\mu}^{n-1} \mapsto (F \alpha_{\mu}^{n-1}, \Pi(F \alpha_{\mu+\frac{1}{2}}^{n-1})),$$

is $\text{aff}(n-1|1)$–isomorphism. So, we get the natural $\text{aff}(n-1|1)$–isomorphism from $\mathbb{F}_{\lambda}^n \otimes \mathbb{F}_{\mu}^n$ to

$$\mathbb{F}_{\lambda}^{n-1} \otimes \mathbb{F}_{\mu}^{n-1} \oplus \mathbb{F}_{\lambda}^{n-1} \otimes \Pi(\mathbb{F}_{\mu+\frac{1}{2}}^{n-1}) \oplus \Pi(\mathbb{F}_{\mu+\frac{1}{2}}^{n-1}) \oplus \Pi(\mathbb{F}_{\mu+\frac{1}{2}}^{n-1})$$

denoted $\psi_{\lambda,\mu}$. Therefore, we deduce a $\text{aff}(n-1|1)$–isomorphism:

$$\Psi_{\lambda,\mu,\nu} : \tilde{D}_{\lambda,\mu,\nu}^{n-1} \rightarrow \mathbb{D}_{\lambda,\mu,\nu}^n,$$

$$A \mapsto \varphi_{\nu}^{-1} \circ A \circ \psi_{\lambda,\mu}.$$ \hfill (2.11)

Here, we identify the $\text{aff}(n-1|1)$–modules via the following isomorphisms:

$$\Pi \left( D_{\lambda,\mu,\nu}^{n-1} \right) \rightarrow \text{Hom}_{\text{diff}}(\mathbb{F}_{\lambda}^{n-1} \otimes \mathbb{F}_{\mu}^{n-1}, \Pi(\mathbb{F}_{\nu}^{n-1})), \quad \Pi(A) \mapsto \Pi \circ A,$$

$$\Pi \left( D_{\lambda+\frac{1}{2},\mu,\nu}^{n-1} \right) \rightarrow \text{Hom}_{\text{diff}}(\mathbb{F}_{\lambda}^{n-1} \otimes \Pi(\mathbb{F}_{\mu+\frac{1}{2}}^{n-1}), \mathbb{F}_{\nu}^{n-1}), \quad \Pi(A) \mapsto A \circ (1 \otimes \Pi),$$

$$\Pi \left( D_{\lambda,\mu+\frac{1}{2},\nu}^{n-1} \right) \rightarrow \text{Hom}_{\text{diff}}(\mathbb{F}_{\lambda}^{n-1} \otimes \Pi(\mathbb{F}_{\mu}^{n-1}), \mathbb{F}_{\nu}^{n-1}), \quad \Pi(A) \mapsto A \circ (\Pi \otimes \sigma),$$

$$\Pi \left( D_{\lambda,\mu+\frac{1}{2},\nu+\frac{1}{2}}^{n-1} \right) \rightarrow \text{Hom}_{\text{diff}}(\mathbb{F}_{\lambda}^{n-1} \otimes \Pi(\mathbb{F}_{\mu+\frac{1}{2}}^{n-1}), \Pi(\mathbb{F}_{\nu+\frac{1}{2}}^{n-1})), \quad \Pi(A) \mapsto A \circ (\Pi \otimes \sigma \circ \Pi),$$

$$\Pi \left( D_{\lambda+\frac{1}{2},\mu+\frac{1}{2},\nu}^{n-1} \right) \rightarrow \text{Hom}_{\text{diff}}(\mathbb{F}_{\lambda}^{n-1} \otimes \Pi(\mathbb{F}_{\mu+\frac{1}{2}}^{n-1}), \Pi(\mathbb{F}_{\nu}^{n-1})), \quad A \mapsto \Pi \circ A \circ (1 \otimes \Pi),$$

$$\Pi \left( D_{\lambda+\frac{1}{2},\mu+\frac{1}{2},\nu+\frac{1}{2}}^{n-1} \right) \rightarrow \text{Hom}_{\text{diff}}(\mathbb{F}_{\lambda}^{n-1} \otimes \Pi(\mathbb{F}_{\mu+\frac{1}{2}}^{n-1}), \Pi(\mathbb{F}_{\nu+\frac{1}{2}}^{n-1})), \quad A \mapsto A \circ (\Pi \otimes \sigma \circ \Pi),$$

where $\lambda' = \lambda + \frac{1}{2}$, $\mu' = \mu + \frac{1}{2}$, $\nu' = \nu + \frac{1}{2}$ and $\sigma(F) = (-1)^{|F|}F$. \hfill \boxed{\phantom{00000000}}
3 \( \mathfrak{aff}(1) \)– and \( \mathfrak{aff}(1|1) \)–invariant bilinear differential operators

In this section we will investigate \( \mathfrak{aff}(1) \)– and \( \mathfrak{aff}(1|1) \)–invariant bilinear differential operators on tensor densities. These results will be useful for the computation of cohomology.

**Proposition 3.1.** [3] There exist only the \( \mathfrak{aff}(1) \)–invariant bilinear differential operators

\[
J^\tau_{k} : \mathcal{F}_{\tau} \otimes \mathcal{F}_{\lambda} \rightarrow \mathcal{F}_{\tau+\lambda+k}, \quad (\varphi dx^\tau, \phi dx^\lambda) \mapsto J^\tau_{k}(\varphi, \phi)dx^{\tau+\lambda+k}
\]

given by

\[
J^\tau_{k}(\varphi, \phi) = \sum_{0 \leq i, j, i+j = k} c_{i,j}^\tau \varphi^{(i)} \phi^{(j)},
\]

where \( k \in \mathbb{N} \) and the coefficients \( c_{i,j}^\tau \) are constants.

**Theorem 3.2.** There are only the following \( \mathfrak{aff}(1|1) \)–invariant bilinear differential operators acting in the spaces \( \mathbb{F}^1_{\lambda} \):

\[
\mathbb{J}^\tau_{k}^{\lambda,1} : \mathbb{F}^1_{\tau} \otimes \mathbb{F}^1_{\lambda} \rightarrow \mathbb{F}^1_{\tau+\lambda+k} \rightarrow \mathbb{F}^1_{\tau+\lambda+k} \rightarrow \mathbb{F}^1_{\tau+\lambda+k},
\]

where \( k \in \frac{1}{2}\mathbb{N} \). The operators \( \mathbb{J}^\tau_{k}^{\lambda,1} \) labeled by semi-integer \( k \) are odd; they are given by

\[
\mathbb{J}^\tau_{k}^{\lambda,1}(F, G) = \sum_{i+j=[k]} \left( \Gamma_{i,j}^{\tau,\lambda,1} F^{(i)} G^{(j)} + \tilde{\Gamma}_{i,j}^{\tau,\lambda,1} (-1)^{|F|} \eta_{1}(F^{(i)}) \eta_{1}(G^{(j)}) \right).
\]

(3.12)

The operators \( \mathbb{J}^\tau_{k}^{\lambda,1} \), where \( k \in \mathbb{N} \), are even; set \( \mathbb{J}^\tau_{0}^{\lambda,1}(F, G) = FG \) and

\[
\mathbb{J}^\tau_{k}^{\lambda,1}(F, G) = \sum_{i+j=k} \Psi_{i,j}^{\tau,\lambda,1} F^{(i)} G^{(j)} + \sum_{i+j=k+1} \tilde{\Psi}_{i,j}^{\tau,\lambda,1} (-1)^{|F|} \eta_{1}(F^{(i)}) \eta_{1}(G^{(j)}),
\]

(3.13)

where \( [k] \) denotes the integer part of \( k \), \( k > 0 \), and \( \Gamma_{i,j}^{\tau,\lambda,1} \), \( \tilde{\Gamma}_{i,j}^{\tau,\lambda,1} \), \( \Psi_{i,j}^{\tau,\lambda,1} \) and \( \tilde{\Psi}_{i,j}^{\tau,\lambda,1} \) are constants.

**Proof.** Let \( \mathbb{T}^1 : \mathbb{F}^1_{\tau} \otimes \mathbb{F}^1_{\lambda} \rightarrow \mathbb{F}^1_{\mu} \) be an \( \mathfrak{aff}(1|1) \)–invariant differential operator. Using the fact that, as \( \text{vect}(1) \)–modules,

\[
\mathbb{F}^1_{\tau} \otimes \mathbb{F}^1_{\lambda} \simeq \mathcal{F}_{\tau} \otimes \mathcal{F}_{\lambda} \oplus \Pi(\mathcal{F}_{\tau+\frac{1}{2}} \otimes \mathcal{F}_{\lambda+\frac{1}{2}}) \oplus \mathcal{F}_{\tau} \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \oplus \Pi(\mathcal{F}_{\tau+\frac{1}{2}}) \otimes \mathcal{F}_{\lambda}
\]

(3.14)

and

\[
\mathbb{F}^1_{\mu} \simeq \mathcal{F}_{\mu} \oplus \Pi(\mathcal{F}_{\mu+\frac{1}{2}}),
\]

we can deduce that the restriction of \( \mathbb{T}^1 \) to each component of the right hand side of (3.14) is \( \mathfrak{aff}(1) \)–invariant. So, the parameters \( \tau, \lambda \) and \( \mu \) must satisfy

\[
\mu = \lambda + \tau + k, \quad \text{where} \quad k \in \frac{1}{2}\mathbb{N}.
\]

The corresponding operators will be denoted \( \mathbb{J}^\tau_{k}^{\lambda,1} \). Obviously, if \( k \) is integer, then the operator \( \mathbb{J}^\tau_{k}^{\lambda,1} \) is even and its restriction to each component of the right hand side of (3.14) coincides (up to a scalar factor) with the respective \( \mathfrak{aff}(1) \)–invariant operators:
operators: $c$ and if right hand side of (3.14) coincides (up to a scalar factor) with the respective 

$$J_k: F_{\tau} \otimes F_\lambda \rightarrow F_\mu, \quad J_{k+1}^{\tau+\frac{1}{2},\lambda+\frac{1}{2}}: \Pi(F_{\tau+\frac{1}{2}}) \otimes \Pi(F_{\lambda+\frac{1}{2}}) \rightarrow F_\mu, \quad J_{k+1}^{\tau+\frac{1}{2},\lambda+\frac{1}{2}}: F_{\tau} \otimes \Pi(F_{\lambda+\frac{1}{2}}) \rightarrow \Pi(F_{\mu+\frac{1}{2}}). \quad (3.15)$$

If $k$ is semi-integer, then the operator $J_k^{\tau,\lambda}$ is odd and its restriction to each component of the right hand side of (3.14) coincides (up to a scalar factor) with the respective $\text{aff}(1)$–invariant operators:

$$J_{[k]+1}^{\tau,\lambda}: F_{\tau} \otimes F_\lambda \rightarrow \Pi(F_{\mu+\frac{1}{2}}), \quad J_{[k]}^{\tau+\frac{1}{2},\lambda+\frac{1}{2}}: \Pi(F_{\tau+\frac{1}{2}}) \otimes \Pi(F_{\lambda+\frac{1}{2}}) \rightarrow \Pi(F_{\mu+\frac{1}{2}}), \quad J_{[k]}^{\tau+\frac{1}{2},\lambda}: F_{\tau} \otimes \Pi(F_{\lambda+\frac{1}{2}}) \rightarrow F_\mu, \quad J_{[k]}^{\tau+\frac{1}{2},\lambda}: \Pi(F_{\tau+\frac{1}{2}}) \otimes F_\lambda \rightarrow F_\mu. \quad (3.16)$$

More precisely, let $F_{\alpha \tau} \otimes G_{\alpha \lambda} \in \mathbb{P}_1^\tau \otimes \mathbb{P}_1^\lambda$, where $F = f_0 + \theta_1 f_1$ and $G = g_0 + \theta_1 g_1$, with $f_0, f_1, g_0, g_1 \in \mathbb{R}[x]$. Then if $k$ is integer, we have

$$J_k^{\tau,\lambda}(\varphi, \psi) = [a_1 J^{\tau,\lambda}_k(f_0, g_0) + a_2 J^{\tau+\frac{1}{2},\lambda+\frac{1}{2}}_k(f_1, g_1) + \theta_1 \left(a_3 J^{\tau,\lambda}_k(f_0, g_0) + a_4 J^{\tau+\frac{1}{2},\lambda+\frac{1}{2}}_k(f_1, g_1)\right)] \alpha_1^\mu, \quad (3.17)$$

and if $k$ is semi-integer, we have

$$J_k^{\tau,\lambda}(\varphi, \psi) = [b_1 J_{[k]}^{\tau,\lambda+\frac{1}{2}}(f_0, g_1) + b_2 J_{[k]}^{\tau+\frac{1}{2},\lambda}(f_1, g_0) + \theta_1 \left(b_3 J_{[k]+1}^{\tau,\lambda}(f_0, g_0) + b_4 J_{[k]}^{\tau+\frac{1}{2},\lambda+\frac{1}{2}}(f_1, g_1)\right)] \alpha_1^\mu, \quad (3.18)$$

where the $a_i$ and $b_i$ are constants. The invariance of $J_k^{\tau,\lambda}$ with respect to $X_{\theta_1}$ reads:

$$\mathbf{L}^{\lambda}_{X_{\theta_1}} \circ J_k^{\tau,\lambda} = (-1)^{2k} J_k^{\tau,\lambda} \circ \mathbf{L}_{X_{\theta_1}} = 0. \quad (3.19)$$

The formula (3.19) allows us to determine the coefficients $a_i$ and $b_i$. More precisely, the invariance property with respect to $X_{\theta_1}$ yields:

1. If $k$ is integer

$$c_{i,k-i}^{\tau,\lambda+\frac{1}{2}} = c_{i-1,k-i}^{\tau+\frac{1}{2},\lambda+\frac{1}{2}} + c_{i,k-i}^{\tau,\lambda} \quad c_{i,k-i}^{\tau+\frac{1}{2},\lambda} = c_{i,k-i}^{\tau,\lambda} + \frac{1}{2} \left(c_{i-1,k-i}^{\tau+\frac{1}{2},\lambda+\frac{1}{2}} - c_{i,k-i}^{\tau+\frac{1}{2},\lambda+\frac{1}{2}}\right).$$

2. If $k$ is semi-integer

$$c_{i,k-i}^{\tau,\lambda+\frac{1}{2}} = -c_{i,k-i}^{\tau+\frac{1}{2},\lambda} + \frac{1}{2} \left(c_{i-1,k-i}^{\tau+\frac{1}{2},\lambda+\frac{1}{2}} - c_{i,k-i}^{\tau+\frac{1}{2},\lambda+\frac{1}{2}}\right), \quad c_{i+1,k-i-\frac{1}{2}}^{\tau,\lambda+\frac{1}{2}} = -c_{i+1,k-i-\frac{1}{2}}^{\tau+\frac{1}{2},\lambda} + \frac{1}{2} \left(c_{i,k-i-\frac{1}{2}}^{\tau+\frac{1}{2},\lambda+\frac{1}{2}} - c_{i,k-i-\frac{1}{2}}^{\tau+\frac{1}{2},\lambda+\frac{1}{2}}\right), \quad c_{i,k-i-\frac{1}{2}}^{\tau+\frac{1}{2},\lambda} = -c_{i,k-i-\frac{1}{2}}^{\tau,\lambda} + \frac{1}{2} \left(c_{i+1,k-i-\frac{1}{2}}^{\tau+\frac{1}{2},\lambda+\frac{1}{2}} + c_{i,k-i-\frac{1}{2}}^{\tau+\frac{1}{2},\lambda+\frac{1}{2}}\right).$$

where $c_{i,j}^{\tau,\lambda} = 0 \forall i < 0$ or $j < 0$. Therefore, we easily check that $J_k^{\tau,\lambda}$ is expressed as in (3.12-3.13).
4 The \(\text{aff}(2|1)\)–invariant bilinear differential operators

In this section we will investigate \(\text{aff}(2|1)\)–invariant bilinear differential operators on tensor densities. These results allow us to compute \(\text{aff}(2|1)\)–relative cohomology.

**Theorem 4.1.** The space of \(\text{aff}(2|1)\)–invariant bilinear differential operators acting in the spaces \(F^2\): 
\[
\mathbb{J}_k^{\tau,\lambda} : F^2 \otimes F^2 \rightarrow F^2 \quad \mathbb{J}_k^{\tau,\lambda+2} : F^2 \rightarrow F^2 \quad \mathbb{J}_k^{\tau,\lambda+k} : (F^\tau, G^\lambda) \rightarrow \mathbb{J}_k^{\tau,\lambda+k}(F, G) = F_G k = 0,
\]
where \(k \in \frac{1}{2}\mathbb{N}\), is purely even and it is spanned by the operator 
\[
\mathbb{J}_0^{\tau,\lambda}(F, G) = \sum_{i+j=1} \Gamma_{i,j,1}^{\tau,\lambda} F^{(i)} G^{(j)} + \Gamma_{0,0}^{\tau,\lambda} (-1)^{|F|} (\eta_1(F) \eta_1(G) + \eta_2(F) \eta_2(G)) + \sum_{i+j=k-1} \Gamma_{i,j,2}^{\tau,\lambda} (-1)^{|F|} (\eta_1(F) \eta_2(G) - \eta_2(F) \eta_1(G)) + \sum_{i+j=k-1} \Gamma_{i,j,3}^{\tau,\lambda} \eta_1(F) \eta_1(G) + \sum_{i+j=k-1} \Gamma_{i,j,4}^{\tau,\lambda} \eta_2(F) \eta_1(G) + \sum_{i+j=k-1} \Gamma_{i,j,5}^{\tau,\lambda} \eta_1(G) \eta_2(F) + \sum_{i+j=k-1} \Gamma_{i,j,6}^{\tau,\lambda} \eta_2(G) \eta_1(F)
\]
for \(k = 1\), and the operator 
\[
\mathbb{J}_k^{\tau,\lambda+2} = \sum_{i+j=k} \Gamma_{i,j}^{\tau,\lambda+2} F^{(i)} G^{(j)} + \sum_{i+j=k} \Gamma_{i,j}^{\tau,\lambda+2} (-1)^{|F|} (\eta_1(F) \eta_1(G) + \eta_2(F) \eta_2(G)) + \sum_{i+j=k} \Gamma_{i,j}^{\tau,\lambda+2} \eta_1(F) \eta_2(G) + \sum_{i+j=k} \Gamma_{i,j}^{\tau,\lambda+2} \eta_2(F) \eta_1(G) + \sum_{i+j=k} \Gamma_{i,j}^{\tau,\lambda+2} \eta_1(G) \eta_2(F) + \sum_{i+j=k} \Gamma_{i,j}^{\tau,\lambda+2} \eta_2(G) \eta_1(F)
\]
for \(k \geq 2\); where \(\Gamma_{i,j}^{\tau,\lambda,s} \geq 1\), \(s \in \{1, \ldots, 6\}\), are constants.

**Proof.** Let \(T : F^2 \otimes F^2 \rightarrow F^2 \) be an \(\text{aff}(2|1)\)–invariant bilinear differential operator. Observe that the \(\text{aff}(2|1)\)–invariance of \(T\) is equivalent to invariance with respect to \(X_{\theta_1, \theta_2}\) and the subsuperalgebras \(\text{aff}(1|1)\) and \(\text{aff}(1|1)_1\). Note that, the \(\text{aff}(1|1)\)–invariant elements of \(\Pi(D^2_{\tau, \lambda, \mu})\) can be deduced from those given in (3.12) and (3.13) by using the following \(\text{aff}(1|1)\)–isomorphism 
\[
\mathbb{D}^1_{\tau, \lambda, \mu} \rightarrow \Pi(D^1_{\tau, \lambda, \mu}), \quad A \mapsto \Pi(A \circ (\sigma \otimes \sigma)).
\]
Now, by isomorphism (2.11) we exhibit the \(\text{aff}(1|1)\)–invariant elements of \(\Pi(D^2_{\tau, \lambda, \mu})\). Of course, these elements are identically zero if \(2(\mu - \tau - \lambda) \notin \mathbb{N}\). So, the parameters \(\tau, \lambda\) and \(\mu\) must satisfy
\[
\mu = \tau + \lambda + k, \quad \text{where} \quad k \in \frac{1}{2}\mathbb{N}.
\]
The corresponding operators will be denoted \(\mathbb{J}_k^{\tau,\lambda,2}\). Obviously, if \(k\) is integer, then the operator \(\mathbb{J}_k^{\tau,\lambda,2}\) is even and if \(k\) is semi-integer, then the operator \(\mathbb{J}_k^{\tau,\lambda,2}\) is odd.

Any \(\text{aff}(2|1)\)–invariant element \(\mathbb{J}_k^{\tau,\lambda,2}\) of \(\Pi(D^2_{\tau, \lambda, \mu})\) with \(\mu = \tau + \lambda + k\), can be expressed as follows:
\[
\Psi_{\tau, \lambda, \mu} \left( \mathbb{J}_k^{\tau,\lambda,1} + \mathbb{J}_k^{\tau,\lambda+1,1} + \mathbb{J}_k^{\tau,\lambda,2} + \mathbb{J}_k^{\tau,\lambda+2,1} + \mathbb{J}_k^{\tau,\lambda,1} \right) \circ (\sigma \otimes \sigma)
\]
(4.23)
where \( J_k^{τ,λ,1} \) are defined by (3.12–3.13). The invariance of \( J_k^{τ,λ,2} \) with respect to \( \text{aff}(1|1)_1 \) imposes some supplementary conditions over the coefficients of the operators \( J_k^{τ,λ,2} \). By a direct computation, we get:

- For \( k \in \mathbb{N} + \frac{1}{2} \),

\[
\begin{align*}
\Gamma_k^{τ,λ,1} & = \Gamma_k^{τ,λ,1} - 1 + \Gamma_k^{τ,λ,1} - 1, \\
\Gamma_k^{τ,λ,1} & = \Gamma_k^{τ,λ,1} - 1 + \Gamma_k^{τ,λ,1} - 1, \\
\Gamma_k^{τ,λ,1} & = \Gamma_k^{τ,λ,1} - 1 + \Gamma_k^{τ,λ,1} - 1, \\
\Gamma_k^{τ,λ,1} & = \Gamma_k^{τ,λ,1} - 1 + \Gamma_k^{τ,λ,1} - 1,
\end{align*}
\]

where \( \Gamma_k^{τ,λ} = \Gamma_k^{τ,λ} = \Gamma_k^{τ,λ} = 0 \forall i < 0 \) or \( j < 0 \).

- For \( k \in \mathbb{N} \),

\[
\begin{align*}
\Gamma_k^{τ,λ,1} & = \Gamma_k^{τ,λ,1} - 1 + \Gamma_k^{τ,λ,1} - 1, \\
\Gamma_k^{τ,λ,1} & = \Gamma_k^{τ,λ,1} - 1 + \Gamma_k^{τ,λ,1} - 1, \\
\Gamma_k^{τ,λ,1} & = \Gamma_k^{τ,λ,1} - 1 + \Gamma_k^{τ,λ,1} - 1, \\
\Gamma_k^{τ,λ,1} & = \Gamma_k^{τ,λ,1} - 1 + \Gamma_k^{τ,λ,1} - 1,
\end{align*}
\]

where \( \Gamma_k^{τ,λ} = \Gamma_k^{τ,λ} = \Gamma_k^{τ,λ} = 0 \forall i < 0 \) or \( j < 0 \).

Finally, the invariance with respect to \( X_0, Y_0 \) completely determines the space of \( \text{aff}(2|1) \)-invariant elements of \( D_2^{τ,λ,μ} \).

## 5 The \( \text{aff}(n|1) \)-Relative Cohomology of \( \mathcal{K}(n) \) Acting on \( D_n^{n,λ,μ} \)

Let us first recall some fundamental concepts from cohomology theory (see, e.g., [5, 7, 8]).

### 5.1 Lie superalgebra cohomology

Let \( g = g_0 \oplus g_1 \) be a Lie superalgebra acting on a superspace \( V = V_0 \oplus V_1 \) and let \( h \) be a subalgebra of \( g \). (If \( h \) is omitted it is assumed to be \( \{0\} \).) The space of \( h \)-relative \( n \)-cochains of \( g \) with values in \( V \) is the \( g \)-module

\[
C^n(g, h; V) := \text{Hom}_h(\Lambda^n(g/h); V).
\]

The **coboundary operator** \( \delta_0 : C^n(g, h; V) \longrightarrow C^{n+1}(g, h; V) \) is a \( g \)-map satisfying \( \delta_0 \circ \delta_0 = 0 \). The kernel of \( \delta_0 \), denoted \( Z^n(g, h; V) \), is the space of \( h \)-relative \( n \)-cocycles, among them, the elements in the range of \( \delta_0 \) are called \( h \)-relative \( n \)-coboundaries. We denote \( B^n(g, h; V) \) the space of \( n \)-coboundaries.
By definition, the $n^{th}$ $h$-relative cohomology space is the quotient space

$$H^n(g, h; V) = Z^n(g, h; V)/B^n(g, h; V).$$

We will only need the formula of $\delta_n$ (which will be simply denoted $\delta$) in degrees 0 and 1: for $v \in C^0(g, h; V) = V^b$, $\delta v := (-1)^{|g|} g \cdot v$, where

$$V^b = \{ v \in V \mid h \cdot v = 0 \quad \text{for all} \quad h \in h \} ,$$

and for $\Upsilon \in C^1(g, h; V)$,

$$\delta(\Upsilon)(g, h) := (-1)^{|g|} g \cdot \Upsilon(h) - (-1)^{|h|(|g|+|\Upsilon|)} h \cdot \Upsilon(g) - \Upsilon([g, h]) \quad \text{for any} \quad g, h \in g.$$

### 5.2 The space $H^1_{\text{diff}}(K(n), \text{aff}(n|1); D^n_{\lambda,\mu})$

In this subsection, we will compute the first differential cohomology spaces $H^1_{\text{diff}}(K(n), \text{aff}(n|1); D^n_{\lambda,\mu})$ for $n = 0, 1$ and 2. Our main tools are the following two results

**Lemma 5.1.** Any 1-cocycle $\Upsilon \in Z^1_{\text{diff}}(K(n); D^n_{\lambda,\mu})$ vanishing on $\text{aff}(n|1)$ is $\text{aff}(n|1)$–invariant.

**Proof.** The 1-cocycle relation of $\Upsilon$ reads:

$$(-1)^{|F||\Upsilon|} \|_{X_F} \lambda \mu \Upsilon(X_G) - (-1)^{|G|(|F|+|\Upsilon|)} \|_{X_G} \lambda \mu \Upsilon(X_F) - \Upsilon([X_F, X_G]) = 0,$$

(5.24)

where $X_F, X_G \in K(n)$. Thus, if $\Upsilon(X_F) = 0$ for all $X_F \in \text{aff}(n|1)$, the equation (5.24) becomes

$$(-1)^{|F||\Upsilon|} \|_{X_F} \lambda \mu \Upsilon(X_G) - \Upsilon([X_F, X_G]) = 0,$$

(5.25)

expressing the $\text{aff}(n|1)$–invariance of $\Upsilon$.

**Theorem 5.1.** [2] Let $A^n_{\lambda,\mu} : \mathfrak{h}_\lambda \rightarrow \mathfrak{h}_\mu$, $(F\alpha^\lambda_n) \mapsto A^n_{\lambda,\mu}(F)\alpha^\mu_n$ be a non-zero $\text{aff}(n|1)$–invariant linear differential operator. Then, up to a scalar factor, the map $A^n_{\lambda,\mu}$ is given by:

$$A^n_{\lambda,\lambda+k}(F) = F^{(k)}, \quad \text{for} \quad k \in \mathbb{N} \quad \text{and} \quad n \in \mathbb{N}$$

$$A^n_{\lambda,\lambda-k}(F) = \overline{\alpha}_1\overline{\alpha}_2\cdots\overline{\alpha}_n(F^{(k)}), \quad \text{for} \quad k \in \mathbb{N} \quad \text{and} \quad n \geq 1.$$

### 5.2.1 The space $H^1_{\text{diff}}(\text{vect}(1), \text{aff}(1); D_{\lambda,\mu})$

The main result of this subsection is the following

**Theorem 5.2.**

$$H^1_{\text{diff}}(\text{vect}(1), \text{aff}(1); D_{\lambda,\mu}) \simeq \begin{cases} \mathbb{R} & \text{if} \quad \begin{cases} \mu - \lambda = 2, 3, 4 \text{ for all } \lambda, \\
\lambda = 0 \text{ and } \mu = 1, \\
\lambda = 0 \text{ or } \lambda = -4 \text{ and } \mu - \lambda = 5, \\
\lambda = -\frac{5\pm\sqrt{10}}{2} \text{ and } \mu - \lambda = 6, \\
0 & \text{otherwise.} \end{cases} \end{cases}$$

(5.27)

For $X \frac{d}{dx} \in \text{vect}(1)$ and $f dx^\lambda \in F_\lambda$, we write

$$C_{\lambda,\lambda+k}(X \frac{d}{dx})(f dx^\lambda) = C_{\lambda,\lambda+k}(X, f) dx^{\lambda+k}.$$
The spaces \( H^1_{\text{diff}}(\text{vect}(1),\text{aff}(1); D_{\lambda,\lambda+k}) \) are generated by the cohomology classes of the following 1-cocycles:

\[
\begin{align*}
C_{0,1}(X,f) & = X''f \\
C_{\lambda,\lambda+2}(X,f) & = X^{(3)}f + 2X''f' \\
C_{\lambda,\lambda+3}(X,f) & = X^{(3)}f' + X''f'' \\
C_{\lambda,\lambda+4}(X,f) & = -\lambda X^{(5)}f + X^{(4)}f' - 6X^{(3)}f'' - 4X''f(3) \\
C_{0,5}(X,f) & = 2X^{(5)}f' - 5X^{(4)}f'' + 10X^{(3)}f(3) + 5X''f(4) \\
C_{-4,1}(X,f) & = 12X^{(6)}f + 22X^{(5)}f' + 5X^{(4)}f'' - 10X^{(3)}f(3) - 5X''f(4) \\
C_{\alpha_i,\lambda+6}(X,f) & = \alpha_i X^{(7)}f - \beta_i X^{(6)}f' - \gamma_i X^{(5)}f'' - 5X^{(4)}f(3) + 5X^{(3)}f(4) + 2X''f(5),
\end{align*}
\]

where
\[
\begin{align*}
a_1 & = -\frac{5+\sqrt{19}}{2}, & \alpha_1 & = -\frac{-22+5\sqrt{19}}{4}, & \beta_1 & = \frac{31+7\sqrt{19}}{2}, & \gamma_1 & = \frac{25+7\sqrt{19}}{2} \\
a_2 & = -\frac{-5-\sqrt{19}}{2}, & \alpha_2 & = -\frac{-22-5\sqrt{19}}{4}, & \beta_2 & = \frac{31-7\sqrt{19}}{2}, & \gamma_2 & = \frac{25-7\sqrt{19}}{2}.
\end{align*}
\]

Proof. Note that, by Lemma 5.1, the \( \text{aff}(1) \)-relative cocycles are \( \text{aff}(1) \)-invariant bilinear differential operators. On the other hand, Feigin and Fuchs [6] calculated \( H^1_{\text{diff}}(\text{vect}(1); D_{\lambda,\mu}) \). The result is as follows

\[
H^1_{\text{diff}}(\text{vect}(1); D_{\lambda,\mu}) \simeq \begin{cases} \\
\mathbb{R} & \text{if } \mu - \lambda = 0, 2, 3, 4 \text{ for all } \lambda, \\
\mathbb{R}^2 & \text{if } \lambda = 0 \text{ and } \mu = 1, \\
\mathbb{R} & \text{if } \lambda = 0 \text{ or } \lambda = -4 \text{ and } \mu - \lambda = 5, \\
\mathbb{R} & \text{if } \lambda = -\frac{5+\sqrt{19}}{2} \text{ and } \mu - \lambda = 6, \\
0 & \text{otherwise}.
\end{cases}
\]

Hence, the cohomology spaces \( H^1_{\text{diff}}(\text{vect}(1),\text{aff}(1); D_{\lambda,\mu}) \) vanish for \( \mu - \lambda \geq 7 \). So we only have to study the \( \text{aff}(1) \)-invariant bilinear differential operators \( J_k^{-1}\lambda \) for \( k \leq 7 \). More precisely, the 1-cocycle condition imposes conditions on the constants \( c_{i,j} \): we get a linear system for \( c_{i,j} \). Thereafter, taking into account these conditions, we eliminate all constants underlying coboundaries. A straightforward but long computation leads to the result.

### 5.2.2 The space \( H^1_{\text{diff}}(K(1),\text{aff}(1|1); D^1_{\lambda,\mu}) \)

The main result of this subsection is the following

**Theorem 5.3.**

\[
H^1_{\text{diff}}(K(1),\text{aff}(1|1); D^1_{\lambda,\mu}) \simeq \begin{cases} \\
\mathbb{R} & \text{if } \begin{cases} \\
\mu - \lambda = \frac{1}{2} & \text{for } \lambda = 0, \\
\mu - \lambda = \frac{3}{2}, 2, \frac{5}{2} & \text{for all } \lambda, \\
\mu - \lambda = 3 & \text{for } \lambda = 0, -\frac{5}{2}, \\
\mu - \lambda = 4 & \text{for } \lambda = -\frac{7+\sqrt{23}}{4}, \\
\end{cases}
\end{cases}
\]

The space \( H^1_{\text{diff}}(K(1),\text{aff}(1|1); D^1_{\lambda,\mu}) \) is spanned by the cohomology classes of the following
1-cocycles:
\[
\begin{align*}
\Upsilon^1_{0, \frac{1}{2}}(X_G)(F) &= \eta_G(F) \alpha_1^{\frac{1}{2}}.
\end{align*}
\]
\[
\begin{align*}
\Upsilon^1_{\lambda, \frac{1}{2}}(X_G)(F \alpha_1^1) &= \begin{cases} 
\eta_G(F) \alpha_1^{\lambda + \frac{1}{2}} & \text{if } \lambda \neq -\frac{1}{2}, \\
(\eta_G(F) + \eta_G(F') + (-1)^G \eta_G(F \alpha_1)) & \text{if } \lambda = -\frac{1}{2}.
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\Upsilon^1_{\lambda, \frac{1}{2}}(X_G)(F \alpha_1^2) &= \begin{cases} 
3 \eta_G(F') - 2 \eta \eta_G(F) + (-1)^G \eta_G(F \alpha_1) & \text{if } \lambda \neq -1,
(-1)^G \eta_G(F) + 2 \eta \eta_G(F') + 2 \eta_G(F') & \text{if } \lambda = -1.
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\Upsilon^1_{\lambda, \frac{1}{2}}(X_G)(F \alpha_1^3) &= \begin{cases} 
\left( (-1)^G \eta_G(F) \eta_G(F') - \frac{2 \lambda + 1}{3} \left( (-1)^G \eta_G(F \eta_G(F') + G'' F'' \right) + \frac{\lambda(\lambda + 1)}{6} G''(F') \right) & \text{if } \lambda = 0, \frac{5}{2}.
(\frac{3}{2} \eta_G(F - (-1)^G \eta_G(F \eta_G(F')) & \text{if } \lambda \neq -1.
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\Upsilon^1_{\lambda, \frac{1}{2}}(X_G)(F \alpha_1^4) &= \begin{cases} 
\left( (-1)^G \eta_G(F) \eta_G(F') - \frac{2 \lambda + 1}{3} \left( (-1)^G \eta_G(F \eta_G(F') + G'' F'' \right) + \frac{\lambda(\lambda + 1)(2\lambda + 1)}{15} \right) & \text{if } \lambda = 0, \frac{5}{2}.
(\frac{3}{2} \eta_G(F - (-1)^G \eta_G(F \eta_G(F')) & \text{if } \lambda \neq -1.
\end{cases}
\end{align*}
\]

To prove Theorem 5.3 we need the following Lemmas

**Lemma 5.4.** [1] The 1-cocycle $\Upsilon$ of $K(1)$ is a coboundary if and only if its restriction $\Upsilon'$ to $\text{vect}(1)$ is a coboundary.

**Lemma 5.5.** As a $K(n - 1)$–module, we have
\[
(D_{\lambda, \mu})_0 \simeq \mathbb{D}^{n-1}_{\lambda, \mu} \oplus \mathbb{D}^{n-1}_{\lambda + \frac{1}{2}, \mu + \frac{1}{2}} \text{ and } (D_{\lambda, \mu})_1 \simeq \Pi(D^{n-1}_{\lambda, \mu} \oplus D^{n-1}_{\lambda + \frac{1}{2}, \mu + \frac{1}{2}}).
\]

Proof. Observe that the $\text{aff}(n - 1)$–isomorphism (2.10) is also a $K(n - 1)$–isomorphism. Thus, by isomorphism (2.10), we deduce a $K(n - 1)$–isomorphism,
\[
\Phi_{\lambda, \mu} : \mathbb{D}^{n-1}_{\lambda, \mu} \oplus \mathbb{D}^{n-1}_{\lambda + \frac{1}{2}, \mu + \frac{1}{2}} \oplus \Pi \left( \mathbb{D}^{n-1}_{\lambda, \mu} \oplus \mathbb{D}^{n-1}_{\lambda + \frac{1}{2}, \mu + \frac{1}{2}} \right) \to \mathbb{D}_n^n
\]

(5.31)

Here, we identify the $K(n - 1)$–modules via the following isomorphisms:
\[
\Pi \left( \mathbb{D}^{n-1}_{\lambda, \mu} \right) \to \text{Hom}_{\text{diff}} \left( \mathbb{P}^{n-1}_{\lambda}, \Pi(\mathbb{P}^{n-1}_{\mu}) \right) \quad \Pi(A) \mapsto \Pi \circ A,
\]
\[
\Pi \left( \mathbb{D}^{n-1}_{\lambda + \frac{1}{2}, \mu + \frac{1}{2}} \right) \to \text{Hom}_{\text{diff}} \left( \Pi(\mathbb{P}^{n-1}_{\lambda + \frac{1}{2}}, \mathbb{P}^{n-1}_{\mu}) \right) \quad \Pi(A) \mapsto A \circ \Pi,
\]
\[
\mathbb{D}^{n-1}_{\lambda, \mu} \to \text{Hom}_{\text{diff}} \left( \Pi(\mathbb{P}^{n-1}_{\lambda}, \mathbb{P}^{n-1}_{\mu}) \right) \quad A \mapsto \Pi \circ A \circ \Pi.
\]

Proof of Theorem 5.3. According to Lemma 5.5, we see that $H^1_{\text{diff}}(\text{vect}(1), \text{aff}(1); D_{\lambda, \mu})$ can be deduced from the spaces $H^1_{\text{diff}}(\text{vect}(1), \text{aff}(1); D_{\lambda, \mu})$:
\[
H^1_{\text{diff}} \left( \text{vect}(1), \text{aff}(1); D^1_{\lambda, \mu} \right) \simeq H^1_{\text{diff}} \left( \text{vect}(1), \text{aff}(1); D_{\lambda, \mu} \right) \oplus H^1_{\text{diff}} \left( \text{vect}(1), \text{aff}(1); D_{\lambda + \frac{1}{2}, \mu + \frac{1}{2}} \right) \oplus H^1_{\text{diff}} \left( \text{vect}(1), \text{aff}(1); \Pi(D_{\lambda, \mu}) \right) \oplus H^1_{\text{diff}} \left( \text{vect}(1), \text{aff}(1); \Pi(D_{\lambda + \frac{1}{2}, \mu + \frac{1}{2}}) \right).
\]
Hence, if $2(\mu - \lambda) \notin \{3, \ldots, 13\}$, then the cohomology space $H^1_{\text{diff}}(K(1), \text{aff}(1|1); D^1_{\lambda, \mu})$ vanishes. Indeed, let $\Upsilon$ be any element of $Z^1_{\text{diff}}(K(1), \text{aff}(1|1); D^1_{\lambda, \mu})$. Then by (5.27) and (5.32), up to a coboundary, the restriction of $\Upsilon$ to $\text{vect}$.

5.2.3 The space $H^1_{\text{diff}}(K(2), \text{aff}(2|1); D^2_{\lambda, \mu})$

The main result of this subsection is the following Theorem 5.6.

**Theorem 5.6.**

$$H^1_{\text{diff}}(K(2), \text{aff}(2|1); D^2_{\lambda, \mu}) \simeq \begin{cases} \mathbb{R} & \text{if } \mu - \lambda = 1, \\ \mathbb{R}^2 & \text{if } \mu - \lambda = 2, \\ 0 & \text{otherwise.} \end{cases}$$

It is spanned by the following 1-cocycles:

$$\Upsilon^2_{\lambda, \lambda+1}(X)(F \alpha^2_{\lambda}) = \begin{cases} \eta_1 \eta_2(G') F \alpha^2_{\lambda+1} & \text{if } \lambda \neq -\frac{1}{2}, \\ \left( \eta_1 \eta_2(G') F + (-1)^{|G|} \sum_{i=1}^2 (-1)^i \eta_{3-i}(G') \eta_i(F) \right) \alpha^2_{\lambda+1} & \text{if } \lambda = -\frac{1}{2}, \end{cases}$$

$$\Upsilon^2_{\lambda, \lambda+2}(X)(F \alpha^2_{\lambda}) = \left( 2 \lambda + 1 \right) \left( \frac{2 \lambda}{3} G'' F - (-1)^{|G|} \sum_{i=1}^2 \eta_i(G'' \eta_i(F)) - 2 \eta_2 \eta_1 (G') \eta_2 \eta_1 (F) \right) \alpha^2_{\lambda+2}.$$

$$\bar{\Upsilon}^2_{\lambda, \lambda+2}(X)(F \alpha^2_{\lambda}) = \begin{cases} \left( (-1)^{|G|} \sum_{i=1}^2 (-1)^i \eta_{3-i}(G'') \eta_i(F) + 2 \lambda \eta_2 \eta_1 (G'') F - \\ 2 \eta_2 \eta_1 (G') \eta_2 F + \right) \alpha^2_{\lambda+2} & \text{if } \lambda \neq -1, \\ \left( (-1)^{|G|} \sum_{i=1}^2 (-1)^i \eta_{3-i}(G') \eta_i(F)' - \eta_2 \eta_1 (G') F' + \\ (-1)^{|G|} \sum_{i=1}^2 (-1)^i \eta_{3-i}(G'') \eta_i(F') - G'' \eta_2 \eta_1 (F) \right) \alpha^2_{\lambda+2} & \text{if } \lambda = -1. \end{cases}$$

To prove the Theorem above, we need first the following Proposition

**Proposition 5.2.** [4] For $\lambda = 0$ or $\lambda \neq \mu$, any element of $Z^1_{\text{diff}}(K(2), \text{aff}(2|1); D^2_{\lambda, \mu})$ is a coboundary over $K(2)$ if and only if at least one of its restrictions to the subalgebras $K(1)$ or $K(1)^1$ is a coboundary.

Proof of Theorem 5.6. Note that, by Lemma 5.1, the $\text{aff}(2|1)$–relative 1-cocycles are $\text{aff}(2|1)$–invariant bilinear differential operators and by Proposition 5.2, they are related to the $\text{aff}(1|1)$–relative 1-cocycles.
Now, let us study the relationship between any 1-cocycle of $\mathcal{K}(2)$ and its restriction to the subalgebra $\mathcal{K}(1)$. By Lemma 5.5, we see that $H^1_{\text{diff}}(\mathcal{K}(1), \text{aff}(1|1); \mathbb{D}^2_{\lambda,\mu})$ can be deduced from the spaces $H^1_{\text{diff}}(\mathcal{K}(1), \text{aff}(1|1); \mathbb{D}^1_{\lambda,\mu})$:

$$H^1_{\text{diff}}(\mathcal{K}(1), \text{aff}(1|1); \mathbb{D}^2_{\lambda,\mu}) \cong H^1_{\text{diff}}(\mathcal{K}(1), \text{aff}(1|1); \mathbb{D}^1_{\lambda,\mu}) \oplus H^1_{\text{diff}}(\mathcal{K}(1), \text{aff}(1|1); \mathbb{D}^1_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}) \oplus H^1_{\text{diff}}(\mathcal{K}(1), \text{aff}(1|1); \Pi(\mathbb{D}^1_{\lambda,\mu+\frac{1}{2}})) \oplus H^1_{\text{diff}}(\mathcal{K}(1), \text{aff}(1|1); \Pi(\mathbb{D}^1_{\lambda+\frac{1}{2},\mu}))$.

(5.34)

Hence, for $\lambda = 0$ or $\lambda \neq \mu$, if $2(\mu - \lambda) \notin \{2, \ldots, 9\}$, the corresponding cohomology $H^1_{\text{diff}}(\mathcal{K}(2), \text{aff}(2|1); \mathbb{D}^2_{\lambda,\mu})$ vanish. Indeed, let $\Upsilon$ be any element of $Z^1_{\text{diff}}(\mathcal{K}(2), \text{aff}(2|1); \mathbb{D}^2_{\lambda,\mu})$. Then by (5.30) and (5.34), up to a coboundary, the restriction of $\Upsilon$ to $\text{aff}(1|1)$ vanishes, so $\Upsilon = 0$ by Proposition 5.2.

For $2(\mu - \lambda) \in \{2, \ldots, 9\}$ or $\mu = \lambda \neq 0$, we study the operators $J_{\mu,\lambda,2}$. To study these operators $J_{\mu,\lambda,2}$ satisfying $\delta(J_{\mu,\lambda,2}) = 0$, we consider the two components of its restriction to $\text{aff}(1|1)$ which we compare with $\Upsilon_{\lambda,\mu}^1$ and $\Upsilon_{\lambda+\frac{1}{2},\mu}^1$ or $\Upsilon_{\lambda,\mu+\frac{1}{2}}^1$ and $\Upsilon_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}^1$ depending on whether $\mu - \lambda$ is integer or semi-integer. A straightforward but long computation leads to the result.

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