Virtual photons in SU(2) chiral perturbation theory and electromagnetic corrections to $\pi\pi$ scattering

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Abstract

We construct the generating functional of two flavor chiral perturbation theory including the effects of virtual photons in the one loop approximation. As an application, we calculate the electromagnetic corrections to the elastic $\pi\pi$ scattering amplitude, in particular to its S–wave threshold parameters. Numerical estimates are given for the reaction $\pi^0\pi^0 \rightarrow \pi^0\pi^0$. These electromagnetic effects are found to be smaller than the hadronic two–loop corrections for the scattering length $a_0$. The effective range $b_0$ increases by 36% due to the unitary cusp.

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1. Elastic pion–pion scattering is the purest reaction to test our understanding of the spontaneous and the explicit chiral symmetry breaking in QCD. This reaction can be calculated to high precision within the framework of chiral perturbation theory (CHPT) and is one tool to pin down the size of the scalar quark condensate $B_0 = |\langle 0 | \bar{q} q | 0 \rangle | / F_0$, with $F_0$ the pion decay constant (in the chiral limit), and to give bounds on the up and down quark masses (for a review, see [1]). CHPT is the effective field theory of the standard model at low energies, which admits an expansion in small external momenta and quark (pion) masses, with $q$ collectively denoting any one of these small parameters. This so-called chiral expansion, which proceeds in steps of powers of $q^2$, can be mapped one-to-one on an expansion in the number of pion loops and higher dimension local operators [3]. At present, elastic $\pi\pi$ scattering in the threshold region has been investigated at tree level ($q^2$) [2], to one loop ($q^4$) [4] and to two loop accuracy ($q^6$) [5]. There exits also a dispersive analysis to order $q^6$ in the framework of generalized CHPT [6]. Of particular interest are the $S$–wave scattering lengths $a_{I0}$, with $I = 0, 2$ the total isospin of the two-pion system, since they vanish in the chiral limit [2]. Consider as an example $a_{00}^0$ in CHPT:

For the central values of the various input parameters, the tree, one– and two–loop results are $a_{00}^0 = 0.156, 0.201$ and $0.217$, in order. While the one–loop corrections are sizeable ($\sim 25\%$), the two–loop corrections are already considerably smaller ($\sim 10\%$). Apart from these strong interaction corrections, there are also electromagnetic contributions. In particular, the charged to neutral pion mass difference, which is an effect of the order of $(M_{\pi^+} - M_{\pi^0})/M_{\pi^+} \sim 4\%$, is almost entirely of electromagnetic origin [7]. It is therefore of importance to get a handle on the electromagnetic corrections for the $\pi\pi$ scattering amplitude to further sharpen the theoretical predictions. Virtual photons have already been included in three flavor CHPT [8][9] and various effects like e.g. the violation of Dashen’s theorem or the electromagnetic (em) corrections to the decay $\eta \to 3\pi$ [10] have been calculated. Since $\pi\pi$ scattering can be described purely within SU(2) CHPT, we construct here the generating functional including virtual photons to one loop for the two flavor case. As proposed in [8], we assign the chiral dimension one to the electric charge, so the lowest order effective Lagrangian coupling the virtual photons to the pions is of $O(e^2) \sim O(q^2)$. The difference to the three flavor case considered in refs. [8][9] comes largely from the fact that for two flavors there are less independent operators. This reduced number of terms is the natural basis to consider purely pionic processes. As an example, we then estimate the em corrections to the $\pi\pi$ scattering amplitude, in particular to the threshold parameters (scattering lengths, effective ranges). In this letter, we concentrate on the reaction $\pi^0\pi^0 \to \pi^0\pi^0$ since space forbids a thorough discussion of the treatment of the infrared divergences appearing in charged pion scattering. Previous work mostly related to the leading em corrections involving at least one $\pi^+\pi^-$ pair can be found in [11][12][13][14]. In most of these calculations, the pion mass difference has not been included systematically.

2. The effective field theory build of pions, collected in $U(x) = u^2(x)$, photons ($A_\mu$) and other scalar ($s$), pseudoscalar ($p$), vector ($v_\mu$) and axial–vector ($a_\mu$) external sources starts

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#5 In particular, $F_\pi = 93.2$ MeV and the charged pion mass are used. For a discussion on the parameter sensitivity, see [3].
at dimension two,
\[ \mathcal{L}^{(2)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2} (\partial^2 A^\mu)^2 + \frac{F_0^2}{4} (D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U) + C \langle Q_R U Q_L U^\dagger \rangle, \tag{1} \]
where \( \langle \rangle \) denotes the trace in flavor space, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) the photon field strength tensor, \( \lambda \) the gauge–fixing parameter (from here on, we work in the Lorentz gauge \( \lambda = 1 \)), \( D_\mu \) the generalized covariant derivative,
\[ D_\mu U = \partial_\mu U - i (v_\mu + a_\mu + QA_\mu) U + i U (v_\mu - a_\mu + QA_\mu), \tag{2} \]
and \( Q \) the quark charge matrix,
\[ Q = \frac{e}{3} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} = \frac{e}{6} \begin{pmatrix} 3 \tau^3 + 1 \end{pmatrix}. \tag{3} \]

In what follows, we work in the so–called \( \sigma \)–model gauge,
\[ U(x) = \sigma(x) \mathbf{1} + i \vec{\tau} \cdot \vec{\pi}(x)/F_0, \quad \sigma(x) = \sqrt{1 - \pi^2(x)/F_0^2}. \tag{4} \]
The third term in Eq.(1) is the standard non–linear \( \sigma \)–model coupled to external sources (we neglect here singlet axial components and set \( \langle a_\mu \rangle = 0 \) \[4\]). In particular, we have \( \chi = 2B_0(s + ip) \) and the external scalar source contains the quark mass matrix, \( s(x) = \text{diag}(m_u, m_d) + \ldots \). The last term in the dimension two Lagrangian is the lowest order chiral invariant term one can construct from pion and photon fields \[13\]. To make it invariant under chiral SU(2)_L×SU(2)_R transformations, we have introduced the spurions \( Q_{L,R} \), which have the following transformation properties,
\[ Q_I \rightarrow g_I Q_I g_I^\dagger, \quad g_I \in SU(2)_I, \quad I = L, R \tag{5} \]
At a later stage, one sets \( Q_L = Q_R = Q \). The constant \( C \) can be calculated from the neutral to charged pion mass difference since this term leads to \( (\delta M^2)_{\text{em}} = 2e^2C/F_0^2 \). This identification is based upon the fact that the quark mass difference \( m_d - m_u \) only gives a tiny contribution to \( M_{\pi^0}^2 - M_{\pi^0}^2 \) \[16\]. It can therefore be expected that these quark mass effects are also tiny for the elastic \( \pi\pi \) scattering amplitude. Consequently, we will work in the isospin limit \( m_d = m_u = \hat{m} \) in what follows. For later convenience, we introduce the dimensionless constant
\[ Z = C/F_0^4 = 0.89, \tag{6} \]
with \( F_0 = 88 \text{ MeV} \) \[4\]. As already stated before, we count the external vector and axial–vector fields as well as the charge matrices \( Q, Q_L, Q_R \) as \( \mathcal{O}(q) \) and the photon field as \( \mathcal{O}(1) \). This has the advantage of a consistent power counting between the strong and electromagnetic interactions, i.e. \( e \sim q \) and one has terms of dimension two, four and so on. Here, dimension two means either order \( q^2 \) or \( e^2 \) and similar at higher orders. The construction of the generating functional proceeds along standard lines making use of heat kernel techniques for elliptic Euclidean differential operators. We follow here the approach outlined in \[8\] and combine the expansion around the classical solutions for the meson and photons fields, \( U = U^{\text{cl}} + i u \xi u/F_0 + \ldots \) and \( A_\mu = A_\mu^{\text{cl}} + \epsilon_\mu \), respectively, in one
set of fluctuations variables η = (ξ_1, . . . , ξ_3, ε_0, . . . , ε_3). One then expands the generating functional

\[ \exp[\int i Z(s, p, v, a)] = N \int [dA][dU] \exp \left\{ i \int d^4 x \left( L_2 + L_4 \right) \right\} \]

up to second order in the fluctuations η. Using dimensional regularization, the divergences can be extracted in a straightforward manner. For the two flavor case at hand, the electromagnetic part of the dimension four counterterm Lagrangian takes the form

\[ L_{\text{em}}^{(4)} = \sum_{i=1}^{13} k_i O_i, \]

with the \( O_i \) monomials in the fields of dimension four. The low-energy constants \( k_i \) absorb the divergences in the standard manner,

\[ k_i = \kappa_i L + k^r_i(\mu), \]

\[ L = \frac{\mu^d}{16\pi^2} \left\{ \frac{1}{d-4} \left[ \ln(4\pi) + \Gamma'(1) + 1 \right] \right\}, \]

with \( \mu \) the scale of dimensional regularization and \( d \) the number of space–time dimensions. The explicit expressions for the operators \( O_i \) and their \( \beta \)-functions \( \kappa_i \) are collected in table 1. The \( k^r_i(\mu) \) are the renormalized, finite and scale–dependent low–energy constants. These can be fixed by data or have to be estimated with the help of some model.

| \( i \) | \( O_i \) | \( \kappa_i \) |
|---|---|---|
| 1 | \( (QUQU^\dagger)^2 \) | \( 3/2 + 3Z + 12Z^2 \) |
| 2 | \( QUQU^\dagger \left( D_\mu U D_\mu U^\dagger \right) \) | \( 2Z \) |
| 3 | \( U^\dagger D_\mu UQ \left( D_\mu UQ^\dagger \right) \) | \( -3/4 \) |
| 4 | \( \langle QUQU^\dagger \rangle \langle DUUU^\dagger \rangle \) | \( -2Z \) |
| 5 | \( \left( [Q, D_\mu, Q]U^\dagger D_\mu U \right) - \left( [Q, D_\mu, Q]D_\mu UU^\dagger \right) \) | \( -1/4 \) |
| 6 | \( \langle D_\mu Q \rangle [D_\mu, Q]U^\dagger \rangle \) | \( 0 \) |
| 7 | \( \langle QUQU \rangle \langle \chi U^\dagger + U \chi^\dagger \rangle \) | \( 1/4 + 2Z \) |
| 8 | \( \langle (U^\dagger \chi - \chi^\dagger U) (U^\dagger QUQ - QU^\dagger QU) \rangle \) | \( 1/8 - Z \) |
| 9 | \( \langle Q^2 \rangle \langle QUQU \rangle \) | \( -3 - 3Z/5 - 12Z^2/5 \) |
| 10 | \( \langle Q^2 \rangle \langle D_\mu U D_\mu U^\dagger \rangle \) | \( -27/20 - Z/5 \) |
| 11 | \( \langle Q^2 \rangle \langle \chi U^\dagger + U \chi^\dagger \rangle \) | \( -1/4 - Z/5 \) |
| 12 | \( \langle [D_\mu, Q_R] [D_\mu, Q_R] + [D_\mu, Q_L] [D_\mu, Q_L] \rangle \) | \( 0 \) |
| 13 | \( \langle Q^2 \rangle \) | \( 3/2 - 12Z/5 + 84Z^2/25 \) |

Table 1: Counterterms and their \( \beta \)-functions.
As it is the case for the hadronic LECs in the two–flavor case \[4\], one can introduce scale–independent couplings, called \( \tilde{k}_i \), via

\[
k_i^r(\mu) = \frac{K_i}{32\pi^2} \left[ \tilde{k}_i + \ln \frac{M_{\pi^0}^2}{\mu^2} \right]. \tag{11}
\]

Notice that we choose the neutral pion mass as the reference scale. This is natural since the neutral pion mass is almost entirely a hadronic effect, in contrast to the charged one. Some remarks concerning these results are in order. To arrive at the structures given in the table, we have made use of the equations of motions for the classical terms. The last term in the table \( \sim \langle Q^2 \rangle^2 \) does not contain any pion field and is only needed for the complete renormalization, i.e. it does not influence physical processes. The same applies to \( \mathcal{O}_{12} \). Similarly, the operators \( \mathcal{O}_{9,10,11} \) only lead to renormalizations of \( C, F_0 \) and \( M_0 \), with \( M_0^2 = 2\hat{m}B_0 \) the leading term in the quark mass expansion of the pion mass. A typical difference between SU(2) and SU(3) is the operator \( \mathcal{O}_{11} \), which has a vanishing \( \beta \)-function in SU(3) \[3\]. Note furthermore that symmetry does allow for terms which are odd in powers of \( Q \), like e.g. a term \( \sim (Q^2 UQU^\dagger) \). Such terms are, however, unphysical since the electric charge always appears in even powers for physical processes. Consequently, we did not further study such terms. We also have repeated the SU(3) calculation and found that the \( \beta \)-functions for the terms \( K_{15,16,17} \) are incorrectly given in the existing literature. The correct ones read (in the notation of Urech \[8\])

\[
\Sigma_{15} = 3/2 + 3Z + 14Z^2, \quad \Sigma_{16} = -3 - 3Z/2 - Z^2, \quad \Sigma_{17} = 3/2 - 3Z/2 + 5Z^2. \tag{12}
\]

This difference in the terms \( \sim Z^2 \) can be traced back to the fact that the operator \( \sigma^{ab} \), which appears in the four–dimensional Euclidean one–loop functional, has to be symmetrized. This was not done in the part \( \sim C \) in \[3\], i.e. when going from Eq.(15) to Eq.(21) in that paper.

3. Before evaluating the pertinent em corrections, we collect here some basic definitions concerning the elastic \( \pi \pi \) scattering amplitude (in the threshold region). Consider the process \( \pi^a(p_a) + \pi^b(p_b) \rightarrow \pi^c(p_c) + \pi^d(p_d) \), for pions with isospin 'a, b, c, d' and momenta \( p_{a,b,c,d} \). The corresponding Mandelstam variables are \( s = (p_a + p_b)^2, \quad t = (p_a - p_c)^2, \quad u = (p_a - p_d)^2 \) with \( s + t + u = M_{\pi^a}^2 + M_{\pi^b}^2 + M_{\pi^c}^2 + M_{\pi^d}^2 \). The scattering amplitude can be expressed in terms of a single function, denoted \( A(s, t, u) \)\#7

\[
T^{cd;ab} = A(s, t, u) \delta^{ab} \delta^{cd} + A(t, s, u) \delta^{ac} \delta^{bd} + A(u, t, s) \delta^{ad} \delta^{bc}. \tag{13}
\]

The chiral expansion of \( A(s, t, u) \) takes the form

\[
A(s, t, u) = A^{(2)}(s, t, u) + A^{(4)}(s, t, u) + \mathcal{O}(q^6), \tag{14}
\]

where \( A^{(m)} \) is of order \( q^m \) and the symbol \( \mathcal{O}(q^6) \) denotes terms like \( s^3, s^2t, s^2e^2, se^4, \ldots \).

The form of \( A^{(2)}(s, t, u) \) was first given by Weinberg \[2\] and the next–to–leading order terms by Gasser and Leutwyler \[4\]. For convenience, one splits this contribution as

\#6 Res Urech has kindly informed us that he has checked his coefficients \( \Sigma_{15,16,17} \) finding agreement with our results.

\#7 This is correct in the isospin–conserving case. The more general form of the amplitude in case of isospin violation will be given in \[22\].
For comparison with the data, one decomposes $T^{(4)}(s, t, u) = B(s, t, u) + C(s, t, u)$, where $B(s, t, u)$ collects the unitarity corrections and $C(s, t, u)$ the (real) tree and tadpole contributions. To evaluate the one–loop graphs, we introduce the modified pion propagator

$$\Delta_{\pi}^{ab}(\ell) = \frac{i\delta^{ab}}{[\ell^2 - M_{\pi}^2 - \delta M^2(1 - \delta^{ab})]}, \quad \delta M^2 = \frac{2e^2C}{F_0^2}, \quad (15)$$

with $\ell$ the pion four–momentum and $'a, b'$ isospin indices. The form of Eq.(15) is due to the fact that in this gauge, the operator $\sim C(QUQ^{\dagger})$ only contributes to terms which are quadratic in pion fields. To evaluate the unitarity corrections, i.e. the diagrams which give rise to the imaginary part of the scattering amplitude, we introduce a generalization of the commonly used "bubble function" $\tilde{J}$,

$$\tilde{J}(q^2, M^2, A) = \frac{1}{16\pi^2} [\frac{1}{2}(\ln \sigma_+ - 1 + \ln \sigma_- - 1) + 2], \quad (16)$$

with

$$\sigma = \sqrt{1 - \frac{4M^2}{s} + \frac{A}{s} \left( \frac{A}{s} + 2 \right)},$$

$$\sigma_- = \sqrt{1 - \frac{4M^2}{s} \left( \frac{A}{s} + 1 \right)^{-2}}, \quad \sigma_+ = \sqrt{1 - \frac{4M^2}{s} \left( 1 - \frac{A}{M} \right) \left( \frac{A}{s} - 1 \right)^{-2}}, \quad (17)$$

with $A$ a quantity of dimension $[mass^2]$ like e.g. $\delta M^2$. For $A = 0$, one recovers the standard form of $\tilde{J}$ [1] since then $\sigma = \sigma_- = \sigma_+ = (1 - 4M^2/s)^{1/2}$. All loop integral can be expressed in terms of $\tilde{J}(q^2, M^2, A)$ and some polynoms (modulo logarithms). In terms of physical processes, we have five reaction channels

(a) $\pi^0\pi^0 \rightarrow \pi^0\pi^0$, \quad (b) $\pi^+\pi^- \rightarrow \pi^0\pi^0$, \quad (c) $\pi^+\pi^- \rightarrow \pi^+\pi^-$, \quad (d) $\pi^0\pi^+ \rightarrow \pi^0\pi^+$, \quad (e) $\pi^+\pi^+ \rightarrow \pi^+\pi^+.$ \quad (19)

For comparison with the data, one decomposes $T^{cd;ab}$ into amplitudes of definite total isospin ($I = 0, 1, 2$) and projects out partial–wave amplitudes $T^I_l(s)$,

$$T^I_l(s) = \sqrt{1 - 4M_{\pi}^2/s} \left[ \exp\{2i[\delta^I_l(s) + i\eta^I_l(s)]\} - 1 \right], \quad (20)$$

with $s = 4(M_{\pi}^2 + q^2)$ and $q$ the pion momentum in the c.m. system.\([\#8]\) Furthermore, $l$ denotes the total angular momentum of the two–pion system. The phase shifts $\delta^I_l(s)$ are real and the inelasticities $\eta^I_l(s)$ set in at four–pion threshold. Below $KK$ threshold, $\approx 1 \text{ GeV}^2$, they are negligible and will be ignored in what follows. Near threshold, the partial–wave amplitudes take the form

$$\text{Re } T^I_l(s) = q^2 \{ a^I_l + q^2 b^I_l + O(q^4) \}.$$ \quad (21)

The coefficients $a^I_l$ are called scattering lengths, the $b^I_l$ are the range parameters. In the following, we will concentrate on these quantities.

\([\#8]\)This holds only for the equal mass case. The generalization to the unequal mass case is obvious.
4. We are now in the position to evaluate the em corrections to the elastic $\pi\pi$ scattering amplitude. To one-loop order, one has of course the standard strong interaction graphs, i.e. tree graphs at orders $q^2$ and $q^4$ as well as one-loop diagrams at $O(q^4)$. In the $\sigma$–model gauge, these can be calculated according to standard methods, the only difference being the modified pion propagator, Eq.(15). The corresponding Feynman diagrams are shown in fig. 1a,b,c,d. There are two tree graphs at order $e^2$. Diagram a) contributes to the charged pion mass shift and b) vanishes for the reasons discussed above. That is also the reason why there is no one-loop graph like c) with the insertion on the four-pion vertex. The next seven graphs in fig.1 are of order $O(e^2q^2)$. From the irreducible photon loop diagrams (fig.1e,f,g) only the first one is non-vanishing since one has no $\gamma^4\pi$ and $2\gamma^4\pi$ vertices in this particular gauge (this point was also stressed by Gasser [17] and Ecker [18]). Clearly, graph h) are only gives rise to wave function renormalization and i) vanishes in dimensional regularization. Finally, the counterterms with exactly one insertion from $L^{(4)}_{\text{em}}$ are depicted in figs.1j,k. From these two, only graph 1j gives rise to a genuine em correction to the $\pi\pi$ scattering amplitude. For charged pions, there are in addition photon exchange graphs as shown in fig.2. At tree level $O(e^2)$ one has the one-photon exchange diagram. To one-loop order $e^4$ and $e^2q^2$, there are six topologically different graphs. While diagram 2a is of order $e^2q^2$, the others are of $O(e^4)$. In particular, there are three types of two-photon exchange graphs with one and two intermediate pion propagators, figs.2b,c,d, in order. The last two diagrams in fig.2 are simply vertex and self-energy corrections. As one easily convinces oneself, the photon exchange graphs like 1e, 2a, . . . , 2f lead to IR divergences. These are cancelled in the standard fashion by considering the radiation of very soft photons in the initial and final states. #9 For comparison with experiment, one can also use the standard Gamov factors [21] to remove the Coulomb enhancement in the initial and the final state for charged pions. What we are really after are the electromagnetic effects once these ”kinematical” em effects are removed. Space forbids here to discuss these matters in detail and we refer to ref. [22] for a comprehensive treatment. In what follows, we consider only the scattering of neutral pions, i.e. the reaction (a) in Eq.(19).

5. For $\pi^0\pi^0 \to \pi^0\pi^0$, we have $s + t + u = 4M_{\pi^0}^2$ and the leading order Weinberg term reads

$$A^{(2)}(s, t, u) = \frac{s - M_{\pi^0}^2}{F_\pi^2}$$

in terms of the physical values $M_{\pi^0} = 134.97$ MeV and $F_\pi = 92.5$ MeV. #10 The respective shifts from the lowest order values $M$ and $F_0$ are accounted for in the next–to–leading order contribution $C(s, t, u)$. It is important to stress that the lowest order amplitude is therefore not affected directly by the em corrections, only indirectly through the pion mass shift. This result is, of course, well–known [17]. The unitarity corrections take a form similar to the purely strong interaction results of [4],

$$B(s, t, u) = \frac{1}{6F_\pi^4} [6(s - M_{\pi^0}^2)^2 J(s, M_{\pi^0}^+, 0) - 3(s^2 - 4sM_{\pi^0}^2 + 3M_{\pi^0}^4)J(s, M_{\pi^0}, 0)]$$

#9 A lucid discussion of such infrared effects is given in chapter 13 of ref. [19]. A detailed study of radiative four–meson amplitudes in CHPT can be found in [20].

#10 Note that the extraction of this value for $F_\pi$ includes one–loop radiative corrections [23].
\[ + [t(t-u) - 2M_{\pi^0}^2 t + 4M_{\pi^0}^2 u - 2M_{\pi^0}^4] \bar{J}(t, M_{\pi^0}, 0) \\
+ [u(u-t) - 2M_{\pi^0}^2 u + 4M_{\pi^0}^2 t - 2M_{\pi^0}^4] \bar{J}(u, M_{\pi^0}, 0) \]  

(23)

The next-to-leading order tree and tadpole contributions read

\[
\frac{1}{96\pi^2 F_\pi^4} \left[ 2(\bar{l}_1 - \frac{4}{3})(s - 2M_{\pi^0}^2)^2 + (\bar{l}_2 - \frac{5}{6})(s^2 + (t-u)^2) - 3M_{\pi^0}^4 \bar{l}_3 \\
+ 12M_{\pi^0}^2 (s - 2M_{\pi^0}^2) \bar{l}_4 - 12M_{\pi^0}^2 s + 15M_{\pi^0}^4 \right] \\
- \frac{1}{16\pi^2 F_\pi^4} \left\{ \ln \frac{M_{\pi^0}^2}{M_{\pi^0}^2} \left[ M_{\pi^+}^2 (3s - 4M_{\pi^0}^2) + (s - M_{\pi^0}^2)^2 \right] \\
+ e^2 F_\pi^2 (3s - 4M_{\pi^0}^2) \left[ \frac{3}{2} \tilde{k}_3 + 2Z\tilde{k}_4 \right] \right\}. \]  

(24)

The last two terms deserve some discussion. First, the term \( \sim \ln(M_{\pi^+}^2/M_{\pi^0}^2) \) is due to the fact that we normalize the scale-independent SU(2) low-energy constants \( \bar{l}_i \) to the neutral pion mass and that in the loops one has neutral as well as charged pion pairs propagating. The last term in Eq.(24) is the novel em contribution from \( \mathcal{L}_{\text{em}}^{(4)} \). To arrive at these results, we have used the low-energy expansions of the pion mass and the pion decay constant.\(^{[11]}\)

\[
M_{\pi^0}^2 = M_0^2 \left\{ 1 - \frac{M_0^2}{32\pi^2 F_0^2} \bar{l}_3 + \frac{M_{\pi^0}^2}{16\pi^2 F_0^2} \ln \frac{M_{\pi^0}^2}{M_0^2} + \frac{5e^2}{72\pi^2} \left[ \left( \frac{2Z + \frac{1}{4}}{Z + \frac{1}{4}} \right) \bar{k}_7 - \left( \frac{Z + \frac{1}{4}}{Z + \frac{1}{4}} \right) \bar{k}_{11} \right] \\
- \frac{5e^2}{72\pi^2} \left[ 2Z \bar{k}_2 - \left( \frac{Z}{5} + \frac{27}{20} \right) \bar{k}_{10} \right] - \frac{e^2}{32\pi^2} \left[ 3\bar{k}_3 + 4Z\bar{k}_4 \right] \right\}, \]  

(25)

\[
F_{\pi} = F_0 \left\{ 1 + \frac{M_0^2}{16\pi^2 F_0^2} \bar{l}_4 - \frac{M_{\pi^0}^2}{32\pi^2 F_0^2} \ln \frac{M_{\pi^0}^2}{M_0^2} \\
+ \frac{5e^2}{144\pi^2} \left[ 2Z \bar{k}_2 - \left( \frac{Z}{5} + \frac{27}{20} \right) \bar{k}_{10} \right] + \frac{e^2}{64\pi^2} \left[ 3\bar{k}_3 + 4Z\bar{k}_4 \right] \right\}. \]  

(26)

where \( M_{\pi^+}^2 = M_0^2 + \delta M^2 \), see Eq.(15). Notice that in \( B(s, t, u) \) and \( C(s, t, u) \) we have set \( F_0 = F_{\pi}, M_0 = M_{\pi^0} \) and \( M_{\pi^+} = M_{\pi^0} \) since these differences are of order \( q^2 \) and thus beyond the accuracy of the calculation presented here. As a check, we recover the result for \( A^{(2)} + A^{(4)} \) of [24] for one common pion mass and setting \( e = 0 \). From the amplitude given in Eqs. (22,23,24), it is straightforward to calculate the pertinent scattering lengths \( a_0(00; 00) \) and effective ranges \( b_0(00; 00) \) numerically or analytically [22]. These are related to the ones in the isospin basis via

\[
a_0(00; 00) = \frac{1}{3} a_0^0 + \frac{2}{3} a_0^2, \quad b_0(00; 00) = \frac{1}{3} b_0^0 + \frac{2}{3} b_0^2. \]  

(27)

One can show analytically that for the process \( \pi^0\pi^0 \rightarrow \pi^0\pi^0 \), the operators \( \sim \bar{k}_i \) do not contribute.

\(^{[11]}\)In this letter, we identify the neutral with the charged pion decay constant. A more thorough discussion of the em effects on the AA correlators is given in [22].
For the numerical evaluation, we use the central values of the strong low–energy constants $\bar{\ell}_i$ from [25], $\bar{\ell}_1 = -1.7$, $\bar{\ell}_2 = 6.0$ together with $\bar{\ell}_3 = 2.9$, $\bar{\ell}_4 = 4.3$ and the values for $F_\pi$, $M_{\pi^0}$ and $M_{\pi^+}$ given above. The corresponding S–wave threshold parameters are given in table 2 in comparison to the strong one–loop results and the experimental data.

|          | $a_0(00;00)$ | $b_0(00;00)$ |
|----------|--------------|--------------|
| $e = 0$  | 0.0360       | 0.0302       |
| $e \neq 0$ | 0.0340       | 0.0412       |
| Exp.     | $0.056 \pm 0.027$ [26,27] | $0.029 \pm 0.044$ [28] |

Table 2: S–wave threshold parameters with em corrections at one loop ($e \neq 0$) compared to the hadronic one–loop results ($e = 0$) in units of the inverse neutral pion mass. The data should only be considered indicative since they mostly stem from processes involving charged pions. The uncertainties are added in quadrature.

For the scattering length $a_0$ the effect of the em corrections is of the order of 5%, still approximately a factor of two smaller than the strong two–loop correction [8]. For the range parameter $b_0$, however, we observe an 36% increase. This is due to unitarity cusp at $s_0 = 4M_{\pi^+}^2$, which is expected to scale as $\sqrt{M_{\pi^+}^2 - M_{\pi^0}^2}/M_{\pi^+} \simeq 26\%$. The dominant isospin–violating effect is thus entirely given through the charged to neutral pion mass difference, similar to the case of the electric dipole amplitude in neutral pion photoproduction off nucleons [29].

6. To summarize, we have constructed the generating functional for two–flavor chiral perturbation theory including the effects of virtual photons in the one–loop approximation. Counting the electric charge as a small momentum, there are in total 13 terms contributing to the em Lagrangian at next–to–leading order (some of these are only needed for renormalization). As an application, we have considered the em corrections to the elastic $\pi\pi$ scattering amplitude and given numerical estimates for $\pi^0\pi^0 \rightarrow \pi^0\pi^0$. The charged to neutral pion mass difference produces an pronounced effect on the S–wave effective range. In a forthcoming publication, we will present results also for the channels involving charged pions [22] and include the effects due to the quark mass difference $\sim m_d - m_u$.

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Figure 1: Graphs contributing to the em corrections to elastic $\pi\pi$ scattering. Solid and wiggly lines denote pions and photons, in order. The heavy dot and the box refer to insertions from the em counterterms at order $e^2$ and $e^4$, respectively. Graphs b), f) and g) vanish in the $\sigma$–model gauge. i) vanishes in dimensional regularization. Crossed graphs are not shown.
Figure 2: One and two–photon exchange graphs contributing to the em corrections for elastic scattering of charged pions at one loop. While diagram a) is of order $e^2 q^2$, the others are of $O(e^4)$. Crossed graphs are not shown. For notations, see fig.1.