On partition polynomials and partition functions

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Abstract. In this paper we revisit the work of E.T. Bell concerning partition polynomials in order to introduce the reciprocal partition polynomials. We give their explicit formulas and apply the result to compute closed formulae for some well-known partition functions.

1. Introduction

Many known and new arithmetical functions are included as special cases of partition polynomials. In this work we consider partition polynomials $P_n(z)$ introduced and studied by E. T. Bell in the work [2]. We introduce and study the reciprocal polynomials $W_n(z)$ which are a generalization of partition function $p(n)$ and restricted partition functions $W(n,d^s)$. Formal calculus allow us to compute $W_n(z)$ and deduce explicit formula for $W(n,d^s)$ and $p(n)$. We end the work by the generalized partition polynomials $WP_n(z)$ including $P_n(n)$ and $W_n(z)$. The result conducts to explicit formula of a large family of partition functions.

2. Partition polynomials and properties

We reproduce here the family of polynomials constructed by E.T. Bell in the work [2]. Let $n > 0$ be an integer and $C_j$ denote a set of distinct integers > 0. $C_j$ may contain any finite or infinite number of elements. The polynomial $\psi^{(j)}_n(z_j)$ for a given $z_j$ and $C_j$ is defined by

$$\psi^{(j)}_n(z_j) = \sum_{d \in C_j \atop d|n} dz_j^{n/d}.$$  

We consider now $j = 1, 2, \cdots, s$ and $a_j$ are integers not all zero. Let $z = (z_1, \cdots, z_s)$, the polynomial $\Psi_n(z)$ is defined by

$$\Psi_n (z) = -\sum_{j=1}^{s} a_j \psi^{(j)}_n(z_j).$$

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The partition polynomial $P_n(z, C, a) = P_n(z)$ of rank $n$, argument $z$, set $C = \{C_1, C_2, \cdots, C_s\}$ and index $a = (a_1, a_2, \cdots, a_s)$ is given by

$$
(2.3) \quad P_n(z) = \sum_{\pi(n)} \prod_{j=1}^{n} \left[ \frac{1}{k_j!} \left( \frac{\Psi_j(z)}{g} \right)^{k_j} \right],
$$

where $\pi(n) = \{(k_1, \cdots, k_n) \in \mathbb{N} \setminus k_1 + 2k_2 + \cdots + nk_n = n\}$. Some recursive formulæ for $P_n(z)$ are

$$
(2.4) \quad P_n(z, C, a + b) = \sum_{j=0}^{n} P_j(z, C, a) P_{n-j}(z, C, b),
$$

$$
(2.5) \quad P_n(z, A + B, a + b) = \sum_{j=0}^{n} P_j(z, A, a) P_{n-j}(z, B, b).
$$

We say that $f(t)$ is a generating function for the sequence $(c_n)_{n \in \mathbb{N}}$ of numbers or polynomials if $f(t)$ is written as a power series, then

$$
(2.6) \quad f(t) = \sum_{n \geq 0} c_n t^n.
$$

We do not have to concern ourselves with questions of convergence of the series (2.6), since we are interested in the coefficients $c_n$. We consider such series as formal power series in $t$; for more information about this theory we refer to account given by Ivan Niven [14]. In the same sense the generating function of polynomials $P_n(z)$ is

$$
(2.7) \quad f(t) = f(t, z, C, a) = \prod_{j=1}^{s} (1 - z_j t^{n_j})^{a_j} \prod_{k=2}^{n_2} (1 - z_2 t^{n_2})^{a_2} \cdots \prod_{s} (1 - z_s t^{n_s})^{a_s},
$$

where $\prod_j$ denotes a product with respect to all $n_j$ such that $n_j \in S_j$. E.T. Bell gave the croquet of the proof by considering $f(t) = \exp \log f(t)$ and indicates to use Maclaurin’s theorem to get corresponding series expansion. Here we revisit the proof by using Fià di Bruno formula (see [6]). If $h(t)$ and $g(t)$ are functions for which all the necessary derivatives are defined, then

$$
(2.8) \quad (h \circ g)^{(n)}(t) = \sum_{k_1 + \cdots + k_n = n} \frac{n!}{k_1! \cdots k_n!} h^{(k)}(g(t)) \prod_{i=1}^{n} \left( \frac{g^{(i)}(t)}{i!} \right)^{k_i},
$$

where $f^{(n)}(t) = \frac{d^n f}{dt^n}$. A detailed proof is given by Steven Roman [16] by using the umbral calculus. Then the coefficients $[t^n]h \circ g(t)$ of the series expansion of $h \circ g(t)$ take the form $[t^0]h \circ g(t) = h(g(0))$ and for $n \geq 1$;

$$
(2.9) \quad [t^n]h \circ g(t) = \sum_{k_1 + 2k_2 + \cdots + nk_n = n} \frac{h^{(k)}(g(0))}{k_1! \cdots k_n!} \prod_{i=1}^{n} \left( \frac{g^{(i)}(0)}{i!} \right)^{k_i}.
$$
We know that
\[ f(t) = \exp \left( \sum_{j=1}^{s} \sum_{n_j \in S_j} a_j \log (1 - z_j t^{n_j}) \right), \]
but we have \( \log (1 - t) = -\sum_{n \geq 1} t^{n} \). Then
\[ f(t) = \exp \left( -\sum_{k \geq 1} \frac{1}{k} \sum_{j=1}^{s} \sum_{n_j \in S_j} a_j z_j^{k n_j} \right). \]
For computing the formal power series of \( f \) let
\[ g(t) = -\sum_{k \geq 1} \frac{1}{k} \sum_{j=1}^{s} \sum_{n_j \in S_j} a_j z_j^{k n_j}, \]
then \( f(t) = e^{g(t)} \). Furthermore for \( n \geq 1 \) we have
\[ [t^{n}] f(t) = \sum_{k=1}^{n} \frac{n!}{k! \cdots n!} \prod_{i=1}^{n} \left( -\sum_{j=1}^{s} \sum_{n_j \in S_j} a_j z_j^{i/n_j} \right)^{k_i}. \]
Since
\[ \sum_{j=1}^{s} \sum_{n_j \in S_j} a_j z_j^{i/n_j} = \sum_{j=1}^{s} a_j \Psi_i(z_j) = \Psi_i(z), \]
then
\[ (2.10) \quad P_n(z) = \sum_{k=1}^{n} \sum_{k_1 + \cdots + k_n = k} \prod_{j=1}^{n} \left[ \frac{1}{k_j} \left( \frac{\Psi_j(z)}{j} \right)^{k_j} \right], \]
identical to the expression (2.3) below.

3. Explicit formula of reciprocal partition polynomials

Let us introducing the family \( W_n(z) = W_n(z, C, a) \) of partition polynomials, where \( W_n(z) \) is generated by the function \( 1/f(t) \). The Cauchy product of generating functions of \( P_n(z) \) and \( W_n(z) \) equal 1. Hence \( W_0(z) = 1 \) and others are obtained from the recursive formula
\[ (3.1) \quad W_n(z) = -\sum_{k=0}^{n-1} W_k(z) P_{n-k}(z). \]
For the prove we refer to [8]. From the definition of \( W_n(z) \), we can write \( W_n(z) = P_n(z, C, -a) \) with \( -a = (-a_1, \cdots, -a_s) \). According to the relation \( 1/f(t) = \exp(-\log f(t)) \), the following theorem is immediate.
\textbf{Theorem 3.1.}

\begin{equation}
W_n(z) = \sum_{r(n)} (-1)^{\Sigma k_j} \prod_{j=1}^{n} \left[ \frac{1}{k_j!} \left( \Psi_j(z) \right)^{k_j} \right].
\end{equation}

We can prove the identity (3.2) with another method; which is based on exponential partial Bell polynomials. To learn more about this technique we refer to recent works [7, 9, 10].

\section{Expression of restricted partition function $W(n,d^s)$.}

Special case, namely, restricted partition function $W(n,d^s) = W(n, \{d_1, d_2, \ldots, d_s\})$ is completely studied, but the given formulas still so much big. $W(n,d^s)$ is a number of partitions of $n$ into positive integers $d_1, d_2, \ldots, d_s$ each not greater than $s$. The corresponding generating function has the form

\begin{equation}
\prod_{i=1}^{s} \frac{1}{1 - x^{d_i}} = \sum_{n \geq 0} W(n,d^s) x^n.
\end{equation}

$W(n,d^s)$ satisfies the basic recursive relation

\begin{equation}
W(n,d^s) - W(n-d_s,d^s) = W(n,d^{s-1}).
\end{equation}

Sylvester ([20, 21]) showed that the restricted partition function may be presented as a sum of Sylvester waves

\begin{equation}
W(n,d^s) = \sum_j W_j(n,d^s),
\end{equation}

where the sum $\sum_j$ is over all distinct factors of the elements in the set $d^s$. B.Y. Rubinstein and L.G. Fel (see [17]) proved that

\begin{equation}
W_j(n,d^s) = \frac{1}{(\omega_j - 1)! \pi_{\omega_j}} \sum_{\rho_j} \rho_j^{-n} \prod_{i=1}^{\omega_j} \left( 1 - \rho_j^{d_i} \right) \times
\end{equation}

\begin{equation}
\sum_{k=0}^{\omega_j-1} B_k^{(w_j)} (n + n_{w_j} | d^{w_j}) H_{w_j-1-k}^{(s-w_j)} (n_{w_j} - n_{w_j}, \rho_j | d^{s-w_j}),
\end{equation}

where

\begin{equation}
\frac{e^{st} \prod_{i=1}^{m} (1 - \rho_i^{d_i})}{\prod_{i=1}^{m} e^{dt} - \rho_i^{d_i}} = \sum_{n \geq 0} H_n^{(m)} (\rho | d_n) \frac{t^n}{n!}, \rho_i^{d_i} \neq 1.
\end{equation}

and

\begin{equation}
\frac{e^{st} \prod_{i=1}^{m} d_i}{\prod_{i=1}^{m} (e^{dt} - 1)} = \sum_{n \geq 0} B_n^{(m)} (s | d_n) \frac{t^n}{n!}.
\end{equation}
$W(n, d^s)$ corresponds to $W_n(z)$ in the case $z = a = 1 = (1, 1, ..., 1)$ and $C_i = \{d_i\}$ for all $i = (1, 2, \cdots, s)$. According to these conditions we have for all $j = 1, \cdots, s$:

$$\Psi_n(1) = - \sum_{j < k \leq s} d_j.$$

We define the restricted divisor function to $S$; $d_S(j) = \sum_{d | j} d_i$, then a simple formula of $W(n, d^s)$ is given by the following theorem.

**Theorem 3.2.**

(3.7) $$W(n, d^s) = \sum_{\pi(n)} \prod_{j=1}^{n} \left[ \frac{1}{k_j!} \left( \frac{d_S(j)}{j} \right)^{k_j} \right].$$

### 3.2. Generating functions of partitions.

Let a set $S \subset \mathbb{N}$ and $p(n|S)$ the number of partitions of $n$ into elements of $S$. Then the generating function of $p(n|S)$ is

(3.8) $$\prod_{k \in S} \frac{1}{1 - t^k} = \sum_{n \geq 0} p(n|S)t^n.$$  

If $p_m(n|S)$ is the number of partitions with exactly $m$-part in $S$, then the generating function is

(3.9) $$\prod_{k \in S} \frac{1}{1 - xt^k} = \sum_{m,n \geq 0} p_m(n|S)x^mt^n.$$  

In fact we have

$$\prod_{k \in S} \frac{1}{1 - t^k} = \prod_{k \in S} \sum_{m, n \geq 0} t^{km} = \sum_{m \geq 0} t^{\sum_{k \in S} mk}.$$  

Then

$$[n! \prod_{k \in S} 1 - t^k] = \sum_{m \geq 0} 1.$$  

For the second, we have

$$\prod_{k \in S} \frac{1}{1 - xt^k} = \prod_{k \in S} \sum_{m \geq 0} x^m t^{mk} = \sum_{m \geq 0} x^{\sum_{k \in S} mk} t^{\sum_{k \in S} km_k}.$$  

Then

$$[x^m t^n] \prod_{k \in S} \frac{1}{1 - xt^k} = \sum_{\sum_{k \in S} mk = m, \sum_{k \in S} km_k = n} 1.$$  

When $S = \mathbb{N}$, the corresponding generating functions may be displayed, respectively, as

(3.10) $$\prod_{k=1}^{\infty} \frac{1}{1 - t^k} = \sum_{n \geq 0} p(n)t^n.$$
The elementary aspects of the theory of partitions are given in detail in [11, Chap. 19]. A partition of a positive integer \( n \) may be thought as an unordered representation of \( n \) as a sum of other positive integers. Thus \( 3 + 2, 2 + 3 \) represent the same partition of 5. Euler gave the first recurrent formula (see [5]) of the arithmetical function \( p(n) \):

\[
p(n) = \sum_{k=1}^{n} (-1)^{k+1} \left[ p \left( n - \frac{1}{2}k(3k - 1) \right) + p \left( n - \frac{1}{2}(3k + 1) \right) \right].
\]

Andrews after proving this formula (see [1]), he says No one has ever found a more efficient algorithm for computing \( p(n) \). It computes a full table of values of \( p(n) \) for \( n > 5 \), in time \( O(n^{3/2}) \). In 1917 Hardy and Ramanujan (see [12]) applied on (3.10) the theory of functions of complex variables and developed a method which yields an asymptotic formula for \( p(n) \):

\[
p(n) = \frac{1}{2\pi \sqrt{2}} \sum_{k \leq \alpha \sqrt{n}} A_k(n) \frac{d}{dn} \left( \frac{\exp \left( C \sqrt{n-1/24} \right)}{\sqrt{n-1/24}} \right) + O(n^{-1/4}),
\]

with \( \alpha \) as an arbitrary constant, and

\[
A_k(n) = \sum_{\substack{h \mod k \\&\&\&\& h \neq 0}} \omega_{h,k} e^{-2\pi i n/k}, C = \pi \sqrt{2/3}.
\]

Rademacher (see [15]) replaced the asymptotic formula (3.13) by the equality

\[
p(n) = \frac{1}{2\pi \sqrt{2}} \sum_{k \geq 1} A_k(n) \frac{d}{dn} \left( \frac{\exp \left( C \sqrt{n-1/24} \right)}{\sqrt{n-1/24}} \right),
\]

in which the series is absolutely convergent. Recently; Aleksa Srdanov (see [19]) investigated the arithmetical function \( p(n) \). First he defines numbers \( p(n,m) \) the number of all possible partitions of the number \( n \) having exactly \( m \) parts, \((1 \leq m \leq n)\). Then \( p(n) = \sum_{k=1}^{n} p(n,k) \). Finally he computed in different way the expression of numbers \( p(n,k) \); for more details, we refer to Theorems 1, 2, 3 and 4 in the work [19].

For \( S = \mathbb{N} \), we have

\[
d_S(j) = \sigma(j) = \sum_{i|j} i
\]

and the following theorem is immediate
Theorem 3.3. We have \( p(0) = 1 \) and for \( n \geq 1 \);

\[
p(n) = \sum_{\pi(n)} \prod_{j=1}^{n} \left[ \frac{1}{k_j} \left( \frac{\sigma(j)}{j} \right)^{k_j} \right].
\]

If there is some difficulties for computing \( \sigma(n) \), we purpose the following formula:

\[
\sigma(n) = \prod_{j=1}^{m} \left[ \left\lfloor \frac{b_j}{2} \right\rfloor \sum_{k=0}^{\left\lfloor \frac{b_j}{2} \right\rfloor} (-1)^k \left( \begin{array}{c} b_j - k \\ k \end{array} \right) \left( p_j \right)^k (1 + p_j)^{b_j - 2k} \right],
\]

when we know the decomposition of \( n \) on prime factors; \( n = p_1^{b_1} \cdot \ldots \cdot p_m^{b_m} \).

4. Generalized partition polynomials

Let \( w = (w_1, \ldots, w_l) \in \mathbb{C}^l \), \( b = (b_1, \ldots, b_l) \in \mathbb{N}^l \) and \( S = (S_1, \ldots, S_l) \); where \( S_j \) contains finite or infinite number of elements. We introduce the generalized partition polynomials \( WP_n(w, z) \) including polynomials \( P_n(z) \) and \( W_n(z) \) by the generating function:

\[
\sum_{n=0}^{\infty} WP_n(w, z) t^n = f(t, w, S, b).
\]

The following corollary is immediate

Corollary 4.1.

\[
WP_n(w, z) = \sum_{m=0}^{n} P_m(z) W_{n-m}(w).
\]

4.1. Application to some partition functions. In the literature finitely many partition functions are studied. We focus our interest in partition functions \( a(n) \), \( \bar{a}(n) \), \( \psi^\star(n) \) and \( \varphi^\star(n) \). The arithmetical function \( a(n) \) counts the number of partitions of weight \( n \) such that the even parts can appear in two colors (see [3, 18]). So, for example, \( a(3) = 4 \) where the colored partitions in question are

\[
3, 2_1 + 1, 2_2 + 1 \quad \text{and} \quad 1 + 1 + 1.
\]

Its generating function takes the form

\[
\prod_{n=1}^{\infty} (1 - t^n) \prod_{n=1}^{\infty} (1 - t^{2n}) = \sum_{n\geq 0} a(n) t^n.
\]

Chan (see [3]) proved that

\[
3 \prod_{n=1}^{\infty} (1 - t^{3n}) \prod_{n=1}^{\infty} (1 - t^{6n}) = \sum_{n\geq 0} a(3n + 2) t^n.
\]
Byungchan Kim (see [13]) introduced the overcubic partition function \( \bar{a}(n) \), which counts all of the overlined versions of the cubic partitions counted by \( a(n) \). Its generating function takes the form

\[
\prod_{n=1}^\infty \frac{1 - t^{4n}}{(1 - t^n)^2} = \sum_{n \geq 0} \bar{a}(n)t^n.
\]

Kim provided that

\[
6 \prod_{n=1}^\infty \frac{1 - t^{3n}}{(1 - t^n)^3} = \sum_{n \geq 0} \bar{a}(3n + 2)t^n.
\]

The Ramanujan’s \( \psi \) and \( \phi \) functions are defined as

\[
\psi(t) := \sum_{n \geq 0} t^{(n+1)/2}
\]

and

\[
\phi(t) = 1 + 2 \sum_{n \geq 1} t^{n^2}.
\]

These functions admit the following reformulations

\[
\psi(t) = \prod_{n=1}^\infty \frac{1 - t^{2n}}{(1 - t^n)^2}
\]

and

\[
\phi(t) = \prod_{n=1}^\infty \frac{1 - t^{2n}}{(1 - t^n)^2} \prod_{n=1}^\infty \frac{1 - t^{4n}}{(1 - t^2n)^2}.
\]

\( \psi^*(n) \) and \( \phi^*(n) \) the partition functions generated respectively by \( \psi \) and \( \phi \). The generating functions of these partition functions are special case of the function

\[
F(t) = \prod_{n=1}^\infty \frac{(1 - t^{r_1n})^{\alpha_1} (1 - t^{r_2n})^{\alpha_2}}{(1 - t^{s_1n})^{\beta_1} (1 - t^{s_2n})^{\beta_2}}
\]

Let the function \( I_i \) such that \( I_i(j) = 1 \) if \( i \mid j \) and zero otherwise. We consider \( WP(n) \) the partition function generated by the function \( F(t) \). According to Theorem we conclude that

\[
WP(n) = \sum \prod_{\pi(n)} \sum_{j=1}^n \left[ \frac{1}{k_j^j} \left( \frac{b_1I_{s_1}(j)\sigma(j/s_1) + b_2I_{s_2}(j)\sigma(j/s_2)}{j} \right) \right]
\]

\[
+ \sum m \sum n \prod_{j=1}^m \left[ \frac{1}{k_j^j} \left( \frac{-a_1I_{r_1}(j)\sigma(j/r_1) - a_2I_{r_2}(j)\sigma(j/r_2)}{j} \right) \right]
\]

\[
\times \sum \prod_{\pi(n-m)} \left[ \frac{1}{k_j^j} \left( \frac{b_1I_{s_1}(j)\sigma(j/s_1) + b_2I_{s_2}(j)\sigma(j/s_2)}{j} \right) \right]
\]
From this identity follow the explicit formula for considered partition functions:

\[(4.13)\quad a(n) = \sum_{\pi(n)} (-1)^{\sum k_j} \prod_{j=1}^{n} \left[ \frac{1}{k_j!} \left( \frac{\sigma(j) + I_2(j)\sigma(j/2)}{j} \right)^{k_j} \right],\]

\[(4.14)\quad a(3n+2) = 3 \sum_{\pi(n)} 4^{\sum k_j} \prod_{j=1}^{n} \left[ \frac{1}{k_j!} \left( \frac{\sigma(j) + I_2(j)\sigma(j/2)}{j} \right)^{k_j} \right] + 3 \sum_{m=1}^{n} \sum_{\pi(m)} (-3)^{\sum k_j} \prod_{j=1}^{m} \left[ \frac{1}{k_j!} \left( \frac{I_3(j)\sigma(j/3) + I_6(j)\sigma(j/6)}{j} \right)^{k_j} \right] \times \sum_{\pi(n-m)} 4^k \prod_{j=1}^{n-m} \left[ \frac{1}{k_j!} \left( \frac{\sigma(j) + I_2(j)\sigma(j/2)}{j} \right)^{k_j} \right],\]

\[(4.15)\quad \bar{a}(n) = \sum_{\pi(n)} \prod_{j=1}^{n} \left[ \frac{1}{k_j!} \left( \frac{2\sigma(j) + I_2(j)\sigma(j/2)}{j} \right)^{k_j} \right] + \sum_{m=1}^{n} \sum_{\pi(m)} (-1)^{\sum k_j} \prod_{j=1}^{m} \left[ \frac{1}{k_j!} \left( \frac{I_4(j)\sigma(j/4)}{j} \right)^{k_j} \right] \times \sum_{\pi(n-m)} \prod_{j=1}^{n-m} \left[ \frac{1}{k_j!} \left( \frac{2\sigma(j) + I_2(j)\sigma(j/2)}{j} \right)^{k_j} \right],\]

\[(4.16)\quad \bar{a}(3n+2) = 6 \sum_{\pi(n)} \prod_{j=1}^{n} \left[ \frac{1}{k_j!} \left( \frac{8\sigma(j) + 3I_2(j)\sigma(j/2)}{j} \right)^{k_j} \right] + \sum_{m=1}^{n} \sum_{\pi(m)} \prod_{j=1}^{m} \left[ \frac{1}{k_j!} \left( \frac{-6I_3(j)\sigma(j/3) - 3I_4(j)\sigma(j/4)}{j} \right)^{k_j} \right] \times \sum_{\pi(n-m)} \prod_{j=1}^{n-m} \left[ \frac{1}{k_j!} \left( \frac{8\sigma(j) + 3I_2(j)\sigma(j/2)}{j} \right)^{k_j} \right],\]

\[(4.17)\quad \psi^*(n) = \sum_{\pi(n)} \prod_{j=1}^{n} \left[ \frac{1}{k_j!} \left( \frac{\sigma(j)}{j} \right)^{k_j} \right] + \sum_{m=1}^{n} \sum_{\pi(m)} (-2)^{\sum k_j} \prod_{j=1}^{m} \left[ \frac{1}{k_j!} \left( \frac{I_4(j)\sigma(j/2)}{j} \right)^{k_j} \right] \times \sum_{\pi(n-m)} \prod_{j=1}^{n-m} \left[ \frac{1}{k_j!} \left( \frac{\sigma(j)}{j} \right)^{k_j} \right],\]
\[
\varphi^\ast (n) = \sum_{n(n)} 2 \sum_{k_j} \prod_{j=1}^{n} \left( \frac{\sigma(j) + I_4(j)\sigma(j/4)}{j} \right)^{k_j} \\
+ \sum_{m=1}^{n} \sum_{\pi(n)} (-5)^{\sum_{j=1}^{m} \prod_{j=1}^{n-m} \left( \frac{\sigma(j) + I_4(j)\sigma(j/4)}{j} \right)^{k_j}} \times \sum_{\pi(n-m)} 2 \sum_{k_j} \prod_{j=1}^{n-m} \left( \frac{\sigma(j) + I_4(j)\sigma(j/4)}{j} \right)^{k_j}.
\]

(4.19)

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