Inhomogeneities and cosmological expansion

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Key words inhomogeneous cosmology, cosmological acceleration, luminosity distance

I review work on the influence of inhomogeneities in the matter distribution on the determination of the luminosity distance of faraway sources, and the connection to the perceived cosmological acceleration.

Talk at the 9\textsuperscript{th} Hellenic School and Workshops: Standard Model and Beyond – Standard Cosmology

1 An inhomogeneous model of the Universe

In inhomogeneous cosmologies the local volume expansion does not necessarily coincide with the expansion rate deduced from the the luminosity distance of faraway sources [1]. An interesting possibility is that the growth of inhomogeneities in the matter distribution affects the astrophysical observations similarly to accelerated expansion in a homogeneous Friedmann-Robertson-Walker (FRW) background. This may happen if the luminosity distance is increased because of the propagation of light through inhomogeneous regions before reaching the observer.

At length scales above $\sim 50 \text{ Mpc}$ the density contrast in the Universe is at most of $O(1)$. A popular modelling of the cosmological background is based on the Lemaitre-Tolman-Bondi (LTB) metric. This geometry has spherical symmetry, but can be inhomogeneous along the radial direction. Several spherical regions, described by the LTB metric, can be embedded in a homogeneous FRW background. This construction is characterized as a LTB Swiss-cheese model. There are two possible choices for the location of an observer, which are consistent with the isotropy of the Cosmic Microwave Background: i) in the interior of a spherical inhomogeneity, near its center; ii) in the homogeneous region, with the light travelling across several inhomogeneities during its propagation from source to observer.

The LTB metric can be written in the form

$$ds^2 = -dt^2 + \frac{R^2(t, r)}{1 + f(r)} dr^2 + R^2(t, r) d\Omega^2,$$

where $d\Omega^2$ is the metric of a two-sphere and $f(r)$ is an arbitrary function. The function $R(t, r)$ describes the location of a shell of matter marked by $r$ at the time $t$. The Einstein equations give

$$\dot{R}^2(t, r) = \frac{1}{8\pi M^2} \frac{\mathcal{M}(r)}{R} + f(r)$$

where $\mathcal{M}'(r) = 4\pi R^2 \rho(t, r) R'$ and $G = (16\pi M^2)^{-1}$. The generalized mass function $\mathcal{M}(r)$ of the pressureless fluid with energy density $\rho(t, r)$ can be chosen arbitrarily.

We parametrize the energy density at some arbitrary initial time as $\rho_i(r) = \rho(0, r) = (1 + \epsilon(r)) \rho_0, i$. The initial energy density of the homogeneous background surrounding the spherical inhomogeneity is $\rho_0, i$. If the size of the inhomogeneity is $r_0$, the matching with the homogeneous metric in the exterior requires $4\pi \int_{r_0}^{-} r^2 \epsilon(r) dr = 0$, so that $\mathcal{M}(r_0) = 4\pi r_0^3 \rho_0, i / 3$. As we assume that the homogeneous metric is flat, we also have $f(r_0) = 0$. The typical evolution of such an inhomogeneous background is depicted

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Fig. 1 The evolution of the density profile for a central underdensity surrounded by an overdensity.

in fig. 1 The configuration models a void, with a central underdensity surrounded by an overdensity. The central density is reduced during the cosmological evolution, while the matter is concentrated in the periphery.

2 Propagation of light beams and luminosity distance

The optical equations describe the evolution of the characteristics of a beam (area and shape of its cross-section) during its propagation in a given gravitational background. For a LTB Swiss-cheese model, with a density contrast not much larger than 1, the relevant equation is

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\frac{1}{\sqrt{A}} \frac{d^2 \sqrt{A}}{d\lambda^2} = -\frac{1}{4M^2 \rho} (k^0)^2,
\]

where \(A\) is the cross section of a light beam, \(\lambda\) an affine parameter along the null trajectory and \(k^i = dx^i/d\lambda\). We neglect the shear tensor, which describes deformations of the beam, because it is important only when the beam passes near regions in which the density exceeds the average one by several orders of magnitude. We assume that the light emission near the source is not affected by the large-scale geometry. By choosing an affine parameter that is locally \(\lambda = t\) in the vicinity of the source, we can set \(d\sqrt{A}/d\lambda \bigg|_{\lambda=0} = \sqrt{\Omega_s}\), where \(\Omega_s\) is the solid angle spanned by the beam when the light is emitted by a point-like isotropic source. This relation and \(\sqrt{A} \bigg|_{\lambda=0} = 0\) provide the initial conditions for eq. 3.

In order to define the luminosity distance, we consider photons emitted within a solid angle \(\Omega_s\) by an isotropic source with luminosity \(L\). These photons are detected by an observer for whom the light beam has a cross-section \(A_o\). The redshift factor is \(1 + z = \omega_s/\omega_o = k^0_s/k^0_o\). The luminosity distance is \(D_L = (1 + z)\sqrt{A_o/\Omega_s}\), with \(A_o\) the beam area measured by the observer for a beam emitted within \(\Omega_s\). The beam area can be calculated by solving eq. 3. We consider light beams that pass through several inhomogeneities. The light is emitted from a point at the edge of the first inhomogeneous region, with a random initial direction, and moves through it. Subsequently, the beam crosses the following inhomogeneity in a similar fashion. The angle of entry into the new inhomogeneity is assumed again to be random. The initial conditions are set by the values of \(\sqrt{A}\) and \(d\sqrt{A}/d\lambda\) at the end of the first crossing. This process is repeated until the light arrives at the observer.
from 0.5 to 2, while the average value of \( w \) is again negative and of 1 \( \pm 1 \) h\(^{-1}\) Mpc the error increases from 0.015 to 0.025 as \( z \rightarrow 1 \) h\(^{-1}\) Mpc or larger. The total integral of the distributions has been normalized to 1 in all cases, so that they are in fact probability densities. They have similar profiles that are asymmetric around zero. Each distribution has a maximum at a value larger than zero and has been normalized to 1 in all cases, so that they are in fact probability densities. They have similar profiles that are asymmetric around zero. Each distribution has a maximum at a value larger than zero and a long tail towards negative values. The average deviation is zero to a good approximation in all cases. This is expected because of flux conservation [4,5]: As long as the light propagation in an inhomogeneous background does not modify significantly the redshift, the energy may be redistributed in various directions through gravitational lensing by inhomogeneities, but the total flux is conserved and remains the same as in a FRW background.

The longer tail of the distribution towards small luminosity distances is a consequence of the presence of a thin and dense spherical shell around each central underdensity. The portion of light beams that cross several shells is small. However, the focusing is substantial for such beams and the resulting luminosity distance much shorter than the average. The effect of the long tail is compensated by the shift of the maximum of the distribution towards positive values. The form of the distribution is very similar to that derived in studies modelling the inhomogeneities through the standard Swiss-cheese model [6]. In that case the strong focusing is generated by the very dense concentration of matter at the center of each spherical inhomogeneity. We emphasize, however, that the two models have a different region of applicability. The standard Swiss-cheese model [6] is appropriate for length scales of \( O(1) h^{-1}\) Mpc or smaller, while the LTB Swiss-cheese model [3,4] for scales of \( O(10) h^{-1}\) Mpc or larger.

The width of the distribution \( \delta_d \) determines the error induced to cosmological parameters derived through the luminosity curve, while the location of its maximum \( \delta_m \) the bias in such determinations. A small sample of data is expected to favour values of the luminosity distance near the maximum of the distribution, and thus generate a bias [3,4]. An important quantity is the effective equation of state \( w = p/\rho \) deduced from astrophysical data. The presence of inhomogeneities induces a statistical error in \( w \), as well as a shift of its average value if the sample is small [4]. For inhomogeneities with a typical size of 40 h\(^{-1}\) Mpc the error is \( \delta w \approx 0.015 \) for all \( z \) between 0.5 and 2, while the average value \( \bar{w} \) for a small sample is negative and of \( O(10^{-3}) \). For a size of 133 h\(^{-1}\) Mpc the error increases from 0.015 to 0.025 as \( z \) increases from 0.5 to 2, while the average value \( \bar{w} \) is again negative and of \( O(10^{-3}) \). For a size of 400 h\(^{-1}\) Mpc the

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**Fig. 2** The distribution of luminosity distances for various redshifts in the LTB Swiss-cheese model if the inhomogeneities have a characteristic scale of 133 h\(^{-1}\) Mpc.
error increases from 0.03 to 0.05 for $z$ increasing from 0.5 and 2, while the average is $\bar{w} \simeq -0.015$. The values of $\delta_d$ and $\delta_m$ can be compared to those generated by the effects of gravitational lensing at scales characteristic of galaxies or clusters of galaxies (modelled through the standard Swiss-cheese model) [6]. The typical values of $\delta_d$ and $\delta_m$ are larger by at least an order of magnitude than the ones we obtained. The reason is the difference in the density contrast.

We conclude that, if the source and the observer have random locations, the presence of inhomogeneities with large length scales - even comparable to the horizon distance - and density contrast of $\mathcal{O}(1)$ does not influence the propagation of light sufficiently in order to explain the supernova data without dark energy. For this to be possible the effect on $w$ would need to be close to 1. However, the errors induced in the measurements of the luminosity distance of high-redshift sources can be substantial, depending on the modelling of the inhomogeneous background. Care must be taken in the extraction of cosmological parameters from such data.

3 Analytical estimates and a central observer

For a smooth density field with a contrast of $\mathcal{O}(1)$, the size of an inhomogeneity $r_0$ determines its effect on quantities such as redshift and luminosity distance of a source. An analytical estimate of the effect is possible [7]. The relevant quantity is the dimensionless ratio $\bar{H} = r_0 H / r_0$ to the horizon distance $1/H$.

If the observer is located at a random position within the homogeneous region, each crossing of an inhomogeneity produces an effect of $\mathcal{O}(\bar{H}^3)$ for the travel time and the redshift. For the beam area and the luminosity distance the effect is of $\mathcal{O}(\bar{H}^2)$. However, flux conservation implies that positive and negative contributions to the beam area cancel during multiple crossings. The size of the maximal average effect of each crossing on the beam area and luminosity distance is set by the effect on the redshift, which is of $\mathcal{O}(\bar{H}^2)$ [4,5]. Photons with redshift $\sim 1$ pass through $\sim (1/H)/r_0 = \bar{H}^{-1}$ inhomogeneities before arrival, assuming that these are tightly packed. As a result, the expectation is that the maximal final effect for a random position of the observer is of $\mathcal{O}(\bar{H}^2)$ for all quantities. Allowing for corrections arising from numerical factors, this conclusion is supported by the detailed analysis of [3,4].

For an observer located at the center of a spherical inhomogeneity, the deviations of travelling time, redshift, beam area and luminosity distance from their values in a homogeneous background are of $\mathcal{O}(\bar{H}^2)$. The luminosity distance is increased by the presence of a central underdensity, while it is reduced by a central overdensity. The increase in the luminosity distance if the observer is located near the center of a large void can be employed for the explanation of the supernova data. An increase of $\mathcal{O}(10\%)$, as required by the data, would imply the existence of a void with size close to $10^3 h^{-1}$ Mpc. Numerical factors can reduce the required size, depending on the details of the particular cosmological model employed. However, a typical void with size of $\mathcal{O}(10) h^{-1}$ Mpc leads to a negligible increase of the luminosity distance.

Acknowledgements  N. T. is supported in part by the EU Marie Curie Network “UniverseNet” (HPRN–CT–2006–035863) and the ITN network “UNILHC” (PITN-GA-2009-237920).

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