Small mass expansion of functional determinants on the generalized cone

Guglielmo Fucci and Klaus Kirsten

Department of Mathematics, Baylor University, Waco, TX 76798, USA
E-mail: Guglielmo_Fucci@Baylor.edu and Klaus_Kirsten@Baylor.edu

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Abstract
In this paper we compute the small mass expansion for the functional determinant of a scalar Laplacian defined on the bounded, generalized cone. In the framework of zeta function regularization, we obtain an expression for the functional determinant valid in any dimension for both Dirichlet and Robin boundary conditions in terms of the spectral zeta function of the base manifold. Moreover, as a particular case, we specify the base to be a $d$-dimensional sphere and present explicit results for $d = 2, 3, 4, 5$.

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1. Introduction

The study of functional determinants of elliptic second-order partial differential operators is of major importance in mathematical physics and quantum field theory [11, 12, 26]. In mathematics they encode particular important information about the spectrum of the operator under consideration. In quantum field theory, instead, functional determinants of elliptic operators are used in order to evaluate the one-loop effective action [11, 12, 20, 26, 29]. In the majority of cases of physical interest, one has to deal with second-order hyperbolic linear partial differential operators describing the dynamics of fields in Minkowski spacetime. By performing a Wick rotation to Euclidian spacetime, the dynamical operator becomes elliptic. The mathematical advantage lies in the fact that for elliptic self-adjoint partial differential operators on compact manifolds the spectral theorem holds. This means, in particular, that the discrete spectrum is bounded from below and that all the eigenvalues can be ordered as follows: $-|C| \leq \lambda_1 \leq \lambda_2 \ldots \rightarrow +\infty$. One can then construct the spectral $\zeta$-function of the operator, say $L$, as

$$
\zeta(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}, \quad (1.1)
$$
which is convergent for $\text{Re } s > D/2$, with $D$ being the dimension of the manifold under consideration. One can analytically continue, in a unique way, $\zeta(s)$ to a meromorphic function in the whole complex plane which coincides with (1.1) in its domain of convergence.

The functional determinant of the operator $L$ is defined as a product of all its eigenvalues. Obviously such an expression is divergent and, thus, needs to be regularized. One way of making sense of this infinite product is using $\zeta$-function regularization. In this framework, one defines the functional determinant of $L$ to be

$$\text{Det } L = \exp \left[ -\zeta'(0) \right]$$

(1.2)

where the derivative of $\zeta(s)$ at $s = 0$ has been obtained by analytic continuation in the way mentioned before. This object is of fundamental importance in quantum field theory because it yields the one-loop effective action $\Gamma_{(1)}$ as

$$\Gamma_{(1)} = \sigma \ln \text{Det } L,$$

(1.3)

where $\sigma = 1/2, -1/2$ according to whether we are dealing with neutral scalar fields or Dirac fields. Spectral zeta function techniques have a variety of applications in physics other than the computation of regularized functional determinants [20, 21]. In fact, zeta function regularization is a rather powerful tool, for instance, in the study of the Casimir effect [6], Kaluza–Klein compactification, strings and $p$-branes etc [21].

In this paper we utilize $\zeta$-function techniques in order to obtain the small mass expansion of the functional determinant for the scalar Laplacian on the generalized cone. This work represents an extension to the massive case of the investigations initiated in the paper [3]; see also [5, 17]. Apart from the mathematical and theoretical interest of computing small mass corrections to the massless functional determinant, the main physical interest lies in the evaluation of the one-loop effective action or the Casimir energy for massive fields on orbifolds. In fact an orbifold is defined locally as the quotient space of a smooth manifold $X$ and a discrete isometry group $G$ acting linearly on $X$. In general the action of the group $G$ on $X$ will have fixed points, and these points are then mapped to conical singularities in the quotient space. We can thus say that orbifolds can be locally represented as generalized cones. These geometries are of fundamental importance in order to compute the one-loop effective action in field theoretical models containing orbifold compactification (see e.g. [31]). It is hoped that by combining a systematic small mass expansion with a large mass expansion from the heat kernel the intermediate range of mass values for which serious numerical work is needed can be minimized. That this can be achieved in principle has recently been demonstrated convincingly in [25] in the context of the QCD instanton determinant.

The outline of this paper is as follows. In section 2 we describe the geometry of a bounded generalized cone and we introduce the basic objects of our study. In particular eigenmodes, eigenvalues and the related zeta function of the Laplacian on the cone are discussed. We consider two boundary conditions, namely Dirichlet and Robin, and the functional determinants for massive scalar fields on the cone are evaluated for each case. Specializing the generalized cone to the case of the ball very explicit answers involving the zeta function of Riemann are given. The appendix contains a list of polynomials needed for the computation and the conclusions point to the most important results of this paper.

2. Geometric background and $\zeta$-function

In this paper we consider a particular bounded manifold which is known as the generalized cone. The generalized cone is defined as the $D = (d + 1)$-dimensional manifold $\mathcal{M} = I \times \mathcal{N}$, where $\mathcal{N}$ is the base manifold, supposed to be a smooth Riemannian manifold possibly with
boundary, and $I = [0, 1] \subseteq \mathbb{R}$. The generalized cone is endowed with the hyperspherical metric \cite{9}

$$ds^2 = dr^2 + r^2 d\Sigma^2,$$

(2.1)

where $d\Sigma^2$ represents the metric on $\mathcal{N}$ and $r \in I$. It is known \cite{3} that the curvatures on $\mathcal{M}$ and on the base $\mathcal{N}$ are conformally related as follows:

$$R^{ij} = \frac{1}{r^2} \left[ \hat{R}^{ij} - (\delta_i^k \delta^j_l - \delta_i^l \delta^j_k) \right], \quad R^i_j = \frac{1}{r^2} \left[ \hat{R}^i_j - (d - 1) \delta^i_j \right],$$

$$R = \frac{1}{r^2} \left[ \hat{R} - d(d - 1) \right],$$

(2.2)

where $R$ and $\hat{R}$ are the curvature tensors, respectively, on $\mathcal{M}$ and $\mathcal{N}$. It is easily seen, from relations (2.2), that in general the manifold under consideration has a singularity at the origin $r = 0$.

Let $\Delta_\mathcal{M}$ be the Laplacian defined on the manifold $\mathcal{M}$ and let $L^2(\mathcal{M})$, the space of square integrable scalar functions $\phi$ on the generalized cone, be its domain. We are interested in the following eigenvalue problem:

$$(-\Delta_\mathcal{M} + m^2)\phi = \alpha^2 \phi,$$

(2.3)

where the parameter $m$ represents the mass of the scalar field. In hyperspherical coordinates, the Laplacian is separable and can be written as

$$\Delta_\mathcal{M} = \frac{\partial^2}{\partial r^2} + \frac{d}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_\mathcal{N},$$

(2.4)

where $\Delta_\mathcal{N}$ denotes the Laplacian on the base manifold $\mathcal{N}$. The solution of the eigenvalue equation (2.3) which is regular at the origin can be written as a product of a radial function and an angular one as follows:

$$\phi = r^{\nu^2/2} J_\nu(\gamma r) Y_\Omega,$$

(2.5)

where $J_\nu$ is the Bessel function of the first kind and we have set $\alpha^2 = \gamma^2 + m^2$, that is, $\gamma^2$ are the eigenvalues of $-\Delta_\mathcal{M}$. (Allowing for functions that are square integrable but singular at the origin results in the possibility of families of self-adjoint extensions of $\Delta_\mathcal{M}$ \cite{27}, a topic not considered in this paper.) The angular part of (2.5) is the solution of the equation

$$\Delta_\mathcal{N} Y_\Omega = -\lambda^2 Y_\Omega,$$

(2.6)

In order to find the index $\nu$ of the Bessel function we substitute the general solution (2.5) into (2.3). By taking into account the angular relation (2.6), the radial equation becomes

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{\gamma^2 - 1}{r^2} \left( \lambda^2 + \frac{(1 - d)^2}{4} \right) \right] J_\nu(\gamma r) = 0,$$

(2.7)

which is satisfied if the following holds:

$$\nu^2 = \lambda^2 + \frac{(1 - d)^2}{4}.$$  

(2.8)

The introduction of a constant curvature on the manifold $\mathcal{M}$ does not add major complications. In fact, by denoting with $\xi$ the coupling constant, the eigenvalue equation (2.3) acquires the term $-\xi R$ with $R$ given in (2.2). In this case, the angular part of the eigenvalue equation remains the same while the radial part leads to the equation

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{\gamma^2 - 1}{r^2} \left( \lambda^2 + \xi \hat{R} - \xi d(d - 1) + \frac{(1 - d)^2}{4} \right) \right] J_\nu(\gamma r) = 0,$$

(2.9)
which is satisfied if the index $\nu$ of the Bessel function takes the form
\[ \nu^2 = \lambda_2 + \xi R + d(d - 1)(\xi_d - \xi), \]
where $\xi_d = (d - 1)/4d$ represents the conformal coupling constant in $d$ dimensions. The case in which the curvature $\hat{R}$ is not constant has also been described in [3].

Let us now turn our attention to the spectral $\zeta$-function associated with $-\Delta_M + m^2$ on the manifold $\mathcal{M}$. This is defined as
\[ \zeta_M(s) = \sum_{\nu} (\nu^2 + m^2)^{-s}, \]
where we will assume that no negative eigenvalues occur so that we can use the standard branch cut of the logarithm. Our aim is to express the zeta function for the whole manifold $\mathcal{M}$ as much as possible in terms of the base manifold zeta function $\zeta_N$ [9] which is defined as
\[ \zeta_N(s) = \sum_{\nu} d(\nu)(\nu^2)^{-s}, \]
where $d(\nu)$ is the degeneracy of the scalar harmonics $Y(\Omega)$ on $\mathcal{N}$. Without specifying the base manifold $\mathcal{N}$ one is still able to impose boundary conditions [3]; in particular we are interested in setting
\[ J_\nu(\gamma) = 0, \]
for Dirichlet, and
\[ \left( 1 - \frac{d + 1}{2} - \beta \right) J_\nu(\gamma) + \gamma J'_\nu(\gamma) = 0, \]
for Robin. Note that for $\beta = 0$ the last relation reduces to the Neumann boundary condition. For the problem under consideration the eigenvalues of the Laplacian are not known explicitly; however, the boundary conditions (2.13) and (2.14) allow their determination in an implicit fashion.

For Dirichlet boundary conditions a convenient way to express $\zeta_M(s)$ in terms of a contour integral in the complex plane is as follows [3–5, 23, 26]:
\[ \zeta_M(s) = \sum_{\nu} d(\nu) \frac{1}{2\pi i} \int_\Gamma dk [k^2 + m^2]^{-s} \frac{\partial}{\partial k} \ln J_\nu(k), \]
where $\Gamma$ is a contour that encircles all the zeros of $J_\nu(k)$ on the positive real axis in the counterclockwise direction. By deforming the contour $\Gamma$ to the imaginary axis one gets [5, 26]
\[ \zeta_M(s) = \sum_{\nu} d(\nu) \frac{\sin(\pi s)}{\pi} \int_{\text{Im} = 0}^\infty dk [k^2 - m^2]^{-s} \frac{\partial}{\partial k} \ln(k^{-s} J_\nu(k)). \]
At this point, it is very useful to split the spectral $\zeta$-function into two parts [4, 5, 26]. In order to do so we exploit the asymptotic expansion of the modified Bessel functions $I_\nu(k)$ for $\nu \to \infty$ and for $z = k/\nu$ fixed as [22, 28]
\[ I_\nu(vz) \sim \frac{1}{\sqrt{2\pi v}} \frac{e^{\nu z}}{(1 + z^2)^{1/4}} \left[ 1 + \sum_{k=1}^\infty \frac{u_k(t)}{\nu^k} \right], \]
where the polynomials $u_k(t)$ are determined by the recurrence relation
\[ u_{k+1}(t) = \frac{1}{2} t^2 (1 - t^2) u_k'(t) + \frac{1}{8} \int_0^t d\tau (1 - 5\tau^2) u_k(\tau), \]
with \( u_0(t) = 1 \) and
\[
t = \frac{1}{\sqrt{1 + z^2}}, \quad \eta = \sqrt{1 + z^2} + \ln \left[ \frac{z}{1 + \sqrt{1 + z^2}} \right].
\] (2.19)

By adding and subtracting \( N \) leading terms of the asymptotic expansion (2.17) one can write [3–5, 26]
\[
\zeta_{\mathcal{M}}(s) = Z(s) + \sum_{i=1}^{N} A_i(s),
\] (2.20)

where
\[
A_{-1}(s) = \frac{1}{4\sqrt{\pi} \Gamma(s)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \Gamma \left( s + j + \frac{1}{2} \right) m^{2j} \zeta_N \left( s + j - \frac{1}{2} \right),
\] (2.21)
\[
A_0(s) = -\frac{1}{4} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{\Gamma \left( s + j \right)}{\Gamma(s)} m^{2j} \zeta_N \left( s + j \right),
\] (2.22)
\[
A_i(s) = -\frac{1}{\Gamma(s)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{m^{2j} \zeta_N \left( s + j + \frac{1}{2} \right)}{\Gamma \left( b + \frac{i}{2} \right)} \sum_{b=0}^{\infty} x_{i,b} \frac{\Gamma(s + b + j + \frac{i}{2})}{\Gamma \left( b + \frac{i}{2} \right)}
\] (2.23)

and
\[
Z(s) = \sum_{\nu} d(\nu) Z_\nu(s),
\] (2.24)

with
\[
Z_\nu(s) = \frac{\sin(\pi s)}{\pi} \int_1^{\infty} \frac{dz}{z} \left[ (z^2)^{-s} \right] \times \frac{\partial}{\partial z} \left\{ \ln[z^{-\nu} I_\nu(z)] - \ln \left[ \frac{z^{-\nu}}{\sqrt{2\pi} \nu (1 + z^2)^{1/4}} \right] \right\} - \sum_{n=1}^{N} D_n(t)^\nu.
\] (2.25)

The terms \( D_n(t) \) appearing in (2.25) are defined through the cumulant expansion [3–5, 26]
\[
\ln \left[ 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right] \sim \sum_{n=1}^{\infty} \frac{D_n(t)}{\nu^n},
\] (2.26)

and have the polynomial structure
\[
D_n(t) = \sum_{i=0}^{n} x_{i,n} t^{n+2i}.
\] (2.27)

Polynomials (2.27) are listed, up to the sixth order, in the appendix.

### 3. Small mass expansion for Dirichlet boundary conditions

We are particularly interested in the evaluation of the first mass correction to the functional determinant of the scalar Laplacian on the generalized cone. Higher orders can be obtained along the same lines. In this case, the relevant formulas for the terms (2.21)–(2.23) are
\[
A_{-1}(s) = \frac{1}{4\sqrt{\pi} \Gamma(s+1)} \left\{ \zeta_N \left( s - \frac{1}{2} \right) - \frac{m^2 s (s - \frac{1}{2})}{s + 1} \zeta_N \left( s + \frac{1}{2} \right) \right\} + O(m^4),
\] (3.1)
\[ A_0(s) = -\frac{1}{4} \{ \zeta_N(s) - m^2 \zeta_N(s + 1) \} + O(m^4), \]

\[ A_j(s) = -\frac{1}{\Gamma(s)} \sum_{b=0}^j x_{j,b} \frac{\Gamma(s + b + \frac{1}{2})}{\Gamma(b + \frac{1}{2})} \times \left\{ \zeta_N \left( s + \frac{i}{2} \right) - m^2 \left( s + b + \frac{i}{2} \right) \zeta_N \left( s + 1 + \frac{i}{2} \right) \right\} + O(m^4) \]

obtained from the series (2.21)–(2.23) by considering only the contributions coming from \( j = 0 \) and \( j = 1 \) which represent, respectively, the terms of zeroth and second order in the mass parameter.

In order to evaluate the functional determinant we need to compute the derivatives of the quantities (2.21)–(2.25). We would like to point out here that for the explicit calculation that will follow it is sufficient to subtract the first \( D - 1 = d \) terms of the asymptotic expansion in (2.17) [3–5]. Therefore, we will set \( N = d \) in equation (2.20).

Let us start with the function \( Z_\nu(s) \). By performing the derivative and then setting \( s = 0 \) one obtains

\[ Z_\nu'(0) = -\ln I_\nu(m) - \frac{1}{2} \ln 2\pi \nu - \frac{1}{4} \ln \left( 1 + \frac{m^2}{\nu^2} \right) + \nu \sqrt{1 + \frac{m^2}{\nu^2}} + \frac{1}{2} \ln \frac{m^2}{\nu^2} - \nu \ln \left( 1 + \sqrt{1 + \frac{m^2}{\nu^2}} \right) + \sum_{n=1}^d \nu^{-n} D_n \left( \left( 1 + \frac{m^2}{\nu^2} \right)^{-\frac{1}{2}} \right). \]

By expanding the last expression in terms of the mass up to the order \( m^2 \), and by noting that

\[ \sum_{n=1}^d \nu^{-n} D_n \left( \left( 1 + \frac{m^2}{\nu^2} \right)^{-\frac{1}{2}} \right) = \sum_{n=1}^d D_n(1) \frac{\nu^n}{\nu^n} - m^2 \sum_{n=1}^d \nu^{-n-2} \left[ \frac{n D_n(1)}{2} + \sum_{b=0}^n x_{n,b} \right] + O(m^4), \]

we get

\[ Z_\nu'(0) = \ln \Gamma(\nu + 1) + \nu - \nu \ln \nu - \frac{1}{2} \ln 2\pi \nu + \sum_{n=1}^d \frac{D_n(1)}{\nu^n} \]

\[ -m^2 \left\{ \frac{1}{4(\nu + 1)} - \frac{1}{4\nu} + \frac{1}{4\nu^2} + \frac{1}{2\nu^2} \sum_{n=1}^d \frac{D_n'(1)}{\nu^n} \right\} + O(m^4), \]

where \( D_n'(1) \) denotes the derivative of the polynomials \( D_n(t) \) evaluated at \( t = 1 \), and we have used the fact that

\[ \frac{n D_n(1)}{2} + \sum_{b=0}^n x_{n,b} b = \frac{1}{2} D_n'(1). \]

In what follows we will write the spectral \( \zeta \)-function on the base manifold \( \mathcal{M} \) in a way that explicitly shows its structure. Namely, at singular points \( \alpha \) of \( \zeta_\nu \) we have the Laurent expansions

\[ \zeta_\nu(s + \alpha) = \frac{1}{s} \text{Res} \zeta_\nu(\alpha) + \text{FP} \zeta_\nu(\alpha) + O(s), \quad \zeta_\nu'(s + \alpha) = -\frac{1}{s^2} \text{Res} \zeta_\nu(\alpha) + O(s^2), \]

\[ (3.8) \]
and at regular points we have
\[ \zeta_{\mathcal{M}}(s + \alpha) = \zeta_{\mathcal{M}}(\alpha) + s \zeta'_{\mathcal{M}}(\alpha) + O(s^2), \tag{3.9} \]
where Res denotes the residue of the function and FP its finite part. We would like to make an important remark here. As already mentioned earlier, the generalized cone is a singular manifold and for this reason the heat kernel asymptotic expansion in general will contain a non-standard logarithmic term \[7\]. This behavior is translated into the appearance of a pole at \(s = 0\) in the spectral \(\zeta\)-function on \(\mathcal{M}\) \[5, 9, 32\]. In what follows, we will assume that the logarithmic term in the heat kernel asymptotic expansion does not appear; this will allow a standard definition of the functional determinant \[3\]. From a more formal point of view, as said, this means that \(\zeta_{\mathcal{M}}(s)\) is regular at \(s = 0\), which is the case if \(\zeta_{\mathcal{M}}(s)\) is regular at \(s = -1/2\).

By performing the derivative of \(A_{-1}(s)\), equation (3.1), and then by taking the limit as \(s \to 0\) we arrive at the following expression:
\[ A'_{-1}(0) = (\ln 2 - 1) \zeta_{\mathcal{M}} \left( -\frac{1}{2} \right) - \frac{1}{2} \zeta'_{\mathcal{M}} \left( -\frac{1}{2} \right) \]
\[ + m^2 \left[ \frac{1}{4} (2 \ln 2 + 1) \text{Res} \zeta_{\mathcal{M}} \left( \frac{i}{2} \right) - \frac{1}{4} \text{FP} \zeta_{\mathcal{M}} \left( \frac{i}{2} \right) \right] + O(m^4), \tag{3.10} \]
which is obtained by utilizing expressions (3.8) and (3.9) together with the well-known properties of the Gamma function \[24\], furthermore by keeping in mind that \(\zeta_{\mathcal{M}}(s)\) is regular at \(s = -1/2\). For the derivative at \(s = 0\) of \(A_0(s)\) one has
\[ A'_0(0) = -\frac{1}{4} \zeta'_{\mathcal{M}}(0) + \frac{m^2}{4} \text{FP} \zeta_{\mathcal{M}}(1) + O(m^4), \tag{3.11} \]
and, finally, for the terms \(A_i(s)\) in (3.3) we get
\[ A'_i(0) = -\frac{\zeta_{\mathcal{M}}(-i)}{i} \left[ \gamma \text{Res} \zeta_{\mathcal{M}} \left( \frac{i}{2} \right) + \text{FP} \zeta_{\mathcal{M}} \left( \frac{i}{2} \right) \right] - \sum_{b=0}^i \gamma_{x,b} \Psi \left( b + \frac{i}{2} \right) \text{Res} \zeta_{\mathcal{M}} \left( \frac{i}{2} \right) \]
\[ + m^2 \left\{ \frac{\zeta_{\mathcal{M}}(-i)}{i} \text{Res} \zeta_{\mathcal{M}} \left( \frac{i}{2} + 1 \right) + \sum_{b=0}^i \gamma_{x,b} \left( b + \frac{i}{2} \right) \left[ \left( \gamma + \Psi \left( b + \frac{i}{2} \right) \right) \right] \right\} + O(m^4), \tag{3.12} \]
where \(\gamma\) is the Euler–Mascheroni constant, \(\zeta_{\mathcal{M}}(s)\) represents the Riemann \(\zeta\)-function and \(\Psi(s)\) is the logarithmic derivative of the Gamma function.

At this point, we exploit the integral representation of the function \(\ln \Gamma(v)\) \[24\],
\[ \ln \Gamma(v) = \left( v - \frac{1}{2} \right) \ln v - v + \frac{1}{2} \ln 2\pi + \int_0^\infty dt \left( \frac{1}{2} \ln \left( \frac{1}{t} + \frac{1}{e^t - 1} \right) \right) e^{-tv}, \tag{3.13} \]
and the integral representation for the inverse powers of \(v\) as follows:
\[ v^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty dt \ t^{n-1} e^{-tv}, \tag{3.14} \]
to find
\[ Z_n(0) = \int_0^\infty \rho \left[ \sum_{\alpha=1}^d \frac{D_{\alpha}(1)}{\Gamma(n)} t^n + \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right] e^{-tv} \]
\[ - m^2 \int_0^\infty \rho \left( \sum_{\alpha=1}^d \frac{D_{\alpha}(1)}{2\Gamma(n+2)} t^{n+2} - \frac{t e^{-t}}{4} - \frac{t^2}{4} \right) e^{-tv} \]
\[ + O(m^4). \tag{3.15} \]
It will be convenient to define the ‘square root’ heat kernel associated with \( \nu \) [3]:

\[
K_{\nu, t}^{1/2}(t) = \sum_{n=0} d(\nu) e^{-i\nu}.
\]

(3.16)

By introducing a regularization parameter \( z \) and by utilizing definition (3.16), one can rewrite (3.15) as

\[
Z'(0, z) = \int_0^\infty dt t^{-1} \left[ \sum_{n=1}^d \frac{D_n(1)}{\Gamma(n)} t^n + \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right] K_{\nu, t}^{1/2}(t)
\]

\[ - m^2 \int_0^\infty dt t^{-1} \left( \sum_{n=1}^d \frac{D_n'(1)}{2\Gamma(n+2)} t^{n+2} + \frac{t e^{-t}}{4} - \frac{t}{4} + \frac{t^2}{4} \right)
\]

\times K_{\nu, t}^{1/2}(t) + O(m^4),
\]

(3.17)

where the limit as \( z \to 0 \) will be taken to recover \( Z'(0) \).

By recalling that the spectral \( \zeta \)-function is obtained from the heat kernel by an inverse Mellin transform as [12, 23, 26, 30]

\[
\zeta_{\nu, s} = \frac{1}{\Gamma(s)} \int_0^\infty ds \int_0^\infty dt t^{s-1} K_{\nu, t}^{1/2}(t),
\]

(3.18)

we can express function (3.17) in the form

\[
Z'(0, z) = \sum_{n=1}^d \frac{D_n(1)}{\Gamma(n)} t^n \Gamma(z + n) \zeta_{\nu, s} \left( \frac{z + n}{2} \right) + \frac{1}{2} \Gamma(z) \zeta_{\nu, s} \left( \frac{z}{2} \right) - \Gamma(z - 1) \zeta_{\nu, s} \left( \frac{z - 1}{2} \right)
\]

\[
+ \Gamma(z) \zeta_{\nu, s+1} \left( \frac{z + 1}{2} \right) - m^2 \left\{ \sum_{n=1}^d \frac{D_n'(1)}{2\Gamma(n+2)} \Gamma(z + n + 2) \zeta_{\nu, s} \left( \frac{z + n + 2}{2} \right) \right. 
\]

\[
- \frac{1}{4} \Gamma(z + 1) \zeta_{\nu, s} \left( \frac{z + 1}{2} \right) + \frac{1}{4} \Gamma(z + 2) \zeta_{\nu, s} \left( \frac{z + 2}{2} \right)
\]

\[
+ \frac{1}{4} \Gamma(z + 1) \zeta_{\nu, s} \left( z + 1, 1 \right) \right\} + O(m^4),
\]

(3.19)

where we have introduced

\[
\zeta_{\nu, s+1}(z) = \frac{1}{\Gamma(z)} \int_0^\infty ds \int_0^\infty dt t^{s-1} K_{\nu, t}^{1/2}(t)
\]

(3.20)

and

\[
\zeta_{\nu, s}(z, u) = \frac{1}{\Gamma(z)} \sum_{n=1}^d d(\nu) \int_0^\infty ds \int_0^\infty dt t^{s-1} e^{-(u+u)t} = \frac{1}{\Gamma(z)} \int_0^\infty ds \int_0^\infty dt t^{s-1} e^{-ut} K_{\nu, t}^{1/2}(t).
\]

(3.21)

As mentioned, the expression of primary interest, namely \( Z'(0) \), is obtained by just taking the limit as \( z \) approaches the zero of (3.19). By recalling the structure of the spectral \( \zeta \)-function on \( \nu \) in (3.8) and (3.9), the limit as \( z \to 0 \) gives

\[
Z'(0) = \sum_{n=1}^d \frac{\xi_{\nu, s}(n)}{n} \left[ 2\Psi(n) \text{Res} \zeta_{\nu, s} \left( \frac{n}{2} \right) + \text{FP} \zeta_{\nu, s} \left( \frac{n}{2} \right) + (1 - y) \zeta_{\nu, s} \left( - \frac{1}{2} \right) - \frac{y}{2} \zeta_{\nu, s} \left( 0 \right) \right]
\]

\[
+ \frac{1}{2} \zeta_{\nu, s} \left( 0 \right) + \frac{1}{2} \zeta_{\nu, s} \left( \frac{1}{2} \right) + \lim_{z \to 0} \left\{ \sum_{n=1}^d \frac{\xi_{\nu, s}(n)}{n} \text{Res} \zeta_{\nu, s} \left( \frac{n}{2} \right) + \frac{1}{2z} \zeta_{\nu, s} \left( 0 \right) \right\}
\]

\[
+ \frac{1}{2} \zeta_{\nu, s} \left( \frac{1}{2} \right) + \Gamma(z) \zeta_{\nu, s+1} \left( z \right)
\]

\[
- m^2 \left\{ \sum_{n=1}^d \frac{D_n'(1)}{2} \left[ 2\Psi(n+2) \text{Res} \zeta_{\nu, s} \right]
\]

(3.22)
\[
\times \left( \frac{n + 1}{2} \right) + \text{FP} \frac{1}{2} + 1 - \frac{1}{2} \text{Res} \frac{1}{2} (1)
\]
\[
- \frac{1}{4} \text{FP} \frac{1}{2} + \frac{1}{4} \text{FP} \frac{1}{2} (1) + \lim_{z \to 0} \sum_{n=1}^{d} \frac{D_n(1)}{z} \text{Res} \frac{n + 1}{2} \]
\[
- \frac{1}{2z} \text{Res} \frac{1}{2} (1) + \frac{1}{2z} \text{Res} \frac{1}{2} (1) + \frac{1}{4} \Gamma(z + 1) \zeta_N(z + 1, 1) \}
\]
\[+ O(m^4). \quad (3.22)\]

As a last step, we have to explicitly compute the remaining limits in equation (3.22). By utilizing the series [3, 24]
\[\frac{1}{e^t - 1} = \frac{1}{t} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{t^n}{n!} \zeta_k(-n), \quad (3.23)\]
and the asymptotic expansion of the heat kernel
\[K_{1/2}(t) \sim \sum \tilde{a}_k t^{k-d}, \quad (3.24)\]
where
\[\tilde{a}_k = 2 \Gamma(d - k) \text{Res} \frac{d - k}{2}, \quad (3.25)\]
for \(k = 0, 1, \ldots, d - 1, k = d + (2l + 1)\) with \(l \in \mathbb{N}^+\), and
\[\tilde{a}_k = \frac{(-1)^{k-d}}{(k-d)!} \zeta_N \left( \frac{d - k}{2} \right), \quad (3.26)\]
for \(k > d\) and \(k \in \mathbb{N}^+\) even, one can prove that [3]
\[\zeta_N(0) = - \zeta_N \left( \frac{1}{2} \right) - \frac{1}{2} \zeta_N(0) - 2 \sum_{n=1}^{d} \frac{\zeta_k(-n)}{n} \text{Res} \frac{n}{2}, \quad (3.27)\]
The value \(\zeta_N(0)\) just computed is needed in the small \(z\) expansion
\[\Gamma(z) \zeta_N(z) = \frac{1}{z} - \zeta_N(0) - \gamma \zeta_N(0) + \zeta_N'(0) + O(z). \quad (3.28)\]
By using this expansion together with (3.27) in the part of zeroth order in the mass of equation (3.22), one observes that the quantities containing the divergence 1/z cancel each other exactly.

In order to present the complete final result for Dirichlet boundary conditions, we also need to consider the terms of quadratic order in the mass. Exactly in the same way as we have done for the terms of order zero in the mass, we exploit the following small \(z\) expansion:
\[\Gamma(z + 1) \zeta_N(z + 1) = \frac{1}{z} - \text{Res} \frac{1}{2} (1, 1) - \gamma \text{Res} \frac{1}{2} (1, 1) + \text{FP} \frac{1}{2} (1, 1) + O(z). \quad (3.29)\]
The value of \(\text{Res} \frac{1}{2} (z + 1, 1)\) at \(z = 0\) can be obtained, in the same way as before, by using the asymptotic expansion (3.24) and definition (3.21). Explicitly, we get
\[\text{Res} \frac{1}{2} (1, 1) = 2 \sum_{n=0}^{d} (-1)^n \text{Res} \frac{n + 1}{2}. \quad (3.30)\]
By substituting this last expression in the \( m^2 \) terms containing \( 1/z \) in (3.22), one finds
\[
\lim_{z \to 0} \left[ \frac{d}{dz} \sum_{n=1}^{d} \frac{D_n'(1)}{z} \text{Res} \xi_{r} \left( \frac{n+2}{2} \right) - \frac{1}{2z} \text{Res} \xi_{r} \left( \frac{1}{2} \right) + \frac{1}{2z} \text{Res} \xi_{r} (1) + \frac{1}{4} \Gamma (z+1) \xi_{r} (z+1, 1) \right] = \lim_{z \to 0} \left[ \frac{1}{z} \sum_{n=1}^{d} \left( D_n'(1) - \frac{(-1)^n}{2} \right) \text{Res} \xi_{r} \left( \frac{n+2}{2} \right) \right]. \tag{3.31}
\]

Using [26]
\[
D_n'(1) = \frac{(-1)^n}{2}, \tag{3.32}
\]
the expression in (3.31) vanishes identically. By combining the results in (3.10)–(3.12) and (3.22) we can finally obtain the expression for the small mass expansion of the functional determinant of the Laplacian on the generalized cone as follows:
\[
\xi_{r}^\prime (0) = \xi_{r+1}^\prime (0) + 2 \sum_{n=1}^{d} \frac{\xi_{r} (-n)}{n} \text{Res} \xi_{r} \left( \frac{n}{2} \right) \sum_{k=1}^{n-1} \frac{1}{k} + 2 \sum_{n=1}^{d} \text{Res} \xi_{r} \left( \frac{n}{2} \right) \int_{0}^{1} \frac{D_n(t) - t^2 D_n(1)}{t(1-t^2)} \, dt + m^2 \left[ - \frac{1}{4} \text{FP} \xi_{r} (1, 1) \right] - \frac{1}{2} \text{Res} \xi_{r} (1) + \frac{1}{4} (2 \ln 2 + 1) \text{Res} \xi_{r} \left( \frac{1}{2} \right) + \sum_{n=1}^{d} \frac{\xi_{r} (-n)}{n} \text{Res} \xi_{r} \left( \frac{n}{2} + 1 \right) - 2 \sum_{n=1}^{d} \frac{D_n'(1)}{2} \text{Res} \xi_{r} \left( \frac{n}{2} + 1 \right) \left( \sum_{k=1}^{n-1} \frac{1}{k} + \frac{2n+1}{n(n+1)} \right) - \sum_{n=1}^{d} \text{Res} \xi_{r} \left( \frac{n}{2} + 1 \right) \int_{0}^{1} \frac{D_n'(t) - t D_n'(1)}{1-t^2} \, dt + O (m^4), \tag{3.33}
\]

where, in order to obtain the last expression, we have used the relations
\[
\Psi \left( z + \frac{n}{2} \right) = -\gamma - 2 \int_{0}^{1} \frac{t^{2z(n+1)} - t^2}{t(1-t^2)} \, dt, \tag{3.34}
\]
and
\[
\Psi (n+2) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k} + \frac{1}{n(n+1)}. \tag{3.35}
\]

We would like to point out that the terms of order zero in the mass in equation (3.33) coincide with the ones obtained in [3]. The terms proportional to \( m^2 \) are, instead, the new massive corrections to the scalar Laplacian on the generalized cone for Dirichlet boundary conditions.

### 4. Small mass expansion for Robin boundary conditions

The calculational procedure to follow in order to compute \( \xi_{r}^\prime (0) \) for Robin boundary conditions closely resembles the one used in the previous section for the Dirichlet case.
We will describe only few necessary changes [3–5]. For Robin boundary conditions we need, in addition to the asymptotic expansion (2.17), the following one for $I'_ν(νz)$ [24, 28]:

$$I'_ν(νz) \sim \frac{1}{\sqrt{2\pi ν}} \frac{e^{ν(1+\frac{z^2}{2})^{1/4}}}{\sqrt{2πν}} \left[ 1 + \sum_{k=1}^{∞} \frac{v_k(t)}{ν^k} \right],$$

(4.1)

where the polynomials $v_k(t)$ are determined by the recurrence relation

$$v_k(t) = u_k(t) + t(t^2 − 1) \left[ \frac{1}{2} u_{k-1}(t) + tu'_{k-1}(t) \right].$$

(4.2)

In analogy with the Dirichlet case, we will make use of the cumulant expansion [3–5, 26]

$$\ln \left[ 1 + \sum_{k=1}^{∞} \frac{v_k(t)}{ν^k} + \frac{u}{ν} \left( 1 + \sum_{k=1}^{∞} \frac{u_k(t)}{ν^k} \right) \right] \sim \sum_{n=1}^{∞} \frac{M_n(t, u)}{ν^n},$$

(4.3)

where the terms $M_k(t, u)$ have a polynomials structure analogous to the $D_n(t)$, namely

$$M_n(t, u) = \sum_{i=0}^{n} z_{i,n}(u)t^{n+2i},$$

(4.4)

where the coefficients $z_{i,n}$ depend on the variable $u = 1 - D/2 - β$. For convenience, polynomials (4.4) are listed, up to the sixth order, in the appendix.

The evaluation of $A_1(s), A_0(s)$ and $A_i(s)$ and their first derivative at zero, in the Robin case, follows exactly the same lines as for the Dirichlet case once the coefficients $x_{i,n}$ are replaced with $z_{i,n}$ [3, 4]. We will focus our attention on the computation of the function $Z_R(s)$ which presents slight modifications from the Dirichlet case. In what follows the subscript $R$ will denote the Robin case. We write $Z_R(s)$, as we did before, in the form

$$Z_R(s) = \sum_{ν} d(ν)Z_ν,R(s).$$

(4.5)

By taking the derivative of $Z_ν,R(s)$ and setting $s = 0$ one obtains

$$Z'_ν,R(0) = \ln Γ(ν + 1) + ν − ν ln ν − \frac{1}{2} ln 2π ν − \ln \left( 1 + \frac{u}{ν} \right) - m^2 \left\{ \frac{1}{4(ν + 1)} - \frac{1}{4ν} - \frac{1}{4ν^2} \right\}$$

$$+ \frac{1}{2(ν + 1)(ν + u)} \left\{ \sum_{n=1}^{d} v^{−n}M_n \left( 1 + \frac{m^2}{ν^2}, u \right) \right\} + O(m^4).$$

(4.6)

By expanding the last term in equation (4.6) in powers of the mass up to the term $m^2$ we obtain

$$\sum_{n=1}^{d} v^{−n}M_n \left( 1 + \frac{m^2}{ν^2}, u \right) = \sum_{n=1}^{d} \frac{D_n(1)}{v^n} + \sum_{n=1}^{d} \frac{(-1)^{n+1} u^n}{n} \left( \frac{m^2}{ν} \right)^n$$

$$− m^2 \sum_{n=1}^{d} v^{−n−2} \left[ \frac{n D_n(1)}{2} + \sum_{b=0}^{n} z_{n,b}b + \frac{(-1)^{n+1} u^n}{2} \right] + O(m^4),$$

(4.7)

where we have used the relations [5, 26]

$$M_n(1, 0) = D_n(1),$$

(4.8)

$$M_n(1, u) − M_n(1, 0) = (-1)^{n+1} \frac{u^n}{n}.$$
At this point, by comparing the small $z$ expansion of $\ln[u I_v(z) + v z I'_v(z)]$ with its Olver expansion we get the following useful relation:

$$M'_n(1, u) = D'_n(1) + (-1)^{n+1} \sum_{k=0}^{n} u^k. \tag{4.10}$$

This relation allows us to write the expression for $Z'_{\nu, R}(0)$ in (4.6) in the form

$$Z'_{\nu, R}(0) = Z'_{\nu}(0) - \ln \left(1 + \frac{1}{v}\right) + \frac{d}{n} \sum_{n=1}^{d} \frac{(-1)^{n+1}}{n} \left(\frac{u}{v}\right)^n \left[-\frac{m^2}{2} \left(\frac{1}{(v+1)(v+u)} - \frac{1}{v^2} + \frac{1}{v^2} \sum_{n=1}^{d} \left(-1\right)^{n+1} \left(\frac{u}{v}\right)^n \left(\sum_{k=0}^{n} u^k\right)\right]\right] + O(m^4). \tag{4.11}$$

One can clearly see from the previous result that in order to study the Robin case we only need to consider the new additional terms. We would like to point out that this property has already been noticed and utilized in the case of the functional determinant for massless scalar fields in [3]. Here, we have found that the same feature appears also for the $m^2$ correction. With the last remark in mind, let us define

$$N(u) = \sum d(v) \left[-\ln \left(1 + \frac{1}{v}\right) + \sum_{n=1}^{d} \frac{(-1)^{n+1}}{n} \left(\frac{u}{v}\right)^n \right], \tag{4.12}$$

and

$$P(u) = \sum d(v) \left[\frac{1}{(v+1)(v+u)} - \frac{1}{v^2} + \frac{1}{v^2} \sum_{n=1}^{d} \left(-1\right)^{n+1} \left(\frac{u}{v}\right)^n \left(\sum_{k=0}^{n} u^k\right)\right]. \tag{4.13}$$

The defining expression for $N(u)$ can be evaluated similarly to the procedure utilized in the Dirichlet case. The calculation for $N(u)$ is shown in detail in [3] and the final result can be explicitly written as

$$N(u) = \zeta'_{\nu}(0, u) - \frac{1}{2} \zeta'_{\nu}(0) + \sum_{k=1}^{d} \frac{(-1)^{k+1}}{k} u^k \left[2 \operatorname{Res} \zeta_{\nu}\left(\frac{k}{2}\right) \left(\Psi(k) + \gamma\right) + \text{FP} \zeta_{\nu}\left(\frac{k}{2}\right)\right], \tag{4.14}$$

where $\zeta'_{\nu}(z, u)$ has been defined in (3.21).

Let us now turn our attention to the term $P(u)$. By utilizing the integral representation (3.14) and a partial fraction decomposition, one immediately finds

$$\frac{1}{(v+1)(v+u)} = \frac{1}{(u-1)} \int_0^\infty \text{d}t \left(e^{-(v+1)t} - e^{-(v+u)t}\right). \tag{4.15}$$

Using (3.14) also for the other terms in (4.13) this shows

$$P(u) = \sum d(v) \int_0^\infty \text{d}t \frac{te^{-t}}{u-1} - \frac{te^{-t}}{u-1} - t^2 - \sum_{n=1}^{d} \frac{(-1)^n}{\Gamma(n+2)} \left(\sum_{k=0}^{n} u^k\right) t^{n+2}. \tag{4.16}$$

In order to evaluate each term individually, we introduce a regularization parameter $z$ and define

$$P(u, z) = \sum d(v) \int_0^\infty \text{d}t \frac{te^{-t}}{u-1} - \frac{te^{-z}}{u-1} - t^2 - \sum_{n=1}^{d} \frac{(-1)^n}{\Gamma(n+2)} \left(\sum_{k=0}^{n} u^k\right) t^{n+2}, \tag{4.17}$$
such that \( P(u) = P(u, 0) \). By using definition (3.21) one can see that

\[
P(u, z) = \frac{1}{u - 1} \Gamma(z + 1) \zeta_{\mathcal{N}}(z + 1, 1) - \frac{1}{u - 1} \Gamma(z + 1) \zeta_{\mathcal{N}}(z + 1, u) - \sum_{n=1}^{d} \frac{(-1)^n}{\Gamma(n + 2)} \times \left( \sum_{k=0}^{n} u^k \right) \Gamma(z + n + 2) \zeta_{\mathcal{N}} \left( \frac{z + n + 2}{2k} \right) - \Gamma(z + 2) \zeta_{\mathcal{N}} \left( \frac{z + 2}{2} \right).
\]

(4.18)

As before, we need to take the limit as \( z \) approaches zero of the last expression. By recalling the analytic structure of the spectral \( \zeta \)-function on the base manifold \( \mathcal{N} \), one obtains

\[
P(u, 0) = \frac{1}{u - 1} \left[ \text{FP} \frac{\zeta_{\mathcal{N}}(1, 1)}{\zeta_{\mathcal{N}}(1, u)} \right] - 2(1 - \gamma) \text{Res} \zeta_{\mathcal{N}}(1) - \text{FP} \zeta_{\mathcal{N}}(1)
- \sum_{n=1}^{d} (-1)^n \left( \sum_{k=0}^{n} u^k \right) \left[ \text{FP} \zeta_{\mathcal{N}} \left( \frac{n + 2}{2} \right) + 2\gamma \zeta_{\mathcal{N}}(1) \right]
- 2\gamma \sum_{n=1}^{d} (-1)^n \text{Res} \zeta_{\mathcal{N}} \left( \frac{n + 1}{2} \right) \left( \frac{1 - u^n}{u - 1} \right)
+ \lim_{z \to 0} \frac{2}{z} \left[ \sum_{n=0}^{d} (-1)^n \text{Res} \zeta_{\mathcal{N}} \left( \frac{n + 1}{2} \right) \left( \frac{1 - u^n}{u - 1} \right)
- \sum_{n=1}^{d} (-1)^n \left( \sum_{k=0}^{n} u^k \right) \text{Res} \zeta_{\mathcal{N}} \left( \frac{n + 2}{2} \right) - \text{Res} \zeta_{\mathcal{N}}(1) \right],
\]

(4.19)

where in order to obtain this formula we have used the fact that

\[
\text{Res} \zeta_{\mathcal{N}}(1, u) = 2 \sum_{k=0}^{d} (-1)^k u^k \text{Res} \zeta_{\mathcal{N}} \left( \frac{k + 1}{2} \right).
\]

(4.20)

In the expression that multiplies \( 1/z \) in (4.19), one can note that the terms corresponding to \( n = 0 \) vanish identically, the terms with \( n = 1 \) cancel the residue of \( \zeta_{\mathcal{N}}(s) \) at \( s = 1 \) and the remaining terms of the sum with \( n \geq 2 \) vanish as well due to the following identity:

\[
\sum_{n=0}^{k} u^n = \frac{u^{k+1} - 1}{u - 1}.
\]

(4.21)

As a consequence, the potentially divergent term in \( 1/z \) vanishes identically.

We can finally write down the expression for the small mass expansion of the functional determinant of the Laplacian for Robin boundary conditions as follows:

\[
\zeta'_{\mathcal{N}, \mathcal{R}}(0) = \zeta'_{\mathcal{N}, \mathcal{R}}(0) + \zeta'_{\mathcal{N}}(0, u) + \zeta_{\mathcal{N}} \left( \frac{-1}{2} \right) \ln 2 + \sum_{n=1}^{d} \text{Res} \zeta_{\mathcal{N}} \left( \frac{n}{2} \right) \left( \sum_{k=0}^{n-1} \frac{1}{k} \right) M_{n}(1)
+ 2 \sum_{n=1}^{d} \text{Res} \zeta_{\mathcal{N}} \left( \frac{n}{2} \right) \int_{0}^{1} dr \frac{M_{n}(t, u) - t^2 M_{n}(1, u)}{r(1 - r^2)} + m^2 \left[ \frac{1}{4} \text{FP} \zeta_{\mathcal{N}}(1, 1)
- \frac{1}{2(u - 1)} \left[ \text{FP} \zeta_{\mathcal{N}}(1, 1) - \text{FP} \zeta_{\mathcal{N}}(1, u) \right] + \frac{1}{2} \text{Res} \zeta_{\mathcal{N}}(1)\right]
+ \frac{1}{4} (2 \ln 2 + 1) \text{Res} \zeta_{\mathcal{N}} \left( \frac{1}{2} \right) + \sum_{n=1}^{d} \frac{1}{n} \text{Res} \zeta_{\mathcal{N}} \left( \frac{n+1}{2} \right) \left[ \zeta_{\mathcal{R}}(n) + (-1)^{n+1} u^n \right]
\]

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\[ + \sum_{n=1}^{d} (-1)^n \text{Res} \zeta_{N} \left( \frac{n}{2} + 1 \right) \frac{2n+1}{2(u-1)} \left( \frac{2n+1}{n(n+1)} + \sum_{k=1}^{n-1} \frac{1}{k} \right) \]

\[ - \sum_{n=1}^{d} \text{Res} \zeta_{N} \left( \frac{n}{2} + 1 \right) \int_{0}^{1} \frac{M'_{n}(t, u) - t M'_{n}(1, u)}{1 - t^2} \right) \left( \frac{2n+1}{n(n+1)} + \sum_{k=1}^{n-1} \frac{1}{k} \right) + O(m^4). \] (4.22)

Here once again we would like to point out that the terms of zeroth order in the mass for \( \zeta'_{M/R}(0) \) coincide with the results obtained in [3]. The terms proportional to \( m^2 \) are, instead, the new massive corrections to the scalar Laplacian on the generalized cone for Robin boundary conditions.

The expressions for \( \zeta'_{M/R}(0) \) for Dirichlet and Robin boundary conditions obtained in (3.33) and (4.22) contain the spectral \( \zeta \)-function and its first derivative on the base manifold \( N \). These expressions represent a very general result, holding for any smooth base manifold \( N \) and in dimension \( D \). Without specifying the manifold \( N \) one cannot go further than (3.33) and (4.22) in the evaluation of the functional determinant. In the next section we specify the base manifold to be a \( d \)-dimensional sphere. This case is of particular importance because the spectral \( \zeta \)-function on the \( d \)-dimensional sphere can be evaluated explicitly in terms of the Barnes \( \zeta \)-function [3, 8].

5. \( d \)-dimensional sphere as the base manifold \( N \)

In this section we assume that the base manifold is a \( d \)-dimensional sphere. In this case the relevant Bessel function index is

\[ \nu = \left( l + \frac{d - 1}{2} \right), \] (5.1)

and the eigenfunctions are hyperspherical harmonics with degeneracy

\[ d(l) = (2l + d - 1) \frac{(l + d - 2)!}{l!(d - 1)!}. \] (5.2)

By using definition (2.12), we can write \( \zeta_{N}(s) \) as

\[ \zeta_{N}(s) = \sum_{l=0}^{\infty} (2l + d - 1) \frac{(l + d - 2)!}{l!(d - 1)!} \left( 1 + \frac{d - 1}{2} \right)^{-2s}. \] (5.3)

The \( \zeta \)-function in (5.3) can be written as a sum of Barnes \( \zeta \)-functions [3, 8]

\[ \zeta_{N}(s) = \zeta_{B} \left( 2s, \frac{d + 1}{2} \right) + \zeta_{B} \left( 2s, \frac{d - 1}{2} \right), \] (5.4)

where the Barnes \( \zeta \)-function is defined as [2, 19]

\[ \zeta_{B}(s, a|\vec{r}) = \sum_{\vec{m}=0}^{\infty} \frac{1}{(a + \vec{m} \cdot \vec{r})^s}. \] (5.5)

valid for \( \text{Re}(s) > d \) where \( \vec{m} \) and \( \vec{r} \) are \( d \)-dimensional vectors, and where the notation \( \zeta_{B}(s, a|1) = \zeta_{B}(s, a) \) has been used.

In our case, we consider the integral representation of the Barnes \( \zeta \)-function, namely [3]

\[ \zeta_{B}(s, a) = \frac{i \Gamma(1-s)}{2\pi} \int_{L} \frac{e^{z(1-s)}(-z)^{s-1}}{2^{s} \sinh^{d} \left( \frac{z}{2} \right)} ds. \] (5.6)
where \( L \) represents the Hankel contour. By using this integral representation in expression (5.4) we can write \( \zeta_N(s) \) as [3]

\[
\zeta_N(s) = \frac{i}{\Gamma(1 - 2s)} \int_L \frac{(-z)^{2s-1} \cosh \left( \frac{z}{2} \right)}{2^{d-1} \sinh^d \left( \frac{z}{2} \right)} \, dz
\]

\[
= \frac{i}{2\pi (d-1)} \frac{1}{\sinh^{d-1} \left( \frac{z}{2} \right)} \frac{1}{-z}.
\]

(5.7)

In particular, we need to compute, for Dirichlet boundary conditions, the residues of \( \zeta_N(s) \) at the points \( s = m/2 \) with \( m \) being a positive integer. We note that in (5.7) the only pole inside the contour occurs at \( z = 0 \). Defining the coefficients \( D(d-1)^{-1} \) as [3, 10]

\[
\left( \frac{z}{\sinh z} \right)^{-d-1} = \sum_{\nu=0}^{\infty} D_{d-1}^{(d-1)} \frac{\nu^\nu}{\nu!}
\]

(5.8)

and by using the residue method we obtain [3]

\[
\text{Res} \zeta_N \left( \frac{m}{2} \right) = \frac{2^{m-d} D_{d-m}^{(d-1)}}{(d-1)(m-2)!}(d-m)!
\]

(5.9)

valid for \( m \geq 2 \) and \( d \geq m \). Another particular value that we will need in the subsequent calculations is \( \zeta_N(-1/2) \). Again from (5.7) one shows as before

\[
\zeta_N \left( -\frac{1}{2} \right) = \frac{2^{1-d} D_{d+1}^{(d-1)}}{(d-1)(d+1)!}.
\]

(5.10)

In addition to result (5.9), we will need, in the case of Robin boundary conditions, the residues of \( \zeta_N(s) \) at \( s = m/2 + 1 \). These follow immediately from (5.9) by replacing \( m \) with \( m+2 \), the result then being valid for \( m \geq 0 \) and \( d \geq m+2 \).

For the \( d \)-dimensional sphere we also have that

\[
\zeta_{N+1}(s) = \sum_{l=0}^{\infty} e(l) \left( l + \frac{d+1}{2} \right)^{-s},
\]

(5.11)

where

\[
e(l) = (2l + d) \frac{(l + d - 1)!}{l!d!}.
\]

By writing [3]

\[
e(l) = \sum_{a=0}^{d} e_a \left( l + \frac{d+1}{2} \right)^a
\]

(5.12)

which defines \( e_a \), we have that

\[
\zeta_{N+1}(s) = \sum_{a=0}^{d} e_a \zeta_H \left( s - a, \frac{d+1}{2} \right).
\]

(5.13)

where the coefficients \( e_a \) depend on the dimension \( d \) and are determined from equation (5.12) and \( \zeta_H \) represents the Hurwitz \( \zeta \)-function. In particular, we have

\[
\zeta_{N+1}(0) = \sum_{a=0}^{d} e_a \zeta_H \left( -a, \frac{d+1}{2} \right).
\]

(5.14)

For our calculations we will also need the following expression:

\[
\zeta_N(s, u) = \sum_{a=0}^{d-1} e_a(u) \zeta_H \left( s - a, \frac{d-1}{2} + u \right).
\]

(5.15)
where the coefficients \( e_d(u) \) are defined by the equality [3]

\[
d(l) = (2l + d - 1) \left( \frac{l + d - 2)!}{l!(d - 1)!} \right) = \sum_{a=0}^{d-1} e_d(u) \left( l + \frac{d - 1}{2} + u \right)^\alpha. \tag{5.16}
\]

### 6. Specific dimensions for Dirichlet boundary conditions

By utilizing the result for \( \zeta^M_N(0) \) and the particular form obtained for \( \zeta_N(s) \) and \( \zeta_N+1(s) \) when the base manifold is a sphere, we can get explicit expressions for specific dimensions. Here, we will present the results for \( d = 2, 3, 4, 5 \). Obviously, results for higher dimensions can be extracted with some amount of work from the formulas presented in the previous sections.

For the base manifold of dimension \( d = 2 \) we obtain

\[
\zeta^M_N(0) = \frac{3}{32} - \frac{1}{12} \ln 2 + \frac{1}{3} \zeta_R(-1) - \frac{3}{4} \zeta_R(-2) + m^2 \left[ -1 + \frac{1}{2}(\gamma + 2 \ln 2) \right] + O(m^4). \tag{6.1}
\]

For the base manifold of dimension \( d = 3 \) we obtain

\[
\zeta^M_N(0) = \frac{173}{30240} + \frac{1}{90} \ln 2 + \frac{1}{3} \zeta_R(-1) - \frac{1}{2} \zeta_R(-2) + \frac{1}{3} \zeta_R(-3) + m^2 \left[ - \frac{5}{24} + \frac{1}{4} (\ln 2 - \gamma) \right] + O(m^4). \tag{6.2}
\]

For the base manifold of dimension \( d = 4 \) we obtain

\[
\zeta^M_N(0) = \frac{47}{9216} + \frac{17}{2880} \ln 2 - \frac{1}{3} \zeta_R(-1) - \frac{1}{2} \zeta_R(-2) + \frac{7}{64} \zeta_R(-3) + \frac{5}{64} \zeta_R(-4) + m^2 \left[ - \frac{7}{576} + \frac{1}{16} (\gamma + 2 \ln 2) \right] + O(m^4). \tag{6.3}
\]

For the base manifold of dimension \( d = 5 \) we obtain

\[
\zeta^M_N(0) = -\frac{4027}{6486480} - \frac{1}{576} \ln 2 - \frac{1}{2} \zeta_R(-1) - \frac{1}{2} \zeta_R(-2) - \frac{1}{2} \zeta_R(-3) + \frac{1}{60} \zeta_R(-5) - m^2 \frac{143}{6048} + O(m^4). \tag{6.4}
\]

In order to obtain the above expressions for even dimensions we have exploited the formula

\[
\zeta_M^N(s, q + \frac{1}{2}) = 2^{s} \ln 2 \left[ \zeta_R(s) - \sum_{n=1}^{2q-1} \frac{1}{n^s} \right] + 2^{s} \left[ \zeta_R(s) + \sum_{n=1}^{2q-1} \ln n \right] - \zeta_R(s) - \sum_{n=1}^{q-1} \ln n \frac{n}{n^s}, \tag{6.5}
\]

valid for any integer \( q \geq 0 \), while for odd dimensions we have used

\[
\zeta_M^N(s, z) = \zeta_R(s) + \sum_{n=0}^{z-2} \ln(n + 1) \frac{1}{(n + 1)^s}, \tag{6.6}
\]

which holds for any integer \( z \geq 2 \).

Relation (6.5) can be proved as follows. The Hurwitz zeta function with second argument being half-integer can be written as

\[
\zeta_M^N(s, q + \frac{1}{2}) = 2^{s} \zeta_M^N(s, 2q) - \zeta_M^N(s, q). \tag{6.7}
\]
where \( q \geq 0 \) is an integer. Now, the resulting Hurwitz zeta functions can be written in terms of Riemann zeta functions as follows:

\[
\zeta_H(s, q) = \zeta_R(s) - \frac{q}{n^s}.
\]  

(6.8)

By substituting the last relation in formula (6.7) one obtains the expression

\[
\zeta_H\left(s, q + \frac{1}{2}\right) = 2^s \left[ \zeta_R(s) - \sum_{n=1}^{q-1} \frac{1}{n^s} \right] - \zeta_R(s) + \sum_{n=1}^{q-1} \frac{1}{n^s}.
\]  

(6.9)

Finally, differentiation of the last equality with respect to \( s \) leads to formula (6.5).

Formula (6.6) can be proved in a similar fashion. In fact, since \( z \) is an integer, one obtains (6.6) by differentiating (6.8) and by shifting the index of the sum.

We would like to point out that the results for the zeroth order in mass obtained for \( d = 2, 3, 4, 5 \) coincide with the ones obtained in [3, 26].

7. Specific dimensions for Robin boundary conditions

For Robin boundary conditions we consider the general result obtained in (4.22) and specialize it to the case in which the base manifold is a sphere of dimension \( d = 2, 3, 4, 5 \). In addition to the spectral functions \( \zeta_{d, h}(s) \) and \( \zeta_{d+1, h}(s) \) obtained in the previous sections, we will also need \( \text{FP}\zeta_{d, h}(1, u) \). This term is evaluated by noticing that (5.15) can be rewritten as

\[
\text{FP}\zeta_{d, h}(1, u) = e_0(u)\text{FP}\zeta_H\left(1, \frac{d-1}{2} + u\right) + \sum_{\alpha=0}^{d-2} e_{\alpha+1}(u)\text{FP}\zeta_H\left(-\alpha, \frac{d-1}{2} + u\right).
\]  

(7.1)

At this point, by using the following relations [24]:

\[
\text{FP}\zeta_H\left(1, \frac{d-1}{2} + u\right) = -\Psi\left(\frac{d-1}{2} + u\right), \quad \text{FP}\zeta_H\left(-\alpha, \frac{d-1}{2} + u\right) = \frac{B_{\alpha+1}\left(\frac{d-1}{2} + u\right)}{\alpha + 1},
\]  

(7.2)

where \( B_{\alpha}(z) \) are the Bernoulli polynomials, one obtains

\[
\text{FP}\zeta_{d, h}(1, u) = -e_0(u)\Psi\left(\frac{d-1}{2} + u\right) - \sum_{\alpha=0}^{d-2} e_{\alpha+1}(u)\frac{B_{\alpha+1}\left(\frac{d-1}{2} + u\right)}{\alpha + 1}.
\]  

(7.3)

For the base manifold of dimension \( d = 2 \) we obtain

\[
\zeta'_{d, h}(0) = \frac{1}{32} - \frac{1}{6} \ln 2 + \frac{u}{2} - \frac{1}{2} \xi(-1) - \frac{3}{4} \xi(-2) - 2u \ln \Gamma\left(\frac{1}{2} + u\right)
\]
\[
+ 2 \int_0^u \ln \Gamma\left(\frac{1}{2} + x\right) + x^2 \left[ \frac{1}{2} (\gamma + 2 \ln 2) \frac{u + 1}{u - 1} - \frac{u}{u - 1} \right] + O(m^4).
\]  

(7.4)

For the base manifold of dimension \( d = 3 \) we have

\[
\zeta'_{d, h}(0) = \frac{11}{4320} + \frac{1}{90} \ln 2 + \frac{u}{30} - \frac{5}{12} u^2 - \frac{u^3}{3} + \frac{1}{6} \xi(-1) + \frac{1}{2} \xi(-2) + \frac{1}{3} \xi(-3)
\]
\[
+ u^2 \ln \Gamma(1 + u) - 2 \int_0^u \ln \Gamma(1 + x) + m^2 \left[ \frac{-5}{24} - \frac{\gamma}{4} - \frac{1}{4} \ln 2 - \frac{u}{2} \ln 2 \right]
\]
\[
- \frac{u}{4} - \frac{1}{2(u - 1)} \left[ 1 + \gamma - \frac{u}{2} + u^2 \left( \Psi(u + 1) - \frac{3}{2} \right) \right] + O(m^4).
\]  

(7.5)
For the base manifold of dimension $d = 4$ we have

$$
\zeta'_{\mathcal{M},R}(0) = -\frac{61}{46080} - \frac{11}{576} u - \frac{1}{16} u^2 + \frac{11}{72} u^3 + \frac{1}{24} u^4 + \frac{7}{720} \ln 2 + \frac{1}{48} \zeta'_{R}(-1) - \frac{1}{32} \zeta'_{R}(-2)
$$

$$
- \frac{7}{48} \zeta'_{R}(-3) - \frac{5}{64} \zeta'_{R}(-4) + \frac{u}{12} \ln \Gamma\left(\frac{3}{2} + u\right) - \frac{1}{3} u^3 \ln \Gamma\left(\frac{3}{2} + u\right)
$$

$$
- \frac{1}{12} \int_0^u dx \ln \Gamma\left(\frac{3}{2} + x\right) + \int_0^u dx x^2 \ln \Gamma\left(\frac{3}{2} + x\right) + m^2 \left\{ \frac{13}{576} + \frac{5}{36} u + \frac{2}{9} u^2
$$

$$
+ \gamma + \frac{1}{8} \ln 2 - \frac{1}{2(u - 1)} \left[ -\frac{1}{24} - \frac{\gamma}{4} - \frac{1}{2} \ln 2 - \frac{17}{72} u + \frac{u^2}{3} + \frac{11}{18} u^3
$$

$$
- \frac{u}{12} (4u^2 - 1) \Psi\left(\frac{3}{2} + u\right) \right\} + O(m^4).
$$

Finally, for the base manifold of dimension $d = 5$ we have

$$
\zeta'_{\mathcal{M},R}(0) = -\frac{9479}{32432400} - \frac{u}{315} + \frac{517}{1512} u^2 + \frac{83}{1512} u^3 - \frac{19}{480} u^4 - \frac{u^5}{45} - \frac{1}{756} \ln 2 - \frac{u^3}{36} \ln 2
$$

$$
+ \frac{u^5}{60} \ln 2 - \frac{1}{60} \zeta'_{R}(-1) - \frac{1}{24} \zeta'_{R}(-2) + \frac{1}{24} \zeta'_{R}(-4) + \frac{1}{60} \zeta'_{R}(-5)
$$

$$
- \frac{1}{12} u^2 \ln \Gamma(u + 2) + \frac{1}{12} u^2 \ln \Gamma(u + 2) + \frac{1}{6} \int_0^u dx x \ln \Gamma(x + 2)
$$

$$
- \frac{1}{3} \int_0^u dx x^3 \ln \Gamma(x + 2) + m^2 \left\{ -\frac{13}{4320} + \frac{u}{40} - \frac{u^2}{96}
$$

$$
- \frac{u^3}{32} + \frac{u^2}{24} \ln 2 - \frac{u^3}{24} \ln 2 - \frac{1}{2(u - 1)} \left[ \frac{1}{24} + \frac{u}{24}
$$

$$
+ \frac{31}{144} u^2 - \frac{u^3}{8} - \frac{25}{144} u^4 + \frac{u^2}{12} (u^2 - 1) \Psi(u + 2) \right\} + O(m^4).
$$

We would like to mention, once again, that the results of zeroth order in mass obtained in this section coincide with the ones obtained in [3, 26]. Moreover, it is worth noticing that although the above results for $\zeta'_{\mathcal{M},R}(0)$ are finite in the limit as $u \to 1$, corresponding to Neuman boundary conditions, they do not reproduce the correct result. This is due to the fact that, when studying Neuman boundary conditions, particular care is needed with zero modes which have to be dealt with separately [5]. The terms involving integrals over $\ln \Gamma(u)$ could be given more explicitly in terms of the zeta function of Riemann and its derivative, [18], but the chosen form is more compact.

8. Conclusions

The paper continues the analysis of the functional determinant of the Laplacian on the generalized cone started in [3, 5, 17]. Whereas in these references results were given for $m = 0$ only, this paper provides a way to evaluate a systematic small-$m$ expansion in powers of $m^2$. It is important to stress that this work shows that the mathematical framework developed in [3] can be effectively used in order to evaluate the massive corrections to the functional determinant without encountering any major technical complications. This is an important point, which was not clear until this work was actually done. The leading-order correction
for Dirichlet and Robin boundary conditions on the generalized cone are given in (3.33) and
(4.22). Specializing to the example of a ball the results in sections 6 and 7 are found. Higher
orders can be computed as needed following the procedure outlined in this work. We would
like to make a few concluding remarks at this point.

It is known that in general the effective action and the Casimir energy cannot be computed
in an analytic way for an arbitrary value of the mass parameter. For this reason, it is crucial to
develop efficient methods in order to obtain results valid at least in a finite range of values of
m. A large mass expansion is obtained thanks to the heat kernel asymptotic expansion. More
explicitly, the massive spectral zeta function can be written, by utilizing the Mellin transform
and the small t asymptotic expansion of the heat kernel, as follows:

\[ \sum_{\gamma} (\gamma^2 + m^2)^{-s} \sim \frac{1}{(4\pi)^{D/2} \Gamma(s)} \sum_{k=0,1/2,1,...}^{\infty} B_k \int_0^{\infty} dt \, t^{s+k-D/2-1} e^{-tm^2} \]

with \( B_k \) representing the coefficients of the heat kernel asymptotic expansion. For other mass
regimes, one needs to use a different approach. To this end, quite recently, a method which
is a combination of analytic and numerical parts has been used in order to evaluate the QCD
instanton determinant for a wide range of the mass parameter and general classes of radial
background fields [13–16, 25].

The main motivation of this work is to provide an analytic expression for the small mass
expansion of the functional determinant on a background which contains a conical singularity.
The results obtained in this paper, combined with large mass expansions coming from the
heat kernel, can help reduce drastically the numerical work necessary for the analysis of
intermediate values of the mass. To stress this point further, let us consider in some detail
the case of Dirichlet boundary conditions in \( D = 3 \) dimensions. The first few heat kernel
coefficients in that case read [3]

\[ B_0 = \frac{3}{4\pi}, \quad B_{1/2} = -2\pi^{3/2}, \quad B_1 = \frac{8\pi}{3}, \quad B_3 = \frac{16\pi}{315}, \quad B_{3/2} = -\frac{\pi^{3/2}}{120} \]

and the large mass expansion of the determinant is

\[ \zeta'_{\text{eff}}(0) = \frac{1}{8} m^3 + \frac{1}{4} m^2 - \frac{2}{3} m + \frac{1}{48} \ln m^2 - \frac{1}{4} m^2 \ln m^2 - \frac{2}{315} m - \frac{1}{960m^2} + O \left( \frac{1}{m^3} \right). \]

Including more terms in this expansion is numerically negligible roughly for \( m > 1 \). The
graphs of the large mass expansion above and of the small mass expansion in equation (6.1)
are given in figure 1(a) and (b) for \( m \in [0, 10] \) and \( m \in [0, 2] \). From the above remarks it
is clear that in order to find the determinant for all masses one needs to interpolate from the
small mass graph starting at about \( m = 0.2 \) to the large mass graph at about \( m = 1 \). Including
higher powers in the small mass expansion the upper graph supposedly would bend down further
making the interpolation even easier.

Finally, the small mass expansion obtained here can be used in order to find the massive
corrections to the one-loop effective action in the ambit of orbifold compactification which is
a topic of great interest in string theory (see e.g. [1] and references therein). The techniques
utilized here, in order to find the functional determinant, represent an alternative approach to
the covariant background method studied in [31].
Figure 1. Graphs of the small and large mass expansions for $\zeta'(0)$ in $D = 3$.

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Appendix. Polynomials $D_n(t)$ and $M_n(t)$ up to the order $n = 6$

We list, here, the polynomials $D_n(t)$ up to the sixth order. By utilizing expression (2.26) and the recurrence relation (2.18) one obtains

\begin{align}
D_1(t) &= \frac{1}{8} t - \frac{5}{24} t^3, \\
D_2(t) &= \frac{1}{16} t^2 - \frac{3}{8} t^4 + \frac{5}{16} t^6, \\
D_3(t) &= \frac{25}{384} t^3 - \frac{531}{640} t^5 + \frac{221}{128} t^7 - \frac{1105}{1152} t^9, \\
D_4(t) &= \frac{13}{128} t^4 - \frac{71}{32} t^6 + \frac{531}{64} t^8 - \frac{339}{32} t^{10} + \frac{565}{128} t^{12}, \\
D_5(t) &= \frac{1073}{5120} t^5 - \frac{50049}{7168} t^7 + \frac{186821}{4608} t^9 - \frac{44899}{512} t^{11} + \frac{82825}{1024} t^{13} - \frac{82825}{3072} t^{15}, \\
D_6(t) &= \frac{103}{192} t^6 - \frac{405}{16} t^8 + \frac{1677}{8} t^{10} - \frac{5389}{64} t^{12} + \frac{65385}{14} t^{14} - \frac{11805}{16} t^{16} + \frac{19675}{96} t^{18}.
\end{align}

The polynomials $M_n(t, u)$ follow by using expression (4.3) together with the recurrence relation (4.2). One can find

\begin{align}
M_1(t, u) &= -\frac{3}{8} t + \frac{7}{24} t^3 + tu, \\
M_2(t, u) &= -\frac{3}{16} t^2 + \frac{5}{8} t^4 - \frac{7}{16} t^6 + \frac{u}{2} t^2 - \frac{u^2}{2} - \frac{u}{2} t^2, \\
M_3(t, u) &= -\frac{21}{128} t^3 + \frac{869}{640} t^5 - \frac{315}{128} t^7 + \frac{1463}{1152} t^9 + \frac{3u}{8} t^3 - \frac{5u}{4} t^5 + \frac{7u}{8} t^7 - \frac{u^2}{2} t^3 + \frac{u^2}{2} + \frac{u^3}{3} t^3.
\end{align}
\[
M_4(t, u) = -\frac{27}{128}t^4 + \frac{109}{32}t^6 - \frac{733}{64}t^8 + \frac{441}{32}t^{10} - \frac{707}{128}t^{12} + \frac{3u}{8}t^4 - \frac{23u}{8}t^6 + \frac{41u}{8}t^8 - \frac{21u}{8}t^{10} - \frac{3u^2}{2}t^6 - t^8u^2 + \frac{u^3}{2}t^4 - \frac{u^3}{4}t^6 - \frac{u^4}{4}t^4.
\]

\[
M_5(t, u) = -\frac{1899}{5120}t^5 + \frac{72003}{7168}t^7 - \frac{247735}{4608}t^9 + \frac{56761}{512}t^{11} - \frac{101395}{1024}t^{13} + \frac{495271}{15360}t^{15} + \frac{63}{2}t^5 - \frac{1537}{16}t^7 + \frac{917}{16}t^9 - \frac{1463}{16}t^{11} - \frac{9u^2}{16}t^5 + \frac{59u^2}{16}t^7 - \frac{99u^2}{16}t^9 + \frac{49u^2}{8}t^{11} + \frac{5u^3}{8}t^7 - \frac{7u^3}{8}t^9 - \frac{u^4}{2}t^5 + \frac{u^4}{2}t^7 + \frac{u^5}{5}t^9.
\]

\[
M_6(t, u) = -\frac{27}{32}t^6 + \frac{69}{2}t^8 - \frac{17163}{64}t^{10} + \frac{4973}{8}t^{12} - \frac{9789}{8}t^{14} + \frac{3465}{4}t^{16} - \frac{45493}{192}t^{18} + \frac{27}{32}u^6 - \frac{681u^2}{32}t^8 + \frac{1793u^2}{16}t^{10} - \frac{3671u^2}{16}t^{12} + \frac{6531u^2}{32}t^{14} - \frac{2121u^2}{32}t^{16} - \frac{3u^2}{4}t^{18} + \frac{75u^2}{8}t^8 - \frac{233u^4}{8}t^{10} + \frac{269u^4}{8}t^{12} - \frac{105u^4}{8}t^{14} + \frac{19u^4}{24}t^{16} - \frac{37u^4}{8}t^{18} + \frac{59u^4}{8}t^{10} - \frac{85u^4}{24}t^{12} + \frac{3u^4}{4}t^6 + \frac{2r^4u^4}{4}t^{10} + \frac{u^5}{2}t^6 - \frac{u^5}{2}t^8 - \frac{u^6}{6}t^6.
\]

For completeness, we will explicitly compute (A.1) and (A.7). From equation (2.26) one finds, by equating like powers of \( v \), the relation

\[
D_1(t) = u_1(t).
\]

In order to compute \( u_1(t) \) we exploit the recurrence relation (2.18) to find

\[
u_1(t) = \int_0^t d\tau (1 - 5\tau^2).
\]

By performing the elementary integration one finally obtains (A.1). In a very similar way one can evaluate (A.7). In fact, from the cumulant expansion (4.3) we get

\[
M_1(t, u) = v_1(t) + ut.
\]

According to (4.2) one can find an expression for \( v_1(t) \) as follows

\[
v_1(t) = u_1(t) + \frac{1}{2}t(t^2 - 1).
\]

By substituting result (A.14) for \( u_1(t) \) in the previous equation one finds (A.7). All the other polynomials can be computed, as many as needed, with the help of a simple computer program.

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