The existence and uniqueness of integral solutions to some nonlinear reaction–diffusion system with nonlocal retarded initial conditions

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ABSTRACT

In this paper, a viability results for nonlinear retarded reaction–diffusion system with the non-local retarded initial condition are studied. The existence, uniqueness and global asymptotic stability of solutions are investigated. Based on the compactness arguments, Tychonoff fixed point theorem and invariance technique, the proof of our main results is presented.

1. Introduction

Let A and B are the m-dissipative operators on real Banach spaces X and Y respectively. We assume the nonlinear retarded reaction–diffusion system with a nonlocal retarded initial condition in order to find \( (u, v) \in C_b((-\tau, +\infty); \mathcal{D}(A)) \times C_b((-\tau, +\infty); \mathcal{D}(B)) \) satisfying the following equations:

\[
\begin{align*}
\tau(t) &= u(t) + f(t, u(t), \ldots, u_n(t)), \\
\tau'(t) &= g(t, u(t), \ldots, u_n(t)), \quad t \in \mathbb{R}_+,
\end{align*}
\]

Here, \( \tau_0 = 0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n = \tau \) to be fixed, \( f : \mathbb{R}_+ \times C((-\tau, 0], \mathcal{D}(A)^{n+1}) \times C((-\tau, 0], \mathcal{D}(B)^{n+1}) \rightarrow X \) and \( g : \mathbb{R}_+ \times C((-\tau, 0], \mathcal{D}(A)^{n+1}) \times C((-\tau, 0], \mathcal{D}(B)^{n+1}) \rightarrow Y \) are continuous functions, namely \( u_0(t) = u(t - \tau_k) \) and \( v_0(t) = v(t - \tau_k) \) for \( k = 0, \ldots, n \), knowing that, \( p : C_b((-\tau, +\infty); \mathcal{D}(A)) \times C_b((-\tau, +\infty); \mathcal{D}(B)) \rightarrow C((-\tau, 0]; \mathcal{D}(A)) \) and \( q : C_b((-\tau, +\infty); X) \times C_b((-\tau, +\infty); \mathcal{D}(B)) \rightarrow C((-\tau, 0]; \mathcal{D}(B)) \) are non-expansive.

The system (1) has been addressed by several authors, and they analyzed various cases, such as Alam and Alam [1], Burlică [2], Burlică and Roșu [3–5], Díaz and Vrabie [6], Meknani and Zhang [7], Neucală and Vrabie [8], Roșu [9,10] been studied by Burlică and Roșu [11], Vrabie [12–14], García and Reich [15] and Paicu and Vrabie [31]. Otherwise, the result for nonlinear evolution inclusions with nonlocal retarded initial conditions was investigated by Vrabie [16] and references therein. All these studies were motivated by the applicability of this type of problem in mathematical modeling, it covers most of the nonlinear evolution equations with nonlocal initial conditions appearing in physics, these examples were presented in Deng [17] and McKibben [18], Section 10.2, p. 394–398. We emphasize that, our result is inspired by Vrabie [16] and our main goal is to extend Theorem 3.1 in Burlică and Vrabie [19] under almost the same general assumptions, and our main result is adapted to the results for nonlinear reaction–diffusion system with nonlocal retarded initial conditions.

We draw our results depending on the compactness method presented in [19,20]. In this study, we prove the approximate problem (6) has at least \( C^0 \)-solutions \( (u_\alpha, v_\alpha) \) converge in \( C_b((-\tau, +\infty); X) \times C_b((-\tau, +\infty); Y) \) to \( (u, v) \in C_b((-\tau, +\infty); X) \times C_b((-\tau, +\infty); Y) \) which is \( C^0 \)-solution of the system (1) if and only if the \( C^0 \)-solutions set \( \{(u_\alpha, v_\alpha) \mid 0 \leq \alpha \leq 1 \} \) is compact and closed. To investigate that, we required to impose auxiliary systems (8)–(9) and more details are shown in Section 4.

The paper is divided into seven sections. In Section 2, we present some concepts and results referring to evolutions governed by m-dissipative operators, needed in the sequel. In Section 3, we state our main result, i.e. Theorem 3.2. Section 4 presents the outline of the proof. Section 5 is mainly devoted to some auxiliary results and in Section 6 we give the complete proof of Theorem 3.2. Finally, in Section 7, we analyze an example illustrating the effectiveness of the abstract theory.

2. Preliminaries

For further background and details pertaining to this section, we refer to the articles [21–24].

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Let $X$ and $Y$ be Banach spaces with norm $\|\cdot\|$. If $I$ is an interval, $C(I, X) \times C(I, Y)$ denotes the product space of all bounded and continuous functions from $I$, equipped with sup-norm
\[
\|(u, v)\|_{C(I, X) \times C(I, Y)} = \max \left\{ \|u(t)\|_{C(I, X)}, \|v(t)\|_{C(I, Y)}, t \in I \right\},
\]
while $C(I, \overline{D(A)}) \times C(I, \overline{D(B)})$ denotes the closed product subset in $C(I, X) \times C(I, Y)$ consisting of all elements $(u, v) \in C(I, X) \times C(I, Y)$ satisfying $(u(t), v(t)) \in \overline{D(A)} \times \overline{D(B)}$ for each $t \in I$. Further, $C([a, b), X) \times C([a, b), Y)$ represents the product space of all continuous functions from $[a, b)$ to $X \times Y$ endowed with the sup-norm
\[
\|(u, v)\|_{C([a, b), X) \times C([a, b), Y)} = \max \left\{ \|u(t)\|_{C([a, b), X)}, \|v(t)\|_{C([a, b), Y)}, t \in [a, b) \right\},
\]
and $C([a, b), \overline{D(A)}) \times C([a, b), \overline{D(B)})$ denote the closed product subset of $C([a, b), X) \times C([a, b), Y)$ containing all $(u, v) \in C([a, b), X) \times C([a, b), Y)$ with $(u(t), v(t)) \in \overline{D(A)} \times \overline{D(B)}$ for each $t \in [a, b)$.

In addition, we assume $C_b((c, +\infty); X) \times C_b((c, +\infty); Y)$ as the product space $C_b((c, +\infty); X) \times C_b((c, +\infty); Y)$ equipped with the family of semi-norms
\[
\|(\cdot, \cdot)\|_{k}, \ k \in \mathbb{N}, \ k > c, \ c \in \mathbb{R},
\]
defined by
\[
\|(u, v)\|_k = \|(u, v)\|_{C_b((c, \infty); X) \times C_b((c, \infty); Y)} \ \forall \ k = 1, 2, \ldots,
\]
such spaces are separated locally convex space, endowed with a family of semi-norms mentioned above.

The operator $A : D(A) \subseteq X \rightharpoonup X$ is called dissipative if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, we have that
\[
\|x_1 - x_2, y_1 - y_2\|_\infty = 0,
\]
where
\[
[x, y]_\infty := \frac{1}{t} \left( \|x + ty\| - \|x\| \right)
\]
and
\[
[x, y]_+ = \lim_{t \to 0} [x, y]_\infty := \inf \left\{ [x, y]_\infty; t > 0 \right\} \ \forall x, y \in X.
\]
Obviously, we can show that
\[
[x, y]_+ \leq \|y\| \ \forall x, y \in X.
\]
If the operator $A : D(A) \subseteq X \rightharpoonup X$ is dissipative and the range of $I - \lambda A$ satisfies $R(I - \lambda A) = X$ for $\lambda > 0$, then $A$ is m-dissipative.

For every $f \in L^1([a, b]; X)$, we say that the function $u : [a, b) \to \overline{D(A)}$ is a $C^0$-solution or integral solution on $[a, b)$ of the following evolution equation:
\[
u'(t) = Au(t) + f(t), \quad (2)
\]
if $u \in C([a, b), X)$ and satisfies the following inequality:
\[
\|u(t) - x\| \leq \|u(\xi) - x\| + \int_\xi^t \|u(s) - x\| \, ds, \quad \forall \xi, x, t \in [a, b),
\]
for each $x \in D(A), y \in Ax$ and $a \leq s \leq t \leq b$.

**Theorem 2.1:** Let $A : D(A) \subseteq X \rightharpoonup X$ be an m-dissipative operator such that $A + \omega I$ is dissipative with $\omega > 0$. For each $\xi \in \overline{D(A)}$ and $f \in L^1([a, b); X)$, the evolution Equation (2) on $[a, b)$ has a unique $C^0$-solution which satisfies $u(\xi) = \xi$. If $u$ and $v$ are two $C^0$-solutions of (2) corresponding to $f$ and $g \in L^1([a, b); X)$, respectively, then
\[
\|u(t) - v(t)\| \leq e^{-\omega(t-\xi)} \|u(\xi) - v(\xi)\| + \int_\xi^t e^{-\omega(t-s)} \|f(s) - g(s)\| \, ds, \quad (3)
\]
In particular, if $x \in D(A)$ and $y \in Ax$, we have
\[
\|u(t) - x\| \leq e^{-\omega(t-\xi)} \|u(\xi) - x\| + \int_\xi^t e^{-\omega(t-s)} \|f(s)\| + y \, ds, \quad (4)
\]
for each $a \leq s \leq t \leq b$.

(cf. Barbu [21, Theorem 4.1, p.130]).

For $\xi \in \overline{D(A)}$, $f \in L^1(a, b; X)$ and $t \in [a, b)$, the problem (2) has a unique $C^0$-solution $v : [\xi, b) \to \overline{D(A)}$ satisfying the initial condition $v(t) = \xi$ denote by $(u(t, \xi, \xi, f)$. We denote $S_A(t) : \overline{D(A)} \to \overline{D(A)}, \ t \geq 0)$ the semigroup generated by $A$ over $\overline{D(A)}$ and defined by $S_A(t) \xi = u(t, 0, \xi, 0)$, for each $\xi \in \overline{D(A)}$ and $t \geq 0$. Moreover, if the operator $(S_A(t))_{t \geq 0}$ is a compact operator, then the semigroup generated by $A$ is compact.

**Definition 2.2:** A subset $F \subseteq L^1([a, b); X)$ is uniformly integrable, if for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that
\[
\int_E \|f(s)\| \, ds \leq \varepsilon, \quad \text{for each } f \in F, \text{ and each measurable subset } E \subseteq [a, b]
\]
whose Lebesgue measure $\lambda(E) < \delta(\varepsilon)$.

**Theorem 2.3:** For each $c \in (a, b)$, the $C^0$-solutions set $\{u(\cdot, a, \xi, f); \xi \in F \}$ is relatively compact in $C([c, b); X)$ if, the m-dissipative operator $A : D(A) \subseteq X \rightharpoonup X$ generates a compact semigroup, $B \subseteq \overline{D(A)}$ be bounded, and $F$ be uniformly integrable in $L^1([a, b); X)$, Furthermore, if $B$ is relatively compact in $X$, then, the $C^0$-solutions set is relatively compact even in $C([a, b); X)$.

Return to [24, Theorem 2.3.3, p.47].
Here, we point to the Tychonoff fixed point theorem, which is very useful in proving our main results.

**Theorem 2.4:** Let $K$ be a non-empty, convex and closed subset in separated locally convex topological vector space $X$. Then, the mapping $\Lambda : K \rightarrow K$ has at least one fixed point, i.e.

$$\exists g \in K \; \text{such that} \; g \in \Lambda(g)$$

provided that $\Lambda$ is continuous function and $\Lambda(K)$ is relatively compact.

See Edwards [22, Theorem 3.6.1, p.161] and Tychonoff [25].

3. Existence and uniqueness of integral solution

In this section, we deal with the existence and uniqueness of $C^0$-solutions for the problem (1) under the following assumptions:

**(H_0)** $A : D(A) \subseteq X \rightarrow X$ is $m$-dissipative operator, $0 \in D(A)$, $0 \in A0$ and there exists $\omega > 0$ such that $A + \omega I$ is dissipative.

**(H_1)** $B$ is $m$-dissipative, $0 \in D(B), 0 \in B0$ and there exists $\gamma > 0$ such that $B + \gamma I$ is dissipative.

**(H_2)** $B$ generates a compact semigroup.

**(H_3)** The constants $\ell_k > 0$, $\tau_k$, $(k = 0, \ldots, n)$, $\omega > 0$ and $\gamma > 0$ satisfy the non-resonance conditions:

$$(c_1) \quad \ell = \sum_{k=0}^{n} \ell_k < \frac{\omega \gamma}{\omega + \gamma}.$$

$$(c_2) \quad a = \sum_{k=0}^{n} \ell_k e^{\omega \tau_k} < \omega \quad \text{and} \quad b = \sum_{k=0}^{n} \ell_k e^{\gamma \tau_k} < \gamma.$$ 

**(H_4)** The function $f : \mathbb{R}_+ \times C([-\tau, 0]; D(A)^{n+1}) \times C([-\tau, 0]; D(B)^{n+1}) \rightarrow X$ is continuous and satisfies:

$$(f_2) \quad \text{There exists } m > 0 \text{ such that }$$

$$\|f(t, 0, \ldots, 0)\|_X \leq m,$$

for each $t \in \mathbb{R}_+$.

$$(H_5) \quad \text{The function } g : \mathbb{R}_+ \times C([-\tau, 0]; D(A)^{n+1}) \times C([-\tau, 0]; D(B)^{n+1}) \rightarrow Y \text{ is continuous and satisfies:}$$

$$(g_1) \quad \text{with } \xi_k > 0, \; k = 0, \ldots, n, \text{ and } m > 0 \text{ is given by }$$

$$f(t) + g(t, u_0, \ldots, u_n, v_0, \ldots, v_n) = (0, \ldots, 0),$$

and each $(u_0, \ldots, u_n) \in C([-\tau, 0]; D(A)^{n+1})$ and $(v_0, \ldots, v_n) \in C([-\tau, 0]; D(B)^{n+1})$, and each $t \geq 0;$

$$(g_2) \quad \text{for each } (u_0, \ldots, u_n, v_0, \ldots, v_n) \in C([-\tau, 0]; D(A)^{n+1}) \text{ and each } t \geq 0;$$

$$g(t, u_0, \ldots, u_n, v_0, \ldots, v_n) \leq n \sum_{k=0}^{n} \ell_k \left[\|u_k - \tilde{u}_k\|_{C([-\tau, 0]; X)} + \|v_k - \tilde{v}_k\|_{C([-\tau, 0]; Y)}\right] + m,$$

and each $(u_0, \ldots, u_n) \in C([-\tau, 0]; D(A)^{n+1})$ and $(v_0, \ldots, v_n) \in C([-\tau, 0]; D(B)^{n+1})$, and each $t \geq 0;$$

$$(p_1) \quad \text{for each } u \in C_b((-\tau, +\infty); D(A)) \text{ and each } v \in C_b((-\tau, +\infty); D(B)),$$

we have

$$\|p(u, v)\|_{C((-\tau, 0]; X)} \leq \|u\|_{C_b((-\tau, +\infty); X)} \cdot \|v\|_{C_b((-\tau, +\infty); X)}$$

and each $(u_0, \ldots, u_n) \in C([-\tau, 0]; D(A)^{n+1})$ and $(v_0, \ldots, v_n) \in C([-\tau, 0]; D(B)^{n+1})$, and each $t \geq 0;$$

$$(q_1) \quad \text{for each } u \in C_b((-\tau, +\infty); X) \text{ and each } v \in C_b((-\tau, +\infty); D(B)),$$

we have

$$\|q(u, v)\|_{C((-\tau, 0]; X)} \leq \|v\|_{C_b((-\tau, +\infty); Y)}.$$
(q₂) with $c > 0$ given in (p₂) for each $u, \hat{u} \in \mathbb{C}_{b}([-\tau, +\infty); X)$ and each $v, \hat{v} \in \mathbb{C}_{b}([-\tau, +\infty); \overline{D(B)})$, we have

$$\|q(u, v) - q(u, \hat{u})\|_{\mathbb{C}([-\tau, 0]; Y)} \leq \max \left\{ \|u - \hat{u}\|_{\mathbb{C}_{b}(\mathbb{R}_{+}; X)}, \|v - \hat{v}\|_{\mathbb{C}_{b}(\mathbb{R}_{+}; Y)} \right\},$$

(q₃) For each bounded set $B$ belong to $\mathbb{C}_{b}([-\tau, +\infty); X)$, and for each bounded set $\tilde{E}$ belong to $\mathbb{C}_{b}([-\tau, +\infty); \overline{D(B)})$ which is relatively compact in $\mathcal{C}_{b}(\overline{\sigma_{r} + \infty}; Y)$ for each $\sigma \in (0, +\infty)$, the set $q(E, \tilde{E})$ is relatively compact in the space $\mathcal{C}(\mathbb{R}_{+}; Y)$.

**Definition 3.1:** We said that a $C^0$-solution $(u, v)$ of (1) is globally asymptotically stable if for each $(\rho, \varrho) \in \mathcal{C}(\mathbb{R}_{+}; \mathcal{A}(t)) \times \mathcal{C}(\mathbb{R}_{+}; \overline{D(B)})$ the unique $C^0$-solution $(w, \tilde{w}) \in \mathcal{C}(\mathbb{R}_{+}; \mathcal{A}(t)) \times \mathcal{C}(\mathbb{R}_{+}; \overline{D(B)})$ of the problem

$$w'(t) \in \mathcal{A}(t) + f(t, w(t), \ldots, w_n(t), \tilde{w}_0(t), \ldots, \tilde{w}_n(t)), \quad t \in \mathbb{R}_{+},$$

$$\tilde{w}'(t) \in B\tilde{w}(t) + g(t, w(t), \ldots, w_n(t), \tilde{w}_0(t), \ldots, \tilde{w}_n(t)), \quad t \in \mathbb{R}_{+},$$

$$w(0) = \rho(0), \quad \tilde{w}(0) = \tilde{\rho}(0), \quad t \in [-\tau, 0],$$

is such that

$$\|w(t) - \tilde{w}(t)\| \to 0, \quad t \to +\infty.$$
5. Auxiliary results

For convenience and clarity, we will divide the proof of Theorem 3.2 into five steps and they are labelled as the following five lemmas.

**Lemma 5.1 ([19]):** If \((H_1), (B_1), (H_B)\) and \((q_1), (q_2)\) are satisfied, then, for all \((f, g) \in C_b([0, +\infty); X) \times C_b([0, +\infty); Y)\) where \(\alpha > 0\), the system

\[
\begin{align*}
-u'(t) & = Au(t) + f(t), \quad t \in \mathbb{R}_+, \\
v'(t) & = Bv(t) + g(t), \quad t \in \mathbb{R}_+, \\
u(t) & = (1 - \alpha)p(u, v)(t), \quad t \in [-\tau, 0], \\
v(t) & = (1 - \alpha)q(u, v)(t), \quad t \in [-\tau, 0],
\end{align*}
\]

has a unique \(C^0\)-solution \((u, v) \in C_b([-\tau, +\infty); \overline{D(A)}) \times C_b([-\tau, +\infty); \overline{D(B)})\). Furthermore, these inequalities

\[
\|u\|_{C_b([-\tau, +\infty); X)} \leq \frac{m}{\omega}, \quad \|v\|_{C_b([-\tau, +\infty); Y)} \leq \frac{m}{\gamma}
\]

are guaranteed, if \(\|f\|_{C_b([0, +\infty); X)} \leq m\) and \(\|g\|_{C_b([0, +\infty); Y)} \leq m\) are satisfied.

Here, let us assume \(\tilde{U}(t) = (\hat{u}_0(t), \ldots, \hat{u}_n(t))\) and \(\tilde{V}(t) = (\hat{v}_0(t), \ldots, \hat{v}_n(t))\) is endowed with the following sup-norm:

\[
\|\tilde{U}\|_{\infty} = \max_{k=0, \ldots, n} \|\hat{u}_k\|_{C_b([-\tau, +\infty); X)} ,
\|\tilde{V}\|_{\infty} = \max_{k=0, \ldots, n} \|\hat{v}_k\|_{C_b([-\tau, +\infty); Y)}
\]

**Lemma 5.2:** If \((H_A), (H_B), (H_Y), (H_B), (H_2)\) and \((c_1)\) in \(\mathcal{H}_2\) are satisfied. Then, for each \((u, h) \in C_b([-\tau, +\infty); X) \times C_b([0, +\infty); Y)\) satisfying

\[
\|u\|_{C_b([-\tau, +\infty); X)} \leq \frac{d}{\omega}, \quad \|h\|_{C_b([0, +\infty); Y)} \leq d
\]

such that

\[
d = \frac{\omega \gamma - \ell}{\omega \gamma}
\]

the pair \((\hat{u}, \hat{v})\), where \(v\) is the unique \(C^0\)-solution of (8) and \(u\) is the unique \(C^0\)-solution of (9), satisfies:

\[
\|\hat{u}\|_{C_b([-\tau, +\infty); X)} \leq \frac{d}{\omega}, \quad \|\hat{v}\|_{C_b([-\tau, +\infty); Y)} \leq \frac{d}{\gamma}
\]

so, for each \(k = 1, \ldots, n\), we have

\[
\|\hat{u}_k\|_{C_b([-\tau, +\infty); X)} \leq \frac{d}{\omega}, \quad \|\hat{v}_k\|_{C_b([-\tau, +\infty); Y)} \leq \frac{d}{\gamma}
\]

therefore,

\[
\|f(t, \tilde{U}, \tilde{V})\|_{X} \leq d \quad \forall t \in [0, +\infty).
\]

**Proof:** First, we prove that, for each \(k = 0, \ldots, n\), we have

\[
\|\hat{u}_k\|_{C_b([-\tau, +\infty); Y)} \leq \frac{d}{\gamma}
\]

In fact, for \(k = 0\), and from inequality (4), we get

\[
\|\hat{v}(t)\| \leq (1 - \varepsilon) e^{-\gamma t} \|\hat{v}(0)\| + \frac{d}{\gamma} (1 - e^{-\gamma t}),
\]

for each \(t \in [0, +\infty).\) By applying Lemma 5.1 in [19], we obtain

\[
\|\hat{v}\|_{C_b([0, +\infty); Y)} \leq \frac{d}{\gamma},
\]

on the other hand, from \((q_1)\), and for each \(t \in [-\tau, 0]\), we have

\[
\|\hat{v}(t)\| = \|q(\hat{u}, \hat{v})(t)\| \leq \|\hat{v}\|_{C_b([0, +\infty); Y)} \leq \frac{d}{\gamma},
\]

hence, for each \(k = 1, \ldots, n\), we obtain

\[
\|\hat{u}_k\| \leq \frac{d}{\omega}, \quad \forall k = 0, \ldots, n.
\]

Actually, for \(k = 0\), from Lemma 5.1 in [19], and Lemma 3.2 in [7], we conclude that

\[
\|\hat{u}(t)\| \leq (1 - \varepsilon) e^{-\omega t} \|\hat{u}(0)\| + \int_0^t e^{-\omega(t-s)} \|f(s, \tilde{U}, \tilde{V})\| ds
\]

\[
\leq (1 - \varepsilon) e^{-\omega t} \|\hat{u}(0)\| + \frac{1}{\omega} (1 - e^{-\omega t}) \times \left[ \sum_{k=0}^n \ell_k \left( \|\hat{u}_k\|_{C_b([-\tau, 0); X)} + \|\hat{v}_k\|_{C_b([-\tau, 0); Y)} \right) + m \right]
\]

\[
\leq (1 - \varepsilon) e^{-\omega t} \|\hat{u}\|_{C_b([0, +\infty); X)} + \frac{1}{\omega} (1 - e^{-\omega t}) \times \left[ \sum_{k=0}^n \ell_k \|\hat{u}_k\|_{C_b([-\tau, 0); X)} + \ell \frac{d}{\gamma} + m \right]
\]

\[
\leq (1 - \varepsilon) e^{-\omega t} \|\hat{u}\|_{C_b([0, +\infty); X)} + \frac{1}{\omega} (1 - e^{-\omega t}) \times \left[ \ell \left( \|\hat{u}\|_{C_b([0, +\infty); X)} + \frac{d}{\gamma} + m \right) \right],
\]
due to the
\[ \|\hat{u}\|_{C_b([-\tau, 0]; X)} = \|\hat{u}(t - \tau_k)\|_{C_b([-\tau, 0]; X)} \]
\[ = \sup_{\theta \in [-\tau, 0]} \|\hat{u}(t - \tau_k + \theta)\| \leq \|\hat{u}\|_{C_b([0, +\infty); X)}, \]
so, by calling Lemma 5.1 in [19], we get
\[ \|\hat{u}\|_{C_b([0, +\infty); X)} \leq \frac{d}{\omega}, \] (21)
otherwise, for \( k = 1, \ldots, n \), and for each \( t \in [-\tau, 0] \), we observe that
\[ \|\hat{u}(t)\| = \|p(\hat{u}, \hat{y})(t)\| \leq \|\hat{u}\|_{C_b([0, +\infty); X)} \leq \frac{d}{\omega}, \]
according to the above results, we obtain that
\[ \|\hat{u}_k\|_{C_b([-\tau, +\infty); X)} \leq \frac{d}{\omega} \quad \forall k = 0, \ldots, n, \]
and then,
\[ \|\hat{u}\|_{\infty} \leq \frac{d}{\omega}. \] (22)
Finally, due to (20), (22) and (H1), we deduce that
\[ \|f(t, \hat{u}, \hat{y})\| \leq d \quad \forall t \in [0, +\infty). \] (23)
Hence, the proof is completed. \( \blacksquare \)

**Lemma 5.3 ([26]):** Let \( \{f_p : \mathbb{R}_+ \times C([-\tau, 0]; \overline{D(A)}^{d+1}) \rightarrow X; p \in \mathbb{N}\} \) be a family of continuous functions satisfying:

(H1) there exists \( 0 < \ell < \omega \) such that \( \|f_p(t, x) - f_p(t, y)\|_X \leq \ell\|x - y\|_{C([-\tau, 0]; \overline{D(A)}^{d+1})}, \) for each \( p \in \mathbb{N} \), \( t \in [0, +\infty) \) and \( x, y \in C([-\tau, 0]; \overline{D(A)}^{d+1}) \);

(H2) there exists \( m > 0 \) such that \( \|f_p(t, 0, \ldots, 0)\|_X \leq m, \) for each \( p \in \mathbb{N} \), \( t \in [0, +\infty) \);

(H3) \( \lim_{p} \|f_p(t, x) - f(t, x)\|_X = 0, \) for \( t \in [0, +\infty) \) (for \( t \) in bounded intervals in \([0, +\infty)\) and \( x \) in bounded subsets in \( C([-\tau, 0]; \overline{D(A)}^{d+1}) \)).

Let \( \{p_p : C([-\tau, +\infty]; \overline{D(A)}) \rightarrow C([-\tau, 0]; \overline{D(A)}); p \in \mathbb{N}\} \) be a family of functions satisfying:

(H4) for each \( p \in \mathbb{N} \) and \( u \in C([-\tau, +\infty); \overline{D(A)}) \), we have
\[ \|p_p(u)\|_{C([-\tau, 0]; X)} \leq \|\hat{u}\|_{C([0, +\infty); X)}; \]

(H5) there exists \( c > 0 \) such that for each \( p \in \mathbb{N} \) and \( u, \hat{u} \in C_b([-\tau, +\infty); \overline{D(A)}) \), we have
\[ \|p_p(u) - p_p(\hat{u})\|_{C([-\tau, 0]; X)} \leq \|u - \hat{u}\|_{C([0, +\infty); X)}; \]

(H6) \( \lim_p \|p_p(u) - p(u)\| = 0, \) for \( u \) in bounded subsets in \( C_b([-\tau, +\infty); \overline{D(A)}) \) (and \( p : C_b([-\tau, +\infty); \overline{D(A)}) \rightarrow C_b([-\tau, 0]; \overline{D(A)}) \) is continuous).

Let us assume further that \( A \) satisfies (H8) and let \( (u_p)_p \) be the sequence of \( C^0 \)-solutions of the problem
\[ u'_p(t) \in Au_p(t) + F(t, u_p(t)), \quad t \in \mathbb{R}_+, \]
\[ u_p(t) = p_p(u_p(t)), \quad t \in [-\tau, 0], \] (24)
whose existence and uniqueness is satisfied by Theorem 3.1 which was presented by Burlică and Rosu [11]. Then, we have
\[ \lim_p \|u_p - u\|_{C_b([-\tau, +\infty); X)} = 0, \]
as far as
\[ \lim_p \|u_p - u\|_{C_b([-\tau, +\infty); X)} = 0, \]
where \( u \) is the unique \( C^0 \)-solution of the problem
\[ u'(t) \in Au(t) + F(t, u(t)), \quad t \in \mathbb{R}_+, \]
\[ u(t) = p(u)(t), \quad t \in [-\tau, 0]. \] (25)

**Lemma 5.4:** Let us take \( K_\rho \subseteq C_b([-\tau, +\infty); X) \) as the closed ball with a centre 0 and radius \( \hat{\rho} = d/\omega \) for each \( d \geq 0 \) which is given by (15) and let \( K_\rho \subseteq C_b([0, +\infty); Y) \) as the closed ball with a centre 0 and radius \( d \) multiply by \( \psi_\alpha \), such that
\[ K_\rho = K_\rho \times K_\rho. \]

If the hypotheses (H1), (H2), (H3), (H4), (H5), (H6) and (H7) are satisfied. Then, the operator \( \Lambda_\alpha : K_\rho \rightarrow K_\rho \) defined in (10) is continuous with respect to the topology on \( C_b([-\tau, +\infty); X) \times C_b([0, +\infty); Y) \) such that \( K_\rho, K_\rho \) and \( R \) is endowed with such topology.

**Proof:** To show that the function \( \Lambda_\alpha \) is continuous, it is enough to prove that for any arbitrary sequence \( (u_p, h_p), \tilde{(u_p, h_p)} \) in \( K_\rho \) with \( \Lambda_\alpha(u_p, h_p) = (\tilde{u}_p, \tilde{h}_p) \) for \( p \in \mathbb{N} \) and
\[ \lim_p \|u_p, h_p\| - (u, h)\|C_b([-\tau, +\infty); X) \times C_b([0, +\infty); Y) = 0, \]
it implies that
\[ \lim_p \Lambda_\alpha(u_p, h_p) - \Lambda_\alpha(u, h)\|C_b([-\tau, +\infty); X) \times C_b([0, +\infty); Y) = 0. \]
The proof was shown in [19]. So, we avoid rewrite it here. \( \blacksquare \)

**Lemma 5.5 ([19,20]):** For each \( \alpha \in (0, 1) \), the set \( \text{conv}(\Lambda_\alpha(K_\rho)) \) is relatively compact in the product space \( C_b([-\tau, +\infty); X) \times C_b([0, +\infty); Y) \), if the assumptions (H1), (H2), (H3), (H4), (H5), (H6) and (H7) are satisfied.

**6. Proof of Theorem 3.2:** Firstly, for an arbitrarily fixed number \( \alpha \in (0, 1) \), we suppose that \( K_\rho = \text{conv}(\Lambda_\alpha(K_\rho)) \subseteq C_b([-\tau, +\infty); X) \times C_b([0, +\infty); Y) \). Using Lemma 5.5, we conclude that the
operator $\Lambda_\alpha : K_\alpha \to K_\alpha$ is attained the assumptions of Tychonoff fixed point theorem in [25]. So, $\Lambda_\alpha$ has at least one fixed point $(u_\alpha, h_\alpha)$ which equal that the approximate problem (6) has at least one $C^0$-solution $(u_\alpha, v_\alpha)$. Secondly, by using the same arguments in [19,20], we can show that: for each real number $\mu \in (0, 1)$, the set of the $C^0$-solutions $\{(u_\alpha, v_\alpha), \alpha \in (0, 1)\}$ is compact and closed in $C^0((\tau, +\infty) ; X) \times C^0((\tau, +\infty) ; Y)$. That is to say, if $\alpha_0 \notin \alpha$ there is a subsequence $(u_{\alpha_0}, v_{\alpha_0})$ satisfy
\[
\lim_{\alpha \to \alpha_0} \| (u_{\alpha_0}, v_{\alpha_0}) - (u, v) \|_{C^0((\tau, +\infty) ; X) \times C^0((\tau, +\infty) ; Y)} = 0, 
\]
such that $(u, v)$ are $C^0$-solutions of (1). Thirdly, by virtue of [11], we get that
\[
\| p(u, v) \|_{C^0((\tau, 0) ; X)} \leq \| u \|_{C^0((\tau, +\infty) ; X)}, \tag{26}
\]
for each $u \in C^0((\tau, +\infty) ; X)$. Moreover, as $0 \notin \alpha_0$, by calling the inequality (4) in Theorem 2.1, (H7) and (26), we obtain that
\[
\| u(t) \| \leq e^{-\sigma t} \| p(u(t)) \|_{C^0((\tau, 0) ; X)}
\]
\[
+ \int_0^t e^{-\sigma (t-s)} \left( \| f(s, u_0, \ldots, u_n) \| + \| f(s, 0, \ldots, 0) \| \right) \, ds,
\]
\[
\leq e^{-\sigma t} \| u \|_{C^0((\tau, +\infty) ; X)} + \frac{\ell}{\omega} \left( 1 - e^{-\sigma t} \right)
\]
\[
\times \left( \| u \|_{C^0((\tau, +\infty) ; X)} + \frac{\ell}{\omega} \right). \tag{27}
\]
In addition, we call Lemma 4.2 in [11] with the initial nonlocal condition (p2) and the same nonlocal condition (p1) in (1), we obtain
\[
\| u \|_{C^0((\tau, +\infty) ; X)} \leq \frac{m}{\ell - \omega},
\]
Similarly for the second inequality of (5).

Finally, we prove that each $C^0$-solution of problem (1) is globally asymptotically stable. To show that, let $u \in C^0((\tau, +\infty) ; D(\tilde{A}))$ be the unique $C^0$-solution of (1), $\rho \in C((-\tau, 0) ; D(\tilde{A}))$ be arbitrary but fixed, and $w \in C^0((\tau, +\infty); D(\tilde{A}))$ be the unique $C^1$-solution of the problem
\[
w'(t) = Bw(t) + g(t, w_0(t), \ldots, w_n(t)), \quad t \in \mathbb{R}^+,
\]
\[
w(t) = \rho(t), \quad t \in [-\tau, 0].
\]
Returning back to (3) in Theorem 2.1 and using $f_1$ in (H7), we get that
\[
\| u(t) - w(t) \|
\]
\[
\leq e^{-\sigma t} \| u(0) - w(0) \| + \int_0^t e^{-\sigma (t-s)}
\]
\[
\times \left( \| f(s, u_0, \ldots, u_n) - f(s, w_0, \ldots, w_n) \| \right) \, ds,
\]
\[
\leq e^{-\sigma t} \| u(0) - w(0) \| + \int_0^t e^{-\sigma (t-s)}
\]
\[
\times \left( \sum_{k=0}^n \xi_k \| u_k - w_k \|_{C^0([-\tau, 0] ; X)} \right) \, ds \quad \forall \ t \geq 0,
\]
this implies that
\[
e^{-\sigma t} \| u(t) - w(t) \| \leq \| u(0) - w(0) \|
\]
\[
+ \int_0^t e^{-\sigma s} \left( \sum_{k=0}^n \xi_k \| u_k - w_k \|_{C([-\tau, 0] ; X)} \right) \, ds \quad \forall \ t \geq 0. \tag{28}
\]

Furthermore, due to (p1), the second term in inequality (28) is simplified as follows:
\[
\int_0^t e^{\sigma s} \left( \sum_{k=0}^n \xi_k \| u_k - w_k \|_{C([-\tau, 0] ; X)} \right) \, ds,
\]
\[
= \int_0^t e^{\sigma s} \left( \sum_{k=0}^n \xi_k \| u_k - w_k \|_{C([-\tau, 0] ; X)} \right) \, ds,
\]
\[
\leq \frac{1 - e^{-\sigma t}}{\omega} \left[ \frac{m}{\omega - \ell} + \| \rho \|_{C([-\tau, 0] ; X)} \right].
\]
we imposed that
\[
M = \| u(0) - w(0) \|
\]
\[
+ \ell \frac{1 - e^{-\sigma t}}{\omega} \left[ \frac{m}{\omega - \ell} + \| \rho \|_{C([-\tau, 0] ; X)} \right].
\]
Using the Gronwall's lemma in Vrabie [27, Lemma 1.5.2, p.44], we conclude that
\[
\| u(t) - w(t) \| \leq M e^{(\sigma - \omega)t} \quad \forall \ t \geq 0,
\]
as $\alpha - \omega < 0$ and going to the limit, we conclude that
\[
\lim_{t \to \infty} \| u(t) - w(t) \| = 0.
\]
Likewise for $v$. So, the proof is over.

## 7. Application

In this section, we present an application for the problem under consideration. So, we propose the following system:

\[
\frac{\partial u}{\partial t}(t, x) = \Delta \phi(u(t, x)) - \omega u(t, x)
\]
\[
+ f(t, x, U(t, x), V(t, x)), \quad (t, x) \in \mathbb{R}^+ \times \Omega,
\]
\[
\frac{\partial v}{\partial t}(t, x) = \Delta \psi(v(t, x)) - \gamma v(t, x)
\]
\[
+ g(t, x, U(t, x), V(t, x)), \quad (t, x) \in \mathbb{R}^+ \times \Omega,
\]
\[
\phi(u(t, x)) = n(v(t, x)) = 0, \quad (t, x) \in \mathbb{R}^+ \times \Gamma,
\]
\[
u(t, x) = \frac{\int_{-\tau}^{t} \int_{-\tau}^{t} h(c) Z(v(t + c, x)) \, dc \, dt}{\int_{-\tau}^{t} h(c) Z(v(t + c, x)) \, dc}, \quad (t, x) \in [-\tau, 0] \times \Omega,
\]
\[
\nu(t, x) = v(t + T, x), \quad (t, x) \in [-\tau, 0] \times \Omega. \tag{29}
\]
Here, we assume that $\Omega$ as a nonempty bounded domain in $\mathbb{R}^s, s \geq 2$, which has a $C^1$- boundary
Given the assumptions described above, we conclude

1. The operator \( \Delta \Theta : D(\Delta \Theta) \subseteq L^1(\Omega) \rightarrow L^1(\Omega) \) is m-dissipative such that
   \[
   D(\Delta \Theta) = \left \{ u \in L^1(\Omega), \exists v \in M_0(u) \cap W_0^{1,1}(\Omega), \right. \\
   \left. \Delta v \in L^1(\Omega) \right \},
   \Delta \Theta(u) = \left \{ \Delta v, v \in M_0(u) \cap W_0^{1,1}(\Omega) \right \}
   \cap L^1(\Omega) \text{ for } u \in D(\Delta \Theta).
   \]

2. The operator \( \Delta \Theta \) generates a compact semigroup, if there exist two constants \( C > 0 \) and \( \hat{\beta} > 0 \) if \( s \leq 2 \) with \( \hat{\beta} > (s-2)/s \) and if \( s \geq 3 \) such that \( \Theta^s(r) \leq C|r|^\hat{\beta} \) for each \( r \in \mathbb{R}/\{0\} \) where \( \Theta : \mathbb{R} \rightarrow \mathbb{R} \) is \( C^0 \) on \( \mathbb{R} \) and \( C^1 \) on \( \mathbb{R}/\{0\} \).

These results have been already demonstrated in [28,29]. Indeed, the readers are directed to review Diaz and Vrabie [30] for more details about the sufficient condition for the semigroup generated by \( \Delta \Theta \) to represent weakly compact maps in \( L^1(\Omega) \) in compact sets in \( L^1(\Omega) \) for \( t > 0 \).

Theorem 7.2: Let us consider \( \Omega \) the nonempty bounded and open subset in \( \mathbb{R}^n, s \geq 1 \) which has \( C^1 \) boundary \( Y \). We impose that the operators \( \phi : D(\phi) \subseteq \mathbb{R} \rightarrow \mathbb{R} \) and \( \eta : D(\eta) \subseteq \mathbb{R} \rightarrow \mathbb{R} \) are maximal monotone such that \( 0 \in D(\phi), 0 \in D(\eta) \) and \( 0 \in \phi(0), 0 \in \eta(0), \) respectively. Let \( f, g : \mathbb{R} \times C([-\tau, 0]; L^1(\Omega)^{n+1}) \times C([-\tau, 0]; L^1(\Omega)^{n+1}) \rightarrow \mathbb{R} \) be continuous. We assume \( \mu \) as a positive \( \sigma \)-finite and complete measure defined on the class of Borel measurable sets in \([c, +\infty), h \in L^1(\mathbb{C}, +\infty; \mu, \mathbb{R}) \) is nonnegative. Let \( Z : C([-\tau, 0]; L^1(\Omega)) \rightarrow \mathbb{R}_+ \) to know that \( (H_2) \) is attained. Then, the problem (29) has a unique \( C^0 \)-solution and globally asymptotically stable, if satisfy the following arguments:

\( (a_1) \) there exist two constants \( C > 0 \) and \( \tilde{\beta} > 0 \) if \( s \leq 2 \) and \( \tilde{\beta} > (s-2)/s \) and if \( s \geq 3 \) we have \( \eta(r) \leq C|r|^\tilde{\beta} \) for each \( r \in \mathbb{R}/\{0\} \) such that \( \eta : \mathbb{R} \rightarrow \mathbb{R} \) is \( C^0 \) on \( \mathbb{R} \) and \( C^1 \) on \( \mathbb{R}/\{0\} \);

\( (a_2) \) there exist \( \xi_k > 0, k = 0, \ldots, n \) and \( m > 0 \) such that
   \[
   \| f(t, u, V) - f(t, \hat{U}, \hat{V}) \|_{L^1(\Omega)} \leq \sum_{k=0}^{n} \xi_k \left( \| u_k - \hat{u}_k \|_{C([-\tau, 0]L^1(\Omega))} + \| v_k - \hat{v}_k \|_{C([-\tau, 0]L^1(\Omega))} \right),
   \]
   \[
   \| g(t, u, V) \|_{L^1(\Omega)} \leq \sum_{k=0}^{n} \xi_k \left( \| u_k \|_{C([-\tau, 0]L^1(\Omega))} + \| v_k \|_{C([-\tau, 0]L^1(\Omega))} + m, \right.
   \]
   \[
   + \left. \| f(t, 0, \ldots, 0) \|_{L^1(\Omega)} \right) \leq m,
   \]
   for each \( (t, U, V), (t, \hat{U}, \hat{V}) \in \mathbb{R}_+ \times C([-\tau, 0]; L^1(\Omega)^{n+1}) \times C([-\tau, 0]; L^1(\Omega)^{n+1}).
   \]

\( (a_3) \) \( \mu \) is a Borel measure on \([c, +\infty) \) with \( \lim_{t \rightarrow 0} \mu \left( [l, c + \alpha t] \right) = 0; \)

\( (a_4) \) \( \| \mu \|_{L^1(\mathbb{C}, +\infty; \mu, \mathbb{R})} \leq 1; \)

\( (a_7) \) \( Z : C([-\tau, 0]; L^1(\Omega)) \rightarrow \mathbb{R}_+ \) is non-expansive with \( Z(0) = 0. \)

8. Conclusion

In this paper, the nonlinear retarded reaction–diffusion system with nonlocal retarded initial condition over \( t \in \mathbb{R}_+ \) is investigated. The compactness arguments and the Tychonoff fixed Point theorem are applied to solve (1). Furthermore, the assumption \( (H_2), (H_6), (c_1), (H_7), (g_1), (p_1), (q_1) \) and \( (q_3) \) guarantees the existence of \( C^0 \)-solutions of problem (1), as well as \( (c_2), (g_2), (p_2) \) and \( (q_2) \) strengthen the presence of uniqueness and global uniform asymptotic stability of \( C^0 \)-solution. Moreover, some imposed auxiliary results with their proofs are introduced which facilitate us to show our main results Theorem (3.2). Besides, our theoretical study, we present an effective example which showing that the theoretical results are fully compatible with their practical results.
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