WEAK CONVERGENCE OF DELAY SDES WITH APPLICATIONS TO CARATHÉODORY APPROXIMATION

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(Communicated by Arnulf Jentzen)

ABSTRACT. In this paper, we consider a fundamental class of stochastic differential equations with time delays. Our aim is to investigate the weak convergence with respect to delay parameter of the solutions. Based on the techniques of Malliavin calculus, we obtain an explicit estimate for the rate of convergence. An application to the Carathéodory approximation scheme of stochastic differential equations is provided as well.

1. Introduction. It is known that the Carathéodory approximation scheme was introduced by Carathéodory in the early part of 20th century for deterministic differential equations, see e.g. [5]. In the context of stochastic equations, the first Carathéodory approximation results were obtained by Bell and Mohammed [2] (also see Section 2.6 in [16] for a general formulation). Let $x_0 \in \mathbb{R}, n \geq 1$ and $(B(t))_{t \in [0,T]}$ be a standard Brownian motion. We consider stochastic differential equation (SDE)

$$x(t) = x_0 + \int_0^t b(s,x(s))ds + \int_0^t \sigma(s,x(s))dB(s), \quad t \in [0,T]$$

(1)

2020 Mathematics Subject Classification. Primary: 65C30, 60H10; Secondary: 60H07.

Key words and phrases. Delay SDEs, weak convergence, Carathéodory approximation, Malliavin calculus.

N. T. Dung and H. T. P. Thao are supported by the Vietnam National University, Hanoi under grant number QG.20.21. T. C. Son, N. V. Tan, T. M. Cuong and P. D. Tung are supported by Viet Nam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.03-2019.08.

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and its Carathéodory approximation

\[
\begin{align*}
X^n(t) &= x_0 + \int_0^t b(s, x^n(s-1/n))ds + \int_0^t \sigma(s, x^n(s-1/n))dB(s), \ t \in [0, T] \\
x^n(t) &= x_0, \ t \in [-1/n, 0].
\end{align*}
\]

(2)

It was proved in [2, 16] that, when the coefficients are Lipschitz and have linear growth, \(x^n(t)\) strongly converges to \(x(t)\) as \(n \to \infty\). Moreover, the following estimate for the strong rate of convergence holds

\[
E \left[ \sup_{0 \leq t \leq T} |x^n(t) - x(t)|^2 \right] \leq \frac{C}{n}, \ n \geq 1,
\]

(3)

where \(C\) is a positive constant not depending on \(n\).

As discussed in Section 2.6 of [16], the advantage of Carathéodory approximation is that we do not need to compute \(x^1(t), \ldots, x^{n-1}(t)\) but compute \(x^n(t)\) directly (this reduces a lot of calculations on stochastic integrals). In addition, the Carathéodory approximation also works well for SDEs with non-Lipschitz coefficients. In the last decades, the Carathéodory approximation has been considered for various stochastic differential equations. Among others, we mention Turo [19] for stochastic functional differential equations, Mao [14, 15] and Liu [12] for semilinear stochastic evolution equations with time delays, Ferrante & Rovira [8] for delay differential equations driven by fractional Brownian motion, Faizullah [7] for linear stochastic evolution equations with time delays, Ferrante & Rovira [8] for various stochastic differential equations. Among others, we mention Turo [19] for doubly perturbed SDEs, etc.

It is also known that the weak convergence rate of numerical approximations is very useful in practical applications (see, e.g. [1] for a short discussion). In fact, for certain numerical schemes such as Milstein scheme, Runge-Kutta scheme and Euler-Maruyama scheme, etc. many weak convergence results can be found in the literature, see e.g. [1, 4, 9, 11]. However, to the best of our knowledge, the weak convergence results of the Carathéodory approximation are scarce even for the system (1)-(2). Motivated by this observation, our aim is to partially fill up this gap.

In this paper, to make the problem more interesting, we consider delay stochastic differential equations of the form

\[
\begin{align*}
X_{\tau_1}(t) &= \varphi(0) + \int_0^t b(s, X_{\tau_1}(s), X_{\tau_1}(s-\tau_1))ds + \int_0^t \sigma(s, X_{\tau_1}(s), X_{\tau_1}(s-\tau_1))dB(s), \ t \in [0, T] \\
X_{\tau_1}(t) &= \varphi(t), \ t \in [-\tau_1, 0]
\end{align*}
\]

and

\[
\begin{align*}
X_{\tau_2}(t) &= \varphi(0) + \int_0^t b(s, X_{\tau_2}(s), X_{\tau_2}(s-\tau_2))ds + \int_0^t \sigma(s, X_{\tau_2}(s), X_{\tau_2}(s-\tau_2))dB(s), \ t \in [0, T] \\
X_{\tau_2}(t) &= \varphi(t), \ t \in [-\tau_2, 0],
\end{align*}
\]

(4) (5)

where \(0 \leq \tau_1, \tau_2 < \infty\) and \(\varphi : (-\infty, 0] \to \mathbb{R}\) is a bounded deterministic function. Our aim is to study the weak convergence of \(X_{\tau_2}(t)\) to \(X_{\tau_1}(t)\) as \(\tau_2 \to \tau_1\). More specifically, we will employ the techniques of Malliavin calculus to provide a quantitative estimate for the quantity

\[
|Eg(X_{\tau_2}(t)) - Eg(X_{\tau_1}(t))|,
\]
where $g$ is a bounded and measurable function, see Theorem 3.6. Our method is different from the existing ones in the literature and based on a new result established recently in [6]. The restatement of this new result and further comments will be given in Lemma 2.1 and Remark 1.

Our results applied to the Carathéodory approximation system (1)-(2) yield

\[ \sup_{0 \leq t \leq T} |Eg(x^n(t)) - Eg(x(t))| \leq \frac{C}{\sqrt{n}}, \quad n \geq 1. \]  

(6)

It should be noted that, when the test function $g$ is Lipschitz continuous, the strong rate (3) implies the weak rate (6). However, for bounded and measurable test functions, there is no such implication and hence, the novelty of our results lies in the fact that (6) holds true for any bounded and measurable test function. The price to pay is that, besides Lipschitz and linear growth conditions, we have to impose additional assumptions on the coefficients, see conditions (ii)-(iii) of Corollary 3.

The rest of this article is organized as follows. In Section 2, we recall some concepts of Malliavin calculus and a general result obtained in our recent work [6]. Our main results are then stated and proved in Section 3. The conclusion and some remarks are given in Section 4.

2. Preliminaries. As we have said in the Introduction, this paper is based on techniques of Malliavin calculus. For the reader’s convenience, let us recall some elements of Malliavin calculus (for more details see [17]). We suppose that $(B(t))_{t \in [0,T]}$ is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is a natural filtration generated by the Brownian motion $B$. For $h \in L^2[0,T]$, we denote by $B(h)$ the Wiener integral

\[ B_h = \int_0^T h(t) dB(t). \]

Let $S$ denote the dense subset of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ consisting of smooth random variables of the form

\[ F = f(B_{h_1}, ..., B_{h_n}), \]  

(7)

where $n \in \mathbb{N}, h_1, ..., h_n \in L^2[0,T]$ and $f$ is an infinitely differentiable function such that together with all its partial derivatives has at most polynomial growth order. If $F$ has the form (7), we define its Malliavin derivative as the process $DF := \{D_tF, t \in [0, T]\}$ given by

\[ D_tF = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(B_{h_1}, ..., B_{h_n})h_k(t). \]

More generally, for each $k \geq 1$, we can define the iterated derivative operator by setting $D^1_{t_1, ..., t_k}F = D_{t_1}...D_{t_k}F$. For any $p, k \geq 1$, we shall denote by $\mathbb{D}^{k,p}$ the closure of $S$ with respect to the norm

\[ \|F\|_{k,p}^p := E|F|^p + E \left[ \int_0^T |D^1_{t}F|^p dt_1 \right] + ... + E \left[ \int_0^T \ldots \int_0^T |D^k_{t_1, ..., t_k}F|^p dt_1 \ldots dt_k \right]. \]

A random variable $F$ is said to be Malliavin differentiable if it belongs to $\mathbb{D}^{1,2}$. For the convenience of the reader, we recall that the derivative operator $D$ satisfy the chain rule, i.e.

\[ D\varphi(F) = \varphi'(F)DF. \]  

(8)
Moreover, we have the following relations between Malliavin derivative and the integrals

\[ D_r \left( \int_0^T u(s)ds \right) = \int_r^T D_r u(s)ds, \]

\[ D_r \left( \int_0^T u(s)dB(s) \right) = u_r + \int_r^T D_r u(s)dB(s) \]

for all \( 0 \leq r \leq T \), where \( (u(t))_{t \in [0,T]} \) is an \( \mathcal{F} \)-adapted and Malliavin differentiable stochastic process.

An important operator in the Malliavin’s calculus theory is the divergence operator \( \delta \), it is the adjoint of the derivative operator \( D \). The domain of \( \delta \) is the set of all functions \( u \in L^2(\Omega, L^2[0,T]) \) such that for all \( F \in \mathbb{D}^{1,2} \) it holds that

\[ E[\langle DF, u \rangle_{L^2[0,T]}] \leq C(u)\|F\|_{L^2[0,T]}, \]

where \( C(u) \) is some positive constant depending on \( u \). In particular, if \( u \in \text{dom}\delta \), then \( \delta(u) \) is characterized by following duality relationship

\[ E[\langle DF, u \rangle_{L^2[0,T]}] = E[F\delta(u)] \quad \text{for any } F \in \mathbb{D}^{1,2}. \quad (9) \]

Let \( \mathcal{B} \) denote the space of measurable functions \( g : \mathbb{R} \to \mathbb{R} \) such that \( \|g\|_{\infty} := \sup_{x \in \mathbb{R}} |g(x)| \leq 1 \). We have the following.

**Lemma 2.1.** Let \( F_1 \in \mathbb{D}^{2,2} \) satisfy that \( \|DF_1\|_{L^2[0,T]} > 0 \) a.s. Then, for any random variable \( F_2 \in \mathbb{D}^{1,2} \) and any \( g \in \mathcal{B} \), we have

\[ |Eg(F_1) - Eg(F_2)| \]

\[ \leq C \left( E\|DF_1\|_{L^2[0,T]}^2 \right)^{\frac{1}{4}} \left( \int_0^T \int_0^T |D_\theta D_r F_1|^2 d\theta dr \right)^{\frac{1}{2}} \|F_1 - F_2\|_{1,2}, \]

provided that the expectations exist, where \( C \) is an absolute constant.

**Proof.** This lemma comes from Theorem 3.1 in our recent paper [6]. Here, for the reader’s convenience, we recall its proof. We write \( \langle .., .. \rangle \) instead of \( \langle .. \rangle_{L^2[0,T]} \) and \( \|\cdot\| \) instead of \( \|\cdot\|_{L^2[0,T]} \). By the routine approximation argument, we can assume that \( g \) is continuous. Indeed, for example, we can approximate \( g \) by bounded and Lipschitz continuous functions defined by

\[ g_\varepsilon(x) = \int_{-\infty}^\infty \mathbf{1}_{|y| < \frac{1}{\varepsilon}} g(y) \rho_\varepsilon(x - y) dy, \quad \varepsilon > 0, \]

where \( \rho_\varepsilon \) is the standard mollifier: \( \rho_\varepsilon(x) = \frac{\rho(x/\varepsilon)}{\varepsilon} \), where \( \rho(x) = C \mathbf{1}_{\{|x| < 1\}} e^{\frac{-x^2}{2}} \) and \( C \) is a constant such that \( \int_{-\infty}^{\infty} \rho(x) dx = 1 \).

The function \( \psi(y) := \int_{-\infty}^{\infty} g(z) dz \) is differentiable with bounded derivative. Hence, \( \psi(F_i) \in \mathbb{D}^{1,2} \) for \( i = 1,2 \). We obtain

\[ \langle D\psi(F_i), DF_i \rangle = g(F_i)\langle DF_i, DF_i \rangle, \quad i = 1,2, \]

or equivalently

\[ \langle D \int_{-\infty}^{F_i} g(z)dz, DF_i \rangle = g(F_i)\langle DF_i, DF_i \rangle, \quad i = 1,2. \]
As a consequence, we get

\[ (D \int_{F_1}^{F_2} g(z) dz, DF_1) = g(F_1)(DF_1, DF_1) - g(F_2)(DF_2, DF_1) \]

\[ = (g(F_1) - g(F_2)) (DF_1, DF_1) + g(F_2)(DF_1 - DF_2, DF_1), \]

and hence,

\[ g(F_1) - g(F_2) = \frac{\langle D \int_{F_1}^{F_2} g(z) dz, DF_1 \rangle}{\|DF_1\|^2} - \frac{g(F_2)\langle DF_1 - DF_2, DF_1 \rangle}{\|DF_1\|^2}. \]

Taking the expectation yields

\[ Eg(F_1) - Eg(F_2) = E \left[ \frac{\langle D \int_{F_1}^{F_2} g(z) dz, DF_1 \rangle}{\|DF_1\|^2} \right] - E \left[ \frac{g(F_2)\langle DF_1 - DF_2, DF_1 \rangle}{\|DF_1\|^2} \right]. \]

(10)

Now we consider the stochastic process

\[ u_r := \frac{D_r F_1}{\|DF_1\|^2}, \quad 0 \leq r \leq T \]

and use Proposition 1.3.1 in [17] to get

\[ E[\delta(u)^2] \leq \int_0^T E[u_r]^2 dr + \int_0^T \int_0^T E[D_\theta u_r]^2 d\theta dr \]

\[ = E\|DF_1\|^2 + \int_0^T \int_0^T E[D_\theta u_r]^2 d\theta dr \]

(11)

By the chain rule (8) for Malliavin derivatives, we have

\[ D_\theta u_r = \frac{D_\theta D_r F_1}{\|DF_1\|^2} - 2 \frac{D_r F_1 \langle D_\theta DF_1, DF_1 \rangle}{\|DF_1\|^4}, \quad 0 \leq \theta, r \leq T. \]

By the Cauchy-Schwarz inequality, we obtain

\[ \int_0^T \int_0^T E[D_\theta u_r]^2 d\theta dr \leq 2E \left[ \int_0^T \int_0^T \frac{|D_\theta D_r F_1|^2}{\|DF_1\|^4} d\theta dr \right] + 8E \int_0^T \int_0^T \frac{|D_r F_1|^2}{\|DF_1\|^6} d\theta dr \]

\[ = 2E \left[ \int_0^T \int_0^T \frac{|D_\theta D_r F_1|^2}{\|DF_1\|^4} d\theta dr \right] + 8E \left[ \int_0^T \int_0^T \frac{|D_\theta D_r F_1|^2}{\|DF_1\|^4} d\theta dr \right] \]

\[ \leq 10 \left( E \left[ \left( \int_0^T \int_0^T |D_\theta D_r F_1|^2 d\theta dr \right)^2 \right] \right)^{\frac{1}{2}} \left( E \left[ \|DF_1\|^{-8} \right] \right)^{\frac{1}{2}}. \]

(12)

Inserting (12) into (11) yields

\[ E[\delta(u)^2] \leq E\|DF_1\|^2 + 10 \left( E \left[ \left( \int_0^T \int_0^T |D_\theta D_r F_1|^2 d\theta dr \right)^2 \right] \right)^{\frac{1}{2}} \left( E \left[ \|DF_1\|^{-8} \right] \right)^{\frac{1}{2}}. \]

(13)
By using the relation (9), the Cauchy-Schwarz inequality and \( \|g\|_\infty \leq 1 \), it follows from (10) that

\[
|Eg(F_1) - Eg(F_2)| \leq E \left| \int_{F_2}^{F_1} g(z) \, dz \delta(u) \right| + E \left| g(F_2) \frac{\langle DF_1 - DF_2, DF_1 \rangle}{\|DF_1\|^2} \right|
\]

\[
\leq (E|F_1 - F_2|^2)^{\frac{1}{2}} \left( E[\delta(u)^2] \right)^{\frac{1}{2}} + E \left[ \frac{\|DF_1 - DF_2\|}{\|DF_1\|^2} \right]
\]

\[
\leq (E|F_1 - F_2|^2)^{\frac{1}{2}} \left( E[\delta(u)^2] \right)^{\frac{1}{2}} + (E\|DF_1 - DF_2\|^2)^{\frac{1}{2}} (E\|DF_1\|^{-2})^{\frac{1}{2}}.
\]

So we can obtain the desired conclusion by inserting (13) into (14) and then using Cauchy-Schwarz inequality.

\[\square\]

**Remark 1.** Lemma 2.1 shows that the weak convergence can be controlled by the norm \( \|\cdot\|_{1,2} \) in the space of Malliavin differentiable random variables. In its statement, we do not require any special structure of \( F_1 \) and \( F_2 \). We therefore believe that Lemma 2.1 can be applied to other numerical schemes such as Euler-Maruyama and Runge-Kutta schemes. In the present paper, the main reason for choosing Carathéodory scheme is due to the lack of weak convergence results for this scheme in the literature. It is also worth mentioning that the proof of Lemma 2.1 heavily relies on dimension one. The generalization to higher dimensions will be a difficult and interesting problem.

3. The main results. In the whole paper, we are going to impose the following assumptions:

**Assumption 1.** The functions \( b, \sigma : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R} \) are Lipschitz and have linear growth, i.e. there exists \( L > 0 \) such that

\[
|b(t, x_1, y_1) - b(t, x_2, y_2)| + |\sigma(t, x_1, y_1) - \sigma(t, x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|)
\]

for all \( t \in [0, T], x_1, x_2, y_1, y_2 \in \mathbb{R} \) and

\[
|b(t, x, y)| + |\sigma(t, x, y)| \leq L(|x| + |y| + 1), \ \forall t \in [0, T], x, y \in \mathbb{R}.
\]

**Assumption 2.** For every \( t \in [0, T] \), \( b(t, \ldots), \sigma(t, \ldots) \) are twice differentiable functions with the partial derivatives bounded by \( L \).

Hereafter, we denote by \( C \) (with or without an index) a generic constant which may vary at each appearance. For any function \( h \) of \( n \) variables, we denote

\[
h'(x_1, \ldots, x_n) := \frac{\partial h}{\partial x_1}(x_1, \ldots, x_n), \quad h''(x_1, \ldots, x_n) := \frac{\partial h}{\partial x_1 \partial x_2}(x_1, \ldots, x_n).
\]

For any \( a, b \in \mathbb{R} \), we denote \( a \vee b = \max\{a, b\} \). In the proofs, we frequently use the fundamental inequality

\[
(a_1 + \cdots + a_n)^p \leq n^{p-1}(a_1^p + \cdots + a_n^p)
\]

for all \( a_1, \ldots, a_n \geq 0 \) and \( p \geq 2 \).
3.1. **Malliavin derivative of delay SDEs.** In this Subsection, given $\tau \geq 0$, we investigate Malliavin derivatives of the solution to the following delay SDE:

$$
\left\{
\begin{array}{l}
X_t = \varphi(0) + \int_0^t b(s,X_t,X_t(s-\tau))ds + \int_0^t \sigma(s,X_t(s),X_t(s-\tau))dB(s), \ t \in [0,T], \\
X_t = \varphi(t), \ t \in [-\tau,0].
\end{array}
\right.
$$

(15)

For the Malliavin differentiability of the solutions, we have the following.

**Proposition 1.** Let Assumption 1 hold. Then, the equation (15) has a unique solution $(X_t(t))_{t \in [-\tau,T]}$ and this solution is Malliavin differentiable. Moreover, the derivative $D_\theta X_t(t)$ satisfies

(i) When $t \in [-\tau,0]$, $D_\theta X_t(t) = 0$ for all $0 \leq \theta \leq T$,

(ii) When $t \in (0,T]$, $D_\theta X_t(t) = 0$ for $\theta > t$ and

$$
D_\theta X_t(t) = \sigma(\theta,X_t(\theta),X_t(\theta-\tau)) + \int_\theta^t b^*_2(s)D_\theta X_t(s)ds + \int_\theta^{t-\tau} \sigma^*_2(s)D_\theta X_t(s-\tau)dB(s) + \int_\theta^t \sigma^*_3(s)D_\theta X_t(s)dB(s), \ 0 \leq \theta \leq t-\tau.
$$

(16)

$$
D_\theta X_t(t) = \sigma(\theta,X_t(\theta),X_t(\theta-\tau)) + \int_\theta^t b^*_2(s)D_\theta X_t(s)ds + \int_\theta^{t-\tau} \sigma^*_2(s)D_\theta X_t(s-\tau)dB(s), \ (t-\tau) \vee 0 < \theta \leq t.
$$

(17)

where $b^*_2(s), b^*_3(s), \sigma^*_2(s), \sigma^*_3(s)$ are adapted stochastic processes and bounded by the Lipschitz constant $L$. Here, we use the convention $[0,t-\tau] = 0$ if $t < \tau$.

**Proof.** For $t \in [-\tau,0]$, we have $X_t(t) = \varphi(t)$ is deterministic. Hence, the Malliavin derivative of $X_t(t)$ always vanishes. On the other hand, since the solution $(X_t(t))_{t \in [0,T]}$ is $\mathbb{F}$-adapted, we always have $D_\theta X_t(t) = 0$ for $\theta > t$. When $\theta \leq t$, by using the same argument as in the proof of Theorem 2.2.1 in [17], we can show that the solution $(X_t(t))_{t \in [0,T]}$ is also Malliavin differentiable. Applying the operator $D$ to the equation (15) we deduce

$$
D_\theta X_t(t) = \sigma(\theta,X_t(\theta),X_t(\theta-\tau)) + \int_\theta^t b^*_2(s)D_\theta X_t(s)ds + \int_\theta^{t-\tau} \sigma^*_2(s)D_\theta X_t(s-\tau)dB(s), \ t \in (0,T].
$$

(18)

By Proposition 1.2.4 in [17] and Lipschitz property of $b$ and $\sigma$, there exist adapted processes $b^*_2(s), b^*_3(s), \sigma^*_2(s)$ and $\sigma^*_3(s)$, uniformly bounded by $L$ such that

$$
D_\theta [b(s,X_t(s),X_t(s-\tau))] = b^*_2(s)D_\theta X_t(s) + b^*_3(s)D_\theta X_t(s-\tau),
$$

(19)

$$
D_\theta [\sigma(s,X_t(s),X_t(s-\tau))] = \sigma^*_2(s)D_\theta X_t(s) + \sigma^*_3(s)D_\theta X_t(s-\tau).
$$

(20)

Because $D_\theta X_t(s-\tau) = 0$ for $\theta > s-\tau$, we obtain the equations (16) and (17) by inserting (19) and (20) into (18). So the proof of Proposition is complete. □

**Remark 2.** If $b(t,\cdot,\cdot)$ and $\sigma(t,\cdot,\cdot)$ are continuously differentiable, then $b^*_2(s) = b^*_2(s,X_t(s),X_t(s-\tau)), b^*_3(s) = b^*_3(s,X_t(s),X_t(s-\tau)), \sigma^*_2(s) = \sigma^*_2(s,X_t(s),X_t(s-\tau)), \sigma^*_3(s) = \sigma^*_3(s,X_t(s),X_t(s-\tau))$. Consequently, under Assumptions 1 and 2, we have

$$
D_\theta X_t(t) = \sigma(\theta,X_t(\theta),X_t(\theta-\tau)) + \int_\theta^t b^*_2(s,X_t(s),X_t(s-\tau))D_\theta X_t(s)ds
$$

$$
+ \int_\theta^{t-\tau} b^*_3(s,X_t(s),X_t(s-\tau))D_\theta X_t(s-\tau)dB(s) + \int_\theta^t \sigma^*_2(s,X_t(s),X_t(s-\tau))D_\theta X_t(s)dB(s) + \int_\theta^t \sigma^*_3(s,X_t(s),X_t(s-\tau))D_\theta X_t(s)dB(s)
$$

$$
+ \int_\theta^{t-\tau} \sigma^*_3(s,X_t(s),X_t(s-\tau))D_\theta X_t(s-\tau)dB(s).
$$

(19)
Let Assumption 1 hold. Then, for every $\theta$, $0 \leq \theta \leq t - \tau$ (21) and

$$D_\theta X_\tau(t) = \sigma(\theta, X_\tau(\theta), X_\tau(\theta - \tau)) + \int_\theta^t b_2^\tau(s, X_\tau(s), X_\tau(s - \tau))D_\theta X_\tau(s)\,ds$$

$$+ \int_\theta^t \sigma_2^\tau(s, X_\tau(s), X_\tau(s - \tau))D_\theta X_\tau(s)\,dB(s), \quad (t - \tau) \lor 0 < \theta \leq t. \quad (22)$$

Moreover, $X_\tau(t)$ is twice Malliavin differentiable. The proof of this fact is similar to that of Theorem 2.2.2 in [17], we omit the details. The second order Malliavin derivative of $X_\tau(t)$ will be computed in Proposition 3 below.

We have the following familiar estimates for the moments.

**Lemma 3.1.** Let Assumption 1 hold. Then, for every $p \geq 2$, we have

$$E|X_\tau(t)|^p \leq C_{p,L,T}, \quad t \in [0,T] \quad (23)$$

and

$$E|X_\tau(t) - X_\tau(s)|^p \leq C_{p,L,T}|t - s|^\frac{p}{2}, \quad s, t \in [0,T], \quad (24)$$

where $C_{p,L,T}$ is a positive constant.

**Proof.** The proof is similar to that of Lemmas 6.1 and 6.2 in [16]. So we omit it. Here, just remark that we use the Burkholder-Davis-Gundy inequality instead of the Itô isometry.

**Lemma 3.2.** Let Assumption 1 hold. Then, for every $p \geq 2$, we have

$$E|D_\theta X_\tau(t)|^p \leq C_{p,L,T}, \quad 0 \leq \theta \leq t \leq T \quad (25)$$

and

$$E|D_\theta X_\tau(t) - D_\theta X_\tau(s)|^p \leq C_{p,L,T}|t - s|^\frac{p}{2}, \quad 0 \leq \theta \leq s \lor t \leq T, \quad (26)$$

where $C_{p,L,T}$ is a positive constant.

**Proof.** By the linear growth property of $\sigma$ and the estimate (23) we have

$$E|\sigma(\theta, X_\tau(\theta), X_\tau(\theta - \tau))|^p \leq C_{p,L,T}, \quad \forall \ 0 \leq \theta \leq T. \quad (27)$$

It follows from equations (16) and (17) that the Malliavin derivative $D_\theta X_\tau(t)$ satisfies

$$D_\theta X_\tau(t) = \sigma(\theta, X_\tau(\theta), X_\tau(\theta - \tau)) + \int_\theta^t \tilde{b}_2^\tau(s)D_\theta X_\tau(s)\,ds$$

$$+ \int_\theta^t \tilde{b}_3^\tau(s)D_\theta X_\tau(s - \tau)\mathbf{1}_{[\theta + \tau, t]}(s)\,ds + \int_\theta^t \tilde{\sigma}_2^\tau(s)D_\theta X_\tau(s)\,dB(s)$$

$$+ \int_\theta^t \tilde{\sigma}_3^\tau(s)D_\theta X_\tau(s - \tau)\mathbf{1}_{[\theta + \tau, t]}(s)\,dB(s), \quad 0 \leq \theta \leq t \leq T. \quad (28)$$

We therefore get

$$E|D_\theta X_\tau(t)|^p$$

$$\leq 5^{p-1} \left( E|\sigma(\theta, X_\tau(\theta), X_\tau(\theta - \tau))|^p + E \left| \int_\theta^t \tilde{b}_2^\tau(s)D_\theta X_\tau(s)\,ds \right|^p \right)$$

$$+ E \left| \int_\theta^t \tilde{b}_3^\tau(s)D_\theta X_\tau(s - \tau)\mathbf{1}_{[\theta + \tau, t]}(s)\,ds \right|^p + E \left| \int_\theta^t \tilde{\sigma}_2^\tau(s)D_\theta X_\tau(s)\,dB(s) \right|^p$$
By the boundedness of $\bar{b}_2^\gamma(s)$, $\bar{b}_3^\gamma(s)$, $\bar{\sigma}_2^\gamma(s)$, $\bar{\sigma}_3^\gamma(s)$, the estimate (27) and the Hölder and Burkholder-Davis-Gundy inequalities, it is easy to see that

\[ E|D_\theta X_\tau(t)|^p \leq C_{p,L,T} + C_{p,L,T} \int_0^t E|D_\theta X_\tau(s)|^p ds + C_{p,L,T} \int_0^t E|D_\theta X_\tau(s - \tau)|^p ds \]

\[ \leq C_{p,L,T} + C_{p,L,T} \int_0^t E|D_\theta X_\tau(s)|^p ds, \quad 0 \leq \theta \leq t \leq T, \]

where $C_{p,L,T}$ is some positive constant. So, we can obtain (25) by using Gronwall’s lemma.

In order to prove (26), we assume, without the loss of generally, that $s < t$. We consider three cases separately.

**Case 1.** $0 \leq \theta \leq s - \tau$. From the equation (16), we have

\[ D_\theta X_\tau(t) - D_\theta X_\tau(s) = \int_s^t \bar{b}_2^\gamma(u)D_\theta X_\tau(u)du + \int_s^t \bar{b}_3^\gamma(u)D_\theta X_\tau(u - \tau)du \]

\[ + \int_s^t \bar{\sigma}_2^\gamma(u)D_\theta X_\tau(u)dB_u + \int_s^t \bar{\sigma}_3^\gamma(u)D_\theta X_\tau(u - \tau)du. \]

Using the Hölder and Burkholder-Davis-Gundy inequalities and (25), we have

\[ E|D_\theta X_\tau(t) - D_\theta X_\tau(s)|^p \leq C_{p,L,T} \left((t - s)^{p-1} \int_s^t E|D_\theta X_\tau(u)|^p du + (t - s)^{\frac{p}{2} - 1} \int_s^t E|D_\theta X_\tau(u)|^p du\right) \]

\[ \leq C_{p,L,T}|t - s|^{p/2}. \]

**Case 2.** $s - \tau < \theta \leq t - \tau$. From the equations (16) and (17) we have

\[ D_\theta X_\tau(t) - D_\theta X_\tau(s) = \int_s^\theta \bar{b}_2^\gamma(u)D_\theta X_\tau(u)du + \int_\theta^t \bar{b}_3^\gamma(u)D_\theta X_\tau(u - \tau)du \]

\[ + \int_s^\theta \bar{\sigma}_2^\gamma(u)D_\theta X_\tau(u)dB_u + \int_\theta^t \bar{\sigma}_3^\gamma(u)D_\theta X_\tau(u - \tau)du. \]

Using the same arguments as in the proof of Case 1, we obtain

\[ E|D_\theta X_\tau(t) - D_\theta X_\tau(s)|^p \leq C_{p,L,T} \left(|t - s|^{p/2} + |t - \theta - \tau|^{p/2}\right) \]

\[ \leq C_{p,L,T}|t - s|^{p/2}. \]

**Case 3.** $t - \tau < \theta \leq t$. From (17) we have

\[ D_\theta X_\tau(t) - D_\theta X_\tau(s) = \int_s^t \bar{b}_2^\gamma(u)D_\theta X_\tau(u)du + \int_s^t \bar{\sigma}_2^\gamma(u)D_\theta X_\tau(u)dB_u, \]

and hence, we also have

\[ E|D_\theta X_\tau(t) - D_\theta X_\tau(s)|^p \leq C_{p,L,T}|t - s|^{p/2}. \]

This finishes the proof of Proposition.
Proposition 2. Let Assumptions 1 hold and, in addition, we assume that
\[ |\sigma(t, x, y)| \geq \sigma_0 > 0, \quad \forall t \in [0, T], x, y \in \mathbb{R}. \]
Then, for every \( p \geq 1 \) and for all \( 0 < t \leq T \), we have
\[
E \left[ \frac{1}{\| DX_r(t) \|_{L^2[0, T]}^{2p}} \right] \leq C_{p, L, T} t^{-p},
\]
where \( C_{p, L, T} \) is a positive constant.

Proof. Fixed \( t \in (0, T] \). By using the fundamental inequality \( (a + b + c)^2 \geq \frac{a^2}{2} - 2(b^2 + c^2) \), we obtain from the equation (28) that
\[
|D_\theta X_r(t)|^2 \\
\geq \frac{1}{2} \sigma^2(\theta, X_r(\theta), X_r(\theta - \tau)) \\
- 2 \left( \int_\theta^t \bar{b}_3^2(s) D_\theta X_r(s)ds + \int_\theta^t \bar{b}_5^2(s) D_\theta X_r(s - \tau) 1_{[\theta + \tau, t]}(s)ds \right)^2 \\
- 2 \left( \int_\theta^t \bar{\sigma}_3^2(s) D_\theta X_r(s)dB(s) + \int_\theta^t \bar{\sigma}_5^2(s) D_\theta X_r(s - \tau) 1_{[\theta + \tau, t]}(s)dB(s) \right)^2,
\]
where \( \sigma^2(\theta, X_r(\theta), X_r(\theta - \tau)) \) belongs to \( (0, 1) \). Hence,
\[
\| DX_r(t) \|_{L^2[0, T]}^2 \\
\geq \int_{t(1 - \epsilon)}^t |D_\theta X_r(t)|^2 d\theta \geq \int_{t(1 - \epsilon)}^t \frac{\sigma^2(\theta, X_r(\theta), X_r(\theta - \tau))}{2} d\theta \\
- 2 \int_{t(1 - \epsilon)}^t \left( \int_\theta^t \bar{b}_3^2(s) D_\theta X_r(s)ds + \int_\theta^t \bar{b}_5^2(s) D_\theta X_r(s - \tau) 1_{[\theta + \tau, t]}(s)ds \right)^2 d\theta \\
- 2 \int_{t(1 - \epsilon)}^t \left( \int_\theta^t \bar{\sigma}_3^2(s) D_\theta X_r(s)dB(s) + \int_\theta^t \bar{\sigma}_5^2(s) D_\theta X_r(s - \tau) 1_{[\theta + \tau, t]}(s)dB(s) \right)^2 d\theta \\
\geq \frac{\sigma_0^2 t \epsilon}{2} - I_y(t) \geq \frac{2}{y} - I_y(t),
\]
where
\[
I_y(t) := 2 \int_{t(1 - \epsilon)}^t \left( \int_\theta^t \bar{b}_3^2(s) D_\theta X_r(s)ds + \int_\theta^t \bar{b}_5^2(s) D_\theta X_r(s - \tau) 1_{[\theta + \tau, t]}(s)ds \right)^2 d\theta \\
+ 2 \int_{t(1 - \epsilon)}^t \left( \int_\theta^t \bar{\sigma}_3^2(s) D_\theta X_r(s)dB(s) + \int_\theta^t \bar{\sigma}_5^2(s) D_\theta X_r(s - \tau) 1_{[\theta + \tau, t]}(s)dB(s) \right)^2 d\theta.
\]
Then, by Markov inequality, we obtain
\[
P \left( \| DX_r(t) \|_{L^2[0, T]}^2 \leq \frac{1}{y} \right) \leq P \left( \frac{2}{y} - I_y(t) \leq \frac{1}{y} \right) = P \left( I_y(t) \geq \frac{1}{y} \right) \\
\leq y^{q/2} E \left( |I_y(t)|^{q/2} \right) \quad \forall \ q \geq 2.
\]
By the inequality $\left(\|a\| + |b|\right)^{q/2} \leq 2^{q/2-1}(\|a\|^{q/2} + |b|^{q/2})$, we get

$$E|I_y(t)|^{q/2} \leq 2^{q-1} \left[ E \left( \left( \int_0^t \tilde{b}_x(s) D_y X_\tau(s) ds + \int_0^t \tilde{b}_y(s) D_y X_\tau(s - \tau) \mathbb{1}_{[\theta+r,t]}(s) ds \right)^2 ds \right)^{q/2} 
+ E \left( \int_0^t \tilde{a}_x^T(s) D_y X_\tau(s) dB(s) + \int_0^t \tilde{a}_y^T(s) D_y X_\tau(s - \tau) \mathbb{1}_{[\theta+r,t]}(s) dB(s) \right)^2 ds \right)^{q/2}.$$

By using the Hölder and Burkholder-Davis-Gundy inequalities, it follows from (25) that

$$E|I_y(t)|^{q/2} \leq C_{q,L,T} (t\varepsilon)^{\frac{q-2}{q}} \left[ \int_{t(1-\varepsilon)}^t (t - \theta)^{\frac{q-2}{2}} \int_\theta^t E|D_y X_\tau(s)|^p ds d\theta \right]$$

$$+ \int_{t(1-\varepsilon)}^t \left( \int_\theta^t E|D_y X_\tau(s)|^2 ds \right)^{q/2} d\theta$$

$$\leq C_{q,L,T} (t\varepsilon)^{\frac{q-2}{q}} \left[ \int_{t(1-\varepsilon)}^t (t - \theta)^{\frac{q}{2}} d\theta + \int_{t(1-\varepsilon)}^t (t - \theta)^{q/2} d\theta \right]$$

$$\leq C_{q,L,T} (t\varepsilon)^{\frac{q-2}{q}} (t\varepsilon)^{\frac{q}{2} + 1}$$

$$= C_{q,L,T} \left( \frac{4}{y\sigma_0^2} \right)^q,$$

(31)

where $C_{q,L,T}$ is a positive constant. Combining (30) and (31) we deduce

$$P \left( \|D_X(t)\|_{L^2[0,T]}^2 \leq \frac{1}{y} \right) \leq C_{q,L,T} y^{p/2} \left( \frac{4}{y\sigma_0^2} \right)^p \forall \ p \geq 2, \ y \geq y_0.$$

For any $p \geq 2$ and $q = 2p + 1$, we obtain the following estimates

$$E \left( \|D_X(t)\|_{L^2[0,T]}^{2p} \right) = \int_0^\infty p y^{p-1} P \left( \|D_X(t)\|_{L^2[0,T]}^2 > y \right) dy$$

$$\leq \int_0^{y_0} p y^{p-1} dy + \int_0^\infty p y^{p-1} P \left( \|D_X(t)\|_{L^2[0,T]}^2 < \frac{1}{y} \right) dy$$

$$\leq y_0^p + p C_{p,L,T} \int_{y_0}^\infty y^{p-1} y^{q/2} \left( \frac{4}{y\sigma_0^2} \right)^q dy$$

$$= y_0^p + 2p C_{p,L,T} \left( \frac{4}{\sigma_0^2} \right)^{2p+1} y_0^{-\frac{q}{2}}.$$

(32)

We recall here that $y_0 = \frac{4}{\sigma_0^2}$. So (29) follows from (32).

**Proposition 3.** Let Assumptions 1 and 2 hold. It holds that

$$E|D_r D_\theta X_\tau(t)|^4 \leq C_{r,L,T} \forall 0 \leq \theta, r \leq t \leq T,$$

where $C_{L,T}$ is a positive constant.
Proof. We rewrite the equations (21) and (22) as follows

\[
D_\theta X_\tau(t) = \sigma(\theta, X_\tau(\theta), X_\tau(\theta - \tau)) + \int_0^t b'_2(s, X_\tau(s), X_\tau(s - \tau)) D_\theta X_\tau(s) ds \\
+ \int_0^t b'_3(s, X_\tau(s), X_\tau(s - \tau)) D_\theta X_\tau(s - \tau) \mathbf{1}_{[\theta + \tau, t]}(s) ds \\
+ \int_0^t \sigma'_2(s, X_\tau(s), X_\tau(s - \tau)) D_\theta X_\tau(s) dB(s) \\
+ \int_0^t \sigma'_3(s, X_\tau(s), X_\tau(s - \tau)) D_\theta X_\tau(s - \tau) \mathbf{1}_{[\theta + \tau, t]}(s) dB(s), \quad 0 \leq \theta \leq t \leq T.
\]

Hence, the second order Malliavin derivative of \(X_\tau(t)\) can be computed by

\[
D_r D_\theta X_\tau(t) = D_r[\sigma(\theta, X_\tau(\theta), X_\tau(\theta - \tau)) + \sigma'_2(r, X_\tau(r), X_\tau(r - \tau)) D_\theta X_\tau(r) \\
+ \sigma'_3(r, X_\tau(r), X_\tau(r - \tau)) D_\theta X_\tau(r - \tau) \mathbf{1}_{[\theta + \tau, t]}(r) \\
+ \int_0^t D_r [b'_2(s, X_\tau(s), X_\tau(s - \tau))] D_\theta X_\tau(s) ds \\
+ \int_0^t b'_3(s, X_\tau(s), X_\tau(s - \tau)) D_r D_\theta X_\tau(s) ds \\
+ \int_0^t \sigma'_2(s, X_\tau(s), X_\tau(s - \tau)) D_r D_\theta X_\tau(s - \tau) \mathbf{1}_{[\theta + \tau, t]}(s) ds \\
+ \int_0^t \sigma'_3(s, X_\tau(s), X_\tau(s - \tau)) D_r D_\theta X_\tau(s - \tau) \mathbf{1}_{[\theta + \tau, t]}(s) dB(s)
\]

where, for \(h \in \{b'_2, b'_3, \sigma'_2, \sigma'_3\}\), we have

\[
D_r[h(s, X_\tau(s), X_\tau(s - \tau))] = b'_2(s, X_\tau(s), X_\tau(s - \tau)) D_r X_\tau(s) + b'_3(s, X_\tau(s), X_\tau(s - \tau)) D_r X_\tau(s - \tau).
\]

Note that the partial derivatives of \(b\) and \(\sigma\) are bounded. By using (25) and the Hölder and Burkholder-Davis-Gundy inequalities, we verify that

\[
E|D_r D_\theta X_\tau(t)|^4 \\
\leq C_{T,L} + C_{T,L} \int_{[\theta, \tau]} E|D_r D_\theta X_\tau(s)|^4 ds + C_{T,L} \int_{[\theta, \tau]} E|D_r D_\theta X_\tau(s - \tau)|^4 ds \\
\leq C_{T,L} + C_{T,L} \int_{[\theta, \tau]} E|D_r D_\theta X_\tau(s)|^4 ds, \quad 0 \leq r, \theta \leq t \leq T,
\]

which, by Gronwall’s lemma, gives us the desired conclusion. \qed
3.2. \(L^p\)-distances and weak convergence. In this Subsection, we consider the equations (4) and (5). We first bound \(L^p\)-distances between the solutions and between their Malliavin derivatives. Then, we use Lemma 2.1 to estimate the weak rate of convergence. In the whole subsection, we assume without loss of generality that \(\tau_1 < \tau_2\).

Lemma 3.3. Let Assumptions 1 hold and let \(h : [0, T] \times \mathbb{R}^2 \to \mathbb{R}\) be such that

\[
|h(t, x_1, y_1) - h(t, x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|)
\]

for all \(t \in [0, T], x_1, x_2, y_1, y_2 \in \mathbb{R}\). Then, for every \(p \geq 2\) and for all \(t \in [0, T]\),

\[
\int_0^t E|\varphi(t) - \varphi(s)|^p ds \leq C_{p,L,T} \left( t^{\tau_1} - \tau_2 \right)^{p/2} + \int_0^{t^{\tau_1}} |\varphi(s) - \varphi(s)|^p ds + \int_0^t E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p ds \right),
\]

(33)

where \(C_{p,L,T}\) is a positive constant.

Proof. We have

\[
\int_0^t E|\varphi(t) - \varphi(s)|^p ds \leq L^p \int_0^t E(|X_{\tau_1}(s) - X_{\tau_2}(s)| + |X_{\tau_1}(s) - X_{\tau_2}(s)|)^p ds
\]

\[
\leq L^p \left( \int_0^t E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p + E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p ds \right)
\]

\[
= L^{p+1} M_p(t) + L^p \int_0^t E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p ds, \quad t \in [0, T],
\]

where

\[
M_p(t) := \int_0^t E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p ds
\]

\[
\leq 2^{p-1} \int_0^t (E|X_{\tau_1}(s) - X_{\tau_2}(s) - X_{\tau_1}(s)|^p + E|X_{\tau_2}(s) - X_{\tau_2}(s) - X_{\tau_2}(s)|^p) ds.
\]

Since \(X_{\tau_1}(s) = X_{\tau_2}(s) = \varphi(s), s \in [-\tau_1, 0]\), it holds that

\[
\int_0^t E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p ds
\]

\[
= \int_0^{t^{\tau_1}} E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p ds + \int_{t^{\tau_1}}^t E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p ds
\]

\[
= \int_0^{t^{\tau_1}} E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p ds = \int_0^t E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p ds
\]

\[
\leq \int_0^t E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p ds, \quad t \in [0, T].
\]
On the other hand, we have
\[
\int_0^t E|X_{\tau_2}(s - \tau_1) - X_{\tau_2}(s - \tau_2)|^p ds = \int_0^{t \land \tau_1} E|X_{\tau_2}(s - \tau_1) - X_{\tau_2}(s - \tau_2)|^p ds \\
+ \int_{t \land \tau_2}^{t \land \tau_1} E|X_{\tau_2}(s - \tau_1) - X_{\tau_2}(s - \tau_2)|^p ds \\
+ \int_t^{t \land \tau_2} E|X_{\tau_2}(s - \tau_1) - X_{\tau_2}(s - \tau_2)|^p ds, \ t \in [0, T].
\]
Then, recalling (24), we obtain
\[
\int_0^t E|X_{\tau_2}(s - \tau_1) - X_{\tau_2}(s - \tau_2)|^p ds \\
\leq \int_0^{t \land \tau_1} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^p ds + 2^{p-1} \int_{t \land \tau_1}^{t \land \tau_2} E|X_{\tau_2}(s - \tau_1) - \varphi(0)|^p ds \\
+ 2^{p-1} \int_{t \land \tau_1}^{t \land \tau_2} |\varphi(0) - X_{\tau_2}(s - \tau_2)|^p ds \\
+ \int_{t \land \tau_2}^{t} E|X_{\tau_2}(s - \tau_1) - X_{\tau_2}(s - \tau_2)|^p ds \\
\leq \int_0^{t \land \tau_1} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^p ds + C_{p,L,T} \int_{t \land \tau_1}^{t \land \tau_2} (s - \tau_1)^{p/2} ds \\
+ 2^{p-1} \int_{t \land \tau_1}^{t \land \tau_2} |\varphi(0) - \varphi(s - \tau_2)|^p ds + C_{p,L,T} \int_{t \land \tau_2}^{t} (\tau_2 - \tau_1)^{p/2} ds \\
\leq C_{p,L,T}(t|\tau_1 - \tau_2|^{p/2} + \int_0^{t \land \tau_1} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^p ds + \int_{t \land \tau_2}^{t} E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p ds), \ t \in [0, T].
\]
Hence,
\[
M_p(t) \leq C_{p,L,T}(t|\tau_1 - \tau_2|^{p/2} + \int_0^{t \land \tau_1} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^p ds \\
+ \int_{t \land \tau_2}^{t} E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p ds), \ t \in [0, T].
\]
As a consequence, we obtain (33) by inserting (35) into (34). The proof of the lemma is complete.

**Proposition 4.** Let Assumption 1 hold. Then, for every \( p \geq 2 \) and for all \( t \in [0, T] \), there exists a positive constants \( C_{p,L,T} \) such that
\[
E|X_{\tau_1}(t) - X_{\tau_2}(t)|^p \leq C_{p,L,T} t^{\frac{p}{2}-1} \left( t|\tau_1 - \tau_2|^{p/2} \\
+ \int_0^{t \land \tau_1} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^p ds + \int_{t \land \tau_2}^{t} E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p ds \right).
\]
\[
(36)
\]
**Proof.** We write
\[
X_{\tau_1}(t) - X_{\tau_2}(t) = I_1(t) + I_2(t), \ 0 \leq t \leq T,
\]
where the terms \( I_1(t), I_2(t) \) are defined by
\[
I_1(t) := \int_0^t [b(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1)) - b(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))] ds,
\]
\[
I_2(t) := \int_0^t [\sigma(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1)) - \sigma(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))] dB(s).
\]
(37)
Using the Hölder inequality and Lemma 3.3, we deduce
\[ E|I_1(t)|^p \leq t^{p-1} \int_0^t E|b(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1)) - b(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))|^p ds \]
\[ \leq C_{p,L,T} t^{p-1} \left( |\tau_1 - \tau_2|^{p/2} + \int_0^{t \wedge \tau_1} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^p ds \right) \]
\[ + \int_{t \wedge \tau_1}^{t \wedge \tau_2} |\varphi(0) - \varphi(s - \tau_2)|^p ds \]
\[ + C_{p,L,T} \int_0^t E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p ds, \quad 0 \leq t \leq T. \]
(38)

Similarly, we also have
\[ E|I_2(t)|^p \]
\[ \leq C_{p,L,T} t^{p-1} \left( |\sigma(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1)) - \sigma(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))|^2 ds \right)^{p/2} \]
\[ \leq C_{p,L,T} t^{p-1} \int_0^t E|\sigma(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1)) - \sigma(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))|^p ds \]
\[ \leq C_{p,L,T} t^{p-1} \left( |\tau_1 - \tau_2|^{p/2} + \int_0^{t \wedge \tau_1} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^p ds \right) \]
\[ + \int_{t \wedge \tau_1}^{t \wedge \tau_2} |\varphi(0) - \varphi(s - \tau_2)|^p ds \]
\[ + C_{p,L,T} \int_0^t E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p ds, \quad 0 \leq t \leq T. \]
(39)

Combining (37), (38) and (39), we obtain
\[ E|X_{\tau_1}(t) - X_{\tau_2}(t)|^p \]
\[ \leq C_{p,L,T} t^{p-1} \left( |\tau_1 - \tau_2|^{p/2} + \int_0^{t \wedge \tau_1} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^p ds \right) \]
\[ + \int_{t \wedge \tau_1}^{t \wedge \tau_2} |\varphi(0) - \varphi(s - \tau_2)|^p ds \]
\[ + C_{p,L,T} \int_0^t E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p ds, \quad 0 \leq t \leq T. \]

Since the function \( t \mapsto |\tau_1 - \tau_2|^{p/2} + \int_0^{t \wedge \tau_1} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^p ds + \int_{t \wedge \tau_1}^{t \wedge \tau_2} |\varphi(0) - \varphi(s - \tau_2)|^p ds \) is non-decreasing, we can use the Gronwall-type lemma (see, Theorem 1.4.2 in [18]) to get
\[ E|X_{\tau_1}(t) - X_{\tau_2}(t)|^p \]
\[ \leq C_{p,L,T} t^{p-1} \left( |\tau_1 - \tau_2|^{p/2} + \int_0^{t \wedge \tau_1} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^p ds \right) \]
\[ + \int_{t \wedge \tau_1}^{t \wedge \tau_2} |\varphi(0) - \varphi(s - \tau_2)|^p ds \]
\[ + C_{p,L,T} \int_0^t E|X_{\tau_1}(s) - X_{\tau_2}(s)|^p ds, \quad 0 \leq t \leq T. \]

The proof of Proposition is complete. \( \square \)
Corollary 1. Under the assumptions of Lemma 3.3, we have

\[
\int_0^t E|h(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1)) - h(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))|^p ds
\leq C_{p,L,T} \left( t|\tau_1 - \tau_2|^p/2 + \int_0^{t \wedge \tau_1} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^p ds \right.
\]
\[
+ \int_{t \wedge \tau_1}^{t \wedge \tau_2} |\varphi(0) - \varphi(s - \tau_2)|^p ds \right)
\]
\[
\leq C_{p,L,T} \left( |\tau_1 - \tau_2|^p/2 + \int_0^{t \wedge \tau_1} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^p ds \right.
\]
\[
+ \int_{t \wedge \tau_1}^{t \wedge \tau_2} |\varphi(0) - \varphi(s - \tau_2)|^p ds \right)
\]

for every \( p \geq 2 \) and for all \( t \in [0, T] \), where \( C_{p,L,T} \) is a positive constant.

Proof. Follows directly from Lemma 3.3 and Proposition 4.

Next, we estimate the distance between the Malliavin derivatives. For this purpose, we set

\[
I_{1,\tau}(\theta, t) := \int_{\theta}^{t} b_2(s, X_{\tau}(s), X_{\tau}(s - \tau)) D_{\theta} X_{\tau}(s) ds,
\]
\[
I_{2,\tau}(\theta, t) := \int_{\theta + \tau}^{t} b_3(s, X_{\tau}(s), X_{\tau}(s - \tau)) D_{\theta} X_{\tau}(s - \tau) ds,
\]
\[
J_{1,\tau}(\theta, t) := \int_{\theta}^{t} \sigma_2(s, X_{\tau}(s), X_{\tau}(s - \tau)) D_{\theta} X_{\tau}(s) dB(s),
\]
\[
J_{2,\tau}(\theta, t) := \int_{\theta + \tau}^{t} \sigma_3(s, X_{\tau}(s), X_{\tau}(s - \tau)) D_{\theta} X_{\tau}(s - \tau) dB(s),
\]

where \((X_\tau(t))_{t \in [\tau, T]}\) is the solution to the equation (15).

Lemma 3.4. Let Assumptions 1 and 2 hold. We have

\[
E|I_{1,\tau_1}(\theta, t) - I_{1,\tau_2}(\theta, t)|^2 + E|J_{1,\tau_1}(\theta, t) - J_{1,\tau_2}(\theta, t)|^2
\]
\[
\leq C_{L,T} \left( |\tau_1 - \tau_2|^2 + \int_0^{t \wedge \tau_1} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^2 ds \right.
\]
\[
+ \int_{t \wedge \tau_1}^{t \wedge \tau_2} |\varphi(0) - \varphi(s - \tau_2)|^2 ds \right)
\]
\[
+ C_{L,T} \int_0^t E|D_{\theta} X_{\tau_1}(s) - D_{\theta} X_{\tau_2}(s)|^2 ds, \quad 0 \leq \theta \leq t \leq T,
\]

(40)

where \( C_{L,T} \) is a positive constant.

Proof. We have

\[
I_{1,\tau_1}(\theta, t) - I_{1,\tau_2}(\theta, t)
\]
\[
= \int_{\theta}^{t} [b_2(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1)) - b_2(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))] D_{\theta} X_{\tau_1}(s) ds
\]
\[
+ \int_{\theta}^{t} b_2(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2)) [D_{\theta} X_{\tau_1}(s) - D_{\theta} X_{\tau_2}(s)] ds, \quad 0 \leq \theta \leq t.
\]
Then, by the Hölder inequality, we obtain
\[
E[I_{1,\tau_1}(\theta,t) - I_{1,\tau_2}(\theta,t)]^2 
\leq 2(t-\theta) \int_\theta^t E[b_2'(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1)) - b_2'(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))] D\theta X_{\tau_1}(s)^2 ds 
+ 2(t-\theta) \int_\theta^t E[b_2'(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2)) [D\theta X_{\tau_1}(s) - D\theta X_{\tau_2}(s)]^2 ds 
\leq C_{L,T} \left( \int_\theta^t E[b_2'(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1)) - b_2'(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))]^4 ds \right)^{1/2} 
\times \left( \int_\theta^t E[D\theta X_{\tau_1}(s)]^4 ds \right)^{1/2} 
+ C_{L,T} \int_\theta^t E[D\theta X_{\tau_1}(s) - D\theta X_{\tau_2}(s)]^2 ds, \quad 0 \leq \theta \leq t. \quad (41)
\]

Note that, by the estimate (25), we have \( \int_\theta^t E[D\theta X_{\tau_1}(s)]^4 ds \leq C_{L,T} \). Furthermore, an application of Corollary 1 to \( h = b_2' \) yields
\[
\int_\theta^t E[b_2'(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1)) - b_2'(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))]^4 ds 
\leq C_{L,T} \left( |\tau_1 - \tau_2|^2 + \int_0^{\tau_1 \wedge \tau_2} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^4 ds 
+ \int_{\tau_1 \wedge \tau_2}^{t \wedge \tau_1} |\varphi(0) - \varphi(s - \tau_2)|^4 ds \right), \quad 0 \leq \theta \leq t.
\]
Hence, we get
\[
E[I_{1,\tau_1}(\theta,t) - I_{1,\tau_2}(\theta,t)]^2 
\leq C_{L,T} \left( |\tau_1 - \tau_2|^2 + \int_0^{\tau_1 \wedge \tau_2} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^4 ds + \int_{\tau_1 \wedge \tau_2}^{t \wedge \tau_1} |\varphi(0) - \varphi(s - \tau_2)|^4 ds \right)^{1/2} 
+ C_{L,T} \int_\theta^t E[D\theta X_{\tau_1}(s) - D\theta X_{\tau_2}(s)]^2 ds, \quad 0 \leq \theta \leq t.
\]
Thus (40) is verified for \( E[I_{1,\tau_1}(\theta,t) - I_{1,\tau_2}(\theta,t)]^2 \). Similarly, we have
\[
E[J_{1,\tau_1}(\theta,t) - J_{1,\tau_2}(\theta,t)]^2 
\leq 2 \int_\theta^t E[|\sigma_2'(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1)) - \sigma_2'(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))|] D\theta X_{\tau_1}(s)^2 ds 
+ 2 \int_\theta^t E[|\sigma_2'(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2)) [D\theta X_{\tau_1}(s) - D\theta X_{\tau_2}(s)]^2 ds 
\leq C \left( \int_\theta^t E[|\sigma_2'(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1)) - \sigma_2'(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))|]^4 ds \right)^{1/2} 
\times \left( \int_\theta^t E[D\theta X_{\tau_1}(s)]^4 ds \right)^{1/2} 
+ C \int_\theta^t E[D\theta X_{\tau_1}(s) - D\theta X_{\tau_2}(s)]^2 ds, \quad 0 \leq \theta \leq t. \quad (42)
\]
The right hand side of (42) has the same form as that of (41). We therefore can conclude that (40) also holds true for \( E[J_{1,\tau_1}(\theta,t) - J_{1,\tau_2}(\theta,t)]^2 \). The proof of the lemma is complete. \( \square \)
Lemma 3.5. Let Assumptions 1 and 2 hold. Then, we have for $t > \tau_2$,
\[
E[I_{2,\tau_1}(\theta, t) - I_{2,\tau_2}(\theta, t)]^2 + E[J_{2,\tau_1}(\theta, t) - J_{2,\tau_2}(\theta, t)]^2 \\
\leq C_{L,T} \left( |\tau_1 - \tau_2|^2 + \int_0^{t\wedge \tau_1} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^4 ds + \int_{t\wedge \tau_1}^{t\wedge \tau_2} |\varphi(0) - \varphi(s - \tau_2)|^4 ds \right)^{1/2} \\
+ C_{L,T} \int_{\theta}^{t} E[D_{\theta}X_{\tau_1}(s) - D_{\theta}X_{\tau_2}(s)]^2 ds, \quad 0 \leq \theta \leq t - \tau_2,
\]
where $C_{L,T}$ is a positive constant.

Proof. When $t > \tau_2$ and $0 \leq \theta \leq t - \tau_2$, we have
\[
I_{2,\tau_1}(\theta, t) - I_{2,\tau_2}(\theta, t) \\
= \int_{\theta + \tau_1}^{t} b_3(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1))D_{\theta}X_{\tau_1}(s - \tau_1)ds \\
- \int_{\theta + \tau_2}^{t} b_3(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))D_{\theta}X_{\tau_2}(s - \tau_2)ds \\
= \int_{\theta + \tau_1}^{t} [b_3(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1)) - b_3(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))]D_{\theta}X_{\tau_1}(s - \tau_1)ds \\
+ \int_{\theta + \tau_1}^{t} b_3(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1))D_{\theta}X_{\tau_1}(s - \tau_1)ds \\
+ \int_{\theta + \tau_2}^{t} b_3(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))D_{\theta}X_{\tau_1}(s - \tau_1)ds \\
+ \int_{\theta + \tau_2}^{t} b_3(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))D_{\theta}X_{\tau_2}(s - \tau_2)ds. \tag{44}
\]
We observe that the first addend in the right hand side of (44) can be estimated as in the proof of Lemma 3.4 and we obtain
\[
E\left| \int_{\theta + \tau_1}^{t} [b_3(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1)) - b_3(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))]D_{\theta}X_{\tau_1}(s - \tau_1)ds \right|^2 \\
\leq C_{L,T} \left( |\tau_1 - \tau_2|^2 + \int_0^{t\wedge \tau_1} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^4 ds + \int_{t\wedge \tau_1}^{t\wedge \tau_2} |\varphi(0) - \varphi(s - \tau_2)|^4 ds \right)^{1/2}.
\]
For the second addend, it follows from (25) that
\[
E\left| \int_{\theta + \tau_1}^{t} b_3(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1))D_{\theta}X_{\tau_1}(s - \tau_1)ds \right|^2 \\
\leq |\tau_1 - \tau_2| \int_{\theta + \tau_1}^{t\wedge \tau_2} E\left| b_3(s, X_{\tau_1}(s), X_{\tau_1}(s - \tau_1))D_{\theta}X_{\tau_1}(s - \tau_1) \right|^2 ds \\
\leq C_{L,T} |\tau_1 - \tau_2|^2 \leq C_{L,T} |\tau_1 - \tau_2|.
\]
For the third addend, we use the Hölder inequality and (26) to get
\[
E\left| \int_{\theta + \tau_2}^{t} b_3(s, X_{\tau_2}(s), X_{\tau_2}(s - \tau_2))D_{\theta}X_{\tau_1}(s - \tau_1)ds \right|^2 \\
\leq C_{L,T} \int_{\theta + \tau_2}^{t} E\left| D_{\theta}X_{\tau_1}(s - \tau_1) - D_{\theta}X_{\tau_2}(s - \tau_2) \right|^2 ds \\
\leq C_{L,T} \int_{\theta + \tau_2}^{t} E\left| D_{\theta}X_{\tau_1}(s - \tau_1) - D_{\theta}X_{\tau_1}(s - \tau_2) \right|^2 ds \\
+ C_{L,T} \int_{\theta + \tau_2}^{t} E\left| D_{\theta}X_{\tau_1}(s - \tau_2) - D_{\theta}X_{\tau_2}(s - \tau_2) \right|^2 ds
\]
\[
\begin{align*}
&\leq C_{L,T} \int_{\theta + \tau_2}^{t} |\tau_1 - \tau_2| ds + C_{L,T} \int_{0}^{t} E|D_{\theta}X_{\tau_1}(s) - D_{\theta}X_{\tau_2}(s)|^2 ds \\
&= C_{L,T}|\tau_1 - \tau_2| + C_{L,T} \int_{0}^{t} E|D_{\theta}X_{\tau_1}(s) - D_{\theta}X_{\tau_2}(s)|^2 ds.
\end{align*}
\]

Hence, we can obtain (43) for \(E|J_{2,\tau_1}(\theta, t) - J_{2,\tau_2}(\theta, t)|^2\). This finishes the proof of Lemma because the estimate for \(E|J_{2,\tau_1}(\theta, t) - J_{2,\tau_2}(\theta, t)|^2\) can be treated similarly.

**Proposition 5.** Let Assumptions 1 and 2 hold. Then, there exists a positive constants \(C_{L,T}\) such that, for all \(t \in [0, T]\),

\[
E\|DX_{\tau_1}(t) - DX_{\tau_2}(t)\|^2_{L^2[0,T]} \leq C_{L,T}\left( t^2 |\tau_1 - \tau_2|^2 \\
+ \int_{0}^{t \wedge \tau_2} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^4 ds + t \int_{t \wedge \tau_1}^{t \wedge \tau_2} |\varphi(0) - \varphi(s - \tau_2)|^4 ds \right)^{1/2},
\]

where \(C_{L,T}\) is a positive constant.

**Proof.** We consider the following cases.

**Case 1.** \(t > \tau_2\). In this case, we write

\[
\begin{align*}
E\|DX_{\tau_1}(t) - DX_{\tau_2}(t)\|^2_{L^2[0,T]} \\
&= \int_{0}^{t - \tau_2} E|D_{\theta}X_{\tau_1}(t) - D_{\theta}X_{\tau_2}(t)|^2 d\theta + \int_{t - \tau_2}^{t - \tau_1} E|D_{\theta}X_{\tau_1}(t) - D_{\theta}X_{\tau_2}(t)|^2 d\theta \\
&+ \int_{t - \tau_1}^{t} E|D_{\theta}X_{\tau_1}(t) - D_{\theta}X_{\tau_2}(t)|^2 d\theta.
\end{align*}
\]

We observe from the equations (21) and (22) that, when \(0 \leq \theta \leq t - \tau_2\), we have

\[
\begin{align*}
D_{\theta}X_{\tau_1}(t) - D_{\theta}X_{\tau_2}(t) &= \sigma(\theta, X_{\tau_1}(\theta), X_{\tau_1}(\theta - \tau_1)) - \sigma(\theta, X_{\tau_2}(\theta), X_{\tau_2}(\theta - \tau_2)) \\
&+ I_{1,\tau_1}(\theta, t) - I_{1,\tau_2}(\theta, t) + I_{2,\tau_1}(\theta, t) - I_{2,\tau_2}(\theta, t) \\
&+ J_{1,\tau_1}(\theta, t) - J_{1,\tau_2}(\theta, t) + J_{2,\tau_1}(\theta, t) - J_{2,\tau_2}(\theta, t).
\end{align*}
\]

When \(t - \tau_2 < \theta \leq t - \tau_1\), we have

\[
\begin{align*}
D_{\theta}X_{\tau_1}(t) - D_{\theta}X_{\tau_2}(t) &= \sigma(\theta, X_{\tau_1}(\theta), X_{\tau_1}(\theta - \tau_1)) - \sigma(\theta, X_{\tau_2}(\theta), X_{\tau_2}(\theta - \tau_2)) \\
&+ I_{1,\tau_1}(\theta, t) - I_{1,\tau_2}(\theta, t) + I_{2,\tau_1}(\theta, t) \\
&+ J_{1,\tau_1}(\theta, t) - J_{1,\tau_2}(\theta, t) + J_{2,\tau_1}(\theta, t).
\end{align*}
\]

When \(t - \tau_1 < \theta \leq t\), we have

\[
\begin{align*}
D_{\theta}X_{\tau_1}(t) - D_{\theta}X_{\tau_2}(t) &= \sigma(\theta, X_{\tau_1}(\theta), X_{\tau_1}(\theta - \tau_1)) - \sigma(\theta, X_{\tau_2}(\theta), X_{\tau_2}(\theta - \tau_2)) \\
&+ I_{1,\tau_1}(\theta, t) - I_{1,\tau_2}(\theta, t) + J_{1,\tau_1}(\theta, t) - J_{1,\tau_2}(\theta, t).
\end{align*}
\]

Hence, using the estimates established in Lemmas 3.4 and 3.5, we deduce

\[
\begin{align*}
E\|DX_{\tau_1}(t) - DX_{\tau_2}(t)\|^2_{L^2[0,T]} \\
&\leq \int_{0}^{t} E|\sigma(\theta, X_{\tau_1}(\theta), X_{\tau_1}(\theta - \tau_1)) - \sigma(\theta, X_{\tau_2}(\theta), X_{\tau_2}(\theta - \tau_2))|^2 d\theta \\
&+ \int_{t - \tau_2}^{t - \tau_1} E|I_{2,\tau_1}(\theta, t)|^2 + E|J_{2,\tau_1}(\theta, t)|^2 d\theta.
\end{align*}
\]
Theorem 3.6. Let Assumption 1 and 2 hold. We assume, in addition, that

\[ |\sigma(t, x, y)| \geq \sigma_0 > 0, \quad \forall t \in [0, T], x, y \in \mathbb{R}. \]

Then, for any \( g \in \mathcal{B} \) and \( t \in (0, T] \), we have

\[
|Eg(X_{t_1}(t)) - Eg(X_{t_2}(t))| \leq C_{T, L} \left( |t_1 - t_2|^2 + t \int_{t_1}^{t_2} |\varphi(s - t_1) - \varphi(s - t_2)|^4 ds + t^{-1} \int_{t_1}^{t_2} |\varphi(0) - \varphi(s - t_2)|^4 ds \right)^{1/4},
\]

where \( C_{T, L} \) is a positive constant.
Proof. Fixed \( t \in (0, T) \), we consider the random variables \( F_1 = X_{\tau_1}(t) \) and \( F_2 = X_{\tau_2}(t) \). Thanks to Propositions 2, 3, 4 and 5 we have the following
\[
E\|DF_1\|_{L^2[0,T]}^8 \leq C_{T,L} t^{-4}, \quad (E\|DF_1\|_{L^2[0,T]}^2)^2 \leq C_{T,L} t^{-2},
\]
\[
E \left( \int_0^T \int_0^T |D_g D_r F_1|^2 d\theta dr \right)^2 = E \left( \int_0^T \int_0^t |D_g D_r E[X_{\tau_1}(t)]|^2 d\theta dr \right)^2 \leq t^2 \int_0^T \int_0^t E|D_g D_r E[X_{\tau_1}(t)]|^4 d\theta dr \leq C_{L,T} t^4,
\]
and
\[
\|F_1 - F_2\|_{1,2} = \left( E|X_{\tau_1}(t) - X_{\tau_2}(t)|^2 + E\|DX_{\tau_1}(t) - DX_{\tau_2}(t)\|_{L^2[0,T]}^2 \right)^{1/2} \leq C_{T,L} \left( t^2 |\tau_1 - \tau_2|^2 + t \int_{t^\wedge \tau_1}^{t \wedge \tau_2} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^4 ds + t \int_{t \wedge \tau_1}^{t \wedge \tau_2} |\varphi(0) - \varphi(s - \tau_2)|^4 ds \right)^{1/4}.
\]
Consequently, in view of Lemma 2.1, we obtain
\[
|Eg(X_{\tau_1}(t)) - Eg(X_{\tau_2}(t))| = |Eg(F_1) - Eg(F_2)| \leq C \left( E\|DF_1\|_{L^2[0,T]}^8 \left( \int_0^T \int_0^T |D_g D_r F_1|^2 d\theta dr \right)^2 + (E\|DF_1\|_{L^2[0,T]}^2)^2 \right)^{1/4} \|F_1 - F_2\|_{1,2} \leq C_{T,L} \left( |\tau_1 - \tau_2|^2 + t^{1/4} \int_{t \wedge \tau_1}^{t \wedge \tau_2} |\varphi(s - \tau_1) - \varphi(s - \tau_2)|^4 ds + t^{-1} \int_{t \wedge \tau_1}^{t \wedge \tau_2} |\varphi(0) - \varphi(s - \tau_2)|^4 ds \right)^{1/4}.
\]
This completes the proof. \( \square \)

Clearly, when the initial data \( \varphi \) is a continuous function, the estimate (46) implies that \( X_{\tau}(t) \) weakly converges to \( X_{\nu}(t) \) as \( \tau \to \nu \). Moreover, when \( \varphi \) is Hölder continuous, we have the following Hölder continuity of solutions with respect to delay parameter.

Corollary 2. Suppose the assumptions of Theorem 3.6. In addition, we assume that the initial data \( \varphi \) is Hölder continuous with exponent \( \beta \in (0, 1) \). Then, for any \( g \in B \), we have
\[
\sup_{0 \leq t \leq T} |Eg(X_{\tau_1}(t)) - Eg(X_{\tau_2}(t))| \leq C_{T,L} |\tau_1 - \tau_2|^{\beta \wedge \frac{1}{2}},
\]
where \( C_{L,T} \) is a positive constant.

Proof. We have
\[
t^{-1} \int_{t \wedge \tau_1}^{t \wedge \tau_2} (\tau_2 - s)^{4\beta} ds \leq t^{-1} \int_{t \wedge \tau_1}^{t \wedge \tau_2} (\tau_2 - \tau_1)^{4\beta} ds \leq (\tau_2 - \tau_1)^{4\beta}.
\]
Hence, it follows from (46) that
\[
|Eg(X_{\tau_1}(t)) - Eg(X_{\tau_2}(t))| \leq C_{T,L} \left( |\tau_1 - \tau_2|^2 + t^{-1} \int_{t \wedge \tau_1}^{t \wedge \tau_2} |\tau_1 - \tau_2|^{4\beta} ds + t^{-1} \int_{t \wedge \tau_1}^{t \wedge \tau_2} (\tau_2 - s)^{4\beta} ds \right)^{1/4} \leq C_{T,L} |\tau_1 - \tau_2|^{\beta \wedge \frac{1}{2}}, \quad 0 < t \leq T.
\]
This, together with the fact that \(|Eg(X_{\tau_1}(t)) - Eg(X_{\tau_2}(t))| = 0\) when \(t = 0\), gives us (48).

When the coefficients \(b(t, x, y), \sigma(t, x, y)\) do not depend on \(x\), the delays \(\tau_1 = 0, \tau_2 = 1/n\) and the initial data \(\varphi(t) = x_0, t \in [-1/n, 0]\), we obtain the following weak rate of convergence for the Carathéodory approximation.

**Corollary 3.** Consider the Carathéodory approximation system (1)-(2). Suppose that

(i) \(b(t, x)\) and \(\sigma(t, x)\) are Lipschitz in \(x\) and satisfy linear growth,

(ii) \(b(t, \cdot)\) and \(\sigma(t, \cdot)\) are twice differentiable with bounded derivatives,

(iii) \(|\sigma(t, x)| \geq \sigma_0 > 0\) for all \(t \in [0, T]\) and \(x \in \mathbb{R}\).

Then, for any \(g \in B\), we have

\[
\sup_{0 \leq t \leq T} |Eg(x^n(t)) - Eg(x(t))| \leq \frac{C_{L,T}}{\sqrt{n}} \quad n \geq 1,
\]

where \(C_{L,T}\) is a positive constant.

4. **Conclusion.** In this paper, we employed the technique of Malliavin calculus to study the weak convergence of delay SDEs. The interesting point of our results lies in the fact that we are able to provide an explicit estimate for the rate of convergence and we only require the test function \(g\) to be bounded. Furthermore, the method introduced in the paper can be used to investigate the weak convergence and the Carathéodory approximation for more general equations such as SDEs with multiple delays

\[
dX(t) = b(t, X(t), X(t - \tau_1), \ldots, X(t - \tau_k))dt + \sigma(t, X(t), X(t - \tau_1), \ldots, X(t - \tau_k))dB(t), \quad t \in [0, T]
\]

or SDEs with variable delays

\[
dX(t) = b(t, X(t), X(t - \tau(t)))dt + \sigma(t, X(t), X(t - \tau(t)))dB(t), \quad t \in [0, T].
\]

However, the computations will be more complex. Hence, in the present paper, we have chosen the simplest class of delay SDEs to illustrate the main features of the method rather than getting bogged down with complex notations.

**Acknowledgments.** The authors would like to thank the anonymous referees for their valuable comments for improving the paper. A part of this paper was done while the authors were visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM). The authors would like to thank the VIASM for financial support and hospitality.

**REFERENCES**

[1] V. Bally and D. Talay, The law of the Euler scheme for stochastic differential equations. I. Convergence rate of the distribution function, *Probab. Theory Related Fields*, 104 (1996), 43–60.

[2] D. R. Bell and S. E. A. Mohammed, On the solution of stochastic ordinary differential equations via small delays, *Stochastics Stochastics Rep.*, 28 (1989), 293–299.

[3] M. Benabdallah and M. Bouzra, Carathéodory approximate solutions for a class of perturbed stochastic differential equations with reflecting boundary, *Stoch. Anal. Appl.*, 37 (2019), 936–954.

[4] E. Buckwar, R. Kuske, S.-E. Mohammed and T. Shardlow, Weak convergence of the Euler scheme for stochastic differential delay equations, *LMS J. Comput. Math.*, 11 (2008), 60–99.
WEAK CONVERGENCE OF DELAY SDES

[5] E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[6] N. T. Dung and T. C. Son, Lipschitz continuity in the Hurst index of the solutions of fractional stochastic Volterra integro-differential equations, Submitted.

[7] F. Faizullah, A note on the Carathéodory approximation scheme for stochastic differential equations under G-Brownian motion, Z. Naturforsch., 67a (2012), 699–704.

[8] M. Ferrante and C. Rovira, Convergence of delay differential equations driven by fractional Brownian motion, J. Evol. Equ., 10 (2010), 761–783.

[9] B. Jourdain and A. Kohatsu-Higa, A review of recent results on approximation of solutions of stochastic differential equations, Stochastic Analysis with Financial Applications, 121–144, Progr. Probab., 65, Birkhäuser/Springer Basel AG, Basel, (2011).

[10] I. Karatzas and S. E. Shreve, Brownian Motion and Stochastic Calculus, Second edition. Graduate Texts in Mathematics, 113. Springer-Verlag, New York, 1991.

[11] P. E. Kloeden and E. Platen, Numerical Solution of Stochastic Differential Equations, Applications of Mathematics (New York), 23. Springer-Verlag, Berlin, 1992.

[12] K. Liu, Carathéodory approximate solutions for a class of semilinear stochastic evolution equations with time delays, J. Math. Anal. Appl., 220 (1998), 349–364.

[13] W. Mao, L. Hu and X. Mao, Approximate solutions for a class of doubly perturbed stochastic differential equations, Adv. Difference Equ., (2018), Paper No. 37, 17 pp.

[14] X. Mao, Approximate solutions for a class of stochastic evolution equations with variable delays, Numer. Funct. Anal. Optim., 12 (1991), 525–533 (1992).

[15] X. Mao, Approximate solutions for a class of stochastic evolution equations with variable delays. II, Numer. Funct. Anal. Optim., 15 (1994), 65–76.

[16] X. Mao, Stochastic Differential Equations and Applications, Second edition. Horwood Publishing Limited, Chichester, 2008.

[17] D. Nualart, The Malliavin Calculus and Related Topics, Second edition. Probability and its Applications (New York). Springer-Verlag, Berlin, 2006.

[18] B. G. Pachpatte, Inequalities for Differential and Integral Equations, Mathematics in Science and Engineering, 197. Academic Press, Inc., San Diego, CA, 1998.

[19] J. Turo, Carathéodory approximation solutions to a class of stochastic functional-differential equations, Appl. Anal., 61 (1996), 121–128.

Received May 2020; revised June 2021; early access October 2021.

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