Deformations of spacetime metrics

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Summary. This work is devoted to study the deformation of spacetime metrics as generalized conformal transformations. Some applications are also considered, in particular the equations of motion in deformed spacetime are studied.

1 Introduction

Spacetime deformations are interesting to study in geometrical as well as in physical situations, as exact solutions of Einstein’s equations can be modified into new metrics which better describe the physical observations. On the other hand since, according to the Gauss law, given a two-dimensional metric \(g\) it is always possible find a scalar field such that \(g\) is locally conformally flat, observing that conformal transformations are a particular case of deformations, we show how Gauss law can be generalized to every couple of n-dimensional metrics \((g, h)\), deforming one metric into other by matrices of scalar fields. Finally, geodetic motions of test particles into deformed manifold are considered, in particular lensing effects.
and gravitational redshift are studied. The equation of gravitational waves is also obtained in the case of “small” deformation of the background.

2 Deformations

In order to define the deformation of a metric $g$ on an $n$-dimensional manifold $M$ \cite{1, 5}, let us decompose it with a vielbein $\omega \equiv (e^a_i)$ as $g_{\mu\nu} = \eta_{AB} \omega^A \omega^B$. If $\phi^A_B$ is a scalar field on $\omega^B$, we can define a deformed vielbein $\omega^A_{\mu} \equiv \omega^A \phi^B_A$ and $e^a_{\mu} \equiv \psi^A_B e^a_B$ where $\psi^A_B \equiv \delta^A_B \phi^B_A$ (first deforming matrix). The following commutation relations for the deformed basis hold

$$[e^a_A, e^b_B] = \psi^a_A \psi^b_B \left[e^a_A, e^b_B\right] + 2\psi^a_A e^b_B \left(\partial_a \psi^b_B\right)$$

A new metric $\tilde{g}$ is obtained as follows

$$g_{\mu\nu} = \eta_{AB} \omega^A \omega^B \text{ or equivalently } \tilde{g}_{\mu\nu} = G_{CD} \omega^A \omega^B, \quad \tilde{g}^{AB} = G^{CD} e^a_C e^b_D,$$

where $G^{AB} \equiv \eta_{AB} \phi^C \phi^D$ and $G^{CD} \equiv \eta_{AB} \psi^C \psi^D$ (second deforming matrix). The set of first deforming matrices form a right coset with Lorentz matrix in vielbein indices; the identity is the class of equivalence composed by the Lorentz matrices. Deformations should not be confused with diffeomorphisms. In order to understand better this point let us write the metric $\tilde{g}_{\mu\nu} = g_{\mu\nu} \phi^a \phi^b$ where $\phi^a \phi^b \equiv \phi^a_B e^b_B$, then when the matrices $\phi^a_B$ define $\phi^a_B$ Jacobian matrices then the metric $\tilde{g}_{\mu\nu}$ and the tensor $g_{\mu\nu}$ are equivalent.

2.1 Deformations of three-dimensional metrics

Coll et al. showed in \cite{2} that any three-dimensional metric can be locally obtained as a conformal transformation of a constant curvature metric $h$ summed to the tensor product of a 1-form by itself, or $g = \sigma h + \epsilon s \otimes s$, where $\epsilon = \pm 1$ and $\sigma$ is a scalar. Moreover, according to Riemann’s theorem, a scalar relation $\Psi(\sigma, \|s\|) = 0$ between $\sigma$ and $s$ has to be imposed as the metric can be defined, at most, by three independent functions. We find the first deforming matrix associated to the deformation $g = \sigma h + \epsilon s \otimes s$, as

$$\phi^A = \sqrt{\sigma} \delta^A_C + \alpha^A s_C, \quad \text{where } \alpha \text{ is an arbitrary function of } s \text{ and } \|s\|.$$

2.2 Properties of deformed metrics

Conformal deformations are a particular simple example of deformations where the first deforming matrix is $\psi^A = \Omega \delta^A_C$. More generally a metric $h$ can be deformed in $g$ by

$$g_{\mu\nu} = h_{\mu\nu} + X_{\mu\nu}, \quad g_{\mu\nu} = \eta_{AB} + X_{\mu\nu},$$

where $X_{\mu\nu}$ is a tensor. In terms of the second deforming matrices $G_{AB}$ and $G^{AB}$, the (2) reads

$$G_{AB} = \eta_{AB} + C_{AB}, \quad G^{AB} = \eta^{AB} + D^{AB} \text{ with } X_{\mu\nu} = D^{AB} e^a_C e^b_D.$$

If $X^{ad} = -g^{ab} \eta^{cd} h_{bc} (D^{AB} = -\eta^{AC} G^{BD} C_{BC})$ then $X^{ab} h_{bc} = \delta^a_c (D^{AB} C_{BC} = \delta^A_C)$, otherwise $X^{ab} h_{bc} \neq \delta^a_c$. In particular, given the matrix $\phi$ defined by $\phi^A_B \equiv \delta^A_B + \alpha^A \phi^B$, the following relation holds

$$G_{AB} = \eta_{CD} \phi^C_A \phi^D_B = \eta_{AB} + C_{AB} \quad \text{with } C_{AB} = \eta_{CD} \phi^C_A \phi^D_B.$$
In order to consider the geodetics in deformed spacetime let us take into account the metric 

2.4 Geodetic motion

approximation we have neglected the terms \( X_{h} \) equation for the perturbation \( h_{ab} \).

After some manipulation, adopting the same gauge choice as in [6] we obtain the wave equation for the perturbation \( h_{ac} \) in the form:

\[
\nabla^b \nabla_b h_{ac} = 2 R^b_{\phantom{b}cd} h_{bd}, \quad \nabla_b g_{ac} = 0. \tag{5}
\]

From (5) we finally find the following wave equation:

\[
(\nabla^b \nabla_b C_{AB}) \omega^A_a \omega^B_c = C_{AB} \left[ 2 R^b_{\phantom{b}ac} \omega^A_d \omega^B_d - \nabla^b \nabla_b (\omega^A_a \omega^B_c) \right] \tag{6}
\]

In particular, for deformation of the flat manifold the (6) is \( (\nabla^b \nabla_b C_{AB}) \omega^A_a \omega^B_b = 0 \).

2.3 Gravitational waves

We consider the metric (2) as perturbation of the exact metric \( g \), solution of \( G_{\alpha\beta} = 0 \), if the scalar fields satisfy the following condition \( |C_{AB}| \ll 1 \) \forall \alpha \in B \), where \( |C_{AB}| = \|g_{CD} \varphi^C \varphi^D_B \| \) is the module of the matrix element \( C_{AB} \)(see also [1]). Therefore the perturbation of the metric \( g \) is given by the tensor \( h_{ab} \) by the following: \( g_{ab} = g_{ab} + h_{ab} \), while, since in the given approximation we have neglected the terms \( X_{ad} h_{dc} \) and \( h_{ab} h_{bd} \), we have \( g_{ab} = g_{ab} - h_{ab} \). After some manipulation, adopting the same gauge choice as in [6] we obtain the wave equation for the perturbation \( h_{ac} \) in the form:

\[
\nabla^b \nabla_b h_{ac} = 2 R^b_{\phantom{b}cd} h_{bd}, \quad \nabla_b g_{ac} = 0. \tag{5}
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In particular, for deformation of the flat manifold the (6) is \( (\nabla^b \nabla_b C_{AB}) \omega^A_a \omega^B_b = 0 \).

2.4 Geodetic motion

In order to consider the geodetics in deformed spacetime let us take into account the metric (2). The geodetic equation in deformed spacetime is

\[
v^a \tilde{\nabla}_a v^d = \frac{1}{2} g^{cd} \left( \nabla_a h_{cb} + \nabla_b h_{ca} - \nabla_c h_{ba} \right) v^a v^b, \quad \tilde{\nabla}_b g_{ac} = 0 \tag{7}
\]

where \( v^a \) is the test–particle four–velocity. The (7) can be written also as

\[
X_{dc} (h_{cb,a} + h_{ca,b} - h_{ba,c}) v^a v^b + X_{dc} (g_{cb,a} + g_{ca,b} - g_{ba,c}) v^a v^b +
+ g^{dc} (h_{cb,a} + h_{ca,b} - h_{ba,c}) v^a v^b = 0. \tag{8}
\]

In particular the geodetic motion in the spacetime endowed with (4) is described by the following equation

\[
v^a \tilde{C}^{ab}_{\phantom{ab}c} v^c = 2 (\ln \Omega) v^b - (g_{ac} v^a v^c) \left[ g^{bd} \nabla_d \ln \Omega + X^{bd} \ln \Omega \right] + 2 v^a (\nabla_a \ln \Omega) g_{dc} X^{db} v^c +
+ \frac{1}{2} v^a \left[ (\nabla_a h_{cd}) + (\nabla_c h_{ad}) - (\nabla_d h_{ac}) \right] v^c \left( g^{bd} \Omega^{-2} + X^{bd} \right).
\]

where \( \tilde{C}^{ab}_{\phantom{ab}c} \) are the tensors defined as \( \tilde{\nabla}_a \omega_b = \nabla_a \omega_b - \tilde{C}^{ac}_{\phantom{ac}b} \omega_c, \tilde{\nabla}_a t^b = \nabla_a t^b + \tilde{C}^{ab}_{\phantom{ab}c} \epsilon^c \).
2.5 Lensing

The light propagation in the \( (M, \tilde{g}) \) can be described by the following lensing equation

\[
\frac{dp^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma = 0, \quad \text{or} \quad \Delta p^\alpha \equiv -\int_{-\infty}^{+\infty} \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \, d\lambda,
\]

where \( \Gamma^\alpha_{\beta\gamma} \) are the connection coefficients compatible with a \( \tilde{g} \) and \( p^\alpha \equiv dx^\alpha/d\lambda \) is the four–vector wave, \( \lambda \) is a parameter on the geodetic. We assume the trajectory is on xy plane, then, if the motion is initially parallel to the x axe, the deflection angle \( \Phi \) can be expressed in terms of \( p^y \) as \( \frac{dp^y}{dx} = \tan \Phi \propto \Phi \simeq 0 \) and \( p^x \) is considered constant (Cf. [6]). We consider the source very far from the observer and from the lens, consequently the four–momentum \( p^h \) is given by the following integral along the trajectory.

\[
\delta p^y = -\frac{\eta Qp}{2} \int_{-\infty}^{+\infty} \bar{g}^{y\delta} \left[ \partial_\delta (\bar{\omega}_x^x \bar{\omega}_y^y) + \partial_\delta (\bar{\omega}_x^y \bar{\omega}_y^x) - \partial_\delta (\bar{\omega}_x^x \bar{\omega}_y^y) \right] p^\beta p^\gamma \, d\lambda \quad \text{or also,}
\]

\[
\delta p^y = \Delta p^y - \frac{1}{2} \int_{-\infty}^{+\infty} \left[ X^{y\delta} (g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta}) + g^{y\delta} (h_{\delta\beta,\gamma} + h_{\delta\gamma,\beta} - h_{\beta\gamma,\delta}) \right] p^\beta p^\gamma \, d\lambda + \frac{1}{2} \int_{-\infty}^{+\infty} X^{y\delta} (h_{\delta\beta,\gamma} + h_{\delta\gamma,\beta} - h_{\beta\gamma,\delta}) p^\beta p^\gamma \, d\lambda .
\]

2.6 Schwarzschild lensing

A stationary distribution of matter constituted by a spherical symmetric body (Schwarzschild lens) [4] is deformed as follows in the cartesian coordinate

\[
\tilde{d}s^2 = A \left( 1 + \frac{2\Phi}{c^2} \right) c^2 dt^2 - A \left( 1 - \frac{2\Phi}{c^2} \right) \delta_{ij} dx^i dx^j .
\]

In the approximation of weak field, holding if the considered distances are very much larger then the Schwarzschild radius of the field, \( \Phi = -GM/|x| \) is the gravitational potential associated to the matter distribution, \(|x|^2 = \delta_{ij} x^i x^j \). We obtain the following lensing equation [4] \( \frac{d\Phi}{dl_{eucl}} = -\frac{c^2}{e} \nabla \Phi \) where \( \nabla \Phi \equiv [\nabla \Phi - e(e, \nabla \Phi)] \), \( e^k = \frac{dl_{eucl}}{dx^k} \) is the spatial vector with \( dl_{eucl}^2 \equiv \delta_{ij} dx^i dx^j \). We considered the following two cases: for \( A = A(t) \) the lensing equation became \( \frac{d\Phi}{dl_{eucl}} = -\frac{c^2}{e} \nabla \Phi - (\delta_{ik} \ln A) e^k \) for \( A = A(r) \) the equation became \( \frac{d\Phi}{dl_{eucl}} = -\nabla \Phi \) where \( \Phi \equiv \frac{c^2}{e} \Phi + \ln A \). It is particularly interesting to note that the last transformation, that is nothing more than a diffeomorphism, reduces itself, in the lensing effect to a redefinition of the gravitational potential.

2.7 Geodetic deviation

A family \( \gamma_s(t) \) at one parameter \( s \) of geodetics of \( (M, \tilde{g}) \) is parameterized by the affine parameter \( t \) in such a way that a point \( p \in \gamma_s(t) \) is well defined by the map \( (t, s) \rightarrow \gamma_s(t) \). The separation among the two curves changes respect to the parameter \( t \) with a velocity \( \dot{v}^a \equiv T^a \nabla_b S^b \) given by the geodetic equation:

\[
a^a \equiv T^a \nabla_b v^b = -\hat{R}_{c;bd} v^c T^d = a^a_0 - \hat{R}_{c;bd} S^b \hat{v}^c T^d, \quad a^a_0 \equiv -\hat{R}_{c;bd} S^b \hat{v}^c T^d .
\]
where $T^a = \left(\frac{\partial}{\partial t}\right)^a$ is four–vector tangent to the family of geodetics and the four–vector $S^b = \left(\frac{\partial}{\partial x^b}\right)^b$ gives the infinitesimal displacement among two infinitesimal close geodetics, $R_{cde}^a$ is the curvature tensor. We finally find

$$\tag{12} a^a = -2(\partial_\nu \tilde{C}^a_{[d|c]} + \tilde{C}^f_{[c|d]} \tilde{T}^{a|f} + F^f_{[c|d]} \tilde{C}^a_{|f]} S^b T^c T^d).$$

### 2.8 Gravitational redshift

Consider two observers $O_1$ and $O_2$ at rest in the spacetime $(M, \tilde{g})$, where $\tilde{g}_{ab} = \tilde{G}_{\alpha\beta} \omega^\alpha_a \omega^\beta_b$ is a static metric (−, +, +, +). $O_1$ emits a light signal received by $O_2$ at $P_2$. In the approximation of the geometric optic the light rays propagate in the spacetime $(M, \tilde{g})$ along null geodetic. The frequency $\nu$, the wave four–vector $p^a$ and the four–velocity $v^a$ are linked by the relation: $\nu = -p_0 v^0$. From this we find the emission frequency $\nu_1$ and the frequency of absorbing $\nu_2$ of the photons traveling between the two observers $\nu_i = -(p_i v_i^0)|_{P_i}$, where $v_1^a$ and $v_2^a$ are the four–velocity of the observers $O_1$ and $O_2$ respectively. Because of the $v_1^a$ and $v_2^a$ are tangent to the Killing time–vector field $\xi^a (\xi^a \xi_a = \tilde{g}_{00})$, we find: $v_i^a = [\xi^a]/(-\xi^0)|_{P_i}$. Then using the property that the internal product between the vector field $\xi^a$ and the tangent field $p^a$ is constant along the geodetic we have in terms of the wave lengths $\lambda_1$ and $\lambda_2$:

$$z \equiv \frac{\lambda_2 - \lambda_1}{\lambda_1}; \quad z = \frac{\nu_1}{\nu_2} - 1 = \frac{\sqrt{-\tilde{G}_{\alpha\beta} \omega^\alpha_0 \omega^\beta_0}|_{P_2}}{\sqrt{-\tilde{G}_{\alpha\beta} \omega^\alpha_0 \omega^\beta_0}|_{P_1}} - 1. \tag{13}$$

### 3 Conclusion

In this work we deformed spacetime metrics using matrices of scalar fields. Such deformations generalize conformal transformations. In particular we considered deformations as a generalization of the Gauss theorem to every couple n-dimensional metrics. We focused on the main features of deformed manifold showing, in particular, in what sense deformations differ by diffeomorphism. The equation of gravitational waves in the case of “small” deformation is obtained. Finally, we recover the equations of motion in deformed spacetime studying, in particular, the lensing effects and gravitational redshift induced by deformations. Details on the gravitational lens produced by deformations of the Schwarzschild metric are given, in particular a conformal deformation of the metric induces a deformation of the gravitational potential. The equation of geodetic deviation is also obtained. These considerations motivated the idea that deformations can be used to adapt solutions of Einstein’s equations to the cosmological observations and more generally to the experimental data.

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