NON-EXISTENCE OF POSITIVE SOLUTIONS FOR A HIGHER ORDER FRACTIONAL EQUATION

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Abstract. In this paper, we consider a nonlinear equation involving fractional Laplacian of higher order on the whole space. We establish the equivalence between the pseudo-differential equation and an integral equation by applying the maximum principle and the Liouville theorem. For positive solutions to the equation, we obtained non-existence by applying the method of moving planes.

1. Introduction. In this paper, we investigate the following problem:

\[
\begin{aligned}
(-\Delta)^s u(x) &= |x|^a u^p(x), & x &\in \mathbb{R}^n, \\
(-\Delta) u(x) &\geq 0, & x &\in \mathbb{R}^n,
\end{aligned}
\]

where \(0 < \alpha < 2, s = \alpha + 2 < n, a > -s, p > 0\), the operator \((-\Delta)^{\frac{\alpha}{2}+1}\) is defined by

\[
(-\Delta)^{\frac{\alpha}{2}+1} u(x) = c_{n,\alpha} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_{\epsilon}(x)} \frac{u(x) + \frac{1}{2n}\Delta u(x)|x-y|^{2} - u(y)}{|x-y|^{n+\alpha+2}} dy
\]

for \(u \in S\), the Schwartz space of rapidly decreasing smooth functions on \(\mathbb{R}^n\), and \(c_{n,\alpha}\) is a normalized positive constant depending on \(n\) and \(\alpha\). Denote

\[L_s = \{ u : \mathbb{R}^n \to \mathbb{R} | \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+\alpha+2}} dx < \infty \},\]

then the operator \((-\Delta)^{\frac{\alpha}{2}+1}\) can be extended on a wider spaces \(L_s \cap C^{s+\epsilon}_{loc}(\mathbb{R}^n), \epsilon > 0\).

For \(u \in S\), the usual fractional Laplacian \((-\Delta)^{\frac{\alpha}{2}}\) as a non-local operator is defined by

\[
(-\Delta)^{\frac{\alpha}{2}} u(x) = c(n,\alpha) \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_{\epsilon}(x)} \frac{u(x) - u(y)}{|x-y|^{n+\alpha}} dy,
\]

where \(c(n,\alpha)\) is positive constant only depending on \(n\) and \(\alpha\). The operator \((-\Delta)^{\frac{\alpha}{2}}\) can also be equivalently described by the Fourier transform:

\[
(-\Delta)^{\frac{\alpha}{2}} u(\xi) = |\xi|^a \hat{u}(\xi), u \in S.
\]

Let

\[L_{\alpha} = \{ u : \mathbb{R}^n \to \mathbb{R} | \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+\alpha}} < \infty \},\]

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then \((-\Delta)^{\frac{\alpha}{2}}\) can be extended on \(L^{\alpha} \cap C^{\alpha+\epsilon}_{loc}(\mathbb{R}^n), \epsilon > 0\).

Caffarelli and Silvestre [7] introduced the extension method which turns nonlocal problems involving the fractional Laplacian \((p = 2)\) into local ones in higher dimensions. For \(u : \mathbb{R}^n \rightarrow \mathbb{R}\), if

\[
U : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}
\]

satisfies the problem:

\[
\begin{cases}
\Delta_x U + \frac{1-\alpha}{y} U_y + U_{yy} = 0, & (x, y) \in \mathbb{R}^n \times [0, \infty), \\
U(x, 0) = u(x), & x \in \mathbb{R}^n,
\end{cases}
\]

then

\[
c(n, \alpha)(-\Delta)^{\frac{\alpha}{2}} u(x) = \lim_{y \to 0^+} -y^{1-\alpha} U_y.
\]

The classical theories for local elliptic partial differential equations can be applied. We refer to [3, 15] and references therein for broad applications of this method.

For \(u \in L^s \cap C^{s+\epsilon}_{loc}(\mathbb{R}^n)\), it follows from [25] that

\[
(-\Delta)^{\frac{\alpha}{2}+1} u(x) = (-\Delta)^{\frac{\alpha}{2}} \circ (-\Delta) u(x). \quad (4)
\]

We say that a function \(u : \mathbb{R}^n \rightarrow \mathbb{R}\) is a classic solution to problem (1), if \(u \in L^s \cap C^{s+\epsilon}_{loc}(\mathbb{R}^n)\) satisfies problem (1) in the point-wise sense.

In the following, we give our main results.

**Theorem 1.1.** For \(1 \leq p < \frac{n+s+a}{n-s}\), problem (1) has no positive solutions.

**Theorem 1.2.** For \(a = 0, p = \frac{n+s}{n-s}\), every positive solution \(u(x)\) of (1) is radially symmetric and decreasing about some point \(x_0\) and therefore assumes the form

\[
c(t^2 + |x-x_0|^2)^{\frac{n-s}{2}}
\]

with some positive constants \(c\) and \(t\).

**Theorem 1.3.** For \(a = 0, 0 < p < \frac{n+s}{n-s}\), problem (1) has no positive solutions.

The fractional Laplacian operator comes from many phenomena, such as quantum mechanics, anomalous diffusion, turbulence, molecular dynamics, phase transitions and crystal dislocation [4, 5, 19, 20, 31, 33]. In probability and finance, it also can be seen as the infinitesimal generator of Lévy stable diffusion processes. See [1, 5, 8, 17, 26, 28, 29, 32, 34] and the references therein.

In recent years, many authors investigated existence [3, 4, 5, 19, 20, 30, 31, 35], regularity [3, 4, 5, 7, 31, 33], symmetry [9, 16, 19, 20, 21, 35] and monotonicity [16, 21, 30] of the elliptic equations involving the fractional Laplacian operator. But there are few results (due to lack of maximum principle) for equations involving higher fractional Laplacian operator \((-\Delta)^{\frac{\alpha}{2}+1}\). An useful method to study the fractional Laplacian is the integral equations method, which turns a given fractional Laplacian equation into its equivalent integral equation, and then various properties of the original equation can be obtained by investigating the integral equation, see [11, 13, 14] and references therein. In [35], Zhuo, Chen, Cui and Yuan established the equivalence between a fractional Laplacian equation and an integral equation, and obtained radial symmetry and non-existence for positive solutions. In order to explore symmetry and non-existence of the solutions to (1), we first give an equivalent equation in integral form. Then we prove non-existence of positive solutions by applying the method of moving planes of integral form.
This paper is organized as follows. Section 2 devotes to establish the equivalence between the equation and an equation in integral form. In Section 3, we prove non-existence of positive solutions to the equation.

2. The equivalent equation in integral form. The following Liouville theorem for \(\alpha\)-harmonic function first appeared in [2]. An alternative proof was given in [35].

**Lemma 2.1.** [35] Assume that \(n \geq 2\). Let \(u\) be a solution of
\[
\begin{cases}
(-\Delta)^\frac{\alpha}{2} u(x) = 0, & x \in \mathbb{R}^n, \\
u(x) \geq 0, & x \in \mathbb{R}^n,
\end{cases}
\]
then \(u \equiv C\).

The following maximum principle is also crucial for us.

**Lemma 2.2.** [31] Let \(\Omega \subset \mathbb{R}^n\) be a bounded open set, and let \(f\) be a lower-semicontinuous function in \(\bar{\Omega}\) such that \((-\Delta)^\frac{\alpha}{2} f(x) \geq 0\) in \(\Omega\) and \(f(x) \geq 0\) in \(\mathbb{R}^n \setminus \Omega\). Then \(f(x) \geq 0\) in \(\mathbb{R}^n\).

**Theorem 2.3.** Assume that \(u\) is a nonnegative solution of (1), then \(u\) also satisfies
\[
u(x) = c_n \int_{\mathbb{R}^n} \frac{|y|^{\alpha} u^p(y)}{|x-y|^{n-\alpha}} dy
\]
and vice versa.

**Proof.** According to Kulczycki[24], the Green function \(G^R_\alpha(x,y)\) of \((-\Delta)^\frac{\alpha}{2}\) on the ball \(B_R\) satisfies
\[
\begin{cases}
(-\Delta)^\frac{\alpha}{2} G^R_\alpha(x,y) = \delta(x-y), & x,y \in B_R(0), \\
G^R_\alpha(x,y) = 0, & x \text{ or } y \notin B_R(0),
\end{cases}
\]
where \(G^R_\alpha(x,y) = \frac{A_{n,\alpha}}{|x-y|^{n-\alpha}} \int_0^{\frac{R^2}{r^2}} b^{\frac{\alpha}{2}-1} \frac{db}{(1+b)^{n+\frac{\alpha}{2}}} x,y \in B_R(0),\)
and \(s_R = \frac{|x-y|^2}{R^2}, t_R = (1 - \frac{|x|^2}{R^2})(1 - \frac{|y|^2}{R^2})\).

The Green functions \(G^R_2(x,y)\) of \((-\Delta)^\frac{\alpha}{2}\) on the ball \(B_R\) satisfies
\[
\begin{cases}
(-\Delta)^\frac{\alpha}{2} G^R_2(x,y) = \delta(x-y), & x,y \in B_R(0), \\
G^R_2(x,y) = 0, & x \text{ or } y \notin B_R(0).
\end{cases}
\]
Let
\[
G_s^R(x,y) = \int_{\mathbb{R}^n} G^R_\alpha(x,z) G^R_2(z,y) dz.
\]
It is easy to verify
\[
(-\Delta)^\frac{\alpha}{2} G_s^R(x,y) = \begin{cases}
\delta(x-y), & x,y \in B_R(0), \\
0, & x \text{ or } y \notin B_R(0).
\end{cases}
\]
Set
\[
v_R(x) = \int_{\mathbb{R}^n} G_s^R(x,y) |y|^\alpha u^p(y) dy.
\]
A direct computation derives that
\[
\begin{cases}
(-\Delta)^\frac{\alpha}{2} (u(x) - v_R(x)) = 0, & x \in B_R(0), \\
u(x) - v_R(x) \geq 0, & x \notin B_R(0).
\end{cases}
\]
Since
\[ \begin{cases} 
(\Delta)^{s/2}((\Delta)(u(x) - v_R(x))) = 0, & x \in B_R(0), \\
(\Delta)(u(x) - v_R(x)) \geq 0, & x \notin B_R(0),
\end{cases} \tag{8} \]
Lemma 2.2 and (7) show that
\[ \begin{cases} 
(\Delta)(u(x) - v_R(x)) \geq 0, & x \in B_R(0), \\
u(x) - v_R(x) \geq 0, & x \notin B_R(0),
\end{cases} \tag{9} \]
By the maximum principles of the Laplace, it follows that
\[ v_R(x) \leq u(x), x \in \mathbb{R}^n. \]
Hence \( v_R(x) \) is well defined and the limit exists in the pointwise sense as \( R \to \infty \). Let
\[ v(x) = \lim_{R \to \infty} v_R(x) = c_n \int_{\mathbb{R}^n} \frac{|y|^a u^p(y)}{|x - y|^{n-s}} dy. \]
Taking the limit for \( R \to \infty \) in (7) and (9), we get
\[ \begin{cases} 
(\Delta)^{s/2}(u(x) - v(x)) = b, & x \in \mathbb{R}^n, \\
(\Delta)(u(x) - v(x)) \geq 0, & x \in \mathbb{R}^n, \\
u(x) - v(x) \geq 0, & x \in \mathbb{R}^n.
\end{cases} \]
By the Liouville theorems of the fractional Laplacian and the fact \( (\Delta)^{s/2} = (\Delta)^{\frac{s}{2}} \circ (\Delta) \), there exists a constant \( b \geq 0 \), such that
\[ \begin{cases} 
(\Delta)(u(x) - v(x)) = b, & x \in \mathbb{R}^n, \\
u(x) - v(x) \geq 0, & x \in \mathbb{R}^n.
\end{cases} \]
It is easy to see that \( b = 0 \) by contradiction. Applying Liouville theorems of the Laplace, there exists a constant \( c \geq 0 \), such that
\[ u(x) - v(x) = c. \]
Assume that \( c > 0 \), then
\[ u(x) \geq c_n \int_{\mathbb{R}^n} \frac{|y|^a c^p}{|x - y|^{n-s}} dy = \infty, \]
which implies that \( c = 0 \). Hence (6) holds.

**Remark 1.** In the proof of Theorem 2.3, the condition \( (\Delta)u(x) \geq 0 \) appearing in (8) is crucial to apply the maximum principle. It would be very interesting to know whether this assumption can be removed.

3. **Proof of main results.** For \( a = 0, p = \frac{n+s}{n-s} \), Chen, Li and Ou\[13, 14\] obtained the following result for integral equation (6).

**Theorem 3.1.** [13, 14] Every positive solution \( u(x) \) of (6) with \( a = 0, 0 < s < n, p = \frac{n+s}{n-s} \) is radially symmetric and decreasing about some point \( x_0 \) and therefore assumes the form
\[ c \left( \frac{t}{t^2 + |x - x_0|^2} \right)^{\frac{n-s}{2}} \]
with some positive constants \( c \) and \( t \).

Combining Theorem 2.3 with Theorem 3.1, we can obtain Theorem 1.2.

For the integral equation (6) with \( a = 0, 0 < p < \frac{n+s}{n-s} \), Y.Y. Li\[27\] proved non-existence of positive solutions by using the method of moving sphere.
Theorem 3.2. [27] Let \( n \geq 1, a = 0, 0 < s < n \). (i) For \( 0 < p < \frac{n}{n-s} \), the equation (6) does not have any positive Lebesgue measurable solution; (ii) For \( \frac{n}{n-s} \leq p < \frac{n+s}{n-s} \), the equation (6) does not have any positive solution \( u \in L_{loc}^{\frac{n(p-1)}{n-s}} \).

For \( a = 0, 0 < p < \frac{n+s}{n-s} \), Theorem 1.3 follows directly from Theorem 3.2 and Theorem 2.3.

Theorem 1.1 can be concluded by applying the method of moving planes, which is also valid for \( a = 0, 1 \leq p < \frac{n+s}{n-s} \). We will present details in the following.

Replacing \( x \) by \( \frac{x}{|x|^2} \) in (6), we get

\[
 u\left(\frac{x}{|x|^2}\right) = c_n \int_{\mathbb{R}^n} \frac{|y|^a u^p(y)}{|x|^2 - |y|^{n-s}} \, dy
 = c_n \int_{\mathbb{R}^n} \frac{x^n}{|x|^2} \left( \frac{|y|^a u^p(y)}{|y|^{n-s}} \right) \, dy
 = c_n \int_{\mathbb{R}^n} |x|^{n-s} u^p \left( \frac{y}{|y|^2} \right) \, dy.
\]

Let

\[
 \tilde{u}(x) = \frac{1}{|x|^{n-s}} u\left(\frac{x}{|x|^2}\right)
\]

be the Kelvin transform of \( u \). Then (6) can be rewritten as

\[
 \tilde{u}(x) = c_n \int_{\mathbb{R}^n} \frac{\tilde{u}^p(y)}{|x-y|^{n-s} |y|^{n+s+a-p(n-s)}} \, dy.
\]

Consider the \( x_1 \) direction. For \( x = (x_1, x_2, \ldots, x_n) = (x_1, x') \in \mathbb{R}^n, \lambda < 0 \), denote

\[
 T_\lambda = \{ x = (x_1, x') \in \mathbb{R}^n | x_1 = \lambda \},
\]

\[
 \Sigma_\lambda = \{ x = (x_1, x') \in \mathbb{R}^n | x_1 < \lambda \}.
\]

Let

\[
 x^\lambda = (2\lambda - x_1, x')
\]

be the reflection of the point \( x = (x_1, x') \) about plane \( T_\lambda \) and

\[
 \Sigma_\lambda^c = \{ x | x^\lambda \in \Sigma_\lambda \},
\]

be the complement of \( \Sigma_\lambda \). Denote

\[
 \tilde{u}_\lambda(x) = \tilde{u}(x^\lambda),
\]

\[
 w_\lambda(x) = \tilde{u}_\lambda(x) - \tilde{u}(x)
\]

and

\[
 \Sigma_\lambda^- = \{ x \in \Sigma | \tilde{u}_\lambda(x) < \tilde{u}(x) \}.
\]

Lemma 3.3. Assume that \( u \) is a positive solution to (1), then

\[
 \tilde{u}_\lambda(x) - \tilde{u}(x)
 = c_n \int_{\Sigma_\lambda^c} \frac{\tilde{u}^p(y^\lambda)}{|y^\lambda|^{n+s+a-p(n-s)} |y|^{n+s+a-p(n-s)}} \, dy
 - \frac{\tilde{u}^p(y)}{|y|^{n+s+a-p(n-s)}} \, dy
 + \frac{1}{|x-y|^{n-s}} - \frac{1}{|x-y^\lambda|^{n-s}} \, dy.
\]

(10)
Step 1. By Lemma 3.3, for

\[ \bar{u}(x) - \bar{u}(x) \]

\[ = c_n \int_{\mathbb{R}^n} \frac{\bar{u}^p(y)}{|x - y|^{n-s}} dy \]

\[ - c_n \int_{\mathbb{R}^n} \frac{\bar{u}^p(y)}{|x - y|^{n-s}} dy \]

\[ = c_n \int_{\Sigma_\lambda} \frac{\bar{u}^p(y) - \bar{u}^p(y)}{|x - y|^{n-s}} dy \]

\[ + c_n \int_{\Sigma_\lambda} \frac{\bar{u}^p(y^\lambda)}{|x - y|^{n-s}} dy \]

\[ - c_n \int_{\Sigma_\lambda} \frac{\bar{u}^p(y)}{|x - y|^{n-s}} dy \]

\[ = c_n \int_{\Sigma_\lambda} \left( \frac{\bar{u}^p(y^\lambda)}{|y|^{n-s}} - \frac{\bar{u}^p(y)}{|y|^{n-s}} \right) \]

\[ \left( \frac{1}{|x - y|^{n-s}} - \frac{1}{|x - y^\lambda|^{n-s}} \right) dy. \]

Hence the lemma is proved. \qed

We need the following Hardy-Littlewood-Sobolev inequality, which is proved in [14].

Lemma 3.4. Let \( g \in L^{\frac{n}{n-s}}(\mathbb{R}^n) \) for \( \frac{n}{n-s} < r < \infty \). Define

\[ Tg(x) = \int_{\mathbb{R}^n} \frac{g(y)}{|x - y|^{n-s}} dy. \]

Then

\[ \|Tg\|_{L^r} \leq C(n, s, r) \|g\|_{L^{\frac{n}{n-s}}(\mathbb{R}^n)}. \]

Proof of Theorem 1.1. We divide the proof into two steps.

Step 1. By Lemma 3.3, for \( x \in \Sigma_\lambda \), we have

\[ \bar{u}(x) - \bar{u}_\lambda(x) \]

\[ = c_n \int_{\Sigma_\lambda} \left( \frac{\bar{u}^p(y) - \bar{u}^p(y^\lambda)}{|y|^{n-s}} - \frac{\bar{u}^p(y^\lambda)}{|y^\lambda|^{n-s}} \right) \]

\[ \left( \frac{1}{|x - y|^{n-s}} - \frac{1}{|x - y^\lambda|^{n-s}} \right) dy \]

\[ = c_n \int_{\Sigma_\lambda} \left( \frac{\bar{u}^p(y) - \bar{u}^p(y^\lambda)}{|y|^{n-s}} + \bar{u}^p(y^\lambda) \left( \frac{1}{|y^\lambda|^{n-s}} - \frac{1}{|y^\lambda|^{n-s}} \right) \right) \]

\[ \left( \frac{1}{|x - y|^{n-s}} - \frac{1}{|x - y^\lambda|^{n-s}} \right) dy \]

\[ \leq c_n \int_{\Sigma_\lambda} \frac{\bar{u}^p(y) - \bar{u}^p(y^\lambda)}{|y|^{n-s}} \left( \frac{1}{|x - y|^{n-s}} - \frac{1}{|x - y^\lambda|^{n-s}} \right) dy. \]
Consequently, \( \lambda > \lambda \)
This means that there exists

\[
\text{Hence for } \lambda
\]

Observing \( 0 < \bar{u}(y^\lambda) \leq \bar{u}(y) \) for \( y \in \Sigma^\lambda \), \( \bar{u}(y^\lambda) \in L^r(\Sigma^\lambda) \) for \( \frac{n}{n-s} < r < \infty \), and using the Hardy-Littlewood-Sobolev inequality, we arrive at

\[
\|w_\lambda\|_{L^r(\Sigma^\lambda)} \leq C(n, s, r, p) \| \frac{\bar{u}^{p-1}(y)(\bar{u}(y) - \bar{u}(y^\lambda))}{|y|^{n+s+a-p(n-s)}} \|_{L_c^{\frac{n}{n+s+a-p(n-s)}}(\Sigma^\lambda)}
\]

The fact \( |\bar{u}(y)| \sim \frac{1}{|y|^{n-s}} \) \( (|y| \to \infty) \) implies that for \( \lambda \) sufficiently negative,

\[
C(n, s, r, p) \| \frac{\bar{u}^{p-1}(y)}{|y|^{n+s+a-p(n-s)}} \|_{L_c^{\frac{n}{n+s+a-p(n-s)}}(\Sigma^\lambda)} < \frac{1}{2}.
\]

It follows from (11) that

\[
|\Sigma^\lambda| = 0.
\]

Hence for \( \lambda \) sufficiently negative and \( x \in \Sigma^\lambda \),

\[
w_\lambda(x) \geq 0.
\]

This provides start points for the method of moving planes.

**Step 2.** Keep Moving the plane \( T_\lambda \) until the limiting position

\[
\lambda_0 = \sup \{ \lambda < 0 | w_\mu(x) \geq 0, x \in \Sigma^\mu, \forall \mu \leq \lambda \}.
\]

We will show that \( \lambda_0 = 0 \). On the contrary, suppose that \( \lambda_0 < 0 \), we claim for \( x \in \Sigma^\lambda_0 \),

\[
w_{\lambda_0}(x) \equiv 0.
\]

If not, it follows from Lemma 3.3 , for \( x \in \Sigma^\lambda_0 \),

\[
w_{\lambda_0}(x) = c_\alpha \int_{\Sigma^\lambda_0} \left( \frac{\bar{u}^p(y^\lambda_0)}{|y^\lambda_0|^{n+s+a-p(n-s)}} - \frac{\bar{u}^p(y)}{|y|^{n+s+a-p(n-s)}} \right)
\]

\[
\left( \frac{1}{|x-y|^{n-s}} - \frac{1}{|x-y^\lambda_0|^{n-s}} \right) dy > 0.
\]

(12)

It shows that \( |\Sigma^\lambda_0| = 0 \). For \( \lambda < 0 \) and \( \lambda > \lambda_0 \) sufficiently closed to \( \lambda_0 \), (11) still holds, i.e.

\[
\|w_\lambda\|_{L^r(\Sigma^\lambda)} \leq C(n, s, r, p) \|w_\lambda\|_{L^r(\Sigma^\lambda)} \| \frac{\bar{u}^{p-1}(y)}{|y|^{n+s+a-p(n-s)}} \|_{L_c^{\frac{n}{n+s+a-p(n-s)}}(\Sigma^\lambda)}.
\]

Since \( \lim_{\lambda \to \lambda_0^-} \Sigma^\lambda = \Sigma^\lambda_0 \) and \( |\bar{u}(y)| \sim \frac{1}{|y|^{n-s}} \) \( (|y| \to \infty) \), for \( \lambda \) close to \( \lambda_0 \), we deduce that

\[
C(n, s, r, p) \| \frac{\bar{u}^{p-1}(y)}{|y|^{n+s+a-p(n-s)}} \|_{L_c^{\frac{n}{n+s+a-p(n-s)}}(\Sigma^\lambda)} < \frac{1}{2}.
\]

Consequently,

\[
|\Sigma^\lambda_0| = 0.
\]

This means that there exists \( \lambda > \lambda_0 \), such that

\[
\bar{u}(x^\lambda) \geq \bar{u}(x),
\]
which is a contradiction with the definition of \( \lambda_0 \). So the claim has been proved, that is to say, for all \( x \in \Sigma_{\lambda_0} \)
\[
\tilde{u}(x^{\lambda_0}) = \tilde{u}(x).
\]
(14)

Moreover, for \( x \in \Sigma_{\lambda_0} \), we get
\[
0 = w_{\lambda_0}(x) = c_n \int_{\Sigma_{\lambda_0}} \left( \frac{\tilde{u}^p(y^{\lambda_0})}{|y^{\lambda_0}|^{n+\alpha-p(n-s)}} - \frac{\tilde{u}^p(y)}{|y|^{n+\alpha-p(n-s)}} \right) \left( \frac{1}{|x-y|^{n-s}} - \frac{1}{|x-y^{\lambda_0}|^{n-s}} \right) dy > 0,
\]
(15)

which is a contradiction. Hence \( \lambda_0 = 0 \) and for \( x \in \Sigma_0 \), we have
\[
\tilde{u}(x^0) \geq \tilde{u}(x).
\]
(16)

Similarly, we can move the plane from \( x_1 = +\infty \) to the left to derive that
\[
\tilde{u}(x^0) \leq \tilde{u}(x), x \in \Sigma_0.
\]

Together with (16), we have
\[
\tilde{u}(x^0) = \tilde{u}(x), x \in \Sigma_0.
\]
(17)

Since the direction of \( x_1 \)-axis is arbitrary, we have proved that \( u \) is symmetric about the origin.

For any point \( x_0 \in \mathbb{R}^n \), Let
\[
\tilde{u}(x) = \frac{1}{|x-x_0|^{n-s}} u \left( \frac{x-x_0}{|x-x_0|^2} + x_0 \right)
\]
be the Kelvin transform centered at \( x_0 \). Through a similar argument, we can prove that \( \tilde{u}(x) \) is symmetric about \( x_0 \). The usual way deduces that \( u(x) \) must be positive constant, details please see reference [35]. This is impossible since \( u \) satisfies (1) or (6). We complete the proof of the theorem. \( \square \)

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