ABELIAN QUOTIENTS OF MONOIDS OF HOMOLOGY CYLINDERS

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ABSTRACT. We discuss abelian quotients of monoids of homology cylinders of a surface. We show that both the monoid of all homology cylinders and that of irreducible homology cylinders are not finitely generated and moreover they have big abelian quotients. The proof is given by applying sutured Floer homology theory to homological fibered knots introduced in a previous paper.

1. Introduction

Let $\Sigma_{g,1}$ be a compact oriented connected surface of genus $g$ with one boundary component. Homology cylinders over $\Sigma_{g,1}$, each of which consists of a homology cobordism $M$ between two copies of $\Sigma_{g,1}$ and markings of both sides of the boundary of $M$, appeared in the context of the theory of finite type invariants for 3-manifolds (see Goussarov [7], Habiro [8], Garoufalidis-Levine [4] and Levine [12]), and play an important role in the systematic study of the set of 3-manifolds. In our previous paper [6], we observed their relationship to knot theory by introducing homological fibered knots.

The set $\mathcal{C}_{g,1}$ of isomorphism classes of homology cylinders over $\Sigma_{g,1}$ becomes a monoid by the natural stacking operation. It is known that the monoid $\mathcal{C}_{g,1}$ contains the mapping class group $\mathcal{M}_{g,1}$ of $\Sigma_{g,1}$ as a unit subgroup (see Example 2.2). Moreover, many techniques and invariants to study $\mathcal{M}_{g,1}$ such as Johnson-Morita homomorphisms and the Magnus representation can be extended to $\mathcal{C}_{g,1}$ (see [4], [17]). By using them, we can observe that $\mathcal{C}_{g,1}$ and $\mathcal{M}_{g,1}$ hold many properties in common.

Taking account of the similarity between $\mathcal{C}_{g,1}$ and $\mathcal{M}_{g,1}$, we now pay our attention to abelian quotients of them. It is known that $\mathcal{M}_{g,1}$ is a perfect group for $g \geq 3$, namely it has no non-trivial abelian quotients. In this paper, however, we will show that the opposite holds for $\mathcal{C}_{g,1}$.

The outline of this paper is as follows. After introducing homology cylinders in Section 2, we see that $\mathcal{C}_{g,1}$ has a big abelian quotient arising from the reducibility of a homology cylinder as a 3-manifold (Theorem 2.4). However this fact seems not to be suitable for our purpose of comparing $\mathcal{M}_{g,1}$ and $\mathcal{C}_{g,1}$ because all homology cylinders coming from $\mathcal{M}_{g,1}$ have irreducible underlying 3-manifolds. Therefore we shall introduce the submonoid $\mathcal{C}^{\text{irr}}_{g,1}$ of $\mathcal{C}_{g,1}$ consisting of irreducible homology cylinders. The main result is Theorem 2.6 that $\mathcal{C}^{\text{irr}}_{g,1}$ also has a big abelian quotient coming from a completely different context from that of $\mathcal{C}_{g,1}$. In fact, we prove it in Section 3 as an application of sutured Floer homology theory.
Sutured Floer homology was defined first in [9] by Juhász, then an alternative definition was given in [15] by Ni. It is a variant of Heegaard Floer homology theory defined by Ozsváth and Szabó (here we only refer to [16], which contains the results we use later, for details). In the last section, we observe our results from the viewpoint of homology cobordisms of homology cylinders.

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2. HOMOLOGY CYLINDERS

The definition of homology cylinders goes back to Goussarov [7], Habiro [8], Garoufalidis-Levine [4] and Levine [12]. Strictly speaking, the definition below is closer to that in [4] and [12].

**Definition 2.1.** A homology cylinder \((M, i_+, i_-)\) over \(\Sigma_{g,1}\) consists of a compact oriented 3-manifold \(M\) with two embeddings \(i_+, i_- : \Sigma_{g,1} \hookrightarrow \partial M\) such that:

(i) \(i_+\) is orientation-preserving and \(i_-\) is orientation-reversing;

(ii) \(\partial M = i_+(\Sigma_{g,1}) \cup i_-(\Sigma_{g,1})\) and \(i_+(\Sigma_{g,1}) \cap i_- (\Sigma_{g,1}) = i_+(\partial \Sigma_{g,1}) = i_- (\partial \Sigma_{g,1})\);

(iii) \(i_+|_{\partial \Sigma_{g,1}} = i_-|_{\partial \Sigma_{g,1}}\); and

(iv) \(i_+, i_- : H_1(\Sigma_{g,1}; \mathbb{Z}) \to H_1(M; \mathbb{Z})\) are isomorphisms.

Two homology cylinders \((M, i_+, i_-)\) and \((N, j_+, j_-)\) over \(\Sigma_{g,1}\) are said to be isomorphic if there exists an orientation-preserving diffeomorphism \(f : M \cong N\) satisfying \(j_+ = f \circ i_+\) and \(j_- = f \circ i_-\). We denote by \(C_{g,1}\) the set of all isomorphism classes of homology cylinders over \(\Sigma_{g,1}\). We define a product operation on \(C_{g,1}\) by

\[(M, i_+, i_-) \cdot (N, j_+, j_-) := (M \cup_{i_+ \circ (j_+)^{-1}} N, i_+, j_-)\]

for \((M, i_+, i_-), (N, j_+, j_-) \in C_{g,1}\), so that \(C_{g,1}\) becomes a monoid with the unit \((\Sigma_{g,1} \times [0, 1], \text{id} \times 1, \text{id} \times 0)\).

**Example 2.2.** For each diffeomorphism \(\varphi\) of \(\Sigma_{g,1}\) which fixes \(\partial \Sigma_{g,1}\) pointwise, we can construct a homology cylinder by setting

\[(\Sigma_{g,1} \times [0, 1], \text{id} \times 1, \varphi \times 0),\]

where collars of \(i_+(\Sigma_{g,1})\) and \(i_-(\Sigma_{g,1})\) are stretched half-way along \((\partial \Sigma_{g,1}) \times [0, 1]\). It is easily checked that the isomorphism class of \((\Sigma_{g,1} \times [0, 1], \text{id} \times 1, \varphi \times 0)\) depends only on the (boundary fixing) isotopy class of \(\varphi\) and that this construction gives a monoid homomorphism from the mapping class group \(\mathcal{M}_{g,1}\) of \(\Sigma_{g,1}\) to \(C_{g,1}\). In fact, it is an injective homomorphism (see Garoufalidis-Levine [4] Section 2.4], Levine [12 Section 2.1] and also [6 Proposition 2.3]).

As seen in Example 2.2, we may regard \(C_{g,1}\) as an enlargement of \(\mathcal{M}_{g,1}\). We recall here that \(\mathcal{M}_{g,1}\) is a perfect group for \(g \geq 3\) (see Korkmaz’s survey [11] and papers listed there for details). In comparing the structures of \(C_{g,1}\) and \(\mathcal{M}_{g,1}\), it seems interesting to discuss abelian quotients of \(C_{g,1}\). Note that we need to be careful when mentioning abelian quotients of \(C_{g,1}\) since it is not a group but a monoid. We avoid this by considering its universal group and its abelian quotients. Recall that every monoid \(S\) has a universal group
\( \mathcal{U}(S) \) characterized as an initial object of all monoid homomorphisms from \( S \) to any group with the usual universality condition. One of constructions of \( \mathcal{U}(S) \) is to regard a monoid presentation of \( S \) as a group presentation.

In our discussion below, the case where \( g = 0 \) is exceptional. We can check that \( C_{0,1} \) is isomorphic to the monoid \( \theta_2^3 \) of all (integral) homology 3-spheres whose product is given by connected sums. Indeed, an isomorphism \( \theta_2^3 \cong C_{0,1} \) is given by assigning to each homology 3-sphere \( X \) the homology cylinder \(((D^2 \times [0,1]) \# X, \text{id} \times 1, \text{id} \times 0)\) with the inverse homomorphism given by closing (see Example 3.1). Consequently, \( C_{0,1} \) is an abelian monoid which is not finitely generated.

We begin our main argument by the following observations.

**Lemma 2.3.** For each \((M, i_+, i_-) \in C_{g,1} \) with \( g \geq 1 \), surfaces \( i_+(\Sigma_{g,1}) \) and \( i_-(\Sigma_{g,1}) \) are incompressible in \( M \).

**Proof.** It suffices to show that \( i_+ : \pi_1(\Sigma_{g,1}) \to \pi_1(M) \) is injective. Since \( i_+ : \Sigma_{g,1} \hookrightarrow M \) induces an isomorphism on homology, it follows from Stallings’ theorem [18, Theorem 3.4] that \( i_+ \) induces isomorphisms on all stages of nilpotent quotients. Combining it with the fact that \( \pi_1(\Sigma_{g,1}) \) is free, in particular residually nilpotent, we see that \( i_+ : \pi_1(\Sigma_{g,1}) \to \pi_1(M) \) is injective.

**Proposition 2.4.** The monoid \( C_{g,1} \) is not finitely generated, for every \( g \geq 0 \). In fact, the abelianization of \( \mathcal{U}(C_{g,1}) \) has infinite rank.

**Proof.** The case where \( g = 0 \) is as mentioned above. We now assume \( g \geq 1 \). For each homology cylinder \((M, i_+, i_-) \in C_{g,1} \), the underlying 3-manifold \( M \) has a prime decomposition of the form

\[
M \cong M_0 \# X_1 \# X_2 \# \cdots \# X_n,
\]

where \( M_0 \) is the unique prime factor having \( \partial M \) and \( X_1, X_2, \ldots, X_n \) are homology 3-spheres. Using this decomposition, we can define the *forgetting* map

\[
F : C_{g,1} \longrightarrow \theta_2^3
\]

by

\[
F(M, i_+, i_-) = S^3 \# X_1 \# X_2 \# \cdots \# X_n.
\]

The uniqueness of the prime decomposition of a 3-manifold shows that \( F \) is well-defined.

We now claim that \( F \) is a surjective monoid homomorphism. Let \((M, i_+, i_-), (N, j_+, j_-) \in C_{g,1} \). We decompose \( M \) into \( M \cong M_0 \# X_M \), where \( M_0 \) is the prime factor having \( \partial M \) and \( X_M \) is a homology 3-sphere. Similarly, we have \( N \cong N_0 \# X_N \). By Lemma 2.3, the underlying 3-manifold of the product \((M, i_+, i_-) \cdot (N, j_+, j_-) \) has a decomposition

\[
(M_0 \cup_{i_- o j_+}^{-1} N_0) \# X_M \# X_N
\]

such that \( M_0 \cup_{i_- o j_+}^{-1} N_0 \) is the prime factor having the boundary. This shows that \( F \) is a monoid homomorphism. The surjectivity of \( F \) follows from the existence of a splitting \( \theta_2^3 \to C_{g,1} \) defined by \( X \mapsto ((\Sigma_{g,1} \times [0,1]) \# X, \text{id} \times 1, \text{id} \times 0) \).

The result follows from the fact that \( \theta_2^3 \) satisfies the conditions mentioned in the statement.
Proposition 2.4 says that the monoid $C_{g,1}$ has a different property about its abelian quotients from the mapping class group $M_{g,1}$, which arises from the reducibility of the underlying 3-manifold of a homology cylinder. However the underlying 3-manifolds of homology cylinders obtained from $M_{g,1}$ are all product $\Sigma_{g,1} \times [0,1]$ and, in particular, irreducible. Therefore it seems reasonable to consider the following subset of $C_{g,1}$.

**Definition 2.5.** A homology cylinder $(M, i_+, i_-)$ is said to be irreducible if the underlying 3-manifold $M$ is irreducible. We denote by $C_{g,1}^{irr}$ the subset of $C_{g,1}$ consisting of all irreducible homology cylinders.

Note that $C_{g,1}^{irr}$ is a submonoid of $C_{g,1}$, for $C_{g,1}^{irr} = \ker(F)$. In particular, $C_{0,1}^{irr}$ is the trivial monoid. Note also that we have an injective monoid homomorphism $M_{g,1} \hookrightarrow C_{g,1}^{irr}$. The following is the main result of this paper, whose proof will be given in the next section.

**Theorem 2.6.** The monoid $C_{g,1}^{irr}$ is not finitely generated, for every $g \geq 1$. In fact, the abelianization of $U(C_{g,1}^{irr})$ has infinite rank.

### 3. Proof of Theorem 2.6

Our proof of Theorem 2.6 will be obtained as an application of sutured Floer homology theory due to Juhász [9, 10] and Ni [14, 15].

For each homology cylinder $(M, i_+, i_-) \in C_{g,1}$, we have a natural decomposition

$$\partial M = i_+(\Sigma_{g,1}) \cup i_-(\partial \Sigma_{g,1})$$

of $\partial M$. Such a decomposition defines a sutured manifold $(M, \gamma)$ with the suture $s(\gamma) = i_+(\partial \Sigma_{g,1}) = i_-(\partial \Sigma_{g,1})$. Sutured manifolds were originally defined by Gabai [3], to which we refer for details.

**Example 3.1.** For a knot $K$ in a closed oriented connected 3-manifold $Y$ with a Seifert surface $S$, let $M$ be the manifold obtained from the knot exterior $Y - N(K)$ by cutting open along $S$ and $\gamma$ the annulus $\partial N(K) \cap \partial M$. Then, $(M, \gamma)$ is called complementary sutured manifold for $S$. The core curve of $\gamma$ is denoted by $s(\gamma)$, and called the suture. The suture $s(\gamma)$ and $K$ is parallel. On the other hand, for each $(M, i_+, i_-) \in C_{g,1}$ (or, more generally, each marked cobordism of $\Sigma_{g,1}$), we have a closed oriented connected 3-manifold

$$C_M := M/(i_+(x) = i_-(x)) \quad (x \in \Sigma_{g,1})$$

called the closing and a knot $i_+ (\partial \Sigma_{g,1}) = i_- (\partial \Sigma_{g,1})$ with a Seifert surface $S = i_+(\Sigma_{g,1}) = i_-(\Sigma_{g,1})$ in $C_M$.

Sutured Floer homology is an invariant of balanced sutured manifolds (see Juhász [9, Definition 2.11]), where all of the sutured manifolds mentioned above satisfy this condition. It assigns a finitely generated abelian group $SFH(M, \gamma)$ to each balanced sutured manifold $(M, \gamma)$. We rely on papers of Juhász [9, 10] and Ni [14, 15] for the definition and fundamental properties of sutured Floer homology, and concentrate on using this theory. Juhász [10, Theorem 1.5] showed that

$$SFH(M, \gamma) = \overline{HFK}(C_M, s(\gamma), g(S))$$

(3.1)
holds for any sutured manifold \((M, \gamma)\) mentioned in Example 3.1 where the right hand side is the genus \(g(S)\) part of the knot Floer homology of the knot \(s(\gamma)\) in \(C_M\) with the Seifert surface \(S\) of genus \(g(S)\).

For each homology cylinder \((M, i_+, i_-) \in C_{g,1}\), we put \(SFH(M, i_+, i_-) := SFH(M, \gamma)\) with \(s(\gamma) = i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})\).

**Proposition 3.2.** \(SFH(M, i_+, i_-)\) contains \(\mathbb{Z}\) for any \((M, i_+, i_-) \in C_{g,1}\).

**Proof.** We first assume that \(M\) is irreducible. By Juhasz [10, Theorem 1.4], all we have to do is to check that \((M, i_+, i_-)\) gives a taut sutured manifold. That is,

(i) \(M\) is irreducible; and

(ii) \(i_+(\Sigma_{g,1})\) and \(i_-(\Sigma_{g,1})\) are incompressible and Thurston norm minimizing in their homology classes in \(H_2(M, \gamma)\) with \(s(\gamma) = i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})\).

The condition (i) is automatic and the first half of (ii) follows from Lemma 2.3. For the latter half of (ii), it suffices to show that \(i_+(\Sigma_{g,1})\) is Thurston norm minimizing.

Suppose that we have a proper embedding \(j : \Sigma_{h,1} \hookrightarrow M\) of a surface \(\Sigma_{h,1}\) of genus \(h < g\) satisfying \([j(\Sigma_{h,1})] = [i_+(\Sigma_{g,1})] \in H_2(M, \gamma)\). We take a basis \(\{\delta_1, \delta_2, \ldots, \delta_{2h}\}\) of \(\pi_1(j(\Sigma_{h,1}))\) such that

\[
(3.2) \quad s(\gamma) = i_+(\partial\Sigma_{g,1}) = j(\partial\Sigma_{h,1}) = \prod_{i=1}^{h} [\delta_{2i-1}, \delta_{2i}] \in [\pi_1(M), \pi_1(M)]
\]

under suitable orientations. Here we set the basepoint on \(s(\gamma)\). By Stallings [18, Lemma 3.1], we see that

\[
i_+ : \frac{[\pi_1(\Sigma_{g,1}), \pi_1(\Sigma_{g,1})]}{[\pi_1(\Sigma_{g,1}), [\pi_1(\Sigma_{g,1}), \pi_1(\Sigma_{g,1})]]} \longrightarrow \frac{[\pi_1(M), \pi_1(M)]}{[\pi_1(M), [\pi_1(M), \pi_1(M)]]}
\]

is an isomorphism, and we pull back (3.2) to an equality

\[
\partial\Sigma_{g,1} = \sum_{i=1}^{h} i_+^{-1}([\delta_{2i-1}]) \land i_+^{-1}([\delta_{2i}]) \in \wedge^2(H_1(\Sigma_{g,1})),
\]

where \([\delta_k] \in H_1(M)\) denotes the homology class of \(\delta_k\). On the other hand, \(\partial\Sigma_{g,1} = \sum_{j=1}^{g} x_j \land y_j\), the symplectic form, for any symplectic basis \(\{x_1, \ldots, x_g, y_1, \ldots, y_g\}\) of \(H_1(\Sigma_{g,1})\). Define an endomorphism of \(H_1(\Sigma_{g,1})\) by

\[
x_i \mapsto i_+^{-1}([\delta_{2i-1}]), \quad y_i \mapsto i_+^{-1}([\delta_{2i}] \quad \text{for } 1 \leq i \leq h,
\]

\[
x_j, y_j \mapsto 0 \quad \text{for } h+1 \leq j \leq g.
\]

By definition, this endomorphism is not injective, but preserves the symplectic form. However, such an endomorphism does not exist since the symplectic form embodies the intersection form on \(H_1(\Sigma_{g,1})\), which is nondegenerate, a contradiction. Therefore \(i_+(\Sigma_{g,1})\) is Thurston norm minimizing and we finish the proof when \(M\) is irreducible.

When \(M\) is not irreducible, we take a prime decomposition \(M = M_0 \# X_1 \# X_2 \# \cdots \# X_n\) as in Section 2 where \(X_1, X_2, \ldots, X_n\) are all homology 3-spheres. Then we obtain the conclusion by an argument similar to the proof of [10, Corollary 8.3] using the connected sum formula [9, Proposition 9.15].
By formulas of Juhász \[10\], Proposition 8.6] and Ni \[15\], Theorem 4.1, 4.5] together with the fact (3.1), we have

\[
SFH((M, i_+, i_-) \cdot (N, j_+, j_-)) \otimes \mathbb{Q} \cong (SFH(M, i_+ \cdot i_-) \otimes SFH(N, j_+ \cdot j_-)) \otimes \mathbb{Q}
\]

for \((M, i_+, i_-), (N, j_+, j_-) \in \mathcal{C}_{g,1}\). Hence by taking the rank of \(SFH\), we obtain a monoid homomorphism

\[
R : \mathcal{C}_{g,1} \rightarrow \mathbb{Z}_{>0}^\times
\]

defined by

\[
R(M, i_+, i_-) = \text{rank}_\mathbb{Z}(SFH(M, i_+ \cdot i_-)),
\]

where \(\mathbb{Z}_{>0}^\times\) is the monoid of positive integers whose product is given by multiplication. We call \(R\) the rank homomorphism. Note that the restriction of \(R\) on \(\mathcal{M}_{g,1}\) is trivial since every element of \(\mathcal{M}_{g,1}\) has its inverse.

By the uniqueness of the prime decomposition of an integer, we can decompose \(R\) into prime factors

\[
R = \bigoplus_{p: \text{prime}} R_p : \mathcal{C}_{g,1} \rightarrow \mathbb{Z}_{>0}^\times = \bigoplus_{p: \text{prime}} \mathbb{Z}_{>0}^{(p)}
\]

where \(\mathbb{Z}_{>0}^{(p)}\) is a copy of \(\mathbb{Z}_{>0}\), the monoid of non-negative integers whose product is given by sums, corresponding to the power of the prime number \(p\). We now restrict the above homomorphisms to \(\mathcal{C}_{g,1}^{\text{irr}}\).

**Proposition 3.3.** For \(g \geq 1\), the set \(\{ R_p : \mathcal{C}_{g,1}^{\text{irr}} \rightarrow \mathbb{Z}_{>0} | p \text{ prime}\}\) contains infinitely many non-trivial homomorphisms that are linearly independent as elements of \(\text{Hom}(\mathcal{C}_{g,1}^{\text{irr}}, \mathbb{Z}_{>0})\).

To prove this proposition, we have to observe the images of the homomorphisms \(R_p\). More specifically, we need many homology cylinders whose ranks of \(SFH\) are known. We now use homological fibered knots defined in the previous paper \[6\] to construct such homology cylinders.

Recall that a knot \(K\) in \(S^3\) is said to be homologically fibered if \(K\) satisfies the following two conditions:

(i) The degree of the normalized Alexander polynomial \(\Delta_K(t)\) of \(K\) is \(2g(K)\); and

(ii) \(\Delta_K(0) = \pm 1\),

where \(g(K)\) is the genus of \(K\) and \(\Delta_K(t)\) is normalized so that its lowest degree is 0. In \[6\], Theorem 3.4], we showed that \(K\) is homologically fibered if and only if \(K\) has a Seifert surface \(S\) whose complementary sutured manifold is a homology product. Here, a homology product means a homology cylinder without markings. Note that Crowell and Trotter observed in \[2\] this essentially. (See also \[13\].) We also showed that if \(K\) is homologically fibered, then any minimal genus Seifert surface gives a homology product.

For a homological fibered knot \(K\) of genus \(g\), we obtain an irreducible homology cylinder \((M, i_+, i_-) \in \mathcal{C}_{g,1}^{\text{irr}}\) by fixing a pair of markings of the boundary of the complementary sutured manifold \((M, K)\) for a minimal genus Seifert surface. By (3.1),

\[
SFH(M, i_+, i_-) = SFH(M, K) \cong \widehat{HFK}(S^3, K, g)
\]

holds for such a homology cylinder.
Proof of Proposition 3.3. We first give a proof of the case where $g = 1$. We now consider pretzel knots $P(2l + 1, 2m + 1, 2n + 1)$ with $2l + 1 < 0$. As depicted in Figure 1, each of such knots has a genus 1 Seifert surface.

![Pretzel knots](image)

**Figure 1.** Standard diagram of Pretzel knots with Seifert surfaces of genus 1

It is well known that the normalized Alexander polynomial of $P(2l + 1, 2m + 1, 2n + 1)$ is given by

$$
\Delta_{P(2l+1,2m+1,2n+1)}(t) = (1 + l + m + n + lm + mn + nl)(t - 1)^2 + t.
$$

Using this formula, we see that the sequence

$$
\{ P_n := P(-2n + 1, 2n + 1, 2n^2 + 1) \}_{n=1}^{\infty}
$$

consists of homological fibered knots of genus 1 since $\Delta_{P_n}(t) = 1 - t + t^2$. Moreover, a computation due to Ozsváth-Szabó [16] gives

$$
\overline{HFK}(S^3, P_n, 1) \cong \mathbb{Z}_{(1)}^{n^2-n} \oplus \mathbb{Z}_{(2)}^{n^2-n+1} \cong \mathbb{Z}^{2n^2-2n+1},
$$

from which we see that $\{ 2n^2 - 2n + 1 \}_{n=1}^{\infty} \subset R(C_{1,1}^{\text{irr}})$.

We now analyze the sequence $\{ 2n^2 - 2n + 1 \}_{n=1}^{\infty}$ of positive integers. Elementary number theory says that there exists a positive integer $m$ such that $m^2 \equiv -1 \mod p$ if odd prime $p$ satisfies $p \equiv 1 \mod 4$. In this case, put

$$
n = \begin{cases} 
\frac{m + 1}{2} & (m: \text{odd}) \\
\frac{m + p + 1}{2} & (m: \text{even})
\end{cases}.
$$

Then, if $m$ is odd, we have

$$
2n^2 - 2n + 1 = \frac{m^2 + 1}{2} \equiv 0 \mod p,
$$

and if $m$ is even, we also have

$$
2n^2 - 2n + 1 = \frac{(m + p)^2 + 1}{2} \equiv 0 \mod p.
$$

Hence we can conclude that $R_p$ is non-trivial if $p \equiv 1 \mod 4$. It is well known that there exist infinitely many such prime numbers.
For a homology cylinder $M \in \mathcal{C}_{1,1}^{\text{irr}}$, let
$$p(M) := \max\{1 \cup \{p \mid R_p(M) \neq 0\}\}.$$  
Take a sequence of homology cylinders $\{M_i\}_{i=1}^{\infty} \subset \mathcal{C}_{1,1}^{\text{irr}}$ such that $p(M_1) < p(M_2) < \cdots$. Then we can see that $\{R_p(M_i)\}_{i=1}^{\infty}$ are linearly independent by evaluating them by $M_i$'s. We finish the proof of the case where $g = 1$.

Let $P_n(k)$ be a homological fibered knot of genus $k + 1$ obtained from $P_n$ by taking connected sums with $k$-tuples of trefoils. By [4, Theorem 1.1] and [10, Corollary 8.8] together with the fact that the trefoil is a fibered knot of genus 1, so that $HF \tilde{K}(S^3, \text{trefoil}, 1) \cong \mathbb{Z}$, we have
$$\text{rank}_\mathbb{Z}(HF \tilde{K}(S^3, P_n, 1)) = \text{rank}_\mathbb{Z}(HF \tilde{K}(S^3, P_n(k), k + 1)).$$
Therefore, the cases where $g \geq 2$ follow from the same argument as above.

Remark 3.4. It was shown in [5] that for all odd numbers $a$, $b$, $c$ satisfying $|a|, |b|, |c| \geq 3$, the pretzel knot $P(a, b, c)$ has a unique incompressible (hence minimal genus) Seifert surface up to isotopy.

Proof of Theorem 2.6. Suppose $\mathcal{C}_{g,1}^{\text{irr}}$ was finitely generated. Then except for finitely many primes, the homomorphisms $R_p$ are trivial on any finite set of generators, and hence on whole $\mathcal{C}_{g,1}^{\text{irr}}$. This contradicts Proposition 3.3 and we have proved the first half of our claim. The latter half follows from the construction that uses infinitely many homomorphisms whose targets are abelian.

4. Observations from the viewpoint of homology cobordism

In [4], Garoufalidis-Levine introduced homology cobordisms of homology cylinders, which give an equivalence relation of homology cylinders. We finish this paper by two observations of our results (Proposition 2.4 and Theorem 2.6) from the viewpoint of this equivalence relation.

Two homology cylinders $(M, i_+, i_-)$ and $(N, i_+, i_-)$ over $\Sigma_{g,1}$ are homology cobordant if there exists a compact oriented smooth 4-manifold $W$ such that:

1. $\partial W = M \cup (-N)/(i_+(x) = j_+(x), i_-(x) = j_-(x)) \quad x \in \Sigma_{g,1}$;
2. the inclusions $M \hookrightarrow W, N \hookrightarrow W$ induce isomorphisms on the integral homology.

We denote by $\mathcal{H}_{g,1}$ the quotient set of $\mathcal{C}_{g,1}$ with respect to the equivalence relation of homology cobordism. The monoid structure of $\mathcal{C}_{g,1}$ induces a group structure of $\mathcal{H}_{g,1}$. It is known that $\mathcal{M}_{g,1}$ can be embedded in $\mathcal{H}_{g,1}$ (see [4, Section 2.4], [12, Section 2.1]).

One important problem is to determine whether $\mathcal{H}_{g,1}$ is perfect or not. In fact, no non-trivial abelian quotients of $\mathcal{H}_{g,1}$ are known at present. We now observe that it is difficult to give an answer to this problem by using the homomorphisms used in this paper. First we consider the forgetting homomorphism $F : C_{g,1} \to \theta_2^g$ discussed in Section 2.

Theorem 4.1. For every abelian group $A$ and every non-trivial monoid homomorphism $\varphi_A : \theta_2^g \to A$, the composite $\varphi_A \circ F : C_{g,1} \to A$ does not factor through $\mathcal{H}_{g,1}$, for all $g \geq 1$.

\footnote{After we wrote the first version of this paper, Cha, Friedl and Kim settled this problem in [1]}. 

\textit{Proof}: Let $\varphi_A : \theta_2^g \to A$ be a homomorphism, and suppose $\varphi_A \circ F : C_{g,1} \to A$ factors through $\mathcal{H}_{g,1}$. Then there exists a homomorphism $\psi : \mathcal{H}_{g,1} \to A$ such that $\varphi_A = \psi \circ F$. Since $\mathcal{H}_{g,1}$ is perfect, $\psi$ is trivial on any finite set of generators. Moreover, $\mathcal{H}_{g,1}$ is not finitely generated, hence $\psi$ is trivial on the whole $\mathcal{H}_{g,1}$. This contradicts the fact that $\varphi_A$ is non-trivial. Thus, $\varphi_A \circ F$ does not factor through $\mathcal{H}_{g,1}$. **Q.E.D.**
Proof. It follows from Myers’ result [13, Theorem 3.2] that every homology cylinder in \( C_{g,1} \) with \( g \geq 1 \) is homology cobordant to an irreducible one, whose image by \( F \) is trivial by definition. Hence if \( \varphi_A \circ F \) factors through \( H_{g,1} \) for a monoid homomorphism \( \varphi_A : \theta^3 \to A \), then \( \varphi_A \) must be trivial.

Next we consider the rank homomorphisms \( R_p : C_{g,1}^{\text{irr}} \to \mathbb{Z}_{\geq 0} \) discussed in Section 3. It induces a group homomorphism \( R_p : \mathcal{U}(C_{g,1}^{\text{irr}}) \to \mathbb{Z} \) on universal groups. Note that the quotient group of \( C_{g,1}^{\text{irr}} \) by homology cobordism relation is also \( H_{g,1} \) as mentioned in the proof of Theorem 4.1.

**Theorem 4.2.** For each \( g \geq 1 \), the homomorphism \( R_p : \mathcal{U}(C_{g,1}^{\text{irr}}) \to \mathbb{Z} \) does not factor through \( H_{g,1} \) if it is non-trivial.

**Proof.** Since \( R_p(C_{g,1}^{\text{irr}}) \subset \mathbb{Z}_{\geq 0} \) and \( H_{g,1} \) is a quotient group of \( C_{g,1}^{\text{irr}} \), the homomorphism \( R_p \) must be trivial if it factors through \( H_{g,1} \). \( \square \)

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