DeepVARwT: Deep Learning for a VAR Model with Trend

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ABSTRACT
Time series modelling and prediction is useful in many fields of application such as economics, finance and engineering. The vector autoregressive (VAR) model has been used to describe the dependence within and across multiple time series. This is a model for stationary time series, which can be extended to allow the presence of a deterministic trend in each series. In this paper, we demonstrate a new approach that employs deep learning methodology for maximum likelihood estimation of the trend and the dependence structure at the same time. A Long Short-Term Memory (LSTM) network is used for this purpose. We provide a simulation study and applications to real data. In the simulation study, we use realistic trend functions generated from real data and compare the estimates with true function/parameter values. In the real data applications, we compare the prediction performance of this model with state-of-the-art models in the literature.

KEYWORDS
Dependence modeling, VAR, Causality condition, Trend, Deep learning

1. Introduction

In practice, many time series exhibit nonstationary characteristics in the mean. For example, Fig. 1 shows three quarterly US macroeconomic series, namely GDP gap, inflation, and federal funds rate, as analyzed by [13]. Each series is nonstationary as the mean is apparently not constant. More examples of time series with trends will be given in Sections 4.2 and 4.3.

![Figure 1. US macroeconomic series spanning 1955Q1 to 2003Q1.](image-url)
A simple approach to detrending a time series is to difference it until it appears to be stationary. This is effective when the trend is a low order polynomial. However, the trend itself may be of interest, and modeling it together with the dependence structure can be preferable. The former can be estimated by smoothing the data, using methods such as Kernel Smoothing [26], Locally Weighted Scatterplot Smoothing (Lowess), or Smoothing Splines, to name just a few. The series after removing the trend in each component can then be analyzed by fitting a stationary model. Inference on model parameters will have to ignore errors in estimating the trend in this semi-parametric approach in two stages.

The vector autoregressive VAR($p$) model

$$y_t = A_1 y_{t-1} + A_2 y_{t-2} + \cdots + A_p y_{t-p} + \epsilon_t, \quad t = 0, \pm 1, \pm 2, \ldots, \tag{1}$$

is for stationary time series $\{y_t\}$, where $A_1, \ldots, A_p$ are constant coefficient matrices, and $\{\epsilon_t\}$ is multivariate white noise. It can be extended to accommodate a polynomial trend in each series. If we assume the mean $\mu_t$ of $y_t$ consists of $k$-th order polynomials, and $\{y_t - \mu_t\}$ satisfies the VAR model (1), then a VAR with trend (VARwT) model can be written as

$$y_t = A_1 y_{t-1} + A_2 y_{t-2} + \cdots + A_p y_{t-p} + Cx_t + \epsilon_t, \quad t = 0, \pm 1, \pm 2, \ldots, \tag{2}$$

where $x_t = (1, t, t^2, \ldots, t^k)'$ and $C$ is a matrix of constants. Both the trend and the dependence parameters can be estimated simultaneously using ordinary least squares [21].

Fig. 2 shows polynomial trends estimated together with VAR(4) coefficients for the series in Fig. 1. We can see that even with a relatively high order $k = 9$, the trend functions missed a few peaks and troughs in the data. In other words, there appears to be over smoothing.

Figure 2. US macroeconomic series and estimated polynomial trends (red lines).
An alternative to fitting polynomial trends is to use B-splines and the results can often get better. A practitioner will face a choice of trend models, while remembering not to ignore dependence in the errors, especially when they have prediction in mind.

Recent advances in machine learning have made available to the statistics community a wealth of network structures and the associated training methodologies for finding patterns in vast quantities of data. There have been attempts at deep learning based statistical forecasting, see [28][25][23]. All these methods require the time series to be independent so that the loss function can be written in a simple additive form, thus leaving out dependence information across the series.

In this paper, we model the mean \( \mu_t \) by a recurrent neural network of the LSTM (Long Short-Term Memory) type, with input \( x_t \) at time \( t \) to be defined later, and simultaneously \( \{ y_t - \mu_t \} \) by the VAR model (1). All the model parameters are estimated at the same time. The exact Gaussian log-likelihood is used, and no assumption is made on the independence between the component series. We enforce the causality condition on the VAR parameters to ensure the stability of the model. This is often overlooked in the literature.

The rest of the paper is organized as follows: Section 2 defines the model and discusses trend generation, VAR parameterization, the Gaussian log-likelihood function, and its use in network training. Section 3 is a simulation study using trends generated from real data. Section 4 shows results of model fitting to three data sets and comparisons with alternative models in terms of forecasting accuracy. Section 5 offers concluding remarks.

2. Model fitting and prediction

The Deep VAR with trend (DeepVARwT) model is given by

\[
y_t - \mu_t = A_1(y_{t-1} - \mu_{t-1}) + A_2(y_{t-2} - \mu_{t-2}) + \cdots + A_p(y_{t-p} - \mu_{t-p}) + \epsilon_t, \quad t = 0, \pm 1, \pm 2, \ldots, (3)
\]

where \( \{ \epsilon_t \} \) is i.i.d. Gaussian vector white noise with mean vector 0 and variance-covariance matrix \( \Sigma \). It is also assumed that as a result of causality, \( \epsilon_t \) is uncorrelated with \( y_{t-1}, y_{t-2}, \ldots \), so that the RHS of (3) consists of the best linear predictor \( \hat{y}_t - \mu_t \) of \( y_t - \mu_t \) in terms of \( y_{t-1}, y_{t-2}, \ldots \) (infinite past) and the prediction error \( \epsilon_t \). The trend \( \mu_t \) as well as \( A_1, \ldots, A_p \) and \( \Sigma \) will all come from an LSTM network which is described below.

The difference between this model and the VARwT model (2) is in the formulation of \( \mu_t \). If we leave \( \mu_t \) unspecified, we have a semi-parametric model.

2.1. Long Short-Term Memory (LSTM)

A neural network takes input \( x_t \) at time \( t \), passes it through layers of neurons (processing units) to produce an output. A weighted average of all the input received at each neuron goes into an activation function to produce output for the next stage. A recurrent network also uses output at time \( t-1 \) as input for time \( t \). An LSTM network has special cells and gates to control information flow. At time \( t \), the memory cell \( c_t \) puts information from the last memory cell \( c_{t-1} \) through the forget gate \( f_t \) and information from the candidate memory cell \( c_t \) through the input gate \( i_t \). The output gate \( o_t \) decides how much information from the memory cell \( c_t \) should contribute to the hidden state \( h_t \). Fig. 3 shows the computation unit for the hidden state \( h_t \) in an LSTM network and the corresponding calculations are as follows [5].
Input gate: $i_t = \sigma \left( W_{xi} x_t + W_{hi} h_{t-1} + b_i \right)$,
Forget gate: $f_t = \sigma \left( W_{xf} x_t + W_{hf} h_{t-1} + b_f \right)$,
Output gate: $o_t = \sigma \left( W_{xo} x_t + W_{ho} h_{t-1} + b_o \right)$,
Candidate memory cell: $c_t = \tanh \left( W_{xc} x_t + W_{hc} h_{t-1} + b_c \right)$,
Memory cell: $c_t = f_t \odot c_{t-1} + i_t \odot c_t$,
Hidden state: $h_t = o_t \odot \tanh (c_t)$,

where $W_{xi}, W_{xf}, W_{xo}, W_{hi}, W_{hf}, W_{ho}, W_{xc}$ and $W_{hc}$ are weight parameters, $b_i, b_f, b_o$ and $b_c$ are bias parameters, $\sigma(\cdot)$ is the sigmoid function and the operator $\odot$ denotes the element-wise product.

2.2. Time-dependent trend generation using LSTM

The hidden state

$$h_t = \text{LSTM}(h_{t-1}, x_t; \phi),$$

is mapped to the trend term

$$\mu_t = W_{\mu} h_t + b_\mu,$$

where $x_t$ is the input, $= (t, t^2, t^3)'$ (say, with $t$ suitably scaled), $\phi$ contains the weight and bias parameters in (4), $W_{\mu}$ and $b_\mu$ are additional weight and bias parameters respectively.

2.3. VAR parameter generation

Let $m$ be the dimension of $y_t$ and $p$ the order of the VAR model for $\{y_t - \mu_t\}$. We allocate $m^2 p$ parameters in the neural network to form candidates for the coefficient matrices $A_1, \ldots, A_p$, and another $m(m + 1)/2$ parameters to form a lower triangular matrix $L$. The latter is used to
construct $\Sigma = LL'$ for the variance-covariance matrix of $\varepsilon_t$. These parameters are initialized and updated by the network together with other network parameters. The candidate coefficient matrices $A_1, \ldots, A_p$ go through the next step of reparameterization.

### 2.4. Reparameterizing VAR(p) to enforce causality

It is usually assumed that model (1) is causal in the sense that $y_t$ can be expressed linearly in terms of $\varepsilon_t$, $\varepsilon_{t-1}$, ..., so that $\varepsilon_t$ is the innovation or one-step-ahead prediction error corresponding to the best linear predictor $\hat{y}_t = A_1 y_{t-1} + A_2 y_{t-2} + \cdots + A_p y_{t-p}$ of $y_t$ in terms of $y_{t-1}, y_{t-2}, \ldots$. The causality condition is that all the roots of $\text{det}(I - A_1 z - A_2 z^2 - \cdots - A_p z^p)$ lie outside the unit circle [7]. It also ensures that the linear system given by (1) is stable in the sense that bounded input leads to bounded output.

The parameter space of a causal VAR model is highly complicated. In the univariate case it can be mapped to $(-1, 1)$ in each dimension using partial autocorrelations, see [2]. Work on the multivariate case include [18], [1], [24] and [9].

Given a set of candidate VAR coefficient matrices $A_1, \ldots, A_p$, we transform them using the Ansley-Kohn transform [1] in the following two steps, so that the causality condition is satisfied.

- **Partial autocorrelation matrix construction.** For $j = 1, \ldots, p$, find the Cholesky factorization $I + A_j A_j' = B_j B_j'$, then compute $P_j = B_j^{-1} A_j$ as partial autocorrelation matrices [1].

- **Causal VAR coefficient generation.** The partial autocorrelation matrices $\{P_j\}$ are mapped into the coefficient matrices $\{A_{sj}\}$ and $\{A_{sj}^*\}$ for forward and backward predictions using $s$ past/future values, with prediction error variance-covariance matrices $\Sigma_s$ and $\Sigma_s^*$ respectively. The answers for $s = p$ are used to calculate new $A_1, \ldots, A_p$.

  **Initialization:** Make $\Sigma_0 = \Sigma_0^* = I$, and $L_0 = L_0^* = I$, $I$ being the identity matrix.

  **Recursion:** For $s = 0, \ldots, p - 1$,

  - Compute
    
    $$A_{s+1,s+1} = L_s P_{s+1} (L_s')^{-1}, \quad A_{s+1,s+1}^* = L_s^* P_{s+1} L_s^{-1}. \quad (8)$$

  - For $i = 1, \ldots, s$ ($s > 0$), compute

    $$A_{s+1,i} = A_{si} - A_{s+1,s+1} A_{s,s-i+1}^*, \quad A_{s+1,i}^* = A_{si}^* - A_{s+1,s+1} A_{s,s-i+1}. \quad (9)$$

  - Compute

    $$\Sigma_{s+1} = \Sigma_s - A_{s+1,s+1} \Sigma_s^* A_{s+1,s+1}^*, \quad (10)$$

    $$\Sigma_{s+1}^* = \Sigma_s^* - A_{s+1,s+1}^* \Sigma_s (A_{s+1,s+1})'. \quad (11)$$

and obtain their Cholesky factorizations $L_{s+1} L_{s+1}'$ and $L_{s+1}^* L_{s+1}''$ respectively.
Causal VAR coefficients: Compute new

\[ A_i = (LL_p^{-1})A_p(LL_p^{-1})^{-1}, \quad i = 1, \ldots, p \]  

(12)

to use as casual VAR\((p)\) coefficient matrices.

2.5. The Gaussian log-likelihood

Given that the time series \(\{y_t\}\) has a Gaussian structure and AR\((p)\) dependence, the likelihood of \(y = (y'_1, \ldots, y'_T)'\) can be written as

\[
L(\theta; y) = f(y_1, \ldots, y_p) \prod_{t=p+1}^T f(y_t|y_{t-1}, \ldots, y_{t-p}),
\]

(13)

using the joint normal density \(f(y_1, \ldots, y_p)\) and the conditional Gaussian densities \(f(y_t|y_{t-1}, \ldots, y_{t-p})\), \(t = p + 1, \ldots, T\), where \(\theta\) consists of the parameters \(\psi_1 = \{\phi, W, b\}\) for trend generation and \(\psi_2\) for the VAR coefficient matrices \(A_1, \ldots, A_p\) and the \(\Sigma\) matrix. The log-likelihood is

\[
\ell(\theta; y) = -\frac{1}{2} \left[ n \log(2\pi) + \log|R_p| + (y_{1:p} - \mu_{1:p})'R_p^{-1}(y_{1:p} - \mu_{1:p}) + (T-p)\log|\Sigma| + \sum_{t=p+1}^T \varepsilon_t'\Sigma^{-1}\varepsilon_t \right],
\]

(14)

where \(n = mT\) \((m\) being the dimension of \(y_i)\), \(R_p\) is the variance-covariance matrix of \(y_{1:p} = (y'_1, \ldots, y'_p)'\) obtained using standard results \([17]\), \(\mu_{p+1}, \ldots, \mu_T\) from the output of the neural network and the VAR coefficient matrices \(A_1, \ldots, A_p\) are used for the recursive calculation of \(\varepsilon_{p+1}, \ldots, \varepsilon_T\) according to (3).

2.6. Network training

We employ the popular tool PyTorch \([20]\) for network training, after setting up its structure. The parameters of the neural network are updated by a modified gradient descent (GD) algorithm AdaGrad (Adaptive Gradient) \([3]\). The basic idea of gradient descent is to follow the opposite direction of the gradient of the loss function at the current point. Compared with conventional gradient descent algorithms, AdaGrad provides individual adaptive learning rates for different parameters. At iteration \(k\), the learning rate is modified by the diagonal elements of \(G = \sum_{\tau=1}^k g^{(\tau)}g^{(\tau)'}\), where \(g^{(\tau)}\) is the gradient of the loss function at iteration \(\tau\).

Details of the GD based training procedure are presented in Algorithm 1, where the initial values \(\psi_1^{(0)}\) for the trend part are obtained by minimizing the sum of squares of differences between \(y_t\) and \(\mu_t\) (nonlinear least squares), and \(\psi_2^{(0)} = \{A_1^{(0)}, \ldots, A_p^{(0)}, L^{(0)}\}\) by fitting \(\text{VAR}(p)\) to the detrended data using OLS and Cholesky factorization. The initial state \(h_0\) and the initial candidate memory cell \(c_0\) are both set to 0. The loss function then becomes minus the log-likelihood for network training. The trend parameters are fine-tuned with a smaller learning rate so that the trend terms get updated in small steps to avoid large changes that affect the estimation of the VAR parameters.
Algorithm 1 GD based network training for VAR with trend

**Input:**
- Time series observations \( y_1, \ldots, y_T \);
- Values of input to the network \( x_1, \ldots, x_T \);
- Learning rates \( \eta_1 \) and \( \eta_2 \);
- Number of iterations \( K \);
- Precision value \( \text{prec} \) for the stopping criteria.

**Output:**
- Optimal network parameter values \( (\psi_1^{(k)}, \psi_2^{(k)}) \).

1. Set \( k = 0, rc_1 = rc_2 = \text{prec} + 1 \).
2. **while** \( k \leq K \) and \( (rc_1 > \text{prec} \text{ or } rc_2 > \text{prec}) \) **do**  
   - Iterations for MLE/network optimization. 
   3. **for** \( t \leftarrow 1 \) to \( T \) **do** 
   4. Compute hidden state \( h_t^{(k)} = \text{LSTM}(h_{t-1}^{(k)}, x_t; \phi^{(k)}) \).
   5. Compute trend term \( \mu_t^{(k)} = W^{(k)}_{\mu} h_t^{(k)} + b^{(k)}_{\mu} \).
   6. **end for**
   7. Compute \( P_1^{(k)}, \ldots, P_p^{(k)} \) from \( A_1^{(k)}, \ldots, A_p^{(k)} \) using (7).
   8. Transform \( P_1^{(k)}, \ldots, P_p^{(k)} \) into new \( A_1^{(k)}, \ldots, A_p^{(k)} \) using (8) to (12).
   9. Compute \( \Sigma^{(k)} = L^{(k)}L^{(k)\prime} \).
   10. Evaluate loss function \( -\ell(\theta^{(k)}, y) \) at \( \theta^{(k)} = (\psi_1^{(k)}, \psi_2^{(k)}) \) using (14).
   11. Compute relative change of log-likelihood \( (k \geq 2) \):
       \[
       rc_1 = \left| \frac{\ell(\theta^{(k-1)}, y) - \ell(\theta^{(k-2)}, y)}{\ell(\theta^{(k-2)}, y)} \right|.
       \]
   12. Compute relative change of log-likelihood \( (k \geq 1) \):
       \[
       rc_2 = \left| \frac{\ell(\theta^{(k)}, y) - \ell(\theta^{(k-1)}, y)}{\ell(\theta^{(k-1)}, y)} \right|.
       \]
   13. Compute gradient of loss function 
       \[
       \mathbf{g}_1^{(k)} = \frac{\partial}{\partial \psi_1}( -\ell(\theta; y) )\big|_{\theta = \theta^{(k)}} \quad \mathbf{g}_2^{(k)} = \frac{\partial}{\partial \psi_2}( -\ell(\theta; y) )\big|_{\theta = \theta^{(k)}}.
       \]
   14. Compute \( G_1 = \sum_{t=0}^k \mathbf{g}_1^{(t)} \mathbf{g}_1^{(t)\prime}, G_2 = \sum_{t=0}^k \mathbf{g}_2^{(t)} \mathbf{g}_2^{(t)\prime} \).
   15. Update trend parameters \( \psi_1^{(k+1)} = \psi_1^{(k)} - \eta_1 \text{diag}(G_1)^{-\frac{1}{2}} \otimes \mathbf{g}_1^{(k)} \).
   16. Update VAR parameters \( \psi_2^{(k+1)} = \psi_2^{(k)} - \eta_2 \text{diag}(G_2)^{-\frac{1}{2}} \otimes \mathbf{g}_2^{(k)} \).
   17. \( k \leftarrow k + 1 \)** end while**

2.7. Prediction from trained network

We continue to run the trained network for \( t = T + 1, T + 2 \) etc to generate future trend values \( \mu_t \) and produce point forecasts using the formula (A9) in the Appendix.

Approximate 95% prediction intervals can be obtained by adding or subtracting 1.96 times the standard deviations of prediction errors using results in the Appendix.
3. Simulation study

To assess the finite sample performance of the deep learning based maximum likelihood estimation method, we simulated 100 samples each of size $T = 800$ from the semi-parametric VAR(2) model

$$y_t - \mu_t = A_1(y_{t-1} - \mu_{t-1}) + A_2(y_{t-2} - \mu_{t-2}) + \epsilon_t,$$

(15)

where

$$A_1 = \begin{pmatrix} -1.0842 & -0.1245 & 0.3137 \\ -0.7008 & -0.3754 & -0.2064 \\ 0.3166 & 0.3251 & 0.2135 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.5449 & -0.3052 & -0.1952 \\ -0.4057 & 0.5129 & 0.3655 \\ 0.0054 & -0.2911 & 0.2066 \end{pmatrix},$$

and

$$\Sigma = \begin{pmatrix} 0.4834 & -0.2707 & 0.1368 \\ -0.2707 & 0.4079 & -0.0221 \\ 0.1368 & -0.0221 & 0.4103 \end{pmatrix},$$

is the variance-covariance matrix of $\epsilon_t$. Values of the trend term $\mu_t$ were obtained by kernel smoothing from daily closing prices of three US stocks from 3rd October 2016 to 5th December 2019.

An example of the simulated multiple series is shown in Fig. 4, each having clearly a trend that looks more realistic than artificial functions.

![Figure 4. Simulated series from VAR(2) model with trend.](image)

We used an LSTM network with one hidden layer of 20 units. The input at time $t$ was $x_t = (t, t^2, t^3, 1/t, 1/t^2, 1/t^3)'$. The learning rates were $\eta_1 = 0.001$ and $\eta_2 = 0.01$, with $K = 600$ iterations and precision $\text{prec} = 10^{-5}$. The computation time for each set of parameter estimates was about 1 hour on an Intel Core i9 2.3 GHz processor with eight cores.
3.1. Simulation results

Following [4], we use mean absolute deviation

$$\text{MAD}_i = \frac{1}{3 \times 800} \sum_{k=1}^{3} \sum_{t=1}^{800} |\hat{\mu}_{kt}^{(i)} - \mu_{kt}|$$

to evaluate the accuracy of trend estimation in the $i$th simulation run, $i = 1, \ldots, 100$. Fig. 5 shows estimated trends from DeepVARwT (left panel) and VARwT (right panel) with MAD at the first quartile (short dashed, black), the median (dotted, red), and the third quartile (long dashed, black) respectively among the 100 simulation runs. The estimated trends from our model follow the true trends very closely while those from VARwT show over-smoothing of local changes.

![Series1](image1)

![Series2](image2)

![Series3](image3)

**Figure 5.** True (solid, blue) and estimated trends from DeepVARwT (left pane) and VARwT (right pane) with MAD at first quartile (short dashed, black), third quartile (long dashed, black), and median (dotted, red).
Table 1 reports summary statistics of 100 estimates of each parameter from our model and VARwT, where $a^{(i)}_{jk}$ refers to the $(j,k)$-th entry of the coefficient matrix $A_i$ and $\sigma_{jk}$ is the $(j,k)$-th entry of the variance-covariance matrix $\Sigma$. We can observe that compared with VARwT, the DeepVARwT model gives rise to reduced biases at the expense of standard deviations (SDs). The parameter estimates are more accurate with smaller mean squared errors (MSEs) than those obtained from the VARwT model.

Table 1. Estimation results of DeepVARwT and VARwT: true value above sample mean, standard deviation, mean squared error of 100 estimates of each parameter and sample bias.

|                | $a^{(1)}_{11}$ | $a^{(1)}_{12}$ | $a^{(1)}_{13}$ | $a^{(1)}_{21}$ | $a^{(1)}_{22}$ | $a^{(1)}_{23}$ | $a^{(1)}_{31}$ | $a^{(1)}_{32}$ | $a^{(1)}_{33}$ |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| **DeepVARwT**  |                |                |                |                |                |                |                |                |                |
| True value     | -1.0842        | -0.1245        | 0.3137         | -0.7008        | -0.3754        | -0.2064        | 0.3166         | 0.3251         | 0.2135         |
| Mean           | -1.0103        | -0.1199        | 0.2671         | -0.6802        | -0.3410        | -0.2240        | 0.2900         | 0.3601         | 0.2891         |
| Bias           | 0.0739         | 0.0046         | -0.0466        | 0.0206         | 0.0344         | -0.0176        | -0.0266        | 0.0353         | 0.0756         |
| SD             | 0.1776         | 0.0816         | 0.0861         | 0.0784         | 0.0813         | 0.0622         | 0.0421         | 0.0465         | 0.0945         |
| MSE            | 0.0367         | 0.0066         | 0.0095         | 0.0065         | 0.0077         | 0.0041         | 0.0025         | 0.0034         | 0.0146         |
| **VARwT**      |                |                |                |                |                |                |                |                |                |
| True value     | -0.5449        | -0.3052        | -0.1952        | -0.4057        | 0.5129         | 0.3635         | 0.0034         | -0.2911        | 0.2086         |
| Mean           | 0.7127         | -0.2808        | -0.1122        | 0.2335         | 0.8189         | 0.4848         | -0.1662        | -0.2880        | 0.3966         |
| Bias           | 0.2086         | 0.0645         | 0.0528         | 0.1039         | 0.0801         | 0.0460         | 0.0566         | 0.0368         | 0.0833         |
| SD             | 0.0499         | 0.0045         | 0.0034         | 0.0121         | 0.0073         | 0.0022         | 0.0041         | 0.0021         | 0.0108         |
| MSE            |                |                |                |                |                |                |                |                |                |
| $\sigma^{(2)}_{11}$ | -0.4674        | -0.3246        | -0.2213        | -0.3677        | 0.5431         | 0.3756         | -0.0245        | -0.2643        | 0.2692         |
| **DeepVARwT**  |                |                |                |                |                |                |                |                |                |
| True value     | 0.4834         | -0.2707        | 0.4079         | 0.1368         | -0.0221        | 0.4103         |                |                |                |
| Mean           | 0.5357         | -0.2652        | 0.4283         | 0.1175         | -0.0285        | 0.4347         |                |                |                |
| Bias           | 0.0523         | 0.0055         | 0.0204         | -0.0193        | -0.0064        | 0.0244         |                |                |                |
| SD             | 0.1321         | 0.0597         | 0.0575         | 0.0275         | 0.0196         | 0.0424         |                |                |                |
| MSE            | 0.0200         | 0.0036         | 0.0037         | 0.0011         | 0.0004         | 0.0024         |                |                |                |
| $\sigma^{(2)}_{21}$ | 1.2291         | -0.0039        | 0.6636         | 0.0714         | -0.0625        | 0.5114         |                |                |                |
| **VARwT**      |                |                |                |                |                |                |                |                |                |
| True value     | 1.2291         | -0.0039        | 0.6636         | 0.0714         | -0.0625        | 0.5114         |                |                |                |
| Mean           | 1.2291         | -0.0039        | 0.6636         | 0.0714         | -0.0625        | 0.5114         |                |                |                |
| Bias           | 0.7457         | 0.2668         | 0.2557         | -0.0654        | -0.0404        | 0.1011         |                |                |                |
| SD             | 0.0811         | 0.0267         | 0.0352         | 0.0312         | 0.0206         | 0.0265         |                |                |                |
| MSE            | 0.5626         | 0.0719         | 0.0666         | 0.0052         | 0.0020         | 0.0109         |                |                |                |
4. Real data applications

4.1. US macroeconomics series 1

For the US macroeconomic series (Fig. 1), we fit a model and make forecasts 20 times, each time using a training sample of size \( T = 166 \). The training samples are \( y_{i:T}^{(i)} = \{y_i, y_{i+1}, \ldots, y_{i+T-1}\} \), \( i = 1, \ldots, 20 \), and we forecast \( h = 1, 2, \ldots, 8 \) quarters ahead.

The order \( p = 4 \) for a VAR model is a common choice in the analysis of quarterly macroeconomic series, for example, [14], [15] and [10]. The first model we fitted was DeepVARwT(4). The number of input \( t \) functions and the hidden state size were the two most crucial hyperparameters. A grid search was conducted to find a set of values with maximum likelihood, among 2, 3 or 4 \( t \) functions and 5, 10 or 15 hidden states. For efficiency, we relied on our experience to set values for the other hyperparameters. The learning rates were \( \eta_1 = 0.0005 \) and \( \eta_2 = 0.01 \), with \( K = 500 \) iterations and precision \( \text{prec} = 10^{-7} \).

The estimated trends (red) are shown in Fig. 6 for the first training sample \( (i = 1) \), which can be seen to follow the observations (black) smoothly.

**Figure 6.** The first training sample (black lines) from 1955Q1 to 1996Q2 and the corresponding estimated trends (red lines).
The sample autocorrelations of residuals are shown in Fig. 7. The results are good for the GDP gap series, reasonable for the federal funds rate series, and a little concerning for the inflation series in terms of the number of values outside the boundaries.

![Figure 7. Sample autocorrelations of residuals.](image)

Fig. 8 contains normal QQ plots of the residuals. There is slight deviation from normality for all the series at both ends.

![Figure 8. Normal QQ plots of residuals.](image)

For comparison, we also fitted a VARwT(4) model, a DeepAR and a DeepState model using default hyperparameter values. The input at time $t$ was $x_t = (t, t^2, t^3, t^4, t^5, t^6, t^7, t^8, t^9)'$ for the VARwT model to account for the number of turning points in the trend. Table 2 gives a summary of these models and the software packages used.

| Model   | Description                                         | Available software                      |
|---------|-----------------------------------------------------|-----------------------------------------|
| VARwT   | Vector autoregressive model with trend [21]          | vars::VAR(exogen=x)                     |
| DeepAR  | Deep learning based autoregressive model [25]       | gluonts.DeepAREstimator()              |
| DeepState| Deep learning based state space model [23]          | gluonts.DeepStateEstimator()            |

To evaluate the accuracy of point forecasts, we computed the $h$-step-ahead Absolute Percentage Error averaged over 20 forecasts

$$APE(h) = \frac{1}{20} \sum_{i=1}^{20} \left| \frac{\hat{y}_{T+h}^{(i)} - y_{T+h}^{(i)}}{\hat{y}_{T+h}^{(i)}} \right| \times 100$$

for each component series $\{y_t^{(i)}\}$ in the $i$th training sample, where $\hat{y}_{T+h}^{(i)}$ is the $h$-step-ahead
All the experiments were conducted on an Intel Core i9 2.3 GHz processor with eight cores. The DeepV ARwT model outperformed the other models for federal funds rate. It gave the best accuracy. Its performance is similar for inflation with a slight drop to second place in terms of prediction intervals for GDP gap over \( h = 1:4 \) and \( h = 1:8 \), while in second place for prediction intervals at all the forecasting horizons for all the series (except \( h = 1 \) for GDP gap and \( h = 1, 2 \) for inflation). Our model also gave more accurate prediction intervals in the long term (\( h = 4, 8 \)) and overall (\( h = 1:4 \) and \( h = 1:6 \)) for all the series.

- **DeepVARwT vs VARwT.** Compared with VARwT, DeepVARwT produced superior point forecasts at almost all the forecasting horizons for all the series (except \( h = 1 \) for GDP gap and \( h = 1, 2 \) for inflation). Our model also gave more accurate prediction intervals in the long term (\( h = 4, 8 \)) and overall (\( h = 1:4 \) and \( h = 1:6 \)) for all the series.

- **DeepVARwT vs other deep learning based models.** Compared with DeepAR and DeepState, our model resulted in better point forecasts and prediction intervals at all the forecasting horizons for federal funds rate. It also gave more precise prediction intervals at all the forecasting horizons for GDP gap.

### Table 3. Performance of DeepVARwT against other models according to APE and SIS.

|                | GDP gap         | Inflation | Federal funds rate |
|----------------|-----------------|-----------|--------------------|
|                | Absolute Percentage Error | Scaled Interval Score |                        |
| \( h = 1 \)    | \( h = 2 \)    | \( h = 4 \) | \( h = 8 \) | \( h = 1 \)    | \( h = 2 \)    | \( h = 4 \) | \( h = 8 \) |                        |
| VARwT          | 665.927        | 2982.609  | 293.199            | 1124.228          | 1042.088       | 897.853     | 1.592               | 4.114               | 34.332       | 244.788      | 13.157      | 81.907      |
| DeepAR         | 333.063        | 389.993   | 173.499            | 260.860           | 258.932        | 240.882     | 8.280               | 15.304              | 35.105      | 65.816       | 21.020      | 39.587      |
| DeepState      | 1023.302       | 1070.784  | 178.982            | 200.245           | 612.331        | 403.614     | 7.903               | 16.995              | 22.545      | 34.583       | 16.800      | 24.950      |
| DeepVARwT      | 671.569        | 877.267   | 162.329            | 202.388           | 466.350        | 326.924     | 8.487               | 8.961               | 14.509      | 29.158       | 10.201      | 17.008      |

The DeepVARwT model outperformed the other models for federal funds rate. It gave the best prediction intervals for GDP gap over \( h = 1:4 \) and \( h = 1:8 \), while in second place for prediction accuracy. Its performance is similar for inflation with a slight drop to second place in terms of SIS over \( h = 1:4 \).

All the experiments were conducted on an Intel Core i9 2.3 GHz processor with eight cores. The number of weight parameters in the LSTM network for the 20 fitted DeepVARwT models ranged from 715 to 1350. The computation time of 20 predictions from VARwT, DeepAR, DeepState and DeepVARwT was approximately 1, 42, 210, and 55 minutes, respectively.

|                | Absolute Percentage Error | Scaled Interval Score |
|----------------|-----------------|----------------------|
| \( h = 1 \)    | \( h = 2 \)    | \( h = 4 \) | \( h = 8 \) | \( h = 1 \)    | \( h = 2 \)    | \( h = 4 \) | \( h = 8 \) |
| VARwT          | 10.521          | 25.868               | 90.407             | 424.838          | 44.681        | 157.662     | 2.226           | 7.145               | 31.160     | 137.790      | 14.374      | 53.498      |
| DeepAR         | 9.685           | 19.915               | 55.689             | 119.189           | 30.633        | 62.829      | 4.759           | 11.914              | 33.627     | 53.807       | 18.122      | 32.993      |
| DeepState      | 33.190          | 36.782               | 59.985             | 128.743           | 44.186        | 70.322      | 6.962           | 10.260              | 13.660     | 22.901       | 10.862      | 15.458      |
| DeepVARwT      | 7.834           | 14.032               | 33.676             | 80.449            | 19.659        | 39.916      | 1.473           | 3.946               | 8.095      | 17.381       | 5.262       | 9.541       |

The Scaled Interval Score [6] is averaged as follows:

\[
SIS(h) = \frac{1}{20} \sum_{i=1}^{20} \left( \frac{1}{\alpha} (y_{T+h}-\hat{y}_i^{(i)}(T+h)) + \frac{1}{\alpha} (\hat{y}_i^{(i)}(T+h)-y_{T+h}) \right) + \frac{2}{\alpha} (\hat{y}_i^{(i)}(T+h)-y_{T+h}) \mathbb{1}_{\hat{y}_i^{(i)}(T+h)<y_{T+h}} + \frac{2}{\alpha} (y_{T+h}-\hat{y}_i^{(i)}(T+h)) \mathbb{1}_{\hat{y}_i^{(i)}(T+h)>y_{T+h}},
\]

to measure the overall accuracy of the \((1 - \alpha) \times 100\%\) prediction intervals \((l_i^{(i)}(T+h), u_i^{(i)}(T+h))\) for the \(i\)th training sample, \(i = 1, \ldots, 20\), where \(\mathbb{1}_A\) is the indicator function for the condition \(A\), \(s\) is the seasonality of the time series (\(s = 4\) for quarterly data).

Table 5 shows the forecasting performances of different models at several horizons \( h = 1, 2, 4, 8 \) and averages over \( h = 1, \ldots, 4 \) and \( h = 1, \ldots, 8 \). When a model performs best, the corresponding number in the table will be bold.

The DeepVARwT model outperformed the other models for federal funds rate. It gave the best prediction intervals for GDP gap over \( h = 1:4 \) and \( h = 1:8 \), while in second place for prediction accuracy. Its performance is similar for inflation with a slight drop to second place in terms of SIS over \( h = 1:4 \).
4.2. Global temperatures

Global warming has attracted significant attention in recent research, as demonstrated by studies such as [8], [12], and [11]. Fig. 9 shows three annual temperature anomaly series from distinct regions: the Northern Hemisphere, the Southern Hemisphere and the Tropics from 1850 to 2021, which are described in detail in [19]. The data are temperature anomalies relative to a reference period of 1961-1990 [19]. Each series consists of 172 yearly observations.

From Fig. 9, we can observe obvious trends in the three series. [11] assumed that the trends in the Northern and Southern Hemispheres series are deterministic and modelled the local changes in data using a vector shifting-mean autoregressive model with order $p = 3$. We continue to fit a DeepVARwT(3) model to the three series and make predictions $h = 1, 2, ..., 6$ steps ahead of $T = 147$. As with our first real data application, this is repeated 19 times, each time moving the training sample forward by one time point. The search ranges for the number of $t$ functions and hidden state size were $2, 3, 4$ and $3, 5, 8$, respectively. The learning rates were $\eta_1 = 0.0005$ and $\eta_2 = 0.01$, with $K = 500$ iterations and precision $\text{prec} = 10^{-7}$.

The forecasts will be compared with those from VARwT(3), DeepAR, and DeepState models with default hyperparameters. The exogenous variables for VARwT are $x_t = (t, t^2, t^3, t^4, t^5)'$ to account for the number of turning points in the series.

From Fig. 10, we can see that the estimated trends (red) for the first training sample ($i = 1$) follow the observations (black) smoothly.

![Temperature anomaly series for the Northern Hemisphere, the Southern Hemisphere and the Tropics from 1850 to 2021.](image-url)
The first training sample (black lines) from 1850 to 1996 and the corresponding estimated trends (red lines).

The sample autocorrelations of residuals are shown in Fig. 11. The results are very good for all the series with all the values within boundaries.

Fig. 12 contains normal QQ plots of the residuals. The results are very good for all the series showing clearly straight line patterns.
Table 4 shows the APE and SIS values of different models at several horizons $h = 1, 2, 4, 6$ and averaged over $h = 1 : 3$ and $h = 1 : 6$.

- **DeepVARwT vs VARwT.** Compared with the time-invariant VAR with trend, our model produced better point forecasts at all forecasting horizons for all the series. It gave better prediction intervals in the long term ($h = 4, 6$) and overall ($h = 1 : 6$) for all the series.

- **DeepTVARwT vs other deep learning based models.** Compared with DeepAR and DeepState, our model produced more accurate point forecasts at almost all forecasting horizons for all the series (except $h = 4$ for Tropics). Our model resulted in better prediction intervals at all forecasting horizons for all the series.

Overall, the DeepVARwT model gave better forecasts and prediction intervals than other models, especially for the Northern and Southern Hemisphere series.

The number of weight parameters in the network for the 20 fitted DeepVARwT models varied between 444 and 508. The computation time for generating 20 predictions using VARwT, DeepAR, DeepState and DeepVARwT was about 1, 45, 80, and 30 minutes, respectively.
4.3. *US macroeconomics series 2*

We continue to apply our model to another set of US macroeconomic data (Fig.13) including inflation rate (year-over-year log growth rate of the GDP price index), unemployment rate and treasury interest rate from 1953Q1 to 2001Q3, as analysed by [22]. The inflation rate differs from the first real data example where it is defined as “the percentage change in the GDP, chain-weighted price index at annual rate” [13]. Each series consists of 195 observations and exhibits a clear trend. We fitted a DeepVARwT(4) model to these series and forecast \( h = 1, 2, \ldots, 8 \) steps ahead of \( T = 168 \). Consistent with our previous real data applications, we repeated 19 times, each time moving the training sample forward by one time point.

The search ranges for the number of \( t \) functions and hidden state size were 2, 3, 4 and 10, 12, 15, respectively. We employed the learning rates \( \eta_1 = 0.0005 \) and \( \eta_2 = 0.01 \), with \( K = 500 \) iterations and precision \( prec = 10^{-7} \). The number of weight parameters in the network for the 20 fitted DeepVARwT models varied between 635 and 2,185.

The forecasts will be compared with those from a VARwT(4) model using \( x_t = (t, t^2, t^3, t^4, t^5, t^6, t^7, t^8, t^9)^\prime \) to account for the number of turning points in the series, DeepAR, [25], and DeepState models with default hyperparameters. The computation time to generate 20 predictions using VARwT, DeepAR, DeepState and DeepVARwT was about 1, 45, 80, and 35 minutes, respectively.

![Inflation rate](image1)

![Unemployment rate](image2)

![Treasury bill interest rate](image3)

*Figure 13.* Inflation rate, unemployment rate and treasury bill interest rate for the US from 1953Q1 to 2001Q3

From Fig. 14, we can see that the estimated trends (red) for the first training sample \((i = 1)\)
follow the observations (black) smoothly.

The sample autocorrelations of residuals are shown in Fig. 15. The results are reasonably good for the inflation rate and treasury bill interest series, and a little concerning for the unemployment rate series in terms of the number of values outside the boundaries.

Fig. 12 contains normal QQ plots of the residuals. There is some deviation from normality for all the series at both ends.
Table 4 shows the APE and SIS values of different models at several horizons $h = 1, 2, 4, 8$ and averaged over $h = 1:4$ and $h = 1:8$.

- **DeepVARwT vs VARwT.** Compared with the time-invariant VAR with trend, DeepT-VARwT produced better point forecasts at almost all forecasting horizons for all the series (except $h = 1$ for inflation rate and $h = 1, 2$ for treasury bill interest rate). It also gave more accurate prediction intervals at almost all forecasting horizons for all the series (except $h = 1, 2$ for inflation rate).

- **DeepTVARwT vs other deep learning based models.** Compared with DeepAR and DeepState, our model produced more accurate point forecasts at almost all forecasting horizons for unemployment rate (except $h = 1$) and treasury bill interest rate (except $h = 1, 2$). Our model resulted in better prediction intervals at all forecasting horizons for unemployment rate and treasury bill interest rate.

Table 5. Performance of DeepVARwT against other models according to APE and SIS.

|                  | Absolute Percentage Error | Scaled Interval Score |
|------------------|---------------------------|-----------------------|
| Inflation rate   |                           |                       |
| $h=1$            | $h=2$         | $h=4$         | $h=8$         | $h=1:4$       | $h=1:8$       |
| VARwT            | 9.961         | 24.991        | 76.114        | 259.015       | 39.129        | 111.141       |
| DeepAR           | 7.763         | 15.281        | 28.017        | 52.398        | 18.497        | 31.416        |
| DeepState        | 18.373        | 22.375        | 29.036        | 40.723        | 24.191        | 31.294        |
| DeepVARwT        | 13.707        | 24.019        | 34.797        | 34.283        | 25.646        | 29.747        |
| Unemployment rate|                           |                       |
| $h=1$            | $h=2$         | $h=4$         | $h=8$         | $h=1:4$       | $h=1:8$       |
| VARwT            | 3.514         | 8.960         | 25.229        | 93.339        | 13.425        | 38.390        |
| DeepAR           | 3.250         | 6.524         | 13.014        | 31.405        | 8.099         | 16.310        |
| DeepState        | 15.144        | 15.941        | 19.629        | 25.726        | 17.515        | 20.596        |
| DeepVARwT        | 3.447         | 5.845         | 9.833         | 17.266        | 6.788         | 10.656        |
| Treasury bill interest rate |                       |                       |
| $h=1$            | $h=2$         | $h=4$         | $h=8$         | $h=1:4$       | $h=1:8$       |
| VARwT            | 5.390         | 10.537        | 20.041        | 75.861        | 12.581        | 30.385        |
| DeepAR           | 4.868         | 10.073        | 18.693        | 24.993        | 12.340        | 18.016        |
| DeepState        | 13.149        | 13.452        | 14.818        | 22.711        | 13.881        | 16.394        |
| DeepVARwT        | 7.024         | 11.045        | 11.688        | 16.699        | 10.130        | 12.626        |

Over $h = 1:4$ and $h = 1:8$, the DeepVARwT model outperformed other models in providing better forecasts and prediction intervals, except for the inflation rate over $h=1:4$ where DeepAR did better.

The number of weight parameters in the network for the 20 fitted DeepVARwT models varied between 849 and 1350. The computation time for generating 20 predictions using VARwT, DeepAR, DeepState and DeepVARwT was about 1, 45, 80, and 59 minutes, respectively.
5. Summary and further discussion

In this work, we proposed a new approach to VAR modeling and forecasting by generating trends as well as model parameters using an LSTM network and the associated deep learning methodology for exact maximum likelihood estimation. A simulation study demonstrated the effectiveness of the proposed approach. Three examples with real data are provided to show that it competes well with existing models in terms of prediction performance.

Default values of the hyper-parameters for the DeepAR and DeepState models were used, which worked reasonably well but can be tweaked for better performance. The python code and data to reproduce forecasting results is available at https://github.com/lixixibj/DeepVARwT-data-code.

The computation becomes more challenging as the number/length of the component series increases. With high dimensional time series, a potential avenue for future research involves incorporating regularization and low-rank structure into the model fitting. One approach is to impose a low-rank assumption on $A_1, \ldots, A_p$ combined into a single matrix. This enables a reduction along one specific direction [27]. Building on this concept, [27] further rearranged $A_i, i = 1, \ldots, p$ into a tensor to reduce the dimension along three directions, allowing each direction to have a different low-rank structure. Incorporating a tensor structure into our DeepVARwT model could be a potential future direction.

The model that has been explored so far rely on the assumption of Gaussianity. However, in practical applications such as demand forecasting, series may exhibit sporadic occurrences with periods of no activity at all. This intermittent behaviour of demand calls for the relaxation of the Gaussian assumption to accommodate discrete data. It might be possible to generalise the digitised Gaussian ARMA model of [16] to the multivariate case.

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Appendix A. Prediction error variances and covariances

First consider the model for \( \{y_t\} \) to be VAR(1) with trend:

\[
y_t - \mu_t = A(y_{t-1} - \mu_{t-1}) + \epsilon_t,
\]

where \( \{\epsilon_t\} \) is white noise, \( \epsilon_t \sim N(0, \Sigma) \) and \( \epsilon_t \) is uncorrelated with \( y_{t-1}, y_{t-2}, \ldots \).

Then, we can decompose \( y_{T+\ell} \) starting with \( y_T \) for \( \ell = 1, \ldots, h \):

\[
\begin{align*}
y_{T+1} - \mu_{T+1} &= A(y_T - \mu_T) + \epsilon_{T+1}, \\
y_{T+2} - \mu_{T+2} &= A(y_{T+1} - \mu_{T+1}) + \epsilon_{T+2} = A^2(y_T - \mu_T) + A\epsilon_{T+1} + \epsilon_{T+2}, \\
y_{T+3} - \mu_{T+3} &= A^3(y_T - \mu_T) + A^2\epsilon_{T+1} + A\epsilon_{T+2} + \epsilon_{T+3}, \\
&\vdots \\
y_{T+h} - \mu_{T+h} &= A^h(y_T - \mu_T) + \sum_{i=0}^{h-1} A^i \epsilon_{T+h-i},
\end{align*}
\]

where \( A^i \) is understood to be the identity matrix when \( i = 0 \).

From (A2), the best linear predictor for \( y_{T+\ell} \) given \( y_T, y_{T-1}, \ldots \) is

\[
\hat{y}_{T+\ell} = E[y_{T+\ell} | y_T, y_{T-1}, \ldots] = A^\ell(y_T - \mu_T) + \mu_{T+\ell},
\]

and the associated prediction error variance-covariance matrix is

\[
\text{Var}(y_{T+\ell} - \hat{y}_{T+\ell}) = \sum_{i=0}^{\ell-1} A^i \Sigma (A^i)',
\]

When \( \{y_t\} \) follows the VAR(\( p \)) model (3) with trend, we use its VAR(1) form

\[
y_t^* - \mu_t^* = A^*(y_{t-1}^* - \mu_{t-1}^*) + \epsilon_t^*,
\]

where \( y_t^* = (y_t', y_{t-1}', \ldots, y_{t-p+1}')' \), \( \mu_t^* = (\mu_t', \mu_{t-1}', \ldots, \mu_{t-p+1}')' \),

\[
A^* = \begin{bmatrix}
A_1 & A_2 & \cdots & \cdots & A_p \\
I & O & \cdots & \cdots & O \\
o & I & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
o & \cdots & O & I & O
\end{bmatrix},
\]

and

\[
y_t = [I, O, \cdots, O]y_t^*.
\]
The variance-covariance matrix of \( \varepsilon_t^* = (\varepsilon_t', 0', ..., 0')' \) is

\[
\Sigma^* = \begin{bmatrix}
\Sigma & \mathbf{0} \\
\mathbf{0} & \mathbf{0}
\end{bmatrix}.
\]

(A8)

Using (A3), the best linear predictor for \( y_{t+h}^* \) given \( y_t^*, y_{t-1}^*, ... \) is

\[
\hat{y}_{t+h}^* = (A^*)^h(y_t^* - \mu_t^*) + \mu_{t+h}^*.
\]

(A9)

Using (A4), the variance-covariance matrix of the prediction error for \( \hat{y}_{t+h}^* \) is

\[
\sum_{i=0}^{h-1} (A^*)^i \Sigma^*( (A^*)^i)'.
\]

(A10)

The prediction for \( y_{t+h} \) can be extracted from that for \( y_{t+h}^* \). The prediction error variance-covariance matrix for \( \hat{y}_{t+h} \) is in the top-left corner of the above.