Gröbner Basis Techniques for Computing Actions of K-Categories *

Anne Heyworth†
School of Informatics,
University of Wales, Bangor
Gwynedd, LL57 1UT, U.K.
a.l.heyworth@bangor.ac.uk

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Abstract

This paper involves categories and computer science. Gröbner basis theory is a branch of computer algebra which has been usefully applied to a wide range of problems. Kan extensions are a key concept of category theory capable of expressing most algebraic structures. The paper combines the two, using Gröbner basis techniques to compute certain kinds of Kan extension.

1 Introduction

The paper is motivated by a question which arises from two pieces of research. Firstly, the work of Brown and Heyworth [2] which extends rewriting techniques to enable the computation of left Kan extensions over the category of sets. It is well known that left Kan extensions can be defined over categories other than 

Sets. Secondly, the ‘folklore’, made explicit in [1] that rewriting theory is a special case of noncommutative Gröbner basis theory. It is therefore natural to ask whether Gröbner bases can provide a method for computing Kan extensions beyond the special case of rewriting.

To answer this question completely, fully exploiting the computational power of Gröbner basis techniques relating to Kan extensions is the ultimate aim. This paper provides a first step by showing how standard noncommutative Gröbner basis procedures can be used to calculate left Kan extensions of K-category actions. In the final section of the paper a number of interesting problems arising from the work are identified.

2 Background

This paper builds on work of Brown and Heyworth [2] on extensions of rewriting methods.

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The standard expression of rewriting is in terms of words \( w \) in a free monoid \( \Delta^* \) on a set \( \Delta \). This may be extended to terms \( x|w \) where \( x \) belongs to a set \( X \) and the link between \( x \) and \( w \) is in terms of an action. More precisely, we suppose a monoid \( A \) acts on the set \( X \) on the right, and there is given a morphism of monoids \( F: A \to B \) where \( B \) is given by a presentation with generating set \( \Delta \). The result of the rewriting will then be normal forms for the induced action of \( B \) on \( F_*(X) \). This gives an important extension of rewrite methods.

In fact monoids may be replaced by categories, and sets by directed graphs. This gives a formulation in terms of left Kan extensions, or induced actions of categories, which is explained in [2]. Further, categories can be replaced by \( K \)-categories, as will be described later.

Let \( A \) be a category. A category action \( X \) of \( A \) is a functor \( X : A \to Sets \). Let \( B \) be a second category and let \( F: A \to B \) be a functor. Then an extension of the action \( X \) along \( F \) is a pair \((E, \varepsilon)\) where \( E : B \to Sets \) is a functor and \( \varepsilon : X \to E \circ F \) is a natural transformation. The left Kan extension of the action \( X \) along \( F \) is an extension of the action \((E, \varepsilon)\) with the universal property that for any other extension of the action \((E', \varepsilon')\) there exists a unique natural transformation \( \alpha : E \to E' \) such that \( \varepsilon' = \alpha \circ \varepsilon \).

The problem that has been introduced is that of “computing a Kan extension”. In order to keep the analogy with computation and rewriting for presentations of monoids we propose a definition of a left Kan extension. The papers [1, 3, 9, 11] were very influential on our choices.

A left Kan extension data \((X', F')\) consists of small categories \( A, B \) and functors \( X' : A \to Sets \) and \( F' : A \to B \). A left Kan extension presentation is a quintuple \( P := kan(\Gamma|\Delta|RelB|X|F) \) where \( \Gamma \) and \( \Delta \) are (directed) graphs; \( X : \Gamma \to Sets \) and \( F : \Gamma \to P\Delta \) are graph morphisms to the category of sets and the free category on \( \Delta \) respectively; and \( RelB \) is a set of relations on the free category \( P\Delta \). Formally, we say \( P \) presents the left Kan extension \((E, \varepsilon)\) of the left Kan extension data \((X', F')\) where \( X' : A \to Sets \) and \( F' : A \to B \) if \( \Gamma \) is a generating graph for \( A \) and \( X : \Gamma \to Sets \) is the restriction of \( X' : A \to Sets \); \( cat(\Delta|RelB) \) is a category presentation for \( B \) and \( F : \Gamma \to P\Delta \) induces \( F' : A \to B \).

We expect that a left Kan extension \((E, \varepsilon)\) is given by a set \( EB \) for each \( B \in Ob\Delta \) and a function \( Eb : EB_1 \to EB_2 \) for each \( b : B_1 \to B_2 \in B \) (defining the functor \( E \)) together with a function \( \varepsilon_A : XA \to EFA \) for each \( A \in ObA \) (the natural transformation).

The main result of [3] defines rewriting procedures on \( T := \bigsqcup_{B \in Ob\Delta} \bigsqcup_{A \in ObA} XA \times P\Delta(FA, B) \) which is basically a set with a partial right action of the arrows of \( P\Delta \).

Two kinds of rewriting are involved here. The first is the familiar \( x|uv \to x|urv \) given by a relation \((l, r)\) – these rules are known as the ‘E-rules’. The second derives from a given action of certain words on elements, so allowing rewriting \( x|F(a)v \to x \cdot a|v \) – these rules are known as the ‘\( \varepsilon \)-rules’. Further, the elements \( x \) and \( x \cdot a \) may belong to different sets. When such rewriting procedures complete, the associated normal form gives in effect a computation of what we call the Kan extension defined by the presentation.
The ‘folklore’ of the relation of rewriting and Gröbner basis techniques, alluded to in [14] and [17] is made explicit in [1].

The polynomial ring $K[X^*]$ consists of all polynomials having coefficients in the field $K$ and terms from $X^*$ together with the usual operations of polynomial addition and (noncommutative) multiplication. Given a generating set for an ideal $I$ in this ring it is a problem to determine whether two given polynomials $f$ and $g$ are equivalent modulo the ideal i.e. whether they occur within the same congruence class. If a Gröbner basis $G$ can be constructed for $I$ from the original generating set then the congruence problem can be solved. The Gröbner basis calculation depends on a well-ordering of $X^*$ and a definition of polynomial reduction, which is determined by comparing leading terms. In the noncommutative case it is not always successful.

The key observation is that the rewriting techniques used in calculating a monoid $M$ from a set of generators $X$ and a rewrite system $R$ compatible with an ordering $>$ corresponds step-by-step to the Gröbner basis techniques used in calculating the congruence classes of the polynomial ring $K[X^*]$ with respect to the ideal generated by the difference binomials $l - r$ for $(l, r)$ in $R$.

This provides the background to our problem of determining whether Gröbner bases can be used to calculate Kan extensions other than in the special case of rewriting systems. The first observation is that Gröbner bases involve polynomials, so we should examine how the addition operation is represented in categories.

## 3 K-Category Actions

We use the definitions of [13]. Let $K$ be a field. A $K$-category is a category whose hom-sets (a hom-set is the set of all morphisms between a given pair of objects) are $K$-modules. A morphism of $K$-categories or $K$-functor $F$ preserves the $K$-module structure of the hom-sets so $F(a + b) = F(a) + F(b)$, $F(ka) = kF(a)$ for all arrows $a, b$ such that $a + b$ is defined and scalars $k$ in $K$.

The free $K$-category on the graph $\Delta$ is the category $P_K\Delta$ whose objects are the objects of $\Delta$ and whose arrows $Arr P_K\Delta$ are all polynomials of the form $p = k_1 m_1 + \cdots + k_n m_n$ where $k_1, \ldots, k_n \in K$ and $m_1, \ldots, m_n \in P\Delta(B_1, B_2)$ for some $B_1, B_2 \in Ob\Delta$. We will refer to $m_1, \ldots, m_n$ as the terms which occur in $f$. Note that functions $src$ and $tgt$ are well-defined as $src(f) := src(m_1) = \cdots = src(m_n)$ and $tgt(f) := tgt(m_1) = \cdots = tgt(m_n)$.

The relations of a $K$-category can be of the form $p = q$ where both sides have the same source and target. Therefore $R$ will be assumed to be a set of polynomials $p - q$ i.e. a subset of $Arr P_K\Delta$. If $R = \{r_1, \ldots, r_n\}$ is such a set of relations on $P_K\Delta$ then the congruence generated by $R$ is defined as follows:

$$f =_R h \text{ if and only if } f = h + k_1 p_1 r_1 q_1 + \cdots + k_n p_n r_n q_n$$

for some $k_1, \ldots, k_n \in K$ and $p_1, \ldots, p_n, q_1, \ldots, q_n \in Arr P_K\Delta$ where $src(f) = src(h) = src(u_1) = \cdots = src(u_n)$ and $tgt(f) = tgt(h) = tgt(v_1) = \cdots = tgt(v_n)$ and $u_1 r_1 v_1, \ldots, u_n r_n v_n$ are defined in
Arr\(P_K\Delta\). The \(K\)-category \(P_K\Delta / \cong_R\) whose elements are the congruence classes of \(ArrP_K\Delta\) with respect to \(F\) is known as the \textit{factor} \(K\)-category.

**Definition 3.1** Let \(K\) be a field. A \textit{\(K\)-category presentation} is a pair \(\text{cat}_K(\Delta|R)\) where \(\Delta\) is a graph and \(R \subseteq ArrP_K\Delta\). The \(K\)-category it presents is the factor category \(P_K\Delta / \cong_R\).

Our first result enables the use of Buchberger’s algorithm to compute Gröbner bases which enable the specification of the morphisms of a \(K\)-category presented in this way.

Let \(>\) be an admissible well-ordering on \(ArrP\Delta\) i.e. \(>\) is Noetherian and compatible with the operation of path concatenation. Define the \textit{leading term} of a polynomial \(f\) to be the term occurring in \(f\) which is the greatest path in \(\Delta\) with respect to \(>\) and denote it \(LT(f)\). Define a \textit{reduction relation} \(\rightarrow_R\) on \(ArrP_K\Delta\) by \(f \rightarrow f − k_iu_i r_i v_i\) when \(u_i(LT(r_i))v_i\) occurs in \(f\) with coefficient \(k_i \in K\) for \(u_i, v_i \in ArrP\Delta\), \(r_i \in R\). The reflexive, symmetric and transitive closure of \(\rightarrow_R\) is denoted \(\leftrightarrow_R\). If the reduction relation \(\rightarrow_R\) is complete (i.e. Noetherian and confluent) then we say that \(R\) is a \textit{Gröbner basis}.

**Lemma 3.2**

\[
\frac{Arr_PK\Delta}{\cong_R} \cong_{\leftrightarrow_R} \frac{ArrP_K\Delta}{\leftrightarrow_R}
\]

**Proof** It is clear from the definitions that the equivalence relation \(\leftrightarrow_R\) is contained in \(=_R\).

For the converse, suppose \(f =_R h\). Then there exist \(r_1, \ldots, r_n \in R\) and \(p_1, \ldots, p_n, q_1, \ldots, q_n \in P_K\Delta\), such that \(f = h + p_1 r_1 q_1 + \cdots + p_n r_n q_n\). By splitting \(p_i\) and \(q_i\) into their component terms for \(i = 1, \ldots, n\) we obtain \(f = h + k_1 u_1 r_1 v_1 + \cdots + k_j u_j r_j v_j + \cdots + k_t u_t r_t v_t\) for some \(k_1, \ldots, k_t \in K\), \(u_1, \ldots, u_t, v_1, \ldots, v_t \in P\Delta\). It follows immediately from this that \(f \leftrightarrow_R h\). \(\Box\)

**Proposition 3.3** The relation \(\rightarrow_R\) is Noetherian on \(ArrP_K\Delta\).

The \textit{matches} of \(R\) are the pairs of polynomials \((r_1, r_2)\) whose leading terms overlap on some subword i.e. \(uLT(r_1)v = LT(r_2)\) or \(LT(r_2) = uLT(r_2)v\) or \(uLT(r_1) = LT(r_2)v\) or \(LT(r_1)v = uLT(r_2)\) for some \(u, v \in ArrP\Delta\). If there is a match between \(r_1\) and \(r_2\) we may write \(u_1LT(r_1)v_1 = u_2LT(r_2)v_2\) for some \(u_1, v_1 \in ArrP\Delta\). The \textit{S-polynomial} resulting from a match is then the difference \(u_1 r_1 v_1 − u_2 r_2 v_2\). The set of S-polynomials of a finite set of polynomials is finite and can be computed.

**Lemma 3.4** If all S-polynomials resulting from matches of \(R\) reduce to zero by \(\rightarrow_R\) then \(\rightarrow_R\) is confluent on \(ArrP_K\Delta\).

**Outline Proof** Observing that \(ArrP_K\Delta\) is a subset of the free \(K\)-algebra on \((ArrP\Delta)^*\) we can deduce that the relation \(\rightarrow_R\) is confluent on the free \(K\)-algebra. The fact that \(\rightarrow_R\) preserves source and target enables us to deduce that \(\rightarrow_R\) cannot reduce an element of \(ArrP_K\Delta\) to anything not defined in \(ArrP_K\Delta\). Thus \(\rightarrow_R\) is confluent on \(ArrP_K\Delta\). \(\Box\)

Buchberger’s algorithm calculates the S-polynomials of a system \(R\) and attempts to reduce them to zero by \(\rightarrow_R\). If an S-polynomial cannot be reduced it is added to the system. The S-polynomials of the modified system \(R'\) are then computed – the process looping until a system is found whose S-polynomials can all be reduced to zero.
Theorem 3.5 (Buchberger’s Algorithm and $K$-category Presentations) If it terminates, then Buchberger’s algorithm applied to $(R, >)$, will return a Gröbner basis for $=_R$ on $\text{Arr}_P\Delta$.

Proof All that remains to be verified is that S-polynomials resulting from matches found in $\text{rem}_R$. We assume all polynomials in $R$ to be monic (possible since $K$ is a field). Now S-polynomials result from two types of overlap.

For the first case let $r_1, r_2$ be polynomials in $R$ such that $u\text{LT}(r_1) = \text{LT}(r_2)v$ for some $u, v \in \text{Arr}_P\Delta$. Then the S-polynomial is $s := \text{rem}(r_2)v - u\text{rem}(r_1) \in \text{Arr}_P\Delta$ where $\text{rem}(r_i) := r_i - \text{LT}(r_i)$ for $i = 1, 2$. Now $\text{rem}(r_2)v - u\text{rem}(r_1) = ur_1 - r_2v$ therefore $s = \text{rem}(r_2)v - u\text{rem}(r_1) = _R 0$, and hence the congruence generated by $R' := R \cup \{s\}$ coincides with $=_R$.

For the second case let $r_1, r_2$ be polynomials in $R$ such that $u\text{LT}(r_1)v = \text{LT}(r_2)$ for some $u, v \in \text{Arr}_P\Delta$. Then the S-polynomial is $s := \text{rem}(r_2) - u\text{rem}(r_1)v \in \text{Arr}_P\Delta$. Now $\text{rem}(r_2) - u\text{rem}(r_1)v = ur_1v - r_2v$ therefore $s = \text{rem}(r_2) - u(r_1)v = _R 0$, and hence the congruence generated by $R' := R \cup \{s\}$ coincides with $=_R$.

An example of an application of the results proved above can be found in Section 5.

4 Left Kan Extensions

We obtain a further result by expressing the presentation of a noncommutative polynomial algebra as a problem of computing a left Kan extension over framed modules $\text{K Mods}$ (modules over a fixed field).

Definition 4.1 A left Kan extension data for $K$-categories $(M', F')$ consists of small categories $A, B$ and functors $M' : A \to \text{K Mods}$ and $F' : A \to B$. A left Kan extension presentation for $K$-categories is a quintuple $\mathcal{P} := \text{kan}(\Gamma|\Delta|\text{Rel}B|M|F')$ where

i) $\Gamma$ and $\Delta$ are (directed) graphs;

ii) $M : \Gamma \to \text{K Mods}$ and $F : \Gamma \to P_K\Delta$ are graph morphisms to the category of $K$-modules and the free $K$-category on $\Delta$ respectively;

iii) $\text{Rel}B$ is a set of relations on the free $K$-category $P_K\Delta$.

Formally, we say $\mathcal{P}$ presents the left Kan extension $(E, \varepsilon)$ of the left Kan extension data $(M', F')$ where $M' : A \to \text{K Mods}$ and $F' : A \to B$ if $\Gamma$ is a generating graph for $A$ and $M : \Gamma \to \text{K Mods}$ is the restriction of $M' : A \to \text{K Mods}$; $\text{cat}_K(\Delta|\text{Rel}B)$ is a $K$-category presentation for $B$ and $F : \Gamma \to P_K\Delta$ induces $F' : A \to B$.

We expect that a left Kan extension $(E, \varepsilon)$ is given by a set $EB$ for each $B \in \text{Ob}\Delta$ and a function $Eb : EB_1 \to EB_2$ for each $b : B_1 \to B_2 \in \text{B}$ (defining the $K$-functor $E$) together with a function $\varepsilon_A : XA \to EFA$ for each $A \in \text{Ob}A$ (the natural transformation).

For the following theorem it is helpful to note that $=_{FQ}$ will denote the right congruence generated by $FQ$. Square brackets $[\cdot]_{FQ}$ denote the corresponding congruence classes.
Theorem 4.2 (Congruences on Algebras are Kan Extensions)

Let $P := \text{kan}(\Gamma \Delta | \text{Rel} B | M | F)$ be a presentation of a Kan extension for $K$-categories where:

i) $\Gamma$ is the graph with one object $A$ and a collection of arrows $Q$,

ii) $\Delta$ is the graph with one object $B$ and a set of arrows $X$,

iii) $\text{Rel} B$ is a set of polynomial relations $R \subseteq K[X^*],$

iv) $M : A \to \text{K Mods}$ maps $A$ to $K[1]$ and the arrows of $A$ to the identity morphism,

v) $F : A \to P_K \Delta$ maps the arrows of $A$ to polynomials of $K[X^*].$

Then the left Kan extension presented by $P$ is $(E, \varepsilon)$ where

i) $E(B)$ is isomorphic to $(K[X^*]/=_{R})/=_{FQ},$

ii) $E(b)$ is defined by $E(b)[p] := [pb]_R,$

iii) $\varepsilon : M \to E \circ F$ is given by $\varepsilon_A M(q) := [[q]_{R}]_{FQ}.$

Outline Proof

It is required to verify that $E$, as defined above, is a well-defined $K$-functor. This is quite routine and comes from the fact that the congruence preserves addition, scalar multiplication and right-multiplication. To verify that $\varepsilon$ is a natural transformation of $K$-functors is straightforward, remembering that $M(q)$ is the identity morphism on $K[1]$. To check the universal property we suppose there is another such pair $(E', \varepsilon')$ and by drawing the commutative diagram we find that there is a unique natural transformation $\alpha : E \to E'$ defined by $\alpha(b) := E'(b)(\varepsilon'(1_A))$ for $b \in X^*$.

It is not claimed that this result is at all deep or difficult, given the results of [2] but it allows the possibility of using Gröbner bases to compute different types of left Kan extensions.

Corollary 4.3 Gröbner bases can be used to compute left Kan extensions of the above type.

Outline Proof

Let $P$ be as above. Define the $P_K \Delta$-set as

$$T := MA \times \text{Arr} P_K \Delta$$

and write the terms $A|p$ where $p \in \text{Arr} P_K \Delta$. Define the system of polynomials $S := (S_E, S_\varepsilon)$ where

$$S_E := R \quad \text{and} \quad S_\varepsilon := \{A|Fq - A|1 : q \in Q\}$$

The results in [10] describe Gröbner basis procedures for one-sided ideals in finitely presented noncommutative algebras over fields. The polynomials defining the $K$-algebra $B$ as a quotient of the free $K$-algebra $P_K \Delta$ are combined with the polynomials defining a right congruence $=_{FQ}$ of $B$, by using a tagging notation. Standard noncommutative Gröbner basis techniques can then be applied to the mixed set of polynomials, thus calculating $B/ =_{FQ}$ whilst working in a free structure, avoiding the complication of computing in $B$.

Suppose $G$ is a Gröbner basis for $S$. Then the Kan extension is given in the following way:
i) $E(B) := \text{IRR}$,

ii) $E(b) : A|p \mapsto \text{irr}(A|pb)$, for $A|p$ in $E(B)$, $b$ in $K[X^*]$.

iii) $\varepsilon_A(1) := A|1$

where $\text{irr}(A|pb)$ is the irreducible result of repeated reduction of $A|pb$ by $\rightarrow_G$ and $\text{IRR}$ is the set of all irreducible terms of $T$.

\[\square\]

Remark 4.4 It is worth remarking that, as with the rewriting methods developed in [3], the Gröbner basis methods developed in [10] which are referred to above do not require changes in the existing programs. The use of tags enables the combination of polynomials giving the conditions for the action of the Kan extension together with the polynomials giving the conditions for the natural transformation.

5 Examples

The first example illustrates the previous section, showing that the standard Gröbner basis computation is the computation of a Kan extension and extending the example to make clear the type of calculation used for right congruences of algebras. The second example demonstrates the use of Gröbner bases to calculate the morphisms of a $K$-category given by a presentation. In each case we consider the left Kan extension given by a presentation $P := \langle \Gamma \Delta | RelB | M | F \rangle$.

Example 5.1 Let $\Gamma$ be the trivial graph with one object $A$. Let $\Delta$ be the graph with one object $B$ and arrows $X := \{e_1, e_2, e_3\}$. Let $RelB$ be the set of polynomials

$R := \{e_1e_1 - e_1, e_2e_2 - e_2, e_3e_3 - e_3, e_3e_1 - e_1e_3, e_2e_1e_2 - e_1e_2e_1 + \frac{2}{9}e_2 - \frac{2}{9}e_1, e_3e_2e_3 - e_2e_3e_2 + \frac{2}{9}e_3 - \frac{2}{9}e_2\}.$

$F : \Gamma \rightarrow P_K \Delta$ be inclusion and define $M(A) := K[1]$. The system $S$ consists only of untagged polynomials $R$ because there are no non-trivial arrows in $A$. We use the length-lexicographic ordering with $e_3 > e_2 > e_1$ to obtain Gröbner basis for the congruence generated by $S$ in $K[X^*]$ by adding

$e_3e_2e_1e_3 - e_2e_3e_2e_1 + \frac{2}{9}e_2e_1 - \frac{2}{9}e_1e_3$

to $R$. The irreducible terms $\text{IRR}$ in this case are sums of $K$-multiples of the following terms

$\{A|1, A|e_1, A|e_2, A|e_3, A|e_1e_2, A|e_1e_3, A|e_2e_1, A|e_2e_3, A|e_3e_2, A|e_1e_2e_1, A|e_1e_2e_3, A|e_1e_3e_2, A|e_2e_1e_3, A|e_2e_1e_3e_2, A|e_1e_3e_2e_1, A|e_2e_1e_3e_2, A|e_2e_3e_2e_1\}.$

In this example the tag “$A|$” is redundant: the $K$-module $EB$ is a $K$-algebra, in fact it is the Hecke algebra $H_4$. Suppose now that $\Gamma$ has one arrow $q$ whose image under $F$ is $e_2e_1$. The system of polynomials $S$ for the Kan extension now has an $e$-polynomial namely $A|e_2e_1 - A|1$. Applying Buchberger’s Algorithm with the length-lexicographic ordering $e_3 > e_2 > e_1 > A|$ results in a Gröbner basis of mixed polynomials:
\{e_1 e_1 - e_1, \; e_2 e_2 - e_2, \; e_3 e_3 - e_3, \; e_1 e_1 - e_1 e_3, \; e_2 e_1 e_2 - e_1 e_2 e_1 + \frac{2}{9} e_2 - \frac{2}{9} e_1, \; e_3 e_2 e_3 - e_2 e_3 e_2 + \frac{2}{9} e_3 - \frac{2}{9} e_2, \\
e_3 e_2 e_1 e_3 - e_2 e_3 e_2 e_1 + \frac{2}{9} e_2 e_1 - \frac{2}{9} e_1 e_3, \; A|e_2 e_1 - A|1, \; A|e_1 - A|1\}.

The right congruence classes of \(e_2 e_1\) on \(H_4\) are represented by sums of \(K\)-multiples of the following irreducible terms, i.e. \(\text{IRR}\) consists of:

\[ \{A|1, \; A|e_2, \; A|e_3, \; A|e_2 e_3, \; A|e_3 e_2, \; A|e_2 e_3 e_1, \; A|e_2 e_3 e_2 e_1\} \]

Here the tag “\(A\)” is necessary in the computation of the Gröbner basis. The final results may be written as right congruence classes \([e_3 e_2]^r\), say, instead of tagged terms \(A|e_3 e_2\) but the representation as tagged terms allows us to determine whether \(e_1 e_2 e_3\) and \(e_2 e_3\) occur in the same class: reducing \(A|e_1 e_2 e_3\) and \(A|e_2 e_3\) has the same result, so they are congruent.

**Example 5.2** Let \(B\) be the \(\mathbb{Q}\)-category generated by the graph \(\Delta\):

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
h & g & j & f & e \\
\end{array}
\]

The arrows of the free category \(P_K \Delta\) are sums of \(\mathbb{Q}\)-multiples of terms occurring in the same column of the following table (the hom-sets consisting solely of identities are omitted):

| \(B_1 \to B_2\) | \(B_1 \to B_3\) | \(B_1 \to B_4\) | \(B_1 \to B_5\) | \(B_2 \to B_2\) | \(B_2 \to B_3\) | \(B_4 \to B_3\) | \(B_5 \to B_4\) | \(B_5 \to B_3\) |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \(a\)          | \(ac\)         | \(h\)          | \(e\)          | \(1_{B_2}\)    | \(c\)          | \(g\)          | \(f\)          | \(j\)          |
| \(ab\)         | \(abc\)        | \(ef\)         |                | \(b\)          | \(bc\)         |                | \(fg\)         |                |
| \(ab^2\)       | \(ab^2c\)      |                | \(b^2\)        | \(b^2c\)       |                |                |                |                |
| \(\vdots\)     | \(\vdots\)     | \(\vdots\)     | \(\vdots\)     | \(\vdots\)     | \(\vdots\)     | \(\vdots\)     | \(\vdots\)     | \(\vdots\)     |
| \(ab^n\)       | \(ab^n c\)     |                | \(b^n\)        | \(b^n c\)      |                |                |                |                |

Let \(R\) be the relations defining \(B\)

\[ R := \{ab^3 - ab^2 - ab + a, \; b^3 c - b^2 c - bc + c, \; abc + d - ef g, \; ac + d - hg, \; fg - j\} \]

Applying the length-lexicographic ordering with \(a < b < c < d < e < f < g < h < j\) it can be checked that \(R\) is a Gröbner basis. It can therefore be immediately deduced that the arrows of \(B\) are uniquely represented by \(\mathbb{Q}\)-multiples of terms occurring in the same column of the following table:
6 Further Questions

6.1 Induced Modules

It would be useful to phrase the results of Section 4 in terms of induced modules, relating it to the commutative case in [7].

6.2 Extensions of Gröbner basis techniques

To apply rewriting to Kan extensions we had to generalise it. We have not yet discovered how precisely to generalise Gröbner bases to apply to any Kan extension of \( K\)-categories over \( K\text{-Mod} \).

6.3 Rings with Many Objects

Mitchell’s classic work, generalises noncommutative homological ring theory to (pre)additive category theory [13]. His work motivates the investigation of Gröbner basis techniques for \( K\)-categorical Kan extensions by the potential for Gröbner bases to provide more powerful methods of computation (of homology or cohomology) in this setting.

6.4 Term rewriting and Monads

Term rewriting systems, widely used throughout computer science, are similar to algebraic theories (algebraic theories declare term constructors, term rewriting systems declare term constructors and rewrite constructors). Algebraic theories can be modelled by finitary monads over \( \text{Sets} \). Term rewriting systems can be modelled by finitary monads over the category of preorders \( \text{Pre} \). This has been useful in providing categorical proofs of rewriting theories. The particularly interesting point is that term rewriting systems can be modelled as monads over a more complex base category. So \( C\)-algebraic theories can be modelled by finitary monoids on \( C \). There is a relation between monads, adjoint functors and Kan extensions. We need to investigate the relation between string rewriting for Kan extensions and the monads modelling algebraic theories and term rewriting systems.

6.5 Petri nets

Gröbner basis procedures can be usefully applied in Petri net analysis. To every Petri net there is an associated category – a Petri category [12]. How does the structure for Petri categories relate to Kan extensions? Are the Gröbner basis techniques usefully extended by relating these two areas or are they in fact the means by which the areas can be related?
6.6 Automatic Structure

For groups, monoids and coset systems there is a well-known concept of an automatic structure. These systems are special cases of Kan extensions so it is natural to ask what would be the definition of an automatic structure for a left Kan extension in general.

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