CORRIGENDUM TO “REGULARITY FOR STABLY PROJECTIONLESS, SIMPLE C*-ALGEBRAS”

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Abstract. An error is identified and corrected in the construction of a non-$\mathcal{Z}$-stable, stably projectionless, simple, nuclear $C^*$-algebra carried out in a paper by the second author.

The problem

The construction in Section 4 of the second author’s paper [?], used to prove [?, Theorem 4.1], contains a vital error. The construction is meant to produce a simple $C^*$-algebra with perforation in its Cuntz semigroup, as an inductive limit of stably projectionless subhomogeneous $C^*$-algebras.

The notation set out in [?] will be reused here, mostly without recalling the definitions.

The idea is to use generalized Razak building blocks $R(X, k) \subseteq C(X, M_{k+1})$ (as defined in [?, Section 4.2]) as the stably projectionless building blocks of the inductive system; the connecting maps are unitary conjugates of restrictions of diagonal maps $D_{\alpha_1, \ldots, \alpha_p} : C(X, M_n) \to C(Y, M_m)$ (as defined in [?, Section 4.1]).

For generalized Razak building blocks $R(X, k) \subseteq C(X, M_{k+1})$ and $R(Y, \ell) \subseteq C(X, M_{\ell+1})$, [?, Proposition 4.3] characterizes when a diagonal map $D_{\alpha_1, \ldots, \alpha_p} : C(X, M_{k+1}) \to C(Y, M_{\ell+1}) \otimes M_m$ is unitarily conjugate to a map which sends $R(X, k)$ into $R(Y, \ell) \otimes M_m$. The characterization includes the equations

1. $ka_0 + (k+1)a_1 = (m - s(k+1))\ell$, and
2. $kb_0 + (k+1)b_1 = (m - s(k+1))(\ell + 1),$

where $a_0, a_1, b_0, b_1$, and $s$ count certain values of the maps $\alpha_1, \ldots, \alpha_p$; they additionally satisfy

3. $p = a_0 + a_1 + s\ell = b_0 + b_1 + s(\ell + 1).$

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In [?, Remark 4.4], a specific (parametrized) solution is provided to the condition in [?, Proposition 4.3], and this solution is used in [?, Section 4.4] to construct the example.

Implicit in the definition of diagonal maps in [?, Section 4.1] is that they are unital (as maps \(C(X, M_n) \to C(Y, M_m)\)). In the case of [?, Proposition 4.3], this means that

\[
p(k + 1) = m(\ell + 1).
\]

However, the solution provided in [?, Remark 4.4] does not satisfy (4). In fact, some algebraic manipulation of the equations in [?, Proposition 4.3] shows that there are not very many solutions at all. Certainly, suppose that \(m, \ell, p, s, a_0, a_1, b_0, b_1\) satisfy (1), (2), (3), and (4). Combining (3) and (4) yields

\[
(b_0 + b_1 + s(\ell + 1))(k + 1) = m(\ell + 1).
\]

Subtracting (2) from this produces \(b_0 = 0\). Likewise, one obtains \(a_0 = m\).

Crucial to the construction in [?] is the use of both coordinate projections and flipped coordinate projections among the eigenmaps in the diagonal map \(D_{\alpha_1, \ldots, \alpha_p}\). As intimated in [?, Remark 4.4], there may be up to \(\max\{a_0, b_1\}\) coordinate projections and \(\max\{a_1, b_0\}\) flipped coordinate projections. To get perforation, the number of coordinate projections and flipped coordinate projections needs to be a very large fraction of the total number of eigenmaps. Since solutions to [?, Proposition 4.3] necessarily have \(b_0 = 0\), it is actually not possible to get perforation in the Cuntz semigroup with this kind of construction.

**The solution**

Here we describe a correction to the construction in [?, Section 4], permitting a correct proof of [?, Theorem 4.1]. The solution is to allow slightly more general diagonal maps which include some copies of the zero representation.

Let \(X, Y\) be compact Hausdorff spaces and let \(\alpha_1, \ldots, \alpha_p : Y \to X\) be continuous functions. Suppose that \(m, n, r \in \mathbb{N}\) satisfy \(np + r = m\). Define \(D_{\alpha_1, \ldots, \alpha_p, r} : C(X, M_n) \to C(Y, M_m)\) by

\[
D_{\alpha_1, \ldots, \alpha_p, r}(f) := \text{diag}(f \circ \alpha_1, f \circ \alpha_2, \ldots, f \circ \alpha_p, 0_r)
\]

\[
:= \begin{pmatrix}
  f \circ \alpha_1 & 0 & \cdots & 0 \\
  0 & f \circ \alpha_2 & \cdots & \vdots \\
  \vdots & \vdots & \ddots & 0 \\
  0 & \cdots & f \circ \alpha_p & 0 \\
  0 & \cdots & 0 & 0_r
\end{pmatrix}.
\]
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We have the following generalization of [?, Proposition 4.2] (the only difference being that the map $D_{\alpha_1^{(i)}, \ldots, \alpha_p^{(i)}}$ is replaced by the more general $D_{\alpha_1^{(i)}, \ldots, \alpha_p^{(i)}, r_i}$). The proof is exactly the same.

Proposition 1. Let

$$A_1 \xrightarrow{\phi^1} A_2 \xrightarrow{\phi^2} \cdots$$

be an inductive limit, such that for each $i$, the algebra $A_i$ is a subalgebra of $C(X_i, M_{m_i})$ and $\phi^i_{t+1} = \text{Ad}(u) \circ D_{\alpha^{(i)}_1, \ldots, \alpha^{(i)}_p, r_i}$ for some unitary $u \in C(X_{i+1}, M_{m_{i+1}})$ (so that $m_{i+1} = m_i p_i + r_i$). Suppose that $X_i$ contains a copy $Y_i$ of $[0, 1]^{d_{i-1}}$ such that

1. $A_i|_{Y_i} = C(Y_i, M_{m_i})$,
2. for $t = 1, \ldots, d_i$, $\alpha^{(i)}_t|_{Y_{i+1}}$ takes $Y_{i+1}$ to $Y_i$ via the $t$th coordinate projection $([0, 1]^{d_{i-1}})^{d_i} \to [0, 1]^{d_{i-1}}$, and
3. for $t = d_i + 1, \ldots, p_i$, $\alpha^{(i)}_t|_{Y_{i+1}} : Y_{i+1} \to X_i$ factors through the interval.

If

$$\prod_{i=1}^{\infty} \frac{d_{i+1}}{p_i} > 0$$

and $p_i > 1$ for all $i$ then for any $n \in \mathbb{N}$, there exists $[a], [b] \in \mathcal{Cu}(\lim A_i)$ and $k \in \mathbb{N}$ such that

$$(k+1)[a] \leq k[b]$$

yet $[a] \not\leq n[b]$.

We have the following generalization of [?, Proposition 4.3]; the diagonal map $D_{\alpha_1, \ldots, \alpha_p}$ of [?, Proposition 4.3] is replaced by the more general $D_{\alpha_1, \ldots, \alpha_p, r}$. This results in a looser condition in (ii) (compare (1), (2) to (??), (??) respectively). The proof is nearly the same and contains no new tricks.

Proposition 2. Let $X = (X, x_0, x_1), Y = (Y, y_0, y_1)$ be double-pointed spaces and let $k, \ell, m, p, r$ be natural numbers such that

$$p(k + 1) + r = m(\ell + 1).$$

Let $\alpha_1, \ldots, \alpha_p : Y \to X$ be continuous maps. Then the following are equivalent:

1. There exists a unitary $u \in C(Y, M_{\ell+1}) \otimes M_m$ such that $uD_{\alpha_1, \ldots, \alpha_p, r}(R(X, k))u^* \subseteq R(Y, \ell) \otimes M_m$; and
2. Counting multiplicity we have

$$\{\alpha_1(y_0), \ldots, \alpha_p(y_0)\} = a_0\{x_0\} \cup a_1\{x_1\} \cup \ell\{z_1\} \cup \cdots \cup \ell\{z_s\}$$

and

$$\{\alpha_1(y_1), \ldots, \alpha_p(y_1)\} = b_0\{x_0\} \cup b_1\{x_1\} \cup (\ell + 1)\{z_1\} \cup \cdots \cup (\ell + 1)\{z_s\}$$

for some $a_0, \ldots, a_p, b_0, \ldots, b_p \geq 0$. If $p = 1$, then

$$\{\alpha_1(y_0), \ldots, \alpha_p(y_0)\} = a\{x_0\} \cup a\{x_1\} \cup a\{x_2\} \cup \cdots \cup a\{x_s\}$$

and

$$\{\alpha_1(y_1), \ldots, \alpha_p(y_1)\} = b\{x_0\} \cup b\{x_1\} \cup b\{x_2\} \cup \cdots \cup b\{x_s\}$$

for some $a, b \geq 0$.
for some points \( z_1, \ldots, z_s \in X \), and some natural numbers \( a_0, a_1, b_0, b_1 \) satisfying

\[
ka_0 + (k + 1)a_1 = (m - s(k + 1) - q)\ell, \quad \text{and}
\]

\[
kb_0 + (k + 1)b_1 = (m - s(k + 1) - q)(\ell + 1),
\]

for some \( q \in \mathbb{N} \).

Here is a solution to (3), (4), (5), and (6), parametrized by \( s, k, u \in \mathbb{N}_{>0} \); it is almost the same as the solution in [?, Remark 4.4] with the notable difference of being correct.

\[
\ell := k + 1 + 2u,
\]
\[
m := (k^2 + 3k + 1)s,
\]
\[
a_0 := (k + 1)(k + 1 + u)s, \quad a_1 := ks,
\]
\[
b_0 := (k + 1)su, \quad b_1 := k(k + 2 + u)s,
\]
\[
r := (k^2 + 2k + ku - u)s,
\]
\[
q := ks,
\]
\[
p := (k^2 + 2ku + 3k + 3u + 2)s.
\]

The construction in [?, Section 4.4] proceeds using this solution in place of the one in [?, Remark 4.4]. In essence, the only difference is that the assignment

\[
m_{i+1} := m_i(k_i + 1)^2 s_i
\]

is replaced by

\[
m_{i+1} := m_i(k_i^2 + 3k_i + 1) s_i.
\]

As opposed to the original (though incorrect) construction in [?], it is not obvious that the algebra \( A \) constructed with these corrections has a tracial state (as opposed to only having a densely defined trace). One need not be concerned that this causes problems in proving the desired properties of this example, since nowhere in the statement or proof of [?, Theorem 4.1] (nor elsewhere in [?]) is it used that \( A \) has a tracial state.

This correction thereby provides a proof of [?, Theorem 4.1].

REFERENCES

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