The growth of dry convection in the conditionally stable troposphere: Non-adiabatic effects

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Abstract

In this work, we study the growth characteristics of the convective instability (CI) in the dry troposphere by relaxing the adiabatic compressibility condition of Oberbeck-Boussinesq (OB) approach. We derive a new non-adiabatic-Boussinesq (NAB) expression for the modified Brunt-Vaisala frequency ($\omega_b$), without considering the adiabatic compressibility condition of OB approach. This NAB expression reduces to the known Oberbeck-Boussinesq (OB) expression under adiabatic compressibility condition. The NAB expression of $\omega_b$ is found to be modified from its OB counterpart such that the stabilizing adiabatic lapse rate in OB expression is replaced by a modified non-adiabatic lapse rate given as $(\eta - 1)$ times the auto-convective lapse rate. Here $\eta$ is the ratio of hydrostatic density to the total density. We perform numerical experiments of CI for the conditionally stable troposphere i.e for the troposphere that has the environmental lapse rate negative but smaller than the adiabatic lapse rate. A novel feature of the present study is that the CI grows under proposed NAB approach in spite the conditionally stable condition and remains suppressed under OB approach. The present study, thus, proposes an alternative NAB approach for the positive growth of the CI in the dry troposphere for which the CI is conditionally stable under OB approach.

Keywords: Tropospheric convection, Non-adiabatic flows, Numerical simulation

1. Introduction

The convection in the troposphere of the terrestrial atmosphere is a natural phenomenon, that determines the meteorological conditions such as convective storms, thunderstorms, small and large-scale circulations etc. The convection arises owing to the convective instability (CI) in the presence of destabilizing negative environmental temperature gradient that competes with the stabilizing adiabatic temperature gradient [1 2 3]. These gradients are respectively, characterized by the environmental lapse rate $\gamma_e = \frac{1}{T} \frac{\partial T}{\partial y}$ and the adiabatic lapse rate $\gamma_{ad} = \frac{g}{c p T}$. 

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Theoretical works concerning the tropospheric CI are in abundance and most of them are based on the Oberbeck-Boussinesq (OB) approach. The four most important assumptions under OB approach are the following [3, 5]:

i. The shallow atmosphere condition for the ambient hydrostatic atmosphere which means that the density scale height is approximately equal to the pressure scale height,

ii. Isothermal incompressible flow (time restrictions on the perturbation such that the buoyancy velocity remains slower than the acoustic velocity),

iii. A linearization of $\frac{1}{\rho}$ where $\rho$ is the total density, consists of hydrostatic equilibrium density and the linearly perturbed density,

iv. Adiabatic compressible flow.

These assumptions simplify the instability treatment and a linear Brunt-Vaisala frequency (or growth rate) expression for the Oberbeck-Boussinesq-Convective-Instability (OBCI) is obtained in the following form [3]:

$$\omega_{\text{B}} = \sqrt{g \left( \gamma_{\text{ad}} + \gamma_e \right)}$$

In this conservative form, the OBCI may not be excited since usually $|\gamma_e| < \gamma_{\text{ad}}$ for the radiative-convective equilibrium temperature profile in the troposphere [6] i.e, the troposphere is conditionally (or adiabatically) stable to the OBCI. Even during maximum surface heating conditions, $|\gamma_e|$ becomes larger than $\gamma_{\text{ad}}$ only within a part of the troposphere but not in the entire troposphere. In order to account for the observed convective dynamics, instead of $\gamma_{\text{ad}}$, the average adiabatic lapse rate $\gamma_{\text{ad}}^\text{av} = 6.5/T$ or the moist or wet adiabatic lapse rate $\gamma_{\text{ad}}^\text{m} < \gamma_{\text{ad}}^\text{av}$ are considered [7] and CI simulation are performed for the moist convection that do not satisfy some or all of the OB assumptions [8, 9, 10]. Also, aforementioned OB assumptions may not hold during non-linear phase of CI and numerous studies have been performed for the CI in the dry troposphere to account for the the non-linear dynamics [12, 13, 14, 15, 16].

Lilly [17] has derived the governing equations for the OBCI incorporating non-adiabatic dynamics i.e. without the fourth OB assumption. These equations are employed to study the non-adiabatic effects for conditionally unstable troposphere. However, to date, the studies concerning the non-adiabatic contributions arising from the non-adiabatic nature of compression are not pursued for the conditionally stable dry troposphere. It should be noted that the conditionally stable condition ($|\gamma_e| < \gamma_{\text{ad}}, \gamma_e < 0$) that prevails in the most part of the troposphere, invalidates the fourth OB assumption since the temperature gradient is not strong enough to support the adiabatic compression which is a rapid process. Thus, the non-adiabatic contributions are expected during linear phase of the CI for conditionally stable troposphere and this aspect is not investigated so far. Moreover, an analytical expression for the Brunt-Vaisala frequency or growth rate under non-adiabatic compressibility condition is always desirable and it is not yet available. These aspects are the focus of the
present study. Hereby, we define a non-adiabatic-Boussinesq (NAB) approach that consider all OB assumptions except the last assumption i.e. adiabatic compressibility condition. The corresponding convective instability is referred as non-adiabatic-Boussinesq-CI (NABCI). The objectives of the present study are the following:

I. Derivation of an analytical expression for $\omega_b$ of NABCI having a form similar to $\omega_{bb}$,

II. To search for the possibilities of positive growth of CI under condition ($|\gamma_e| < \gamma_{ad}$, $\gamma_e < 0$) i.e., under stable OBCI condition,

With these objectives, we develop a theoretical framework for the CI in the dry troposphere in appendices A-C. In appendix A, we derive the governing equations for the CI using inviscid hydrodynamic equations. These governing equations include the growth (or wave) equation for the vertical wind $w_y$, temperature equation and the perturbed density (total density minus hydrostatic density) equation. The growth equation contains the general expression of Brunt-Vaisala frequency $\omega_b$. In appendices B and C respectively, the $\omega_b$ expression is derived without the adiabatic compressibility condition (non-adiabatic-Boussinesq or NAB approach) and with all four OB assumptions (OB approach). Below, in section 2.1, we analyze the derived equations qualitatively. In section 2.2, a one dimensional (in the vertical direction representing the altitude) simulation of the CI is presented to study the NAB contributions for chosen initially adiabatically stable temperature profile.

2. Results and Discussion

2.1. The growth of CI under proposed NAB approach: a qualitative analysis

In the appendix B, we have derived a non-adiabatic expression (B9) of the Brunt-Vaisala frequency in the following form:

$$\omega_b^2 = \eta g (\gamma_d + \gamma_e)$$

where $\gamma_d = (\eta - 1) \gamma_{ac}$ and $\eta = \frac{\rho_h}{\rho} \approx 1 - \frac{\rho}{\rho_h}$ \quad (1)

where $\gamma_d$ is defined as the non-adiabatic lapse rate, $(\rho_h, \rho_t, \rho)$ are the hydrostatic density, perturbed density and the total density. Equation (1) is obtained under shallow atmosphere, isothermal incompressible and linearization conditions which are the first three OB assumptions. Remaining OB assumption, i.e. adiabatic compressibility is not employed and thus this expression is a non-adiabatic expression of $\omega_b$ representing the NAB approach.

$\eta$ in (1) is the measure of degree of density change within a heated fluid parcel inside which $\rho \leq \rho_h \Rightarrow \eta \geq 1$. Thus, the non-adiabatic lapse rate $\gamma_d$ is always positive within a heated fluid parcel i.e. this term has the stabilizing effects. For example, $\eta = (1.01, 1.3, 2.10)$ correspond to the degree of density change equals to (1%,20%,50%,90%) respectively representing the initiation, linear, weakly non-linear and highly non-linear phases of the convection, respectively. The
expression (1) is, however, valid only in the linear phase of convection when the shallow convection condition is satisfied.

In the appendix C, we have employed the last OB assumption (i.e. the adiabatic compressibility condition) and found that $\eta = \gamma$, where $\gamma$ is the ratio of specific heats. Thus the adiabatic compressibility assumption in OB imposes the condition on the degree of density change such that it should be approximately equals to $\frac{\gamma - 1}{\gamma} \sim 30\%$ which can be taken as the upper limit for the linear phase of the convection. The question is what happens to the CI from beginning, when the degree of density change is insignificant, up to the upper limit of the linear phase i.e., during the phase when $\eta$ increases from 1 to $\gamma$. It is evident from (1) that as $\eta$ increases from 1 to $\gamma$, the stabilizing term also increases. Therefore, the last OB assumption i.e. the adiabatic compressibility enforces the stabilizing effects. It is thus expected that from the initiation to the upper limit of linear phase of convection i.e from the NAB phase to the OB phase, the required threshold $\gamma_e$ for the instability increases from $\sim 0.01 \gamma_{ac} \ll \gamma_{ad}$ to $(\gamma - 1) \gamma_{ac} = \gamma_{ad}$ as shown in appendix C. It is thus evident that during the NAB phase, the convection convincingly grows in the troposphere since the stabilizing non-adiabatic lapse rate is much smaller than the adiabatic lapse rate.

It should be pointed out that the present study deals with the dry convection and thus the non-adiabatic effects discussed above are not owing to the presence of moist or another phase of matter than the gas. The non-adiabatic effects, in the present study, are owing to the linear phase ($\eta \sim 1$) of the convection which is not fast enough (owing to $\gamma_e < \gamma_{ad}$ for the radiative-convective equilibrium temperature profile) to undergo the spontaneous or adiabatic compression.

2.2. The growth of CI under proposed NAB approach: a quantitative analysis

To study aforementioned aspects quantitatively, we develop a simulation model of the CI where the following set of equations are simultaneously solved:

$$\frac{\partial^2 w_y}{\partial t^2} = -\omega_b^2 w_y - \eta g \frac{\partial w_y}{\partial y},$$

$$\frac{\partial \rho_t}{\partial t} = w_y (\rho_h + \rho_t) \gamma_e, \quad \frac{\partial \rho_h}{\partial t} = -w_y \frac{\partial \rho_h}{\partial y},$$

$$\frac{\partial T}{\partial t} = -w_y T \gamma_e, \quad \gamma_e = \frac{1}{T} \frac{\partial T}{\partial y}$$

(2)

Here, $(w_y, \rho_t, \rho_h, T)$ are the wind, perturbed density inside the heated fluid parcel, the hydrostatic density and the total temperature respectively. The first equation in (2) is the equation (A3) in the appendix A. Using continuity equation (A1) and equation (B8) in appendices A-B, two simultaneous equations for $(\rho_t, \rho_h)$ are obtained in (2). The temperature equation in (2) is obtained using the pressure and continuity equations in (A1).

It should be mentioned that the incompressibility condition decouples the mechanical and thermodynamical processes [17] and leads to the continuity equations for $(\rho)$ without the non-adiabatic term and the continuity equation for
\( \rho_t \) with the non-adiabatic term, as derived by Lilly \[17\] as equations (2.2), (2.9) and (2.11). In the present study, the derived continuity equations for \( \rho \) in equation (A1) is identical to equation (2.2, 2.9) derived by Lilly \[17\]. Moreover, the continuity equation for \( \rho_t \) as derived above in (2) is similar to the equation (2.11) of Lilly \[17\] such that it retains the non-adiabatic contribution through \( \gamma_e \) term.

The set of equations in (2) are solved using FTCS finite-difference, Crank-Nicolson implicit integration scheme and Successive-Over-Relaxation method. This numerical scheme is discussed in detail by \[18, 19\] to solve the convective instability in the ionospheric plasma and to solve the acoustic-gravity wave equation in the atmosphere. The non-local term in the growth equation of \( w_y \) is not considered in the present study since our objective is to understand the non-adiabatic effects that is owing to the non-adiabatic lapse rate \( \gamma_d \) in \( \omega^2_b \) term.

The simulation domain is one-dimensional, consists of altitude covering 0-20 km altitude with grid resolution \( \Delta y = 0.5 \text{km} \). The initial hydrostatic density \( \rho_h \) profile is considered to be the exponential decreasing with height with scale height equals to 20km. In Figure 1a, the chosen radiative-convective equilibrium temperature profile is shown. In Figure 1b, the initial environmental lapse rate \( \gamma_e \) multiplied by the initial temperature \( (T_o) \), is plotted. For comparison, the \( \gamma_{ad} \) and \( \gamma_{ac} \) multiplied by \( T_o \) are also plotted. It is evident from figure 1b that the chosen temperature profile satisfies \( |\gamma_e| < \gamma_{ad} \) i.e the profile is adiabatically stable in the entire troposphere.

We present the simulation results by performing three numerical experiments corresponding to three approaches which are the NAB, OB and a general approach. These three approaches correspond to three different expression for \( \omega^2_b \) in (2) and is respectively given by expressions (1), (C1) and (A4) as follows:

\[
\omega^2_b = \eta g [ (\eta - 1) \gamma_{ac} + \gamma_e ] \quad \text{where} \quad \eta = \frac{\rho_h}{\rho} \approx 1 - \frac{\rho_t}{\rho_h} \quad \text{(NAB approach)} \tag{3}
\]

\[
\omega^2_{B,B} = g (\gamma_{ad} + \gamma_e) \quad \text{(OB approach)}
\]

\[
\omega^2_{B,g} = \frac{1}{\rho^2} \frac{\partial p}{\partial y} \frac{\partial \rho}{\partial y} - \frac{1}{\rho} \frac{\partial^2 p}{\partial y^2} \quad \text{(a general approach)}
\]

In this section, we present the results corresponding to NAB approach while in the next section, the results from other two approaches are presented and compared with the results from the NAB approach.

The source of perturbation can be either of wind type or thermal type. We study a wind type perturbation such that a uniform (over altitude) amplitude \( = 10^{-3} \text{m/s} \) at \( t=0 \) is chosen. With this perturbation, the temporal-altitude evolution of the \((w_y, \rho_t, \rho_h \text{ and } T)\) are simulated. In Figures 2, the results are presented. In Figures 2a-2c respectively, the temporal-altitude evolution of \( \eta = \frac{\rho_h}{\rho_h + \rho_t} \), \( T \gamma_e = \frac{\partial T}{\partial y} \) and \( w_y \) are shown. In 2d, the temporal evolution of the maximum (over altitude) value \( (w_{y}^{\text{max}}) \) of \( w_y \) is plotted. We may note the following characteristics from Figure 2:
A. \( w_{\text{max}} \) grows linearly and then exponentially for first 10 minutes and then becomes constant,

B. With time, \((\eta, T \text{ and } w_y)\) grow in amplitudes in the altitude region where \(\gamma_e < 0\) initially and they attain the largest amplitudes near the altitude where the initial \(\gamma_e\) has maximum negative value,

C. At \(\eta = \gamma\) that occurs at t=10 minutes, \(|\gamma_e|\) becomes equal to \(\gamma_{\text{ad}}\) and \(w_{\text{max}}\) stops growing and becomes saturated to the value of 20 m/s.

D. After this time, though, both \(\eta\) and \(\gamma_e\) continue to grow.

The exponential growth of \(w_{\text{max}}\) under characteristic (A) is an indication of the linear phase of the CI. The growth suggests that the small wind perturbation of \(\sim 10^{-3}\) m/s has grown to 20 m/s within 10 minutes under the action of CI. The characteristics under (A-B) suggest that in spite of the initial adiabatic stable temperature profile, the CI grows in the troposphere. The growth is owing to the non-adiabatic nature of the CI during linear or NAB phase \((\eta < \gamma)\) when the stabilizing lapse rate \((\eta - 1)\gamma_{\text{ac}}\) remains smaller than the \(|\gamma_e|\). As time progress, both \(\eta\) and \(\gamma_e\) and thus both stabilizing and destabilizing lapse rates grow, leading to the saturation, as indicated by characteristic (C).

The characteristic under (C) also suggests that as the tropospheric state approach to the adiabatic state of gas, \(|\gamma_e|\) approaches \(\gamma_{\text{ad}}\). This suggests the consistent adjustment of the temperature profile following the nature of compression within the heated fluid parcel and consistent transition of CI from NAB to OB type. As \(|\gamma_e|\) approaches \(\gamma_{\text{ad}}\), the CI begins to slow down but the growth rate remains positive, as suggested by characteristic (D).

It is evident from Figure 2(b) that \(|\gamma_e| < \gamma_{\text{ac}}\) is maintained in the troposphere. Thus, the shallow convection condition, which is employed to derive the expression (1) is always satisfied, during the simulation. The present simulation is not applicable for the non-linear phase \((\eta > 2)\) of the convection since the shallow convection condition may not be satisfied under non-linear phase. Figures 2(a-b) suggest that during the simulation, the linearity and the shallow atmosphere conditions are maintained i.e. the ambient scales satisfy the condition of linear-first order approximation. This classifies the proposed NAB approach as of Boussinesq in nature.

2.3. The proposed NAB approach vs. OB and a general approach

In this section, we present the results from three numerical experiments corresponding to three approaches which are the NAB, OB and the general approach. As mentioned in the last section, these three approaches are different in terms of \(\omega_b^2\) expression which are respectively given in (3).

In Figure 3, the time evolution of \(w_{\text{max}}\) is shown under three numerical experiments. The evolution under NAB approach, as shown in Figure 2d, is replotted as a blue curve in Figure 3. We may note that under OB and the general approaches, \(w_{\text{max}}\) shows periodic variation in time such that the mean amplitude is decreasing with time. Under OB approach, the periodic variation is expected
since the ambient atmosphere is adiabatically stable leading to the positive $\omega^2_{NB}$ in expression (3). Under general approach, similar oscillating behavior as under OB approach suggests that instability is not excited when none of the four OB assumptions are implemented. In contrast to OB and general approaches, the proposed NAB approach with the new expression for $\omega^2_b$, reveals an exponential growth, as discussed in the last section. Therefore, the proposed mechanism assists the growth of the CI for the troposphere which is stable under OB and the general approaches.

The possible reason for different growth characteristics under NAB and OB approaches is as follows: Under OB approach, that imposes the adiabatic (or rapid) condition, the instability does not have sufficient time to extract the free energy from the negative temperature gradient. On contrary, under proposed NAB approach, the instability acquires sufficient time to extract energy since no adiabatic or rapid condition is imposed.

A new set of governing equations of the CI, given by equation (2), provides an alternative mechanism for the positive growth of the instability in the adiabatically stable troposphere. These set of equations do not incorporate moist convective dynamics, but rather retains the non-adiabatic effects of the dry convection. The non-adiabatic effects, here, are arising owing to the small amplitude of density perturbation $\eta < \gamma$ in the presence of weak $|\gamma_e| < \gamma_{ad}$. The convection of two types, i.e. NAB and OB can be studied with the identical set of equations, as shown in the present study. In future, we intend to extend this simulation study for the two and three dimensional convective instability, whose results may directly be compared with the observations.

3. Summary

In this work, we present an alternative Non-adiabatic Boussinesq (NAB) approach of the convective instability (CI) without imposing the adiabatic compressibility condition in the dry troposphere. Lilly [17] had presented the NAB framework for the CI which is often applied for the conditionally unstable dry and moist troposphere. However, for the conditionally stable dry troposphere, such studies are not pursued and is the focus of the present study. In the conditionally stable dry troposphere, non-adiabatic effects may arise since the temperature gradient or the environmental lapse rate is weaker than the adiabatic lapse rate and may not sustain the rapid dynamics arising owing to the adiabatic compressibility condition of OB approach.

We have derived a new Brunt-Vaisala frequency ($\omega_b$) expression for the instability of NAB type. Under proposed NAB approach, $\omega_b$ is found to be modified from its OB counterpart such that the stabilizing adiabatic lapse rate $\gamma_{ad}$ is replaced by a modified lapse rate $(\eta - 1) \gamma_{ac}$. Here, $(\gamma_{ac}, \eta)$ are the auto-convective lapse rate and the ratio of hydrostatic density to the total density. It is shown that with the adiabatic compressibility condition, $\eta$ becomes equal to the specific heat ratio ($\gamma$) and the new NAB expression of $\omega_b$ reduces to the known OB expression.
Therefore, the adiabatic compressibility condition under OB approach imposes the condition on $\eta$ which, in turn, imposes the condition on the degree of density change to be approximately equals to 30% (for $\gamma \sim 1.4 - 1.6$) of the hydrostatic density. The question is what happens to the CI during 0%–30% of degree of density change, i.e., during the initial phase when $\eta$ increases from 1 to $\gamma$? In this work, we seek to answer this question by proposing the NAB approach with a new expression for $\omega_b$ that retains the $\eta$ without imposing compressibility condition on it and allows the instability to evolve self-consistently from $\eta \approx 1$ to $\eta = \gamma$ i.e from NAB to OB.

We further obtain the governing equations for the wind, temperature and density and perform numerical experiment under NAB approach for the conditionally stable dry troposphere. The novel features of these experiments are as follows: (a) In spite the initial conditional stable temperature profile, the CI grows exponentially in the troposphere owing to the non-adiabatic nature of the CI during linear phase ($1 < \eta < \gamma$), (b) As the tropospheric state approach to the adiabatic state of gas i.e. $\eta \rightarrow \gamma$, $|\gamma_e|$ approaches towards $\gamma_{ad}$ suggesting the consistent adjustment of the temperature profile and consistent transition of CI from NAB to OB type.

The simulation is also carried out for two other approaches which use a general and OB expressions of $\omega_b$. In contrast to the exponential growth under proposed NAB approach, the general and OB approaches reveal periodic variation in time suggesting the stability of CI. In the absence of adiabatic compressibility condition, the CI acquires sufficient time to extract the energy from the unstable negative temperature gradient and subsequently grows under NAB approach. Therefore, the proposed NAB mechanism assists the growth of the CI for the troposphere which, otherwise, is stable under OB and the general approaches.

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Appendix A. Governing growth equation of the instability

We consider the atmospheric density ($\rho$) and pressure ($p$) to have small variation in the horizontal direction ($\hat{x}$), in comparison to the large variation in the vertical direction ($\hat{y}$) so that we may neglect the horizontal fluid advection in comparison to the vertical fluid convection. On the other hand, the vertical ($\hat{y}$) and horizontal ($\hat{x}$) variations of the wind ($\vec{w}$) is governed by the incompressible flow condition. Under such horizontally-stratified-incompressible conditions, the hydrodynamic equations may be written in the following form:

$$\frac{\partial \rho}{\partial t} = -w_y \frac{\partial \rho}{\partial y} ; \quad \frac{\partial p}{\partial t} = -w_y \frac{\partial p}{\partial y} ; \quad p = \rho RT$$  \hfill (A.1)

$$\frac{\partial w_y}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - g ; \quad \frac{\partial w_x}{\partial y} + \frac{\partial w_z}{\partial x} = 0$$  \hfill (A.2)

Taking a time derivative of the momentum equation (A2) and substituting expressions from (A1), leads to the following equation for the vertical wind $w_y$:

$$\frac{\partial^2 w_y}{\partial t^2} = \frac{1}{\rho} \left[ w_y \frac{\partial^2 p}{\partial y^2} + \frac{\partial p}{\partial y} \frac{\partial w_y}{\partial y} \right] - \frac{1}{\rho^2} \frac{\partial p}{\partial y} \left[ \frac{\partial p}{\partial y} \right] w_y$$  

or

$$\frac{\partial^2 w_y}{\partial t^2} = -\omega_b^2 w_y + \frac{1}{\rho} \frac{\partial p}{\partial y} \frac{\partial w_y}{\partial y}$$  \hfill (A.3)

where $\omega_b$ is the Brunt-Vaisala frequency defined by following expression:

$$\omega_b^2 = \frac{1}{\rho^2} \frac{\partial p}{\partial y} \frac{\partial p}{\partial y} - \frac{1}{\rho^2} \frac{\partial^2 p}{\partial y^2}$$  \hfill (A.4)

Depending on whether $\omega_b$ is negative (positive), $w_y$ may grow (oscillate) in time, leading to the instability (gravity wave). Kherani et al [18, 19] have used compressible form of (A3) to study the acoustic-gravity wave dynamics that was earlier derived by [20]. The second term in (A3) is a non-local term that determines the preferred wavelength of the growing instability or the gravity wave. The governing growth equation (A3) of similar form is derived in the past under OB assumptions and without the non-local term [21, 22].

Appendix B. Non-adiabatic-Boussinesq (NAB) expression of $\omega_b$

In this section, we derive the expressions for density and pressure gradients appearing in (A4), by applying first three OB assumptions mentioned in the introduction. In other words, except the adiabatic compressibility condition, other three OB assumptions are considered.
We consider $\rho$ to be composed of the hydrostatic equilibrium density $\rho_h$ and a non-hydrostatic (or perturbed) density $\rho_t$, i.e.,

$$\rho = \rho_h + \rho_t; \quad p = p_h + p_t; \quad \rho_h = -\frac{1}{g} \frac{\partial p_h}{\partial y}; \quad \rho_h + \rho_t = \frac{p_h + p_t}{RT} \quad (B.1)$$

1. Isothermal-incompressible condition (first OB assumption): The isothermal incompressibility is defined as follows:

$$\frac{1}{\rho} \frac{\partial \rho}{\partial y} = 0 \quad (B.2)$$

2. Shallow-atmosphere condition (second OB assumption): Using the state of gas equation for the hydrostatic atmosphere $\rho_h = \frac{p_h}{RT}$, the hydrostatic density scale height may be written as follows:

$$\frac{1}{\rho_h} \frac{\partial \rho_h}{\partial y} = \frac{1}{\rho_h} \frac{\partial p_h}{\partial y} - \frac{1}{T} \frac{\partial T}{\partial y}$$

Substitution of the hydrostatic condition $\frac{\partial p_h}{\partial y} = -\rho_h g$ together with $p_h = \rho_h RT$, leads to the following expression:

$$\frac{1}{\rho_h} \frac{\partial \rho_h}{\partial y} = -\gamma_{ac} - \gamma_e$$

Here we define $\gamma_{ac} = \frac{g}{RT}$ and $\gamma_e = \frac{1}{T} \frac{\partial T}{\partial y}$ where $\gamma_{ac}$ and $\gamma_e$ are the auto-convection and environmental lapse rates respectively.

The shallow atmosphere condition is defined as $\gamma_{ac} > |\gamma_e|$ that leads to the following condition:

$$\frac{1}{\rho_h} \frac{\partial \rho_h}{\partial y} \approx -\gamma_{ac} \quad (B.3)$$

i.e., the hydrostatic density scale height equals to the hydrostatic pressure scale height

3. Expression for $\frac{1}{\rho} \frac{\partial \rho}{\partial y}$ in $\omega_b$: In general, using the state of gas equation $\rho = \frac{p}{RT}$, the expression for $\frac{1}{\rho} \frac{\partial \rho}{\partial y}$ may be written as follows:

$$\frac{1}{\rho} \frac{\partial \rho}{\partial y} = \frac{1}{p} \frac{\partial p}{\partial y} - \frac{1}{T} \frac{\partial T}{\partial y} \quad (B.4)$$

Substituting $\rho = \rho_h + \rho_t$ and $p = p_h + p_t$ from (B1), above equation reduces to the following expression:

$$\frac{1}{\rho} \frac{\partial \rho}{\partial y} = \frac{1}{p} \frac{\partial p_h}{\partial y} + \frac{1}{p} \frac{\partial p_t}{\partial y} - \frac{1}{T} \frac{\partial T}{\partial y}$$
Substitution of the hydrostatic condition \( \frac{\partial p_h}{\partial y} = -\rho_h g \) together with \( p = \rho RT \) from (B1), leads to the following expression:

\[
\frac{1}{\rho} \frac{\partial \rho}{\partial y} = -\frac{\rho_h g}{RT} + \frac{1}{p} \frac{\partial p_t}{\partial y} - \frac{1}{T} \frac{\partial T}{\partial y}
\]

Here we define \( \frac{\rho_h}{\rho} = \eta \), \( \gamma_{ac} = \frac{g}{RT} \) and \( \gamma_e = \frac{1}{T} \frac{\partial T}{\partial y} \) where \( \gamma_{ac} \) and \( \gamma_e \) are the auto-convection and environmental lapse rates respectively. With these definitions, above expression reduces to the following form:

\[
\frac{1}{\rho} \frac{\partial \rho}{\partial y} = -\eta \gamma_{ac} - \gamma_e + \frac{1}{p} \frac{\partial p_t}{\partial y}
\]

Imposing the isothermal-incompressible condition (B2), following expression for \( \frac{1}{\rho} \frac{\partial \rho}{\partial y} \) in \( \omega_b \) is obtained:

\[
\frac{1}{\rho} \frac{\partial \rho}{\partial y} = -(\eta \gamma_{ac} + \gamma_e)
\]  

4. Expression for \( \frac{1}{\rho} \frac{\partial p}{\partial y} \) in \( \omega_b \): Using \( p = p_h + p_t \), the pressure gradient may be written as follows:

\[
\frac{\partial p}{\partial y} = \frac{\partial p_h}{\partial y} + \frac{\partial p_t}{\partial y}
\]

or

\[
\frac{1}{\rho} \frac{\partial p}{\partial y} = \frac{1}{\rho} \frac{\partial p_h}{\partial y} + \frac{1}{\rho} \frac{\partial p_t}{\partial y}
\]

or

\[
\frac{1}{\rho} \frac{\partial p}{\partial y} = -\frac{\rho_h g}{\rho} + \frac{1}{\rho} \frac{\partial p_t}{\partial y}
\]

Imposing the isothermal incompressibility condition (B2), following expression for \( \frac{1}{\rho} \frac{\partial p}{\partial y} \) in \( \omega_b \) may be obtained:

\[
\frac{1}{\rho} \frac{\partial p}{\partial y} = -\frac{\rho_h g}{\rho} = -\eta g
\]  

(B.6)

Further differentiating (B6) leads to the following expression:

\[
\frac{\partial^2 p}{\partial y^2} = -g \frac{\partial \rho_h}{\partial y}
\]

Using shallow atmosphere condition (B3), we obtain the following expression for \( \frac{1}{\rho} \frac{\partial^2 p}{\partial y^2} \) in \( \omega_b \):

\[
\frac{1}{\rho} \frac{\partial^2 p}{\partial y^2} \approx g \frac{\rho_h g}{\rho} \gamma_{ac} = \eta g \gamma_{ac}
\]  

(B.7)
5. Incompressible condition: At this point, it is important to explain why condition (B2) is referred as the isothermal-incompressible condition in the present study? We begin with equation (B5) which is obtained by imposing isothermal-incompressible condition and can be written as follows:

\[
\frac{1}{\rho} \frac{\partial (\rho h + \rho_t)}{\partial y} = - (\eta \gamma_{ac} + \gamma_e)
\]

or

\[
\frac{1}{\rho} \frac{\partial \rho h}{\partial y} + \frac{1}{\rho} \frac{\partial \rho_t}{\partial y} = - (\eta \gamma_{ac} + \gamma_e)
\]

Imposing the shallow-atmosphere condition (B3), following equation is obtained:

\[
- \eta \gamma_{ac} + \frac{1}{\rho} \frac{\partial \rho_t}{\partial y} = - (\eta \gamma_{ac} + \gamma_e)
\]

or

\[
\frac{1}{\rho} \frac{\partial \rho_t}{\partial y} = - \gamma_e \tag{B.8}
\]

which is the governing equation for the isothermal incompressibility stating that the density change is owing to the thermal expansion. For this reason, the condition (B2) is termed as the isothermal incompressible condition. The shallow atmosphere condition together with the isothermal incompressible condition (B2) ensures the incompressibility in the troposphere [5].

6. NAB expression of \(\omega_b^2\): Substituting (B5-B7) in (A4), we obtain following expression of \(\omega_b^2\):

\[
\omega_b^2 = \eta g \left[ (\eta - 1) \gamma_{ac} + \gamma_e \right] \text{ where } \eta = \frac{\rho_h}{\rho} \tag{B.9}
\]

This is a general NAB expression for the Brunt-Vaisala frequency, obtained in the present study.

7. Linearization of \(\eta\) appearing with \(\gamma_{ac}\): The third OB assumption is the linearization of \(\frac{1}{\rho}\) or \(\eta\) in (B9) in a following manner [3]:

\[
\eta = \frac{\rho_h}{\rho} = \frac{\rho_h}{\rho_h \left( 1 + \frac{\rho_t}{\rho_h} \right)} \approx 1 - \frac{\rho_t}{\rho_h}
\]

This linearization is done only for \(\eta\) appearing with \(\gamma_{ac}\) where gravity is coming with \(\gamma_{ac}\) i.e. with the hydrostatic pressure gradient force. For \(\eta\) appearing only with gravity can be taken as unity. With these linearization scheme, (B9) reduces to the following form:

\[
\omega_b^2 = - g \left( \frac{\rho_t}{\rho_h} \gamma_{ac} - \gamma_e \right) \tag{B.10}
\]
Appendix C. Oberbeck-Boussinesq (OB) expression of $\omega_b$

1. Adiabatic Compressibility condition: The adiabatic compressible condition, which is the last remaining assumption under OB approach, is described by the following law of thermodynamics:

$$\frac{1}{\rho} \frac{\partial \rho}{\partial y} = \frac{c_p}{\gamma R} \gamma_e$$

Substitution of $\gamma_e$ from the isothermal-incompressible equation (B8) reduces the above equation of the adiabatic compressible equation into following condition on $\eta$:

$$\frac{\rho_t}{\rho} = -\frac{\gamma R}{c_p} \Rightarrow \eta = \frac{\rho_t}{\rho} = \gamma$$

or

$$\frac{\rho_t}{\rho_h} = -\frac{R}{c_p}$$

whose substitution into (B10) leads to the known OB expression for $\omega_{bB}^2$:

$$\omega_{bB}^2 = g \left( \gamma_{ad} + \gamma_e \right)$$  \hspace{1cm} \text{(C.1)}$$

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Figure 1: (a) The ambient radiative-convective equilibrium temperature ($T_o$) profile, (b) The environmental lapse rate $\gamma_e$ multiplied by $T_o$. For comparison, the adiabatic and auto-convective lapse rates ($\gamma_{ad}, \gamma_{ac}$), multiplied by $T_o$ are also plotted.
Figure 2: Simulation of CI using governing equations (2) with a wind source: The temporal-altitude evolution of the (a) $\eta = \frac{\rho_h}{\rho_h + \rho_t}$, (b) $T_{ce}$, (c) the wind $w_y$ are shown respectively. Here $(\rho_h, \rho_t)$ are the hydrostatic equilibrium and perturbed densities, respectively. In (d) the temporal evolution of the maximum (over altitude) value $w_y^{max}$ of $w_y$ is plotted. The green line is drawn at the time when $\eta$ approaches $\gamma$. 

$\frac{\rho_h}{\rho_h + \rho_t}$
Figure 3: Comparative study from three different approaches: the temporal evolution of the maximum (over altitude) value \( w_y^{\text{max}} \) of \( w_y \) is plotted for NAB (blue), OB (green) and the general approach (red).