The abelian fibration on the Hilbert cube of a K3 surface of genus 9

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Abstract

In this paper we construct an abelian fibration over \( \mathbb{P}^3 \) on the Hilbert cube of the primitive K3 surface of genus 9. After the abelian fibration constructed by Mukai on the Hilbert square on the primitive K3 surface of genus 5, this is the second example where the abelian fibration on such a Hilb\(_n\)S is directly constructed. Our example is also the first known abelian fibration on a Hilbert scheme Hilb\(_n\)S of a primitive K3 surface \( S \) which is not the Hilbert square of \( S \); the primitive K3 surfaces on the Hilbert square of which such a fibration exists are known by a recent result of Hassett and Tschinkel.

1 Introduction

Generalities. The smooth complex projective variety \( X \) is a hyperk"ahler manifold if \( X \) is simply connected and \( H^0(\Omega^2_X) = \mathbb{C} \omega \) for an everywhere non-degenerate form \( \omega \), the symplectic form on \( X \). In particular, by the non-degeneracy of \( \omega \), a hyperk"ahler manifold is always even-dimensional and with a trivial canonical class, see e.g. [4] or [5] for a survey of the basic properties of hyperk"ahler manifolds.

A fibre space structure, or a fibration on the smooth projective manifold \( X \) is a projective morphism, with connected equidimensional fibers, from \( X \) onto a normal projective variety \( Y \) such that \( 0 < \text{dim}(Y) < \text{dim}(X) \). By the theorems of Matsushita (see [10]), any fibration \( f : X \to Y \) on a hyperk"ahler 2n-fold \( X \) is always a Lagrangian abelian fibration, i.e. the general fiber \( F_y = f^{-1}(y) \subset X \) must be an abelian n-fold which is also a Lagrangian submanifold of \( X \) with respect to the form \( \omega \); in addition \( Y \) has to be a Fano n-fold with the same Betti numbers as \( \mathbb{P}^n \).

The question when a hyperk"ahler 2n-fold \( X \) admits a structure of a fibration in the above sense is among the basic problems for hyperk"ahler manifolds (cf. [4], p. 171). An additional question is whether the base \( Y \) of any such fibration is always the projective space \( \mathbb{P}^n \), ibid. For a more detailed discussion on the problem, see the paper [16] of J. Sawon.

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As shown by Beauville, the Hilbert schemes $\mathcal{H}ilb_n S$ of length $n$ zero-subschemes of smooth K3 surfaces $S$ are hyperkähler manifolds, see [1].

In the special case when $S$ is an elliptic K3 surface, the elliptic fibration $S \to \mathbb{P}^1$ on $S$ induces naturally a structure of an abelian fibration $\mathcal{H}ilb_n S \to \mathbb{P}^n$ for any $n \geq 2$, see e.g. [16], Ex. 3.5. Another candidate is Beauville’s fibration $\mathcal{H}ilb_g S \to \mathbb{P}^g$ for a K3 surface $S$ containing a smooth curve of genus $g \geq 2$, in particular if $S = S_{2g-2}$ is a primitive K3 surface of genus $g \geq 2$, ibid. Ex. 3.6. But Beauville’s fibration is not regular, i.e. it is not a fibration in the above sense.

Until now, the only abelian fibration constructed directly on a Hilbert power of a primitive K3 surface $S = S_{2g-2}, g \geq 2$ is the abelian fibration due to Mukai ([11]) on the Hilbert square of the primitive K3 surface $S_8$ of genus 5. On the base of this example, and by using a deformation argument, Hassett and Tschinkel manage to prove the existence of an abelian fibration over $\mathbb{P}^2$ on the Hilbert square of the primitive K3 surface of degree $2m^2$, for any $m \geq 2$, see [3].

**The main result of the paper and a conjecture.** In this paper we construct an abelian fibration over $\mathbb{P}^3$ on the Hilbert cube of the general primitive K3 surface of genus 9. After Mukai’s example, this is the second case where the abelian fibration on $\mathcal{H}ilb_n S$ is constructed directly. This is also the first known abelian fibration on a Hilbert power of a primitive K3 surface, which is not a Hilbert square.

In addition, we consider the conjecture that if $S$ is a primitive K3 surface of genus $g \geq 2$, then the Hilbert scheme $X = \mathcal{H}ilb_n S$ admits a structure of an abelian fibration if and only if $2g - 2 = m^2(2n - 2)$ for some integer $m \geq 2$. This conjecture is posed as a question by several authors (cf. [4], [3]), so we claim no originality.

The Hassett-Tschinkel result covers the cases $n = 2, m \geq 2$ of this conjecture, and Mukai’s example corresponds to the particular case $n = m = 2$. Our example proves the conjecture in the case $n = 3, m = 2$.

**The construction of the abelian fibration in brief.** To explain our construction, we compare it with Mukai’s construction of the abelian fibration on the Hilbert square of a primitive K3 surface $S_8$ of genus 5.

The general $S = S_8$ is a complete intersection of three quadrics in $\mathbb{P}^5$. Let $\mathbb{P}(H^0(I_S(2))) = \tilde{\mathbb{P}}^2_S$ be the plane of quadratic equations of $S$ in $\mathbb{P}^5$. Inside $\tilde{\mathbb{P}}^2_S$ the subset $\Delta_S$ of singular quadrics is a smooth plane sextic, and the double covering of $\tilde{\mathbb{P}}^2_S$ branched along $\Delta_S$ defines uniquely a K3 surface $F_S$ of genus 2, the dual K3 surface of $S$ (cf. [14]). Any subscheme $\xi \in \mathcal{H}ilb_2 S$ spans a line $l_\xi = \langle \xi \rangle \subset \mathbb{P}^5$, and the set $L_\xi = \mathbb{P}(H^0(I_{S\cup l_\xi}(2)))$ of quadrics containing $S$ and $l_\xi$ is a line in $\tilde{\mathbb{P}}^2_S$. If $\mathbb{P}^2_S$ is the dual plane of $\tilde{\mathbb{P}}^2_S$, the association $f : \xi \mapsto L_\xi$ defines a regular map

$$f : \mathcal{H}ilb_2 S \to \mathbb{P}^2_S.$$ 

In turn, a fixed line $L \in \mathbb{P}^2_S$ defines uniquely a 3-fold $X_L \supset S$ in $\mathbb{P}^5$ as the common zero locus of the quadrics $Q \in L \subset \tilde{\mathbb{P}}^2_S$. By the definition of $f$, the 0-schemes $\xi$ in the
preimage \( A_L = f^{-1}(L) \) are intersected on \( S \) by the lines \( L \subset X_L \), thus the fiber \( A_L \) is isomorphic to the Fano family \( F(X_L) \) of lines on the threefold \( X_L \). The theorems in \[10\] imply that the map \( f \) is an abelian fibration, in particular for the general \( L \) the family \( F(X_L) \) is an abelian surface. The last is classically known: the general \( X_L \) is the quadratic complex of lines, and the family of lines \( F(X_L) \) is the jacobian of a genus 2 curve \( F_L \), the dual curve to \( X_L \); the last turns out to be a hyperplane section of \( F_S \) – the double covering \( F_L \) of \( L \) branched along the 6-tuple \( \Delta_L = \Delta_S \cap L \), see e.g. \[2\].

Let now \( S = S_{16} \subset P^9 \) be a general primitive K3 surface of genus 9, and let \( \mathcal{Hilb}_3 S \) be the Hilbert scheme of length 3 zero-schemes \( \xi \subset S \). Following the above example, we shall define step-by-step a map

\[
\begin{align*}
\text{Structure of the paper.} & \quad \text{The crucial step in our construction is to find the analog of lines through 2 points on } S_8. \text{ It turns out that these are twisted cubics on } \\
& \quad \Sigma = LG(3, 6), \text{ and the crucial property of these cubics is: Through the general triple } \xi \text{ of points on } \Sigma \text{ passes a unique twisted cubic } C_\xi \text{ that lies in } \Sigma. \\
& \quad \text{In Sections 2 and 3 we show that on the general K3 surface } S = S_{16} \subset LG(3, 6), \text{ if } \\
& \quad \xi \text{ is a 0-scheme on } S, \text{ then there exists on } LG(3, 6) \text{ a unique connected rational cubic curve } C_\xi \text{ intersecting on } S \text{ the zero-scheme } \xi. \text{ This identifies } \mathcal{Hilb}_3 S \text{ with the 6-fold } \\
& \quad \text{Hilbert scheme } \mathcal{C}(S) \text{ of twisted cubic curves } C \subset LG(3, 6) \text{ that lie in Fano 3-folds } \\
& \quad X_h, h \in P^3_S. \\
& \quad \text{If } F_S \subset \hat{P}_S^3 \text{ is the } Sp(3)-\text{dual quartic surface of } S, \text{ any } h \in P^3_S = \hat{P}_S^3* \text{ defines a } \\
& \quad \text{hyperplane section } F_h \subset F_S, \text{ the dual plane quartic of } X_h. \text{ For the general } h, F_h \text{ is a } \\
& \quad \text{smooth plane quartic and } X_h \text{ is a smooth prime Fano 3-fold of genus 9; one can regard this } X_h \text{ as the analog of the general quadratic complex of lines } X_L \text{ through the K3 surface } S_8. \text{ For } h' \neq h'' \text{ the families of cubics } \mathcal{C}(X_{h'}) \text{ and } \mathcal{C}(X_{h''}) \text{ do not intersect each}
\end{align*}
\]
other, thus giving a regular map

\[ f : \text{Hilb}_3S \cong \mathcal{C}(S) \to \mathbb{P}^3_S, \]

with \( h = f(\xi) \) identified with the 10-space \( \mathbb{P}^{10}_\xi = \text{Span}(S \cup C_\xi) \). The results of [10] imply that \( f \) is an abelian fibration, and the construction of \( f \) identifies the fiber \( A_h = f^{-1}(h) \) of \( f \) with the Hilbert scheme \( \mathcal{C}(X_h) \) of twisted cubic curves on the 3-fold \( X_h \).

In Section 4 we show that for the general \( h \) the family \( \mathcal{C}(X_h) \) is nothing else but the jacobian \( J(F_h) \) of the dual plane quartic \( F_h \) to the 3-fold \( X_h \). Just as in the case for the quadratic complex of lines, the abelian threefold \( A_h = \mathcal{C}(X_h) \) can be identified with the intermediate jacobian \( J(X_h) \) of the Fano 3-fold \( X_h \).

At the end, in Section 5 we describe the group law on the general fiber \( A_h \) of \( f \), in the interpretation of \( A_h \) as the Hilbert scheme \( \mathcal{C}(X_h) \) of twisted cubic curves on \( X_h \).

More precisely, any twisted cubic \( C_\circ \subset X_h \) defines on the abelian 3-fold \( A_h = \mathcal{C}(X_h) \) an additive group structure with \( C_\circ = 0 \), and we identify which cubic curve on \( X = X_h \) is the sum under this group operation of two general cubics \( C', C'' \subset X \). Notice that this is the analog of the Donagi’s group law on the family \( F(X_L) \) of lines on the 3-fold quadratic complex of lines (see [2]), identified with the general fiber of Mukai’s abelian fibration on \( \text{Hilb}_2S_8 \).

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### 2 Twisted cubic curves on \( \text{LG}(3, 6) \)

Let \( V \) be 6-dimensional vector space, let \( \alpha \in \wedge^2 V^* \) be a nondegenerate 2-form on \( V \). The natural linear map \( d_\alpha : \wedge^3 V \to V \) induced by \( \alpha \) has a 14-dimensional kernel \( W = \ker d_\alpha \). Consider the Plücker embedding \( \mathbf{G}(2, V) \subset \mathbb{P}(\wedge^3 V) \). Then \( \Sigma = \text{LG}(3, V) = \mathbb{P}(W) \cap \mathbf{G}(2, V) \subset \mathbb{P}(\wedge^3 V) \) is the 6-dimensional Grassmannian of Lagrangian planes in \( \mathbb{P}^5 = \mathbb{P}(V) \) with respect to \( \alpha \). If \( H \subset \Sigma \) is the hyperplane divisor in this Plücker embedding, then the degree of \( \Sigma \) is \( H^6 = 16 \), while the canonical divisor \( K_\Sigma = -4H \). Thus a general linear section \( \mathbb{P}^{10} \cap \Sigma \subset \mathbb{P}(W) \) is a Fano threefold of genus 9, while a general linear section \( \mathbb{P}^9 \cap \Sigma \subset \mathbb{P}(W) \) is a K3 surface of genus 9.

**Proposition 2.1.** (Mukai [12]) A general prime Fano threefold of genus 9 is the linear section \( \mathbb{P}^{10} \cap \text{LG}(3, V) \) for some \( \mathbb{P}^{10} \subset \mathbb{P}(W) \). A general K3-surface of genus 9 is the linear section \( \mathbb{P}^9 \cap \text{LG}(3, V) \) for some \( \mathbb{P}^9 \subset \mathbb{P}(W) \).

Let \( Sp(3) \subset SL(V) \) be the symplectic group of linear transformations that leaves \( \alpha \) invariant. Then \( W \) is an irreducible \( Sp(3) \)-representation and the four orbit closures of \( Sp(3) \) on \( \mathbb{P}(W) \) are

\[ \Sigma \subset \Omega \subset F \subset \mathbb{P}(W) \]
Their dimensions are 6, 9, 12 and 13. In particular, $F$ is a quartic hypersurface, the union of projective tangent spaces to $\Sigma$. Similarly, there are four orbit closures of $Sp(3)$ in the dual space:

$$\Sigma^* \subset \Omega^* \subset F^* \subset P(W^*),$$

isomorphic to the orbit closures in $P(W)$. In particular, the quartic hypersurface $F^*$ is the dual variety of $\Sigma$ and parametrizes tangent hyperplane sections to $\Sigma$. If $X = \Sigma \cap L$ is a linear section of $\Sigma$, and $L^\perp \subset P(W^*)$ is the orthogonal linear space, then we denote by $F_X$ the intersection $L^\perp \cap F^*$ and call it the $Sp(3)$-dual variety to $X$.

The four orbits in $P(W)$ are characterized by secant properties of $\Sigma$ (cf. [8]):

**Proposition 2.2.** Let $\omega \in P(W)$, then:

(a) If $\omega \in P(W) - \Omega$, then through $\omega$ passes a unique bisecant or tangent line $l_\omega$ to $\Sigma$. The line $l_\omega$ is tangent to $\Sigma$ if and only if $\omega \in F$.

(b) If $\omega \in \Omega - \Sigma$ then the set of lines which pass through $\omega$ and are bisecant or tangent to $\Sigma$ sweep out a 4-space $P^4_\omega \subset P(W)$, and the intersection $Q_\omega = P^4_\omega \cap \Sigma$ is a smooth 3-fold quadric. In the space $P^5$, there exists a point $x = x(\omega)$ such that the quadric $Q_\omega = Q_{x(\omega)} \subset \Sigma$ coincides with the set of Lagrangian planes that pass through the point $x(\omega)$.

Furthermore

**Lemma 2.3.** (a) A line not contained in $\Sigma$ intersect $\Sigma$ in a 0-scheme of length $\leq 2$.

(b) $\Sigma$ does not contain planes, and a plane that intersects $\Sigma$ along a conic section is contained in $\Omega$.

(c) A plane $P \cong P^2$ such that the intersection scheme $P \cap \Sigma$ contains a 0-scheme $Z$ of length 3 either intersects $\Sigma$ exactly at $Z$ or $P \cap \Sigma$ contains a line or a conic.

(d) A 3-space $P \cong P^3$ such that the intersection $P \cap \Sigma$ contains a curve $C$ of degree 3 either intersects $\Sigma$ exactly along $C$ and $C$ is defined by a determinantal net of quadrics or $P \subset \Omega$ and $P \cap \Sigma$ is a quadric surface of rank $\geq 3$.

**Proof.** Since $\Sigma$ is defined by quadrics, (a) follows. The planes that are parameterized by a conic have a common point $p \in P^5$, so the conic lies in the 3-dimensional smooth quadric $Q_p$ [22]. Since this quadric is smooth, it and therefore also $\Sigma$, contains no planes, and the conic is a plane section of $Q_p$. Hence $P \subset Q_p \subset \Omega$ and (b) follows. For (c), consider a plane $P$ whose intersection with $\Sigma$ contains a scheme $Z$ of length 4, and assume that the intersection is zero-dimensional. Then $Z$ must be a complete intersection of two conics, and the intersection $P \cap \Sigma$ is precisely $Z$. Let $S$ be a general $P^9$ that contains $P$, and let $S = \Sigma \cap P^9$. Then, by Bertini, $S$ is a smooth surface. In fact $S$ is a $K3$-surface. Since $Z$ define dependent conditions on hyperplanes $h$, there is a nontrivial extension of $I_{Z,S}$ by $O_S$ which define a rank 2 sheaf $E$ on $S$. Since no length three subscheme on $S$ is contained in a line, $E$ is a vector bundle (cf. [15]), and since no plane intersects $S$ in a scheme of length five $E$ is base point free. Therefore a general section of $E$ is a smooth subscheme of length four contained in a plane. But through
this subscheme there are two lines that meet in a point outside \( S \), contradicting the above proposition.

Let \( P \) be a \( \mathbb{P}^3 \) that intersects \( \Sigma \) in a curve \( C \). By (c) this curve has degree at most 3, so for (d) we may assume that the degree is three. Again by (c), the intersection is pure: There are no zero-dimensional components. Now, \( C \) cannot be a plane curve by part a). Furthermore, since \( \Sigma \) is the intersection of quadrics, \( C \) is contained in at least three independent quadrics. Pick two without common component, then for degree reasons alone they link \( C \) to a line and \( C \) is defined by a determinantal net of quadrics. If \( P \) intersects \( \Sigma \) in a surface, this surface, by (a) is a quadric of rank at least 3, and by (b) lies in \( \Omega \).

Lemma 2.4. Let \( M \) be a \( 2 \times 3 \)-matrix of linear forms in \( \mathbb{P}^3 \) whose rank 1 locus is a curve \( C_M \) of degree 3, and let \( p \in \mathbb{P}^3 \setminus C_M \). Then there is a unique line passing through \( p \) that intersects \( C_M \) in a scheme of length 2, unless \( p \) lies in a plane that intersects \( C_M \) in a curve of degree 2.

Proof. Let \( M(p) \) be \( M \) evaluated in \( p \). Then \( M(p) \) has rank 2, so let \((a_1, a_2, a_3)\) be the unique (up to scalar) solution to \( M(p) \cdot a = 0 \). Then \( M \cdot a \) defines two linear forms that vanish on \( p \). If they are independent, they define a line that intersects \( C_M \) in a scheme of length 2, and if they are dependent, then they define a plane that intersect \( C_M \) in a curve of degree 2. The uniqueness of the line in the first case follows by construction.

Lemma 2.5. Let \( \mathbb{P}^9 \subseteq \mathbb{P}(W) \) be general. In particular assume that \( S = \Sigma \cap \mathbb{P}^9 \) is a smooth K3 surface of genus 9 with no rational curve of degree less than 4. Then:

(a) A line \( l \) can intersect \( S \) in at most a 0-scheme of length \( \leq 2 \).

(b) If a plane \( \mathbb{P}^2 \) intersects \( S \) in a scheme containing a 0-scheme \( \xi \) of length 3 then \( \mathbb{P}^2 =< \xi > \) and \( \mathbb{P}^2 \cap S = \xi \).

(c) If a 3-space \( \mathbb{P}^3 \) is such that the intersection scheme \( \mathbb{P}^3 \cap S \) contains a 0-scheme \( \xi \) of length 3 and the intersection \( \mathbb{P}^3 \cap \Sigma \) contains a curve \( C \) of degree 3, then \( \mathbb{P}^3 \cap \mathbb{P}^9 =< \xi > \), \( \mathbb{P}^3 \cap S = \xi \) and \( \mathbb{P}^3 \cap \Sigma = C \); here \( \mathbb{P}^9 =< S > \).

Proof. (a) follows immediately from Lemma 2.3. By (a), \( < \xi > = \mathbb{P}^2 \). If the intersection scheme \( Z = \mathbb{P}^2 \cap S \supset \xi \) contains \( \xi \) properly, then by Lemma 2.3 \( Z \) will contain a line or a conic. But \( S \) contains no curves of degree less than 4, so (b) follows.

(c) Since \( \mathbb{P}^3 \cap S \supset \xi \) then \( \mathbb{P}^3 \) contains the plane \( \mathbb{P}^2 =< \xi > \subset \mathbb{P}^9 \), by (b). Therefore either \( \mathbb{P}^3 \cap \mathbb{P}^9 =< S > \) or \( \mathbb{P}^3 \cap \mathbb{P}^9 =< \xi > \). But if \( \mathbb{P}^3 \cap \mathbb{P}^9 \) then the curve \( C \subset \mathbb{P}^3 \cap \Sigma \) will be contained in \( S = \Sigma \cap \mathbb{P}^9 \), which is impossible since by assumption \( S = S_{16} \) does not contain curves of degree 3. Therefore \( \mathbb{P}^3 \cap \mathbb{P}^9 =< \xi > \) and \( \mathbb{P}^3 \cap S =< \xi > \cap S = \xi \).
If $P^3 \cap \Sigma$ contains more than the curve $C$ then by Lemma 2.3 the intersection $Q = P^3 \cap \Sigma$ will be a quadric surface of rank at least 3. But then $S \cap Q$ is a conic, contradicting the assumption that $S$ has no curve of degree less than 4.

**Definition 2.6.** Let $\text{Hilb}_{3t+1}(\Sigma)$ be the Hilbert scheme of twisted cubic curves contained in $\Sigma$ and let $\text{Hilb}_3(\Sigma)$ be the Hilbert scheme of length three subschemes of $\Sigma$.

Since $\Sigma$ contains no planes, every member $C$ of $\text{Hilb}_{3t+1}(\Sigma)$ is a curve of degree 3 defined by a determinantal net of quadrics: $C$ lies in at least 3 quadrics, and since there are no planes in $\Sigma$ two general ones intersect in a curve of degree 4 that links $C$ to a line.

Consider the incidence

$$I_3 = \{(\xi, C)|\xi \subset C\} \subset \text{Hilb}_3(\Sigma) \times \text{Hilb}_{3t+1}(\Sigma)$$

and the restriction to $S$:

$$I_3(S) = \{(\xi, C) \in I_3|\xi \subset S\}$$

**Definition 2.7.** Let $\mathcal{C}(S) \subset \text{Hilb}_{3t+1}(\Sigma)$ be the image of the projection

$$I_3(S) \rightarrow \text{Hilb}_{3t+1}(\Sigma),$$

i.e. the Hilbert scheme of twisted cubic curves in $\Sigma$ that intersect $S$ in a scheme of length three.

**Corollary 2.8.** Let $S = P^9 \cap \Sigma \in P(W)$ be a smooth linear section with no rational curves of degree less than four.

(a) $\mathcal{C}(S) \subset \text{Hilb}_{3t+1}(\Sigma)$ is a closed subscheme.

(b) The intersection map

$$\sigma : \mathcal{C}(S) \rightarrow \text{Hilb}_3S, \ C \mapsto C \cap S$$

is well defined on any $C \in \mathcal{C}(S)$.

**Proof.** Consider the map $\text{Hilb}_{3t+1}(\Sigma) \rightarrow G(4, 14)$ defined by $C \mapsto <C>$. By Lemma 2.6 the subset $\mathcal{C}(S)$ is simply the pullback under this map of the closed variety of spaces that intersect $P^9$ in codimension one. Therefore (a) follows, while (b) follows since no component of $C$ is contained in $S$.

**Lemma 2.9.** The intersection map $\sigma : \mathcal{C}(S) \rightarrow \text{Hilb}_3S$ is surjective.
Proof. By Corollary 2.8 the image $\sigma(\mathcal{C}(S))$ is a closed subscheme of $\text{Hilb}_3S$. Since $\text{Hilb}_3S$ is irreducible, to prove that $\sigma$ is surjective it is enough to see that $\sigma$ is dominant.

For a point $x \in S$ denote by $P^2_x \subset \mathcal{P}^5 = \mathcal{P}(V)$ the Lagrangian plane of $x$, and let

\[ \mathcal{U} = \{ \xi \in \mathfrak{H}ilb_3S : \xi = x + y + z \text{ is reduced and such that } \text{the Lagrangian planes } P^2_x, P^2_y \text{ and } P^2_z \text{ are mutually disjoint } \} . \]

Clearly $\mathcal{U} \subset \mathfrak{H}ilb_3S$ is open and dense, so it rests to see that

For any $\xi = x + y + z \in \mathcal{U}$ there exists a smooth twisted cubic $C \subset \Sigma$ that passes through $x$, $y$ and $z$.

Let $U_0$, $U_\infty$ and $U_y$ be the Lagrangian 3-spaces of $x$, $y$ and $z$ in the 6-space $V$, i.e. $P^2_x = \mathcal{P}(U_0)$, $P^2_z = \mathcal{P}(U_\infty)$ and $P^2_y = \mathcal{P}(U)$.

Since $P^2_x$ and $P^2_z$ do not intersect each other, we may write:

\[ V = U_0 \oplus U_\infty. \]

Furthermore, since $U_0$ and $U_\infty$ are Lagrangian, we may choose coordinates $(e_i, x_i)$ on $U_0$ and $(e_{3+i}, x_{3+i})$ on $U_\infty$, $i = 1, 2, 3$ such that in these coordinates the form $\alpha$ can be written as

\[ \alpha = x_1 \wedge x_4 + x_2 \wedge x_5 + x_3 \wedge x_6. \]

Since $P^2_y = \mathcal{P}(U)$ is also Lagrangian, and since $P^2_y$ does not intersect $P^2_z$, there exists a symmetric non-singular $3 \times 3$ matrix $B$ such that the Plücker coordinates of $y$ in the system $(e_i, x_i)$, $i = 1, ..., 6$ are uniquely written in the form

\[ y = \exp(B) = (1 : B : \wedge^2 B : \text{det}(B)). \]

For a parameter $t \in \mathbb{C}$, the matrix $tB$ correspond to a Lagrangian plane that does not intersect $P^2_z$. Thus

\[ C = C_{x,y,z} = \{ \exp(t) = (1 : tB : t^2 \wedge^2 B : t^3 \text{det}(B)), \ t \in \mathbb{C} \cup \infty \} \]

is a smooth twisted cubic on $\Sigma$ through $x = \exp(0)$, $y = \exp(1)$ and $z = \exp(\infty)$. 

\begin{proposition}
Let $S = \mathbb{P}^9 \cap \Sigma$ be a smooth linear section with no rational curves of degree less than four, and let $\mathcal{C}(S)$ be the Hilbert scheme of twisted cubic curves on $\Sigma$ that intersect $S$ in a scheme of length three. Then the restriction map

\[ \sigma : \mathcal{C}(S) \to \mathfrak{H}ilb_3S \quad C \mapsto C \cap S \]

is an isomorphism. In particular $\mathcal{C}(S)$ is a smooth projective variety.
\end{proposition}
Proof. First we show that \( \sigma \) is bijective. Let \( \xi \in \mathcal{Hilb}_3(S) \) let \( P \) be the span of \( \xi \) and assume that \( C_1, C_2 \in \mathcal{C}(S) \) with \( C_i \cap S = \xi \) for \( i = 1, 2 \).

Let \( q \in P \) be a general point. Since \( S \) contains no plane curves, \( P \) is not contained in \( \Omega \), so we may assume that \( q \) does not lie in \( \Omega \). Therefore there is a unique line through \( P \) that intersects \( \Sigma \) in a scheme of length 2. On the other hand for each \( i \), there is a unique line through \( q \) that meets \( C_i \) in a scheme of length 2. Hence these lines must coincide, and lie in \( P \). But the only way the general point \( q \) in \( P \) can lie on a unique line through \( \xi \), that intersects \( \xi \) in a scheme of length 2, is that \( \xi \) is the first order neighborhood of a point, i.e. when \( P \) is the tangent plane to \( S \) at the support \( q \) of \( \xi \). In this case both \( C_1 \) and \( C_2 \) must be singular at \( q \) and have at least one line component. In fact, if the tangent cone to \( C_i \) is planar, then this plane must contain the line component through \( q \). Since \( S \) does not contain lines, the tangent cone must span the 3-space, and each \( C_i \) is contained in the tangent cone to \( \Sigma \) at \( q \). But this tangent cone is the cone over a Veronese surface (cf. [8]). In the projection from \( \xi \) a unique line through \( q \) of \( \mathcal{G} \) restricted to \( \text{Im} \) is mapped to a line \( L \) through \( q \) that intersects \( \Sigma \) in a scheme of length 2. On the other hand, the ramification means that the doubling of \( q \) does not intersect the Veronese surface, so it is not contained in the cubic hypersurface secant variety of the Veronese surface. It is a well known classical fact that there is a unique plane through \( L \) that meet \( V \) in a scheme of length 3. This plane and the plane \( P \) spans a \( \mathbb{P}^3 \) that intersects \( \Sigma \) along three lines through \( q \). Therefore, also in this case \( C_1 \) and \( C_2 \) must coincide.

Next, we consider ramification of the map \( \sigma \). Consider the morphism \( g : \mathcal{C}(S) \rightarrow \mathbb{G}(4, 14) \), defined by \([C] \mapsto [< C >]\), and similarly \( g_3 : \mathcal{Hilb}_3 S \rightarrow \mathbb{G}(3, 10) \) defined by \([\xi] \mapsto [< \xi >]\). By Lemma 2.3, both \( g \) and \( g_3 \) are embeddings. Therefore the restriction map \( \sigma \) factors through \( g \), the restriction map \([\mathbb{P}^3] \mapsto [\mathbb{P}^3 \cap \mathbb{P}^9]\) and the inverse of \( g_3^{-1} \) restricted to \( \text{Im} g_3 \). Hence a point of ramification for \( \sigma \) is also a point of ramification for the restriction map \([< C >] = [\mathbb{P}^3] \mapsto [\mathbb{P}^3 \cap \mathbb{P}^9] = [< \xi >]\). If \([< C >]\) is such a point, then there is a tangent line to \( \text{Im} g \) at the point \([< C >]\) in \( \mathbb{G}(4, 14) \) that is collapsed by the restriction map. But this occurs only if the tangent line is contained in \( \mathbb{G}(4, 14) \) and parameterizes \( \mathbb{P}^3 \)s through \( < \xi > \) in a \( \mathbb{P}^4 \). Since \( < C > \cap \Sigma = C \), and \( \sigma \) is a bijective morphism, no other \( \mathbb{P}^3 \) in this pencil can intersect \( \Sigma \) in a twisted cubic curve. On the other hand, the ramification means that the doubling of \( < C > \) in \( \mathbb{P}^4 \) intersect \( \Sigma \) in a doubling \( D \) of \( C \), i.e. \( D \) is a nonreduced curve of degree 6. For each point \( p \in C \) consider the span \( P_p \) of the tangent cone to \( D \) at \( p \). On the one hand, any line in \( P_p \) through \( p \) is tangent to \( D \). On the other hand, the dimension of \( P_p \) is one more than the dimension of the span of the tangent cone to \( C \) at \( p \). Notice that if \( p \) and \( q \) are distinct points on \( C \), and the line \( \overline{pq} \) is not a component of \( C \), then \( P_p \cap P_q \cap \overline{pq} = \emptyset \); otherwise \( \overline{pq} \) is tangent to \( D \) at \( p \) or \( q \) and is therefore contained in \( C \) by Lemma 2.3.

Now, either \( C \) is reduced, and two general tangent lines span \( < C > \), or \( C \) has a singular point whose tangent cone spans \( < C > \).

In the first case, let \( p \) and \( q \) are smooth points on \( C \) such that their tangent lines to \( C \) span \( \mathbb{P}^3 \), then \( P_p \) and \( P_q \) are planes and \( P_p \cap P_q \) meet in a point \( r \). The lines \( \overline{mr} \) and \( \overline{mr} \) are both tangent to \( D \) so the plane spanned by \( p, q, r \) is by Lemma 2.3 contained in
Ω. As \( p \) and \( q \) moves, the lines \( \overline{pq} \) fill \( \langle C \rangle \), so \( \langle C \rangle \subset \Omega \) against the assumption. In the second case, let \( p \) be a singular point on \( C \) such that the tangent cone to \( C \) at \( p \) spans \( \langle C \rangle \), then \( P_p = P^4 \) is contained in the tangent cone to \( \Sigma \) at \( p \). In particular \( D \) is contained in this tangent cone. But this tangent cone is a cone over a Veronese surface, so if \( P_p \) intersects this cone in a curve of degree 6, the intersection cannot be proper. In fact, the only improper intersections of a \( P^4 \) with a cone over a Veronese surface that contains the vertex, is the union of a quadric cone surface and a line. But in our case, the quadric cone would then have to be contained in \( \langle C \rangle \), contradicting the assumption.

This concludes the argument that \( \sigma \) is bijective and unramified, i.e. an isomorphism.

3 The abelian fibration on \( \text{Hilb}_3 S \)

The Hilbert scheme \( \mathcal{C}(S) \) of twisted cubic curves on \( \Sigma \) that intersect \( S \) in a scheme of length three, admits a natural fibration over \( P^3 \). Composed with the inverse of the restriction map \( \sigma : \mathcal{C}(S) \to \text{Hilb}_3 S \), we obtain an fibration on \( \text{Hilb}_3 S \). In this section we describe this fibration and explain how it may also be obtained by a vector bundle approach as suggested in the introduction. An upshot of the alternative approach is the existence of genus three curves in the fibers (cf. Proposition 3.8). Let \( C \subset \Sigma \) be a twisted cubic curve that intersect \( S \) in a scheme of length three. Then the union \( C \cup S \) spans a \( P^{10}(C) \subset \langle \Sigma \rangle = P(W) \). Since

\[ \{P^{10}|S \subset P^{10} \subset P(W)\} \cong P^4_S \]

the assignment \( C \mapsto P^{10}(C) \) defines a map

\[ \tilde{f} : \mathcal{C}(S) \to P^3_S. \]

For any \( h \in P^3_S \), let \( X_h = \Sigma \cap P^{10}_h \supset S \). Then \( X_h \) is a threefold, in fact, the general \( X_h \supset S \) is a smooth Fano threefold of genus 9. The fiber \( \tilde{f}^{-1}(h) \) of the map \( \tilde{f} \) is

\[ \mathcal{C}(X_h) = \{C \in \text{Hilb}_{3t+1}(X_h)\}. \]

Assume that \( S \) contains no curves of degree \( \leq 3 \). Then the map \( \tilde{f} \) is a morphism. On the other hand the restriction map \( \sigma : \mathcal{C}(S) \to \text{Hilb}_3 S \) is in his case an isomorphism, so the inverse of \( \sigma \) composes with \( \tilde{f} \) to define a morphism

\[ f = \tilde{f} \circ \sigma^{-1} : \text{Hilb}_3 S \to P^3_S. \]

By the theorems of Matsushita (cf. [10]), \( f \) is a fibration on the hyperkähler 6-fold \( \text{Hilb}_3 S \), in particular \( f \) is a Lagrangian abelian fibration, i.e. the general fiber \( A_h \) is an abelian 3-fold which is a Lagrangian submanifold with respect to the nondegenerate 2-form on the hyperkähler 6-fold \( \text{Hilb}_3 S \). Thus
Theorem 3.1. Let \( S = \mathbb{P}^9 \cap \mathcal{L}G(3, V) \subset \mathbb{P}(W) \) be a smooth linear section with no rational curves of degree less than four, then \( \text{Hilb}_3 S \) admits a fibration \( f : \text{Hilb}_3 S \to \mathbb{P}^3_S \).

(a) For any \( h \in \mathbb{P}^3_S \) the fiber \( f^{-1}(h) = A_h = \sigma(\mathcal{C}(X_h)) \) where \( \mathcal{C}(X_h) \) is the Hilbert scheme of twisted cubic curves on the threefold \( X_h = \Sigma \cap \mathbb{P}^{10}_h \supset S \) and \( \sigma \) is the isomorphism above, defined by \( C \mapsto C \cap \mathbb{P}^9 \).

(b) For the general \( h \in \mathbb{P}^3_S \) the fiber \( A_h \) of \( f \) is an abelian 3-fold which is a Lagrangian submanifold of the hyperkähler 6-fold \( \text{Hilb}_3 S \), i.e.

\[
f : \text{Hilb}_3 S \to \mathbb{P}^3_S
\]

is a Lagrangian abelian fibration on \( \text{Hilb}_3 S \).

By the general choice of \( S \), the general \( X_h \supset S \) is a general smooth prime Fano 3-fold of genus 9 (cf. Proposition 3.1), so the above theorem implies

Corollary 3.2. The Hilbert scheme of twisted cubic curves \( \mathcal{C}(X) = \text{Hilb}_{3i+1}(X) \) of the general prime Fano threefold \( X \) of genus 9 is isomorphic to a 3-dimensional abelian variety.

In the next section we shall show that the abelian variety \( \mathcal{C}(X_h) \cong A_h \) coincides with the Jacobian of the \( \text{Sp}(3) \)-dual plane quartic curve \( F_X \) to \( X \).

Consider also an alternative approach.

First recall from [8] section 3.2 that zeros of a general 2-form \( \beta \in \wedge^2 V^* \) on \( \Sigma = \mathcal{L}G(3, V) \) is a Segre threefold \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \). The two 2-forms \( \alpha \) and \( \beta \) define a unique triple of conjugate lines with respect to the two forms. More precisely, \( \alpha \) defines a correlation \( L = L_\alpha \) which assigns to a point a hyperplane in \( \mathbb{P}^5 \) through the point. This correlation is symmetric, in the sense that \( y \in L(x) \) if and only if \( x \in L(y) \). A triple of lines \( L_1, L_2, L_3 \) in \( \mathbb{P}^5 \) are conjugate with respect to \( \alpha \) if \( \cap_{x \in L_i} L(x) = \langle L_j, L_k \rangle \) when \( 1, 2, 3 = i, j, k \). If \( \alpha \) and \( \beta \) are nondegenerate, and \( L_1, L_2, L_3 \) are conjugate w.r.t. both \( \alpha \) and \( \beta \), then the planes in \( \mathbb{P}^5 \) that are Lagrangian with respect to both forms are precisely the planes that meet the three lines. Clearly these planes are parameterized by a Segre threefold in \( \Sigma \), and any Segre threefold appear this way.

Lemma 3.3. A general twisted cubic curve \( C \) in \( \Sigma \) and a general tangent hyperplane section \( H_i \) to \( \Sigma \) that contains \( C \), determines uniquely a Segre threefold \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) in \( \Sigma \) that contains \( C \) and this threefold lies in \( H_i \).

The first general means that \( C \) parameterizes planes of a 3-fold \( V_C \cong \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5 \). The second general means that the plane \( P \) of the point of tangency of \( H_i \) meets this 3-fold in 3 distinct points.

Proof. The 3 points \( x_1, x_2, x_3 \) in \( P \cap V_C \) determines three lines \( L_1, L_2, L_3 \) in \( V_C \) that meet each plane in \( V_C \) in a point. It is enough to prove that these three lines are conjugate w.r.t. \( \alpha \) since uniqueness follows from the above. Notice that for each point
x in \( V_C \) the hyperplane \( L(x) \) intersects \( V_C \) in the plane \( P_x \) (through \( x \)) and a quadric surface \( Q_x = \mathbb{P}^1 \times \mathbb{P}^1 \). So for each point in \( Q_x \) there is a unique line in \( Q_x \) that meet every plane in \( V_C \) in a point. Consider the three points \( x_i \) and the corresponding quadrics \( Q_i = \mathbb{P}^1 \times \mathbb{P}^1 \). Since \( x_i \in P \), the hyperplane \( L(x_i) \) contains \( P \), so \( x_j \) and \( x_k \) also lies in \( L(x_i) \). Therefore \( x_j, x_k \in Q_i \), and \( L_j, L_k \) lies in \( Q_i \). Let \( x \in L(x_j) \) be a general point, then \( L(x) \) by symmetry contains \( x_i \), so \( L_i \) must lie in the quadric \( Q_x \). In particular the hyperplane \( L(x) \) contains \( L_i \). Therefore the three lines \( L_1, L_2, L_3 \) are conjugate with respect to \( \alpha \). The corresponding Segre threefold in \( \Sigma \) contains \( C \) and the point of \( P \), so by [IR] Lemma 3.2.5 it is contained in \( H_t \).

Recall from \( \Box \) the following proposition

**Proposition 3.4.** For each \( t \in F^* \setminus \Omega^* \subset \mathbb{P}(W^*) \), the tangent hyperplane section \( H_t \subset \Sigma \) projects from its point \( v(t) \) of tangency to a linear section of \( G(2, 6) \), hence it admits a rank 2 vector bundle \( E_t \). Every Segre threefolds in \( \Sigma \) that pass through \( v(t) \) is the zero locus of a section of this vector bundle, and the general section of \( E_t \) is of this kind.

**Remark 3.5.** For the general \( K3 \)-surface section \( S \) of \( \Sigma \), the set of tangent hyperplane sections \( H_t \) that contain \( S \) form the \( Sp(3) \)-dual quartic surface \( T = F_S \), hence the same surface \( T \) parameterizes rank 2 vector bundles \( E \) on \( S \) with determinant \( H \) and \( H^0(S, E) = 6 \). The general section of a vector bundle \( E \) vanishes along a subscheme of length 6 on \( S \).

**Remark 3.6.** Similarly for a Fano threefold section \( X \) of \( \Sigma \), the set of tangent hyperplane sections \( H_t \) that contain \( X \) form the \( Sp(3) \)-dual quartic curve \( F_X \), hence the same curve \( F_X \) parameterizes rank 2 vector bundles \( E \) on \( X \) with determinant \( H \) and \( H^0(X, E) = 6 \). The general section of a vector bundle \( E \) vanishes along an elliptic curve of degree 6, a codimension 2 linear section of a Segre threefold.

**Lemma 3.7.** Consider a Fano threefold section \( X = \mathbb{P}^{10} \cap \Sigma \subset \mathbb{P}(W) \). Let \( C \) be a twisted cubic curve contained in \( X \), and let \( H_t \) be a singular hyperplane section of \( \Sigma \) that contains \( X \). Then there is a unique twisted cubic curve \( C' \subset X \) with length\((C \cap C') = 2\) determined by \( t \). In particular, \( X \) admits a rank 2 vector bundle \( E_t \) and a unique section \( s \in \Gamma(X, E_t) \), that \( Z(s) = C \cup C' \).

**Proof.** Let \( Y_{C,t} \) be the unique Segre 3-fold through \( C \) that is contained in the tangent hyperplane section \( H_t \) of \( \Sigma \). Then \( Y_{C,t} \cap X = C \cup C' \) is a codimension 2 linear section of a Segre 3-fold. In particular \( C' \) is a twisted cubic curve and length\((C \cap C') = 2\). The final statement follows from Proposition 3.4 and Remark 3.6.

Consider the incidence

\[
I_{S,T} = \{ (\xi, t) \in \mathcal{H}ilb_3S \times T | H^0(S, E_t \otimes I_\xi) > 0 \}
\]

Clearly, by the above

\[
I_{S,T} = \{ (\xi, t) | C_\xi \subset H_t \}
\]
where $C_\xi$ is the unique twisted cubic curve through $\xi$. Now, $T = F_S$ is the $Sp(3)$-dual quartic surface to $S$, hence

$$T_\xi = \{ t \in T | (\xi, t) \in I_{S,T} \}$$

is the plane quartic curve on $T$ which is $Sp(3)$-dual to the threefold $S \cup C_\xi \cap \Sigma$. Therefore

$$\xi \mapsto T_\xi$$

defines again the map

$$f : Hilb_3 S \to |T_\xi| = \mathbb{P}^3_S.$$  

Let $A_\xi$ be the fiber $f^{-1}(f(\xi))$. Let $\eta \in A_\xi$ be a general point in the fiber. By Lemma 3.7 there is a morphism

$$\tau_\eta : T_\xi \to A_\xi$$

which assigns to a point $t \in T_\xi$ the element $(C' \cap S) \in A_\xi$, where $C'$ is the unique twisted cubic curve residual to $C_\eta$ in the Segre 3-fold defined by $C_\eta$ and $t$. By uniqueness, this map is injective: The curve $C_\eta$ is the unique twisted cubic curve residual to $C'$ determined by $t$.

**Proposition 3.8.** Let $S = \mathbb{P}^9 \cap \Sigma$ be a smooth linear section with no rational curves of degree less than four. Let $T = F_S$ be the $Sp(3)$-dual quartic surface, and consider the abelian fibration $f : Hilb_3 S \to |T_\xi| = \mathbb{P}^3_S$, where $T_\xi$ is the plane section of $T$ determined by $C_\xi$, the unique twisted cubic curve in $\Sigma$ through $\xi$. Let $A_\xi$ be the fiber $f^{-1}(f(\xi))$. When $T_\xi$ is smooth, then any element $\eta \in A_\xi$ defines an embedding $\tau_\eta : T_\xi \to A_\xi$.

## 4 Twisted cubic curves on the prime Fano threefolds of genus 9

In the previous section we concluded that the fibers of the fibration

$$f : Hilb_3 S \to \mathbb{P}^3_S$$

are abelian threefolds. From the construction we note that the fiber is identified with the subset of $\mathcal{C}(S)$ of twisted cubic curves $C$ that are contained in a fixed $P^9_h \supset S$. In particular, the general fiber coincides with the Hilbert scheme of twisted cubic curves in the Fano threefold $X_h = \Sigma \cap P^9_h$. The $Sp(3)$-dual variety to $X_h$ is a plane quartic curve $F_{X_h}$, the plane section $T_h = T \cap P^2_h$ of the $Sp(3)$-dual surface $T$ to $S$, where $P^2_h = (P^9_h)^{-1} \subset \mathbb{P}(W^*)$. In this section we shall prove the following theorem:

**Theorem 4.1.** Let $S = \mathbb{P}^9 \cap \Sigma \subset \mathbb{P}(W)$ be a smooth linear section with no rational curves of degree less than four. Let $T = F_S$ be the $Sp(3)$-dual quartic surface, and consider the abelian fibration $f : Hilb_3 S \to |T_h| = \mathbb{P}^3_S$. The general fiber $A_h = f^{-1}(h)$ is isomorphic to the jacobian of the $Sp(3)$-dual plane quartic curve $T_h$ to the Fano 3-fold $X_h \supset S$.  

13
For the proof we shall need to know some additional properties of the Hilbert scheme \( \mathcal{C}(X) \) of twisted cubic curves on the general prime Fano threefold \( X \) of genus 9. We begin with:

**Proposition 4.2.** Let \( X = \mathbb{P}^{10} \cap \Sigma \) be a general prime Fano threefold of index 1 and genus 9, and let \( F_X \) be its \( Sp(3) \)-dual plane quartic curve. Then the intermediate jacobian of \( X \) is isomorphic to the jacobian \( J(F_X) \) of \( F_X \).

**Proof.** In [13], Mukai identifies the intermediate jacobian of \( X \) with the Jacobian of a curve of genus 3, and in [7] this curve is identified with \( F_X \). \( \square \)

We shall show that the family \( \mathcal{C}(X) \) is isomorphic to the intermediate jacobian \( J(X) \). For this we shall use the birational properties of the Fano 3-fold \( X \) of genus 9 related to twisted cubic curves on \( X \).

Let \( B \subset X \subset \mathbb{P}^{10} \) be a smooth rational normal cubic curve, and consider the rational projection

\[ \pi : X \to \mathbb{P}^6 \]

from the space \( \mathbb{P}^3 = \langle B \rangle \). Let \( X' \subset \mathbb{P}^6 \) be the proper \( \pi \)-image of \( X \), and let \( \beta : \tilde{X} \to X \) be the blowup of \( X \) at \( B \). Since we may assume that \( \langle B \rangle \cap X = B \), the blowup \( \beta \) resolves the indeterminacy locus of \( \pi \), so the projection \( \pi \) extends to a morphism

\[ \tilde{\pi} : \tilde{X} \to X' \]

Recall from §4.1 in [6] that the cubic \( B \subset X \) fulfills the conditions (*)-(**) if:

- (*) the anticanonical divisor \( -K_{\tilde{X}} \) is numerically effective (nef) and \( (\xi - K_{\tilde{X}})^3 > 0 \) (i.e. \( -K_{\tilde{X}} \) is big);
- (**) there are no effective divisors \( D \) on \( X \) such that \( (\xi - K_{\tilde{X}})^3.D = 0 \).

By the Remark on page 66 in [6], the twisted cubic \( B \subset X \) will satisfy the conditions (*)-(**) if the morphism \( \tilde{\pi} \) has only a finite number of fibers of positive dimension.

**Lemma 4.3.** Suppose that \( X = X_{16} \) is general and the smooth twisted cubic \( B \subset X \) is general. Then the morphism \( \tilde{\pi} \) has only a finite number of irreducible fibers of positive dimension, and these are precisely the proper \( \beta \)-preimages of the \( e = e(B) = 12 \) lines on \( X \) that intersect the twisted cubic curve \( B \).

**Proof.** First, we may assume that \( \langle B \rangle \cap X = B \) by Lemma 2.5 and that \( B \) is irreducible. The family of planes in \( \mathbb{P}^5 \) parameterized by \( B \) sweeps out an irreducible 3-fold of degree 3. If this 3-fold is singular, it is a cone, and \( B \subset Q_p \) for some \( p \in \mathbb{P}^5 \), contrary to the assumption. Therefore this 3-fold is smooth, isomorphic to the smooth Segre embedding of \( \mathbb{P}^1 \times \mathbb{P}^2 \). Thus any two planes representing points of \( B \) are disjoint; in particular no conic in \( X \) intersects \( B \) in a scheme of length 2, since the Lagrangian planes representing the points of a conic in \( X \) always have a common point.
A positive dimensional fiber of $\tilde{\pi}$ is mapped by $\beta$ to a $\mathbb{P}^4 \supset <B>$ such that $\mathbb{P}^4 \cap X$ contains a curve $D$ which strictly contains $B$.

For a $\mathbb{P}^4$ as above the intersection $<D> \cap X$ can’t be of dimension $\geq 2$. Indeed the coincidence $<B> \cap X = B$ implies that if $<D> \cap X$ contains a surface $S$, then $S$ would have degree 3 since its intersection with $<B>$ is $B$. But then the span of $S$ must lie in $\Omega$ and $<B> \cap X$ must be a quadric surface, which contradicts to $\text{Pic}(X) = \mathbb{Z}$. We may assume that $<D> \cap X = D$. First note that by the ramification argument in the proof of Proposition 2.10, $D$ is reduced along $B$. Thus $D = B \cup C$, and $\deg C = \text{length} C \cap B$.

If $\deg C > 3$, then $<C> = \mathbb{P}^4$, and for $p \in B \setminus C$ the projection of $C$ from $p$ will have a 1-parameter family of trisecant lines. Therefore $D = B \cup C$ will have at least a 1-parameter family of 4-secant planes, and by Lemma 2.5 each of these planes will intersect $X$ in a conic. Hence the intersection $X \cap \mathbb{P}^4$ will contain at least a surface, which again contradicts to $\text{Pic}(X) = \mathbb{Z}$. Therefore $\deg C \leq 3$. By the preceding no component of $C$ can be a conic (a 1-cycle of degree 2), otherwise $X$ will have a 2-secant conic which is impossible – see above. Therefore either $C$ is an irreducible twisted cubic or any connected component of $C$ is a line.

But if $C$ is an irreducible irreducible twisted cubic then by the preceding the zero-scheme $C \cap B$ must be of length 3, i.e. $C$ and $B$ will be two twisted cubic curves that pass through the length 3 zero-scheme $C \cap B$, thus contradicting to Proposition 2.10.

Therefore any connected component of $C$ must be a line which is simply secant to $B$, i.e. any curve on $X$ contracted by $\pi$ must be a secant line to $B$.

Finally, since the Fano 3-fold $X = X_{16}$ is general, then by Theorem 4.2.7 in [6], the family of lines on $X$ is smooth, 1-dimensional, and the lines on $X$ sweep out on $X$ a divisor $R \in |\mathcal{O}_X(4)|$.

Consider a general linear section $S = \Sigma \cap \mathbb{P}^9$ and the 3-dimensional family of Fano 3-folds $X_t = \Sigma \cap \mathbb{P}^10_t$ that contain $S$. Let $R_t$ be the divisor swept out by the lines on $X_t$, and let $D_t = R_t \cap S$. Assume that every twisted cubic on $X_t$ is contained in $R_t$. By Theorem 3.1 the family of twisted cubic curves on any $X_t$ must be 3-dimensional, by the preceding any of these twisted cubic curves intersects the curve $D_t$ in a scheme of length 3, and by Proposition 2.10 any length 3 subscheme of $D_t$ is the intersection on $S$ of a twisted cubic curve on $X_t$. Now, $D_t$ is a curve in $|\mathcal{O}_S(4)|$, and two curves $D_t$ and $D_{t'}$ either have a common component or intersect each other in a zero scheme of length 256. Let $Z$ be a subscheme of length 3 of this intersection. Then there is a unique twisted cubic curve on both $X_t$ and $X_t'$, that intersects $S$ in $Z$. But these two curves must be distinct, contradicting to Proposition 2.10.

Therefore the general twisted cubic $C$ in the general $X = X_t$ lies outside $R = R_t$, in particular $C$ intersects a finite number of lines in $X$. Since $R \in |\mathcal{O}_X(4)|$ this finite number is 12.

□
Let $B \subset X = X_{16}$ be a general twisted cubic curve on a general $X = X_{16}$ that satisfies the conclusion of the proposition, in particular properties (*)-(**). Then by Lemma 4.1.1 in [6], there exists a commutative diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\xi} & \tilde{X}^+ \\
\beta \downarrow & \searrow & \pi^+ \downarrow \\
X' & \xrightarrow{\varphi} & Y
\end{array}
\]

where $\tilde{\pi}$ is as above and $\pi^+$ is another small contraction, $\xi: \tilde{X} \to \tilde{X}^+$ is a flop and $\varphi: \tilde{X}^+ \to Y$ is a Mori extremal morphism on the smooth threefold $\tilde{X}^+$, ibid.

**Lemma 4.4.** (see [Iskovskikh-Prokhorov, Chapter 4, Proposition 4.6.3 (iv)]). Let $B \subset X = X_{16}$ be a smooth twisted cubic curve in a smooth 3-fold which fulfills the properties (*)-(**). Then in the above diagram the extremal morphism $\varphi: \tilde{X}^+ \to Y$ is of type $\textbf{E1}$, i.e. a contraction of a divisor $D \subset \tilde{X}^+$ to a smooth curve $\Gamma \subset Y$ such that:

(a) $Y = Y_5 \subset \mathbb{P}^6$ is a smooth Fano threefold of index $r = r(Y) = 2$ and of degree 5;
(b) $\Gamma = \Gamma^3_9 \subset Y$ is a smooth curve of genus 3 and of degree 9.

Following the notation from §4 in [6], we denote by $H$ and $L$ the hyperplane classes of $X$ and $Y$, and by $H^* = \beta^*(H)$ and $L^* = \varphi^*(L)$ their preimages – correspondingly in $\text{Pic}(\tilde{X})$ and in $\text{Pic}(\tilde{X}^+)$.

Let $E = \beta^{-1}(B) \subset \tilde{X}$ be the exceptional divisor of $\beta$, and let $E^+ = \xi(E)$ be its proper transform in $\tilde{X}^+$. Let also $D = \varphi^{-1}(\Gamma) \subset \tilde{X}^+$ be the exceptional divisor of $\varphi$, and denote by $\tilde{D} \subset \tilde{X}$ its proper transform on $\tilde{X}$, ibid.:
Proposition 4.5. Let $X = X_{16}$ be general and let $B \subset X$ be a general twisted cubic on $X$. Let $\psi : X \to Y$ be the birationality in the above diagram. Denote by $\mathcal{C}(X)$ and by $\mathcal{C}(Y)$ the families of twisted cubic curves on $X$ and $Y$. Then, in the above notation:

(a) The birational map $\psi : X \to Y$ is defined by the non-complete linear system $|2H - 3B|$.

(b) The extremal divisor of $\psi$, the proper transform $D_X \subset X$ of the exceptional divisor $D$ of $\varphi$, is the unique effective divisor of the non-complete linear system $|3H - 5B|$ on $X$.

(c) The general extremal curve of $\psi$, the general smooth irreducible curve on $X$ contracted by $\psi$, is the general element of the compactified family $C[2](X; B) \subset \mathcal{C}(X)$ of twisted cubic curves $C \subset X$ bisecant to $B$.

Proof. We shall prove (1.a)-(1.b)-(1.c); as seen below, the proof of (2.a)-(2.b)-(2.c) is similar.

Proof of (1.a). Since $\varphi$ is an extremal morphism of type $E_1$, then by Lemma 4.1.2 (i) of [4] and the data $g = g(X) = 9$, $g(B) = 0$ and $deg(B) = 3$, one obtains:

\[
(-K_{\tilde{X}^+})^3 = 2(g(X) + g(B) - deg(B) - 1) - 2 = 8
\]

\[
(-K_{\tilde{X}^+})^2 E^+ = deg(B) + 2 - 2g(B) = 5
\]

\[
(-K_{\tilde{X}^+}) (E^+)^2 = 2g(B) - 2 = -2.
\]

\[
E^3 = -deg(B) + 2 - 2g(B) = -1
\]

Since, by Lemma 4.4, the morphism $\varphi : \tilde{X}^+ \to Y$ is extremal of type $E_1$ onto the Fano threefold $Y$ of index $r = 2$, then by Lemma 4.1.5 (ii) of [4], there exists an integer $\alpha \geq 1$ such that $\alpha + 1 \equiv 1(\text{mod } r)$, i.e. $\alpha$ is odd, and

\[
D = \alpha(-K_{\tilde{X}^+}) - rE^+ = \alpha(-K_{\tilde{X}^+}) - 2E^+
\]

in $Pic(\tilde{X}^+) = ZK_{\tilde{X}^+} + ZE^+$. Next, by taking in mind that $g(\Gamma) = 3$ (see Lemma 4.4), the third identity for $\varphi$ on p. 69 in [4] gives

\[
D^2 (-K_{\tilde{X}^+}) = 2g(\Gamma) - 2 = 4.
\]
Since by the preceding
\[ D^2.(-K_{\bar{X}^+})^2 = (\alpha(-K_{\bar{X}^+}) - 2E^+)^2.(-K_{\bar{X}^+}) \]
\[ = \alpha^2(-K_{\bar{X}^+})^3 - 4\alpha(-K_{\bar{X}^+}).E^+ - 4(-K_{\bar{X}^+})^2.(E^+)^2 = 8\alpha^2 - 20\alpha - 8, \]
then the identity \( D^2.(-K_{\bar{X}^+}) = 4 \) yields
\[ 2\alpha^2 - 5\alpha - 3 = 0. \]

Since \( 2\alpha^2 - 5\alpha - 3 = (2\alpha + 1)(\alpha - 3) \), and since by Lemma 4.1.5 (ii) of \[6\] the integer \( \alpha \) must be positive, then the only possibility left is \( \alpha = 3 \), i.e.
\[ D = 3(-K_{\bar{X}^+}) - 2E^+; \]
in particular \( \alpha \) is odd, ibid.

Since \( Y \) is a Fano threefold of index \( r = 2 \) then \( K_Y = -2L \), and since \( \varphi : \bar{X} \to Y \) is a blowup of a curve then
\[ K_{\bar{X}^+} = \varphi^*(K_Y) + D = -2L^* + D \]
in \( Pic(\bar{X}^+) = \mathbb{Z}L^* + \mathbb{Z}D; \) notice that \( D \) is the exceptional divisor of the blowup \( \varphi \) of the curve \( \Gamma \subset Y \). This, together with the previous formula for \( D \), yields \( K_{\bar{X}^+} = -2L^* + D = -2L^* + 3(-K_{\bar{X}^+}) - 2E^+ \). Therefore
\[ L^* = 2(-K_{\bar{X}^+}) - E^+; \]
and since \( \xi : \bar{X} \to \bar{X}^+ \) is a flop, then in \( Pic(\bar{X}) \)
\[ \bar{L} = 2(-K_{\bar{X}}) - E \]
for the proper \( \xi \)-preimage \( \bar{L} \) of \( L^* \).

Now we use that \( X \) is a Fano threefold of index 1, i.e. \( -K_X = H \) where \( H \) is the hyperplane class of \( X \), and that \( \beta : \bar{X} \to X \) is a blowup of a curve – the curve \( B \subset X \). Therefore
\[ K_{\bar{X}} = \beta^*(K_X) + E = -H^* + E; \]
recall that \( H^* = \beta^*(H) \) is the preimage of \( H \) and that \( E \) is the exceptional divisor of \( \beta \). Then by the above formulas
\[ L^* = 2(-K_{\bar{X}}) - E = 2(H^* - E) - E = 2H^* - 3E, \]
i.e. the composition map \( \varphi \circ \xi : \bar{X} \to Y \) is given by the linear system \( L^* = |2H^* - 3E|; \) notice that \( L \) is the hyperplane class on \( Y \). Since \( E \) is the exceptional divisor of the blowup of \( B \subset X \), the rational map \( \psi : X \to Y \) is given by the non-complete linear system \( |2H - 3B| \).
Proof of (1.b). We proceed as above: For the proper $\xi$-preimage $\tilde{D} \subset \tilde{X}$ of the exceptional divisor $D \subset \tilde{X}^+$ one obtains

$$\tilde{D} = 3(-K_{\tilde{X}}) - 2E = 3(H^* - E) - 2E = 3H^* - 5E,$$

i.e. the extremal divisor $D_X = \beta(\tilde{D}) \subset X$ belongs to the non-complete linear system $|3H - 5B|$ on $X$.

Proof of (1.c). The general extremal curve $C$ on $X$ is the same as the proper transform of the general fiber of $\varphi$. Since the map $\tilde{X} \to Y$ is given by the linear system $\tilde{L}$ then the proper preimage on $\tilde{X}$ of such curve is the same as the general irreducible curve $\tilde{C} \subset \tilde{X}$ such that $\tilde{C}.\tilde{L} = 0$, i.e.

$$\tilde{C}.(2H^* - 3E) = 0.$$ 

Therefore the general extremal curve $C \subset X$ is a smooth curve of degree $3n$ such that $\text{deg}(C.B) = 2n$ for some integer $n = n(C) \geq 1$. Since the compactified family of extremal curves on $X$ is a 1-dimensional algebraic family of 1-cycles on $X$ (with a smooth irreducible base isomorphic to the curve $\Gamma$), then the integer $n(C) = n$ does not depend on $C$, and it remains to see that $n = 1$.

The equality $n = 1$ is equivalent to say that $\text{deg}(C.B) = 2$ for the extremal curve $C \subset X$. By their definition, the extremal curves $C \subset X$ are the proper transforms on $X$ of the (extremal) curves $C^+ \subset \tilde{X}$ generating the extremal ray $R = R[C^+]$ defining the extremal morphism $\varphi$. Since $E$ is the exceptional divisor of $\beta$ over $B$ and $E^+$ is its proper transform on $\tilde{X}^+$, then $\text{deg}(C.B) = C^+.E^+$, and it rests to see that $C^+.E^+ = 2$.

For this recall that $\varphi : \tilde{X}^+ \to Y$ is an extremal contraction of type $E_1$; in particular the extremal ray $R = R[C^+]$ (generated by any of the extremal curves $C^+ \subset \tilde{X}^+$) has length 1, i.e.

$$\text{length}(R) = -K_{\tilde{X}^+}.C^+ = 1,$$

see e.g. Theorem 1.4.3 (Mori-Kollár) in [3]. Next we use the formula

$$E^+ = -2K_{\tilde{X}^+} - L^*$$

(see above), and the fact that the extremal curves $C^+ \subset \tilde{X}^+$, the fibers of the ruled surface $\varphi|_D : D \to \Gamma$, are orthogonal to the hyperplane class of $Y$, i.e. $C^+.L^* = 0$. Therefore

$$C^+.E^+ = C^+.(-2K_{\tilde{X}^+} - L^*) = 2C^+.(-K_{\tilde{X}^+}) = 2 \cdot \text{length}(R) = 2.$$ 

Hence $n = 1$, i.e. the extremal curves $C \subset X$ of $\psi$ are the twisted cubic curves on $X$ bisecant to $B$.

The proof of (2.a)-(2.b)-(2.c) is similar; notice that $\beta : \tilde{X} \to X$ is, just as $\varphi$, an extremal contraction of type $E_1$. \hfill $\square$

**Proposition 4.6.** The Hilbert scheme $\mathcal{C}(X)$ of twisted cubic curves on the general $X = X_{16}$ is isomorphic to the jacobian $J(\Gamma)$. 

19
Proof. By Corollary 3.2, the Hilbert scheme \( \mathcal{C}(X) \) is a 3-dimensional abelian variety. Therefore to prove the isomorphism between the abelian varieties \( \mathcal{C}(X) \) and \( J(\Gamma) \) it will be enough to see that they are birationally equivalent. For this we shall construct a natural birationality \( \phi = \phi_B : \mathcal{C}(X) \to Pic^5(\Gamma) \) defined by the choice of a general twisted cubic \( B \subset X \).

Let the twisted cubic \( B \subset X \) be as in Proposition 4.5 let \( \psi : X \to Y \) be the birationality defined by \( B \). Let \( C \subset X \) be another general twisted cubic on \( X = X_{16} \subset \mathbb{P}^{10} \); in particular \( C \) is smooth, and \( C \) spans, together with \( B \), a 7-space \( P^7_C = \langle C \cup B \rangle \subset \mathbb{P}^{10} \). In the space \( \mathbb{P}^{10} = \langle X \rangle > \), the family of codimension 2 subspaces \( P^8 \) that contain \( P^7_C \) is parameterized by the projective plane \( \hat{P}^2_C(t) = \mathbb{P}^{10}/P^7_C \). The general \( P^8_t \in \hat{P}^2_C(t) \) intersects on \( X \) a reduced canonical 1-cycle

\[
C^9_{16,t} = B + C + C_{10,t} \subset X
\]

of degree 16 and of arithmetical genus 9. For the general \( t \), the residual component \( C_{10,t} \) is a smooth elliptic curve of degree 10 on \( X \) intersecting any of the curves \( B \) and \( C \) at 5 points.

In the notation of Proposition 4.5, the proper \( \psi \)-image \( C_{5,t} \subset Y \) of \( C_{10,t} \subset X \) is a smooth projectively normal elliptic quintic intersecting \( X = X_{16} \subset \mathbb{P}^{10} \) at an effective 0-cycle \( D_t \) of degree 5. When \( P^8_t \) moves in \( \hat{P}^2_C(t) \), the 0-cycles \( D_t \) describe a \( P^2 \)-family of effective divisors of degree 5 on the genus 3 curve \( \Gamma \). Therefore all divisors \( D_t \) belong to the same linear system on \( \Gamma \). Since by the Riemann-Roch theorem the dimension of a complete linear system of degree 5 on a curve of genus 3 is always 2, the family \( \{D_t\} \) is in fact a complete linear system on \( \Gamma \), and we denote this system by \( |D_C| \).

Thus we have defined a rational map:

\[
\phi : \mathcal{C}(X) \to Pic^5(\Gamma), \quad C \mapsto |D_C|.
\]

Next we shall see that the rational map \( \phi \) is a birational. For this we shall construct the rational inverse to \( \phi \).

Let \( |D| = \{D_t : t \in \mathbb{P}^2\} \in Pic^5(\Gamma) \) be general, and let \( D_t \in |D| \) be general. The effective 0-cycle \( D_t \) of degree 5 on \( \Gamma \subset \hat{Y} \) spans a 4-space \( P^4_t \subset \mathbb{P}^6 = \langle Y \rangle > \). Since \( Y = Y_5 \) is the smooth del Pezzo threefold of degree 5 (i.e. the smooth codimension 3 transversal linear section of \( G(2, 5) \), see Theorem 3.3.1 in [6]), then the 4-space \( P^4_t \) intersects on \( Y \) a projectively normal elliptic quintic curve \( C_{5,t} \).

On the del Pezzo threefold \( Y \), there exists a \( P^1 \)-family (a pencil) of quadratic sections \( \{S_s : s \in \mathbb{P}^1\} \) containing the 1-cycle \( C_{21}^8 = \Gamma^3_9 + C_{5,t} \) of degree 14 and of arithmetical genus 8. The base locus of this pencil is a half-canonical 1-cycle \( C_{21}^{20} = \Gamma^3_9 + C_{5,t} + C_{6,t} \), where \( C_{6,t} \) is an elliptic sextic on \( Y \) intersecting \( \Gamma \) at a 0-cycle of degree 9. Now it is not hard to see that the residue 1-cycle \( C_{6,t} \) does not depend on the choice of \( t \); therefore

\[
C_{21}^{20} = \Gamma^3_9 + C_{5,t} + C_6,
\]

where \( C_6 = C_{6,t} \) for any \( t \).
By Proposition 4.5, the proper $\psi$-preimage of the curve $C_6 \subset Y$ is a twisted cubic curve $C \subset X$, while the proper preimage of $C_{5,t} \subset Y$ is an elliptic curve $C_{10,t} \subset X$ of degree 10 intersecting both $B$ and $C$ at 5 points. Turn back to the construction of the rational map $\phi$, we see that such $C_{10,t}$ can’t be other than one of the $\mathbb{P}^2$-family of residue elliptic curves of degree 10 in the construction of $\phi(C)$. Now the construction of $\phi$ implies that $\phi(C) = |D|$.

The existence of $\phi$ and $\phi^{-1}$ at the general points shows that the Hilbert scheme $\mathcal{C}(X)$ of twisted cubic curves on $X$ is birational to $\text{Pic}^5(\Gamma) \cong J(\Gamma)$. This proves the birationality between the abelian threefolds $\mathcal{C}(X)$ and $\text{Pic}^5(\Gamma) \cong J(\Gamma)$, and the proposition follows.

Let $F = F_X$ be the $Sp(3)$-dual curve parameterizing singular hyperplane sections of $\Sigma$ that contain the Fano threefold $X$, and let $J(F)$ be the jacobian of $F$. The embeddings from Proposition 3.8 translate into embeddings

$$\tau_C : F \to A = \mathcal{C}(X) \quad x \mapsto C(x)$$

defined for a general twisted cubic curve $C$ on $X$ by associating to a point $x \in F$ the unique twisted cubic curve $C(x) \subset X$ which is bisecant to $C$ and determined by the singular hyperplane section of $\Sigma$ defined by $x$.

**Proposition 4.7.** The Hilbert scheme $\mathcal{C}(X)$ is isomorphic to the jacobian $J(F)$ of the $Sp(3)$-dual quartic curve $F$ to $X$. Moreover, for the general $C \in A = \mathcal{C}(X)$ the set $\tau_C(F) = F_C \subset \mathcal{C}(X)$ is a translate of the Abel-Jacobi image of the curve $F$ in the abelian threefold $J(F) \cong \mathcal{C}(X)$.

**Proof.** By the criterion of Matsusaka, for the abelian 3-fold $A$ the intersection of any ample divisor $D \subset A$ with an effective 1-cycle $F_C \subset A$ is at least 3, and the equality $(D \cdot F_C) = 3$ is only possible if $A = J(F_C)$, $D$ is a copy of the theta divisor in $J(F_C)$, and $F_C \subset A$ is an Abel-Jacobi translate of $F_C$ in its jacobian $J(F_C) = A$.

Therefore to prove the proposition it will be enough to find an ample divisor $D \subset A = \mathcal{C}(X)$ such that $D \cdot F_C = 3$.

In order to describe such a divisor, recall that the Fano threefold $X$ contains a 1-dimensional family of lines, see e.g. Proposition 4.2.2 and Theorem 4.4.13 and Theorem 4.2.7 in [6]. In particular this 1-dimensional family of lines is parameterized by a smooth curve of genus 17. Fix a general line $\lambda \subset X$, and consider the subfamily

$$D_\lambda = \{ C \in A : C \cap \lambda \neq \emptyset \}.$$ 

We first need to show that $D_\lambda$ is an ample divisor. For this consider a curve $E$ on $\mathcal{C}(X)$ and the corresponding union of twisted cubic curves $S_E \subset X$. Since the family of lines in $X$ is parameterized by a smooth curve of genus 17, the general twisted cubic curve of $E$ cannot intersect every line on $X$. Therefore the intersection $D_\lambda \cdot E$
is finite. On the other hand $S_E$ is a hypersurface section of $X$, so the intersection 
$D_\lambda \cdot E \geq \text{length} S_E \cap \lambda > 0$. Hence $D_\lambda$ is ample. 

We proceed to show that 

$$D_\lambda \cdot F_C = 3$$

in $A$. Now, $F_C \subset \mathcal{C}(X)$ is simply the family of twisted cubic curves on $X$ bisecant to $C$. Therefore one has to see that among these there are exactly three twisted cubics that intersect the general line $\lambda \subset X$. 

By (1.c) in Proposition 4.3 the family $F_C$ of twisted cubic curves is just the family of extremal curves of the birationality $\psi$ defined by the twisted cubic $C \subset X$. But the extremal twisted cubic curves from the family $F_C$ sweep out the extremal divisor $D_X \in |3 - 5C|$, in particular $D_X$ is a cubic hypersurface section of $X$, see (1.b) in 4.5. As a cubic section of $X$, the surface $D_X$ intersects the line $\lambda \subset X$ at three points. Whence there are three twisted cubic curves bisecant to $C$ and secant to $\lambda$ on $X$. 

**Corollary 4.8.** Let the K3 surface $S = S_{16}$ be general, and let $T = F_S$ be the $Sp(3)$-dual quartic surface to $S$. Let $h \in P_S^3$ be general and let $P^2_h \subset P(W^*)$ and $P^{10}_h \subset P(W)$ be the corresponding orthogonal linear spaces. Then the Hilbert scheme $\mathcal{C}(X_h)$ of twisted cubic curves on the Fano $3$-fold $X_h = LG(3,6) \cap P^{10}_h$ is isomorphic to the jacobian $J(T_h)$, where $T_h = T \cap P^2_h$.

**Proof.** First, note that the general $X_h \supset S$ is a general prime Fano threefold of genus $9$. The hyperplane section $T_h = T \cap P^2_h \subset T$ is exactly the $Sp(3)$-orthogonal curve to $X_h$, so the corollary follows from Proposition 4.7. 

Finally, since the general fiber $A_h$ of $f$ is isomorphic to $\mathcal{C}(X_h)$ we have completed the proof of Theorem 4.1.

### 5 The group law on the Hilbert scheme of twisted cubic curves in a Fano threefold of genus $9$

We now fix a Fano threefold $X = P^{10} \cap \Sigma$ and its $Sp(3)$-dual curve $F = F_X$ parameterizing singular hyperplane sections of $\Sigma$ that contain $X$, and consider the Jacobian $J(F)$ which by Proposition 4.7 is identified with the Hilbert scheme $A = \mathcal{C}(X)$ of twisted cubic curves on $X$. Of crucial importance for our definition of the group law on $A = \mathcal{C}(X)$ are the embeddings 

$$\tau_C : F \rightarrow A = \mathcal{C}(X) \quad x \mapsto C(x)$$

defined for a general twisted cubic curve $C$ on $X$, by associating to a point $x \in F$ the unique twisted cubic curve $C(x) \subset X$ bisecant to $C$ and determined by the singular hyperplane section of $\Sigma$ defined by $x$. We denote by $F_C \subset A$ the image $\tau_C(F)$. 

22
To study the various curves $F_C$ in $A$, we fix, once and for all, a point $x_0 \in F$. Let

$$\alpha : F \to F_0 \subset A \quad x \mapsto C_x$$

be the Abel-Jacobi map with base point $x_0$, and fix the image $\alpha(x_0) = C_0 \in F_0 \subset A$ as the origin of the group structure on $A$. Recall from Proposition 3.4 that each $x \in F$ determines a rank 2 vector bundle $E_x$ on $X$.

**Lemma 5.1.** For each twisted cubic curve $C \subset X$, the curve $C \cup C(x)$ is the zero-locus of a section of the rank 2 vector bundle $E_x$ on $X$.

**Proof.** This follows immediately from Lemma 3.7. □

**Lemma 5.2.** The translate $F_\infty := F_C + C$ is independent of $C$.

**Proof.** By Lemma 5.1, the point $C + C(x) = C' + C'(x)$ for any $x \in F$, since $C \cup C(x)$ and $C' \cup C'(x)$ are both the zero locus of a section of the vector bundle $E_x$ on $X$. In particular $F \to A \quad x \mapsto C + C(x)$ is well defined and independent of $C$. The image $\{C + C(x)|x \in F\} \subset A$ is clearly the translation $F_C + C$. □

By the identification of $A$ with $J(F) = \text{Pic}_0(F)$, there exists a constant $c_\infty \in \text{Pic}_1(F)$ such that the Abel-Jacobi translate

$$F_\infty = \{x - c_\infty : x \in F\} \subset \text{Pic}_0(F) = A,$$

where $x - c_\infty$ denotes the rational equivalence class of the divisor.

**Corollary 5.3.** For any $x \in F$ and any $C \in A$, the following identity holds

$$C(x) + C = x - c_\infty \in \text{Pic}_0F = A.$$
Now we are ready to describe the group law on the threefold $A$. For this we shall need the following elementary technical lemma

**Lemma 5.5.** For a plane quartic curve $F$, any element $D \in \text{Pic}_1(F)$ can be represented as the rational equivalence class of a divisor $x - y + z$ for some $x, y, z \in F$.

*Proof.* Follows from the Riemann-Roch theorem. \hfill \Box

**The sum of two twisted cubic curves in $X$**

Let $A$ be the abelian 3-fold above, identified with the jacobian $J(F)$ of the plane quartic curve $F$, see Theorem [4.1].

Let $(A, +; C_0)$ be the additive group structure on $A$, with the element $C_0$ declared to be the neutral element. Below we shall find the curve $C'' \in A$ representing the sum $C' + C''$ of two general elements $C'$ and $C''$ of $A$.

By Lemmas 5.4 and 5.5, there exists a triple of points $x, y, z \in F$ such that

\[
C'' = C'(xyz)
\]

is the result of applying the chain of transformations

\[
C' \rightarrow C'(x) \rightarrow C'(xy) \rightarrow C'(xyz)
\]

as above. Notice that the result is independent of which representative $x - y + z$ in its rational equivalence class of divisors one chooses. Let $C_0(xyz)$ be the result of the same chain of transformations to $C_0$:

\[
C_0 \rightarrow C_0(x) \rightarrow C_0(xy) \rightarrow C_0(xyz),
\]

i.e. $C_0(x) \in F_{C_0}$ is the point representing the point $x \in F$ under the isomorphism $F \rightarrow F_{C_0}$, $C_0(xy) \in F_{C_0(x)}$ is the point representing the point $y \in F$ under the isomorphism $F \rightarrow F_{C_0(x)}$, and $C_0(xyz) \in F_{C_0(xy)}$ is the point representing the point $z \in F$ under the isomorphism $F \rightarrow F_{C_0(xy)}$. 

![Diagram](attachment:image.png)
We shall see that the point
\[ C'''' = C_0(xyz) \in A \]
is the sum of the points \( C' \) and \( C'' \) in \((A_h, \hat{+}; C_0)\).

For this, we observe that since \( C_0(xyz) = C'''' \) is obtained from \( C_0 \) by the same chain of transformations as \( C'' = C'(xyz) \) from \( C' \) then, by Lemma 5.4
\[ C' + C'' = C' + C'(xyz) = x - y + z - c_\infty = C_0 + C_0(xyz) = C_0 + C'''' . \]
Therefore \( C'''' + C_0 = C' + C'' \), and since \( C_0 \) was chosen to be the zero in \((A_h, \hat{+}; C_0)\) then
\[ C'''' = C' \hat{+} C'' , \]
in the addition law of the group \((A_h, \hat{+}; C_0)\).

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