An improved performance of greedy heuristic solutions for a bi-criteria mixture packaging problem of two types of items with bounded weights

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Received: 8 August 2019; Revised: 25 December 2019; Accepted: 22 January 2020

Abstract
In this paper, we treat a lexicographic bi-criteria combinatorial optimization model of mixture packaging of two types of items, which arises in actual food packing systems, so-called multi-head weighers. The primary objective is to minimize the total weight of chosen items for a package, and the second objective is to maximize the total priority of them. The constraints are that the total weight must be no less than a given target weight, and that the weight sum of chosen items of each type must also be no less than a given necessity minimum. The weight of an item of each type is bounded by the necessity minimum from the above. We show that a greedy heuristic solution with the total weight at most twice the minimum attains the total priority at least the conditionally maximum, which is an improved performance guarantee of greedy heuristic solutions for the lexicographic bi-criteria mixture packaging problem of two types of items.

Keywords: Combinatorial optimization, Mixture packaging, Subset sum, Greedy heuristic algorithms, Performance guarantee

1. Introduction

In this paper, we revisit the performance guarantee of greedy heuristic solutions for a lexicographic bi-criteria mixture packaging problem of two types of items (see Karuno and Nakahama, 2017). The combinatorial optimization problem arises in actual food packing systems, known as multi-head weighers, e.g., see Wurdemann, et al. (2011), Mechanical Engineering Heritage (2017). In this paper, we are interested in the mathematical aspect of the lexicographic bi-criteria mixture packaging problem of two types of items, and please see the literature for the detail of the repetitive motion of multi-head weighers, e.g., Morinaka (2000). We have also explained the repetitive motion of multi-head weighers in some our previous papers, e.g., see Imahori et al. (2014), Imahori et al. (2016), Karuno and Nakahama (2018).

We are given a finite set $I$ of items with their positive integer weights and positive integer priorities, and its partition $I = I_1 \cup I_2$ with $I_1 \cap I_2 = \emptyset$, where $I_k$ is the set of items of type $k \in \{1, 2\}$. In practice, the priority of each item can be defined to be the duration of the item in the food packing system. The duration of an item is caused by the repetitive nature of packaging operation (see Karuno et al., 2007, Karuno and Nakahama, 2017). The primary objective is to minimize the total weight of chosen items for a package, and the second objective is to maximize the total priority of them. The constraints are that the total weight of chosen items for a package must be no less than a prescribed integral target weight, and that the weight sum of chosen items of each type must also be no less than a prescribed integral necessity minimum. The primary objective aims at making the total weight of a package as close to the target weight as possible, together with the target weight constraint. The second objective expects the duration of an item in the food packing system to be reduced in an operation run.
The lexicographic bi-criteria mixture packaging problem of two types of items is NP-hard, while it can be solved in pseudo-polynomial time (see Imahori et al., 2014, Imahori et al., 2016). The pseudo-polynomial time algorithm based on dynamic programming is also applied to a heuristic algorithm design for the problem of minimizing the total weight of chosen items in a feasible solution (see Karuno and Nakahama, 2018). That is, for a given real \( \epsilon > 0 \) (with an upper limit), the performance factor of the heuristic algorithm on the minimum total weight is at most \( (1 + \epsilon) \), and the time complexity is polynomial in \( n = |I| \) and \( 1/\epsilon \). Such a heuristic algorithm is known as a fully polynomial time approximation scheme (FPTAS), e.g., see Vazirani (2001).

A greedy heuristic performance has been considered for the lexicographic bi-criteria mixture packaging problem of two types of items, where the weight of an item of each type is bounded by the necessity minimum from the above (see Karuno and Nakahama, 2017). It has been showed that a greedy heuristic solution with the total weight at most twice the minimum attains the total priority at least the conditionally maximum. In this paper, we also assume that the weight of an item of each type is bounded by the necessity minimum from the above. We then show that a greedy heuristic solution with the total weight at most twice the minimum attains the total priority at least the conditionally maximum, which is an improved performance guarantee of greedy heuristic solutions for the lexicographic bi-criteria mixture packaging problem of two types of items.

2. Preliminaries

2.1. Problem Description

We first review an instance of the lexicographic bi-criteria mixture packaging problem of two types of items. Let \( I = \{i | i = 1, 2, \ldots, n\} \) denote a set of current items with their integer weights \( w_i > 0 \) and integer priorities \( p_i > 0 \), where \( I_1 = \{i | i = 1, 2, \ldots, r\} \subseteq I \) is the set of \( r (\leq n) \) items of the first type, and \( I_2 = I \setminus I_1 = \{i | i = r + 1, r + 2, \ldots, n\} \) is the set of \( (n - r) \) items of the second type. Let \( t > 0 \) denote a target weight of a package, and let \( b_k > 0 \) denote a necessity minimum of the weight sum of chosen items of type \( k \in \{1, 2\} \) in a package. The three weight parameters \( t, b_1, \) and \( b_2 \) are integral, and they satisfy

\[
\begin{align*}
t &\leq \sum_{i=1}^{n} w_i, \quad (1) \\
\max_{1 \leq i \leq r} \left\{w_i\right\} &\leq b_1 \leq \sum_{i=1}^{r} w_i, \quad (2) \\
\max_{r+1 \leq i \leq n} \left\{w_i\right\} &\leq b_2 \leq \sum_{i=r+1}^{n} w_i. \quad (3)
\end{align*}
\]

In order to omit some trivial cases, we assume that

\[
t \geq b_1 + b_2, \quad (4)
\]

which implies

\[
t \geq \max_{1 \leq i \leq n} \left\{w_i\right\}. \quad (5)
\]

In this paper, we further renumber the items of each type so that they meet

\[
\frac{p_1}{w_1} \geq \frac{p_2}{w_2} \geq \cdots \geq \frac{p_r}{w_r}, \quad (6)
\]

\[
\frac{p_{r+1}}{w_{r+1}} \geq \frac{p_{r+2}}{w_{r+2}} \geq \cdots \geq \frac{p_n}{w_n}. \quad (7)
\]

Let \( \sigma = \sigma(1), \sigma(2), \ldots, \sigma(n) \) denote a permutation on the entire item set \( I \) such that it satisfies

\[
\frac{p_{\sigma(1)}}{w_{\sigma(1)}} \geq \frac{p_{\sigma(2)}}{w_{\sigma(2)}} \geq \cdots \geq \frac{p_{\sigma(n)}}{w_{\sigma(n)}}. \quad (8)
\]

Also, let

\[
\kappa = \min \left\{j \in \{1, 2, \ldots, n\} | \sum_{i=1}^{j} w_{\sigma(i)} \geq t \right\} \quad (9)
\]

denote the critical item index, and let

\[
\gamma = \frac{p_{\sigma(\kappa)}}{w_{\sigma(\kappa)}} \quad (10)
\]
denote the critical priority rate. Note that $\sum_{i=1}^{r} w_{ri} < 2r$ holds under the assumption of $w_{ij} < t$ for any item $i \in I$ (see Eq. (5)).

For a package to be produced from the current $n$ items, we now provide the following 0-1 integer programming formulation:

**Problem P**

\[
\begin{align*}
\text{minimize} & \quad f(x) = \sum_{i=1}^{n} w_{i}x_{i} \quad \text{as the primary objective,} \\
\text{subject to} & \quad \sum_{i=1}^{n} w_{i}x_{i} \geq t, \\
& \quad \sum_{i=1}^{n} w_{i}x_{i} \geq b_{1}, \\
& \quad \sum_{i=r+1}^{n} w_{i}x_{i} \geq b_{2}, \\
& \quad x_{i} \in \{0, 1\}, \quad i = 1, 2, \ldots, n,
\end{align*}
\]

where each 0-1 variable $x_{i}$ is defined to be

\[
x_{i} = \begin{cases} 
1 & \text{if the } i\text{-th item is chosen,} \\
0 & \text{otherwise.}
\end{cases}
\]

A vector $x = (x_{1}, x_{2}, \ldots, x_{n})$ of the 0-1 variables satisfying Eqs. (13)–(16) is referred to as a feasible solution of problem P. For an instance of problem P, let $f^{*}$ denote the minimum of the total weight of chosen items in a feasible solution, and let $x = x^{*}$ denote a feasible solution which attains the minimum of the total weight, i.e., $f^{\ast} = f(x^{*})$. An optimal solution $x = x^{*}$ is defined to be a feasible solution such that it satisfies $f(x^{*}) = f^{*}$ and it maximizes the total priority among feasible solutions with the minimum total weight $f^{*}$, i.e., it satisfies $g(x^{*}) \geq g(x)$ for any feasible solution $x^{*}$ with $f(x^{*}) = f^{*}$. We call $g^{\ast} = g(x^{*})$ the conditionally maximum total priority. Problem P asks to find an optimal solution.

### 2.2. Heuristic Performance Evaluation

For the maximization of the total priority (i.e., the second objective of problem P), we remark that choosing all the items in the set $I$ for a package can just attain the true maximum value. Let $g_{\text{max}} = \sum_{i \in I} p_{i}$ denote the true maximum value, i.e., the priority sum over all the items in the set $I$. Clearly, it meets $g_{\text{max}} \geq g^{\ast}$ to the conditionally maximum total priority.

Consider the following instance of problem P:

- $I = I_{1} \cup I_{2}$ where $I_{1} = \{1, 2, \ldots, r\}$ and $I_{2} = \{r + 1, r + 2, \ldots, 2r\}$ with $r \geq 4$ and $n = 2r$,
- $w_{1} = w_{r+1} = 45$, $w_{2} = w_{r+2} = 45$, $w_{i} = w_{r+i} = 25$ for $i = 3, 4, \ldots, r$,
- $p_{1} = p_{r+1} = 2$, $p_{2} = p_{r+2} = 2$, $p_{i} = p_{r+i} = 1$ for $i = 3, 4, \ldots, r$,
- $t = 100$, $b_{1} = b_{2} = 50$.

Obviously, the minimum of the total weight in a feasible solution can be equal to the target weight, i.e., $f^{\ast} = t = 100$, and we see an optimal solution $x^{*} = (x_{1}^{*}, x_{2}^{*}, \ldots, x_{n}^{*})$ to be

\[
x_{i}^{*} = \begin{cases} 
1 & \text{if } i \in \{3, 4, r + 3, r + 4\}, \\
0 & \text{otherwise},
\end{cases}
\]

which indicates

\[
f(x^{*}) = w_{3} + w_{4} + w_{r+3} + w_{r+4} = 25 + 25 + 25 + 25 = 100
\]

of the total weight and

\[
g(x^{*}) = p_{3} + p_{4} + p_{r+3} + p_{r+4} = 1 + 1 + 1 + 1 = 4
\]
of the total priority. On the other hand, the priority sum over all the items is obtained to be

$$g_{\text{max}} = \sum_{i \in I} p_i = 2 \times 2 + (r - 2) + 2 \times 2 + (r - 2) = 2r + 4 = n + 4.$$  

Also, construct another feasible solution $x' = (x'_1, x'_2, \ldots, x'_r)$ to be

$$x'_i = \begin{cases} 
1 & \text{if } i \in \{1, 2, r + 1, r + 2\}, \\
0 & \text{otherwise},
\end{cases}$$

which indicates

$$f(x') = w_1 + w_2 + w_{r+1} + w_{r+2} = 45 + 45 + 45 + 45 = 180$$

of the total weight and

$$g(x') = p_1 + p_2 + p_{r+1} + p_{r+2} = 2 + 2 + 2 + 2 = 8$$

of the total priority. Notice that it satisfies $f(x') \leq 2f(x^*)$ and $g(x') = 8 \geq g(x^*) = 4$, while $g(x') = 8 < g_{\text{max}} = n + 4$ holds and $g(x')/g_{\text{max}} \to 0$ when $n \to \infty$.

A performance guarantee analysis for an optimization problem with multiple criteria is to design an FPTAS, e.g., see Bazgan, et al. (2009), Erlebach, et al. (2002), which delivers a feasible solution such that all objective function values of the feasible solution satisfy a given accuracy $\varepsilon > 0$ from each optimum. However, if we approach problem P in such a manner, then we need to introduce a mathematical transformation, e.g., scaling and shifting, into the problem so that a heuristic solution meets a given accuracy for both of the two objective functions. Notice that for the problem instance provided in the above, there are no feasible solutions with the total weight at most $2f(x^*)$ such that they attain the total priority at least $g_{\text{max}}/2$ when the number $n$ of items is relatively large. Instead of introducing some mathematical transformation, we here evaluate the performance guarantee of the total priority of a greedy heuristic solution by the ratio to the conditionally maximum $g^*$. Note that the conditionally maximum of the total priority is brought us by the lexicographic optimization modeling.

Let $m$ denote the number of item types contained in the item set $I$. In this paper, we are treating the problem with $m = 2$. Karuno and Saito (2017) have considered the problem with $m = 1$, and showed that a greedy heuristic solution with the total weight at most twice the minimum attains the total priority at least the conditionally maximum. As mentioned in the previous section, Karuno and Nakahama (2017) have started the performance guarantee analysis of greedy heuristic solutions for the problem with $m = 2$, but they have only illustrated an instance-dependent ratio of the greedy heuristic total priority to the conditionally maximum, which is unfortunately less than one. Therefore, the performance guarantee which we are going to show in this paper is a meaningful improvement. On the other hand, we do not know at present whether the problem with a general $m \geq 3$ admits such a polynomial time greedy heuristic algorithm.

Further, an allowable error in the primary objective function may be related to a tolerance from the target weight for a package. From the viewpoint, numerical results have been reported in Hashiguchi et al. (2015), Karuno et al. (2007) for the problem with $m = 1$. For the problem with $m = 2$, some numerical results of greedy heuristic solutions with the total weight at most twice the minimum have also been reported in Karuno and Nakahama (2017), where the total weight of a greedy heuristic solution has empirically been demonstrated how far from the minimum of the total weight obtained by an optimal solution. On the other hand, we would like to concentrate our attention on the theoretical improvement of the performance guarantee in this paper.

### 3. Greedy Item Selection

#### 3.1. Preprocessing

Recall the permutation $\sigma$ on the entire item set $I$ satisfying Eq. (8). From the $\sigma$, we construct a greedy solution $\tilde{x}$ for a given instance of problem P such that for each item $i \in I$,

$$\tilde{x}_i = \begin{cases} 
1 & \text{if } i \in \{\sigma(1), \ldots, \sigma(\kappa)\} \subseteq I, \\
0 & \text{otherwise},
\end{cases}$$

(18)

which meets $t \leq \sum_{i=1}^{\kappa} w_i \tilde{x}_i < 2t$ (see Eq. (9)). Suppose that the greedy solution $\tilde{x}$ satisfies $\sum_{i=1}^{\kappa} w_i \tilde{x}_i \geq b_1$ and $\sum_{i=\kappa+1}^{\eta} w_i \tilde{x}_i \geq b_2$ both. Then, the $\tilde{x}$ is a feasible solution with the total weight at most twice the minimum for the given instance, i.e., $f(\tilde{x}) \leq 2f^*$, since the minimum total weight clearly meets $f^* \geq t$ (see Eq. (13)).
Also, from the definition of the critical priority rate (see Eq. (10)), we see that
\[
g(x^*) = \sum_{i=1}^{n} p_i x_i^* = \sum_{i=1}^{n} p_{\sigma(i)} x_{\sigma(i)}^* = \sum_{i=1}^{k} p_{\sigma(i)} x_{\sigma(i)}^* + \sum_{i=k+1}^{n} p_{\sigma(i)} x_{\sigma(i)}^*
\leq \sum_{i=1}^{k} p_{\sigma(i)} x_{\sigma(i)}^* + \gamma \sum_{i=k+1}^{n} w_{\sigma(i)} x_{\sigma(i)}^*
= \sum_{i=1}^{k} (p_{\sigma(i)} - \gamma w_{\sigma(i)}) x_{\sigma(i)}^* + \gamma \sum_{i=k+1}^{n} w_{\sigma(i)} x_{\sigma(i)}^*
\leq \sum_{i=1}^{k} (p_{\sigma(i)} - \gamma w_{\sigma(i)}) \bar{x}_{\sigma(i)} + \gamma \sum_{i=k+1}^{n} w_{\sigma(i)} x_{\sigma(i)}^*
= \sum_{i=1}^{k} (p_{\sigma(i)} - \gamma w_{\sigma(i)}) x_{\sigma(i)} + \gamma \sum_{i=k+1}^{n} w_{\sigma(i)} x_{\sigma(i)}^*,
\]
since \( p_{\sigma(i)} - \gamma w_{\sigma(i)} \geq 0 \) and \( 0 \leq x_{\sigma(i)}^* \leq \bar{x}_{\sigma(i)} = 1 \) hold for each \( i = 1, 2, \ldots, k \). We rewrite the above relation
\[
g(x^*) \leq g(\bar{x}) - \gamma f(\bar{x}) + \gamma f(x^*),
\]
and we reach our goal, i.e., we obtain \( g(\bar{x}) \geq g(x^*) = g^* \), since \( f(\bar{x}) \geq f(x^*) = f^* \) holds by the minimality of the \( x^* \) on the total weight when the \( \bar{x} \) is a feasible solution.

Therefore, in the following, we focus on the case in which \( \Sigma_{i=1}^{r} w_i \bar{x}_i \geq b_1 \) and \( \Sigma_{i=r+1}^{n} w_i \bar{x}_i < b_2 \) hold. Note that the solution \( \bar{x} \) satisfies at least one of the two necessity minimum constraints (see Eqs. (14) and (15)). That is, suppose that it meets \( \Sigma_{i=1}^{r} w_i \bar{x}_i < b_1 \) and \( \Sigma_{i=r+1}^{n} w_i \bar{x}_i < b_2 \) both. Then, the solution \( \bar{x} \) satisfies \( \Sigma_{i=1}^{r} w_i \bar{x}_i < b_1 + b_2 \leq t \) (see Eq. (4)), which contradicts the construction manner of the \( \bar{x} \) (see Eqs. (9) and (18)).

3.2. Modification Procedures

For the sequence \( \sigma \) defined in Eq. (8), and for each item type \( k \in \{1, 2\} \), let
\[
S_k = I_k \cap \{\sigma(1), \sigma(2), \ldots, \sigma(k)\}
\]
denote the item set of type \( k \) chosen in the solution \( \bar{x} \) (see Eq. (18)). Also, let
\[
\lambda = \max\{j \in S_1\}
\]
be the last chosen item of the first type in the solution \( \bar{x} \), and let
\[
\mu = \max\{j \in S_2\}
\]
be the last chosen item of the second type in the solution \( \bar{x} \). If the set \( S_2 \) in the above is empty, let \( \mu = r \) for notational convenience. The \( \mu \) satisfies \( r + 1 \leq \mu \leq n \) if \( S_2 \neq \emptyset \). Further, let
\[
\alpha = \min\{j \in \{1, 2, \ldots, r\} | \sum_{i=1}^{j} w_i \geq b_1\},
\]
and let
\[
\tau = \min\{j \in \{r+1, \ldots, n\} | \sum_{i=r+1}^{j} w_i \geq b_2\}.
\]
Then, from the assumption of \( \Sigma_{i=1}^{r} w_i \bar{x}_i = \Sigma_{i=1}^{\lambda} w_i \geq b_1 \), \( \lambda \geq \alpha \) obviously holds. Similar to this, \( \mu < \tau \) also holds from the assumption of \( \Sigma_{i=r+1}^{\mu} w_i \bar{x}_i = \Sigma_{i=r+1}^{\mu} w_i < b_2 \). Let
\[
\gamma_{r} = \frac{p_{\tau}}{w_{\tau}} \leq \frac{p_{\sigma(r)}}{w_{\sigma(r)}} = \gamma
\]
denote the priority per weight of the item \( \tau \in I_2 \). Notice the relation of \( \gamma_{r} \leq \gamma \) in the above (see Eq. (10) for the critical priority rate \( \gamma \)), since \( \tau \in \{\sigma(r+1), \ldots, \sigma(n)\} \) holds from \( \mu < \tau \).

First, we are going to show the following lemma:

**Lemma 1.** For an instance of problem P such that it meets \( \Sigma_{i=1}^{r} w_i \bar{x}_i \geq b_1 \) and \( \Sigma_{i=r+1}^{n} w_i \bar{x}_i < b_2 \) with respect to the permutation \( \sigma \), if \( \Sigma_{i=r+1}^{n} w_i \leq t \) holds, then there exists a greedy heuristic solution such that the total weight is at most twice the minimum, and also the total priority is at least the conditionally maximum.

After proving the lemma, we are going to remove the assumption of \( \Sigma_{i=r+1}^{n} w_i \leq t \) from the statement.
3.2.1. The First Modification

The first modification of the solution $\tilde{x}$ is simple. For each item $i \in I$, let

$$x'_i = \begin{cases} 1 & \text{if } \tilde{x}_i = 1 \text{ or } i \in \{\mu + 1, \ldots, \tau\}, \\ 0 & \text{otherwise.} \end{cases} \quad (26)$$

Notice that $f(x') = \sum_{i=1}^{n} w_i x'_i > \sum_{i=1}^{\mu} w_i \tilde{x}_i \geq t$ and $\sum_{i=1}^{\mu} w_i \tilde{x}_i = \sum_{i=1}^{\mu} w_i \tilde{x}_i \geq b_1$ hold from the property of the solution $\tilde{x}$, and also that $\sum_{i=1}^{\mu} w_i \tilde{x}_i \geq b_2$ clearly holds by the definition of the item $\tau$ (see Eq. (24)). That is, the newly created $x'$ is absolutely a feasible solution for the given instance of problem $P$.

For further case analysis, we additionally introduce the following notation: For a solution $x = (x_1, x_2, \ldots, x_n)$, define

$$f_1(x) = \sum_{i=1}^{r} w_i x_i, \quad f_2(x) = \sum_{i=r+1}^{n} w_i x_i, \quad g_1(x) = \sum_{i=1}^{r} p_i x_i, \quad g_2(x) = \sum_{i=r+1}^{n} p_i x_i,$$

which imply $f(x) = f_1(x) + f_2(x)$ and $g(x) = g_1(x) + g_2(x)$.

Case 1-1: $f_1(x') < f_1(x')$ and $f_2(x') \geq f_2(x')$. We have known that $f_1(x') = f_1(\tilde{x}) \geq b_1$ and $f_2(x') \geq b_2$ hold. From the construction manner of the solution $\tilde{x}$, we have

$$f_1(x') + f_2(x') = f_1(x) + f_2(x') \geq b_1 + b_2 > f_1(\tilde{x}) + f_2(\tilde{x}) \geq t,$$

which contradicts the minimality of the $x'$ on the total weight.

Case 1-2: $f_1(x') \geq f_1(x')$ and $f_2(x') < f_2(x')$. From the minimality of the $x'$ on the total weight, we see that

$$f_1(x') + f_2(x') = f(x') \geq f'(x') = f_1(x') + f_2(x'),$$

and directly from this, we obtain

$$f_1(x') - f_1(x') \geq f_2(x') - f_2(x') > 0.$$

Then, from the greedy nature of the $\tilde{x}$ (see Eq. (19) with respect to $f_1(\tilde{x}) = f_1(x')$ and $g_1(\tilde{x}) = g_1(x')$), we have

$$g_1(x') - g_1(x') \geq \gamma \left( f_1(x') - f_1(x') \right) \geq \gamma \left( f_2(x') - f_2(x') \right) > 0,$$

and also have

$$g_2(x') - g_2(x') \leq \gamma \left( f_1(x') - f_1(x') \right) \leq \gamma \left( f_2(x') - f_2(x') \right),$$

since the critical priority rate meets $\gamma \geq \gamma > 0$ (see Eq. (25)). Hence, we obtain

$$g(x') = g_1(x') + g_2(x') \geq g_1(x') + g_2(x') = g'.$$

Case 1-3: $f_1(x') \geq f_1(x')$ and $f_2(x') \geq f_2(x')$. Again, from the greedy nature of the $\tilde{x}$, we see that

$$g_1(x') - g_1(x') \geq \gamma \left( f_1(x') - f_1(x') \right) \geq 0,$$

and also see that

$$g_2(x') - g_2(x') \geq \gamma \left( f_2(x') - f_2(x') \right) \geq 0.$$

Hence, we have

$$g(x') = g_1(x') + g_2(x') \geq g_1(x') + g_2(x') = g'.$$

Suppose that $f(x') = \sum_{i=1}^{n} w_i x'_i \leq 2 \tau$ holds. Then, we can immediately reach our goal, i.e., $f(x') \leq 2 \tau$ and $g(x') \geq g'$. In the following, we therefore assume that $f(x') = \sum_{i=1}^{n} w_i x'_i > 2 \tau$ holds.
3.2.2. The Second Modification

The current assumption is rewritten as follows:

\[ f(x') = \sum_{i=1}^{2} w_i + \sum_{i=r+1}^{2} w_i > 2t. \]  

(27)

For this case, we further modify the solution \( \bar{x} \) (through the first modification \( x' \)) to the following \( x'' \):

\[ x'' = \begin{cases} 
1 & \text{if } x'_i = 1 \text{ and } i \neq \lambda, \\
0 & \text{otherwise}.
\end{cases} \]  

(28)

That is, we remove the item \( \lambda \in I_1 \) from the solution \( x' \) to make the \( x'' \).

From the current assumption of Eq. (27), we have

\[ f(x') = f(x') - w_1 > 2t - b_1 = t + (t - b_1) > t, \]  

and from \( b_2 \leq \sum_{i=r+1}^{t} w_i < 2b_2 \) (see Eq. (24)), we also have

\[ f_1(x'') = \sum_{i=1}^{r-1} w_i + \sum_{i=r+1}^{2} w_i > 2t - b_1 - 2b_2 \geq 2(b_1 + b_2) - b_1 - 2b_2 = b_1. \]

Hence, the second modification \( x'' \) is a feasible solution under the assumption of Eq. (27).

Case 2-1: \( f_1(x'') < f_1(x') \) and \( f_2(x'') \geq f_2(x') \). We have known that \( f_1(x'') > 2t - b_1 - 2b_2 \geq b_1 \) and \( f_2(x') \geq b_2 \) hold. From Eq. (27) and the assumption of this case, we have

\[ f_1(x') + f_2(x') > f_1(x'') + f_2(x') > (2t - b_1 - 2b_2) + b_2 = 2t - b_1 - b_2 \geq t, \]

which contradicts the minimality of the \( x' \) on the total weight.

Case 2-2: \( f_1(x'') \geq f_1(x') \) and \( f_2(x'') < f_2(x') \). From the minimality of the \( x' \) on the total weight, we see that

\[ f_1(x'') + f_2(x'') = f(x'') \geq f(x') = f_1(x') + f_2(x'), \]

and we obtain

\[ f_1(x'') - f_1(x') \geq f_2(x') - f_2(x'') > 0. \]

As in Case 1-2, from the greedy nature of the \( \bar{x} \), we have

\[ g_1(x'') - g_1(x') \geq \gamma \left( f_1(x'') - f_1(x') \right) \geq \gamma \left( f_2(x') - f_2(x'') \right) > 0, \]

and also have

\[ g_2(x') - g_2(x'') \leq \gamma \left( f_2(x') - f_2(x'') \right) \leq \gamma \left( f_2(x') - f_2(x'') \right). \]

Hence, we obtain

\[ g(x'') = g_1(x'') + g_2(x'') \geq g_1(x') + g_2(x') = g'. \]

Further, again from the construction manner of the solution \( \bar{x} \), we see that \( \sum_{i=1}^{t} w_i < t \) holds (see Eq. (9)). Hence, together with the assumption of this case, we obtain

\[ f(x'') = f_1(x'') + f_2(x'') = \sum_{i=1}^{r-1} w_i + \sum_{i=r+1}^{2} w_i < t + f_2(x') < f(x') + f(x') = 2f'. \]

Case 2-3: \( f_1(x'') \geq f_1(x') \) and \( f_2(x'') \geq f_2(x') \). As in Case 1-3, from the greedy nature of the \( \bar{x} \), we see that

\[ g_1(x'') - g_1(x') \geq \gamma \left( f_1(x'') - f_1(x') \right) \geq 0, \]

and also see that

\[ g_2(x'') - g_2(x') \geq \gamma \left( f_2(x'') - f_2(x') \right) \geq 0, \]
which clearly imply
\[ g(x') = g_1(x'') + g_2(x'') \geq g_1(x') + g_2(x') = g'. \]

It is left for us to show \( f(x') \leq 2f^* \). We have known that \( \sum_{i=1}^{t-1} w_i < t \) holds from the construction manner of the \( \tilde{x} \) (see again Eq. (9)). When \( \sum_{i=r+1}^{t} w_i \leq t \) holds (that is, the assumption in the lemma statement), the solution \( x'' \) satisfies
\[ f(x'') = f_1(x'') + f_2(x'') = \sum_{i=1}^{t-1} w_i + \sum_{i=r+1}^{t} w_i < t + t \leq 2f^*, \]
by which we complete the proof of Lemma 1.

Note again that \( f_2(x') = f_2(x'') = \sum_{i=r+1}^{t} w_i \leq 2b_2 \) holds by the assumption of bounded weights (see Eqs. (3) and (24)). Hence, if \( f_1(x') = \sum_{i=1}^{t} w_i \leq 2b_1 \) holds, then the \( x' \) satisfies \( f(x') = f_1(x') + f_2(x') \leq 2(b_1 + b_2) \leq 2t \leq 2f^* \) even under the assumption of Eq. (27).

### 3.3. Heuristic Performance without the Assumption

For proving Lemma 1, we have utilized the assumption of \( \sum_{i=r+1}^{t} w_i \leq t \) only in Case 2-3. In order to show the performance guarantee for a general instance of problem P without the assumption, we consider the remaining case such that it satisfies
\[ t < \sum_{i=r+1}^{t} w_i < 2b_2. \] (30)

**Case 3-1:** \( \sum_{i=1}^{r} w_i < t - b_2 \). Let
\[ \tilde{x}_i = \begin{cases} 1 & \text{if } i \in \{1, 2, \ldots, r\} \cup \{r + 1, \ldots, r', r\}, \\ 0 & \text{otherwise}. \end{cases} \] (31)

Then, \( f_1(\tilde{x}) = \sum_{i=1}^{r} w_i \geq f_1(x') \geq b_1 \) and \( g_1(\tilde{x}) = \sum_{i=1}^{r} w_i \geq g_1(x') \) clearly hold. Since \( f_2(\tilde{x}) = \sum_{i=r+1}^{t} w_i > t > b_2 \) by assumption, we easily understand that
\[ f(\tilde{x}) = f_1(\tilde{x}) + f_2(\tilde{x}) \geq t, \]
which implies the feasibility of the solution \( \tilde{x} \) in this case. We also have
\[ f(\tilde{x}) = f_1(\tilde{x}) + f_2(\tilde{x}) < (t - b_2) + 2b_2 = t + b_2 < 2t \leq 2f^* \]
as an upper bound on the total weight of the \( \tilde{x} \).

As in Case 1-3 and Case 2-3, if \( f_2(\tilde{x}) \geq f_2(x') \) holds, then we obtain \( g_2(\tilde{x}) \geq g_2(x') \) due to the greedy nature of the item renumbering of each type (see Eqs. (6) and (7)), which implies \( g(\tilde{x}) \geq g(x') = g^* \), together with \( g_1(\tilde{x}) \geq g_1(x') \). Otherwise (i.e., if \( f_2(\tilde{x}) < f_2(x') \) holds), we see that
\[ f_1(x') + f_2(x') > f_1(x') + f_2(\tilde{x}) > b_1 + t > t, \]
which contradicts the minimality of the \( x' \) on the total weight.

**Case 3-2:** \( \sum_{i=1}^{r} w_i \geq t - b_2 \). Let
\[ \alpha' = \min \left\{ j \in \{1, 2, \ldots, r\} \mid \sum_{i=1}^{j} w_i \geq t - b_2 \right\}, \] (32)
and let
\[ \tilde{x}_i = \begin{cases} 1 & \text{if } i \in \{1, 2, \ldots, \alpha'\} \cup \{r + 1, \ldots, r\}, \\ 0 & \text{otherwise}. \end{cases} \] (33)

Then, \( f_1(\tilde{x}) = \sum_{i=1}^{r} w_i \geq t - b_2 \geq b_1 \) holds (see Eq. (4)), and we have known \( f_2(\tilde{x}) = \sum_{i=r+1}^{t} w_i > t > b_2 \) by assumption. Obviously,
\[ f(\tilde{x}) = f_1(\tilde{x}) + f_2(\tilde{x}) > (t - b_2) + b_2 \geq t. \]
holds. Hence, the $\tilde{x}$ is a feasible solution also in this case. From Eqs. (4) and (32), we have

$$f(\tilde{x}) = f_1(\tilde{x}) + f_2(\tilde{x}) < (t - b_2 + b_1) + 2b_2 = t + b_1 + b_2 \leq 2t \leq 2f^*$$

as an upper bound on the total weight of the $\tilde{x}$.

Suppose that $f_1(\tilde{x}) < f_1(x^*)$ holds. Then, since $f_1(\tilde{x}) \geq t - b_2 \geq b_1$ and $f_2(x^*) \geq b_2$ hold, we have

$$f_1(x^*) + f_2(x^*) > f_1(\tilde{x}) + f_2(x^*) \geq (t - b_2) + b_2 = t,$$

which contradicts the minimality of the $x^*$ on the total weight. Hence, we obtain $f_1(\tilde{x}) \geq f_1(x^*)$. Also, suppose that $f_2(\tilde{x}) < f_2(x^*)$ holds. Then, we have

$$f_1(x^*) + f_2(x^*) > f_1(x^*) + f_2(\tilde{x}) > b_1 + t > t,$$

which contradicts the minimality of the $x^*$ on the total weight. As in Case 1-3 and Case 2-3, from $f_1(\tilde{x}) \geq f_1(x^*)$ and $f_2(\tilde{x}) \geq f_2(x^*)$, we again see that the $\tilde{x}$ satisfies $g_1(\tilde{x}) \geq g_1(x^*)$ and $g_2(\tilde{x}) \geq g_2(x^*)$ due to the greedy nature of the item renumbering of each type (see Eqs. (6) and (7)). Therefore, we obtain $g(\tilde{x}) \geq g(x^*) = g^*$ also in this case.

Obviously, the greedy solutions $\tilde{x}$, $x'$, $\tilde{x}'$ and $\tilde{x}$ can be obtained in polynomial time, more precisely, in $O(n \log n)$ time.

**Theorem 1.** For an instance of problem P, there exists a polynomial time greedy heuristic solution such that the total weight is at most twice the minimum, and also the total priority is at least the conditionally maximum.

**4. Conclusions**

In this paper, we revisited a lexicographic bi-criteria combinatorial optimization model of mixture packaging of two types of items, which arises in actual food packing systems, so-called multi-head weighers. The primary objective was to minimize the total weight of chosen items for a package, and the second objective was to maximize the total priority of them. The constraints were that the total weight of chosen items for a package must be no less than a given target weight, and that the weight sum of chosen items of each type must also be no less than a given necessity minimum. The weight of an item of each type was assumed to be bounded by the necessity minimum from the above. In this paper, we showed that a greedy heuristic solution with the total weight at most twice the minimum attains the total priority at least the conditionally maximum, which is an improved performance guarantee of greedy heuristic solutions for the lexicographic bi-criteria mixture packaging problem of two types of items.

For future research, it would be significant to examine a performance guarantee of greedy heuristic solutions for the lexicographic bi-criteria mixture packaging problem of more than two types of items. Of course, it would be interesting to consider whether the problem discussed in this paper admits another polynomial time heuristic algorithm such that it has a smaller factor than two on the primary objective of the minimum total weight and also a constant factor on the second objective of the conditionally maximum total priority. Performance guarantee analysis for an optimization problem with multiple criteria is an important research topic, and it is still attractive.

**Acknowledgment**

We are grateful to two anonymous referees for their constructive suggestions which were truly helpful to improve the readability of the proof. We also thank the Committees of International Symposium on Scheduling 2019 (ISS 2019), which was sponsored by the Scheduling Society of Japan (SSJ), and was co-sponsored by the Manufacturing Systems Division of the Japan Society of Mechanical Engineers (JSME). We gave our presentation on a part of the results at the symposium (Karuno and Nakahama, 2019). This refined version includes a new section with a numerical example to illustrate our evaluation manner of performance guarantee. This research was partially supported by JSPS KAKENHI Grant JP16K01241 and Grant JP19K04880.

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