Do firms share the same functional form of their growth rate distribution? A statistical test

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Abstract

We propose a hypothesis testing procedure to investigate whether the same growth rate distribution is shared by all the firms in a balanced panel or, more generally, whether they share the same functional form for this distribution, without necessarily sharing the same parameters. We apply the test to panels of US and European Union publicly quoted manufacturing firms, both at the sectoral and at the subsectoral NAICS levels. We consider the following null hypotheses about the growth rate distribution of the individual firms: i) an unknown shape common to all firms, with all the firms sharing also the same parameters, or with the firm variance related to its firm size through a scaling relationship, and ii) several functional shapes described the Subbotin family of distributions. Our empirical results indicate that firms do not have a common shape of the growth rate distribution at the sectorial NAICS level, whereas firms may typically be described by the same shape of the distribution at the subsectorial level, even if the specific shape may not be the same for different subsectors.

Keywords: growth rate distribution of individual firm, heterogeneous firms, EDF tests

\textit{JEL:} C12, C15, D22
1. Introduction

Since R. Gibrat proposed a stochastic model to describe the growth of a firm, see Gibrat (1931), an important branch of the literature has been concerned with the empirical testing of its consequences (for some recent reviews see, for example, Sutton (1997), Santarelli et al (2006), and Coad (2007)). Gibrat’s model deals with the growth of a individual firm and it is based upon two assumptions, namely (i) the Law of Proportionate Effect (also known as Gibrat’s law), stating that the proportionate growth of a firm in a given period is a random variable independent of the initial firm’s size and (ii) the assumption of statistical independence of successive growths. The main features of the model are that, after a long period, the logarithmic growth rates are normally distributed and independent of the initial firm’s size.

Recently, the empirical investigations on the validity of Gibrat’s model or of alternative growth models are receiving an increasing and renewed interest motivated by the availability of extensive data sets containing a large number of firms which could in principle allow to scrutinize alternative models with high statistical accuracy, see Stanley et al (1996), Bottazzi et al (2001), Bottazzi and Secchi (2003), Bottazzi and Secchi (2003b), Lotti et al (2003), Bottazzi and Secchi (2006), Bottazzi et al (2006b), Fu et al (2006), Riccaboni et al (2008). However even in such data sets it is hard to empirically test the model at the level of individual firms because the available data sets typically contain a large number of firms which are sampled over a small number of time periods each. Moreover for balanced panels, the longer is the time period considered, the smaller is the number of firms in the panel.

In this paper we propose a hypothesis testing procedure able to discriminate whether a given null hypothesis about the distribution of the growth rate of a individual firm is compatible with panel data characterized by a large number of firms sampled for a limited number of time records. The main motivation for the present work arises from the awareness that it is important to discriminate between the following alternative hypotheses that the non Gaussianity of the growth rate distribution is due to (i) the intrinsic nature of the stochastic process or (ii) to the heterogeneity of the firms analyzed in the panels.

The traditional approach to circumvent the difficulty of short time series has been to assume that the growth time series of each individual firm in the panel

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1Balanced panels contains only firms for which there are data in the whole period under study. As a consequence, firms that enter or exit the market in that period are not considered.

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is an independent realization of the same stochastic process which governs the
growth dynamics of all the firms. In other words, such an approach assumes
that each firm is statistically identical to the *model firm* (MF) described by that
stochastic process. Here we address as the MF hypothesis the assumption that
all firms in the panel are described by the same stochastic process, which results
in the same growth rate distribution, with the same parameters, for all the firms
in the panel. Under this hypothesis, the statistical properties of the growth rate
distribution of the model firm at a given instant of time can be inferred from
the statistical properties of the set of growth rates of all firms in the panel at the
same time. Moreover, if the MF stochastic process is stationary, the statistical
properties of each firm and therefore of the pooled sample of all the firms is
time independent. By assuming the MF hypothesis, earlier empirical investiga-
tions have in general corroborated Gibrat’s model, whereas many recent studies
carried over large data sets claim that it must be rejected, see Sutton (1997),
Santarelli et al (2006), and Coad (2007). For instance, several recent works find
a non Gaussian, “tent shaped”, i.e. Laplace distributions for the aggregate of
growth rates ² (see for instance Stanley et al (1996), Bottazzi et al (2001), Bott-
azzi and Secchi (2006), Bottazzi et al (2006b)), and also evidence in some data
sets for a dependence of the growth rate distributions on the firms initial size (see,
for example, Stanley et al (1996), Riccaboni et al (2008) and references therein).
These tests of the Gibrat’s model are implicitly based on the MF hypothesis and
it is therefore important to investigate whether this hypothesis itself is supported
by empirical data.

The above mentioned shortness of the firm growth time series also makes
it difficult to test the MF hypothesis directly on the time series of individual
firms. In this article we investigate the MF null hypothesis and a variant of it,
denoted by MFₘ, which assumes that there is a scaling relationship between the
variance of the growth rate distribution of each individual firm and its average
size. Finally, we also test the null hypothesis that all firms in a balanced panel
share a given known *functional form* (or *shape*) for their growth rate distribu-
tion, although the parameters that characterize that distribution may be different
from firm to firm. We choose five such functional forms from the Subbotin fam-
ily of distributions, which have often been used in the characterization of the
growth rates distribution of firms (see Bottazzi and Secchi (2006) and references
therein). In some data sets we consider also the testing of some shapes belonging
to the Asymmetric Exponential Power (AEP) family of distributions introduced

²Interestingly a recent paper (Alfarano et al 2012) suggests that also profit rate of firms fol-
 lows a Laplace distribution.
In the hypotheses tested here it is assumed that all the idiosyncratic parameters of the growth rate distribution of an individual firm are included into the first two moments of the distribution. The main idea behind the test is that if the null hypothesis were to be valid, i.e., if in fact there exists a single shape that fits well the growth rate distribution of all the individual firms in the panel, then after the application of a suitable procedure removing the firm idiosyncrasies across the panel the growth rate distributions must be the same for all the individual firms. We perform several studies which indicate that the test has generally a high power to reject a false null even when the time series are short, provided that the number of firms in the panel is relatively large.

We apply our method to balanced panels of publicly quoted manufacturing firms from the European Union and United States of America. The firms are selected from Amadeus Top 250,000 and Compustat databases respectively. Both these databases comprise only large firms. We choose to investigate balanced data because preliminary investigation we performed with unbalanced ones have put in evidence the presence of a high degree of asymmetry in the growth rate distribution. The role of entry-and-exit dynamics leads to a considerable asymmetry also in the profit rate distribution (Alfarano et al (2012)). The presence of such a pronounced asymmetry adds a further dimension in the space of the parameters and makes the approach much more complex and computationally demanding. We have therefore decided to preliminary deal with balanced panels that are described by almost symmetric or slightly asymmetric growth rate distributions. The case of the presence of strong asymmetry is left for a future work.

In the selection of the panel data, we considered two levels according to the NAICS code. Specifically, we consider the manufacturing sector defined by the NAICS 2-digit classification code with values ranging from 31 to 33, and the three major subsectors of the manufacturing sector, characterized by a 3-digit NAICS code. Given the limitations in constructing balanced panels with a relatively large number of firms from these databases we choose to work with growth periods of 8 years (from 1996 to 2004) for the Amadeus Top 250,000 panels and 9 and 18 years (from 1981 to 1999) for the Compustat’s panels. We find that the statistical validation of the null hypotheses depends on the nature of the investigated panel (the results of our tests depend both on the time length of the panel and on the level of aggregation considered - sectoral or subsectoral).

The article is organized as follows. In Section 2 we present our basic assumptions and ideas which motivate the hypothesis test proposed in this work.
In Section 3 we present the test procedure and discuss its power. In Section 4 we report and discuss the results of the applications of the test on balanced panels containing publicly quoted manufacturing firms extracted from the Compustat and Amadeus Top 250,000 databases, considering both sectoral and subsectoral levels of aggregation, where we test the seven null hypotheses mentioned above (MF, MFm, and the five chosen members of the Subbotin family). In the last section we present our conclusions. In the Appendices we present some technical details about the assumptions and the power of the tests.

2. Basic assumptions behind the test procedure

In this Section we present the basic set of assumptions characterizing the null hypotheses which will be considered in this work, as well as the rationale behind the construction of a hypothesis test concerning these nulls.

We shall consider balanced panels of data containing \(N\) firms whose sizes \(S^j_i\) \((i = 1, 2, \cdots, N)\) are recorded for \(T + 1\) instants of time \(j (j = 0, 1, 2, \cdots, T)\). The logarithmic growth rate of the \(i\)-th firm at time \(j (j > 0)\) is defined as 
\[
    r^j_i = \ln \frac{S^j_i}{S^{j-1}_i}.
\]

The basic assumptions characterizing the investigated null hypotheses are:

B1: individual firms grow independently among themselves;

B2: the time series of the growth rate of each firm consists of statistically independent records;

B3: the time series of growth rates of each firm is stationary.

B4: let \(R_i\) be the (time independent) random variable (RV) describing the growth rate of firm \(i\) and let \(p_i\) be its corresponding probability density function (pdf). The idiosyncratic parameters of the growth rate distribution of firm \(i\) are its mean \(\mu_i\) and its standard deviation \(\sigma_i\). The pdf of the standardized variable
\[
    X_i = \frac{R_i - \mu_i}{\sigma_i}
\]
will be denoted by \(p^\text{std}_i\). We assume that \(p^\text{std}_i = p^\text{std}\) for all \(i\), i.e., we assume that such standard distribution is the same for all firms, and in this work it is precisely \(p^\text{std}\) which will be referred to as the common functional form, or shape, shared by all the growth rate distributions of individual firms in the panel.
Below we shall comment upon the plausibility of these assumptions in the context of the panels analyzed in this work.

Although there are claims in the literature about a strong inter-firm growth dependency, such strong correlations are generally not corroborated by empirical evidences [see Sutton (2007)], which makes our assumption B1 plausible. Moreover, to corroborate such a statement we have performed some tests to detect the presence of statistically significant non zero cross-correlations for all the firm pairs in each of the panels considered by us. The results of these investigations are summarized in Appendix A.1, where in Table B.8 we can observe that at the significance level of 5% there is statistically significant evidence for non zero cross-correlations for only a very small percentage of firm pairs. In the worst case – see dataset D1 defined in section 4 – that percentage is only about 2% greater than the expected rate of false rejections of the zero correlation null, which is the significance level 5%. Such results make assumption B1 plausible in the context of the panels investigated in this work.

Regarding assumption B2, we remind that Coad (2007b), when considering a balanced panel of French manufacturing firms, reported evidence of significant serial growth rate correlations. By using a different approach, which focused upon the autocorrelations observed directly in the short time series of each individual firm in the panel, we test for statistically significant non zero serial autocorrelation in each panel considered here. Our results are summarized in Table B.9 of Appendix A.2, where we observe statistically significant evidence for a non zero serial autocorrelation at the significance level of 5% for only a small fraction of firms in each panel. In the worst case – see panel D2, subsector 325 defined in section 4 – we find significant non zero autocorrelations for a fraction of about 21% of the firms in the panel. Again, taking into account the probability of false rejections of a true null in the test (5%), we argue that the observed results are, as a rule, compatible with the hypothesis asserting that the autocorrelations are null for all the firms in the panel. It is worth mentioning that such results, which concern only panels containing large firms, are not qualitatively at odds with those reported by Coad (2007), which observed statistically significant results only for slight serial growth rate correlations for large firms. Thus we conclude that our Assumption B2 is justifiable as a working assumption.

Assumption B3 seems plausible especially when one deals with time series of small length. However, as we shall see in the applications of Section 4, in our panels we shall observe some evidence that this might not be a good assumption when longer time series are considered.

Finally, B4 is the key assumption characterizing the null hypotheses of this
work. It states that after the elimination of the firm idiosyncrasies across the panel, the growth rate distribution must be the same for all firms. Stating more precisely, the standardized growth rates $X_i (i = 1, \cdots, N)$ introduced in Equation (1) must be independent and identically distributed (i.i.d.) random variables, with density $p_{\text{std}}$. The main goal of this work is to construct a statistical test to investigate whether such an assumption (together with B1, B2, and B3) is compatible with the empirical evidences. Although Gibrat’s model and the MF hypothesis are particular cases of Assumption B4, such an assumption extends Gibrat’s model by allowing the possibility of non Gaussian shapes $p_{\text{std}}$, as well as extending the MF hypothesis by allowing the possibility of the existence of some restricted idiosyncrasies among the firms in the panel.

Notwithstanding the above arguments in support of the plausibility of assumptions B1-B3, their usefulness, together with assumption B4, as good premises for the characterization of the growth rates distributions of the firms in a given panel will be ultimately judged on the basis of the results obtained through the hypothesis test which will be discussed in Section 3.

Let $R$ be the random variable whose values are obtained first by randomly selecting a firm in the panel and after by observing its growth rate. Under B1-B3 the corresponding pdf is given by

$$P_R(r) = \frac{1}{N} \sum_{i=1}^{N} p_i(r). \quad \text{(2)}$$

Now, suppose one knows a priori the actual idiosyncratic parameters $\mu_i$ and $\sigma_i$ ($i = 1, \cdots N$) and standardizes the growth rate random variables $R_i$ according to Equation (1), obtaining $X_i$. We can therefore define another random variable $X$, analogous to $R$, whose pdf is obtained just by putting $p_{i,\text{std}}$ in the place of $p_i$ in the above Equation

$$P_X(x) = \frac{1}{N} \sum_{i=1}^{N} p_{i,\text{std}}(x) = p_{\text{std}}(x), \quad \text{(3)}$$

where the variable $X$ is such that its values are obtained by randomly selecting any firm in the panel and after observing its standardized growth rate (it should not be confused with the standardization of the variable $R$). For the last equality in the above equation we taken assumption B4 into account. Both the random variables $R$ and $X$ refer to the aggregate of firms in the panel. The pdf $P_R$ is that obtained when we pool together the original growth rates data of all the individual firms of the panel, whereas $P_X$ is the distribution which we would obtain
when pooling together all the corresponding standardized data of all the single firms. Equation (3) states that this latter distribution replicates the functional form $p_{\text{std}}$ which is common to the growth rate distribution of all the individual firms in the panel. This means also that the variables $X_i$ are identically distributed to the variable $X$. Such a standardization procedure has the effect of removing the idiosyncrasies among individual firms, and its main consequence is that the common functional form $p_{\text{std}}$ could be observed, in principle, by simply pooling together the standardized data corresponding to all the individual firms in the panel.

As a particular case, if the individual firm growth rates $R_i$ are all identically distributed according to a pdf $p(r)$ then not only $P_X(x) = p_{\text{std}}(x)$, but also

$$P_R(r) = p(r),$$

as it can be straightforwardly seen from Eq. (2). In this particular case the aggregate distribution of all the original (non standardized) data replicates the exact distribution $p(r)$ which is common to all the individual firms in the panel, as well as the aggregation of the corresponding standardized data replicates the functional form $p_{\text{std}}$, which in this case is obviously the standardized version of the distribution $p(r)$, or, equivalently, in this case $X$ is the standardization of the variable $R$.

The above considerations suggest that under assumptions B1-B4 it would be enough, in principle, to observe the aggregate distribution of all the standardized empirical growth rates of the firms in the panel to infer the shape $p_{\text{std}}$, or functional form, common to all the growth distributions of the individual firms. Unfortunately, in practical situations things are not so simple. The main difficulty is that one does not know a priori the actual idiosyncratic parameters $\mu_i$ and $\sigma_i$ of the individual firms. Usually these parameters must be estimated from the empirical time series of each individual firm. As we already mentioned in the Introduction, available data sets typically contain a small number of time records for each firm and consequently the best we can obtain are very imprecise estimates for these parameters. Even under the validity of assumptions B1-B4, if we use such noisy estimates to “standardize” the empirical growth rates the corresponding aggregate distribution may deviate significantly with respect to the theoretically expected distribution $p_{\text{std}}$. From now on we shall name as standardization only the procedure using the actual parameters mean and standard deviation of the individual firms, whereas we shall refer to as a z-transformation the analogous procedure which uses the corresponding time series estimates. Accordingly, we shall refer to a z-transformed growth rate as a z-growth rate. Below
we report some numerical simulations we have performed in order to illustrate this point.

We simulated $N = 10,000$ independent time series of length $T$ from a given distribution $p^{\text{std}}$ by taking $T$ independent outcomes from each time series. We then z-transformed the simulated data of each time series and plotted the histogram corresponding to all these z-data pooled together. We considered different time series lengths ranging from $T = 8$ to $T = 60$. These values are typically observed in empirical investigations of real firm data. As it is illustrated in Figure 1, the distribution of the aggregate z-transformed data may be quite different from the functional form which was used to simulate the data. Such a deviation is more significant for short time series. Whereas for time series simulated from a Gaussian shape (Figure 1a) the z-transformation does not cause an appreciable deviation with respect to the functional form actually used in the data simulation, for non Gaussian variables the z-transformation may dramatically change that distribution. For example, in Figure 1b we observe that for time series of length $T = 8$ simulated from a Laplacian functional form, the distribution of the aggregate z-data has a shape which looks more similar to a Gaussian than to a Laplacian itself. Thus, from a visual inspection of this plot, one could be led to incorrectly conclude that the data were generated from a Gaussian functional form. Customary statistical tests corroborate our conclusion that the z-transformation procedure on short time series leads to aggregate z-data whose distribution may be artificially closer to a Gaussian than to the functional form from which the data were actually simulated. This example illustrates the need for an accurate test for discriminating among alternative hypotheses for the common functional form of the growth rate distribution.

Since for short time series, and even under the validity of assumptions B1-B4, the distribution of the aggregate z-growth rates may be very different from the underlying common functional form $p^{\text{std}}$, in order to construct a consistent test of hypothesis based on the ideas of this section one should compare the empirical aggregate z-growth rates distribution, denoted by $\tilde{p}^{z-\text{tr}}$, with the distribution $p^{z-\text{tr}}$ expected under the validity of the null hypothesis and which takes into account the deformations caused by the noisy nature of parameters estimators in the z-transformation procedure. This is the subject of the next Section, where we shall set our test procedure.

3. Testing procedure

In addition to the RV’s already discussed in the last Section we start this Section by considering some other RV’s in the context of assumptions B1-B4,
which will be used in the test procedure. We shall consider the time series (of length $T$) of each firm $i$ in the panel as a particular realization of a random sample of the variable $R_i$, with $T$ elements, denoted by $\{R_i^j\}_{j=1}^T$. We then introduce the RV which describes the z-transformed growth rate of firm $i$

$$Z_i = \frac{R_i - \hat{\mu}_i}{\hat{\sigma}_i},$$

(5)

where the sample estimates $\hat{\mu}_i$ and $\hat{\sigma}_i^2$ are the RV’s given by $\hat{\mu}_i = \frac{1}{T} \sum_{j=1}^{T} R_i^j$ and $\hat{\sigma}_i^2 = \frac{1}{N-1} \sum_{j=1}^{T} [R_i^j - \hat{\mu}_i]^2$. Writing $R_i = \sigma_i X_i + \mu_i$, where $X_i$ was introduced in Equation (1), it is straightforward to see that $\hat{\mu}_i = \sigma_i \overline{X}_i + \mu_i$ and $\hat{\sigma}_i = \sigma_i s_{X_i}$, where $\overline{X}_i$ and $s_{X_i}$ are respectively the sample estimators for the mean and standard deviation of the corresponding variable $X_i$. Substituting these expressions into Equation (5) we get

$$Z_i = \frac{X_i - \overline{X}_i}{s_{X_i}} = \frac{X - \overline{X}}{s_X},$$

(6)

where in the last equality we used the fact that the i.i.d. random variables $X_i$ ($i = 1, \cdots, N$) are identically distributed to the variable $X$, which has density $p^{std}$ and describes the common shape under the null hypothesis. This last relation is crucial to the development of our hypothesis test. It says that the z-transformation procedure, similarly to the standardization, also eliminates the idiosyncrasies among the firms in the panel, because the random variables $\{Z_i\}_{i=1}^N$ are independent identically distributed. Thus, the expected distribution $p^{z-tr}$ of the aggregate z-growth rates under the validity of the null hypothesis will be the distribution of the RV $Z$, identically distributed to the variables $Z_i$ ($i = 1, \cdots, N$) given by the last equality in Equation (6).\(^3\) It is not straightforward to obtain the distribution $p^{z-tr}$ from a given functional form $p^{std}$. Below we suggest a procedure to obtain such a distribution by Monte Carlo simulations, through the following steps:

S1. simulate a large number $M$ ($M \gg N$) of independent surrogate time series of length $T$, each one independently drawn from the hypothesized distribution $p^{std}$;

S2. apply the procedure of z-transformation on each of these surrogate time series;

\(^3\)We observe that when $T \to \infty$ the variable $Z$ tends in distribution to the variable $X$. This was illustrated in Figures 1a) and 1b).
S3. construct the distribution of all the above simulated z-growth rates pooled together. This will be the simulation of the distribution $p_{\tilde{z}^{tr}}$.

Now we propose the following procedure to test whether the shape $p_{\text{std}}$ associated with assumptions B1-B4 fits well or not to the common functional form assumed to be shared by all the growth rate distributions of the individual firms in a given balanced panel:

T1. construct the distribution $p_{\tilde{z}^{tr}}$, expected under the validity of the null hypothesis, according to the steps S1-S3 above;

T2. construct the distribution of the aggregate z-transformed growth rates of the empirical panel, denoted as $\tilde{p}_{\tilde{z}^{tr}}$;

T3. set the significance level $\alpha$ and apply usual goodness of fit tests to decide if the distribution $p_{\tilde{z}^{tr}}$ fits well or not to the empirical distribution $\tilde{p}_{\tilde{z}^{tr}}$.

In what concerns step T3 of the above test procedure we follow Stephens (1974) and use goodness of fit tests based on the empirical cumulative distribution function (EDF tests). The EDF statistics considered here are $A^2$ (Anderson-Darling), $W^2$ (Cramer-Von Mises), $U^2$ (Watson), and $D$ (Kolmogorov). As EDF tests require that the cumulative distribution obtained under the null hypothesis be a continuous function, in this work we did linear interpolations between each successive pair of discrete points of the discrete distribution obtained in step S3 in order to make it continuous.

In this work we consider two groups of tests about the functional form $p_{\text{std}}$. The first group refers to the case in which the functional form is unknown and must be estimated from the empirical data. This case will correspond to our null hypotheses MF and MF$_{m}$. The critical values to test the MF null against a panel is computed in the following way. First we simulate $N_s = 10,000$ bootstrap resamplings of the panel by randomly choosing its elements, with replacements. Then, for each of these resampled panels we compute the distribution according to steps S1-S3, and then evaluate the corresponding value for each EDF statistic. Then, the critical value for each statistic at the significance level $\alpha$ is identified with the corresponding $(1-\alpha)$-quantile.

In our treatment of the MF null hypothesis, the common distribution $p$ shared by all the firms in the panel (recall Equation 4) will be estimated by the empirical distribution of the aggregate of all original growth rates in the panel. Accordingly, the common shape $p_{\text{std}}$ to be used in the above step S1 will be the “standardized” version of the distribution $p$ (by using the sample global mean and
standard deviation of the entire panel, which are often accurate estimates). On the other hand, the MF$_m$ null hypothesis assumes that there is a scaling relationship between $S_i$, the average size of the firm $i$ in the period under consideration, and the variance $\sigma_i^2$ of its growth rate distribution, which can be cast in the form

$$\sigma_i \propto S_i^{-d_0}, \quad i = 1, \cdots, N, \quad (7)$$

where $d_0$ is the scaling exponent assumed to be the same for all the firms in the panel. The value of a constant of proportionality in this relation will be immaterial after the z-transformation procedure is taken. The MF$_m$ hypothesis states that, after rescaling the growth rates of each firm in the panel by the idiosyncratic factors $S_i^{-d_0}$, all the individual firms will share the same growth rate distribution with the same parameters. Therefore, to test a panel for the MF$_m$ with a scaling exponent $d_0$ it will be enough to rescale each individual firm time series by the factor $S_i^{-d_0}$ and after simply apply the test for the MF null in the resulting panel. Again, the average sizes $S_i (i = 1, \cdots, N)$ will be estimated from the empirical data.

The value used for the exponent $-d_0$ in the rescaling procedure was the slope $-\hat{d}_0$ estimated from a least squares linear fit of the empirical data $\{(\log \hat{S}_i, \log \hat{\sigma}_i)\}_{i=1}^N$ in each considered set. For each of the four EDF statistics considered, and when the null was the MF or the MF$_m$ hypothesis, the critical value (which defines the critical region for a one-sided test at a significance level $\alpha$) was computed by a Monte Carlo procedure using 10,000 samples, according to the procedure detailed in Appendix B.1.

The second one is the case in which in the functional form $p_{std}$ is specified a priori as a known standard distribution. In this case we shall choose $p_{std}$ from the Subbotin family of symmetric densities, which generalizes the Gaussian and the Laplacian densities (see Bottazzi and Secchi (2003) and Fagiolo et al (2008)). This case comprises five null hypotheses corresponding to functional forms $p_{std}$ specified a priori from a subset of shapes belonging to the Subbotin family of distributions. The probability density (pdf) of the Subbotin family has the following

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4 It is worth noticing that the presence of a scaling law between size and variance is not exclusive to the MF$_m$ hypothesis, since it may occur also in the cases when the functional form $p_{std}$ is a priori specified. Such a scaling is nothing more than a particular way the idiosyncratic $\sigma_i$ ($i = 1, \cdots, N$) are distributed among the firms in the panel. The $z$-transformations are taken such idiosyncrasies are all eliminated, and such a scaling will not be relevant to the test of a specific known shape $p_{std}$.

5 We shall also test some asymmetric functional forms from the Asymmetric Exponential Power family (Bottazzi and Secchi (2011)).
general expression:

\[ p(r) = \frac{1}{2\gamma \beta \Gamma \left( \frac{1}{\beta} + 1 \right)} \exp \left( -\frac{1}{\beta} \left| \frac{r - \mu}{\gamma} \right|^\beta \right), \]  

(8)

where parameter \( \beta \) characterizes the shape of the distribution, \( \mu \) stands for the mean and, for a fixed \( \beta \), \( \gamma \) is proportional to the standard deviation \( \sigma \), namely \( \gamma = \sigma \beta^{-\frac{1}{\beta}} \sqrt{\frac{\Gamma \left( \frac{1}{\beta} \right)}{\Gamma \left( \frac{3}{\beta} \right)}} \). The smaller the value of the shape parameter \( \beta \), the more leptokurtic is the distribution. The functional forms associated to this family correspond to the standard versions of the above pdf, which reduces the number of free parameters to just one. We choose \( \beta \) as the free parameter which, by its turn, will identify the specific null shape \( p_{\text{std}} \) being tested. Specifically, we investigated Subbotin null shapes characterized by \( \beta = 1/2, 3/4, 1 \) (Laplace), 3/2, and 2 (Gaussian) (see Bottazzi and Secchi (2003) and Fagiolo et al (2008)). We performed Monte Carlo simulations with 10,000 simulated samples to compute the critical values at a 5% significance level for each of the considered EDF statistics, according to the procedures illustrated in Appendix B.2.

In the Appendices B.1, B.2, and B.3 we report on several Monte Carlo studies concerning the power of the proposed test. We considered null hypotheses in which \( p_{\text{std}} \) is both a priori known or unknown. We built panels with \( N \) ranging from 5 to 350, and for \( T = 9 \) and \( T = 18 \). In these studies of power we rejected the null hypothesis at the level of significance \( \alpha \) if it is rejected by at least one of the four EDF statistics at that significance level; such a “global” test tends to increase the probability to reject the null when it is valid (Type I Error) beyond the value \( \alpha \). On the other hand, this criterion increases the power of the global test in comparison with each single test applied separately. For the case of nulls in which the functional form is specified a priori we considered some members of the symmetric Subbotin family and some members of the asymmetric AEP family of distributions (Bottazzi and Secchi (2011)) and studied the power of the global test to discriminate between members of a same family. In the case of nulls in which the functional form is not specified a priori, we considered only the MF_{\text{m}} null (because the MF null emerges as the particular case of it characterized by \( d_0 = 0 \)) and studied the power of the global test in discriminating a difference between the “true” scaling exponent \( d_a \) in the panel and the exponent \( d_0 \) related to the null. Summarizing the results of those studies, we observed an often high power of the global test to reject a false null, with the power highly increasing when \( N \) or \( T \) increase in the considered ranges.
4. Empirical analyses of balanced panels of EU and US manufacturing firms

In this section we apply the hypothesis testing procedure proposed in the previous Section on six balanced panels extracted from Compustat and Amadeus Top 250,000, which are two databases widely investigated in the recent literature. The Compustat database is issued by Standard & Poor’s and contains data derived from publicly quoted North American companies. The Amadeus Top 250,000 is issued by Bureau van Dijk and contains data about the largest European firms. The Amadeus Top 250,000 database also gives the information about whether or not a firm is publicly quoted. In order to be consistent with respect to the Compustat database, in our investigations we considered only those firms from the Amadeus Top 250,000 database which are publicly quoted. For both databases we considered only manufacturing firms according to the NAICS classification. Specifically, the manufacturing sector comprises firms whose NAICS codes at the 2-digit level range from 31 to 33. Because Compustat allows to consider balanced panels covering somewhat larger time periods than Amadeus Top 250,000 we considered a time partition of the Compustat panel corresponding to a larger time period, aiming to detect evidence for a time dependence of our results.

Below we describe the six main balanced panels considered here, which contain publicly quoted firms aggregated at the whole manufacturing sectoral level:

D1: contains all the publicly quoted manufacturing firms in Amadeus Top 250,000 database which belong to any European Union (EU) country and for which there exist data on annual revenue/turnover for the whole period from 1996 to 2004 ($N = 698$, $T = 8$);
D2: contains all the US manufacturing firms in Compustat database for which there exist data on annual sales for the whole period from 1981 to 1999 (\(N = 764, T = 18\));

D3: contains the same set of firms of panel D2, but with a time restriction to the first half period, 1981-1990 (\(N = 764, T = 9\));

D4: contains the same set of firms of panel D2, but with a time restriction to the last half period, 1990-1999 (\(N = 764, T = 9\));

D5: contains all the US manufacturing firms in Compustat database for which there exist data on annual sales for the whole period from 1990 to 1999 (\(N = 1,412, T = 9\));

D6: contains all firms which are present in panel D5 but which are not present in panel D4, for the same time period common to both these panels (\(N = 648, T = 9\)).

The rationale behind our choice is to investigate broadly used databases of two major economic regions the U.S. and the European Union. The study focuses on balanced panels and therefore it is a natural choice to consider publicly quoted firms that are among the most stable firms. In order to maintain heterogeneity of the firms to a tractable level we decided to consider only firms of the manufacturing sector. From the Amadeus database we extracted 8 yearly time records. For this reason we are not considering subsets of it covering shorter time periods. From the Compustat database we extracted 18 yearly time records and therefore such a choice gave us the possibility to divide the entire time period in two non overlapping successive sets. This is done by dividing the panel D2 into the panels D3 and D4. We also investigate a balanced panel of Compustat covering only for the time period 1990-1999 (set D5). In this period the number of firms of the balanced panels is 1412. Set D5 therefore contains firms that may or may not be present in set D2 (and therefore in set D4). We track separately those firms that are in set D2 but are not in set D2 in set D6 to see whether they bear a specific profile.

In all our investigations, the proxy used for the firm size is given by (i) the annual revenue/turnover converted to the Euro value of 2000 for panel D1 (Amadeus Top 250,000) and (ii) the annual sales converted to the US Dollar of 2000 for all the other panels obtained from Compustat. As a consequence, the growth rates obtained from these data are already detrended for the GDP annual variation of the corresponding country.
The role of subsectoral specificities characterizing the growth rates of firms in the manufacturing sector has been considered, for example, in Bottazzi and Secchi (2006) and Bottazzi et al (2006b). In order to investigate heterogeneities among different subsectors of the manufacturing sector we have also considered the balanced subpanels corresponding to the three major subsectors of each of the six main panels. For panel D1 (Amadeus) these major subsectors are: 325 (Chemical Manufacturing), 311 (Food Manufacturing), and 334 (Computer and Electronic Product Manufacturing). For the remaining panels (Compustat) they are: 334, 325 and 333 (Machinery Manufacturing).

For each of the panels (at both the sectoral and subsectoral levels) we test seven null hypotheses, namely the MF and MF_m model, and the symmetric Subbotin distribution with \( \beta = 1/2, 3/4, 1, 3/2, \) and 2. The observed values for each of the four EDF statistics (\( A^2, W^2, U^2, \) and \( D^2 \)) are shown in Tables 1-6. Each of these tables refers to a main panel and the three subpanels corresponding to its largest subsectors. Each panel and its three largest subpanels were tested against the seven nulls mentioned above. In those Tables we printed in boldface those cases for which the observed value of the EDF statistic \( \hat{S}_{obs} \) does not reject the corresponding null at the significance level of 5%. In other words, EDF tests measure the goodness of fit between two distributions, which in our hypothesis test procedure are the distributions \( p_{z-tr} \) and the corresponding empirical distribution \( \tilde{p}_{z-tr} \). 11 In the next two subsections we discuss in details such results at the two levels of aggregation considered here (sectoral and subsectoral).

4.1. Discussion of the results at the sectoral level

Below we discuss in details the results of our hypothesis testing procedure for each of the six main panels characterized at the sectoral level.

Our first observation is that at the significance level of 5% both the MF and the MF_m nulls are rejected for all the panels at the sectoral level (see the first and second columns of results in Tables 1-6). This means that, if B1-B4 were valid assumptions, then these panels contains firms which are heterogeneous with respect to the growth rate mean or standard deviation (such that a scaling following Equation (7) is also excluded). Therefore, with respect to the question whether

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11In the case of the MF null the cases in boldface indicate that the empirical distribution of all the original panel data pooled together fits well to the common distribution assumed to be shared by all the firms in the panel. In the case of the MF_m null those cases suggest that the empirical distribution of the rescaled panel data pooled together fits well the rescaled distribution assumed to be shared by all the individual firms in the panel.
the MF hypothesis, or its variation which allows a size-variance scaling, is sup-
ported or not by the empirical data, our test gives a negative answer at the sectoral
level and in the context of the balanced panels investigated here.

In what regards the nulls corresponding to shapes selected from the Subbotin
family, our tests reject all of them for almost all the main panels, with only
one exception, which is panel D5, where only the Laplacian shape ($\beta = 1$) is
not rejected at the 5% significance level. However, this seems to be a puzzling
result, since panel D5 is a disjoint union of the firms in panels D4 and D6, and
the Laplacian null was rejected for these two panels. Therefore, the absence of
rejection of a Laplacian shape in D5 seems to be an effect due to the merging of
two heterogeneous panels (D4 and D6).

The selection criterion for the firms in the panels D4 and D6 can be viewed
in terms of their past business activity. In fact, firms in panel D4 were active in
the whole period 1981-1999 and therefore had a business activity documented
well before the starting data (1990) of recording in such panel, whereas those in
panel D6 were active for the shorter time sub-period 1990-1999, and therefore
entered in the database after 1981 and before 1990.\footnote{In this article “activity”
refers to the presence or absence of a firm in the database. A firm
that starts the activity in a given year could be either a new firm created in that year or a firm that
entered the database in that year because, for example, started to be publicly quoted.}

The above considerations suggest that the different business history of the
firms in the panels D4 and D6 might be responsible for the suggested hetero-
geneity among these two panels. Figure 2a) shows the histogram for the em-
pirical distribution $\tilde{\rho}^{x-tr}$ corresponding to the panel D5, as well as the expected
distributions $\rho^{x-tr}$ corresponding to the seven null hypotheses considered here. A
visual inspection of that Figure corroborates the results of Table 5, in the sense
that at the sectoral level the Laplacian shape ($\beta = 1$) fits well the empirical distri-
bution $\tilde{\rho}^{x-tr}$. On the other hand, Figure 2b) illustrates the difference between the
two panels D4 and D6. It is clear in that Figure that the panels D4 and D6 are
asymmetric with respect to zero and that their asymmetries are opposite. When
these two panels are merged to form panel D5 the opposite asymmetry cancels
out. It is worth noticing that the main source of the discrepancy between panels
D4 and D6 comes from the central part of the shown distributions, whereas the
tails are well described by the Laplacian null.

Our working hypothesis is therefore that both D4 and D6 are described by
some sort of asymmetric laplacian shapes and – since the asymmetries of these
two panels were opposite - this has the effect of suppressing the overall asym-
metry of the merged panel D5, which turned to be consistent with a (symmet-
ric) Laplacian null. This line of reasoning is illustrated in Figure 3 where we show the result of the aggregation of two variables, both fulfilling an asymmetric Laplace distribution, but with opposite asymmetries. Figure 3a) shows that when the asymmetry is large then the distribution of the merged variables deviates strongly from a Laplacian. Rather one may get a clearly bimodal distribution. On the other hand, Figure 3b) illustrates that when the asymmetry is not so large then the distribution of the merged variables resemble a symmetric Laplacian, with a slight deviation from it in its very central part.

We applied our hypothesis testing to investigate also the above working hypothesis. In Table 7 we show the results of our hypothesis testing when applied to the Panels D4 and D6, where we consider asymmetric shapes corresponding to some selected members of the AEP family, characterized by the asymmetry parameter $m$ and with $\beta_l = \beta_r = \beta = 1$, i.e., we considered Laplacian shapes with “central” asymmetries, according to Appendix B.3. The results in that Table clearly show that AEP shapes are accepted for three of four statistics at the 5% significance level, with $m = +0.07$ for panel D4 and with $m = -0.07$ for panel D6. For larger values of $m$ the AEP null is rejected. This result shows that moderate and opposite asymmetric Laplacian distributions fit well the common shapes assumed to be shared by the growth rate distribution of firms in the panels D4 and D6. Therefore, by using the argument illustrated in the Figure 3b), one could explain why the observation about panel D5 being described with a Laplacian shape is compatible with the fact that panels D4 and D6 are not described by such a functional form.

4.2. Discussion of the results at the subsectoral level

Now we discuss in details the results of Tables 1-6 at the subsectoral level, where we consider the manufacturing subsectors corresponding to the three largest subpanels of each main panel. Before analyzing each Table separately, we first observe that in several cases two or more nulls were accepted for the same subpanel by all the four EDF statistics at the 5% significance level. This occurs for all the subpanels of Table 1, for subpanels D3s325 and D3s334 of Table 3, for subpanels D4s325 and D4s333 of Table 4, for subpanel D5s333 of Table 5, and for subpanel D6s333 of Table 6.

The above cases of more-than-one null absence of rejection can be understood by taking into account the studies of power of tests discussed in Appendix

\footnote{However, for both panels D4 and D6 these nulls are not rejected by any of the four EDF statistics at the significance level of 1% (this result is not shown in Table 7, but is available under request).}
As we already mentioned, those studies suggest that for relatively small $T$ and $N$ our test could not have sufficient power to discriminate between a null shape which is close to the true shape of the panel. For example, Figure B.4 shows that for small $T$ and $N$ the test has a low power to discriminate a small ($\lesssim 0.2$) discrepancy between $d_0$, which is the scaling exponent of the null, and the “true” exponent $d_a$. As another example, Figure B.8b shows that, for small $T$ and $N$, if the true Subbotin shape in the investigated panel is $\beta = 1$, the test has a low power to reject the closest nulls $\beta = 3/4$ and $\beta = 3/2$. On the other hand, in both these examples, when the null deviates more and more from the true shape (or from the true scaling exponent, in the MF case), the power to reject such nulls increases towards high values. These considerations suggest that the acceptance of two or more close Subbotin nulls indicates that may have some Subbotin shape, characterized by a value of $\beta$ which is close to those of the corresponding accepted nulls, which provides a good fitting to the common functional form assumed to be shared by all the firms in the panel. The same applies for the acceptance of both the nulls MF and the MF with an exponent close to zero (which is the case of subpanel D1s334).

The results of Table 1 for the subpanel D1s325 show that a Subbotin shape with $\beta \gtrsim 1$ fits well the functional form assumed to be shared by the firms in this subpanel. Similarly, they indicate that a Subbotin shape close to a Laplacian ($\beta \approx 1$) fits well the common functional form associated with D1s311. Finally, all the Subbotin shapes considered in our tests were rejected for the D1s334 subpanel. On the other hand, both the nulls MF and MF were not rejected for this subpanel. Since the scaling exponent associated with the MF is small ($d_0 = 0.077$) the considerations about the power of our tests suggest that these results are compatible with the MF null.

Table 2 shows that only the Laplacian shape ($\beta = 1$) provides a good fit to the common functional form associated to the D2s325 subpanel. For the other subsectors of this panel all the tested nulls were rejected. We remember that Panel D2 corresponds to the largest time period considered in this work, namely $T = 18$ years. We shall discuss the results for this panel together with the results observed in the panels D3 and D4, which correspond to the division of this panel into the two half-periods of 9 years.

Table 3 shows the results for Panel D3, which is the first half-period of 9 years of panel D2. This Table shows similar results for the two largest subsectors of this panel D3s325 and D3s334, namely that a Subbotin shape with $\beta \gtrsim 1$ fits well the growth rate distribution of the firms in both of them. On the other hand, the subpanel D3s333 rejects all the tested nulls, with the exception of the MF.
null with a scaling exponent $\hat{d}_0 = 0.106$.

Table 4 shows the results for Panel D4, which corresponds to the second half-period of 9 years of panel D2. The results for D4s325 and D4s333 suggest that a Subbotin shape close to the Laplacian with $\beta \approx 1$ fits well the growth rate distribution in both these subpanels. However, for D4s325 besides the Subbotin shape, also the MF$_m$ null (with an exponent $\hat{d}_0 = 0.130$) was not rejected. For the subsector D4s334 the MF$_m$ null (with a scaling exponent $\hat{d}_0 = 0.119$) is the only accepted null.

A joint interpretation of the results for panels D2, D3 and D4 at the subsectoral level suggest that the Chemical Manufacturing subsector (NAICS 325) is the more persistent in time in what concerns the shape describing well the growth rate distributions. Specifically, our results show that a Laplacian shape fits well the common functional form associated with all these panels both in the whole period of 18 years and in the two half-periods of 9 years.

In Table 5 the Laplacian null is accepted by almost all the four EDF statistics at the level of 5% in all the three largest subsectors. Since panel D5 is the disjoint union of the firms in the panels D4 and D6 we shall discuss the results for these three panels together (see also the corresponding Tables 4 and 6). In what regards the largest subsector of panel D5 (NAICS 334), we observe that the two corresponding disjoint subpanels D4s334 and D6s334 are not described by the same shape, because D4s344 accepts only an MF$_m$ null, whereas D6s344 accepts only a Subbotin (Laplacian). This situation is similar to the one of the second largest subsector of D5 (NAICS 325), where D4s325 accepts a Subbotin shape (and a MF$_m$), whereas D6s325 rejects all the tested nulls. Only the third largest subsector of D5 (NAICS 333) presents similar conclusions of its two disjoint counterparts D4s333 and D6s333, which may both be described by a Subbotin shape close to the Laplacian. Thus, these results indicate that even if a Laplacian shape may be assumed to describe the growth rate distribution of all the firms in the subsectors of Panel D5, such a functional form can result from the effect of aggregation of variables with slight different shapes associated with the corresponding disjoint subpanels.

Summing up the above discussions, our first overall observation is that, differently to what was observed at the level of the manufacturing sector, at the subsectoral level (and at the significance level of 5%) the subpanels as a rule may be described by a common shape. However, such common shapes may differ among subsectors of the same panel and may also differ for the same subsector in different panels. Such a general observation indicates that the manufacturing sector is in general a heterogeneous set of subsectors which are formed by firms
that can be well fitted by a common functional form. Such a conclusion seems to be valid for time periods of approximately a decade. In fact, when applied to panels encompassing a longer time span (roughly two decades), our hypothesis testing procedure often rejected all the nulls considered here (an exception is the subpanel D2s325).

These observations might be related to the fact that (i) different sub-sectors have sizable idiosyncratic growth components and aggregating sub-sectors together distorts the form of the distribution; (ii) there are slow time dependent factors which affect the dynamics of the moments of the distribution. Thus even if for short time periods our null hypothesis cannot be rejected, when the investigated period is longer the distribution of growth rates is no longer described by a common functional form; and (iii) for large data sets, or panels with a long time span, the test has a high power, rejecting nulls which deviate even slightly from the actual distribution. In these cases it could be instructive, for instance, to consider a more refined set of nulls, perhaps other members of the Subbotin, or of the AEP family, with a more extensive set of values for the parameters $\beta$ and $m$.

5. Conclusions

We have performed a hypothesis testing procedure to investigate the nature of the growth rate stochastic process of individual firms belonging to a balanced panel of firms. As a particular case, our procedure allows to test the model firm hypothesis, which assumes that all firms follow the same stochastic process with the same parameters (MF) or with a variance which is power law depending on the size of the firm (MF$_m$). More generally, the test takes the decision whether a certain shape fits well to the common functional form assumed to be shared by the growth rate distribution of the firms in the panel. The relevance of the test relies in its ability to cope with the common problem in firm growth analysis consisting in having data sets with a large number of firms but only a small number of time records for each firm.

We have applied our test to empirical data panels of US and European Union publicly quoted manufacturing firms. Our results indicate that both the MF and the MF$_m$ hypotheses must be rejected for all the considered panels comprising firms characterized at the NAICS 2 digit level of the manufacturing sector. Moreover, still at this level of firm characterization, our results lead to the rejection of most of our tested hypotheses on a common functional form of growth rate distribution belonging to the Subbotin family. Among the rejected hypotheses
there is the Gaussian, which is the functional form consistent with the Gibrat’s description in terms of a multiplicative shocks model.

In the data sets of manufacturing sector considered here, there is also a case of no rejection (by all the four statistics at the 5% significance level) occurred with Panel D5 and only for the null hypothesis consisting of a Laplacian shape, which is often considered in the literature as a valid alternative to the Gaussian one (see for instance Stanley et al (1996), Bottazzi et al (2001) and Bottazzi and Secchi (2006)). However, quite surprisingly our test rejected the same null hypothesis for the two Panels D4 and D6, which are disjoint panels of firms whose union is the Panel D5. Specifically, growth rate distribution of firms with longer business activity (D4) have more mass for small positive rates, whereas growth rate distribution of firms with a relatively shorter business activity (D6) have more mass for small negative rates. One might think that the observation of a symmetric Laplacian in the panel D5 but not in D4 and D6 might be due to differences in the power of the test for these three panels. In fact sets D4 and D6 contain approximately half of the records of panel D5 and therefore the power is greater for panel D5 than for panels D4 and D6. However, as it is illustrated in panels b), c) and d) of Figure B.11, the power of the test is sufficient in all the cases. Therefore we interpret the observation of a symmetric Laplacian in the panel D5 as due to the aggregation into a single panel of the two sub panels D4 and D6, which present slight and opposite asymmetric Laplacian shapes, as discussed in Section 4.1.

The investigation of panels of firms belonging to the same subsector (characterized by a 3-digit NAICS code) indicates that our null hypothesis of a common functional form cannot be rejected at the 5% significance level for several panels and for different nulls, including the MF, the MF\textsubscript{m} and the Subbotin distribution with different β values. The absence of rejections is more pronounced when the investigated time period is approximately one decade.

Specifically, the MF hypothesis was almost always rejected, the only exception being subpanel D1s334. The MF\textsubscript{m} hypothesis was not rejected in 4 cases, which refer to subpanels D1s334, D3s333, D4s325 and D4s334. The hypothesis which is mostly not rejected among the Subbotin family of distributions was the Laplacian, which was not rejected in 12 out of the 18 cases investigated at the sub panels level. The Gaussian shape was almost always rejected in our panels.\textsuperscript{14}

Summarizing our results, for time periods of approximately one decade we observed that balanced panels of firms belonging to the manufacturing sector are not characterized by a specific shape of the growth rate distribution, because they

\textsuperscript{14}The only exceptions are the Kolmogorov tests in the subpanels D1s325 and D1s311.
are heterogeneous aggregate of firms belonging to different subsectors. The subsectors, however, appear to be more homogeneous, in the sense that firms in the same subsector generally have their growth rates distribution well described by a common functional form, or shape. Our results suggest that the Gaussian and the MF hypotheses can not be considered as useful approximations to describe the shape of the growth rates distributions of firms belonging to the same subsector. The MF hypothesis can provide a useful description in a few cases. Among the distributions of the Subbotin family, the Laplacian distribution is the one that was not rejected in the largest number of cases, and can therefore be considered as a useful approximation of the growth rate distribution when investigating panels of firms at the subsector level. When the time period is longer than one decade (two decades in our investigations), all the investigated common shapes are usually rejected by our test.

In summary, we believe that one should consider that the heterogeneity of the panel has an important impact on how realistic is a model describing the growth rate distribution of empirical firms. In other words, a hypothesis as the modified model firm, the Laplacian distribution and, to a lesser extent, the model firm can be used as a fruitful approximation describing real data provided that the heterogeneity of the firms composing the panel under investigation is not too large, e.g. firms are belonging to the same economic subsector, and the considered time interval does not exceed a decade. Among the possible model assumptions the Laplacian distribution is the one which is typically in better agreement with empirical data. When the requirements on firms heterogeneity and investigation time period are not fulfilled, one should carefully estimate whether simplifying models like the Gaussian distribution or the model firm are crucial for the conclusions obtained. If they were then the robustness of the conclusions reached might be questioned.

Finally, we would like to emphasize that our results are relative to balanced panels of publicly quoted firms. One has therefore to consider, for instance, other databases including small firms and the obvious survivorship biases (see Elton et al (1996) for a discussion) that might prevent a straightforward generalization of our results to a general set of firms. The results reported in this article confirm that the empirical characterization and the theoretical modeling of the growth rates distribution of individual firms and of a panel of firms is still an open and challenging subject.
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Table 1: Results of the hypothesis testing introduced in Section 3, for the set of the four EDF statistics considered (column “S” in this Table), for the Panel D1 and its three largest sub-sectors D1s325, D1s334, and D1s311 (corresponding to the NAICS codes 325, 334, and 311, respectively). In each entry of this table the first number (\( \hat{S}_{obs} \)) is the observed value of the corresponding EDF statistic, whereas the second number between parenthesis (\( S_{5\%} \)) is the corresponding critical value at the 5% significance level. The null is rejected at this significance level when \( \hat{S}_{obs} > S_{5\%} \). Different nulls are reported in different columns of the table. All the entries printed in boldface indicate no rejection of the corresponding null, which means that with respect to the corresponding EDF statistic the null provides a good fit (at the significance level of 5%) to the common functional form assumed to be shared by the growth rate distribution of all the firms in the corresponding panel. In the first column (“Panel”) of this Table it is also shown the estimated value \( \hat{d}_0 \) which was used in the tests for the MF\(_m\) null hypothesis.

| Panel | S  | MF\(_m\)  | MF\(_m\)  | \( \beta = \frac{1}{2} \) \( \hat{S}_{obs} (S_{5\%}) \) | \( \beta = \frac{1}{2} \) \( \hat{S}_{obs} (S_{5\%}) \) | \( \beta = \frac{1}{2} \) (Laplacian) \( \hat{S}_{obs} (S_{5\%}) \) | \( \beta = \frac{1}{2} \) (Gaussian) \( \hat{S}_{obs} (S_{5\%}) \) | \( \beta = \frac{1}{2} \) (Gaussian) \( \hat{S}_{obs} (S_{5\%}) \) |
|-------|----|-----------|-----------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| D1    | A\(^2\) | 5.993(0.625) | 5.402(0.615) | 30.540(1.692) | 9.796(1.228) | 3.041(1.017) | 2.896(0.852) | 7.699(0.806) |
|       | N=698  | 0.849(0.108) | 0.771(0.106) | 4.741(0.328) | 1.569(0.221) | 0.478(0.172) | 0.417(0.136) | 1.155(0.123) |
|       | T=8    | 0.846(0.100) | 0.768(0.098) | 4.649(0.256) | 1.505(0.185) | 0.425(0.151) | 0.372(0.128) | 1.113(0.118) |
|       | \( d_0 = 0.077 \) | D | 1.814(0.873) | 1.730(0.862) | 3.956(1.314) | 2.357(1.116) | 1.508(1.014) | 1.359(0.929) | 2.073(0.903) |
| D1s325 | A\(^2\) | 2.292(0.682) | 1.542(0.638) | 4.673(1.623) | 1.706(1.224) | 0.706(1.017) | 0.623(0.855) | 1.248(0.799) |
|       | N=94   | 0.379(0.121) | 0.262(0.111) | 0.788(0.315) | 0.308(0.219) | 0.127(0.173) | 0.078(0.136) | 0.155(0.123) |
|       | T=8    | 0.378(0.109) | 0.260(0.103) | 0.775(0.247) | 0.299(0.183) | 0.120(0.152) | 0.074(0.126) | 0.152(0.117) |
|       | \( d_0 = 0.111 \) | D | 1.117(0.897) | 0.981(0.882) | 1.589(1.291) | 1.157(1.113) | 0.923(1.011) | 0.696(0.923) | 0.785(0.887) |
| D1s334 | A\(^2\) | 0.510(0.583) | 0.493(0.588) | 6.120(1.636) | 3.219(1.242) | 2.014(1.019) | 1.310(0.861) | 1.385(0.797) |
|       | N=69   | 0.086(0.095) | 0.082(0.095) | 1.028(0.315) | 0.564(0.221) | 0.356(0.173) | 0.215(0.136) | 0.205(0.122) |
|       | T=8    | 0.084(0.091) | 0.081(0.091) | 0.964(0.247) | 0.519(0.187) | 0.321(0.152) | 0.192(0.126) | 0.188(0.117) |
|       | \( d_0 = 0.070 \) | D | 0.633(0.823) | 0.652(0.827) | 1.957(1.282) | 1.553(1.123) | 1.296(1.009) | 1.159(0.922) | 1.101(0.883) |
| D1s311 | A\(^2\) | 0.987(0.611) | 0.898(0.605) | 3.250(1.638) | 1.076(1.236) | 0.365(1.027) | 0.380(0.870) | 0.924(0.813) |
|       | N=75   | 0.149(0.105) | 0.132(0.104) | 0.477(0.317) | 0.160(0.219) | 0.053(0.174) | 0.057(0.138) | 0.140(0.125) |
|       | T=8    | 0.147(0.096) | 0.131(0.095) | 0.477(0.248) | 0.160(0.184) | 0.053(0.155) | 0.057(0.129) | 0.140(0.119) |
|       | \( d_0 = 0.021 \) | D | 0.980(0.855) | 0.967(0.851) | 1.489(1.287) | 1.015(1.110) | 0.712(1.010) | 0.556(0.927) | 0.800(0.887) |
Table 2: Results of the hypothesis testing for the set of the four considered EDF statistics, for the Panel D2 and its three largest sub-sectors D2s325, D2s334, and D2s333 (corresponding to the NAICS codes 325, 334, and 333, respectively). The meaning of the entries is the same as in Table 1.
Table 3: Results of the hypothesis testing for the set of the four considered EDF statistics, for the Panel D3 and its three largest sub-sectors D3s325, D3s334, and D3s333 (corresponding to the NAICS codes 325, 334, and 333, respectively). The meaning of the entries is the same as in Table 1.

| Panel | $S$ | $\beta = \frac{1}{2}$ | $\beta = \frac{3}{2}$ | $\beta = \frac{5}{2}$ | $\beta = 1$ (Laplacian) | $\beta = \frac{7}{2}$ (Gaussian) |
|-------|-----|---------------------|---------------------|---------------------|-----------------------|-----------------------|
|       | $\hat{S}_{obs} (S\%)$ | $\hat{S}_{obs} (S\%)$ | $\hat{S}_{obs} (S\%)$ | $\hat{S}_{obs} (S\%)$ | $\hat{S}_{obs} (S\%)$ | $\hat{S}_{obs} (S\%)$ |
| D3    | A$^2$ | 17.210(0.664) | 12.040(0.645) | 43.780(1.832) | 12.920(1.298) | 2.568(1.042) | 2.876(0.836) | 10.760(0.795) |
| N=764 | W$^2$ | 2.750(0.119) | 2.980(0.138) | 7.022(0.361) | 2.179(0.241) | 0.441(0.178) | 0.350(0.136) | 1.484(0.124) |
| T=9   | U$^2$ | 2.690(0.106) | 1.845(0.103) | 6.994(0.273) | 2.152(0.197) | 0.418(0.155) | 0.332(0.126) | 1.471(0.118) |
| $d_0$ = 0.127 | D | 2.969(0.907) | 2.482(0.882) | 4.553(1.366) | 2.690(1.155) | 1.500(1.029) | 1.162(0.929) | 2.170(0.901) |
| D3s325 | A$^2$ | 6.042(0.802) | 2.551(0.689) | 4.847(1.855) | 1.476(1.279) | 0.391(1.034) | 0.521(0.845) | 1.467(0.798) |
| N=87  | W$^2$ | 0.974(0.148) | 0.401(0.124) | 0.779(0.276) | 0.244(0.191) | 0.059(0.155) | 0.064(0.125) | 0.204(0.118) |
| T=9   | U$^2$ | 0.949(0.130) | 0.400(0.110) | 0.779(0.276) | 0.244(0.191) | 0.059(0.155) | 0.064(0.125) | 0.204(0.118) |
| $d_0$ = 0.199 | D | 1.771(0.981) | 1.198(0.914) | 1.616(1.364) | 1.004(1.147) | 0.635(1.021) | 0.715(0.925) | 1.062(0.892) |
| D3s334 | A$^2$ | 2.207(0.597) | 2.232(0.597) | 10.290(1.880) | 3.341(1.296) | 0.815(1.043) | 0.431(0.857) | 1.823(0.796) |
| N=161 | W$^2$ | 0.329(0.103) | 0.333(0.102) | 1.619(0.370) | 0.538(0.234) | 0.130(0.181) | 0.068(0.138) | 0.286(0.123) |
| T=9   | U$^2$ | 0.321(0.095) | 0.304(0.095) | 1.612(0.278) | 0.534(0.192) | 0.120(0.157) | 0.066(0.127) | 0.285(0.117) |
| $d_0$ = 0.126 | D | 1.124(0.859) | 1.179(0.851) | 2.252(1.369) | 1.475(1.141) | 0.893(1.024) | 0.563(0.930) | 1.039(0.892) |
| D3s333 | A$^2$ | 0.837(0.625) | 0.518(0.606) | 5.745(1.811) | 2.879(1.270) | 2.045(1.040) | 2.383(0.855) | 3.397(0.796) |
| N=82  | W$^2$ | 0.122(0.109) | 0.074(0.104) | 0.941(0.357) | 0.497(0.235) | 0.353(0.181) | 0.377(0.136) | 0.509(0.125) |
| T=9   | U$^2$ | 0.111(0.099) | 0.062(0.096) | 0.818(0.272) | 0.404(0.192) | 0.270(0.158) | 0.326(0.126) | 0.471(0.119) |
| $d_0$ = 0.106 | D | 0.907(0.875) | 0.688(0.857) | 1.788(1.348) | 1.395(1.143) | 1.231(1.044) | 1.346(0.929) | 1.607(0.892) |
Table 4: Results of the hypothesis testing for the set of the four considered EDF statistics, for the Panel D4 and its three largest sub-sectors D4s325, D4s334, and D4s333 (corresponding to the NAICS codes 325, 334, and 333, respectively). The meaning of the entries is the same as in Table 1.
Table 5: Results of the hypothesis testing for the set of the four considered EDF statistics, for the Panel D5 and its three largest sub-sectors D5s325, D5s334, and D5s333 (corresponding to the NAICS codes 325, 334, and 333, respectively). The meaning of the entries is the same as in Table 1.
Table 6: Results of the hypothesis testing for the set of the four considered EDF statistics, for the Panel D6 and its three largest sub-sectors D6s325, D6s334, and D6s333 (corresponding to the NAICS codes 325, 334, and 333, respectively). The meaning of the entries is the same as in Table 1.

Table 7: Results of the hypothesis testing for Panels D4 and D6 with null shapes from the Asymmetric Exponential Power (AEP) family, for the four considered EDF statistics. The nulls are characterized by the asymmetry parameter $m$, with $\beta_1 = \beta_2 = \beta = 1$, which correspond to asymmetric Laplacian shapes. The meaning of the entries is the same as in Table 1.
Appendix A. Investigation on growth rate correlations in the empirical data

Appendix A.1. Test of statistically significant cross-correlations

In this Appendix we report the results of our investigations about the plausibility, in the context of the panels considered in this work, of the Assumption B1 of Section 2 which states that different firms in the panel grow independently. We perform a numerical test to identify if there are significant cross-correlations among the growth rates of different firms in each of the panels studied in this paper, restricting us to the sectoral level of aggregation (panels D1 to D6). These tests are performed as follows:

1. we first compute the cross-correlation coefficients $\hat{\rho}_{ij}$ for each pair of firms $(i, j)$ in the empirical panel. For a panel of $N$ firms, there exist $N(N-1)/2$ distinct such coefficients.\footnote{For $\hat{\rho}_{ij}$ we use the usual sample Pearson’s cross correlation coefficient:

$$\hat{\rho}_{ij} = \frac{1}{\hat{\sigma}_i \hat{\sigma}_j} \left[ \frac{1}{T-1} \sum_{k=1}^{T} (r^i_k - \hat{\mu}_i)(r^j_k - \hat{\mu}_j) \right],$$

where $r^i_k$, $k = 1, \cdots, T$ is the number of empirical records of the time series of firm $i$ and $\hat{\mu}_i$ and $\hat{\sigma}_i$ are the corresponding sample mean and standard deviation respectively.}

2. we simulate $N_s$ surrogate replicas of the panel by bootstrapping, in each of these replicas, the data belonging to each individual firm;

3. for each replica $k$ ($k = 1, \cdots, N_s$) we compute the $N(N-1)/2$ cross-correlation coefficients $R^{(k)}_{ij}$;

4. we then fix a significance level $t$ and consider, for each pair $(i, j)$, the distribution of the cross correlation values $\{R^{(k)}_{ij}\}_{k=1}^{N_s}$, from which we identify the critical values $R_{ij}^{(L)}$ and $R_{ij}^{(H)}$, corresponding respectively to the $t/2$ and $1 - t/2$ quantiles of this distribution;

5. we rejected the null hypothesis corresponding to a zero cross correlation for the pair of firms $(i, j)$ in the given panel if the observed $\hat{\rho}_{ij}$ were located in the critical region $[-1, R_{ij}^{(L)}] \cup (R_{ij}^{(H)}, 1]$.

In our test we use a thousand bootstrap replications. In Table B.8 we report, for each of the panels considered, the percentage of pairs of firms for which it was observed a statistically significant cross correlation at the significance level of 5%. Because for all the investigated panels such a percentage is close to 5%, which is also the probability of a false rejection of the null hypothesis (Type I Error), we conclude that such results support the plausibility of the assumption B1 in the context of these panels.
Appendix A.2. Test of statistically significant autocorrelations.

Here we report the results of the numerical test we perform to verify the plausibility of assumption B2 of Section 2. Assumption B2 assumes that successive growth rates in the time series of each individual firm in a given panel are statistically independent. We test such hypothesis by estimating the statistically significant autocorrelations in the time series of the individual firms of the panel. The tests are performed as follows:

1. in each of the considered panels for the $i$-th firm we compute the 1-lag autocorrelation function $\hat{\rho}_i$;\(^{16}\)
2. for the $i$-th firm in each panel we take $N_r$ bootstrap resamplings and compute, for each of the bootstrap replications, the 1-lag autocorrelation function $\rho_i^{(k)} (k = 1, \ldots, N_r)$;
3. we then set a confidence level $t$ and consider, for each firm $i$ in the panel, the distribution of the autocorrelation values $\{\rho_i^{(k)}\}_{k=1}^{N_r}$ from which we identified the critical values $\rho_i^{(L)}$ and $\rho_i^{(H)}$, which are respectively the $t/2$ and $1 - t/2$ quantiles of this distribution;
4. we reject, at the significance level $t$, the null hypothesis of zero 1-lag autocorrelation in the time series of firm $i$ when the observed 1-lag autocorrelation $\hat{\rho}_i$ are located in the critical region $[-1, \rho_i^{(L)}] \cup (\rho_i^{(H)}, 1]$.

For each of the considered panels, in Table B.9 we show the percentage of firms for which we observe a significant 1-lag autocorrelation at the significance level of 5%. The maximum deviation of that percentage with respect to 5% was about 12%, in panel D2. Thus, even taking into account the worst case, we may argue that for the majority of time series in each of the panels considered we observe only weak evidence of statistically significant autocorrelations. This result makes plausible the Assumption B2.

Appendix B. Studies of the power of the hypothesis test.

In the appendices below we report on studies of the power concerning the hypothesis testing procedure introduced by steps T1 to T3 in Section 3.

\(^{16}\)We use the autocorrelation estimate proposed by Huitema and McKeen (1991), which is more suitable for short time series and is non biased when the time series has actually zero autocorrelation:

\[
\hat{\rho}_i = \frac{1}{T} + \frac{\sum_{j=1}^{T-1} (r_j^i - \hat{\mu}_i)(r_{j+1}^i - \hat{\mu}_i)}{(T-1) \hat{\sigma}_i^2},
\]

where $r_j^i (j = 1, \ldots, T)$ stands for the firm $i$ time series data and $\hat{\mu}_i$ and $\hat{\sigma}_i^2$ are the corresponding sample estimates for the mean and variance.
Appendix B.1. Power test of the MF$_{m}$ hypothesis.

Here we present the results of our studies concerning the power of our hypothesis test to reject the MF$_{m}$ null with a scaling exponent $d_0$ (see Equation 7) when the panel upon which this null is being tested is actually simulated under the validity of such a null, but with a scaling exponent $d_a$ eventually different from $d_0$.

First we show that the test of the MF$_{m}$ null does not depend on the particular values of $d_0$ and $d_a$, but only on the difference $D = d_0 - d_a$ between these exponents. Suppose we have a panel of $N$ firms whose characteristic sizes $S_i$ ($i = 1, \cdots, N$) are drawn from a given size distribution. We are assuming that the actual standard deviations of the growth rates of each firm is related to its characteristic size according to the scaling law $\sigma_i \propto S_i^{-d_a}$ ($i = 1, \cdots, N$), where $d_a$ is the “true” exponent associated to the panel. Thus, given the set of sizes of the firms $\{S_i\}_{i=1}^N$, the set of standard deviations of their growth rates $\{\sigma_i\}_{i=1}^N$ is completely determined.\(^{17}\) Now suppose that we simulate $T$ growth rates for each firm $i$ by independently drawing the outcomes from a particular zero mean distribution having standard deviation $\sigma_i$, thus obtaining a simulated $N \times T$ panel $P_a = [r_{ij}]$. Denote by $G^{\text{std}}$ the standardized version (zero mean and unit variance) of such a distribution. So, this panel can be written as $P_a = [r_{ij}] = [x_{ij}\sigma_i] = [x_{ij}S_i^{-d_a}]$, where the elements $x_{ij}$ can be viewed as independent outcomes of the standardized distribution $G^{\text{std}}$. According to the paragraph after Equation (7) of Section 3, to test the null MF$_{m}$ with exponent $d_0$ all one must to do is to rescale each row of the panel $P_a$ by the factor $S_i^{-d_0}$, thus obtaining the panel

$$\begin{align*}
P_a' &= \begin{bmatrix} r_{ij} \\ S_i^{-d_0} \end{bmatrix} = \begin{bmatrix} x_{ij}S_i^{d_0-d_a} \end{bmatrix},
\end{align*}$$

(B.1)

which must now be tested against the MF null by following the procedure presented in the paragraph before equation (7) of Section 3. Summing up, to test the MF$_{m}$ null with a scaling exponent $d_0$ on the panel $P_a$ is equivalent to test the MF null on the above panel $P_a'$. Thus, the relevant quantity to address the test sensibility in discriminating a difference between $d_0$ and $d_a$ is just that difference $D = d_0 - d_a$. Accordingly, the probability that our hypothesis test will reject a MF$_{m}$ null with scaling exponent $d_0$ when the “true” exponent in the panel is $d_a$ equals the probability that the test will reject the MF null ($d_0 = 0$) when the “true” exponent in the panel is $-D$. We investigated such probability through Monte Carlo simulations, according to the procedure below:

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\(^{17}\)As mentioned in the main text, the value of a constant of proportionality in this scaling relation is immaterial, and it will be set as unity.
i) for the sake of our simulations, we assumed that the characteristic sizes of the firms are log-normally distributed in the panel, which, together with the scaling law $\sigma_i \propto S_i^{D_i}$, implies that the standard deviations $\{\sigma_i\}_{i=1}^{N}$ will be also log-normally distributed in the panel;

ii) first we simulated a panel $\mathcal{P}_k^{(0)}$ by drawing the elements of an $N \times T$ matrix from a given standard distribution $G_{std}$;

iii) we then draw $N$ outcomes $\{\sigma_i\}_{i=1}^{N}$ from the $\sigma$ distribution of step i) and multiplied each row of the panel $\mathcal{P}_k^{(0)}$ by one of these values, thus obtaining a panel $\mathcal{P}_k$, which by its turn simulates a panel of firms satisfying an MF with a scaling exponent $-D$;

iv) we then applied the test procedure (steps T1-T3 of Section 3) to test the null MF on the panel $\mathcal{P}_k$. We rejected the MF null at the significance level $\alpha$ if it was rejected by at least one of the four goodness of fit EDF statistics at that significance level;

v) we repeated steps ii) to iv) 500 times (i.e., $k = 1, \ldots, 500$), and recorded the fraction of rejections in all these simulations, which was associated with the power of our global test to reject the MF null with scaling exponent $d_0$ when the simulated panel has a “true” exponent $d_a$, which equivalently is also the rate of rejection of the MF null ($d_0 = 0$) when the simulated panel has a “true” exponent $D = d_0 - d_a$.

We show our results in the Figures B.4-B.7. We can observe that the power to reject the MF null with an exponent $d_0$ increases when the difference $D = d_0 - d_a$ gets larger. The power also increases with the number of firms in the panel and with the increasing of the time period. We also observe that the power is higher when the actual data come from a Gaussian functional form $G_{std}$, when compared with a Laplacian one. This could be expected because the scaling of the variance

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18 This is a reasonable approximation for many observed industries (see the reviews cited in the Introduction).

19 In these studies of power we used $N_s = 5,000$, whereas in the empirical applications of Section 4 we used $N_s = 10,000$. The power test procedure is very time consuming procedure, because it must be repeated $N_s$ times for each panel $\mathcal{P}_k$, where $k = 1, \ldots, 500$. In these studies of power we saved a significant amount of computational time by using a slightly modified version of the procedure of Section 3, which runs faster than the original one, without a sensible loss of the quality of the results. Such a modification consists in computing the cumulated distribution of steps S1-S3 just once for each panel $\mathcal{P}_k$.
tends to lead to fatter tails in the aggregate distribution, and it is more difficult to detect deviations in the tails of a Laplacian than in a Gaussian. It is worth noticing that in the literature it is observed a scaling exponent typically around 0.2 (see Stanley et al. 2006)). Our studies of power show that our global test is able to detect a difference of this order in the exponents $d_0$ and $d_a$ for sufficiently large panels or sufficiently long time periods (in our simulations the global test already detects a difference $D \approx \pm 0.2$ in panels with $N \gtrsim 60$ firms and $T \approx 18$ years).

Appendix B.2. Power test of the Subbotin family of distributions

Here we report on the results of our investigations of the power of our proposed test to reject a null corresponding to an a priori specified functional form $p^{\text{std}}$ belonging to the Subbotin family of distributions when the “true” functional form describing the simulated panel is another (eventually different) member of the same family of distributions.

The Subbotin family are described by a general density given by Equation (8). The standard form of this general pdf can be completely characterized by the shape parameter $\beta$, as mentioned in Section 4. Therefore, we shall identify the functional form $p^{\text{std}}$ corresponding to the null by the value $\beta_0$ of the shape parameter, as well as we shall identify the “true” Subbotin shape in the panel by $\beta_a$.

Through a Monte Carlo procedure we calculated the critical values for each of the four EDF statistics at the significance level $\alpha$ (we used $\alpha = 5\%$) and for a set of null hypotheses characterized by $\beta_0 = \frac{1}{2}, \frac{2}{3}, 1, \frac{3}{2}$ and 2. These critical values were computed in the following way. For each null $p^{\text{std}}$ (characterized by the $\beta_0$ parameter) we first constructed the expected distribution $p^{z-tr}$ through steps S1-S3 of Section 3. We then simulated $N_s = 10,000$ panels by sampling $N \times T$ matrices by drawing independently their elements from the null $p^{\text{std}}$. We then z-transformed the data in each of these simulated panels, computed the values of the four EDF statistics for each simulated panel, and identified the 95% quantile for each one of the four EDF statistics in the set of the corresponding $N_s$ values. The quantile 95% of each statistic is the critical value defining the critical region for a one-sided goodness of fit EDF test at the 5% significance level (more details about the Monte Carlo procedure to calculate critical values for EDF goodness of fit statistics can be found in Capasso et al. (2009)).

Finally, to determine the power of our hypothesis test to reject the null $\beta_0$ when the “true” common shape shared by the firms in the panel is $\beta_a$, we proceeded as follows. For the “true” Subbotin shapes $\beta_a$ we considered exactly the same set of values used for $\beta_0$. For each value of $\beta_a$ we simulate 1,000 panels,
and upon each of these simulated panels we applied the global EDF test for each null $\beta_0$. The rate of rejections of each null $\beta_0$ in this set of simulations with a fixed $\beta_a$ is thus identified with the power of our global test to reject the null $\beta_0$ when the panel is actually described by a common shape given by $\beta_a$. When $\beta_0 = \beta_a$, this rate of rejection corresponds to the probability of a Type I Error in the global test.

Figure B.8 shows the results for some of the cases studied, but it illustrates the general behavior for all the cases we considered\(^{20}\). We can observe that when $T$ and $N$ are both relatively small, the test shows a low power to reject null shapes $\beta_0$ which are close to the “true” shapes $\beta_a$. For instance, Figure B.8a) shows that when $\beta_a = \frac{3}{4}$ there is a high probability that the nulls corresponding to $\beta = \frac{1}{2}$ and $\beta = 1$ be accepted if $N$ is relatively small. For fixed $N$ and $T$, the power increases as $\beta_0$ deviates more and more from $\beta_a$. On the other hand, if $N$ or $T$ is relatively large, the power is very high in general. For a high value of $T$ and a not so small value of $N$ ($N \gtrsim 60$, for instance), the test has a very high probability to reject a null $\beta_0$ which deviates even slightly from $\beta_a$. For instance, plots B.8d), B.8e), and B.8f), corresponding to $T = 18$, show that the valleys are narrower when compared with those in the plots at the first column, which means that even the nulls $\beta_0$ closer to the “true” $\beta_a$ have a high probability to be rejected. The same observation applies for $T = 9$ and $N \gtrsim 350$.

Appendix B.3. Power test with asymmetric distributions

We now consider the power of our hypothesis testing in discriminating between functional forms having some degree of asymmetry. We chose as the set of nulls and the “true” (simulated) shapes the same subset of members of the family of asymmetric exponential power distributions (AEP) introduced by Bottazzi and Secchi (2011), which is a five parameters generalization of the Subbotin family of exponential distributions. The general pdf of the AEP family is given by

$$f_{AEP}(x) = \frac{1}{C} e^{-\left\{\frac{|x-m|^{\beta_l} \theta_l(m-x) + |x-m|^{\beta_r} \theta_r(x-m)}{\gamma_l \gamma_r}\right\}}, \quad (B.2)$$

where it is allowed that the decay parameters $\beta_l, \beta_r > 0$ may be different at different sides of the distribution, as well as the dispersions, $\gamma_l, \gamma_r > 0$. The parameter $m$ now refers to the mode of the distribution. In the above expression

$$C = \gamma_l A_0(\beta_l) + \gamma_r A_0(\beta_r), \quad (B.3)$$

\(^{20}\)An enlarged set of results is available upon request.
where
\[ A_k(x) = x^{\frac{k+1}{x}} \Gamma \left( \frac{k+1}{x} \right). \]  
(B.4)

The mean and variance are given by
\[
\mu = m + \frac{1}{C} \left[ \gamma_r^2 A_1(\beta_r) - \gamma_l^2 A_1(\beta_l) \right],
\]
(B.5)

\[
\sigma^2 = \frac{1}{C} \left[ \gamma_r^3 A_2(\beta_r) + \gamma_l^3 A_2(\beta_l) \right].
\]
(B.6)

When the mean and the variance are given, the above equations reduce the number of free parameters to three. We choose these free parameters to be \( m, \beta_l \) and \( \beta_r \), in a way that the dispersions \( \gamma_l \) and \( \gamma_r \) are completely given in terms of them. Assumption B4 of Section 2 thus implies that if all firms in a panel have a common shape belonging to the AEP family, then such a shape \( p_{\text{std}} \) may be completely characterized by these three parameters.

In these studies of power we restricted us only to AEP shapes for which \( \beta_r = \beta_l = \beta \), and considered only null and “true” shapes having the same \( \beta \). Therefore, for a fixed \( \beta \) both the nulls and the “true” shapes will be characterized completely in terms of just one parameter, \( m \). In such a case the consistency of equations (B.5-B.6) implies that such a parameter must assume only values in the symmetric interval
\[
-m^*(\beta) < m < m^*(\beta),
\]
(B.7)
where \( m^*(\beta) > 0 \) is given in the Table B.10, for several values of \( \beta \). When \( m \rightarrow m^* \) the right dispersion \( \gamma_r \) tends to zero, whereas when \( m \rightarrow -m^* \) we have \( \gamma_l \rightarrow 0 \), and these are the two opposite limiting situations of extreme “central” asymmetries. As \( m \) varies within the above interval the central “mass” of the distribution moves to the left or to the right (recall that \( m \) is the mode of the distribution). When \( m = 0 \) the left and right dispersions become equal \( \gamma_l = \gamma_r = \gamma \), and we recover the Subbotin family of symmetric shapes. So, the parameter \( m \) in this case measures the strength of the asymmetry. Figure B.9 illustrates such a behavior for a Laplacian (\( \beta = 1 \)) with central asymmetry.

To study the power of our hypothesis test to reject an asymmetric null with shape \( m_0 \) when the “true” shape in the panel is \( m_n \) we simulated 1,000 balanced panels with all the individual firms being drawn from an AEP shape characterized by a given value \( m_n \) of the asymmetry parameter and decay \( \beta \) (in these studies we considered \( \beta = \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \text{and} 2 \)). The null and the “true” shapes were allowed to assume different values in the same set of values. For each \( T \) (\( T = 9 \) and \( T = 18 \) years) we built several power plots, each corresponding to a different
value for the parameter $m_a$ associated with the simulated data. Figures B.10-B.12 show some illustrative plots we obtained, in which the rejection rates of the global test is plotted against the several nulls considered. In each figure we plotted together the power plots for panels containing $N = 60, 120, \text{ and } 350$ firms. The significance level used for each individual EDF test in all these plots was $\alpha = 5\%$.

From our results we observe that: i) for all the cases studied we find a very high power in discriminating somewhat large differences between $m_0$ and $m_a$ [$|m_0 - m_a| \gtrsim 0.2 m^*(\beta)$], with a slightly worse sensibility for higher values of $\beta$; ii) for panels containing the largest number of firms ($N = 320$ in the figures) or with the largest time span ($T = 18$) the test shows a high power in discriminating even slight differences between $m_0$ and $m_a$ [$\lesssim 0.2 m^*(\beta)$], and iii) the power to discriminate slight differences between $m_0$ and $m_a$ tends to decrease as $\beta$ increases, especially for the smaller panels and for smaller time spans. Summing up, we may say that the our global test is able to discriminate with a high power even slight differences in the central asymmetry, especially in the cases of $\beta$ being not so large ($\lesssim 1.5$) and $N$ and $T$ being somewhat large ($N \gtrsim 120$ and $T \gtrsim 18$).

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21 The complete set of plots is available on request.
### Table B.8:
In each entry, the number on the top indicates the percentage of pairs of firms in the panel for which it was observed a non zero cross correlation which was significant at the 5% level. The number in round brackets is the number \( N(N-1)/2 \) of pairs of firms in each panel.

| dataset | D1 \((T=8)\) | D2 \((T=18)\) | D3 \((T=9)\) | D4 \((T=9)\) | D5 \((T=9)\) | D6 \((T=9)\) |
|---------|--------------|--------------|--------------|--------------|--------------|--------------|
| percentage of pairs with significant non zero cross correlation (total of pairs) | 6.99 \((243253)\) | 6.46 \((291466)\) | 6.48 \((291466)\) | 6.18 \((291466)\) | 5.99 \((996166)\) | 6.04 \((209628)\) |

### Table B.9:
The numbers on the left indicate the percentage of firms in each panel whose time series show significant 1-lag autocorrelations at the significance level of 5%. The number in round brackets indicate the total number of firms in each panel.

| dataset | D1 \((T=8)\) | D2 \((T=18)\) | D3 \((T=9)\) | D4 \((T=9)\) | D5 \((T=9)\) | D6 \((T=9)\) |
|---------|--------------|--------------|--------------|--------------|--------------|--------------|
| sector | 7.9 \(\text{(698)}\) | 12.2 \(\text{(764)}\) | 5.9 \(\text{(764)}\) | 4.3 \(\text{(764)}\) | 6.4 \(\text{(1412)}\) | 8.3 \(\text{(648)}\) |

### Table B.10: Boundary values \( m^*(\beta) \) for the “central” asymmetry parameter \( m \), for some values of \( \beta \).

| \( \beta \) | \( 1/2 \) | \( 3/4 \) | 1 | \( 5/4 \) | \( 3/2 \) | 2 |
|-----------|--------|--------|---|--------|--------|---|
| \( m^*(\beta) \) | \( \sqrt{\frac{3}{10}} \) | \( \frac{\Gamma(\frac{3}{4})}{\sqrt{\Gamma(\frac{3}{4})}} \) | \( \frac{1}{\sqrt{2}} \) | \( \frac{\Gamma(\frac{5}{4})}{\sqrt{\Gamma(\frac{5}{4})}} \) | \( \frac{\Gamma(\frac{3}{2})}{\sqrt{\Gamma(\frac{3}{2})}} \) | \( \frac{\sqrt{2}}{\sqrt{\pi}} \) |
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Figure 1: pdf of all standardized artificial data for three time series lengths. Data are generated according to a Gaussian (panel a) and Laplace (panel b) distribution. In each panel we plot for comparison the corresponding distribution $p_{\text{std}}(z)$ which would be the expected one if the mean and standard deviation estimates were accurate (solid line). In panel b) we plot also the standard Gaussian distribution (dashed line), for the sake of comparison.
Figure 2:  

a) Comparison between the pdf of the empirical $z$-growth rates $\tilde{p}^{\tau - \tau}$ for panel D5 and the expected shapes $p^{z - \tau}$ corresponding to the seven null hypotheses considered.  
b) pdf of empirical $z$-growth rates $\tilde{p}^{\tau - \tau}$ for the panel D5 and its subpanels D4 and D6.
Figure 3: Example of the composition of two stochastic processes described by asymmetric Laplace distributions with opposite asymmetries (gray curves with open and closed circles). These distributions are of the kind referred as AEP distributions with “central” asymmetries. When the asymmetry is large (top panel), we observe that the resulting distribution (black solid line) describing the overall data turns out to be clearly bimodal, and can depart very much from a symmetric Laplace distribution (dotted black line). On the other hand, if the asymmetry is not very large (bottom panel) the resulting distribution is very close to a symmetric laplacian distribution (with exception in its very central part). In both panels $\beta_l = \beta_r = 1$. In panel a) we have $\gamma_l = 0.416441$, $\gamma_r = 0.816441$ for the $m = -0.4$ curve and $\gamma_l = 0.816441$, $\gamma_r = 0.416441$ for the $m = +0.4$ curve. In panel a) we have $\gamma_l = 0.651783$, $\gamma_r = 0.751783$ for the $m = -0.1$ curve and $\gamma_l = 0.751783$, $\gamma_r = 0.651783$ for the $m = +0.1$ curve.
Figure B.4: Power of the global test to reject a null MF\textsubscript{m} with scaling exponent $d_0$ against a panel of data actually satisfying MF\textsubscript{m} with an exponent $d_a$. We observe that for large enough panels ($N \gtrsim 350$) the global test is able to discriminate a difference of about 0.2 between $d_0$ and $d_a$. In this plot $T=9$ and the simulated panels of data are drawn from a distribution having a Laplacian functional form.

Figure B.5: Power of the test to reject a null MF\textsubscript{m} with scaling exponent $d_0$ against a panel of data actually satisfying MF\textsubscript{m} with an exponent $d_a$. We observe that for large enough panels ($N \gtrsim 120$) the global test is able to discriminate a difference of about 0.2 between $d_0$ and $d_a$. In this plot $T=9$. These plots show a greater power of the MF\textsubscript{m} test in detecting asymmetries when the data actually comes from a Gaussian, in comparison with the Laplacian functional form.
Figure B.6: Power of the test to reject a null MF with scaling exponent $d_0$ against a panel of data actually satisfying MF with an exponent $d_a$. We observe that for $N \gtrsim 60$ the global test is able to discriminate a difference of about 0.2 between the exponents $d_0$ and $d_a$. In these plots $T = 18$ and the simulated data come from a distribution with a Laplacian functional form.

Figure B.7: Power of the test to reject a null MF with scaling exponent $d_0$ against a panel of data actually satisfying MF with an exponent $d_a$. We observe that for $N \gtrsim 120$ the global test is able to discriminate a difference of about 0.1 between the exponents $d_0$ and $d_a$. In these plots $T = 18$ and all the data come from a Gaussian functional form. Again the power is higher when the data come from a gaussian functional form, when compared with the Laplacian.
Figure B.8: Analysis of the power of the global test for a significance level $\alpha = 0.05$. The horizontal axis corresponds to the values of the shape parameter $\beta_0$, associated with the null $H_0$. The alternative distribution $H_a$, from which the data were actually drawn, is a Subbotin with $\beta = 3/4$ [panels a) and d)], $\beta = 1$ [panels b) and (e)], and $\beta = 3/2$ [panels c) and (f)]. Horizontal dashed and dotted lines at rejection rates of respectively 100% and 80% are shown for reference. In each plot the value of the minimum gives the effective probability of a Type I Error in the global test (which is in general slightly greater than $\alpha$).
Figure B.9: (a) Generalized Laplacian distribution with “central” asymmetry. In this plot $m = 0$ corresponds to the Laplace distribution (green curve). All curves correspond to zero mean and unit variance distributions. (b) the same distributions plotted in a log scale. We considered here the same parameters as in Figure 3.
Figure B.10: In all these plots $\beta = 0.75$ for both the simulated panels and the tested nulls. Each plot correspond to panels having the same simulated asymmetry parameter $m$. The abscissae correspond to different nulls, characterized by their values of the asymmetry parameter $m_0$. The point of minimum in each plot correspond to the value of parameter $m$ of the simulated panels (and the rate of rejection of the null at these points correspond to the Type I Error of the global test, considering together all the four EDF tests considered in this work). The rates of rejections (power) were calculated with respect to a set of 1,000 simulated panels, and considered a significance level of $\alpha = 0.05$ for each individual EDF test. In each instance of application of our tests the critical values were found by Monte Carlo simulation by using 10,000 samples. The considered $m$ values are: $m = -0.318503$ for panel a) and panel f; $m = -0.130001$ for panel b) and panel g); $m = 0$. for panel c) and panel h); $m = +0.130001$ for panel d) and panel i); $m = +0.318503$ for panel e) and panel j).
Figure B.11: Plots similar to Figure B.10, but now with $\beta = 1$. The considered $m$ values are: $m = -0.346482$ for panel a) and panel f); $m = -0.141421$ for panel b) and panel g); $m = 0.$ for panel c) and panel h); $m = +0.141421$ for panel d) and panel i); $m = +0.346482$ for panel e) and panel j).
Figure B.12: Plots similar to Figures B.10 and B.11, but now with $\beta = 1.5$. The considered $m$ values are: $m = -0.376019$ for panel a) and panel f); $m = -0.153477$ for panel b) and panel g); $m = 0.$ for panel c) and panel h); $m = +0.153477$ for panel d) and panel i); $m = +0.376019$ for panel e) and panel j).