On the regular-convexity of Ricci shrinker limit spaces

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Abstract

In this paper we study the structure of the pointed-Gromov-Hausdorff limits of sequences of Ricci shrinkers. We define a regular-singular decomposition following the work of Cheeger-Colding for manifolds with a uniform Ricci curvature lower bound, and prove that the regular part of any Ricci shrinker limit space is convex, inspired by Colding-Naber’s original idea of parabolic smoothing of the distance functions.

Contents

1 Introduction 2

2 Background 5

2.1 Basic estimates for Ricci shrinkers and manifolds in $N_m(F, K)$ 5

2.2 Weak-compactification of the moduli spaces 10

2.3 Regular-singular decomposition of the Gromov-Hausdorff limits 13

3 Parabolic smoothing of the distance function 15

3.1 Heat kernel on manifolds in $N_m(F, K)$ 16

3.2 Smoothing the distance function 23

4 Convexity of the regular part in Gromov-Hausdorff limits 24

4.1 Gromov-Hausdorff distance between nearby metric balls 24

4.2 Extension of limit minimal geodesics 31

5 Discussion 36

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1 Introduction

A Ricci shrinker is a triple \((M, g, f)\) where \((M, g)\) is a complete Riemannian manifold, and \(f\) is a \(C^2\) potential function on \(M\) such that its Ricci curvature \(Rc\) satisfies

\[ Rc + Hess_f = \frac{1}{2} g, \]  

(1.1)

where \(f\) is normalized by adding a constant, if necessary, so that the scalar curvature \(R\) satisfies

\[ R + |\nabla f|^2 = f. \]  

(1.2)

We will always fix some minimal point \(p \in M\) (whose existence guaranteed by Lemma 2.1) as a base point, making a pointed Ricci shrinker \((M, p, g, f)\). We also recall the following fundamental fact (due to Binglong Chen [8]) for the scalar curvature on a Ricci shrinker:

\[ R \geq 0. \]  

(1.3)

Ricci shrinkers, usually regarded as generalizations of positive Einstein manifolds, form an important collection of objects for our understanding of the singularities of Ricci flows. Indeed, Ricci shrinkers are critical points of Perelman’s \(\mu\)-functional, see [27]. Up to dimension three, all Ricci shrinkers are classified up to isometry, see [15], [27], [28], [21], [22], and [3]. However, the higher, even four, dimensional cases are much more complicated and a usual approach is to consider the whole collection of a given dimension as a moduli space. Important questions immediately arise: Is this moduli space compact with respect to some reasonable topology? If not, is there a standard model for the added points in the moduli space as a result of compactification? Systematic studies of the moduli space of complete Ricci shrinkers are initiated in [23], where the above questions are partially answered: it is shown that a sequence of non-collapsed smooth Ricci shrinkers is expected to subconverge, in the pointed-\(\hat{C}^{\infty}\)-Cheeger-Gromov topology, to a metric space called a conifold Ricci shrinker, see [23, Theorem 8.6].

There is yet another nice property that a conifold Ricci shrinker is defined to satisfy: The regular part \(\mathcal{R}\), which is an open manifold, should be strongly convex relative to the whole limit space \(X\). Here we say that \(\mathcal{R}\) is strongly convex if any limit minimal geodesic intersecting \(\mathcal{R}\) non-trivially has its entire interior contained in \(\mathcal{R}\). Note that this is a slightly stronger concept compared to the usual geodesic convexity.

The main purpose of the current paper is then to prove the desired regular-convexity, therefore justifying the limit space to be indeed a conifold Ricci shrinker. Let us denote by \(M_m(A)\) the moduli space of \(m\)-dimensional Ricci shrinkers with a uniform \(\mu\)-entropy lower bound by \(-A\) \((A > 0\) fixed), and our first result is the following regular-convexity theorem:

**Theorem 1.1** (Regular-convexity of Ricci shrinker limits). Let \(((M_i, p_i, g_i, f_i)) \subset M_m(A)\) be a sequence of Ricci shrinkers that converges, in the pointed-\(\hat{C}^{\infty}\)-Cheeger-Gromov topology, to a metric space with potential \((X, p, d, f)\), then \(X\) has a regular-singular decomposition such that the regular part is strongly convex. Therefore, \((X, p, d, f)\) is a conifold Ricci shrinker.

**Remark 1.2.** The concept of conifold Ricci shrinkers has its origin in the Kähler-Ricci flat setting [9, Definition 1.2], where the collection of certain Calabi-Yau conifolds was shown to be compact in the pointed-\(\hat{C}^{\infty}\)-Cheeger-Gromov topology, and such compactness played a fundamental role in the resolution of the Hamilton-Tian conjecture for Kähler-Ricci flows in [10].
We refer the readers to [23] and Section 2.2 for detailed discussions of the concepts involved. This theorem generalizes Colding-Naber’s fundamental regular-convexity theorem in [11] when the Ricci curvature is uniformly bounded from below. In view of the close relation between the geometry of Ricci shrinkers and of manifolds with a uniform Bakry-Émery Ricci curvature lower bound, we pursue a similar path that leads to Colding-Naber’s theorem.

An alternative approach to Theorem 1.1, as one may suggest, would be applying a suitable conformal transformation, so that the resulted manifold will locally acquire Ricci curvature bounds (see [38], [18] and [23, Lemma 3.7]), and Colding-Naber’s Hölder continuity theorem could be directly applied on an increasing sequence of exhausting domains to prove the global convexity result. The technique of taking suitable conformal transformations has actually been utilized in [23] to improve the regularity of convergence. But it fails in the current context, due to the simple fact that the conformal transformations involved do not preserve the minimal geodesics.

We need to emphasize, however, that the proof of Theorem 1.1 could not be achieved by a simple application of Colding-Naber’s argument to the comparison geometry of the Bakry-Émery Ricci curvature: It is well-known that the comparison geometry in such setting depends, not only on the tensor lower bound, but also on the gradient bound of the potential function. Basic properties of complete Ricci shrinkers, tell that the potential function has its gradient controlled, in magnitude, by a linear function of the distance to the base point: from (1.2), (1.3) and Lemma 2.1, we see

\[ \forall x \in M, \quad 2|\nabla f(x)| \leq d(p, x) + \sqrt{2m}. \] (1.4)

This growing gradient bound implies that the estimates one could obtain from the Bakry-Émery Ricci curvature lower bound become worse as one moves further and further away from the base point.

While there have been extensive studies in the literature (see, for instance, [35], [34] and [37]) for manifolds with a uniform Bakry-Émery Ricci curvature lower bound and a uniform bound on the gradient of the potential function, they are not directly applicable to the case we are handling: The assumption of a uniform bound on the gradient of the potential functions is valid when studying the metric tangent cone of a fixed point in a Ricci shrinker limit space (see [34] and Section 2.3), yet from (1.4) we see that in order to study the global properties of the regular part in the limit space, such as the strong convexity we just mentioned, it is necessary to develop estimates adapted to changing gradient control on the potential functions.

Therefore, we follow Colding-Naber’s original idea, but we also have to start from rebuilding the most basic estimates. Recall that the major new technique in Colding-Naber’s proof is a parabolic smoothing of the distance function. This technique relies on a uniform Ricci curvature lower bound and its resulting Li-Yau heat kernel estimates [24]. On a Ricci shrinker, since the Ricci curvature lower bound is replaced by a Bakry-Émery Ricci curvature bound, it is more natural to consider, instead, the $f$-heat kernel, which is the heat kernel with respect to the weighted measure $\mu_f$ whose density is defined as $d\mu_f := e^{-f}dV_g$. Then the corresponding $f$-heat kernel bounds roughly follow from a local volume doubling property and a local $L^2$-Poincaré inequality of $\mu_f$.

In fact, we will give a new $f$-heat kernel estimate (Theorem 3.1), and its applications to the smoothing of distance functions, on more general classes of manifolds, defined by only extracting the necessary analytic properties:
Definition 1.3. Given a positive smooth nondecreasing function $F(r)$ on $[0, \infty)$, and a number $K \geq 0$, $\mathcal{N}_m(F, K)$ is defined to be the class of pointed smooth metric measure space $(M^m, p, g, \mu_f)$, where $\forall U \subset M$ open, $\mu_f(U) := \int_U e^{-f} dV_g$, such that

(a). $(M^m, g)$ is a complete $m$-dimensional smooth Riemannian manifold.

(b). There exists a $C^2$ function $f$ on $M$ such that $f$ achieves a global minimum at some $p \in M$, and $\forall x \in M$, $\max[|f|(x), |\nabla f|^2(x)] \leq F^2(d_g(p, x))$.

(c). The Bakry-Émery Ricci curvature satisfies $Rc_f := Rc + Hess_f \geq -Kg$.

In addition, the subclass $\mathcal{N}_m(F, K; V_0) \subset \mathcal{N}_m(F, K)$ consists of all manifolds satisfying the

(d). Non-collapsing condition: $\mu_f(B(p, 1)) \geq V_0$.

We also define a subclass $\mathcal{M}_m(F, K; V_0) \subset \mathcal{N}_m(F, K; V_0)$ (see [23 Definition 10.1]) as all manifolds further satisfying the following:

(e). $\forall x \in M$, $Rc_f(x) \leq Kg(x)$ and $|R|(x) + |\nabla f|^2(x) \leq F^2(d_g(p, x))$.

Obviously, if we consider the above mentioned weighted measure, then by (1.4) any Ricci shrinker $(M^m, p, g, f)$ belongs to $\mathcal{N}_m(F_{RS}, 1/2)$ for the linear function $F_{RS}(t) := \frac{1}{2}(t + \sqrt{2m})$ (for any $t > 0$). By [23 Lemma 2.5], we have for $c_m := (4\pi)^{\frac{m}{2}} e^{-\frac{m}{2}}$,

$\mathcal{M}_m(A) \subset \mathcal{M}_m(F_{RS}, \frac{1}{2}; c_m e^{-\frac{m}{2}})$.

For the sake of simplicity, we will also denote any manifold $(M, p, g, \mu_f) \in \mathcal{N}_m(F, K)$ by $(M, p, g, f)$.

Remark 1.4. We point out any $(M^m, p, g, f) \in \mathcal{N}_m(F, K)$ is a smooth $RCD(K, \infty)$ space, whose definition can be found in [2], see also [26], [31] and [32]. Theorem 1.1 shows that the boundary of the moduli space $\mathcal{M}_m(A)$, in the pointed-$\mathcal{C}^{\infty}$-Cheeger-Gromov topology, consists of conifold Ricci shrinkers. By [12], we see that these boundary points of $\mathcal{M}_m(A)$ provide natural and novel examples of non-trivial $RCD(1, \infty)$ spaces.

In fact, on a pointed-Gromov-Hausdorff limit of a sequence of manifolds in $\mathcal{N}_m(F, K)$, one could define the regular part purely in terms of metric tangent cones: By blowing-up the metric, the effect of the potential function will be neglected, and the metric tangent cone could be defined by Gromov’s compactness theorem, see Section 2.3. Following a similar manner as [6], the concept of regular-singular decomposition in Theorem 1.1 could be generalized to the pointed-Gromov-Hausdorff limits of sequences of manifolds in $\mathcal{N}_m(F, K)$. There is also, as discussed in Section 2.2, a natural limit measure on a pointed-Gromov-Hausdorff limit, making the convergence a pointed-measured-Gromov-Hausdorff convergence, see Definition 2.11. In the last section, we will follow Colding-Naber’s idea to prove a Hölder continuity theorem (Theorem 4.5) for manifolds in $\mathcal{N}_m(F, K)$, and supplement the necessary details in extending limit minimal geodesics (Lemma 4.7) on the pointed-measured-Gromov-Hausdorff limits of manifolds in $\mathcal{N}_m(F, K)$. Both of these results become indispensable ingredients in proving the main theorem of the paper.
Theorem 1.5. Let a sequence \(((M_i, p_i, g_i, f_i)) \subset N_m(F, K)\) converge to \((X, p_\infty, d_\infty, \nu_\infty)\) in the pointed-measured-Gromov-Hausdorff topology. Then there is a unique natural number \(k \leq m\), such that \(\nu_\infty(\mathcal{R}_k) > 0\). Moreover, \(\mathcal{R}_k\) is both \(\nu_\infty\)-almost everywhere convex and weakly convex in \(X\).

Here \(\nu_\infty\) is a limit measure as just mentioned (see Proposition 2.12), and we call a set \(S \subset X\) to be \(\nu_\infty\)-almost everywhere convex if for \(\nu_\infty \times \nu_\infty\) almost every pair of points \((x, y) \in S \times S\), there is a minimal geodesic entirely contained in \(S\) that connects them. Also we call a set \(S \subset X\) to be weakly convex if

\[
\forall x, y \in S, \quad d_X(x, y) = \inf_{\sigma \subset S} |\sigma|,
\]

where the infimum is taken over all curves \(\sigma\) connecting \(x\) and \(y\), and entirely contained in \(S\).

Remark 1.6. The main theorem of Colding-Naber [11] can be regarded as a special case of Theorem 1.5 by letting \(F \equiv 0\). Also, Theorem 1.1 is a natural consequence of Theorem 1.5 after invoking the regularity improvement made in [23]. The proof of Theorem 1.5 however, is independent of [23]. Actually, Theorem 4.5 as the main step in proving Theorem 1.5 is by itself a certain regularity improvement result — it takes care of the lowest level of regularity, while the major concern of [23] focuses on higher regularities. Also compare Remark 2.15.

The paper is arranged as following: after recalling the necessary background in Section 2, we will present the useful analytic properties about the manifolds in \(N_m(F, K)\) in Section 3, and in Section 4 we finish the proof of the main theorem, Theorem 1.5 and its consequence Theorem 1.1.

The following notations are employed throughout the paper:

1. \(D\) denotes a large positive constant, say \(D > 10m\);
2. \(Rc, R\) denote the Ricci and scalar curvature respectively;
3. \(B_d(x, r)\) denote the geodesic \(r\)-ball centered at \(x\), with metric structure induced by \(d\) (the dependence of \(d\) is sometimes omitted when no confusion is caused);
4. \(A(x, r_1, r_2) := B(x, r_2) \setminus B(x, r_1)\) for \(r_2 \geq r_1 > 0\);
5. \(\mathcal{R}_k\) denotes the \(k\)-stratum of the regular part, \(\mathcal{R}\) the entire regular part, and \(S\) the singular part.

2 Background

In this section we recall the basic analytic properties of Ricci shrinkers and manifolds in the moduli \(N_m(F, K)\), and discuss various concepts related to the regular-singular decomposition of the pointed-Gromov-Hausdorff limits.

2.1 Basic estimates for Ricci shrinkers and manifolds in \(N_m(F, K)\)

It is immediate from the definition that the function \(f\) must be smooth. Moreover, we have the following point wise estimate of \(f\) by [17] Lemma 2.1:
Lemma 2.1. Let $(M^m, g, f)$ be a Ricci shrinker. Then there exists a point $p \in M$ where $f$ attains its infimum and $f$ satisfies the quadratic growth estimate

$$\frac{1}{4} (d(x, p) - 5m)^2 \leq f(x) \leq \frac{1}{4} (d(x, p) + \sqrt{2m})^2$$

for all $x \in M$, where $a_+ := \max \{0, a \}$. Moreover, if $p_1, p_2 \in M$ are two distinct minima of $f$, then $d(p_1, p_2) \leq \sqrt{2m + 5m}$.

In other words, $f$ increases like a quadratic function. Moreover, it follows from (1.2) that $f$ is nonnegative and $|\nabla f|$ increases at most linearly. From now on, whenever we talk about a pointed Ricci shrinker, we fix one of the minima of $f$ as the base point. Recall that associated to the Ricci shrinker metric structure, there is a natural measure $\mu_f$, with density $d\mu_f = e^{-f} dV_g$. It is clear from Lemma 2.1 and the next lemma that $\mu_f$ is a finite measure, see [4, Corollary 1.1].

Lemma 2.2 (Lemma 2.2 of [17] and Theorem 1.2 of [4]). For each dimension $m$, and for any non-compact Ricci shrinker $(M^m, p, g, f)$ with $p \in M$ being a minimal point of $f$,

$$\frac{|B(p, D)|}{|B(p, D_0(m))|} \leq C_1(m) D^m, \quad (2.1)$$

for any $D \geq D_0(m)$, with $D_0(m) := \sqrt{2m + 4 + 5m}$ and $C_1(m) := 2 \left( \frac{2}{m+2} \right)^{m/2}$.

Proof. It follows from [35, Theorem 1.1,(a)] that

$$\Delta f r \leq \frac{m-1}{r} + (m-1) \sqrt{K} + F(2D)$$

on $B(q, D)$, where $r(\cdot) := d(q, \cdot)$.

Proof. It follows from [35, Theorem 1.1,(a)] that

$$\Delta f r \leq m_K(r) + F(2D),$$

where $m_K(r) = \frac{m}{r} + (m-1) \sqrt{K}$.
where $m_K(r)$ is the mean curvature of the geodesic sphere in $M^m_K$. Since the mean curvature is bounded as

$$m_K(r) \leq (m - 1) \sqrt{K} \coth \sqrt{K}r \leq \frac{m - 1}{r} + (m - 1) \sqrt{K},$$

where we have used the elementary inequality $x \coth x \leq 1 + x$ for $x \geq 0$.

**Corollary 2.4.** Let $(M, p, g, f) \in N_m(F, K)$ and fix $q \in B(p, D)$. If we denote the weighted volume form on sphere with respect to $q$ by $\mathcal{A}_f(r, v)$ with $v \in S_q M$ (the unit tangent vectors at $q \in M$), then

$$\frac{\mathcal{A}_f(r, v)}{\mathcal{A}_f(s, v)} \leq e^{F(2D)r \mathcal{A}^{-1}_K(r)} \leq \left( \frac{r}{s} \right)^{m-1} e^{(m-1) \sqrt{K} + F(2D)(r-s)}.$$

**Proof.** It is clear from the definition $\mathcal{A}_f(r, v) = e^{-f} \mathcal{A}(r, v)$ that

$$(\ln \mathcal{A}_f(t, v))' = m_f(t, v) = \Delta_f r(t, v) \leq m_K(t) + F(2D) = (\ln \mathcal{A}^{-1}_K(t))' + F(2D)$$

$$\leq (m-1)r^{-1} + (m-1) \sqrt{K} + F(2D).$$

Then integrating from $0 < s$ to $r \leq D$, we have the desired estimates. $\square$

By [39] Lemma 3.2, we could further estimate, for $0 < r_1 \leq r_2, 0 < s_1 \leq s_2, s_1 < r_1$ and $s_2 < r_2$, that $\forall v \in S_q M$ (the unit sphere in the tangent space of $q \in M$),

$$\int_{s_2}^{r_2} \mathcal{A}_f(t, v) \, dt \leq \int_{s_1}^{r_2} e^{F(2D)t \mathcal{A}^{-1}_K(t)} \, dt,$$

and by the mean value theorem,

$$\int_{s_1}^{r_2} \mathcal{A}_f(r_2, v) \, dt \leq \int_{r_1}^{r_2} e^{F(2D)r \mathcal{A}_f(v, t)} \, dt.$$

Integrating the above inequalities in all tangent directions and arguing as [39] Theorem 3.1, we have the following volume comparison theorem (compare also [35] Theorem 1.2)):

**Theorem 2.5.** Let $(M^m, p, g, f) \in N_m(F, K)$, and fix $q \in M$ such that $B(q, r_2) \subset B(p, D)$. Then for $s_2 \geq s_1 > 0$ and $r_1 \in (0, r_2)$ such that $s_2 \leq r_2$ and $s_1 \leq r_1$, we have the following estimates:

$$\frac{\mu_f(A(q, r_2, r_1))}{\mu_f(A(q, s_2, s_1))} \leq e^{F(D)(r_2-s_1)} \frac{Vol^m_K(r_2) - Vol^m_K(s_2)}{Vol^m_K(r_1) - Vol^m_K(s_1)},$$

$$\frac{\mu_f(B(q, r_2))}{\mu_f(B(q, s_2))} \leq e^{F(D)r_2} \frac{Vol^m_K(r_2)}{Vol^m_K(s_2)},$$

and $$\frac{\mu_f(\partial B(q, r_2))}{\mu_f(B(q, s_2))} \leq e^{F(D)r_2} \frac{Area^m_K(r_2)}{Vol^m_K(s_2)},$$

where for any $r > 0$, $Vol^m_H(r)$ is the volume of the geodesic $r$-ball in $M^m_K$, and $Area^m_K(r)$ is the area of its boundary.
Especially, when \( r_2 = 2s_2 \) in the second inequality above, we have the following volume doubling property: when \( B(q, 2r) \subset B(p, D) \),

\[
\mu_f(B(q, 2r)) \leq C_2(m, F, K, D)\mu_f(B(q, r)),
\]

where the doubling constant \( C_2(m, F, K, D) := 2^{m-1}e^{D(m-1)\sqrt{K_f}(D)} \) is uniform for \( D \).

For later applications, we also need the following segment inequality, originally due to Cheeger-Colding \[5\] for manifolds with uniform Ricci lower bound.

**Theorem 2.6** (Segment inequality). Let \((M, p, g, f) \in \mathcal{N}_m(F, K)\). For any \( D > 0 \), there exists a constant \( C_{seg} = C_{seg}(m, F, K, D) \) such that if \( U \) is a geodesically convex set in \( B(q, D) \subset B(p, 2D) \), then for any set \( V \subset U \),

\[
\int_{V \times U} \mathcal{F}_u(x, y) \, d\mu_f(x) d\mu_f(y) \leq C_{seg} \mu_f(V) (\text{diam } U) \int_U u \, d\mu_f
\]

where \( u \) is a nonnegative continuous function on \( U \) and

\[
\mathcal{F}_u(x, y) := \inf_{\gamma} \int_0^{d(x, y)} u(\gamma(t)) \, dt,
\]

with infimum being taken over all minimal geodesics connecting \( x \) and \( y \).

**Proof.** It follows from Corollary [2.4] and an argument based on the original one of Cheeger-Colding’s. For the sake of completeness we write down the technical details, see also [16] for a version only for Ricci shrinkers.

We may consider \( \mathcal{F}_u(x, y) = \mathcal{F}_u^+(x, y) + \mathcal{F}_u^-(x, y) \) where

\[
\mathcal{F}_u^+(x, y) := \inf_{\gamma_{xy}} \int_0^{d(x, y)} u(\gamma_{xy}(t)) \, dt \quad \text{and} \quad \mathcal{F}_u^-(x, y) := \inf_{\gamma_{xy}} \int_0^{d(x, y)} u(\gamma_{xy}(t)) \, dt.
\]

Since \( \mathcal{F}_u^+(x, y) = \mathcal{F}_u^-(y, x) \), by Fubini’s theorem,

\[
\int_{V \times V} \mathcal{F}_u^+(x, y) \, d\mu_f(x) d\mu_f(y) = \int_{V \times V} \mathcal{F}_u^-(x, y) \, d\mu_f(x) d\mu_f(y),
\]

and so we only need to do the estimate for \( \mathcal{F}_u^+ \). For any \( x \in V \) and any \( v \in S_x M \) fixed, define \( d_{x,v} := \min\{t > 0 : \exp_x(tv) \in \partial U\} \), also denote \( \gamma_v(t) = \exp_x(tv) \). Then \( \forall t \in (0, d_{x,v}) \), by Corollary [2.4]

\[
\mathcal{F}_u^+(\gamma_v(t/2), \gamma_v(t)) \, d\mu_f(\gamma_v(t)) \leq C \left( \int_{\gamma_v(t)/2}^t u(\gamma_v(s)) \, ds \right) \mathcal{A}_f(v, t) \, dt
\]

\[
\leq C \left( \int_{\gamma_v(t)/2}^t u(\gamma_v(s)) \mathcal{A}_f(v, s) \, ds \right) \, dt.
\]
By the assumption on $V \subset U$, for almost every $y \in X$, there exists some $v \in S_x M$ such that $\gamma_s(d(x,y)) = y$, we have

$$
\int_V F_u^+(x,y) \, d\mu_f(y) \leq \int_{S_x M} \int_0^{d_{xy}} F_u^+(\gamma_s(t/2), \gamma_s(t)) \mathcal{A}_f(v, t) \, dr \, dv
$$

$$
\leq C \text{diam} U \int_{S_x M} \int_0^{d_{xy}} u(\gamma_s(t)) \, \mathcal{A}_f(v, s) \, ds \, dv
$$

$$
\leq C \text{diam} U \int_U u \, d\mu_f.
$$

Finally, integrating the above inequality for $x \in X$, we get

$$
\int_V \int_V F_u^+(x,y) \, d\mu_f(y) \, d\mu_f(x) \leq C \mu_f(V) \text{diam} U \int_U u \, d\mu_f.
$$

\[ \square \]

With the help of the volume doubling property and the segment inequality, the following local $L^2$-Poincaré inequality holds, see [7] for a proof.

**Proposition 2.7** (Local $L^2$-Poincaré inequality). Let $(M, p, g, f) \in \mathcal{N}_m(F, K)$. For any $D > 0$, there exists a constant $C_p = C_p(m, F, K, D)$ such that for any $B(q, r) \subset B(p, D)$,

$$
\int_{B(q, r)} \left| u - \int_{B(q, r)} u \, d\mu_f \right|^2 \, d\mu_f \leq C_p r^2 \int_{B(q, r)} |\nabla u|^2 \, d\mu_f
$$

for any $u \in C^1(B(q, r))$.

**Remark 2.8.** Throughout this paper we will let $\bar{f}$ denote the average over a set whose total mass is weighted against the measure in the integral, that is to say, for any integrable function $u$ on $B(q, r)$,

$$
\int_{B(q, r)} u \, d\mu_f := \frac{1}{\mu_f(B(q, r))} \int_{B(q, r)} u \, d\mu_f.
$$

Moreover, the local volume doubling with the local $L^2$-Poincaré inequality will imply the following local Sobolev inequality, see [29].

**Proposition 2.9** (Local $L^2$-Sobolev inequality). Let $(M, p, g, f) \in \mathcal{N}_m(F, K)$. For any $D > 0$, there exists a constant $C_{Sob} = C_{Sob}(m, F, K, D)$ such that for any $B(q, r) \subset B(p, D)$,

$$
\left( \int_{B(q, r)} u^{2m} \, d\mu_f \right)^{\frac{2m}{m-2}} \leq \frac{C_{Sob} r^2}{\mu_f(B(q, r))^{\frac{m}{2}}} \int_{B(q, r)} |\nabla u|^2 + r^{-2} u^2 \, d\mu_f
$$

for any $u \in C^1(B(q, r))$. 

9
2.2 Weak-compactification of the moduli spaces

To begin this section, we first present the following weak-compactness theorem of $N_m(F, K)$.

**Theorem 2.10 (Weak-compactness of $N_m(F, K)$).** Let $\{(M^m_i, p_i, g_i, f_i)\}$ be a sequence in $N_m(F, K)$, and let $d_i$ denote the length structure induced by $g_i$. By passing to a subsequence if necessary, we have

$$\lim_{k \to \infty} (M^m_i, p_i, d_i, f_i) \overset{\text{pointed-Gromov-Hausdorff}}{\to} (X, p_\infty, d_\infty, f_\infty),$$

where $(X, d_\infty)$ is a length space, $f_\infty$ is a Lipschitz function on $X$.

**Proof.** For any $D > 0$, we have a uniform volume doubling constant on $B(p_i, D) \subset M_i$ by (2.2). Then it follows from the standard ball packing argument of Gromov, see [13] Proposition 5.2 that

$$\lim_{k \to \infty} (M^m_i, p_i, d_i) \overset{\text{pointed-Gromov-Hausdorff}}{\to} (X, p_\infty, d_\infty).$$

In addition, since $\|\nabla f_i\| \leq F(D)$ on $B(p_i, D) \subset M_i$, it is clear from the Arzela-Ascoli theorem that $f_i$ converge to a locally Lipschitz limit function $f_\infty$. Moreover, $\|f_\infty\|_{\text{Lip}} \leq F(D)$ on $B(p_\infty, D) \subset X$. ∎

Besides the pure metric structure, we also have a limit measure on the pointed-Gromov-Hausdorff limit, and we define the pointed-measured-Gromov-Hausdorff convergence as following:

**Definition 2.11.** Let $\{(M_i, p_i, d_i, \mu_i)\}$ be a sequence of metric measure spaces, we say this sequence pointed-measured-Gromov-Hausdorff converges to a metric measure space $(X, p, d, \mu)$ if there exist a sequence of radii $D_k \uparrow \infty$, and pointed-Gromov-Hausdorff approximations $\Phi_k : B_d(p_i, D_k) \to B_d(p, D) \subset X$, such that $(\Phi_k)_* \mu_i \to \mu$ in $C_0(B_d(p, D_k))'$, the dual space of all continuous functions on $B_d(p, D_k)$, vanishing on $\partial B_d(p, D_k)$.

Now for each $(M, p, g, f) \in N_m(F, K)$ we define the renormalized measure $\nu_f := \mu_f(B(p, 1))^{-1} \mu_f$, and have the following proposition in analogy to the case of manifolds with a uniform Ricci curvature lower bound [6]:

**Proposition 2.12.** Consider a sequence $\{(M_i, p_i, g_i, f_i)\} \subset N_m(F, K)$ that converges to a pointed metric space $(X, p_\infty, d_\infty, f_\infty)$ in the pointed-Gromov-Hausdorff topology. Then there is a subsequence, denoted by $\{(M_{i_j}, p_{i_j}, g_{i_j}, f_{i_j})\}$, and a Randon measure $\nu_\infty$ such that $\{(M_{i_j}, p_{i_j}, g_{i_j}, \nu_{f_{i_j}})\}$ converges to $(X, p_\infty, d_\infty, \nu_\infty)$ in the pointed-measured-Gromov-Hausdorff topology. Moreover, $\nu_\infty$ satisfies the following conditions:

1. $\forall x \in X$ and $\forall r > 0$, suppose $M_{i_j} \ni x_{i_j} \overset{GH}{\to} x$, then

$$\nu_\infty(B(x, r)) = \lim_{j \to \infty} \nu_{f_{i_j}}(B(x_{i_j}, r));$$

2. $\forall x \in X$ and $\forall r_2 \geq r_1 > 0$,

$$\frac{\nu_\infty(B(x, r_2))}{\nu_\infty(B(x, r_1))} \leq c_F(x, r_2) \frac{\text{Vol}_K^m(r_2)}{\text{Vol}_K^m(r_1)}.$$

where $c_F(x, r_2) := F(r_2 + d_\infty(x, p_\infty))(r_2 + d_\infty(x, p_\infty))$. 

10
Furthermore, any Randon measure on $X$ satisfying the above two conditions agrees with $\nu_{\infty}$.

In the case of Gromov-Hausdorff convergence of Riemannian manifolds with a uniform Ricci curvature lower bound, this was shown in [6, Section 1], where the natural measure was renormalized by the volume of a unit ball (centered at some base point chosen on the manifold), and the renormalized measure was shown to converge, uniformly on compact subsets, to a limit Randon measure on the pointed-Gromov-Hausdorff limit. By Theorem 2.5, we could easily see the following estimates: 

$$\frac{\mu_f(B(x, r))}{\mu_f(B(y, s))} \leq e^{2Df(D)} \frac{Vol^m_K(r + d(x, y))}{Vol^m_K(s)},$$

and

$$\frac{\mu_f(B(x, r))}{\mu_f(B(y, s))} \geq \begin{cases} e^{-2Df(D)} \frac{Vol^m_K(r)}{Vol^m_K(s)} & \text{when } r \leq s + d(x, y), \\ 1 & \text{when } r \geq s + d(x, y). \end{cases}$$

Obviously, these estimates have [6, Estimates (1.2)-(1.4)] as their counterparts for manifolds with a uniform Ricci curvature lower bound, and consequently, the same constructions as [6, Theorem 1.6, Theorem 1.10] work in our situation to deduce the last proposition.

If we further focus on the sub-collection of all $m$-dimensional non-compact Ricci shrinkers, the natural measure $\mu_f$, as pointed out by [4, Corollary 1.1], has finite total mass. Due to the growth property of $f$ (Lemma 2.1), $\mu_f$ has an essentially canonical choice of base point — one of the minima of $f$. We will therefore consider a Ricci shrinker $(M, p, g, f)$ together with the canonical probability measure $\rho := \mu_f(M)^{-1}\mu_f$. In fact we have the following:

**Proposition 2.13.** Let $\{(M_i, p_i, g_i, f_i)\}$ be a sequence of $m$-dimensional non-compact Ricci shrinkers that pointed Gromov-Hausdorff converges to a metric space $(X, p_{\infty}, d_{\infty}, f_{\infty})$, then there is a subsequence and a unique Randon measure $\rho_{\infty}$ satisfying conditions (1) and (2) in Proposition 2.12. Moreover, $\rho_{\infty}$ is a probability measure.

**Proof.** It only remains to show that the limit measure $\rho_{\infty}$ is a probability measure. To see this we turn to the estimates in Lemma 2.1 and Lemma 2.2. For each $k \in \mathbb{N}$, define $D_k := 2^k D_0(m)$, and for the sake of simplicity let $B_{i,k}$ denote $B(p_i, D_k) \subset M_i$. Then we could estimate for each $i$ and each $k \geq K_m := [\frac{1}{2} \log_2(18m)]$:

$$\mu_f(M_i \setminus B_{i,k}) = \sum_{j=k}^{\infty} \int_{B_{i,j} \setminus B_{i,j-1}} e^{-f_i} dV_{g_i} \leq \sum_{j=k}^{\infty} |B_{i,j}| D_j^{m-1} e^{-\frac{1}{2} (D_j - 5m)^2} \leq |B_{i,0}| (2D_0(m))^m \int_k^{\infty} e^{-u} du = e^{-k (2D_0(m))^m |B_{i,0}|}.$$

This estimate, combined with the inequality

$$\mu_f(B_{i,0}) \geq e^{-\frac{1}{2}(D_0(m) + \sqrt{2m})^2} |B_{i,0}|,$$
implies
\[
\frac{\mu_f(M_i \setminus B_{i,k})}{\mu_f(M_i)} \leq e^{-k(2D_0(m))^m} e^{\frac{1}{2}(D_0(m)+\sqrt{2}m)^2}.
\]

Therefore, for each \(i\) and each \(k \geq K_m\),
\[
1 - e^{-k(2D_0(m))^m} e^{\frac{1}{2}(D_0(m)+\sqrt{2}m)^2} \leq \rho_i(B_{i,k}) \leq 1.
\]

Taking the pointed-Gromov-Hausdorff convergence with the radii \(\{D_k\}_{k \geq K_m}\), it is easy to see that the limit measure \(\rho_\infty\) has unit total mass, whence a probability measure. \(\Box\)

If we only consider the class \(\mathcal{M}_m(F, K; V_0)\), it follows from [23, Theorem 10.1] that

**Theorem 2.14.** Consider a sequence \(\{(M_i^m, p_i, g_i, f_i)\} \subset \mathcal{M}_m(F, K; V_0)\) such that

\[
(M_i^m, p_i, d_i, f_i) \xrightarrow{\text{pointed-Gromov-Hausdorff}} (M_\infty, p_\infty, d_\infty, f_\infty).
\]

Then \(M_\infty\) has a regular-singular decomposition \(M_\infty = R \cup S\) with the following properties.

(a). The singular part \(S\) is a closed set of Minkowski codimension at least 4.

(b). The regular part \(R\) is an \(m\)-dimensional open manifold with a \(C^{1,\alpha}\) metric \(g_\infty\) and \(f_\infty\) is a \(C^{1,\alpha}\) function on \(R\).

The convergence can be improved to

\[
(M_i, p_i, g_i, f_i) \xrightarrow{\text{pointed-}\hat{C}^{1,\alpha}-\text{Cheeger-Gromov}} (M_\infty, p_\infty, g_\infty, f_\infty),
\]

and the metric structure induced by smooth curves in \((R, g_\infty)\) coincides with \(d_\infty\).

Moreover, the limit renormalized measure \(\nu_\infty\) on \(M_\infty\) is defined as following: \(\nu_\infty\) vanishes on \(S\), and on \(R\) it has density \(V^{-1}_\infty d\mu_{f_\infty}\), where

\[
V_\infty := \lim_{i \to \infty} \mu_f(B(p_i, 1)).
\]

**Remark 2.15.** In fact, combining the work of Wang-Zhu [34] and Zhang-Zhu [37], the pointed-Gromov-Hausdorff convergence could already be improved to the pointed-\(\hat{C}^{1,\alpha}\)-Cheeger-Gromov convergence. However, without the endeavors made in [23], one cannot directly improve the regularity to pointed-\(\hat{C}^{1,\alpha}\)-Cheeger-Gromov convergence, let alone the pointed-\(\hat{C}_\infty^{\alpha}\)-Cheeger-Gromov convergence for Ricci shrinkers in Theorem 1.1.

Note that the limit measure identities

\[
\nu_\infty = \begin{cases} V^{-1}_\infty \mu_{f_\infty} & \text{on } R, \\ 0 & \text{on } S. \end{cases}
\]
amounts to say that in the sub-collection $\mathcal{M}_m(F, K; V_0)$, pointed-Gromov-Hausdorff topology is equivalent to pointed-measured-Hausdorff topology. Therefore we will only need to discuss the pointed-Gromov-Hausdorff topology on $\mathcal{M}_m(F, K; V_0)$.

If the sequence in consideration actually consists of complete Ricci shrinkers, we could promote the convergence to pointed-$C^\infty$-Cheeger-Gromov convergence, by the usual elliptic bootstrapping argument (see [17] and [23]). Also, the limit measure could be shown to be a probability measure $\rho_\infty$ such that with $U_\infty := \lim_{j \to \infty} \mu_f(M_j)$, we have

$$\rho_\infty = \begin{cases} U_\infty^{-1} \mu_f & \text{on } \mathcal{R}, \\ 0 & \text{on } \mathcal{S}. \end{cases}$$

### 2.3 Regular-singular decomposition of the Gromov-Hausdorff limits

The regular-singular decomposition of the pointed-Gromov-Hausdorff limit in Theorem [2.14] could be discussed in the more general setting for manifolds in $\mathcal{N}_m(F, K)$.

The definition of the regular part in the pointed-Gromov-Hausdorff limit $(X, p_\infty, d_\infty, f_\infty)$ of a sequence in $\mathcal{N}_m(F, K)$ is based on the concept of metric tangent cones, as done in the case of manifolds with a uniform Ricci curvature lower bound, see [6, Section 0].

To see the existence of a metric tangent cone for any point $x \in X$, we fix any sequence of scales $r_j \to 0$ (assuming $r_j \in (0, 1)$), and assume that $M_i \ni x \xrightarrow{GH} X \ni x$. We could then consider the sequence of pointed metric spaces $\{(X, x, \tilde{d}_j)\}$, with the rescaled metrics $\tilde{d}_j := r_j^{-1}d_\infty$. Clearly, we have $(M_i, p_i, r_j^{-2}g_i) \in \mathcal{N}_m(\tilde{F}_j, r_j K) \subset \mathcal{N}_m(F, K)$, where $\tilde{F}_j := r_j F$. Now for any $D > 0$ and any $j$ fixed, we have

$$B_{r_j^{-2}g_i}(x_i, D) \xrightarrow{\text{Gromov-Hausdorff}} B_{\tilde{d}_j}(x, D).$$

Therefore regarding $B_{r_j^{-2}g_i}(x_i, D) \subset (M_i, p_i, r_j^{-2}g_i)$, we have, by Proposition [2.12] that there is a limit renormalized measure $\tilde{\nu}_j$ such that $\forall B_{\tilde{d}_j}(x', 2r) \subset B_{\tilde{d}_j}(x, D)$,

$$\tilde{\nu}_j(B_{\tilde{d}_j}(x', 2r)) \leq C_j(x, r, D) \tilde{\nu}_j(B_{\tilde{d}_j}(x', r)),$$

where the sequence

$$C_j(x, r, D) := e^{F(d_\infty(x, p_\infty)+D)DVol_{\tilde{r}_j K/2}^m(2r)/Vol_{\tilde{r}_j K/2}^m(r)}$$

is uniformly bounded since $r_j \to 0$. Therefore the sequence of pointed metric spaces $\{(X, x, \tilde{d}_j)\}$, equipped with measures $\tilde{\nu}_j$, have a uniform doubling constant within the fixed distance $D$ to the base point. This implies that the maximal number $N_j(x, r, D)$ of disjoint $r$-balls fitting into $B_{\tilde{d}_j}(x, D) \subset (X, \tilde{d}_j)$ is uniformly bounded in $j \in \mathbb{N}$, and Gromov’s compactness theorem [13] Proposition 5.2 guarantees the existence of a complete metric space to which a subsequence in $\{(X, x, \tilde{d}_j)\}$ converges in the pointed-Gromov-Hausdorff topology. This limit metric space defines a metric tangent cone of $X$ at $x$. 

13
Now for each \( k = 1, 2, \cdots, m \), we define, following [6] Definition 0.1, the \( k \)-regular part of \( X \):
\[
\mathcal{R}_k := \{ x \in X : \text{any metric tangent cone at } x \text{ is isometric to the Euclidean } k\text{-space} \}.
\] (2.7)
We also call \( \bigcup_{k=1}^m \mathcal{R}_k \) the regular part of \( X \), denoted by \( \mathcal{R} \), and \( \mathcal{S} := X \setminus \mathcal{R} \).

To justify the notation, we have the following characterization of the regular part in the non-collapsing case:

**Theorem 2.16.** Let \( \{(M^m_i, p_i, g_i, f_i)\} \) be a sequence of manifolds in \( \mathcal{N}_m(F, K; V_0) \) such that
\[
(M^m_i, p_i, d_i, f_i) \xrightarrow{\text{pointed-Gromov-Hausdorf}} (M_{\infty}, p_{\infty}, d_{\infty}, f_{\infty}).
\]
Then \( y \in \mathcal{R} \) if and only if there exists a tangent cone at \( y \) which is isometric to \( (\mathbb{R}^m, g_{\text{Euc}}) \).

**Proof.** We first prove that if \( y \in B(p_{\infty}, D) \) is a regular point, then any tangent cone at \( y \) is \( (\mathbb{R}^m, g_{\text{Euc}}) \).
Otherwise, all tangent cones at \( y \) are isometric to \( (\mathbb{R}^l, g_{\text{Euc}}) \) for some integer \( l < m \). Then for any \( \epsilon > 0 \) there exists a small \( r = r(\epsilon) > 0 \) such that
\[
d_{GH}\left(\left\{B_{r^{-1}d_{\infty}}(y, 2), r^{-1}d_{\infty}\right\}, (B_{d_{\text{Euc}}}(0, 2), d_{\text{Euc}})\right) < \epsilon.
\]
Now for \( i \) large enough such that
\[
d_{GH}\left(\left\{B_{r^{-1}d_{i}}(y, 2), r^{-1}d_{i}\right\}, (B_{d_{\text{Euc}}}(0, 2), d_{\text{Euc}})\right) < \epsilon\] (2.8)
where \( y_i \rightarrow y \). We fix \( k = 10/\epsilon \) and consider a family of disjoint balls \( \{B(x_k, k^{-1})\} \) such that \( \{B(x_k, 2k^{-1})\} \) cover \( B(0, 1) \subset \mathbb{R}^l \). It is clear that \( N_k \leq k^l \). If we take \( x_{k,i} \rightarrow x_k \), then it is clear from (2.8) that \( B_{r^{-1}d_{i}}(y_{k,i}, 1) \) is covered by \( \left\{B_{r^{-1}d_{i}}(x_{k,i}, 3k^{-1})\right\} \) if \( i \) is sufficiently large.

We next estimate the volume of \( B_{r^{-1}d_{i}}(x_{k,i}, 3k^{-1}) \) by using Theorem 2.5. Let \( \bar{f}_i = f_i - f_i(x_{k,i}) \) and \( \bar{g}_i = r^{-2}g_i \), then \( R\bar{f}_i = Rf_i \geq r^2\bar{g}_i \). In addition, \( |\nabla g, \bar{f}_i| = r|\nabla g, f_i| \leq rF(D) \). It is clear from Theorem 2.5 that
\[
|B_{r^{-1}d_{i}}(x_{k,i}, 3k^{-1})|_{\bar{g}_i} \leq Ck^{-m}
\]
for some \( C \) independent of \( r \) and \( k \) if \( r \) and \( k^{-1} \) are sufficiently small. Therefore
\[
|B_{r^{-1}d_{i}}(y_{k,i}, 1)|_{\bar{g}_i} \leq N_k|B_{r^{-1}d_{i}}(x_{k,i}, 3k^{-1})|_{\bar{g}_i} \leq CN_kk^{-m} \leq Ck^{-m}.
\] (2.9)
However, \( |B_{r^{-1}d_{i}}(y_{k,i}, 1)|_{\bar{g}_i} = r^{-m}|B_d(y_{k,i}, 1)|_{\bar{g}_i} \), which by Theorem 2.5 again, is greater than a constant \( C = C(m, K, F(2D), D, V_0) \). If we let \( \epsilon \rightarrow 0 \), then we get a contradiction from (2.9).

Conversely, for any point \( y \in B(p_{\infty}, D) \) such that there exists a sequence \( r_k \rightarrow 0 \) satisfying
\[
d_{GH}\left(\left\{B_{r_k^{-1}d_{\infty}}(y, 2), r_k^{-1}d_{\infty}\right\}, (B_{d_{\text{Euc}}}(0, 2), d_{\text{Euc}})\right) < \epsilon
\]
for any \( \epsilon > 0 \) if \( k \) is sufficiently large. As before, with \( r_k \) fixed, we have
\[
d_{GH}\left(\left\{B_{r_k^{-1}d_{i}}(y, 2), r_k^{-1}d_{i}\right\}, (B_{d_{\text{Euc}}}(0, 2), d_{\text{Euc}})\right) < \epsilon
\] (2.10)
if $i$ is sufficiently large. Now we can apply [34 Lemma 4.11] to conclude that
\[ |B_{r_i^{-1}d_i}(y_i, 1)|_{\bar{g}_i} \geq (1 - \Psi(\epsilon))\omega_m \]
for $\Psi(\epsilon) \to 0$ if $\epsilon \to 0$. By Theorem [2.5], it is clear that for any $s \leq 1$,
\[ |B_{r_i^{-1}d_i}(y_i, s)|_{\bar{g}_i} \geq (1 - \Psi(\epsilon))\omega_ms^m. \]
In other words, for any $r \leq r_k$,
\[ |B_{r^{-1}d_i}(y_i, 1)|_{r^{-2}\bar{g}_i} \geq (1 - \Psi(\epsilon))\omega_m. \]
From [34 Corollary 4.8] which we apply to the metric $r^{-2}\bar{g}_i$ and $\bar{f}_i$, it implies that
\[ d_{GH}\left(\left(B_{r^{-1}d_i}(y_i, 1), B_{d_{\text{Euc}}}(0, 1)\right)\right) < \Psi(\epsilon). \]
Note that the above inequality holds uniformly for any $r \leq r_k$. By taking $i \to \infty$, we have
\[ d_{GH}\left(\left(B_{r^{-1}d_{\infty}}(y_{\infty}, 1), B_{d_{\text{Euc}}}(0, 1)\right)\right) < \Psi(\epsilon). \]
We can conclude immediately that all tangent cones at $y_{\infty}$ is $(\mathbb{R}^m, g_{\text{Euc}})$.

In general, we notice that for any fixed $D > 0$, the concepts of (weakly) $k$-Euclidean points in $B(p_{\infty}, D)$ are defined indifferently from the case with a uniform Ricci curvature lower bound (see Definition 0.3, Definition 2.2 and Definitions of $\mathcal{WE}_k$ and $(\mathcal{WE}_k)_e$ in [6]). Therefore, the concepts involved in proving [5] Theorem 2.1] are parallel to the case of $\mathcal{N}_m(F, K)$, and the very same proof leads to the following

**Proposition 2.17 (Nelibility of the singular set).** Suppose a sequence $\{(M_i, p_i, g_i, f_i)\} \subset \mathcal{N}_m(F, K)$ converges to a limit metric space $(X, p_{\infty}, d_{\infty})$ in the pointed-Gromov-Hausdorff topology, together with a limit function $f_{\infty}$ and a limit measure $\mu_{\infty}$, then
\[ \mu_{\infty}(S) = 0. \]

In the case of the measured, pointed Gromov-Hausdorff limits of a sequence of Ricci shrinkers, we of course have the limit probability measure satisfying $\rho_{\infty}(S) = 0$.

### 3 Parabolic smoothing of the distance function

This section contains the analytic core of the paper: the $f$-heat kernel bounds on manifolds in $\mathcal{N}_m(F, K)$, Theorem 3.1, and their applications in the parabolic smoothing of the distance functions (Lemma 3.15 and Lemma 3.16).
3.1 Heat kernel on manifolds in $\mathcal{N}_m(F, K)$

Given a metric measure space $(M, p, g, \mu_f)$ in $\mathcal{N}_m(F, K)$, note that the weighted Laplacian operator $\Delta_f$ is self-adjoint with respect to the measure $\mu_f$. Moreover, we have the following Bochner formula for any smooth function $u$ on $M$,

$$\frac{1}{2} \Delta_f |\nabla u|^2 = |\text{Hess}_f u|^2 + \text{Rc}_f(\nabla u, \nabla u) + \langle \nabla \Delta_f u, \nabla u \rangle.$$  \hfill (3.1)

If $u$ is defined on the spacetime $M \times [0, T)$ and satisfies the weighted heat equation

$$\square_f u \equiv (\partial_t - \Delta_f) u = 0,$$

then a parabolic version of (3.1) is

$$\frac{1}{2} \square_f |\nabla u|^2 = -|\text{Hess}_f u|^2 - \text{Rc}_f(\nabla u, \nabla u).$$ \hfill (3.2)

Now we denote the heat kernel by $H(x, y, t)$ or $H_f(x, y, t)$ if we want to emphasize the role of $f$. The existence and uniqueness of $H$ can be found, for example in [14, Theorem 7.7, Corollary 9.6]. To apply [14, Corollary 9.6], we must check the stochastic completeness of $(M, g, \mu_f)$. By our definition, $(M, g, \mu_f)$ is a smooth $\text{CD}(K, \infty)$ space, then there exists a constant $C$ such that

$$\mu_f(B(p, r)) \leq C e^{Cr^2}.$$  \hfill (3.3)

The proof of the above inequality can be found in [33, Theorem 18.12]. Then the stochastic completeness follows immediately from [14, Theorem 11.8].

We have the following upper and lower bound of $H$, see also [36, Theorem 1.1]:

**Theorem 3.1.** Let $(M, p, g, \mu_f)$ be a space in $\mathcal{N}_m(F, K)$. For any $D > 0$, there exists a constant $C = C(m, F(2D), K, D) > 1$ such that

$$\frac{C^{-1}}{\mu_f(B(x, \sqrt{t}))} \exp\left(-\frac{d^2(x, y)}{C^{-1}t}\right) \leq H(x, y, t) \leq \frac{C}{\mu_f(B(x, \sqrt{t}))} \exp\left(-\frac{d^2(x, y)}{Ct}\right),$$  \hfill (3.3)

for any $x, y \in B(q, D/3)$ with $d(p, q) \leq D$ and $0 < t < D^2/4$.

**Proof.** The upper and lower bound of the weighted heat kernel $H$ follow essentially from [30, Theorem 4.1, Theorem 4.8]. In our setting, it is clear that the Dirichlet form is defined as

$$\mathcal{E}(u, v) = \int \langle \nabla u, \nabla v \rangle \, d\mu_f.$$  \hfill (3.2)

Then the Markov semigroup $(P_t)_{t \geq 0}$ satisfies for any $t > 0$ and $u \in L^2(M, \mu_f)$,

$$\frac{dP_t u}{dt} = \Delta_f P_t u.$$

Since we have the local volume doubling property (2.2) and $L^2$-Poincaré inequality (Proposition 2.7), then conclusion follows immediately. \hfill $\Box$
Corollary 3.2. With the same conditions as those in Theorem 3.1, there exists a constant \( C = C(m, F(2D), K, D) > 1 \) such that

\[
\int_{B(q,D/3)\setminus B(q,r)} H(q,y,t) \, d\mu_f(y) \leq Cr^{-2}t
\]

for any \( r \leq D/10 \) and \( t \leq D^2/4 \).

Proof. By computation

\[
\int_{B(q,D/3)\setminus B(q,r)} H(q,y,t) \, d\mu_f(y) \\
\leq \sum_{k=N_0}^{N_1} \int_{B(q,2^k \sqrt{t}) \setminus B(q,2^{k-1} \sqrt{t})} H(q,y,t) \, d\mu_f(y) \\
\leq \sum_{k=N_0}^{N_1} Ce^{-C^{-1}4^k} \frac{\mu_f(B(x, 2^k \sqrt{t}))}{\mu_f(B(x, 2^{k-1} \sqrt{t}))} \\
\leq C \sum_{k=N_0}^{\infty} e^{-C^{-1}4^k} \leq C r^{-2}t
\]

where \( N_0 = \lceil \log_2 \frac{r}{\sqrt{t}} \rceil \) and \( N_1 = \lceil \log_2 \frac{D}{3 \sqrt{t}} \rceil \). Here we have used the elementary inequality

\[
\sum_{k=0}^{\infty} e^{-4^k} \leq Cr^{-1}
\]

for any \( l > 0 \).

We need the following Li-Yau gradient estimate from [20, Theorem 3.1, (a)]:

Theorem 3.3. With the same conditions, there exists a constant \( C = C(m, K, F(2D), D) > 1 \) such that

\[
C^{-1} \frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{C}{t}
\]

on \( B(q,D) \times [0,D^2] \).

Next we need the following Harnack inequality which is a special case of [36, Theorem 3.1]:

Theorem 3.4. With the same conditions, for any \( D > 0 \), there exists a constant \( C = C(m, K, F(2D), D) > 1 \) such that

\[
\sup_{B(q,r/2)} u(\cdot, r^2/2) \leq Cu(q, r^2)
\]

where \( u \) is a positive solution of the weighted heat equation on \( B(q,r) \times [0,r^2] \).
For later applications, we now show the following gradient estimate:

**Lemma 3.5** (Cheng-Yau estimate). Let \((M, p, g, \mu_f)\) be a space in \(\mathcal{N}_m(F, K)\). Consider a smooth function \(u\) on \(B(q, r) \times \{s, s - r^2\}\) with \(r \leq D\) which satisfies the weight heat equation and is bounded. Then there exists a constant \(C = C(m, K, F(2D), D) > 0\) such that for any \(r \leq D\),

\[
\left| \nabla u \right|_{B(q, r/2) \times \{s - r^2/2, s\}} \leq Cr^{-1}\text{osc}_{{B(q, r) \times \{s - r^2, s\}}} u.
\]

**Proof.** Without loss of generality, we assume that \(u \geq 0\). We choose a cutoff function \(\psi\) on \(\mathbb{R}\) such that \(\psi = 1\) on \((-\infty, r]\) and \(\psi = 0\) on \([2r, \infty)\). Moreover, we assume that \(|\psi'| \leq C(n)r^{-1}\). We set \(\eta(x, t) = \psi(d(q, x))\psi(s - t)\). Multiplying both sides of \(\Box f = 0\) by \(\eta^2 u\) and integrating by parts, we obtain

\[
\iint |\nabla (\eta u)|^2 \, d\mu_f \, dt \leq \iint (|\nabla \eta|^2 + \eta_t^2/2)u^2 \, d\mu_f \, dt - \int u^2\eta_t^2/2 \, d\mu_f \big|_{t=s} \leq Cr^{-2} \iint_{B(q, r) \times \{s - r^2, s\}} u^2 \, d\mu_f \, dt.
\]

That is,

\[
\iint_{B(q, r/2) \times \{s - r^2/2, s\}} |\nabla u|^2 \, d\mu_f \, dt \leq C\mu_f(B(q, r))\text{osc}_{{B(q, r) \times \{s - r^2, s\}}} u^2.
\]

On the other hand, by computation,

\[
\Box f |\nabla u|^2 = -2|Hess u|^2 - 2Rc_f(\nabla u, \nabla u) \leq 2K|\nabla u|^2
\]

or

\[
\Box f \left(e^{-2K|\nabla u|^2}\right) \leq 0.
\]

Now we apply the Moser iteration on \(e^{-2K|\nabla u|^2}\), see [36, Proposition 2.7], that

\[
\left| \nabla u \right|_{B(q, r/4) \times \{s - r^2/2, s\}, 4s} \leq \frac{C}{r^2 \mu_f(B(q, r)/2)} \iint_{B(q, r/2) \times \{s - r^2/4, s\}} |\nabla u|^2 \, d\mu_f \, dt.
\]

Therefore,

\[
\left| \nabla u \right|_{B(q, r/4) \times \{s - r^2/2, s\}, 4s} \leq \frac{C\mu_f(B(q, r))}{r^2 \mu_f(B(q, r)/2)} \text{osc}_{{B(q, r) \times \{s - r^2, s\}}} u^2 \leq Cr^{-2}\text{osc}_{{B(q, r) \times \{s - r^2, s\}}} u^2
\]

by the volume doubling. \(\square\)

Now we prove the existence of a local cut-off function, see also [5, Theorem 6.33].

**Theorem 3.6.** Let \((M, p, g, \mu_f)\) be a space in \(\mathcal{N}_m(F, K)\). For any \(D > 0\) and \(q\) with \(d(p, q) \leq D\), there exists a smooth cutoff function \(\phi\) which is supported in \(B(q, r)\) and \(\phi = 1\) on \(B(q, r/2)\) such that

\[
r|\nabla \phi| + r^2|\Delta_f \phi| \leq C
\]

for some \(C = C(m, K, F(2D), D) > 0\) and \(r \leq D/3\).
Proof. Consider the function $u(x, t) = H(x, q^* , t)$, then it follows from (3.3) that there exist $C_0 = C_0(m, K, F(2D), D) > 1$ and $C_1 = C_1(m, K, F(2D), D) > 1$ such that
\[
\begin{align*}
  u &\geq C_0^{-1} a & \text{on } B(q^*, \rho/2) \times [\rho^2/3, \rho^2], \\
  u &\leq C_0^{-1} a/2 & \text{on } B(q^*, 2C_1\rho) \setminus B(q^*, C_1\rho) \times [\rho^2/3, \rho^2], \\
  u &\leq Ca & \text{on } B(q^*, 2C_1\rho) \times [\rho^2/4, \rho^2],
\end{align*}
\]
where $a = \nu^{-1}_f (B(q^*, \rho))$. Here we require that all sets considered are contained in $B(q, D/6)$.

Now it follows from Lemma 3.5 that
\[
|\nabla u| \leq C\rho^{-1} a
\]
on $B(q^*, C_1 r) \times [\rho^2/3, \rho^2]$.

Now we take a nonnegative cut-off function $\eta$ on $\mathbb{R}$ supported in $[\rho^2/2, 2\rho^2/3]$ such that $|\eta'| \leq C\rho^{-2}$ and
\[
\int_{\rho^2/2}^{2\rho^2/3} \eta \geq C^{-1}\rho^2.
\]
Then we define a function
\[
\psi(x) = C_0 \frac{\partial}{\partial \rho^2} \int_{\rho^2/2}^{2\rho^2/3} u(x, t)\eta(t) \, dt.
\]
Then we have
\[
\begin{align*}
  \psi &\geq 1 & \text{on } B(q^*, \rho/2), \\
  \psi &\leq 1/2 & \text{on } B(q^*, 2C_1\rho) \setminus B(q^*, C_1\rho), \\
  |\nabla \psi| &\leq C\rho^{-1} & \text{on } B(q^*, C_1\rho).
\end{align*}
\]
Moreover,
\[
\Delta_f \psi(x) = C_0 \frac{\partial}{\partial \rho^2} \int_{\rho^2/2}^{2\rho^2/3} \Delta_f u(x, t)\eta(t) \, dt
\]
\[
= C_0 \frac{\partial}{\partial \rho^2} \int_{\rho^2/2}^{2\rho^2/3} \partial_t u(x, t)\eta(t) \, dt
\]
\[
= - C_0 \frac{\partial}{\partial \rho^2} \int_{\rho^2/2}^{2\rho^2/3} u(x, t)\partial_t \eta(t) \, dt,
\]
therefore on $B(q^*, C_1 r)$,
\[
\rho^2 |\Delta_f \psi| \leq C.
\]
Now we construct a smooth nondecreasing function $F(t) = 0$ if $t \leq 1/2$ and $F(t) = 1$ if $t \geq 1$. Then by considering the composite function $F(\psi(x))$ we have proved that for any $B(q^*, r) \subset B(q, D)$, there exists a cutoff function $\phi^*$ supported in $B(q^*, r)$ such that $\phi^* = 1$ on $B(q^*, C_1^{-1} r)$ and
\[
|r| |\nabla \phi^*| + r^2 |\Delta_f \phi^*| \leq C.
\]
The rest proof is a standard covering argument. By the local volume doubling, there exists an integer \( N = N(m, K, F(2D)) > 1 \) such that we can find \( q_1, q_2, \ldots, q_N \in M \) such that

\[
B(q, 1/2) \subset \bigcup_{i=1}^{N} B(q_i, C_1 r).
\]

Then the function \( \phi := F(\sum_{i=1}^{N} \phi_i) \) will satisfy all conditions.

Now the theorem follows from a standard argument by [2.2]. \( \square \)

**Remark 3.7.** In [34, Lemma 1.5], the same conclusion is proven using Green’s function.

Now we need the following space-time control of the Hessian term of a heat equation solution:

**Lemma 3.8.** With the same assumptions as in Lemma 3.5 there exists \( C = C(m, K, F(2D), D) > 0 \) such that

\[
\int_{s-t^2/4}^{s} \int_{B(q, r/4)} |\text{Hess}_u|^2 \, d\mu_f dt \leq C r^{-2} \left( \frac{\text{osc}}{B(q, x) \times [s-t^2/4, s]} \right)^2 u^2.
\]

**Proof.** We have as before

\[
\square / |\nabla u|^2 = -2 |\text{Hess}_u|^2 - 2 R_{x_f}(\nabla u, \nabla u) \leq -2 |\text{Hess}_u|^2 + 2 K |\nabla u|^2. \tag{3.4}
\]

We choose a nonincreasing cutoff function \( \eta \) on \( \mathbb{R} \) such that \( \eta(x) = 1 \) if \( x \leq 1/2 \) and \( \eta(x) = 0 \) if \( x \geq 1 \). Let \( \phi \) be a cutoff function constructed in the last theorem such that \( \phi = 1 \) on \( B(q, r/2) \) and is supported in \( B(q, r) \). We also set \( \psi(x, t) := \phi(x) \eta(-tr^2) \). By multiplying both sides of (3.4) by \( \psi \) and integrating, we have

\[
\iint 2|\text{Hess}_u|^2 \psi \, d\mu_f dt \leq \iint -\psi \square / |\nabla u|^2 + 2 K |\nabla u|^2 \psi \, d\mu_f dt
\]

\[
= \iint (\partial_t + \Delta) \psi |\nabla u|^2 + 2 K |\nabla u|^2 \psi \, d\mu_f dt - \left. \int |\nabla u|^2 \psi \, d\mu_f \right|_{t=\infty}
\]

\[
\leq C r^{-2} \iint_{B(q, r/2) \times [s-t^2/2, s]} |\nabla u|^2 \, d\mu_f dt.
\]

Then the conclusion follows from Lemma 3.5 and the local volume doubling. \( \square \)

For a closed set \( X \subset M \) and \( 0 < r_0 < r_1 \), the annulus \( A_{r_0, r_1}(X) \) is defined as \( T_{r_1} \setminus T_{r_0} \), where \( T_r(X) \) is the \( r \)-tubular neighborhood of \( X \). Then by using Theorem 3.4 and a similar argument in [11, Lemma 2.6] we have

**Corollary 3.9.** Let \( (M, p, g, \mu_f) \) be a space in \( N_m(F, K) \). For any \( D > 0 \), \( 0 < 10 r_0 < r_1 < D/10 \) and a compact set \( X \subset B(q, D/10) \) with \( d(p, q) \leq D \), there exists a smooth nonnegative cutoff function \( \phi \) such that for some constant \( C = C(m, K, F(2D), D) > 0 \),

1. \( \phi = 1 \) on \( A_{3r_0, r_1}(X) \) and \( \phi = 0 \) on \( M \setminus A_{2r_0, r_1/2}(X) \).
2. \( r_0 |\nabla \phi| + r_0^2 |\Delta_f \phi| \leq C \) on \( A_{2r_0, 3r_0}(X) \).

3. \( r_1 |\nabla \phi| + r_1^2 |\Delta_f \phi| \leq C \) on \( A_{r_1/3, r_1/2}(X) \).

Next we prove the following mean value inequality which is similar to the Lemma 2.1 of [11].

**Lemma 3.10.** Let \((M, p, g, \mu_f)\) be a space in \( N_m(F, K) \). For any \( D > 0 \) and \( q \) with \( d(p, q) \leq D \), there exists a constant \( C = C(m, K, F(2D), D) > 1 \) such that the following holds. If \( u_t = u(x, t) \) is nonnegative continuous function on \( M \times [0, r^2] \) with compact support on each time slice in \( B(q, D/5) \), \( r \leq D/10 \) and \( \square_f u \geq -c_0 \) in the distribution sense, then

\[
\int_{B(x,r)} u_0 \, d\mu_f \leq C(u(x) + c_0 r^2).
\]

**Proof.** We fix \( x \) and \( r \), then for any \( t \in [0, r^2] \), we have

\[
\begin{align*}
\partial_t \int H(x, y, r^2 - t) u(y, t) \, d\mu_f(y) & = \int \partial_t Hu + H \partial_t u \, d\mu_f \\
& = \int -\Delta_f Hu + H \Delta_f u + H \square_f u \, d\mu_f \\
& = \int H \square_f u \, d\mu_f \\
& \geq -c_0 \int H \, d\mu_f = -c_0.
\end{align*}
\]

As \( t \to r^2 \), it follows from the definition of \( H \) that

\[
\lim_{t \to r^2} \int H(x, y, r^2 - t) u(y, t) \, d\mu_f(y) = u(x, r^2).
\]

From the heat kernel lower bound (3.3), we have

\[
\int H(x, y, r^2) u_0(y) \, d\mu_f(y) \geq C \int_{B(x,r)} u_0(y) \, d\mu_f(y).
\]

The proof is complete if we integrate (3.5) from 0 to \( r^2 \). \( \square \)

In particular, if \( u \) is independent of \( t \), we have

**Corollary 3.11.** Let \((M, p, g, \mu_f)\) be a space in \( N_m(F, K) \). For any \( D > 0 \) and \( q \) with \( d(p, q) \leq D \), there exists a constant \( C = C(m, K, F(2D), D) > 1 \) such that the following holds. If \( u(x) \) is nonnegative continuous function on \( M \) with compact support in \( B(q, D/5) \), \( r \leq D \) and \( \Delta_f u \leq -c_0 \), then

\[
\int_{B(x,r)} u \, d\mu_f \leq C(u(x) + c_0 r^2).
\]
3.2 Smoothing the distance function

In this subsection, we fix a manifold \( (M^m, p, g, f) \in N_m(F, K) \) and two points \( q', q \) in \( B(p, D/2) \) with \( d(q', q) = d \). Recall that the excess function of \( q' \) and \( q \) is defined as

\[
e(x) := d(x, q') + d(x, q) - d(q', q).
\]

We also set \( d^-(x) = d(q', x) \) and \( d^+(x) = d - d(q, x) \).

Now we have

**Theorem 3.12.** Assume that \( d \leq 1 \) and a constant \( 0 < \epsilon < 1 \), if \( x \in A_{ed,2d}([q', q]) \) satisfies \( e(x) \leq r^2d \leq \bar{r}^2(m, \epsilon)d \), then

\[
\int_{B_{rd}(x)} e \, d\mu_f \leq Cr^2d
\]

for some \( C = C(m, K, F(2D)) > 0 \).

**Proof.** By Corollary 3.9 with \( X = [q', q] \), there exists a cutoff function \( \phi \). If \( u = \phi e \), then

\[
\Delta_f u = \Delta_f \phi e + \phi \Delta_f e + 2(\nabla \phi, \nabla e) \leq |\Delta_f \phi e + \phi \Delta_f e + 4|\nabla \phi| \leq \frac{C}{d}
\]

where we have used Theorem 2.3 and \( d \leq 1 \) for the last inequality. Then the theorem follows from Corollary 3.11. \( \square \)

Similar to [11, Section 2], we evolve the distance functions to \( q' \) and \( q \) by the weighted heat equation. For a fixed \( \delta > 0 \), by using Corollary 3.9, there exists a cutoff function \( \phi \) such that \( \phi = 1 \) on \( M_{\delta d/4,8d} \) and \( \phi = 0 \) on \( M \setminus M_{\delta d/16,16d} \) where

\[
M_{r,s} := A_{rd,sd}(q') \cap A_{rd,sd}(q).
\]

Then we define \( h^\pm_t \) and \( e_t \) to be solutions to the equation \( \Box_f h^\pm_t = 0 \) and \( \Box_f e_t = 0 \) with initial values \( h^\pm_0 = \phi d^\pm \) and \( e_0 = \phi e \), respectively.

**Lemma 3.13.** There exists a constant \( C = C(m, K, F(2D), \delta) > 0 \) such that

\[
\Delta_f e_t, \Delta_f h^-_t, -\Delta_f h^+_t \leq \frac{C}{d}.
\]

**Proof.** We only prove the conclusion for \( e_t \), others are similar. As before, we have

\[
\Delta_f e_0 = \Delta_f (\phi e) \leq \frac{C}{d}.
\]

Moreover, for any \( t > 0 \),

\[
\Delta_f e_t = \int \Delta_{f,x} H(x,y,t)\phi(y)e(y) \, d\mu_f(y) = \int \Delta_{f,y} H(x,y,t)\phi(y)e(y) \, d\mu_f(y) \leq \frac{C}{d}.
\]

\( \square \)
**Lemma 3.14.** There exists a constant $C = C(m, K, F(2D), \delta) > 0$ such that for any $e \leq \varepsilon(m, \delta)$ and $x \in M_{3/2, 4}$, the following holds for each $y \in B_{10e}(x)$.

(i) $|e_{\epsilon^2, e^2}(y)| \leq C(e^2 d + e(x))$.

(ii) $|\partial_1 e_{\epsilon^2, e^2}(y) + |\Delta f e_{\epsilon^2, e^2}(y) \leq C(\frac{1}{d} + e(x) \epsilon^2 d^2)$.

(iii) $\int_{B_{\epsilon^2, e^2}(y)} |\text{Hess}_{\epsilon^2, e^2}| \, d\mu_f \leq C\left(\frac{1}{d} + e(x) \epsilon^2 d^2\right)^2$ for some $r \in [1/2, 1]$.

(iv) $|h^e_{\epsilon^2, e^2} - d^e(x) \leq C(e^2 d + e(x))$.

(v) $|\nabla h^e_{\epsilon^2, e^2}(y) \leq 1 + C e^2 d^2$.

**Proof.** From Lemma 3.13 we have

$$e_r(x) = e_0(x) + \int_{0}^{1} \Delta f e_r \, ds \leq e(x) + C t$$

and hence

$$e_r(x) \leq e_0(x) + C e^2 d$$

for any $t \in [e^2 d^2, 100 e^2 d^2]$. Then it follows from Theorem 3.4 that if $y \in B_{10e}(x)$,

$$e_{\epsilon^2, e^2}(y) \leq C(e^2 d + e(x))$$

and the (i) follows. (ii) follows from Lemma 3.3 and (iii) follows from Lemma 3.13 and the gradient estimate Theorem 3.4. In addition, (iv) follows from Lemma 3.8. Indeed, since we have

$$\int_{e^2 d^2/2}^{e^2 d} \int_{B_{\epsilon^2, e^2}(y)} |\text{Hess}_{\epsilon^2, e^2}|^2 \, d\mu_f \, dt \leq C\left(\frac{1}{d} + e(x) \epsilon^2 d^2\right)^2 .$$

Therefore such $r$ must exist. Next (v) follows from an identical proof in [11] Lemma 2.3. Finally, (vi) follows the same as [11] Lemma 2.17 by using Corollary 3.6. $\Box$

Recall that an $\epsilon$-geodesic between $q'$ and $q$ is a unit speed curve $\sigma$ such that $||\sigma| - d(q', q)| \leq \epsilon$. In particular, it implies that for any $x \in \sigma$, $e(x) \leq e^2 d^2$.

**Lemma 3.15.** There exists a constant $C_3 = C_3(m, F, K, D, \delta) > 0$ such that for any $e \leq \varepsilon(m, \delta)$, $x \in M_{3/2, 4}$ with $e(x) \leq e^2 d$, and any $\epsilon$-geodesic $\sigma$ connecting $q'$ and $q$, there exists $r \in [1/2, 2]$ with

(i) $|h^e_{\epsilon^2, e^2} - d^e(x) \leq C_3 e^2 d$.

(ii) $\int_{B_{\epsilon^2, e^2}(x)} |\nabla h^e_{\epsilon^2, e^2}|^2 - 1 |d\mu_f \leq C_3 e$.

(iii) $\int_{B_{\epsilon^2, e^2}(x)} \left(\int_{B_{\epsilon^2, e^2}(x)} |\nabla h^e_{\epsilon^2, e^2}|^2 - 1 |d\mu_f \right) \leq C_3 e^2 d$.

(iv) $\int_{B_{\epsilon^2, e^2}(\sigma(x))} |\text{Hess}_{\epsilon^2, e^2}| \, d\mu_f \leq C_3 d^{-2}$.
Proof. The proof is the same as [11] Theorem 2.19. □

Similar to [11] Theorem 2.20, we also have

**Lemma 3.16.** There exists a constant $C_4 = C_4(m, F, K, D, \delta) > 0$ such that for any $x \in M_{\delta/2}$ and $\delta \leq s < t \leq d_x = d(q', x)$, the following estimates hold,

\[
\begin{align*}
(i) & \int_0^{d_x} \|\nabla h_{r,t}^s \|^2 \leq \frac{C_4}{\delta}(e(x) + r^2), \\
(ii) & \int_0^{d_x} |\langle \nabla h_{r,t}^s, \nabla d^s \rangle - 1| \leq \frac{C_4}{\delta}(e(x) + r^2), \\
(iii) & \int_0^{d_x} |\nabla h_{r,t}^s - \nabla d^s| \leq \frac{C_4}{\sqrt{\delta}}(e(x) + r).
\end{align*}
\]

4 Convexity of the regular part in Gromov-Hausdorff limits

In this section we prove our main improvement of the previous structural results in [23] about the pointed-Gromov-Hausdorff limits of manifolds in $N_m(F, K)$: we will show that the regular part, as defined in Section 2.3, is both weakly convex and almost everywhere convex with respect to the limit measure. In conjunction with the regularity improvements obtained in [23], we also show that the regular part on a pointed-Gromov-Hausdorff limit of manifolds in $N_m(F, K)$ is actually a strongly convex open subset. Results about Ricci shrinkers are summarized in Theorem 4.11.

4.1 Gromov-Hausdorff distance between nearby metric balls

In this subsection, we prove a Gromov-Hausdorff distance control of nearby geodesic balls of the same size. The proof follows from the original idea in [11] Section 3, but since the Ricci lower bound, the basic assumption underlying essentially everywhere of their arguments, is unavailable for manifolds in $N_m(F, K)$, we have to rework most of the details there and fit them into our setting.

Fix $\gamma : [0, l] \to M$ a minimal geodesic of length $l$ and unit speed. Let $\gamma(0) = q$ and $\gamma(l) = q'$. For $d_x:=d(q, -)$, let $\psi_s$ be the homeomorphism generated by the vector field $-\nabla d_q$. We notice that $\psi_s$ is smooth away from $q$ and the cut locus $C_q$ of $q$. Moreover, if $x \neq q$ and $x \notin C_q$, then the curve $s \mapsto \psi_s(x)$ is the unique minimal geodesic connecting $x$ to $q$. Throughout this section we fix some $\delta \in (0, 1/10)$. We will show the Gromov-Hausdorff continuity in a Hölder manner for balls centered on $\gamma([\delta l, l - \delta l])$, with $\delta \in (0, 1/10)$.

Fixing $t \in [\delta l, l - \delta l]$, for each $r \in [0, \delta/10]$, we consider the following core neighborhood of $\gamma(t)$:

\[
H_r := \left\{ y \in B(\gamma(t), r) : \text{ for all } s \in [0, t - \delta l], \ d(\psi_s(y), \gamma(t - s)) \leq \exp \left( C_5 \sqrt{\delta} \right) r \right\},
\]

where

\[
C_5 = C_5(m, F, K, D, \delta, l) := \frac{1}{2} \left( 2(m - 1)(\delta l)^{-1} + 2(m - 1) \sqrt{K} + 2F(2R) + (1 - 2\delta)Kl \right) \frac{1}{2}.
\]

Intuitively speaking, such neighborhood of $\gamma(t)$ consists of points in $B(\gamma(t), r)$ that are carried by $\psi_s$ up to a controllable distance for all $s \leq t - \delta l$. On a manifold in $N_m(F, K)$, we could in fact conclude that when $r$ is sufficiently small, almost every point of $B(\gamma(t), r)$ are in $H_r$.
Lemma 4.1. Fix $\delta \in (0, 1/10)$ and $t \in [\delta l, l - \delta l]$. Let $\gamma : [0, l] \to M$ be a minimal geodesic of unit speed, with $\gamma(0) = q$ and $\gamma(l) = q'$, then for $r > 0$ sufficiently small, 

$$
\mu_f(H'_r) = \mu_f(B(\gamma(t), r)).
$$

(4.1)

Proof. Let $\iota(x)$ denote the injectivity radius of $x \in M$, and define

$$
r := \min \left\{ \delta l/10, \min_{s \in [0,l]} \iota(\gamma(s)) \right\}.
$$

Since $\gamma([0, l])$ is compact and $\iota$ is positive and continuous on $M$, we see $r > 0$.

For any $x \in B(\gamma(t), r) \setminus C_q$, there exists some tangent vector $v \in T_{\gamma(t)}M$ such that $exp_{\gamma(t)} v = x$. Then consider the Jacobi field $J$ along $\gamma$ such that $J(0) = 0$ and $J(t) = v$. Since $x \notin C_q$, $J$ never vanishes on $\gamma([\delta l, l])$. Moreover, we have $exp_{\gamma(t-s)}J(t-s) = \psi_s(x)$ and $d(\gamma(t-s), \psi_s(x)) = |J(t-s)|$. Therefore it suffices to examine $|J|$ along $\gamma$.

In order to estimate $|J|$, we notice that $\frac{d}{dt} |J|^2 = Hess_{d_q,J,J}$, and thus

$$
\left| \frac{d}{dt} \log |J|^2 \right| \leq |Hess_{d_q}|.
$$

Integrating from any $s \in [0, t - \delta l]$ to $t$ we see

$$
\left| \log \frac{|J|^2(t)}{|J|^2(t-s)} \right| \leq \int_{t-s}^t |Hess_{d_q}|(\gamma(u)) \, du \leq \|Hess_{d_q}\|_{L^2(\gamma([\delta l, l - \delta l]))} \sqrt{s}.
$$

Therefore, $\forall s \in [0, t - \delta l]$, we have

$$
d(\psi_s(x), \gamma(t-s)) \leq \exp \left( \frac{1}{2} \|Hess_{d_q}\|_{L^2(\gamma([\delta l, l - \delta l]))} \sqrt{s} \right) r,
$$

(4.2)

and the following claim guarantees that $B(x, r) \setminus C_q \subset H'_r$, whence the weighted volume estimate.

Claim: For the given $\gamma$, we have $\|Hess_{d_q}\|_{L^2(\gamma([\delta l, l - \delta l]))}^2 \leq 4C^2 \delta$, where

$$
D := \sup_{r \in [0,l]} d(p, \gamma(t)).
$$

Proof of claim: By Theorem 2.3, we have

$$
\Delta_f d_q \leq \frac{m - 1}{d_q} + (m - 1) \sqrt{K} + F(2R) \quad \text{and} \quad \Delta_f d_q' \leq \frac{m - 1}{d_q'} + (m - 1) \sqrt{K} + F(2R).
$$

On the other hand, since the function $(d_q + d_q')(\gamma(t))$ attains a smooth minimum on $\gamma$, we see $\Delta_f (d_q + d_q')(\gamma(t)) \geq 0$; moreover, we notice that $\nabla d_q(\gamma(t)) = -\nabla d_q'(\gamma(t))$. Therefore, we have $\Delta_f d_q(\gamma(t)) \geq -\Delta_f d_q'(\gamma(t)) \geq -\frac{m - 1}{d_q(\gamma(t))} - (m - 1) \sqrt{K} - F(2R)$, and thus

$$
\max \left\{ |\Delta_f d_q(\gamma(\delta l))|, |\Delta_f d_q(\gamma(l - \delta l))| \right\} \leq \frac{m - 1}{\delta l} + (m - 1) \sqrt{K} + F(2R).
$$
Plugging $u = d_q$ in to the Bochner formula (3.1), we immediately have:

$$\forall t \in [\delta l, l - \delta l], \quad 0 = \Delta f |\nabla d_q|^2(\gamma(t)) \geq 2|Hess d_q|^2(\gamma(t)) + 2\partial_t \left(\Delta f_d(\gamma(t))\right) - K.$$ 

Therefore, integrating $t \in [\delta l, l - \delta l]$ we obtain

$$\int_{\delta l}^{l-\delta l} |Hess d_q|^2 \, dt \leq \frac{2(m-1)}{\delta l} + 2(m-1) \sqrt{K} + 2F(2R) + (1 - 2\delta)Kl, \quad (4.3)$$

whence the desired $L^2$-estimate. \hfill \Box

From the proof of the proposition, we could clearly see that $H'_l$ is determined by the specific $M$ and $\gamma$, rather than a uniform neighborhood that we would like to get. In fact, it is impossible to get such neighborhood in a uniform way; however, in the sequel, we will see that there is a sufficiently large (in volume) set, which is not necessarily a neighborhood of $\gamma$, but which resembles the key property of $H'_l$: the gradient flow lines of $-\nabla d_q$ with initial data in this set does not spread too far away from $\gamma$. Moreover, this set is defined analytically and its properties depend on the estimates uniformly.

Define

$$\mathcal{B}'_s(r) := \{z \in B(\gamma(t), r) : \forall u \in [0, s l], \psi_u(z) \in B(\gamma(t-u), 2r)\}. \quad (4.4)$$

Clearly, $\mathcal{B}'_0(r) = B(\gamma(t), r)$, since $\psi_0$ is the identity map; by the continuity of $-\nabla d_q$ outside the cut-locus of $q$, we know that there is a small $\varepsilon > 0$ such that,

$$\forall s \in [0, \varepsilon l], \quad \frac{\mu_f(\mathcal{B}'_s(r))}{\mu(B(\gamma(t-s), r))} \geq \frac{1}{2}. \quad (4.5)$$

Clearly, this $\varepsilon$ may vary from one specific manifold to another, it may also depend on $r$.

However, let this $\varepsilon$ be chosen as the maximal possible value that satisfies (4.5), and we will show its irrelevance of specific manifolds and $r$ provided $r \leq r_0$, some fixed constant.

Now let $c'_t$ be the characteristic function of $\mathcal{B}'_s(r) \times \mathcal{B}'_s(r)$ in $B(\gamma(t), r) \times B(\gamma(t), r)$, then for any $s \in [0, \varepsilon l]$ and $\eta \in (0, 1/2)$, we define

$$\mathcal{F}'_s(x, y) := \int_0^s c'_t(x, y) \left(\int_{B(\gamma(t), r)} |Hess h_{t^l}| \right) \, du, \quad (4.6)$$

and

$$I'_s(r) := \int_{B(\gamma(t), r) \times B(\gamma(t), r)} \mathcal{F}'_s(x, y) \, d\mu_f(x) d\mu_f(y) \quad (4.7)$$

$$T'_\eta := \left\{x \in B(\gamma(t), r) : e_{q,q'}(x) \leq C_E r^2 (\eta \delta l)^{-1}, \text{ and } \int_{B(\gamma(t), r)} \mathcal{F}'_s(x, y) \, d\mu_f(y) \leq \eta^{-1} I'_s(r)\right\}, \quad (4.8)$$

and for each $x \in T'_\eta$ we define

$$T'_\eta(x) := \left\{y \in B(\gamma(t), r) : e_{q,q'}(y) \leq C_E r^2 (\eta \delta l)^{-1}, \text{ and } \mathcal{F}'_s(x, y) \leq \eta^{-2} I'_s(r)\right\}. \quad (4.9)$$
By the excess function estimate in Theorem \[3.12\] and Chebyshev’s inequality, we have
\[
\frac{\mu_f(T^r_{\eta})}{\mu_f(B(\gamma(t), r))} \geq 1 - 2\eta, \quad \text{and} \quad \forall x \in T^r_{\eta}, \quad \frac{\mu_f(T^r_{\eta}(x))}{\mu_f(B(\gamma(t), r))} \geq 1 - 2\eta.
\tag{4.10}
\]
Notice that these estimates are uniform. Now we come to the following

**Lemma 4.2.** Fix \(\xi \in (0, 1/20)\) and \(\eta \in (0, 1/100)\), and consider a minimal geodesic \(\gamma : [0, l] \rightarrow B(p, D)\) of unit speed. There is an \(\varepsilon_0 = \varepsilon_0(\eta | m, F, K, D, \delta)\) and a \(C_\delta = C_\delta(m, F, K, D, \delta)\) such that whenever \(\varepsilon < \varepsilon_0\), then for any fixed \(r \in (0, \delta/10)\), once \([4.3]\) holds on \([t - \varepsilon, t]\), then \(\forall s \in [t - \varepsilon, t]\), \(\forall x_1 \in T^r_{\eta}\) and \(\forall x_2 \in T^r_{\eta}(x) \cap \mathcal{B}_s^r(\xi r)\),
\[
|d(\psi_s(x_1), \psi_s(x_2)) - d(x_1, x_2)| \leq C_\delta \eta^{-2} r \sqrt{s/l}.
\tag{4.11}
\]
Moreover, \(x_1 \in \mathcal{B}_s^r\) for all \(s \in [0, \varepsilon]\).

**Proof.** Fix any \(x_1 \in T^r_{\eta}\setminus C_q\) and denote
\[
\varepsilon(x_1) := \min \{1, \sup \{s \leq \varepsilon l : \forall u \in [0, s], \ psi_u(x_1) \in B(\gamma(t - u), 2r)\}\}.
\]
Clearly, when \(s \leq \varepsilon(x_1)\), \(x_1 \in \mathcal{B}_s^r\); moreover, \(\mathcal{B}_s^r(\xi r) \subset \mathcal{B}_s^r(r)\). Therefore \(c_j^s(x_1, x_2) = 1\). By the continuity of the mapping \(u \mapsto \psi_u(x_1)\), we also see that
\[
\psi_{\varepsilon(x_1)}(x_1) \not\in B(\gamma(t - \varepsilon(x_1)), \frac{3}{2} r)\tag{4.12}
\]
We will show that \(\varepsilon(x_1) = \varepsilon l\) for suitably chosen \(\varepsilon_0\). Now for any \(x_2 \in (T^r_{\eta}(x_1) \cap \mathcal{B}_s^r(r))\setminus C_q\) fixed, we let \(\sigma_1\) and \(\sigma_2\) denote the integral curves of \(-\nabla d_q\) starting from \(x_1\) and \(x_2\), respectively. These are minimal geodesics, and integrating (3.6) in \(11\) Lemma 3.4 we get for any \(s \leq \varepsilon(x_1),\)
\[
|d(\psi_s(x_1), \psi_s(x_2)) - d(x_1, x_2)| \leq \int_0^s |\nabla h_{s, 2} - \nabla d_q|((\sigma_1(u))) \ du
\tag{4.13}
+ \int_0^s |\nabla h_{s, 2} - \nabla d_q|((\sigma_2(u))) \ du + T_s^r(x_1, x_2).
\]
We now estimate each term in the right-hand side of (4.13). By Lemma \(3.16\) and the choice of \(x_1, x_2\), we see for \(i = 1, 2,\)
\[
\forall s \in [0, \varepsilon(x_1)], \int_0^s |\nabla h_{s, 2} - \nabla d_q|((\sigma_i(u))) \ du \leq C_4 \eta^{-\frac{4}{2}} r \sqrt{s/l}.
\tag{4.14}
\]
The last term on the right-hand side of (4.13) is by definition bounded by \(\eta^{-2} I_s^r(r)\). By the segment inequality of Theorem \(2.6\) and the definition of \(\mathcal{B}_s^r(r)\), for any \(s \in [0, \varepsilon(x_1)]\) we could estimate \(I_s^r(r)\) as:
\[
I_s^r(r) \leq \int_0^s \left( \frac{1}{\mu_f(\mathcal{B}_s^r(r))^2} \int_{\psi_s(\mathcal{B}_s^r(r)) \times \psi_s(\mathcal{B}_s^r(r))} \left( \int_{\gamma, s, t} |Hess h_{s, 2}| \right) \ du \right)^{1/2} \ du
\leq \int_0^s \left( \frac{10 C_{Seg}}{\mu_f(\mathcal{B}_s^r(r))^2} \int_{B(\gamma(t - s), 5r)} |Hess h_{s, 2}| \ du \right) \ du
\leq \int_0^s \left( \frac{10 C_{Seg}}{\mu_f(\mathcal{B}_s^r(r))^2} \int_{B(\gamma(t - s), 5r)} |Hess h_{s, 2}| \ du \right) \ du.
\]
Moreover, by the volume doubling property (2.2) within \( B(p_0, D) \), assumption (4.5) and Lemma 8.15 we could continue to estimate: \( \forall s \in [0, \varepsilon(x_1)] \),

\[
I_s \leq \int_0^r \left( 10r C_2 C_{seg} \left( \frac{\mu_f(\gamma(u), r))}{\mu_f(B_s(r))} \right)^2 \int_{B(\gamma(u), slr)} |Hess_{h_2}| \, d\mu_f \right) \, du
\leq 10r C_2 C_{seg} \left( \int_{slr}^{u-s\delta} \int_{B(\gamma(u), slr)} |Hess_{h_2}|^2 \, d\mu_f \right)^{\frac{1}{2}} \sqrt{s}
\[
\leq 10C_2 C_{seg} r \sqrt{s/l},
\]

(4.15)

Now (4.13), (4.16) and (4.15) together imply that for almost every \( x_1 \in T_{\eta}^r \) and \( x_2 \in T_{\eta}^r(x_1) \cap B'_q(r) \),

\[
\forall s \in [0, \varepsilon(x_1)], \quad |d(\psi_s(x_1), \psi_s(x_2)) - d(x_1, x_2)| \leq C_6 \eta^{-2} \sqrt{s/l},
\]

(4.16)

where \( C_6 = C_6(m, F, K, D, \delta) := 2C_4 + 10C_2 C_{seg} \sqrt{C_3} \). Here we emphasize that in proving this estimate we only need \( x_1 \in T_{\eta}^r \cap B'_q(r) \) and \( x_2 \in T_{\eta}^r(x) \cap B'_q(r) \). The stronger assumption that \( x_2 \in B'_q(\xi r) \) is not needed yet.

Now we put \( \varepsilon_0 = \eta^4/(16C_2^2) \). Suppose, for the purpose of a contradiction argument, that the inequalities \( \varepsilon(x_1) < \varepsilon \leq \varepsilon_0 \) hold, then since \( x_2 \in B'_q(\xi r) \), we have \( d(\psi_s(x_2), \gamma(\eta - s)) \leq 2\xi r \leq r/10 \) whenever \( s \in [0, \varepsilon(x_1)] \), and the triangle inequality implies that

\[
d(\psi_{\varepsilon(x_1)}(x_1), \gamma(\eta - s(x_1))) \leq \frac{5}{4}r,
\]

whence a desired contradiction to (4.12).

Therefore \( \varepsilon(x_1) = \varepsilon \), and (4.16) is valid for all \( s \in [0, \varepsilon] \). Especially, this is (4.11) holding for all \( s \in [0, \varepsilon] \), as claimed by the lemma. \( \square \)

**Remark 4.3.** Let us emphasis that the estimate (4.11) depends on (4.5) whose range of validity depends on the specific manifold, geodesic and scale \( r \). But with these estimates we are now ready to remove the such dependence of \( \varepsilon \) in (4.5).

**Lemma 4.4.** There exists a small \( \varepsilon_1 = \varepsilon_1(\eta) \mid m, F, K, D, \delta > 0 \) such that for any fixed \( r \in (0, \delta l/10) \), if \( \gamma : [0, l] \rightarrow B(p, D) \) is a minimal geodesic of unit speed and \( t \in (\delta l, (1 - \delta)l) \), then (4.5) holds for some \( \varepsilon \geq \varepsilon_1 \). In fact, \( \forall s \in [0, \varepsilon_1], (T_{\eta}^r)_{C_q} \subset B'_q(r) \).

**Proof.** Fix \( r \in (0, \delta l/10) \), and let \( \varepsilon \) be the largest possible such number that (4.5) holds. Again, this \( \varepsilon \) is positive but its value depends on the specific \( M \) and \( \gamma \). We will choose an \( \varepsilon_1 \) depending only on \( m, F, K, D, \delta \) and \( \eta \) such that \( \varepsilon < \varepsilon_1 \) to hold then a contradiction will be deduced.

**Step 1: Connecting to the good core neighborhood.** Recall that Lemma 4.1 tells that there are a small \( r' \sim r'(M, \gamma) > 0 \) and a core neighborhood \( H_{r'} = B(\gamma(t), r') \mid C_q \), such that it stays close to \( \gamma \) under the geodesic flow. Let us now fix this neighborhood of \( \gamma(t) \), which depends on specific \( M \) and \( \gamma \). Notice that if we set \( \varepsilon' := (\ln 2/C_3)^2 \delta \), then by the definition of \( H'_{\varepsilon'} \) and the proof of Lemma 4.1 we have

\[
\forall s \in [0, \varepsilon'_1], \forall \varepsilon'' \in [0, r'], \quad H'_{\varepsilon''} \subset B'_q(r''),
\]

(4.17)

28
We also let \( \xi \) be some small positive number, say \( \xi \leq \frac{1}{50} \), and let \( r_i := \xi^{-i} r \) for \( i = 0, 1, 2, \ldots, I \), where \( I := \left\lfloor \log_\xi \frac{r}{2} \right\rfloor \) is defined to be the first natural number such that \( r_I \leq r'/2 \).

Now for an arbitrary \( x_0 \in T_{\eta}^\iota \setminus \mathcal{C}_q \) fixed, our plan is to connect it to \( H_{\eta}^\iota \) by selecting \( \{x_i\}_{i=0}^I \) inductively: suppose \( x_i \) is chosen, then pick any \( x_{i+1} \in (T_{\eta}^\iota(x_i) \cap T_{\eta}^{\iota+1}) \setminus \mathcal{C}_q \). This is doable because (4.10) is independent of \( r \): as long as we choose \( \eta \leq (\xi/4)^m \), then we have

\[
\mu_f(T_{\eta}^\iota(x_i)) + \mu_f(T_{\eta}^{\iota+1}) \geq (1 - 2\eta) \left( \mu_f(B(\gamma(t), r_i)) + \mu_f(B(\gamma(t), r_{i+1})) \right) \\
\geq (1 - 2\eta)(1 + \xi^m)\mu_f(B(\gamma(t), r_i)) \\
> \mu_f(B(\gamma(t), r_i)),
\]

i.e. \( T_{\eta}^\iota(x_i) \cap T_{\eta}^{\iota+1} \) has positive weighted volume, and especially a non-empty intersection outside the cut-locus of \( q \). We denote by \( \sigma_i \) the integral curve of \(-\nabla d_q\) with initial value \( x_i \). Each \( \sigma_i \) is a minimal geodesics.

**Step 2: Estimating the distance.** According to (4.17), \( x_i \in B^\iota_{\kappa}(r_I) \) whenever \( s \in [0, \xi^\iota I] \). Therefore, applying Lemma 4.2 to \( x_{I-1} \in T_{\eta}^{\iota-1} \) and \( x_I \in T_{\eta}^{\iota-1}(x_{I-1}) \cap B^\iota_{\kappa}(r_I) \), we could obtain

\[
\forall s \in [0, \min\{eI, e_0I, \xi^\iota I\}], \quad d(\psi_s(x_I), \psi_s(x_{I-1})) \leq \left( \frac{5}{4} + \xi \right) r_{I-1}.
\]

This further implies that for any \( s \leq \min\{eI, e_0I, \xi^\iota I\} \),

\[
d(\psi_s(x_{I-1}), \gamma(t-s)) \leq d(\psi_s(x_I), \gamma(t-s)) + \left( \frac{5}{4} + \xi \right) r_{I-1} \\
\leq \left( \frac{5}{4} + 3\xi \right) r_{I-1}.
\]

Especially, \( x_{I-1} \in B^\iota_{\kappa}(r_{I-1}) \) whenever \( s \leq \min\{eI, e_0I, \xi^\iota I\} \).

We could now apply Lemma 4.2 to the pair of points \( x_{I-2} \) and \( x_{I-1} \), and conclude that \( x_{I-2} \in B^\iota_{\kappa}(r_{I-2}) \) whenever \( s \leq \min\{eI, e_0I, \xi^\iota I\} \). Repeating the same argument another \( I-2 \) steps, we will get for any \( s \leq \min\{eI, e_0I, \xi^\iota I\} \),

\[
d(\psi_s(x_0), \gamma(t-s)) \leq d(\psi_s(x_I), \gamma(t-s)) + \left( \frac{5}{4} + \xi \right) \sum_{i=0}^{I-1} \xi^i \\
\leq \frac{3}{2} r,
\]

by the choice of \( \xi \). Especially, this implies that \( x_0 \in B^\iota_{\kappa}(r) \) whenever \( s \leq \min\{eI, e_0I, \xi^\iota I\} \).

**Step 3: Bounding \( \varepsilon \) from below.** Setting \( \varepsilon_1 := \min\{e_0I, \xi^\iota I\} \), we show that whenever \( \eta \leq 100^{-m} \), \( \varepsilon \geq e_0 \) by a contradiction argument:

Otherwise, notice that \( \mu_f(B^\iota_{\kappa}(r))/\mu_f(B(\gamma(t-s), r)) \) varies continuously with respect to \( s \), then by (4.5) and the maximality of \( \varepsilon \), we have

\[
\frac{\mu_f(B^\iota_{\kappa}(r))}{\mu_f(B(\gamma(t-s), r))} = \frac{1}{2}.
\]

29
However, since $\epsilon < \epsilon_1$, we have $(T^r_\eta \setminus C_p) \subset B_2^i(r)$, therefore
\[
\frac{\mu_f(B^i_\eta(r))}{\mu_f(B(\gamma(1 - \epsilon l), r))} \geq 1 - 2\eta,
\]
a contradiction, since $\eta \leq 1/100$. Therefore $\epsilon \geq \epsilon_1$, a constant solely determined by $m, F, K, D, \delta$ and $\eta$. We notice here that $\lim_{\eta \to 0} \epsilon_1(\eta|m, F, D, \delta) = 0$ by the definition of $\epsilon_1$. □

We are now ready to estimate the Gromov-Hausdorff distance of metric balls of arbitrarily small size $r$:

**Theorem 4.5** (Gromov-Hausdorff distance between nearby metric balls). Fix a space $(M, p, g, f)$ in the moduli $N_\eta(F, K)$, then for any $\delta \in (0, 1/10)$ and $\epsilon \in (0, \delta/10)$ fixed, there are constants $C_7, C_8$ only depending on $m, F, K, D, \delta$ such that on any minimal geodesic $\gamma$ contained in $B(p, D)$ with $|\gamma| = l$, and for any $x, y$ on $\gamma([\delta l, (1 - \delta)l])$,
\[
\frac{d(x, y)}{l} \leq C_7 \quad \Rightarrow \quad d_{\text{GH}}(B(x, r), B(y, r)) \leq C_8 \left(\frac{d(x, y)}{l}\right)^{1/2} r.
\]

*Proof.* Since the estimate is symmetric in terms of $x$ and $y$, we only argue in one direction. Now fix any $s \in [0, \epsilon l]$. We immediately have $\mu_f(B_2^i(s)) \geq (1 - 2\eta)\mu_f(B(\gamma(t), r))$, in view of Lemma 4.4 and (4.10). Moreover, we have $\mu_f(B_2^i(s) \cap T^r_\eta(x)) \geq (1 - 4\eta)\mu_f(B(\gamma(t), r))$ for any $x \in T^r_\eta$, still because of (4.10). Such volume estimates, together with the volume comparison within $B(p_0, D)$, imply that $T^r_\eta$ and $T^r_\eta(x) \cap B_2^i(r)$ are $4C_2^{-\frac{1}{2}}\eta^{\frac{1}{2}}r$-dense subsets of $B(\gamma(t), r)$, whenever $x \in T^r_\eta$.

Recall that (4.16) tells that for any $x_1 \in T^r_\eta$ and any $x_2 \in T^r_\eta(x) \cap B_2^i(r)$,
\[
|d(\psi_s(x_1), \psi_s(x_2)) - d(x_1, x_2)| \leq C_5 s^{-2} r \sqrt{s/l}.
\]
Now for any $x_1, x_2 \in T^r_\eta$, since $\mu_f(T^r_\eta(x_1) \cap T^r_\eta(x_2) \cap B_2^i(r)) \geq (1 - 8\eta)\mu_f(B(\gamma(t), r))$, we could select some
\[
y \in T^r_\eta(x_1) \cap T^r_\eta(x_2) \cap B_2^i(r) \cap B(x_1, 8C_2^{-\frac{1}{2}}\eta^{\frac{1}{2}}r),
\]
and estimate
\[
|d(\psi_s(x_1), \psi_s(x_2)) - d(x_1, x_2)| \leq |d(\psi_s(x_2), \psi_s(y)) - d(x_2, y)| + d(\psi_s(x_1), \psi_s(y)) + d(x_1, y) \\
\leq 2C_5 s^{-2} r \sqrt{s/l} + 2d(x_1, y) \\
\leq 2r(C_5 s^{-2} \sqrt{s/l} + 8C_2^{-\frac{1}{2}}\eta^{\frac{1}{2}}).
\]
In order to make this last line of the last estimate having only $s$ as the variable, we would like to choose $\eta$ according to the value of $s$. Notice that in order for this estimate to hold, we cannot violate $s \leq \epsilon_1 l$; on the other hand, let us recall that $\epsilon_1$ depends on $\eta$, which ultimately comes into play via $\epsilon_0 = \eta^4/(16C_0^2)$. Therefore, we will first choose $\eta(s)$, then check that $s/l \leq \eta(s)^4/(16C_0^2)$ and $\eta(s) \leq 10^{-2m}$ (see Step 1 of Lemma 4.2): let
\[
\eta(s) := C_6^{\frac{1}{2m}} C_2^{\frac{1}{2m}} (s/(8l))^{1/2m},
\]
the right-hand side of which, being a constant only depending on

Moreover, the requirements that $s/l \leq \eta(s)/(16C_0^2)$ and $\eta(s) \leq 10^{-2m}$ translate as

the right-hand side of which, being a constant only depending on $m, D,$ and $\delta$. Therefore, once $s/l$ is below this $C_7$, the previous requirements of $\eta(s)$ are meet, and all previous estimates go through with no problem.

Therefore, whenever $d(x, y) \leq C_7(m, D, \delta)l$, we have, by the density estimate and (4.19), the desired estimate, with the constant

$$C_8(m, F, K, D, \delta) := 24(8^m C_6/C_2^2)^{1/2m+1}.$$  \hfill \Box

### 4.2 Extension of limit minimal geodesics

Throughout this subsection, we fix a sequence $\{(M_i, p_i, g_i, f_i)\} \subset N_m(F, K)$ that converges in the pointed-Gromov-Hausdorff topology to a limit $(X, p_\infty, d_\infty, f_\infty)$. Our focus will be on the non-compact case, which is more complicated and natural to consider. We will provide detailed proofs for the necessary adjustments to generalize Colding-Naber’s argument in [11] Sections 1.2 and 1.4 to $N_m(F, K)$ limits.

In view of Theorem 4.3 we could only compare geodesic balls centered at two points that are away from the endpoints of a minimal geodesic connecting them. For a fixed complete Riemannian manifold $(M, g)$, and any pair of points $x, y \in M$, we let $\gamma_{xy}$ denote a minimal geodesic connecting them. Due to the possible existence of the cut-locus, not every pair of points $(x, y) \in M \times M$ sees their $\gamma_{xy}$ minimally extensible to both ends. But the minimal extensibility holds for almost every pair of points, with respect to the natural product measure on $M \times M$.

In order to prove the almost everywhere extensibility of limit minimal geodesics on a pointed-Gromov-Hausdorff limit, we have to show that the problematic cut-loci do not accumulate to acquire positive limit measure during the convergence. The key observation, due to Shouhei Honda [19], is that the cut-loci could be characterized by an inequality — the non-vanishing of the excess function [11] — whose effective version persists to the Gromov-Hausdorff limit. The proof of what we need for $N_m(F, K)$ limits is the same as the original one in [11] Appendix A, except at one point: the crucial estimate in [11] Lemma A.2] relies on the Laplacian comparison with a uniform Ricci curvature lower bound, which is not available for manifolds in $N_m(F, K)$. The following lemma fills this only gap in carrying Colding-Naber’s original argument to our setting:

**Lemma 4.6.** Suppose $(M, p, g, f) \in N_m(F, K)$. For each $\delta, r \in (0, 1)$, $D > 0$ and $k \in \mathbb{N}$, there exists a constant $C_9 = C_9(m, F, K, D, \delta)$ such that

$$\frac{\mu_{f,f} \left( C_M(r, k) \cap A_{\delta,\delta^{-1}}(D) \right)}{\mu_f(B(p, l))^2} \leq C_9 r. \hspace{1cm} (4.20)$$
Here we define, as \([11] (A.5)\),

\[
C_M(r, k) := \{(x, y) \in M \times M : \forall z, w \in M, d^2(x, z) + d^2(y, w) \geq 2r^2 \Rightarrow e_{(z, w)}(x, y) \geq k^{-2}\},
\]

and

\[
e_{(z, w)}(x, y) := 2^{-\frac{1}{4}}d(x, y) + (d(x, z)^2 + d(y, w)^2)^{\frac{1}{4}} - 2^{-\frac{1}{4}}d(z, w),
\]

is the excess function on the isometric product manifold \((M \times M, g \oplus g)\), see \([11] (A.2)\). Notice that \(C_M(r, k)\) is just a quantitative version of the \(r\)-cut-loci \(C_M(r)\) of the manifold \((M \times M, g \oplus g)\), which is actually \(\bigcup_{k \in \mathbb{N}} C_M(r, k)\). Moreover, \((x, y) \in C_M(r)\) if and only if the geodesics emanating from the midpoint towards \(x\) and \(y\) respectively are minimal till they reach \(x\) and \(y\), but at least one of them could not be extended beyond \(x\) or \(y\) as a minimal geodesic for at least a distance of \(r\).

Furthermore, the set \(A_{\delta^{-1}}(D)\) is defined for any \(\delta > 0\) as

\[
A_{\delta^{-1}}(D) := \{(x, y) \in M \times M : p_{xy} \in B(p, D), \delta \leq d_A(x, y) \leq \delta^{-1}\},
\]

with \(p_{xy}\) denoting the midpoint of a minimal geodesic connecting \(x\) and \(y\), and \(d_A(x, y) = 2^{-\frac{1}{4}}d(x, y)\) denoting the distance between \((x, y)\) and the diagonal \(\Delta\) of \(M \times M\), in the product metric.

We also notice that the product of \((M, p, g, f)\) by itself is \((M \times M, (p, p), g \oplus g, f \cdot f)\), where \(\forall x, y \in M, f \cdot f(x, y) := f(x)f(y)\). We have the product an element of \(N_{2m}(\tilde{F}, K)\), where \(\tilde{F}(x, y) := \sqrt{F(d(p, x))^2 + F(d(p, y))^2}\), since clearly we have

\[
\forall x, y \in M, \quad |\nabla f \cdot f|^2(x, y) \leq F(d(p, x))^2 + F(d(p, y))^2,
\]

and \(Rc_{g \oplus g} + \nabla^2(f \cdot f) \geq -Kg \oplus g\).

Due to the metric product structure, the distance to the diagonal \(d_A\) enjoys the following Laplace comparison inequality: \(\forall x, y \in B(p, D)\),

\[
\Delta f \cdot f d_A(x, y) = \frac{1}{\sqrt{2}}(\Delta f)_x d(x, y) + \frac{1}{\sqrt{2}}(\Delta f)_y d(x, y) 
\leq \sqrt{2} \left(\frac{m - 1}{d(x, y)} + (m - 1) \sqrt{K} + F(2R)\right).
\]

By \([11\text{ Lemma A.1}]\), the distance of \((x, y)\) to the diagonal \(D\) is realized by the distance to \((p_{xy}, p_{xy})\), the midpoint of the minimal geodesic connecting \(x\) and \(y\). The geodesic ray realizing the distance to the diagonal is then given as:

\[
\forall (x, y) \in M \times M, \quad s \mapsto \exp_{(p_{xy}, p_{xy})} s(\mathbf{v}_{xy}, -\mathbf{v}_{xy}),
\]

where \(\mathbf{v}_{xy} := \dot{\gamma}_{xy}(\frac{1}{4}d(x, y))\), and \((\mathbf{v}_{xy}, -\mathbf{v}_{xy}) \in S_{p_{xy}}M \times S_{p_{xy}}M\), the product of unit tangent vectors. Associated to this exponential map, we could consider

\[
T_r(D) := \{(x, y) \in M \times M : d_A(x, y) \leq r \text{ and some } p_{xy} \in B(p, D)\}
= \left\{\exp_{(q, q)} s(\mathbf{v}, -\mathbf{v}) : s \in [0, r/\sqrt{2}], \; p \in B(p, D), \; \mathbf{v} \in S_qM\right\},
\]

32
which is the open \( r \)-tubular neighborhood of the diagonal \( \Delta \subset B(p_0, D) \times B(p_0, D) \). Clearly, \( A_{\delta, \delta^{-1}}(D) = T_{\delta^{-1}}(D) \setminus T_{\delta}(D) \), and

\[
\partial T_{\delta}(D) = \left\{ (x, y) \in M \times M : d_A(x, y) = s \text{ and some } p_{xy} \in B(p, D) \right\}
\]

\[
= \left\{ \exp_{(q, q)} s(v, -v) : (q, v) \in SB(p, D), \right\},
\]

where \( SB(p, D) \) is the sphere bundle of \( B(p, D) \).

We denote the area form of \( \partial T_{\delta}(D) \) at \( \exp_{(q, q)} s(v, -v) \) by \( \mathcal{A}^2(q, v, s) \), and the \( e^{-f} \)-weighted product measure density on \( \partial T_{\delta}(D) \) by \( \mathcal{A}^2_{f, \delta}(q, v, s) := e^{-f} \mathcal{A}^2(q, v, s) \). We notice that

\[
\partial s \ln \left( \mathcal{A}^2_{f, \delta}(q, v, s) \right) = \Delta_{f, \delta} d_A(\exp_q s v, \exp_q -sv)
\]

\[
\leq \sqrt{2} \left( (m-1)s^{-1} + (m-1) \sqrt{F(D + \sqrt{2}\delta)} \right),
\]

therefore, for any fixed \((q, v) \in SB(p, D)\) and \( s \in [\delta, \delta^{-1}] \), the ratio

\[
\frac{\mathcal{A}^2_{f, \delta}(q, v, s)}{s \sqrt{2(m-1)e^{(m-1)\sqrt{F(D + \sqrt{2}\delta^{-1})}}} s}
\]

is monotone non-increasing with respect to \( s \).

Recall that \((x, y) \in C_M(r)\) if and only if the geodesics \( s \mapsto \exp_{p, sv} \) and \( s \mapsto \exp_{p, -sv} \) are minimal for \( s \in (0, \frac{1}{2} d(x, y)) \), but cease to be so for some \( s \in (\frac{1}{2} d(x, y), \frac{1}{2} d(x, y) + r) \). Therefore

\[
\forall \delta > 0, \text{ } \forall (q, v) \in SB(p, D), \quad \left| \left\{ s \in (\delta, \delta^{-1}) : \exp_q s v, -v \in C_M(r) \right\} \right| \leq r.
\]

We now put the estimates together to see:

\[
\mu_{f, f} \left( C_M(r) \cap A_{\delta, \delta^{-1}}(D) \right) = \int_{\delta}^{\delta^{-1}} \left( \int_{\partial T_{\delta}(D)} \chi_{C_M(r)} e^{-f} \right) ds
\]

\[
= \int \int_{SB(p, D)} \chi_{C_M(r)} \mathcal{A}^2_{f, \delta}(q, v, s) ds d\sigma(q, v)
\]

\[
\leq \int \int_{SB(p, D)} \chi_{C_M(r)} ds \frac{\mathcal{A}^2_{f, \delta}(q, v, \delta)}{\sqrt{2(m-1)e^{(m-1)\sqrt{F(D + \sqrt{2}\delta^{-1})}}} \delta} d\sigma(q, v)
\]

\[
\leq \int \int_{SB(p, D)} \mathcal{A}^2_{f, \delta}(q, v, \delta) \frac{\mu_{f, f}(\partial T_{\delta}(D))}{\sqrt{2(m-1)e^{(m-1)\sqrt{F(D + \sqrt{2}\delta^{-1})}}} \delta} d\sigma(q, v)
\]

\[
= \frac{\mu_{f, f}(\partial T_{\delta}(D))}{\sqrt{2(m-1)e^{(m-1)\sqrt{F(D + \sqrt{2}\delta^{-1})}}} \delta} r.
\]

Finally, since \( \forall x \in B(p, D + 1) \) fixed \( \{ y \in B(p, D + 1) : (x, y) \in \partial T_{\delta}(D) \} = \partial B(x, \sqrt{2}\delta) \cap B(p, D + 1) \), we have, by Theorem 2.5

\[
\mu_{f, f}(\partial T_{\delta}(D)) \leq \int_{B(p, D + \sqrt{2}\delta)} \mu_{f, f}(\partial B(x, \sqrt{2}\delta)) e^{-f(x)} dV_p(x)
\]

\[
\leq \frac{e^{F(D + \sqrt{2}\delta^{-1})}(x) \text{Area}^{m-1}(\sqrt{2}\delta)}{\text{Vol}^{m-1}(\sqrt{2}\delta)} \int_{B(p, D + \sqrt{2}\delta)} \mu_{f, f}(B(x, \sqrt{2}\delta)) e^{-f(x)} dV_p(x),
\]

33
and applying Theorem 2.5 again we have

$$
\mu_{ff} \left( C_M(r) \cap A_{\delta, \delta^{-1}}(D) \right) \leq C_9 \mu_{f}(B(p, 1))^2 r,
$$

where

$$
C_9 = C_9(m, F, K, D, \delta) := \frac{e^{F(D + \sqrt{2}\delta^{-1})}(D + \delta^{-1})}{\delta \sqrt{(m - 1)} e^{(m - 1)\sqrt{2K + \sqrt{2}F(D + \sqrt{2}\delta^{-1})}} \frac{\text{Area}_{F}^{m-1}(\sqrt{2}\delta)}{\text{Vol}_{K}^{m}(\sqrt{2}\delta)} \left( \frac{\text{Vol}_{K}^{m}(D + \sqrt{2})}{\text{Vol}_{K}^{m}(1)} \right)^2}.
$$

Therefore we get the desired estimate (4.20), since $C_M(r) = \bigcup_{k=0}^{\infty} C_M(r, k)$.

The rest of the argument in showing the almost everywhere extensibility follows verbatim as the rest of [11, Appendix A], as well as [19]. So we have shown that with respect to the limit measure, almost every pair of points lie in a minimal geodesic that minimally extends to both ends:

**Lemma 4.7** (Extension of limit minimal geodesics). Let a sequence $\{(M_i, p_i, g_i, f_i)\} \subset N_m(F, K)$ converge to $(X, p_\infty, d_{\infty}, f_\infty)$ in the pointed-Gromov-Hausdorff topology, such that their associated renormalized measures $\nu_{f_i}$ also converge to a limit measure $\nu_\infty$ on $X$, then $\nu_\infty \times \nu_\infty$ almost every pair of point $(x, y) \in X \times X$ lies in the interior of some limit minimal geodesic.

Now we are in a position to state and prove the (weak) convexity of the regular part in the Gromov-Hausdorff limit of a sequence of Ricci shrinkers. To start the discussion, let us state an immediate consequence of the Hölder continuity of the geodesic balls along a geodesic segment:

**Proposition 4.8** (Hölder continuity of tangent cones). Let $(X, p_\infty, d_{\infty}, f_{\infty})$ be a pointed-Gromov-Hausdorff limit of a sequence in $N_m(F, K)$, then the tangent cones resulted from the same scaling sequence varies Hölder-continuously in the Gromov-Hausdorff topology, as the point varies in the interior of limit minimal geodesics.

**Proof.** Let $\gamma_{\infty} : [0, l] \to X$ be a limit minimal geodesic of unit speed, and set

$$
D = 2 \max_{s \in [0, l]} d(p, \gamma_{\infty}(s)).
$$

Now for any $s, t \in (\delta l, (1 - \delta)l) (\delta \in (0, 1/10))$ such that $|s - t| \leq C_7(m, D, \delta)l$, and for any $r \in (0, \delta l/10)$, we have

$$
d_{GH}(B_X(\gamma_{\infty}(s), r), B_X(\gamma_{\infty}(t), r)) \leq C_8(n, D, \delta) \left( \frac{|s - t|}{l} \right)^{1/\alpha} r.
$$

Notice that this estimate is scaling invariant, therefore we could push it to the tangent cone: let $r_i \to 0$ be a sequence of positive numbers that determines tangent cones $X_{\gamma_{\infty}(s)}$ and $X_{\gamma_{\infty}(t)}$, then the above inequalities give

$$
d_{GH}(B_{X_{\gamma_{\infty}(s)}}(o, 1), B_{X_{\gamma_{\infty}(t)}}(o, 1)) \leq C_8(m, D, \delta) \left( \frac{|s - t|}{l} \right)^{1/\alpha} r,
$$

whence the desired Hölder continuity of tangent cones. □
Remark 4.9. Especially, we see that for $t_i \to t \in (0, l)$, taking
\[
\delta = \frac{1}{2l} \min \{ d_X(\gamma_\infty(t), \gamma_\infty(0)), d_X(\gamma_\infty(t), \gamma_\infty(l)), 1/10 \},
\]
the above estimate gives:
\[
X_{\gamma_\infty(t_i)} \xrightarrow{\text{pointed-Gromov-Hausdorff}} X_{\gamma_\infty(t)},
\]
and if $X_{\gamma(t)} = \mathbb{R}^k$, then so is $X_{\gamma_\infty(t)}$: $\gamma_\infty((0, l)) \cap \mathbb{R}^k$ is a closed subset of $\gamma_\infty((0, l))$.

Given the extension lemma (Lemma 4.7), the Hölder continuity of tangent cones (Proposition 4.8), and the $\nu_\infty$-negligibility of the singular set (Proposition 2.17), we could now prove Theorem 1.5 in a way identical to the original one in [11, Sections 1.2 and 1.4]. For the sake of simplicity, we will not repeat the argument here, but refer the readers to the proofs of Theorems 1.7, 1.18 and 1.20 in [11].

If we consider the sub-collection $\mathcal{M}_m(F, K; V_0)$, then together with Theorem 2.14, we have

**Theorem 4.10.** Let a sequence $\{(M_i, p_i, g_i, f_i)\} \subset \mathcal{N}_m(F, K; V_0)$ converge to a limit metric space $(X, p_\infty, d_\infty, f_\infty)$ in the pointed-Gromov-Hausdorff topology, then the regular part $\mathcal{R} \subset X$ is a strongly convex open set, equipped with a limit $C^{1,\alpha}$ metric $g_\infty$ such that $(\mathcal{R}, g_\infty)$ becomes a metric subspace of $(X, d_\infty)$.

**Proof.** It has already proven in Theorem 2.14 that the regular part $\mathcal{R}$ is open in $X$, that the convergence is $C^{1,\alpha}$ on $\mathcal{R}$, and that the limit metrics (in the metric sense and in the tensor sense) coincide. We only need to prove the strong convexity. Now if a minimal geodesic $\gamma : [0, 1] \to X$ intersects $\mathcal{R}$ non-trivially, then set
\[
I_{\gamma, \mathcal{R}} := \{ t \in [0, 1] : \gamma(t) \in \mathcal{R} \}
\]
is non-empty and is open relative to $[0, 1]$. But by Remark 4.9, we know that $I_{\gamma, \mathcal{R}}$ is also closed relative to $(0, 1)$, therefore $I_{\gamma, \mathcal{R}} = (0, 1)$ and therefore the entire interior of $\gamma$ is contained $\mathcal{R}$. The strong convexity is thus proven.

If we further restrict our attention to the collection of all $m$-dimensional complete Ricci shrinkers, then we have the following

**Theorem 4.11** (Regular-convexity of Gromov-Hausdorff limits). Let $\{(M_i, p_i, g_i, f_i)\}$ be a sequence of pointed $m$-dimensional Ricci shrinkers which converges to a limit metric space $(X, p_\infty, d_\infty, f_\infty)$ equipped with a limit potential function $f_\infty$, in the pointed-Gromov-Hausdorff topology, such that the associated probability measures $p_i$ converges to $p_\infty$ on $X$, then the following holds:

1. Assuming the sequence is contained in $\mathcal{M}_m(A)$ for some fixed positive constant $A$, then the Hausdorff dimension of $X$ is $m$, and $\mathcal{R} \subset X$ is a strongly convex open set, which, when equipped with the limit metric $d_\infty$, becomes an $m$-dimensional Riemannian manifold with a $C^\infty$ metric tensor that satisfies the Ricci shrinker equation;
2. Without assuming a uniform positive lower bound of the set \( \{ \mu_f(M_i) \} \), then there is a unique natural number \( k \leq m \), such that \( \rho_\infty(X \setminus R_k) = 0 \); moreover, \( R_k \) is both \( \rho_\infty \)-a.e. convex and weakly convex.

Clearly, after applying a usual elliptic regularity argument, the first alternative in this theorem is a special case of Theorem 4.10. Moreover, by the equivalence of the uniform \( \mu \)-entropy lower bound and the uniform volume non-collapsing property (see [27], [17] and [23, Lemma 2.5]), the first alternative of this theorem states the same as Theorem 1.1. The second alternative is a restatement of Theorem 1.5 for the special case of Ricci shrinkers.

5 Discussion

In geometric analysis, the compactness of the moduli of certain collection of spaces, in an appropriate topology, is a fundamental problem. In the setting of Ricci shrinkers, we would like to ask whether the collection of all conifold Ricci shrinkers with a given dimension and a uniform lower bound of the \( \mu \)-entropy is compact in the pointed-\( \hat{C}^\infty \)-Cheeger-Gromov topology.

This question is not a direct consequence of Theorem 1.1 since it is not true that all conifold Ricci shrinkers arise as the pointed-\( \hat{C}^\infty \)-Cheeger-Gromov limits of elements in \( M_m(A) \). In the Kähler setting, orbifold Kähler-Ricci solitons (of complex dimension at least 2) whose quotient singularities are of real codimension at least 4 and non-smoothable provide examples of conifold Ricci shrinkers not in the closure of \( \mathcal{K}M_n(A) \), the moduli space of (complex) \( n \)-dimensional Kähler-Ricci shrinkers with \( \mu \)-entropy bounded below by \( -A \). The Riemannian setting is even more complicated. It is by itself an interesting problem to understand those conifold Ricci shrinkers that are not on the boundary of \( M_m(A) \), and partial progress towards this direction has already been made in [25] and [23].

We believe that the compactness question could be answered affirmatively, in view of the previous work done in the Kähler-Ricci flat setting [9, Theorem 1.3], especially considering that many of the analytical tools developed in [9] only assume the Riemannian setting.

References

[1] U. Abresch and D. Gromoll, On complete manifolds with nonnegative Ricci curvature, J. Amer. Math. Soc., 3 (1990), 355-374.

[2] L. Ambrosio, N. Gigli, and G. Savare, Metric measure spaces with Riemannian Ricci curvature bounded from below, Duke Math. J. 163 (2014), no. 7, 1405-1490, DOI 10.1215/00127094-2681605. MR3205729.

[3] H.-D. Cao, B.-L. Chen and X.P. Zhu, Recent developments on Hamilton’s Ricci flow, Surveys in differential geometry, Vol. XII. Geometric flows, 47-112, Surv. Diff. Geom., 12, Int. Press, Somerville, MA, 2008.

[4] H.-D. Cao and D. Zhou, On complete gradient shrinking Ricci solitons, J. Diff. Geom. 85 (2010), no. 2, 175-186.
[5] J. Cheeger and T. H. Colding, *Lower Bounds on Ricci Curvature and the Almost Rigidity of Warped Products*, Ann. of Math. 144 (1996), 189-237.

[6] J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below. I*, J. Diff. Geom. 45 (1997), 406-480.

[7] J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below. III*, J. Diff. Geom. 54 (2000), no. 1, 37-74.

[8] B.-L. Chen, *Strong uniqueness of the Ricci flow*, J. Diff. Geom. 82 (2009), no. 2, 362-382.

[9] X.X. Chen and B. Wang, *Space of Ricci flows (II)—Part A: moduli of Singular Calabi-Yau spaces*, Forum Math. Sigma 5 (2017), DOI 10.1017/fms.2017.28.

[10] X.X. Chen and B. Wang, *Space of Ricci flows (II)—Part B: weak compactness of the flows*, arXiv: 1405.6797, to appear in J. Diff. Geom.

[11] T. H. Colding and A. Naber, *Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications*, Ann. of Math. 176 (2012), 1173-1229.

[12] N. Gigli, A. Mondino, G. Savaré, *Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows*, Proceedings of the London Mathematical Society, volume 111(2015), issue 5, 1071-1129.

[13] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, 152 (1999), Birkhäuser Boston.

[14] A. Grigor’yan, *Heat Kernel and Analysis on Manifolds*, AMS/IP Studies in Advanced Mathematics Volume: 47, 2009.

[15] R. S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in Differential Geom. 2 (1995), 7-136, International Press.

[16] S. Huang, *ε-Regularity and structure of four-dimensional shrinking Ricci solitons*, Int. Math. Res. Not. (2018), DOI 10.1093/imrn/rny069.

[17] R. Haslhofer and R. Müller, *A compactness theorem for complete Ricci shrinkers*, Geom. Funct. Anal. 21 (2011), 1091-1116.

[18] R. Haslhofer and R. Müller, *A note on the compactness theorem for 4d Ricci shrinkers*, Proc. Amer. Math. Soc. 143 (2015), 4433-4437.

[19] S. Honda, *Bishop-Gromov type inequality on Ricci limit spaces*, J. Math. Soc. Japan 63 (2011), 419-442.

[20] N. N. Khanh, *Gradient estimates of Li Yau type for a general heat equation on Riemannian manifolds*, Archivum Math. 52 (2016), no. 4, 207-219.

[21] A. Naber, *Noncompact shrinking four solitons with nonnegative curvature*, J. Reine Angew. Math. 645 (2010), 125-153.
[22] L. Ni and N. Wallach, *On a classification of gradient shrinking solitons*, Math. Res. Lett. 15 (2008), no. 5, 941-955.

[23] H. Z. Li, Y. Li and B. Wang, *On the structure of Ricci shrinkers*, arXiv:1809.04049.

[24] P. Li and S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. 156 (1986), 153-201.

[25] Y. Li and B. Wang, *The rigidity of Ricci shrinkers of dimension four*, arXiv:1701.01989, to appear in Trans. Amer. Math. Soc.

[26] J. Lott and C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. 169 (2009), 903-991.

[27] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math.DG/0211159.

[28] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv:math.DG/0303109.

[29] L. Saloff-Coste, *A note on Poincaré, Sobolev and Harnack inequality*, Int. Math. Res. Not. no.2 (1992), 27-38.

[30] K. T. Sturm, *Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality*, J. Math. Pures Appl. 75(9) (1996), 273-297.

[31] K. T. Sturm, *On the geometry of metric measure spaces. I*, Acta Math. 196 (2006), no. 1, 65-131, DOI 10.1007/s11511-006-0002-8.

[32] K. T. Sturm, *On the geometry of metric measure spaces. II*, Acta Math. 196 (2006), no. 1, 133-177, DOI 10.1007/s11511-006-0003-7.

[33] C. Villani, *Optimal Transport, Old and New*, Grundlehren Math. Wiss., vol. 338, Springer, 2008.

[34] F. Wang and X. H. Zhu, *The structure of spaces with Bakry-Émery Ricci curvature bounded below*, J. Reine Angew. Math., DOI 10.1515/crelle-2017-0042.

[35] G. Wei and W. Wylie, *Comparison geometry for the Bakry-Émery Ricci tensor*, J. Diff. Geom. 83 (2009), 377-406.

[36] J. Y. Wu and P. Wu, *Heat kernel on smooth metric measure spaces and applications*, Math. Ann. 365 (2016), Issue 1-2, 309-344.

[37] Q. S. Zhang and M. Zhu, *Bounds on harmonic radius and limits of manifolds with bounded Bakry-Émery Ricci curvature*, J. Geom. Anal. (2018), https://doi.org/10.1007/s12220-018-0072-9

[38] Z. L. Zhang, *Degeneration of shrinking Ricci solitons*, Int. Math. Res. Not. 21 (2010), 4137-4158.
[39] S.-H. Zhu, The comparison geometry of Ricci curvature. In: Comparison geometry (Berkeley, CA, 1993-94), volume 30 of Math. Sci. Res. Inst. Publ, pages 221-262. Cambridge Univ. Press, Cambridge, 1997, MR 1452876, Zbl 0896.53036.

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