Gravitational energy from a combination of a tetrad expression and Einstein’s pseudotensor

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Abstract
The energy–momentum for a gravitating system can be considered using the tetrad teleparallel gauge current in orthonormal frames, instead of the more commonly used Einstein pseudotensor, which makes use of holonomic frames. The tetrad expression itself gives a better result for gravitational energy than Einstein’s in that it gives a positive gravitational energy in the small sphere approximation. Inspired by an idea of Deser, we propose an alternative quasilocal gravitational energy expression in the small sphere limit which also enjoys the positive energy property by combining the tetrad expression and the Einstein pseudotensor, such that the connection coefficient has a form appropriate to a suitable intermediate between orthonormal and holonomic frames.

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1. Introduction

Identifying an appropriate expression for the gravitational energy at the quasilocal level is still an interesting and fundamental open question in general relativity. The existence of gravitational field energy is demonstrated in certain physical phenomena, a dramatic example is the heating of Io, in which the tidal force of Jupiter acts on its satellite Io. Owing to the equivalence principle, it is not meaningful to study the gravitational energy at a point, but the quasilocal concept can get around this difficulty. Positivity is a desired quasilocal property, but it is difficult to prove in general. Therefore we consider positivity for just a small region in the vacuum.

Consider the small sphere approximation in the vacuum (taking for simplicity units such that \( c = 1 \)). The leading term of the energy–momentum quasilocal expression should be proportional to \( r^5 B_{\alpha\beta\mu\nu} t^\alpha t^\beta t^\mu t^\nu \) [1], where \( B_{\alpha\beta\mu\nu} \) is the Bel–Robinson tensor and \( t^\alpha \) is the timelike unit normal. This is sufficient because the Bel–Robinson ‘energy’ is non-negative. It should be mentioned that the physical dimension of the Bel–Robinson ‘energy-density’ \( B_{\alpha\beta\mu\nu} t^\alpha t^\beta t^\mu t^\nu \) is cm\(^{-4}\). The quasilocal value for the gravitational energy should be a multiple
of $\frac{4}{9}\pi r^3 (r^2 B_{\rho \mu \nu \tau} t^\rho t^\mu t^\nu)$ in the small sphere limit, which includes a factor of the Euclidean volume of the three-ball of radius $r$. Then the dimension of $r^2 B_{\rho \mu \nu \tau} t^\rho t^\mu t^\nu$ fits the energy-density dimension. (In the non-vacuum case the energy–momentum should be proportional to $\frac{4}{9}\pi r^3 T_{\mu \nu} t^\nu$ to lowest order, where $T_{\mu \nu}$ is the stress–energy–momentum tensor.)

Gravitational energy has been studied using the Einstein pseudotensor in holonomic frames for a long time [2]. Recently, Deser et al [3] used a similar method for calculating the Landau–Lifschitz pseudotensor and obtained the Bel–Robinson tensor in terms of a specific combination of these two classical pseudotensors. In [4], a tetrad expression evaluated in orthonormal frames was also shown to give a positive multiple of the Bel–Robinson tensor in a small region. In this paper, inspired by the idea of Deser et al, we propose an expression for gravitational energy that exhibits the positive energy property by employing a specific combination of the tetrad expression and the Einstein pseudotensor. This means that the connection coefficients are selected by a unique specific intermediate between orthonormal and holonomic frames.

2. Ingredient

The curvature 2-form, in terms of differential forms, is

$$R^\alpha_{\beta \mu} := d\Gamma^\alpha_{\beta \lambda} + \Gamma^\alpha_{\lambda \beta} \wedge \Gamma^\lambda_{\mu \beta}. \quad (1)$$

Here we are using the standard Levi-Civita connection, which is metric compatible and torsion free. Consequently since the torsion 2-form

$$T^\alpha_{\beta \mu} := d\theta^\alpha_{\beta \mu} + \Gamma^\alpha_{\beta \lambda} \wedge \theta^\lambda_{\mu \beta}. \quad (2)$$

vanishes, then

$$D\eta^\alpha_{\beta \mu} := d\eta^\alpha_{\beta \mu} + \eta^\alpha_{\lambda \beta} \wedge \eta^\lambda_{\mu \beta} - \Gamma^\alpha_{\lambda \beta} \wedge \eta^\lambda_{\mu \beta} - \Gamma^\beta_{\lambda \mu} \wedge \eta^\lambda_{\alpha \beta} = 0, \quad (3)$$

where the dual basis $\eta^\alpha_{\beta \mu} := (\theta^\alpha \wedge \ldots)$ and $\theta^\alpha_{\beta \mu}$ is the co-frame.

At a single point, it is well known that one can choose Riemann normal coordinates in which the connection coefficient satisfies

$$\Gamma^\alpha_{\beta \mu}(0) = 0, \quad -3\partial_\nu \Gamma^\alpha_{\beta \mu} = R^\alpha_{\beta \mu \nu} + R^\alpha_{\mu \nu \beta}. \quad (4)$$

Similarly, one can choose orthonormal frames [4] such that

$$\Gamma^\alpha_{\beta \mu}(0) = 0, \quad 2\partial_\lambda \Gamma^\alpha_{\beta \mu} = R^\alpha_{\beta \mu \lambda}. \quad (5)$$

We define the Bel–Robinson tensor $B_{\alpha \beta \mu \nu}$ and the tensor $S_{\alpha \beta \mu \nu}$ in empty spacetime [2] as follows:

$$B_{\alpha \beta \mu \nu} := R_{\alpha \lambda \mu \sigma} R^\lambda_{\beta \nu} R^\lambda_{\sigma \mu \nu} + R_{\alpha \lambda \mu \sigma} R^\lambda_{\beta \nu} R^\lambda_{\sigma \mu \nu} - \frac{1}{4} S_{\alpha \beta \mu \nu} R_{\lambda \sigma \rho \tau} R^{\lambda \sigma \rho \tau}, \quad (6)$$

$$S_{\alpha \beta \mu \nu} := R_{\alpha \lambda \sigma \mu} R^\lambda_{\beta \nu} R^\lambda_{\sigma \mu \nu} + R_{\alpha \lambda \sigma \mu} R^\lambda_{\beta \nu} R^\lambda_{\sigma \mu \nu} + \frac{1}{4} S_{\alpha \beta \mu \nu} R_{\lambda \sigma \rho \tau} R^{\lambda \sigma \rho \tau}. \quad (7)$$

3. The tetrad and the Einstein superpotentials

Consider the first-order Lagrangian (4-form) density [5, 6]

$$L := dq \wedge p - \Lambda(q, p). \quad (8)$$
where \( q \) is an \( f \)-form, \( p \) is a \((3-f)\) form and \( \Lambda \) is the potential. The corresponding Hamiltonian 3-form (density) is
\[
\mathcal{H}(N) = \mathcal{L}_N q \wedge p - i_N \mathcal{L},
\]
where the Lie derivative \( \mathcal{L}_N = i_N d + di_N \). The interior product of the Lagrangian with an arbitrary vector field \( N \) is
\[
i_N \mathcal{L} = \mathcal{L}_N q \wedge p - \epsilon i_N q \wedge dp - \epsilon dq \wedge i_N p - i_N \Lambda - d(i_N q \wedge p),
\]
where \( \epsilon = (-1)^f \). The Hamiltonian 3-form (density) is
\[
\mathcal{H}(N) = N^\mu \mathcal{H}_\mu + dB(N),
\]
where
\[
N^\mu \mathcal{H}_\mu = \epsilon i_N q \wedge dp + \epsilon dq \wedge i_N p + i_N \Lambda,
\]
and the natural boundary term is
\[
B(N) = i_N q \wedge p.
\]

This is called the quasilocal boundary expression, because when one integrates the Hamiltonian density over a finite region to get the Hamiltonian, the boundary term leads to an integral over the boundary of the region.

For GR we may take (with \( \kappa = 8\pi G \))
\[
q \rightarrow -\frac{1}{2\kappa} \eta_{\alpha}^\beta, \quad p \rightarrow \Gamma^\alpha_\beta_\gamma,
\]
and rewrite (13) as
\[
2\kappa B(N) = \Gamma^\alpha_\beta_\gamma i_N \eta_{\alpha}^\beta = -\frac{1}{2} N^\sigma U_{\alpha}^{[\mu\nu]} \epsilon_{\mu\nu},
\]
where \( N^\nu \) is the timelike vector field and the superpotential is
\[
U_{\alpha}^{[\mu\nu]} = -\sqrt{-g} g^\beta_\sigma \Gamma^\alpha_\rho_\delta \delta_{\mu\nu}^\rho_\sigma.
\]

We can choose to evaluate the components of the metric and connection in any frame as long as the superpotential gives physically sensible results; in particular, inside matter (mass density) and at spatially infinity (ADM mass). It is called tetrad [4] if the connection is in terms of orthonormal frames, while Freud [7] used holonomic frames. The corresponding expressions for the gravitational energy were studied in [4] and [2], respectively. One can use differential forms to derive the boundary expression instead of using the superpotential components. Furthermore, selectively combining orthonormal and holonomic frames through the connection can produce the desired result for gravitational energy. This is the main point of this paper and the details are presented in the next section.

4. Combination of tetrad and Einstein expressions

The pseudotensor can be obtained from the superpotential as
\[
t^\mu_\sigma = \partial_\nu U_{\alpha}^{[\mu\nu]}.
\]

Deser et al [3] considered a certain combination of the Einstein \( E_{\alpha\beta} \) and Landau–Lifschitz \( L_{\alpha\beta} \) pseudotensors so that in Riemann normal coordinates at the origin
\[
\partial_\mu_\nu \left( L_{\alpha\beta} + \frac{1}{2} E_{\alpha\beta} \right) = R_{\alpha\beta_\mu\nu}.
\]

This combination is satisfactory because the Bel–Robinson tensor guarantees the desired property of providing a positive gravitational energy density within a very small region. Namely
\[
B_{\alpha\beta_\mu\nu} i^\sigma_\nu i^\mu_\rho \epsilon_\nu_\nu = E_{\alpha\beta} E_{\alpha\beta} + H_{\alpha\beta} H_{\alpha\beta} \geq 0,
\]
where $E_{ab}$ and $H_{ab}$ are the electric and magnetic parts, given in terms of the Weyl curvature $C_{abcd}$ by

$$E_{ab} = C_{abmn} t^m t^n, \quad H_{ab} = s C_{abmn} t^m t^n.$$  \hspace{1cm} (20)

Using differential forms, consider the middle term of (15) taking $dN^\mu = 0$, i.e., the components of the vector field $N^\mu$ in the normal coordinates are taken to be constant. Using equations (15), (1) and (3), we find

$$dB(N) = N^\mu 2 \kappa \left( \frac{s}{2} R^\rho_{\mu \nu \xi \kappa} + \frac{k}{3} (R^\rho_{\mu \nu \xi \kappa} + R^\rho_{\mu \nu \xi \kappa}) \right) x^\xi x^\kappa + \mathcal{O}(x^3).$$  \hspace{1cm} (21)

Inspired by [3], let us take the connection in the neighborhood of the origin to have the following expansion in Riemann normal coordinates:

$$\Gamma^\alpha_{\mu \rho} = \left\{ \frac{s}{2} R^\alpha_{\mu \rho \nu} - \frac{k}{3} (R^\alpha_{\rho \nu \mu} + R^\alpha_{\rho \nu \mu}) \right\} x^\nu dx^\mu + \mathcal{O}(x^2).$$  \hspace{1cm} (22)

The multipliers $s$ and $k$ are real numbers such that $s + k = 1$ for consistent normalization. Implicitly for each $s = 1 - k$ this relation is a prescription for the choice of frame in the small neighborhood.

Using (22), after a tedious calculation (21) becomes

$$2 \kappa dB(N) = - N^\mu 2 G^\rho_{\mu \rho} \eta^\rho - N^\mu \frac{1}{4} \left[ s \frac{3}{2} B^\rho_{\mu \xi \kappa} + \frac{k}{3} \left( 5 B^\rho_{\mu \xi \kappa} - \frac{1}{2} S^\rho_{\mu \xi \kappa} \right) \right] x^\xi x^\kappa \eta^\rho + \mathcal{O}(\text{Ricci}, x) + \mathcal{O}(x^3).$$  \hspace{1cm} (23)

The first term of this expression is dominated by the Einstein tensor $G^\rho_{\mu \rho}$, which means that it satisfies the desired condition inside a matter at the origin to match the equivalence principle. The higher order terms carry the information about the gravitational energy. As mentioned before, $s + k = 1$. We will consider three cases.

Case (i). $k = 0$. This gives the tetrad teleparallel gauge current energy–momentum expression $M_{ab}$ for the pure orthonormal frames. The associated second derivatives at the origin [4] are

$$\partial^2_{\xi \kappa} M^\rho_{\mu} = \frac{1}{2} B^\rho_{\mu \xi \kappa}.$$  \hspace{1cm} (24)

Case (ii). $k = 1$. This is the Einstein pseudotensor in pure holonomic frames and the corresponding second derivatives [2, 3] are

$$\partial^2_{\xi \kappa} E^\rho_{\mu} = \frac{1}{5} (4 B^\rho_{\mu \xi \kappa} - S^\rho_{\mu \xi \kappa}).$$  \hspace{1cm} (25)

Case (iii). $k = -3$. The energy–momentum density of this particular combination is the expression $t_{ab}$ whose small vacuum region value is governed by the second derivatives

$$\partial^2_{\xi \kappa} t^\rho_{\mu} = 2 B^\rho_{\mu \xi \kappa}.$$  \hspace{1cm} (26)

This is the desired result: it contains only the Bel–Robinson tensor, and this combination intermediate between the orthonormal and holonomic expressions is unique. Using a similar calculation [4], the gravitational energy–momentum for this case is

$$(-E, P_\rho) = P_\mu = - \frac{1}{2 \kappa} \int B_{\mu \rho \xi \kappa} x^\xi x^\kappa dx - \frac{r^5}{60 G} B_{\mu \rho \xi \kappa} x^\xi x^\kappa t^\rho,$$  \hspace{1cm} (27)

where $B_{\mu \rho \xi \kappa} = B_{\mu \rho \xi \kappa} = (E_{ab} E_{ab} + H_{ab} H_{ab}, 2 \xi_{a b} E_{a b} H_{a b})$. The tensor $S_{\mu \rho \xi \kappa}$ has been eliminated simply because $S_{\mu \rho \xi \kappa} = -10(E_{a b} E_{a b} - H_{a b} H_{a b}, 0)$ which cannot guarantee the positivity.
5. Conclusion

Finding a suitable expression for gravitational energy remains an outstanding open problem. The energy–momentum for a gravitating system has been considered using the tetrad teleparallel gauge current in orthonormal frames; it gives a good result, namely the Bel–Robinson tensor which has the positive gravitational energy property. Likewise, the classical Einstein pseudotensor has been used in holonomic frames to investigate the same subject; however, it does not have the desired positivity result. Deser et al used a combination of the second derivatives of the Einstein and Landau–Lifschitz pseudotensors to obtain the Bel–Robinson tensor. Inspired by their work, we have found a quasilocal gravitational energy expression in the small sphere limit which enjoys the positive energy property from a combination of the tetrad expression and the Einstein pseudotensor, such that the connection components are selected as a uniquely specific intermediate between the orthonormal and holonomic frame expressions is unique.

We have found an expression which satisfies the Bel–Robinson small region positivity condition. Proving positivity on a larger scale is a much greater challenge. We hope it can be done for our expression.

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