The triviality of a certain invariant of link maps in the four-sphere

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Abstract

It is an open problem whether Kirk’s $\sigma$ invariant is the complete obstruction to a link map $S^2 \cup S^2 \to S^4$ being link homotopically trivial. With the objective of constructing counterexamples, Li proposed a link homotopy invariant $\omega$ that is defined on the kernel of $\sigma$ and also obstructs link nullhomotopy. We show that $\omega$ is determined by $\sigma$, and is a strictly weaker invariant.

1 Introduction

A link map is a (continuous) map

$$f : S^{p_1} \cup S^{p_2} \cup \ldots \cup S^{p_n} \to S^m$$

from a union of spheres into another sphere such that $f(S^{p_i}) \cap f(S^{p_j}) = \emptyset$ for $i \neq j$. Two link maps are said to be link homotopic if they are connected by a homotopy through link maps, and the set of link homotopy classes of link maps as above is denoted $LM_{p_1,p_2,\ldots,p_n}^m$. It is a familiar result that $LM_{1,1,\ldots,1}^3$ is classified by the linking number, and in his foundational work Milnor [15] described invariants of $LM_{1,1,\ldots,1}^3$ which classified $LM_{1,1,\ldots,1}^3$. These invariants (the $\mu$-invariants) were refined much later by Habegger and Lin [6] to achieve an algorithmic classification of $LM_{1,1,\ldots,1}^3$.

Higher dimensional link homotopy began with a study of $LM_{p,q}^m$ when $p, q \leq m - 3$, first by Scott [18] and later by Massey and Rolfsen [16]. Both papers made particular use of a generalization of the linking number, defined as follows. Given a link map $f : S^p \cup S^q \to S^m$, choose a point $\infty \in S^m \setminus f(S^p \cup S^q)$ and identify $S^m \setminus \infty$ with $\mathbb{R}^m$. When $p, q < m$, the map

$$S^p \times S^q \to S^{m-1}, (x,y) \mapsto \frac{f(x) - f(y)}{||f(x) - f(y)||}$$

is nullhomotopic on the subspace $S^p \vee S^q$ and so determines an element $\alpha(f) \in \pi_{p+q}(S^{m-1})$. When $m = p + q + 1$, the link homotopy invariant $\alpha$ is the integer-valued linking number. In a certain dimension range, $\alpha$ was shown in [16] to classify embedded link maps $S^p \cup S^q \to S^m$ up to link homotopy. Indeed, historically, link homotopy roughly separated into settling two problems.
1. Decide when an embedded link map is link nullhomotopic.

2. Decide when a link map is link homotopic to an embedding.

In a large metastable range this approach culminated in a long exact sequence which reduced the problem of classifying $LM_{p,q}^m$ to standard homotopy theory questions (see [11]).

On the other hand, four-dimensional topology presents unique difficulties, and link homotopy of 2-spheres in the 4-sphere requires different techniques. In this setting, the first problem listed above was solved by Bartels and Teichner [2], who showed that an embedded link $S^2 \cup S^2 \cup \ldots \cup S^2 \to S^4$ is link nullhomotopic. In this paper we are interested in invariants of $LM_{2,2}^4$ which have been introduced to address the second problem.

Fenn and Rolfsen [3] showed that $\alpha$ defines a surjection $LM_{2,2}^4 \to \mathbb{Z}_2$ and in doing so constructed the first example of a link map $S^2 \cup S^2 \to S^4$ which is not link nullhomotopic. Kirk [9] generalized this result, introducing an invariant $\sigma$ of $LM_{2,2}^4$ which further obstructs embedding and surjects onto an infinitely generated group.

To a link map $f : S_2^2 \cup S_2^2 \to S^4$, where we use signs to distinguish component 2-spheres, Kirk defined a pair of integer polynomials $\sigma(f) = (\sigma_+(f), \sigma_-(f))$ such that each component is invariant under link homotopy of $f$, determines $\alpha(f)$ and vanishes if $f$ is link homotopic to a link map that embeds either component. It is an open problem whether $\sigma$ is the complete obstruction; that is, whether $\sigma(f) = (0, 0)$ implies that $f$ is link homotopic to an embedding. By [2, Theorem 5], this is equivalent to asking if $\sigma$ is injective on $LM_{2,2}^4$. Seeking to answer in the negative, Li proposed an invariant $\omega(f) = (\omega_+(f), \omega_-(f))$ to detect link maps in the kernel of $\sigma$.

When $\sigma_\pm(f) = 0$, after a link homotopy the restricted map $f|_{S_2^2} : S_2^2 \to S^4 \setminus f(S_2^2)$ may be equipped with a collection of Whitney disks, and $\omega_\pm(f) \in \mathbb{Z}_2$ obstructs embedding by counting weighted intersections between $f(S_2^2)$ and the interiors of these disks. Precise definitions of these invariants will be given in Section 2.

By [13] (and [14]), $\omega$ is an invariant of link homotopy, but the example produced in [13] of a link map $f$ with $\sigma(f) = (0, 0)$ and $\omega(f) \neq (0, 0)$ was found to be in error by Pilz [17]. The purpose of this paper is to prove that $\omega$ cannot detect such examples; indeed, it is a weaker invariant than $\sigma$.

**Theorem 1.1.** Let $f$ be a link map with $\sigma_-(f) = 0$ and let $a_1, a_2, \ldots$ be integers so that $\sigma_+(f) = \sum a_n(s^n - 1)$. Then

$$\omega_-(f) = \sum a_n \mod 2,$$

where the sum is over all $n$ equal to 2 modulo 4.

Consequently, there are infinitely many distinct classes $f \in LM_{2,2}^4$ with $\sigma_+(f) = 0$, $\omega_+(f) = 0$ but $\sigma_-(f) \neq 0$ (see Proposition 3.11). In particular, the following corollary answers Question 6.2 of [13].
Corollary 1.2. If a link map $f$ has $\sigma(f) = (0, 0)$, then $\omega(f) = (0, 0)$.

By [14, Theorem 1.3] and [19, Theorem 2], Theorem 1.1 may be interpreted geometrically as follows.

Corollary 1.3. Let $f$ be a link map such that $\sigma_+(f) = 0$. Then, after a link homotopy, the self-intersections of $f(S^2_+)$ may be paired up with framed, immersed Whitney disks in $S^4 \setminus f(S^2_-)$ whose interiors are disjoint from $f(S^2_+)$. This paper is organized as follows. In Section 2 we first review Wall intersection theory in the four-dimensional setting. The geometric principles thus established underlying the link homotopy invariants $\sigma$ and $\omega$, which we subsequently define. In Section 3 we exploit that, up to link homotopy, one component of a link map is unknotted, immersed, to equip this component with a convenient collection of Whitney disks which enable us to relate the invariants $\sigma$ and $\omega$. A more detailed outline of the proof may be found at the beginning of that section.

2 Preliminaries

Let us first fix some notation. For an oriented path or loop $\alpha$, let $\overline{\alpha}$ denote its reverse path; if $\alpha$ is a based loop, let $[\alpha]$ denote its based homotopy class. Let $\ast$ denote composition of paths, and denote the interval $[0,1]$ by $I$. Let $\equiv$ denote equivalence modulo 2.

In what follows assume all manifolds are oriented and equipped with basepoints; specific orientations and basepoints will usually be suppressed.

2.1 Intersection numbers in 4-manifolds

The link homotopy invariants investigated in this paper are closely related to the algebraic “intersection numbers” $\lambda$ and $\mu$ introduced by Wall [20]. For a more thorough exposition of the latter invariants in the four-dimensional setting, see Chapter 1 of [5], from which our definitions are based.

Suppose $A$ and $B$ are properly immersed, self-transverse 2-spheres or 2-disks in a connected 4-manifold $Y$. (By self-transverse we mean that self-intersections arise precisely as transverse double points.) Suppose further that $A$ and $B$ are transverse and that each is equipped with a path (a whisker) connecting it to the basepoint of $Y$. For an intersection point $x \in A \cap B$, let $\lambda(A, B)[x] \in \pi_1(Y)$ denote the homotopy class of a loop that runs from the basepoint of $Y$ to $A$ along its whisker, then along $A$ to $x$, then back to the basepoint of $Y$ along $B$ and its whisker. Define $\text{sign}_{A,B}[x]$ to be 1 or $-1$ depending on whether or not, respectively, the orientations of $A$ and $B$ induce the orientation of $Y$ at $x$. The Wall intersection number $\lambda(A, B)$ is defined by the sum

$$\lambda(A, B) = \sum_{x \in A \cap B} \text{sign}_{A,B}[x] \lambda(A, B)[x]$$

3
in the group ring $\mathbb{Z}[\pi_1(Y)]$, and is invariant under homotopy rel boundary of $A$ or $B$ \footnote{Since $A$ and $B$ are both whiskered, we permit ourselves to confuse them with their respective homotopy classes in $\pi_2(Y)$.}, but depends on the choice of basepoint of $Y$ and the choices of whiskers and orientations. For an element $h = \sum_i n_i g_i$ in $\mathbb{Z}[\pi_1(Y)]$ $(n_i \in \mathbb{Z}, g_i \in \pi_1(Y))$, define $\overline{h} \in \mathbb{Z}[\pi_1(Y)]$ by $\sum_i n_i \overline{g_i}$. From the definition it is readily verified that $\lambda(B, A) = \overline{\lambda(A, B)}$ and that the following observations, which we record for later reference, hold.

**Proposition 2.1.** If $x, y \in A \cap B$, then the product of $\pi_1(Y)$-elements $\lambda(A, B)[x] (\lambda(A, B)[y])$ is represented by a loop that runs from the basepoint to $A$ along its whisker, along $A$ to $x$, then along $B$ to $y$, and back to the basepoint along $A$ and its whisker. Moreover, if $Y$ has abelian fundamental group, then this group element does not depend on the choice of whiskers and basepoint. \hfill \Box

**Proposition 2.2.** If $D_A \subset A$ is an immersed 2-disk that is equipped with the same whisker and oriented consistently with $A$, then for each $x \in D_A \cap B$ we have $\lambda(A, B)[x] = \lambda(D_A, B)[x]$ and $\text{sign}_{A, D}[x] = \text{sign}_{D_A, B}[x]$. \hfill \Box

The intersection numbers respect sums in the following sense. Suppose that $A$ and $B$ as above are 2-spheres, and suppose there is an embedded arc $\gamma$ from $A$ to $B$, with interior disjoint from both. Let $\iota_A$ be a path that runs along the whisker for $A$, then along $A$ to the initial point of $\gamma$, and let $\iota_B$ be a path that runs from the endpoint of $\gamma$, along $B$ and its whisker to the basepoint of $Y$. Form the connect sum $A \# B$ of $A$ and $B$ along $\gamma$ in such a way that the orientations of each piece agree with the result. Equipped with the same whisker as $B$, the 2-sphere $A \# B$ represents the element $[A + gB] \in \pi_2(Y)$, where $g = [\iota_A \ast \gamma \ast \iota_B] \in \pi_1(Y)$. If $C$ is an immersed 2-disk or 2-sphere in $Y$ transverse to $A$ and $B$, then $\lambda(A + gB, C) = \lambda(A, C) + g\lambda(B, C)$. The additive inverse $-A \in \pi_2(Y)$ is represented by reversing the orientation of $A$.

Allowing $A$ again to be a self-transverse 2-disk or 2-sphere, the Wall self-intersection number $\mu(A)$ is defined as follows. Let $f_A : D \to Y$ be a map with image $A$, where $D = D^2$ or $S^2$. Let $x$ be a double point of $A$, and let $x_1$, $x_2$ denote its two preimage points in $D$. If $U_1, U_2$ are disjoint neighborhoods of $x_1$, $x_2$ in $\text{int} \ D$, respectively, that do not contain any other double point preimages, then the embedded 2-disks $f_A(U_1)$ and $f_A(U_2)$ in $A$ are said to be two different \textbf{branches} (or sheets) intersecting at $x$. Let $\mu(A)[x] \in \pi_1(Y)$ denote the homotopy class of a loop that runs from the basepoint of $Y$ to $A$ along its whisker, then along $A$ to $x$ through one branch $f_A(U_1)$, then along the other branch $f_A(U_2)$ and back to the basepoint of $Y$ along the whisker of $A$. (Such a loop is said to \textit{change branches} at $x$.) Define $\text{sign}_A[x]$ to be $1$ or $-1$ depending on whether or not, respectively, the orientations of the two branches of $A$ intersecting at $x$ induce the orientation of $Y$ at $x$. In the group ring $\mathbb{Z}[\pi_1(Y)]$, let

$$
\mu(A) = \sum_x \text{sign}_A[x] \mu(A)[x],
$$
where the sum is over all such self-intersection points. (Note that it may sometimes be more convenient to write \(\mu(f_A) = \mu(A)\).) For a fixed whisker of \(A\), changing the order of the branches in the above definition replaces \(\mu(A)[x]\) by its \(\pi_1(Y)\)-inverse, so \(\mu(A)\) is only well-defined in the quotient \(Q(Y)\) of \(\mathbb{Z}[\pi_1(Y)]\), viewed as an abelian group, by the subgroup \(\{a - \pi : a \in \pi_1(Y)\}\). The equivalence class of \(\mu(A)\) in this quotient group is invariant under regular homotopy rel boundary of \(A\). Note also that if the 4-manifold \(Y\) has abelian fundamental group, then \(\mu(A)\) does not depend on the choice of whisker.

Let \(\text{self}(A) \in \mathbb{Z}\) denote the signed self-intersection number of \(A\). The reduced Wall self-intersection number \(\tilde{\mu}(A)\) may be defined by

\[
\tilde{\mu}(A) = \mu(A) - \text{self}(A) \in Q(Y).
\]

It is an invariant of homotopy rel boundary \([3\ Proposition\ 1.7]\); this observation derives from the fact that non-regular homotopy takes the form of local “cusp” homotopies which may each change \(\mu(A)\) by \(\pm 1\) (see \([3\ Section\ 1.6]\).)

### 2.2 The link homotopy invariants

We now recall the definitions of the link homotopy invariants \(\sigma\) of Kirk \([9]\) and \(\omega\) of Li \([13]\).

Let \(f : S^2_+ \cup S^2_- \rightarrow S^4\) be a link map. After a link homotopy (in the form of a perturbation) of \(f\) we may assume the restriction \(f_\pm = f|_{S^2_\pm} : S^2_\pm \rightarrow S^4 \setminus f(S^2_\pm)\) to each component is a self-transverse immersion. Let \(X_- = S^4 \setminus f(S^2_-)\) and choose a generator \(s\) for \(H_1(X_-) \cong \mathbb{Z}\), which we write multiplicatively. For each double point \(p\) of \(f(S^2_+),\) let \(\alpha_p\) be a simple circle on \(f(S^2_+)\) that changes branches at \(p\) and does not pass through any other double points. We call \(\alpha_p\) an accessory circle for \(p\). Letting \(n(p) = \text{lk}(\alpha_p, f(S^2_+))\), one defines

\[
\sigma_+(f) = \sum_p \text{sign}(p)(s^{n(p)\pm 1})
\]

in the ring \(\mathbb{Z}[s]\) of integer polynomials, where the sum is over all double points of \(f(S^2_+)\), and to simplify notation we write \(\text{sign}(p) = \text{sign}_{f_+(S^2_+)}(p)\).

Reversing the roles of \(f_+\) and \(f_-\), we similarly define \(\sigma_-(f)\) and write \(\sigma(f) = (\sigma_+(f), \sigma_-(f)) \in \mathbb{Z}[s] \oplus \mathbb{Z}[s]\). Kirk showed in \([9]\) that \(\sigma\) is a link homotopy invariant, and in \([10]\) that if \(f\) is link homotopic to a link map for which one component is embedded, then \(\sigma(f) = (0, 0)\).

Let \(\rho : \pi_1(X_-) \rightarrow H_1(X_-) = \mathbb{Z}[s]\) denote the Hurewicz map. Referring to the definition of \(\mu\) in the preceding section as applied to the map \(f_+ : S^2_+ \rightarrow X_-\), observe that \(\rho\) carries \(Q(X_-)\) to the ring of integer polynomials \(\mathbb{Z}[s]\) and Kirk’s invariant \(\sigma_+\) is given by

\[
\sigma_+(f) = \rho(\tilde{\mu}(f_+)) \in \mathbb{Z}[s]. \tag{2.1}
\]

As in \([13]\), we say that \(f\) is \(\pm\)-good if \(\pi_1(X_+) \cong \mathbb{Z}\) and the restricted map \(f_\pm\) is a self-transverse immersion with \(\text{self}(f_\pm) = 0\). We say that \(f\) is \(\text{good}\) if it is both \(\pm\)- and \(\mp\)-good. Equation (2.1) has the following consequence.
Proposition 2.3. If \( f \) is a \( \pm \)-good link map, then \( \sigma_\pm(f) = \mu(f_\pm) \).

The invariant \( \sigma_\pm(f) \) obstructs, up to link homotopy, pairing up double points of \( f(S^2_+) \) with Whitney disks in \( X_+ \). While the essential purpose of Whitney disks is to embed (or separate) surfaces (see \[3\] Section 1.4), our focus will be on their construction for the purposes of defining certain invariants. In the setting of link maps, the following standard result (phrased in the context of link maps) is the key geometric insight behind all the invariants we discuss in this paper and will find later application.

Lemma 2.4. Let \( f \) be a link map such that \( f_- \) is a self-transverse immersion, and suppose \( \{p^+, p^-\} \) are a pair of oppositely-signed double points of \( f(S^2_+) \). Let \( U \) and \( V \) each be an embedded 2-disk neighborhood of \( \{p^+, p^-\} \) on \( f(S^2_+) \) such that \( U \) and \( V \) intersect precisely at these two points. On \( f(S^2_+) \), let \( \alpha^+, \alpha^- \) be loops based at \( p^+, p^- \) (respectively) that leave along \( U \) and return along \( V \). Let \( \gamma_U, \gamma_V \) be oriented paths in \( U, V \) (respectively) that run from \( p^+ \) to \( p^- \). Then the oriented loop \( \gamma_U \cup \gamma_V \) satisfies

\[
\|\text{lk}(\gamma_U \cup \gamma_V, f(S^2_+))\| = |\text{lk}(\alpha^+, f(S^2_+)) - \text{lk}(\alpha^-, f(S^2_+))|.
\]

Wishing to obtain a “secondary” obstruction, in \[13\] Li proposed the following \((\mathbb{Z}_2 \oplus \mathbb{Z}_2)\)-valued invariant to measure intersections between \( f(S^2_+) \) and the interiors of these disks.

Suppose \( f \) is a \( \pm \)-good link map with \( \sigma_-(f) = \mu(f_-) = 0 \). The double points of \( f(S^2_+) \) may then be labeled \( \{p^+_i, p^-_i\}_i \) so that \( \text{sign}(p^+_i) = -\text{sign}(p^-_i) \) and \( n_i := |n(p^+_i)| = |n(p^-_i)| \); consequently, by Lemma 2.4 we may let \( f^{-1}(p^+_i) = \{x^+_i, y^+_i\} \) and \( f^{-1}(p^-_i) = \{x^-_i, y^-_i\} \) so that if \( \gamma_i \) is an arc on \( S^2_+ \) connecting \( x^+_i \) to \( x^-_i \) (and missing all other double point preimages) and \( \gamma'_i \) is an arc on \( S^2_+ \) connecting \( y^+_i \) to \( y^-_i \) (and missing \( \gamma_i \) and all other double point preimages), then the loop \( f(\gamma_i) \cup f(\gamma'_i) \subset f(S^2_+) \) is nullhomologous, hence nullhomotopic, in \( X_+ \). Let \( U_i \) (respectively, \( U'_i \)) be a neighborhood of \( \gamma_i \) (respectively, \( \gamma'_i \)) in \( S^2_+ \). The arcs \( \{\gamma_i, \gamma'_i\}_{i=1}^k \) and neighborhoods \( \{U_i, U'_i\}_{i=1}^k \) may be chosen so that the collection \( \{U_i\}_{i=1}^k \cup \{U'_i\}_{i=1}^k \) is mutually disjoint, and so that the resulting Whitney circles \( \{f(\gamma_i) \cup f(\gamma'_i)\}_{i=1}^k \) are mutually disjoint, simple circles in \( X_+ \) such that each bounds an immersed Whitney disk \( W_i \) in \( X_+ \) whose interior is transverse to \( f(S^2_+) \).

Since the two branches \( f(U_i) \) and \( f(U'_i) \) of \( f(S^2_+) \) meet transversely at \( \{p^+_i, p^-_i\} \), there are a pair of smooth vector fields \( v_1, v_2 \) on \( \partial W_i \) such that \( v_1 \) is tangent to \( f(S^2_+) \) along \( \gamma_i \) and normal to \( f(S^2_+) \) along \( \gamma'_i \), while \( v_2 \) is normal to \( f(S^2_+) \) along \( \gamma_i \) and tangent to \( f(S^2_+) \) along \( \gamma'_i \). Such a pair defines a normal framing of \( W_i \) on the boundary. We say that \( \{v_1, v_2\} \) is a correct framing of \( W_i \), and that \( W_i \) is framed, if the pair extends to a normal framing of \( W_i \). By boundary twisting \( W_i \) (see page 5 of \[3\]) if necessary, at the cost of introducing more interior intersection points with \( f(S^2_+) \), we can choose the collection of Whitney disks \( \{W_i\}_{i=1}^k \) such that each is correctly framed.

Let \( 1 \leq i \leq k \). To each point of intersection \( x \in f(S^2_+) \cap \text{int} W_i \), let \( \beta_x \) be a loop that first goes along \( f(S^2_+) \) from its basepoint to \( x \), then along \( W_i \).
to \( f(\gamma_i') \subset \partial W_i \), then back along \( f(S^2) \) to the basepoint of \( f(S^2) \) and let 
\[
m_i(x) = \text{lk}(f(S^2), \beta_x).
\]
Let 
\[
\mathcal{L}_i^-(x) = n_i + n_i m_i(x) + m_i(x) \mod 2;
\] summing over all such points of intersection, let 
\[
\mathcal{L}^{-}(W_i) = \sum_{x \in f(S^2) \cap \text{int} W_i} \mathcal{L}_i^-(x) \mod 2.
\]
Then Li’s \( \omega^- \)-invariant applied to \( f \) is defined by 
\[
\omega^-(f) = \sum_{i=1}^{k} \mathcal{L}^{-}(W_i) \mod 2.
\] (2.3)

We record two observations about this definition for later use. The latter is a special case of Proposition 2.1.

**Remark 2.5.** If \( n_i \) is odd then \( \mathcal{L}^{-}(W_i) = f(S^2) \cdot \text{int} W_i \mod 2 \), while if \( n_i \) is even then 
\[
\mathcal{L}^{-}(W_i) = \sum_{x \in f(S^2) \cap \text{int} W_i} m_i(x).
\]

**Remark 2.6.** Suppose \( x, y \in f(S^2) \cap \text{int} W_i \), and let \( \beta \) be a loop that runs from \( x \) to \( y \) along \( f(S^2) \), then back to \( x \) along \( \text{int} W_i \). We have 
\[
m_i(x) + m_i(y) = \text{lk}(f(S^2), \beta) \mod 2.
\]

Now suppose \( f \) is an arbitrary link map with \( \sigma_-(f) = 0 \). By standard arguments we may choose a \( -- \)-good link homotopy representative \( f' \) of \( f \), and \( \omega(f) \) is defined by setting \( \omega_-(f) = \omega_-(f') \). By a result in [14], in [13] it was shown that this defines an invariant of link homotopy; Theorem 1.1 gives a new proof. By interchanging the roles of \( f_+ \) and \( f_- \) (and instead assuming \( \sigma_+(f) = 0 \)), we obtain \( \omega_+(f) \in \mathbb{Z}_2 \) similarily, and write \( \omega(f) = (\omega_+(f), \omega_-(f)) \).

Based on similar geometric principles, Teichner and Schneiderman [19] defined a secondary obstruction with respect to the homotopy invariant \( \mu \). When adapted to the context of link homotopy, however, their invariant reduces to \( \omega \) [14].

### 2.3 Surgering tori to 2-spheres

A common situation to arise in this paper is the following. Suppose we have a torus (punctured torus, respectively) \( T \) in the 4-manifold \( Y \), on which we wish to perform surgery along a curve so as to turn it into a 2-sphere (2-disk, respectively) whose self-intersection and intersection numbers may be calculated. Our device for doing so is the following lemma, which is similar to [14, Lemma 4.1] and so its proof is omitted. A similar construction may be found at the bottom of page 86 of [4].

If \( D_A \subset A \) is an immersed 2-disk, let \( \mu(A)|_{D_A} \) denote the contribution to \( \mu(A) \) due to self-intersection points on \( D_A \).
Lemma 2.7. Suppose that $Y$ is a codimension-0 submanifold of $S^4$ and $\pi_1(Y)$ is abelian. Let $B$ be a properly immersed 2-disk or 2-sphere in $Y$, and suppose $T$ is an embedded torus (or punctured torus) in $Y \setminus \text{int} B$. Let $\delta_0$ be a simple, non-separating curve on $T$, let $\delta_1$ be a normal pushoff of $\delta_0$ on $T$ and let $\hat{T}$ denote the annulus on $T$ bounded by $\delta_0 \cup \delta_1$. Suppose there is a map $J : D^2 \times I \to Y$

such that $J(D^2 \times I) \cap T = J(\partial D^1 \times I) = \hat{T}$ and $J(\partial D^1 \times \{i\}) = \delta_i$ for $i = 0, 1$. Then, after a small perturbation,

$$S = (T \setminus \text{int} \hat{T}) \cup_{\delta_0 \cup \delta_1} J(D^2 \times \{0, 1\})$$

is a properly immersed, self-transverse 2-sphere (or 2-disk, respectively) in $Y$ such that

(i) $\lambda(S, B) = (1 - [\delta_2])\lambda(D, B)$ in $\mathbb{Z}[\pi_1(Y)]$, and

(ii) $\mu(S) \equiv \mu(S)|_{D'} + \mu(S)|_{D''} + (D \cdot D')|_{\delta_2}$ in $Q(Y)$ mod 2,

where $D = J(D^2 \times 0) \subset S$ is oriented consistently with and shares a whisker with $S$, $D' = J(D^2 \times 1)$, and $\delta_2$ is a dual curve to each of $\delta_0$ and $\delta_1$ on $T$ such that $\delta_2 \cap \hat{T}$ is a simple arc running from $\delta_0$ to $\delta_1$.

The hypotheses of this lemma will frequently be encountered in the following form. We have $T$ and the nullhomotopic curve $\delta_0$; we then choose an immersed 2-disk $D \subset Y$ bounded by $\delta_0$ and let $J$ be its “thickening” along a section (which is not necessarily non-vanishing) obtained by extending over the 2-disk a normal section to $\delta_0$ that is tangential to $T$.

3 Proof of Theorem 1.1

Let us first outline the steps of our proof. Up to link homotopy, one component $f(S_2^2)$ of a link map $f$ is unknotted, immersed, and in Section 3.1 we exploit this to construct a collection of mutually disjoint, embedded, framed Whitney disks $\{V_i\}$ for $f(S_2^2)$ with interiors in $S^4 \setminus f(S_2^2)$, such that each has nullhomotopic boundary in the complement of $f(S_2^2)$. We show how these disks (along with disks bounded by accessory circles for $f_-$) can be used to construct 2-sphere generators of $\pi_2(X_-)$. The algebraic intersections between $f(S_2^2)$ and these 2-spheres are then computed in terms of the intersections between $f(S_2^2)$ and the aforementioned disks.

In Section 3.2 we surger the disks $\{V_i\}$, so to exchange their intersections with $f(S_2^2)$ for intersections with $f(S_2^2)$; In this way we obtain immersed, framed Whitney disks $\{W_i\}$ for $f_-$ in $S^4 \setminus f(S_2^2)$, such that the algebraic

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2Note that we shall sometimes exclude curly brackets for a one-point set that occurs in a Cartesian product.
intersections between \( f(S^2) \) and \( W_i \), measured by \( \omega_-(f) \), are related to the algebraic intersections between \( f(S^2) \) and \( V_i \).

In Section 3.3 we complete the proof by combining the results of these two sections: the intersections between \( f(S^2) \) and generators of \( \pi_2(X_-) \) of the former section are related to \( \sigma_+(f) \), which by the latter section can be related to \( \omega_-(f) \).

3.1 Unknotted immersions and Whitney disks in \( X_- \)

A notion of unknottedness for surfaces in 4-space was introduced by Hosokawa and Kawauchi in [7]. A connected, closed, orientable surface in \( \mathbb{R}^4 \) is said to be unknotted if it bounds an embedded 3-manifold in \( \mathbb{R}^4 \) obtained by attaching (3-dimensional) 1-handles to a 3-ball. They showed that by attaching (2-dimensional) 1-handles only, any embedded surface in \( \mathbb{R}^4 \) can be made unknotted. Kamada [8] extended their definition to immersed surfaces in \( \mathbb{R}^4 \) and gave a notion of equivalence for such immersions. In that paper it was shown that an immersed 2-sphere in \( \mathbb{R}^4 \) can be made equivalent, in this sense, to an unknotted, immersed one by performing (only) finger moves.

It was noted in [14] that we may perform a link homotopy to “unknot” one immersed component of a link map (see Lemma 3.2). The algebraic topology of the complement of this unknotted (immersed) component is greatly simplified, making the computation of the invariants defined in the previous section more tractable.

Let us begin with a precise definition of an unknotted, immersed 2-sphere. To do so we construct cusp regions which have certain symmetry properties; our justification for these specifications is to follow. Using the moving picture method, Figures 1(a), (b) (respectively) illustrate properly immersed, oriented 2-disks \( D^+ \), \( D^- \) (respectively) in \( \mathbb{R}^4 \), each with precisely one double point \( r^+, r^- \) (respectively) of opposite sign. In those figures we have indicated coordinates \( (x_1, x_2, x_3, x_4) \) of \( \mathbb{R}^4 \); our choice of the \( x_2 \)-ordinate to represent “time” is a compromise between ease of illustration and ease of subsequent notation.

As suggested by these figures, we construct \( D^\pm \) so that it has boundary \( \partial D^\pm = \partial D^2 \times 0 \times 0 \) and so that it intersects \( D^1 \times 0 \times D^1 \times D^1 \) in an arc lying in the plane \( D^1 \times 0 \times D^1 \times 0 \). Further, letting \( \theta^\pm \) denote the loop on \( D^\pm \) in this plane that is based at \( r^\pm \) and oriented as indicated in those figures, we have that the reverse loop \( \overline{\theta^\pm} \) is given by

\[
\Sigma \circ \theta^\pm = \overline{\theta^\pm},
\]

where \( \Sigma \) is the orientation-preserving self-diffeomorphism of \( \mathbb{R}^4 \) given by

\[\text{More precisely, we can choose a parameterization } \theta^\pm : I \to D^1 \times 0 \times D^1 \times 0 \text{ and paths } \theta_i^\pm : I \to D^1, i = 1, 3, \text{ so that for } t \in I \text{ we have } \theta^\pm(t) = (\theta_1^\pm(t), 0, \theta_3^\pm(t), 0) \text{ and hence } \Sigma(\theta^\pm(t)) = (-\theta_1^\pm(t), 0, \theta_3^\pm(t), 0) = \overline{\theta^\pm}(t).\]
(a) The immersed 2-disk $D^+$ with one double point $r^+$.

(b) The immersed 2-disk $D^-$ with one double point $r^-$. 

Figure 1

$(x_1, x_2, x_3, x_4) \mapsto (-x_1, -x_2, x_3, x_4)$. Lastly, we may suppose that

\[ \Sigma(D^\pm) = D^\pm. \]  \hspace{1cm} (3.2)

After orienting $D^4$ appropriately, the immersed 2-disk $D^+$ ($D^-$, respectively) has a single, positively (negatively, respectively) signed double point $r^+$ ($r^-$, respectively), and $\theta^\pm$ is an oriented loop on $D^\pm$ based at $r^\pm$ which changes branches there. (Note also that in this construction we may suppose that $D^-$ is the image of $D^+$ under the orientation-reversing self-diffeomorphism of $D^4$ given by $(x_1, x_2, x_3, x_4) \mapsto (-x_1, x_2, x_3, x_4)$.) We call $D^+$ and $D^-$ *cusps*.

Roughly speaking, an unknotted immersion is obtained from an unknotted embedding by “grafting on” cusps of this form. The purpose of the above specifications is so that, by a manoeuvre resembling the Disk Theorem, we may more conveniently move these cusps around on the 2-sphere so that accessory circles of the form $\theta^\pm$ are permuted and perhaps reversed as oriented loops.

Formally, let $d \geq 0$, let $\varepsilon : \{1, 2, \ldots, d\} \to \{+, -\}$ be a map which associates a + sign or a - sign to each $i \in \{1, 2, \ldots, d\}$, and write $\varepsilon_i = \varepsilon(i)$. Let $U$ denote the image of an oriented, unknotted embedding $S^2 \to S^4$; that is, an embedding that extends to the 3-ball (which is unique up to ambient isotopy). Suppose there are a collection of mutually disjoint, equi-oriented embeddings $b_i : D^4 \to S^4$, $i = 1, \ldots, d$, such that

\[ b_i^{-1}(U) = D^2 \times 0 \times 0 \]

for each $1 \leq i \leq d$. By removing the interiors of the 2-disks $\{b_i(D^2 \times 0 \times 0)\}_i$ from $U$ and attaching, for each $1 \leq i \leq d$, the cusp $b_i(D^2)$ along $b_i(\partial D^2 \times 0 \times 0)$,
we obtain an unknotted, immersed 2-sphere in $S^4$:

$$U_\varepsilon = [U \setminus \bigcup_{i=1}^d \text{int } b_i(D^4)] \cup \bigcup_{i=1}^d b_i(D^{\varepsilon_i}). \quad (3.3)$$

Note that we use the function $\varepsilon$ in (3.3) only for convenience of notation in the proofs that follow. Since the embeddings $\{b_i\}$ can always be relabeled, one sees that the definition of $U_\varepsilon$ depends precisely on the choice of unknotted 2-sphere $U$, the embeddings $\{b_i\}$, and two non-negative integers $d_+$ and $d_-$, where $d_\pm$ is the number of $1 \leq i \leq d$ such that $\varepsilon_i = \pm$.

The following lemma will allow us to perform an ambient isotopy of $S^4$ which carries the model $U_\varepsilon$ back to itself such that accessory circles are permuted (and perhaps reversed in orientation) in a prescribed manner.

**Lemma 3.1.** Let $\rho$ be a permutation of $\{1, 2, \ldots, d\}$. For each $1 \leq i \leq d$, let $\mu_i \in \{-1, 1\}$. There is an ambient isotopy $\hat{\varphi} : S^4 \times I \rightarrow S^4$ such that $\hat{\varphi}_1$ fixes $U_\varepsilon \setminus \bigcup_{i=1}^d \text{int } b_i(D^4)$ set-wise and, for each $1 \leq i \leq d$,

$$\hat{\varphi}_1 \circ b_i(x, y) = b_{\rho(i)}(\mu_i x, y)$$

for all $(x, y) \in D^2 \times D^2$. In particular, if $\varepsilon_{\rho(i)} = \varepsilon_i$, then $\varphi_1(U_\varepsilon \cap b_i(D^4)) = U_\varepsilon \cap b_{\rho(i)}(D^4)$ and

$$\hat{\varphi}_1 \circ b_i \circ \theta_{\varepsilon_i} = \begin{cases} b_{\rho(i)} \circ \theta_{\varepsilon_i} & \text{if } \mu_i = 1, \\ b_{\rho(i)} \circ \theta_{-\varepsilon_i} & \text{if } \mu_i = -1. \end{cases}$$

The proof consists of using the Disk Theorem [12, Corollary 3.3.7] to transport 4-ball neighborhoods of the cusps around the 2-sphere, and is deferred to Appendix A. We proceed instead to apply the lemma to equip one component of a link map, viewed as an immersion into the 4-sphere, with a particularly convenient collection of mutually disjoint, embedded, framed Whitney disks.

For this purpose it will be useful to give a particular construction of an unknotted immersion of a 2-sphere in $S^4$ with $d \geq 0$ pairs of opposite-signed double points.

### 3.1.1 A model, unknotted immersion

For $A \subset \mathbb{R}^3$ and real numbers $a < b$, write $A[a, b] = A \times [a, b] \subset \mathbb{R}^3 \times \mathbb{R}$, and $A[a] = A \times a$. Choose an increasing sequence

$$0 = t_1^- < t_1^+ < t_2^- < t_2^+ < \cdots < t_{d-1}^- < t_{d-1}^+ < t_d^- < t_d^+ = 1,$$

and for each $1 \leq i \leq d$, write $I_i = [t_i^-, t_i^+]$ and let $t_i = (t_i^+ + t_i^-)/2$. On the unit circle, oriented clockwise, let $x^+, x^-, y^-$ be distinct, consecutive points
and let $D_1^+, D_1^-$ be disjoint neighborhoods of $\{x^+, y^+\}, \{x^-, y^-\}$, respectively. Let $\hat{\alpha}^\pm : I \to D_1^\pm$ be a path in $D_1^\pm$ running from $x^\pm$ to $y^\pm$; let $\eta_x$ and $\eta_y$ be simple paths on $S^1$ running $x^+$ to $x^-$ and from $y^-$ to $y^+$, respectively. Let $\hat{\eta}_x$ and $\hat{\eta}_y$ be disjoint neighborhoods of $\eta_x$ and $\eta_y$ in $S^1$, respectively. See Figure 2.

For each $1 \leq i \leq d$, let $\hat{\Theta}_i : D^1 \to I_i$ be a linear map such that $\hat{\Theta}_i(0) = t_i$ and $\hat{\Theta}_i(\pm 1) = t_i^\pm$, and let $\Theta_i : D^3 \times D^1 \to D^3 \times I_i$ be the map $\Theta_i(x, t) = (x, \hat{\Theta}_i(t))$. Let $G : S^1 \times D^1 \to D^3 \times D^1$ be an oriented, self-transverse immersion with image as shown in Figure 3 (ignoring the shadings), with two double-points $p^\pm = G(x^\pm, 0) = G(y^\pm, 0)$, such that $G(x, t) \subset D^3[t]$ for each $x \in S^1$ and $t \in D^1$. Then $\alpha^\pm = G(\hat{\alpha}^\pm \times 0)$ is an oriented loop on $G(S^1 \times D^1)$ based at $p^\pm$ which changes branches there. Note that $G$ is the trace of a regular homotopy from the circle in $D^3$ to itself that Figure 3 illustrates.

For each $1 \leq i \leq d$, define a map $G_i : S^1 \times I_i \to D^3 \times I_i$ by

$$G_i(\Theta_i(x, t)) = \Theta_i(G(x, t))$$

for $(x, t) \in S^1 \times D^1$. Write the 2-sphere as the capped off cylinder

$$S^2 = (D^2 \times \{-1, 1\}) \cup \bigcup_{i=1}^d \Theta_i(S^1 \times I)$$

in $D^3 \times D^1$, and define a map $\hat{u}_d : S^2 \to D^3 \times D^1 \subset S^4$ by the identity on $D^2 \times \{-1, 1\}$ and by $G_i$ on $S^1 \times I_i = \Theta_i(S^1 \times D^1)$.

After smoothing corners, $\hat{U}_d = \hat{u}_d(S^2)$ is an immersed 2-sphere in $S^4$. Let $\alpha_i^\pm$ be the oriented loop on $\hat{U}_d$ given by $\alpha_i^\pm(s) = \Theta_i(G(\hat{\alpha}^\pm(s), 0)) = \Theta_i(\alpha_i^\pm(s))$ for $s \in I$. Observe that $\alpha_i^\pm$ is based at the $\pm$-signed double point $p_i^\pm = \Theta_i(G(x^\pm, 0)) = \Theta_i(G(y^\pm, 0))$ and changes branches there.
Referring to Figure 3, let $V[0] \subset D^3[0]$ ($V \subset D^3$) be the obvious, embedded Whitney disk for the immersed annulus $G(S^1 \times D^1)$ in $D^3 \times D^1$, bounded by $G((\eta_x \cup \eta_y) \times 0)$. For arbitrarily small $\varepsilon > 0$, by pushing a neighborhood of $G(\eta_y \times 0)$ in $G(\eta_y \times (-\varepsilon, \varepsilon))$ into $D^3 \times 0$ (as in Figure 4), we may assume that the Whitney disk is framed: the constant vector field $(0, 0, 1, 0)$ that points out of the page in each hyperplane $R^3[t_0]$ and the constant vector field $(0, 0, 0, 1)$ are a correct framing. Thus the maps $\{\Theta_i\}_{i=1}^d$ carry $V[0]$ to a complete collection of mutually disjoint, embedded, framed Whitney disks $V_i \subset R^3[t_i]$, $i = 1, \ldots, d$, for $\hat{U}_d$ in $S^4$. In particular, the boundary of $V_i$ (equipped with an orientation) is given by

$$\partial V_i = \Theta_i(G((\eta_x \cup \eta_y) \times 0)) = G_i(\eta_x \times 0) \cup G_i(\eta_y \times 0).$$

By Lemma 4.2 of [14], we may assume after a link homotopy that one component of a link map is of the form of this model of an unknotted immersion.

**Lemma 3.2.** A link map $f$ is link homotopic to a good link map $g$ such that $g(S^2) = \hat{U}_d$ for some non-negative integer $d$. \qed

We proceed to generalize the proof of [14 Lemma 4.4] to construct representatives of a basis of $\pi_2(X_-)$ and compute their algebraic intersections with $f(S^2)$ in terms of the algebraic intersections between $f(S^2)$ and $\{V_i\}_i$. While parts (i) and (iii) of Proposition 3.3 may be deduced directly from that paper, we include the complete proof for clarity.

**Proposition 3.3.** Let $f$ be a good link map such that $f(S^2) = \hat{U}_d$. Equip $f(S^2)$ with a whisker in $X_-$ and fix an identification of $\pi_1(X_-)$ with $\mathbb{Z}[s]$ so as to write $\mathbb{Z}[\pi_1(X_-)] = \mathbb{Z}[s, s^{-1}]$. Then $\pi_2(X_-) \cong \mathbb{Z}[s, s^{-1}]$ and there is a $\mathbb{Z}[s, s^{-1}]$-basis represented by mutually disjoint, self-transverse, immersed, whiskered 2-spheres $\{A^+_i, \hat{A}^-_i\}_{i=1}^d$ in $X_-$ with the following properties. For each $1 \leq i \leq d$, there is an integer Laurent polynomial $q_i(s) \in \mathbb{Z}[s, s^{-1}]$ such that

(i) $n_i := q_i(1) = \text{lk}(f(S^2), \alpha^+_i)$,

(ii) $\lambda(A^+_i, \hat{A}^-_i) = s + s^{-1}$ mod 2,

(iii) $\lambda(f(S^2), A^+_i) = (1 - s)^2 q_i(s)$, and

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(iv) \( \lambda(f(S_2^3), A_i^-) - \lambda(f(S_2^3), A_i^+) = (1 - s)^2 \lambda(f(S_2^3), V_i^c) \),

where \( V_i^c \) is a 2-disk in \( X_- \) obtained from \( V_i \) by removing a collar in \( X_+ \), and \( A_{i+}^j \) is the image of a section of the normal bundle of \( A_{i+}^j \). Moreover, if for any \( 1 \leq j \leq d \) and \( \varepsilon \in \{-, +\} \) the loop \( \alpha_i^\varepsilon \) bounds a 2-disk in \( S^4 \) that intersects \( f(S_2^3) \) exactly once, then we may choose \( A_i^\varepsilon \) so that

\[
\lambda(f(S_2^3), A_i^\varepsilon) = (1 - s)^2.
\]

**Proof.** For each \( 1 \leq i \leq d \), the double points \( p_i^+ \) and \( p_i^- \) of \( f(S_2^3) \) lie in \( D^3[t_i] \) and \( f(S_2^3) \) intersects the 4-ball \( D^3[t_i^-, t_i^+] \subset D^3 \times D^1 \) precisely along \( \Theta_i(G(S^1 \times D^1)) \). In what follows we shall denote \( \hat{G} = G(S^1 \times D^1) \); note that for \( t \in D^1 \), \( G \cap (D^3[t]) = G(S^1 \times t) \) and \( \Theta_i(G(S^1 \times t)) = f(S_2^3) \cap (D^3[\hat{\Theta}_i(t)]) \). Observe that by the construction of the annulus \( G \) we may assume there are integers \(-1 < a^- < b^- < 0 < a^+ < b^+ < 1 \) such that \( G(S^1 \times [a^\pm, b^\pm]) = G_0[a^\pm, b^\pm) \) for some (embedded circle) \( G_0 \subset D^3 \).

Let \( E^\pm \) denote the 2-disk bounded by \( a^\pm \) in \( D^3[0] \), and choose a 3-ball \( N^\pm \) so that \( N^\pm[-1, 1] \) is a 4-ball neighborhood of \( p^\pm \) and \( \Theta_i(N^\pm[-1, 1]) \) is disjoint from \( f(S_2^3) \). There is an embedded torus \( T^\pm \) in \( N^\pm[-1, 1] \) \( \backslash \hat{G} \) that intersects \( E^\pm \) exactly once; see Figure 5 for an illustration of \( T^+ \) and \( T^- \) in \( D^3 \times D^1 \). The torus \( T^\pm \) appears as a cylinder in each of 3-balls \( N^\pm[-1] \) and \( N^\pm[1] \), and as a pair of circles in \( N^\pm[t] \) for \( t \in (-1, 1) \). For each \( 1 \leq i \leq d \), let

![Figure 5](image)

\( T_i^\pm = \Theta_i(T^\pm) \subset N^\pm[t_i^-, t_i^+] \backslash f(S_2^3) \). By Alexander duality the linking pairing

\[
H_2(X_-) \times H_1(f(S_2^3)) \rightarrow \mathbb{Z}
\]

defined by \( (R, v) \mapsto R \cdot \gamma \), where \( v = \partial \gamma \subset S^4 \), is nondegenerate. Thus, as the loops \( \{\alpha_i^+, \alpha_i^-\} \) represent a basis for \( H_1(f(S_2^3)) \cong \mathbb{Z}^{2d} \), we have that \( H_2(X_-) \cong \mathbb{Z}^{2d} \) and (after orienting) the so-called linking tori \( \{T_i^+, T_i^-\} \) represent a basis. We proceed to apply the construction of Section 2.3 (twice, successively) to turn these tori into 2-spheres.

Let \( \hat{\Delta}^\pm \subset N^\pm \) be the embedded 2-disk so that \( \hat{\Delta}^\pm[a^\pm] \) appears in \( N^\pm[a^\pm] \) as in Figure 6, and let \( \hat{\delta}^\pm = \partial \hat{\Delta}^\pm \). The disk \( \hat{\Delta}^\pm[a^\pm] \) intersects \( G \) at two points, which are the endpoints of an arc of the form \( \gamma^\pm[a^\pm] (\gamma^\pm \subset G_0) \) in \( G(S^1 \times a^\pm) \), also illustrated.
In Figure 6 we have also illustrated in \( D^3[a^\pm] \) the restriction of a tubular neighborhood of \( \hat{G} \) to \( \gamma^\pm[a^\pm] \), which we may write in the form \((\gamma^\pm \times D^2)[a^\pm]\) and choose so that \( \Theta_1 \) carries \((\gamma^\pm \times D^2)[a^\pm]\) to a tubular neighborhood of \( f(S^2_1) \) restricted to \( \Theta_1(\gamma^\pm[a^\pm]) \) in \( S^4 \setminus f(S^2_1) \). Let \( \Delta^\pm \) be the embedded punctured torus in \( D^3 \) obtained from \( \hat{\Delta}^\pm \) by attaching a 1-handle along \( \gamma^\pm \); that is, let

\[
\Delta^\pm = \left[ \hat{\Delta}^\pm \setminus (\partial \gamma \times \text{int } D^2) \right] \cup (\gamma^\pm \times S^1).
\]

Note that \( \Delta^\pm \) has boundary \( \delta^\pm \).

Let \( \beta^\pm \) be a loop on \( \Delta^\pm \) formed by connecting the endpoints of a path on \( \gamma^\pm \times S^1 \) by an arc on \( \hat{\Delta}^\pm \setminus (\partial \gamma \times \text{int } D^2) \) so that \( \beta^\pm[a^\pm] \) links \( G(S^1 \times a^\pm) \) zero times; see Figure 7. Let \( \hat{\beta}^\pm \) be a pushoff of \( \beta^\pm \) along a normal vector field tangent to \( \Delta^\pm \). We see that \( \hat{\beta}^\pm \) and \( \hat{\beta}^\pm[a^\pm] \) bound embedded 2-disks \( F^\pm \) and \( \hat{F}^\pm \) in \( D^3 \), respectively, which are disjoint (that is to say, the aforementioned normal vector field extends to a normal vector field of \( F^\pm \)). Then \( \beta^\pm[a^\pm] \) and \( \hat{\beta}^\pm[a^\pm] \) bound the disjoint, embedded 2-disks \( F^\pm[a^\pm] \) and \( \hat{F}^\pm[a^\pm] \), respectively, which lie in \( D^3[a^\pm] \setminus G(S^1 \times a^\pm) \).

Since \( f(S^2_1) \) is disjoint from \( \Theta_1(N^+[-1,1]) \) and (we may assume) from a collar of the 2-disk \( \Theta_1(F^+[a^\pm]) \), observe that \( F^+ \) may be constructed from a normal pushoff \( \hat{E}^+ \) of \( E^+ \) in \( D^3 \) by attaching a collar so that the intersections
between $\Theta_i(F^+[a^+])$ and $f(S^2_+) \cap E^+_i$ occur entirely on $\Theta_i(\hat{E}^+[a^+])$. Similarly, since $f(S^2_+) \cap E^+_i$ is disjoint from $\Theta_i(N^+[−1, 1])$ and $\Theta_i(N^-[−1, 1])$, and (we may assume) from a collar of the 2-disk $\Theta_i(F^−[a^-])$, the 2-disk $F^−$ may be chosen to contain a normal pushoff $\hat{V}$ of $V$ in $D^3$ and a normal pushoff $\hat{E}^+$ of $E^+$ in $D^3$ so that the intersections between $\Theta_i(F^−[a^-])$ and $f(S^2_+) \cap E^+_i$ occur entirely on $\Theta_i(\hat{E}^+[a^+])$ and $\Theta_i(\hat{V}[a^-])$. From these observations, by Proposition 2.2 we have (by an appropriate choice of whiskers and orientations in $X_-$)
\[
\lambda(f(S^2_+), \Theta_i(F^+[a^+])) = \lambda(f(S^2_+), \Theta_i(\hat{E}^+[a^+])) \tag{3.6}
\]
and
\[
\lambda(f(S^2_+), \Theta_i(F^−[a^-])) = \lambda(f(S^2_+), \Theta_i(\hat{E}^+[a^+])) + \lambda(f(S^2_+), \Theta_i(\hat{V}[a^-])) \tag{3.7}
\]
in $\mathbb{Z}[s, s^{-1}]$. But, as $\Theta_i(\hat{E}^+[a^+])$ and $\Theta_i(\hat{E}^+[a^-])$ are each normal pushoffs in $X_-$ of $E^+_i := \Theta_i(E^+[0])$, the 2-disk bounded by $a^+_i$, we deduce from Equation (3.6) that
\[
\lambda(f(S^2_+), \Theta_i(F^+[a^+])) = q_i^{(1)}(s) \tag{3.8}
\]
for some integer Laurent polynomial $q_i^{(1)}(s) \in \mathbb{Z}[s, s^{-1}]$ such that
\[
q_i^{(1)}(1) = f(S^2_+) \cdot E^+_i = \text{lk}(f(S^2_+), a^+_i). \tag{3.9}
\]
Moreover, if $|f(S^2_+) \cap E^+_i| = 1$ for some $i$, then (as we are free to choose the orientation and whisker of $\Theta_i(F^+[a^+])$) we may take $\hat{q}_i^{(1)}(s) = 1$. Similarly, as $\Theta_i(\hat{V}[a^-])$ is a normal pushoff of $V_i$, from Equation (3.7) we have (by an appropriate choice of orientations and whiskers in $X_-$)
\[
\lambda(f(S^2_+), \Theta_i(F^−[a^-])) = \hat{q}_i^{(1)}(s) + \lambda(f(S^2_+), V_i^-), \tag{3.10}
\]
where $V_i^- \subset X_-$ is obtained from $V_i$ (which has boundary on $f(S^2_+)$) by removing a collar in $S^4 \setminus f(S^2_+)$.  

From the (embedded) punctured torus $\Delta^\pm$, we remove the interior of the annulus bounded by $\beta^\pm \cup \hat{\beta}^\pm$ and attach the 2-disks $F^\pm \cup \hat{F}^\pm$. We thus obtain an embedded 2-disk $\hat{A}^\pm \subset D^3$ which has boundary $\delta^\pm$ and is such that $\hat{A}^\pm[a^\pm]$ lies in $D^3[a^\pm] \setminus G(S^1 \times a^\pm)$. Consequently, $\Theta_i(\hat{A}^\pm[a^\pm])$ is an embedded 2-disk in $X_-$ with boundary $\Theta_i(\delta^\pm[a^\pm]), \Theta_i(\Delta^\pm[a^\pm]) \subset X_- \setminus f(S^2_+)$ by applying the construction of Lemma 2.7 with the 2-disk $\Theta_i(F^\pm[a^\pm])$ and its normal pushoff $\Theta_i(\hat{F}^\pm[a^\pm])$.  

Now, for an interior point $z^\pm$ on $\gamma^\pm$, the loop $z^\pm \times S^1 \subset \gamma^\pm \times S^1$ is dual to $\beta^\pm$ and $\hat{\beta}^\pm$ on $\Delta^\pm$, and is meridinal to $\hat{G}$ in $D^3 \times D^1$. Thus the loop $\Theta_i(z^\pm \times S^1)$ is dual to $\partial \Theta_i(F^\pm[a^\pm])$ and $\partial \Theta_i(F^\pm[a^\pm])$ on $\Theta_i(\Delta^\pm[a^\pm]), \Theta_i(\Delta^\pm[a^\pm])$, and represents $s$ or $s^{-1}$ in $\pi_1(X_-) = \mathbb{Z}(s)$. By Lemma 2.7(i) and Equations (3.8)-(3.10), then, we have (by an appropriate choice of orientation and whisker in $X_-$)
\[
\lambda(f(S^2_+), \Theta_i(\hat{A}^\pm[a^\pm])) = (1 - s)q_i^{(2)}(s) \tag{3.11}
\]
\[ \lambda(f(S^2_+), \Theta_i(\hat{\Delta}^\pm[a^\pm])) = (1 - s)q_i^{(2)}(s) + (1 - s)\lambda(f(S^2_+), V^c_i) \]  

for some integer Laurent polynomial \( q_i^{(2)}(s) \in \mathbb{Z}[s, s^{-1}] \) such that \( q_i^{(2)}(1) = q_i^{(1)}(1) \).

We proceed to attach an annulus to each of the 2-disks \( \hat{\Delta}^\pm[a^\pm] \) and \( \hat{\Delta}^\pm[b^\pm] \) so to obtain 2-disks with which to surger the linking torus \( T^\pm \) using the construction of Lemma 2.7.

In Figure 8 we have illustrated an oriented circle \( \delta^+_a \) on \( T^\pm \subset N^\pm[-1, 1] \) which intersects \( D^3[-1] \) and \( D^3[1] \) each in an arc, and as a pair of points in \( D^3[t] \) for \( t \in (-1, 1) \). We have also illustrated a normal pushoff \( \delta^+_b \) of \( \delta^+_a \) on \( T^\pm \). Let \( \hat{T}^\pm \) denote the annulus on \( T^\pm \) bounded by \( \delta^+_a \cup \delta^+_b \). Notice that the pair \( \delta^+_a \cup \delta^+_b \) is isotopic in \( N^\pm[-1, 1] \) to the link \( \delta^+_a' \cup \delta^+_b' \) in \( N^\pm[-1, 1] \) which we have also illustrated in Figure 8. We then see that the annulus \( \hat{T}^\pm \) is homotopic in \( N^\pm[-1, 1] \) to the annulus \( \delta^\pm[a^\pm, b^\pm] \) by a homotopy of \( S^1 \times I \) whose restriction to \( S^1 \times \{0, 1\} \) is a homotopy from \( \delta^+_a \cup \delta^+_b \) to the link \( \delta^\pm[a^\pm, b^\pm] \) through a sequence of links, except for one singular link where the two components pass through each other. That is, we can find a regular homotopy \( K^\pm : (S^1 \times I) \times I \to N^\pm[-1, 1] \) such that

1. \( K^\pm_0(S^1 \times I) = \hat{T}^\pm, \ K^\pm_0(S^1 \times 0) = \delta^+_a, \ K^\pm_0(S^1 \times 1) = \delta^+_b; \)
2. \( K^\pm_1(S^1 \times I) = \delta^\pm[a^\pm, b^\pm], \ K^\pm_1(S^1 \times 0) = \delta^\pm[a^\pm], \ K^\pm_1(S^1 \times 1) = \delta^\pm[b^\pm]; \)

and

3. \( K^\pm_t(S^1 \times \{0, 1\}) \) is a link for all \( t \in I \) except at one value \( t' \), where \( K^\pm_{t'}(S^1 \times \{0, 1\}) \) has precisely one transverse self-intersection point, arising as an intersection point between \( K^\pm_{t'}(S^1 \times 0) \) and \( K^\pm_{t'}(S^1 \times 1) \).

Denote this intersection point by \( w^\pm \).

We may further suppose that the image of the homotopy \( K^\pm \) lies in a tubular neighborhood \( \approx (S^1 \times I) \) of \( \delta^+_a \) in \( N^\pm[-1, 1] \). Hence, in particular, we may assume that for each \( t \in (0, 1) \) the image of \( K^\pm_t \) is disjoint from each of \( \hat{\Delta}^\pm[a^\pm] \) and \( \hat{\Delta}^\pm[b^\pm] \).

Attaching the annuli \( \hat{\Delta}^\pm((S^1 \times 0) \times I) \) and \( \hat{\Delta}^\pm((S^1 \times 1) \times I) \) to the 2-disks \( \hat{\Delta}^\pm[a^\pm] \) and \( \hat{\Delta}^\pm[b^\pm] \) along \( \delta^\pm[a^\pm] = K^\pm(S^1 \times 0 \times 1) \) and \( \delta^\pm[b^\pm] = K^\pm(S^1 \times 1 \times 1) \), respectively, we obtain embedded 2-disks

\[ \Omega^\pm_a = \hat{\Delta}^\pm[a^\pm] \cup_{\delta^\pm[a^\pm]} K^\pm((S^1 \times 0) \times I) \]

and

\[ \Omega^\pm_b = \hat{\Delta}^\pm[b^\pm] \cup_{\delta^\pm[b^\pm]} K^\pm((S^1 \times 1) \times I) \]

in \( D^4 \setminus \hat{G} \). Observe that their respective boundaries \( \partial \Omega^\pm_a = \delta^+_a \) and \( \partial \Omega^\pm_b = \delta^+_b \) lie on \( T^\pm \).
By construction, \( \Omega^\pm_a \) and \( \Omega^\pm_b \) intersect precisely once, transversely, at the intersection point \( w^\pm \) between \( K^\pm((S^1 \times 0) \times I) \) and \( K^\pm((S^1 \times 1) \times I) \). Also, since the image of \( K^\pm \) lies in \( N^\pm[-1, 1] \), the intersections between \( \Theta_i(\Omega^\pm_a) \) and \( f(S^2_2) \) lie precisely on \( \Theta_i(\hat{A}^\pm[a^\pm]) \). Consequently, by Proposition 2.2 we have (after an appropriate choice of orientations and whiskers in \( X^- \))

\[
\lambda(f(S^2_2), \Theta_i(\Omega^\pm_a)) = \lambda(f(S^2_2), \Theta_i(\hat{A}^\pm[a^\pm])).
\]  

(3.15)

Now, let \( A^\pm \) be the 2-sphere in \( D^4 \setminus \hat{G} \) obtained by removing from the (embedded) torus \( T^\pm \) the interior of the annulus \( \hat{T}^\pm \) (bounded by \( \delta^+_a \cup \delta^+_b \)) and attaching the embedded 2-disks \( \Omega^\pm_a \cup \Omega^\pm_b \). Then \( A^\pm \) is immersed and self-transverse in \( D^4 \), with a single double point: the point \( w^\pm \).

Wishing to apply Lemma 2.7, define a homotopy from \( \Omega^\pm_a \) to \( \Omega^\pm_b \) as follows. Let \( c : S^1 \times I \to D^2 \) be a collar with \( c(S^1 \times 1) = \partial D^2 \), and let \( F_{\hat{A}^\pm} : D^2 \to D^3 \) be an embedding with image \( \hat{A}^\pm \). Since \( K^\pm(S^1 \times I) \) and \( \hat{A}^\pm[a, b] \) each lie in \( D^4 \setminus \hat{G} \), from Equations (3.13)-(3.14) it is readily seen that

\[
J^\pm_t(c(x, s)) = K^\pm(x, t, s) \quad \text{for } (x, s) \in S^1 \times I,
\]

\[
J^\pm_t(y) = F_{\hat{A}^\pm}(y)[a^\pm + t(b^\pm - a^\pm)] \quad \text{for } y \in D^2,
\]

defines a homotopy \( J^\pm : D^2 \times I \to D^4 \setminus \hat{G} \) from \( \Omega^\pm_a \) to \( \Omega^\pm_b \). Then

\[
A^\pm = (T^\pm \setminus \text{int } \hat{T}^\pm) \cup_{\delta^+_a \cup \delta^+_b} J^\pm(D^2 \times \{0, 1\}).
\]  

(3.16)
Now, let \( A_i^{\pm} = \Theta_i(A^{\pm}) \); then \( A_i^{\pm} \) is an immersed, self-transverse 2-sphere in \( D^3[\hat{t}^{\pm}, t^{\pm}] \setminus f(S^2_i) \) constructed by surgering the embedded torus \( T_\pm \subset X_- \setminus f(S^2_i) \) using the 2-disks \( \Theta_i(\Omega^+_0) \cup \Theta_i(\Omega^-_0) \subset X_- \). Furthermore, observe that a dual curve to \( \delta^+_a \) and \( \delta^-_b \) on the torus \( T_\pm \) is meridinal to \( \hat{G} \) in \( D^4 \), so a dual curve to \( \Theta_i(\delta^+_a) \) and \( \Theta_i(\delta^-_b) \) on the linking torus \( \Theta_i(T_\pm) \) represents \( s \) or \( s^{-1} \) in \( \pi_1(X_-) = \mathbb{Z}(s) \). Thus, by Equation (3.16) and Lemma 2.7(i) we have (after an appropriate choice of whiskers and orientations in \( X_- \))

\[
\lambda(f(S^2_i), A_i^+) = (1 - s)\lambda(f(S^2_i), \Theta_i(\Omega^+_0)).
\]

From Equations (3.11) and (3.15) we therefore have

\[
\lambda(f(S^2_i), A_i^+) = (1 - s)^2 q_i(s)
\]

and

\[
\lambda(f(S^2_i), A_i^-) = (1 - s)^2 q_i(s) + (1 - s)^2 \lambda(f(S^2_i), V_i^-)
\]

for some \( q_i(s) \in \mathbb{Z}[s, s^{-1}] \) such that \( q_i(1) = \text{lk}(f(S^2_i), \alpha_i^+) \). As observed earlier, if \( |f(S^2_i) \cap E_i^+| = 1 \) for some \( i \), then we may take \( q_i(s) = 1 \). Moreover, since \( \Theta_i(\Omega^+_a) \) and \( \Theta_i(\Omega^-_b) \) are embedded and intersect precisely once, we have from Lemma 2.7(ii) that

\[
\mu(A_i^{\pm}) = s \mod 2
\]

in \( \mathbb{Z}_2[s] \). Hence \( \lambda(A_i^{\pm}, A_i^{\pm}) = s + s^{-1} \mod 2 \).

Finally, by construction, \( A_i^{\pm} \) is homologous to \( T_i^{\pm} \) for each \( i \), so by Lemma 4.3 the immersed 2-spheres \( \{A_i^+, A_i^-\}_{i=1}^d \) represent a \( \mathbb{Z}[s, s^{-1}] \)-basis for \( \pi_2(X_-) \).

The rest of this section will be devoted to applying Lemma 3.1 to prove the following proposition, which will allow us to surger out the intersections between each \( V_i \) and \( f(S^2_i) \) (in exchange for intersections with \( f(S^2_i) \)).

**Proposition 3.4.** Let \( f \) be a good link map such that \( \sigma_-(f) = 0 \) and \( f(S^2_i) = \hat{U}_d \) for some \( d \geq 0 \). Then, perhaps after an ambient isotopy, we may assume that \( f(S^2_i) = \hat{U}_d \) and the embedded Whitney disks \( \{V_i\}_{i=1}^d \) in \( S^4 \) are framed and satisfy \( V_i \cdot f(S^2_i) = 0 \) for each \( 1 \leq i \leq d \).

The remainder of this section shall be devoted to proving this result. Recall from the beginning of the present section that on the immersed circle \( G(S^1 \times 0) \) in \( D^3[0] \), the arc \( G(D^1 \pm 0) \) contains the loop \( \alpha^\pm = G(\hat{\alpha}^\pm \times 0) \) in its interior.

**Lemma 3.5.** Let \( d \geq 0 \). For each \( i \in \{1, 2, \ldots, d\} \) let \( \mu_i \in \{-1, 1\} \), and let \( \varsigma \) be a permutation of \( \{1, 2, \ldots, d\} \). There are 4-ball neighborhoods \( B_i^\pm \) of \( G_i(D^1 \pm \times S^1) \) in \( S^4 \), \( i = 1, \ldots, d \), and an ambient isotopy \( \varphi : S^4 \times I \rightarrow S^4 \) such that \( \varphi_1(\hat{U}_d) = \hat{U}_d \), and for each \( 1 \leq i \leq d \),

(i) \( \varphi_1 \) restricts to the identity on \( B_i^+ \) (so \( \varphi_1 \circ \alpha_i^+ = \alpha_i^+ \)),

(ii) \( \varphi_1 \) carries \( (B_i^-, G_i(D^1 \times D^1)) \) to \( (B_i^-, G_i(D^1 \times D^1)) \) and
Lemma 3.1 yields an ambient isotopy $\phi : D^2 \times D^2 \to D^2 \times D^2$ carrying the cusp $D^2 \times D^2$ of Section 3.1 to $G(D^2 \times D^2)$, the double point $r^2$ to $p^2$, and the oriented loop $\theta^2$ to $\alpha^2$.

For each $1 \leq i \leq d$, let $\hat{\mu}_i \in \{-, +\}$ denote the sign of $\mu_i$, let $\Psi^+_i$ be the orientation-preserving diffeomorphism given by

$$
\Psi^+_i = \Theta_i \circ \Pi^+_i : D^2 \times D^2 \to B^3_1 \times I_i,
$$

and let $\overline{\Psi}^+_i = \Psi^+_i \circ \Sigma$, where $\Sigma$ is the orientation-preserving diffeomorphism of $D^2 \times D^2$ defined in Section 3.1 by $\Sigma(x, y) = (-x, y)$. Recalling Equations (3.1)-(3.2), observe that $\Psi^+_i(D^2) = G_1(D^2_1 \times D^2_1) = \overline{\Psi}^+_i(D^2)$,

$$
\Psi^+_i \circ \theta^2 = \alpha^2_i \text{ and } \overline{\Psi}^+_i \circ \theta^2 = \alpha^2_i.
$$

(3.19)

Furthermore, by construction, $\hat{U}_d$ is obtained from the unknotted, embedded 2-sphere $(S^1 \times D^2) \cup (D^2 \times \{\pm 1\}) \subset D^3 \times D^1$ by removing its intersections with the 4-balls $B^3_1 \times I_i = \Psi^+_i(D^4)$, and attaching the cusps $G_1(D^2_1 \times D^2_1) = \Psi^+_i(D^2)$, for $i = 1, 2, \ldots, d$. For each $1 \leq i \leq d$, let $b_{2i} = \Psi^+_i$, $b_{2i-1} = \overline{\Psi}^-_i$, and define a permutation $\rho$ on $\{1, 2, \ldots, 2d\}$ by $\rho(2i) = 2i$ and $\rho(2i - 1) = 2\zeta(i) - 1$. Then Lemma 3.1 yields an ambient isotopy $\varphi : S^1 \times I \to S^1$ such that $\varphi_1$ fixes

$$
\hat{U}_d \setminus \bigcup_{i=1}^{d} (b_{2i}(D^4) \cup b_{2i-1}(D^4)) = \hat{U}_d \setminus \bigcup_{i=1}^{d} ((B^3_1 \cup B^3_2) \times I_i)
$$

set-wise, satisfies

$$
\varphi_1 \circ \Psi^+_i = \varphi_1 \circ b_{2i} = b_{2i} = \Psi^+_i \quad \text{and} \quad \varphi_1 \circ \overline{\Psi}^-_i(x, y) = \varphi_1 \circ b_{2i-1}(x, y) = b_{2i-1}(\hat{\mu}_i x, y) = \overline{\Psi}^-_{\zeta(i)}(\hat{\mu}_i x, y).
$$

(3.21)

(3.22)

Now, putting $B^i_1 = B^i \times I_i$, Equation (3.21) gives part (i) of the lemma; Equation (3.22) gives part (ii), and part (iii) follows from Equation (3.19) and by noting that Equation (3.22) implies $\varphi_1 \circ \overline{\Psi}^-_i = \overline{\Psi}^-_{\zeta(i)}$ if $\mu_i = 1$ and $\varphi_1 \circ \overline{\Psi}^-_i = \overline{\Psi}^-_{\zeta(i)}$ if $\mu_i = -1$. Since $\overline{\Psi}^-_i(D^2) = \Psi^-_{\zeta(i)}(D^2)$, $\varphi_1$ sends $\bigcup_{i=1}^{d} (\Psi^+_i(D^2) \cup \Psi^-_{\zeta(i)}(D^2))$ to

$$
\bigcup_{i=1}^{d} (\Psi^+_i(D^2) \cup \overline{\Psi}^-_{\zeta(i)}(D^2)) = \bigcup_{i=1}^{d} (\Psi^+_i(D^2) \cup \overline{\Psi}^-_{\zeta(i)}(D^2)),
$$

so $\varphi_1(\hat{U}_d) = \hat{U}_d$ by Equation (3.20).
We may now perform an ambient isotopy which carries \( f(S_+^2) = \hat{U}_d \) back to itself in such a way that the accessory circles \( \{ \alpha_1^+, \alpha_1^- \}_{i=1}^d \) are rearranged into canceling pairs with respect to their linking numbers with \( f(S_+^2) \).

**Lemma 3.6.** Let \( f \) be a link map such that \( \sigma_- (f) = 0 \) and \( f(S_+^2) = \hat{U}_d \) for some \( d \geq 0 \). Then \( f \) is link homotopic (in fact, ambient isotopic) to a link map \( g \) such that \( g(S_+^2) = \hat{U}_d \) and, for each \( 1 \leq i \leq d \),

\[
\text{lk}(\alpha_i^+, g(S_+^2)) = \text{lk}(\alpha_i^-, g(S_+^2)).
\]

**Proof.** Since \( \sigma_- (f) = 0 \), there is a function \( \mu : \{1, 2, \ldots, d\} \rightarrow \{-1, 1\} \) and a permutation \( \varsigma \) on \( \{1, 2, \ldots, d\} \) such that

\[
\text{lk}(\alpha_i^+, f(S_+^2)) = \mu_i \cdot \text{lk}(\alpha_{\varsigma^{-1}(i)}^-, f(S_+^2))
\]

for each \( 1 \leq i \leq d \). By Lemma 3.5, there is an ambient isotopy \( \varphi : S^4 \times I \rightarrow S^4 \) such that \( \varphi_1(U_d) = \hat{U}_d \) and, for each \( 1 \leq i \leq d \), \( \varphi_1 \circ \alpha_i^+ = \alpha_i^+ \), \( \varphi_1 \circ \alpha_i^- = \alpha_{\varsigma(i)}^- \) if \( \mu_i = 1 \), and

\[
\varphi_1 \circ \alpha_i^- = \alpha_{\varsigma(i)}^-
\]

if \( \mu_i = -1 \). Then, for each \( 1 \leq i \leq d \),

\[
\varphi_1^{-1}(\alpha_i^-) = \begin{cases} 
\alpha_{\varsigma^{-1}(i)}^- & \text{if } \mu_i = 1, \\
\alpha_{\varsigma^{-1}(i)}^+ & \text{if } \mu_i = -1,
\end{cases}
\]

and hence

\[
\text{lk}(\varphi_1^{-1}(\alpha_i^-), f(S_+^2)) = \mu_i \cdot \text{lk}(\alpha_{\varsigma^{-1}(i)}^-, f(S_+^2)) = \text{lk}(\alpha_i^+, f(S_+^2)).
\]

Thus, taking \( g = \varphi_1 \circ f \), we have

\[
\text{lk}(\alpha_i^-, g(S_+^2)) = \text{lk}(\varphi_1(\varphi_1^{-1}(\alpha_i^-)), \varphi_1(f(S_+^2))) = \text{lk}(\varphi_1^{-1}(\alpha_i^-), f(S_+^2)) = \text{lk}(\alpha_i^+, f(S_+^2)) = \text{lk}(\varphi_1(\alpha_i^+), \varphi_1(f(S_+^2))) = \text{lk}(\alpha_i^+, g(S_+^2)).
\]

Having established a means to permute the accessory circles of \( \hat{U}_d \) in a prescribed way, we may now complete the proof of Proposition 3.4.

**Proof of Proposition 3.4.** By Lemma 3.6 we may assume, after an ambient isotopy, that \( f(S_+^2) = \hat{U}_d \) and the accessory circles \( \{ \alpha_1^+, \alpha_1^- \}_{i=1}^d \) on \( f(S_+^2) \) satisfy \( \text{lk}(\alpha_i^+, f(S_+^2)) = \text{lk}(\alpha_i^-, f(S_+^2)) \) for each \( 1 \leq i \leq d \). Recall the notation of Figure 2 and that we let \( \eta_x \) and \( \eta_y \) denote disjoint neighborhoods of \( \eta_x \) and \( \eta_y \), respectively, on the circle \( S^1 \). For each \( 1 \leq i \leq d \), \( G_i(\eta_x \times [-\frac{1}{2}, \frac{1}{2}]) \)
and $G_i(\tilde{\eta}_y \times [-\frac{1}{2}, \frac{1}{2}])$ are embedded 2-disk neighborhoods of $\{p_i^+, p_i^-\}$ on $f(S^2_+) \subset X_+$ which intersect precisely at these two points, and the accessory circles $\{\alpha_i^+, \alpha_i^-\}$ leave along $G_i(\tilde{\eta}_x \times [-\frac{1}{2}, \frac{1}{2}])$ and return along $G_i(\tilde{\eta}_y \times [-\frac{1}{2}, \frac{1}{2}])$. Thus, as the arc $G_i(\eta_x \times 0) \subset G_i(\tilde{\eta}_x \times [-\frac{1}{2}, \frac{1}{2}])$ runs from $p^+$ to $p^-$, and the arc $G_i(\eta_y \times 0) \subset G_i(\tilde{\eta}_y \times [-\frac{1}{2}, \frac{1}{2}])$ runs from $p^+$ to $p^-$, by Lemma 2.4 and Equation (3.5) we have

$$\|\text{lk}(\partial V_i, f(S^2_+))\| = \|\text{lk}(G_i(\eta_x \times 0) \cup G_i(\eta_y \times 0), f(S^2_+))\| = \|\text{lk}(\alpha_i^+, f(S^2_+)) - \text{lk}(\alpha_i^-, f(S^2_+))\| = 0.$$\hfill\qed

### 3.2 Whitney disks in $X_+$

Referring to the notation of Proposition 3.4 and Proposition 3.3, we next show that by altering the interiors of the 2-disks $\{V_i\}_i$ so to exchange their intersections with $f(S^2_+)$ for intersections with $f(S^2_-)$, we are able to compute $\omega_-$ as follows.

**Proposition 3.7.** For each $1 \leq i \leq d$, the pair $\{p_i^+, p_i^-\}$ of double points of $f(S^2_-) \subset X_+$ may be equipped with a framed, immersed Whitney disk $W_i$ in $X_+ \subset X$ such that $\partial W_i = \partial V_i$. Furthermore, there are integer Laurent polynomials $\{u_i(s)\}_{i=1}^d$ such that

$$\omega_-(f) = \sum_{i: n_i \text{ even}} u_i(1) \mod 2$$

and for each $1 \leq i \leq d$,

$$\lambda(f(S^2_*), V_i^*) = (1 + s)u_i(s) \mod 2.$$ 

Define a ring homomorphism $\varphi : Z[s, s^{-1}] \to Z_2$ by

$$Z[s, s^{-1}] \xrightarrow{\varphi} Z[s, s^{-1}] \xrightarrow{s \mapsto 1} Z \xrightarrow{\mod 2} Z_2,$$

where $\partial$ is the formal derivative defined by setting $\partial(s^n) = ns^{n-1}$ (for $n \in Z$) and extending by linearity. Recall from Section 2 that we use $\equiv$ to denote equivalence modulo 2, and for an integer Laurent polynomial $g(s) \in Z[s, s^{-1}]$, we write $\overline{g(s)} = g(s^{-1})$. The following properties of $\varphi$ are readily verified.

**Lemma 3.8.** If $g(s) \in Z[s, s^{-1}]$, then

(i) $\varphi(\overline{g(s)}) \equiv \varphi(g(s))$,

(ii) $\varphi(s \cdot g(s)) \equiv g(1) + \varphi(g(s))$, and

(iii) $\varphi((1 + s^n)g(s)) \equiv n \cdot g(1)$.
Let $1 \leq i \leq d$. Since $f(S^2_+) \cdot V_i = 0$ the intersections between $f(S^2_+)$ and $\text{int} \, V^c_i$ (which may be assumed transverse after a small homotopy of $f_+$) may be decomposed into pairs of opposite sign $\{x^j_i, y^j_i\}_{j=1}^{J_i}$ for some $J_i \geq 0$ (for any choice of orientation of $S^2_+$ and $V^c_i$). For each $1 \leq j \leq J_i$, choose a simple path $\alpha^j_i$ on $f(S^2_+)$ from $x^j_i$ to $y^j_i$ whose interior is disjoint from $\bigcup_{k=1}^{d} V_k$, let $\beta^j_i$ be a simple path in $\text{int} \, V^c_i$ from $y^j_i$ to $x^j_i$ whose interior misses $f(S^2_+)$, and let $\rho^j_i = \alpha^j_i \cup \beta^j_i$. The resulting collection of loops $\{\rho^j_i\}_{j=1}^{J_i}$ in $X_-$ may be chosen to be mutually disjoint. For each $1 \leq j \leq J_i$ define the $\mathbb{Z}_2$-integer

$$m^j_i = \text{lk}(f(S^2_+), \rho^j_i) \mod 2.$$ 

Note that $m^j_i$ is well-defined because $f(S^2_+)$ and $V^c_i$ are simply-connected (c.f. Proposition 2.1).

**Lemma 3.9.** There are integer Laurent polynomials $\{u_i(s)\}_{i=1}^{d}$ in $\mathbb{Z}[s, s^{-1}]$ such that for each $1 \leq i \leq d$ we have

$$\lambda(f(S^2_+), V^c_i) \equiv (1 + s)u_i(s)$$

and $u_i(1) \equiv \sum_{j=1}^{J_i} m^j_i$.

**Proof.** Choose whiskers connecting $f(S^2_+)$ and $\{V^c_i\}_{i=1}^{d}$ to the basepoint of $X_-$. Let $1 \leq i \leq d$ and $1 \leq j \leq J_i$. Since $\rho^j_i$ is a loop in $X_-$ that runs from $x^j_i$ to $y^j_i$ along $f(S^2_+)$, and back to $x^j_i$ along $V^c_i$, by Proposition 2.1 we have

$$\lambda(f(S^2_+), V^c_i)[x^j_i] \cdot (\lambda(f(S^2_+), V^c_i)[y^j_i])^{-1} = s^{\hat{m}^j_i} \in \pi_1(X_-),$$

where $\hat{m}^j_i$ is an integer such that $\hat{m}^j_i \equiv \text{lk}(f(S^2_+), \rho^j_i) \equiv m^j_i$. Thus

$$\lambda(f(S^2_+), V^c_i)[x^j_i] = s^{\hat{m}^j_i} \cdot \lambda(f(S^2_+), V^c_i)[y^j_i],$$

so the mod 2 contribution to $\lambda(f(S^2_+), V^c_i)$ due to the pair of intersections $\{x^j_i, y^j_i\}$ is

$$\lambda(f(S^2_+), V^c_i)[x^j_i] + \lambda(f(S^2_+), V^c_i)[y^j_i] \equiv (1 + s^{\hat{m}^j_i})s^{l^j_i},$$

for some $l^j_i \in \mathbb{Z}$. Choose $u^j_i(s) \in \mathbb{Z}[s, s^{-1}]$ such that

$$(1 + s)u^j_i(s) \equiv (1 + s^{\hat{m}^j_i})s^{l^j_i};$$

applying $\varphi$ to both sides (c.f. Lemma 3.8) yields

$$u^j_i(1) \equiv m^j_i. \tag{3.24}$$

Summing over all such pairs $\{x^j_i, y^j_i\}_{j=1}^{J_i}$ we have

$$\lambda(f(S^2_+), V^c_i) \equiv \sum_{j=1}^{J_i} (1 + s)u^j_i(s) \equiv (1 + s)u_i(s),$$

where $u_i(s) = \sum_{j=1}^{J_i} u^j_i(s)$ satisfies $u_i(1) \equiv \sum_{j=1}^{J_i} m^j_i$ by Equation (3.24). \qed
Let $1 \leq i \leq d$. We now remove the intersections between $V_i$ and $f(S^2_2)$ by surgering $V_i$ along the paths $\{\alpha_j^i\}_{j=1}^{J_i}$, obtaining an embedded $J_i$-genus, once-punctured surface $\hat{V}_i$ in $X_+$ which has interior in $X_-$ and coincides with $V_i$ near the boundary.

Since $f(S^2_2)$ is transverse to $V_i$, for each $1 \leq j \leq J_i$ the restriction of a tubular neighborhood of $f(S^2_2)$ to the arc $\alpha_j^i$ may be identified with a 3-ball $h_j^i : D^1 \times D^2 \to X_-$ such that $h_j^i(D^1 \times 0) = \alpha_j^i$ and $h_j^i(D^1 \times D^2)$ intersects $V_i$ in two embedded 2-disks $h_j^i(1 \times D^2)$ and $h_j^i(-1 \times D^2)$ neighborhoods of $x_j^i$ and $y_j^i$ in $\hat{V}_i$, respectively. Attaching handles to $V_i$ along the arcs $\alpha_j^i$, $j=1,\ldots,J_i$, yields the surface

$$\hat{V}_i = [V_i \setminus \bigcup_{j=1}^{J_i} \text{int } h_j^i(\partial D^1 \times D^2)] \bigcup_{j=1}^{J_i} h_j^i(D^1 \times \partial D^2),$$

which is disjoint from both $f(S^2_2)$ and $f(S^2_2)$. See Figure 9.

![Figure 9](image)

Now, for each $1 \leq j \leq J_i$ we may assume that $\beta_j^i$ intersects $h_j^i(1 \times \partial D^2)$ and $h_j^i(-1 \times \partial D^2)$ exactly once, at points $\hat{x}_j^i$ and $\hat{y}_j^i$, respectively. Let $\hat{\beta}_j^i$ be the subarc of $\beta_j^i$ on $\hat{V}_i$ running from $\hat{y}_j^i$ to $\hat{x}_j^i$, let $\hat{\alpha}_j^i$ be a path on $h_j(D^1 \times \partial D^2)$ connecting $\hat{x}_j^i$ to $\hat{y}_j^i$, and put $\hat{\rho}_j^i = \hat{\alpha}_j^i \cup \hat{\beta}_j^i$. By band-summing $\hat{\alpha}_j^i$ with meridinal circles of $f(S^2_2)$ of the form $h_j(z \times \partial D^2)$ (for a point $z$ in $\text{int } D^1$) if necessary, we may assume that $\text{lk}(\hat{\rho}_j^i, f(S^2_2)) = 0$ (see Figure 9). Hence, as $\pi_1(X_+)$ is abelian, there is an immersed 2-disk $\hat{Q}_j^i$ in $X_+$ bound by $\hat{\rho}_j^i$. We may further assume that $\hat{Q}_j^i$ misses a collar of $\partial V_i = \partial \hat{V}_i$ and is transverse to $f(S^2_2)$.

By boundary twisting $\hat{Q}_j^i$ along $\hat{\beta}_j^i$ (and so introducing intersections between the interior of $\hat{Q}_j^i$ and $\hat{V}_i$) if necessary we may further assume that a normal section of $\hat{\rho}_j^i$ that is tangential to $\hat{V}_i$ extends to a normal section of $\hat{Q}_j^i$ in $X_+$. Hence there is a normal pushoff $\tilde{Q}_j^i$ of $\hat{Q}_j^i$ and an annulus $g_j^i$ on $\hat{V}_i$ with boundary...
\[ \partial g_i \cup \partial \hat{Q}_i \cup \partial \hat{Q}_i \] (see Figure 10). Iterating the construction of Lemma 2.7 we may then surger \( \hat{V}_i \) along \( \hat{\rho}_i \), using \( \hat{Q}_i \) and its pushoff \( \hat{Q}_i \), for all \( 1 \leq j \leq J_i \), to obtain an immersed 2-disk \( W_i \) in \( X_+ \) such that the framing of \( V_i \) (which agrees with \( W_i \) near the boundary) along its boundary extends over \( W_i \). But \( V_i \) is a framed Whitney disk for \( f(S^2_+) \) in \( S^4 \), so \( W_i \subset X_+ \) is a framed Whitney disk for \( f(S^2_-) \) in \( X_+ \subset S^4 \). That is,

\[
W_i = (\hat{V}_i \setminus \text{int } \bigcup_{j=1}^{J_i} g_i) \cup \bigcup_{j=1}^{J_i} \partial g_i \cup \hat{Q}_i \cup \hat{Q}_i
\]
is a framed, immersed Whitney disk for the immersion \( f_- : S^2 \rightarrow X_+ \). Let \( W_i^c \) denote the complement in \( W_i \) of a half-open collar it shares with \( V_i \) so that \( \partial W_i^c = \partial V_i^c \) and \( f(S^2_-) \) intersects \( W_i^c \) in its interior.

![Figure 10](image_url)

The first step in relating \( \omega_-(f) \) to the intersections between \( f(S^2_+) \) and the \( V_i \)'s is the following lemma.

**Lemma 3.10.** The contribution to \( \omega_-(f) \) due to intersections between \( f(S^2_-) \) and the interior of \( W_i \) is

\[
\mathcal{L}^-(W_i) = \begin{cases} 
0 & \text{if } n_i \text{ is odd,} \\
\sum_{j=1}^{J_i} m_j & \text{if } n_i \text{ is even.}
\end{cases}
\]

**Proof.** Referring to the constructions preceding the lemma statement, since \( \text{int } \hat{V}_i \subset X_- \), the only intersections between \( \text{int } W_i \) and \( f(S^2_-) \) lie on the immersed 2-disks \( \{\hat{Q}_i, \hat{Q}_i\}_{j=1}^{J_i} \). Indeed, since \( \hat{Q}_i \) is the pushoff of \( \hat{Q}_i \) along a section of its normal bundle that is tangent to the annulus \( g_i \) on \( \hat{V}_i \), there is an immersion of a 3-ball

\[
H_i : D^2 \times I \rightarrow X_+
\]
such that \( H^2_i(D^2 \times 0) = \hat{Q}^i_1, H^2_i(D^2 \times 1) = \hat{Q}^i_2 \) and \( H^2_i(S^1 \times I) = \varrho^i_2 \). Furthermore, since \( \hat{Q}^i_1 \) is transverse to \( f(S^2_i) \) we may assume that if we let

\[
K^j_i = \# \{ f(S^2_i) \cap \hat{Q}^j_1 \},
\]

then there are distinct points \( x_k \in \text{int} D^2 \), \( 1 \leq k \leq K^j_i \), such that \( f(S^2_i) \) intersects \( H^2_i(D^2 \times I) \) precisely along the arcs \( \{ H^2_i(x_k \times I) \} \). Whence the intersections between \( f(S^2_i) \) and \( \text{int} W_i \) consist precisely of pairs

\[
\hat{x}^j_{i,k} := H^2_i(x_k \times 0) \subset \hat{Q}^j_1 \quad \text{and} \quad \hat{x}^j_{i,k} := H^2_i(x_k \times 1) \subset \hat{Q}^j_2,
\]

for \( 1 \leq k \leq K^j_i \). Thus, in particular, if \( n_i \) is odd then from Remark 2.5 we have \( \mathcal{L}^-(W_i) = 0 \).

Suppose now that \( n_i \) is even. Note that since the loop \( \tilde{\rho}^j_i \) on \( \tilde{V}_i \) is freely homotopic in \( X_\omega \) to \( \rho^i_1 \) (to see this, collapse \( h^j_i \) onto its core \( \alpha^j_i \subset f(S^2_i) \) in \( X_\omega \)) we have \( \text{lk}(f(S^2_i), \tilde{\rho}^j_i) \equiv m^j_i \), so

\[
K^j_i \equiv f(S^2_i) \cdot \hat{Q}^j_1 \equiv m^j_i.
\]  

We may arrange that there are points \( z \in \text{int} D^1 \) and \( d \in S^1 \) so that the meridinal circle \( h^j_i(z \times S^1) \) of \( f(S^2_i) \) on \( \tilde{V}_i \) intersects \( \varrho^i_2 \) along the arc \( H^2_i(d \times I) \).

Let \( \iota_I \) denote the interval \([0, 1]\) oriented from 0 to 1, and let \( \zeta^j_i \) be the path on \( h^j_i(z \times S^1) \setminus \text{int} \varrho^i_1 \) that runs from \( H^2_i(d \times 0) \) to \( H^2_i(d \times 1) \). Then the loop based at \( H^2_i(d \times 1) \) and spanning \( h^j_i(z \times S^1) \) given by

\[
\eta^j_i := H^2_i(d \times \iota_I) \ast \zeta^j_i
\]

is a meridinal loop of \( f(S^2_i) \), so \( \text{lk}(f(S^2_i), \eta^j_i) \equiv 1 \).

**Claim.** For fixed \( i, j \): if \( n_i \) is even then for each \( 1 \leq k \leq K^j_i \) the contribution to \( \mathcal{L}^-(W_i) \) due to the pair \( \{ \hat{x}^j_{i,k}, \hat{x}^j_{i,k} \} \) is

\[
\mathcal{L}^-(\hat{x}^j_{i,k}) + \mathcal{L}^-(\hat{x}^j_{i,k}) \equiv 1.
\]

**Proof of claim.** Let \( \gamma^j_{i,k} \) be a path in \( D^2 \) connecting \( x_k \) to \( d \). Then

\[
\beta^j_{i,k} := H^2_i(x_k \times \iota_I) \ast H^2_i(\gamma^j_{i,k} \times 0) \ast \zeta^j_i \ast H^2_i(\overline{\gamma^j_{i,k}} \times 1)
\]

is a loop that runs from \( \hat{x}^j_{i,k} \) to \( \hat{x}^j_{i,k} \) along \( H^2_i(x_k \times I) \subset f(S^2_i) \) and then back to \( \hat{x}^j_{i,k} \) along \( W_i \), so by Remarks 2.5 and 2.6 we have

\[
\mathcal{L}^-(\hat{x}^j_{i,k}) + \mathcal{L}^-(\hat{x}^j_{i,k}) \equiv m_i(\hat{x}^j_{i,k}) + m_i(\hat{x}^j_{i,k})
\]

\[
\equiv \text{lk}(f(S^2_i), \beta^j_{i,k}).
\]

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Now, the loop $\beta_{i,k}^j$ is homotopic in $X_+$ to the loop

$$H_i^j(x_k \times \tau) * H_i^j(\gamma_{i,k}^j \times 0) * H_i^j(d \times \iota_I) * \eta_i^j * H_i^j(\gamma_{i,k}^j \times 1),$$

but the loop

$$H_i^j(x_k \times \tau) * H_i^j(\gamma_{i,k}^j \times 0) * H_i^j(d \times \iota_I) * H_i^j(\gamma_{i,k}^j \times 1)$$

bounds the 2-disk $H_i^j(\gamma_{i,k}^j \times I) \subset X_+$. Thus $\beta_{i,k}^j$ is homotopic in $X_+$ to

$$H_i^j(\gamma_{i,k}^j \times 1) * \eta_i^j * H_i^j(\gamma_{i,k}^j \times 1),$$

and so $\text{lk}(f(S_d^2), \beta_{i,k}^j) \equiv \text{lk}(f(S_d^2), \eta_i^j) \equiv 1$. □

Applying the claim to all such pairs of intersections $\{\hat{x}_{i,k}^j, \check{x}_{i,k}^j\}_{k=1}^{K_i^j}$ between $f(S_d^2)$ and $\hat{Q}_i^j \cup \check{Q}_i^j$, over all $1 \leq j \leq J_i$, yields the total contribution

$$\mathcal{L}^-(W_i) \equiv \sum_{x \in f(S_d^2) \cap \text{int } W_i} \mathcal{L}^-(x)$$

$$\equiv \sum_{j=1}^{J_i} \sum_{x \in \hat{Q}_i^j \cup \check{Q}_i^j} \mathcal{L}^-(x)$$

$$\equiv \sum_{j=1}^{J_i} \sum_{k=1}^{K_i^j} (\mathcal{L}^-(\hat{x}_{i,k}^j) + \mathcal{L}^-(\check{x}_{i,k}^j))$$

$$\equiv \sum_{j=1}^{J_i} K_i^j$$

$$\equiv \sum_{j=1}^{J_i} m_i^j,$$

where the last equality is by Equation (3.25). This completes the proof of Lemma 3.10. □

Applying the lemma, we have

$$\omega_-(f) \equiv \sum_{i: n_i \text{ odd}} \mathcal{L}^-(W_i) + \sum_{i: n_i \text{ even}} \mathcal{L}^-(W_i)$$

$$\equiv \sum_{i: n_i \text{ even}} \sum_{j=1}^{J_i} m_i^j,$$

where $1 \leq i \leq d$. Proposition 3.7 now follows from Lemma 3.9. □

### 3.3 Relating $\sigma_+$ and $\omega_-$

We now bring Kirk’s invariant $\sigma_+(f)$ into the picture by noting its relationship with the homotopy class of $f(S_d^2)$ as an element of $\pi_2(X_-)$. 27
Referring to Proposition 3.3, since $\pi_2(X_-)$ is generated as a $\mathbb{Z}[s, s^{-1}]$-module by the 2-spheres $\{A^+_i, A^-_i\}_{i=1}^d$, there are integer Laurent polynomials $\{c^+_i(s), c^-_i(s)\}_{i=1}^d$ such that, as a (whiskered) element of $\pi_2(X_-)$, $f(S^2_+)$ is given by

$$f(S^2_+) = \sum_{i=1}^d c^+_i(s)A^+_i + c^-_i(s)A^-_i. \quad (3.26)$$

By the sesquilinearity of the intersection form $\lambda(\cdot, \cdot)$ we have from Proposition 3.3 that

$$\lambda(f(S^2_+), f(S^2_+)) \equiv \sum_{i=1}^d c^+_i(s)c^+_i(s)\lambda(A^+_i, A^+_i) + c^-_i(s)c^-_i(s)\lambda(A^-_i, A^-_i)$$

$$\equiv (s + s^{-1})\sum_{i=1}^d \left[ c^+_i(s)c^+_i(s) + c^-_i(s)c^-_i(s) \right]. \quad (3.27)$$

In [9], Kirk showed that $\sigma$ has the following image.

**Proposition 3.11 ([9]).** If $g$ is a link map, then

$$\sigma_+(g) + \sigma_-(g) = a_0 + \sum_{n=2}^m a_n(n^2s - s^n)$$

for some integer $m \geq 0$ and integers $a_0, a_2, a_3, \ldots, a_m$.

Now, since $\sigma_-(f) = 0$ and $f$ is a good link map, by Proposition 2.3 we have

$$\lambda(f(S^2_+), f(S^2_+)) \equiv \sigma_+(f) + \sigma_+(f)$$

$$\equiv \sum_{n=2}^m a_n\left[ s^n + s^{-n} + n(s + s^{-1}) \right] \quad (3.28)$$

for some integers $a_2, \ldots, a_m$. The following observation about the terms in the right-hand side of this equation will be useful in performing some arithmetic in $\mathbb{Z}_2[s, s^{-1}]$.

**Lemma 3.12.** Let $n \geq 2$ be an integer. Then

$$s^n + s^{-n} + n(s + s^{-1}) \equiv (1 + s)^2r_n(s)$$

for some integer Laurent polynomial $r_n(s) \in \mathbb{Z}[s, s^{-1}]$ such that

$$r_n(1) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

**Proof.** If $n = 2k$ for some $k \geq 1$, then modulo 2 we have

$$s^n + s^{-n} \equiv s^{-2k}(1 + s^{4k})$$

$$\equiv s^{-2k}(1 + s^4)((s^4)^{k-1} + (s^4)^{k-2} + \ldots + (s^4)^{1} + 1).$$
On the other hand, if \( n = 2k + 1 \) for some \( k \geq 1 \), by direct expansion one readily verifies that, modulo 2,

\[
s^n + s^{-1} + s^{-n} + s \equiv (s + s^{-1})(s^{2k} + s^{2k-2} + \ldots + s^2 + 1) + (s + s^{-1})(s^{-2k} + s^{-2k+2} + \ldots + s^{-2} + 1)
\equiv (s + s^{-1}) \sum_{l=1}^{k} (s^{2l} + s^{-2l})
\equiv s^{-1}(1 + s^2) \sum_{m=1}^{k} s^{-2m}(s^{4m} + 1)
\equiv s^{-1}(1 + s^2)^2 \sum_{m=1}^{k} s^{-2m}(s^{4m} + 1)^4
\equiv (1 + s)^6 \tilde{r}_n(s)
\]

for some \( \tilde{r}_n(s) \in \mathbb{Z}[s, s^{-1}] \). \( \square \)

Now, from Equation (3.26), Proposition 3.3(ii) and the sesquilinearity of \( \lambda(\cdot, \cdot) \), we have

\[
\lambda(f(S^2_n), A^\pm) \equiv c_1^\pm(s)\lambda_2(A^\pm, A^\pm) \equiv c_1^\pm(s)(s + s^{-1}).
\]

Comparing with Proposition 3.3(iii),(iv) and Proposition 3.7, we see that there are integer Laurent polynomials \( \{q_i(s), u_i(s)\}_{i=1}^{d} \) such that

\[
\omega_-(f) \equiv \sum_{i: \text{n_i even}} u_i(1) \quad (3.29)
\]

and for each \( 1 \leq i \leq d \),

\[
c_i^+(s) \equiv q_i(s),
\]

\[
c_i^-(s) \equiv q_i(s) + (1 + s)u_i(s),
\]

where \( q_i(1) \equiv n_i \). Thus Equation (3.27) becomes

\[
\lambda(f(S^2_n), f(S^2_n))
\equiv (s + s^{-1}) \sum_{i=1}^{d} [q_i(s)\overline{q_i(s)} + (q_i(s) + (1 + s)u_i(s))\overline{q_i(s) + (1 + s)u_i(s)}]
\equiv (s + s^{-1}) \sum_{i=1}^{d} [(1 + s)u_i(s)\overline{q_i(s)} + (1 + s^{-1})q_i(s)\overline{u_i(s)}
+ (1 + s)(1 + s^{-1})u_i(s)\overline{u_i(s)}].
\]

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Comparing with Equation (3.28) and applying Lemma 3.12 we then have

\[
(s + s^{-1}) \sum_{i=1}^{d} [(1 + s)u_i(s)q_i(s) + (1 + s^{-1})q_i(s)\overline{u}_i(s)] + (1 + s)(1 + s^{-1})u_i(s)\overline{u}_i(s) \equiv (1 + s) \sum_{n=2}^{k} a_n r_n(s) \\
≡ (s + s^{-1}) (1 + s^{-1})^2 \sum_{n=2}^{k} a_n \hat{r}_n(s)
\]

for some integer Laurent polynomials \( \{r_n(s), \hat{r}_n(s)\}_{n=2}^{k} \) (here, \( \hat{r}_n(1) = r_n(1) = n/2 \) if \( n \) is even, and \( \hat{r}_n(1) = 0 \) if \( n \) is odd. Since \( \mathbb{Z}_2[s, s^{-1}] \) is an integral domain, we may divide both sides of Equation (3.30) by \( (s + s^{-1})(1 + s^{-1}) \) to obtain

\[
(1 + s^{-1}) \sum_{n=2}^{k} a_n \hat{r}_n(s) \equiv \sum_{i=1}^{d} u_i(s)\overline{q}_i(s)s + q_i(s)\overline{u}_i(s) + (1 + s)u_i(s)\overline{u}_i(s).
\]

Applying the homomorphism \( \varphi \) of Lemma 3.8 to both sides of Equation (3.32) then yields the following equality in \( \mathbb{Z}_2 \):

\[
\sum_{n=2}^{k} a_n \hat{r}_n(1) \equiv \sum_{i=1}^{d} [q_i(1)u_i(1) + \varphi(u_i(s)q_i(s))] + \varphi(q_i(s)\overline{u}_i(s)) + u_i(1)\overline{u}_i(1)
\]

\[
≡ \sum_{i=1}^{d} q_i(1)u_i(1) + u_i(1)
\]

\[
≡ \sum_{i=1}^{d} u_i(1)(n_i + 1)
\]

\[
≡ \sum_{i: n_i \text{ is even}} u_i(1).
\]

Thus, as \( \hat{r}_n(1) \equiv 1 \) if and only if \( n \) is even and \( n/2 \equiv 1 \) (i.e., \( n = 2 \mod 4 \)), from Equation (3.29) we have

\[
\omega_{-}(f) \equiv \sum_{n} \{a_n : n = 2 \mod 4\},
\]

completing the proof of Theorem 1.1. \( \square \)

### A Appendix: Proof of Lemma 3.1

We break the proof into the following lemmas.

**Lemma A.1.** Fix an orientation of \( S^2 \). Let \( \hat{b}_1, \hat{b}_2 : D^2 \to S^2 \) be a pair of mutually disjoint, equi-oriented embeddings. Let \( N_1, N \) be 2-disk neighborhoods of \( \hat{b}_1(D^2) \) and \( \hat{b}_1(D^2) \cup \hat{b}_2(D^2) \) in \( S^2 \), respectively.
Lemma A.1. Denote the closed 2-disk in $\mathbb{R}^2$ with support on $N_1$ such that $h_1 \circ \hat{b}_1(x, y) = \hat{b}_1(-x, -y)$ for $(x, y) \in D^2$.

(ii) There is an ambient isotopy $h : S^2 \times I \to S^2$ with support on $N$ such that $h_1 \circ \hat{b}_1 = \hat{b}_2$ and $h_1 \circ \hat{b}_2 = \hat{b}_1$.

Proof. We prove (ii) only; (i) is easier (note that the transformation from $D^2 = D^3 \times D^1$ to itself given by $(x, y) \mapsto (-x, -y)$ is orientation-preserving).

Let $\hat{N}$ be a 2-disk neighborhood of $\hat{b}_1(D^2) \cup \hat{b}_2(D^2)$ in the interior of $N$, and choose a collar $c : \partial N \times I \to N$ of $N$ such that $c(x, 0) = x$ for $x \in \partial N$ and $c(\partial N \times 1) = \partial N$. Since the embeddings $\hat{b}_1$ and $\hat{b}_2$ are equi-oriented, by the Disk Theorem [12, Corollary 3.3.7] and the Isotopy Extension Theorem [12, Theorem 2.5.2], there is an ambient isotopy $\hat{h} : \hat{N} \times I \to \hat{N}$ such that $\hat{h}_1 \circ \hat{b}_1 = \hat{b}_2$ and $\hat{h}_1 \circ \hat{b}_2 = \hat{b}_1$. Choose a smooth function $m : I \to I$ satisfying $m(0) = 1$ and $m(1) = 0$, and define $h : S^2 \times I \to S^2$ as follows. For each $t \in I$, let $h_t = \hat{h}_t$ on $\hat{N}$, let

$$h(c(x, s), t) = c(\hat{h}(x, m(s)t), s)$$

for $(x, s) \in \partial \hat{N} \times I$, and let $h_t = \text{id}_{S^2}$ elsewhere. It is readily verified that $h_0 = \text{id}_{S^2}$, and that for each $t \in I$, the map $h_t$ is well-defined on $\partial \hat{N} = c(d\hat{N} \times 0)$ and constant on the complement of $S^2 \setminus \text{int} N$.

We extend these isotopies to $S^4$ as follows.

Lemma A.2. Suppose that $b_1, b_2 : D^4 \to S^4$ are a pair of equi-oriented embeddings with mutually disjoint images such that, if $S^2 \subset S^4$ denotes the standard embedding, we have $b^{-1}_i(S^2) = D^3 \times 0 \times 0$ for $i = 1, 2$. Let $N_1, N$ be 2-disk neighborhoods of $b_1(D^4) \cap S^2$ and $(b_1(D^4) \cup b_2(D^4)) \cap S^2$, respectively.

(i) There is an ambient isotopy $F : S^4 \times I \to S^4$ with support on an arbitrarily small 4-ball neighborhood of $b_1(D^4) \cup N_1$ such that $F_1$ fixes $S^2$ set-wise, and $F_1(b_1(x, y)) = b_1(-x, -y)$ for $(x, y) \in D^2 \times \hat{D}^2$.

(ii) There is an ambient isotopy $H : S^4 \times I \to S^4$ with support on an arbitrarily small 4-ball neighborhood of $b_1(D^4) \cup b_2(D^4) \cap N$ such that $H_1$ fixes $S^2 \setminus \text{int}(b_1(D^4) \cup b_2(D^4))$ set-wise, and $H_1 \circ b_1 = b_2$ and $H_1 \circ b_2 = b_1$.

Proof. We prove (ii) only; (i) is an analogous application of part (i) of Lemma A.1. Denote the closed 2-disk in $\mathbb{R}^2$ of radius $\frac{1}{2}$ by $\hat{D}^2$. Since for $i = 1, 2$, $b_i(D^4)$ intersects the standard 2-sphere along $b_i(D^2 \times 0 \times 0)$, we may identify a tubular neighborhood of $S^2 = S^2 \times 0 \times 0$ with $S^2 \times D^2$ so that there are equi-oriented, disjoint embeddings $\hat{b}_i : D^2 \to b_i(D^4) \cap S^2$ such that $b_i(D^4) = \hat{b}_i(D^2) \times \hat{D}^2$ and $\hat{b}_i$ is given by $b_i(x, y) = (\hat{b}_i(x), \frac{1}{2} y)$ for $(x, y) \in D^2 \times \hat{D}^2$.

By Lemma A.1(ii) there is an ambient isotopy $h : S^2 \times I \to S^2$ with support on $N$ such that $h_1 \circ \hat{b}_1 = \hat{b}_2$ and $h_1 \circ \hat{b}_2 = \hat{b}_1$; in particular, $h_1$ fixes $N \setminus \text{int}(b_1(D^2) \cup b_2(D^2))$ set-wise. We construct an isotopy $H$ of $N \times D^2 \subset S^2 \times D^2$ such that, for each $t \in I$:

$$...$$
Choose a smooth function $m : I \to I$ such that $m(1) = 0$ and $m(s) = 1$ for all $s \in [\frac{1}{2}, 1]$. For each $t \in I$ and $(x, y) \in N \times D^2$, let $H_t(x, y) = (h_{m(|y|)}(x, y), y)$, where $| \cdot |$ denotes the Euclidean norm on $D^2$. Note that on $N \times D^2$, $H_t$ has inverse given by $H_t^{-1}(x, y) = (h_{-m(|y|)}^{-1}(x), y)$. To verify (1), observe that for $x \in \partial N$ and $y \in D^2$ we have $(h_{m(|y|)}(x), y) = (x, y)$; for $x \in N$ and $y \in \partial D^2$, we have $(h_{m(|y|)}(x), y) = (h_0(x), y) = (x, y)$. To verify (2), observe that if $y \in \bar{D}^2$ then $m(|y|) = 1$ and so $(h_{m(|y|)}(x), y) = (h_t(x), y)$. Regarding (3), for $(x, y) \in D^2 \times D^2$ we have

$$H_1(b_1(x, y)) = H_1(b_1(x), \frac{1}{2}y) = (h_1(b_1(x), \frac{1}{2}y)) = (b_2(x), \frac{1}{2}y) = b_2(x, y),$$

and we have $H_1 \circ b_2 = b_1$ similarly.

Now, by (1) we may extend $H$ to an isotopy of $S^4$ that is constant on the complement of $N \times D^2$. Since $h_1$ fixes $N \setminus \text{int}(b_1(D^2) \cup b_2(D^2)) = [N \setminus \text{int}(b_1(D^2) \cup b_2(D^2))] \times 0$ set-wise, so does $H_1$ by property (2); hence $H_1$ fixes $S^2$ (since $H_1$ is the identity outside $N^2 \times D^2$).

We may now prove Lemma 3.1.

**Proof of Lemma 3.1.** Choose an orientation-preserving diffeomorphism $\Phi : S^4 \to S^4$ that takes $U$ to the standard embedding $S^2 \subset S^4$; then $b'_i = \Phi \circ b_i$, $i = 1, \ldots, d$, is a collection of mutually disjoint, equi-oriented embeddings $D^4 \to S^4$ whose images intersect $S^2$ precisely along $b'_i(D^2 \times 0 \times 0)$, respectively.

If $\rho$ is non-trivial, write it as a product of non-trivial transpositions $\rho = \tau_1 \tau_2 \cdots \tau_n$ for some $n \geq 1$. For each $1 \leq k \leq n$, write $\tau_k = (a_k b_k)$ for some $a_k, b_k \in \{1, 2, \ldots, d\}$, and let $N^{(k)}$ be a 2-disk neighborhood of $(b'_a(D^4) \cup b'_b(D^4)) \cap S^2 \setminus \bigcup_{i \neq a, b} b'_i(D^4)$. By Lemma A.2(ii) there is an ambient isotopy $H^{(k)} : S^4 \times I \to S^4$ with support on $N^{(k)} \times D^2$ in $S^4 \setminus \bigcup_{i \neq a, b} b'_i(D^4)$ such that $H^{(k)}_1$ fixes

$$S^2 \setminus \text{int}(b_{a_k}(D^4) \cup b_{b_k}(D^4))$$

set-wise and is such that $H^{(k)}_1 \circ b'_i = b'_{\tau_k(i)}$ for $i = a_k, b_k$. Define $H : S^4 \times I \to S^4$ by

$$H(x, t) = H^{(k)}(x, n(t - \frac{k-1}{n}))$$

for $x \in S^4$ and $t \in [\frac{k-1}{n}, \frac{k}{n}]$, where $k = 1, 2, \ldots, n$. Then $H$ is an ambient isotopy which fixes

$$S^2 \setminus \bigcup_{i=1}^d b'_i(D^4)$$

set-wise and satisfies

$$H_1 \circ b'_i = b'_{\rho(i)}$$

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for each $1 \leq i \leq d$. Now, by Lemma A.2(i), for each $1 \leq i \leq d$ there is an ambient isotopy $F^{(i)} : S^4 \times I \to S^4$ with support on a 4-ball neighborhood of $N(\rho(i))$ in $S^4 \setminus \bigcup_{k \neq \rho(i)} b'_k(D^4)$ such that $F_1^{(i)}$ fixes $S^2$ set-wise and is such that

$$F_1^{(i)} \circ b'_\rho(i)(x, y) = b'_\rho(i)(\mu_i x, y)$$

for $(x, y) \in D^2 \times D^2$. Define $F : S^4 \times I \to S^4$ by

$$F(x, t) = F^{(i)}(x, d(t - \frac{i-1}{d}))$$

for $x \in S^4$ and $t \in [\frac{i-1}{d}, \frac{i}{d}]$, where $i = 1, 2, \ldots, d$. Then $F$ is an ambient isotopy which fixes $S^2$ set-wise and satisfies

$$F_1 \circ b'_\rho(i)(x, y) = b'_\rho(i)(\mu_i x, y)$$

for $(x, y) \in D^2 \times D^2$. Thus if $K : S^4 \times I \to S^4$ is the ambient isotopy defined by $H_{2t}$ for $t \in [0, \frac{1}{2}]$ and $F_{2t-1}$ for $t \in [\frac{1}{2}, 1]$, then $\hat{\phi}_t = \Phi^{-1} \circ K_t \circ \Phi$ is the required isotopy.

\[ \square \]

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