The idea of non-commutative space coordinates started with Heisenberg back in the 1930’s by means of a letter addressed to Peierls. It contained the notion of an uncertain relation in space-time as a possibility to avoid singularities which can appear on the pontual particles self-energy terms. Based on such piece of advice, Peierls worked on the Landau problem. Then the idea propagated as a chain until Snyder since Peierls talked to Pauli, who included Oppenheimer in the discussion, who passed those ideas to his PhD student Hartrand Snyder [1–3]. Thus Snyder was the first to develop the idea that the space coordinates could not commute in too close distances scales. In his work, published in 1947 [4, 5], he proposed a new approach to understand space-time, which, at the highly enough energies, should be switched for a geometry in non-abelian theories [6], in standard model [10 – 12] and in the understanding of the quantal Hall effect [13].

Contemporary to the initial studies on non-commutative geometry, Wigner introduced in 1932 the first formalism for quantum mechanics in phase-space in order to develop the quantum kinetic theory [15]. In the Wigner formalism, each operator, say $A$, defined in the Hilbert space, $H$, is associated with a function, say $a_W(q,p)$, defined in phase space, $\Gamma$ [15, 18]. Then there is an application $\Omega_W : A \rightarrow a_W(q,p)$, such that, the associative algebra of operators defined in $H$ turns out to be an associative (but not commutative) algebra is $\Gamma$, given by $\Omega : AB \rightarrow a_W(q,p)\star b_W(q,p)$, where the star-product (Moyal product) $\star$ is defined by

$$a_W(q,p)\star b_W(q,p) = a_W(q,p)\exp\left[\frac{i}{\hbar} \left( \frac{\partial}{\partial q} \hat{\partial}_p + \frac{\partial}{\partial p} \hat{\partial}_q \right) \right]|_{\hbar=0} b_W(q,p).$$

It should be noted that the above equation can be seen as an operator $\hat{A} = a_W \star$ acting in functions $b_W$. From a physical and mathematica standpoint, the phase space and Moyal product have been explored in different ways [16, 34], in particular to study the irreducible unitary representations of kinematical groups considering operators of the type $a_W \star$. In this sense, in a recent our work, using

\begin{equation}
\Delta \hat{\partial}_\mu \Delta \hat{\partial}_\nu \geq \frac{1}{2} |\theta^{\mu \nu}|.
\end{equation}
the notion of symplectic struture and of Weyl product of a not commutative geometry, we studied unitary representations of Galilei group and we showed as to write the Schroedinger equation in phase space \[ \text{Eq. (35)} \]. This approach gives rise to a new procedure to derive Wigner function without the use of the Liouville-von Neumman equation. In others works of our, we showed as to extend this representation to the relativistic case using the Poincaré group \[ \text{Eq. (35, 37)} \]. In other words, using the notion of symplectic struture and the Weyl product, we derive the Klein-Gordon and the Dirac equations in phase space, and we presented the concoction of this formalism with the Wigner function \[ \text{Eq. (30)} \].

In this work we restrict in the no relativistic case. We study eigenvalue problems of Schroedinger equation in phase space \[ \text{Eq. (35)} \]. This approach is used here as the carrier space for representation of Lie algebras.

The presentation is organized in the following way. In the section 2, we define a Hilbert space \( H(\Gamma) \) over a phase space with its natural symplectic struture. \( H(\Gamma) \) will turn out to be the space of representation of the Galilei group. In the section 3, we construct the representations of symplectic struture and the Weyl product, we constructed unitary representations for Lie algebra for the Poincaré group and from this representations we derive the Klein-Gordon and the Dirac equations in phase space, and we presented the concoction of this formalism with the Wigner function \[ \text{Eq. (30)} \].

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In this work we restrict in the no relativistic case. We study eigenvalue problems of Schroedinger equation in phase space \[ \text{Eq. (35)} \]. This approach is used here as the carrier space for representation of Lie algebras.

Consider an analytical manifold \( \mathbb{M} \) where each point is specified by coordinates \( q \). The coordinates of each point in the cotangent-bundle \( \Gamma = T^*\mathbb{M} \) are denoted by \( (q, p) \). The \( 2N \)-dimensional manifold \( \Gamma \) is equipped with 2-form, that is defined by

\[
\omega = dq \wedge dp,
\]

and is called the symplectic form. The operator

\[
\Lambda = \frac{\partial}{\partial q} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial q}
\]

(1)

with the symplectic form lead to the Poisson bracket,

\[
\{ f, g \} = \omega(f\Lambda, g\Lambda) = f\Lambda g,
\]

where

\[
\{ f, g \} = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q},
\]

and \( f = f(q, p), g = g(q, p) \). The manifold \( \Gamma \) constituted by space \( T^*\mathbb{M} \) endowed with this symplectic structure is then called the phase space, and the set of analytical functions \( f(q, p) \) is denoted by \( C^\infty(\Gamma) \). The vector fields over \( \Gamma \) are given by

\[
X_f = f\Lambda = \frac{\partial f}{\partial q} \frac{\partial}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial}{\partial q}
\]

The Hilbert space associated with \( \Gamma \) is introduced by a set of complex functions \( \psi(q, p) \), which are square integrable in \( C^\infty(\Gamma) \), i.e.

\[
\int dpdq |\psi|^2 < \infty.
\]

Then the functions may be defined as \( \psi(q, p) = \langle q, p | \psi \rangle \), with

\[
\langle \psi | \phi \rangle = \int dpdq \psi^\dagger(q, p)\phi(q, p),
\]

where \( \langle \psi \rangle \) is a dual vector of \( |\psi\rangle \). This Hilbert space, denoted by \( H(\Gamma) \), is used here as the carrier space for representation of Lie algebras.

Consider \( \ell = \{ a_i, i = 1, 2, 3, ... \} \) a Lie algebra over the (real) field \( \mathbb{R} \), of a Lie group \( \mathcal{G} \), characterized by the algebraic relations \( [a_i, a_j] = C_{ijk}a_k \), where \( C_{ijk} \in \mathbb{R} \) are the structure constants and \( (, ) \) is the Lie product. We construct unitary symplectic representations for \( \ell \), denoted by \( \ell_{sp} \), using the star-product, as given in Eq. (35). The associative product in \( H(\Gamma) \) is introduced from \( \Lambda \), Eq. (11), as a mapping \( e^{ia\Lambda} = : \Gamma \times \Gamma \to \Gamma \), defined by

\[
(f \ast g)(q, p) = f(q, p)e^{ia\Lambda}g(q, p)
\]

\[
= \exp [ia(\partial_{q'}\partial_{x'} - \partial_{p'}\partial_{q'})] f(q, p)g(q', p')|_{q'=q, p'=p},
\]

(2)

(3)

where \( f \) and \( g \) are functions in \( C^\infty(\Gamma) \) and \( \partial_x = \partial/\partial x \) (\( x = p, q \)). The constant \( a \) fixes units, without any special meaning at this level. The usual associative product is obtained by taking \( a = 0 \). In addition, to each function, say \( f(q, p) \), we introduce an operators in the form \( \hat{f} = f(q, p)* \). Such an operator will be used as the generator of unitary transformations.
III. GALILEI-LIE ALGEBRA IN $\mathcal{H}(\Gamma)$ AND SCHÖDINGER EQUATION IN PHASE SPACE

In this section, we study the representation of Galilei group in $\mathcal{H}(\Gamma)$. This procedure leads us to the Schrödinger equation in phase space. Then a connection between this representation and the Wigner formalism is established.

Using the star-operator, $\hat{A} = a^*\ast$, the momentum and position operators, respectively, are defined by

$$\hat{Q} = q^* = q + \frac{i\hbar}{2}\partial_q,$$  \hspace{1cm} (4)

$$\hat{P} = p^* = p - \frac{i\hbar}{2}\partial_q.$$

(5)

In this sense, the following operators are introduced,

$$\hat{K} = m\hat{Q} - t\hat{P},$$

(6)

$$\hat{L}_i = \epsilon_{ijk}\hat{Q}_j\hat{P}_k = \epsilon_{ijk}q_jp_k - \frac{i\hbar}{2}\epsilon_{ijk}q_j\frac{\partial}{\partial p_k} + \frac{i\hbar}{2}\epsilon_{ijk}p_k\frac{\partial}{\partial q_j} + \frac{\hbar^2}{4}\frac{\partial^2}{\partial q_j\partial p_k}$$

(7)

and

$$\hat{H} = \frac{\hat{P}^2}{2m} = \frac{1}{2m}(\hat{P}_1^2 + \hat{P}_2^2 + \hat{P}_3^2) = \frac{1}{2m}((p_1 - \frac{i\hbar}{2}\frac{\partial}{\partial q_1})^2 + (p_2 - \frac{i\hbar}{2}\frac{\partial}{\partial q_2})^2 + (p_3 - \frac{i\hbar}{2}\frac{\partial}{\partial q_3})^2).$$

(8)

From this set of unitary operators we obtain, after some long but simple calculations, the following set of commutation relations:

$$[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k,$$

$$[\hat{L}_i, \hat{K}_j] = i\hbar\epsilon_{ijk}\hat{K}_k,$$

$$[\hat{L}_i, \hat{P}_j] = i\hbar\epsilon_{ijk}\hat{P}_k,$$

$$[\hat{K}_i, \hat{P}_j] = i\hbar m\delta_{ij}1,$$

$$[\hat{K}_i, \hat{H}] = i\hbar\hat{P}_i,$$

with all other commutation relations being null. This is the Galilei-Lie algebra with a central extension characterized by $m$. The operators defining the Galilei symmetry $\hat{P}$, $\hat{K}$, $\hat{L}$ and $\hat{H}$ are then generators of translations, boost, rotations and time translations, respectively.

The physical content of this representation can be derived if we first observe that $\hat{Q}$ and $\hat{P}$ are transformed by the boost as

$$\exp(-iv\frac{\hat{K}}{\hbar})\hat{P}_j\exp(iv\frac{\hat{K}}{\hbar}) = \hat{P}_j + mv_j1,$$

(9)

$$\exp(-iv\frac{\hat{K}}{\hbar})\hat{Q}_j\exp(iv\frac{\hat{K}}{\hbar}) = \hat{Q}_j + v_jt1.$$  

(10)

Furthermore

$$[\hat{Q}_i, \hat{P}_j] = i\hbar\delta_{ij}1.$$  

Therefore, $\hat{Q}$ and $\hat{P}$ can be taken to be the physical observables of position and momentum. The Galilei boost transforms them according to Eqs. (9) and (10). To be consistent, generators $\hat{L}$ are interpreted as the angular momentum operator, and $\hat{H}$ is taken as the Hamiltonian operator. The Casimir invariants of the Lie algebra are given by

$$I_1 = \hat{H} - \frac{\hat{P}^2}{2m} \text{ and } I_2 = \hat{L} - \frac{1}{m}\hat{K} \times \hat{P},$$

where $I_1$ describes the Hamiltonian of a free particle and $I_2$ is associated with the spin degrees of freedom. First, we study the scalar representation; i.e. spin zero.

Defining the operators

$$\overline{Q} = q1 \text{ and } \overline{P} = p1,$$

we observe that, under the boost, $\overline{Q}$ and $\overline{P}$ transform as

$$\exp(-iv\frac{\hat{K}}{\hbar})2\overline{Q}\exp(iv\frac{\hat{K}}{\hbar}) = 2\overline{Q} + vt1,$$

and

$$\exp(-iv\frac{\hat{K}}{\hbar})2\overline{P}\exp(iv\frac{\hat{K}}{\hbar}) = 2\overline{P} + mv1.$$  

This shows that, $\overline{Q}$ and $\overline{P}$ transform as position and momentum variables, respectively. These operators satisfy $[\overline{Q}, \overline{P}] = 0$. Then $\overline{Q}$ and $\overline{P}$ cannot be interpreted as observables. Nevertheless, they can be used to construct a frame in Hilbert space with the content of phase space. Then we define an orthogonal basis in $\mathcal{H}(\Gamma)$ by

$$\overline{Q}|q,p\rangle = |q,p\rangle \text{ and } \overline{P}|q,p\rangle = p|q,p\rangle,$$

with

$$\langle q,p|q',p'\rangle = \delta(q - q')\delta(p - p'),$$

such that $\int dqdp|q,p\rangle\langle q,p| = 1$. It is worth noting that the wave function $\psi(q,p,t) = \langle q,p|\psi(t)\rangle$ is associated
with the state of the system, but not have the content of the usual quantum mechanics state.

The time evolution equation for \( \psi(q, p, t) \) is derived by using the generator of time translations, such that

\[
\psi(t) = e^{-\frac{itH}{\hbar}} \psi(0),
\]

and the Hermitian adjoint is

\[
\psi^\dagger(t) = \psi^\dagger(0)e^{\frac{itH}{\hbar}}.
\]

Now we get

\[
\imath \hbar \partial_t \psi(q, p; t) = \hat{H}(q, p)\psi(q, p; t),
\]

or

\[
\imath \hbar \partial_t \psi(q, p; t) = H(q, p) \star \psi(q, p; t),
\]

which is the Schrödinger equation in phase space \([33]\).

The average of a physical observable \( \hat{A}(q, p) = a(q, p; t) \dagger \), in the state \( \psi(q, p) \) is given by

\[
\langle A \rangle = \int dqdp \psi^\dagger(q, p) \hat{A}(q, p) \psi(q, p)
= \int dqdp \psi^\dagger(q, p)[a(q, p) \star \psi(q, p)]
= \int dqdp a(q, p)[\psi(q, p) \star \psi^\dagger(q, p)].
\]

The association of \( \psi(q, p, t) \) with the Wigner function is given by \([33]\),

\[
f_W(q, p) = \psi(q, p, t) \star \psi^\dagger(q, p, t).
\]

This function satisfies the Liouville-von Neumann equation \([33]\), and the probability density in configuration space is

\[
\rho(q) = \int dp \left[ \psi(q, p) \star \psi^\dagger(q, p) \right] = \int dp \psi(q, p)\psi^\dagger(q, p),
\]

while in momentum space it is

\[
\rho(p) = \int dq \left[ \psi(q, p) \star \psi^\dagger(q, p) \right] = \int dq \psi(q, p)\psi^\dagger(q, p).
\]

It is important to emphasize that the average of an observable is consistent with the Wigner formalism, i.e. from Eqs. \((14)\) and \((15)\), we have

\[
\langle A \rangle = \int dqdp a(q, p)f_W(q, p; t).
\]

This provides a complete set of physical rules to interpret representations and opens the way to study other improvements. For example, we consider in the next sections the three dimensional harmonic oscillator and the noncommutative oscillator.

### IV. 3D Harmonic Oscillator in Phase Space

In this section we intent to show the solution of harmonic oscillator when expressed in phase space. With this example, our aim is to familiarize the reader to the formalism used to determine the Wigner function. Thus we start with the following 3-dimensional hamiltonian

\[
H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2,
\]

where \( p^2 = p_x^2 + p_y^2 + p_z^2 \) and \( q^2 = x^2 + y^2 + z^2 \).

In order to write such a system in phase space we need to replace the coordinates and momenta by

\[
q_i = q_i + \frac{\imath}{2} \frac{\partial}{\partial p_i}, \quad p_i^\star = p_i + \frac{\imath}{2} \frac{\partial}{\partial q_i},
\]

respectively. We have adopted the unities system where \( m = \omega = 1 \). Thus using the prescription \((19)\) into expression \((18)\) we get the hamiltonian operator

\[
H = \frac{1}{2} \left[ p_x^2 + p_y^2 + p_z^2 + x^2 + y^2 + z^2 + \imath \left( z \frac{\partial}{\partial p_z} - p_z \frac{\partial}{\partial x} \right) + \right.
+ \imath \left( y \frac{\partial}{\partial p_y} - p_y \frac{\partial}{\partial y} \right) + \imath \left( x \frac{\partial}{\partial p_x} - p_x \frac{\partial}{\partial z} \right) -
- \frac{1}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial p_x^2} + \frac{\partial^2}{\partial p_y^2} + \frac{\partial^2}{\partial p_z^2} \right) \right].
\]

Now we are able to write the Schrödinger equation in phase space. The solutions of such an equation will yield the Wigner functions. To solve the equation \( H \star \Psi = E \Psi \), let us perform a change of variables. We will use

\[
\zeta = \frac{1}{2} \left( p_x^2 + p_y^2 + p_z^2 + x^2 + y^2 + z^2 \right),
\]

then it yields

\[
\frac{\partial \Psi}{\partial x} = x \frac{\partial \Psi}{\partial \zeta}, \quad \frac{\partial \Psi}{\partial p_x} = p_x \frac{\partial \Psi}{\partial \zeta},
\]
\[
\frac{\partial \Psi}{\partial y} = y \frac{\partial \Psi}{\partial \zeta}, \quad \frac{\partial \Psi}{\partial p_y} = p_y \frac{\partial \Psi}{\partial \zeta},
\]
\[
\frac{\partial \Psi}{\partial z} = z \frac{\partial \Psi}{\partial \zeta}, \quad \frac{\partial \Psi}{\partial p_z} = p_z \frac{\partial \Psi}{\partial \zeta},
\]

and

\[
\frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial \Psi}{\partial \zeta} + x^2 \frac{\partial^2 \Psi}{\partial \zeta^2}, \quad \frac{\partial^2 \Psi}{\partial y^2} = \frac{\partial \Psi}{\partial \zeta} + y^2 \frac{\partial^2 \Psi}{\partial \zeta^2}, \quad \frac{\partial^2 \Psi}{\partial z^2} = \frac{\partial \Psi}{\partial \zeta} + z^2 \frac{\partial^2 \Psi}{\partial \zeta^2},
\]
\[
\frac{\partial^2 \Psi}{\partial p_x^2} = \frac{\partial \Psi}{\partial \zeta} + p_x^2 \frac{\partial^2 \Psi}{\partial \zeta^2}, \quad \frac{\partial^2 \Psi}{\partial p_y^2} = \frac{\partial \Psi}{\partial \zeta} + p_y^2 \frac{\partial^2 \Psi}{\partial \zeta^2}, \quad \frac{\partial^2 \Psi}{\partial p_z^2} = \frac{\partial \Psi}{\partial \zeta} + p_z^2 \frac{\partial^2 \Psi}{\partial \zeta^2}.
\]

As a consequence the imaginary part of \( H \star \Psi = E \Psi \) vanishes which leads to
yields the solution of Schrödinger equation in phase space is \( \hbar \) with \( n \) being an integer. We recall that we are using \( \Psi = \exp \left( \frac{u}{\hbar} \right) \) where \( u \) is a geometric function, defined as

\[
F = \frac{\alpha(a+1)z^2}{\gamma(\gamma+1)2} + \frac{\alpha(a+1)(a+2)z^3}{\gamma(\gamma+1)(\gamma+2)6} + \ldots,
\]

if we compare it to eq. (22), we conclude that \( \chi = F \left( -E - \frac{3}{2} ; r \right) \). It is important to note that the confluent hypergeometric function is finite only if the parameter \( \alpha \) is a negative integer, therefore such a constraint yields

\[
E = E_n = n + \frac{3}{2},
\]

with \( n \) being an integer. We recall that we are using \( \hbar = 1 \), hence the energy has the expected behavior. Then the solution of Schrödinger equation in phase space is

\[
\Psi_n(\zeta) = \exp \left( -2\zeta \right) F(-n, 3, 4\zeta),
\]

where \( F(-n, 3, 4\zeta) \) is the confluent hypergeometric function with proper parameters. The Wigner function can be calculated using Eq. (13), thus it yields

\[
f_{W}^{n}(q,p) = C_n \exp \left( -2\zeta \right) F(-n, 3, 4\zeta),
\]

where \( C_n = \exp \left( -2E_n \right) F(-n, 3, 4E_n) \).

V. 2D NONCOMMUTATIVE OSCILLATOR IN PHASE SPACE

In this section, we solve the Schrödinger equation for two-dimensional noncommutative oscillator in phase space. Our aim is to derive the Wigner function for this problem. So, consider then the Hamiltonian of a two-dimensional oscillator (we take \( \hbar = 1, m = 1 \) and \( \omega = 1 \),

\[
H = \frac{1}{2}(\dot{x}^2 + \dot{p}_x^2) + \frac{1}{2}(\dot{y}^2 + \dot{p}_y^2).
\]

For our aim we define the star product

\[
* = *_{\hbar \theta} = \exp(i\hbar \sum_{i=1}^{2} \left( \frac{\partial}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_i} \frac{\partial}{\partial q_i} \right) + \frac{i\theta}{2} \left( \frac{\partial}{\partial x} \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \frac{\partial}{\partial x} \right)),
\]

where \( q_i = (x, y) \) and \( p_i = (p_x, p_y) \). And the following equations

\[
q_i* = q_i + \frac{i}{2} \partial_{q_i} + \frac{i}{2} \theta_{ij} \partial_{p_j}, \quad (26)
\]

and

\[
p_i* = p_i + \frac{i}{2} \partial_{p_i} + \frac{i}{2} \theta_{ij} \partial_{q_j}. \quad (27)
\]

The operators defined in equations Eq. (26) and Eq. (27) satisfies the following commutation relations

\[
[q_i, p_j] = i\delta_{ij},
\]

and

\[
[p_i, q_j] = -i\theta_{ij}.
\]

In this way, we have a noncommutative phase space. Using the Eq. (26) and Eq. (27), Schrödinger equation in phase space is write as

\[
\frac{1}{2}(x + \frac{i}{2} \partial_{p_x} + \frac{i}{2} \theta \partial_{p_y})^2 + (p_x + \frac{i}{2} \partial_x - \frac{i}{2} \theta \partial_{p_y})^2 + (y + \frac{i}{2} \partial_{p_y} - \frac{i}{2} \theta \partial_{p_x})^2 + (p_y + \frac{i}{2} \partial_y + \frac{i}{2} \theta \partial_{p_x})^2 \psi(x, y, p_x, p_y) = E\psi(x, y, p_x, p_y).
\]

(28)

Note that we take \( \theta = \theta_{12} = -\theta_{21} \).

For solve the Eq. (28), we define the new coordinates \( \tilde{x} = x, \tilde{y} = (1 + \theta^2)^{-1/2}(y - \theta p_x), \tilde{p}_x = (1 + \theta^2)^{-1/2}(p_x + \theta y), \) and \( \tilde{p}_y = p_y \).

From this new coordinates, we define the new star-operators,

\[
\tilde{x}* = x*,
\]

\[
\tilde{y}* = (1 + \theta^2)^{-1/2}(y* - \theta p_x*),
\]

\[
\tilde{p}_x* = (1 + \theta^2)^{-1/2}(p_x* + \theta y*),
\]

and

\[
\tilde{p}_y* = p_y *.
\]

Is possible to show the following relations
\[ [\vec{x}^{\star}, \vec{p}^{\star}] = (1 + \theta^2)^{1/2}, \]
\[ [\vec{y}^{\star}, \vec{p}_y^{\star}] = (1 + \theta^2)^{1/2}. \]

So, defining
\[ \tilde{a}_x^{\star} = \frac{1}{\sqrt{2}}(\vec{x}^{\star} + i\vec{p}_x^{\star}), \]
\[ \tilde{a}_x^{\dagger} = \frac{1}{\sqrt{2}}(\vec{x}^{\star} - i\vec{p}_x^{\star}), \]
\[ \tilde{a}_y^{\star} = \frac{1}{\sqrt{2}}(\vec{y}^{\star} + i\vec{p}_y^{\star}), \]
we can write the Schrödinger equation as
\[ (\tilde{a}_x^{\star} \tilde{a}_x + \tilde{a}_y^{\dagger} \tilde{a}_y + (1 + \theta^2)^{1/2})\psi(\vec{x}, \vec{p}_x, \vec{y}, \vec{p}_y) = E\psi(\vec{x}, \vec{p}_x, \vec{y}, \vec{p}_y), \]
where \([\tilde{a}_i, \tilde{a}_j^\dagger] = i(1 + \theta^2)^{1/2}\delta_{ij}.\]

The operators defined by \( \tilde{a}_x^{\star} \) and \( \tilde{a}_x^{\dagger} \) are the annihilation and creation operators, respectively. In this way, we can write the energies corresponding as
\[ E_{n_x n_y} = (1 + \theta^2)^{1/2}(n_x + n_y + 1). \]

For ground state \( \psi_{00}(\vec{x}, \vec{p}_x, \vec{y}, \vec{p}_y) = \phi_0(\vec{x}, \vec{p}_x)\chi_0(\vec{y}, \vec{p}_y) \) we have \( \tilde{a}_x^{\star} \phi_0 \tilde{a}_x \phi_0 = 0, \) which is explicitly written as
\[ \frac{1}{\sqrt{2}}(\vec{x} + \frac{1}{2}\partial_x)\phi_0 = 0, \]
and
\[ \frac{1}{\sqrt{2}}(\vec{y} + \frac{1}{2}\partial_y)\phi_0 = 0. \]

Looking for real solutions of Eq. (30) and Eq. (31) we have to solve the following set of equations:
\[ (\vec{x} + \frac{1}{2}\partial_x)\phi_0 = 0, \]
\[ (\vec{y} + \frac{1}{2}\partial_y)\chi_0 = 0, \]
\[ (\vec{p}_x + \frac{1}{2}\partial_{p_x})\phi_0 = 0, \]
and
\[ (\vec{p}_y + \frac{1}{2}\partial_{p_y})\chi_0 = 0. \]

A general solution is given by
\[ \psi_{00} = C_0 \exp\left(-\left(\vec{x}^2 + \vec{p}_x^2 + \vec{y}^2 + \vec{p}_y^2\right)\right), \]
where \( C_0 = \frac{1}{\pi} \) is a normalization constant. For \( n \geq 1 \) the functions \( \psi_n \) are determined by using the creation operator, that is,
\[ \psi_{n_x n_y} = \frac{1}{\sqrt{n!}}(\tilde{a}_x^{\dagger} \tilde{a}_y^{\dagger})^n \psi_{00}. \]

From Eq. (15) we can write the Wigner function associated with each \( \psi_{n_x n_y} \), that is
\[ f_W(\vec{x}, \vec{p}_x, \vec{y}, \vec{p}_y) = \psi_{n_x n_y} \star \psi_{n_x n_y}^{\dagger}. \]

For \( n_x = 1, n_y = 1 \), we find
\[ f_W(\vec{x}, \vec{p}_x, \vec{y}, \vec{p}_y) \sim [1 - 2((\vec{x}^2 + (\vec{p}_x)^2))e^{-((\vec{x}^2)+(\vec{p}_x)^2)}][1 - 2((\vec{y}^2 + (\vec{p}_y)^2))e^{-((\vec{y}^2)+(\vec{p}_y)^2)}], \]
For \( n_x = 2, n_y = 2 \), we find
\[ f_W^2(\vec{x}, \vec{p}_x, \vec{y}, \vec{p}_y) \sim [2 - 4((\vec{x}^2 + (\vec{p}_x)^2)) + ((\vec{x}^2 + (\vec{p}_x)^2))^2] \times e^{-((\vec{x}^2)+(\vec{p}_x)^2)}[2 - 4((\vec{y}^2 + (\vec{p}_y)^2)) + ((\vec{y}^2 + (\vec{p}_y)^2))^2] \times e^{-((\vec{y}^2)+(\vec{p}_y)^2)}], \]
For arbitrary \( n_x \) and \( n_y \) we have the result
\[ f_W^2 \sim Ln[2((\vec{p}_x)^2 + (\vec{x})^2)] Ln[2((\vec{p}_y)^2 + (\vec{y})^2)] e^{-((\vec{x}^2)+(\vec{p}_x)^2) + (\vec{y}^2) + (\vec{p}_y)^2)}, \]
where \( Ln \) are the Laguerre polynomials. In terms of original variables, we can write
\[ f_W^2(\vec{x}, \vec{y}; \theta) \sim Ln[(\vec{x}^2 + (1 + \theta^2)^{-1}(\vec{p}_x + \theta \vec{y})^2)] \times \times Ln[(\vec{y}^2 + (1 + \theta^2)^{-1}(\vec{y} - \theta \vec{p}_x)^2 + \vec{p}_y^2) \times \times e^{-((\vec{x}^2) + \vec{p}_x^2) + \vec{y}^2 + \vec{p}_y^2)}, \]
where \( \beta = (1 + \theta^2)^{-1} \). Once the noncommutative parameter \( \theta = 0 \), the equation above back as that in commutative space. This result is the same as the one derived in works \[38\] and \[39\], following a different method.

In Eq. (30) we realize that the Wigner function corresponding to the noncommutative oscillator depends on the parameter \( \theta \). However, in theoretical point of view, on of the main problems in non-commutative models is the determination of parameter \( \theta \). In most models this parameter is arbitrary. A question of great relevance to the acceptance of non-commutative models as candidate for the description of physical phenomena is therefore how the parameter may be related to the observable physical quantities \[40\] [42].
VI. CONCLUDING REMARKS

In this letter we have set forth a symplectic representations of the Galilei group, that give rise to quantum theories in phase space. The Schrödinger equation is derived. As an application, we study the 3D harmonic oscillator and the noncommutative oscillator in phase space, and obtain the Wigner function. The symplectic representations are constructed by using the notion of the Moyal product (or star-product), as a noncommutative geometrical content. Then a Hilbert space is defined from a manifold with characteristics of phase space. The states are represented by quasi-probability amplitudes. This is important since it gives rise to a connection with the Wigner function. One central point to be emphasized is that the approach developed here permit the calculation of Wigner functions by a method based on unitary representations.

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[1] W. Pauli. Scientific Correspondence, Vol II, p.15, Ed. K. von Meyenn, (Spring-Verlag, Berlin, 1985).
[2] W. Pauli. Scientific Correspondence, Vol III, p.380, Ed. K. von Meyenn, (Spring-Verlag, Berlin, 1993).
[3] R. Jackiw. Physical instances of noncommuting coordinates. Nucl. Phys. Proc. Suppl. 108 (2002) 30.
[4] H. S. Snyder. Quantized Spacetime, Phys. Rev. 71 (1947) 38.
[5] H. S. Snyder. The Eletromagnetic Field in Quantized Spacetime. Phys. Rev. 72 (1947) 68.
[6] A. H. Chanseddine, G. Felder. J. Frohlich. Gravity in Noncommutative Geometry. Commun. Math. Phys. 155 (1993) 205.
[7] W. Kalau, M. Walze. Gravity Noncommutative Geometry and the Wodzicki Residue. J. Geom. Phys. 16 (1995) 327.
[8] D. Kastler. The Dirac Operator and Gravitation Commun. Math Phys. 106 (1991).
[9] A. Connes. J. Loot. Particle Models and noncommutative Geometry. Nucl. Phys. Proc. Suppl. 18B (1991) 29.
[10] J. C. Varilly, J. M. Garcia-Bondia. Noncommutative Differenial Geometry and the Standard Model. J. Geom. Phys. 12 (1993) 223.
[11] C. P Martin, J. C. Varilly, J. M. Garcia-Bondia. The Standard Model as a Noncommutative Geometry: the low energy regime. Phys. Rept. 294 (1998) 363.
[12] J. Belissard, A. van Elst, H. Schulz-Baldes. The Noncommutative Geometry of the Quantum Hall Effect. J. Math. Phys. 35 (1994) 53.
[13] N. Seiberg, E. Witten, JHEP 9909 (1999) 32, hep-th/9908142.
[14] E.P. Wigner, Z. Phys. Chem. B 19 (1932) 749.
[15] M. Hillery, R. F. O’Connell, M. O. Scully, E. P. Wigner, Phys. Rep. 106 (1984) 121.
[16] Y.S. Kim, M.E. Noz, Phase Space Picture and Quantum Mechanics - Group Theoretical Approach (W. Scientific, London, 1991).
[17] T. Curtright, D. Fairlie, C. Zachos, Phys. Rev. D 58 (1998) 25002.
[18] C.K. Zachos, Int. J. Mod. Phys. A 17 (2002) 297.
[19] F.C. Khanna, A.P.C. Malbouisson, J.M.C. Malbouisson, A.E. Santana, Thermal Quantum Field Theory: Algebraic Aspects and Applications (W. Scientific, Singapore, 2009).
[20] H. Weyl, Z. Phys. 46 (1927) 1.
[21] J.E. Moyal, Proc. Camb. Phil. Soc. 45 (1949) 99.
[22] S.A. Smolyansky, A.V. Prozorkevich, G. Maino, S.G. Mashnic, Ann. Phys. (N.Y.) 277 (1999) 193.
[23] T. Curtright, C. Zachos, J. Phys. A 32 (1999) 771.
[24] I. Galaviz, H. García-Compeán, M. Przanowski, F.J. Turrubiates, Weyl-Wigner-Moyal for Fermi Classical Systems, arXiv: hep-th/0612245v1.
[25] J. Dito, J. Math. Phys. 33 (1992) 791.
[26] Go. Torres-vega, J.H. Frederick, J. Chem. Phys. 93 (1990) 8862.
[27] M.C.B. Fernandes, J.D.M. Vianna, Braz. J. Phys. 28 (1998) 22.
[28] D. Galetti, A.F.R.T. Piza, Physica A 280 (2001) 405.
[29] M.C.B. Fernandes, A. E. Santana, J. D. M. Vianna, J. Phys. A: Math. Gen. 36 (2003) 3841.
[30] A.E. Santana, A. Matsuo Neto, J.D.M. Vianna, F.C. Khanna, Physica A 280 (2001) 405.
[31] D. Bohm, B.J. Hiley, Found. Phys. 11 (1981) 179.
[32] M.C.B. Andrade, A.E. Santana, J.D.M. Vianna, J. Phys. A: Math. Gen. 33 (2000) 4015.
[33] M.A. Alonso, G.S. Pogosyan, K.B. Wolf, J. Math. Phys. 43 (2002) 5857.
[34] M.D. Oliveira, M.C.B. Fernandes, F.C. Khanna, A.E. Santana, J.D.M. Vianna, Ann. Phys. (N.Y.) 312 (2004) 492.
[35] R.G.G. Amorim, M.C.B. Fernandes, F.C. Khanna, A.E. Santana, J.D.M. Vianna, Phys. Lett. A 361 (2007) 464.
[36] R.G.G. Amorim, F.C. Khanna, A.E. Santana, J.D.M. Vianna, Physica A 388 (2009) 3771.
[37] L. Kang et al, Int. J. Theor. Phys. 49 (2010) 134.
[38] A. Hatzinikitas, I. Smyrnakis, J. Math. Phys. 43 (2002) 113.
[39] X. Calmet, Eur. Phys. J. C 41 (2005) 269.
[40] A. Kokado, T. Okamura, T. Saito, Phys. Rev. D 69 (2004) 128007.
[41] A. Jellal, J. Phys. A 34 (2001) 10159.