IDEMPOTENTS WITH SMALL NORMS

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Abstract. Let Γ be a locally compact group. We answer two questions left open in [7] and [9]:
(i) For abelian Γ, we prove that if \( \chi_S \in B(\Gamma) \) is an idempotent with norm \( \| \chi_S \| < \frac{4}{3} \), then \( S \) is the union of two cosets of an open subgroup of \( \Gamma \).
(ii) For general Γ, we prove that if \( \chi_S \in M_{cb}A(\Gamma) \) is an idempotent with norm \( \| \chi_S \|_{cb} < \frac{1+\sqrt{2}}{2} \), then \( S \) is an open coset in \( \Gamma \).

1. Introduction

In his 1968 papers, Saeki determined idempotent measures on a locally compact abelian group \( G \) with small norms. These are equivalent to determining idempotent functions in the Fourier–Stieltjes algebras \( B(\Gamma) \) on a locally compact abelian group \( \Gamma \) with small norms (where \( \Gamma \) and \( G \) could be taken as Pontryagin duals of each other). The statements of Saeki’s results in the Fourier–Stieltjes setting are:

**Theorem 1.1** (Saeki). Let \( \Gamma \) be a locally compact abelian group, and let \( \varphi \) be an idempotent function in \( B(\Gamma) \) so that \( \varphi = \chi_S \) for some nonempty \( S \subseteq \Gamma \). Then

(i) \( \| \varphi \| < \frac{1+\sqrt{2}}{2} \), then \( S \) is an open coset of \( \Gamma \).

(ii) \( \| \varphi \| \in (1, \frac{\sqrt{17}+1}{4}) \), then \( S \) is the union of two cosets of an open subgroup of \( \Gamma \) but is not a coset itself.

For abelian \( \Gamma \), it is well-known (see [5], page 73) that if \( S \) is an open coset of \( \Gamma \), then \( \| \chi_S \| = 1 \), and whereas if \( S \) is the union of two cosets of an open subgroup of \( \Gamma \) but is not a coset itself, then

\[
\| \chi_S \| = \begin{cases} 
\frac{2}{q \sin(\pi/2q)} & \text{if } q \text{ is odd} \\
\frac{2}{q \tan(\pi/2q)} & \text{if } q \text{ is even} \\
\frac{4}{\pi} & \text{if } q = \infty
\end{cases}
\]

where \( q \) is the “relative order” of the two cosets forming \( S \). The largest value in (1.1) is \( \frac{4}{3} \) when \( q = 3 \) and the smallest one is \( \frac{1+\sqrt{2}}{2} \) when \( q = 4 \). In particular, the number \( \frac{1+\sqrt{2}}{2} \) in Theorem 1.1 (i) is sharp.

The paper [7] asked whether or not the interval \((1, \frac{\sqrt{17}+1}{4})\) in Theorem 1.1 (ii) could be increased to \((1, \frac{4}{3})\), and we answer this in affirmative in Theorem 3.4. Note that the interval \((1, \frac{4}{3})\) is sharp because of the discussion in the previous paragraph.

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and also since there are idempotents $\chi_S$ of $B(\Gamma)$ with $\|\chi_S\| = 4/3$ but $S$ is not any union of two cosets of open subgroup of $\Gamma$ (see the last paragraph of [7]).

Lesser is known about idempotents in $B(\Gamma)$ with small norms for general locally compact group $\Gamma$. Ilie and Spronk [3] proved that $\chi_S$ is an idempotent in $B(\Gamma)$ with $\|\chi_S\| = 1$ if and only if $S$ is an open coset of $\Gamma$. More generally, Stan proved the following.

**Theorem 1.2** (Stan [9]). Let $\Gamma$ be a locally compact group, and let $\varphi$ be an idempotent function in $M_{cb}A(\Gamma)$ so that $\varphi = \chi_S$ for some nonempty $S \subseteq \Gamma$. If $\|\varphi\|_{cb} < \frac{2}{\sqrt{3}}$, then $S$ is an open coset of $\Gamma$, and in which case $\|\varphi\|_{cb} = 1$.

Here $M_{cb}A(\Gamma)$ is the completely bounded multiplier algebra $M_{cb}A(\Gamma)$ of the Fourier algebra $A(\Gamma)$ and is defined as follows. Since the Fourier algebra $A(\Gamma)$ is the predual of the group von Neumann algebra $VN(\Gamma)$, it has the canonical operator space structure, which makes it a completely contractive operator algebra (see the monograph [2] for more details). The completely bounded multiplier algebra $M_{cb}A(\Gamma)$ of $A(\Gamma)$ consists of those continuous functions $\varphi : \Gamma \to \mathbb{C}$ such that the mapping $f \mapsto \varphi : f, A(\Gamma) \to A(\Gamma)$, is completely bounded, and its completely bounded norm is denoted as $\|\varphi\|_{cb}$ (whereas the Fourier–Stieltjes norm on $\Gamma$ will be simply denoted as $\|\cdot\|$ in this paper). In general, we have

$$B(\Gamma) \subseteq M_{cb}A(\Gamma),$$

but, for amenable locally compact groups $\Gamma$,

$$B(\Gamma) = M_{cb}A(\Gamma),$$

isometrically.

Thus an idempotent of $B(\Gamma)$ with a small norm is always an idempotent of $M_{cb}A(\Gamma)$ with a small(er) norm.

In Theorem 2.2 we increase the number $\frac{2}{\sqrt{3}}$ in Stan’s Theorem 1.2 to the sharp bound of $\frac{1 + \sqrt{2}}{2}$, and so obtaining a generalisation of the first mentioned result of Saeki, Theorem 1.1 (i), to general locally compact groups.

2. Idempotents of $M_{cb}A(\Gamma)$ with norm lesser than $\frac{1 + \sqrt{2}}{2}$

In this section, let $\Gamma$ be any locally compact group, and let $\chi_S$ be an idempotent of $M_{cb}A(\Gamma)$ with $\|\chi_S\|_{cb} < \frac{1 + \sqrt{2}}{2}$. Our aim is to show that $S$ is an open coset of $\Gamma$. It is obvious that $S$ is open, and so, it remains to show that $S$ is a coset of $\Gamma$. By [8] Corollary 6.3 (i), it is sufficient for us to consider the case where $\Gamma$ is discrete.

We first make a simple observation.

**Lemma 2.1.** For any $s \in S$ and $t \in \Gamma$, if $st \in S$ (resp. $ts \in S$), then $st^n \in S$ for every $n \in \mathbb{Z}$ (resp. $t^n s \in S$ for every $n \in \mathbb{Z}$).

**Proof.** By translation, we may (and shall) suppose that $s = e$, the identity of $\Gamma$. Consider $\Gamma_0$ be the (abelian) group generated by $t$, then

$$\|\chi_{S \cap \Gamma_0}\| = \|\chi_{S \cap \Gamma_0}\|_{cb} \leq \|\chi_S\|_{cb} < \frac{1 + \sqrt{2}}{2}.$$ 

So by Saeki’s Theorem 1.1 (i), we see that $S \cap \Gamma_0 = \Gamma_0$. This gives the lemma. □
To get more information out of the assumption on $\|\chi_S\|_{cb}$, we shall follow in the footsteps of [9] and use the connection shown in [1] between the norm $\|\cdot\|_{cb}$ of $M_{cb}A(\Gamma)$ and the Schur multiplier norms described below.

Denote by $K_0$ the space of matrices that have only finitely many nonzero entries whose rows and columns are indexed by $\Gamma$. Then $K_0$ is identified with a subspace of $B(\ell^2(\Gamma))$. Recall that the Schur multiplication of two matrices $A$ and $X$, indexed by $\Gamma$, is defined as

$$(A \bullet X)(s,t) := A(s,t)X(s,t) \quad (s,t \in \Gamma),$$

and for each matrix $A$, indexed by $\Gamma$, its Schur multiplier norm is

$$\|A\|_{\text{Schur}} := \sup \left\{ \frac{\|A \bullet X\|_{B(\ell^2(\Gamma))}}{\|X\|_{B(\ell^2(\Gamma))}} : X \in K_0 \right\}.$$ 

Of course, this discussion works for any index set $\Gamma$, and a particular matrix that is useful for us is the following $3 \times 3$ matrix

$$F_0 := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.\tag{2.1}$$

Using the orthogonal matrix $U := \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1 & -1 \\ \sqrt{2} & -1 & 1 \end{pmatrix}$ and the vector $\xi := \frac{1}{2} \begin{pmatrix} \sqrt{2} \\ 1 \\ 1 \end{pmatrix}$, we see that

$$\|F_0\|_{\text{Schur}} \geq \|A \bullet U\|_{B(\ell^2)} \geq \frac{\|A \bullet U\|_{\ell^2}}{\|\xi\|_{\ell^2}} = \frac{\sqrt{26}}{4} > \frac{1 + \sqrt{2}}{2}.$$ 

As a matter of fact, it is proved in [4, Proposition 5.1(8)] that $\|F_0\|_{\text{Schur}} = \frac{9}{7}$, but the above simple calculation is sufficient for our purpose. Hence, any matrix $A$ that has a submatrix of the form $F_0$ in (2.1) must satisfy

$$\|A\|_{\text{Schur}} > \frac{1 + \sqrt{2}}{2}.\tag{2.2}$$

Returning to our problem on the group $\Gamma$, each function $\varphi : \Gamma \to \mathbb{C}$ defines a matrix $M_\varphi$, indexed by $\Gamma$, by setting

$$M_\varphi(s,t) := \varphi(s^{-1}t) \quad (s,t \in \Gamma).$$

An important fact shown in [1] is that

$$\|\varphi\|_{cb} = \|M_\varphi\|_{\text{Schur}}.\tag{2.3}$$

Hence, the previous paragraph implies that $M_{\chi_S}$ cannot have (2.1) as a submatrix.

Our main result of this section is the following.

**Theorem 2.2.** Let $\Gamma$ be a locally compact group, and let $\varphi$ be an idempotent function in $M_{cb}A(\Gamma)$ so that $\varphi = \chi_S$ for some nonempty $S \subseteq \Gamma$. If $\|\varphi\|_{cb} < \frac{1 + \sqrt{2}}{2}$, then $S$ is an open coset of $\Gamma$.

**Proof.** As discussed above, we may (and shall) suppose that $\Gamma$ is discrete. Also, applying a translation if necessary, we suppose that $e \in S$. So it remains to prove that $S$ is a subgroup of $\Gamma$.

By Lemma (2.1) we see that if $u \in S$, then $u^n \in S$ for every $n \in \mathbb{Z}$. Thus it remains to show that $S$ is closed under multiplication.
We next claim that if \( u, v \in S \), then either \( uv \in S \) or \( vu \in S \). Indeed, assume towards a contradiction that both \( uv \notin S \) and \( vu \notin S \). Then the submatrix of \( M_S \) with rows \( e, u, v \) and columns \( u, v, w \) is

\[
\begin{pmatrix}
\chi_S(e) & \chi_S(u) & \chi_S(v) \\
\chi_S(s) & \chi_S(u^2) & \chi_S(uv) \\
\chi_S(v) & \chi_S(vu) & \chi_S(v^2)
\end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}
\]

by the previous paragraph. This contradicts the previous discussion.

Finally, suppose that \( u, v \in S \), the proof is completed if we can show that \( uv \in S \). The claim shows that either \( uv \in S \) or \( vu \in S \). Assume the latter holds, then from Lemma 2.1 with \( s = v \) and \( t = u \), we obtain that \( vu^{-1} \in S \). Since we must have \( u^{-1} \in S \), this in turn implies, by a similar argument, that \( v^{-1}u^{-1} \in S \). But then, since \( v^{-1}u^{-1} = (uv)^{-1} \), we must have \( uv \in S \). Hence, in any case, \( uv \in S \), and the proof is completed.

\[ \square \]

3. Idempotents of \( B(\Gamma) \) with norm lesser than \( \frac{4}{3} \), for abelian \( \Gamma \)

In this section, let \( \Gamma = (\Gamma, 0, +) \) be a locally compact abelian group. We aim to strengthen Saeki’s Theorem [3, (ii)] by enlarging its range of \((1, \frac{\sqrt{7} + 1}{4})\) to the optimal \((1, \frac{4}{3})\). So let \( \chi_S \) be an idempotent function in \( B(\Gamma) \) with \( \|\chi_S\| \leq \frac{4}{3} \).

Actually, in [7], Saeki works with idempotents of the measure algebra \( M(G) \) on a locally compact abelian group \( G \), and so, our \( \Gamma \) and his \( G \) could be considered the Pontryagin’s duals of each other. Thus \( B(\Gamma) \cong M(G) \) isometrically, and we denote by \( \mu \) the idempotent measure in \( M(G) \) that corresponds to \( \chi_S \).

As in the previous section, we may reduce our problem to the case where \( \Gamma \) is discrete. Thus suppose that \( \Gamma \) is discrete, and so \( G \) is compact.

Saeki’s proof of Theorem 1.1(ii) in [7] invokes the following lemma several times.

**Lemma 3.1** (Saeki). Assume as above. Suppose there exists \( u \) and \( v \) in \( S \) and \( w \) in \( \Gamma \) such that \( u + w \) belongs to \( S \) but neither \( v + w \) nor \( v - w \) belongs to \( S \). Then we have \( \|\mu\| \geq \frac{\sqrt{7} + 1}{4} \).

In the main argument, Saeki uses this lemma to show that if \( S \) is not the union of any two cosets of a subgroup in \( \Gamma \), then \( \|\mu\| \geq \frac{\sqrt{7} + 1}{4} \). The argument used breaks the problem up into many cases, and in the cases where this lemma is not used, it is always shown that in fact \( \|\mu\| \geq \frac{4}{3} \). Thus if we can strengthen this lemma, then Theorem 1.1(ii) is strengthened also. In fact, we prove the following.

**Lemma 3.2**. Assume as above. Suppose there exists \( u, v \in S \), and \( w \in \Gamma \), such that \( u + w \in S \), and \( v + w, v - w \notin S \). Then \( \|\chi_S\| = \|\mu\| \geq \frac{4}{3} \).

**Proof.** Let us define a function \( f \in C(G) \) to be

\[
f(x) = (x, u)[2 + 2(x, w) + \frac{1}{2}(x, w)] + (x, v)[2 - (x, w) - (x, w)].
\]

If \( u - w \in S \), we get \( \int f(x) \, d\mu(x) = \frac{13}{2} \), otherwise it will simply be 6. Next, we calculate the uniform norm \( |f|_{\ell^\infty} \) of \( f \), by taking \( x \in G \), and set \( (x, w) =: e^{i\theta} \). Then we see

\[
|f(x)| \leq \left(2 + 2e^{i\theta} + \frac{1}{2}e^{-i\theta}\right) + 2e^{i\theta} - e^{-i\theta}
= \sqrt{\frac{25}{4} + 10\cos(\theta) + 4\cos^2(\theta) + 2 - 2\cos(\theta)} = \frac{9}{2}.
\]
Thus $|f|_G \leq \frac{9}{2}$. Hence

$$\|\mu\| \geq \frac{|\int_G f(x) \, d\mu(x)|}{|f|_G} \geq \frac{6}{9/2} = \frac{4}{3}.$$ 

This proves our lemma.

Now we can get our desired result.

**Theorem 3.3.** Let $\Gamma$ be a locally compact abelian group, and let $\varphi$ be an idempotent function in $B(\Gamma)$ so that $\varphi = \chi_S$ for some nonempty $S \subseteq \Gamma$. If $\|\varphi\| \in (1, \frac{4}{3})$, then $S$ is the union of two cosets of some open subgroup of $\Gamma$ but is not a coset itself.

**Proof.** This follows from the previous discussion with the argument of [7].

Equivalently, the above translates into the following:

**Theorem 3.4.** Let $G$ be a locally compact abelian group, and let $\mu$ be an idempotent measure on $G$ with $\|\mu\| \in (1, \frac{4}{3})$. Then

$$d\mu(x) = \left[(-x, \gamma_1) + (-x, \gamma_2)\right] \, dm(x)$$

where $m$ is the Haar measure of some compact subgroup $H$ of $G$, and $\gamma_1, \gamma_2$ are distinct characters of $H$.

**References**

[1] M. Bożejko, and G. Fendler, ‘Herz-Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group’, *Boll. Un. Mat. Ital. A* (6) 3 (1984) 297–302.

[2] E.G. Effros, and Z.-J. Ruan, *Operator spaces*, (London Mathematical Society Monographs. New Series, 23, The Clarendon Press, Oxford University Press, New York) 2000

[3] M. Ilie, and N. Spronk, ‘Completely bounded homomorphisms of the Fourier algebras’, *J. Functional Analysis* 225 (2005) 480–499.

[4] R. H. Levene, ‘Norms of idempotent Schur multipliers’, *New York J. Math.* 20 (2014) 325–352.

[5] W. Rudin, *Fourier analysis on groups*, (Interscience Tracts in Pure and Applied Mathematics, No. 12, Interscience Publishers (a division of John Wiley and Sons), New York-London).1962

[6] S. Saeki, ‘On norms of idempotent measures’, *Proc. Amer. Math. Soc.* 19 (1968) 600–602.

[7] S. Saeki, ‘On norms of idempotent measures. II’, *Proc. Amer. Math. Soc.* 19 (1968) 367–371.

[8] N. Spronk, ‘Measurable Schur multipliers and completely bounded multipliers of the Fourier algebras’, *Proc. London Math. Soc. (3)* 89 (2004) 161–192.

[9] A.-M. P. Stan, ‘On idempotents of completely bounded multipliers of the Fourier algebra $A(G)$’, *Indiana Univ. Math. J.* 58 (2009) 523–535.

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