SOME IRREDUCIBLE COMPONENTS OF THE VARIETY OF COMPLEX
n + 1-DIMENSIONAL LEIBNIZ ALGEBRAS

A.KH. KHUDOYBERDIYEV, M. LADRA, K.K. MASUTOVA, AND B.A. OMirov

Abstract. In the present paper we indicate some Leibniz algebras whose closures of orbits under the natural action of GL_n form an irreducible component of the variety of complex n-dimensional Leibniz algebras. Moreover, for these algebras we calculate the bases of their second groups of cohomologies.

1. Introduction

Jean-Louis Loday introduced Leibniz algebras because of considerations in algebraic K-theory [19]. We know that the Lie algebra homology involves the Chevalley-Eilenberg chain complex, which in turns involves exterior powers of the Lie algebra. Loday found that there is a non-antisymmetric generalization where roughly speaking one has the tensor and not the exterior powers of the Lie algebra in the complex; this new complex defines the Leibniz homology of Lie algebras. The Leibniz homology is related to the Hochschild homology in the same way the Lie algebra homology is related to the cyclic homology.

In many cases where the Leibniz algebra involved may depend on the parameters it is useful to know the structure of the set of all Leibniz algebras of a given dimension. The aim of this work is to establish some results from a geometrical point of view in the study of Leibniz algebras. Any Leibniz algebra law is considered as a point of an affine algebraic variety defined by the polynomial equations coming from the Leibniz identity for a given basis. This way provides an description of the difficulties in classification problems referring to the classes of nilpotent and solvable Leibniz algebras. The orbits relative to the action of the general linear group correspond to the isomorphism classes of Leibniz algebras and so classification problems (up to isomorphism) can be reduced to the classification of these orbits. An affine algebraic variety is a union of a finite number of irreducible components and the Zariski open orbits provide interesting classes of Leibniz algebras to be classified. The Leibniz algebras of this class are called rigid.

The research of varieties of Lie algebras laws over the field C complex numbers have been extensively studied, establishing various important structural results and properties. On the contrary, the problem for varieties of Leibniz algebras has not been considered in detail. The research of varieties of Lie and Leibniz algebra laws is essentially based on the cohomological study of Leibniz algebras and on deformation theory. Deformations of arbitrary rings and associative algebras, results about rigid Lie algebras and related cohomology questions, were first investigated in 1964 by Gerstenhaber [12]. Later, the notion of deformation was applied to Lie algebras by Nijenhuis and Richardson [22], where they transform the topological problem related to rigidity into a cohomological problem, proving that a Lie algebra g is rigid if the second group $H_2(g,g)$ of the Chevalley-Eilenberg cohomology vanishes.

In this paper, we are concerned with the structure of the variety Leib$_{n+1}$, the variety of the $(n + 1)$-dimensional Leibniz algebras, in particular, with answers to the following question: What irreducible components do Leib$_{n+1}$ fall into? The answers to this question would allow to describe partially the structures of some Leibniz algebras of dimension $n + 1$. We shall obtain general results on some irreducible components of the variety of finite-dimensional Leibniz algebras and indicate representatives of solvable Leibniz algebras, whose closures of orbits form irreducible components. We hope to develop this line of research in the next works.

The paper is organized as follows. In Section 2 we recall some necessary notions about Leibniz algebras, cohomology and degenerations of Leibniz algebras. In Section 3 we describe derivations of the solvable Leibniz algebras whose nilradical is a filiform algebra of type $F_1^n$ (see below Theorem 2.9), present (1,1)-invariants for various types of solvable algebras and give representatives of irreducible
components of the variety $\text{Lieb}_{n+1}$ of the $(n+1)$-dimensional Leibniz algebras. Finally, in the last subsection, give descriptions of the second cohomology group of the solvable Leibniz algebras whose nilradical is a filiform algebra of type $F_n^1$.

Throughout the paper, we denote by $L$ a finite-dimensional Leibniz algebra over the field of complex numbers. Moreover, in the multiplication table of a Leibniz algebra the omitted products and in the expansion of 2-cocycles the omitted values are assumed to be zero.

2. Preliminaries

In this section we give necessary definitions on Leibniz algebras, cohomology, degenerations and known results.

We present the definition of the main object of our study.

**Definition 2.1.** [19] A Leibniz algebra over a field $F$ is a vector space $L$ equipped with a bilinear map, called bracket,

$$[-,-]: L \times L \rightarrow L$$

satisfying the Leibniz identity:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y],$$

for all $x, y, z \in L$.

The set $\text{Ann}_r(L) = \{x \in L \mid [y, x] = 0, \forall y \in L\}$ is called the right annihilator of the Leibniz algebra $L$. Note that $\text{Ann}_r(L)$ is an ideal of $L$ and for any $x, y \in L$, the elements $[x, x], [x, y] + [y, x] \in \text{Ann}_r(L)$.

2.1. Solvable Leibniz algebras. For a Leibniz algebra $L$ we consider the following central lower and derived series:

$$L^1 = L, \quad L^{k+1} = [L^k, L^1], \quad k \geq 1;$$

$$L^{[i]} = 1, \quad L^{[s+1]} = [L^{[s]}, L^{[s]}], \quad s \geq 1.$$  

**Definition 2.2.** A Leibniz algebra $L$ is said to be nilpotent (respectively, solvable), if there exists $n \in \mathbb{N}$ ($m \in \mathbb{N}$) such that $L^n = 0$ (respectively, $L^{[m]} = 0$). The minimal number $n$ (respectively, $m$) with such property is said to be the index of nilpotency (respectively, of solvability) of the algebra $L$.

Obviously, the index of nilpotency of an $n$-dimensional nilpotent Leibniz algebra is not greater than $n + 1$.

**Definition 2.3.** An $n$-dimensional Leibniz algebra is said to be null-nilpotent if $\dim L^i = n - i - 1, 1 \leq i \leq n + 1$.

Remark that a null-nilpotent Leibniz algebra has maximal index of nilpotency.

**Theorem 2.4** ([2]). An arbitrary $n$-dimensional null-nilpotent Leibniz algebra is isomorphic to the algebra:

$$NF_n : \quad [e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq n - 1,$$

where $\{e_1, e_2, \ldots, e_n\}$ is a basis of the algebra $NF_n$.

From Theorem 2.4 it is easy to see that a nilpotent Leibniz algebra is null-nilpotent if and only if it is a one-generated algebra, i.e., an algebra generated by unique element. Note that this notion has no sense in the Lie algebras case, because they are at least two-generated.

It should be noted that the sum of any two nilpotent (solvable) ideals is nilpotent (solvable).

**Definition 2.5.** The maximal nilpotent (solvable) ideal of a Leibniz algebra is said to be a nilradical (solvable radical) of the algebra.

Below, we present the description of solvable Leibniz algebras whose nilradical is isomorphic to the algebra $NF_n$.

**Theorem 2.6** ([8]). Let $R$ be a solvable Leibniz algebra whose nilradical is $NF_n$. Then there exists a basis $\{e_1, e_2, \ldots, e_n, x\}$ of the algebra $R$ such that the multiplication table of $R$ with respect to this basis has the following form:

$$RNF_n : \begin{cases}
[e_i, e_1] = e_{i+1}, & 1 \leq i \leq n - 1, \\
[x, e_1] = -e_1, & \\
e_i, x = ie_i, & 1 \leq i \leq n.
\end{cases}$$
Definition 2.7. An \( n \)-dimensional Leibniz algebra \( L \) is said to be filiform if \( \dim L^i = n - i \) for \( 2 \leq i \leq n \).

Now let us define a natural graduation for a filiform Leibniz algebra.

Definition 2.8. Given a filiform Leibniz algebra \( L \), put \( L_i = L^i/L^{i+1}, \ 1 \leq i \leq n-1 \), and \( \text{gr}(L) = L_1 \oplus L_2 \oplus \cdots \oplus L_{n-1} \). Then \( [L_i, L_j] \subseteq L_{i+j} \) and we obtain the graded algebra \( \text{gr}(L) \). If \( \text{gr}(L) \) and \( L \) are isomorphic, then we say that the algebra \( L \) is naturally graded.

Thanks to [2] and [24] it is well known that there are three types of naturally graded filiform Leibniz algebras. In fact, the third type encloses the class of naturally graded filiform Lie algebras.

Theorem 2.9. Any complex \( n \)-dimensional naturally graded filiform Leibniz algebra is isomorphic to one of the following pairwise non-isomorphic algebras:

\[
F^1_n: [e_i, e_1] = e_{i+1}, \quad 2 \leq i \leq n-1,
\]

\[
F^2_n: [e_1, e_1] = e_3, \ [e_i, e_1] = e_{i+1}, \quad 3 \leq i \leq n-1,
\]

\[
F^3_n(\alpha) : \begin{cases}
[e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq n-1, \\
[e_i, e_{n+1-i}] = -[e_{n+1-i}, e_i] = \alpha(-1)^{i+1}e_n, & 2 \leq i \leq n-1,
\end{cases}
\]

where \( \alpha \in \{0, 1\} \) for even \( n \) and \( \alpha = 0 \) for odd \( n \).

The following theorem decomposes all \( n \)-dimensional filiform Leibniz algebras into three families of algebras.

Theorem 2.10 ([2]). Any complex \( n \)-dimensional filiform Leibniz algebra admits a basis \( \{e_1, e_2, \ldots, e_n\} \) such that the table of multiplication of the algebra has one of the following forms:

\[
F_1(\alpha_1, \ldots, \alpha_n, \theta) = \begin{cases}
[e_1, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\
[e_1, e_2] = \theta e_n,
\end{cases}
\]

\[
F_2(\beta_1, \ldots, \beta_n, \gamma) = \begin{cases}
[e_1, e_1] = e_3, & 3 \leq i \leq n-1, \\
[e_1, e_2] = \beta e_4 + \beta_5 e_5 + \cdots + \beta_n e_n,
\end{cases}
\]

\[
F_3(\theta_1, \theta_2, \theta_3) = \begin{cases}
[e_1, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\
[e_1, e_i] = -e_{i+1}, & 3 \leq i \leq n-1, \\
[e_1, e_1] = \theta_1 e_n,
\end{cases}
\]

where \( \alpha \in \{0, 1\} \) for even \( n \) and \( \alpha = 0 \) for odd \( n \).

Below we present the description of solvable Leibniz algebras whose nilradical is isomorphic to the algebra \( F^1_n \).

Theorem 2.11 ([9]). An arbitrary \( (n+1) \)-dimensional solvable Leibniz algebra with nilradical \( F^1_n \) is isomorphic to one of the following pairwise non-isomorphic algebras:

\[
R_1 : \begin{cases}
[e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\
x, e_1 = -e_1 - e_2,
\end{cases}
\]

\[
R_2(\alpha) : \begin{cases}
[e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\
x, e_1 = -e_1,
\end{cases}
\]

\[
[e_1, x] = e_1,
\]

\[
[e_i, x] = (i-1)e_i, & 2 \leq i \leq n,
\]

\[
[e_i, x] = (i-1+\alpha)e_i, & 2 \leq i \leq n,
\]
In particular, any (to the description of the variety of Leibniz algebras (similar to Lie algebras case) we refer the reader

4 A.KH. KHUDOYBERDIYEV, M. LADRA, K.K. MASUTOV AND B.A. OMIROV

\[ R_3 : \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n - 1, \\
x, e_1 = -e_1, \\
e_1, x = e_1, \\
x, x = e_n. \end{cases} \]

\[ R_4 : \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n - 1, \\
x, e_1 = -e_1, \\
e_1, x = e_1 + e_n, \\
x, x = (i - n) e_i. \end{cases} \]

\[ R_5(\alpha_i) = R_5(\alpha_4, \ldots, \alpha_n): \begin{cases} [e_1, e_1] = e_3, & 2 \leq i \leq n - 1, \\
x, e_1 = -e_1, \\
e_1, x = e_1, \\
x, x = e_i + \sum_{j=i+2}^n \alpha_j e_j, & 2 \leq i \leq n. \end{cases} \]

Moreover, the first non-vanishing parameter \( \{\alpha_4, \ldots, \alpha_n\} \) in the algebra \( R_5(\alpha_4, \ldots, \alpha_n) \) can be scaled to 1.

Due to the work [6] we conclude that there is no \((n + 1)\)-dimensional solvable Leibniz algebra whose nilradical is an algebra from the family \( F_1(\alpha_4, \ldots, \alpha_n, \theta) \) except the algebra \( F_n^1 \). Moreover, any \((n + 1)\)-dimensional solvable Leibniz algebra whose nilradical is an algebra from the family \( F_3(\theta_1, \theta_2, \theta_3) \) is Lie algebra. Concerning the second family, from [6] we know that there exists a solvable Leibniz algebra of dimension \((n + 1)\) only when the nilradical is one of the following:

\[ L_1 = F_2(0, 0, \ldots, 0, 1), \quad L_2^{\beta + 3} = F_2(0, 0, \ldots, 0, \beta + 3, 0, \ldots, 0), \quad n \text{ is odd}, \]

\[ L_3^{j} = F_2^j(0, 0, \ldots, 0, 1, 0, \ldots, 0, 0), \quad 4 \leq j \leq n. \]

In particular, any \((n + 1)\)-dimensional solvable Leibniz algebra whose nilradical is either \( L_1 \), \( L_2^{\beta + 3} \) or \( L_3^{j} \) is isomorphic, respectively, to the algebra with the following table of multiplication:

\[ R(L_1) : \begin{cases} [e_1, e_1] = e_3, & 3 \leq i \leq n - 1, \\
x, e_1 = -e_1, \\
e_1, x = e_1, \\
x, x = (i - 1)e_i. \end{cases} \]

\[ R(L_2^{\beta + 3}) : \begin{cases} [e_1, e_1] = e_3, & 3 \leq i \leq n - 1, \\
x, e_1 = -e_1, \\
e_1, x = e_1, \\
x, x = (i - 1)e_i. \end{cases} \]

\[ R(L_3^{j}) : \begin{cases} [e_1, e_1] = e_3, & 3 \leq i \leq n - 1, \\
x, e_1 = -e_1, \\
e_1, x = e_1, \\
x, x = (i - 1)e_i. \end{cases} \]

For acquaintance with the definition of cohomology group of Leibniz algebras and its applications to the description of the variety of Leibniz algebras (similar to Lie algebras case) we refer the reader.
to the papers [3] [12] [13] [19] [21] [22]. Here just recall that the second cohomology group of a Leibniz algebra \( L \) with coefficients in a corepresentation \( M \) is the quotient space
\[ HL^2(L, M) = ZL^2(L, M) / BL^2(L, M), \]
where the 2-cocycles \( \varphi \in ZL^2(L, M) \) and the 2-coboundaries \( f \in BL^2(L, M) \) are defined as follows
\[
(d^2 \varphi)(x, y, z) = [x, \varphi(y, z)] - [\varphi(x, z), y] + [\varphi(x, y), z] - \varphi([x, y], z) - \varphi([x, z], y) = 0 \quad (2.1)
\]
and
\[
f(x, y) = [d(x), y] + [x, d(y)] - d([x, y]) \quad \text{for some linear map} \ d. \quad (2.2)
\]

The following proposition summarizes results regarding derivations and the second group cohomology for the algebras \( RFN_n \) and \( R_1 \) from the works [1] and [18].

**Proposition 2.12.**
\[
\begin{align*}
\dim \text{Der}(RFN_n) &= 2, & \dim \text{Der}(R_1) &= 2, \\
\dim BL^2(RFN_n, RFN_n) &= (n + 1)^2 - 2, & \dim BL^2(R_1, R_1) &= (n + 1)^2 - 2, \\
\dim ZL^2(RFN_n, RFN_n) &= (n + 1)^2 - 2, & \dim ZL^2(R_1, R_1) &= (n + 1)^2 - 2, \\
\dim HL^2(RFN_n, RFN_n) &= 0, & \dim HL^2(R_1, R_1) &= 0.
\end{align*}
\]

### 2.2. Degeneration of Leibniz algebras.

The bilinear maps \( V \times V \to V \) form a vector space \( \text{Hom}(V \otimes V, V) \) of dimension \( \dim(V)^3 \), which can be viewed with its natural structure of an affine algebraic variety over the field \( F \). An \( n \)-dimensional Leibniz algebra \( L \) of the variety \( \text{Leib}_n \) may be considered as an element \( \lambda(L) \) via the bilinear mapping \( \lambda \): \( L \otimes L \to L \) satisfying Leibniz identity.

The group \( \text{GL}_n(F) \) naturally acts on \( \text{Leib}_n \) via change of basis, i.e.,
\[
(g * \lambda)(x, y) = g\left(\lambda(g^{-1}(x), g^{-1}(y))\right), \quad g \in \text{GL}_n(F), \quad \lambda \in \text{Leib}_n.
\]

The orbits \( \text{Orb}(\cdot) \) under this action are the isomorphism classes of algebras.

Note that solvable Leibniz algebras of the same dimension also form an invariant subvariety of the variety of Leibniz algebras under the mentioned action.

**Definition 2.13.** It is said that an algebra \( \lambda \) degenerates to an algebra \( \mu \), if \( \text{Orb}(\mu) \) lies in the Zariski closure of \( \text{Orb}(\lambda) \) (denoted by \( \overline{\text{Orb}(\lambda)} \)). We denote this by \( \lambda \to \mu \).

It is remarkable that \( \text{Orb}(NF_n) \) and \( \text{Orb}(RFN_n) \) are open sets in the subvariety of \( n \)-dimensional nilpotent Leibniz algebras [2] and the variety of \( (n + 1) \)-dimensional Leibniz algebras [1], respectively (they are so-called rigid algebras).

In the case when the ground field is the complex numbers \( \mathbb{C} \), we give an equivalent definition of degeneration.

**Definition 2.14.** Let \( g \): \( (0, 1] \to \text{GL}_n(V) \) be a continuous mapping. We construct a parameterized family of Leibniz algebras \( g_t = (V, [-, -]_t), t \in (0, 1] \) isomorphic to \( L \). For each \( t \) the new Leibniz bracket \( [-, -]_t \) on \( V \) is defined via the old one as follows: \( [x, y]_t = g_t[g_t^{-1}(x), g_t^{-1}(y)] \), \( \forall x, y \in V \). If for any \( x, y \in V \) there exists the limit
\[
\lim_{t \to 0^+} [x, y]_t = \lim_{t \to 0^+} g_t[g_t^{-1}(x), g_t^{-1}(y)] =: [x, y]_0,
\]
then \( [-, -]_0 \) is a well-defined Leibniz bracket. The Leibniz algebra \( L_0 = (V, [-, -]_0) \) is called a degeneration of the algebra \( L \).

For given Leibniz algebras \( \lambda, \mu \in \text{Leib}_{n+1} \) sometimes it is quite difficult to establish the existence of degeneration \( \lambda \to \mu \). It is helpful to obtain some necessary invariant conditions for the existence of a degeneration. The complete list of the invariants conditions can be found in the works [5], [14], [16], [23]. Here we give some of them which we shall use.

We denote by \( \text{Der}(\lambda) \), \( \lambda^m, \text{Lie}(\lambda) \) the space of derivations, powers and maximal Lie subalgebra of the algebra \( \lambda \), respectively.

Let \( \lambda \to \mu \) be a nontrivial degeneration. Then the following inequalities hold:
\[
\dim \text{Der}(\lambda) < \dim \text{Der}(\mu), \quad \dim \lambda^m \geq \dim \mu^m \quad \text{for any} \ m \in \mathbb{N}, \quad \dim \text{Lie}(\lambda) \leq \dim \text{Lie}(\mu). \quad (2.3)
\]

Further we shall use \((i, j)\)-invariant. This invariant was given for Lie algebras in [5] and it is also applicable for Leibniz algebras.
Let $\lambda \in \mathbf{Leib}_{n+1}$ with structure constants $(\gamma^k_{i,j})$, and $(i,j)$ be a pair of positive integers such that

$$c_{i,j} = \frac{\text{tr}(ad x)^i \cdot \text{tr}(ad y)^j}{\text{tr}((ad x)^i \circ (ad y)^j)}$$

is independent of the elements $x, y$ of the Leibniz algebra $\lambda$. Then $c_{i,j}(\lambda) = c_{i,j}(\lambda)$ is a quotient of two polynomials in $C[\gamma^k_{i,j}]$. If neither of these polynomials is zero, we call $c_{i,j} \in C[\gamma^k_{i,j}]$ an $(i,j)$-invariant of $\lambda$. Suppose $\lambda \in \mathbf{Leib}_{n+1}$ has an $(i,j)$-invariant $c_{i,j}$. Then all $\mu \in \text{Orb}(\lambda)$ have the same $(i,j)$-invariant.

Denote by $\mathcal{LR}_n(N)$ the set of all $n$-dimensional solvable Leibniz algebras whose nilradical is $N$. In the paper [7] it is proved that a given degeneration between two solvable Leibniz algebras implies some restriction on their nilradicals. In particular, in the case of equality of dimensions of nilradicals the existence of degeneration between solvable Leibniz algebras implies the existence of degeneration between their nilradicals.

**Proposition 2.15.** Let $R_1, R_2$ be $n$-dimensional solvable Leibniz algebras and let $R_1 \in \mathcal{LR}_n(N_1), R_2 \in \mathcal{LR}_n(N_2)$. If $R_1 \rightarrow R_2$, then $\dim N_2 \geq \dim N_1$. Moreover, if $\dim N_1 = \dim N_2$ and $R_1 \rightarrow R_2$, then $N_1 \rightarrow N_2$.

3. Some irreducible components of the variety $\mathbf{Leib}_{n+1}$

In this section we present some irreducible components of the variety $\mathbf{Leib}_{n+1}$ in terms of closures of orbits of some Leibniz algebras. The following equality $\dim \text{Orb}(\lambda) = (n + 1)^2 - \dim \text{Der}(\lambda)$ gives us the dimensional relation between orbits and derivations of an algebra $\lambda$.

Let $L$ be an $(n+1)$-dimensional solvable Leibniz algebra whose nilradical is the filiform algebra $F^1_n$. The proposition below describes derivations of the algebras from $R_2(\alpha) - R_3(\alpha)$ in the list of Theorem 2.11.

**Proposition 3.1.** Derivations of the algebras $R_2(\alpha) - R_3(\alpha)$ have the following form:

$$\begin{align*}
\text{Der}(R_2(\alpha)) : & \left\{ \begin{array}{l}
d_1(e_1) = e_1, \quad d_1(e_i) = (i - 2)e_i, \quad 3 \leq i \leq n, \\
d_2(e_i) = e_i, \quad 2 \leq i \leq n, \\
d_3(x) = -e_1, \quad d_3(e_i) = e_{i+1}, \quad 2 \leq i \leq n - 1; \\
\end{array} \right.
\text{Der}(R_2(2-n)) : & \left\{ \begin{array}{l}
d_1(e_1) = e_1, \quad d_1(e_i) = (i - 2)e_i, \quad 3 \leq i \leq n, \\
d_2(e_i) = e_i, \quad 2 \leq i \leq n, \\
d_3(x) = -e_{n-1}, \quad d_3(e_i) = e_{n+1}, \quad 2 \leq i \leq n - 1, \\
\end{array} \right.
\text{Der}(R_2(1-n)) : & \left\{ \begin{array}{l}
d_1(e_1) = e_1, \quad d_1(e_i) = (i - 2)e_i, \quad 3 \leq i \leq n, \\
d_2(e_i) = e_i, \quad 2 \leq i \leq n, \\
d_3(x) = -e_1, \quad d_3(e_i) = e_{i+1}, \quad 2 \leq i \leq n - 1, \\
\end{array} \right.
\text{Der}(R_3) : & \left\{ \begin{array}{l}
d_1(e_1) = e_1, \quad d_1(e_i) = (i - n)e_i, \quad 2 \leq i \leq n - 1, \\
d_2(x) = e_1, \quad d_2(e_i) = -e_{i+1}, \quad 2 \leq i \leq n - 1, \\
d_3(x) = e_n; \\
\text{Der}(R_4) : & \left\{ \begin{array}{l}
d_1(e_1) = e_1, \quad d_1(e_i) = (i + 1 - n)e_i, \quad 2 \leq i \leq n, \\
d_2(x) = -e_1, \quad d_2(e_i) = e_{i+1}, \quad 2 \leq i \leq n - 1, \\
d_3(e_1) = e_n, \quad d_3(x) = -e_{n-1}; \\
\text{Der}(R_5(0)) : & \left\{ \begin{array}{l}
d_1(e_1) = e_1, \quad d_1(e_i) = (i - 1)e_i, \quad 2 \leq i \leq n, \\
d_2(e_i) = e_i, \quad 2 \leq i \leq n, \\
d_3(e_1) = e_j, \quad d_j(e_i) = e_{i+j-2}, \quad 3 \leq j \leq n, \quad 2 \leq i \leq n - j + 2; \\
\text{Der}(R_5(\alpha_i)) : & \left\{ \begin{array}{l}
d_1(e_1) = e_1, \quad d_2(e_i) = e_2 + \alpha_n e_n, \quad d_1(e_1) = e_i, \quad 3 \leq i \leq n, \\
d_2(e_i) = e_{i+1}, \quad d_j(e_i) = e_{i+j-1}, \quad 2 \leq j \leq n - 1, \quad 2 \leq i \leq n - j + 1,
\end{array} \right.
\end{align*}$$

where in the case of $R_5(\alpha_i)$ one of the parameters $\alpha_i$ is nonzero.

**Proof.** The proof of the proposition is carrying out by direct computations. \hfill $\Box$

Due to equality 2.2 defining the space $BL^2$ we have
Corollary 3.2.
\[
\dim BL^2(R_2(\alpha), R_2(\alpha)) = \begin{cases} 
(n+1)^2 - 4, & \alpha = 2 - n \text{ or } 1 - n, \\
(n+1)^2 - 3, & \alpha \neq 2 - n, 1 - n;
\end{cases}
\]
\[
\dim BL^2(R_3, R_3) = (n+1)^2 - 3;
\]
\[
\dim BL^2(R_4, R_4) = (n+1)^2 - 3;
\]
\[
\dim BL^2(R_5(\alpha_i), R_5(\alpha_i)) = \begin{cases} 
\alpha_i = 0 \text{ for all } i, \\
\alpha_i \neq 0 \text{ for some } i.
\end{cases}
\]

The following result presents values of \((1, 1)\)-invariants for the algebras \(RNF_n, R_1 - R_5(\alpha_i), R(L_1), R(L_2^{\frac{a+3}{2}}), R(L_3^a)\).

Proposition 3.3.
\[
c_{1,1}(RNF_n) = \frac{(1 + 2 + 3 + \cdots + n)^2}{2(2n+1)},
\]
\[
c_{1,1}(R_1) = \frac{(1 + 2 + 3 + \cdots + n - 1)^2}{1 + 1 + 2^2 + 3^2 + \cdots + (n-1)^2} = \frac{3(n^2 - n + 2)}{2(6 + n(n-1)(2n-1))},
\]
\[
c_{1,1}(R(L_1)) = \frac{(1 + 2 + 3 + \cdots + n - 1 + \frac{n-1}{2})^2}{1 + 2^2 + 3^2 + \cdots + (n-1)^2 + \left(\frac{n-1}{2}\right)^2} = \frac{3(n^2 - 1)}{4n-3},
\]
\[
c_{1,1}(R(L_2^{\frac{a+3}{2}})) = \frac{(1 + 2 + 3 + \cdots + n - 1 + \frac{n-1}{2})^2}{1 + 2^2 + 3^2 + \cdots + (n-1)^2 + \left(\frac{n-1}{2}\right)^2} = \frac{3(n^2 - 1)}{4n-3},
\]
\[
c_{1,1}(R(L_3^a)) = \frac{(1 + 2 + 3 + \cdots + n - 1 + j - 2)^2}{1 + 2^2 + 3^2 + \cdots + (n-1)^2 + (j-2)^2} = \frac{3(n^2 - n + 2j - 4)}{4(n^3 - 2n^2 + n + 3(j-2))},
\]
\[
c_{1,1}(R_2(\alpha)) = \frac{(1 + (1 + \alpha) + (2 + \alpha) + \cdots + (n - 1 + \alpha))^2}{1^2 + (1 + \alpha)^2 + (2 + \alpha)^2 + \cdots + (n - 1 + \alpha)^2},
\]
\[
c_{1,1}(R_3) = \frac{(1 + (2 - n) + (3 - n) + \cdots + (-1))^2}{1^2 + (2 - n)^2 + (3 - n)^2 + \cdots + 1^2} = \frac{3(n^2 - 3)}{2(2n^2 - 9n + 15)},
\]
\[
c_{1,1}(R_4) = \frac{(1 + (3 - n) + (4 - n) + \cdots + (-1) + 1)^2}{1^2 + (3 - n)^2 + (4 - n)^2 + \cdots + 1^2 + 0 + 1} = \frac{3(n^2 - 5n + 10)}{2(2n^3 - 15n^2 + 37n - 18)},
\]
\[
c_{1,1}(R_5(\alpha_i)) = n - 1.
\]

Proof. The proof of the proposition is carried out by direct calculations.

Remark 3.4. From Proposition 3.3 it is easy to see that
\[
c_{1,1}(R_2(1)) = c_{1,1}(RNF_n), \quad c_{1,1}(R_2(0)) = c_{1,1}(R_1),
\]
\[
c_{1,1}(R_2(1-n)) = c_{1,1}(R_3), \quad c_{1,1}(R_2(2-n)) = c_{1,1}(R_4).
\]

Below, we give some degenerations.

Example 3.5. There exist the following degenerations
\[
F_n^1 \to F_n^2 \quad \text{via} \quad g_t(e_1) = e_1 - t^{-1}e_2, \quad g_t(e_2) = t^{-1}e_2, \quad g_t(e_i) = e_i, \quad 3 \leq i \leq n;
\]
\[
RNF_n \to R_2(1) \quad \text{via} \quad g_t(x) = x, \quad g_t(e_1) = t^{-1}e_1, \quad g_t(e_1) = t^{-i+2}e_i, \quad 2 \leq i \leq n;
\]
\[
R_1 \to R_2(0) \quad \text{via} \quad g_t(x) = x, \quad g_t(e_1) = e_1, \quad g_t(e_i) = te_i, \quad 2 \leq i \leq n;
\]
\[
R_3 \to R_2(1-n) \quad \text{via} \quad g_t(x) = x, \quad g_t(e_1) = e_1, \quad g_t(e_i) = te_i, \quad 2 \leq i \leq n;
\]
\[
R_4 \to R_2(2-n) \quad \text{via} \quad g_t(x) = x, \quad g_t(e_1) = e_1, \quad g_t(e_i) = te_i, \quad 2 \leq i \leq n;
\]
\[
R_5(\alpha) \to R_5(0) \quad \text{via} \quad g_t(x) = x, \quad g_t(e_1) = te_1, \quad g_t(e_i) = te_i, \quad 2 \leq i \leq n.
\]

Now we present representatives of irreducible components of the variety \(\text{Leib}_{n+1}\).

Theorem 3.6. \(\text{Orb}(R_3), \text{Orb}(R_4)\) and \(\bigcup_{\alpha \in K} \text{Orb}(R_2(\alpha))\), where \(K = \mathbb{C} \setminus \{0, 1, 1 - n, 2 - n\}\), are irreducible components of \(\text{Leib}_{n+1}\).

Proof. Firstly, we will prove the assertion of the theorem for the algebra \(R_3\), that is, we will prove the non-existence of degeneration from any algebra \(X \in \text{Leib}_{n+1}\) to the algebra \(R_3\). Let us assume the contrary, i.e., \(X \to R_3\), then from Proposition 3.1 and inequalities (2.3) we conclude that \(\dim(\text{Der }X) < \)
dim(Der $R_3$) = 3. Actually, the algebra $X$ is a solvable Leibniz algebra. Indeed, if $X$ is not solvable then by Levi’s Theorem \[4\] it decomposes into a semidirect sum of a semi-simple Lie algebra $S$ and a solvable radical $Rad(X)$. Since in a semi-simple Lie algebra (which, clearly, has dimension greater or equal to 3) an operator $ad(x)$ for any $x \in S$ is a derivation of the algebra $X$, then we obtain dim(Der $X$) $\geq$ 3. Therefore, the algebra $X$ is solvable.

Taking into account that dim $R_2^2$ = $n$ and solvability of $X$ from inequalities \[2.3\], we derive that dim $X^2$ = $n$. Since the square of a solvable algebra belongs to the nilradical, then the solvable algebra $X$ has nilradical of dimension $n$. Therefore, the nilradical of $X$ degenerates to $F_2^1$ (because nilradical of the algebra $R_3$ is $F_2^1$). Again applying inequality \[2.3\] we derive that the nilradical of $X$ is one of the following algebras:

$$NF_n, \quad F_2^1(\alpha_4, \alpha_5, \ldots, \alpha_n, \theta), \quad F_2(\beta_4, \ldots, \beta_n, \gamma), \quad F_3(\theta_1, \theta_2, \theta_3).$$

The non-trivial degeneration $F_2^1 \rightarrow F_2^2$ from Example 3.3 implies the non-existence of degeneration from the algebra $F_2^2$ to the algebra $F_2^1$.

From arguments above we conclude that the possibilities for the solvable algebra $X$ are the following:

$$RNF_n, \quad R_1-R_5(\alpha_i), \quad R(L_1), \quad R(L_2^{\beta_4+2}), \quad R(L_3^{\gamma}).$$

Comparing $c_{1,1}$ invariants from Proposition 2.3 dimensions of the spaces of derivations from Propositions 2.12, 3.1 and applying inequalities (2.3), we conclude that none of the above algebras degenerates to the algebra $R_3$. Hence, $\text{Orb}(R_3)$ is an irreducible component of the variety $\text{Leib}_{n+1}$.

The assertion of theorem for the algebra $R_3$ and the family of algebras $R_2(\alpha)$, with $\alpha \in K$, is proved applying the same arguments as used for the algebra $R_3$ and degenerations from Example 3.3.

Let us assume that $X \rightarrow R_5(\alpha_4, \ldots, \alpha_n)$. Since dim Der($R_5(\alpha_4, \ldots, \alpha_n)$) $>$ 3 then from dimensions arguments as they were used for the algebra $R_3$ we cannot assert the solvability of $X$. Nevertheless applying similar arguments as in Theorem 3.6 we obtain the following result.

**Proposition 3.7.** $U_{n \in \mathbb{C}} \text{Orb}(R_5(\alpha_4, \ldots, \alpha_n))$ forms an irreducible component in the subvariety of $(n + 1)$-dimensional solvable Leibniz algebras of the variety $\text{Leib}_{n+1}$.

### 3.1. Second cohomology of the algebras $R_2 - R_3$.

In this subsection we give descriptions of the second cohomology group of the algebras $R_2 - R_3$ by presenting their bases. In fact, we find bases of the space $BL^2$ and $ZL^2$ for these algebras. These descriptions can be applied in the study of infinitesimal deformations and extensions of mentioned algebras (see works [10, 11, 15, 17, 21, 22] and references therein).

In the next theorem we present the general form of 2-cocycles for the algebra $R_3$.

**Theorem 3.8.** An arbitrary $\varphi \in ZL^2(R_3, R_3)$ has the following form:

$$\varphi(e_1,e_1) = \sum_{i=1}^{n} a_{1,i}e_i,$$

$$\varphi(e_i,e_1) = a_{i,0}x + \sum_{s=1}^{n} a_{i,s}e_s, \quad 2 \leq i \leq n - 1,$$

$$\varphi(e_n,e_1) = a_{n-1,0}e_1 - \sum_{i=1}^{n-1} (\sum_{t=2}^{i-1} a_{n+i-t-1,i})e_i + a_{n,n}e_n,$$

$$\varphi(e_1,e_2) = b_{1,1}e_1,$$

$$\varphi(e_i,e_2) = (i-n)b_{1,1}e_1 + b_{2,3}e_{i+1}, \quad 2 \leq i \leq n,$$

$$\varphi(e_1,e_3) = -a_{1,0}e_1, \quad 3 \leq i \leq n,$$

$$\varphi(e_{i},e_3) = (n-i)a_{2,0}e_i - (a_{2,1} + b_{1,1})e_{i+1}, \quad 2 \leq i \leq n - 1,$$

$$\varphi(e_{i},e_j) = (n-i)a_{j-1,0}e_i + (a_{j-2,0} - a_{j-1,1})e_{i+1}, \quad 2 \leq i \leq n - 1, \quad 4 \leq j \leq n,$$

$$\varphi(e_1,x) = \frac{2}{(n-2)(n-1)} \sum_{i=2}^{n} a_{1,i}x + c_1e_1 + \sum_{i=2}^{n-1} (i-n-1)a_{1,i+1}e_i + c_1 e_n,$$

$$\varphi(e_2,x) = (n-2)b_{1,1}x + (n-1)b_{2,3}e_1 + \sum_{i=2}^{n} c_{2,i}e_i.$$
\[ \varphi(e_3, x) = (3 - n)a_{2,0}x + (2 - n)(a_{2,1} + b_{1,1})e_1 + (a_{2,2} - \frac{2}{n - 1} \sum_{t=2}^{n} a_{t,t})e_2 + (c_{2,2} + c_{1,1})e_3 \]
\[ + \sum_{i=4}^{n-1} (c_{2,i-1} + (3 - i)a_{2,i})e_i + (c_{2,n-1} + (3 - n)a_{2,n} - a_{2,0})e_n. \]
\[ \varphi(e_i, x) = (i - n)a_{i-1,0}x + (n - i + 1)(a_{i-2,0} - a_{i-1,1})e_1 + \sum_{s=2}^{i-2} (i - s) \sum_{t=2}^{n} a_{i+t-1-t,s}e_s \]
\[ + \sum_{t=2}^{i-1} a_{t,t} + \frac{(i + 1 - 2n)(i - 2)}{(n - 2)(n - 1)} \sum_{t=2}^{n} a_{t,t}e_i + (c_{2,2} + (i - 2)c_{1,1})e_i \]
\[ + \sum_{s=i+1}^{n-1} (c_{2,s-i+2} + (i - s) \sum_{t=2}^{i-1} a_{t,s-i+t+1})e_s \]
\[ + (c_{2,n-i+2} - a_{i-1,0} + (i - n) \sum_{i=2}^{i-1} a_{t,n-i+t+1})e_n, \quad 4 \leq i \leq n, \]
\[ \varphi(x, e_1) = -\frac{2}{(n - 2)(n - 1)} \sum_{t=2}^{n} a_{t,t}x - c_{1,1}e_1 + a_{1,3}e_2 + \sum_{i=3}^{n} d_{1,i}e_i, \]
\[ \varphi(x, e_2) = -b_{2,3}e_1 + b_{1,1}e_n, \]
\[ \varphi(x, e_3) = (a_{2,1} + b_{1,1})e_1 - a_{2,0}e_n, \]
\[ \varphi(x, e_i) = (a_{i-1,1} - a_{i-2,0})e_1 - a_{i-1,0}e_n, \quad 4 \leq i \leq n, \]
\[ \varphi(x, a_{n-1,0}x + (a_{n-1,1} - a_{n-2,0})e_1 + \sum_{i=2}^{n-2} ((n - i)(a_{1,i+2} - d_{1,i+1}) + \sum_{s=2}^{i} a_{n-i+s-1,s})e_i \]
\[ - (2c_{1,0} + d_{1,n} + c_{1,n} + a_{n,n})e_{n-1} + \beta_ne_n. \]

Proof. We set

\[ \varphi(e_i, e_1) = a_{i,0}x + \sum_{s=1}^{n} a_{i,s}e_s, \quad 1 \leq i \leq n, \quad \varphi(e_1, x) = c_{1,0}x + \sum_{i=1}^{n} c_{1,i}e_i, \]
\[ \varphi(e_1, e_2) = b_{1,0}x + \sum_{i=1}^{n} b_{1,i}e_i, \quad \varphi(e_2, x) = c_{2,0}x + \sum_{i=1}^{n} c_{2,i}e_i, \]
\[ \varphi(e_2, e_2) = b_{2,0}x + \sum_{i=1}^{n} b_{2,i}e_i, \quad \varphi(x, e_1) = d_{1,0}x + \sum_{i=1}^{n} d_{1,i}e_i, \]
\[ \varphi(x, x) = \beta_0x + \sum_{i=1}^{n} \beta_ie_i, \quad \varphi(x, e_2) = d_{2,0}x + \sum_{i=1}^{n} d_{2,i}e_i. \]

Applying equality (2.1) for the triple \((e_i, e_1, e_1)\), we obtain \([e_i, \varphi(e_1, e_1)] = 0\). Hence, \(\varphi(e_i, e_1) = \sum_{i=2}^{n} a_{1,i}e_i\). Similarly, the equation \((d^2\varphi)(e_i, e_j, e_k) = 0\), for \(2 \leq j, k \leq n\), leads to \([e_i, \varphi(e_j, e_k)] = 0\). Consequently, we have \(\varphi(e_j, e_k) \in \text{span} \{e_2, e_3, \ldots, e_n\}\).

Considering \((d^2\varphi)(e_1, e_1, e_2) = 0\), we derive

\[ \varphi(e_1, e_2) = b_{1,0}x + b_{1,1}e_1 + b_{1,n}e_n. \]

Moreover, from \((d^2\varphi)(e_i, e_1, e_2) = 0\) with \(2 \leq i \leq n - 1\), we deduce

\[ \varphi(e_{i+1}, e_j) = [e_i, \varphi(e_1, e_2)] + [\varphi(e_1, e_2), e_1], \]

which inductively get

\[ \varphi(e_i, e_2) = b_{1,0} \frac{(i - 2)(i + 1 - 2n)}{2} e_i + ((i - 2)b_{1,1} + b_{2,2})e_i + \sum_{s=i+1}^{n} b_{2,s-i+2}e_s, \quad 2 \leq i \leq n. \]

Similarly, we have

\[ (d^2\varphi)(e_0, e_1, e_2) = 0 \Rightarrow b_{1,0} = 0, \]
\[ (d^2\varphi)(e_1, e_2, x) = 0 \Rightarrow b_{1,n} = 0, \quad c_{2,0} = (n - 2)b_{1,1}, \]
\[ (d^2\varphi)(e_2, e_2, x) = 0 \Rightarrow b_{2,2} = (2 - n)b_{1,1}, \quad c_{2,1} = (n - 1)b_{1,3}, \quad b_{2,i} = 0, \quad 4 \leq i \leq n. \]
Now we consider \((d^2 \varphi)(e_i, e_j, e_1) = 0\) with \(2 \leq j \leq n - 1\). Then we have
\[
\varphi(e_i, e_{j+1}) = [\varphi(e_i, e_j), e_1] - [e_i, \varphi(e_j, e_1)] - \varphi([e_i, e_1], e_j).
\]
Applying the induction on \(j\) for any \(i\) and the equality above we obtain the following:
\[
\begin{align*}
\varphi(e_i, e_j) &= -a_{j-1,0}e_1, & 3 \leq j \leq n, \\
\varphi(e_i, e_3) &= (n-i)a_{2,0}e_i - (a_{2,1} + b_{1,1})e_{i+1}, & 2 \leq i \leq n - 1, \\
\varphi(e_i, e_j) &= (n-i)a_{j-1,0}e_i + (a_{j-2,0} - a_{j-1,1})e_{i+1}, & 2 \leq i \leq n - 1, \quad 4 \leq j \leq n.
\end{align*}
\]
On the other hand, the condition \((d^2 \varphi)(e_i, e_n, e_1) = 0\) implies
\[
a_{n,0} = 0, \quad a_{n,1} = a_{n-1,0}.
\]
We consider equality (2.1) for the triple \((e_1, x, e_1)\), then we get
\[
d_{1,0} = -c_{1,0}, \quad a_{1,2} = 0, \quad e_{1,t} = (i - 1 - n)a_{1, t+1}, \quad 2 \leq i \leq n - 1.
\]
Thus, we have
\[
\begin{align*}
\varphi(e_1, x) &= c_{1,0}x + c_{1,1}e_1 + \sum_{i=2}^{n-1} (i - n - 1)a_{1, i+1}e_i + c_{1,n}e_n, \\
\varphi(e_2, x) &= (n-2)b_1,x + (n-1)b_{2,3}e_1 + \sum_{i=2}^{n} c_{2,i}e_i
\end{align*}
\]
and \([e_1, \varphi(e_i, x)] = (i-n)a_{1,0}e_1 \) for \(3 \leq i \leq n\).

From the equality \((d^2 \varphi)(e_i, e_1, x) = 0\), \(2 \leq i \leq n - 1\), we have
\[
\begin{align*}
\varphi(e_{i+1}, x) &= \varphi(e_i, e_1) + [e_i, \varphi(e_1, x)] - [\varphi(e_i, e_1), x] + \varphi(e_i, [e_1, x]) + \varphi([e_i, x], e_1) \\
&= [\varphi(e_i, e_1)] + (i + 1 - n)x + c_{1,0}e_i + c_{1,1}e_{i+1} + (i - n)a_{1,1}e_1 + \sum_{s=2}^{n} (i + 1 - s)a_{i,s}e_s.
\end{align*}
\]
Hence, we obtain inductively that
\[
\begin{align*}
\varphi(e_3, x) &= (3-n)a_{2,0}x + (2-n)(a_{2,1} + b_{1,1})e_1 + (a_{2,2} + (2-n)c_{1,0})e_2 + (c_{2,2} + c_{1,1})e_3 \\
&+ \sum_{i=4}^{n-1} (c_{2,i-1} + (3-i)a_{2,i})e_i + (c_{2,n-1} + (3-n)a_{2,n} - a_{2,0})e_n, \\
\varphi(e_i, x) &= (i-n)a_{i-1,0}x + (n-i+1)(a_{i-2,0} - a_{i-1,1})e_1 + \sum_{s=2}^{i-2} (i-s) \sum_{t=2}^{s} a_{i+t-s-1,t}e_s \\
&+ \left(\frac{(i+1-2n)(i-2)}{2}\right)c_{1,0} + \sum_{i=2}^{i-1} a_{i, t}e_{i-1} + (c_{2,2} + (i - 2)c_{1,1})e_i \\
&+ \sum_{s=i+1}^{n-1} (c_{2,s-i+2} + (i-s) \sum_{t=2}^{i-1} a_{i+s-t+t+1}e_s + (c_{2,n-i+2} - a_{i-1,0} + (i-n) \sum_{t=2}^{i-1} a_{t,n-i+t+1})e_n,
\end{align*}
\]
where \(4 \leq i \leq n\).

Moreover, the condition \((d^2 \varphi)(e_n, e_1, x) = 0\) implies \([\varphi(e_n, x), e_1] - [\varphi(e_n, e_1), x] + \varphi(e_n, [e_1, x]) = 0\), which derives
\[
(n-1)a_{n,2}e_2 + \sum_{s=3}^{n-1} (n-s+1)(a_{n,s} + \sum_{t=2}^{s-1} a_{n+t-s,t})e_s + \left(-\frac{(n-1)(n-2)}{2}\right)c_{1,0} + \sum_{s=2}^{n} a_{s,t}e_s = 0.
\]
Thus, we get
\[
a_{n,2} = 0, \quad c_{1,0} = \frac{2}{(n-1)(n-2)} \sum_{t=2}^{n} a_{t,t}, \quad a_{n,s} = -\sum_{t=2}^{s-1} a_{n+t-s,t}, \quad 3 \leq s \leq n - 1.
\]
Considering equality (2.1) for the following triples \((e_2, x, e_1), (e_1, x, e_2), (x, e_1, e_2), (e_2, x, e_2)\) we obtain:
\[
d_{1,1} = -c_{1,1}, \quad d_{2,0} = 0, \quad d_{2,s} = 0, \quad 2 \leq s \leq n - 1, \quad d_{2,1} = -b_{2,3}.
\]
From \((d^2 \varphi)(x, e_1, e_1) = 0\) with \(2 \leq i \leq n - 1\), we have
\[
\varphi(x, e_{i+1}) = [\varphi(x, e_i), e_1] - [x, \varphi(e_i, e_1)] + \varphi(e_i, e_i).
\]
which inductively implies
\[
\varphi(x, e_3) = (a_{2,1} + b_{1,1})e_1 - a_{2,0}e_n, \\
\varphi(x, e_i) = (a_{i-1,1} - a_{i-2,0})e_1 - a_{i-1,0}e_n, \quad 4 \leq i \leq n.
\]
Finally, the equalities \((d^2\varphi)(e_1, x, x) = (d^2\varphi)(x, e_1, x) = (d^2\varphi)(x, e_2, x) = 0\) imply
\[
d_1,2 = a_{1,3}, \quad d_2, n = b_{1,1},
\]
\[
\beta_0 = a_{n-1,0}, \quad \beta_1 = a_{n-1,1} - a_{n-2,0},
\]
\[
\beta_{n-1} = -d_{1,n} - c_{1,n} - a_{n,n} - \frac{4}{(n-1)(n-2)} \sum_{i=2}^{n} a_{i,t},
\]
\[
\beta_i = (n-i)(a_{1,i+2} - d_{1,i+1}) + \sum_{s=2}^{i} a_{n-i+s-1}, \quad 2 \leq i \leq n-2,
\]
which complete the proof of the theorem. \(\square\)

**Corollary 3.9.** \(\dim ZL^2(R_3, R_3) = (n+1)^2 - 2\) and \(\dim HL^2(R_3, R_3) = 1.\)

Now we present a basis of \(HL^2(R_3, R_3).\)

**Proposition 3.10.** The adjoin class \(\xi\) forms a basis of \(HL^2(R_3, R_3),\) where
\[
\xi : \begin{cases}
\xi(e_1, x) = e_1, \\
\xi(e_i, x) = (i-2)e_i, & 3 \leq i \leq n, \\
\xi(x, e_1) = -e_1.
\end{cases}
\]

**Proof.** In order to find a basis of \(HL^2(R_3, R_3)\) we need to describe linear independent elements which lie in \(ZL^2(R_3, R_3)\) and do not lie in \(BL^2(R_3, R_3).\) For achieve this purpose we will find a basis of 2-cocyles and 2-coboundaries.

Since an arbitrary element of \(ZL^2(R_3, R_3)\) has the form of Theorem 3.4, we shall use this description.

Note that there are parameters \((a_{i,j}, b_{1,1}, b_{2,3}, c_{1,1}, c_{1,n}, \beta, c_{2,k}, d_{1,s})\) in the general form of elements \(ZL^2(R_3, R_3).\) One of the natural basis of the space \(ZL^2\) is a basis whose basis elements are obtained by the instrumentality of these parameters. For the fixed pair \((i, j)\) we denote by \(\varphi(a_{i,j})\) a cocyle which has \(a_{i,j} = 1\) and all other parameters are equal to zero. Define such type of notation for other parameters.

Set
\[
\varphi_{i,j} = \varphi(a_{i,j}), \quad \psi_1 = \varphi(b_{1,1}), \quad \psi_2 = \varphi(b_{2,3}), \quad \psi_3 = \varphi(c_{1,1}),
\]
\[
\psi_4 = \varphi(c_{1,n}), \quad \psi_5 = \varphi(\beta), \quad \eta_k = \varphi(c_{2,k}), \quad \rho_s = \varphi(d_{1,s}),
\]
where \(1 \leq i \leq n, 0 \leq j \leq n, 2 \leq k \leq n, 3 \leq s \leq n,\) and
\[
(i, j) \notin \{(1, 0), (1, 1), (1, 2), (n, 0), (n, 1), \ldots, (n, n-1)\}.
\]

In order to find a basis of \(BL^2(R_3, R_3)\) we consider the endomorphisms \(f_{j,k} : R_3 \rightarrow R_3\) defined as follow
\[
f_{i,j}(e_i) = e_j, \quad 1 \leq i, j \leq n, \\
f_{i,n+1}(e_i) = x, \quad 1 \leq i \leq n, \\
f_{n+1,j}(x) = e_j, \quad 1 \leq j \leq n, \\
f_{n+1,n+1}(x) = x,
\]
where in the expansion of endomorphisms the omitted values are assumed to be zero.

Consider
\[
g_{i,j}(x, y) = [f_{i,j}(x), y] + [x, f_{i,j}(y)] - f_{i,j}([x, y]).
\]

Note that \(g_{i,j} \in BL^2(R_3, R_3)\) and now we separate a basis from these elements. Since the dimension of the space \(\text{Der}(R_3)\) is equal to 3, then to take a basis we should exclude three elements \(g_{i,j}.\) The description of \(\text{Der}(R_3)\) allow us to release the elements \(g_{1,1}, g_{2,3}\) and \(g_{n+1,n}.\)
By direct computation we obtain

\[
\begin{align*}
g_{j,k} & : \\
g_{j,k} & = \begin{cases} 
  g_{1,2} = -\varphi_{1,3}, \\
  g_{1,i} = -\rho_i - \varphi_{1,i+1}, & 3 \leq i \leq n-1, \\
  g_{1,n} = \psi_4 - \rho_n, \\
  g_{1,n+1} = \sum_{k=2}^{n-1} (n-k)\varphi_{k,k} - \psi_4 - \rho_n, \\
  g_{2,1} = -\psi_2, \\
  g_{2,i} = -\varphi_{2,i+1} + (2-i)\eta_i, & 2 \leq i \leq n-1, \; i \neq 3, \\
  g_{2,n} = (2-n)\eta_n, \\
  g_{2,n+1} = \varphi_{2,n+1} - \psi_1 - \eta_n, \\
  g_{i,k} = \varphi_{i-1,k} - \varphi_{i,k+1}, & 3 \leq i \leq n-1, \; 1 \leq k \leq n-1, \\
  g_{i,n} = \varphi_{i+1,n}, & 3 \leq i \leq n-1, \\
  g_{i,n+1} = \varphi_{i-1,n} + \varphi_{i,1}, & 3 \leq i \leq n-1, \\
  g_{n,1} = \varphi_{n-1,1}, \quad g_{n,2} = \varphi_{n-1,1}, \\
  g_{n,n} = \varphi_{n-1,n} + \psi_5, \\
  g_{n,n+1} = \varphi_{n-1,0}, \\
  g_{n+1,1} = -\eta_3, \\
  g_{n+1,i} = -\rho_{i+1}, & 2 \leq i \leq n-1, \\
  g_{n+1,n+1} = \psi_3 - 2\psi_5 + (n-2)\eta_2.
\end{cases}
\end{align*}
\]

From these equalities, it is not difficult to check that $\psi_3$ and $\eta_2$ do not belong to $BL^2(R_3, R_3)$, but $\psi_3 + (n-2)\eta_2 \in BL^2(R_3, R_3)$. Thus, we can take the adjoint class of $\psi_3$ as a basis of $ZL^2(R_3, R_3)$. 

\[\square\]

**Theorem 3.11.** An arbitrary $\varphi \in ZL^2(R_4, R_4)$ has the following form:

\[
\begin{align*}
\varphi(e_1,e_1) & = \sum_{i=3}^{n} a_{1,i}e_i, \\
\varphi(e_i,e_1) & = a_{i,0}x + \sum_{s=1}^{n} a_{i,s}e_s, \; 2 \leq i \leq n-1, \\
\varphi(e_n,e_1) & = a_{n-1,0}e_1 - \sum_{i=3}^{n-1} \left( \sum_{t=2}^{i-1} a_{n+i-t-1,t}e_1 - \sum_{t=2}^{n-1} a_{i,t} + \frac{(n-4)(n-1)}{2}d_{1,0} \right)e_n, \\
\varphi(e_1,e_2) & = b_{1,1}e_1 + b_{1,1}e_n, \\
\varphi(e_i,e_2) & = (i-n+1)b_{1,1}e_i + b_{2,3}e_{i+1}, \; 2 \leq i \leq n, \\
\varphi(e_1,e_3) & = -a_{1,0}e_1 - a_{i-1,0}e_n, \; 3 \leq i \leq n, \\
\varphi(e_i,e_3) & = (n-i-1)a_{2,0}e_i - (a_{2,1} + b_{1,1})e_{i+1}, \; 2 \leq i \leq n-1, \\
\varphi(e_n,e_j) & = a_{j-1,0}e_n, \; 3 \leq j \leq n, \\
\varphi(e_i,e_j) & = (n-i-1)a_{j-1,0}e_i + (a_{j-2,0} - a_{j-1,1})e_{i+1}, \; 2 \leq i \leq n-1, \; 4 \leq j \leq n, \\
\varphi(e_1,x) & = (a_{n-1,0} - d_{1,0})x + (a_{n-1,1} - a_{n-2,0} - d_{1,1})e_1 + \sum_{i=2}^{n} \left( \sum_{t=2}^{i} a_{n+t-i-1,t} - (n-i)a_{1,i+1} \right)e_i \\
& + \sum_{t=2}^{n-1} a_{i,t} - a_{1,n} + \frac{(n-2)(n-3)}{2}d_{1,0}e_{n-1} + c_{1,n}e_n, \\
\varphi(e_2,x) & = (n-3)b_{1,1}x + (n-2)b_{2,3}e_1 + \sum_{i=2}^{n} c_{2,i}e_i.
\end{align*}
\]
\[ \varphi(e_3, x) = (4 - n)a_{2,0}x + (3 - n)(a_{2,1} + b_{1,1})e_1 + (a_{2,2} + (n - 3)d_{1,0})e_2 + (c_{2,2} - d_{1,1})e_3 \]
\[ + \sum_{i=4}^{n-2} (c_{2,i-1} + (3 - i)a_{2,i})e_i + (c_{2,n-2} + (4 - n)a_{2,n-1} + a_{2,0})e_{n-1} \]
\[ + (c_{2,n-1} + (3 - n)a_{2,n} - a_{2,1})e_n, \]
\[ \varphi(e_i, x) = (i - n + 1)a_{i-1,0}x + (n - i)(a_{i-2,0} - a_{i-1,1})e_1 \]
\[ + \sum_{s=2}^{i-2} (i - s) \sum_{t=2}^{s} a_{i+t-s-1,t}e_s + \sum_{t=2}^{i-1} a_{t,t} + \frac{(2n - i - 3)(i - 2)}{2} d_{1,0}e_{i-1} \]
\[ + (c_{2,2} + (2 - i)d_{1,1})e_1 + \sum_{s=i+1}^{n-2} (c_{2,s-i+2} + (i - s) \sum_{t=2}^{i-1} a_{t,s-i+t+1})e_s \]
\[ + (c_{2,n-i+1} + a_{i-1,0} + (i - n + 1) \sum_{t=2}^{i-1} a_{t,n-i+t})e_{n-1} \]
\[ + (c_{2,n-i+2} + a_{i-2,0} - a_{i-1,1} + (i - n) \sum_{t=2}^{i-1} a_{t,n-i+t+1})e_n, \quad 4 \leq i \leq n - 2, \]
\[ \varphi(e_{n-1}, x) = (a_{n-3,0} - a_{n-2,1})e_1 + \sum_{s=2}^{n-3} (n - 1 - s) \sum_{t=2}^{s} a_{n+t-s-2,t}e_s \]
\[ + \sum_{i=2}^{n-2} a_{t,t} + \frac{(n - 2)(n - 3)}{2} d_{1,0}e_{n-2} + (c_{2,2} + a_{n-2,0} - (n - 3)d_{1,1})e_{n-1} \]
\[ + (c_{2,3} + a_{n-3,0} - a_{n-2,1} - \sum_{t=2}^{n-2} a_{t,t+2})e_n, \]
\[ \varphi(e_n, x) = a_{n-1,0}x + \sum_{s=2}^{n-2} (n - s) \sum_{t=2}^{s} a_{n+t-s-1,t}e_s + \sum_{t=2}^{n-1} a_{t,t} + \frac{(n - 2)(n - 3)}{2} d_{1,0} + a_{n-1,0})e_{n-1} \]
\[ + (c_{2,2} + a_{n-2,0} - a_{n-1,1} - (n - 2)d_{1,1})e_n, \]
\[ \varphi(x, e_1) = d_{1,0}x + d_{1,1}e_1 + a_{1,3}e_2 + \sum_{i=3}^{n} d_{1,i}e_i, \]
\[ \varphi(x, e_2) = -b_{2,3}e_1 - b_{1,1}e_{n-1}, \]
\[ \varphi(x, e_3) = (a_{2,1} + b_{1,1})e_1 + a_{2,0}e_{n-1}, \]
\[ \varphi(x, e_i) = (a_{i-1,1} - a_{i-2,0})e_1 + a_{i-1,0}e_{n-1}, \quad 4 \leq i \leq n, \]
\[ \varphi(x, x) = -a_{n-2,0}x + (a_{n-3,0} - a_{n-2,1})e_1 + \sum_{i=2}^{n-3} ((i - n + 1)(d_{1,i+1} - a_{1,i+2}) - \sum_{t=2}^{i} a_{n-i+t-2,t})e_i \]
\[ + (a_{1,n} - d_{1,n-1} - \frac{n^2 - 5n + 10}{2} d_{1,0} - \sum_{t=2}^{n-2} a_{t,t})e_{n-2} + (a_{n-1,n} + d_{1,1} - c_{1,n})e_{n-1} + \beta_n e_n. \]

**Proof.** The proof of this theorem is carrying out by applying similar arguments as in the proof of Theorem 5.8. \( \square \)
Remark 3.12. It should be noted that in the case \( n = 3 \) an arbitrary 2-cocycle for the algebra \( R_4 \) is different from the description of Theorem 3.11 and it has the form:

\[
\begin{align*}
\varphi(e_1, e_1) &= a_{1,3}e_3, \\
\varphi(e_2, e_1) &= a_{2,0}x + \sum_{s=1}^{3} a_{i,s}e_s, \\
\varphi(e_3, e_1) &= a_{2,0}e_1 - (a_{2,2} - d_{1,0})e_3, \\
\varphi(e_1, e_2) &= b_{1,1}e_1 + b_{1,1}e_3, \\
\varphi(e_2, e_2) &= b_{2,3}e_3, \\
\varphi(e_3, e_2) &= b_{1,1}e_3, \\
\varphi(e_1, e_3) &= -a_{2,0}e_1 - a_{2,0}e_3, \\
\varphi(e_2, e_3) &= -(a_{2,1} + b_{1,1})e_3, \\
\varphi(e_3, e_3) &= -(a_{2,0}e_3), \\
\varphi(e_1, x) &= (a_{2,0} - d_{1,0})x + (a_{2,1} + b_{1,1} - d_{1,1})e_1 + (a_{2,2} - a_{1,3})e_2 - e_1c_{1,3}e_3, \\
\varphi(e_2, x) &= b_{2,3}e_1 + c_2e_2 + e_{2,3}e_3, \\
\varphi(e_3, x) &= a_{2,0} + (a_{2,2} + a_{2,0})e_2 - (a_{2,1} + c_{2,2} - d_{1,1})e_3, \\
\varphi(x, e_1) &= d_{1,0}x + d_{1,1}e_1 + (a_{1,3} - 2d_{1,0})e_2 + d_{1,3}e_3, \\
\varphi(x, e_2) &= b_{2,3}e_1 - b_{1,1}e_2, \\
\varphi(x, e_3) &= (a_{2,1} + b_{1,1})e_1 + a_{2,0}e_2, \\
\varphi(x, x) &= b_{1,1}x + b_{2,3}e_1 + (a_{2,3} - c_{1,3} + d_{1,1})e_2 + \beta_3e_3.
\end{align*}
\]

Corollary 3.13. \( \dim ZL^2(R_4, R_4) = (n+1)^2 - 2 \), \( \dim HL^2(R_4, R_4) = 1 \) and the adjoint class \( \rho \) forms a basis of \( HL^2(R_4, R_4) \), where

\[
\rho : \begin{cases}
\rho(e_1, x) = e_1, \\
\rho(e_i, x) = (i-2)e_i, & 3 \leq i \leq n, \\
\rho(x, e_1) = -e_1, \\
\rho(x, x) = -e_{n-1}.
\end{cases}
\]

Theorem 3.14. An arbitrary \( \varphi \in ZL^2(R_2(\alpha), R_2(\alpha)) \) has the following form:

\[
\begin{align*}
\varphi(e_1, e_1) &= \sum_{i=2}^{n} a_{1,i}e_i, \\
\varphi(e_i, e_1) &= \sum_{s=1}^{n} a_{i,s}e_s + a_{i,0}x, & 2 \leq i \leq n - 1, \\
\varphi(e_n, e_1) &= a_{n-1,0}e_1 - \sum_{i=3}^{n-1} \left( \sum_{t=2}^{i-1} a_{n+i-t-1,t} \right) e_i + \left( (n-1)(2\alpha + n) \right) d_{1,0} - \sum_{t=2}^{n-1} a_{i,t}e_n, \\
\varphi(e_1, e_2) &= b_{1,1}e_1 + b_{1,1}e_3, \\
\varphi(e_i, e_2) &= ((i-2)b_{1,1} + b_{2,2})e_i + b_{2,3}e_{i+1}, & 2 \leq i \leq n, \\
\varphi(e_1, e_i) &= -a_{i-1,0}e_1, & 3 \leq i \leq n, \\
\varphi(e_i, e_3) &= -(a + i - 1)a_{2,0}e_1 - (a_{2,1} + b_{1,1})e_{i+1}, & 2 \leq i \leq n, \\
\varphi(e_i, e_j) &= -(a + i - 1)a_{j-1,0}e_i + (a_{j-2,0} - a_{j-1,1})e_{i+1}, & 2 \leq i \leq n, & 4 \leq j \leq n.
\end{align*}
\]
\[ \varphi(e_1, x) = -d_{1,1}e_1 + \sum_{i=2}^{n-1} (\alpha + i - 2)a_{1,i+1}e_i + c_{1,n}e_n - d_{1,0}x, \]
\[ \varphi(e_2, x) = -ab_{2,3}e_1 + \sum_{i=2}^{n} c_{i}e_i - (\alpha + 1)b_{1,1}x, \]
\[ \varphi(e_3, x) = (\alpha + 1)(a_{2,1} + b_{1,1})e_1 + (a_{2,2} - (\alpha + 1)d_{1,0})e_2 + (c_{2,2} - d_{1,1})e_3 \]
\[ + \sum_{i=4}^{n} (c_{2,i} - 1) + (3 - i)a_{2,i}e_i + (\alpha + 2)a_{2,0}x, \]
\[ \varphi(e_i, x) = (\alpha + i - 2)(a_{i-1,1} - a_{i-2,0})e_1 + \sum_{s=2}^{i-1} (i - s) \sum_{t=2}^{s} a_{s-t+1, s-t}e_s \]
\[ + \left( \sum_{s=1}^{i-1} \frac{(2\alpha + i - 1)(i - 2)}{2} d_{1,0} \right) e_{i-1} + (c_{2,2} - (i - 2)d_{1,1})e_i \]
\[ + \sum_{s=1}^{n-2} (c_{2,s-i+2} + (i - s) \sum_{t=2}^{s-1} a_{s-t-i+1}e_s + (\alpha + i - 1)a_{i-1,0}x, \quad 4 \leq i \leq n, \]
\[ \varphi(x, e_i) = \sum_{i=1}^{n} d_{1,i}e_i + d_{1,0}x, \]
\[ \varphi(x, e_2) = -b_{2,3}e_1 - b_{1,n}e_{n-1}, \]
\[ \varphi(x, e_3) = (a_{2,1} + b_{1,1})e_1, \]
\[ \varphi(x, e_i) = (a_{i-1,1} - a_{i-2,0})e_1, \quad 4 \leq i \leq n, \]
\[ \varphi(x, x) = \sum_{i=2}^{n-2} (\alpha + i - 1)(d_{3,i}e_1 - a_{1,i+2})e_i + ((\alpha + n - 2)d_{1,n} - c_{1,n})e_{n-1} + \beta_n e_n \]

with restrictions
\[
\begin{align*}
  (\alpha - 1)a_{1,2} & = 0, \quad (\alpha + 1)b_{1,n} = 0, \quad (\alpha + 1)(b_{2,2} - (\alpha + 1)b_{1,1}) = 0, \\
  (n - 3)b_{1,n} & = 0, \quad \alpha(d_{1,2} - a_{1,3}) = 0.
\end{align*}
\]

**Proof.** The proof of this proposition is carrying out by applying similar arguments as in the proof of Theorem 3.8. \[\square\]

From the equalities \ref{eq:3.11} it implies that if \( n \neq 3 \), then \( b_{1,n} = 0 \). Thus, we distinguish the cases \( n = 3 \) and \( n > 3 \). Moreover, the general form of infinitesimal deformations also depends on the value of \( \alpha \). Therefore, we have

**Corollary 3.15.**

\[
\dim ZL^2(R_2(\alpha), R_2(\alpha)) = \begin{cases}
(n + 1)^2 - 1, & \alpha = 0; \pm 1, \\
(n + 1)^2 - 2, & \alpha \neq 0; \pm 1,
\end{cases} \quad \text{for } n > 3;
\]
\[
\dim ZL^2(R_2(\alpha), R_2(\alpha)) = \begin{cases}
15, & \alpha = 0; 1, \\
16, & \alpha = -1, \\
14, & \alpha \neq 0; \pm 1,
\end{cases} \quad \text{for } n = 3.
\]

**Corollary 3.16.**

\[
\dim HL^2(R_2(\alpha), R_2(\alpha)) = \begin{cases}
2, & \alpha = 0; \pm 1; 1 - n; 2 - n, \\
1, & \alpha \neq 0; \pm 1; 1 - n; 2 - n,
\end{cases} \quad \text{for } n > 3;
\]
\[
\dim HL^2(R_2(\alpha), R_2(\alpha)) = \begin{cases}
4, & \alpha = -1, \\
2, & \alpha = 0; 1; -2, \\
1, & \alpha \neq 0; \pm 1; -2,
\end{cases} \quad \text{for } n = 3.
\]

In the following proposition similarly to the proof of Proposition 3.10 we find a basis of \( HL^2(R_2(\alpha), R_2(\alpha)) \).
Proposition 3.17. The basis of $HL^2(R_2(\alpha), R_2(\alpha))$ consists of the following adjoint classes

$$
\left\{ \begin{array}{l}
\overline{\psi_1}, \psi_1, \\
\psi_1, \psi_2,
\end{array} \right. \quad \alpha = 0; 1 - n; 2 - n, \\
\overline{\psi_2}, \quad \alpha = -1, \\
\overline{\psi_1}, \psi_2, \psi_3, \psi_4, \\
\quad \alpha = -1, \\
\overline{\psi_2}, \psi_3, \psi_4, \\
\quad \alpha \neq 0; \pm 1; 1 - n; 2 - n, \\
\overline{\psi_1}, \quad \alpha = 0; -1; -2, \\
\psi_3: \psi_3(e_1, x) = e_n, \\
\psi_4: \psi_4(e_1, e_2) = e_n, \\
\quad \psi_3, \psi_4(x, x) = -e_{n-1}, \\
\psi_4: \psi_4(x, e_2) = -e_{n-1}.
\right. $$

where

$$
\rho : \left\{ \begin{array}{l}
\rho(e_1, x) = e_1, \\
\rho(e_i, x) = (i - 2)e_i, \quad 3 \leq i \leq n, \\
\rho(x, e_i) = -e_1.
\end{array} \right.
$$

Theorem 3.18. An arbitrary 2-cocycle of $ZL^2(R_5(\alpha_4, \alpha_5, \ldots, \alpha_n), R_5(\alpha_4, \alpha_5, \ldots, \alpha_n))$ has the following form:

\begin{align*}
\varphi(e_1, e_1) &= a_{2,0}x + a_{2,1}e_1 + (a_{2,2} - \alpha_n a_{n,2})e_2 + \sum_{i=3}^{n} a_{1,i} e_i, \\
\varphi(e_i, e_1) &= a_{i,0}x + \sum_{s=1}^{n} a_{i,s} e_s, \quad 2 \leq i \leq n - 1, \\
\varphi(e_n, e_1) &= \sum_{i=2}^{n} a_{n,i} e_i, \\
\varphi(e_1, e_2) &= (-c_0 + \sum_{j=4}^{n} \alpha_j a_{j-1,0})e_2 + (-c_1 + \sum_{j=4}^{n} \alpha_j a_{j-1,1})e_3 + \sum_{i=4}^{n-1} \alpha_i (-c_0 + \sum_{j=4}^{n} \alpha_j a_{j-1,0})e_i, \\
\varphi(e_i, e_2) &= (-c_0 + \sum_{j=4}^{n} \alpha_j a_{j-1,0})e_i + (-c_1 + \sum_{j=4}^{n} \alpha_j a_{j-1,1})e_{i+1} + \sum_{k=i+2}^{n} \alpha_i (-c_0 + \sum_{j=4}^{n} \alpha_j a_{j-1,0})e_k, \quad 2 \leq i \leq n, \\
\varphi(e_1, e_j) &= -a_{j-1,0}e_2 - a_{j-1,1}e_3 - a_{j-1,0} \sum_{k=4}^{n-1} a_k e_k, \quad 3 \leq j \leq n, \\
\varphi(e_i, e_j) &= -a_{j-1,0}e_i - a_{j-1,1}e_{i+1} - a_{j-1,0} \sum_{k=i+2}^{n} a_k e_k, \quad 2 \leq i \leq n, \quad 3 \leq j \leq n, \\
\varphi(e_1, x) &= (c_0 - \alpha_n a_{n-1,0})x + (c_1 - d_2 - \alpha_n a_{n-1,1})e_1 + (e_2 + a_{1,3} - a_{2,3} + d_2 + \alpha_n a_{n-1,1} + \alpha_n a_{n,3})e_2 + (c_3 + a_{1,4} - a_{2,4} + \alpha_n (a_{n,3} - \alpha_4 a_{n,2}))e_3 \\
&\quad + \sum_{i=4}^{n-1} (c_i + a_{1,i+1} - a_{2,i+1} + \alpha_n (a_{n,i} - \alpha_{i+1} a_{n,2})) + \sum_{j=4}^{i} \alpha_j (a_{1,i-j+3} - a_{2,i-j+3})e_i + c_{n+1} e_n.
\end{align*}
\( \varphi(e_2, x) = c_0 x + \sum_{i=1}^{n} c_i e_i, \)
\( \varphi(x, e_1) = -d_2 e_1 + \sum_{i=2}^{n} d_i e_i, \)
\( \varphi(e_3, x) = (a_{2,0} + \sum_{j=4}^{n} \alpha_j a_{j,0}) x + (a_{2,1} + \sum_{j=4}^{n-1} \alpha_j a_{j,1}) e_1 + (-a_{2,1} + \sum_{j=4}^{n} \alpha_j a_{j,2}) e_2 + \sum_{k=4}^{n} (c_{k-1} - a_{2,1} \alpha_k - \sum_{j=2}^{k-2} \alpha_k a_{j+2} a_{2,j} + \sum_{j=4}^{n} \alpha_j a_{j,k}) e_k + a_{2,1} \alpha_n e_n, \)
\( \varphi(e_{i+1}, x) = (a_{i,0} + \sum_{j=4}^{n} \alpha_{j-i-2} a_{j,0}) x + (a_{i,1} + \sum_{j=4}^{n-1} \alpha_{j-i-2} a_{j,1}) e_1 + (-a_{i,1} + \sum_{j=4}^{n} \alpha_{j-i-2} a_{j,2}) e_2 + \sum_{k=4}^{n} (c_{k-1-i} + \sum_{j=i+1}^{k-1} \alpha_{k-i} a_{j,1} + \sum_{s=2}^{n} \alpha_{k-s} + \sum_{j=i+2}^{n} a_{j-i-2} a_{j,k} e_k + a_{2,1} a_{j,1} e_n, \)
\( \varphi(x, x) = d_3 e_2 + d_4 e_3 + \sum_{i=4}^{n-2} (d_{i+1} + \sum_{j=4}^{i} \alpha_j d_{i+3-j}) e_i + (d_n + \sum_{j=4}^{n} \alpha_j d_{n+2-j}) e_{n-1} + \beta e_n. \)

**Proof.** The proof of this proposition is carrying out by applying similar arguments as in the proof of Theorem 3.8. \(\square\)

**Corollary 3.19.**

\[ \dim ZL^2(R_3(\alpha_i), R_5(\alpha_i)) = n^2 + 3n - 3, \]
\[ \dim HL^2(R_3(\alpha_i), R_5(\alpha_i)) = \begin{cases} 2n - 4, & \alpha_i = 0 \text{ for all } i, \\ 2n - 5, & \alpha_i \neq 0 \text{ for some } i. \end{cases} \]

Let us introduce the notations
\[
\rho : \begin{cases} \rho(e_1, x) = e_1 + e_2, \\ \rho(e_i, x) = (i - 3) e_i, & \text{for } 4 \leq i \leq n, \\ \rho(x, e_1) = e_1 - e_2, \end{cases}
\]
\[
\psi_k (4 \leq k \leq n - 1) : \begin{cases} \psi_k(e_1, x) = e_k, \\ \psi_k(e_i, x) = e_{k+i-2}, & \text{for } 2 \leq i \leq n - k + 2, \end{cases}
\]
\[
\psi_n : \begin{cases} \psi_n(e_2, x) = e_n, \\ \varphi_{n,2}(e_n, e_1) = e_2, \\ \varphi_{n,2}(e_1, x) = -\alpha_n \sum_{j=3}^{n-1} \alpha_j e_j, \\ \varphi_{n,2}(e_i, x) = \sum_{j=2}^{i-1} \alpha_{n+j+1-i} e_j, & \text{for } 3 \leq i \leq n - 1, \\ \varphi_{n,2}(e_n, x) = \sum_{j=1}^{n-1} \alpha_j e_j, \end{cases}
\]
\[ \forall n,k \ (3 \leq k \leq n - 1) : \left\{ \begin{array}{l}
\varphi_{n,k}(e_n, e_1) = e_k, \\
\varphi_{n,k}(e_1, x) = -\alpha_ne_k, \\
\varphi_{n,k}(e_i, x) = \sum_{j=k}^{i+k-3} \alpha_{n+j+3-i-k}e_j, \quad 3 \leq i \leq n + 2 - k, \\
\varphi_{n,k}(e_i, e_j) = \sum_{j=k}^{n-j} \alpha_{n+j+3-i-k}e_j, \quad n + 3 - k \leq i \leq n - 1, \\
\varphi_{n,k}(e_n, x) = \sum_{j=k}^{n} \alpha_{j+3-k}e_j.
\end{array} \right. \]

**Proposition 3.20.** The adjoint classes \( \psi_k \) (4 \leq k \leq n) and \( \overline{\varphi_{n,k}} \) (2 \leq k \leq n - 1) form a basis of \( HL^2(R_5(0), R_5(0)) \). The basis of \( HL^2(R_5(\alpha_4, \ldots, \alpha_n), R_5(\alpha_4, \ldots, \alpha_n)) \) with \( (\alpha_4, \ldots, \alpha_n) \neq (0, \ldots, 0) \), is also the same except one cocycle \( \psi_k \) with \( \alpha_k \neq 0 \).

**Proof.** Since there are parameters \( (\alpha_{i,j}, c_k, \beta, d_s) \) in the general form of 2-cocycles for the algebra \( R_5(\alpha_4, \ldots, \alpha_n) \), we consider the natural basis of the space \( ZL^2(\alpha) \) whose basis elements are obtained by the instrumentality of these parameters.

Similarly as in the proof of Proposition 3.10 we denote by \( \varphi(\alpha_{i,j}) \) the cocycle which satisfies \( \alpha_{i,j} = 1 \) and all other parameters are equal to zero. We define such type of notation for other parameters by notations

\[ \varphi_{i,j} = \varphi(\alpha_{i,j}), \quad \psi_k = \varphi(c_k), \quad \rho_s = \varphi(d_{1,s}), \quad \eta = \varphi(\beta), \]

where \( 1 \leq i \leq n, \ 0 \leq j \leq n, \ 0 \leq k \leq n + 1, \ 2 \leq s \leq n \) and

\( (i, j) \notin \{ (1, 0), (1, 1), (1, 2) \} \).

To define the basis of \( BL^2(R_5(\alpha_i), R_5(\alpha_i)) \) we consider the endomorphisms \( f_{j,k} : R_5(\alpha_i) \to R_5(\alpha_i) \) defined as follow

\[ f_{i,j}(e_i) = e_j, \quad 1 \leq i, j \leq n, \]
\[ f_{i,n+1}(e_i) = x, \quad 1 \leq i \leq n, \]
\[ f_{n+1,j}(x) = e_j, \quad 1 \leq j \leq n, \]
\[ f_{n+1,n+1}(x) = x, \]
where in the expansion of endomorphisms the omitted values are assumed to be zero.

Consider

\[ g_{i,j}(x, y) = [f_{i,j}(x), y] + [x, f_{i,j}(y)] - f_{i,j}([x, y]). \]

Note that \( g_{i,j} \in BL^2(R_5(\alpha_i), R_5(\alpha_i)) \) and by direct computation we express \( g_{i,j} \) via the elements \( \varphi_{i,j}, \psi_k, \rho_s \) and \( \eta \).

In the case of \( \alpha_4 = \alpha_5 = \cdots = \alpha_n = 0 \) we obtain that any linear combination of elements \( \rho_2, \psi_3, 4 \leq k \leq n \) and \( \varphi_{n,j}, 2 \leq j \leq n - 1 \) does not belong to \( BL^2(R_5(\alpha_i), R_5(\alpha_i)) \). Therefore, the adjoint classes of these elements form a basis of \( HL^2(R_5(\alpha_i), R_5(\alpha_i)) \).

However, in the case of \( (\alpha_4, \alpha_5, \ldots, \alpha_n) \neq (0, 0, \ldots, 0) \) we obtain that

\[ 2\alpha_4\psi_4 + 3\alpha_5\psi_5 + \cdots + (n - 2)\alpha_n\psi_n \in BL^2(R_5(\alpha_i), R_5(\alpha_i)). \]

Hence, in this case we get that the basis of \( HL^2(R_5(\alpha_i), R_5(\alpha_i)) \) also consists from \( \overline{\psi_k} \) (4 \leq k \leq n - 1), except one cocycle \( \psi_k \) with \( \alpha_k \neq 0 \).

**Acknowledgments**

This work was partially supported by Ministerio de Economía y Competitividad (Spain), grant MTM2013-43687-P (European FEDER support included); by Xunta de Galicia, grant GRC2013-045 (European FEDER support included) and by the Grant No.0251/GF3 of Education and Science Ministry of Republic of Kazakhstan.

**References**

[1] Ancochea Bermúdez J.M., Campoamor-Stursberg R. *On a complete rigid Leibniz non-Lie algebra in arbitrary dimension*, Linear Algebra Appl., vol. 438(8), 2013, p. 3397–3407.

[2] Ayupov Sh.A., Omirov B.A. *On some classes of nilpotent Leibniz algebras*, Sib. Math. J., vol. 42(1), 2001, p. 18–29.

[3] Balavoine D. *Déformations et rigidité géométrique des algèbres de Leibniz*, Comm. Algebra, vol. 24, 1996, p. 1017–1034.

[4] Barnes, D.W. *On Levi’s theorem for Leibniz algebras*, Bull. Aust. Math. Soc., vol. 86(2), 2012, p. 184–185.
[5] Burde D., Steinhoff C. Classification of orbit closures of 4-dimensional complex Lie algebras, J. Algebra, vol. 214(2), 1999, p. 729-739.

[6] Camacho L.M., Masutova K.K., Omirov B.A. Solvable Leibniz algebras with filiform nilradical. arXiv:1307.1665, 2013.

[7] Casas J.M., Khudoyberdiyev A.Kh., Ladra M., Omirov B.A. On the degenerations of solvable Leibniz algebras, Linear Algebra Appl., vol. 439(2), 2013, p. 472–487.

[8] Casas J.M., Ladra M., Omirov B.A., Karimjanov I.K. Classification of solvable Leibniz algebras with null-filiform nilradical, Linear and Multilinear Algebra, vol. 61(6), 2013, p. 758–774.

[9] Casas J.M., Ladra M., Omirov B.A., Karimjanov I.K. Classification of solvable Leibniz algebras with naturally graded filiform nilradical, Linear Algebra Appl., vol. 438(7), 2013, p. 2973–3000.

[10] Fialowski A., Millionschikov D.V. Cohomology of Graded Lie Algebras of Maximal Class, J. Algebra, 296 (2006), p. 157–170.

[11] Fialowski A., Penkava M. Formal deformations, contractions and moduli spaces of Lie algebras, Internat. J. Theoret. Phys., vol. 47, 2008, p. 561–582.

[12] Gerstenhaber M. On the deformation of rings and algebras, I, III, Ann. of Math. (2), vol. 79, 1964, p. 59–103; vol. 88, 1968, p. 1–34.

[13] Goze M., Khakhimdjanov Yu. Nilpotent Lie algebras, Kluwer Academic Publishers, Dordrecht, vol. 361, 1996, 336 pp.

[14] Grunewald F., O’Halloran J. Varieties of nilpotent Lie algebras of dimension less than six, J. Algebra, vol. 112, 1988, p. 315–325.

[15] Khakimdjanov Yu., Navarro R.M. Deformations of filiform Lie algebras and superalgebras, J. Geom. Phys., vol. 60, 2010, p. 1156–1169.

[16] Khudoyberdiyev A.Kh., Omirov B.A. The classification of algebras of level one, Linear Algebra Appl., vol. 439(11), 2013, p. 3460–3463.

[17] Khudoyberdiyev A.Kh., Omirov B.A. Infinitesimal deformations of null-filiform Leibniz superalgebras, J. Geom. Phys., vol. 74, 2013, p. 370–380.

[18] Khudoyberdiyev A.Kh., Omirov B.A. Infinitesimal deformations of naturally graded filiform Leibniz algebras, J. Geom. Phys., vol. 86, 2014, p. 149–163.

[19] Loday J.-L. Une version non commutative des algèbres de Lie: les algèbres de Leibniz, Enseign. Math. (2), vol. 39(3–4), 1993, p. 269–293.

[20] Loday J.-L., Pirashvili T. Universal enveloping algebras of Leibniz algebras and (co)homology, Math. Ann., vol. 296, 1993, p. 139–158.

[21] Millionschikov D.V. Deformations of filiform Lie algebras and symplectic structures, Proc. Steklov Inst. Math., vol. 252(1), 2006, p. 182–204.

[22] Nijenhuis A., Richardson R.W. Cohomology and deformations in graded Lie algebras, Bull. Amer. Math. Soc., vol. 72, 1966, p. 1–29.

[23] Seeley C. Degenerations of 6-dimensional nilpotent Lie algebras over C, Comm. Algebra, vol.18, 1990, p. 3493–3505.

[24] Vergne M. Cohomologie des algèbres de Lie nilpotentes. Application à l’étude de la variété des algèbres de Lie nilpotentes, Bull. Soc. Math. France, vol. 98, 1970, p. 81–116.