THE NONCOMMUTATIVE GEOMETRY
OF SQUARE SUPERPOTENTIAL ALGEBRAS

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Abstract

A superpotential algebra is square if its quiver admits an embedding into a two-
torus such that the image of its underlying graph is a square grid, possibly with diag-
nal edges in the unit squares (examples are provided by brane tilings in physics). Such
an embedding is special since it reveals much of the algebras representation theory
through a device we introduce called an impression. Using impressions, we classify all
simple representations of maximal k-dimension of all homogeneous square superpoten-
tial algebras and show that the localization of each algebra is a noncommutative
crepant resolution with a 3 dimensional normal Gorenstein center, and hence a local
Calabi-Yau algebra. Another special property of these algebras, equipped with an
impression, is that crystal melting (a type of stability change) and quiver mutation
may be regarded as a single operation.

A particular class of square superpotential algebras, the $Y_{p,q}$ algebras, is consid-
ered in detail. We show that the Azumaya and smooth loci of the centers coincide, and
we make the proposal that the “stack-like” maximal ideals sitting over the singular
locus are exceptional divisors of a blowup of the center shrunk to zero size.

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1 Introduction

1.1 Overview

Superpotential algebras are a class of quiver algebras that have arisen in string theory and have found mathematical interest in their own right. They are currently studied in the context of noncommutative crepant resolutions, Calabi-Yau algebras, tilting theory, and cluster algebras, and our focus here will be on the first two of these. More specifically, we will consider resolving singularities through the use of matrix-valued functions, where we view the superpotential algebras as noncommutative coordinate rings over their singular centers.

To motivate this viewpoint, recall that the points of an affine variety $X$ may always be identified with the simple modules over the (commutative) ring of polynomial functions on $X$, and a point $p$ in $X$ is singular (resp. smooth) if and only if the projective dimension of the corresponding simple module is infinite (resp. equals the topological dimension of $X$ at $p$). It is therefore natural to extend this idea to noncommutative coordinate rings, deeming a point (i.e., simple module) smooth if its projective dimension equals the topological dimension of its center. Also as in the commutative case, the evaluation of a function at a simple module is simply its corresponding representation.

A certain subvariety of the center—the Azumaya locus—plays an important part in this story since it often coincides with the smooth locus of the center $[21, 13]$. In regards to
noncommutative singularity resolution, the compliment of this locus is known in physics as the locus where “brane fractionation” occurs [7]. We propose the geometric interpretation that the maximal ideals of the noncommutative algebra that sit over the non-Azumaya locus are the exceptional divisors of a resolution of the center shrunk to zero size.

We introduce a device called an “impression” to aid in our analysis (def. 2.1). Simply put, an impression of a noncommutative algebra may be thought of as a closely related commutative algebra containing the center as a subalgebra. For example, an impression of a McKay quiver algebra for an abelian quotient singularity $\mathbb{C}^n/G$ is simply $\mathbb{C}[x_1, \ldots, x_n]$. When such a device exists, it may be useful because it may help determine the algebras center; it may enable symplectic geometric concepts to be related to the representation theory of the algebra; and if the algebra is noetherian and a finitely generated module over its center, then its impression explicitly determines all its simple modules of maximal $k$-dimension—what we call the “large simples”. These modules are important because they, under suitable hypotheses, are parameterized by the Azumaya locus of the algebra.

The paper is organized as follows: In sec. 2, impressions are defined and a few of their key properties are established. A general impression is then given for square superpotential algebras, and using this impression it is shown that each homogeneous algebra (def. 3.10) is prime, noetherian, and a finitely generated module over its center. Letting $A$ denote such an algebra and $Z$ its center, we then classify all large simple $A$-modules and show that if $V$ is a large simple with annihilator $p$, and $m := p \cap Z \in \mathrm{Max} Z$, then as $A$-modules,

$$V \oplus (\# \text{ of vertices in quiver}) \cong A/p,$$

and

$$\text{pd}_A(V) = \text{pd}_{Z_m}(k_m),$$

where $A/p$ generalizes the residue field $k_m \cong Z/m$ at $m$. Equipped with an impression, we then observe that quiver mutating and crystal melting may be regarded as a single operation over a square superpotential algebra. In sec. 4 we show that $\mathrm{Max} Z$ is a 3-dimensional normal Gorenstein toric algebraic variety, and that the localization of $A$ at the origin of $\mathrm{Max} Z$ is a noncommutative crepant resolution. This generalizes work of Ueda and Yamazaki who proved, using different techniques, that the $Y_{1,0}$ algebra is a noncommutative crepant resolution of the conifold [27, thm. 1]. As a corollary, these algebras are locally Calabi-Yau.

A special class of square superpotential algebras conjecturally related to Sasaki-Einstein manifolds, namely the $Y^{p,q}$ algebras, is then considered in detail in sec. 5. Letting $A$ be a (non-localized) $Y^{p,q}$ algebra, all simple $A$-modules are classified; the Azumaya locus of $A$ and the smooth locus of $Z$ are determined and found to coincide; and $A$ is shown to have global dimension 3. Finally, we justify our proposal regarding “point-like” exceptional divisors by using symplectic reduction on the impression.

We also introduce a few explicit examples with interesting algebraic properties: an example of a square superpotential algebra with a non-noetherian center is given in ex. 3.9
examples of superpotential algebras with non-trivial noetherian centers and infinite global dimension are given in ex. 4.12 and examples of superpotential algebras (with trivial centers) having global dimension any odd number at least 5 are given in ex. 4.14.

Section 4.3 is based on joint work with Alex Dugas, and I thank him for kindly allowing me to publish it here. The main results of this section were independently discovered by Mozgovoy \cite{25} in a different context, also generalizing Ueda and Yamazaki’s results.

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Conventions: $k$ denotes an algebraically closed field of characteristic zero; all algebras are finitely generated over $k$. By module we mean left module. For brevity the term “quiver algebra” is used in place of “path algebra modulo relations”. By a cycle in a quiver we mean an oriented cycle. The set of paths in a quiver $Q$ of length $n$ is denoted $Q_n$. $h(p)$ and $t(p)$ denote the head vertex and tail vertex of a path $p$, respectively. Path concatenation is read right to left (following the composition of maps). The term “superpotential algebra” is synonymous with “vacualgebra”, and the completion is often called a “Jacobian algebra”.

1.2 Square superpotential algebras

A superpotential algebra is a type of quiver algebra where the relations are derived from certain “equations of motion” in a physical theory. Stated concisely, a quiver algebra is a quotient of a path algebra, that is, an algebra whose basis consists of all paths in a quiver (i.e., directed graph) and multiplication is given by path concatenation. A representation of (or module over) a quiver algebra is obtained by associating a vector space to each vertex of the quiver, representing each arrow by a linear map from the vector space at its tail to the vector space at its head, and requiring these linear maps satisfy the relations of the algebra.

We now define a superpotential algebra. Let $Q$ be a quiver and $kQ$ its path algebra. Two paths $p$ and $p'$ are cyclically equivalent if $p$ is a cyclic permutation of the arrows of $p'$, so all non-cyclic paths are cyclically equivalent to zero. The trace space of $kQ$, denoted $\text{tr}(kQ)$, is the $k$-vector space spanned by the paths of $Q$ up to cyclic equivalence\footnote{In other words, spanned by the “necklaces” of $Q$.} and
an element of $\text{tr}(kQ)$ is called a superpotential. For each $a \in Q_1$, define a $k$-linear map $\partial_a : \text{tr}(kQ) \to kQ$ as follows: for each path $b_n \cdots b_1 \in Q_{\geq 1}$ with $b_1, \ldots, b_n \in Q_1$, set
\[
\partial_a (b_n \cdots b_1) := \sum_{1 \leq j \leq n} \delta(a, b_j) e_{\text{tl}(b_j)} b_{j-1} \cdots b_n \cdots b_{j+1},
\]
for each $e \in Q_0$, set $\partial_a e := 0$, and extend $k$-linearly to $\text{tr}(kQ)$. For $W \in \text{tr}(kQ)$, set
\[
\partial W := \langle \partial_a W \mid a \in Q_1 \rangle.
\]
The superpotential algebra with quiver $Q$ and superpotential $W$ is then the quiver algebra $kQ/\partial W$. In this paper we are interested in a particularly simple class of superpotential algebras that arise from what are called brane tilings—specifically, “brane boxes” and “brane diamonds”—in string theory (see [19] and references therein). Embedding these relatively simple superpotential algebras into a two-torus is standard; what we introduce here that is new is a relationship between a particular choice of embedding (when it exists) and the representation theory of the corresponding algebras.

**Definition 1.1.** We say a superpotential algebra $kQ/\partial W$ is square if

- The underlying graph $\tilde{Q}$ of $Q$ admits an embedding $\iota$ into a two-torus $T^2$ such that
  - $\iota(Q_0) \subset T^2$ is a square lattice, denoted $Q_0 = \{1, \ldots, n\} \times \{1, \ldots, m\}$;
  - for each $(i, j) \in Q_0$, $\tilde{Q}$ has edges
    \[
    \{(i, j), (i+1, j)\}, \quad \{(i, j), (i, j+1)\},
    \]
    and at most one of the following two diagonal edges:
    \[
    \{(i, j), (i+1, j+1)\}, \quad \{(i+1, j), (i, j+1)\}.
    \]
  
  $Q$ has an orientation such that the following are oriented cycles: (i) each unit square not containing a diagonal, and (ii) each triangle with two unit length sides. We call these the unit cycles of $Q$.

- The superpotential is given by
  \[
  W = \sum_{c \in \{\text{clockwise unit cycles, up to cyclic equiv}\}} c - \sum_{c' \in \{\text{counterclockwise unit cycles, up to cyclic equiv}\}} c' \in \text{tr}(kQ)
  \]
**Example 1.2.** The following two examples are perhaps the simplest square superpotential algebras. The center of the second example is the coordinate ring for the conifold. The quivers on the right are drawn in the plane.

\[
W = xyz - zyx, \\
A = Z(A) \cong k[x, y, z]
\]

**Example 1.3.** The \(Y^{p,q}\) algebras, which are conjecturally related to a class of Sasaki-Einstein manifolds, form a class of square superpotential algebras. The \(Y^{p,q}\) quivers all have base of length \(n = 2\), arbitrary height \(m = p\), and are constructed by vertically “stacking” any of the following three graphs,

identifying vertices \((0, j) = (2, j)\) and \((i, 0) = (i + i_0, m)\) for each \(i, j\) and some \(0 \leq i_0 < n\), and then choosing a compatible orientation. The label \(q\) is given by

\[
q = p - \# \left\{ \begin{array}{c} \begin{array}{c} \end{array} \end{array} \right\} - 2 \cdot \# \left\{ \begin{array}{c} \begin{array}{c} \end{array} \end{array} \right\}.
\]

In the examples in fig. 1, the quivers on the right are drawn in the plane. Note that the bottom two quivers are related by a quiver mutation; this will be discussed in sec. 3.4.

### 2 Impressions

We introduce the following definition as our main tool for analyzing square superpotential algebras\(^2\)

\(^2\)A related definition in a different context has been given independently by Broomhead [12, def. 4.1.1]. In his definition the center of the noncommutative algebra is necessarily toric; examples of algebras with non-toric centers that admit impressions are considered in [3].
Definition 2.1. Let $k$ be an algebraically closed field and suppose $A$ is a finitely generated $k$-algebra whose simple modules have maximal $k$-dimension $d \leq \aleph_0$. If there exists a finitely generated commutative $k$-algebra $B$, an open dense subset $U \subseteq \text{Max } B$, and an algebra homomorphism $\tau : A \rightarrow \text{End}_B \left( B^d \right)$ such that the composition

$$\tau_m : A \xrightarrow{\tau} \text{End}_B \left( B^d \right) \xrightarrow{\epsilon_m} \text{End}_B \left( \left( B/m \right)^d \right) \cong \text{End}_k \left( k^d \right)$$

is a simple representation of $A$ for each $m \in U$, then $(\tau, B)$ is called an impression of $A$.

As we will see, an impression of an algebra $A$ may be useful because

- it may help determine the center of $A$;
• it may enable symplectic geometric concepts to be related to the representation theory of $A$;

• and if $A$ is noetherian and a finitely generated module over its center, then its impression explicitly determines all $A$-modules of maximal $k$-dimension, up to isomorphism.

Note that if $A$ is prime, noetherian, and module-finite over its center, its PI-degree equals $d$ \([13, 3.1.a]\) (that is, the least possible degree of a monic polynomial identity on $A$ is $2d$). Moreover, if $A$ is a prime (resp. semiprime) PI ring of minimal degree $2d$, then $A$ admits an embedding into $M_d(B) \cong \text{End}_B(B^d)$ for some field $B$ (resp. product of fields) \([23, 13.4.2]\), though here we will work in the reverse direction, using similar embeddings to show that a class of algebras are prime PI rings.

Let $Z$ denote the center of $A$.

**Lemma 2.2.** Let $(\tau, B)$ be an impression of an algebra $A$ whose simple modules have maximal $k$-dimension $d \leq \aleph_0$. Then

$$Z \cong \{ f \in B \mid f1_d \in \text{im} \tau \} \subset B,$$

and thus if $B$ is reduced there is a surjective morphism

$$\text{Max } B \xrightarrow{\phi} \text{Max } Z.$$

**Proof.** Suppose $a \in Z$ and consider the matrix $\tau(a) = (b_{ij})_{1 \leq i,j \leq d}$. Since $(\tau, B)$ is an impression of $A$, for each $m \in \text{Max } B$ we have $\tau_m(a) = (b_{ij}(m))$, with $b_{ij}(m) \in B/m \cong k$. By Shur’s lemma (since $k = \bar{k}$ and $d \leq \aleph_0$), for $m \in U$ we have $\tau_m(a) \in k1_d$, so $b_{ij}(m) = 0$ whenever $i \neq j$, and $b_{ii}(m) = b_{jj}(m)$ for each $i, j$. Since $U$ is dense in $\text{Max } B$ it follows that $b_{ii} \equiv 0$ for $i \neq j$ and $b_{ii} \equiv b_{jj}$ for each $i, j$, that is, $\tau(a) = b_{11}1_d$.

Now suppose $f1_d \in \text{im } \tau$, say $\tau(a) = f1_d$ for some $a \in A$. For any $b \in A$, $\tau(ab - ba) = \tau(a)\tau(b) - \tau(b)\tau(a) = 0$, so $ab = ba$ since $\tau$ is a monomorphism, and thus $a \in Z$. \(\square\)

Recall that a ring $R$ is prime if for each $a, b \in R$, if $arb = 0$ for each $r \in R$ then $a = 0$ or $b = 0$.

**Lemma 2.3.** Let $(\tau, B)$ be an impression of an algebra $A$ that is module-finite over its center $Z$. If $B$ is a prime ring then $Z$ and $A$ are both prime rings, and consequently if $Z$ is noetherian then $\text{Max } Z$ is an algebraic variety.

**Proof.** First note that $d < \infty$ since $A$ is module-finite over $Z$. Since $B$ is prime, $M_d(B) \cong \text{End}_B(B^d)$ is prime, and thus $A$ is prime since $\tau$ is an algebra monomorphism. $Z$ is also prime since by lemma \([12]\) it is a subring of an integral domain, and so it too is an integral domain. \(\square\)
We call a simple module of maximal $k$-dimension a “large simple” because of the important role they play in noncommutative singularity resolution. Under suitable hypotheses given in sec. 3.15 the large simples parameterize the smooth locus of the algebras center.

**Proposition 2.4.** Let $(τ, B)$ be an impression of a prime noetherian algebra $A$ that is module-finite over its center. If $V$ is a large simple $A$-module, then there is some $r ∈ \text{Max } B$ such that $V ∼ = (B/r)^d$.

**Proof.** Let $V$ be a large simple $A$-module and $p = \text{ann}_A V$ it’s annihilator. Since $A$ is module-finite over its center, $m := p ∩ Z$ is a maximal ideal of $Z$. By lemma 2.2 there is some $r ∈ \text{Max } B$ such that $φ(r) = m$. But then $m = φ(r) = \text{ann}_Z ((B/r)^d)$. Let $W$ be a simple direct summand of $(B/r)^d$ with annihilator $q$. Then $\text{ann}_Z W ⊇ \text{ann}_Z(B/r)^d = m$, and so since $m$ is a maximal ideal it must be that $q ∩ Z = \text{ann}_Z W = m$. But $m$ is in the Azumaya locus of $A$ and $A$ is prime, noetherian, and module-finite over its finitely generated center, so $W$ must be of maximal $k$-dimension, namely $d$, so $W ∼ = (B/r)^d$. Finally, since $V$ is a large simple module, $p ∩ Z = m = q ∩ Z$ implies $p = q$, and hence $(B/r)^d ∼ = W ∼ = V$ since $A$ is again module-finite over its center.

3 Impressions of square superpotential algebras

3.1 An impression

In this subsection we give an explicit impression for all square superpotential algebras. Let $E_{ji}$ denote the matrix with a 1 in the $(ji)$-th slot and zeros elsewhere.

**Proposition 3.1.** Let $A = kQ/∂W$ be a square superpotential algebra and set $B := k[x_1, x_2, y_1, y_2]$. For each $a ∈ Q_1$, define $\bar{τ}(a)$ to be the monomial corresponding to the orientation of $a$ as follows:

\[
\begin{array}{ccccccc}
  \rightarrow & \rightarrow & \uparrow & \uparrow & \rightarrow & \rightarrow & \rightarrow \\
  x_1 & x_2 & y_1 & y_2 & x_1y_1 & x_2y_2 & x_1y_2 \\
  \end{array}
\]

There is then an algebra monomorphism

$$\tau : A → M_{|Q_0|}(B) ∼ = \text{End}_B (B^{[Q_0]})$$

defined on the generating set $Q_0 ∪ Q_1$ by $e_i ↦ E_{ii}$ for each $i ∈ Q_0$ and $a ↦ \bar{τ}(a)E_{h(a),t(a)}$ for each $a ∈ Q_1$. 

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For each $i, j \in Q_0$ and $a \in e_j A e_i$, let $\bar{\tau}(a) \in B$ denote the $(ji)$-th matrix component of $\tau(a)$ viewed as an element in $M_{|Q_0|}(B)$. Then $c \bar{\tau}(a) = \bar{\tau}(ca)$ whenever $c \in k$, so for each $i, j \in Q_0$, $\bar{\tau}$ is a well-defined $k$-homomorphism $e_j A e_i \rightarrow B$ since the labeling of arrows is preserved by the relations of $\partial W$. In the following proof we will show that for each $i, j$, $\bar{\tau}$ is also injective. Note that $\bar{\tau}$ cannot be an algebra homomorphism (unless $|Q_0| = 1$).

Proof. First, $\tau$ is well-defined since the labeling of arrows is preserved by the generators of $\partial W$. $\tau$ is an algebra homomorphism since $A$ is isomorphic to the tiled matrix ring

$$A \cong \begin{bmatrix} e_1 A e_1 & e_1 A e_2 & \cdots & e_1 A e_n \\ e_2 A e_1 & e_2 A e_2 & & \\ & \vdots & \ddots & \\ e_n A e_1 & & & e_n A e_n \end{bmatrix},$$

where $n := |Q_0|$, and for each $i, j \in Q_0$, $\bar{\tau}$ is a $k$-homomorphism $e_j A e_i \rightarrow B$.

We now show $\tau$ is injective. Suppose $p, p' \in A$ satisfy $\tau(p) = \tau(p')$. Then the corresponding matrix entries must be equal, so we may assume $p, p' \in e_j A e_i$ for some $i, j \in Q_0$. In this case, $\tau(p) = \tau(p')$ is equivalent to $\bar{\tau}(p) = \bar{\tau}(p')$. Since $\bar{\tau}$ is a $k$-homomorphism, we may assume $p, p'$ are elements in a basis of $e_j A e_i$, so suppose $p$ and $p'$ are paths. With this assumption, $\tau(p) = 0$ implies $p = 0$ since $B$ is an integral domain, so we may also suppose $\tau(p) \neq 0$.

We may denote any path $p \in kQ$ by its ordered monomial labeling $m$ in the non-commuting variables $x_1, x_2, y_1, y_2$, together with its tail vertex $t(p)$. If $x_{\alpha} y_{\beta}$ denotes a diagonal arrow, then let $x_{\alpha} y_{\beta} = y_{\beta} x_{\alpha}$. By abuse of notation, $\bar{\tau}(m) = m$. We want to show that if $\bar{\tau}(m) = \bar{\tau}(m')$, then $m = m'$ modulo $\partial W$, hence $p = p'$, and this is done in a case-by-case analysis in appendix A.

In physics terms, given any configuration of vev’s satisfying the $F$-flatness constraints there is a choice of gauge so that all fields with the same orientation on the two-torus have the same vev.

Remark 3.2. If we relax the condition that a superpotential algebra be square and only insist that its quiver admits an embedding into a two-torus with superpotential relations, then it would be natural to define

$$\sigma(a) := x_1^{\langle a, (1,0) \rangle} x_2^{\langle a, (-1,0) \rangle} y_1^{\langle a, (0,1) \rangle} y_2^{\langle a, (0,-1) \rangle}$$

on each arrow $a$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^2$ (viewed as the covering space for the two-torus). But then in general $\tau$ would not be well-defined.
Lemma 3.3. Let $A$ be a square superpotential algebra. Then $e_ia_i$ is a commutative subring for each $i \in Q_0$.

Proof. Let $a, b \in e_ia_i$. Then $\tau(ab) = \tau(a)\tau(b) = \tau(b)\tau(a) = \tau(ba)$, and so $ab = ba$ since $\tau$ is injective by prop. 3.1.

The following proposition holds for any (noncommutative) ring $S$ with a complete set of orthogonal idempotents $L$, that is, $\sum_{j \in L} e_j = 1_S$ and $e_ie_j = \delta_{ij}e_i$ for each $e_i, e_j \in L$. A quiver algebra $kQ/I$ has a complete set of orthogonal idempotents as long as $|Q_0| < \infty$.

Lemma 3.4. Let $S$ be a ring with a complete set of orthogonal idempotents $L$ and suppose $e_i \in L$. If $V$ is a simple $S$-module, then $e_iV$ is a simple $e_iSe_i$-module or zero.

Proof. Let $V$ be a simple $S$-module and assume to the contrary that $e_iV \neq 0$ is not a simple $e_iSe_i$-module. Then there exists a nonzero proper $e_iSe_i$-submodule $W \subseteq e_iV$. Let $u \in e_iV \setminus W$ and $w \in W$, with $w$ nonzero. Since $V$ is a simple $S$-module there is an $a \in S$ such that $aw = u$. Since $L$ is a complete set of idempotents, $S = \sum_{j,k \in J} e_jSe_k$, so we may write $a = \sum_{j,k \in J} a_{jk}$ with $a_{jk} \in e_jSe_k$. But then

$$u = e_iu = e_i(aw) = e_i\left(\sum_{jk} a_{jk}\right)w = e_i\left(\sum_{jk} e_ja_{jk}e_k\right)(e_iw) = a_{ii}w,$$

so $a_{ii}w = u$ and $a_{ii} \in e_iSe_i$, contradicting our choice of $w$.

Proposition 3.5. Let $k$ be an algebraically closed field and let $S$ be a $k$-algebra with a complete set of orthogonal idempotents $L$. Supposing $e_i \in L$, if the corner ring $e_iSe_i$ is a commutative subring then for every simple left $S$-module $V$, $\dim_k e_iV \leq 1$.

Proof. Suppose $e_iSe_i$ is a commutative ring and let $V$ be a simple $S$-module. By lemma 3.4, $e_iV$ is a simple $e_iSe_i$-module, from which it follows that $\dim_k e_iV \leq 1$ since $k$ is algebraically closed.

We may easily check prop. 3.5 when $S$ is an artinian path algebra: the corner rings $e_iSe_i = \{e_i\}$ are trivially commutative, and so by lemma 3.4 we have $\dim_k e_iV \leq 1$ for every simple $S$-module $V$ and all $i \in Q_0$, and this agrees with the fact that the only simple $S$-modules are the vertex simples.

Returning to the case when $S = A$ is a square superpotential algebra, the following corollary is immediate from lemma 3.3.

Corollary 3.6. Let $V$ be a simple $A$-module. Then $\dim_k e_iV \leq 1$ for each $i \in Q_0$. 

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Consequently, the maximal $k$-dimension of a simple $A$-module is at most $|Q_0|$. But it is also at least $|Q_0|$:  

**Lemma 3.7.** Let $m$ be any point in the dense open set

$$U = \{x_1x_2y_1y_2 \neq 0\} \subset \text{Max } B.$$  

Then $(B/m)^{|Q_0|}$ is a simple $A$-module with action $a \cdot v = \tau(a)v$.

**Proof.** Each arrow is represented by a nonzero scalar, and there is a path from $i$ to $j$ for each $i, j \in Q_0$. 

**Theorem 3.8.** Let $A$ be square superpotential algebra. Then $A$ admits an impression $(\tau, B)$, where $\tau$ is given by the labeling of arrows (2), $B = k[x_1, x_2, y_1, y_2]$, $d = |Q_0|$, and $U = \{x_1x_2y_1y_2 \neq 0\} \subset \text{Max } B$.

### 3.2 Corollaries

It is possible for a square superpotential algebra to have vertices $i, j \in Q_0$ satisfying $\bar{\tau}(e_i Ae_i) \neq \bar{\tau}(e_j Ae_j)$, as the following example demonstrates. Consequently, the algebra has a non-noetherian center.

**Example 3.9.** Let $A = kQ/\partial W$ be a square superpotential algebra with quiver

\[
\begin{array}{ccccccc}
1 & 5 & 6 & 7 & 1 \\
4 & 8 & 10 & 9 & 4 \\
3 & 7 & 11 & 8 & 3 \\
2 & 6 & 10 & 7 & 2 \\
1 & 5 & 6 & 7 & 1 \\
\end{array}
\]

Set $s := y_1^4x_1^4$ and $t := x_1x_2y_1y_2$. One may check that $s$ is in $\bar{\tau}(e_2 Ae_2)$ but not in $\bar{\tau}(e_1 Ae_1)$, while the infinite set of monomials $st, s^2t, s^3t, \ldots$ is in $\bar{\tau}(e_i Ae_i)$ for each $i \in Q_0$. It follows from prop. 3.11 below that the center $Z$ of $A$ is non-noetherian since the ring $k[t, st, s^2t, s^3t, \ldots]$ is non-noetherian.

**Definition 3.10.** Let $A$ be a superpotential algebra that admits an impression. We say $A$ is **homogeneous** if $\bar{\tau}(e_i Ae_i) = \bar{\tau}(e_j Ae_j)$ for each $i, j \in Q_0$. 

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Proposition 3.11. If $A$ is a square superpotential algebra then

$$Z = k \left[ \sum_{i \in Q_0} \gamma_i \in \bigoplus_{i \in Q_0} e_i A e_i \mid \tilde{\tau}(\gamma_i) = \tilde{\tau}(\gamma_j) \text{ for each } i, j \in Q_0 \right],$$

and if $A$ is also homogeneous then $e_i A e_i = Z e_i$ for all $i \in Q_0$.

Proof. This follows immediately from lemma 2.2 since $1 = \sum_{i \in Q_0} e_i$.

In the following lemma we show that any homogeneous square superpotential algebra is noetherian and a finitely generated $Z$-module. Let $\ell(p)$ denote the length of a path $p \in kQ$.

Lemma 3.12. Let $A = kQ/I$ be a quiver algebra with center $Z$, and suppose $e_i A e_i = Z e_i$ for each $i \in Q_0$; in particular $A$ may be a homogeneous square superpotential algebra. Then $A$ is a noetherian ring and a finitely generated $Z$-module.

Proof. Let $m := |Q_0|$. We claim $Z A = Z Q_{\leq m}$: if $a \in Q_{>m}$ is a path then $a$ must have a non-vertex cyclic subpath, say $a = b' cb$ where $c$ is a cycle and $b, b'$ are paths. Since $e_{t(c)} A e_{t(c)} = Z e_{t(c)}$, there exists a $\tilde{c} \in Z$ such that $\tilde{c} e_{t(c)} = c$, so

$$a = b' cb = b' \tilde{c} e_{t(c)} b = b' \tilde{c} b = \tilde{c} b b.'$$

But viewing $b' b$ and $a$ as elements of $kQ$ we have $\ell(b' b) < \ell(a)$, and so this may be repeated a finite number of times until $a \in Z b'^m$ with $\ell(b'^m) \leq m$. More generally, if $a = a_1 + \cdots + a_t$ with each $a_j \in Q_{\geq 0}$, then $a \in Z b_1 + \cdots + Z b_t$ for some $b_1, \ldots, b_t \in Q_{\leq m}$, proving our claim. $A$ is therefore noetherian as well by the Artin-Tate lemma [23. 13.9.7] (A is a finitely generated $k$-algebra and module-finite over a central subalgebra).

Theorem 3.13. Both a square superpotential algebra $A$ and its center $Z$ are prime rings, and if $A$ is homogeneous then $\text{Max } Z$ is a toric algebraic variety.

Proof. Since $B = k[x_1, x_2, y_1, y_2]$ is an integral domain we may apply lemma 2.3 to conclude that $A$ and $Z$ are prime. By lemma 3.12 if $A$ is homogeneous then $Z$, viewed as a subalgebra of $B$ (lemma 2.2), is generated by a finite set of monomials, and thus $\text{Max } Z$ is a toric algebraic variety.

The following corollary will be used in later sections.

---

A path algebra (without relations) is rarely noetherian: it is left (right) noetherian if and only if each vertex $i$ contained in an oriented cycle has only one arrow whose tail (resp., head) coincides with $i$ [16, p. 5].
**Corollary 3.14.** Let $A$ be prime quiver algebra satisfying $e_i A e_i = Z e_i$ for each $i \in Q_0$, such as a homogeneous square superpotential algebra. If $a \in e_i A$ and $b \in A e_i$ are nonzero, then $ba$ is nonzero as well.

*Proof.* Suppose $ba = 0$. Since $A$ is prime, $bra \neq 0$ for some $r \in e_i A e_i = Z e_i$. But then there is a $z \in Z$ such that $ze_i = r$, so $bra = bza = zba = 0$, a contradiction. \qed

### 3.3 Large simples

Recall that we are only considering finitely generated algebras over an algebraically closed field $k$. Simple modules of maximal $k$-dimension play an important role in noncommutative singularity resolution, since such modules often parameterize the smooth locus of the algebras center. For instance, if $S$ is a prime noetherian algebra with finite global dimension, finitely generated as a module over its center $Z$, then a simple $A$-module has maximal $k$-dimension only if the maximal ideal $\text{ann}_Z(V)$ is contained in the smooth locus of $\text{Max } Z$ [13, prop. 3.1, lemma 3.3]. In addition, often for a generic stability parameter $\theta \in \mathbb{Z} | Q_0 |$, the $\theta$-stable simple modules are precisely the simples of maximal $k$-dimension.

For brevity then we call these the *large simples*.

In the following theorem we classify all large simples over homogeneous square superpotential algebras. Letting $V$ be a large simple, $p := \text{ann}_A(V)$, and $m := p \cap Z$, we then show that the projective dimension of $V$ is equal to the projective dimension of the corresponding residue field $Z/m$ over $Z$, and we give an explicit description of the “noncommutative residue field” $A/p$ in terms of $V$.

**Theorem 3.15.** Let $A$ be a homogeneous square superpotential algebra with the impression $(\tau, B)$ given in theorem 3.8. Then for each large simple $A$-module $V$ there is a point $m \in \text{Max } B$ such that $V \cong (B/m)^{|Q_0|}$, where the module structure of $(B/m)^{|Q_0|}$ is given by $av := \tau_m(a)v$.

*Proof.* This follows from propositions 3.13 2.4 and lemma 3.12. \qed

The relationship between simples of maximal $k$-dimension and the smooth locus is through the Azumaya locus, which we now describe. Suppose $S$ is a noetherian algebra, module-finite over a commutative subalgebra $Z$; then the intersection of a primitive ideal of $S$ with $Z$ is a maximal ideal of $Z$ [20 thm. 6.3.3]\(^4\) and so we may define a surjective map (with finite fibers) $\pi : \text{Prim } S \to \text{Prim } Z = \text{Max } Z$ by $p \mapsto p \cap Z$. The subset of

\(^4\)More generally, $S$ may be a finitely generated PI algebra over a commutative Jacobson ring $Z$. \hfill 14
Max $Z$ consisting of those points whose pre-image is a single primitive ideal is known as the “Azumaya locus” [23, 13.7.9], and is defined to be the set

$$\{m \in \text{Max } Z \mid S_m \text{ is Azumaya over } Z_m\},$$

where $S_m$ is Azumaya over $Z_m$ if $S_m$ is a finitely generated projective $Z_m$-module and the natural map $S_m \otimes_{Z_m} (S_m)^{\text{op}} \to \text{End}_{Z_m}(S_m)$ given by $(a \otimes b) \cdot s = asb$ is an isomorphism [23, def. 5.3.24].

This locus often coincides with the smooth locus of Max $Z$, as first discovered by Le Bruyn in the case where $S$ is graded [21, thm. 1], and then in the ungraded case by Brown and Goodearl [13, sec. 3]. Moreover, the Azumaya locus parameterizes the isoclasses of simples whose $k$-dimension equals the PI-degree $d$, and these are precisely the simples of maximal $k$-dimension when $S$ is prime [13, prop. 3.1.a]. As realized by Berenstein [7], the compliment of this locus—the “ramification locus”—captures the well-known notion of “fractional branes” in string theory: if there exists a distinct pair $p, p' \in \text{Prim } S$ such that $p \cap Z = p' \cap Z =: m$, then we say $p$ and $p'$ are fractional branes or fractional points, and brane fractionation occurs at $m \in \text{Max } Z$. $p$ and $p'$ are thus stack-like points in Prim $S$ since they map to the same point in Max $Z$. We note however that primitive spectra are always affine.

In the following, $\phi$ is as defined in lemma 2.2.

**Proposition 3.16.** Let $A$ be a homogeneous square superpotential algebra. A simple $A$-module $V$ is large if and only if $\text{ann}_Z(V) \in \text{Max } Z$ is in the Azumaya locus of $A$.

**Proof.** As noted above by [13, prop. 3.1], if $S$ is prime then a simple $S$-module $V$ is large if and only if $m = \text{ann}_Z(V)$ is in the Azumaya locus of $S$. This then holds for any homogeneous square superpotential algebra by lemma 3.12 and thm. 3.13.

Given a variety $X$, a point $p \in X$ is smooth if and only if the projective dimension of the residue field at $p$ equals the topological dimension of $X$ at $p$. We want to generalize this fact to the case when $X$ has a noncommutative coordinate ring, and so we need some way to localize in this setting. For our purposes it suffices to consider the Ore localizations $A_m := Z_m \otimes_Z A$ with $m \in \text{Max } Z$. When $m$ is in the Azumaya locus, $A_m$ is local with unique maximal ideal $m_mA_m$ [23, 13.7.9]. An $A$-module $V$ can then be localized so that it

---

To be precise, they showed that if $S$ is prime, noetherian, Auslander-regular, Macaulay, and module-finite over its center, and if the compliment of the Azumaya locus has codimension at least 2 in Max $Z$, then the Azumaya and smooth loci coincide [13, thm. 3.8].

Since primitive ideals are prime, any primitive spectrum admits the Zariski topology, where just as in the commutative case the closed sets are of the form $V(I) := \{p \in \text{Prim } S \mid I \subseteq p\}$ for $I$ an ideal. Note that $\text{Prim } S = \text{Max } S$ whenever $S$ is commutative, artinian, or module-finite over a noetherian central subring.
is an $A_m$-module by setting $V_m := Z_m/m_m \otimes k V$, where $Z_m/m_m \cong k$ is the residue field at $m$.

**Remark 3.17.** If $\text{ann}_Z(V) = m \in \text{Max } Z$, then a simple $A$-module $V$ and the localized $A_m$-module $V_m$ are isomorphic both as $A$-modules and $A_m$-modules. To see this, first note that any $A$-module is an $A_m$-module by setting $aw := \psi(a)w$, where $w \in V_m$ and $\psi : A \to A_m$ is the algebra monomorphism given by $\psi(a) = 1 \otimes a$. Define $\phi : V \to V_m$ by $\phi(v) = 1 \otimes v$; then $\phi$ is an $A$-module isomorphism, $\phi(aw) = \psi(a)\phi(w) = a\phi(v)$, injective since $\frac{1}{t} \cdot Z_m/m_m \otimes tv = 1 \otimes v = \phi(v) = 0$ implies $tv = 0$, which implies $v = 0$ since $t \not\in m = \text{ann}_Z(V)$. But then $V$ is an $A_m$-module by setting $bv := \phi^{-1}(b\phi(v))$, and so $\phi^{-1} : V_m \to V$ is an $A_m$-module isomorphism: $\phi^{-1}(bw) = \phi^{-1}(b\phi(v)) = b\phi^{-1}(w)$.

The evaluation of a function $f \in A$ at a point $p \in \text{Prim } A = \text{Max } A$ sitting over the Azumaya locus may be taken to either be its image in $A_m/p_m$, or its corresponding representation in $\text{End}_k(V)$, and in the following proposition we find that these are equivalent notions, just as in the commutative case.

**Proposition 3.18.** Let $A$ be a prime noetherian algebra, module-finite over its center, let $V$ be a large simple $A$-module with annihilator $p$ and set $m := p \cap Z$. Then there are $A$-module and $A_m$-module isomorphisms

$$A_m/p_m \cong A/p \cong V^\oplus d \cong (V_m)^{\oplus d},$$

where $d := \text{dim}_k(V)$. Consequently, if $A$ is a homogeneous square superpotential algebra then

$$A_m/p_m \cong A/p \cong V^\oplus |Q_0| \cong (V_m)^{\oplus |Q_0|}.$$

**Proof.** First note that the PI-degree of $A$ is $d$ [13, 3.1.a]. The factor $A/p$ is then a primitive PI ring, so by Kaplansky’s theorem [23, 13.3.8] it is a central simple algebra whose only simple is $V$, so the PI-degree of $A/p$ is also $d$, and thus it has dimension $d^2$ over its center. By the Artin-Wedderburn theorem there is an isomorphism of $A$-modules, $A/p \cong V^\oplus d$. The remaining isomorphisms (4) follow from remark 3.17.

The consequence follows from lemma 3.12 and theorems 3.13, 3.15.

In the following, recall that a ring is semiperfect if every finitely generated left (right) module $V$ admits a projective cover $P$, that is, there is an epimorphism $P \to V$ such that, for any submodule $L \subseteq P$, $\ker \phi + L = P$ implies $L = P$. If a projective resolution is constructed from projective covers then its length will give the precise projective dimension, rather than just an upper bound. In the proof of thm. [3.20, specifically (13)], we determine the (unique) projective covers of the large simple $A_m$-modules.

---

7It is possible that when these modules are over $A$ they do not admit projective covers.
Proposition 3.19. Suppose $A$ is a quiver algebra satisfying $e_i A e_i = Z e_i$ for each $i \in Q_0$, such as a homogeneous square superpotential algebra, and let $V$ be a large simple $A$-module. Set $p := \text{ann}_A(V)$ and $m := p \cap Z \in \text{Max } Z$. Then $A_m$ is semiperfect and
\[ A_m e_i \cong A_m e_j, \quad p_m e_i \cong p_m e_j, \quad A e_i \not\cong A e_j, \quad p e_i \not\cong p e_j, \]
while as $A$-modules,
\[ A e_i / p e_i \cong A e_j / p e_j \cong V. \quad (6) \]

Consequently, any single vertex forms a basic set of idempotents for $A_m$, while the set of all vertices forms a basic set for $A$.\footnote{A set of idempotents in $R$ is basic if it is a complete set of orthogonal idempotents $e_1, \ldots, e_n$ such that $R e_1, \ldots, R e_n$ is a complete irredundant set of representatives of the $R$-modules of the form $R e$ for some primitive idempotent $e$.}

Proof. By prop. 3.16, $m$ (which is a maximal ideal by lemma 3.12) is in the Azumaya locus of $A$, so $A_m$ contains only one primitive ideal, namely $p_m$, and thus the Jacobson radical of $A_m$ is $J = p_m$. Moreover, $A_m$ has a complete set of orthogonal idempotents $e_1, \ldots, e_n$, and for each $i \in Q_0$, the corner ring $e_i A_m e_i$ is local:
\[ e_i A_m e_i = Z_m \otimes_Z e_i A e_i \cong Z_m \otimes_Z Z \cong Z_m. \quad (7) \]

It follows [1, thm. 27.6] that $A_m$ is semiperfect and the set $A_m e_1 / p_m e_1, \ldots, A_m e_n / p_m e_n$ is the set of all simple $A_m$-modules, with
\[ A_m / p_m = A_m e_1 / p_m e_1 \oplus \cdots \oplus A_m e_n / p_m e_n. \]

Since there is only one simple $A_m$-module,
\[ A_m e_i / p_m e_i \cong A_m e_j / p_m e_j \cong V_m. \quad (8) \]

In thm. 3.20 below we show that there is a projective cover $A_m e_i \xrightarrow{\phi} V_m \cong A_m e_i / p_m e_i$, and since $V_m \cong A_m e_j / p_m e_j$, $\phi$ must factor through $A_m e_j$, so by symmetry
\[ A_m e_i \cong A_m e_j. \quad (9) \]

Of course, $A e_i \not\cong A e_j$ when $i \neq j$ (argument in [16] p. 4): otherwise there would be some $f \in e_i A e_j$ and $g \in e_j A e_i$ with $f g = e_i$, $g f = e_j$, so $e_i = f g \in A e_j A$, a contradiction. We remark that $A_m e_i$ is indecomposable since its endomorphism ring $\text{End}_{A_m} (A_m e_i) \cong e_i A_m e_i$ is local by (7).\footnote{\label{fn:local} $A e_i$ is indecomposable as well since the only idempotents in its endomorphism ring are 0 and 1.}
By [11, cor. 17.20] $J(A_m)e_i$ is the unique maximal submodule of $A_m e_i$, and so by (9),

$$p_m e_i = J(A_m)e_i \cong J(A_m)e_j = p_m e_j.$$ 

Now since $m = \text{ann}_Z(V)$, it follows by remark 3.17 and (8) that the following are isomorphic both as $A$-modules and $A_m$-modules:

$$A e_i/p e_i \cong A_m e_i/p_m e_i \cong A_m e_j/p_m e_j \cong A e_j/p e_j,$$

and these are also isomorphic to $V$ and $V_m$.

Note that an alternative proof of (5), namely $A/p \cong V \oplus |Q_0|$, is immediate from prop. 3.19.

**Theorem 3.20.** Suppose $A$ is a PI algebra and let $V$ be a simple $A$-module such that $m := \text{ann}_Z(V) \in \text{Max } Z$ is in the Azumaya locus of $A$ and $A_m$ is semiperfect with a basic set of idempotents. Then

$$\text{pd}_A(V) = \text{pd}_{A_m}(V_m) = \text{pd}_A(A/p) = \text{pd}_{A_m}(A_m/p_m) = \text{pd}_{Z_m}(Z_m/m_m), \quad (10)$$

where $p := \text{ann}_A(V)$.

**Proof.** We first claim $\text{pd}_{A_m}(A_m/p_m) = \text{pd}_{Z_m}(Z_m/m_m)$: Suppose the following is a projective resolution of the residue field $Z_m/m_m$ over the local ring $Z_m$:

$$\cdots \longrightarrow (Z_m)^{\oplus n} \longrightarrow Z_m \overset{1}{\longrightarrow} Z_m/m_m \longrightarrow 0. \quad (11)$$

By assumption $m$ is in the Azumaya locus of $A$, so the localization $A_m$ is an Azumaya algebra, and thus (by definition) $A_m$, and hence the direct summand $A_m e$ for any $e$ in a basic set of idempotents for $A_m$, is a projective $Z_m$-module [23, 13.7.6]. But then $A_m e$ is a flat $Z_m$-module as well, so the functor $- \otimes_{Z_m} A_m e$ is exact. Applying this functor to the resolution (11) we obtain the exact sequence

$$\cdots \longrightarrow (Z_m)^{\oplus n} \otimes_{Z_m} A_m e \cong (A_m e)^{\oplus n} \longrightarrow Z_m \otimes_{Z_m} A_m e \cong (A_m e)^{\oplus n} \longrightarrow Z_m/m_m \otimes_{Z_m} A_m e \longrightarrow 0. \quad (12)$$

The modules in this sequence are now over $Z_m \otimes_{Z_m} A_m \cong A_m$. By [23, 13.7.9], the ideal $p_m \subset A_m$ is generated by $m$, that is, $p_m = \langle m \rangle = mA_m$, and so

$$Z_m/m_m \otimes_{Z_m} A_m e \cong Z_m \otimes_{Z_m} A_m e/(m_m(A_m e)) \cong A_m e/p_m e.$$
The following is a characterization of projective covers \[1\, 27.13\]: Suppose \( R \) is a semiperfect ring with a basic set of idempotents \( e_1, \ldots, e_n \) and Jacobson radical \( J \), and let \( M \) be a finitely generated \( R \)-module. Then if 
\[
M/J \cong (Re_1/Je_1)^{(k_1)} \oplus \cdots \oplus (Re_n/Je_n)^{(k_n)},
\]
there is a unique projective cover \( Re_1^{(k_1)} \oplus \cdots \oplus Re_n^{(k_n)} \to M/J \to 0 \). Consider the case \( R = A_m \) and \( M = A_m e \). Since \( A_m \) is an Azumaya algebra, it contains only one primitive ideal, namely \( p_m \), and thus \( J(A_m) = p_m \). Consequently 
\[
A_m e \xrightarrow{1} A_m e/p_m e \to 0 \quad (13)
\]
is a projective cover. Our claim follows then by the exactness of (12).

Since \( A/p \) is a primitive PI ring, it is central simple by Kaplansky’s theorem \[23, 13.3.8\], so \( A/p \cong V^{\otimes n} \) for some \( n \geq 1 \) by the Artin-Wedderburn theorem, and similarly \( A_m/p_m \cong (V_m)^{\otimes n} \). It follows that \( \text{pd}_A(A/p) = \text{pd}_A(V) \) and \( \text{pd}_{A_m}(A_m/p_m) = \text{pd}_{A_m}(V_m) \) since for any ring \( S \) and family of \( S \)-modules \( M_i \), \( \text{pd}_S(\bigoplus_i M_i) = \sup \{ \text{pd}_S(M_i) \} \) [26, prop. 5.1.20].

Finally, \( \text{pd}_A(V) = \text{pd}_{A_m}(V_m) \): since \( Z_m \) is a projective \( Z_m \)-module, by \[23, 7.4.2.iii\] \( \text{fl}_Z(Z_m) = \text{fl}_{Z_m}(Z_m) = 0 \), so we may tensor a projective resolution of \( V \) over \( A \) with the exact functor \( Z_m \otimes Z - \), giving a projective resolution of \( Z_m \otimes_Z V \cong Z_m/m_m \otimes_Z V = V_m \) over \( Z_m \otimes_Z A = A_m \).

The following remark characterizes the primitive ideals of these algebras.

**Remark 3.21.** In the above proof we used the following fact, shown in \[23, 13.7.9\]: If \( A \) is an Azumaya algebra with center \( Z \) then any ideal \( I \) of \( A \) is generated by \( I \cap Z \). When \( A \) is a quiver algebra satisfying \( e_i Ae_i = Ze_i \) for each \( i \in Q_0 \), this is particularly easy to see: Let \( h \in p_m \), and without loss of generality assume \( h \subseteq e_i A_m e_i \) for some \( i, j \in Q_0 \). Since \( V \) is large (lemma 3.12), there are paths \( c \in e_j Ae_j, d \in e_i Ae_i \) which do not annihilate \( V \), and thus the cycle \( cd \in e_j Ae_j = Ze_j \) is not in \( m e_j \). Consequently,
\[
h = \frac{cd}{dh} h = \frac{c}{cd}(dh) \in \langle m \rangle,
\]
where \( h \in p \) implies \( dh \in p \cap e_i Ae_i = p \cap Ze_i = me_i \).

**Corollary 3.22.** Let \( A \) be a homogeneous square superpotential algebra and \( V \) a large simple \( A \)-module. Then the equalities (10) hold.

**Proof.** By lemma 3.12 such algebras are PI rings; by prop. 3.19 \( A_m \) is semiperfect with a basic set of idempotents; and by prop. 3.16 \( \text{ann}_Z(V) \) is in the Azumaya locus. Apply thm. 3.20.
3.4 “Quiver mutating = crystal melting”

Using the impression of a square superpotential algebra $A$ given in thm. 3.8, we give an algorithm which reduces to a quiver mutation on $Q$ and to a specific change of stability parameter—crystal melting—on the subquiver of $Q$ that is the support of a family of modules parameterized by the exceptional divisor in a (commutative) resolution of $Y \to \text{Max } Z$, where $Y$ is a stable moduli space.

Recall that a mutated quiver $\mu_i(Q)$ at a vertex $i$ is obtained from $Q$ by first reversing the direction of all arrows with either head or tail at $i$; then for each length 2 path $be_a$ in $Q$, add a new arrow from $t(a)$ to $h(b)$; and lastly removing all 2-cycles that may have formed.

We observe that a definition can be formulated that reduces to both quiver mutating and crystal melting (defined below) when $A$ is a square superpotential algebra.

**Definition 3.23.** Let $A = kQ/\partial W$ be a square superpotential algebra with impression $(\tau, B)$, and let $i \in Q_0$. Suppose the quiver $\tilde{Q}$ is obtained from $Q$ as follows:

1. Replace each $a \in Q_1 e_i$ (resp. $a \in e_i Q_1$) by an arrow $\tilde{a}$ with head (resp. tail) at $i$ satisfying $\tau(\tilde{a}) = \tau(a)$;

2. If the orientation of a path $a \in Q_1 e_i Q_1$ has been reversed after step 1, add an arrow $\tilde{a}$ satisfying $\tau(\tilde{a}) = \tau(a)$.

3. Remove any 2-cycles in the quiver that may have formed.

Then we say $\tilde{Q}$ is the $\tau$-mutation of $Q$ at vertex $i$.

Note that the impression (2) specifies the tail (resp. head) of $\tilde{a}$ in the first step and both the head and tail of $\tilde{a}$ in the second step.

If we apply a $\tau$-mutation to a vertex with in/out degree 2, then $\tilde{Q}$ is again a quiver for a square superpotential algebra,

In this case the $\tau$-mutation is also a quiver mutation; note that $i$ must have in/out degree 2 for the mutated quiver $\mu_i(Q)$ to also be a quiver of a square superpotential algebra. If
we apply a $\tau$-mutation to a vertex such as

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {$\cdot$};
  \node (b) at (1,0) {$\cdot$};
  \node (c) at (1,1) {$\cdot$};
  \node (d) at (0,1) {$\cdot$};
  \node (e) at (2,0) {$i$};
  \draw (a) -- (b); \draw (b) -- (c); \draw (c) -- (d); \draw (d) -- (a);
  \draw[->] (a) to (b);
  \draw[->] (b) to (c);
  \draw[->] (c) to (d);
  \draw[->] (d) to (a);
\end{tikzpicture}
\end{center}

\[ \tau \text{-mutate at } i \]

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {$\cdot$};
  \node (b) at (1,0) {$\cdot$};
  \node (c) at (1,1) {$\cdot$};
  \node (d) at (0,1) {$\cdot$};
  \node (e) at (2,0) {$\tau$};
  \draw (a) -- (b); \draw (b) -- (c); \draw (c) -- (d); \draw (d) -- (a);
  \draw[->] (a) to (b);
  \draw[->] (b) to (c);
  \draw[->] (c) to (d);
  \draw[->] (d) to (a);
\end{tikzpicture}
\end{center}

then $\tilde{Q} = Q$, but if we apply it to the subquiver

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {};\node (b) at (1,0) {$\cdot$};\node (c) at (1,1) {$\cdot$};\node (d) at (0,1) {$\cdot$};\node (e) at (2,0) {$\tau$};\node (f) at (2,1) {$i$};
  \draw (a) -- (b); \draw (b) -- (c); \draw (c) -- (d); \draw (d) -- (a);
  \draw[->] (a) to (b);
  \draw[->] (b) to (c);
  \draw[->] (c) to (d);
  \draw[->] (d) to (a);
\end{tikzpicture}
\end{center}

\[ \tau \text{-mutate at } i \]

then we find that the source $i$ becomes a sink (and a sink becomes a source). Such an operation is a type of stability change called “crystal melting”, where the subquiver is the support of a family of modules parameterizing an exceptional divisor in a (commutative) resolution of the center of the algebra. This stability change (should) correspond to a flop of the resolution.

It would be interesting to see if this definition could be generalized to other quiver algebras admitting impressions, and if it plays a role in tilting theory.

4 Noncommutative crepant resolutions

Before presenting the main result of this section, we recall two definitions. First, a noncommutative crepant resolution of a normal Gorenstein domain $R$ is a homologically homogeneous $R$-algebra of the form $A = \text{End}_R(M)$, where $M$ is a reflexive $R$-module [28 def. 4.1]. Secondly, a ring $R$ which is a finitely generated module over a central normal Gorenstein subdomain $C$ is Calabi-Yau of dimension $n$ if (i) $\text{gl. dim } R = \text{K. dim } C = n$; (ii) $R$ is a maximal Cohen-Macaulay module over its center $Z(R)$; and (iii) $\text{Hom}_C(R, C) \cong R$, as $R$-bimodules [11 introduction]. The main result of this section is the following theorem, which will be proved in the course of the next 3 subsections.

**Theorem 4.1.** A homogeneous square superpotential algebra localized at the origin of its center is a noncommutative crepant resolution, and consequently a local Calabi-Yau algebra of dimension 3.
Sec. 4.3 is based on joint work with Alex Dugas. In sec. 4.2, superpotential algebras with non-trivial noetherian centers and infinite global dimension, as well as superpotential algebras with trivial centers and global dimension any odd number at least 5, are introduced.

4.1 Normal Gorenstein centers

For the following, let \( A \) be a homogeneous square superpotential algebra; then \( e_i A e_i \cong Z \) for each \( i \in Q_0 \) by prop. 3.11. Let \( m \) and \( n \) be as in def. 1.1 and suppose \( m \geq n \).

\( Ze_i \) is generated by cycles in \( e_i A e_i \) with \( \bar{\tau} \)-images of the form

\[
\bar{\tau}(\eta) = x_{\alpha_{\pm 1}}^{nu_\pm 1} (x_1 x_2)^{s_\pm 1} (y_1 y_2)^t
\]

for certain values of \( u, v \geq 0, t \leq nu, \) and \( s \leq mv \). Such a cycle “wraps” the horizontal \((x_{\alpha})\) direction of the torus \( u \) times and the vertical direction \((y_{\beta}) v \) times. If \( \eta \in e_i A e_i \) has \( \bar{\tau} \)-image (14), then it is easy to see that there is a cycle \( \eta' \in e_i A e_i \), oriented in the opposite direction, satisfying

\[
\bar{\tau}(\eta') = x_{\alpha_{\pm 1}}^{nu_\pm 1} y_{\beta_{\pm 1}}^{mv_{\pm 1}} (x_1 x_2)^{mv_{\pm 1} - s_{\pm 1}} (y_1 y_2)^{nu_{\pm 1} - t_{\pm 1}},
\]

so letting \( \sigma \) denote the unit cycle at \( i \) (i.e., \( \bar{\tau}(\sigma) = x_1 x_2 y_1 y_2 \)), we have \( \bar{\tau}(\eta \eta') = \bar{\tau}(\sigma^{nu+mv}) \).

Since \( \bar{\tau} \) is a \( k \)-monomorphism on \( e_i A e_i \),

\[
\eta \eta' = \sigma^{nu+mv}.
\]

Now let \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in e_i A e_i \) be the cycles satisfying

\[
\bar{\tau}(\alpha_1) = x_1^{nu} (y_1 y_2)^t, \quad \bar{\tau}(\alpha_2) = x_2^{nu} (y_1 y_2)^{nu-t},
\]

\[
\bar{\tau}(\beta_1) = y_1^{mv} (x_1 x_2)^s, \quad \bar{\tau}(\beta_2) = y_2^{mv} (x_1 x_2)^{mv-s},
\]

with \( s, t \) as small as possible (so the cycles respectively head right, left, up, and down).

Lemma 4.2. The sequence \((\alpha_1, \alpha_2, \beta_1)\) is a \( Z \)-regular sequence.

Proof. The \( \bar{\tau} \)-images of \( \alpha_1, \alpha_2, \) and \( \beta_1 \) are algebraically independent, and since \( \bar{\tau} \) is injective on \( e_i A e_i \), the cycles themselves are algebraically independent.

Proposition 4.3. The (classical Krull) dimension of the center of a homogeneous square superpotential algebra is 3.

Proof. By (15) there are only 3 algebraically independent cycles in \( e_i A e_i \), which we may take to be \( \alpha_1, \alpha_2, \beta_1 \).
The following lemma will be used in prop. 4.5, lemma 4.6 and prop. 4.15.

**Lemma 4.4.** If \( a \in e_i A e_k, \ b \in e_j A e_k \) are paths and \( \tau(b) = s \tau(a) \) for some nonzero monomial \( s \in B \), then there is a path \( d \in e_j A e_i \) such that \( \tau(d) = s \).

**Proof.** We proceed by induction on the degree of \( s \). Suppose \( \deg(s) = 1 \), say \( \tau(s) = y_\beta \). By the proof of prop. 3.1, we may further suppose \( a = x_\alpha r \) and \( b = x_\alpha y_\beta r \), modulo \( \partial W \), for some path \( r \in A e_k \). Thus either

\[
\begin{align*}
\text{either} & \quad \begin{array}{c} \downarrow \hspace{1cm} \uparrow \end{array}, \text{ which is not a possible configuration, or} \\
\end{align*}
\]

in which case \( b \) also equals \( y_\beta x_\alpha r \). Thus \( b = da \) with \( d = y_\beta \).

Now assume there is such a \( d \) whenever \( \deg(s) < n \), and suppose \( \deg(s) = n \). Choose a representative of \( b \) in \( kQ \), and let \( y_\beta \) be the leftmost subpath of \( b \) such that \( y_\beta^n \tau(a) \), \( y_\beta^{n+1} \not\mid \tau(a) \), and \( y_\beta^{n+1} \tau(b) \). Again by the proof of prop. 3.1, we may assume \( b = x_\alpha y_\beta r' \) and \( a = x_\alpha r \). Thus by induction \( r' = d' r \) for some path \( d' \), so \( b = x_\alpha y_\beta d' r \), which again by prop. 3.1 equals \( d' y_\beta x_\alpha r \) or \( d' x_\alpha y_\beta r \), so we may apply the above argument to deduce that there is a path \( d = d' y_\beta \in e_j A e_i \) such that \( \tau(d) = s \).

**Proposition 4.5.** The center of a homogeneous square superpotential algebra is normal.

**Proof.** By prop. 3.13 Max \( Z \) is a toric algebraic variety, and thus it suffices to check that if \( c_1, c_2 \in e_i A e_i \) are cycles with \( \tau(c_1) = m \), \( \tau(c_2) = m' \), say \( m' \geq m \), then there is a cycle \( c_3 \in e_i A e_i \) such that \( \tau(c_3) = m' - m \). But this follows from lemma 4.4 by setting \( i = j = k \) and noting that \( \tau(c_3) = m' - m', \tau(c_1) \). \( \square \)

Set \( \tilde{Z} := Z e_i / (\alpha_1, \alpha_2, \beta_1) \). \( \tilde{Z} \) is a zero-dimensional local ring since the dimension of \( Z \) and the length of the regular sequence are equal, and we now show that this ring is Gorenstein by showing that it has a simple socle. Recall that the socle of a \( \tilde{Z} \)-module \( M \) is the annihilator in \( M \) of the unique maximal ideal of \( \tilde{Z} \) (or equivalently, the sum of its simple submodules).

**Lemma 4.6.** The socle of \( \tilde{Z} \) is the principal ideal \( (\sigma^{n-1}) \), and is thus simple.

**Proof.** Note that \( \sigma^{n-1} \not\in \tilde{Z} \) since the generating relations are binomial and

- \( \tau(\sigma^{n-1}) \) has \( (n-1) \) \( x_1 \) (resp. \( x_2 \)) arrows while \( \alpha_1 \) (resp. \( \alpha_2 \)) has \( n \) \( x_1 \) (resp. \( x_2 \)) arrows;
- \( \tau(\sigma^{n-1}) \) has \( (n-1) \) \( y_1 \) arrows while \( \beta_1 \) has \( m \geq n \) \( (n-1) \) \( y_1 \) arrows.
(If \( n > m \), then exchange the roles of \( \{ x_1, x_2 \} \) and \( \{ y_1, y_2 \} \).)

Since \( A \) is homogeneous, if \( t \in \{0, n\} \) (resp. \( s \in \{0, m\} \)) then in each row (resp. each column) each unit square is identical and contains a diagonal arrow. In this case the quiver can be redrawn so that there are only three orientations of arrows, namely one horizontal, one vertical, and one diagonal:

Let \( n' \) be the width and \( m' \) the height of the new fundamental region of the two-torus in the plane. Then such a quiver is the McKay quiver for the representation of the abelian group \( \mathbb{Z}/r\mathbb{Z} \), where \( r := \text{lcm}(m', n') \), and \( 1 \) is represented by the diagonal matrix

\[
\text{diag} \left( \omega^{r/n'}, \omega^{r/m'}, \omega^{-r/n'-r/m'} \right) \in \text{SL}_3(k),
\]

with \( \omega \) a primitive \( r \)-th root of unity. It follows that \( Z \) is Gorenstein.

Otherwise suppose \( 0 < t < n \) and \( 0 < s < m \). Let \( \eta \in e_i A e_i \) be a cycle; we want to show that \( \eta \sigma^{n-1} \in \tilde{Z} \), and it suffices to show that \( \overline{\tau}(\alpha_1), \overline{\tau}(\alpha_2) \), or \( \overline{\tau}(\beta_1) \) divides \( \overline{\tau}(\eta \sigma^{n-1}) \) by prop. \[4.3\] (with \( i = j = k \)). If \( x_1|\overline{\tau}(\eta) \) then \( x_1^n y_1^{n-1} y_2^{n-1} \overline{\tau}(\eta \sigma^{n-1}) \), so \( t < n \) implies \( \overline{\tau}(\alpha_1)|\overline{\tau}(\eta \sigma^{n-1}). \) If \( y_1|\overline{\tau}(\eta) \) then since \( s < m \) at least one of \( x_1, x_2 \) divides \( \overline{\tau}(\eta), \) and so at least one of \( x_1^n y_1^n y_2^{n-1}, x_2^n y_1^n y_2^{n-1} \) divides \( \overline{\tau}(\eta \sigma^{n-1}). \) But again since \( t < n, \) at least one of \( \overline{\tau}(\alpha_1), \overline{\tau}(\alpha_2) \) divides \( \overline{\tau}(\eta \sigma^{n-1}). \)

\[\Box\]

**Theorem 4.7.** The center of a square superpotential algebra is Gorenstein.

Proof. \( \tilde{Z} \) is Gorenstein since it is a local zero-dimensional ring with a simple socle [18 prop. 21.5], and thus since the sequence \( (\alpha_1, \alpha_2, \beta_1) \) is regular, \( Z \) is Gorenstein as well [18 def. 21.3].

4.2 Homological homogeneity

In this section we show that the localization of any square superpotential algebra at the origin of its center is homologically homogenous with global dimension 3. Recall that if \( S \) is a commutative noetherian equidimensional \( k \)-algebra and \( A \) is a module-finite \( S \)-algebra, then \( A \) is “homologically homogeneous” if all simple \( A \)-modules have the same projective dimension (see [14, 28 sec. 3]).

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Letting $m_0$ denote the maximal ideal at the origin of Max $Z$, the only simple modules over the localization $A_{m_0}$ are the vertex simples. A vertex simple is a simple module $V^i$ in which every path, with the exception of vertex $i$, is represented by zero. In physics terms, the vertex simples are (often) the fractional branes that probe the apex of a tangent cone on a singular Calabi-Yau.

4.2.1 The Berenstein-Douglas complex

We first establish notation. If $g, h \in Q_1$, set $\delta_{h,g} = e_{t(g)}$ if $g = h$ and 0 otherwise. For $p = g_n \cdots g_1 \in Q_{\geq 1}$, $g_i, h \in Q_1$, set

$$\delta_{h,p} := \delta_{h,g_n}g_{n-1}\cdots g_1,$$

and for any $i \in Q_0$,

$$\delta_{h} e_i = e_i \delta_{h} := 0.$$

Extend $k$-linearly to $kQ$.

Lemma 4.8. Let $Q$ be a quiver and $W \in \text{tr}(kQ_{\geq 2})$ a superpotential. Then for each $i \in Q_0$, $g \in Q_1e_i$, and $h \in e_iQ_1$,

$$\delta_{h}(\partial_{g}W) = (\partial_{h}W) \delta_{g} =: W_{h,g}.$$

Consequently

$$W_{h} = \sum_{g \in Q_{1}e_i} W_{h,g}g \quad \text{and} \quad W_{g} = \sum_{h \in e_iQ_1} hW_{h,g}. \quad (17)$$

Proof. Let $i \in Q_0$ and $p = (d_ng) \cdots (d_2gh)(d_1gh) \in e_iQ_{\geq 1}e_i$, with $g, h \in Q_1$ and $gh$ not a subpath of $d_j$ for each $1 \leq j \leq n$ (though $g$ or $h$ separately may be). Then

$$\partial_{g}\sum_{\text{cyc}} p = (hd_{n}gh \cdots d_1) + (hd_{n-1}gh \cdots d_n) + \cdots + (hd_1gh \cdots d_2) + A,$$

$$\partial_{h}\sum_{\text{cyc}} p = (d_ngh \cdots d_1g) + (d_{n-1}gh \cdots d_ng) + \cdots + (d_1gh \cdots d_2g) + B,$$

where $\delta_{h} A = B \delta_{g} = 0$. Thus

$$\delta_{h}\left(\partial_{g}\sum_{\text{cyc}} p\right) = (d_ngh \cdots d_1) + (d_{n-1}gh \cdots d_n) + \cdots + (d_1gh \cdots d_2) = \left(\partial_{h}\sum_{\text{cyc}} p\right) \delta_{g}.$$
This brings us to the following proposition.

**Proposition 4.9.** Let $A = kQ/\partial W$ be a superpotential algebra and $V^i$ the vertex simple $A$-module at $i \in Q_0$. Write $Q_1e_i = \{g_1, \ldots, g_m\}$, $e_iQ_1 = \{h_1, \ldots, h_n\}$, and set $p_i := \text{ann}_AV^i$. Then the sequence

$$0 \rightarrow Ae_i \xrightarrow{\delta_2 := \begin{bmatrix} h_1 & \cdots & h_n \end{bmatrix}} \bigoplus_{1 \leq k \leq n} Ae_{e_i(h_k)} \xrightarrow{\delta_1 := \begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix}} \bigoplus_{1 \leq j \leq m} Ae_{e_i(g_j)} \xrightarrow{\phi} A/p_i \cong V^i \rightarrow 0;$$

is a projective complex and $\phi$ is a projective cover.

**Proof.** $Ae_i \xrightarrow{\phi} A/p_i$ is a projective cover since $\phi$ is superfluous: $e_i \notin \ker\phi$, so if $\ker\phi +_A L = Ae_i$ then $e_i \in L$, hence $L \supseteq Ae_i$. Each term is a direct summand of a free $A$-module and so is projective. The sequence is a complex by lemma 4.8. \qed

We call (18) the “Berenstein-Douglas” complex. In [8, sec. 5.5], Berenstein and Douglas constructed this complex and raised the question: under what conditions is this complex a projective resolution of a vertex simple $A$-module? We will show by example that in general the complex may fail to be exact in both the second and third terms. However, we will also show that when $A$ is a square superpotential algebra (not necessarily homogeneous (def. 3.10)), the complex is indeed a projective resolution of any vertex simple module. The following theorem is proved in the course of the next two subsections.

**Theorem 4.10.** Let $A$ be a square superpotential algebra. If $V$ is a vertex simple $A$-module with annihilator $p$ then

$$\text{pd}_A(V) = \text{pd}_A(A/p) = 3,$$

where (18) is a minimal projective resolution of $V$ and $A/p \cong V$.

### 4.2.2 Exactness at the second term

**Lemma 4.11.** Let $A$ be a square superpotential algebra and $V$ a vertex simple $A$-module. Then $\text{im}\delta_2 = \ker\delta_1$ in the Berenstein-Douglas complex.
Proof. Order the sets $Q_1e_i = \{g_1, \ldots, g_n\}$ and $e_iQ_1 = \{h_1, \ldots, h_n\}$ both clockwise, such that $g_1h_1$ a subpath of a term of $W$. Then

$$
\delta_1 = \left[ W_{h_kg_j} \right]_{k,j} = \begin{bmatrix}
a_1 & 0 & \ldots & -b_m
-b_1 & a_2 & 0
0 & -b_2 & 0
\vdots & \ddots & \ddots
0 & 0 & a_m
\end{bmatrix},
$$

where each $a_\ell, b_\ell$ is nonzero. Suppose $(d_1 \cdots d_n) \in \ker \delta_1$. Then $d_\ell a_\ell - d_{\ell+1}b_\ell = 0$ for each $1 \leq \ell \leq n$. By lemma 4.13 below, $d_\ell a_\ell$ and $d_{\ell+1}b_\ell$ are each nonzero, so they must be in the same corner ring, hence $d_\ell, d_{\ell+1} \in e_jA$ for some $j \in Q_0$, and thus $\{d_1, \ldots, d_n\} \subset e_jA$. Furthermore,

$$(\bar{\tau}(d_\ell)\bar{\tau}(a_\ell))E_{j,t(a_\ell)} = \tau(d_\ell)\tau(a_\ell) = \tau(d_\ell a_\ell) = \tau(d_{\ell+1}) = \bar{\tau}(d_{\ell+1})\tau(b_\ell) = \bar{\tau}(d_{\ell+1})\bar{\tau}(b_\ell)E_{j,t(b_\ell)},$$

so $\bar{\tau}(d_\ell)\bar{\tau}(a_\ell) = \bar{\tau}(d_{\ell+1})\bar{\tau}(b_\ell)$ since $t(a_\ell) = h(g_\ell) = t(b_\ell)$. There are variables $\alpha_\ell, \beta_\ell \in \{x_1, x_2, y_1, y_2\}$ such that $\alpha_\ell|\bar{\tau}(a_\ell), \beta_\ell|\bar{\tau}(b_\ell)$, and

$$\alpha_\ell \nmid \bar{\tau}(b_\ell), \beta_\ell \nmid \bar{\tau}(a_\ell),$$

so both $\alpha_{\ell-1}$ and $\beta_\ell$ divide $\bar{\tau}(d_\ell)$. In particular, $\alpha_{\ell-1}, \beta_\ell|\bar{\tau}(h_\ell)$ since $(h_1 \cdots h_n) \in \ker \delta_1$.

We claim $\bar{\tau}(h_\ell)|\bar{\tau}(d_\ell)$: Since $h_\ell$ is an arrow, the degree of $\bar{\tau}(h_\ell)$ is 1 or 2. If the degree is 1, then $\bar{\tau}(h_\ell) = \alpha_{\ell-1} = \beta_\ell|\bar{\tau}(d_\ell)$. Otherwise suppose $\bar{\tau}(h_\ell) = x_1y_1$ (resp. $x_1y_2, x_2y_1, x_2y_2$). Then $(\alpha_{\ell-1}, \beta_\ell) = (x_1, y_1)$ (resp. $(y_2, x_1), (y_1, x_2), (x_2, y_2)$), so $\bar{\tau}(h_\ell)|\bar{\tau}(d_\ell)$, verifying our claim. Since $d_\ell \in Ae_{h_\ell}$, it follows by the proof of prop. 3.1 that for each $\ell$ there is some $d'_\ell \in e_jAe_i$ such that $d_\ell = d'_\ell h_\ell$, and so

$$\tau(d'_\ell)\tau(h_\ell)\bar{\tau}(a_\ell) = \tau(d_{\ell+1})\tau(h_{\ell+1})\bar{\tau}(b_\ell).$$

We now claim $\bar{\tau}(d'_\ell) = \bar{\tau}(d'_{\ell+1})$. This holds if $\bar{\tau}(b_\ell) = \bar{\tau}(h_\ell)$ and $\bar{\tau}(a_\ell) = \bar{\tau}(h_{\ell+1})$ since $B$ is an integral domain, so suppose $\bar{\tau}(b_\ell) \neq \bar{\tau}(h_\ell)$; then $\bar{\tau}(h_{\ell+1})$ has degree 2, say $\bar{\tau}(h_\ell) = x_1y_1$. Then $\bar{\tau}(h_{\ell+1}) = x_1y_2$ or $y_2$ since the set $\{h_1, \ldots, h_n\} = e_iQ_1$ is labeled clockwise. But then $\bar{\tau}(a_\ell) = y_2$ and $\bar{\tau}(b_\ell) = y_1$ or $x_1y_1$ respectively. It follows that $\bar{\tau}(d'_\ell) = \bar{\tau}(d'_{\ell+1})$. The other 3 cases are similar, verifying our claim. Since $d'_\ell$ and $d'_{\ell+1}$ are both elements of $e_jAe_i$, we have $\bar{\tau}(d_\ell) = \bar{\tau}(d_{\ell+1})$, and since $\bar{\tau}$ is injective, $d_\ell = d'_{\ell+1}$. Since this holds for each $\ell$,

$$(d_1 \cdots d_n) = (d_1 h_1 \cdots d_n h_n) = d'_1(h_1 \cdots h_n).$$

\[\square\]
The following three superpotential algebras are examples where exactness fails in the second term, \( \text{im} \delta_2 \subset \text{ker} \delta_1 \), of the Berenstein-Douglas resolution. Each algebra has a nontrivial noetherian center—specifically 1 dimensional—and infinite global dimension. In each case we show how the second connecting map may be realized in two different ways, which is why the exactness fails. \( V^i \) denotes the vertex simple at \( i \in Q_0 \), and the map \( Ae_i \rightarrow V^i \) is defined via the \( A \)-module isomorphism \( V^i \cong A/\text{ann}_A V^i \).

**Example 4.12.**

- **Q:** \( \begin{array}{ccc} W &=& a^2 b, \text{ so } Z = k[t^2]. \end{array} \)

  \( A \) has infinite global dimension since the vertex simple \( V \) has projective resolution

  \[
  \cdots \rightarrow \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \rightarrow A^2 \rightarrow \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \rightarrow A^2 \rightarrow \begin{bmatrix} a \\ b \end{bmatrix} \rightarrow A \rightarrow V \rightarrow 0.
  \]

  The second map, \( \delta_1 \), satisfies

  \[
  \begin{bmatrix}
  W_{aa} & W_{ab} \\
  W_{ba} & W_{bb}
  \end{bmatrix} = \begin{bmatrix} * & * \\
  W_{ab} & 0 
  \end{bmatrix}
  \]

- **Q:** \( \begin{array}{ccc} W &=& c_1 ba - c_2 ab, \text{ so } Z = k[c_1 + c_2]. \end{array} \)

  \( A \) has infinite global dimension:

  \[
  \cdots \rightarrow \begin{bmatrix} a & -c_2 \\ 0 & b \end{bmatrix} \rightarrow Ae_1 \oplus Ae_2 \rightarrow \begin{bmatrix} b & c_1 \\ 0 & a \end{bmatrix} \rightarrow Ae_2 \oplus Ae_1 \rightarrow \begin{bmatrix} a & -c_2 \\ 0 & b \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ a \end{bmatrix} \rightarrow Ae_1 \rightarrow V^1 \rightarrow 0.
  \]

  \( \delta_1 \) satisfies

  \[
  \begin{bmatrix}
  W_{bc_1} & W_{ba} \\
  W_{c_1c_1} & W_{c_1a}
  \end{bmatrix} = \begin{bmatrix} * & * \\
  0 & W_{ac_2} 
  \end{bmatrix}
  \]

- **Q:** \( W \) given by (1), with

  \[
  \begin{array}{ccc}
  x_2 & y_5 & 2 \\
  x_2 & y_6 & 3 \\
  x_3 & x_4 & 2 \\
  x_1 & y_1 & 4 \\
  x_2 & y_1 & 1 \\
  x_3 & x_4 & 3 \\
  y_2 & y_1 & 2 \\
  \end{array}
  \]
$A$ has infinite global dimension:

$$
\cdots \to Ae_5 \oplus Ae_3 \xrightarrow{\cdots} Ae_5 \oplus Ae_3 \xrightarrow{\cdots} Ae_6 \oplus Ae_4 \xrightarrow{\cdots} Ae_1 \oplus Ae_2 \xrightarrow{\cdots} A_1 \xrightarrow{-1} V^1 \to 0.
$$

$\delta_1$ satisfies

$$
\begin{bmatrix}
W_{x_5x_1} & W_{x_5y_1} \\
W_{y_3x_1} & W_{y_3y_1}
\end{bmatrix} = \begin{bmatrix}
W_{y_5y_2} & W_{y_5x_2} \\
W_{x_3y_2} & W_{x_3x_2}
\end{bmatrix}
$$

### 4.2.3 Exactness at the third term

In the following lemma we find that the kernel of $\delta_2$ in the complex (18) is zero whenever $W_{a,b} \neq W_{a}$ for each $a, b \in Q_1$ and $\text{im} \delta_2 = \ker \delta_1$. Recall from the proof of thm. 4.10 that this lemma is not necessary when $A$ is homogeneous.

**Lemma 4.13.** Let $A$ be a superpotential algebra with superpotential $W \in \text{tr} (kQ_{\geq 2})$ and $g \in Q_1$. Consider the left $A$-module homomorphism $Ae_{h(g)} \xrightarrow{g} Ae_{t(g)}$ given by the multiplication map $m \mapsto mg$. Then

$$
\ker(g) = \langle W_{h,g} \in A \mid h \in e_{t(g)}Q_1 \text{ and } W_{h,g}g = W_h \text{ in } kQ \rangle.
$$

In particular, $\ker(g) = 0$ whenever $W_{h,g}g \neq W_h$ for each $h \in e_{t(g)}Q_1$.

**Proof.** Note $\langle W_{h,g} \mid W_{h,g}g = W_h \rangle \subseteq \ker(g)$, so suppose $f \in kQ$ is such that $fg \in \partial W := \langle W_h := \partial_h W \mid h \in Q_1 \rangle$, that is,

$$
fg = \sum_{h \in Q_1} l_h W_h r_h,
$$

for some coefficients $l_h, r_h \in kQ$. We may assume $l_h, r_h \in \{0, 1\}$: first, suppose there is a right coefficient $r_h$ containing a path $p \in Q_{\geq 1}$. Then $p = p'g$ for some path $p' \in Q_{\geq 0}$, so $l_h W_h p' \in \partial W$, so is zero in $A$, and thus each right coefficient $r_h$ may be taken to be 0 or 1. Now consider $fg = \sum l_h W_h$ with left coefficients $l_h$. By definition, $W \in kQ_{\geq 2}$, so $W_h \in kQ_{\geq 1}$. Thus the arrow $g$ must be a rightmost subpath of $W_h$ whenever $l_h \neq 0$. But then there is an $f' \in kQ$ such that $f'g = \sum l'_h W_h \in \partial W$ with each $l'_h$ either 0 or 1. Thus we take $l_h, r_h \in \{0, 1\}$. 

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Let \( \{h_1, \ldots, h_m\} \) be the set of arrows \( h_j \) such that \( l(h_j) = r_{h_j} = 1 \). Denote \( f = \sum_{j=1}^{m} \sum_{k=1}^{h_j} f_k^j \), where each \( f_k^j \in Q_{\geq 0} \) satisfies \( t(f_k^j) = h(g) = i \) and \( h(f_k^j) = t(h_j) \). Then

\[
f g = \sum_{j=1}^{m} W_{h_j}
\]

implies

\[
W = \left( \sum_{j=1}^{m} h_j \left( f_1^j + \cdots + f_{n_j}^j \right) \right) g + \cdots,
\]

where \( \partial h_j(\cdots) = 0 \). But then for each such \( j \),

\[
W_{h_j} = (f_1^j + \cdots + f_{n_j}^j) g = W_{h_j,gg}.
\]

We now give examples of superpotential algebras where exactness fails in the third term, \( \ker \delta_2 \neq 0 \). In the first set of examples each algebra has a nontrivial noetherian center (again 1 dimensional) and infinite global dimension, while in the second set of examples each algebra has trivial center but global dimension any odd number at least 5.

**Example 4.14.**

- Let \( Q \) be the cycle quiver, consisting of a single oriented cycle \( c = a_n \cdots a_2 a_1 \), \( a_i \in Q_1 \), up to cyclic equivalence, and let \( W \in k[c] \). Then \( Z \cong k[c]/(\partial W) \).

  If not both \( n = 1 \) and \( W = c^2 \) then the global dimension of \( A \) is infinite:

  \[
  \cdots \rightarrow Ae_{h(a_{n-1})} \xrightarrow{a_{n-1}} Ae_{t(a_{n-1})} \xrightarrow{W_{a_{n-1},a_n}} Ae_{h(a_n)} \xrightarrow{a_n} Ae_{t(a_n)} \xrightarrow{W_{a_n,1}} Ae_{h(a_1)} \xrightarrow{a_1} Ae_{t(a_1)} \xrightarrow{1} V^{t(a_1)} \rightarrow 0.
  \]

- \( Q: \)

\[
W = (a_1 \cdots d_1) - (a_2 \cdots d_1).
\]

\( A \) has global dimension \( 2n + 3 \), \( n \geq 1 \):

\[
0 \rightarrow Ae_{h(a_1)} \xrightarrow{[a_1, a_2]} Ae_{t(a_1)} \oplus Ae_{t(a_2)} \xrightarrow{\left[ \begin{array}{c} W_{a_1,d_1} \\ W_{a_2,d_1} \end{array} \right]} Ae_{h(d_1)} \xrightarrow{d_1} Ae_{t(d_1)} \xrightarrow{W_{d_1,d_2}} Ae_{h(d_2)}
\]
\[
\begin{align*}
\xrightarrow{d_2} \ A_{e(t(d_2))} & \xrightarrow{W_{d_2,d_3}} A_{e(h(d_3))} \rightarrow \cdots \rightarrow A_{e(h(d_n))} \\
\xrightarrow{d_n} \ A_{e(t(d_n))} & \xrightarrow{[W_{d_n,b_1} \ W_{d_n,b_2}]} A_{e(h(b_1))} \oplus A_{e(h(b_2))} \xrightarrow{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}} A_{e(h(d_n))} \xrightarrow{\lambda} V_{h(d_n)} \rightarrow 0.
\end{align*}
\]

We now prove thm. 4.10.

Proof of Theorem 4.10. First, im $\delta_2 = \ker \delta_1$ by lemma 4.11. Secondly, if $A$ is homogeneous then cor. 3.14 implies $\ker \delta_2 = 0$. If $A$ is not homogeneous, then apply lemma 4.13, noting that $W_{h,g} \neq W_h$ for any $g, h \in Q_1$: if $W_{h,g} = 0$ then $W_{h,g} = 0 \neq W_h$; otherwise there exists a term of $W$ with subpath $gh$. But $|Q_1e_t(g)| \geq 2$ implies there is a $g' \in Q_1e_t(g) \setminus \{g\}$ such that $g'h$ is a subpath of a term of $W$ as well, so in either case $W_{h,g} \neq W_h$. □

4.3 Endomorphism rings

This section is based on joint work with Alex Dugas. We show that any homogeneous square superpotential algebra (def. 3.10) is an endomorphism ring of a reflexive module over its center. A related result was independently discovered by Mozgovoy [25] in a different context.

If $Y \rightarrow X$ is a resolution of a singular algebraic variety $X$, then $X$ and $Y$ are birational and hence have the same function field $k(Y) \cong k(X)$. In hopes of replacing $k(Y)$ by a noncommutative ring $A$, such a requirement is clearly too strong, so instead we only ask that $A$ and $k(X)$ be Morita equivalent, and this holds if $A$ is an endomorphism ring of a $k[X]$-module. To see why, note that if $A \cong \text{End}_R(M)$ where $M$ is a finitely generated projective generator for the category $R$-mod, then $R$ and $A$ are Morita equivalent. In Van den Bergh’s definition, $M$ is not a projective $k[X]$-module, but it is a projective $k[X]$-module since any module over a field is free.\[10\]

But by propositions 3.11 and 3.13, any homogeneous square superpotential algebra $A$ is prime and satisfies $e_iAe_i = Ze_i$ for each $i \in Q_0$. First note that $e_iAe_k$ is a $Z$-module for each $i \in Q_0$: if $z \in Z$, $a \in e_iAe_k$, then $za = ze_ia = e_iza \in e_iAe_k$.

**Proposition 4.15.** Let $A$ be a homogeneous square superpotential algebra. Then for each $i, j, k \in Q_0$, there is an isomorphism

\[
e_jAe_i \xrightarrow{\cong} \text{Hom}_Z(e_iAe_k,e_jAe_k)
\]

\[10\]If $\text{Max} Z$ has singularities then $Z$ has infinite global dimension, whereas $A$ is supposed to have finite global dimension, so the two rings should not be Morita equivalent.
where \( f_d(a) = da \).

**Proof.** Surjectivity: Suppose \( f \in \text{Hom}_Z (e_iAe_k, e_jAe_k) \). Let \( a_1, a_2 \in e_iAe_k \), \( b_1 := f(a_1), b_2 := f(a_2) \in e_jAe_k \), and \( h \in e_kAe_i \). Assuming \( a_1 \neq a_2 \), it follows that \( \bar{\tau}(a_1) \neq \bar{\tau}(a_2) \) since \( \bar{\tau} \) is injective. Then

\[
0 = f(a_2(ha_1) - (a_2h)a_1) = f(a_2\alpha_1 - \alpha_2a_1) \quad \text{with} \quad \alpha_1, \alpha_2 \in Z \\
= f(\alpha_1a_2 - \alpha_2a_1) = \alpha_1b_2 - \alpha_2b_1,
\]

so

\[
\bar{\tau}(\alpha_1e_j)E_{jj} \cdot \bar{\tau}(b_2)E_{j1} = \tau(\alpha_1e_j)\tau(b_2) = \tau(\alpha_1b_2) = \tau(\alpha_2b_1) = \bar{\tau}(\alpha_2e_j)\tau(b_1) = \bar{\tau}(\alpha_2b_1) = \bar{\tau}(a_2h)\bar{\tau}(b_1).
\]

hence

\[
\bar{\tau}(ha_1)\bar{\tau}(b_2) = \bar{\tau}(\alpha_1e_j)\bar{\tau}(b_2) = \bar{\tau}(\alpha_2e_j)\bar{\tau}(b_1) = \bar{\tau}(a_2h)\bar{\tau}(b_1).
\]

But

\[
\bar{\tau}(ha_\alpha)E_{11} = \tau(ha_\alpha) = \tau(h)\tau(a_\alpha) = \bar{\tau}(h)E_{11} \cdot \bar{\tau}(a_\alpha)E_{11},
\]

so \( \bar{\tau}(a_1)\bar{\tau}(b_2) = \bar{\tau}(a_2)\bar{\tau}(b_1) \) since \( B \) is an integral domain. Since \( \bar{\tau}(a_1) \neq \bar{\tau}(a_2) \), it follows that \( t\bar{\tau}(b_1) = s\bar{\tau}(a_1) \) and \( t\bar{\tau}(b_2) = s\bar{\tau}(a_2) \) for some coprime elements \( s, t \in B \). Consequently \( t \) divides both \( \bar{\tau}(a_1) \) and \( \bar{\tau}(a_2) \), hence the \( \bar{\tau} \)-image of each \( a \in e_iAe_k \) since \( a_1 \) and \( a_2 \) were arbitrary, and so \( t = 1 \).

Since the set of paths forms a basis for \( A \), it is sufficient to consider a path \( a \in e_iAe_k \). Write \( f(a) = b_1 + \cdots + b_m \) in terms of paths \( b_i \) and \( s = s_1 + \cdots + s_m \) in terms of monomials \( s_\ell \in B \). Since \( \bar{\tau} \) is a \( k \)-homomorphism,

\[
\bar{\tau}(b_1) + \cdots + \bar{\tau}(b_m) = \bar{\tau}(f(a)) = s\bar{\tau}(a) = (s_1 + \cdots + s_m') \bar{\tau}(a).
\]

Since the \( \bar{\tau} \)-image of a path is a monomial in \( B = k[x_1, x_2, y_1, y_2] \), we have \( m = m' \) and so (by possibly re-indexing) \( \bar{\tau}(b_i) = s_\ell \bar{\tau}(a) \). Thus we may suppose \( f(a) \) is a path and \( s \) is a monomial.

First, \( s \) is nonzero since \( A \) is prime and so \( 1 = \text{ann}_A(e_iAe_k) = 0 \) (if \( d \in \text{ann}_A(e_iAe_k) \) then \( f_d \in \text{Hom}_Z (e_iAe_k, e_jAe_k) \)). Thus by lemma 4.4 there is a \( d \in e_jAe_i \) satisfying \( \bar{\tau}(d) = s \). It follows that \( \bar{\tau}(f(a)) = s\bar{\tau}(a) = \bar{\tau}(d)\bar{\tau}(a) = \bar{\tau}(da) \) since \( d \in e_jAe_i \) and \( a \in e_iAe_k \) are paths and \( da \neq 0 \) by cor. 3.14. Since \( \bar{\tau} \) is injective, \( \bar{\tau} \) is injective on the corner rings, and thus \( f(a) = da \). Since \( a \) was arbitrary, \( f = f_d \).

**Injectivity:** Let \( d \in e_jAe_i \) be nonzero. Since \( B \) is an integral domain, \( da \neq 0 \) for any nonzero \( d \in e_iA \) by cor. 3.14 so \( f_d \) is injective, and in particular \( f_d \neq 0 \). \( \square \)
**Proposition 4.16.** Let $A$ be a homogeneous square superpotential algebra. Then for any $i \in Q_0$, $e_i Ae_k$ is a reflexive $Z$-module, and so $Ae_k$ is a reflexive $Z$-module.

**Proof.** For $i, j \in Q_0$,

$$
\text{Hom}_Z(e_i Ae_j, Z) = \text{Hom}_Z(e_i Ae_j, Ze_j) = \text{Hom}_Z(e_i Ae_j, e_j Ae_j) \cong e_j Ae_i,
$$

where the last isomorphism follows from prop. 4.15 with $k = j$. Thus

$$
\text{Hom}_{Z}(e_i Ae_k, Z) \cong e_k Ae_i \quad \text{and} \quad \text{Hom}_Z(e_k Ae_i, Z) \cong e_i Ae_k.
$$

\[\square\]

**Theorem 4.17.** Let $A$ be a homogeneous square superpotential algebra. Then for any $i \in Q_0$, $Ae_i$ is a reflexive $Z$-module and

$$
A \cong \text{End}_Z(Ae_i).
$$

**Proof.**

$$
A = \bigoplus_{i,j \in Q_0} e_i Ae_j
\cong \bigoplus_{i,j \in Q_0} \text{Hom}_Z(e_i Ae_k, e_j Ae_k) \quad \text{by prop. 4.15}
\cong \text{Hom}_Z \left( \bigoplus_i e_i Ae_k, \bigoplus_j e_j Ae_k \right)
= \text{End}_Z(Ae_k).
$$

\[\square\]

Since $A$ is prime, $Z_m$ is the center of $A_m$, and so thm. 4.17 holds after localization:

$$
A_m = Z_m \otimes_Z A \cong Z_m \otimes_Z \text{End}_Z(Ae_i) \cong \text{End}_{Z_m}(A_m e_i).
$$

**Example 4.18.** Consider the $Y^{4,0}$ algebra $A$ with two “double impurities”. $A$ is homogeneous by prop. 5.1 below. Label vertex $i$ by the commutative ring $\bar{\tau}(e_i Ae_1) \subset B$ for each $i \in Q_0$, and denote by $R$ by the subalgebra of $B$ isomorphic to $Z$.
A is then isomorphic to the endomorphism ring of the direct sum of the above $R$-modules, each of which is reflexive. Note that the free $R$-module can be placed at any vertex.

**Example 4.19.** The conifold quiver algebra $A$ given in example 1.2 is the $Y^{1,0}$ algebra, homogeneous with center

$$Z \cong R = k[x_1y_1, x_2y_2, x_1y_2, x_2y_1] \cong k[a, b, c, d]/(ab - cd),$$

again by prop. 5.1 below. It is standard [29, sec. 1, example] to view $A$ as the endomorphism ring

$$A \cong \text{End}_R(R \oplus I) = \begin{pmatrix} R & I \\ I^{-1} & R \end{pmatrix},$$

where $I = (a, c) = (x_1y_1, x_1y_2)$ and $I^{-1} = (a, d) = (x_1y_1, x_2y_1)$. Our method realizes $A$ as a slightly different endomorphism ring:

$$A \cong \text{End}_R(R \oplus R[x_1, x_2]) \cong \text{End}_R(R \oplus R[y_1, y_2]).$$

**Example 4.20.** Consider the McKay quiver algebra for the representation

$$\text{diag} \left( e^{\pi i/4}, e^{3\pi i/4} \right) \subseteq \text{GL}_2(\mathbb{C})$$

of $\mathbb{Z}/8$. One may check that $e_j A e_i \cong \text{Hom}_Z(e_i A e_k, e_j A e_k)$ for each $i, j, k \in Q_0$, and that $A$ admits an impression $(\tau, B)$, where $B = k[x, y]$ and

$$\tau : A \to M_{|Q_0|}(B) \cong \text{End}_B(B^{|Q_0|}).$$

The center of $A$ is isomorphic to $R := B^G = k[x^8, y^4, x^2y]$. Using this impression we again label vertex $i$ by the commutative ring $\bar{\tau}(e_i A e_1)$ for each $i \in Q_0$: 

Then

$$A \cong \text{End}_R \left( R \oplus R[x] \oplus R[x^2, y^3] \oplus R[x^3, x^3] \oplus R[x^4, y^2] \oplus R[x^5, xy^2] \oplus R[x^6, y] \oplus R[x^7, xy] \right).$$
Remark 4.21. In algebraic geometry (with derived categories) it is standard to associate line bundles over a smooth projective variety to the vertices of an appropriate quiver. However, here we are associating line bundles over a singular affine variety (i.e., rank 1 modules over the coordinate ring $R \cong \mathbb{Z}$) to the vertices.

We are now in a position to prove the main result of this section.

Proof of Theorem 4.1. It follows from theorems 3.13, 4.7, 4.10, and 4.17, that any homogeneous square superpotential algebra localized at the origin of its center is a noncommutative crepant resolution, since the only simples over the localized algebra are the vertex simples. Moreover, by a result of Braun [11, ex. 2.22], a noncommutative crepant resolution $R$ is locally Calabi-Yau if $Z(R)$ is a finitely generated $k$-algebra (or local and complete) and char $k = 0$. □

5 The $Y_{p,q}$ algebras

We now consider a particular class of square superpotential algebras in detail, namely the $Y_{p,q}$ algebras defined in ex. 1.3. These algebras are conjecturally related to certain Sasaki-Einstein manifolds in the $\mathcal{N} = 1, d = 4$ AdS/CFT correspondence in string theory.

The only simple modules over a square superpotential algebra $A$ localized at the origin of its center $Z$ are the vertex simples, and classifying the simples over the corresponding non-local algebra $A$ is non-trivial.

5.1 A classification of simples

Proposition 5.1. A $Y_{p,q}$ algebra $A$ is homogeneous (see def. 3.10) with center

$$
Z \cong k \left[ x_{\alpha_1} x_{\alpha_2} y_1 y_2, \prod_{\ell=1}^{p+q} x_{\alpha_\ell}^p, \prod_{\ell=1}^{p-q} x_{\alpha_\ell}^q \mid \alpha_\ell \in \{1, 2\} \right].
$$

\begin{equation}
(19)
\end{equation}

Proof. By lemma 2.2, $Ze_i$ is generated by the cycles without cyclic proper subpaths at vertex $i$. □

For fixed $p$ and $q$, the different $Y_{p,q}$ algebras are related by sequences of quiver mutations at vertices with in/out degree 2, and it is clear from (19) that the centers do not change under such mutations. In physics terms, the chiral ring $Z$ is invariant under these Seiberg dualities (i.e., quiver mutations) in the low energy limit [6].
We expect that the center of any homogeneous square superpotential algebra satisfying the following lemma has only an isolated singularity at the origin, and we will find that this indeed the case for the $Y^{p,q}$ algebras when $p \neq q$.

**Lemma 5.2.** Let $V$ be a simple module over a $Y^{p,q}$ algebra $A$, with $p \neq q$. If at least one arrow does not annihilate $V$, then no vertex will annihilate $V$.

**Proof.** Since $V$ is a non-vertex simple there is an arrow $a$ which does not annihilate $V$, and since $V$ is simple $a$ must be a subpath of a cycle $c \in e_iAe_i$ that does not annihilate $V$. Since $p \neq q$ we may assume that both $x_1$ and $y_1$ divide $\bar{\tau}(c)$. We claim that for each vertex $j$ there are paths $p \in e_jAe_i$, $p' \in e_iAe_j$ such that $\bar{\tau}(p)$ and $\bar{\tau}(p')$ are functions of only $x_1$ and $y_1$: one checks that there is a path from $s_a$ to $t_\beta$, $\alpha, \beta \in \{0,1\}$, whose $\bar{\tau}$-image is a function of only $x_1$ and $y_1$ in the following cases, with the exception of such a path from $s_2$ to $t_1$ in the third case:

In regards to the exceptional case, since $p \neq q$ we have

The arrows denoted $\sim\sim$ form a path from $s'_\alpha$ to $t_1$ whose $\bar{\tau}$-image is a function of only $x_1$ and $y_1$. It follows that there is also such a path from $s_2$ to $t_1$ by making a complete (vertical) cycle around the two-torus from $t_2$ to $t_1$. This proves the claim, and so $\bar{\tau}(p') = (x_1^{p+q} y_1^q)^n$ for some $n \geq 1$ by prop. 5.1.
Now assuming $c$ has no proper cyclic subpath, and since we assumed $x_1, y_1$ divide $\bar{\tau}(c)$, $\bar{\tau}(c)$ must equal $x_1^2 y_1 y_2$, $x_1 x_2 y_1 y_2$, or $x_1^{p+q-r} x_2^2 y_1^p$ for some $0 \leq r \leq p + q - 1$. But then in these respective cases,

\[
\bar{\tau}(c^{2n}) = (x_1^2 y_1 y_2)^{2n} = (x_1^{p+q} y_1^{p})^{2n} = \bar{\tau}(p'd), \\
\bar{\tau}(c^{2m}) = (x_1 x_2 y_1 y_2)^{2m} = ((x_1^{p+q} y_1^{p})(x_2^{p+q} y_1^{p})(x_1^{p-q} y_2^{p})(x_2^{p-q} y_2^{p}))^{2m} = \bar{\tau}(p'd), \\
\bar{\tau}(c^{(p+q)n}) = (x_1^{p+q-r} x_2^2 y_1^{p})^{(p+q)n} = (x_1^{p+q} y_1^{p})^{(p+q-r)n}(x_2^{p+q} y_1^{p})^{rn} = \bar{\tau}((p')^{p+q-r} d),
\]

where in each case $d \in c_i A e_i$ by prop. 5.1. Since $c, d, p' \in c_i A e_i$, the above equalities hold when $\bar{\tau}$ is replaced by $\tau$, and since $\tau$ is injective the respective cycles are equal. But $c$ acts on $V$ by a non-zero scalar multiple of the matrix $E_{ii}$, so $c^{mn}$, and hence $e_j$, does not annihilate $V$.

In what follows, recall that the large simples over any homogeneous square superpotential algebra were classified in prop. 3.15.

**Proposition 5.3.** If $V$ is a simple module over a $Y^{p,q}$ algebra, then $V$ will either be a large simple or a vertex simple.

**Proof.** By cor. 3.6 $\dim_k e_i V \leq 1$ for each $i \in Q_0$. Thus if $V$ is a non-vertex simple there must be some arrow represented by a nonzero scalar, so by lemma 5.2 each vertex will be nonzero, and hence $\dim_k e_i V = 1$ for each $i \in Q_0$.

### 5.2 Azumaya loci and (non-local) global dimensions

Recall the definition of $\phi$ from lemma 2.2.

**Proposition 5.4.** If $V = (B/m)^d$ is a large simple $A$-module then $x_1$ and $x_2$ (resp. $y_1$, $y_2$) cannot both vanish. If $p \neq q$, then the only singular point in $\text{Max } Z$ is the origin.

**Proof.** The first claim follows since $(B/m)^d$ is a simple of maximal $k$-dimension, and so if $x_1 = x_2 = 0$ (resp. $y_1 = y_2 = 0$) then $V$ cannot be simple.

Let $R$ denote the subring of $B$ isomorphic to $Z$. There are four cases to consider in showing that the origin is the only singular point of $\text{Max } R$:

- The locus in $\text{Max } R$ where no coordinates vanish in $B$ is smooth since $R$ is the ring of invariants of a torus action, and the torus acts freely on the locus where no coordinates vanish.
The locus in Max $R$ where the only vanishing coordinate in $B$ is $y_1$ (resp. $y_2$) is smooth: the restriction of $R$ to this locus is generated by the affine coordinates for the cone over the rational normal curve of degree $r$ (resp. $2p - r$), that is, the projectivization of the homogeneous coordinates $\prod_{\ell=1}^{r} x_{\alpha_{\ell}}$ (resp. $\prod_{\ell=1}^{2p-r} x_{\alpha_{\ell}}$). Since the rational normal curve is smooth, the cone minus the apex is also smooth.

The locus in Max $R$ where the only vanishing coordinate in $B$ is $x_\alpha$ is smooth: the restriction of $R$ to this locus is generated by
\[
k \left[ y_1^p x_{\beta}^{2p-r}, \ y_2^p x_{\beta}^r, \ x_\beta^2 y_1 y_2 \right] \cong k[t_1, t_2, t_3] / (t_1 t_2 - t_3^p),\]
where $\beta \neq \alpha$, and the rank of the Jacobian away from the vanishing of $t_1, t_2, t_3$ is 1.

The locus in Max $R$ where the only vanishing coordinates in $B$ are $x_\alpha$ and $y_1$ (resp. $y_2$) is smooth since the restriction of $R$ to this locus is freely generated by a single variable, namely $y_1^p x_{\beta}^{2p-r}$ (resp. $y_2^p x_{\beta}^r$).

\[\Box\]

**Theorem 5.5.** Let $A$ be a $Y_{np}$ algebra with center $Z$. Then the Azumaya locus of $A$ coincides with the smooth locus of Max $Z$. Specifically, the Azumaya locus of $A$ is the dense open set
\[
W = \begin{cases} 
\text{Max } Z \setminus \phi(\{x_1 = x_2 = 0\}) & \text{when } p = q \\
\text{Max } Z \setminus \{0\} & \text{otherwise}
\end{cases}.
\]

**Proof.** By prop. 3.16 a simple $A$-module $V$ is large if and only if $\text{ann}_Z(V)$ is in the Azumaya locus of $A$. In the case $p \neq q$, if $V$ is a non-vertex simple $A$-module then $V$ is large by prop. 5.3, so the maximal ideal $\text{ann}_Z V$ is in the Azumaya locus of $A$. The case $p = q$ is similar, noting that there are clearly two distinct isoclasses of simple modules whose $Z$-annihilators are in the locus $\phi(\{x_1 = x_2 = 0\})$ (with $\phi$ as in lemma 2.2). \[\Box\]

We note that the case $p = q$ is already well understood since the $Y_{np}$ algebras are McKay quiver algebras for certain finite abelian subgroups of $\text{SL}_3(\mathbb{C})$.

**Theorem 5.6.** Let $A$ be a (non-localized) $Y_{np}$ algebra with $p \neq q$. Then $A$ is homologically homogeneous of global dimension 3. Moreover, if $V$ is a vertex simple $A$-module with annihilator $p$ and $m = p \cap Z$, then as both $A$ and $A_m$-modules,
\[
A_m/p_m \cong A/p \cong V, \tag{20}
\]
while if $V$ is a non-vertex simple then
\[
A_m/p_m \cong A/p \cong \bigoplus_{i \in Q_0} V. \tag{21}
\]

38
Proof. The projective dimension of any vertex simple $A$-module is 3 by theorem 4.10 and we claim the projective dimension of any non-vertex simple is also 3. Let $V$ be a non-vertex simple with annihilator $p$; then $m := p \cap Z$ is in the Azumaya locus of $A$ by prop. 5.3; hence the smooth locus of Max $Z$ by theorem 5.5. Consequently,

$$\text{pd}_{A}(V) = \text{pd}_{Z_{m}}(Z_{m}/m_{m}) = 3,$$

where the first equality follows from theorem 3.20 and the second equality follows since $m$ is a smooth point. By a theorem of Bass [2, prop. III.6.7(a)], if $S$ is any noetherian ring module-finite over its center then

$$\text{gl. dim}(S) = \sup \{ \text{pd}_{S}(M) \mid M \text{ simple} \}.$$

Thus $\text{gl. dim}(A) = 3$ by lemma 3.12. Moreover, by prop. 3.13 Max $Z$ is irreducible, hence equidimensional, and thus $A$ is homologically homogeneous.

The isomorphisms (20) are clear and the isomorphisms (21) follow from propositions 5.5 and 3.18.

To conclude this section, we show that the “$R$-charge” of an arrow determined by $a$-maximization is consistent with its impression (2). Let $A$ be a superpotential algebra module-finite over its center $Z$; then the $R$-charge of an arrow $a \in Q_{1}$ is conjectured to be the volume of the “zero locus” of $a$ in Max $Z$, that is, the locus consisting of the maximal ideals $m \in \text{Max } Z$ such that $a \in m_{m}A_{m}$. In physics terms, the $R$-charge of a field is conjectured to be the volume of the locus where symmetry is not broken in the vev moduli space. This has been explored in [15, 14], for example. We first check that when $A$ is a $Y^{p,q}$ algebra the labeling of arrows (2) is consistent with the numerical $R$-charge assignments determined by $a$-maximization (20), as first computed for the $Y^{2,1}$ quiver in [10], and then for general $(p,q)$ in [5].

**Proposition 5.7.** The (numerical) $R$-charge assignments of the arrows in a $Y^{p,q}$ quiver determined by $a$-maximization are consistent with the labels given in (2).

**Proof.** Denote an arrow $a$ by its label $\tau(a)$ given in (2). The $R$-charge assignments as
computed in [6] are
\[r(x_1 y_1) = r(x_2 y_1) = \frac{3q - 2p + \sqrt{4p^2 - 3q^2}}{3q}\]
\[= \frac{1}{3} (-1 + \sqrt{13})\]
\[r(x_1) = r(x_2) = 2p(2p - \sqrt{4p^2 - 3q^2})/3q^2\]
\[= \frac{4}{3} (4 - \sqrt{13})\]
\[r(y_2) = (-4p^2 + 3q^2 + 2pq + (2p - q)\sqrt{4p^2 - 3q^2})/3q^2\]
\[= -3 + \sqrt{13}\]
\[r(y_1) = (-4p^2 + 3q^2 - 2pq + (2p + q)\sqrt{4p^2 - 3q^2})/3q^2\]
\[= \frac{1}{3} (-17 + 5\sqrt{13})\]
\[r(x_1 y_2) = r(x_2 y_2) = \frac{3q + 2p - \sqrt{4p^2 - 3q^2}}{3q}\]

To check consistency, one verifies that \(r(x_\alpha y_\beta) = r(x_\alpha) + r(y_\beta)\) in each of the four cases \(\alpha, \beta \in \{1, 2\}\).

5.3 Exceptional divisors with zero volume: a proposal

In this section we introduce a proposal that provides a geometric reason as to why the fractional points are smooth (i.e., \(pd_A(V_i) = 3 = \dim Z_0\) for each \(i \in Q_0\)). This work is extended in [3]. Specifically, we propose that the points in Max\(A\) that sit over the non-Azumaya locus of Max\(Z\) are exceptional divisors of a resolution \(Y \to \text{Max} Z\) shrunk to zero size. In physics terms, the fractional branes probing a singularity see the variety they are embedded in as smooth since they are wrapping exceptional divisors that have been shrunk to “stack-like” points. In what follows we use the symplectic quotient construction on the impression of a \(Y^{p,q}\) algebra, and we take \(k = \mathbb{C}\).

In thm. 3.13 Max\(Z\) was shown to be a toric algebraic variety. \(Z\) is therefore the ring of invariants of some torus action on \(B = \mathbb{C}[x_1, x_2, y_1, y_2]\), which we determine in the following lemma. It is straightforward to verify with prop. 5.1.

Lemma 5.8. The center \(Z \subset B\) of a \(Y^{p,q}\) algebra is the ring of invariants of the torus action
\[\begin{align*}
(x_1, x_2, y_1, y_2) \mapsto (\lambda^{-p} \omega^{-1} x_1, \lambda^{-p} \omega^{-1} x_2, \lambda^{p+q} \omega^2 y_1, \lambda^{p-q} y_2)
\end{align*}\]
with \((\lambda, \omega) \in \mathbb{C}^* \times \mathbb{Z}/(p - q)\mathbb{Z}\).

In the special case \((p, q) = (1, 0), \omega = 1\) and so \(Z\) is the coordinate ring for the conifold (i.e., quadric cone) given in ex. 1.2 Moreover, in the case \((p, q) = (2, 1)\) again \(\omega = 1\), and it is straightforward to check that \(Z\) is the complex cone over the first del Pezzo surface
\(dP_1\) (i.e., \(\mathbb{CP}^2\) blownup at one point), verifying an argument that this should indeed be the case given in [9, sec. 2].

Before considering the associated \(Y^{p,q}\) moment map, recall the following standard construction. The symplectic manifold \((\mathbb{C}^2, \omega = \frac{1}{2}(dx \wedge d\bar{x} + dy \wedge d\bar{y}))\) admits a hamiltonian action of the maximal compact subgroup \(T := \{t \in \mathbb{C}^* \mid |t| = 1\}\) of \(\mathbb{C}^*\) given by \((x,y) \mapsto (tx, ty)\). The dual \(\mathfrak{g}^*\) of the Lie algebra of \(T\) is then \(\mathbb{R}\), so there is a moment map

\[\mu : \mathbb{C}^2 \to \mathfrak{g}^* \cong \mathbb{R}, \quad \mu(x, y) = \frac{1}{2}(|x|^2 + |y|^2).\]

It follows that

\[\mu^{-1}(1/2)/T = \{(x, y) \in M \mid |x|^2 + |y|^2 = 1\} / T = \{\mathbb{CP}^1\ \text{with radius } 1\},\]

and more generally

\[\mu^{-1}(|a|^2/2)/T = \{(ax, ay) \in M \mid |x|^2 + |y|^2 = 1\} / T = \{\mathbb{CP}^1\ \text{with radius } |a|\} .\]

Varying \(|a|\) is then equivalent to varying the radius of the \(\mathbb{CP}^1\). In particular, \(|a| \to 0\) is equivalent to the radius vanishing.

Now since the center of a \(Y^{p,q}\) algebra is a normal toric variety (prop. [1,5]), it is also a symplectic variety with a (non-degenerate) symplectic form obtained by pulling back the standard symplectic form on \(\text{Max } B = \mathbb{C}^4\). There is a hamiltonian action on \(\text{Max } B\) by the maximal compact subgroup \(T := U(1) \times \mathbb{Z}/(p - q)\mathbb{Z}\),

\[(x_1, x_2, y_1, y_2) \mapsto (t^{-p}x_1, t^{-p}x_2, t^{p+q}y_1, t^{p-q}y_2).\]

Again the dual of the Lie algebra of \(T\) is \(\mathfrak{g}^* \cong \mathbb{R}\), and so there is a moment map

\[\mu : \text{Max } B \to \mathbb{R}, \quad \mu(x_1, x_2, y_1, y_2) = \frac{1}{2}(-p|x_1|^2 - p|x_2|^2 + (p + q)|y_1|^2 + (p - q)|y_2|^2).\]

The singular variety \(\text{Max } Z\) is then the symplectic reduction at the origin,

\[\text{Max } Z = \mu^{-1}(0)/T,\]

\[11\text{Martelli and Sparks proved that the real cone over the } Y^{2,1}\text{ manifold is the complex cone over } dP_1, \quad [22], \text{ and so it follows that the real cone over the } Y^{2,1}\text{ manifold coincides with primitive spectrum of the } Y^{2,1}\text{ algebra away from the origin.}\]

\[12\text{In physics terms, in the Lagrangian of an } N = 1\text{ physical theory, } a \text{ is a Fayet-Iliopoulos parameter and the moment map constraints are the D-terms; see for example } [24], \text{ and in the case of the } Y^{p,q}\text{ manifolds (starting from a metric), } [22].\]
and two different blowups of Max $Z$ are given by $\mu^{-1}(r^2/2)/T$ for $r > 0$ and $r < 0$, respectively. From the previous example, the absolute value of $r \in g^*$ may be viewed as the radius of the exceptional divisor in the respective blowup of Max $Z$.

For the following, suppose $p \neq q$ (the case $p = q$ is similar). Let $\mathcal{M}_r$ denote the space of all representations $\tau_m$ where $m \in \mu^{-1}(r^2/2) \subset \text{Max } B$. We find the following:

- If $r > 0$, $\mathcal{M}_r$ is parameterized by a blowup of Max $Z$ at the origin (a $\mathbb{CP}^1$ family together with Max $Z \setminus \{0\}$);
- If $r < 0$, $\mathcal{M}_r$ is parameterized by the “flopped” blowup; and
- If $r = 0$, $\mathcal{M}_0$ is parameterized Max $Z \setminus \{0\}$, together with the direct sum of smooth fractional points, i.e., vertex simples.

Specifically, $r = 0$ determines the constraints

$$x_1 = x_2 = 0 \iff y_1 = y_2 = 0 \quad \text{when } p \neq q,$$

$$x_1 = x_2 = 0 \iff y_1 = 0 \quad \text{when } p = q.$$

and we observe that these are the same constraints obtained by requiring the $A$-modules be simple, since the simples are either vertex simples or are large when $p \neq q$ (prop. 5.3). (The fractional points are smooth by thm. 4.10)

As an example of what happens when $r \neq 0$, consider the $Y^{4,2}$ algebra given in ex. 1.3. The dotted arrows in the respective quivers denote the arrows that annihilate each (non-simple) module in the $\mathbb{CP}^1$ family, and the $\tau$-images of the remaining arrows give explicit coordinates for the $\mathbb{CP}^1$'s (given below).

\[
\begin{align*}
\text{r > 0:} & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
8 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\text{r < 0:} & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
8 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
[y_1 : y_2^2] & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
8 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
[x_1 : x_2] & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
8 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

**Remark 5.9.** Using symplectic reduction on the impression is very similar to quiver stability conditions with a generic stability parameter in $\mathbb{Z}^{[Q_0]}$. However, we note that the
semisimple module consisting of the direct sum of vertex simples cannot be stable for any
generic stability parameter, and so the usual methods of quiver stability are not sufficient
to capture the fractional points, in contrast to our method.

Our proposal provides a geometric view of Van den Bergh’s idea that the noncommu-
tative resolution, loosely speaking, lies at the intersection between the various flops [29].
However, to make the identification precise it would need to be known how many fractional
points (if more than one) correspond to a given exceptional divisor, and to what resolutions
of Max Z the correspondence holds. Progress in this direction is given in [3], but many
questions remain.

A Proof of injectivity in prop. 3.1

The map $\tau$ given in prop. 3.1 is injective:

First, the set of paths in $A$ is $\mathbb{Z}$-graded, where the horizontal and vertical arrows (the
first four arrows in (2)) have degree 1 while the diagonal arrows (the latter four arrows in
(2)) have degree 2. This grading is preserved under $\bar{\tau}$, so $p$ and $p'$ have the same degree. We
proceed by induction on the degree of $p$. If $p$ has degree 1 then it is an arrow so $\bar{\tau}(p) = \bar{\tau}(p')$
implies $p = p'$. So suppose the lemma holds for all paths of degree $< n$, and suppose $p$ has
degree $n$.

View $p$ and $p'$ as elements of $kQ$, let $a$ be the leftmost arrow of $p$, and say $y_1$ divides
$\bar{\tau}(p)$. If $a$ is also the leftmost arrow of $p'$ then we are done by induction, so suppose $a'$
is the leftmost arrow in $p'$ such that $y_1$ divides $\bar{\tau}(a')$. We claim that modulo $\partial W$,

$$p' = t_n \cdots t_2(t_1y_1)s_m \cdots s_1 = t_n \cdots t_2(y_1t_1)s_m \cdots s_1 \quad \text{or} \quad t_n \cdots t_3(y_1t_1t_2)s_m \cdots s_1,$$

where $s_\ell, t_\ell \in \{x_1, x_2, y_1, y_2\}$, in a case-by-case analysis. The following argument will be
used repeatedly: if $u = y_\alpha x_\beta y_\gamma$ is a path with $\alpha \neq \gamma$, that is, $u$ equals say

$$\begin{array}{c}
\begin{array}{c}
\downarrow \quad \uparrow \\
1 \quad 2
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\downarrow \\
1
\end{array} \quad \begin{array}{c}
\downarrow \\
2
\end{array}
\end{array}, \quad \text{or} \quad \begin{array}{c}
\begin{array}{c}
\downarrow \quad \uparrow \\
1 \quad 2
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\downarrow \\
1
\end{array} \quad \begin{array}{c}
\downarrow \\
2
\end{array}
\end{array}, \quad \text{or} \quad \begin{array}{c}
\begin{array}{c}
\downarrow \quad \uparrow \\
1 \quad 2
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\downarrow \\
1
\end{array} \quad \begin{array}{c}
\downarrow \\
2
\end{array}
\end{array}, \quad \text{or} \quad \begin{array}{c}
\begin{array}{c}
\downarrow \quad \uparrow \\
1 \quad 2
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\downarrow \\
1
\end{array} \quad \begin{array}{c}
\downarrow \\
2
\end{array}
\end{array}, \quad \text{or} \quad \begin{array}{c}
\begin{array}{c}
\downarrow \quad \uparrow \\
1 \quad 2
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\downarrow \\
1
\end{array} \quad \begin{array}{c}
\downarrow \\
2
\end{array}
\end{array}, \quad \text{or} \quad \begin{array}{c}
\begin{array}{c}
\downarrow \quad \uparrow \\
1 \quad 2
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\downarrow \\
1
\end{array} \quad \begin{array}{c}
\downarrow \\
2
\end{array}
\end{array} \quad \text{or} \quad \begin{array}{c}
\begin{array}{c}
\downarrow \quad \uparrow \\
1 \quad 2
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\downarrow \\
1
\end{array} \quad \begin{array}{c}
\downarrow \\
2
\end{array}
\end{array}, \quad \text{so}

$$y_\alpha x_\beta y_\gamma = y_\gamma x_\beta y_\alpha.$$

Additionally, if say the factor $x_\beta y_\gamma$ is a diagonal arrow, then we also have

$$y_\alpha y_\gamma x_\beta = y_\alpha x_\beta y_\gamma = y_\gamma x_\beta y_\alpha.$$ Similarly for $x \leftrightarrow y$.

• $t_1 = y_2$:
If $t_2 = 1$ or $y_2$ then $s_1 = x_\alpha$, hence $y_1x_\alpha$ is a diagonal arrow and $t_1 = y_2$ is a vertical arrow. Apply (23).

If $t_2 = x_\alpha$ then at least one of the factors $(x_\alpha y_2)$, $(y_1 s_m)$, must be a diagonal arrow. Both factors cannot be diagonal arrows since is not a possible configuration. Apply (23).

- $t_1 = x_\alpha$:

  - If $t_2 = y_2$, either $y_1$ or $t_2 = y_2$ is a vertical arrow since otherwise either the configuration or would occur.
    - If $y_1$ is a vertical arrow, apply (23).
    - Suppose $y_1$ is not a vertical arrow.
      - If $x_\alpha \neq s_m = x_\beta$ then $y_2x_\alpha y_1s_m$ is a unit cycle and we are done.
      - If $x_\alpha = s_m$ then the configuration must occur since
        
        
        
        is not possible, and so
        
        
        
        where the last equality holds since $y_1$ is a vertical arrow.

  - Suppose $t_2 = x_\beta$.
    - If $y_1 s_m$ is a diagonal arrow then $t_1 = x_\alpha$ must be a horizontal arrow. Apply (23).
    - If $x_\alpha y_1$ is a diagonal arrow then $t_2 = x_\beta$ must be a horizontal arrow since otherwise the configuration would occur. Apply (23).
    - Suppose $y_1$ is a vertical arrow.
      - If $\alpha \neq \beta$ then $t_3x_\beta$ is a diagonal arrow, hence $t_3 = y_2$. Apply (23) twice:
        
        
        where the last equality holds since $y_1$ is a vertical arrow.
· Suppose $\alpha = \beta$, so $\tilde{\tau}(t_2t_1a') = x_\alpha^2y_1$. If the path $t_2t_1a'$ is in the configuration $a'$ then $x_\alpha y_1 = y_1x_\alpha$, otherwise $t_2t_1a'$ is in the configuration $a'$ since $t_2' \rightarrow t_1 \rightarrow \rightarrow$ is not possible (and $a'$ is the leftmost $y_1$ variable in $p'$). Repeating this argument we find that $p' = x_\alpha^ny_1s_m \cdots s_1$ with $x_\alpha y_1$ in the configuration

\[
\begin{array}{cccc}
a' & t_1 & t_2 & \ldots \\
\end{array}
\]

It follows that the leftmost arrow in $p$, namely $a$, must be diagonal with $\tilde{\tau}(a) = y_1x_\gamma$, $\gamma \neq \alpha$, so since $\tilde{\tau}(p) = \tilde{\tau}(p')$ there is an $x_\gamma$ factor in $\tilde{\tau}(p')$. Let $b$ be the leftmost arrow in $p'$ such that $x_\gamma$ divides $\tilde{\tau}(b)$. We may apply the arguments in all the above cases with $y_1 \mapsto x_\gamma$, with the exception of configuration (24), to show that $x_\gamma$ can be moved leftward so that it is adjacent to $y_1$ in $p'$ modulo $\partial W$, and the result follows. But it is not possible that both $y_1$ and $x_\gamma$ are in the configuration (24):

\[
\begin{array}{cccc}
a' & t_1 & t_2 & \ldots \\
\end{array}
\]

- Suppose $t_2 = 1$.
  * If $x_\alpha y_1$ is a diagonal arrow then $x_\alpha y_1 = y_1x_\alpha$.
  * If $y_1s_m$ is a diagonal arrow, then apply (23).
If $y_1$ is a vertical arrow, then apply the above case $p' = x_n^s y_1 s_m \cdots s_1$ with $n = 1$. Thus $p$ and $p'$ have the same leftmost arrow modulo $\partial W$, and by induction we are done.

B A math-physics dictionary for quivers

In reverse geometric engineering [7], a type of quiver algebra called a superpotential algebra is constructed from the (classical) equations of motion of an $\mathcal{N} = 1$ supersymmetric quiver gauge theory. In the original physics proposal/conjecture of Berenstein, Douglas, and Leigh (see [7, 8] and references therein), the center of a superpotential algebra is the coordinate ring for an affine tangent cone (or at least some affine chart) on a 3 complex-dimensional singular Calabi-Yau variety—the “hidden internal space” of our universe.

The algebra itself is then viewed as a noncommutative ring of functions on the space of its simple modules, just as is the case in “commutative” algebraic geometry (when $k = \overline{k}$). They conjectured that, at least in physically relevant examples, this space is a “noncommutative resolution” of the algebra’s singular center since $D$-branes supposedly see the variety they are embedded in as smooth [17].

The $Y^{p,q}$ quivers are of interest to physicists since they encode the gauge theory in the conjectured AdS/CFT correspondence when the horizon is a $Y^{p,q}$ Sasaki-Einstein 5-manifold (given by metric data on the topological space $S^2 \times S^3$). The $Y^{p,q}$ quiver gauge theories were constructed to model these geometries using symmetry arguments in a process known as geometric engineering, by Benvenuti, Franco, Hanany, Martelli, Sparks, and Kazakopoulos [5, 6]. In this paper we instead start with the $Y^{p,q}$ quiver gauge theories and derive their dual geometries by the methods of reverse geometric engineering; such a geometry is conjectured to coincide with the real cone over a $Y^{p,q}$ manifold (the horizon), but this is still unknown for $p > 2$.

The following is a partial dictionary between quiver gauge theories (specifically in regards to the mesonic branch since that is the focus of this paper) and quiver representation theory. We begin with the following:

- quiver gauge theory $\leftrightarrow$ a quiver algebra and its representations;
  in particular, a $d = 4, \mathcal{N} = 1$ supersymmetric quiver gauge theory $\leftrightarrow$ a path algebra

\footnote{The Calabi-Yau variety need not be singular, but often theories with singularities are able to more closely model nature by, for example, breaking supersymmetry; see [9].

14 According to the AdS/CFT correspondence, this variety does not necessarily need to be actual physical space, but may instead just be a parameter space for something similar to mass (“vacuum expectation values”) for certain fields that live in our $(3 + 1)$-dimensional spacetime manifold.

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modulo $F$-flatness constraints, namely a superpotential algebra

- complexified $U(n)$ gauge group $\Leftrightarrow$ general linear group
  (In this context, by $U(n)$ physicists usually mean $U(n)$ complexified, that is, if $H_1$ and $H_2$ are elements of the Lie algebra $u(n)$, then $\exp(H_1 + iH_2) \in \text{GL}_n(\mathbb{C})$, and for any $L \in \text{GL}_n(\mathbb{C})$ there is some such $H_1$ and $H_2$ such that $L = \exp(H_1 + iH_2)$.)

- gauge invariance (under complexified gauge group) $\Leftrightarrow$ isomorphism classes of quiver representations

- Seiberg dual gauge theories, that is, different gauge theories in the UV which flow to the same fixed point in the IR $\Leftrightarrow$ different superpotential algebras that have the same centers or different superpotential algebras whose bounded derived categories of modules are equivalent \[8\]

In the following table we sketch a $d = 4$, $\mathcal{N} = 1$ AdS/SCFT correspondence, or more generally a procedure for geometric and reverse geometric engineering, for a superpotential algebra $A$. Note that the universe is thought to be a product $\mathcal{M} \times X$, where $\mathcal{M}$ is $3 + 1$ dimensional Minkowski space and $X$ is a compact 3 complex-dimensional (possibly singular) Calabi-Yau variety, the classical vacuum moduli space. A $Dn$-brane (with $n$ odd) fixes the endpoints of a string; mathematically it is a sheaf (or a complex of sheaves) supported on an $n + 1$ (real) dimensional subvariety of $\mathcal{M} \times X$. Here we only consider $D3$-branes which extend into $\mathcal{M}$ and are point-like, i.e., sky scraper sheaves, on $X$. More generally though one also includes various 5- and 7-branes (such as in the physical realizations of dimer models), and $D3$-branes are allowed to wrap nontrivial cycles in $X$. 

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Gauge theory on $\mathcal{M}$  
Geometry and physics of $X$  
Quiver representations

| $U(1)$ gauge group on the fractional brane at vertex $i$ | a stack of $|Q_0|$ fractional branes at the apex of the tangent cone $C_p(X)$ at a point $p \in X$ | vertices of $Q$ |
|-------------------------------------------------------|-----------------------------------------------------------------|------------------|
| (complexified) $U(n)$ gauge group at vertex $i$ | open oriented string stretching from the fractional brane at vertex $i$ to the fractional brane at vertex $j$ | $A$-module $V_\rho$ with $\dim_\mathbb{C} e_i V_\rho = n$ |
| bifundamental field transforming in the fundamental representation of the gauge group at vertex $j$ and the antifundamental representation of the gauge group at vertex $i$ | a point in $C_p(X)$ (or a bulk $D3$-brane at a point in $C_p(X)$) | an arrow with tail at $i$ and head at $j$ |
| vev of a bifundamental field | matrix representation of the corresponding arrow | |
| a possible configuration of vev’s modulo the $F$-flatness constraints | an isoclass of simple $A$-modules (or the corresponding primitive ideal) | |
| (mesonic) chiral ring | coordinate ring for $C_p(X)$ | center of $A$ |
| mesonic field | | cycle in the quiver |

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