Separability and Killing Tensors in Kerr-Taub-NUT-de Sitter Metrics in Higher Dimensions

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ABSTRACT

A generalisation of the four-dimensional Kerr-de Sitter metrics to include a NUT charge is well known, and is included within a class of metrics obtained by Plebanski. In this paper, we study a related class of Kerr-Taub-NUT-de Sitter metrics in arbitrary dimensions $D \geq 6$, which contain three non-trivial continuous parameters, namely the mass, the NUT charge, and a (single) angular momentum. We demonstrate the separability of the Hamilton-Jacobi and wave equations, we construct a closely-related rank-2 Stäckel-Killing tensor, and we show how the metrics can be written in a double Kerr-Schild form. Our results encompass the case of the Kerr-de Sitter metrics in arbitrary dimension, with all but one rotation parameter vanishing. Finally, we consider the real Euclidean-signature continuations of the metrics, and show how in a limit they give rise to certain recently-obtained complete non-singular compact Einstein manifolds.

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1 Introduction

Four-dimensional solutions of the Einstein equations have been extensively studied for many decades. In relatively recent times, since the discovery of supergravity and superstring theory, solutions of the Einstein equations, or the coupled Einstein-matter equations, in higher dimensions have also been found to be of physical interest. An encyclopaedic classification of all the known four-dimensional solutions can be found in [1], but the higher-dimensional cases have been less extensively investigated. Various classes of higher-dimensional solution have been obtained, including black holes that generalise the four-dimensional Schwarzschild, Reissner-Nordström and Kerr solutions.

There are few general methods available for solving the Einstein equations. Almost always, one is forced to make a symmetry assumption. If the isometry group has orbits of codimension one, the problem then reduces to solving non-linear ordinary differential equations. A much more challenging task results if the orbits have higher codimension, since then non-linear partial differential equations are involved. Two techniques have proved useful in the past for tackling cases like these, where the number of independent variables is two or greater. One technique is to adopt the Kerr-Schild ansatz, which in effect reduces the Einstein equations to linear equations, and this has proved useful recently in obtaining the general higher-dimensional version of the Kerr-de Sitter metrics [2]. Another technique, pioneered by Carter, is to require of the metric that it admit separation of variables for the Hamilton-Jacobi equation, or for the wave equation or Laplace equation [3]. This has the further advantage that having obtained the metric, one is actually in a position to do something with it; namely, to study its geodesics and eigenfunctions explicitly.

In a remarkable paper, Carter exploited this idea to obtain a general class of metrics in four dimensions which include the Kerr-Taub-NUT-de Sitter solutions [3]. This general class, in the formalism given by Plebanski [4], takes the form

$$ds^2 = \frac{p^2 + q^2}{X} dp^2 + \frac{p^2 + q^2}{Y} dq^2 + \frac{X}{p^2 + q^2} (d\tau + q^2 d\sigma)^2 - \frac{Y}{p^2 + q^2} (d\tau - p^2 d\sigma)^2,$$

where

$$X = \gamma - g^2 - \epsilon p^2 - \lambda p^4 + 2\ell p,$$

$$Y = \gamma + e^2 + \epsilon q^2 - \lambda q^4 - 2mq. \quad (2)$$

They are solutions of the coupled Einstein-Maxwell equations with a cosmological constant $\lambda$, and with electric and magnetic charges given by $e$ and $g$. We shall restrict attention to the case of pure Einstein metrics in this paper, and so we set $e = g = 0$. The remaining constants $(\gamma, m, \ell, \epsilon)$ effectively comprise 3 real continuous parameters and one discrete
parameter, since one can always make coordinate scaling transformations to absorb the magnitude of, say, the dimensionless constant \( \epsilon \). Thus one may view \((\gamma, m, \ell)\) as continuous parameters, and take \( \epsilon = +1, -1 \) or 0. The constants \((\gamma, m, \ell)\) are related to the angular momentum, mass and NUT charge. Special cases of (1) include the Kerr-de Sitter solution and the Taub-NUT-de Sitter solution.

These four-dimensional metrics have a simple higher-dimensional generalisation [5], which, as we shall show, also has the property that both the Hamilton-Jacobi equation and the wave equation may be solved by separation of variables. Thus, for these very special metrics, which encompass the Kerr-de Sitter metrics in arbitrary dimension with all but one rotation parameter vanishing, the geodesic flow on the cotangent bundle is a completely integrable system in the sense of Liouville. Such integrable dynamical systems are comparatively rare, and when arising from the metric, are associated with the existence of special tensor fields, called Stäckel-Killing tensor fields, on the manifold. We shall construct these explicitly for the higher-dimensional metrics. Although the Säckel-Killing tensor in the four-dimensional metrics admits a Yano-Killing tensor square root, it appears rather unlikely that this feature will extend to the higher-dimensional generalisations.

A feature of the four-dimensional metrics is that they may be, upon analytic continuation to \((2, 2)\) metric signature, be cast into a double Kerr-Schild form, and this has played some rôle in their construction, and that of more generalised metrics, by Plebanski. We find that the higher-dimensional metrics may also be cast in double Kerr-Schild form, and that, although it is not true in general, that the double Kerr-Schild form renders the Einstein equations linear, in our case we find that it does result in linear equations.

An important application of the original four-dimensional metrics was the first construction of a complete, non-singular, compact, inhomogeneous Einstein manifold [6]. This was done by analytically continuing the metrics to positive-definite (Euclidean) signature, and making a careful study of regularity conditions near coordinate singularities. We show that the same can be done for the higher-dimensional metrics discussed in this paper.

2 Higher-dimensional Generalisation

In this paper, we consider higher-dimensional generalisations of the form

\[
ds^2 = \frac{p^2 + q^2}{X} \, dp^2 + \frac{p^2 + q^2}{Y} \, dq^2 + \frac{X}{p^2 + q^2} (d\tau + q^2 \, d\sigma)^2 - \frac{Y}{p^2 + q^2} (d\tau - p^2 \, d\sigma)^2 + \frac{p^2 q^2}{\gamma} \, d\Omega_k^2,
\]  

(3)
where

\[ X = \gamma - \epsilon p^2 - \lambda p^4 + 2\ell p^{1-k}, \quad Y = \gamma + \epsilon q^2 - \lambda q^4 - 2m q^{1-k}, \]  

(4)

and \( d\Omega_k^2 = g_{ij} dx^i dx^j \) is an Einstein metric on a space of dimension \( k \), normalised so that its Ricci tensor satisfies \( R_{ij} = (k - 1) g_{ij} \). One might, for example, take \( d\Omega_k^2 \) to be the metric on the unit sphere \( S^k \). It was shown in [5] that these metrics satisfy the \( D = k + 4 \) dimensional Einstein equation

\[
\hat{R}_{MN} = (k + 3) \lambda \hat{g}_{MN}.
\]  

(5)

The verification of the Einstein equations can be performed rather straightforwardly in a coordinate basis. From (3), and decomposing the coordinate indices as \( M = (\mu, i) \), we may write the components of the \( D \)-dimensional metric as

\[
\hat{g}_{\mu\nu} = g_{\mu\nu}, \quad \hat{g}_{ij} = \frac{p^2 q^2}{\gamma} g_{ij}, \quad \hat{g}_{\mu i} = 0,
\]  

(6)

where \( g_{\mu\nu} \) is the four-dimensional Plebanski-type metric with the modified functions \( X \) and \( Y \) given in (4). It is easily seen that the non-vanishing components of the affine connection \( \hat{\Gamma}^M_{NP} \) are given by

\[
\hat{\Gamma}^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho}, \quad \hat{\Gamma}^i_{jk} = \Gamma^i_{jk},
\]

\[
\hat{\Gamma}^i_{\mu j} = \delta^i_j \partial_\mu \log(pq) = \delta^i_j \left( \frac{1}{q} \delta^1_{\mu} + \frac{1}{p} \delta^2_{\mu} \right),
\]

\[
\hat{\Gamma}^\mu_{ij} = -\frac{pq}{\gamma (p^2 + q^2)} (pq \delta^1_{\mu} + q X \delta^2_{\mu}) g_{ij}.
\]  

(7)

Here, the explicit index values 1 and 2 refer to the \( q \) and \( p \) coordinates respectively. From these expressions, it is straightforward to substitute into the expression for the curvature, and hence to verify that (3) satisfies (5).

The arbitrary-dimensional Kerr-de Sitter metrics with a single rotation parameter \( a \), which were obtained in [7], arise as special cases of the more general Einstein metrics (3). Specifically, if we take the parameters in (4) to be

\[
\gamma = a^2, \quad \epsilon = 1 - \lambda a^2, \quad m = M, \quad \ell = 0,
\]  

(8)

and define new coordinates according to

\[
p = a \cos \theta, \quad q = r, \quad \tau = t - \frac{a}{\Xi} \phi, \quad \sigma = -\frac{1}{a \Xi} \phi,
\]  

(9)

where \( \Xi \equiv 1 + \lambda a^2 \), then (3) reduces precisely to the metrics obtained in [7]. The more general solutions that we have obtained include the NUT charge \( \ell \) as an additional non-trivial parameter, when \( D \neq 5 \).
The case $D = 5$ is somewhat degenerate in the above construction, in that the ostensibly additional NUT parameter $\ell$ is fictitious in this case. This is easily seen from the expressions for the metric functions $X$ and $Y$ in (4) when $k = 1$:

$$X = \gamma - \epsilon p^2 - \lambda p^4 + 2\ell, \quad Y = \gamma + \epsilon q^2 - \lambda q^4 - 2m,$$

(10)

One can absorb the parameter $\ell$ by means of additive shifts in the constants $\gamma$ and $m$. Since a constant scaling of the $S^k$ metric $d\Omega^2_k$ in (3) is irrelevant when $k = 1$, the upshot is that our construction in $D = 5$ is equivalent to the one where the NUT charge $\ell$ is set to zero, thus reducing to a case already considered in [7]. In all other dimensions $D \geq 4$, the NUT charge is a non-trivial additional parameter.

3 Separability

The covariant Hamiltonian function on the cotangent bundle of the metrics (3) is given by

$$\mathcal{H}(P_M, x^M) \equiv \frac{1}{2} g^{MN} P_M P_N$$

$$= \frac{1}{2(p^2 + q^2)} \left[ \frac{1}{X} (P_\sigma + p^2 P_\tau)^2 - \frac{1}{Y} (P_\sigma - q^2 P_\tau)^2 + X P^2_p + Y P^2_q \right]$$

$$+ \frac{\gamma}{2p^2 q^2} g^{ij} P_i P_j. \quad (11)$$

The coordinates $\tau$ and $\sigma$ are ignorable, and their conjugate momenta $P_\tau$ and $P_\sigma$ are constants. The Hamiltonian-Jacobi equation

$$\mathcal{H}(\partial_M S, x^M) = -\frac{1}{2} \mu^2$$

(12)

has separable solutions of the form

$$S = P_\tau \tau + P_\sigma \sigma + F(p) + G(q) + W(x^i), \quad (13)$$

where

$$2\kappa = \frac{1}{X} (P_\sigma + p^2 P_\tau)^2 + X \left( \frac{dF}{dp} \right)^2 + \frac{2\gamma c}{p^2} + \mu^2 p^2,$$

$$-2\kappa = -\frac{1}{Y} (P_\sigma - q^2 P_\tau)^2 + Y \left( \frac{dG}{dq} \right)^2 + \frac{2\gamma c}{q^2} + \mu^2 q^2,$$

$$c = \frac{1}{2} g^{ij} \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j}. \quad (14)$$

The three quantities $\mu^2$, $c$ and $\kappa$ are separation constants, associated to the three mutually Poisson-commuting functions $\mathcal{H}$ and

$$\mathcal{C} = \frac{1}{2} g^{ij} P_i P_j,$$
\[ K = \frac{1}{2(p^2 + q^2)} \left[ \frac{q^2}{X} (P_\sigma + p^2 P_\tau)^2 + \frac{p^2}{Y} (P_\sigma - q^2 P_\tau)^2 + q^2 X P_\sigma^2 - p^2 Y P_\tau^2 \right] \\
+ \frac{1}{2} \gamma \left( \frac{1}{p^2} - \frac{1}{q^2} \right) g^{ij} P_i P_j. \]  

(15)

If equations (14) hold, then \( C \) takes the value \( c \) and \( K \) takes the value \( \kappa \). The function \( W \) satisfies the Hamilton-Jacobi equation governing geodesic motion on the \( k \)-dimensional Einstein manifold with metric \( d\Omega^2_k \). For a general Einstein manifold, this is as far as one can go with finding the geodesics. However, in special cases, such as the sphere \( S^k \), there will be further constants of the motion. If there are \( k-1 \) such constants arising from \( k-1 \) mutually-commuting independent functions on the Einstein manifold, then the geodesic flow on the \((k + 4)\)-dimensional Einstein manifold will be completely integrable. Another way to say this is that while the product of two manifolds with completely-integrable geodesics gives a new manifold with completely-integrable geodesics, this property will not in general be true for warped products such as we are considering here. However, for the very special choice of warp function \( p^2 q^2 / \gamma \) which arises in the metrics (3), together with the simple \( k \)-dependent modifications of the \( X \) and \( Y \) functions in (4), the property of complete integrability is maintained.

These complete integrability properties may be viewed as the classical limit of the quantum-mechanical statement that the Schrödinger equation \( \hat{\nabla}^2 \psi = \mu^2 \psi \) is also separable. This can be seen from the Laplacian

\[ \hat{\nabla}^2 = \frac{1}{p^2 + q^2} \left[ p^{-k} \frac{\partial}{\partial p} \left( p^k X \frac{\partial}{\partial p} \right) + q^{-k} \frac{\partial}{\partial q} \left( q^k Y \frac{\partial}{\partial q} \right) \right] \\
+ \frac{1}{X} \left( \frac{\partial}{\partial \sigma} + p^2 \frac{\partial}{\partial \tau} \right)^2 - \frac{1}{Y} \left( \frac{\partial}{\partial \sigma} - q^2 \frac{\partial}{\partial \tau} \right)^2 \right] + \frac{\gamma}{p^2 q^2} \nabla^2. \]  

(16)

Multiplying \( \hat{\nabla}^2 \psi = \mu^2 \psi \) by \((p^2 + q^2)\) immediately reveals the separability.

Associated with \( K \) is a rank-2 symmetric Stäckel-Killing tensor \( K^{MN} \), given by \( K = \frac{1}{2} K^{MN} P_M P_N \), where \( K \) is given in (15). It satisfies the Killing-tensor equation

\[ \hat{\nabla}_{(M} K_{NP)} = 0, \]  

(17)

by virtue of the fact that \( K \) Poisson-commutes with the Hamiltonian. We may also then define the second-order differential operator

\[ \hat{K} = -\frac{1}{2} \hat{\nabla}_M (K^{MN} \hat{\nabla}_N), \]  

(18)

analogous to the operator \( \hat{H} = -\frac{1}{2} \hat{\nabla}_M (\hat{g}^{MN} \hat{\nabla}_N) = -\frac{1}{2} \hat{\nabla}^2. \) General theory [3] shows that \( \hat{K} \) and \( \hat{H} \) commute, and, moreover, they obviously commute with the operator \( \hat{C} \equiv -\frac{1}{2} \nabla_i (g^{ij} \nabla_j) = -\frac{1}{2} \nabla^2. \)
In the Carter class of four-dimensional Kerr-Taub-NUT-de Sitter metrics, it is known \cite{8} that the Stäckel-Killing tensor \( K_{MN} \) can be written as the square of a Yano-Killing 2-form \( Y_{MN} \):

\[
K_{MN} = Y_{MP} Y_{NP},
\]

where \( Y_{MN} \) satisfies

\[
\hat{\nabla}_{(M} Y_{N)P} = 0.
\]

This is equivalent to the statement that \( \hat{\nabla}_M Y_{NP} = \partial_{[M} Y_{NP]} \), or \( 3\hat{\nabla} Y = dY \). It follows straightforwardly from (20) that \( K_{MN} \) satisfies the Stäckel-Killing equation (17).

In four dimensions, i.e. \( k = 0 \), one has from (15) that

\[
Y = q dp \wedge (d\tau + q^2 d\sigma) + p dq \wedge (d\tau - p^2 d\sigma),
\]

whence

\[
*Y = q dq \wedge (d\tau - p^2 d\sigma) - p dp \wedge (d\tau + q^2 d\sigma).
\]

Thus one can write \cite{8} \( *Y = dA \), where

\[
A = \frac{q^4}{2(p^2 + q^2)} (d\tau - p^2 d\sigma) - \frac{p^4}{2(p^2 + q^2)} (d\tau + q^2 d\sigma).
\]

One might wonder whether in the higher-dimensional metrics (3), the Stäckel-Killing tensor we have found might also be expressible as the square of a Yano-Killing tensor. For a general dimension \( D = 4 + k \) this looks unlikely, because there is no obvious 2-form available on the additional \( k \)-dimensional Einstein manifold. If, however, the higher-dimensional manifold is Kähler-Einstein, the Kähler form \( J_{ij} \) becomes available, and an obvious generalisation of (21) is

\[
\hat{Y} = Y \pm \sqrt{q^2 - p^2} J.
\]

This is, by construction, one possible square root of the Killing tensor \( K_{MN} \). However, a simple calculation shows that, for example, \( \hat{\nabla}_\mu \hat{Y}_{jk} + \hat{\nabla}_j \hat{Y}_{\mu k} \neq 0 \), and thus \( \hat{Y}_{MN} \) is not a Yano-Killing tensor. There is no other obvious candidate for \( \hat{Y}_{MN} \) that might yield a generalisation of the four-dimensional Yano-Killing tensor.

It is interesting to note that the more general class of accelerating type-D metrics of Plebanski and Demianski \cite{9} does not admit a separation of variables, for either the Hamiltonian-Jacobi equation or the wave equation. These metrics are of the form

\[
ds^2 = \frac{1}{(1 - pq)^2} \left[ \frac{p^2 + q^2}{X} dp^2 + \frac{p^2 + q^2}{Y} dq^2 + \frac{X}{p^2 + q^2} (d\tau + q^2 d\sigma)^2 - \frac{Y}{p^2 + q^2} (d\tau - p^2 d\sigma)^2 \right],
\]
where $X$ and $Y$ are certain polynomial functions of $p$ and $q$ respectively [9]. The conformal prefactor $(1 − pq)^{-2}$ spoils the separability of the Hamilton-Jacobi equation, except in the massless case. Thus the Plebanski-Demianski metrics admit conformal Stäckel-Killing tensors but not Stäckel-Killing tensors in the strict sense.

4 Double Kerr-Schild Metric

It is of interest to note that the higher-dimensional metrics (3) may be cast in a double Kerr-Schild form. This is most conveniently done by analytically continuing to a real form of the metric with signature $(2, 2 + k)$. This continued metric can then be written in the form

$$ds^2 = ds^2 + U (k_M dx^M)^2 + V (l_M dx^M)^2,$$

where the fiducial “base” metric $ds^2$ is the de Sitter metric, and $k^M$ and $l^M$ are two linearly-independent mutually-orthogonal affinely-parameterised null geodesic congruences:

$$k_M k^M = l_M l^M = k_M l^M = 0,$$

$$k^M \nabla_M k_N = l^M \nabla_M l_N = 0.$$

Note that the indices on $k_M$ and $l_M$ can be raised with either $\hat{g}^{MN}$ or $\bar{g}^{MN}$.

Specifically, the analytically-continued metric is obtained from (3) by sending

$$p \to ip, \quad \ell \to i \ell^{k-1}, \quad \gamma \to -\gamma,$$

$$X \to -\Delta_p, \quad Y \to \Delta_q,$$

resulting in the metric

$$ds^2 = \frac{q^2 - p^2}{\Delta_p} dp^2 + \frac{q^2 - p^2}{\Delta_q} dq^2 - \frac{\Delta_p}{q^2 - p^2} (d\tau + q^2 d\sigma)^2 - \frac{\Delta_q}{q^2 - p^2} (d\tau + p^2 d\sigma)^2 + \frac{p^2 q^2}{\gamma} d\Omega_k^2,$$

where

$$\Delta_p = \gamma - \epsilon p^2 + \lambda p^4 - 2\ell p^{1-k}, \quad \Delta_q = -\gamma + \epsilon q^2 - \lambda q^4 - 2m q^{1-k}.$$

If we now define new coordinates $\tilde{\tau}$ and $\tilde{\sigma}$ by

$$d\tilde{\tau} = d\tau + \frac{q^2}{\Delta_p} dp - \frac{q^2}{\Delta_q} dq, \quad d\tilde{\sigma} = d\sigma - \frac{dp}{\Delta_p} + \frac{dq}{\Delta_q},$$

then a straightforward calculation shows that (29) can be written as (26), where

$$ds^2 = \frac{1}{q^2 - p^2} \left[ \Delta_p (d\tilde{\tau} + q^2 d\tilde{\sigma})^2 - \Delta_q (d\tilde{\tau} + p^2 d\tilde{\sigma})^2 \right]$$
where \( \frac{\partial}{\partial s} \) that \( \hat{k} \)

It is easily verified that \( k_M dx^M = \bar{d} \bar{\tau} + q^2 \bar{d} \bar{\sigma} \), \( l_M dx^M = d \bar{\tau} + p^2 \bar{d} \bar{\sigma} \),

\[
\begin{align*}
U &= \frac{2 \ell p^{1-k}}{q^2 - p^2}, \quad V = \frac{2m q^{1-k}}{q^2 - p^2}.
\end{align*}
\]

It is easily verified that \( k_M \) and \( l_M \) satisfy (27). Note that as vectors, one has

\[
k^M \partial_M = -\frac{\partial}{\partial q}, \quad l^M \partial_M = -\frac{\partial}{\partial p}.
\]

The metric \( d\bar{s}^2 \) appearing in (32) is the \((4+k)\)-dimensional de Sitter metric (since it is the Kerr-Taub-NUT de Sitter metric with the mass and NUT parameters \( m \) and \( \ell \) set to zero). Its inverse is given simply by

\[
\left( \frac{\partial}{\partial s} \right)^2 = \frac{1}{q^2 - p^2} \left[ 2(\frac{\partial}{\partial \sigma} - q^2 \frac{\partial}{\partial \bar{\tau}}) \frac{\partial}{\partial q} - 2(\frac{\partial}{\partial \bar{\sigma}} - p^2 \frac{\partial}{\partial \bar{\tau}}) \frac{\partial}{\partial p} + \tilde{\Delta}_p (\frac{\partial}{\partial p})^2 + \tilde{\Delta}_q (\frac{\partial}{\partial q})^2 \right] + \frac{\gamma}{p^2 q^2} \left( \frac{\partial}{\partial s} \right)^2,
\]

where \((\partial/\partial s)^2\) is the inverse of \(d\Omega_k^2\). Note that if one defines \( h_{MN} = U k_M k_N + V l_M l_N \), so that \( \hat{g}_{MN} = \bar{g}_{MN} + h_{MN} \), then the inverse full metric is given by \( \hat{g}^{MN} = \bar{g}^{MN} - h^{MN} \), since \( h_{MN} h^{NP} = 0 \).

In a metric of single Kerr-Schild form, where \( d\bar{s}^2 = d\bar{s}^2 + U (k_M dx^M)^2 \) and \( k^M \) is a null geodesic congruence, a straightforward calculation shows that the Ricci tensor of the full metric, written with mixed indices \( \hat{R}^M_N \), is given exactly by [1,10]

\[
\hat{R}^M_N = \hat{R}^M_N - h^M_P \hat{R}^P_N + \frac{1}{2} \nabla_P \nabla_N h^{MP} + \frac{1}{2} \nabla^P \nabla^M h_{NP} - \frac{1}{2} \nabla^P \nabla_P h^M_N.
\]

In other words, the “linearised approximation” is exact in this case.

In the case of the double Kerr-Schild metrics (26), i.e. \( \hat{g}_{MN} = \bar{g}_{MN} + h_{MN} \) with \( h_{MN} = U k_M k_N + V l_M l_N \) and \( k_M \) and \( l_M \) satisfying (27), the expression for \( \hat{R}^M_N \) in terms of \( \hat{R}^M_N \) is still, of course, purely of finite polynomial order in \( h_{MN} \), because of the exact property that \( \hat{g}^{MN} = \bar{g}^{MN} - h^{MN} \). However, the weaker conditions satisfied by \( h_{MN} \) in the double Kerr-Schild case imply that in general, terms higher than linear order in \( h_{MN} \) contribute to \( \hat{R}^M_N \). Interestingly, however, in the specific case of the double Kerr-Schild form (32) of the higher-dimensional Kerr-Taub-NUT-de Sitter metrics (3), with \( U \) and \( V \) as in (32), the linear expression (35) is still exact. In fact, more generally we find that if one takes the functions \( U \) and \( V \) to be given by

\[
\begin{align*}
U &= \frac{f(p)}{q^2 - p^2}, \quad V = \frac{g(q)}{q^2 - p^2},
\end{align*}
\]
where \( f(p) \) and \( g(q) \) are arbitrary functions, then the Ricci tensor \( \hat{R}^{MN} \) is given exactly by (35).

It is perhaps worth remarking that one can always choose to view a double Kerr-Schild metric of the form (26) as a single Kerr-Schild metric, by including one or other of the added null terms \( U (k_M dx^M)^2 \) or \( V (l_M dx^M)^2 \) as part of the fiducial “base” metric \( ds^2 \).

Thus in our present example one can view the higher-dimensional Kerr-Taub-NUT-de Sitter metrics (3) as either Kerr-de Sitter with the NUT charge added via a Kerr-Schild term, or else as massless Kerr-Taub-NUT-de Sitter with the mass added via a Kerr-Schild term.

5 Euclidean-signature Metrics

It is also of interest to consider Einstein metrics of positive-definite signature. We can perform such a “Euclideanisation” of the metric (3) by making the following analytic continuation:

\[
p \to \iota p, \quad \tau \to i \tau, \quad \sigma \to i \sigma, \quad \ell \to i^{k-1} \ell,
\]

\[
X \to -\Delta_p, \quad Y \to \Delta_q, \quad \gamma \to -\gamma.
\]

The metric (3) then becomes

\[
ds^2 = \frac{q^2 - p^2}{\Delta_p} dp^2 + \frac{q^2 - p^2}{\Delta_q} dq^2 + \frac{\Delta_p}{q^2 - p^2} (d\tau + q^2 d\sigma)^2 + \frac{\Delta_q}{q^2 - p^2} (d\tau + p^2 d\sigma)^2 + \frac{p^2 q^2}{\gamma} d\Omega_k^2,
\]

where

\[
\Delta_p = \gamma - \epsilon p^2 + \lambda p^4 - 2\ell p^{1-k}, \quad \Delta_q = -\gamma + \epsilon q^2 - \lambda q^4 - 2m q^{1-k}.
\]

It was shown recently in [11] that after Euclideanisation, the higher-dimensional Kerr-de Sitter metrics with a single rotation parameter that were found in [7] yield, in a special limiting case, complete Einstein metrics that extend smoothly onto non-singular manifolds. Since our more general Einstein metrics encompass those in [7], they certainly admit the same non-singular complete metrics as special limiting cases. However, the additional NUT charge parameter that we have in our new metrics in \( D \geq 6 \) provides additional possibilities for obtaining complete, compact Einstein metrics, as we shall now show.

The metrics (38) are of cohomogeneity two, since the metric functions depend on both the coordinates \( p \) and \( q \).\(^1\) In a compact metric, the endpoints in the ranges \( p_1 \leq p \leq p_2 \)

\(^1\)We are not concerned here with any cohomogeneity that might be associated with the \( k \)-dimensional Einstein metric \( d\Omega_k^2 \) if it were not taken to be a sphere or any other homogeneous metric. For simplicity, and without losing any essential generality, we shall consider \( d\Omega_k^2 \) to be the round metric on \( S^k \) in what follows.
and \( q_1 \leq q \leq q_2 \) of the \( p \) and \( q \) coordinates will be defined by degenerations of the metric, corresponding to collapsing of the principal orbits. This can occur at zeros of \( \Delta_p \) or \( \Delta_q \), or at \( p = 0 \) or \( q = 0 \). Although the possibility of obtaining compact non-singular metrics of cohomogeneity two cannot be immediately excluded, it is certainly the case that non-singularity is most easily achieved by reducing the cohomogeneity to degree one, and so we shall make this assumption in the discussion that follows. The reduction of cohomogeneity can be achieved by choosing the parameters so that the coordinate range for either \( p \) or \( q \) shrinks to zero; i.e. \( p_1 = p_2 \), or \( q_1 = q_2 \). It turns out that for regular solutions, we should arrange to shrink the coordinate range for \( q \), by choosing the parameters so that \( \Delta_q \) has two roots that coalesce at \( q = q_0 \):

\[
\Delta_q(q_0) = \Delta_q'(q_0) = 0, \tag{40}
\]

where \( \Delta_q' \) denotes the derivative of \( \Delta_q \) with respect to \( q \).

It is useful to re-express \((\gamma, m, \ell, \lambda)\) in terms of dimensionless parameters \((g, \tilde{m}, \tilde{\ell}, L)\):

\[
\gamma = g q_0^2, \quad \lambda = L q_0^{-2}, \quad m = \tilde{m} q_0^{k+1}, \quad \ell = \tilde{\ell} q_0^{k+1}. \tag{41}
\]

The conditions (40) for the double root can conveniently be used to solve for \( \epsilon \) and \( \tilde{m} \) in terms of \( q_0 \):

\[
\epsilon = \frac{(k + 3)L - (k - 1)g}{k + 2}, \quad \tilde{m} = \tilde{m}_0 = \frac{g - L}{k + 1}. \tag{42}
\]

Moving slightly away from the case of the double root, by displacing \( \tilde{m} \) away slightly from \( \tilde{m}_0 \), we now define

\[
\tilde{m} = \tilde{m}_0 - \frac{\delta^2}{2c q_0}, \quad q = q_0 + \delta \cos \theta, \quad \tau = \frac{q_0^2 \delta}{2c} \phi, \quad \sigma = \frac{2c}{q_0} \psi - \frac{c}{\delta} \phi, \quad p = q_0r \tag{43}
\]

where \( c^{-1} \equiv (k + 3)L - (k - 1)g \), and then send \( \delta \to 0 \). We find that in this limit the metric becomes

\[
d s^2 \bigg|_{q_0} = \frac{1 - r^2}{Q(r)} d r^2 + \frac{4c^2 Q(r)}{1 - r^2} (d \psi + \cos \theta d \phi)^2 + c(1 - r^2)(d \theta^2 + \sin^2 \theta d \phi^2) + \frac{r^2}{g} d \Omega_k^2, \tag{44}
\]

where

\[
Q(r) = g - \frac{(k + 3)L - (k - 1)g}{k + 1} r^2 + L r^4 - 2\tilde{\ell} r^{1-k}. \tag{45}
\]

This metric can be recognised as the \( n = 1 \) special case of the class of cohomogeneity-one Einstein metrics obtained in [12], having a base Einstein-Kähler manifold \( K_{2n} \) of dimension \( 2n \), with fibres that involve a complex line bundle over \( K_{2n} \) and an additional \( S^k \) warped-product factor. The conditions for regularity of such metrics were analysed in detail in [12].
Applied to our $n = 1$ case where $K_2 = S^2$, these results show that compact non-singular Einstein metrics can be achieved by choosing the parameters so that $r$ ranges either from 0 to $r_0$, where $\Delta_p(r_0) = 0$, or else by choosing the parameters such that $r$ ranges between two distinct positive roots $r_1$ and $r_2$ of $\Delta_p$. The former requires $\bar{\ell} = 0$, and was in fact obtained in [11] as a limit of the single-rotation Kerr-de Sitter metrics; in this case, it is essential for regularity that the metric $d\Omega^2_k$ be the round metric on $S^k$. The latter requires $\bar{\ell} \neq 0$. It was obtained in [12] by starting with a cohomogeneity-one metric ansatz, but here we have shown how it arises as a limiting case of the more general higher-dimensional Kerr-Taub-NUT-de Sitter metrics that we have constructed in this paper. (In this case, since the coefficient of $d\Omega^2_k$ is everywhere non-vanishing, one can choose any regular Einstein metric for $d\Omega^2_k$.)

6 Conclusions

In this paper, we have studied some properties of a class of higher-dimensional generalisations of the Kerr-Taub-NUT-de Sitter metrics. We have shown that they share many, but not all, of the remarkable properties of their four-dimensional progenitors. For example, they admit separation of variables for both the Hamilton-Jacobi and wave equations, and we have exhibited the associated second-rank Säckel-Killing tensors. By contrast to the four-dimensional case, however, the Säckel-Killing tensor appears not to have a Yano-Killing tensor square root. It should be emphasised that the separability of the higher-dimensional solutions, which leads in many cases to completely-integrable geodesics flows, is a rather non-trivial consequence of the detailed form of the solutions, which is mandated by the Einstein equations. A recent study of separability in the higher-dimensional Kerr-de Sitter metrics showed that this is possible at least in the special case of all rotation parameters equal, when there is an enhanced isometry group [13]. In our case, the equations can be separated in situations where there is only a single rotation parameter, together with a NUT charge. Previous results on separability, in the absence of the NUT charge and cosmological constant, were obtained in [14].

Like their four-dimensional progenitors, the metrics (3) may be cast in double Kerr-Schild form, which may provide a fruitful ansatz for the further study of higher-dimensional solutions of the Einstein equations. This is because the solutions cast in this form may be regarded as their own linear approximations. This, unlike the single Kerr-Schild ansatz, is not a general feature in the double Kerr-Schild case, but it does hold for the metrics we have
considered. More generally, the double Kerr-Schild ansatz leads to quartic non-linearity in
the Einstein equations.

The double Kerr-Schild form requires us to consider an analytically-continued form of the
metric with two time directions. Analytic continuation will also produce metrics of positive-
definite (Euclidean) signature. This we have done, and by considering special limiting cases,
obtained complete non-singular compact Einstein manifolds that were previously obtained
in [12].

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