THE FRACTIONAL LAPLACIAN IN POWER-WEIGHTED $L^p$ SPACES:
INTEGRATION-BY-PARTS FORMULAS AND SELF-ADJOINTNESS

MATTEO MURATORI

Abstract. We consider the fractional Laplacian operator $(-\Delta)^s$ (let $s \in (0, 1)$) on Euclidean space and investigate the validity of the classical integration-by-parts formula that connects the $L^2(\mathbb{R}^d)$ scalar product between a function and its fractional Laplacian to the nonlocal norm of the fractional Sobolev space $\dot{H}^s(\mathbb{R}^d)$. More precisely, we focus on functions belonging to some weighted $L^p$ space whose fractional Laplacian belongs to another weighted $L^p$ space: we prove and disprove the validity of the integration-by-parts formula depending on the behaviour of the weight $\rho(x)$ at infinity. The latter is assumed to be like a power both near the origin and at infinity (the two powers being possibly different). Our results have direct consequences for the self-adjointness of the linear operator formally given by $\rho^{-1}(-\Delta)^s$. The generality of the techniques developed allows us to deal with weighted $L^p$ spaces as well.

1. Introduction

Given $d \in \mathbb{N}$ and any $s \in (0, 1)$, the fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^d$ is a nonlocal operator defined on test functions by

$$(-\Delta)^s(\phi)(x) := C_{d,s} \text{p.v.} \int_{\mathbb{R}^d} \frac{\phi(x) - \phi(y)}{|x-y|^{d+2s}} \, dy \quad \forall x \in \mathbb{R}^d, \ \forall \phi \in D(\mathbb{R}^d),$$

where p.v. denotes the principal value of the integral about $x$ and $C_{d,s}$ is a suitable positive constant depending only on $d$ and $s$, such that $\lim_{s \to 1^-}(-\Delta)^s(\phi) = -\Delta \phi$ (see for instance [13, Sections 3, 4]). An alternative representation of $(-\Delta)^s$ is the one involving the celebrated extension of Caffarelli and Silvestre [10], where the fractional Laplacian of $\phi$ is seen as the trace of the normal derivative of the harmonic extension of $\phi$ in the upper half-plane (at least for $s = \frac{1}{2}$, while for a general $s \in (0, 1)$ one has to introduce a suitable degenerate or singular elliptic operator). Even though it has proved to be a very powerful tool in dealing with issues related to the fractional Laplacian, we shall no further consider the aforementioned extension, since our arguments need not take advantage of it.

A Sobolev space naturally associated with the fractional Laplacian is $\dot{H}^s(\mathbb{R}^d)$, namely the closure of $D(\mathbb{R}^d)$ endowed with the norm

$$\|\phi\|_{\dot{H}^s(\mathbb{R}^d)} := \left\|(-\Delta)^{s/2}(\phi)\right\|_{L^2(\mathbb{R}^d)} \quad \forall \phi \in D(\mathbb{R}^d).$$

A well-known result (see [13, Proposition 3.6]) asserts that

$$\|\phi\|^2_{\dot{H}^s(\mathbb{R}^d)} = \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(|\phi(x) - \phi(y)|^2)}{|x-y|^{d+2s}} \, dxdy \quad \forall \phi \in D(\mathbb{R}^d),$$

so that we can equivalently define $\dot{H}^s(\mathbb{R}^d)$ by means of the nonlocal (squared) norm appearing in the r.h.s. of (1.1). Let us point out that by $\dot{H}^s(\mathbb{R}^d)$ one usually means the space of functions $v \in L^2(\mathbb{R}^d)$ such that $\|v\|_{\dot{H}^s(\mathbb{R}^d)} < \infty$, which in fact coincides with $L^2(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$. However,
since below we shall deal with functions belonging to some weighted $L^2$ spaces ($L^p$ in general), throughout the paper we shall never make use of $H^s(\mathbb{R}^d)$.

By means of classical Fourier-transform arguments (we refer again to [13, Section 3]), it is straightforward to show that if $v \in L^2(\mathbb{R}^d)$ and $(-\Delta)^s(v) \in L^2(\mathbb{R}^d)$ (to be understood in the distributional sense), then $v \in H^s(\mathbb{R}^d)$. Moreover, since $\|\cdot\|_{H^s(\mathbb{R}^d)}$ naturally induces an inner product $\langle \cdot, \cdot \rangle_{H^s(\mathbb{R}^d)}$, the following integration-by-parts formulas hold:

$$\langle v, w \rangle_{H^s(\mathbb{R}^d)} = C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{d+2s}} \, dx \, dy$$

$$= \int_{\mathbb{R}^d} (-\Delta)^{s/2}(v)(x)(-\Delta)^{s/2}(w)(x) \, dx$$

$$= \int_{\mathbb{R}^d} v(x)(-\Delta)^s(w)(x) \, dx = \int_{\mathbb{R}^d} (-\Delta)^s(v)(x) \, w(x) \, dx$$

(1.2)

for all $v, w \in L^2(\mathbb{R}^d)$ such that $(-\Delta)^s(v), (-\Delta)^s(w) \in L^2(\mathbb{R}^d)$. We referred to (1.2) as formulas for “integration by parts” having in mind the case $s = 1$, where the second term is replaced by the $[L^2(\mathbb{R}^d)]^d$ scalar product between gradients. It is worth mentioning that the last line of (1.2) entails the self-adjointness of the fractional Laplacian with domain $\{v \in L^2(\mathbb{R}^d) : (-\Delta)^s(v) \in L^2(\mathbb{R}^d)\}$. We shall resume this point shortly.

The main purpose of the paper is to establish (or disprove) the validity of (1.2) in a suitable weighted framework. More precisely, let $\rho(x)$ be a weight (i.e. a nonnegative, measurable function) in $\mathbb{R}^d$ such that

$$c |x|^{-\gamma_0} \leq \rho(x) \leq C |x|^{-\gamma_0} \text{ a.e. in } B_1 \quad \text{and} \quad c |x|^{-\gamma} \leq \rho(x) \leq C |x|^{-\gamma} \text{ a.e. in } B_1'$$

(1.3)

for some positive constants $c < C$ and exponents $\gamma_0, \gamma \in \mathbb{R}^+$, where $B_1$ denotes the ball of radius one centred at the origin. In other words, we assume that $\rho(x)$ behaves like a nonpositive power both near the origin and at infinity, the two powers being possibly different. We focus on functions $v, w \in L^2(\mathbb{R}^d; \rho(x) \, dx)$ such that $(-\Delta)^s(v), (-\Delta)^s(w) \in [L^2(\mathbb{R}^d; \rho(x)^{-1} \, dx)]^d$ (to be understood in the distributional sense) and at least one of the functions $v, w$ is in $H^s(\mathbb{R}^d; \rho(x)^{-2} \, dx)$.

Note that, within such class of functions, at least the last line of (1.2) makes sense. In fact we shall prove that (1.2) does hold provided $\gamma_0 \in [0, d)$ and $\gamma \in [0, 2s]$. The power $\gamma = 2s$ is referred to as critical since it is precisely the one corresponding to the scaling of the fractional Laplacian, see Section 3 below and in particular Lemmas 3.3 and 3.4. As we work with weighted Lebesgue spaces, establishing the validity of (1.2) is not a trivial task since we cannot exploit direct Fourier-transform techniques. Indeed our methods of proof will only make use of regularisation-by-mollification and cut-off arguments. In this regard, we devote Section 2 to prove a result that may also have an independent interest, namely the fact that one can approximate functions in the power-weighted Lebesgue spaces above by means of standard mollifications (Theorem 2.1). This is important to our ends since the mollification operator commutes with translation-invariant operators such as the fractional Laplacian, so that, for instance, a function $v \in L^2(\mathbb{R}^d; \rho(x) \, dx)$ with $(-\Delta)^s(v) \in L^2(\mathbb{R}^d; \rho(x)^{-1} \, dx)$ can be approximated alongside its fractional Laplacian by means of its mollifications (see Proposition 3.3). Hence, we start from the validity of (1.2) in $D(\mathbb{R}^d)$, mollify $v$ and $w$, cut them off and let the cut-off parameter tend to infinity: in order to make remainder terms vanish, it is essential that $\gamma \leq 2s$, i.e. that the power of $\rho(x)$ at infinity is subcritical or at most critical.

Somewhat surprisingly, at least in the case $d > 2s$, we are able to extend the validity of the integration-by-parts formulas to any $\gamma \in [2s, d]$ as well (that is, to some supercritical $\gamma$). However, since cut-off techniques fail, we have to proceed by means of completely different
arguments. More precisely, we shall prove that under our assumptions $v$ and $w$ coincide with their Riesz potentials, namely $v = I_{d,s} \ast (-\Delta)^s(v)$ and $w = I_{d,s} \ast (-\Delta)^s(w)$, where $I_{d,s}$ is the Riesz kernel or Green function of the fractional Laplacian in $\mathbb{R}^d$ (see the beginning of Section 5 below and the monograph \[30\] as a general reference). Loosely speaking, this means that they have much better integrability properties than expected, which is crucial.

Our results can actually be generalised to any $p \in [2, \infty)$ for $d \leq 2s$ and to any $p \in [2, 2d/(d - 2s)]$ for $d > 2s$. Accordingly, the critical power $\gamma = 2s$ must be replaced by $\gamma = d - \frac{s}{2}(d - 2s)$. The precise statements and introduction of the underlying functional setting are given in Sections 1.1–1.2. We preferred to prove the subcritical and supercritical cases separately, since as explained above the techniques are rather different.

Nevertheless, the case $p = 2$ is by itself interesting. Indeed, the validity of \[1.2\] for all $v, w \in L^2(\mathbb{R}^d; \rho(x)dx)$ such that $(-\Delta)^s(v), (-\Delta)^s(w) \in L^2(\mathbb{R}^d; [\rho(x)]^{-1}dx)$ is equivalent to the self-adjointness of the linear operator formally given by $\rho^{-1}(-\Delta)^s$ in $L^2(\mathbb{R}^d; \rho(x)dx)$. As a consequence, this operator generates a continuous semigroup in $L^2(\mathbb{R}^d; \rho(x)dx)$, so that the Cauchy problem for the weighted, fractional heat-type equation

$$
\begin{aligned}
\rho(x)u_t &= -(-\Delta)^s(u) & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\
u(x, 0) &= \mu(x) \in L^2(\mathbb{R}^d; \rho(x)dx) & \text{in } \mathbb{R}^d,
\end{aligned}
$$

is well posed. In addition, such semigroup turns out to be Markov and can therefore be extended in a consistent way to a contraction semigroup in $L^p(\mathbb{R}^d; \rho(x)dx)$ for all $p \in [1, \infty]$, the latter being analytic for $p \in (1, \infty)$. The precise statement is provided by Theorem \[1.3\].

We finally prove that, still under the assumption $d > 2s$, formulas \[1.2\] fail as soon as $\gamma > d$ (see Theorem \[1.3\]). In particular, we deduce that in this case the operator $\rho^{-1}(-\Delta)^s$ is not self-adjoint in $L^2(\mathbb{R}^d; \rho(x)dx)$. This is due to the presence of nontrivial constants, since $\rho \in L^1(\mathbb{R}^d)$. Hence, the only set of parameters left undetermined is $d \leq 2s$ and $\gamma > d - \frac{s}{2}(d - 2s)$, that is $d = 1$, $s \in [1/2, 1)$ and $\gamma > 1 + \frac{s}{2}(2s - 1)$. There our techniques prevent us from establishing whether or not \[1.2\] holds (see Remark \[1.4\]).

The major motivation for investigating the validity of \[1.2\] in weighted Lebesgue spaces came from \[25\], a recent paper in which the author and collaborators studied the weighted, fractional porous medium equation with initial measure data, that is

$$
\begin{aligned}
\rho(x)u_t &= -(-\Delta)^s(u^m) & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\
u(x, 0) &= \mu \in L^1(\mathbb{R}^d) & \text{in } \mathbb{R}^d,
\end{aligned}
$$

where $m > 1$ and $\mu$ is a positive, finite Radon measure on $\mathbb{R}^d$. In particular, uniqueness is established by suitably adapting a “duality method” originally developed in \[35\], which basically consists in proving that the equation solved by the difference of the Riesz potentials of two possibly different solutions admits zero as its unique solution. In order to apply such method, it is essential to justify rigorously some integration by parts that involve functions belonging to $L^2(\mathbb{R}^d; \rho(x)dx)$ whose fractional Laplacian belongs to $L^2(\mathbb{R}^d; [\rho(x)]^{-1}dx)$ (actually in low dimensions one has to cope with analogous issues in weighted $L^p$ spaces, for some $p > 2$). Moreover, the well-posedness of \[1.4\], which is related to the dual problem, is also crucial. The interest in taking measures as initial data comes in turn from the analysis of the asymptotic behaviour of general solutions, see \[26\].

The study of nonlinear diffusion equations involving fractional Laplacians has received an increasing amount of interest recently. In \[33\]–\[34\] the authors investigate the fractional porous medium equation (PME from here on) in $\mathbb{R}^d$ with $L^1(\mathbb{R}^d)$ initial data. A thorough
asymptotic analysis is then carried out in [11]. Delicate \textit{a priori} estimates, both from above and below, are the main concern of [8]. As for the weighted, fractional PME (1.5), a first well-posedness analysis (for more regular data) is performed by [36]. Fractional diffusions of porous medium type on \textit{bounded domains} are deeply analysed in [7, 9]. We refer to [42] for an excellent overview of the state of the art in the theory of nonlinear fractional diffusion.

Weighted \textit{local} nonlinear diffusions have also been investigated in the last few years. In the series of papers [37, 38, 29] the authors study the so-called \textit{inhomogeneous} PME in Euclidean space, namely (1.5) with $s = 1$. They consider regular weights (or \textit{densities}) such that $\rho(x) \approx |x|^{-\gamma}$ for some $\gamma > 0$ as $|x| \to \infty$. It is remarkable that the asymptotics of solutions changes considerably depending on whether $\gamma$ is lower or greater than the critical value $\gamma = 2$ (which corresponds to the natural scaling of the Laplacian). The critical case is then addressed in [32]. A further analysis is carried out by [28], where $\rho(x) = |x|^{-2}$ for all $x \in \mathbb{R}^d$. For a general theory of weighted PME's we refer the reader e.g. to [23, 24].

In the linear context, besides the classical reference [2], there are some relatively recent works involving both fractional heat-type equations and weighted, local parabolic equations in $\mathbb{R}^d$. In [1] the authors focus on uniqueness issues for (1.4) with $\rho \equiv 1$: they look for a sharp class of positive solutions that can be written as the convolution between their initial datum and the heat kernel. The paper [27] is the linear counterpart of [28]. In [14] a general weighted, second-order parabolic problem is studied: the density $\rho$ can depend on time as well, and uniqueness results are discussed as the behaviour of $\rho(x, t)$ as $|x| \to \infty$ varies.

In dimension one, the celebrated paper [19] provides an exhaustive analysis of second-order parabolic equations, possibly degenerate or singular at the boundary, by exploiting the (spectral) theory of one-parameter semigroups due to Hille and Yoshida. With no claim for completeness, we also quote the more recent works [16, 17], where semigroups generated by linear and quasilinear one-dimensional, weighted, elliptic operators with quite general boundary conditions are investigated, and [22], where the authors study a one-dimensional, elliptic operator degenerating of first order at the boundary. In several space dimensions some degenerate, elliptic operators on domains (with homogeneous Dirichlet boundary conditions) are considered by [20, 21]. More precisely, in [20] the authors deal with a weight proportional to the distance to the boundary. In [21] a thorough analysis of elliptic operators (and of the associated semigroups) whose diffusion coefficients degenerate linearly at the boundary only in tangential directions is performed. In [18] similar operators with nonhomogeneous boundary conditions are analysed.

The connection between fractional Laplacians and symmetric, $2s$-stable \textit{Lévy processes} is by now a well-established issue. In this regard, we refer the reader to the nice survey [10], where such connection is made apparent by resorting to simple Random walks with jumps.

The probabilistic interpretation of the fractional Laplacian, and of similar nonlocal diffusion operators, has successfully been exploited to give pointwise bounds on the corresponding heat kernel (\textit{i.e.} the transition density function of the underlying Lévy process). In [3] the authors study a quite general class of Markov processes of pure jump type, with Dirichlet forms that extend (1.1) and allow for anisotropic processes, obtaining lower and upper bounds for the associated heat kernels. By means of real-analytic arguments, two-sided sharp estimates of the heat kernel are given also in [6], where isotropic, unimodal Lévy processes are considered. In [11], by taking advantage of probabilistic methods, the authors provide two-sided sharp estimates for the heat kernel of the (regional) fractional Laplacian in $C^{1,1}$ domains. Less regular domains are dealt with by [5].
Let us remark that in the present paper we make no use of stochastic representations for $\rho^{-1}(-\Delta)^s$. In principle, it is not even clear whether it is possible to associate such an operator with some Lévy-type process (the role of the weight appears to be non trivial). Nevertheless, the purely analytic problem of providing suitable two-sided bounds for its heat kernel (in terms of $d, s, \gamma_0, \gamma$) is also left open.

1.1. Notations and basic definitions. For any nonnegative, nontrivial measurable function $\rho$ and measurable set $\Omega \subset \mathbb{R}^d$, we denote by $L^p_\rho(\Omega)$ (let $p \in [1, \infty)$) the Lebesgue space of all measurable functions $f$ such that

$$\|f\|_{L^p_\rho(\Omega)} := \int_{\mathbb{R}^d} |f(x)|^p \rho(x)\,dx < \infty.$$ 

If $\Omega = \mathbb{R}^d$ we set $\|f\|_{p,\rho} := \|f\|_{L^p_\rho(\mathbb{R}^d)}$. Moreover, in the special case of power weights, namely $\rho(x) = |x|^\lambda$ for some $\lambda \in \mathbb{R}$, we set $L^p_\lambda(\Omega) := L^p_{|x|\lambda}(\Omega)$ and $\|f\|_{p,\lambda} := \|f\|_{p,|x|\lambda}$. In the non-weighted case $\lambda = 0$ we use the standard notations $L^p(\Omega) := L^p_0(\Omega)$ and $\|f\|_p := \|f\|_{p,0}$.

In the sequel we shall mostly choose $\Omega = B_r(x_0)$, that is the Euclidean ball of radius $r > 0$ centred at $x_0 \in \mathbb{R}^d$, or its complement $B_r^c(x_0)$. To simplify notation, we adopt the convention $B_r := B_r(0)$.

For weights satisfying appropriate assumptions, we provide a functional space that will be very useful in the sequel.

**Definition 1.1.** Let $p \in (1, \infty)$ with $p' := \frac{p}{p-1}$. Suppose that $\rho$ satisfies \((1.3)\) for some $\gamma_0 \in [0, d)$ and $\gamma \in [0, d+ps]$. We denote by $X_{p,s,\rho}$ the space of all functions $v \in L^p_\rho(\mathbb{R}^d)$ such that $(-\Delta)^s(v)$ (as a distribution) belongs to $L^{p'}_{\rho'}(\mathbb{R}^d)$, where $\rho' := \rho^{-p'/(p'-1)}$.

In the special case $p = 2$, we set $X_{s,\rho} := X_{2,s,\rho}$.

According to the above definition, a function $v \in L^p_\rho(\mathbb{R}^d)$ belongs to $X_{p,s,\rho}$ if and only if there exists an element $\mathcal{V} \in L^{p'}_{\rho'}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} v(x) (-\Delta)^s(\phi)(x)\,dx = \int_{\mathbb{R}^d} \mathcal{V}(x) \phi(x)\,dx \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d). \tag{1.6}$$

We stress that the assumptions $\gamma_0 \in [0, d)$ and $\gamma \in [0, d+ps]$ ensure, in particular, that both the left- and the right-hand side of \((1.6)\) are in fact distributions.

1.2. Statement of the main results. Our most important contribution to the validity, or otherwise, of the integration-by-parts formulas \((1.2)\) is the following.

**Theorem 1.2.** Let either $d \leq 2s$ and $p \in [2, \infty)$ or $d > 2s$ and $p \in [2, 2d/(d-2s)]$. Suppose that $\rho$ satisfies \((1.3)\) for some $\gamma_0 \in [0, d)$ and $\gamma \in \left[0, d \vee \left(d - \frac{2s}{d-2s}\right)\right]$. Then formulas \((1.2)\) hold for all $v, w \in X_{p,s,\rho}$.

On the other hand, if $d > 2s$ and $\gamma \in (d, d+ps)$ then formulas \((1.2)\) fail in $X_{p,s,\rho}$.

As mentioned above, Theorem 1.2 entails some crucial consequences concerning the self-adjointness of the operator $\rho^{-1}(-\Delta)^s$.

**Theorem 1.3.** Suppose that $\rho$ satisfies \((1.3)\) for some $\gamma_0 \in [0, d)$ and $\gamma \in [0, d+2s]$. Let us define the linear operator $A : D(A) := X_{s,\rho} \subset L^2_\rho(\mathbb{R}^d) \to L^2_\rho(\mathbb{R}^d)$ as follows:

$$A(f) := \rho^{-1}(-\Delta)^s(f) \quad \forall f \in D(A).$$
Then $A$ is a densely defined, nonnegative self-adjoint operator in $L^2_p(\mathbb{R}^d)$, whose associated quadratic form is

$$Q(v,v) := \|v\|^2_{H^s(\mathbb{R}^d)} = \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (v(x) - v(y))^2 \, dx \, dy,$$

with domain $D(Q) = L^2_p(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$. Moreover, $Q$ is a Dirichlet form and $A$ generates a Markov semigroup $S_2(t)$ on $L^2_p(\mathbb{R}^d)$. In particular, for all $p \in [1, \infty]$ there exists a contraction semigroup $S_p(t)$ on $L^2_p(\mathbb{R}^d)$, consistent with $S_2(t)$ on $L^2_p(\mathbb{R}^d) \cap L^p_0(\mathbb{R}^d)$, which is furthermore analytic with a suitable angle $\theta_p > 0$ for all $p \in (1, \infty)$.

In the case $d > 2s$ and $\gamma \in (d, d + 2s)$, the operator $A$ is no more self-adjoint in $L^2_p(\mathbb{R}^d)$.

**Remark 1.4.** We point out that, upon requiring $d > 2s$, we only exclude the case $d = 1$ with $s \in (1/2, 1)$. More precisely, in the light of Theorem 1.2, in the set of parameters $d = 1$, $s \in [1/2, 1)$, $p \in [2, \infty)$ and $1 + \frac{2}{p}(2s - 1) < \gamma \leq 1 + ps$, the validity (or the failure) of (1.2) in $X_{p,s,\rho}$ is left as an open problem, since for such choices neither cut-off nor potential techniques work (see Sections 3–5).

### 1.3. Organization of the paper.

Section 2 is entirely devoted to the proof of the fact that mollifications are dense in the weighted $L^p$ spaces we consider (Theorem 2.1). We then briefly show that our assumptions on the weight for such a result to hold are to some extent sharp (Remark 2.2). In Section 3 first we collect some straightforward decay and scaling properties of fractional Laplacians of test functions (Lemmas 3.1–3.4), then we establish some fundamental intermediate steps, involving the space $X_{p,s,\rho}$, which are essential to prove the integration-by-parts formulas (Proposition 3.6 and Lemmas 3.5, 3.7). The proof of Theorems 1.2, 1.3 is split between Sections 4 and 5. In Section 4, after providing a continuous-embedding result (Lemma 4.1), we give the proof of the validity of the integration-by-parts formulas for subcritical-critical powers. The special case $p = 2$ is discussed afterwards. Finally, Section 5 deals with supercritical powers under the assumption $d > 2s$: by means of potential techniques, upon establishing some preliminary results (Lemmas 5.1, 5.3), we prove and disprove the validity of the integration-by-parts formulas.

### 2. Density of mollifications in power-weighted $L^p$ spaces

In the following we shall assume that $\rho$ is a weight that behaves like a power, not necessarily negative, both near the origin and at infinity, namely that

$$c |x|^{\lambda} \leq \rho(x) \leq C |x|^{\lambda} \quad \text{a.e. in } B_1 \quad \text{and} \quad c |x|^{\Lambda} \leq \rho(x) \leq C |x|^{\Lambda} \quad \text{a.e. in } B_1^c$$

(2.1)

for some positive constants $c < C$ and exponents $\lambda, \Lambda \in \mathbb{R}$.

Our aim in the present section is to show that the standard mollification of a function $f \in L^p_0(\mathbb{R}^d)$ converges to $f$ in $L^p_0(\mathbb{R}^d)$, under suitable assumptions on $p$ and $\lambda$. This result will frequently be exploited through Sections 5–5, but we believe it can also have an independent interest.

**Theorem 2.1.** Let $p \in (1, \infty)$. Suppose that (2.1) holds for some $\lambda \in (-d, (p - 1)d)$ and $\Lambda \in \mathbb{R}$. Let $f \in L^p_0(\mathbb{R}^d)$ and consider the mollification

$$f_\varepsilon(x) := \int_{\mathbb{R}^d} \eta_\varepsilon(x - y) f(y) \, dy \quad \forall x \in \mathbb{R}^d,$$

(2.2)

where

$$\eta_\varepsilon(x) := \varepsilon^{-d} \eta \left( \frac{x}{\varepsilon} \right) \quad \forall x \in \mathbb{R}^d, \quad \forall \varepsilon > 0,$$
and $\eta$ is a nonnegative, regular function supported in $B_1$, such that $\int_{\mathbb{R}^d} \eta(x) \, dx = 1$. Then, $f_\varepsilon \in C^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ and
\[
\lim_{\varepsilon \to 0} \| f_\varepsilon - f \|_{p,\rho} = 0. \tag{2.3}
\]

**Proof.** To simplify readability we shall only deal with the model case $\rho(x) = |x|^\lambda$. Minor modifications are required to deal with a more general weight as in the statement, which will be discussed in the end of the proof.

In order to give sense to (2.2) and to prove that $f_\varepsilon \in C^\infty(\mathbb{R}^d)$, we first need to show that $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. To this end, by means of Hölder’s inequality, for any $R > 0$ we get:
\[
\int_{B_R} |f(y)| \, dy \leq \left( \int_{B_R} |y|^{-\frac{\lambda}{p-1}} \, dy \right)^{\frac{p-1}{p}} \left( \int_{B_R} |f(y)|^p |y|^{\lambda} \, dy \right)^{\frac{1}{p}} \leq \frac{|S_{d-1}|^{\frac{p-1}{p}} R^{\frac{(p-1)d-\lambda}{p}}}{(d-\lambda)^{\frac{p-1}{p}}} \| f \|_{p,\lambda}, \tag{2.4}
\]
where $S_{d-1}$ is the unitary $(d-1)$-dimensional sphere. Note that the r.h.s. of (2.4) is finite in view of the assumption $\lambda < (p-1)d$.

The validity of (2.3) is actually implied by the validity of the estimate
\[
\| f_\varepsilon \|_{p,\lambda} \leq K \| f \|_{p,\lambda} \quad \forall f \in L^p_\lambda(\mathbb{R}^d) \tag{2.5}
\]
for a suitable positive constant $K$ independent of $\varepsilon$ and $f$. Indeed, once we have established (2.5), we can proceed in a standard way. First of all, we pick a sequence of functions $\{f_n\}$ that are compactly supported in $\mathbb{R}^d \setminus \{0\}$ such that
\[
\lim_{n \to \infty} \| f_n - f \|_{p,\lambda} = 0. \tag{2.6}
\]
This is always possible: for any given $n \in \mathbb{N}$ one can consider the truncated functions $f_n(x) := f(x) \chi_{\{1/n \leq |x| \leq n\}}$. It is plain that each $f_n \in L^p(\mathbb{R}^d)$ is by definition compactly supported in $\mathbb{R}^d \setminus \{0\}$ and that (2.6) holds. By standard results (see e.g. [11] Chapters 2, 3 or [15] Appendix C.4) the mollification $(f_n)_\varepsilon$ of $f_n$ converges to $f_\varepsilon$ in $L^p(\mathbb{R}^d)$ as $\varepsilon \to 0$. Since $(f_n)_\varepsilon$ is eventually supported in $B_{2n} \cap B_{1/2n}^c$ and the weight $|x|^\lambda$ is equivalent to 1 in such region, we deduce that
\[
\lim_{\varepsilon \to 0} \| (f_n)_\varepsilon - f_n \|_{p,\lambda} = 0. \tag{2.7}
\]
By using the triangular inequality, the linearity of the mollification operator and (2.5), we get:
\[
\| f_\varepsilon - f \|_{p,\lambda} \leq (K + 1) \| f_n - f \|_{p,\lambda} + \| (f_n)_\varepsilon - f_n \|_{p,\lambda}. \tag{2.8}
\]
Thanks to (2.6), for any $\delta > 0$ we can pick $n_\delta$ so large that $\| f_n - f \|_{p,\lambda} \leq \delta$. By letting $\varepsilon \to 0$ in (2.8) with $n = n_\delta$ and using (2.7), we end up with
\[
\limsup_{\varepsilon \to 0} \| f_\varepsilon - f \|_{p,\lambda} \leq (K + 1) \delta,
\]
whence (2.3) follows from the arbitrariness of $\delta$.

We are therefore left with proving (2.5). To this aim, let us split $\| f_\varepsilon \|_{p,\lambda}$ in a convenient way:
\[
\| f_\varepsilon \|_{p,\lambda}^p = \int_{B_{2\varepsilon}} |f_\varepsilon(x)|^p |x|^\lambda \, dx + \int_{B_{2\varepsilon}^c} |f_\varepsilon(x)|^p |x|^\lambda \, dx. \tag{2.9}
\]
We shall estimate the two integrals above separately. As for the first one, we have (recall that \( \lambda > -d \)):
\[
\int_{B_{2\varepsilon}} |f_\varepsilon(x)|^p |x|^\lambda \, dx \leq \frac{2^{d+\lambda} \varepsilon^{d+\lambda} |S_{d-1}|}{d+\lambda} \|f_\varepsilon\|_{L^\infty(B_{2\varepsilon})}^p, \tag{2.10}
\]
where, by virtue of (2.2) and (2.4) (with \( R = 3\varepsilon \)),
\[
\|f_\varepsilon\|_{L^\infty(B_{2\varepsilon})} \leq \frac{\|\eta\|_{\infty}}{\varepsilon^d} \int_{B_{3\varepsilon}} |f(y)| \, dy \leq \frac{\left( \varepsilon^{d+\lambda} \right)}{(d+\lambda) \left( d - \frac{\lambda}{p-1} \right)^{p-1} \|\eta\|_{\infty}} \|f\|_{p,\lambda}. \tag{2.11}
\]
From (2.10) and (2.11) we obtain
\[
\int_{B_{2\varepsilon}} |f_\varepsilon(x)|^p |x|^\lambda \, dx \leq \frac{2^{d+\lambda} \varepsilon^{d+\lambda} |S_{d-1}|}{d+\lambda} \|f_\varepsilon\|_{L^\infty(B_{2\varepsilon})}^p, \tag{2.12}
\]
We now turn to the second integral in the r.h.s. of (2.9). We have:
\[
\int_{B_{2\varepsilon}} |f_\varepsilon(x)|^p |x|^\lambda \, dx \leq \int_{\mathbb{R}^d} |f(y)|^p \left( \int_{B_{2\varepsilon}} \eta_\varepsilon(x-y) |x|^\lambda \, dx \right) \, dy, \tag{2.13}
\]
where we exploited Hölder’s inequality, for any fixed \( x \in B_{2\varepsilon}^c \), with respect to the probability measure \( \eta_\varepsilon(x-y) \, dy \). Thanks to (2.13), it is enough to show that there exists a positive constant \( K' \), independent of \( \varepsilon \), such that
\[
\int_{B_{2\varepsilon}} \eta_\varepsilon(x-y) |x|^\lambda \, dx \leq K' |y|^\lambda \quad \forall y \in \mathbb{R}^d. \tag{2.14}
\]
To this aim, first of all note that
\[
\int_{B_{2\varepsilon}} \eta_\varepsilon(x-y) |x|^\lambda \, dx \leq \|\eta\|_{\infty} \int_{B_{2\varepsilon}} \chi_{\{|x-y|\leq \varepsilon\}} |x|^\lambda \, dx \tag{2.15}
\]
It is apparent that for \( |y| < \varepsilon \) the integral in the r.h.s. of (2.15) is identically zero, while for \( |y| > 2\varepsilon \) we have:
\[
\int_{B_{2\varepsilon}} \chi_{\{|x-y|\leq \varepsilon\}} |x|^\lambda \, dx \leq \int_{B_{2\varepsilon}} |x|^\lambda \, dx \leq \frac{|S_{d-1}|}{2\lambda d} \max \left\{ 3^{\lambda}, 1 \right\} |y|^\lambda. \tag{2.16}
\]
Hence, it remains to estimate the r.h.s. of (2.15) as \( y \) varies in \( \overline{B}_{2\varepsilon} \setminus B_\varepsilon \). We point out that in such region the following inequality holds:
\[
\int_{B_{2\varepsilon}} \chi_{\{|x-y|\leq \varepsilon\}} |x|^\lambda \, dx \leq \int_{B_{2\varepsilon} \setminus B_\varepsilon} |x|^\lambda \, dx \leq \frac{|S_{d-1}|}{\varepsilon^d} \left( \left| y \right| + \varepsilon \right)^{d+\lambda} - 2^{d+\lambda} \varepsilon^{d+\lambda} \tag{2.17}
\]
It is then direct to see that there exists a positive constant \( M \), independent of \( \varepsilon \), such that
\[
\left( \left| y \right| + \varepsilon \right)^{d+\lambda} - 2^{d+\lambda} \varepsilon^{d+\lambda} \leq M |y|^\lambda \quad \forall y \in \overline{B}_{2\varepsilon} \setminus B_\varepsilon. \tag{2.18}
\]
Hence, by gathering (2.15)−(2.18), we can deduce that (2.14) does hold with
\[
K' = |S_{d-1}| \|\eta\|_{\infty} \max \left\{ \frac{\max \left\{ 3^{\lambda}, 1 \right\}}{2^{\lambda d}}, \frac{M}{d+\lambda} \right\}, \tag{2.19}
\]
whence inequality (2.3) follows in view of (2.12) and (2.13), which completes the proof.
In order to handle a weight \(\rho\) whose power-type behaviours near the origin and at infinity are different, one can split \(f\) in the sum \(f = f_1 + f_2\), with \(f_1 := f\chi_{B_1^c}\) and \(f_2 := f\chi_{B_1^c}\). By

linearity, \(f_\varepsilon = (f_1)_\varepsilon + (f_2)_\varepsilon\); it is therefore enough to show that (2.3) holds for \(f_1\) and \(f_2\) separately. As concerns \(f_1\), since the latter and its mollifications are (eventually) supported in \(B_{3/2}\), one can modify \(\rho(x)\) so that it behaves like \(|x|^\lambda\) also in \(B_{3/2}^c\) and then apply the first part of the proof. Similarly, because \(f_2\) and its mollifications are eventually supported in \(B_{1/2}^c\), the validity of the analogue of (2.3) (and so of (2.4)) is now implied by the validity of (2.14) in the region \(|y| > 1/2\), which holds for all \(\lambda = \Lambda \in \mathbb{R}\) in view of (2.16). □

Remark 2.2. Note that the above assumption \(\lambda \in (-d, (p - 1)d)\) is necessary. Indeed, consider the following function:

\[
g(x) := \frac{\chi_{B_{1/2}}(x)}{|x|^{d_1} \log|x|} \quad \forall x \in \mathbb{R}^d \setminus \{0\}.
\]

It is apparent that \(g \notin L^{1}_{loc}(\mathbb{R}^d)\) and its mollification \(g_{\varepsilon}\) is equal to \(-\infty\) in a set of positive measure, for all \(\varepsilon > 0\). However, \(g\) belongs to \(L^{p}_{\lambda}(\mathbb{R}^d)\) for all \(\lambda \geq (p - 1)d\).

As concerns the bound from below over \(\lambda\), let

\[
h(x) := \chi_{B_1} |x|^{-\frac{\lambda}{p}} \in L^{p}_{\lambda}(\mathbb{R}^d).
\]

The mollification \(h_{\varepsilon}\) is strictly positive in a neighbourhood of the origin, for all \(\varepsilon > 0\). In particular, if \(\lambda \leq -d\) then \(h_{\varepsilon} \notin L^{p}_{\lambda}(\mathbb{R}^d)\), so that (2.3) (with \(f = h\)) cannot hold.

3. Fractional Laplacians and power-weighted \(L^p\) spaces

In this section we first discuss some elementary properties concerning the fractional Laplacian, and a similar related operator, applied to standard test functions. We then investigate

the precise functional setting where we shall prove or disprove the validity of the integration-by-parts formulas.

The proofs of the first two lemmas are omitted, since they follow e.g. by exploiting arguments similar to those used in [3, Lemma 2.1].

Lemma 3.1. The fractional Laplacian \((-\Delta)^s(\phi)(x)\) of any \(\phi \in \mathcal{D}(\mathbb{R}^d)\) is a regular function that decays (at least) like \(|x|^{-d - 2s}\) as \(|x| \to \infty\).

Lemma 3.2. Let \(p \in (1, \infty)\). For any \(\phi \in \mathcal{D}(\mathbb{R}^d)\) the function

\[
l_{p,s}(\phi)(x) := \int_{\mathbb{R}^d} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{d + ps}} \, dy \quad \forall x \in \mathbb{R}^d
\]

is continuous and decays (at least) like \(|x|^{-d - ps}\) as \(|x| \to \infty\).

In the special case \(p = 2\) we set \(l_s := l_{2,s}\).

Lemma 3.3. For any \(R > 0\), let \(\xi_R\) be the cut-off function

\[
\xi_R(x) := \xi \left(\frac{x}{R}\right) \quad \forall x \in \mathbb{R}^d,
\]

where \(\xi\) is a nonnegative, regular function such that \(\|\xi\|_{\infty} = 1\), \(\equiv 1\) in \(B_1\) and \(\equiv 0\) in \(B_2\). Then, \((-\Delta)^s(\xi_R)\) and \(l_{p,s}(\xi_R)\) enjoy the following scaling properties:

\[
(-\Delta)^s(\xi_R)(x) = R^{-2s} (-\Delta)^s(\xi)(x/R), \quad l_{p,s}(\xi_R)(x) = R^{-ps} l_{p,s}(\xi)(x/R) \quad \forall x \in \mathbb{R}^d. \quad (3.1)
\]
Proof. We only show the result for \( l_{p,s} \), since the proof for \((-\Delta)^s\) is analogous. Upon letting \( \tilde{y} = y/R \), one has
\[
l_{p,s}(\xi_R)(x) = \int_{\mathbb{R}^d} \frac{|\xi_R(x) - \xi_R(y)|^p}{|x - y|^{d+ps}} \, dy = R^{-ps} \int_{\mathbb{R}^d} \frac{|\xi(x/R) - \xi(\tilde{y})|^p}{|x/R - \tilde{y}|^{d+ps}} \, d\tilde{y} = R^{-ps} l_{p,s}(\xi)(x/R)
\]
for all \( x \in \mathbb{R}^d \). □

The following lemma displays some consequences of the above properties.

Lemma 3.4. Let \( \xi \) and \( \xi_R \) be as in Lemma 3.3. Let \( q \in [1,\infty) \) and \( \gamma \in [0,d+2qs] \). Then the norms
\[
\| |x|^\gamma (-\Delta)^s(\xi) \|_{q,-\gamma} \quad \text{and} \quad \| |x|^\gamma l_s(\xi) \|_{q,-\gamma}
\]
are finite. If in addition \( \gamma \in [0,d+2s] \), then also the norms
\[
\| |x|^\gamma (-\Delta)^s(\xi) \|_{\infty} \quad \text{and} \quad \| |x|^\gamma l_s(\xi) \|_{\infty}
\]
are finite. Moreover, the following identities hold:
\[
\| |x|^\gamma (-\Delta)^s(\xi_R) \|_{q,-\gamma} = \frac{\| |x|^\gamma (-\Delta)^s(\xi) \|_{q,-\gamma}}{R^{2s-\gamma - \frac{d}{q}}}, \tag{3.4}
\]
\[
\| |x|^\gamma (-\Delta)^s(\xi_R) \|_{\infty} = \frac{\| |x|^\gamma (-\Delta)^s(\xi) \|_{\infty}}{R^{2s-\gamma - \frac{d}{q}}}, \tag{3.5}
\]
\[
\| |x|^\gamma l_s(\xi_R) \|_{q,-\gamma} = \frac{\| |x|^\gamma l_s(\xi) \|_{q,-\gamma}}{R^{2s-\gamma - \frac{d}{q}}}, \tag{3.6}
\]
\[
\| |x|^\gamma l_s(\xi_R) \|_{\infty} = \frac{\| |x|^\gamma l_s(\xi) \|_{\infty}}{R^{2s-\gamma - \frac{d}{q}}} \tag{3.7}
\]

Proof. The finiteness of (3.2) and (3.3) is ensured by the decay properties of \((-\Delta)^s(\xi)(x)\) and \( l_s(\xi)(x) \) recalled by Lemmas 3.1 and 3.2. Identities (3.4)–(3.7) just follow from (3.1). □

For a function \( f \) belonging to \( L^1_{\text{loc}}(\mathbb{R}^d) \cap L^{1,d-2s}(B_1^c) \), a property that any element of \( L^p_{\text{loc}}(\mathbb{R}^d) \) (let \( p \in (1,\infty) \)) enjoys provided \( p \) satisfies (2.1) with
\[
\lambda < (p-1)d \quad \text{and} \quad \Lambda > -d - 2ps,
\]
the action
\[
\phi \mapsto \int_{\mathbb{R}^d} f(x) (-\Delta)^s(\phi)(x) \, dx \quad \forall \phi \in D(\mathbb{R}^d) \tag{3.8}
\]
is indeed an element of \( D'(\mathbb{R}^d) \). This is an immediate consequence of the fact that the notion of convergence of a sequence \( \{\phi_n\} \subset D(\mathbb{R}^d) \) to \( \phi \) in \( D(\mathbb{R}^d) \) implies, in particular, the pointwise convergence of \( \{(-\Delta)^s(\phi_n)\} \) to \((-\Delta)^s(\phi)\) and the validity of the bound
\[
|(-\Delta)^s(\phi_n)(x)| \leq K (1 + |x|)^{-d-2s} \quad \forall x \in \mathbb{R}^d
\]
for a suitable positive constant \( K \) independent of \( n \) (recall Lemma 3.1). Since \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \cap L^{1,d-2s}(B_1^c) \), we can pass to the limit in (3.8) (with \( \phi = \phi_n \)) as \( n \to \infty \) by dominated convergence.

In the next lemma we show that, for regular functions having suitable integrability properties at infinity, the distributional fractional Laplacian and the classical one do coincide.
Lemma 3.5. Let \( v \in C^\infty(\mathbb{R}^d) \cap L^p(\Lambda^+ \mathbb{R}^n) \), with \( p \in (1, \infty) \) and \( \Lambda \geq -d - ps \). Then the classical fractional Laplacian of \( v \), defined by

\[
(-\Delta)^s(v)(x) := C_{d,s} \text{p.v.} \int_{\mathbb{R}^d} \frac{v(x) - v(y)}{|x - y|^{d+2s}} \, dy \quad \forall x \in \mathbb{R}^d,
\]

is a continuous function which coincides with its distributional fractional Laplacian, in the sense that

\[
\int_{\mathbb{R}^d} v(x) (-\Delta)^s(\phi)(x) \, dx = \int_{\mathbb{R}^d} (-\Delta)^s(v)(x) \phi(x) \, dx \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d).
\]

Proof. To begin with, let us prove that formula (3.9) provides us with a locally bounded function of \( x \). To this end, fix \( R > 0 \) and let \( x \) vary in \( B_R \). It is direct to check that

\[ x \mapsto \text{p.v.} \int_{B_{2R}} \frac{v(x) - v(y)}{|x - y|^{d+2s}} \, dy \quad \forall x \in B_R \]

is bounded in modulus by a constant (depending on \( R \)) times \( \| \nabla^2 v \|_{L^\infty(B_{2R})} \). Moreover, still for \( x \) varying in \( B_R \), we have:

\[
2^{-d-2s} \int_{B_{2R}} \frac{|v(x) - v(y)|}{|x - y|^{d+2s}} \, dy \leq \| v \|_{L^\infty(B_R)} \int_{B_{2R}^c} |y|^{-d-2s} \, dy + \| v \|_{L^p(\Lambda^+ \mathbb{R}^n)} \left( \int_{B_{2R}^c} |y|^{-s(d+2s)+\Lambda} \, dy \right)^{\frac{1}{p'}},
\]

where \( p' := \frac{1}{p} \). Note that the r.h.s. is finite since \( v \in C^\infty(\mathbb{R}^d) \cap L^p(\Lambda^+ \mathbb{R}^n) \) with \( \Lambda \geq -d - ps \). We have therefore proved that \( (-\Delta)^s(v) \) is locally bounded. Continuity follows by similar arguments that we omit.

Now we must prove that \( (-\Delta)^s(v) \) is in fact the distributional fractional Laplacian of \( v \), namely the validity of (3.10). Let us first consider the truncated function \( \xi_{Rv} \in \mathcal{D}(\mathbb{R}^d) \) (with \( \xi_R \) as in Lemma 3.3 and observe that, for the latter, the identity

\[
\int_{\mathbb{R}^d} \xi_{Rv}(x) v(x) (-\Delta)^s(\phi)(x) \, dx = \int_{\mathbb{R}^d} (-\Delta)^s(\xi_{Rv})(x) \phi(x) \, dx \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d)
\]

does hold (recall the related discussion in the Introduction). Using the product formula

\[
(-\Delta)^s(\xi_{Rv})(x) = \xi_{Rv}(x) (-\Delta)^s(v)(x) + (\xi_{Rv})(x) (-\Delta)^s(v)(x)
\]

\[
+ 2 C_{d,s} \int_{\mathbb{R}^d} \frac{(v(x) - v(y))(\xi_{Rv}(x) - \xi_{Rv}(y))}{|x - y|^{d+2s}} \, dy
\]

and plugging it in (3.11), we get:

\[
\int_{\mathbb{R}^d} \xi_{Rv}(x) v(x) (-\Delta)^s(\phi)(x) \, dx \\
= \int_{\mathbb{R}^d} \phi(x) \xi_{Rv}(x) (-\Delta)^s(v)(x) \, dx + \int_{\mathbb{R}^d} \phi(x) v(x) (-\Delta)^s(\xi_{Rv})(x) \, dx \\
+ 2 C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(x) \frac{(v(x) - v(y))(\xi_{Rv}(x) - \xi_{Rv}(y))}{|x - y|^{d+2s}} \, dx \, dy.
\]

By letting \( R \to \infty \) we can pass to the limit safely in the l.h.s. and in the first term of the r.h.s. of (3.13), since \( v \) is locally regular and integrable at infinity with respect to the weight.
\[ |x|^{-d-2s} \text{, while } (-\Delta)^s(v) \text{ is locally bounded as shown above. The second term vanishes: indeed, by taking advantage of (3.5) (with } \gamma = 0 \text{, we obtain:}
\]
\[ \int_{\mathbb{R}^d} |\phi(x) v(x) (-\Delta)^s(\xi_R(x))| \, dx \leq R^{-2s} \|(-\Delta)^s(\xi)\|_\infty \int_{\mathbb{R}^d} |\phi(x) v(x)| \, dx . \]

In order to handle the last term in the r.h.s. of (3.13), we have to work a bit more. First of all, let us prove that also \( t_{p,s}(v)(x) \) is locally bounded (actually continuous). Indeed, for all \( R > 0 \) the function
\[ x \mapsto \int_{B_{2R}} |v(x) - v(y)|^p \frac{dy}{|x-y|^{d+ps}} \quad \forall x \in B_R \quad (3.14) \]
is bounded in modulus by a constant (depending on \( R \)) times \( \|\nabla v\|_{L^\infty(B_{2R})} \). Moreover,
\[ 2^{-[d-1+p(s+1)]} \int_{B_{2R}^c} |v(x) - v(y)|^p \frac{dy}{|x-y|^{d+ps}} \leq \|v\|_{L^\infty(B_R)} \int_{B_{2R}^c} |y|^{-d-ps} \, dy + \|v\|_{L^p_{\rho,-d-ps}(B_{2R})} , \]
and the r.h.s. is finite thanks to the assumption \( \Lambda \geq -d-ps \). By applying Hölder’s inequality w.r.t. \( dy \) and using Lemma 3.3, it is not difficult to infer the estimate
\[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(x) \frac{|v(x) - v(y)|}{|x-y|^{d+2s}} \, dx dy \leq \frac{\|t_{p',s}(\xi)\|_{L^\infty}}{R^s} \int_{\mathbb{R}^d} \phi(x) \|t_{s}(v(x))\|_{L^\infty} \, dx . \]
By letting \( R \to \infty \) we then deduce that also the last term in the r.h.s. of (3.13) vanishes, so that (3.10) is finally proved. \( \square \)

Although most of our results will also hold in the case of positive \( \lambda \) and \( \Lambda \), from here on we shall mainly focus on negative powers (i.e. on weights \( \rho \) as in Definition 1.1), since we aim at considering weights that can be singular near the origin (\( \lambda \) negative), and the matter of the validity of (1.2) becomes less and less trivial the faster \( \rho(x) \) decays as \( |x| \to \infty \) (\( \Lambda \) negative).

The following proposition shows some useful properties of the space \( X_{p,s,\rho} \).

\textbf{Proposition 3.6.} Let \( p, p', \rho, \rho' \) and \( X_{p,s,\rho} \) be as in Definition 1.1. Then:
\begin{enumerate}[(a)]  
\item \( \mathcal{D}(\mathbb{R}^d) \subset X_{p,s,\rho} ; \)
\item \( X_{p,s,\rho} \) endowed with the norm \( \|v\|_{X_{p,s,\rho}} := \left( \|v\|_{p,p}^2 + \|(-\Delta)^s(v)\|_{p',\rho'}^2 \right)^{1/2} \quad \forall v \in X_{p,s,\rho} \) is a reflexive Banach space (Hilbert if \( p = 2 \)) ;
\item the subspace \( C^\infty(\mathbb{R}^d) \cap X_{p,s,\rho} \) is dense in \( X_{p,s,\rho} \) ;
\item the map \( B: \times X_{p,s,\rho} \to \mathbb{R} \), defined by \( B(v,w) := \int_{\mathbb{R}^d} (-\Delta)^s(v)(x) w(x) \, dx \quad \forall v, w \in X_{p,s,\rho} , \)
is a continuous bilinear form on \( X_{p,s,\rho} \).
\end{enumerate}

\textbf{Proof.} In order to prove (a) it is enough to check that, for any \( \phi \in \mathcal{D}(\mathbb{R}^d) \), we have \( \phi \in L^{p'}_{\rho'}(\mathbb{R}^d) \) and \( (-\Delta)^s(\phi) \in L^p_{\rho}(\mathbb{R}^d) \). This is straightforward: \( \rho \) is locally integrable since \( \gamma_0 \in [0,d) \) and \( (-\Delta)^s(\phi)(x) \) is a regular function decaying at least like \( |x|^{-d-2s} \) as \( |x| \to \infty \) (Lemma 3.1), which in particular implies that it belongs to \( L^{p'}_{\rho'}(\mathbb{R}^d) \) since \( \gamma \leq d + ps \).

As for (b), let us take a Cauchy sequence \( \{v_n\} \subset X_{p,s,\rho} \). By the definition of \( \|\cdot\|_{X_{p,s,\rho}} \) and by the completeness of \( L^p_{\rho}(\mathbb{R}^d) \) and \( L^{p'}_{\rho'}(\mathbb{R}^d) \), there exist \( v \in L^p_{\rho}(\mathbb{R}^d) \) and \( \mathcal{V} \in L^{p'}_{\rho'}(\mathbb{R}^d) \) such
that \(v_n \to v\) in \(L^p_{\rho}(\mathbb{R}^d)\) and \((-\Delta)^s(v_n) \to \mathcal{V}\) in \(L^{p'}_{\rho}(\mathbb{R}^d)\) as \(n \to \infty\), respectively. Showing the completeness of \(X_{p,s,\rho}\) is therefore equivalent to showing that \(\mathcal{V} = (-\Delta)^s(v)\), which holds provided we can pass to the limit in the identity

\[
\int_{\mathbb{R}^d} v_n(x)(-\Delta)^s(\phi)(x) \, dx = \int_{\mathbb{R}^d} (-\Delta)^s(v_n)(x) \, \phi(x) \, dx \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d).
\]

This is indeed the case because \((-\Delta)^s(\phi) \in L^p_{\rho}(\mathbb{R}^d)\) reads \(\rho^{-1}(-\Delta)^s(\phi) \in L^p_{\rho}(\mathbb{R}^d)\), and the same is true for \((-\Delta)^s(v_n)\). The fact that \(X_{s,\rho}\) is Hilbert just follows upon defining the scalar product

\[
\langle v, w \rangle_{X_{s,\rho}} := \int_{\mathbb{R}^d} v(x) \, w(x) \, \rho(x) \, dx + \int_{\mathbb{R}^d} (-\Delta)^s(v)(x) \, (-\Delta)^s(w)(x) \, [\rho(x)]^{-1} \, dx \quad \forall v, w \in X_{s,\rho}.
\]

In general \(X_{p,s,\rho}\) is reflexive because so are \(L^p_{\rho}(\mathbb{R}^d)\) and \(L^{p'}_{\rho}(\mathbb{R}^d)\).

Now let us deal with (c). We shall exploit the key result provided by Theorem 2.1. Thanks to the latter, given any \(v \in X_{p,s,\rho}\) its mollification \(v_\varepsilon\) belongs to \(C^\infty(\mathbb{R}^d) \cap L^p_{\rho}(\mathbb{R}^d)\) and converges to \(v\) in \(L^p_{\rho}(\mathbb{R}^d)\) as \(\varepsilon \to 0\). We claim that the fractional Laplacian of \(v_\varepsilon\), which is well defined both in the classical and in the distributional sense in view of Lemma 3.5, is in fact the mollification of \((-\Delta)^s(v)\), that is

\[
(-\Delta)^s(v_\varepsilon) = [-(-\Delta)^s(v)]_\varepsilon.
\] 

(3.15)

Indeed, for any \(\phi \in \mathcal{D}(\mathbb{R}^d)\) the following identities hold:

\[
\int_{\mathbb{R}^d} (-\Delta)^s(\phi)(x) \, v_\varepsilon(x) \, dx = \int_{\mathbb{R}^d} (-\Delta)^s(\phi)(x) \left( \int_{\mathbb{R}^d} \eta_\varepsilon(x-y) \, v(y) \, dy \right) \, dx \\
= \int_{\mathbb{R}^d} \phi(x) \left( \int_{\mathbb{R}^d} \eta_\varepsilon(x-y) \, (-\Delta)^s(v)(y) \, dy \right) \, dx \\
= \int_{\mathbb{R}^d} \phi(x) \, [(-\Delta)^s(v)]_\varepsilon(x) \, dx.
\]

The above exchanges of order of integration are justified by Fubini’s Theorem since

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| (-\Delta)^s(\phi)(x) \eta_\varepsilon(x-y) \, v(y) \right| \, dx \, dy + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \phi(x) \eta_\varepsilon(x-y) \, (-\Delta)^s(v)(y) \right| \, dx \, dy < \infty
\]

as a consequence of the fact that \(|v|_\varepsilon, \phi \in L^p_{\rho}(\mathbb{R}^d)\) and \((-\Delta)^s(\phi), |(-\Delta)^s(v)|_\varepsilon \in L^{p'}_{\rho}(\mathbb{R}^d)\).

Having established (3.15) we can use again Theorem 2.1 to deduce that \(v_\varepsilon \in C^\infty(\mathbb{R}^d) \cap X_{p,s,\rho}\) and

\[
\lim_{\varepsilon \to 0} \|v_\varepsilon - v\|_{X_{p,s,\rho}} = \lim_{\varepsilon \to 0} \left( \|v_\varepsilon - v\|^2_{p,\rho} + \|(-\Delta)^s(v)\|_{p',\rho}^2 \right)^{\frac{1}{2}} = 0,
\]

which does prove (c).

The only nontrivial point of (d) is the continuity of \(\mathcal{B}\), which follows as a direct application of Hölder’s inequality w.r.t. the measure \(\rho(x)\, dx:\)

\[
|\mathcal{B}(v, w)| \leq \left( \int_{\mathbb{R}^d} |(-\Delta)^s(v)(x)|^{p'} \rho(x) \, dx \right)^{\frac{1}{p'}} \left( \int_{\mathbb{R}^d} |w(x)|^p \rho(x) \, dx \right)^{\frac{1}{p}} \leq \|v\|_{X_{p,s,\rho}} \|w\|_{X_{p,s,\rho}}.
\]

We finally establish some crucial integral estimates for functions in \(X_{p,s,\rho}\).
Lemma 3.7. Let either \( p = 2 \) and \( \gamma \in [0, d + 2s] \) or \( p \in (2, \infty) \) and \( \gamma \in [0, d + ps) \). Suppose that \( \rho \) satisfies (3.3) for some \( \gamma_0 \in [0, d) \) and for such a \( \gamma \). Let \( I_s \) be as in Lemma 3.2 and \( \xi, \xi_R \) be as in Lemma 3.5. Let \( v_1, v_2 \in C^\infty(\mathbb{R}^d) \cap X_{p,s,\rho}^c \). Then the integral

\[
I_R (v_i) := \int_{\mathbb{R}^d} \xi_R^2 (x) \left( \int_{\mathbb{R}^d} \frac{(v_1 (x) - v_1 (y))^2}{|x - y|^{d+2s}} \, dy \right) \, dx \quad i = 1, 2
\]

is finite for all \( R > 0 \). Moreover, the following estimates hold for all \( R \geq 1 \):

\[
\int_{\mathbb{R}^d} |v_1 (x) v_2 (x) \xi_R (x) (\Delta)^{\gamma} (\xi_R) (x)| \, dx \leq T (R, v_1, v_2),
\]

\[
\int_{\mathbb{R}^d} |v_1 (x) v_2 (y)| \left( \int_{\mathbb{R}^d} \frac{(\xi_R (x) - \xi_R (y))^2}{|x - y|^{d+2s}} \, dx \right) \, dy \leq T (R, v_1, v_2),
\]

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\xi_R (x) v_1 (x) (v_2 (x) - v_2 (y)) (\xi_R (x) - \xi_R (y))| \frac{1}{|x - y|^{d+2s}} \, dx \, dy \leq \left[ T (R, v_1, v_1) \right]^{\frac{1}{2}} \left[ I_R (v_2) \right]^{\frac{1}{2}},
\]

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\xi_R (x) v_1 (y) (v_2 (x) - v_2 (y)) (\xi_R (x) - \xi_R (y))| \frac{1}{|x - y|^{d+2s}} \, dx \, dy \leq \left[ T (R, v_1, v_1) \right]^{\frac{1}{2}} \left[ I_R (v_2) \right]^{\frac{1}{2}},
\]

where

\[
T (R, v_1, v_2) := K \left( R^{-2s} \alpha (R) \frac{2d + (p-2)d}{p} \| v_1 \|_{p,\rho} \| v_2 \|_{p,\rho} + R^{-2d + p(2s-d) - 2s} \frac{2d + (p-2)d - 2s}{p} \| v_1 \|_{L^p_B (\mathcal{B}_R \cap \mathcal{B}_R \cap \mathcal{B}_R)} \| v_2 \|_{L^p_B (\mathcal{B}_R \cap \mathcal{B}_R \cap \mathcal{B}_R)} \right),
\]

\( \alpha : [1, \infty) \mapsto [1, \infty) \) being any monotone function of \( R \) with \( \lim_{R \to \infty} \alpha (R) = \infty \) and \( K \) being a suitable positive constant that depends only on \( d, s, p, \gamma_0, \gamma, c, \xi \).

Proof. In order to show that \( I_R (v_i) \) is finite for all \( R > 0 \) note that, since \( \xi_R \) is supported in \( B_{2R} \), it is enough to prove that the function

\[
x \mapsto \int_{\mathbb{R}^d} \frac{(v_1 (x) - v_1 (y))^2}{|x - y|^{d+2s}} \, dy
\]

stays bounded as \( x \) varies in \( B_{2R} \). To this end one can proceed exactly as we did in establishing the local boundedness of (3.14). We point out that here it is necessary to ask \( \gamma < d + ps \) in the case \( p > 2 \).

Now let us deal with estimates (3.17)–(3.20). Since \( \| \xi \|_{\infty} = 1 \), by using (3.4) with \( q = \frac{p}{p-2} \), with \( \gamma = 0 \), the left-hand inequalities in (1.3), and exploiting a three-point Hölder’s inequality, we find:

\[
\int_{\mathbb{R}^d} |v_1 (x) v_2 (x) \xi_R (x) (\Delta)^{\gamma} (\xi_R) (x)| \, dx \leq R^{-2s} \| v_1 \|_{p,\rho} \| v_2 \|_{p,\rho} \| (\Delta)^{\gamma} (\xi_R) \|_{\infty} \left( \rho^{-2/p} \right) \| v_1 \|_{L^p_B (\mathcal{B}_R \cap \mathcal{B}_R \cap \mathcal{B}_R)} \| v_2 \|_{L^p_B (\mathcal{B}_R \cap \mathcal{B}_R \cap \mathcal{B}_R)} \| x \|_{\gamma (\Delta)^{\gamma} (\xi_R \cap \mathcal{B}_R \cap \mathcal{B}_R)},
\]

where for \( p = 2 \) it is understood that \( \frac{p}{p-2} = \infty \). In view of Lemma 3.4 and of the left-hand inequalities in (1.3), it is apparent that (3.22) implies (3.17). Similarly (use (3.5) and (3.7).
instead), we have:
\[
\int_{\mathbb{R}^d} |v_1(y)v_2(y)| l_s(\xi R)(y) \, dy \leq R^{-2s} \|v_1\|_{p,s,\rho} \|v_2\|_{p,s,\rho} \|l_s(\xi)\|_{\infty} \left\|\frac{\rho^{-2/p}}{L^p_{\rho}(B_{\alpha(R)})}\right\|_{\frac{2d+\rho(2d-\alpha)-2s}{p}} - \gamma
\]

and
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \frac{\xi R(x)v_1(\xi R(x) - \xi R(y))}{|x-y|^{d+2s}} \right| \, dx \, dy \leq \left( \frac{\|v_1\|_{p,s,\rho} \|l_s(\xi)\|_{\infty} \left\|\frac{\rho^{-2/p}}{L^p_{\rho}(B_{\alpha(R)})}\right\|_{\frac{2d+\rho(2d-\alpha)-2s}{p}}}{R^{2s}} + \frac{\|v_1\|_{p,s,\rho} \|l_s(\xi)\|_{\infty} \left\|\frac{\rho^{-2/p}}{L^p_{\rho}(B_{\alpha(R)})}\right\|_{\frac{2d+\rho(2d-\alpha)-2s}{p}}}{R^{2s}} \right)^{\frac{1}{2}} [I_{R}(v_2)] \frac{1}{2},
\]

whence (3.18) and (3.19). The proof of (3.20) is completely analogous. \hfill \Box

4. Subcritical-critical powers

This section is devoted to the proof of our main results, namely the validity of (1.2) in $X_{p,s,\rho}$ and a consequent self-adjointness property for the operator $\rho^{-1}(-\Delta)^s$ in $L^p_{\rho}(\mathbb{R}^d)$, under the additional assumption that $\rho$ satisfies (1.3) for some $\gamma$ smaller than or equal to the critical value $d - \frac{\alpha}{2}(d-2s)$. Note that such assumption is restrictive only in the case $d > 2s$. To our purposes, we first need a preliminary continuous-embedding result.

Lemma 4.1. Let either $d \leq 2s$ and $p \in [2, \infty)$ or $d > 2s$ and $p \in [2, 2d/(d-2s)]$. Suppose that $\rho$ satisfies (1.3) for some $\gamma_0 \in [0, d)$ and $\gamma \in [0, d - \frac{\alpha}{2}(d-2s)]$. Then $X_{p,s,\rho}$ is continuously embedded in $H^s(\mathbb{R}^d)$ and the following inequality holds:
\[
\|v\|_{H^s(\mathbb{R}^d)} \leq \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(x) - v(y))^2}{|x-y|^{d+2s}} \, dx \, dy \leq \int_{\mathbb{R}^d} v(x) (-\Delta)^s(v)(x) \, dx \quad \forall v \in X_{p,s,\rho}.
\]

Proof. We shall first prove (4.1) for the elements of a sequence $\{v_n\} \subset C_\infty(\mathbb{R}^d) \cap X_{p,s,\rho}$ converging to $v$ in $X_{p,s,\rho}$, which exists in view of Proposition 3.6, and then pass to the limit as $n \to \infty$. To this end, take a family of cut-off functions $\{\xi R\}_{R \geq 1}$ as in Lemma 3.3. Since $\xi R v_n$ belongs to $C_\infty(\mathbb{R}^d)$, identity (1.2) with $v = v = \xi R v_n$ holds, that is
\[
\|\xi R v_n\|_{H^s(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \xi^2_R(x) v_n(x) (-\Delta)^s(v_n)(x) \, dx + \int_{\mathbb{R}^d} \xi R(x) (-\Delta)^s(\xi R(x))(v_n^2)(x) \, dx
\]
\[
+ 2C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi R(x) v_n(x) \left( \frac{v_n(x) - v_n(y)}{|x-y|^{d+2s}} \right) \xi R(y) \, dx \, dy
\]
(recall the product formula (3.12)). The l.h.s. of (4.2) reads
\[
2 \|\xi R v_n\|_{H^s(\mathbb{R}^d)}^2 = C_{d,s} \int_{\mathbb{R}^d} \xi^2_R(x) \left( \int_{\mathbb{R}^d} \frac{(v_n(x) - v_n(y))^2}{|x-y|^{d+2s}} \, dy \right) \, dx
\]
\[
+ C_{d,s} \int_{\mathbb{R}^d} v_n^2(y) \left( \int_{\mathbb{R}^d} \frac{(\xi R(x) - \xi R(y))^2}{|x-y|^{d+2s}} \, dx \right) \, dy
\]
\[
+ 2C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi R(x) v_n(y) \left( \frac{v_n(x) - v_n(y)}{|x-y|^{d+2s}} \right) \xi R(x) \xi R(y) \, dx \, dy.
\]
By exploiting inequality (3.19) from Lemma 3.7 with \( v_1 = v_2 = v_n \), and taking advantage of Young’s inequality, we estimate the third term in the r.h.s. of (4.2) as follows:

\[
2 C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi_R(x)v_n(x) \frac{(v_n(x) - v_n(y)) (\xi_R(x) - \xi_R(y))}{|x-y|^{d+2s}} \, dx \, dy \leq \delta C_{d,s} I_R(v_n) + \delta^{-1} C_{d,s} \mathcal{T}(R, v_n, v_n)
\]  

for all \( \delta > 0 \), where \( \mathcal{T} \) is defined by (3.21). The same can be done for the third term in the r.h.s. of (4.3) upon using (3.20):

\[
2 C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi_R(x)v_n(y) \frac{(v_n(x) - v_n(y)) (\xi_R(x) - \xi_R(y))}{|x-y|^{d+2s}} \, dx \, dy \leq \delta C_{d,s} I_R(v_n) + \delta^{-1} C_{d,s} \mathcal{T}(R, v_n, v_n).
\]

Thanks to (3.17) and (3.18) with \( v_1 = v_2 = v_n \) we can also estimate the second term in the r.h.s. of (4.2) and the second term in the r.h.s. of (4.3):

\[
\int_{\mathbb{R}^d} |\xi_R(x) (-\Delta)^s (\xi_R(x)) v_n^2(x)| \, dx \leq \mathcal{T}(R, v_n, v_n),
\]

\[
C_{d,s} \int_{\mathbb{R}^d} v_n^2(y) \int_{\mathbb{R}^d} \frac{(|\xi_R(x) - \xi_R(y)|)^2}{|x-y|^{d+2s}} \, dx \, dy \leq C_{d,s} \mathcal{T}(R, v_n, v_n).
\]

By combining (4.2), (4.3) and (4.4) we thus infer that

\[
\left| \int_{\mathbb{R}^d} \xi_R(x)v_n(x) (-\Delta)^s (\xi_R v_n)(x) \, dx - \int_{\mathbb{R}^d} \xi_R^2(x) v_n(x) (-\Delta)^s (v_n)(x) \, dx \right| \leq \delta C_{d,s} I_R(v_n) + (\delta^{-1} C_{d,s} + 1) \mathcal{T}(R, v_n, v_n),
\]

where \( I_R \) is defined by (3.16). Similarly, by gathering (4.3), (4.5) and (4.7) we get

\[
2 \|\xi_R v_n\|_{H^s(\mathbb{R}^d)}^2 - C_{d,s} I_R(v_n) \leq \delta C_{d,s} I_R(v_n) + (\delta^{-1} + 1) C_{d,s} \mathcal{T}(R, v_n, v_n).
\]

Hence, (4.2), (4.3), (4.8) and (4.9) yield

\[
C_{d,s} I_R(v_n) \leq 2 \int_{\mathbb{R}^d} \xi_R^2(x) v_n(x) (-\Delta)^s (v_n)(x) \, dx + 3 \delta C_{d,s} I_R(v_n) + (3 \delta^{-1} C_{d,s} + C_{d,s} + 2) \mathcal{T}(R, v_n, v_n),
\]

that is

\[
\frac{1 - 3\delta}{2} C_{d,s} I_R(v_n) \leq \int_{\mathbb{R}^d} \xi_R^2(x) v_n(x) (-\Delta)^s (v_n)(x) \, dx + \frac{3 \delta^{-1} C_{d,s} + C_{d,s} + 2}{2} \mathcal{T}(R, v_n, v_n).
\]

It is straightforward to verify that, in view of the hypotheses on \( \gamma \), there holds

\[
\lim_{R \to \infty} \mathcal{T}(R, v_n, v_n) = 0
\]

provided \( \alpha(R) = o(R) \) as \( R \to \infty \). Hence, for any fixed \( \delta \in (0, 1/3) \), we can pass to the limit in (4.10) as \( R \to \infty \) to get, by means e.g. of Fatou’s Lemma and dominated convergence,

\[
\frac{1 - 3\delta}{2} C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v_n(x) - v_n(y))^2}{|x-y|^{d+2s}} \, dx \, dy \leq \int_{\mathbb{R}^d} v_n(x) (-\Delta)^s (v_n)(x) \, dx.
\]

Note that the r.h.s. of (4.11) is finite thanks to Proposition 3.6 (4). By letting \( \delta \to 0 \) we therefore end up with (4.1) for \( v = v_n \); the fact that \( v_n \) belongs to \( H^s(\mathbb{R}^d) \) is a consequence.
of the boundedness of the family $\{\xi_{Rv_n}\}_{R \geq 1}$ in $\dot{H}^s(\mathbb{R}^d)$ (we leave it to the reader to check this assertion). A further passage to the limit as $n \to \infty$ yields the result. □

We are now in position to prove the validity of the integration-by-parts formulas in the case where $\gamma$ is at most critical.

Proof of Theorem 1.2 (case $\gamma \leq d - \frac{d}{2} (d - 2s)$). We shall proceed along the lines of proof of Lemma 1.1, i.e. we start from the validity of the identity
\[
\langle \xi_{Rv_n}, \xi_{Rw_n} \rangle_{\dot{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \xi_R(x) v_n(x) (-\Delta)^s (\xi_R w_n)(x) \, dx,
\]
where $\{v_n\}, \{w_n\} \subset C^\infty(\mathbb{R}^d) \cap X_{p,s,p}$ are sequences converging to $v$ and $w$, respectively, in $X_{p,s,p}$. The analogues of (4.12) and (4.13) read
\[
\langle \xi_{Rv_n}, \xi_{Rw_n} \rangle_{\dot{H}^s(\mathbb{R}^d)}
\]
\[
= \int_{\mathbb{R}^d} \xi_R(x) v_n(x) (-\Delta)^s (\xi_R w_n)(x) \, dx
\]
\[
= \int_{\mathbb{R}^d} \xi_R^2(x) v_n(x) (-\Delta)^s (w_n)(x) \, dx + \int_{\mathbb{R}^d} \xi_R(x) (-\Delta)^s (\xi_R(x) v_n(x) w_n(x)) \, dx
\]
\[
+ 2 C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi_R(x) v_n(x) \frac{(w_n(x) - w_n(y)) (\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} \, dx \, dy.
\]
and
\[
2 \langle \xi_{Rv_n}, \xi_{Rw_n} \rangle_{\dot{H}^s(\mathbb{R}^d)} = C_{d,s} \int_{\mathbb{R}^d} \xi_R^2(x) \left( \int_{\mathbb{R}^d} \frac{(v_n(x) - v_n(y)) (w_n(x) - w_n(y))}{|x - y|^{d+2s}} \, dy \right) \, dx
\]
\[
+ C_{d,s} \int_{\mathbb{R}^d} v_n(y) w_n(y) \left( \int_{\mathbb{R}^d} \frac{(\xi_R(x) - \xi_R(y))^2}{|x - y|^{d+2s}} \, dx \right) \, dy
\]
\[
+ C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi_R(x) v_n(y) \frac{(w_n(x) - w_n(y)) (\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} \, dx \, dy
\]
\[
+ C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi_R(x) w_n(y) \frac{(v_n(x) - v_n(y)) (\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} \, dx \, dy.
\]
By exploiting Lemma 1.1 together with the trivial inequality
\[
C_{d,s} I_R(v) \leq 2 \|v\|^2_{\dot{H}^s(\mathbb{R}^d)} \quad \forall v \in X_{p,s,p},
\]
we obtain:
\[
2 C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \xi_R(x) v_n(x) \frac{(w_n(x) - w_n(y)) (\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} \right| \, dx \, dy
\]
\[
\leq 2 \sqrt{2} C_{d,s}^2 \|w_n\|_{\dot{H}^s(\mathbb{R}^d)} \left[ I(R, v_n, v_n) \right]^{\frac{1}{2}},
\]
\[
C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \xi_R(x) v_n(y) \frac{(w_n(x) - w_n(y)) (\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} \right| \, dx \, dy
\]
\[
\leq \sqrt{2} C_{d,s}^2 \|w_n\|_{\dot{H}^s(\mathbb{R}^d)} \left[ I(R, v_n, v_n) \right]^{\frac{1}{2}},
\]
Finally, we let \( n \to \infty \) and pass to the limit in (4.21) to get (1.2): Proposition 3.6 and Lemma 4.1 ensure that both the left- and the right-hand side are continuous bilinear forms on \( X_{p,s,\rho} \).

By taking advantage of Theorem 1.2 with \( p = 2 \), we are then able to prove Theorem 1.3 under the additional constraint \( \gamma \leq 2s \).

**Proof of Theorem 1.3 (case \( \gamma \leq 2s \)).** It is direct to check that \( A \) acts from \( D(A) \) to \( L^2_{p}(\mathbb{R}^d) \), since \( A(f) \in L^2_{p}(\mathbb{R}^d) \) is equivalent to \( (-\Delta)^s(f) \in L^2_{p,\gamma}(\mathbb{R}^d) \), which is true by the definition of \( X_{s,\rho} \). The fact that \( A \) is densely defined just follows from the inclusion \( D(\mathbb{R}^d) \subset X_{s,\rho} \) (Proposition 3.6 and Lemma 3.1). The nonnegativity of \( A \) is a trivial consequence of Theorem 1.2 and the second line of (1.2) with \( v = w = f \).

In order to prove that \( A \) is a symmetric operator, note that the last line of (1.2) can be rewritten as

\[
\int_{\mathbb{R}^d} f(x) A(g)(x) \rho(x) dx = \int_{\mathbb{R}^d} A(f)(x) g(x) \rho(x) dx \quad \forall f, g \in D(A),
\]

which means \( D(A) \subset D(A^*) \) and \( A = A^* \) on \( D(A) \). Hence, proving that \( A \) is self-adjoint in \( L^2_{p}(\mathbb{R}^d) \) amounts to establishing the opposite inclusion \( D(A^*) \subset D(A) \). To this end we point out that, by the definition of \( D(A^*) \), one has that \( h \in D(A^*) \) if and only if \( h \in L^2_{p}(\mathbb{R}^d) \) and there exists a positive constant \( M_h \) such that

\[
\int_{\mathbb{R}^d} h(x) A(g)(x) \rho(x) dx \leq M_h \| g \|_{L^2_{p}} \quad \forall g \in D(A).
\]
As recalled above, \(\mathcal{D}(\mathbb{R}^d) \subset D(A)\): in particular (4.22) holds for all \(g = \phi \in \mathcal{D}(\mathbb{R}^d)\). Since \(\mathcal{D}(\mathbb{R}^d)\) is dense in \(L^2_{\rho}(\mathbb{R}^d)\), we can infer the existence of a unique \(E \in L^2_{\rho}(\mathbb{R}^d)\) such that

\[
\int_{\mathbb{R}^d} h(x) A(\phi)(x) \rho(x) \, dx = \int_{\mathbb{R}^d} h(x) (-\Delta)^s(\phi)(x) \, dx = \int_{\mathbb{R}^d} E(x) \phi(x) \rho(x) \, dx \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d). 
\]

(4.23)

The last identity in (4.23) implies \(\rho E = (-\Delta)^s(h)\), whence \(h \in X_{p,\rho} = D(A)\). We have therefore established the inclusion \(D(A^*) \subset D(A)\), and self-adjointness is proved.

Let us finally deal with the quadratic form \(Q\) associated with \(A\). Thanks to Theorem 1.2 we have that

\[
Q(v, v) = \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(x) - v(y))^2}{|x - y|^{d+2s}} \, dx dy
\]

for all \(v \in D(A)\). As it is well known (see e.g. [12] Section 1.2), the domain \(D(Q)\) of \(Q\) is precisely the closure of \(D(A)\) endowed with the norm

\[
\|v\|_Q := \sqrt{\|v\|^2_{L^2_{p,\rho}} + Q(v, v)} = \sqrt{\|v\|^2_{L^2_{p,\rho}} + \|v\|^2_{L^2_{\rho}(\mathbb{R}^d)}} \quad \forall v \in D(A).
\]

It is straightforward to check that such closure is nothing but \(L^2_{\rho}(\mathbb{R}^d) \cap H^s(\mathbb{R}^d)\), and that the quadratic form on \(D(Q) = L^2_{\rho}(\mathbb{R}^d) \cap H^s(\mathbb{R}^d)\) is still represented by (4.24).

By classical results (see e.g. [12] Sections 1.3, 1.4), proving that \(A\) generates a Markov semigroup is equivalent to proving that if \(v\) belongs to \(D(Q)\) then both \(v \vee 0\) and \(v \wedge 1\) belong to \(D(Q)\) and satisfy

\[
Q(v \vee 0, v \vee 0) \leq Q(v, v) \quad \text{and} \quad Q(v \wedge 1, v \wedge 1) \leq Q(v, v),
\]

which is true in view of the characterization of \(Q\) given above. The last assertions of the statement then follow from the general theory of symmetric Markov semigroups (see in particular [12] Theorems 1.4.1, 1.4.2).

\[\square\]

**Remark 4.2.** *A posteriori*, for \(p\) and \(p\) complying with the hypotheses of Theorem [12] under the additional constraint \(\gamma \leq d - \frac{d}{2}(d - 2s)\), the subspace \(\mathcal{D}(\mathbb{R}^d)\) is dense in \(X_{p,s,\rho}\).

Indeed, in view of Proposition [4.6] (1), it is enough to prove that for any \(\varphi \in C^\infty(\mathbb{R}^d) \cap X_{p,s,\rho}\), the cut-off family \(\{\xi_{R,R} \varphi\}_{R \geq 1}\) converges to \(\varphi\) in \(X_{p,s,\rho}\) as \(R \to \infty\). The only nontrivial point to cope with is the convergence of \(\{(\Delta)^s(\xi_{R,R} \varphi)\}\) to \((-\Delta)^s(\varphi)\) in \(L^p_{\rho}(\mathbb{R}^d)\), which can be tackled by arguing as in the proof of Lemma 3.7 (we omit the technical details).

5. Supercritical powers in the case \(d > 2s\)

In this section we deal with supercritical values of \(\gamma\), namely \(\gamma > d - \frac{d}{2}(d - 2s)\), for \(d > 2s\) and \(p \in [2, 2d/(d - 2s)]\). *A priori* it is not clear whether or not (1.2) continues to hold. Indeed, from the one hand the space \(X_{p,s,\rho}\) should get larger if one looks at \(\|v\|_{p,\rho}\), on the other hand it should get smaller if one looks at \(\|(-\Delta)^s(v)\|_{p'_{\rho},\rho'}\). In fact, in agreement with Theorems 1.2 [1.3] we shall see that the actual value that separates the region where (1.2) is valid from the one where it is not is \(\gamma = d\).

As mentioned in the Introduction, the assumption \(d > 2s\) ensures the existence of the following *Riesz kernel* (or *Green function*) of the fractional Laplacian in \(\mathbb{R}^d\):

\[
l_{d,s}(x) := \frac{R_{d,s}}{|x|^{d-2s}} \quad \forall x \in \mathbb{R}^d \setminus \{0\},
\]
where \( \kappa_{d,s} > 0 \) is an explicit positive constant (see e.g. [30 Chapter I, Section 1]). As it is well known, the Riesz kernel solves
\[
(-\Delta)^s (l_{d,s}) = \delta \quad \text{in} \quad \mathbb{R}^d,
\]
in the sense that
\[
\int_{\mathbb{R}^d} l_{d,s} (x) (-\Delta)^s (\phi)(x) \, dx = \phi(0) \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d).
\]
As a consequence, for any \( f \) having good integrability properties, there exists only one solution to the equation \((-\Delta)^s(F) = f \) in \( \mathbb{R}^d \), which is given by \( F = l_{d,s} * f \). Such a solution is referred to as the Riesz potential of \( f \).

By taking advantage of potential techniques, we shall then prove Theorem [12] for supercritical powers. Before, we need some technical lemmas.

**Lemma 5.1.** Let \( d > 2s \) and \( \phi \in \mathcal{D}(\mathbb{R}^d) \). Then the Riesz potential of \( \phi \), namely \((l_{d,s} * \phi)(x)\), is a regular function decaying (at least) like \(|x|^{-d+2s}\) as \(|x| \to \infty\). Moreover, \(|\nabla (l_{d,s} * \phi)(x)|\) decays (at least) like \(|x|^{-d-1+2s}\) as \(|x| \to \infty\).

**Proof.** These properties are rather standard, hence we omit the proof. As a reference, see e.g. [31 Lemma 2.2.13].

**Lemma 5.2.** Let \( d > 2s \), \( \phi \in \mathcal{D}(\mathbb{R}^d) \) and \( l_{p,s} \) be defined as in Lemma 5.1. Then, under the assumption \( p \in (1, d/(d-2s)) \), the function \( l_{p,s} (l_{d,s} * \phi)(x) \) is continuous and decays (at least) like \(|x|^{-p(d-s)}\) as \(|x| \to \infty\).

**Proof.** Again, we shall give a rigorous proof of the decay behaviour only, since continuity easily follows from the properties of \( l_{d,s} \) and \( \phi \). To this aim, let us set \( \Phi(x) := (l_{d,s} * \phi)(x) \) and observe that
\[
l_{p,s} (\Phi)(x) = \int_{B_{2|x|}^c} \frac{|\Phi(x) - \Phi(y)|^p}{|x-y|^{d+ps}} \, dy + \int_{B_{|x|/2}(x)} \frac{|\Phi(x) - \Phi(y)|^p}{|x-y|^{d+ps}} \, dy
+ \int_{B_{2|x|} \cap B_{|x|/2}(x) \cap B_{|x|/2}^c} \frac{|\Phi(x) - \Phi(y)|^p}{|x-y|^{d+ps}} \, dy + \int_{B_{|x|/2}} \frac{|\Phi(x) - \Phi(y)|^p}{|x-y|^{d+ps}} \, dy
=: F_1(x) + F_2(x) + F_3(x) + F_4(x).
\]
Since \(|x| \leq |y|/2\) as \( y \in B_{2|x|}^c \), by exploiting the decay behaviour of \( \Phi \) (Lemma 5.1) we easily deduce that \( F_1(x) \leq C |x|^{-p(d-s)} \) for \(|x| \) large (from here on we denote by \( C \) a generic positive constant independent of \( \phi \)). As for \( F_2 \), upon noting that \(|\Phi(x) - \Phi(y)|^p \leq |\nabla \Phi(z)|^p |x-y|^p \) for all \( y \in B_{|x|/2}(x) \), for some \( z \in B_{|x|/2}(x) \), and recalling the decay properties of \(|\nabla \Phi|\) (Lemma 5.1), we can infer that
\[
F_2(x) \leq \max_{z \in B_{|x|/2}(x)} \frac{1}{|x-y|^{d-p(1-s)}} \int_{B_{|x|/2}(x)} |\nabla \Phi(z)|^p \, dy \leq C |x|^{-p(d-s)}.
\]
The integral \( F_3 \) can be dealt with as \( F_1 \), with inessential modifications. Finally, we have:
\[
F_4(x) \leq \frac{C}{|x|^{d+ps}} \left( |x|^{d-p(d-2s)} + \int_{B_{|x|/2}} |\Phi(y)|^p \, dy \right)
\]
for \(|x| \) large, namely \( F_4(x) \leq C |x|^{-p(d-s)} \) since \( p < d/(d-2s) \).
Lemma 5.3. Let $d > 2s$ and $p \in [2, 2d/(d - 2s)]$. Suppose that $\rho$ satisfies \([1.3]\) for some $\gamma_0 \in [0, d)$ and $\gamma \in (d - \frac{2}{d}(d - 2s), d]$. Then $v = l_{d,s} \ast (-\Delta)^s(v)$ for all $v \in X_{p,s,\rho}$.

Proof. Given $v \in X_{p,s,\rho}$ and any test function $\phi \in D(\mathbb{R}^d)$, let $\Phi = l_{d,s} \ast \phi$. For all $R \geq 1$, it is plain that $\xi_R \Phi$ is also a test function. Hence, by the definition of $(-\Delta)^s(v)$, in view of the product formula \([3.12]\) (with $v = \Phi$ there) and the fact that $(-\Delta)^s(\Phi) = \phi$, there holds

$$
\int_{\mathbb{R}^d} (-\Delta)^s(v) \xi_R(x) \Phi(x) \, dx = \int_{\mathbb{R}^d} v(x) \xi_R(x) \phi(x) \, dx + \int_{\mathbb{R}^d} v(x) (-\Delta)^s(\xi_R)(x) \Phi(x) \, dx
+ 2 C_{d,s} \int_{\mathbb{R}^d} v(x) \int_{\mathbb{R}^d} \frac{(\xi_R(x) - \xi_R(y))(\Phi(x) - \Phi(y))}{|x - y|^{d + 2s}} \, dy \, dx.
$$

It is apparent that the first term converges to $\int_{\mathbb{R}^d} v(x) \phi(x) \, dx$ as $R \to \infty$. As for the second term, upon taking advantage of Lemma 3.1, it turns out that

$$
Q \int_{\mathbb{R}^d} \int_{B(\xi_R(x))} |(-\Delta)^s(\xi(x))|^{\frac{p}{p-1}} \frac{|x|^{-\frac{p-d}{p-1}}}{c} \, dx
$$

where $Q$ is any positive constant such that $|\Phi(x)| \leq Q |x|^{-d+2s}$ for all $x \in B_1^c$. Let $\alpha(R) = o(R)$ as $R \to \infty$. Thanks to Lemma 5.1 and the assumptions on $\rho$, it is not difficult to infer that $\sigma(R)$ behaves like $\alpha(R)(2ps + \gamma - d)/p$, like $\log[\alpha(R)](p-1)/p$ or tends to a constant as $R \to \infty$ depending on whether $\gamma > d - 2ps$, $\gamma = d - 2ps$ or $\gamma < d - 2ps$, respectively. Similarly, in view of Lemma 5.1, it turns out that $\varsigma(R)$ tends to a constant, behaves like $\log[R/\alpha(R)](p-1)/p$ or like $[R/\alpha(R)](d - 2ps - \gamma)/p$ as $R \to \infty$ depending on whether $\gamma > d - 2ps$, $\gamma = d - 2ps$ or $\gamma < d - 2ps$, respectively. In any case, these properties ensure that the r.h.s. of \([5.2]\) vanishes as $R \to \infty$ for all $\gamma \in (d - \frac{2}{d}(d - 2s), d]$.

As for the third term in \([5.1]\), we get

$$
\int_{\mathbb{R}^d} v(x) L_R(x) \, dx \leq \frac{(||f_{q,s}(\xi))||_{L^p}}{R^s} \int_{B_1(\xi)} \left|\frac{l_{q,s}(\Phi)(x)}{\rho(x)} \right|^{\frac{p}{p-1}} \, dx
+ \frac{Q}{R^{\frac{d-2}{p}}} \int_{B_1(\xi)} \left|\frac{l_{q,s}(\Phi)(x)}{\rho(x)} \right|^{\frac{p}{p-1}} \, dx
$$

\([5.3]\).
provided \( q > \frac{d}{25} \), upon recalling the definition of \( l_{q,s} \) given in Lemma 3.2, using Lemma 3.3 and arguing similarly to the proof of Lemma 3.7 Here \( Q_0 \) is any positive constant such that \( |l_{q,s}(\Phi)(x)| \leq Q_0 |x|^{-q(d-s)} \) for all \( x \in B_1^R \). Let \( \alpha(R) = o(R) \) as \( R \to \infty \). Thanks to Lemma 3.3, \( q > \frac{d}{25} \) is equivalent to \( q' < d/(d-2s) \) and the assumptions on \( \rho \), one can deduce that \( \sigma_0(R) \) behaves like \( \alpha(R)^{(p s + \gamma - d)/p} \), like \( \log[\alpha(R)]^{(p-1)/p} \) or tends to a constant as \( R \to \infty \) depending on whether \( \gamma > d - ps \), \( \gamma = d - ps \) or \( \gamma < d - ps \), respectively. Similarly, by virtue of Lemma 3.2 one infers that \( \sigma_0(R) \) tends to a constant, behaves like \( \log[\alpha(R)]^{(p-1)/p} \) or like \( (R/\alpha(R))^{(d-ps-\gamma)/p} \) as \( R \to \infty \) depending on whether \( \gamma > d - ps \), \( \gamma = d - ps \) or \( \gamma < d - ps \), respectively. As a consequence, also the r.h.s. of (5.3) vanishes as \( R \to \infty \) for all \( \gamma \in (d - \frac{d}{25}(d-2s), d) \).

Since \( \rho \) satisfies (1.3) for some \( \gamma > d - \frac{d}{25}(d-2s) \), from Lemma 5.1 it is straightforward to check that \( \Phi \) belongs to \( L^p_\rho(\mathbb{R}^d) \), so that \( (-\Delta)^s \Phi \in L^1(\mathbb{R}^d) \) and the l.h.s. of (5.1) converges to \( \int_{\mathbb{R}^d} (-\Delta)^s(\nu(x)) \Phi(x) \, dx \) as \( R \to \infty \). Hence, by letting \( R \to \infty \) in (5.1) and using Fubini’s Theorem, the assertion follows given the arbitrariness of \( \Phi \).

We are now in position to prove Theorem 1.2 in the case \( \gamma \) is supercritical.

**Proof of Theorem 1.2 (case \( d > 2s \) and \( \gamma > d - \frac{d}{25}(d-2s) \)).** Thanks to Lemma 5.3 for \( \gamma \) supercritical and smaller than or equal to \( d \) we only need to prove the identity

\[
\int_{\mathbb{R}^d} \left| (-\Delta)^{s/2} (l_{d,s} * f)(x) \right|^2 \, dx = \int_{\mathbb{R}^d} (l_{d,s} * f)(x) \, \Phi(x) \, dx \quad \forall f \in L^p_\rho(\mathbb{R}^d) : \, l_{d,s} * f \in L^p_\rho(\mathbb{R}^d) .
\]

(5.4)

To this end, let us consider the Green function of \( (-\Delta)^s \) on \( B_R \), namely the unique positive solution \( G_{y,R} \) to

\[
(-\Delta)^s_{B_R}(G_{y,R}) = \delta_y \quad \text{in} \ B_R ,
\]

where by \( (-\Delta)^s_{B_R} \) we mean the spectral fractional Laplacian in \( B_R \) and by \( \delta_y \) we denote the Dirac delta centred at \( y \in B_R \). Given \( h \in L^2(B_R) \), the unique solution \( u \in H^s_\rho(B_R) \) to

\[
\begin{cases}
(-\Delta)^s_{B_R}(u) = h & \text{in} \ B_R , \\
u = 0 & \text{on} \ \partial B_R ,
\end{cases}
\]

is provided by \( u_R(x) = \int_{B_R} G_{y,R}(x) \, h(y) \, dy \), in the sense that

\[
\int_{B_R} (-\Delta)^{s/2}_{B_R}(u_R)(x) \, (-\Delta)^{s/2}_{B_R}(\psi)(x) \, dx = \int_{B_R} h(x) \, \psi(x) \, dx \quad \forall \psi \in H^s_\rho(B_R) .
\]

(5.5)

By setting \( h = f_\epsilon \), with \( f_\epsilon \) as in (2.2), and plugging \( \psi = u_R \) in (5.5), we find the identity

\[
\int_{B_R} \left| (-\Delta)^{s/2}_{B_R}(u_R)(x) \right|^2 \, dx = \int_{B_R} f_\epsilon(x) \, u_R(x) \, dx .
\]

(5.6)

As it is well known (see e.g. [9] Appendix 11 and references therein),

\[
\lim_{R \to \infty} G_{y,R}(x) = l_{d,s}(x-y) \quad \text{and} \quad G_{y,R}(x) \leq l_{d,s}(x-y) \quad \forall x \neq y , \ \forall R > 0 ,
\]

so that by dominated convergence we can pass to the limit in (5.6) as \( R \to \infty \) to get

\[
\int_{\mathbb{R}^d} \left| (-\Delta)^{s/2}(l_{d,s} * f_\epsilon)(x) \right|^2 \, dx \leq \int_{\mathbb{R}^d} f_\epsilon(x) \, (l_{d,s} * f_\epsilon)(x) \, dx
\]

(5.7)

along with the weak convergence of \( (-\Delta)^{s/2}_{B_R}(u_R) \) to \( (-\Delta)^{s/2}(l_{d,s} * f_\epsilon) \) in \( L^2(\mathbb{R}^d) \). Actually, in order to make sure that the r.h.s. of (5.7) is finite, we first have to show that \( l_{d,s} * f_\epsilon \in L^p_\rho(\mathbb{R}^d) \) as a consequence of the fact that \( f \in L^p_\rho(\mathbb{R}^d) \) (recall that also \( f_\epsilon \in L^p_\rho(\mathbb{R}^d) \) by Theorem


To this end, first of all note that \( l^t \in L^\infty_{\text{loc}}(\mathbb{R}^d) \). Indeed, for any \( \delta \geq \frac{1}{2} \) and all \( x \in B_\delta \), we have:

\[
|l^t(x)| \leq (l^t * |f^t\chi_{B_{2\delta}}|)(x) + 2^{d-2s} \kappa_{d,s} C_{\beta} \|f^t\|_{p',\rho'} \left( \int_{B_{2\delta}} \frac{1}{|y|^{(d-2s)p+\gamma}} \, dy \right)^\frac{1}{\rho'}.
\]

Since \( f^t \chi_{B_{2\delta}} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \), we deduce that \( F \in L^\infty(\mathbb{R}^d) \); moreover \((d-2s)p + \gamma >d\) thanks to the hypotheses on \( \gamma \). As for \textit{global} integrability properties of \( l^t \), it is readily seen that from \( f^t \in L^\infty_{\text{loc}}(\mathbb{R}^d) \cap L^{p'}(B_1^t) \) there follows \( f^t \in L^p(\mathbb{R}^d) \) for all \( q \) such that \( p'd/(d+(p'-1)\gamma) < q \leq p' \). In particular, thanks to Theorem 1 at p. 119], we infer that \( l^t \) for all \( r > d - \frac{2}{2}(d-2s) \), it is immediate to check that the latter condition and \( (5.8) \) meet for some \( r \).

Let us denote by \( H^s(\mathbb{R}^d) \) the space of \textit{compactly supported} functions belonging to \( \dot{H}^s(\mathbb{R}^d) \). Given any \( \psi \in H^s(\mathbb{R}^d) \), it is direct to see that the energy \( \int_{B_R} |(-\Delta)^{s/2}(\psi)(x)|^2 \, dx \) is eventually nonincreasing w.r.t. \( R \) and converges to \( \int_{\mathbb{R}^d} |(-\Delta)^{s/2}(\psi)(x)|^2 \, dx \) as \( R \to \infty \). In particular, we have that \( (-\Delta)^{s/2}(\psi) \) (set to be zero in \( B_R^c \)) converges strongly in \( L^2(\mathbb{R}^d) \) to \( (-\Delta)^{s/2}(\psi) \), so that the aforementioned weak convergence holds as a consequence of (5.5) (with \( h = f^t \)) and the just established finiteness of the r.h.s. of (5.7). In a similar way one can deduce the weak convergence of \( (-\Delta)^{s/2}(u_R) \) to \( (-\Delta)^{s/2}(l^t) \) in \( L^2(\mathbb{R}^d) \). We are therefore allowed to pass to the limit in (5.5) (with \( h = f^t \)) as \( R \to \infty \) to get

\[
\int_{\mathbb{R}^d} (-\Delta)^{s/2}(l^t * f^t)(x)(-\Delta)^{s/2}(\psi)(x) \, dx = \int_{\mathbb{R}^d} f^t(x) \, \psi(x) \, dx \quad \forall \psi \in H^s(\mathbb{R}^d).
\]

By letting \( \varepsilon \to 0 \) in (5.7) we infer that \( (-\Delta)^{s/2}(l^t * f^t) \) converges weakly in \( L^2(\mathbb{R}^d) \) to \( (-\Delta)^{s/2}(l^t) \). Indeed, Theorem 2.1 and the identity \( l^t * f^t = (l^t * f)^t \) ensure that

\[
\lim_{\varepsilon \to 0} \left( \|f^t - f^t\|_{p',\rho'} + \|l^t * f^t - l^t * f\|_{p',\rho'} \right) = 0.
\]

Hence, by letting \( \varepsilon \to 0 \) in (5.9), we end up with

\[
\int_{\mathbb{R}^d} (-\Delta)^{s/2}(l^t * f)(x)(-\Delta)^{s/2}(\psi)(x) \, dx = \int_{\mathbb{R}^d} f(x) \, \psi(x) \, dx \quad \forall \psi \in H^s(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d).
\]

We can then plug \( \psi = u_R \) (set to be zero outside \( B_R \)) in (5.10) and let \( R \to \infty \) to obtain

\[
\int_{\mathbb{R}^d} (-\Delta)^{s/2}(l^t * f)(x)(-\Delta)^{s/2}(l^t * f^t)(x) \, dx = \int_{\mathbb{R}^d} f(x) \, (l^t * f^t)(x) \, dx,
\]

so that (5.5) finally follows by passing to the limit in (5.11) as \( \varepsilon \to 0 \). The fact that \( l^t * f \in \dot{H}^s(\mathbb{R}^d) \) is just a consequence of the method of proof. Note that, in order to establish the validity of the integration-by-parts formulas (1.2), it is enough to use (5.4) with \( f = (-\Delta)^s(v) \pm (-\Delta)^s(w) \).
It remains to prove that, in the case where $\gamma$ is larger than the spatial dimension $d$, formulas (1.2) always fail. This is due to the presence of nontrivial constants in the space $X_{p,s,\rho}$ (note that $\rho$ is in $L^1(\mathbb{R}^d)$). In fact, since $(-\Delta)^s(1) \equiv 0$, should (1.2) hold then
\[
\int_{\mathbb{R}^d} (-\Delta)^s(v)(x) \, dx = 0 \quad \forall v \in X_{p,s,\rho}.
\] (5.12)

Given any nonnegative, nontrivial function $\phi \in D(\mathbb{R}^d)$, let us consider its Riesz potential $\Phi = I_{d,s} \ast \phi$. Thanks to Lemma 5.1, it is plain that $\Phi \in L^p_\rho(\mathbb{R}^d)$. Moreover, it is apparent that $(-\Delta)^s(\Phi) = \phi \in L^{p'}_{\rho'}(\mathbb{R}^d)$. Hence, $\Phi \in X_{p,s,\rho}$ and we can plug $v = \Phi$ in (5.12) to get
\[
\int_{\mathbb{R}^d} \phi(x) \, dx = 0,
\]
which is absurd unless $\phi \equiv 0$. \qed

Finally, the assertions of Theorem 1.3 for $d > 2s$ and $\gamma > 2s$ are direct consequences of the results just proved and of the method of proof of Theorem 1.3 itself for subcritical-critical powers (see Section 4).

References

[1] R. A. Adams, “Sobolev Spaces”, Pure and Applied Mathematics, Vol. 65. Academic Press, New York-London, 1975.
[2] A. V. Balakrishnan, Fractional powers of closed operators and the semigroups generated by them, Pacific J. Math. 10 (1960), 419–437.
[3] M. T. Barlow, R. F. Bass, Z.-Q. Chen, M. Kassmann, Non-local Dirichlet forms and symmetric jump processes, Trans. Amer. Math. Soc. 361 (2009), 1963–1999.
[4] B. Barrios, I. Peral, F. Soria, E. Valdinoci, A Widder’s type theorem for the heat equation with nonlocal diffusion, Arch. Ration. Mech. Anal. 213 (2014), 629–650.
[5] K. Bogdan, T. Grzywny, M. Ryznar, Heat kernel estimates for the fractional Laplacian with Dirichlet conditions, Ann. Probab. 38 (2010), 1901–1923.
[6] K. Bogdan, T. Grzywny, M. Ryznar, Density and tails of unimodal convolution semigroups, J. Funct. Anal. 266 (2014), 3543–3571.
[7] M. Bonforte, Y. Sire, J. L. Vázquez, Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains, Discrete Contin. Dyn. Syst. 35 (2015), 5725–5767.
[8] M. Bonforte, J. L. Vázquez, Quantitative local and global a priori estimates for fractional nonlinear diffusion equations, Adv. Math. 250 (2014), 242–284.
[9] M. Bonforte, J. L. Vázquez, A priori estimates for fractional nonlinear degenerate diffusion equations on bounded domains, Arch. Ration. Mech. Anal. 218 (2015), 317–362.
[10] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32 (2007), 1245–1260.
[11] Z.-Q. Chen, P. Kim, R. Song, Heat kernel estimates for the Dirichlet fractional Laplacian, J. Eur. Math. Soc. 12 (2010), 1307–1329.
[12] E. B. Davies, “Heat Kernels and Spectral Theory”, Cambridge Tracts in Mathematics, 92. Cambridge University Press, Cambridge, 1990.
[13] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012), 521–573.
[14] S. Eidelman, S. Kamin, F. Porper, Uniqueness of solutions of the Cauchy problem for parabolic equations degenerating at infinity, Asymptot. Anal. 22 (2000), 349–358.
[15] L. C. Evans, “Partial Differential Equations”, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.
[16] A. Favini, G. Ruiz Goldstein, J. A. Goldstein, S. Romanelli, $C_0$-semigroups generated by second order differential operators with general Wentzell boundary conditions, Proc. Amer. Math. Soc. 128 (2000), 1981–1989.
[17] A. Favini, G. Ruiz Goldstein, J. A. Goldstein, S. Romanelli, Degenerate second order differential operators generating analytic semigroups in $L^p$ and $W^{1,p}$, Math. Nachr. 238 (2002), 78–102.
[18] A. Favini, G. Ruiz Goldstein, J. A. Goldstein, S. Romanelli, The heat equation with generalized Wentzell boundary condition, J. Evol. Equ. 2 (2002), 1–19.
[19] W. Feller, The parabolic differential equations and the associated semi-groups of transformations, Ann. of Math. 55 (1952), 468–519.
[20] S. Fornaro, G. Metafune, D. Pallara, J. Prüss, $L^p$-theory for some elliptic and parabolic problems with first order degeneracy at the boundary, J. Math. Pures Appl. 87 (2007), 367–393.
[21] S. Fornaro, G. Metafune, D. Pallara, R. Schnaubelt, Degenerate operators of Tricomi type in $L^p$-spaces and in spaces of continuous functions, J. Differential Equations 252 (2012), 1182–1212.
[22] S. Fornaro, G. Metafune, D. Pallara, R. Schnaubelt, One-dimensional degenerate operators in $L^p$-spaces, J. Math. Anal. Appl. 402 (2013), 308–318.
[23] G. Grillo and M. Muratori, Sharp short and long time $L^\infty$ bounds for solutions to porous media equations with homogeneous Neumann boundary conditions, J. Differential Equations 254 (2013), 2261–2288.
[24] G. Grillo, M. Muratori and M. M. Porzio, Porous media equations with two weights: smoothing and decay properties of energy solutions via Poincaré inequalities, Discrete Contin. Dyn. Syst. 33 (2013), 3599–3640.
[25] G. Grillo, M. Muratori, F. Punzo, Fractional porous media equations: existence and uniqueness of weak solutions with measure data, Calc. Var. Partial Differential Equations 54 (2015), 3303–3335.
[26] G. Grillo, M. Muratori and F. Punzo, On the asymptotic behaviour of solutions to the fractional porous medium equation with variable density, Discrete Contin. Dyn. Syst. 35 (2015), 5097–5062.
[27] R. G. Iagar, A. Sánchez, Asymptotic behavior for the heat equation in nonhomogeneous media with critical density, Nonlinear Anal. 89 (2013), 24–35.
[28] R. G. Iagar, A. Sánchez, Large time behavior for a porous medium equation in a nonhomogeneous medium with critical density, Nonlinear Anal. 102 (2014), 226–241.
[29] S. Kamin, G. Reyes, J. L. Vázquez, Long time behavior for the inhomogeneous PME in a medium with rapidly decaying density, Discrete Contin. Dyn. Syst. 26 (2010), 521–549.
[30] N. S. Landkof, “Foundations of Modern Potential Theory”, Die Grundlehren der mathematischen Wissenschaften, Band 180. Springer-Verlag, New York-Heidelberg, 1972.
[31] M. Muratori, “Weighted Functional Inequalities and Nonlinear Diffusions of Porous Medium Type”, Ph.D Thesis in Mathematical Models and Methods in Engineering, Politecnico di Milano (Italy), 2015.
[32] S. Nieto, G. Reyes, Asymptotic behavior of the solutions of the inhomogeneous porous medium equation with critical vanishing density, Commun. Pure Appl. Anal. 12 (2013), 1123–1139.
[33] A. de Pablo, F. Quirós, A. Rodríguez, J. L. Vázquez, A fractional porous medium equation, Adv. Math. 226 (2011), 1378–1409.
[34] A. de Pablo, F. Quirós, A. Rodríguez, J. L. Vázquez, A general fractional porous medium equation, Comm. Pure Appl. Math. 65 (2012), 1242–1284.
[35] M. Pierre, Uniqueness of the solutions of $u_t - \Delta \varphi(u) = 0$ with initial datum a measure, Nonlinear Anal. 6 (1982), 175–187.
[36] F. Punzo, G. Terrone, On the Cauchy problem for a general fractional porous medium equation with variable density, Nonlinear Anal. 8 (2014), 27–47.
[37] G. Reyes, J. L. Vázquez, The inhomogeneous PME in several space dimensions. Existence and uniqueness of finite energy solutions, Commun. Pure Appl. Anal. 7 (2008), 1275–1294.
[38] G. Reyes, J. L. Vázquez, Long time behavior for the inhomogeneous PME in a medium with slowly decaying density, Commun. Pure Appl. Anal. 8 (2009), 493–508.
[39] E. Stein, “Singular Integrals and Differentiability Properties of Functions”, Princeton Mathematical Series, Princeton University Press, Princeton, N.J., 1970.
[40] E. Valdinoci, From the long jump random walk to the fractional Laplacian, Bol. Soc. Esp. Mat. Apl. 49 (2009), 33–44.
[41] J. L. Vázquez, Barenblatt solutions and asymptotic behaviour for a nonlinear fractional heat equation of porous medium type, J. Eur. Math. Soc. 16 (2014), 769–803.
[42] J. L. Vázquez, Recent progress in the theory of nonlinear diffusion with fractional Laplacian operators, Discrete Contin. Dyn. Syst. Ser. S 7 (2014), 857–885.

Dipartimento di Matematica “F. Casorati”, Università degli Studi di Pavia, via A. Ferrata 5, 27100 Pavia, Italy
E-mail address: matteo.muratori@unipv.it