BLOWUP AND ILL-POSEDNESS RESULTS FOR A DIRAC EQUATION WITHOUT GAUGE INVARIANCE

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Abstract. We consider the Cauchy problem for a nonlinear Dirac equation on $\mathbb{R}^n$, $n \geq 1$, with a power type, non gauge invariant nonlinearity $\sim |u|^p$. We prove several ill-posedness and blowup results for both large and small $H^s$ data. In particular we prove that: for (essentially arbitrary) large data in $H^{\frac{n}{2} + \epsilon} (\mathbb{R}^n)$ the solution blows up in a finite time; for suitable large $H^s (\mathbb{R}^n)$ data and $s < \frac{n}{2} - \frac{1}{p - 1}$ no weak solution exist; when $1 < p < 1 + \frac{1}{n}$ (or $1 < p < 1 + \frac{2}{n}$ in $n = 1, 2, 3$), there exist arbitrarily small initial data data for which the solution blows up in a finite time.

1. Introduction. We consider the Cauchy problem for the Dirac equation

$$i\partial_t u = Du + F(u),$$
$$u(0, x) = u_0(x),$$

(1)

where $u = u(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}^N$ is an unknown function, $N = 2^{\lfloor \frac{n+1}{2} \rfloor}$, $[x]$ is the integer part of $x \in \mathbb{R}$, $u_0$ is a given function, and $D = \sum_{j=1}^n \alpha_j \partial_j$. Here, $\partial_j$ are the partial derivatives, $\beta$ and $\alpha_j$ are Dirac matrices which satisfy

$$\beta^* = \beta, \quad \beta^2 = I_N, \quad \beta \alpha_j + \alpha_j \beta = 0 \text{ for } j = 1, \ldots, n,$$

$$\alpha_j^* = \alpha_j \text{ for } j = 1, \ldots, n, \quad \alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk}I_N \text{ for } j, k = 1, \ldots, n,$$

$\delta_{jk}$ is the Kronecker delta, and $I_N$ the identity matrix in $M_N(\mathbb{C})$. Throughout this paper, we assume that the function $F = \left( \begin{array}{c} F_1 \\ \vdots \\ F_N \end{array} \right) : \mathbb{C}^N \to \mathbb{C}^N$ satisfies the following assumption (A):

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Let $p > 1$. The function $F$ satisfies the estimates

\[(A) \quad |F(u)| \lesssim |u|^p, \quad |F(u_1) - F(u_2)| \lesssim |u_1 - u_2|(|u_1| + |u_2|)^{p-1}.\]

The most important examples of nonlinearities satisfying (A) are

\[|u|^{p-1}u, \quad \|u, \beta u\|^{\frac{p-2}{2}} \beta u, \quad |u|^p.\]

In these cases, equation (1) is invariant under the scaling

\[u(t, x) \mapsto \sigma^{\frac{1}{p-1}} u(\sigma t, \sigma x)\]

for $\sigma > 0$. Thus we see that the scale invariant Sobolev exponent is $s_c := \frac{n}{2} - \frac{1}{p-1}$.

We summarize the results for (1) with $F(u) = \langle u, \beta u \rangle \beta u$. For $n = 1$, Candy [4] proved global (in time) well-posedness in $L^2(\mathbb{R})$. Pecher [14] proved local well-posedness in $H^s(\mathbb{R}^2)$ with $s > 3/4$. Escobedo and Vega [6] proved local well-posedness in $H^s(\mathbb{R}^3)$ with $s > 1$; they also considered more general nonlinearities. Machihara et. al. [12] proved small data scattering in $H^1(\mathbb{R}^3)$ with some additional regularity in the angular variables. Note that $H^1(\mathbb{R}^3)$ is the scale critical space in the 3D case. Recently, Bejenaru and Herr [1], [2] and Bournaveas and Candy [3] obtained small data scattering in the scale critical Sobolev spaces for $n = 2, 3$. These results rely on the null structure of the nonlinearity $\langle u, \beta u \rangle \beta u$, hence it is not clear if they can be extended to more general nonlinearities.

In this paper, we prove several blowup and ill-posedness results for (1) in the case when the nonlinear term is not gauge invariant. Before stating our results, we define our notion of weak solution:

**Definition 1.1.** We say $u$ is a weak solution of (1) on $[0, T)$, if $u_0 \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $u \in L^p_{\text{loc}}([0, T) \times \mathbb{R}^n)$ satisfies

\[
-i \int_0^T \int_{\mathbb{R}^n} t \partial_t \psi(t, x) u(t, x) dx dt
\]

\[= i \int_{\mathbb{R}^n} t \psi(0, x) u_0(x) dx - \sum_{j=1}^n \int_0^T \int_{\mathbb{R}^n} t \partial_j \psi(t, x) \alpha_j u(t, x) dx dt \]

\[+ \int_0^T \int_{\mathbb{R}^n} t \psi(t, x) F(u(t, x)) dx dt\]

for any $\psi \in C^\infty_0([0, T) \times \mathbb{R}^n; \mathbb{C}^N)$. Here we denote by $t^j \psi$ the transpose of $\psi$.

Writing for brevity

\[e_1 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\]

we shall consider for simplicity data and nonlinearities which are scalar multiples of the vector $e_1$; it is clear that several extensions are possible.

In our first result, we prove that for suitable large data the life span of any weak solution is finite, and satisfies an explicit bound.

**Theorem 1.2.** Let $n \geq 1$, $p > 1$, $\mu \in \mathbb{C} \setminus \{0\}$, $F(u) = \mu |u|^p e_1$ and $u_0(x) = \lambda f(x) e_1 \in L^1_{\text{loc}}(\mathbb{R}^n)$. Let $u$ be a weak solution of (1) on $[0, T(\lambda))$. Let $z \in \mathbb{C}$ be such that $\Re(\mu z) > 0$, and assume

\[\Re(iz f(x)) \geq C_0 |x|^{-k} \text{ in a nbd of } x = 0 \quad (2)\]
where \( k < \min(n, \frac{1}{p-1}) \), \( C_0 > 0 \). Then there exists \( \lambda_0, C > 0 \) such that
\[
T(\lambda) \leq C\lambda^{-1/\delta} \text{ for all } \lambda \geq \lambda_0,
\]
where \( \delta = \frac{1}{p-1} - k > 0 \).

For instance, we can choose \( z = \Re \mu \) so that condition (2) becomes
\[
-\Re \mu \cdot \Im f(x) \geq C_0 |x|^{-k} \text{ in a nbd of } x = 0
\]
(3)
or we can choose \( z = -i\Im \mu \), in which case condition (2) becomes
\[
\Im f(x) \geq C_0 |x|^{-k} \text{ in a nbd of } x = 0.
\]

It is trivial to construct explicit examples satisfying (2). Indeed, let \( \chi \in C_0^\infty([0, \infty)) \) such that
\[
0 \leq \chi(r) \leq 1 \text{ for } r \geq 0 \text{ and } \chi(0) > 0,
\]
and define
\[
f(x) = -i\Re f(x) |x|^{-k} \chi(|x|).
\]
(4)
Then, \( f \) satisfies (2). Moreover, \( f \) belongs to \( H^s(\mathbb{R}^n) \) in the range \( s < \frac{n}{2} - k \) if \( k < n \), see [10, Example 5.1] or [15, Lemma 1 in §2.3] for example.

Remark 1. A byproduct of standard proofs of local existence in \( H^s(\mathbb{R}^n) \) is the blowup alternative: if \( T \) is the maximal existence time of the solution, then either \( T = \infty \), or \( T < \infty \) and \( \lim_{t \to T} \|u(t, \cdot)\|_{H^s} = \infty \). In cases when the blowup alternative holds, the previous result yields blowup of solutions for large data.

Since \( H^s(\mathbb{R}^n) \) for \( s > n/2 \) is a Banach algebra, local in time well-posedness with blowup alternative holds in \( H^s(\mathbb{R}^n) \) for \( n/2 < s < p \). Thus a local strong solution \( u(t) \), continuous in time with values in \( H^s(\mathbb{R}^n) \), exists on some maximal time interval \( [0, T(\lambda)) \), and if \( T(\lambda) = \infty \) the \( H^s(\mathbb{R}^n) \) norm of the solution goes to infinity as \( t \nearrow T(\lambda) \). A similar result holds in the case \( n = 3, p \geq 3 \) for \( \frac{3}{2} - \frac{1}{p-1} < s < p \), thanks to the Strichartz estimate
\[
\|e^{-itD}u_0\|_{L_x^2 L_t^\infty} \lesssim \|u_0\|_{H^s},
\]
(5)
which is valid on \( \mathbb{R}^3 \) provided \( s > 1 \) ([12]). In both results, we can remove the constraint \( s < p \) if the nonlinearity \( F \) is smooth. Note that these solutions are also weak solutions in the sense of Definition 1.1 (see Appendix A at the end of the paper for a quick summary of local existence).

For large data in \( H^s \) with \( s > \frac{n}{2} \) we prove the following general blowup result:

**Corollary 1.** Let \( n \geq 1, p > n/2, \lambda > 0, \mu \in \mathbb{C}\setminus\{0\}, \) \( F(u) = \mu |u|^p e_1 \) and \( u_0(x) = \lambda f(x) e_1 \neq 0 \). Assume \( f \) belongs to \( H^s \) for some \( \frac{n}{2} < s < p \), and there exists \( x \) such that \( i\mu f(x) \notin (-\infty, 0] \). Let \( T(\lambda) \) be the maximal life span of the strong \( H^s \) solution of (1). Then there exists \( \lambda_0 > 0 \) such that \( T(\lambda) \lesssim \lambda^{-\frac{n}{p-1}} \) for all \( \lambda \geq \lambda_0 \), and we have
\[
\lim_{t \to T(\lambda)} \|u(t)\|_{H^s} = \infty.
\]

For the solutions obtained via Strichartz estimates in the 3D case, we obtain a similar result provided the data satisfy the additional constraint (2). Note that the condition \( \frac{3}{2} - \frac{1}{p-1} < s < p \) is compatible with (2) provided \( k < \frac{1}{p-1} \).

**Corollary 2.** Let \( n = 3, p \geq 3, \lambda > 0, \mu \in \mathbb{C}\setminus\{0\}, \) \( F(u) = \mu |u|^p e_1 \) and \( u_0(x) = \lambda f(x) e_1 \neq 0 \). Assume \( f \) belongs to \( H^s(\mathbb{R}^n) \) for some \( \frac{3}{2} - \frac{1}{p-1} < s < \frac{3}{2} \) and satisfies (2) (for some \( k < \frac{3}{2} - s \)). Let \( u(t) \) be the strong solution of (1) in \( H^s(\mathbb{R}^n) \) on the
Let Proposition 1. If \( u \in H^s(\mathbb{R}^n) \) is a global weak solution to \( u_t - \Delta u + u |u|^p = 0 \) for \( s > \frac{n}{2} \), we can prove by the same methods nonexistence of global weak solutions. Note that in the following Theorem no condition is imposed on the size of \( f \).

**Theorem 1.3.** Let \( n \geq 1, \ p > 1 + \frac{1}{n}, \ \mu \in \mathbb{C}\setminus\{0\} \) and \( F(u) = \mu |u|^p e_1 \). Then Problem (1) is ill-posed in \( H^s(\mathbb{R}^n) \) if \( s < \frac{n}{2} - \frac{1}{p-1} \). More precisely, one can construct initial data in \( H^s(\mathbb{R}^n) \) for which no weak solution exists.

The previous results do not include the case of small initial data. However, if \( p \) is sufficiently close to 1 we can prove by the same methods nonexistence of global weak solutions. Note that in the following Theorem no condition is imposed on the size of \( f \).

**Theorem 1.4.** Let \( n \geq 1, \ 1 < p < 1 + \frac{1}{n}, \ \mu \in \mathbb{C}\setminus\{0\} \), \( F(u) = \mu |u|^p e_1 \) and \( u_0(x) = f(x)e_1 \in L^1_{loc}(\mathbb{R}^n) \). Let \( z \in \mathbb{C} \) be such that \( \Re(\mu z) > 0 \), and assume that for a.e. \( x \in \mathbb{R}^n \)

\[
\Re(izf(x)) \geq 0. \tag{6}
\]

If \( u \) is a global weak solution to (1), then \( u = 0 \) almost everywhere on \( [0, \infty) \times \mathbb{R}^n \).

For instance, as before, we can choose \( z = \Re \mu \) so that condition (6) becomes

\[-\Re \mu \cdot \Im f(x) \geq 0 \text{ for a.e. } x\]

or we can choose \( z = -i \Im \mu \), in which case condition (6) becomes

\[\Im \mu \cdot \Re f(x) \geq 0 \text{ for a.e. } x.\]

Finally, in the low dimensional cases \( n = 1, 2, 3 \) we can construct small blowup solution for a larger range of \( p \). Note that the initial data (7) belong to any \( H^s(\mathbb{R}^n) \) space with \( s > 0 \) thanks to the assumption \( k > \frac{n}{2} \), and this ensures the local existence of a strong solution in \( H^s(\mathbb{R}^n) \) for \( s > \frac{n}{2} \).

**Proposition 1.** Let \( 1 \leq n \leq 3, \ 1 < p < 1 + \frac{2}{n}, \ \frac{n}{2} < s < p, \ \mu \in \mathbb{C}\setminus\{0\}, \ \varepsilon > 0, \ F(u) = \mu |u|^p e_1 \) and \( u_0(x) = \varepsilon f(x)e_1 \). Let \( z \in \mathbb{C} \) be such that \( \Re(\mu z) > 0 \). For some \( \frac{n}{2} < k < \frac{1}{p-1} \), define

\[f(x) = -i\varepsilon(1 - \chi(|x|))|x|^{-k}, \tag{7}\]

where \( \chi \in C_0(0, \infty) \) with \( 0 \leq \chi \leq 1, \ \chi(r) = 1 \) for \( 0 \leq r \leq 1 \), and \( \chi(r) = 0 \) for \( r \geq 2 \). Let \( T(\varepsilon) \) be the maximal lifespan of the local strong solution of Problem (1), continuous in time with values in \( H^s(\mathbb{R}^n) \). Then there exists \( \varepsilon_0 \) and \( C > 0 \) such that

\[T(\varepsilon) \leq C \varepsilon^{-1/\delta}\]

for any \( 0 < \varepsilon < \varepsilon_0 \), where \( \delta := \frac{1}{p-1} - k > 0 \). Moreover we have

\[\lim_{t \to T(\varepsilon)} \|u(t, \cdot)\|_{H^s} = \infty.\]

Blowup results for the Schrödinger equation without gauge invariance have been proved by Ikeda and Wakasugi [8], Ikeda and Inui [9], [10] and Oh [13]. They used so-called test function method, see [17], [18], which is also used in this paper. Ikeda and Inui [9], [10] showed that the maximal existence time \( T(\lambda) \) for suitable initial data (satisfying conditions similar to (3)) is bounded by \( \lambda^{-1/\kappa} \) with \( \kappa = \frac{1}{p-1} - \frac{k}{2} \).
The different rate obtained here is of course due to the fact that in the Dirac equation the Laplacian is replaced by a first order operator;

Blowup for the wave equation in absence of gauge invariance, \( \Box u = |u|^p \), is a classical and well studied problem (see for instance [7], [11], [16]). In this case the sharp blowup range of \( p \) is known; note however that one can exploit some positivity properties of the fundamental solution or its averages, which are ruled out for the Dirac equation. It is also clear that blowup results for (1) can not be proved by reduction to the wave equation case, since in the process we obtain a derivative nonlinearity.

2. Proof of the main results. Our argument is inspired by Ikeda and Inui [10]. Let \( \mu \in \mathbb{C}\backslash\{0\} \), \( \lambda \geq 0 \) and assume

\[
F(u) = \mu |u|^p e_1, \quad u_0(x) = \lambda f(x) e_1 \tag{8}
\]

We define two functions \( \eta(t) \in C_0^\infty([0, \infty)) \) and \( \phi(x) \in C_0^\infty(\mathbb{R}^n) \) such that \( 0 \leq \eta, \phi \leq 1 \) and

\[
\eta(t) = \begin{cases} 1 & (0 \leq t < 1/2), \\ 0 & (t \geq 1) \end{cases}, \quad \phi(x) = \begin{cases} 1 & (0 \leq |x| < 1/2), \\ 0 & (|x| \geq 1) \end{cases}.
\]

For \( R > 0 \), we also define the cut-off functions

\[
\psi(t,x) = \eta(t)\phi(x), \quad \psi_R(t,x) = \psi(t/R, x/R) = \eta_R(t)\phi_R(x),
\]

where \( \eta_R(t) = \eta(t/R) \), \( \phi_R(x) = \phi(x/R) \). Finally, for \( q = \frac{p}{p-1} \), we introduce the quantities

\[
I(R) := \int_0^T \int_{\mathbb{R}^n} |u(t,x)|^p \psi_R^{[q]+1}(t,x) dx dt, \quad J(R) := \int_{\mathbb{R}^n} f(x) \phi_R^{[q]+1}(x) dx,
\]

where \([q]\) denotes the inter part of \( q \).

We show the following bound:

**Lemma 2.1.** Let \( n \geq 1 \), \( p > 1 \), and let \( F, u_0 \) be of the form (8) for some \( \mu \in \mathbb{C}\backslash\{0\} \) and \( \lambda > 0 \). Let \( z \in \mathbb{C} \) be such that \( \Re(\mu z) > 0 \). Then there exists a constant \( C = C(n, p, \psi) > 0 \) such that

\[
\lambda \Re(iz J(R)) \leq C(\Re(\mu z))^{-\frac{1}{p-1}} R^n - \frac{1}{p-1}
\]

for any \( R \in (0, T(\lambda)) \).

**Proof.** By the definition of a weak solution with the test function \( \psi_R^{[q]+1}(t,x) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \),

we get

\[
\mu I(R) + i\lambda J(R) = -i \int_0^T \int_{\mathbb{R}^n} \partial_t (\psi_R^{[q]+1})(t,x)(1, \ldots, 1) u(t,x) dx dt
\]

\[
+ \sum_{j=1}^n \int_{\mathbb{R}^n} \partial_j (\psi_R^{[q]+1})(t,x)(1, \ldots, 1) \alpha_j u(t,x) dx.
\]
Multiplying by \( z \) and taking the real part we have

\[
\Re(\mu z)I(R) + \lambda \Re(izJ(R)) = \int_0^T \int_{\mathbb{R}^n} \partial_t(\psi_R^{[q]+1})(t, x)(1, \ldots, 1)3z \mu u(t, x)dxdt
\]

\[
+ \sum_{j=1}^n \int_0^T \int_{\mathbb{R}^n} \partial_j(\psi_R^{[q]+1})(t, x)(1, \ldots, 1)\Re(\alpha_j z \mu u(t, x))dxdt
\]

\[
\leq C \frac{1}{R} \int_0^T \int_{\mathbb{R}^n} |u(t, x)||\psi_R^{[q]}(t, x)\nabla_{t,x}\psi(t/R, x/R)|dxdt
\]

\[
\leq C \frac{1}{R} \int_0^T \int_{\mathbb{R}^n} |u(t, x)|\psi_R^{[q]}(t, x)dxdt
\]

\[
\leq C \frac{1}{R} \left( \int_0^T \int_{\mathbb{R}^n} |u(t, x)|^p \psi_R^{[q]}(t, x)dxdt \right)^{1/p} \left( \int_0^R \int_{|x|\leq R} dxdt \right)^{1/q} \tag{9}
\]

\[
\leq CR^{\frac{n+1}{q}-1}I(R)^{1/p},
\]

where we used \([q]p \geq [q] + 1\) in the last inequality, which follows from \([q] \geq q - 1\).

We therefore arrive at

\[
\lambda \Re(izJ(R)) \leq C R^{\frac{n+1}{q}-1}I(R)^{1/p} - \Re(\mu z)I(R)
\]

with \( C = C(n, p, \psi). \) Using the inequality \( ab \leq a^p/p + b^q/q \), we obtain

\[
CR^{\frac{n+1}{q}-1}I(R)^{1/p} - \Re(\mu z)I(R)
\]

\[
= C(\Re(\mu z))^{-1/p} R^{\frac{n+1}{q}-1}(\Re(\mu z)I(R))^{1/p} - \Re(\mu z)I(R)
\]

\[
\leq C(\Re(\mu z))^{-\frac{n}{p+1}} R^{n+1-q}.
\]

\[\Box\]

**Lemma 2.2.** Let \( n \geq 1, p > 1, \) and let \( F, u_0 \) be of the form (8) for some \( \mu \in \mathbb{C}\setminus \{0\} \). Let \( z \in \mathbb{C} \) be such that \( \Re(\mu z) > 0 \). Assume in addition that for some \( R_0 > 0 \) one has \( \Re(izJ(R_0)) > 0 \). Then there exists \( \lambda_0 = \lambda_0(n, p, \mu, z, R_0) > 0 \) such that for all \( \lambda \geq \lambda_0 \) we have \( T_\lambda \leq 2R_0 \).

**Proof.** If \( T_\lambda \geq 2R_0 \), we can apply Lemma 2.1 with \( R = R_0 \) and we get

\[
\lambda \leq C(n, p, \psi) R_0^{n-\frac{1}{p+1}} \Re(izJ(R_0))^{-\frac{1}{p+1}} \Re(\mu z)^{-\frac{1}{p+1}} =: \lambda_0.
\]

Equivalently, we have proved that if \( \lambda \geq \lambda_0 \) then \( T_\lambda \leq 2R_0 \). \(\Box\)

**Lemma 2.3.** Let \( n, p, F, u_0, \mu, z, \lambda_0, R_0 \) be as in Lemma 2.2. In addition, assume that for some \( c_0 > 0, \gamma \in \mathbb{R} \) and all \( 0 < R \leq R_0 \) one has \( \Re(izJ(R)) \geq c_0 R^\gamma \).

Then for all \( \lambda \geq \lambda_0 \) we have \( \lambda \leq CT_\lambda^{n-\gamma-\frac{1}{p+1}} \) with \( C = C(n, p, \mu, z, \gamma, c_0) \). In particular if \( \delta := \gamma + \frac{1}{p+1} - n > 0, \) this implies \( T_\lambda \leq \lambda^{-1/\delta} \) for \( \lambda \geq \lambda_0 \).

**Proof.** We know already that \( T_\lambda \leq 2R_0 \) by the previous Lemma. Applying Lemma 2.1 with \( R = T_\lambda/2 \) and the assumption on \( J(R) \) we get immediately \( \lambda \leq C(n, p, \mu, z, \gamma, c_0)T_\lambda^{n-\frac{1}{p+1}-\gamma}. \) \(\Box\)

2.1. **Proof of Theorem 1.2.** If \( f(x) \) satisfies (2), we have for \( R > 0 \) small enough

\[
\Re(izJ(R)) = \int_{\mathbb{R}^n} \Re(izf(x))\psi_R^{[q]+1}(x)dx \geq C_0 \int_{|x| \leq R/2} |x|^{-k} dx \geq c_0 R^{n-k}
\]

for some \( c_0 > 0. \) Theorem 1.2 then follows from Lemma 2.3 with \( \gamma = n - k. \)
2.2. Proof of Corollaries 1 and 2. Corollary 2 follows by a direct application of Theorem 1.2, hence we focus on the proof of Corollary 1.

By assumption, there exists a point \( x \) such that \( f(x) \) is not a positive multiple of \( i\mu \), and by translation invariance of the problem we can suppose \( x = 0 \). Now we need the following elementary fact:

**Lemma 2.4.** Let \( \mu, w \in \mathbb{C} \setminus \{0\} \). If \( w \) is not of the form \( \alpha \cdot i\mu \) for some \( \alpha > 0 \), then there exists \( z \in \mathbb{C} \) such that \( \Re(\mu z) > 0 \) and \( \Re(izw) > 0 \).

**Proof.** By assumption, \( iw \) and \( \mu \) are contained in an open half-plane \( H \) of \( \mathbb{C} \) with \( 0 \in \partial H \), and any such half-plane can be written \( H = \{ v \in \mathbb{C} : \Re(vz) > 0 \} \) for some \( z \).

By the Lemma, there exists \( z \in \mathbb{C} \) such that \( \Re(\mu z) > 0 \) and \( \Re(izw) > 0 \). By continuity of \( f \), we have \( \Re(izf(x)) \geq C_0 > 0 \) for \( x \) in a nbd of 0. Thus the assumptions of Theorem 1.2 are satisfied with \( k = 0 \) and we obtain that the life span \( T(\lambda) \) is finite for all \( \lambda \geq \lambda_0 \) and satisfies the bound in the statement. The final claim concerning the growth of the \( H^s(\mathbb{R}^n) \) norm follows from the blowup alternative.

2.3. Proof of Theorem 1.3. It is not restrictive to assume \( s > -\frac{n}{2} \) since the result for this case includes the result for lower values of \( s \). Let \( z \) be such that \( \Re(\mu z) > 0 \), and let \( u_0 = \lambda f(x)e_1 \) with \( f \) of the form (4), \( \lambda > 0 \) and \( \frac{k}{1-p} < k < \frac{n}{2} - s \) (so that \( f \in H^s \) as noticed above). In Section 2.1 we proved that \( \Re(izJ(R)) \geq c_0 R^{n-k} \) for \( R \) small. Now assume there exists a weak solution \( u \) defined on some maximal time interval \( [0, T(\lambda)) \) with \( T(\lambda) > 0 \); by Lemma 2.1 we can write

\[
\lambda c_0 R^{n-k} \leq CR^a \quad \Rightarrow \quad \lambda \leq CR^{k-\frac{a}{p-1}}
\]

for \( R \) small. Letting \( R \to 0 \) we obtain \( \lambda = 0 \) since \( k > \frac{1}{p-1} \), and this concludes the proof.

2.4. Proof of Theorem 1.4. Recalling inequality (9) and using the assumption \( \Re(izf) \geq 0 \), we obtain

\[
\Re(\mu z)I(R) \leq CR^{n+1-q}I(R)^{1/p}
\]

which leads to

\[
I(R) \leq CR^{n+1-q} = CR^{\frac{1}{p-1}}
\]

for all \( R > 0 \). Since \( n - \frac{1}{p-1} < 0 \), letting \( R \to \infty \) we obtain

\[
\lim_{R \to \infty} I(R) = 0
\]

and this implies \( u = 0 \).

2.5. Proof of Proposition 1. By the local existence result in \( H^s(\mathbb{R}^n) \), \( s > \frac{n}{2} \), we know that there exists \( \epsilon_0 > 0 \) such that \( T(\epsilon) > 16 \) for \( 0 < \epsilon < \epsilon_0 \). We also note that, for \( R \geq 8 \),

\[
\Re(izJ(R)) = \int_{\mathbb{R}^n} \Re(izf(x))e^{[\cdot]^{n+1}}(x)dx \geq |x|^2 \int_{2|z| \leq R/2} |x|^{-k}dx \geq CR^{n-k}.
\]

Applying Lemma 2.1 with \( \lambda = \epsilon \) we get

\[
\epsilon \leq CR^{k-\frac{1}{p-1}}
\]
for all $8 \leq R < T(\epsilon)$ and $0 < \epsilon < \epsilon_0$. Choosing $R = T(\epsilon)/2$ and noticing that by assumption $k - \frac{1}{p - 1} = -\delta < 0$, we obtain the required bound on $T(\epsilon)$. The claim on the norm of the solution follows as usual from the blowup alternative.

**Appendix A. Well-posedness.** For the convenience of the reader, we briefly recall here the arguments for the local well posedness of the Cauchy problem (1). As usual, we rewrite (1) in integral form:

$$u(t, x) = e^{-it\mathcal{D}}u_0(x) - i \int_0^t e^{-i(t-t')\mathcal{D}}F(u)(t', x)dt'.$$

Let $\mathcal{U}_{u_0}[u](t, x)$ be the right hand side of this equality. Our goal is to prove that there exists a unique fixed point of $\mathcal{U}$ in a suitable complete metric space $X_T$. We denote by $\| \cdot \|_{L^p_T K}$ the norm of the space $L^p([0, T); K)$, where $K$ is a Banach space on $\mathbb{R}^n$, for example $K = H^s(\mathbb{R}^n)$ or $L^q(\mathbb{R}^n)$.

**Lemma A.1.** Let $n = 3$, $2 \leq a \leq \infty$, and $s > \frac{3}{2} - \frac{1}{a}$. Then, we have

$$\|e^{-it\mathcal{D}}u_0\|_{L^a_T L^\infty} \lesssim \|u_0\|_{H^s}.$$

**Proof.** By the Sobolev embedding theorem, we have

$$\|e^{-it\mathcal{D}}u_0\|_{L^a_T L^\infty} \lesssim \|u_0\|_{H^{3/2+s}}$$

for $\varepsilon > 0$. Interpolating it with (5), we obtain the desired bound. \hfill \Box

Here, we assume that $F$ satisfies the following fractional chain rule: Let $0 \leq s < p$. For all $u \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ we have

$$\|F(u)\|_{H^s} \lesssim \|u\|_{H^s} \|u\|_{L^p}^{p-1}. \hspace{1cm} \text{(B)}$$

Thanks to [15, Theorem 1 in §5.4.3], the nonlinearity $F(u) = |u|^p$ fulfills this assumption.

**Proposition 2.** Let $n = 3$, $p \geq 3$, and $s \in (\frac{3}{2} - \frac{1}{p-1}, p)$ Assume that $F$ satisfies (A) and (B). Then, (1) is local in time well-posed in $H^s(\mathbb{R}^3)$. Moreover, $u$ is a weak solution to (1) in the sense of Definition 1.1 Furthermore, the following blowup alternative holds: if $T$ is the maximal existence time of the solution, then either $T = \infty$ or $T < \infty$ and $\lim_{t \to T} \|u(t, \cdot)\|_{H^s} = \infty$.

**Proof of Proposition 2.** We only consider the case $\frac{3}{2} - \frac{1}{p-1} < s \leq \frac{3}{2}$ since the proof for the case $s > \frac{3}{2}$ follows by a similar but simpler argument, based on Sobolev embedding instead of Strichartz estimates. Pick $\gamma > p - 1 \geq 2$ with $\gamma < \frac{2}{3 - 2s}$, which is possible because of the assumption. Form Lemma A.1, the linear part is estimated as follows:

$$\|e^{-it\mathcal{D}}u_0\|_{L^\infty_T H^s \cap L^\infty_T L^\infty} \leq C\|u_0\|_{H^s} =: M. \hspace{1cm} \text{(10)}$$

Set

$$X_T := \{u \in L^\infty([0, T); H^s(\mathbb{R}^3)) \cap L^\gamma([0, T); L^\infty(\mathbb{R}^3)) : \|u\|_{X_T} \leq 2M\},$$

where

$$\|u\|_{X_T} := \|u\|_{L^\infty_T H^s} + \|u\|_{L^\gamma_T L^\infty}.$$

We note that $X_T$ is a complete metric space endowed with the metric

$$d(u_1, u_2) := \|u_1 - u_2\|_{L^\infty_T L^2}.$$
Remark 2. For the case $s \leq 3/2$ it is not necessary to introduce the distance $d$ but one could estimate directly in the $X_T$ norm, thanks to the condition $p \geq 3$. Indeed, roughly speaking, we can proceed as follows. If $s < p - 1$, then, (B) yields

$$\|\mathcal{U}_u[1] - \mathcal{U}_u[2]\|_{X_T}$$

$$\lesssim \|F(u_1) - F(u_2)\|_{L^1_T H^s} \lesssim \|(u_1 - u_2)(p^{-1} + |u_2|^{p-1})\|_{L^1_T H^s}$$

$$\lesssim \|u_1 - u_2\|_{L^p_T H^s} \left(\|u_1\|_{L^1_T L^\infty} |u_2|^{p-1} + \|u_2\|_{L^1_T L^\infty} \right)$$

$$+ \|u_1 - u_2\|_{L^{p-1}_T L^\infty} \left(\|u_1\|_{L^{p-1}_T H^s} + \|u_2\|_{L^{p-1}_T H^s} \right)$$

$$\lesssim \|u_1 - u_2\|_{L^p_T H^s} \left(\|u_1\|_{L^{p-1}_T L^\infty} + \|u_2\|_{L^{p-1}_T L^\infty} \right)$$

$$+ \|u_1 - u_2\|_{L^{p-1}_T L^\infty} \left(\|u_1\|_{L^{p-1}_T H^s} + \|u_2\|_{L^{p-1}_T H^s} \right)$$

$$\lesssim T^{(p-2)(\frac{1}{p - s} - \frac{1}{2})} \|u_1 - u_2\|_{X_T} \left(\|u_1\|_{X_T} + \|u_2\|_{X_T} \right)$$

Thus, for $s < p - 1$, we have

$$\|\mathcal{U}_u[1] - \mathcal{U}_u[2]\|_{X_T} \lesssim T^{(p-2)(\frac{1}{p - s} - \frac{1}{2})} \|u_1 - u_2\|_{X_T} \left(\|u_1\|_{X_T} + \|u_2\|_{X_T} \right)$$

If we take $T$ so small that $CT^{1 - \frac{p-1}{p}} M^{p-1} < 1/4$, we get $\mathcal{U}_u[u] \in X_T$. Next, we show that $\mathcal{U}_u$ is a contraction on $X_T$. This follows by computations similar to the above ones:

$$d(\mathcal{U}_u[u_1], \mathcal{U}_u[u_2])$$

$$\leq \int_0^T \|e^{-i(t-t')\mathcal{D}} (F(u_1) - F(u_2))(t', \cdot)\|_{L^p_T L^2} dt' \leq \|F(u_1) - F(u_2)\|_{L^1_T L^2}$$

$$\leq C\|u_1 - u_2\|_{L^p_T L^2} \left(\|u_1\|_{L^{p-1}_T L^\infty} + \|u_2\|_{L^{p-1}_T L^\infty} \right)$$

$$\leq 2CT^{1 - \frac{1}{p}} M^{p-1} \|u_1 - u_2\|_{L^p_T L^2} < \frac{1}{2} \|u_1 - u_2\|_{L^p_T L^2}.$$
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