QED vacuum between an unusual pair of plates

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Abstract

We consider the photon field between an unusual configuration of infinite parallel plates: a perfectly conducting plate ($\epsilon \to \infty$) and an infinitely permeable one ($\mu \to \infty$). After quantizing the vector potential in the Coulomb gauge, we obtain explicit expressions for the vacuum expectation values of field operators of the form $<\hat{E}_i\hat{E}_j>_0$ and $<\hat{B}_i\hat{B}_j>_0$. These field correlators allow us to reobtain the Casimir effect for this set up and to discuss the light velocity shift caused by the presence of plates (Scharnhorst effect \cite{1, 2, 3}) for both scalar and spinor QED.

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1 Introduction

Ordinary QED deals with processes in unbounded spacetime, with no boundary conditions whatsoever or external fields imposed on and without compactification of any spatial dimension. Nonetheless, a number of physical interesting processes involving photons and electrons (bound or not) occur within the confines of physical boundaries, that is, within a cavity. As an example consider the spontaneous emission by an atom. This process is due to the coupling of electromagnetic vacuum oscillations to the bound electron in the atom and in free space is a position-independent observable. However, inside a cavity the vacuum electromagnetic field modes can change substantially and as a consequence the spontaneous emission rate is affected and can become position-dependent [4, 5, 6] (see also the textbook by Milonni [7] and references therein). For a “cavity” comprised by a single metallic wall, for instance, the spontaneous emission rate goes with the reciprocal of the fourth power of the distance of the atom to the wall. In a broader sense, we can say that inside the cavity we can think of the atom as probing the local fluctuations of the electromagnetic vacuum.

The influence of the atom-cavity interaction on the atomic spontaneous emission rate is one among a large number of effects of the so-called cavity QED, a specific branch of QED that basically deals with the influences of the surroundings of a physical system on its radiative properties (see ref(s) [9, 10] for recent reviews). Although the first cavity QED effect is attributed to Purcell [11], who pointed out that the spontaneous emission process associated with nuclear magnetic moment transitions at radio frequencies could be enhanced if the system were coupled to a resonant external electric circuit, we can say that the first detailed papers on this subject were those written by Casimir and Polder [12] in which, among other things, forces between polarizable atoms and metallic walls were treated, and by Casimir in his seminal work [13]. In its electromagnetic version, the Casimir effect is the macroscopic attraction force between two parallel perfectly conducting infinite surfaces due to the redistribution of normal modes of the vacuum electromagnetic field between them. Experimentally, the Casimir effect between metallic surfaces was first observed by Sparnaay [14] and recently with remarkable accuracy by Lamoreux [15] and Mohideen and Roy [16]. The various Casimir effects have been the subject of many studies, for a review see [17, 18].

Still another spectacular instance of cavity QED is the Scharnhorst effect [1, 2]. This effect is basically the velocity shift caused by the change in the zero-point energy density of the quantized electromagnetic field induced by the presence of Casimirlike plates. Recall that an external electromagnetic field such as that of a propagating light couples to the quantized radiation field through fermionic loops. The Scharnhorst effect is not the only example where non-trivial vacua affects the speed of light. In fact this subject has attracted the attention of many physicists in the last years [19, 20, 21, 22, 23, 24].
It is clear from what was stated above that an analysis of the QED vacuum inside cavities is crucial for an understanding of its observable properties. Here we shall consider the QED vacuum confined by an unusual pair of mirrors. Specifically, we shall place an infinite perfectly conducting ($\epsilon \to \infty$) surface parallel to a second infinite perfectly permeable ($\mu \to \infty$) surface held at fixed distance $L$ from the first. This setup was first considered by Boyer in order to compute the corresponding Casimir effect in the framework of random electrodynamics [25] and leads to a repulsive force. This result is somewhat intriguing, since it seems to contradict the explanation given for the usual attractive Casimir effect which suggests that there is a greater number of modes outside the plates than inside [7]. In fact, this is not true: there is only a rearrangement of modes, for a nice explanation of this problem see [8]. For the generalized $\zeta$-function approach applied to the repulsive Casimir effect for parallel plates geometry see [27, 28].

This paper is organized as follows: in section 2 we determine the photon field $A(r, t)$ in the region between Boyer’s plates making use of the Coulomb gauge. Next we also evaluate the field operator correlators $\langle \hat{E}_i \hat{E}_j \rangle_0$ and $\langle \hat{B}_i \hat{B}_j \rangle_0$ with the aid of a simple but efficient regularization prescription. In section 3 we apply our results to reobtain the repulsive Casimir pressure of this setup. In section 4 we discuss the Scharnhorst effect but for this different situation. In particular, we show that, contrary to the case with of the usual pair of Casimir plates considered by Scharnhorst [1] and Barton [4], Boyer’s plates lead to a decrease in the speed of a light for propagation perpendicular to the plates. In section 5 we discuss the Scharnhorst effect for the case of scalar QED trying to keep as much as possible a close analogy with the spinorial QED case. Section 6 is left for the final remarks and conclusions.

We use natural units so that Planck’s constant $\hbar$ and the speed of light $c$ are set equal to one. For the electromagnetic fields we employ the unrationalized gaussian system. The fine structure constant reads $\alpha = e^2 \approx 1/137$.

## 2 Vacuum electromagnetic field between Boyer’s plates

The setup we will consider consists of two infinite parallel surfaces (the plates) one of which will be considered to be a perfect conductor ($\epsilon \to \infty$) while the other is supposed to be perfectly permeable ($\mu \to \infty$). Also, we will choose Cartesian axes in such a way that the axis $OZ$ is perpendicular to both surfaces. The perfectly conducting surface will be placed at $z = 0$ and the permeable one at $z = L$. The electromagnetic fields must satisfy the following boundary conditions: (a) the tangential components $E_x$ and $E_y$ of the electric field as well as the normal component $B_z$ of the magnetic field must vanish on the metallic plate at $z = 0$. (b) The tangential components $B_x$ and $B_y$ of the magnetic
field must vanish on the permeable plate at \( z = L \). It is convenient to work with the vector potential \( \mathbf{A}(r, t) \) in the Coulomb gauge in which \( \nabla \cdot \mathbf{A}(r, t) = 0 \), \( \mathbf{E}(r, t) = -\partial \mathbf{A}(r, t)/\partial t \) and \( \mathbf{B}(r, t) = \nabla \times \mathbf{A}(r, t) \). Then the physical boundary conditions combined with our choice of gauge permit us to translate the boundary conditions in terms of the vector potential components. At \( z = 0 \) we have:

\[
A_x(x, y, 0, t) = 0; \quad A_y(x, y, 0, t) = 0; \quad \frac{\partial}{\partial z} A_z(x, y, 0, t) = 0,
\]

On the other hand, at \( z = L \) we have:

\[
\frac{\partial}{\partial x} A_x(x, y, L, t) = 0; \quad \frac{\partial}{\partial y} A_y(x, y, L, t) = 0; \quad A_z(x, y, L, t) = 0.
\]

The appropriate vector potential \( \mathbf{A}(r, t) \) that satisfies the wave equation, the Coulomb gauge condition and the previous boundary conditions can be written in the form:

\[
\mathbf{A}(r, t) = \frac{1}{\pi} \left( \frac{\pi}{L} \right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \int d^2 \kappa \frac{\sqrt{\omega}}{\kappa} \left\{ a^{(1)}(\kappa, n) \hat{\kappa} \times \hat{z} \sin \left[ \left( n + \frac{1}{2} \right) \frac{\pi z}{L} \right] \right\} e^{i(\kappa \cdot \rho - \omega t)}
\]

\[
+ a^{(2)}(\kappa, n) \left\{ i \left( n + \frac{1}{2} \right) \omega L \sin \left[ \left( n + \frac{1}{2} \right) \frac{\pi z}{L} \right] - \frac{\omega}{\omega L} \cos \left[ \left( n + \frac{1}{2} \right) \frac{\pi z}{L} \right] \right\} e^{i(\kappa \cdot \rho - \omega t)}
\]

\[
+ \text{Hermitian conjugate},
\]

where \( \kappa = (k_x, k_y) \) and \( \rho \) is the position vector in the \( xy \)-plane. The normal frequencies are given by

\[
\omega = \omega(\kappa, n) = \sqrt{\kappa^2 + \left( n + \frac{1}{2} \right)^2 \frac{\pi^2}{L^2}}.
\]

The Fourier coefficients \( a^{(\lambda)}(\kappa, n) \) where \( \lambda = 1, 2 \) is the polarization index, are operators acting on the photon state space and satisfy the commutation relation

\[
\left[ a^{(\lambda)}(\kappa, n), a^{(\lambda')}(\kappa', n') \right] = \delta_{\lambda\lambda'} \delta_{nn'} \delta(\kappa - \kappa') .
\]

It is convenient to write the vector potential in the general form:

\[
\mathbf{A}(r, t) = \sum_{\alpha} a^{\alpha}(0) \mathbf{A}^{\alpha}(r) e^{-i\omega_{\alpha} t} + \text{H. c.},
\]

where \( \mathbf{A}^{\alpha}(r) \) denotes the mode functions. The mode functions for each polarization state obey the Helmholtz equation and satisfy the boundary conditions stated above. In our case they are given by:

\[
\mathbf{A}_{\kappa n}^{(1)}(r) = \frac{1}{\pi} \left( \frac{\pi}{L} \right)^{\frac{1}{2}} \frac{1}{\omega} \sin \left[ \left( n + \frac{1}{2} \right) \frac{\pi z}{L} \right] e^{-i\kappa \cdot \rho} \hat{\kappa} \times \hat{z},
\]

\[
\mathbf{A}_{\kappa n}^{(2)}(r) = \frac{1}{\pi} \left( \frac{\pi}{L} \right)^{\frac{1}{2}} \frac{i}{\omega} \sin \left[ \left( n + \frac{1}{2} \right) \frac{\pi z}{L} \right] e^{-i\kappa \cdot \rho} \hat{\kappa} \cdot \hat{\kappa},
\]
and

\[ A_{\kappa n}^{(2)}(r) = \left( \frac{\pi}{\omega} \right)^{\frac{3}{2}} \frac{1}{\omega^2} \left[ \hat{\kappa} \sin \left( \frac{n + \frac{1}{2}}{\omega} \frac{\pi z}{L} \right) - \frac{\kappa}{\omega} \cos \left( \frac{n + \frac{1}{2}}{\omega} \frac{\pi z}{L} \right) \right] e^{-i\kappa \cdot \rho}. \quad (8) \]

Next we evaluate the electric field operator \( E(r, t) \). Recalling that \( a^\dagger|0\rangle = 0 \) and \( a^\dagger|0\rangle = 0 \) we first write for the correlators \( <E_i(r, t)E_j(r, t)_0> \) a general expression of the form:

\[ <E_i(r, t)E_j(r, t)_0> = \sum_\alpha E_{i\alpha}(r)E_{j\alpha}^*(r). \quad (9) \]

In our case (7) and (8) yield

\[ E_{i\kappa n}^{(1)}(r) = \left( \frac{\pi}{\omega} \right)^{\frac{3}{2}} \frac{1}{\omega^2} \left[ \hat{\kappa} \sin \left( \frac{n + \frac{1}{2}}{\omega} \frac{\pi z}{L} \right) - \frac{\kappa}{\omega} \cos \left( \frac{n + \frac{1}{2}}{\omega} \frac{\pi z}{L} \right) \right] e^{-i\kappa \cdot \rho} (\hat{\kappa} \times \hat{z})_i, \quad (10) \]

and

\[ E_{i\kappa n}^{(2)}(r) = \left( \frac{\pi}{\omega} \right)^{\frac{3}{2}} \frac{1}{\omega^2} \left[ \hat{\kappa} \sin \left( \frac{n + \frac{1}{2}}{\omega} \frac{\pi z}{L} \right) - \frac{\kappa}{\omega} \cos \left( \frac{n + \frac{1}{2}}{\omega} \frac{\pi z}{L} \right) \right] e^{-i\kappa \cdot \rho}. \quad (11) \]

Now we substitute (10) and (11) into (9), write \( \hat{\kappa} = \cos \phi \delta_{iz} + \sin \phi \delta_{iy}, \hat{z} = \delta_{iz} \) and \( (\hat{\kappa} \times \hat{z})_i = \sin \phi \delta_{iz} - \cos \phi \delta_{iy} \), where \( \phi \) is the azimuthal angle in the \( xy \)-plane and compute all angular integrals. In this way we wind up with

\[ <\hat{E}_i(r, t)\hat{E}_j(r, t)_0> = \left( \frac{2}{\pi} \right) \left( \frac{\pi}{L} \right)^{\frac{3}{2}} \sum_{n=0}^{\infty} \sin^2 \left( \frac{n + \frac{1}{2}}{\omega} \frac{\pi z}{L} \right) \int_0^\infty dk \kappa \omega(\kappa, n) \]

\[ + \left( \frac{2}{\pi} \right) \left( \frac{\pi}{L} \right)^{\frac{5}{2}} \sum_{n=0}^{\infty} \sin^2 \left( \frac{n + \frac{1}{2}}{\omega} \frac{\pi z}{L} \right) (n + \frac{1}{2})^2 \int_0^\infty dk \kappa \omega^{-1}(\kappa, n) \]

\[ + \left( \frac{2}{\pi} \right) \left( \frac{\pi}{L} \right)^{\frac{5}{2}} \sum_{n=0}^{\infty} \cos^2 \left( \frac{n + \frac{1}{2}}{\omega} \frac{\pi z}{L} \right) \int_0^\infty dk \kappa^3 \omega^{-1}(\kappa, n), \quad (12) \]

where \( \delta_{ij} := \delta_{iz}\delta_{jz} + \delta_{iy}\delta_{jy} \) and \( \delta_{ij} := \delta_{iz}\delta_{jz} \). The previous equation is only a formal expression for the field correlator \( <\hat{E}_i(r, t)\hat{E}_j(r, t)_0> \), since it is an ill-defined expression plagued by divergent terms. Therefore, it lacks of physical meaning unless we adopt a regularization prescription. We will first regularize the integrals in equation (12) by using a method based on analytical extension in the complex plane. The idea is the following: take for example the first integral that appears on the r.h.s. of (12),

\[ I_1(n, L) := \int_0^\infty dk \kappa \left( \kappa^2 + \left( n + \frac{1}{2} \right)^2 \frac{\pi^2}{L^2} \right)^{1/2}. \]
Since this integral diverges for large $\kappa$, it is natural to modify the integrand so that the integral becomes finite. Our choice will be simply

$$I_1(n, L) \rightarrow I_{1, reg}(n, L; s) := \int_0^{\infty} d\kappa \kappa \left( \kappa^2 + \frac{(n + \frac{1}{2})^2 \pi^2}{L^2} \right)^{1/2-s}$$

and after the calculations we will take the limit $s \rightarrow 0$. For the moment, let us assume that $\Re s$ is large enough to give a precise mathematical meaning for the previous integral. Then, making use of the following integral representation of the Euler beta function, c.f. formula 3.251.2 [36]:

$$\int_0^\infty dx x^{\mu-1} \left( x^2 + a^2 \right)^{\nu-1} = \frac{B(\mu, 1-\nu)}{a^{\mu+2\nu-2}},$$

where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$, which holds for $\Re \left( \nu + \frac{\mu}{2} \right) < 1$ and $\Re \mu > 0$, we get

$$I_{1, reg}(n, L; s) = \frac{1}{2} \left[ (n + \frac{1}{2})^{3-2s} \frac{\Gamma(s-3/2)}{\Gamma(s-1/2)} \right] \left[ (n + \frac{1}{2})^{3-2s} \cos \left( \frac{2(n + 1/2)\pi z}{L} \right) \right]$$

Inserting this result into the first term of the r.h.s. of (12) (call it $T_1$), it takes the form:

$$T_1 = \left( \frac{1}{2s-3} \right) \left( \frac{\pi}{L} \right)^{3-2s} \frac{\delta_{ij}^\parallel}{2L} \left[ \zeta_H(2s-3, 1/2) - \sum_{n=0}^{\infty} \left( n + \frac{1}{2} \right)^{3-2s} \cos \left( \frac{2(n + 1/2)\pi z}{L} \right) \right]$$

where $\zeta_H(z, a)$ is the well known Hurwitz zeta function. Making the analytical extension to the $s$-complex plane and taking the limit $s \rightarrow 0$, we get

$$T_1 = -\frac{1}{6\pi} \left( \frac{\pi}{L} \right)^4 \frac{\delta_{ij}^\parallel}{L} \left\{ \left( -\frac{7}{8} \right) \times \frac{1}{120} - G(\pi z/L) \right\},$$

where we made use of $\zeta_H(-3, 1/2) = (-7/8) \times (1/120)$ and defined

$$G(\xi) = -\frac{1}{8} \times \frac{d^3}{d\xi^3} \left( \frac{1}{2\sin \xi} \right)$$

$$= \frac{1}{8} \left( \frac{\cos^3 \xi}{\sin^4 \xi} + \frac{5 \cos \xi}{2\sin^2 \xi} \right).$$

Analogous calculations can be performed with the other terms of the r.h.s. of (12). It is then straightforward to show that

$$\langle \hat{E}_i(r, t) \hat{E}_j(r, t) \rangle_0 = \left( \frac{\pi}{L} \right)^4 \frac{2}{3\pi} \left[ \left( -\frac{7}{8} \right) \left( -\delta_{ij}^\parallel + \delta_{ij}^\perp \right) \right] \frac{1}{120} + \delta_{ij} G(\pi z/L),$$

$$\langle \hat{E}_i(r, t) \hat{E}_j(r, t) \rangle_0 = \left( \frac{\pi}{L} \right)^4 \frac{2}{3\pi} \left[ \left( -\frac{7}{8} \right) \left( -\delta_{ij}^\parallel + \delta_{ij}^\perp \right) \right] \frac{1}{120} + \delta_{ij} G(\pi z/L),$$

(18)
and proceeding in the same way we did in the evaluation of the electric field correlators we obtain

\[
\langle \hat{B}_i(r, t) \hat{B}_j(r, t) \rangle_0 = \left( \frac{\pi}{L} \right)^4 \frac{2}{3\pi} \left[ \left( -\frac{7}{8} \right) \left( -\delta^i + \delta^j \right)_{ij} \frac{1}{120} - \delta_{i,j}G(\pi z/L) \right], \tag{19}
\]

for the magnetic field correlators. A straightforward calculation along the lines given here or the use of time-reversal invariance shows that the correlators \( \langle E_i(r, t)B_j(r, t) \rangle_0 = 0 \).

In passing, observe that no substractions whatsoever were required in our regularization procedure. This is a common feature of regularization prescriptions based on the analytical extension. However, other methods where the subtraction of the field correlators involving no boundary conditions are present can be used yielding the same results.

### 3 The Casimir effect between Boyer’s plates

In order to get confidence in the previous results for the field operator correlators between Boyer’s plates, let us reobtain Boyer’s result [25] concerning the Casimir effect for this unusual set up. First, recall that the zero-point energy density \( \rho_0 \) for the electromagnetic fields is defined by the following vacuum expectation value:

\[
\rho_0 = \frac{1}{8\pi} \langle E^2 + B^2 \rangle_0. \tag{20}
\]

Making use of (18) and (19) we obtain the position-dependent correlators:

\[
\langle E^2 \rangle_0 = \left( \frac{\pi}{L} \right)^4 \frac{2}{3\pi} \left[ \frac{7}{8 \times 120} + 3G(\xi) \right], \tag{21}
\]

\[
\langle B^2 \rangle_0 = \left( \frac{\pi}{L} \right)^4 \frac{2}{3\pi} \left[ \frac{7}{8 \times 120} - 3G(\xi) \right]. \tag{22}
\]

If we add these two equations the position-dependent terms will cancel out and if we substitute the result into (20) we will obtain:

\[
\rho_0 = \frac{7}{8} \times \frac{\pi^2}{720L^4}, \tag{23}
\]

which is the position-independent and positive Casimir energy density leading to a repulsive force per unit area between the plates [25, 27, 28].

It is also convenient to analyze the behavior of the correlators \( \langle E^2 \rangle_0 \) and \( \langle B^2 \rangle_0 \) in the situations where one of the plates is removed. Let us first consider the limit of a
single metal plate located at \( z = 0 \). This means that we are taking the limit \( L \to \infty \) in our previous results. The results are:

\[
< E^2 >_0 \approx + \frac{3}{4 \pi z^4},
\]  

(24)

and

\[
< B^2 >_0 \approx - \frac{3}{4 \pi z^4},
\]  

(25)

in agreement with the literature [26]. On the other hand, the limit of a single infinitely permeable plate is obtained by removing the metal plate. This can be accomplished if we consider the limits \( L \to \infty, \; z \to \infty \) in the previous results, but with \( L - z \ll L \). For this case we obtain:

\[
< E^2 >_0 \approx - \frac{3}{4 \pi (z - L)^4},
\]  

(26)

and

\[
< B^2 >_0 \approx + \frac{3}{4 \pi (z - L)^4},
\]  

(27)

Equations (26) and (27) are new results. Let us turn our attention now to one of the most intriguing properties of the QED vacuum: its anisotropy and the concomitant consequences on the speed of light.

4 The Scharnhorst effect for the spinor QED

The Scharnhorst effect [1, 2] is basically the light velocity shift in the QED vacuum caused by the presence of two parallel plates for propagation inside the plates and perpendicular to them. This was shown to occur for small frequencies \( \omega \ll m \) (soft photon approximation) and in the weak field limit. For the case of metallic plates, Scharnhorst [1] and later on Barton [2] showed that the phase velocity, which for this case coincides with the phase velocity for small frequencies, is greater than its value in free space (\( c \)) for propagation perpendicular to the plates. However, this does not mean that the signal velocity can be greater than \( c \) because to determine the wave front velocity it is necessary to investigate the dispersion relation in the infinite frequency limit (see reference [30, 3, 29, 31] for some discussion on this issue). The Scharnhorst effect with a boundary condition other than the standard one for perfect metallic plates has also been considered [32]. It can be understood as follows: the external field as that describing the propagation of a plane wave interacts with the quantized electromagnetic fields through the fermionic loops and hence, any change in the quantized field modes, as for example caused by imposition of boundary conditions, can in principle modify the wave propagation. In references [4, 4] this change was induced by the presence of two perfect parallel conducting plates. Since these authors
assumed that the plates do not impose any boundary condition on the fermionic field, the Scharnhorst effect appears only at the two-loop level. Also, because it is a perturbative effect, it can be obtained by direct computation of the relevant Feynman diagrams that contribute to the effective action, namely: the two possible diagrams for the photon polarization tensor at two-loop level. This was precisely Scharnhorst’s approach, who after using a previous representation for the photon propagator between two metallic plates obtained by Bordag, Robaschik and Wieczorek \cite{33} found for propagation perpendicular to the plates that

$$v_\perp = 1 + \frac{11\pi^2}{2^2.3^4.5^2} \frac{\alpha^2}{(mL)^4}. \quad (28)$$

Later on, the same result was rederived by Barton \cite{2} in a more economic way, where the connection to the Casimir energy density is more apparent. The starting point in Barton’s approach is the addition to the electromagnetic field lagrangian density of a correction term represented by the Euler-Heisenberg \cite{34} effective lagrangian density, so that the full lagrangian density reads:

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} = \frac{1}{8\pi} \left( E^2 - B^2 \right) + g \left[ \left( E^2 - B^2 \right)^2 + 7 \left( E \cdot B \right)^2 \right], \quad (29)$$

where $g := \frac{\alpha^2}{5 \cdot 3^2 \cdot 2^3 \cdot \pi^2 m^4}$. The lagrangian density represented by (29) describes the first vacuum polarization effects on slowly varying fields for which the condition $\omega \ll m$ holds and is valid only in the weak field approximation. In other words, the first non-linear effects to Maxwell equations coming from QED are described by the quartic terms added to the usual Maxwell lagrangian density in the above formula. The corresponding vacuum polarization $P$ and magnetization $M$ are given by:

$$P = \frac{\partial \mathcal{L}^{(1)}}{\partial E} = 4g \left( E^2 - B^2 \right) E + 14g \left( E \cdot B \right) B, \quad (30)$$

and

$$M = \frac{\partial \mathcal{L}^{(1)}}{\partial B} = -4g \left( E^2 - B^2 \right) B + 14g \left( E \cdot B \right) E. \quad (31)$$

In order to include a radiative correction into the formalism, we can follow reference \cite{2} and rewrite the fields in equations (30) and (31) as the sum of two parts, one describing the quantized fields and the other one describing the classical fields, that is, we write: $E \rightarrow E_q + E_c$ and $B \rightarrow B_q + B_c$ and substitute into (30) and (31). This procedure is tantamount to the coupling of the external fields to the quantized ones by means of the intermediary action of a fermionic loop. Keeping only the terms which are linear in the
classical fields, we obtain the following expressions for the electric susceptibility $\chi^{(e)}_{ij}$ and magnetic susceptibility $\chi^{(m)}_{ij}$ tensors of the vacuum:

$$\chi^{(e)}_{ij} = 4g \left[ <E_q^2 - B_q^2 >_0 \delta_{ij} + 2 <E_{qi}E_{qj} >_0 \right] + 14g <B_{qi}B_{qj} >_0 ,$$

$$\chi^{(m)}_{ij} = 4g \left[ -<E_q^2 - B_q^2 >_0 \delta_{ij} + 2 <B_{qi}B_{qj} >_0 \right] + 14g <E_{qi}E_{qj} >_0 .$$

The dielectric and permittivity tensors of the vacuum are:

$$\epsilon_{ij} = \delta_{ij} + 4\pi \chi^{(e)}_{ij} = \delta_{ij} + \Delta \epsilon_{ij} ,$$

$$\mu_{ij} = \delta_{ij} + 4\pi \chi^{(m)}_{ij} = \delta_{ij} + \Delta \mu_{ij} ,$$

The vacuum expectation values in (32) and (33) can be easily calculated with the correlators given by (18) and (19). If we do this, we obtain for $\Delta \epsilon_{ij}$ and $\Delta \mu_{ij}$ the results:

$$\Delta \epsilon_{ij} = g \left( \frac{\pi}{L} \right) \frac{16}{3} \left[ \left( -\frac{7}{8} \right) \left( -\delta^\parallel + \delta^\perp \right)_{ij} \left( \frac{11}{120} \right) + 3\delta_{ij}G(\xi) \right] ,$$

and

$$\Delta \mu_{ij} = g \left( \frac{\pi}{L} \right) \frac{16}{3} \left[ \left( -\frac{7}{8} \right) \left( -\delta^\parallel + \delta^\perp \right)_{ij} \left( \frac{11}{120} \right) - 3\delta_{ij}G(\xi) \right] .$$

We can also derive single plate limits for $\Delta \epsilon_{ij}$ and $\Delta \mu_{ij}$. Making use of the approximations to $G(\xi)$ in the limits $\xi \to 0$ and $\xi \to \pi$ we have near the conducting plate at $z = 0$:

$$\Delta \epsilon_{ij} = -\Delta \mu_{ij} = 18g \frac{\delta_{ij}}{z^4} ,$$

and near the permeable plate at $z = L$:

$$\Delta \epsilon_{ij} = -\Delta \mu_{ij} = -18g \frac{\delta_{ij}}{(z - L)^4} .$$

Now, we are interested in the refraction index $n = \sqrt{\epsilon\mu}$ and its first order shift:

$$\Delta n = \frac{1}{2} \left( \Delta \epsilon + \Delta \mu \right) ,$$

for directions of propagation defined by the cartesian axis. Let us consider first a plane wave propagating in the $OX$-direction with the electric field vibrating in the $OZ$-direction. Then $\Delta \epsilon \to \Delta \epsilon_{33}$ and $\Delta \mu \to \Delta \mu_{22}$, and from (34), (37) and (40) we can easily verify that $\Delta n = \frac{1}{2} \left( \Delta \epsilon_{33} + \Delta \mu_{22} \right) = 0$. We obtain the same result in all instances in which the propagation is parallel to the plane of the plates. As a consequence the speed of light
remains unchanged for propagation parallel to the plates. Now consider a plane wave propagating along the $OZ$-axis, perpendicularly to the pair of plates. Consider the wave polarized in the $OX$-direction, for instance. Then $\Delta \epsilon \rightarrow \Delta \epsilon_{11}$ and $\Delta \mu \rightarrow \Delta \mu_{22}$, and from (36), (37) and (40) we now obtain:

$$\Delta n_\perp \approx \frac{1}{2} (\Delta \epsilon_{11} + \Delta \mu_{22})$$

$$= + \frac{7}{8} \times \frac{\alpha^2}{(mL)^4} \frac{11\pi^2}{2^2 \cdot 3^4 \cdot 5^2}$$

which is the result obtained by Scharnhorst [1] and reobtained by Barton [2] multiplied by the factor $-7/8$. The speed of light in that direction will be:

$$v_\perp \approx 1 - \frac{7}{8} \times \frac{\alpha^2}{(mL)^4} \frac{11\pi^2}{2^2 \cdot 3^4 \cdot 5^2} < 1,$$

as anticipated in the beginning of this work. The direction-averaged light velocity between Boyer’s plates also satisfies the unifying formula written down by Latorre, Pascual and Tarrach [23] for spinor QED which reads

$$\langle v \rangle = 1 - \frac{44\alpha^2}{135m_e^4} \rho_0.$$ (43)

It can be shown that this formula can be obtained in the weak field limit of Dittrich and Gies’ approach to the study of non-trivial vacua [24]. We will return to this in the next section.

### 4.1 The Scharnhorst effect in scalar QED

Although the interaction of charged fermions of spin 1/2 with themselves and with the photon field is described in a very satisfactory way by spinor QED, we are not prohibited of thinking on other theories. It may be very instructive to study other theories that, though not realistic, respect all important physical principles as for instance, the gauge principle and relativistic invariance. This is the case of the so-called scalar QED, which describes charged bosons interacting with themselves and with the radiation field. Naively, we could think that the interaction between the pseudoscalars charged mesons $\pi^\pm$ and $K^\pm$ could be described by scalar QED, but this is not true, mainly because these mesons have an inner structure and their interaction is dominated by the strong interaction. In fact, since there are no fundamental charged bosons in Nature, scalar QED is of limited application. However, scalar QED can be viewed as a toy model in many situations and hence it may shed some light on interesting physical processes, as we shall see. Without
further apologies, we shall consider in this section the Scharnhorst effect in the framework of scalar QED. In the case of scalar QED the analogue of the Euler-Heisenberg effective lagrangian reads [35]:

\[ L^{(1)}_0 = g_0 \left[ \frac{7}{4} (E^2 - B^2)^2 + (E \cdot B)^2 \right], \tag{44} \]

with \( g_0 := \frac{\alpha^2}{5 \cdot 3^2 \cdot 2^5 \cdot \pi^2 \cdot m_o^4} \), where \( m_o \) is the mass of the hypothetical charged boson associated with 1-loop scalar QED. As before, the polarization \( P \) and the magnetization \( M \) are defined by equations (30) and (31), and as before we make use of the substitutions \( E \rightarrow E_q + E_c \) and \( B \rightarrow B_q + B_c \) and keep only terms linear in the classical fields to obtain the corrections \( \Delta \epsilon_{ij} \) and \( \Delta \mu_{ij} \) to the dielectric and permittivity tensors of the scalar QED vacuum. The results are

\[ \Delta \epsilon_{ij} = 28\pi g_0 < E^2 - B^2 >_0 \delta_{ij} + 56\pi g_0 < E_i E_j >_0 + 8\pi g_0 < B_i B_j >_0, \tag{45} \]

\[ \Delta \mu_{ij} = -28g_0 < E^2 - B^2 >_0 \delta_{ij} + 56\pi g_0 < B_i B_j >_0 + 8\pi g_0 < E_i E_j >_0. \tag{46} \]

Now we can make use of these results and analyze the speed of light in confined scalar QED vacuum. Since the Scharnhorst effect for scalar QED has never been discussed before, we will evaluate the light velocity shifts for two cases, to wit, for Casimir’s plates and for Boyer’s plates.

**Casimir’s plates.** We shall consider the two perfectly conducting plates at \( z = 0 \) and \( z = L \) respectively. Expressions for the electric and magnetic field correlators for the Casimir’s plates can be found in, for instance, [2], here we merely state the results

\[ < E_i(r,t) E_j(r,t) >_0 = \left( \frac{\pi}{L} \right)^4 \frac{2}{3\pi} \left[ (-\delta^\parallel + \delta^\perp)_{ij} \frac{1}{120} + \delta_{ij} F(\pi z/L) \right], \tag{47} \]

and

\[ < B_i(r,t) B_j(r,t) >_0 = \left( \frac{\pi}{L} \right)^4 \frac{2}{3\pi} \left[ (-\delta^\parallel + \delta^\perp)_{ij} \frac{1}{120} - \delta_{ij} F(\pi z/L) \right], \tag{48} \]

where \( F(\xi) \) is defined by:

\[ F(\xi) := -\frac{1}{8} \times \frac{d^3}{d\xi^3} \left( \frac{1}{2} \cot \xi \right). \tag{49} \]

Now we take (47) and (48) into (45) and (46) and after some simple manipulations we end up with

\[ \Delta \epsilon_{ij} = \frac{16}{3} g_0 \left( \frac{\pi}{L} \right)^4 \left[ (-\delta^\parallel + \delta^\perp)_{ij} \left( \frac{1}{15} \right) + 27\delta_{ij} F(\xi) \right], \tag{50} \]

and,

\[ \Delta \mu_{ij} = \frac{16}{3} g_0 \left( \frac{\pi}{L} \right)^4 \left[ (-\delta^\parallel + \delta^\perp)_{ij} \left( \frac{1}{15} \right) - 27\delta_{ij} F(\xi) \right]. \tag{51} \]
With these results we can now calculate the first correction to the refraction index $\Delta n$ and, consequently, the correction to the speed of light between Casimir’s plates in scalar QED. As in the corresponding case of spinor QED, we find that the speed of light parallel to the plates remains unchanged, but the speed of light perpendicular to the plates is modified by an amount given by

$$\Delta v_\perp = -\Delta n = + \frac{16}{45} g_0 \left( \frac{\pi}{L} \right)^4 > 0.$$  \hspace{1cm} (52)

It is interesting to compare this result with the analogous effect that took place in spinor QED. Assuming the same charge for the particles (bosons and fermions), we see that the ratio between the light velocity shifts for scalar and usual QED is given by

$$\frac{\Delta v^h}{\Delta v_\perp} = 8 \times \left( \frac{m}{m_o} \right)^4.$$  \hspace{1cm} (53)

Boyer’s plates. Now we repeat the procedure for the unusual pair of plates that we are discussing here. The electric and magnetic field correlators we need are given by equations (18) and (19). Substituting into (45) and (46) we obtain

$$\Delta \epsilon_{ij} = \frac{16}{3} g_0 \left( \frac{\pi}{L} \right)^4 \left[ \left( -\frac{7}{8} \right) \left( -\delta^\| + \delta^\perp \right)_{ij} \left( \frac{1}{15} \right) + 27 \delta_{ij} G(\xi) \right],$$  \hspace{1cm} (54)

and,

$$\Delta \mu_{ij} = \frac{16}{3} g_0 \left( \frac{\pi}{L} \right)^4 \left[ \left( -\frac{7}{8} \right) \left( -\delta^\| + \delta^\perp \right)_{ij} \left( \frac{1}{45} \right) - 27 \delta_{ij} G(\xi) \right].$$  \hspace{1cm} (55)

Hence, the speed of light between Boyer’s plates in the direction perpendicular to the plates is modified by the amount

$$\Delta v_\perp = -\Delta n = -\frac{7}{8} \times \frac{16}{45} g_0 \left( \frac{\pi}{L} \right)^4 < 0.$$  \hspace{1cm} (56)

The results given by equations (52) and (56) can be unified by considering the average taken over all directions of propagation of the speed of light between the plates. To accomplish this first we write, for instance, for Casimir’s plates:

$$v(\theta) = 1 - \frac{16}{45} g_0 \left( \frac{\pi}{L} \right)^4 \cos^2 \theta,$$  \hspace{1cm} (57)

where $\theta$ is the angle between the direction of propagation and the $OZ$-axis. Next we take the average over all direction. The result is:

$$\langle v \rangle = \frac{1}{4\pi} \int v(\theta) d\Omega = 1 + \frac{8\alpha^2}{135m_o^4} \left( \frac{\pi^2}{720L^4} \right) = 1 + \frac{8\alpha^2}{135m_o^4} \rho_0.$$  \hspace{1cm} (58)
The same result can be obtained from equation (54) with \( \rho_0 = \left( -\frac{7}{8} \right) \times \left( \frac{\pi^2}{720} L^4 \right) \). This is the scalar QED version of the unifying formula obtained by Pascual, Latorre and Tarrach for spinor QED [23], and also as in the spinor QED case, it corresponds to the weak field limit of a more general approach due to Dittrich and Gies [24].

**Final remarks**

In this work we have endeavoured to give another example of the consequences of imposing boundary conditions on QED vacuum oscillations by discussing these oscillations confined by a somewhat unusual pair of plates. In particular, we have obtained through a simple regularization procedure the correlators for the vacuum oscillations of the electromagnetic field sandwiched between these plates, the associated Casimir energy density and the natural converse of the original Scharnhorst effect at zero temperature. Incidentally, observe that contrary to the case of Casimir’s plates, in the case we discussed here there is no critical temperature for which the Scharnhorst effect would vanish. Also, as in the case of Casimir’s plates, the refraction index is frequency-independent, for the Euler-Heisenberg lagrangian density holds only for slowly varying fields. We have also examined the scalar QED version of the Scharnhorst effect and produced a a formula that plays the role of the unifying formula due to Latorre, Pascual and Tarrach for the case of spinor QED.

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