COUNTEREXAMPLES TO THE DISCRETE AND CONTINUOUS WEIGHTED WEISS CONJECTURES

ANDREW WYNN

ABSTRACT. Counterexamples are presented to weighted forms of the Weiss conjecture in discrete and continuous time. In particular, for certain ranges of α, operators are constructed that satisfy a given resolvent estimate, but fail to be α-admissible. For α ∈ (−1, 0) the operators constructed are normal, while for α ∈ (0, 1) the operator is the unilateral shift on the Hardy space $H^2(D)$.

1. Introduction

Suppose that $(T(t))_{t \geq 0} \subset \mathcal{L}(X)$ is a $C_0$-semigroup with infinitesimal generator $A$ on a Hilbert space $X$. Let $C \in \mathcal{L}(D(A), \mathbb{C})$ be a linear operator which is bounded with respect to the graph norm $\| \cdot \|_{D(A)} := \|A \cdot \|_X + \| \cdot \|_X$ on $D(A)$. Consider the linear system given by

$$
\begin{align*}
\dot{x}(t) &= Ax(t), \quad t > 0; \\
x(0) &= x_0 \in X; \\
y(t) &= Cx(t), \quad t > 0.
\end{align*}
$$

If $x_0 \notin D(A)$, it is not necessarily the case that mild solution $x(t) = T(t)x_0$ lies in $D(A)$ for each $t > 0$ and hence, the output map $y(\cdot)$ is not properly defined. However, if it is assumed that $C$ is admissible for $A$ in the sense that there exists a constant $M > 0$ such that

$$
\int_0^\infty |CT(t)x_0|^2 dt \leq M^2 \|x_0\|^2_X, \quad x_0 \in D(A),
$$

then the operator $\Psi : D(A) \to L^2(\mathbb{R}^+) \times \mathbb{C}$ given by $(\Psi x)(\cdot) := CT(\cdot)x$ extends continuously to the whole space $X$. In this case, the output map is considered to be given by $y = \Psi x_0$.

A generalisation of this idea, studied in [7, 8], is to require that $\Psi$ is bounded from $D(A)$ to a weighted $L^2$-space. For $\alpha \in (-1, 1)$, the functional $C$ is said to be $\alpha$-admissible for $A$ if there exists a constant $M > 0$ such that

$$
\int_0^\infty t^\alpha |CT(t)x_0|^2 dt \leq M^2 \|x_0\|^2_X, \quad x_0 \in D(A)
$$

and it is not difficult to show [7] that $\alpha$-admissibility implies the resolvent condition

$$
\sup_{\lambda \in \mathbb{C}_+} (\text{Re}\lambda)^{-\alpha} \|CR(\lambda, A)\|_{X^*} < \infty.
$$

An interesting problem is to attempt to characterise the operators $A, C$ and weights $\alpha$ for which the reverse implication $\alpha \Rightarrow \alpha$ is true. The continuous weighted Weiss conjecture is said to hold for a class of operators if, given a generator $A$ of
that class, \( \alpha \)-admissibility of any observation operator \( C \in \mathcal{L}(D(A), \mathbb{C}) \) is equivalent to \( \mathbf{(2)} \).

Initially, the case \( \alpha = 0 \) was considered and in this situation it has been shown that the Weiss conjecture holds whenever \( A \) is the generator of a \( C_0 \)-semigroup of contractions \( \mathbf{(11)} \). However, counterexamples to the unweighted conjecture also exist \( \mathbf{[9, 14, 15]} \). For a survey of the subject see \( \mathbf{[12]} \). The weighted form of the conjecture was introduced in \( \mathbf{[7]} \) for generators of analytic \( C \)-semigroups and for \( \alpha \in (-1, 1) \) the weighted Weiss conjecture holds in this situation whenever \( A^{1/2} \) is admissible for \( A \).

If \( A \) is a normal operator generating an analytic \( C_0 \)-semigroup it is easy to check that \( A^{1/2} \) is admissible for \( A \). Furthermore, if \( \alpha \in [0, 1) \) and \( A \) is a normal operator generating a contractive \( C_0 \)-semigroup, the weighted Weiss conjecture holds without the assumption of analyticity \( \mathbf{[24]} \). In \( \mathbf{[2]} \) it is shown that, even for normal operators, the weighted Weiss conjecture fails in the case \( \alpha \in (-1, 0) \).

A discrete form of the weighted Weiss conjecture can also be formulated \( \mathbf{[10, 24]} \). If \( X \) is a Hilbert space and \( A \in \mathcal{L}(X) \) with spectrum \( \sigma(A) \subset \overline{\mathbb{D}} := \{ z \in \mathbb{C} : |z| \leq 1 \} \) and \( C \in X^* \), the linear functional \( C \) is said to be discrete \( \alpha \)-admissible for \( A \) if

\[
\sum_{n=0}^{\infty} (1 + n)^\alpha |CA^n x|^2 \leq M \|x\|^2_X, \quad x \in X
\]

If \( C \) is discrete \( \alpha \)-admissible for \( A \) it can be shown \( \mathbf{[24]} \) that

\[
\sup_{\omega \in \mathbb{D}} (1 - |\omega|^2)^{\frac{1}{2}} \|C(I - \omega A)^{-1}\|_{X^*} < \infty.
\]

The discrete weighted Weiss conjecture is said to hold for a class of operators if, given a generator \( A \) of that class, discrete \( \alpha \)-admissibility of any \( C \in X^* \) is equivalent to \( \mathbf{(1)} \). If \( \alpha = 0 \), the discrete weighted Weiss conjecture holds for contraction operators \( \mathbf{[10]} \) and it is shown in \( \mathbf{[24]} \) that for \( \alpha \in (0, 1) \) the discrete weighted Weiss conjecture holds for contractive, normal operators.

In \( \mathbf{[2]} \) counterexamples are given to the discrete conjecture. It is shown that, even for normal operators, the discrete weighted Weiss conjecture fails for \( \alpha \in (-1, 0) \). In the case \( \alpha \in (0, 1) \), the unilateral shift on \( H^2(\mathbb{D}) \) fails the discrete weighted Weiss conjecture, in contrast to the unweighted case \( \alpha = 0 \).

## 2. Counterexamples to the continuous weighted Weiss conjecture

Let \( \alpha \in (-1, 0) \). Suppose that \( \mu \) is a finite, positive measure such that \( \text{supp}(\mu) \) is a bounded subset of the closed upper half plane \( \Pi_+ := \{ z \in \mathbb{C} : \text{Im} z \geq 0 \} \) and \( \mu(\mathbb{R}) = 0 \). Let \( X := L^2(\Pi_+, \mu) \) and define operators \( A \in \mathcal{L}(X) \), \( C \in X^* \) by

\[
(Af)(z) := izf(z), \quad f \in X, z \in \Pi_+, \quad Cf := \int_{\Pi_+} f(z)d\mu(z), \quad f \in X.
\]

Notice that \( A \in \mathcal{L}(X) \) is a normal operator, generating a contractive \( C_0 \)-semigroup \( \mathbf{(T(t))_{t \geq 0}} \) on \( X \) given by

\[
(T(t)f)(z) = e^{itz}f(z), \quad f \in X, z \in \Pi_+, t > 0.
\]

For an interval \( I \subset \mathbb{R} \) define \( R(I) := \{ x + iy \in \Pi_+ : x \in I, y \in (0, |I|/2) \} \). The resolvent estimate \( \mathbf{(2)} \) can be characterised in terms of a bound on \( \mu \) on the sets \( R(I) \).

\[ \text{ADDITIONAL CONTENT HERE} \]
Lemma 2.1. Let $\alpha \in (-1,0)$ and assume that $X, A, C$ and $\mu$ are as above. Then (2) holds if and only if there exists a constant $c > 0$ such that $\mu(R(I)) \leq c |I|^{1+\alpha}$, for any interval $I \subset \mathbb{R}$.

Proof. For any $\lambda \in \mathbb{C}_+$ and $x \in X$, $CR(\lambda, A)x = (R(\lambda, A)x, 1)_X$ where $1(z) = 1, z \in \Pi_+$. Hence,

$$||CR(\lambda, A)||^2_X = ||R(\lambda, A)^*1||^2_X = \int_{\Pi_+} \frac{d\mu(z)}{|\lambda - iz|^2}, \quad \lambda \in \mathbb{C}_+.$$ 

The result follows from [24], Lemma 5.8. \hfill \square

The following result provides a condition on $\mu$ that is necessary for weighted admissibility.

Proposition 2.2. Let $\alpha \in (-1,0)$ and suppose that $A, C, X$ and $\mu$ are as above. If $C$ is $\alpha$-admissible for $A$ there exists a constant $M > 0$ such that

$$\left(\int_{\Pi_+} \int_0^\infty e^{izt} t^{\alpha/2} v(t) dt \right)^2 \leq M ||v||_{L^2(\mathbb{R}_+)}^2, \quad v \in L^2(\mathbb{R}_+).$$

Proof. Suppose that $C$ is $\alpha$-admissible for $A$. Let $v \in L^2(\mathbb{R}_+)$ and

$$x \in D := \left\{ y \in X : \int_{\Pi_+} \frac{|y(z)|^2}{(\text{Im} z)^{1+\alpha}} d\mu(z) < \infty \right\}.$$ 

Then,

$$\int_{\Pi_+} \int_0^\infty |e^{izt} x(z) t^{\alpha/2} v(t) dt d\mu(z) = \int_{\Pi_+} \left( \int_0^\infty e^{-i\text{Im} z t^{\frac{\alpha}{2}}} |v(t)| dt \right) |x(z)| d\mu(z)$$

(by Cauchy-Schwarz) \leq c_\alpha ||v||_2 \int_{\Pi_+} \frac{|x(z)|}{(\text{Im} z)^{\frac{\alpha}{2}}} d\mu(z)

(by Cauchy-Schwarz) \leq c_\alpha ||v||_2 \left( \mu(\Pi_+) \int_{\Pi_+} \frac{|x(z)|^2}{(\text{Im} z)^{1+\alpha}} d\mu(z) \right)^{\frac{1}{2}} < \infty

and Fubini’s theorem may be applied. Now,

$$\left\| \left( \int_0^\infty e^{i(\cdot)t^{\alpha/2}} v(t) dt \right), \mathcal{F}(\cdot) \right\|_X = \left\| \int_{\Pi_+} \left( \int_0^\infty e^{izt} t^{\alpha/2} v(t) dt \right) x(z) d\mu(z) \right\|

(by Fubini) \leq \left( \int_0^\infty \left( \int_{\Pi_+} e^{izt} x(z) d\mu(z) \right) t^{\alpha/2} v(t) dt \right)^{1/2}

(by Cauchy-Schwarz) \leq \left( \int_0^\infty t^{\alpha/2} |CT(t)x|^2 dt \right)^{1/2} ||v||_{L^2(\mathbb{R}_+)}

(by $\alpha$-admissibility) \leq M \|x\|_X ||v||_{L^2(\mathbb{R}_+)}.$$ 

Since $D$ is dense in $X$,

$$\left\| \int_0^\infty e^{i(\cdot)t^{\alpha/2}} v(t) dt \right\|_X \leq M ||v||_{L^2(\mathbb{R}_+)}.\]
If $\alpha > -1$ then, upon identifying functions that differ by a constant, the weighted Dirichlet space $D^2_{1+\alpha}(\Pi_+)$ contains those analytic functions $F : \Pi_+ \to \mathbb{C}$ for which

$$\|F\|^2_{D^2_{1+\alpha}} := \int_0^\infty \int_{-\infty}^{\infty} y^{1+\alpha}|F'(x+iy)|^2\,dx\,dy < \infty.$$  

Furthermore, by [5, Theorem 3], $F \in D^2_{1+\alpha}(\Pi_+)$ if and only if there exists a function

$$w \in L^2(\mathbb{R}_+, t^{-\alpha}dt) := \left\{ f : \mathbb{R}_+ \to \mathbb{C} : f \text{ measurable, } \int_0^\infty t^{-\alpha}|f(t)|^2dt < \infty \right\}$$

and a constant $c \in \mathbb{C}$ with

$$(6) \quad F(z) = \int_0^\infty e^{itz}w(t)\,dt + c, \quad z \in \Pi_+.$$  

In this case there exists a constant $k > 0$ with $\|F\|_{D^2_{1+\alpha}} = k\|w\|_{L^2(\mathbb{R}_+, t^{-\alpha}dt)}$. Proposition 2.2 now states that for $\alpha \in (-1, 0)$, the embedding

$$(7) \quad D^2_{1+\alpha}(\Pi_+) \hookrightarrow L^2(\Pi_+, d\mu)$$

is necessary for $\alpha$-admissibility of $C$ with respect to $A$. Hence, in order to create a counterexample, it is enough to find a measure $\mu$ satisfying $\mu(R(I)) \leq c|I|^{1+\alpha}$ but for which (7) does not hold.

In the unweighted case $\alpha = 0$, Lemma 2.1 and Proposition 2.2 are still true. However, since the unweighted Weiss conjecture holds for normal operators, a counterexample cannot be created in this case. Indeed, Proposition 2.2 implies that $H^2(\Pi_+) \hookrightarrow L^2(\Pi_+, d\mu)$ is necessary for $0$-admissibility, but by the Carleson measure theorem (see e.g. [3]), this embedding is equivalent to the bound $\mu(R(I)) \leq c|I|$. By Lemma 2.1 this bound on $\mu$ is the same as (2) with $\alpha = 0$. In fact, it is for exactly this reason that the unweighted Weiss conjecture is true for normal operators [22].

The reason that counterexamples can be found in the case $\alpha \in (-1, 0)$ is that measures satisfying (4) are characterised by a bound involving the Riesz capacity of certain sets (see e.g. [23, Theorem 4.4]) but not by a simple condition of the form $\mu(R(I)) \leq c|I|^{1+\alpha}$.

**Riesz capacities.** Let $\beta \in (0, 1)$. The $\beta$-Riesz capacity of a subset $A \subset \mathbb{R}$ is given by

$$(8) \quad \text{Cap}_\beta(A) := \inf \{ \|g\|_{L^2(\mathbb{R})}^2 : g \in L^2(\mathbb{R}), I_{\beta} * g \geq 1 \text{ on } A, \ g \geq 0 \}$$

where the **Riesz kernel** $I_{\beta}$ is defined by $I_{\beta} := |x|^\beta - 1$ (see e.g. [1], p.8). If $O \subset \mathbb{R}$ is an open set define

$$R(O) := \bigcup_{i=1}^\infty R(I_i),$$

where $O = \bigcup_{i=1}^\infty I_i$ is the decomposition of $O$ into disjoint open intervals of $\mathbb{R}$. If $\alpha \in (-1, 0)$, it is shown in [3, Theorem 6.1] that there exists a measure $\mu$ on $\Pi_+$ for which $\mu(R(I)) \leq c|I|^{1+\alpha}$ for any interval $I \subset \mathbb{R}$, but for which there does not exist a constant $c > 0$ with

$$(9) \quad \mu(R(O)) \leq c \cdot \text{Cap}_{-\alpha/2}(O), \quad O \subset \mathbb{R} \text{ open.}$$
Such a measure will be used to construct the counterexamples. With respect to the operators $A$ and $C$ introduced in [2] it is shown in Lemma 2.1 that the resolvent bound (2) is equivalent to the one-box condition $\mu(R(I)) \leq c|I|^{1+\alpha}$, while it will be shown later that the capacity estimate (11) is necessary for $\alpha$-admissibility. However, it will be useful to determine the possible structure of such a measure.

**Theorem 2.3.** Let $\alpha \in (-1, 0)$. Then there exists a finite, positive measure $\mu$ on $\Pi_+$ with compact support such that

(i) There exists $c > 0$ such that $\mu(R(I)) \leq c|I|^{1+\alpha}$, for any interval $I \subset \mathbb{R}$.

(ii) There does not exist $c > 0$ such that $\mu(R(O)) \leq c \cdot \text{Cap}_{-\alpha/2}(O)$ holds for every open set $O \subset \mathbb{R}$.

**Proof.** From the proof of [3, Theorem 6.1] there exists a non-trivial positive measure $\nu$ on $\mathbb{R}$ and a compact set $K \subset \mathbb{R}$ such that: there exists $c > 0$ such that $\nu(I) \leq c|I|^{1+\alpha}$, for any interval $I \subset \mathbb{R}$; $\text{Cap}_{-\alpha/2}(K) = 0$ and $\text{supp}(\nu) \subset K$. Since $\text{Cap}_{-\alpha/2}(K) = 0$, [1, Theorem 2.3.11] implies that there exists a sequence $(O^{(n)})_{n=1}^{\infty}$ of open sets $O^{(n)} \subset \mathbb{R}$ such that for each $n \in \mathbb{N}$, $O^{(n)} \supseteq O^{(n+1)}$, $K \subset O^{(n)}$ and additionally,

\[
\text{Cap}_{-\alpha/2}(O^{(n)}) \leq \frac{1}{n^2}, \quad n \in \mathbb{N}.
\]

The set $O^{(1)}$ can be expressed as a disjoint union of open intervals which form an open cover for $K$. Since $K$ is compact there exists a finite subcover $I_1^{(1)}, \ldots, I_{N_1}^{(1)}$ of intervals such that $\hat{O}^{(1)} := \bigcup_{i=1}^{N_1} I_i^{(1)} \supset K$ and since $\hat{O}^{(1)} \subset O^{(1)}$ it follows that $\text{Cap}_{-\alpha/2}(\hat{O}^{(1)}) \leq \text{Cap}_{-\alpha/2}(O^{(1)})$. Since $K \subset O^{(2)} \cap \hat{O}^{(1)}$, compactness can again be applied and there exist open intervals $I_1^{(2)}, \ldots, I_{N_2}^{(2)}$ such that $\hat{O}^{(2)} := \bigcup_{i=1}^{N_2} I_i^{(2)} \supset K$, $\hat{O}^{(2)} \subset O^{(2)} \cap \hat{O}^{(1)}$ and $\text{Cap}_{-\alpha/2}(\hat{O}^{(2)}) \leq \text{Cap}_{-\alpha/2}(O^{(2)})$. In this way it is possible to inductively define open sets $\hat{O}^{(n)} \subset \mathbb{R}$, such that for each $n \in \mathbb{N}$:

(a) $\hat{O}^{(n)} = \bigcup_{i=1}^{N_n} I_i^{(n)}$, for disjoint open intervals $I_1^{(n)}, \ldots, I_{N_n}^{(n)} \subset \mathbb{R}$;

(b) $K \subset \hat{O}^{(n+1)} \subset \hat{O}^{(n)} \subset O^{(n)}$;

(c) $\text{Cap}_{-\alpha/2}(\hat{O}^{(n)}) \leq \text{Cap}_{-\alpha/2}(O^{(n)}) \leq 1/n^2$.

For each $n \in \mathbb{N}$, define $\gamma_n := \frac{1}{n} \min_{i=1}^{N_n} |I_i^{(n)}| > 0$ and notice that without loss of generality the sets $\hat{O}^{(n)}$ can be picked in such a way that $(\gamma_n)_{n=1}^{\infty}$ is monotone decreasing. Furthermore, since $\text{Cap}_{-\alpha/2}(K) = 0$ it must be that case that $\gamma_n \to 0$ as $n \to \infty$. The measure $\mu$ on $\Pi_+$ is then defined by

\[
\mu := \sum_{m=1}^{\infty} \frac{1}{m^2} \cdot (\nu \times \delta_{\gamma_m}),
\]

where $\delta_x$ is the point mass at $x \in \mathbb{R}$.

(i) Let $I \subset \mathbb{R}$ be an interval. Then,

\[
\mu(R(I)) = \sum_{m=1}^{\infty} \frac{1}{m^2} \nu(R(I)) \leq \sum_{m=1}^{\infty} \frac{\nu(I)}{m^2} \leq \frac{c\pi^2|I|^{1+\alpha}}{6}.
\]
(ii) For a contradiction suppose that there exists a constant \( c > 0 \) such that 
\[ \mu(R(O)) \leq c \cdot \text{Cap}_{-\alpha/2}(O) \] for any open set \( O \subset \mathbb{R} \). For each \( n \in \mathbb{N} \),
\[ (12) \quad R(I_i^{(n)}) \supset \{ x + iy \in \Pi_+ : x \in I_i^{(n)}, 0 < y \leq \gamma_n \}, \quad 1 \leq i \leq N_n \]
and hence,
\[ \mu(R(\tilde{O}^{(n)})) = \mu \left( \bigcup_{i=1}^{N_n} R(I_i^{(n)}) \right) = \sum_{m=1}^{\infty} \frac{1}{m^2} \left( \nu \times \delta_{\gamma_n} \right) \left( \bigcup_{i=1}^{N_n} R(I_i^{(n)}) \right) \]
(by (12)) \[ \geq \sum_{m=n}^{\infty} \frac{1}{m^2} \nu \left( \bigcup_{i=1}^{N_n} I_i^{(n)} \right) \]
(by (a) and (b)) \[ \geq \nu(K) \sum_{m=n}^{\infty} \frac{1}{m^2} \]
\[ \geq \nu(K) \frac{1}{2n} \).
Hence, from the above inequality and (c),
\[ \frac{\nu(K)}{2n} \leq \mu(R(\tilde{O}^{(n)})) \leq c \cdot \text{Cap}_{-\alpha/2}(\tilde{O}^{(n)}) \leq \frac{c}{n^2}, \quad n \in \mathbb{N}, \]
contradicting the assumption. \( \square \)

In view of Proposition 2.2 to link the measure (11) with \( \alpha \)-admissibility requires linking the capacity estimate (9) with a weighted Dirichlet space. In other words, it is useful to relate each function \( g \in L^2(\mathbb{R}^+) \) with some \( G \in D^2_{1+\alpha}(\Pi_+) \). To provide this link (see Proposition 2.6) it is of interest to derive some properties of the harmonic extension of \( I^\beta \ast g \) to the upper half plane \( \Pi_+ \). The harmonic extension \( u_f \) of a function \( f \in L^p(\mathbb{R}) \) is given by
\[ (13) \quad u_f(x + iy) := (f \ast P_y)(x), \quad x + iy \in \Pi_+ \]
where \( P_y(x) := y/\pi(x^2 + y^2) \) is the Poisson kernel which satisfies
\[ (FP_y)(t) := \int_{-\infty}^{\infty} e^{-ist} P_y(s) ds = e^{-|t|y}, \quad t \in \mathbb{R}, y > 0. \]
For suitable functions \( g \), the function \( M g \) is the Hardy-Littlewood maximal function of \( g \) defined by
\[ (Mg)(x) := \sup_{I} \frac{1}{|I|} \int_I |g(x)| dx, \quad x \in \mathbb{R}. \]
It is well known (see e.g. [1], p.3) that if \( g \in L^p(\mathbb{R}) \) for \( 1 < p \leq \infty \) then there exists a constant \( c > 0 \), depending only on \( p \), for which
\[ (14) \quad \|Mg\|_{L^p(\mathbb{R})} \leq c \cdot \|g\|_{L^p(\mathbb{R})}. \]

**Proposition 2.4** ([1, Proposition 3.1.2]). Let \( \beta \in (0, 1) \). Then there exists a constant \( A > 0 \), depending only upon \( \beta \) and \( p \), such that for any measurable function \( g \geq 0 \) and any \( x \in \mathbb{R} \),
\[ (I^\beta \ast g)(x) \leq A\|g\|_{L^p}^{\beta p} \cdot ((Mg)(x))^{1-\beta p}, \quad 1 \leq p < \frac{1}{\beta}. \]
Proof. Applying Proposition 2.4 with \( \beta = -\alpha/2 \) and \( p = 2 \) gives

\[
|f(x)| = |(I_{-\alpha/2} * g)(x)| \leq (I_{-\alpha/2} * |g|)(x) \leq A \|g\|_2^{-\alpha} \cdot ((M|g|)(x))^{1+\alpha}, \quad x \in \mathbb{R}.
\]

Hence,

\[
\int_{-\infty}^{\infty} \frac{|f(x)|^2}{\pi x^2} \, dx = \int_{-\infty}^{\infty} \frac{|(I_{-\alpha/2} * g)(x)|^2}{\pi x^2} \, dx \leq A \frac{\pi}{\alpha} \cdot \|g\|_2^{-\alpha} \int_{-\infty}^{\infty} |(M|g|)(x)|^2 \, dx.
\]

(by 14) \leq A \frac{\pi}{\alpha} \cdot \|g\|_2^{-\alpha} \cdot c^2 \cdot \|g\|_2^2 = A \cdot \|g\|_2^{-\alpha} < \infty.

Therefore, \( f \in L^{\frac{2}{1+\alpha}}(\mathbb{R}) \), where \( \frac{2}{1+\alpha} \in (2, \infty) \). It is shown in ([9], p.17) that \( u_f \) must then satisfy

\[
|u_f(x + iy)| \leq \left( \frac{2}{\pi y} \right)^{\frac{1}{2+\alpha}} \sup_{\eta > 0} \left( \int_0^\infty |u_f(x + i\eta)|^{\frac{2}{1+\alpha}} \, dx \right)^{\frac{1}{2+\alpha}} \to 0, \quad y \to \infty.
\]

\( \square \)

Proposition 2.6. Let \( \alpha \in (-1, 0) \). Suppose that \( g \in L^2(\mathbb{R}) \), \( f := I_{-\alpha/2} * g \) and let \( u_f \) be the harmonic extension of \( f \) to \( \Pi_+ \). Then there exists an analytic function \( G \in D_{1+\alpha}^2(\Pi_+) \) and a function \( w \in L^2(\mathbb{R}, t^{-\alpha} \, dt) \) for which \( \text{Re}G = u_f, \|G\|_{D_{1+\alpha}} = c \|g\|_{L^2(\mathbb{R})} \) and

\[
(16) \quad G(z) = \int_0^\infty e^{izt} w(t) \, dt, \quad z \in \Pi_+.
\]

Proof. Since \( u_f \) is harmonic in \( \Pi_+ \), there exists an analytic function \( \tilde{G} : \Pi_+ \to \mathbb{C} \) with \( \text{Re} \tilde{G} = u_f \). It is shown in ([20], p.83) that for any \( x + iy \in \Pi_+ \),

\[
\int_{-\infty}^{\infty} \frac{|\tilde{G}'(x + iy)|^2}{\pi x^2} \, dx = \int_{-\infty}^{\infty} \frac{|\nabla u_f(x + iy)|^2}{\pi x^2} \, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} |t|^2 |(\mathcal{F}f)(t)|^2 e^{-2\pi t^2} \, dt.
\]

Furthermore, it is shown in [17] that \( |(\mathcal{F}f)(t)| = \cot \left( \frac{\pi \alpha}{4} \right) |t|^{\alpha/2} |\mathcal{F}g(t)| \) for almost every \( t \in \mathbb{R} \). An application of Fubini’s theorem implies that

\[
\int_0^\infty \int_{-\infty}^{\infty} y^{1+\alpha} |\tilde{G}'(x + iy)|^2 \, dy \, dx = c \int_{-\infty}^{\infty} |t|^{-\alpha} |(\mathcal{F}f)(t)|^2 \, dt
\]

(17) \( \text{(by Plancherel)} = 2\pi c \|g\|_{L^2(\mathbb{R})}^2 < \infty. \)

Hence, \( \tilde{G} \in D_{1+\alpha}^2(\Pi_+) \) and by [6] there exists a function \( w \in L^2(\mathbb{R}, t^{-\alpha} \, dt) \) and a constant \( K \in \mathbb{C} \) for which

\[
\tilde{G}(z) = \int_0^\infty e^{izt} w(t) \, dt + K, \quad z \in \Pi_+.
\]
Furthermore, if \( G(z) := \int_0^\infty e^{itz} w(t) dt \), then \( G(x + iy) \) has the property that for each \( x \in \mathbb{R} \), \( G(x + iy) \to 0, y \to \infty \). By Proposition 2.5, \( \text{Re}(\tilde{G})(x + iy) = u_f(x + iy) \to 0, y \to \infty \) and hence, \( \text{Re}(K) = \text{Re}(\tilde{G} - G)(x + iy) \to 0, y \to \infty \). Since \( K \) is constant,

\[
\text{Re}(G) = \text{Re}(\tilde{G}) = u_f.
\]

Finally, since \( G' = \tilde{G}' \), it follows from (17) that \( \|G\|_{D^2_{1+\alpha}} = \sqrt{2\pi c \cdot \|g\|_{L^2(\mathbb{R})}} \). \( \square \)

The counterexample. It is now possible to prove the main result of this section. Recall that \( \alpha \in (-1, 0) \), \( X := L^2(\Pi_+, \mu), (Af)(z) := izf(z) \) and \( C = \int_{\Pi_+} f(z) d\mu(z) \). The argument to show that (9) is necessary for \( \alpha \)-admissibility of \( C \) with respect to \( A \) is similar to [23, Theorem 4.4].

**Theorem 2.7.** Let \( \alpha \in (-1, 0) \). Suppose that \( \mu \) is as in Theorem 2.3 and that \( X, C \) and \( A \) are as above. Then \( C \) is not \( \alpha \)-admissible for \( A \) but

\[
\sup_{\lambda \in \mathbb{C}_+} (\text{Re}\lambda)^{1-\alpha} \|CR(\lambda, A)\|_{X^*} < \infty.
\]

**Proof.** Since \( \mu \) satisfies property (i) of Theorem 2.3, it follows from Lemma 2.1 that the resolvent estimate (2) holds.

Assume for a contradiction that \( C \) is \( \alpha \)-admissible for \( A \) and let \( O \subset \mathbb{R} \) be an open set. From the definition of \( \text{Cap}_{-\alpha/2}^+(O) \), there exists a function \( g \in L^2(\mathbb{R}) \), \( g \geq 0 \) for which \( I_{-\alpha/2} \ast g \geq 1 \) on \( O \) and \( \|g\|_{L^2(\mathbb{R})} \leq 2\text{Cap}_{-\alpha/2}^+(O) \). Define \( f := I_{-\alpha/2} \ast g \), and let \( u_f \) be the harmonic extension of \( f \) to the upper half plane. From Proposition 2.6 there exists an analytic function \( G : \Pi_+ \to \mathbb{C} \) and a function \( w \in L^2(\mathbb{R}, t^{-\alpha} dt) \) for which \( \text{Re}G = u_f, \|w\|_{L^2(\mathbb{R}, t^{-\alpha} dt)} = \|G\|_{D^2_{1+\alpha}} = c\|g\|_{L^2(\mathbb{R})} \) and

\[
G(z) = \int_0^\infty e^{itz} w(t) dt, \quad z \in \Pi_+.
\]

Let \( w^0(t) := w(t)^{-\alpha/2}, t \in \mathbb{R}_+ \). Then \( w^0 \in L^2(\mathbb{R}_+) \) with \( \|w^0\|_{L^2(\mathbb{R}_+)} = c\|g\|_{L^2(\mathbb{R})} \) and Proposition 2.2 implies that there exists a constant \( M > 0 \) such that

\[
\int_{\Pi_+} \left| \int_0^\infty e^{itz} t^{\alpha/2} w^0(t) dt \right|^2 d\mu(z) \leq M^2\|w^0\|_{L^2(\mathbb{R}_+)}^2,
\]

\[
\implies \int_{\Pi_+} \left| \int_0^\infty e^{itz} w(t) dt \right|^2 d\mu(z) \leq c^2 M^2\|g\|_{L^2(\mathbb{R})}^2.
\]

(19) \( \implies \int_{\Pi_+} |G(z)|^2 d\mu(z) \leq c^2 M^2\|g\|_{L^2(\mathbb{R})}^2. \)

Now suppose that \( O = \bigcup O_j \), where each \( O_j \) is an open interval in \( \mathbb{R} \). If \( x + iy \in R(O_j) \), for \( O_j = (a, b) \), then since \( f = I_{-\alpha/2} \ast g \geq 1 \) on \( O_j \),

\[
u_f(x + iy) = (f \ast P_y)(x) \geq (\chi_{O_j} \ast P_y)(x) \geq \int_{x-a}^{x+b} P_y(u) du \geq c \cdot \arctan(2) := \delta > 0.
\]

This holds for any \( x + iy \in R(O_j) \) and hence,

\[
1 \leq \delta^{-2}(f \ast P_y)(x)^2 = \delta^{-2}|u_f(x + iy)|^2,
\]

\( x + iy \in R(O) \).
Therefore,
\[
\mu(R(O)) = \int_{R(O)} d\mu(z) \leq \delta^{-2} \int_{R(O)} |u_f(z)|^2 d\mu(z)
\]
\[
(\text{Re} G = u_f) \leq \delta^{-2} \int_{\Pi_+} |\text{Re} G(z)|^2 d\mu(z)
\]
\[
\leq \delta^{-2} \int_{\Pi_+} |G(z)|^2 d\mu(z)
\]
(by definition of \(g\)) \leq 2\delta^{-2}c^2M^2\|g\|_{L^2(\mathbb{R})}^2.

This contradicts property (ii) of Theorem 23 and hence \(C\) is not \(\alpha\)-admissible for \(A\). \(\square\)

3. Counterexamples to the Discrete Weighted Weiss Conjecture

Suppose that \(X\) is a Hilbert space, \(A \in \mathcal{L}(X)\) with \(\sigma(A) \subset \overline{\mathbb{D}}\) and \(C \in X^\ast\). For \(\alpha \in (-1, 1)\), it is shown in [24] that if \(C\) is discrete \(\alpha\)-admissible for \(A\) then
\[
\sup_{\omega \in \mathbb{D}} (1 - |\omega|^2)^{\frac{1}{2} - \alpha} \|C(I - \omega A)^{-1}\|_{X^\ast} < \infty.
\]
This result fails for \(\alpha \in (0, 1);\) the unilateral shift on \(H^2(\mathbb{D})\), a contractive, non-normal operator, does not satisfy the discrete weighted Weiss conjecture conjecture. In the case \(\alpha = 0\), Harper proved in [10] that any contraction operator satisfies the (unweighted) discrete Weiss conjecture.

Discrete \(\alpha\)-admissibility is related to Carleson measures for weighted Dirichlet spaces. For \(\alpha \in (-1, 1)\), the weighted Dirichlet space \(D^2_\alpha(\mathbb{D})\) contains those analytic functions \(f(z) = \sum_{n=0}^{\infty} f_n z^n\) on \(\mathbb{D}\) for which
\[
\|f\|_{D^2_\alpha} := \sum_{n=0}^{\infty} (1 + n)^\alpha |f_n|^2 < \infty.
\]
A positive measure \(\mu\) on \(\mathbb{D}\) is said to be an \(\alpha\)-Carleson measure if
\[
D^2_\alpha(\mathbb{D}) \hookrightarrow L^2(\mathbb{D}, \mu) := \{f : \mathbb{D} \to \mathbb{C} : f\ \text{measurable}, \int_{\mathbb{D}} |f(z)|^2 d\mu(z) < \infty\}.
\]
If \(\alpha \in [0, 1)\) a measure \(\mu\) is a \((-\alpha)\)-Carleson measure if and only if there exists a constant \(c > 0\) such that \(\mu(S(I)) \leq c|I|^{1+\alpha}\), for any arc \(I \subset \mathbb{T}\) (see, e.g. [19]). Here,
\[
S(I) := \{z = re^{i\theta} : e^{i\theta} \in I, \ 1 - |I| \leq r < 1\}.
\]
The following result will be useful in constructing the counterexamples.

**Lemma 3.1.** Let \(\alpha \in (-1, 1)\). Then there exists a constant \(c > 0\) such that
\[
\mu(S(I)) \leq c|I|^{1+\alpha}
\]
for any arc \(I \subset \mathbb{T}\) if and only if there exists a constant \(k > 0\) such that
\[
\int_{\mathbb{D}} \frac{d\mu(z)}{|1 - \omega z|^2} \leq \frac{k}{(1 - |\omega|^2)^{1-\alpha}}, \quad \omega \in \mathbb{D}.
\]

Proof. It is shown in [24, §3] that (21) implies \( \mu(S(I)) \leq c|I|^{1+\alpha} \) for any arc \( I \subset \mathbb{T} \).
For the converse, notice first that by rotational invariance and the fact that \( \mu(\mathbb{D}) < \infty \), it is sufficient to show that (21) holds for \( \omega \in (a, 1) \), for some fixed \( a \in (0, 1) \). Let \( \omega > 1/2 \). Define arcs \( I_n \subset \mathbb{T} \) by \( I_n := \{ e^{i\theta} : \theta \in (-2^n\pi(1-\omega), 2^n\pi(1-\omega)) \} \) and sets \( A_0 := S(I_0), A_n := S(I_n) \setminus S(I_{n-1}), n \geq 1 \). Notice that for a given \( \omega \in (0, 1) \), there exists \( N_\omega \in \mathbb{N} \) such that \( A_n = \emptyset \) for \( n \geq N_\omega \). Since \( \mu \) satisfies \( \mu(S(I)) \leq c|I|^{1+\alpha} \),
\[
\mu(S(I_n)) \leq c|I_n|^{1+\alpha} = c2^n(1+\alpha)(1-\omega)^{1+\alpha}.
\]
A simple geometric argument shows that there exists a constant \( m > 0 \), independent of \( \omega \), such that
\[
|1-\omega z| \geq \frac{1}{2} |\omega^{-1} - z| \geq \frac{m}{2}(1-\omega)2^{n-1}, \quad z \in A_n, n \geq 1.
\]
Hence, from (22) and (23),
\[
\int_{\mathbb{D}} \frac{d\mu(z)}{|1-\omega z|^2} = \sum_{n=0}^{\infty} \frac{m^2(1-\omega)^{2+2n}}{2^n} \leq \frac{c}{m^2(1-\omega)^{1-\alpha}} \sum_{n=0}^{\infty} 2^n(\alpha-1)
\]
\[
(\alpha \in (-1, 1)) \leq \frac{k}{(1-\omega^2)^{1-\alpha}}.
\]
\[\square\]

The case \( \alpha \in (-1, 0) \). The idea is the same as in case of continuous weighted admissibility. If \( \alpha \in (-1, 0) \), there exists a finite, positive Borel measure \( \mu \) on \( \mathbb{D} \) such that [24 Theorem 19]:
(a) There exists a constant \( c > 0 \) such that \( \mu(S(I)) \leq c|I|^{1+\alpha} \), any arc \( I \subset \mathbb{T} \);
(b) \( \mathcal{D}^2_{\omega}(\mathbb{D}) \not\subset L^2(\mathbb{D}, \mu) \).
The space \( \mathcal{O}(\mathbb{D}) \) of analytic functions introduced in [10] will be useful. For \( \alpha \in (-1, 1) \),
\[
\mathcal{O}(\mathbb{D}) := \{ f : \mathbb{D} \to \mathbb{C} : f \text{ analytic, } \exists R > 1 \text{ with } \sum_{n=0}^{\infty} R^n |f_n| < \infty \}
\]
is a dense subspace of \( \mathcal{D}^2_{\omega}(\mathbb{D}) \) ([24, Lemma 2.1]). Furthermore, if \( A \) is a bounded linear operator on a Hilbert space \( X \) with \( \sigma(A) \subset \overline{\mathbb{D}} \), then for a function \( f(z) = \sum_{n=0}^{\infty} f_n z^n \in \mathcal{O}(\mathbb{D}) \) it is possible to define \( f(A) := \sum_{n=0}^{\infty} f_n A^n \in \mathcal{L}(X) \).

Theorem 3.2. Let \( \alpha \in (-1, 0) \). Suppose that \( \mu \) is a finite, positive measure on \( \mathbb{D} \) satisfying (a) and (b) as above. Let \( X := L^2(\mathbb{D}, \mu), (Af)(z) := z f(z), f \in X \) and \( C f := \int_{\mathbb{D}} f(z) d\mu(z), f \in X \). Then \( C \) is not discrete \( \alpha \)-admissible for \( A \) but
\[
\sup_{\omega \in \mathbb{D}} (1 - |\omega|^2)^{\frac{1+\alpha}{2}} \| C(I - \omega A)^{-1} \|_{X^*} < \infty.
\]

Proof. It is not difficult to show that
\[
\| C f(A) \|_{X^*}^2 = \int_{\mathbb{D}} |f(z)|^2 d\mu(z), \quad f \in \mathcal{O}(\mathbb{D}).
\]
Since \( \mu \) satisfies (a), Lemma 3.1 implies that the resolvent estimate holds.
Suppose for a contradiction that $C$ is discrete $\alpha$-admissible for $A$. Then there exists a constant $M > 0$ such that for any $f = (f_n) \in \ell^2$,

$$
\left(\sum_{n=0}^{\infty} (1 + n)^{\alpha/2} f_n C A^n x \right)^2 \leq M^2 \|x\|^2 \left(\sum_{n=0}^{\infty} \|f_n\|^2\right), \quad x \in X.
$$

(25)

Suppose that $g(z) := \sum_{n=0}^{\infty} g_n z^n \in D_{\alpha}(\mathbb{D})$ and let $a_n := (1 + n)^{-\alpha/2} g_n, n \in \mathbb{N}$. Then $(a_n)_{n=0}^{\infty} \in \ell^2$ and (25) implies that

$$
\left(\sum_{n=0}^{\infty} g_n C A^n x \right)^2 \leq M^2 \|x\|^2 \|g\|_{-\alpha}^2, \quad x \in X.
$$

Since $\mathcal{O}(\mathbb{D}) \subset D_{\alpha}(\mathbb{D})$ it follows that $\|Cf(A)\|_X \leq M \|f\|_{-\alpha}$, for each $f \in \mathcal{O}(\mathbb{D})$. From [24],

$$
\int_0^1 |f(z)|^2 d\mu(z) \leq M^2 \|f\|_{-\alpha}^2, \quad f \in \mathcal{O}(\mathbb{D}).
$$

(26)

Since $g \in D_{\alpha}(\mathbb{D})$, and $\mathcal{O}(\mathbb{D})$ is dense in $D_{\alpha}(\mathbb{D})$ [24, Lemma 2.1], there exist $g^{(n)} \in \mathcal{O}(\mathbb{D})$ such that $\|g - g^{(n)}\|_{-\alpha} \to 0$ as $n \to \infty$. By Fatou’s lemma,

$$
\int_{\mathbb{D}} |g(z)|^2 d\mu(z) = \int_{\mathbb{D}} \liminf_{n \to \infty} |g^{(n)}(z)|^2 d\mu(z)
\leq \liminf_{n \to \infty} \int_{\mathbb{D}} |g^{(n)}(z)|^2 d\mu(z)
\leq M^2 \liminf_{n \to \infty} \|g^{(n)}\|_{-\alpha}^2 \quad \text{(by (26))}
\leq M^2 \|g\|_{-\alpha}^2.
$$

Since $g \in D_{\alpha}(\mathbb{D})$ was arbitrary, this contradicts the fact that $\mu$ satisfies (b).

**The case** $\alpha \in (0, 1)$. A simple example of a non-normal contraction operator on a Hilbert space is the unilateral shift $S$ on $H^2(\mathbb{D})$ given by

$$(Sf)(z) := zf(z), \quad f \in H^2(\mathbb{D}), z \in \mathbb{D}.$$ Since $S$ is a contraction it satisfies the unweighted discrete Weiss conjecture. However, for $\alpha \in (0, 1)$, the resolvent bound

$$
\sup_{\omega \in \partial \mathbb{D}} (1 - |\omega|^2)^{1-\alpha} \|C(I - \omega S)^{-1}\| < \infty
$$

is not sufficient for discrete $\alpha$-admissibility of an observation operator $C \in H^2(\mathbb{D})^*$ with respect to $S$—see Theorem [3, 8]. In other words, for $\alpha \in (0, 1)$, the discrete weighted Weiss conjecture does not hold for contraction operators. It is possible to translate the counterexample from Theorem [3, 8] to continuous time operators and deduce that for $\alpha \in (0, 1)$, the continuous weighted Weiss conjecture is not true for contractive $C_0$-semigroups. In particular, the right-shift semigroup on $L^2(\mathbb{R}_+)$ does not satisfy the continuous weighted Weiss conjecture for $\alpha \in (0, 1)$, which is in contrast to the unweighted case [13]. This result will be published in a separate paper.

The proof of Theorem [3, 8] depends upon linking a number of areas of function space theory which are introduced in the following section.
Multipliers of Dirichlet spaces, Carleson measures and BMOA. If $\beta < 0$, the Dirichlet space norm $\| \cdot \|_\beta^2$ is equivalent to the expression

$$\int_\mathbb{D} |f(z)|^2(1 - |z|^2)^{-(1+\beta)}dA(z), \quad f \in D_\alpha^2(\mathbb{D})$$

where $dA(z) = dx dy$, $z = x + iy \in \mathbb{D}$ is Lebesgue area measure on $\mathbb{D}$. A function $f$ is said to be a multiplier from $D_\beta^2(\mathbb{D})$ into $D_\gamma^2(\mathbb{D})$, written $f \in M(D_\beta^2(\mathbb{D}), D_\gamma^2(\mathbb{D}))$, if $fg \in D_\gamma^2(\mathbb{D})$ whenever $g \in D_\beta^2(\mathbb{D})$. Multipliers of Dirichlet spaces are closely related to Carleson measures. The following result [19, Theorem 1.1] is a consequence of the equivalence of (28) to the norm $\| \cdot \|_\beta$.

**Theorem 3.3.** Let $\gamma < \beta \leq 0$. Then $f \in M(D_\beta^2(\mathbb{D}), D_\gamma^2(\mathbb{D}))$ if and only if $f$ is analytic and

$$|f(z)|^2(1 - |z|^2)^{-(1+\gamma)}dA(z)$$

is a $\beta$-Carleson measure.

For $\gamma < \beta < 0$ it is shown in [26] that $M(D_\beta^2(\mathbb{D}), D_\gamma^2(\mathbb{D})) = B^{\delta,\beta}(\mathbb{D})$, where for $\delta > 1$, $B^\delta(\mathbb{D})$ is the weighted Bloch space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ for which

$$\sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2)^{\delta} < \infty.$$

The situation is different for multipliers from the Hardy space $D_0^2(\mathbb{D}) = H^2(\mathbb{D})$ into a Dirichlet space $D_\beta^2(\mathbb{D})$ for $\beta < 0$. In particular, it is shown in [26] that $M(D_0^2(\mathbb{D}), D_\beta^2(\mathbb{D})) = F(2, -\beta, 1)$ where the $F$-space $F(p, q, s)$, introduced in [24], contains those analytic functions $f : \mathbb{D} \to \mathbb{C}$ for which

$$\sup_{a \in \mathbb{D}} \int_\mathbb{D} |f'(z)|^p(1 - |z|^2)^q g(z, a)^s dA(z) < \infty.$$

Here $g(z, a)$ is the Green function on $\mathbb{D}$ given by

$$g(z, a) := -\log \left| \frac{a - z}{1 - \bar{a}z} \right|, \quad a, z \in \mathbb{D}.$$

In addition to multipliers and Carleson measures, discrete $\alpha$-admissibility with respect to $S$ is related to functions of bounded mean oscillation. For a locally integrable function $f : \mathbb{T} \to \mathbb{C}$, let $f_I := \frac{1}{|I|} \int_I f$ denote the mean value of $f$ over the arc $I \subset \mathbb{T}$. Then $f$ is said to have bounded mean oscillation if

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|} \int_I |f(z) - f_I|^2 dz < \infty$$

and the space $BMOA$ contains those functions in $H^2(\mathbb{D})$ whose boundary functions have bounded mean oscillation. It should be noted (28, p.266) that the space $BMOA$ is unchanged if the $L^2$-norm in (29) is replaced by an $L^p$-norm, for any $1 \leq p < \infty$. Also, $F(2, 0, 1) = BMOA$ and for $\beta > 0$, the spaces $F(2, \beta, 1)$ provide natural generalisations of $BMOA$. The following theorem links $BMOA$ to Carleson measures.

**Theorem 3.4 (16 Theorem 2).** For $f \in H^2(\mathbb{D})$ the following are equivalent:

- $f \in BMOA$;
- For one/all $\beta > 0$, the measure $|(I_\beta f)(z)|^2(1 - |z|^2)^{2\beta-1}dA(z)$ is a 0-Carleson measure.
In the above theorem, the fractional derivative operator $I_{\beta} : H^2(\mathbb{D}) \to \mathcal{D}^2_{2\beta}(\mathbb{D})$, see \cite{[27]}, p.18, is defined for any $\beta > 0$ by

\[
(I_{\beta}f)(z) := \sum_{n=0}^{\infty} (1 + n)^{\beta} f_n z^n, \quad f(z) := \sum_{n=0}^{\infty} f_n z^n \in H^2(\mathbb{D}).
\]

It is also of interest to note that

\[
(zf(z))' = \sum_{n=0}^{\infty} (1 + n)f_n z^n = (I_1 f)(z), \quad f \in H^2(\mathbb{D}), z \in \mathbb{D}.
\]

It will be shown below that for a linear functional $Cf := \langle f, c \rangle_{H^2}$, each of the following conditions is equivalent to \cite{[27]} if $\alpha \in (0, 1)$:

1. $|I_{\alpha}c(z)|^2 (1 - |z|^2)^{1-\alpha} dA(z)$ is a $(-\alpha)$-Carleson measure on $\mathbb{D}$;
2. $I_{\alpha}c \in M(\mathcal{D}_{-\alpha}(\mathbb{D}), \mathcal{D}^2_{2\alpha}(\mathbb{D})) = \mathcal{B}^{2-\alpha/2}(\mathbb{D})$;

and that each of the following conditions is equivalent to discrete $\alpha$-admissibility of $C$ with respect to $S$:

1. $I_{\alpha/2}c \in \text{BMOA}$;
2. $|I_{\alpha}c(z)|^2 (1 - |z|^2)^{1-\alpha} dA(z)$ is a 0-Carleson measure on $\mathbb{D}$;
3. $I_{\alpha}c \in M(D_0(\mathbb{D}), D^2_{\alpha-\alpha}(\mathbb{D})) = F(2, 2 - \alpha, 1)$.

For $\alpha \in (0, 1)$, it is shown in \cite{[27]} that $F(2, 2 - \alpha, 1) \subseteq \mathcal{B}^{2-\alpha/2}(\mathbb{D})$.

**Discrete $\alpha$-admissibility of the unilateral shift.** The first step is to provide an alternative expression for the norm of the operator $C(I - \bar{\omega}S)^{-1}$. In the following, whenever $C \in H^2(\mathbb{D})$ is an observation operator, $c := C^*$ is the function in $H^2(\mathbb{D})$ for which $Cf = \langle f, c \rangle_{H^2}$, $f \in H^2(\mathbb{D})$. As pointed out by the referee, Proposition \cite{[27]} is essentially known; a ‘folklore’ result. I would also like to thank the referee for providing the following short proof.

**Proposition 3.5.** Let $C \in H^2(\mathbb{D})^\ast$. Then for any $\omega \in \mathbb{D}$,

\[
\|C(I - \bar{\omega}S)^{-1}\|_{H^2(\mathbb{D})^\ast} = \left\| \frac{z \omega(z) - \omega \omega(z)}{z - \omega} \right\|_{H^2(\mathbb{D})}.
\]

**Proof.** Let $\omega \in \mathbb{D}$, $f \in H^2(\mathbb{D})$ and define $k_{\omega}(z) := (1 - \bar{\omega}z)^{-1}, z \in \mathbb{D}$. Then $C(I - \bar{\omega}S)^{-1}f = \langle f, k_{\omega}c \rangle_{H^2}$ and hence, if $P_+ : L^2(\mathbb{T}) \to H^2(\mathbb{D})$ is the Hilbert space orthogonal projection onto $H^2(\mathbb{D})$, it follows that

\[
C(I - \bar{\omega}S)^{-1}f = \langle f, k_{\omega}c \rangle_{H^2} = \langle f, P_+(\bar{k}_{\omega}c) \rangle_{H^2}, \quad f \in H^2(\mathbb{D}).
\]

Now, $P_+(\bar{k}_{\omega}c) = \bar{k}_{\omega}c - g$, where $g \in H^2(\mathbb{D})^\perp \subset L^2(\mathbb{T})$ is the unique vector for which $\bar{k}_{\omega}c - g \in H^2(\mathbb{D})$. It is easy to check that $g(z) := \frac{\omega(z) - \omega(z)}{z - \omega} \in H^2(\mathbb{D})^\perp$ has these properties since

\[
(\bar{k}_{\omega}c)(z) = \frac{c(z)}{1 - \omega z} = \frac{zc(z)}{z - \omega}, \quad z \in \mathbb{T}
\]

and $z \mapsto \frac{zc(z) - \omega \omega(z)}{z - \omega} \in H^2(\mathbb{D})$. Therefore,

\[
P_+(\bar{k}_{\omega}c) = \frac{zc(z) - \omega \omega(z)}{z - \omega}
\]

and the result follows from (32). \hfill \square

**Proposition 3.6.** Let $\alpha \in (0, 1)$ and suppose that $C \in H^2(\mathbb{D})^\ast$. Then \cite{[27]} holds if and only if $I_{1\alpha}c \in \mathcal{B}^{2-\frac{\alpha}{2}}(\mathbb{D})$. 


Proof. Proposition 3.5 implies that (27) holds if and only if
\begin{equation}
\sup_{\omega \in \mathbb{D}} (1 - |\omega|^2)^{1-\alpha} \int_0^{2\pi} \left| \frac{e^{i\theta} - e^{i\phi}}{e^{i\theta} - \omega} \right|^2 d\theta < \infty
\end{equation}
where $c(e^{i\theta}) \in L^2(\mathbb{T})$ is the boundary function of $c \in H^2(\mathbb{D})$. It is shown in (27), p.165) that for $f \in H^2(\mathbb{D})$,
\[ \int_0^{2\pi} \left| \frac{f(e^{i\theta}) - f(\omega)}{e^{i\theta} - \omega} \right|^2 d\theta \sim \int_{\mathbb{D}} \frac{|f'(z)|^2(1 - |z|^2)}{|1 - \omega z|^2} dA(z), \quad \omega \in \mathbb{D}. \]
Hence, from the above equivalence, (30) and (33), the resolvent bound (27) holds if and only if
\[ \sup_{\omega \in \mathbb{D}} (1 - |\omega|^2)^{1-\alpha} \int_{\mathbb{D}} \frac{|(\mathcal{I}_1 c)(\omega)|^2(1 - |\omega|^2) dA(z)}{|1 - \omega|^2} < \infty. \]
Lemma 3.1 implies that this holds if and only if $|(\mathcal{I}_1 c)(\omega)|^2(1 - |\omega|^2) dA(z)$ is a $(-\alpha)$-Carleson measure and by Theorem 3.3 this is equivalent to
\[ \mathcal{I}_1 c \in M(D_{\alpha}^2(\mathbb{D}), D_{2-\alpha}^2(\mathbb{D})) = \mathcal{B}^{2-\alpha/2}(\mathbb{D}). \]
\[ \square \]

Proposition 3.7. Let $\alpha \in (0, 1)$ and suppose that $C \in X^*$ is an observation operator. Then $C$ is discrete $\alpha$-admissible for $A$ if and only if $\mathcal{I}_1 c \in F(2, 2 - \alpha, 1)$.

Proof. Since $F(2, 2 - \alpha, 1) = M(D_{\alpha}^2(\mathbb{D}), D_{2-\alpha}^2(\mathbb{D}))$, it follows from Theorem 3.5 that $\mathcal{I}_1 c \in F(2, 2 - \alpha, 1)$ if and only if $|(\mathcal{I}_1 c)(\omega)|^2(1 - |\omega|^2)^{1-\alpha} dA(z)$ is a 0-Carleson measure. Since $\mathcal{I}_{1-\alpha/2} \mathcal{I}_{\alpha/2} = \mathcal{I}_1$, this is the same as saying
\[ |(\mathcal{I}_{1-\alpha/2}(\mathcal{I}_{\alpha/2} c))(\omega)|^2(1 - |\omega|^2)^{2(1-\alpha/2)-1} dA(z) \]
is a 0-Carleson measure. By Theorem 3.3 this is equivalent to $\mathcal{I}_{\alpha/2} c \in \text{BMOA}$. It is shown in (18), p.284) that $\mathcal{I}_{\alpha/2} c \in \text{BMOA}$ if and only if the generalised Hankel operator $\Gamma_c : \ell^2 \to \ell^2$ represented by the matrix
\[
\begin{pmatrix}
0 & c_1 & c_2 & \cdots \\
2\pi c_1 & 2\pi c_2 & 2\pi c_3 & \cdots \\
3\pi c_2 & 3\pi c_3 & 3\pi c_4 & \cdots \\
4\pi c_3 & 4\pi c_4 & 4\pi c_5 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
is bounded. Now, if $f \in H^2(\mathbb{D})$ is given by $f(z) := \sum_{n=0}^{\infty} f_n z^n$, $z \in \mathbb{D}$, then
\[ \sum_{n=0}^{\infty} (1 + n)^{\alpha} |CS^n f|^2 = \sum_{n=0}^{\infty} (1 + n)^{\alpha} |(S^n f, c)_H|^2 \]
\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{\alpha/2} f_m \bar{c}_{m+n} \]
\[ = \|\Gamma_c((f_n)_{n=0}^{\infty})\|^2. \]
Hence, boundedness of $\Gamma_c$ is equivalent to discrete $\alpha$-admissibility of $C$ with respect to $S$. \[ \square \]

Theorem 3.8. Let $\alpha \in (0, 1)$. Then,
(i) If \( \{27\} \) holds for an observation operator \( C \in X^* \), \( C \) is discrete \( \beta \)-admissible for \( S \) for any \( \beta \in [0, \alpha) \).

(ii) There exists an observation operator \( C \in X^* \) which satisfies \( \{27\} \) but for which \( C \) is not discrete \( \alpha \)-admissible for \( S \).

Proof. (i) Let \( \beta \in [0, \alpha) \). It is shown in [25] that \( B^{2-\frac{2}{\beta}} \subseteq F(2, 2 - \beta, 1) \) and hence, by Propositions 3.6 and 3.7, \( C \) is discrete \( \beta \)-admissible for \( S \).

(ii) Since \( F(2, 2 - \alpha, 1) \not\subseteq B^{2-\frac{2}{\beta}} \), there exists a function

\[
f(z) := \sum_{n=0}^{\infty} f_n z^n \in B^{2-\frac{2}{\beta}} \setminus F(2, 2 - \alpha, 1).
\]

In particular, \( f \) is analytic on \( \mathbb{D} \),

\[
\int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2) dA(z) \leq k \int_{\mathbb{D}} (1 - |z|^2)^{\alpha-1} dA(z) < \infty
\]

and by \( \{28\} \), \( f \in D_2^2(\mathbb{D}) \). Since \( I_1 : H^2(\mathbb{D}) \to D_2^2(\mathbb{D}) \) is an isomorphism,

\[
c(z) := \sum_{n=0}^{\infty} f_n \frac{1}{1+n} z^n \in H^2(\mathbb{D})
\]

and \( I_1 c = f \). Hence, \( Cg := \langle g, c \rangle_{H^2}, g \in H^2(\mathbb{D}) \) defines a bounded linear functional on \( H^2(\mathbb{D}) \). By Proposition 3.6 \( C \) satisfies \( \{27\} \) but by Proposition 3.7 \( C \) is not discrete \( \alpha \)-admissible for \( S \). □

REFERENCES

[1] D Adams and L. Hedberg, Function spaces and potential theory, 1st ed., Grundlehren der mathematischen Wissenschaften, vol. 314, Springer, 1996.
[2] N. Arcozzi, R. Rochberg, and E. Sawyer, Carleson measures for analytic Besov spaces, Rev. Mat. Iberoamerican 18 (2002), no. 2, 443–510.
[3] G. Dafni, G.E. Karadzhov, and J. Xiao, Classes of Carleson-type measures generated by capacities, Math. Z. 258 (2008), no. 4, 827–844.
[4] P. Duren, Theory of \( H^p \) spaces, Dover Publications, 2000.
[5] P. Duren, E.A. Gallardo-Gutiérrez, and A. Montes-Rodríguez, A Paley-Wiener theorem for Bergman spaces with application to invariant subspaces, Bull. Lon. Math. Soc. 39 (2007), no. 3, 459–466.
[6] J.B. Garnett, Bounded analytic functions, revised 1st ed., Graduate texts in mathematics, vol. 236, Springer, 2007.
[7] B.H. Haak and C. Le Merdy, \( \alpha \)-admissibility of observation and control operators, Houston J. Math. 31 (2005), no. 4, 1153–1167.
[8] B.H. Haak and P.C. Kunstmann Weighted admissibility and wellposedness of linear systems in Banach spaces, SIAM J. Control Optim. 45 (2007), no. 6, 2094–2118.
[9] Z.M. Harper, Weighted norm inequalities for convolution operators and links with the Weiss conjecture, J. Evol. Eq. 5 (2005), no. 3, 387–405.
[10] , Applications of the discrete Weiss conjecture in operator theory, Integral Equations Operator Theory 54 (2006), no. 1, 69–88.
[11] B. Jacob and J.R. Partington, The Weiss conjecture on admissibility of observation operators for contraction semigroups, Integral Equations Operator Theory 40 (2001), no. 2, 231–243.
[12] , Admissibility of control and observation operators for semigroups: a survey, Current Trends in Operator Theory and its Applications, Operator Theory: Advances and Applications, Vol. 149, 2004, Proceedings of IWOTA 2002, pp. 199–221.
[13] B. Jacob, J.R. Partington, and S. Pott, Admissible and weakly admissible observation operators for the right shift semigroup, Proc. Edinb. Math. Soc. 45 (2002), no. 2, 353–362.
[14] B. Jacob, O. Staffans, and H. Zwart, Weak admissibility does not imply admissibility for analytic semigroups, Systems Control Lett. 48 (2003), no. 3–4, 341–350.
[15] B. Jacob and H. Zwart, *Counterexamples for observation operators*, SIAM J. Control Optim. 43 (2004), no. 1, 137–153.

[16] M. Jevtić, *On the Carleson measure characterization of BMOA functions on the unit ball*, Proc. Amer. Math. Soc. 114 (1992), no. 2, 379–386.

[17] G. Okikiolu, *Fourier transforms and the operator $H_\alpha$*, Proc. Cambridge Philos. Soc. 62 (1966), 73–78.

[18] V. Peller, *Hankel operators and their applications*, Springer, 2003.

[19] D.A. Stegenga, *Multipliers of the Dirichlet space*, Illinois J. Math. 24 (1980), no. 1, 113–139.

[20] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton mathematical series, vol. 30, Princeton University Press, 1970.

[21] G.D. Taylor, *Multipliers on $D_\alpha$*, Trans. Amer. Math. Soc. 123 (1966), 229–240.

[22] G. Weiss, *A powerful generalisation of the Carleson measure theorem*, Open problems in mathematical systems theory and control (E. Sontag, M. Vidyasagar, and J. Willerns, eds.), Springer, 1998.

[23] Z. Wu, *Clifford analysis and commutators on the Besov spaces*, J. Funct. Anal. 169 (1999), no. 1, 121–147.

[24] A. Wynn, *$\alpha$-admissibility of observation operators in discrete and continuous time*, Complex Anal. Oper. Theory, online first, 2008.

[25] R. Zhao, *On a general family of function spaces*, Ann. Acad. Sci. Fenn. Math. Diss. 105 (1996), 1–56.

[26] ———, *Pointwise multipliers from weighted Bergman spaces and Hardy spaces to weighted Bergman spaces*, Ann. Acad. Sci. Fenn. Math. 29 (2004), no. 1, 139–150.

[27] K. Zhu, *Spaces of holomorphic functions in the unit ball*, Graduate texts in mathematics, vol. 226, Springer, 2005.

[28] ———, *Operator theory in function spaces*, Mathematical surveys and monographs, vol. 138, American Mathematical Society, 2007.

St John’s College, Oxford, OX1 3JP
E-mail address: andrew.wynn@sjc.ox.ac.uk