Erratum to ”A unified approach to the plus-construction, Bousfield localization, Moore spaces and zero-in-the-spectrum examples”

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In Corollary 4.10 of [2], we have to assume that the group $G$ is finitely presented. This is due to a gap in the proof of Lemma 2.2 when claiming that the real reduced group $C^*$-algebra $C^*_R(G)$ is $G$-dense. The proof only works for finitely generated free modules. We need to modify Definition 2.1 as follows.

**Definition 1 (Definition 2.1)** A $G$-dense (resp. finitely $G$-dense) ring $(R, \phi)$ is a unital ring $R$ together with a ring homomorphism $\phi : \mathbb{Z}[G] \to R$ such that, when $R$ is regarded as a left $\mathbb{Z}[G]$-module via $\phi$, then, for any right $\mathbb{Z}[G]$-module $M$, free (resp. finitely generated free) right $R$-module $F$ and $R$-module surjection $f : M \otimes_{\mathbb{Z}[G]} R \to F$, the module $F$ has an $R$-basis in $f(M \otimes 1)$.

It is not hard to see that a $G$-dense ring is finitely $G$-dense. Lemma 2.2 should read as follows.

**Lemma 2 (Lemma 2.2)** The set of finitely $G$-dense rings contains the real group $C^*$-algebra $C^*_R(G)$, the real group von Neumann algebra $N_R(G)$, the real Banach algebra $l^1_R(G)$. The set of $G$-dense rings contains the rings $\mathbb{Z}/p$ for any prime $p$, $k \subseteq \mathbb{Q}$ a subring of rationals (with trivial $G$-actions), and the group rings $k[G]$. 

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The main result Theorem 1.1 should read as follows, where we add the case of finitely $G$-dense rings.

**Theorem 3 (Theorem 1.1)** Assume that $G$ is a group and $(R, \phi)$ is a $G$-dense ring. Let $X$ be a CW complex with fundamental group $\pi = \pi_1(X)$. Suppose that $\alpha : \pi \to G$ is a group homomorphism such that

$$H_1(\alpha) : H_1(\pi; R) \to H_1(G; R) \text{ is injective, and}$$

$$H_2(\alpha) : H_2(\pi; R) \to H_2(G; R) \text{ is surjective.}$$

Assume either that $R$ is a principal ideal domain or that the relative homology group $H_1(G, \pi; R)$ is a stably free $R$-module. Then there exist a CW complex $Y$ and an inclusion $g : X \to Y$ with the following properties:

(i) $Y$ is obtained from $X$ by adding 1-cells, 2-cells and 3-cells, such that

(ii) $\pi_1(Y) = G$ and $\pi_1(g) = \alpha : \pi_1(X) \to \pi_1(Y)$, and

(iii) for any $q \geq 2$ the map $g$ induces an isomorphism

$$g_* : H_q(X; R) \xrightarrow{\cong} H_q(Y; R).$$

(1)

When $X$ is finite, $G$ is finitely presented and $R$ is finitely $G$-dense, then $Y$ is also finite.

In Corollary 4.10, we have to assume that the group $G$ is finitely presented, which is the same assumption as that of Higson, Roe and Schick [1].

**Corollary 4 (Corollary 4.10)** For a finitely presented group $G$ with

$$H_0(G; C^*_r(G)) = H_1(G; C^*_r(G)) = H_2(G; C^*_r(G)) = 0,$$

there is a finite CW complex $Y$ such that $\pi_1(Y) = G$ and for each integer $n \geq 0$, the homology group $H_n(Y; C^*_r(G)) = 0$.

All other results are still true.

The author would like to thank Professor T. Schick for noting a gap in a previous claim that $C^*_R(G)$ is $G$-dense.
References

[1] N. Higson, J. Roe and T. Schick, *Spaces with vanishing $l^2$-homology and their fundamental groups (after Farber and Weinberger)*, Geometriae Dedicata, 87 (2001), 335-343.

[2] S. Ye, *A unified approach to the plus-construction, Bousfield localization, Moore spaces and zero-in-the-spectrum examples*, Israel Journal of Mathematics. 192 (2012) 699-717.

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A unified approach to the plus-construction, Bousfield localization, Moore spaces and zero-in-the-spectrum examples

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Abstract

We introduce a construction adding low-dimensional cells to a space that satisfies certain low-dimensional conditions; it preserves high-dimensional homology with appropriate coefficients. This includes as special cases Quillen’s plus construction, Bousfield’s integral homology localization, the existence of Moore spaces $M(G,1)$ and Bousfield and Kan’s partial $k$-completion of spaces. We also use it to generalize counterexamples to the zero-in-the-spectrum conjecture found by Farber and Weinberger, and by Higson, Roe and Schick.

1 Introduction

The aim of this article is to give a unified treatment of Quillen’s plus-construction, Rodríguez and Scevenels’ work on Bousfield’s integral localization, Varadarajan’s theorem on the existence of Moore spaces, the partial $k$-completion of Bousfield and Kan, and counterexamples to the zero-in-the-spectrum conjecture by Farber and Weinberger, and Higson, Roe and Schick. We introduce a method for adding low-dimensional cells to a space satisfying certain low-dimensional conditions and preserving high-dimensional homology with appropriate coefficients. First, we briefly review these existing works.

Let $X$ be a CW complex with fundamental group $G$ and $P$ a perfect normal subgroup of $G$, i.e., $P = [P,P]$. Quillen [23] shows that there exists
a CW complex $X^+_P$, whose fundamental group is $G/P$, and an inclusion $f: X \to X^+_P$ such that

$$H_n(X; f_*M) \cong H_n(X^+_P; M)$$

for any integer $n$ and local coefficient system $M$ over $X^+_P$. Here $X^+_P$ is called the plus-construction of $X$ with respect to $P$ and is unique up to homotopy equivalence. The plus-construction can be used to define higher algebraic $K$-theory, as follows. Let $R$ be a unital associative ring with $n$-th general linear group $\text{GL}_n(R)$ and $E_n(R)$ its subgroup generated by elementary matrices. Let $\text{GL}(R) = \cup_n \text{GL}_n(R)$. Then $E(R) = \cup_n E_n(R)$ is the maximal perfect normal subgroup of $\text{GL}(R)$. For the classifying space $B\text{GL}(R)$, let $B\text{GL}(R)^+$ denote its plus-construction with respect to $E(R)$. The algebraic $K$-groups are defined as $K_i(R) = \pi_i(B\text{GL}(R)^+)$ ($i \geq 1$).

While the plus-construction preserves all ordinary homology with coefficients, for each generalized homology theory there is a Bousfield localization preserving it. More precisely, write $\text{Ho}$ for the pointed homotopy category of CW complexes. Then Bousfield [3] shows that each generalized homology theory $h_*$ determines an $h_*$-localization functor $E: \text{Ho} \to \text{Ho}$ and a natural transformation $\eta : \text{Id} \to E$. This localization is characterized by the universal property that $\eta_X : X \to EX$ is the terminal $h_*$-homology equivalence going out of $E$, i.e.,

(i) $\eta_X : X \to EX$ induces $h_*(X) \cong h_*(EX)$, and

(ii) for any map $f : X \to Y \in \text{Ho}$ inducing $h_*(X) \cong h_*(Y)$ there is a unique map $r : Y \to EX \in \text{Ho}$ with $rf = \eta_X$.

For ordinary homology theory $HZ$ with $\mathbb{Z}$ as coefficients, Bousfield’s $HZ$-localization $X_{HZ}$ of a space $X$ is homotopy equivalent to localization of $X$ with respect to a map of classifying spaces $Bf : BF_1 \to BF_2$ induced by a certain homomorphism $f : F_1 \to F_2$ of free groups (cf. [10] [11]). This implies that a space $X$ is $HZ$-local if and only if the induced map $Bf^* : \text{map}(BF_2, X) \to \text{map}(BF_1, X)$ is a weak homotopy equivalence. Rodríguez and Scevenels [24] show that there is a topological construction that, while leaving the integral homology of a space unchanged, kills the intersection of the transfinite lower central series of its fundamental group. Moreover, this is the maximal subgroup that can be factored out of the fundamental group without changing the integral homology of a space. For more information
on $HZ$-localization and homology equivalence with other coefficients, see \cite{2,4,5,10,20} and references therein.

The plus-construction shares some common features with the construction of Moore spaces in \cite{27}. Given an integer $n \geq 1$ and a group $G$ (abelian if $n \geq 2$), a Moore space $M(G, n)$ is a space $X$ such that $\pi_j(X) = 0$ for $j < n$, $\pi_n(X) = G$ and $H_i(X; \mathbb{Z}) = 0$ for $i > n$. For $n \geq 2$, such a space always exists. For $n = 1$, Varadarajan \cite{27} proves that there exists a Moore space $M(G, 1)$ if and only if $H_2(G; \mathbb{Z}) = 0$.

Let $k$ be the constant ring $\mathbb{Z}/p$ (prime $p$) or $k \subseteq \mathbb{Q}$ a subring of the rationals. For a space $X$, let $P_{\pi_1(X)}$ be the largest subgroup of $\pi_1(X)$ for which $H_1(P_{\pi_1(X)}; k) = 0$. Bousfield and Kan \cite{6} show that there exists a space $C^k(X)$, which is called the partial $k$-completion of $X$, and a map $\phi : X \to C^k(X)$ such that the fundamental group $\pi_1(C^k(X)) = \pi_1(X)/P_{\pi_1(X)}$ and for any integer $q \geq 0$ the map $\phi$ induces an isomorphism $H_q(X; k[\pi/P]) \cong H_q(Y; k[\pi/P])$. (Here, $k[\pi/P]$ is the group ring over $k$ of $\pi/P$.)

The zero-in-the-spectrum conjecture goes back to Gromov, who asked whether for a closed, aspherical, connected and oriented Riemannian manifold $M$ there always exists some $p \geq 0$, such that zero belongs to the spectrum of the Laplace-Beltrami operator $\Delta_p$ acting on the square integrable $p$-forms on the universal covering $\tilde{M}$ of $M$. Farber and Weinberger \cite{13} show that the conjecture is not true if the condition that $M$ is aspherical is dropped. More generally, Higson, Roe and Schick \cite{14} show that for a finitely presented group $G$ satisfying $H_0(G; C^*_r(G)) = H_1(G; C^*_r(G)) = 0$, there always exists a finite CW complex $Y$ with $\pi_1(Y) = G$ such that $Y$ is a counterexample to the conjecture if $M$ is not required to be aspherical.

In this note, a more general construction is provided to preserve homology theory. For this, we have to introduce the notion of a $G$-dense ring (for details, see Definition \cite{2.11}). Examples of $G$-dense rings include the real reduced group $C^*$-algebra $C^*_r(G)$, the real group von Neumann algebra $N_rG$, the real Banach algebra $l^1_r(G)$, the constant rings $k = \mathbb{Z}/p$ (prime $p$) and $k \subseteq \mathbb{Q}$ a subring of the rationals, the group ring $k[G]$, and so on.

Notation. Let $\pi$ and $G$ be two groups. Suppose $R$ is a $\mathbb{Z}[G]$-module and $BG, B\pi$ are the classifying spaces. For a group homomorphism $\alpha : \pi \to G$, we will denote by $H_1(G, \pi; R)$ the relative homology group $H_1(BG, B\pi; R)$ with coefficients $R$. All spaces are assumed to be connected.

**Theorem 1.1.** Assume that $G$ is a group and $(R, \phi)$ is a $G$-dense ring. Let $X$ be a CW complex with fundamental group $\pi = \pi_1(X)$. Assume $\alpha : \pi \to G$
is a group homomorphism such that
\[ \alpha_* : H_1(\pi; R) \to H_1(G; R) \text{ is injective, and} \]
\[ \alpha_* : H_2(\pi; R) \to H_2(G; R) \text{ is surjective.} \]

Suppose either that \( R \) is a principal ideal domain or that the relative homology group \( H_1(G, \pi; R) \) is a stably free \( R \)-module. Then there exists a CW complex \( Y \) with the following properties:

(i) \( Y \) is obtained from \( X \) by adding 1-cells, 2-cells and 3-cells, such that

(ii) \( \pi_1(Y) = G \) and the inclusion map \( g : X \to Y \) induces the same fundamental group homomorphism as \( \alpha \), and

(iii) for any \( q \geq 2 \) the map \( g \) induces an isomorphism
\[
g_\ast : H_q(X; R) \xrightarrow{\cong} H_q(Y; R). \tag{1}\]

Theorem 1.1 has many important consequences, including the following.

(1) \( \alpha \) surjective.

• When \( R = \mathbb{Z} \) and \( \ker \alpha \) is perfect, Proposition 4.2 shows that Quillen’s plus-construction is a special case of this theorem.

• When \( R = \mathbb{Z} \), we obtain the result of Rodríguez and Scevenels [24] on Bousfield integral localization (cf. Corollary 4.3).

• When \( k = \mathbb{Z}/p \) or \( k \subseteq \mathbb{Q} \) a subring of the rationals and \( R = k[G] \), the theorem yields the partial \( k \)-completion of Bousfield and Kan [6] (see Corollary 4.7).

(2) \( \pi = 1 \).

• When \( R = \mathbb{Z} \), we obtain in Corollary 4.4 the existence of the Moore space \( M(G, 1) \), which was first proved by Varadarajan in [27].

• When \( R = C^*_R(G) \), the theorem yields the results obtained by Farber-Weinberger [13] and Higson-Roe-Schick [14] on the zero-in-the-spectrum conjecture (actually our result is more general, see Corollary 4.10).

In Section 2, we give the definition and some examples of \( G \)-dense rings. The main result Theorem 1.1 is proved in Section 3. Several applications are presented in the last section.
2 G-dense rings

Let G be a group. In this section, we introduce a kind of rings which includes as special cases the real reduced group C*-algebra $C^*_R(G)$, the real group von Neumann algebra $\mathcal{N}_R G$, the real Banach algebra $l^1_R(G)$, $k = \mathbb{Z}/p$ for some prime $p$ or $k \subseteq \mathbb{Q}$ a subring of the rationals, and the group ring $k[G]$.

**Definition 2.1.** A G-dense ring $(R, \phi)$ is a unital ring $R$ together with a ring homomorphism $\phi : \mathbb{Z}[G] \to R$ such that, when $R$ is regarded as a left $\mathbb{Z}[G]$-module via $\phi$, then, for any right $\mathbb{Z}[G]$-module $M$, free right $R$-module $F$ and $R$-module surjection $f : M \otimes_{\mathbb{Z}[G]} R \to F$, the module $F$ has an $R$-basis in $f(M \otimes 1)$.

When $\phi$ is obvious, it is omitted from the notation. Some examples of G-dense rings are as follows. Recall that for a group $G$, the space $l^2(G) = \{ f : G \to \mathbb{C} \mid \sum_{g \in G} |f(g)|^2 < +\infty \}$ is a Hilbert space with inner product $\langle f_1, f_2 \rangle = \sum_{x \in G} f_1(x)\overline{f_2(x)}$. Let $B(l^2(G))$ be the set of all bounded linear operators of the Hilbert space $l^2(G)$. By definition, the real reduced group C*-algebra $C^*_R(G)$ is the completion of $\mathbb{R}[G]$ in $B(l^2(G))$ with respect to the operator norm, while the real group von Neumann algebra $\mathcal{N}_R G$ is the completion of $\mathbb{R}[G]$ in $B(l^2(G))$ with respect to the weak operator norm. The real Banach algebra $l^1_R(G)$ is the completion of the group ring $\mathbb{R}[G]$ with respect to the $l^1$-norm.

**Lemma 2.2.** The set of G-dense rings contains the real reduced group C*-algebra $C^*_R(G)$, the real group von Neumann algebra $\mathcal{N}_R G$, the real Banach algebra $l^1_R(G)$, the constant rings $k = \mathbb{Z}/p$ for any prime $p$, $k \subseteq \mathbb{Q}$ a subring of the rationals, and the group rings $k[G]$.

*Proof.* We prove the lemma case by case. Let $M$ be a right $\mathbb{Z}[G]$-module, $F$ a free right $R$-module and $f : M \otimes_{\mathbb{Z}[G]} R \to F$ a surjection of $R$-modules. Choose a basis $(b_i)_{i \in S}$ of $F$ for some index set $S$. Since $f$ is surjective and $R$-linear, we can assume

\[ b_i = \sum f(x_{ik} \otimes 1)a_{ik} \]

for some $x_{ik} \in M$ and $a_{ik} \in R$. 

(i) \( k = \mathbb{Z}/p \) for some prime \( p \), and \( R = k \) or \( k[G] \).

The ring homomorphism \( \phi : \mathbb{Z}[G] \to R \) is induced from the natural map \( \mathbb{Z} \to \mathbb{Z}/p \). There is a surjection \( \beta : \mathbb{Z}[G] \to R \). Choose some \( a_{ik} \in \beta^{-1}(a_{ik}) \). Then we have
\[
b_i = \sum f(x_{ik} \cdot \tilde{a}_{ik} \otimes 1),
\]
which is in the image of \( f(M \otimes 1) \).

(ii) \( k \subseteq Q \) a subring of the rationals, and \( R = k \) or \( k[G] \).

The ring homomorphism \( \phi : \mathbb{Z}[G] \to R \) is induced from the natural map \( \mathbb{Z} \to k \). There is an inclusion \( \beta : R \hookrightarrow \mathbb{Q}[G] \). Then there exists an integer \( n_i \), which is invertible in \( R \), such that \( n_i a_{ik} \in \mathbb{Z}[G] \) and
\[
n_i b_i = \sum f(x_{ik} \otimes 1)n_i a_{ik} = \sum f(x_{ik} \cdot n_i a_{ik} \otimes 1),
\]
which is in the image of \( f(M \otimes 1) \). Since \( n_i \) is invertible, \( (n_i b_i)_{i \in S} \) is still a basis.

(iii) \( R = C_{\mathbb{R}}^*(G), l_{\mathbb{R}}^1(G) \) or \( \mathcal{N}_{\mathbb{R}}G \).

The ring homomorphism \( \phi : \mathbb{Z}[G] \to R \) is the natural inclusion. The proof is similar to that of Proposition 4.4 in [14]. We just briefly repeat here. Assume \( R = C_{\mathbb{R}}^*(G) \), while the other cases are similar. Choose \( a_{ik} \in C_{\mathbb{R}}^*(G) \). Since \( F \) is a free \( C_{\mathbb{R}}^*(G) \) module, there is a natural product topology on \( F \). As the set of all bases in \( F \) is open and the module multiplication operation
\[
F \times C_{\mathbb{R}}^*(G) \to F
\]
is continuous, for each pair \((i, k)\) we can choose \( a'_{ik} \in \mathbb{Q}[G] \) sufficiently close to \( a_{ik} \) such that the elements
\[
b'_i = \sum f(x_{ik}) \otimes a'_{ik}
\]
form a new basis for \( F \). Since the tensor is over \( \mathbb{Z}[G] \), a similar argument as in the case of \( k \subseteq \mathbb{Q} \), and \( R = k \) or \( k[G] \) shows that the image of \( f(M \otimes 1) \) contains a basis.

The following lemma provides more examples of \( G \)-dense rings.

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Lemma 2.3. Let $G$ be a group and $N$ a normal subgroup of $G$ inducing the canonical surjection $\psi : \mathbb{Z}[G] \to \mathbb{Z}[G/N]$. Assume that $(R, \phi)$ is a $G/N$-dense ring. Then $(R, \phi \circ \psi)$ is $G$-dense.

Proof. Suppose that, for a right $\mathbb{Z}[G]$-module $M$ and a free right $R$-module $F$, there is a right $R$-module surjection $f : M \otimes_{\mathbb{Z}[G]} R \twoheadrightarrow F$. Since

$$M \otimes_{\mathbb{Z}[G]} R \cong (M \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/N]) \otimes_{\mathbb{Z}[G/N]} R,$$

the module $F$ has an $R$-basis in $f(M \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/N] \otimes_{\mathbb{Z}[G/N]} 1)$. The quotient map $\psi$ gives

$$f(M \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/N] \otimes_{\mathbb{Z}[G/N]} 1)$$

as a subset of $f(M \otimes_{\mathbb{Z}[G]} 1)$. \hfill \qed

Recall from \cite{22, 8, 9, 16} that a ring $R$ is right Steinitz if $R$ has the property that any linearly independent subset of a free right $R$-module $F$ can be extended to a basis of $F$. It is also known that right Steinitz rings are precisely the right perfect local rings (cf. \cite{22}).

Proposition 2.4. Let $G$ be a group. Any right Steinitz ring $R$ with a ring homomorphism $\phi : \mathbb{Z}[G] \to R$ is $G$-dense.

Proof. Let $R$ be a right Steinitz ring, $M$ a right $\mathbb{Z}[G]$-module, $F$ a free right $R$-module and $f : M \otimes_{\mathbb{Z}[G]} R \twoheadrightarrow F$ an $R$-module surjection. By Theorem 1.1 in \cite{22}, any generating set of a free right $R$-module $F$ contains a basis of $F$. Since the set $f(M \otimes 1)$ is a generating set of $F$, this shows that $R$ is $G$-dense. \hfill \qed

Before we give an example of a ring which is not $G$-dense, let’s present a matrix property of $G$-dense rings. For a matrix $A = (a_{ij}) \in M_n(R)$, denote by $A_k$ the submatrix spanned by the first $k$ columns.

Proposition 2.5. Let $G$ be a group, $R$ a unital ring with invariant basis number (IBN, cf. \cite{13}) and $n$ a positive integer. Assume that $(R, \phi)$ is $G$-dense. Then for every integer $n \geq 1$, any matrix $A \in \text{GL}_n(R)$ and any integer $k$ with $1 \leq k \leq n$, there exists a matrix $B \in M_{k \times n}(\mathbb{Z}[G])$ such that $\phi(B)A_k \in \text{GL}_k(R)$.\hfill \qed
Proof. Let \( n \geq 1 \) be an integer and the standard basis of \( R^n \) be denoted as \( \{f_1, f_2, \ldots, f_n\} \). Write the element \( r_1f_1 + r_2f_2 + \cdots + r_nf_n \) in \( R^n \) as the vector \([r_1, r_2, \ldots, r_n]\). Let \( \{e_1, e_2, \ldots, e_n\} \) be the standard basis of \((\mathbb{Z}[G])^n\). Given \( A = (a_{ij}) \in \text{GL}_n(R) \), define the map

\[
\alpha : (\mathbb{Z}[G])^n \otimes_{\mathbb{Z}[G]} R \to R^n
\]

by \( \alpha(\sum e_i \otimes r_i) = [r_1, r_2, \ldots, r_n]A \). For each integer \( k \) such that \( 1 \leq k \leq n \), let \( p \) be the standard projection to the first \( k \) components

\[
p : R^n \to R^k.
\]

Since \( A \) is invertible, we have a surjection

\[
p \circ \alpha : (\mathbb{Z}[G])^n \otimes_{\mathbb{Z}[G]} R \to R^k.
\]

According to the definition of \( G \)-dense rings and the assumption that \( R \) has IBN, we can find elements \( x_1, x_2, \ldots, x_k \in (\mathbb{Z}[G])^n \) such that \( \{p \circ \alpha(\phi(x_i)) = \phi(x_i)A \mid i = 1, 2, \ldots, k\} \) is a basis for \( R^k \). Let \( B = [x_1, x_2, \ldots, x_k]^T \). By the definitions of \( \alpha \) and \( p \), the matrix \( \phi(B)A_k \) is invertible. \( \Box \)

According to Proposition 2.5, the following example shows that the ring of Gauss integers \( \mathbb{Z}[i] \) is not \( G \)-dense for the trivial group \( G \).

Example 2.6. Let \( \mathbb{Z}[i] \) be the Gauss integers. Note that the matrix

\[
\begin{bmatrix}
3 & 2 - i \\
i + 2 & 2
\end{bmatrix}
\]

lies in \( \text{SL}_2(\mathbb{Z}[i]) \). But we are not able to find two integers \( a, b \) such that \( 3a + (i + 2)b \) is invertible in \( \mathbb{Z}[i] \), since the only units are \( 1, -1, i, -i \). This shows \( \mathbb{Z}[i] \) is not \( G \)-dense for the trivial group \( G \).

3 The generalized plus-construction

In this section, we will prove the main result, Theorem 1.1. Theorem 1.1 shows that for certain homology theories, there is a construction that preserves the higher homology groups.

In order to prove Theorem 1.1, we use the following lemma, which is a more general version of Hopf’s exact sequence.
Lemma 3.1 (Lemma 2.2 in [13]). Let $G$ be a group and $V$ be a left $\mathbb{Z}[G]$-module. For any CW complex $X$ with fundamental group $G$ and universal covering space $\tilde{X}$, there is an exact sequence

$$H_2(\tilde{X}) \otimes_{\mathbb{Z}[G]} V \to H_2(X; V) \to H_2(G; V) \to 0.$$ 

Proof of Theorem 1.1. For the group homomorphism $\alpha : \pi_1(X) = \pi \to G$, we construct a CW complex $W$ such that $\pi_1(W) = G$ as follows. Let $S$ be a set of normal generators of ker($\alpha$), i.e., ker($\alpha$) is generated by elements of the form $gsg^{-1}$ for $s \in S$ and $g \in \pi$. For each element in $S$, attach a 2-cell ($D^2, S^1$) to $X$ to kill the corresponding element in $\pi$. Extend the presentation of $\pi/\ker(\alpha)$ by generators and relations to a presentation of $G$. This can be obtained by adding generators and relations to the presentation of $\pi/\ker(\alpha)$. When $f$ is surjective, we only need to add relations. For each such generator (resp. relation), we continue to add a 1-cell (resp. 2-cell) to $X$ (see 5.1 in [1] for more details). Let $W$ denote the resulting space.

We consider the homology groups of the pair $(W, X)$. By Lemma 3.1, there is a commutative diagram

$$
\begin{array}{ccc}
H_2(\tilde{X}) \otimes_{\mathbb{Z}[G]} R & \to & H_2(\tilde{W}) \otimes_{\mathbb{Z}[G]} R \\
\downarrow & & \downarrow \downarrow \downarrow j_4 \\
H_2(X; R) & \xrightarrow{j_2} & H_2(W; R) \\
\downarrow j_3 & & \downarrow j_5 \\
H_2(\pi; R) & \xrightarrow{\alpha_*} & H_2(G; R)
\end{array}
$$

where the middle horizontal chain is the long exact sequence of homology groups for the pair $(W, X)$ and the two vertical lines are the exact sequences as in Lemma 3.1. Notice that

$$H_1(X; R) \cong H_1(\pi; R) \to H_1(W; R) \cong H_1(G; R)$$

is injective by assumption. This implies $j_1 : H_2(W; R) \to H_2(W, X; R)$ is surjective in the above diagram. For any element $a \in H_2(W, X; R)$, choose its preimage $b \in H_2(W; R)$. Since $\alpha_* : H_2(\pi; R) \to H_2(G; R)$ is surjective by assumption, there exists some element $c \in H_2(\pi; R)$ such that $\alpha_*(c) = j_5(b)$. Let $d \in H_2(X; R)$ be a preimage of $c$, i.e. $j_3(d) = c$. By the commutativity of the diagram, $j_5(b - j_2(d)) = 0$. Therefore, there exists an element $e \in$
It can be checked that \( j_1 \circ j_4(e) = b = j_2(d) \). Therefore, there is a surjection

\[
j_1 \circ j_4 : H_2(\tilde{W}) \otimes_{\mathbb{Z}[G]} R \to H_2(W, X; R)
\]

by this diagram chase.

We show that the relative homology group \( H_2(W, X; R) \) can be taken to be a free \( R \)-module. Let

\[
0 \to C_2(\tilde{W}, \tilde{X}; R) \xrightarrow{i} C_1(\tilde{W}, \tilde{X}; R) \xrightarrow{j} C_0(\tilde{W}, \tilde{X}; R) \to 0
\]

be the chain of relative complexes (for details, see Section 4.5). Since \( H_0(X; R) \cong R/\langle \alpha(g)x - x \mid g \in \pi, x \in R \rangle \to H_0(W; R) \cong R/\langle gx - x \mid g \in G, x \in R \rangle \) is surjective, \( H_0(W, X; R) = 0 \). Therefore, the following sequences

\[
0 \to H_2(W, X; R) \to C_2(\tilde{W}, \tilde{X}; R) \to \text{Im } i \to 0;
0 \to \text{Im } i \to \text{ker } j \to H_1(W, X; R) \to 0;
0 \to \text{ker } j \to C_1(\tilde{W}, \tilde{X}; R) \to C_0(\tilde{W}, \tilde{X}; R) \to 0
\]

are exact. When \( R \) is a principal ideal domain, \( H_2(W, X; R) \) is always a free \( R \)-module, viewed as a submodule of the free module \( C_2(\tilde{W}, \tilde{X}; R) \). When \( R \) is not a principal ideal domain, \( H_1(W, X; R) = H_1(G, \pi; R) \) is a stably free \( R \)-module by assumption. According to the exact sequences above, \( H_2(W, X; R) \) is also stably free as an \( R \)-module. By wedging \( W \) with some 2-spheres, which does not change the fundamental group \( G \), we can further assume that \( H_2(W, X; R) \) is a free \( R \)-module.

Since \( H_2(\tilde{W}) \otimes_{\mathbb{Z}[G]} R \to H_2(W, X; R) \) is surjective and \( H_2(W, X; R) \) is a free \( R \)-module, by the definition of \( G \)-dense rings we have a set \( S \) of elements in \( \pi_2(W) = H_2(\tilde{W}) \) whose image forms a basis for \( H_2(W, X; R) \). Then there are maps \( b_\lambda : S^2_\lambda \to W \) with \( \lambda \in S \) such that for all \( q \geq 2 \), the composition of maps

\[
H_q(\vee_{\lambda \in S} S^2_\lambda; R) \to H_q(W; R) \to H_q(W, X; R)
\]

is an isomorphism. For each such \( \lambda \), attach a 3-cell \( (D^3, S^2) \) to \( W \) along \( b_\lambda \). Let \( Y \) denote the resulting space. We see that the diagram

\[
\begin{array}{ccc}
\vee_{\lambda} S^2 & \longrightarrow & W \\
\downarrow & & \downarrow \\
\vee_{\lambda} D^3 & \longrightarrow & Y
\end{array}
\]
is a pushout diagram. By the van Kampen theorem, the fundamental group of $Y$ is still $G$. Denoting by $H_*(\cdot)$ the homology groups $H_*(-; R)$, we have the following commutative diagram:

\[
\cdots \to H_3(\vee D^3, \vee S^2) \to H_2(\vee S^2, \text{pt}) \to H_2(\vee D^3, \text{pt}) \to H_2(\vee D^3, \vee S^2) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\cdots \to H_3(Y, W) \to H_2(W, X) \to H_2(Y, X) \to H_2(Y, W).
\]

By a five lemma argument, for any $q \geq 2$ the relative homology group $H_q(Y, X; R) = 0$, which shows that $H_q(X; R) \cong H_q(Y; R)$. \qed

# 4 Applications

## 4.1 Quillen’s plus-construction

In this subsection, we show that Quillen’s plus-construction is a special case of Theorem 1.1. For this, recall that a fibration $F \to E \xrightarrow{p} B$ with connected fiber $F$, total space $E$ and base space $B$ is said to be quasi-nilpotent if the action of $\pi_1(B)$ on $H_*(F; \mathbb{Z})$ is nilpotent. The following result was proved in [1].

**Lemma 4.1** (4.3(xii) in [1]). The following properties of a map $f : X \to Y$ are equivalent.

1. $f$ is acyclic, i.e. $\tilde{H}_*(F_f; \mathbb{Z}) = 0$ for the homotopy fiber $F_f$ of $f$.
2. $f$ is quasi-nilpotent and $H_*(f)$ is an isomorphism.

The following result shows Quillen’s plus-construction is a special case of Theorem 1.1 when $R = \mathbb{Z}$ and ker $\alpha$ is perfect. According to Lemma 2.2, $\mathbb{Z}$ is a $G$-dense ring.

**Proposition 4.2.** Let $R = \mathbb{Z}$. Among all spaces $Y$ formed from $X$ by adjoining cells of dimension at most 3 and satisfying (1) in Theorem 1.1, it is possible to choose one such that the map $g : X \to Y$ is Quillen’s plus-construction with respect to ker$(\alpha)$ if and only if $\alpha : \pi \to G$ is surjective and ker$(\alpha)$ is perfect.

**Proof.** Assume $R = \mathbb{Z}$ and the map $g : X \to Y$ in Theorem 1.1 is Quillen’s plus-construction. Then $g$ is acyclic, i.e. $\tilde{H}_*(F_g; \mathbb{Z}) = 0$ for the homotopy fiber $F_g$ of $g$. This implies that the homotopy fiber $F_g$ is an acyclic space.
Since $H_1(F_g; \mathbb{Z}) = 0$, we have that $\pi_1(F_g)$ is perfect. By the fact that $F_g \to X \to Y$ is a fiber sequence and $\tilde{H}_0(F_g) = 0$, then $\alpha$ is surjective and $\ker(\alpha) = \text{im}[\pi_1(F_g) \to \pi]$ is perfect.

Conversely, when $\alpha$ is surjective and $\ker(\alpha)$ is perfect, let $\bar{X}$ be the covering space of $X$ with fundamental group $\pi_1(\bar{X}) = \ker(\alpha)$. Since $\ker(\alpha)_{\text{ab}} = 0$, there is a simply connected space $\bar{Y}$ and a map $\bar{g} : \bar{X} \to \bar{Y}$ such that (1) in Theorem 1.1 holds for any $q \geq 1$ with $R = \mathbb{Z}$ and $G$ the trivial group. Since $\bar{Y}$ is simply connected, the map $\bar{g} : \bar{X} \to \bar{Y}$ is quasi-nilpotent and thus acyclic by Lemma 4.1. Let $Y$ be the pushout of the diagram

$$
\begin{array}{ccc}
X & \longrightarrow & \bar{Y} \\
\downarrow & \searrow \cdot & \downarrow \\
X & \longrightarrow & Y.
\end{array}
$$

By the van Kampen Theorem, $\pi_1(Y) = \pi_1(X)/\pi_1(\bar{X}) = \pi/\ker(\alpha) = G$. According to (4.20) in [1], the map $X \to Y$ is still an acyclic map.

4.2 Bousfield’s integral localization

In this subsection, we show that the Bousfield’s integral localization is a special case of Theorem 1.1 when $R = \mathbb{Z}$ and $\alpha$ is surjective, as presented in the following lemma. Again note that $\mathbb{Z}$ is a $G$-dense ring by Lemma 2.2.

**Corollary 4.3 ([24]).** Let $X$ be a CW complex with fundamental group $\pi$ and $N$ a relatively perfect normal subgroup of $\pi$, i.e. $[\pi, N] = N$. Then there is a CW complex $Y$ obtained from $X$ by adding 2-cells and 3-cells such that $\pi_1(Y) = \pi/N$ and for any $q \geq 0$ we have

$$H_q(X; \mathbb{Z}) \cong H_q(Y; \mathbb{Z}).$$

Conversely, for some CW complex $Y$ let $f : X \to Y$ be an integral homology equivalence of spaces that induces an epimorphism on the fundamental groups. Then $\ker[f_* : \pi_1(X) \to \pi_1(Y)]$ is relatively perfect in $\pi_1(X)$.

**Proof.** By [15], there is a long exact sequence

$$H_2(\pi; \mathbb{Z}) \to H_2(\pi/N; \mathbb{Z}) \to N/[\pi, N] \to H_1(\pi; \mathbb{Z}) \to H_1(\pi/N; \mathbb{Z}) \to 0. \quad (2)$$

When $N = [\pi, N]$, we have that the map $H_2(\pi; \mathbb{Z}) \to H_2(\pi/N)$ is surjective and $H_1(\pi; \mathbb{Z}) \to H_1(\pi/N)$ is an isomorphism. According to Theorem 1.1
with \( R = \mathbb{Z} \), there exists a CW complex \( Y \) and a map \( g : X \to Y \) such that for any \( q \geq 0 \) we have \( H_q(X; \mathbb{Z}) \cong H_q(Y; \mathbb{Z}) \). By the proof of Theorem 1.1, \( Y \) is obtained from \( X \) by attaching 2-cells and 3-cells. Now assume that \( f : X \to Y \) is an integral homology equivalence that induces an epimorphism on the fundamental groups. Then

\[
H_1(X; \mathbb{Z}) = H_1(\pi; \mathbb{Z}) \cong H_1(\pi_1(Y); \mathbb{Z}) = H_1(Y; \mathbb{Z}).
\]

There is a commutative diagram

\[
\begin{array}{ccc}
H_2(X; \mathbb{Z}) & \to & H_2(\pi; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H_2(Y; \mathbb{Z}) & \to & H_2(\pi_1(Y); \mathbb{Z}),
\end{array}
\]

where the left vertical map is an isomorphism. This shows that the right vertical map is an epimorphism. According to the same long exact sequence above, \( \ker f = [\pi, \ker f] \), i.e., \( \ker f \) is relatively perfect.

### 4.3 Moore spaces

In this subsection, we give an application of Theorem 1.1 to the existence of Moore spaces. Recall the definition of Moore spaces from the Introduction. For \( n \geq 2 \), such a space always exists. For \( n = 1 \), we get the following result, which was first proved by Varadarajan in [27].

When \( \pi = 1 \) in Theorem 1.1, we have the following condition for existence of \( M(G, 1) \).

**Corollary 4.4 ([27]).** There exists a Moore space \( M(G, 1) \) if and only if \( H_2(G; \mathbb{Z}) = 0 \).

**Proof.** According to Lemma 2.2 \( \mathbb{Z} \) is a \( G \)-dense ring. If \( H_2(G; \mathbb{Z}) = 0 \), let \( X = \text{pt} \) and \( \alpha : 1 \to G \) be the trivial group homomorphism in Theorem 1.1. It is clear that \( \alpha \) induces a surjection on 2-dimensional homology groups and an injection on 1-dimensional homology groups. Then there exists a space \( Y \) with \( \pi_1(Y) = G \) and \( H_i(Y; \mathbb{Z}) = 0 \) for any \( i \geq 2 \). This makes \( Y \) an \( M(G, 1) \).

Conversely, let \( Y = M(G, 1) \) be a Moore space. According to the Hopf exact sequence (cf. Lemma 3.1)

\[
\pi_2(Y) \to H_2(Y; \mathbb{Z}) \to H_2(\pi_1(Y); \mathbb{Z}) \to 0,
\]

we now have \( H_2(G; \mathbb{Z}) = H_2(\pi_1(Y); \mathbb{Z}) = 0 \). \( \square \)
Remark 4.5. It has been noted (cf. [20]) that the plus-construction shares some common features with the construction of Moore spaces in [27]. We have actually shown in Proposition 4.2 and Corollary 4.4 that both the plus-construction and the existence of $M(G, 1)$ are just two extreme cases of Theorem 1.1 with $\mathbb{Z}$ as coefficients.

Similarly, we can consider Moore spaces with coefficients. Suppose $G$ is a group and $R$ is a $\mathbb{Z}[G]$-module. Let $M = M(G, 1; R)$ be the Moore space with coefficients $R$, i.e. $\pi_0(M) = 0, \pi_1(M) = G$ and $H_i(M; R) = 0$ for $i > 1$. The following new result characterizes when there exists a Moore space with coefficients $R$.

**Proposition 4.6.** Let $R$ be a $G$-dense ring. Suppose that $R$ is a principal ideal domain or that the relative homology group $H_1(G, 1; R)$ is a stably free $R$-module. Then there exists a Moore space $M(G, 1; R)$ if and only if $H_2(G; R) = 0$.

**Proof.** The proof is similar to the proof of Corollary 4.4. □

### 4.4 Partial $k$-completion of Bousfield and Kan

Let $k = \mathbb{Z}/p$ be the constant ring for any prime $p$ or $k \subseteq \mathbb{Q}$ a subring of the rationals. A group $G$ is called $k$-perfect, if $H_1(G; k) = k \bigotimes_{\mathbb{Z}} G_{ab} = 0$. The following result was first obtained by Bousfield and Kan [9]; it is a special case of Theorem 1.1 when $\alpha$ is surjective.

**Corollary 4.7** (Prop. 6.3 in [9], p. 219). Assume $k = \mathbb{Z}/p$ is the constant ring for any prime $p$ or $k \subseteq \mathbb{Q}$ a subring of rationals. Let $X$ be a CW complex with fundamental group $\pi$ with $P$ its maximal $k$-perfect subgroup. Then there exists a CW complex $Y$ and a map $g : X \to Y$ such that $\pi_1(Y) = \pi/P$ and for any $q \geq 0$ we have

$$H_q(X; k[\pi/P]) \cong H_q(Y; k[\pi/P]).$$

**Proof.** By [15], there is a long exact sequence

$$H_2(\pi; k[\pi/P]) \to H_2(\pi/P; k[\pi/P]) \to k[\pi/P] \bigotimes_{\mathbb{Z}[\pi/P]} P_{ab} \to H_1(\pi; k[\pi/P]) \to H_1(\pi/P; k[\pi/P]) \to 0.$$

When $k[\pi/P] \bigotimes_{\mathbb{Z}[\pi/P]} P_{ab} \cong k \bigotimes_{\mathbb{Z}} P_{ab} = 0$, we can see

$$H_2(\pi; k[\pi/P]) \to H_2(\pi/P; k[\pi/P]).$$

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is surjective and $H_1(\pi; k[\pi/P]) \to H_1(\pi/P; k[\pi/P])$ is an isomorphism. According to the long exact sequence of homology groups

$$\cdots \to H_1(\pi; k[\pi/P]) \to H_1(\pi/P; k[\pi/P]) \to H_1(\pi/P, \pi; k[\pi/P]) \to H_0(\pi; k[\pi/P]) \to H_0(\pi/P; k[\pi/P]) \to 0,$$

we have $H_1(\pi/P, \pi; k[\pi/P]) = 0$. Therefore, applying Theorem 1.1 with $G = \pi/P$ and the $G$-dense ring $R = k[\pi/P]$, we get a CW complex $Y$ and a map $g : X \to Y$ such that for any $q \geq 0$ there is an isomorphism

$$H_q(X; k[\pi/P]) \cong H_q(Y; k[\pi/P]).$$

\[ \square \]

4.5 Zero-in-the-spectrum conjecture

In this subsection, we give an application of Theorem 1.1 to the zero-in-the-spectrum conjecture. This conjecture is stated in $L^2$-homology, whose definition and basic properties are presented as follows.

$L^2$-homology.

Let $G$ be a group and $l^2(G)$ be the Hilbert space spanned by $G$ (cf. Section 2). By definition, the reduced group $C^*$-algebra $C^*_r(G)$ (resp. $C^*_r(G)$) is the completion of $\mathbb{C}[G]$ (resp. $\mathbb{R}[G]$) in $B(l^2(G))$ with respect to the operator norm. Suppose $\tilde{Y}$ is a $G$-CW complex and $C_\ast(Y)$ the cellular chain complex of $Y$. The $L^2$ $n$-th homology group with coefficients $C^*_r(G)$ of $Y$ is defined as $H_n(C_\ast(Y) \bigotimes_{\mathbb{Z}[G]} C^*_r(G))$. When $Y$ is a CW complex with fundamental group $G$, the universal covering space $\tilde{Y}$ is a $G$-CW complex with the group action as deck transformation. The $L^2$-homology groups with coefficients $C^*_r(G)$ of $Y$ are defined as the homology groups of the complex $C_\ast(\tilde{Y}) \bigotimes_{\mathbb{Z}[G]} C^*_r(G)$, where $C_\ast(\tilde{Y})$ is the cellular chain complex of $\tilde{Y}$.

Let $f : X \to Y$ be a cellular map of CW complexes and assume $\pi_1(Y) = G$. We view $C_\ast(\tilde{X})$ as a free right $\mathbb{Z}[\pi_1(X)]$-module spanned by the $i$-th dimensional cells in $X$ and define the $L^2$-homology groups of $X$ with coefficients $C^*_r(G)$ as

$$H_\ast(X; C^*_r(G)) = H_\ast((C_\ast(\tilde{X}) \bigotimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z}[G]) \bigotimes_{\mathbb{Z}[G]} C^*_r(G))$$

$$= H_\ast(C_\ast(\tilde{X}) \bigotimes_{\mathbb{Z}[\pi_1(X)]} C^*_r(G)).$$

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Suppose \( f_1 : \pi_1(X) \to \pi_1(Y) \) is the group homomorphism induced by \( f \). Then there is a well-defined chain map
\[
f_* : C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z}[G] \to C_*(\tilde{Y}), \quad xh \otimes g \mapsto f(x)f_1(h)g
\]
for any cell \( x \in X, h \in \pi_1(X) \) and \( g \in G \). Taking the mapping cylinder if necessary, we can assume \( f_* \) is an inclusion of free \( \mathbb{Z}[G] \)-modules. We can define the relative \( L^2 \)-homology as the homology of the cokernel chain complex \( C_*(\tilde{Y}, \tilde{X}; C_*(G)) \) of the chain map \( f_* \) in the usual sense to get a long exact sequence
\[
\cdots \to H_n(X; C^*_r(G)) \to H_n(Y; C^*_r(G)) \to H_n(Y, X; C^*_r(G)) \to 0.
\]
When we consider the group von Neumann algebra \( \mathcal{N}G \) or the Hilbert space \( l^2(G) \) instead of the reduced group \( C^* \)-algebra \( C^*_r(G) \), the \( L^2 \)-homology with coefficients \( \mathcal{N}G \) or \( l^2(G) \) can be defined in a similar way. For more details on \( L^2 \)-homology, we refer the reader to the book of Lück [17] and the article [14].

**Proposition 4.8.** Let \( X \) be a CW complex with fundamental group \( G \). Assume \( R' \subseteq R \) are two \( \mathbb{Z}[G] \)-modules with \( R' \) a unital ring such that \( R \) is an \( R' \)-module and \( R/R' \) is a flat \( R' \)-module. Then for any nonnegative integer \( n \), the following are equivalent:

(i) For any \( 0 \leq i \leq n \), the homology group \( H_i(X; R') = 0 \).

(ii) For any \( 0 \leq i \leq n \), the homology group \( H_i(X; R) = 0 \).

In particular, the homology groups \( H_i(X; C^*_r(G)) \) are zero in degrees \( 0 \) through \( n \) if and only if the homology groups \( H_i(X; C^*_r(G)) \) are zero in degrees \( 0 \) through \( n \).

**Proof.** Suppose that for any \( 0 \leq i \leq n \), we have \( H_i(X; R') = 0 \). Let \( (C_*(\tilde{X})) \) be the chain complex of the universal covering space \( \tilde{X} \), which is viewed as a chain complex of free \( \mathbb{Z}[G] \)-modules. Thus \( C_*(\tilde{X}) \otimes_{\mathbb{Z}[G]} R' \) is a chain complex of free \( R' \)-modules. According to the Künneth spectral sequence (cf. page 143 in [26]), we have
\[
\Tor^R_{p}(H_q(X; R'), R) \Rightarrow H_{p+q}(X; R),
\]
which shows that for any $0 \leq i \leq n$, the homology group $H_i(X; R) = 0$.

Conversely, assume that any for $0 \leq i \leq n$, we have $H_i(X; R) = 0$. Since $\text{Tor}_0^R(M, N) = M \otimes_R N$ for any $R'$-modules $M$ and $N$, \[ H_0(X; R') \otimes_{R'} R = H_0(X; R) = 0 \]
by the Künneth spectral sequence. According to the long exact sequence
\[ \cdots \rightarrow \text{Tor}_1^R(H_0(X; R'), R/R') \rightarrow H_0(X, R') \otimes_{R'} R' = H_0(X; R') \rightarrow H_0(X, R') \otimes_{R'} R = 0 \rightarrow H_0(X, R') \otimes_{R'} R' \rightarrow \cdots , \]
there is a surjection \[ \text{Tor}_1^R(H_0(X; R'), R/R') \twoheadrightarrow H_0(X; R'). \]
Since $R/R'$ is a flat $R'$-module by assumption, $H_0(X; R') = 0$. By the same spectral sequence, we can get
\[ \text{Tor}_0^R(H_1(X; R'), R) = H_1(X; R) = 0. \]
Similar arguments prove inductively that for any $0 \leq i \leq n$, the homology group $H_i(X; R') = 0$. Clearly, $C_r^*(G) \cong C_r^*(G) \bigoplus iC_r^*(G)$ if we consider the real and imaginary parts. The fact that $C_r^*(G)/C_r^*(G)$ is a free $C_r^*(G)$-module proves the last part of the proposition. \[ \square \]

**Zero-in-the-spectrum conjecture.**

In the notation of the Introduction, zero not belonging to the spectrum of $\Delta = \Delta_*$ can also be expressed as the vanishing of $H_*(M; C_r^*(\pi_1(M)))$. For more details, we refer the reader to [17].

**Conjecture 4.9.** Let $M$ be a finite aspherical $CW$-complex (or weakly a closed, connected, oriented and aspherical Riemannian manifold) with fundamental group $\pi$. Then for some $i \geq 0$, $H_i(X; C_r^*(\pi)) \neq 0$.

If the condition that $X$ is aspherical is dropped, the following corollary, which is a special case of Theorem [14] when $R = C_r^*(G)$ and $\pi = 1$, shows the above conjecture is not true. This result is a generalization of the results obtained by Farber-Weinberger [13] and Higson-Roe-Schick [14]. Here we do not assume $G$ is finitely presented. Recall from Lemma [22] that $C_r^*(G)$ is a $G$-dense ring.
Corollary 4.10 ([14]). For a group $G$ with
\[ H_0(G; C_r^*(G)) = H_1(G; C_r^*(G)) = H_2(G; C_r^*(G)) = 0, \]
there is a CW complex $Y$ such that $\pi_1(Y) = G$ and for each integer $n \geq 0$, the homology group $H_n(Y; C_r^*(G)) = 0$. If $G$ is finitely presented, $Y$ can be chosen to be a finite CW complex.

Proof. According to Proposition 4.8,
\[ H_0(G; C_r^*(G)) = H_1(G; C_r^*(G)) = H_2(G; C_r^*(G)) = 0. \]
Let $\alpha : \pi = 1 \to G$, $R = C_r^*(G)$ and $X = \text{pt}$ in Theorem 1.1. By the long exact sequence of homology groups
\[
\cdots \to H_1(1; C_r^*(G)) \to H_1(G; C_r^*(G)) \to H_1(G, 1; k[\pi/P]) \to H_0(1; C_r^*(G)) \to H_0(G; C_r^*(G)) \to 0,
\]
we have
\[ H_1(G, \pi; C_r^*(G)) = H_0(1; C_r^*(G)) = C_r^*(G), \]
which is a free $C_r^*(G)$-module. Therefore, there exists $Y$ by Theorem 1.1 and Proposition 4.8 such that $\pi_1(Y) = G$ and for all $n \geq 0$ we have $H_n(Y; C_r^*(G)) = 0$. When $G$ is finitely presented, the proof of Theorem 1.1 shows the number of cells added to $X$ is finite. \qed

The groups satisfying the condition of Corollary 4.10 have to be non-amenable (for more details, see [7] and Rosenberg’s note in [25]).

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