The order of complex numbers

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Abstract: In this paper, we define an ordering relation for a set of complex numbers, and research the properties and theorems of the ordering, solve some simple complex inequalities with the ordering.

Key words: complex number, ordering, inequality.

1 Introduction

In general, it is impossible to judge the order of two complex numbers. In fact, how to judge the order of two complex numbers is how to define their sequencing. We think it is possible to define the order of two complex numbers. Since the order of complex numbers can’t bring more useful interest than that of real numbers, no one is willing to define the order of the complex numbers.

It is assumed that the properties of the inequalities of real numbers hold. In this paper, we define the order of the complex numbers, which is the extension of the order of real number, then get some properties and theorem about this order in complex. At the end, using this order, we resolve some inequalities.

Let small letters denote real numbers, and let capital letters denote complex numbers. We use \( \mathbb{R} \) and \( \mathbb{C} \) to denote the set of real numbers and the set of complex numbers respectively. We also use Greece small letter to denote the argument of complex number, and use script capital letter to denote the number set.

2 The ordering relation (\( \leq \)) of the set of complex numbers

Let complex numbers \( A = a_1 + a_2i, B = b_1 + b_2i, C = c_1 + c_2i, \ldots \)

Definition 1  \( A \leq B \iff (a_1 < b_1) \cup [(a_1 = b_1) \cap (a_2 \leq b_2)], \) that is \( a_1 < b_1, \) or \( a_1 = b_1 \) and \( a_2 \leq b_2. \)

\( A \geq B \iff (a_1 > b_1) \cup [(a_1 = b_1) \cap (a_2 \geq b_2)]. \)

From the above definition, we get the following properties.

Property 1  (reflexivity) \( A \leq A. \)

Property 2  \( [(A \geq B) \ \text{and} \ (A \leq B)] \iff (A = B). \)

Proof  \{ A \geq B \} \cap \{ A \leq B \}

\[ \iff \{(a_1 > b_1) \cup [(a_1 = b_1) \cap (a_2 \geq b_2)]\} \cap \{(a_1 < b_1) \cup [(a_1 = b_1) \cap (a_2 \leq b_2)]\} \]

\[ \iff \{(a_1 > b_1) \cap (a_1 < b_1)\} \cup \{(a_1 = b_1) \cap (a_2 \geq b_2)\} \cap \{(a_1 = b_1) \cap (a_2 \leq b_2)\} \]

\[ \cup \{(a_1 > b_1) \cap [(a_1 = b_1) \cap (a_2 \leq b_2)]\} \cup \{(a_1 = b_1) \cap (a_2 \geq b_2)\} \cap (a_1 < b_1) \]

\[ \iff \{(a_1 = b_1) \cap (a_2 \geq b_2)\} \cap (a_1 = b_1) \cap (a_2 \leq b_2)\} \]
\[\Leftrightarrow (a_1 = b_1) \cap (a_2 = b_2) \Leftrightarrow (A = B)\].

**Property 3** (transitivity) \(A \leq B\) and \(B \leq C\) \(\Rightarrow\) \(A \leq C\).

**Proof** \(\{A \leq B\} \cap \{B \leq C\}\)

\[\Leftrightarrow \{(a_1 < b_1) \cup [(a_1 = b_1) \cap (a_2 \leq b_2)]\} \cap \{(b_1 < c_1) \cup [(b_1 = c_1) \cap (b_2 \leq c_2)]\}\]

\[\Leftrightarrow \{(a_1 < b_1) \cap [b_1 < c_1]\} \cup \{(a_1 = b_1) \cap (a_2 \leq b_2)\} \cap \{(b_1 = c_1) \cap (b_2 \leq c_2)\}\]

\[\cup \{(a_1 < b_1) \cap (b_1 = c_1) \cap (b_2 \leq c_2)\} \cup \{(a_1 = b_1) \cap (a_2 \leq b_2) \cap [b_1 < c_1]\}\]

\[\Rightarrow \{(a_1 < c_1)\} \cup \{(a_1 < b_1) \cap (b_2 \leq c_2)\} \cup \{(a_1 = c_1) \cap (a_2 \leq c_2)\} \cup \{(a_1 = b_1) \cap (a_2 \leq c_2)\}\]

\[\Rightarrow \{(a_1 < c_1)\} \cup \{(a_1 = b_1) \cap (a_2 \leq c_2)\} \Leftrightarrow \ A \leq C\].

For any two complex numbers \(A, B \in \mathcal{C}\), at least one between the two relation expressions \(A \leq B\), \(A \geq B\) holds. Combing with properties 1, 2, 3, we know that the set of complex numbers \(\mathcal{C}\) is a totally ordered set under the condition of the above defined ordering relation \(\leq\). It is easy to see that the above defined ordering relation \(\leq\) is the extension of that of real numbers.

**Theorem 1** \(A \leq B \Leftrightarrow A + C \leq B + C\).

**Proof** \((A + C \leq B + C)\)

\[D.1 \Leftrightarrow \{(a_1 + c_1 < b_1 + c_1) \cup [(a_1 + c_1 = b_1 + c_1) \cap (a_2 + c_2 \leq b_2 + c_2)]\}\]

\[R.P \Leftrightarrow \{(a_1 < b_1) \cup [a_1 = b_1) \cap (a_2 \leq b_2)]\} \quad D.1 \Leftrightarrow (A \leq B)\].

The notation \(D.1\) in the first equivalent symbol denote the following result is deduced by definition 1. The notation \(R.P\) in the second equivalent symbol denote the following result is deduced by the properties of the inequalities of real numbers. In the following, we also use the similar remark.

**Theorem 2** The term in the inequalities can be moved from one side to the other, that is \(A + B \leq C \Leftrightarrow A \leq C - B\).

**Proof** \(A + B \leq C \quad D.1 \Leftrightarrow A + B - B \leq C - B \Leftrightarrow A \leq C - B\).

**Theorem 3** \(A \leq B\), and \(C \leq D\) \(\Rightarrow\) \(A + C \leq B + D\).

**Proof** Since \(A \leq B \quad D.1 \Leftrightarrow A + C \leq B + C\) and \(C \leq D \quad D.1 \Leftrightarrow B + C \leq B + D\), by the transitivity, we get \(A + C \leq B + D\).

**Theorem 4** Let \(r > 0\), then \(A \leq B \Leftrightarrow rA \leq rB\).

**Proof** \((rA \leq rB) \quad D.1 \Leftrightarrow (ra_1 < rb_1) \cup [(ra_1 = rb_1) \cap (ra_2 \leq rb_2)]\]

\[R.P \Leftrightarrow (a_1 < b_1) \cup [(a_1 = b_1) \cap (a_2 \leq b_2)] \quad D.1 \Leftrightarrow (A \leq B)\].

**Theorem 5** Let \(r < 0\), then \(A \leq B \Leftrightarrow rA \geq rB\). Especially, \((A \leq B) \Leftrightarrow (-A \geq -B)\).

**Proof** \((rA \geq rB) \quad D.1 \Leftrightarrow (ra_1 > rb_1) \cup [(ra_1 = rb_1) \cap (ra_2 \geq rb_2)]\]

\[R.P \Leftrightarrow (a_1 < b_1) \cup [(a_1 = b_1) \cap (a_2 \leq b_2)] \quad D.1 \Leftrightarrow (A \leq B)\].

3 The operation of the set

**Definition 2** Let the set of complex numbers \(\mathcal{B} \subset \mathcal{C}\), \(\theta \in \mathcal{R}\). We define the rotation set of \(\mathcal{B}\) by \(e^{i\theta}\mathcal{B} := \{Ze^{i\theta}; Z \in \mathcal{B}\}\).

**Remark 1** \(e^{i\theta}\mathcal{B}\) denote the set \(\mathcal{B}\) rotates \(\theta\) radian around the origin.

**Theorem 6** \(W e^{i\theta} \in \mathcal{B} \Leftrightarrow W \in e^{-i\theta}\mathcal{B}\).
Proof Let $W := Z e^{-i\theta} \leftrightarrow Z = W e^{i\theta}$. By the definition, we get

\[ e^{-i\theta} B := \{ Z e^{-i\theta} \in \mathbb{C}; Z \in B \} = \{ W \in \mathbb{C}; W e^{i\theta} \in B \} \]

From the last equality we deduce that $W \in e^{-i\theta} B \leftrightarrow W e^{i\theta} \in B$.

**Example 1** When $B = \{ Z; Z \geq A \in \mathbb{C} \}$ is a semi-open and semi-closed half plane, $e^{i\theta} B$ is also a semi-open and semi-closed half plane, that is $(A = a_1 + a_2 i, Z = z_1 + z_2 i)$

\[ e^{i\theta} B \overset{D.2}{=} \{ Z e^{i\theta}; Z \geq A \} = \{ Z; e^{-i\theta} Z \geq A \} \]

\[ = \{ Z; (z_1 \cos \theta + z_2 \sin \theta) + (z_2 \cos \theta - z_1 \sin \theta)i \geq a_1 + a_2 i \} \]

\[ \overset{D.1}{=} (Z; z_1 \cos \theta + z_2 \sin \theta > a_1) \cup [(z_1 \cos \theta + z_2 \sin \theta = a_1) \cap (z_2 \cos \theta - z_1 \sin \theta \geq a_2)]. \]

The first part is an open half plane, the part in $[\ ]$ is a half line.

**Definition 3** Let the set of complex numbers $B \subset \mathbb{C}, r \geq 0$. We define the dilation set of $B$ by $rB := \{ rZ \in \mathbb{C}; Z \in B \}$.

**Remark 2** $rB$ denotes $B$ makes a stretching at the extension ratio $r$.

**Theorem 7** $rW \in B \leftrightarrow W \in B/r$.

**Proof** Let $W := Z/r \leftrightarrow Z = rW$. By the definition, we get

\[ B/r := \{ Z/r \in \mathbb{C}; Z \in B \} = \{ W \in \mathbb{C}; rW \in B \}. \]

Namely, $W \in B/r \leftrightarrow rW \in B$.

**Definition 4** Let the set of complex numbers $B \subset \mathbb{C}, A \in \mathbb{C}$. We define the translation set of $B$ by $B + A := \{ Z + A \in \mathbb{C}; Z \in B \}$.

**Remark 3** $A + B$ denotes $B$ is translated.

**Theorem 8** $A + W \in B \leftrightarrow W \in B - A$.

**Proof** Let $W := Z - A \leftrightarrow Z = W + A$. By the definition, we get

\[ B - A := \{ Z - A \in \mathbb{C}; Z \in B \} = \{ W \in \mathbb{C}; W + A \in B \}. \]

Namely, $W \in B - A \leftrightarrow W \in B + A$.

**Definition 5** Let the set of complex numbers $B \subset \mathbb{C}$. We define the inversion set of $B$ by $1/B := \{ 1/Z \in \mathbb{C}; Z \in B \}$.

**Remark 4** $1/B$ and $B$ is symmetric about the unit circumference $\{|Z| = 1\}$, or is called the inversion transform.

**Theorem 9** $1/W \in B \leftrightarrow W \in 1/B$.

**Proof** Let $W := 1/Z \leftrightarrow Z = 1/W$. By the definition, we get

\[ 1/B := \{ 1/Z \in \mathbb{C}; Z \in B \} = \{ W \in \mathbb{C}; 1/W \in B \}. \]

Namely, $W \in 1/B \leftrightarrow 1/W \in B$.

**Example 2** When $B = \{ Z; Z \geq A \in \mathbb{C} \}$ is a semi-open and semi-closed half plane, $1/B$ is a semi-open and semi-closed disc which is symmetric about the real axis, that is $(Z := z_1 + z_2 i \neq 0)$

\[ 1/B \overset{D.5}{=} \{ 1/Z; Z \geq A \} = \{ Z; 1/Z \geq A \} \]

\[ = \{ Z; \frac{z_1}{z_1^2 + z_2^2} - \frac{z_2}{z_1^2 + z_2^2} \geq a_1 + a_2 i \} \]

\[ \overset{D.1}{=} \left( \frac{z_1}{z_1^2 + z_2^2} > a_1 \right) \cup [(\frac{z_1}{z_1^2 + z_2^2} = a_1) \cap (\frac{z_2}{z_1^2 + z_2^2} \geq a_2)] \]

\[ = (a_1 z_1^2 + a_2 z_2^2 > z_1) \cup [(a_1 z_1^2 + a_1 z_2^2 = z_1) \cap (a_2 z_1^2 + a_2 z_2^2 \leq -z_2)]. \]
The first part is an open disc, its center is \( \frac{1}{2a} + 0i \), its radius is \( \frac{1}{2a} \). The part in \( [ ] \) is a circular arc whose point isn’t the origin.

**Definition 6** Let the set of complex numbers \( \mathcal{B} \subset \mathbb{C} \). We define the radication set of \( \mathcal{B} \) by \( \mathcal{B}^{1/2} := \{ Z^{1/2}; Z \in \mathcal{B} \} \).

**Theorem 10** \( W^2 \in \mathcal{B} \Leftrightarrow W \in \mathcal{B}^{1/2} \).

**Proof** Let \( Z^{1/2} := \{ -W, W \} \Leftrightarrow Z = W^2 \). By the definition, we get

\[
\mathcal{B}^{1/2} := \{ Z^{1/2}; Z \in \mathcal{B} \} = \{ -W; W^2 \in \mathcal{B} \} \cup \{ W; W^2 \in \mathcal{B} \} = \{ W; W^2 \in \mathcal{B} \}.
\]

Namely, \( W \in \mathcal{B}^{1/2} \Leftrightarrow W^2 \in \mathcal{B} \).

**Example 3** When \( \mathcal{B} = \{ Z; Z \geq A \} \) is a semi-open and semi-closed half plane, its radication set \( \mathcal{B}^{1/2} \) is a semi-open and semi-closed domain whose boundary is hyperbola, that is

\[
\mathcal{B}^{1/2} := \{ Z^{1/2}; Z \geq A \} = \{ Z; Z^2 \geq A \}
\]

\[
= \{ Z; (z_1 + z_2i)^2 \geq a_1 + a_2i \}
\]

\[
= \{ Z; (z_1^2 - z_2^2) + 2z_1z_2i \geq a_1 + a_2i \}
\]

\[
\overset{D.1}{=} [z_1^2 - z_2^2 \geq a_1] \cup [(z_1^2 - z_2^2 = a_1) \cap (2z_1z_2 \geq a_2)].
\]

The hyperbola \( z_1^2 - z_2^2 = a_1 \) splits the plane into three parts (When \( a_1 = 0 \), it splits the plane into four parts.). If \( a_1 > 0 \), \( \mathcal{B}^{1/2} \) is two non-neighbor parts that doesn’t contain the origin. If \( a_1 < 0 \), \( \mathcal{B}^{1/2} \) is a connected part that contains the origin. We call it the hyperbola domain.

4 Solving inequalities

**Definition 7** To solve the inequality \( f(Z) \geq g(Z) \) means to find the set \( \{ Z \in \mathbb{C}; f(Z) \geq g(Z) \} \).

**Definition 8** Define \( \mathcal{D}(A) := \mathcal{D}(Z \geq A) := \{ Z \in \mathbb{C}; Z \geq A \} = \{ Z \in \mathbb{C}; Z \in \mathcal{D}(Z \geq A) \} \).

\( \mathcal{D}(A) \) is a semi-open and semi-closed perpendicular half plane that is split by the perpendicular line \( \{ Z \in \mathbb{C}; ReZ = ReA \} \).

1. **Solving the linear inequality with one unknown** Let \( A = re^{i\theta}(r \geq 0) \), solve the inequality \( AZ - B \geq 0 \).

   **Solution** By the definition and properties of inequalities, we get the solution set \( \mathcal{J} \) is the following.

\[
\mathcal{J} \overset{D.7}{=} \{ Z \in \mathbb{C}; AZ - B \geq 0 \} \overset{T.2.4}{=} \{ Z \in \mathbb{C}; e^{i\theta}Z \geq B/r \}
\]

\[
W = e^{i\theta}Z \overset{D.2}{=} \{ We^{-i\theta} \in \mathbb{C}; W \geq B/r \} \overset{D.7}{=} e^{-i\theta} \{ W \in \mathbb{C}; W \geq B/r \}
\]

\[
\overset{D.8}{=} e^{-i\theta} \mathcal{D}(Z \geq B/r).
\]

Let a perpendicular half plane rotate \( \theta \) radian anticlockwise around the origin, we get a half plane whose boundary is an oblique line, that is the solution set \( \mathcal{J} \).

2. **Solving the linear inequalities with one unknown** Let \( A = re^{i\theta}, C = ue^{i\phi}(r, u \geq 0) \), solve the inequalities \( AZ - B \geq 0, CZ - D \geq 0 \).
**Solution** Since the solution sets of two inequalities are
\[ e^{-i\theta} \mathcal{D}(B/r), \quad e^{-i\phi} \mathcal{D}(B/u), \]
respectively. The solution set of the inequalities \( \mathcal{S} \) is the following.
\[ \mathcal{S} = [e^{-i\theta} \mathcal{D}(B/r)] \cap [e^{-i\phi} \mathcal{D}(B/u)]. \]
The solution set \( \mathcal{S} \) denotes the intersection of two half planes.

3. **Solving the linear fractional inequality** Let \( B - AC \neq 0 \), solve the inequality \( \frac{AZ+B}{Z+C} \geq D \).

**Solution** Let \( B - AC := re^{i\theta} \), then the solution set \( \mathcal{S} \) is the following.
\[ \mathcal{S} \overset{D.7}{=} \{ Z \in \mathcal{C}; \frac{AZ+B}{Z+C} \geq D \} \overset{T.14}{=} \{ Z \in \mathcal{C}; \frac{1}{e^{-i\theta}(Z+C)} \geq \frac{D-A}{r} \} \]
\[ \overset{D.8}{=} \{ Z \in \mathcal{C}; \frac{1}{e^{-i\theta}(Z+C)} \in \mathcal{D}(\frac{D-A}{r}) \} \overset{T.9}{=} \{ Z \in \mathcal{C}; e^{-i\theta}(Z+C) \in \frac{1}{\mathcal{D}(\frac{D-A}{r})} \} \]
\[ \overset{T.6}{=} \{ Z \in \mathcal{C}; (Z+C) \geq \frac{1}{\mathcal{D}(\frac{D-A}{r})} \cdot e^{i\theta} \cdot e^{-i\theta} - C \}
\]
The solution set \( \mathcal{S} \) is a semi-open and semi-closed disc. Let a semi-open and semi-closed perpendicular half plane \( \mathcal{D}(\frac{D-A}{r}) \) become a disc \( \frac{1}{\mathcal{D}(\frac{D-A}{r})} \cdot e^{i\theta} \) by the inversion transform defined as example 2, then let it rotate \( \theta \) radian around the origin, we get the disc \( \frac{1}{\mathcal{D}(\frac{D-A}{r})} \cdot e^{i\theta} - C \), finally translating the disc we get the solution set \( \frac{1}{\mathcal{D}(\frac{D-A}{r})} \cdot e^{i\theta} - C \).

4. **Solving the inequality of the second order** Let \( A \neq 0 \), solve the inequality of the second order \( AZ^2 + BZ + C \geq 0 \).

**Solution** Let \( A := re^{i\theta} \), then the solution set \( \mathcal{S} \) is the following.
\[ \mathcal{S} \overset{D.7}{=} \{ Z; AZ^2 + BZ + C \geq 0 \} \overset{T.14}{=} \{ Z; e^{i\theta}(Z + \frac{B}{2A})^2 \geq \frac{B^2 - 4AC}{4rA} \} \]
\[ \overset{D.8}{=} \{ Z; [e^{i\theta/2}(Z + \frac{B}{2A})]^2 \in \mathcal{D}(\frac{B^2 - 4AC}{4rA}) \} \overset{T.10}{=} \{ Z; e^{i\theta/2}(Z + \frac{B}{2A}) \in \mathcal{D}^{1/2}(\frac{B^2 - 4AC}{4rA}) \} \]
\[ \overset{T.6}{=} \{ Z; Z + \frac{B}{2A} \in \mathcal{D}^{1/2}(\frac{B^2 - 4AC}{4rA}) \cdot e^{-i\theta/2} \}
\]
\[ \overset{T.8}{=} \{ Z; Z \in \mathcal{D}^{1/2}(\frac{B^2 - 4AC}{4rA}) \cdot e^{-i\theta/2} - \frac{B}{2A} \} = \mathcal{D}^{1/2}(\frac{B^2 - 4AC}{4rA}) \cdot e^{-i\theta/2} - \frac{B}{2A} \]
The solution set \( \mathcal{S} \) is a semi-open and semi-closed hyperbola domain. Let a semi-open and semi-closed perpendicular half plane \( \mathcal{D}(\frac{B^2 - 4AC}{4rA}) \) become a hyperbola domain \( \mathcal{D}^{1/2}(\frac{B^2 - 4AC}{4rA}) \) by the radication defined as example 3, then let the hyperbola domain rotate \( \theta \) radian anticlockwise around the origin, we get the hyperbola domain \( \mathcal{D}^{1/2}(\frac{B^2 - 4AC}{4rA}) \cdot e^{-i\theta/2} \), finally translating it we get the solution set \( \mathcal{D}^{1/2}(\frac{B^2 - 4AC}{4rA}) \cdot e^{-i\theta/2} - \frac{B}{2A} \).
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