Spontaneous Orbifold Symmetry Breaking and Generation of Mass Hierarchy

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Abstract

A very simple mechanism is proposed that stabilizes the orbifold geometry within the context of the Randall-Sundrum proposal for solving the hierarchy problem. The electro-weak TeV scale is generated from the Planck scale by spontaneous breaking of the orbifold symmetry.

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M-theory represents the most remarkable theoretical success of the end of the millennium. The moduli space of this theory contains all five, anomaly free, ten-dimensional superstring theories and the eleven-dimensional supergravity. In this context the most successful cosmological construction is that of Horava and Witten [1]. Compactification of the Horava-Witten theory on an eleven-dimensional orbifold $R^{10} \times S^1/Z_2$ to four dimensions on a deformed Calabi-Yau manifold yields that the resulting theory has $N = 1$ supersymmetry. This implies, in turn, that the early universe has undergone a phase where it was five-dimensional [2]. In Ref. [3] an effective five-dimensional theory was derived by direct compactification of the Horava-Witten theory on a Calabi-Yau space. A static solution to the field equations of this theory exists that may be interpreted as a pair of parallel three-branes that are located at the fixed points of the circle that represent the boundaries of the orbifold $S^1/Z_2$.

A set-up with two three-branes located at the boundaries of a five-dimensional $AdS_5$ spacetime has been used recently by Randall and Sundrum to address the hierarchy problem [4]. They proposed an scenario where the metric is not factorizable. The four-dimensional metric, in this scenario, is multiplied by a "warp" factor which is a function of the additional dimension

$$ds^2 = e^{-2k|y|} \eta_{mn}dx^ndx^m + dy^2,$$

where $k$ is a scale of order the Planck scale, $x^a$ are the usual 4-d coordinates and $0 \leq y \leq \pi \lambda$ is the coordinate of the extra dimension. This line-element is consistent with orbifold symmetry ($y \rightarrow -y$) and with four-dimensional Poincare invariance [4]. Randall and Sundrum have shown that this metric is a solution to Einstein’s equations in a simple set-up with two three-branes and appropriate cosmological terms. The two three-branes are located at the orbifold fixed points $y = 0$ and $y = \pi \lambda$. These represent the boundaries of the five-dimensional spacetime.

Working out the consequences of the localized energy density inherent to the brane set-
up, Randall and Sundrum found a new solution to the hierarchy problem. In the Randall-Sundrum proposal four-dimensional mass scales are related to five-dimensional input mass parameters and the "warp" factor, $e^{-2k|y|}$. This small exponential factor is the source of the large hierarchy between the observed Planck and weak scales \[^4\]. In this set-up the relation between the Planck scale and the fundamental scale is found to be

$$M_{Pl}^2 = 2M^3 \int_0^{\pi \lambda} dy e^{-2k|y|} = \frac{M^3}{k}(1 - e^{-2k\pi\lambda}), \quad (2)$$

and $M_{Pl}^2$ is a well-defined value even in the limit $\lambda \rightarrow \infty$, in contrast to the product-space expectation that $M_{Pl}^2 = M^3\lambda\pi$. By taking the second "regulator" brane at infinity and considering the coordinate $y = 0$ to be the location of the Planck brane, one can derive \[^4\]

$$M_{Pl}^2 = 2M^3 \int_0^{\infty} dy e^{-2k|y|} = \frac{M^3}{k}, \quad (3)$$

so that if $M$ and $k$ are of order $M_{Pl} = 10^{19}$GeV, the graviton zero mode is coupled correctly to generate four-dimensional gravity. For a brane located a distance $y_0$ from the Planck brane we have $M_{Pl} e^{-k|y_0|} \sim TeV$, i. e., the electroweak scale (of order TeV) is reproduced by physics confined to the brane located a distance $y_0$ from the brane where the graviton is localized. The generation of this hierarchy requires an exponential of order 30. The advantage of taking a five-dimensional spacetime with infinite extension in the $y$-direction is that one has a better chance of addressing issues such as the cosmological constant problem and black-hole physics \[^5\].

In this letter we further exploit the ideas of Randall and Sundrum. We present a very simple mechanism of orbifold geometry stabilization that allows determining the location of the TeV brane. It is literally a mechanism of spontaneous orbifold symmetry breaking. Our starting point is a generalization of the Randall-Sundrum 5-d line-element in the form

$$ds^2 = e^{-2k\sigma(y)} g_{mn}(x) dx^m dx^n + e^{l\gamma(y)+h(x)} dy^2, \quad (4)$$

where $l$ is an arbitrary constant factor, $\sigma$ and $\gamma$ are arbitrary functions of the additional
$y$-coordinate, while the four-dimensional metric $g_{ab}$ and $h$ are functions of the familiar 4-d coordinates $x^a$. For our purposes it suffices to study a general five-dimensional spacetime without fixing the matter content. Neither standard model nor hidden matter sectors are specified. Therefore, we do not pretend to give any realistic particle picture, but a general five-dimensional gravity set-up. The effective 5-d action we shall study is,

$$S = \int d^5x \sqrt{-G}(2M^3 R - \Lambda),$$  \hspace{1cm} (5)

where $G$ is the determinant of the five-dimensional metric $G_{AB} (A, B = 0, 4)$, $R$ is the Ricci scalar obtained from the 5-d Ricci tensor $R_{AB}$ ($R = G^{MN}R_{MN}$) and $\Lambda$ is a cosmological constant that plays the role of a vacuum energy density in the five-dimensional spacetime. The field equations derivable from (5) are

$$R_{AB} - \frac{1}{2} G_{AB} R = - \frac{\Lambda}{4M^3} G_{AB},$$ \hspace{1cm} (6)

These equations can be split into the following set of equations:

$$R_{ab} - \frac{1}{2} g_{ab} e^{-2k\sigma} R = - \frac{\Lambda}{4M^3} e^{-2k\sigma} g_{ab},$$ \hspace{1cm} (7)

$$R_{44} - \frac{1}{2} e^{t\gamma + h} R = - \frac{\Lambda}{4M^3} e^{t\gamma + h},$$ \hspace{1cm} (8)

and

$$R_{a4} = - \frac{3}{2} k \sigma' h_{,a} = 0,$$ \hspace{1cm} (9)

where the prime denotes derivative with respect to the additional $y$-coordinate. Eq.(9) implies two possibilities; either $\sigma' = 0$ ($\sigma = const.$) or $h_{,a} = 0$ ($h = const.$). In this letter we shall interested in the 2nd possibility and we shall set $h = const. = 0$ so the line-element (4) can be written as,

$$ds^2 = e^{-2k\sigma(y)} g_{mn}(x) dx^m dx^n + e^{t\gamma(y)} dy^2.$$ \hspace{1cm} (10)
Therefore the field equations (7) and (8) yield

\[ 4R_{ab} - \frac{1}{2}g_{ab} 4R = 3k e^{-2k\sigma - l\gamma}(\sigma'' - \frac{l}{2}\gamma'\sigma' - 2k\sigma'^2)g_{ab} - \frac{\Lambda}{4M^3}e^{-2k\sigma}g_{ab}, \quad (11) \]

and

\[ \sigma'^2 = \frac{e^{l\gamma}}{12k^2}(e^{2k\sigma} 4R - \frac{\Lambda}{2M^3}), \quad (12) \]

respectively. \(4R_{ab}\) refers to the familiar four-dimensional Ricci tensor made out of the 4-d Christoffel symbols \(\{^a_{bc}\} = \frac{1}{2}g^{an}(g_{bn,c} + g_{cn,b} - g_{bc,n})\) and \(4R = g^{mn} 4R_{mn}\) is the four-dimensional Ricci scalar. The 5-d and 4-d Ricci scalars are related through \(R = e^{2k\sigma} 4R + 2k e^{-l\gamma}(4\sigma'' - 2l\gamma'\sigma' - 10k\sigma'^2)\). Combining the trace of Eq.(11) with Eq.(12) one obtains:

\[ \sigma'' - \frac{l}{2}\gamma'\sigma' - 2k\sigma'^2 = \frac{\Lambda}{24kM^3}e^{l\gamma}. \quad (13) \]

For further simplification of our analysis let us set \(g_{ab} = \eta_{ab}\) - the usual four-dimensional Minkowski metric. In this case \(4R_{ab} = 4R = 0\) so, from Eq.(11), one gets

\[ \sigma'' - \frac{l}{2}\gamma'\sigma' - k\sigma'^2 = \frac{\Lambda}{12kM^3}e^{l\gamma}. \quad (14) \]

Combining Eqs.(13) and (14) yields

\[ \sigma'' = \frac{l}{2}\gamma'\sigma', \quad (15) \]

which, after integrating, gives

\[ \sigma' = Ce^{\frac{l}{2}\gamma}, \quad (16) \]

where \(C\) is some integration constant. If we put Eq.(15) into Eq.(13) or (14), we obtain

\[ \sigma'^2 = -\frac{\Lambda}{24k^2M^3}e^{l\gamma}, \quad (17) \]
so the integration constant \( C = \sqrt{-\frac{\Lambda}{24k^2M^3}} \). Eq.(17) (or Eq.(16)) makes sense only for negative \( \Lambda < 0 \) (i.e., \( C^2 > 0 \) and the constant \( C \) is real). This leads that our model five-dimensionsal spacetime is \( AdS_5 \).

If we introduce a new coordinate through \( \frac{d\sigma}{\gamma} = dy \), it is encouraging noting that the line-element (10) can then be written as

\[
ds^2 = e^{-2k|\sigma|} \eta_{mn} dx^m dx^n + r_c^2 d\sigma^2,
\]

where \( r_c = \sqrt{\frac{24k^2M^3}{-\Lambda}} \) and the orbifold symmetry \( \sigma \rightarrow -\sigma \) has been taken into account. While deriving Eq.(18) we have used the following chain of equalities: \( e^{\ell \gamma} dy^2 = \frac{e^{\ell \gamma}}{\sigma^2} d\sigma^2 = \frac{24k^2M^3}{-\Lambda} d\sigma^2 \). The line-element (18) exactly coincides with that of Randall and Sundrum if we set \( \Lambda = -24k^2M^3 \).

The most interesting feature of our set-up is contained in Eq.(17) (or Eq.(16)). Since both \( \sigma \) and \( \gamma \) are functions of the coordinate \( y \), Eq.(17) can be written in the general form

\[
\sigma'^2 + U(\sigma) = \mathcal{E},
\]

where \( \mathcal{E} \) is an arbitrary positive constant and \( U(\sigma) \) is an arbitrary function of \( \sigma \). Then Eq.(19) implies that (see Eq.(17))

\[
e^{\ell \gamma} = r_c^2 [\mathcal{E} - U(\sigma)].
\]

Therefore, any solution \( \sigma(y) \) of the differential equation (19) in the form

\[
\pm \int \frac{d\sigma}{\sqrt{\mathcal{E} - U(\sigma)}} = y + C_1,
\]

implies, in virtue of Eq.(20), a solution \( \gamma(y) \). This feature enables us some freedom in the choice of the function \( U(\sigma) \). This is, precisely, the feature we shall exploit in searching for a mechanism of orbifold geometry stabilization. In fact, Eq.(19) may be given the following particle interpretation: it represents a scalar \( \sigma \)-particle with kinetic energy \( \sigma'^2 \) and total
energy $E$, that moves along the $y$-direction in a potential $U(\sigma)$. In other words this can be put as follows. The derivative of Eq.(20) yields

$$l\gamma' = -\frac{dU}{d\sigma}\sigma'/(E - U), \quad (22)$$

so, if we put Eq.(22) into Eq.(15) and, taking into account Eq.(19), one gets

$$\sigma'' = -\frac{1}{2} \frac{dU}{d\sigma}. \quad (23)$$

This equation of motion can also be obtained with the help of the variational principle from the action $S_y = \int dy L[\sigma', \sigma]$, where $L[\sigma', \sigma] = \sigma'^2 - U(\sigma)$.\footnote{From Eq.(23) one sees that the potential should respect orbifold symmetry, i.e., $U(\sigma) = U(-\sigma)$} Solutions to the equation of motion (23) that correspond to states of least energy are those with $\sigma = \sigma_i$ such that $U(\sigma_i)$ is a minimum. For these solutions the classical hamiltonian $H \sim \int dy[\sigma'^2 + U(\sigma)]$ is a minimum too. Therefore, solutions to Eq.(23) for which $\sigma = \sigma_i$ represent the ground $\sigma$-states of the system. These ground states stabilize the orbifold geometry in the sense that the points $\sigma = \sigma_i$ correspond to stable (ground) configurations of the scalar field $\sigma$, yielding that branes located at these points are stable against small perturbations of $\sigma$.

One instructive example is provided by the potential $U(\sigma) = \lambda \sigma^2$, where $\lambda$ is an arbitrary constant. For positive $\lambda > 0$ Eq.(21) can be readily integrated to give

$$\sigma(y) = \sqrt{\frac{E}{\lambda}} \sin \sqrt{\lambda} y, \quad (24)$$

where $-\frac{\pi}{2\sqrt{\lambda}} \leq y \leq \frac{\pi}{2\sqrt{\lambda}}$, so $-\sqrt{\frac{E}{\lambda}} \leq \sigma \leq \sqrt{\frac{E}{\lambda}}$ (we have set $C_1 = 0$). In this case the five-dimensional spacetime has two boundaries that are located at $\sigma = -\sqrt{\frac{E}{\lambda}}$ and $\sigma = \sqrt{\frac{E}{\lambda}}$ respectively. However these are not stable configurations in $\sigma$. The only stable configuration may be localized at $\sigma = 0$. For negative $\lambda < 0$, we would have, instead,

$$\sigma(y) = \sqrt{\frac{E}{-\lambda}} \sinh \sqrt{-\lambda} y, \quad (25)$$
where now $-\infty \leq y \leq \infty$ and $-\infty \leq \sigma \leq \infty$ so, the $\sigma$-direction is unbounded. In this case there is no stable configuration supporting four-dimensional physics. We recall that this is just an instructive example.

A potential that supports the Randall-Sundrum mechanism for generating the mass hierarchy is the following

$$U(\sigma) = \lambda(\sigma_0^2 - \sigma^2)^2,$$  \hspace{1cm} (26)

where $-\infty \leq \sigma \leq \infty$, i. e., we have an orbifold of $AdS_5$ geometry with infinite extent in the $\sigma$-direction. Following Ref. [5] we locate the Planck brane at the origin of the $\sigma$-coordinate $\sigma = 0$. The constant $\sigma_0$ is taken in such a way that $e^{-k|\sigma_0|} = T eV/M_{Pl}$ ($k \sim 10^{19}$GeV). For positive $\lambda > 0$, the potential $U(\sigma)$ in Eq.(26) has two minima at $\sigma = \sigma_0$ and $\sigma = -\sigma_0$ respectively. The origin $\sigma = 0$ is a local maximum. In other words, the stable ground $\sigma$-states are located at the minimuma $\sigma = \pm \sigma_0$. In this case the electroweak TeV scale is generated from the Planck scale by spontaneous orbifold symmetry breaking. In fact, if the system was initially at $\sigma = 0$ (the Planck brane) it "rolls down" until the ground state at $\sigma = \sigma_0$ or at $\sigma = -\sigma_0$ is reached. Once the ground state is reached, say at $\sigma = -\sigma_0$, the orbifold symmetry inherent to the line-element (18) is not a symmetry of this ground state.

This mechanism also applies in the case of an arbitrary metric $g_{ab}$ if we make the identification

$$4\Lambda \equiv \left( \frac{\Lambda}{4M^3} - 3ke^{-l^2/2}\gamma\sigma'' - \frac{l}{2}\gamma'\sigma' - 2k\sigma'^2 \right)e^{-2k\sigma},$$  \hspace{1cm} (27)

where $4\Lambda$ is a four-dimensional cosmological constant. In this case Eq.(11) can be written as

\footnote{For negative $\lambda < 0$, instead, this potential has two maxima at $\sigma = \pm \sigma_0$ and a local minimum at $\sigma = 0$. This theory has no bound $\sigma$-states since there may be tunneling from the region $|\sigma| \leq \sigma_0$ into the unbounded region $|\sigma| > \sigma_0$}
therefore, the requirement that the 4-d cosmological constant should be zero ($^4\Lambda = 0$) enables our mechanism to work since, in this case, Eq.(14) and the subsequent set of equations (including Eq.(17)) hold true.

What we have proposed here is a very simple mechanism, literally a mechanism of spontaneous orbifold symmetry breaking, that allows stabilization of the orbifold geometry while solving the hierarchy problem a la Randall-Sundrum. However it is not addressed to give a realistic particle picture. A more realistic approach must contain matter sectors. This will be the subject of forthcoming work.

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