Single-loop interferometer for minimal ellipsometry

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Abstract
We present a simple polarizing Mach-Zehnder interferometer that can be used for optimal minimal ellipsometry: Only four intensities are measured to determine the three Stokes parameters, and an optimal choice for the four polarization projections can be achieved for any sufficiently small wavelength range of interest.

Dedicated to Professor Herbert Walther
— grandmaster of optics, classical and quantum —
on the occasion of his 70th birthday.
The polarization properties of light — be it emitted by a laser source, for instance, reflected from a surface under study, or emanating from some sample tissue of interest — need to be determined in many applications. It is, therefore, a common and frequent task in an optics laboratory to establish the values of the three Stokes parameters that quantify the polarization in a standard and convenient way. The usual procedure is to measure them one by one, which is straightforward but not very efficient. We present here a simple interferometric setup by which one can get all three Stokes parameters simultaneously and efficiently.

All standard ellipsometers (or polarimeters) are essentially employing a setup of the kind depicted in Fig. 1. In this compact design, all six intensities are measured simultaneously, but it is, of course, also possible to carry out three consecutive measurements of two intensities each, for which Figs. 5, 7, and 8 in Ref. [2] give a recent example. One pair of detectors measures the intensities for vertical and horizontal linear polarization, $I_V$ and $I_H$, and so

![Figure 1: Six-output setup for standard ellipsometry. One third of the incoming light intensity is analyzed by a polarizing beam splitter (PBS) at $0^\circ$ to establish the value of Stokes parameter $Q$. The remaining two thirds are distributed evenly to two more PBSs, one set at $45^\circ$ for determining Stokes parameter $U$, the other behind a quarter-wave plate (QWP) at $45^\circ$ for Stokes parameter $V$.](image-url)
determines the first Stokes parameter in accordance with

\[ Q = \frac{I_V - I_H}{I_V + I_H}. \] (1)

Another pair measures the intensities for linear polarization half-way between horizontal and vertical, denoted by ±45°, yielding the second Stokes parameter

\[ U = \frac{I_{+45} - I_{-45}}{I_{+45} + I_{-45}}. \] (2)

And the third pair measures the intensities for right-circular and left-circular light to establish the third Stokes parameter,

\[ V = \frac{I_R - I_L}{I_R + I_L}. \] (3)

Since the inequality

\[ Q^2 + U^2 + V^2 \leq 1 \] (4)

is necessarily obeyed, the Stokes vector

\[ \vec{S} = \begin{pmatrix} Q \\ U \\ V \end{pmatrix} \] (5)

identifies a point inside the so-called Poincaré sphere, \(|\vec{S}| \leq 1\). On the surface of the sphere, we have pure polarization states, linear polarization on the equator and circular polarization at the poles, and points inside the sphere mark states of mixed polarization, with “completely mixed” (that is: \(Q = U = V = 0\)) at the center of the sphere. All of this is standard textbook wisdom.

There are just three Stokes parameters, so that one should be able to establish their values by measuring four intensities only, rather than six. The interferometric setup of Fig. 2 achieves this indeed. The intensities \(I_1, \ldots, I_4\) measured by the four photodiodes are related to the Stokes parameters by

\[ \frac{I_1}{I_2} = \frac{I}{4} \left(1 - \frac{U \pm \sqrt{2}Q}{\sqrt{3}}\right), \]
Figure 2: Four-output single-loop interferometer for minimal ellipsometry. The light passes through a Mach-Zehnder interferometer that has a half-wave plate (HWP) at 45° in one arm and a path-length difference that corresponds to a relative phase $\phi$ of $e^{i\phi} = (\sqrt{2} + i)/\sqrt{3}$. The light of one output port is analyzed directly by a polarizing beam splitter (PBS), while that emerging from the other port is first sent through a quarter-wave plate (QWP) at 45°. The values of the three Stokes parameters are then obtained as linear combinations of the four relative intensities measured by the photodiodes 1, 2, 3, and 4. The input rotator (IR) is a set of wave plates for a global unitary polarization transformation.

\[
\begin{align*}
\begin{pmatrix} I_3 \\ I_4 \end{pmatrix} &= \frac{I}{4} \left( 1 + \frac{U \pm \sqrt{2}V}{\sqrt{3}} \right), \\
\end{align*}
\]

where $I = I_1 + I_2 + I_3 + I_4$ is the total intensity $I$. Accordingly, the Stokes parameters are readily available,

\[
\begin{align*}
Q &= \sqrt{6} (I_2 - I_1)/I, \\
U &= \sqrt{3} (I_3 + I_4 - I_1 - I_2)/I, \\
V &= \sqrt{6} (I_3 - I_4)/I.
\end{align*}
\] (7)

The relative intensities $I_j/I$ are essentially projections of the Stokes vector onto four particular directions, $4I_j/I = 1 + \vec{a}_j \cdot \vec{S}$, that are given by

\[
\begin{align*}
\begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \end{pmatrix} &= \begin{pmatrix} \mp \sqrt{2/3} \\ -\sqrt{1/3} \end{pmatrix}, & \begin{pmatrix} \vec{a}_3 \\ \vec{a}_4 \end{pmatrix} &= \begin{pmatrix} 0 \\ \pm \sqrt{1/3} \end{pmatrix}.
\end{align*}
\] (8)
The angle between any two of them is the same,
\[
\vec{a}_j \cdot \vec{a}_k = \frac{4}{3} \delta_{jk} - \frac{1}{3} = \begin{cases} 
1 & \text{for } j = k, \\
-1/3 & \text{for } j \neq k.
\end{cases}
\]
(9)

This is to say that they realize the perfect tetrahedron geometry, which is known to be optimal for minimal ellipsometry [4]. An easy way to think of these vectors is that they point from the center of a cube to nonadjacent corners, with the cube inscribed into the Poincaré sphere. These matters are illustrated in Fig. 3.

By a suitably chosen combination of wave plates for the unitary polarization transformation labeled by IR in Fig. 2, an overall rotation of the vector

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{tetrahedron_vectors.png}
\caption{The tetrahedron vectors of Eqs. (8) point to nonadjacent corners of a cube that is inscribed to the Poincaré sphere. Four corners of the cube, those for vectors \( \vec{a}_1 \) and \( \vec{a}_2 \) and opposite to them, are on the equator where Stokes parameter \( V \) vanishes. The other four corners, those for vectors \( \vec{a}_3 \) and \( \vec{a}_4 \) and opposite to them, are on the vertical great circle where Stokes parameter \( Q \) vanishes. On the axis \( H \rightarrow V \) we have the polarization states with \( U = V = 0 \) that can be mixed by blending horizontal and vertical polarization only. The other equatorial axis \(-45^\circ \rightarrow +45^\circ \) marks the \( Q = V = 0 \) states that result from mixing the linear polarizations that are half-way between horizontal and vertical. On the vertical axis \( L \rightarrow R \) we have \( Q = U = 0 \), corresponding to polarization states that one gets when mixing left-circular with right-circular polarization.}
\end{figure}
quartet (8) can be performed. This enables the experimenter to work with the tetrahedron of her choosing.

It should be clear that the setup of Fig. 2 is not unique for the purpose of implementing minimal ellipsometry of this optimal kind. For example, there is also a setup that uses polarizing beam splitters at the entry and exit ports of the Mach-Zehnder interferometer instead of the polarization-insensitive elements in Fig. 2.

Further we note that the interferometer of Fig. 2 has a single loop and two output ports, whereas some alternative setups have two loops [5], or a single loop with more output ports, among them the interferometer of the experiment by Clarke et al. [6]. Yet another setup has no loop at all [7].

The perfect tetrahedron quartet of Eqs. (8) and (9) is realized by the setup of Fig. 2 only if all optical elements are just right, that is: the beam splitters split 1:1 for all polarizations, the wave plates introduce phase differences of exactly $\pi$ and $\pi/2$ and are precisely set at $45^\circ$, the path difference corresponds truly to the desired interferometer phase, the polarizing beam splitters have ideal properties as well, and the four photodiodes have identical efficiencies. In practice, all these conditions can be met for a small wavelength range only, if at all, so that distorted tetrahedrons, one for each wavelength range, will typically be obtained in a real experiment. Rather than Eqs. (6)–(9), we then have

$$I_j = \frac{I}{4}(w_j + \vec{b}_j \cdot \vec{S}) \quad \text{for } j = 1, \ldots, 4$$

with $\sum_{j=1}^{4} w_j = 4$ and $\sum_{j=1}^{4} \vec{b}_j = 0$ (10)

for the wavelength range in question, where the $w_j$s determine the output intensities for unpolarized input, and the vector quartet of the $\vec{b}_j$s form a distorted tetrahedron [8].

Even when the $w_j$s deviate much from their ideal unit value and the distortion of the tetrahedron borders on disfigurement, the proper functioning
as an ellipsometer is assured as long as one can solve the four equations of (10) for the Stokes vector $\vec{S}$. This is achieved by [9]

$$\vec{S} = \frac{1}{4} \sum_{j=1}^{4} w_j \vec{c}_j - \frac{1}{4} \sum_{j=1}^{4} I_j \vec{e}_j,$$

(11)

where

$$\vec{c}_1 = \frac{\vec{b}_2 \times \vec{b}_3 + \vec{b}_3 \times \vec{b}_4 + \vec{b}_4 \times \vec{b}_2}{\vec{b}_2 \cdot (\vec{b}_3 \times \vec{b}_4)}$$

(12)

and cyclic permutations $1 \to 2 \to 3 \to 4 \to 1$ give $\vec{c}_2$, $\vec{c}_3$, and $\vec{c}_4$. As a consequence, we just need that the denominator in (12) does not vanish, which is the basic geometrical requirement that the distorted tetrahedron has a nonzero volume. But one should try to stay close to the ideal tetrahedron geometry because it minimizes statistical errors [4].

In summary, we have presented a simple interferometric setup for minimal ellipsometry. It consists of a Mach-Zehnder interferometer with polarization-changing optical elements and polarization-sensitive intensity measurements at the output ports. The distribution of the incoming intensity to the four partial intensities at the output is uniquely related to the polarization properties of the incident light, and the three Stokes parameters can be inferred in a very simple manner from the measured output intensities. There is an ideal tetrahedron geometry, for the corresponding vectors in the Poincaré sphere, but the setup is fully functional even when the actual geometry deviates much from the ideal one.

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Notes and references

[1] By convention, a polarizing beam splitter at 0° reflects vertically polarized light and transmits horizontally polarized light. Likewise, a quarter-wave plate at 0° introduces a phase difference of π/2 between vertically and horizontally polarized light.

[2] N. Korolkova, G. Leuchs, R. Loudon, T. C. Ralph, and Ch. Silberhorn, Phys. Rev. A 65, 052306 (2002).

[3] A detailed description of setups like the one in Fig. 2 will be given elsewhere [7]. We are content here with a brief account and a statement of the most important facts.

[4] J. Řeháček, B.-G. Englert, and D. Kaszlikowski, Minimal qubit tomography, eprint quant-ph/0405084.

[5] See, for example, J. M. Renes, Frames, Designs, and Spherical Codes in Quantum Information Theory (Dissertation, University of New Mexico, 2004), Fig. 6.5.

[6] R. B. M. Clarke, V. M. Kendon, A. Chefles, S. M. Barnett, E. Riis, and M. Sasaki, Phys. Rev. A 64, 012303 (2001); see also A. Chefles, “Quantum States: Discrimination and Classical Information Transmission. A Review of Experimental Progress,” in Quantum State Estimation, edited by M. Paris and J. Řeháček, Lecture Notes in Physics, Vol. 649 (Springer Verlag, 2004).

[7] B.-G. Englert, Goh C. G., Ch. Kurtsiefer, A. Lamas Linares, Ng H. K., Tin K. M., in preparation.

[8] There are altogether 12 real parameters that specify the four \( w_j \)s and the four \( \vec{b}_j \)s. They can be determined experimentally by measuring \( I_1, \ldots, I_4 \) for four suitably chosen, known polarizations of the input light. In this sense, then, the setup is “self-calibrating.”

[9] Equations (11) and (12) apply if there are no polarization-dependent losses.