Existence of periodic orbits for sectional Anosov flows

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Abstract

We prove that every sectional Anosov flow (or, equivalently, every sectional-hyperbolic attracting set of a flow) on a compact manifold has a periodic orbit. This extends the previous three-dimensional result obtained in [6].

1 Introduction

A well-known problem in dynamics is to investigate the existence of periodic orbits for flows on compact manifolds. This problem has a satisfactory solution under certain circumstances. In fact, every Anosov flow of a compact manifold has not only one but infinitely many periodic orbits instead. In this paper we shall investigate this problem not for Anosov but for the sectional Anosov flows introduced in [20]. It is known for instance that every sectional Anosov flow of a compact 3-manifold has a periodic orbit, this was proved in [6]. In the transitive case (i.e. with a dense orbit in the maximal invariant set) it is known that the maximal invariant set consists of a homoclinic class and, therefore, the flow has infinitely many periodic orbits [4]. Another relevant result by Reis [24] proves the existence of infinitely many periodic orbits under certain conditions. Our goal here is to extend [6] to the higher-dimensional setting. More precisely, we shall prove that every sectional Anosov flow (or, equivalently, every sectional hyperbolic attracting set of a flow) on a compact manifold has a periodic orbit. Let us state our result in a precise way.

Consider a compact manifold $M$ of dimension $n \geq 3$ with a Riemannian structure $\| \cdot \|$ (sometimes we write compact $n$-manifold for short). We denote by $\partial M$ the boundary of $M$. Fix $X$ a vector field on $M$, inwardly transverse to

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the boundary $\partial M$ (if nonempty) and denotes by $X_t$ the flow of $X$, $t \in \mathbb{R}$. The maximal invariant set of $X$ is defined by

$$M(X) = \bigcap_{t \geq 0} X_t(M).$$

Notice that $M(X) = M$ in the boundaryless case $\partial M = \emptyset$. A subset $\Lambda \subset M(X)$ is called invariant if $X_t(\Lambda) = \Lambda$ for every $t \in \mathbb{R}$. We denote by $m(L)$ the minimum norm of a linear operator $L$, i.e.,

$$m(L) = \inf_{v \neq 0} \frac{\|Lv\|}{\|v\|}.$$

**Definition 1.1.** A compact invariant set $\Lambda$ of $X$ is partially hyperbolic if there is a continuous invariant splitting $T\Lambda M = E^s \oplus E^c$ such that the following properties hold for some positive constants $C, \lambda$:

1. $E^s$ is contracting, i.e., $\|DX_t(x)\big|_{E^s_x}\| \leq Ce^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.

2. $E^s$ dominates $E^c$, i.e., $\frac{\|DX_t(x)\big|_{E^c_x}\|}{m(DX_t(x)\big|_{E^c_x})} \leq Ce^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.

We say the central subbundle $E^c_x$ of $\Lambda$ is sectionally expanding if

$$\dim(E^c_x) \geq 2 \quad \text{and} \quad \left|\det(DX_t(x)\big|_{L_x})\right| \geq C^{-1}e^{\lambda t}, \text{ for all } x \in \Lambda, \quad t > 0$$

and all two-dimensional subspace $L_x$ of $E^c_x$. Here $\det(DX_t(x)\big|_{L_x})$ denotes the jacobian of $DX_t(x)$ along $L_x$.

Recall that a singularity of a vector field is a zero of it. We say that it is hyperbolic if the eigenvalues of its linear part have non zero real part. On the other hand, a point $p$ is periodic if there is a minimal $T > 0$ (called period) such that $X_T(p) = p$. By a periodic orbit we mean the full orbit $\{X_t(p) : t \in \mathbb{R}\}$ of a periodic point $p$.

**Definition 1.2.** A sectional hyperbolic set is a partially hyperbolic set whose singularities are hyperbolic and whose central subbundle is sectionally expanding.

With these definitions we can state our main results.

**Theorem A.** Every sectional Anosov flow on a compact manifold has a periodic orbit.

An equivalent version of this result is as follows.

Given $\Lambda \in M$ compact, recall that $\Lambda$ is an attracting set if $\Lambda = \cap_{t > 0} X_t(U)$ for some compact neighborhood $U$ of it, where this neighborhood is often called isolating block. It is well known that the isolating block $U$ can be chosen to be positively invariant, i.e., $X_t(U) \subset U$ for all $t > 0$. We call a sectional hyperbolic set with the above property as a sectional hyperbolic attracting.

Thus, the Theorem A is equivalent to the result below.
Theorem B. Every sectional hyperbolic attracting of a $C^1$ vector field on a compact manifold has a periodic orbit.

Theorem A will be proved extending the arguments in [6] to the higher dimensional setting. Indeed, in Section 2 we provide useful definitions for Lorenz-like singularity, singular-cross sections and triangular maps for higher dimensional case ($n$-triangular maps for short) in the sectional hyperbolic context. Also, we extend some definition, lemmas and propositions necessaries for the next sections. In Section 3 we give sufficient conditions for a hyperbolic $n$-triangular map to have a periodic point. In Section 4 we proved that hyperbolic $n$-triangular maps satisfying these hypotheses have a periodic point and we proved the Theorem A.

2. Singular cross-sections and triangular maps in higher dimensions

In this section, we provide a definition of Lorenz-like singularity in the sectional hyperbolic context. Also, in the same way of [15], [16], let us to define singular cross-sections for the higher dimensional case. We set certain maps defined on a finite disjoint union of these singular cross-sections. As in [6], they are discontinuous maps still preserving the continuous foliation (but not necessarily constant). Thus, on compact manifolds of dimension $n \geq 3$, particularly to group of these maps we shall call them $n$-triangular maps.

2.1 Lorenz-like singularities and useful results

Let $M$ be a compact $n$-manifold, $n \geq 3$. Fix $X \in X^1(M)$, inwardly transverse to the boundary $\partial M$. We denotes by $X_t$ the flow of $X$, $t \in \mathbb{R}$.

In our context, the singular cross-sections are strongly associated with the Lorenz-like singularities. This leads to provide a Lorenz-like singularity’s definition in our context too.

For this purpose, we start by presenting the standard definition of hyperbolic set and some preliminary definitions.

**Definition 2.1.** A compact invariant set $\Lambda$ of $X$ is hyperbolic if there are a continuous tangent bundle invariant decomposition $T_\Lambda M = E^s \oplus E^X \oplus E^u$ and positive constants $C, \lambda$ such that

- $E^X$ is the vector field’s direction over $\Lambda$.
- $E^s$ is contracting, i.e., $\| DX_t(x) |_{E^s_x} \| \leq Ce^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$. 
\( E^u \) is expanding, i.e., \( \| DX_{-t}(x) \|_{E^u_p} \| \leq C e^{-\lambda t} \), for all \( x \in \Lambda \) and \( t > 0 \).

A closed orbit is hyperbolic if it is also hyperbolic, as a compact invariant set. An attractor is hyperbolic if it is also a hyperbolic set.

It follows from the stable manifold theory [14] that if \( p \) belongs to a hyperbolic set \( \Lambda \), then the following sets
\[
W^s_X(p) = \{ x : d(X_t(x), X_t(p)) \to 0, t \to \infty \} \quad \text{and} \quad W^u_X(p) = \{ x : d(X_t(x), X_t(p)) \to 0, t \to -\infty \}
\]
are \( C^1 \) immersed submanifolds of \( M \) which are tangent at \( p \) to the subspaces \( E^s_p \) and \( E^u_p \) of \( T_p M \) respectively. Similarly,
\[
W^s_X(p) = \bigcup_{t \in \mathbb{R}} W^s(X_t(p)) \quad \text{and} \quad W^u_X(p) = \bigcup_{t \in \mathbb{R}} W^u(X_t(p))
\]
are also \( C^1 \) immersed submanifolds tangent to \( E^s_p \oplus E^s_p \) and \( E^u_p \oplus E^u_p \) at \( p \) respectively. Moreover, for every \( \epsilon > 0 \) we have that
\[
W^s_X(p, \epsilon) = \{ x : d(X_t(x), X_t(p)) \leq \epsilon, \forall t \geq 0 \} \quad \text{and} \quad W^u_X(p, \epsilon) = \{ x : d(X_t(x), X_t(p)) \leq \epsilon, \forall t \leq 0 \}
\]
are closed neighborhoods of \( p \) in \( W^s_X(p) \) and \( W^u_X(p) \) respectively.

There is also a stable manifold theorem in the case when \( \Lambda \) is sectional hyperbolic set. Indeed, denoting by \( T_{\Lambda} M = E^s_\Lambda \oplus E^u_\Lambda \) the corresponding the sectional hyperbolic splitting over \( \Lambda \) we have from [14] that the contracting subbundle \( E^s_\Lambda \) can be extended to a contracting subbundle \( E^s_U \) in \( M \). Moreover, such an extension is tangent to a continuous foliation denoted by \( W^{ss} \) (or \( W^{ss}_X \) to indicate dependence on \( X \)). By adding the flow direction to \( W^{ss} \) we obtain a continuous foliation \( W^s \) (or \( W^s_X \)) now tangent to \( E^s_U \oplus E^u_U \). Unlike the Anosov case \( W^s \) may have singularities, all of which being the leaves \( W^{ss}(\sigma) \) passing through the singularities \( \sigma \) of \( X \). Note that \( W^s \) is transverse to \( \partial M \) because it contains the flow direction (which is transverse to \( \partial M \) by definition). So, note the following remark

It turns out that every singularity \( \sigma \) of a sectional hyperbolic set \( \Lambda \) satisfies \( W^{ss}(\sigma) \subset W^s(\sigma) \).

Furthermore, there are two possibilities for such a singularity, namely, (1)
\[
dim(W^{ss}(\sigma)) = \dim(W^s(\sigma)) \quad \text{(and so } W^{ss}(\sigma) = W^s(\sigma))
\]
or
\[
dim(W^s(\sigma)) = \dim(W^{ss}(\sigma)) + 1.
\]
In the later case we call it Lorenz-like according to the following definition.

**Definition 2.2.** We say that a singularity \( \sigma \) of a sectional-Anosov flow \( X \) is Lorenz-like if \( \dim(W^s(\sigma)) = \dim(W^{ss}(\sigma)) + 1 \).
Considering the above definitions, we show the following lemmas presenting an elementary but very useful dichotomy for the singularities of a sectional hyperbolic sets.

First, we consider the called hyperbolic lemma.

**Lemma 2.3.** Let $\Lambda$ be a sectional hyperbolic set of a $C^1$ vector field $X$ of $M$. Then, there is a neighborhood $U \subset X^1(M)$ of $X$ and a neighborhood $V \subset M$ of $\Lambda$ such that if $Y \in U$, every nonempty, compact, non singular, invariant set $H$ of $Y$ in $V$ is hyperbolic saddle-type (i.e. $E^s \neq 0$ and $E^u \neq 0$).

**Proof.** See ([21]).

This following result examining the sectional hyperbolic splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ of a sectional Anosov flow $X \in X^1(M)$ appears in [5].

**Theorem 2.4.** Let $X$ be a sectional-Anosov flow $C^1$ for $M$. Then, every $\sigma \in \text{Sing}(X) \cap M(X)$ satisfies

$$M(X) \cap W^{ss}_X(\sigma) = \{\sigma\}.$$ 

The following results appear in [6], but this version is a modification for sectional hyperbolic sets in the higher dimensional case.

**Theorem 2.5.** Let $X$ be a sectional Anosov flow of $M$. Let $\sigma$ be a singularity of $M(X)$. If there is $x \in M(X) \setminus W^s(\sigma)$ such that $\sigma \in \omega_X(x)$, then $\sigma$ is Lorenz-like and satisfies

$$M(X) \cap W^{ss}_X(\sigma) = \{\sigma\}.$$ 

**Proof.** The equality follows from Theorem 2.4. We assume that $M(X)$ is connected for, otherwise, we consider the connected components. Suppose that $\sigma \in M(X) \cap \text{Sing}(X)$ satisfies $\sigma \in \omega_X(x)$ for some $x \in M(X) \setminus W^{ss}(\sigma)$. Let us prove that $\sigma$ is Lorenz-like. Since $M(X)$ is maximal invariant of $X$ we have $\omega_X(x) \in M(X)$ and so $\sigma \in M(X)$. Assume by contradiction that $\sigma$ is not Lorenz-like. Then, by (1) we have that $\dim(W^{ss}(\sigma)) = \dim(W^s(\sigma))$ and so $W^{ss}(\sigma) = W^s(\sigma)$. Since $x \notin W^{ss}(\sigma)$, one has $\omega_X(x) \cap (W^s(\sigma) \setminus \{\sigma\}) \neq \emptyset$. But recall that $\omega_X(x) \in M(X)$ and beside with $W^{ss}(\sigma) = W^s(\sigma)$, we obtain that

$$M(X) \cap (W^{ss}_X(\sigma) \setminus \{\sigma\}) \neq \emptyset$$

contradicting the equality in Theorem 2.4. This proves the result.

**Proposition 2.6.** Let $X$ be a sectional Anosov flow of $M$. If $M(X)$ has no Lorenz-like singularities, then $M(X)$ has a periodic orbit.
Proof. Let \( x \) be a point in \( M(X) \setminus Sing(X) \). We claim that \( \omega_X(x) \) has no singularities. Indeed, suppose by contradiction that \( \omega_X(x) \) has a singularity \( \sigma \). By hypothesis \( M(X) \) has no Lorenz-like singularities and so \( \sigma \) is not Lorenz-like too. Hence \( W^{ss}(\sigma) = W^s(\sigma) \) by (1), and by using the Theorem 2.4 one has

\[
M(X) \cap (W^s_X(\sigma) \setminus \{\sigma\}) = \emptyset
\]

It follows in particular that \( (W^s_X(\sigma) \setminus \{\sigma\}) \) is no belongs to \( M(X) \). Since \( x \in (M(X) \setminus Sing(X)) \) we conclude that \( x \notin W^s_X(\sigma) = W^s_X(\sigma) \). It follows from Theorem 2.5 that \( \sigma \) is Lorenz-like. This is a contradiction and the claim follows.

Now we conclude the proof of the proposition. Clearly \( \omega_X(x) \subset M(X) \) since \( M(X) \) is compact. The claim and 2.3 imply that \( \omega_X(x) \) is a hyperbolic set. It follows from the Shadowing Lemma for flows [13] that there is a periodic orbit of \( X_t \) close to \( \omega_X(x) \). Since \( M(X) \) is the maximal invariant, we have that such a periodic orbit is contained in \( M(X) \). Then we obtain the result. \( \square \)

### 2.2 Singular cross-section in higher dimension

Here, we will define singular cross-section in the higher dimensional context. First, we will denote a cross-section by \( \Sigma \) and its boundary by \( \partial \Sigma \). Also, the hypercube \( I^k = [-1, 1]^k \) will be submanifold of dimension \( k \) with \( k \in \mathbb{N} \).

Let \( \sigma \) be a Lorenz-like singularity. Hereafter, we will denote \( \dim(W^{ss}_X(\sigma)) = s \), \( \dim(W^s_X(\sigma)) = u \) and therefore \( \dim(W^s_X(\sigma)) = s + 1 \) by definition. Moreover \( W^{ss}_X(\sigma) \) separates \( W^s_{\text{loc}}(\sigma) \) in two connected components denoted by \( W^{s,t}_{\text{loc}}(\sigma) \) and \( W^{s,b}_{\text{loc}}(\sigma) \) respectively.

Thus, we begin by considering \( B^u[0, 1] \approx I^u \) and \( B^{ss}[0, 1] \approx I^s \) where \( B^{ss}[0, 1] \) is the ball centered at zero and radius 1 contained in \( \mathbb{R}^{\dim(W^{ss}(\sigma))} = \mathbb{R}^s \) and \( B^u[0, 1] \) is the ball centered at zero and radius 1 contained in \( \mathbb{R}^{\dim(W^u(\sigma))} = \mathbb{R}^{n-s-1} = \mathbb{R}^u \).

**Definition 2.7.** A singular cross-section of a Lorenz-like singularity \( \sigma \) consists of a pair of submanifolds \( \Sigma^t, \Sigma^b \), where \( \Sigma^t, \Sigma^b \) are cross-sections such that

\[
\Sigma^t \text{ is transversal to } W^{s,t}_{\text{loc}}(\sigma) \quad \text{and} \quad \Sigma^b \text{ is transversal to } W^{s,b}_{\text{loc}}(\sigma).
\]

Note that every singular cross-section contains a pair singular submanifolds \( l^t, l^b \) defined as the intersection of the local stable manifold of \( \sigma \) with \( \Sigma^t, \Sigma^b \) respectively and, additionally, \( \dim(l^*) = \dim(W^{ss}(\sigma)) \) \( ( \ast = t, b ) \).

Thus a singular cross-section \( \Sigma^* \) will be a hypercube of dimension \( (n-1) \), i.e., diffeomorphic to \( B^u[0, 1] \times B^{ss}[0, 1] \). Let \( f : B^u[0, 1] \times B^{ss}[0, 1] \rightarrow \Sigma^* \) be the diffeomorphism, such that

\[
f(\{0\} \times B^{ss}[0, 1]) = l^*
\]
and \( \{0\} = 0 \in \mathbb{R}^u \). Define
\[
\partial \Sigma^* = \partial^h \Sigma^* \cup \partial^v \Sigma^*
\]
by:
\[
\partial^h \Sigma^* = \{ \text{union of the boundary submanifolds which are transverse to } l^* \}
\]
\[
\partial^v \Sigma^* = \{ \text{union of the boundary submanifolds which are parallel to } l^* \}.
\]

From this decomposition we obtain that
\[
\partial^h \Sigma^* = (I^u \times \bigcup_{j=0}^{s-1}(I^j \times \{-1\} \times I^{s-j-1})) \bigcup (I^u \times \bigcup_{j=0}^{s-1}(I^j \times \{1\} \times I^{s-j-1}))
\]
\[
\partial^v \Sigma^* = ([\cup_{j=0}^{u-1}(I^j \times \{-1\} \times I^{u-j-1}) \times I^s) \bigcup ([\cup_{j=0}^{u-1}(I^j \times \{1\} \times I^{u-j-1}) \times I^s),
\]
where \( I^0 \times I = I \).

Hereafter we denote \( \Sigma^* = B^u[0,1] \times B^{ss}[0,1] \).

### 2.2.1 Refinement of singular cross-sections and induced foliation

Hereafter, \( X \) denotes a sectional Anosov flow of a compact \( n \)-manifold \( M, n \geq 3 \), \( X \in \mathcal{X}^\infty(M) \). Let \( M(X) \) be the maximal invariant of \( X \).

In the same way of [6], we obtain an induced foliation \( F \) on \( \Sigma \) by projecting \( F^{ss} \) onto \( \Sigma \), where \( F^{ss} \) denotes the invariant continuous contracting foliation on a neighborhood of \( M(X) \) [14].

For the refinement, since \( \Sigma^* = B^u[0,1] \times B^{ss}[0,1] \), we will set up a family of singular cross-sections as follows: Given \( 0 < \Delta \leq 1 \) small, we define \( \Sigma^{*,\Delta} = B^u[0,\Delta] \times B^{ss}[0,1] \), such that
\[
l^* \subset \Sigma^{*,\Delta} \subset \Sigma^*, \text{ i.e.} \hspace{1cm} (l^* = \{0\} \times B^{ss}[0,1]) \subset (\Sigma^{*,\Delta} = B^u[0,\Delta] \times B^{ss}[0,1]) \subset (\Sigma^* = B^u[0,1] \times B^{ss}[0,1]),
\]
where we fix a coordinate system \((x^*, y^*)\) in \( \Sigma^* \) \((* = t, b)\). We will assume that \( \Sigma^* = \Sigma^{*,1} \).

### 2.3 \( n \)-Triangular maps

We begin by reminding the three-dimensional case [6], where the authors choose the cross-sections as copies of \([0,1] \times [0,1]\) and they define maps on a finite disjoint union of these one copies called triangular maps. This concept is frequently used by maps on \([0,1] \times [0,1]\) preserving the constant vertical foliation. Also, they assume two hypotheses imposing certain amount of differentiability close to the points whose iterates fall eventually in the interior of \( \Sigma \).
For the higher dimensional case, we will define a certain maps on a finite disjoint union of singular cross-sections $\Sigma$. Here, we will modify the triangular map’s definition and we will impose some suitable properties in order to define a \textit{triangular hyperbolic map}. These maps could be discontinuous and they will preserve still the continuous foliation (but not necessarily constant). Thus, on compact manifolds of dimension $n \geq 3$, particularly to group of these maps we shall call them $n$-triangular maps.

By using the above definitions about singular cross-sections, we will provide the following definitions.

\textbf{Definition 2.8.} Let $\Sigma$ be a disjoint union of finite singular cross-sections $\Sigma_i$, $i = 1, \ldots, k$. We denote by $l_0$ to the singular leaf of the singular cross-section $\Sigma_i$. Here $L_0$ stands the union of singular leaves, i.e.,

$$L_0 = \bigcup_{i=1}^{n} l_0_i.$$ 

Recall that $\partial^v \Sigma_i$ is the union of the boundary submanifolds which are parallel to $l_0$. In the same way we set by $\partial^v \Sigma$ as

$$\partial^v \Sigma = \bigcup_{i=1}^{n} \partial^v \Sigma_i$$

Hereafter, given a map $F$ we will denote its domain by $\text{Dom}(F)$.

\textbf{Definition 2.9.} Let $F : \text{Dom}(F) \subset \Sigma \rightarrow \Sigma$ be a map $x$ a point in $\text{Dom}(F)$. We say that $x$ is a periodic point of $F$ if there is $n \geq 1$ such that $F^j(x) \in \text{Dom}(F)$ for all $0 \leq j \leq n - 1$ and $F^n(x) = x$.

\textbf{Definition 2.10.} We say that a submanifold $c$ of $\Sigma$ is a $k$-surface if it is the image of a $C^1$ injective map $c : \text{Dom}(c) \subset \mathbb{R}^k \rightarrow \Sigma$, with $\text{Dom}(c)$ being $I^k$ and $k \leq n - 1$. For simplicity, hereafter $c$ stands the image of this one map. A $k$-surface $c$ is vertical if it is the graph of a $C^1$ map $g : I^{n-k-1} \rightarrow I^k$, i.e., $c = \{(g(y), y) : y \in I^k\} \subset \Sigma$.

\textbf{Definition 2.11.} A continuous foliation $\mathcal{F}_i$ on a component $\Sigma_i$ of $\Sigma$ is called \textit{vertical} if its leaves are vertical $s$-surfaces and $\partial^v \Sigma \subset \mathcal{F}_i$, where $s = \text{dim}(B^{s}[0,1])$. A \textit{vertical foliation} $\mathcal{F}$ of $\Sigma$ is a foliation which restricted to each component $\Sigma_i$ of $\Sigma$ is a vertical foliation.

It follows from the definition above that the leaves $L$ of a vertical foliation $\mathcal{F}$ are vertical $s$-surfaces hence differentiable ones. In particular, the tangent space $T_xL$ is well defined for all $x \in L$. 

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Remark 2.12. Note that, given a singular cross-section $\Sigma$ equipped with a vertical foliation $\mathcal{F}$, one has that $\dim(\mathcal{F}) = \dim(B^s[0,1]) = s$, and each leaf $L$ of $\mathcal{F}$ has the same dimension of $W^s(\sigma)$, being $\sigma$ the Lorenz-like singularity associated to $\Sigma$.

Definition 2.13. Let $F : \text{Dom}(F) \subset \Sigma \to \Sigma$ a map and $\mathcal{F}$ be a vertical foliation on $\Sigma$. We say that $F$ preserves $\mathcal{F}$ if for every leaf $L$ of $\mathcal{F}$ contained in $\text{Dom}(F)$ there is a leaf $f(L)$ of $\mathcal{F}$ such that $F(L) \subset f(L)$ and the restricted map $F/_{L} : L \to f(L)$ is continuous.

If $\mathcal{F}$ is a vertical foliation on $\Sigma$ a subset $B \subset \Sigma$ is a saturated set for $\mathcal{F}$ if it is union of leaves of $\mathcal{F}$. We shall write $\mathcal{F}$-saturated for short.

Definition 2.14 (n-Triangular map). A map $F : \text{Dom}(F) \subset \Sigma \to \Sigma$ is called $n$-triangular if it preserves a vertical foliation $\mathcal{F}$ on $\Sigma$ such that $\text{Dom}(F)$ is $\mathcal{F}$-saturated and $\dim(\Sigma) = n - 1$, with $n \geq 3$. Note that a 3-triangular map is the classical triangular map.

2.4 Hyperbolic n-triangular maps

In the same way of [6], in order to find periodic points for $n$-triangular maps, also we introduce some kind of hyperbolicity for these maps. The hyperbolicity will be defined through cone fields in $\Sigma$: We denote by $T\Sigma$ the tangent bundle of $\Sigma$. Given $x \in \Sigma$, $\alpha > 0$ and a linear subspace $V_x \subset T_x\Sigma$ we denote by $C_\alpha(x, V_x) \equiv C_\alpha(x)$ the cone around $V_x$ in $T_x\Sigma$ with inclination $\alpha$, namely

$$C_\alpha(x) = \{v_x \in T_x\Sigma : \angle(v_x, V_x) \leq \alpha\}.$$ 

Here $\angle(v_x, V_x)$ denotes the angle between a vector $v_x$ and the subspace $V_x$. A cone field in $\Sigma$ is a continuous map $C_\alpha : x \in \Sigma \to C_\alpha(x) \subset T_x\Sigma$, where $C_\alpha(x)$ is a cone with constant inclination $\alpha$ on $T_x\Sigma$. A cone field $C_\alpha$ is called transversal to a vertical foliation $\mathcal{F}$ on $\Sigma$ if $T_xL$ is not contained in $C_\alpha(x)$ for all $x \in L$ and all $L \in \mathcal{F}$.

Now we can define hyperbolic $n$-triangular map.

Definition 2.15 (Hyperbolic n-triangular map). Let $F : \text{Dom}(F) \subset \Sigma \to \Sigma$ be a $n$-triangular map with associated vertical foliation $\mathcal{F}$. Given $\lambda > 0$ we say that $F$ is $\lambda$-hyperbolic if there is a cone field $C_\alpha$ in $\Sigma$ such that:

1. $C_\alpha$ is transversal to $\mathcal{F}$.

2. If $x \in \text{Dom}(F)$ and $F$ is differentiable at $x$, then

$$DF(x)(C_\alpha(x)) \subset \text{Int}(C_{\alpha/2}(F(x)))$$
\[ \| DF(x) \cdot v_x \| \geq \lambda \| v_x \|, \]
for all \( v_x \in C_\alpha(x) \).

3 Periodic points for hyperbolic \( n \)-triangular maps

In the three dimensional case \([6]\), the authors impose certain conditions or properties so-called hypotheses (H1) and (H2) on triangular maps. With this conditions the periodic point arose on the triangular map. Therefore, the general tools for searching our definitions is trying and reproducing these hypotheses, in the higher dimensional setting. Here, the topology turns out to play a significant role in this extension, imposing certain restrictions on the manifolds, kind of foliations and singular cross-sections one may have.

In this section we give sufficient conditions for a hyperbolic \( n \)-triangular map to have a periodic point.

3.1 Hypotheses (A1)-(A2)

They impose some regularity around those leaves whose iterates eventually fall into \( \Sigma \setminus (\partial^r \Sigma) \). To state them we will need the following definition. If \( \mathcal{F} \) is foliation we use the notation \( L \in \mathcal{F} \) to mean that \( L \) is a leaf of \( \mathcal{F} \).

**Definition 3.1.** Let \( F : \text{Dom}(F) \subset \Sigma \to \Sigma \) be a triangular map such that \( \partial^r \Sigma \subset \text{Dom}(F) \). For all \( L \in \mathcal{F} \) contained in \( \text{Dom}(F) \) we define the (possibly \( \infty \)) number \( n(L) \) as follows:

1. If \( F(L) \subset \Sigma \setminus (\partial^r \Sigma) \) we define \( n(L) = 0 \).

2. If \( F(L) \subset \partial^r \Sigma \) we define

\[ n(L) = \sup \{ n \geq 1 : F^i(L) \subset \text{Dom}(F) \text{ and } F^{i+1}(L) \subset \partial^r \Sigma, \forall 0 \leq i \leq n - 1 \}. \]

Essentially \( n(L) + 1 \) gives the first non-negative iterate of \( L \) falling into \( \Sigma \setminus (\partial^r \Sigma) \).

**Remark 3.2.** On the other hand, note that if \( n(L_*) \geq 1 \) for \( L_* \in \mathcal{F} \) contained in \( \text{Dom}(F) \), for every neighborhood \( S \) of \( L_* \), \( F(S) \cap \partial^r \Sigma \neq \emptyset \). We denote

\[ V_{L_*}(S) \equiv V_{L_*} = F(S) \cap \partial^r \Sigma. \]
Therefore \( V_{L*} \neq \emptyset \) and if \( L_* \notin \partial^w \Sigma, \ V_{L*} \) splits \( F(S) \) in two connected components \( S'_1, S'_2 \). It shows that there exists three connected components \( S_0, S_1, S_2 \) of \( S \) such that

\[
S = S_0 \cup S_1 \cup S_2 \quad \text{and} \quad F(S_0) = V_{L*}, \quad F(S_1) = S'_1, \quad \text{and} \quad F(S_2) = S'_2.
\]

Given \( L \in \mathcal{F} \) contained in \( \text{Dom}(F) \), the number \( n(L) \) and the neighborhood \( V_{L*} \) play fundamental role in the following definition.

**Definition 3.3 (Hypotheses (A1)-(A2)).** Let \( F: \text{Dom}(F) \subset \Sigma \rightarrow \Sigma \) be a \( n \)-triangular map such that \( \partial^w \Sigma \subset \text{Dom}(F) \). We say that \( F \) satisfies:

(A1) If \( L \in \mathcal{F} \) satisfies \( L \subset \text{Dom}(F) \) and \( n(L) = 0 \), then there is a \( \mathcal{F} \)-saturated neighborhood \( S \) of \( L \) in \( \Sigma \) such that the restricted map \( F|_S \) is \( C^1 \).

(A2) If \( L_* \in \mathcal{F} \) satisfies \( L_* \subset \text{Dom}(F), \ 1 \leq n(L_*) < \infty \) and

\[
F^{n(L_*)}(L_*) \subset \text{Dom}(F),
\]

then there is a connected neighborhood \( S \subset \text{Dom}(F) \) of \( L_* \) such that \( S = S_0 \cup S_1 \cup S_2 \) (see Remark 3.2) and the connected components \( S_1, S_2 \) (possibly equal if \( L_* \subset \partial^w \Sigma \)) beside \( V_{L_*} \) satisfying the properties below:

1. Both \( F(S_1) \) and \( F(S_2) \) are contained in \( \Sigma \setminus (\partial^w \Sigma) \).

2. \( F^i(V_{L_*}) \subset \partial^w \Sigma \) for all \( 0 \leq i \leq n(L_*) \), i.e., as \( V_{L_*} \) is \( \mathcal{F} \)-saturated, for any \( L' \in V_{L_*} \) one has that \( n(L') = n(L_*) - 1 \) (or \( n(L') = n(L_*) \)) for the case \( L_* \in \partial^w \Sigma \)

3. For each \( j \in \{0,1,2\} \), there is \( 0 \leq n^j(L_*) \leq n(L_*) + 1 \) such that if \( y_i \in S_j \) is a sequence converging to \( y \in L_* \), then \( F(y_i) \) is a sequence converging to \( F^{n(L_*)}(y) \). If \( n^1(L_*) = 0 \), then \( F \) is \( C^1 \) in \( S_j \cup V_{L_*} \). Note that \( n^0(L_*) = n(L_*) \) by \( \mathcal{A} \).

4. If \( L_* \subset \Sigma \setminus (\partial^w \Sigma) \) (and so \( S_1 \neq S_2 \)), then either \( n^1(L_*) = 0 \) and \( n^2(L_*) \geq 1 \) or \( n^1(L_*) \geq 1 \) and \( n^2(L_*) = 0 \).

### 3.2 Existence on hyperbolic \( n \)-triangular maps

The following theorem will deal with the existence of periodic points for hyperbolic \( n \)-triangular maps satisfying (A1) and (A2). Since the conditions (A1) and (A2) generalize the conditions in [6], also we have that the three dimensional Lorenz attractor return map is an example for us. Recall the Lorenz attractor return map has a periodic point and it is a \( \lambda \)-hyperbolic \( n \)-triangular map satisfying (A1) and (A2) with \( \lambda \) large and \( \text{Dom}(F) = \Sigma \setminus L_0 \). Indeed, the main motivation is show the following theorem for higher dimensional case. More precisely,
Theorem 3.4. Let $F$ be a $\lambda$-hyperbolic $n$-triangular map satisfying (A1) and (A2) with $\lambda > 2$ and $\text{Dom}(F) = \Sigma \setminus L_0$. Then, $F$ has a periodic point.

Since the proof of Theorem 3.4 is technical, we include some preliminaries for the proof and the proof in the next section.

4 Existence of the periodic point

In this section we shall prove the Theorem 3.4. The proof follow the same way of [6]. We will extend and modify some results for the higher dimensional case. In Subsection 4.1 we present preliminary lemmas for the proof of the Theorem 3.4. In Subsection 4.2 we prove the theorem.

4.1 Preliminary lemmas

Hereafter we fix $\Sigma$ as in Subsection 2.2. Let $k$ be the number of components of $\Sigma$. We shall denote by $SL$ the leaf space of a vertical foliation $F$ on $\Sigma$. It turns out that $SL$ is a disjoint union of $k$-copies $B^u_1[0,1], \ldots, B^u_k[0,1]$ of $B^u[0,1]$. We denote by $F_B$ the union of all leaves of $F$ intersecting $B$. If $B = \{x\}$, then $F_x$ is the leaf of $F$ containing $x$. If $S, B \subset \Sigma$ we say that $S$ cover $B$ whenever $B \subset F_S$.

The lemma below quotes some elementary properties of $n(L)$ in Definition 3.1.

Lemma 4.1. Let $F: \text{Dom}(F) \subset \Sigma \to \Sigma$ be a triangular map with associated vertical foliation $F$. If $L \in F$ and $L \subset \text{Dom}(F)$, then:

1. If $F$ has no periodic points and $\partial^u \Sigma \subset \text{Dom}(F)$, then
   $$n(L) \leq 2k.$$

2. $n(L) = 0$ if and only if $F(L) \subset \Sigma \setminus (\partial^u \Sigma)$.

3. $F^i(L) \subset \partial^u \Sigma$ for all $1 \leq i \leq n(L)$.

4. If $F^{n(L)}(L) \subset \text{Dom}(F)$, then $F^{n(L)+1}(L) \subset \Sigma \setminus (\partial^u \Sigma)$.

If $F: \text{Dom}(F) \subset \Sigma \to \Sigma$ is a $n$-triangular map with associated foliation $F$, then we also have an associated $u$-dimensional map

$$f: \text{Dom}(f) \subset SL \to SL.$$

This map allows us to obtain certain geometric properties for the singular cross-section whole.

We use this map in the definition below.
Definition 4.2. Let $F : \text{Dom}(F) \subset \Sigma \to \Sigma$ a triangular map with associated foliation $\mathcal{F}$ and $f : \text{Dom}(f) \subset SL \to SL$ its associated $u$-dimensional map. Then we define the following limit sets:

$$\mathcal{V} = \{f(B) : B \in \mathcal{F}, B \subset \text{Dom}(F) \text{ and } B \subset \partial^u \Sigma\}.$$ 

$$\mathcal{L} = \bigcup \left\{K_i : i \in \{1, \ldots, k\}, \lim_{L \to L_0} f(L) \text{ exists and such that } K_i = \lim_{L \to L_0} f(L) \right\}.$$ 

The lemma below is a direct consequence of (A2).

Lemma 4.3. Let $F : \text{Dom}(F) \subset \Sigma \to \Sigma$ a n-triangular map satisfying (A2) and $\mathcal{F}$ be its associated foliation. Let $L_*$ be a leaf of $\mathcal{F}$, $L_* \subset \text{Dom}(F)$, $1 \leq n(L_*) < \infty$ and $F^{n(L_*)}(L_*) \subset \text{Dom}(F)$. If there is a sequence $(L_k)_{k \in \mathbb{N}}$ such that $L_k \to L_*$, then:

1. If $\#\{L : L \in (V_{L_*} \cap (F(L_k))_{k \in \mathbb{N}})\} = \infty$, then there exists a subsequence $(L_{k_i})_{i \in \mathbb{N}}$ such that $\text{Lim}_{i \to \infty} F(L_{k_i}) = F(L_*)$.

2. If $\#\{L : L \in (S_1 \cap (F(L_k))_{k \in \mathbb{N}})\} = \infty$, then there exists a subsequence $(L_{k_i})_{i \in \mathbb{N}}$ such that $\text{Lim}_{i \to \infty} F(L_{k_i}) = F^{n_1(L_*)}(L_*)$.

3. If $\#\{L : L \in (S_2 \cap (F(L_k))_{k \in \mathbb{N}})\} = \infty$, then there exists a subsequence $(L_{k_i})_{i \in \mathbb{N}}$ such that $\text{Lim}_{i \to \infty} F(L_{k_i}) = F^{n_2(L_*)}(L_*)$.

In each case the corresponding limits belong to

$$\partial^u \Sigma \cup \mathcal{V}.$$ 

Proof. By hypotheses $F$ satisfies the property (A2), so there is a connected neighborhood $S = S_0 \cup S_1 \cup S_2$ of $L_*$. If we have (1), since $L_k \to L_*$, one has that $\#\{L : L \in (S_0 \cap (L_k)_{k \in \mathbb{N}})\} = \infty$. Then, there exists a subsequence $(L_{k_i})_{i \in \mathbb{N}} \subset S_0$ such that $L_{k_i} \to L_*$. Thus, $\text{Lim}_{i \to \infty} F(L_{k_i}) = F(L_*)$.

If we have (2) beside the (A2) property, in the same way one has that there exists a subsequence $(L_{k_i})_{i \in \mathbb{N}} \subset S_1$ such that $L_{k_i} \to L_*$ and $\text{Lim}_{i \to \infty} F(L_{k_i}) = F^{n_1(L_*)}(L_*)$. Analogously for (3).

Given a map $F : \text{Dom}(F) \subset \Sigma \to \Sigma$ we define its discontinuity set $D(F)$ by

$$D(F) = \{x \in \text{Dom}(F) : F \text{ is discontinuous in } x\}.$$ 

In the sequel we derive useful properties of $\text{Dom}(F)$ and $D(F)$.

Lemma 4.4. Let $F$ be a n-triangular map satisfying (A1), $F : \text{Dom}(F) \subset \Sigma \to \Sigma$ and $\mathcal{F}$ be its associated foliation. If $L \in \mathcal{F}$ and $L \subset D(F)$, then $F(L) \subset \partial^u \Sigma$. 


Proof. Suppose by contradiction that $L \subset D(F)$ and $F(L) \subset \Sigma \setminus (\partial^v \Sigma)$. These properties are equivalent to $n(L) = 0$ by Lemma 4.1(2). Then, by using (A1), there is a neighborhood of $L$ in $\Sigma$ restricted to which $F$ is $C^1$. In particular, $F$ would be continuous in $L$ which is absurd.

Lemma 4.5. Let $F$ be a $n$-triangular map satisfying (A1)-(A2), $F : Dom(F) \subset \Sigma \to \Sigma$ and $\mathcal{F}$ be its associated foliation. If $F$ has no periodic points and $\partial^v \Sigma \subset Dom(F)$, then $Dom(F) \setminus D(F)$ is $\mathcal{F}$-saturated, open in $Dom(F)$ and $F/(Dom(F)\setminus D(F))$ is $C^1$.

Proof. In order to prove the lemma, it suffices to show that $\forall x \in Dom(F) \setminus D(F)$ there is a neighborhood $S$ of $\mathcal{F}_x$ in $\Sigma$ such that $F/S$ is $C^1$. To find $S$ we proceed as follows. Fix $x \in Dom(F) \setminus D(F)$. As $Dom(F)$ is $\mathcal{F}$-saturated, one has $\mathcal{F}_x \subset Dom(F)$ and so $n(\mathcal{F}_x)$ is well defined. By using the Lemma 4.1(1), one has $n(\mathcal{F}_x) < \infty$.

If $n(\mathcal{F}_x) = 0$, then the neighborhood $S$ of $L = \mathcal{F}_x$ in (A1) works.

By simplicity, if $n(\mathcal{F}_x) \geq 1$ let us denote $L_x = \mathcal{F}_x$. Clearly $1 \leq n(L_x) < \infty$ and Definition 3.2 of $n(L_x)$ implies $f^{n(L_x)}(L_x) \subset \partial^v \Sigma$. By hypothesis $\partial^v \Sigma \subset Dom(F)$ and then $F^{n(L_x)}(L_x) \subset Dom(F)$. So, we can choose $S$ as the neighborhood of $L_x$ in (A2). Let us prove that this neighborhood works.

First we claim that $L_x \subset \partial^v \Sigma$. Indeed, if $L_x \subset \Sigma \setminus (\partial^v \Sigma)$, then $S$ has three different connected components $S_0, S_1, S_2$. By (A2)-(4) we can assume $n^1(L_x) > 1$ where $n_1(L)$ comes from (A2)-(3). Choose sequence $x^i_1 \in S_1 \to x$ then $F(x^i_1) \to F^{n(L_x)}(x)$ by (H2)-(3). As $F$ is continuous in $x$ we also have $F(x^i_1) \to F(x)$ and then $F^{n(L_x)}(x) = F(x)$ because limits are unique. Thus, $F^{n(L_x)}(x^i_1) = x$ because $F$ is injective and so $x$ is a periodic point of $F$ since $n^1(L_x) - 1 \geq 1$. This contradicts the non-existence of periodic points for $F$. The claim is proved.

The claim implies that $S$ has two components, i.e, $S_0$ and $S_1 = S_2$. By using (A2), for the component $S_0$ one has $n^0(L_x) = 0$ and $n^1(L_x) = n^2(L_x) = 1$ since $F$ is continuous in $x \in L_x$. Then, $F/S$ is $C^1$ by the last part of (A2)-(3). This finishes the proof.

Lemma 4.6. Let $F : Dom(F) \subset \Sigma \to \Sigma$ be a triangular map satisfying (A1)-(A2). If $F$ has no periodic points and $Dom(F) = \Sigma \setminus L_0$, then $Dom(F) \setminus D(F)$ is open in $\Sigma$.

Proof. $Dom(F)$ is open in $\Sigma$ because $Dom(F) = \Sigma \setminus L_0$ and $L_0$ is closed in $\Sigma$. $Dom(F) \setminus D(F)$ is open in $Dom(F)$ by Lemma 4.5 because $F$ has no periodic points and $\partial^v \Sigma \subset \Sigma \setminus L_0 = Dom(F)$. Thus $Dom(F) \setminus D(F)$ is open in $\Sigma$. 

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Now, we need introduced some definitions for the next result. We say that \(L\) is related with \(L' (L \sim L')\) if we have that \(n(L), n(L') \geq 1\) and \(L, L' \subset S_0(L) \cap S_0(L')\).

**Definition 4.7.** Let \(L\) be a leaf of \(\mathcal{F}\) and \(L \in \text{Dom}(F)\). If \(n(L) \geq 1\), we define the leaf class associated to the leaf \(L\) as
\[
\langle L \rangle = \{ L' \in \text{Dom}(F) | L' \sim L \}.
\]
If \(n(L) = 0\), we say that \(\langle L \rangle = \{ L \} \).

**Remark 4.8.** If \(X\) is a vector field of codimension 1, i.e., \(\text{dim}(E^c) = 2\), one has that \(\langle L \rangle = \{ L \}\) and so \(\text{dim}(\langle L \rangle) = \text{dim}(\{ L \}) = s\).

**Lemma 4.9.** Let \(L\) be a leaf in \(D(F)\). Then \(\langle L \rangle\) is a \((s + 1)\)-submanifold (or \(s\)-submanifold if \(\langle L \rangle = \{ L \}\)) of \(\Sigma\), and whose boundary belong to \(\partial^s \Sigma\).

**Proof.** By using the Lemma [4.3], \(D(F)\) is closed in \(\Sigma \setminus L_0\). So, given \(L \in \text{Dom}(F)\) by (A2) there is \(S(L) = S(L)_0 \cup S(L)_1 \cup S(L)_2\). Then, \(\text{Cl}(S(L)_0) \subset D(F)\), and as \(D(F)\) is \(\mathcal{F}\)-saturated, one has that \(\text{Cl}(S(L)_0) \setminus S(L)_0 = \partial S(L)_0\) are leaves. Thus, for each \(L' \in \partial S(L)_0 \subset D(F)\), by using the Lemma [4.3] we obtain that \(F(L') \subset \partial^s \Sigma\). Then, again by (A2) there exists \(S(L') = S(L')_0 \cup S(L')_1 \cup S(L')_2\). In the same way, we proceeds analogously for \(S(L')\). Since \(\Sigma\) has finite diameter, we conclude that \(\langle L \rangle\) has boundary in \(\partial^s \Sigma\). \(\square\)

We have that if \(L \in D(F)\), then \(F(L) \in \partial^s \Sigma\) (Lemma [4.4]), and this motivate the following definition.

**Definition 4.10.** We define the discontinuous class of leaves by the set
\[
\langle D(F) \rangle = \{ \langle L \rangle | L \in D(F) \}\]

**Definition 4.11.** A subset \(B \) of \(\Sigma\) is \(\mathcal{F}\)-discrete if it corresponds to a set of leaves whose only points of accumulations are the leaves in \(L_0\).

**Lemma 4.12.** If \(F\) has no periodic points, then \(\langle D(F) \rangle\) is discrete.

**Proof.** By contradiction, we suppose that \(\langle D(F) \rangle\) is not \(\mathcal{F}\)-discrete. Then, there is an open neighborhood \(U\) of \(L_0\) in \(\Sigma\) such that \(\langle D(F) \rangle \setminus A\) contains infinitely many classes \(\langle L_n \rangle\), where
\[
A = \{ \langle L \rangle \in \langle D(F) \rangle | \langle L \rangle \cap U \neq \emptyset \}.
\]

By using Lemma [4.3], \(D(F)\) is closed in \(\text{Dom}(F) = \Sigma \setminus L_0\), and so \(\langle D(F) \rangle\) is closed in \(\text{Dom}(F) = \Sigma \setminus L_0\) too. Since \(D(F) \setminus U\) is closed in \(\text{Dom}(F) \setminus U\), one has
that \( \langle D(F) \rangle \setminus A \) is closed. As \( U \) is an open neighborhood of \( L_0 \) and \( \text{Dom}(F) = \Sigma \setminus L_0 \) we obtain that \( \text{Dom}(F) \setminus U \) is compact in \( \Sigma \). Henceforth \( \langle D(F) \rangle \setminus A \) is compact. So, without loss of generality, we can assume that \( \langle L_n \rangle \) converges to a class \( \langle L_\ast \rangle \) of \( \langle D(F) \rangle \setminus A \). Clearly \( \langle L_\ast \rangle \subset \text{Dom}(F) \). Since \( \langle L_n \rangle \subset D(F) \) we have \( F(\langle L_n \rangle) \subset \partial^f\Sigma \) by Lemma 4.4. It follows that \( n(W_\ast) \geq 1 \), for all \( W_\ast \subset \langle L_\ast \rangle \).

We also have \( n(W_\ast) \leq 2k < \infty \) by Lemma 4.1-(2) since \( F \) has no periodic points and \( \partial^e\Sigma \subset \Sigma \setminus L_0 = \text{Dom}(F) \), for all \( W_\ast \subset \langle L_\ast \rangle \). By Definition 3.1 we have \( f^n(L_\ast)(L_\ast) \subset \partial^e\Sigma \subset \text{Dom}(F) \).

By definition \( \langle L_n \rangle \cap \langle L_\ast \rangle = \emptyset \) for all \( n \in \mathbb{N} \). Now, by using the property (A2), for each \( L_n \) and for \( L_\ast \) we can choose the following neighborhood associated to \( \langle L_n \rangle \) and \( \langle L_\ast \rangle \) as follows:

\[
CS_n = \bigcup_{W \in \langle L_n \rangle} S(W) \quad \text{and} \quad CS_\ast = \bigcup_{W_\ast \in \langle L_\ast \rangle} S(W_\ast).
\]

Since \( \langle L_n \rangle \) is compact, one has that
\[
CS_n = \bigcup_{i=1}^k S(W_i) \quad \text{and} \quad CS_\ast = \bigcup_{i=1}^k S(W_\ast,i).
\]

As \( \langle L_n \rangle \rightarrow \langle L_\ast \rangle \) and \( \langle L_n \rangle \cap \langle L_\ast \rangle = \emptyset \) we can assume \( \langle L_n \rangle \subset CS_\ast \setminus \langle L_\ast \rangle \) for all \( n \). As \( \langle L_n \rangle \cap \langle L_\ast \rangle = \emptyset \) for all \( n \) we can further assume that \( \langle L_n \rangle \subset CS_{1,\ast} \) where \( CS_{1,\ast} \) is one of the (possibly equal) connected components of \( CS_\ast \setminus \langle L_\ast \rangle \), i.e.,

\[
CS_{1,\ast} = \bigcup_{i=1}^k S_1(W_i)
\]

As \( F(S_1(L_n)) \subset \Sigma \setminus (\partial^e\Sigma) \) for all \( n \in \mathbb{N} \) by (A2)-(1) we conclude that \( F(\langle L_n \rangle) \subset \Sigma \setminus (\partial^e\Sigma) \) for all \( n \). However, \( F(\langle L_n \rangle) \subset \partial^e\Sigma \) by Lemma 4.4 since \( L_n \subset D(F) \) a contradiction. This proves the lemma.

\[\square\]

We need to extend some definitions for next lemmas and propositions. A vertical band in \( \Sigma \) between two vertical s-surfaces \( L, L' \) in the same component \( \Sigma \) is nothing but a cylinder \( H \) such that \( L, L' \in \partial H \) and whose diameter is \( l \), where \( l \) represent the distance of \( L \) to \( L' \), i.e., \( l = \text{dist}(L, L') \). Let us denote by \( H(l, L') \) and \( H[L, L'] \) the open and vertical band respectively.

Given a \( u \)-surface \( c \), we say that \( c \) is tangent to \( C_\alpha \) if \( Dc(t) \in C_\alpha(c(t)) \) for all \( t \in \text{Dom}(c) \subset \mathbb{R}^u \). A \( C_\alpha \)-spine of a vertical band \( H(L, L') \) (or \( H[L, L'] \) ) is a \( u \)-surface \( c \subset H(L, L') \) tangent to \( C_\alpha \), such that \( \partial c \subset \partial H(L, L') \) and \( \text{int}(c) \subset H(L, L') \).
Lemma 4.13. Let \( c \subset \text{Dom}(F) \setminus D(F) \) be an open \( u \)-surface transversal to \( F \). If there is \( n \geq 1 \) and open \( C^1 \) \( u \)-surface \( c^* \) whose closure \( \text{Cl}(c^*) \subset c \) and such that \( F^n(c^*) \subset (\text{Dom}(F) \setminus D(F)) \) for all \( 0 \leq i \leq n - 1 \) and \( F^n(c^*) \) covers \( c \), then \( F \) has a periodic point.

Proof. We prove the lemma by contradiction. Then, we suppose that \( F \) has no periodic point. So, by using the Lemma 4.5, \( \text{dom} \) \( F \) is denoted by \( c \). Suppose that \( \text{Lemma} \ 4.15 \). \( F \) satisfies the assumptions imply that \( f^i(c) \) is defined for all \( 0 \leq i \leq n - 1 \) and

\[
\text{Cl}(c^*) \subset c \subset f^n(c^*).
\]

Then, \( f^n \) has a periodic point \( L_{ss} \). As \( F^n(L_{ss}) \subset f(L_{ss}) = L_{ss} \) and \( F^n|_{L_{ss}} \) is continuous, then the Brower fixed point Theorem implies that \( F^n \) has a fixed point. This fixed point represents a periodic point of \( F \). \( \Box \)

Lemma 4.14. \( F \) carries a \( u \)-surface \( c \subset \text{Dom}(F) \setminus D(F) \) tangent to \( C_\alpha \) (with volume \( V(c) \)) into a \( u \)-surface tangent to \( C_\alpha \) (with volume \( \geq \lambda \cdot V(c) \)).

Proof. See \[6\]. \( \Box \)

Lemma 4.15. Suppose that \( F \) has no periodic points. Let \( L, L' \) be different leaves in \( D(F) \) such that the open vertical band \( H(L, L') \subset \text{Dom}(F) \setminus D(F) \). If \( c \) is a \( C_\alpha \)-spine of \( H(L, L') \), then \( F(\text{Int}(c)) \) covers a vertical band \( H(W, W') \) with

\[
W, W' \subset \partial \Sigma \cup \mathcal{V}.
\]

Proof. By using Lemma \[4.3\] we have that \( F/H(L, L') \) is \( C^1 \) because \( H(L, L') \subset \text{Dom}(F) \setminus D(F) \). And Lemma \[4.3\] implies

\[
F(L), F(L') \subset \partial \Sigma
\]

because \( L, L' \subset D(F) \). Clearly \( L, L' \subset \text{Dom}(F) \) and then \( n(L), n(L') \) are defined. By \( \text{Lemma} \ 4.1 \) we have \( n(L), n(L') \geq 1 \). Then, \( 1 \leq n(L), n(L') < \infty \) by Lemma \[4.1 \] (1) since \( F \) has no periodic points and \( \partial \Sigma \subset \Sigma \setminus L_0 = \text{Dom}(F) \). By the same reason

\[
F^n(L)(L), F^n(L')(L') \subset \text{Dom}(F).
\]

If there exists a sequence converging to \( L \) or \( L' \), by using the Lemma \[4.3\] exist the limit \( F(L_n) \) and this limit belong to \( \partial \Sigma \cup \mathcal{V} \). Let \( W \) and \( W' \) be these limits respectively.

Now, let \( c \) be a \( C_\alpha \)-spine of \( H(L, L') \). To fix ideas we assume \( \partial c \subset \partial H(L, L') \), and this implies that there are \( p, q \in \partial c \) such that \( p \in L \) and \( q \in L' \). As \( \text{Int}(c) \subset H(L, L') \subset \text{Dom}(F) \setminus D(F) \) we have that \( F(\text{Int}(c)) \) is defined. As
We have that $F(\text{Int}(c))$ is a $u$-surface whose boundary containing points that belong in $\partial'\Sigma \cup \mathcal{V}$.

Clearly $W \neq W'$ because $F$ preserves $\mathcal{F}$. Then, $F_{F(\text{Int}(c))} = H(W, W')$ is an open vertical band, where $W, W' \subset \partial'\Sigma \cup \mathcal{V}$.

**Definition 4.16.** Let $p$ be a point of $M$. We define the radius of injectivity $\text{inj}(p)$ at a point $p$ as the largest radius for which the exponential map at $p$ is an injective map, i.e.,

$$\text{inj}(p) = \sup \{ r > 0 : \exp_p : B(0, r) \to M \text{ is injective} \}.$$ 

Also, we say that the radius of injectivity of the manifold $M$ is the infimum of the radius at all points, i.e.,

$$\text{inj}(M) = \inf \{ \text{inj}(p) | p \in M \}.$$ 

**Lemma 4.17.** Suppose that $F$ has no periodic points. For every open $u$-surface $c \subset \text{Dom}(F) \setminus D(F)$ tangent to $C_{\alpha}$ there are an open $u$-surface $c^* \subset c$ and $n'(c) > 0$ such that $F^j(c^*) \subset \text{Dom}(F) \setminus D(F)$ for all $0 \leq j \leq n'(c) - 1$ and $F^{n'(c)}(c^*)$ covers a vertical band $H(W, W')$ with $W, W' \subset \partial'\Sigma \cup \mathcal{V} \cup \mathcal{L}$.

**Proof.** Let $c \subset \text{Dom}(F) \setminus D(F)$ be an $u$-surface tangent to $C_{\alpha}$.

In the same way of [6], the proof is based on the following, that it will be modified for the higher dimensional case. This claim will be proved by modifying the same arguments used in [6,12] and a similar argument used by [7] beside the radius of injectivity definition.

**Claim 4.18.** There are an open $u$-surface $c^{**} \subset c$ and $n''(c) > 0$ such that $F^j(c^{**}) \subset \text{Dom}(F) \setminus D(F)$ for all $0 \leq j \leq n''(c) - 1$ and $F^{n''(c)}(c^{**})$ covers an open vertical band $H(L, L') \subset \text{Dom}(F) \setminus \langle D(F) \rangle$, where $L, L'$ are different leaves in $D(F) \cup L_0$.

**Proof.** For every open $u$-surface $c' \subset \text{Dom}(F) \setminus D(F)$ tangent to $C_{\alpha}$ we define

$$N(c') = \sup \{ n \geq 1 : F^j(c') \subset \text{Dom}(F) \setminus \langle D(F) \rangle, \forall 0 \leq j \leq n - 1 \}.$$ 

Note that $1 \leq N(c') < \infty$ because $\lambda > 1$ and $\Sigma$ has finite diameter. In addition, $F^N(c')(c')$ is a $u$-surface tangent to $C_{\alpha}$ with

$$F^N(c')(c') \cap (D(F) \cup L_0) \neq \emptyset$$
because $Dom(F) = \Sigma \setminus L_0$.

Define the number $\beta$ by

$$\beta = (1/2) \cdot \lambda.$$ 

Then, $\beta > 1$ since $\lambda > 2$. Define $c_1 = c$ and $N_1 = N(c_1)$.

Since $F^{N_1}(c_1)$ is a open $u$-surface, if $F^{N_1}(c_1)$ intersects $\langle D(F) \rangle \cup L_0$ in a unique leaf class $\langle L_1 \rangle$, then $F^{N_1}(c_1) \setminus \langle L_1 \rangle$ has two connected components. In this case we define

- $c_2^* = \text{the connected component of } F^{N_1}(c_1) \setminus \langle L_1 \rangle \text{ that contain a ball whose}$
  - $\text{radius of injectivity is greater or equal to any ball of the complement.}$
- $c_2 = F^{-N_1}(c_2^*)$.

The following properties hold,

1) $c_2 \subset c_1$ and then $c_2$ is an open $u$-surface tangent to $C_\alpha$.

2) $F^j(c_2) \subset Dom(F) \setminus \langle D(F) \rangle$, for all $0 \leq j \leq N_1$.

3) $V(F^{N_1}(c_2)) \geq \beta \cdot V(c_1)$.

In fact, the first property follows because $F^{N_1} /_{c_2}$ is injective and $C^1$. The second one follows from the definition of $N_1 = N(c_1)$ and from the fact that $c_2^* = F^{N_1}(c_2)$ does not intersect any leaf in $\langle D(F) \rangle \cup L_0$. The third one follows from Lemma 4.14 because

$$V(F^{N_1}(c_2)) = V(c_2^*) \geq (1/2) \cdot V(F^{N_1}(c_1)) \geq$$

$$\geq (1/2) \cdot \lambda^{N_1} V(c_1) \geq (1/2) \cdot \lambda V(c_1) = \beta \cdot V(c_1)$$

since $\lambda > 2$ and $N_1 \geq 1$.

Next we define $N_2 = N(c_2)$. The second property implies $N_2 > N_1$. As before, if $F^{N_2}(c_2)$ intersects $\langle D(F) \rangle \cup L_0$ in a unique leaf class $\langle L_2 \rangle$, then $F^{N_2}(c_2) \setminus \langle L_2 \rangle$ has two connected components. In such a case we define analogously $c_3^*$ and also $c_3 = F^{-N_2}(c_3^*)$.

As before

$$V(F^{N_1}(c_3)) = V(c_3^*) \geq (1/2) \cdot V(F^{N_2}(c_2)) \geq (1/2) \cdot \lambda^{N_2-N_1} V(F^{N_1}(c_2)) \geq \beta^2 V(c_1)$$

because of the third property. So,

1) $c_3 \subset c_2$ and $c_3$ is an open $u$-surface tangent to $C_\alpha$.

2) $F^j(c_3) \subset Dom(F) \setminus \langle D(F) \rangle$ for all $0 \leq j \leq N_2$. 

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3) $V(F^N(c_3)) \geq \beta^2 \cdot V(c_1)$.

In this way we get a sequence $N_1 < N_2 < N_3 < \cdots < N_l < \cdots$ of positive integers and a sequence $c_1, c_2, c_3, \cdots c_l, \cdots$ of open $u$-surfaces (in $c$) such that the following properties hold $\forall l \geq 1$

1) $c_{l+1} \subset c_l$ and $c_{l+1}$ is an open $u$-surface tangent to $C_\alpha$.

2) $F^j(c_{l+1}) \subset Dom(F) \setminus \langle D(F) \rangle$ for all $0 \leq j \leq N_l$.

3) $V(F^{N_l}(c_{l+1})) \geq \beta^l \cdot V(c_1)$.

The sequence $c_l$ must stop by Property (3) since $\Sigma$ has finite diameter. So, there is a first integer $l_0$ such that $F^{N(c_{l_0})}(c_{l_0})$ intersects $\langle D(F) \rangle \cup L_0$ in two different leaves class $\langle L \rangle \cup \langle L' \rangle$. Note that these classes must be contained in the same component of $\Sigma$ since $F^{N(c_{l_0})}(c_{l_0})$ is connected. Hence, we can suppose that the vertical band $H(L, L')$ bounded by $L, L'$ is well defined. We can assume that $H(L, L') \subset Dom(F) \setminus \langle D(F) \rangle$ because $\langle D(F) \rangle$ is $\mathcal{F}$-discrete by Lemma 4.12. Choosing $c^{**} = c_{l_0}$ and $n''(c) = N_{l_0}$ we get the result.

Now we finish the proof of Lemma 4.17. Let $c^{**, n''(c)}$ and $L, L' \subset D(F) \cup L_0$ be as in Claim 4.18. We have three possibilities: $L, L' \subset D(F)$; $L \subset L_0$ and $L' \subset D(F)$; $L \subset D(F)$ and $L' \subset L_0$. We only consider the two first cases since the later is similar to the second one.

First we assume that $L, L' \subset D(F)$. As $F^{n''(c)}(c^{**})$ is tangent to $C_\alpha$, and covers $H(L, L')$, we can assume that $F^{n''(c)}(c^{**})$ itself is a $C_\alpha$-spine of $H(L, L')$. Then, applying Lemma 4.15 to this spine, one gets that $F^{n''(c)+1}(c^{**})$ covers a vertical band $H(W, W')$ with $W, W' \subset \partial^v \Sigma \cup \mathcal{V}$.

In this case the choices $c^* = c^{**}$ and $n'(c) = n''(c) + 1$ satisfy the conclusion of Lemma 4.17.

Finally we assume that $L \subset L_0$ and $L' \subset D(F)$. As $L \subset L_0$ we have $L = L_{0i}$ for some $i = 1, \cdots, k$.

On the one hand, $H(L_{0i}, L') = H(L, L') \subset Dom(F) \setminus \langle D(F) \rangle$ and then $\langle D(F) \rangle \cap H(L_{0i}, L') = \emptyset$. So, Lemma 4.12 implies that exists the limit $K_i$. Consequently

$K_i \in \mathcal{L}$.

On the other hand, $F(L') \subset \partial^v \Sigma$ by Lemma 4.1 since $L' \subset D(F)$. It follows that $1 \leq n(L')$ and also $n(L') \leq 2k$ by Lemma 4.1(1) since $F$ has no periodic
points and $\partial^n \Sigma \subset \Sigma \setminus L_0 = \text{Dom}(F)$. Since $F^{n(L')}(L') \subset \partial^n \Sigma$ by the definition of $n(L')$ we obtain

$$F^{n(L')}(L') \subset \text{Dom}(F).$$

Then, Lemma 4.3 applied to $L'$ implies that $\lim_{L \to L'} f(L) = f^*(L')$ exists and satisfies

$$f^*(L') \subset (\partial^n \Sigma) \cup \mathcal{L}.$$

But $F(H(L_0i, L'))$ (and so $F^{n''(c)}(c^{**})$) covers $H(K_i, f^*(L'))$ since $H(L_0i, L') \subset \text{Dom}(F) \setminus \langle D(F) \rangle$. Setting $W = K_i$ and $W' = f^*(L')$ we get

$$W, W' \subset (\partial^n \Sigma) \cup \mathcal{V} \cup \mathcal{L}.$$

(Recall the definition of $\mathcal{V}$ in Definition 4.2) Then, $F^{n''(c)}(c^{**})$ covers $(W, W')$ as in the statement. Choosing $c^* = c^{**}$ and $n'(c) = n''(c) + 1$ we obtain the result.

4.2 Proof of the Theorem 3.4

Finally we prove Theorem 3.4. Let $F$ be a $\lambda$-hyperbolic $n$-triangular map satisfying (A1)-(A2) with $\lambda > 2$ and $\text{Dom}(F) = \Sigma \setminus L_0$. We assume by contradiction that the following property holds:

(P) $F$ has no periodic points.

Since $\partial^n \Sigma \subset \Sigma \setminus L_0$ and $\Sigma \setminus L_0 = \text{Dom}(F)$ we also have

$$\partial^n \Sigma \subset \text{Dom}(F).$$

Then, the results in the previous subsections apply. In particular, we have that $\text{Dom}(F) \setminus D(F)$ is open in $\Sigma$ (by Lemma 4.6) and that $\langle D(F) \rangle$ is $\mathcal{F}$-discrete (by Lemma 4.12-(1)). All together imply that $\text{Dom}(F) \setminus D(F)$ is open-dense in $\Sigma$.

Now, let $\mathcal{B}$ be a family of open vertical bands of the form $H(W, W')$ with

$$W, W' \subset (\partial^n \Sigma) \cup \mathcal{V} \cup \mathcal{L}.$$

It is clear that $\mathcal{B} = \{B_1, \cdots, B_m\}$ is a finite set. In $\mathcal{B}$ we define the relation $B \leq B'$ if and only if there are an open $u$-surfaces $c \subset B$ tangent to $C_\alpha$ with closure $\text{CL}(c) \subset \text{Dom}(F) \setminus \langle D(F) \rangle$, an open $u$-surface $c^* \subset c$ and $n > 0$ such that

$$F^j(c^*) \subset \text{Dom}(F) \setminus \langle D(F) \rangle, \quad \forall 0 \leq j \leq n - 1,$$

and $F^n(c^*)$ covers $B'$.
As $\text{Dom}(F) \setminus D(F)$ is open-dense in $\Sigma$, and the bands in $\mathcal{B}$ are open, we can use Lemma 4.17 to prove that for every $B \in \mathcal{B}$ there is $B' \in \mathcal{B}$ such that $B \leq B'$. Then, we can construct a chain

$$B_{j_1} \leq B_{j_1} \leq B_{j_2} \leq \cdots,$$

with $j_i \in \{1, \cdots, m\}$ ($\forall i$) and $j_1 = 1$. As $\mathcal{B}$ is finite it would exist a closed sub-chain

$$B_{j_i} \leq B_{j_{i+1}} \leq \cdots \leq B_{j_{i+s}} \leq B_{j_i}.$$

Hence there a positive integer $n$ such that $F^n(B_{j_i})$ covers $B_{j_i}$. Applying Lemma 4.13 to suitable $u$-surfaces $c^* \subset Cl(c^*) \subset c \subset B_{j_i}$ we obtain that $F$ has a periodic point. This contradicts (P) and the proof follows.

### 4.3 Proof of the Main Theorem

**Proof.** Let $X$ be a sectional Anosov flow on a compact $n$-manifold $M$. We prove the Theorem by contradiction, i.e, to prove that $M(X)$ has a periodic orbit we assume that this is not so. Fix $\lambda > 2$. As there is a singular cross-section $\Sigma$ close to $M(X)$ such that if $F$ is the return map of the refinement $\Sigma(\delta)$ of $\Sigma$, then there is $\delta > 0$ such that $F$ is a $\lambda$-hyperbolic $n$-triangular map with $\text{Dom}(F) = \Sigma L_0$. We also have that $F$ satisfies (A1) and (A2) in Subsection 3.1 since $\Sigma$ is close to $\Lambda$. Then, $F$ has a periodic point by Theorem 3.4 since $\lambda > 2$. This periodic point belongs to a periodic orbit of $X_t$ which in turns belongs to $M(X)$ since it is maximal invariant. Consequently $X_t$ has a periodic orbit in $M(X)$, a contradiction. This contradiction proves the result. \qed

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