New ideas about multiplication of tensorial distributions

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Abstract

There is a huge need in general relativity for a consistent and useful mathematical theory defining the multiplication of tensor distributions in a geometric (diffeomorphism invariant) way. Significant progress has been made through the concept of Colombeau algebras, and the construction of full Colombeau algebras on differential manifolds for arbitrary tensors. Despite the fact that this goal was achieved, it does not incorporate clearly enough the concept of covariant derivative and hence is of a limited use.

We take a different approach: we consider any type of preference for smooth distributions (on a smooth manifold) as nonintuitive, which means all our approach must be based fully on the Colombeau equivalence relation as the fundamental feature of the theory. After taking this approach we very naturally obtain a canonical and geometric theory defining tensorial operations with tensorial distributions, including covariant derivative. This also happens because we no longer need any explicit canonical geometric construction of Colombeau algebras. The big advantage of our approach lies also in the fact that it brings a physical insight into the mathematical concepts used and naturally leads to formulation of physics on (what we call) piecewise smooth manifolds, rather than on smooth manifold. This brings to the language of physics much higher symmetry (in the same way as turning from Poincare invariance to diffeomorphism invariance), and is compatible with our intuition, that “pointwise” properties in some metaphorical sense “do not matter”.

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1 Introduction

In our view there are the following basic reasons why one shall work with the language of distributions rather than with the old language of functions:

First there is a deep physical meaning to work with distributions rather than with smooth tensor fields. Distributions should not only be a convenient tool to do computations in the cases, in which one cannot use the old standard differential geometry, but in comparison to the old language of functions we consider them to be mathematical objects much more accurately expressing what one measures in physics. The point is that the question: “What is the ‘amount’ of physical quantity at some open set?” is in our view a much more reasonable physical question (reasonable from the point of view, what we can ask the experimentalists to measure), as the question: “How is the quantity at a given point?” But “point values” as “recovered” by delta distributions do seem to give the different and reasonable meaning to the last question. It is also a strong intuition that the “amount” of physical quantity on the open set $A_1 \cup A_2$, where $A_1, A_2$ are disjoint is the sum of “amounts” of that quantity on $A_1$ and $A_2$, hence it is appropriate to speak rather about distributions than about general smooth mappings from functions to real numbers (the mappings should also be linear).

The second reason is that the real applications confirm the need for a much richer language as is the language of smooth tensor fields. Actually when we look for physically interesting solutions it might be always a matter of importance to have a much larger class of objects available as is the class of smooth tensor fields.

The third reason (which is a bit more speculative) is the relation of the language defining the multiplication of distributions to quantum gravity. This is a question of considerable importance, since usually the problems requiring distributions, that means problems going beyond the language of the old differential geometry, are the short scale ones. But understanding some operations with distributions, specifically their multiplication has a large impact on quantum field theory, particularly on the problem with interacting fields. Hence solving these questions can have significant consequences for quantizing gravity as well.

The considerations about language intuitiveness lead us to an interesting conclusion: the language of distributions (being connected with our intuitions) strongly suggests that properties of what we can see as classical tensor fields should not matter on the sets having any $L$-measure 0 (we de-

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1It might be expected in the case of Algebraic Quantum Field Theory, where they work with distributional operators.
fine it further, it is an analogue of Lebesgue measure in $\mathbb{R}^n$). But following our intuition that in the specific sense “points do not matter”, or “points do not have any real physical meaning”, we shall naturally expect that we will be able to generalize our language from a smooth manifold into a piecewise smooth manifold (which will bring higher symmetry to our concepts). The first traces of piecewise smooth coordinate transformations are already seen for example in [17].

Now it is worth noticing how strange is the situation within the theory of distributions: we have a useful and meaningful language of distributions, which can be geometrically generalized, but this language works only for linear physics. But linear physics is only a starting point (or at best rough approximation) to describe the real interactions, and hence the nonlinear physics. So one naturally expects that the language of distributions will be a result of some mathematical language defining their multiplication. Moreover, at the same time we want this language to contain the old language of differential geometry (as its specific case), as it is in the case of distributions and linear theories. It is quite obvious that one vertex of this square diagram should be resolved. The practical need of this language is obvious as well, as we see in the numerous applications [10, 14, 15, 2, 28, 30, 31] (but this is not the main motivation of our work).

The main motivation of this work is the development of the language of distributional tensors, strongly connected with the physical intuition (this means it has to also be based on the concept of piecewise smooth manifold), defining all the basic tensorial operations in the generalized way, enabling us to understand the previous results and the problems attached to them. It is worth stressing that most of this motivation results from a shift in views in the field of foundations of physics (so it is given by “deeper” philosophical reasons), by having an impact on the practical physical questions as well, but the size of this impact has still to be explored. We claim that the goals described as our motivation are achieved in this paper. Particularly we generalized all the basic concepts from smooth tensor field calculus (including the fundamental concept of covariant derivative) in the two basic directions:

a) First generalization goes in the direction of the class of objects being constructed from the maps in every chart given as integrals from some piecewise continuous functions, and this class we call $D_{n,E,A}^\mu(M)$.

b) Second generalization is a generalization to the class of objects naturally connected with a smooth manifold belonging to our piecewise smooth manifold (in the sense that the smooth atlas of the smooth manifold is a subatlas of our piecewise smooth atlas). This is a good
analogy to the generalization known from the classical distribution theory. The class of such objects we call $D_{n(A')}^n(M)$.

Our calculus is in our opinion the most natural and straightforward construction achieving these two particular goals. The fact, that such a natural construction seems to exist supports our faith in practical meaning of the mathematical language later developed. But apart from that fact, part of the goal of this paper is also to suggest much more ambitious, natural generalizations, which are unfortunately at present only in the form of conjectures. We provide the reader later in the text with such conjectures.

Now one of the main points: What is the relation of our approach to the achievements of the standard Colombeau algebra approach [4, 5, 31] (being already geometrically formulated and incorporating arbitrary rank tensorial distributions in the canonical way [19, 20, 18, 16, 31])? The basic point is that our approach is fully based on the Colombeau equivalence relation reformulated to our language. This means we take and use only that particular feature of the Colombeau theory, completely avoiding the Colombeau algebra construction.

What are the advantages of this approach comparing to Colombeau theory? First of all, by avoiding the factorization (which is the case of Colombeau algebra) we fulfill the physical intuitiveness condition of the language constructed. Second, we can naturally and easily generalize the concept of covariant derivative in our formalism, which has not been completely satisfactorily achieved by Colombeau algebra approach. This must be taken as a absolutely necessary condition to achieve any goal of generalization of a fundamental and physically significant mathematical formalism. It is specifically worth discussing the third advantage: Why is the classical approach so focused on Colombeau algebras? The answer is simple: we want to get algebra of $C^{\infty}(M)$ functions as a subalgebra of our algebra (that is why we need to factorize by negligible functions, and we need to get the biggest space where they form an ideal, which is the space of moderate functions). But there is one strange thing: all our efforts are aimed to reach the goal of getting a more rich space than is the space of smooth functions. But there is no way to get a larger differential subalgebra than the algebra of smooth functions, as it is shown by Schwartz impossibility result (see for example [31]). That is why we use only the equivalence relation instead of straight equality. But then the question remains: Why should be there still preference of smooth functions? Is not the key part of all the theory the equivalence

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There is some work done in this direction in [32] and [24], but it is minimally far from obvious how (by taking this given approach) one can reconstruct the expected results outside the class of smooth tensor fields.
relation? So why are we not satisfied with the way the equivalence relation recovers the smooth objects multiplication (we require something stronger), but we are satisfied with the way it recovers multiplication within the larger class? Unlike the Colombeau algebra based approach, we are simply taking seriously the idea that one shall treat all the objects in an equal way, which means we do not see any reason to try to achieve "something more" with smooth objects than we do with objects outside this class. And in the fact that we treat all the objects in the same way lies the third advantage of our theory; it makes the theory much more natural than the Colombeau algebra approach. The fourth advantage is that it naturally works with much more general concept of piecewise smooth manifold, so the generalization of physical laws within such language brings to the language of physics a much higher symmetry. The fifth and last, “small” advantage comes from the fact that by avoiding the Colombeau algebras construction we automatically remove the problem of canonical embedding of arbitrary tensorial distributions. But one has to acknowledge that this problem was already resolved also within the Colombeau theory approach [20].

What are the disadvantages of this approach comparing to Colombeau theory? A conservative person might be not satisfied with the fact, that we do not have smooth tensor algebra as a subalgebra of our algebra. This means that also the classical smooth tensorial fields have to be considered to be solutions of equivalencies only (as opposed to classical, “stronger” view to take them as solutions of the equations). But in my view, this is not a problem at all. If we consider physical laws as using equivalence only, they have many solutions in the algebra we constructed containing all the classical solutions for smooth tensor distributions (equivalently smooth tensor fields) and mostly leaving open only the question of physical meaning of those various particular solutions (in general we refer also to nonlinear objects), which is nothing new in physics. This is one of the main reasons why we see no problem in abandoning the conservative requirement to obtain the smooth tensor fields algebra as subalgebra of our algebra. Our previous considerations suggest that we shall look only for distributional solutions (as those having physical relevance) and in such case it can be seen that we will get (after taking some additional intuitive requirements) a unique solution (up to initial values obviously), if it exists. And as already mentioned, it can be shown that all the classical equations related to smooth tensor fields (if we restrict ourselves to a smooth manifold) are a particular consequence of these equivalencies with the given conditions and all the main results computed by

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3Under “solution” we mean here any object fulfilling the relations given by the equivalence.
various authors can be reinterpreted in that approach.

The structure of this paper is the following:

In the first part we want to present the current state of the Coulombeau algebra theory and its geometric formulations. We want to indicate where are its weaknesses.

This part is followed by technical parts, in which we simply define our theory and prove the basic theorems. At the beginning we define basic concepts underlying our theory, then we define the concept of generalized tensor fields, their important subclasses and basic operations between the generalized tensor fields, like the tensor multiplication. This is followed by the definition of the basic concept of our theory, the concept of equivalence between two generalized tensor fields. The last technical part deals with the definition of the covariant derivative operator and formulation of the initial value problem in our theory.

All the technical parts are followed by sections, which discuss our results and show how our theory relates to the practical results already achieved and described in the first part of the paper.
2 Overview of the present state of the theory

2.1 The standard $\mathbb{R}^n$ theory of Colombeau algebras

The so called special Colombeau algebra is on $\mathbb{R}^n$ defined as:

$$G(\mathbb{R}^n) = \mathcal{E}_M(\mathbb{R}^n)/\mathcal{N}(\mathbb{R}^n),$$

(1)

where $\mathcal{E}_M(\mathbb{R}^n)$ (moderate functions) is defined as the algebra of functions:

$$\mathbb{R}^n \times (0,1] \rightarrow \mathbb{R}$$

(2)

that are smooth on $\mathbb{R}^n$ (this is usually called $\mathcal{E}(\mathbb{R}^n)$), and for any compact subset $K$ of $\mathbb{R}^n$ (in our notation $K \subset \subset \mathbb{R}^n$) it holds that:

$$\forall \alpha \in \mathbb{N}_0^n, \exists p \in \mathbb{N} \text{ such that } \sup_{x \in K} |D^\alpha f_\epsilon(x)| \leq O(\epsilon^{-p}) \text{ as } \epsilon \to 0.$$  

(3)

The $\mathcal{N}(\mathbb{R}^n)$ (negligible functions) are functions from $\mathcal{E}(\mathbb{R}^n)$ where for any $K \subset \subset \mathbb{R}^n$ holds that:

$$\forall \alpha \in \mathbb{N}_0^n, \exists p \in \mathbb{N} \text{ such that } \sup_{x \in K} |D^\alpha f_\epsilon(x)| \leq O(\epsilon^p) \text{ as } \epsilon \to 0.$$  

(4)

This simple formulation can be straightforwardly generalized into general manifolds just by substituting the concept of Lie derivative for the “naive” derivative used before.

It can be shown that by using convolution with an arbitrary smoothing kernel, we can embed a distribution into Colombeau algebra. By a smoothing kernel we mean in the widest formulation a smooth function $\rho_\epsilon \in (0,1]$ such that:

$$\text{supp}(\rho_\epsilon) \to \{0\} \text{ for } (\epsilon \to 0)$$

$$\int \rho_\epsilon(x)dx \to 1 \text{ for } (\epsilon \to 0)$$

and $\forall \eta > 0 \exists C \forall \epsilon \in (0,\eta) \sup_{x \in K} |\rho_\epsilon| < C.$  

(5)

This most generic embedding approach is mentioned for example in [17] (in some sense also in [23]).

Then there are used more “restricted” embeddings to $G(\mathbb{R}^n)$. We can choose for instance a subclass of mollifiers called $A_0(\mathbb{R}^n)$, which are smooth functions being from $D(\mathbb{R}^n)$ (smooth, compact supported) and (i.e. [13])

$$\forall \epsilon \text{ holds } \int \rho_\epsilon(x)dx = \int \frac{1}{\epsilon^n} \rho \left( \frac{x}{\epsilon} \right) dx = 1.$$  

(6)
Sometimes the class is even more restricted. To do that, we shall define classes $A_n(\mathbb{R}^n)$ as classes of smooth, compact supported functions, such that

$$\int x_1^i \cdots x_j^l \phi(x) dx = \delta_{0k} \quad \text{for} \quad i + \cdots + j = k \leq n. \quad (7)$$

Obviously, we have $A_{n+1}(\mathbb{R}^n) \subset A_n(\mathbb{R}^n)$. Then the most restricted class of mollifiers is taken to be the class $A^\infty(\mathbb{R}^n)$. This approach is taken in the references [28], [30], [31], [26], [15].

The question remains, whether that embedding is canonical. It would be, if the difference of two mollifier choices will belong to the class of negligible functions. This is the case for embeddings of regular distributions given by a smooth function (we use the shorthand “smooth distributions”) and using the third, most restricted class of mollifiers [31]. But you cannot extend this property to $D'(\mathbb{R}^n)$ (see [31]). A similar statement saying that by taking the $A^0(\mathbb{R}^n)$ class of mollifiers one obtains noncanonical embedding, unless that class is not properly restricted, we can find in [13].

But what is usually considered to be a canonical formulation of Colombeau algebras in $\mathbb{R}^n$ is the following: The theory is formulated in terms of functions

$$\mathbb{R}^n \times A_0(\mathbb{R}^n) \to \mathbb{R} \quad \text{(call them $F$)} \quad (8)$$

and Colombeau algebra is defined in such way that it is a factor algebra of moderate functions over negligible functions, where:

Moderate function is such function from $F$ that:

$$\forall n, m \in \mathbb{N} \ \exists N \in \mathbb{N} \quad \text{such that if} \quad \phi \in A^N(\mathbb{R}^n), \quad \text{there are} \ \alpha, \rho > 0,$$

$$\text{such that} \quad \sup_{|x| \leq n} \left| \frac{d^m}{dx^m} F(\phi, x) \right| \leq \alpha \varepsilon^{-N} \quad \text{if} \quad 0 < \varepsilon < \rho. \quad (9)$$

Negligible functions are such that are from $F$ and:

$$\forall n, m \in \mathbb{N} \ \forall p \in \mathbb{N} \ \exists q \in \mathbb{N} \quad \text{such that if} \quad \phi \in A^p(\mathbb{R}^n), \ \exists \alpha, \rho > 0,$$

$$\text{such that} \quad \sup_{|x| \leq n} \left| \frac{d^m}{dx^m} F(\phi, x) \right| \leq \alpha \varepsilon^p \quad \text{if} \quad 0 < \varepsilon < \rho. \quad (10)$$

Then distributions automatically define such a function by mapping (see [4], [7], [23], [31] etc.):

$$A \to \left( A_x, \frac{1}{\varepsilon^n} \phi \left( \frac{y - x}{\varepsilon} \right) \right). \quad (11)$$
Moreover, Colombeau shows that if we take a different embedding of smooth distributions, the “straightforward” embedding by identity, the difference is a negligible function (which is a similar result to what is in [4]).

In all of these formulations, there are two important consequences given by the Colombeau algebra:

a) smooth functions form a subalgebra of the Colombeau algebra \( (C(f)C(g) = C(f \cdot g) \) for \( f, g \) being smooth distributions).

b) distributions form a subspace of Colombeau algebra (this means for instance that \( C(f') = C'(f) \)).

We can formulate a relation of equivalence between an element of the special Colombeau algebra \( f_{\epsilon}(x) \) and a distribution \( A \), if for any \( \phi \in D(\mathbb{R}^n) \) we get

\[
\lim_{\epsilon \to 0} \int f_{\epsilon}(x)\phi(x)dx = (A, \phi).
\]

Then two elements of Colombeau algebra \( f_{\epsilon}(x), g_{\epsilon}(x) \) are equivalent, if for any \( \phi(x) \in D(\mathbb{R}^n) \)

\[
\lim_{\epsilon \to 0} \int (f_{\epsilon} - g_{\epsilon})\phi(x)dx = 0.
\]

In the original theory, which will be described below, one can easily see that for the choice of \( A^\infty(\mathbb{R}^n) \) mollifiers \(^4\) all the classical relations must be respected by the equivalence. It respects multiplication of distribution by a smooth distribution \([31]\) in the sense that: \( A(f \cdot g) \approx A(f)A(g) \), where \( f \) is a smooth distribution and \( g \in D'(\mathbb{R}^n) \). From \([4]\), it follows that it must respect in that sense a multiplication of piecewise continuous functions (in the sense of regular distributions given by a piecewise continuous functions). The other important theorem says \([7]\), that if \( g \) is a distribution and \( f \approx g \), then for arbitrary natural number \( n \) holds \( D^n f \approx D^n g \). The fourth important theorem states \([7]\) that if \( f \) is equivalent to distribution \( g \), and if \( h \) is a smooth distribution, then \( f \cdot h \) is equivalent to \( g \cdot h \).

In the canonical formulation, there is a clear relation of equivalence formulated either between an element of the Colombeau algebra and a distribution, or analogically between two elements of the Colombeau algebra.

This relation is such, that if there \( \exists m \), such that for any \( \phi \in A^m(\mathbb{R}^n) \) and for any \( \psi(x) \in D(\mathbb{R}^n) \) holds that

\[
\lim_{\epsilon \to 0} \int (f(\phi_{\epsilon}, x) - g(\phi_{\epsilon}, x))\psi(x)dx = 0
\]

\(^4\)In case of special algebras, if we do not take the most reduced class of mollifiers, it is not completely clear what kind of results can be generally proven with this concept of equivalence, but it is used in particular cases.
where $\phi_{\epsilon} = \epsilon^{-n}\phi(\frac{x}{\epsilon})$, than we say that $f$ and $g$ are equivalent ($f \approx g$).

For this embedding obviously hold the same commutation relations as for the previous one (according to differentiation). There can be also proven that for $f_1 \ldots f_n$ being regular distributions given by piecewise continuous functions holds that

$$A(f_1) \ldots A(f_n) \approx A(f_1 \ldots f_n)$$

and for $f$ being arbitrary distribution and $g$ smooth distribution holds

$$A(f)A(g) \approx A(f \cdot g).$$

In older Colombeau papers all these things are formulated equivalently as the relations between elements of Colombeau algebra, being a subalgebra of $C^\infty(D(\mathbb{R}^n))$. The definitions of moderate and negligible elements are exactly the same as in canonical formulation, apart of taking the $D(\mathbb{R}^n) \times \mathbb{R}^n$ class as the domain of our mappings (being a wider domain than $A^\infty(\mathbb{R}^n) \times \mathbb{R}^n$). We shall also realize that this formulation speaks equivalently about the elements of $C^\infty(D(\mathbb{R}^n))$ class, which can be seen by taking convolution of those elements with the objects from $D(\mathbb{R}^n)$ class. It is really simple to see that you can formulate all the previous relations as relations between elements of this algebra (pointwise operations), containing distributions. This two algebras are apparently isomorphic. This approach was taken by [5], and in some older review of Colombeau as well.

### 2.2 Geometric approach

The first point should be to define what is meant by arbitrary rank tensorial distributions on arbitrary manifolds, by avoiding to refer to preferred charts. Usually we mean by distribution representing an $(m, n)$ tensor field an element from dual to the space of objects given by the tensor multiplication of $(m, n)$ tensor fields and smooth compact supported $k$-form fields (on $k$ dimensional space), hence

$$T_{\nu_1 \ldots \nu_k}^{\mu_1 \ldots \alpha} \times \omega_g \rightarrow \int (A_{\mu_1 \ldots \alpha}^{\nu_1 \ldots \beta} \cdot T_{\nu_1 \ldots \nu_k}^{\mu_1 \ldots \alpha} \omega_g)$$

This is very much the same as to say that the test space are smooth compactly supported tensor densities $T_{\nu_1 \ldots \nu_k}^{\mu_1 \ldots \alpha}$. This is referring to the paper [10] (and it is mentioned and used in [9] as well). We have to mention here, that the topology taken on that space is the usual topology of uniform convergence for arbitrary derivatives related to arbitrary chart (so the old property should be valid in all charts). In [10] the authors add also the definition of the derivative
operator acting on the given space\(^5\). Some simplified scalar versions use our first definition equivalent to Geroch and Traschen, but they use the concept of Lie derivative instead of covariant derivative, which does make sense, since:

a) to use derivatives of distributions we automatically need derivatives along vector fields (but this condition is not very important)

b) Lie derivative is a map from \(p\)-forms to \(p\)-forms

c) in case of Lie derivatives, we do not need any additional structure (as connection in case of covariant derivative)

This is due to the paper of Balasin \[1\]. There is supposedly an equivalent formulation to Geroch and Traschen \[10\], given by Vickers and Steinbauer \[31\], which says that it can be taken as \(D'(M) \otimes T^m_n(M)\), where \(D'(M)\) is the dual to the space of smooth, compactly supported \(k\)-form fields (\(k\) dimensional space), or in other words the space of sections with distributional coefficients. The authors use Lie derivatives as a derivative operator, presumably for the reasons I have given. More similar formulations to Vickers and Steinbauer \[31\] can be found in \[18\]. The authors just generalize the whole construction and use the formulation of the dual to the space of compactly supported sections of the bundle \(E^* \otimes Vol^{1-q}\), where \(Vol^{1-q}\) is a space of \((1-q)\)-densities and \(E^*\) is a dual to a tensor bundle \(E\) (hence there belong objects such as \(E \otimes Vol^p\)). As a differential operator on that algebra is taken again the Lie derivative operator along a smooth vector field. We can refer also to another formulation, which comes from the old book of deRham \[8\] (it uses the expression “current”). It takes the space of distributions as a dual space to space of all compactly supported form fields. It is clear that tensors having range higher than \(n\) (\(n\) is the dimension of space) cannot be obtained directly in this way, but can be given by tensor (current) multiplication.

Let us mention here, that there is one thing we consider to be unsatisfactory on these constructions and this is the fact that to do physics we need much more to incorporate the concept of covariant derivative, than the concept of the Lie derivative.

Now, let us move back to the Colombeau algebras:

\(^5\)It is quite unclear considering the way Geroch and Traschen \[10\] incorporate the derivative operator, by using an intuitive definition:

\[
(\nabla_m T^\mu...\nu_{\alpha...\beta}, p^m_{\mu...\nu}) \doteq -\left(T^\mu...\nu_{\alpha...\beta}, \nabla_m p^m_{\mu...\nu}\right),
\]

what is the meaning of that derivative operator on a tensor density. Is it a mapping to a different tensor density and being relative to some fixed coordinates? If the expression on the other side is not a tensor density, what is its meaning? There are no explicit answers to this questions in Geroch and Traschen \[10\].
It is easy to formulate the first noncanonical construction of Colombeau algebras on general manifolds in covariant form; you obtain a space of \( \epsilon \)-sequences of functions on general manifold \( M \). Note that in the noncanonical case you only substitute for \( \mathbb{R}^n \) derivative the concept of Lie derivative, while keeping the rest of the definitions basically the same. Thus the definition of the Colombeau algebra will be again:

\[
\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)
\]  

(19)

where \( \mathcal{E}_M(\Omega) \) (moderate functions) is defined as algebra of functions:

\[
\Omega \times (0, 1] \to \mathbb{R}
\]  

(20)

such that are smooth on \( \Omega \) (this is usually called \( \mathcal{E}(\Omega) \)) and for any \( K \subset\subset \Omega \) holds that:

\[
\forall \alpha \in \mathbb{N}_0^n, \exists p \in \mathbb{N} \text{ such that } \forall \xi_1...\xi_k \text{ which are smooth vector fields} \text{ holds: } \sup_{x \in K} |L_{\xi_1}...L_{\xi_k}f_\epsilon(x)| \leq O(\epsilon^{-p}) \text{ as } \epsilon \to 0.
\]  

(21)

Negligible functions we define exactly in this analogy to our first definiton\(^6\).

After having this definition, it is easy to define the generalized Colombeau tensor algebra by the tensor product of sections of a tensor bundle and Colombeau algebra. This is equivalent to the usual formulations (see \([18]\)), which are more based on a generalization of mappings, not just from \( M \) to \( \mathbb{R}^n \), but to arbitrary manifold. To achieve such formulation you have to change in the usual definitions the absolute value to the expression “any Riemann measure on the target space” and then you get the algebra

\[
\Gamma_C(X, Y) = \Gamma_M(X, Y)/\mathcal{N}(X, Y)
\]  

(22)

as in \([22], [18], [31]\). Then we can easily introduce a map to a \( TM \) manifold which means generalized tensor fields on \( M \). The question remains how should be distributions embedded into those algebras. Now it is clear this will be noncanonical embedding from various reasons. First, it is noncanonical

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\(^6\)This is due to Heinzle and Steinbauer \([15]\). There are also different definitions: in a paper of Kunzinger and Steinbauer \([18]\), the authors use instead of “for every number of Lie derivatives along all the possible smooth vector fields” the expression “for every linear differential operator”, but they prove that these definitions are equivalent. What is also interesting to mention, according to their paper, this is equivalent to the statement saying that for any chart holds: \( \Phi \in \mathcal{E}_M(\mathbb{R}^n) \), at that chart (see \([18]\)).

\(^7\)As Kunzinger and Steinbauer say, this is equivalent to \( \mathcal{G}(X) \otimes \Gamma(X, E) \) tensor valued Colombeau generalized functions \([22]\).
even on $\mathbb{R}^n$. But another, second reason is that this embedding depends on the preferred class of charts on $M$. The embedding is defined in [18] as: we pick an atlas and take a smooth partition of unity subordinate to $V_{\alpha_j}$, supp$(\Theta) \subseteq V_{\alpha_j}$, $j \in \mathbb{N}$ and we choose for every $j$, $\xi_j \in D(V_{\alpha_j})$, such that $\xi_j = 1$ on supp$(\xi_j)$, then we can fix in fixed charts an $A^\infty(\mathbb{R}^n)$ element $\rho$ and the embedding is given by

$$\sum_{j=1}^{\infty} \left( ((\xi_j(\Theta_j \circ \psi_{\alpha_j}^{-1})u_{\alpha_j}) \ast \rho) \circ \psi_{\alpha_j} \right)_\epsilon,$$

where $\Psi_{\alpha_j}$ is a coordinate mapping and $\ast$ is a convolution.

Now let us here define the equivalence relation in the analogy to the $\mathbb{R}^n$ case. Since in [18] the authors take the strongest constraint on the mollifier, one would expect that strong results will be obtained, but the definition is more complicated. And in fact, standard results (such as embedding of smooth function multiplying distribution is equivalent to multiplication of their embeddings) are not valid here [18]. That is why the authors formulate stronger concept of $k$-association, saying that $U \in \Gamma_C$ is $k$ associated to function $f$, if

$$\lim_{\epsilon \to 0} L_{\xi_1} \ldots L_{\xi_l}(U_\epsilon - f) \to 0$$

uniformly on compact sets for all $l \leq k$. The cited paper does not contain definition of $k$ equivalence between two generalized functions, but it is quite easy to see, that the given definitions can be easily adapted for this case.

Now all this concepts can be easily generalized and adapted for arbitrary generalized tensor fields, understood in the way as defined before.

Now if we want to get a canonical formulation, we certainly cannot generalize it straight from the $\mathbb{R}^n$ case (the reason is that the definition of $A^n(\mathbb{R}^n)$ classes is not diffeomorphism invariant). There is an approach, which provides us with a canonical formulation of generalized scalar fields taken in [12]. The authors define the space $\mathcal{E}(M)$ as a space of $C^\infty(M \times A^0(M))$, where $A^0(M)$ is the space of n-forms (n-dimensional space), such that $\int \omega = 1$. Now the authors define a smoothing kernel as $C^\infty$ map from:

$$M \times I \to A^0(M)$$

such that it satisfies:

(i) $\forall K \Subset M \exists \epsilon_0, C > 0 \forall p \in K \forall \epsilon \leq \epsilon_0$, supp $\phi(\epsilon, p) \subseteq B_C(p)$,

(ii) $\forall K \Subset M, \forall k, l \in \mathbb{N}_0, \forall X_1, \ldots X_k, Y_1, \ldots Y_l$ smooth vector fields, $\sup_{p \in K, q \in M} \|L_{Y_1} \ldots L_{Y_l}(L_{X_1} \ldots L_{X_k}) \cdots (L_{X_k} \ldots L_{X_1}) \Phi(\epsilon, p)(q)\| = O(\epsilon^{-(n+1)})$
where $L'$ is defined as:

$$L'_X f(p, q) = L_X (p \rightarrow f(p, q)) = \frac{d}{dt} f((F^t_L)(p), q)|_0$$

(26)

and $B_{eC}$ is a ball centered at $C$ having radius $\epsilon$ measured relatively to a Riemannian metric. If (i) holds for one Riemannian metric, then it holds for any Riemannian metric, so although this definition seems to be related to a particular Riemannian metric, it is not. Let us call the class of such smoothing kernels $A^0(M)$. Then the authors define classes $A^m(M)$ as the set of all $\Phi \in A^0(M)$ such that $\forall f \in C^\infty(M)$ and $\forall K \subset M$ (compact subset) holds:

$$\sup_{p \in K} \| f(p) - \int_M f(q) \Phi(\epsilon, p)(q) \| = O(\epsilon^{m+1})$$

(27)

Then one defines the moderate and the negligible functions in the following way:

$R \in \mathcal{E}(M)$ is moderate if $\forall K \subset M \ \forall k \in \mathbb{N}_0 \ \exists N \in \mathbb{N} \ \forall X_1, ..., X_k$ smooth vector fields $\forall \Phi \in A^0(M)$ holds

$$\sup_{p \in K} \| L_{X_1} ... L_{X_k} (R(\Phi(\epsilon, p), p)) \| = O(\epsilon^{-N}),$$

(28)

and $R \in \mathcal{E}(M)$ is negligible if $\forall K \subset M, \forall k, l \in \mathbb{N}_0 \ \exists m \in \mathbb{N} \ \forall X_1, ..., X_k$ smooth vector fields, $\forall \Phi \in A_m(M)$ holds:

$$\sup_{p \in K} \| L_{X_1} ... L_{X_k} (R(\Phi(\epsilon, p), p)) \| = O(\epsilon^l).$$

(29)

Now we can define the Colombeau algebra in the usual way as a factor algebra of moderate functions over negligible functions. Distributions being defined as a dual to $n$-forms can be embedded into this algebra in a complete analogy to the canonical $\mathbb{R}^n$ formulation. Also association is in this case defined in the “usual” way (integral with compact supported smooth $n$-form field). As expected, this formulation makes no exception in the fact that association reproduces both: the multiplication of a distribution by a smooth function and the classical multiplication of continuous functions.

However attempting a straightforward generalization from scalars to tensors brings immediate problems, since the embedding does not commute with the diffeomorphism action. The problem was finally resolved in [20]. The authors realized that diffeomorphism invariance can be achieved by adding some background structure defining how tensors transport from point to point, hence a transport operator. Colombeau $(m, n)$ rank tensors are then
taken from the class of smooth maps $C^\infty(\omega, q, A)$ having values in $(T^m_q)_q M$, where $\omega \in A_0(M)$, $q \in M$ and $A$ is from the class of compactly supported transport operators. After defining the concept of Lie derivative on those objects and the concept of the core of a transport operator, the authors define in a slightly complicated analogy to the previous case the moderate and the negligible tensor fields, and by factorization they obtain the canonical version of the generalized tensor fields (for more details see [20]). The canonical embedding of tensorial distributions is the following:

The smooth tensorial objects are embedded as

$$\tilde{t}(p, \omega, A) = \int t(q) A(p, q) \omega(q) dq$$

where as expected $\omega \in A_0(M)$, $t$ is the smooth tensor field and $A$ is the transport operator.

After that arbitrary tensorial distribution $s$ is embedded (to $\tilde{s}$) by the condition

$$\tilde{s}(\omega, p, A) \cdot t(p) = (s, A(p, ) \cdot t(p) \otimes \omega(,))$$

where on the left side we are contracting the embedded object with a smooth tensor field $t$, and on the right side we are applying the given tensorial distribution $s$ in the variable assigned by the dot.

The authors show in [20] that this embedding fulfills all the important properties, for example commutes with the Lie derivative operator. All the other results related to equivalence relation (etc.) are obtained in the analogy to the previous cases.

In [22], [21] and [18] the given authors generalized all the basic geometric structures, like connection, covariant derivative, curvature, or geodesics into the geometric formulation of the special Colombeau algebra. Hence they defined the whole generalized geometry. However in our view, the crucial part is missing. What we would like to see, is an intuitive and clear definition of the covariant derivative operator acting on the distributional objects in the canonical Colombeau algebra formulation, on one hand reproducing all the classical results and on the other hand extending them in the same natural way as it is in the classical distributional theory with the classical derivative operator. Whether there is any way how to achieve this goal by the concept of generalized covariant derivative acting on the generalized tensor fields as being defined in [22], or [24] is unclear. And if this generalized geometry can be formulated also within the canonical Colombeau algebra construction is also not immediately clear.
There cannot be any hope of finding a more appropriate, generalized formulation of physics within this framework without finding such clear and intuitive definition of the covariant derivative; all that can be in this situation achieved is to use these constructions to solve some specific problems within the area of physics. But as we see, such more ambitious goal can be very naturally achieved by our own construction, which follows after this overview.

2.3 Practical application of those results

Now we can quickly pass through various applications of the theoretical background already presented. The first application we can possibly mention is the nongeneral relativistic one. In [6] the authors provide us with weak solutions of nonlinear partial differential equations (using Colombeau algebra) representing shock waves. They use explicitly the relation

\[ H^n H' \approx \frac{1}{n+1} H' \]

(32)

using the special version of Colombeau algebra (related to mollifiers from \( A^0(\mathbb{R}^n) \) class). This types of equations are analysed by Obeguggenberger in his paper [26], giving us some mathematical results concerning existence and uniqueness of weak solutions of nonlinear partial differential equations.

In general relativity we have various results using Colombeau algebras. First, we can focus on the distributional Schwarzschild geometry, which is analysed for example in [15]. The authors start to work in Schwarzschild coordinates using the special Colombeau algebra and \( A^\infty(\mathbb{R}^n) \) classes of mollifiers. They obtain the delta-functional results (as expected) for the Einstein tensor, and hence stress-energy tensor. But in Schwarzschild coordinates they got into troubles with the given embedding of their distributional tensors, since this coordinates do not contain the 0 point. As a result, if we look for smooth embeddings, we do not obtain an inverse element in Colombeau algebra on the surroundings of 0 for values of \( \epsilon \) going to 0 (but if we require that the inverse relation should apply only in the sense of equivalence, it makes no real problem). But the authors (see [15]) are able to come to significant results by turning to Eddington-Finkelstein coordinates and getting metric in a Kerr-Schild form, in which they are able to compute \( R_{ab}, G_{ab} \) and hence \( T_{ab} \) (in their (1,1) form, because their dependence on \( g^{\mu\nu} \) has in these coordinates relatively simple form) as delta-functional objects (which is expected). This result is not depending on the mollifier (see also [31]), but the relation between different embeddings given by the different coordinate systems is not really analysed there [3].

\[ \text{It seems that the authors use relations between } R_{\mu\nu}, \text{ and components of } g^{\mu\nu} \text{ obtained} \]
Even in the case of Kerr geometry there is a computation of $\sqrt{(g')^\mu_\nu R^\mu_\nu}$ (where $g^{ab} = g'^{ab} + fk^a k^b$) done by Balasin, but this is mollifier dependent (see [2], or [31]). The coordinate dependence of the results is even more unclear here.

There exists a ultrarelativistic weak limit of the Schwarzschild metric (turning from Eddington, Finkelstein coordinates to $u = t + r$, $v = t - r$ coordinates and taking boost with the weak $v \to c$ limit). We obtain the “delta functional” Aichelburg metric. Steinbauer and Kunzinger provided us with a computation of geodesics in that geometry [23]. The authors take the special Colombeau algebra (and take $A^0(\mathbb{R}^n)$ as their class of mollifiers), and they prove that geodesics are given by the refracted lines (results are mollifier independent). This is again expected. Moreover, what seems to be really interesting is that there is a continuous metric which is connected with the Aichelburg metric by a generalized coordinate transform (see [17], [31]).

The last thing we want to mention are the conical spacetimes. They are analysed for example in an old paper from Geroch and Traschen [10]. The authors define the $gt$-metrics first. These are metrics which provide us with a distributional Ricci tensor in a very naive sense, where the multiplication is given just by simple multiplication of functions giving us the regular distributions. But the conical spacetime metric does not belong to this case. There was a calculation provided by Clarke, Vickers and Wilson [31] giving us the stress energy tensor, but this is mollifier dependent (they are coordinate independent though [31]).

in that coordinates by using algebraic tensor computations, but it is not obvious, whether these results can be obtained by computations in the Colombeau algebra using $\approx$ relation, since some simple tricks cannot be used here in the general cases (such as substitution).
3 The basic concepts

Before we would like to say anything about distributional tensor fields, we have to define the basic concepts, which will be used in all the following mathematical constructions. This task is provided in this section, hence this part is crucially important for understanding all the next theory. An attentive reader, having read through the introduction and abstract, will understand, why we are particularly interested in defining these concepts.

3.1 Definitions

3.1.1 Definition of (M,A)

Definition 3.1. By piecewise smooth function we mean a function from an open set $A \subseteq \mathbb{R}^4 \rightarrow \mathbb{R}^m$ such, that there exists an open set $B$ (in the usual “open ball” topology on $\mathbb{R}^4$) on which this function is smooth and $B = A \setminus N$ (where $N$ has a Lebesgue measure 0).

Take $M$ as a 4D paracompact, Hausdorff locally Euclidean space, on which there exists a smooth atlas $A'$; hence $(M, A')$ is a smooth manifold. Now take a ordered couple $(M, A)$, where $A$ is the biggest possible atlas, where all the maps are connected by piecewise smooth transformations such that:

1) the transformations and their inverses have on every compact subset of $\mathbb{R}^4$ all the first derivatives (on the domains where they exist) bounded (hence Jacobians, inverse Jacobians are on every compact set bounded)

2) it contains at least one biggest smooth subatlas $A' \subseteq A$ (coordinate transformations between maps are smooth there).

Notation:
Every subset of $M$ on which exists a chart from our atlas $A$, we call $\Omega_{Ch}$. Arbitrary chart on $\Omega_{Ch}$ from our atlas $A$ we call $Ch(\Omega_{Ch})$.

Definition 3.2. Take some set $\Omega_{Ch}$. Take some open subset of that set $\Omega' \subset \Omega_{Ch}$. Then $Ch(\Omega_{Ch})_{\Omega'}$ means the $\Omega'$ boundary of a chart $Ch(\Omega_{Ch})$ and is defined simply as $Ch(\Omega')$ (which means we “ignore” the area outside $\Omega'$).

Note that the letter $A'$ will be from now automatically used for some smooth subatlas of $A$. 

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Definition 3.3. We define by $L$-measure on $M$ (on Borel sets obviously) a measure which can be on every $\Omega_{Ch}$ expressed in every chart as integral from some piecewise continuous function which is everywhere nonzero, except of a set having Lebesgue measure 0 (at that chart).

3.1.2 Jacobians and algebraic operations with Jacobians

Now it is obvious that since transformations between maps do not have to be everywhere once differentiable, Jacobian and inverse Jacobian can be always undefined on the set having Lebesgue measure 0. Now consider, that for example the relation $J_{\alpha}^\mu (J^{-1})_{\nu}^\beta = \delta^\mu_{\nu}$ might hold even at the points where both Jacobian and inverse Jacobian are undefined, when we understand this multiplication in the sense of limit. This would generally mean that if we can define these expressions on the places where they are technically undefined (this set is having Lebesgue measure 0) by values which make them continuous at those places (this gives a unique specification of those values, if they exist) we understand them as having those values there. From this follows that the limit of the matrix product should be at $\mu = \nu$ equal to 1 and at $\mu \neq \nu$ 0.

3.1.3 Tensor fields on $M$

Then we understand by the tensor field on $M$ an object which is:

1) defined relatively to the 1-differentiable subatlas of $A$ everywhere except of a set having $L$-measure 0 (so this set is a function of the given 1-differentiable subatlas),

2) if it is undefined at some 1-differentiable subatlas of $A$ at some set (having $L$-measure 0), it means that if we express them in arbitrary chart from that subatlas, there is no way how to attribute it at this set such values, that the expressions would be at that set continuous,

3) it transforms between charts in the old prescribed tensorial way

\[ T^{\mu_{\ldots}}_{\nu_{\ldots}} = J_{\alpha}^\mu (J^{-1})_{\nu}^\beta \cdots T_{\beta_{\ldots}}^\mu_{\ldots} \]  

(33)

again by taking into account what was previously said: even at the points where the expression $T^{\mu_{\ldots}}_{\nu_{\ldots}} = J_{\alpha}^\mu (J^{-1})_{\nu}^\beta \cdots T_{\beta_{\ldots}}^\mu_{\ldots}$ is technically undefined, if there exist values by which it can be on this set defined and continuous, we understand it as having those values there \[10\].

\[10\] As was already mentioned, because of the uniqueness of these values (if they exist)
3.1.4 Important classes of test objects

Notation:
Take a class of such 4-form fields defined in every chart from $A$ on the set $M \setminus S$ ($S$ having any L-measure 0), which are compact supported and their support lies within some $\Omega_{Ch}$, call it $C^p(M)$.

By $C^p(\Omega_{Ch})$ call a subclass of $C^p(M)$, such that for these 4-forms it holds, that their support is a subset of the given $\Omega_{Ch}$. Note that only the $C^p(\Omega_{Ch})$ subclasses form linear spaces.

Such subclass of those 4-forms, that there exists a subatlas $\tilde{A} \subset A$ ($\tilde{A}$ being a subatlas containing at least one biggest smooth subatlas $A'$) in which they are expressed by densities, being in every chart from $\tilde{A}$ everywhere defined and smooth, we call $C^p_S(M)$.

Under $C^p_{S(\tilde{A})}(M)$ we mean a subclass of $C^p_S(M)$ such that these are 4-forms being in every chart from this particular $\tilde{A}$ expressed by everywhere smooth density and this density is nonsmooth in every chart outside $\tilde{A}$.

$C^p_{S(\tilde{A})}(\Omega_{Ch})$ means $C^p_{S(\tilde{A})}(M)$ element having support in the given $\Omega_{Ch}$.

3.1.5 Topology on $C^p_{S(\tilde{A})}(\Omega_{Ch})$

Take a following topology on each $C^p_{S(\tilde{A})}(\Omega_{Ch})$: a sequence from $\Phi_n \in C^p_{S(\tilde{A})}(\Omega_{Ch})$ converges to an element $\Phi$ from that set if all the supports of $\Phi_n$ lie in a single compact set and in arbitrary chart $Ch(\Omega_{Ch}) \in \tilde{A}$ for derivative of arbitrary degree $k$ taken at that chart it holds, that $\partial_k \Phi_n'$ converges uniformly (at that chart) to $\partial_k \Phi'$; where $\Phi_n'$ and $\Phi'$ are densities belonging to that forms expressed at that chart.

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this definition makes sense. If for example $T_{\mu...}^{\nu...}$ is at some given point undefined and the functions expressing this tensor field in some chart $Ch_1$ do not have a defined limit, this limit can still exist in $Ch_2$, since Jacobians and inverse Jacobians of the transformation from $Ch_1$ to $Ch_2$ might be undefined at that point as well.

From now the letter $\tilde{A}$ everywhere in the text automatically means such subatlas.
4 Scalars

This section deals with the definition of scalar as the easiest particular example of generalized tensor field. This is the reason why we deal with scalars separately, instead of taking more “logical”, straightforward way to tensor fields of arbitrary rank.

4.1 Definitions

4.1.1 Definition of $D'_A(M)$

Notation:
Now take the space of such linear maps $\Psi (\in S) \to \mathbb{R}$, when $S$ is such that $\exists \tilde{A}$ such that $C^p_{S(\tilde{A})}(M) \subseteq S \subseteq C^p(M)$ and these linear maps should for every $\Omega_{Ch}$ give on $C^p_{S(\tilde{A})}(\Omega_{Ch})$ subset of $C^p_{S(\tilde{A})}(M)$ a continuous mapping (relatively to the topology taken in 3.1.5). Call this set $D'(M)$.

Notation:
Now take subsets of $D'(M)$ such that they have some common set $\cup_n C^p_{S(\tilde{A}_n)}(M)$ belonging to their domains, and are $\forall \Omega_{Ch}, \forall n$, continuous on $C^p_{S(\tilde{A}_n)}(\Omega_{Ch})$ (when we say “continuous” on some set, we mean obviously also fully defined there). Call such subsets $D'_{(\cup_n \tilde{A}_n)}(M)$.

By $D'_{(\cup_n \tilde{A}_n)}(M)$ we mean such objects that belong to $D'_{(\cup_n \tilde{A}_n)}(M)$ and are defined only on $\cup_n C^p_{S(\tilde{A}_n)}(M)$ (so this class is their full domain).

Obviously $T \in D'_{(\cup_n \tilde{A}_n)}(M)$ means $T \in \cap_n D'_{(\tilde{A}_n)}(M)$.

Notation:
For arbitrary element of $D'_{(\tilde{A})}(M)$, construct an algebra $D'_{(\tilde{A})}A(M)$ from its elements by pointwise multiplication and linear combination of elements from $D'_{(\tilde{A})}(M)$.

Pointwise multiplication gives in that case mapping from $\Psi$ into multiplication of the images (real numbers) of those maps: $(A \cdot B, \psi) = (A, \psi) \cdot (B, \psi)$ and the domain on which the product (and the linear combination as well) is defined is an intersection of domains of $A$ and $B$ (trivially always nonvoid, containing $C^p_{S(\tilde{A})}(M)$ at least).

\textsuperscript{12}The meaning of linear here is such, that whenever the map is defined on some $\Psi_1, \Psi_2$ (both belonging to one class $C^p(\Omega_{Ch})$), $\lambda_1 \Psi_1, \lambda_2 \Psi_2$ and $\lambda_1 \Psi_1 + \lambda_2 \Psi_2$, then there holds the usual relation of linearity.

\textsuperscript{13}The union means here just trivially taking only one element $\tilde{A}$.
Alternatively we can say that the set $D'_{(\tilde{\mathcal{A}})A}(M)$ can be understood as set of multivariable arbitrary degree polynomials (where different “variables” are different $D'_{(\tilde{\mathcal{A}})}(M)$ objects).

Note also that the resulting arbitrary element from $D'_{(\tilde{\mathcal{A}})A}(M)$ has in general all the properties defining $D'(M)$ objects except of being necessarily linear.

**Definition 4.1.** The set of objects obtained by union $\cup_{\tilde{\mathcal{A}}}D'_{(\tilde{\mathcal{A}})A}(M)$ we denote $D'_{\tilde{\mathcal{A}}}(M)$, and call them generalized scalar fields (GSF).

### 4.1.2 Important subclasses of $D'(M)$ and $D'_{\tilde{\mathcal{A}}}(M)$

**Notation:**

Now take a subset of $D'(M)$ such that it can be expressed as an integral from a piecewise smooth function defined on the subset of $C^P(M)$, where this integral converges and there exists a smooth atlas $A'$, where the function is smooth and call this class $D'_S(M)$.

The larger class of such $D'(M)$, such that it is expressible by integral from piecewise continuous function (again everywhere on $C^P(M)$, where it converges) is denoted as $D'_E(M)$.

If we use the notation $D'_{E(\cup_{n} \tilde{\mathcal{A}}n)}(M)$, we mean $D'(M)$ objects defined only on $\cup_{n}C^P_{S(\tilde{\mathcal{A}}n)}(M)$ and on those 4-forms being expressible by integral of piecewise continuous function.

By using $D'_{E(\cup_{n} \tilde{\mathcal{A}}n)}(M)$ we automatically mean that it is such subclass of $D'_{E(\cup_{n} \tilde{\mathcal{A}}n)}(M)$, that it is expressible by an integral from a smooth function at some smooth subatlas $A' \subseteq \cup_{n} \tilde{\mathcal{A}}n$.

### 4.1.3 Topology on $D'_{(\tilde{\mathcal{A}}o)}(M)$

If we take objects from $D'_{(\tilde{\mathcal{A}}o)}(M)$ and we take a weak ($\sigma-$, but from now we call it only weak) topology on that set, we know that any object is a limit of some sequence from $D'_{S(\tilde{\mathcal{A}}o)}(M)$ objects from that set (they form a dense subset of that set). That is what we know from the classical theory. Such space is complete.

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14Actually it holds that if and only if the function is integrable in every chart on every compact set in $\mathbb{R}^n$, than integral from this function defines an $D'(M)$ object and is defined at least on the whole $C^P_S(M)$ class.
5 Generalized tensor fields

This section is of crucial importance. It provides us with definitions of all the basic objects we are interested in, hence the generalized tensor fields and all their subclasses of special importance as well.

5.1 Definitions

5.1.1 $D_n^m(M)$ class

First let us clearly state how to interpret $J_{\mu}^\nu$ symbol in all the following definitions. It is a matrix from piecewise smooth functions $S \setminus N \to \mathbb{R}$, $S$ being a open set from $\mathbb{R}^4$ and $N$ having Lebesgue measure 0. But we can map them by the inverse of the first coordinate mapping to $\Omega_{Ch}$ and it will be a matrix of functions $\Omega_{Ch} \setminus K \to \mathbb{R}$, $K$ having an L-measure 0. $J_{\mu}^\nu \Psi$ is then understood as a matrix of 4-forms from $C^P(\Omega_{Ch})$, in other words that means that outside $\Omega_{Ch}$ we trivially define them to be 0.

Definition 5.1. By a $D_n^m(M)$ object we mean a linear mapping from $V \to \mathbb{R}^{4m+n}$, where $S \subseteq V \subseteq S'$ and $S$ is such set that $\exists \tilde{A}$ and $\exists A' \subseteq \tilde{A}$ ($A'$ is some biggest smooth atlas), such that:

$S$ is given as:

$$\bigcup_{\Omega_{Ch}} \left\{ (\Psi, Ch(\Omega_{Ch})) : \Psi \in C^P_{S(\tilde{A})}(\Omega_{Ch}), Ch(\Omega_{Ch}) \in A' \right\}$$

$S'$ is given as:

$$\bigcup_{\Omega_{Ch}} \left\{ (\Psi, Ch(\Omega_{Ch})) : \Psi \in C^P(\Omega_{Ch}), Ch(\Omega_{Ch}) \in A \right\}$$

and:

1) $\forall \Omega_{Ch}$ it is in every chart from $A' \subset \tilde{A}$ continuous on that $C^P_{S(\tilde{A})}(\Omega_{Ch})$ objects (appearing in $S$ class).

2) This map also transforms with charts as:

$$A_{\mu...\beta}(Ch_1, \Psi) = A_{\alpha...\lambda}(Ch_2, J^\mu_{\alpha}...J^\gamma_{\lambda}(J^{-1})_{\nu}...,(J^{-1})_{\delta}^\nu \Psi)$$

We could also choose for our basic objects maps taking ordered couples from $C^P(\Omega_{Ch}) \times Ch(\Omega_{Ch}')$ ($\Omega_{Ch} \neq \Omega_{Ch}'$) as well. The linearity condition then automatically determines their values, since for $\Psi \in C^P(\Omega_{Ch})$, whenever it holds that $\Omega_{Ch} \cap \text{supp} \Psi = \{0\}$, they must automatically give 0 for any chart argument. Hence these two definitions are trivially connected and choice between them is just purely formal (only a matter of “taste”).

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3) The following consistency condition holds: If $\tilde{\Omega}_{Ch} \subset \Omega_{Ch}$, then $A^{\mu_1...\mu_n\nu_1...\nu_m}$ gives
$\Psi \times Ch(\Omega_{Ch}) |_{\tilde{\Omega}_{Ch}} \Psi \in C^P(\tilde{\Omega}_{Ch})$ the same results\(^{16}\) as than on
$\Psi \times Ch(\Omega_{Ch}) \Psi \in C^P(\Omega_{Ch})$.

Now we can formally extend this notation also for the case $m = n = 0$. This would mean scalars, exactly as defined before. So from now on $m, n$ take also the 0 value, which means the theory in the following sections holds also for the scalar objects, which also means it contains the results of the previous section as its specific case.

5.1.2 Important subclasses of $D^{m,n}_{\pi}(M)$ classes

**Definition 5.2.** Completely by analogy to scalars we define classes $D^{m,n}_{\pi E}(M)$: on arbitrary $\Omega_{Ch}$, being fixed in arbitrary chart $Ch_1(\Omega_{Ch}) \in A$, we can express it in arbitrary chart $Ch_2(\Omega_{Ch}) \in A$ as an integral from a multi-dimensional matrix of piecewise continuous functions on the subset from $C^P(M)$, where the integral is defined\(^{17}\).

Analogically class $D^{m,n}_{\pi S}(M) \subset D^{m,n}_{\pi E}(M)$ is such, that those objects can be at some biggest smooth atlas $A'$ and arbitrary charts\(^{16}\) $Ch_1(\Omega_{Ch}), Ch_2(\Omega_{Ch}) \in A'$ expressed by an integral from a multi-index matrix of smooth functions.

$D^{m,n}_{\pi (\cup_{\tilde{A}_l}At(A_l))}(M)$ (where $At(\cdot)$ stands for some subatlas of $A$ being a function of some argument, expressed by $(\cdot)$) means a class of objects being for every $\Omega_{Ch}$ continuous in every $Ch(\Omega_{Ch}) \in At(\tilde{A}_l)$ on $C^P_{S(\tilde{A}_l)}(\Omega_{Ch})$.

$D^{m,n}_{\pi (\cup_{\tilde{A}_l}At(A_l))}(M)$ means class of objects from $D^{m,n}_{\pi (\cup_{\tilde{A}_l}At(A_l))}(M)$ being defined only on the $\cup_{\Omega_{Ch}} \cup_l \{ (\Psi, Ch(\Omega_{Ch})) : \Psi \in C^P_{S(\tilde{A}_l)}(\Omega_{Ch}), Ch(\Omega_{Ch}) \in At(\tilde{A}_l) \}$ domain.

If we have $D^{m,n}_{\pi (\cup_{\tilde{A}_l}At(A_l))}(M), D^{m,n}_{\pi (\cup_{\tilde{A}_l}At(A_l))}(M)$ classes where $At(\tilde{A}_l) = A' \subset \tilde{A}_l$, we use simple $D^{m,n}_{\pi (\cup_{\tilde{A}_l}A_l)}(M), D^{m,n}_{\pi (\cup_{\tilde{A}_l}A_l)}(M)$ notation\(^{19}\).

\(^{16}\)By the results we mean defined/undefined and when defined, than having the same value.

\(^{17}\)Actually we will use the expression “multi-index matrix” also later in the text and it means nothing more than just specifically ordered set of functions.

\(^{18}\)The first chart is an argument of this generalized tensor field and the second chart is the one in which we express the given integral.

\(^{19}\) We have to realize that the subatlas $A'_{\pi}$ specifies completely the atlas $\tilde{A}_{\pi}$, since taking forms smooth in $A'_{\pi}$ determines automatically the whole set of charts in which they are still smooth. This fact contributes to the simplicity of this notation.
5.1.3 Definition of $D^m_{nA}(M)$, hence generalized tensor fields

Definition 5.3. Now define $D^m_{n(A')}A(M)$ to be the algebra constructed from $D^m_{n(A')}\{M\}$ objects by tensor multiplication, exactly in the analogy to the case of scalars (reduces for scalars to multiplication already defined). The object being a result of the tensor multiplication is again a mapping $V \to \mathbb{R}^{m+n}$ and we define it to be such, that runs the indexes of the object being multiplied first and of the object which multiplies second.

Now call by $D^m_{nA}(M)$ a set given as $\bigcup_{A'}D^m_{n(A')}A(M)$, meaning a union of all possible $D^m_{n(A')}A(M)$. Call the objects belonging to this set the generalized tensor fields (GTF).

Definition 5.4. Furthermore if we use the same procedure than in the previous definition, just instead of constructing the algebras from $D^m_{n(A')}\{M\}$ objects, we construct them only from $D^m_{n(\cup_{i=1}l)(\tilde{A}_i)}\{M\}\cup D^m_{n(\cup_{i=1}l)(\tilde{A}_i)}\{M\}$ objects (again by tensor multiplication), we use for the resulting union the $D^m_{n(\cup_{i=1}l)(\tilde{A}_i)}A(M)$ sign.

5.1.4 Definition of $\Gamma$–objects, their classes and algebras

Notation:

Now let us define the generalized space of objects $\Gamma^l(M)$. These objects are defined exactly in the same way than $D^m_n(M)$ ($m+n=l$) objects, we just do not require that they should transform from chart to chart in the given prescribed way, as it is given in the definition of generalized vector/tensors (point 2).\footnote{The definition of $\Gamma^l(M)$ includes also the case $m = 0$. Now we see, that the scalars can be taken as subclass of $\Gamma^0(M)$ being for every element from their domains defined at the whole $A$ as constants with respect to the chart argument (independent on charts). Note also that for a general $\Gamma^m(M)$ object there is no meaningful differentiation between “upper” and “lower” indices, but we will still use formally the $T^m_{\nu\ldots\nu}$ notation (for all cases).}

In the same way (just by not putting requirements on the transformation properties) we can simply generalize $D^m_{n(\cup_{i=1}l)(\tilde{A}_i)}\{M\}$, $D^m_{n(\cup_{i=1}l)(\tilde{A})_0}(M)$, $D^m_{n(\cup_{i=1}l)(\tilde{A})_0}(M)$, $D^m_{n(\cup_{i=1}l)(\tilde{A})_0}(M)$ and $D^m_{nA}(M)$ into $\Gamma^m_{(\cup_{i=1}l)(\tilde{A}_i)}(M)$, $\Gamma^m_{(\cup_{i=1}l)(\tilde{A})_0}(M)$, $\Gamma^m_{(\cup_{i=1}l)(\tilde{A}_0)}(M)$, $\Gamma^m_{(\cup_{i=1}l)(\tilde{A}_0)}(M)$, $\Gamma^m_{(\cup_{i=1}l)(\tilde{A}_0)}(M)$ and $\Gamma^m_{A}(M)$ classes.

It is obvious that all the latter classes contain all the former classes as their subclasses (this is a result of what we called a “generalization”).

Note that when we fix $\Gamma^m_{E}(M)$ objects in arbitrary chart from $A$, they must be expressed by integrals from multi-index matrix of on every compact set integrable functions. In the case of $D^m_{nE}(M)$ subclass it can be required
at only one chart, since the transformation properties together with boundedness of Jacobians and inverse Jacobians provide that it must hold in any other chart from $A$.

The specific subclass of $\Gamma^m_E(M)$ is $\Gamma^m_S(M)$, being a subclass of distributions, which is possible to express in any chart from $A$ (being an argument of the given $\Gamma$ object) by integrals from multi-index matrix of smooth functions (when we express them in the same chart, which is taken as the argument). If we use $\Gamma^m_S(\cup_n A'_n o) (M)$, this means that we just limit its chart domain to $\cup_{\Omega Ch} \cup_n \{(\Psi, Ch(\Omega_{Ch})) : \Psi \in C^p_{\Omega Ch} (\Omega_{Ch}), Ch(\Omega_{Ch}) \in A'_n \}$, where $\hat{A}_n$ is given by the condition $A'_n \subset \hat{A}_n$.

5.1.5 Topology on $\Gamma^m_{(\cup_n A'_n) o} (M)$

If we take the class of $\Gamma^m_{(\cup_n A'_n) o} (M)$ and we put on this class the weak (point or $\sigma$-) topology, then the subclass of those objects defined as $\Gamma^m_{S(\cup_n A'_n) o} (M)$, is dense in that class. The same holds for $D^m_{n(\cup_n A'_n) o} (M)$ and $D^m_{nS(\cup_n A'_n) o} (M)$.

5.1.6 Definition of contraction

**Definition 5.5.** The last thing we want to mention here is to define the contraction of a $\Gamma^m_A (M)$ object, $(C)$ in the obvious way: $C(A_{\mu...}^{\nu...}) = A_{\mu...}^{\nu...}$.

Now we have to realize, it is a mapping $D^m_{n} (M) \to D^{m-1}_{n} (M)$ and $\Gamma^m (M) \to \Gamma^{m-2} (M)$, but it is not in general the mapping $D^m_{nA} (M) \to D^{m-1}_{n-1A} (M)$, only $D^m_{nA} (M) \to \Gamma^{m+n-2} (M)$. 

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6 The relation of equivalence ($\approx$)

This part provides us with the fundamental concept of the theory, the concept of equivalence of generalized tensor fields. Most of the first part is devoted to fundamental definitions, the beginning of the second part deals with the basic, important theorems, which are just generalizations of Colombeau multiplication of distributions to tensor multiplication of generalized tensor fields and adds important conjectures as well. The first part ends with the subsection “additional definitions” and the second part with the subsection “additional theory”, both deal with much less important theoretical results, but they serve very well to put light on what equivalence of generalized tensor fields “physically” means.

6.1 Definitions

6.1.1 The necessary concepts to define the relation of equivalence

Notation:
Take some subatlas of our atlas, and this will be a biggest subatlas of charts, which are maps to the whole $\mathbb{R}^4$. Such maps exist on each set $\Omega_{Ch}$ and they will be denoted as $Ch'(\Omega_{Ch})$. We say that a chart $Ch'(\Omega_{Ch})$ is centered at the point $q \in \Omega_{Ch}$, if this point is mapped by this chart mapping to 0 (in $\mathbb{R}^4$). We will use the $Ch'(q, \Omega_{Ch})$ notation. Further we will work with those classes.

Definition 6.1. Take some $\Omega_{Ch}$, $q \in \Omega_{Ch}$ and $Ch'(q, \Omega_{Ch}) \in \tilde{A}$. The set of 4-forms $\phi(\epsilon, x) \in A^n(\tilde{A}, Ch'(q, \Omega_{Ch}))$ is defined in such way that $\phi_\epsilon \in C^P(\Omega_{Ch})$ belongs to this class if:

a) In the given $Ch'(q, \Omega_{Ch})$, $\forall \epsilon$ holds that:

$$\int_{\sigma(\Omega_{Ch})} \prod_i x_i^{k_i} \phi'(x, \epsilon) = \delta_{k0}, \quad \sum_i k_i = k, \quad k \leq n, \quad n \in \mathbb{N}$$

b) dependence on $\epsilon$ is given as $\epsilon^{-4}\phi'(\frac{x}{\epsilon})$ in that given chart.

Definition 6.2. Take arbitrary $q$, $\Omega_{Ch}$ ($q \in \Omega_{Ch}$), $Ch'(q, \Omega_{Ch}) \in \tilde{A}$ and natural number $n$, and for any $\phi_\epsilon \in A^n(Ch'(q, \Omega_{Ch}), \tilde{A})$ we can in the $Ch'(\Omega_{Ch})$ define a continuous set of maps (depending on the parameter $y$)

$$A^n(Ch'(q, \Omega_{Ch}), \tilde{A}) \rightarrow C^P(\Omega_{Ch}) \quad (34)$$

as:
a) on $\Omega_{\text{Ch}}$ and in $\text{Ch}'(\Omega_{\text{Ch}})$, it is given by: $\phi'(z)\epsilon^{-4} \to \phi'(\frac{x-y}{\epsilon})\epsilon^{-4}$ (again $\phi'$ being density expressing $\phi$ in this chart), which depending on the parameter $y \in \mathbb{R}^4$ gives us on $\Omega_{\text{Ch}}$ various 4-form fields (by inverse $\text{Ch}'(\Omega_{\text{Ch}})$ mapping a particular function $\Phi': B(\subseteq \mathbb{R}^4) \to \mathbb{R}$ gives us a 4-form field $\Phi$ on $\Omega_{\text{Ch}}$, defined by $\Phi'(x)dx \wedge \ldots \wedge dz$ at a given chart).

b) outside $\Omega_{\text{Ch}}$ it will be 0.

Call these 4-form fields $\phi_\epsilon(y)$.

**Definition 6.3.** Now, take any $A_{\mu\nu\ldots}^m \in \Gamma^m_A(At(\hat{A}))(M)$, and by applying this given $A_{\mu\nu\ldots}^m$ in arbitrary fixed $\text{Ch}(\Omega_{\text{Ch}}) \in At(\hat{A},\Omega_{\text{Ch}})$ on the given 4-form $\phi_\epsilon(y)$ created from $\phi_\epsilon \in A^n(\text{Ch}'(q,\Omega_{\text{Ch}}),\hat{A})$, we get a function $\mathbb{R}^4 \to \mathbb{R}^{m+n}$ (depending on $A_{\mu\nu\ldots}^m \in \Gamma^m_A(M)$, a given chart and on a given 4-form). As a result, such function is function of these objects: $A_{\mu\nu\ldots}^m(\in \Gamma^m_A(M)), \phi_\epsilon(\in A^n(q,\Omega_{\text{Ch}},\text{Ch}'(\Omega_{\text{Ch}})))$, $y$, $\text{Ch}(\Omega_{\text{Ch}})$ and we denote it as:

$$F_{\mu\nu\ldots}^m(A_{\mu\nu\ldots},\hat{A},\Omega_{\text{Ch}},\text{Ch}'(q,\Omega_{\text{Ch}}),n,\phi_\epsilon(y),\text{Ch}(\Omega_{\text{Ch}})).$$

**6.1.2 Definition of the relation of equivalence**

**Definition 6.4.** $A_{\mu\nu\ldots}^m, T_{\mu\nu\ldots}^m \in \Gamma^m_A(M)$ are called equivalent $((A_{\mu\nu\ldots}^m \approx T_{\mu\nu\ldots}^m))$, if they belong to the same classes $\Gamma^m_A(At(\hat{A}))(M)$ and if:

$\forall \Omega_{\text{Ch}} \, \forall q \, (q \in \Omega_{\text{Ch}}) \, \forall \text{Ch}'(q,\Omega_{\text{Ch}}) \in \hat{A}$ (such that $A_{\mu\nu\ldots}^m, T_{\mu\nu\ldots}^m \in \Gamma^m_A(At(\hat{A}))(M)$) and $\forall \text{Ch}(\Omega_{\text{Ch}}) \in At(\hat{A},\Omega_{\text{Ch}})$ $\exists n$ such that $\forall \phi_\epsilon \in A^n(\text{Ch}'(q,\Omega_{\text{Ch}}),\hat{A})$ holds that for any compact supported, smooth function $\mathbb{R}^4 \to \mathbb{R}$ $\psi$ it holds:

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^4} \{ F_{\mu\nu\ldots}^m(A_{\mu\nu\ldots},q,\Omega_{\text{Ch}},\text{Ch}'(\Omega_{\text{Ch}}),n,\phi_\epsilon(y),\text{Ch}(\Omega_{\text{Ch}})) - F_{\mu\nu\ldots}^m(T_{\mu\nu\ldots},q,\Omega_{\text{Ch}},\text{Ch}'(\Omega_{\text{Ch}}),n,\phi_\epsilon(y),\text{Ch}(\Omega_{\text{Ch}})) \} \psi(y) d^4y = 0.$$  \hspace{1cm} (35)

Note that for $A, B, C, D$ from the same $\Gamma^n(At(\hat{A}))(M)$, being defined on the same domains, should obviously hold: $A \approx B$, $C \approx D$ then $\lambda_1 A + \lambda_2 C \approx \lambda_1 B + \lambda_2 D$ for $\lambda_1, \lambda_2 \in \mathbb{R}$.

**Notation:**

Now since we defined the equivalence relation, it divides $\Gamma^n_A(M)$ objects naturally to equivalence classes and those equivalence classes will be denoted as $\Gamma^n_A(M)$. In the next considerations we may also use more limited classes.
of equivalence \( \tilde{D}_{m,n}^{A}(M) \), \( \tilde{D}_{m,n}^{E,A}(M) \) etc., as equivalence classes build only from the elements belonging to those given classes.

A careful reader now understands what is the relation between our concept of equivalence and Colombeau equivalence relation. It is simple: this definition just translates the Colombeau equivalence relation (see [5]) into our language with the equivalence classes preserving all the features of Colombeau equivalence classes (what will be proven in the following theorems).

6.1.3 Additional definitions (concepts of associated field and \( \Lambda \) class)

We define the following concept (of association) just to bring some insight to what is happening in the most simple (but most important and useful) cases and to be able see better the relation between the calculus we defined (concerning equivalence) and the “old” tensor calculus. It should bring us also better view on what equivalence means (at least in these cases) in terms of physics. It just means that the quantities differ on the large scales, but take the same small scale limit (for the small scales approach each other).

**Definition 6.5.** Take \( T^{\mu\nu\ldots}_{\nu\ldots} \in \Gamma^{m}_{A}(M) \). If it holds that:

1. \( \forall \tilde{A} \) for which it holds that \( T^{\mu\nu\ldots}_{\nu\ldots} \in \Gamma^{m}_{A(At(\tilde{A}))}(M) \) and \( \forall A' \) such that \( A' \subset \tilde{A}, \forall \Omega_{Ch}, \exists \Omega_{Ch} \setminus N(A') \) (set \( N \) being 0 in any \( L \)-measure), such that \( \forall q \in \Omega_{Ch} \setminus N, \forall Ch'(q,\Omega_{Ch}) \in A', \forall Ch(\Omega_{Ch}) \in At(\tilde{A}), \exists n \) such that \( \forall \phi_{e} \in A^{n}(Ch'(q,\Omega_{Ch}),\tilde{A}) \) exists the limit: \( \lim_{\epsilon \to 0}(T^{\mu\nu\ldots}_{\nu\ldots},\phi_{e}) \), independent on \( \phi_{e} \),

2. for arbitrary \( A'_{1} \subset \tilde{A}_{1}, A'_{2} \subset \tilde{A}_{2} \), such that \( T^{\mu\nu\ldots}_{\nu\ldots} \in \Gamma^{m}_{A(At(\tilde{A}_{1}))}(M) \) and for \( Ch(\Omega_{Ch}) \in At(\tilde{A}_{1},\Omega_{Ch}) \cap At(\tilde{A}_{2},\Omega_{Ch}) \) is this limit for every \( q \in \Omega_{Ch} \setminus (N(A'_{1}) \cup N(A'_{2})) \) the same,

then the given object obtained is mapping: \( Ch(\Omega_{Ch})(\in At(\tilde{A})) \times \Omega_{Ch} \setminus N(A') \to \mathbb{R}^{m+n} \) (\( A' \subset \tilde{A} \)) depending on class \( T^{\mu\nu\ldots}_{\nu\ldots} \in \Gamma^{m}_{A}(M) \) and we call this map (which fulfills necessarily the same consistency conditions for \( \tilde{\Omega}_{Ch} \subset \Omega_{Ch} \) than the \( \Gamma^{m}_{A}(M) \) objects) a field associated to the \( T^{\mu\nu\ldots}_{\nu\ldots} \in \Gamma^{m}_{A}(M) \), and we use the expression \( A_{s}(T^{\mu\nu\ldots}_{\nu\ldots}) \).

**Definition 6.6.** Call by class \( \Lambda \subset \Gamma^{E}_{E}(M) \) such objects that they are in every chart expressible by an integral of such piecewise continuous multi-index matrix of functions (integral expressed in the chart, which is argument of that object) that for every point \( z_{0} \) where it is continuous it holds:
\[ \exists \delta > 0, \ \exists K > 0, \ \forall \epsilon \ \text{such that} \ (0 \leq \epsilon \leq \delta), \ \text{that for arbitrary unit vector } n \ \text{(in the standard metric on } \mathbb{R}^4) \ T_{\mu \cdots}^\nu(z_0) - K\epsilon \leq T_{\mu \cdots}^\nu(z_0 + n\epsilon) \leq T_{\mu \cdots}^\nu(z_0) + K\epsilon \]

6.2 Theory

6.2.1 Reproduction of the basic results by the equivalence relation

Theorem 6.1. Any class \( \tilde{\Gamma}_m^{m\mathcal{A}(\cup A_n)\mathcal{A}(\tilde{A}_n)\mathcal{A}}(M) \) contains maximally one linear element.

Proof. We need to prove that there do not exist such two elements of \( \Gamma^m(M) \), which are equivalent. Take two elements \( B \) and \( T \) of this kind (both with the given domains and continuity), take arbitrary \( \Omega_{Ch} \) and arbitrary \( \hat{A} \) from their domains, and arbitrary \( Ch'(\Omega_{Ch}) \in \hat{A} \), and map all the \( C^{\mathcal{P}}_{S(\hat{A})}(\Omega_{Ch}) \) objects to smooth, compact supported functions on \( \mathbb{R}^4 \) through this chosen chart mapping. Now both \( B \) and \( T \) give at fixed, but arbitrary \( Ch(\Omega_{Ch}) \in At(\hat{A}) \) a linear, continuous map on those functions (the only difference from Colombeau distributions is that it is in general a map to \( \mathbb{R}^m \), rather than only to \( \mathbb{R} \), which is “cosmetic”). Now at every given \( Ch(\Omega_{Ch}) \in At(\hat{A}) \), by taking this construction, our concept of equivalence reduces to Colombeau equivalence from [5] and hence the same results should hold. One of the results says that there are no two different distributions being equivalent. We have to realize that all these parameters are fixed but arbitrary and all the 4-forms from their domains have to be mapped to this \( \mathbb{R}^4 \) functions for some proper fixing of \( \Omega_{Ch} \) and \( \hat{A} \). Further we have to realize that they are arguments of \( B \) and \( T \) only in the charts, in which \( B \) and \( T \) were compared as maps on the spaces of \( \mathbb{R}^4 \) functions to which they were mapped (this “arbitrary chart fixing” covers all their domain). The result must be then that \( B \) and \( T \) must be identical and that is what we wanted to prove.

Theorem 6.2. Any class of equivalence \( \tilde{\Gamma}_m^{E\mathcal{A}}(M) \) contains maximally one linear element.

Proof. We have to understand that \( \Gamma^m_{E\mathcal{A}}(M) \) are continuous and defined on every \( C^{\mathcal{P}}_{S(\hat{A})}(M) \) in every \( A \), so now they are compared in arbitrary chart from \( A \). After realizing this, we can repeat the previous proof and understand that if there exists the given element from \( \Gamma^m_{E\mathcal{A}}(M) \) belonging to the resulting class of equivalence, it is automatically determined if and how is this element defined for the 4-forms outside \( C^{\mathcal{P}}(M) \).
The expression appearing in the following theorem (and also later on for example in the definition of covariant derivative) $T_{\nu...}^{\mu...}(A_{\beta...}^a; \Psi)$ will be understood in the following way: at chart $Ch_1(\Omega_{Ch}) \in At(\hat{A}_n)$ we express $A_{\beta...}^a$ as multi-index matrix of functions (obtained relatively to $Ch_1(\Omega_{Ch})$), multiply $\Psi \in C^P(\Omega_{Ch})$ by this multi-index matrix of functions and the resulting object we understand as multi-index matrix of $C^P(\Omega_{Ch})$ objects. From this follows that we define them outside $\Omega_{Ch}$ set to be trivially 0. This multi-index matrix of $C^P(\Omega_{Ch})$ objects we substitute to $T_{\nu...}^{\mu...}$ fixed at that given chart $Ch_1(\Omega_{Ch})$.

**Theorem 6.3.** The following statements hold:

a) for any $T_{\nu...}^{\mu...}(\alpha_{...}; \Psi) \in \Gamma^a_{EA}(M)$, such that for any $\Omega_{Ch}$ and any two fixed arbitrary charts on $\Omega_{Ch}$ it is given as $T_{\nu...}^{\mu...}(\alpha_{...}; \Psi) = \int T_{1\nu...}^{\mu...}(\Psi) \ldots \int T_{N\nu...}^{\mu...}(\Psi)$, it holds that in the class of equivalence $\tilde{\Gamma}^a_{EA}(M)$ where it belongs, exists a linear element and is defined (on that given arbitrary $\Omega_{Ch}$, in those arbitrary charts on $\Omega_{Ch}$) as $\int T_{1\nu...}^{\mu...}(\Psi) \ldots \int T_{N\nu...}^{\mu...}(\Psi)$ (as being applied on $\Psi \in C^o_S(\Omega_{Ch})$), both only if the integral $\int (T_{1\nu...}^{\mu...} \ldots T_{N\nu...}^{\mu...})$ for every $\Omega_{Ch}$ and for every two charts $Ch_1(\Omega_{Ch})$, $Ch_2(\Omega_{Ch}) \in A$ converges on every compact subset of $\Omega_{Ch}$.

Exactly the same statement holds, if we take instead of $\Gamma^m_{EA}(M)$ its subclass $D^m_{bEA}(M)$ and instead of the equivalence class $\Gamma^m_{EA}(M)$, the equivalence class $\tilde{D}^m_{bEA}(M)$.

The same statement holds if we take instead of $\Gamma^m_{EA}(M)$ and $D^m_{nEA}(M)$ classes, the classes $\Gamma^m_{EA(\cup_n At(\hat{A}_n))}(M)$ and $D^m_{EA(\cup_n At(\hat{A}_n))}(M)$ (with the exception that the given convergence property shall be considered only at charts $Ch_1(\Omega_{Ch}), Ch_2(\Omega_{Ch}) \in \cup_n At(\hat{A}_n)$).

b) For any distribution $A_{\beta...}^a \in \Gamma^a_{S(\cup_n At_n)(\hat{A}_n)}(M)$ and an element $T_{\nu...}^{\mu...}(\alpha_{...}; \Psi) \in \Gamma^m_{(\cup_n At_n)(\hat{A}_n)}(M)$, it holds that $A_{\beta...}^a T_{\nu...}^{\mu...}$ is equivalent to an element of $\Gamma^m_{(\cup_n At_n)(\hat{A}_n)}(M)$ (and for subclass $D^m_{b(\cup_n At_n)}(M) \in \Gamma^m_{(\cup_n At_n)(\hat{A}_n)}(M)$ and $D^m_{I S(\cup_n At_n o)}(M) \in \Gamma^m_{(\cup_n At_n o)}(M)$ to an element $D^k_{I+1(\alpha o)}(M)$). The element is given as $T_{\nu...}^{\mu...}(A_{\beta...}^a; \Psi)$.

c) for any tensor distribution $A_{\beta...}^a \in \Gamma^a_{S}(M)$ and an element $T_{\nu...}^{\mu...}(\alpha_{...}; \Psi) \in \Gamma^m_{(\cup_n At_n o)(\hat{A}_n)}(M)$, it holds that $A_{\beta...}^a T_{\nu...}^{\mu...}$ is equivalent to an element of $\Gamma^m_{(\cup_n At_n o)(\hat{A}_n)}(M)$.

The element is given as $T_{\nu...}^{\mu...}(A_{\beta...}^a; \Psi)$.

**Proof.** a) Use exactly the same construction as in the previous proof. We fix $T_{\nu...}^{\mu...}(\alpha_{...}; \Psi)$ at arbitrary chart from $A$ (as it is continuous in arbitrary chart on every $C^P(\hat{A}(\hat{A}))$) and express it in the chart used during the first step of the construction as a map $\Psi \rightarrow \int T_{1\nu...}^{\mu...}(\Psi) \ldots \int T_{N\nu...}^{\alpha_{...}}(\Psi)$.  

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Then it is a result of Colombeau theory that, if \( \int T_{\mu_1, \ldots, \nu}^\alpha \Psi \ldots T_{\nu_\cdot, \beta}^{\cdot \alpha} \Psi' \) is defined as a linear mapping on given \( \mathbb{R}^4 \) functions \( \Psi \) (in our case they are related by fixed (but arbitrary) chart mapping to given \( C^\infty_\mathcal{S}(M) \) objects), than these two objects are equivalent. Now everything was fixed, but arbitrary, so in this way the result is proven. From this proof we immediately see that the simple transformation properties of the \( D_{n}^m(M) \) objects\(^{21}\) are fulfilled by this linear object, if the objects we multiply are from \( D_{nA}^m(M) \), so the second result can be proven immediately. The last two results concerning the classes with limited domains are completely obvious after we proved these results, we just have to realize how they are exactly defined.

b) is proven completely in the same way, we just have to realize that because of the “limited” domain of the \( D_{bS(\cup_n A')_o}^m(M) \) objects, we can effectively use the concept of smoothness in this case.

c) is just the same than b), the only difference is, that the domain of the product is limited because of the “second” term in the product.

Note that this means that tensor multiplication gives on appropriate subclasses of \( D_{nE}^m(M) \) the mapping \( \tilde{D}_{bE}(M) \times \tilde{D}_{nE}^m(M) \rightarrow \tilde{D}_{b+nE}^{m+1}(M) \) and on appropriate subclasses the \( \tilde{\Gamma}_{bE}(M) \times \tilde{\Gamma}_{nE}^m(M) \rightarrow \tilde{\Gamma}_{b+nE}^{m+1}(M) \) mapping. What is quite sad, this cannot be extended to \( D_{nA}^m(M) \).

**Theorem 6.4.** Take \( T_{\mu_1, \ldots, \nu}^\alpha \in \tilde{\Gamma}_{bE}(M) \), \( A_{\mu_1, \ldots, \nu}^\alpha \in \tilde{\Gamma}_{bE}(M) \) and \( L_{\mu_1, \ldots, \nu}^\alpha \in \tilde{\Gamma}_{bE}(M) \), than \( T_{\mu_1, \ldots, \nu}^\alpha \approx A_{\mu_1, \ldots, \nu}^\alpha \) implies \(^{22}\) \( L \otimes T \approx L \otimes A \).

**Proof.** Use the same method than previously and trivially follows from the results of Colombeau theory (especially from the theorem saying that if a Colombeau algebra object is equivalent to a distribution, then after multiplying them by a smooth distribution, they remain equivalent).

**Theorem 6.5.** Contraction (of \( \mu \) and \( \nu \) index) is for such \( T_{\mu_1, \ldots, \nu}^\alpha \in D_{nE}^m(M) \), that they are from the equivalence class containing a linear element (in this case the element is given as mapping \( \Psi \rightarrow \int T_{\mu_1, \ldots, \nu}^\alpha \Psi \) ) a map to \( \tilde{\Gamma}_{E}^{m+n-2}(M) \) and the class \( \tilde{\Gamma}_{E}^{m+n-2}(M) \) is such, that contains (exactly) one element from \( D_{n-1E}^{m-1}(M) \), which is given as mapping \( \Psi \rightarrow \int T_{\alpha_1, \ldots, \nu}^\alpha \Psi \).

**Proof.** The proof is obvious after we realize that operations of contraction and changing the object from the class of equivalence commute (this trivially follows from our previous note about addition and equivalence).

\(^{21}\)We include also the scalar objects here.

\(^{22}\)It is obvious, that we can extend the definition domains either of \( T \) and \( A \), or of \( L \).
6.2.2 Conjectures

**Conjecture 6.1.** Tensor multiplication gives everywhere on \( D_{nEA}^m(M) \) a mapping \( \tilde{D}_{a} \times \tilde{D}_{m} \rightarrow \tilde{D}_{a+m} \) and everywhere on \( \Gamma_{EA}^a(M) \) a map \( \tilde{\Gamma}_{a} \times \tilde{\Gamma}_{b} \rightarrow \tilde{\Gamma}_{a+b} \).

**Conjecture 6.2.** Take an element \( A_{\alpha...\beta...} \in \Gamma_{(\cup_n A_n)}(M) \) and such \( T_{\mu...\nu...} \in \Gamma_{(\cup_n A_n)}(M) \) element, that \( \forall \tilde{A}_l \subseteq \cup_n \tilde{A}_n \) on every \( \Omega_{Ch} \), in every chart \( \Omega_{Ch} \in At(\tilde{A}_n, \Omega_{Ch}) \) is for \( \Psi \in \mathcal{C}^P_{S(\tilde{A}_m)}(M) \) \( T_{\mu...\nu...} \Psi \) still some element of the class \( \mathcal{C}^P_{S(\tilde{A}_m)}(M) \) and also holds \( \Omega_{Ch} \in At(\tilde{A}_m, \Omega_{Ch}) \), where \( T_{\mu...\nu...} \) is a multi-index matrix of functions in \( \Omega_{Ch} \). Then it holds that \( A_{\beta...\alpha...} T_{\mu...\nu...} \) is equivalent to an element of \( \Gamma_{(\cup_n A_n)}(M) \) (and for subclasses \( D_{a+b} \rightarrow D_{a+b} \)). The element is given as \( T_{\mu...\nu...}(A_{\alpha...\beta...}) \).

6.2.3 Additional theory

**Theorem 6.1.** Any \( T_{\mu...\nu...} \in \Lambda \) (as was defined by Definition 6.6) defines an \( A_\alpha(T_{\mu...\nu...}) \) object on \( M \). But the same field can be associated to more elements of \( \Gamma_{EA}^n(M) \).

**Proof.** For \( \forall \Omega_{Ch}, \forall \tilde{A} \) and \( \forall Ch'(\Omega_{Ch}) \), we can in fixed but arbitrary \( Ch(\Omega_{Ch}) \) write the \( T_{\mu...\nu...} \) object as a multi-dimensional matrix of piecewise continuous functions. We see in \( Ch'(\Omega_{Ch}) \), that \( \Psi_\epsilon = A_\alpha(\tilde{A}, Ch'(q, \Omega_{Ch})) \) are in fact delta-sequences, so we just have to show, that on the set where those functions are continuous in the stronger “upper” sense, they give the value of this multi-index matrix. So write the integral:

\[
\int_{\epsilon}^{b} T_{\mu...\nu...}(x) \frac{1}{\epsilon^4} \Phi \left( \frac{x}{\epsilon} \right) d^4x.
\]

By substitution \( x = \epsilon z \) we obtain:

\[
\int_{a}^{b} T_{\mu...\nu...}(\epsilon z) \Phi(z) d^4z.
\]

But from the properties of \( T_{\mu...\nu...}(x) \) it follows:

\[
\int_{a}^{b} (T_{\mu...\nu...}(z_0) - K\epsilon) \Phi(z) d^4z \leq \int_{a}^{b} (T_{\mu...\nu...}(z_0 + n\epsilon) \Phi(z) d^4z \leq \int_{a}^{b} (T_{\mu...\nu...}(z_0) + K\epsilon) \Phi(z) d^4z
\]

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for some $\epsilon$ small enough.

But we are taking the limit $\epsilon \to 0$ which, together with the fact that $\Phi(\chi)$ are normed to 1, means that the integral must go to $T_{\nu...}^{\alpha...}(z_0)$. The “rest”, where it is not continuous in this sense, has Lebesgue measure 0, hence the $\Omega_{Ch}$ part, which is mapped to this “rest” has L-measure 0. But the values of that multi-index matrix in the given chart at this arbitrary, fixed point give us a given associated field (are independent on delta sequence obviously). The other statement will be proven by a particular example.

**Theorem 6.2.** The objects from field associated to a $T_{\nu...}^{\alpha...} \in \Lambda \cap D_{\mu...}^{\nu...}(M)$, for each $\Omega_{Ch}$, transform for every couple of charts from their domains $Ch_1(\Omega_{Ch}) \in A_1, Ch_2(\Omega_{Ch}) \in A_2$ on some $M/(N(A_1') \cup N(A_2'))$ (set $N(A_1') \cup N(A_2')$ having L-measure 0) as ordinary tensor fields with piece-wise smooth transformations. This means that if we take $J_{\alpha...}(J^{-1})^\beta_{\nu...} T_{\beta...}^{\nu...}$ as the multi-index matrix which appears under the integrals, then on the domains where both this object and the object $T_{\beta...}^{\nu...}$ are continuous in the previous sense (everywhere apart of a set having Lebesgue measure 0), the associated fields transform as the tensor fields with those Jacobians.

**Proof.** The part where the immediately follows from what we said and did in the previous proof. Of course we see immediately, that $N$ has L-measure 0, since both sets, set where the domain of the given associated field in the first chart is undefined and the domain where is Jacobian nonsmooth (which is even stronger condition than we need) are both having L-measure 0, which means that their union has obviously L-measure 0.

Note, that if for $\Gamma^m_{\alpha}(M)$ object it holds

$$\lim_{\epsilon \to 0} T_{\nu...}^{\alpha...}(\phi_\epsilon) = (-)\infty$$

at least at one point, then the field associated to this object, can be associated to another object, which is nonequivalent to this object. This is explicitly shown in the proof of the next statement.

**Theorem 6.3.** Take $\delta(q, Ch_k(\Omega_{Ch})) \in D'_{(\alpha_0)}(M)$ being defined as mapping from each 4-form $C_{S(\tilde{A})}^P(M)$ ($A' \subseteq \tilde{A}$) to the value of the density expressing this form at point $q$, as being expressed in chart $Ch_k(\Omega_{Ch}) \in A'$ ($q \in \Omega_{Ch}$). Then any power $\in \mathbb{N}_+$ of $\delta(q, Ch_k(\Omega_{Ch}))$ is associated to function defined on $M \setminus \{q\}$ and being 0. Note that this function is associated to any power

---

23 Of course, some transformations in a generalized sense might be defined also on the $N$ set.
(∈ N_+) (being nonzero natural number) of δ(q), but different powers of δ(q) are mutually nonequivalent \[24\].

**Proof.** It is obvious, that contracting powers of δ(q, Ch_k(Ω_Ch)) with sequence of 4-forms from arbitrary \( A^n(Ch'\,(q, Ω_Ch), \tilde{A}) \), (q′ ≠ g) (they have a support stretching to another point than q) will give simply 0. At q a given \( δ^n(q, Ch_k(Ω_Ch)) \) applied on the given smooth compact supported function (in the given chart \( Ch_k(Ω_Ch) \) (in fact it is a density)) \( Ψ(x) ∈ A^n(Ch_k'(q, Ω_Ch), \tilde{A}) \) will lead to the expression \( e^{-4n}Ψ^n(\frac{x}{ε}) \). Then if we want to compute

\[
\lim_{ε→0} \int_{-∞}^{∞} \left( \frac{1}{ε^{4m}}Ψ^n \left( \frac{x}{ε} \right) - \frac{1}{ε^{4m}}Ψ^m \left( \frac{1}{ε} \right) \right) Φ(x)dx
\]

then this leads to

\[
\lim_{ε→0} \frac{1}{ε^{4m-4}}Φ(0) \int_{-∞}^{∞} \left( Ψ^n(x) - \frac{1}{ε^{4(n-m)}}Ψ^m(x) \right) dx
\]

which is for \( n ≠ m, n, m ∈ N_+ \) apparently divergent, hence nonzero. \( \square \)

Note, that any (natural number higher than one) power of delta function is nonequivalent to any distribution.

**Theorem 6.4.** We see that the map \( A_s \) is linear \[25\] and for arbitrary number of \( g^α_{β…}h^μ_{ν…} ∈ Λ \) it holds: \( A_s(g^α_{β…} ⊗ ... ⊗ h^μ_{ν…}) = A_s(g^α_{β…}) ⊗ ... ⊗ A_s(h^μ_{ν…}) \). Of course, it is trivial that in the first term it is a product between Λ objects and in the second the classical tensor product.

**Proof.** Trivially connected with previous proofs: we have to realize that what the tensor product means for the objects from Λ class, and that at the fixed charts it gives integrals of multi-index matrices, which for some choice of indexes componentwise multiply. Because you obtain the associated field by substituting \( Ψ, \) to each of them and taking the limit, we see, that this gives in fact the “classical” tensor multiplication. \( \square \)

\[24\]We have to realize that it is hard to find in our theory a more “natural” definition generalizing the concept of delta function from \( \mathbb{R}^n \). But there is still another natural generalization. The result is an object from \( T^n_{(\epsilon, A_\Lambda_\omega)}(M) \), defined as: \( δ(Ch_k(Ω_Ch), q, Ψ) = Ψ(Ch_k(Ω_Ch), q), Ch_k(Ω_Ch) ∈ A'_n, Ψ ∈ C_P(\tilde{A}_n) (A'_n ⊂ \tilde{A}_n), \) where \( Ψ(Ch_k(Ω_Ch), q) \) means value of the density given by \( Ψ \) at the point \( q \), when expressed at the chart \( Ch_k(Ω_Ch) \).

\[25\]Linear in this case means, that being defined on \( T, L \) implies being defined on \( λ_1T + λ_2L \ (λ_1, λ_2 ∈ \mathbb{R}) \) and linear in the usual sense of the word.
This should mean, that by any tensor multiplication of elements from class $\Lambda$ (important multiplication of two scalars as a subcase) we obtain an element which is equivalent to an element from class $\Lambda$, but something similar was already expressed in wider form by the statement that $\otimes$ is a map for given constrained elements (but less constrained than $\Lambda \cap \mathcal{D}_{n}^{m}(M)$)

$D_{nE,A}^{m}(M) \times D_{iE,A}^{k}(M) \rightarrow \mathcal{D}^{m+k}_{n+iE,A}(M)$ (containing the $\mathcal{D}^{m}_{nE}(M)$ element).

This statement just shows, that it holds also for the subclass $\Lambda$ (which after understanding theory from the previous section, must be obvious anyway).

**Conjecture 6.3.** If $T_{\mu...}^{\nu...} \in \mathcal{D}^{mn}_{n(\mathcal{A}')}^{m}(M)$ has an associated field, then it transforms on its domains as a tensor field.

**Conjecture 6.4.** If $T_{\mu...}^{\nu...} \in \Gamma^{m}_{(\mathcal{A}')}^{n}(M)$, having an associated field and $L_{\beta...}^{\alpha...} \in \Gamma^{n}_{\mathcal{S}}(M)$, then $A_{\alpha}(T \otimes L) = A_{\alpha}(T) \otimes A_{\alpha}(L)$, again with different, but obvious meaning of $\otimes$ on the opposite sides of the equation.

**Conjecture 6.5.** If for $C, D, F, B \in \Gamma^{m}_{\mathcal{A}}(M)$, belonging to the same classes $\Gamma^{m}_{\mathcal{A}(\mathcal{A}')}^{n}(M)$, it holds that $\forall \mathcal{A}$ such that they belong to $\Gamma^{m}_{\mathcal{A}(\mathcal{A}')}^{n}(M)$ and $\forall \mathcal{A}' \subseteq \mathcal{A}$ all $C, D, F, B$ have associated fields defined on the whole $M$ for every $\mathcal{A}$, then $F \approx B$ and $C \approx D$ implies $C \otimes F \approx D \otimes B$. 
7 Covariant derivative

The last fundamental concept, which is missing until now, is some derivative operator on GTF, which is obviously necessary for us to formulate appropriate language for physics and generalize physical laws. Such operator must obviously reproduce our concept of covariant derivative on the smooth tensor fields (through the given association relation to the smooth manifold).

This is provided in this following chapter. The beginning of the first part is devoted again to fundamental definitions, the beginning of the second part gives us fundamental theorems, again just generalizing Colombeau results for our case. After these theorems we similarly to previous section formulate conjectures representing the very important and natural extensions of these results (bringing a lot of new significance to our results). The last subsection in the second part, which is called again “additional theory”, brings analogically to previous section just physical insight to what is happening in our abstract calculus and is of lower importance.

7.1 Definitions

7.1.1 Definition of simple derivative and connection coefficients

Definition 7.1. We define a map, called simple derivative given by smooth vector field \( U^i \) (smooth in the atlas \( A' \)) as a mapping:

\[
\Gamma^m_{\{A\}}(M) \rightarrow \Gamma^m_{\{A\}}(M)
\]

and:

\[
T^{\mu...}_{\nu...}(U) (Ch_k(\Omega_{Ch}), \psi) = -T^{\mu...}_{\nu...}(Ch_k(\Omega_{Ch}), (U^\alpha\Psi)_\alpha) \quad \Psi \in C^P(\Omega_{Ch})
\]

where \( (U^\alpha\Psi)_\alpha \) is understood in the following way: we express \( U^\alpha \) at the chart \( Ch_k(\Omega_{Ch}) \), \( \Psi \) by the density function at that chart \( (\Psi') \) and take the derivatives \( (U^\alpha\Psi')_\alpha \) at that chart. They give us some function at that chart, which expresses in that chart some object from \( C^P(\Omega_{Ch}) \), hence this means that it is trivially extended to \( M \) by being 0 everywhere outside \( \Omega_{Ch} \).

This object we use as an argument in \( T^{\mu...}_{\nu...} \).

To make a consistency check: \( T^{\mu...}_{\nu...}(U) \) is object which is defined at least on the domain \( C^P_{S(A)}(M) \) objects in \( A' \subseteq \tilde{A} \) (\( A' \) related to \( U^i \)) and is continuous on the same domain, which means it is a \( \Gamma^m(M) \) object. To see this, just realize that when we take arbitrary \( \Omega_{Ch} \) and arbitrary chart \( Ch(\Omega_{Ch}) \) from \( A' \), we see that at that chart \( (U^\alpha\Psi')_\alpha \) gives us by inverse chart
mapping 4-forms, which are compact supported at $\Omega_{Ch}$ and smooth. Hence they are from the domain of $T_{\nu...}^{\mu...}$ at any chart from $A'$. At arbitrary chart from $A'$ we see trivially (from the theory of distributions), that if $\Psi_n \to \Psi$, then $T_{\nu...}^{\mu...}(\Psi_n) \to T_{\nu...}^{\mu...}(\Psi)$, hence it is continuous.

**Definition 7.2.** Now, we can call by a generalized connection such object from $\Gamma^3(M)$, that it is defined and continuous on the whole $C^\infty_a(\tilde{A}_m)$ in every chart from $A$, and transforms as:

$$
\Gamma_{\beta\gamma}(Ch_2, \Psi) = \Gamma_{\nu\delta}^\mu(Ch_1, ((J^{-1})^\nu_\beta(J^{-1})^\delta_\gamma J^\mu_m - J^\mu_m(J^{-1})^m_\beta_\gamma)\Psi) \quad (36)
$$

### 7.1.2 Definition of covariant derivative

**Definition 7.3.** By a covariant derivative (on $\Omega_{Ch}$) in the direction of an in $A'$ smooth vector field $U^i(M)$, of an object $A_{\nu...}^{\mu...} \in \Gamma^m_{\nu...}(\Omega_{Ch})$, we call an object:

$$
D_{C(U)}A_{\nu...}^{\mu...}(\psi) = A_{\nu...}^{\mu...}(U\psi) + \Gamma_{\alpha\rho}^\mu(\psi)A_{\nu...}^{\rho...}(U^\alpha\psi) - \Gamma_{\alpha\nu}^\mu(\psi)A_{\rho...}^{\alpha...}(U^\rho\psi) \quad (37)
$$

We explained already in the previous section how to understand such expressions as $T_{\nu...}^{\mu...}(U^\alpha\Psi)$.

Covariant derivative is in this way defined everywhere on $\Gamma^m_{(\tilde{A}_m)}(M)$. This result we can see easily, since simple derivative gives us still an object from $\Gamma^m_{(\tilde{A}_m)}(M)$. The second term containing generalized connection is from $\Gamma^m_{(\tilde{A}_m)}(M)$ trivially too.

**Definition 7.4.** Furthermore, extend the definition of covariant derivative to $\Gamma^m_{(A'_A)}(M)$ class just by stating that on every nonlinear object (note that every such object is constructed by tensor multiplication of linear objects) it is defined by the condition that it should follow the Leibniz rule (so it is a standard derivative operator, since it is trivially linear as well).

### 7.2 Theory

#### 7.2.1 Basic equivalences related to differentiation and some of the interesting conjectures

**Notation**

Take as $A_n$ some $n$-times continuously differentiable subatlas of $A$, and by class $S'_n$ denote such $\Gamma^a_{EA(\cup_\tilde{A}_m\cup(\tilde{A}_m)_m)}(M)$ objects that:

- $\cup_\tilde{A}_m\tilde{A}_m = A_n$ and $\forall m \ At(\tilde{A}_m) = A_n$
b) it is given for arbitrary chart from $A_n$, taken as its argument, by integral from multi-index matrix of $n$-times continuously differentiable functions (if the integral is also expressed at arbitrary chart from $A_n$)

**Theorem 7.1.** The following statements hold:

a) Take a vector field $U^i$, which is smooth at $A' \subset A_{n+1} \subset A_n$ for $n \geq 1$ (formally including also $n = n + 1 = \infty$, hence $A'$), generalized connection $\Gamma^\mu_{\nu\alpha} \in \Gamma_3^3(M)$ and $T^\nu_{\mu\ldots}(U^a\psi) \in D_n^m(M) \cap S_n', \text{ where } S'_n$ is related to the given $A_n$, then following holds:

Covariant derivative is a map from this object to an object from $S'_{n+1}$ (being related to the given $A_{n+1}$). Moreover, the equivalence class $\tilde{\Gamma}^m_{A(A')}(M)$ of the image contains exactly one linear element being expressible on its domain in every two charts from its domain as integral from some multi-index matrix of piecewise continuous functions and this element is given as a map $\Psi \rightarrow \int U^a T^\nu_{\mu\ldots}(U^a\psi)$ (where “;” means the classical covariant derivative related to the “classical” connection appearing under the integral defining the $\Gamma^\mu_{\nu\alpha}$ object).

b) Take $U^i$ being smooth particularly at $A'$ and $\Gamma^\mu_{\nu\alpha} \in \Gamma_3^3(M)$, then the following holds:

Covariant derivative is a map: $D^m_{n(A')} \rightarrow \tilde{\Gamma}^m_{A(A')}(M)$ and the classes $\tilde{\Gamma}^m_{A(A')}(M)$ of the image contain (exactly) one element of $D^m_{n(A')}$. 

**Proof.** a) Covariant derivative is

$$T^\nu_{\mu\ldots}(U)(\psi) + \Gamma^\mu_{\nu\rho}(\psi)T^\nu_{\mu\ldots}(U^a\psi) - \Gamma^\rho_{\nu\rho}(\psi)T^\mu_{\nu\ldots}(U^a\psi).$$

Take the first term $T^\nu_{\mu\ldots}(U)$, express it at a given $Ch_k(\Omega_{Ch}) \in A_n$ as the following map: $\Psi \rightarrow \int_{\sigma(\Omega_{Ch})} T^\nu_{\mu\ldots}(U^a\psi) \Psi^{Ch_k}$, where the $T^\nu_{\mu\ldots}(U^a\psi)$ symbol means multi-index matrix of functions representing $T^\nu_{\mu\ldots}$ at that chart $Ch_k(\Omega_{Ch})$. Furthermore the symbol $\int_{\sigma(\Omega_{Ch})}$ means that the integral is expressed at the chart $Ch_k(\Omega_{Ch})$ and $\Psi^{Ch_k}$ a density related to given 4-form expressed at that chart. So if analogically $U^a(Ch_k)\alpha$ means the given object expressed at $Ch_k(\Omega_{Ch})$, then:

$$T^\nu_{\mu\ldots}(U)(Ch_k, \Psi) = -\int_{\sigma(\Omega_{Ch})} (T^\nu_{\mu\ldots}(U(Ch_k)\alpha\psi(Ch_k)), (Ch_k)\alpha).$$

Here $\Psi$ is in arbitrary chart from $A_n$ $n$-times continuously differentiable (the domain is limited to such objects by the second covariant
derivative term, which is added to simple derivative) and such that 
\((U^i \Psi)_i\) is at any chart from \(A_n\) \(n\)-times continuously differentiable.

Now take the given expression and by using the integration per parts (since all the objects under the integral are at least continuously differentiable \((n \geq 1)\), we can use it), together with compactness of support, we see that:

\[
T_{\nu,\ldots}(U)(Ch(k)(\Omega Ch), \Psi) = - \int_{\sigma(\Omega Ch)} T_{\nu,\ldots}(U)(Ch(k)\alpha \Psi(Ch(k)))(Ch(k)\alpha) = \int_{\sigma(\Omega Ch)} T_{\nu,\ldots}(U)(Ch(k)\alpha \Psi(Ch(k)))(Ch(k)\alpha).
\]

Then it holds that:

\[
T_{\nu,\ldots}(U)(Ch(m)(\Omega Ch), \Psi) = \int_{\sigma(\Omega Ch)} T_{\nu,\ldots}(U)(Ch(m)\alpha \Psi(Ch(m)))(Ch(m)\alpha).
\]

We see that this is defined and continuous at arbitrary chart from \(A_{n+1}\), for every \(C^P_{\mathcal{A}}(\tilde{A})\) \((A' \subset \tilde{A} \subset A_{n+1})\).

Now we see that for the rest from the covariant derivative expression it holds, that it is equivalent to the map

\[
\Psi \rightarrow \int (\Gamma^\mu_{\alpha\rho} U^\alpha T^\rho_{\nu,\ldots} - \Gamma^\alpha_{\nu\rho} U^\rho T^\mu_{\alpha,\ldots}) \Psi
\](see theorem 6.3) and between charts it can be rewritten as an integral, where the objects appearing under the integral transform exactly as their classical analogues. This must hold, since \(T^\mu_{\nu,\ldots}\) is everywhere continuous (in every chart considered), hence on every compact set bounded, so the given object is well defined. This means that taking this object at chart \(Ch_m(\Omega Ch)\) and expressing it through the chart \(Ch_k(\Omega Ch)\) and Jacobians (with the integral expressed at chart \(Ch_k(\Omega Ch)\)), as in the previous case, we discover (exactly as in the classical case), that the resulting object under the integral transforms as some \(D^u_{\alpha\beta}(M)\) object, with the classical expression for the covariant derivative of a tensor field appearing under the integral.

b) The resulting object is defined particularly only at \(C^P_{\mathcal{A}}(\tilde{A})\) \((A' \subset \tilde{A})\).

We have to realize that \(T^\mu_{\nu,\ldots}\) can be written as a \((N \to \infty)\) weak limit (in every chart from \(A'\)) of \(T^\mu_{\nu,\ldots} \in D^m_{\nu,\ldots}(\mathcal{A}'(\tilde{A}'))(M)\). It is an immediate result of previous constructions and Colombeau theory, that

\[
-\Gamma^\alpha_{\nu\rho}(\Psi) T^\mu_{\alpha,\ldots}(U^\rho \psi) \approx -T^\mu_{\alpha,\ldots}(\Gamma^\alpha_{\nu\rho} U^\rho \psi)
\]
and just write those objects (that one, as well as $T_{\mu \ldots \nu}(U)$ object) as limits of integrals of some sequence of “smooth” objects in a fixed, but arbitrary chart (of course the chart where the integrals are expressed is fixed, arbitrary, but quite trivially non-interesting). But we see, that we can obviously use per parts integration again (in the first expression) and from the “old” tensorial relations, we get the “tensorial” transformation properties under the limit, which means that the resulting object, which is a limit of those objects transforms in the way the $D_n^m(M)$ objects transform.

Theorem 7.2. The a) part of the theorem 7.1 can be formulated also with some generalized concept of covariant derivative, where we simply do not require $U^i$ vector field to be smooth at some $A'$, only we require that it should be $n+1$ differentiable in $A_{n+1}$.

Proof. We just have to follow our proof and realize, that we used the smoothness of $U^i$ in $A'$, only because it is required by our definition of covariant derivative (it has its reasons in other cases).

This statement has a crucial importance, since it shows that all the classical calculus of smooth tensor fields with all the basic operations is contained in our language (if we take the equivalence instead of equality being the crucial part of our theory), but it can be even extended to arbitrary objects from $S'$ (if we connect the concept of covariant derivative with given $\Gamma^3_E(M)$ connection), hence it is better than the classical tensor calculus.

Of course, we can think about conjectures extending our results in a very important way:

Conjecture 7.1. Take arbitrary at any chart piecewise smooth on every compact set bounded\(^{26}\) vector field $U^i$, $\Gamma_{\mu \alpha}^\nu \in \Gamma^3_E(M)$ and $T_{\mu \ldots \nu}^\alpha \in D_n^m(M)$, such that integrals $\int T_{\mu \ldots \nu}^\alpha \Gamma^\alpha_{\beta \delta}$\(^{27}\) are at every chart finite on every compact set, then following holds:

Covariant derivative along this vector field is a map from this object to an object from $\Gamma_{A}^{m+n}(M)$ and the class $\Gamma_{A}^{m+n}(M)$ of the image contains (exactly) one element of $D_n^m(M)$.

\(^{26}\)To be exact, the expression “covariant derivative” is used in this and the following conjecture in a more general way, since we do not put on $U^i$ the condition of being smooth at some subatlas $A'$.

\(^{27}\)This means we are trivially integrating $T_{\mu \ldots \nu}^\alpha \Gamma^\alpha_{\beta \delta}$ on compact sets in subset of $\mathbb{R}^4$ given as image of the given chart mapping.
Conjecture 7.2. Take \( U^i \) being piecewise smooth vector field and \( \Gamma^\alpha_\nu \in \Gamma^\nu_\alpha(M) \), following holds:

Covariant derivative along this vector field is a map: \( D_n^m(M) \to \tilde{\Gamma}^{m+n}_{A(\cup_n\A' o)}(M) \) and the classes \( \tilde{\Gamma}^{m+n}_{A(\cup_n\A' o)}(M) \) of the image contain (exactly) one element of \( D_n^m(M) \).

Theorem 7.3. For \( U^i \) being smooth tensor field at \( A' \), connection taken from \( \Gamma^3_S(M) \), \( T^{\mu\cdots}_{\nu\cdots} \in \Gamma^m_{A(A'o)}(M) \) and \( A^{\mu\cdots}_{\nu\cdots} \in \Gamma^m_{(A'o)}(M) \), holds that \( T^{\mu\cdots}_{\nu\cdots} \approx A^{\mu\cdots}_{\nu\cdots} \) implies \( D^n_C(U) T \approx D^n_C(U) A \) for arbitrary natural number \( n \).

Proof. Pick arbitrary \( \Omega_{Ch} \) and arbitrary fixed chart \( Ch'(\Omega_{Ch}) \in A' \). In that chart we get all the 4-forms from the domain of these \( \Gamma^m_A(M) \) objects as smooth compact supported functions (given by densities expressed at that chart). The objects \( T^{\mu\cdots}_{\nu\cdots}((U^a\Psi)_a) \) and \( A^{\mu\cdots}_{\nu\cdots}((U^a\Psi)_a) \) as objects of Colombeau algebra (realize that the connection, fixed at that chart is also an object of Colombeau algebra) are equivalent to \( U^a(\Psi)T^{\mu\cdots}_{\nu\cdots}_a(\Psi) \) and \( U^a(\Psi)A^{\mu\cdots}_{\nu\cdots}_a(\Psi) \), where the derivative means the "distributional derivative" as used in the Colombeau theory (fulfilling the Leibniz rule) and \( U^a(\Psi) \) is simply a \( D^m_C(A'o)(M) \) object with the given vector field appearing under integral. But as we know, in the Colombeau theory it holds that if an object is equivalent to a distributional object, then their derivatives of arbitrary degree are equivalent and also it holds that if arbitrary object is equivalent to a distributional object, then if we multiply them by arbitrary smooth distribution, they remain equivalent. Since at that fixed chart holds (still in the Colombeau theory sense),

\[
T^{\mu\cdots}_{\nu\cdots} \approx A^{\mu\cdots}_{\nu\cdots}
\]

and since their simple derivatives we obtain only by the operations previously described, also their simple derivatives must remain equivalent. The same holds about the second covariant derivative term (containing connection). So those classical Colombeau theory objects (classical theory just trivially extended to what we call multi-index matrices of functions) given by the covariant derivative of those tensor fields fixed and expressed at that chart are in the sense of the Colombeau theory equivalent. But the \( \Omega_{Ch} \) set was arbitrary and also the chart was an arbitrary chart from their domain, so they are equivalent also in our sense.

We can try to extend this statement to a conjecture:

Conjecture 7.3. For \( U^i \) being piecewise smooth tensor field, connection taken from \( \Gamma^3_S(M) \), \( T^{\mu\cdots}_{\nu\cdots} \in \Gamma^m_{A(\cup_n\A' o)}(M) \) and \( A^{\mu\cdots}_{\nu\cdots} \in \Gamma^m_{(\cup_n\A' o)}(M) \), holds
that $T_{\mu...}^{\nu...} \approx A_{\mu...}^{\nu...}$ implies $D_{C(U)}^n T \approx D_{C(U)}^n A$ for arbitrary natural number $n$, if this covariant derivative exists.

That conjecture in fact means that covariant derivative using this type of generalized connection is a map from $\tilde{\Gamma}_n^m(M)$ containing a linear element to $\tilde{\Gamma}_n^m(A_n)$. Note that we can also try to prove more modestly extended version, taking the same statement and just extending $\Gamma_{A(A')}(U)$, $\Gamma_{A(A')}(M)$ to $\Gamma_{A(A')}(M)$, $\Gamma_{A(A')}(M)$.

7.2.2 Additional theory

**Theorem 7.4.** Take $U^i$ smooth at some $A' \subset A$, $\Gamma_{\mu\alpha}^{\nu} \in \Lambda$ and $T_{\mu...}^{\nu...} \in S'_n \cap D_b^m(M)$ ($n \geq 1$, $S'_n$ is related to $A_n$), then $D_{C(U)}^n T_{\mu...}^{\nu...}$ has an associated field which is on $M \setminus N$ ($N$ having $L^1$ measure 0) the classical covariant derivative of the associated (tensor) field to $T_{\mu...}^{\nu...}$, defined on the whole $A_{n+1}$ ($A' \subset A_{n+1}$).

*Proof.* This is really trivial, just take the definition of the classical covariant derivative and define the linear mapping given at a particular chart argument $Ch_2$ as

$$\Psi \rightarrow \int^{Ch_1} U^{\nu}(Ch_2) As(T_{\nu...}^{\mu...}(Ch_2))_{\mu} \Psi^{(Ch_1)} Ch_1, Ch_2 \in A_n.$$

This is appropriate, since $As(T_{\nu...}^{\mu...}(Ch_2))_{\mu}$ is defined everywhere apart of a set having Lebesgue measure 0. Now from our previous results it is obvious that the given linear mapping is equivalent to the $D_{C(U)}^n T_{\mu...}^{\nu...}$ object and that $U^{\nu} As(T_{\nu...}^{\mu...})_{\mu} = As(D_{C(U)}^n T_{\mu...}^{\nu...})$, apart of a given set $N$.

**Theorem 7.5.** Extended analogy of theorem 7.3, can be proven, if we use generalized concept of covariant derivative, without assuming that vector field is smooth at some $A'$, but only $n + 1$ continuously differentiable on $A_{n+1}$.

*Proof.* Exactly in the same analogy as before.

This statement means, that we have fulfilled our aim, to define a concept of covariant derivative “lifted” from the smooth manifold and smooth tensor algebra to GTF in sense of association. It completes the required connection with the old tensor calculus.

28Notice, that the $N$ set is the same for every chart from $\tilde{A}_c$ and since we are limited only to the $\tilde{A}_c$ atlas, it is in this context the only set considered.
Conjecture 7.4. Take $U^i$ being smooth particularly at $A'$, $T^\mu_{\nu\alpha\beta} \in D_{\alpha(\omega)}(M)$ having associated field and $\Gamma_{\nu\alpha}^{\mu} \in \Gamma_2^{3}(M)$, than $D_{C(U)}T^\mu_{\nu\alpha\beta}$ has associated field which is a covariant derivative of associated (tensor) field to $T^\mu_{\nu\alpha\beta}$ (hence as $\otimes$ covariant derivative operator commutes in some sense).

Note that except of covariant derivative we can easily generally define for every $\Gamma_{\alpha(\omega)}^{\mu}(M)$ the Lie derivative along arbitrary, in $A'$ atlas smooth vector field (not generalized, but we still did not prove anything about larger classes of vector fields, than smooth vector fields in case of covariant derivative). This is simply defined as $(L_VA, \Psi) \doteq (A, L_V\Psi)$, since the Lie derivative along the given vector field is a mapping from $n$-forms to $n$-forms, preserving their required properties for such $C_{(\omega)}^{\mu}(M)$ class, that holds $A' \subset \tilde{A}$. 
8 Basic discussion of previous results and open questions

8.0.3 Interpretation of physical “laws” and their solutions

We build up the algebra of GTF, being able to incorporate the concept of covariant derivative (with the given presuppositions about vector fields and connection) for set of algebras constructed from specific distributional objects. The use in physics is meaningful where the operations of tensor multiplication and covariant derivative give a map from appropriate subclass of \( D^m_{\alpha}(M) \) class to the elements of \( \tilde{\Gamma}^m(M) \) containing a \( D^m_{\alpha}(M) \) element. This is always provided to work between appropriate subclass of piecewise continuous distributional objects, but a given physical equivalence might specify a larger set of objects for which these operations provide such mapping. Note that the whole problem lies in the multiplication of distributions outside the \( D^m_{\alpha E}(M) \) class (for instance it can be easily seen that square root of \( \delta(Ch_k, q) \) as introduced above is not equivalent to any distribution), since the product does not have to be necessarily equivalent to a distributional object and even worse, in such cases the multiplication is not necessarily a mapping between equivalence classes of the given algebra elements (\( \tilde{D}^m_{nA}(M) \)). The same holds about contraction.

But even in such cases there can be a further hope. For example we can abandon the requirement that certain quantities must be linear (for example metric connection) and only some results of their multiplication are really physical (meaning linear). Then it is a question whether they should be constructed (constructed from the linear objects as for example metric connection from the metric tensor) through the exact equality or only through the equivalence. If we put the weaker (equivalence) condition, then there is a vast number of objects we can choose, and many other important questions can be posed. Even in the case the mathematical operations do depend on particular representatives of the equivalence classes there is no reason to be worried, in such situation it might be an interesting question if there are any specific “paths” which can be used to solve the physical equivalencies. The other point is, that if these operations do depend on members of those classes, then we can reverse this process, and for example in the case of multiplication of two delta functions we can find their nonlinear equivalents first and then take their square, thus obtaining possibly an object belonging to an equivalence class of a distribution.

As I mentioned in the introduction, these are not attempts to deal with physical problems in the random, ad hoc way. Rather I want to give the following interpretation to what is happening: Physical differential equations
should be changed into equivalencies. For that reason they have plenty solutions ("solution" means any object fulfilling given relations) in the given algebra (after will be proven that covariant derivative is well defined for example for all the GTF elements, then it is the whole GTF algebra). It is much "more" solutions than it was in case of classical partial differential equations (but all the smooth distributions representing "classical" solutions of the "classical" initial value problem are there), but what is under the question is the possibility to formulate the initial value problem for larger classes of objects than $D_{nE}^{tm}(M)$ and $D_{n(A')o}^{tm}(M)$ (see the next section) and if this is possible than there is another question about the physical meaning of those solutions. But even in the case we get nonlinear objects as solutions of some general initial value problem formulation, it does not have to be necessarily something surprising, the case where physical laws are solved also by physically meaningless solutions is nothing new. What is suggested to be usually a physically meaningful solution of the given equivalence is given by most of the distributive solutions (that is why classical calculus is so successful), but they do not have to be necessarily the only ones.

8.0.4 Interpretation of physical quantities

Now there is another topic to discuss, the interpretation of physical observables as "amounts" of quantities on the open sets is dependent on our notion of volume. So how shall we get the notion of volume in such context? First, by volume we mean a volume of an open set. We will consider only the open sets belonging to some $\Omega_{Ch}$ though. Let us now assume that we have a metric tensor from $D_{nE}^{tm}(M)$. This induces a (volume) 4-form. By volume of an open set $\Omega' \subset \Omega_{Ch}$ we understand an integral of this 4-form, multiplied by noncontinuous function which is equal to 1 inside $\Omega'$ and 0 outside $\Omega'$. Such a 4-form is definitely an object from $CP(M)$ (particularly from $CP(\Omega_{Ch})$). The "amounts" of physical quantities we obtain, when our $D_{nA}^{tm}(M)$ objects act on that resulting 4-form.
9 Some notes on initial value problem within the partial differential $\approx$ equivalencies on $D_{n}^{m}(M)$

In this section we want to complete our mathematical structure, to get a clear idea, how a physical problem can be clearly formulated in our language. As we see, the last part missing is a clear definition, what we mean by initial value problem in our theory. That is the purpose of this part of our paper. It is again divided into what we call “basic ideas” and “additional ideas”. The first part is of a big importance, the second part is less important, it just gives a suggestion how to recover the classical geometric concept of geodesics in our theory.

9.1 Basic ideas

9.1.1 The approach giving the definition of the initial value problem

Take a hypersurface (this can be obviously generalized to any submanifold of lower dimension) $N \subset M$, which is such that it is at some subatlas $\tilde{A}' \subset A$ a piecewise smooth submanifold. $\tilde{A}'$ is also such that $\exists A'$ such that $A' \subset \tilde{A}'$.

If we consider simply space of 3-form fields on that hypersurface separately from our space of 4-forms on $M$ (we give up on the idea of relating them), we get certainly in two different cases important maps:

Take the case of such $D_{nE}^{m}(M)$ objects, that at every chart from $\tilde{A}'$ their associated (tensor) fields are defined everywhere on $N$, apart of a set having any 3 dimensional L-measure on $N$ equal to 0. At every smooth subatlas $A' \subset \tilde{A}'$ we define the map from $D_{nE}^{m}(M)$ to $D_{nE}^{m}(N)$ simply by embedding of the given associated tensor field to that $D_{nE}^{m}(M)$ object$^{29}$ to $N$. This defines a tensor field on a piecewise smooth manifold $N$ and in this way (as a map from 3-forms to integral from that 3-form multiplied by the tensor field) an $D_{nE}^{m}(N)$ object. What remains to be proven is that at any smooth subatlas we map the same $D_{nE}^{m}(M)$ object to $D_{nE}^{m}(N)$, otherwise this formulation is meaningless.

The other case is a map $D_{nE}^{m}(M) \rightarrow D_{nE}^{m}(N)$ $(A' \subset \tilde{A}')$, defined in a simple way: The objects from $D_{nE}^{m}(M)$ are mapped simply as associated smooth tensor fields (in the previous sense) and the rest of distributional

$^{29}$As we said the given $D_{nE}^{m}(M)$ object we can define by integral taken from a 4-form, being argument of this object, multiplied by this tensor field.
objects from \( D'_{n(A')} M \) is mapped through the fact that they are weak limits of smooth distributions \( D'_{n(A')} (M) \) (which is coordinate independent for arbitrary tensor distributions). This is achieved by embedding the smooth distributions first and taking the limit afterwards (exchanging the order of operations). The basic conjecture is that if this limit exists on \( M \), it will exist on \( N \) (in the weak topology), by taking embedded smooth distributions.

Now we can say that initial value conditions of (for example) second order partial differential equations are given by two distributional objects from \( D'_{n(A')} (M') \) on two hypersurfaces \( N_1, N_2 \) (not intersecting each other) and the solution is a distributional object from the same class, which fulfills the \( \approx \) equation and is mapped as defined before to this two distributional objects.

9.1.2 Useful conjecture related to our approach

Note, that we can possibly (if the limit commutes) extend this “initial value” approach through the \( D'_{nS}(M) \) class to all the weak limits of those objects (if we take the weak topology). This means extension to the class of objects belonging to \( D'_{n}(M) \), being for any chart from \( A \) defined on the whole \( C^P(M) \) and the class being such that \( D'_{nS}(M) \) is dense there.

9.2 Additional ideas

9.2.1 “Null geodesic solution” conjecture

**Conjecture 9.1.** Pick some atlas \( A' \). Pick some GTF \( g_{\mu\nu} \in D^0_{2S(A')} (M) \), such that it has associated field to \( g_{\mu\nu} \), being a Lorentzian signature metric tensor field. Take some \( \Omega_{Ch} \) and two spacelike hypersurfaces \( H_1, H_2 \), \( H_1 \cap H_2 = \{0\}, H_1 \cap \Omega_{Ch} \neq \{0\}, H_2 \cap \Omega_{Ch} \neq \{0\} \), being such that there exist two points \( q_1, q_2 \), \( q_1 \in H_1 \cap \Omega_{Ch}, q_2 \in H_2 \cap \Omega_{Ch} \) separated by a null curve geodesics (relatively to a given tensor field associated to \( g_{\mu\nu} \), and the geodesics lies within \( \Omega_{Ch} \). Construct a chart \( Ch_k(\Omega_{Ch}) \in A' \), which fulfills that both of the hypersurfaces are hypersurfaces (they are smooth manifolds relatively to \( A' \)) given by \( u = const. \) condition and given geodesics is representing \( u \)-coordinate curve.

Take classical free field equation with equivalence: \( \Box g \Phi \approx 0 \) (with \( g^{\mu\nu} \) as previously defined).

Then look for the distributional solution of our equation with the initial value conditions being \( \delta(Ch'_k(\Omega_{Ch} \cap H_1), q_1) \in D'_{n(A')} (H_1) \) on first hypersurface and \( \delta(Ch''_k(\Omega_{Ch} \cap H_2), q_2) \in D'_{n(A')} (H_2) \) on the second hypersurface \( Ch'_k(\Omega_{Ch} \cap H_1), Ch''_k(\Omega_{Ch} \cap H_2) \) are coordinate charts being the
same on intersection of given hypersurface and \( \Omega_{Ch} \) than original coordinates just with removed \( u \). Then the solution of this initial value problem is a mapping \((\Phi, \Psi)\) which is such that it is in \( Ch_k(\Omega_{Ch}) \) expressed as \( \int du \prod_i dx^i(\delta(x^i(q_1)))\Psi'(u, x^j) \) (where \( x^i(q_1) \) is image of \( q_1 \) in chart mapping \( Ch_k(\Omega_{Ch}) \) and \( \Psi' \) is density expressing \( \Psi \) form in \( Ch_k(\Omega_{Ch}) \)).

We can formulate similar conjectures for timelike and spacelike geodesics, just exchanging point separation by null curve by separation by timelike or spacelike curve, and exchanging our equation by equation \((\Box_g \pm m^2)\Phi \approx 0\) (\( m \) being arbitrary nonzero real number) with \( \pm \) depending on which signature we take and whether we look for timelike or spacelike geodesics. All the rest should be the same.

Some insight to our conjectures can be brought by calculating the massless case for flat Minkowski space, using modified cartesian coordinates \((u = x - ct, x, y, z)\). We get the expected results.
10 Relation to approaches described and to particular applications

Our approach is in some sense a generalization of Colombeau approach from [4], which is equivalent to canonical $\mathbb{R}^n$ approach. So for $\Omega_{Ch}$, after picking some $Ch'(\Omega_{Ch}) \in A'$ (which determines classes $A^n(M)$ related to those coordinates), and taking into account only $C^p_{S(A')}(\Omega_{Ch})$ ($A' \subset \tilde{A}$) objects (hence considering $D^m_{n(A'o)}(M)$ class only), we obtain from our construction the mathematical language used in [4]. But all the basic equivalence relations from Colombeau approach have been generalized first to $D^m_{n(A'o)}(M)$ objects and second to appropriate subclasses of $D^m_{nA'E}(M)$ objects.

Now there are certain statements in $\mathbb{R}^n$, where one has to check whether they are not just a result of this specific reduction. A good example is a statement

$$H^n \delta \approx \frac{1}{n+1} \delta$$ (38)

($H$ is Heaviside distribution obviously). The only thing we have to do is to interpret the symbols inside this equation geometrically. This is a $\mathbb{R}^1$ relation. $H$ is understood as a $D'_{(A'o)}(M)$ element and defined on the manifold (one dimensional, so the geometry would be quite trivial) by integral containing a function on $M$ given by inverse coordinate mapping substituted to $H$. The derivative is a covariant derivative along the smooth vector field, which is constant and unit in that fixed coordinates. Then $\delta$ can be reinterpreted as $\delta(Ch_k, q)$, where $Ch_k$ is the given fixed chart which we are using and $q$ is the 0 point in our chart. Then the given relation holds also in our case, since it is obvious that (see our theorem 8.2 in the covariant derivative section) $H' = \delta(q, Ch_k)$ and

$$H' = L(H)' = L(H^{n+1})' \approx (H^{n+1})' = (n+1) H^n H'$$ (39)

By $L$ we mean here a regular distribution using the object in the brackets as a function under integral, hence in this case an object from $D'_E(M)$ (to avoid confusion in the notation, from how we used it, follows $L(H) = H$). This is a nice, but rather trivial illustration.

This can be generalized to more nontrivial cases. Take the $\mathbb{R}^n$ topological manifold. Fix the chart covering the whole manifold, where we can express Heaviside distribution as $H(x) \in D'_{(A'o)}(M)$, hence the hypersurface where it is nonsmooth is given at that chart by $x = 0$. Now the derivative sign will be in fact a covariant derivative taken along a smooth vector field being
perpendicular (relatively to this metric\textsuperscript{30}) to the hypersurface on which is $H$ in those coordinates (and in every others) noncontinuous. We easily see that the given covariant derivative of $H$ gives a distribution (call it $\tilde{\delta} \in D'(A_0(M))$) which is at that chart expressed as

$$\tilde{\delta}(\Phi) = \left(\delta(x), \int \Phi(x, y, z...t) dydz...dt\right)$$  \hspace{1cm} (40)

(the hypersurface where is $H$ nonsmooth is expressed as $x = 0$ obviously). In some sense this distribution reminds us the “geodesic” distribution from the part above. Then holds the above relation:

$$H^n \tilde{\delta} \approx \frac{1}{n+1} \tilde{\delta}$$  \hspace{1cm} (41)

This generalized form of our previous statement can be useful in case of computation with Heaviside functional metrics (computation of connection in fixed coordinates).

But the considerations for example mean, that the result derived in canonical Colombeau algebras on $\mathbb{R}^n$ by [4] can be achieved in our formalism as well. This holds also for geodesic computation in curved space geometry in [23].

The results derived in special Colombeau algebras (in geometrically non-trivial cases) are more complicated, since the authors use the strongest, $A^\infty(\mathbb{R}^n)$ version of the theory and this is not contained in our particular chart representations (since we are using only $A^n(\mathbb{R}^n)$ with finite $n$). It is obvious that the equivalences must work for this strong formulation (since obviously $A^\infty(M) \subset A^n(M) \forall n \in \mathbb{N}$), and uniqueness of distribution solution must hold as well, so at this stage there seems to be no obstacle to reformulate our theory by using $A^\infty(Ch_k, q, \Omega_{Ch})$ classes (and taking elements from $D'(A_0(M)$ at least), if necessary. But it is unclear whether one can make the calculations (provided by those authors) also in our weaker formulation.

This strong formulation was used in the Schwarzschild case [15], but there is a problem. The fact that authors regularize various functions piece by piece should not be any problem in Colombeau theory\textsuperscript{31}. But the problem lies as I already mentioned in the first part, in the use of formula for $R^\mu_{\nu\rho\sigma}$ derived within smooth tensor field algebra. In fact it is clear, that if we want

\textsuperscript{30}Note that since it is flat space it makes sense to speak about a perpendicular vector field, since we can uniquely transport vectors to the hypersurface.

\textsuperscript{31}Although the authors use in the first part, not being necessarily connected with the results, quite problematic embeddings.
to derive in that case Ricci tensor straight from its definition, we cannot avoid multiplications of delta function by a non-smooth function, which is in Colombeau theory deeply non-trivial.

In the other cases, the case of Kerr’s theory and the case of conical spacetimes theory, this problem appears as well and as a consequence of that, results are mollifier dependent, not being consequences of our theory in the strict sense anymore (but there is no reasonable mathematical theory where they make better sense). This means that the better understanding of these results will be necessary. By better understanding provided by our theory we mean derivation of these results by a net of equivalencies, by taking some quantities to be nonlinear. So these results should follow from our basic principle, saying that the equivalencies are the fundamental part of all the mathematical formulation of physics.

11 Conclusions

The main objectives of this work were to build foundations of a mathematical language, which would reproduce the old language of smooth tensor calculus and extend it in the same time, where it is appropriate. The reasons for these objectives were given at the beginning of this paper. This paper is a first step to such theory, but it already achieves its basic goals, so these results we consider as worthwhile independently on the fact how successful the later work on that topic will be.

Just to summarize, the result of our work is a theory based purely on equivalencies instead of equalities, using a well defined concept of generalized tensor field and the covariant derivative operator, which is well defined at least on the proper subclass of generalized tensor fields. We also defined (using some conjectures) what we mean in our theory by initial value problem for partial differential equivalencies. Our theory naturally relates to many results beyond the classical smooth tensor calculus, already derived.

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