On a $\psi_1$ - norm estimate of sum of dependent random variables using simple random walk on graph

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Abstract

We obtained a $\psi_1$ estimate for the sum of Rademacher random variables under condition that they are dependent.

1 Introduction

Let $X_1, \ldots, X_N$ be a sequence of independent real valued random variables and let $\Sigma = \sum_{i=1}^{N} X_i$. The estimate of moments of $\Sigma$, that is of the quantities $\|\Sigma\|_p = \mathbb{E}(\Sigma^p)^{1/p}$, are often appear in many areas of mathematics. The growth of moments is closely related to the behavior of the tails of $\Sigma$.

Probabilists have been interested in the moments of sums of random variables since the early part of last century. Khinchine 1923 paper appears to make the rst significant contribution to this problem [8]. It provides inequalities for the moments of a sum of Rademacher random variables. In 1970, Rosenthal generalised Khinchine result to the case of positive or mean-zero random variables [12]. Further renements to these bounds have been made by Latala and Hitczenko, Montgomery-Smith and Oleszkiewiez in more recent times [9, 7]. Nowadays, it appears that in the different applications of mathematics, statistics, computer science and engineering similar estimates for the case when random variables are not independent are important (see for example [6, 11]).

Our aim in the present work is to find bound on the sum of random variables, $\Sigma = \sum_{i=1}^{2n} X_i$, in the case when $X_i = a_i \varepsilon_i$, where $a \in \mathbb{R}^{2n}$ and $\varepsilon_i, i = 1, \ldots, 2n$, under an additional assumption on the Rademacher random variables, namely

$$S = \sum_{i=1}^{2n} \varepsilon_i = 0. \tag{1}$$

To shorter notation, by $\mathbb{E}_S$ we denote an expectation with assumption (1).

Recall, the Rademacher random variables satisfying the following condition: $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}$, for $i = 1, \ldots, 2n$. As usual for $\varepsilon \in \{\pm 1\}^{2n}$ by $\varepsilon_1, \ldots, \varepsilon_{2n}$ we denote coordinates of $\varepsilon$. 

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Consider the following set

$$\Omega = \left\{ \varepsilon \in \{-1, 1\}^{2^n} \mid \sum_{i=1}^{2^n} \varepsilon_i = 0 \right\} = \left\{ \varepsilon \in \{-1, 1\}^{2^n} \mid \text{card} \{ i : \varepsilon_i = 1 \} = n \right\}. \quad (2)$$

Thus, for $\varepsilon \in \Omega$ the sequence of its coordinates is a sequence of a weekly dependent Rademacher random variables.

For set $\Omega$ we put into correspondence the group $\Pi_{2n}$ of all permutations of set $\{1, ..., 2n\}$ as

$$\sigma \in \Pi_{2n} \longleftrightarrow A_{\sigma} = \{ \varepsilon \in \Omega \mid \varepsilon_i = 1 \text{ if } \sigma(i) \leq n; \varepsilon_i = -1 \text{ if } \sigma(i) > n \}.$$  

Define $f : \Pi_{2n} \rightarrow \mathbb{R}$ by

$$f(\sigma) := \left| \sum_{i=1}^{n} a_{\sigma(i)} - \sum_{i=n+1}^{2n} a_{\sigma(i)} \right|. \quad (3)$$

Note, that $\mathbb{E}_{\mathcal{S}} \left| \sum_{i=1}^{n} a_i \varepsilon_i \right|^p = \mathbb{E} |f|^p$. Thus, it is enough to estimate $p$-th moments of $f$.

In the present paper we obtained the following result.

**Theorem 1.1.** Let $f$ defined as above. Then, for $p \geq 2$,

$$(\mathbb{E} f^p)^{1/p} \leq \mathbb{E} |f| + C p \|a\|_2.$$

The paper is organized as following. In the next section we provide the necessary known tools and definitions. In Section 3, we will establish bounds on $\psi_1$-norm.

## 2 Preliminaries

### 2.1 Orlicz norms and $\psi_\alpha$-estimates.

**Definition 2.1.** An Orlicz function is a convex, increasing function $\psi : [0, \infty) \rightarrow [0, \infty]$, such that $\psi(0) = 0$ and $\psi(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Classical examples of Orlicz functions are

$$\varphi_p(x) = \frac{x^p}{p}, \quad p \geq 1, \forall x \geq 0 \quad (4)$$

and

$$\psi_\alpha(x) = e^{x^\alpha} - 1, \quad \alpha \geq 1, \forall x \geq 0. \quad (5)$$
Definition 2.2. Let $\psi$ be an Orlicz function. For any real random variable $X$ on a measurable space $(\Omega, \sigma, \mu)$, define its $L_\psi$-norm by

$$\|X\|_\psi := \inf\{c > 0 : E \psi(|X|/c) \leq 1\}.$$  

We say $X$ is $\psi$-variable if $\|X\|_\psi < \infty$.

The following well-known theorem describes the behaviour of a random variable with bounded $\psi_\alpha$-norm (see for example [3]).

Theorem 2.3. Let $X$ be real-valued random variable and $\alpha \geq 1$. The following assertions are equivalent:

1. There exists $K_1 > 0$, such that $\|X\|_{\psi_\alpha} \leq K_1$.

2. There exists $K_2 > 0$, such that for every $p \geq \alpha$,

$$\left(E|X|^p\right)^{1/p} \leq K_2 p^{1/\alpha}.$$  

3. There exists $K_3, K'_3 > 0$, such that for every $t > K'_3$,

$$P(|X| \geq t) \leq \exp\left(-t^{\alpha}/K_3^{\alpha}\right).$$

Note, $K_2 \leq 2eK_1$, $K_3 \leq eK_2$, $K'_3 \leq e^2K_2$, $\alpha \leq 2\max(K_2, K'_3)$.

4. In the case $\alpha > 1$, let $\beta$ be such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. There exist $K_4, K'_4 > 0$ such that for every $\lambda \geq 1/K'_4$,

$$E \exp \left(\lambda|X|\right) \leq \exp\left(\lambda K_4\right)^{\beta}.$$  

Note, $K_4 \leq K_1$, $K'_4 \leq K_1$, $K'_3 \leq 2K_4^{\beta}/(K'_4)^{\beta-1}$.

The space $L_\psi(\Omega, \sigma, \mu) = \{X : \|X\|_\psi < \infty\}$ is the Orlicz space associated to $\psi$. Note that the Orlicz space associated to function $\varphi_p$, defined by (4), is the classical $L_p$-space.

2.2 Simple random walk on graph.

Let $G(V, E)$ be a connected undirected graph, where $V$ stays for a set of vertices and $E$ is a set of edges. A simple random walk is a sequence of vertices $v_0, v_1, \ldots, v_t$, where $v_i \sim v_{i+1}$ (that is $\{v_i, v_{i+1}\} \in E$) for $i = 0, 1, \ldots, t - 1$. That is, given an initial vertex $v_0$, select randomly an adjacent vertex $v_1$, and move to this neighbor. Then, select randomly a neighbor $v_2$ of $v_1$, and move to it, etc. The probability it moves from vertex $v_i$ to $v_{i+1}$ (assuming it sits at $v_i$) is given by

$$p(v_i, v_{i+1}) = \begin{cases} 
\frac{1}{\deg(v_i)}, & \text{if } v_i \sim v_{i+1} \\
0, & \text{otherwise},
\end{cases}$$

(6)
where \( deg(v_i) \) denotes the degree of vertex \( v_i \). This is a walk using a transition probability matrix, \( P = (p(v_i, v_{i+1}))_{v_i, v_{i+1} \in V} \). The transition probability \( (\ref{eq:transition_probability}) \) has a reversible equilibrium probability distribution \( \mu(v_i) \). That is,

\[
\mu(v_i) p(v_i, v_{i+1}) = \mu(v_{i+1}) p(v_{i+1}, v_i)
\]

and \( \mu(v_i) \) is proportional to \( deg(v_i) \).

Let \( I \) be the \( V \times V \) identity matrix. The discrete Laplacian is the matrix \( L = P - I \) with its eigenvalues \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \), ordered in non-increasing order. The smallest eigenvalue, \( \lambda_1 > 0 \), is called the spectral gap of the random walk.

For \( f : V \rightarrow \mathbb{R} \) define

\[
|||f|||_\infty^2 = \frac{1}{2} \sup_{v_i \in V} \sum_{v_{i+1} \in V} |f(v_i) - f(v_{i+1})|^2 p(v_i, v_{i+1}).
\] (7)

We will use the following concentration inequality (see \cite{1} or \cite{10}):

**Theorem 2.4.** Assume that \((p, \mu)\) is reversible on the finite graph \( G(V, E) \), and let \( \lambda_1 > 0 \) be the spectral gap. Then, if \( |||f|||_\infty^2 < \infty \), we have

\[
\mu(\{ f > \int f \, d\mu + t \}) \leq 3 \exp \left( \frac{-t \sqrt{\lambda_1}}{2|||f|||_\infty^2} \right).
\] (8)

Let us now specialize to \( V = \Pi_{2n} \), the group of all permutations \( \sigma \) of the set \( \{1, \ldots, 2n\} \), and to \( E = \{(\sigma, \sigma \tau) \mid \tau \text{ is a transposition on } \Pi_{2n}\} \). The transition probability \( p(\sigma, \sigma \tau) \) on \( G = (\Pi_{2n}, E) \) is

\[
p(\sigma, \sigma \tau) = \frac{2}{(2n)^2},
\] (9)

and reversible equilibrium distribution \( \mu \) on \( \Pi_{2n} \) is a unique invariant measure for \( p \) (see for example \cite{4} for these facts). Also, as proved in \cite{5}, the spectral gap of the random transposition walk on \( \Pi_{2n} \) is \( \lambda_1 = \frac{2}{2n} = \frac{1}{n} \). Thus, the concentration inequality \( (\ref{eq:concentration_inequality}) \) for simple random walk on \( G(\Pi_{2n}, E) \) can be rewritten as

\[
\mu(\{ \sigma : f(\sigma) - Ef \geq t \}) \leq \exp \left( \frac{-t}{2|||f|||_\infty^2 \sqrt{n}} \right).
\] (10)

### 3 Proof of Theorem 1.1

We are going to use inequality \( (\ref{eq:concentration_inequality}) \). We calculate first

\[
|||f|||_\infty^2 = \frac{1}{2} \sup_{\sigma \in \Pi_{2n}} \sum_{\tau, \sigma \tau \in \Pi_{2n}} |f(\sigma) - f(\sigma \tau)|^2 p(\sigma, \sigma \tau),
\]
where \( p(\sigma, \sigma \tau) \) is defined in [1].

Consider \( g(\sigma) = \sum_{i=1}^{n} a_{\sigma(i)} - \sum_{i=n+1}^{2n} a_{\sigma(i)} \). Since \( \tau(i, j) \) is a random transposition with \( i, j \) chosen uniformly from the set \( \{1, \ldots, 2n\} \), we obtain
\[
g(\sigma) - g(\sigma \tau) = 2(a_i - a_j)h(i, j),
\]
where
\[
h(i, j) = \begin{cases} 
1, & \text{if } j \leq n < i \leq 2n \\
-1, & \text{if } i \leq n < j \leq 2n \\
0, & \text{otherwise.}
\end{cases}
\]

Thus, \(|f(\sigma) - f(\sigma \tau)|^2 = 4(a_i - a_j)^2 h^2(i, j)\). And we can calculate
\[
\|\| f \|\|_\infty^2 = \frac{1}{n^2} \sum_{\tau(i, j)} (a_i - a_j)^2 h^2(i, j)
\]
\[
= \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=n+1}^{2n} (a_i - a_j)^2 h^2(i, j)
\]
\[
= \frac{2}{n^2} \left( n \|a\|_2^2 - 2 \sum_{i=1}^{n} \sum_{j=n+1}^{2n} a_i a_j \right)
\]

Since
\[
- \sum_{i=1}^{n} \sum_{j=n+1}^{2n} a_i a_j \leq \sum_{i=1}^{n} \sum_{j=n+1}^{2n} \frac{a_i^2 + a_j^2}{2} = \frac{n}{2} \|a\|_2^2,
\]
the last equation can be bounded by
\[
\|\| f \|\|_\infty^2 \leq \frac{4}{n} \|a\|_2^2. \tag{11}
\]

Now, using (10), (11) and an upper bound \( \Gamma(x) = x^{x-1} \), for all \( x \geq 1 \) (see for example [2]), we obtain
\[
\mathbb{E}(f - \mathbb{E}f)^p = \int_{0}^{\infty} \mu((f(\sigma) - \mathbb{E}f)^p \geq t^p) dt^p \leq 6p \int_{0}^{\infty} e^{-t/(4\|a\|_2^2)} t^{p-1} dt
\]
\[
= 6p \Gamma(p) \|a\|_2^p \leq 4p \|a\|_2^2.
\]

Hence
\[
(\mathbb{E}f^p)^{1/p} \leq \mathbb{E}|f| + 24p \|a\|_2.
\]

**Remark:** Note that \( \mathbb{E}|f| \leq (\mathbb{E}|f|^2)^{1/2} \), where \( \mathbb{E}|f|^2 \) can be directly calculated (see [13]).

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