EVERY TRANSCENDENTAL OPERATOR HAS A NON-TRIVIAL INVARIANT SUBSPACE

YUN-SU KIM

ABSTRACT. In this paper, to solve the invariant subspace problem, contraction operators are classified into three classes; (Case 1) completely non-unitary contractions with a non-trivial algebraic element, (Case 2) completely non-unitary contractions without a non-trivial algebraic element, or (Case 3) contractions which are not completely non-unitary.

We know that every operator of (Case 3) has a non-trivial invariant subspace. In this paper, we answer to the invariant subspace problem for the operators of (Case 2). Since (Case 1) is simpler than (Case 2), we leave as a question.

INTRODUCTION

An important open problem in operator theory is the invariant subspace problem. The invariant subspace problem is the question whether the following statement is true or not:

Every bounded linear operator $T$ on a separable Hilbert space $H$ of dimension $\geq 2$ over $\mathbb{C}$ has a non-trivial invariant subspace.

Since the problem is solved for all finite dimensional complex vector spaces of dimension at least 2, in this note, $H$ denotes a separable Hilbert space whose dimension is infinite. It is enough to think for a contraction $T$, i.e., $\|T\| \leq 1$ on $H$. Thus, in this note, $T$ denotes a contraction.

If $T$ is a contraction, then

(Case 1) $T$ is a completely non-unitary contraction with a non-trivial algebraic element, or

(Case 2) $T$ is a transcendental operator; that is, $T$ is a completely non-unitary contraction without a non-trivial algebraic element, or

(Case 3) $T$ is not completely non-unitary.

In this note, we discuss the invariant subspace problem for operators of (Case 2). By using fundamental properties (Proposition 2.2, Proposition 2.3, and Corollary 2.4) of transcendental operators, we answer to the invariant subspace problem for the operators of (Case 2) in Theorem 2.5.

Every transcendental operator defined on a separable Hilbert space $H$ has a non-trivial invariant subspace.

Thus, we answered to the invariant subspace problem for the (Case 2) in Theorem 2.5 and, clearly, we know that every operator of (Case 3) has a non-trivial invariant subspace. Thus, to answer to the invariant subspace problem, it suffices to answer for (Case 1).

Key words and phrases. Algebraic elements; $C_0$-Operators; The Invariant Subspace Problem; Transcendental elements; Transcendental Operators; MSC(2000) 47A15; 47S99.
We do not consider project (Case 1) in this note, and leave as a question:

**Question.** Let $T \in L(H)$ be a completely non-unitary contraction such that $T$ has a non-trivial algebraic element. Then, does the operator $T$ have a non-trivial invariant subspace?

The author would like to appreciate the advice of Professor Ronald G. Douglas.

1. **Preliminaries and Notation**

In this note, $\mathbb{C}$, $\overline{M}$ and $L(H)$ denote the set of complex numbers, the (norm) closure of a set $M$, and the set of bounded linear operators from $H$ to $H$ where $H$ is a separable Hilbert space whose dimension is not finite, respectively.

For a set $A = \{a_i : i \in I\} \subset H$, $\bigvee A$ denotes the closed subspace of $H$ generated by $\{a_i : i \in I\}$.

If $T \in L(H)$ and $M$ is an invariant subspace for $T$, then $T|_M$ is used to denote the restriction of $T$ to $M$, and $\sigma(T)$ denotes the spectrum of $T$.

1.1. **A Functional Calculus.** Let $H^\infty$ be the Banach space of all (complex-valued) bounded analytic functions on the open unit disk $D$ with supremum norm [4]. A contraction $T$ in $L(H)$ is said to be **completely non-unitary** provided its restriction to any non-zero reducing subspace is never unitary.

B. Sz.-Nagy and C. Foias introduced an important functional calculus for completely non-unitary contractions.

**Proposition 1.1.** Let $T \in L(H)$ be a completely non-unitary contraction. Then there is a unique algebra representation $\Phi_T$ from $H^\infty$ into $L(H)$ such that:

- (i) $\Phi_T(1) = I_H$, where $I_H \in L(H)$ is the identity operator;
- (ii) $\Phi_T(g) = T$, if $g(z) = z$ for all $z \in D$;
- (iii) $\Phi_T$ is continuous when $H^\infty$ and $L(H)$ are given the weak$^*$-topology.
- (iv) $\Phi_T$ is contractive, i.e. $\|\Phi_T(u)\| \leq \|u\|$ for all $u \in H^\infty$.

We simply denote by $u(T)$ the operator $\Phi_T(u)$.

B. Sz.-Nagy and C. Foias [4] defined the class $C_0$ relative to the open unit disk $D$ consisting of completely non-unitary contractions $T$ on $H$ such that the kernel of $\Phi_T$ is not trivial. If $T \in L(H)$ is an operator of class $C_0$, then

$$\ker \Phi_T = \{u \in H^\infty : u(T) = 0\}$$

is a weak$^*$-closed ideal of $H^\infty$, and hence there is an inner function generating $\ker \Phi_T$. The **minimal function** $m_T$ of an operator $T$ of class $C_0$ is the generator of $\ker \Phi_T$; that is, $\ker \Phi_T = m_T H^\infty$. Also, $m_T$ is uniquely determined up to a constant scalar factor of absolute value one [1].

1.2. **Algebraic Elements.** In this section, we provide the notion of **algebraic elements** for a completely non-unitary contraction $T$ in $L(H)$.

**Definition 1.2.** [3] Let $T \in L(H)$ be a completely non-unitary contraction. An element $h$ of $H$ is said to be **algebraic with respect to $T$** provided that $\theta(T)h = 0$ for some $\theta \in H^\infty \setminus \{0\}$. If $h \neq 0$, then $h$ is said to be a **non-trivial algebraic element** with respect to $T$.

If $h$ is not algebraic with respect to $T$, then $h$ is said to be **transcendental with respect to $T$**.
2. The Main Results

If $T$ is a contraction, then

(Case 1) $T$ is a completely non-unitary contraction with a non-trivial algebraic element, or

(Case 2) $T$ is a completely non-unitary contraction without a non-trivial algebraic element; that is, every non-zero element in $H$ is transcendental with respect to $T$, or

(Case 3) $T$ is not completely non-unitary.

It is clear for (Case 3). To answer to the invariant subspace problem for (Case 2), we provide the following definition;

**Definition 2.1.** If $T$ is a completely non-unitary contraction without a non-trivial algebraic element; that is, every non-zero element in $H$ is transcendental with respect to $T$, then $T$ is said to be a transcendental operator.

**Proposition 2.2.** If $T : H \to H$ is a transcendental operator, then, for any $\theta \in H^\infty \setminus \{0\}$, $\theta(T)$ is one-to-one.

*Proof.* Suppose that $\theta(T)$ is not one-to-one for a function $\theta \in H^\infty \setminus \{0\}$. Then, there is a non-zero element $h$ in $H$ such that $\theta(T)h = 0$; that is, $h$ is a non-trivial algebraic element with respect to $T$. This, however, is a contradiction, since $T$ is a transcendental operator. Thus, $\theta(T)$ is one-to-one for any $\theta \in H^\infty \setminus \{0\}$.

□

Recall that an arbitrary subset $M$ of $H$ is said to be linearly independent if every nonempty finite subset of $M$ is linearly independent.

**Proposition 2.3.** If $T : H \to H$ is a transcendental operator, then, for any non-zero element $h$ in $H$, $M = \{T^n h : n = 0, 1, 2, \cdots\}$ is linearly independent.

*Proof.* Let $h \in H \setminus \{0\}$ be given. Suppose that $M = \{T^n h : n = 0, 1, 2, \cdots\}$ is not linearly independent. Then, there is a polynomial $p \in H^\infty \setminus \{0\}$ such that $p(T)h = 0$. Thus $h$ is a non-trivial algebraic element with respect to $T$. This, however, is a contradiction, since $T$ is a transcendental operator. Therefore, $M$ is linearly independent.

□

**Corollary 2.4.** Under the same assumption as Proposition 2.3, for a given function $\theta \in H^\infty \setminus \{0\}$, $M' = \{\theta(T)^n h : n = 1, 2, \cdots\}$ is linearly independent.

*Proof.* In the same way as Proposition 2.3, it is proven.

□

By a densely defined operator $K$ in $H$, we mean a linear mapping $K$ on the domain $D(K)$ (which is a subspace of $H$ and dense in $H$) of $K$ into $H$ [5]. Recall that

$$D(SK) = \{x \in D(K) : Kx \in D(S)\},$$

where $S$ and $K$ are unbounded operators [5].

Finally, the time has come to answer to the invariant subspace problem for operators of (Case 2).
Theorem 2.5. If $T$ is a transcendental operator in $L(H)$ and $\lambda \in \sigma(T)$, then

(i) $S = T - \lambda I_H$ has a non-trivial invariant subspace, where $I_H$ is the identity operator on $H$, and

(ii) $T$ also has a non-trivial invariant subspace.

Proof. (i) Since $T$ is transcendental, $S \neq 0$ (Note that if $S = 0$, then $p(T) = 0$ where $p(z) = z - \lambda$), and by Proposition 2.2, $S (= p(T))$ is one-to-one. Since $S$ is not invertible, $S$ is not onto. Let $h$ be a non-zero element in $H$ such that $h$ does not belong to the range of $S$ and

\begin{equation}
M = \bigvee \{S^n h : n = 0, 1, 2, \ldots\}.
\end{equation}

If $M \neq H$, then $M$ is a non-trivial invariant subspace for $S$, and so we assume that $M = H$. If

\begin{equation}
M' = \bigvee \{S^n h : n = 1, 2, \ldots\},
\end{equation}

then we will show that $h \notin M'$. Since $S$ is one-to-one and $h \neq 0$,

\begin{equation}
M' \neq \{0\}.
\end{equation}

Suppose that $h \in M'$.

By Corollary 2.4, we conclude that $\{S^n h : n = 1, 2, \ldots\}$ is linearly independent, and so, by Gram-Schmidt process, we have an orthonormal basis $B$ of $M'$ such that

\begin{equation}
B = \{P_i(S)h : i = 1, 2, \ldots\},
\end{equation}

where $P_i(i = 1, 2, \ldots)$ is a polynomial satisfying $\bigvee\{S^n h : n = 1, 2, \ldots, m\} = \bigvee\{P_i(S)h : i = 1, 2, \ldots, m\}$ for any $m \in \{1, 2, \ldots\}$.

Then,

\begin{equation}
h = \sum_{i=1}^{\infty} a_i P_i(S)h
\end{equation}

where $a_i \in \mathbb{C}$.

It follows that

\begin{equation}
\sum_{i=0}^{\infty} a_i P_i(S)h = 0
\end{equation}

where $a_0 = -1$, and $P_0(S) = I_H$.

For any $k \in \{0, 1, 2, 3, \ldots\}$, by equation (2.7),

\begin{equation}
\sum_{i=0}^{\infty} a_i P_i(S)(S^k h) = \lim_{m \to \infty} \sum_{i=0}^{m} a_i P_i(S)(S^k h) = \lim_{m \to \infty} S^k (\sum_{i=0}^{m} a_i P_i(S)h) = 0
\end{equation}

By the same way as above, since $\{S^n h : n = 0, 1, 2, \ldots\}$ is linearly independent, by Gram-Schmidt process, we have an orthonormal basis $B'$ of $H$ such that

\begin{equation}
B' = \{e_i = f_i(S)h : i = 0, 1, 2, \ldots\},
\end{equation}

where $f_i(i = 0, 1, 2, \ldots)$ is a polynomial satisfying $\bigvee\{S^n h : n = 0, 1, 2, \ldots, m\} = \bigvee\{f_i(S)h : i = 0, 1, 2, \ldots, m\}$ for any $m \in \{0, 1, 2, \ldots\}$. Note that $c_0 = \frac{h}{\|h\|}$.

Clearly, $\sum_{i=0}^{\infty} a_i P_i(S)$ is linear on \{\(c_n S^n h : c_n \in \mathbb{C} \text{ and } n = 0, 1, 2, \ldots\)\}, and by equation (2.5), $\sum_{i=0}^{\infty} a_i P_i(S)$ is a densely defined operator in $H$. It is not assumed that $\sum_{i=0}^{\infty} a_i P_i(S)$ is bounded or continuous.
By equations (2.8) and (2.9), we have that, for any $e_i \in B'$,

\begin{equation}
\sum_{i=0}^{\infty} a_i P_i(S) \epsilon_i = 0.
\end{equation}

Thus, if

\begin{equation}
A = \{ \sum_{i=0}^{m} c_i e_i : c_i \in \mathbb{C}, \text{ and } m = 0, 1, 2, \cdots \},
\end{equation}

then, by equation (2.10),

\begin{equation}
\sum_{i=0}^{\infty} a_i P_i(S) x \equiv 0
\end{equation}

for any $x \in A \subset D(\sum_{i=0}^{\infty} a_i P_i(S))$.

Let

\[ K = \sum_{i=1}^{\infty} a_i g_i(S), \]

where $g_i$ is a polynomial such that

\begin{equation}
P_i(z) = zg_i(z)
\end{equation}

for $i \in \{1, 2, 3, \cdots \}$. Note that by the definition of $P_i$, we can easily find the polynomial $g_i$ satisfying equation (2.13) for $i \in \{1, 2, 3, \cdots \}$.

Since $\{S^n h : n = 1, 2, 3, \cdots \} \subset D(K)$ (by equation (2.6)) and $h \in M'$ (defined in (2.3)) by assumption, $K$ is a densely defined operator in $H$. It is not assumed that $K$ is bounded or continuous.

By equation (2.12), since $a_0 = -1$,

\begin{equation}
(SK)(x) = S(\sum_{i=1}^{\infty} a_i g_i(S))(x) = \sum_{i=1}^{\infty} a_i P_i(S)(x) = x,
\end{equation}

for any $x \in A \subset D(K)$ (Note that $A \subset D(SK) \subset D(K)$ by equation (2.1) [5]).

Since $e_0 = \frac{h}{\|h\|}$ and $e_0 \in A$, by equation (2.14), we have that $SK(e_0) = \frac{h}{\|h\|}$. Thus, $h$ belongs to the range of $S$, but it is a contradiction.

Therefore,

\begin{equation}
h \notin M'
\end{equation}

Thus, by (2.4) and (2.15), we conclude that $M'$ is a non-trivial invariant subspace for $S$.

(ii) In (i), $T(M') = (S + \lambda I_H)M' \subset M'$

Therefore, $M'$ is also a non-trivial invariant subspace for $T$.

\[ \square \]

References

[1] H. Bercovici, Operator theory and arithmetic in $H^\infty$, Amer. Math. Soc., Providence, Rhode Island (1988).

[2] S.W. Brown, Some invariant subspaces for subnormal operators, Integral Equations Operator Theory 1 (1978), 310-333.

[3] Yun-Su Kim, $C^0$-Hilbert Modules, preprint, 2007.

[4] B. Sz.-Nagy and C. Foias, Harmonic analysis of operators on Hilbert space, North-Holland, Amsterdam (1970).

[5] W. Rudin, Functional Analysis, McGraw-Hill, Second Edition (1991).
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TOLEDO, TOLEDO, OHIO, U.S.A.
E-mail address: Yun-Su.Kim@utoledo.edu