High order difference schemes for a time fractional differential equation with Neumann boundary conditions

Seakweng Vong∗ Zhibo Wang†

Abstract

Based on our recent results, in this paper, a compact finite difference scheme is derived for a time fractional differential equation subject to the Neumann boundary conditions. The proposed scheme is second order accurate in time and fourth order accurate in space. In addition, a high order alternating direction implicit (ADI) scheme is also constructed for the two-dimensional case. Stability and convergence of the schemes are analyzed using their matrix forms.

Keywords: Time fractional differential equation, Neumann boundary conditions, compact ADI scheme, weighted and shifted Grünwald difference operator, convergence

1 Introduction

Fractional differential equations have grown to be the focus of many studies due to their various applications. Readers can refer to the books [1, 2] for background of these equations. One of the key features of the fractional derivatives is the nonlocal dependence which causes difficulties when numerical schemes are designed for solving fractional differential equations. However, with the efforts of numerous researchers, great progress along this direction has been made in recent years. Interested readers can refer to [3]–[25] for a brief review. We remark here that the list does not mean to be complete but we try to include those that are more related to the present study.

In this article, we consider high order finite difference schemes for the following time fractional differential equation in a region Ω:

\[ C_0^\gamma D^\gamma_t u(x, t) = \kappa_1 \Delta u(x, t) - \kappa_2 u(x, t) + g(x, t), \quad x \in \Omega, \quad 0 < t \leq T, \quad 1 < \gamma < 2, \quad (1) \]

subject to the initial conditions:

\[ u(x, 0) = \psi(x), \quad \frac{\partial u(x, 0)}{\partial t} = \phi(x), \quad x \in \bar{\Omega} = \Omega \cup \partial \Omega, \]

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and the zero flux boundary condition:
\[
\frac{\partial u(x,t)}{\partial n} = 0, \quad x \in \partial \Omega, \quad 0 < t \leq T,
\]
where \(\partial \Omega\) is the boundary of \(\Omega\), \(\frac{\partial}{\partial n}\) is the differentiation in the normal direction and \(\kappa_1, \kappa_2\) are some positive constants. We further suppose that \(\frac{\partial \psi(x)}{\partial n} = 0\), for \(x \in \partial \Omega\). We have used \(\frac{C}{0}D_t^\gamma u\) to denote the Caputo fractional derivative of \(u\) with respect to the time variable \(t\), which is
\[
\frac{C}{0}D_t^\gamma u(x,t) = \frac{1}{\Gamma(2-\gamma)} \int_0^t (t-s)^{1-\gamma} ds, 
\]
with \(\Gamma(\cdot)\) being the gamma function. Theoretical results such as existence and uniqueness of solutions to fractional differential equations can be found in \([1,2]\). In recent years, there are growing interests on the study of numerical solutions for time fractional differential equations subject to the Neumann boundary condition \([21]-[25]\).

We note that equation \([1]\) can be written equivalently as \([15]\):
\[
\frac{\partial u(x,t)}{\partial t} = \phi(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ \kappa_1 \frac{\partial^2 u(x,s)}{\partial x^2} - \kappa_2 u(x,s) \right] ds + f(x,t), \quad x \in \Omega, \quad 0 < t \leq T, 
\]
where \(0 < \alpha = \gamma - 1 < 1\), \(f(x,t) = \alpha_\gamma^\alpha g(x,t)\), and \(\alpha_\gamma^\alpha\) is the Riemann-Liouville fractional integral operator of order \(\alpha\), defined as
\[
\alpha_\gamma^\alpha g(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(x,s) ds. 
\]

By applying the weighted and shifted Grünwald difference (see \([13,14,16]\)) to the Riemann-Liouville fractional integral, we establish compact schemes with second order temporal accuracy and fourth order spatial accuracy. Our analysis is based on the matrix form of the schemes and it turns out to give intuitive ideas of some norms and inner products defined in previous related works.

This paper is organized as follows. We first consider the one-dimensional problem in Section 2 and 3, where we propose a high order scheme and study its convergence respectively. In Section 4, a high order alternating direction implicit scheme is proposed for the two-dimensional problem. Numerical examples are given in the last section.

2 The proposed compact difference scheme

In this section, we develop a high order scheme for the following one-dimensional problem:
\[
\frac{C}{0}D_t^\gamma u(x,t) = \kappa_1 \frac{\partial^2 u(x,t)}{\partial x^2} - \kappa_2 u(x,t) + g(x,t), \quad 0 \leq x \leq L, \quad 0 < t \leq T, \quad 1 < \gamma < 2, \quad (2)
\]
\[
u(x,0) = \psi(x), \quad \frac{\partial u(x,0)}{\partial t} = \phi(x), \quad 0 \leq x \leq L, \quad (3)
\]
\[
\frac{\partial u(0,t)}{\partial x} = 0, \quad \frac{\partial u(L,t)}{\partial x} = 0, \quad 0 < t \leq T. \quad (4)
\]
We assume that \(\psi \equiv 0\) in \([3]\) without loss of generality since we can solve the equation for \(v(x,t) = u(x,t) - \psi(x)\) in general.
An equivalent form of (2) can be read as:
\[
\frac{\partial u(x, t)}{\partial t} = \phi(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \left[ \kappa_1 \frac{\partial^2 u(x, s)}{\partial x^2} - \kappa_2 u(x, s) \right] ds + f(x, t),
\]
where \(0 \leq x \leq L\), \(0 < t \leq T\), \(0 < \alpha = \gamma - 1 < 1\), \(f(x, t) = 0 I_t^\gamma g(x, t)\).

For given integers \(M\) and \(N\), we discretize the equation using spatial step size \(h = \frac{L}{M}\) and temporal step size \(\tau = \frac{T}{N}\) respectively. For \(i = 0, 1, \ldots, M\) and \(k = 0, 1, \ldots, N\), denote \(x_i = ih\), \(t_k = k\tau\), \(u^k = (u^k_0, u^k_1, \ldots, u^k_M)^T\). To study the grid function \(u = \{u^i_0 \leq i \leq M, 0 \leq k \leq k \leq N\}\) that approximates the solution, the following notations are needed:

\[
\delta_x u^k_{i-\frac{1}{2}} = \frac{1}{h} (u^k_i - u^k_{i-1}), \quad 1 \leq i \leq M,
\]

\[
\delta_x^2 u^k_i = \begin{cases} \frac{2}{h} \delta_x u^k_{i+\frac{1}{2}}, & i = 0, \\ \frac{1}{h} (\delta_x u^k_{i+\frac{1}{2}} - \delta_x u^k_{i-\frac{1}{2}}), & 1 \leq i \leq M - 1, \\ -\frac{2}{h} \delta_x u^k_{M-\frac{1}{2}}, & i = M. \end{cases}
\]

\[
\mathcal{H}u_i = \begin{cases} \frac{1}{h} (5u_0 + u_1), & i = 0, \\ \frac{1}{h} (u_{i-1} + 10u_i + u_{i+1}), & 1 \leq i \leq M - 1, \\ \frac{1}{h} (u_{M-1} + 5u_M), & i = M. \end{cases}
\]

\[
\langle u, v \rangle = h \sum_{i=0}^{M} u_i v_i, \quad \|u\|^2 = \langle u, u \rangle, \quad \|u\|_{\infty} = \max_{0 \leq i \leq M} |u_i|.
\]

Discretization of \(\frac{\partial^2 u}{\partial x^2}\) is based on the following lemma:

**Lemma 2.1** (23) Denote \(\zeta(s) = (1 - s)^3[5 - 3(1 - s)^2]\).

(I) If \(f(x) \in C^6[0, 1]\), then we have

\[
\frac{5}{6} f'''(x_0) + \frac{1}{6} f''(x_1) - \frac{2}{h} \int f(x_1) - f(x_0) h - f'(x_0) h = -\frac{h}{6} f'''(x_0) + \frac{h^3}{90} f^{(5)}(x_0) + \frac{h^4}{180} \int_0^1 f^{(6)}(x_0 + sh) \zeta(s) ds.
\]

(II) If \(f(x) \in C^6[M-1, M]\), then we get

\[
\frac{1}{6} f'''(x_{M-1}) + \frac{5}{6} f''(x_M) - \frac{2}{h} [f'(x_M) - f(x_M) - f(x_{M-1})] h = \frac{h}{6} f'''(x_M) - \frac{h^3}{90} f^{(5)}(x_M) + \frac{h^4}{180} \int_0^1 f^{(6)}(x_M - sh) \zeta(s) ds.
\]

(III) If \(f(x) \in C^6[i-1, i+1]\), \(1 \leq i \leq M - 1\), then it holds that

\[
\frac{1}{12} [f'''(x_{i-1}) + 10 f''(x_i) + f''(x_{i+1})] - \frac{1}{h^2} [f(x_{i-1}) - 2 f(x_i) + f(x_{i+1})] h = \frac{h^4}{360} \int_0^1 [f^{(6)}(x_i - sh) + f^{(6)}(x_i + sh)] \zeta(s) ds.
\]
where \( \omega_k = (-1)^k \left( \frac{\tau}{k} \right) \). By the idea of [13], the following second order approximation for Riemann-Liouville fractional integrals is derived very recently in [16].

**Lemma 2.2** Let \( f(t), -\infty I_t^{2-\alpha} f \) and \( (i\omega)^{2-\alpha} \mathcal{F}[f](\omega) \) belong to \( L^1(\mathbb{R}) \). Define the weighted and shifted difference operator by

\[
\mathcal{I}_{\tau,p,q}^\alpha f(t) = \frac{2q + \alpha}{2(q-p)} A_{\tau,q}^\alpha f(t) + \frac{2p + \alpha}{2(p-q)} A_{\tau,p}^\alpha f(t),
\]

then we have

\[
\mathcal{I}_{\tau,p,q}^\alpha f(t) = -\infty I_t^{\alpha} f(t) + O(\tau^2)
\]

for \( t \in \mathbb{R} \), where \( p \) and \( q \) are integers and \( p \neq q \).

With \( (p,q) = (0,-1) \), which yields \( \frac{2q+\alpha}{2(q-p)} = 1 - \frac{\alpha}{2} \); \( \frac{2p+\alpha}{2(p-q)} = \frac{\alpha}{2} \) in Lemma 2.2, we get that

\[
0I_t^{\alpha} u(x_i,t_{n+1}) = \tau^\alpha \left[ \left( 1 - \frac{\alpha}{2} \right) \sum_{k=0}^{n+1} \omega_k u_i^{n+1-k} + \frac{\alpha}{2} \sum_{k=0}^{n} \omega_k u_i^{n-k} \right] + O(\tau^2)
\]

and

\[
0I_t^{\alpha} u_{xx}(x_i,t_{n+1}) = \tau^\alpha \left[ \left( 1 - \frac{\alpha}{2} \right) \sum_{k=0}^{n+1} \omega_k \delta_x^2 u_i^{n+1-k} + \frac{\alpha}{2} \sum_{k=0}^{n} \omega_k \delta_x^2 u_i^{n-k} \right] + O(\tau^2 + \tau^2)
\]

where

\[
\lambda_0 = (1 - \frac{\alpha}{2})\omega_0, \quad \lambda_k = \left( 1 - \frac{\alpha}{2} \right)\omega_k + \frac{\alpha}{2}\omega_{k-1}, \quad k \geq 1.
\]

Therefore, a weighted Crank-Nicolson scheme for equation (5) can be given by

\[
\frac{u_i^{n+1} - u_i^n}{\tau} = \phi_1 + \frac{\tau^\alpha}{2} \left[ \sum_{k=0}^{n+1} \lambda_k (\kappa_1 \delta_x^2 u_i^{n+1-k} - \kappa_2 u_i^{n+1-k}) + \sum_{k=0}^{n} \lambda_k (\kappa_1 \delta_x^2 u_i^{n-k} - \kappa_2 u_i^{n-k}) \right] + \frac{1}{2} (f_i^n + f_i^{n+1}).
\]

To derive a higher order scheme, we follow the idea in [25]. Beginning with \( i = 0 \), one has

\[
\mathcal{H}(u_0^{n+1} - u_0^n) = \frac{\tau^{\alpha+1}}{2} \sum_{k=0}^{n+1} \lambda_k \left( \frac{2}{h} \delta_x^2 u_0^{n+1-k} - \frac{2}{h} \partial u(0,t_{n+1-k}) \right) + h^3 \frac{\delta^5 u(0,t_{n+1-k})}{90} - \kappa_2 \mathcal{H}(u_0^{n+1-k})
\]

\[
+ \frac{\tau^{\alpha+1}}{2} \sum_{k=0}^{n} \lambda_k \left( \frac{2}{h} \delta_x^2 u_0^{n-k} - \frac{2}{h} \partial u(0,t_{n-k}) \right) + h^3 \frac{\delta^5 u(0,t_{n-k})}{90} - \kappa_2 \mathcal{H}(u_0^{n-k})
\]

\[
+ \tau \mathcal{H} \phi_0 + \frac{\tau^2}{2} \mathcal{H}(f_0^n + f_0^{n+1}) + \tau R_0^{n+1},
\]

(7)
where \( R_0^{n+1} = O(\tau^2 + h^4) \).

We can now differentiate equation (2) with respect to \( x \) to give

\[
C_0 D_t^\gamma \frac{\partial u(x, t)}{\partial x} = \kappa_1 \frac{\partial^2 u(x, t)}{\partial x^2} - \kappa_2 \frac{\partial u(x, t)}{\partial x} + g_x(x, t).
\]

Letting \( x \to 0^+ \) and noticing the boundary condition (4), we have

\[
\kappa_1 \frac{\partial^2 u(0, t)}{\partial x^2} = -g_x(0, t).
\]

With the Caputo fractional derivative operator \( C_0 D_t^\gamma \) acting on (8), it follows that

\[
C_0 D_t^\gamma \frac{\partial^2 u(0, t)}{\partial x^2} = -\frac{1}{\kappa_1} C_0 D_t^\gamma g_x(0, t).
\]

Meanwhile, differentiating equation (2) three times with respect to \( x \) yields

\[
C_0 D_t^\gamma \frac{\partial^3 u(x, t)}{\partial x^3} = \kappa_1 \frac{\partial^4 u(x, t)}{\partial x^4} - \kappa_2 \frac{\partial^3 u(x, t)}{\partial x^3} + g_{xxx}(x, t).
\]

Once again, let \( x \to 0^+ \) in (10). We can then substitute (8) and (9) to (10) to achieve

\[
\kappa_1 \frac{\partial^3 u(0, t)}{\partial x^3} = -g_{xxx}(0, t) - \frac{\kappa_2}{\kappa_1} g_x(0, t) - \frac{1}{\kappa_1} C_0 D_t^\gamma g_x(0, t).
\]

Inserting (8), (11) into (7) and noticing the boundary condition (4), the compact scheme for \( i = 0 \) can be given, by omitting small terms, as:

\[
\mathcal{H}(u_0^{n+1} - u_0^n) = \tau^{\alpha+1} \left[ \sum_{k=0}^{n+1} \lambda_k (\kappa_1 \delta_x^2 u_0^{n+1-k} - \kappa_2 \mathcal{H} u_0^{n+1-k}) + \sum_{k=0}^{n} \lambda_k (\kappa_1 \delta_x^2 u_0^{n-k} - \kappa_2 \mathcal{H} u_0^{n-k}) \right]
+ \tau^{\alpha+1} \left[ \sum_{k=0}^{n} \lambda_k \left( \frac{h}{6} (g_x)_0^{n+1-k} - \frac{h^3}{360} (g_{xxx})_0^{n+1-k} + \frac{\kappa_2}{\kappa_1} (g_x)_0^{n+1-k} + \frac{1}{\kappa_1} (C_0 D_t^\alpha g_x)_0^{n+1-k} \right) \right]
+ \tau^{\alpha+1} \left[ \sum_{k=0}^{n} \lambda_k \left( \frac{h}{6} (g_x)_0^{n-k} - \frac{h^3}{360} (g_{xxx})_0^{n-k} + \frac{\kappa_2}{\kappa_1} (g_x)_0^{n-k} + \frac{1}{\kappa_1} (C_0 D_t^\alpha g_x)_0^{n-k} \right) \right]
+ \tau \mathcal{H} \phi_0 + \frac{\tau}{2} \mathcal{H}(f_0^n + f_0^{n+1}), \quad 0 \leq n \leq N - 1.
\]

The scheme at the other end can be similarly derived as

\[
\mathcal{H}(u_M^{n+1} - u_M^n) = \tau^{\alpha+1} \left[ \sum_{k=0}^{n+1} \lambda_k (\kappa_1 \delta_x^2 u_M^{n+1-k} - \kappa_2 \mathcal{H} u_M^{n+1-k}) + \sum_{k=0}^{n} \lambda_k (\kappa_1 \delta_x^2 u_M^{n-k} - \kappa_2 \mathcal{H} u_M^{n-k}) \right]
- \tau^{\alpha+1} \left[ \sum_{k=0}^{n+1} \lambda_k \left( \frac{h}{6} (g_x)_M^{n+1-k} - \frac{h^3}{360} (g_{xxx})_M^{n+1-k} + \frac{\kappa_2}{\kappa_1} (g_x)_M^{n+1-k} + \frac{1}{\kappa_1} (C_0 D_t^\alpha g_x)_M^{n+1-k} \right) \right]
- \tau^{\alpha+1} \left[ \sum_{k=0}^{n} \lambda_k \left( \frac{h}{6} (g_x)_M^{n-k} - \frac{h^3}{360} (g_{xxx})_M^{n-k} + \frac{\kappa_2}{\kappa_1} (g_x)_M^{n-k} + \frac{1}{\kappa_1} (C_0 D_t^\alpha g_x)_M^{n-k} \right) \right]
+ \tau \mathcal{H} \phi_M + \frac{\tau}{2} \mathcal{H}(f_M^n + f_M^{n+1}), \quad 0 \leq n \leq N - 1.
\]
One can readily see that, at the internal grid, the scheme can be written as

\[
\mathcal{H}(u_i^{n+1} - u_i^n) = \frac{\tau}{2} \left[ \sum_{k=0}^{n+1} \lambda_k (\kappa_1 \delta_x^2 u_i^{n+1-k} - \kappa_2 \mathcal{H} u_i^{n+1-k}) + \sum_{k=0}^{n} \lambda_k (\kappa_1 \delta_x^2 u_i^{n-k} - \kappa_2 \mathcal{H} u_i^{n-k}) \right] + \tau \mathcal{H} \phi_i + \frac{\tau}{2} \mathcal{H} (f_i^n + f_i^{n+1}), \quad 1 \leq i \leq M - 1, \quad 0 \leq n \leq N - 1.
\]

The approximate solution is solved with

\[ u_i^0 = 0, \quad 0 \leq i \leq M. \tag{15} \]

It is easy to see that at each time level, the difference scheme which consists of (12)–(15) is a linear tridiagonal system with strictly diagonal dominant coefficient matrix. Thus the difference scheme has a unique solution.

3 Stability and convergence analysis of the compact scheme

We give the convergence of the proposed scheme in this section. The main result can be established by the following lemmas:

**Lemma 3.1** \([16]\) Let \(\{\lambda_n\}_{n=0}^\infty\) be defined as \([6]\), then for any positive integer \(k\) and real vector \((v_1, v_2, \ldots, v_k)^T \in \mathbb{R}^k\), it holds that

\[
\sum_{n=0}^{k-1} \left( \sum_{p=0}^{n} \lambda_p v_{n+1-p} \right) v_{n+1} \geq 0.
\]

**Lemma 3.2** \([26]\) Assume that \(\{k_n\}\) and \(\{p_n\}\) are nonnegative sequences, and the sequence \(\{\phi_n\}\) satisfies

\[
\phi_0 \leq g_0, \quad \phi_n \leq g_0 + \sum_{l=0}^{n-1} p_l + \sum_{l=0}^{n-1} k_l \phi_l, \quad n \geq 1,
\]

where \(g_0 \geq 0\). Then the sequence \(\{\phi_n\}\) satisfies

\[
\phi_n \leq \left( g_0 + \sum_{l=0}^{n-1} p_l \right) \exp \left( \sum_{l=0}^{n-1} k_l \right), \quad n \geq 1.
\]

Our compact difference scheme consisting of (12)–(15) has high order convergence. To be more precise, we have

**Theorem 3.1** Assume that \(u(x,t) \in C_x^{6,2}([0, L] \times [0, T])\) is the solution of (2)–(4) and \(\{u_i^k\}_{0 \leq i \leq M, \quad 0 \leq k \leq N}\) is a solution of the finite difference scheme (12)–(15), respectively. Denote

\[
e_i^k = u(x_i, t_k) - u_i^k, \quad 0 \leq i \leq M, \quad 0 \leq k \leq N.
\]

Then there exists a positive constant \(c\) such that

\[
\|e^k\| \leq c (\tau^2 + h^4), \quad 0 \leq k \leq N.
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\[
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\]

Then there exists a positive constant \(c\) such that

\[
\|e^k\| \leq c (\tau^2 + h^4), \quad 0 \leq k \leq N.
\]
Proof. We can easily get the following error equation:

\[
C(e^{k+1} - e^k) = -\frac{\kappa_1 \tau^\alpha + 1}{2h^2} \sum_{l=0}^{k} \lambda_l \bar{Q}(e^{k+1-l} + e^{k-l}) - \frac{\kappa_2 \tau^\alpha + 1}{2} \sum_{l=0}^{k} \lambda_l \bar{C}(e^{k+1-l} + e^{k-l}) + \tau \bar{R}^{k+1},
\]

where \(\|\bar{R}^{k+1}\| \leq c_1(\tau^2 + h^4),\)

\[
\bar{C} = \frac{1}{12} \begin{pmatrix} 10 & 2 \\ 1 & 10 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 10 & 1 \\ 2 & 10 \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} 2 & -2 \\ -1 & 2 & -1 \\ \vdots & \vdots & \vdots \\ -1 & 2 & -1 \\ -2 & 2 \end{pmatrix}.
\] (16)

Multiplying the equation (16) with \(\frac{1}{2} I \oplus \frac{1}{2},\) where \(I\) is the identity matrix, we get

\[
C(e^{k+1} - e^k) = -\frac{\kappa_1 \tau^\alpha + 1}{2h^2} \sum_{l=0}^{k} \lambda_l Q(e^{k+1-l} + e^{k-l}) - \frac{\kappa_2 \tau^\alpha + 1}{2} \sum_{l=0}^{k} \lambda_l C(e^{k+1-l} + e^{k-l}) + \tau R^{k+1},
\] (17)

where \(\|R^{k+1}\| \leq c_2(\tau^2 + h^4),\)

\[
C = \frac{1}{12} \begin{pmatrix} 5 & 1 \\ 1 & 10 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 10 & 1 \\ 1 & 5 \end{pmatrix} = E^2, \text{ with } E \text{ being the square root of } C,
\] (18)

\[
Q = \begin{pmatrix} 1 & -1 \\ -1 & 2 & -1 \\ \vdots & \vdots & \vdots \\ -1 & 2 & -1 \\ -1 & 1 \end{pmatrix} = S^T S, \text{ with } S = \begin{pmatrix} -1 & 1 \\ \vdots & \vdots \\ 1 & -1 \end{pmatrix} \in \mathbb{R}^{M \times M+1}.
\] (19)

Here we have used the fact that \(C = \frac{1}{12} \text{tri}[1, 5, 1] + \frac{5}{12} (0 \oplus I \oplus 0)\) is positive definite.

Multiplying (18) by \(h(e^{k+1} + e^k)^T,\) we obtain

\[
h(e^{k+1} + e^k)^T C(e^{k+1} - e^k)
= -\frac{\kappa_1 \tau^\alpha + 1}{2h} \sum_{l=0}^{k} \lambda_l (e^{k+1} + e^k)^T S^T S(e^{k+1-l} + e^{k-l}) - \frac{\kappa_2 h^\tau^\alpha + 1}{2} \sum_{l=0}^{k} \lambda_l (e^{k+1} + e^k)^T E^2(e^{k+1-l} + e^{k-l}) + \tau h(e^{k+1} + e^k)^T R^{k+1}.
\]

Summing up for \(0 \leq k \leq n - 1\) and noting that

\[
h(e^{k+1} + e^k)^T C(e^{k+1} - e^k) = h[(e^{k+1})^T C e^{k+1} - (e^k)^T C e^k], \quad h(e^n)^T C e^n \geq \frac{1}{4} \|e^n\|^2,
\]
we have, by Lemma 3.1,

\[
\frac{1}{4} \| e^n \|^2 \leq \tau h (e^n + e^{n-1})^T R^n + \tau h \sum_{k=0}^{n-2} (e^{k+1} + e^k)^T R^{k+1}
\]

\[
\leq \frac{1}{5} \| e^n \|^2 + \frac{5\tau^2}{4} \| R^n \|^2 + \frac{\tau}{2} \| e^{n-1} \|^2 + \frac{\tau}{2} \| R^n \|^2 + \frac{\tau}{2} \sum_{k=1}^{n-1} \| e^k \|^2 + \frac{\tau}{2} \sum_{k=1}^{n-2} \| e^k \|^2 + \tau \sum_{k=1}^{n-1} \| R^k \|^2,
\]

which gives

\[
\| e^n \|^2 \leq 25\tau^2 \| R^n \|^2 + 20\tau \sum_{k=1}^{n-1} \| e^k \|^2 + 20\tau \sum_{k=1}^{n} \| R^k \|^2 
\leq 20\tau \sum_{k=1}^{n-1} \| e^k \|^2 + c_3 (\tau^2 + h^4)^2,
\]

then the desired result follows by Lemma 3.2 \( \Box \)

**Remark 3.1** One can adopt the idea of the proof for Theorem 3.1 to show that the proposed compact scheme \([12] - [15]\) is unconditionally stable. In fact, consider the solution \{v^k_i\} of

\[
\mathcal{H}(v^{k+1}_i - v^k_i) = \frac{\tau^{\alpha+1}}{2} \left[ \sum_{l=0}^{k+1} \lambda_l (\kappa_1 \delta_x^2 v_0^{k+1-l} - \kappa_2 \mathcal{H} v_0^{k+1-l}) + \sum_{l=0}^{k} \lambda_l (\kappa_1 \delta_x^2 v_0^{k-l} - \kappa_2 \mathcal{H} v_0^{k-l}) \right] 
\]

\[
+ \frac{\tau^{\alpha+1}}{2} \sum_{l=0}^{k+1} \lambda_l \left( \frac{h}{6} (g_x)_0^{k+1-l} - \frac{h^3}{90} (g_{xxx})_0^{k+1-l} + \frac{k_2}{k_1} (g_x)_0^{k+1-l} + \frac{1}{k_1} c_D (g_x)_0^{k+1-l} \right) 
\]

\[
+ \frac{\tau^{\alpha+1}}{2} \sum_{l=0}^{k} \lambda_l \left( \frac{h}{6} (g_x)_0^{k-l} - \frac{h^3}{90} (g_{xxx})_0^{k-l} + \frac{k_2}{k_1} (g_x)_0^{k-l} + \frac{1}{k_1} c_D (g_x)_0^{k-l} \right) 
\]

\[
+ \tau \mathcal{H}(\phi_i + \bar{\rho}_i) + \frac{\tau}{2} \mathcal{H}(f^{k+1}_M), \quad 0 \leq k \leq N - 1,
\]

\[
\mathcal{H}(v^{k+1}_M - v^k_M) = \frac{\tau^{\alpha+1}}{2} \left[ \sum_{l=0}^{k+1} \lambda_l (\kappa_1 \delta_x^2 v_M^{k+1-l} - \kappa_2 \mathcal{H} v_M^{k+1-l}) + \sum_{l=0}^{k} \lambda_l (\kappa_1 \delta_x^2 v_M^{k-l} - \kappa_2 \mathcal{H} v_M^{k-l}) \right] 
\]

\[
- \frac{\tau^{\alpha+1}}{2} \sum_{l=0}^{k+1} \lambda_l \left( \frac{h}{6} (g_x)_M^{k+1-l} - \frac{h^3}{90} (g_{xxx})_M^{k+1-l} + \frac{k_2}{k_1} (g_x)_M^{k+1-l} + \frac{1}{k_1} c_D (g_x)_M^{k+1-l} \right) 
\]

\[
- \frac{\tau^{\alpha+1}}{2} \sum_{l=0}^{k} \lambda_l \left( \frac{h}{6} (g_x)_M^{k-l} - \frac{h^3}{90} (g_{xxx})_M^{k-l} + \frac{k_2}{k_1} (g_x)_M^{k-l} + \frac{1}{k_1} c_D (g_x)_M^{k-l} \right) 
\]

\[
+ \tau \mathcal{H}(\phi_M + \bar{\rho}_M) + \frac{\tau}{2} \mathcal{H}(f^{k+1}_M), \quad 0 \leq k \leq N - 1,
\]

\[
\mathcal{H}(v^{k+1}_i - v^k_i) = \frac{\tau^{\alpha+1}}{2} \left[ \sum_{l=0}^{k+1} \lambda_l (\kappa_1 \delta_x^2 v_i^{k+1-l} - \kappa_2 \mathcal{H} v_i^{k+1-l}) + \sum_{l=0}^{k} \lambda_l (\kappa_1 \delta_x^2 v_i^{k-l} - \kappa_2 \mathcal{H} v_i^{k-l}) \right] 
\]

\[
+ \tau \mathcal{H}(\phi_i + \bar{\rho}_i) + \frac{\tau}{2} \mathcal{H}(f^{k+1}_i + f^{k+1}_i), \quad 1 \leq i \leq M - 1, \quad 0 \leq k \leq N - 1,
\]
with \( v_i^0 = \rho_i \), \( 0 \leq i \leq M \). Then, by (12)–(14) and (21)–(23), one can check that \( \varepsilon_i^1 = v_i^1 - u_i^1 - \rho_i \) satisfy
\[
\mathcal{H}_k \varepsilon_{i+1}^k - \varepsilon_i^k = \frac{\tau^\alpha + 1}{2} \left[ \sum_{l=0}^{k+1} \lambda_l (\kappa_1 \delta_x^2 z_{i+1}^k - \kappa_2 \mathcal{H} z_{i+1}^k) + \sum_{l=0}^k \lambda_l (\kappa_1 \delta_x^2 z_i^k - \kappa_2 \mathcal{H} z_i^k) \right] + \tau \mathcal{H} \rho_i
\]
\[
+ \frac{\tau^\alpha + 1}{2} \left[ \sum_{l=0}^{k+1} \lambda_l (\kappa_1 \delta_x^2 \rho_i - \kappa_2 \mathcal{H} \rho_i) + \sum_{l=0}^k \lambda_l (\kappa_1 \delta_x^2 \rho_i - \kappa_2 \mathcal{H} \rho_i) \right], \quad 0 \leq i \leq M, \quad 0 \leq k \leq N - 1,
\]
\( \varepsilon_i^0 = 0, \quad 0 \leq i \leq M. \) (24)

By following the proof for Theorem 3.1 and noting \( \tau^\alpha \sum_{l=0}^{k+1} \lambda_l = \frac{1}{\Gamma(\alpha + 1)} + O(\tau) \), we then have the estimate
\[
\| \varepsilon^k \|^2 \leq 20 \tau \sum_{l=0}^{k-1} \| \varepsilon^l \|^2 + \left[ \frac{5}{\Gamma(\alpha + 1)} + 1 \right] \left[ \| \kappa_1 \delta_x^2 \rho \|^2 + \| \kappa_2 \rho \|^2 + \| \rho \| \right]^2 \leq e^{20\tau} \left[ \frac{5}{\Gamma(\alpha + 1)} + 1 \right] \left[ \| \kappa_1 \delta_x^2 \rho \|^2 + \| \kappa_2 \rho \|^2 + \| \rho \| \right]^2.
\]
This implies
\[
\| v^k - u^k \| \leq \| v^k - u^k - \rho \| + \| \rho \|
\leq e^{10\tau} \left[ \frac{5}{\Gamma(\alpha + 1)} + 1 \right] \left[ \| \kappa_1 \delta_x^2 \rho \|^2 + \| \kappa_2 \rho \|^2 + \| \rho \| \right],
\]
concluding the stability of the scheme.

4 The compact ADI scheme for the two-dimensional problem

In this section, we turn to study the two-dimensional problem:
\[
\mathcal{O}^\gamma_D \mathcal{D}^\gamma u = \Delta u - u + g(x,y,t), \quad (x,y) \in \Omega, \quad 0 < t \leq T, \quad 1 < \gamma < 2, \quad (26)
\]
\[
u(x,y) = 0, \quad \frac{\partial u(x,y,0)}{\partial t} = \phi(x,y), \quad (x,y) \in \bar{\Omega} = \Omega \cup \partial \Omega, \quad (27)
\]
\[
\frac{\partial u(x,y,t)}{\partial n} |_{\partial \Omega} = 0, \quad (x,y) \in \partial \Omega, \quad 0 < t \leq T, \quad (28)
\]
where \( \Delta \) is the two-dimensional Laplacian, \( n \) is the unit outward normal vector of the domain \( \Omega = (0,L_1) \times (0,L_2) \) with boundary \( \partial \Omega \).

An equivalent form of (26) read as:
\[
\frac{\partial u(x,y,t)}{\partial t} = \phi(x,y) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\Delta u(x,y,s) - u(x,y,s)] ds + f(x,y,t), \quad (29)
\]
where \( (x,y) \in \bar{\Omega}, \quad 0 < t \leq T, \quad 0 < \alpha = \gamma - 1 < 1, \quad f(x,y,t) = \mathcal{O}^\alpha_D \mathcal{D}^\alpha u(x,y,t). \)

Discretization of (29) are carried out with steps similar to that of the one-dimensional problem. To this end, we let \( h_1 = \frac{L_1}{M_1}, \quad h_2 = \frac{L_2}{M_2} \) and \( \tau = \frac{T}{N} \) be the spatial and temporal step sizes respectively, where \( M_1, M_2 \) and \( N \) are some given integers. For \( i = 0,1, \ldots, M_1, \quad j = 0,1, \ldots, M_2 \)
and $k = 0, 1, \ldots, N$, denote $x_i = ih_1$, $y_j = jh_2$, $t_k = k\tau$. We introduce the following notations on a grid function $u = \{u_{ij}^k|0 \leq i \leq M_1, 0 \leq j \leq M_2, 0 \leq k \leq N\}$:

$$\delta_x u_{i-\frac{1}{2},j} = \frac{1}{h_1}(u_{ij} - u_{i-1,j}),$$

$$\delta^2_x u_{ij} = \begin{cases} 
\frac{2}{h_1^2}\delta_x u_{i+\frac{1}{2},j}, & i = 0, \ 0 \leq j \leq M_2, \\
\frac{1}{h_1}(\delta_x u_{i+1,j} - \delta_x u_{i-\frac{1}{2},j}), & 1 \leq i \leq M_1 - 1, \ 0 \leq j \leq M_2, \\
-\frac{2}{h_1}\delta_x u_{M_1-\frac{1}{2},j}, & i = M_1, \ 0 \leq j \leq M_2,
\end{cases}$$

$$\mathcal{H}_x u_{ij} = \begin{cases} 
\frac{1}{2}(5u_{0,j} + u_{1,j}), & i = 0, \ 0 \leq j \leq M_2, \\
\frac{1}{12}(u_{i-1,j} + 10u_{i,j} + u_{i+1,j}), & 1 \leq i \leq M_1 - 1, \ 0 \leq j \leq M_2, \\
\frac{1}{12}(u_{M_1-1,j} + 5u_{M_1,j}), & i = M_1, \ 0 \leq j \leq M_2.
\end{cases}$$

One can defined similar notations in the $y$ direction. We further denote:

$$\mathcal{H}u_{ij} = \mathcal{H}_x \mathcal{H}_y u_{ij}, \quad \Lambda u_{ij} = (\mathcal{H}_y \delta^2_x + \mathcal{H}_x \delta^2_y)u_{ij},$$

$$\langle u, v \rangle = h_1h_2\sum_{i=0}^{M_1}\sum_{j=0}^{M_2}u_{ij}v_{ij}, \quad \|u\|^2 = \langle u, u \rangle, \quad \|u\|_\infty = \max_{0 \leq i \leq M_1, 0 \leq j \leq M_2} |u_{ij}|.$$
where \((R_1)^{n+1} = O(\tau^2 + h_1^4 + h_2^4)\).

Denoting \(F^n_{ij} = \frac{1}{2}(\tau H f^n_{ij} + G^n_{ij})\), and adding a small term \(\frac{\mu^2 \lambda^2}{1 + \mu \lambda_0} \delta_x^2 \delta_y^2 (u_{ij}^{n+1} - u_{ij}^n)\) on both sides of (30), we have

\[
\begin{align*}
\mathcal{H}(u_{ij}^{n+1} - u_{ij}^n) + \frac{\mu^2 \lambda^2}{1 + \mu \lambda_0} \delta_x^2 \delta_y^2 (u_{ij}^{n+1} - u_{ij}^n) \\
= \tau \phi_{ij} + \mu \sum_{k=0}^{n+1} \lambda_k (\Lambda - \mathcal{H}) u_{ij}^{n+1-k} + \sum_{k=0}^{n} \lambda_k (\Lambda - \mathcal{H}) u_{ij}^{n-k} + F^n_{ij} + F^{n+1}_{ij} + \tau R^{n+1}_{ij} \quad (31)
\end{align*}
\]

\(u_{ij}^0 = 0, \quad (x_i, y_j) \in \Omega,\)

with \(R^{n+1}_{ij} = O(\tau^2 + h_1^4 + h_2^4)\). Omitting the truncation error in (31), we reach the following scheme in the ADI setting:

\[
\begin{align*}
(\sqrt{1 + \mu \lambda_0} \mathcal{H}_x - \frac{\mu \lambda_0}{\sqrt{1 + \mu \lambda_0} \delta_x^2} \sqrt{1 + \mu \lambda_0} \mathcal{H}_y - \frac{\mu \lambda_0}{\sqrt{1 + \mu \lambda_0} \delta_y^2}) u_{ij}^{n+1} \\
= \mathcal{H} u_{ij}^n + \frac{\mu^2 \lambda^2}{1 + \mu \lambda_0} \delta_x^2 \delta_y^2 u_{ij}^n + \mu \sum_{k=1}^{n+1} \lambda_k (\Lambda - \mathcal{H}) u_{ij}^{n+1-k} + \sum_{k=0}^{n} \lambda_k (\Lambda - \mathcal{H}) u_{ij}^{n-k} \\
+ \tau \phi_{ij} + F^n_{ij} + F^{n+1}_{ij},
\end{align*}
\]

\((x_i, y_j) \in \Omega, \quad 0 \leq n \leq N - 1, \quad u_{ij}^0 = 0, \quad (x_i, y_j) \in \bar{\Omega}.\)

For ADI methods (see [27] for example), the solution \(\{u_{ij}^{n+1}\}\) is determined by solving two independent one-dimensional problems. Specifically, the intermediate variables

\[
u_{ij}^* = (\sqrt{1 + \mu \lambda_0} \mathcal{H}_y - \frac{\mu \lambda_0}{\sqrt{1 + \mu \lambda_0} \delta_y^2}) u_{ij}^{n+1}, \quad 0 \leq i \leq M_1, \quad 0 \leq j \leq M_2,
\]

are first solved from the following system with fixed \(j \in \{0, 1, \ldots, M_2\}:

\[
(\sqrt{1 + \mu \lambda_0} \mathcal{H}_x - \frac{\mu \lambda_0}{\sqrt{1 + \mu \lambda_0} \delta_x^2}) u_{ij}^* \\
= \mathcal{H} u_{ij}^n + \frac{\mu^2 \lambda^2}{1 + \mu \lambda_0} \delta_x^2 \delta_y^2 u_{ij}^n + \mu \sum_{k=1}^{n+1} \lambda_k (\Lambda - \mathcal{H}) u_{ij}^{n+1-k} + \sum_{k=0}^{n} \lambda_k (\Lambda - \mathcal{H}) u_{ij}^{n-k} \\
+ \tau \phi_{ij} + F^n_{ij} + F^{n+1}_{ij},
\]

\(0 \leq i \leq M_1, \quad u_{ij}^* = u_{ij}^{n+1}, \quad 0 \leq j \leq M_2.
\]

When \(\{u_{ij}^*\}\) is ready, the approximate solution \(\{u_{ij}^{n+1}\}\) is solved from the following system for fixed \(i \in \{0, 1, \ldots, M_1\}:

\[
(\sqrt{1 + \mu \lambda_0} \mathcal{H}_y - \frac{\mu \lambda_0}{\sqrt{1 + \mu \lambda_0} \delta_y^2}) u_{ij}^{n+1} = u_{ij}^*, \quad 0 \leq j \leq M_2.
\]

By implementing the ADI method, the computational cost for solving a two-dimensional problem can be greatly reduced.

We now proceed to give the convergence result of our compact ADI scheme (31). We remark that, with the convergence of the scheme, one can show that the scheme is stable in the same sense as that given in Remark 3.1.

**Theorem 4.1** Assume that \(u(x, y, t) \in C^{6,6,2}_{x,y,t}([0, T])\) is the solution of (26)–(28) and \(u_{ij}^k|0 \leq i \leq M_1, \quad 0 \leq j \leq M_2, \quad 0 \leq k \leq N\) is a solution of the finite difference scheme (31), respectively. Denote

\[
e_{ij}^k = u(x_i, y_j, t_k) - u_{ij}^k, \quad 0 \leq i \leq M_1, \quad 0 \leq j \leq M_2, \quad 1 \leq k \leq N.
\]
Then there exists a positive constant \( \bar{c} \) such that

\[
\|e^k\| \leq \bar{c}(\tau^2 + h_1^4 + h_2^4), \quad 0 \leq k \leq N,
\]

where \( e^k = [e^k_{0,0}, e^k_{0,1}, \cdots, e^k_{1,0}, e^k_{1,1}, \cdots, e^k_{M_1,0}, e^k_{M_1,1}, \cdots, e^k_{0,M_2}, e^k_{1,M_2}, \cdots, e^k_{M_1,M_2}]^T. \)

**Proof.** One can easily check that the following error equation holds:

\[
(C_{M_2+1} \otimes \bar{C}_{M_1+1})(e^{k+1} - e^k) + \frac{\mu^2 \lambda_0^2}{(1 + \mu \lambda_0)h_1 h_2^2} (Q_{M_2+1} \otimes \bar{Q}_{M_1+1})(e^{k+1} - e^k)
\]

\[
= -\frac{\tau^{\alpha+1}}{2h_1^2} \sum_{l=0}^{k} \lambda_l (C_{M_2+1} \otimes \bar{C}_{M_1+1})(e^{k+1-l} + e^{-l}) - \frac{\tau^{\alpha+1}}{2h_2^2} \sum_{l=0}^{k} \lambda_l (Q_{M_2+1} \otimes \bar{C}_{M_1+1})(e^{k+1-l} + e^{-l})
\]

\[
- \frac{\tau^{\alpha+1}}{2} \sum_{l=0}^{k} \lambda_l (C_{M_2+1} \otimes \bar{C}_{M_1+1})(e^{k+1-l} + e^{-l}) + \tau R^{k+1},
\]

\[
e_{ij}^0 = 0, \quad 0 \leq i \leq M_1, \quad 0 \leq j \leq M_2,
\]

where \( \|R^{k+1}\| \leq \bar{c}_1(\tau^2 + h_1^4 + h_2^4) \), and the matrices \( \bar{C}_{M_1+1}, \bar{C}_{M_1+1}, \bar{Q}_{M_1+1}, \bar{Q}_{M_1+1} \) are given in (17) with the corresponding sizes given by the subscripts.

Multiplying the equation (32) with \((\frac{1}{2} \otimes I_{M_2-1} \oplus \frac{1}{2}) \otimes (\frac{1}{2} \otimes I_{M_1-1} \oplus \frac{1}{2})\), we get

\[
(C_{M_2+1} \otimes C_{M_1+1})(e^{k+1} - e^k) + \frac{\mu^2 \lambda_0^2}{(1 + \mu \lambda_0)h_1 h_2^2} (Q_{M_2+1} \otimes Q_{M_1+1})(e^{k+1} - e^k)
\]

\[
= -\frac{\tau^{\alpha+1}}{2h_1^2} \sum_{l=0}^{k} \lambda_l (C_{M_2+1} \otimes Q_{M_1+1})(e^{k+1-l} + e^{-l}) - \frac{\tau^{\alpha+1}}{2h_2^2} \sum_{l=0}^{k} \lambda_l (Q_{M_2+1} \otimes C_{M_1+1})(e^{k+1-l} + e^{-l})
\]

\[
- \frac{\tau^{\alpha+1}}{2} \sum_{l=0}^{k} \lambda_l (C_{M_2+1} \otimes C_{M_1+1})(e^{k+1-l} + e^{-l}) + \tau R^{k+1},
\]

\[
e_{ij}^0 = 0, \quad 0 \leq i \leq M_1, \quad 0 \leq j \leq M_2,
\]

where \( \|R^{k+1}\| \leq \bar{c}_2(\tau^2 + h_1^4 + h_2^4) \) and

\[
C_{M_1+1} = E_{M_1+1}^2, \quad C_{M_2+1} = E_{M_2+1}^2, \quad Q_{M_1+1} = S_{M_1+1}^T S_{M_1+1}, \quad Q_{M_2+1} = S_{M_2+1}^T S_{M_2+1}
\]

are as in (19), (20) respectively. Once again, we have used the subscripts to indicate the sizes of the matrices.

We can now multiply (33) by \( h_1 h_2 (e^{k+1} + e^k)^T \) and add up the equations for \( 0 \leq k \leq n - 1 \). Noting that

\[
(e^{k+1} + e^k)^T (C_{M_2+1} \otimes C_{M_1+1})(e^{k+1} - e^k) = (e^{k+1})^T (C_{M_2+1} \otimes C_{M_1+1}) e^{k+1} - (e^k)^T (C_{M_2+1} \otimes C_{M_1+1}) e^k,
\]

\[
(e^{k+1} + e^k)^T (Q_{M_2+1} \otimes Q_{M_1+1})(e^{k+1} - e^k) = (e^{k+1})^T (Q_{M_2+1} \otimes Q_{M_1+1}) e^{k+1} - (e^k)^T (Q_{M_2+1} \otimes Q_{M_1+1}) e^k,
\]

\[
h_1 h_2 (e^n)^T (C_{M_2+1} \otimes C_{M_1+1}) e^n \geq \frac{1}{16} \|e^n\|^2, \quad (e^n)^T (Q_{M_2+1} \otimes Q_{M_1+1}) e^n \geq 0,
\]

\[
(C_{M_2+1} \otimes Q_{M_1+1}) = (E_{M_2+1} \otimes S_{M_1+1}^T) (E_{M_2+1} \otimes S_{M_1+1}),
\]

\[
(Q_{M_2+1} \otimes C_{M_1+1}) = (S_{M_2+1}^T \otimes E_{M_1+1}) (S_{M_2+1} \otimes E_{M_1+1}),
\]
Furthermore, the temporal convergence order and spatial convergence order, denoted by numerical examples given below, the maximum norm errors are reported. We have tested (we remark here that we have similar observations in [16]). Therefore, in the between the exact and the numerical solutions also match the proposed order for the examples results are given by the discrete $L^2$ norm, we find that the maximum norm errors

$$E_\infty(h, \tau) = \max_{0 \leq k \leq N} \|U^k - u^k\|_\infty$$

from which we can conclude the theorem just as in the one dimensional case. \qed

5 Numerical experiments

In this section, we carry out numerical experiments for the finite difference scheme to illustrate our theoretical statements. All our tests were done in MATLAB. Although our theoretical results are given by the discrete $L^2$ norm, we find that the maximum norm errors

$$\|e_n\| \leq 18^6 E_\infty(h, \tau) \leq 18^6 \frac{\sum_{k=0}^{n-2} \|e^{k+1} + e^k\| R^{k+1}}{18^6}$$

between the exact and the numerical solutions also match the proposed order for the examples we have tested (we remark here that we have similar observations in [16]). Therefore, in the numerical examples given below, the maximum norm errors are reported.

We first consider the following one-dimensional problem:

**Example 5.1**

$$\frac{C}{\partial} D_\gamma u = \frac{\partial^2 u}{\partial x^2} - u + g(x, t), \quad 0 \leq x \leq 1, \quad 0 < t \leq 1, \quad 1 \leq \gamma < 2,$$

$$u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = 0, \quad 0 \leq x \leq 1,$$

$$\frac{\partial u(0, t)}{\partial x} = 0, \quad \frac{\partial u(L, t)}{\partial x} = 0, \quad 0 < t \leq 1,$$

where $g(x, t) = \frac{\Gamma(\gamma+3)}{2} \alpha t^2 e^x x^2 (1 - x)^2 - e^x t^{\gamma+2} (2 - 8x + 8x^3)$.

Note that the equation can be equivalently written as

$$\frac{\partial u(x, t)}{\partial t} = a I_t^\alpha [u_{xx}(x, t) - u(x, t)] + f(x, t), \quad 0 \leq x \leq 1, \quad 0 < t \leq 1,$$

where $\alpha = \gamma - 1$, $f(x, t) = (\alpha + 3) e^x x^2 (1 - x)^2 t^{\alpha+2} - \frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)} e^x (2 - 8x + 8x^3) t^{2\alpha+3}$. The exact solution is $u(x, t) = e^x x^2 (1 - x)^2 t^{\alpha+3}$.

Figure 1 plots the curves of the exact solution and numerical solution for the problem at $t = 1$ with $\alpha = 0.5$, $h = \tau = \frac{1}{100}$. The maximum norm errors are shown in Table I and Table II. Furthermore, the temporal convergence order and spatial convergence order, denoted by

$$Rate_1 = \log_2 \left( \frac{E_\infty(h, 2\tau)}{E_\infty(h, \tau)} \right) \quad \text{and} \quad Rate_2 = \log_2 \left( \frac{E_\infty(2h, \tau)}{E_\infty(h, \tau)} \right),$$

respectively, are reported.
Figure 1: The exact solution and numerical solution for Example 5.1 at $t = 1$, when $\alpha = 0.5$, $h = \tau = \frac{1}{100}$.

Table 1: Numerical convergence orders in temporal direction with $h = \frac{1}{50}$ for Example 5.1

| $\tau$ | $\alpha = 0.3$ | $\alpha = 0.5$ | $\alpha = 0.7$ |
|--------|----------------|----------------|----------------|
|        | $E_\infty(h, \tau)$ | Rate1 | $E_\infty(h, \tau)$ | Rate1 | $E_\infty(h, \tau)$ | Rate1 |
| 1/5    | 1.6417e-3       | *     | 2.3844e-3       | *     | 3.1904e-3       | *     |
| 1/10   | 4.1558e-4       | 1.9820| 6.0481e-4       | 1.9791| 7.9961e-4       | 1.9964|
| 1/20   | 1.0441e-4       | 1.9929| 1.5221e-4       | 1.9904| 2.0066e-4       | 1.9945|
| 1/40   | 2.6115e-5       | 1.9993| 3.8133e-5       | 1.9970| 5.0254e-5       | 1.9975|
| 1/80   | 6.4822e-6       | 2.0103| 9.5080e-6       | 2.0038| 1.2551e-5       | 2.0015|

Table 2: Numerical convergence orders in spatial direction with $\tau = \frac{1}{2000}$ when $\alpha = 0.5$ for Example 5.1

| $h$    | $E_\infty(h, \tau)$ | Rate2 |
|--------|----------------------|-------|
| 1/2    | 2.2688e-2            | *     |
| 1/4    | 1.3235e-3            | 4.0995|
| 1/8    | 8.2429e-5            | 4.0050|
| 1/16   | 5.1310e-6            | 4.0058|
| 1/32   | 3.0926e-7            | 4.0524|

Next we turn to consider the stability of the scheme by testing (25) numerically. We note that the bound in (25) has been magnified to a certain extent when it is derived theoretically. In our test, we find that the mere quantity $B \geq \left[ \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)} + 1 \right] \sqrt{\|\delta_x^2 \rho\|^2 + \|\rho\|^2 + \|\bar{\rho}\|^2}$ already serves as a good bound for $\|e^N\|$. We have considered two kinds of perturbation given by the discretization of some functions $\rho$, $\bar{\rho}$ and the results are given in Table 3.
Table 3: Stability of the scheme for Example 5.1 when $T = 1$, $\alpha = 0.5$.

| $\rho = \tilde{\rho} = 0.1x$ | $\rho = \tilde{\rho} = 0.1 \sin(x)$ |
|-----------------------------|-----------------------------------|
| $M = \frac{1}{h}$ | $N = \frac{1}{\tau}$ | $\|\varepsilon^N\|$ | $B$ | $M = \frac{1}{h}$ | $N = \frac{1}{\tau}$ | $\|\varepsilon^N\|$ | $B$ |
| 100 | 500 | 0.3097 | 5.4637 | 100 | 500 | 0.2668 | 6.0461 |
| 1000 | 500 | 0.3102 | 5.4637 | 1000 | 500 | 0.2672 | 6.0461 |
| 2000 | 500 | 0.3105 | 5.4637 | 2000 | 500 | 0.2674 | 6.0461 |
| 4000 | 500 | 0.3107 | 5.4637 | 4000 | 500 | 0.2675 | 6.0461 |

The next example is a two-dimensional problem.

**Example 5.2**

$$\int_0^T \mathcal{D}_t^\alpha u = \Delta u - u + \cos(x) \cos(y) \left[ \frac{\Gamma(\gamma+4)}{6} t^3 + 3t^{\gamma+3} \right], \quad (x, y) \in \Omega = (0, \pi) \times (0, \pi), \quad 0 < t \leq 1,$$

$$u(x, y, 0) = 0, \quad \partial_t u(x, y, 0) = 0, \quad (x, y) \in \bar{\Omega}, \quad \frac{\partial u(x, y, t)}{\partial n} \bigg|_{\partial \Omega} = 0, \quad (x, y) \in \partial \Omega, \quad 0 < t \leq 1.$$

Note that the equation can be equivalently written as

$$\frac{\partial u(x, y, t)}{\partial t} = 0 I_t^\alpha (\Delta u - u) + \cos(x) \cos(y) \left[ (\alpha + 4) t^{\alpha+3} + \frac{3\Gamma(\alpha+5)}{\Gamma(2\alpha+5)} t^{2\alpha+4} \right],$$

where $\alpha = \gamma - 1$. The exact solution for this problem is $u(x, t) = \cos(x) \cos(y) t^{\alpha+4}$.

We let $h_1 = h_2 = h$, in this example. Figure 2 shows the exact solution (left) and numerical solution (right) for Example 5.2, when $\alpha = 0.5$, $h = \tau = \frac{1}{100}$. In addition, the maximum norm errors between the exact and the numerical solutions

$$E_\infty(h, \tau) = \max_{0 \leq k \leq N} \max_{(x_i, y_j) \in \Omega} |u(x_i, y_j, t_k) - u_{ij}^k|$$

are shown in Table 4 and Table 5. Meanwhile, the temporal convergence order and spatial convergence order, denoted by

$$Rate_1 = \log_2 \left( \frac{E_\infty(h, 2\tau)}{E_\infty(h, \tau)} \right) \quad \text{and} \quad Rate_2 = \log_2 \left( \frac{E_\infty(2h, \tau)}{E_\infty(h, \tau)} \right),$$

respectively, are reported. These tables confirm the theoretical analysis.

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Figure 2: The exact solution (left) and numerical solution (right) for Example 5.2 when $\alpha = 0.5$, $h = \tau = \frac{1}{50}$.

Table 4: Numerical convergence orders in temporal direction with $h = \frac{\pi}{50}$ for Example 5.2.

| $\tau$  | $\alpha = 0.3$ | $\alpha = 0.5$ | $\alpha = 0.7$ |
|---------|----------------|----------------|----------------|
|         | $E_{\infty}(h, \tau)$ Rate1 | $E_{\infty}(h, \tau)$ Rate1 | $E_{\infty}(h, \tau)$ Rate1 |
| 1/5     | 1.5208e-2 *    | 2.0989e-2 *    | 2.9859e-2 *    |
| 1/10    | 3.7508e-3 2.0196 | 5.2056e-3 2.0115 | 7.5354e-3 1.9864 |
| 1/20    | 9.3437e-4 2.0051 | 1.2874e-3 2.0155 | 1.8780e-3 2.0045 |
| 1/40    | 2.3411e-4 1.9968 | 3.1958e-4 2.0102 | 4.6777e-4 2.0053 |
| 1/80    | 5.8774e-5 1.9939 | 7.9583e-5 2.0057 | 1.1666e-4 2.0035 |

Table 5: Numerical convergence orders in spatial direction with $\tau = \frac{1}{20000}$ when $\alpha = 0.5$ for Example 5.2.

| $h$     | $E_{\infty}(h, \tau)$ Rate2 |
|---------|-----------------------------|
| $\pi/2$ | 3.4342e-3 *                 |
| $\pi/4$ | 2.0348e-4 4.0770            |
| $\pi/8$ | 1.2502e-5 4.0247            |
| $\pi/16$| 7.7904e-7 4.0043            |
| $\pi/32$| 4.9832e-8 3.9665            |

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