A FDR-preserving field theory for interacting Brownian particles: one-loop theory and MCT

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We develop a field theoretical treatment of a model of interacting Brownian particles. We pay particular attention to the requirement of the time reversal invariance and the fluctuation-dissipation relationship (FDR). The method used is a modified version of the auxiliary field method due originally to Andreanov, Biroli and Lefevre [J. Stat. Mech. P07008 (2006)]. We recover the correct diffusion law when the interaction is dropped as well as the standard mode coupling equation in the one-loop order calculation for interacting Brownian particle systems.

I. INTRODUCTION

The only existing successful first-principle theory of structural glass transition [1, 2, 3, 4, 5], the mode coupling theory (MCT) [6, 7, 8, 9, 10], is beset with absence of controllable approximation characterized by smallness parameter. Some years ago one of us attempted to remedy the situation by introducing and working out a dynamical fluid model with a Kac-type long range interaction [11] of appropriate form among elements of the reference fluid, which is anticipated to exhibit glassy behavior [12]. As is well-known, this model has a smallness parameter which is the inverse force range of the Kac potential measured in units of inverse microscopic length scale of the reference fluid. However, the difficulty with this work is the inadequacy of the expansion scheme which violated the detailed balance originating from the time-reversal (TR) invariance of the model equation.

Recently a great deal of attention [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28] is being paid to go beyond the standard MCT. This stimulated developments of satisfactory perturbative calculational methods. Particularly noteworthy is the work of Andreanov, Biroli and Lefevre (ABL) [29] where complications associated with the nonlinear TR transformation of the variable set (namely, the density field $\rho$ and its conjugate $\hat{\rho}$ [30]) are avoided by introducing auxiliary variable set $\theta$, $\hat{\theta}$
which linearizes the TR transformation. The theory was applied to interacting Brownian particle system [32]. The theory was also applied to continuum nonlinear fluid model[31] where the variable set had to be supplemented with the momentum density and velocity fields and their conjugate fields. In this case the theory becomes enormously complicated and will not be considered here. However, although ingenious is the whole approach, consequences of the theory worked out so far have yielded some unsatisfactory features as follows.

- The equation for the nonergodicity parameter gives non-trivial result even for non-interacting Brownian particle systems.

- The memory integrals entering the equation for the density-density correlation function are ill-behaved.

In our recent communication [33], we have proposed a new set of auxiliary fields still denoted as $[\theta], [\hat{\theta}]$ which are defined slightly differently from ABL. However, consequences are drastically different so that the two unsatisfactory features mentioned above now disappear. Here we present a detailed account of our short communication paper. The paper is organized as follows. In Section II the dynamical density field model of interacting Brownian particle system is introduced which is expressed as an action integral containing the density field $[\rho]$ and its conjugate field $[\hat{\rho}]$. This action is shown to be invariant under a certain nonlinear TR transformation. This transformation can be converted into a linear one by introducing a conjugate pair of auxiliary fields $[\theta]$ and $[\hat{\theta}]$. The resulting action integral is divided into the Gaussian and non-Gaussian parts, each of which is separately TR-invariant. However, certain terms coming from the both parts cancel when summed over. Consideration of this fact is essential to recover a simple diffusion law for the nonequilibrium averaged density in non-interacting case. In Sections III and IV we develop a renormalized perturbation theory for interacting cases, and recover within one-loop order the standard MCT equation for the density-density correlation function by invoking the irreducible memory function approach. Section IV summarizes the paper and gives discussion.
II. THE DYNAMIC DENSITY FUNCTIONAL THEORY

A. The dynamic equation for the density fluctuations

We start with the following Langevin equation for the density field \( \rho(r, t) \) of interacting Brownian particles

\[
\partial_t \rho(r, t) = \nabla \cdot \left( \rho(r, t) \nabla \frac{\delta F[\rho]}{\delta \rho(r, t)} \right) + \eta(r, t) \tag{II.1}
\]

where the Gaussian thermal noise \( \eta(r, t) \) has zero mean and variance of the form

\[
< \eta(r, t) \eta(r', t') > = 2T \nabla \cdot \nabla' \left( \rho(r, t) \delta(r - r') \delta(t - t') \right) \tag{II.2}
\]

where the Boltzmann constant \( k_B \) is set to unity, and \( T \) is the temperature of the system. Note that the noise correlation depends on the density variable, i.e., the noise is multiplicative. This is necessary for the Langevin equation (II.1) to satisfy the detailed balance condition so that the system is guaranteed to evolve into the equilibrium state governed by the free energy \( F[\rho] \). In (II.1), \( F[\rho] \) is the free energy density functional which takes the following form:

\[
F[\rho] = T \int d\rho(r) \left( \ln \frac{\rho(r)}{\rho_0} - 1 \right) + \frac{1}{2} \int d\rho \int d\rho' \delta \rho(r) U(r - r') \delta \rho(r') \tag{II.3}
\]

where \( \delta \rho(r, t) \equiv \rho(r, t) - \rho_0 \) is the density fluctuation around the equilibrium density \( \rho_0 \). In (II.3) the first term is the ideal gas part [34] of the free energy, \( F_id[\rho] \), and the second term the interaction part of the free energy, \( F_{int}[\rho] \). Using Ito calculus, Dean [35] has derived the above nonlinear Langevin equation for the (microscopic) density of system of interacting Brownian particles with pair potential \( U(r) \). Earlier, Kawasaki [36] has also obtained the same form of Langevin equation for the coarse-grained density with \( U(r) \) replaced by \(-Tc(r)\), \( c(r) \) being the direct correlation function, by adiabatically eliminating the momentum field in the fluctuating hydrodynamic equations [31] of the simple dense liquids. For this case, the (effective) free energy density functional (II.3) takes the Ramakrishnan-Yussouff (RY) form [37]. For further discussions regarding the nature (and controversy) of the dynamic equation (II.1)-(II.3), we refer to Ref. [38].

B. The dynamic action and the time-reversal invariance

We consider the corresponding action integral \( S[\rho, \dot{\rho}] \) [39] which governs the stochastic dynamics of the coarse-grained density variable. The dynamic action can be derived as follows. The average of
a dynamic quantity of density variable, \(A[\rho]\), should be taken over the thermal-noise driven density fluctuations satisfying the dynamic equation (II.1) and (II.2):

\[
< A[\rho] > = \int \mathcal{D}\rho \ A[\rho] \left\langle \delta \frac{\partial \rho}{\partial \rho}(\mathbf{r}, t) - \nabla \cdot \left( \rho \frac{\delta F}{\delta \rho} \right) \right\rangle_{\eta} \\
= \int \mathcal{D}\rho \int \mathcal{D}\hat{\rho} \ A[\rho] \ \exp \left( \int d\mathbf{r} \int dt \ i \hat{\rho} \left[ \frac{\partial \rho}{\partial \rho} - \nabla \cdot \left( \rho \frac{\delta F}{\delta \rho} \right) \right] \right) \left\langle \exp \left( - \int d\mathbf{r} \int dt \ i \hat{\rho}(\mathbf{r}, t) \hat{\rho}(\mathbf{r}, t) \right) \right\rangle_{\eta} \\
= \int \mathcal{D}\rho \int \mathcal{D}\hat{\rho} \ A[\rho] \ \exp \left( S[\rho, \hat{\rho}] \right),
\]

\[
S[\rho, \hat{\rho}] = \int d\mathbf{r} \int dt \ \left\{ i \hat{\rho} \left[ \frac{\partial \rho}{\partial \rho} - \nabla \cdot \left( \rho \frac{\delta F}{\delta \rho} \right) \right] - T \rho \nabla \cdot \left( \rho \frac{\delta F}{\delta \rho} \right) \right\} \\
= \int d\mathbf{r} \int dt \ \left\{ i \hat{\rho} \left[ \frac{\partial \rho}{\partial \rho} + iT \nabla \cdot \left( \rho \frac{\delta F}{\delta \rho} \right) \right] \right\} (II.4)
\]

where the auxiliary field \(\hat{\rho}\) is real and the last term involving quadratic \(\hat{\rho}\) comes from the average over multiplicative thermal noise \(\eta\). In the first line of (II.4), employing the Ito calculus makes the Jacobian of transformation constant. The dynamic action of this form appearing in the above equation with the RY free energy functional was first written down in [32].

The TR symmetry of the dynamics should be manifested in the dynamic action. In particular, under the TR the two fields \(\rho\) and \(\hat{\rho}\) should transform in such a way that the dynamic action (II.4) remains invariant under these transformations. In order to see this TR invariance of the action, one can rearrange the dynamic action (II.4) as

\[
S[\rho, \hat{\rho}] = \int d\mathbf{r} \int dt \ \left\{ i \hat{\rho} \left[ \frac{\partial \rho}{\partial \rho} - \nabla \cdot \left( \rho \frac{\delta F}{\delta \rho} \right) \right] - T \rho \nabla \cdot \left( \rho \frac{\delta F}{\delta \rho} \right) \right\} \\
= \int d\mathbf{r} \int dt \ \left\{ i \hat{\rho} \left[ \frac{\partial \rho}{\partial \rho} + iT \nabla \cdot \left( \rho \frac{\delta F}{\delta \rho} \right) \right] \right\} (II.5)
\]

This last form of the action suggests that the dynamic action becomes invariant under the field transformation

\[
\rho(\mathbf{r}, -t) = \rho(\mathbf{r}, t) \\
\hat{\rho}(\mathbf{r}, -t) = -\hat{\rho}(\mathbf{r}, t) + i \frac{T}{\delta \rho} \frac{\delta F}{\delta \rho} (\mathbf{r}, t) (II.6)
\]

The invariance of the action under the transformation (II.6) can be shown as follows. Using the transformation of \(\hat{\rho}\) in (II.6) one can rewrite (II.5) as

\[
S[\rho, \hat{\rho}] = \int d\mathbf{r} \int dt \ i \hat{\rho}(\mathbf{r}, t) \frac{\partial \rho}{\partial \rho}(\mathbf{r}, t) + T \int d\mathbf{r} \int dt \ i \hat{\rho}(\mathbf{r}, t) \nabla \cdot \left( \rho(\mathbf{r}, t) \nabla i \hat{\rho}(\mathbf{r}, -t) \right) (II.7)
\]

With the transformation (II.6) the first integral in (II.7) is shown to be TR invariant as

\[
\int d\mathbf{r} \int dt \ i \hat{\rho}(\mathbf{r}, t) \frac{\partial \rho}{\partial \rho}(\mathbf{r}, t) = - \int d\mathbf{r} \int dt \ i \hat{\rho}(\mathbf{r}, -t) \frac{\partial \rho}{\partial \rho}(\mathbf{r}, t) \\
= - \int d\mathbf{r} \int dt \ i \left( - \hat{\rho}(\mathbf{r}, t) + i \frac{T}{\delta \rho} \frac{\delta F}{\delta \rho} (\mathbf{r}, t) \right) \frac{\partial \rho}{\partial \rho}(\mathbf{r}, t) = \int d\mathbf{r} \int dt \ i \hat{\rho}(\mathbf{r}, t) \frac{\partial \rho}{\partial \rho}(\mathbf{r}, t) + \frac{1}{T} \int dt \ \frac{\partial \delta F[\rho]}{\partial \rho} (II.8)
\]
where the first line results from the change of integration variable $t \rightarrow -t$, and we used the fact that the surface term $\int dt \partial_t F[\rho]/T$ vanishes. The second term in (II.7) is manifestly TR invariant since when the time is reversed the spatial integration by parts twice recovers its original form. This proves the invariance of the action (II.4) under the TR transformation (II.6). Note that the TR invariance holds for the generalized form of dynamic action

$$S[\rho, \dot{\rho}] = \int dr \int dt \left\{ i\dot{\rho} \left[ \partial_t \rho - \nabla \cdot \left( D(\rho) \frac{\delta F}{\delta \rho} \right) \right] - T i\dot{\rho} \nabla \cdot (D(\rho) \nabla \dot{\rho}) \right\}$$  \hspace{1cm} (II.9)

where $D(\rho(r,t))$ is a local function of the density field. This action corresponds to the dynamic equation of the form [23]

$$\partial_t \rho(r,t) = \nabla \cdot \left( D(\rho(r,t)) \nabla \frac{\delta F}{\delta \rho(r,t)} \right) + \eta(r,t),$$

$$<\eta(r,t)\eta(r',t')> = 2T \nabla \cdot \nabla' \left( D(\rho(r,t)) \delta(r-r') \delta(t-t') \right)$$  \hspace{1cm} (II.10)

There exists another field-transformation leading to the time-reversal invariance of the action. To see this, one can again rearrange the dynamic action (II.4) as

$$S[\rho, \dot{\rho}] = \int dr \int dt \left\{ i\dot{\rho}(r,t)(-iT) \left( \frac{i}{T} \partial_t \rho(r,t) + \nabla \cdot (\rho(r,t) \nabla \dot{\rho}(r,t)) \right) - i\dot{\rho}(r,t) \nabla \cdot \left( \rho(r,t) \nabla \frac{\delta F}{\delta \rho} \right) \right\}$$  \hspace{1cm} (II.11)

This form of the action suggests the following transformation

$$\rho(r,-t) = \rho(r,t)$$

$$\nabla \cdot (\rho(r,t) \nabla \dot{\rho}(r,-t)) = \frac{i}{T} \partial_t \rho(r,t) + \nabla \cdot (\rho(r,t) \nabla \dot{\rho}(r,t))$$  \hspace{1cm} (II.12)

or equivalently,

$$\rho(r,-t) = \rho(r,t)$$

$$\dot{\rho}(r,-t) = \dot{\rho}(r,t) + iA(\rho(r,t))$$  \hspace{1cm} (II.13)

with the local function $A(\rho(r,t))$ defined as

$$\nabla \cdot (\rho(r,t) \nabla A(\rho(r,t))) = \frac{1}{T} \partial_t \rho(r,t)$$  \hspace{1cm} (II.14)

One can rewrite (II.11) using the transformation (II.12) as

$$\mathcal{S}[\rho, \dot{\rho}] = \int dr \int dt \left\{ i\dot{\rho}(r,t)(-T) \left( \nabla \cdot (\rho(r,t) \nabla \dot{\rho}(r,-t)) \right) - i\dot{\rho}(r,t) \nabla \cdot \left( \rho(r,t) \nabla \frac{\delta F}{\delta \rho} \right) \right\}$$  \hspace{1cm} (II.15)
We have previously noted that the first term is manifestly TR invariant. We thus only need to look at the last term in (II.15):

\[ -\int dr \int dt i\dot{\rho}(r,t)\nabla \cdot \left( \rho(r,t)\nabla \frac{\delta F}{\delta \rho} \right) = -\int dr \int dt i\dot{\rho}(r,-t)\nabla \cdot \left( \rho(r,t)\nabla \frac{\delta F}{\delta \rho} \right) \]

\[ = -\int dr \int dt i \frac{\delta F}{\delta \rho} \nabla \cdot \left( \rho(r,t)\nabla \rho(r,-t) \right) = -\int dr \int dt i \frac{\delta F}{\delta \rho} \left[ \frac{i}{T} \partial_t \rho(r,t) + \nabla \cdot (\rho(r,t)\nabla \rho(r,t)) \right] \]

\[ = -\int dr \int dt i\dot{\rho}(r,t)\nabla \cdot \left( \rho(r,t)\nabla \frac{\delta F}{\delta \rho} \right) \]  

(II.16)

Therefore the dynamic action (II.4) is TR invariant under the second type of the transformation (II.12) as well. The invariance of the action under the second transformation will also hold for the generalized form of the dynamic action (II.9). As will be shown in the next subsection, the FDR is readily derived from these TR transformations.

Note that both (II.6) and (II.12) are intrinsically nonlinear transformations. The equation (II.6) is nonlinear because of the noninteracting contribution \( F_{id} \), whereas the transformation (II.12) is nonlinear due to the 'extra' factor of the density field in our dynamic equation (i.e. the multiplicative nature of the Langevin equation). As discussed in the work of ABL [29], this nonlinearity is the underlying reason why the FDR, obeyed by the action, is not preserved order by order in the renormalized perturbation theory developed for the dynamic action (II.4).

C. Fluctuation-dissipation relation (FDR)

The FDR is a hallmark of the equilibrium dynamics, which provides a fundamental relationship between the correlation of equilibrium fluctuations and the linear response to external perturbation. Here it is shown that the FDR is a direct consequence of the TR symmetry.

The response function \( R(r,t; r',t') \) is defined as a link between induced density change \( \Delta <\rho(r,t)> \) and an external infinitesimal field \( h_e(r',t') \) added to \( F \) (not to the Langevin equation):

\[ \Delta <\rho(r,t)> \equiv \int dr' \int dt' R(r,t; r',t') h_e(r',t') \]  

(II.17)

The contribution of the external field to the free energy, \( \Delta F \equiv -\int dr \int dt \delta \rho(r,t)h_e(r,t) \), will bring the corresponding change in the action, \( \Delta S \), which is given by

\[ \Delta S = \int dr \int dt i\dot{\rho}(r,t)\nabla \cdot \left( \rho(r,t)\nabla h_e(r,t) \right) \]  

(II.18)

The induced density change is then given by

\[ \Delta <\rho(r,t)> = \left\langle \rho(r,t)\Delta S \right\rangle = i \int dr' \int dt' \left\langle \rho(r,t)\nabla' \cdot \left( \rho(r',t')\nabla' \dot{\rho}(r',t') \right) \right\rangle h_e(r',t') \]  

(II.19)
where the integration by parts was performed twice. We see from (II.17) and (II.19) that the response function \( R(r, t; r', t') \) is given by \[16\]
\[
R(r, t; r', t') = i \langle \rho(r, t) \nabla' \cdot \left( \rho(r', t') \nabla' \hat{\rho}(r', t') \right) \rangle
\] (II.20)

Note that the form of the response function differs from the conventional form of the response function 
\( i \rho(r, t) \nabla^2 \hat{\rho}(r', t') \) which holds for the Langevin dynamics with additive noise: (II.20) reflects the multiplicative nature of the original Langevin equation (II.1) and (II.2).

Now we show that the FDR follows directly from the TR transformations. We use the following identity
\[
0 = \langle \rho(r, t) \frac{\delta S}{\delta \hat{\rho}(r', t')} \rangle = \langle \rho(r, t) \left[ i \partial_t \rho(r', t') - i \nabla' \cdot \left( \rho(r', t') \nabla' \frac{\delta F}{\delta \hat{\rho}(r', t')} \right) \right] \rangle
\]
\[
+ T \nabla' \cdot \left( \rho(r', t') \nabla' \hat{\rho}(r', t') \right) + T \nabla' \cdot \left( \rho(r', t') \nabla' \hat{\rho}(r', t') \right) \rangle
\]
\[
= \langle \rho(r, t) \left[ i \partial_t \rho(r', t') - T \nabla' \cdot \left( \rho(r', t') \nabla' \hat{\rho}(r', t') \right) \right] + T \nabla' \cdot \left( \rho(r', t') \nabla' \hat{\rho}(r', t') \right) \rangle \rangle \] (II.21)

where the TR transformation (II.6) was used. The equations (II.20) and (II.21) give the FDR \[40\]
\[
- \frac{1}{T} \partial_t G_{\rho \rho}(r - r', t - t') = -R(r - r', t' - t) + R(r - r', t - t')
\] (II.22)

where \( G_{\rho \rho}(r - r', t - t') \equiv \langle \delta \rho(r, t) \delta \rho(r', t') \rangle \) is the density correlation function. Since the causality requires \( R(r - r', t' - t) = 0 \) for \( t > t' \), (II.22) leads to the standard form of the FDR
\[
R(r - r', t - t') = -\Theta(t - t') \frac{1}{T} \partial_t G_{\rho \rho}(r - r', t - t')
\] (II.23)

where \( \Theta(t) \) is the Heaviside step function. The FDR (II.22) is more directly obtained from the second transformation (II.12) when the second member of (II.12) (denoting the space and time coordinates \( r' \) and \( t' \), respectively) is multiplied by \( \rho(r, t) \) and is taken average.

D. Linearization of the time-reversal transformation

From now on, we focus on the first TR transformation (II.6). Since the nonlinearity of the TR transformation makes the perturbation expansion inconsistent with the FDR [29], this inconsistency would be resolved if the transformation is properly linearized. With the form of the free energy given
in (II.3), one can explicitly write \((1/T)\delta F/\delta \rho\) as

\[
\frac{1}{T} \frac{\delta F_{id}[\rho]}{\delta \rho(r,t)} = \ln \frac{\rho(r,t)}{\rho_0} \equiv \frac{\delta \rho(r,t)}{\rho_0} + f(\delta \rho(r,t)),
\]

\[
\frac{1}{T} \frac{\delta F_{\text{int}}[\rho]}{\delta \rho(r,t)} = \frac{1}{T} \int dr'u(r-r')\delta \rho(r',t),
\]

\[
f(\delta \rho(r,t)) \equiv -\sum_{n=2}^{\infty} \frac{1}{n!} (-\delta \rho(r,t)/\rho_0)^n
\]

(II.24)

where \(f(\delta \rho(r,t))\) is the contribution of the non-Gaussian (higher than the quadratic) part of \(F_{id}[\rho]\). The equation (II.24) leads to an explicit form of the TR transformation (II.6) as

\[
\rho(r,-t) = \rho(r,t)
\]

\[
\dot{\rho}(r,-t) = -\dot{\rho}(r,t) + i\dot{K} * \delta \rho(r,t) + if(\delta \rho(r,t)),
\]

\[
\dot{K} * \delta \rho(r,t) \equiv \int dr' K(r-r')\delta \rho(r',t)
\]

(II.25)

where the kernel \(K(r)\) is defined as \(K(r) \equiv (\delta(r)/\rho_0 + U(r)/T)\). Note that the transformation (II.25) is nonlinear due to the non-Gaussian nature of \(F_{id}[\rho]\), the ideal-gas part of the free energy.

1. **The Gaussian approximation**

If one entirely neglects \(f(\delta \rho(r,t))\) in (II.24), this is tantamount to approximating \(F_{id}[\rho]\) to the Gaussian form

\[
F_{id}^{G}[\rho] \simeq T^{2} \rho_0 \int dr (\delta \rho(r,t))^2
\]

(II.26)

In this case, the transformation (II.25) becomes linear and consequently the FDR would be preserved by the perturbation theory order by order. However, when (II.26) is substituted to the original equation (II.1), the dynamic equation generates the following two terms

\[
\nabla \cdot \left( \rho(r,t) \nabla \frac{\delta F_{id}^{G}[\rho]}{\delta \rho(r,t)} \right) = T \nabla^2 \rho(r,t) + \frac{T}{\rho_0} \nabla \cdot (\delta \rho \nabla \rho)
\]

(II.27)

While the first term is the linear diffusion term, the term which is expected for the noninteracting case, the second nonlinear term gives rise to a spurious contribution, incorrectly yielding a nontrivial result even in the absence of particle interaction [16, 29, 41]. In order to obtain the correct behavior for the noninteracting system, one really should take into account the full logarithmic form of \(F_{id}[\rho]\).
2. Introduction of the auxiliary fields

Fully incorporating $F_{id}[\rho]$ without making truncation, a natural way to make the transformation (II.25) linear is to introduce a new field $\theta(r,t)$ defined as

$$\theta(r,t) \equiv f(\delta \rho(r,t)) = \frac{1}{T} \frac{\delta F_{id}}{\delta \rho} - \frac{\delta \rho}{\rho_0}$$

Note that the definition (II.28) differs from that of ABL in that whereas in the work of ABL, the auxiliary field is defined as the functional derivative of the full free energy with respect to density

$$\theta_{ABL}(r,t) \equiv \frac{1}{T} \frac{\delta F}{\delta \rho}(r,t) = \hat{K} * \delta \rho(r,t) + f(\delta \rho(r,t)),$$

(II.29)

(II.28) limits the new variable $\theta(r,t)$ to the nonlinear part of the transformation.

With the nonlinear constraint (II.28), using the first member of (II.24) we obtain the ideal-gas contribution to the body force as

$$\nabla \cdot \left( \rho \nabla \frac{\delta F_{id}}{\delta \rho} \right) = T \nabla \cdot \left( \rho \nabla \left( \frac{\delta \rho}{\rho_0} + \theta \right) \right) = T \nabla^2 \rho + \frac{T}{\rho_0} \nabla \cdot (\delta \rho \nabla \rho) + \rho_0 T \nabla^2 \theta + T \nabla \cdot (\delta \rho \nabla \theta)$$

(II.30)

We have seen that the first two terms are the contributions from the Gaussian part of $F_{id}$. On the other hand, since due to cancellation of the two nonlinear effects the entire ideal-gas contribution to the dynamics should be of pure diffusion

$$\nabla \cdot \left( \rho \nabla \delta F_{id}/\delta \rho \right) = T \nabla \cdot \left( \rho \nabla \ln(\rho/\rho_0) \right) = T \nabla^2 \rho,$$

the sum of the last three terms in (II.30) should vanish:

$$\frac{T}{\rho_0} \nabla \cdot (\delta \rho \nabla \rho) + \rho_0 T \nabla^2 \theta + T \nabla \cdot (\delta \rho \nabla \theta) = 0$$

(II.31)

As shown in the subsection G, this cancellation is used to obtain the correct dynamic behavior for the noninteracting case.

E. New form of the dynamic action

With the constraint (II.28), the action (II.4) takes the following new form

$$S[\rho, \dot{\rho}] = \int dr \int dt \left\{ i \dot{\rho} \left[ \partial_t \rho - \nabla \cdot \left( \rho \nabla \frac{\delta F_{id}}{\delta \rho} \right) - \nabla \cdot \left( \rho \nabla \frac{\delta F_{int}}{\delta \rho} \right) \right] - T \rho (\nabla \dot{\rho})^2 \right\}$$

$$\rightarrow S[\rho, \dot{\rho}, \theta, \dot{\theta}] = \int dr \int dt \left\{ i \dot{\rho} \left[ \partial_t \rho - T \nabla \cdot \left( \rho \nabla \left( \frac{\delta \rho}{\rho_0} + \theta \right) \right) - T \nabla \cdot \left( \rho \nabla \hat{U} * \delta \rho \right) \right]$$

$$- T \rho (\nabla \dot{\rho})^2 + i \dot{\theta} \left( \theta - f(\delta \rho) \right) \right\}$$

(II.32)
where the last term comes from the exponentiation of the delta functional \( \delta[\theta(r, t) - f(\rho(r, t))] \).

We now identify the linear transformation under which the new action \( S[\rho, \dot{\rho}, \theta, \dot{\theta}] \) becomes invariant. The transformation of \( \rho(r, t) \) and \( \dot{\rho}(r, t) \) are already given in (II.25) with \( f(\delta \rho(r, t)) \) being replaced by \( \theta(r, t) \). The transform of \( \theta(r, t) \) must be the same as that of \( \rho(r, t) \) since \( \theta \) is a local function of \( \rho \). Thus we only need to know how \( \theta(r, t) \) should transform. Since as shown in (II.7), the terms involving \( \dot{\rho}(r, -t) \) becomes manifestly TR invariant, we only have to consider the term \( v_\alpha(r, t) \equiv i\dot{\rho}(r, t)\partial_\alpha \rho(r, t) \), together with the term \( v_\beta(r, t) \equiv i\dot{\theta}(r, t)\theta(r, t) \). When time is reversed, \( v_\alpha(r, -t) \) generates a term \( \theta(r, t)\partial_\alpha \rho(r, t) \). This term should be cancelled by the term generated by the \( v_\beta(r, -t) \). This requires that \( \theta \) transform as \( \dot{\theta}(r, -t) = \dot{\theta}(r, t) + i\partial_\alpha \rho(r, t) \). Thus the linear transformation under which the new action (II.32) becomes invariant is given by

\[
\begin{align*}
\rho(r, -t) &= \rho(r, t) \\
\dot{\rho}(r, -t) &= -\dot{\rho}(r, t) + iK \ast \delta \rho(r, t) + i\theta(r, t) \\
\theta(r, -t) &= \theta(r, t) \\
\dot{\theta}(r, -t) &= \dot{\theta}(r, t) + i\partial_\alpha \rho(r, t)
\end{align*}
\]  
(II.33)

It is easy to show that the modulus of the associated transformation matrix \( O \) is unity (\( \det O = -1 \)).

The new action \( S[\rho, \dot{\rho}, \theta, \dot{\theta}] \) can be decomposed into the Gaussian part \( S_g[\rho, \dot{\rho}, \theta, \dot{\theta}] \) and the non-Gaussian part \( S_{ng}[\rho, \dot{\rho}, \theta, \dot{\theta}] \) [44]:

\[
S[\rho, \dot{\rho}, \theta, \dot{\theta}] = S_g[\rho, \dot{\rho}, \theta, \dot{\theta}] + S_{ng}[\rho, \dot{\rho}, \theta, \dot{\theta}]
\]

\[
S_g[\rho, \dot{\rho}, \theta, \dot{\theta}] = \int dr \int dt \left\{ i\dot{\rho} \left[ \partial_\alpha \rho - T \nabla^2 \rho - \rho_0 T \nabla^2 \theta - \rho_0 \nabla^2 \dot{U} + \delta \rho \right] - T \rho_0 (\nabla \dot{\rho})^2 + i\dot{\theta} \right\}
\]

\[
S_{ng}[\rho, \dot{\rho}, \theta, \dot{\theta}] = \int dr \int dt \left\{ i\dot{\rho} \left[ -\nabla \cdot (\delta \rho \nabla \dot{U} + \delta \rho) - \frac{T}{\rho_0} \nabla \cdot (\delta \rho \nabla \theta) - T \nabla \cdot (\delta \rho \nabla \theta) \right]
- T \delta \rho (\nabla \dot{\rho})^2 - i\dot{\theta} f(\delta \rho) \right\}
\]  
(II.34)

The actions \( S_g[\rho, \dot{\rho}, \theta, \dot{\theta}] \) and \( S_{ng}[\rho, \dot{\rho}, \theta, \dot{\theta}] \) are shown to be separately invariant under the transformation (II.33). This is apparent since the transformation (II.33) is linear in \( \delta \rho \), \( \dot{\rho} \), \( \theta \), and \( \dot{\theta} \). Though with the constraint (II.28) the three underlined terms in (II.34) vanish when summed together, one should keep each of them in the explicit calculation of renormalized perturbation theory since their presence is crucial for the separate invariance of the actions \( S_g[\rho, \dot{\rho}, \theta, \dot{\theta}] \) and \( S_{ng}[\rho, \dot{\rho}, \theta, \dot{\theta}] \) which enables one to construct the FDR-preserving renormalized perturbation theory from these actions. Nevertheless we explicitly show in the one-loop order that the ultimate effect of these three underlined terms is their cancellation. We also note that the linearization of the TR transformation inevitably brings back the non-polynomial nonlinearity \(-i\dot{\theta} f(\delta \rho)\) in the action.
F. New form of the response function

The presence of the external infinitesimal field \( h_e(r, t) \) leads to shifting \( f(\delta \rho(r, t)) \) to \( f(\delta \rho(r, t)) - h_e(r, t)/T \) in (II.25). This gives rise to a change in action \( \Delta S[\rho, \dot{\rho}, \theta, \dot{\theta}] = \int dr \int dt \dot{\theta}(r, t) h_e(r, t)/T \). Then

\[
\Delta < \rho(r, t) > = \left\langle \rho(r, t) \Delta S[\rho, \dot{\rho}, \theta, \dot{\theta}] \right\rangle = \int dr' \int dt' \frac{i}{T} \left\langle \delta \rho(r, t) \dot{\theta}(r', t') \right\rangle h_e(r', t')
\]  

(II.35)

Therefore with the new form of the action (II.32), the response function takes the form

\[
R(r, t; r', t') = \frac{i}{T} \left\langle \delta \rho(r, t) \dot{\theta}(r', t') \right\rangle
\]  

(II.36)

Taking correlation of the last member of (II.33) with \( i\delta \rho(r, t)/T \) we obtain

\[
\frac{i}{T} \left\langle \delta \rho(r, t) \dot{\theta}(r', -t') \right\rangle = \frac{i}{T} \left\langle \delta \rho(r, -t) \dot{\theta}(r', t') \right\rangle = \frac{i}{T} \left\langle \delta \rho(r, t) \dot{\theta}(r', t') \right\rangle - \frac{1}{T} \partial_r C_{\rho\theta}(r, t; r', t')
\]  

(II.37)

which is the FDR (II.22).

G. Dynamics for the noninteracting case: nonperturbative result

The noninteracting case \((U = 0)\) is an important guide in dealing with the \( \theta \) field since one has to recover the linear diffusion law when \( U = 0 \). In the absence of the particle interaction, the full action (II.32) reduces to \( S_{id}[\psi] \equiv S[\psi; U = 0] \)

\[
S_{id}[\psi] \equiv \int dr \int dt \left\{ i\dot{\psi} \left[ \partial_t \rho - T \nabla^2 \rho - \rho_0 T \nabla^2 \theta - \frac{T}{\rho_0} \nabla \cdot (\delta \rho \nabla \rho) - T \nabla \cdot (\delta \rho \nabla \theta) \right] - T \rho_0 (\nabla \rho)^2 - T \delta \rho (\nabla \dot{\theta})^2 + i\partial_t \theta - i\partial_t f(\delta \rho) \right\}
\]  

(II.38)

where \( \psi \equiv (\delta \rho, \dot{\rho}, \theta, \dot{\theta}) \). We show that the action \( S_{id}[\psi] \) yields the dynamic behavior consistent for the noninteracting system. We make use of the following identities

\[
\left\langle \delta \rho(2) \frac{\delta S_{id}[\psi]}{\delta \rho(1)} \right\rangle = 0, \quad \left\langle \delta \rho(2) \frac{\delta S_{id}[\psi]}{\delta \theta(1)} \right\rangle = 0
\]  

(II.39)

where \( 1 \equiv (r, t) \) and \( 2 \equiv (0, 0) \). The first identity can be written explicitly as

\[
0 = \left\langle \delta \rho(2) \frac{\delta S_{id}[\psi]}{\delta \rho(1)} \right\rangle = i \left( \frac{\partial}{\partial t} - T \nabla^2 \right) G_{\rho\rho}(1 - 2) + 2T \rho_0 \nabla^2 \left\langle \dot{\rho}(1) \delta \rho(2) \right\rangle + 2T' \left\langle \delta \rho(2) \nabla \cdot (\delta \rho(1) \nabla \dot{\rho}(1)) \right\rangle
\]  

(II.40)

where we used the fact that the sum of the three underlined terms in (II.38) vanishes (see (II.31)). Similarly, using the second identity in (II.39), we obtain

\[
0 = \left\langle \delta \rho(2) \frac{\delta S_{id}[\psi]}{\delta \theta(1)} \right\rangle = -i\rho_0 T \nabla^2 \left\langle \dot{\rho}(1) \delta \rho(2) \right\rangle + \left\langle \dot{\theta}(1) \delta \rho(2) \right\rangle - i T \left\langle \delta \rho(2) \nabla \cdot (\delta \rho(1) \nabla \dot{\rho}(1)) \right\rangle
\]  

(II.41)
where cancellation of the underlined terms was not used. Since in (II.41), \( \langle \dot{\rho}(1) \delta \rho(2) \rangle = \langle \dot{\theta}(1) \delta \rho(2) \rangle = 0 \) for \( t > 0 \) by causality, we obtain
\[
\langle \delta \rho(2) \nabla \cdot (\delta \rho(1) \nabla \dot{\rho}(1)) \rangle = 0 \quad \text{for} \quad t > 0 \tag{II.42}
\]
The eqs. (II.40), (II.42), and causality lead to
\[
\partial_t G_{\rho\rho}(r,t) = T \nabla^2 G_{\rho\rho}(r,t), \quad \text{for} \quad t > 0. \tag{II.43}
\]
This result, the diffusion equation for the density correlation function, is the anticipated dynamic behavior for the noninteracting system \( \diamondsuit[42]\diamondsuit \).

On the other hand, if one employs the ABL approach ((II.29)),
\[
\theta_{\text{ABL}}(r,t) \equiv \frac{1}{T} \frac{\delta F}{\delta \rho(r,t)} = \hat{K} * \delta \rho(r,t) + f(\delta \rho(r,t)),
\]
one obtains the following action
\[
S_{\text{ABL}}[\psi] = \int dr \int dt \left\{ i \dot{\rho} \left[ \partial_t \rho - T \nabla \cdot \left( \rho \nabla \theta \right) \right] - T \rho \nabla^2 \rho \right\} + \int dr \int dt \left\{ i \dot{\theta} \left[ \theta - \frac{\delta \rho}{\rho_0} - f(\delta \rho) \right] \right\} \tag{II.44}
\]
where the subscript ABL of \( \theta \) is dropped for simplicity. In the absence of interaction, \( K(r) = \delta(r)/\rho_0 \), and hence \( \hat{K} * \delta \rho(r,t) = \delta \rho(r,t)/\rho_0 \). Therefore
\[
S_{\text{ABL,id}}[\psi] = \int dr \int dt \left\{ i \dot{\rho} \left[ \partial_t \rho - T \nabla \cdot \left( \rho \nabla \theta \right) \right] - T \rho \nabla^2 \rho \right\} + \int dr \int dt \left\{ i \dot{\theta} \left[ \theta - \frac{\delta \rho}{\rho_0} - f(\delta \rho) \right] \right\} \tag{II.45}
\]
Applying the above two identities and additional one \( \langle \delta \rho(2) \delta S_{\text{ABL,id}}[\psi]/\delta \dot{\theta}(1) \rangle = 0 \) to \( S_{\text{ABL,id}}[\psi] \), we obtain for \( t > 0 \)
\[
\partial_t G_{\rho\rho}(r,t) = T \nabla^2 G_{\rho\rho}(r,t) + \rho_0 T \nabla^2 \left\{ f(\delta \rho(r,t)\delta \rho(0,0)) \right\} + T \nabla \cdot \left\{ \delta \rho(r,t) \nabla \theta(r,t) \delta \rho(0,0) \right\} \tag{II.46}
\]
The ABL action (II.45) thus does not appear to yield the anticipated dynamic behavior for the noninteracting particles. This discrepancy is puzzling since no approximation has been made to reach (II.46). In fact, however, a careful treatment as detailed in Appendix A leads to the diffusion law (II.43).

III. THE RENORMALIZED PERTURBATION THEORY: THE LOOP EXPANSION

We are now ready to develop a renormalized perturbation theory \( [39] \) for the new form of the action (II.34), which preserves the FDR order by order.
A. Action

Four field variables $\psi_\alpha(r,t)$, $\alpha = 1, 2, 3, 4$ are defined where

$$\psi_1(r,t) \equiv \delta \rho(r,t) \equiv \rho(r,t) - \rho_0, \quad \psi_2(r,t) \equiv \dot{\rho}(r,t), \quad \psi_3(r,t) \equiv \theta(r,t), \quad \psi_4(r,t) \equiv \dot{\theta}(r,t) \quad \text{(III.1)}$$

We arrange fields $\psi_\alpha(r,t)$ in column vector $\Psi(j)$ and row vector $\Psi^T(j)$ where $j$ stands for a set of indices and space-time variable: $j \equiv \alpha_j, x_j$ with $x_j \equiv (r_j, t_j)$. Thus we have

$$\Psi(j) = \left( \psi_{\alpha_j}(x_j) \right) = \begin{pmatrix} \delta \rho(x_j) \\ \dot{\rho}(x_j) \\ \theta(x_j) \\ \dot{\theta}(x_j) \end{pmatrix}, \quad \Psi^T(j) = \left( \psi_{\alpha_j}(x_j) \right)^T = \begin{pmatrix} \delta \rho(x_j) \\ \dot{\rho}(x_j) \\ \theta(x_j) \\ \dot{\theta}(x_j) \end{pmatrix} \quad \text{(III.2)}$$

Also introduce the notation

$$\partial_j \equiv \frac{\partial}{\partial t_j}, \quad \nabla_j \equiv \frac{\partial}{\partial x_j} \quad \text{(III.3)}$$

We formally write the action integral (II.32) as

$$S[\psi] = -\frac{1}{2} G_0^{-1}(12) \psi(1) \psi(2) + \sum_{n=3}^{\infty} \frac{1}{n!} V_s(12\cdots n) \psi(1) \psi(2) \cdots \psi(n) \quad \text{(III.4)}$$

where $V_s(12\cdots n)$ is the fully symmetrized vertices, and we have used the convention that repeated arguments $j = \alpha_j, x_j$ imply summation over $j = \alpha_j$ and integration over $x_j$.

B. The one-particle irreducible (1PI) loop expansion

1. Generating functionals and Legendre transform

We introduce a generating functional defined as

$$W[J] \equiv \ln < e^{J(1)\psi(1)} >, \quad < \cdots > \equiv \frac{\int d[\psi] \cdots e^{S[\psi]} }{\int d[\psi] e^{S[\psi]} } \quad \text{(III.5)}$$

where $J(1)$ is the local source field. The functional $W[J]$ is the generating functional for the connected (cumulant) correlation functions: averages and pair correlations without source field are obtained from functional derivatives of $W[J]$ with respect to the source field $J$ in the limit of zero source

$$G(12\cdots n) \equiv \frac{\delta^n W[J]}{\delta J(1) \delta J(2) \cdots \delta J(n)} \bigg|_{J=0},$$

$$G(1) \equiv \phi(1) = \frac{\delta W[J]}{\delta J(1)} \bigg|_{J=0} = < \psi(1) >,$$

$$G(12) \equiv \frac{\delta^2 W[J]}{\delta J(1) \delta J(2)} \bigg|_{J=0} = \left( < \psi(1) \psi(2) > - \phi(1) \phi(2) \right), \quad \text{etc.} \quad \text{(III.6)}$$
where response functions are included by using hatted variables, and will not be explicitly mentioned.

We define a new generating functional known as the vertex functional $\Gamma[\phi]$ via the Legendre transform

$$\Gamma[\phi] \equiv W[J] - J(1)\phi(1) \quad (III.7)$$

We then obtain

$$\frac{\delta \Gamma[\phi]}{\delta \phi(1)} = \frac{\delta W[J]}{\delta \phi(1)} - \frac{\delta J(2)}{\delta \phi(1)} \phi(2) = -J(1) \quad (III.8)$$

where the first two terms in the rhs cancels since $\frac{\delta W[J]}{\delta J(2)} = \phi(2)$.

The vertex functions $\Gamma_n(12 \cdots n)$ are defined as the derivatives of the vertex functional $\Gamma[\phi]$ with respect to $\phi$’s:

$$\Gamma_n(12 \cdots n) \equiv -\frac{\delta^n \Gamma[\phi]}{\delta \phi(1) \delta \phi(2) \cdots \delta \phi(n)} \quad (III.9)$$

Let us compute the first few vertex functions. We already obtained the first one,

$$\Gamma_1(1) = -\frac{\delta \Gamma[\phi]}{\delta \phi(1)} = J(1).$$

The second vertex function is given by the inverse of the propagator:

$$\Gamma_2(12) = -\frac{\delta^2 \Gamma[\phi]}{\delta \phi(1) \delta \phi(2)} = \frac{\delta J(1)}{\delta \phi(2)} = \left[ \frac{\delta \phi(2)}{\delta J(1)} \right]^{-1} = \frac{\delta^2 W[J]}{\delta J(1) \delta J(2)} = G^{-1}(12) \quad (III.10)$$

In order to obtain the higher-order vertex functions, we use the following general equation

$$\delta G^{-1}(12) = -G^{-1}(11')\delta G(1'2')G^{-1}(2'2) = -\Gamma_2(11')\delta G(1'2')\Gamma_2(2'2) \quad (III.11)$$

Now the third vertex function $\Gamma_3(123)$ is given by

$$\Gamma_3(123) \equiv \frac{\delta \Gamma_2(12)}{\delta \phi(3)} = \frac{\delta G^{-1}(12)}{\delta \phi(3)} = -\Gamma_2(11')\frac{\delta G(1'2')}{\delta \phi(3)} \Gamma_2(2'2) \quad (III.12)$$

Using

$$\frac{\delta G(12)}{\delta \phi(3)} = \frac{\delta G(12)}{\delta J(3')} \frac{\delta J(3')}{\delta \phi(3)} = G(123')\Gamma_2(3'3), \quad (III.13)$$

we obtain

$$\Gamma_3(123) = -\Gamma_2(11')\Gamma(22')\Gamma_2(3'3)G(1'2'3') \quad (III.14)$$

or equivalently

$$G(123) = -G(11')G(22')G(33')\Gamma_3(1'2'3') \quad (III.15)$$
Likewise, the 4-point correlation function is given by

\[ G(1234) = GG\Gamma_3 \Gamma_3 GG + (2 \text{ sym. terms}) - GGGG \Gamma_4 \]  
(III.16)

where the indices are suppressed for brevity. Therefore one can obtain the higher order correlation functions from the vertex functions, which in turn can be obtained from the vertex functional \( \Gamma[\phi] \) by functional differentiation with respect to \( \phi(1) \).

2. 1PI Loop expansion

One can systematically calculate the vertex functional \( \Gamma[\phi] \) via the loop expansion [45, 46, 47, 48]. We first rewrite (III.5) as

\[ W[J] = \lambda \ln \left( c \int d[\psi] \exp \left[ \frac{1}{\lambda} \left( S[\psi] + J(1)\psi(1) \right) \right] \right) \]  
(III.17)

where \( c \) is the normalization factor: it is determined by the requirement \( W[J = 0] = 0 \) which leads to

\[ c^{-1} = \int d\psi \exp \left( S[\psi]/\lambda \right) \]  

One can perform a formal expansion of \( W[J] \) in powers of \( \lambda \) (and can set \( \lambda = 1 \) at the end of calculation). This is an expansion in the strength of fluctuations and is called the loop expansion. We expand the integrand in (III.17) around the field \( \psi_c \) which extremizes the action

\[ S[\psi] + J(1)\psi(1) \]  
for a given source field \( J(1) \):

\[ \frac{\delta S[\psi_c]}{\delta \psi_c(1)} = -J(1) \]  
(III.18)

Writing \( \psi(1) = \psi_c(1) + \sqrt{\lambda}\chi(1) \), we expand \( S[\psi] + J(1)\psi(1) \) as

\[ S[\psi] + J(1)\psi(1) = S[\psi_c + \sqrt{\lambda}\chi] + J(1)\left( \psi_c(1) + \sqrt{\lambda}\chi(1) \right) \]

\[ = \left( S[\psi_c] + J(1)\psi_c(1) \right) + \sum_{n=2}^{\infty} \frac{\lambda^{n/2}}{n!} S^{(n)}_c(12 \cdots n)\chi(1)\chi(2) \cdots \chi(n) \]  
(III.19)

where \( S^{(n)}_c(12 \cdots n) \equiv \delta^n S[\psi_c]/\delta\psi_c(1)\delta\psi_c(2) \cdots \delta\psi_c(n) \), and the term linear in \( \chi \) vanishes due to the relation (III.18).

The first two terms in (III.19) give the leading order results

\[ W[J] = W_0[J] \equiv S[\psi_c] + J(1)\psi_c(1), \]

\[ \phi(1) = \phi_0(1) \equiv \frac{\delta W_0[J]}{\delta J(1)} = \frac{\delta \psi_c(2)}{\delta J(1)} \frac{\delta \psi_c(2)}{\delta \psi_c(2)} \left( S[\psi_c] + J(1')\psi_c(1') \right) \]

\[ = \frac{\delta \psi_c(2)}{\delta J(1)} \psi_c(1') \frac{\delta J(1')}{\delta \psi_c(2)} = \psi_c(1) \]

\[ \Gamma[\phi] = \Gamma_0[\phi] \equiv W_0[J] - J(1)\phi_0(1) = S[\psi_c] = S[\phi] \]  
(III.20)
The functional $W_0[J]$ contains the connected tree (no loop) diagrams only: it is the generating functional of the connected tree diagrams. The zeroth order result of $\Gamma[\phi]$ is therefore given by the 'average' action $S[\phi]$.

Considering the expansion (III.19) up to the term proportional to $\lambda$, we have the Gaussian integral

$$
\int d\chi \exp \left[ -\frac{1}{2}|S_c^{(2)}(12)|\chi(1)\chi(2) \right] = \left[ \det |S_c^{(2)}(12)| \right]^{-1/2} = \exp \left( -\frac{1}{2} \text{Tr} \ln |S_c^{(2)}(12)| \right)
$$

(III.21)

where the identity $\ln \det M = \text{Tr}(\ln M)$ is used for the last equality. This means that

$$
W[J] = W_0[J] + \lambda W_1[J] + O(\lambda^2)
$$

$$
W_1[J] = -\frac{1}{2} \left[ \text{Tr} \ln |S_c^{(2)}(12)| - \text{Tr} \ln |S_c^{(2)}(12)|_{J=0} \right]
$$

(III.22)

where the last term comes from the normalization.

With the general form of the action (III.4), $S_c^{(2)}(12)$ is given by

$$
S_c^{(2)}(12) = -G^{-1}(12; \psi_c) \equiv -G^{-1}_0(12) + V_s(123)\psi_c(3) + \frac{1}{2!}V_s(1234)\psi_c(3)\psi_c(4) + \cdots
$$

(III.23)

Since from (III.18) $\psi_c = 0$ for $J = 0$ (excluding any symmetry breaking solution of (III.18) with $J = 0$), we obtain

$$
W_1[J] = -\frac{1}{2} \text{Tr} \ln \left( G^{-1}(11'; \psi_c)G_0(1'2) \right)
$$

$$
= -\frac{1}{2} \text{Tr} \ln \left( \delta(12) - V_s(11'3)\psi_c(3)G_0(1'2) - \frac{1}{2!}V_s(11'34)\psi_c(3)\psi_c(4)G_0(1'2) + \cdots \right)
$$

$$
= \frac{1}{2} \left\{ V_s(11'3)\psi_c(3)G_0(1'1) + \frac{1}{2}V_s(11'34)\psi_c(3)\psi_c(4)G_0(1'1)
$$

$$
- \frac{1}{2}V_s(11'2)\psi_c(2)G_0(1'2')V_s(2'3'4)\psi_c(4)G_0(3'1) + \cdots \right\}
$$

(III.24)

Thus the trace operation generates a set of 1PI [49] one-loop diagrams. This expression leads to

$$
\Gamma[\phi] = W_0[J] + \lambda W_1[J] - J(1)\phi(1) = S[\psi_c] + J(1)(\psi_c(1) - \phi(1)) + \lambda W_1[J]
$$

(III.25)

where we have not yet included the second order contribution to $W[J]$. We need to express $\psi_c(1)$ and $J(1) = -\delta S[\psi_c]/\delta \psi_c(1)$ in terms of $\phi$ field. Noting that

$$
\psi_c(1) = \phi(1) + O(\lambda)
$$

$$
-J(1) = S^{(1)}(1; \psi_c) \equiv \frac{\delta S[\psi_c]}{\delta \psi_c(1)} = S^{(1)}(1; \phi) + S^{(2)}(12; \phi)(\psi_c(2) - \phi(2)) + O(\lambda^2),
$$

$$
S[\psi_c] = S[\phi] + S^{(1)}(1; \phi)(\psi_c(1) - \phi(1)) + O(\lambda^2),
$$

(III.26)
we obtain for the first two terms in (III.25)

\[
S[\psi_c] + J(1)(\psi_c(1) - \phi(1)) = S[\phi] + S^{(1)}(1; \phi)(\psi_c(1) - \phi(1)) + \frac{1}{2}S^{(2)}(12; \phi)(\psi_c(1) - \phi(1))(\psi_c(2) - \phi(2)) \\
- \left( S^{(1)}(1; \phi) + S^{(2)}(12; \phi)(\psi_c(2) - \phi(2)) \right)(\psi_c(1) - \phi(1)) + O(\lambda^3)
\]

\[
= S[\phi] - \frac{1}{2}S^{(2)}(12; \phi)(\psi_c(1) - \phi(1))(\psi_c(2) - \phi(2)) + O(\lambda^3)
\]

where the two linear contributions cancel. Therefore, up to the one-loop order, the eq. (III.25) leads to

\[
\Gamma[\phi] = S[\phi] + \lambda W_1[\psi_c = \phi] = S[\phi] - \frac{\lambda}{2} \text{Tr} \ln \left( G^{-1}(\phi) \cdot G_0 \right) + O(\lambda^2)
\]

(III.28)

where the last term in the last line of (III.27) was not yet included because of its being \(O(\lambda^2)\).

We now come to the two-loop calculation. Considering (III.17) and (III.19), we have the following two-loop contributions for \(W[J]\)

\[
W[J] = W_0[J] + \lambda W_1[J] + \lambda^2 W_2[J] + O(\lambda^3)
\]

\[
W_2[J] = \frac{1}{4!} S_c^{(4)}(1234) < \chi(1)\chi(2)\chi(3)\chi(4) >_0
\]

\[
+ \frac{1}{2!} \left( \frac{1}{3!} \right)^2 S_c^{(3)}(123)S_c^{(3)}(456) < \chi(1)\chi(2)\chi(3)\chi(4)\chi(5)\chi(6) >_0
\]

(III.29)

where \(W_0[J]\) and \(W_1[J]\) are given respectively by (III.20) and (III.24). In (III.29), \(< \cdots >_0\) denotes the average over the Gaussian distribution

\[
< \cdots >_0 \equiv \frac{\int d[\chi](\cdots) \exp \left( -\frac{1}{2} |S_c^{(2)}(12)|\chi(1)\chi(2) \right)}{\int d[\chi] \exp \left( -\frac{1}{2} |S_c^{(2)}(12)|\chi(1)\chi(2) \right)}.
\]

Using the Wick’s theorem and symmetry of \(S_c^{(3)}(123)\) and \(S_c^{(4)}(1234)\), we obtain

\[
W_2[J] = \frac{1}{8} S_c^{(4)}(1234)G(12)G(34) + \frac{1}{12} S_c^{(3)}(123)S_c^{(3)}(456)G(14)G(25)G(36)
\]

\[
+ \frac{1}{8} S_c^{(3)}(123)S_c^{(3)}(456)G(12)G(56)G(34)
\]

(III.30)

The diagrammatic expression for \(W_2[J]\) is shown in Fig. 1. While the first two diagrams in Fig. 1 are 1PI diagrams, the last one in Fig. 1 is 1PR (one-particle reducible) diagram. We will see that the Legendre transform to \(\Gamma[\phi]\) eliminates this 1PR diagram, which makes \(\Gamma[\phi]\) diagrammatically simpler than \(W[J]\).

Now going back to (III.25) and adding \(W_2[J]\), the second-order contribution of \(W[J]\), we have for the Legendre transform of \(W[J]\)

\[
\Gamma[\phi] = S[\psi_c] + J(1)(\psi_c(1) - \phi(1)) + \lambda W_1[J] + \lambda^2 W_2[J]
\]

(III.31)
We have seen in (III.27) that the first two terms in the rhs. of (III.31) has no $O(\lambda)$-contribution. Up to the second-order in $\lambda$, they are then given by

$$S[\psi_c] + J(1)(\psi_c(1) - \phi(1)) = S[\phi] - \frac{1}{2} S^{(2)}(12; \phi)(\psi_c(1) - \phi(1))(\psi_c(2) - \phi(2)) + O(\lambda^3) \quad (III.32)$$

Now we compute $\psi_c(1) - \phi(1)$. Note that

$$\phi(1) = \frac{\delta W[J]}{\delta J(1)} = \psi_c(1) + \lambda \frac{\delta W_1[J]}{\delta J(1)} + \cdots$$

$$\psi_c(1) - \phi(1) = -\lambda \frac{\delta W_1[J]}{\delta J(1)} = -\lambda \frac{\delta \psi_c(2)}{\delta J(1)} \frac{\delta W_1[J]}{\delta \psi_c(2)} = \lambda[S_c^{(2)}]^{-1}(12) \frac{\delta W_1[J]}{\delta \psi_c(2)}$$

where we used (III.18) and (III.23). Using

$$\frac{\delta W_1[J; \phi]}{\delta \phi(2)} = -\frac{1}{2} \frac{\delta}{\delta \phi(2)} \left[ \text{Tr} \ln \left( G^{-1}(\phi) \cdot G_0 \right) \right]$$

we obtain

$$\psi_c(1) - \phi(1) = -\lambda G(12; \phi) S^{(3)}(234; \phi) G(34; \phi) + O(\lambda^2) \quad (III.35)$$

Substitution of (III.35) into (III.32) gives

$$S[\psi_c] + J(1)(\psi_c(1) - \phi(1))$$

$$= S[\phi] - \frac{1}{2} S^{(2)}(12; \phi) \cdot \left( -\lambda \frac{1}{2} G(13; \phi) S^{(3)}(345; \phi) G(45; \phi) \right) \cdot \left( -\lambda \frac{1}{2} G(26; \phi) S^{(3)}(678; \phi) G(78; \phi) \right)$$

$$= S[\phi] + \frac{\lambda^2}{8} S^{(3)}(245; \phi) G(45; \phi) G(26; \phi) S^{(3)}(678; \phi) G(78; \phi) + O(\lambda^3) \quad (III.36)$$

where $S^{(2)}(12; \phi) G(13; \phi) = -\delta(23)$ is used. The last term in (III.36) has the same structure as the 1PR diagram in (III.30). There is one additional term of the same structure which comes from $\lambda W_1[J]$ in (III.31). Since $W_1[J]$ is a functional of $\psi_c(J)$ via $J(1) = -\delta S[\psi_c]/\delta \psi_c(1)$, one expands $\lambda W_1[J]$ as

$$\lambda W_1[J; \psi_c] = \lambda W_1[\phi] + \lambda \frac{\delta W_1[\phi]}{\delta \phi(1)} (\psi_c(1) - \phi(1)) + O(\lambda^3)$$

$$= \lambda W_1[\phi] - \frac{\lambda^2}{4} S^{(3)}(123; \phi) G(23; \phi) G(14; \phi) S^{(3)}(456; \phi) G(56; \phi) + O(\lambda^3) \quad (III.37)$$
where the last equality is obtained by use of (III.34) and (III.35). Finally, we consider the last term in (III.31):

\[
\lambda^2 W_2[J; \psi_c] = \lambda^2 W_2[\phi] + O(\lambda^3)
\]

\[
= \frac{\lambda^2}{8} S^{(4)}(1234; \phi)G(12; \phi)G(34; \phi) + \frac{\lambda^2}{12} S^{(3)}(123; \phi)S^{(3)}(456; \phi)G(14; \phi)G(25; \phi)G(36; \phi)
\]

\[
+ \frac{\lambda^2}{8} S^{(3)}(123; \phi)S^{(3)}(456; \phi)G(12; \phi)G(56; \phi)G(34; \phi) + O(\lambda^3)
\]

(III.38)

where (III.30) was used. Adding up (III.36)-(III.38), we obtain up to the two-loop order

\[
\Gamma[\phi] = S[\phi] - \frac{\lambda}{2} \text{Tr} \ln (G^{-1}(\phi) \cdot G_0) + \Gamma_{1PI}[\phi],
\]

\[
\Gamma_{1PI}[\phi] \equiv \lambda^2 \left\{ \frac{1}{8} S^{(4)}(1234; \phi)G(12; \phi)G(34; \phi) + \frac{1}{12} S^{(3)}(123; \phi)S^{(3)}(456; \phi)G(14; \phi)G(25; \phi)G(36; \phi) \right\}
\]

(III.39)

Note that the 1PR diagram was eliminated in \(\Gamma[\phi]\). Therefore, up to the second order in \(\lambda\), \(\Gamma_{1PI}[\phi]\) turns out to be the sum of the two-loop 1PI diagrams. It was shown that this feature is indeed the general structure: \(\Gamma_{1PI}[\phi]\) is the sum of all (two and higher loop) 1PI-diagrams with propagator \(G(\phi)\) and vertices dictated by the interaction potential. Shown in Fig. 2 are the diagrams for \(\Gamma_{1}[\phi]\) up to the three-loop order [46, 47, 48, 51].

C. The two-particle irreducible (2PI) loop expansion

Although \(\Gamma_{1PI}[\phi]\) consists of 1PI diagrams only, some of the diagrams in \(\Gamma_{1PI}[\phi]\) are 2PR diagrams (the diagrams which are disconnected by cutting two lines). Introduction of the bilocal source field \(K(12)\) can eliminate all 2PR diagrams in the vertex functional obtained via the double Legendre transform [46, 47, 50, 51, 52]. Thus the resulting vertex functional \(\Gamma[\phi, G]\) has simpler structure than \(\Gamma[\phi]\). A specific example is the three-loop result for \(\Gamma_{1}[\phi]\). As shown in Fig. 2, in the three-loop order \(\Gamma_{1}[\phi]\) has six 1PI diagrams. Among these diagrams, three (the third, fifth, and sixth) diagrams are 2PI ones (the diagrams which are not disconnected by cutting the two lines), and the remaining three are 2PR ones. The two-loop diagrams in Fig. 2 are 2PI diagrams as well. These 2PR diagrams were shown [46, 47, 50, 51] to be eliminated in the functional \(\Gamma[\phi, G]\) obtained via the double Legendre transform.

The generating functional \(W[J, K]\) for the connected correlation functions is defined as

\[
W[J, K] = \ln \left( c \int d[\psi] \exp \left( S[\psi] + J(1)\psi(1) + \frac{1}{2} \psi(1)K(12)\psi(2) \right) \right)
\]

(III.40)
where the normalization constant is determined by the condition \( W[J = 0, K = 0] = 0 \). Averages and correlations are generated from \( W[J, K] \) as

\[
\phi(1) \equiv \langle \psi(1) \rangle = \frac{\delta W[J, K]}{\delta J(1)},
\]

\[
G(12) = \frac{\delta^2 W[J, K]}{\delta J(1) \delta J(2)} = \left( \langle \psi(1) \psi(2) \rangle - \phi(1) \phi(2) \right),
\]

\[
\frac{\delta W[J, K]}{\delta K(12)} = \frac{1}{2} \langle \psi(1) \psi(2) \rangle = \frac{1}{2} \left( G(12) + \phi(1) \phi(2) \right)
\]

\[
\frac{\delta W[J, K]}{\delta K(12) \delta K(34)} = \frac{1}{4} \langle \psi(1) \psi(2) \psi(3) \psi(4) \rangle - \frac{1}{4} \langle \psi(1) \psi(2) \rangle \langle \psi(3) \psi(4) \rangle
\]

The double Legendre transform is then given by

\[
\Gamma[\phi, G] = W[J, K] - \langle \psi(1) \rangle \phi(1) - \frac{1}{2} K(12) \left( G(12) + \phi(1) \phi(2) \right)
\]

where the source fields \( J \) and \( K \) are should be eliminated in favor of \( \phi \) and \( G \). We obtain the derivatives

\[
\frac{\delta \Gamma[\phi, G]}{\delta \phi(1)} = \frac{\delta W[J, K]}{\delta J(2)} \frac{\delta J(2)}{\delta \phi(1)} + \frac{\delta W[J, K]}{\delta K(23)} \frac{\delta K(23)}{\delta \phi(1)} - \frac{\delta J(2)}{\delta \phi(1)} \phi(2) - J(1)
\]

\[
- \frac{1}{2} \frac{\delta K(23)}{\delta \phi(1)} \left( G(23) + \phi(2) \phi(3) \right) - K(12) \phi(2) = -J(1) - K(12) \phi(2),
\]

\[
\frac{\delta \Gamma[\phi, G]}{\delta G(12)} = \frac{\delta W[J, K]}{\delta J(3)} \frac{\delta J(3)}{\delta G(12)} + \frac{\delta W[J, K]}{\delta K(34)} \frac{\delta K(34)}{\delta G(12)} - \frac{\delta J(3)}{\delta G(12)} \phi(3) - \frac{1}{2} \frac{\delta K(34)}{\delta G(12)} \left( G(34) + \phi(3) \phi(4) \right)
\]

As before, we first define \( W[J, K] \) as

\[
W[J, K] = \lambda \ln \left( e \int d[\psi] \exp \left[ \frac{1}{\lambda} \left( S[\psi] + J(1) \psi(1) + \frac{1}{2} \psi(1) K(12) \psi(2) \right) \right] \right)
\]

We then expand the integrand (III.44) around the minimum field \( \psi_c(1) \) of the exponential for a given \( J \) and \( K \):

\[
\frac{\delta S[\psi_c]}{\delta \psi_c(1)} = -\left( J(1) + K(12) \psi_c(2) \right)
\]

Writing \( \psi(1) = \psi_c(1) + \sqrt{\lambda} \chi(1) \), we obtain

\[
S[\psi] + J(1) \psi(1) + \frac{1}{2} K(12) \psi(1) \psi(2) = \left( S[\psi_c] + J(1) \psi_c(1) + \frac{1}{2} K(12) \psi_c(1) \psi_c(2) \right)
\]

\[
+ \frac{\lambda}{2} \left( S_c^{(2)}(12) + K(12) \right) \chi(1) \chi(2) + \sum_{n=3}^{\infty} \frac{\lambda^{n/2}}{n!} S_c^{(n)}(12 \cdots n) \chi(1) \chi(2) \cdots \chi(n)
\]
where the linear terms in $\chi$ are eliminated due to (III.45). We see from (III.45) and (III.46) that the formal structure of the expansion is the same as before, provided that the following replacements are made:

$$
S[\psi_c] \rightarrow S[\psi_c] + \frac{1}{2} K(12) \psi_c(1) \psi_c(2),
$$

$$
S_c^{(2)}(12) \rightarrow S_c^{(2)}(12) + K(12)
$$

$$
G(12; \psi_c) \equiv -(S_c^{(2)}(12) + K(12))^{-1} \quad (\text{III.47})
$$

There is no change in the higher order terms in (III.46). Therefore the analysis in the previous section still holds, and we have

$$
W[J, K] = W_0[J, K] + W_1[J, K] + W_2[J, K] + \cdots
$$

$$
W_0[J, K] = S[\psi_c] + J(1) \psi_c(1) + \frac{1}{2} K(12) \psi_c(1) \psi_c(2)
$$

$$
W_1[J, K] = -\frac{\lambda}{2} \text{Tr} \ln \left( G^{-1}(\psi_c) \cdot G_0 \right)
$$

$$
W_2[J, K] = \lambda^2 \left[ \frac{1}{8} S_c^{(4)}(1234) G(12; \psi_c) G(34; \psi_c) + \frac{1}{12} S_c^{(3)}(123) S_c^{(3)}(456) G(14; \psi_c) G(25; \psi_c) G(36; \psi_c) + \frac{1}{8} S_c^{(3)}(123) S_c^{(3)}(456) G(12; \psi_c) G(56; \psi_c) G(34; \psi_c) \right] \quad (\text{III.48})
$$

where the normalization constant is left out.

The double Legendre transform is now given by

$$
\Gamma[\phi, G] = -J(1) \phi(1) - \frac{1}{2} K(12) \left( \lambda G(12) + \phi(1) \phi(2) \right) + W_0[J, K] + W_1[J, K] + W_2[J, K] + \cdots
$$

$$
= -J(1) \phi(1) - \frac{1}{2} K(12) \left( \lambda G(12) + \phi(1) \phi(2) \right)
$$

$$
+ S[\psi_c] + J(1) \psi_c(1) + \frac{1}{2} K(12) \psi_c(1) \psi_c(2) + W_1[J, K] + W_2[J, K] + \cdots
$$

$$
= S[\psi_c] + (J(1) + K(12) \psi_c(2)) (\psi_c(1) - \phi(1)) - \frac{1}{2} K(12) (\psi_c(1) - \phi(1)) (\psi_c(2) - \phi(2))
$$

$$
- \frac{\lambda}{2} K(12) G(12) + W_1[J, K] + W_2[J, K] + \cdots \quad (\text{III.49})
$$

Let us look at the first three terms of the rhs in the last line of (III.49). Using (III.45) and expanding
them around $\phi$, we obtain

$$S[\psi_c] + (J(1) + K(12)\psi_c(2)) (\psi_c(1) - \phi(1)) - \frac{1}{2} K(12) (\psi_c(1) - \phi(1)) (\psi_c(2) - \phi(2))$$

$$= S[\psi_c] - \delta S[\psi_c] \delta \psi_c(1) (\psi_c(1) - \phi(1)) - \frac{1}{2} K(12) (\psi_c(1) - \phi(1)) (\psi_c(2) - \phi(2))$$

$$= S[\phi] + S^{(1)}(1; \phi) (\psi_c(1) - \phi(1)) + \frac{1}{2} S^{(2)}(12; \phi) (\psi_c(1) - \phi(1)) (\psi_c(2) - \phi(2)) + \cdots$$

$$- \left(S^{(1)}(1; \phi) + S^{(2)}(12; \phi) (\psi_c(2) - \phi(2)) + \cdots\right) (\psi_c(1) - \phi(1))$$

$$- \frac{1}{2} K(12) (\psi_c(1) - \phi(1)) (\psi_c(2) - \phi(2))$$

$$= S[\phi] - \frac{1}{2} \left(S^{(2)}(12; \phi) + K(12)\right) (\psi_c(1) - \phi(1)) (\psi_c(2) - \phi(2)) + \cdots$$  \hspace{1cm} (III.50)

We see that there is no one-loop contribution from these three terms since the terms linear in $(\psi_c - \phi)$ cancel in (III.50); they only have the zero-loop, two-loop and higher contributions for $\Gamma[\phi, G]$. Substituting (III.50) into (III.49), we obtain

$$\Gamma[\phi, G] = S[\phi] - \frac{1}{2} \left(S^{(2)}(12; \phi) + K(12)\right) (\psi_c(1) - \phi(1)) (\psi_c(2) - \phi(2))$$

$$- \frac{\lambda}{2} K(12) G(12) + W_1 [J, K] + W_2 [J, K] + \cdots$$  \hspace{1cm} (III.51)

In order to simplify (III.51), we first consider the two-point correlations function. It is given by using (III.35) and (III.41)

$$\phi(1) = \psi_c(1) + \frac{\lambda}{2} G(12; \phi) S^{(3)}(234; \phi) G(34; \phi) + O(\lambda^2)$$

$$G(12) = \frac{\delta \phi(1)}{\delta J(2)} = \frac{\delta \psi_c(1)}{\delta J(2)} + \frac{\lambda}{2} \frac{\delta \phi(1')}{\delta J(2)} \frac{\delta}{\delta \phi(1')} \left\{ G(12'; \phi) S^{(3)}(2'34; \phi) G(34; \phi) \right\}$$

$$= G(12; \psi_c) + \frac{\lambda}{2} F(11'; \phi) G(1'2) + O(\lambda^2)$$  \hspace{1cm} (III.52)

where \( F(11'; \phi) \equiv \delta \frac{\delta}{\delta \phi(1')} \left\{ G(12'; \phi) S^{(3)}(2'34; \phi) G(34; \phi) \right\} \). From (III.52), the inverse of $G(12; \psi_c)$ is given by

$$G^{-1}(12; \psi_c) = G^{-1}(12) + \frac{\lambda}{2} G^{-1}(11') F(1'2; \phi) + O(\lambda^2)$$  \hspace{1cm} (III.53)

Using (III.53), we obtain the following expression for the term $-\frac{\lambda}{2} K(12) G(12)$ in (III.51).

$$-\frac{\lambda}{2} K(12) G(12) = \frac{\lambda}{2} G(12) \left( S^{(2)}(12) + G^{-1}(12; \psi_c) \right)$$

$$= \frac{\lambda}{2} G(12) \left[ S^{(2)}(12; \phi) + S^{(3)}(123; \phi) (\psi_c(3) - \phi(3)) + G^{-1}(12) + \frac{\lambda}{2} G^{-1}(11') F(1'2; \phi) \right] + O(\lambda^3)$$

$$= \frac{\lambda}{2} G(12) \left( S^{(2)}(12; \phi) + G^{-1}(12) \right) + \frac{\lambda}{2} G(12) S^{(3)}(123; \phi) (\psi_c(3) - \phi(3)) + \frac{\lambda^2}{4} F(22; \phi) + O(\lambda^3)$$  \hspace{1cm} (III.54)
We now look at the term $W_1[J,K]$, (III.48). Using (III.53), we have

$$W_1[J,K] = -\frac{\lambda}{2} \text{Tr} \ln \left( G^{-1}(\psi_c) \cdot G_0 \right) = -\frac{\lambda}{2} \text{Tr} \left[ \ln \left( G^{-1} \cdot G_0 \right) + \ln (I + \frac{\lambda}{2} F) \right]$$

$$= -\frac{\lambda}{2} \text{Tr} \ln \left( G^{-1} \cdot G_0 \right) - \frac{\lambda^2}{4} \text{Tr} F + O(\lambda^3)$$

(III.55)

where $I$ is the unit matrix. We find that when (III.54) and (III.55) are added, the last terms in (III.54) and (III.55) cancel against each other.

Thus we have obtained the following expression for $\Gamma[\phi, G]$ up to the two-loop order

$$\Gamma[\phi, G] = S[\phi] - \frac{\lambda}{2} \text{Tr} \ln \left( G^{-1} \cdot G_0 \right) + \frac{\lambda}{2} \text{Tr} \left( G \cdot S^{(2)}(\phi) + I \right)$$

$$- \frac{1}{2} \left( S^{(2)}(12; \phi) + K(12) \right) (\psi_c(1) - \phi(1)) (\psi_c(2) - \phi(2))$$

$$+ \frac{\lambda}{2} G(12) S^{(3)}(123; \phi) (\psi_c(3) - \phi(3)) + W_2[J,K] + O(\lambda^3)$$

(III.56)

For the last three terms, the same analysis in the previous section that led to (III.39) is applied, and the 1PR diagram in $W_2[J,K]$ is eliminated. Consequently, the final expression for $\Gamma[\phi, G]$ up to the two-loops with $\lambda = 1$ is given by

$$\Gamma[\phi, G] = S[\phi] - \frac{1}{2} \text{Tr} \ln \left( G^{-1} \cdot G_0 \right) + \frac{1}{2} \text{Tr} \left( G \cdot S^{(2)}(\phi) + I \right) + \Gamma_{2PI}[\phi, G]$$

$$\Gamma_{2PI}[\phi, G] \equiv \frac{1}{8} S^{(4)}(1234; \phi) G(12) G(34) + \frac{1}{12} S^{(3)}(123; \phi) S^{(3)}(456; \phi) G(14) G(25) G(36) + O(\lambda^3)$$

(III.57)

where $G$ appears in place of $G(\phi)$ since from (III.52) $G = G(\phi)$ at the lowest order. We can obtain the three-loop calculation to explicitly show that the 2PR diagrams contained in $\Gamma_{2PI}[\phi, G]$ are eliminated. It has also been shown [47, 50] that $\Gamma_{2PI}[\phi, G]$ is the sum of the two-loop and higher 2PI diagrams. Figure 3 shows the diagrams for $\Gamma_{2PI}[\phi, G]$ up to the three-loop order.

In the absence of the source fields $J = K = 0$, we have $\phi = 0$, $S[\phi] = 0$, $S^{(2)}(\phi) = -G_0^{-1}$, and $\delta\Gamma[\phi, G]/\delta G(12) = 0$. Also $S^{(3)}(123; \phi)$ etc. reduce to the ordinary vertices $V_s(123)$ etc. Therefore we have in the absence of sources

$$\Gamma[0, G] = \frac{1}{2} \text{Tr} \ln \left( G \cdot G_0^{-1} \right) - \frac{1}{2} \text{Tr} \left( G \cdot G_0^{-1} - I \right) + \Gamma_{2PI}[0, G]$$

$$0 = \frac{\delta\Gamma[0, G]}{\delta G(12)} = \frac{1}{2} G^{-1}(12) - \frac{1}{2} G_0^{-1}(12) + \frac{\delta\Gamma_{2PI}[0, G]}{\delta G(12)}$$

(III.58)

The last line is the Schwinger-Dyson equation which can be cast into the form with the self-energy $\Sigma$

$$G^{-1}(12) = G_0^{-1} - \Sigma(12), \quad \Sigma(12) \equiv 2 \frac{\delta\Gamma_{2PI}[0, G]}{\delta G(12)}$$

$$\Sigma(12) = \frac{1}{2} V_s(1234) G(34) + \frac{1}{2} V_s(134) V_s(256) G(35) G(46)$$

(III.59)
where the last line, the one-loop expression for the self-energy \( \Sigma \), is obtained by use of (III.57). Figure 4 shows the self-energy diagrams up to the two-loop order. These diagrams are obtained from the diagrams of \( \Gamma_2[\phi,G] \) (Fig. 4) by cutting a line (corresponding to taking a derivative of \( \Gamma_{2PI}[\phi,G] \) with respect to \( G \)).

### D. Unperturbed propagator

Recalling we wrote the Gaussian part of the action as

\[
S_g[\psi] = \frac{1}{2} G_0^{-1}(12) \psi(1) \psi(2) = -\frac{1}{2} \Psi^T(1) \cdot G_0^{-1}(12) \cdot \Psi(2),
\]

we read off the inverse of the unperturbed propagator from \( S_g[\psi] \) in (II.34) as follows.

\[
G_0^{-1}(12) = \begin{pmatrix}
0 & iD_1 \delta(12) + i\rho_0 \nabla^2 U(12) & 0 & 0 \\
i\hat{D}_1 \delta(12) + i\rho_0 \nabla^2 U(12) & -2\rho_0 T \nabla^2 \delta(12) & i\rho_0 T \nabla^2 \delta(12) & 0 \\
0 & iT \rho_0 \nabla^2 \delta(12) & 0 & -i\delta(12) \\
0 & 0 & -i\delta(12) & 0
\end{pmatrix}
\]

(III.60)

where \( D_1 \equiv (\partial/\partial_1 + T \nabla_1^2) \) and \( \hat{D}_1 \equiv (-\partial/\partial_1 + T \nabla_1^2) \).

### E. Vertices

The non-Gaussian part of the action \( S_{ng}[\psi] \) given in (II.34) can be written as

\[
S_{ng}[\psi] = \frac{1}{3!} V_{abc}^S(123) \psi_a(1) \psi_b(2) \psi_c(3) + \frac{1}{4!} V_{abcd}^S(1234) \psi_a(1) \psi_b(2) \psi_c(3) \psi_d(4) + \cdots
\]

(III.61)

The fully symmetrized vertices \( V_{abc}^S(123) \) etc. are given by

\[
V_{abc}(123) = \frac{\delta^3 S_{ng}[\psi]}{\delta \psi_a(1) \delta \psi_b(2) \delta \psi_c(3)},
\]

\[
V_{abc}^S(123) = \left[ V_{abc}(123) + V_{acb}(132) + V_{bca}(231) + V_{bca}(213) + V_{cab}(312) + V_{cba}(321) \right]
\]

(III.62)
From the explicit expression of \( S_{ng} \), we have the following 5 types of cubic vertices

\[
V_{\hat{\rho}\hat{\rho}\hat{\rho}}(123) = \frac{\delta^3 S_{ng}[\psi]}{\delta \hat{\rho}(1)\delta \rho(2)\delta \rho(3)} = V_{\hat{\rho}\hat{\rho}\hat{\rho}}^{int}(123) + V_{\hat{\rho}\hat{\rho}\hat{\rho}}^{id}(123)
\]

\[
V_{\hat{\rho}\hat{\rho}\hat{\rho}}^{int}(123) = (-i)\nabla_1 \cdot \left[ \delta(12)\nabla_1 U(13) + \delta(13)\nabla_1 U(12) \right],
\]

\[
V_{\hat{\rho}\hat{\rho}\hat{\rho}}^{id}(123) = \frac{i T}{\rho_0} \nabla_1^2 \left[ \delta(12)\delta(13) \right],
\]

\[
V_{\hat{\rho}\hat{\rho}\theta}(123) = \frac{\delta^3 S_{ng}[\psi]}{\delta \hat{\rho}(1)\delta \rho(2)\delta \theta(3)} = 2T\nabla_1 \cdot \left[ \delta(12)\nabla_1 \delta(13) \right],
\]

\[
V_{\hat{\theta}\hat{\rho}\rho}(123) = \frac{\delta^3 S_{ng}[\psi]}{\delta \theta(1)\delta \rho(2)\delta \rho(3)} = \frac{i}{\rho_0} \delta(12)\delta(13)
\]

(III.63)

where \( \delta(12) \equiv \delta(r_1 - r_2)\delta(t_1 - t_2) \), etc. and the indices \( \text{int} \) and \( \text{id} \) stand for particle interaction and ideal gas, respectively. Note that the vertices \( V_{\hat{\rho}\hat{\rho}\hat{\rho}}^{int}(123) \), \( V_{\hat{\rho}\hat{\rho}\hat{\rho}}^{id}(123) \), and \( V_{\hat{\rho}\hat{\rho}\theta}(123) \) are already symmetric under exchange of 2 and 3, whereas \( V_{\hat{\rho}\hat{\rho}\theta}(123) \) and \( V_{\hat{\theta}\hat{\rho}\rho}(123) \) are not. This fact should be kept in mind when we compute the one-loop diagrams in order to avoid double-counting of the symmetry factor generated from the diagrams. Also \( U(12) \) in (III.63) actually means

\[
U(12) \equiv U(r_1 - r_2)\delta(t_1 - t_2).
\]

Thus the above 3-point vertices are nonzero only at the same time \( t_1 = t_2 = t_3 \). Finally, we observe an interesting relationship between the two vertices \( V_{\hat{\rho}\hat{\rho}\rho}^{id}(123) \) and \( V_{\hat{\theta}\rho\rho}(123) \):

\[
\rho_0 T\nabla_1^2 V_{\hat{\rho}\hat{\rho}\rho}^{id}(123) + V_{\hat{\rho}\hat{\rho}\rho}^{id}(123) = 0
\]

(III.64)

which will be useful in computing the one-loop diagrams.

The quartic and higher order vertices come from the term \(-i\hat{\theta}(1)f(\delta \rho(1))\) in \( S_{ng}[\psi] \). For example, the nonvanishing quartic vertex is given by

\[
V_{\hat{\rho}\hat{\rho}\rho\rho}(1234) = -2\frac{i}{\rho_0} \delta(12)\delta(13)\delta(14)
\]

(III.65)

F. Time-reversal symmetry and FDR for \( G \) and \( \Sigma \)

The time-reversed variable set, (II.33), now denoted by \( \tilde{\Psi}(j) \), is given by

\[
\tilde{\Psi}(j) = \tilde{\psi}_{\alpha_j}(x_j) = \begin{pmatrix} \delta \rho(x_j) \\ -\hat{\rho}(x_j) + i\theta(x_j) + i\tilde{K} \cdot \delta \rho(x_j) \\ \hat{\theta}(x_j) + i\partial_t \rho(x_j) \end{pmatrix} \equiv O(\partial_t) \cdot \Psi(j)
\]

(III.66)
which also defines $O(\partial_t)$. The transformation property of the propagator $G$ under TR with the spatial coordinates being suppressed is then given by

$$G(t - t') = \langle \Psi(t)\bar{\Psi}(t') \rangle$$
$$G(t' - t) = \langle \tilde{\Psi}(t)\tilde{\Psi}(t') \rangle = \langle \left( O(\partial_t)\Psi(t) \left( O(\partial_{t'})\Psi(t') \right)^T \right) = O(\partial_t) \cdot G(t - t') \cdot O^T(\partial_{t'})$$
$$= O(\partial_t) \cdot G(t - t') \cdot O^T(-\partial_t)$$

(III.67)

By setting $t' = 0$ in (III.67), we have

$$G(-t) = O(\partial_t) \cdot G(t) \cdot O^T(-\partial_t)$$

(III.68)

As for $\Sigma$, using the fact that $\Sigma$ transforms like $G^{-1}$ we obtain

$$\Sigma(-t) = [O^T(-\partial_t)]^{-1} \cdot \Sigma(t) \cdot O^{-1}(\partial_t) = O^T(\partial_t) \cdot \Sigma(t) \cdot O(-\partial_t)$$

(III.69)

where we have used the fact that $O$, $O^T$ represent time reversal, $O^{-1}$, $(O^T)^{-1}$ represent just another time reversal which is obtained by changing signs of $t$. The transformation matrices appearing in (III.68) and (III.69) are explicitly given by

$$O(\partial_t) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ i\tilde{K}^* & -1 & i & 0 \\ 0 & 0 & 1 & 0 \\ i\partial_t & 0 & 0 & 1 \end{pmatrix}, \quad O^T(-\partial_t) \equiv \begin{pmatrix} 1 & i \tilde{K} & 0 & -i\partial_t \\ 0 & -1 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$O(-\partial_t) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ i\tilde{K} & -1 & i & 0 \\ 0 & 0 & 1 & 0 \\ -i\partial_t & 0 & 0 & 1 \end{pmatrix}, \quad O^T(\partial_t) \equiv \begin{pmatrix} 1 & i\tilde{K}^* & 0 & i\partial_t \\ 0 & -1 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(III.70)

where $\partial_t$ is a differential operator acting to the left, and $\tilde{K}(\ast \tilde{K})$ means the spatial convolution of $K(r)$ to whatever comes to the right (left).
1. FDR for the propagator $G$

Straightforward algebra gives the expressions for the elements of the matrices on both sides of (III.68):

$$
G_{\rho\rho}(r,-t) = G_{\rho\rho}(r,t) \\
G_{\rho\theta}(r,-t) = i\tilde{K} \ast G_{\rho\rho}(r,t) - G_{\rho\rho}(r,t) + iG_{\rho\theta}(r,t) \\
G_{\theta\rho}(r,-t) = G_{\rho\theta}(r,t) \\
G_{\rho\theta}(r,-t) = -i\partial_t G_{\rho\rho}(r,t) + G_{\rho\theta}(r,t)
$$

$$
G_{\rho\rho}(r,-t) = i\tilde{K} \ast G_{\rho\rho}(r,t) - G_{\rho\rho}(r,t) + iG_{\rho\theta}(r,t) \\
G_{\tilde{\rho}\tilde{\rho}}(r,-t) = i\tilde{K} \ast \left( i\tilde{K} \ast G_{\rho\rho}(r,t) - G_{\rho\rho}(r,t) + iG_{\rho\theta}(r,t) \right) - i\left( \tilde{K} \ast G_{\rho\rho}(r,t) + G_{\rho\theta}(r,t) \right) \\
G_{\rho\theta}(r,-t) = i\tilde{K} \ast \left( -i\partial_t G_{\rho\rho}(r,t) + G_{\rho\theta}(r,t) \right) + i\partial_t G_{\rho\rho}(r,t) + \partial_t G_{\rho\theta}(r,t) + iG_{\theta\theta}(r,t) = 0
$$

$$
G_{\theta\rho}(r,-t) = G_{\theta\rho}(r,t) \\
G_{\theta\theta}(r,-t) = i\tilde{K} \ast G_{\rho\rho}(r,t) - G_{\rho\rho}(r,t) + iG_{\theta\rho}(r,t) \\
G_{\theta\theta}(r,-t) = G_{\theta\theta}(r,t) \\
G_{\tilde{\theta}\rho}(r,-t) = -i\partial_t G_{\theta\rho}(r,t) + G_{\tilde{\theta}\rho}(r,t)
$$

$$
G_{\tilde{\theta}\rho}(r,-t) = i\partial_t G_{\rho\rho}(r,t) + G_{\tilde{\theta}\rho}(r,t) \\
G_{\tilde{\theta}\theta}(r,-t) = i\partial_t \left( i\tilde{K} \ast G_{\rho\rho}(r,t) - G_{\rho\rho}(r,t) + iG_{\rho\theta}(r,t) \right) + i\tilde{K} \ast G_{\tilde{\rho}\rho}(r,t) + iG_{\tilde{\theta}\theta}(r,t) = 0 \\
G_{\tilde{\theta}\theta}(r,-t) = i\partial_t G_{\rho\rho}(r,t) + G_{\tilde{\theta}\rho}(r,t) \\
G_{\tilde{\theta}\theta}(r,-t) = i\partial_t \left( -i\partial_t G_{\rho\rho}(r,t) + G_{\rho\theta}(r,t) \right) - i\partial_t G_{\rho\theta}(r,t) = 0
$$

(III.71)

where we used the property of $G_{\tilde{\alpha}\tilde{\beta}}(t) = 0$.

Now, various FDRs are obtained from relevant members of (III.71). First we look at the $[\rho\tilde{\theta}]$ element:

$$
G_{\rho\tilde{\theta}}(r,-t) - G_{\rho\tilde{\theta}}(r,t) = -i\partial_t G_{\rho\rho}(r,t)
$$
Since $G_{\rho\theta}(r,-t) = 0$ for $t > 0$ by causality we obtain the standard FDR in view of the response function (II.36) as

$$R(r,t) = i\frac{t}{T}G_{\rho\theta}(r,t) = -\Theta(t)\frac{1}{T}\partial_t G_{\rho\theta}(r,t) \quad \text{(III.72)}$$

In the same manner we derive the following FDR’s for $G$:

$$G_{\rho\rho}(r,t) = i\Theta(t)\left(\hat{K} * G_{\rho\rho}(r,t) + G_{\rho\theta}(r,t)\right)$$
$$G_{\theta\rho}(r,t) = i\Theta(t)\left(\hat{K} * G_{\theta\rho}(r,t) + G_{\theta\theta}(r,t)\right)$$
$$G_{\theta\theta}(r,t) = i\Theta(t)\partial_t G_{\theta\rho}(r,t) \quad \text{(III.73)}$$

Using the causality requirement $G_{\tilde{\alpha}\beta}(t) = 0$ for $t > 0$, we obtain the FDRs for the adjoint elements:

$$G_{\tilde{\rho}\rho}(r,-t) = i\Theta(t)\left(\hat{K} * G_{\rho\rho}(r,t) + G_{\theta\rho}(r,t)\right) = G_{\rho\rho}(r,t)$$
$$G_{\tilde{\theta}\rho}(r,-t) = i\Theta(t)\partial_t G_{\rho\rho}(r,t) = G_{\rho\theta}(r,t)$$
$$G_{\tilde{\rho}\theta}(r,-t) = i\Theta(t)\left(\hat{K} * G_{\rho\theta}(r,t) + G_{\theta\theta}(r,t)\right) = G_{\theta\rho}(r,t)$$
$$G_{\tilde{\theta}\theta}(r,-t) = i\Theta(t)\partial_t G_{\rho\theta}(r,t) = G_{\theta\theta}(r,t) \quad \text{(III.74)}$$

where $G_{\mu\theta}(r,t) = G_{\theta\rho}(r,t)$ was used. Four correlations involving only hatted variables vanish.

The second line of (III.74) is nothing but the FDR (II.37).
As before, we first write out 16 matrix elements of (III.69):

\[
\begin{align*}
\Sigma_{\rho\rho}(r, -t) &= i\hat{K} \ast \Sigma_{\rho\rho}(r, t) - i\partial_t \Sigma_{\rho\rho}(r, t) + i\hat{K} \ast \left( \Sigma_{\rho\rho}(r, t) + i\hat{K} \ast \Sigma_{\rho\rho}(r, t) - i\partial_t \Sigma_{\rho\rho}(r, t) \right) \\
&\quad + i\partial_t \left( \Sigma_{\rho\rho}(r, t) + i\hat{K} \ast \Sigma_{\rho\rho}(r, t) - i\partial_t \Sigma_{\rho\rho}(r, t) \right) = 0 \\
\Sigma_{\rho\theta}(r, -t) &= -\Sigma_{\rho\theta}(r, t) - i\hat{K} \ast \Sigma_{\rho\theta}(r, t) - i\partial_t \Sigma_{\rho\theta}(r, t) \\
\Sigma_{\theta\rho}(r, -t) &= i\Sigma_{\rho\theta}(r, t) + i\hat{K} \ast \left( i\Sigma_{\rho\theta}(r, t) + \Sigma_{\rho\theta}(r, t) \right) + i\partial_t \left( i\Sigma_{\rho\theta}(r, t) + \Sigma_{\rho\theta}(r, t) \right) \\
\Sigma_{\theta\theta}(r, -t) &= \Sigma_{\rho\theta}(r, t) + i\hat{K} \ast \Sigma_{\rho\theta}(r, t) + i\partial_t \Sigma_{\rho\theta}(r, t) \\
\Sigma_{\theta\theta}(r, -t) &= -\Sigma_{\rho\theta}(r, t) - \Sigma_{\rho\theta}(r, t) \\
\Sigma_{\phi\phi}(r, -t) &= -\Sigma_{\rho\phi}(r, t) - \Sigma_{\rho\phi}(r, t) \\
\Sigma_{\phi\phi}(r, -t) &= i\Sigma_{\rho\phi}(r, t) + \Sigma_{\rho\phi}(r, t) \\
\Sigma_{\phi\phi}(r, -t) &= \Sigma_{\rho\phi}(r, t) + i\hat{K} \ast \Sigma_{\rho\phi}(r, t) - i\partial_t \Sigma_{\rho\phi}(r, t) \\
\Sigma_{\phi\phi}(r, -t) &= -\Sigma_{\rho\phi}(r, t) \\
\Sigma_{\phi\phi}(r, -t) &= i\Sigma_{\rho\phi}(r, t) + \Sigma_{\rho\phi}(r, t) \\
\Sigma_{\phi\phi}(r, -t) &= \Sigma_{\rho\phi}(r, t)
\end{align*}
\]

(III.75)

where we used the property \( \Sigma_{\alpha\beta}(r, t) = 0 \) with unhatted \( \alpha \) and \( \beta \) indices.
Using the causality property $\Sigma_{\hat{\alpha}\hat{\beta}}(-t) = 0$ for $t > 0$, we obtain from (III.75) the following FDRs:

\[
\begin{align*}
\Sigma_{\hat{\rho}\hat{\rho}}(r, t) &= i\Theta(t) \left(-\hat{K} * \Sigma_{\hat{\rho}\hat{\rho}}(r, t) + \partial_t \Sigma_{\hat{\rho}\hat{\theta}}(r, t)\right) \\
\Sigma_{\hat{\rho}\hat{\theta}}(r, t) &= -i\Theta(t) \Sigma_{\hat{\rho}\hat{\rho}}(r, t) \\
\Sigma_{\hat{\theta}\hat{\rho}}(r, t) &= i\Theta(t) \left(-\hat{K} * \Sigma_{\hat{\theta}\hat{\rho}}(r, t) + \partial_t \Sigma_{\hat{\theta}\hat{\theta}}(r, t)\right) \\
\Sigma_{\hat{\theta}\hat{\theta}}(r, t) &= -i\Theta(t) \Sigma_{\hat{\rho}\hat{\rho}}(r, t)
\end{align*}
\]  

(III.76)

We obtain the similar FDRs for the adjoint elements using $\Sigma_{\alpha\beta}(t) = 0$ for $t > 0$:

\[
\begin{align*}
\Sigma_{\rho\rho}(r, -t) &= -i\Theta(t) \left(-\hat{K} * \Sigma_{\rho\rho}(r, t) + \partial_t \Sigma_{\rho\rho}(r, t)\right) \\
\Sigma_{\rho\theta}(r, -t) &= i\Theta(t) \left(-\hat{K} * \Sigma_{\rho\theta}(r, t) + \partial_t \Sigma_{\rho\rho}(r, t)\right) \\
\Sigma_{\theta\rho}(r, -t) &= -i\Theta(t) \Sigma_{\rho\rho}(r, t) = \Sigma_{\rho\rho}(r, t) \\
\Sigma_{\theta\theta}(r, -t) &= i\Theta(t) \Sigma_{\rho\rho}(r, t)
\end{align*}
\]  

(III.77)

Note from (III.25) that while $\Sigma_{\rho\rho}(r, t)$ and $\Sigma_{\theta\theta}(r, t)$ are symmetric under time-reversal, $\Sigma_{\rho\theta}(r, t)$ and $\Sigma_{\theta\rho}(r, t)$ are anti-symmetric under time-reversal. The unhatted diagonal elements in (III.75) vanish.

G. Dynamical Equation

The dynamic equations for the correlation and response functions are formally given by the matrix Schwinger-Dyson (SD) equation (see (III.59)):

\[
G^{-1}_0(13) \cdot G(32) = \delta(12) + \Sigma(13) \cdot G(32)
\]  

(III.78)

where the unperturbed propagator $G^{-1}_0(12)$ is given in (III.60). Setting $1 \equiv (r, t)$, $2 \equiv (0, 0)$, and $3 \equiv (r_s, s)$, and introducing the space Fourier-transform

\[
\Sigma(r, t) = \int_k \Sigma(k, t)e^{ikr}
\]  

(III.79)

where $\int_k \equiv \int d^3k/(2\pi)^3$, we can express the above matrix SD equation as

\[
\int ds G^{-1}_0(k, t - s) \cdot G(k, s) = \delta(t)I + \int ds \Sigma(k, t - s) \cdot G(k, s)
\]  

(III.80)
where $I$ is the $4 \times 4$ unit matrix, and the time-integration ranges within $(-\infty, \infty)$. The unperturbed inverse propagator $G_0^{-1}(k, t)$ is calculated from (III.60) as

$$
G_0^{-1}(k, t) = \begin{pmatrix}
0 & i\left( -\partial_t - \rho_0 T k^2 K(k) \right)\delta(t) & 0 & 0 \\
i\left( -\partial_t - \rho_0 T k^2 K(k) \right)\delta(t) & 2\rho_0 T k^2 \delta(t) & i\rho_0 T k^2 \delta(t) & 0 \\
0 & -i\rho_0 T k^2 \delta(t) & 0 & -i\delta(t) \\
0 & 0 & -i\delta(t) & 0
\end{pmatrix}
$$

(III.81)

Using (III.81), we write down matrix element of lhs of (III.80).

$$
\int ds \, G_0^{-1}(k, t - s) \cdot G(k, s) = 
\begin{pmatrix}
X_+ G_{\rho\rho} & 0 & X_+ G_{\rho\theta} & 0 \\
X_- G_{\rho\rho} + R(2G_{\rho\rho} - iG_{\theta\rho}) & X_- G_{\rho\theta} - iRG_{\theta\rho} & X_- G_{\rho\theta} + R(2G_{\rho\theta} - iG_{\theta\theta}) & X_- G_{\rho\theta} - iRG_{\theta\theta} \\
-iRG_{\rho\rho} - iG_{\theta\rho} & 0 & -iRG_{\rho\theta} - iG_{\theta\theta} & 0 \\
-iG_{\theta\rho} & -iG_{\theta\rho} & -iG_{\theta\theta} & -iG_{\theta\theta}
\end{pmatrix}
$$

(III.82)

with

$$X_+ \equiv i\left( -\partial_t - \rho_0 T k^2 K(k) \right), \quad X_- \equiv i\left( -\partial_t - \rho_0 T k^2 K(k) \right), \quad R \equiv \rho_0 T k^2 \quad \text{(III.83)}$$

We now find the matrix elements of the first term in the rhs of (III.80). First of all, suppressing the wave number and time integration for the moment and denoting $\alpha, \beta, \gamma = \rho, \theta$, we note the following

$$[\Sigma \cdot G]_{\tilde{\alpha}\beta} = \Sigma_{\tilde{\alpha}\gamma} G_{\gamma\beta} + \Sigma_{\tilde{\alpha}\tilde{\gamma}} G_{\tilde{\gamma}\beta} \neq 0,$$

whereas

$$[\Sigma \cdot G]_{\alpha\tilde{\beta}} = \Sigma_{\alpha\gamma} G_{\gamma\tilde{\beta}} + \Sigma_{\alpha\tilde{\gamma}} G_{\tilde{\gamma}\tilde{\beta}} = 0$$

since $\Sigma_{\alpha\gamma} \equiv 0$ and $G_{\tilde{\gamma}\tilde{\beta}} \equiv 0$ by causality. Therefore we get

$$
\Sigma \cdot G = 
\begin{pmatrix}
\Sigma_{\rho\rho} \cdot G_{\rho\rho} + \Sigma_{\rho\theta} \cdot G_{\rho\theta} & 0 & \Sigma_{\rho\rho} \cdot G_{\rho\theta} + \Sigma_{\rho\theta} \cdot G_{\rho\theta} & 0 \\
\Sigma_{\rho\theta} \cdot G_{\rho\rho} + \Sigma_{\rho\theta} \cdot G_{\rho\theta} & \Sigma_{\rho\rho} \cdot G_{\rho\theta} + \Sigma_{\rho\theta} \cdot G_{\rho\theta} & \Sigma_{\rho\rho} \cdot G_{\rho\theta} + \Sigma_{\rho\theta} \cdot G_{\rho\theta} & 0 \\
\Sigma_{\theta\rho} \cdot G_{\rho\rho} + \Sigma_{\theta\theta} \cdot G_{\rho\theta} & 0 & \Sigma_{\theta\rho} \cdot G_{\rho\theta} + \Sigma_{\theta\theta} \cdot G_{\rho\theta} & 0 \\
\Sigma_{\theta\rho} \cdot G_{\rho\rho} + \Sigma_{\theta\theta} \cdot G_{\rho\theta} & \Sigma_{\theta\rho} \cdot G_{\rho\theta} + \Sigma_{\theta\theta} \cdot G_{\rho\theta} & \Sigma_{\theta\rho} \cdot G_{\rho\theta} + \Sigma_{\theta\theta} \cdot G_{\rho\theta} & 0
\end{pmatrix}
$$

(III.84)
where

\[
\begin{align*}
\left[ \Sigma \cdot G \right]_{\hat{\rho} \rho} &= \Sigma_{\hat{\rho} \rho} \cdot G_{\rho\rho} + \Sigma_{\rho \hat{\rho}} \cdot G_{\hat{\rho} \rho} + \Sigma_{\rho \hat{\theta}} \cdot G_{\hat{\theta} \rho} + \Sigma_{\hat{\rho} \hat{\theta}} \cdot G_{\hat{\theta} \rho} \\
\left[ \Sigma \cdot G \right]_{\rho \hat{\theta}} &= \Sigma_{\rho \hat{\theta}} \cdot G_{\rho \rho} + \Sigma_{\rho \hat{\rho}} \cdot G_{\hat{\rho} \theta} + \Sigma_{\rho \hat{\theta}} \cdot G_{\hat{\theta} \theta} + \Sigma_{\hat{\rho} \hat{\theta}} \cdot G_{\hat{\theta} \theta} \\
\left[ \Sigma \cdot G \right]_{\hat{\theta} \rho} &= \Sigma_{\hat{\theta} \rho} \cdot G_{\rho \rho} + \Sigma_{\hat{\rho} \hat{\theta}} \cdot G_{\rho \hat{\rho}} + \Sigma_{\hat{\theta} \rho} \cdot G_{\hat{\rho} \theta} + \Sigma_{\hat{\rho} \hat{\theta}} \cdot G_{\rho \hat{\theta}} \\
\left[ \Sigma \cdot G \right]_{\hat{\theta} \theta} &= \Sigma_{\hat{\theta} \theta} \cdot G_{\rho \rho} + \Sigma_{\hat{\rho} \hat{\theta}} \cdot G_{\rho \hat{\rho}} + \Sigma_{\hat{\theta} \theta} \cdot G_{\hat{\rho} \theta} + \Sigma_{\hat{\rho} \hat{\theta}} \cdot G_{\rho \hat{\theta}}
\end{align*}
\]  

(III.85)

Using the equations of various matrices explicitly displayed above we list below the relevant equations of motion where the matrix element is shown as \([\alpha \beta]\)

\[
\begin{align*}
[21] \quad i \left( -\partial_t - \rho_0 T k^2 K(k) \right) G_{\rho \rho}(k, t) &= 2 \rho_0 T k^2 G_{\hat{\rho} \rho}(k, t) - i \rho_0 k^2 G_{\theta \rho}(k, t) \\
&- \int ds \left[ \Sigma_{\hat{\rho} \rho}(k, t-s) G_{\rho \rho}(k, s) + \Sigma_{\rho \hat{\rho}}(k, t-s) G_{\hat{\rho} \rho}(k, s) + \Sigma_{\rho \hat{\theta}}(k, t-s) G_{\hat{\theta} \rho}(k, s) \right]
\end{align*}

\[
\begin{align*}
[22] \quad i \left( -\partial_t - \rho_0 T k^2 K(k) \right) G_{\rho \theta}(k, t) &= 2 \rho_0 T k^2 G_{\rho \hat{\rho}}(k, t) - i \rho_0 T k^2 G_{\rho \theta}(k, t) \\
&- \int ds \left[ \Sigma_{\hat{\rho} \rho}(k, t-s) G_{\rho \theta}(k, s) + \Sigma_{\rho \hat{\rho}}(k, t-s) G_{\rho \hat{\theta}}(k, s) + \Sigma_{\rho \hat{\theta}}(k, t-s) G_{\hat{\rho} \rho}(k, s) \right]
\end{align*}

\[
\begin{align*}
[23] \quad i \left( -\partial_t - \rho_0 T k^2 K(k) \right) G_{\theta \theta}(k, t) &= 2 \rho_0 T k^2 G_{\rho \hat{\rho}}(k, t) - i \rho_0 T k^2 G_{\theta \theta}(k, t) \\
&- \int ds \left[ \Sigma_{\hat{\rho} \rho}(k, t-s) G_{\theta \rho}(k, s) + \Sigma_{\rho \hat{\rho}}(k, t-s) G_{\theta \hat{\rho}}(k, s) + \Sigma_{\rho \hat{\theta}}(k, t-s) G_{\rho \rho}(k, s) \right]
\end{align*}

\[
\begin{align*}
[41] \quad -i G_{\theta \rho}(k, t) - \int ds \left[ \Sigma_{\hat{\rho} \rho}(k, t-s) G_{\rho \rho}(k, s) + \Sigma_{\rho \hat{\rho}}(k, t-s) G_{\hat{\rho} \rho}(k, s) \right]
\end{align*}

\[
\begin{align*}
[42] \quad -i G_{\theta \hat{\rho}}(k, t) - \int ds \left[ \Sigma_{\hat{\rho} \rho}(k, t-s) G_{\rho \hat{\rho}}(k, s) + \Sigma_{\rho \hat{\rho}}(k, t-s) G_{\rho \hat{\theta}}(k, s) \right]
\end{align*}

\[
\begin{align*}
[43] \quad -i G_{\theta \rho}(k, t) - \int ds \left[ \Sigma_{\hat{\rho} \rho}(k, t-s) G_{\rho \rho}(k, s) + \Sigma_{\rho \hat{\rho}}(k, t-s) G_{\hat{\rho} \rho}(k, s) \right]
\end{align*}

\[
\begin{align*}
[44] \quad -i G_{\theta \hat{\rho}}(k, t) - \int ds \left[ \Sigma_{\hat{\rho} \rho}(k, t-s) G_{\rho \hat{\rho}}(k, s) + \Sigma_{\rho \hat{\rho}}(k, t-s) G_{\rho \hat{\theta}}(k, s) \right] = \delta(t)
\end{align*}

(III.86)

We consider \([21]\) element in (III.86) for \(t > 0\). Since \(G_{\hat{\rho} \rho}(k, t) = 0\) for \(t > 0\) by causality, one can...
rewrite [21] element as

\[
\partial_t G_{\rho\rho}(k, t) = -\rho_0 T k^2 \left( K(k) G_{\rho\rho}(k, t) + G_{\theta\rho}(k, t) \right) \\
+ i \int_{-\infty}^{t} ds \left[ \Sigma_{\rho\rho}(k, t-s) G_{\rho\rho}(k, s) + \Sigma_{\rho\theta}(k, t-s) G_{\theta\rho}(k, s) \right] \\
+ i \int_{-\infty}^{0} ds \left[ \Sigma_{\rho\rho}(k, t-s) G_{\rho\rho}(k, s) + \Sigma_{\rho\theta}(k, t-s) G_{\theta\rho}(k, s) \right]
\]  

(III.87)

where we set the upper limits of the integrals using the causality properties of the self-energies (response functions) in the first (last) two integrations: \( \Sigma_{\alpha\beta}(k, t-s) = 0 \) for \( s > t \) and \( G_{\alpha\beta}(k, s) = 0 \) for \( s > 0 \) where \( \alpha, \beta = \rho, \theta \). One can simplify the integrals in (III.87) using the FDRs for the self-energies. For the first integral, using (III.76) we obtain

\[
i \int_{-\infty}^{t} ds \Sigma_{\rho\rho}(k, t-s) G_{\rho\rho}(k, s) = \int_{-\infty}^{t} ds \left( K(k) \Sigma_{\rho\rho}(k, t-s) - \partial_t \Sigma_{\rho\rho}(k, t-s) \right) G_{\rho\rho}(k, s)
\]  

(III.88)

The second term in the rhs of (III.88) can be rewritten by integration by parts as

\[
- \int_{-\infty}^{t} ds \partial_t \Sigma_{\rho\rho}(k, t-s) G_{\rho\rho}(k, s) = \int_{-\infty}^{t} ds \left[ \partial_s \Sigma_{\rho\rho}(k, t-s) \right] G_{\rho\rho}(k, s)
\]

(III.89)

where the boundary terms vanish since \( \Sigma_{\rho\rho}(k, 0) = 0 \) due to the oddness of \( \Sigma_{\rho\rho}(k, t) \) in time (see the \( \hat{\rho}\theta \)-element in (III.75)), and \( G_{\rho\rho}(k, -\infty) = 0 \). The second integral in (III.87) can be simplified by use of (III.76) as

\[
i \int_{-\infty}^{t} ds \Sigma_{\rho\theta}(k, t-s) G_{\theta\rho}(k, s) = \int_{-\infty}^{t} ds \Sigma_{\rho\theta}(k, t-s) G_{\theta\rho}(k, s)
\]  

(III.90)

Therefore we obtain for the first two integrals in (III.87)

\[
i \int_{-\infty}^{t} ds \left[ \Sigma_{\rho\rho}(k, t-s) G_{\rho\rho}(k, s) + \Sigma_{\rho\theta}(k, t-s) G_{\theta\rho}(k, s) \right]
\]

\[
= \int_{-\infty}^{t} ds \left[ \Sigma_{\rho\rho}(k, t-s) \left( K(k) G_{\rho\rho}(k, s) + G_{\theta\rho}(k, s) \right) - \Sigma_{\rho\theta}(k, t-s) \partial_s G_{\rho\rho}(k, s) \right]
\]  

(III.91)

We now show that \( (-\infty, 0) \)-contributions of (III.91) cancel the last two integrals in (III.87). The third integral in (III.87) involves the response function \( G_{\rho\rho}(s) \) with \( s < 0 \). Using the \( \hat{\rho}\rho \)-element of (III.71), we have

\[
G_{\rho\rho}(k, -t) = i \left( K(k) G_{\rho\rho}(k, t) + G_{\theta\rho}(k, t) \right), \quad \text{for} \quad t > 0
\]  

(III.92)
Again setting \( t = -s (s < 0) \) in (III.92), we obtain
\[
G_{\hat{\rho}\rho}(k, s) = i\left( K(k)G_{\rho\rho}(k, -s) + G_{\theta\rho}(k, -s) \right), \quad \text{for} \quad s < 0
\]
\[
= i\left( K(k)G_{\rho\rho}(k, s) + G_{\theta\rho}(k, s) \right), \quad \text{for} \quad s < 0
\]  
(III.93)

where we use the fact that the \( \rho \) and \( \theta \) field do not change under time reversal. Substituting (III.93) into the third integral in (III.87), we obtain
\[
\int_{-\infty}^{0} ds \Sigma_{\hat{\rho}\hat{\rho}}(k, t-s)G_{\hat{\rho}\rho}(k, s) = -\int_{-\infty}^{0} ds \Sigma_{\hat{\rho}\hat{\rho}}(k, t-s)\left( K(k)G_{\rho\rho}(k, s) + G_{\theta\rho}(k, s) \right)
\]  
(III.94)

This cancels the \((-\infty, 0)\) parts of the first two integrals in (III.91). Similarly, for the last integral in (III.87), we use the FDR, the Fourier transform of the \([\hat{\theta}\rho]\) element of (III.71)
\[
G_{\hat{\theta}\rho}(k, -t) = i\partial_t G_{\rho\rho}(k, t) \quad \text{for} \quad t > 0
\]  
(III.95)

Again setting \( t = -s \) in (III.95), we have
\[
G_{\hat{\theta}\rho}(k, s) = -i\partial_s G_{\rho\rho}(k, -s) = -i\partial_s G_{\rho\rho}(k, s)
\]  
(III.96)

Substituting (III.96) into the last integral in (III.87), we obtain
\[
\int_{-\infty}^{0} ds \Sigma_{\hat{\rho}\hat{\theta}}(k, t-s)G_{\hat{\rho}\theta}(k, s) = \int_{-\infty}^{0} ds \Sigma_{\hat{\rho}\hat{\theta}}(k, t-s)\partial_s G_{\rho\rho}(k, s),
\]  
(III.97)

which cancels the \((-\infty, 0)\) part of the last integral in (III.91). Using these cancellations, we can write down the final form for the dynamic equation for \( G_{\rho\rho}(k, t) \) for \( t > 0 \) as
\[
\partial_t G_{\rho\rho}(k, t) = -\rho_0 Tk^2 \left( K(k)G_{\rho\rho}(k, t) + G_{\theta\rho}(k, t) \right)
\]
\[
+ \int_{0}^{t} ds \left[ \Sigma_{\hat{\rho}\hat{\rho}}(k, t-s)\left( K(k)G_{\rho\rho}(k, s) + G_{\theta\rho}(k, s) \right) - \Sigma_{\hat{\rho}\hat{\rho}}(k, t-s)\partial_s G_{\rho\rho}(k, s) \right]
\]  
(III.98)

In similar ways, one can obtain dynamic equations for the remaining elements of \( G \) in (III.86). The eq. (III.98) is coupled to the eq. for \( G_{\theta\rho}(k, t) \) which is given by
\[
G_{\theta\rho}(k, t) = \Sigma_{\hat{\rho}\hat{\rho}}(k, 0)G_{\rho\rho}(k, t) + \int_{0}^{t} ds \Sigma_{\hat{\rho}\hat{\theta}}(k, t-s)\left( K(k)G_{\rho\theta}(k, s) + G_{\theta\theta}(k, s) \right)
\]
\[
- \int_{0}^{t} ds \Sigma_{\hat{\rho}\hat{\rho}}(k, t-s)\partial_s G_{\rho\rho}(k, s)
\]  
(III.99)

Likewise, we can obtain the two coupled eqs. for \( G_{\rho\theta}(k, t) \) and \( G_{\theta\theta}(k, t) \):
\[
\partial_t G_{\rho\theta}(k, t) = -\rho_0 Tk^2 \left( K(k)G_{\rho\theta}(k, t) + G_{\theta\theta}(k, t) \right) + \int_{0}^{t} ds \Sigma_{\hat{\rho}\hat{\theta}}(k, t-s)\left( K(k)G_{\rho\theta}(k, s) + G_{\theta\theta}(k, s) \right)
\]
\[
- \int_{0}^{t} ds \Sigma_{\hat{\rho}\hat{\theta}}(k, t-s)\partial_s G_{\rho\theta}(k, s),
\]  
(III.100)
\[
G_{\theta\theta}(k, t) = \Sigma_{\hat{\rho}\hat{\theta}}(k, 0)G_{\rho\theta}(k, t) + \int_{0}^{t} ds \Sigma_{\hat{\theta}\hat{\theta}}(k, t-s)\left( K(k)G_{\rho\theta}(k, s) + G_{\theta\theta}(k, s) \right)
\]
\[
- \int_{0}^{t} ds \Sigma_{\hat{\theta}\hat{\theta}}(k, t-s)\partial_s G_{\theta\theta}(k, s)
\]  
(III.101)
In the same manner, we obtain two sets of coupled dynamic eqs. for the response functions:

\[
\partial_t G_{\rho\rho}(k, t) = -\rho_0 T k^2 K(k) G_{\rho\rho}(k, t) - \rho_0 T k^2 G_{\theta\theta}(k, t) \\
+ i \int_0^t ds \left[ \Sigma_{\rho\rho}(k, t - s) \left( K^2(k) G_{\rho\rho}(k, s) + 2K(k) G_{\rho\theta}(k, s) + G_{\theta\theta}(k, s) \right) \right] \\
- \partial_t \Sigma_{\rho\rho}(k, t - s) \left( K(k) G_{\rho\rho}(k, s) + G_{\rho\theta}(k, s) \right)
\]

\[
G_{\theta\theta}(k, t) = i \int_0^t ds \left[ \Sigma_{\theta\theta}(k, t - s) \left( K(k) G_{\rho\rho}(k, s) + 2K(k) G_{\rho\theta}(k, s) + G_{\theta\theta}(k, s) \right) \right] \\
- \partial_t \Sigma_{\theta\theta}(k, t - s) \left( K(k) G_{\rho\rho}(k, s) + G_{\rho\theta}(k, s) \right)
\]

\[
G_{\rho\theta}(k, t) = -\rho_0 T k^2 K(k) G_{\rho\theta}(k, t) - \rho_0 T k^2 G_{\theta\theta}(k, t) \\
+ i \int_0^t ds \left[ \Sigma_{\rho\theta}(k, t - s) \left( K(k) \partial_s G_{\rho\rho}(k, s) + \partial_s G_{\theta\rho}(k, s) \right) \right] \\
- \partial_t \Sigma_{\rho\theta}(k, t - s) \partial_s G_{\rho\rho}(k, s)
\]

\[
\left(III.101\right)
\]

H. Static input

In the present work we are considering the situation in which the system (from some arbitrary initial state) has already evolved into the equilibrium state excluding, however, the crystalline state here and after. That is, the system is in equilibrium state for \( t \geq 0 \), and the present theory aims to describe the dynamics of the equilibrium fluctuations given the static information as input. Therefore the static informations of the system enter through the initial values of the dynamic correlation functions:

\[
G_{\rho\rho}(k, 0) = S_{\rho\rho}(k), \quad G_{\theta\rho}(k, 0) = S_{\theta\rho}(k), \quad G_{\theta\theta}(k, 0) = S_{\theta\theta}(k)
\]

\[
\left(III.102\right)
\]

where \( S_{\rho\rho}(k) \) etc. are the equilibrium correlation functions.

In Appendix A, we derived the following relation between the equilibrium correlation functions:

\[
K(k) S_{\rho\rho}(k) + S_{\theta\rho}(k) = 1.
\]

\[
\left(III.103\right)
\]

We also obtain the following relations by setting \( t = 0 \) in the dynamic equations (III.99) and (III.100)

\[
G_{\theta\rho}(k, 0) = \Sigma_{\rho\rho}(k, 0) G_{\rho\rho}(k, 0)
\]

\[
G_{\theta\theta}(k, 0) = \Sigma_{\rho\theta}(k, 0) G_{\rho\theta}(k, 0)
\]

\[
\left(III.104\right)
\]
Using the static input (III.102) and the equilibrium relation (III.103), one can express \( \Sigma_{\theta\theta}(k,0) \) and \( G_{\theta\theta}(k,0) \) from (III.104) in terms of the static structure factor \( S_{\rho\rho}(k) \equiv S(k) \) as

\[
\Sigma_{\theta\theta}(k,0) = \frac{S_{\theta\rho}(k)}{S_{\rho\rho}(k)} = \frac{1}{S(k)} - K(k),
\]

\[
G_{\theta\theta}(k,0) = S_{\theta\theta}(k) = \left( \frac{1}{S(k)} - K(k) \right)^2 S(k)
\]  

(III.105)

In Appendix A, we showed that \( S_{\rho\rho}(k) = S_{\theta\theta}(k) = 0 \) in the absence of interaction, \( U = 0 \). This means that the initial value of the self-energy \( \Sigma_{\theta\theta}(k,0) \) should vanish for the noninteracting case:

\[
\Sigma_{\theta\theta}(k,0) = 0 \quad \text{for} \quad U = 0
\]

(III.106)

If one uses the RY free energy (II.3), then as shown in Appendix A, \( K(k) \) is equal to the inverse of the static structure factor: \( K(k) = S^{-1}(k) \). For this case, it implies from (III.104) and (III.105) that \( \Sigma_{\theta\theta}(k,0) = G_{\theta\rho}(k,0) = G_{\theta\theta}(k,0) = 0 \) even for the interacting case.

### I. Further nonperturbative results

We first note that defining

\[
X(k,t) \equiv K(k)G_{\rho\rho}(k,t) + G_{\theta\rho}(k,t)
\]

\[
Y(k,t) \equiv K(k)G_{\rho\theta}(k,t) + G_{\theta\theta}(k,t)
\]

(III.107)

one can rewrite the sets of eqs. (III.98), (III.99) and (III.100) in terms of \( G_{\rho\rho}(k,t) \) and \( X(k,t) \), and \( G_{\rho\theta}(k,t) \) and \( Y(k,t) \) respectively as

\[
\partial_t G_{\rho\rho}(k,t) = -\rho_0 T k^2 X(k,t) + \int_0^t ds \left[ \Sigma_{\rho\rho}(k,t-s) X(k,s) - \Sigma_{\rho\theta}(k,t-s) \partial_s G_{\rho\rho}(k,s) \right],
\]

\[
X(k,t) = \frac{1}{S(k)} G_{\rho\rho}(k,t) + \int_0^t ds \left[ \Sigma_{\theta\rho}(k,t-s) X(k,s) - \Sigma_{\theta\theta}(k,t-s) \partial_s G_{\rho\theta}(k,s) \right]
\]  

(III.108)

and

\[
\partial_t G_{\rho\theta}(k,t) = -\rho_0 T k^2 Y(k,t) + \int_0^t ds \left[ \Sigma_{\rho\rho}(k,t-s) Y(k,s) - \Sigma_{\rho\theta}(k,t-s) \partial_s G_{\rho\theta}(k,s) \right],
\]

\[
Y(k,t) = \frac{1}{S(k)} G_{\rho\theta}(k,t) + \int_0^t ds \left[ \Sigma_{\theta\rho}(k,t-s) Y(k,s) - \Sigma_{\theta\theta}(k,t-s) \partial_s G_{\rho\theta}(k,s) \right]
\]  

(III.109)

In (III.108) and (III.109), the first member of (III.105) was used. One can notice that these two sets of equations share the same structure, which leads to interesting results, as shown below. Defining the Laplace transform

\[
G_{\rho\rho}^L(k,z) \equiv \int_0^\infty dt \ e^{-zt} G_{\rho\rho}(k,t), \quad \text{etc.,}
\]

(III.110)
we readily obtain the Laplace transform $G^L_{\rho\rho}(k, z)$ from (III.108) as
\[
G^L_{\rho\rho}(k, z) = S(k) \cdot \left[ z + \frac{\rho_0 T k^2}{S(k)} \cdot \frac{1}{D(\Sigma^L(z))} \right]^{-1},
\]
\[
D(\Sigma^L(z)) \equiv \frac{(1 + \Sigma^L_{\rho\rho}(k, z))(1 - \Sigma^L_{\theta\theta}(k, z))}{1 - \Sigma^L_{\rho\rho}(k, z)/\rho_0 T k^2} - \rho_0 T k^2 \Sigma^L_{\theta\theta}(k, z) \quad (\text{III.111})
\]
Likewise from another set (III.109) we obtain
\[
G^L_{\rho\theta}(k, z) = S_{\rho\theta}(k) \cdot \left[ z + \frac{\rho_0 T k^2}{S(k)} \cdot \frac{1}{D(\Sigma^L(z))} \right]^{-1} \quad (\text{III.112})
\]
The two equations (III.111) and (III.112) imply that the two correlation functions $G_{\rho\rho}(k, t)$ and $G_{\rho\theta}(k, t)$ turn out to be the same when each is normalized with its own initial value (i.e., equilibrium correlation):
\[
G_{\rho\theta}(k, t) = \frac{S_{\rho\theta}(k)}{S(k)} \cdot G_{\rho\rho}(k, t) = \left( \frac{1}{S(k)} - K(k) \right) G_{\rho\rho}(k, t) \quad (\text{III.113})
\]
where (III.103) is used. This relation in turn leads to
\[
X(k, t) \equiv K(k) G_{\rho\rho}(k, t) + G_{\theta\rho}(k, t) = K(k) G_{\rho\rho}(k, t) + G_{\rho\theta}(k, t) = \frac{1}{S(k)} G_{\rho\rho}(k, t) \quad (\text{III.114})
\]
where the time-reversal symmetric relation $G_{\theta\rho}(k, t) = G_{\rho\theta}(k, t)$ is used. In similar manner, one obtains
\[
Y(k, t) \equiv K(k) G_{\rho\rho}(k, t) + G_{\theta\theta}(k, t) = \frac{1}{S(k)} G_{\rho\rho}(k, t) = \frac{1}{S(k)} \left( \frac{1}{S(k)} - K(k) \right) G_{\rho\rho}(k, t),
\]
\[
G_{\theta\theta}(k, t) = \left( \frac{1}{S(k)} - K(k) \right) G_{\rho\theta}(k, t) = \left( \frac{1}{S(k)} - K(k) \right)^2 G_{\rho\rho}(k, t) \quad (\text{III.115})
\]
The relations (III.114) and (III.115) with $t = 0$ are of course fully consistent with (III.105). Note also that (III.113) implies that
\[
G_{\rho\theta}(k, t) = 0 \quad \text{for} \quad U = 0 \quad (\text{III.116})
\]
since the relation $K(k) = 1/S(k)$ holds for the noninteracting case. Due to the time reversal symmetric relation, we also have
\[
G_{\theta\rho}(k, t) = G_{\rho\theta}(k, t) = G_{\theta\theta}(k, t) = 0 \quad \text{for} \quad U = 0 \quad (\text{III.117})
\]
We further point out that if the RY free energy is employed from the outset, then we have from (III.113) and (III.115) that $G_{\rho\theta}(k, t) = G_{\theta\theta}(k, t) = 0$ even for the interacting case since $K(k) = 1/S(k)$. 

We also note in passing that the second line of (III.108) and (III.114) leads to a nonperturbative identity

$$\int_0^t ds \left[ \Sigma_{\hat{\theta}\hat{\theta}}(\textbf{k}, t - s) \frac{G_{\rho\rho}(\textbf{k}, s)}{S(\textbf{k})} - \Sigma_{\hat{\theta}\hat{\rho}}(\textbf{k}, t - s) \partial_s G_{\rho\rho}(\textbf{k}, s) \right] = 0 \quad (III.118)$$

Using (III.114), one can rewrite the dynamic eq. for $G_{\rho\rho}(\textbf{k}, t)$, (III.108) as

$$\partial_t G_{\rho\rho}(\textbf{k}, t) = -\frac{\rho_0 T k^2}{S(\textbf{k})} G_{\rho\rho}(\textbf{k}, t) + \int_0^t ds \left[ \Sigma_{\hat{\rho}\hat{\rho}}(\textbf{k}, t - s) \frac{G_{\rho\rho}(\textbf{k}, s)}{S(\textbf{k})} - \Sigma_{\hat{\rho}\hat{\theta}}(\textbf{k}, t - s) \partial_s G_{\rho\rho}(\textbf{k}, s) \right] \quad (III.119)$$

Note that the equation now becomes a closed equation for the density correlation function $G_{\rho\rho}(\textbf{k}, t)$ alone since the self-energies $\Sigma_{\hat{\rho}\hat{\rho}}(\textbf{k}, t)$ and $\Sigma_{\hat{\rho}\hat{\theta}}(\textbf{k}, t)$ can be expressed solely in terms of $G_{\rho\rho}(\textbf{k}, t)$ via the FDRs (III.73) and the relations (III.113) and the second relation of (III.115).

**IV. ONE-LOOP RESULTS AND MCT EQUATION**

A. One-loop calculations of the self-energies

We now need to express the self energies $\Sigma_{\hat{\alpha}\hat{\beta}}$ appearing in (III.119) in terms of $G_{\rho\rho}(\textbf{k})$. Here we obtain this expression up to the one-loop order in the loop expansion of the dynamic action. As shown in Fig. 4, we have two types of one-loop diagrams. But the first diagram which involves 4-point vertex does not contribute since self-energies appearing in (III.119) are of the form $\Sigma_{\hat{\rho}\hat{\rho}}(\textbf{k}, t)$ and there is no 4-point vertex involving two hatted variables. Therefore we only need to consider the second one-loop diagrams in Fig. 4, which can be generically written as

$$\Sigma_{\hat{\alpha}\hat{\beta}}(\textbf{12}) = \frac{1}{2} V_{\gamma\gamma'}(\textbf{134}) V_{\gamma'\delta'}(\textbf{256}) G_{\gamma\gamma'}(\textbf{35}) G_{\delta\delta'}(\textbf{46}) \quad (IV.1)$$

where indices $\gamma, \gamma', \delta$ and $\delta'$ include hatted variable indices as well.

We write down individual expressions of $\Sigma_{\hat{\alpha}\hat{\beta}}(\textbf{12})$ diagrams, starting with $\Sigma_{\hat{\theta}\hat{\theta}}(\textbf{12})$

$$\Sigma_{\hat{\theta}\hat{\theta}}(\textbf{12}) = \frac{1}{2} V_{\hat{\theta}\hat{\rho}}(\textbf{134}) V_{\hat{\rho}\hat{\rho}}(\textbf{256}) G_{\rho\rho}(\textbf{35}) G_{\rho\rho}(\textbf{46}) \quad (IV.2)$$

The diagrammatic expression for (IV.2) is shown in Fig. 5. Note that the symmetry factor generated from the exchange $\textbf{3} \leftrightarrow \textbf{4}$ and $\textbf{5} \leftrightarrow \textbf{6}$ is already contained in the definition of the vertex $V_{\hat{\theta}\hat{\rho}}$. We keep this convention for other symmetry-possessing vertices $V_{\hat{\rho}\rho}(\textbf{123})$ and $V_{\hat{\theta}\hat{\theta}}(\textbf{123})$. We also note that since the vertex $V_{\hat{\theta}\hat{\rho}}$ does not involve $U$, one-loop expression (IV.2) at $t = 0$ does not satisfy the nonperturbative requirement (III.106) which should hold for the noninteracting case. Next we write
down the expressions for the self-energy \( \Sigma_{\hat{\rho}\hat{\rho}}(12) \):

\[
\Sigma_{\hat{\rho}\hat{\rho}}(12) = \sum_{j=1}^{4} \Sigma^{(j)}_{\hat{\rho}\hat{\rho}}(12)
\]

\[
\Sigma^{(1)}_{\hat{\rho}\hat{\rho}}(12) = \frac{1}{2} V^{int}_{\hat{\rho}\hat{\rho}}(134) V_{\hat{\theta}\hat{\rho}}(256) G_{\hat{\rho}\hat{\rho}}(35) G_{\hat{\rho}\hat{\rho}}(46)
\]

\[
\Sigma^{(2)}_{\hat{\rho}\hat{\rho}}(12) = \frac{1}{2} V^{id}_{\hat{\rho}\hat{\rho}}(134) V_{\hat{\theta}\hat{\rho}}(256) G_{\hat{\rho}\hat{\rho}}(35) G_{\hat{\rho}\hat{\rho}}(46)
\]

\[
\Sigma^{(3)}_{\hat{\rho}\hat{\rho}}(12) = \frac{1}{2} V_{\hat{\rho}\hat{\rho}}(134) V_{\hat{\theta}\hat{\rho}}(256) G_{\hat{\rho}\hat{\rho}}(35) G_{\hat{\theta}\hat{\rho}}(46) \times 2
\]

\[
\Sigma^{(4)}_{\hat{\rho}\hat{\rho}}(12) = \frac{1}{2} V_{\hat{\rho}\hat{\rho}}(134) V_{\hat{\theta}\hat{\rho}}(256) G_{\hat{\rho}\hat{\rho}}(35) G_{\hat{\rho}\hat{\theta}}(46) \times 2 \quad (IV.3)
\]

We first point out that the diagram \( \Sigma^{(4)}_{\hat{\rho}\hat{\rho}}(12) \) vanishes since \( G_{\hat{\rho}\hat{\rho}}(35) = 0 \) by causality for \( t_1 = t_3 = t_4 = t \) and \( t_2 = t_5 = t_6 = 0 \) with \( t > 0 \). Hence we will omit this type of diagrams from now on. The multiplication factor 2 in the last two diagrams comes from, e.g.,

\[
\Sigma^{(3)}_{\hat{\rho}\hat{\rho}}(12) = \frac{1}{2} V_{\hat{\rho}\hat{\rho}}(134) V_{\hat{\theta}\hat{\rho}}(256) G_{\hat{\rho}\hat{\rho}}(35) G_{\hat{\theta}\hat{\rho}}(46) + \frac{1}{2} V_{\hat{\rho}\hat{\rho}}(134) V_{\hat{\theta}\hat{\rho}}(256) G_{\hat{\theta}\hat{\rho}}(35) G_{\hat{\rho}\hat{\rho}}(46)
\]

\[
= \frac{1}{2} V_{\hat{\rho}\hat{\rho}}(134) V_{\hat{\theta}\hat{\rho}}(256) G_{\hat{\rho}\hat{\rho}}(35) G_{\hat{\theta}\hat{\rho}}(46) \times 2 \quad (IV.4)
\]

The last line follows from the fact that under the exchanges of dummy variables \( 3 \leftrightarrow 4 \) and \( 5 \leftrightarrow 6 \) in the second term, the two terms in rhs of the first lines become the same since \( V_{\hat{\rho}\hat{\rho}}(143) = V_{\hat{\rho}\hat{\rho}}(134) \) and \( V_{\hat{\theta}\hat{\rho}}(265) = V_{\hat{\theta}\hat{\rho}}(256) \). The diagrams for non-vanishing \( \Sigma_{\hat{\rho}\hat{\rho}}(12) \) is shown in Fig. 6.

We now move on to the self-energies \( \Sigma_{\hat{\theta}\hat{\rho}}(12) \) and \( \Sigma_{\hat{\rho}\hat{\theta}}(12) \).

\[
\Sigma_{\hat{\theta}\hat{\rho}}(12) = \sum_{j=1}^{4} \Sigma^{(j)}_{\hat{\theta}\hat{\rho}}(12)
\]

\[
\Sigma^{(1)}_{\hat{\theta}\hat{\rho}}(12) = \frac{1}{2} V_{\hat{\theta}\hat{\rho}}(134) V^{int}_{\hat{\rho}\hat{\rho}}(256) G_{\hat{\rho}\hat{\rho}}(35) G_{\hat{\rho}\hat{\rho}}(46)
\]

\[
\Sigma^{(2)}_{\hat{\theta}\hat{\rho}}(12) = \frac{1}{2} V_{\hat{\theta}\hat{\rho}}(134) V^{id}_{\hat{\rho}\hat{\rho}}(256) G_{\hat{\rho}\hat{\rho}}(35) G_{\hat{\rho}\hat{\rho}}(46)
\]

\[
\Sigma^{(3)}_{\hat{\theta}\hat{\rho}}(12) = \frac{1}{2} V_{\hat{\rho}\hat{\theta}}(134) V_{\hat{\theta}\hat{\rho}}(256) G_{\hat{\rho}\hat{\rho}}(35) G_{\hat{\theta}\hat{\rho}}(46) \times 2
\]

\[
\Sigma^{(4)}_{\hat{\theta}\hat{\rho}}(12) = \frac{1}{2} V_{\hat{\rho}\hat{\theta}}(134) V_{\hat{\theta}\hat{\rho}}(256) G_{\hat{\rho}\hat{\rho}}(35) G_{\hat{\theta}\hat{\rho}}(46) \times 2 \quad (IV.5)
\]
The self-energy $\Sigma_{\tilde{p}\tilde{p}}(12)$ has 14 diagrams:

$$
\Sigma_{\tilde{p}\tilde{p}}(12) = \sum_{j=1}^{14} \Sigma_{\tilde{p}\tilde{p}}^{(j)}(12)
$$

$$
\Sigma_{\tilde{p}\tilde{p}}^{(1)}(12) = \frac{1}{2} V_{\tilde{p}\tilde{p}}^{\text{int}}(134) V_{\tilde{p}\tilde{p}}^{\text{int}}(256) G_{\tilde{p}\tilde{p}}(35) G_{\tilde{p}\tilde{p}}(46)
$$

$$
\Sigma_{\tilde{p}\tilde{p}}^{(2)}(12) = \frac{1}{2} V_{\tilde{p}\tilde{p}}^{\text{int}}(134) V_{\tilde{p}\tilde{p}}^{\text{id}}(256) G_{\tilde{p}\tilde{p}}(35) G_{\tilde{p}\tilde{p}}(46)
$$

$$
\Sigma_{\tilde{p}\tilde{p}}^{(3)}(12) = \frac{1}{2} V_{\tilde{p}\tilde{p}}^{\text{int}}(134) V_{\tilde{p}\tilde{p}}^{\text{int}}(256) G_{\tilde{p}\tilde{p}}(35) G_{\tilde{p}\tilde{p}}(46) \times 2
$$

$$
\Sigma_{\tilde{p}\tilde{p}}^{(4)}(12) = \frac{1}{2} V_{\tilde{p}\tilde{p}}^{\text{int}}(134) V_{\tilde{p}\tilde{p}}^{\text{int}}(256) G_{\tilde{p}\tilde{p}}(35) G_{\tilde{p}\tilde{p}}(46)
$$

$$
\Sigma_{\tilde{p}\tilde{p}}^{(5)}(12) = \frac{1}{2} V_{\tilde{p}\tilde{p}}^{\text{int}}(134) V_{\tilde{p}\tilde{p}}^{\text{id}}(256) G_{\tilde{p}\tilde{p}}(35) G_{\tilde{p}\tilde{p}}(46)
$$

$$
\Sigma_{\tilde{p}\tilde{p}}^{(6)}(12) = \frac{1}{2} V_{\tilde{p}\tilde{p}}^{\text{id}}(134) V_{\tilde{p}\tilde{p}}^{\text{int}}(256) G_{\tilde{p}\tilde{p}}(35) G_{\tilde{p}\tilde{p}}(46) \times 2
$$

$$
\Sigma_{\tilde{p}\tilde{p}}^{(7)}(12) = \frac{1}{2} V_{\tilde{p}\tilde{p}}^{\text{int}}(134) V_{\tilde{p}\tilde{p}}^{\text{id}}(256) G_{\tilde{p}\tilde{p}}(35) G_{\tilde{p}\tilde{p}}(46) \times 2
$$

$$
\Sigma_{\tilde{p}\tilde{p}}^{(8)}(12) = \frac{1}{4} V_{\tilde{p}\tilde{p}}^{\text{int}}(134) V_{\tilde{p}\tilde{p}}^{\text{int}}(256) G_{\tilde{p}\tilde{p}}(35) G_{\tilde{p}\tilde{p}}(46) \times 2
$$

$$
\Sigma_{\tilde{p}\tilde{p}}^{(9)}(12) = \frac{1}{2} V_{\tilde{p}\tilde{p}}^{\text{int}}(134) V_{\tilde{p}\tilde{p}}^{\text{int}}(256) G_{\tilde{p}\tilde{p}}(35) G_{\tilde{p}\tilde{p}}(46) \times 2
$$

$$
\Sigma_{\tilde{p}\tilde{p}}^{(10)}(12) = \frac{1}{2} V_{\tilde{p}\tilde{p}}^{\text{int}}(134) V_{\tilde{p}\tilde{p}}^{\text{int}}(256) G_{\tilde{p}\tilde{p}}(35) G_{\tilde{p}\tilde{p}}(46) \times 2
$$

$$
\Sigma_{\tilde{p}\tilde{p}}^{(11)}(12) = \frac{1}{2} V_{\tilde{p}\tilde{p}}^{\text{int}}(134) V_{\tilde{p}\tilde{p}}^{\text{int}}(256) G_{\tilde{p}\tilde{p}}(35) G_{\tilde{p}\tilde{p}}(46) \times 2
$$

$$
\Sigma_{\tilde{p}\tilde{p}}^{(12)}(12) = \frac{1}{2} V_{\tilde{p}\tilde{p}}^{\text{int}}(134) V_{\tilde{p}\tilde{p}}^{\text{int}}(256) G_{\tilde{p}\tilde{p}}(35) G_{\tilde{p}\tilde{p}}(46) \times 2
$$

$$
\Sigma_{\tilde{p}\tilde{p}}^{(13)}(12) = \frac{1}{2} V_{\tilde{p}\tilde{p}}^{\text{int}}(134) V_{\tilde{p}\tilde{p}}^{\text{int}}(256) G_{\tilde{p}\tilde{p}}(35) G_{\tilde{p}\tilde{p}}(46) \times 2
$$

$$
\Sigma_{\tilde{p}\tilde{p}}^{(14)}(12) = \frac{1}{2} V_{\tilde{p}\tilde{p}}^{\text{int}}(134) V_{\tilde{p}\tilde{p}}^{\text{int}}(256) G_{\tilde{p}\tilde{p}}(35) G_{\tilde{p}\tilde{p}}(46) \times 2
$$

(IV.6)

The diagrams for $\Sigma_{\tilde{p}\tilde{p}}(12)$ and $\Sigma_{\tilde{p}\tilde{p}}(12)$ are shown respectively in Fig. 7 and Fig. 8.

It is convenient to classify the entire one-loop diagrams into three classes: the diagrams whose vertex biproducts possessing no $U$ (Class 0), the ones possessing single $U$ (Class 1), and the ones quadratic in $U$ (Class 2) where we note that $U$ occurs only in $V_{\tilde{p}\tilde{p}}^{\text{int}}$ (the vertex with filled circle in Fig. 6 through Fig. 8):

Class 0 = $\Sigma_{\tilde{p}\tilde{p}}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(2)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(3)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(2)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(3)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(4)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(5)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(6)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(8)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(10)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(11)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(12)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(13)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(14)}(12)$

Class 1 = $\Sigma_{\tilde{p}\tilde{p}}^{(1)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(1)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(2)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(3)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(4)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(7)}(12)$, $\Sigma_{\tilde{p}\tilde{p}}^{(9)}(12)$

Class 2 = $\Sigma_{\tilde{p}\tilde{p}}^{(1)}(12)$

(IV.7)
Therefore the kernels $\Sigma^{\hat{\rho}\hat{\rho}}(k, t - s)\frac{G_{\rho\rho}(k, s)}{S(k)} - \Sigma_{\hat{\rho}\hat{\theta}}(k, t - s)\partial_s G_{\rho\rho}(k, s)$ can now be rewritten as

$$
\int_0^t ds \left[ \Sigma_{\hat{\rho}\hat{\rho}}(k, t - s)\frac{G_{\rho\rho}(k, s)}{S(k)} - \Sigma_{\hat{\rho}\hat{\theta}}(k, t - s)\partial_s G_{\rho\rho}(k, s) \right] = 
\int_0^t ds \left[ \left( \Sigma_{\hat{\rho}\hat{\rho}} - \rho_0 T^2 \Sigma_{\theta\theta} \right)(k, t - s)\frac{G_{\rho\rho}(k, s)}{S(k)} + \left( \rho_0 T^2 \Sigma_{\theta\theta} - \Sigma_{\hat{\theta}\hat{\rho}} \right)(k, t - s)\partial_s G_{\rho\rho}(k, s) \right]
$$

(IV.8)

Next, we note the following relation between the two vertices $V_{\hat{\theta}\rho\rho}(\mathbf{123})$ and $V_{\hat{\rho}\rho\rho}(\mathbf{123})$ (see (III.64)):

$$
\rho_0 T V_1^2 V_{\hat{\theta}\rho\rho}(\mathbf{123}) + V_{\hat{\rho}\rho\rho}(\mathbf{123}) = 0
$$

(IV.9)

This relation reflects the fact that the auxiliary field is designed to take care of the nonlinearity of the ideal-gas contribution. As shown below, this identity leads to the mutual cancellation of the 5 pairs of diagrams in the convolution integral in (III.80). We note the following relation between $\Sigma_{\theta\theta}(\mathbf{12})$ and $\Sigma^{(2)}_{\hat{\rho}\hat{\theta}}(\mathbf{12})$ in the convolution integral:

$$
\rho_0 T V_1^2 \Sigma_{\theta\theta}(\mathbf{12}) + \Sigma^{(2)}_{\hat{\rho}\hat{\theta}}(\mathbf{12}) = \frac{1}{4} \left( \rho_0 T V_1^2 V_{\hat{\theta}\rho\rho}(\mathbf{134}) + V_{\hat{\rho}\rho\rho}(\mathbf{134}) \right) V_{\hat{\theta}\rho\rho}(\mathbf{256}) G_{\rho\rho}(\mathbf{35}) G_{\rho\rho}(\mathbf{46}) = 0
$$

(IV.10)

The Fourier transform of (IV.10) can be written as

$$
\Sigma^{(2)}_{\hat{\rho}\hat{\theta}}(k, t) = \rho_0 T k^2 \Sigma_{\theta\theta}(k, t)
$$

(IV.11)

In the same manner we obtain the same type of relations for the other 4 pairs of one-loop self energies as consequences of (IV.9):

$$
\Sigma^{(4)}_{\hat{\rho}\hat{\theta}}(k, t) = \rho_0 T k^2 \Sigma^{(1)}_{\theta\theta}(k, t), \quad \Sigma^{(5)}_{\hat{\rho}\hat{\theta}}(k, t) = \rho_0 T k^2 \Sigma^{(2)}_{\theta\theta}(k, t),
\Sigma^{(6)}_{\hat{\rho}\hat{\theta}}(k, t) = \rho_0 T k^2 \Sigma^{(4)}_{\theta\theta}(k, t), \quad \Sigma^{(8)}_{\hat{\rho}\hat{\theta}}(k, t) = \rho_0 T k^2 \Sigma^{(5)}_{\theta\theta}(k, t)
$$

(IV.12)

Therefore the kernels $\Sigma_{\theta\theta}(k, t)$ and $\Sigma_{\theta\theta}(k, t)$ are eliminated from the convolution integrals in (IV.8). There will be the corresponding cancellations in the higher loop calculations as well. Hence the convolution integral (IV.8) can now be rewritten as

$$
\int_0^t ds \left[ \Sigma_{\hat{\rho}\hat{\rho}}(k, t - s)\frac{G_{\rho\rho}(k, s)}{S(k)} - \left( \Sigma^{(1)}_{\theta\theta} + \Sigma^{(3)}_{\theta\theta} \right)(k, t - s)\partial_s G_{\rho\rho}(k, s) \right]
$$

(IV.13)

where the superscript $R$ denotes the remaining diagrams in the kernel $\Sigma_{\hat{\rho}\hat{\rho}}$.

We now compute the remaining diagrams and list the results below. Details of calculation are
given in Appendix B. We first write down the results for Class 1 and Class 2 diagrams.

\[
\begin{align*}
\Sigma_{p\bar{p}}^{(1)}(k, t) &= -\frac{1}{\rho_0} \int_q k \cdot q U(q) G_{pp}(q, t) G_{pp}(k - q, t) \\
\Sigma_{p\bar{p}}^{(2)}(k, t) &= -\frac{T}{\rho_0} k^2 \int_q k \cdot q U(q) G_{pp}(q, t) G_{pp}(k - q, t) \\
\Sigma_{p\bar{p}}^{(3)}(k, t) &= -2iT \int_q [(k \cdot q)^2 U(q) + (k \cdot q)(k \cdot (k - q)) U(q(U(k - q)))] G_{pp}(q, t) G_{pp}(k - q, t) \\
\Sigma_{p\bar{p}}^{(4)}(k, t) &= -T \int_q [(k \cdot q)^2 U(q) + (k \cdot q)(k \cdot (k - q)) U(q)] G_{pp}(q, t) G_{pp}(k - q, t) \\
\Sigma_{p\bar{p}}^{(5)}(k, t) &= -T \int_q [(k \cdot q)^2 U(q) + (k \cdot q)(k \cdot (k - q)) U(q(U(k - q)))] G_{pp}(q, t) G_{pp}(k - q, t) \\
\end{align*}
\]

where \( \int_q \equiv \int d^3q/(2\pi)^3 \). Noting that \( \Sigma_{p\bar{p}}^{(7)}(k, t) = \Sigma_{p\bar{p}}^{(9)}(k, t) \) (since \( G_{p\theta}(q, t) = G_{\theta p}(q, t) \)), and \( G_{p\bar{p}}(q, t) = i \left( K(q) G_{pp}(q, t) + G_{\theta p}(q, t) \right) \) with \( K(q) \equiv 1/\rho_0 + U(q)/T \), (FDR (III.73)), we obtain

\[
\begin{align*}
\left( \Sigma_{p\bar{p}}^{(3)} + \Sigma_{p\bar{p}}^{(7)} + \Sigma_{p\bar{p}}^{(9)} \right)(k, t) &= 2T \int_q [(k \cdot q)^2 U(q) + (k \cdot q)(k \cdot (k - q)) U(q(U(k - q)))] K(q) G_{pp}(q, t) G_{pp}(k - q, t) \\
&= 2T \rho_0 \int_q [(k \cdot q)^2 U(q) + (k \cdot q)(k \cdot (k - q)) U(q(U(k - q)))] G_{pp}(q, t) G_{pp}(k - q, t) \\
&+ \int_q [(k \cdot q)^2 U^2(q) + (k \cdot q)(k \cdot (k - q)) U(q(U(k - q)))] G_{pp}(q, t) G_{pp}(k - q, t) \\
\end{align*}
\]

where the last integral has the same structure as that of \( \Sigma_{p\bar{p}}^{(1)}(k, t) \), (IV.14). The first integral of (IV.15) can be simplified by use of

\[
\int_q (k \cdot q)(k \cdot (k - q)) U(k - q) G_{pp}(q, t) G_{pp}(k - q, t) = \int_q (k \cdot q)(k \cdot (k - q)) U(q) G_{pp}(q, t) G_{pp}(k - q, t)
\]

which is obtained by shifting the integration variable \( q \to k - q \). Using this feature, the first integral in (IV.15) is simplified as

\[
2T \rho_0 \int_q [(k \cdot q)^2 + (k \cdot q)(k \cdot (k - q))] U(q) G_{pp}(q, t) G_{pp}(k - q, t) = 2T \rho_0 k^2 \int_q k \cdot q U(q) G_{pp}(q, t) G_{pp}(k - q, t)
\]

(IV.16)

which is the same integral as that of \( \Sigma_{p\bar{p}}^{(2)}(k, t) \), (IV.14). Therefore, we obtain

\[
\begin{align*}
\left( \Sigma_{p\bar{p}}^{(1)} + \Sigma_{p\bar{p}}^{(2)} + \Sigma_{p\bar{p}}^{(3)} + \Sigma_{p\bar{p}}^{(7)} + \Sigma_{p\bar{p}}^{(9)} \right)(k, t) &= \frac{T}{\rho_0} k^2 \int_q k \cdot q U(q) G_{pp}(q, t) G_{pp}(k - q, t) \\
&+ \int_q [(k \cdot q)^2 U^2(q) + (k \cdot q)(k \cdot (k - q)) U(q(U(k - q)))] G_{pp}(q, t) G_{pp}(k - q, t)
\end{align*}
\]
Now we calculate and examine Class 0 diagrams.

\[
\Sigma_{\hat{\rho}\hat{\rho}}^{(3)}(k, t) = -\frac{T}{\rho_0^2} \int q \cdot q G_{\hat{\rho}\hat{\rho}}(q, t) G_{\rho\rho}(k - q, t)
\]

\[
\Sigma_{\hat{\rho}\hat{\rho}}^{(10)}(k, t) = -\frac{T^2}{\rho_0^3} k^2 \int q \cdot q G_{\hat{\rho}\hat{\rho}}(q, t) G_{\rho\rho}(k - q, t)
\]

\[
\Sigma_{\hat{\rho}\hat{\rho}}^{(11)}(k, t) = -T^2 \int q (k \cdot q) (k \cdot (k - q)) G_{\hat{\rho}\hat{\rho}}(q, t) G_{\rho\rho}(k - q, t)
\]

\[
\Sigma_{\hat{\rho}\hat{\rho}}^{(12)}(k, t) = -2T^2 \int q (k \cdot q) (k \cdot (k - q)) G_{\hat{\rho}\hat{\rho}}(q, t) G_{\rho\rho}(k - q, t)
\]

\[
\Sigma_{\hat{\rho}\hat{\rho}}^{(13)}(k, t) = -2iT^2 \int q (k \cdot q) (k \cdot (k - q)) G_{\hat{\rho}\hat{\rho}}(q, t) G_{\rho\rho}(k - q, t)
\]

\[
\Sigma_{\hat{\rho}\hat{\rho}}^{(14)}(k, t) = -2iT^2 \int q (k \cdot q) (k \cdot (k - q)) G_{\hat{\rho}\hat{\rho}}(q, t) G_{\rho\rho}(k - q, t)
\]

Using the FDR relation, \(G_{\hat{\rho}\hat{\rho}}(q, t) = i \left( K(q) G_{\hat{\rho}\rho}(q, t) + G_{\theta\theta}(q, t) \right)\), given in (III.73), we have

\[
\left( \Sigma_{\hat{\rho}\hat{\rho}}^{(11)} + \Sigma_{\hat{\rho}\hat{\rho}}^{(13)} \right)(k, t) = T^2 \int q (k \cdot q) G_{\theta\rho}(q, t) G_{\rho\rho}(k - q, t) + 2T^2 \int q (k \cdot q)^2 K(q) G_{\theta\rho}(q, t) G_{\rho\rho}(k - q, t)
\]

\[
= T^2 \int q (k \cdot q)^2 \left( \frac{1}{S^2(q)} - K^2(q) \right) G_{\rho\rho}(q, t) G_{\rho\rho}(k - q, t)
\]

where the last line follows from using (III.113) and (III.115):

\[
2K(q) G_{\theta\rho}(q, t) + G_{\theta\theta}(q, t) = \left[ 2K(q) \left( \frac{1}{S(q)} - K(q) \right) + \left( \frac{1}{S(q)} - K(q) \right)^2 \right] G_{\rho\rho}(q, t)
\]

\[
= \left( \frac{1}{S^2(q)} - K^2(q) \right) G_{\rho\rho}(q, t)
\]

Likewise using the FDR \(G_{\rho\rho}(q, t) = i \left( K(q) G_{\rho\rho}(q, t) + G_{\rho\rho}(q, t) \right)\), (III.71), we have

\[
\left( \Sigma_{\rho\hat{\rho}}^{(12)} + \Sigma_{\rho\hat{\rho}}^{(14)} \right)(k, t) = T^2 \int q (k \cdot q) (k \cdot (k - q)) G_{\theta\rho}(q, t) G_{\rho\rho}(k - q, t)
\]

\[
+ 2T^2 \int q (k \cdot q) (k \cdot (k - q)) K(k - q) G_{\theta\rho}(q, t) G_{\rho\rho}(k - q, t)
\]

\[
= T^2 \int q (k \cdot q) (k \cdot (k - q)) \left( \frac{1}{S(q)} - K(q) \right) \cdot \left( \frac{1}{S(k - q)} + K(k - q) \right) G_{\rho\rho}(q, t) G_{\rho\rho}(k - q, t)
\]

\[
= T^2 \int q (k \cdot q) (k \cdot (k - q)) \left[ \frac{1}{S(q)} - K(q) \right] \cdot \left( \frac{1}{S(k - q)} + K(k - q) \right) G_{\rho\rho}(q, t) G_{\rho\rho}(k - q, t)
\]

\[
= T^2 \int q (k \cdot q) (k \cdot (k - q)) \left( \frac{1}{S(q) S(k - q)} - K(q) K(k - q) \right) G_{\rho\rho}(q, t) G_{\rho\rho}(k - q, t)
\]

where the relation (III.113) was used.

Now we are going to add up all the diagrams of \(\Sigma_{\hat{\rho}\hat{\rho}}(k, t)\) where \(\Sigma_{\hat{\rho}\hat{\rho}}^{(j)} \quad j = 4, 5, 6, 8\) are already taken care of through (IV.12). One can recognize the terms sharing with the same structure in the integrand:
(a) $\Sigma_{\rho p}^{(10)}(k, t)$ with the first integral of (IV.17),
(b) $(\Sigma_{\rho p}^{(11)} + \Sigma_{\rho p}^{(13)})(k, t)$ with the first term in the second integral of (IV.17),
(c) $(\Sigma_{\rho p}^{(12)} + \Sigma_{\rho p}^{(14)})(k, t)$ with the second term in the second integral of (IV.17).

We first compute the following terms:

\begin{align*}
(a) \quad & U(q) - T \left( \frac{1}{S(q)} - K(q) \right) = U(q) - T \left( \frac{1}{\rho_0} - c(q) - \left( \frac{1}{\rho_0} + \frac{U(q)}{T} \right) \right) = 2U(q) + Tc(q), \\
(b) \quad & U^2(q) + T^2 \left( \frac{1}{S(q)} - K^2(q) \right) = U^2(q) + T^2 \left[ \left( \frac{1}{\rho_0} - c(q) \right)^2 - \left( \frac{1}{\rho_0} + \frac{U(q)}{T} \right)^2 \right] \\
& = T^2c(q)^2 - 2 \frac{T^2}{\rho_0} \left( c(q) + \frac{U(q)}{T} \right), \\
(c) \quad & U(q)U(k - q) + T^2 \left( \frac{1}{S(q)S(k - q)} - K(q)K(k - q) \right) \\
& = T^2c(q)c(k - q) - \frac{T^2}{\rho_0} \left[ \left( c(q) + \frac{U(q)}{T} \right) + \left( c(k - q) + \frac{U(k - q)}{T} \right) \right] \\
& = (IV.22)
\end{align*}

where the direct correlation function $c(q)$ is related to the static structure factor $S(q)$ as $S(q) = \rho_0/(1 - \rho_0c(q))$. Using (IV.22), we obtain

\begin{align*}
\Sigma_{\rho p}(k, t) = T^2 \int_q \left[ (k \cdot q)^2 c(q) + (k \cdot q)(k \cdot (k - q))c(q)c(k - q) \right] G_{\rho p}(q, t)G_{\rho p}(k - q, t) \\
- 2 \frac{T^2}{\rho_0} \int_q (k \cdot q)^2 \left( c(q) + \frac{U(q)}{T} \right) G_{\rho p}(q, t)G_{\rho p}(k - q, t) \\
- \frac{T^2}{\rho_0} \int_q (k \cdot q)(k \cdot (k - q)) \left[ \left( c(q) + \frac{U(q)}{T} \right) + \left( c(k - q) + \frac{U(k - q)}{T} \right) \right] G_{\rho p}(q, t)G_{\rho p}(k - q, t) \\
+ \frac{T}{\rho_0} k^2 \int_q (k \cdot q) \left[ 2U(q) + Tc(q) \right] G_{\theta p}(q, t)G_{\rho p}(k - q, t) \\
& = (IV.23)
\end{align*}

One can further simplify the last three integrals by noting the third integral becomes by symmetry

\begin{align*}
\int_q (k \cdot q)(k \cdot (k - q)) \left[ \left( c(q) + \frac{U(q)}{T} \right) + \left( c(k - q) + \frac{U(k - q)}{T} \right) \right] G_{\rho p}(q, t)G_{\rho p}(k - q, t) \\
= 2 \int_q (k \cdot q)(k \cdot (k - q)) \left( c(q) + \frac{U(q)}{T} \right) G_{\rho p}(q, t)G_{\rho p}(k - q, t) \\
& = (IV.24)
\end{align*}

Then the sum of the last three integrals is given by

\begin{align*}
& \left( \text{the sum of the last three integrals of (IV.23)} \right) \\
& = -2 \frac{T^2}{\rho_0} k^2 \int_q (k \cdot q) \left( c(q) + \frac{U(q)}{T} \right) G_{\rho p}(q, t)G_{\rho p}(k - q, t) \\
& + \frac{T}{\rho_0} k^2 \int_q (k \cdot q) \left( 2U(q) + Tc(q) \right) G_{\rho p}(q, t)G_{\rho p}(k - q, t) \\
& = -\frac{T^2}{\rho_0} k^2 \int_q (k \cdot q)G_{\rho p}(q, t)G_{\rho p}(k - q, t) \\
& = (IV.25)
\end{align*}
One can also rewrite the first integral in (IV.23) as

\[ T^2 \int_q \left[ (k \cdot q)^2 c^2(q) + (k \cdot q)(k \cdot (k - q))c(q)c(k - q) \right] G_{\rho \rho}(q, t)G_{\rho \rho}(k - q, t) \]

\[ = T^2 \int_q \frac{1}{2}(k \cdot q)^2 c^2(q) + \frac{1}{2}(k \cdot (k - q))^2 c^2(k - q) + (k \cdot q)(k \cdot (k - q))c(q)c(k - q) \] \[ \cdot G_{\rho \rho}(q, t)G_{\rho \rho}(k - q, t) \]

\[ = \frac{T^2}{2} \int_q \left[ (k \cdot q)c(q) + (k \cdot (k - q))c(k - q) \right]^2 G_{\rho \rho}(q, t)G_{\rho \rho}(k - q, t) \quad \text{(IV.26)} \]

Using (IV.25) and (IV.26), we obtain final expression for \( \Sigma_{\hat{\rho} \hat{\rho}}(k, t) \) as

\[ \Sigma_{\hat{\rho} \hat{\rho}}(k, t) = \frac{T^2}{2} \int_q \left[ (k \cdot q)c(q) + (k \cdot (k - q))c(k - q) \right]^2 G_{\rho \rho}(q, t)G_{\rho \rho}(k - q, t) \]

\[ - \frac{T^2}{\rho_0^2} k^2 \int_q (k \cdot q)c(q)G_{\rho \rho}(q, t)G_{\rho \rho}(k - q, t) \quad \text{(IV.27)} \]

There are two remaining kernels, \( \Sigma_{\hat{\rho} \hat{\rho}}^{(1)} \), (IV.14) and \( \Sigma_{\hat{\rho} \hat{\rho}}^{(2)} \), (IV.18) in the convolution integral (IV.13). The sum of these kernels is given by

\[ \left( \Sigma_{\hat{\rho} \hat{\rho}}^{(1)} + \Sigma_{\hat{\rho} \hat{\rho}}^{(2)} \right)(k, t) = \frac{1}{\rho_0} \int_q k \cdot q U(q)G_{\rho \rho}(q, t)G_{\rho \rho}(k - q, t) - \frac{T}{\rho_0} \int_q k \cdot q G_{\hat{\rho} \hat{\rho}}(q, t)G_{\rho \rho}(k - q, t) \]

\[ = \frac{T}{\rho_0} \int_q k \cdot q c(q)G_{\rho \rho}(q, t)G_{\rho \rho}(k - q, t) \quad \text{(IV.28)} \]

where (III.113) and \( U(q) + T \left( \frac{1}{\rho_0^2} - K(q) \right) = -Tc(q) \) is used.

Summing up, we have obtained the following one-loop expressions of the self-energies in the non-perturbative dynamic equation for \( G_{\rho \rho}(k, t) \)

\[ \partial_t G_{\rho \rho}(k, t) = -\rho_0 \frac{T k^2}{S(k)} G_{\rho \rho}(k, t) + \int_0^t ds \left[ \Sigma_{\hat{\rho} \hat{\rho}}(k, t - s)G_{\rho \rho}(k, s) - \Sigma_{\hat{\rho} \hat{\rho}}(k, t - s)G_{\rho \rho}(k, s) \right] \]

\[ \Sigma_{\hat{\rho} \hat{\rho}}(k, t) = \frac{T^2}{2} \int_q \left[ V^2(k, q) - \frac{k^2}{\rho_0^2} V(k, q) \right] G_{\rho \rho}(q, t)G_{\rho \rho}(k - q, t), \]

\[ \Sigma_{\hat{\rho} \hat{\rho}}(k, t) = \frac{T}{2 \rho_0^2} \int_q V(k, q)G_{\rho \rho}(q, t)G_{\rho \rho}(k - q, t), \]

\[ V(k, q) = \left[ (k \cdot q)c(q) + (k \cdot (k - q))c(k - q) \right] \quad \text{(IV.29)} \]

B. The MCT equation

The equation (IV.29) takes a different form from the standard MCT equation. While the convolution integral in MCT contains only the time derivative of the density correlation function, the first part of the convolution integral of (IV.29) involves the density correlation function itself, instead
of its time derivative. Note that this structure is a nonperturbative feature, independent of one-loop calculations. It is also very interesting that in the convolution integral of (IV.29) there is an additional term which involves the time derivative of the density correlation function. One can appreciate below the importance of this term in recovering the standard MCT equation.

The difference between (IV.29) and the standard MCT is quite analogous to that between the reducible and irreducible memory functions [53] appearing in the projection operator approach to the dissipative stochastic systems. It has been shown for a general class of dissipative stochastic systems obeying the detailed balance condition [54] that the conventional memory function in the exact equation for the correlation function, which is obtained from the projection operator method, can be further reduced to the so-called irreducible memory function. In particular, the derived exact dynamic equation for the correlation function of the slow variable \(A(t)\) is given by

\[
\partial_t C_A(t) = -|E_A|C_A(t) + \int_0^t ds M_A(t-s) C_A(s),
\]

\(C_A^L(z) = C_A(0) \left[ z + |E_A| - M_A^L(z) \right]^{-1}
\]

(IV.30)

where \(|E_A|^{-1}\) is a characteristic short time scale in the system, and \(M_A(t)\) is the conventional memory function. The memory function \(M_A(t)\) turns out to be further reducible to the irreducible memory function \(M_{irr}^A(t)\):

\[
M_A(t) = M_{irr}^A(t) - |E_A|^{-1} \int_0^t ds M_A(t-s) M_{irr}^A(s),
\]

\[M_A^L(z) = \frac{M_{irr}^A(z)}{1 + |E_A|^{-1} M_{irr}^A(z)}
\]

(IV.31)

Note that \(M_{irr}^A(z = 0)\) can grow indefinitely when the global relaxation time grows indefinitely in contrast to \(M_A^L(z = 0)\). The above two eqs. lead to the dynamic eq. for \(C_A(t)\)

\[
\partial_t C_A(t) = -|E_A|C_A(t) - |E_A|^{-1} \int_0^t ds M_{irr}^A(t-s) \dot{C}_A(s)
\]

\[C_A^L(z) = C_A(0) \left[ z + \frac{|E_A|}{1 + |E_A|^{-1} M_{irr}^A(z)} \right]^{-1}
\]

(IV.32)

For dissipative systems with detailed balance like the one under consideration, the mode coupling approximation directly applied to the usual memory kernel \(M_A(t)\) in (IV.30) can lead to absurd results, which is not the case for the irreducible memory kernel \(M_{irr}^A(t)\) in (IV.32). See [54]. This is expected to be the same for loop expansions [55]: the dynamic equation for \(G_{\rho \rho}(k,t)\) with one-loop self-energies, (IV.29), is indeed likely to become unstable in the long-time region.
Invoking the irreducible memory function formulation, we rewrite (IV.29) into the corresponding form of (IV.32) where the convolution integral involves only the time derivative of the density correlation function:

\[ \partial_t G_{\rho\rho}(k, t) = -\frac{\rho_0 T k^2}{S(k)} G_{\rho\rho}(k, t) - \int_0^t ds \mathcal{M}(k, t - s) \dot{G}_{\rho\rho}(k, s) \]  

(IV.33)

The kernel \( \mathcal{M}(k, t) \), corresponding to the irreducible memory function (\( \mathcal{M}(k, t) = |E_A|^{-1} M_{ir}^A(t) \) with \( |E_A| = \rho_0 T k^2 / S(k) \)), satisfies the following equation:

\[ \mathcal{M}(k, t) = \tilde{\Sigma}_{\hat{\rho}\hat{\rho}}(k, t) + \Sigma_{\hat{\rho}\hat{\theta}}(k, t) + \int_0^t ds \mathcal{M}(k, t - s) \tilde{\Sigma}_{\hat{\rho}\hat{\rho}}(k, s) \]  

(IV.34)

where \( \tilde{\Sigma}_{\hat{\rho}\hat{\rho}}(k, t) \equiv \Sigma_{\hat{\rho}\hat{\rho}}(k, t) / (\rho_0 T k^2) \). It is interesting to recognize the sum of the first two terms in (IV.34) is nothing but the standard mode coupling kernel:

\[ \Sigma_{MC}(k, t) \equiv \tilde{\Sigma}_{\hat{\rho}\hat{\rho}}(k, t) + \Sigma_{\hat{\rho}\hat{\theta}}(k, t) = \frac{T}{2\rho_0} \int_q \left[ (\hat{k} \cdot q)c(q) + (\hat{k} \cdot (k - q))c(k - q) \right]^2 G_{\rho\rho}(q, t) G_{\rho\rho}(k - q, t) \]  

(IV.35)

where \( \hat{k} \equiv k/k \).

Now in (IV.34) when \( \mathcal{M}(k, t) \) is iterated, the convolution integral generates the terms \( \int_0^t ds \Sigma_{MC}(k, t - s) \tilde{\Sigma}_{\hat{\rho}\hat{\rho}}(k, s) + \cdots \). All these terms are higher-loop terms. We thus see that the structure of the theory becomes so simple that only the first contribution \( \mathcal{M}(k, t) = \Sigma_{MC}(k, t) \) retains the one-loop two-particle irreducible structure. Therefore up to the one-loop order, it is perfectly legitimate to take this one-loop contribution only, ignoring the terms generated by the convolution integral in (III.34). With \( \mathcal{M}(k, t) = \Sigma_{MC}(k, t) \), the equation (IV.33) reduces to the standard MCT equation:

\[ \partial_t G_{\rho\rho}(k, t) = -\frac{\rho_0 T k^2}{S(k)} G_{\rho\rho}(k, t) - \int_0^t ds \Sigma_{MC}(k, t - s) \dot{G}_{\rho\rho}(k, s) \]  

(IV.36)

V. SUMMARY AND DISCUSSION

A. Summary

A field theoretical model for interacting Brownian particle systems is analyzed with a view to develop a renormalized perturbation theory consistent with the FDR. This is made particularly transparent by introduction of a conjugate pair of auxiliary field variables thereby linearizing the TR transformation. There is a price to pay for this linearization: a logarithmic nonlinearity reappears in
the new dynamical action, the second memeber of (II.32), whereas the original dynamical action, the first memeber of (II.32), contains only polynomial nonlinearities. For non-interacting particle systems we recover a simple diffusion law as one expects, which is not the case in some recent works [16, 29]. For interacting particle cases, we recover the standard MCT in the one-loop order of the renormalized perturbation theory.

B. Discussion

Having worked out the one loop calculation, the next natural step is to undertake higher loop calculations. This is anticipated to be an audacious task: one needs to pay meticulous attention to all possible terms (or diagrams) up to desired higher order. Still this is worthwhile since no systematic calculation of corrections to the standard MCT is yet available which is crucial to theoretically assess successes and inadequacies of the standard MCT, and to push the theory beyond the current MCT.

In order to systematize higher order calculations, one needs to identify a proper smallness parameter (or parameters) of the expansion. Such an expansion parameter is denoted as $\lambda$ in (III.17), but is equated to unity afterwards [56].

For a Kac-type system the inverse force range is such a smallness parameter. However, this cannot be a whole story as indicated by failure of this approach to meet requirements of TR-invariance and FDR that ensues. Non-trivial effects proportional to the absolute temperature arising from nonlinearities of the non-interacting part of the action have to be properly included. Development of systematic expansion scheme for such model systems would be quite instructive. In future we may undertake extensions of our approach to treat multibody correlations [14, 19, 24, 25, 26]. And we may also attempt to extend our approach to genuinely non-equilibrium problems like aging [57] and colloids under external shear flow [58].

Theory of dynamic aspects, not static aspects, is common to old MCT. Its spirit is to predict dynamics with statics as input since dynamics is profoundly more difficult. Even this modest aim has not been achieved to our satisfaction.

Now, any satisfactory nonequilibrium theory must describe correctly the dynamics of equilibrium fluctuations. Our work is only a first step towards genuine nonequilibrium theory.

We can compare situation with well-understood critical dynamics. There the dynamical renormalization group theory (DRG) successfully combines Wilson’s RG theory of statics with the old MCT of critical dynamics [60]. The resulting DRG is the final theory where indeed statics and dynamics can
be treated on equal footing and is capable of predicting even minute details. Extension of the present formulation to genuine non-equilibrium situation should be possible. The problems like nonlinear rheology are now within our reach.

It is our hope to develop non-equilibrium theory of glassy dynamics where statics and dynamics can be treated on equal footing eventually. However, for that purpose one first needs a theory which correctly describes dynamics of equilibrium fluctuations. Sophisticated stage of statics of the current liquid state theory [61] gives us a hope to bring dynamics to comparable stage in future.

Before we end our discussion, it is worthwhile to make comments on the recent related works. Szamel reports in a recent paper [28] a diagrammatic formulation of interacting Brownian particles. It is a weak coupling expansion scheme (i.e., the expansion in terms of the bare propagator) for the hierarchical equations for the multi-density correlation functions. We understand possible usefulness of Szamel’s approach which does not seem to have problem with the FDR. In some way his (Andersen’s [59] as well) is closer to the original MCT where equilibrium properties are included as input. Nonetheless, we wonder how convenient is the calculation. Moreover, some uncontrolled aspects enter in obtaining the final hierarchical set of equations. Also the higher order structure is not revealed in the formulation. Although the formulation uses ‘one-loop approximation’, it is not a genuine loop expansion theory in contrast to our field theory formulation.

APPENDIX A: DERIVATION OF THE DIFFUSION EQUATION FROM THE ABL ACTION IN THE ABSENCE OF PARTICLE INTERACTION

We carefully look at the ABL action in the absence of interaction:

\[ S_{\text{ABL, id}}[\psi] = \int d\mathbf{r} \int dt \left\{ i\dot{\rho} \left[ \partial_t \rho - T \nabla \cdot \left( \rho \nabla \theta \right) \right] - T \rho (\nabla \dot{\rho})^2 + i\dot{\theta} \left( \theta - \frac{\delta \rho}{\rho_0} - f(\delta \rho) \right) \right\} \] (A.1)

First note that

\[ T \nabla \cdot (\rho \nabla \theta) \equiv \nabla \cdot (\rho \nabla \frac{\delta F_{\text{id}}}{\delta \rho} ) = T \nabla^2 \rho \] (A.2)

We rewrite the action (A.1) as

\[ S_{\text{ABL, id}}[\psi] = \int d\mathbf{r} \int dt \left\{ i\dot{\rho} \left[ \partial_t \rho - T \nabla^2 \rho + T \nabla^2 \rho - T \nabla \cdot (\rho \nabla \theta) \right] - T \rho (\nabla \dot{\rho})^2 + i\dot{\theta} \left( \theta - \frac{\delta \rho}{\rho_0} - f(\delta \rho) \right) \right\} \] (A.3)

where the two underlined terms cancel due to (A.2).

We make use of the following identities

\[ \langle \delta \rho(\mathbf{2}) \frac{\delta S_{\text{ABL, id}}[\psi]}{\delta \rho(\mathbf{1})} \rangle = 0, \quad \langle \delta \rho(\mathbf{2}) \frac{\delta S_{\text{ABL, id}}[\psi]}{\delta \theta(\mathbf{1})} \rangle = 0 \] (A.4)
where \( \mathbf{1} \equiv (r, t) \) and \( \mathbf{2} \equiv (0, 0) \). The first identity can be written explicitly as

\[
0 = \langle \delta \rho(\mathbf{2}) \frac{\delta S_{\text{id}}[\psi]}{\delta \rho(\mathbf{1})} \rangle = i \left( \frac{\partial}{\partial t} - T \nabla^2 \right) G_{\rho \rho}(\mathbf{1} - \mathbf{2}) + 2T \rho_0 \nabla^2 \langle \hat{\rho}(1) \delta \rho(\mathbf{2}) \rangle + 2T \langle \delta \rho(\mathbf{2}) \nabla \cdot (\delta \rho(\mathbf{1}) \nabla \hat{\rho}(1)) \rangle
\]  

(A.5)

where we used the fact that the sum of the two underlined terms in (A.3) vanishes. Similarly, using the second identity in (A.4), we obtain

\[
0 = \langle \delta \rho(\mathbf{2}) \frac{\delta S_{\text{id}}[\psi]}{\delta \theta(\mathbf{1})} \rangle = -i \rho_0 T \nabla^2 \langle \hat{\theta}(1) \delta \rho(\mathbf{2}) \rangle + i \langle \hat{\theta}(1) \delta \rho(\mathbf{2}) \rangle - i T \langle \delta \rho(\mathbf{2}) \nabla \cdot (\delta \rho(\mathbf{1}) \nabla \hat{\rho}(1)) \rangle
\]  

(A.6)

where cancellation of the underlined terms was not used. Since in (A.6), \( \langle \hat{\rho}(1) \delta \rho(\mathbf{2}) \rangle = \langle \hat{\theta}(1) \delta \rho(\mathbf{2}) \rangle = 0 \) for \( t > 0 \) by causality, we obtain

\[
\langle \delta \rho(\mathbf{2}) \nabla \cdot (\delta \rho(\mathbf{1}) \nabla \hat{\rho}(1)) \rangle = 0 \quad \text{for} \quad t > 0
\]  

(A.7)

The eqs (A.6), (A.7), and causality leads to

\[
\partial_t G_{\rho \rho}(r, t) = T \nabla^2 G_{\rho \rho}(r, t), \quad \text{for} \quad t > 0.
\]  

(A.8)

In this way, we recover the diffusion eq. from the ABL action.

In Sec. II.G, we have obtained ((II.46))

\[
\partial_t G_{\rho \rho}(r, t) = T \nabla^2 G_{\rho \rho}(r, t) + \rho_0 T \nabla^2 \left\langle f(\delta \rho(r, t) \delta \rho(0, 0)) \right\rangle + T \nabla \cdot \left\langle \delta \rho(r, t) \nabla \theta(r, t) \delta \rho(0, 0) \right\rangle
\]  

(A.9)

Here if the constraint \( \theta \equiv \delta \rho/\rho_0 + f(\delta \rho) \) is used, the last two terms cancel:

\[
\rho_0 T \nabla^2 f(\delta \rho) + T \nabla \cdot (\delta \rho \nabla \theta) = \rho_0 T \nabla \cdot \left( \nabla f + \frac{\delta \rho}{\rho_0} \nabla \theta \right) = \rho_0 T \nabla \cdot \left( \nabla \theta - \frac{\nabla \rho}{\rho_0} + \frac{\delta \rho}{\rho_0} \nabla \theta \right)
\]

\[
= \rho_0 T \nabla \cdot \left( \frac{\rho_0 \nabla \theta - \nabla \rho}{\rho_0} \right) = 0
\]  

(A.10)

where the last equality holds due to (A.2). Therefore (A.9) reduces to the diffusion eq. (A.8). It is extremely puzzling then why ABL’s own analysis does not yield (A.8) in the absence of particle interaction.
APPENDIX B: EQUAL TIME CORRELATIONS

1. Non-interacting particles

We begin with the microscopic density $\hat{\rho}(r) \equiv \sum_j \delta(r - r_j)$. This should not be confused with the field conjugate to $\rho$. Then

$$< \hat{\rho}(r) \hat{\rho}(r') > = \sum_{jl} < \delta(r - r_j) \delta(r' - r_l) > = \sum_{j=l} \cdots + \sum_{j \neq l} \cdots$$

$$= N \delta(r - r') < \delta(r - r_1) > + N(N - 1) < \delta(r - r_1) < \delta(r - r_2) >$$

$$= \rho_0 \delta(r - r') + \rho_0^2$$

(B.1)

Since $< \hat{\rho}(r) > = \rho_0$ we get

$$S_{\rho\rho}(r - r') = < \delta\hat{\rho}(r)\delta\hat{\rho}(r') > = \rho_0 \delta(r - r')$$

(B.2)

2. Interacting particles

We use (II.24)

$$\theta(r) = \beta \frac{\delta F_{id}}{\delta \rho(r)} - \frac{\delta \rho(r)}{\rho_0}$$

(B.3)

where $\beta \equiv 1/T$. We first show that the variable $\theta$ has zero average:

$$< \theta(r) > = \beta < \frac{\delta F_{id}}{\delta \rho(r)} > = \beta < \frac{\delta F}{\delta \rho(r)} > - \beta < \frac{\delta F_{int}}{\delta \rho(r)} > = \beta < \frac{\delta F}{\delta \rho(r)} >$$

$$= - \int d\{\rho\} \delta \frac{\delta}{\delta \rho(r)} e^{-\beta F} = 0$$

(B.4)

where we have used the fact that $\delta F_{int}/\delta \rho(r)$ is linear in $\delta \rho(r)$ whose average vanishes. Next we turn to another correlation:

$$S_{\rho\theta}(r - r') \equiv < \delta \rho(r) \theta(r') > = \left< \delta \rho(r) \left( \beta \frac{\delta F_{id}}{\delta \rho(r')} - \frac{\delta \rho(r')}{\rho_0} \right) \right>$$

$$= \left< \delta \rho(r) \beta \frac{\delta F}{\delta \rho(r')} \right> - \left< \delta \rho(r) \hat{K} * \delta \rho(r') \right>$$

$$= \delta(r - r') - \hat{K} * S_{\rho\rho}(r - r')$$

(B.5)

where $K(r) \equiv \left( \delta(r)/\rho_0 + \beta U(r) \right)$. The Fourier transform of (B.6) is given by

$$K(k)S_{\rho\rho}(k) + S_{\rho\theta}(k) = 1$$

$$K(k) = \frac{1}{\rho_0} + \frac{1}{T} U(k)$$

(B.6)
Note that since $S_{\rho\rho}(k) = \rho_0$ ((B.2)) in the noninteracting system ($U \equiv 0$), $S_{\rho\theta}(k)$ vanishes for the noninteracting case:

$$S_{\rho\theta}(k) = 1 - \frac{1}{\rho_0}\rho_0 = 0 \quad \text{for} \quad U = 0 \quad (B.7)$$

If at the outset the RY free energy functional (II.3) was used, then the function $K(k)$ in (B.6) is the inverse of the static structure factor $S_{\rho\rho}(k)$ since

$$K(k) = \frac{1}{\rho_0} - \frac{c(k)}{1} = \frac{1}{S_{\rho\rho}(k)} \quad (B.8)$$

This relation implies from (B.6) that the equal-time correlation $S_{\rho\theta}(k)$ vanishes even in the presence of the particle interaction:

$$S_{\rho\theta}(k) = 0 \quad (B.9)$$

Finally we consider

$$S_{\theta\theta}(r - r') = \langle \theta(r)\theta(r') \rangle = \langle \left( \frac{\beta}{\delta \rho(r)} - \hat{K} \ast \delta \rho(r) \right) \left( \frac{\beta}{\delta \rho(r')} - \hat{K} \ast \delta \rho(r') \right) \rangle$$

$$= \beta^2 \langle \frac{\delta F}{\delta \rho(r)} \frac{\delta F}{\delta \rho(r')} \rangle - 2\hat{K} \ast \delta(r - r') + \hat{K} \ast \hat{K} \ast S_{\rho\rho}(r - r')$$

$$= \beta \langle \frac{\delta^2 F}{\delta \rho(r) \delta \rho(r')} \rangle - 2\hat{K}(r - r') + \hat{K} \ast \hat{K} \ast S_{\rho\rho}(r - r') \quad (B.10)$$

This is different from the direct correlation functions as defined by

$$c^{(2)}(r - r') \equiv -\beta \frac{\delta^2 F_{\text{ex}}}{\delta \rho(r) \delta \rho(r')} \quad (B.11)$$

where $F_{\text{ex}} = F - F_{\text{id}}$ is the interaction part (excess part) of the full (i.e. renormalized with respect to fluctuations) density functional.

**APPENDIX C: CALCULATIONS OF $\Sigma_{\rho\rho}$ AND $\Sigma_{\rho\theta}$**

Here we derive the equations (IV.14) and (IV.18). Referring to (IV.3) for $\Sigma_{\rho\rho}$ and to (III.63) for the $V$’s we find
1. Derivation of (IV.14)

\[
\Sigma^{(1)}_{\bar{\rho}\bar{\theta}}(12) = \frac{1}{2} \int_{3456} V^\text{int}_{\bar{\rho}\bar{\theta}pp}(134)V^\text{int}_{\bar{\theta}pp}(256)G_{\rho\rho}(35)G_{\rho\rho}(46) \\
= \frac{1}{2} \int_{3456} (-i) \nabla_1 \cdot [\delta(13)\nabla_1 U(14) + \delta(14)\nabla_1 U(13)] \cdot \frac{i}{\rho_0^2}\delta(25)\delta(26)G_{\rho\rho}(35)G_{\rho\rho}(46) \\
= \frac{1}{2\rho_0^2} \int_{34} \nabla_1 \cdot [\delta(13)\nabla_1 U(14)]G_{\rho\rho}(32)G_{\rho\rho}(42) \\
= \frac{1}{\rho_0^2} \int_{34} \nabla_1 \cdot \left[ G_{\rho\rho}(12)\nabla_1 \hat{U} + G_{\rho\rho}(12) \right] (C.1)
\]

where \( \int_{3456} \equiv \int d3 \int d4 \int d5 \int d6 \), etc. With \( 1 = (r, t) \) and \( 2 = (0, 0) \), the spatial Fourier transform of (C.1) is given by

\[
\Sigma^{(1)}_{\bar{\rho}\bar{\theta}}(k, t) = -\frac{1}{\rho_0^2} \int_{q} k \cdot q U(q)G_{\rho\rho}(q, t)G_{\rho\rho}(k - q, t) (C.2)
\]

which is the first member of (IV.14). Next we find similarly from (IV.6) and (III.63)

\[
\Sigma^{(1)}_{\rho\rho}(12) = \frac{1}{2} \int_{3456} V^\text{int}_{\rho\rho}(134)V^\text{int}_{\rho\rho}(256)G_{\rho\rho}(35)G_{\rho\rho}(46) \\
= \frac{1}{2} \int_{3456} (-i) \nabla_1 \cdot [\delta(13)\nabla_1 U(14) + \delta(14)\nabla_1 U(13)] \\
\cdot (-i) \nabla_2 \cdot [\delta(25)\nabla_2 U(26) + \delta(26)\nabla_2 U(25)]G_{\rho\rho}(35)G_{\rho\rho}(46) \\
= -\int_{3456} \nabla_1^j [\delta(13)\nabla_1^j U(14)] \nabla_2^l [\delta(25)\nabla_2^l U(26)] \left[ G_{\rho\rho}(35)G_{\rho\rho}(46) + G_{\rho\rho}(36)G_{\rho\rho}(45) \right] (C.3)
\]

The first term of (C.3) is computed as

\[
\Sigma^{(1,a)}_{\rho\rho}(12) \equiv -\int_{3456} \nabla_1^j [\delta(13)\nabla_1^j U(14)] \nabla_2^l [\delta(25)\nabla_2^l U(26)] G_{\rho\rho}(35)G_{\rho\rho}(46) \\
= -\int_{56} \nabla_1^j [G_{\rho\rho}(15)\nabla_1^j \hat{U} + G_{\rho\rho}(16)] \nabla_2^l [\delta(25)\nabla_2^l U(26)] \\
= -\nabla_1^j \nabla_2^l [G_{\rho\rho}(12)\nabla_1^j \hat{U} + \hat{G}_{\rho\rho} * U(12)] \\
= -\nabla_1^j \nabla_2^l [G_{\rho\rho}(12)\nabla_1^j \hat{U} + \hat{G}_{\rho\rho} * U(12)] (C.4)
\]
where \( \hat{G} \) implies convolution with the function \( G(r) \) as in the case of \( \hat{U} \). The second term of (C.3) is similarly computed as

\[
\Sigma_{\hat{p}\hat{p}}^{(1,b)}(12) = -\int_{3456} \nabla_1^j \left[ \delta(13) \nabla_1^j U(14) \right] \nabla_2^l \left[ \delta(25) \nabla_2^l U(26) \right] G_{pp}(36) G_{pp}(45)
\]

\[
= -\int_{56} \nabla_1^j \left[ G_{pp}(16) \nabla_1^j \hat{U} * G_{pp}(15) \right] \nabla_2^l \left[ \delta(25) \nabla_2^l U(26) \right]
\]

\[
= -\nabla_1^j \nabla_2^l \left[ \delta(25) \nabla_2^l \hat{U} * G_{pp}(12) \right] \nabla_2^l G_{pp} * U(12)
\]

Adding up the above two terms we obtain

\[
\Sigma_{\hat{p}\hat{p}}^{(1)}(12) = \Sigma_{\hat{p}\hat{p}}^{(1,a)}(12) + \Sigma_{\hat{p}\hat{p}}^{(1,b)}(12)
\]

\[
= -\nabla_1^j \nabla_2^l \left[ G_{pp}(12) \nabla_1^j \hat{U} * G_{pp} * U(12) \right] - \nabla_1^j \nabla_2^l \left[ \nabla_2^l \hat{U} * G_{pp}(12) \nabla_1^j G_{pp} * U(12) \right]
\]

whose Fourier transform is given by

\[
\Sigma_{\hat{p}\hat{p}}^{(1)}(k,t) = -\int_q \left[ (k \cdot q)^2 U^2(q) + (k \cdot q)(k \cdot (k - q)) U(q) U(k - q) \right] G_{pp}(q,t) G_{pp}(k - q,t)
\]

which is the second member of (IV.14). Next we have

\[
\Sigma_{\hat{p}\hat{p}}^{(2)}(12) = \frac{1}{2} \int_{3456} V^{int}_{\hat{p}\hat{p}}(134) V^{id}_{\hat{p}\hat{p}}(256) G_{pp}(35) G_{pp}(46)
\]

\[
= \frac{1}{2} \int_{3456} (-i) \nabla_1 \left[ \delta(13) \nabla_1 U(14) + \delta(14) \nabla_1 U(13) \right]
\]

\[
\times \left( \frac{-iT}{\rho_0} \right) \nabla_2^l \left[ \delta(25) \delta(26) \right] G_{pp}(35) G_{pp}(46)
\]

\[
= \left( -\frac{T}{\rho_0} \right) \nabla_1 \left[ \delta(13) \nabla_1 U(14) \right] \nabla_2^l \left[ G_{pp}(32) G_{pp}(42) \right]
\]

\[
= \left( -\frac{T}{\rho_0} \right) \nabla_1 \nabla_2^l \left[ G_{pp}(12) \nabla_1 \hat{U} * G_{pp}(12) \right]
\]

\[
= \left( -\frac{T}{\rho_0} \right) \nabla_1^2 \left[ G_{pp}(12) \nabla_1 \hat{U} * G_{pp}(12) \right]
\]

The Fourier transform of (C.8) is given by

\[
\Sigma_{\hat{p}\hat{p}}^{(2)}(k,t) = -\frac{T}{\rho_0} k^2 \int_q k \cdot q U(q) G_{pp}(q,t) G_{pp}(k - q,t)
\]

which is the third member of (IV.14).
\[ \Sigma_{\rho \rho}^{(3)}(12) = \int_{3456} V_{\rho \rho}^{int}(134) V_{\rho \rho}(256) G_{\rho \rho}(35) G_{\rho \rho}(46) \]
\[ = \int_{3456} (-i) \nabla_1 \cdot [\delta(13) \nabla_1 U(14) + \delta(14) \nabla_1 U(13)] \]
\[ \times (-2T) \nabla_2 \delta(25) \cdot [\nabla_0 \delta(56)] G_{\rho \rho}(35) G_{\rho \rho}(46) \]
\[ = 2iT \int_{3456} \nabla_1 \cdot [\delta(13) \nabla_1 U(14)] \nabla_2 \delta(25) \cdot [\nabla_0 \delta(56)] \]
\[ \times [G_{\rho \rho}(35) G_{\rho \rho}(46) + G_{\rho \rho}(45) G_{\rho \rho}(36)] \quad (C.10) \]

The first integral of (C.10) is calculated as
\[ \Sigma_{\rho \rho}^{(3,a)}(12) = 2iT \int_{3456} \nabla_1^I \left[ \delta(13) \nabla_1^I U(14) \right] \nabla_2^I \delta(25) [\nabla_0^I \delta(56)] G_{\rho \rho}(35) G_{\rho \rho}(46) \]
\[ = -2iT \int_{3456} \nabla_1^I \left[ \delta(13) \nabla_1^I U(14) \right] \nabla_2^I \delta(25) G_{\rho \rho}(35) \nabla_5^I G_{\rho \rho}(45) \]
\[ = -2iT \int_{3456} \nabla_1^I \left[ \delta(13) \nabla_1^I U(14) \right] \nabla_2^I \left[ G_{\rho \rho}(32) \nabla_2^I G_{\rho \rho}(42) \right] \]
\[ = -2iT \nabla_1^I \nabla_2^I \left[ G_{\rho \rho}(12) \nabla_2^I \nabla_1 \hat{U} + G_{\rho \rho}(12) \right] \quad (C.11) \]

The second integral of (C.10) is similarly calculated as
\[ \Sigma_{\rho \rho}^{(3,b)}(12) = 2iT \int_{3456} \nabla_1^I \left[ \delta(13) \nabla_1^I U(14) \right] \nabla_2^I \delta(25) [\nabla_0^I \delta(56)] G_{\rho \rho}(45) G_{\rho \rho}(36) \]
\[ = -2iT \int_{3456} \nabla_1^I \left[ \delta(13) \nabla_1^I U(14) \right] \nabla_2^I \delta(25) G_{\rho \rho}(45) \nabla_5^I G_{\rho \rho}(35) \]
\[ = -2iT \int_{3456} \nabla_1^I \left[ \delta(13) \nabla_1^I U(14) \right] \nabla_2^I \left[ G_{\rho \rho}(42) \nabla_2^I G_{\rho \rho}(32) \right] \]
\[ = -2iT \nabla_1^I \nabla_2^I \left[ \nabla_1^I \hat{U} + G_{\rho \rho}(12) \nabla_2^I G_{\rho \rho}(12) \right] \quad (C.12) \]

Adding up (C.11) and (C.12), we obtain
\[ \Sigma_{\rho \rho}^{(3)}(12) = \Sigma_{\rho \rho}^{(3,a)}(12) + \Sigma_{\rho \rho}^{(3,b)}(12) \]
\[ = -2iT \nabla_1^I \nabla_2^I \left[ G_{\rho \rho}(12) \nabla_1 \nabla_2 \hat{U} + G_{\rho \rho}(12) \right] - 2iT \nabla_1^I \nabla_2^I \left[ \nabla_1^I \hat{U} + G_{\rho \rho}(12) \nabla_2^I G_{\rho \rho}(12) \right] \quad (C.13) \]

The Fourier transform of (C.13) is given by
\[ \Sigma_{\rho \rho}^{(3)}(k, t) = -2iT \int_{q} \left[ (k \cdot q)^2 U(q) + (k \cdot q)(k \cdot (k - q)) U(k - q) \right] G_{\rho \rho}(q, t) G_{\rho \rho}(k - q, t) \quad (C.14) \]

This is the 4th member of (IV.14).
\[ \sum_{\hat{\rho}\hat{\theta}}^{(7)}(12) = \int_{3456} V_{\hat{\rho}\hat{\theta}}^{int}(134)V_{\hat{\rho}\rho}(256)G_{\rho\rho}(35)G_{\rho\theta}(46) \]

\[
= \int_{3456} (-i)\nabla_1 \cdot \left[ \delta(13)\nabla_1 U(14) + \delta(14)\nabla_1 U(13) \right] \cdot (-iT)\nabla_2 \left[ \delta(25)\nabla_2 \delta(26) \right] G_{\rho\rho}(35)G_{\rho\theta}(46) \\
= (-T) \int_{3456} \nabla_2^l \left[ \delta(13)\nabla_2^l U(14) \right] \nabla_2^l \left[ \delta(25)\nabla_2^l \delta(26) \right] G_{\rho\rho}(35)G_{\rho\theta}(46) + G_{\rho\phi}(45)G_{\rho\phi}(36) \tag{C.15}
\]

The first integral of (C.15) is computed as

\[
\sum_{\hat{\rho}\hat{\theta}}^{(7,a)}(12) = (-T) \int_{3456} \nabla_2^l \left[ \delta(13)\nabla_2^l U(14) \right] \nabla_2^l \left[ \delta(25)\nabla_2^l \delta(26) \right] G_{\rho\rho}(35)G_{\rho\theta}(46) \\
= (-T) \int_{56} \nabla_2^l \left[ G_{\rho\rho}(15)\nabla_2^l \hat{U} * G_{\rho\theta}(16) \right] \nabla_2^l \left[ \delta(25) \nabla_2^l \delta(26) \right] \\
= (-T) \nabla_2^l \left[ \delta(13)\nabla_2^l U(12) \right] \nabla_2^l \left[ \delta(25) \nabla_2^l \delta(26) \right] G_{\rho\rho}(12)G_{\rho\theta}(12) \tag{C.16}
\]

Likewise the second integral of (C.15) is computed as

\[
\sum_{\hat{\rho}\hat{\theta}}^{(7,b)}(12) = (-T) \int_{3456} \nabla_2^l \left[ \delta(13)\nabla_2^l U(14) \right] \nabla_2^l \left[ \delta(25)\nabla_2^l \delta(26) \right] G_{\rho\rho}(45)G_{\rho\theta}(36) \\
= (-T) \nabla_2^l \left[ \delta(13)\nabla_2^l U(12) \right] \nabla_2^l \left[ \delta(25) \nabla_2^l \delta(26) \right] G_{\rho\rho}(12)G_{\rho\theta}(12) \tag{C.17}
\]

Adding up (C.16) and (C.17), we obtain

\[
\sum_{\hat{\rho}\hat{\theta}}^{(7)}(12) = \sum_{\hat{\rho}\hat{\theta}}^{(7,a)}(12) + \sum_{\hat{\rho}\hat{\theta}}^{(7,b)}(12) \\
= (-T) \nabla_2^l \left[ \delta(13)\nabla_2^l U(12) \right] \nabla_2^l \left[ \delta(25) \nabla_2^l \delta(26) \right] G_{\rho\rho}(12)G_{\rho\theta}(12) + (-T) \nabla_2^l \left[ \delta(13)\nabla_2^l U(12) \right] \nabla_2^l \left[ \delta(25) \nabla_2^l \delta(26) \right] G_{\rho\rho}(12)G_{\rho\theta}(12) \tag{C.18}
\]

The Fourier transform of (C.18) is given by

\[
\sum_{\hat{\rho}\hat{\theta}}^{(7)}(k, t) = -T \int_q \left[ (k \cdot q)^2 U(q) + (k \cdot q)(k \cdot (k - q))U(k - q) \right] G_{\rho\phi}(q, t)G_{\rho\rho}(k - q, t) \tag{C.19}
\]

This is the 5th member of (IV.14).
Adding up (C.21) and (C.22) leads to

\[
\sum^{(9.a)}_{\rho \theta}(12) = (-T) \int_{3456} \nabla^j_1 \left[ \delta(13) \nabla^j_1 \delta(14) \right] \nabla^j_2 \left[ \delta(25) \nabla^j_2 U(26) \right] G_{\rho \rho}(35) G_{\theta \rho}(46)
\]

\[
= (-T) \int_{56} \nabla^j_1 \left[ G_{\rho \rho}(15) \nabla^j_1 G_{\theta \rho}(16) \right] \nabla^j_2 \left[ \delta(25) \nabla^j_2 U(26) \right]
\]

\[
= (-T) \nabla^j_1 \nabla^j_2 \left[ G_{\rho \rho}(12) \nabla^j_1 \hat{G}_{\theta \rho} * U(12) \right]
\]

\[
= (-T) \nabla^j_1 \nabla^j_1 \left[ G_{\rho \rho}(12) \nabla^j_1 \hat{G}_{\theta \rho} * U(12) \right]
\]  

(C.21)

The second integral of (C.20) is calculated as

\[
\sum^{(9.b)}_{\rho \theta}(12) = (-T) \int_{3456} \nabla^j_1 \left[ \delta(13) \nabla^j_1 \delta(14) \right] \nabla^j_2 \left[ \delta(25) \nabla^j_2 U(26) \right] G_{\rho \rho}(36) G_{\theta \rho}(45)
\]

\[
= (-T) \int_{56} \nabla^j_1 \left[ G_{\rho \rho}(16) \nabla^j_1 G_{\theta \rho}(15) \right] \nabla^j_2 \left[ \delta(25) \nabla^j_2 U(26) \right]
\]

\[
= (-T) \nabla^j_1 \nabla^j_2 \left[ \nabla^j_1 G_{\theta \rho}(12) \nabla^j_2 \hat{G}_{\rho \rho} * U(12) \right]
\]

\[
= (-T) \nabla^j_1 \nabla^j_1 \left[ \nabla^j_1 G_{\theta \rho}(12) \nabla^j_1 \hat{G}_{\rho \rho} * U(12) \right]
\]  

(C.22)

Adding up (C.21) and (C.22) leads to

\[
\sum^{(9)}_{\rho \theta}(12) = \sum^{(9.a)}_{\rho \theta}(12) + \sum^{(9.b)}_{\rho \theta}(12)
\]

\[
= (-T) \nabla^j_1 \nabla^j_1 \left[ G_{\rho \rho}(12) \nabla^j_1 \hat{G}_{\theta \rho} * U(12) \right] + (-T) \nabla^j_1 \nabla^j_1 \left[ \nabla^j_1 G_{\theta \rho}(12) \nabla^j_1 \hat{G}_{\rho \rho} * U(12) \right]
\]  

(C.23)

The Fourier transform of (C.23) is given by

\[
\sum^{(9)}_{\rho \theta}(k, t) = -T \int_{q} \left[ (k \cdot q)^2 U(q) + (k \cdot q)(k \cdot (k - q))U(k - q) \right] G_{\rho \theta}(q, t) G_{\rho \rho}(k - q, t)
\]  

(C.24)

This completes the derivation of (IV.14). We move on to the derivation of (IV.18).

2. Derivation of (IV.18)

We start with \( \sum^{(3)}_{\rho \theta}(12) \), (IV.4).

\[
\sum^{(3)}_{\rho \theta}(12) = \int_{3456} V_{\rho \rho}(134) V_{\theta \theta}(256) G_{\rho \rho}(35) G_{\theta \rho}(46)
\]

\[
= \int_{3456} (-iT) \nabla^j_1 \left[ \delta(13) \nabla^j_1 \delta(14) \right] \cdot \frac{i}{\rho_0^2} \delta(25) \delta(26) G_{\rho \rho}(35) G_{\theta \rho}(46)
\]

\[
= \frac{T}{\rho_0^2} \int_{34} \nabla^j_1 \left[ \delta(13) \nabla^j_1 \delta(14) \right] G_{\rho \rho}(32) G_{\theta \rho}(42)
\]

\[
= \frac{T}{\rho_0^2} \nabla_1 \left[ G_{\rho \rho}(12) \nabla_1 G_{\theta \rho}(12) \right]
\]  

(C.25)
The Fourier transform of (C.25) is given by

$$\Sigma_{\tilde{\rho}\tilde{\rho}}^{(3)}(k, t) = -\frac{T}{\rho_0} \int_{q} k \cdot q G_{\theta\rho}(q, t) G_{\rho\rho}(k - q, t) \quad (C.26)$$

which is the first member of (IV.18).

The quantity $\Sigma_{\tilde{\rho}\tilde{\rho}}^{(10)}(12)$ can be obtained from $\Sigma_{\tilde{\rho}\tilde{\rho}}^{(9)}(12)$, (C.4), since the vertex $V^id(256)$ is obtained from $V^int(256)$ with $U(12)$ being replaced by $(T/\rho_0)\delta(12)$. We thus have

$$\Sigma_{\tilde{\rho}\tilde{\rho}}^{(10)}(k, t) = \left[ \Sigma_{\tilde{\rho}\tilde{\rho}}^{(9)}(k, t) \right]_{U(q) = U(k - q) = \frac{T}{\rho_0}}$$

$$= -\frac{T^2}{\rho_0} \int_{q} \left[ (k \cdot q)^2 + (k \cdot q)(k \cdot (k - q)) \right] G_{\theta\rho}(q, t) G_{\rho\rho}(k - q, t)$$

$$= -\frac{T^2}{\rho_0} k^2 \int_{q} k \cdot q G_{\theta\rho}(q, t) G_{\rho\rho}(k - q, t) \quad (C.27)$$

which is the second member of (IV.18).

From (IV.6)

$$\Sigma_{\tilde{\rho}\tilde{\rho}}^{(11)}(12) = \int_{3456} V_{\tilde{\rho}\tilde{\rho}}(134) V_{\tilde{\rho}\tilde{\rho}}(256) G_{\rho\rho}(35) G_{\theta\theta}(46)$$

$$= \int_{3456} (-iT) \nabla^l_1 \left[ \delta(13) \nabla^l_1 \delta(14) \right] (-iT) \nabla^l_2 \left[ \delta(25) \nabla^l_2 \delta(26) \right] G_{\rho\rho}(35) G_{\theta\theta}(46)$$

$$= (-T^2) \int_{34} \nabla^l_1 \left[ \delta(13) \nabla^l_1 \delta(14) \right] \nabla^l_2 \left[ G_{\rho\rho}(32) \nabla^l_2 G_{\theta\theta}(42) \right]$$

$$= (-T^2) \nabla^l_1 \nabla^l_1 \left[ G_{\rho\rho}(12) \nabla^l_1 \nabla^l_1 G_{\theta\theta}(12) \right] \quad (C.28)$$

whose Fourier transform is given by

$$\Sigma_{\tilde{\rho}\tilde{\rho}}^{(11)}(k, t) = -T^2 \int_{q} (k \cdot q)^2 G_{\theta\theta}(q, t) G_{\rho\rho}(k - q, t) \quad (C.29)$$

This is the third member of (IV.18).

Again from (IV.6)

$$\Sigma_{\tilde{\rho}\tilde{\rho}}^{(12)}(12) = \int_{3456} V_{\tilde{\rho}\tilde{\rho}}(134) V_{\tilde{\rho}\tilde{\rho}}(256) G_{\rho\rho}(35) G_{\theta\theta}(46)$$

$$= \int_{3456} V_{\tilde{\rho}\tilde{\rho}}(134) V_{\tilde{\rho}\tilde{\rho}}(256) G_{\rho\rho}(36) G_{\theta\rho}(45)$$

$$= (-T^2) \int_{3456} \nabla^l_1 \left[ \delta(13) \nabla^l_1 \delta(14) \right] \nabla^l_2 \left[ \delta(25) \nabla^l_2 \delta(26) \right] G_{\rho\rho}(36) G_{\theta\rho}(45)$$

$$= (-T^2) \int_{34} \nabla^l_1 \left[ \delta(13) \nabla^l_1 \delta(14) \right] \nabla^l_2 \left[ G_{\theta\rho}(42) \nabla^l_2 G_{\rho\rho}(32) \right]$$

$$= (-T^2) \nabla^l_1 \nabla^l_1 \left[ G_{\theta\rho}(12) \nabla^l_1 G_{\rho\rho}(12) \right] \quad (C.30)$$
whose Fourier transform is given by

\[ \Sigma_{\rho\rho}^{(12)}(k,t) = -T^2 \int_q (k \cdot q)(k \cdot (k - q))G_{\rho\rho}(q,t)G_{\rho\rho}(k - q,t) \quad (C.31) \]

which is the 4th member of (IV.18).

\[ \Sigma_{\rho\rho}^{(13)}(12) = \int_{3456} V_{\rho\rho\theta}(134) V_{\rho\rho\rho}(256) G_{\rho\rho}(35) G_{\theta\rho}(46) \]
\[ = \int_{3456} (-iT) \nabla_1^2 \left[ \delta(13) \nabla_1^2 \delta(14) \right] \cdot (-2T) \nabla_2^2 \delta(25) \nabla_6^2 \delta(56) G_{\rho\rho}(35) G_{\theta\rho}(46) \]
\[ = (-2iT^2) \int_{34} \nabla_1^2 \left[ \delta(13) \nabla_1^2 \delta(14) \right] \nabla_2^2 \left[ G_{\rho\rho}(32) \nabla_2 G_{\rho\rho}(42) \right] \]
\[ = (-2iT^2) \nabla_1^2 \left[ G_{\rho\rho}(12) \nabla_1 G_{\theta\rho}(12) \right] \quad (C.32) \]

The Fourier transform of (C.32) is given by

\[ \Sigma_{\rho\rho}^{(13)}(k,t) = -2iT^2 \int_q (k \cdot q)^2 G_{\rho\rho}(q,t)G_{\rho\rho}(k - q,t) \quad (C.33) \]

which is the 5th member of (IV.18).

Finally,

\[ \Sigma_{\rho\rho}^{(14)}(12) = \int_{3456} V_{\rho\rho\theta}(134) V_{\rho\rho\rho}(256) G_{\rho\rho}(35) G_{\theta\rho}(46) \]
\[ = \int_{3456} V_{\rho\rho\theta}(134) V_{\rho\rho\rho}(256) G_{\rho\rho}(36) G_{\theta\rho}(45) \]
\[ = \int_{3456} (-iT) \nabla_1^2 \left[ \delta(13) \nabla_1^2 \delta(14) \right] \cdot (-2T) \left[ \nabla_2^2 \delta(25) \nabla_6^2 \delta(56) \right] G_{\rho\rho}(36) G_{\theta\rho}(45) \]
\[ = (-2iT^2) \int_{34} \nabla_1^2 \left[ \delta(13) \nabla_1^2 \delta(14) \right] \nabla_2^2 \left[ G_{\theta\rho}(42) \nabla_2 G_{\rho\rho}(32) \right] \]
\[ = (-2iT^2) \nabla_1^2 \left[ \nabla_1 G_{\rho\rho}(12) \nabla_1 G_{\theta\rho}(12) \right] \quad (C.34) \]

The Fourier transform of (C.34) is given by

\[ \Sigma_{\rho\rho}^{(14)}(k,t) = -2iT^2 \int_q (k \cdot q)(k \cdot (k - q))G_{\theta\rho}(q,t)G_{\rho\rho}(k - q,t) \quad (C.35) \]

which is the last member of (IV.18). This completes the derivation of (IV.18).

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If $F[\rho]$ were independent of $T$, $\lambda$ can be identified with $T$ by change of variable as $\hat{\rho} \rightarrow \hat{\rho}/T$. However, this does not work since $F_{id}[\rho]$ depends on $T$ which brings in complications. If we can find some new non-perturbative results involving ideal gas such as the one we met around (III.26), that would be most helpful to discover smallness parameter (parameters).

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FIGURE CAPTIONS

Figure 1. Diagrammatic expression for $W_2[J]$ ((III.30)). The line denotes the propagator $G(\psi_c)$, and the crossing points with 3-branching lines and with 4-branching lines denote respectively the vertices $S_c^{(3)}$ and $S_c^{(4)}$. While the first two diagrams are 1PI diagrams, the last one is 1PR diagram.

Figure 2. Two and three loop diagrams for $\Gamma_{1PI}[\phi]$. The line denotes the propagator $G(\phi)$, and the crossing points with 3-branching lines and with 4-branching lines denote respectively the vertices $S^{(3)}(\phi)$ and $S^{(4)}(\phi)$. The two loop diagrams are both 1PI and 2PI diagrams. Other 2PI diagrams are the 3rd, 5th, and 6th diagrams.

Figure 3. Two and three loop diagrams for $\Gamma_{2PI}[\phi,G]$. Via the double Legendre transform the 2PR diagrams in Fig. 2 are eliminated in $\Gamma_{2PI}[\phi,G]$. Here the line denotes the full propagator $G$. In all figures, the vertices with more than 4-legs are not shown. These higher vertices do not contribute to the two-loop results for both $\Gamma_{1PI}[\phi]$ and $\Gamma_{2PI}[\phi]$, and hence do not contribute to the one-loop result for the self-energy.

Figure 4. One and two loop diagrams for the self-energy $\Sigma(12)$. These diagrams are obtained from those in Fig. 3 by cutting a single line.

Figure 5. The one-loop diagram for $\Sigma_{\hat{\theta}\hat{\theta}}(12)$.

Figure 6. The one-loop diagrams for $\Sigma_{\hat{\rho}\rho}(12)$. The filled circle in the vertex $\hat{\rho}\rho\rho\rho$ denotes the vertex $V^{int}_{\hat{\rho}\rho\rho\rho}$, which is to be distinguished from the vertex $V^{id}_{\hat{\rho}\rho\rho\rho}$ (without filled circle).

Figure 7. The one-loop diagrams for $\Sigma_{\hat{\theta}\rho}(12)$.

Figure 8. The one-loop diagrams for $\Sigma_{\hat{\rho}\rho}(12)$.
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