Spherical characters: the supercuspidal case

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This paper is dedicated to the memory of George Mackey.

ABSTRACT. We exhibit a basis for the space of spherical characters of a distinguished supercuspidal representation \( \pi \) of a connected reductive \( p \)-adic group, subject to the assumption that \( \pi \) is obtained via induction from a representation of an open compact mod centre subgroup. We derive an integral formula for each spherical character belonging to the the basis. This formula involves integration of a particular kind of matrix coefficient of \( \pi \). We also obtain a similar formula for the function realizing the spherical character. In addition, we determine, subject to some conditions, which of these spherical characters vanish identically on an open neighbourhood of the identity. We verify that the requisite conditions are always satisfied for distinguished tame supercuspidal representations of groups that split over tamely ramified extensions.

1. Introduction

The reader may refer to later sections for specific information regarding definitions and notation. Let \( G \) be a connected reductive \( p \)-adic group.

Let \( \theta \) be an involution of \( G \) and let \( H \) be a reductive \( p \)-adic group that is a subgroup of the group of fixed points \( G^\theta \) of \( \theta \) and contains the identity component of \( G^\theta \). Then \( G/H \) is a reductive \( p \)-adic symmetric space. Harmonic analysis on reductive \( p \)-adic symmetric spaces involves the study of \( H \)-biinvariant distributions on \( G \). Unlike harmonic analysis on real and complex reductive symmetric spaces, which has been studied extensively, there are many open questions in this area.

Many results in harmonic analysis on \( G \) have some kind of analogue in harmonic analysis on \( G/H \). In [RR], Rader and Rallis carried over to reductive \( p \)-adic symmetric spaces some results of Howe and many results of Harish-Chandra (mainly from the Queen’s lectures [HC2]). The \( H \)-biinvariant distributions that play a role analogous to that played in harmonic analysis on \( G \) by characters of irreducible admissible representations are the spherical characters of class one irreducible admissible representations. (See Section 5 for the definition of spherical character.)

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In this paper we study the basic properties of spherical characters of class one irreducible supercuspidal representations. Let $\pi$ be such a representation. In Theorem 6.1 we attach an $H$-biinvariant distribution $D_\varphi$ to each matrix coefficient $\varphi$ of $\pi$. If $\varphi$ does not have certain invariance properties, then $D_\varphi = 0$. Each $D_\varphi$ is a spherical character of $\pi$. Using results from [RR] (as stated in Theorem 5.1 of this paper), we obtain an integral formula for each spherical character $D_\varphi$ as a function on the $\theta$-regular set. Our integral formulas for $D_\varphi$, as a distribution and as a function on the $\theta$-regular set, are analogues of results of Harish-Chandra that express the ordinary character of an irreducible supercuspidal representation in terms of integration of a matrix coefficient of the representation (see Theorem 3.2).

Harish-Chandra's integral formulas for the ordinary characters of irreducible supercuspidal representations have proved useful both in computing character values and in deriving qualitative properties of characters. Our hope is that the formulas for spherical characters that we obtain in this paper will prove useful in a similar way. In related work, currently in progress, these formulas are being used in the study of germ expansions of spherical characters of distinguished supercuspidal representations.

Assume that $\pi$ is a class one irreducible supercuspidal representation that can be realized as an induced representation (as discussed in Section 7). A lemma of Hakim and Mao ([HMa]) describing the space of $H$-invariant linear functionals on the space of $\pi$ is used to prove that the space of spherical characters of $\pi$ is spanned by the distributions $D_\varphi$ as $\varphi$ ranges over matrix coefficients of $\pi$. We exhibit a set of matrix coefficients that give rise to a set of linearly independent spherical characters, thus obtaining a basis for the space of spherical characters of $\pi$ (see Theorem 8.5). We remark that certain of the examples discussed in [HM3] are examples of supercuspidal representations for which the space of spherical characters has dimension greater than one (see Remark 8.6).

In Section 9 subject to some conditions, we determine which of the spherical characters of $\pi$ vanish identically on the intersection of an open neighbourhood of the identity with the $\theta$-regular set. In particular, if the dimension of $H$-invariant linear functionals on the space of $\pi$ has dimension greater than one, there exist spherical characters of $\pi$ that exhibit this vanishing property. Note that this contrasts with the behaviour of ordinary characters of irreducible admissible representations, which never vanish identically on any open neighbourhood of the identity. At the end of the section, we state results from [HM3] which imply that the conditions needed for the vanishing results always hold for distinguished tame supercuspidal representations of groups that split over tamely ramified extensions of $F$.

2. General notation and definitions

Let $G$ be a reductive $p$-adic group. That is, $G$ is the group of $F$-rational points $G(F)$ of a reductive algebraic $F$-group $G$, where $F$ is a nonarchimedean local field. Because some of the results that we use in this paper have not been proved in the positive characteristic setting, we will assume that $F$ has characteristic zero. Thus $F$ is a finite extension of $\mathbb{Q}_p$, where $p$ is the residual characteristic of $F$.

We will assume that $G$ is connected. However, certain reductive subgroups of $G$ that appear in this paper may not be connected.

Let $\pi$ be a complex representation of $G$. A vector in the space $V$ of $\pi$ is said to be smooth if it is fixed by an open subgroup of $G$. A smooth representation
π is a representation having the property that every vector in V is smooth. If π is smooth and has the additional property that the space of K-fixed vectors in V is finite-dimensional for every compact open subgroup K of G, we say that π is admissible. An admissible representation π is supercuspidal whenever the matrix coefficients of π are compactly supported modulo the centre Z of G.

Let V∗ be the dual space of π and let π∗ be the representation of G that is dual to π. Let π be a smooth representation of G. The dual representation π∗ is not necessarily smooth. The contragredient ˜π of π is the restriction of π∗ to the subspace ˜V of smooth vectors in V∗, and is clearly a smooth representation. It is easy to see that π is admissible, resp. supercuspidal, if and only if ˜π is admissible, resp. supercuspidal.

The usual pairing on ˜V × V will be denoted by ⟨·, ·⟩. We identify V and ˜V via the isomorphism that takes a vector v in V to the smooth linear functional ˜v → ⟨v, ·⟩ on V. The pairing ⟨·, ·⟩ extends in the obvious way to the union of V∗ × V and ˜V × ˜V. The extension will also be denoted by ⟨·, ·⟩.

Let C∞ c(G) be the space of complex-valued, locally constant, compactly supported functions on G. A distribution on G is a linear functional on C∞ c(G). A G-invariant distribution on G is one that takes the same value on f and the function g → f(xgx−1) for all f ∈ C∞ c(G) and all g ∈ G.

If θ : G → G is an automorphism of order two that is defined over F, we say that θ is an involution of G. Given such an involution, let Gθ be the group of fixed points of θ in G, and let (Gθ)◦ be the identity component of Gθ. Let H be an F-subgroup of G such that (Gθ)◦ ⊂ H ⊂ Gθ. Let H = H(F). The space G/H is called a (reductive) p-adic symmetric space. A distribution on G is said to be H-invariant if it takes the same value on f and on the function g → f(h1gh2) for all f ∈ C∞ c(G) and all h1 and h2 ∈ H.

Let Greg be the set of regular elements of G. An element g of G belongs to Greg if and only if the identity component of the centralizer of g in G is a maximal torus. The set Greg is open and dense in G.

A torus T in G is called θ-split if θ(t) = t−1 for all t ∈ T. Let θ be an involution of G. The set Gθ-reg of θ-regular elements in G consists of the elements g in G having the property that the intersection of the centralizer of gθ(g)−1 in G with the connected component of the identity in {x ∈ G | θ(x) = x−1} is a maximal θ-split torus in G. The set Gθ-reg is open and dense in G.

Haar measure on a unimodular locally compact group G will be denoted by dg. If G1 is a closed unimodular subgroup of G, then dg G1 will denote a G-invariant measure on the coset space G/G1.

3. Ordinary characters

Let π be an admissible representation of G. If f ∈ C∞ c(G), then the operator π(f) = ∫G f(g)π(g) dg has finite rank. The character Θπ of π is the G-invariant distribution defined by Θπ(f) = trace π(f), f ∈ C∞ c(G).

The first part of the following theorem was originally proved by Harish-Chandra (HC2) for G connected. Although we will not need it here, we remark that it has since been generalized to disconnected G by Clozel (C). The second part of the theorem was proved by Harish-Chandra for K a good maximal compact subgroup, and was generalized to arbitrary open compact subgroups by Rader and Silberger (RS).
Theorem 3.1. Let $\pi$ be an admissible finite-length representation of $G$.

1. The character distribution $\Theta_\pi$ is given by integration against a locally integrable function (also denoted by $\Theta_\pi$ and called the character of $\pi$) on $G$. The function $\Theta_\pi$ is locally constant on $G^{\text{reg}}$.

2. Let $K$ be an open compact subgroup of $G$. Then $\int_K \pi(\kappa g k^{-1}) dk$ has finite rank whenever $g \in G^{\text{reg}}$. If Haar measure on $K$ is normalized so that $K$ has volume one, then

$$\Theta_\pi(g) = \text{trace} \left( \int_K \pi(\kappa g k^{-1}) dk \right), \quad g \in G^{\text{reg}}.$$ 

Theorem 3.2. (HC1) Let $\pi$ be an irreducible supercuspidal representation of $G$ and let $d(\pi)$ be the formal degree of $\pi$. If $\varphi$ is a matrix coefficient of $\pi$, then

1. $\varphi(1) \Theta_\pi(g) = d(\pi) \int_{G/Z} \int_K \varphi(x \kappa g x^{-1}) dx \, dk, \quad g \in G^{\text{reg}}.$

2. $\varphi(1) \Theta_\pi(f) = d(\pi) \int_{G/Z} \int_G f(g) \varphi(x g x^{-1}) dg \, dx, \quad f \in C_c^\infty(G).$

Remark 3.3. The above theorem was generalized to discrete series characters by Rader and Silberger ([RS]).

4. Distinguished representations and $p$-adic symmetric spaces

Let $\pi$ be a smooth representation of $G$. Let $\text{Hom}_H(\pi, 1)$ be the space of $H$-invariant elements in the dual $V^*$ of the space $V$ of $\pi$. The representation $\pi$ is said to be distinguished (or $H$-distinguished) if $\text{Hom}_H(\pi, 1)$ is nonzero.

Irreducible admissible representations $\pi$ having the property that both $\pi$ and $\tilde{\pi}$ are distinguished are referred to as class one representations. The supercuspidal representations that we study in this paper have the property that $\text{Hom}_H(\pi, 1)$ is isomorphic to $\text{Hom}_H(\tilde{\pi}, 1)$ (see Remark 8.2). Hence such representations are distinguished if and only if they are class one.

Information on the structure of reductive $p$-adic symmetric spaces may be found in the papers [HH], [HW] and [RR]. It is customary to identify $H$-invariant distributions on $G/H$ with $H$-biinvariant distributions on $G$ via composition with the map $f \mapsto \bar{f}$ where $f(gH) = \int_H f(gh) \, dh, \ f \in C_c^\infty(G), \ g \in G$.

The spherical characters of class one representations play a key role in harmonic analysis on $G/H$. Because the dimension of $\text{Hom}_H(\pi, 1)$ can be greater than one, the dimension of the space of spherical characters of a representation $\pi$ can be greater than one. Some examples of supercuspidal representations for which $\dim \text{Hom}_H(\pi, 1)$ exceeds one are given in [HM3].

The reader may consult the introduction of [RR] for a discussion of the connections between harmonic analysis on $p$-adic symmetric spaces and the relative trace formula, including nonvanishing of period integrals of automorphic forms.

A study of $\text{Hom}_H(\pi, 1)$ for representations $\pi$ arising via parabolic induction has been carried out by Blanc and Delorme ([BD]).

Distinguished supercuspidal representations have been studied by various researchers. The papers [H1], [H2], [H3], [HMa], [HM1], [HM2], [HM3], [Pr1] and [Pr2] (as well as others not mentioned here) study distinguished supercuspidal...
representations using approaches that involve realizing supercuspidal representations as induced representations (in the sense discussed in Section 2). The most general cases are treated in [HM3], which makes a detailed study of $\text{Hom}_H(\pi, 1)$ for the supercuspidal representations constructed by Yu ([Y]).

Some authors (see, for example, [AKT] and [Ka]) show that in certain cases existence of poles of $L$-functions can determine whether representations are distinguished. This is related to the fact that in some contexts the set of distinguished representations arising via some sort of functorial lift. See, for example, [AR], [AT], [E] and [HM1].

The papers [H1], [H2] and [H3] contain results on spherical characters for particular examples.

We remark that, because of our focus on the $p$-adic case, we are not mentioning papers that treat distinguished automorphic representations exclusively.

5. Spherical characters - definition and basic properties

Let $\pi$ be an irreducible admissible representation of $G$ and let $V$ be the space of $\pi$. Let $\lambda_{\pi} \in \hat{V}$. If $f \in C_c^\infty(G)$, define $\pi(f)\lambda_{\pi} \in \hat{V}$ by

$$\langle \hat{v}, \pi(f)\lambda_{\pi} \rangle = \langle \pi(\hat{f})\hat{v}, \lambda_{\pi} \rangle, \quad \hat{v} \in \hat{V},$$

where $\hat{f} \in C_c^\infty(G)$ is defined by $\hat{f}(g) = f(g^{-1})$, $g \in G$. It is a simple matter to check that $\pi(f)\lambda_{\pi}$ is smooth for every $f \in C_c^\infty(G)$, that is, $\pi(f)\lambda_{\pi} \in \hat{V}$. Hence, since we are identifying $V$ and $\hat{V}$, we may (and do) identify $\pi(f)\lambda_{\pi}$ with an element of $V$. Let $\lambda_{\pi} \in V^*$. The map

$$f \mapsto \langle \lambda_{\pi}, \pi(f)\lambda_{\pi} \rangle, \quad f \in C_c^\infty(G),$$

defines a distribution on $G$. These types of distributions can be viewed as generalizations of those distributions given by integration against matrix coefficients of representations.

If $\pi$ is a class one representation, the spherical characters of $\pi$ are the linear combinations of distributions of the above form that are $H$-biinvariant. Let $\lambda_{\pi}$ and $\lambda_{\pi}$ be nonzero elements of $\text{Hom}_H(\pi, 1)$ and $\text{Hom}_H(\pi, 1)$, respectively. Set

$$\Phi_\pi(f) = \langle \lambda_{\pi}, \pi(f)\lambda_{\pi} \rangle, \quad f \in C_c^\infty(G).$$

It follows from $H$-invariance of $\lambda_{\pi}$ and $\lambda_{\pi}$ that $\Phi_\pi$ is $H$-biinvariant. We will refer to $\Phi_\pi$ as the spherical character of $\pi$ associated to the pair of linear functionals $\lambda_{\pi}$ and $\lambda_{\pi}$.

The statements in the second part of the following theorem can be viewed as analogues for spherical characters of the results for ordinary characters stated in Theorem 3.1. However, as indicated in [RR], the function that realizes the spherical character is not generally locally integrable on $G$. Let $C_c^\infty(G^{\theta-reg})$ be the space of complex-valued, compactly supported, locally constant functions on $G^{\theta-reg}$.

**Theorem 5.1.** ([RR]) Let $\Phi_\pi$ be the spherical character of $\pi$ that is associated to elements $\lambda_{\pi} \in \text{Hom}_H(\pi, 1)$ and $\lambda_{\pi} \in \text{Hom}_H(\pi, 1)$. Let $K$ be a compact open subgroup of $G$.

1. If $g \in G^{\theta-reg}$, then $\int_{K \cap H} \pi^*(kg^{-1})\lambda_{\pi} \, dk$ lies in $\hat{V}$. The function $g \mapsto \int_{K \cap H} \pi^*(kg^{-1})\lambda_{\pi} \, dk$ is a $C^\infty$ function from $G^{\theta-reg}$ to $\hat{V}$. 
6. Matrix coefficients and spherical characters of supercuspidal representations

Throughout this section we assume that $\pi$ is an irreducible supercuspidal representation having the property that the central quasicharacter $\chi_{\pi}$ of $\pi$ is trivial on $H\cap Z$. Note that $\pi$ cannot be distinguished if it does not have this property.

The restriction of a matrix coefficient of $\pi$ to $H$ is $H \cap Z$-biinvariant and compactly supported modulo $H \cap Z$. Let $V$ be the space of $\pi$. Given $v \in V$, define $\tilde{v} \in V^{\ast}$ by

$$\langle \lambda_{\tilde{v}}, v \rangle = \int_{H/H \cap Z} \langle \tilde{v}, \pi(h)v \rangle \, dh^{\times}, \quad v \in V.$$  

Note that $\lambda_{\tilde{v}} \in \text{Hom}_{H}(\pi, 1)$. Later in the paper we describe which vectors $\tilde{v}$ give rise to nonzero elements of $\text{Hom}_{H}(\pi, 1)$ (when $\pi$ is an induced representation).

Similarly, if we fix $v \in V$, we can define $\lambda_{v} \in \text{Hom}_{H}(\tilde{\pi}, 1)$ by

$$\langle \tilde{v}, \lambda_{v} \rangle = \int_{H/H \cap Z} \langle \tilde{\pi}(h)\tilde{v}, v \rangle \, dh^{\times}, \quad \tilde{v} \in \tilde{V}.$$  

Fix $v_{0} \in V$ and $\tilde{v}_{0} \in \tilde{V}$. Let $\varphi(g) = \langle \tilde{v}_{0}, \pi(g)v_{0} \rangle$, $g \in G$. As we show below, we may use the matrix coefficient $\varphi$ to define an $H$-biinvariant distribution on $G$, and this distribution is a spherical character of $\pi$ which is nonzero if and only if $\lambda_{\tilde{v}_{0}}$ and $\lambda_{v_{0}}$ are both nonzero. Furthermore, the formula of Rader and Rallis (Theorem 3.2.12) converts into an integral formula for this spherical character (see (4) below). The latter formula can be viewed as an analogue for spherical characters of Harish-Chandra’s integral formula for the ordinary character of a supercuspidal representation (see Theorem 3.2.11). Similarly, the expression for the distribution $\varphi(\varphi)$ in terms of integration of a matrix coefficient is an analogue of the one in Theorem 3.2.12.

**Theorem 6.1.** Let $\pi$ and $\varphi$ be as above.

1. The map $f \mapsto D_{\varphi}(f) = \int_{H/H \cap Z} \int_{H/H \cap Z} \int_{G} f(g) \varphi(h_{1}gh_{2}) \, dg \, dh_{1}^{\times} \, dh_{2}^{\times}$, $f \in C_{c}^{\infty}(G)$, defines an $H$-biinvariant distribution on $G$.

2. Fix $v_{1} \in V$ and $\tilde{v}_{1} \in \tilde{V}$. Let $\varphi(g) = \langle \tilde{v}_{1}, \pi(g)^{-1}v_{1} \rangle$, $g \in G$. Suppose that $f \in C_{c}^{\infty}(G)$ has the property that $\varphi(g) = \int_{Z} f(gz)\chi_{\pi}(z) \, dz$ for all $g \in G$. Then

$$D_{\varphi}(f) = d(\pi)^{-1} \langle \lambda_{v_{0}}, v_{1} \rangle \langle \tilde{v}_{1}, \lambda_{\tilde{v}_{0}} \rangle.$$  

3. Let $\Phi_{\pi}$ be the spherical character of $\pi$ associated to $\lambda_{v_{0}}$ and $\lambda_{\tilde{v}_{0}}$. Then $\Phi_{\pi} = D_{\varphi}$. Moreover, $\Phi_{\pi}$ is nonzero if and only if $\lambda_{v_{0}}$ and $\lambda_{\tilde{v}_{0}}$ are nonzero.

4. Let $K$ be a compact open subgroup of $G$. Normalize Haar measure on $K \cap H$ so that $K \cap H$ has volume one. Then

$$\Phi_{\pi}(g) = \int_{H/H \cap Z} \int_{K \cap H} \int_{H/H \cap Z} \varphi(h_{2}gh_{1}) \, dh_{2}^{\times} \, dk \, dh_{1}^{\times}, \quad g \in G^{0\text{-reg}}.$$  

Remark 6.2. (1) If we combine (2) and the first part of (3), we obtain a special case of Lemma 4 of [14].

(2) Clearly we may extend the definition of $D_\varphi$ to any function $\varphi$ belonging to the span of the matrix coefficients of $\pi$. For each such function $\varphi$, $D_\varphi$ belongs to the space of spherical characters of $\pi$.

Proof. Fix $f \in C^\infty_c(G)$. Define $\psi(h_1, h_2) = \langle \tilde{\psi}_0, \pi(h_1)\pi(f)\pi(h_2)v_0 \rangle$, $h_1, h_2 \in H$. Note that $\psi$ is $H \cap Z$-biinvariant. Since $\pi(f)$ has finite rank, the span of the vectors $\pi(f)\pi(h_2)v_0$, as $h_2$ ranges over $H$, is finite-dimensional. Thus there is a finite set of matrix coefficients of $\pi$ such that the span of their restrictions to $H$ contains all of the functions $h_1 \mapsto \psi(h_1, h_2)$, as $h_2$ varies in $H$. It follows that there exists a compact subset $C_1$ of $H$ such that the support of each of the functions $h_1 \mapsto \psi(h_1, h_2)$ lies inside $C_1(H \cap Z) \times H$.

Since $\psi(h_1, h_2) = \langle \tilde{\pi}(\tilde{f})\tilde{\pi}(h_1)\tilde{v}_0, \pi(h_2)v_0 \rangle$, we may use the fact that $\tilde{\pi}(\tilde{f})$ has finite rank to see that there exists a compact subset $C_2$ of $H$ such that the support of each function $h_2 \mapsto \psi(h_1, h_2)$ lies inside $C_2(H \cap Z)$.

It now follows that $\psi$ is supported in $C_1(H \cap Z) \times C_2(H \cap Z)$. That is, $\psi$ has compact support modulo $(H \cap Z) \times (H \cap Z)$. Therefore the integral

$$\int_{H/H \cap Z} \int_{H/H \cap Z} \psi(h_1, h_2) \, dh_1^x \, dh_2^x$$

converges. Because $\psi(h_1, h_2) = \int_G \varphi(h_1gh_2) \, dg$, the above integral is equal to $D_\varphi(f)$. It is clear from the form of $D_\varphi(f)$ that $D_\varphi$ is $H$-biinvariant.

Next, fix $f \in C^\infty_c(G)$ as in (2). Let $h_1, h_2 \in H$. Then

$$\int_G \varphi(h_1gh_2) \, dg = \int_{G/Z} \int_Z \varphi(gz) \varphi(h_1gh_2) \, dz \, dg^x = \int_{G/Z} \varphi(h_1gh_2) \, \varphi(g) \, dg^x$$

$$= \int_{G/Z} \langle \tilde{\pi}(h_1^{-1})\tilde{v}_0, \pi(g)\pi(h_2)v_0 \rangle \langle \tilde{v}_1, \pi(g^{-1})v_1 \rangle \, dg^x$$

$$= d(\pi)^{-1} \langle \tilde{\pi}(h_1^{-1})\tilde{v}_0, v_1 \rangle \langle \tilde{v}_1, \pi(h_2)v_0 \rangle.$$  

Note that the fourth equality above is obtained via an application of the orthogonality relations for matrix coefficients of quasi-discrete series representations. In view of the above, it now follows from the definitions of $D_\varphi$, $\lambda_{\tilde{\psi}_0}$ and $\lambda_{v_0}$ that (for this particular choice of $f$) $D_\varphi(f)$ has the form given in (2).

Now we fix an arbitrary $f \in C^\infty_c(G)$. Then

$$\Phi_\pi(f) = \langle \lambda_{\tilde{\psi}_0}, \pi(f)\lambda_{v_0} \rangle = \int_{H/H \cap Z} \langle \tilde{\psi}_0, \pi(h_1)\pi(f)\lambda_{v_0} \rangle \, dh_1^x$$

$$= \int_{H/H \cap Z} \langle \tilde{\pi}(\tilde{f})\pi(h_1^{-1})\tilde{v}_0, \lambda_{v_0} \rangle \, dh_1^x$$

$$= \int_{H/H \cap Z} \int_{H/H \cap Z} \langle \tilde{\pi}(\tilde{f})\pi(h_1^{-1})\tilde{v}_0, \pi(h_2)v_0 \rangle \, dh_1^x \, dh_2^x = D_\varphi(f).$$

The assertion about nonvanishing of $\Phi_\pi$ is an immediate consequence of $\Phi_\pi = D_\varphi$ and (2).
Fix $g \in G^{\text{reg}}$. Let $K$ be as in the statement of the theorem. Applying Theorem 5.1(2) and using the definitions of $\lambda_{\tilde{v}_0}$ and $\lambda_{v_0}$, we find that

$$\Phi_\pi(g) = \left\langle \int_{K \cap H} \pi^*(kg^{-1})\lambda_{\tilde{v}_0} \, dk, \lambda_{v_0} \right\rangle$$

$$= \int_{H/H \cap Z} \left\langle \int_{K \cap H} \pi^*(kg^{-1})\lambda_{\tilde{v}_0} \, dk, \pi(h_1)v_0 \right\rangle \, dh_1$$

$$= \int_{H/H \cap Z} \int_{K \cap H} \langle \lambda_{\tilde{v}_0}, \pi(gk^{-1}h_1)v_0 \rangle \, dk \, dh_1$$

$$= \int_{H/H \cap Z} \int_{H/H \cap Z} \langle \tilde{v}_0, \pi(h_2gk^{-1}h_1)v_0 \rangle \, dh_2 \, dk \, dh_1.$$

Upon making the change of variables $k \mapsto k^{-1}$, we obtain the desired expression for $\Phi_\pi(g)$. \qed

7. Supercuspidal representations as induced representations

Before discussing spherical characters of supercuspidal representations in more detail, we make some remarks about irreducible supercuspidal representations that can be realized as induced representations.

Let $J$ be an open subgroup of $G$ that contains $Z$ and has the property that $J/Z$ is compact. Let $\kappa$ be an irreducible smooth representation of $J$. Because $J/Z$ is compact, $\kappa$ is finite-dimensional. We denote the space of $\kappa$ by $W$. Let $\pi = \text{Ind}_J^G \kappa$ be the smooth representation of $G$ that is obtained via compact induction from the representation $\kappa$. The space $V$ of $\pi$ is the set of functions $f$ from $G$ to $W$ satisfying

- $f(jg) = \kappa(j)f(g)$ for all $j \in J$ and $g \in G$
- The support of $f$ lies inside a finite union of right cosets of $J$ in $G$.

If $f \in V$ and $g \in G$, then $(\pi(g)f)(x) = f(xg)$ for all $x \in G$.

If $g \in G$, let $J^g = J \cap gjg^{-1}$. Define a representation $\kappa^g$ of $J^g$ by $\kappa^g(x) = \kappa(g^{-1}xg)$, $x \in J^g$. It is known that $\pi$ is irreducible if and only if for every $g \in G - J$ the representations $\kappa^g$ and $\kappa | J^g$ have no constituents in common. We remark that this irreducibility criterion takes the same form as Mackey’s irreducibility criterion for induced representations of finite groups.

Any smooth irreducible representation is admissible. Observe that $\pi$ has matrix coefficients that are compactly supported modulo $Z$. Hence, whenever $\pi$ is irreducible, $\pi$ is supercuspidal.

It is conjectured that every irreducible supercuspidal representation of a connected reductive $p$-adic group $G$ has the above form (for some choice of $J$ and $\kappa$). This conjecture has been verified for many groups. We do not take the time for a detailed discussion of cases for which the conjecture has been verified. We remark that if $G$ is a general linear group, the conjecture follows from work on Bushnell and Kutzko (BK) on parametrizing the admissible dual of $G$. Also, if $G$ is a connected reductive $p$-adic group that splits over a tamely ramified extension of $F$ and satisfies some tameness hypotheses, J.-L. Kim (K) has shown that the tame supercuspidal representations of $G$ constructed by Yu (Y) exhaust the irreducible supercuspidal representations of $G$. 
8. Spherical characters: the supercuspidal case

Suppose that \( \pi \) is an irreducible supercuspidal representation of \( G \) that is of the form \( \text{Ind}_J^G \kappa \) (as discussed in the previous section). In the first part of this section, we state a result due to Hakim and Mao that gives a direct sum decomposition of the space \( \text{Hom}_H(\pi, 1) \). Let \( W \) and \( \widetilde{W} \) be the spaces of \( \kappa \) and of \( \widetilde{\kappa} \), respectively. If \( J_1 \) is a subgroup of \( J \), we denote the spaces of \( J_1 \)-fixed vectors in \( W \) and \( \widetilde{W} \) by \( W^{J_1} \) and \( \widetilde{W}^{J_1} \), respectively. Each of the summands in the decomposition of \( \text{Hom}_H(\pi, 1) \) is isomorphic to \( \widetilde{W}^{J \cap gHg^{-1}} \) for some \( g \in G \). As indicated below, this implies that \( \text{Hom}_H(\pi, 1) \) is spanned by elements of the form \( \lambda_{\tilde{v}_0} \) (as defined in Section 6) for particular kinds of vectors \( \tilde{v}_0 \in \widetilde{V} \). As a consequence, each spherical character of \( \pi \) is of the form \( D_\varphi \), where \( \varphi \) is a finite linear combination of matrix coefficients of \( \pi \).

At the end of the section, we demonstrate that certain spherical characters of \( \pi \) are linearly independent. This is used to show that the dimension of the space of spherical characters of \( \pi \) is the square of the sum of the dimensions of the spaces \( W^{J \cap gHg^{-1}} \) as \( g \) ranges over a set of representatives for the \( J-H \) double cosets in \( G \).

The contragredient representation \( \pi^\vee \) may (and will) be realized as \( \text{Ind}_J^G \widetilde{\kappa} \). Hence \( \widetilde{V} \) is realized as the set of functions \( f \) from \( G \) to \( \widetilde{W} \) that have the same support properties as functions in \( V \), and satisfy \( f(jg) = \widetilde{\kappa}(j)f(g) \) for all \( j \in J \) and \( g \in G \).

Let \( \langle \cdot, \cdot \rangle_W \) be the usual pairing of \( \widetilde{W} \) and \( W \). We normalize the invariant measure on \( G/J \) in such a way that the pairing \( \langle \cdot, \cdot \rangle \) of \( \widetilde{V} \) with \( V \) is given by

\[
\langle \tilde{f}, f \rangle = \int_{G/J} \langle \tilde{f}(g), f(g) \rangle_W \, dg^X, \quad \tilde{f} \in \widetilde{V}, f \in V.
\]

**Lemma 8.1.** (HMa) Then

\[
\text{Hom}_H(\pi, 1) \simeq \bigoplus_{g \in J \cap G/H} \widetilde{W}^{J \cap gHg^{-1}},
\]

where the sum is over a set of representatives for the \( J-H \) double cosets in \( G \).

**Remark 8.2.** As \( \widetilde{W} \simeq W \) and \( \text{Ind}_J^G \kappa \) is isomorphic to \( \widetilde{W}^{J \cap gHg^{-1}} \) for all \( g \in G \), it follows from the above lemma that \( \text{Hom}_H(\pi, 1) \simeq \text{Hom}_H(\widetilde{\pi}, 1) \). In particular, \( \pi \) is class one if and only if \( \pi \) is distinguished.

Suppose \( g \in G \) is such that \( \widetilde{W}^{J \cap gHg^{-1}} \neq 0 \). Let \( \tilde{w} \) be a nonzero element of \( \text{Ind}_J^G \kappa \). The element \( \lambda_{\tilde{w}} \in \text{Hom}_H(\pi, 1) \) that corresponds to \( \tilde{w} \) under the isomorphism of Lemma 8.1 is defined by

\[
\langle \lambda_{\tilde{w}}, f \rangle = \int_{H \cap H \cap Z} \langle \tilde{w}, f(gh) \rangle_W \, dh^X, \quad f \in V.
\]

We remark that the formula given in (HMa) involves an integral over \( (H \cap g^{-1}Jg) \setminus H \). As \( g^{-1}Jg/Z \) is compact, the group \( H \cap gJg^{-1} \) is compact modulo \( H \cap Z \). Thus, as long as we normalize measures appropriately, the above formula agrees with the one from (HMa).

Let \( f_{\tilde{w}} \) be the unique element of \( \widetilde{V} \) that is supported on \( J \) and satisfies \( f_{\tilde{w}}(1) = \tilde{w} \). It follows from the definition of \( \lambda_{\tilde{w}} \) that \( \lambda_{\tilde{w}} = \lambda_{\tilde{v}_0} \) for \( \tilde{v}_0 = \widetilde{\pi}(g^{-1})f_{\tilde{w}} \).
Similarly, if we fix a nonzero \( w \in W^{J \cap g'H'g^{-1}} \) the associated \( \lambda_w \in \text{Hom}_H(\pi, 1) \) is of the form \( \lambda_w \) with \( v_0 = \pi(g^{-1})w \), where \( f_w \) is the function in \( V \) that is supported on \( J \) and satisfies \( f_w(1) = w \).

In view of the above discussion, the space of spherical characters of \( \pi \) may be described as follows.

**Lemma 8.3.** Each spherical character of \( \pi \) is of the form \( D_\phi \) for some function \( \phi \) that is a linear combination of matrix coefficients of \( \pi \). Each matrix coefficient in the linear combination can be taken to be of the form \( x \mapsto \langle \tilde{v}_0, \pi(x) v_0 \rangle \) where \( \tilde{v}_0 = \tilde{\pi}(g^{-1})f_{\tilde{w}}, g \in G, \tilde{w} \in \tilde{W}^{J \cap g'H'g^{-1}}, \) and \( v_0 = \pi(g^{-1})f_w, g' \in G, \) and \( w \in W^{J \cap g'H'g^{-1}} \).

Let \( \{ g_i \mid i \in I \} \) be a set of representatives for a set of distinct \( J-H \) double cosets in \( G \) for which \( W^{J \cap g'H'g^{-1}} \neq 0, i \in I \). For each \( i \in I \), let \( J_i = J \cap g_iHg_i^{-1} \) and \( n_i = \dim W^{J_i} \). Choose a basis \( \beta_i \) of \( W \) in such a way that the first \( n_i \) vectors \( w_1^{(i)}, \ldots, w_{n_i}^{(i)} \) of \( \beta_i \) form a basis of \( W^{J_i} \), and the first \( n_i \) vectors \( w_1^{(i)*}, \ldots, w_{n_i}^{(i)*} \) of the basis of \( \tilde{W} \) dual to \( \beta_i \) form a basis of \( \tilde{W}^{J_i} \).

Let \( I' = \{ \alpha = (i, j) \mid i \in I, 1 \leq j \leq n_i \} \). If \( \alpha = (i, j) \in I' \), set \( v_\alpha = \pi(g_i^{-1})w_{i*}^{(j)} \) and \( \tilde{v}_\alpha = \tilde{\pi}(g_i^{-1})f_{\tilde{w}_{i*}^{(j)}} \). If \( \alpha, \beta \in I' \), let \( \Phi^\alpha_\beta \) be the spherical character associated to \( \lambda_\alpha \) and \( \lambda_\beta \).

**Lemma 8.4.** The distributions \( \Phi^\alpha_\beta, \alpha, \beta \in I' \), are linearly independent.

**Proof.** Let \( \alpha, \beta \in I' \). According to Theorem 6.1.3, \( \Phi^\alpha_\beta = D_{\phi^\alpha_\beta} \) where \( \phi^\alpha_\beta(g) = \langle \tilde{v}_\alpha, \pi(g) v_\beta \rangle \). Choose \( f_{\alpha, \beta} \in C_c(G) \) such that

\[
\int_Z f_{\alpha, \beta}(g) \chi(\pi(z) d^\times z = \langle \tilde{v}_\alpha, \pi(g^{-1})v_\beta \rangle = \phi^\alpha_\beta(g^{-1}), \quad g \in G.
\]

Let \( \gamma, \delta \in I' \). Applying Theorem 6.1.2, we have

\[
\Phi^\alpha_\beta(f_{\gamma, \delta}) = d(\pi)^{-1}(\lambda_\alpha, v_\delta, \gamma, \lambda_\beta, v_\beta).
\]

Suppose that \( \alpha = (i, j) \) and \( \delta = (\ell, m) \). Then

\[
\langle \lambda_\alpha, v_\delta \rangle = \int_{H/H \cap Z} \langle \pi(g_i^{-1}) f_{w_{i*}^{(j)}} , \pi(h g_i^{-1}) f_{w_{m*}^{(i)}} \rangle dh \times
\]

\[
= \int_{H/H \cap Z} \langle w_{i*}^{(j)*}, f_{w_{m*}^{(i)}(g_i h g_i^{-1})} \rangle \times dh \times.
\]

If \( h \in H \), then \( g_i h g_i^{-1} \in J \) implies \( g_i \in Jg_iH \). Thus \( \langle \lambda_\alpha, v_\delta \rangle = 0 \) if \( i \neq \ell \). If \( i = \ell \), then \( f_{w_{i*}^{(j)}(g_i h g_i^{-1})} \neq 0 \) if and only if \( h \in g_i^{-1}J g_i \cap H \). For such \( h \), \( f_{w_{i*}^{(j)}(g_i h g_i^{-1})} = w_{m*}^{(i)} \), since \( w_{m*}^{(i)} \in W^{J_i} \). Let \( c_i \) be the volume of \( (g_i^{-1}J g_i \cap H)/Z \) relative to the measure \( dh \times \). When \( i = \ell \),

\[
\langle \lambda_\alpha, v_\delta \rangle = c_i \langle w_{i*}^{(j)*}, w_{m*}^{(i)} \rangle,
\]

which, by definition of \( w_{i*}^{(j)*} \), is equal to \( c_i \) when \( m = j \) and zero when \( m \neq j \). We conclude that \( \langle \lambda_\alpha, v_\delta \rangle \) is nonzero if and only if \( \alpha = \delta \).

A similar argument shows that \( \langle \tilde{v}_\gamma, \lambda_\nu \rangle \) is nonzero if and only if \( \gamma = \beta \).

Thus \( \Phi^\alpha_\beta(f_{\gamma, \delta}) \) is nonzero if and only if \( \alpha = \gamma \) and \( \beta = \delta \). This proves the lemma. \( \square \)
The following theorem is a consequence of the above lemma.

**Theorem 8.5.**

1. Let $g_1, g_2, \ldots$ be representatives for the set of all distinct $JH$ double cosets in $G$ that contain elements $g \in G$ for which $W^{Jg}H^{-1}$ is nonzero. Let $I' = \{ \alpha = (i, j) \mid i \in I, 1 \leq j \leq n_i \}$. Define spherical characters $\Phi^{(x, \alpha)}_s$, $\alpha, \beta \in I'$, as above. Then $\{ \Phi^{(x, \alpha)}_s \mid \alpha, \beta \in I' \}$ is a basis for the space of spherical characters of $\pi$.

2. If $\dim \text{Hom}_H(\pi, 1) = s < \infty$, then the dimension of the space of spherical characters of $\pi$ is $s^2$.

**Remark 8.6.** In [HM3], we showed that $\text{Hom}_H(\pi, 1)$ is finite-dimensional for each of the irreducible supercuspidal representations constructed in [Y]. Furthermore, we exhibit some examples of such representations $\pi$ for which $\dim \text{Hom}_H(\pi, 1)$ is greater than one.

## 9. Vanishing properties of spherical characters

Let $\pi = \text{Ind}_J^G \kappa$ be as in the previous section. Throughout this section we assume that the residual characteristic of $F$ is odd. Let $G^{tu}$ be the set of topologically unipotent elements in $G$. Each element of $G^{tu}$ belongs to the pro-unipotent radical of some parahoric subgroup of $G$. Define $\tau : G \to G$ by $\tau(g) = g \theta(g)^{-1}, g \in G$.

**Lemma 9.1.** Suppose that $\theta(J) = J$. Let $g, g' \in G$ be such that $\theta(g), \tau(g') \in J$ and $JgH \neq Jg'H$. Let $U$ be an open neighbourhood of the identity having the property that $\tau(U) \subset G^{tu}$. Then $U \cap (Hg^{-1}Jg'H) = \emptyset$.

**Proof.** Let $\tilde{J} = g^{-1}Jg$. Then

$$\theta(\tilde{J}) = \theta(g^{-1}) \theta(J) \theta(g) = g^{-1} \tau(g) \theta(J) \theta(g) = \tilde{J},$$

since $\tau(g) \in J$ and $\theta(J) = J$. Note that $\theta(g^{-1}g') = g^{-1} \tau(g') \theta(g) g \in g^{-1}Jg = \tilde{J}$. Furthermore, $HgJg'H = H\tilde{J}g^{-1}g'H$ and $JgH \cap Jg'H = \emptyset$ is equivalent to $JH \cap \tilde{J}g^{-1}g'H = \emptyset$. Thus, after replacing $J$ by $\tilde{J}$ and $g$ by 1, if necessary, it suffices to prove that $HJg'H \cap U = \emptyset$ whenever $g' \notin JH$ and $\tau(g') \in J$.

Suppose that $h_1, h_2 \in H, k \in J$. Let $u = h_1 k g'h_2$. If $u \in U$, then $\tau(u) = h_1 \tau(kg')h_1^{-1} \in \tau(U) \subset G^{tu}$. Since $G^{tu}$ is stable under conjugation by elements of $G$, we have $\tau(kg') \in G^{tu}$. Thus there exists a point $x$ in the Bruhat-Tits building of $G$ and a positive real number $t$ such that $\tau(kg') \in G_{x,t}$, where $G_{x,t}$ is the open compact subgroup of $G$ attached to the pair $(x, t)$ by Moy and Prasad ([MP]). Note that

$$\theta(\tau(kg')) = \tau(kg')^{-1} \text{ and } \tau(kg') = k_t \theta(k)^{-1} \in J',$$

where $J' = J \cap G_{x,t} \cap H(G_{x,t})$. Applying Proposition 2.12 of [HM3] to the $\theta$-stable group $J'$, we have

$$\tau(J') = \{ k' \in J' \mid \tau(k') = k'^{-1} \}.$$

Hence $\tau(kg') = \tau(k_1)$ for some $k_1 \in J'$. This implies that $k_1^{-1}kg' \in H$. As $k_1, k \in J$, we have $g' \in JH$, which is impossible. Therefore $u \in Hg'JH$ implies $u \notin U$.

We remark that neighbourhoods $U$ as in the lemma are easy to find. For example, if $K$ is a compact open subgroup of $G$ such that $K \subset G^{tu}$, we could take $U = K \cap \theta(K)$.
Proposition 9.2. Let \( \{ g_i \mid i \in I \} \), \( I' \), and \( \{ \Phi^{(\alpha,\beta)}_\pi \mid \alpha, \beta \in I' \} \) be as in Theorem [X.7.1]. Assume that \( \theta(J) = J \) and \( \tau(g_i) \in J \) for all \( i \in I \). Let \( \alpha = (i,j) \), \( \beta = (\ell,m) \in I' \). Choose \( \mathcal{U} \) as in Lemma [Y.1]. Then, if \( i \neq \ell \),

\[
\Phi^{(\alpha,\beta)}_\pi(g) = 0 \quad \forall \; g \in \mathcal{U} \cap G^{\theta,\text{reg}},
\]

Proof. Assume that \( i \neq \ell \). It suffices to show that \( \Phi_\pi(f) = 0 \) for all \( f \in C_c^\infty(G) \) such that the support of \( f \) lies inside \( \mathcal{U} \). Let \( f \) be such a function. Since \( \Phi^{(\alpha,\beta)}_\pi = D_{\rho^{(\alpha,\beta)}} \), we have

\[
\Phi^{(\alpha,\beta)}_\pi(f) = \int_{H/H \cap Z} \int_{H/H \cap Z} \int_G f(g) \langle w_j(i), f_{w_m(i)}(g_i h_1 g h_2 g_i^{-1}) \rangle_W \, dg \, dh_1 \, dh_2.
\]

According to Lemma [Z.2], if \( g \in \mathcal{U} \) and \( h_1, h_2 \in H \), then \( g_i h_1 g h_2 g_i^{-1} \notin J \). Because \( f_{w_m(i)} \) is supported on \( J \) and \( f \) is supported in \( \mathcal{U} \), this forces \( \Phi^{(\alpha,\beta)}_\pi(f) = 0 \). \( \square \)

Next, suppose that \( G \) splits over a tamely ramified extension of \( F \). Let \( \pi \) be one of the irreducible supercuspidal representations of \( G \) constructed in [Y]. Each such \( \pi \) is of the form \( \text{Ind}_{J}^{G} \kappa \) for some \( J \) and \( \kappa \). The results of [HM3] give a description of \( \text{Hom}_H(\pi,1) \) for \( H = G^{\theta}(F) \), subject to some hypotheses on quasicharacters of the \( F \)-rational points of certain reductive \( F \)-subgroups of \( G \). The hypotheses are satisfied under mild constraints on the residual characteristic of \( F \), and they always hold when \( G \) is a general linear group. (The reader may refer to Section 2.6 of [HM3] for a precise statement of the hypotheses.) The following proposition is a restatement of some parts of Theorem 5.26 of [HM3]. It follows from the proposition that the vanishing results of Proposition [Z.2] hold for the spherical characters of those supercuspidal representations of \( G \) for which \( \text{dim} \text{Hom}_H(\pi,1) \) is greater than one.

Proposition 9.3. (HM3) Let \( \pi \) be one of the supercuspidal representations constructed in [Y]. Assume that the hypotheses of [HM3] are satisfied. Then

1. \( \text{Hom}_H(\pi,1) \) is finite-dimensional.
2. If \( \text{Hom}_H(\pi,1) \) is nonzero then \( J \) may be chosen so that \( \theta(J) = J \) and \( \tau(g) \in J \) for all \( g \in G \) such that \( W_{\theta}^{\circ} \cap g H g^{-1} = \{ 0 \} \).

Corollary 9.4. If \( \text{dim} \text{Hom}_H(\pi,1) > 1 \), then there exist spherical characters of \( \pi \) that vanish on the intersection of \( G^{\theta,\text{reg}} \) with some open neighbourhood of the identity.

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