New exact solution for the (3+1) conformable space–time fractional modified Korteweg–de-Vries equations via Sine-Cosine Method

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**ABSTRACT**

In this research work, we established exact solution for the conformable space–time fractional (3+1) dimensional modified Korteweg De Vries equations (mKdV). A Sine-Cosine method is used for obtaining travelling wave solutions for these models with minimal algebra. We can conclude that the proposed scheme is reliable and efficient as its required minimal algebra without using sophisticated Mathematical tools (maple, Mathematica and others). The goal has been achieved with minimal computational cost and the present solutions obtained will serve as new solutions to the modified Korteweg–de-Vries (mKdV) equations.

**1. Introduction**

Nonlinear partial differential equations are known in fluid mechanics, plasma physics and nonlinear dispersive wave among others due to their vital roles and applications in many engineering and science application problems. However, a very good and well known example of the nonlinear partial differential equation describing shallow water waves is the Korteweg–de-Vries equation (KdV) given by

\[ u_t + uu_x + u_{xxx} = 0 \]  \hspace{1cm} (1)

The KdV equation also plays vital role in modelling blood pressure pulses and internal gravity waves in oceans among many other applications. Thus, efforts of investigating exact travelling wave solutions to problem (1) is of tremendous benefit since they help in understanding the physics behind. In addition, as higher dimensional models turn to be more realistic; some modifications to Eq. (1) have been demonstrated in literature among which the (3+1)-dimensional modified KdV equation given in Eq. (2) by Hereman [1,2] as

\[ ut + 6u^2ux + u_{xxy} = 0 \]  \hspace{1cm} (2)

Another recent (3 + 1)-dimensional modified KdV equations given by Wazwaz that read in [3] as;

\[ ut + 6u^2uy + u_{xyy} = 0, \]
\[ ut + 6u^2u_z + u_{xyy} = 0 \]  \hspace{1cm} (3)

These equations play important role in three-dimensional non-linear dispersion problems. However, this research aims to study this modified KdV equations given in Equations (1)–(3) with the introduction of the fractional order derivative in both the space and time variables using the new conformable fractional derivative [4] by Wazwaz [5]. Also, it is important to note that several analytical methods have been used in this regard in treating such problems, read [6–17]. The definition of the conformable fractional derivative [4] read; let \( u: [0, \infty) \rightarrow \mathbb{R} \), the \( \alpha \)’s order conformable derivative of \( u \) is defined by

\[ D^\alpha_t (u(t)) = \lim_{\epsilon \to 0} \frac{u(t+\epsilon t^{1-\alpha}) - u(t)}{\epsilon}, \quad t > 0, \alpha \in (0, 1) \]  \hspace{1cm} (4)

The paper is organized as follows: Section 2 devoted to properties of the conformable fractional derivative and method. Section 3 tackles the main problems and Section 4 is for conclusion.

**2. The properties of the conformable fractional derivative and methodology of solution**

Some properties of the conformable fractional derivative are given using the following definition:

**Definition [6]:** Let \( \alpha \in (0, 1) \) and suppose \( u(t) \) and \( v(t) \) are \( \alpha \)-differentiable at \( t > 0 \).

(a) \( D^\alpha_t (ct^\alpha) = c t^{\alpha-\alpha} \) for all \( c \in \mathbb{R} \).
(b) \( D^\alpha_t (a) = 0 \), \( a \) for all constant function \( u(t) = a \).
(c) \( D^\alpha_t (au(t)) = a D^\alpha_t (u(t)) \), for all \( a \) constant.
(d) \( D^\alpha_t (u(t)v(t)) = D^\alpha_t (u(t))v(t) + u(t)D^\alpha_t (v(t)) \), for all \( a, b, c \in \mathbb{R} \).
(e) \( D^\alpha_{t} (v(t)u(t)) = D^\alpha_t (v(t))u(t) + u(t)D^\alpha_t (v(t)) \).

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Let’s consider a nonlinear partial differential equation of the form

\[ P(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0 \] (5)

which describes the dynamical wave solution \( u(x, t) \). It is vital to summarize the main steps of the Sine-Cosine method given by Wazwaz in [5] as follows:

Step 1: To find the travelling wave solution of equation (5), we introduce the wave variable

\[ \xi = (x - ct) \] (6)

\[ u(x, t) = u(\xi) \] (7)

Step 2: Based on this, we use the following changes:

\[ \frac{\partial}{\partial t} = -c \frac{d}{d\xi}, \quad \frac{\partial^2}{\partial t^2} = c^2 \frac{d^2}{d\xi^2}, \]
\[ \frac{\partial}{\partial x} = \frac{d}{d\xi}, \quad \frac{\partial^2}{\partial x^2} = \frac{d^2}{d\xi^2} \] (8)

and so on for other derivatives. Using equation (8) changes the PDE equation (5) to an ODE

\[ P(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \ldots) = 0 \] (9)

where \( u_\xi \) denotes \( \frac{du}{d\xi} \).

Step 3: We then integrate the ordinary differential equation (9) as many times as possible, and setting the constant of integration to be zero.

Step 4: The solution may be set in the form

\[ u(x, t) = \chi \sin^\beta(\mu \xi) \] (10)

or in the form

\[ u(x, t) = \chi \cos^\beta(\mu \xi) \] (11)

where \( \chi, \mu, \) and \( \beta \) are parameters to be determined.

Step 5: As a consequence, the derivatives of equation (10) becomes

\[ u(\xi) = \chi \sin^\beta(\mu \xi) \] (12)
\[ u^\theta(\xi) = \chi^\theta \sin^{\theta \beta}(\mu \xi) \] (13)
\[ (u^\theta)_\xi = n\mu \beta \chi^\theta \sin^{\theta \beta}(\mu \xi) \] (14)
\[ (u^\theta)_{\xi\xi} = -n^2 \mu^2 \beta^2 \chi^\theta \sin^{\theta \beta}(\mu \xi) + n\mu^2 \chi^\theta (n\beta - 1) \sin^{\theta \beta - 2}(\mu \xi) \] (15)

And the derivatives of equation (11) becomes

\[ u(\xi) = \chi \cos^\beta(\mu \xi) \] (16)

\[ u^\theta(\xi) = \chi^\theta \cos^{\theta \beta}(\mu \xi) \] (17)
\[ (u^\theta)_\xi = -n\mu \beta \chi^\theta \sin^{\theta \beta}(\mu \xi) \] (18)
\[ (u^\theta)_{\xi\xi} = -n^2 \mu^2 \beta^2 \chi^\theta \sin^{\theta \beta}(\mu \xi) + n\mu^2 \chi^\theta (n\beta - 1) \sin^{\theta \beta - 2}(\mu \xi) \] (19)

and so on for other derivatives.

We substitute equations (10) to (20) in to reduced equation obtained above in equation (9), balance the terms of the cosine functions when equation (12) is used, or balance the sine functions when equation (17) is used, and solving the resulting system of algebraic equations by using the computerized symbolic calculations to obtain all possible values of the parameters \( \chi, \mu, \) and \( \beta \).

3. Application

In this section, we the application of Sine-Cosine method for the exact solutions of (3 + 1)-dimensional conformable space–time fractional modified Korteweg–de-Vries (mKdV) equations.

3.1. The first conformable space–time fractional mKdV equation

We consider the first (3 + 1) – dimensional conformable space time fractional mKdV equation given by

\[ D^\alpha_x U + 6 D^\alpha_x U^3 + D^\alpha_{xy} U \] (20)

Applying the following wave transformation

\[ U(x, y, z, t) = U(\xi), \text{ where} \]
\[ \xi = \left( \frac{\alpha x}{\alpha} + \frac{by}{\alpha} + \frac{cz}{\alpha} - \frac{dt}{\alpha} \right) \] (21)

We have;

\[ -dU' + 6a(U^3)' + abcU'' = 0 \] (22)

on integrating equation (23) we have;

\[ -dU + 6aU^3 + abcU' = 0 \] (23)

Having use of sine-cosine method, we are to consider

\[ U(x, t) = \lambda \sin^\beta(\mu \xi), \] (24)
\[ U(x, y, z, t) = U(\xi), \] where
\[ \lambda = \left( \frac{\alpha x}{\alpha} + \frac{by}{\alpha} + \frac{cz}{\alpha} - \frac{dt}{\alpha} \right) \]
\[ U = \lambda \sin^\beta(\mu \xi) \] (25)
\[ U'' = (-\lambda \sin^{\beta - 2}(\mu \xi) \beta \mu^2 (-\beta + \beta \sin^2(\mu \xi) + 1)) \] (26)
Solving this system, we obtained solutions method, in view of the above equation, we obtain the

\[ -d(\lambda \sin^2(\mu \xi) + 6\alpha \lambda \sin^2(\mu \xi))^3 + abc(-\lambda \sin^2(\mu \xi) \beta \mu^2 \times (-\beta + \beta \sin^2(\mu \xi) + 1))) = 0. \]  
\[ (27) \]

The equation (28) is satisfied if the following system of algebraic equations holds:

\[ \beta - 1 \neq 0, \]  
\[ (28) \]
\[ 3\beta = \beta - 2, \]  
\[ (29) \]
\[ -d = abc\mu^2 \beta^2, \]  
\[ (30) \]
\[ 6\alpha \lambda^2 = -abc\mu^2 \beta^2. \]  
\[ (31) \]
Solving this system, we obtained

\[ \beta = -1 \]  
\[ (32) \]
\[ \mu = \sqrt{\frac{-d}{abc}} \]  
\[ (33) \]
The same results are obtained if we use the cosine method, in view of the above equation, we obtain the solutions

\[ U(x, t) = \sqrt{\frac{d}{3a}} \csc \left( \sqrt{\frac{-d}{abc}} \left( \frac{ax^\alpha + by^\alpha}{\alpha} + \frac{cz^\alpha}{\alpha} - \frac{dt^\alpha}{\alpha} \right) \right), \ldots d < 0, \]  
\[ (34) \]
\[ U(x, t) = \sqrt{\frac{d}{3a}} \sec \left( \sqrt{\frac{-d}{abc}} \left( \frac{ax^\alpha + by^\alpha}{\alpha} + \frac{cz^\alpha}{\alpha} - \frac{dt^\alpha}{\alpha} \right) \right) \]  
\[ d < 0 \text{ and } i = \sqrt{-1}. \]  
\[ (35) \]

For \( d > 0 \), the following solutions

\[ U(x, t) = \sqrt{\frac{d}{3a}} \csc \left( \sqrt{\frac{-d}{abc}} \left( \frac{ax^\alpha + by^\alpha}{\alpha} + \frac{cz^\alpha}{\alpha} - \frac{dt^\alpha}{\alpha} \right) \right), \ldots d > 0 \]  
\[ (36) \]
\[ U(x, t) = \sqrt{\frac{d}{3a}} \sec \left( \sqrt{\frac{-d}{abc}} \left( \frac{ax^\alpha + by^\alpha}{\alpha} + \frac{cz^\alpha}{\alpha} - \frac{dt^\alpha}{\alpha} \right) \right), \ldots d > 0 \text{ and } i = \sqrt{-1}. \]  
\[ (37) \]

3.2. The second conformable space–time fractional mKdV equation

We consider the second \((3 + 1)\)-dimensional conformable space–time fractional mKdV equation given by

\[ D^\alpha_x U + 6 D^\alpha_y U^3 + D^\alpha_{xyz} U = 0. \]  
\[ (38) \]

Using the wave transformation

\[ U(x, y, z, t) = U(\xi), \text{ where } \]  
\[ = \left( \frac{ax^\alpha}{\alpha} + \frac{by^\alpha}{\alpha} + \frac{cz^\alpha}{\alpha} - \frac{dt^\alpha}{\alpha} \right) \]  
\[ (39) \]
By differentiating \( U(\xi) \) with respect to \( x, y, z \) and \( t \) and substituting them in equation (39) we have;

\[-dU' + 6bU^3' + abcU'' = 0. \]  
\[ (40) \]

Integrating equation (41) once, we have;

\[-dU + 6bU^3 + abcU' = 0 \]  
\[ (41) \]
Letting

\[ U(x, t) = \lambda \cos^\beta (\mu \xi) \]  
\[ (42) \]
\[ U'' = (-\lambda \cos^\beta - 2(\mu \xi) \beta \mu^2 (\beta + \beta \cos^2(\mu \xi) + 1)). \]  
\[ (43) \]
By substituting \( U' \) and \( U'' \) in equation (42), we get \(-d(\lambda \cos^\beta (\mu \xi) + 6b(\lambda \cos^\beta (\mu \xi))^3) + abc(-\lambda \cos^\beta - 2(\mu \xi) \beta \mu^2 (\beta + \beta \cos^2(\mu \xi) + 1)) = 0. \)

This equation is satisfied if the system of algebraic equations holds:

\[ \beta - 1 \neq 0, \]  
\[ (44) \]
\[ 3\beta = \beta - 2, \]  
\[ (45) \]
\[ -d = abc\mu^2 \beta^2, \]  
\[ (46) \]
\[ 6\alpha \lambda^2 = -abc\mu^2 \beta^2. \]  
\[ (47) \]
Solving this system leads to

\[ \beta = -1 \]  
\[ (48) \]
\[ \mu = \sqrt{\frac{-d}{abc}} \]  
\[ (49) \]
Consequently, the following solutions were obtained

\[ U(x, t) = \sqrt{\frac{d}{3a}} \csc \left( \sqrt{\frac{-d}{abc}} \left( \frac{ax^\alpha + by^\alpha}{\alpha} + \frac{cz^\alpha}{\alpha} - \frac{dt^\alpha}{\alpha} \right) \right), \ldots d > 0, \]  
\[ (50) \]
Applying the wave transformation, we have
\[
U(x, t) = \sqrt{\frac{\alpha}{3}} \csc \left( \sqrt{\frac{-d}{abc}} \left( \frac{\alpha}{\alpha} \frac{\alpha}{\alpha} + \frac{\alpha}{\alpha} + \frac{\alpha}{\alpha} \right) - \frac{d t}{d} \right), \ldots d < 0 \text{ and } i = \sqrt{-1}. \tag{51}
\]
For \( d > 0 \), the following solutions
\[
U(x, t) = \sqrt{\frac{d}{3b}} \sec h \left( \sqrt{\frac{-d}{abc}} \left( \frac{\alpha}{\alpha} \frac{\alpha}{\alpha} + \frac{\alpha}{\alpha} \right) + \frac{cz}{\alpha} - \frac{d t}{d} \right), \ldots d > 0, \tag{52}
\]
\[
U(x, t) = \sqrt{\frac{d}{3c}} \csc h \left( \sqrt{\frac{-d}{abc}} \left( \frac{\alpha}{\alpha} \frac{\alpha}{\alpha} + \frac{\alpha}{\alpha} \right) + \frac{cz}{\alpha} - \frac{d t}{d} \right), \ldots d > 0 \text{ and } i = \sqrt{-1}. \tag{53}
\]

### 3.3. The third conformable space–time fractional mKdV equation

We consider the third \((3 + 1)\)-dimensional conformable space–time fractional mKdV equation given by
\[
D^\alpha_{t} U + 6 D^\alpha_{x} U^3 + D^\alpha_{yy} U = 0. \tag{54}
\]
Applying the wave transformation, we have
\[
U(x, y, z, t) = U(\xi) \tag{55}
\]
where \( \xi = \left( \frac{\alpha x}{\alpha} + \frac{by}{\alpha} + \frac{cz}{\alpha} - \frac{d t}{d} \right) \).

By differentiating we have;
\[
-d U + 6 c(U^3)' + abcU'' = 0. \tag{56}
\]
On integrating equation (57) we have;
\[
-d U + 6 cU^3 + abcU'' = 0. \tag{57}
\]
Proceeding as before gives the following set of solutions.

\[
U(x, t) = \sqrt{\frac{d}{3c}} \sec h \left( \sqrt{\frac{-d}{abc}} \left( \frac{\alpha}{\alpha} \frac{\alpha}{\alpha} + \frac{\alpha}{\alpha} \right) + \frac{cz}{\alpha} - \frac{d t}{d} \right), \ldots d < 0, \tag{58}
\]
\[
U(x, t) = \sqrt{\frac{d}{3c}} \csc h \left( \sqrt{\frac{-d}{abc}} \left( \frac{\alpha}{\alpha} \frac{\alpha}{\alpha} + \frac{\alpha}{\alpha} + \frac{\alpha}{\alpha} \right) - \frac{d t}{d} \right), \ldots d < 0 \text{ and } i = \sqrt{-1}. \tag{59}
\]
For \( d > 0 \), the following solutions
\[
U(x, t) = \sqrt{\frac{d}{3c}} \sec h \left( \sqrt{\frac{-d}{abc}} \left( \frac{\alpha}{\alpha} \frac{\alpha}{\alpha} + \frac{\alpha}{\alpha} \right) + \frac{cz}{\alpha} - \frac{d t}{d} \right), \ldots d > 0, \tag{60}
\]
\[
U(x, t) = \sqrt{\frac{d}{3c}} \csc h \left( \sqrt{\frac{-d}{abc}} \left( \frac{\alpha}{\alpha} \frac{\alpha}{\alpha} + \frac{\alpha}{\alpha} + \frac{\alpha}{\alpha} \right) - \frac{d t}{d} \right), \ldots d > 0 \text{ and } i = \sqrt{-1}. \tag{61}
\]

### 4. Conclusion

This paper presents new exact solution to the old and the newly introduced \((3 + 1)\)-dimensional modified Korteweg–de-Vries equations (mKdV) coupled with the conformable fractional derivative orders in both the space and time variables using Sine-Cosine approach. Also, the solutions obtained will serve as new solutions for the conformable space–time fractional \((3 + 1)\) dimensional modified Korteweg de Vries equations (mKdV). The goal of research has been achieved with minimal computational cost without using the well-known mathematical tools (Matlab, Maple, Mathematica and others).

### Disclosure statement

No potential conflict of interest was reported by the authors.

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