Some properties of \{k\}-packing function problem in graphs

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Abstract

The recently introduced \{k\}-packing function problem is considered in this paper. Special relation between a case when \(k = 1\), \(k \geq 2\) and linear programming relaxation is introduced with sufficient conditions for optimality. For arbitrary simple connected graph \(G\) there is construction procedure for finding values of \(k\) for which \(L_{\{k\}}(G)\) can be determined in the polynomial time. Additionally, relationship between \{1\}-packing function and independent set number is established. Optimal values for some special classes of graphs and general upper and lower bounds are introduced.

Keywords: \{k\}-packing function problem, independent set, dominating set, integer linear programming.

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1. Introduction

In this paper, we will consider simple, finite and undirected graphs. For a graph \(G\), \(V(G)\) and \(E(G)\) denote its vertex and edge sets, respectively. Further, for any \(v \in V(G)\), its open neighborhood \(N_G(v)\) is the

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set of all vertices that are adjacent to $v$, and its closed neighborhood is $N_G[v] = N_G(v) \cup \{v\}$. For a function $f : V(G) \to \mathbb{N} \cup \{0\}$, and $A \subseteq V(G)$ it should be denoted $f(A) = \sum_{v \in A} f(v)$. Let $|V(G)| = n$ and $A_G = [a_{ij}]_{n \times n}$, where $a_{ij} = \begin{cases} 1, i = j \lor (i, j) \in E(G) \\ 0, \text{otherwise} \end{cases}$

For a graph $G$ and a positive integer $k$, a function $f : V(G) \to \mathbb{N} \cup \{0\}$, is a $\{k\}$-packing function of graph $G$, if for each vertex $v \in V(G)$ value $f(N_G[v])$ is at most $k$. The maximum possible value of $f(V(G))$ over all $\{k\}$-packing functions of graph $G$ is denoted as $L_{\{k\}}(G)$. Formally, $L_{\{k\}}(G) = \max_{f : V(G) \to \mathbb{N} \cup \{0\}} \{f(V(G)) | (\forall v \in V(G)) f(N_G[v]) \leq k\}$.

The distance between vertices $u$ and $v$, denoted as $d_G(u, v)$ is the length of the shortest $u - v$ path. The square of a graph $G$, named $G^2$, is the graph obtained from $G$ by adding all edges between vertices from $V(G)$ that have a common neighbor, i.e. $G^2 = (V(G), E(G^2))$, where $E(G^2) = \{(u, v) \in V(G) \times V(G) | d_G(u, v) \leq 2\}$. The complement of a graph $G$, named $\overline{G}$, is defined as $\overline{G} = (V(G), \overline{E(G)})$, where $\overline{E(G)} = \{(u, v) \in V(G) \times V(G) | u \neq v \land (u, v) \notin E(G)\}$. The independent set $I(G)$ of a graph is a set of vertices, subset of $V(G)$, such that there are no edges between them, i.e. $(u, v) \in I(G) \Rightarrow (u, v) \notin E(G))$. Independence number of a graph, named $\alpha(G)$ is the cardinality of a maximal independent set $I(G)$.

For $k$ being fixed positive integer Meir and Moon [10] introduced $k$-packing set $P \subseteq V(G)$ as a set of vertices such that distance between $u$ and $v$ is greater than $k$ for distinct $u, v \in P$, and $k$-packing number ($\rho_k(G)$) as the number of vertices of such largest set. It stands that $\rho_1(G) = \alpha(G)$ is the independence number.

Gallant et al. in [8] introduced $k$-limited packing as a modification of packing number problem allowing that intersection of each closed neighborhood with a given set contains no more than $k$ vertices. In [3, 6] Dobson et al. proved that $k$-limited packing is NP-complete for split and bipartite graphs. It was also shown that $P_4$-tidy graphs are solvable in polynomial time.

The notion of $\{k\}$-packing function was introduced by Leoni and Hinrichsen [1] as a variation of $k$-limited packing in order to solve the problem of locating garbage dumps in a given city. In this scenario, it is possible to place more than one dump in a certain location, requesting that no more than $k$ dumps are placed in each vertex and its neighborhood. Although notation is
similar, for \( k \geq 2 \) it must be clearly distinguished \( k \)-limited packing function and \( \{k\} \)-packing function. Relationship between \( k \)-limited packing and \( \{k\} \)-packing function is established in [3]. It was stated that \( L_{\{k\}}(G) \geq L_k(G) \). Additionally, in [1] where it is shown that \( L_{\{k\}}(G) = L_k(G \otimes K_k) \) (\( \otimes \) is a strong product of graphs).

**Proposition 1.** ([12]) For a graph \( G \) and a positive integer \( k \) it holds \( L_{\{k\}}(G) \geq k \cdot L_1(G) \)

**Proposition 2.** ([13]) For any connected graph \( G \) and integer \( k \in \{1, 2\} \)
\[ L_k(G) \geq \left\lceil \frac{k \cdot \text{diam}(G) + k}{3} \right\rceil \]

**Proposition 3.** ([8, 11, 12]) For path \( P_n \) holds \( L_{\{k\}}(P_n) = \left\lceil \frac{n}{3} \right\rceil \cdot k \).

**Proof.** The proposition directly holds from the following statements:

- In ([8]) in Lemma 3 it was proven that \( L_1(P_n) = \left\lceil \frac{n}{3} \right\rceil \);
- From [11] Theorem 1. it holds that \( \gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil \);
- Finally in ([12]) Theorem 3.1 it was proven that \( \gamma(G) = L_1(G) \Rightarrow L_{\{k\}}(P_n) = k \cdot L_1(P_n) \).

Therefore \( L_{\{k\}}(P_n) = k \cdot L_1(P_n) = \left\lceil \frac{n}{3} \right\rceil \cdot k \).

**Theorem 1.** ([12]) The \( \{k\} \)-packing function problem is NP-complete for all integer \( k \) fixed.

The polynomial equivalence between \( \{k\} \)-packing function problem and \( k \)-limited packing in graphs is discussed in [3].

2. **New theoretical properties**

In this section, relationship between \( \{k\} \)-packing, \( \{1\} \)-packing problem and relaxation of \( \{1\} \)-packing will be established as well as some properties of \( \{k\} \)-packing function problem for certain classes of graphs. Without loss of generality, we will assume that considered graphs are connected and have at least two vertices since if the graph is not connected we can consider connected components instead, using the following simple property.
Property 1. If $G$ is not connected and has connected components $Con_1(G), Con_2(G), \ldots Con_{nc}(G)$ then $L_{\{k\}}(G) = \sum_{i=1}^{nc} L_{\{k\}}(Con_i(G))$

Proof. Let $v \in V$ be an arbitrary vertex from a connected component $Con_j(G)$. Since $v \in Con_j(G) \Rightarrow N[v] \subseteq Con_j(G)$, then all constraints $f(N_G[v]) \leq k$ can be grouped by connected components and considered independently.

Let $Z_{rlx}(G)$ be an optimal solution of the relaxed $\{1\}$-packing problem. Relaxation is performed by $Z_{rlx}^*(G) = \max_{f:V(G) \to [0, +\infty)} \{f(V(G))|(\forall v \in V(G))f(N_G[v]) \leq 1\}$, i.e. relaxed packing function can take fractional (real) values.

Now we can formulate simple, but effective, relation among $L_{\{k\}}(G)$, $L_{\{1\}}(G)$ and $Z_{rlx}^*(G)$.

Proposition 4. For arbitrary $k \in \mathbb{N}$ it stands $k \cdot L_{\{1\}}(G) \leq L_{\{k\}}(G) \leq k \cdot Z_{rlx}^*(G)$

Proof. It should be noted that proof cannot be based on Proposition 1 and fact that $L_{\{k\}}(G) \geq L_k(G)$.

Let $f$ be a $\{1\}$-packing function of $G$ with the maximum value of all such functions. Then function $g : V(G) \to \mathbb{N} \cup \{0\}$ such that $g(v) = k \cdot f(v)$ is obviously a $\{k\}$-packing function of $G$. Consequently, $L_{\{k\}}(G) = \max_{h:V(G) \to \mathbb{N} \cup \{0\}} \{h(V(G))|(\forall v \in V(G))h(N_G[v]) \leq k\} \geq g(V(G))$. Therefore, $L_{\{k\}}(G) \geq k \cdot L_{\{1\}}(G)$.

It should be noted that $L_{\{k\}}(G) \geq k \cdot L_{\{1\}}(G)$ directly follows from Proposition 1 and fact that $L_{\{1\}}(G) = L_1(G)$.

Let $f_{rlx} : V(G) \to [0, +\infty)$ be a relaxed $\{1\}$-packing function with maximum value of all such functions. As it stands that

$$(\forall v \in V(G))f_{rlx}(N_G[v]) \leq 1 \Rightarrow k \cdot f_{rlx}(N_G[v]) \leq k$$

and $\{k\}$-packing function has non negative integer values, then

$L_{\{k\}}(G) = \max_{h:V(G) \to \mathbb{N} \cup \{0\}} \{h(V(G))|(\forall v \in V(G))h(N_G[v]) \leq k\} \leq k \cdot \max_{f:V(G) \to [0, +\infty)} \{f(V(G))|(\forall v \in V(G))f(N_G[v]) \leq 1\} = k \cdot Z_{rlx}^*(G)$.
It is interesting to find when equalities hold, i.e. when \( k \cdot L_{\{1\}}(G) = L_{\{k\}}(G) \) or \( k \cdot L_{\{1\}}(G) = k \cdot Z^*_r(G) \). Sufficient condition for both equalities will be given in the following theorem.

**Theorem 2.** If \( A_G \) is a totally unimodular matrix, then \( L_{\{k\}}(G) = k \cdot L_{\{1\}}(G) = k \cdot Z^*_r(G) \) holds.

**Proof.** Let \( G = (V, E) \) be a graph whose \( A_G \) is a totally unimodular matrix. Let us consider \( \{k\} \)-packing function problem. The problem can be formulated as a following integer linear program. Let us denote the variables \( x_i, i = 1, \ldots, |V| \) such that \( x_i = f(i) \). Then, \( \{k\} \)-packing function problem can be formulated as

\[
\begin{align*}
\max & \sum_{i=1}^{\vert V \vert} x_i \\
\text{subject to} & \sum_{j \in N_G[i]} x_j \leq k, \quad i = 1, \ldots, |V| \\
& x_i \in \{0, 1, \ldots, k\}, \quad i = 1, \ldots, |V|
\end{align*}
\]

It is easy to see that condition \( \sum_{i \in N_G[j]} x_i \leq k \) could be replaced with \( \sum_{j=1}^{\vert V \vert} a_{ij} x_j \leq k \) where \( a_{ij} \) are elements of matrix \( A_G \). Now, the formulation is

\[
\begin{align*}
\frac{\max}{\sum_{i=1}^{\vert V \vert} x_i} \\
\text{subject to} & \frac{\sum_{j=1}^{\vert V \vert} a_{ij} x_j \leq k, \quad i = 1, \ldots, |V|} \\
& \frac{x_i \in \{0, 1, \ldots, k\}, \quad i = 1, \ldots, |V|}{0}
\end{align*}
\]

Since this is Integer Linear Programming (ILP) formulation, it is natural to consider its relaxation. Instead of integer constraint \( x_i \in \{1, \ldots, k\} \), let
us consider non-negativity constraint \( x_i \geq 0 \). From the first constraint, it is obvious that for every vertex \( i \) will be \( x_i \leq k \). Let us now consider linear programming formulation

\[
\text{max} \sum_{i=1}^{|V|} x_i \tag{7}
\]

subject to

\[
\sum_{j=1}^{|V|} a_{ij} x_j \leq k, \quad i = 1, \ldots, |V| \tag{8}
\]

\[
x_i \geq 0, \quad i = 1, \ldots, |V| \tag{9}
\]

Note that this formulation for \( k = 1 \) is exactly Linear Programming (LP) formulation of \( Z_{rtx}(G) \):

\[
\text{max} \sum_{i=1}^{|V|} x_i \tag{10}
\]

subject to

\[
\sum_{j=1}^{|V|} a_{ij} x_j \leq 1, \quad i = 1, \ldots, |V| \tag{11}
\]

\[
x_i \geq 0, \quad i = 1, \ldots, |V| \tag{12}
\]

Since at least one feasible solution of the formulation above exists, \( x_i = 0, i = 1, \ldots, |V| \), and all variables have upper bound, an optimal solution also exists. From the theory of integer linear programming, it is known that polyhedron \( X(b) \), defined as \( X(b) = \{ x | Ax \geq b \} \) for any integer vector \( b \), is an integer if and only if the matrix \( A \) is totally unimodular. Since polyhedron of relaxation of our problem is \( X(b) = \{ x | A_G x \leq k \cdot e_{|V|} \} \), where \( e_{|V|} = (1, \ldots, 1)^T \) is vector of ones and dimension equal to \( |V| \), has totally unimodular matrix \( A_G \), it can be concluded that all of polyhedron nodes are integer. This means that all optimal solutions of the relaxation problem are integer. As ILP and LP formulations differ only in the condition of
integrality, it can be concluded that optimal solutions of the relaxation and ILP formulation are the same under the conditions of this theorem.

We have proved that \( L_{\{1\}}(G) = Z^*_{rlx}(G) \). From Proposition which states that \( k \cdot L_{\{1\}}(G) \leq L_{\{k\}}(G) \leq k \cdot Z^*_{rlx}(G) \) and equality of the first and the third term directly holds \( k \cdot L_{\{1\}}(G) = L_{\{k\}}(G) = k \cdot Z^*_{rlx}(G) \).

From the well-known fact that any LP problem has a polynomial complexity, the following assertion holds.

**Corollary 1.** If \( A_G \) is a totally unimodular matrix, then \( \{k\} \)-packing function problem can be solved in polynomial time.

However, total unimodularity of matrix \( A_G \) is not necessary condition for \( k \cdot L_{\{1\}}(G) = L_{\{k\}}(G) = k \cdot Z^*_{rlx}(G) \) to hold, which is illustrated by the following example.

**Example 1.** Let graph \( G \) be a claw graph with four vertices, i.e. \( G = (V, E) \), where \( V = \{1, 2, 3, 4\} \) and \( E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\} \). Matrix \( A_G \) is not totally unimodular since \( \det(A_G) = -2 \). Since \( N[1] = V(G) \) taking into consideration \( L_{\{1\}}(G) \) we have \( f(V(G)) = f(N[1]) \leq 1 \). We can construct \( \{1\} \)-packing function \( f \) where \( f(V(G)) = 1: f(1) = 1 \) and \( f(2) = f(3) = f(4) = 0 \). It is obvious that constructed function \( f \) is also maximum \( Z^*_{rlx}(G) \) of the relaxation problem. From the previous facts, clearly \( L_{\{1\}}(G) = Z^*_{rlx}(G) = 1 \). Therefore, by Proposition it holds \( k \cdot L_{\{1\}}(G) = L_{\{k\}}(G) = k \cdot Z^*_{rlx}(G) = k \).

The following example illustrates the case when \( k \cdot L_{\{1\}}(G) < L_{\{k\}}(G) \).

**Example 2.** Let us consider graph \( G \) given in Figure.

For graph \( G \) presented in Figure \( 2 \cdot L_{\{1\}}(G) = 2 < L_{\{2\}}(G) = 3 \) holds, since values \( L_{\{1\}}(G) = 1 \) and \( L_{\{2\}}(G) = 3 \) are obtained by a total enumeration. For \( k = 1 \), \( \{1\} \)-packing function with maximal value is defined as follows: \( f_1(1) = 1; f_1(2) = f_1(3) = f_1(4) = f_1(5) = f_1(6) = 0 \). For \( k = 2 \), \( \{2\} \)-packing function with maximal value is defined as follows: \( f_2(2) = f_2(3) = f_2(6) = 1; f_2(1) = f_2(4) = f_2(5) = 0 \).

Next, it will be presented an example where \( L_{\{k\}}(G) < k \cdot Z^*_{rlx}(G) \).
Example 3. Let graph $G$ be given $V(G) = \{1, 2, \ldots, 30\}$ and adjacency matrix $A_G$ given in Figure 2. For graph $G$ presented in Figure 2, $L_{\{1\}}(G) = 1 < \lfloor Z_{rlx}(G) \cdot k \rfloor$ holds. Values $L_{\{1\}}(G) = 1$ can be obtained using ILP formulation (4)-(6), while $Z_{rlx}(G) = \frac{1}{3}$ can be obtained from relaxed LP formulation (10)-(12). Values of function $f$ which correspond to $Z_{rlx}(G)$ are: $f(3) = f(4) = \frac{1}{7}$; $f(5) = f(6) = \frac{2}{21}$; $f(8) = f(13) = f(20) = f(24) = \frac{4}{21}$; $f(10) = f(16) = f(18) = \frac{5}{21}$; $f(19) = \frac{1}{21}$ and $f(11) = \frac{1}{3}$. For any other vertex $v$, $f(v) = 0$. For graph $G$ given in Figure 1, $k \cdot L_{\{1\}}(G) < L_{\{k\}}(G)$. 

![Figure 1: An example of a graph $G$ where $k \cdot L_{\{1\}}(G) < L_{\{k\}}(G)$.

![Figure 2: An example of a graph $G$ where $L_{\{k\}}(G) < [k \cdot Z_{rlx}(G)]$.](image-url)
In the sequel, we will prove that equality $L_{\{k_1\}}(G) = k \cdot Z_{rlx}^*(G)$ holds for all graphs, but only for certain values of $k$.

**Theorem 3.** For arbitrary graph $G$, $(\exists q \in \mathbb{N})(\forall k_1 \in \mathbb{N})$ $L_{\{k_1 \cdot q\}}(G) = k_1 \cdot q \cdot Z_{rlx}^*(G)$.

**Proof.** For arbitrary graph $G$, let $(x_1^*, \ldots, x_n^*)$ is an optimal solution of linear programming formulation (10)-(12), with objective function value $Z_{rlx}^*(G)$. Since constraint matrix $A_G$ is an integer matrix and right-hand side vector $b = (1 \ 1 \ \ldots \ 1)^T$ is also the integer vector, then each feasible solution must be a vector with rational coordinates. Therefore, it also holds for optimal solution, i.e. $(\forall i) (x_i^* = \frac{p_i}{q_i}$ where $p_i \in \mathbb{Z}$, $q_i \in \mathbb{N}$ and $gcd(p_i, q_i) = 1$ where $gcd(a, b)$ is the greatest common divisor of $a$ and $b$. Let us introduce $q = lcm(q_1, \ldots, q_n)$ where $lcm$ is the least common multiple. From the definition it is obvious that $q_1, \ldots, q_n \in \mathbb{N} \Rightarrow q \in \mathbb{N}$. If $x_i^* = 0$ then $p_i = 0$, let fix $q_i = 1$ in that case. If (10)-(12) has multiple optimal solutions we will assume that we can arbitrarily choose one of them.

Let $k = k_1 \cdot q$ and let $(y_1^*, \ldots, y_n^*)$ is optimal solution of the dual problem of the linear programming formulation (10)-(12). It satisfies $A_G \cdot (y_1^* \ldots y_n^*)^T \geq (1 \ 1 \ \ldots \ 1)^T$. Since $(x_1^*, \ldots, x_n^*)$ and $(y_1^*, \ldots, y_n^*)$ are optimal solutions of the mutually dual problems it follows that values of corresponding objective functions are equal, that is $\sum_{i=1}^{n} x_i^* = \sum_{i=1}^{n} y_i^*$. Dual problem of the problem (7)-(9) is

$$\max \sum_{i=1}^{V} k \cdot Y_i = k \cdot \sum_{i=1}^{V} Y_i$$

subject to

$$\sum_{i=1}^{V} a_{ij} Y_i \geq 1, \quad j = 1, \ldots, |V|$$

$$Y_i \geq 0, \quad i = 1, \ldots, |V|$$

As it can be seen value of objective function is equal to $k$ times of objective function of the dual of problem (10)-(12). Now, it can be concluded that
optimal value of objective function (7) is equal to \( k \cdot \sum_{i=1}^{n} x_i^* \) and consequently that \( (k \cdot x_1^*, \ldots, k \cdot x_n^*) \) is optimal solution of linear programming formulation (7)-(9). As \( k = q \cdot k_1 \), such that \( q = \text{lcm}(q_1, \ldots, q_n) \) and \( (\forall i) x_i^* = \frac{p_i}{q_i} \) then \( k_1 \cdot q \cdot x_i^* = k_1 \cdot q \cdot \frac{p_i}{q_i} \in \mathbb{Z} \). Since \( (k_1 \cdot q \cdot x_1^*, \ldots, k_1 \cdot q \cdot x_n^*) \) is vector of integers, and it is optimal solution of linear programming formulation (7)-(9) then it is also optimal solution of integer linear programming formulation (10)-(12) with optimal value \( k_1 \cdot q \cdot Z_{rlx}^* \). Therefore, \( L_{(k_1 \cdot q)}(G) = k_1 \cdot q \cdot Z_{rlx}^* \) which confirms the statement of the theorem.

**Corollary 2.** \( \lim_{k \to +\infty} \frac{L_{(k)}(G)}{k} = Z_{rlx}^*(G) \)

**Proof.** For a given graph \( G \) let us consider sequence \( (L_{(k)}(G))_{k \in \mathbb{N}} \) and its subsequence \( (L_{(l \cdot q)}(G))_{l \in \mathbb{N}} \) and \( q \in \mathbb{N} \) as defined in Theorem 3. From Property 4 follows that \( (\forall k)L_{(k)}(G) \leq k \cdot Z_{rlx}^*(G) \) implying \( (\forall k)\frac{L_{(k)}(G)}{k} \leq Z_{rlx}^*(G) \). For subsequence \( (L_{(l \cdot q)}(G))_{l \in \mathbb{N}} \) from Theorem 3 it holds \( (\forall l)L_{(l \cdot q)}(G) = l \cdot q \cdot Z_{rlx}^*(G) \), so \( (\forall l)\frac{L_{(l \cdot q)}(G)}{l \cdot q} = Z_{rlx}^*(G) \), implying \( \lim_{l \to +\infty} \frac{L_{(l \cdot q)}(G)}{l \cdot q} = Z_{rlx}^*(G) \), which directly confirms the statement.

**Corollary 3.** For any graph \( G \) there exists \( q \in \mathbb{N} \) such that \( L_{(k \cdot q)}(G) \) can be found in polynomial time for any \( k_1 \in \mathbb{N} \).

**Proof.** Let us consider \( q \) as defined in Theorem 3. If \( k = q \cdot q_1 \) then by Theorem 3 optimal solution of \( L_{(k)}(G) \) can be obtained as optimal solution of linear programming formulation (7)-(9). Since it can be achieved in polynomial time, then in this case \( L_{(k)}(G) \) can be obtained in polynomial time.

**Observation 1.** It should be noted that in Theorem 3 (12) word ”fixed” is necessary. Although for each simple connected graph \( G \) and for some values of \( k \), \( L_{(k \cdot q)}(G) \) can be determined in polynomial time, considered problem is still NP-complete for \( k \) fixed.

**Observation 2.** It should be noted that \( q \) defined in Theorem 3 is not necessarily minimal in the case with multiple optimal solution of (10)-(12). The number of optimal solutions can be in worst case infinite (even uncountable), though all have the same optimal value, the minimal value of \( q \) defined in Theorem 3 may not be obtained in polynomial time.

Even in the case with single optimal solution of (10)-(12), \( q = \text{lcm}(q_1, \ldots, q_n) \) may not be the minimal \( k \) for which (10)-(12) has integer optimal solution.
Previous considerations were based on the Integer Linear Programming formulation of the proposed problem and its relaxation. Now, let us present several properties of \( \{k\}\)-packing function problem which are not derived from ILP formulation. In the following proposition, it will be proven that \( \{1\}\)-packing function problem of an arbitrary graph \( G \) can be reduced to vertex independence number problem on a graph \( G^2 \).

**Proposition 5.** \( L_{\{1\}}(G) = \alpha(G^2) \).

**Proof.** (\( \Rightarrow \)) Let \( f \) be a 1-packing function whose value \( f(V(G)) = L_{\{1\}}(G) \). We define \( I = \{ v \in V(G) \mid f(v) = 1 \} \). Let \( u, v \in V(G), u \neq v \) and \( (u, v) \in E(G^2) \), i.e. \( d(u, v) \leq 2 \). Then we have two cases:

- **case 1:** \( v \in N(u) \). Since \( f \) is 1-packing function then \( f(N[u]) = 1 \) implying \( f(u) + f(v) \leq 1 \).
- **case 2:** \( u, v \in N(w) \). Since \( f \) is 1-packing function then \( f(N[w]) \leq 1 \) implying \( f(u) + f(v) \leq 1 \).

In both cases we have \( f(u) + f(v) \leq 1 \) implying that \( (u \notin I \lor v \notin I) \). Since for each edge from \( E(G^2) \) has at least one endpoint in \( I \), then \( I \) is independent set of \( G^2 \).

(\( \Leftarrow \)) Let \( I \) be an independent set of \( G^2 \). We define \( f(v) = \begin{cases} 1, & v \in I \\ 0, & v \notin I \end{cases} \).

Let \( v \) be an arbitrary vertex from \( V(G) \), and \( u, w \in N(v) \) and \( u \neq w \). Then, \( d(u, w) \leq 2 \). Since \( I \) is an independent set of \( G^2 \) at most one of vertices \( u, w \) is in \( I \), so \( f(u) + f(v) + f(w) \leq 1 \). Since \( u \) and \( w \) are arbitrary vertices from \( N(v) \), then \( f(N[v]) = \sum_{w \in N[v]} f(w) \leq 1 \). In the case when \( v \) has only one neighbor \( u \), holds \( f(N[v]) = f(u) + f(v) \leq 1 \). Since \( v \) is an arbitrary vertex from \( V(G) \) it follows that \( f \) is 1-packing function of \( G \).

**Corollary 4.** \( L_{\{1\}}(G) = \rho_2(G) \)

**Corollary 5.** If \( Diam(G) = 2 \), then \( L_{\{1\}}(G) = 1 \).

**Proof.** If \( Diam(G) = 2 \), then \( G^2 = K_{|V(G)|} \), and consequently, \( L_{\{1\}}(G) = \alpha(K_{|V(G)|}) = 1 \).

Next, it will be proposed computationally simple lower bound based upon the graph diameter.
Proposition 6. $L_{\{k\}}(G) \geq \left\lceil \frac{1+Diam(G)}{3} \right\rceil \cdot k$

Proof. From Proposition 2 it stands that $L_1(G) \geq \left\lceil \frac{diam(G)+1}{3} \right\rceil$. On the other hand, from Proposition 1 it stands that $L_{\{k\}} \geq k \cdot L_1$. By combining mentioned inequalities we obtain $L_{\{k\}} \geq k \cdot L_1 \geq k \cdot \left\lceil \frac{diam(G)+1}{3} \right\rceil$

This lower bound is tight as it can be seen from Proposition 3.

Next, it will be introduced upper bound based on the vertices’ degree.

Proposition 7. $L_{\{k\}}(G) \leq \left\lfloor \frac{nk}{1+\delta(G)} \right\rfloor$.

Proof. For each vertex $v \in V(G)$ it holds that $f(N[v]) \leq k$. Summing previous inequalities on all vertices from $V$ we obtain:

$$n \cdot k \geq \sum_{v \in V} f(N[v]) = \sum_{v \in V} \sum_{w \in N[v]} f(w)$$

On the other hand, for arbitrary vertex $u$ from $V$, in previous sums $f(u)$ appears exactly $1+\deg(u)$ times: once for vertex $u$ and $\deg(u)$ times for each vertex that is adjacent to the vertex $u$. Therefore, we get:

$$\sum_{v \in V} \sum_{w \in N[v]} f(w) = \sum_{u \in V} (1+\deg(u)) \cdot f(u) \geq \sum_{u \in V} (1+\delta) \cdot f(u) = (1+\delta) \cdot \sum_{u \in V} f(u) = (1+\delta) \cdot f(V(G))$$

As a consequence, it holds

$$f(V(G)) \leq \frac{n \cdot k}{1+\delta} \Rightarrow f(V(G)) \leq \left\lfloor \frac{nk}{1+\delta} \right\rfloor$$

The previous inequality holds because $f(V(G)) \in \mathbb{N} \cup \{0\}$.

Corollary 6. If $G$ is a regular graph of degree $r$, then $L_{\{k\}}(G) \leq \left\lfloor \frac{nk}{1+r} \right\rfloor$

Bounds in Proposition 7 are tight as it can be seen from the two following statements.
Property 2. For complete graph (clique) $K_n$ holds $L_{\{k\}}(K_n) = k$.

Proposition 8. For cycle $C_n$ holds $L_{\{k\}}(C_n) = \lfloor \frac{nk}{3} \rfloor$.

Proof. Let graph $C_n$ be a cycle, i.e. $C_n = (V, E)$ where $V = \{0, 1, 2, \ldots, n-1\}$ and $E = \{\{0,1\}, \{1,2\}, \{2,3\}, \ldots, \{n-2,n-1\}, \{n-1,0\}\}$.

Let us define function $f$ as

$$f(v_i) = 
\begin{cases}
\left\lfloor \frac{k}{3} \right\rfloor, & i \equiv 0 \pmod{3}, \\
\left\lfloor \frac{k}{3} + 0.5 \right\rfloor, & i \equiv 1 \pmod{3}, \\
\left\lceil \frac{k}{3} \right\rceil, & i \equiv 2 \pmod{3}.
\end{cases}$$

All possible cases are presented in Table 1.

From Table it is obvious that in each case $f(N[w]) \leq k$ and $f(V(G)) = \lfloor \frac{nk}{3} \rfloor$. Therefore we proved that $L_{\{k\}}(C_n) \geq \lfloor \frac{nk}{3} \rfloor$. Since $C_n$ is regular graph with $r = 2$ it holds that $L_{\{k\}}(G) \leq \lfloor \frac{nk}{1+2} \rfloor = \lfloor \frac{nk}{3} \rfloor$. Consequently, equality $L_{\{k\}}(G) = \lfloor \frac{nk}{3} \rfloor$ holds.

3. Conclusions

In this paper the $\{k\}$-packing function problem is studied. First, special relation was established between cases when $k = 1$, $k \geq 2$, and the optimal solution of the linear programming relaxation. Second, sufficient conditions for optimality were introduced. It was proven that, for arbitrary simple connected graph $G$ and some values of $k$, $L_{\{k\}}(G)$ can be determined in the polynomial time. Next, $\{1\}$-packing function problem was studied and its connection with the independent set number and 2-packing problem. Finally, lower and upper bound was introduced as well as optimal values for some special classes of graphs.

The future work could be directed to considering the $\{k\}$-packing function number of some challenging classes of graphs.

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Table 1: \( f(N[v]) \) for \( C_n \)

| \( n \) | \( k \) | \( v \) | \( f(N[v]) \) |
|-------|-------|-------|----------------|
| 3m    | 3l    | \( v_{3m} \) | \( \lceil \frac{3l}{3} \rceil + | \frac{3l}{3} \rceil + | \frac{3l}{3} + 0.5 \rceil = l + l + l = 3l \leq k \) |
| 3m    | 3l+1  | \( v_{3m} \) | \( \lceil \frac{3l+1}{3} \rceil + | \frac{3l+1}{3} \rceil + | \frac{3l+1}{3} + 0.5 \rceil = l + l + l + 1 = 3l + 1 \leq k \) |
| 3m    | 3l+2  | \( v_{3m} \) | \( \lceil \frac{3l+2}{3} \rceil + | \frac{3l+2}{3} \rceil + | \frac{3l+2}{3} + 0.5 \rceil = l + l + l + 1 + 1 = 3l + 2 \leq k \) |
| 3m+1  | 3l    | \( v_{3m+1} \) | \( \lceil \frac{3l}{3} \rceil + | \frac{3l}{3} \rceil + | \frac{3l}{3} + 0.5 \rceil = l + l + l = 3l \leq k \) |
| 3m+1  | 3l+1  | \( v_{3m+1} \) | \( \lceil \frac{3l+1}{3} \rceil + | \frac{3l+1}{3} \rceil + | \frac{3l+1}{3} + 0.5 \rceil = l + l + l + 1 = 3l + 1 \leq k \) |
| 3m+1  | 3l+2  | \( v_{3m+1} \) | \( \lceil \frac{3l+2}{3} \rceil + | \frac{3l+2}{3} \rceil + | \frac{3l+2}{3} + 0.5 \rceil = l + l + l + 1 + 1 = 3l + 2 \leq k \) |
| 3m+1  | 3l+3  | \( v_{3m+1} \) | \( \lceil \frac{3l+3}{3} \rceil + | \frac{3l+3}{3} \rceil + | \frac{3l+3}{3} + 0.5 \rceil = l + l + l + 1 + 1 + 1 = 3l + 3 \leq k \) |
| 3m+2  | 3l    | \( v_{3m+2} \) | \( \lceil \frac{3l}{3} \rceil + | \frac{3l}{3} \rceil + | \frac{3l}{3} + 0.5 \rceil = l + l + l = 3l \leq k \) |
| 3m+2  | 3l+1  | \( v_{3m+2} \) | \( \lceil \frac{3l+1}{3} \rceil + | \frac{3l+1}{3} \rceil + | \frac{3l+1}{3} + 0.5 \rceil = l + l + l + 1 = 3l + 1 \leq k \) |
| 3m+2  | 3l+2  | \( v_{3m+2} \) | \( \lceil \frac{3l+2}{3} \rceil + | \frac{3l+2}{3} \rceil + | \frac{3l+2}{3} + 0.5 \rceil = l + l + l + 1 + 1 = 3l + 2 \leq k \) |
| 3m+2  | 3l+3  | \( v_{3m+2} \) | \( \lceil \frac{3l+3}{3} \rceil + | \frac{3l+3}{3} \rceil + | \frac{3l+3}{3} + 0.5 \rceil = l + l + l + 1 + 1 + 1 = 3l + 3 \leq k \) |

\[ f(V(C_3m+1)) = (m+1) \cdot l + m \cdot l + m(l+1) = 3ml + l = (3m+1) \cdot \frac{3 \cdot m + 1}{3} = \lceil \frac{3m+1}{3} \rceil \]
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