Localization and holography in $\mathcal{N} = 2$ gauge theories

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Abstract

We compare exact results from Pestun’s localization [1] of $SU(N)$ $\mathcal{N} = 2^*$ gauge theory on $S^4$ with available holographic models. While localization can explain the Coulomb branch vacuum of the holographic Pilch-Warner flow [2], it disagrees with the holographic Gauntlett et.al [3] vacuum of $\mathcal{N} = 2$ super Yang-Mills theory. We further compute the free energy of the Pilch-Warner flow on $S^4$ and show that it disagrees with the localization result both for a finite $S^4$ radius, and in the $S^4$ decompactification limit. Thus, neither model represents holographic dual of supersymmetric $S^4$ localization of [1].

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1 Introduction and motivation

Original work of Maldacena [4] established a duality between $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills (SYM) theory and type IIb string theory. A lot of subsequent work was devoted to generalizing the holographic correspondence to non-conformal theories, and theories with reduced supersymmetry. Extensions of gauge/gravity correspondence to gauge theories with eight supercharges (the $\mathcal{N} = 2$ supersymmetric models) play a special role, as they allow for a direct check of the correspondence with the field-theoretic Seiberg-Witten solution [5,6]. The two notable examples of $\mathcal{N} = 2$ dualities are the Pilch-Warner [2] (PW) and the Gauntlett et.al [3] (GKMW) holographic renormalization group (RG) flows. In the former, one considers a planar limit of $\mathcal{N} = 4$ $SU(N)$ SYM at large ’t Hooft coupling, deformed by $\mathcal{N} = 2$ hypermultiplet mass term (the so called $\mathcal{N} = 2^*$ gauge theory); in the latter, one starts with the $S^2$-compactified Little String Theory in the ultraviolet, and flows in the infrared to large-$N \mathcal{N} = 2$ $SU(N)$ SYM. $\mathcal{N} = 2$ gauge theories have quantum Coulomb branch vacua $\mathcal{M}_C$, parameterized by the expectation values of the complex scalar $\Phi$ in the $\mathcal{N} = 2$ vector multiplet, taking values in the Cartan subalgebra of the gauge group. For the $SU(N)$
gauge group,
\[ \Phi = \text{diag}(a_1, a_2, \ldots, a_N), \quad \sum_i a_i = 0, \]  
resulting in complex dimension of the moduli space
\[ \dim_{\mathbb{C}} \mathcal{M}_C = N - 1. \]  

In the large-\(N\) limit, and for strong 't Hooft coupling, the holographic duality reduces to the correspondence between the gauge theory and type IIb supergravity. Since supergravities have finite number of light modes, one should not expect to see the full moduli space of vacua in \(\mathcal{N} = 2\) examples of gauge/gravity correspondence. This is indeed what is happening: the PW flow localizes on a semi-circle distribution of (1.1) with a linear number density [7],
\[ \text{Im}(a_i) = 0, \quad a_i \in [-a_0, a_0], \quad a_0^2 = \frac{m^2 g_Y^2 N}{4\pi^2}, \]  
where \(m\) is the hypermultiplet mass; the GKMW flow localizes on a circular distribution of eigenvalues, centered about the origin with radius \(u_0 \propto \sqrt{N}\) [3],
\[ \rho(a) = \frac{N}{2\pi u_0} \delta(|a| - u_0), \quad \int \int_{\mathbb{C}} d^2a \rho(a) = N. \]  

An outstanding question is the mechanism of the holographic localization on the moduli space of \(\mathcal{N} = 2\) vacua in the large-\(N\) limit.

A possible mechanism of holographic localization was proposed in [8]. In [1] Pestun pointed out that the partition function of \(\mathcal{N} = 2\) gauge theory on \(S^4\), and some supersymmetric observables, can be computed exactly from the corresponding matrix model. This \(S^4\) compactification does not twist the supersymmetry — around the trivial background the compactified theory does not have zero modes. In the large-\(N\) limit, the gauge theory partition function naturally localizes on the saddle-point of the corresponding matrix model, thus providing a selection mechanism for the Coulomb branch vacuum in the \(S^4\) decompactification limit. Such argument indeed explains the PW vacuum (1.3), as well as correctly reproduces the holographic computation of the supersymmetric Wilson loop in PW geometry [2]. So, does it mean that \(\mathcal{N} = 2\)
PW consistent truncation of type IIb supergravity contain large-$N$ holographic dual to Pestun’s supersymmetric compactification of $\mathcal{N} = 2^*$ gauge theory on $S^4$?

To answer this question we compute the free energy of the compactified Euclidean PW flow on the four-sphere of radius $R$. Ambiguities of the holographic renormalization imply that for $mR \ll 1$ only $\mathcal{O}(m^6 R^6)$ (and subleading terms) are renormalization-scheme independent. We show that at order $\mathcal{O}(m^6 R^6)$ there is a disagreement between the holographic and the matrix model free energies. Such disagreement indicates that PW truncation is inconsistent with the $\mathcal{N} = 2^*$ gauge theory $S^4$ supersymmetries of [1].

Still, motivated by the success of [8], the possibility remains that the holographic and the matrix model free energies agree in the $S^4$ decompactification limit. Unfortunately, we study the free energy of $S^4$-compactified PW flow in the limit $mR \gg 1$ and find that it is in conflict with the matrix model result. Thus, Pestun’s localization can not explain holographic localization of $\mathcal{N} = 2$ supergravity flows. The last point is further stressed by pointing out that large-$N$ localization of $\mathcal{N} = 2 \ SU(N)$ SYM [9] is different from the holographic vacuum localization of GKMW supergravity flow.

In section 2 we collect relevant localization results for $\mathcal{N} = 2^*$ gauge theory [8], and for $\mathcal{N} = 2$ SYM [9]. While the corresponding matrix model to $\mathcal{N} = 2^*$ gauge theory localizes on holographic PW vacuum (1.3), it fails to localize on GKMW vacuum (1.4) in the case of $\mathcal{N} = 2$ SYM. In section 3 we compute the free energy of $S^4$-compactified holographic PW flow. We compare the results with the matrix model computation. We conclude in section 4.

Note added: After the paper was published, I was informed by Francesco Bigazzi about his relevant work [11] (see also [12] ). The authors of [11] consider a more general ansatz than GKMW for wrapped D5-branes, resulting in a different $\mathcal{N} = 2$ Coulomb branch vacuum of large-$N \ SU(N)$ SYM. Unlike the GKMW vacuum, this vacuum does agree with the localization vacuum of [9]. It would be interesting to compare the expectation values of circular Wilson loops and the free energy in supergravity background of [11] with the computations in [9].

2 Localization of $\mathcal{N} = 2$ gauge theories on $S^4$

According to [1], the partition function of $\mathcal{N} = 2$ gauge theories on $S^4$ of radius $R$ reduces to $(N - 1)$ dimensional integral over $\{\hat{a}_i \equiv a_i R\}$ (see (1.1)) of an effective matrix model:

\[
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for $\mathcal{N} = 2^*$ $SU(N)$ gauge theory, $Z_{\mathcal{N}=2^*}$,

$$Z_{\mathcal{N}=2^*} = \int d^{N-1}\hat{a} \prod_{i<j} \frac{(\hat{a}_i - \hat{a}_j)^2 H^2(\hat{a}_i - \hat{a}_j)}{H(\hat{a}_i - \hat{a}_j - 2mR)H(\hat{a}_i - \hat{a}_j + 2mR)} e^{-\frac{8\pi^2N}{\lambda} \sum_j \hat{a}_j^2 |Z_{\text{inst}}|^2}, \quad (2.1)$$

with $\lambda \equiv g^2_{YM} N$;

for $\mathcal{N} = 2$ $SU(N)$ SYM, $Z_{\text{SYM}}$,

$$Z_{\text{SYM}} = \int d^{N-1}\hat{a} \prod_{i<j} [(\hat{a}_i - \hat{a}_j)^2 H^2(\hat{a}_i - \hat{a}_j)] e^{-\frac{8\pi^2N}{\lambda} \sum_j \hat{a}_j^2 |Z_{\text{inst}}|^2}, \quad (2.2)$$

with the running 't Hooft coupling $\lambda$ evaluated at the cut-off set by the $S^4$ radius $R$, [9]:

$$\frac{4\pi^2}{\lambda} = -\ln(\Lambda R). \quad (2.3)$$

In (2.3) $\Lambda$ is the strong coupling scale of the SYM. The function $H(x)$ is expressed as an infinite product over the spherical harmonics,

$$H(x) \equiv \prod_{n=1}^\infty \left(1 + \frac{x^2}{n^2}\right)^n e^{-\frac{x^2}{n^2}}. \quad (2.4)$$

Matrix integrals (2.1) and (2.3) dramatically simplify in the large-$N$ limit. First, as the instantons are suppressed in the planar limit, we can set $|Z_{\text{inst}}| = 1$. Second, the saddle point approximation becomes exact [13].

In what follows, it is convenient to introduce

$$K(x) \equiv -(\ln H(x))'. \quad (2.5)$$

We further set $R = 1$ (and drop the caret)— along with $m$ (for $\mathcal{N} = 2^*$ gauge theory) or $\Lambda$ (for the SYM) the $S^4$ radius is the only other dimensionful scale; thus the $R$-dependence can always be restored from dimensional analysis.

### 2.1 $\mathcal{N} = 2^*$ gauge theory

The saddle-point equations derived from (2.1) take form [8,13]

$$\frac{1}{N} \sum_{k>j} \left( \frac{1}{a_j - a_k} - K(a_j - a_k) + \frac{1}{2} K(a_j - a_k + m) + \frac{1}{2} K(a_j - a_k - m) \right) = \frac{8\pi^2}{\lambda} a_j. \quad (2.6)$$

Assuming$^2$ $\text{Im}(a_i) = 0$, $a_i \in [-\mu, \mu]$, and introducing a linear eigenvalue density

$$\rho(x) = \frac{1}{N} \sum_i \delta(x - a_i), \quad \int_{-\mu}^\mu dx \rho(x) = 1, \quad (2.7)$$

$^2$This is justified a posteriori.
we find
\[\int_{-\mu}^{\mu} dy \rho(y) \left( \frac{1}{x-y} - K(x-y) + \frac{1}{2} K(x-y+m) + \frac{1}{2} K(x-y-m) \right) = \frac{8\pi^2}{\lambda} x. \quad (2.8)\]

In the limit\(^3\) \(\lambda \to \infty\) the solution is given by [8]
\[\rho(x) = \frac{2}{\pi \mu^2} \sqrt{\mu^2 - x^2}, \quad \mu = \frac{\sqrt{\lambda (\frac{1}{R^2} + m^2)}}{2\pi}, \quad (2.9)\]
where we restored the \(R\)-dependence. In the \(S^4\) decompactification limit, i.e., \(mR \to \infty\), the distribution (2.9) reproduces the PW vacuum (1.3).

The free energy \(F_{N=2^*}^{loc}\),
\[F_{N=2^*}^{loc} = -\ln Z_{N=2^*}, \quad (2.10)\]
can be computed by first differentiating it with respect to \(m\):
\[\frac{\partial}{\partial m} F_{N=2^*}^{loc} = \left\{ \frac{1}{2} \sum_{i,j} \left( K(a_i - a_j - m) - K(a_i - a_j - m) \right) \right\} \int \int dxdy \rho(x) \rho(y) \left( K(x-y-m) - K(x-y+m) \right). \quad (2.11)\]

The leading contribution in the limit \(\lambda \to \infty\) then becomes [8]
\[\frac{\partial}{\partial m} F_{N=2^*}^{loc} = -N^2 m \ln \frac{\lambda (1+m^2) e^{2\gamma + \frac{3}{2}}}{16\pi^2}, \quad (2.12)\]
where \(\gamma = -\psi(1)\) is the Euler’s constant. Ensuring that the free energy agrees with that of the \(N=4\) SYM in the limit \(m \to 0\) [9] we find,
\[F_{N=2^*}^{loc} = -\frac{N^2}{2} (1+m^2 R^2) \ln \frac{\lambda (1+m^2 R^2) e^{2\gamma + \frac{3}{2}}}{16\pi^2}. \quad (2.13)\]

From (2.13) it is easy to extract the small- and large-\(R\) limits:
\[F_{N=2^*}^{loc} = N^2 \times \begin{cases} C + (-\frac{1}{2} + C) m^2 R^2 - \frac{1}{4} m^4 R^4 + \frac{1}{12} m^6 R^6 + O(m^8 R^8), & (mR \ll 1), \\ -m^2 R^2 \ln(mR), & (mR \gg 1), \end{cases} \quad (2.14)\]
where we denoted
\[C = -\frac{1}{2} \ln \frac{\lambda e^{2\gamma + \frac{3}{2}}}{16\pi^2}. \quad (2.15)\]

In section 3 we compute the free energy of \(S^4\)-compactified PW flow. We find that the holographic free energy disagrees with (2.13), (2.14).

\(^3\)As emphasized in [10] this limit has an irregular fuzzy fine structure.
2.2 \( N = 2 \) SYM

The saddle-point equations derived from (2.2) take form [9]

\[
\frac{1}{N} \sum_{k \neq j} \left( \frac{1}{a_j - a_k} - K(a_j - a_k) \right) = \frac{8\pi^2}{\lambda} a_j = -2a_j \ln \Lambda. \tag{2.16}
\]

Assuming\(^4\) \( \text{Im}(a_i) = 0 \), \( a_i \in (-\mu, \mu) \), and introducing a linear eigenvalue density as in (2.7), we find

\[
\int_{-\mu}^{\mu} dy \rho(y) \left( \frac{1}{x-y} - K(x-y) \right) = -2x \ln \Lambda. \tag{2.17}
\]

Analytic solutions to (2.17) are available in the limiting cases \( \Lambda \ll 1 \) and \( \Lambda \gg 1 \) [9]. As we will be interested in the \( S^4 \) decompactification limit, we will focus on the latter case:

\[
\rho(x) = \frac{1}{\pi \sqrt{\mu^2 - x^2}}; \quad \mu = 2e^{-1-\gamma} \Lambda R. \tag{2.18}
\]

Solution (2.18) has to be taken with a grain of salt:

- first, the boundary conditions for the eigenvalue density \( \rho(x) \) are not satisfied, \( i.e., \)
  \[
  \lim_{x \rightarrow \pm \mu} \rho(x) = 0; \tag{2.19}
  \]

- second, the \( S^4 \) decompactification limit is problematic, as in this case \( \lambda \rightarrow 0_- \) (see (2.3)). We return to these points later.

The free energy \( \mathcal{F}_{\text{SYM}}^{\text{loc}} \),

\[
\mathcal{F}_{\text{SYM}}^{\text{loc}} = -\ln Z_{\text{SYM}}, \tag{2.20}
\]

can be computed by first differentiating it with respect to \( \Lambda \):

\[
\frac{\partial}{\partial \ln \Lambda} \mathcal{F}_{\text{SYM}}^{\text{loc}} = -2N \left( \sum_j a_j^2 \right) = -2N^2 \int dx \rho(x) x^2 = -N^2 \mu^2 = -4e^{-2-2\gamma} N^2 \Lambda^2 R^2, \quad \Lambda R \gg 1, \tag{2.21}
\]

leading to

\[
\mathcal{F}_{\text{SYM}}^{\text{loc}} = -2e^{-2-2\gamma} N^2 \Lambda^2 R^2, \quad \Lambda R \gg 1. \tag{2.22}
\]

We would like to compare matrix model results of [9] reviewed above with the holographic computation of GKMW [3]. In the latter holographic RG flow one starts

\(^4\)Again, this is justified \textit{a posteriori}. 
with large number of NS5 branes wrapping a 2-cycle in the UV, and flows in the IR to $SU(N)$ SYM. The strong coupling scale of the SYM is set by the $S^2$–compactification scale of Little String Theory on the world-volume of the five-branes. Clearly, the field theory dual to GKMW flow at high-energy can not be a SYM. Thus, there is no reason to expect that the small $S^4$ radius localization results would agree with results from the small-$R$ $S^4$-compactified GKMW flow. One would expect, however, that the moduli space properties of GKMW would agree with the matrix model computations in the limit $\Lambda R \to \infty$. Unfortunately, this is not the case: compare (1.4) and (2.18). We can identify the disagreement a bit more precisely. Following [3], the probe $U(1)$ gauge coupling $\tau$ on the GKMW moduli space parameterized by $u$ takes the form

$$\tau(u) = i \frac{2N}{\pi} \ln \frac{u}{\Lambda},$$

(2.23)

where $\tilde{\Lambda}$ is the IR cutoff, related to the strong coupling scale of the SYM. The $\mathcal{N} = 2$ supersymmetry relates the coupling to the metric on the moduli space as

$$\tau(u) = i \frac{N}{\pi} \int dx \rho(x) \ln \frac{(uR - x)^2}{\mu^2},$$

$$= i \frac{2N}{\pi} \ln \frac{\sqrt{u^2R^2 - \mu^2 + uR}}{2\mu}, \quad uR > \mu,$$

(2.24)

where we used the saddle-point vacuum of the matrix model (2.18). Notice that (2.23) and (2.24) agree in the limit $u \gg \Lambda$, provided we relate

$$\tilde{\Lambda} = 2e^{-1-\gamma} \Lambda,$$

(2.25)

but disagree in general. This discrepancy might be attributed to the fact that the RG equation in the matrix model (2.3) makes sense only up to $R \sim \frac{1}{\Lambda}$, which in turn implies that the moduli space coupling (2.24) can be accurate only when evaluated at $|u| \gg \frac{1}{R} \sim \Lambda$.

Having established that the Coulomb branch vacua in the saddle-point matrix model for $SU(N)$ SYM and the GKMW flow differ, there is no point to proceed with the detailed comparison of the corresponding free energies.

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5 We use $R$ simply as a dimensionful parameter to facilitate the comparison with the matrix model result (2.18).
3 $\mathcal{N} = 2^*$ free energy on $S^4$ from holography

In this section we compute the free energy $F$ of $S^4$-compactified Pilch-Warner holographic RG flow. We compare the latter with the matrix model result $F_{\mathcal{N} = 2^*}^{loc}$ (2.13) — we stress that here, unlike the relation between the $\mathcal{N} = 2$ SYM and the GKMW holographic RG flow, it makes sense to compare the free energies for generic values of $mR$.

3.1 Effective action and equations of motion

The supergravity background dual to $S^4$ compactification of $\mathcal{N} = 2^*$ gauge theory [14] is a deformation of the original $AdS_5 \times S^5$ geometry\(^6\) induced by a pair of scalars $\alpha$ and $\chi$ of the five-dimensional gauge supergravity. (At zero temperature, such a deformation was constructed by Pilch and Warner (PW) [2]\(^7\).) According to the general scenario of a holographic RG flow, the asymptotic boundary behavior of the supergravity scalars is related to the bosonic and fermionic mass parameters of the relevant operators inducing the RG flow in the boundary gauge theory. Based on such a relation, and the fact that $\alpha$ and $\chi$ have conformal dimensions two and one, respectively, we call the supergravity scalar $\alpha$ a bosonic deformation, and the supergravity scalar $\chi$ a fermionic deformation of the D3-brane geometry.

The action of the effective five-dimensional gauged supergravity including the scalars $\alpha$ and $\chi$ is given by

$$S = \int_{M_5} d\xi^5 \sqrt{-g} \mathcal{L}_5$$

$$= \frac{1}{4\pi G_5} \int_{M_5} d\xi^5 \sqrt{-g} \left[ \frac{1}{4} R - 3(\partial \alpha)^2 - (\partial \chi)^2 - \mathcal{P} \right],$$

where the potential\(^8\)

$$\mathcal{P} = \frac{1}{16} \left[ \frac{1}{3} \left( \frac{\partial W}{\partial \alpha} \right)^2 + \left( \frac{\partial W}{\partial \chi} \right)^2 \right] - \frac{1}{3} W^2$$

is a function of $\alpha$ and $\chi$, and is determined by the superpotential

$$W = -e^{-2\alpha} - \frac{1}{2} e^{4\alpha} \cosh(2\chi).$$

\(^6\)With $S^4$ slicing of $AdS_5$ — see [15].

\(^7\)See [7, 16] for the gauge theory interpretation of the PW geometry.

\(^8\)We set the five-dimensional gauged supergravity coupling to one. This corresponds to setting the radius $L$ of the five-dimensional sphere in the undeformed metric to 2.
In our conventions, the five-dimensional Newton’s constant is

\[ G_5 \equiv \frac{G_{10}}{2^5 \text{vol}_{S^5}} = \frac{4\pi}{N^2} . \]

(3.4)

The action (3.1) yields the Einstein equations

\[ R_{\mu\nu} = 12 \partial_{\mu}\alpha \partial_{\nu}\alpha + 4 \partial_{\mu}\chi \partial_{\nu}\chi + \frac{4}{3} g_{\mu\nu} \mathcal{P} , \]

(3.5)

as well as the equations for the scalars

\[ \Box \alpha = \frac{1}{6} \frac{\partial \mathcal{P}}{\partial \alpha} , \quad \Box \chi = \frac{1}{2} \frac{\partial \mathcal{P}}{\partial \chi} . \]

(3.6)

To construct the \( S^4 \) compactification of the Pilch-Warner flow, we choose an ansatz for the metric respecting \( SO(5) \) rotational invariance

\[ ds^2_5 = c_1^2(r) (dS^4)^2 + dr^2 , \]

(3.7)

where \( (dS^4)^2 \) is a round metric on \( S^4 \) of unit radius. With this ansatz, the equations of motion for the background become

\[ \alpha'' + \alpha' \left( \ln c_1^4 \right)' - \frac{1}{6} \frac{\partial \mathcal{P}}{\partial \alpha} = 0 , \quad \chi'' + \chi' \left( \ln c_1^4 \right)' - \frac{1}{2} \frac{\partial \mathcal{P}}{\partial \chi} = 0 , \]

(3.8)

\[ c_1'' + c_1' \left( \ln c_1^3 \right)' - \frac{3}{c_1} + \frac{4}{3} c_1 \mathcal{P} = 0 , \]

where the prime denotes a derivative with respect to the radial coordinate \( r \). In addition, there is a first-order constraint

\[ (\alpha')^2 + \frac{1}{3} (\chi')^2 - \frac{1}{3} \mathcal{P} - (\left( \ln c_1 \right)')^2 + \frac{1}{c_1^2} = 0 . \]

(3.9)

It was shown in [14] that any solution to (3.8) and (3.9) can be lifted to a full ten-dimensional solution of type IIb supergravity. This includes the metric, the three- and five-form fluxes, the dilaton and the axion. In particular, the ten-dimensional Einstein frame metric is given by Eq. (32) in [14].

For \( S^4 \)-compactified flow, we find it convenient to introduce a new radial coordinate \( x \) as follows :

\[ x(r) = c_1(r) , \quad x \in [0, +\infty) . \]

(3.10)
With this new coordinate, the background equations of motion (3.8) become

\[
0 = \rho'' - 4x(-144\rho^4c^4 + \rho^{12}x^2c^8 - 16\rho^6x^2c^6 - 2\rho^{12}x^2c^4 + \rho^{12}x^2) \\
- 16\rho^6x^2c^2(\rho')^3/(\rho^2(-16\rho^6x^2c^2 - 192\rho^4c^4 - 16\rho^6x^2c^4 - 2\rho^{12}x^2c^6)) \\
- 2\rho^{12}x^2c^4 + \rho^{12}x^2) - (32\rho^6x^2c^2 + 16x^2c^4 - 192\rho^4c^4 + 5\rho^{12}x^2c^6 - 32\rho^6x^2c^6 \\
- 10\rho^{12}x^2c^4 + 5\rho^{12}x^2)(\rho')^2/(\rho(-16\rho^6x^2c^2 - 192\rho^4c^4 - 16\rho^6x^2c^4 + \rho^{12}x^2c^8 - 16\rho^6x^2c^6 \\
- 2\rho^{12}x^2c^4 + \rho^{12}x^2) + (1/3)(-240\rho^6x^2c^4 - 240x^2c^6 + 15\rho^{12}x^2c^{10} - 240\rho^6x^2c^8 \\
- 30\rho^{12}x^2c^6 + 15\rho^{12}x^2c^2 - 4(\rho')^2\rho^{12}x^4 - 2304\rho^6x^2c^2 + 576\rho^4(c')^2x^2 \\
+ 64(c')^2x^4 + 8(\rho')^2\rho^{12}x^4 + 64(c')^2\rho^6x^4 - 4(\rho')^2\rho^{12}x^4/xc^2(-16\rho^6x^2c^2 \\
- 192\rho^4c^4 - 16\rho^6x^2c^2 - 16\rho^6x^2c^4 - 2\rho^{12}x^2c^4 + \rho^{12}x^2)) + (4/3)\rho(-x^2\rho^{12}(c')^2 \\
- 6\rho^{12}c^2 + 3\rho^{12}c^2 + 3\rho^{12}c^{10} + 24\rho^6 - 8\rho^4c^4 - 8\rho^6x^2c^4 - x^2\rho^{12}(c')^2c^8 \\
+ 2x^2\rho^{12}(c')^2c^4 + 4x^2\rho^6(c')^2c^2 + 4x^2\rho^6(c')^2c^2)/(c^2(-16\rho^6x^2c^2 - 192\rho^4c^4 - 16\rho^6x^2c^4 \\
+ \rho^{12}x^2c^8 - 16\rho^6x^2c^6 - 2\rho^{12}x^2c^4 + \rho^{12}x^2)) \right) \right) \\
(3.11)
\]

where the prime now denotes a derivative with respect to \(x\), and we further introduced

\[
\rho \equiv e^\alpha, \quad c \equiv e^\chi. \tag{3.13}
\]

We demand that a physical RG flow should correspond to a background geometry without naked singularities. To ensure regularity, it is necessary to impose the following
Figure 1: (Color online) RG flow coefficients \( \{r_0^o, c_0^o\} \) as a function of \( k \) (blue curves). The green curves represent perturbative approximations \((3.18)\) to orders \(O(k^2)\) and \(O(k^3)\) correspondingly; the red curves represent full perturbative approximations \((3.18)\).

asymptotic conditions:

- at the origin, i.e., as \( x \to 0_+ \),
  \[
  \rho = r_0^o + O(x^2), \quad c = c_0^o + O(x^2),
  \]  
  \[(3.14)\]
  with \( r_0^o c_0^o \neq 0 \);
- and at the boundary, i.e., as \( y \equiv \frac{1}{x} \to 0_+ \),
  \[
  \rho = 1 + y^2 \left( r_{1,0}^b + r_{11}^b \ln y \right) + O(y^4 \ln^2 y),
  
  c = 1 + y \left[ c_{1,0}^b + \frac{1}{2} y^2 (c_{1,0}^b)^2 + y^3 c_{2,0}^b \left( c_{2,0}^b + \left( 4 + \frac{4}{3} (c_{1,0}^b)^2 \right) \ln y \right) \right] + O(y^4 \ln y). \]  
  \[(3.15)\]

The asymptotic coefficients \( c_{1,0}^b \) and \( r_{1,1}^b \) are related to masses of the bosonic and fermionic components of \( \mathcal{N} = 2^* \) hypermultiplet. The precise relation can be established as in [17] (see Appendix A for specific details):

\[
  c_{1,0}^b = \frac{k}{2}, \quad r_{1,1}^b = \frac{1}{6} k^2, \quad k \equiv mL.
  \]  
  \[(3.16)\]

Given \( k \) (and thus \( c_{1,0}^b \) and \( r_{1,1}^b \) via \((3.16)\) ) there is a unique nonsingular RG flow specified by

\[
  \{r_0^o, c_0^o, r_{1,0}^b, c_{2,0}^b\}. \]  
  \[(3.17)\]
5
10
15
20
k
5
10
15
20
k
Figure 2: (Color online) RG flow coefficients \( \{r_{1,0}^{b}, c_{2,0}^{b}\} \) as a function of \( k \) (blue curves). The green curves represent perturbative approximations (3.18) to order \( \mathcal{O}(k^2) \); the red curves represent full perturbative approximations (3.18).

While it is difficult to construct analytic solutions to (3.11)-(3.15) for generic \( k \), and thus determine (3.17), it is possible to do so perturbatively in \( k \). We find\(^9\)

\[
\begin{align*}
    r_{0}^{o} &= 1 - \frac{1}{72}k^2 + \frac{1}{16}k^4 r_{0,4}^{o} + \mathcal{O}(k^6), \quad r_{0,4}^{o} = 0.0178(5), \\
    c_{0}^{o} &= 1 + \frac{1}{6}k + \frac{1}{72}k^2 - \frac{1}{648}k^3 - \frac{11}{31104}k^4 + \frac{1}{32}k^5 c_{0,5}^{o} + \mathcal{O}(k^6), \quad c_{0,5}^{o} = -0.0003(2), \\
    r_{1,0}^{b} &= \frac{1}{6}k^2 + \frac{1}{16}k^4 r_{1,0,4}^{b} + \mathcal{O}(k^6), \quad r_{1,0,4}^{b} = 0.1333(3), \\
    c_{2,0}^{b} &= 2 + \frac{1}{2}k^2 + \frac{1}{16}k^4 c_{2,0,5}^{b} + \mathcal{O}(k^6), \quad c_{2,0,5}^{b} = -0.0542(2).
\end{align*}
\] 

(3.18)

Furthermore, the solution can always be found numerically, using the 'shooting method' introduced in [18]. Results of the latter numerical analysis are presented in Fig. 1 and Fig. 2.

### 3.2 \( \mathcal{F} \) via holographic renormalization

Following gauge-gravity correspondence, the free energy \( \mathcal{F} \) of the boundary theory is given by the Euclidean gravitational action of its holographic dual \( S_{E} \). As usual, ultraviolet divergences in the field theory in computing \( \mathcal{F} \) are reflected in the infrared divergences of the dual gravitational bulk geometry. Both must be regularized and

\(^9\)We illustrate the solution to order \( \mathcal{O}(k^2) \) inclusive in Appendix B.
renormalized. In the context of PW flow the holographic renormalization has discussed in details in [19, 20]. Below, we present the necessary details.

Let \( r_c \) be the position of the boundary, and \( S_E^{r_c} \) be the Euclidean gravitational action on the cut-off space

\[
\lim_{r_c \to \infty} S_E^{r_c} = S_E ,
\]

where \( S_E \) is the on-shell Euclidean version of (3.1). Using equations of motion (3.8), the regularized action takes form

\[
S_E^{r_c} = \frac{1}{4\pi G_5} \left( \frac{1}{8} (c_1(r))^4 \right)^\prime \bigg|_{0}^{r_c} - \frac{3}{2} \int_0^{r_c} c_1(r)^2 \, dr \bigg) \vol(S^4) ,
\]

(3.20)

Notice that the integral contribution in (3.20) arises entirely from \( S^4 \) curvature. Besides the standard Gibbons-Hawking term \( S_{GH} = -\frac{1}{8\pi G_5} \vol(S^4) \sqrt{h} \nabla_{\mu} n^{\mu} = -\frac{1}{8\pi G_5} \left[ (c_1(r))^4 \right]^\prime \bigg|_{0}^{r_c} \vol(S^4) ,
\]

(3.21)

we supplement the combined regularized action \( (S_E^{r_c} + S_{GH}) \) by the appropriate boundary counterterms which are needed to get a finite action. These boundary counterterms must be constructed from the local metric and \( \{ \alpha = \ln \rho, \chi \} \) scalar invariants on the boundary \( \partial M_5 \), except for the terms associated with the conformal anomaly which include an explicit dependence on the position of the boundary \([19, 20], \)

\[
S_{\text{counter}} = \frac{1}{4\pi G_5} \vol(S^4) c_1(r_c)^4 \left[ \frac{3}{4} + \frac{1}{4} R_{S^4} + \frac{1}{2} \chi^2 + 3 \alpha^2 + \frac{3}{2} \frac{\alpha^2}{\ln \epsilon} \right.
\]

\[
+ \ln \epsilon \left( \frac{1}{3} R_{S^4} \chi^2 + \frac{2}{3} \chi^4 - \frac{1}{2} \left( R_{S^4} R_{S^4} - \frac{1}{3} R^2_{S^4} \right) \right) + \mathcal{L}_{\text{ambiguity}} \bigg] ,
\]

(3.22)

Here \( R_{S^4} \) and \( R_{S^4 \ ij} \) are the \( S^4 \) Ricci scalar and tensor; the coefficients \( \{ \delta_1 \cdots \delta_5 \} \) parameterize ambiguities of the holographic renormalization scheme. The conformal anomaly terms depend on the position of the boundary; we choose to parameterize this position by the physical quantity

\[
\epsilon \equiv \frac{1}{\sqrt{g_{S^4 S^4}}} \bigg|_{r_c}^{r_c} = c_1(r_c)^{-1} .
\]

(3.23)

The counterterms (3.22) are fixed in such a way that the renormalized Euclidean action \( I_E \) is finite

\[
I_E \equiv \lim_{r_c \to \infty} \left( S_E^{r_c} + S_{GH} + S_{\text{counter}} \right) , \quad |I_E| < \infty .
\]

(3.24)
In what follows we construct each term in (3.24) explicitly. Since the RG flow is parameterized by \(x\), (3.10), we use the same coordinate in computing \(I_E\). Starting from (3.9),

\[
0 = \left( \frac{\rho'(x)}{\rho(x)} \right)^2 + \frac{1}{3} \left( \frac{c'(x)}{c(x)} \right)^2 - 1 \left( \frac{dx(r)}{dr} \right)^2 - \frac{1}{3} \mathcal{P}(\rho(x), c(x)) + \frac{1}{x^2},
\]

(3.25)

where the prime denotes derivative with respect to \(x\), we find

\[
\left( \frac{dx(r)}{dr} \right)^2 = \left( 3\rho(x)^2 c(x)^2 - 3x^2 c(x)^2 (\rho'(x))^2 - x^2 \rho(x)^2 (c'(x))^2 \right)^{-1}
\]

\[
\times \left( \frac{1}{4} \rho(x)^4 x^2 + 3\rho(x)^2 c(x)^2 + \frac{c(x)^2 x^2}{4\rho(x)^2} - \frac{c(x)^6 \rho(x)^{10} x^2}{64} + \frac{1}{4} c(x)^4 \rho(x)^{4} x^2 \right.
\]

\[
+ \frac{c(x)^2 \rho(x)^{10} x^2}{32} - \rho(x)^{10} x^2 \bigg) \]

(3.26)

Denoting \(x_c = x(r_c)\) and using \(\text{vol}(S^4) = 8\pi^2/3\), we rewrite (3.20) as

\[
S^r_E = \frac{2\pi}{3G_5} \left( J_1 + J_2 \right),
\]

\[
J_1 \equiv \frac{1}{2} x^3 \left( \frac{dx(r)}{dr} \right)_{x_c}^x, \quad J_2 \equiv -\frac{3}{2} \int_0^{x_c} x^2 \left( \frac{dx(r)}{dr} \right)^{-1} dx.
\]

(3.27)

Notice that using (3.26), both \(J_1\) and \(J_2\) are expressed through \(x\) radial coordinate. Also, given (3.23), \(\epsilon = x_c^{-1}\). Using asymptotic expansions (3.14) and (3.15) we find

\[
J_1 \equiv J_1^{\text{singular}} + J_1^{\text{finite}},
\]

\[
J_1^{\text{singular}} = \frac{1}{4} \epsilon^{-1} + \left( \frac{1}{2} + \frac{1}{6} (c_{1,0}^b)^2 \right) \epsilon^{-2} + \left( r_{1,1}^b \right)^2 \ln^2 \epsilon + \left( 2(c_{1,0}^b)^2 + \frac{2}{3} (c_{1,0}^b)^4 \right)
\]

\[
+ 2r_{1,1}^b r_{1,0}^b + \frac{1}{2} \left( r_{1,1}^b \right)^2 \ln \epsilon,
\]

\[
J_1^{\text{finite}} = \frac{1}{2} r_{1,0}^b r_{1,1}^b + \frac{1}{2} (c_{1,0}^b)^2 c_{2,0}^b + \frac{1}{8} \left( r_{1,1}^b \right)^2 + \left( r_{1,0}^b \right)^2 + \frac{1}{36} (c_{1,0}^b)^4 + \frac{1}{6} (c_{1,0}^b)^2 - \frac{1}{2}
\]

\[+ \mathcal{O}(\epsilon^2 \ln^3 \epsilon),
\]

(3.28)

where we explicitly separated the singular and the finite parts of \(J_1\) as \(\epsilon = x_c^{-1} \to 0\). Note that both \(J_1^{\text{singular}}\) and \(J_1^{\text{finite}}\) receive contribution only from the boundary. Unlike
\( J_1, J_2 \) can not be computed in closed form analytically. Here we find:

\[
J_2 \equiv J_2^{\text{singular}} + J_2^{\text{finite}},
\]

\[
J_2^{\text{singular}} = -\frac{3}{2} \int_0^1 dx \left( 2x - \frac{4(c_{1,0}^b)^2 + 12}{3x} \right) = -\frac{3}{2} \epsilon^2 - (2(c_{1,0}^b)^2 + 6) \ln \epsilon + \frac{3}{2},
\]

\[
J_2^{\text{finite}} = -\frac{3}{2} \int_0^1 x^2 \left( \frac{dx(r)}{dr} \right)^{-1} dx - \frac{3}{2} \int_1^\epsilon x^2 \left( \frac{dx(r)}{dr} \right)^{-1} dx - \left( 2x - \frac{4(c_{1,0}^b)^2 + 12}{3x} \right)
\]

Next, we represent GH term (3.21) as

\[
S_{GH} = \frac{2\pi}{3G_5} J_3, \quad J_3 = -2x^3 \left( \frac{dx(r)}{dr} \right) \bigg|_{x \epsilon}.
\]

Using (3.14) and (3.15) we find

\[
J_3 \equiv J_3^{\text{singular}} + J_3^{\text{finite}},
\]

\[
J_3^{\text{singular}} = -\epsilon^4 - \left( \frac{2}{3} (c_{1,0}^b)^2 + 2 \right) \epsilon^2 - 4(r_{1,1}^b)^2 \ln^2 \epsilon
\]

\[
- \left( 8r_{1,0}^b r_{1,1}^b + 2(r_{1,1}^b)^2 + 8(c_{1,0}^b)^2 + \frac{8}{3} (c_{1,0}^b)^4 \right) \ln \epsilon,
\]

\[
J_3^{\text{finite}} = -2r_{1,0}^b r_{1,1}^b - 2(c_{1,0}^b)^2 c_{2,0}^b - \frac{2}{3} (c_{1,0}^b)^2 - \frac{1}{9} (c_{1,0}^b)^4 - \frac{1}{2} (r_{1,1}^b)^2 - 4(r_{1,0}^b)^2 + 2
\]

\[
+ O(\epsilon^2 \ln^3 \epsilon).
\]

Finally,

\[
S_{\text{counter}} = \frac{2\pi}{3G_5} J_4, \quad J_4 \equiv J_4^{\text{singular}} + J_4^{\text{finite}} + J_4^{\text{finite, ambiguity}},
\]

\[
J_4^{\text{singular}} = \frac{3}{4} \epsilon^{-4} + \left( 3 + \frac{1}{2} (c_{1,0}^b)^2 \right) \epsilon^{-2} + 3(r_{1,1}^b)^2 \ln^2 \epsilon
\]

\[
+ \left( 8(c_{1,0}^b)^2 + 2(c_{1,0}^b)^2 + 6 + \frac{3}{2} (r_{1,1}^b)^2 + 6r_{1,0}^b r_{1,1}^b \right) \ln \epsilon,
\]

\[
J_4^{\text{finite}} = 3(r_{1,0}^b)^2 + (c_{1,0}^b)^2 c_{2,0}^b - \frac{1}{6} (c_{1,0}^b)^4 + 3r_{1,0}^b r_{1,1}^b + O(\ln^{-1} \epsilon),
\]

\[
J_4^{\text{finite, ambiguity}} = \delta_1 (r_{1,1}^b)^2 + \delta_2 (c_{1,0}^b)^4 + 12\delta_3 (c_{1,0}^b)^2 + 144\delta_4 + O(\ln^{-1} \epsilon).
\]

Note that irrespectively as to whether or not supersymmetry is preserved along the RG flow, i.e., \( r_{1,1}^b = \frac{2}{3} (c_{1,0}^b)^2 \) — see (3.16), all the singularities in \( I_E \) cancel:

\[
J_1^{\text{singular}} + J_2^{\text{singular}} + J_3^{\text{singular}} + J_4^{\text{singular}} = \frac{3}{2}.
\]
Collecting the finite pieces (and accounting for (3.33)), we find

\[
I_E = \frac{2\pi}{3G_5} \left( 3 + \frac{3}{2} c_{1,0}^b k^b + \left( \delta_1 - \frac{3}{8} \right) (r_{1,1}^b)^2 + (c_{1,0}^b)^2 \left( 12\delta_3 - \frac{1}{2} c_{2,0}^b - \frac{1}{2} \right) \right)
\]

\[+ \left( \delta_2 - \frac{1}{4} \right) (c_{1,0}^b)^4 + 144\delta_4 - 12\delta_5 \ r_{1,1}^b + \lim_{\epsilon \to 0} J_2^{finite} \right),
\]  

(3.34)

where \(\delta_i\) are renormalization scheme-dependent ambiguities. For supersymmetric RG flows (see (3.16)) we have

\[
I_{E_susy}^{\text{finite}} = \frac{2\pi}{3G_5} \left( 3 + \left( \frac{1}{8} - \frac{1}{8} c_{2,0}^b + \frac{1}{4} r_{1,0}^b \right) k^2 - \frac{5}{192} k^4 + \lim_{\epsilon \to 0} J_2^{finite} \right)
\]

\[+ I_{E_susy,ambiguity}, \quad I_{E_susy,ambiguity} = \frac{2\pi}{3G_5} (A_0 + A_2 k^2 + A_4 k^4),
\]

(3.35)

where \(A_i\) are ambiguity (purely numerical — mass independent) coefficients. Notice that an immediate consequence of (3.35) is that from the dual gravitational perspective, \(I_{E_susy}^{\text{finite}}\) is ambiguous up to an order-two polynomial in \((mL)^2\).

In section 3.1 we constructed perturbative in \(k\), and fully nonlinear in \(k\), gravitational RG flows corresponding to compactification of \(N = 2^+\) gauge theory on \(S^4\). In the rest of this section we outline perturbative computations of \(I_{E_susy}^{\text{finite}}\) to order \(O(k^2)\), present perturbative results to order \(O(k^6)\) inclusive\(^{10}\), and also present full nonperturbative in \(k\) results.

Using (B.4)-(B.6) and (3.26), we find

\[ - \frac{3}{2} \int_0^1 x^2 \left( \frac{dx(r)}{dr} \right)^{-1} = -\frac{3\sqrt{5}}{2} + 6 \arcsinh \frac{1}{2} + \left( \frac{k}{2} \right)^2 \left( -\frac{24}{\sqrt{5}} \arctanh \frac{1}{\sqrt{5}} \right)
\]

\[+ 12 \arctanh \frac{1}{\sqrt{5}} - \frac{19\sqrt{5}}{10} + 2 \arcsinh \frac{1}{2} \right) + O(k^4),
\]

\[\lim_{\epsilon \to 0} \left[ - \frac{3}{2} \int_1^{1/\epsilon} x \left( x^2 \left( \frac{dx(r)}{dr} \right)^{-1} - \left( 2x - \frac{4(c_{1,0}^b)^2 + 12}{3x} \right) \right) \right] = -\frac{9}{2} + \frac{3\sqrt{5}}{2}
\]

\[- 6 \arctanh \frac{1}{\sqrt{5}} + \left( \frac{k}{2} \right)^2 \left( \frac{24}{\sqrt{5}} \arctanh \frac{1}{\sqrt{5}} + 14 \arctanh \frac{1}{\sqrt{5}} - \frac{7}{2} + \frac{19\sqrt{5}}{10} \right) + O(k^4),
\]

(3.36)

leading to

\[\lim_{\epsilon \to 0} J_2^{finite} = -\frac{9}{2} - \frac{7}{2} \left( \frac{k}{2} \right)^2 .
\]

(3.37)

\(^{10}\)As we emphasized in (3.35) this is the first renormalization scheme-independent contribution to \(I_{E_susy}^{\text{finite}}\).
Figure 3: (Color online) Blue curves represent free energy $\mathcal{F}$ of the holographic PW flow compactified on $S^4$ of radius $R$, see (3.35), in the scheme with $I_{\text{E}}^{\text{susy, ambiguity}} = 0$. The red curve in the left panel is perturbative in $(mR)^2$ approximation to $\mathcal{F}$, see (3.39). The red curve in the right panel is the best fit to data with the ansatz (3.42).

Further using (3.35) we find

$$I_{\text{E}}^{\text{susy}} = \frac{2\pi}{3G_5} \left( -\frac{3}{2} - 5 \left( \frac{k}{2} \right)^2 + \mathcal{O}(k^4) \right) + I_{\text{E}}^{\text{susy, ambiguity}}. \quad (3.38)$$

A straightforward, albeit tedious computation extends (3.38) to order $\mathcal{O}(k^6)$:

$$I_{\text{E}}^{\text{susy}} = \frac{2\pi}{3G_5} \left( -\frac{3}{2} - 5 \left( \frac{k}{2} \right)^2 - \frac{3}{4} \left( \frac{k}{2} \right)^4 + \mathcal{I}_6 \left( \frac{k}{2} \right)^6 + \mathcal{O}(k^8) \right) + I_{\text{E}}^{\text{susy, ambiguity}}, \quad (3.39)$$

$$\mathcal{I}_6 = 0.10696(3).$$

From (3.39), the leading unambiguous contribution to $I_{\text{E}}^{\text{susy}} = \mathcal{F}$ is

$$\mathcal{F} = \frac{N^2}{6} \left( \text{ambiguous} + \mathcal{I}_6 (mR)^6 + \mathcal{O}((mR)^8) \right). \quad (3.40)$$

Given numerical solution to RG flow of $\mathcal{N} = 2^*$ gauge theory on $S^4$ as discussed in section 3.1, we can evaluate its free energy (3.35) in the scheme with $I_{\text{E}}^{\text{susy, ambiguity}} = 0$. Results are presented in Fig. 3. Blue curves represent $\mathcal{F}$ as a function of $(mR)^2$. The red curve in the left panel is perturbative in $(mR)^2$ approximation to $\mathcal{F}$, see (3.39). A polynomial fit of 30 first points to a blue curve produces

$$\frac{6}{N^2} \frac{\mathcal{F}}{I_{\text{fit}}} = -1.5 - 5.0(mR)^2 - 0.749996(mR)^4 + 0.106829(mR)^6 + \mathcal{O}((mR)^8), \quad (3.41)$$

$^{11}$We use (3.4), and $L = 2$, $R = 1$ to express the results in gauge theory variables.
in excellent agreement with (3.39). The red curve in the right panel is the matrix model motivated fit to data (see (2.13)):

\[
\frac{1}{N^2} \mathcal{F}_{fit} = \mathcal{F}_0 + \mathcal{F}_1 (mR)^2 + \mathcal{F}_2 (mR)^4 + \mathcal{F}_3 (1 + (mR)^2) \ln(1 + (mR)^2),
\]

\[
\mathcal{F}_0 = -2.00(7), \quad \mathcal{F}_1 = -0.18(3), \quad \mathcal{F}_2 = -0.02(8), \quad \mathcal{F}_3 = -0.36(9).
\]

We used 2000 points in the fit.

We would like to compare (3.41) and (3.42) with the matrix model result (2.13). As we emphasized earlier, while the holographic free energy \(\mathcal{F}\) is ambiguous, these ambiguities are completely parameterized by a second-order polynomial in \((mR)^2\). A choice of the latter polynomial is equivalent to a choice of the renormalization scheme. Thus, for \(mR \ll 1\) the leading unambiguous coefficient is that of \((mR)^6\), i.e., \(N^2\mathcal{I}_6/6\). Since

\[
\frac{1}{6} \mathcal{I}_6 \neq \frac{1}{12},
\]

where the RHS is the matrix model prediction (2.14), we conclude that \(S^4\)-compactified PW flow can not represent a holographic dual to Pestun’s large-\(N\) matrix model [1] for arbitrary values of \(mR\).

Is it possible to recover the holographic result in the \(S^4\) decompactification limit? Holographic renormalization scheme generically differs from the scheme implicit in the matrix model computation. To account for possible differences, we fit the large \(mR\) data points of the holographic free energy with the ansatz (3.42). Notice that the coefficients \(\{\mathcal{F}_0, \cdots, \mathcal{F}_2\}\) encode the generic scheme dependence, on top of the expected result \(\mathcal{F}_{loc}^{N=2^*}, \ (2.13)\). Since

\[
\mathcal{F}_3 \neq -\frac{1}{2},
\]

where the RHS is the \(mR \gg 1\) matrix model prediction (2.14), we conclude that Pestun’s large-\(N\) matrix model [1] can not reproduce the decompactification limit of the PW flow on \(S^4\).

4 Conclusion

In this paper we address the question whether matrix model localization method developed by Pestun for \(\mathcal{N} = 2\) gauge theories can be used to explain the selection of the Coulomb branch vacuum in holographic \(\mathcal{N} = 2\) renormalization group flows. We focus on two examples: the \(\mathcal{N} = 2^*\) RG flow [2], and the RG flow to the SYM [3]. While in
the former case the matrix model correctly identifies the PW vacuum [8], the matrix model analysis [9] fails to reproduce the GKMW vacuum — there is an agreement though in the extreme high-energy limit, i.e., at energy scales much higher than the compactification scale set by the $S^4$ radius. A possible reason for the discrepancy in the SYM case can be attributed to the fact that the matrix model saddle-point in the $S^4$ decompactification limit is unphysical, as it is identified in the regime with negative running $g_{YM}^2$ coupling.

Further detailed comparison demonstrates that the free energy of the $S^4$ compactified PW flow presented here disagrees with the matrix model computation reported in [8]. The disagreement occurs both for $mR \ll 1$ and in the decompactification limit $mR \gg 1$. We argued that disagreement can not be an artifact of the (potential) difference in holographic and matrix model renormalization schemes.

It is important to understand a reason why the five-dimensional PW effective action [2], which represents a consistent truncation of type IIb supergravity [14], and agrees (at least in the asymptotic boundary region) with the holographic rules of constructing massive deformations of ${\cal N} = 4$ SYM, apparently, does not contain a holographic flow dual to Pestun’s $N = 2^* S^4$ supersymmetric compactification. If there is a different (unknown as advocated in [10]) gravitational dual, how precisely is it different from the PW model? We believe that resolving this issue will lead to a deeper understanding of non-conformal holographic RG flows.

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A  Matching $\mathcal{N} = 2^*$ masses to SUGRA RG flow non-normalizable coefficients

PW solution [2] represents a gravitational dual to $\mathcal{N} = 2^*$ gauge theory on $R^{3,1}$. Given appropriate metric ansatz,

$$ds_5^2 = c_1(r)^2(dR^{3,1})^2 + dr^2,$$

(A.1)

solutions of supersymmetric RG flows from effective action (3.1)-(3.3) can be parameterized as [2]

$$c_1 = \frac{k\rho^2}{\sinh(2\chi)},$$

$$\rho^6 = \cosh(2\chi) + \sinh(2\chi) \ln \frac{\sinh(\chi)}{\cosh(\chi)},$$

(A.2)

where the single integration constant $k$ is related to the hypermultiplet mass $m$ according to [7]

$$k = mL = 2m.$$

(A.3)

Introducing a new radial coordinate as in (3.10), the leading boundary, i.e., $y \equiv \frac{1}{2} \to 0_+$, asymptotics of the flows (A.2) are given by

$$\rho = 1 + \frac{1}{12} k^2 \left( 2 \ln \frac{k}{2} + 1 + 2 \ln y \right) y^2 + \mathcal{O}(y^4 \ln^2 y),$$

$$e^\chi = 1 + \frac{1}{2} y \frac{1}{8} k^2 y^2 + y^2 k^3 \frac{1}{48} \left( 8 \ln \frac{k}{2} + 1 + 8 \ln y \right) + \mathcal{O}(y^4 \ln y).$$

(A.4)

Matching (A.4) with (3.15) we identify

$$c_{b,0} = \frac{k}{2}, \quad r_{b,1} = \frac{1}{6} k^2.$$

(A.5)

B  Perturbative solution of (3.11)-(3.15) to order $\mathcal{O}(k^2)$

It is straightforward to solve (3.11)-(3.15) perturbatively in $\mathcal{O}(k^2)$. Specifically, assuming

$$c(y) = 1 + \sum_{n=1}^{\infty} \left( \frac{k}{2} \right)^n c_n(y), \quad \rho(y) = 1 + \sum_{n=1}^{\infty} \left( \frac{k^2}{6} \right)^n \rho_n(y),$$

(B.1)

with the asymptotic boundary conditions

$$c_n(y) = \delta_n y + \mathcal{O}(y^2), \quad \rho_n(y) = \delta_n y^2 \ln y + \mathcal{O}(y^2),$$

(B.2)
and regularity as $x \to 0_+$, we find from (3.11)-(3.12) two sets of ODEs:

\begin{align}
0 &= c''(n) - \frac{3 + 8y^2}{y(1 + 4y^2)} c'(n) + \frac{3}{y^2(1 + 4y^2)} c(n) + S_{c,n}, \\
0 &= \rho''(2n) - \frac{3 + 8y^2}{y(1 + 4y^2)} \rho'(2n) + \frac{3}{y^2(1 + 4y^2)} \rho(2n) + S_{\rho,n},
\end{align}

(B.3)

where the source terms at order $n$, i.e., $S_{c,n}$ and $S_{\rho,n}$, are functionals of solutions at previous orders. To order $O(k^2)$ inclusive we find

\begin{align}
c(1) &= y \sqrt{1 + 4y^2} - 4 y^3 \arctanh \frac{1}{\sqrt{1 + 4y^2}}, \\
c(2) &= 8 y^6 \arctanh^2 \frac{1}{\sqrt{1 + 4y^2}} - 4 y^4 \sqrt{1 + 4y^2} \arctanh \frac{1}{\sqrt{1 + 4y^2}} + \frac{1}{2} y^2 + 2 y^4, \\
\rho(2) &= y^2 - y^2 \sqrt{1 + 4y^2} \arctanh \frac{1}{\sqrt{1 + 4y^2}}.
\end{align}

(B.4) (B.5) (B.6)

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