ON THE DYNAMICAL SYMMETRIC ALGEBRA OF AGEING: LIE STRUCTURE, REPRESENTATIONS AND APPELL SYSTEMS

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The study of ageing phenomena leads to the investigation of a maximal parabolic subalgebra of $\text{conf}_3$ which we call $\text{alt}$. We investigate its Lie structure, prove some results concerning its representations and characterize the related Appell systems.

1. Introduction

Ageing phenomena occur widely in physics: glasses, granular systems or phase-ordering kinetics are just a few examples. While it is well-accepted that they display some sort of dynamical scaling, the question has been raised whether their non-equilibrium dynamics might possess larger symmetries than merely scale-invariance. At first sight, the noisy terms in the Langevin equations usually employed to model these systems might appear to exclude any non-trivial answer, but it was understood recently that provided the deterministic part of a Langevin equation is Galilei-invariant, then all observables can be exactly expressed in terms of multipoint correlation functions calculable from the deterministic part only. It is therefore of interest to study the dynamical symmetries of non-linear partial differential equations which extend dynamical scaling. In this context, the so-called Schrödinger algebra $\mathfrak{sh}$ has been shown to play an important rôle in phase-ordering kinetics. In what follows we shall restrict to one space dimension and we recall in figure 1 through a root diagram the definition of $\mathfrak{sh}$ as a parabolic subalgebra of the conformal algebra $\text{conf}_3$.6

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2. A brief perspective on the algebra \( \mathfrak{alt} \)

There is a classification of semi-linear partial differential equations with a parabolic subalgebra of \( \mathfrak{conf}_3 \) as a symmetry.\(^9\) Here we shall study the abstract Lie algebra \( \mathfrak{alt} \), which is the other maximal parabolic subalgebra of \( \mathfrak{conf}_3 \) (see figure 1) and its representations; we shall also see that, like the algebra \( \mathfrak{sch} \), it can be embedded naturally in an infinite-dimensional Lie algebra \( W \) which is an extension of the algebra \( \text{Vect}(S^1) \) of vector fields on the circle. Quite strikingly, we shall find on our way a 'no-go theorem' that proves the impossibility of a conventional extension of the embedding \( \mathfrak{alt} \subset \mathfrak{conf}_3 \).

2.1. The abstract Lie algebra \( \mathfrak{alt} \)

Elementary computations make it clear that (see figure 1 and \( D = 2X_0 - N \))

\[
\mathfrak{alt} = \langle V_+, D, Y_{-\frac{1}{2}} \rangle \ltimes \langle X_1, Y_{\frac{1}{2}}, M_0 \rangle := g \ltimes h
\]

(1)

is a semi-direct product of \( g \simeq \mathfrak{sl}(2, \mathbb{R}) \) by a three-dimensional commutative Lie algebra \( h \); the vector space \( h \) is the irreducible spin-1 real representation of \( \mathfrak{sl}(2, \mathbb{R}) \), which can be identified with \( \mathfrak{sl}(2, \mathbb{R}) \) itself with the adjoint action. So one has the following

Proposition 2.1:

1. \( \mathfrak{alt} \simeq \mathfrak{sl}(2, \mathbb{R}) \otimes \mathbb{R}[\varepsilon]/\varepsilon^2 \), where \( \varepsilon \) is a 'Grassmann' variable;
2. \( \mathfrak{alt} \simeq \mathfrak{p}_3 \) where \( \mathfrak{p}_3 \simeq \mathfrak{so}(2,1) \ltimes \mathbb{R}^3 \) is the relativistic Poincaré algebra in \( (2+1) \)-dimensions.
Proof: The linear map $\Phi : \text{alt} \to \mathfrak{sl}(2, \mathbb{R}) \otimes \mathbb{R}[\varepsilon]/\varepsilon^2$ defined by

$$
\Phi(V_+) = L_1, \quad \Phi(D) = L_0, \quad \Phi(Y_{-1}) = L_{-1} \quad \Phi(X_1) = 1/2 L_1, \quad \Phi(Y_{-2}) = L_0, \quad \Phi(M_0) = L_{-1}
$$

is easily checked to be a Lie isomorphism. □

In particular, the representations of $\text{alt} \simeq \mathfrak{p}_3$ are well-known since Wigner studied them in the 30’es.

2.2. Central extensions: an introduction

Consider any Lie algebra $\mathfrak{g}$ and an antisymmetric real two-form $\alpha$ on $\mathfrak{g}$. Suppose that its Lie bracket $[,]$ can be 'deformed' into a new Lie bracket $\tilde{[,]}$ on $\mathfrak{g} := \mathfrak{g} \times \mathbb{R}K$, where $[K, \mathfrak{g}] = 0$, by putting $\tilde{[X, 0], [Y, 0]} = ([X,Y], \alpha(X,Y))$. Then $\mathfrak{g}$ is called a central extension of $\mathfrak{g}$. The Jacobi identity is equivalent with the nullity of the totally antisymmetric three-form $d\alpha : \Lambda^3(\mathfrak{g}) \to \mathbb{R}$ defined by

$$d\alpha(X,Y,Z) = \alpha([X,Y],Z) + \alpha([Y,Z],X) + \alpha([Z,X],Y).$$

Now we say that two central extensions $\mathfrak{g}_1, \mathfrak{g}_2$ of $\mathfrak{g}$ defined by $\alpha_1, \alpha_2$ are equivalent if $\mathfrak{g}_2$ can be gotten from $\mathfrak{g}_1$ by substituting $(X, c) \mapsto (X, c + \lambda(X))$ $(X \in \mathfrak{g})$ for a certain 1-form $\lambda \in \mathfrak{g}^*$, that is, by changing the non-intrinsic embedding of $\mathfrak{g}$ into $\mathfrak{g}_1$. In other words, $\alpha_1$ and $\alpha_2$ are equivalent if $\alpha_2 - \alpha_1 = d\lambda$, where $d\lambda(X,Y) = \langle \lambda, [X,Y] \rangle$. The operator $d$ can be made into the differential of a complex (called Chevalley-Eilenberg complex), and the preceding considerations make it clear that the classes of equivalence of central extensions of $\mathfrak{g}$ make up a vector space $H^2(\mathfrak{g}) = Z^2(\mathfrak{g})/B^2(\mathfrak{g})$, where $Z^2$ is the space of cocycles $\alpha \in \Lambda^2(\mathfrak{g}^*)$ verifying $d\alpha = 0$, and $B^2$ is the space of coboundaries $d\lambda, \lambda \in \mathfrak{g}^*$. We have the well-known

**Proposition 2.2:** The Lie algebra $\text{alt}$ has no non-trivial central extension: $H^2(\text{alt}) = 0$.

All this becomes very different when one embeds $\text{alt}$ into an infinite-dimensional Lie algebra.

2.3. Infinite-dimensional extension of $\text{alt}$

The Lie algebra $\text{Vect}(S^1)$ of vector fields on the circle has a long story in mathematical physics. It was discovered by Virasoro in the 70’es that
Vect(S^1) has a one-parameter family of central extensions which yield the so-called Virasoro algebra

\[ \text{vir} := \text{Vect}(S^1) \oplus \mathbb{R}K = \langle (L_n)_{n \in \mathbb{Z}}, K \rangle \]  

(2)

with Lie brackets

\[ [K, L_n] = 0, \quad [L_n, L_m] = (n - m)L_{n+m} + \delta_{n+m,0} c \frac{n(n^2 - 1)K}{2} \quad (c \in \mathbb{R}) \]

When \( c = 0 \), one retrieves Vect(S^1) by identifying the \((L_n)\) with the usual Fourier basis \((e^{i n \theta} d\theta)_{n \in \mathbb{Z}}\) of periodic vector fields on \([0, 2\pi]\), or with \(-z^{n+1} \frac{d}{dz}\) with \(z := e^{i \theta}\). Note in particular that \(\langle L_{-1}, L_0, L_1 \rangle\) is isomorphic to \(\mathfrak{sl}(2, \mathbb{R})\), and that the Virasoro cocycle restricted to \(\mathfrak{sl}(2, \mathbb{R})\) is 0, as should be (since \(\mathfrak{sl}(2, \mathbb{R})\) has no non-trivial central extensions).

It is tempting to embed \(\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{sl}(2, \mathbb{R}) \otimes \mathbb{R}[\varepsilon]/\varepsilon^2\) into the Lie algebra

\[ \mathcal{W} := \text{Vect}(S^1) \otimes \mathbb{R}[\varepsilon]/\varepsilon^2 = \langle L_n \rangle_{n \in \mathbb{Z}} \ltimes \langle L_\varepsilon \rangle_{n \in \mathbb{Z}}, \]

(3)

with Lie brackets

\[ [L_n, L_m] = (n - m)L_{n+m}, \quad [L_n, L_\varepsilon] = (n - m)L_{\varepsilon}^{n+m}, \quad [L_\varepsilon, L_\varepsilon] = 0. \]

These brackets come out naturally putting \(\mathcal{W}\) in the 2 \(\times\) 2-matrix form

\[ L_n \mapsto \begin{pmatrix} L_n & 0 \\ 0 & L_n \end{pmatrix}, \quad L_\varepsilon \mapsto \begin{pmatrix} 0 & L_\varepsilon \\ 0 & 0 \end{pmatrix} \]

(4)

leading to straightforward generalizations (there exists a deformation of \(\mathfrak{sv}\) that can be represented as upper-triangular 3 \(\times\) 3 Virasoro matrices). In terms of the standard representations of Vect(S^1) as modules of \(\alpha\)-densities \(\mathcal{F}_\alpha = \{ u(z)(dz)\alpha \}\) with the action

\[ f(z) \frac{d}{dz} (u(z)(dz)\alpha) = (fu' + \alpha f'u)(z)(dz)\alpha, \]

(5)

we have

**Proposition 2.3:** \(\mathcal{W} \simeq \text{Vect}(S^1) \ltimes \mathcal{F}_{-1}\).

There are two linearly independent central extensions of \(\mathcal{W}\):

1. the natural extension to \(\mathcal{W}\) of the Virasoro cocycle on Vect(S^1), namely \([ , ] = [ , ]\) except for \([\tilde{L}_n, \tilde{L}_{-n}] = n(n^2 - 1)K + 2nL_0\). In other words, Vect(S^1) is centrally extended, but its action on \(\mathcal{F}_{-1}\) remains unchanged;

2. the cocycle \(\omega\) which is zero on \(\Lambda^2(\text{Vect}(S^1))\) and \(\Lambda^2(\mathcal{F}_{-1})\), and defined by \(\omega(L_n, L_\varepsilon) = \delta_{n+m,0} n(n^2 - 1)K\) on Vect(S^1) \(\times\) \(\mathcal{F}_{-1}\).

A natural related question is: can one deform the extension of Vect(S^1) by the Vect(S^1)-module \(\mathcal{F}_{-1}\)? The answer is: no, thanks to the triviality.
of the cohomology space $H^2(\text{Vect}(S^1), \mathcal{F}_{-1})$. Hence, any Lie algebra structure $[\cdot, \cdot]$ on the vector space $\text{Vect}(S^1) \oplus \mathcal{F}_{-1}$ such that

\[
[(X, \phi), (Y, \psi)] = ([X, Y]_{\text{Vect}(S^1)}, \text{ad}_{\text{Vect}(S^1)} X. \psi - \text{ad}_{\text{Vect}(S^1)} Y. \phi + B(X, Y))
\]

is isomorphic to the Lie structure of $\mathcal{W}$ (where $B$ is an antisymmetric two-form on $\text{Vect}(S^1)$). So one may say that $\mathcal{W}$ and its central extensions are natural objects to look at.

2.4. Some results on representations of $\mathcal{W}$

We now state two results which may deserve deeper thoughts and will be developed in the future.

**Proposition 2.4 (‘no-go theorem’):** There is no way to extend the usual representation of $\mathfrak{alt}$ as conformal vector fields into an embedding of $\mathcal{W}$ into the Lie algebra of vector fields on $\mathbb{R}^3$.

**Proposition 2.5:** The infinite-dimensional extension $\mathcal{W}$ of the algebra $\mathfrak{alt}_1$ is a contraction of a pair of commuting Virasoro algebras $\mathfrak{vir} \oplus \mathfrak{vir} \to \mathcal{W}$. In particular, we have the explicit differential operator representation

\[
L_n = -t^{n+1} \partial_t + (n+1)t^n r \partial_r - (n+1)x t^n - n(n+1)\gamma t^{n-1} r \\
L_n^\gamma = -t^{n+1} \partial_r - (n+1)\gamma t^n
\]

where $x$ and $\gamma$ are parameters and $n \in \mathbb{Z}$.

3. Appell systems

**Definition 3.1.** Appell polynomials $\{h_n(x); n \in \mathbb{N}\}$ on $\mathbb{R}$ are usually characterized by the two conditions

- $h_n(x)$ are polynomials of degree $n$,
- $Dh_n(x) = nh_{n-1}(x)$, where $D$ is the usual derivation operator.

Interesting examples are furnished by the shifted moment sequences

\[
h_n(x) = \int_{-\infty}^{\infty} (x + y)^n p(dy)
\]

where $p$ is a probability measure on $\mathbb{R}$ with all moments finite.

This definition generalizes to higher dimensions. On non-commutative algebraic structures, the shifting corresponds to left or right multiplication and in general, $\{h_n\}$ is not a family of polynomials. We shall call it **Appell systems**.
Appell systems of the Schrödinger algebra \( \mathfrak{sch} \) have been investigated\(^1\) but the algebra \( \mathfrak{alt} \) requires a specific study. \( \mathfrak{alt} \) has the following Cartan decomposition:

\[
\mathfrak{alt} = \mathfrak{P} \oplus \mathfrak{R} \oplus \mathfrak{L} = \{Y_1, X_1\} \oplus \{Y_0, X_0\} \oplus \{Y_{-1}, X_{-1}\}
\]

and there is a one to one correspondence between the subalgebras \( \mathfrak{P} \) and \( \mathfrak{L} \). Write \( X \in \mathfrak{alt} \) in the form \( X = \sum_{i=1}^{6} \alpha_i b_i \), where \( \{b_i, i = 1, \ldots, 6\} \) is a basis of \( \mathfrak{alt} \). The \( \alpha_i \) are called coordinates of the first kind.

Here we use the basis \( b_1 = Y_1, b_2 = X_1, b_3 = Y_0, b_4 = X_0, b_5 = Y_{-1}, b_6 = Y_1 \). Let ALT be the simply connected Lie group corresponding to \( \mathfrak{alt} \). Group elements in a neighbourhood of the identity can be expressed as

\[
e^{X} = e^{A_1 b_1} \cdots e^{A_6 b_6}
\]

The \( A_i \) are called coordinates of the second kind.

Referring to the decomposition (7), we specialize variables, writing \( V_1, V_2, B_1, B_2 \) for \( A_1, A_2, A_5, A_6 \) respectively. Basic for our approach is to establish the partial group law:

\[
e^{B_1 Y_{-1} + B_2 X_{-1}} e^{V_1 Y_1 + V_2 X_1} = ?.
\]

We get

\[
B_1 Y_{-1} + B_2 X_{-1} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
-B_2 & 0 & -B_1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -B_2 & 0
\end{pmatrix}, \quad V_1 Y_1 + V_2 X_1 = \begin{pmatrix}
0 & V_2 & 0 & V_1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

and finally:

\[
e^{B_1 Y_{-1} + B_2 X_{-1}} e^{V_1 Y_1 + V_2 X_1} = \begin{pmatrix}
1 & V_2 & 0 & V_1 \\
-B_2 & 1 - B_2 V_2 & -B_1 & -B_2 V_2 - B_1 V_2 \\
0 & 0 & 1 & V_2 \\
0 & 0 & -B_2 & 1 - B_2 V_2
\end{pmatrix}
\]

**Proposition 3.1.** In coordinates of the second kind, we have the Leibnitz formula,

\[
g(0,0,0,0,B_1,B_2)g(V_1,V_2,0,0,0,0) = g(A_1,A_2,A_3,A_4,A_5,A_6) =
\frac{B_1 V_2^2 + V_1}{(1-B_2 V_2)}, \frac{V_2}{(1-B_2 V_2)}, -\frac{2B_1 V_2 + B_2 V_1}{(1-B_2 V_2)}, 2\ln(1-B_2 V_2), (10)
\]

Now we are ready to construct the representation space and basis-the canonical Appell system. To start, define a vacuum state \( \Omega \). The elements \( Y_1, X_1 \) of \( \mathfrak{P} \) can be used to form basis elements

\[
|jk\rangle = Y_1^j X_1^k \Omega, j,k \geq 0
\]
of a Fock space $\mathcal{F} = \text{span}\{|jk\rangle\}$ on which $Y_1, X_1$ act as raising operators, $Y_{-1}, X_{-1}$ as lowering operator and $Y_0, X_0$ as multiplication with the constants $\gamma, x$ (up to the sign) correspondingly. That is,

$$Y_1\Omega = |10\rangle, \quad X_1\Omega = |01\rangle$$
$$Y_{-1}\Omega = 0, \quad X_{-1}\Omega = 0$$
$$Y_0\Omega = -\gamma|00\rangle, \quad X_0\Omega = -x|00\rangle \quad (12)$$

The goal is to find an abelian subalgebra spanned by some selfadjoint operators acting on representation space, just constructed. Such a two-dimensional subalgebra can be obtained by an appropriate “turn” of the plane $\mathfrak{P}$ in the Lie algebra, namely via the adjoint action of the group element formed by exponentiating $X_{-1}$. The resulting plane, $\mathfrak{P}_\beta$ say, is abelian and is spanned by

$$\bar{Y}_1 = e^{\beta X_{-1}}e^{z_1}Y_1e^{-\beta X_{-1}} = Y_1 - 2\beta Y_0 + \beta^2 Y_{-1}$$
$$\bar{X}_1 = e^{\beta X_{-1}}X_1e^{-\beta X_{-1}} = X_1 - 2\beta X_0 + \beta^2 X_{-1} \quad (13)$$

Next we determine our canonical Appell systems. We apply the Leibniz formula (10) with $B_1 = 0, B_2 = \beta, V_1 = z_1, V_2 = z_2$ and (12). This yields

$$e^{z_1 Y_1}e^{z_2 X_1}\Omega = e^{\beta X_{-1}}e^{z_1 Y_1}e^{z_2 X_1}e^{-\beta X_{-1}}\Omega = e^{\beta X_{-1}}e^{z_1 Y_1}e^{z_2 X_1}\Omega =
\frac{e^{z_1 Y_1}}{e^{z_1 Y_1}}\frac{e^{z_2 X_1}}{e^{z_2 X_1}}e^{\beta X_{-1}}e^{z_1 Y_1}e^{z_2 X_1}\Omega = e^{\left(1 - \beta z_2\right)z_2}e^{(1 - \beta z_2)}(1 - \beta z_2)^{-2x}\Omega \quad (14)$$

To get the generating function for the basis $|jk\rangle$ set in equation (14)

$$v_1 = \frac{z_1}{(1 - \beta z_2)^2}, \quad v_2 = \frac{z_2}{(1 - \beta z_2)} \quad (15)$$

Substituting throughout, we have

**Proposition 3.2.** The generating function for the canonical Appell system, $|jk\rangle = Y_j^l X_k^l \Omega$ is

$$e^{v_1 Y_1 + v_2 X_1}\Omega = \exp(y_1 \frac{v_1}{(1 + \beta v_2)^2})\exp(y_2 \frac{v_2}{(1 + \beta v_2)})\exp(\frac{2\gamma v_1}{(1 - \beta v_2)^2})\frac{(1 + \beta v_2) - 2x}{(1 - \beta v_2)^2}\Omega \quad (16)$$

where we identify $Y_1\Omega = y_1 \cdot 1$ and $X_1\Omega = y_2 \cdot 1$ in the realization as function of $y_1, y_2$.

With $v_1 = 0$, we recognize the generating function for the Laguerre polynomials, while $v_2 = 0$ reduces to the generating function for Hermite polynomials.
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