SUPER DUALITY FOR QUANTUM AFFINE ALGEBRAS OF TYPE A

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Abstract. We introduce a new approach to the study of finite-dimensional representations of the quantum group of the affine Lie superalgebra $Lgl_{M|N} = \mathbb{C}[t, t^{-1}] \otimes gl_{M|N}$ ($M \neq N$). We explain how the representations of the quantum group of $Lgl_{M|N}$ are directly related to those of the quantum affine algebra of type $A$, using an exact monoidal functor called truncation. This can be viewed as an affine analogue of super duality of type $A$.

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1. Introduction

1.1. Lie superalgebra and super duality. Let $gl_{M|N}$ be the general linear Lie superalgebra over $\mathbb{C}$. The character of a finite-dimensional irreducible representation of $gl_{M|N}$ has been obtained in [3, 8, 40] by various independent methods.

One of these methods is the notion of super duality, which was conjectured in [12], and proved in [8] (see also [6] for another proof). Roughly speaking, the super duality provides a concrete connection between the category of finite dimensional $gl_{M|N}$-modules and a maximal parabolic BGG category of $gl_{M+N}$, which is induced from an equivalence between these two categories at infinite ranks. This naturally explains why the theory of finite-dimensional $gl_{M|N}$-module is not parallel to that of $gl_{M+N}$ at all, but how it is still determined by the Kazhdan-Lusztig theory of $gl_{M+N}$. We also refer the reader to [3, 10] for its generalization to the BGG category of $gl_{M|N}$.

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1.2. Quantum affine superalgebra. Let \( \mathfrak{gl}_{M|N} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}l_{M|N} \) be the affine Lie superalgebra associated to \( \mathfrak{g}l_{M|N} \). In this paper, we introduce a new approach to understanding finite-dimensional representations of the quantum group of \( \mathfrak{g}l_{M|N} \). In other words, we explain how they are directly related to or determined by finite-dimensional representations of the quantum affine algebra of type \( A \), which can be viewed as an affine analogue of super duality of type \( A \) due to Cheng-Lam \([5]\). For this purpose, we adopt the notion of a generalized quantum group introduced by Kuniba-Okado-Sergeev \([31]\) as the quantum group for \( \mathfrak{g}l_{M|N} \).

Let \( q \) be an indeterminate. The generalized quantum group \( \mathcal{U}(\epsilon) \) of type \( A \) is a Hopf algebra over \( k = \mathbb{Q}(q) \) associated to a sequence \( \epsilon = (\epsilon_1, \ldots, \epsilon_n) \) with \( n \geq 4 \) and \( \epsilon_i \in \{0, 1\} \), which originates in the study of the three-dimensional Yang-Baxter equation (see \([31]\) and references therein). It is given by an analogue of Drinfeld-Jimbo’s presentation including quantum Serre relation for the odd generator corresponding to \((\epsilon_i, \epsilon_{i+1}) = (0, 1) \) or \((1, 0) \) with respect to cyclic order. The sequence \( \epsilon \) represents the type of non-conjugate Borel subalgebra of \( \mathfrak{g}l_{M|N} \) under the Weyl group action of \( \mathfrak{g}l_{M|N} \), where \( M \) and \( N \) are the numbers of \( 0 \) and \( 1 \) in \( \epsilon \), respectively.

We assume that \( M \neq N \) in this paper. Note that \( \mathcal{U}(\epsilon) \) is different from the previously known quantum affine superalgebras introduced by Yamane \([41]\) (say \( U(\epsilon) \) with a Drinfeld-Jimbo type presentation) and by Zhang \([42, 43, 44]\), where he gives a classification of irreducible modules with respect to Drinfeld realization (when \( \epsilon \) is standard) and studies certain properties of tensor product of fundamental representations under \( R \) matrix including its irreducibility. However, no direct connection with the representation theory of the quantum affine algebra of type \( A \) is given as far as we understand.

1.3. Fusion construction of irreducible modules. Let \( \mathcal{C}(\epsilon) \) be the category of finite-dimensional \( \mathcal{U}(\epsilon) \)-modules with polynomial weights. When \( \epsilon \) is homogeneous, that is, \( \epsilon_{n|0} = (0, \ldots, 0) \) or \( \epsilon_{0|n} = (1, \ldots, 1) \), it is the category of finite-dimensional modules over the quantum group of type \( A_{n-1}^{(1)} \) with polynomial weights. As a module over the subalgebra \( \mathcal{U}(\epsilon) \) corresponding to \( \mathfrak{g}l_{M|N} \), any object in \( \mathcal{C}(\epsilon) \) is completely reducible and decomposes into a direct sum of irreducible polynomial representations \( V_\epsilon(\lambda) \) parametrized by \((M|N)\)-hook partitions \( \lambda \).

For \( l \geq 1 \), let \( \mathcal{W}_{l,\epsilon}(x) \) denote a \( q \)-analogue of the \( l \)-supersymmetric power of the natural representation of \( \mathcal{U}(\epsilon) \) with spectral parameter \( x \in k^\times \), which is the \( l \)-th fundamental representation in case of \( \epsilon_{0|n} \). We denote by \( \mathcal{W}_{l,\epsilon}(x)_{\text{aff}} \) the affinization of \( \mathcal{W}_{l,\epsilon}(x) \).
We construct a large family of irreducible \( \mathcal{U}(\epsilon) \)-modules in \( \mathcal{C}(\epsilon) \) by fusion construction. More precisely, we show that there exists a well-defined renormalized \( R \) matrix in the sense of [22] on arbitrary tensor product of fundamental representations\(^{(1.1)}\)

\[
\mathbf{r}_{l,\epsilon}(\mathbf{c}) : \mathcal{W}_{l_{1},\epsilon}(c_{1}) \otimes \cdots \otimes \mathcal{W}_{l_{t},\epsilon}(c_{t}) \rightarrow \mathcal{W}_{l_{t},\epsilon}(c_{t}) \otimes \cdots \otimes \mathcal{W}_{l_{1},\epsilon}(c_{1}) ,
\]

for \( l = (l_{1}, \ldots, l_{t}) \in \mathbb{Z}_{+}^{t} \), and \( \mathbf{c} = (c_{1}, \ldots, c_{t}) \in (\mathbb{K}^{	imes})^{t} \), and that the image of \( \mathbf{r}_{l,\epsilon}(\mathbf{c}) \), say \( \mathcal{W}_{\epsilon}(l, \mathbf{c}) \), is irreducible if it is not zero. The proof is based on the properties of the \( R \) matrix on real simple modules developed in [22] together with the universal \( R \) matrix for \( \mathcal{U}(\epsilon) \) and the spectral decomposition of its normalized form on \( \mathcal{W}_{l,\epsilon}(x)_{\text{aff}} \otimes \mathcal{W}_{m,\epsilon}(y)_{\text{aff}} \).

1.4. **Truncation functor.** Suppose that \( \epsilon' \) is a subsequence of \( \epsilon \), say \( \epsilon' < \epsilon \). Motivated by the super duality for general linear and ortho-symplectic Lie superalgebra due to Cheng-Lam-Wang [8, 9, 11], the first author introduced a functor \( \mathbf{t}_{\epsilon'}^{\epsilon} : \mathcal{C}(\epsilon) \rightarrow \mathcal{C}(\epsilon') \), which is induced from a homomorphism from \( \mathcal{U}(\epsilon') \) to \( \mathcal{U}(\epsilon) \) [33]. It sends \( \mathcal{W}_{l,\epsilon'}(x) \) and \( V_{\epsilon'}(\lambda) \) to \( \mathcal{W}_{l,\epsilon'}(x) \) and \( V_{\epsilon'}(\lambda) \), respectively. Above all, its most important feature is that it is exact and preserves tensor products.

We show that the fusion construction in (1.1) is compatible with the truncation, that is, \( \mathbf{t}_{\epsilon'}^{\epsilon}(\mathbf{r}_{l,\epsilon}(\mathbf{c})) = \mathbf{r}_{l,\epsilon'}(\mathbf{c}) \) and hence \( \mathbf{t}_{\epsilon'}^{\epsilon}(\mathcal{W}_{l,\epsilon}(l, \mathbf{c})) = \mathcal{W}_{l,\epsilon'}(l, \mathbf{c}) \). Hence if we take \( \epsilon'' \) such that \( \epsilon_{0/n} < \epsilon'' \) and \( \epsilon < \epsilon'' \), then we have the following:

\[
\begin{array}{ccc}
\mathcal{C}(\epsilon) & \xrightarrow{\mathbf{r}_{l,\epsilon}^{\epsilon''}} & \mathcal{C}(\epsilon'') \\
\downarrow & & \downarrow \\
\mathcal{C}(\epsilon) & \xrightarrow{\mathbf{t}_{\epsilon'}^{\epsilon(0/n)}} & \mathcal{C}(\epsilon(0/n))
\end{array}
\]

In particular, this implies that the composition multiplicity of \( \mathcal{W}_{l,\epsilon}(l, \mathbf{c}) \) in a standard module, and the branching multiplicity of a polynomial \( \mathcal{U}(\epsilon) \)-module in \( \mathcal{W}_{l,\epsilon}(l, \mathbf{c}) \) are equal to those in \( \mathcal{C}(\epsilon(0/n)) \) whenever they are non-zero. In this sense, the structure of irreducible \( \mathcal{W}_{l,\epsilon}(l, \mathbf{c}) \) is completely determined by its counterpart in \( \mathcal{C}(\epsilon(0/n)) \).

1.5. **Duality.** In order to have a more concrete connection, we consider the full subcategory \( \mathcal{C}_{J}(\epsilon) \) of \( \mathcal{C}(\epsilon) \), whose composition factors are the irreducible \( \mathcal{U}(\epsilon) \)-modules appearing as composition factors in a tensor product of \( \mathcal{W}_{1,j}(q^{2j}) \) for \( j \in \mathbb{Z} \). Then we construct and apply the generalized quantum affine Schur-Weyl duality functor by Kang-Kashiwara-Kim [20] on \( \mathcal{C}_{J}(\epsilon) \).

Let \( R^{J}_{-gm} \) denote the category of finite-dimensional graded modules over the quiver Hecke algebra \( R^{J} \) associated to a Dynkin quiver of type \( A_{\infty} \). Here \( J \) denotes the set of vertices of the quiver which is equal to \( \mathbb{Z} \) as a set. As was observed in [33] (also in [31] when \( \epsilon \) is standard), the spectral decomposition of the normalized \( R \) matrix on \( \mathcal{W}_{l,\epsilon}(x)_{\text{aff}} \otimes \mathcal{W}_{m,\epsilon}(y)_{\text{aff}} \) does not depend on the choice of \( \epsilon \) up to decomposition into irreducible \( \mathcal{U}(\epsilon) \)-modules. Thus for any \( \epsilon \), we can construct an analogue of generalized quantum affine Schur-Weyl duality functor \( \mathcal{F}_{\epsilon} : R^{J}_{-gm} \rightarrow \mathcal{C}_{J}(\epsilon) \) as in [20], which is associated to \( S : J \rightarrow \{ \mathcal{W}_{l,\epsilon}(1) \} \) and \( X : J \rightarrow \mathbb{K} \) with \( X(j) = q^{-2j} \). The functor \( \mathcal{F}_{\epsilon} \) is exact and preserves tensor products. Furthermore, it is compatible with the truncation so that we
have $\text{tr}_{\epsilon'} \circ F_\epsilon \cong F_{\epsilon'}$ for $\epsilon' < \epsilon$. We remark that $F_\epsilon$ can be also defined for other choice of $S$ and $J$ as in [20].

Let $C^J_\ell(\epsilon)$ be the full subcategory consisting of objects which are direct sums of polynomial $\mathcal{U}(\epsilon)$-modules of degree $\ell$. For the subalgebra $\mathcal{U}(\ell)$ of $\mathcal{U}$ of degree $\ell$, let $\mathcal{U}(\ell)$-mod$_0$ be the category of finite-dimensional $\mathcal{U}(\ell)$-modules on which $x_k$ acts nilpotently. We show that the functor $F_\epsilon$ gives an equivalence from $\mathcal{U}(\ell)$-mod$_0$ to $C^J_\ell(\epsilon)$ if $n > \ell$ by adapting similar arguments for the affine Schur-Weyl duality due to Chari-Pressley [7] and then using the isomorphism between the quiver Hecke algebra and affine Hecke algebra of type $A$ (after a suitable completion) by Brundan-Kleshchev [4] and Rouquier [38].

If we take $\epsilon''$ and $\epsilon'$ such that $\epsilon_{0|n} < \epsilon' < \epsilon''$ and $\epsilon < \epsilon''$ in (1.2) and the lengths of $\epsilon'$, $\epsilon''$ are greater than $\ell$, then we obtain

$$
\begin{array}{ccc}
C^J_\ell(\epsilon) & \xrightarrow{\text{tr}_{\epsilon''}} & C^J_\ell(\epsilon') \\
\cong & & \cong \\
C^J_\ell(\epsilon'') & \xrightarrow{\text{tr}_{\epsilon|n}} & C^J_\ell(\epsilon_{0|n})
\end{array}
$$

where $\text{tr}_{\epsilon''}$ is an equivalence. Now the above relation induces an equivalence between the inverse limit categories by taking limits of the sequences $\epsilon''$ and $\epsilon'$ (cf. [14]). Let $\epsilon^\infty = (\epsilon_i)_{i \geq 1}$ be an infinite sequence with infinitely many 0’s and 1’s such that $\epsilon$ is a subsequence of $\epsilon^\infty$, and let $\epsilon^\infty = (1, 1, 1, \ldots)$. Then we have an equivalence of monoidal categories

$$
\begin{array}{ccc}
C^J(\epsilon) & \xrightarrow{\text{tr}_{\epsilon|n}} & C^J(\epsilon_{0|n}) \\
\cong & & \cong \\
\text{tr}_{\epsilon^\infty} & \xrightarrow{\epsilon^\infty} & \text{tr}_{\epsilon^\infty|n}
\end{array}
$$

where $C^J(\epsilon^\infty)$ (resp. $C^J(\epsilon^\infty|n)$) is the restricted inverse limit category associated to $\epsilon^\infty$ (resp. $\epsilon^\infty|n$), and $\text{tr}_\epsilon$ (resp. $\text{tr}_{\epsilon_{0|n}}$) denotes the truncation associated to the corresponding subsequence.

The equivalence between $C^J(\epsilon^\infty)$ and $C^J(\epsilon^\infty|n)$ induced from (1.3) with $\epsilon^\infty = (0, 0, 0, \ldots)$ gives a kind of duality on the finite-dimensional representations of the quantum affine algebra of type $A_{n-1}^{(1)}$, which looks interesting in itself. This correspondence at the level of Grothendieck rings can be viewed as an affine analogue of the involution on the ring of symmetric functions sending a Schur function to another one of conjugate shape.

1.6. Applications. We present a couple of applications. First, we immediately have that the Grothendieck ring of $C^J(\epsilon)$ is a quotient of the polynomial ring generated by the variables corresponding to the fundamental representations in $C^J(\epsilon)$. In particular, the polynomial, which yields the isomorphism class of a given irreducible module in $C^J(\epsilon)$, is the same as the one in $C^J(\epsilon_{0|n})$ for a sufficiently large $n$ (up to truncation of variables).

Next, we consider a family of irreducible modules in $C^J(\epsilon)$, which corresponds to the Kirillov-Reshetikhin modules over the usual quantum affine algebras. Recall that there exists a short exact sequence consisting of tensor products of Kirillov-Reshetikhin modules [16, 33, 36], which produces an equation called $T$-system [30]. It plays an important role in
the study of $q$-characters [15] and the monoidal categorification of a cluster algebra [18, 19]. Now the diagram (1.3) also implies the existence of such a short exact sequence in $C_{\ell}(\epsilon)$, which yields the same $T$-system.

We expect that our approach (1.3) can be used to apply various other properties of $C(\epsilon|\ell)$ to finite-dimensional representations in $C(\epsilon)$.

1.7. Organization. The paper is organized as follows. In Section 2 we recall the definition of $U(\epsilon)$ and show that the positive (or negative) half of it has a non-degenerate bilinear form. We also construct an isomorphism from $U(\epsilon)$ to the quantum affine superalgebra in [11] after a suitable extension of both of algebras. In Section 3 we define a renormalized $R$ matrix on a tensor product of fundamental representations and construct a family of irreducible objects in $C(\epsilon)$ by fusion construction (1.1). In Section 4 we show that the fusion construction is compatible with the truncation so that the truncation $tr^c_{\epsilon}$ sends $W(\ell, c)$ to $W_{\epsilon}(\ell, c)$. To consider an object which is a limit of $W(\ell, c)$ with respect to truncation, we introduce the inverse limit category associated to $\{C(\epsilon^{(k)})\}_{k \geq 1}$ with $\epsilon^{(k)} < \epsilon^{(k+1)}$. In Section 5 we verify that an analogue of the generalized quantum affine Schur-Weyl duality functor is well-defined for any $\epsilon$. In Section 6 we prove the equivalence of $\mathcal{F}_c$ on $R^q(\ell)$-mod $\epsilon$ with rank $n > \ell$, which implies the equivalence of $tr^c_{\epsilon'}$ for $\epsilon'$ with rank $n' > \ell$.

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2. Generalized quantum groups of type $A$

2.1. Generalized quantum group $U(\epsilon)$. In this paper, we assume that $n$ is a positive integer such that $n \geq 4$ and $q$ is an indeterminate. Let $\mathbb{Z}_+$ denote the set of non-negative integers. We put

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}} \quad (m \in \mathbb{Z}_+).$$

We assume the following notations:

- $k = \mathbb{Q}(q)$, $k^* = k \setminus \{0\}$,
- $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ : a sequence with $\epsilon_i \in \{0, 1\}$ $(i = 1, \ldots, n)$,
- $M = \{|i| \epsilon_i = 0\}$ and $N = \{|i| \epsilon_i = 1\}$,
- $\epsilon_{M|N}$ : a sequence with $\epsilon_1 = \cdots = \epsilon_M = 0$, $\epsilon_{M+1} = \cdots = \epsilon_{M+N} = 1$ $(M + N = n)$,
- $\mathbb{I} = \{1 < 2 < \cdots < n\}$ : a linearly ordered set with $\mathbb{Z}_2$-grading $\mathbb{I} = \mathbb{I}_0 \cup \mathbb{I}_1$ such that

$$\mathbb{I}_0 = \{i \mid \epsilon_i = 0\}, \quad \mathbb{I}_1 = \{i \mid \epsilon_i = 1\},$$

- $P$ : the free abelian group with a basis $\{\delta_i \mid i \in \mathbb{I}\}$,
- $(\cdot, \cdot)$ : a bilinear form on $P$ such that $(\delta_i, \delta_j) = (-1)^{i+j} \delta_{ij} (i, j \in \mathbb{I})$,
- $P' = \text{Hom}_{\mathbb{Z}_2}(P, \mathbb{Z})$ with a basis $\{\delta_i' \mid i \in \mathbb{I}\}$ such that $(\delta_i, \delta_j') = \delta_{ij} (i, j \in \mathbb{I})$,
- $I = \{0, 1, \ldots, n-1\}$,
- $\alpha_i = \delta_i - \delta_{i+1} \in P$ $(i \in I)$,
Definition 2.1. We define $\delta_\nu = \delta^\nu - (-1)^{\epsilon_1^i + \epsilon_{i+1}} \delta^\nu_{i+1} \in P^\nu \ (i \in I)$,
$J_{\text{even}} = \{ i \in I \mid (\alpha_i | \alpha_i) = \pm 2 \}$, $J_{\text{odd}} = \{ i \in I \mid (\alpha_i | \alpha_i) = 0 \}$,
$q_i = (-1)^{\epsilon_1^i} q (-1)^{\epsilon_1^i} \ (i \in \mathbb{I})$, that is,
$$q_i = \begin{cases} q & \text{if } \epsilon_i = 0, \\
-q^{-1} & \text{if } \epsilon_i = 1, \end{cases} (i \in \mathbb{I}),$$
$q(\cdot, \cdot) :$ a symmetric biadditive function from $P \times P$ to $k^*$ given by
$$q(\mu, \nu) = \prod_{\alpha \in \mathbb{P}} q_\alpha^{(\mu, \nu)(\alpha, \delta)},$$
$P_{\text{af}} = \mathbb{Z}\delta_1 \oplus \cdots \oplus \mathbb{Z}\delta_n \oplus \mathbb{Z}\delta :$ a free abelian group of rank $n + 1$,
$\alpha_i = \delta_i - \delta_{i-1} + \delta_0 \delta \in P_{\text{af}} \ (i \in I)$,
$Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset P_{\text{af}}$,
$Q_+ = \sum_{i \in I} \mathbb{Z}\alpha_i$, $Q_- = -Q_+$,
$\text{cl} : P_{\text{af}} \to P$ : the linear map given by $\text{cl}(\delta_i) = \delta_i$ for $i \in \mathbb{I}$ and $\text{cl}(\delta) = 0$,
$i : P \to P_{\text{af}} :$ a section of $\text{cl}$ given by $i(\delta_i) = \delta_i$ for $i \in \mathbb{I}$.

Throughout the paper, we understand the subscript $i \in I$ modulo $n$.

Definition 2.1. We define $\mathcal{U}(\epsilon)$ to be the associative $k$-algebra with 1 generated by $k_\mu, e_i, f_i$ for $\mu \in P$ and $i \in I$ satisfying

\begin{align*}
(2.1) & \quad k_0 = 1, \quad k_\mu + k_{\nu} = k_{\mu + \nu} \quad (\mu, \nu \in P), \\
(2.2) & \quad k_\mu e_i k_{-\mu} = q(\mu, \alpha_i)e_i, \quad k_\mu f_i k_{-\mu} = q(\mu, \alpha_i)^{-1} f_i \quad (i \in I, \mu \in P), \\
(2.3) & \quad e_i f_j - f_j e_i = \delta_{ij} k_{\alpha_i - k_{-\alpha_i}} \quad (i, j \in I), \\
(2.4) & \quad e_i^2 = f_i^2 = 0 \quad (i \in I_{\text{odd}}),
\end{align*}

and

\begin{align*}
& e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0 \quad (i - j \neq \pm 1 \ (\text{mod } n)), \\
& e_i^2 e_j - (-1)^{\epsilon_1^i} [2] e_i e_j e_i + e_j e_i^2 = 0 \quad (i \in I_{\text{even}} \text{ and } i - j \equiv \pm 1 \ (\text{mod } n)), \\
& f_i^2 f_j - (-1)^{\epsilon_1^i} [2] f_i f_j f_i + f_j f_i^2 = 0 \quad (i \in I_{\text{even}} \text{ and } i - j \equiv \pm 1 \ (\text{mod } n)), \\
& e_i e_{i-1} e_i e_{i+1} - e_{i-1} e_i e_{i+1} e_i + e_{i+1} e_i e_{i-1} e_i \\
& \quad - e_{i-1} e_i e_{i+1} e_i + (-1)^{\epsilon_1^i} [2] e_i e_{i-1} e_i e_{i+1} e_i = 0, \quad (i \in I_{\text{odd}}), \\
& f_i f_{i-1} f_i f_{i+1} - f_i f_{i+1} f_i f_{i-1} + f_{i+1} f_i f_{i-1} f_i \\
& \quad - f_{i-1} f_i f_{i+1} f_i + (-1)^{\epsilon_1^i} [2] f_i f_{i-1} f_i f_{i+1} f_i = 0.
\end{align*}

We call $\mathcal{U}(\epsilon)$ the generalized quantum group of affine type $A$ associated to $\epsilon$ (see [31]).

Let $\mathcal{U}(\epsilon)^+$ (resp. $\mathcal{U}(\epsilon)^-$) be the subalgebra of $\mathcal{U}(\epsilon)$ generated by $e_i$ (resp. $f_i$) for $i \in I$, and $\mathcal{U}(\epsilon)^0$ be the one generated by $k_\mu$ for $\mu \in P$. Note that $\mathcal{U}(\epsilon)^\pm$ is naturally graded by $Q_\pm$. 
For simplicity, we put \( k_i = k_{\alpha_i} \) for \( i \in I \). We have
\[
k_i e_j k_{i}^{-1} = q(\alpha_i, \alpha_j) e_j, \quad k_i f_j k_{i}^{-1} = q(\alpha_i, \alpha_j)^{-1} f_j,
\]
\[
e_i f_j - f_j e_i = \delta_{ij} k_{i}^{-1} - k_{i}^{-1}.
\]
for \( i, j \in I \). Note that
\[
q(\alpha_i, \alpha_i) = \begin{cases} q^2 & \text{if } i \in I_{\text{even}} \text{ and } (\epsilon_i, \epsilon_{i+1}) = (0, 0), \\ q^{-2} & \text{if } i \in I_{\text{even}} \text{ and } (\epsilon_i, \epsilon_{i+1}) = (1, 1), \\ -1 & \text{if } i \in I_{\text{odd}}, \end{cases}
\]
for \( i \in I \). If \( \epsilon_i = 0 \) (resp. \( \epsilon_i = 1 \)) for all \( i \in I \), then the subalgebra of \( U(\epsilon) \) generated by \( e_i, f_i, k_i \) for \( i \in I \) is isomorphic to \( U_q' (A_{n-1}^{(1)}) \) (resp. \( U_{-q^{-1}}' (A_{n-1}^{(1)}) \)), the quantum affine algebra of type \( A_{n-1}^{(1)} \) without derivation, more precisely its quotient by a central element.

There is a Hopf algebra structure on \( U(\epsilon) \), where the comultiplication \( \Delta \) is given by
\[
\Delta(k_\mu) = k_\mu \otimes k_\mu, \\
\Delta(e_i) = 1 \otimes e_i + e_i \otimes k_i^{-1}, \\
\Delta(f_i) = f_i \otimes 1 + k_i \otimes f_i,
\]
for \( \mu \in P \) and \( i \in I \).

Let \(- : U(\epsilon) \rightarrow U(\epsilon)\) be the involution of \( \mathbb{Q} \)-algebras given by
\[
\tilde{q} = q^{-1}, \quad \tilde{e}_i = e_i, \quad \tilde{f}_i = f_i, \quad \tilde{k}_\mu = k_\mu^{-1},
\]
for \( i \in I \) and \( \mu \in P \). Let \( \epsilon^c = (\epsilon_1^c, \ldots, \epsilon_n^c) \) be given by \( \epsilon_i^c = 1 - \epsilon_i \) for \( 1 \leq i \leq n \). Then there is an isomorphism of \( \mathbb{Q} \)-algebras \( \sim : U(\epsilon) \rightarrow U(\epsilon^c) \) given by
\[
(2.7) \quad \tilde{q} = q^{-1}, \quad \tilde{e}_i = e_i, \quad \tilde{f}_i = f_i, \quad \tilde{k}_\mu = k_\mu,
\]
for \( i \in I \) and \( \mu \in P \).

2.2. Quantum group \( U(\epsilon) \). Let \( \mathfrak{f}(\epsilon) \) be the free associative \( k \)-algebra with 1 generated by \( \theta_i \) for \( i \in I \). Then \( \mathfrak{f}(\epsilon) \) is naturally graded by \( Q_+ \), that is, \( \mathfrak{f}(\epsilon) = \bigoplus_{\beta \in Q_+} \mathfrak{f}(\epsilon)_{\beta} \) with \( \theta_i \in \mathfrak{f}(\epsilon)_{\alpha_i} \) for \( i \in I \). We let \( |x| = \text{cl}(\beta) \) for \( x \in \mathfrak{f}(\epsilon)_{\beta} \).

We regard \( \mathfrak{f}(\epsilon) \otimes \mathfrak{f}(\epsilon) \) as an associative \( k \)-algebra with multiplication
\[
(x_1 \otimes x_2)(y_1 \otimes y_2) = q(|x_2|, |y_1|)^{-1}(x_1 y_1) \otimes (x_2 y_2)
\]
for homogeneous \( x_i, y_i \in \mathfrak{f}(\epsilon) \) (\( i = 1, 2 \)). Let \( r : \mathfrak{f}(\epsilon) \rightarrow \mathfrak{f}(\epsilon) \otimes \mathfrak{f}(\epsilon) \) be the unique \( k \)-algebra homomorphism given by \( r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i \) for \( i \in I \).

**Proposition 2.2.** There exists a unique symmetric bilinear form \( (\cdot, \cdot) \) on \( \mathfrak{f}(\epsilon) \) with values in \( k \) such that
\[
(1) \quad (1, 1) = 1,
(2) \quad (\theta_i, \theta_j) = \delta_{ij} \text{ for } i, j \in I,
(3) \quad (x, y'y'') = (r(x), y' \otimes y'') \text{ for } x, y', y'' \in \mathfrak{f}(\epsilon),
(4) \quad (x'x'', y) = (x' \otimes x'', r(y)) \text{ for } x', x'', y \in \mathfrak{f}(\epsilon).
\]
Proof. The proof is the same as in [34, Proposition 1.2.3].

Let \( \mathcal{I} \) be the radical of \(( , )\) in Proposition 2.2 and define

\[
f(\epsilon) = f(\epsilon)/\mathcal{I}.
\]

Note that \( f(\epsilon) \) is also \( Q_+ \)-graded since \( \mathcal{I} \) is \( Q_+ \)-graded. For \( i \in I \), let \( i^r : f(\epsilon) \rightarrow f(\epsilon) \) be the unique \( k \)-linear map such that

1. \( i^r(1) = 0 \) and \( i^r(\theta_j) = \delta_{ij} \) for \( j \in I \),
2. \( i^r(xy) = i^r(x)y + q(|x|, \alpha_i)^{-1}x, i^r(y) \) for homogeneous \( x, y \in f(\epsilon) \).

Similarly, let \( r_i : f(\epsilon) \rightarrow f(\epsilon) \) be the \( k \)-linear map such that

1. \( r_i(1) = 0 \) and \( r_i(\theta_j) = \delta_{ij} \) for \( j \in I \),
2. \( r_i(xy) = xr_i(y) + q(|y|, \alpha_i)^{-1}r_i(x)y \) for homogeneous \( x, y \in f(\epsilon) \).

By the same arguments as in [34, 1.2.13], we have

\[
(\theta, y, x) = (y, i^r(x)), \quad (y\theta_i, x) = (y, r_i(x)),
\]

for \( x, y \in f(\epsilon) \). Then we can check that \( \mathcal{I} \) is invariant under \( i^r \) and \( r_i \). We also denote by \( i^r \) and \( r_i \) the induced maps on \( f(\epsilon) \).

Lemma 2.3. For \( i \in I_{odd} \), we have \( i^r(\theta_i^2) = 0 \) and \( \theta_i^2 \in \mathcal{I} \).

Proof. We have \( q(\alpha_i, \alpha_i) = q_iq_{i+1} = -1 \) since \( \epsilon_i \neq \epsilon_{i+1} \), and hence

\[
i^r(\theta_i^2) = \theta_i + q(\alpha_i, \alpha_i)\theta_i = \theta_i - \theta_i = 0.
\]

By [28], we have \( (\theta_i^2, \theta_i^2) = (\theta_i, i^r(\theta_i^2)) = 0 \), which implies that \( \theta_i^2 \in \mathcal{I} \) since \( (f(\epsilon)_\mu, f(\epsilon)_\nu) = 0 \) for \( \mu \neq \nu \).

Proposition 2.4. The generators \( \theta_i \) (\( i \in I \)) of \( f(\epsilon) \) satisfy the relations (2.8) and (2.6).

Proof. The proof of (2.5) is the same as in [34, Proposition 1.4.3].

Let us prove the relation (2.6) for \( \theta_1, \theta_2, \theta_3 \) when \( \epsilon = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \) with \( \epsilon_2 \neq \epsilon_3 \). We consider only the case when \( \epsilon = (0, 0, 1, 1) \) since the other cases can be proved similarly.

Let

\[
S = \theta_2\theta_1\theta_2\theta_3 - \theta_2\theta_3\theta_2\theta_1 + \theta_3\theta_2\theta_1\theta_2 - \theta_1\theta_2\theta_3\theta_2 + [2]\theta_2\theta_1\theta_3\theta_2.
\]

We claim that \( (S, \theta_1\theta_2\theta_3\theta_4) = 0 \) for all \( (i_1, i_2, i_3, i_4) \) such that \( |\theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4}| = \alpha_1 + 2\alpha_2 + \alpha_3 \).

Note that

\[
(q(\alpha_i, \alpha_j))_{1 \leq i, j \leq 3} = \begin{pmatrix}
q^2 & q^{-1} & 1 \\
q^{-1} & -1 & -q \\
1 & -q & q^{-2}
\end{pmatrix}.
\]

For example, if \( (i_1, i_2, i_3, i_4) = (2, 1, 2, 3) \), then we have

\[
(\theta_2\theta_1\theta_2\theta_3, \theta_2\theta_1\theta_2\theta_3) = 1 - q^2,
(\theta_2\theta_3\theta_2\theta_1, \theta_2\theta_1\theta_2\theta_3) = 0,
(\theta_3\theta_2\theta_1\theta_2, \theta_2\theta_1\theta_2\theta_3) = q^{-2}(1 - q^2),
(\theta_1\theta_2\theta_3\theta_2, \theta_2\theta_1\theta_2\theta_3) = 0,
(\theta_2\theta_1\theta_3\theta_2, \theta_2\theta_1\theta_2\theta_3) = q - q^{-1},
\]

\]
and hence
\[(S, \theta_2 \theta_1 \theta_2 \theta_3) = (1 - q^2) + q^{-2}(1 - q^2) + (q + q^{-1})(q - q^{-1}) = 0.\]
The other cases can be also proved in a straightforward manner. Therefore, we conclude that \(S \in I\). □

Let \(\mathcal{U}(\varepsilon)\) be the associative \(k\)-algebra with 1 generated by \(K_\mu, E_i, F_i\) for \(\mu \in P\) and \(i \in I\) subject to the following relations:
\[
\begin{align*}
K_0 &= 1, \quad K_\mu K_{\mu'} = K_{\mu + \mu'} \quad (\mu, \mu' \in P), \\
K_\mu E_i K_{-\mu} &= q(\mu, \alpha_i) E_i, \quad K_\mu F_i K_{-\mu} = q(\mu, \alpha_i)^{-1} F_i \quad (i \in I, \mu \in P), \\
E_i F_j - F_j E_i &= \delta_{ij} \frac{K_{\alpha_i^\vee} - K_{-\alpha_i}}{q - q^{-1}} \quad (i, j \in I).
\end{align*}
\]
Then we define \(U(\varepsilon)\) to be the quotient of \(\mathcal{U}(\varepsilon)\) by the two-sided ideal generated by \(h(E_0, \ldots, E_n)\) and \(h(F_0, \ldots, F_n)\) for any \(h(\theta_0, \ldots, \theta_n) \in I\).

Let \(U(\varepsilon)^+\) (resp. \(U(\varepsilon)^-\)) be the subalgebra of \(U(\varepsilon)\) generated by \(E_i\) (resp. \(F_i\)) for \(i \in I\), and \(U(\varepsilon)^0\) be the one generated by \(K_\mu\) for \(\mu \in P\). By similar arguments as in [34, 3.2], it is not difficult to see that we have an isomorphism of \(k\)-vector spaces
\[
U(\varepsilon)^+ \otimes U(\varepsilon)^0 \otimes U(\varepsilon)^- \longrightarrow U(\varepsilon),
\]
sending \(u \otimes K_\mu \otimes v\) to \(u K_\mu v\).

Let \(\pm : f(\varepsilon) \longrightarrow U(\varepsilon)^\pm\) be the homomorphism of \(k\)-algebras given by \(\theta_i^+ = E_i\) and \(\theta_i^- = F_i\) for \(i \in I\), respectively. By Lemma 2.3 and Proposition 2.4 there exists a surjective homomorphism of \(k\)-algebras
\[
\pi : \mathcal{U}(\varepsilon) \longrightarrow U(\varepsilon),
\]
such that \(\pi(k_\mu) = K_\mu, \pi(e_i) = E_i, \pi(f_i) = F_i\) for \(\mu \in P\) and \(i \in I\).

2.3. Quantum affine superalgebra. Let us first recall the quantized enveloping algebra associated to the affine Lie superalgebra corresponding to the Cartan matrix \(A = (\langle \alpha_j, \alpha_i^\vee \rangle)_{i,j \in I}\).

Definition 2.5. Let \(U(\varepsilon)\) be the associative \(k\)-algebra with 1 generated by \(K_\mu, E_i, F_i\) for \(\mu \in P\) and \(i \in I\) satisfying
\[
\begin{align*}
K_0 &= 1, \quad K_{\mu + \mu'} = K_\mu K_{\mu'} \quad (\mu, \mu' \in P), \\
K_\mu E_i K_{-\mu} &= q(\mu, \alpha_i) E_i, \quad K_\mu F_i K_{-\mu} = q^{-1}(\mu, \alpha_i) F_i \quad (i \in I, \mu \in P), \\
E_i F_j - (-1)^{p(i)p(j)} F_j E_i &= (-1)^{i'j} \delta_{ij} \frac{K_{\alpha_i^\vee} - K_{-\alpha_i}}{q - q^{-1}} \quad (i, j \in I), \\
E_i^2 &= F_i^2 = 0 \quad (i \in I_{\text{odd}}), \\
E_i E_j - (-1)^{p(i)p(j)} E_j E_i &= F_i F_j - (-1)^{p(i)p(j)} F_j F_i = 0, \quad (i, j \in I, i - j \equiv 1 \ (\text{mod} \ n)), \\
E_i^2 E_j E_i - [2] E_i E_j E_i + E_j E_i^2 &= 0 \quad (i \in I_{\text{even}} \text{ and } i - j \equiv 1 \ (\text{mod} \ n)), \\
F_i^2 F_j - [2] F_i F_j F_i + F_j F_i^2 &= 0 \quad (i \in I_{\text{odd}} \text{ and } i - j \equiv 1 \ (\text{mod} \ n)),
\end{align*}
\]
(2.17) \[
E_i, \left[ E_{i-1}, E_i \right](-1)^{p(i-1)+1}q^{-1}, E_{i+1} \right](-1)^{p(i)+1}q^{-1} - \left[ E_{i-1}, E_i \right](-1)^{p(i)+1}q^{-1} + \left[ E_{i+1}, E_i \right](-1)^{p(i-1)+1}q^{-1} = 0
\]
\[
F_i, \left[ F_{i-1}, F_i \right](-1)^{p(i+1)+1}q^{-1}, F_{i+1} \right](-1)^{p(i)+1}q^{-1} - \left[ F_{i-1}, F_i \right](-1)^{p(i+1)+1}q^{-1} + \left[ F_{i+1}, F_i \right](-1)^{p(i-1)+1}q^{-1} = 0
\]
(i \in I_{\text{odd}}),
where \( p(i) = \epsilon_i + \epsilon_{i+1} \) (\( i \in I \)), and \([X,Y]_t = XY - tYX \) for \( t \in k \).

Let \( \Sigma \) be the bialgebra over \( k \) generated by \( \sigma_j \) for \( j \in \mathbb{I} \), which commute with each other and satisfy \( \sigma_j^2 = 1 \). Here the comultiplication is given by \( \Delta(\sigma_j) = \sigma_j \otimes \sigma_j \) for \( j \in \mathbb{I} \). Then \( U(\epsilon) \) is a \( \Sigma \)-module algebra where \( \Sigma \) acts on \( U(\epsilon) \) by

(2.18) \[
\sigma_j K_\mu = K_\mu, \quad \sigma_j E_i = (-1)^{\epsilon_j(\delta_j|\alpha_i)} E_i, \quad \sigma_j F_i = (-1)^{\epsilon_j(\delta_j|\alpha_i)} F_i,
\]
for \( j \in \mathbb{I} \), \( \mu \in P \) and \( i \in I \). Let \( U(\epsilon)[\sigma] \) be the semidirect product of \( U(\epsilon) \) and \( \Sigma \).

Let

(2.19) \[
P^\epsilon_{af} = P^\epsilon_{af} \oplus Z\Lambda_0 = \bigoplus_{j \in \mathbb{I}} Z\delta_i \oplus Z\delta \oplus Z\Lambda_0,
\]
with a symmetric bilinear form

(2.20) \[
(\delta_i|\delta_j) = (-1)^{\epsilon_i \epsilon_j} \delta_{ij}, \quad (\delta_i|\delta) = (\delta_i|\Lambda_0) = (\Lambda_0|\Lambda_0) = (\delta|\delta) = 0, \quad (\Lambda_0|\delta) = 1.
\]

We define \( U(\epsilon)[\sigma] \) to be the \( k \)-algebra defined in the same way as in Definition 2.5 with \( P \), \((\cdot | \cdot)\) and \( \alpha_i \) replaced by (2.19), (2.20), and \( \alpha_i \), respectively. Let \( U(\epsilon)[\sigma] \) be the semidirect product of \( U(\epsilon)[\sigma] \) and \( \Sigma \) with respect to (2.18).

**Remark 2.6.** Let \( \sigma = \sigma_1 \cdots \sigma_n \). Then the subalgebra of \( U(\epsilon)[\sigma] \) generated by \( K_\mu (\mu \in Q + Z\Lambda_0 \subset P^\epsilon_{af}) \), \( E_i, F_i \) (\( i \in I \)) and \( \sigma \) is equal to the quantized enveloping algebra \( U^\sigma_q \) [11, Section 6.4] of the affine Lie superalgebra associated to \( A = ((\alpha_j, \alpha_j'))_{i,j \in \mathbb{I}} \). Note that

(2.21) \[
\sigma K_\mu = K_\mu \sigma, \quad \sigma E_i = (-1)^{p(i)} E_i \sigma, \quad \sigma F_i = (-1)^{p(i)} F_i \sigma,
\]
for \( \mu \) and \( i \in I \). Indeed, the algebra \( U^\sigma_q \) is defined in [11, Section 6.4] as a quotient of the algebra \( \tilde{U}^\sigma_q \) satisfying the relations (2.11)-(2.13) and (2.21) by the 2-sided ideal generated by the radical of a symmetric bilinear form. Then it is shown [11, Theorem 6.8.2] that \( U^\sigma_q \) is isomorphic to the one defined by the relations in Definition 2.5 and (2.21) when \( M \neq N \).

We also define \( \mathcal{U}(\epsilon)[\sigma] \) to be the \( k \)-algebra defined in the same way as in Definition 2.1 where \( P \) and \( \mathcal{q}(\cdot, \cdot) \) replaced by \( P^\epsilon_{af} \) and

\[
\mathcal{q}(\mu, \nu) = \mathcal{q}(\mu_0, \nu_0)q^{ab' + a'b},
\]
for \( \mu = \mu_0 + a\Lambda_0 + b\delta, \nu = \nu_0 + a'\Lambda_0 + b'\delta \in P^\epsilon_{af} \) with \( \mu_0, \nu_0 \in P \). Then we define \( \mathcal{U}(\epsilon)[\sigma] \) to be the semidirect product of \( \mathcal{U}(\epsilon)[\sigma] \) and \( \Sigma \), where \( \Sigma \) acts on \( \mathcal{U}(\epsilon)[\sigma] \) by

\[
\sigma_j K_\mu = K_\mu, \quad \sigma_j E_i = (-1)^{\epsilon_j(\delta_j|\alpha_i)} E_i, \quad \sigma_j F_i = (-1)^{\epsilon_j(\delta_j|\alpha_i)} F_i,
\]
for \( j \in \mathbb{I}, \mu \in P^\epsilon_{af} \) and \( i \in I \).

We may assume that \( M, N \neq 0 \). By (2.17), we may also assume that \( \epsilon_1 = 0 \). Let \( 1 = i_0 < i_1 < \cdots < i_l < i_{l+1} = n \) be the unique sequence such that
Now, let us define a map $\tau$. Note that $I$ is an isomorphism generated by $K_i$ also ([31, Section 3.3]), where the map is defined only on the subalgebra of finite type generated by $K_i$ for each $1 \leq k \leq l + 1$.

Suppose that $\tau(E_i) = e_i$, $\tau(F_i) = f_i$, and $\tau(K_i) = k_i$.

(1) If $i \in \text{I}_{\text{even}}$ with $(\epsilon_i, \epsilon_{i+1}) = (0, 0)$, then we define
\[ \tau(E_i) = e_i, \quad \tau(F_i) = f_i, \quad \tau(K_i) = k_i. \]

(2) If $i \in \text{I}_{\text{odd}}$ with $(\epsilon_i, \epsilon_{i+1}) = (0, 1)$, then we define
\[ \tau(E_i) = e_i \sigma_{\leq i}, \quad \tau(F_i) = f_i \sigma_{\leq i}, \quad \tau(K_i) = k_i \sigma_i. \]

(3) If $i \in \text{I}_{\text{even}}$ with $(\epsilon_i, \epsilon_{i+1}) = (1, 1)$, then we have $\{i, i + 1\} \subset \mathbb{I} \cap \mathbb{I}_1$ for some $1 \leq k \leq l + 1$, and define
\[ \tau(E_i) = e_i \sigma_{i}^{\leq i-k-1}, \quad \tau(F_i) = f_i \sigma_{i}^{\leq i-k-1}, \quad \tau(K_i) = -k_i \sigma_i. \]

(4) If $i \in \text{I}_{\text{odd}} \setminus \{0\}$ with $(\epsilon_i, \epsilon_{i+1}) = (1, 0)$, then we have $i = i_k$ for some $1 < k \leq l$, and define
\[ \tau(E_i) = e_i \sigma_{i}^{\leq i-k-1}, \quad \tau(F_i) = f_i \sigma_{i}^{\leq i-k-1}, \quad \tau(K_i) = (-1)^{i-k} k_i \sigma_i. \]

(5) If $i = 0$ and $i \in \text{I}_{\text{odd}}$, then $i = \epsilon_1$, $0$, and define
\[ \tau(E_0) = e_0 \sigma_{\leq n}^{\leq n-i}, \quad \tau(F_0) = f_0 \sigma_{\leq n}^{\leq n-i-1}, \quad \tau(K_0) = (-1)^{n-i} k_0 \sigma_n. \]

Theorem 2.7. Suppose that $M, N \neq 0$. Let $\mathcal{X}$ be the $\mathbb{k}$-subalgebra of $U(\epsilon)_{\text{al}}^\mathbb{A} [\sigma]$ generated by $E_i, F_i, K_i, K_i^{-1}$ ($i \in \mathbb{I}$), $K_+ \Lambda_0$, and $\sigma_j$ ($j \in \mathbb{I}$). Let $\mathcal{Y}$ be the $\mathbb{k}$-subalgebra of $\mathcal{U}(\epsilon)_{\text{al}}^\mathbb{A} [\sigma]$ generated by $e_i, f_i, k_i, k_i^{-1}$ ($i \in \mathbb{I}$), $k_+ \Lambda_0$, and $\sigma_j$ ($j \in \mathbb{I}$). Then $\tau$ extends to an isomorphism of $\mathbb{k}$-algebras $\tau : \mathcal{X} \rightarrow \mathcal{Y}$ satisfying
\[ \tau(\sigma_j) = \sigma_j \quad (j \in \mathbb{I}), \quad \tau(K_\Lambda_0) = k_\Lambda_0. \]

Proof. It is straightforward to check that $\tau$ yields a well-defined homomorphism of $\mathbb{k}$-algebras $\tau : \mathcal{X} \rightarrow \mathcal{Y}$. The inverse map is defined in a similar way, and hence it is an isomorphism. □

Remark 2.8. The isomorphism is a generalization of the one in [32, Proposition 4.4] (see also [31, Section 3.3]), where the map is defined only on the subalgebra of finite type generated by $K_\mu$ ($\mu \in P$), $E_i, F_i$ ($i \in \mathbb{I} \setminus \{0\}$) and $\sigma_j$ ($j \in \mathbb{I}$), and $\epsilon$ is assumed to be standard, that is, $\epsilon = \epsilon_{M|N}$. We also remark that the algebra $U(\epsilon)_{\text{al}}^\mathbb{A}$ is a Hopf algebra [31], but the isomorphism $\tau$ on $\mathcal{X}$ does not preserve the comultiplication.
2.4. Isomorphism from $\mathcal{U}(\varepsilon)$ to $U(\varepsilon)$.

**Theorem 2.9.** Suppose that $M \neq N$. Then the map $\pi : \mathcal{U}(\varepsilon) \rightarrow U(\varepsilon)$ in (2.10) is an isomorphism of $k$-algebras.

**Proof.** We may define $U(\varepsilon)_{af}$ in the same way as in $U(\varepsilon)_{af}$ (see Section 2.3). There is a well-defined action of $\Sigma$ on $U(\varepsilon)_{af}$ given by

$$\sigma_j K_\mu = K_\mu, \quad \sigma_j E_i = (-1)^{\epsilon_j(\delta_j|\alpha)} E_i, \quad \sigma_j F_i = (-1)^{\epsilon_j(\delta_j|\alpha)} F_i,$$

for $j \in \mathbb{I}$, $\mu \in \mathfrak{P}_{\alpha}^0$ and $i \in I$, since the action of $\Sigma$ on $\mathfrak{f}(\varepsilon)$ is well-defined and it preserves $\mathcal{I}$. Then $U(\varepsilon)_{af}[\varepsilon]$ is defined in a similar way. The projection $\pi$ (2.10) can be extended naturally to

$$\pi : \mathcal{U}(\varepsilon)_{af}[\varepsilon] \rightarrow U(\varepsilon)_{af}[\varepsilon],$$

where $\pi(\sigma_j) = \sigma_j$ for $j \in \mathbb{I}$. Let $\mathcal{X}$ be the $k$-subalgebra of $U(\varepsilon)_{af}[\varepsilon]$ generated by $E_i, F_i, K_i, K_i^{-1}$ ($i \in I$), $K_{\pm \lambda_0}$, and $\sigma_j$ ($j \in \mathbb{I}$), where $K_i := K_{\alpha_i}$. Consider

$$\mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{Z}.$$

Let $\mathcal{X}^+$ be the subalgebra of $\mathcal{X}$ generated by $E_i$ for $i \in I$ and let $\mathcal{X}^0$ be the subalgebra generated by $\sigma_j$ ($j \in \mathbb{I}$), $K_i, K_i^{-1}$ ($i \in I$) and $K_{\pm \lambda_0}$. We define $\mathcal{Y}^+, \mathcal{Y}^0$ and $\mathcal{X}^+, \mathcal{X}^0$ similarly. Recall that $\mathcal{X}^+, \mathcal{Y}^+, \mathcal{X}^0$ are $\mathcal{Q}_+$-graded. For $\beta = \sum_{i \in I} c_i \alpha_i \in \mathcal{Q}_+$, let $\text{ht}(\beta) = \sum_{i \in I} c_i$.

Suppose that $y \in \text{Ker} \pi \cap \mathcal{Y}^+ \setminus \{0\}$ is given. Since $(\mathcal{X}^+_\beta, \mathcal{Y}^+_\beta) = 0$ for $\beta, \beta' \in \mathcal{Q}_+$ ($\beta \neq \beta'$), we may assume that $y$ is homogeneous, that is, $y \in \mathcal{Y}^+_\beta$ for some $\beta \in \mathcal{Q}_+$. We further assume that $\text{ht}(\beta)$ is minimal such that $\text{Ker} \pi \cap \mathcal{Y}^+_\beta \neq 0$.

By Theorem 2.7, there exist a unique non-zero $X \in \mathcal{X}^+_{\beta}$ and a monomial $\zeta$ in $\sigma_i$ such that $\tau(X \zeta) = y$. Since $\text{Ker} \pi \circ \tau$ is a two-sided ideal and $\zeta$ is invertible, we have $X \in \text{Ker} \pi \circ \tau$ as well.

Consider $[X, F_i] := XF_i - (-1)^{\rho(\beta)} F_i X$ for $i \in I$, where $\rho(\beta) = \sum_{i \in I} c_i$. Note that $[X, F_i] \in \text{Ker} \pi \circ \tau$. Applying (2.13), we have

$$[X, F_i] = X' K_i + X'' K_i^{-1}$$

for some $X', X'' \in \mathcal{X}^+_{\beta-\alpha_i}$. Write $\tau(X') = Y' \sigma'$ and $\tau(X'') = Y'' \sigma''$ for some $Y', Y'' \in \mathcal{Y}^+$ and $\sigma', \sigma'' \in \Sigma$. Let $Z' = \pi(Y')$ and $Z'' = \pi(Y'')$. Applying $\pi \circ \tau$ to (2.22), we get

$$Z' K_i \zeta' + Z'' K_i^{-1} \zeta'' = 0,$$

for some $\zeta', \zeta'' \in \Sigma$. Since the vectors $Z' K_i \zeta', Z'' K_i^{-1} \zeta''$ are linearly independent by (2.19), we have $Z' = Z'' = 0$. Hence $Y', Y'' = 0$ by the minimality of $\text{ht}(\beta)$, which implies $X' = X'' = 0$ and $[X, F_i] = 0$.

Finally, we have $X = 0$ by Proposition 6.5.1, and then $y = 0$, which contradicts the fact that $y$ is non-zero. Therefore we conclude that $\pi_{|\mathcal{Y}^+}$ is injective (and the same holds for $\mathcal{Y}^-$, where $\mathcal{Y}^-$ is the subalgebra generated by $F_i$ ($i \in I$)). Since $\pi_{|\mathcal{Y}^0}$ is an isomorphism, so is $\pi$ by (2.20). \qed
3. R matrix

3.1. Finite-dimensional modules and affinization. Let $V$ be a $U(\epsilon)$-module, which is a $k$-vector space. For $\lambda \in P$, let

$$V_\lambda = \{ u \in V \mid k_\mu u = q(\lambda, \mu) u \ (\mu \in P) \}$$

be the $\lambda$-weight space of $V$. We have

$$e_i V_\lambda \subset V_{\lambda + \alpha_i}, \quad f_i V_\lambda \subset V_{\lambda - \alpha_i} \quad (i \in I).$$

Put $\text{wt}(V) = \{ \mu \in P \mid V_\mu \neq 0 \}$. Let $P_{\geq 0} = \sum_{i \in I} \mathbb{Z}_+ \delta_i$. For $\lambda \in P_{\geq 0}$ with $\lambda = \sum_i \lambda_i \delta_i$, let $\deg(\lambda) = \sum_i \lambda_i$. Let $\hat{U}(\epsilon)$ be the $k$-subalgebra of $U(\epsilon)$ generated by $k_\mu, e_i$ and $f_i$ for $\mu \in P$ and $i \in I \setminus \{0\}$.

**Definition 3.1.**

1. Define $\mathcal{C}(\epsilon)$ to be the category of finite-dimensional $U(\epsilon)$-modules $V$ such that $V$ has a weight space decomposition with $\text{wt}(V) \subset P_{\geq 0}$, that is, $V = \bigoplus_{\lambda \in P_{\geq 0}} V_\lambda$.
2. For $\ell \in \mathbb{Z}_+$, define $\mathcal{C}^\ell(\epsilon)$ to be the full subcategory of $\mathcal{C}(\epsilon)$ consisting of $V$ such that $\deg(\lambda) = \ell$ for all $\lambda \in \text{wt}(V)$.
3. Define $\check{\mathcal{C}}(\epsilon)$ to be the category of finite-dimensional $\hat{U}(\epsilon)$-modules $V$ such that $V$ has a weight space decomposition with $\text{wt}(V) \subset P_{\geq 0}$. The subcategory $\check{\mathcal{C}}^\ell(\epsilon)$ is defined in the same way.

Note that $\mathcal{C}(\epsilon)$ is closed under taking submodules, quotients and tensor products, while $\mathcal{C}^\ell(\epsilon)$ is closed under taking submodules and quotients. We also have

$$\mathcal{C}(\epsilon) = \bigoplus_{\ell \in \mathbb{Z}_+} \mathcal{C}^\ell(\epsilon).$$

**Remark 3.2.** Our notion of weight space is slightly different from that of usual quantum affine algebras. However, if we restrict the action of $U(\epsilon)^0$ to that of subalgebra corresponding to $\mathfrak{sl}(M|N)$, then the finite-dimensional representations of $\hat{U}(\epsilon)$ and $U(\epsilon)$ are directly related by the isomorphism $\tau$ in Theorem 2.7 as follows.

Let $X$ (resp. $Y$) be the subalgebra of $U(\epsilon)[\sigma]$ (resp. $U(\epsilon)[\sigma]$) generated by $E_i, F_i, K_i, K_i^{-1}$ $(i \in I)$ and $\sigma_j$ $(j \in \mathbb{I})$ (resp. $e_i, f_i, k_i, k_i^{-1}, \sigma_j$). Then the isomorphism in Theorem 2.7 restricts to $\tau : X \rightarrow Y$ and induces an equivalence between two categories of modules over $X$ and $Y$. Let $V^\tau$ denote the $X$-module obtained from a $Y$-module $V$ by $\tau$.

Let $V$ be a $U(\epsilon)$-module with weight space decomposition $V = \bigoplus_{\lambda \in P} V_\lambda$. Then we have the weight space decomposition as a $Y$-module $V = \bigoplus_{\lambda \in P} V_\lambda$, where

$$\mathcal{T} = P/\mathbb{Z}(\delta_1 + \delta_2 + \cdots + \delta_n),$$

$$V_\lambda = \bigoplus_{\mu \in \mathcal{T}} V_\mu = \{ u \in V \mid k_\mu u = q(\alpha_i, \lambda) u \ (i \in I) \} \quad (\lambda \in P).$$

One can extend the $U(\epsilon)$-action to $U(\epsilon)[\sigma]$ by $\sigma_i u = (-1)^{e_i(\delta_i, \lambda)} u$ for $i \in \mathbb{I}$ and $u \in V_\lambda$. On the other hand, for a $U(\epsilon)$-module $W$ with the weight space decomposition $W = \bigoplus_{\lambda \in P} W_\lambda$,
where
\[ W_\lambda = \{ u \in W | K_\mu u = q^{(\mu|\lambda)} u \ (\mu \in P) \} , \]
we have a similar weight space decomposition as an \( X \)-module, and extend \( W \) to a \( U(\epsilon)[\sigma] \) module in the same way. Then the decomposition \( V^\tau = \bigoplus_{\lambda \in \mathcal{P}} V_\lambda^\tau \) coincides with the weight space decomposition as an \( X \)-module, that is,
\[ V_\lambda^\tau = \{ u \in V^\tau | K_i u = q^{(\alpha_i|\lambda)} u \ (i \in I) \} . \]

Let us introduce an important family of \( U(\epsilon) \)-modules in \( \mathcal{C}(\epsilon) \). Let \[ \mathbb{Z}_+^n(\epsilon) = \{ \mathbf{m} = (m_1, \ldots, m_n) | \epsilon_i = 0 \Rightarrow m_i \in \mathbb{Z}_+, \epsilon_i = 1 \Rightarrow m_i \in \{0, 1\} \} . \]
For \( \mathbf{m} \in \mathbb{Z}_+^n(\epsilon) \), let \( |\mathbf{m}| = m_1 + \cdots + m_n \). For \( i \in \mathbb{I} \), put \( \mathbf{e}_i = (0, \cdots, 1, \cdots, 0) \) where 1 appears only in the \( i \)-th component. For \( l \in \mathbb{Z}_+ \), let \[ W_{l,\epsilon} = \bigoplus_{\mathbf{m} \in \mathbb{Z}_+^n(\epsilon), |\mathbf{m}| = l} k|\mathbf{m}| \]
be the \( k \)-vector space spanned by \( |\mathbf{m}| \) for \( \mathbf{m} \in \mathbb{Z}_+^n(\epsilon) \) with \( |\mathbf{m}| = l \).

For \( x \in \mathbb{k}^\times \), we denote by \( W_{l,\epsilon}(x) \) a \( U(\epsilon) \)-module \( V \), where \( V = W_{l,\epsilon} \) as a \( \mathbb{k} \)-space and the actions of \( k_\mu, \epsilon_i, f_i \) are given by
\[
k_\mu|\mathbf{m}| = q^{\left(\mu, \sum_{j \in \mathbb{I}} m_j \delta_j\right)} |\mathbf{m}| ,
\]
\[
\epsilon_i|\mathbf{m}| = \begin{cases} x^{\delta_i,0}[m_{i+1}] |\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1} | & \text{if } \mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1} \in \mathbb{Z}_+^n(\epsilon) , \\ 0 & \text{otherwise} , \end{cases}
\]
\[
f_i|\mathbf{m}| = \begin{cases} x^{-\delta_i,0}[m_i] |\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1} | & \text{if } \mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1} \in \mathbb{Z}_+^n(\epsilon) , \\ 0 & \text{otherwise} , \end{cases}
\]

for \( \mu \in P, i \in I \) and \( \mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n(\epsilon) \). Here we understand \( \mathbf{e}_0 = \mathbf{e}_n \). It is clear that \( W_{l,\epsilon}(x) \in \mathcal{C}(\epsilon) \). We regard \( W_{l,\epsilon} = W_{l,\epsilon}(1) \). We call \( W_{l,\epsilon}(x) \) a fundamental representation with spectral parameter \( x \).

**Remark 3.3.** If \( \epsilon = \epsilon_0\eta \) and \( 1 \leq l < n \), then \( W_{l,\epsilon}(x) \) is the \( l \)-th fundamental representation of \( U_{q^{-1}}(A_{n-1}^{(1)}) \). If \( \epsilon = \epsilon_n\eta \) and \( \ell \geq 1 \), then \( W_{l,\epsilon}(x) \) is the Kirillov-Reshetikhin module corresponding to the partition \( (l) \).

**Remark 3.4.** The map \( \phi(|\mathbf{m}|) = q^{\sum m_i^2 - \sum_{i\leq j} m_im_j} |\mathbf{m}| \) gives an isomorphism of \( U(\epsilon) \)-modules from \( W_{l,\epsilon}(x) \) to itself with another \( U(\epsilon) \)-action defined in \[ (2.15)] \), where a different comultiplication is used. Note that the \( A_0 \)-span of \( |\mathbf{m}| \) is a crystal lattice of \( W_{l,\epsilon} \) \[ 33 \], where \( A_0 \) is the subring of \( f(q) \in \mathbb{k} \) which are regular at \( q = 0 \).

Let \( \mathcal{P} \) be the set of partitions. A partition \( \lambda = (\lambda_i)_{i \geq 1} \in \mathcal{P} \) is called an \((M|N)\)-hook partition if \( \lambda_{M+1} \leq N \). We denote the set of all \((M|N)\)-hook partitions by \( \mathcal{P}_{M|N} \). For \( \lambda \in \mathcal{P}_{M|N} \), let \( V_\epsilon(\lambda) \) be the irreducible highest weight \( U(\epsilon) \)-module with the highest weight.
\[\sum_{i \geq 1} m_i \delta_i.\] Here \(m_i\) denotes the number of \(i\)'s in the tableau \(H_\lambda\) of shape \(\lambda\), which is defined inductively as follows (see also [11] (2.37)):

1. If \(\epsilon_1 = 0\) (resp. \(\epsilon_1 = 1\)), then fill the first row (resp. column) of \(\lambda\) with 1.
2. After filling a subdiagram \(\mu\) of \(\lambda\) with \(1, \ldots, k\), fill the first row (resp. column) of \(\lambda/\mu\) with \(k + 1\) if \(\epsilon_{k+1} = 0\) (resp. \(\epsilon_{k+1} = 1\)).

If we put \(V = V_\epsilon((1))\), then \(V^\otimes \ell\) is semisimple and its irreducible components are \(V_\epsilon(\lambda)\)'s for \((M|N)\)-hook partitions \(\lambda\) of \(\ell\) [2] [3]. In particular, \(V_\epsilon(\lambda) \in \hat{C}(\epsilon)\). Note that \(W_{l,\epsilon}(x) \cong V_\epsilon((l))\) for \(l \geq 1\) as a \(\hat{U}(\epsilon)\)-module. Hence, any tensor product of a finite number of \(W_{l,\epsilon}(x)\)'s is completely reducible as a \(\hat{U}(\epsilon)\)-module, and decomposes into \(V_\epsilon(\lambda)\)'s for \(\lambda \in \mathcal{P}_M\).

Let \(V\) be a \(\hat{U}(\epsilon)\)-module in \(C(\epsilon)\) and \(z\) an indeterminate. We define a \(\hat{U}(\epsilon)\)-module \(V_{\sl\sl\sl}\) by

\[V_{\sl\sl\sl} = \mathbb{k}[z, z^{-1}] \otimes V,\]

where \(k_\mu\), \(e_i\), and \(f_i\) acts as \(1 \otimes k_\mu\), \(z^{\delta_i \alpha} \otimes e_i\), and \(z^{-\delta_i \alpha} \otimes f_i\), respectively. For \(\lambda \in P_{\sl\sl\sl}\), we define the \(\lambda\)-weight space of \(V_{\sl\sl\sl}\) by

\[(V_{\sl\sl\sl})_\lambda = z^k \otimes V_{\cl(\lambda)},\]

where \(\lambda - \iota \circ \cl(\lambda) = k\delta\). Then we have

\[V_{\sl\sl\sl} = \bigoplus_{\lambda \in P_{\sl\sl\sl}} (V_{\sl\sl\sl})_\lambda,\]

and \(e_i(V_{\sl\sl\sl})_\lambda \subset (V_{\sl\sl\sl})_{\lambda + \alpha_i}\), \(f_i(V_{\sl\sl\sl})_\lambda \subset (V_{\sl\sl\sl})_{\lambda - \alpha_i}\), for \(i \in I\) and \(\lambda \in P_{\sl\sl\sl}\). The map sending \(g(z) \otimes m\) to \(z g(z) \otimes m\) for \(m \in V\) and \(g(z) \in \mathbb{k}[z, z^{-1}]\) gives an isomorphism of \(\hat{U}(\epsilon)\)-modules

\[z : V_{\sl\sl\sl} \longrightarrow V_{\sl\sl\sl},\]

such that \((V_{\sl\sl\sl})_\lambda \subset (V_{\sl\sl\sl})_{\lambda + \delta}\) for \(\lambda \in P_{\sl\sl\sl}\). For \(x \in \mathbb{k}^\times\), we define

\[V_x = V_{\sl\sl\sl}/(z - x)V_{\sl\sl\sl}.\]

Note that \(W_{l,\epsilon}(x) \cong (W_{l,\epsilon}(x))_1 \otimes (W_{l,\epsilon}(x))_{1}^*\) for \(x \in \mathbb{k}^\times\) as a \(\hat{U}(\epsilon)\)-module.

3.2. Universal R matrix. In this subsection, we assume that \(M \neq N\). Let \(\overline{\Delta}\) be the comultiplication on \(\hat{U}(\epsilon)\) given by \(\overline{\Delta}(u) = \overline{\Delta}(u)\) for \(u \in \hat{U}(\epsilon)\), that is,

\[\overline{\Delta}(k_\mu) = k_\mu \otimes k_\mu,\]

\[\overline{\Delta}(e_i) = 1 \otimes e_i + e_i \otimes k_i,\]

\[\overline{\Delta}(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i,\]

for \(\mu \in P\) and \(i \in I\). Let

\[\hat{U}(\epsilon)^+ \otimes \hat{U}(\epsilon)^- = \bigoplus_{\xi \in Q} \prod_{\xi \neq \nu} (\hat{U}(\epsilon)^+ \otimes \hat{U}(\epsilon)^-),\]

which is a ring under the multiplications induced from \(\hat{U}(\epsilon)^\pm\). Similarly, let \(\hat{U}(\epsilon)^- \otimes \hat{U}(\epsilon)^+\) be the one defined by replacing \(x \otimes y \in \hat{U}(\epsilon)^+ \otimes \hat{U}(\epsilon)^-\) with \(y \otimes x\).

For \(\beta \in Q^+\), let \(B_\beta\) be a basis of \(f(\epsilon)_{\beta}\) and let \(B_\beta^* = \{ b^* \mid b \in B_\beta \}\) be the dual basis of \(B_\beta\), that is, \((b^*, b') = \delta_{b,b'}\) for \(b, b' \in B_\beta\). We may identify \(U(\epsilon)\) with \(U(\epsilon)\) by Theorem [2]. By
slightly modifying the arguments in the proof of [34, Theorem 4.1.2], we have the following analogue.

**Theorem 3.5.**

1. There is a unique \( \Theta_\beta \in \mathcal{U}(\epsilon)_\beta^+ \otimes \mathcal{U}(\epsilon)_\beta^- \) for each \( \beta \in Q_+ \) such that \( \Theta_0 = 1 \otimes 1 \) and \( \Theta \Delta(u) = \Delta(u) \Theta \)

   for \( u \in \mathcal{U}(\epsilon) \), where \( \Theta = \sum_{\beta \in Q_+} \Theta_\beta \in \mathcal{U}(\epsilon)_+ \otimes \mathcal{U}(\epsilon)_- \).

2. For \( \beta = \sum_{i \in I} a_i \alpha_i \in Q_+ \), we have

   \[ \Theta_\beta = (q-q^{-1})^{ht(\beta)} \sum_{b \in B_\beta} b^+ \otimes (b^*)^- , \]

   where \( ht(\beta) = \sum_{i \in I} a_i \).

3. Let \( \overline{\Theta} = \sum_{\beta \in Q_+} \overline{\Theta}_\beta \), where \( \overline{\Theta}_\beta := (- \otimes -)(\Theta_\beta) \). Then

   \[ \overline{\Theta} \Theta = \Theta \overline{\Theta} = 1. \]

**Remark 3.6.** We can prove that \( \Theta \) also satisfies the properties in [34, 4.2] by slight modification of the arguments.

Let \( V, W \) be \( \mathcal{U}(\epsilon) \)-modules in \( \mathcal{C}(\epsilon) \). Let \( z_1, z_2 \) be indeterminates and let

\[ V_{\text{aff}} = k[z_1, z_1^{-1}] \otimes V; \quad W_{\text{aff}} = k[z_2, z_2^{-1}] \otimes W. \]

We consider two completions given in [27, Section 7]:

\[ V_{\text{aff}} \hat{\otimes} W_{\text{aff}} = \sum_{\lambda, \mu \in P_\text{aff} \beta \in Q_+} \prod_{\beta \in Q_+} (V_{\text{aff}})_\lambda^+ \otimes (W_{\text{aff}})_\mu^- , \]

\[ W_{\text{aff}} \hat{\otimes} V_{\text{aff}} = \sum_{\lambda, \mu \in P_\text{aff} \beta \in Q_+} \prod_{\beta \in Q_+} (W_{\text{aff}})_\lambda^- \otimes (V_{\text{aff}})_\mu^+ , \]

which are invariant under the action of \( \mathcal{U}(\epsilon)_+ \hat{\otimes} \mathcal{U}(\epsilon)_- \) and \( \mathcal{U}(\epsilon)_- \hat{\otimes} \mathcal{U}(\epsilon)_+ \), respectively. Let \( \Pi_q : V_{\text{aff}} \hat{\otimes} W_{\text{aff}} \longrightarrow V_{\text{aff}} \hat{\otimes} W_{\text{aff}} \) be given by

\[ \Pi_q(v \otimes w) = q(cl(\mu), cl(\nu)) v \otimes w, \]

for \( v \in (V_{\text{aff}})_\mu \) and \( w \in (W_{\text{aff}})_\nu \). Let \( s : V_{\text{aff}} \hat{\otimes} W_{\text{aff}} \longrightarrow W_{\text{aff}} \hat{\otimes} V_{\text{aff}} \) be the map given by \( s(v \otimes w) = w \otimes v \). Then

**Theorem 3.7.** We have an isomorphism of \( \mathcal{U}(\epsilon) \)-modules

\[ R_{\text{univ}} := \Theta \circ \Pi_q \circ s : V_{\text{aff}} \hat{\otimes} W_{\text{aff}} \longrightarrow W_{\text{aff}} \hat{\otimes} V_{\text{aff}} \]

**Proof.** The proof is almost the same as the one in [34, Theorem 32.1.5], where we replace \( f \Pi \) in [34] with \( \Pi_q \). The inverse is given by \( s \circ \Pi_q^{-1} \circ \overline{\Theta} \). □

The operator \( R_{\text{univ}} \) is called the universal \( R \) matrix. Since \( \text{wt}(V) \) and \( \text{wt}(W) \) are finite subsets of \( P_{\geq 0} \), we have as \( \mathcal{U}(\epsilon) \)-modules

\[ V_{\text{aff}} \hat{\otimes} W_{\text{aff}} = k[[z_1/z_2]] \otimes_{k[[z_1/z_2]]} (V_{\text{aff}} \otimes W_{\text{aff}}) , \]

\[ W_{\text{aff}} \hat{\otimes} V_{\text{aff}} = k[[z_1/z_2]] \otimes_{k[[z_1/z_2]]} (W_{\text{aff}} \otimes V_{\text{aff}}) . \]
In particular, we have a $\mathcal{U}(\epsilon)$-linear map

$$\mathcal{R}^{\text{univ}} : \mathcal{V}_{\text{aff}} \otimes \mathcal{V}_{\text{aff}} \longrightarrow \mathbb{k}[\mathbb{Z}_1/\mathbb{Z}_2] \otimes \mathbb{k}[\mathbb{Z}_1/\mathbb{Z}_2] (\mathcal{V}_{\text{aff}} \otimes \mathcal{V}_{\text{aff}}).$$

### 3.3. Truncation

Let $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_{n-1})$ be the sequence obtained from $\epsilon$ by removing $\epsilon_i$ for some $i \in \mathbb{I}$. We further assume that

$\epsilon'$ is homogeneous when it is of length 3, that is, $\epsilon' = (000)$ or $(111)$. Let $\mathbb{I}' = \{1, \ldots, n-1\}$ with the $\mathbb{Z}_2$-grading given by $\mathbb{I}' = \{i \mid e'_i = \varepsilon\}$ ($\varepsilon = 0, 1$). We assume that the weight lattice for $\mathcal{U}(\epsilon')$ is $P' = \bigoplus_{i \in \mathbb{I}'} \mathbb{Z} e'_i$. Put $I' = \{0, 1, \ldots, n-2\}$. Let us denote by $k'_{\mu}, e'_i, f'_j$ the generators of $\mathcal{U}(\epsilon')$ for $\mu \in P'$ and $j \in I'$. Put $k'_j = k'_{\delta'_j - \delta'_{j+1}}$ for $j \in I'$. Then we define $\hat{k}'_j, \hat{e}_j, \hat{f}_j \in \mathcal{U}(\epsilon)$ for $l \in I'$ and $j \in I'$ as follows:

$$\hat{k}'_j = \begin{cases} k_l & \text{for } 1 \leq l \leq i - 1, \\ k_{i+l} & \text{for } i \leq l \leq n-1. \end{cases}$$

**Case 1.** If $2 \leq i \leq n-1$, then

$$(\hat{e}_j, \hat{f}_j) = \begin{cases} (e_j, f_j) & \text{for } j \leq i-2, \\ \left(e_{i-1}, e_i \mid q_{i-1,1}, f_i, f_{i-1} \right)_{q_{i-1,1}} & \text{for } j = i-1, \\ (e_{j+1}, f_{j+1}) & \text{for } j \geq i. \end{cases}$$

**Case 2.** If $i = n$, then

$$(\hat{e}_j, \hat{f}_j) = \begin{cases} (e_j, f_j) & \text{for } j \neq 0, \\ \left(e_{n-1}, e_0 \mid q_{n-1,0}, f_0, f_{n-1} \right)_{q_{n-1,0}} & \text{for } j = 0. \end{cases}$$

**Case 3.** If $i = 1$, then

$$(\hat{e}_j, \hat{f}_j) = \begin{cases} (e_0, e_1 \mid q_{0,1}, f_1, f_0 \right)_{q_{0,1}} & \text{for } j = 0, \\ (e_{j+1}, f_{j+1}) & \text{for } j \neq 0. \end{cases}$$

Here $q_{a,b} = q(a_\alpha, b_\alpha)$ for $a, b \in I$, and $[X, Y]_t = XY - t YX$ for $X, Y \in \mathcal{U}(\epsilon)$, $t \in \mathbb{k}^\times$.

**Theorem 3.8 (\cite{KK}, Theorem 4.3).** Under the above hypothesis, there exists a homomorphism of $\mathbb{k}$-algebras

$$\phi_{\epsilon'} : \mathcal{U}(\epsilon') \longrightarrow \mathcal{U}(\epsilon),$$

such that for $l \in \mathbb{I}'$ and $j \in I'$

$$\phi_{\epsilon'}(k'_l) = \hat{k}'_l, \quad \phi_{\epsilon'}(e'_l) = \hat{e}_j, \quad \phi_{\epsilon'}(f'_l) = \hat{f}_j.$$ 

More generally, suppose that the sequence $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_{n-1})$ with $1 \leq r \leq n - 3$ is obtained from $\epsilon$ by removing $\epsilon_{i_1}, \ldots, \epsilon_{i_r}$ for some $i_1 < \cdots < i_r$. For $0 \leq k \leq r$, let $\epsilon^{(k)}$ be a sequence such that

1. $\epsilon^{(0)} = \epsilon, \epsilon^{(r)} = \epsilon'$,
2. $\epsilon^{(k)}$ is obtained from $\epsilon^{(k-1)}$ by removing $\epsilon_{i_k}$ for $1 \leq k \leq r$. 




We define a homomorphism of \(\mathbb{k}\)-algebras
\[
\phi_{e'} : \mathcal{U}(e') \longrightarrow \mathcal{U}(\epsilon),
\]
by \(\phi_{e'} = \phi_{e'}^{(0)} \circ \phi_{e'}^{(1)} \circ \cdots \circ \phi_{e'}^{(r-1)}\).

For a \(\mathcal{U}(\epsilon)\)-module \(V\) in \(\mathcal{C}(\epsilon)\), let
\[
\text{tr}_{e'}(V) = \bigoplus_{\mu \in \text{wt}(V)} V_{\mu}.
\]
We denote by \(\pi_{e'} : V \longrightarrow \text{tr}_{e'}(V)\) the natural projection. For any \(\mathcal{U}(\epsilon)\)-modules \(V, W\) in \(\mathcal{C}(\epsilon)\) and \(f \in \text{Hom}_{\mathcal{U}(\epsilon)}(V, W)\), let
\[
\text{tr}_{e'}(f) : \text{tr}_{e'}(V) \longrightarrow \text{tr}_{e'}(W)
\]
be the \(\mathbb{k}\)-linear map given by \(\text{tr}_{e'}(f)(v) = f(v)\) for \(v \in \text{tr}_{e'}(V)\). Then we have the following commutative diagram of \(\mathbb{k}\)-vector spaces:

\[
\begin{array}{ccc}
V & \xrightarrow{f} & W \\
\downarrow{\pi_{e'}} & & \downarrow{\pi_{e'}} \\
\text{tr}_{e'}(V) & \xrightarrow{\text{tr}_{e'}(f)} & \text{tr}_{e'}(W)
\end{array}
\]

**Proposition 3.9** ([33 Propositions 4.4]). Under the above hypothesis,

1. the space \(\text{tr}_{e'}(V)\) is invariant under the action of \(\mathcal{U}(e')\) via \(\phi_{e'}\), and hence a \(\mathcal{U}(e')\)-module in \(\mathcal{C}(e')\),
2. the map \(\text{tr}_{e'}(f) : \text{tr}_{e'}(V) \longrightarrow \text{tr}_{e'}(W)\) is \(\mathcal{U}(e')\)-linear,
3. the space \(\text{tr}_{e'}(V \otimes W)\) is naturally isomorphic to the tensor product of \(\text{tr}_{e'}(V)\) and \(\text{tr}_{e'}(W)\) as a \(\mathcal{U}(e')\)-module.

Hence we have a functor
\[
\text{tr}_{e'} : \mathcal{C}(\epsilon) \longrightarrow \mathcal{C}(e'),
\]
which we call truncation. Note that \(\text{tr}_{e'}\) is exact by its definition and monoidal by Proposition 3.9 in the sense of [20 Appendix A.1]. We may also define \(\tilde{\text{tr}}_{e'} : \tilde{\mathcal{C}}(\epsilon) \longrightarrow \tilde{\mathcal{C}}(e')\) in the same way.

**Proposition 3.10** ([33 Propositions 4.5 and 4.6]). Let \(M'\) and \(N'\) be the numbers of \(j\)'s with \(\epsilon'_j = 0\) and \(\epsilon'_j = 1\) in \(e'\), respectively.

1. For \(\lambda \in \mathcal{P}_{M'N'}\), \(\text{tr}_{e'}(V_{\epsilon'}(\lambda))\) is non-zero if and only if \(\lambda \in \mathcal{P}_{M'|N'}\). In this case, we have \(\text{tr}_{e'}(V_{\epsilon'}(\lambda)) \cong V_{\epsilon'}(\lambda)\), as a \(\mathcal{U}(e')\)-module.
2. For \(l \in \mathbb{Z}_+\) and \(x \in \mathbb{k}^\times\), \(\text{tr}_{e'}(W_{l, \epsilon}(x))\) is non-zero if and only if \(l \in \mathcal{P}_{M'|N'}\). In this case, we have \(\text{tr}_{e'}(W_{l, \epsilon}(x)) \cong W_{l, \epsilon'}(x)\) as a \(\mathcal{U}(e')\)-module.

**Remark 3.11.** We may assume that the isomorphism
\[
\text{tr}_{e'}(W_{l, \epsilon}(x_1) \otimes \cdots \otimes W_{l, \epsilon}(x_l)) \cong W_{l, \epsilon'}(x_1) \otimes \cdots \otimes W_{l, \epsilon'}(x_l)
\]
induced from Proposition 3.10(2), when restricted as a $\hat{U}(e')$-linear map, gives the isomorphism in Proposition 3.10(1) on each $\tau_{e'}(V_\epsilon(\lambda))$.

3.4. **Fundamental representations and $R$ matrices.** Let $l, m \in \mathbb{Z}_+$ and $x, y \in \mathbb{k}^\times$ be given. As a $\hat{U}(\epsilon)$-module,

\[(3.4) \quad W_{l,\epsilon}(x) \otimes W_{m,\epsilon}(y) \cong \bigoplus_{t \in H(l, m)} V_\epsilon((l + m - t, t)),\]

where $H(l, m) = \{ t \mid 0 \leq t \leq \min\{l, m\}, (l + m - t, t) \in \mathcal{P}_{M|N} \}$ (cf. [33 Remark 3.5]). Let $\epsilon'' = (\epsilon''_1, \ldots, \epsilon''_n)$ be a sequence of 0,1’s with $n'' \geq n$ such that

1. $\epsilon$ is a subsequence of $\epsilon''$,
2. as a $\hat{U}(\epsilon'')$-module

\[W_{l,\epsilon''}(x) \otimes W_{m,\epsilon''}(y) \cong \bigoplus_{0 \leq t \leq \min\{l, m\}} V_\epsilon''((l + m - t, t)),\]

3. if $\epsilon' = \epsilon_{M''|0}$ with $M'' = \{| i \mid \epsilon''_i = 0 \}$, then as a $\hat{U}(\epsilon')$-module

\[(3.5) \quad W_{l,\epsilon'}(x) \otimes W_{m,\epsilon'}(y) \cong \bigoplus_{0 \leq t \leq \min\{l, m\}} V_{\epsilon'}((l + m - t, t)).\]

Note that we have the following:

\[
\begin{array}{ccc}
W_{l,\epsilon''}(x) \otimes W_{m,\epsilon''}(y) & \xrightarrow{\pi_{\epsilon''}} & W_{l,\epsilon'}(x) \otimes W_{m,\epsilon'}(y) \\
W_{l,\epsilon}(x) \otimes W_{m,\epsilon}(y) & \xrightarrow{\pi_{\epsilon'}} & W_{l,\epsilon'}(x) \otimes W_{m,\epsilon'}(y)
\end{array}
\]

For $0 \leq t \leq \min\{l, m\}$, let $v'(l, m, t)$ be the highest weight vector of $V_{\epsilon'}((l + m - t, t))$ in $W_{l,\epsilon'}(x) \otimes W_{m,\epsilon'}(y)$ such that

\[v'(l, m, t) \in L_{l,\epsilon'} \otimes L_{m,\epsilon'}, \quad \epsilon'((l, m, t)) \equiv |le_1| \otimes |(m - t)e_1 + te_2| \pmod{qL_{l,\epsilon'} \otimes L_{m,\epsilon'}},\]

where $L_{s,\epsilon'}$ is the lower crystal lattice of $W_{s,\epsilon'}$ spanned by $|m\rangle$. We also define $v'(m, l, t)$ in the same manner. By Proposition 3.10(1), we may regard

\[V_{\epsilon'}((l + m - t, t)) \subset V_{\epsilon''}((l + m - t, t)) \quad (0 \leq t \leq \min\{l, m\}), \quad V_{\epsilon}((l + m - t, t)) \subset V_{\epsilon''}((l + m - t, t)) \quad (t \in H(l, m)),\]

as a $\mathbb{k}$-space.

For $0 \leq t \leq \min\{l, m\}$, let $\mathcal{P}^{l,m}_t : W_{l,\epsilon''}(x) \otimes W_{m,\epsilon''}(y) \rightarrow W_{m,\epsilon''}(y) \otimes W_{l,\epsilon''}(x)$ be a $\hat{U}(\epsilon'')$-linear map given by $\mathcal{P}^{l,m}_t(v'(l, m, t')) = \delta_{\epsilon'' \epsilon'} v'(m, l, t')$. For $t \in H(l, m)$, let

\[(3.6) \quad \mathcal{P}^{l,m}_t : W_{l,\epsilon}(x) \otimes W_{m,\epsilon}(y) \longrightarrow W_{m,\epsilon}(y) \otimes W_{l,\epsilon}(x)\]

be its restriction onto $W_{l,\epsilon}(x) \otimes W_{m,\epsilon}(y)$. Note that $\mathcal{P}^{l,m}_t$ on $W_{l,\epsilon}(x) \otimes W_{m,\epsilon}(y)$ is independent of the choice of $\epsilon''$ and $x, y$. 

Remark 3.12. When we define $P_{l,m}^{0,1}$ (3.6), we may take $c' = c_{0|N''}$ with $N'' = \{i | c''_i = 1\}$ such that (3.5) holds. Recall that $U(c') \cong U_{q^{-1}}(A_{N''}^{(1)})$ (up to a central element). In this case, we take $v'(l, m, t)$ to be the highest weight vector of $V_{c'}((l + m - t, t))$ in $W_{l, c'}(x) \otimes W_{m, c'}(y)$ such that
\[ v'(l, m, t) \in L_{l, c'} \otimes L_{m, c'}, \]
\[ v'(l, m, t) = (e_{1, t} + e_{m+1, t+m-t}) \otimes (e_{1, m}) \mod q^{-1} L_{l, c'} \otimes L_{m, c'}, \]
where $e_{a,b} = \sum_{0 \leq i \leq b} e_i$ and $L_{l, c'}$ is the lower crystal lattice over the subring of $f(q) \in k$ regular at $q = \infty$.

Let $z_1, z_2$ be indeterminates and let
\[ (W_{l, c'})_{\text{aff}} = k[z_1^{\pm 1}] \otimes W_{l, c'}, \quad (W_{m, c'})_{\text{aff}} = k[z_2^{\pm 1}] \otimes W_{m, c'}, \]
\[ (W_{l, c'}) = k(z_1) \otimes W_{l, c'}, \quad (W_{m, c'}) = k(z_2) \otimes W_{m, c'}. \]
Note that
\[ k(z_1, z_2) \otimes k[z_1^{\pm 1}, z_2^{\pm 1}]((W_{l, c'})_{\text{aff}} \otimes (W_{m, c'})_{\text{aff}}) \cong ((W_{l, c'})_{\text{aff}} \otimes (W_{m, c'})_{\text{aff}}). \]

Theorem 3.13. For $l, m \in \mathbb{Z}_+$, $(W_{l, c'})_{\text{aff}} \otimes (W_{m, c'})_{\text{aff}}$ is an irreducible representation of $k(z_1, z_2) \otimes U(\varepsilon)$.

Proof. By (3.3) Theorem 4.7, $W_{l, c'} \otimes W_{m, c'}$ is irreducible. Hence the irreducibility of $k(z_1, z_2) \otimes k[z_1^{\pm 1}, z_2^{\pm 1}]((W_{l, c'})_{\text{aff}} \otimes (W_{m, c'})_{\text{aff}})$ follows from (25) Lemma 3.4.2. □

Suppose that $M \neq N$. Let $R_{l,m}^{\text{univ}}$ denote the universal $R$ matrix acting on $(W_{l, c'})_{\text{aff}} \otimes (W_{m, c'})_{\text{aff}}$ (3.2):
\[ (W_{l, c'})_{\text{aff}} \otimes (W_{m, c'})_{\text{aff}} \xrightarrow{R_{l,m}^{\text{univ}}} k[[z_1/z_2]] \otimes k[[z_1/z_2]]((W_{m, c'})_{\text{aff}} \otimes (W_{l, c'})_{\text{aff}}). \]
Let $s = \max H(l, m) = \min\{l, m, n - 1\}$ and $V_s = V_{c'}((l + m - s, s))$. Then
\[ R_{l,m}^{\text{univ}}|_{V_s} = \varphi_{l,m}(z_1/z_2)\text{id}_{V_s}, \]
for some $\varphi_{l,m}(z_1/z_2) \in k[[z_1/z_2]] \setminus \{0\}$. Put $z = z_1/z_2$ and let
\[ c_s(z) = \prod_{i=0}^{\min\{l,m\}} \frac{1 - q^{l+m-2i+2}z}{q^{l+m-2i+2}z}, \]
where we understand $c_s(z) = 1$ in case of $s = \min\{l, m\}$. Note that we have $s = \min\{l, m\}$ for $M \geq 2$.

Now, we define the normalized $R$ matrix by
\[ R_{l,m}^{\text{norm}} = \varphi_{l,m}(z)^{-1}c_s(z)R_{l,m}^{\text{univ}}. \]
Here we understand $\varphi_{l,m}(z)^{-1}$ as an element in the quotient field $k((z_1/z_2))$ of $k[[z_1/z_2]]$.

Since $R_{l,m}^{\text{norm}}|_{V_s} = c_s(z)\text{id}_{V_s}$, we have by Theorem 3.13 a $k[z_1^{\pm 1}, z_2^{\pm 1}] \otimes U(\varepsilon)$-linear map
\[ (W_{l, c'})_{\text{aff}} \otimes (W_{m, c'})_{\text{aff}} \xrightarrow{R_{l,m}^{\text{norm}}} k(z_1, z_2) \otimes k[z_1^{\pm 1}, z_2^{\pm 1}]((W_{m, c'})_{\text{aff}} \otimes (W_{l, c'})_{\text{aff}}), \]
or a \( k(z_1, z_2) \otimes \mathcal{U}(\epsilon) \)-linear map

\[
(W_{l, \epsilon})_{\text{aff}} \otimes (W_{m, \epsilon})_{\text{aff}} \xrightarrow{\mathcal{R}^\text{norm}_{l, m}} (W_{m, \epsilon})_{\text{aff}} \otimes (W_{l, \epsilon})_{\text{aff}},
\]

where \( (W_{l, \epsilon})_{\text{aff}} \otimes (W_{m, \epsilon})_{\text{aff}} \xrightarrow{\mathcal{R}^\text{norm}_{l, m}} k(z_1, z_2) \otimes [z_1^{\pm 1}, z_2^{\pm 1}] (W_{l, \epsilon})_{\text{aff}} \otimes (W_{m, \epsilon})_{\text{aff}} \).

**Corollary 3.14.** Under the above hypothesis, \( \mathcal{R}^\text{norm}_{l, m} \) is a unique \( k(z_1, z_2) \otimes \mathcal{U}(\epsilon) \)-linear map \( (3.8) \) or \( (3.9) \) such that \( \mathcal{R}^\text{norm}_{l, m} |_{V_z} = c_s(z) \text{id}_{V_z} \).

**Proof.** It follows immediately from Theorem 3.13. \( \square \)

We have the following spectral decomposition of \( \mathcal{R}^\text{norm}_{l, m} \).

**Theorem 3.15.** For \( l, m \in \mathbb{Z}_+ \), we have

\[
\mathcal{R}^\text{norm}_{l, m} = \sum_{t \in H(l, m)} \rho_t(z) \mathcal{P}^l_{t, m}, \quad \rho_t(z) = \prod_{i=t+1}^{\min(l, m)} \frac{1 - q^{l+m-2i+2}z}{1 - q^{l+m-2i+2}},
\]

where \( z = z_1/z_2 \) and \( \rho_{\min(l, m)}(z) = 1 \).

**Proof.** It is shown in [31] that there is a \( k(z_1, z_2) \otimes \mathcal{U}(\epsilon) \)-linear map \( R_{l, m} : (W_{l, \epsilon})_{\text{aff}} \otimes (W_{m, \epsilon})_{\text{aff}} \xrightarrow{\mathcal{R}^\text{norm}_{l, m}} (W_{m, \epsilon})_{\text{aff}} \otimes (W_{l, \epsilon})_{\text{aff}} \) such that \( R_{l, m} |_{V_z} = c_s(z) \text{id}_{V_z} \), where \( s = \min\{l, m, n-1\} \). The spectral decomposition of \( R_{l, m} \) is given in [33, Theorem 5.2] as follows:

\[
R_{l, m} = \sum_{t \in H(l, m)} \rho_t(z) \mathcal{P}^l_{t, m}, \quad \rho_t(z) = \prod_{i=t+1}^{\min(l, m)} \frac{1 - q^{l+m-2i+2}z}{1 - q^{l+m-2i+2}}.
\]

By Corollary 3.14 we have \( R_{l, m} = \mathcal{R}^\text{norm}_{l, m} \), which implies the decomposition of \( \mathcal{R}^\text{norm}_{l, m} \). \( \square \)

**Remark 3.16.** If we replace \( z \) and \( q \) with \( z^{-1} \) and \( -q^{-1} \), respectively in Theorem 3.15 then it recovers the formula [13] when \( \epsilon = \epsilon_0 |_n \). It is more important to observe that the spectral decomposition of \( \mathcal{R}^\text{norm}_{l, m} \) is independent of the choice of \( \epsilon \) if \( n \) is large enough. This will play a crucial role in the remaining of the paper.

For \( t \geq 2 \), let \( z_1, \ldots, z_t \) be indeterminates and let \( l_1, \ldots, l_t \in \mathbb{Z}_+ \) given. Let \( W_i = (W_{l_i, \epsilon})_{\text{aff}} = k[z_i^{\pm 1}] \otimes W_{l_i, \epsilon} \) for \( 1 \leq i \leq t \). Let \( \mathcal{S}_t \) be the group of permutations on \( t \) letters. Since \( \mathcal{R}^\text{norm}_{l, m} \) satisfies the Yang-Baxter equation, we can define a \( k[z_1^{\pm 1}, \ldots, z_t^{\pm 1}] \otimes \mathcal{U}(\epsilon) \)-linear map

\[
\mathcal{R}^\text{norm}_{l_1, \ldots, l_t, w} : W_1 \otimes \cdots \otimes W_t \xrightarrow{\mathcal{R}^\text{norm}_{l_1, \ldots, l_t, w}} k(z_1, \ldots, z_t) \otimes k[z_1^{\pm}, \ldots, z_t^{\pm}] W_{w(1)} \otimes \cdots \otimes W_{w(t)}
\]

or a \( k(z_1, \ldots, z_t) \otimes \mathcal{U}(\epsilon) \)-linear map

\[
\mathcal{R}^\text{norm}_{l_1, \ldots, l_t, w} : (W_1 \otimes \cdots \otimes W_t)^\wedge \xrightarrow{\mathcal{R}^\text{norm}_{l_1, \ldots, l_t, w}} (W_{w(1)} \otimes \cdots \otimes W_{w(t)})^\wedge,
\]

for \( w \in \mathcal{S}_t \) by a composition of \( \mathcal{R}^\text{norm}_{l, m} \)'s with respect to a reduced expression of \( w \), where \( (W_{w(1)} \otimes \cdots \otimes W_{w(t)})^\wedge = k(z_1, \ldots, z_t) \otimes [z_1^{\pm}, \ldots, z_t^{\pm}] W_{w(1)} \otimes \cdots \otimes W_{w(t)} \). We put

\[
\mathcal{R}^\text{norm}_{l_1, \ldots, l_t} = \mathcal{R}^\text{norm}_{l_1, \ldots, l_t, w_0},
\]

where \( w_0 \) is the longest element in \( \mathcal{S}_t \).
4. Irreducible modules

From now on, we assume that $M \neq N$ for $\epsilon$.

4.1. Renormalized $R$ matrix and fusion construction. Let $V, W$ be $U(\epsilon)$-modules in $\mathcal{C}(\epsilon)$. Let $R^{\text{univ}}_{V, W}$ be the universal $R$ matrix on $V_{\text{aff}} \otimes W_{\text{aff}}$ (3.2). Following [22], we say that $R^{\text{univ}}_{V, W}$ is rationally renormalizable if there exists $a \in \mathbb{k}(\mathbb{z}_1/\mathbb{z}_2)^{\times}$ such that

$$aR^{\text{univ}}_{V, W} : V_{\text{aff}} \otimes W_{\text{aff}} \longrightarrow W_{\text{aff}} \otimes V_{\text{aff}}.$$ 

Suppose that $k$ is the algebraic closure of $\mathbb{Q}(q)$ in $\bigcup_{m > 0} \mathbb{C}(q^{-m})$. If such $a$ exists, then we can choose $a$ (unique up to multiplication by a power of $z_1/z_2$) such that $aR^{\text{univ}}_{V, W}|_{z_1 = c_1, z_2 = c_2}$ is nonzero for any $c_1, c_2 \in \mathbb{k}^{\times}$. Then we have a nonzero $U(\epsilon)$-linear map

$$r_{V, W} = aR^{\text{univ}}_{V, W}|_{z_1 = z_2 = 1} : V \otimes W \longrightarrow W \otimes V,$$

which we call the renormalized $R$ matrix.

The renormalized $R$ matrices $r_{V, W}$’s also satisfy the hexagon property and the Yang-Baxter equation up to a multiplication by (possibly zero) constant, since they are specializations of scalar multiples of $R^{\text{univ}}$ [22].

Let $V$, $W_1$, and $W_2$ be $U(\epsilon)$-modules in $\mathcal{C}(\epsilon)$, and let $f \in \text{Hom}_{U(\epsilon)}(W_1, W_2)$. By the definition of $R^{\text{univ}}$, the following diagram

$$
\begin{array}{ccc}
V \otimes W_1 & \xrightarrow{r_{V, W_1}} & W_1 \otimes V \\
\downarrow{\text{id}_V \otimes f} & & \downarrow{f \otimes \text{id}_V} \\
V \otimes W_2 & \xrightarrow{r_{V, W_2}} & W_2 \otimes V
\end{array}
$$

is commutative up to a constant multiple, provided $r_{V, W_1}$ and $r_{V, W_2}$ exist.

**Remark 4.1.** When $U(\epsilon)$ is the usual quantum affine algebra, that is, either $\epsilon = \epsilon_{n|0}$ or $\epsilon_{0|n}$, $R^{\text{univ}}_{V, W}$ is always rationally renormalizable for any irreducible $V$ and $W$ [22 Section 2.2]. But we do not know yet whether it is also true in case of $\mathcal{C}(\epsilon)$ for arbitrary $\epsilon$.

**Lemma 4.2** (cf. [23] Propositions 2.11 and 2.12]). Under the above hypothesis, $R^{\text{univ}}_{V, W}$ is rationally renormalizable in one of the following cases:

1. $V$ (resp. $W$) is a submodule or a quotient of $V_0$ (resp. $W_0$) and $R^{\text{univ}}_{V_0, W}$ (resp. $R^{\text{univ}}_{V, W_0}$) is rationally renormalizable,

2. $V = V_1 \otimes V_2$ (resp. $W = W_1 \otimes W_2$) and both $R^{\text{univ}}_{V_1, W}$ and $R^{\text{univ}}_{V_2, W}$ (resp. $R^{\text{univ}}_{V, W_1}$ and $R^{\text{univ}}_{V, W_2}$) are rationally renormalizable.

**Proof.** (1) It is clear. (2) It follows from the hexagon property of $R^{\text{univ}}$ (cf. [23] 32.2] and Remark 3.6).

**Theorem 4.3** (cf. [22] Theorems 3.2 and 3.12]). Let $V$, $W$ be irreducible $U(\epsilon)$-modules in $\mathcal{C}(\epsilon)$. Suppose that $R^{\text{univ}}_{V, V}$, $R^{\text{univ}}_{W, W}$ and $R^{\text{univ}}_{V, W}$ are rationally renormalizable and

$$r_{V, V} \in k \text{id}_{V \otimes V} \quad \text{or} \quad r_{W, W} \in k \text{id}_{W \otimes W}.$$
Then \( \text{Im}(r_{V,W}) \) is irreducible, and isomorphic to the head of \( V \otimes W \) and the socle of \( W \otimes V \).

**Proof.** We follow the argument in [22]. But we give a self-contained proof here for the reader’s convenience. Let us assume \( r_{W,W} \in \mathbb{k}\text{id}_{W \otimes 2} \) since a similar argument works for the other case.

Let \( S \subset W \otimes V \) be a non-zero submodule. By assumption, there exist \( a, b \in \mathbb{k}((z_1/z_2))^{\times} \) such that we have \( r_{V,W} = a\mathcal{R}_{V,W}^{\text{univ}}|_{z_1 = z_2 = 1} \) and \( r_{W,W} = b\mathcal{R}_{W,W}^{\text{univ}}|_{z_1 = z_2 = 1} \). By Lemma 4.2, \( \mathcal{R}_{S,W}^{\text{univ}} \) is also rationally renormalizable by \( ab \), that is,

\[
ab\mathcal{R}_{S,W}^{\text{univ}} : S_{\text{aff}} \otimes W_{\text{aff}} \to W_{\text{aff}} \otimes S_{\text{aff}}.
\]

Multiplying the rows of the diagram by \( ab \) and specializing at 1, we obtain a commutative diagram

\[
\begin{array}{ccc}
S \otimes W & \xrightarrow{cr_{S,W}} & W \otimes S \\
W \otimes V \otimes W & \xrightarrow{id_W \otimes r_{V,W}} & W \otimes W \otimes V & \xrightarrow{id_W \otimes 2 \otimes id_V} & W \otimes W \otimes V \\
W \otimes V \otimes W & \xrightarrow{id_W \otimes r_{V,W}} & W \otimes W \otimes V & \xrightarrow{id_W \otimes 2 \otimes id_V} & W \otimes W \otimes V \\
\end{array}
\]

for some \( c \), where we use the assumption \( r_{W,W} = \text{id}_{W \otimes 2} \) up to a multiplication by a non-zero scalar. Therefore we have \( S \otimes W \subset W \otimes r_{V,W}^{-1}(S) \).

By [22, Lemma 3.10], there exists a submodule \( K \) of \( V \) such that \( S \subset W \otimes K \) and \( K \otimes W \subset r_{V,W}^{-1}(S) \). Since \( S \neq 0 \), we have \( K \neq 0 \) and hence \( K = V \), which implies that \( V \otimes W \subset r_{V,W}^{-1}(S) \) and \( \text{Im}(r_{V,W}) \subset S \). It follows that \( \text{Im}(r_{V,W}) \) is a unique simple submodule of \( W \otimes V \) and so equals to the socle of \( W \otimes V \). \( \square \)

**Theorem 4.4.** Let \( V_1, \ldots, V_t \) be irreducible \( U(\epsilon) \)-modules in \( \mathcal{C}(\epsilon) \). Suppose that \( \mathcal{R}_{V_i,V_j}^{\text{univ}} \) are rationally renormalizable and \( r_{V_i,V_j} \in \mathbb{k}\text{id}_{V_i \otimes 2} \) for any \( 1 \leq i, j \leq t \). Let

\[
r : V_1 \otimes \cdots \otimes V_i \longrightarrow V_t \otimes \cdots \otimes V_1
\]

be the composition of \( r_{V_i,V_j} \) associated to a reduced expression of the longest element in \( \mathfrak{S}_t \). Then \( \text{Im} r \) is irreducible if it is not zero.

**Proof.** Use induction on \( t \). It is true for \( t = 2 \) by Theorem 4.3.

Suppose that \( t \geq 3 \). Let \( r = r_t \) be the map in the statement and \( r_{t-1} \) denote the map corresponding to the first \( t - 1 \) factors. Consider the following commutative diagram:

\[
\begin{array}{ccc}
V_1 \otimes \cdots \otimes V_t & \xrightarrow{r_t} & V_t \otimes \cdots \otimes V_1 \\
\downarrow r_{t-1} \otimes \text{id}_{V_t} & & \downarrow r_{t-1} \otimes \text{id}_{V_t} \\
\text{Im}(r_{t-1}) \otimes V_t & \xrightarrow{r_{t-1} \otimes \text{id}_{V_t}} & \text{Im}(r_{t-1}) \otimes V_t \\
\downarrow & & \downarrow \\
V_{t-1} \otimes \cdots \otimes V_1 \otimes V_t & & V_{t-1} \otimes \cdots \otimes V_1 \otimes V_t \\
\end{array}
\]

where \( r_i = \text{id}^{\otimes t-i-1} \otimes r_{V_i,V_t} \otimes \text{id}^{t-1} \) for \( 1 \leq i \leq t - 1 \).
Assume \( r_t \neq 0 \). Then \( r_{t-1} \) has a nonzero image \( L \), which is irreducible by the induction hypothesis. Applying the hexagon property repeatedly, we have the following diagram:

\[
\begin{array}{ccc}
L \otimes V_l & \xrightarrow{r_{l,V_l}} & V_l \otimes L \\
\downarrow & & \downarrow \\
V_{l-1} \otimes \cdots \otimes V_1 \otimes V_l & \xrightarrow{r_{l-1} \circ \cdots \circ F_1} & V_l \otimes V_{l-1} \otimes \cdots \otimes V_1
\end{array}
\]

Note that it commutes up to multiplication by a non-zero scalar since both horizontal maps are nonzero. Thus the image of \( r_t \) is equal to that of \( r_{L,V_l} \), which is simple as seen in Theorem 4.3.

Let \( l, m \in \mathbb{Z}_+ \) be given. By Theorem 3.15 we have

\[(4.1) \quad r_{l,m} := d_{l,m}(z) \mathcal{R}^\mathrm{norm}_{l,m^2} : (W_{l,e})_{\text{aff}} \otimes (W_{m,e})_{\text{aff}} \longrightarrow (W_{m,e})_{\text{aff}} \otimes (W_{l,e})_{\text{aff}},
\]

where \( z = z_1/z_2 \) and

\[(4.2) \quad d_{l,m}(z) = \prod_{i=1}^{\min\{l,m\}} (z - q^{1+m-2i+2}).
\]

Note that \( r_{l,m} \) is not zero for any specialization at \((z_1, z_2) = (x, y) \in (k^\times)^2\). Hence \( \mathcal{R}^\mathrm{univ}_{V_{l,e}} \) is rationally renormalizable when \( V = W_{l,e} \) and \( W = W_{m,e} \).

Suppose that \( V_i = W_{l_i,e}(c_i) \) for some \( l_i \in \mathbb{Z}_+ \) and \( c_i \in k^\times \) for \( 1 \leq i \leq t \). We have \( r_{V_i,V_j} = r_{l_i,l_j} \) for \( 1 \leq i, j \leq t \) given in (4.1). Note that \( r_{l_i,l_j} \in k \text{id}_{V_i \otimes V_j} \) by putting \( z = c_i/c_j = 1 \) in the spectral decomposition of Theorem 3.15. The following follows immediately from Theorem 4.4.

**Corollary 4.5.** Suppose that \( c_1, \ldots, c_t \in k^\times \) are given such that \( c_i/c_j \) is not a zero of \( d_{l_i,l_j}(z_i/z_j) \) for any \( 1 \leq i < j \leq t \). Let

\[(4.3) \quad R : W_{l_1,e}(c_1) \otimes \cdots \otimes W_{l_t,e}(c_t) \longrightarrow W_{l_1,e}(c_1) \otimes \cdots \otimes W_{l_t,e}(c_t)
\]

be the specialization of \( \mathcal{R}^\mathrm{norm}_{l_1,\ldots,l_t} \) in (3.10) at \((z_1, \ldots, z_t) = (c_1, \ldots, c_t)\). Then \( \operatorname{Im} R \) is irreducible if it is not zero.

As another application, we have the following, which plays an important role in later sections.

**Proposition 4.6** (cf. [23 Proposition 3.2.9]). Let \( V, W \) be as in Theorem 4.3. Then for \( f \in \text{Hom}_{\hat{\mathcal{H}}(e)}(V \otimes W, W \otimes V) \), the image of \( f \) lies in the socle of \( W \otimes V \).

**Proof.** The same argument in the proof of [23 Proposition 3.2.9] for modules over quiver Hecke algebras also applies to our case.

**Corollary 4.7.** For \( l, m \geq 1 \) and \( x, y \in k^\times \), we have

\[
\text{Hom}_{\hat{\mathcal{H}}(e)}(W_{l,e}(x) \otimes W_{m,e}(y), W_{m,e}(y) \otimes W_{l,e}(x)) = k \cdot r_{l,m}.
\]
Proof. Again by Theorem 4.15 we have $r_{l,m} \in \text{Ker}_{W_{l,e}(x) \otimes W_{m,e}(y)}$. Let $f$ be a non-zero $U(\epsilon)$-linear map from $W_{l,e}(x) \otimes W_{m,e}(y)$ to $W_{m,e}(y) \otimes W_{l,e}(x)$. Then $\text{Im} f$ is isomorphic to the socle of $W_{m,e}(y) \otimes W_{l,e}(x)$ by Proposition 4.16 which is isomorphic to the head of $W_{l,e}(x) \otimes W_{m,e}(y)$ and irreducible by Theorem 4.3. Since the head of $W_{l,e}(x) \otimes W_{m,e}(y)$ is isomorphic to $\text{Ker} r_{l,m}$. Hence by Schur’s lemma, $f$ is equal to $r_{l,m}$ up to a multiplication by a non-zero scalar.

4.2. Irreducible modules in $C(\epsilon)$ and truncation. Let $\mathcal{P}^+$ be the set of $(l, c)$ such that

1. $l = (l_1, \ldots, l_t) \in \mathbb{Z}^t_+$, and $c = (c_1, \ldots, c_t) \in (k^\times)^t$ for some $t \geq 1$,
2. $c_i/c_j$ is not a zero of $d_{l_i,l_j}(z_i/z_j)$ for any $1 \leq i < j \leq t$, when $t \geq 2$.

For $(l, c) \in \mathcal{P}^+$, we put

- $W_{l,e}(c) = W_{l_1,e}(c_1) \otimes \cdots \otimes W_{l_t,e}(c_t)$,
- $W_{l,e}(c^\text{rev}) = W_{l_1,e}(c_t) \otimes \cdots \otimes W_{l_t,e}(c_1)$,
- $R_{l,e}(c) : W_{l,e}(c) \rightarrow W_{l,e}(c^\text{rev})$ : the $U(\epsilon)$-linear map in (4.3),
- $W_{l,c} : (l, c)$ : the image of $R_{l,e}(c)$,
- $\mathcal{P}^+(\epsilon) = \{(l, c) \mid W_{l,c} \neq 0\}$.

We assume that $W_{l,c} = W_{l_1,e}(c_1)$ when $t = 1$. Recall that $W_{l,c}$ is irreducible if it is not zero by Corollary 4.5. Put

$$\mathcal{E} = \bigsqcup_{r \geq 4} \mathcal{E}_r,$$

where $\mathcal{E}_r$ is the set of $\epsilon = (\epsilon_1, \ldots, \epsilon_r)$ such that (1) $\epsilon_i \in \{0, 1\}$ for $1 \leq i \leq r$, (2) $|\{i \mid \epsilon_i = 0\}| \neq |\{i \mid \epsilon_i = 1\}|$. Let $\prec$ be a partial order on $\mathcal{E}$ such that for $\epsilon \in \mathcal{E}_r$ and $\epsilon' \in \mathcal{E}_{r'}$

$$\epsilon' \prec \epsilon \iff r' < r \text{ and } \epsilon' \text{ is a subsequence of } \epsilon.$$

Lemma 4.8. Let $\epsilon, \epsilon' \in \mathcal{E}$ be given such that $\epsilon' \prec \epsilon$. For $(l, c) \in \mathcal{P}^+(\epsilon)$, we have

$$\text{tr}_{\epsilon'}^\epsilon(R_{l,e}(c)) = R_{l,e}(c).$$

Proof. If we define $\text{tr}_{\epsilon'}^\epsilon$ on the weight spaces in $P_{af}$ in the same way as in (5.3), then we also have $\text{tr}_{\epsilon'}^\epsilon((W_{l,e})_{\text{aff}}) \cong (W_{l,e'})_{\text{aff}}$. By (3.4), we have the following commuting diagram:

\[
\begin{array}{ccc}
(W_{l,e})_{\text{aff}} \otimes \cdots \otimes (W_{l,e})_{\text{aff}} & \overset{R_{l,e}^\text{norm}}{\longrightarrow} & ((W_{l,e})_{\text{aff}} \otimes \cdots \otimes (W_{l,e})_{\text{aff}}) \wedge \\
\pi_{\epsilon'} & & \pi_{\epsilon'} \\
(W_{l,e})_{\text{aff}} \otimes \cdots \otimes (W_{l,e'})_{\text{aff}} & \overset{\text{tr}_{\epsilon'}^\epsilon(R_{l,e}^\text{norm})}{\longrightarrow} & ((W_{l,e'})_{\text{aff}} \otimes \cdots \otimes (W_{l,e'})_{\text{aff}}) \wedge
\end{array}
\]

where $R_{l,e} = R_{l_1,\ldots,l_t}^\text{norm}$ (3.10) with respect to $\epsilon$. By Corollary 3.14 we have

$$\text{tr}_{\epsilon'}^\epsilon(R_{l,e}^\text{norm}) = R_{l,e}^\text{norm},$$

where $R_{l,e}^\text{norm}$ denotes $R_{l,e}^\text{norm}$ (3.8) with respect to $\epsilon$. Since $R_{l,e}^\text{norm}$ is a composition of $R_{l_i,\ldots,l_t}^\text{norm}$’s and $\text{tr}_{\epsilon'}^\epsilon$ preserves composition, we have

$$\text{tr}_{\epsilon'}^\epsilon(R_{l,e}^\text{norm}) = R_{l,e}^\text{norm}.$$
Specializing (4.5) at $c = (c_1, \ldots, c_l)$, we have the following commuting diagram:

\[
\begin{array}{ccc}
\mathcal{W}_{l,c}(c) & \overset{R_{l,c}(c)}{\longrightarrow} & \mathcal{W}_{l,c',c}(c^{\text{rev}}) \\
\pi^*_c & & \pi^*_c \\
\mathcal{W}_{l,c}(c) & \overset{R_{l,c}(c)}{\longrightarrow} & \mathcal{W}_{l,c',c}(c^{\text{rev}})
\end{array}
\]

by (4.6). This implies $\text{tr}^c_\epsilon(R_{l,c}(c)) = R_{l,c}(c)$. \qed

**Theorem 4.9.** Let $\epsilon, \epsilon' \in \mathcal{E}$ be given such that $\epsilon' < \epsilon$. For $(l, c) \in \mathcal{P}^+(\epsilon)$, we have

\[
\text{tr}^c_\epsilon(\mathcal{W}_c(l, c)) \cong \mathcal{W}_c(l, c).
\]

**Proof.** It follows immediately from Lemma 4.8 \qed

Let $(l, c) \in \mathcal{P}^+(\epsilon)$ be given. As a $\mathcal{U}(\epsilon)$-module, $\mathcal{W}_c(l, c)$ is completely reducible and it is a direct sum of $V_c(\lambda)$ for $\lambda \in \mathcal{P}_{M|N}$. Let

\[
m^{(l,c)}_\lambda(\epsilon) = \dim_k \text{Hom}_{\mathcal{U}(\epsilon)}(\mathcal{W}_c(l, c), V_c(\lambda))
\]

be the multiplicity of $V_c(\lambda)$ in $\mathcal{W}_c(l, c)$ for $\lambda \in \mathcal{P}_{M|N}$.

**Lemma 4.10.** Let $(l, c) \in \mathcal{P}^+$ and $\lambda \in \mathcal{P}$ be given. If $m^{(l,c)}_\lambda(\epsilon) \neq 0$ for some $\epsilon \in \mathcal{E}$, then we have

\[
m^{(l,c)}_\lambda(\epsilon) = m^{(l,c)}_\lambda(\epsilon'),
\]

for any $\epsilon' \in \mathcal{E}$ such that $\epsilon \leq \epsilon'$.

**Proof.** Since $m^{(l,c)}_\lambda(\epsilon) \neq 0$, we have $\lambda \in \mathcal{P}_{M|N}$. This implies $\lambda \in \mathcal{P}_{M'|N'}$, where $M'$ and $N'$ are the numbers of 0 and 1 in $\epsilon'$, respectively. By (4.7) (with $\epsilon$ and $\epsilon'$ exchanged), Proposition 4.10 and Theorem 4.9, we have

\[
\dim_k \text{Hom}_{\mathcal{U}(\epsilon)}(\mathcal{W}_c(l, c), V_c(\lambda)) = \dim_k \text{Hom}_{\mathcal{U}(\epsilon')}(\mathcal{W}_c(l, c), V_c(\lambda)).
\]

**Theorem 4.11.** For $(l, c) \in \mathcal{P}^+$ and $\lambda \in \mathcal{P}$, there exists $m^{(l,c)}_\lambda \in \mathbb{Z}_+$ such that

1. if $m^{(l,c)}_\lambda(\epsilon) = 0$, then $m^{(l,c)}_\lambda(\epsilon) = 0$ for all $\epsilon \in \mathcal{E}$,
2. if $m^{(l,c)}_\lambda(\epsilon) \neq 0$, then $m^{(l,c)}_\lambda(\epsilon) = m^{(l,c)}_\lambda(\epsilon')$ for all $\epsilon \in \mathcal{E}$ with $m^{(l,c)}_\lambda(\epsilon) \neq 0$.

**Proof.** Define $m^{(l,c)}_\lambda(\epsilon) = 0$ if $m^{(l,c)}_\lambda(\epsilon) = 0$ for all $\epsilon \in \mathcal{E}$. Suppose that both $m^{(l,c)}_\lambda(\epsilon')$ and $m^{(l,c)}_\lambda(\epsilon'')$ are non-zero for some $\epsilon', \epsilon'' \in \mathcal{E}$. Let us take $\epsilon''' \in \mathcal{E}$ such that $\epsilon' < \epsilon''$ and $\epsilon'' < \epsilon'''$. By Lemma 4.10 we have $m^{(l,c)}_\lambda(\epsilon') = m^{(l,c)}_\lambda(\epsilon'') = m^{(l,c)}_\lambda(\epsilon''')$. Hence $m^{(l,c)}_\lambda(\epsilon)$ is constant for any $\epsilon \in \mathcal{E}$ with $m^{(l,c)}_\lambda(\epsilon) \neq 0$, which we denote by $m^{(l,c)}_\lambda(\epsilon)$. \qed

By Theorem 4.11 the branching multiplicity (4.8) is independent of the choice of $\epsilon$ if it is non-zero. In particular, if $m^{(l,c)}_\lambda(\epsilon) \neq 0$, then

\[
m^{(l,c)}_\lambda(\epsilon) = m^{(l,c)}_\lambda(\epsilon_{M'|0}) = m^{(l,c)}_\lambda(\epsilon_{0|N'}),
\]
for some $M'$ or $N'$. Therefore, finding the usual character of $\mathcal{W}_c(I, c)$ is equivalent to finding the one of the corresponding irreducible module over the usual quantum affine algebra of type $A$.

4.3. **Inverse limit category.** Suppose that

\begin{equation}
\epsilon^\infty = (\epsilon_i)_{i \geq 1} = (\epsilon_1, \ldots, \epsilon_k, \ldots)
\end{equation}

is given, where $\epsilon_i \in \{0, 1\}$ for $i \geq 1$. We also assume that $\epsilon^\infty$ has infinitely many 1’s for convenience. Take an ascending chain $(\epsilon^{(k)})_{k \geq 1}$ of subsequences of $\epsilon^\infty$ in $\mathcal{E}$ with respect to the partial order \[\text{(4.4)}\] such that

\begin{equation}
\epsilon^\infty = \lim_k \epsilon^{(k)}.
\end{equation}

For example, one may choose $\epsilon^{(k)} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{2k+3})$. Let $M_k$, $N_k$ denote the number of 0’s, 1’s in $\epsilon^{(k)}$ respectively. Let us consider the inverse limit category associated to $\{ \mathcal{C}(\epsilon^{(k)}) \}_{k \geq 1}$ (cf. [14], [39, Section 5.1]). We follow the presentation [14].

**Definition 4.12.** Let

\[ \mathcal{C}(\epsilon^\infty) = \lim_k \mathcal{C}(\epsilon^{(k)}) \]

be the category defined as follows:

1. an object is a pair $\mathcal{V} = (\mathcal{V}_k)_{k \geq 1}, (f_k)_{k \geq 1}$ such that $V_k \in \mathcal{C}(\epsilon^{(k)})$, $f_k : \operatorname{tr}_{k+1}^{k+1}(V_{k+1}) \to V_k$ ($k \geq 1$),

where $\operatorname{tr}_{k+1}^{k+1} = \operatorname{tr}_{\epsilon^{(k+1)}}$ and $f_k$ is an isomorphism of $\mathcal{U}(\epsilon^{(k)})$-modules,

2. a morphism from $\mathcal{V} = (\mathcal{V}_k)_{k \geq 1}, (f_k)_{k \geq 1}$ to $\mathcal{W} = (\mathcal{W}_k)_{k \geq 1}, (g_k)_{k \geq 1}$ is a sequence $\phi = (\phi_k)_{k \geq 1}$ such that $\phi_k \in \operatorname{Hom}_{\mathcal{U}(\epsilon^{(k)})}(V_k, W_k)$ and the following diagram is commutative for all $k$:

\[
\begin{array}{ccc}
\operatorname{tr}_{k+1}^{k+1}(V_{k+1}) & \xrightarrow{f_k} & V_k \\
\downarrow \operatorname{tr}_{k+1}^{k+1}(\phi_{k+1}) & & \downarrow \phi_k \\
\operatorname{tr}_{k+1}^{k+1}(W_{k+1}) & \xrightarrow{g_k} & W_k 
\end{array}
\]

Then $\mathcal{C}(\epsilon^\infty)$ is an abelian category. Moreover, it is monoidal (or a tensor category) with respect to $\otimes$, where we define

\[ \mathcal{V} \otimes \mathcal{W} = ((V_k \otimes W_k)_{k \geq 1}, (f_k \otimes g_k)_{k \geq 1}), \]

for $\mathcal{V} = ((V_k)_{k \geq 1}, (f_k)_{k \geq 1})$ and $\mathcal{W} = ((W_k)_{k \geq 1}, (g_k)_{k \geq 1})$. Given $k \geq 1$, let

\[ \operatorname{tr}_k : \mathcal{C}(\epsilon^\infty) \to \mathcal{C}(\epsilon^{(k)}) \]

be a functor given by $\operatorname{tr}_k(\mathcal{V}) = V_k$ and $\operatorname{tr}_k(\phi) = \phi_k$ for $\mathcal{V} = ((V_k)_{k \geq 1}, (f_k)_{k \geq 1})$ and a morphism $\phi = (\phi_k)_{k \geq 1} : \mathcal{V} \to \mathcal{W}$.

The following can be proved by standard arguments.
Lemma 4.13. Given a category \( C \) with a family of functors \( \{ F_k : C \to C(\epsilon^{(k)}) \}_{k \geq 1} \) such that \( \operatorname{tr}_k F_{k+1} \cong F_k \) for \( k \geq 1 \), there exists a functor

\[
F := \lim_{\leftarrow k} F_k : C \to C(\epsilon^\infty),
\]

such that \( \operatorname{tr}_k F \cong F_k \) for all \( k \geq 1 \). Moreover, if \( F_k \) is exact, then so is \( F \). If \( C \) is a monoidal category and \( F_k \) is monoidal, then so is \( F \).

Remark 4.14. For any finite subsequence \( \epsilon' \) of \( \epsilon^\infty \), there exists \( k \) such that \( \epsilon' < \epsilon^{(k)} \) so that we may define a truncation functor \( \operatorname{tr}_{\epsilon'} := \operatorname{tr}_{\epsilon^{(k)}} \circ \operatorname{tr}_k \). Here the choice of \( k \) is irrelevant because truncation functors are transitive by definition.

If we take another ascending chain \( (\epsilon^{(k)})_{k \geq 1} \) of subsequences of \( \epsilon^\infty \) satisfying (4.10) and construct the inverse limit category \( \tilde{C}(\epsilon^\infty) = \lim_{\leftarrow} C(\epsilon^{(k)}) \), then we have an equivalence

\[
C(\epsilon^\infty) \cong \tilde{C}(\epsilon^\infty)
\]

using the truncation functor above and Lemma 4.13. Therefore the inverse limit category \( C(\epsilon^\infty) \) does not depend on the choice of an ascending chain, and the requirement \( \epsilon^{(k)} \in \mathcal{E} \) is not a strong one in this sense.

Lemma 4.15. Suppose that there is a family of functors \( \{ F_k : C \to C(\epsilon^{(k)}) \}_{k \geq 1} \) satisfying

1. \( \operatorname{tr}_k F_{k+1} \cong F_k \) for \( k \geq 1 \),
2. there exists \( N \) such that \( F_n \) is an equivalence of categories for \( n \geq N \).

Then the resulting functor \( F = \lim_{\leftarrow} F_k \) is an equivalence of categories. In particular, if \( \operatorname{tr}_n^N \) is an equivalence for all \( n \geq N \), then \( \operatorname{tr}_n \) is an equivalence for \( n \geq N \).

Proof. Take a quasi-inverse \( G_k \) of \( F_k \) so that \( G_k \circ \operatorname{tr}_k^{k+1} \cong G_{k+1} \) for \( k \geq N \). We claim that \( G \) is a quasi-inverse of \( F \). Clearly \( G \circ F \cong \operatorname{id}_C \).

Given an object \( V \) in \( C(\epsilon^\infty) \), we have

\[
\operatorname{tr}_k (F \circ G(V)) = \operatorname{tr}_k (F(G_N(\operatorname{tr}_N(V)))) = F_k (G_N(\operatorname{tr}_N(V))).
\]

For \( k \geq N \), we have \( G_N \cong G_N \circ \operatorname{tr}_N^k \) and hence \( \operatorname{id}_{C(\epsilon^{(k)})} \cong F_k \circ G_N \circ \operatorname{tr}_N^k \), that is,

\[
\operatorname{tr}_k (V) \cong F_k \circ G_N \circ \operatorname{tr}_N^k (\operatorname{tr}_k(V)) \cong F_k (G_N(\operatorname{tr}_N(V))).
\]

For \( k < N \), as \( F_N \circ G_N \cong \operatorname{id} \),

\[
F_k (G_N(\operatorname{tr}_N(V))) \cong \operatorname{tr}_k^N \circ F_N \circ G_N (\operatorname{tr}_N (V)) \cong \operatorname{tr}_k^N \circ \operatorname{tr}_N (V) \cong \operatorname{tr}_k (V).
\]

From the definition of morphisms in \( C(\epsilon^\infty) \), these isomorphisms are natural. Hence \( F \circ G \cong \operatorname{id}_{C(\epsilon^\infty)} \).

Lemma 4.16. Let \( V \) be an object in \( C(\epsilon^\infty) \). If \( V \) is non-zero and \( \operatorname{tr}_k (V) \) is irreducible or zero for all \( k \), then \( V \) is a simple object.

Proof. Let \( U \) be a non-zero subobject of \( V \). We may assume that \( \operatorname{tr}_k (U) \) is a \( U(\epsilon^{(k)}) \)-submodule of \( \operatorname{tr}_k (V) \). Let \( k_0 \) (resp. \( k_1 \)) be the smallest one such that \( \operatorname{tr}_k (U) \neq 0 \) (resp. \( \operatorname{tr}_k (V) \neq 0 \)). Since \( \operatorname{tr}_k (V) \) is irreducible for \( k \geq k_1 \) and \( \operatorname{tr}_k (U) \) is a submodule of \( \operatorname{tr}_k (V) \), we have \( k_0 \geq k_1 \), and \( \operatorname{tr}_k (V) = \operatorname{tr}_k (U) \) for \( k \geq k_0 \), which also implies that \( \operatorname{tr}_k^N(\operatorname{tr}_k (U)) = \operatorname{tr}_k^N(\operatorname{tr}_k (V)) \) for \( k \leq k_0 \). Therefore, we have \( k_0 = k_1 \) and \( U \) is isomorphic to \( V \).  

\[\square\]
Remark 4.17. We expect that the category $\mathcal{C}(\epsilon^\infty)$ when $\epsilon$ is homogeneous is closely related to the representations of the quantum affine algebra $U_q(\mathfrak{sl}_\infty)$ introduced in \cite{17}.

4.4. $R$ matrix and simple objects in $\mathcal{C}(\epsilon^\infty)$. For $l \in \mathbb{Z}_+$ and $x \in k^\times$, we let
\[
W_{l, \epsilon^\infty}(x) = ((W_{l, \epsilon^k}(x))_{k \geq 1}, (f_k)_{k \geq 1}),
\]
where $f_k$ is an isomorphism in Proposition 3.10(2). Then it is an object in $\mathcal{C}(\epsilon^\infty)$. Define its affinization by
\[
(W_{l, \epsilon^\infty})_{\text{aff}} = (((W_{l, \epsilon^k})_{\text{aff}})_{k \geq 1}, (\text{id}_{k\geq 0^{\pm 1}} \otimes f_k)_{k \geq 1}).
\]
By 4.10 and 4.11, there exists a well-defined morphism (in the sense of Definition 4.12(2))
\[
(W_{l, \epsilon^\infty})_{\text{aff}} \otimes (W_{m, \epsilon^\infty})_{\text{aff}} \xrightarrow{\mathcal{R}^\text{norm}_{(l,m), \epsilon^\infty}} (W_{m, \epsilon^\infty})_{\text{aff}} \otimes (W_{l, \epsilon^\infty})_{\text{aff}},
\]
for $l, m \in \mathbb{Z}_+$, given by $\mathcal{R}^\text{norm}_{(l,m), \epsilon^\infty} = (\mathcal{R}^\text{norm}_{(l,m), \epsilon^k})_{k \geq 1}$, where $\mathcal{R}^\text{norm}_{(l,m), \epsilon^k}$ is the map given in (3.9) with respect to $\epsilon^k$.

For $(l, c) \in \mathcal{P}^+$, let $W_{l, \epsilon^\infty}(c) = W_{l, \epsilon^\infty}(c_1) \otimes \ldots \otimes W_{l, \epsilon^\infty}(c_2)$. By using 4.11, we have a morphism
\[
R_{l, \epsilon^\infty}(c) : W_{l, \epsilon^\infty}(c) \longrightarrow W_{l, \epsilon^\infty}(c^{rev}),
\]
given by $R_{l, \epsilon^\infty}(c) = (R_{l, \epsilon^k}(c))_{k \geq 1}$. Let
\[
W_{\epsilon^\infty}(l, c) = \text{Im}R_{l, \epsilon^\infty}(c),
\]
which is a well-defined object in $\mathcal{C}(\epsilon^\infty)$ by Theorem 4.9. Then we have the following analogue of Theorem 4.5:

Theorem 4.18. For $(l, c) \in \mathcal{P}^+$, $W_{\epsilon^\infty}(l, c)$ is a nonzero simple object in $\mathcal{C}(\epsilon^\infty)$.

Proof. We have $W_{\epsilon^\infty}(l, c) = \left((W_{\epsilon^k}(l, c))_{k \geq 1}, (h_k)_{k \geq 1}\right)$, where $(h_k)_{k \geq 1}$ is induced from the one associated to $W_{\text{rev}^\infty}(c^{rev})$. Suppose that $W_{\epsilon^\infty}(l, c)$ is not zero, and let $k_0$ be the smallest one such that $W_{\epsilon^k}(l, c)$ is not zero. Since $W_{\epsilon^k}(l, c)$ is simple for $k \geq k_0$ by Theorem 4.5, $W_{\epsilon^\infty}(l, c)$ is simple by Lemma 4.16.

Next, we claim that $W_{\epsilon^\infty}(l, c)$ is non-zero for all $(l, c) \in \mathcal{P}^+$. Take a sufficiently large $k$ such that $l_i < N_k$ for $1 \leq i \leq t$, and put $\epsilon' = \epsilon_{0|N_k}$. Since $\epsilon' < \epsilon^{(k)}$, we have by Theorem 4.9 that
\[
\text{tr}_{\epsilon'}^{(k)}(W_{\epsilon^k}(l, c)) = W_{\epsilon'}(l, c).
\]
Since $W_{\epsilon'}(l, c) \neq 0$ by our choice of $\epsilon^{(k)}$ and $\|27\] 27], we have $W_{\epsilon^{(k')}}(l, c) \neq 0$ for all $k' \geq k$. Hence $W_{\epsilon^\infty}(l, c)$ is non-zero. $\square$

We define a category $\hat{\mathcal{C}}(\epsilon^\infty)$ in a similar way, where we replace $\mathcal{U}(\epsilon^{(k)})$ and $\mathcal{C}(\epsilon^{(k)})$ with $\hat{\mathcal{U}}(\epsilon^{(k)})$ and $\hat{\mathcal{C}}(\epsilon^{(k)})$, respectively. For $\lambda \in \mathcal{P}$, let
\[
V_{\epsilon^\infty}(\lambda) = \left((V_{\epsilon^{(k)}}(\lambda))_{k \geq 1}, (\hat{f}_k)_{k \geq 1}\right),
\]
where $\hat{f}_k$ is the isomorphism in Proposition 3.10(1), and we assume that $V_{\epsilon^{(k)}}(\lambda) = 0$ if $\lambda \notin \mathcal{P}_{M_k|N_k}$. By Lemma 4.10, $V_{\epsilon^\infty}(\lambda)$ is a simple object in $\hat{\mathcal{C}}(\epsilon^\infty)$. 
**Theorem 4.19.** For \((l, c) \in P^+, \mathcal{W}_{c,\infty}(l, c)\) is a semisimple object in \(\hat{\mathcal{U}}(e^{\infty})\), and
\[
\mathcal{W}_{c,\infty}(l, c) = \bigoplus_{\lambda \in \mathcal{P}} V_{c,\infty}(\lambda)^{\oplus m_{\lambda}^{(l, c)}},
\]
where \(\lambda\) runs over the partitions of \(l_1 + \cdots + l_\ell\) for \(l = (l_1, \ldots, l_\ell)\), and \(m_{\lambda}^{(l, c)}\) is the multiplicity in Theorem 4.11.

**Proof.** We have as a \(\mathcal{U}(e^{(k)})\)-module
\[
\mathcal{W}_{c,(k)}(l, c) = \bigoplus_{\lambda \in \mathcal{P}_k} V_{c,(k)}(\lambda)^{\oplus m_{\lambda}^{(l, c)}} (k \geq 1).
\]
Hence the decomposition of \(\mathcal{W}_{c,\infty}(l, c)\) follows from Theorem 4.11 (cf. Remark 5.11). \(\Box\)

5. **Generalized quantum affine Schur-Weyl duality functor**

5.1. **Quiver Hecke algebras.** Let us briefly review necessary background for (symmetric) quiver Hecke algebras following the convention in [20][22], with a slight modification of notations.

Let \(k\) be a field. Let \(J\) be a set, \(\mathbb{Z}[J]\) the free abelian group generated by \(J\), and \(\mathbb{N}[J]\) the subset of \(\mathbb{Z}[J]\) consisting of \(\sum_{i \in J} c_i \cdot i\) with \(c_i \in \mathbb{Z}_+\). For \(\ell \geq 2\), we assume that the symmetric group \(S_\ell\), which is generated by \(s_i = (i, i + 1)\) for \(i = 1, \ldots, \ell - 1\), acts on \(J^\ell\) by place permutation.

For \(\beta = \sum_{i \in J} c_i \cdot i \in \mathbb{N}[J]\) with \(\text{ht}(\beta) := \sum_{i \in J} c_i = \ell\), set
\[
J^\beta = \{ \nu = (\nu_1, \ldots, \nu_\ell) \in J^\ell \mid \nu_1 + \cdots + \nu_\ell = \beta \}.
\]
Let \(u, v\) be formal variables. Let \((Q_{ij})_{i,j \in J}\) be the matrix with entries \(Q_{ij} = Q_{ij}(u, v) \in k[u, v]\) such that \(Q_{ij}(u, v) = Q_{ji}(v, u)\) for \(i \neq j\), and \(Q_{ii}(u, v) = 0\).

Suppose that \(A = (a_{ij})_{i,j \in J}\) is a symmetrizable generalized Cartan matrix. Then there exist positive integers \(s_i (i \in J)\) such that \(s_i a_{ij} = s_j a_{ji} : a_{ij}\) for \(i, j \in J\). We further assume that the coefficient of \(u^p v^q (p, q \in \mathbb{Z}_+)\) in \(Q_{ij}(u, v)\) vanishes unless \(a_{ii} p + a_{jj} q = -2 a_{ij}\).

**Definition 5.1.** The quiver Hecke algebra or Khovanov-Lauda-Rouquier algebra \(R(\beta)\) at \(\beta\) associated with \((Q_{ij})_{i,j \in J}\) is the associative \(k\)-algebra with 1 generated by \(e(\nu) (\nu \in J^\beta)\), \(x_k (k = 1, \ldots, \ell)\), and \(\tau_m (m = 1, \ldots, \ell - 1)\) subject to the following relations:
\[
e(\nu)e(\nu') = \delta_{\nu
u'}e(\nu), \quad \sum_{\nu \in J^\beta} e(\nu) = 1,
\]
\[
x_k x_k' = x_k' x_k, \quad x_k e(\nu) = e(\nu) x_k, \quad \tau_m e(\nu) = e(s_m(\nu)) \tau_m,
\]
\[
(\tau_m x_k - x_{s_m(k)} \tau_m) e(\nu) = \begin{cases} -e(\nu) & \text{if } k = m \text{ and } \nu_m = \nu_{m+1}, \\ e(\nu) & \text{if } k = m + 1 \text{ and } \nu_m = \nu_{m+1}, \\ 0 & \text{otherwise}, \end{cases}
\]
\[
\tau_m^2 e(\nu) = Q_{\nu_m,\nu_{m+1}}(x_m, x_{m+1}) e(\nu), \quad \tau_m \tau_m' = \tau_m' \tau_m \quad \text{if } |m - m'| > 1,
\]
\[
(\tau_{m+1} \tau_m \tau_{m+1} - \tau_m \tau_{m+1} \tau_m) e(\nu) = \]
\[
\begin{cases}
Q_{\nu_m,\nu_{m+1}}(x_m, x_{m+1}) - Q_{\nu_m,\nu_{m+1}}(x_{m+2}, x_{m+1}) \\ x_m - x_{m+2}
\end{cases}
\frac{e(\nu)}{c(\nu)} \quad \text{if } \nu_m = \nu_{m+2},
\]
\[0 \quad \text{otherwise.}\]

The algebra \( R(\beta) \) carries a \( \mathbb{Z} \)-grading defined by \( \deg e(\nu) = 0 \), \( \deg x_k e(\nu) = 2 \), and \( \deg \tau_m e(\nu) = -a_{\nu_m,\nu_{m+1}} \). We set
\[
R(\ell) = \bigoplus_{\beta \in J^\ell, \text{ht}(\beta) = \ell} R(\beta) \quad (\ell \geq 1), \quad R = \bigoplus_{\ell \geq 0} R(\ell),
\]
where \( R(0) = k \). For \( \beta \in \mathbb{N}[J] \) with \( \text{ht}(\beta) = \ell \), we have the following decomposition:
\[
R(\beta) = \bigoplus_{\nu \in J^\beta, w \in \mathfrak{S}_\ell} k \left[ x_1, \ldots, x_\ell \right] e(\nu) \tau_w,
\]
where \( \tau_w = \tau_i \cdots \tau_j \) is defined after fixing a reduced expression \( w = s_i \cdots s_j \) for each \( w \in \mathfrak{S}_\ell \).

For \( \beta_1, \beta_2 \in \mathbb{N}[J] \) with \( \text{ht}(\beta_1) = \ell_1 \) and \( \text{ht}(\beta_2) = \ell_2 \), let
\[
e(\beta_1, \beta_2) = \sum_{\nu} e(\nu) \in R(\beta_1 + \beta_2),
\]
where the sum is over \( \nu = (\nu_1, \ldots, \nu_{\ell_1+\ell_2}) \in J^{\beta_1+\beta_2} \) such that \( (\nu_1, \ldots, \nu_{\ell_1}) \) \( \in J^{\beta_1} \) and \( (\nu_{\ell_1+1}, \ldots, \nu_{\ell_1+\ell_2}) \) \( \in J^{\beta_2} \). Then we have a homomorphism of \( k \)-algebras
\[
R(\beta_1) \otimes R(\beta_2) \longrightarrow e(\beta_1, \beta_2)R(\beta_1 + \beta_2)e(\beta_1, \beta_2),
\]
given by sending
\[
e(\nu) \otimes e(\eta) \longrightarrow e(\nu \ast \eta),
\]
\[
x_{k_1} \otimes 1 \longrightarrow x_{k_1} e(\beta_1, \beta_2), \quad 1 \otimes x_{k_2} \longrightarrow x_{k_1+k_2} e(\beta_1, \beta_2),
\]
\[
\tau_{m_1} \otimes 1 \longrightarrow \tau_{m_1} e(\beta_1, \beta_2), \quad 1 \otimes \tau_{m_2} \longrightarrow \tau_{m_1+m_2} e(\beta_1, \beta_2),
\]
for \( 1 \leq k_i \leq \ell_i \) and \( 1 \leq m_i \leq \ell_i - 1 \) (\( i = 1, 2 \)), where \( \nu \ast \eta = (\nu_1, \ldots, \nu_{\ell_1}, \eta_1, \ldots, \eta_{\ell_2}) \) is the concatenation of \( \nu = (\nu_1, \ldots, \nu_{\ell_1}) \) and \( \eta = (\eta_1, \ldots, \eta_{\ell_2}) \). Let \( M_i \) be an \( R(\beta_i) \)-module for \( i = 1, 2 \). The convolution product of \( M_1 \) and \( M_2 \) is defined by
\[
M_1 \circ M_2 = R(\beta_1 + \beta_2) \otimes_{R(\beta_1) \otimes R(\beta_2)} (M_1 \otimes M_2),
\]
which is an \( R(\beta_1 + \beta_2) \)-module.

Let \( R(\beta) \)-gmod denote the category of finite-dimensional graded \( R(\beta) \)-modules for \( \beta \in \mathbb{N}[J] \), and let \( R \)-gmod = \( \bigoplus_{\beta \in \mathbb{N}[J]} R(\beta) \)-gmod. Then \( R \)-gmod is monoidal with respect to the convolution product with the unit \( R(0) = k \). Moreover, there is a degree shifting functor \( q \) on \( R \)-gmod defined by \( (qM)_k = M_{k-1} \) for \( M = \bigoplus_{k \in \mathbb{Z}} M_k \).

Next, let us recall the notion of (renormalized) \( R \) matrix for quiver Hecke algebras. Let \( \beta \in \mathbb{N}[J] \) be given with \( \text{ht}(\beta) = \ell \geq 2 \). For \( 1 \leq m \leq \ell - 1 \), let
\[
\varphi_m e(\nu) = \begin{cases}
(\tau_m x_m - x_m \tau_m) e(\nu) & \text{if } \nu_m = \nu_{m+1}, \\
\tau_m e(\nu) & \text{if } \nu_m \neq \nu_{m+1},
\end{cases}
\]
for $\nu = (\nu_1, \ldots, \nu_t) \in J^3$. For $\ell_1, \ell_2 \geq 1$ with $\ell_1 + \ell_2 = \ell$, put

$$\varphi_{\ell_1, \ell_2} = \varphi_{k_1} \cdots \varphi_{k_{\ell_1} k_{\ell_2}},$$

where $w = s_{k_1} \cdots s_{k_{\ell_1} k_{\ell_2}}$ is a reduced expression of the permutation $w$ such that $w(a) = a + \ell_2$ for $1 \leq a \leq \ell_1$ and $w(b) = b - \ell_1$ for $\ell_1 + 1 \leq b \leq \ell$.

Suppose that $\beta_1, \beta_2 \in \mathbb{N}[J]$ are given such that $\beta = \beta_1 + \beta_2$ and $ht(\beta_i) = \ell_i$ $(i = 1, 2)$. Let $M_i$ be an $R(\beta_i)$-module for $i = 1, 2$. Then we have a homomorphism of $R(\beta)$-modules:

$$R_{M_1, M_2} : M_1 \circ M_2 \longrightarrow M_2 \circ M_1$$

for $a \in R(\beta)$ and $u_i \in M_i$ $(i = 1, 2)$. The renormalized $R$ matrix on $M_1 \circ M_2$ can be defined by using an affinization of $M_i$'s. More precisely, for an indeterminate $z$ and $R(\gamma)$-module $N$, let $N_z := k[z] \otimes N$ be the $k[z] \otimes R(\gamma)$-module defined by

$$e(\nu)(a \otimes u) = a \otimes (e(\nu)u), \quad x_k(a \otimes u) = (za) \otimes u + a \otimes (x_k u), \quad \tau_m(a \otimes u) = a \otimes (\tau_m u),$$

for $a \otimes u \in k[z] \otimes N$. Then we define $r_{M_1, M_2} : M_1 \circ M_2 \longrightarrow M_2 \circ M_1$ by

$$r_{M_1, M_2} = z^{-s} R_{(M_1)_z, (M_2)_z} \mid_{z=0},$$

where $s$ is the largest non-negative integer such that $\text{Im} R_{(M_1)_z, (M_2)_z} \subset z^s R_{(M_1)_z, (M_2)_z}$. We call $r_{M_1, M_2}$ the renormalized $R$ matrix on $M_1 \circ M_2$.

Note that the renormalized $R$ matrices satisfy the Yang-Baxter equation. In particular, this implies that given $\beta_i \in \mathbb{N}[J]$ and $R(\beta_i)$-module $M_i$ $(i = 1, \ldots, t)$, we have a well-defined homomorphism of $R(\beta_1 + \cdots + \beta_t)$-modules

$$r_w : M_1 \circ \cdots \circ M_t \longrightarrow M_{w(1)} \circ \cdots \circ M_{w(t)},$$

for each $w \in \mathcal{S}_t$, which is given by a composition of $r_{M_i, M_j}$'s with respect to a reduced expression of $w$.

### 5.2. Quiver Hecke algebras of type $A$

Let $\Gamma$ be the quiver given by

$$
\begin{array}{ccccccc}
\vdots & \circ & \circ & \circ & \circ & \circ & \circ & \vdots \\
-2 & -1 & 0 & 1 & 2 & & \\
\end{array}
$$

Suppose that $J = \mathbb{Z}$. Let $P_{ij}(u, v) = (u - v)^{d_{ij}}$, where $d_{ij}$ is the number of arrows from $i$ to $j$ in $\Gamma$, and take $(Q_{ij})_{i,j\in J}$ as

$$Q_{ij} = Q_{ij}(u, v) = P_{ij}(u, v)P_{ji}(v, u) \quad (i \neq j).$$

The resulting quiver Hecke algebra $R'(\beta)$ at $\beta \in \mathbb{N}[J]$ is associated to $(Q_{ij})_{i,j\in J}$ and the generalized Cartan matrix $(a_{ij})_{i,j\in \mathbb{Z}}$ with

$$a_{ij} = \begin{cases} 
2 & \text{if } i = j, \\
-1 & \text{if } i = j \pm 1, \\
0 & \text{otherwise.}
\end{cases}$$
One may regard a pair of integers \((a, b)\) with \(a \leq b\) as a positive root of type \(A_\infty\), say 
\[
\beta_{(a,b)} := 1 \cdot a + 1 \cdot (a+1) + \cdots + 1 \cdot b \in \mathbb{N}[J].
\]
Let us call such \((a, b)\) a segment and \(\ell = b - a + 1\) its length. Define the lexicographic order on the set of segments, that is,
\[
(a, b) \geq (a', b') \iff a > a' \text{ or } (a = a', b \geq b').
\]
We call a finite sequence of segments \(((a_1, b_1), \ldots, (a_t, b_t))\) a multisegment, and say that it is ordered if \((a_k, b_k) \geq (a_{k+1}, b_{k+1})\) for all \(1 \leq k \leq t - 1\).

For a segment \((a, b)\) of length \(\ell\), let \(L(a, b) = ku(a, b)\) be a one-dimensional \(R^J(\beta_{(a,b)})\)-module given by
\[
x_k u(a, b) = \tau_m u(a, b) = 0, \quad e(\nu) u(a, b) = \begin{cases} u(a, b) & \text{if } \nu = (a, a+1, \ldots, b), \\ 0 & \text{otherwise}, \end{cases}
\]
for \(1 \leq k \leq \ell, 1 \leq m \leq \ell - 1, \) and \(\nu \in J^{\beta_{(a,b)}}\). We also denote \(L(a, a)\) by \(L(a)\).

Fix \(\ell \geq 2\). Suppose that \(((a_1, b_1), \ldots, (a_t, b_t))\) is an ordered multisegment with \(\ell_k\) the length of \((a_k, b_k)\) and \(\sum \ell_k = \ell\). By (\ref{eq:5.2}), we have a homomorphism of \(R\)-modules
\[
\mathbf{r}_{w_0} : L(a_1, b_1) \circ \cdots \circ L(a_1, b_t) \longrightarrow L(a_t, b_t) \circ \cdots \circ L(a_1, b_1),
\]
where \(\beta = \sum_{i=1}^t \beta_{(a_i, b_i)}\) and \(w_0\) is the longest element in \(S_t\).

**Proposition 5.2** ([\textit{20}, Proposition 4.7]). Under the above hypothesis, we have
\begin{enumerate}
\item the correspondence from \(((a_1, b_1), \ldots, (a_t, b_t))\) to \(\text{hd} (L(a_1, b_1) \circ \cdots \circ L(a_t, b_t))\), the head of \(L(a_1, b_1) \circ \cdots \circ L(a_t, b_t)\), is a bijection from the set of ordered multisegments to the set of isomorphism classes (up to grading shifts) of finite-dimensional irreducible graded \(R^J(\ell)\)-modules,
\item the image of \(\mathbf{r}_{w_0}\) in (\ref{eq:5.4}) is irreducible and isomorphic to \(\text{hd} (L(a_1, b_1) \circ \cdots \circ L(a_t, b_t))\).
\end{enumerate}

Finally we recall some particular exact sequences involving renormalized \(R\)-matrices.

**Proposition 5.3** ([\textit{20}, Proposition 4.3]). Let \((a, b), (a', b')\) be two segments with lengths \(\ell\) and \(\ell'\) respectively such that \((a, b) \geq (a', b')\).
\begin{enumerate}
\item If one of the following holds: \(a' < b \leq b', a > b' + 1, a = a' < b' \leq b,\) then 
\[
\xymatrix{ L(a, b) \circ L(a', b) \ar[r] & L(a', b') \circ L(a, b) }
\]
is an isomorphism.
\item If \(a' < a < b' < b,\) then we have the following exact sequence
\[
\xymatrix{ 0 \ar[r] & L(a', b) \circ L(a, b') \ar[r] & L(a, b) \circ L(a', b') \ar[r] & L(a', b) \circ L(a, b') \ar[r] & 0 }
\]
\item If \(a = b' + 1,\) then we have an exact sequence
\[
\xymatrix{ 0 \ar[r] & L(a', b) \ar[r] & L(a, b) \circ L(a', b') \ar[r] & L(a', b') \circ L(a, b) \ar[r] & L(a', b) \ar[r] & 0. }
\]
Here we ignore the grading and denote by \(r\) the renormalized \(R\)-matrix on \(L(a, b) \circ L(a', b').\)
5.3. Generalized quantum affine Schur-Weyl duality functor. Let $R^J(\beta)$ be the quiver Hecke algebra at $\beta \in \mathbb{N}[J]$ associated to $(Q_{ij})_{i,j \in J}$ in (5.3) and let $R^J(\ell)$ and $R^J$ denote the corresponding ones in (5.1). The goal of this subsection is to show that there is a generalized quantum affine Schur-Weyl duality functor $\mathcal{F}_\epsilon$ from $R^J$-gmod to $\mathcal{C}(\epsilon)$, which can be constructed by the same method as in [20, Section 3] when $\epsilon = \epsilon_{0\vert N}$.

Suppose that $k = \mathbb{k}$ is the algebraic closure of $\mathbb{Q}(q) \subset \bigcup_{m \geq 0} \mathbb{C}((q^{-m}))$. Let $X : J \to \mathbb{k}^\times$ be given by $X(i) = q^{-2i}$ for $i \in J$. Note that $X(i)/X(j) = q^{-2(i-j)}$ is a zero of the denominator of $d_{1,1}(z) = z - q^2$ if and only if $j = i + 1$ for $i \in J$.

Fix $\ell \geq 2$. Let $X_1, \ldots, X_\ell$ be indeterminates. For $\nu = (\nu_1, \ldots, \nu_\ell) \in J^\ell$, put $X(\nu) = (X(\nu_1), \ldots, X(\nu_\ell)) \in (k^\times)^\ell$ and

$$\mathcal{O}_\nu = k \left[ X_1 - X(\nu_1), \ldots, X_\ell - X(\nu_\ell) \right].$$

For $\beta \in \mathbb{N}[J]$ with $\text{ht}(\beta) = \ell$, we define $\mathcal{O}_\beta$ to be a $k$-algebra with a $k$-basis $\{fe(\nu) \mid f \in \mathcal{O}_\nu (\nu \in J^\beta)\}$, where $e(\nu)$ commutes with $\mathcal{O}_\nu$ and $e(\nu)e(\nu') = \delta_{\nu\nu'}e(\nu)$ for $\nu, \nu' \in J^\beta$. Similarly, we define $\mathbb{K}_\beta$, where $\mathcal{O}_\nu$ is replaced by its field of quotients, say $\mathbb{K}_\nu$. We have

$$\mathcal{O}_\beta = \bigoplus_{\nu \in J^\beta} \mathcal{O}_\nu e(\nu), \quad \mathbb{K}_\beta = \bigoplus_{\nu \in J^\beta} \mathbb{K}_\nu e(\nu).$$

Let $k[\mathfrak{S}_\ell]$ denote the group algebra of $\mathfrak{S}_\ell$ over $k$ with a basis $\{r_w \mid w \in \mathfrak{S}_\ell\}$. We assume that $\mathfrak{S}_\ell$ acts on $\mathbb{K}_\beta$ by $w(X_i) = X_{w^{-1}(i)}$ for $w \in \mathfrak{S}_\ell$ and $1 \leq i \leq \ell$, and hence $\mathbb{K}_\beta \otimes_k k[\mathfrak{S}_\ell]$ is a $k$-algebra with $r_w f = w(f)r_w$ for $f \in \mathbb{K}_\beta$ and $w \in \mathfrak{S}_\ell$. Recall that we have an embedding of $k$-algebras

$$R^J(\beta) \hookrightarrow \mathbb{K}_\beta \otimes_k k[\mathfrak{S}_\ell] \ ,$$

or we may identify $R^J(\beta)$ as a subalgebra of $\mathbb{K}_\beta \otimes_k k[\mathfrak{S}_\ell]$ by letting

$$e(\nu) = e(\nu), \quad e(\nu)x_k = e(\nu)X(\nu_k)^{-1}(X_k - X(\nu_k)),
(5.7)\quad e(\nu)\tau_m = \begin{cases} e(\nu)(r_m - 1) \left( \frac{1}{x_m - x_{m+1}} \right) & \text{if } \nu_m = \nu_{m+1}, \\ e(\nu)r_m P_{\nu_{m},\nu_{m+1}}(x_{m+1}, x_m) & \text{if } \nu_m \neq \nu_{m+1}, \end{cases}$$

for $1 \leq k \leq \ell$ and $1 \leq m \leq \ell - 1$, where $r_m = r_{s_m}$ [38, Proposition 3.12].

Now, suppose that $\epsilon$ is given. Let $V = (W_{1,\epsilon})_{\text{aff}}$ and regard $V \otimes^\ell$ as a $k[X_1^{\pm 1}, \ldots, X_\ell^{\pm 1}] \otimes U(\epsilon)$-module, where $X_i$ acts as $z$ on the $i$th component (3.1). Put

$$V_{0}^\otimes_{0} = \mathcal{O}_0 \otimes_k V \otimes^\ell, \quad V_{0}^\otimes_{0} = \mathbb{K}_0 \otimes_{\mathcal{O}_0} V_{0}^\otimes_{0},$$

where $K = k[X_1^{\pm 1}, \ldots, X_\ell^{\pm 1}]$. We have

$$V_{0}^\otimes_{0} = \bigoplus_{\nu \in J^\beta} V_{0}^\otimes_{\nu}, \quad \text{where } V_{0}^\otimes_{\nu} = \mathcal{O}_\nu e(\nu) \otimes_k V \otimes^\ell.$$
Then $V^{\otimes \beta}_K$ is also a right $K \otimes_k k[\mathfrak{g}]$-module, where $1 \otimes r_m$ $(1 \leq m \leq \ell - 1)$ acts on $V^{\otimes \beta}_K$ by $R^{\text{norm}}_{1,1}$ on the $(m, m + 1)$-component in $V^{\otimes \ell}$, that is,

$$
\K_{\nu}(\nu) \otimes_K V^{\otimes \ell} \longrightarrow \K_{s_m(\nu)}(s_m(\nu)) \otimes_K V^{\otimes \ell},
$$

$$
e \otimes (v_1 \otimes \ldots \otimes v_\ell) \longrightarrow s_m(f) \otimes (s_m(\nu)) \otimes (\ldots \otimes R^{\text{norm}}_{1,1}(v_m \otimes v_{m+1}) \otimes \ldots)
$$

for $\nu \in J^\beta$, $f \in \K$ and $v_1 \otimes \ldots \otimes v_\ell \in V^{\otimes \ell}$. Hence $V^{\otimes \beta}_K$ is a right $R^{I}(\beta)$-module by (5.6). Therefore, $V^{\otimes \beta}_K$ is a $(U(\epsilon), R^{I}(\beta))$-bimodule since $V^{\otimes \beta}_K$ is a left $U(\epsilon)$-module and the action of $U(\epsilon)$ commutes with that of $R^{I}(\beta)$.

**Proposition 5.4.** For $\beta \in \mathbb{N}[J]$, $V^{\otimes \beta}_O$ is invariant under the action of $R^{I}(\beta)$. Hence it is a $(U(\epsilon), R^{I}(\beta))$-bimodule.

**Proof.** By Theorem 3.15, we have

$$
R^{\text{norm}}_{1,1}(z) = P_{1}^{1,1} + \frac{1 - q^2 z}{z - q^2} P_{0}^{1,1},
$$

which is the same as in the case of $\epsilon = \epsilon_0$. This enables us to apply literally the same argument as in [20, Theorem 3.3] by replacing $q$ in [20] with $-q^{-1}$ (cf. Remark 3.10).

For a graded $R^{I}(\beta)$-module $M$ (not necessarily finite-dimensional), we define

$$
\mathcal{F}_{\epsilon,\beta}(M) = V^{\otimes \beta}_O \otimes R^{I}(\beta) M.
$$

Then $\mathcal{F}_{\epsilon,\beta}$ is a functor from the category of graded $R^{I}(\beta)$-modules to that of $U(\epsilon)$-modules. We also let

$$
\mathcal{F}_{\epsilon,\ell} = \bigoplus_{\beta \in \mathbb{N}[J], \text{ht}(\beta) = \ell} \mathcal{F}_{\epsilon,\beta}, \quad \mathcal{F}_{\epsilon} = \bigoplus_{\ell \geq 0} \mathcal{F}_{\epsilon,\ell},
$$

which is a functor from the category of graded modules over $R^{I}(\ell)$ (resp. $R^{I}$) to that of $U(\epsilon)$-modules.

**Theorem 5.5** (cf. [20, Theorem 3.4]). The functors in (5.8) induce exact functors

$$
\mathcal{F}_{\epsilon,\ell} : R^{I}(\ell)-\text{gmod} \longrightarrow \mathcal{C}^{\ell}(\epsilon), \quad \mathcal{F}_{\epsilon} : R^{I}-\text{gmod} \longrightarrow \mathcal{C}(\epsilon).
$$

respectively, and $\mathcal{F}_{\epsilon}$ is monoidal.

**Proof.** Thanks to Theorem 3.15, we may apply the same arguments in the proof of [20, Theorem 3.4] as in Proposition 5.4.

Next, let us describe the image of irreducible $R^{I}$-modules under $\mathcal{F}_{\epsilon}$. To compute $\mathcal{F}_{\epsilon}(L(a, b))$ first, we need the following lemma generalizing [1, Lemma B.1] of quantum affine algebras.

**Lemma 5.6.** For $\ell \geq 4$, we have the following exact sequence of $U(\epsilon)$-modules:

$$
0 \rightarrow W_{\ell, \epsilon}(1) \rightarrow W_{\ell, \epsilon}(q^{1-\ell}) \otimes W_{\ell-1, \epsilon}(q) \rightarrow W_{\ell-1, \epsilon}(q) \otimes W_{1, \epsilon}(q^{1-\ell}) \rightarrow W_{\ell, \epsilon}(1) \rightarrow 0.
$$

**Proof.** See Appendix A.1.
Proposition 5.7 (cf. [20] Proposition 4.9). For a segment \((a, b)\) of length \(\ell\), we have
\[ \mathcal{F}_\ell(L(a, b)) \cong W_{\ell, c}(q^{-a-b}). \]

**Proof.** With the help of Proposition 5.3, Lemma 5.6 and Corollary 4.7, the proof of [20, Proposition 4.9] applies in the same manner except that the induction argument proceeds in \(\ell\).

Lemma 5.8. Let \((a, b)\) and \((a', b')\) be segments of lengths \(\ell\) and \(\ell'\), respectively, such that \((a, b) \geq (a', b')\). Set \(c = q^{-a-b}\) and \(c' = q^{-a'-b'}\). Then \(c/c'\) is not a zero of the denominator \(d_{\ell, \ell'}(z)\) of \(\mathcal{R}^{\text{norm}}_{\ell, \ell'}(z)\), and the map
\[ W_{\ell, c}(c') \xrightarrow{F_{\ell}(r_{L(a,b), L(a',b')})} W_{\ell', c}(c') \otimes W_{\ell, c}(c) \]
is equal to a nonzero constant multiple of \(\mathcal{R}^{\text{norm}}_{\ell, \ell'}(c/c')\) except the following case:
\[ a' < a \leq b' < b, \quad M = 1, \quad N \leq b' - a + 1, \]
in which case \(F_{\ell}(r_{L(a,b), L(a',b')}) = 0\).

**Proof.** Let \(r = r_{L(a,b), L(a',b')}\). The assumption \((a, b) \geq (a', b')\) implies \(a' + b' - a - b \leq \ell' - \ell\). Hence \(c/c' = q^a \cdot b' - a - b\) is not a zero of \(d_{\ell, \ell'}(z)\) since
\[ d_{\ell, \ell'}(z) = (z - q^\ell b') (z - q^{\ell+b'-2}) \cdots (z - q^{|\ell'-\ell|+2}). \]
The second assertion follows once we determine when \(F_{\ell}(r_{L(a,b), L(a',b')})\) is nonzero since
\[ \text{Hom}_{U(\ell)}(W_{\ell, c}(c) \otimes W_{\ell', c}(c'), W_{\ell, c}(c) \otimes W_{\ell, c}(c)) = k \mathcal{R}^{\text{norm}}_{\ell, \ell'}(c/c'), \]
by Corollary 4.7.

According to Proposition 5.3, \(r\) is not an isomorphism if and only if \(a' < a \leq b' < b\) or \(a = b' + 1\). In such cases, we apply \(F_{\ell}\) to the exact sequences in Proposition 5.3(2), (3) to get an exact sequence
\[
\begin{array}{c}
0 \\ \rightarrow W_{\ell_1, c}(q^{-a'-b}) \otimes W_{\ell_2, c}(q^{-a-b'}) \\ \rightarrow W_{\ell, c}(c) \otimes W_{\ell, c}(c') \\
\end{array}
\]
where \(\ell_1 = b - a' + 1\) and \(\ell_2 = b' - a + 1\). Now \(F_{\ell}(r) = 0\) if and only if
\[ \dim \left(W_{\ell_1, c}(q^{-a'-b}) \otimes W_{\ell_2, c}(q^{-a-b'})\right) = \dim \left(W_{\ell, c}(c) \otimes W_{\ell', c}(c')\right), \]
and this equality holds if and only if the decompositions of the modules on both sides in the above equation as a \(U(\ell)\)-module coincide (cf. (3.3)). This happens exactly when \(a' < a \leq b' < b\), \(M = 1\) and \(N \leq b' - a + 1\).

Finally we obtain our version of [20, Theorem 4.11].

**Theorem 5.9.** Let \(((a_1, b_1), \ldots, (a_n, b_n))\) be an ordered multisegment with \(\ell_k\) the length of \((a_k, b_k)\) and \(\sum_k \ell_k = \ell\), and let \(L\) be the corresponding irreducible \(R^J(\ell)\)-module in \(R^J(\ell)\)-gmod. If \(N = \{ i \mid \epsilon_i = 1 \}\) is greater than \(\max\{\ell_1, \ldots, \ell_k\}\), then \(W_{\ell}(L, c)\) is non-zero and
\[ F_{\ell}(L) \cong W_{\ell}(L, c), \]
for \((l, c) \in \mathcal{P}^+(e)\) with \(l = (l_1, \ldots, l_t)\) and \(c = (q^{-a_1-b_1}, \ldots, q^{-a_t-b_t})\).

**Proof.** Let \(r\) be the map in \([5.3]\). Recall that \(L \cong \text{Im}(r)\) by Proposition \([5.2]\). Since \(\mathcal{F}\) is exact, we have

\[
\mathcal{F}(L) \cong \mathcal{F}(\text{Im}(r)) \cong \text{Im}\mathcal{F}(r).
\]

By Lemma \([5.8]\), \(\mathcal{F}(r)\) is equal to \(R_{l,r}(e)\) up to multiplication by a nonzero scalar. Note that the condition \(N > \max\{l_1, \ldots, l_t\}\) excludes the exceptional case in Lemma \([5.8]\). Indeed, whenever \(a_j < a_i \leq b_j < b_i\) for some \(i < j\), we have \(b_j - a_i + 1 < b_j - a_j + 1 = \ell_j < N\). This implies that \(\mathcal{F}(r)\) is non-zero and hence \(\text{Im}\mathcal{F}(r) = W_e(l, c)\) is non-zero.

**Theorem 5.10.** Let \(e, e' \in \mathcal{E}\) be given such that \(e' < e\). Then for an \(R^I\)-module \(M\) in \(R^I\text{-mod}\), we have an isomorphism of \(\mathcal{U}(e')\)-modules

\[
\text{tr}_{e'}^* \circ \mathcal{F}_e(M) \cong \mathcal{F}_{e'}(M),
\]

which is natural in \(M\).

**Proof.** Let \(\beta \in \mathbb{N}[J]\) be given with \(\text{ht}(\beta) = \ell\). Let \(M\) be an \(R^I(\beta)\)-module. Since the weight space decomposition of \(\mathcal{F}_{e,\beta}(M)\) is given by

\[
\mathcal{F}_{e,\beta}(M) = \bigoplus_{\lambda \in P} (\mathcal{F}_{e,\beta}(M))_\lambda, \quad (\mathcal{F}_{e,\beta}(M))_\lambda = (V^{\otimes \beta}_0)_\lambda \otimes R^I(\beta) M,
\]

we have

\[
(5.9) \quad \text{tr}_{e'}^* (\mathcal{F}_{e,\beta}(M)) = (\text{tr}_{e'}^* (V^{\otimes \beta}_0)) \otimes R^I(\beta) M.
\]

Since \(V^{\otimes \beta}_0 = \bigoplus_{\nu \in J^\beta} \bigoplus_{e(\nu) \otimes K} V^{\otimes \ell}\), we have as a \(\mathcal{U}(e')\)-module

\[
(5.10) \quad \text{tr}_{e'}^* (V^{\otimes \beta}_0) = \bigoplus_{\nu \in J^\beta} \bigoplus_{e(\nu) \otimes K} \big(\text{tr}_{e'}^* (W_{1,e})_{\text{aff}} \otimes \cdots \otimes \text{tr}_{e'}^* (W_{1,e})_{\text{aff}}\big)
\]

where \(V' = (W_{1,e})_{\text{aff}}\) and \(K = k[X_1^{\pm 1}, \ldots, X_\ell^{\pm 1}]\). Therefore it follows from \((5.9)\) and \((5.10)\) that \(\text{tr}_{e'}^* \circ \mathcal{F}_{e,\beta}(M) \cong \mathcal{F}_{e',\beta}(M)\), which is also natural in \(M\).

Hence we have the following diagram:

\[
\begin{array}{ccc}
\mathcal{F}_e & \xrightarrow{\mathcal{F}_e} & C(e) \\
R^I\text{-mod} & \xrightarrow{\text{tr}_{e'}^*} & C(e') \\
\mathcal{F}_{e'} & \xrightarrow{\mathcal{F}_{e'}} & C(e')
\end{array}
\]

such that \(\text{tr}_{e'}^* \circ \mathcal{F}_e \cong \mathcal{F}_{e'}\).
Corollary 5.11. Let $\epsilon^\infty$ and $(\epsilon^{(k)})_{k \geq 1}$ be as in (11.9) and (11.10), respectively. There exists an exact monoidal functor

$$F_{\epsilon^\infty} = \lim_{\leftarrow} F_{\epsilon^{(k)}} : RJ\text{-gmod} \longrightarrow \mathcal{C}(\epsilon^\infty),$$

such that $\text{tr}_k \circ F_{\epsilon^\infty} \cong F_{\epsilon^{(k)}}$ for all $k \geq 1$.

Proof. It follows from Lemma 4.13 and Theorem 5.10. □

Remark 5.12. Note that a generalized quantum affine Schur-Weyl duality functor for usual quantum affine algebras is given in a more general setting [20, Section 3.1]. We remark that this also applies to the case of $U(\epsilon)$-modules for arbitrary $\epsilon$.

In other words, as in [20, Section 3.1] we can define functors to subcategories of $\mathcal{C}(\epsilon)$ associated to various choices of the triple $(J, X : J \to k^\times, S : J \to S)$, where $S$ is a set of simple objects in $\mathcal{C}(\epsilon)$. For instance, given any orientation $Q$ of the Dynkin diagram of $A_{m-1}$ and a height function $\xi$ on $Q$, one may use the following data:

$$J \subset \{1, \ldots, m-1\} \times \mathbb{Z},$$

$$X(i, p) = q^{-p-m}, \quad S(i, p) = W_{i, \epsilon}(1) \text{ for } (i, p) \in J,$$

as given in [21, Section 3] to obtain a functor from $RJ\text{-gmod}$ to $\mathcal{C}_Q(\epsilon)$. Here $\mathcal{C}_Q(\epsilon)$ is a category of $U(\epsilon)$-modules, which can be viewed as an analog of the category $\mathcal{C}_Q$ for the quantum affine algebra of type $A^{(1)}_{m-1}$ introduced by Hernandez-Leclerc [18]. We should remark that $m$ is not necessarily equal to $n$, the length of $\epsilon$, whenever $S$ is well-defined. The functor is defined in a natural way for $\epsilon$ and compatible with the truncation due to Theorem 3.15 and Lemma 4.8. The type of the corresponding quiver Hecke algebra $RJ$ is of type $Q^{rev}$, the quiver obtained by reversing all the arrows of $Q$, as in [21, Theorem 4.3.1] since the denominators of normalized $R$ matrices $[4.22]$ associated to simple objects in $S$ coincide with those as in the case of $U_q^{'}\left(\mathit{A}_{m-1}^{(1)}\right)$ (cf. Remark 3.16).

We also have a functor from the category of not necessarily graded finite-dimensional modules over quiver Hecke algebras. This is directly related to the more classical quantum affine Schur-Weyl duality [7], which is dealt with in the next section.

6. Duality

6.1. Subcategory $\mathcal{C}_J(\epsilon)$. Let us introduce a subcategory of $\mathcal{C}(\epsilon)$ to describe the image of the generalized quantum affine Schur-Weyl duality functor $F_{\epsilon}$.

Definition 6.1.

1. Let $\mathcal{C}_J(\epsilon)$ be the full subcategory of $\mathcal{C}(\epsilon)$ whose objects have composition factors appearing as a composition factor of a tensor product of modules of the form $W_{1, \epsilon}(q^{-2j})$ for $j \in J$. For $\ell \in \mathbb{Z}_+$, let $\mathcal{C}_J^\ell(\epsilon) = \mathcal{C}_J(\epsilon) \cap \mathcal{C}_J^{\ell}(\epsilon)$.

2. Let $\mathcal{P}^{+}_J(\epsilon)$ be a subset of $\mathcal{P}^{+}(\epsilon)$ consisting of elements $(l, c)$ with $l = (l_1, \ldots, l_t) \in \mathbb{Z}_+^t$, and $c = (c_1, \ldots, c_t) \in (k^\times)^t$ for some $t \geq 1$ such that

$$c_i \in q^{l_i-1+2\mathbb{Z}} \quad (i = 1, \ldots, t).$$


Lemma 6.2. For an $R^j$-module $M$ in $R^j$-gmod, we have $\mathcal{F}_e(M) \in C_J(e)$. Hence we have a functor

$$\mathcal{F}_e : R^j$-gmod \longrightarrow C_J(e).$$

Proof. Recall that the convolution product $M_1 \circ M_2$ is exact both in $M_1$ and $M_2$ by [28, Proposition 2.16], and $L(a, b)$ is a composition factor of a convolution of product of $L(i)$’s for $i \in J$ by Proposition 5.3. By Proposition 6.7, we have $\mathcal{F}_e(L(a, b)) = W_i, \epsilon(q^{-a-b}) \in C_J(e)$, and hence $W_i(l, c) \in C_J(e)$ for $(l, c) \in \mathcal{P}_J^+(\epsilon)$.

Let $0 \subset M_1 \subset \cdots \subset M_r = M$ be a composition series of $M$. Since $\mathcal{F}_e$ is exact, $0 \subseteq \mathcal{F}_e(M_1) \subseteq \cdots \subseteq \mathcal{F}_e(M_r) = \mathcal{F}_e(M)$ is a filtration of $\mathcal{F}_e(M)$ whose subquotients are either zero or of the form $W_i(l, c)$ for $(l, c) \in \mathcal{P}_J^+(\epsilon)$ by Proposition 5.7 and Theorem 5.9. Now note that $W_i(l, c)$ is a quotient of $W_i, \epsilon(c)$, and hence a composition factor of a tensor product of $W_1, \epsilon(q^{-2j})$’s.

By Theorem 6.5 and Lemma 6.2 we have $\mathcal{F}_e = \bigoplus_{\ell \in \mathbb{Z}_+} \mathcal{F}_{e, \ell}$, where

$$(6.2) \quad \mathcal{F}_{e, \ell} : R^j(\ell)$-gmod \longrightarrow C_J^\ell(\epsilon).$$

Proposition 6.3. Let $V$ be a $\mathcal{U}(\epsilon)$-module in $C_J(\epsilon)$. Then $V$ is irreducible if and only if such that $V \cong W_i(l, c)$ for some $(l, c) \in \mathcal{P}_J^+(\epsilon)$.

Proof. Suppose that $V$ is a composition factor of $W_i, \epsilon(q^{-2j_1}) \otimes \cdots \otimes W_i, \epsilon(q^{-2j_t})$ for some $j_1, \ldots, j_t \in J$. Then $V$ is isomorphic to $\mathcal{F}_e(M)$ for some composition factor $M$ of $L(j_1) \otimes \cdots \otimes L(j_t)$. By Proposition 6.2 and Theorem 5.9, $V \cong W_i(l, c)$ for some $(l, c) \in \mathcal{P}_J^+(\epsilon)$.

Conversely, it is clear by definition that $W_i(l, c)$ belongs to $C_J(\epsilon)$.

The category $C_J(\epsilon)$ is compatible with the truncation in the following sense.

Lemma 6.4. Let $\epsilon, \epsilon' \in E$ be given such that $\epsilon' < \epsilon$. For a $\mathcal{U}(\epsilon)$-module $V$ in $C_J(\epsilon)$, let $V' = \text{tr}_{\epsilon'}(V)$. Then we have

(1) $V'$ is a $\mathcal{U}(\epsilon')$-module in $C_J(\epsilon')$.

(2) $\ell(V') \leq \ell(V)$, where $\ell(V)$ denotes the length of the composition series of $V$.

Proof. Since tr~\_\_ is exact, it follows from Theorem 4.9 and Proposition 6.3.

6.2. Affine Hecke algebras and quantum affine Schur-Weyl duality functor. There is a quantum affine Schur-Weyl duality between the affine Hecke algebra and quantum affine algebra of type $A$ [7]. In this subsection, we prove an analogue for $\mathcal{U}(\epsilon)$ and explain how to identify it with our functor $\mathcal{F}_{e, \ell}$.

First, we briefly recall that a quiver Hecke algebra of type $A$ is isomorphic to an affine Hecke algebra (of type $A$) after suitable completions on both sides [4, 38]. Our exposition is based on [29].

Definition 6.5. Let $\ell \geq 2$. The affine Hecke algebra $H_\ell^{\text{aff}}(q^2)$ (of the symmetric group $S_\ell$) is the associative $k$-algebra with 1 generated by $X^\pm_k$ $(1 \leq k \leq \ell)$ and $h_m$ $(1 \leq m \leq \ell - 1)$
subject to the following relations:
\[ h_m h_{m+1} h_m = h_{m+1} h_m h_{m+1}, \quad h_m h_{m'} = h_{m'} h_m \quad (|m - m'| > 1), \]
\[ (h_m - q^2)(h_m + 1) = 0, \]
\[ X_k X_{k'} = X_{k'} X_k, \quad X_k X_k^{-1} = X_k^{-1} X_k = 1, \]
\[ h_m X_m h_m = q^2 X_{m+1} \quad h_m X_k = X_k h_m \quad (k \neq m, m + 1). \]

For convenience, we assume \( H_0^\text{aff}(q^2) = k \) and \( H_1^\text{aff}(q^2) = k[X^\pm 1] \).

The subalgebra \( H_\ell(q^2) \) generated by \( h_m \) (1 \( \leq \ell \leq \ell - 1 \)) is called the finite Hecke algebra.

Note that given a reduced expression \( w = s_{i_1} \cdots s_{i_n} \in \mathcal{S}_\ell \), the element \( h_w := h_{i_1} \cdots h_{i_n} \) does not depend on the reduced expression, and the set \( \{ h_w \}_{w \in \mathcal{S}_\ell} \) is a basis for \( H_\ell(q^2) \).

Put
\[ \mathcal{O}_\ell = \bigoplus_{\beta \in J^\prime} \mathcal{O}_\beta, \quad \mathbb{K}_\ell = \bigoplus_{\beta \in J^\prime} \mathbb{K}_\beta \]
(cf. (5.5)), and consider the following completions:
\[ \mathcal{O} H_\ell^\text{aff}(q^2) := \mathcal{O}_\ell \otimes_k [X_1^\pm 1, \ldots, X_\ell^\pm 1] H_\ell^\text{aff}(q^2) \equiv \mathcal{O}_\ell \otimes_k H_\ell(q^2), \]
\[ \mathbb{K} H_\ell^\text{aff}(q^2) := \mathbb{K}_\ell \otimes \mathcal{O}_\ell \otimes_k H_\ell^\text{aff}(q^2) \equiv \mathbb{K}_\ell \otimes_k H_\ell(q^2), \]
as \( k \)-vector spaces. We simply write \( fe(\nu) h = f e(\nu) \otimes h \) for \( f e(\nu) \in \mathcal{O}_\ell \) and \( h \in H_\ell(q^2) \). We regard \( \mathcal{O} H_\ell^\text{aff}(q^2) \) and \( \mathbb{K} H_\ell^\text{aff}(q^2) \) as associative \( k \)-algebras, where
\[ h_m f e(\nu) = s_m(f) e(s_m(\nu)) h_m + \frac{(q^2 - 1) X_{m+1} f e(\nu) - s_m(f) e(s_m(\nu))}{X_{m+1} - X_m} \]
for \( 1 \leq m \leq \ell - 1 \) and \( f e(\nu) \in \mathcal{O}_\ell \) or \( \mathbb{K}_\ell \). Note that we have a sequence of subalgebras
\[ H_\ell^\text{aff}(q^2) \subset \mathcal{O} H_\ell^\text{aff}(q^2) \subset \mathbb{K} H_\ell^\text{aff}(q^2). \]

For \( 1 \leq m \leq \ell - 1 \), define the intertwiner \( \Phi_m \in \mathbb{K} H_\ell^\text{aff}(q^2) \) by
\[ \Phi_m = h_m - \frac{(q^2 - 1) X_{m+1}}{X_{m+1} - X_m}. \]
The following properties of \( \Phi_m \) can be checked in a straightforward way:
\[ \Phi_m f(X_m, X_{m+1}) e(\nu) = f(X_{m+1}, X_m) e(s_m(\nu)) \Phi_m, \]
\[ \Phi_m \Phi_{m'} = \Phi_{m'} \Phi_m \quad (|m - m'| > 1), \]
\[ \Phi_m \Phi_{m+1} \Phi_m = \Phi_{m+1} \Phi_m \Phi_{m+1} \]
for \( f(X_m, X_{m+1}) \in k(X_m, X_{m+1}) \) and \( \nu \in J^\ell \). Moreover, we have
\[ \Phi_m^2 e(\nu) = \frac{X_{m+1} - q^2 X_m}{X_m - X_{m+1}} \frac{X_m - q^2 X_{m+1}}{X_{m+1} - X_m} e(\nu). \]

Hence if we normalize \( \Phi_m \) by
\[ \bar{\Phi}_m = \frac{X_m - X_{m+1}}{X_{m+1} - q^2 X_m} \Phi_m, \]
then we have \( \bar{\Phi}_m^2 = 1 \).
Theorem 6.6 ([H [28]). Let \( \hat{R}^J(\ell) = k[[x_1, \ldots, x_\ell]] \otimes_k k_{[x_1, \ldots, x_\ell]} R^J(\ell) \) be the completion of \( R^J(\ell) \) with multiplication naturally extended from \( R^J(\ell) \). Then there is an isomorphism of \( k \)-algebras

\[
\Psi : \hat{R}^J(\ell) \longrightarrow \mathcal{O} H^\text{aff}_H(q^2),
\]

given by

\[
\Psi(e(\nu)) = e(\nu),
\]
\[
\Psi(e(\nu)x_k) = e(\nu)X(\nu_k)^{-1}(X_k - X(\nu_k)),
\]
\[
\Psi(e(\nu)\tau_m) = \begin{cases} 
  e(\nu)\Phi_m - 1(X_m - X_{m+1})^{-1} & \text{if } \nu_{m+1} = \nu_m, \\
  e(\nu)\Phi_m - 1(X_m - X_{m+1})^{-1} & \text{if } \nu_{m+1} = \nu_m + 1, \\
  e(\nu)\Phi_m & \text{otherwise},
\end{cases}
\]

for \( \nu \in J^\ell, 1 \leq k \leq \ell \) and \( 1 \leq m \leq \ell - 1 \).

**Proof.** Let us give a proof briefly for the reader’s convenience. First we see that all \( \Psi(e(\nu)\tau_m) \) indeed belong to \( \mathcal{O} H^\text{aff}_H(q^2) \). By (6.3), the embedding (5.7) implies the existence of \( \Psi \) since we have in (5.7)

\[
e(\nu)P_{\nu_m, \nu_{m+1}}(x_m, x_{m+1}) = \begin{cases} 
  e(\nu)(x_m - x_{m+1}) & \text{if } \nu_{m+1} = \nu_m + 1, \\
  e(\nu) & \text{otherwise},
\end{cases}
\]

and \( e(\nu)x_k = e(\nu)(X_k/X(\nu_k) - 1) \).

Recall that \( \{\tau_w\}_{w \in \mathfrak{S}_\ell} \) is an \( \mathcal{O}_H \)-basis for \( \hat{R}^J(\ell) \). Since \( \Psi \) is \( \mathcal{O}_H \)-linear by definition, it is enough to show that \( \{\Psi(\tau_w)\}_{w \in \mathfrak{S}_\ell} \) is an \( \mathcal{O}_H \)-basis for \( \mathcal{O} H^\text{aff}_H(q^2) \), which implies that \( \Psi \) is bijective.

Note that \( \{h_w\}_{w \in \mathfrak{S}_\ell} \) is also an \( \mathcal{O}_H \)-basis for \( \mathcal{O} H^\text{aff}_H(q^2) \). We claim that each \( h_w \) is given by an \( \mathcal{O}_H \)-linear combination of \( \Psi(\tau_m) \)’s. Moreover, since \( \Psi \) is an algebra homomorphism, it is enough to consider \( e(\nu)h_m \).

Suppose \( \nu_m = \nu_{m+1} \). Then

\[
\Psi(e(\nu)\tau_m) = e(\nu)\Phi_m - 1(X_m - X_{m+1})^{-1}X(\nu_m)
\]
\[
= X(\nu_m)e(\nu)\left((X_m - X_{m+1})^{-1}\Phi_m - (X_m - X_{m+1})^{-1}\right)
\]
\[
= X(\nu_m)e(\nu)(X_m + X_{m+1})^{-1}\left(X_m + X_{m+1} - q^2X_m h_m + \frac{(q^2 - 1)X_m + 1}{X_m + q^2X_m}\right)
\]
\[
= X(\nu_m)e(\nu)(X_m + X_{m+1})^{-1}\left(X_m + X_{m+1} - q^2X_m h_m + \frac{q^2(X_m + X_{m+1})}{X_m + q^2X_m}\right)
\]
\[
= q^{-2\nu_m}e(\nu)\left(-\frac{1}{X_m + q^2X_m}h_m + \frac{q^2}{X_m + q^2X_m}\right)
\]
so that

\[
e(\nu)h_m = q^{2\nu_m}(q^2X_m - X_{m+1})\Psi(e(\nu)\tau_m) + q^2e(\nu)
\]
as desired. The other cases can be checked similarly. This proves the claim. \( \square \)
Now, let us prove an analogue of quantum affine Schur-Weyl duality \([7]\) for \(U(\ell)\) and explain how to identify it with our functor \(\mathcal{F}_{e,\ell} \).

Recall that \(V = (W_{1,\ell})^{\text{aff}}\). Put

\[ V = W_{1,\ell} \subset V. \]

Assume that \(V^{\otimes \ell} = (V^{\text{aff}})^{\otimes \ell}\) is a \(k[X_1^{\pm 1}, \ldots, X_{\ell}^{\pm 1}]\)-module where \(X_k\) acts on the \(k\)-th component as the automorphism \(z\). For \(1 \leq m \leq \ell - 1\), let

\[ R_m : V^{\otimes \ell} \to k(X_m, X_{m+1}) \otimes k V^{\otimes \ell} \]

denote the map given by applying \(R_m\) (the projection to the generalized eigenspace), and \(\Phi_m\) as the automorphism \(z\) as an \(H^{\text{aff}}(q^2)\)-module via the following isomorphisms of \(k\)-vector space,

\begin{align*}
V^{\otimes \ell} & \cong V^{\otimes \ell} \otimes k \otimes k \otimes V^{\otimes \ell} \cong (V^{\otimes \ell} \otimes H^{\text{aff}}(q^2)) \otimes k \otimes \ell \\
& \cong V^{\otimes \ell} \otimes H^{\text{aff}}(q^2) \cong V^{\otimes \ell} \otimes H^{\text{aff}}(q^2) \otimes \ell \\
& = V^{\otimes \ell} \otimes H^{\text{aff}}(q^2) \otimes H^{\text{aff}}(q^2) \\
& \equiv V^{\otimes \ell} \otimes H^{\text{aff}}(q^2) \otimes H^{\text{aff}}(q^2).
\end{align*}

Hence \(V^{\otimes \ell}\) becomes a \((U(\ell), \otimes H^{\text{aff}}(q^2))\)-bimodule. We remark that the action of \(\Phi_m\) on \(V^{\otimes \ell}\) coincides with that of \(R_m\). Moreover we can check the following.

**Lemma 6.7.** We have the following.

1. \(V^{\otimes \ell}\) has a right \(H^{\text{aff}}(q^2)\)-module structure given by

\[
(f \otimes v) X_k = (X_k f) \otimes v,
\]

\[
(f \otimes v) h_m = (f \otimes v) \left( \frac{X_{m+1} - q^2 X_m}{X_m - X_{m+1}} R_m + \frac{(q^2 - 1) X_{m+1}}{X_{m+1} - X_m} \right)
\]

for \(1 \leq m \leq \ell\), and \(1 \leq m \leq \ell - 1\).

2. \(V^{\otimes \ell} \cong V^{\otimes \ell} \otimes H^{\text{aff}}(q^2)\) as an \(H^{\text{aff}}(q^2)\)-module.

So we may regard \(V^{\otimes \ell} := \bigotimes_k \otimes k[X_1^{\pm 1}, \ldots, X_{\ell}^{\pm 1}] V^{\otimes \ell}\) as a right \(\otimes H^{\text{aff}}(q^2)\)-module via the following isomorphisms of \(k\)-vector space,

\[
(4.4) \quad V^{\otimes \ell} \cong V^{\otimes \ell} \otimes k \otimes \ell \cong (V^{\otimes \ell} \otimes H^{\text{aff}}(q^2) \otimes k \otimes \ell) \cong V^{\otimes \ell} \otimes H^{\text{aff}}(q^2) \otimes H^{\text{aff}}(q^2)
\]

We can check the following.

**Lemma 6.8.** The \((U(\ell), \tilde{R}^{\ell}(\ell))\)-bimodule structure on \(V^{\otimes \ell}\) induced from \(\Psi\) in Theorem 6.6 coincides with the one defined in Section 5.2.

Let \(H^{\text{aff}}(q^2)\)-mod, be the category of finite-dimensional \(H^{\text{aff}}(q^2)\)-modules such that eigenvalues of \(X_k\) \((1 \leq k \leq \ell)\) are in the set \(\{ X(j) = q^{-2j} | j \in J \}\). Then

\[ H^{\text{aff}}(q^2)\)-mod, \equiv \otimes H^{\text{aff}}(q^2)\)-mod,

where \(\otimes H^{\text{aff}}(q^2)\)-mod is the category of finite-dimensional \(\otimes H^{\text{aff}}(q^2)\)-modules (\(e(\nu)\) acts as the projection to the generalized eigenspace), and \(\equiv\) means an equivalence of categories.

On the other hand, let \(\tilde{R}^{\ell}(\ell)\)-mod be the category of (not necessarily graded) finite-dimensional \(\tilde{R}^{\ell}(\ell)\)-modules. Note that

\[ \tilde{R}^{\ell}(\ell)\)-mod \equiv R^{\ell}(\ell)\)-mod,
where $R^J(\ell)\text{-mod}_0$ is the category of finite-dimensional $R^J(\ell)$-modules such that $x_k$ acts nilpotently.

Since we have an equivalence

$$\mathcal{O}H^\text{aff}(q^2)\text{-mod} \cong \hat{R}^J(\ell)\text{-mod},$$

which is induced from the isomorphism $\Psi$ in Theorem 6.6, we have an equivalence

$$\Psi^* : H^\text{aff}(q^2)\text{-mod}_J \cong R^J(\ell)\text{-mod}_0.$$

Summarizing the above arguments, we have the following.

**Proposition 6.9.** We have a functor

$$F^*_{\epsilon,\ell} : H^\text{aff}(q^2)\text{-mod}_J \to C^\ell_j(\epsilon) \to V^{\otimes \ell} \otimes_{\mathcal{O}H^\text{aff}(q^2)} M.$$

We also have

$$F^*_{\epsilon,\ell}(M) \cong F_{\epsilon,\ell} \circ \Psi^*(M),$$

where $F_{\epsilon,\ell}$ is given in (5.8) and the isomorphism is natural in $M$.

We see from (6.4)

$$F^*_{\epsilon,\ell}(M) = V^{\otimes \ell} \otimes_{\mathcal{O}H^\text{aff}(q^2)} M \cong V^{\otimes \ell} \otimes H_\ell(q^2) M$$

for a left $H^\text{aff}(q^2)$-module $M$, which is related to the Schur-Weyl duality of finite type as follows (note that [33] deals with $H_\ell(q^{-2})$ where $h^{-1}_i$ is used as a generator instead of $h_i$).

**Proposition 6.10** ([33 Theorem 3.1]). Let $R : V^{\otimes 2} \to V^{\otimes 2}$ be a $\hat{U}(\epsilon)$-linear map given by

$$R(|e_i\rangle \otimes |e_j\rangle) = \begin{cases} q|e_i\rangle \otimes |e_i\rangle & \text{if } i = j, \\ q|e_j\rangle \otimes |e_i\rangle & \text{if } i > j, \\ (q^2 - 1)|e_i\rangle \otimes |e_j\rangle + q|e_i\rangle \otimes |e_i\rangle & \text{if } i < j. \end{cases}$$

Then $V^{\otimes \ell}$ has a $(\hat{U}(\epsilon), H_\ell(q^2))$-bimodule structure given by $v \cdot h_m = \mathcal{R}_m(v)$ for $1 \leq m \leq \ell - 1$, where $\mathcal{R}_m$ denotes the map given by applying $\mathcal{R}$ on the $m$-th and $(m + 1)$-st components of $V^{\otimes \ell}$ and the identity elsewhere. Hence we have a functor

$$J_\ell : H_\ell(q^2)\text{-mod} \to \tilde{C}^\ell(\epsilon) \to V^{\otimes \ell} \otimes H_\ell(q^2) M,$$

which is an equivalence for $\ell \leq n$.

**Remark 6.11.** One can directly check that the right $H_\ell(q^2)$-action on $V^{\otimes \ell}$ in Proposition 6.10 coincides with the one in Lemma 6.7. Hence, as $\hat{U}(\epsilon)$-modules,

$$F^*_{\epsilon,\ell}(M) \cong J_\ell(M).$$

The following is an analogue of the result [7 Section 4] for $\hat{U}(\epsilon)$.

**Theorem 6.12.** For $\ell < n$, the functor $F^*_{\epsilon,\ell}$ is an equivalence of categories.
Proof. The proof is given in Appendix A.2 \[\square\]

**Corollary 6.13.** For \(\ell < n\), we have an equivalence of categories
\[
F_{\ell,t} : R^J(\ell)\text{-mod}_0 \longrightarrow C^J_f(\ell).
\]

**Corollary 6.14.** Suppose that \(\epsilon = (\epsilon_1, \ldots, \epsilon_n), \epsilon' = (\epsilon'_1, \ldots, \epsilon'_n) \in E\) are given such that \(\epsilon' < \epsilon\) and \(\ell < n' < n\). Then \(\text{tr}_{\epsilon'}\) induces an equivalence of categories
\[
\text{tr}_{\epsilon'} : C^J_f(\epsilon) \longrightarrow C^J_f(\epsilon').
\]

**Proof.** By Theorem 5.10, we have \(\text{tr}_{\epsilon'} \circ F_{\ell,t} \cong F_{\ell',t}\). Since \(F_{\ell,t}\) and \(F_{\ell',t}\) are equivalences by Corollary 6.13, \(\text{tr}_{\epsilon'}\) is an equivalence. \(\square\)

### 6.3. Equivalence between \(R^J\)-mod\(_0\) and \(C_J(\epsilon^\infty)\)

In this subsection, we reformulate the generalized quantum affine Schur-Weyl duality functor as an equivalence between \(R^J\)-mod\(_0\) and a certain subcategory of \(C(\epsilon^\infty)\) for any given \(\epsilon^\infty\).

**Definition 6.15.** Let \(\epsilon^\infty\) and \((\epsilon(k))_{k \geq 1}\) be as in (4.9) and (4.10), respectively.

1. Let \(C_J(\epsilon^\infty)\) be the full subcategory of \(C(\epsilon^\infty)\) with objects \(V\) such that
   (i) \(\text{tr}_k(V) \in C_J(\epsilon(k))\) for \(k \geq 1\),
   (ii) \(\ell(\text{tr}_k(V)) = \ell(\text{tr}_{k+1}(V))\) for all sufficiently large \(k\).
2. For \(\ell \in \mathbb{Z}_+\), let \(C_J^\ell(\epsilon^\infty)\) be the full subcategory of \(C_J(\epsilon^\infty)\) with objects \(V\) such that \(\text{tr}_k(V) \in C_J^\ell(\epsilon(k))\) for \(k \geq 1\).
3. Let \(P^+_J\) be the set of \((I,c) \in P^+\) satisfying (6.1).

Note that \(C_J(\epsilon^\infty)\) is monoidal. We remark that \(C_J(\epsilon^\infty)\) is not the inverse limit category associated to \(\{C_J^\ell(\epsilon(k))\}_{k \geq 1}\). Instead, we have the following.

**Lemma 6.16.**

1. We have the following decomposition
   \[
   C_J(\epsilon^\infty) = \bigoplus_{\ell \in \mathbb{Z}_+} C_J^\ell(\epsilon^\infty).
   \]
2. For \(\ell \in \mathbb{Z}_+\), we have
   \[
   C_J^\ell(\epsilon^\infty) = \varprojlim C_J^\ell(\epsilon(k)),
   \]
   which is the inverse limit category associated to \(\{C_J^\ell(\epsilon(k))\}_{k \geq 1}\) defined as in Definition 4.12.

**Proof.** (1) Suppose that \(V\) is an object in \(C_J(\epsilon^\infty)\). If we put \(V_k = \text{tr}_k(V)\) for \(k \geq 1\), then we have \(V_k = \bigoplus_{\ell} V^\ell_k\) where \(V^\ell_k \in C_J^\ell(\epsilon(k))\). Let \(V^\ell\) be an object in \(C_J^\ell(\epsilon^\infty)\) such that \(\text{tr}_k(V^\ell) = V^\ell_k\) for \(k \geq 1\).

For \(k \geq 1\), let \(\ell_k = \max\{\ell \mid V^\ell_k \neq 0\}\). If the sequence \(\{\ell_k\}_{k \geq 1}\) is unbounded, then so is \(\{\ell(V_k)\}_{k \geq 1}\), which is a contradiction. Hence \(V = \bigoplus_{\ell} V^\ell\), where \(V^\ell\) is zero except for a finitely many \(\ell\)’s.

(2) Let \(A_1\) and \(A_2\) denote the categories on the lefthand side and the righthand side of the statement respectively. It is clear that \(A_1\) is a subcategory of \(A_2\).
Let $V$ be an object in $A_2$, and let $V_k = \text{tr}_k(V)$ for $k \geq 1$. Given $k$, suppose that $W_{\varepsilon(n)}(l, c)$ is a composition factor of $V_k$. Note that $W_{\varepsilon(n)}(l, c)$ is a direct sum of $V_{\varepsilon(n)}(\lambda)$’s for partitions $\lambda$ of $l$. If $k$ is sufficiently large, then we see from Proposition 6.10 that $\text{tr}_{k-1}^k(W_{\varepsilon(n)}(\lambda))$ is non-zero for all partitions $\lambda$ of $l$, and then $\text{tr}_{k-1}^k(W_{\varepsilon(n)}(l, c))$ is non-zero. This implies that $\ell(V_k) = \ell(V_{k'})$ for $k' \geq k$, and hence $V$ is an object in $A_1$. Therefore $A_1 = A_2$.

Lemma 6.17. Let $V$ be a nonzero object in $C_J(\varepsilon^\infty)$. Then $V$ is simple if and only if $\text{tr}_k(V)$ is irreducible or zero for all $k \geq 1$.

Proof. By Lemma 4.16 the converse is true. Suppose that $V$ is simple. Then by [14, Lemma 4.15], any non-zero $\text{tr}_k(V)$ is irreducible.

Lemma 6.18. If $W_{\varepsilon(n)}(l, c) \cong W_{\varepsilon(n)}(l^*, c^*)$ for some $(l, c)$ and $(l^*, c^*) \in P^+_J$, then $(l, c)$ and $(l^*, c^*)$ are equal up to permutation.

Proof. Take a sufficiently large $k$ such that the number of occurrence of $1$ in $\varepsilon := \varepsilon(k)$, say $N$, is large enough, and put $\varepsilon' = \varepsilon_{0\mid N} < \varepsilon$. Then $W_{\varepsilon(n)}(l, c)$ and $W_{\varepsilon'(n)}(l, c)$ are non-zero, and $\text{tr}_{\varepsilon(n)}(W_{\varepsilon(n)}(l, c)) = W_{\varepsilon(n)}(l, c)$. Similarly, we have $\text{tr}_{\varepsilon'(n)}(W_{\varepsilon'(n)}(l, c)) = W_{\varepsilon'(n)}(l, c)$. Since $W_{\varepsilon(n)}(l, c) \cong W_{\varepsilon'(n)}(l^*, c^*)$, we have $W_{\varepsilon(n)}(l, c) \cong W_{\varepsilon'(n)}(l^*, c^*)$.

Then by the classification of finite-dimensional irreducible modules over $U(\varepsilon')$, the quantum affine algebra of type $A_{N-1}^{(1)}$, we have $t = t^*$ where $l \in \mathbb{Z}_1$, $l^* \in \mathbb{Z}_1^c$, and $(l^*, c^*)$ is a permutation of $(l, c)$.

By [6.2] and Lemmas 4.13 (with $C(\varepsilon(k))$ replaced by $C_J(\varepsilon^\infty)$) and 4.16(2), we have functors

\[
\mathcal{F}_{c^\infty, l} : R^l(\ell)\text{-mod}_0 \longrightarrow C_J^l(\varepsilon^\infty),
\]

\[
\mathcal{F}_{c^\infty} = \bigoplus_l \mathcal{F}_{c^\infty, l} : R^l\text{-mod}_0 \longrightarrow C_J(\varepsilon^\infty),
\]

which are exact and monoidal.

Proposition 6.19. The functor $\mathcal{F}_{c^\infty, l}$ is an equivalence of categories, and hence so is $\mathcal{F}_{c^\infty}$.

Proof. It follows directly from Corollaries 6.13 and 4.16(2), and Lemma 4.15 with $C(\varepsilon(k))$ replaced by $C_J^l(\varepsilon^\infty)$.

Corollary 6.20. The functor $\mathcal{F}_{c^\infty}$ induces a one-to-one correspondence between the sets of isomorphism classes of irreducible objects in $R^l\text{-mod}_0$ and $C_J(\varepsilon^\infty)$ by sending

\[
\text{hd } (L(a_1, b_1) \circ \cdots \circ L(a_t, b_t)) \longrightarrow W_{\varepsilon(n)}(l, c),
\]

where $((a_1, b_1), \ldots, (a_t, b_t))$ is an ordered multisegment and $(l, c) \in P^+_J$ is given by $l = (b_1 - a_1 + 1, \ldots, b_t - a_t + 1)$ and $c = (q^{-a_1-b_1}, \ldots, q^{-a_t-b_t}).$

6.4. Super duality. Let $c^\infty$ and $(\varepsilon^{(k)})_{k\geq 1}$ be as in [4.9] and 4.10, respectively. Let us describe more explicitly a connection between $C_J(\varepsilon^\infty)$ and the category of finite-dimensional representations of the usual quantum affine algebras $U(\varepsilon)$ of type $A$, that is, $\varepsilon = \varepsilon_{n\mid 0}$ and $c = \varepsilon_{n\mid 0}$. 

Let \( \xi^\infty = (\xi_i)_{i \in \mathbb{Z}_{\geq 0}} = (0, 0, 0, \ldots) \), \( \tau^\infty = (\tau_i)_{i \in \mathbb{Z}_{\geq 0}} = (1, 1, 1, \ldots) \).

Recall that \( M_k \) and \( N_k \) are the number of 0’s and 1’s in \( \epsilon^{(k)} \), respectively. Let \( \{r_k\}_{k \geq 1} \) and \( \{s_k\}_{k \geq 1} \) be the strictly increasing sequences satisfying the following:

\[
\epsilon_{k|0} = \xi^{(k)} < \epsilon^{(rk)}, \quad \epsilon_{0|k} = \tau^{(k)} < \epsilon^{(sk)}.
\]

Given \( \ell \in \mathbb{Z}_+ \), let us define functors \( S_{k|0} \) and \( S_{0|k} \)

(6.5)

\[
\begin{align*}
S_{k|0} & : C^\ell_f(\epsilon^\infty) \to C^\ell_f(\epsilon_{k|0}) \\
S_{0|k} & : C^\ell_f(\epsilon_{0|k}) \to C^\ell_f(\epsilon^\infty)
\end{align*}
\]

for \( k \geq 1 \) by

\[
S_{k|0} = \text{tr}^{(rk)}_{k|0} \circ \text{tr}_{rk}, \quad S_{0|k} = \text{tr}^{(rk)}_{0|k} \circ \text{tr}_{sk}.
\]

Applying Lemma 4.13 to \( (6.5) \) with \( C(\epsilon^{(k)}) \) replaced by \( C^\ell_f(\epsilon_{k|0}) \) and \( C^\ell_f(\epsilon_{0|k}) \), we obtain functors \( S_{\infty|0} \) and \( S_{0|\infty} \).

\[
\begin{align*}
S_{\infty|0} & : C^\ell_f(\epsilon^\infty) \to C^\ell_f(\epsilon_{k|0}) \\
S_{0|\infty} & : C^\ell_f(\epsilon_{0|k}) \to C^\ell_f(\epsilon^\infty)
\end{align*}
\]

such that \( \text{tr}_k \circ S_{\infty|0} \cong S_{k|0} \) and \( \text{tr}_k \circ S_{0|\infty} \cong S_{0|k} \) for \( k \geq 1 \). Again by Lemma 6.10(1), we have functors of monoidal categories

\[
\begin{align*}
S_{\infty|0} & : C_f(\epsilon^\infty) \to C_f(\epsilon_{k|0}) \\
S_{0|\infty} & : C_f(\epsilon_{0|k}) \to C_f(\epsilon^\infty)
\end{align*}
\]

Then we have the following, which can be viewed as a quantum affine analogue of super duality of type \( A_\infty \).

**Theorem 6.21.** The functors \( S_{\infty|0} \) and \( S_{0|\infty} \) are equivalences of monoidal categories.

**Proof.** It is enough to show that \( S_{\infty|0} : C^\ell_f(\epsilon^\infty) \to C^\ell_f(\epsilon_{k|0}) \) is an equivalences of categories for each \( \ell \in \mathbb{Z}_+ \). First, note that \( C^\ell_f(\epsilon^\infty) \cong C^\ell_f(\epsilon^{(k)}) \) for \( k > \ell \) by Lemma 4.13 and Corollary 6.14. Also, \( C^\ell_f(\epsilon^{(k)}) \cong C^\ell_f(\epsilon_{k|0}) \) for all sufficiently large \( k \) by Corollary 6.14. Hence \( S_{k|0} : C^\ell_f(\epsilon^\infty) \to C^\ell_f(\epsilon_{k|0}) \) is an equivalence for all sufficiently large \( k \). This implies that \( S_{\infty|0} = \lim_{\ell \to \infty} S_{k|0} : C^\ell_f(\epsilon^\infty) \to C^\ell_f(\epsilon_{k|0}) \) is an equivalence of categories by Lemma 4.15.

**Corollary 6.22.** Given \( \ell \in \mathbb{Z}_+ \), the following are equivalences of categories for a sufficiently large \( k \):

\[
\begin{align*}
S_{k|0} & : C^\ell_f(\epsilon^\infty) \to C^\ell_f(\epsilon_{k|0}) \\
S_{0|k} & : C^\ell_f(\epsilon_{0|k}) \to C^\ell_f(\epsilon^\infty)
\end{align*}
\]
6.5. Grothendieck rings. Let $e^\infty$ be the sequence given as in (4.9). Let $K(C_J(e^\infty))$ be the Grothendieck group of $C_J(e^\infty)$. Then

$$K(C_J(e^\infty)) = \bigoplus_{\ell \in \mathbb{Z}_+} K(C^\ell_J(e^\infty))$$

by Lemma 6.16 and it has a well-defined ring structure with multiplication

$$[V] \cdot [W] = [V \otimes W] \in K(C^{\ell+\ell'}_J(e^\infty)),$$

for $[V] \in K(C^\ell_J(e^\infty))$ and $[W] \in K(C^{\ell'}_J(e^\infty))$. By Theorem 6.21, we have ring isomorphisms from $K(C_J(e^\infty))$ to $K(C_J(\mathbb{C}^\infty))$ and $K(C_J(\mathbb{C}^{\infty}))$.

Let $S = \{ (l,a) \mid l \in \mathbb{N}, a \in l-1+2\mathbb{Z} \}$ and

$$R = \mathbb{Z}[t_{l,a} \mid (l,a) \in S]$$

be the polynomial ring generated by $t_{l,a} \in S$. Put $\deg(t_{l,a}) = l$ and let $R^\ell$ denote the subgroup of $R$ generated by monomials of degree $\ell \in \mathbb{Z}_+$.

**Proposition 6.23.** There is an isomorphism of rings

$$\begin{array}{ccc}
K(C_J(e^\infty)) & \longrightarrow & R \\
[W_{l,e^\infty}(q^a)] & \longmapsto & t_{l,a}
\end{array}$$

where $K(C_J^\ell(e^\infty))$ maps onto $R^\ell$. In particular, $K(C_J(e^{(k)}))$ is a homomorphic image of $R$ given by sending $t_{l,a} \in R$ to $[W_{l,e^{(k)}}(q^a)]$ for $k \geq 1$, where $e^{(k)}$ is given as in (4.10).

**Proof.** For $k \geq 1$, it is well-known [15 Corollary 2] that $K(C_J(e^{(0|k+1)}))$ is isomorphic to

$$R_k := \mathbb{Z}[t_{l,a} \mid l = 1, \ldots, k, a \in l-1+2\mathbb{Z} \subset R,$$

as a ring, where $[W_{l,e^{(0|k+1)}}(q^a)]$ is mapped to $t_{l,a}$, and $K(C_J^\ell(e^{(0|k+1)})) \cong R_k^\ell := R^\ell \cap R_k$ for $\ell \in \mathbb{Z}_+$. Since $t_{l,a} \in R_k$ induces a map $R_k \longrightarrow R_{k-1}$ given by $t_{k,a} = 0$ for $a \in l-1+2\mathbb{Z}$ and its restriction $R_k^\ell \longrightarrow R_{k-1}^\ell$ stabilizes for all sufficiently large $k$, we have

$$K(C_J^\ell(\mathbb{C}^{\infty})) = \lim_{\ell \to \infty} K(C_J^\ell(e^{(0|k)})) \cong R^\ell.$$

Hence we have an isomorphism of rings $K(C_J(\mathbb{C}^{\infty})) \longrightarrow R$ which maps $[W_{l,e^{\infty}}(q^a)]$ to $t_{l,a}$ for $(l,a) \in S$. Therefore, the first isomorphism from $K(C_J(e^{\infty}))$ to $R$ follows from $S_{(0|\infty)}(W_{l,e^{\infty}}(q^a)) = W_{l,e^{\infty}}(q^a)$. The second homomorphism from $R$ to $K(C_J(e^{(k)}))$ is induced from $tr_k$. \hfill \square

**Corollary 6.24.** Let $e \in \mathcal{E}$ and $(l,c) \in \mathcal{P}_J^+(e)$ be given. If $\chi$ is a polynomial in $R$ such that

$$[W_{(c|k)}(l,c)] = \chi([W_{l,e^{(k)}}(q^a)] \mid (l,a) \in S)$$

for a sufficiently large $k$, then the following holds in $K(C_J(e))$:

$$[W_{c}(l,c)] = \chi([W_{l,e}(q^a)] \mid (l,a) \in S).$$
Remark 6.25. The isomorphism of $\mathbb{Q}$-algebras in Eq. induces an equivalence between $\mathcal{C}_J(\mathfrak{L}^\infty)$ and $\mathcal{C}_J(\mathfrak{T}^\infty)$, by which we identify their Grothendieck groups. Under this identification, the equivalence $S_{\infty,0} \circ S_{0,\infty}^{-1}$ induces an involution on $K(\mathcal{C}_J(\mathfrak{L}^\infty))$, which can be viewed as an affine analogue of the involution on the ring of symmetric functions sending a Schur function to another one of conjugate shape.

6.6. KR modules and $T$-system. Let $\epsilon \in \mathcal{E}$ be given. For $r, s \in \mathbb{N}$ and $c \in \mathbb{k}^\times$, let

$$W_{r,s}^\epsilon(c) := W_{r}(l, c) \text{ where } l = (r, \ldots, r) \in \mathbb{Z}_+^r \text{ and } c = c(q^{-2(s-1)}, \ldots, q^{-2}, 1).$$

If $\epsilon = \epsilon_{0|n}$ or $\epsilon_{n|0}$, then $W_{r,s}^\epsilon(c)$ is the usual Kirillov-Reshetikhin module of type $A_{n-1}^{(1)}$ (see Remark 6.27 for more detail).

We note that $W_{r,s}^\epsilon(c) \cong V((s^*)^r)$ as a $\tilde{U}(\epsilon)$-module if is not zero. This can be seen easily from the fact in case of $\epsilon_{0|n}$, Theorem 6.11 and Corollary 6.22. Moreover, $W_{r,s}^\epsilon(c) \in \mathcal{C}_J(\epsilon)$ if and only if $c < q^{-1} + 2\mathbb{Z}$.

Proposition 6.26. There exists a short exact sequence in $\mathcal{C}(\epsilon)$

$$0 \longrightarrow \bigotimes_{r' = r \pm 1} W_{r',s}^{\epsilon}(cq^{-1}) \longrightarrow W_{r,s}^{\epsilon}(cq^{-2}) \longrightarrow W_{r,s+1}^{\epsilon}(cq^{-2}) \longrightarrow 0,$$

where $c \in \mathbb{k}^\times$. Hence the following holds in $K(\mathcal{C}(\epsilon))$:

$$[W_{r,s}^{\epsilon}(c)] [W_{r,s}^{\epsilon}(cq^{-2})] = [W_{r,s+1}^{\epsilon}(c)] [W_{r,s-1}^{\epsilon}(cq^{-2})] + [W_{r-1,s}^{\epsilon}(cq^{-1})] [W_{r+1,s}^{\epsilon}(cq^{-1})].$$

Proof. First we take $c = q^{-1}$ so that all the modules in the sequence belong to $\mathcal{C}_J(\epsilon)$. In case of $\epsilon = \epsilon_{0|n}$, the existence of such short exact sequence is well-known, which is called the $T$-system [10, 30, 35]. Suppose that $\epsilon$ is arbitrary. Take $\epsilon^{\infty}$ as in Eq. with an ascending chain of subsequence $(\epsilon^{(k)})_{k \geq 1}$ such that $\epsilon^{(k)} = \epsilon$ for some $k$. We consider $W_{r,s}^{\epsilon^{(k)}}(c) := W_{r,s}^{\epsilon}(l, c) \in \mathcal{C}_J(\epsilon^{\infty})$. Since we have such a short exact sequence for $\epsilon_{0|m}$ for a sufficiently large $m$, and hence for $\epsilon^{\infty}$ by Corollary 6.22, the result follows by applying $\mathcal{C}_J(\epsilon^{\infty})$. Then by applying an automorphism $\tau_\epsilon$ of $\mathcal{U}(\epsilon)$ ($c \in \mathbb{k}^\times$) given by $\tau_\epsilon(e_i) = c^{\delta_{0,0}}e_i, \tau_\epsilon(f_i) = c^{-\delta_{0,0}}f_i$ and $\tau_\epsilon(k\lambda) = k\lambda$ for $i \in I$ and $\mu \in P$ to the above exact sequence, we may conclude the same result for arbitrary $c \in \mathbb{k}^\times$. (D)

Remark 6.27. (1) Let $W_{s,a}^{(r)}$ denote the Kirillov-Reshetikhin module over $U_q(A_{n-1}^{(1)})$ associated to $r = 1, \ldots, n-1$, $s \geq 1$ and $a \in \mathbb{k}^\times$. The Drinfeld polynomial of $W_{s,a}^{(r)}$ is given by

$$P_i(u) = \begin{cases} \prod_{k=1}^{s}(1 - aq^{2k-2}u) & \text{if } i = r, \\ 1 & \text{if } i \neq r. \end{cases}$$

Note that if $\epsilon = \epsilon_{0|n}$ and $r < n$, then $W_{r,\epsilon}(c)$ corresponds to $W_{1,a}^{(r)}$, where

$$a = -o(r)(-q)^{-n}\tilde{c}$$

by Remark 6.23 and Remark 3.3. Here $\tilde{c} : \mathbb{k} \rightarrow \mathbb{k}$ is an automorphism such that $\tilde{q} = -q^{-1}$ and $o : I \setminus \{0\} \rightarrow \{\pm 1\}$ is a map such that $o(i) = -o(j)$ if $a_{ij} = -1$. Therefore, $W_{r,\epsilon}^\epsilon(c)$ corresponds to $W_{s,a}^{(r)}$. \hfill \square
(2) When $\epsilon = \epsilon_{n|0}$, we have a short exact sequence of tensor products of KR modules where the shapes corresponding to the $U(\epsilon)$-highest weight are the conjugate of the ones appearing in case of $\epsilon = \epsilon_{0|n}$.

(3) In [32], it is shown that the Kirillov-Reshetikhin module $W_{\epsilon:0}(\epsilon)$ has a crystal base with a suitable choice of $c$, when $\epsilon$ is standard, that is, $\epsilon = \epsilon_{M|N}$. An explicit description of its crystal and the associated combinatorial $\mathbb{R}$ matrix is also given in [32].

Appendix A.

A.1. Proof of Lemma 5.6 In this section we prove Lemma 5.6. The proof consists of direct calculations as indicated in [1, Lemma B.1], but we give details for the reader’s convenience as it is little more involved.

We claim that there exists an exact sequence of the following form for each $\ell \geq 2$:

\begin{equation}
0 \rightarrow W_{\ell,\epsilon}(1) \xrightarrow{\psi_1} W_{1,\epsilon}(q^{1-\ell}) \otimes W_{2-1,\epsilon}(q^1) \xrightarrow{R} W_{\ell-1,\epsilon}(q^{1-\ell}) \otimes W_{1,\epsilon}(q^{1-\ell}) \xrightarrow{\psi_2} W_{\ell,\epsilon}(1) \rightarrow 0,
\end{equation}

for some $U(\epsilon)$-linear maps $\psi_1$ and $\psi_2$, where $R = R^\text{norm}_{1,\ell-1}(q^{-\ell})$. Recall from Theorem 3.15 that

\begin{equation}
R^\text{norm}_{1,\ell-1}(z) = P_1 + \frac{1 - z q^\ell}{z - q^\ell} P_0,
\end{equation}

which is equal to $P_1$ when $z = q^{-\ell}$. We may assume that $\epsilon_1 = 0$. Indeed, the result for arbitrary $\epsilon$ follows once we choose $\epsilon' > \epsilon$ with $\epsilon'_1 = 0$ and apply the truncation functor $tr_{\epsilon'}^\epsilon$ to the exact sequence for $\epsilon'$, keeping Propositions 3.9, 3.10 and Lemma 4.8 in mind.

Recall that when $\epsilon_1 = 0$, the $U(\epsilon)$-highest weight vectors of $V((\ell))$ and $V((\ell - 1, 1))$ in the decomposition $W_{1,\epsilon}(x) \otimes W_{\ell-1,\epsilon}(y) = V_c((\ell)) \oplus V_c((\ell - 1, 1))$ are given by

\begin{equation}
|e_1\rangle \otimes |(\ell - 1)e_1\rangle, \quad |e_1\rangle \otimes |(\ell - 2)e_1 + e_2\rangle - q^{\ell - 1}|e_2\rangle \otimes |(\ell - 1)e_1\rangle
\end{equation}

respectively. On the other hand, when $\epsilon_1 = 1$ the highest weight vectors become more complicated.

Let us define $\psi_1$ and $\psi_2$ by

\begin{align*}
\psi_1(|m\rangle) &= \sum_{1 \leq k \leq n} |e_k\rangle \otimes |m - e_k\rangle \left(m_k \prod_{k < j \leq n} q^{m_j}\right), \\
\psi_2(|m\rangle \otimes |e_k\rangle) &= |m + e_k\rangle \prod_{k < j \leq n} q^{-m_j}
\end{align*}

for $|m\rangle$ and $1 \leq k \leq n$. Here we also understand $|m\rangle = 0$ whenever $m \notin \mathbb{Z}^n_+(\epsilon)$. Note that when $\epsilon = (1^N)$, $\psi_1$ and $\psi_2$ coincide with the maps in [1, Lemma B.1] up to a constant multiple.

Lemma A.1. The maps $\psi_1$ and $\psi_2$ are $U(\epsilon)$-linear.

Proof. Since the proof is rather straightforward, let us show that $\psi_1$ commutes with $e_i$ ($i \in I$), and leave the other details to the reader.
Case 1. Suppose that \( i \in I \setminus \{ 0 \} \). First we have

\[
e_{i} \psi_{1} |m\rangle = \sum [m_{k}] \prod_{j \geq k} q^{m_{j}} e_{i} (|e_{k}\rangle \otimes |m - e_{k}\rangle) = \sum [m_{k}] \prod_{j \geq k} q^{m_{j}} |m_{i+1} - 1\rangle \otimes |m + e_{i} - e_{i+1}\rangle
\]

(A.4)

\[
+ [m_{i+1}] \prod_{j > i+1} q^{m_{j}} |m_{i+1} - 1\rangle \otimes |m + e_{i} - 2e_{i+1}\rangle
\]

\[
+ [m_{i+1}] \prod_{j > i+1} q^{m_{j}} q_{i}^{-m_{i}} q_{i+1}^{m_{i+1} - 1} |e_{i}\rangle \otimes |m - e_{i+1}\rangle
\]

\[
+ [m_{i}] \prod_{j > i} q^{m_{j}} |m_{i+1} - 1\rangle \otimes |m - e_{i+1}\rangle.
\]

Let \((\ast)\) denote the sum of last two terms, that is,

\[
(\ast) = [m_{i+1}] \prod_{j > i+1} q^{m_{j}} (|m_{i} q_{i}^{m_{i+1} - 1} + q_{i}^{-m_{i}} q_{i+1}^{m_{i+1} - 1}|e_{i}\rangle \otimes |m - e_{i+1}\rangle).
\]

Suppose first that \( e_{i} |m\rangle = 0 \). Note that \( e_{i} |m\rangle = 0 \) if and only if \( m_{i+1} = 0 \) or \( m_{i+1} \neq 0 \), \( m_{i} = 1 = e_{i} \). If \( m_{i+1} = 0 \), then \( e_{i} \psi_{1} |m\rangle = 0 \). In the other case, \( |m = e_{k} + e_{i} - e_{i+1}\rangle \) is nonzero if and only if \( k = i \). So \( A.3 \) is equal to

\[
(\ast) = [m_{i+1}] \prod_{j > i+1} q^{m_{j}} (|m_{i} q_{i}^{m_{i+1} + 1} + q_{i}^{-m_{i}} q_{i+1}^{m_{i+1} - 1}|e_{i}\rangle \otimes |m - e_{i+1}\rangle).
\]

Since

\[
[m_{i}] q_{i}^{m_{i+1} + 1} + q_{i}^{-m_{i}} q_{i+1}^{m_{i+1} - 1} = [1] q_{i}^{m_{i+1} + 1} + (-q) q_{i+1}^{m_{i+1} - 1}
\]

\[
= \begin{cases} 
q_{i}^{m_{i+1} + 1} + (-q) q_{i+1}^{m_{i+1} - 1} & \text{if } e_{i+1} = 0, \\
q + (-q)(-q^{-1})^{0} & \text{if } e_{i+1} = 1 = m_{i+1}, \\
0 & \text{otherwise}.
\end{cases}
\]

we have \( e_{i} \psi_{1} |m\rangle = 0 \) whenever \( e_{i} |m\rangle = 0 \).

Next suppose that \( e_{i} |m\rangle \neq 0 \) (necessarily \( m_{i+1} \neq 0 \)). We have

\[
\psi_{1} e_{i} |m\rangle = [m_{i+1}] \psi_{1} |m + e_{i} - e_{i+1}\rangle
\]

\[
= [m_{i+1}] \sum_{k \neq i, i+1} [m_{k}] \prod_{j \geq k} q^{m_{j}} |e_{k}\rangle \otimes |m - e_{k} + e_{i} - e_{i+1}\rangle
\]

\[
+ [m_{i+1}] [m_{i+1} - 1] \prod_{j > i+1} q^{m_{j}} |e_{i+1}\rangle \otimes |m - 2e_{i+1} + e_{i}\rangle
\]

\[
+ [m_{i+1}] [m_{i} + 1] q^{-1} \prod_{j > i} q^{m_{j}} |e_{i}\rangle \otimes |m - e_{i+1}\rangle.
\]

It is equal to \( A.4 \) if

\[
(\ast) = [m_{i+1}] [m_{i} + 1] q^{-1} \prod_{j > i} q^{m_{j}} |e_{i}\rangle \otimes |m - e_{i+1}\rangle.
\]

(A.5)
Indeed, we have two possibilities: either $m_i = 0$ or $m_i \neq 0$ with $\epsilon_i = 0$. In the first case, we have

$$(* \Rightarrow \text{A.5})$$

and the product can be written as

$$\prod_{j>i+1} q^{m_j} \cdot q^{-m_{i+1}-1} \text{ if } \epsilon_{i+1} = 0$$
$$\prod_{j>i+1} q^{m_j} \text{ if } \epsilon_{i+1} = 1 = m_{i+1}.$$  

which implies (A.5). In the other case, as $\prod_{j>i+1} q^{m_j} \cdot q^{-m_{i+1}-1} = \prod_{j>i} q^{m_j} \cdot q^{-1}$ by the same reason, we have

$$(* \Rightarrow \text{A.5})$$

Hence (A.5) holds.

Case 2. Suppose that $i = 0$. The proof is similar except that we should consider spectral parameters. First, we have

$$e_0 \psi_1 | \mathbf{m} \rangle = \sum [m_k] \prod_{j[k]} q^{m_j} e_0 ((|e_k \rangle \otimes |m - e_k\rangle)$$
$$= \sum [m_k] \prod_{j[k]} q^{m_j} [m_1] |e_k \rangle \otimes |m - e_k + e_n - e_1\rangle \cdot q$$
$$+ [m_1] \prod_{j>1} q^{m_j} [m_1 - 1] |e_1 \rangle \otimes |m - 2e_1 + e_n\rangle \cdot q$$
$$+ [m_1] \prod_{j>1} q^{m_j} \cdot q^{-m_a} q^{m_{1-1}} |e_n \rangle \otimes |m - e_1\rangle \cdot q^{1-\ell}$$
$$+ [m_n] [m_1] |e_n \rangle \otimes |m - e_1\rangle \cdot q.$$  

Note that since $\ell = \sum m_j$, we have in the third term above

$$\prod_{j>1} q^{m_j} \cdot q^{-m_a} q^{m_{1-1}} q^{1-\ell} = q^{m_a} q^{-m_a} q^{m_{1-1}} q.$$  

Similarly we have

$$\psi_1 e_0 | \mathbf{m} \rangle = [m_1] \psi_1 | \mathbf{m} + e_n - e_1\rangle \cdot q$$
$$= [m_1] \sum [m_k] \prod_{j[k]} q^{m_j} q |e_k \rangle \otimes |m - e_k + e_n - e_1\rangle$$
$$+ [m_1] [m_1 - 1] \prod_{j>1} q^{m_j} q |e_1 \rangle \otimes |m - 2e_1 + e_n\rangle$$
$$+ [m_1] [m_n + 1] |e_n \rangle \otimes |m - e_1\rangle.$$
Now the same argument applies as in Case 1. If \( e_0 | m \rangle = 0 \), then either \( m_1 = 0 \) or \( m_1 \neq 0 \) with \( m_n = 1 = \epsilon_n \). In the first case, we clearly have \( \psi_1 e_0 | m \rangle = 0 \). In the latter case, we have

\[
ee_0 \psi_1 | m \rangle = [m_1] (|1\rangle q + q^{-m_1}(-q^{-1})^{-1} q_{1}^{m_1-1} q |e_n\rangle \otimes |m - e_1\rangle = 0
\]
as \( q^{-1} q_{1}^{m_1-1} \) vanishes regardless of \( \epsilon_1 \).

Next, if \( e_0 | m \rangle \neq 0 \) and \( m_1 \neq 0 \), then again we have \( \psi_1 e_0 | m \rangle = 0 \) since

\[
[m_1] (|m_n\rangle q + q^{-m_1} q_{n}^{-m_n} q_{1}^{m_1-1} q |m_{n+1}\rangle = \begin{cases} [m_1] (0 + q^{-m_1} q_{1}^{m_1-1} q) & \text{if } m_n = 0 \\ [m_1] (|m_n\rangle q + q^{-m_n} q_{1}^{m_1-1} q) & \text{if } m_n \neq 0, \epsilon_n = 0 \\ = [m_1] [m_n + 1]. \end{cases}
\]

This completes the proof. \( \square \)

Lemma A.2. We have

(1) \( \psi_1 \) is injective and \( R \circ \psi_1 = 0 \),

(2) \( \psi_2 \) is surjective and \( \psi_2 \circ R = 0 \).

Proof. (1) It is clear that \( \psi_1 \) is injective since \( \psi_1 \) is non-zero and \( W_{\ell,\epsilon}(1) \) is irreducible.

By definition, we have \( \psi_1((|n_1\rangle) = |C \langle e_1\rangle \otimes (|\ell - 1\rangle e_1 \rangle = C_{v_1} \) for a nonzero constant \( C \).

(A.3) The \( U(\epsilon) \)-highest weight vector \( v_1 \) is sent to zero by \( R \) since \( R = P_1 \) by (A.2). This implies that \( R \circ \psi_1 = 0 \).

(2) Since \( \psi_2 \) is non-zero and \( W_{\ell,\epsilon}(1) \) is irreducible, it is surjective. Note that \( v_2 = |\langle (\ell - 1) e_1 \rangle \otimes e_2 \rangle - q (\ell - 2) e_1 + e_2 \rangle \otimes e_1 \rangle \) generates \( \text{Im} R \), which is isomorphic to \( V((\ell - 1, 1)) \).

Since \( \psi_2(v_2) = 0 \), we have \( \psi_2 \circ R = 0 \). \( \square \)

Lemma A.3. The sequence (A.1) is exact.

Proof. By the previous lemmas and the universal mapping properties of Ker and Coker, we have the following commutative diagram of \( U(\epsilon) \)-modules:

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{Ker} R & \rightarrow & W_{\ell,\epsilon}(x) \otimes W_{\ell-1,\epsilon}(y) & \xrightarrow{R} & W_{\ell-1,\epsilon}(y) \otimes W_{\ell,\epsilon}(x) & \rightarrow & \text{Coker} R & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & \leftarrow & \uparrow & & \leftarrow \\
0 & \rightarrow & W_{\ell,\epsilon}(1) & \rightarrow & W_{\ell,\epsilon}(1) & \rightarrow & 0 \\
\end{array}
\]

Hence two vertical arrows are isomorphisms. This implies that (A.1) is exact. \( \square \)

A.2. Proof of Theorem 6.12. We assume that \( \ell < n \). Put \( F = F_{\ell,\epsilon}^{*} \). We first show that

(A.6) \( F_{\ell,\epsilon}^{*} : H_{\ell}^{\text{aff}}(q^{2})-\text{mod} \rightarrow C(\epsilon) \)

\[
\begin{array}{cccc}
M & \rightarrow & Y^{\otimes \ell} \otimes H_{\ell}(q^{2}) M
\end{array}
\]
is an equivalence of categories. We almost follow the arguments in [7, Section 4.3 - 4.6] except a part of Lemma A.3. The following easy lemma is essential for the later computation.

**Lemma A.4** (cf. [7, Lemma 4.3]).

1. Let $M$ be a finite-dimensional $H_\ell(q^2)$-module. If $v \in V^{\otimes \ell}$ has non-zero components in each isotypical component of $\mathcal{F}_\ell(M)$, then the $k$-linear map

$\begin{align*}
M & \longrightarrow \ V^{\otimes \ell} \otimes_{H_\ell(q^2)} M = \mathcal{F}_\ell(M), \\
M & \mapsto v \otimes m
\end{align*}$

is injective.

2. Let $\{ v_i := |e_i \rangle \mid i = 1, \ldots, n \}$ be the standard basis of $V$. If $i_1, \ldots, i_\ell \in \{1, \ldots, n\}$ are distinct, then the $\mathcal{U}(\epsilon)$-module $V^{\otimes \ell}$ is generated by a single vector $v_{i_1} \otimes \cdots \otimes v_{i_\ell}$.

In particular, the vector satisfies the condition in (1).

We first prove that $F$ is essentially surjective. Suppose that $W \in C^\ell(\epsilon)$ is given. By Theorem 6.10 there exists a $H_\ell(q^2)$-module $M$ for which $W \cong \mathcal{F}_\ell(M) = V^{\otimes \ell} \otimes M$ as a $\mathcal{U}(\epsilon)$-module. We shall extend the $H_\ell(q^2)$-action on $M$ to $H_\ell(q^2)$ so that $W \cong V^{\otimes \ell} \otimes_{H_\ell(q^2)} M \cong V^{\otimes \ell} \otimes_{\mathcal{U}(\epsilon)} M$ as a $\mathcal{U}(\epsilon)$-module.

For $1 \leq j \leq \ell$, set $v^{(j)} = v_1 \otimes \cdots \otimes v_j \otimes v_{j+1} \otimes \cdots \otimes v_\ell$. Regarding $V^{\otimes \ell} \otimes_{H_\ell(q^2)} M$ as a $\mathcal{U}(\epsilon)$-module, the weight of $f_0(v^{(j)} \otimes m)$ is $\delta_1 + \cdots + \delta_\ell \in P$. As

\begin{equation}
\{ v_{i_1} \otimes \cdots \otimes v_{i_\ell} \mid 1 \leq i_1, \ldots, i_\ell \leq \ell \text{ are distinct} \}
\end{equation}

is a basis of $(V^{\otimes \ell})_{\delta_1 + \cdots + \delta_\ell}$, we can write as

\begin{equation}
f_0(v^{(j)} \otimes m) = \sum_i (v_{i_1} \otimes \cdots \otimes v_{i_\ell}) \otimes m_i,
\end{equation}

where the sum is over $i = (i_1, \ldots, i_\ell)$ such that $v_{i_1} \otimes \cdots \otimes v_{i_\ell}$ belongs to $(A.7)$, and $m_i \in M$.

In fact, considering the $H_\ell(q^2)$-action by $R$ in Proposition 6.10 for each $i = (i_1, \ldots, i_\ell)$ in $(A.8)$, there exists $h_i \in H_\ell(q^2)$ such that

\begin{equation}
v_{i_1} \otimes \cdots \otimes v_{i_\ell} = (v_{2} \otimes \cdots \otimes v_{j} \otimes v_{1} \otimes v_{j+1} \otimes \cdots \otimes v_{\ell}) h_i.
\end{equation}

Hence $(A.8)$ is reduced to

\begin{equation}
f_0(v^{(j)} \otimes m) = (v_2 \otimes \cdots \otimes v_j \otimes v_1 \otimes v_{j+1} \otimes \cdots \otimes v_\ell) \otimes m',
\end{equation}

for some $m' \in M$. By Lemma A.3 such $m'$ is unique. Therefore we obtain a $k$-linear endomorphism $\alpha_j^- \in \text{End}(H_\ell(q^2))$ sending $m$ to $m'$. Considering $e_0$-action instead yields $\alpha_j^+$. So we have

\begin{equation}
e_0 \left(v^{(j)} \otimes m\right) = \left(\Delta_j(e_0)v^{(j)}\right) \otimes \alpha_j^+(m) = \sum_{1 \leq i \leq \ell} \left(\Delta_i(e_0)v^{(j)}\right) \otimes \alpha_i^+(m),
\end{equation}

\begin{equation}f_0 \left(v^{(j)} \otimes m\right) = \left(\Delta_j(f_0)v^{(j)}\right) \otimes \alpha_j^-(m) = \sum_{1 \leq i \leq \ell} \left(\Delta_i(f_0)v^{(j)}\right) \otimes \alpha_i^-(m),
\end{equation}

where $\Delta_i(e_0)v^{(j)}$ and $\Delta_i(f_0)v^{(j)}$ are defined in (A.10).
where $\Delta_i(e_0)$ and $\Delta_i(f_0)$ are given by

$$\Delta_i(e_0) = 1^\otimes i \otimes e_0 \otimes (k_0^{-1})^\otimes (\ell - i),$$

$$\Delta_i(f_0) = k_0^\otimes i \otimes f_0 \otimes 1^\otimes (\ell - i),$$

acting on $\mathcal{V}^\otimes \ell$. Note that $\Delta_i(e_0)v^{(j)} = 0$ unless $i = j$. Indeed, $v^{(j)}$ in (A.10) can be replaced by arbitrary $v \in \mathcal{V}^\otimes \ell$.

**Lemma A.5.** For $v \in \mathcal{V}^\otimes \ell$ and $m \in M$, we have

$$e_0(v \otimes m) = \sum_{1 \leq j \leq \ell} (\Delta_j(e_0)v) \otimes \alpha^+_j(m),$$

$$f_0(v \otimes m) = \sum_{1 \leq j \leq \ell} (\Delta_j(f_0)v) \otimes \alpha^-_j(m).$$

**Proof.** We only prove the case for $f_0$ since the other case is similar. Take $v = v_{i_1} \otimes \cdots \otimes v_{i_\ell}$. If none of $i_j$ is equal to $n$, then $\Delta_j(f_0)v = 0$ for any $j$. On the other hand, we have $f_0(v \otimes m) = 0$ since $\delta_{i_1} + \cdots + \delta_{i_\ell} - \delta_n + \delta_1$ is not a weight of $\mathcal{V}^\otimes \ell$. Hence the identity holds.

For each pair of sequences

$$j = (j_1 < j_2 < \cdots < j_r), \quad j' = (j'_1 < j'_2 < \cdots < j'_s)$$

in $\{1, \ldots, \ell\}$, which are disjoint, let $\mathcal{V}^{(j,j')}$ be the subspace of $\mathcal{V}^\otimes \ell$ spanned by vectors of the form $v_{i_1} \otimes \cdots \otimes v_{i_\ell}$, where $i_{j_t} = 1$ ($1 \leq t \leq r$), $i_{j'_t} = n$ ($1 \leq t \leq s$) and $i_j \neq 1, n$ for others. Clearly $\mathcal{V}^\otimes \ell = \bigoplus \mathcal{V}^{(j,j')}$, so that we may prove the identity for $v$ in each $\mathcal{V}^{(j,j')}$. In addition, it is enough to check the identity for $v = v_{i_1} \otimes \cdots \otimes v_{i_\ell} \in \mathcal{V}^{(j,j')}$ with no $v_{t_2}, \ldots, v_{t_{n-1}}$ appearing more than once, because of Lemma A.4 (with respect to the subalgebra of $\mathcal{U}(e)$ generated by $e_i$, $f_i$ and $k_i^{\pm 1}$ for $i = 2, \ldots, n - 1$). There is always such a vector since $\ell < n$.

We shall prove the identity by induction on $s$. We start with $s = 1$, and use induction on $r$. The case when $r = 0$ and $s = 1$ has already been done when we define $\alpha^+_j$ with $v = v^{(j)}$.

Suppose that it is true for $r - 1$. Choose $v = v_{i_1} \otimes \cdots \otimes v_{i_\ell} \in \mathcal{V}^{(j,j')}$ such that only $v_{3}, \ldots, v_{n-1}$ appear as a factor of $v$ without repetition (which is possible as $s, r \geq 1$). Let $v'$ be the vector obtained from $v$ by replacing the last $v_1$ (that is, $v_{j_1}$) by $v_2$ so that $v'$ has one less $v_1$ than $v$. By our choice of $v$, $e_1v' = v$. Then we compute as

$$f_0(v \otimes m) = f_0e_1(v' \otimes m) = e_1f_0(v' \otimes m) = e_1 \sum_j (\Delta_j(f_0)v') \otimes \alpha^-_j(m)$$

$$= e_1 \left( q_1 \left[ \frac{1}{[\{t \mid t \leq r, j_t < j'_t\}] v''} \right] \otimes \alpha^-_j(m) \right)$$

$$= q_1 \left[ \frac{1}{[\{t \mid t \leq r, j_t < j'_t\}] (e_1v' \otimes m) \otimes \alpha^-_j(m) \right)$$

$$= q_1 \left[ \frac{1}{[\{t \mid t \leq r, j_t < j'_t\}] \delta_{j'_t < j_1} \left[ (1 \otimes e_1 \otimes 1^\otimes (\ell - j_r)) v'' \right] \otimes \alpha^-_j(m) \right)$$

$$= q_1 \left[ \frac{1}{[\{t \mid t \leq r, j_t < j'_t\}] \delta_{j'_t < j_1} \left[ (1 \otimes e_1 \otimes 1^\otimes (\ell - j_r)) v'' \right] \otimes \alpha^-_j(m) \right)$$

$$= \left( \Delta_{j'_1}(f_0)v \right) \otimes \alpha^-_j(m) = \sum_j (\Delta_j(f_0)v) \otimes \alpha^-_j(m).$$
Here the third equality follows from induction hypothesis on \( r \), \( v'' \) is the resulting vector of replacing (the unique) \( v_n \) factor of \( v' \) by \( v_1 \), the last equality holds since \( v \) has exactly one \( v_n \) factor, and \( \delta(P) \) is 1 if the statement \( P \) is true and 0 otherwise.

Now assume the result for \( s - 1 \) and let us prove it for \( s \geq 2 \). Choose \( v = v_{i_1} \otimes \cdots \otimes v_{i_t} \in \mathcal{V}(j_j j'_j) \) such that \( v_{n-1} \) does not appear as a factor of \( v \) and for each \( i = 2, \ldots, n - 2, v_i \) occurs at most once (which is possible as \( s \geq 2 \)). We shall compute \([e_{n-1}, f_{n-1}] f_0(v \otimes m)\) in two different ways.

We first have

(A.11) \[ [e_{n-1}, f_{n-1}] f_0(v \otimes m) = \frac{q_{n}^{1-s} - q_{n}^{-s-1}}{q - q^{-1}} f_0(v \otimes m), \]

since \([e_{n-1}, f_{n-1}] = \frac{k_{n-1} - k_{n-1}^{-1}}{q - q^{-1}}\) and the weight of \( f_0(v \otimes m) \) is

\[ \sum_{i_k \neq n-1, n} \delta_{i_k} + s \delta_n + \delta_1 - \delta_n. \]

Next, by similar arguments for (A.9), \( f_0(v \otimes m) \) can be written as a sum of \( v_k \otimes m_k \) for some \( m_k \in M \) and \( v_k = v_{k_1} \otimes \cdots \otimes v_{k_t} \) with none of \( v_{k_i} \) is equal to \( v_{n-1} \). Hence \( f_{n-1} f_0(v \otimes m) = 0 \) and so

(A.12) \[ [e_{n-1}, f_{n-1}] f_0(v \otimes m) = -f_{n-1} e_{n-1} f_0(v \otimes m) = -f_{n-1} f_0 e_{n-1}(v \otimes m). \]

We first compute

\[ e_{n-1}(v \otimes m) = (e_{n-1} v) \otimes m = \left(q_{n}^{s-1} v^{j_1} + q_{n}^{s-2} v^{j_2} + \cdots + v^{j_s}\right) \otimes m, \]

where \( v^{j_p} \) is obtained from \( v \) by replacing \( j'_p \)-th factor (which is \( v_{n} \)) by \( v_{n-1} \). The vector \( v^{j_p} \) has one less \( v_n \)’s than \( v \), so that the induction hypothesis deduces

\[ f_0 e_{n-1}(v \otimes m) = \sum_{p=1}^{s} q_{n}^{s-p} f_0(v^{j_p} \otimes m) = \sum_{p=1}^{s} q_{n}^{s-p} \sum_{t} (\Delta_t(f_0)v^{j_p}) \otimes \alpha_t^{-1}(m). \]

By definition of \( \Delta_t(f_0) \),

\[ \Delta_t(f_0)v^{j_p} = \begin{cases} q_{n}^{u-\delta(u>p)} q_1^{-1} \left| \{ k | k \ll t \} \right| v^{u',p,u} & \text{if } t = j'_u \text{ for some } u \neq p, \\ 0 & \text{otherwise}. \end{cases} \]

where \( v^{u',p,u} \) is obtained from \( v^{j_p} \) by replacing \( j'_u \)-th factor (which is \( v_{n} \)) by \( v_1 \). Since any nonzero \( v^{u',p,u} \) has exactly one \( v_{n-1} \),

\[ f_{n-1} f_0 e_{n-1}(v \otimes m) = f_{n-1} \sum_{p=1}^{s} \sum_{\ell} (\Delta_t(f_0)v^{j_p}) \otimes \alpha_t^{-1}(m) \]

\[ = \sum_{p=1}^{s} q_{n}^{s-p} \sum_{u \neq p} q_{n}^{u-\delta(u>p)} q_1^{-1} \left| \{ k | k \ll t \} \right| f_{n-1} v^{u',p,u} \otimes \alpha_t^{-1}(m) \]

\[ = \sum_{p=1}^{s} q_{n}^{s-p} \sum_{u \neq p} q_{n}^{u-\delta(u>p)} q_1^{-1} \left| \{ k | k \ll t \} \right| q_{n}^{1+\delta(u<p)-p} v^{u} \otimes \alpha_t^{-1}(m) \]
where $v^n$ is obtained from $v$ by replacing $j'_n$-th factor (which is $v_n$) by $v_1$. Now for $1 \leq u \leq s$, the coefficient of $v^u \otimes \alpha_{\ell'}^-(m)$ is

\[
\sum_{p<u} q_n^{s-p} q_n^{u-2} - \left| \{ j < j' \} \right| q_n^{1-p} + \sum_{p>u} q_n^{s-p} q_n^{u-1} - \left| \{ j < j' \} \right| q_n^{2-p} = q_1^{-|\{ j < j' \}|} \sum_{p=1}^{u-1} q_n^{s-p} q_n^{u-2} q_n^{1-p} + \sum_{p=u}^{s-1} q_n^{s-p} q_n^{u-1} q_n^{1-p} = q_1^{-|\{ j < j' \}|} \sum_{p=1}^{s-1} q_n^{s+p-1} q_n^{1-p} = q_1^{-|\{ j < j' \}|} \frac{q_n^{s-1} - q_n^{1-s}}{q_n - q_n^{-1}}.
\]

Finally, combining the computation of (A.11) and (A.12) we obtain

\[
f_0 (v \otimes m) = \frac{q - q^{-1}}{q_n^{s-1} - q_n^{-s}} [f_{n-1}, e_{n-1}] f_0 (v \otimes m) = \frac{q - q^{-1}}{q_n^{s-1} - q_n^{-s}} f_{n-1} f_0 e_{n-1} (v \otimes m) = \sum_{t=1}^{s} (\Delta_{\ell'}^t (f_0) v) \otimes \alpha_{\ell'}^-(m) = \sum_{t=1}^{s} (\Delta_{\ell} (f_0) v) \otimes \alpha_{\ell}^-(m).
\]

since $\Delta_{\ell'}^t (f_0) v = q_n^{s-1} q_1^{-|\{ j < j' \}|} v^u$ and $\Delta_{\ell} (f_0) v = 0$ for $t \neq j'_u$. This completes the induction. \qed

Now, we define

(A.13) \quad $X_j^\pm m = \alpha_j^\pm (m)$ \quad ($m \in M$, $1 \leq j \leq \ell$).

**Lemma A.6.** $M$ is a $H^{\text{aff}}(q^2)$-module with respect to (A.13), and $W$ is isomorphic to $V^{\otimes \ell} \otimes_{H^{\text{aff}}(q^2)} M$ as a $U(\epsilon)$-module.

**Proof.** The proof is almost identical to the one in [7]. So we leave it to the reader. \qed

This completes the proof for essential surjectivity of $\mathcal{F}$.

**Lemma A.7.** The functor $\mathcal{F}$ is fully faithful.

**Proof.** First, $\mathcal{F}$ is faithful since $\mathcal{J}_\ell$ is faithful. So it suffices to show that $\mathcal{F}_\ell$ is surjective on morphisms.

Suppose that $F : \mathcal{F}_\ell (M) \to \mathcal{F}_\ell (M')$ is a $U(\epsilon)$-linear map for $M, M' \in H^{\text{aff}}(q^2)$-mod. Since $\mathcal{J}_\ell$ is an equivalence, there is a $H_{\ell}(q^2)$-linear map $f : M \to M'$ such that $\mathcal{J}_\ell (f) = F$.

Since $F$ is $U(\epsilon)$-linear, $e_0 F (v \otimes m) = F (e_0 (v \otimes m))$. The left hand side is equal to

\[
e_0 F (v \otimes m) = e_0 (v \otimes f(m)) = \sum_j \Delta_j (e_0) v \otimes X_j f(m),
\]

while the right hand side is

\[
F (e_0 (v \otimes m)) = \sum_j \Delta_j (e_0) v \otimes X_j m = \sum_j \Delta_j (e_0) v \otimes f(X_j m).
\]

Now for each $i$, we can choose a vector $v(i)$ so that $\Delta_j (e_0) v = 0$ unless $j = i$, and at the same time $\Delta_i (e_0) v$ is of the form $v_{i_1} \otimes \cdots \otimes v_{i_{\ell}}$, whose factors are all distinct $v_k$’s. For
example, we may take \( v(1) = v_1 \otimes v_2 \otimes \cdots \otimes v_\ell \). Putting \( v = v(i) \) in the above identities, we obtain \( X_i f(m) = f(X_i m) \) by Lemma A.3. Hence \( f \) is \( H^\aff_\ell(q^2) \)-linear as well. □

Therefore, \( \mathcal{F} \) in (A.6) is an equivalence of categories. Since every simple object in \( H^\aff_\ell(q^2) \)-mod is a quotient of \( L(a_1) \circ \cdots \circ L(a_\ell) \) for some \( a_1, \ldots, a_\ell \in \mathbf{k} \), where \( \circ \) is a convolution product, and \( \mathcal{F}(L(a_1) \circ \cdots \circ L(a_\ell)) = W_1(a_1) \otimes \cdots \otimes W_1(a_\ell) \), \( \mathcal{F} \) induces the equivalence

\[
\mathcal{F}_{\epsilon, \ell}^*: H^\aff_\ell(q^2)-\text{mod} \rightarrow \mathcal{C}_\ell^\epsilon(\epsilon).
\]

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