STABILIZERS, MEASURES AND IP-SETS

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Abstract. The purpose of this article is to provide elementary model-theoretic proofs to some existing results on sumset phenomena and IP sets, motivated by Hrushovski’s work on the stabilizer theorem.

INTRODUCTION

A subset of the integers is called an IP set (this stands for infinite-dimensional parallelepiped) if it contains all finite sums of elements (without repetitions) of an infinite subset. More generally, a subset $A$ of an infinite group $G$ (possibly non-abelian) is an IP set if there exists an enumeration $(x_i)_{i \in \mathbb{N}}$ of an infinite subset $B$ of $A$ such that $A$ contains

$$FP(B) = \{ x_{i_1} \cdot \ldots \cdot x_{i_k} | i_1 < \ldots < i_k \}_{k \in \mathbb{N}}.$$ 

Hindman’s influential theorem [15] states that for any finite coloring on the natural numbers, there is a monochromatic IP set. Hindman’s original proof was elementary, yet rather technical. Galvin and Glazer gave a beautiful proof (see the account in [7]) using the topology of the Stone space of ultrafilters on $\mathbb{N}$ as well as Ellis-Numakura’s lemma on the existence of idempotent ultrafilters [27, 11]. Every IP subset $A$ of $G$ has the productset property, that is, there are two infinite subsets $B$ and $C$ of $G$ such that

$$B \cdot C = \{ b \cdot c | b \in B, c \in C \} \subseteq A,$$

which is sometimes referred to as the sumset property when the group $G$ is abelian.

Nathanson [26] attributes to Erdős the so-called $B+C$ conjecture, which was stated by Erdős and Graham in [12] as an old problem:

**Conjecture.** Every subset $A$ of $\mathbb{N}$ of positive lower density, that is,

$$\liminf_{n \to \infty} \frac{|A \cap [1, n]|}{n} > 0,$$

must have the sumset property.

A stronger version of the conjecture has been recently positively answered in work of Moreira, Richter and Robertson [25] using ergodic theory as well as ultrafilters, which also played a fundamental role in the paper of Di Nasso et al. [10]. In
the latter, an instance of the above conjecture was solved for subsets $A$ of Banach density
\[
\text{BD}(A) = \lim_{n \to \infty} \sup_{m \in \mathbb{N}} \frac{|A \cap [m + 1, m + n]|}{n} > \frac{1}{2}.
\]
Di Nasso showed that Erdős’s $B + C$ conjecture can be phrased in terms of independent sums of ultrafilters (see \cite[Lemma 5.1]{DiNasso}). This reformulation carries over to coheir sum of types (that is, ultrafilters on the boolean algebra of definable sets), as proved in \cite[Theorem 3.4]{AndrewsConantGoldbring} by Andrews, Conant and Goldbring. For a group definable in a stable theory, the principal generic type is indeed idempotent with respect to the coheir (or independent) product. Outside of stability, there need not be a single principal generic type, yet in simple theories a partial result was obtained by Pillay, Scanlon and Wagner in \cite[Proposition 2.2]{PillayScanlonWagner}: Given three principal generic (complete) types $p$, $q$ and $r$ over an elementary substructure $M$, we have that there are independent realizations $a$ of $p$ and $b$ of $q$ with $a \cdot b$ realizing $r$, or equivalently, using the notation of section \ref{section2}, the type-definable set $r \cap (p \ast q)$ is not empty. Partial types $\Sigma$ with $\Sigma \cap (\Sigma^{-1} \ast \Sigma) \neq \emptyset$ are not product-free (see Lemma \ref{lemma2.5}), a notion which was already considered in the finitary context by Gowers \cite{Gowers} in order to give a negative answer to a question of Babai and Soós \cite{BabaiSos} on the existence of large product-free sets in finite groups. The second author recently provided \cite{Palacin} a model-theoretic proof of Gowers’s result, relating the existence of large product-free sets to Hrushovski’s stabilizer theorem \cite{HrushovskiStabilizer}.

Hrushovski’s stabilizer theorem allows one to generalise some of the techniques from geometric stability theory to an arbitrary context, replacing in a clever way the role of forking independence and genericity by the ideal of null-sets with respect to a nonstandard counting measure. A fundamental notion present in Hrushovski’s original proof is equationality, as introduced by Srour \cite{Srour} in order to develop a local approach to noetherianity for the category of definable sets in any structure. However, Hrushovski’s proof does not use the full strength of equationality. In section \ref{section2}, we take the opportunity to further explore the role of equationality in some consequences of the stabilizer theorem, and relate it to a local version for large product-free sets and type-definable connected components, proving the following result:

**Theorem** (Corollary \ref{corollary2.6}). Let $G$ be a definable group with an $\emptyset$-type-definable left-translation ideal $I$ of definable subsets of $G$. Assume that $I$ is indiscernibly prime (see Definition \ref{definition1.3}). The following conditions are equivalent for every elementary submodel $M$:

- If a partial type $\Sigma$ over $M$ is wide (i.e. avoids all definable sets from $I$), then $\Sigma$ is product-free, that is, the partial type $\Sigma \cup \Sigma \cdot \Sigma^{-1}$ is inconsistent.
- The group $G$ equals the connected component $G^0_M$, where $G^0_M$ denotes the intersection of all type-definable subgroups over $M$ of bounded index in $G$.

Moreover, if the second condition holds for some elementary submodel $M$, then it holds for every elementary submodel.

The results from section \ref{section2} will then be applied in section \ref{section3} to study the asymptotics of IP sets as well as of product-free sets. In particular, we obtain in Proposition \ref{proposition3.3} an elementary model-theoretic proof of an existing result of Nathanson \cite[Theorem 6]{Nathanson}: every subset $A$ of the integers of positive Banach density contains, for every two natural numbers $n$ and $m$, a subset of the form $F_1 + \cdots + F_n + B$, for every...
for some finite subsets \(F_1, \ldots, F_n\) of \(A\) of size \(m\) and a subset \(B\) of \(A\) of positive Banach density. In the last part of this note (Corollary 3.12), we obtain a non-quantitative finitary version of a result of Gowers [14] (in a weaker form). The infinite version of it (Corollary 3.11) resonates to a certain extent with previous work of Bergelson and Tao [9] on ultra quasirandom groups. We believe that this hints to further connections, yet to be explored, between the techniques used in the stabilizer theorem and the combinatorial approach to complete \(n\)-amalgamation in terms of mixing and multiple recurrence. The research we conducted here has been further developed in a subsequent article [23] for a wider class of groups (including pseudo-finite groups) to treat the case of complete \(3\)-amalgamation, which yields as a by-product a model-theoretic proof of Roth’s theorem on the existence of arithmetic progressions of length 3 for subsets of positive density.

1. Keisler Measures and Ideals

Most of the material in this section can be found in [17]. We work inside a sufficiently saturated model \(U\) of a complete first-order theory (with infinite models) in a language \(\mathcal{L}\). All sets and tuples are taken inside \(U\).

A Keisler measure \(\mu\) is a finitely additive probability measure on some boolean algebra of definable subsets of the ambient model \([U]\). Archetypal examples are measures \(\mu_p\) with two possible values 0 and 1 given by global types \(p\) in \(S(U)\), that is, for every definable set \(X\),

\[\mu_p(X) = 1 \iff p \in [X],\]

where \([X]\) is the clopen space in \(S(U)\) determined by the definable set \(X\).

Given a Keisler measure \(\mu\), the collection of definable sets of measure zero forms an ideal, that is, it is closed under subsets and finite unions. An ideal \(I\) of definable sets is invariant over the parameter set \(A\) (or \(A\)-invariant) if it is invariant under the action of \(\text{Aut}_A(U)\). The ideal \(I\) is type-definable over the parameter set \(A\) if for every formula \(\varphi(x, y)\) there is a partial type \(\Sigma_\varphi(y)\) over \(A\) such that for all \(b\) in \(U\) the following equivalence holds:

\[\Sigma_\varphi(b) \iff \varphi(x, b) \in I.\]

A Keisler measure \(\mu\) is said to be definable over \(A\) (see [17, Definition 3.19]) if for every \(\mathcal{L}\)-formula \(\varphi(x, y)\) the following two properties hold:

- the set of parameters \(b\) in \(U[y]\) with \(\varphi(x, b)\) measurable is definable by an \(\mathcal{L}_A\)-formula \(\rho(y)\).
- for every \(\epsilon > 0\), there is a partition of \(\rho(U)\) into \(\mathcal{L}_A\)-formulae \(\rho_1(y), \ldots, \rho_m(y)\) such that for all pairs \((b, b')\) realizing \(\rho_i(y) \land \rho_j(z)\), we have that

\[|\mu(\varphi(x, b)) - \mu(\varphi(x, b'))| < \epsilon.\]

Every finitely additive probability measure \(\mu\) on all subsets admits an expansion of the original language \(\mathcal{L}\) in which it becomes definable without parameters, see for example [17, Section 2.6]. Namely, add a predicate \(Q_{r, \varphi}(y)\) for each \(r\) in \(\mathbb{Q} \cap [0, 1]\) and every formula \(\varphi(x, y)\) in \(\mathcal{L}\) such that \(Q_{r, \varphi}(b)\) holds if and only if \(\mu(\varphi(U, b)) \leq r\). These predicates \(Q_{r, \varphi}\) give rise to new definable sets, which will also be measurable. Iterating this process countably many times and replacing the ambient model (if necessary), we obtain an expansion of the language \(\mathcal{L}\) such that the corresponding Keisler measure given by \(\mu\) is definable.
Note that for an $A$-definable Keisler measure $\mu$, the ideal of sets of measure zero (called the null-ideal of $\mu$) is type-definable over $A$ and thus $A$-invariant.

**Remark 1.1.** Let $\mathcal{I}$ be a type-definable ideal over an elementary submodel $M$. If $\varphi(x, b)$ does not belong to $\mathcal{I}$ for some $b \in U$, then there is some $\mathcal{L}_M$-formula $\theta(y)$ such that $\varphi(x, c)$ does not belong to $\mathcal{I}$, whenever $c$ realizes $\theta(y)$. In particular, there is some $m \in M$ such that $\varphi(x, m)$ does not belong to $\mathcal{I}$.

**Definition 1.2.** An $\emptyset$-invariant ideal $\mathcal{I}$ of definable sets is **indiscernibly prime** (or has the $S1$-property) if $\varphi(x, b_1)$ belongs to $\mathcal{I}$, whenever $(b_i)_{i \in \mathbb{N}}$ is an indiscernible sequence such that the conjunction $\varphi(x, b_1) \land \varphi(x, b_2)$ lies in $\mathcal{I}$.

By invariance of $\mathcal{I}$, we can replace the roles of $b_1$ and $b_2$ above by any two $b_i$ and $b_j$ with $i \neq j$. The null-ideal of a definable Keisler measure is indiscernibly prime, since the measure of the whole space is bounded above by 1.

A partial type is said to be **wide** (with respect to $\mathcal{I}$) if it contains no definable set in $\mathcal{I}$. The definable set $X$ is wide if the partial type $X(x)$ is. Every wide partial type $\pi(x)$ over a parameter set $A$ can be completed to a wide complete type over any arbitrary set $B$ containing $A$, since the collection of formulae

$$\pi(x) \cup \{ \neg \varphi(x) \mid \varphi(x) \text{ $\mathcal{L}_B$-formula in } \mathcal{I} \}$$

is finitely consistent.

**Remark 1.3.** Let $\mathcal{I}$ be an indiscernibly prime $\emptyset$-invariant ideal containing all the formulas of the form $x = a$ with $a$ in $\text{acl}(\emptyset)$. Every wide definable set $X$ is infinite. In fact, the set $X$ does not fork over $\emptyset$ [2], Lemma 2.9].

In the presence of an ambient (type-)definable group $G$, it is meaningful to require that the ideal $\mathcal{I}$ is invariant under left translation by elements of the group. Note that natural action of $G$ on its definable sets induces an action of $G$ on the set of (partial) types. We will identify a partial type with the subset of its realizations in the structure $U$.

From now on, assume that the $\emptyset$-invariant indiscernibly prime ideal $\mathcal{I}$ of $G$ is closed under left translations. Given a partial type $\pi(x)$ over $A$, define the set

$$\text{St}(\pi) = \{ g \in G \mid \pi(x) \cup g \cdot \pi(x) \text{ is wide} \}$$

and its stabilizer $\text{Stab}(\pi)$ as the subgroup of $G$ generated by $\text{St}(\pi)$. Clearly, the set $\text{St}(\pi)$ is not empty if and only if $\pi$ is wide itself. Since $\mathcal{I}$ is closed under left translations, the set $\text{St}(\pi)$ is closed under taking inverses in $G$. As $\text{St}(\pi)$ is contained in the set $\pi \cdot \pi^{-1}$, it follows that

$$\text{St}(\pi) \subseteq \text{Stab}(\pi) = \bigcup_{n \in \mathbb{N}} \text{St}(\pi) \cdots \text{St}(\pi) \subseteq \langle \pi \rangle_G,$$

whenever $\pi$ is wide, where $\langle \pi \rangle_G$ is the subgroup of $G = G(U)$ generated by $\pi(U)$.

For our purposes, let us now state a summarized version Hrushovski’s stabilizer theorem.

**Fact 1.4.** ([17, Theorem 3.5] & [24, Theorem 2.12]) Let $G$ be a definable group and $\mathcal{I}$ an $\emptyset$-invariant indiscernibly prime ideal on the boolean algebra of all definable subsets of $G$ such that $\mathcal{I}$ is translation-invariant under left multiplication. Given a wide complete type $p$ over an elementary submodel $M$, the subgroup $\text{Stab}(p)$ is type-definable and wide with

$$\text{Stab}(p) = \text{St}(p)^2 = (p \cdot p^{-1})^2.$$
Moreover, every wide type over \( M \) in \( \text{Stab}(p) \) belongs to \( \text{St}(p) \) and \( \text{Stab}(p) \) equals \( G^p_M \), the intersection of all subgroups of bounded index in \( G \) which are type-definable over \( M \).

The above result of Hrushovski generalises a well-known situation for stable groups. Recall that a relation \( R(x, y) \) is stable with respect to the partition of the variables into the tuples \( x \) and \( y \) if there is no sequence \( (a_i, b_i)_{i \in \mathbb{N}} \) in \( \mathbb{U} \) such that \( R(a_i, b_i) \) holds if and only if \( i \leq j \). A formula \( \varphi(x, y) \) is stable if the induced relation is.

Stable formulae are closed under boolean combinations. We will now introduce some notions and properties arising from the local treatment of stability done by Hrushovski and Pillay in [16] (see also [29]).

Given a formula \( \varphi(x, y) \), a subset \( A \) of \( M \) is \( \varphi \)-definable over \( A \) if it is definable by a boolean combination of instances \( \varphi(x, a) \) with \( a \) in \( A \). By a \( \varphi \)-type over \( A \) we mean a maximal finitely consistent collection of instances of the form \( \varphi(x, a) \) or \( \neg \varphi(x, a') \) for \( a, a' \) in \( A \).

We say that a definable set \( X \) is left-generic if finitely many left translates of \( X \) cover \( G \). A partial type is left-generic if it only contains formulae which are left-generic. If \( \varphi(x, y) \) is stable, then left-generic \( \varphi \)-types exist [16, Lemma 5.16(i)].

Whenever the formula \( \varphi(x, y) \) is stable, every \( \varphi \)-type \( p(x) \) over an elementary submodel \( M \) is definable, that is, there is a formula \( \theta(y) \) with parameters over \( M \) such that for all \( m \) in \( M \)

\[
\varphi(x, m) \in p \iff \theta(m).
\]

Furthermore, the definable set \( \theta(y) \) above is unique and can be defined by a positive boolean combination of instances \( \varphi(a, y) \) with parameters in \( M \) (cf. [16, Lemma 5.4]). We refer to this definable set as the \( \varphi \)-definition \( (d_p \varphi)(y) \) of \( p \). Given a superset \( B \supseteq M \) of \( \mathbb{U} \), there is a unique \( \varphi \)-type over \( B \) extending \( p \) which is again definable over \( M \), namely

\[
\{\varphi(x, b) \mid (d_p \varphi)(b)\} \cup \{-\varphi(x, b') \mid - (d_p \varphi)(b')\}.
\]

We refer to this type as the non-forking extension \( p_{1B}(x) \) of \( p(x) \) to \( B \). The global non-forking extension of \( p \) is the \( \varphi \)-type \( p_{\mathbb{U}} \).

Whenever \( \varphi(x, y) \) is equivariant (see [16, Definition 5.13]), that is, every left-translate of an instance of \( \varphi \) is again an instance of \( \varphi \), the group \( G \) acts naturally on the space of global \( \varphi \)-types by left multiplication, so given a \( \varphi \)-type \( p(x) \) over the elementary submodel \( M \), it makes sense to consider its \( \varphi \)-stabilizer to be the group-theoretic stabilizer of the global extension \( p_{\mathbb{U}} \) with respect to this action. Note that

\[
\text{Stab}_\varphi(p) = \{g \in G \mid \forall u ((d_p \varphi)(u) \leftrightarrow (d_p \varphi)(u \cdot g))\}
\]

is clearly a definable subgroup of \( G \) with parameters from \( M \). However, this group need not be definable using instances of \( \varphi \), cf. [16, Remark 2.1]. Conant, Pillay and Terry have provided alternative candidates for the \( \varphi \)-stabilizers in [3, Theorem 2.3].

2. Equations and product-free sets

Srour [32] proposed a local version of noetherianity in terms of equations. Recall that a relation \( R(x; y) \) is an equation (for a given partition of the free variables into
x and y) if there is no sequence \((a_i, b_i)_{i \in \mathbb{N}}\) in \(\bigcup\) such that
\[
R(a_i, b_j) \text{ for all } i < j, \text{ but } \neg R(a_i, b_i) \text{ for all } i.
\]

Equationality is a strengthening of stability, though not every stable relation need be an equation, since stability is preserved under boolean combinations, yet equationality is not. If the relation \(R\) is \(\emptyset\)-invariant, we may assume that the above sequence is indiscernible with respect to any possible order type. In particular, equationality is symmetric in \(x\) and \(y\) for an \(\emptyset\)-invariant relation. Furthermore, it is easy to verify that an \(\emptyset\)-invariant relation \(R\) is equational if and only if, whenever the indiscernible sequence \((a_i)_{i \in \mathbb{N}}\) satisfies that \(R(a_i, b_i)\) holds for all \(i > 0\) with respect to some tuple \(b\), then \(R(a_0, b)\) holds as well. Note that this last condition is a priori not symmetric in \(x\) and \(y\).

**Remark 2.1.** ([17, Remark 2.1] & [22, Remark 3.6]) Consider an \(\emptyset\)-invariant equational relation \(R(x, y)\) such that \(R(a, b)\) holds for some \(a\) and \(b\) such that \(tp(a/M, b)\) and \(tp(b/M, a)\) does not divide over the elementary submodel \(M\). For any \(a' \equiv_M a\) and \(b' \equiv_M b\), we have that \(R(a', b')\).

**Proof.** By symmetry of equationality, it suffices to consider the case that \(R(a, b)\) holds for some \(a\) and \(b\) such that \(tp(a/M, b)\) does not divide over \(M\). By invariance, the above holds for all such \(a_1\) and \(b_1\) with \((a_1, b_1) \equiv_M (a, b)\). Hence, we may assume that \(b' = b\).

Let \(a_0\) be \(a'\) and choose for \(i \geq 1\) a realization \(a_i\) of \(tp(a'/M)\) such that the type \(tp(a_i/M, b, a_0, \ldots, a_{i-1})\) is a coheir extension of its restriction to \(M\). In particular, the type \(tp(a_i/M, b)\) does not divide over \(M\), for \(i \geq 1\), so \(R(a_i, b)\) holds for \(i \geq 1\), since the relation \(R\) is stable [17, Lemma 2.3] (cf. [20, Lemma 3.3]).

By construction, the sequence \((a_i)_{i \in \mathbb{N}}\) is \(M\)-indiscernible. The equivalent characterization of equationality yields that \(R(a_0, b)\), as desired. \(\Box\)

Fix for the rest of this section an \(\emptyset\)-invariant indiscernibly prime ideal \(\mathcal{I}\) on the boolean ring of definable subsets contained in \((X^{-1} \cdot X)^2\), where \(X\) is a wide subset of a group \(G\) definable over an elementary submodel \(M\). In particular, the set-theoretic complements are taken within the ambient set \((X^{-1} \cdot X)^2\). Moreover, we assume that the ideal \(\mathcal{I}\) is translation-invariant under left multiplication by elements of the subgroup generated by \(X\) (in the proofs of Corollary 2.4 and Lemma 2.5, we will only need to consider products of at most four elements in \(X \cup X^{-1}\): if \(Z\) is a definable subset of \((X^{-1} \cdot X)^2\) and \(g\) is an element of the subgroup generated by \(X\) such that \(g \cdot Z\) lies again in \((X^{-1} \cdot X)^2\), then \(Z\) belongs to \(\mathcal{I}\) if and only if \(g \cdot Z\) does.

**Notation.** Given two types \(r\) and \(s\) over \(M\) containing the formula \(G(x)\), we consider the set
\[
r \ast s = \{ b \cdot c \in G \mid b \models r, c \models s \text{ and } tp(c/M, b) \text{ does not divide over } M \}.
\]
In particular, we write \(r \ast r^{-1}\) for \(r \ast tp(a^{-1}M)\), where \(a \models r\).

**Fact 2.2.** ([17, Lemma 2.10]) Whenever the \(\emptyset\)-invariant ideal \(\mathcal{I}\) is indiscernibly prime, for any two partial types \(\Phi(x, y)\) and \(\Psi(x, z)\) containing \(X(x)\), the relation
\[
R_{\Phi, \Psi}(a, b) \iff \Phi(x, a) \cup \Psi(x, b)\]
is not wide.
In particular, the relation \( x \cdot y \in \text{St}(p) \) is an equation, hence stable, for any type \( p \) containing \( X \). Thus, if \( a \cdot b \) belongs to \( \text{St}(p) \) for some \( a \cdot b \) in \( r \cdot s \), then \( r \cdot s \subseteq \text{St}(p) \).

The above fact yields the following immediate consequence for the equational relation \( R_a, \Phi \), which already appeared in [18, Lemma 2.26(1)]:

**Corollary 2.3.** If a partial type \( \Phi(x, a) \) is wide, so is \( \Phi(x, b) \cup \Phi(x, c) \), whenever \( b \) and \( c \) are realizations of \( \text{tp}(a/M) \) such that \( \text{tp}(c/M, b) \) does not divide over \( M \). \( \square \)

**Corollary 2.4.** Let \( q \) be a wide type over \( M \) containing the definable set \( X \). Given any (possibly non-wide) type \( p \) over \( M \) containing the definable set \( X \), the set \( p \cdot p^{-1} \) is contained in \( \text{St}(q) \). In particular, if \( p \) is wide over \( M \), then \( p \cdot p^{-1} \subseteq \text{St}(p) \), so \( b \cdot c^{-1} \) (and thus \( c \cdot b^{-1} \)) belongs to \( \text{St}(q) \), whenever \( b \) and \( c \) realize \( p \) and \( \text{tp}(c/M, b) \) does not divide over \( M \).

**Proof.** Since \( q \) is wide, so is the type \( \Phi(x, a) = a^{-1} \cdot q(x) \) for any \( a \) in \( X \). Fact 2.2 and Corollary 2.3 yield that \( b \cdot c^{-1} \) belongs to \( \text{St}(q) \) for any \( b \) and \( c \) realizing \( p \) with \( \text{tp}(c/M, b) \) non-dividing over \( M \). Since \( \text{St}(q) \) is closed under inverses, we deduce that the product \( c \cdot b^{-1} \) also lies in \( \text{St}(q) \). We conclude that the set (of realizations of) \( p \cdot p^{-1} \subseteq \text{St}(q) \), as desired. \( \square \)

Motivated by the second author’s model-theoretic approach in [28] to Gowers’s solution [14] to the asymptotics of product-free sets [2] in finite quasi-random groups, we say that a partial type \( \Sigma \) is product-free if \( \Sigma \cup (\Sigma^{-1} \cdot \Sigma) \) is inconsistent. Note that \( \Sigma \) is not product-free if and only if some realization \( a \) of \( \Sigma \) is of the form \( a = b \cdot c \), where \( b \) and \( c \) both realize \( \Sigma \). If the complete type \( p \) is not product-free, then every realization \( a \) of \( p \) is a product of two realizations of \( p \). A partial type is not product-free if and only if it is the intersection of definable sets which are not product-free. Therefore, if there exists a product-free (partial) type over an elementary submodel, there is a product-free (partial) type over every elementary submodel.

The next result can be easily proved along the lines of Fact 1.4, as shown in [28, Lemma 3.3]. For the sake of the presentation, we will provide a direct proof, which essentially follows the outline of the proof of [27, Theorem 2.12].

**Lemma 2.5.** Given a wide type \( p \) containing the definable set \( X \) over \( M \), the following conditions are equivalent:

1. The type \( p \) is contained in \( \text{St}(p) \).
2. The type \( p \) is contained in \( (p^{-1} \cdot p) \).
3. The type \( p \) is not product-free.

**Proof.** For (1) \( \Rightarrow \) (2), suppose that a realization \( b \) of \( p \) is contained in \( \text{St}(p) \), so \( b^{-1} \) belongs to \( \text{St}(p) \), and thus \( p \cup b^{-1} \cdot p \) is wide with respect to \( \mathcal{I} \). In particular, there is a realization \( a \) of \( p \cup b^{-1} \cdot p \) which is wide over \( M \cup \{b\} \). That is, the element \( c = b \cdot a \) realizes \( p \) and its type \( \text{tp}(c/M, b) \) does not divide over \( M \), by Remark 1.3.

Hence \( a = b^{-1} \cdot c \) belongs to \( p^{-1} \cdot p \), so every realization of \( p \) does, as desired.

The implication (2) \( \Rightarrow \) (1) is clear, for \( (p^{-1} \cdot p) \subseteq (p^{-1} \cdot p) \).

For (2) \( \Rightarrow \) (3), observe that \( p^{-1} \) is not product-free whenever \( p \) is not product-free. In particular, the complete type \( p^{-1} \) is contained in \( \text{St}(q) \cdot \text{St}(q) \); indeed, choose realizations \( a, b \) and \( c \) of \( p \) such that \( a^{-1} = b \cdot c^{-1} \). Take now \( d \) realizing \( p \) wide over \( M, a, b, c \) and notice that

\[
\text{St}(q) \cdot \text{St}(q)
\]

\[
= (b \cdot d^{-1}) \cdot (d \cdot c^{-1}) \subseteq \text{St}(q) \cdot \text{St}(q),
\]
by Corollary 2.4. In particular, the type \( p \) is contained in \( \text{St}(q) \cdot \text{St}(q) \), for \( \text{St}(q) \) is closed under inverses.

Choose now \( b \) and \( c \) realizing \( p \) such that \( \text{tp}(c/M, b) \) is wide. Set \( \eta = b \cdot c^{-1} \). Notice that \( q = \text{tp}(\eta/M) \) is wide, for it is the restriction of the wide type \( \text{tp}(\eta/M, b) \).

Moreover, the type \( q \) is contained in \( \text{St}(p) \), by Corollary 2.4.

**Claim.** The set \(( \text{St}(q) \cdot \text{St}(q)) \ast q \) is contained in \( \text{St}(p) \), that is, given \( \xi \) in \(( \text{St}(q) \cdot \text{St}(q)) \ast q \) and \( \eta_i \) realizing \( q \) such that \( \text{tp}(\eta_i/M, \xi) \) does not divide over \( M \), then \( \xi \cdot \eta_i \) belongs to \( \text{St}(p) \).

**Proof of Claim.** We first show that \( \text{St}(q) \ast q \) is contained in \( \text{St}(p) \). Given \( \xi_1 \) in \( \text{St}(q) \) arbitrary, the type \( q \cup \xi_1 \cdot q \) is wide, so choose some realization \( \eta_1 \) of \( q \) such that \( \xi_1 \cdot \eta_1 \) realizes \( q \) and \( \text{tp}(\eta_1/M, \xi_1) \) is wide, hence non-dividing over \( M \) by Remark 2.3. As remarked above, the type \( q \) is contained in \( \text{St}(p) \), so the element \( \xi_1 \cdot \eta_1 \) of \( \text{St}(q) \ast q \) belongs to \( \text{St}(p) \). The relation \( x \cdot y \in \text{St}(p) \) is stable, by Fact 2.2, so we deduce that every element of \( \text{St}(q) \ast q \) belongs to \( \text{St}(p) \).

Now we prove that \(( \text{St}(q) \cdot \text{St}(q)) \ast q \subseteq \text{St}(p) \). Since the relation \( x \cdot y \in \text{St}(p) \) is stable (see Fact 2.2), we need only show that some product \( \xi \cdot \eta_2 \) belongs to \( \text{St}(p) \) with \( \text{tp}(\eta_2/M, \xi) \) not dividing over \( M \). Given a realization \( \xi = \xi_1 \cdot \xi_2 \) in \( \text{St}(q) \), find as above some realization \( \eta_2 \) of \( q \) such that the type \( \text{tp}(\eta_2/M, \xi_1, \xi_2) \) is wide and \( \xi_2 \cdot \eta_2 \) realizes \( q \), since \( \xi_2 \) belongs to \( \text{St}(q) \). In particular, the realization \( \xi_2 \cdot \eta_2 \) of \( q \) is wide over \( M, \xi_1 \), and hence \( \text{tp}(\xi_2 \cdot \eta_2/M, \xi_1) \) does not divide over \( M \), so the element \( \xi_1 \cdot (\xi_2 \cdot \eta_2) = \xi \cdot \eta_2 \) of \( \text{St}(q) \ast q \) belongs to \( \text{St}(p) \), by the first part of the proof of this claim. This yields the desired result. \( \square \)

To conclude the proof, recall that \( p^{-1} \) is contained in \( \text{St}(q) \cdot \text{St}(q) \). Hence, the above realization \( b^{-1} \) of \( p^{-1} \) lies in \( \text{St}(q) \cdot \text{St}(q) \). As the realization \( \eta = b \cdot c^{-1} \) of \( q \) is wide over \( M \cup \{ b \} \), we deduce from the previous Claim that the element \( b^{-1} \cdot \eta = c \) belongs to \( \text{St}(p) \). Since \( c \) realizes \( p \), this finishes the proof. \( \square \)

To conclude this section, assume now that the group \( G \) equals the definable set \( X \), and that both \( G \) as well as the ideal \( I \) are defined without parameters, thus the ideal \( I \) is indiscernibly prime on the boolean algebra of all definable subsets of \( G \). We equip the quotient \( G/G^0 \) with the logic topology, that is, a subset in the quotient is closed if and only if its preimage in \( G \) is type-definable over \( M \). The natural map from \( G(M) \) to the compact Hausdorff group \( G/G^0 \) is the universal definable compactification of the group \( G(M) \) with respect to the elementary structure \( M \) [15, Proposition 3.4]. We will establish an equivalence between the triviality of the universal definable compactification and the existence of wide product-free sets.

**Corollary 2.6.** The following are equivalent:

(1) Every wide type \( p \) over an elementary submodel is contained in \( \text{St}(p) \).

(2) No wide type \( p \) over an elementary submodel is product-free.

(3) For every elementary submodel \( N \), we have that \( G = G^0_N \).

Furthermore, if \( I \) is type-definable over \( \emptyset \), then each of the above conditions is equivalent to:

(4) For some elementary submodel \( M \), we have that \( G = G^0_M \).

**Proof.** Lemma 2.5 yields that \((1) \Rightarrow (2) \). For the implication \((2) \Rightarrow (3) \), observe that any left coset of \( G^0_N \) contains a wide type, by Lemma 3.3. Therefore, there are no proper cosets of \( G^0_N \), since a proper coset clearly defines a product-free
partial type. Hence, all is left to show is the implication (R) \implies (S). To see this, given a wide complete type \( p \) over an elementary submodel \( N \), by the last remark in Fact 1.4, the group \( \text{Stab}(p) \) equals \( G^{00}_{N} \), which is \( G \). Hence, the wide type \( p \) is contained in \( G = \text{Stab}(p) \), so it must belong to \( \text{St}(p) \), again by Fact 1.4, which yields the desired conclusion.

Suppose now that \( I \) is type-definable over \( \emptyset \). The implication (R) \implies (O) is immediate, so we need only show that (O) \implies (M). Suppose that \( G = G^{00}_{M} \) for some fixed elementary submodel \( M \) and choose now a wide product-free type \( p \) over an elementary submodel \( N \). By compactness, there is a wide product-free \( N \)-definable set \( A \) in \( p \). Write \( A = \varphi(x, n) \) for some tuple of parameters \( n \) in \( N \). Since \( I \) is type-definable, the Remark 1.1 yields that there is some \( m \) in \( M \) such that \( \varphi(x, m) \) is wide and product-free. Any wide type \( q \) over \( M \) completing \( \varphi(x, m) \) is product-free. The wide type \( q \) lies in \( G = G^{00}_{M} \), so it must lie in \( St(q) \), by Fact 1.4, contradicting Lemma 2.5. \( \Box \)

A particular example of groups with trivial universal definable compactifications are non-principal ultraproducts of the finite simple groups \( \text{PSL}_2(\mathbb{F}_q) \), since such ultraproducts are again simple groups. More generally, ultra quasirandom groups, studied by Bergelson and Tao [6, Definition 31], satisfy all of the above conditions, as shown in [28, Theorem 4.8], for ultra quasirandom groups are particular ultraproducts of finite groups, and hence can be seen as non-standard finite groups (see [30, Definition 2.1] & [28, Definition 2.4]). In the following section, we will explore the connection between product-free types and IP sets. This connection was thoroughly studied in the work of Bergelson and Tao [6] for ultra quasirandom groups, motivated by a quantitative version in the finite setting first exhibited for quasirandom groups by Gowers [14].

3. IP Sets and Measures

The notions of lower, upper and Banach density (as stated in the introduction) can be generalized to countable groups using Følner sequences. A sequence \((F_n)_{n \in \mathbb{N}}\) of finite sets of a countable group \( G \) is a (left) Følner sequence if

\[
\lim_{n \to \infty} \frac{|F_n \cap g \cdot F_n|}{|F_n|} = 1,
\]

for all \( g \) in \( G \). Notice that a subsequence of a Følner sequence is again Følner.

A countable group is amenable if it can be equipped with a Følner sequence. The archetypal example of an amenable group is \( \mathbb{Z} \) with the Følner sequence consisting of the bounded intervals \( F_n = [-n, n] \). Every countable solvable (in particular, every abelian countable) group is amenable.

Given an amenable group \( G \) with a distinguished Følner sequence \( \mathcal{F} = (F_n)_{n \in \mathbb{N}} \), we define the upper density (with respect to \( \mathcal{F} \)) as

\[
\tilde{d}(A) = \limsup_{n \to \infty} \frac{|A \cap F_n|}{|F_n|} \quad \text{for all } A \subseteq G.
\]

Note that \( \tilde{d}(A) = \tilde{d}(g \cdot A) \) for all \( g \) in \( G \), since the sequence \( \mathcal{F} \) is Følner. The upper (left) Banach density is then defined as

\[
d^*(A) = \sup \{ \tilde{d}(A) \mid \mathcal{F} \text{ is a (left) Følner sequence in } G \}.
\]
Remark 3.1. For $G = \mathbb{Z}$, this notion of upper Banach density and the one of the introduction are equivalent (see [3, Remark 1.1 and Lemma 3.3]).

A version of Furstenberg’s correspondence principle for amenable groups [4, Theorem 5.8], or alternatively a clever use of Hahn-Banach [4, Proposition 4.19], produces from a set $A$ of positive upper (Banach) density on an amenable group a left translation invariant measure, which bounds from below the density of any intersection of finitely many translates of $A$. For the sake of the presentation, we will include a proof with no additional input from our side.

Fact 3.2. Let $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ be Følner sequence in a countable amenable group $G$ such that $d_{\mathcal{F}}(A) > 0$ for a fixed subset $A$ of $G$. There exists a left translation invariant finitely additive probability measure $\mu$ on $G$ with $\mu(A) = d_{\mathcal{F}}(A)$ such that for all $g_1, \ldots, g_r$ in $G$ we have that

$$d_{\mathcal{F}}(g_1 \cdot A \cap \ldots \cap g_r \cdot A) \geq \mu(g_1 \cdot A \cap \ldots \cap g_r \cdot A).$$

Proof. By definition of $\limsup$, there is a subsequence $\mathcal{F}_0 = (F'_{n_k})_{k \in \mathbb{N}}$ of $\mathcal{F}$ such that

$$d_{\mathcal{F}_0}(A) = d_{\mathcal{F}}(A) = \lim_{k \to \infty} \frac{|A \cap F'_{n_k}|}{|F'_{n_k}|}.$$

Notice that the sequence $\mathcal{F}_0$ is again Følner. Since $G$ is countable, choose an enumeration $(A_m)_{m \in \mathbb{N}}$ of all finite intersections of translates of $A$ by elements of $G$ with $A_0 = A$ and find for each subset $A_m$ a Følner subsequence $\mathcal{F}_m = (F'_{n_k})_{k \in \mathbb{N}}$ of $\mathcal{F}_{m-1}$ such that the limit

$$\lim_{k \to \infty} \frac{|A_m \cap F'_{n_k}|}{|F'_{n_k}|}$$

exists. By a standard diagonal procedure, consider now $\mathcal{F}'$ with $F'_m = F'_{n_k}$. By construction, for every $g_1, \ldots, g_r$ in $G$, the limit

$$\lim_{m \to \infty} \frac{\left| \bigcap_{i=1}^r (g_i \cdot A) \cap F'_m \right|}{|F'_m|}$$

exists, and bounds $\hat{d}_{\mathcal{F}}(g_1 \cdot A \cap \ldots \cap g_r \cdot A)$ from below.

Choose now a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and define the ultralimit with respect to $\mathcal{U}$

$$\mu(B) = \lim_{\mathcal{U}} \frac{|B \cap F'_m|}{|F'_m|}$$

for each subset $B$ of $G$. By construction, the function $\mu$ is a finitely additive probability measure on $G$ satisfying that $\mu(A) = d_{\mathcal{F}}(A)$. Furthermore, for all $g_1, \ldots, g_r$ in $G$, we also have that $\hat{d}_{\mathcal{F}}(g_1 \cdot A \cap \ldots \cap g_r \cdot A) \geq \mu(g_1 \cdot A \cap \ldots \cap g_r \cdot A)$. Finally, since the sequence $\mathcal{F}'$ is Følner, the above measure is left translation invariant. \hfill \Box

Every measure can be made definable over $\emptyset$ after possibly expanding the language (see the discussion before the Remark 1.1). Thus, we deduce from Fact 3.2 the following result:

Corollary 3.3. For each set of positive upper (Banach) density $A$ of an amenable (countable) group $G$, there is an indiscernibly prime left translation invariant $\emptyset$-type-definable (hence, invariant) ideal $\mathcal{I}$ on the boolean algebra of all definable subsets of $G$ such that $A$ is wide with respect to $\mathcal{I}$. Furthermore, for any $g_1, \ldots, g_r$
in $G$, if the intersection $\bigcap_{i=1}^{n}(g_i \cdot A)$ is wide, then it has positive upper (Banach) density.

As first proved in [5, Theorem 2.6] (see also [5, Proposition 5.3]), in the presence of stability, a subset $A$ of positive upper Banach density in an amenable discrete group has the productset property, that is, there are infinite sets $B$ and $C$ such that $B \cdot C$ is contained in $A$. We say that a set $A$ is stable if the equivariant relation $A(y \cdot x)$ is stable. We would first like to clarify the relation between wide and generic sets under the assumption of stability (We thank the referee for pointing out that our original proof of Proposition 3.5 could be reformulated in a simpler context).

**Remark 3.4.** Let $G$ be an infinite group. Left-generic definable subsets are wide for every left translation invariant ideal $I$ on the Boolean algebra of all definable subsets of $G$. Furthermore, if the definable subset $A$ is stable and wide for some indiscernibly prime left translation invariant $\emptyset$-invariant ideal $I$ on the Boolean algebra of all definable subsets of $G$, then $A$ is left-generic. Indeed, by [5, Remark 5.17 (i)], it suffices to show that the formula $g \cdot A$ does not fork over $\emptyset$, which follows immediately from Remark 3.3.

We now observe that a slightly stronger result than [5, Theorem 2.6] holds:

**Proposition 3.5.** Let $G$ be an infinite group and $A$ a stable left-generic definable subset of $G$. Then there are an infinite definable subgroup $H$ of $G$ with $H \subseteq A \cdot A^{-1}$ as well as sequences $(b_n)_{n \in \mathbb{N}}$ in $H \subseteq A \cdot A^{-1}$ and $(c_n)_{n \in \mathbb{N}}$ in $A$ such that $b_n \cdot c_n$ lies in $A$ for all $n$ and $m$.

In particular, stable wide sets have the productset property, by Remark 3.4.

**Proof.** Let $\varphi(x, y)$ be the stable formula $A(y \cdot x)$, and choose by [10, Claim in the proof of Lemma 5.16] a generic $\varphi$-type $p$, over the ambient model, containing $A(x) = \varphi(x, \text{id}_G)$. The $\varphi$-stabilizer $H \subset A \cdot A^{-1}$ of $p$ is definable and has finite index in $G$ by [10, Lemma 5.16 (ii)]. Therefore, the subgroup $H$ is infinite.

By stability of $\varphi$ and a standard Ramsey argument, we need only show that there are infinite sequences $(b_n)_{n \in \mathbb{N}}$ in $H \subseteq A \cdot A^{-1}$ and $(c_n)_{n \in \mathbb{N}}$ in $A$ such that $b_k \cdot c_m$ lies in $A$ if $k \leq m$. Suppose the elements $b_0, \ldots, b_{n-1}$ and $c_0, \ldots, c_{n-1}$ have already been constructed. Since $H$ is infinite, choose $b_n$ in $H$ different from all $b_i$, with $0 \leq i \leq n - 1$. By definition of the $\varphi$-stabilizer, the $\varphi$-type $b_k \cdot p$ equals $p$ for all $0 \leq k \leq n$. In particular, the $\varphi$-type $p$ contains the intersection $A \cap \bigcap_{k=0}^{n} b_k^{-1} \cdot A$, so this intersection cannot be finite. Choose therefore an element $c_n$ in $A \cap \bigcap_{k=0}^{n} b_k^{-1} \cdot A$ different from all $c_i$, for $0 \leq i \leq n - 1$. By construction, the product $b_k \cdot c_n$ belongs to $A$ for $k \leq n$, as desired.

**Remark 3.6.** An inspection of the above proof yields that it suffices to assume that the stable set $A$ contains a type $p$ whose $\varphi$-stabilizer is infinite. If the group is abelian, this is always the case if the infinite definable set $A$ is a boolean combination of weakly normal formulae by [5, Lemma 2.4].

We do not know whether there are any further situations in which this weaker case can be applied.

By Corollary 3.3, a subset of positive upper Banach density in an amenable group $G$ can be assumed to be definable and wide with respect to a suitable ideal $I$, after possibly expanding the language. Consequently, we obtain the following
immediate observation and thank Gabe Conant for pointing out a mistake in a previous version:

**Corollary 3.7.** Every stable subset of positive upper Banach density in an infinite amenable group $G$ has the productset property for some infinite sets $B$ and $C$, where $B$ lies in $A \cdot A^{-1}$ and $C$ is a subset of $A$. □

Nathanson used a result of Khazdan in order to prove in [26, Theorem 6] that a set $A$ of positive upper density in the integers contains subsets of the form

$$F_1 + \ldots + F_n + B,$$

where each $F_i$ is finite of size at least $m$ and $B$ has positive upper density, for every $n$ and $m$ in $\mathbb{N}$. By Corollary 3.3, we can extend his result to any (possibly non-abelian) amenable group $G$.

**Proposition 3.8.** Let $G$ be an amenable group. Given any two natural numbers $n$ and $m$, and a subset $A$ of positive upper density, there are finite subsets $F_1, \ldots, F_n$ of $A$ of size $m$ and a subset $B$ of $A$ of positive upper density such that

$$F_1 \cdot \ldots \cdot F_n \cdot B \subseteq A.$$

It will follow from the proof that $B$ can be chosen to be a finite intersection of left translates of $A$.

**Proof.** It suffices to show the above result for $n = 1$. Fix $m$ some natural number. By Corollary 3.3, equip $G$ with an $\emptyset$-type-definable left translation invariant indiscernibly prime ideal $I$ on the boolean algebra of all definable subsets of $G$ (in an appropriate language) with $A$ wide and $\emptyset$-definable, such that for any $g_1, \ldots, g_r$ in $G$, if the intersection $\bigcap_{i=1}^{r} (g_i \cdot A)$ is wide, then it has positive upper (Banach) density.

Choose now $g_1, \ldots, g_m$ in $A(\mathcal{U})$ starting an indiscernible sequence. Since $I$ is indiscernibly prime, the set $\bigcap_{i=1}^{m} (g_i^{-1} \cdot A(\mathcal{U}))$ is wide. By the Remark 1.1, we may find elements $h_1, \ldots, h_m$ in $A$ with $B(\mathcal{U}) = \bigcap_{i=1}^{m} (h_i^{-1} \cdot A(\mathcal{U}))$ wide. Since the latter intersection is definable over $G$, we conclude that $B$ has positive upper density (as a subset of $G$). Furthermore, notice that $F_1 \cdot B \subseteq A$ with $F_1 = \{h_1, \ldots, h_m\}$, which finishes the proof. □

We will finish this section with a result on the IP property for wide types in ultra quasirandom groups, along the lines of the work of Bergelson and Tao [6]. In the aforementioned work, the authors take over work of Gowers [14] in the finitary context, in order to prove striking results on deterministic mixing of progressions and IP systems. We will prove a much weaker result, concentrating solely on inclusion in a given (wide) set, yet with an elementary proof from the model-theoretic point of view.

**Theorem 3.9.** Let $G$ be a group equipped with a left translation invariant $\emptyset$-type-definable indiscernibly prime ideal $I$ on the boolean algebra of all definable subsets of $G$, and $A$ be a wide subset of $G$ definable over an elementary substructure $M$ of $\mathcal{U}$. If no wide type containing $A$ is product-free, then $A(M)$ contains an IP set.
Proof. Let $p$ be a wide type over $M$ containing $A(x)$. Since $p$ is not product-free, Lemma 2.5 implies that $p$ belongs to $St(p)$. Choose some $a_0$ realizing $p$ in $U$, so $p \cup a_0^{-1} \cdot p$ is wide. In particular, the subset $A \cap a_0^{-1} \cdot A$ is wide. As the ideal $I$ is type-definable over $\emptyset$, we may find $a_0$ in $A(M)$ with $A_1 = A \cap a_0^{-1} \cdot A$ wide, by Remark 4.4. Every wide type (over $M$) containing the $M$-definable set $A_1$ must contain $A$, so no wide type containing $A_1$ is product-free either. We can iterate this process to find sequences $(a_i)_{i \in \mathbb{N}}$ of elements of $A(M)$ and $(A_k)_{k \in \mathbb{N}}$ of $M$-definable wide subsets of $A$ such that $$a_{k+1} \text{ belongs to } A_{k+1} = A_k \cap a_k^{-1} \cdot A_k.$$ It is now easy to show by induction on $m$ that a finite product $a_{i_1} \cdots a_{i_m}$ belongs to $A_{i_1} \cap \cdots \cap A_{i_m}$ when $i_1 < \ldots < i_m$. Thus, the set $A(M)$ contains all finite ordered products of the sequence $(a_i)_{i \in \mathbb{N}}$, and hence is IP, as desired. □

The above result and Corollary 2.9 yield the immediate consequence.

Corollary 3.10. Let $G$ be a group equipped with a left translation invariant $\emptyset$-type-definable indiscernibly prime ideal $I$ on the boolean algebra of all definable subsets of $G$. If $G = G^M_M$ for some elementary substructure $M$ of $U$, then every wide subset of $G$ contains an IP set. □

Recall the following notions in [14, Definition before Theorem 4.6] & [15, Definition 31]. A finite group $G$ is $d$-quasirandom if it has no non-trivial representations over $\mathbb{C}$ of dimension strictly less than $d$. An ultra quasirandom group is an ultraproduct of finite groups $G_n$ with respect to a non-principal ultrafilter $U$ such that for every $d$ in $N$, the collection of $d$-quasirandom $G_n$ is $U$-large. It was proved in [28, Theorem 4.8] that ultra quasirandom groups are precisely the ultraproducts of finite groups satisfying the conditions in Corollary 2.6. Therefore, Corollary 3.10 yields a weaker version of [15, Lemma 40].

Corollary 3.11. Any internal set $A$ in a ultra quasirandom group $G$ of positive (nonstandard) normalised counting measure contains an IP set. □

A standard Łoś argument yields the following finitary version, which is a non-quantitative weaker version of [14, Theorem 5.3], with $A_F$ constant in Gowers’s terminology.

Corollary 3.12. For every natural number $n$ and every $\epsilon > 0$, there is some $d = d(n, \epsilon)$ such that every finite subset $A$ in a $d$-quasirandom group $G$ with $|A| \geq \epsilon|G|$ contains all possible products of a sequence of length $n$. □

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