The Complexity of Quantum Systems on a One-dimensional Chain

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Abstract

We prove that adiabatic computation is equivalent to standard quantum computation even when the adiabatic quantum system is restricted to be a set of particles on a one-dimensional chain. We give a construction that uses a 2-local Hamiltonian on nearest neighbors using particles that can have ten distinct states. This implies a construction of a one-dimensional chain of qubits in which the Hamiltonian is 6-local. We adapt this construction to show that the 2-local Hamiltonian for 13-state particles is QMA-complete which in turn implies that the 8-local Hamiltonian restricted to a one-dimensional chain of qubits is QMA-complete.

1 Introduction

Adiabatic computation was introduced by Farhi, Goldstone, Gutmann and Sipser [?] as a means of solving difficult classical optimization problems. Although it appears likely that algorithms for this model require exponential time to solve NP-hard problems, adiabatic computation remains an appealing alternative to standard computation since it may be more robust against certain types of quantum errors [?].

The idea behind adiabatic computation is to define two Hamiltonians $H_{init}$ and $H_{final}$. The ground state of $H_{init}$ (the eigenstate corresponding to the smallest eigenvalue) should be easy to prepare, like a tensor product states. The desired output is the ground state of $H_{final}$ which should somehow encode the solution to the problem. We also require that both Hamiltonians be local in that they are the sums of Hamiltonians, each of which operates on a constant number of particles. Thus, they are efficiently described by enumerating the matrices associated with each component. The Hamiltonian is slowly varied from $H_{init}$ to $H_{final}$ by increasing $s$ from 0 to 1 in the following expression: $H(s) = (1-s)H_{init} + sH_{final}$. The adiabatic theorem (expressed more formally in the next section) states that if a system begins in the ground state of $H_{init}$ and the system is varied slowly enough, it will remain in the ground state. Thus, the final state will be the desired ground state of $H_{final}$. The required time to vary $s$ from 0 to 1 depends on the minimum spectral gap of $H(s)$. In order to think of adiabatic quantum computation as computing a classical function (as opposed to quantum states), a measurement of one or more of the particles is then performed to yield a classical output.

van Dam, Mosca and Vazirani proved that the standard model of quantum computation is at least as strong as the adiabatic quantum model [?]. Subsequently, Aharonov et al. showed that

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1Similar results have also been obtained by Aharonov, Gottesman and Kempe [?].
an arbitrary quantum circuit can be simulated by an adiabatic computation \[\text{[?]},\] thus establishing polynomial equivalence of the two models. Furthermore, the construction in \[\text{[?]}.\] showed that any quantum computation could be efficiently simulated by an adiabatic computation with 2-local nearest-neighbor Hamiltonians operating on six-state particles on a two dimensional grid. This was later improved by Oliveira and Tehral \[\text{[?]}.\] to a two-dimensional grid of qubits (two state particles).

We prove the following theorem for 1-dimensional systems:

**Theorem 1**  
An arbitrary quantum circuit can be simulated by an adiabatic computation with 2-local nearest neighbor Hamiltonians operating on a 1-dimensional chain of ten-state particles. This in turn implies that the construction will work on a 1-dimensional chain of qubits with 6-local Hamiltonians, each of which operates on contiguous sets of qubits.

A slight modification of our construction has implications for the study of the complexity of quantum systems. The study of quantum systems from the point of view of their computational complexity is an important part of the more general research program aimed at understanding the power of quantum computing. In particular, many one-dimensional quantum systems appear to be more tractable in some ways than their two-dimensional counterparts \[\text{[?]}.\]. This paper seeks to answer whether certain hardness results for two-dimensional quantum systems can be carried over to the one-dimensional case. In particular, can a one-dimensional quantum system have a sufficiently rich structure to implement universal computation?

Proving that a given question about a quantum system is QMA-complete provides strong evidence that it is computationally difficult, even with the benefit of a quantum computer. The class QMA is the quantum analog of NP and MA. That is, it is the set of all languages that can be probabilistically verified by a quantum verifier in polynomial time. Kitaev initiated the study of QMA-complete problems by defining the local hamiltonian problem, the quantum analog of SAT \[\text{[?]}.\]. In this problem, one is given a Hamiltonian and a guarantee that the lowest eigenvalue for \(H\) is either greater than some value \(b\) or less than another values \(a\), where \(b - a\) is at least an inverse polynomial in the number of qubits in the system. The output of the problem is to determine which alternative is the case for \(H\). Kitaev gave the first QMA-complete problem by showing that the 5-local Hamiltonian is QMA-complete \[\text{[?]}.\]. This was later improved by Kempe and Regev who showed that the 3-local Hamiltonian is QMA-complete \[\text{[?]}.\] and then by Kempe, Kitaev and Regev who showed that the 2-local Hamiltonian is QMA-complete \[\text{[?]}.\]. Finally Oliveira and Tehral \[\text{[?]}.\] showed that this latter result holds even when the 2-local Hamiltonian is constrained to be nearest-neighbor interactions on a grid of qubits. They conjecture that it is not possible to extend this result for a 1-dimensional chain. We give the following result:

**Theorem 2**  
The 2-local Hamiltonian on a 1-dimensional chain of 13-state particles is QMA-complete. This in turn implies that the 8-local Hamiltonian on a 1-dimensional chain is QMA-complete.

Certainly an intriguing question left open here is whether the locality can be reduced and, in particular, whether the 2-local Hamiltonian on a 1-dimensional chain is QMA-complete.

Similar results to these have been independently obtained by Aharonov, Gottesman and Kempe \[\text{[?]}.\].
2 Preliminaries

2.1 The Model of Adiabatic Computation

The Adiabatic Theorem, stated here, is the foundation of adiabatic computation.

Theorem 3 The Adiabatic Theorem (as adapted from [?] and quoted from [?]) Let $H_{\text{init}}$ and $H_{\text{final}}$ be two Hamiltonians acting on a quantum system and consider the time-dependent Hamiltonian $H(s) = (1-s)H_{\text{init}} + sH_{\text{final}}$. Assume that for all $s$, $H(s)$ has a unique ground state. Then for any fixed $\delta > 0$, if

$$T \geq \Omega \left( \frac{\|H_{\text{final}} - H_{\text{init}}\|^{1+\delta}}{\epsilon^3 \min_{s \in [0,1]} \{\Delta_{2+\delta}(H(s))\}} \right),$$

the final state of an adiabatic evolution according to $H$ for time $T$ (with an appropriate setting of global phase) is $\epsilon$-close in $l_2$-norm to the ground state of $H_{\text{final}}$. The matrix norm is the spectral norm $\|H\| = \max_w \|Hw\|/\|w\|$. 

$\delta$ will be a constant and the constant in the $\Omega$ will go to infinity as $\delta$ goes to 0. We use the model of adiabatic computation as described in [?]:

Definition 1 A $k$-local adiabatic computation $AC(n,d,H_{\text{init}},H_{\text{final}},\epsilon)$ is specified by two $k$-local Hamiltonians, $H_{\text{init}}$ and $H_{\text{final}}$ acting on $n$ $d$-dimensional particles, such that both Hamiltonians have unique ground states. The ground state of $H_{\text{init}}$ is a tensor product state. The output is a state that is $\epsilon$-close in $l_2$-norm to the ground state of $H_{\text{final}}$. Let $T$ be the smallest time such that the final state of an adiabatic evolution according to $H(s) = (1-s)H_{\text{init}} + sH_{\text{final}}$ for time $T$ is $\epsilon$-close in $l_s$-norm to the ground state of $H_{\text{final}}$. The running time of the adiabatic algorithm is defined to be $T \cdot \max_s \|H(s)\|$.

Thus, the running time of the computation will be upper bounded by the norm of the Hamiltonians and the lower bound given in Inequality [1].

It is possible in some cases to restrict our attention to a subspace $S$. Suppose that $H(s)$ leaves $S$ invariant. $H_S(s)$ is the restriction of $H(s)$ to $S$. If we start the adiabatic evolution at a state inside $S$ then an adiabatic evolution according to $H_S(s)$ is identical to an adiabatic evolution according to $H(s)$. Therefore, we can use $\Delta H_S(s)$ in Theorem [3] instead of $\Delta H(s)$.

2.2 The Class QMA and the Local Hamiltonian Problem

The class QMA is defined in terms of promise problems defined by a pair $(L_{\text{yes}}, L_{\text{no}})$ of disjoint sets of strings corresponding to YES and NO instances of the problem. The input is guaranteed to be in $L_{\text{yes}} \cup L_{\text{no}}$ and solving the problem entails determining whether a given input string $x$ is in $L_{\text{no}}$ or $L_{\text{yes}}$. $B$ is defined to be the Hilbert space of a single qubit.

Definition 2 (QMA) Fix $\epsilon = \epsilon(|x|)$ such that $\epsilon = 2^{-O(|x|)}$. Then, a promise problem $L \in QMA$ if there exists a quantum polynomial time verifier $V$ and a polynomial $p$ such that

$$\forall x \in L_{\text{yes}}, \exists \xi \in B^{\otimes p(|x|)}, Pr(V(|x\rangle, |\xi\rangle) = 1) \geq 1 - \epsilon$$

$$\forall x \in L_{\text{no}}, \forall \xi \in B^{\otimes p(|x|)}, Pr(V(|x\rangle, |\xi\rangle) = 1) \leq \epsilon$$

where $Pr(V(|x\rangle, |\xi\rangle) = 1)$ denotes the probability that $V$ outputs 1 given $|x\rangle$ and $|\xi\rangle$. 


The first known QMA-complete problem is the local Hamiltonian problem and is a natural analog of SAT.

**Definition 3** We say that an operator \( H : \mathcal{B}^{\otimes n} \to \mathcal{B}^{\otimes n} \) on \( n \) qubits is \( k \)-local if \( H \) can be expressed as the sum of terms, where each term is a Hermitian operator acting on at most \( k \) bits.

**Definition 4** (The (promise) problem \( k \)-local Hamiltonian) A \( k \)-local Hamiltonian on \( n \) qubits \( H = \sum_{j=1}^{r} H_j \) with \( r = \text{poly}(n) \). Each \( H_j \) has a bounded operator norm \( \|H_j\| \leq \text{poly}(n) \) and its entries are specified by \( \text{poly}(n) \) bits. In addition, we are given two constants \( a \) and \( b \) with \( a < b \). In YES instances, the smallest eigenvalue of \( H \) is at most \( a \). In NO instances, it is larger than \( b \).

### 2.3 Previous Results on Spectral Gaps

We make use of some previous results on spectral gaps which we state here in a form that is particularly suited for our purposes. Define \( P_r \) to be the \( r \times r \) matrix of the following form:

\[
\begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} & 0 & \cdots & 0 \\
-\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \cdots \\
0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\
0 & \cdots & 0 & 1 & -\frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

**Lemma 4** Consider a Hamiltonian \( H = H_1 + H_2 \) acting on a subspace \( S \). Consider a basis for \( S, \gamma_1, \ldots, \gamma_r \). Suppose that when \( H_1 \) is restricted to \( S \) and expressed in this basis, the matrix is \( P_r \). Suppose also that when \( H_2 \) is restricted to \( S \) and expressed in this basis, the matrix is diagonal with non-negative integer entries, at least one of which is nonzero. Then the lowest eigenvalue of \( H \) when restricted to \( S \) is \( \Omega(1/r^4) \).

For the proof of this lemma, we make use of the following lemma proven by Kitaev (Lemma 14.4 in [?]).

**Lemma 5** Let \( H_1 \) and \( H_2 \) be two Hamiltonians with ground energies \( a_1 \) and \( a_2 \), respectively. Suppose that for both Hamiltonians the difference between the energy of the (possibly degenerate) ground space and the next highest eigenvalue is larger than \( \Lambda \), and that the angle between the two ground spaces is \( \theta \). Then the ground energy of \( H_1 + H_2 \) is at least \( a_1 + a_2 + 2\Lambda \sin^2(\theta/2) \).

**Proof of Lemma 4** Suppose that the diagonal entries are all non-zero. Since the entries are integral, the lowest eigenvalue of \( H_2 \) is at least 1. Since \( H_1 \) and \( H_2 \) are both positive semi-definite, the ground energy of \( H_1 + H_2 \) is at least 1. Now suppose that \( H_2 \) has at least one non-zero entry. It’s spectral gap is at least 1 since it has at least one non-zero entry on its diagonal. Using standard techniques, one can show that the spectral gap of \( H_1 \) is at least \( \Omega(1/r^2) \). The ground energies of \( H_1 \) and \( H_2 \) are both 0. Furthermore since the ground state of \( H_1 \) is a uniform superposition of all the basis vectors and \( H_2 \) has at least one non-zero entry, the cosine of the angle between the two
ground spaces is at most $1 - 1/r$. Invoking Lemma 5 we have that the ground energy of $H_1 + H_2$ is at least $\Omega(1/r^3)$.

The next lemma is instrumental in providing a spectral gap for the adiabatic computation:

**Lemma 6** Let $H_{\text{final}}$ be $P_r$ for some $r$. Let $H_{\text{init}}$ be an $r \times r$ matrix of the following form:

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]

Define $H(s) = (1 - s)H_{\text{init}} + sH_{\text{final}}$. Then the spectral gap of $H(s)$ is $\Omega(r^{-2})$ for all $s \in [0, 1]$.

This lemma is proven in [?].

3 A One-dimensional Universal Quantum Adiabatic Computer

The proof of the following theorem shows that we can simulate a quantum circuit that uses $L$ gates on $n$ qubits with an adiabatic computation on a chain of $(L + 2)$ ten-state particles on a line.

**Theorem 7** Given a quantum circuit on $n$ qubits with $L$ two-qubit gates implementing a unitary $U$, and $\epsilon > 0$, there is a 2-local adiabatic computation between nearest-neighbor particles on a line $AC(L + 2, 10, H_{\text{init}}, H_{\text{final}}, \epsilon)$ whose running time is polynomial in $L$ and $1/\epsilon$ and whose output is $\epsilon$ close to $U|0^n\rangle$. Moreover, $H_{\text{init}}$ and $H_{\text{final}}$ can be computed by a polynomial time Turing machine.

We first show a construction that uses particles with thirteen states. We will add one extra state at the very end and then reduce the number of states by four by identifying pairs of states. The final construction will use ten states. Our proof makes use of the observation in Section 2.1 that one need only prove a lower bound on the spectral gap of $H(s)$ when restricted to a subspace as long as the adiabatic computation starts in that same subspace.

We follow the convention in [?] which assumes without loss of generality that the quantum circuit to be simulated has a particular layout of the gates. It consists of a sequence of $R$ rounds. Each round is composed of $n$ nearest neighbor gates. The first gate in each round is a one-qubit gate applied to the first qubit. For $i = 2, \ldots, n$, the $i^{th}$ gate is a two-qubit gate applied to qubits $i - 1$ and $i$. It will be convenient to assume that the first round of gates consists entirely of identity gates. This will just serve to increase the number of rounds by one. The value of $R$ is adjusted accordingly. The construction in [?] also has a sequence of $n$ identity gates at the end of each round but we omit that here. The total number of gates then is $L = nR$. They observe that any circuit can be transformed to fit this format by adding additional identity and swap gates.

We now define a Hamiltonian $H_{\text{prop}}$ which will enforce the propagation of the quantum computation. In this section $H_{\text{final}}$ will simply be $H_{\text{prop}}$. The ground state of the $H_{\text{final}}$ will be a uniform superposition of some number of different configurations which we call *templates*. Each template represents a subspace of $2^n$ possible states and the particular state within a given template will encode the state of the computation at some particular point in time. We will think of these templates as a sequence and that the state of the system changes through time from one template...
to the next. In fact, the ground state of $H_{\text{final}}$ will be a superposition of these different snapshots of the computation. We start by giving an overview of $H_{\text{prop}}$. In the course of this description, we will describe the states for each particle and their significance.

We have a chain of $L + 2$ particles, where $L$ is the number of gates in the quantum circuit. The particles will be labelled 0 through $L + 1$. We will think of the particles as organized into $L/n$ contiguous blocks of $n$ particles with a single particle on either end (particles 0 and $L + 1$). The first block of particles are 1 through $n$, the second block is $n + 1$ through $2n$, etc. At any point in time, there are $n$ particles whose state represent the state of the computation. We will call these the \textit{computation particles}. These are either all contiguous or possibly separated by a single control particle somewhere in the middle. In general, $H_{\text{prop}}$ will enforce that the computation qubits shift from the left to the right. When the computation particles are all aligned within the boundaries of a single block (i.e. are located in positions $in + 1, \ldots, (i + 1)n$ for some $i$), then the gates are applied for the $i^{\text{th}}$ round.

There are 13 possible states for the qubits.

- \textbf{F}: F stands for finished. Particles in this state are to the left of the computation particles and will no longer change state.
- \textbf{N}: N stands for new. Particles in this state are to the right of the computation particles and have not yet been reached.
- \textbf{L}: L stands for left. A particle in this state is used for control and indicates that the particle in the control state is propagating to the left.
- \textbf{T}_L, \textbf{T}_R: \textbf{T}_L$ and $\textbf{T}_R$ stand for turn left and right, respectively. A particle in this state is used for control and indicates that the control particle will start moving in the opposite direction. $\textbf{T}_L$ indicates that it will start moving to the left and $\textbf{T}_R$ indicates that it will start moving to the right.
- \textbf{Q}_0, \textbf{Q}_1, \textbf{B}_0, \textbf{B}_1: These states represent qubits. Particles in these states are computation qubits. The subscript represents the state of a qubit in the computation. The $B$ states are to the left of the particle in the control state and the $Q$ states are to the right of the particle in the control state.
- \textbf{R}_0, \textbf{R}_1: R stands for right. A particle in this state is a computation particle as well as a control particle. The subscript represents the state of a qubit in the computation. A particle in this state also indicates that the control particle is moving to the right.
- \textbf{G}_0, \textbf{G}_1: G stands for gate. A particle in this state is a computation particle as well as a control particle. The subscript represents the state of a qubit in the computation. A particle in this state also indicates that the control particle is moving to the right and a gate is being applied.

We will explain the basic construction of $H_{\text{prop}}$ by showing an iteration in which the control particle propagates to the right and then to the left. At the end of the iteration, the computation particles will have shifted one position to the right. At each step, we indicate the component of $H_{\text{prop}}$ that enforces the propagation. If $A$, $B$, $X$, $Y$ are possible particle states, we will define

$$H_{AB \rightarrow XY}^{i} = |AB\rangle\langle AB|_{i,i+1} + |XY\rangle\langle XY|_{i,i+1} - |AB\rangle\langle XY|_{i,i+1} - |XY\rangle\langle AB|_{i,i+1}.$$
The superscript \( i \) indicates that the Hamiltonian is applied to particles \( i \) and \( i+1 \). This particular Hamiltonian enforces the condition that if a state is an eigenstate with eigenvalue 0, then for each configuration with \( AB \) in positions \( i \) and \( i+1 \) there must be a state of equivalent amplitude with \( AB \) replaced by \( XY \).

It will be useful in our exposition for each \( H^i_{AB \rightarrow XY} \) to define two companion transformations. These correspond to transforming any state with \( AB \) in locations \( i \) and \( i+1 \) by changing the \( AB \) to \( XY \). This is represented by \( \mathcal{F} \), where the \( \mathcal{F} \) stands for propagation in the forward direction. Similarly \( B \) will correspond to replacing \( XY \) by \( AB \) which represents propagation in the backward direction.

\[
\mathcal{F}^i_{AB \rightarrow XY} = |XY\rangle\langle AB|_{i,i+1}, \quad B^i_{AB \rightarrow XY} = |AB\rangle\langle XY|_{i,i+1}.
\]

These will not be part of the final Hamiltonian but will be helpful for describing certain states.

For the time being, we will not use that gate states \( (G_0 \text{ and } G_1) \). This means that the state of the represented qubits will not change and there is no change of state between \( \{Q_0, R_0\} \) and \( \{Q_1, R_1\} \). For ease of notation, we will omit the subscripts. For example, \( H_{BL \rightarrow LQ} \) actually represents \( H_{B_0L \rightarrow LQ_0} + H_{B_1L \rightarrow LQ_1} \). Similarly the expression \( H_{RQ \rightarrow BR} \) would be the sum of four different terms for the possible subscripts on the pair of states. We will use the term template to be a representation of a set of states that all have the same letter value with the subscripts on the computation bits unspecified.

We start with the particles in the following template:

\[
\text{F} \cdots \text{F} \quad \text{T} \quad \text{R} \quad \text{Q} \quad \text{Q} \quad \text{Q} \quad \text{Q} \quad \text{N} \quad \text{N} \quad \cdots \quad \text{N}
\]

where \( n = 5 \). Let particle \( i \) be the location of the \( TR \) state. \( H^i_{TRQ \rightarrow FR} \) will result in the following template:

\[
\text{F} \cdots \text{F} \quad \text{F} \quad \text{F} \quad \text{R} \quad \text{Q} \quad \text{Q} \quad \text{Q} \quad \text{Q} \quad \text{N} \quad \text{N} \quad \cdots \quad \text{N}
\]

For \( j = i + 1 \) through \( i + n - 1 \) we have the Hamlitonian \( H^j_{RQ \rightarrow BR} \) which results in

\[
\text{F} \cdots \text{F} \quad \text{F} \quad \text{F} \quad \text{B} \quad \text{B} \quad \text{B} \quad \text{B} \quad \text{R} \quad \text{N} \quad \text{N} \quad \cdots \quad \text{N}
\]

Then \( H^{i+n}_{RN \rightarrow BT_L} \), results in

\[
\text{F} \cdots \text{F} \quad \text{F} \quad \text{F} \quad \text{B} \quad \text{B} \quad \text{B} \quad \text{B} \quad \text{B} \quad \text{T}_L \quad \text{N} \quad \cdots \quad \text{N}
\]

The presence of the particle in the \( T_L \) (turn left) state indicates that propogation will switch from moving right to moving left. \( H^{i+n}_{T_{LN} \rightarrow LN} \), results in

\[
\text{F} \cdots \text{F} \quad \text{F} \quad \text{F} \quad \text{B} \quad \text{B} \quad \text{B} \quad \text{B} \quad \text{B} \quad \text{L} \quad \text{N} \quad \cdots \quad \text{N}
\]

The presence of the particle in state \( L \) indicates propogation to the left via \( H^j_{BL \rightarrow LQ} \) for \( j = i + n \) down to \( i + 1 \) until

\[
\text{F} \cdots \text{F} \quad \text{F} \quad \text{F} \quad \text{L} \quad \text{Q} \quad \text{Q} \quad \text{Q} \quad \text{Q} \quad \text{Q} \quad \text{N} \quad \cdots \quad \text{N}
\]

Finally, \( H^{i+1}_{FL \rightarrow FR} \) brings us back to the original template with the computation particles shifted one position to the right:

\[
\text{F} \cdots \text{F} \quad \text{F} \quad \text{F} \quad \text{T} \quad \text{Q} \quad \text{Q} \quad \text{Q} \quad \text{Q} \quad \text{Q} \quad \text{N} \quad \cdots \quad \text{N}
\]
There are a total of $2n + 3$ configurations in one iteration that shifts the states one position to the right.

The application of the gates is triggered when the computation qubits align within the boundaries of a set because we have a slightly different Hamiltonian at these locations. At the locations $i$ such that $i$ is a multiple of $n$, we have $H_{T_i R Q \rightarrow FG}$ instead of $H_{T_i R Q \rightarrow FR}$. In addition, we apply the qubit gate to $Q_0/1$. If the $Q$ is in position $i$, then this is the $i^{th}$ gate in the computation and we denote the $2 \times 2$ unitary matrix associated with this 1-bit gate as $U_i$. We will express this Hamiltonian matrix over the subspace spanned by $\{T_R Q_0, T_R Q_1, F G_0, F G_1\}$

$$H_{T_i R Q \rightarrow FG}^i = \begin{bmatrix} I & -U_i \\ -U_i^\dagger & I \end{bmatrix}, \quad F_{T_i R Q \rightarrow FG}^i = \begin{bmatrix} 0 & U_i \\ 0 & 0 \end{bmatrix}, \quad B_{T_i R Q \rightarrow FG}^i = \begin{bmatrix} 0 & 0 \\ U_i^\dagger & 0 \end{bmatrix}.$$

Similarly for the 2-qubit gates, we apply the $i^{th}$ gate to $G Q$ in locations $i - 1$ and $i$ and the result is a state of the form $B G$. We express this Hamiltonian as a matrix over the basis

$$\{G_0 Q_0, G_0 Q_1, G_1 Q_0, G_1 Q_1, B_0 G_0, B_0 G_1, B_1 G_0, B_1 G_1\},$$

where the $i^{th}$ gate is expressed by the $4 \times 4$ unitary matrix $U_i$:

$$H_{G Q \rightarrow B G}^{i-1} = \begin{bmatrix} I & -U_i \\ -U_i^\dagger & I \end{bmatrix}, \quad F_{G Q \rightarrow B G}^{i-1} = \begin{bmatrix} 0 & U_i \\ 0 & 0 \end{bmatrix}, \quad B_{G Q \rightarrow B G}^{i-1} = \begin{bmatrix} 0 & 0 \\ U_i^\dagger & 0 \end{bmatrix}.$$

Finally we need the term that turns the propagation of the control state when $G$ reaches the rightmost end of the computation bits. For $i \mod n = 0$, we have $H_{G N \rightarrow B T L}$ instead of $H_{R N \rightarrow B T L}$. This just preserves the value of the represented qubit as before. To summarize,

$$H_{prop} = \sum_{i=0}^{L} H_{prop}^i.$$

For $i \mod n \neq 0$

$$H_{prop}^i = H_{T_i R Q \rightarrow FR}^i + H_{R Q \rightarrow BR}^i + H_{R N \rightarrow B T L}^i + H_{T L N \rightarrow LN}^i + H_{B L \rightarrow L Q}^i + H_{G Q \rightarrow B G}^i + H_{F L \rightarrow F T R}^i.$$

For $i \mod n = 0$, we replace $H_{T_i R Q \rightarrow FR}^i$ with $H_{T_i R Q \rightarrow FG}^i$ and $H_{R N \rightarrow B T L}^i$ with $H_{G N \rightarrow B T L}^i$, to get

$$H_{prop}^i = H_{T_i R Q \rightarrow FG}^i + H_{R Q \rightarrow BR}^i + H_{G N \rightarrow B T L}^i + H_{T L N \rightarrow LN}^i + H_{B L \rightarrow L Q}^i + H_{G Q \rightarrow B G}^i + H_{F L \rightarrow F T R}^i.$$

Finally, if $i = L$, we remove $H_{G N \rightarrow B T L}^i$ to end the sequence. We can define $F$ by using the corresponding $F$ term in place each $H_{prop}^i$ term. Similarly we can define $B$ by using the corresponding $B$ term in place each $H_{prop}^i$ term.

Now we would like to define the target ground state for $H_{final}$. In order to define the target ground state, we define a series of templates. These templates are linearly ordered. The label $t$ for each template ranges from 0 through $T = (2n + 3)(L - n) + n + 1$. The template is represented by a string of length $L + 2$ indicating the state for each particle, omitting the subscript for any
computation bits. In this section, the templates will always have \( n \) computation bits. Thus, the remaining degrees of freedom represented by the omitted subscripts results in a subspace of dimension \( 2^n \). We say that a term \( H_{AB \rightarrow XY} \) in \( H_{prop} \) applies in the forward direction to a template if \( AB \) appears in string corresponding to the template. This means that \( F_{AB \rightarrow XY} \) has a non-zero result when applied to a state in the corresponding subspace for that template. The result will be a state in the template with \( AB \) replaced by \( XY \). Similarly \( H_{AB \rightarrow XY} \) applies in the backward direction if \( XY \) appears in the template. \( B_{AB \rightarrow XY} \) applied to a state in that template results in a state in the template with \( XY \) replaced by \( AB \). Figure 3 shows a chart of the templates. The string representing the templates will depend on the value of \( t \) as indicated by the column labelled Conditions. We also include a column that indicates for each template the set of terms in \( H_{prop} \) that apply in the forward direction and a column for the terms that apply in the backwards direction. The condition \( \text{Boundary} \) holds for a given term if the pair straddles a boundary between blocks of \( n \) particles. That is, if it applies to particles \( p \) and \( p + 1 \) and \( p \mod n = 0 \). For each \( t \), let \( i = \lceil t/(2n+3) \rceil \) and \( j = t \mod (2n + 3) \).

**Lemma 8** For each \( t \in \{0, \ldots, T - 1\} \), exactly one term in \( H_{prop} \) applies to the \( t \)th template. Furthermore, when \( F_{prop} \) is applied to a state in that template, it results in a state in the next
template in the sequence. For each \( t \in \{1, \ldots, T\} \), exactly one term in \( H_{\text{prop}} \) applies to the \( t^{\text{th}} \) template in the backward direction. Furthermore, when \( B_{\text{prop}} \) is applied to a state in that template, it results in a state in the previous template in the sequence. \( B \) is zero on a state in template 0 and \( F \) is 0 on a state in template \( T \).

Proof: The proof consists of verifying that for each template in Figure 11, exactly one term applies in the forward direction (and in only one place) and exactly one term applies in the backward direction (again in only one place). When the term is applied in the forward direction, the result is the next template in the sequence and when the term is applied in the backward direction, the result is the previous template in the sequence. Template 0 is \( TRQ^nN^{L-n+1} \) and there are no terms in \( H_{\text{prop}} \) that apply in the backward direction. The \( T^{\text{th}} \) template is \( F^{L-n+1}B^{n-1}GN \). Since \( H_{GN-BT}^L \) is removed from \( H_{\text{prop}} \), there is no term in \( H_{\text{prop}} \) that applies in the forward direction.

Each template with \( n \) computation particles corresponds to a particular point in the computation to be simulated. However, the correspondence is not one to one. A particular point in the computation can be represented by many templates. The \( t^{\text{th}} \) template corresponds to the point in the computation after \( g(t) \) gates have been performed, where \( g(t) \) is the number of templates in the sequence from 0 through \( t \) containing a particle in a \( G \) state. As each new template with a particle in a \( G \) state is reached, one more gate is performed. If \( i = \lfloor t/(2n+3) \rfloor \) and \( j = t \mod (2n+3) \), then

\[
g(t) = \begin{cases} 
i + \min\{n, j\} & \text{if } i \mod n = 0, \\
\lfloor n[i/n] \rfloor & \text{if } i \mod n \neq 0
\end{cases}
\]

Define \( |\phi_t \rangle \) to be the state in the \( t^{\text{th}} \) template such that the superposition of states defined by the 0/1 subscripts of the computation bits corresponds to the superposition of states in the circuit after \( g(t) \) gates have been applied, assuming that the input to the quantum computation is \( |0 \rangle \). Thus, \( |\phi_0 \rangle = T(Q_0)^nN^{L-n+1} \).

For each \( t \in \{0, \ldots, T-1\} \), \( F_{\text{prop}}|\phi_t \rangle = |\phi_{t+1} \rangle \) and \( F|\phi_T \rangle = 0 \). Also, for each \( t \in \{1, \ldots, T\} \), \( B|\phi_t \rangle = |\phi_{t-1} \rangle \) and \( B|\phi_0 \rangle = 0 \). Now define \( S \) to be the subspace spanned by the states \( |\phi_t \rangle \) for all \( t \in \{0, \ldots, T\} \). These states form an orthonormal basis of \( S \). \( S \) is closed under \( H_{\text{prop}} \). The matrix representation of \( H_{\text{prop}} \) in the \( |\phi \rangle \) basis is exactly \( P_{T+1} \). Recall the definition of \( P_t \) from Section 2.3. The unique ground state of \( H_{\text{final}} \) (\( = H_{\text{prop}} \)) is

\[
|\phi_{\text{final}} \rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\phi_t \rangle.
\]

Now we need to define \( H_{\text{init}} \):

\[
H_{\text{init}} = (I - |TR \rangle \langle TR|) 0.
\]

Note however, that while \( H_{\text{init}} \) in its current form can distinguish between \( |\phi_0 \rangle \) and the other \( |\phi \rangle \)’s, it does not ensure that the initial configuration starts out in \( S \). To address this problem, we add an additional state \( S \) (for Start) and some extra constraints. The initial configuration will be \( TRS^nN^{L-n+1} \). We will add a constraint to \( H_{\text{init}} \) that enforces the condition that if particle 0 is in state \( TR \), then particle 1 is in state \( S \). This is achieved by adding in \( |TRX \rangle \langle TRX| 0 \), where we sum over all states \( X \) such that \( X \neq S \). Similarly for \( i = 1 \) through \( n - 1 \), we add the constraint that if particle \( i \) is in state \( S \), then \( i+1 \) is also in state \( S \). We also add the constraint that if particle \( n \) is in state \( S \), then particle \( n+1 \) is in state \( N \). Finally for all \( i = n+1 \) through \( L \), we add the
constraint that if particle $i$ is in state $N$, then particle $i+1$ is in state $N$. All of these constraints are added into $H_{\text{init}}$.

We need to alter $H_{\text{prop}}$ slightly in order to work with the $S$ states. In $H_{\text{prop}}^0$, we replace $H_{T,RQ\rightarrow FG}^0$ with $H_{T,RQ\rightarrow FG_0}^0$. (Recall that the input is all 0’s). Then for $i = 1$ through $n-1$, we replace $H_{GQ\rightarrow BG}^0$ with $H_{GQ\rightarrow BG_0}^0$. Templates 1 through $n$ are now of the form $F(B_0)G_0S^jN^{n-j-1}N^Ln+1$. Lemma 8 still holds. Furthermore, $H_{\text{init}}$ ensures that there is only one eigenstate with eigenvalue 0 and it is our desired initial configuration.

**Lemma 9** The matrix representation of $H_{\text{init}}$ restricted to $S$ and expressed in the basis of $|\phi_t\rangle$’s is a diagonal matrix. The first diagonal entry is 0 and the others are 1.

*Proof:* All templates start with $F$ except the first one which starts with $T$. All the templates satisfy the other conditions in $H_{\text{init}}$. □

To establish the spectral gap of $H_S(s)$, we simply invoke Lemma 8 which says that the spectral gap of the restriction of $H(s)$ to $S_0$ satisfies $\Delta(H_S(s)) = \Omega((Ln)^{-2})$.

Using the Adiabatic Theorem, we get that the running time of the algorithm will be $O(||H(s)||\epsilon^{-\delta}(Ln)^{4+2\delta})$. Note that $||H(s)||$ is $O(L)$ which gives an overall running time of $O(\epsilon^{-\delta}L(Ln)^{4+2\delta})$. However, this only produces a final state that is $\epsilon$ close to $|\phi_{\text{final}}\rangle$. This is a superposition of the $|\phi_t\rangle$’s and only $|\phi_T\rangle$ encodes the desired output. This can be corrected by adding $L/\epsilon$ identity gates to the computation so that only a fraction of $\epsilon$ of the $|\phi_{t,j}\rangle$’s will encode partial points in the computation. This makes the final running time $O(\epsilon^{-5+3\delta}L(Ln)^{4+2\delta})$.

In order to reduce the number of states from 14 to 10, we observe that we can identify pairs $(N,F)$, $(T_R,T_L)$, $(B_0,Q_0)$ and $(B_1,Q_1)$ and Lemma 8 still holds.

### 4 8-local Hamiltonian on a Chain is QMA-complete

We now turn to the problem of the promise local Hamiltonian. We will continue to work with 13-state particles as in the previous section. (We will not use the $S$ state added at the end of the previous section). The ground state will now encode the computation performed by a quantum verifier $V$ which works on input $|x\xi\rangle$ for input $x$ and witness $\xi$. The total length of the input is $n$ qubits. We will construct a Hamiltonian such that if there exists a witness $\xi$ such that $V$ accepts with probability at least $1-\epsilon$ on input $|0\xi\rangle$, then the lowest eigenvalue of $H$ will be less than $\epsilon$. However, if for every $\xi$, $V$ accepts with probability at most $\epsilon$, then the lowest eigenvalue will be at least 1 over a polynomial in $n$ and $L$.

For this problem, we need to show that there is a large spectral gap over the entire space of the particles, not just when restricted to a particular subspace. The Hamiltonian $H$ will consist of five components:

$$H = H_{\text{prop}} + H_{\text{init}} + H_{\text{out}} + H_{\text{valid}} + H_{\text{legal}}.$$ 

$H_{\text{prop}}$ will the same as in the previous section (with the stipulation that we do not use the changes made at the end of the previous section to incorporate the additional state $S$).

We will use the term *template* here to refer to any $L + 2$-character string over the alphabet

$$\{F,N,B,Q,L,R,G,T_R,T_L\}.$$
As before, each template will designate a subspace according to the unspecified 0/1 subscripts for the computation states. This subspace will have dimension $2^m$ if the template has $m$ computation bits.

We will use various hamiltonians to enforce that only certain templates will be valid or legal. We start with a set of constraints enforced in $H_{\text{valid}}$. We will enforce through $H_{\text{valid}}$ that a template must have a form which is a string of length $L + 2$ and is specified by one of the three regular expressions:

$$F^*B^*(R + G + L)Q^*N^+, \quad F^*T_RQ^*N^+, \quad F^*B^+T_RN.$$ 

Another way to denote these constraints is that a string corresponding to a valid template must be a path in the graph in Figure 2 from the Start node to the End node consisting of $L + 2$ internal nodes. $R$, $L$, $G$ are grouped together for clarity. A path through this node can use either $R$, $L$ or $G$.

These constraints are enforced by having a set of allowed pairs where an edge in the graph corresponds to an allowed pair. We will need some additional constraints to enforce that the first character must be $F$ or $T_R$ and that the last characters must be $N$. For ease of notation, we will omit the subscripts. Therefore, the pair $RQ$ represents four possible pairs for the four possible combinations of subscripts. When summing over a set, summing over all possible combinations of subscripts will be assumed.

$$S = \{FF, FB, BB, FR, FL, FG, BR, BL, BG, RQ, GQ, LQ, QQ, QN, NN, RN, GN, LN, FT_R, T_RQ, BT_L, T_LN\}$$
Another way to express the set $S$ is that it is the set below, where the character $X$ can be either $R$, $L$, or $G$:

$$S = \{ FF, FB, BB, FX, BX, XQ, QQ, QN, NN, XN, FT, TR, BT, TL, T_LN \}$$

There is a one-to-one correspondence between the above pairs and the edges in Figure 4.

For any $i$ from 0 to $T-1$, we have

$$H_{\text{valid}}^i = I - \sum_{\alpha\beta \in S} |\alpha\beta\rangle\langle\alpha\beta|_{i,i+1}.$$  

We then sum these together and add some additional constraints at the beginning and end of the chain.

$$H_{\text{valid}} = (I - |F\rangle\langle F| - |TR\rangle\langle TR|)_{0} + (I - |N\rangle\langle N|)_{L+1} + \sum_{i=0}^{L} H_{\text{legal}}^i.$$  

Note that $H_{\text{valid}}$ is a diagonal matrix with non-negative integers along the diagonal.

**Lemma 10** Consider a state $|\phi\rangle$ contained in a subspace specified by a template. $H_{\text{valid}}|\phi\rangle = \lambda|\phi\rangle$ for some non-negative integer $\lambda$. If $\lambda = 0$, then the template must have a form specified by one of the following three regular expressions:

$$F^+B^+(R+G+L)Q^+N^+, \quad F^+T_LQ^+N^+, \quad F^+B^+TRN.$$  

**Proof:** If $H_{\text{valid}}|\phi\rangle = 0$, then every consecutive pair of characters in the template containing $|\phi\rangle$ must be an allowable pair. Furthermore, the first character must be $F$ or $T_R$ and the last character must be $N$. The graph in Figure 4 has every allowable pair labelled as a directed edge. Furthermore, the graph enforces that any path from Start to End must begin with an $F$ or a $T_R$ and must end with an $N$. Therefore, a valid template must correspond to a path that starts at the Start node, ends at the End node and traverses $L+2$ intermediate nodes. The length of the template is enforced by the physical length of the chain of particles. The set of all such paths correspond to the three regular expressions above with the constraint that the length must be $L+2$.  

We will label each valid template with a pair $(m,t)$. $m$ will designate the number of computation bits in a template. We describe here how to determine $t$ for a particular $m$. We describe how to determine two integers $i$ and $j$ and then let $t = i(2m + 3) + j$. The labelling will have four distinct cases, depending on the form of the template:

| Template | $i \leftarrow k$ | $j \leftarrow 0$ |
|----------|------------------|------------------|
| $F^kT_LQ^mN^{L-m-k+1}$ | $i \leftarrow k$ | $j \leftarrow l + 1$ |
| $F^{k+1}B^t(G+R)Q^mN^{L-m-k+1}$ | $i \leftarrow k$ | $j \leftarrow m + 1$ |
| $F^{k+1}B^mT_LN^{L-m-k+1}$ | $i \leftarrow k$ | $j \leftarrow m + 2 + l$ |

Conversely, Figure 4 shows for a given pair $(m,t)$ the form of the templates corresponding to that pair, where $i = \lfloor t/(2m + 3) \rfloor$ and $j = t \mod (2m + 3)$. Note that for each pair, there is exactly one template unless $1 \leq j \leq m$ in which case there are two templates, depending on whether the particle in the control state is in a $G$ or an $R$ state. Define $T_m = (2m + 3)(L - m) + m + 1$. Also note that for a given $m$, and any $t \in \{0, \ldots, T_m\}$, $(m,t)$ corresponds to a valid template, but having
| Condition | Template | Forward | Backward |
|-----------|----------|---------|----------|
| \(j = 0\) | \(F \cdots FT_R Q \cdots Q NN \cdots N\) | \(T_RQ \rightarrow FR\) if (not boundary) \(T_RQ \rightarrow FG\) | \(FL \leftarrow FT_R\) |
| \(j = 1\) | \(F \cdots F F R Q \cdots QQ \cdots Q NN \cdots N\) | \(RQ \rightarrow BR\) if (not boundary) | \(T_RQ \leftarrow FR\) |
| \(1 < j < n\) | \(F \cdots F F B j Q \cdots Q j m \cdots j \cdots N N \cdots N\) | \(RQ \rightarrow BR\) if (boundary) \(RQ \leftarrow BR\) | \(GQ \rightarrow BG\) \(GQ \leftarrow BG\) |
| \(j = n\) | \(F \cdots F F B \cdots BR N N \cdots N\) | \(RN \rightarrow BT\) if (not boundary) | \(RQ \leftarrow BR\) |
| \(j = n\) | \(F \cdots F F B \cdots BG N N \cdots N\) | \(GN \rightarrow BT\) if (boundary) | \(GQ \leftarrow BG\) |
| \(j = n + 1\) | \(F \cdots F F B \cdots B T_LN \cdots N\) | \(T_LN \rightarrow LN\) if (not boundary) \(RN \leftarrow BT_L\) \(GN \leftarrow BT_L\) | |
| \(j = n + 2\) | \(F \cdots F F B \cdots B \cdots BL N \cdots N\) | \(BL \rightarrow LQ\) \(T_LN \leftarrow LN\) | |
| \(n + 2 < j < 2n + 2\) | \(F \cdots F F B j Q \cdots Q j m \cdots j \cdots LQ \cdots N N \cdots N\) | \(BL \rightarrow LQ\) | \(BL \leftarrow LQ\) |
| \(j = 2n + 2\) | \(F \cdots F F LQ \cdots Q j m \cdots m \cdots N N \cdots N\) | \(FL \rightarrow FT_R\) | \(LQ \leftarrow BL\) |

Figure 3: Generalized Templates

\(t > T_m\) implies that the last particle will not be in state \(N\), which makes the template invalid. If a template is labelled \((m, 0)\), then we will call it an *initial* template. If it is labelled \((m, T_m)\), then we will call it a *final* template.

Figure 4 also shows for each template which terms in \(H_{\text{prop}}\) apply in the forward direction and which apply in the backward direction. The condition (boundary) indicates whether a pair straddles two blocks of particles. That is, if a pair of particles are located in positions \(i\) and \(i + 1\) and \(i \mod n = 0\).

**Lemma 11** Consider a valid template \(T\) labelled with \((m, t)\). At most one term in \(H_{\text{prop}}\) applies to \(T\) in the forward direction. Furthermore, when \(F\) is applied to a state in that template, the result is \(0\) or a state in an \((m, t + 1)\)-template. At most one term in \(H_{\text{prop}}\) applies to \(T\) in the backward direction. Furthermore, when \(B\) is applied to a state in that template, the result is \(0\) or a state in an \((m, t − 1)\)-template.
Proof: The proof consists of verifying that for each template in Figure 4, at most one term applies in the forward direction (and in only one place) and exactly one term applies in the backward direction (again in only one place). When the term is applied in the forward direction, the result is the next template in the sequence and when the term is applied in the backward direction, the result is the previous template in the sequence. □

Lemma 12 Consider a state $|\phi\rangle$ contained in the subspace corresponding to a template $T$. If $T$ is an initial template, then $B|\phi\rangle = 0$. If $T$ is a final template, then $F|\phi\rangle = 0$.

Proof: There is no term in $H_{\text{prop}}$ that applies to an initial template in the backward direction. Furthermore, since $H_{GN\rightarrow BT}^L$ is removed from $H_{\text{prop}}^L$, there is no term in $H_{\text{prop}}$ that applies to a final template in the forward direction. □

We define another Hamiltonion $H_{\text{legal}}$ which penalizes any template for which there is no term in $H_{\text{prop}}$ that applies in the forward direction or for which there is no term that applies in the forward direction (unless it is an initial or final template, respectively). Using the table in Figure 4, we want to forbid pairs $RN$ and $FR$ from crossing a boundary. We will also forbid pairs $GQ$ and $BG$ from crossing a boundary. In addition, we want to forbid pairs $GN$ and $FG$ unless they cross a boundary.

For any $i$ such that $i \mod n = 0$, we have
$$H_{i\text{legal}} = |RN\rangle\langle RN|_i + |FR\rangle\langle FR|_i + |GQ\rangle\langle GQ|_i + |BG\rangle\langle BG|_i.$$  
For any $i$ such that $i \mod n \neq 0$, we have
$$H_{i\text{legal}}^i = |GN\rangle\langle GN|_i + |FG\rangle\langle FG|_i.$$  
As usual, summing over all the subscripts of the computation states is assumed. Finally, we sum these together
$$H_{\text{legal}} = \sum_{i=0}^{L} H_{i\text{legal}}.$$  
We say that a template is legal if any state $|\phi\rangle$ in that template has $H_{\text{legal}}|\phi\rangle = 0$. Otherwise, $H_{\text{legal}}|\phi\rangle \geq 1$ and we say the template is illegal.

Lemma 13 Consider a template $T$ that is both valid and legal. If $T$ is not a final template, then there is a term in $H_{\text{prop}}$ that applies to $T$ in the forward direction. Similarly, if $T$ is not an initial template, there is a term in $H_{\text{prop}}$ that applies to $T$ in the backward direction.

Proof: The proof consists of the observation that any template in Figure 4 for which there is no term in $H_{\text{prop}}$ that applies in the forward direction is made illegal by $H_{\text{legal}}$. Similarly, any template for which there is no term in $H_{\text{prop}}$ that applies in the backward direction is made illegal by $H_{\text{legal}}$. □

We can think of all the valid templates as nodes in a directed graph. There is a directed edge from templates $T$ to $T'$ if applying $F$ to some state in $T$ results in a state in $T'$. (By definition then, applying $B$ to a state in $T'$ results in a state in $T$). We will call this graph the template graph and will refer to nodes and templates interchangeably. Lemmas 11, 12 and 13 imply that this graph
consists of a set of disjoint chains. All the nodes in a chain correspond to templates with the same number of computation particles. Furthermore, the starting node in a maximal chain is either an initial template or an illegal one. The last node in a maximal chain is either a final template or an illegal one.

**Lemma 14** There is exactly one chain in the template graph that contains no illegal nodes. Furthermore, templates in this chain have \( n \) computation particles.

**Proof:** Consider a maximal chain with no illegal nodes. This chain must begin with an initial template and end with a final one. The initial template has the form \( TRQ^mN^{L-m+1} \) for some \( m \). By the forward rule \( TRQ \rightarrow FG \), the next node in the chain is \( FGQ^{m-1}N^{L-m+1} \). If \( m < n \), then \( m - 1 \) applications for the forward rule \( GQ \rightarrow BG \) will result in \( FB^{m-1}GN^{L-m-1} \). Since \( m < n \), this will result in a pair \( GN \) in locations \( m \) and \( m + 1 \) which does not straddle a block boundary. This is made illegal in \( H_{\text{legal}} \). If \( m > n \), then \( n - 1 \) applications of the forward rule \( GQ \rightarrow BG \) to \( FGQ^{m-1}N^{L-m+1} \) will result in \( FB^{n-1}GQ^{m-n}N^{L-m+1} \). This will result in a pair \( GQ \) in locations \( n \) and \( n + 1 \) which is also disallowed in \( H_{\text{legal}} \).

Finally, if \( m = n \), there is exactly one chain from an initial template to a final one (because there is exactly one initial template). The application of the forward rules do not result in any illegal templates. \( \blacksquare \)

It will be convenient at this point to define the remaining two terms in \( H \). The input to the quantum verifier will be \( n \) qubits. \( n_1 \) qubits will be ancillary qubits that are initialized to 0 and \( n_2 \) qubits will be the witness \( \xi \). \( n_1 + n_2 = n \). We force \( x = 0 \) with the following Hamiltonian:

\[
H_{\text{input}} = \sum_{i=1}^{n_1} |Q_1\rangle\langle Q_1|_i + |R_1\rangle\langle R_1|_i + |G_1\rangle\langle G_1|_i + |B_1\rangle\langle B_1|_i.
\]

We will assume that the output will be present in the rightmost qubit in the computation. Therefore, we will have \( H_{\text{out}} \) defined to be \( |G_0\rangle\langle G_0|_L \). Observe that \( H_{\text{input}} \) and \( H_{\text{out}} \) are both closed over the subspace defined by each template.

A maximal chain in the template graph defines a subspace which is just the subspace spanned by all the subspaces defined by the templates along the chain. \( H_{\text{prop}} \) is closed over the subspace defined by any maximal chain in the template graph. All the other terms in \( H \) are closed over the subspace defined by each template.

Define \( S_{\text{legal}} \) to be the subspace defined by the unique chain containing only legal nodes. Let \( S_{\text{legal}}^\perp \) be the orthogonal subspace to \( S_{\text{legal}} \). Since \( H \) is closed under \( S_{\text{legal}} \), it is also closed under \( S_{\text{legal}}^\perp \) and any eigenvector of \( H \) must be contained in \( S_{\text{legal}} \) or \( S_{\text{legal}}^\perp \).

**Lemma 15** Any eigenvector of \( H \) in \( S_{\text{legal}}^\perp \) will have an eigenvalue of at least \( \Omega(1/L^4) \).

**Proof:** Define \( S_{\text{valid}} \) to be the subspace spanned by all the valid templates. Since \( H \) is closed over \( S_{\text{valid}} \), it is also closed under the orthogonal space \( S_{\text{valid}}^\perp \). Any state eigenstate of \( H_{\text{valid}} \) in \( S_{\text{valid}}^\perp \) will have an eigenvalue of at least 1. Since the remaining terms in \( H \) are positive semi-definite, any eigenstate of \( H \) in \( S_{\text{valid}} \) will also have an eigenvalue of at least 1.

Now we can carve up \( H_{\text{valid}} \) into the subspaces defined by the maximal chains and \( H \) is closed on each such subspace. We focus on an arbitrary such maximal chain (except the one containing only legal nodes) and the subspace it defines. The chain goes from an \((m, t_1)\) template to an
\((m, t_2)\)-template for some \(m, t_1\) and \(t_2\). To specify a state in the first template, we specify an \(m\)-bit \(x\) string which will determine the subscripts of particles in the computation states. We call this state \(|\phi_{x,m,t_1}\rangle\). The set of these states for all \(x\) forms a basis of the first template. Since \(\mathcal{F}\) is unitary, when we apply \(\mathcal{F}\) to all the \(|\phi\rangle\)'s, we get a basis of the next template in the chain. Applying \(\mathcal{F}\) \(i - 1\) times gives a basis of the \(i\)th template in the chain. We will focus on a sequence \(|\phi_{x,m,t_1}, \ldots, |\phi_{x,m,t_2}\rangle\), where \(\mathcal{F}^i|\phi_{x,m,t_1}\rangle = |\phi_{x,m,t_1+i}\rangle\). The subspace spanned by these states is closed under \(H_{\text{prop}} + H_{\text{legal}}\). Furthermore \(H_{\text{prop}}\) when restricted to this subspace and expressed in the basis of \(\phi\)'s is \(P_r\), for \(r = t_2 - t_1 + 1\). \(H_{\text{legal}}\) when expressed in this basis is diagonal with non-negative integer entries. Furthermore, we know that there is at least one positive entry on the diagonal because the chain has at least one illegal node. By Lemma 4, we know that the lowest eigenvalue of any eigenstate in the subspace defined by the chain must have an eigenvalue that is at least \(\Omega(1/\langle T_n\rangle^2)\). Since the remaining terms in \(H\) are all positive semi-definite, the lower bound holds for \(H\) as well. □

We will prove the following theorem:

**Theorem 16** If the circuit \(V\) accepts with probability at least \(1 - \epsilon\) on some input \(|0\xi\rangle\), then \(H\) has an eigenvalue smaller than \(\epsilon\). If the circuit \(V\) accepts with probability less than \(\epsilon\) on all inputs \(|0\xi\rangle\), then all eigenvalues of \(H\) are larger than \(1\) over a polynomial in \(n\) and \(L\).

The input to \(V\) consists of \(n_1\) auxiliary qubits and \(n_2\) witness qubits, where \(n_1 + n_2 = n\). Together a string \(x\) of \(n_1\) bits and \(\xi\) of \(n_2\) bits defines an input to \(V\). Let \(|\phi_{x,\xi,0}\rangle\) be the state in template \((n, 0)\) such that the input bits are set to \(x\) and \(\xi\). Let \(|\phi_{x,\xi,t}\rangle = \mathcal{F}^t|\phi_{x,\xi,0}\rangle\) for \(t \in \{1, \ldots, T_n\}\). The space \(S_{\text{legal}}\) is spanned by these \(|\phi\rangle\)'s.

We define

\[
|\nu_{x,\xi}\rangle = \frac{1}{\sqrt{T_n + 1}} \sum_{i=0}^{T_n} |\phi_{x,\xi,i}\rangle.
\]

The following lemma established Theorem 16 in one direction.

**Lemma 17** If there is a \(\xi\) such that \(V\) accepts with probability at least \(1 - \epsilon\) on some input \(|0\xi\rangle\), then the smallest eigenvalue of \(H\) is at most \(\epsilon\).

**Proof:** Let \(|\nu\rangle = |\nu_{0,\xi}\rangle\).

\[
\langle \nu | H_{\text{prop}} | \nu \rangle = \langle \nu | H_{\text{legal}} | \nu \rangle = \langle \nu | H_{\text{valid}} | \nu \rangle = \langle \nu | H_{\text{in}} | \nu \rangle = 0.
\]

For \(t \in \{0, \ldots, T_n - 1\}\), particle \(L\) is not in a \(G\) state, so the conditions of \(H_{\text{out}}\) are satisfied. For \(t = T_n\), if the system is in state \(|\phi_{0,\xi,t}\rangle\) and the \(L^{th}\) particle is measured, the probability that the outcome is \(G_0\) is at most \(\epsilon\). Therefore \(\langle \nu | H_{\text{out}} | \nu \rangle \leq \epsilon\). □

The final step is the following lemma:

**Lemma 18** If for all \(\xi\), \(V\) accepts with probability at most \(\epsilon \leq 1/2\) on input \(|0\xi\rangle\), then the smallest eigenvalue of \(H\) in \(S_{\text{legal}}\) is at least an inverse polynomial in \(L\).

**Proof:** We will use Lemma 5. With \(H_1 = H_{\text{prop}}\) and \(H_2 = H_{\text{in}} + H_{\text{out}}\). All of \(S_{\text{legal}}\) is zero on the other two terms, \(H_{\text{legal}}\) and \(H_{\text{valid}}\). The ground space of \(H_{\text{prop}}\) is spanned by the \(|\nu_{x,\xi}\rangle\)'s, and the smallest non-zero eigenvalue is \(\Omega(1/(T_n)^2)\). Similarly for \(H_2\), the ground space can be expressed as
any linear combination of states that have $G_1$ in particle $L$ and do not have $Q_1$, $R_1$, $G_1$ or $B_1$ in particles 1 through $n$. Any non-zero eigenvalue is at least 1.

Let $P_2$ be the projection onto the ground space of $H_2$. The cosine of the angle between any state $|\phi\rangle$ and the ground state of $H_2$ is just $\langle \phi | P_2 | \phi \rangle$. We will show that for any $|\nu_{x,\xi}\rangle$, the cosine of the angle between $|\nu_{x,\xi}\rangle$ and the ground space of $H_2$

$$\langle \nu_{x,\xi} | P_2 | \nu_{x,\xi} \rangle \leq 1 - \frac{1}{T_n + 1}.$$  

Thus, for small enough $\epsilon$, the sine of the angle between the ground space of $H_1$ and the ground space of $H_2$ is at least $\Omega(1/T_n)$.

The amplitude of $|\phi_{x,\xi,0}\rangle$ in $|\nu_{x,\xi}\rangle$ is at least $1/\sqrt{T_n + 1}$. If $x \neq 0$, this state has a $Q_1$ somewhere in the first $n$ particles and $\langle \phi_{x,\xi,0} | P_2 | \phi_{x,\xi,0} \rangle = 0$. Therefore, $\langle \nu_{x,\xi} | P_2 | \nu_{x,\xi} \rangle \leq 1 - \frac{1}{T_n + 1}$. If $x = 0$, then we know that for all $\xi$,

$$\langle \Phi_{0,\xi,T_n} | |G_1\rangle \langle G_1|_L | \Phi_{0,\xi,T_n} \rangle \leq \epsilon,$$

because this is the probability that the outcome on input $\xi$ is 1. This means that

$$\langle \nu_{0,\xi} | P_2 | \nu_{0,\xi} \rangle \leq 1 - \frac{1}{T}.$$  

$\blacksquare$