A SPHERE THEOREM FOR BACH-FLAT MANIFOLDS WITH
POSITIVE CONSTANT SCALAR CURVATURE

YI FANG AND WEI YUAN

ABSTRACT. We show a closed Bach-flat Riemannian manifold with a fixed positive constant
scalar curvature has to be locally spherical if its Weyl and traceless Ricci tensors are small
in the sense of either \( L^\infty \) or \( L^2 \)-norm. Compared with the complete non-compact case done
by Kim, we apply a different method to achieve these results. These results generalize a
rigidity theorem of positive Einstein manifolds due to M.-A. Singer. As an application, we
can partially recover the well-known Chang-Gursky-Yang’s 4-dimensional conformal sphere
theorem.

1. Introduction

The notion of Bach tensor was first introduced by Rudolf Bach in 1921 (see [1]) when
studying the so-called conformal gravity. That is, instead of using the Hilbert-Einstein
functional, one consider the functional

\[ W(g) = \int_{M^4} |W(g)|^2 dv_g \]
on 4-dimensional manifolds. The corresponding critical points of this functional are charac-
terized by the vanishing of certain symmetric 2-tensor \( B_g \). The tensor \( B_g \) is usually referred
as Bach tensor and the metric is called Bach-flat, if \( B_g \) vanishes.

Let \((M^n, g)\) be an \( n \)-dimensional Riemannian manifold \((n \geq 4)\). The Bach tensor is
defined to be

\[ B_{jk} = \frac{1}{n-3} \nabla^i \nabla^l W_{ijkl} + W_{ijkl} S^{il}, \tag{1.1} \]

where

\[ S_{jk} = \frac{1}{n-2} \left( R_{jk} - \frac{1}{2(n-1)} R g_{jk} \right) \tag{1.2} \]
is the Schouten tensor.

Using the Cotton tensor

\[ C_{ijk} = \nabla_i S_{jk} - \nabla_j S_{ik} \tag{1.3} \]

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1
and the relation
\[ \nabla^i W_{ijkl} = (n - 3) C_{ijk}, \]
we can extend the definition of Bach tensor such that it can be defined for 3-dimensional manifolds:

**Definition 1.1.** For any \( n \geq 3 \), the Bach tensor is defined to be
\[ B_{jk} = \nabla^i C_{ijk} + W_{ijkl} S_{il}. \]
We say a metric is *Bach-flat*, if its Bach tensor vanishes.

Typical examples of Bach flat metrics are Einstein metrics and locally conformally flat metrics. Due to the conformal invariance of Bach-flatness on 4-manifolds, metric conformal to Einstein metrics are also Bach-flat. For 4-dimensional manifolds, it also includes half-locally conformally flat metrics. In general, Tian and Viaclovsky studied the module space of 4-dimensional Bach-flat manifolds (cf. \[7, 8\]). Besides these known "trivial" examples, there are not many examples known about generic Bach-flat manifolds so far. In fact, in some particular situations, one would expect rigidity phenomena occur.

In \[5\], Kim shows that on a complete non-compact 4-dimensional Bach-flat manifold \((M, g)\) with zero scalar curvature and positive Yamabe constant has to be flat, if the \(L^2(M, g)\)-norm of its Riemann curvature tensor is sufficiently small. This result can be easily extended to any dimension \( n \geq 3 \).

Kim’s proof is based on a classic idea that one can get global rigidity from local estimates: applying the ellipticity of Bach-flat metric, the Sobolev’s inequality and together the smallness of \( ||Rm||_{L^2(M, g)} \), one can get the estimate
\[ ||Rm||_{L^1(B_r, g)} \lesssim \frac{C}{r} ||Rm||_{L^2(M, g)} \]
for any fixed \( p \in M \) and \( r > 0 \). Now the conclusion follows by letting \( r \to \infty \).

This method can also be used in various problems, for example, see \[3\]. However, note that the assumption of non-compactness is essential here. One cannot get the rigidity by simply letting \( r \to \infty \), when the manifold is compact without boundary for instance.

Is it possible for us to have a result similar to Kim’s but on closed manifolds? Here by *closed manifolds*, we mean compact manifolds without boundary. In fact, Singer proved that even dimensional closed positive Einstein manifolds with non-vanishing Euler characteristic have to be locally spherical, provided the \( L^+ \)-norm of its Weyl tensor is small (cf. \[6\]). As a special case of Bach-flat metric, this result suggests that this phenomenon might occur in a larger class.

Applying a global estimate for symmetric 2-tensors (see Proposition \[2, 3\]), we can prove the following result:
**Theorem A.** Suppose \((M^n, g)\) is a closed Bach-flat Riemannian manifold with constant scalar curvature
\[ R_g = n(n - 1). \]

If
\[(1.6) \quad ||W||_{L^\infty(M,g)} + ||E||_{L^\infty(M,g)} < \varepsilon_0(n) := \frac{n - 1}{4}, \]
then \((M, g)\) is isometric to a quotient of the round sphere \(\mathbb{S}^n\).

**Remark 1.2.** Note that in Theorem A we do not assume the Yamabe constant is uniformly positively lower bounded. This assumption will be needed in Theorem B. It is equivalent to the existence of a uniform Sobolev’s inequality (see section 4), which was applied frequently in the proof of Theorem B.

Another one by assuming integral conditions:

**Theorem B.** Suppose \((M^n, g)\) is a closed Bach-flat Riemannian manifold with constant scalar curvature
\[ R_g = n(n - 1). \]

Assume that there is a constant \(\alpha_0\) such that its Yamabe constant satisfies that
\[(1.7) \quad Y(M, [g]) \geq \alpha_0 > 0. \]

Then \((M, g)\) is isometric to a quotient of the round sphere \(\mathbb{S}^n\), if
\[(1.8) \quad ||W||_{L^{\frac{2}{\alpha_0}}(M,g)} + ||E||_{L^{\frac{2}{\alpha_0}}(M,g)} < \tau_0(n, \alpha_0) := \frac{3\alpha_0}{32n(n - 1)}. \]

**Remark 1.3.** Bach-flat metrics is one of the typical examples of the so-called critical metrics (cf. [7]). By replacing the presumption Bach-flatness with harmonic curvature, which refers to the vanishing of Cotton tensor when the scalar curvature is a constant, the corresponding version of Theorem A and B are still valid without any essential difficulty.

In particular, applying Theorem B for 4-dimensional manifolds, we can partially recover the well-known 4-dimensional conformal sphere theorem by Chang-Gursky-Yang (cf. [2]; for a generalization see [4]):

**Theorem C.** Suppose \((M^4, g)\) is a closed Bach-flat Riemannian manifold. Assume that there is a constant \(\alpha_0\) such that its Yamabe constant satisfies that
\[(1.9) \quad Y(M, [g]) \geq \alpha_0 > 0. \]

Then \((M, g)\) is conformal to the round sphere \(\mathbb{S}^4\) or its canonical quotient \(\mathbb{R}P^4\), if
\[(1.10) \quad \int_{M^4} |W_g|^2 dv_g < \frac{32}{3} \pi^2 (\chi(M^4) - 2) + \frac{\alpha_0}{192}. \]

**Remark 1.4.** It was shown in [2] that \((M^4, g)\) is conformal to \((\mathbb{C}P^2, g_{FS})\) or a manifold covered isometrically by \(S^1 \times S^3\) endowed with the canonical product metric, if we assume
\[(1.11) \quad \int_{M^4} |W_g|^2 dv_g = 16\pi^2 \chi(M^4) \]
instead.
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2. $\theta$-Codazzi Tensor and Related Inequality

We define a concept which generalizes the classic Codazzi tensor:

**Definition 2.1.** For any $\theta \in \mathbb{R}$, we say a symmetric 2-tensor $h \in S_2(M)$ is a $\theta$-Codazzi tensor if

$$C_{\theta}(h)_{ijk} := \nabla_i h_{jk} - \theta \nabla_j h_{ik} = 0.$$ (2.1)

In particular, $h$ is referred to be a Codazzi tensor or anti-Codazzi tensor if $\theta = 1$ or $\theta = -1$ respectively.

The motivation for us to define this notion is the following identity associated to it:

**Lemma 2.2.** Suppose $(M, g)$ is a closed Riemannian manifold with constant scalar curvature $R_g = n(n-1)\lambda$.

Then for any $h \in S_2(M)$ and $\theta \in \mathbb{R}$,

$$\int_M \left( |\nabla h|^2 - \frac{1}{1+\theta^2} |C_{\theta}(h)|^2 \right) dvol_g$$

$$= \frac{2\theta}{1+\theta^2} \int_M \left[ |\delta h|^2 + W(\circ h, \circ h) + \frac{2}{n-2}(tr h) E \cdot h - \frac{n}{n-2}tr(E \times h^2) - n\lambda |\hat{h}|^2 \right] dvol_g,$$

where $\hat{h} := h - \frac{1}{n}(tr h)g$ is the traceless part of the tensor $h$.

**Proof.** We have

$$\int_M \nabla_i h_{jk} \nabla^j h^{ik} dvol_g$$

$$= - \int_M \nabla_j \nabla_i h^j_{ik} dvol_g$$

$$= - \int_M (\nabla_i \nabla_j h_{k}^{j} + R_{ju}^{j} h_{k}^{l} - R_{ji}^{l} h_{k}^{j} ) h^{ik} dvol_g$$

$$= - \int_M (\nabla_i (\delta h)_{k} + R_{id} h_{k}^{l} - R_{jikl} h^{jl} ) h^{ik} dvol_g$$

$$= \int_M \left[ |\delta h|^2 - (E_{id} h_{k}^{l} + (n-1)\lambda g_{id} h_{k}^{l} ) h^{ik} \right] dvol_g$$

$$+ \int_M \left( W_{jikt} + \frac{2}{n-2} (E_{jikg} - E_{jkig}) + \lambda (g_{jigk} - g_{jkgi}) \right) h^{jl} h^{ik} dvol_g$$

$$= \int_M \left[ |\delta h|^2 + W(h, h) + \lambda ((tr h)^2 - n|h|^2) + \frac{2}{n-2}(tr h) E \cdot h - \frac{n}{n-2}tr(E \times h^2) - n\lambda |\hat{h}|^2 \right] dvol_g.$$


Thus for any $\theta \in \mathbb{R}$,
\[
\int_M |C_\theta(h)|^2 dv_g
= \int_M |\nabla_i h_{jk} - \theta \nabla_j h_{ik}|^2 dv_g
= \int_M \left[ (1 + \theta^2)|\nabla h|^2 - 2\theta \nabla_i h_{jk} \nabla^j h^{ik} \right] dv_g
= \int_M \left[ (1 + \theta^2)|\nabla h|^2 - 2\theta \left( |\delta h|^2 + W(\hat{h}, \hat{h}) + \frac{2}{n-2}(tr h) E \cdot h - \frac{n}{n-2} tr(E \times h^2) - n\lambda |\hat{h}|^2 \right) \right] dv_g.
\]
That is,
\[
\int_M \left( |\nabla h|^2 - \frac{1}{1 + \theta^2} |C_\theta(h)|^2 \right) dv_g
= \frac{2\theta}{1 + \theta^2} \int_M \left[ |\delta h|^2 + W(\hat{h}, \hat{h}) + \frac{2}{n-2}(tr h) E \cdot h - \frac{n}{n-2} tr(E \times h^2) - n\lambda |\hat{h}|^2 \right] dv_g.
\]

From this, we get the following inequality:

**Proposition 2.3.** Suppose $(M, g)$ is a closed Riemannian manifold with constant scalar curvature $R_g = n(n-1)\lambda$.

Then for any $h \in S_2(M)$ and $\theta \in \mathbb{R},$
\[
(2.3) \quad \int_M |\nabla h|^2 dv_g \geq \frac{2\theta}{1 + \theta^2} \int_M \left[ |\delta h|^2 + W(\hat{h}, \hat{h}) + \frac{2}{n-2}(tr h) E \cdot h - \frac{n}{n-2} tr(E \times h^2) - n\lambda |\hat{h}|^2 \right] dv_g,
\]
where equality holds if and only if $h$ is a $\theta$-Codazzi tensor.

In particular, we have

**Corollary 2.4.** Suppose $(M, g)$ is a closed Riemannian manifold with constant scalar curvature $R_g = n(n-1)\lambda$.

Then the traceless part of Ricci tensor satisfies
\[
(2.4) \quad \int_M |\nabla E|^2 dv_g \geq \frac{2\theta}{1 + \theta^2} \int_M \left[ W(E, E) - \frac{n}{n-2} tr E^3 - n\lambda |E|^2 \right] dv_g,
\]
In particular when $\theta = 1$, the equality holds if and only if $g$ is of harmonic curvature.

**Proof.** By the second Bianchi identity, we can easily see that
\[
\delta E = -\frac{n-2}{2n} dR_g = 0.
\]
Note that $tr E = 0$, thus the conclusion follows.
When $\theta = 1$, $E$ is a Codazzi tensor if and only if the Cotton tensor vanishes:

$$C_{ijk} = \frac{1}{n - 2} Alt_{i,j} \left( \nabla_i R_{jk} - \frac{1}{2(n-1)} g_{jk} \nabla_i R \right) = 0.$$ 

\[\square\]

3. $L^\infty$-sphere theorem

We can rewrite the Bach tensor in terms of traceless Ricci tensor:

**Lemma 3.1.** The Bach tensor can be expressed as follow

$$B_g = \frac{1}{n-2} \Delta g E - \frac{1}{2(n-1)} \left( \nabla^2 g R - \frac{1}{n} g \Delta g R \right) + \frac{2}{n-2} \hat{W} \cdot E$$

$$- \frac{n}{(n-2)^2} \left( E \times E - \frac{1}{n} |E|^2 g \right) - \frac{1}{(n-1)(n-2)} R E,$$

where $(\hat{W} \cdot E)_{jk} := W_{ijkl} E^{il}$.

**Proof.** By definition,

$$\nabla^i C_{ijk} = \nabla^i (\nabla_i S_{jk} - \nabla_j S_{ik})$$

$$= \Delta g S_{jk} - (\nabla_j \nabla_i S^i_k + R^i_{ijp} S^p_k - R^p_{ijk} S^i_p)$$

$$= \Delta g S_{jk} - \nabla_j \nabla_k tr S - (Ric \times S)_{jk} + (\hat{Rm} \cdot S)_{jk},$$

where we used the fact

$$\nabla_i S^i_k = \nabla_k tr S$$

by the contracted second Bianchi identity.

Since

$$S = \frac{1}{n-2} E + \frac{R}{2n(n-1)} g$$

and

$$Rm = W + \frac{1}{n-2} E \otimes g + \frac{R}{2n(n-1)} g \otimes g,$$

the conclusion follows by substituting them into

$$B_{jk} = \nabla^i C_{ijk} + W_{ijkl} E^{il}.$$ 

\[\square\]

As the first step, we show the metric has to be Einstein under given presumptions:

**Proposition 3.2.** For $n \geq 3$, there exists a constant $\Lambda_n > 0$ only depends on $n$, such that any closed Bach flat Riemannian manifold $(M^n, g)$ with constant scalar curvature

$$R_g = n(n-1)$$

and

$$||W_g||_{L^\infty(M,g)} + ||E_g||_{L^\infty(M,g)} < \Lambda_n := \frac{n}{3}$$

has to be Einstein.
Proof. Since the scalar curvature $R_g$ is a constant, by Lemma 3.1,

$$B_g = \frac{1}{n-2} \Delta_g E + \frac{2}{n-2} \overset{\circ}{W} \cdot E - \frac{n}{(n-2)^2} \left( E \times E - \frac{1}{n} |E|^2 g \right) - \frac{n}{n-2} E = 0.$$ 

That is,

$$\Delta_g E + 2 \overset{\circ}{W} \cdot E - \frac{n}{n-2} \left( E \times E - \frac{1}{n} |E|^2 g \right) - nE = 0.$$ 

Thus,

$$-E \Delta_g E = 2W(E, E) - \frac{n}{n-2} \text{tr}(E^3) - n|E|^2$$

and hence

$$(3.2) \quad \int_M |\nabla E|^2 dv_g = - \int_M E \Delta_g Edv_g = \int_M \left( 2W(E, E) - \frac{n}{n-2} \text{tr}(E^3) - n|E|^2 \right) dv_g.$$ 

On the other hand, from Corollary 2.4,

$$\int_M |\nabla E|^2 dv_g \geq \frac{2\theta}{1 + \theta^2} \int_M \left( W(E, E) - \frac{n}{n-2} \text{tr}E^3 - n|E|^2 \right) dv_g,$$

for any $\theta \in \mathbb{R}$. Therefore,

$$(3.3) \quad \frac{2(1 - \theta + \theta^2)}{(1 - \theta)^2} \int_M W(E, E) dv_g \geq \frac{n}{n-2} \int_M \left( \text{tr}E^3 + (n-2)|E|^2 \right) dv_g.$$ 

Since

$$\int_M W(E, E) dv_g \leq \|W\|_{L^\infty(M, g)} \int_M |E|^2 dv_g,$$

by taking $\theta = -1$, we get

$$\frac{n}{n-2} \int_M (\text{tr}E^3 + (n-2)|E|^2) dv_g \leq \frac{3}{2} \|W\|_{L^\infty(M, g)} \int_M |E|^2 dv_g.$$ 

That is,

$$\frac{n}{n-2} \int_M \text{tr}E^3 dv_g \leq \left( \frac{3}{2} \|W\|_{L^\infty(M, g)} - n \right) \int_M |E|^2 dv_g.$$ 

From the inequality

$$\int_M \text{tr}E^3 dv_g \geq - \int_M |E|^3 dv_g \geq -\|E\|_{L^\infty(M, g)} \int_M |E|^2 dv_g,$$

we have

$$\left( \frac{3}{2} \|W\|_{L^\infty(M, g)} + \frac{n}{n-2} \|E\|_{L^\infty(M, g)} - n \right) \int_M |E|^2 dv_g \geq 0.$$ 

Therefore for any metric $g$ satisfies

$$\|W\|_{L^\infty(M, g)} + \|E\|_{L^\infty(M, g)} < \Lambda_n := \frac{n}{3},$$

we have $E = 0$. □

It is well-known that the Weyl tensor satisfies an elliptic equation on Einstein manifolds (cf. [6]):
Lemma 3.3. Let \((M^n, g)\) be an Einstein manifold with scalar curvature
\[ R_g = n(n - 1)\lambda, \]
then its Weyl tensor satisfies
\[ \Delta_g W - 2(n - 1)\lambda W - 2Q(W) = 0, \]
where \(Q(W) := B_{ijkl} - B_{jikl} + B_{ikjl} - B_{jkil} \) is a quadratic combination of Weyl tensors with \(B_{ijkl} := g^{pq}g^{rs}W_{prij}W_{qkls}.\)

Now we finish this section by proving one of our main theorem:

Proof of Theorem A. We take
\[ \varepsilon_0 := \min\{\Lambda_n, \frac{n - 1}{4}\} = \frac{n - 1}{4}. \]
From Proposition 3.2, we conclude that \(g\) is an Einstein metric. Applying Lemma 3.3, we have
\[ - \int_M \langle \Delta_g W - 2(n - 1)W, W \rangle dv_g = -2 \int_M \langle Q(W), W \rangle dv_g \leq 8 \int_M |W|^3 dv_g. \]
That is,
\[ \int_M (|\nabla W|^2 + 2(n - 1)|W|^2) dv_g \leq 8 \int_M |W|^3 dv_g. \]
(3.5)
Now we have
\[ 2(n - 1) \int_M |W|^2 dv_g \leq 8 \int_M |W|^3 dv_g \leq 8\|W\|_{L^\infty(M,g)} \int_M |W|^2 dv_g. \]
Thus the Weyl tensor vanishes, since
\[ \|W\|_{L^\infty(M,g)} < \varepsilon_0 = \frac{n - 1}{4}. \]
Therefore, the metric \(g\) is locally spherical. \(\Box\)

4. \(L^{\frac{n}{2}}\)-SPHERE THEOREM

Let \((M, g)\) be an Riemannian manifold. Suppose the Yamabe constant associated to it satisfies that
\[ Y(M, [g]) := \inf_{0 \neq u \in C^\infty(M)} \frac{\int_M \left( \frac{4(n-1)}{n-2} |\nabla u|^2 + R_g u^2 \right) dv_g}{\left( \int_M u^{\frac{2n}{n-2}} dv_g \right)^{\frac{n-2}{n}}} \geq \alpha_0 > 0. \]
By normalizing the scalar curvature such that \( R_g = n(n-1) \), we get
\[
\left( \int_M u^{n-2} dv_g \right)^{\frac{n-2}{n}} \leq \frac{1}{Y(M,[g])} \int_M \left( \frac{4(n-1)}{n-2} |\nabla u|^2 + R_g u^2 \right) dv_g
\]
\[
= \frac{n(n-1)}{Y(M,[g])} \int_M \left( \frac{4}{n(n-2)} |\nabla u|^2 + u^2 \right) dv_g
\]
\[
\leq \frac{4n(n-1)}{3Y(M,[g])} \int_M (|\nabla u|^2 + u^2) dv_g
\]
\[
\leq \frac{4n(n-1)}{3\alpha_0} \int_M (|\nabla u|^2 + u^2) dv_g
\]

Denote \( C_S := \frac{4n(n-1)}{3\alpha_0} > 0 \), we get the Sobolev’s inequality
\[
(\int_M u^{2n} dv_g)^{\frac{n-2}{n}} \leq C_S \int_M (|\nabla u|^2 + u^2) dv_g
\]

Note that, the constant \( C_S > 0 \) only depends on \( n \) and \( \alpha_0 \) and is independent of the metric \( g \).

**Lemma 4.1.** Let \((M^n, g)\) be a Bach flat Riemannian manifold with constant scalar curvature \( R_g = n(n-1) \).

Suppose there is a constant \( \alpha_0 \) such that its Yamabe constant satisfies that
\[
Y(M,[g]) \geq \alpha_0 > 0.
\]

Then \((M^n, g)\) is Einstein, if
\[
||W||_{L^\frac{2n}{n-2}(M,g)} + ||E||_{L^\frac{2n}{n-2}(M,g)} < \delta_0 := \frac{\alpha_0}{4n(n-1)} = \frac{1}{3C_S}.
\]

**Proof.** From equation (3.2) and Hölder’s inequality,
\[
\int_M |\nabla E|^2 dv_g = \int_M \left( 2W(E,E) - \frac{n}{n-2} tr(E^3) - n|E|^2 \right) dv_g
\]
\[
\leq \left( 2||W||_{L^\frac{2n}{n-2}(M,g)} + \frac{n}{n-2}||E||_{L^\frac{2n}{n-2}(M,g)} \right) ||E||_{L^\frac{2n}{n-2}(M,g)}^2 - n||E||_{L^2(M,g)}^2
\]
\[
\leq 3\delta_0 ||E||_{L^\frac{2n}{n-2}(M,g)}^2 - n||E||_{L^2(M,g)}^2.
\]

By Sobolev’s inequality (4.1) and the Kato’s inequality,
\[
||E||_{L^\frac{2n}{n-2}(M,g)}^2 \leq C_S \left( ||\nabla E||_{L^2(M,g)}^2 + ||E||_{L^2(M,g)}^2 \right) \leq C_S \left( ||\nabla E||_{L^2(M,g)}^2 + ||E||_{L^2(M,g)}^2 \right).
\]

Thus, we have
\[
||\nabla E||_{L^2(M,g)}^2 \leq 3\delta_0 C_S \left( ||\nabla E||_{L^2(M,g)}^2 + ||E||_{L^2(M,g)}^2 \right) - n||E||_{L^2(M,g)}^2
\]
\[
= ||\nabla E||_{L^2(M,g)}^2 - (n-1) ||E||_{L^2(M,g)}^2.
\]

Therefore, \( E \) vanishes identically on \( M \) and hence \((M,g)\) is Einstein. \( \square \)
Now we can show

**Proof of Theorem B.** From Lemma 4.1, \((M, g)\) has to be Einstein. Now from Sobolev’s inequality (4.1), Kato’s inequality and inequality (3.5), we have

\[
||W||^2_{L^{\frac{2n}{n-2}}(M,g)} \leq C_S \int_M (|
abla|W|^2 + |W|^2) \, dv_g \leq C_S \int_M (|
abla W|^2 + |W|^2) \, dv_g \leq 8C_S \int_M |W|^3 \, dv_g.
\]

Applying Hölder’s inequality,

\[
\int_M |W|^3 \, dv_g \leq ||W||_{L^\frac{2n}{n-2}(M,g)} ||W||_{L^\frac{2n}{n-2}(M,g)}^2
\]

and hence

\[
(1 - 8C_S ||W||_{L^\frac{2n}{n-2}(M,g)}) \, ||W||_{L^\frac{2n}{n-2}(M,g)}^2 \leq 0,
\]

which implies that \(W\) vanishes identically on \(M\) since

\[
||W||_{L^\frac{2n}{n-2}(M,g)} < \tau_0 := \frac{3\alpha_0}{32n(n-1)} = \frac{1}{8C_S}.
\]

Therefore, \((M, g)\) is isometric to a quotient of \(S^n\).

As for \(n = 4\), we have

**Proof of Theorem C.** Let \(\hat{g} \in [g]\) be the Yamabe metric, which means

\[
R_{\hat{g}} \left( Vol(M^4, \hat{g}) \right)^\frac{1}{2} = Y(M^4, [g]).
\]

We can also normalize it such that

\[
R_{\hat{g}} = 12.
\]

According to the solution of Yamabe problem,

\[
Y(M^4, [g]) \leq Y(S^4, g_{S^4}) = 12 \cdot \left( \frac{8}{3} \pi^2 \right)^\frac{1}{2} = 8\sqrt{6}\pi
\]

and hence

\[
Vol(M^4, \hat{g}) \leq Vol(S^4, g_{S^4}) = \frac{8}{3} \pi^2.
\]

From the Gauss-Bonnet-Chern formula,

\[
\int_{M^4} \left( Q_{\hat{g}} + \frac{1}{4} |E_{\hat{g}}|^2 \right) \, dv_{\hat{g}} = 8\pi^2 \chi(M^4),
\]

where

\[
Q_{\hat{g}} := -\frac{1}{6} \Delta_{\hat{g}} R_{\hat{g}} - \frac{1}{2} |E_{\hat{g}}|^2 + \frac{1}{24} R_{\hat{g}}^2
\]

is the Q-curvature for metric \(\hat{g}\). Thus,

\[
||E_{\hat{g}}||^2_{L^2(M, \hat{g})} = \frac{1}{2} ||W_{\hat{g}}||^2_{L^2(M, \hat{g})} + 12 Vol(M^4, \hat{g}) - 16\pi^2 \chi(M^4) \leq \frac{1}{2} ||W_{\hat{g}}||^2_{L^2(M, \hat{g})} + 16\pi^2 (2 - \chi(M^4)).
\]
and hence
\[
\|W_\hat{g}\|_{L^2(M,\hat{g})}^2 + \|E_\hat{g}\|_{L^2(M,\hat{g})}^2 \leq \frac{3}{2}\|W_\hat{g}\|_{L^2(M,\hat{g})}^2 + 16\pi^2(2 - \chi(M^4)) \\
= \frac{3}{2}\|W_\hat{g}\|_{L^2(M,\hat{g})}^2 + 16\pi^2(2 - \chi(M^4)) \\
< \frac{\alpha_0}{128},
\]
where we used the fact that \(\|W_\hat{g}\|_{L^2(M,\hat{g})}\) is conformally invariant for 4-dimensional manifolds.

On the other hand, the metric \(\hat{g}\) is also Bach-flat, since Bach-flatness is conformally invariant for 4-dimensional manifolds. Applying Theorem B to the Yamabe metric \(\hat{g}\), we conclude that \((M^4, \hat{g})\) is isometric to a quotient of the round sphere \(\mathbb{S}^4\).

For the quotient of an even dimensional sphere, only identity and \(\mathbb{Z}_2\)-actions make it a smooth manifold. Therefore, \((M^4, g)\) is conformal to \(\mathbb{S}^4\) or \(\mathbb{R}P^4\) with canonical metrics. \(\square\)

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(YI FANG) DEPARTMENT OF APPLIED MATHEMATICS, ANHUI UNIVERSITY OF TECHNOLOGY, MA’ANSHAN, ANHUI 243002, CHINA
E-mail address: flxy85@163.com

(WEI YUAN) DEPARTMENT OF MATHEMATICS, SUN YAT-sen UNIVERSITY, GUANGZHOU, GUANGDONG 510275, CHINA
E-mail address: gnr-x@163.com