COMPLETE REPRESENTATION BY PARTIAL FUNCTIONS
FOR COMPOSITION, INTERSECTION AND ANTIDOMAIN

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ABSTRACT. For representation by partial functions in the signature with intersection, composition and antidomain, we show that a representation is meet complete if and only if it is join complete. We show that a representation is complete if and only if it is atomic, but that not all atomic representable algebras are completely representable. We show that the class of completely representable algebras is not universally axiomatisable by first-order sentences. By giving an explicit representation, we show that the completely representable algebras form a basic elementary class.

1. INTRODUCTION

Whenever we have a concrete class of algebras whose operations are set-theoretic operations, we have a notion of a representation: an isomorphism from an abstract algebra to a concrete algebra. Then the representation class—the class of representable algebras—becomes an object of interest itself.

One possibility is for the concrete algebras to be algebras of partial functions, and for this scenario various signatures have been considered. Often, the representation classes have turned out to be finitely-axiomatisable varieties or quasi-varieties [6, 1, 4, 5].

Extra conditions we can impose on a representation are to require that it be meet complete or to require that it be join complete. A representation is meet complete if it turns any existing infima into intersections and join complete if it turns any existing suprema into unions. Hence we can define meet-complete representation classes and join-complete representation classes. In many important cases these two classes coincide. Bounded distributive lattices represented by rings of sets is an example where they do not [2].

In [3], Hirsch and Hodkinson showed that when the representation class is elementary, the complete representation class may (as is the case for Boolean algebras represented as fields of sets) or may not (relation algebras by binary relations) also be elementary.

In this paper we investigate complete representation by partial functions for the signature $(; \land, A)$ of composition, intersection and antidomain. In Section 2 we show that a representation is meet complete if and only if it is join complete. In Section 3 we show that a representation is complete if and only if it is atomic and we use this to show that the class of completely representable algebras is not universally axiomatisable by first-order sentences.

In Section 4 we investigate the validity of various distributive laws with respect to the classes of representable and completely representable $(; \land, A)$-algebras. This

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enables us to give an example of an algebra that is representable and atomic, but not completely representable.

In Section 5 we present an explicit representation, which we use, in Section 6, to prove our main result: the class of completely representable algebras is a basic elementary class.

2. Representations and Complete Representations

In this section we give preliminary definitions and then proceed to show that for the signature \((; \land, A)\), a representation by partial functions is meet complete if and only if it is join complete.

Given an algebra \(\mathfrak{A}\), when we write \(a \in \mathfrak{A}\) or say that \(a\) is an element of \(\mathfrak{A}\), we mean that \(a\) is an element of the domain of \(\mathfrak{A}\). Similarly for the notation \(S \subseteq \mathfrak{A}\) or saying that \(S\) is a subset of \(\mathfrak{A}\). If \(S\) is a subset of the domain of a map \(\theta\) then \(\theta[S]\) denotes the set \(\{\theta(s) \mid s \in S\}\).

**Definition 2.1.** Let \(\sigma\) be an algebraic signature whose symbols are a subset of \(\{; \land, 0, 1', D, R, A\}\). An algebra of partial functions of the signature \(\sigma\) is an algebra of the signature \(\sigma\) whose elements are partial functions and with operations given by the set-theoretic operations on those partial functions described in the following.

Let \(X\) be the union of the domains and ranges of all the partial functions. We call \(X\) the base. The binary operation \(;\) should be composition of partial functions. The binary operation \(\land\) should be intersection. (We identify partial functions with their graphs.) The constants 0 and 1’ should be the nowhere-defined function and the identity function on \(X\) respectively. The unary operations D and R should be the operations of taking the diagonals of the domain and the range of a function respectively. The unary operation A should be the operation of taking the diagonal of the antidomain of a function—those points of \(X\) where the function is not defined.

The list of operations in Definition 2.1 does not exhaust those that have been considered for partial functions, but does include the most commonly appearing operations.

**Definition 2.2.** Let \(\mathfrak{A}\) be an algebra of one of the signatures specified by Definition 2.1. A representation of \(\mathfrak{A}\) by partial functions is an isomorphism from \(\mathfrak{A}\) to an algebra of partial functions of the same signature. If \(\mathfrak{A}\) has a representation then we say it is representable.

**Theorem 2.3** (Jackson and Stokes [5]). The class of \((; \land, A)\)-algebras representable by partial functions is a finitely-based variety.

In fact in [5] a finite equational axiomatisation of the representation class is given, implicitly. So there exist known examples of such axiomatisations.

The next two definitions apply to any situation where the concept of a representation has been defined.

**Definition 2.4.** A representation \(\theta\) of a poset \(\mathcal{P}\) over the base \(X\) is **meet complete** if, for every nonempty subset \(S\) of \(\mathcal{P}\), if \(\land S\) exists, then

\[\theta(\land S) = \bigcap \theta[S].\]
Definition 2.5. A representation \( \theta \) of a poset \( \mathcal{P} \) over the base \( X \) is **join complete** if, for every subset \( S \) of \( \mathcal{P} \), if \( \bigvee S \) exists, then
\[
\theta(\bigvee S) = \bigcup \theta(S).
\]

Note how \( S \) is required to be nonempty in Definition 2.4, but not in Definition 2.5. For representations of Boolean algebras as fields of sets, the notions of meet complete and join complete are equivalent, so in this case we may simply use the adjective **complete**.

If an algebra of the signature \((;\land,A)\) is representable by partial functions, then it forms a \(\land\)-semilattice. Whenever we treat such an algebra as a poset, we are using the order induced by this semilattice.

Note that if \( \mathcal{A} \) is an algebra of the signature \((;\land,A)\) and \( \mathcal{A} \) is representable by partial functions, then \( \mathcal{A} \) must have a least element, 0, given by \( A(a);a \) for any \( a \in \mathcal{A} \) and any representation must represent 0 with the empty set. Similarly \( D := A^2 \) must be represented by the set-theoretic domain operation.

We will make use of the following lemma.

Lemma 2.6. Let \( \mathcal{A} \) be an algebra of the signature \((;\land,A)\). If \( \mathcal{A} \) is representable by partial functions, then for every \( a \in \mathcal{A} \), the set \( \downarrow a \), with least element 0, greatest element \( b \), meet given by \( \land \) and complementation given by \( \overline{b} := A(b);a \) is a Boolean algebra. Any representation \( \theta \) of \( \mathcal{A} \) by partial functions restricts to a representation of \( \downarrow a \) as a field of sets over \( \theta(a) \). If \( \theta \) is a meet-complete or join-complete representation, then the representation of \( \downarrow a \) is complete.

Proof. If \( \theta \) is a representation of \( \mathcal{A} \) by partial functions, then \( b \leq a \implies \theta(b) \subseteq \theta(a) \), so \( \theta \) does indeed map elements of \( \downarrow a \) to subsets of \( \theta(a) \). We have \( b,c \in \downarrow a \implies b \land c \in \downarrow a \) and \( \theta(b \land c) = \theta(b) \cap \theta(c) \) is always true by the definition of functional representability. For \( b \leq a \)
\[
\theta(b) = \theta(A(b);a) = A(\theta(b));a = \theta(a) \setminus \theta(b),
\]
so \( \overline{b} \in \downarrow a \) and \( \theta(\overline{b}) = \theta(b)^c \), where the set complement is taken relative to \( \theta(a) \). Hence the restriction of \( \theta \) to \( \downarrow a \) is a representation of \( \downarrow a,0,\land,\overline{\cdot} \) as a field of sets over \( \theta(a) \) (from which it follows that \( \downarrow a \) is a Boolean algebra).

Suppose \( \theta \) is meet complete. If \( S \) is a nonempty subset of \( \downarrow a \), then all lower bounds for \( S \) in \( \mathcal{A} \) are also in \( \downarrow a \). Hence if \( \bigwedge_{\downarrow a} S \) exists then it equals \( \bigwedge_{\mathcal{A}} S \), and so \( \theta(\bigwedge_{\downarrow a} S) = \bigwedge_{\theta(S)} \). So the representation of \( \downarrow a \) is complete.

Suppose that \( \theta \) is join complete, \( S \subseteq \downarrow a \) and \( \bigvee_{\downarrow a} S \) exists. If \( c \in \mathcal{A} \) and \( c \) is an upper bound for \( S \), then \( c \geq c \land a \geq \bigvee_{\downarrow a} S \). Hence \( \bigvee_{\downarrow a} S = \bigvee_{\mathcal{A}} S \), giving \( \theta(\bigvee_{\downarrow a} S) = \theta(\bigvee_{\mathcal{A}} S) = \bigcup \theta(S) \). So the representation of \( \downarrow a \) is complete.

\( \square \)

Corollary 2.7. Let \( \mathcal{A} \) be an algebra of the signature \((;\land,A)\) and \( \theta \) be a representation of \( \mathcal{A} \) by partial functions. If \( \theta \) is meet complete, then it is join complete.

Proof. Suppose that \( \theta \) is meet complete. Let \( S \) be a subset of \( \mathcal{A} \) and suppose that \( \bigvee_{\mathcal{A}} S \) exists. Let \( a = \bigvee_{\mathcal{A}} S \). Then
\[
\theta(\bigvee_{\mathcal{A}} S) = \theta(a) = \bigcup \theta(S).
\]

\( \square \)

Corollary 2.8. Let \( \mathcal{A} \) be an algebra of the signature \((;\land,A)\) and \( \theta \) be a representation of \( \mathcal{A} \) by partial functions. If \( \theta \) is join complete, then it is meet complete.
Proof. Suppose that $\theta$ is join complete. Let $S$ be a nonempty subset of $\mathfrak{A}$ and suppose that $\bigwedge S$ exists. As $S$ is nonempty, we can find $s \in S$. Then

$$\theta(\bigwedge S) = \theta(\bigwedge(S \wedge \{s\})) = \theta(\bigwedge(S \wedge \{s\})) = \bigcap \theta[S \wedge \{s\}] = \bigcap \theta[S].$$

\[\square\]

Corollaries 2.7 and 2.8 tell us that, just as for representations of Boolean algebras, we can describe representations of $(\cdot, \wedge, A)$-algebras by partial functions as complete, without any risk of confusion about whether we mean meet complete or join complete.

3. Atomicity

We begin our investigation of the complete representation class by considering the property of being atomic, both for algebras and for representations.

**Definition 3.1.** Let $\mathfrak{P}$ be a poset with a least element, 0. An atom of $\mathfrak{P}$ is a minimal nonzero element of $\mathfrak{P}$. We say that $\mathfrak{P}$ is atomic if every nonzero element is greater than or equal to an atom.

If $\mathfrak{P}$ is a poset, then $\text{At}(\mathfrak{P})$ denotes the set of atoms of $\mathfrak{P}$.

**Definition 3.2.** Let $\mathfrak{P}$ be a poset with a least element and let $\theta$ be a representation of $\mathfrak{P}$. Then $\theta$ is atomic if $x \in \theta(a)$ for some $a \in \mathfrak{P}$ implies $x \in \theta(b)$ for some atom $b$ of $\mathfrak{P}$.

We will need the following theorem.

**Theorem 3.3** (Hirsch and Hodkinson [3]). Let $\mathfrak{B}$ be a Boolean algebra. A representation of $\mathfrak{B}$ as a field of sets is atomic if and only if it is complete.

**Proposition 3.4.** Let $\mathfrak{A}$ be an algebra of the signature $(\cdot, \wedge, A)$ and $\theta$ be a representation of $\mathfrak{A}$ by partial functions. Then $\theta$ is atomic if and only if it is complete.

**Proof.** Suppose that $\theta$ is atomic, $S$ is a nonempty subset of $\mathfrak{A}$ and $\bigwedge S$ exists. It is always true that $\theta(\bigwedge S) \subseteq \bigcap \theta[S]$, regardless of whether or not $\theta$ is atomic. For the reverse inclusion, we have

$$\begin{align*}
(x, y) &\in \bigcap \theta[S] \\
\implies (x, y) &\in \theta(s) \text{ for all } s \in S \\
\implies (x, y) &\in \theta(a) \text{ for some atom } a \text{ such that } (\forall s \in S) \ a \leq s \\
\implies (x, y) &\in \theta(a) \text{ for some atom } a \text{ such that } a \leq \bigwedge S \\
\implies (x, y) &\in \theta(\bigwedge S).}
\end{align*}$$

The third line follows from the second because, taking an $s_0 \in S$ and an atom $a$ below $s_0$ with $(x, y) \in \theta(a)$, we have $(x, y) \in \theta(a \wedge s)$ for any $s \in S$. So for all $s \in S$, the element $a \wedge s$ is nonzero, so equals $a$, by atomicity of $a$, giving $a \leq s$.

Conversely, suppose that $\theta$ is complete. Let $(x, y)$ be a pair contained in $\theta(a)$ for some $a \in \mathfrak{A}$. By Lemma 2.6, the map $\theta$ restricts to a complete representation of $\downarrow a$ as a field of sets. Hence, by Theorem 3.3, $(x, y) \in \theta(b)$ for some atom $b$ of the Boolean algebra $\downarrow a$. Since an atom of $\downarrow a$ is clearly an atom of $\mathfrak{A}$, the representation $\theta$ is atomic. \[\square\]

**Corollary 3.5.** Let $\mathfrak{A}$ be an algebra of the signature $(\cdot, \wedge, A)$. If $\mathfrak{A}$ is completely representable by partial functions then $\mathfrak{A}$ is atomic.
Proof. Let $a$ be a nonzero element of $\mathfrak{A}$. Let $\theta$ be any complete representation of $\mathfrak{A}$. Then $\emptyset = \theta(0) \neq \theta(a)$, so there exists $(x, y) \in \theta(a)$. By Proposition 3.4, the map $\theta$ is atomic, so $(x, y) \in \theta(b)$ for some atom $b$ in $\mathfrak{A}$. Then $(x, y) \in \theta(a \wedge b)$, so $a \wedge b > 0$, from which we may conclude that the atom $b$ satisfies $b \leq a$. □

We can use Corollary 3.5 to show that the class of $(\cdot, \wedge, A)$-algebras that are completely representable by partial functions is not closed under subalgebras and so is not a universally axiomatisable class.

**Proposition 3.6.** The class of $(\cdot, \wedge, A)$-algebras that are completely representable by partial functions is not universally axiomatisable.

Proof. Let $B$ be any non-atomic Boolean algebra, for example the countable atomless Boolean algebra, which is unique up to isomorphism. By Stone’s representation theorem we may assume that $B$ is a field of sets, with base $X$ say.

Let $\mathfrak{F}$ be the set of all partial functions on $X$. Then $\mathfrak{F}$ is closed under the set-theoretic operations of composition, antidepdomain and arbitrary nonempty intersections and so forms an algebra of partial functions of the signature $(\cdot, \wedge, A)$ for which the identity function is a complete representation.

The subset $\mathfrak{G}$ of $\mathfrak{F}$ consisting of all restrictions of the identity function to an element of $B$ is easily seen to be closed under composition, antidepdomain and binary intersections. So $\mathfrak{G}$ forms a subalgebra of $\mathfrak{F}$. But $\mathfrak{G}$, being order-isomorphic to $B$, is not atomic and so, by Corollary 3.5, not completely representable by partial functions.

We have demonstrated that the class of $(\cdot, \wedge, A)$-algebras that are completely representable by partial functions is not closed under subalgebras. It follows that the class is not universally axiomatisable. □

4. Distributivity

We now turn our attention to the validity of various distributive laws with respect to the classes of representable and completely representable $(\cdot, \wedge, A)$-algebras. We give the first definition that we will use. Other distributive properties that we refer to later are defined similarly. For distributive properties ‘over meets’ it should be assumed that definitions only require that the relevant equation holds when nonempty subsets are used.

**Definition 4.1.** Let $\mathfrak{P}$ be a poset and $*$ be a binary operation on $\mathfrak{P}$. We say that $*$ is completely right-distributive over joins if, for any subset $S$ of $\mathfrak{P}$ and any $a \in \mathfrak{P}$, if $\bigvee S$ exists, then

$$\bigvee S \ast a = \bigvee \{ S \ast \{ a \} \}.$$ 

**Proposition 4.2.** Let $\mathfrak{A}$ be an algebra of the signature $(\cdot, \wedge, A)$ that is representable by partial functions. Then composition is completely right-distributive over joins.

Proof. As $\mathfrak{A}$ is representable, we may assume the elements of $\mathfrak{A}$ are partial functions. Let $S$ be a subset of $\mathfrak{A}$ such that $\bigvee S$ exists and let $a \in \mathfrak{A}$.

Firstly, for all $s \in S$ we have $\bigvee S \ast s \geq s \ast a$ and so $\bigvee S \ast a$ is an upper bound for $S \ast \{ a \}$.

Now suppose that for all $s \in S$, the element $b \in \mathfrak{A}$ satisfies $b \geq s \ast a$. For $s \in S$, suppose $s$ is defined on $x$ and let $s(x) = y$. If $a$ is defined on $y$, then $s \ast a$ is defined on $x$, so, since $b \geq s \ast a$ and $\bigvee S \ast a \geq s \ast a$, in this case $b \wedge (\bigvee S \ast a)$ is defined on $x$. 


If \( a \) is not defined on \( y \) then, as \( \bigvee S ; a \) is not defined on \( x \). Hence the sub-identity function \( D(b \land (\bigvee S ; a)) \lor A(\bigvee S ; a) \) is defined on the entire domain of \( s \). Therefore

\[
(D(b \land (\bigvee S ; a)) \lor A(\bigvee S ; a)) ; \bigvee S \geq s.
\]

Since \( s \) was an arbitrary element of \( S \), we have

\[
(D(b \land (\bigvee S ; a)) \lor A(\bigvee S ; a)) ; \bigvee S \geq \bigvee S
\]

and so

\[
(D(b \land (\bigvee S ; a)) \lor A(\bigvee S ; a)) ; \bigvee S = \bigvee S.
\]

Therefore

\[
D(b \land (\bigvee S ; a)) ; \bigvee S ; a = (D(b \land (\bigvee S ; a)) \lor A(\bigvee S ; a)) ; \bigvee S ; a
\]

which says that wherever the function \( \bigvee S ; a \) is defined, it agrees with the function \( b \), that is to say \( b \geq \bigvee S ; a \). So \( \bigvee S ; a \) is the least upper bound for \( \bigvee (S ; \{a\}) \). \( \square \)

**Remark 4.3.** For \((; \land, A)\)-algebras representable by partial functions it is easy to see that the following two laws hold.

(i) For finite \( S \), if \( \bigvee S \) exists, then

\[
a ; \bigvee S = \bigvee \{a \} ; S.
\]

(composition is left-distributive over joins)

(ii) For finite, nonempty \( S \),

\[
a ; \bigwedge S = \bigwedge \{a \} ; S.
\]

(composition is left-distributive over meets)

We now give an example that shows that the these distributive laws cannot, in general, be extended to arbitrary joins and meets. We will use this example to show that there exist \((; \land, A)\)-algebras that are representable as partial functions, and atomic, but have no atomic representation.

**Example 4.4.** Consider the following concrete algebra of partial functions, \( \mathfrak{F} \). Its domain is the disjoint union of a one element set, \( \{p\} \), and \( \mathbb{N} \). Let \( S \) be all the subsets of \( \mathbb{N} \) that are either finite and do not contain 1, or cofinite and contain 1. The elements of \( \mathfrak{F} \) are precisely the following functions.

1. Restrictions of the identity to \( A \cup B \) where \( A \subseteq \{p\} \) and \( B \in S \).

2. The function \( f \), defined only on \( p \) and taking \( p \) to 1.

One can check that \( \mathfrak{F} \) is closed under the operations of intersection, composition and antidomain, that \( \mathfrak{F} \) is atomic and that \( f \) is an atom.

For \( i \geq 2 \), let \( g_i \) be the restriction of the identity to \( \{2, \ldots, i\} \). Then \( \bigvee_i g_i \) exists and is equal to the identity restricted to \( \mathbb{N} \). So

\[
f ; \bigvee_{i \geq 2} g_i = f \supset \emptyset = \bigvee_{i \geq 2} (f ; g_i).
\]

For \( i \geq 2 \), let \( h_i \) be the restriction of the identity to \( \{1\} \cup \{i, \ldots\} \). Then \( \bigwedge_i h_i \) exists and is equal to the nowhere-defined function. So

\[
f ; \bigwedge_{i \geq 2} h_i = \emptyset \subset f = \bigwedge_{i \geq 2} (f ; h_i).
\]
Lemma 4.5. Let $\mathfrak{A}$ be an algebra of the signature $(; \land, \lor)$ that is completely representable by partial functions. Then composition in $\mathfrak{A}$ is completely left-distributive over joins and completely left-distributive over meets.

Proof. First we prove that composition is completely left-distributive over joins. Let $S$ be a subset of $\mathfrak{A}$ such that $\bigvee S$ exists and let $a \in \mathfrak{A}$. Let $\theta$ be any complete representation of $\mathfrak{A}$. Suppose that for all $s \in S$ the element $b \in \mathfrak{A}$ satisfies $b \geq s ; a$. Then for all $s \in S$ we have $\theta(b) \supseteq \theta(a ; s)$. Hence

$$\theta(b) \supseteq \bigcup \theta[\{a\} ; S]$$
$$= \bigcup (\{\theta(a)\} ; \theta[S])$$
$$= \theta(a) ; \bigcup \theta[S]$$
$$= \theta(a ; \bigvee S).$$

The second equality is a true property of any collection of functions, indeed of any collection of relations. We conclude that $b \geq a ; \bigvee S$ and hence $a ; \bigvee S$ is the least upper bound for $\{a\} ; S$.

The proof that composition is completely left-distributive over meets is similar. Let $S$ be a nonempty subset of $\mathfrak{A}$ such that $\bigwedge S$ exists and let $a \in \mathfrak{A}$. Let $\theta$ be any complete representation of $\mathfrak{A}$. Suppose that for all $s \in S$, the element $b \in \mathfrak{A}$ satisfies $b \leq s ; a$. Then for all $s \in S$, we have $\theta(b) \subseteq \theta(a ; s)$. Hence

$$\theta(b) \subseteq \bigcap \theta[\{a\} ; S]$$
$$= \bigcap (\{\theta(a)\} ; \theta[S])$$
$$= \theta(a) ; \bigcap \theta[S]$$
$$= \theta(a ; \bigwedge S).$$

This time the second equality holds only because we are working with functions. It is not, in general, a true property of relations. We conclude from the above that $b \geq a ; \bigwedge S$ and hence $a ; \bigwedge S$ is the greatest lower bound for $\{a\} ; S$. □

Proposition 4.6. There exist $(; \land, \lor)$-algebras that are representable by partial functions, and atomic, but have no atomic representation.

Proof. Let $\mathfrak{P}$ be the algebra of Example 4.4. Since $\mathfrak{P}$ is an algebra of partial functions, it is certainly representable by partial functions. We have already mentioned that $\mathfrak{P}$ is atomic. We have demonstrated that composition in $\mathfrak{P}$ is neither completely left-distributive over joins nor over meets. Hence, by Lemma 4.5, $\mathfrak{P}$ has no complete representation. So, by Proposition 3.4, $\mathfrak{P}$ has no atomic representation. □

5. A Representation

We have seen that for an algebra of the signature $(; \land, \lor)$ to be completely representable by partial functions it is necessary for it to be representable by partial functions and atomic and for composition to by completely left-distributive over joins. Next we show that these conditions are also sufficient.
Proposition 5.1. Let $\mathfrak{A}$ be an algebra of the signature $(\cdot, \land, A)$. Suppose $\mathfrak{A}$ is representable by partial functions and atomic and that composition is completely left-distributive over joins. For each $a \in \mathfrak{A}$, let $\theta(a)$ be the following partial function on $\text{At}(\mathfrak{A})$.

$$
\theta(a)(x) = \begin{cases} 
    x : a & \text{if } x : a \neq 0 \\
    \text{undefined} & \text{otherwise}
\end{cases}
$$

Then $\theta$ is a complete representation of $\mathfrak{A}$ by partial functions, with base $\text{At}(\mathfrak{A})$.

Proof. We first need to show that, for each $a \in \mathfrak{A}$, the partial function $\theta(a)$ maps into $\text{At}(\mathfrak{A})$. Let $x$ be an atom and suppose that $x : a$ is nonzero. Let $b \in \mathfrak{A}$ and suppose $b \leq x : a$. Then $D(b) \leq D(x : a) \leq D(x)$. Hence if $D(x) : x = 0$ then $b = 0$. If $D(x) : x > 0$, then we must have $D(b) : x = x$ and hence $D(b) = D(x : a) = D(x)$. Therefore $b = x : a$. So $x : a$ is an atom.

To show that $\theta$ represents composition correctly, let $a, b \in \mathfrak{A}$ and $x \in \text{At}(\mathfrak{A})$. Then clearly $\theta(a ; b)(x) = \theta(a) ; \theta(b)(x)$ if both sides are defined. The left-hand side is defined precisely when $x : a ; b$ is nonzero and the right-hand side when $x : a$ and $x : a ; b$ are both nonzero. Since $x : a ; b \neq 0$ implies $x : a \neq 0$, the domains of definition are the same.

To show that $\theta$ represents binary meet correctly, let $a, b \in \mathfrak{A}$ and $x, y \in \text{At}(\mathfrak{A})$. Then

$$
(x, y) \in \theta(a \land b) \\
\implies (x, y) \in \theta(a) \text{ and } (x, y) \in \theta(b) \quad \text{as } a, b \geq a \land b \\
\implies (x, y) \in \theta(a) \cap \theta(b)
$$

and

$$
(x, y) \in \theta(a) \cap \theta(b) \\
\implies x ; a = y \text{ and } x ; b = y \\
\implies (x ; a) \land (x ; b) = y \\
\implies x ; (a \land b) = y \\
\implies (x, y) \in \theta(a \land b).
$$

To show that antidiomain is represented correctly, let $a \in \mathfrak{A}$ and $x \in \text{At}(\mathfrak{A})$. Then $0 < \theta(A(a))(x) = x ; A(a) \leq x$ if $\theta(A(a))(x)$ is defined. Since $x$ is an atom we have, in this case, $\theta(A(a))(x) = x$. The partial function $A(\theta(a))(x)$ is also a restriction of the identity function. The domains of $\theta(A(a))$ and $A(\theta(a))$ are the same, since we have seen that $\theta(A(a))(x)$ is defined precisely when $x ; A(a) = x$, which is when $x ; a = 0$, which is precisely when $A(\theta(a))$ is defined. This completes the proof that $\theta$ is a representation of $\mathfrak{A}$ by partial functions.

Finally, we show that the representation $\theta$ is complete. Let $S$ be a subset of $\mathfrak{A}$ such that $\bigvee S$ exists. Let $x, y \in \text{At}(\mathfrak{A})$. Then

$$
(x, y) \in \bigcup \theta[S] \\
\implies (x, y) \in \theta(s) \quad \text{for some } s \in S \\
\implies (x, y) \in \theta(\bigvee S) \quad \text{as } \bigvee S \geq s
$$
and

\[(x, y) \in \theta(\bigvee S) \implies x : \bigvee S = y \implies \bigvee (\{x\} \cup S) = y \text{ as composition is completely left-distributive over joins} \implies x : s = y \text{ for some } s \in S, \text{ since } y \text{ is an atom} \implies (x, y) \in \theta(s) \text{ for some } s \in S \implies (x, y) \in \bigcup \theta[S].\]

Hence \(\theta(\bigvee S) = \bigcup \theta[S]\).

\[\square\]

6. Axiomatising the Class

In this final section, we use the conditions for complete representability that we have uncovered to obtain a finite first-order axiomatisation of the complete-representation class.

**Definition 6.1.** A poset \(\mathcal{P}\) is **atomistic** if its atoms are join dense in \(\mathcal{P}\). That is to say that every element of \(\mathcal{P}\) is the join of the atoms less than or equal to it.

Clearly any atomistic poset is atomic. For \((\cdot, \wedge, \mathcal{A})\)-algebras representable by partial functions, the converse is also true.

**Lemma 6.2.** Let \(\mathfrak{A}\) be an algebra of the signature \((\cdot, \wedge, \mathcal{A})\) that is representable by partial functions. If \(\mathfrak{A}\) is atomic, then it is atomistic.

**Proof.** Suppose \(\mathfrak{A}\) is atomic and let \(a \in \mathfrak{A}\). By Lemma 2.6, the algebra \(\downarrow a\) is a Boolean algebra and clearly it is atomic. It is well-known that atomic Boolean algebras are atomistic. So we have

\[a = \bigvee_{\mathfrak{A}} \{x \in \text{At}(\downarrow a) \mid x \leq a\} = \bigvee_{\mathfrak{A}} \{x \in \text{At}(\downarrow a) \mid x \leq a\} = \bigvee_{\mathfrak{A}} \{x \in \text{At}(\mathfrak{A}) \mid x \leq a\}.\]

The second equality holds because \(\mathfrak{A}\) is a meet semilattice and \(\downarrow a\) is a down set of \(\mathfrak{A}\). \(\square\)

**Lemma 6.3.** Let \(\mathfrak{A}\) be an algebra of the signature \((\cdot, \wedge, \mathcal{A})\) that is representable by partial functions and atomic. Let \(\varphi\) be the first-order sentence saying that for any \(a, b, c\), if \(c \geq a \cdot x\) for all atoms \(x\) less than or equal to \(b\), then \(c \geq a \cdot b\). Then composition is completely left-distributive over joins if and only if \(\mathfrak{A} \models \varphi\).

**Proof.** Suppose first that composition is completely left-distributive over joins. As \(\mathfrak{A}\) is atomic it is atomistic. So for any \(a, b \in \mathfrak{A}\) we have

\[a \cdot b = a \cdot \bigvee \{x \in \text{At}(\mathfrak{A}) \mid x \leq b\} = \bigvee \{a \cdot \{x \in \text{At}(\mathfrak{A}) \mid x \leq b\}\}\]

and so \(\varphi\) holds.

Now suppose that \(\mathfrak{A} \models \varphi\). Let \(a \in \mathfrak{A}\) and let \(S\) be a subset of \(\mathfrak{A}\) such that \(\bigvee S\) exists. Then certainly \(a \cdot \bigvee S\) is an upper bound for \(\{a\} \cup S\). To show it is the least...
upper bound, let \( c \) be an arbitrary upper bound for \( \{a\}; S \). Then

\[
\begin{align*}
\text{for all } s \in S & \quad c \geq a; s \\
\implies \text{for all } s \in S \text{ and } x \in \text{At}(\downarrow \bigvee S) \text{ with } x \leq s & \quad c \geq a; x \\
\implies \text{for all } x \in \text{At}(\downarrow \bigvee S) & \quad c \geq a; x \\
\implies \text{for all } x \in \text{At}(\mathfrak{A}) \text{ with } x \leq \bigvee S & \quad c \geq a; x \\
\implies & \quad c \geq a; \bigvee S.
\end{align*}
\]

The third line follows from the second because \( \downarrow \bigvee S \) is a Boolean algebra, so that \( x \in \text{At}(\downarrow \bigvee S) \) implies \( x \leq s \) for some \( s \in S \). The fifth line can be seen to follow from the fourth by first writing \( \bigvee S \) as the join of the atoms below it and then using complete left-distributivity.

We now have everything we need to prove our main result.

**Proposition 6.4.** The class of \((\cdot, \wedge, \Lambda)\)-algebras that are completely representable by partial functions is a basic elementary class.

**Proof.** By Corollary 3.5, Lemma 4.5 and Proposition 5.1, an algebra of the signature \((\cdot, \wedge, \Lambda)\) is completely representable by partial functions if and only if it is representable by partial functions, atomic and composition is completely left-distributive over joins. By Theorem 2.3, the property of being representable by partial functions is characterised by a finite set of first-order sentences. The property of being atomic is easily written as a first-order sentence. By Lemma 6.3, in the presence of the axioms for the first two properties, the property that composition is completely left-distributive over joins can be written as a first-order sentence.

\(\square\)

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