A bijection between permutation matrices and descending plane partitions without special parts, which respects the quadruplet of statistics considered by Behrend, Di Francesco and Zinn–Justin.

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Abstract
We present a bijection between permutation matrices and descending plane partitions without special parts, which respects the quadruplet of statistics considered by Behrend, Di Francesco and Zinn–Justin. This bijection involves the inversion words of permutations and the “usual” representation of descending plane partitions as families of non–intersecting lattice paths.

1 Introduction
It is a well–known fact [10, 5, 4] that the enumeration

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of descending plane partitions with parts not exceeding $n$ (let us denote the set of these objects by $\mathcal{D}_n$)

and of alternating sign matrices of dimension $n$ (let us denote the set of these objects by $\mathcal{A}_n$)
gives the same number:

$$|\mathcal{A}_n| = |\mathcal{D}_n| = \prod_{j=0}^{n-1} \frac{(3 \cdot n + 1)!}{(n + j)!}.$$  \hfill (1)

It appears to be quite difficult to find some “natural” bijection

$$\Phi : \mathcal{A}_n \to \mathcal{D}_n.$$  

However, there are two additional informations which might help in the search for such bijection:

- There is a quadruplet $(i, p, s, q)$ of statistics for both $\mathcal{A}_n$ and $\mathcal{D}_n$, such that all subsets with the same quadruplet of statistics–values are equinumerous (see [3, Theorem 1], the details are given in the next section),

- There are certain subsets of $\mathcal{A}_n$ and of $\mathcal{D}_n$, namely
  - alternating sign matrices with statistic $s = 0$ (let us denote this set by $\mathcal{A}_n^0$; it is, in fact, the set of $n \times n$ permutation matrices),
  - descending plane partitions with statistic $s = 0$ (let us denote this set by $\mathcal{D}_n^0$),

  which are much simpler to understand and for which it is, in fact, quite easy to give “natural” bijections (see below).

So one obvious idea would be to search for a bijection $\Phi$ which respects the quadruplet of statistics $(i, p, s, q)$; i.e., for all $A \in \mathcal{A}_n$ there should hold:

$$(i(A), p(A), s(A), q(A)) = (i(\Phi(A)), p(\Phi(A)), s(\Phi(A)), q(\Phi(A))).$$  \hfill (2)

Clearly, such $\Phi$ restricted to $\mathcal{A}_n^0$ would give a bijection $\Psi : \mathcal{A}_n^0 \to \mathcal{D}_n^0$: So if we find some “restricted” bijection $\Psi$ which respects the quadruplet of statistics in the sense of (2), then we might hope to “extend” it somehow to the desired “full” bijection $\Phi$. 

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The purpose of this note is to present a simple bijection $\Psi$ which indeed respects the quadruplet of statistics in the sense of (2): Our function $\Psi$ relies on the (obvious) representation of descending plane partitions as families of non-intersecting lattice paths and on a certain (obvious) “visualization” of the statistic $i$ (as number of certain cells in an alternating sign matrix).

### 1.1 Other bijections

It should be noted that there are other bijections $A_n^0 \rightarrow D_n^0$. Maybe the simplest one was already mentioned by Lalonde [7, p.981], who referred to the inversion table of a permutation: This table (also called inversion word) gives a (well-known) unique encoding of permutations. The missing link from inversion words to $D_n^0$ was explained by Striker [9, Lemma 5]. Striker uses monotone triangles as intermediate objects to establish the bijection between inversion words and $A_n^0$ (we shall call this Striker’s bijection), but this intermediate step is not necessary: Instead, we can employ directly the well-known encoding of permutations by inversion words (we shall call this Lalonde’s bijection). Unfortunately, none of these two simple bijections respects the statistic $q$ (see Figure 2). Ayyer [1] presented another (inductively constructed) bijection, which does not respect the statistic $i$.

### 1.2 Organization of this note

This note is organized as follows:

- Section 2 contains basic definitions and background information,
- Section 3 presents the (usual) interpretation of descending plane partitions as families of non-intersecting lattice paths (for the expert it will suffice to look at Figure 1),
- Section 4 presents a “visualization” of inversions in alternating sign matrices used for our bijection,
- Section 5 presents a bijection $\Psi : A_n^0 \rightarrow D_n^0$ which respects the quadruplet $(i, p, s, q)$ of statistics.
2 Background information

For reader’s convenience, we recall some background information.

2.1 Descending plane partitions

Here is the definition of descending plane partitions as given by Mills, Robbins and Rumsey [8, Definition 4]:

Definition 1 (descending plane partition). A {\textit{descending plane partition}} is an array $\pi = (a_{i,j})$, $1 \leq i \leq j < \infty$, of \textit{positive integers} $a_{1,1}, a_{1,2}, a_{1,3}, \ldots, a_{1,\mu_1}, a_{2,2}, a_{2,3}, \ldots, a_{2,\mu_2}, \ldots, a_{k,k}, \ldots, a_{k,\mu_k}$ such that

1. rows are \textit{weakly decreasing}, i.e., $a_{i,j} \geq a_{i,j+1}$ for all $i = 1, \ldots, k$ and $i \leq j < \mu_i$,
2. columns are \textit{strictly decreasing}, i.e., $a_{i,j} > a_{i+1,j}$ for all $i = 1, \ldots, k-1$ and $i < j \leq \mu_{i+1}$,
3. $a_{i,i} > \mu_i - i + 1$ for all $i = 1, \ldots, k$,
4. $a_{i,i} \leq \mu_{i-1} - (i-1) + 1$ for all $i = 2, \ldots, k$.

It is easy to see that these conditions imply

$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq k$.

The \textit{parts} of a descending plane partition are the \textit{numbers} (with repetitions) that appear in the array. The \textit{empty} array, which we denote by $\emptyset$, is explicitly allowed.

A descending plane partition $\pi$ where no part is greater than $n$ (i.e., $\pi$ has at most $n - 1$ rows) is said to have dimension $n$.

We denote the $i$–th row of a descending plane partition by $r_i$. The \textit{length} of $r_i$ is the number of parts it contains, which is $\mu_i - i + 1$. So we may rephrase the last two conditions as
(A) The first part of $r_i$ is greater than the length of $r_i$ for $i = 1, \ldots, k$.

(B) The first part of $r_i$ is less or equal than the length of the preceding row $r_{i-1}$ for $i = 2, \ldots, k$.

A part $a_{i,j}$ in a descending plane partition is called *special* if it does not exceed the number of parts to its left (in its row $r_i$), i.e.,

$$a_{i,j} \leq j - i.$$ 

**Example 2.** A typical example is the array

$$
\begin{array}{cccc}
6 & 6 & 6 & 4 \\
5 & 3 & 2 & 1 \\
2 & & & \\
\end{array}
$$

with 3 rows and 10 parts (written in descending order)

$$6, 6, 6, 5, 4, 3, 2, 2, 2, 1,$$

three of which are special parts (indicated as underlined numbers; note that the 2 in the last row is *not* a special part):

$$2, 2, 1.$$

(This is the example $D_0$ considered by Lalonde, see [6, Fig. 1].)

From now on, we shall use the shortcut $DPP$ for descending plane partitions.

### 2.2 Alternating sign matrices

Here is the definition of alternating sign matrices as given by Mills, Robbins and Rumsey, see [8, Definition 1]:

**Definition 3** (alternating sign matrix). An *alternating sign matrix* of dimension $n$ is an $n \times n$ square matrix which satisfies

- all entries are 1, $-1$ or 0,
- every row and column has sum 1,
in every row and column the nonzero entries alternate in sign.

Suppose that $A = (A_{i,j})_1^n$ is an alternating sign matrix of dimension $n$. Then the number of inversions in $M$ is defined to be

$$\sum_{1 \leq i < k \leq n \atop 1 \leq l < j \leq n} A_{i,j} \cdot A_{k,l}. \quad (3)$$

(See [8, p. 344].)

**Example 4.** The following matrix is an example of an alternating sign matrix of dimension 5:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

From now on, we shall use the shortcut $ASM$ for alternating sign matrices.

### 2.3 The Mills–Robbins–Rumsey conjecture

Here is the Conjecture of Mills, Robbins and Rumsey [8, Conjecture 3], slightly rephrased to fit our exposition:

**Conjecture 5.** Suppose that $n, i, p, s$ are nonnegative integers, $0 \leq p \leq n-1$. Let $\mathcal{A}(n, i, p, s)$ be the set of ASMs such that

1. the size of the matrix is $n \times n$,
2. the number of 0’s to the left of the 1 in the first row is $p$,
3. the number of $-1$’s in the matrix is $s$,
4. the number of inversions in the matrix is $i + s$.

On the other hand, let $\mathcal{D}(n, i, p, s)$ be the set of DPPs such that

1. no part exceeds $n$,
2. there are exactly $p$ parts equal to $n$. 

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3. there are exactly \( i \) special parts,

4. there are a total of \( i + s \) parts.

Then \( A(n, i, p, s) \) and \( D(n, i, p, s) \) have the same cardinality.

This conjecture was proved by Behrend, Di Francesco and Zinn–Justin [2, Theorem 1].

### 2.4 The fourth statistic given by Behrend, Di Francesco and Zinn–Justin

In [3], Behrend, Di Francesco and Zinn–Justin presented a fourth statistic \( q \) for ASMs and DPPs and showed that ASMs and DPPs are also equidistributed with respect to the quadruplet of statistics \((i, p, s, q)\) [3, Theorem 1].

This statistic \( q \) is

- for \( n \)-dimensional ASMs equal to the number of 0’s to the right of the 1 in the last row,

- for \( n \)-dimensional DPPs equal to the number of parts \( n - 1 \) plus the number of rows of length \( n - 1 \)

### 2.5 Permutation matrices and inversions

Let \( \sigma \in S_n \) be a permutation of the first \( n \) natural numbers \( \{1, 2, \ldots, n\} \).

#### 2.5.1 Inversions of a permutation

Recall that an inversion of \( \sigma \) is a pair \((i, j)\) such that \( i < j \) but \( \sigma(i) > \sigma(j) \).

For the number \( \text{inv}(\sigma) \) of all inversions of \( \sigma \) we have \( 0 \leq \text{inv}(\sigma) \leq \frac{n(n+1)}{2} \).

We may assign to \( \sigma \) its inversion word \((a_1, a_2, \ldots, a_{n-1})\), where \( a_k \) is the number of inversions \((i, j)\) with \( \sigma(j) = k \), \( k = 1, 2, \ldots, n - 1 \). Clearly we have \( 0 \leq a_k \leq n - k \) and \( a_1 + a_2 + \cdots + a_{n-1} = \text{inv}(\sigma) \).

Considering the permutation word

\[
(\sigma(1), \sigma(2), \ldots, \sigma(n)),
\]

of \( \sigma \), the inversion word’s \( k \)-th entry \( a_k \) is simply the number of elements to the left of \( k \) (in the permutation word) which are greater than \( k \), and
it is easy to see that every word \((b_1, b_2, \ldots, b_{n-1})\) with \(0 \leq b_k \leq n - k\) determines a unique permutation: Inversion words are, in this sense, just another “encoding” for permutations.

### 2.5.2 Permutation matrices

A permutation \(\sigma \in \mathfrak{S}_n\) can be represented by an \(n \times n\)–matrix \(M\) with entries

\[ M_{i,j} = \delta_{\sigma(i),j} \]

(where \(\delta_{x,y}\) denotes Kronecker’s delta: \(\delta_{x,y} = 1\) if \(x = y\), \(\delta_{x,y} = 0\) if \(x \neq y\)). We call this matrix the permutation matrix of \(\sigma\): Clearly, it contains precisely one entry 1 in every row and column.

**Example 6.** Let \(n = 6\) and \(\sigma \in \mathfrak{S}_6\) be the permutation with permutation word

\[ \sigma = (352461). \]

The corresponding permutation matrix is

\[
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

and the corresponding inversion word is

\((2, 3, 1, 1, 1).\)

Note that every permutation matrix is an ASM (which does not contain entries \(-1\)), and that the definition of inversions \((3)\) for ASMs is a generalization of the number of inversions of a permutation (see also Section 4.2).

### 3 Representation of DPPs as lattice paths

If some row \(i\) in a DPP \(\pi = (a_{i,j})\) is shorter than \(a_{i,i} - 1\), i.e.,

\[ \delta = (a_{i,i} - 1) - (\mu_i - i + 1) > 0, \]

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then we shall pad this row with δ trailing zeroes (so the length of some row is the number of non–zero parts in that row.)

Now we employ the well–known encoding of (shifted) tableaux as non–intersecting lattice paths; i.e., we encode a DPP π = (a_{i,j}) of dimension n with r rows as an r–tuple non–intersecting lattice paths in the lattice Z^2. For reader’s convenience, we shall describe the details of this encoding below, but the idea can easily be obtained by looking at the illustrative example in Figure [1].

These lattice paths shall only use horizontal steps to the right or vertical steps downwards, i.e., steps leading from lattice point (x, y) to lattice point (x + 1, y) or to lattice point (x, y − 1).

The starting points of these r lattice paths are the points

\[ S_i := (0, a_{i,i}) , \]

i.e., the lattice path corresponding to row i starts on the vertical axis at height equal to the first part of row i.

The ending points of these lattice paths are the points

\[ E_i := (a_{i,i} − 1, 0) , \]

i.e., the lattice path corresponding to row i ends on the horizontal axis and consists of \( a_{i,i} − 1 \) horizontal steps at heights corresponding to the parts of row i (including the zero parts, which correspond to steps at height 0).

It is easy to see that the set \( D_n \) of n–dimensional DPPs is in bijection with the set of nonintersecting lattice paths as defined above, with no starting point higher than n. (But note that in this representation, the number of horizontal steps of a path is equal to the length of the corresponding row plus the number of horizontal steps at height zero.)

4 Inversions in ASMs

4.1 Orientation of cells

Definition 7 (cells and their partial row– and column–sums). By a cell in some ASM A we simply mean the entry \( A_{i,j} \) (at row i and column j): If \( A_{i,j} = 0 \), we call this a zero–cell, otherwise a non–zero cell.
Figure 1: Lattice–path–encoding for DPPs.
Consider the following two DPPs of dimension 6:

\[
\begin{array}{cccccc}
6 & 6 & 6 & 4 & 2 \\
5 & 3 & 2 & 1 \\
2 & & & & & \\
\end{array}
\quad
\begin{array}{cccccc}
6 & 6 & 6 & 5 & 0 \\
4 & 3 & 1 & & & \\
2 & & & & & \\
\end{array}
\]

The pictures below show the non–intersecting lattice paths corresponding to the above DPPs:

Note that the right DPP has a zero–padded first row, and that the special parts of these DPPs correspond to the horizontal steps (at heights > 0) in the “special range” below the main diagonal $y = x$ (indicated by the gray triangle).
By the \textit{row–sum} \( rsum(A_{i,j}) \) of some cell \( A_{i,j} \) we mean the sum of all entries “weakly to the left” of \( A_{i,j} \):

\[
\text{rsum}(A_{i,j}) := \sum_{k=1}^{j} A_{i,k}.
\]

Likewise, by the \textit{column–sum} \( csum(A_{i,j}) \) of some cell \( A_{i,j} \) we mean the sum of all entries “weakly above” \( A_{i,j} \):

\[
\text{csum}(A_{i,j}) := \sum_{k=1}^{i} A_{k,j}.
\]

Clearly, for every ASM \( A \), the row– and column–sums can only assume values 0 or 1. Observe that

- if \( \text{rsum}(A_{i,j}) = 1 \), then either \( A_{i,j} = 1 \) or \( A_{i,j} = 0 \) and the closest non–zero cell \textit{to the left} of \( A_{i,j} \) is 1,
- if \( \text{rsum}(A_{i,j}) = 0 \), then either \( A_{i,j} = -1 \) or \( A_{i,j} = 0 \) and the closest non–zero cell \textit{to the right} of \( A_{i,j} \) is 1,
- if \( \text{csum}(A_{i,j}) = 1 \), then either \( A_{i,j} = 1 \) or \( A_{i,j} = 0 \) and the closest non–zero cell \textit{above} \( A_{i,j} \) is 1,
- if \( \text{csum}(A_{i,j}) = 0 \), then either \( A_{i,j} = -1 \) or \( A_{i,j} = 0 \) and the closest non–zero cell \textit{below} \( A_{i,j} \) is 1.

\textbf{Definition 8} \textit{(pair of signs of a cell)}. Given some ASM \( A \), for every cell \( A_{i,j} \) of \( A \) we consider the pair

\[(2 \cdot \text{rsum}(A_{i,j}) - 1, 2 \cdot \text{csum}(A_{i,j}) - 1) :\]

According to the possible signs of this pair (plus or minus), we introduce for cells with signs

- \((+, +)\) the shortcut \textit{pp–cs},
- \((+, -)\) the shortcut \textit{pm–cs},
- \((- , +)\) the shortcut \textit{mp–cs},
• \((-\, -\)\) the shortcut \(mm-cs\).

Note that the pair of signs of a zero-cell indicates the position of the nearest 1 in horizontal/vertical direction. E.g., for a zero–pp–c the nearest 1 lies

• in horizontal positive direction (i.e., to the right)
• and in vertical positive direction (i.e., below).

4.2 Inversions in ASMs

Observe that the zero–pp–cs in some permutation matrix \(A = (\delta_{j,\sigma(i)})^{(n,n)}_{(i,j)=(1,1)}\) are in one–to–one correspondence with the inversions of \(\sigma\): \(A_{i,j} = 0\) is a pp–c if

• entry 1 in row \(i\) is in column \(y = \sigma(i) > j\) (to the right of column \(j\)),
• and entry 1 in column \(j\) is in row \(x = \sigma^{-1}(j) > i\) (below row \(i\)).

Together, this gives

\[ (\sigma(i) = y) > (j = \sigma(x)) \text{ and } x > i, \]

i.e., \((i, x)\) is an inversion of the permutation \(\sigma\).

More generally, observe that the fourfold sum (3) defining inversions of an ASM may be rewritten as two double sums:

\[
\sum_{(k,j) = (1,1)}^{(n,n)} \sum_{(i,l) = (1,1)}^{(k-1,j-1)} A_{i,j} \cdot A_{k,l},
\]

and that the inner double sum is simply the product

\[
\left( \sum_{i=1}^{k-1} A_{i,j} \right) \cdot \left( \sum_{l=1}^{j-1} A_{k,l} \right) = \begin{cases} 1 & \text{if } A_{k,l} = -1 \text{ or } A_{k,l} \text{ is a zero–pp–c}, \\ 0 & \text{otherwise.} \end{cases}
\]

Hence the inversions of an ASM \(A\) (according to (3)) are equal to the number of zero–pp–cs of \(A\) plus the number of entries \(-1\) in \(A\).
4.3 Quadruplet of statistics, reformulated

These considerations immediately lead to the following reformulation of the quadruplet of statistics \((i, p, s, q)\):

|   | for \(D \in D_n\): | for \(A \in A_n\): |
|---|-----------------|-----------------|
| \(i\) | \# (non–special parts) | \# (zero–pp–cs) |
| \(p\) | \# (parts equal to \(n\)) | \# (zero–pp–cs in first row) |
| \(s\) | \# (special parts) | \# (entries \(-1\)) |
| \(q\) | \# (parts \((n-1)\)) + \# (rows of length \((n-1)\)) | \# (zero–mm–cs in last row) |

Now it is easy to illustrate by a simple example that neither Lalonde’s nor Striker’s bijection respect statistic \(q\): See Figure 2, where we indicated

- the zero–pp–cs by small equilateral triangles pointing “north–west” (upwards and to the left)
- and the zero–mm–cs by small equilateral triangles pointing “south–east” (downwards and to the right).

4.4 pp–cs and mm–cs are equinumerous

Observe that for every ASM \(A\),

- the number of zero–pp–cs equals the number of zero–mm–cs,
- and the number of zero–mp–cs equals the number of zero–pm–cs.

To show this, we construct a bijection (see Figure 3): For a zero–pp–c \(A_{i,j}\), we construct two paths,

- both starting at \(A_{i,j}\)
- both proceeding only horizontally to the right or vertically downwards,
- and both changing horizontal/vertical direction of movement whenever they encounter a non–zero cell.
Figure 2: Neither Lalonde’s nor Striker’s bijection respect statistic $q$.
Consider the 4–dimensional DPP $D = \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} 
4 & 4 & 4 & 3 & 2 \\
\hline 
3 & 3 & 3 & 3 & 3 \\
\hline 
2 & 2 & 2 & 2 & 2 \\
\hline 
1 & 1 & 1 & 1 & 1 \\
\hline 
0 & 0 & 0 & 0 & 0 \\
\end{array}$: $D$’s $q$–statistic is 2, since it has one part 3 and one row of length 3.
The lattice path representation of $D$ is shown to the left, the inversion word (needed for both Lalonde’s and Striker’s bijection) corresponding to $D$ is $(3, 1, 1)$, and the monotone triangle (needed for Striker’s bijection) corresponding to $D$ is shown on the top.
Both Striker’s and Lalonde’s bijection map $D$ to the ASM shown to the right, with statistic $q$ equal to 3 (the number of mm–cs, indicated by south–east pointing triangles, in the last row is 3). However, $D$ should be mapped to the ASM shown at the bottom, which is the only ASM of dimension 4 with quadruplet statistic $(3, 5, 0, 2)$. 

\[
\begin{array}{cccc}
4 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
\end{array}
\]

\[
\begin{array}{cccc}
3 & 5 & 0 & 2 \\
4 & 4 & 3 & 2 \\
3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
3 & 5 & 0 & 3 \\
4 & 4 & 3 & 2 \\
3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
3 & 5 & 0 & 2 \\
4 & 4 & 3 & 2 \\
3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
\]
Figure 3: The bijection mapping pp–cs to mm–cs.
The left picture shows the two paths constructed for pp–c $A_{1,1}$: The first cell of crossing of these paths is mm–c $A_{4,6}$, so $A_{1,1}$ is mapped to $A_{4,6}$.
The right picture shows the two paths constructed for pp–c $A_{3,2}$: The first cell of crossing of these paths is mm–c $A_{6,5}$, so $A_{3,2}$ is mapped to $A_{6,5}$. Observe that these paths also meet (but do not cross) at the cell $A_{5,4} = -1$.

One of these paths starts horizontally (to the right), the other starts vertically (downwards): The pictures in Figure 3 illustrate this simple idea. Now observe that the paths starting horizontally must necessarily end vertically, and vice versa. Hence the paths must eventually cross at some cell, and any such cell of crossing must necessarily be a zero–mm–c (the paths could meet but not cross at some cell with entry $-1$, see the right picture in Figure 3): Map $A_{i,j}$ to the first cell of crossing $A_{k,l}$ thus obtained. By symmetry (reflection at the second diagonal of $A$) it is immediately clear that this construction gives a bijection.

(By symmetry again, it is clear that the same idea works for pm–cs and mp–cs.)

5 The bijection between DPPs without special parts and permutation matrices

5.1 The statistic $q$ for DPPs without special parts

Observe that if a DPP $D \in \mathcal{D}_n$ without special parts has a row of length $n - 1$, this row

- must start with part $n$
• and must not contain parts smaller than $n - 1$

(this fact is immediately seen from the lattice path encoding of $D$).

Stated otherwise: The somewhat complicated condition “existence of a path of length $n - 1$” can be simply expressed as

$$
\# \text{(parts equal to } n) + \# \text{(parts equal to } n - 1) \geq n - 1.
$$

in the case of DPPs without special parts.

5.2 The bijection

Here is the construction: Given some permutation matrix $A$ (i.e., some ASM without entries $-1$) of dimension $n$, set $k = 1$, $A_1 = A$, and repeat the following step $n - 1$ times:

• note down the number $a_k$ of pp–cs in the first row of $A_k$,
• delete the first row and the column containing the 1 (i.e., column $a_k + 1$),
• rotate the matrix by $180^\circ$,
• let $A_{k+1}$ be the $(n - k) \times (n - k)$–matrix thus obtained, and increase $k$ by 1.

We claim that the sequence of $(n - 1)$ numbers $(a_1, a_2, \ldots, a_{n-1})$ thus obtained

• is an inversion word, i.e., $0 \leq a_i \leq n - i$ for $i = 1, 2, \ldots, n - 1$,
• $a_1 = p(A)$,
• $a_2 = q(A) + \begin{cases} -1 & \text{if } q(A) + p(A) \geq n \\ 0 & \text{otherwise,} \end{cases}$
• $a_1 + a_2 + \cdots + a_{n-1} = i(A)$.

If we succeed in showing this, we may simply employ Striker’s bijection between inversions words and DPPs without special parts: This bijection maps an inversion word $(a_1, a_2, \ldots, a_{n-1})$ to a DPP $D$ of dimension $n$ with precisely $a_i$ parts $(n + 1 - i)$. But this implies
\( \bullet i(D) = a_1 + a_2 + \cdots + a_{n-1} , \)

\( \bullet p(D) = a_1 , \)

\( \bullet q(D) = a_2 + \begin{cases} 1 & \text{if } a_2 + a_1 \geq n - 1 , \\ 0 & \text{otherwise} . \end{cases} \)

(the last assertion is due to the simple characterization (4)) whence we see, that our (yet prospective) bijection does, in fact, respect the quadruplet of statistics \((i, p, s, q)\).

So let us finish our argument: It is clear that \(0 \leq a_k \leq n - k\), since \(a_i\) is (by construction) the number of pp–cs in the first row of an \((n - k + 1) \times (n - k + 1)\) permutation matrix, and this number cannot exceed \(n - k\).

By construction, we also have \(a_1 = p(A)\). Of the \(q(A)\) mm–cs in the last row of \(A\), precisely one will be deleted in the first step of our construction if and only if \(q(A) + p(A) \geq n\): By the rotation at the end of the first step, precisely these “surviving” mm–cs will turn up as the pp–cs in the first row of the matrix at the beginning of the second step, whose number will give \(a_2\).

Finally, if we counted \(a_k\) pp–cs in the first row of the \((n + 1 - k) \times (n + 1 - k)\)–matrix \(A_k\) at the beginning of step \(k\), then the column \(a_k + 1\) of \(A_k\) contains precisely \(a_k\) mm–cs, since the submatrix given by rows 2 to \(n + 1 - k\) and columns 1 to \(a_k\) must contain precisely \(a_k\) entries 1. So in every step, the number of deleted pp–cs equals the number of deleted mm–cs, and after \(n - 1\) steps, all of these cells are deleted. (Figure 4 illustrates this bijective construction.)

### 6 An open question

Our construction was successful because we have the “translation” (4) of the (somewhat complicated) condition

DPP \(D\) of dimension \(n\) has a row of length \(n - 1\)

for DPPs without special parts to an obvious corresponding condition for ASMs without entries \(-1\).

It might be helpful to identify such “corresponding condition” for general ASMs.
Figure 4: Illustration of the steps of the bijective construction. Starting with the $7 \times 7$ permutation matrix corresponding to the permutation $\sigma = (5, 2, 1, 7, 3, 6, 4)$, the pictures show the 6 steps of the bijective construction: The rows and columns to be deleted are indicated by thick gray lines. The inversion word thus obtained is $(4, 2, 1, 1, 0, 1)$.

Step 1: $a_1 = 4$.

Step 2: $a_2 = 2$.

Step 3: $a_3 = 1$.

Step 4: $a_4 = 1$.

Step 5: $a_5 = 0$.

Step 6: $a_6 = 1$. 

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References

[1] Arvind Ayyer. A natural bijection between permutations and a family of descending plane partitions. *European Journal of Combinatorics*, 31(7):1785 – 1791, 2010.

[2] Roger E. Behrend, Philippe Di Francesco, and Paul Zinn-Justin. On the weighted enumeration of alternating sign matrices and descending plane partitions. *Journal of Combinatorial Theory, Series A*, 119(2):331 – 363, 2012.

[3] Roger E. Behrend, Philippe Di Francesco, and Paul Zinn-Justin. A doubly-refined enumeration of alternating sign matrices and descending plane partitions. *Journal of Combinatorial Theory, Series A*, 120(2):409 – 432, 2013.

[4] David M. Bressoud. *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture*. Cambridge University Press, 1999.

[5] Greg Kuperberg. Another proof of the alternative-sign matrix conjecture. *International Mathematics Research Notices*, 1996(3):139–150, 1996.

[6] Pierre Lalonde. Lattice paths and the antiautomorphism of the poset of descending plane partitions. *Discrete Mathematics*, 271(1–3):311 – 319, 2003.

[7] Pierre Lalonde. Alternating sign matrices with one −1 under vertical reflection. *Journal of Combinatorial Theory, Series A*, 113(6):980 – 994, 2006.

[8] W.H. Mills, David P Robbins, and Howard Rumsey Jr. Alternating sign matrices and descending plane partitions. *Journal of Combinatorial Theory, Series A*, 34(3):340 – 359, 1983.

[9] Jessica Striker. A direct bijection between descending plane partitions with no special parts and permutation matrices. *Discrete Mathematics*, 311(21):2581 – 2585, 2011.

[10] D. Zeilberger. Proof of the alternating sign matrix conjecture. *Electronic J. Comb.*, 3(#R13), 1995.