Transport phenomena of multi-Weyl semimetals in co-planar setups

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In addition to the well known topological semimetals (Dirac and Weyl semimetals), a new type of Weyl semimetal with a topological charge \((n)\) larger than one, namely, multi-Weyl semimetal (m-WSM), has been realized in condensed matter experiments. Though the chiral anomaly induced magneto-transport phenomena have been extensively studied in single Weyl semimetal \((n = 1)\), it has not been addressed in the context of multi-Weyl semimetals \((n > 1)\) so far. Using semiclassical Boltzmann transport formalism with the relaxation time approximation, we investigate several intriguing transport properties such as longitudinal magnetoconductivity (LMC), planar Hall effect (PHE), thermo-electric coefficients (TEC) and planar Nernst coefficient (PNE) in m-WSMs considering co-planar magnetic field and electric field or temperature gradient setup. Starting from the linearized model, we show analytically that at zero temperature both LMC and planar Hall conductivity (PHC) vary cubically with topological charge \((n^3)\) while the finite temperature \((T \neq 0)\) correction is proportional to \((n + n^2)T^2\). Interestingly, we find that both the longitudinal and transverse TECs vary quadratically with topological charge (i.e., \(n^2\)). We find universal magnetic field and angular dependencies of all the above transport coefficients. Moreover, in order to verify the analytical findings, we simultaneously investigate their behavior with magnetic field, angle, temperature and chemical potential numerically using the lattice model for m-WSMs. Our analysis with temperature and chemical potential suggests that the chiral anomaly and chiral magnetic effect terms dominate in the transverse part of electrical conductivity and TEC, respectively, while Drude contribution becomes significant for the longitudinal coefficients. We comment also on the possible lattice effects for the deviation of numerical results from the analytical one.

I. INTRODUCTION

In the field of three dimension (3D) topological systems, Weyl semimetal (WSM) has emerged as prime topic of interest. In condensed matter physics, Weyl Fermion appears as a low energy excitation of gapless chiral Fermion near the touching of a pair of nondegenerate bands\textsuperscript{1-9}. The non-trivial topological properties of the WSMs appear due to Weyl nodes. The Weyl node describing the singularity in k-space acts as a source or sink of the Berry curvature. According to no-go theorem, the Weyl nodes always come in pairs of positive and negative topological charges (also called chirality) and total topological charge in the Brillouin zone vanishes\textsuperscript{10,11}. In order to have a topological charge \((n)\) associated with the Weyl node, WSM has to break either time reversal symmetry (TRS) or the space inversion symmetry (IS). The topological charge whose strength is related to Chern number (Gauss’s theorem) is quantized to integer values\textsuperscript{12}.

The WSM phase has been realized experimentally in several inversion asymmetric compounds (TaAs, MoTe\textsubscript{2}, WTe\textsubscript{2}) without breaking TRS\textsuperscript{13-17}. However, all of these materials mentioned above are belong to single-Weyl semimetal, whose energy dispersions are linear in wave vectors and topological charge equals \(\pm 1\). Recently, it has been proposed that the multi-Weyl fermions can also be realized in condensed matter physics\textsuperscript{18-20}. The m-WSMs are referred to those materials which contain Weyl nodes with topological charge higher than 1 (i.e. \(n > 1\)). The quasi-particle dispersion for \(n > 1\) shows natural anisotropy in dispersion. The double-WSM \((n = 2)\) and triple-WSM \((n = 3)\) show linear dispersion along one symmetry direction and quadratic and cubic energy dispersion relations for other two directions respectively. From the DFT calculations, it has been suggested that HgCr\textsubscript{2}Se\textsubscript{4} and SrSi\textsubscript{2} can be the candidate materials for double-WSM\textsuperscript{18-20} whereas the triple-WSM can be realized in A(MoX)\textsubscript{3} (with \(A = Rb, Tl; X = Te\)) kind of materials\textsuperscript{21}. Discrete rotational symmetry in a lattice imposes a strict restriction that only the Weyl nodes with topological charges \(n \leq 3\) can be permitted in real materials\textsuperscript{22}. Moreover, the single-WSM can be viewed as 3D analogue of graphene whereas the double-WSM and triple-WSM can be represented as 3D counterparts of bilayer\textsuperscript{23} and ABC-stacked trilayer graphene\textsuperscript{24,25}, respectively.

The elementary WSM exhibits several fascinating transport properties in the presence as well as absence of external fields. Negative longitudinal magnetoresistance (LMR) and planar Hall effect (PHE) are two most important transport properties which appear due to the non-conservation of separate electron numbers of opposite chirality for relativistic massless fermions, an effect known as the chiral or Adler-Bell-Jackiw anomaly\textsuperscript{2,8-11,26-30}. In recent years, these two transport properties in single WSM are extensively studied both theoretically and experimentally\textsuperscript{31-49}. It is now natural question to ask that how LMR and PHE behave in m-WSMs. In particular, the effects of enhancement of the density of states, anisotropic nonlinear energy dispersion, and modified spin-momentum locking structure on LMR and PHE in m-WSMs are not addressed so far.

In general, the generation of a transverse electric field in the presence of a transverse temperature gradient refers to the Nernst effect. The conventional Nernst effect appears due to Lorentz force in a system in the presence of an external magnetic field applied perpendicular to the temperature gradient. This setup will have an induced electric field normal to both. On the other hand, anomalous Nernst effect (ANE) appears only due to the anomalous velocity of the quasi-particle generated by non-trivial Berry curvature. Using the Nernst setup, the thermo-electric phenomena such as Peltier
coefficient, Nernst effect and longitudinal magneto-thermal conductivity are well studied in the context of single-Weyl semimetal.\textsuperscript{86,50–56} Inspired by the PHE, one can consider a situation where both the thermal gradient $\nabla T$ and magnetic field $\mathbf{B}$ are in-plane but not parallel to each other. These result in an in-plane transverse voltage and this phenomena is referred as the planar Nernst effect (PNE). This effect is different from the conventional Nernst effect as well as ANE. Actually, PNE is the thermal counterparts of the PHE where electric field $\mathbf{E}$ is replaced by $\nabla T$. In the planar Nernst setup, the response of thermo-electric coefficients in the context of m-WSMs has not been explored yet.

Having stated the current research trends, we focus on the transport coefficients in m-WSMs considering the co-planar setups. To be precise, first, we calculate the LMC and PHC analytically considering a linearized model. We find that zero temperature LMC and PHC go as $n^3$ while the finite temperature correction is $O((n + n^3)T^2)$. Next, we find longitudinal thermo-electric coefficient (LTEC) and transverse thermo-electric coefficient (TTEC) (usually referred as the Peltier coefficient) both go as $n^2 T$. Therefore, the multi-Weyl nature shows up distinctly for LMC, PHC and LTEC, TTEC. We find that LMC and LTEC show $B^2 \cos^2 \gamma$ dependence for all the cases $(n = 1, 2, 3)$ whereas PHC and TTEC are proportional to $B^2 \sin \gamma \cos \gamma$. Here, $\gamma$ is the angle between $\mathbf{B}$ and $\mathbf{E}$ for the setup of PHE or between $\mathbf{B}$ and $\nabla T$ for the measurement of thermo-electric coefficients. Using all these coefficients, we calculate the functional form of planar Nernst coefficient which is also proportional to $B^2 \sin \gamma \cos \gamma$. Moreover, in order to verify our analytical findings, we numerically investigate the magnetic field dependence, angular dependence, temperature dependence and the gate voltage dependence of all the above mentioned coefficients considering the lattice models for m-WSM. We systematically show by varying temperature and gate voltage that chiral anomaly and chiral magnetic effect maximally govern the behavior of transverse transport coefficients while Drude part dominates for longitudinal coefficients. We also comment about the lattice possible effects in these quantities.

The rest of the paper is organized as follows. In the next section (Sec. II), we introduce the linearized Hamiltonian as well as TRS breaking lattice Hamiltonian for m-WSMs. Sec. III is devoted to the general expressions of LMC, PHC, TECs and PNE. In Sec. IV, analytical expressions using linearized model and numerical results considering the lattice models of m-WSMs are presented for magneto-transport and thermo-electric transport coefficients (LMC, PHC, TECs and PNE). We discuss the magnetic field dependence, angular dependence, chemical potential dependence and temperature dependence of all the above mentioned quantities in detail. Finally, we summarize our results and discuss possible future directions in Sec. V.

\section{Model Hamiltonian}

\subsection{Linearized Hamiltonian}

The low energy effective Hamiltonian describing the Weyl node with topological charge $(n)$ can be written as\textsuperscript{86,19,57}

\begin{equation}
H_n(\mathbf{k}) = \alpha_n k_+^n [\cos (n\phi_k) \sigma_x + \sin (n\phi_k) \sigma_y] + n k_z \sigma_z \label{eq1}
\end{equation}

where $k_\perp = \sqrt{k_x^2 + k_y^2}$ and $\phi_k = \arctan(k_y/k_x)$. Here, in general, $\alpha_n$ bears the connection to the Fermi velocity. For example, $\alpha_1$ has the dimension of Fermi velocity, while $\alpha_2$ has the dimension of mass. $v$ is equivalent to the velocity associated with the $z$-direction. Here, $\sigma_i (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices representing the pseudo-spin indices. The above Hamiltonian can be written in a compact form as $H = \mathbf{n}_k \cdot \sigma$ with $\mathbf{n}_k = (\alpha_n k_+^n \cos (n\phi_k), \alpha_n k_+^n \sin (n\phi_k), n k_z)$. The energy dispersion of the Weyl node is given by

\begin{equation}
E_\pm^\pm (\mathbf{k}) = \pm \sqrt{\alpha_n^2 k_+^{2n} + v^2 k_z^2}. \label{eq2}
\end{equation}

where $\pm$ represents conduction and valence bands respectively. It is clear from the Eq. (2) that the topological charge determines not only the topological nature of the wave function but also the anisotropic energy dispersion of the system.

The Berry curvature of the $m^{th}$ band for a Bloch Hamiltonian $H(k)$, defined as the Berry phase per unit area in the $k$ space, is given by\textsuperscript{12}

\begin{equation}
\Omega^m_\alpha (\mathbf{k}) = (-1)^m \frac{1}{4} n_{\mathbf{k}}^{abc} \mathbf{n}_k \cdot \left( \frac{\partial \mathbf{n}_k}{\partial k_b} \times \frac{\partial \mathbf{n}_k}{\partial k_c} \right) \label{eq3}
\end{equation}

where $m$ is the band index. The explicit form of the Berry curvature associated with the Weyl node is given by

\begin{equation}
\Omega^\pm_\mathbf{k} = \pm \frac{1}{2} \frac{n v \alpha_n^2 (k^{2n-2})}{\epsilon_k^3} \left( k_x, k_y, n k_z \right) \label{eq4}
\end{equation}

The quasi-particle velocities associated with Weyl node are given by

\begin{equation}
\mathbf{v}_k = \frac{1}{\epsilon_k} \left( k_x \alpha_1^2 k_+^{2(n-1)} k_y, n \alpha_1^2 k_+^{2(n-1)} k_z, v k_z \right) \label{eq5}
\end{equation}

\subsection{Lattice Hamiltonian}

We now discuss a prototype lattice model for multi-Weyl semimetal that breaks TRS but remains invariant under inversion. Generalizing the low-energy effective Hamiltonian of a multi-WSM with broken TR symmetry, the corresponding lattice model can be written as\textsuperscript{57}

\begin{equation}
H = N_\mathbf{k} \cdot \sigma \label{eq6}
\end{equation}

For the single-WSM with $n = 1$, $N_\mathbf{k}$ takes the form $N_x = t \sin k_x, N_y = t \sin k_y$ and $N_z = t_z \cos k_z - m_z + t_0 (2 - \epsilon_{k}^2)$.
\[ \cos k_x - \cos k_y \]. In this model, the Weyl nodes are located at \( \mathbf{K} = (0, 0, \pm k_0) \) with

\[ \cos(k_0) = \frac{t_0}{t_z} \left[ \frac{m_z}{t_0} + \cos k_x + \cos k_y - 2 \right] \]  

(7)

One can expand the above Hamiltonian for \( k_0 = \pm \pi/2 \) with \( m_z = 0 \) to obtain the low energy Weyl Hamiltonian:

\[ H_{Weyl,n=1} \simeq v(\tau_0 \sigma_x k_x + \tau_0 \sigma_y k_y) + v_z \tau_z \sigma_z k_z. \]  

Here, \( \tau_\nu \) represents the chirality of the model.

For the double-Weyl semimetal \( (n = 2) \), the form of \( N_k \) becomes \( N_x = t(\cos k_x - \cos k_y) \), \( N_y = t \sin k_x \sin k_y \) and \( N_z = t_z \cos k_z - m_z + t_0(6 + \cos 2k_x + \cos 2k_y - 4 \cos k_x - 4 \cos k_y) \). The system described by the above contains two Weyl nodes at \( (0, 0, \pm k_0) \) with

\[ \cos(k_0) = \frac{t_0}{t_z} \left[ \frac{m_z}{t_0} - (6 + \cos 2k_x + \cos 2k_y - 4 \cos k_x - 4 \cos k_y) \right] \]  

(8)

One can similarly expand the above Hamiltonian around \( k_0 = 0 \) and \( m_z = 0 \). In this case, the low energy Hamiltonian for double WSM is given by \( H_{Weyl,n=2} \approx \frac{1}{m}(\tau_0 \sigma_x (k_x^2 - k_y^2) + \tau_0 \sigma_y k_x k_y) + v_z \tau_z \sigma_z k_z. \)

Similarly, for a triple-Weyl semimetal with the topological charge \( n = 3 \), one should replace \( N_k \) by \( N_x = t \sin k_x (1 - \cos k_x - 3(1 - \cos k_y)) \), \( N_y = -t \sin k_y (1 - \cos k_y - 3(1 - \cos k_x)) \), and \( N_z = t_z \cos k_z - m_z + t_0(6 + \cos 2k_x + \cos 2k_y - 4 \cos k_x - 4 \cos k_y) \). Here, the Weyl points are appeared at \( k = (0, 0, \pm k_0) \) with \( k_0 \) followed by the Eq.(8). The low energy triple WSM Hamiltonian is given by \( H_{Weyl,n=3} \approx \frac{1}{m}(\tau_0 \sigma_x (k_x^2 - 3k_x k_y^2\tau_z) - \tau_0 \sigma_y (k_y^3 - 3k_y^2 k_z\tau_z) + v_z \tau_z \sigma_z k_z. \)

The energy dispersions of the WSM for \( n = 1, 2, 3 \) are shown in Fig. 1 along various high symmetry directions. It is clear from the figure that the dispersion around a Weyl node with \( n = 1 \) is isotropic in all direction whereas different for double- and triple-Weyl node. In particular, the dispersion along the \( z \) is different from the dispersion along \( x \) or \( y \) direction for both \( n = 2 \) and \( n = 3 \) cases.

III. SEMICLASSICAL FORMALISM FOR CALCULATING TRANSPORT COEFFICIENTS

It has been shown that in the presence of electric field and magnetic field, transport properties get substantially modified due to presence of non-trivial Berry curvature which acts as a fictitious magnetic field in the momentum space. In this section, we will focus on some specific topological responses, namely, longitudinal magnetocconductivity, planar Hall effect and thermo-electric coefficients that could be observed in all Dirac and Weyl semimetals. At the outset, we discuss the planar Hall and planar Nernst set up. In the low field regime, we start from the quasi-classical Boltzmann transport equation. We make resort to kinetic theory, a semiclassical framework approach considering the assumption \( T \ll \sqrt{B} \ll \mu \), where \( \mu \) is the chemical potential, measured from the band-touching point. In this regime, one can ignore the Landau quantization of the energy levels and use semiclassical Boltzmann equation.

In the presence of external perturbative fields (for example, electric field \( \mathbf{E} \) and temperature gradient \( \nabla T \)), the charge current \( (\mathbf{J}) \) and thermal current \( (\mathbf{Q}) \) from linear response theory, can be written as

\[ J_\alpha = L_{\alpha\beta}^1 E_\beta + L_{\alpha\beta}^{12} (-\nabla_\beta T) \]  

(9)

\[ Q_\alpha = L_{\alpha\beta}^{21} E_\beta + L_{\alpha\beta}^{22} (-\nabla_\beta T) \]  

(10)

where \( \alpha \) and \( \beta \) are spatial indices running over \( x, y, z \). Here, \( L_{\alpha\beta}^1 \) and \( L_{\alpha\beta}^{12} \) define the charge conductivity tensor and thermo-electric tensor respectively. The tensors \( L_{\alpha\beta}^{12} \) and \( L_{\alpha\beta}^{21} \) are related to each other by the Onsager’s relation: \( L_{\alpha\beta}^{21} = T L_{\beta\alpha}^{12} \). In the low temperature, the transport coefficients obey the Mott relation as \( L_{\alpha\beta}^{12} = -\frac{e^2}{3\pi} \frac{L_{\alpha\beta}^{21}}{T} \).

The Boltzmann transport equation in its phenomenological form can be written as\(^{58}\)

\[ \left( \frac{\partial}{\partial t} + \mathbf{\hat{r}} \cdot \nabla \mathbf{r} + \mathbf{\hat{k}} \cdot \nabla \mathbf{k} \right) f_{k,r,t} = I_{coll}(f_{k,r,t}) \]  

(11)

where on the right side \( I_{coll}(f_{k,r,t}) \) is the collision integral which incorporates the effects of electron correlations and impurity scattering. We are interested in computing the electron distribution function which is given by \( f_{k,r,t} \). Under the relaxation time approximation with the parameter \( \tau \) that quantifies the average time between two successive collision, the steady-state solution to the Boltzmann equation described in Eq. (11) can be rewritten as

\[ (\mathbf{\hat{r}} \cdot \nabla \mathbf{r} + \mathbf{\hat{k}} \cdot \nabla \mathbf{k}) f_k = \frac{f_0 - f_k}{\tau(\mathbf{k})} \]  

(12)

where \( f_0 \) is the equilibrium Fermi-Dirac distribution function. We here ignore the momentum dependence of \( \tau \) as the qualitative behavior of the result does not change. Now we shall revisit the semiclassical equation of motion for an electron in presence of Berry curvature\(^{59,60}\)

\[ \mathbf{\dot{r}} = D(\mathbf{B}, \Omega_k)(\mathbf{v}_k + \frac{e}{\hbar} (\mathbf{E} \times \Omega_k) + \frac{e}{\hbar} (\mathbf{v}_k \cdot \Omega_k) \mathbf{B}) \]  

(13)
\[ \hbar \dot{\mathbf{k}} = D(\mathbf{B}, \Omega_k)[e \mathbf{E} + \frac{e}{\hbar}(\mathbf{v}_k \times \mathbf{B}) + \frac{e^2}{\hbar}(\mathbf{E} \cdot \mathbf{B})\Omega_k] \] (14)

Here, \( D(\mathbf{B}, \Omega_k) = (1 + \frac{e}{\hbar}(\mathbf{B} \cdot \Omega_k))^{-1} \) is the phase space factor as the Berry curvature \( \Omega_k \) modifies the phase space volume element \( d\mathbf{k} \rightarrow D(\mathbf{B}, \Omega_k) d\mathbf{k} \). and \( \mathbf{v}_k = \frac{1}{\hbar} \partial f_{eq}/\partial \mathbf{v}_k \) is the group velocity. Hereafter, we denote \( D(\mathbf{B}, \Omega_k) \) by \( D \). The term \((\mathbf{E} \cdot \mathbf{B})\Omega_k\) represents the anomalous velocity perpendicular to the applied electric field, while the chiral magnetic effect is due to \((\mathbf{v}_k \cdot \Omega_k)\mathbf{B}\). Moreover, the term \((\mathbf{E} \cdot \mathbf{B})\) is responsible for chiral anomaly which arises in axion-electrodynamics of WSM.

**A. Setup 1: LMC and PHC**

The setup 1 is considered for calculating transport coefficients in PHE. The PHE is defined through an induction of in-plane transverse voltage when the co-planar electric and magnetic fields are not perfectly aligned with each other. In order to get the general expression for planar Hall conductivity, we first consider that the electric field \((\mathbf{E})\) is applied along the \(x\)-axis and \(\mathbf{B}\) is rotated in \(x-y\) plane at a finite angle \(\gamma\) from the \(x\)-axis, i.e. \(\mathbf{B} = B \cos \gamma \hat{x} + B \sin \gamma \hat{y}\), \(\mathbf{E} = E \hat{x}\).

Using the equation of motion described in Eq. (13) and Eq. (14), the general expression for the PHC in the above configuration from the semiclassical Boltzmann equation can be written as \[^{36,39,40}\]

\[ \sigma_{yx} = e^2 \int \frac{d^3k}{(2\pi)^3} \int \left[ (v_y + \frac{eB \sin \gamma}{\hbar} \mathbf{v}_k \cdot \Omega_k) \right] \left( \frac{\partial f_{eq}}{\partial \epsilon} \right) d\epsilon \] (15)

Using the above setup, the expression for the longitudinal magneto-conductivity is given by

\[ \sigma_{xx} = e^2 \int \frac{d^3k}{(2\pi)^3} \int \left[ D(v_x + \frac{eB \cos \gamma}{\hbar} \mathbf{v}_k \cdot \Omega_k)^2 \right] \left( \frac{\partial f_{eq}}{\partial \epsilon} \right) d\epsilon \] (16)

**B. Setup 2: Thermo-Electric Coefficient and Planar Nernst Coefficient**

In order to compute the thermo-electric coefficients (TECs), we apply the temperature gradient \((\nabla T)\) is applied along the \(x\) axis and the magnetic field is rotated in the \(x-y\) plane in the absence of electric field i.e. \(\mathbf{B} = B \cos \gamma \hat{x} + B \sin \gamma \hat{y}\), \(\nabla T = \nabla T \hat{x}\), \(\mathbf{E} = 0\). This setup is inspired by the PHE only electric field is replaced by thermal gradient. From the linear response theory, one can write the LTEC in this setup as

\[ \alpha_{xx} = e \int \frac{d^3k}{(2\pi)^3} \frac{(\epsilon - \mu)}{T} \left( \frac{\partial f_{eq}}{\partial \epsilon} \right) \left( v_x + \frac{eB \cos \gamma}{\hbar} \mathbf{v}_k \cdot \Omega_k \right)^2 \] (17)

and the TTEC i.e., Peltier coefficient can be written as

\[ \alpha_{yx} = e \int \frac{d^3k}{(2\pi)^3} \frac{(\epsilon - \mu)}{T} \left( \frac{\partial f_{eq}}{\partial \epsilon} \right) \left( v_y + \frac{eB \sin \gamma}{\hbar} \mathbf{v}_k \cdot \Omega_k \right) \] (18)

Moreover, using all the coefficients one can study the planar Nernst effect which refers to an in-plane transverse voltage when the \(\nabla T\) and the magnetic field \(\mathbf{B}\) are not aligned with each other. The Planar Nernst coefficient in this setup can be written as

\[ \nu = \frac{E_y}{dT/dx} = \frac{\alpha_{xy} \sigma_{xx} - \alpha_{xx} \sigma_{xy}}{\sigma_{xx}^2 + \sigma_{xy}^2} \] (19)

**IV. RESULTS**

In this section, we will first discuss our analytical results on LMC and PHC for the linearized model of multi-Weyl semimetals. Then we explain our numerical results on these quantities using lattice Hamiltonian. Next we will discuss the dependence of thermo-electric coefficient \((L_{Q,\alpha})\) on various parameters using both linearized and lattice model of m-WSMs. Finally we will comment on the functional dependence of planar Nernst coefficient for the linearized model. We finally verify our analytical findings by considering the TRS breaking lattice model of m-WSM.

**A. Analytical results for linearized Model**

In order to distinctly characterize the transport coefficients, one can break down the complete expression term-wise. For the LMC, as given in Eq. (16), we can break it into three terms, \((1) \sigma_{xx}^{(1)}\): pure velocity term consisting of only \(v_x\), \((2) \sigma_{xx}^{(2)}\): pure chiral anomaly term consisting of only \(\Omega_k, v_k\), \((3) \sigma_{xx}^{(3)}\): consisting of \(v_x(\Omega_k, v_k)\). We note that term \((2)\) and \((3)\) give together the chiral anomaly induced positive LMC in m-WSMs. We shall refer \(\sigma_{xx}^{(2)}\) term as \(\sigma_{xx}\) \((C.A.)\) in all the subsequent figures.

Similarly, for the planar Hall conductivity, as given in Eq. (15), one can break it in following form: \((1) \sigma_{yx}^{(1)}\) containing velocity term \(v_x v_y\), and \((2) \sigma_{yx}^{(2)}\): pure chiral anomaly term \((\Omega_k, v_k)^2\), \((3) \sigma_{yx}^{(3)}\) and \((4) \sigma_{yx}^{(4)}\) contain \(v_x(\Omega_k, v_k)\) and \(v_y(\Omega_k, v_k)\), respectively. We shall below refer the pure chiral anomaly term in PHC as \(\sigma_{yx}\) \((C.A.)\). Moreover, the terms \((2)\), \((3)\) and \((4)\) yield the chiral anomaly induced positive PHC in m-WSMs. At the same time, we present the functional dependence of the complete LMC and PHC, in the leading order. Please see the appendix for the detailed calculation of all the quantities.

A detailed calculation shows that the chiral anomaly term (i.e., term \((ii)\), \(\sigma_{zy}^{(2)}\) in LMC is given by
whereas the total longitudinal magneto-conductivity, $\sigma_{xx}$ is proportional to $n\mu^2 + nT^2 + B^2 \cos^2 \gamma(n^3 \mu^{-2/n} + (n + n^2)T^2 \mu^{-2-2/n})$ in its leading order. First we discuss the the case when $T = 0$. The term containing linear power of $n$ is B-dependent and coming from the velocity term $\sigma_{(1)}^{(1)}$; this is the Drude contribution from the Ohmic conductivity. The multi-Weyl nature that shows up in the pure chiral anomaly is the Drude contribution from the Ohmic conductivity. The total $\sigma_{xx}$ nature that shows up in the pure chiral anomaly part as well as total $\sigma_{xx}$ are the following: the scaling with $B$ and $\gamma$ are now combined with $n$; these are now $n^3 \cos^2 \gamma$ and $n^3 B^2$. The chemical potential dependence is also not unique; it varies as $n$ changes. The zero temperature conductivity varies as $n^3$. On the other hand, the finite temperature correction appears as $nT^2$ and $n^2 T^2$. This is in contrast to the finite temperature correction appearing in the Drude conductivity where the correction and the main term both vary linearly with $n$.

Now, the pure chiral anomaly term of the PHC $\sigma_{yx}^{(2)}$ is given by

$$\sigma_{yx}(C.A.) \simeq \frac{\Gamma(2 - 1/n)}{16\pi^{3/2}h^2\Gamma(5/2 - 1/n)} \cos \gamma \sin \gamma e^{3\tau n^3 B^2 v_{\alpha \nu_3}^{2/n}} (\mu^{-2/n} + \frac{K_B^2 T^2 (2 + n) \mu^{-2-2/n}}{3n^2})$$

Further calculation exhibits that total $\sigma_{yx}$ is also proportional to $B^2 \cos \gamma \sin \gamma (n^3 \mu^{-2/n} + (n + n^2)T^2 \mu^{-2-2/n})$. This is in contrast to the total LMC ($\sigma_{xx}$) because the $B$ independent Drude contribution is zero for the case of PHC. It is clear that the PHC is different from the Lorentz force mediated regular Hall conductivity and even from the Berry phase mediated anomalous Hall conductivity. Similar to the above case, PHC ($\sigma_{yx}$), and its pure chiral anomaly part $\sigma_{yx}(C.A.)$, both accommodate $n^3$ as par the scaling of $B^2$ and $\sin \gamma \cos \gamma$ are concerned. Similar to the case for LMC, the temperature correction remain quadratic in $T$ and with $n$ and $n^2$. Hence, The multi-Weyl nature clearly shows up as one consider higher $n$ ($n > 1$) WSMs.

We also calculate the LMC and PHC ($\sigma_{yx}$ and $\sigma_{yx}$) for the linearized model while the electric field is applied in the $\hat{z}$ direction. The functional dependence with $B$, and $\gamma$ remain unaltered. The $n^3$ scaling is observed. The complete calculation is presented in Appendix. Though the qualitative behavior is same, the qualitative behavior of LMC becomes different for $\sigma_{xx}$ and $\sigma_{xx}$ in the case of double and triple Weyl semimetals due to the presence of anisotropy in dispersion. On the other hand, $\sigma_{yz} = \sigma_{xx}$ for single-WSM.

Next, we study the thermo-electric responses in m-WSMs using the linearized model. We will follow the same prescription for the term-wise breakdown of LTEC and TTEC. $\alpha_{xx}^{(1)}$ is the quadratic velocity term $v_{xx}^2$, $\alpha_{xx}^{(2)}$ contains ($\Omega_k, \nu_k$) $^2$ term coming from chiral magnetic effect, and $\alpha_{xx}^{(3)}$ involves $v_{xx} (\Omega_k, \nu_k)$ term. Similarly, $\alpha_{yx}^{(2)}$ is coming from chiral magnetic effect. A detailed calculation shows that the second term in longitudinal thermo-electric coefficient ($\alpha_{xx}^{2}$), coming from the chiral magnetic effect, is given by

$$\alpha_{xx}(C.M.E.) \simeq \frac{\Gamma(2 - 1/n)}{6\pi^{3/2}h^2\Gamma(5/2 - 1/n)} (n^2 K_B^2 T \mu^{-1-2/n})$$

The total $\alpha_{xx}$ is proportional to $T(-\mu n + B^2 \mu^{-1-2/n} n^2 \cos^2 \gamma)$ in its leading order. The term containing linear power of $n$ is coming from the first term in $\alpha_{xx}^{(2)}$ containing only the velocity. The multi-Weyl nature that shows up in $\alpha_{xx}$ is the following: the scaling with $B$ and $\gamma$ are now combined with $n$; these are now $n^2 \cos^2 \gamma$ and $n^2 B^2$. The chemical potential dependence is also not unique. It varies as $n$ changes. One can note that in the limit $T \rightarrow 0$, Eq. (20) and Eq. (22) are related by the Mott relation. However, it is worth mentioning that the scaling with topological charge changes for $\alpha_{xx}$ compare to $\sigma_{xx}$.

Now, the second term of the transverse thermo-electric coefficient i.e. the purely CME induced contribution of $\alpha_{yx}$ is given by

$$\alpha_{yx}(C.M.E.) \simeq \frac{\Gamma(2 - 1/n)}{48\pi^{3/2}h^2\Gamma(5/2 - 1/n)} \cos \gamma \sin \gamma e^{3\tau n^3 B^2 v_{\alpha \nu_3}^{2/n}} (K_B^2 T \pi^2 \mu^{-1-2/n})$$

Further calculation exhibits that total $\alpha_{yx}$ is also proportional to $TB^2 n^2 \cos \gamma \sin \gamma \mu^{-1-2/n}$. Similar to the above case,
both the total transverse TEC and its dominant part, $\alpha_{yx}$ and $\alpha_{yxx}(C.M.E)$, accommodate $n^2$ as par the scaling of $B^2$ and $\sin \gamma \cos \gamma$ and $T$ are concerned. Hence, The multi-Weyl nature clearly shows up as one considers higher case i.e. $n > 1$ WSMs. One can also note that in the limit $T \to 0$, Eq. (21) and Eq. (23) are related by the Mott relation.

Like the LMC and PHC with an electric field applied applied to $z$-direction, we can also calculate the longitudinal TEC and transverse TEC $\alpha_{xx}$ and $\alpha_{yx}$ when the temperature gradient is along $z$ direction. The functional dependence on $B$ and $\gamma$ remains unaltered as compared to $\alpha_{xx}$ and $\alpha_{yx}$. The detail analytical formulation is presented in the Appendix.

We shall now compute the functional dependence of the Nernst coefficient as $\sigma$ and $\alpha$ are known in their leading orders. The planar Nernst coefficient $\nu$ (19) is given by

$$\nu \sim \frac{O \left( B^2 n^2 T \mu^{-1} - 2/n \right) \cos \gamma \sin \gamma \left( O \left( B^{0.5} T^2 \right) + O \left( B^{2.5} n^{-2} - 2/n \right) \sin \gamma \right) - O \left( B^{2.5} n^{-2} - 2/n \right) \cos \gamma \sin \gamma \left( O \left( B^{0.5} T^2 \right) + O \left( B^{2.5} n^{-2} - 2/n \right) \sin \gamma \right)}{O \left( B^{0.5} T^2 \right) + O \left( B^{2.5} n^{-2} - 2/n \right) \sin \gamma \sin \gamma \left( \cos \gamma \right)^2}$$

One can get a functional form of $\nu$ in the low temperature limit as $O \left(B^2\right) (f_1(n, T, \mu) + f_2(n, T, \mu) \cos \gamma \sin \gamma$ with $f_{1,2}$ being complicated functions of $n$, $T$ and $\mu$. Unlike $\sigma$ and $\alpha$, we can understand from the functional form of $\nu$ that the topological charge dependence is not monotonous. This is due to the fact that both the numerator and denominator of $f_1$ and $f_2$ have non-linear products consisting of $\mu$, $n$ and $T$. Hence, one can expect that the behavior of $\nu$ for different $n$ would strongly depend on the values of $\mu$ and $T$ chosen.

![FIG. 2. (Color online) (a) The behavior of chiral anomaly induced $(E,B)$ LMC $\sigma_{xx}(C.A.)$ and total normalized LMC $\bar{\sigma}_{xx} = \text{norm}(\sigma_{xx}(B \neq 0) - \sigma_{xx}(B = 0))$ as a function of $B$. (c) shows the dominant contribution of chiral anomaly induced PHC $\sigma_{yx}(C.A.)$ and (d) total PHC $\sigma_{yx}$ as a function of applied magnetic field for $m$-WSMs. In all the above cases, the quadratic dependence on $B$ is clearly visible. LMC and PHC both increase with topological charge in a non-linear fashion for a given value of $B$. The parameters chosen are the following: $\gamma = \pi/3$, $T = 10K$, $\mu = 0.05$ (for (a) and (b)) and $\mu = 0.07$ (for (c) and (d)). The $y$ axis of each figure is normalized by its maximum value.](image)

![FIG. 3. (Color online) (a), (b) show chiral anomaly induced $(E,B)$ LMC $\sigma_{xx}(C.A.)$ and total normalized LMC $\bar{\sigma}_{xx} = \text{norm}(\sigma_{xx}(B \neq 0) - \sigma_{xx}(B = 0))$, respectively, as a function of angle $\gamma$ for $n = 1, 2, 3$ at $B = 3 T$, $\mu = 0.05$ and $T = 10 K$. (c) shows the dominant contribution of chiral anomaly induced PHC $\sigma_{yx}(C.A.)$ and (d) total PHC $\sigma_{yx} = \text{norm}(\sigma_{yx}(B, \gamma) - \sigma_{yx}(B, \gamma = 0))$ as a function of $\gamma$ for $m$-WSMs. In all the above cases, LMC and PHC follow $\cos^2 \gamma$ and $\sin 2\gamma$ dependence respectively. LMC and PHC both increase with topological charge in a non-linear fashion for a given value of $\gamma$. The $y$ axis of each figure is normalized by its maximum value.](image)

**B. Numerical results in Lattice Model**

In order to discuss the transport properties in a physical multi-Weyl system, it is always good to consider a lattice model of Weyl fermions with the lattice regularization providing a physical ultra-violet smooth cut-off to the low energy Dirac spectrum. This is because the responses of the transport properties using linearized continuum theory at a finite density turns out to be insufficient.

Here, in setup 1, we first investigate the LMC and PHC under the application of electric field along $x$ direction and discuss its dependence on magnetic field $B$, rotation angle $\gamma$, chemical potential $\mu$ and temperature $T$. We will then discuss the thermo-electric responses LTEC and TTEC using setup 2. Finally, we study the behavior of planar Nernst coefficient with magnetic field and angle.
1. Magnetic Field: In order to properly identify the behavior of complete LMC in multi-WSMs, we plot \( \sigma_{xx}(B) - \sigma_{xx}(0) \) as a function of \( B \) for single, double and triple WSMs as shown in Fig. 2(b). It is clear from the Fig. 2(b) that LMC increases quadratically with magnetic field for all cases. The magnitude of LMC also increases with the topological charge \((n)\) of the WSMs. For detailed investigation, we compute each term of \( \sigma_{xx} \) and find that the main \( B \)-dependent contribution is coming from the chiral anomaly term. The dominant chiral anomaly contribution \( \sigma_{xx}(C.A.) \) to the LMC is also plotted as a function of \( B \) in Fig. 2(a) which shows that this contribution increases with \( n \) in multi-WSMs and agrees with our analytical results (i.e., \( B^2 \) dependence) of linearized model.

The total planar Hall conductivity \( \sigma_{yx} \) as a function of magnetic field is shown in Fig. 2(d) for \( n = 1, 2 \) and 3. One can see that PHC increases in non-linear fashion with \( n \) for a particular magnetic field and also shows \( B^2 \) dependence for all WSMs. The chiral anomaly term \( \sigma_{yx}(C.A.) \) is shown in Fig. 2 (c). It is evident that the chiral anomaly is the origin for the PHC for m-WSMs. Our result for lattice model qualitatively agrees with the results obtained from linearized model. We shall latter show the variation of LMC and PHC with topological charge.

FIG. 4. (Color online) Plots depict the dependence of (a) chiral anomaly induced LMC \( \sigma_{xx}(C.A.) \) and (b) total LMC \( \sigma_{xx} \) on chemical potential for single, double and triple WSMs. (c) shows the dominant contribution of chiral anomaly induced PHC \( \sigma_{yx}(C.A.) \) and (d) total PHC \( \sigma_{yx} \) as a function of chemical potential for mWSMs. The parameters chosen are the following: \( B = 2, \gamma = \pi/3, \mu = 0.05 \). The y axis of each figure is normalized by its maximum value.

Angle: Now we shall focus on the angular dependence of both LMC and PHC. The magnitude of LMC and PHC as a function of the angle \( \gamma \) for a particular magnetic field is depicted in Fig. 3(a)-(d). We find that chiral anomaly term in LMC \( \sigma_{xx}(C.A.) \) shows \( \cos^2 \gamma \) dependence where chiral anomaly part of PHC \( \sigma_{yx}(C.A.) \) exhibits \( \sin \gamma \cos \gamma \) dependence for all \( n \) values (see Fig. 3(a), (c)). The magnitude of both conductivities increases with the topological charge associated with the Weyl node. This numerical finding is in full congruence with the analytical results. Therefore, one can infer that the Weyl nodes play the pivotal role in determining the behavior of LMC and PHC in the presence of a rotated magnetic field and constant electric field. To investigate the angular dependence of the complete LMC and PHC further, we compute \( \sigma_{xx} = \sigma_{xx}(\gamma) - \sigma_{xx}(\gamma = \pi/2) \) (see Fig. 3(b)) and \( \sigma_{yx} = \sigma_{yx}(\gamma) - \sigma_{yx}(\gamma = 0) \) (see Fig. 3(d)) as a function of \( \gamma \) to exclude any off-setting from velocity terms. We would like to emphasize that chiral anomaly term primarily contributes to PHC. The multi-Weyl nature is reflected here that the oscillation amplitude of \( \sigma_{yx} \) for \( n = 1 \) and \( n = 2 \) almost coincides with each other whereas the magnitude for \( n = 3 \) is much greater than the other two.

Chemical potential: Now we shall probe the behavior of LMC and PHC as a function of chemical potential \( \mu \). From the analytical calculation based on the linearized model, it has been shown that chiral anomaly induced LMC behaves as \( \mu^{-2/n} \) while the \( B \) independent term, coming from Drude contribution, goes as \( \mu^2 \). We plot total normalized \( \sigma_{xx} \) along with its chiral anomaly counterparts in Fig. 4(b), (a), respectively. In the lattice model, \( \sigma_{xx}(C.A.) \) decreases with \( \mu \) for all \( n \). However, it is clear from the Fig. 4(a) that for \( 0.37 < \mu < 0.77 \), the chiral anomaly term for \( n = 1 \) decreases most rapidly and \( n = 3 \) decreases most slowly with \( \mu \) indicating the fact that \( \mu^{-2} \) for \( n = 1 \) decays faster than \( \mu^{-2/3} \) for \( n = 3 \). Moreover, the velocity term goes as \( \mu^2 \) according to analytical calculation, this feature is also reflected in \( \sigma_{xx} \) (see Fig. 4(b)) where quadratic behavior is observed. Here m-Weyl nature is clearly reflected while the variation with \( n \) is studied; \( n = 3 \) stays above \( n = 1 \) until \( \mu < 0.77 \). On the other hand, \( \sigma_{yx}(C.A.) \) and \( \sigma_{yx} \) depict the chiral anomaly and total contribution of PHC in Fig. 4 (c) and (d). The qualitative
feature of $\sigma_{yx}$ is similar to $\sigma_{yx}(C.A)$ reflecting the fact that chiral anomaly term is the origin of PHC as $B$ independent velocity term vanishes. Similar to the case of chiral anomaly part of LMC, here also for $\mu > 0.37$, the numerical findings i.e., $n = 1$ falls below $n = 2$ and $n = 3$ appears above $n = 2$ (see Fig. 4 (c) and (d)), is supported by the analytical argument of $\mu^{-2/n}$ dependence. For all of the above cases, the lattice effect is visible for $\mu < 0.37$ where $n = 2$ stays at the bottom.

Temperature: Now, we will discuss the temperature dependence of both LMC and PHC for m-WSMs (see Fig. (5)). Analytical calculation of linearized model shows that $\sigma_{xx}$ and $\sigma_{yz}$ vary as $T^2$ in multi-Weyl semimetals. The chiral anomaly term exhibits both linear and quadratic variation with $n$; $n = 1$ stays at the bottom for low temperature while $n = 3$ appears at the top for $T < 60$ K; moreover, for a given $T$, numerical results for $\sigma_{xx}(C.A)$ (see Fig. 5(a)) clearly suggests a non-linear dependence on $n$. On the other hand, the total $\sigma_{xx}$ (see Fig. 5(b)) shows a strong non-linear $T$ dependence. Although, for a given $T$, the non-linearity with $n$ is heavily reduced as the relative distance between $n = 1, 2$ and $3$ becomes almost same. Both of these observations point to the fact that $B$ independent velocity term which is proportion to $nT^2$, contributes maximally. On the other hand, $\sigma_{yx}$ is controlled by the chiral anomaly term; the $\sigma_{yx}(C.A)$ ( see Fig. 5(d)) and therefore, complete $\sigma_{yx}$ (see Fig. 5(c)) behave in an identical manner with $(n + n^2)T^2$. The multi-Weyl nature is clearly reflected in terms of the non-linear increment with $n$ for $\sigma_{xx}$ and $\sigma_{yx}$.

Scaling with $n$: Figure 6 shows the variation of LMC and PHC with topological charge. We consider data from magnetic field and angular variation to investigate the $n$ dependence. One can see that cubic variation is clearly obtained for $2 \leq n \leq 3$ while for $1 \leq n \leq 2$, a slow rise is visible. The underlying reason might be related to chiral anomaly term which controls the LMC and PHC maximally for higher values of topological charge. On the other hand, velocity term might be responsible for this deviation for low topological charges. Moreover, lattice effect coming from the high energy states is also substantial here.

2. setup 2: Thermo-electric coefficients and planar Nernst coefficient

We now investigate the LTEC and TTEC (also known as Peltier coefficient) as a function of magnetic field $B$, rotation angle $\gamma$, chemical potential $\mu$ and temperature $T$ when the temperature gradient is in $x$ direction.

Magnetic field: In order to properly identify the behavior of LTEC $\alpha_{xx}(B \neq 0)$ and TTEC $\alpha_{yx}(B \neq 0)$, we study the term for chiral magnetic effect (CME), $\alpha_{xx}(C.M.E)$ (see Fig. 7(a)) and $\alpha_{yx}(C.M.E)$ (see Fig. 7(c)) and the normalized total contribution, $\tilde{\alpha}_{xx} = \text{norm}(\alpha_{xx}(B \neq 0) - \alpha_{xx}(B = 0))$ (see Fig. 7(c)) and $\tilde{\alpha}_{yx} = \text{norm}(\alpha_{yx}(B \neq 0) - \alpha_{yx}(B = 0))$ (see Fig. 7(d)). Similar to the LMC and Hall conductivity, we here show here that LTEC and TTEC vary quadratically with magnetic field. This observation can be verified using the linearized model. Therefore, one can say that the maximum contribution is coming from Weyl node even for the lattice model. For detailed investigation, we show that the the terms (Eq.(22) and Eq.(23)), varying as $B^2$, are associated with the chiral magnetic effect. The important point to note here is that this term is contributing maximally. The other terms involving velocity and Drude contribution have an order of magnitude lesser that the chiral magnetic effect term. Interestingly, for a given value of $B$, $\alpha_{xx}$ and $\alpha_{yx}$ increase non-linearly with $n$; this reflects the multi-Weyl nature of the TECs. We note that the absolute value of $\alpha_{xx}$ and $\alpha_{yx}$ are $10^{17}$ times higher than $\sigma_{xx}$ and $\sigma_{yx}$, respectively.

Angle: In order to correctly quantify the behavior of TECs in the lattice model, we first plot the normalized chiral magnetic effect term present in $\alpha_{xx}$ and $\alpha_{yx}$ as shown in Fig. 8(a) and Fig. 8(c), respectively. The numerical finding again satisfy the analytical results, based on the linearized model, i.e., $\alpha_{xx}(C.M.E)$ varies as $\cos^2 \gamma$ and $\alpha_{yx}(C.M.E)$ varies as $\sin \gamma \cos \gamma$. The behavior of total TECs are then shown in Fig. 8(b) and Fig. 8(d) where $\tilde{\alpha}_{xx}$ (i.e., normalized ( $\alpha_{xx}(B, \gamma) - \alpha_{xx}(B, \gamma = \pi/2)$)) and $\tilde{\alpha}_{yx}$ (i.e., normalized $\alpha_{yx}(B, \gamma) - \alpha_{yx}(B, \gamma = 0)$)) exhibit $\cos^2 \gamma$ and $\sin \gamma \cos \gamma$ dependence, respectively. The numerical findings again satisfy the analytical results obtained from the linearized model. The multi-Weyl character is reflected in the non-linear enhancement of the amplitude of oscillation for $\alpha_{xx}$ and $\alpha_{yx}$ with $n$. We note that the absolute value of $\alpha_{xx}$ and $\alpha_{yx}$ are $10^{17}$ times higher than $\sigma_{xx}$ and $\sigma_{yx}$, respectively.
Chemical potential: Here, we study the behavior of LTEC $\alpha_{xx}$ and TTEC $\alpha_{yx}$ as a function of $\mu$ (see Fig. (9)). We actually plot the normalized values of coefficients to correctly understand their behavior. In Fig. 9(a) and Fig. 9(c), we plot the chiral magnetic effect term $\alpha_{xx} (C.M.E)$ and $\alpha_{yx} (C.M.E)$, respectively. The general tendency of its decreasing nature is common for longitudinal as well as transverse TECs; $n = 2$ exhibits the sharpest fall that persists up to $\mu \approx 0.06$. On the other hand, $n = 1$ shows slowest fall up to a certain values of $\mu$ that depends upon the types of the coefficients whether it is longitudinal or transverse. This observation can not be explained using the analytical calculation, hence, the lattice effect might be responsible for the above numerical outcome. The band bending nature at high energy might non-trivially control the behavior. In the large $\mu$ limit for $0.06 < \mu < 0.085$, $n = 3$ decays most slowly and $n = 1$ decays most rapidly. This observation can be analytically supported in the sense that the analytical calculation shows that chiral magnetic effect term goes as $\mu^{-1/2-n}$; therefore, $n = 1$ decays most rapidly while $n = 3$ decreases most slowly.

Now, we shall investigate the total contribution of LTEC coefficients (normalized $\alpha_{xx}(B, \gamma = \pi/3)$) and TTEC (normalized $\alpha_{yx}(B, \gamma = \pi/3)$) as shown in Fig. 9(b) and Fig. 9(d), respectively. In the case LTEC, it shows almost a linear decrease with $\mu < 0.06$. This feature might be caused by the $B$ independent velocity term proportional to $-n\mu$. For $\mu > 0.06$, this feature goes away rather there is a dip and again it increases with $\mu$; this is a multi-Weyl phenomena not present for $n = 1$. This finding can be related to the fact that $\alpha_{xx}$ has both positive and negative powers of $\mu$, i.e., $\mu$ and $\mu^{-1-2/n}$, as shown in the linearized model. Although, the underlying lattice might be controlling the characteristics of LTEC in the large $\mu$ limit. On the other hand, the characteristics $\alpha_{yx} (C.M.E)$ and $\alpha_{yx}$ are qualitatively similar; this again conveys that the chiral magnetic effect term is the key contributor for the TTEC. However, the lattice is also playing a role as the oscillatory behavior of $\alpha_{yx}$ in large $\mu$ limit for $n = 3$ can not be fully explained by the linearized model.

Temperature Dependence:

Now we probe the TECs as a function of temperature (see Fig. (10)). First, we closely investigate the influence of chiral magnetic effect term present in $\alpha_{xx}$ and $\alpha_{yx}$, as shown in Fig. 10(a) and Fig. 10(c), respectively. Analytically it has been shown that all the terms including chiral magnetic effect term goes as $n^2T$; one can see that at the low temperature limit $0 < T < 40K$, TECs $\alpha_{xx}$ and $\alpha_{yx}$ exhibit a linear rise with $T$. For a given $T$, the non-linear spacing between $n = 1, 2, 3$ clearly suggests that TECs have a non-linear $n$ dependence. At high temperature $n = 1$ increases non-linearly while $n = 2$ and $n = 3$ remain linear referring to the multi-Weyl nature. Although, at the high temperature limit, the analytical arguments no longer holds.

On the other hand, complete behavior shows identical temperature profile as shown in Fig. 10(b) for $\alpha_{xx}$ and Fig. 10(d) for $\alpha_{yx}$, respectively. Based on the linearized model, $B$ independent velocity term in $\alpha_{xx}$ varies as $-nT$; this is also reflected in the lattice calculation as shown in $\alpha_{xx}$ where linear decrement with temperature is clearly observed. On the other hand, $\alpha_{yx}$ is maximally controlled by the chiral magnetic effect term. The multi-Weyl nature is reflected as $n = 2$ and $n = 3$ continue to grow linearly with $T$ unlike the $n = 1$
is more clear than the cubic dependence of \(\alpha\) shown in Fig. 6, one can see that quadratic dependence of \(\alpha\) is distinctly different from that of the for \(n=2\) and 3. Moreover, the lattice effect might have a role to play here.

\[\text{scaling with } n: \text{ In Fig. 11, we show that longitudinal and transverse TEC, } \alpha_{xx} \text{ and } \alpha_{yx}, \text{ respectively, varies quadratically with } n. \text{ This reflects the fact that the linearized model is able to capture the underlying physics qualitatively. Although, lattice effect also important as we discussed above. To be precise, the chiral magnetic effect term governs the behavior of } \alpha_{yx}, \text{ while, } \alpha_{xx} \text{ is maximally governed by velocity term. This velocity term deviates from } n^2 \text{ as we see it from the linearized model. Comparing the LMC and PHC with } n \text{ as shown in Fig. 6, one can see that quadratic dependence of } \alpha's \text{ is more clear than the cubic dependence of } \sigma's \text{ for } 1 \leq n \leq 2.\]

**Planar Nernst Coefficient (PNC):** We shall now compute the planar Nernst coefficient \(\nu\) for the lattice model for \(n=1, 2 \text{ and } 3\). We show that \(\nu\) varies quadratically with \(B\) for all \(n\) as shown in Fig. IV B 2(a). This behavior is similar to the behavior of all the transport coefficients in both the setups. On the other hand, angle dependence of the Nernst coefficient appears to be \(\sin \gamma \cos \gamma\) which is similar with the behavior of PHC and TTEC (see Fig. IV B 2(b)). The important point to note here is that for given \(B\) and \(\gamma\), unlike the case for \(\sigma\) and \(\alpha\), here \(\nu\) does not exhibit a monotonic behavior with topological charge. This non-monotonic feature can be explained from analytical functional form obtained for \(\nu\) (24). One can see that \(\nu\) increases with temperature (see Fig. IV B 2(c)) while it decreases with \(\mu\) (see Fig. IV B 2(d)). These two characteristics is directly connected to the PHC and TTEC. To be more precise, \(\nu\) is maximally governed by the chiral anomaly and chiral magnetic effect term. Although, one can notice that multi-Weyl nature is clearly reflected as the behavior of \(\nu\) for \(n=1\) is distinctly different from that of the for \(n=2\) and 3.

\[\text{V. CONCLUSIONS}\]

In this work, we study several intriguing transport coefficients for m-WSMs (which are characterized by the topological charge \(n\) being more that unity) using semiclassical Boltzmann transport equation with the relaxation time approximation. We mainly focus on the co-planar setups where magnetic field \(B\) and electric field \(E\) or temperature gradient \(\nabla T\) lie in the same plane. It is known that the electric and thermoelectric transport properties are extremely useful quantities to probe the non-trivial features associated with the m-WSMs in a co-planar setup. From the practical point of view, these setups are experimentally viable and our theoretical predictions can hence be verified. During the course of our work, we find that there are a few universal behavior in the transport coefficients which could serve as an indication of chiral anomaly or chiral magnetic effect. Additionally, our findings on the m-WSMs using the co-planar setups can be distinctly distinguished from the single-Weyl behavior considering the conventional setups.

In the presence of co-planar electric and magnetic fields, not perfectly aligned with each other (with \(\gamma\) being the angle between \(E\) and \(B\)), we investigate the multi Weyl nature of longitudinal magneto-conductivity (LMC) and planar Hall conductivity (PHC) using the low-energy model. We first study the characteristics (i.e., magnetic field, angle, \(\mu\) and
temperature dependencies) of LMC and PHC. Our analytical calculation shows that PHC appears due to chiral anomaly in m-WSMs. On the other hand, the LMC has a significant $B$-independent Drude contribution whereas this contribution vanishes completely in the case of PHC. We find that at zero temperature both LMC and PHC go as $n^3$ indicating the fact that anisotropy in the energy dispersion leads an enhancement of the magnitude of these quantities as we go from single-WSM to double and triple WSMs. In the low but finite temperature limit, LMC and PHC receive a quadratic temperature correction with linear and bi-linear scaling of topological charge. Moreover, they are quadratically dependent on $B$ except for the Drude part of LMC which is linear with $n$. The chemical potential shows a $n$ dependent scaling for both LMC and PHC. The universal angle dependence is mainly caused by the chiral anomaly; we show that LMC follows as $\cos^2 \gamma$ whereas PHC varies as $\sin \gamma \cos \gamma$.

Moving on the thermo-electric responses, we consider a setup (planar Nernst setup) with co-planar thermal gradient and magnetic field not perfectly aligned with each other (with $\gamma$ being angle between $B$ and $\nabla T$). Here, we investigate the thermo-electric coefficients (TECs) and planar Nernst coefficients (PNE) in m-WSMS using the low-energy model. We similarly characterize the TEC and PNE as a function of magnetic field, angle, $\mu$ and temperature. We find that transverse TEC arises due to the chiral magnetic effect in m-WSMS. Similar to LMC, longitudinal TEC has a significant $B$-independent contribution other than the chiral magnetic effect. Interestingly, in this setup, the longitudinal and transverse TECs vary quadratically with topological charge and linearly with temperature. Hence the anisotropic energy dispersion has two distinct counterparts for planar Hall setup and planar Nernst setup. Although, there a few universal behavior in both the setups; the magnetic field dependence remains same as obtained for LMC and PHC. The longitudinal TEC varies as $\cos^2 \gamma$ while transverse TEC goes as $\sin \gamma \cos \gamma$. Multi Weyl nature is also reflected in the $n$ dependent chemical potential scaling of the TECs. We find that the planar Nernst coefficient (PNC) behaves qualitatively in an identical manner with $B$ and $\gamma$ as compared to the transverse transport coefficient. Surprisingly, unlike the all the other transport coefficients, PNC does not exhibit a monotonic variation with $n$.

In order to verify the analytical findings, we numerically study the TRS broken lattice model of m-WSMs (single-, double- and triple WSM). The quadratic $B$ dependence, non-linear sinusoidal $\gamma$ dependence, obtained from lattice model, are supported by analytical calculations. One can note by investigating $\mu$ and $T$ dependencies that chiral anomaly and chiral magnetic effect determine the behavior of transverse transport coefficients while Drude part (i.e., velocity term) contributes maximally for longitudinal transport responses. On the other hand, the band bending at finite/ high energy leads to lattice effect that causes an apparent deviation from $n^3$ for LMC and PHC in the limit of small topological charge. Additionally, quantitative features with $\mu$ and $T$ that is not captured by the linearized model might be controlled by the underlying lattice model. These numerical results predict experimental observations of LMC, PHC, TECs and PNE of the m-WSMs. It is noteworthy that anomalous response and second order responses might be interesting future problems in this context.
We sincerely thank Renato M. A. Dantas for fruitful discussions.

Now we shall compute the LMC and electrical Hall conductivity for the continuum model (1). We present this calculation to clearly mention the calculation details which we follow for Sec. B. C. Here we assume the electric and magnetic field to have the following form: \( \mathbf{E} = E^j \) and \( \mathbf{B} = B^j \). We refer \( \partial f_{eq}/\partial \epsilon = \tilde{f}_{eq} \). This is the coefficient of electric charge current along \( j \) direction for an applied electric field in \( j \) direction: \( \sigma_{ij} \)

\[
\sigma_{jj} = e^2 \int \frac{d^3k}{(2\pi)^3} \tau D[(v_j + eB_j/h)(\mathbf{v}_k \cdot \mathbf{\Omega}_k)]^2 \left( \frac{\partial f_{eq}}{\partial \epsilon} \right) \tag{A1}
\]

We now decompose the above expression term by term to investigate it more rigorously: \( \sigma_{jj} = \sigma_{jj}^{(1)} + \sigma_{jj}^{(2)} + 2\sigma_{jj}^{(3)} \) where

\[
\sigma_{jj}^{(1)} = \tau e^2 \int \frac{d^3k}{(2\pi)^3} \frac{(v_j)^2}{1 + eB_j/h} \left( \frac{\partial f_0}{\partial \epsilon} \right), \tag{A2}
\]

\[
\sigma_{jj}^{(2)} = \tau e^2 \frac{B^2}{\hbar^2} \int \frac{d^3k}{(2\pi)^3} \frac{(\mathbf{\Omega}_k \cdot \mathbf{v}_k)^2}{1 + eB_j/h} \left( \frac{\partial f_0}{\partial \epsilon} \right), \tag{A3}
\]

\[
\sigma_{jj}^{(3)} = \tau e^3 \frac{B_j}{\hbar} \int \frac{d^3k}{(2\pi)^3} \frac{(v_j)(\mathbf{\Omega}_k \cdot \mathbf{v}_k)}{1 + eB_j/h} \left( \frac{\partial f_0}{\partial \epsilon} \right), \tag{A4}
\]

We note here that two LMCs are given by \( \sigma_{xx} \) and \( \sigma_{xj}, J_j = \sigma_{xx} E_j \) and \( J_z = \sigma_{zz} E_z \).

We make resort to cylindrical polar co-ordinate to do the analytical calculation. \( \int \frac{d^3k}{(2\pi)^3} = \frac{1}{(2\pi)^3} \int_0^\infty k_\perp dk_\perp \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\phi \). We need to compute the following momentum integral for finite temperature; we use the Sommerfeld expansion.

\[
\tilde{f}(\epsilon)_{eq} = \beta \int_0^\infty \left( \frac{h(\epsilon)}{1 + e^\beta(\epsilon - \mu)} - \frac{h(\epsilon)}{(1 + e^\beta(\epsilon - \mu))^2} \right) d\epsilon \tag{A6}
\]

We use the change of variable \( \beta(\epsilon - \mu) = x \) and above integral becomes

\[
\tilde{f}(\epsilon)_{eq} = \int_{-\beta\mu}^0 \left( \frac{h(\mu + x/\beta)}{1 - e^{x}} - \frac{h(\mu + x/\beta)}{(1 - e^{x^2})} \right) dx
+ \int_0^\infty \left( \frac{h(\mu + x/\beta)}{1 + e^{x}} - \frac{h(\mu + x/\beta)}{(1 + e^{x^2})} \right) dx \tag{A7}
\]

In the first integral we use \( x \to -x \) and using the fact that \( (e^{-x} + 1)^{-1} = 1 - (e^{-x} + 1)^{-1} \). We assume \( \beta \mu \gg 1 \) and obtain

\[
\tilde{f}(\epsilon)_{eq} = \int_0^\infty \left( \frac{1}{1 + e^x} - \frac{1}{(1 + e^{x^2})} \right) (h(\mu - x/\beta) + h(\mu + x/\beta)) dx \tag{A8}
\]

Now one can expand \( h \) around \( \mu \) as the integrand decreases exponentially with increasing \( x \)

\[
\tilde{f}(\epsilon)_{eq} = \int_0^\infty \left( \frac{1}{1 + e^x} - \frac{1}{(1 + e^{x^2})} \right) (2h(\mu) + K_B^2 T^2 x^2 h''(\mu)) \tag{A9}
\]

In our case, \( h(k) = (k - \mu)k'' \) and \( \tilde{f}(\epsilon)_{eq} = \pi^2 K_B^2 T^2 \nu \mu''^{-1} \) and when \( h(k) = k'' \) then \( \tilde{f}(\epsilon)_{eq} = \mu'' + \pi^2 K_B^2 T^2 \nu(\nu - 1) \mu''^2 \).

We shall derive the analytical form of LMC in finite temperature by considering \( -\partial f_{eq}/\partial \epsilon = \beta f_{eq}(1 - f_{eq}) = \tilde{f}(\epsilon)_{eq} \).

\[
\sigma_{zz}^{(1)} = \frac{\tau e^2}{(2\pi)^3} \int_0^\infty dk_\perp \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\phi \frac{k_\perp v^4 k_\perp^2/\epsilon^2}{1 + \epsilon n^2 \alpha n^2 k_\perp^{2n-2} B k_\perp/(2e^3 h)} \tilde{f}(\epsilon)_{eq} \tag{A10}
\]

Now we perform the variable substitution \( k_\perp \to k_\perp/\nu \) and \( k_\perp \to k_\perp/\nu^{1/2} \) Hence the energy becomes \( \epsilon \to \epsilon' = \sqrt{k_\perp^2 + k_\perp^2} \).

\[
\sigma_{zz}^{(1)} = \frac{\tau e^2}{(2\pi)^2} \int_0^\infty dk_\perp \int_{-\infty}^{\infty} dz \frac{k_\perp\alpha n^{2n-1} v k_\perp^{2n}/\epsilon^2}{1 + \epsilon n^2 \alpha n^{2n-2} k_\perp^{2n-2} B k_\perp/(2e^3 h)} \tilde{f}(\epsilon')_{eq} \tag{A11}
\]
We then use another change of variable $k_\perp \rightarrow k_\perp^{1/n}$ and $\epsilon' \rightarrow \epsilon'' = \sqrt{k_\perp^2 + k_z^2}$:

$$\sigma_{zz}^{(1)} = \frac{\tau \epsilon'}{(2\pi)^2} \int_{0}^{\infty} dk_\perp \int_{0}^{\infty} dk_z \frac{k_\perp^{2/n-1} \sigma_{n}^{-2/n} v k_z^2/(n \epsilon'^2) f(\epsilon'')}{1 + eB^2 \sigma_{n}^{-2/n} k_\perp^{-2/n} B_k)/(2n^3 \hbar)}$$

Finally, one can perform another transformation $k_\perp = k \sin \theta$ and $k_z = k \cos \theta$ and hence $\epsilon'' \rightarrow \epsilon''' = k$. \( f_0 \int_{0}^{\infty} d\theta \int_{0}^{\infty} dk.

$$\sigma_{zz}^{(1)} = \frac{\tau \epsilon'}{n(2\pi)^2} \int_{0}^{\infty} dk \int_{0}^{\pi} \frac{k^{2/n} \sigma_{\sin \theta}^{2/n} \cos \theta (\sin \theta)^{2/n-1} f(k)}{1 + eB^2 \sigma_{n}^{2/n} k^{-2/n} \cos \theta (2\hbar)}$$

In order to evaluate the integrals we need to perform a series expansion in terms of $eB/\hbar \mu^{2/n}$. We use the series expansion $(1 + x)^{-1} = \sum_{i=0}^{\infty}(-x)^i$ with $x \ll 1$ for the denominator. Therefore, the integral becomes

$$\sigma_{zz}^{(1)} = \frac{\tau \epsilon'}{n(2\pi)^2} \sum_{i} (-eB n^{2} \sigma_{n}^{2/n} / 2\hbar)^i \int_{0}^{\pi} d\theta (\cos \theta)^{2+i}(\sin \theta)^{2/n-1+i(2-2/n)} \int_{0}^{\infty} dkk^{2/n} k^{-2/i} f(k)$$

The leading order terms are given by

$$\sigma_{zz}^{(1)} \simeq \frac{\tau \epsilon'}{n(2\pi)^2} \left( \frac{\Gamma(1/n)\Gamma(3/2)}{\Gamma(3/2 + 1/n)} \left( \mu^{2/n} + \frac{\pi^2 K_B^2 T^2 (2n - \mu^{2/n})}{6n^2} \right) + \frac{eB^2 \sigma_{n}^{4/n}}{\hbar^2} \frac{\Gamma(2 - 1/n)\Gamma(5/2)}{4\Gamma(9/2 - 1/n)} \left( \mu^{-2/n} + \frac{\pi^2 K_B^2 T^2 (2n + \mu^{-2/n})}{6n^2} \right) \right)$$

Similarly, $\sigma_{zz}^{(2)}$ is given by

$$\sigma_{zz}^{(2)} = \frac{\tau \epsilon''}{4\pi^2} \sum_{i} (-eB n^{2} \sigma_{n}^{2/n} / 2\hbar)^i \int_{0}^{\pi} d\theta (\cos \theta)^{1+i}(\sin \theta)^{1+i(2-2/n)} \int_{0}^{\infty} dkk^{-2/i} f(k)$$

Similarly, $\sigma_{zz}^{(3)}$ is given by

$$\sigma_{zz}^{(3)} = \frac{\tau \epsilon''}{2\pi^2} \sum_{i} (-eB n^{2} \sigma_{n}^{2/n} / 2\hbar)^i \int_{0}^{\pi} d\theta (\cos \theta)^{1+i}(\sin \theta)^{1+i(2-2/n)} \int_{0}^{\infty} dkk^{-2/i} f(k)$$

Therefore, the chiral anomaly term in $\sigma_{zz}$ is proportional to $B^{2}(n^{3}\mu^{-2/n} + n^{2}T^{2}\mu^{-2-2/n})$. The complete $\sigma_{zz}$ is also proportional to $\mu^{2/n}/n + T^{2}/n^{3} + B^{2}(n^{3}\mu^{-2/n} + n^{2}T^{2}\mu^{-2-2/n})$.

Similarly, $\sigma_{xx}^{(1)}$ is given by

$$\sigma_{xx}^{(1)} = \frac{\tau \epsilon''}{4\pi^2} \sum_{i} (-eB n^{2} \sigma_{n}^{1/n} / 2\hbar)^i \int_{0}^{\pi} d\theta (\cos \theta)^{2+i}(\sin \theta)^{2(1-2/n)} \int_{0}^{\infty} dkk^{-i(1+1/n)} f(k)$$

Similarly, $\sigma_{xx}^{(2)}$ is given by

$$\sigma_{xx}^{(2)} = \frac{\tau \epsilon''}{2\pi^2} \sum_{i} (-eB n^{2} \sigma_{n}^{1/n} / 2\hbar)^i \int_{0}^{\pi} d\theta (\cos \theta)^{2+i}(\sin \theta)^{2(1-2/n)} \int_{0}^{\infty} dkk^{-i(1+1/n)} f(k)$$
Similarly, \( \sigma_{xx}^{(3)} \) is given by

\[
\sigma_{xx}^{(3)} = \frac{\tau e^4 n^3 \alpha_{1/n} B}{2h^2 (2\pi)^3} \sum_i (-eBn\alpha_{1/n}/2h) \int_0^\pi d\theta \int_0^{2\pi} (\cos \theta)^{1+i} (\sin \theta)^{3-1/n+i(2-1/n)} \int_0^\infty dk k^{-1/n-i(1+1/n)} f(k)_{eq}
\]

\[
\approx -\frac{\nu\tau e^4 n^3 \alpha_{1/n}^2 B^2 \pi}{2h^2 (2\pi)^3} \frac{\Gamma(3-1/n)\Gamma(3/2)}{\Gamma(7/2-1/n)} \left( \mu^{-2/n} + \frac{\pi^2 K_B^2 T^2 (2+n)\mu^{-2-2/n}}{6n^2} \right)
\]

(A15)

Therefore, the chiral anomaly term in \( \sigma_{xx} \) is proportional to \( B^2 (n^3 \mu^{-2/n} + n^2 T^2 \mu^{-2-2/n}) \). The complete \( \sigma_{xx} \) is also proportional to \( n\mu^2 + nT^2 + B^2 (n^3 \mu^{-2/n} + n^2 T^2 \mu^{-2-2/n}) \).

Appendix B: Calculation for LMC and PHC in setup 1: PHE

Having discussed the LMC in normal regular setup, we shall now turn our attention to PH setup. Here we compute two main quantities \( \sigma_{jj} \) and \( \sigma_{ij} \). The magnetic field here is assumed to have the form: \( \mathbf{B} = B \cos \gamma \hat{j} + B \sin \gamma \hat{i} \) and \( \mathbf{E} = E \hat{j} \) with \( j = x, z \). Therefore, the LMC is given by:

\[
\sigma_{jj} = \sigma_{jj}^{(1)} + \sigma_{jj}^{(2)} + 2\sigma_{jj}^{(3)}
\]

\[
\sigma_{jj} = \tau e^2 \int \frac{d^3 k}{(2\pi)^3} \mathcal{D} \left[ (v_j + \frac{eB_j \cos \gamma}{\hbar} (\mathbf{v}_k \cdot \Omega_k))^2 \right] \left( -\frac{\partial f_{eq}}{\partial \epsilon} \right)
\]

(B1)

where

\[
\sigma_{jj}^{(1)} = \tau e^2 \int \frac{d^3 k}{(2\pi)^3} 1 + eB (\cos \gamma \Omega_j + \sin \gamma \Omega_y) \left( -\frac{\partial f_0}{\partial \epsilon} \right),
\]

(B2)

\[
\sigma_{jj}^{(2)} = \tau e^2 \int \frac{d^3 k}{h^2} (\mathbf{v}_j \cdot \mathbf{k})^2 \left( -\frac{\partial f_0}{\partial \epsilon} \right),
\]

(B3)

\[
\sigma_{jj}^{(3)} = \tau e^2 B \cos \gamma \int \frac{d^3 k}{(2\pi)^3} \left( -\frac{\partial f_0}{\partial \epsilon} \right),
\]

(B4)

The first term becomes

\[
\sigma_{xx}^{(1)} = \frac{\tau e^2 n}{v(2\pi)^3} \sum_i (-eBn\alpha_{1/n}/2h) \int_0^\pi d\theta \int_0^{2\pi} d\phi (\cos \phi)^2 (\cos (\phi - \gamma))^i (\sin \theta)^{3+i(2-1/n)} \int_0^\infty dk k^{2-i(1+1/n)} f(k)_{eq}
\]

\[
\approx \frac{\tau e^2 n}{v(2\pi)^3} \frac{\pi^{3/2} \Gamma(2)}{\Gamma(5/2)} \left( \mu^2 + \frac{\pi^2 K_B^2 T^2}{3} \right)
\]

\[
+ (eBn\alpha_{1/n} \mu^{-1-1/n}/2)^2 \frac{\pi^{3/2} (2\cos^2 \gamma + 1) \Gamma(4-1/n)}{16\Gamma(9/2-1/n)} (\mu^{-2/n} + \frac{\pi^2 K_B^2 T^2 (2+n)\mu^{-2-2/n}}{3n^2})
\]

(B6)

Similarly, \( \sigma_{xx}^{(2)} \) is given by

\[
\sigma_{xx}^{(2)} = \frac{\nu\tau e^4 n^3 \alpha_{1/n}^2 B^2 \cos \gamma \Gamma(2-1/n)}{4h^2 (2\pi)^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi (\cos (\phi - \gamma))^i (\sin \theta)^{3-i(2-1/n)}
\]

\[
\int_0^\infty dk k^{2-n-i(1+1/n)} f(k)_{eq}
\]

\[
\approx \frac{\nu\tau e^4 n^3 \alpha_{1/n}^2 B^2 \pi \cos \gamma}{2h^2 (2\pi)^3} \frac{\Gamma(2-1/n)}{\Gamma(5/2-1/n)} (\mu^{-2/n} + \frac{\pi^2 K_B^2 T^2 (2+n)\mu^{-2-2/n}}{3n^2})
\]

(B7)

Similarly, \( \sigma_{xx}^{(3)} \) is given by

\[
\sigma_{xx}^{(3)} = \frac{\tau e^4 n^3 \alpha_{1/n} B \cos \gamma}{2h^2 (2\pi)^3} \sum_i (-eBn\alpha_{1/n}/2h) \int_0^\pi d\theta \int_0^{2\pi} d\phi (\cos (\phi - \gamma))^i (\sin \theta)^{3-i(2-1/n)} \int_0^\infty dk k^{1-n-i(1+1/n)} f(k)_{eq}
\]

\[
\approx -\frac{\nu\tau e^4 n^3 \alpha_{1/n}^2 B^2 \pi \cos \gamma}{2h^2 (2\pi)^3} \frac{\Gamma(3-1/n)\Gamma(3/2)}{\Gamma(7/2-1/n)} (\mu^{-2/n} + \frac{\pi^2 K_B^2 T^2 (2+n)\mu^{-2-2/n}}{3n^2})
\]

(B8)
Therefore, the chiral anomaly term in $\sigma_{xx}$ is proportional to $B^2 \cos^2 \gamma (n^3 \mu_{-2/n}^2 + n^2 T^2 \mu_{-2-2/n}^2)$. The complete $\sigma_{xx}$ is proportional to $n\mu_{-2}^2 + n^2 T^2 + B^2 \cos^2 \gamma (n^3 \mu_{-2/n}^2 + n^2 T^2 \mu_{-2-2/n}^2)$.

In the same fashion, one can compute $\sigma_{zz} = \sigma_{zz}^{(1)} + \sigma_{zz}^{(2)} + \sigma_{zz}^{(3)}$.

\[
\sigma_{zz}^{(1)} = \frac{\tau e^2 v_0 n^{2/3}}{2(2\pi)^3} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \int_0^\infty dk \sum_i (-eB(n^2 \alpha_n^2 k^{-1/2}(\sin \theta)^2 - \sin \theta \cos \gamma) + n\nu c_{1/n} k^{-1/2}(\sin \theta)^2 - \sin \theta \cos \gamma) / 2\hbar \tilde{f}(k)_{eq}
\]

\[
\rho = \frac{\tau e^2 v_0 n^{2/3}}{2(2\pi)^3} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \int_0^\infty dk \sum_i (-eB(n^2 \alpha_n^2 k^{-1/2}(\sin \theta)^2 - \sin \theta \cos \gamma) + n\nu c_{1/n} k^{-1/2}(\sin \theta)^2 - \sin \theta \cos \gamma) / 2\hbar \tilde{f}(k)_{eq}
\]

\[
\sigma_{zz}^{(2)} \text{ is given by}
\]

\[
\sigma_{zz}^{(2)} = \frac{\tau e^4 B^2 v_0 n^{2/3}}{4(2\pi)^3} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \int_0^\infty dk \sum_i (-eB(n^2 \alpha_n^2 k^{-1/2}(\sin \theta)^2 - \sin \theta \cos \gamma) + n\nu c_{1/n} k^{-1/2}(\sin \theta)^2 - \sin \theta \cos \gamma) / 2\hbar \tilde{f}(k)_{eq}
\]

\[
\rho = \frac{\tau e^4 B^2 v_0 n^{2/3}}{4(2\pi)^3} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \int_0^\infty dk \sum_i (-eB(n^2 \alpha_n^2 k^{-1/2}(\sin \theta)^2 - \sin \theta \cos \gamma) + n\nu c_{1/n} k^{-1/2}(\sin \theta)^2 - \sin \theta \cos \gamma) / 2\hbar \tilde{f}(k)_{eq}
\]

\[
\sigma_{zz}^{(3)} \text{ is given by}
\]

\[
\sigma_{zz}^{(3)} = \frac{\tau e^4 B v_0 \cos \gamma}{2(2\pi)^3} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \int_0^\infty dk \sum_i (-eB(n^2 \alpha_n^2 k^{-1/2}(\sin \theta)^2 - \sin \theta \cos \gamma) + n\nu c_{1/n} k^{-1/2}(\sin \theta)^2 - \sin \theta \cos \gamma) / 2\hbar \tilde{f}(k)_{eq}
\]

\[
\rho = \frac{\tau e^4 B v_0 \cos \gamma}{2(2\pi)^3} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \int_0^\infty dk \sum_i (-eB(n^2 \alpha_n^2 k^{-1/2}(\sin \theta)^2 - \sin \theta \cos \gamma) + n\nu c_{1/n} k^{-1/2}(\sin \theta)^2 - \sin \theta \cos \gamma) / 2\hbar \tilde{f}(k)_{eq}
\]

\[
\sigma_{zz}^{(4)} \text{ is given by}
\]

\[
\sigma_{zz}^{(4)} = \frac{\tau e^4 B^2 v_0 \cos \gamma}{2(2\pi)^3} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \int_0^\infty dk \sum_i (-eB(n^2 \alpha_n^2 k^{-1/2}(\sin \theta)^2 - \sin \theta \cos \gamma) + n\nu c_{1/n} k^{-1/2}(\sin \theta)^2 - \sin \theta \cos \gamma) / 2\hbar \tilde{f}(k)_{eq}
\]

\[
\rho = \frac{\tau e^4 B^2 v_0 \cos \gamma}{2(2\pi)^3} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \int_0^\infty dk \sum_i (-eB(n^2 \alpha_n^2 k^{-1/2}(\sin \theta)^2 - \sin \theta \cos \gamma) + n\nu c_{1/n} k^{-1/2}(\sin \theta)^2 - \sin \theta \cos \gamma) / 2\hbar \tilde{f}(k)_{eq}
\]

Therefore, the chiral anomaly term in $\sigma_{zz}$ is proportional to $B^2 \cos^2 \gamma (n^3 \mu_{-2/n}^2 + n^2 T^2 \mu_{-2-2/n}^2)$. The complete $\sigma_{zz}$ is also proportional to $\mu_{2/n}^2 / n + T^2 \mu_{2-2/n}^3 + B^2 \cos^2 \gamma (n^3 \mu_{-2/n}^2 + n^2 T^2 \mu_{-2-2/n}^2)$.

The expression for chiral anomaly induced PH conductivity $\sigma_{yj}$ can similarly be decomposed into four parts: $\sigma_{yj} = \sigma_{yj}^{(1)} + \sigma_{yj}^{(2)} + \sigma_{yj}^{(3)} + \sigma_{yj}^{(4)}$. The complete form is given by

\[
\sigma_{yj} = e^2 \int \frac{d^3k}{(2\pi)^3} D\tau \left( \frac{-\partial f_0}{\partial \epsilon} \right) \left[ (v_y + \frac{eB}{\hbar n^2} (\mathbf{v}_k \cdot \mathbf{\Omega}_k)) (v_j + \frac{eB}{\hbar c} \cos \gamma (\mathbf{v}_k \cdot \mathbf{\Omega}_k)) \right]
\]

and the individual terms are represented as

\[
\sigma_{yj}^{(1)} = \tau e^2 \int \frac{d^3k}{(2\pi)^3} \frac{(\mathbf{v}_k \mathbf{v}_j)}{1 + eB(\cos \gamma \Omega_j + \sin \gamma \Omega_y)} \left( -\frac{\partial f_0}{\partial \epsilon} \right),
\]

\[
\sigma_{yj}^{(2)} = \tau e^4 \frac{B^2 \cos \gamma \sin \gamma}{\hbar^2} \int \frac{d^3k}{(2\pi)^3} \frac{(\mathbf{\Omega}_k \cdot \mathbf{v}_k)^2}{1 + eB(\cos \gamma \Omega_j + \sin \gamma \Omega_y)} \left( -\frac{\partial f_0}{\partial \epsilon} \right),
\]

\[
\sigma_{yj}^{(3)} = \tau e^4 \frac{B \cos \gamma}{\hbar} \int \frac{d^3k}{(2\pi)^3} \frac{(v_y)(\mathbf{v}_k \cdot \mathbf{\Omega}_k)}{1 + eB(\cos \gamma \Omega_j + \sin \gamma \Omega_y)} \left( -\frac{\partial f_0}{\partial \epsilon} \right),
\]

\[
\sigma_{yj}^{(4)} = \tau e^4 \frac{B \sin \gamma}{\hbar} \int \frac{d^3k}{(2\pi)^3} \frac{(v_j)(\mathbf{v}_k \cdot \mathbf{\Omega}_k)}{1 + eB(\cos \gamma \Omega_j + \sin \gamma \Omega_y)} \left( -\frac{\partial f_0}{\partial \epsilon} \right),
\]

First we compute $\sigma_{y\pi}$ where

\[
\sigma_{y\pi}^{(1)} = \frac{\tau e^2 n^2}{2v(2\pi)^3} \sum_i (-eBn\nu a_{1/n} / 2\hbar )^i \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi (\cos(\phi - \gamma))^i \sin(2\phi)(\sin \theta)^{2i(1-1/n)} \int_0^\infty dkk^{2i(1+1/n)} \tilde{f}(k)_{eq}
\]

\[
\rho = \frac{\tau e^2 n^2}{2v(2\pi)^3} \sum_i (-eBn\nu a_{1/n} / 2\hbar )^i \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi (\cos(\phi - \gamma))^i \sin(2\phi)(\sin \theta)^{2i(1-1/n)} \int_0^\infty dkk^{2i(1+1/n)} \tilde{f}(k)_{eq}
\]
\[ \sigma_{yz}^{(2)} \text{ is given by} \]
\[
\sigma_{yz}^{(2)} = \frac{\tau e^4 B^2 n^3 \alpha_n^2/n \sin \gamma \cos \gamma}{4h^2(2\pi)^3} \sum_i (-e B n \nu \alpha_n^{1/n} / 2h)^i \int_0^\pi d\theta \int_0^{2\pi} d\phi (\cos(\phi - \gamma))^i (\sin \theta)^{3-2/n+i(2-1/n)} \\
\int_0^\infty dk k^{-2/n-i(1+1/n)} \tilde{f}(k)_{eq} \\
\simeq \frac{\Gamma(2 - 1/n)}{16\pi^{3/2}h^2 \Gamma(5/2 - 1/n)} \cos \gamma \sin \gamma e^4 \tau^3 B^2 \nu \alpha_n^{2/n} (\mu^{-2/n} + \frac{K_B^2 T^2 \pi^2 (2 + n) \mu^{-2-2/n}}{3n^2}) \tag{B19}
\]

\[ \sigma_{yz}^{(3)} \text{ is given by} \]
\[
\sigma_{yz}^{(3)} = \frac{\tau e^3 B n^2 \alpha_n^{1/n} \sin \gamma}{2h(2\pi)^3} \sum_i (-e B n \nu \alpha_n^{1/n} / 2h)^i \int_0^\pi d\theta \int_0^{2\pi} d\phi (\cos(\phi - \gamma))^i \sin \phi (\sin \theta)^{3-1/n+i(2-1/n)} \\
\int_0^\infty dk k^{-1/n-i(1+1/n)} \tilde{f}(k)_{eq} \\
\simeq -\frac{\Gamma(3 - 1/n)}{32\pi^{3/2}h^2 \Gamma(7/2 - 1/n)} \cos \gamma \sin \gamma e^4 \tau^3 B^2 \nu \alpha_n^{2/n} (\mu^{-2/n} + \frac{K_B^2 T^2 \pi^2 (2 + n) \mu^{-2-2/n}}{3n^2}) \tag{B20}
\]

\[ \sigma_{yz}^{(4)} \text{ is given by} \]
\[
\sigma_{yz}^{(4)} = \frac{\tau e^3 B n^2 \alpha_n^{1/n} \sin \gamma}{2h(2\pi)^3} \sum_i (-e B n \nu \alpha_n^{1/n} / 2h)^i \int_0^\pi d\theta \int_0^{2\pi} d\phi (\cos(\phi - \gamma))^i \cos \phi (\sin \theta)^{3-1/n+i(2-1/n)} \\
\int_0^\infty dk k^{-1/n-i(1+1/n)} \tilde{f}(k)_{eq} \\
\simeq -\frac{\Gamma(3 - 1/n)}{32\pi^{3/2}h^2 \Gamma(7/2 - 1/n)} \cos \gamma \sin \gamma e^4 \tau^3 B^2 \nu \alpha_n^{2/n} (\mu^{-2/n} + \frac{K_B^2 T^2 \pi^2 (2 + n) \mu^{-2-2/n}}{6n^2}) \tag{B21}
\]

Therefore, the chiral anomaly term in \( \sigma_{yz} \) is proportional to \( B^2 \cos \gamma \sin \gamma (n^3 \mu^{-2/n} + nT^2 \mu^{-2-2/n}) \). The complete \( \sigma_{yx} \) is also proportional to \( B^2 \cos \gamma \sin \gamma (n^3 \mu^{-2/n} + nT^2 \mu^{-2-2/n}) \).

Now, we shall now compute \( \sigma_{yz} \) considering electric field \( E = E \hat{z} \); \( \sigma_{yz} = \sigma_{yz}^{(1)} + 2\sigma_{yz}^{(2)} + \sigma_{yz}^{(3)} + \sigma_{yz}^{(4)} \), with

\[ \sigma_{yz}^{(1)} = \frac{\tau e^2 \alpha_n^{1/n}}{(2\pi)^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^\infty dk \sum_i (-e B n \nu \alpha_n^{1/n} / k^{2/n} (\sin \theta)^{2-2/n} \cos \theta \cos \gamma \\
+ n \nu \alpha_n^{1/n} k^{-1/n} (\sin \theta)^{2-1/n} \sin \phi \sin \gamma / 2h)^i \cos \theta \sin \phi (\sin \theta)^{1+i/n} k^{3+1/n} \tilde{f}(k)_{eq} \\
\simeq \frac{\pi \Gamma(3 - 1/n) \Gamma(3/2)}{h^2 \Gamma(9/2 - 1/n)} \cos \gamma \sin \gamma e^4 \tau^3 B^2 \nu \alpha_n^{2/n} (\mu^{-2/n} + \frac{K_B^2 T^2 \pi^2 (2 + n) \mu^{-2-2/n}}{6n^2}) \tag{B22}
\]

\[ \sigma_{yz}^{(2)} = \frac{\tau e^4 \alpha_n^{2/n} B^2 n^3 \nu \sin \gamma \cos \gamma}{(4h^2 \pi)^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^\infty dk \sum_i (-e B n \nu \alpha_n^{2/n} k^{-2/n} (\sin \theta)^{2-2/n} \cos \theta \cos \gamma \\
+ n \nu \alpha_n^{1/n} k^{-1-1/n} (\sin \theta)^{2-1/n} \sin \phi \sin \gamma / 2h)^i (\sin \theta)^{3-2/n} k^{-2/n} \tilde{f}(k)_{eq} \\
\simeq \frac{\tau e^4 \alpha_n^{2/n} B^2 n^3 \nu \sin \gamma \cos \gamma \Gamma(2 - 1/n) \Gamma(1/2)}{4h^2 \Gamma(5/2 - 1/n)} (\mu^{-2/n} + \frac{K_B^2 T^2 \pi^2 (n + 2) \mu^{-2-2/n}}{3n^2}) \tag{B23}
\]

\[ \sigma_{yz}^{(3)} = \frac{\tau e^3 \alpha_n^{1/n} B n^2 \cos \gamma}{(2h^2 \pi)^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^\infty dk \sum_i (-e B n \nu \alpha_n^{2/n} k^{-2/n} (\sin \theta)^{2-2/n} \cos \theta \cos \gamma \\
+ n \nu \alpha_n^{1/n} k^{-1-1/n} (\sin \theta)^{2-1/n} \sin \phi \sin \gamma / 2h)^i (\sin \theta)^{3-1/n} \sin \phi k^{-1-1/n} \tilde{f}(k)_{eq} \\
\simeq \frac{\tau e^3 \alpha_n^{1/n} B n^2 \nu \sin \gamma \cos \gamma \Gamma(3 - 1/n) \Gamma(1/2)}{4h^2 \Gamma(7/2 - 1/n)} (\mu^{-2/n} + \frac{K_B^2 T^2 \pi^2 (n + 2) \mu^{-2-2/n}}{3n^2}) \tag{B24}
\]
Therefore, the chiral anomaly term in $\sigma_{yz}^H$ is proportional to $B^2 \cos \gamma \sin \gamma (n^3 \mu_{2-n} + nT^2 \mu_{2-n})$. The complete $\sigma_{yz}$ is proportional to $B^2 \cos \gamma \sin \gamma (n^3 \mu_{2-n} + nT^2 \mu_{2-n} + n^2 \mu_{2-n} + nT^2 \mu_{2-n})$.

We note that $\gamma = 0$ corresponds to the regular setup where $B$ and $E$ lie in two different planes. The above results reduce to the regular result for LMC and PHC if $\gamma$ is set to zero.

Appendix C: Calculation for TECs and PNC in setup 2: PNE

We shall now compute the thermo-electrical coefficients for transport. The magnetic field here is assumed to have the form: $B = B \cos \gamma j + B \sin \gamma j$ and $\nabla T = \nabla T_j$ with $j = x, z$. The longitudinal transport coefficient are given by $\alpha_{jj}$

$$\alpha_{jj} = e \int \frac{d^3k}{(2\pi)^3} \sigma \frac{(e - \mu)}{T} [(v_j + \frac{eB \cos \gamma}{\hbar} (\mathbf{v}_k \cdot \Omega_k))^2] \left( \frac{\partial f_{eq}}{\partial \epsilon} \right) \quad (C1)$$

the individual terms are represented as

$$\alpha^{(1)}_{jj} = e \int \frac{d^3k}{(2\pi)^3} \frac{(e - \mu)}{T} \frac{(v_j)^2}{1 + eB \cos \gamma \Omega_j + \sin \gamma \Omega_y} \left( \frac{\partial f_0}{\partial \epsilon} \right), \quad (C2)$$

$$\alpha^{(2)}_{jj} = e^3 \frac{B^2 \cos^2 \gamma}{\hbar^2} \int \frac{d^3k}{(2\pi)^3} \frac{(v_j)(\Omega_k \cdot v_k)}{T} \left( \frac{\partial f_0}{\partial \epsilon} \right), \quad (C3)$$

$$\alpha^{(3)}_{jj} = e^3 \frac{B \cos \gamma}{\hbar} \int \frac{d^3k}{(2\pi)^3} \frac{(v_j)(\Omega_k \cdot v_k)}{T} \left( \frac{\partial f_0}{\partial \epsilon} \right), \quad (C4)$$

$$\alpha^{(1)}_{xx} \quad (C5)$$

$$\alpha^{(1)}_{xx} = \frac{\tau en}{T v (2\pi)^3} \sum_i (-eBnv\alpha_n^{1/n}/2\hbar)^i \int_0^\pi \right d\theta \int_0^{2\pi} d\phi (\cos \phi)^i (\sin \theta)^{3+i(2-1/n)}$$

$$\int_0^\infty dk (k - \mu)^{i(1+1/n)} \hat{f}(k)_{eq} \approx \frac{\tau en}{Tv(2\pi)^3} \frac{\pi^{3/2} \Gamma(2)}{\Gamma(5/2)} \left( \frac{-\pi^2 \mu K_B T^2}{3} \right)$$

$$+ n(eBnv\alpha_n^{1/n} \mu^{-1-1/n}/2)^2 \frac{\pi^{3/2}(2 \cos^2 \gamma + 1) \Gamma(4-1/n)}{16 \Gamma(9/2-1/n)} \left( \frac{\pi^2 K_B T^2 \mu_{-1-2/n}}{3} \right) \quad (C6)$$

Similarly, $\alpha^{(2)}_{xx}$ is given by

$$\alpha^{(2)}_{xx} = \frac{\tau en}{4T \hbar^2 (2\pi)^3} \sum_i (-eBnv\alpha_n^{1/n}/2\hbar)^i \int_0^\pi \right d\theta \int_0^{2\pi} d\phi (\cos \phi)^i (\sin \theta)^{3-2/n+i(2-1/n)}$$

$$\int_0^\infty dk (k - \mu)^{-2/n-i(1+1/n)} \hat{f}(k)_{eq} \approx \frac{\tau en}{2T \hbar^2 (2\pi)^3} \frac{\pi^{3/2} \alpha_n^{2/n} B^2 \cos^2 \gamma}{\Gamma(5/2-1/n)} \left( \frac{-\pi^2 K_B T^2 \mu_{-1-2/n}}{3} \right) \quad (C7)$$
Similarly, $\alpha_{zz}^{(3)}_{PN}$ is given by

$$\alpha_{zz}^{(3)} = \frac{\tau e^2 n^2 \alpha_n^1 B \cos \gamma}{2T \theta^2 (2\pi)^3} \sum_i (-eBn \alpha_n^1/2h) \int_0^\pi d\theta \int_0^{2\pi} d\phi (\cos(\phi - \gamma))^{-1/n+i(2-1/n)}$$

$$f_0^\infty dk(k - \mu)k^{1-1/n-i(1+1/n)} \tilde{f}(k)_{eq}$$

$$\sim -\frac{\nu e^3 n^2 \alpha_n^2 B^2 \pi \cos^2 \gamma \Gamma(3 - 1/n)\Gamma(3/2)}{2T \theta^2 (2\pi)^3} \left( \frac{\pi^2 K_B T^2 \mu^{-1/2}}{3} \right)$$  \hspace{1cm} (C8)

Therefore, the chiral anomaly term in $\alpha_{xx}$ is proportional to $n^2 B^2 T \cos^2 \gamma \mu^{-1-2/n}$. The complete $\alpha_{xx}$ is proportional to $T(-\mu n + B^2 \mu^{-1-2/n} n^2 \cos^2 \gamma)$.

In the same fashion, one can compute $\sigma_{zz} = \sigma_{zz}^{(1)} + \sigma_{zz}^{(2)} + \sigma_{zz}^{(3)}$.

$$\alpha_{zz}^{(1)} = \frac{\tau e^2 n^2 \alpha_n^1 B \cos \gamma}{nT(2\pi)^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^\infty dk \sum_i (-eBn \alpha_n^1 k^{-2/n} \cos \theta \cos \gamma$$

$$+ n \alpha_n^1 k^{-1-1/n} \sin \phi \sin \gamma)/2h) \tilde{f}(k)_{eq}$$

$$\sim \tau e^2 n^2 \alpha_n^2 B^2 \pi \cos^2 \gamma \Gamma(2 - 1/n)\Gamma(5/2) \left( \frac{K_B^2 T^2 \mu^{-2/n}}{3h(9/2 - 1/n)} \right)$$  \hspace{1cm} (C9)

$\alpha_{zz}^{(2)}$ is given by

$$\alpha_{zz}^{(2)} = \frac{\tau e^3 B^2 \nu^2 n^2 \alpha_n^1 \cos^2 \gamma}{4T}(2\pi)^3 \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^\infty dk \sum_i (-eBn \alpha_n^1 k^{-2/n} \cos \theta \cos \gamma$$

$$+ n \alpha_n^1 k^{-1-1/n} \sin \phi \sin \gamma)/2h) \tilde{f}(k)_{eq}$$

$$\sim \tau e^3 B^2 \nu^2 \alpha_n^2 B^2 \pi \cos^2 \gamma \Gamma(2 - 1/n)\Gamma(5/2) \left( \frac{K_B^2 T^2 \mu^{-2/n}}{3} \right)$$  \hspace{1cm} (C10)

$\alpha_{zz}^{(3)}$ is given by

$$\alpha_{zz}^{(3)} = \frac{\tau e^3 B^2 \nu \cos \gamma}{2T(2\pi)^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^\infty dk \sum_i (-eBn \alpha_n^1 k^{-2/n} \cos \theta \cos \gamma$$

$$+ n \alpha_n^1 k^{-1-1/n} \sin \phi \sin \gamma)/2h) \tilde{f}(k)_{eq}$$

$$\sim \tau e^3 B^2 \nu \alpha_n^2 B^2 \pi \cos^2 \gamma \Gamma(7/2 - 1/n)(2\pi)^3 \left( \frac{K_B^2 T^2 \mu^{-2/n}}{3} \right)$$  \hspace{1cm} (C11)

Therefore, the chiral anomaly term in $\alpha_{zz}$ is proportional to $n^2 B^2 T \cos^2 \gamma \mu^{-1-2/n}$. The complete $\alpha_{zz}$ is proportional to $T(-\mu^2/n - n B^2 \mu^{-1-2/n} n^2 \cos^2 \gamma)$.

The expression for chiral anomaly induced transverse thermo-electrical coefficients in PN setup $\alpha_{yj}$ can similarly be decomposed into four parts: $\alpha_{yj}^{(3)}_{PN} = \alpha_{yj}^{(1)} + \alpha_{yj}^{(2)} + 2 \alpha_{yj}^{(3)} + \alpha_{yj}^{(4)}$. The complete form is given by

$$\alpha_{yj} = e \int \frac{dk}{(2\pi)^3} D(k - \mu) \frac{\partial f_{eq}}{\partial \epsilon} \left[ (v_y + eB \sin \gamma / h) (v_y + eB \cos \gamma / h) (v_y + eB \sin \gamma / h) (v_y + eB \cos \gamma / h) \right]$$  \hspace{1cm} (C12)
and the individual terms are represented as

\[ \alpha_{y}^{(1)} = \tau e \int \frac{d^3k}{(2\pi)^3} \frac{(\epsilon - \mu)}{T} \left( \frac{\mathbf{v}_y \mathbf{v}_k}{1 + eB(\cos \gamma \Omega_j + \sin \gamma \Omega_y)} \right) \left( \frac{\partial f_0}{\partial \epsilon} \right), \]

\[ \alpha_{y}^{(2)} = \tau e^2 \frac{B^2 \cos \gamma \sin \gamma}{\hbar^2} \int \frac{d^3k}{(2\pi)^3} \frac{(\epsilon - \mu)}{T} \left( \frac{(\mathbf{v}_y)(\mathbf{\Omega_k} \cdot \mathbf{v}_k)}{1 + eB(\cos \gamma \Omega_j + \sin \gamma \Omega_y)} \right) \left( \frac{\partial f_0}{\partial \epsilon} \right), \]

\[ \alpha_{y}^{(3)} = \tau e^2 \frac{B \sin \gamma}{\hbar} \int \frac{d^3k}{(2\pi)^3} \frac{(\epsilon - \mu)}{T} \left( \frac{(\mathbf{v}_y)(\mathbf{\Omega_k} \cdot \mathbf{v}_k)}{1 + eB(\cos \gamma \Omega_j + \sin \gamma \Omega_y)} \right) \left( \frac{\partial f_0}{\partial \epsilon} \right), \]

\[ \alpha_{y}^{(4)} = \tau e^2 \frac{B \sin \gamma}{\hbar} \int \frac{d^3k}{(2\pi)^3} \frac{(\epsilon - \mu)}{T} \left( \frac{(\mathbf{v}_y)(\mathbf{\Omega_k} \cdot \mathbf{v}_k)}{1 + eB(\cos \gamma \Omega_j + \sin \gamma \Omega_y)} \right) \left( \frac{\partial f_0}{\partial \epsilon} \right), \]

First we compute \( \alpha_{y}^{(1)} \) where

\[ \alpha_{y}^{(1)} = \frac{\tau e n}{2 T \Omega (2\pi)^3} \sum_i (-eBn\alpha_1^1/2\hbar)^i \int_0^\pi d\theta \int_0^{2\pi} d\phi (\cos(\phi - \gamma))^i \sin(2\phi)(\sin \theta)^{3+i(2-1/n)} \]

\[ \int_0^\infty (k - \mu)dk^2 - i(1+1/n) \bar{f}(k)_{eq} \]

\[ \simeq \frac{\pi^{-3/2} \Gamma(4 - 1/n)}{16\hbar^2 \Gamma(9/2 - 1/n)} \cos \gamma \sin \gamma e^3 \tau n^2 B^2 v_\alpha^2/3 \]

\[ \bar{\alpha}_{y}^{(2)} \text{ is given by} \]

\[ \alpha_{y}^{(2)} = \frac{\tau e^2 B^2 n^3 \alpha_2^2/n \sin \gamma \cos \gamma}{4 T \hbar^2 (2\pi)^3} \sum_i (-eBn\alpha_1^1/2\hbar)^i \int_0^\pi d\theta \int_0^{2\pi} d\phi (\cos(\phi - \gamma))^i \sin(\sin \theta)^{3+i(2-1/n)} \]

\[ \int_0^\infty (k - \mu)dk^2 - i(1+1/n) \bar{f}(k)_{eq} \]

\[ \simeq \frac{\Gamma(2 - 1/n)}{32\pi^{3/2} \hbar^2 \Gamma(7/2 - 1/n)} \cos \gamma \sin \gamma e^4 \tau n^2 B^2 v_\alpha^2/3 \]

\[ \bar{\alpha}_{y}^{(3)} \text{ is given by} \]

\[ \alpha_{y}^{(3)} = \frac{\tau e^2 B n^2 \alpha_1^1/n \sin \gamma}{2 T \hbar (2\pi)^3} \sum_i (-eBn\alpha_1^1/2\hbar)^i \int_0^\pi d\theta \int_0^{2\pi} d\phi (\cos(\phi - \gamma))^i \sin(\sin \theta)^{3+i(2-1/n)} \]

\[ \int_0^\infty (k - \mu)dk^2 - i(1+1/n) \bar{f}(k)_{eq} \]

\[ \simeq \frac{\Gamma(3 - 1/n)}{32\pi^{3/2} \hbar^2 \Gamma(7/2 - 1/n)} \cos \gamma \sin \gamma e^3 \tau n^2 B^2 v_\alpha^2/3 \]

\[ \bar{\alpha}_{y}^{(4)} \text{ is given by} \]

\[ \alpha_{y}^{(4)} = \frac{\tau e^2 B n^2 \alpha_1^1/n \sin \gamma}{2 T \hbar (2\pi)^3} \sum_i (-eBn\alpha_1^1/2\hbar)^i \int_0^\pi d\theta \int_0^{2\pi} d\phi (\cos(\phi - \gamma))^i \cos(\sin \theta)^{3+i(2-1/n)} \]

\[ \int_0^\infty (k - \mu)dk^2 - i(1+1/n) \bar{f}(k)_{eq} \]

\[ \simeq \frac{\Gamma(3 - 1/n)}{32\pi^{3/2} \hbar^2 \Gamma(7/2 - 1/n)} \cos \gamma \sin \gamma e^3 \tau n^2 B^2 v_\alpha^2/3 \]

Therefore, the chiral anomaly term and complete \( \alpha_{y} \) are both proportional to \( TB^2 n^2 \cos \gamma \sin \gamma \mu^{-1-2/n} \).

On the other hand, \( \alpha_{y}^{(1)} \) is given by

\[ \alpha_{y}^{(1)} = \frac{\tau e n^{1/n}}{2 T \pi^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^\infty (k - \mu) \sum_i (-eBn^2 \alpha_1^1/n k^{-2/n}(\sin \theta)^{2-2/n} \cos \theta \cos \gamma + n \alpha_1^1/n k^{-1/n}(\sin \theta)^{2-1/n} \cos \phi \sin \gamma) / 2\hbar \cos \theta \sin \phi(\sin \theta)^{1/n} k^{1/n} \bar{f}(k)_{eq} \]

\[ \simeq -\frac{\pi \Gamma(3 - 1/n) \Gamma(3/2)}{h^2 \Gamma(9/2 - 1/n)} \cos \gamma \sin \gamma e^3 \tau n^2 B^2 v_\alpha^2/3 \]

\[ \left( \frac{K_B^2 T^2 \mu^{-1-2/n}}{6} \right) \]

(C22)
\[ \alpha_{yz}^{(2)} = \frac{\tau e^2 \alpha_n^{2/n} B^2 n^2 v \sin \gamma \cos \gamma}{(4T \hbar^2 \pi )^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^\infty dk(k - \mu) \sum_i (-eB(n^2 \alpha_n^{2/n} r^{-2/n} \sin \theta)^{2-2/n} \cos \theta \cos \gamma \\
+ n \nu \alpha_n^{1/n} k^{-1-1/n} (\sin \theta)^{2-1/n} \sin \phi \sin \gamma / 2 \hbar)^{i} \Gamma \left( \frac{1}{2} \right) (\tilde{f}(k)_{eq} \right. \\
\simeq \frac{\tau e^2 \alpha_n^{2/n} B^2 n^2 v \sin \gamma \cos \gamma \Gamma(2 - 1/n) \Gamma(1/2)}{4\hbar^2 \Gamma(5/2 - 1/n)} \left( K_B T \pi^2 \mu^{-1-2/n} \right) \] (C23)

\[ \alpha_{yz}^{(3)} = \frac{\tau e^2 \alpha_n^{2/n} B^2 n^2 v \sin \gamma}{(4T \hbar^2 \pi )^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^\infty dk(k - \mu) \sum_i (-eB(n^2 \alpha_n^{2/n} r^{-2/n} \sin \theta)^{2-2/n} \cos \theta \cos \gamma \\
+ n \nu \alpha_n^{1/n} k^{-1-1/n} (\sin \theta)^{2-1/n} \sin \phi \sin \gamma / 2 \hbar)^{i} \sin \theta \cos \gamma \tilde{f}(k)_{eq} \\
\simeq \frac{\tau e^2 \alpha_n^{2/n} B^2 n^2 v \sin \gamma \cos \gamma \Gamma(2 - 1/n) \Gamma(1/2)}{4\hbar^2 \Gamma(7/2 - 1/n)} \left( K_B T \pi^2 \mu^{-1-2/n} \right) \] (C24)

\[ \alpha_{yz}^{(4)} = \frac{\tau e^2 B n \sin \gamma}{(4T \hbar^2 \pi )^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^\infty dk(k - \mu) \sum_i (-eB(n^2 \alpha_n^{2/n} r^{-2/n} \sin \theta)^{2-2/n} \cos \theta \cos \gamma \\
+ n \nu \alpha_n^{1/n} k^{-1-1/n} (\sin \theta)^{2-1/n} \sin \phi \sin \gamma / 2 \hbar)^{i} \sin \theta \cos \gamma \tilde{f}(k)_{eq} \\
\simeq \frac{\tau e^2 \alpha_n^{2/n} B^2 n^2 v \sin \gamma \cos \gamma \Gamma(2 - 1/n) \Gamma(3/2)}{4\hbar^2 \Gamma(7/2 - 1/n)} \left( K_B T \pi^2 \mu^{-1-2/n} \right) \] (C25)

Therefore, the chiral anomaly term in \( \alpha_{yz}^{P,N} \) is proportional to \( TB^2 n^2 \cos \gamma \sin \gamma \mu^{-1-2/n} \), the complete \( \alpha_{yz} \propto n^2 TB^2 \cos \gamma \sin \gamma (\mu^{-1-2/n} + \mu^{1-2/n}) \).

We note that \( \gamma = 0 \) corresponds to the regular setup where \( B \) and \( \nabla T \) lie in two different planes. The above results reduce to the regular result for LTEC and TTEC if \( \gamma \) is set to zero.

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