PAPER

Free vibration analysis of cracked functionally graded non-uniform beams

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Abstract

This paper presents the free vibration analysis of an edge cracked non-uniform symmetric beam made of functionally graded material. The Timoshenko beam theory is used for the finite element analysis of the multi-layered sandwich beam and the cantilever beam is modeled by 50 layers of material. The material properties vary continuously along the thickness direction according to the exponential and power laws. A MATLAB code is used to find the natural frequencies of two types of non-uniform beams, having a constant height but an exponential or linear width variation along the length of the beam. The natural frequencies of the beam are verified with ANSYS software as well as with available literature and good agreement is found. In the study, the effects of different parameters such as crack location, crack depth, power-law index, geometric index and taper ratio on natural frequencies are analyzed in detail.

1. Introduction

Non-uniform cross-section structural members like beams, plates and shells are widely used in complex structures due to their possibilities of efficiently optimizing weight, being more economical and also fulfilling architectural requirements [1]. In recent years, there is an increasing need to fulfill the new modern technology requirements by substituting conventional materials with more advanced ones such as Functionally Graded (FG) materials. The use of FG materials eliminates interlaminar stress concentration and due to its unique properties provides strength and toughness of the structure [2]. A large number of studies related to vibration characteristics of intact and cracked FG uniform beams are available [3–22]. Furthermore the studies have been extended in mechanical analysis of small-sized structures [23–31]. Studies on non-uniform FG beams can also be found in literature [32–47], while studies on non-uniform FG beams with an imperfection are scarce.

The existence of a crack causes a reduction in rigidity and this affects the dynamic behavior of the structure. The dynamic behavior of the structure must be known on forward if resonance of the structure has to be prevented, thus one must take into account the natural frequencies of the structure during the design stage. For that reason in this study, an edge cracked multi-layered symmetric sandwich FG non-uniform beam containing a crack is investigated in terms of free vibration. The considered problem is carried out with Timoshenko beam elements which take first order shear deformation into account, by using the finite element method. It is assumed that the material properties distribution change along the beam height according to the power and exponential laws. To the authors’ knowledge, there are no studies on cracked functionally graded non-uniform beams. As a result, the novelty of this work is to estimate the natural frequencies of an edge cracked FG non-uniform beam with constant height and exponential or linear width variation along the length of the beam. Classical lamination theory is used to find the effective mass density and Young’s modulus of the cantilever FG beam. A detailed study is carried out to observe the effects of crack location, crack depth, power-law index, different material distribution, different geometric index and different taper ratio on the first four natural frequencies.
2. Non-uniform cracked FGM beams

Two types of cracked non-uniform symmetric FG sandwich Timoshenko beams are considered to analyze the vibration characteristics in this study. As shown in Figure 1, the beams are considered to be of constant height but have an exponentially or linearly variable width. Here: L, b, Lc and a, represent length, height, crack location and crack depth, respectively. The width of the beam $d(x)$ in equation (1) is defined by an exponentially varying equation along the length of the beam, while $d_0$ is the half width of the beam. For the tapered beam $b_1$ and $b_2$ are the width of the beam at the fixed end and the free end, respectively.

$$d(x) = d_0 e^{\delta x}$$  (1)

To obtain a more solution, the FGM is modeled by fifty layers of material. Each layer is composed of a mixture of aluminum (Al) and alumina phases (Al₂O₃), arranged symmetrically to the neutral axis of the beam as shown in figure 1. The variations of the material properties through the whole thickness of the beam for exponential law and various power-index values of power law functions are shown in figure 2. The strain between the layers is assumed to be linear. The upper and the lower part of the beam are made of pure ceramic (Al₂O₃), the middle part is pure metal (Al), while the layers between them are made of FGM. For the mixture ratio, a polynomial or an exponential function is chosen, varying continuously from the upper and lower part toward the beam neutral plane.

Young’s modulus and the density of each layer of the beam for the upper half of the beam are given by the exponential and power law equations [10]:

$$E(y) = E_c e^{(\delta (1-2y))}, \quad \delta = \frac{1}{2} \ln \left( \frac{E_c}{E_m} \right)$$  (2)

$$E(y) = (E_c - E_m) \left( y + \frac{1}{2} \right)^n + E_m$$  (3)

where $E_c$ and $E_m$ are Young’s modulus of the ceramic and metal, $y$ the width coordinate and $n$ the power-law index, respectively.
\[ \rho(y) = \rho_c e^{-(1-2y)\delta} = \frac{1}{2} \ln \left( \frac{\rho_c}{\rho_m} \right) \] (4)

\[ \rho(y) = (\rho_c - \rho_m)y + \rho_m \] (5)

Here \( \rho_c \) and \( \rho_m \) are the density for the ceramic and the metal. The variable \( y \) is defined as:

\[ y = -\frac{1}{2}, -\frac{1}{2} + \frac{1}{\eta}, -\frac{1}{2} + \frac{2}{\eta}, \ldots, -\frac{1}{2} \]

where \( \eta = m - 1 \) and \( m \) is the number of layers.

The effective Young's modulus and mass density of the laminated beam are calculated using lamination theory [48]:

\[ E_{ef} = \frac{8}{b^3} \sum_{i=1}^{N/2} (E_i)(y_i^3 - y_{i-1}^3) \] (6)

\[ E_{ef} = \frac{8}{b^3} \sum_{i=1}^{N/2} (E_i)(y_i^3 - y_{i-1}^3) \] (7)

### 3. Stiffness and mass matrices of the Timoshenko beam element

Stiffness and mass matrices derivation for a beam element having two nodes with 2 degrees of freedoms \( \{v, \theta\} \) at each node, and for a bar element having two nodes with 1 degree of freedom \( \{u\} \), are given by Petyt [49] as:

\[
[K_i] = \frac{E I_z}{2k^3(1 + 3\beta)} \begin{bmatrix}
3 & 3k & -3 & 3k \\
3k & (4 + 3\beta)k^2 & -3k & (2 - 3\beta)k^2 \\
-3 & -3k & 3 & -3k \\
3k & (2 - 3\beta)k^2 & -3k & (4 + 3\beta)k^2
\end{bmatrix}
\]

\[
[M_i] = \frac{\rho A k}{210(1 + 3\beta)^2} \begin{bmatrix}
m_1 & m_2 & m_3 & m_4 \\
m_2 & m_3 & m_5 & m_6 \\
m_3 & m_5 & m_7 & m_8 \\
m_4 & m_6 & m_8 & m_5
\end{bmatrix} + \frac{\rho I_z}{30k(1 + 3\beta)^2} \begin{bmatrix}
m_7 & m_8 & -m_7 & m_8 \\
m_8 & m_9 & -m_8 & m_7 \\
-m_7 & m_8 & m_7 & -m_8 \\
-m_8 & m_7 & m_8 & m_7
\end{bmatrix}
\]

\[
[K_L] = \frac{E A}{k} \begin{bmatrix}
1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & 1 & 1 & 1 \\
-\frac{1}{2} & 1 & 1 & 1 \\
-\frac{1}{2} & 1 & 1 & 1
\end{bmatrix}
\]

\[
[M_L] = \rho \cdot A \cdot k \begin{bmatrix}
3 & 1 & 2 \\
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix}
\]

where \( k = 0.5L \), \( E \) is Young's modulus of elasticity and \( I_z \) is the section moment of inertia with respect to the \( z \)-axis. The shear deformation factor \( \beta \) is given as:
\[ \beta = \frac{EI}{\kappa GAk^2} \]  

where \( \kappa \) is the shear correction factor, \( G \) is the shear modulus and \( A \) is the cross sectional area and

\[
\begin{align*}
m_1 &= 156 + 882\beta + 1260\beta^2 \\
m_2 &= (44 + 231\beta + 315\beta^2) a \\
m_3 &= 54 + 378\beta + 630\beta^2 \\
m_4 &= (-26 - 189\beta - 315\beta^2) a \\
m_5 &= (16 + 84\beta + 126\beta^2) a^2 \\
m_6 &= (-12 - 84\beta - 126\beta^2) a \\
m_7 &= 18 \\
m_8 &= (3 - 45\beta) a \\
m_9 &= (8 + 30\beta + 180\beta^2) a^2 \\
m_{10} &= (-2 - 30\beta + 90\beta^2) a 
\end{align*}
\]

using \( K_1, K_2, M_1 \) and \( M_2 \) matrices given above, the total stiffness and mass matrices for 3 degrees of freedoms at each node becomes:

\[
K_{el} = \begin{bmatrix}
K_{11} & K_{12} \\
K_{13} & K_{14} \\
K_{21} & K_{22} \\
K_{23} & K_{24} \\
K_{31} & K_{32} \\
K_{33} & K_{34} \\
K_{41} & K_{42} \\
K_{43} & K_{44}
\end{bmatrix}_{(6\times6)}
\]

\[
M_{el} = \begin{bmatrix}
M_{11} & M_{12} \\
M_{13} & M_{14} \\
M_{21} & M_{22} \\
M_{23} & M_{24} \\
M_{31} & M_{32} \\
M_{33} & M_{34} \\
M_{41} & M_{42} \\
M_{43} & M_{44}
\end{bmatrix}_{(6\times6)}
\]

4. The stiffness matrix for the crack

The cracked node is considered as a cracked element of zero length and zero mass [50]. The strain energy due to the crack leads to flexibility coefficients, derived by Kisa and Brandon [51]. Using the flexibility coefficient according to the displacement vector \( \delta = \{u, v, \theta\} \), the compliance coefficient matrix is as follows:

\[
C = \begin{bmatrix}
c_{11} & c_{12} & c_{13} \\
c_{21} & 0 & c_{23} \\
c_{31} & 0 & c_{33}
\end{bmatrix}_{(3\times3)}
\]

The inverse of the compliance matrix is the stiffness matrix due to the crack. As a result, the crack stiffness matrix is as follows:

\[
K_{cr} = \begin{bmatrix}
[C]^{-1} & -[C]^{-1} \\
-[C]^{-1} & [C]^{-1}
\end{bmatrix}_{(6\times6)}
\]

The eigenvalue problem of free vibration of an intact [52] and cracked beam [9] are as follows;

\[
([K] - \omega^2[M])\{\ddot{\delta}\} = 0 \tag{17}
\]

\[
([K] + [K_{cr}]) - \omega^2[M])\{\ddot{\delta}\} = 0 \tag{18}
\]

where \([K]\) is the global stiffness matrix, \([M]\) is the global mass matrix, \(\omega\) is the natural angular frequency in radians per second, and \(\{\ddot{\delta}\}\) is the mode shape.

5. Verification

In order to verify the correctness of the approach and the MATLAB code, numerical results are compared with the results of an example given by Demir et al [53]. In the literature example results are calculated according to Euler–Bernoulli beam theory. The validation in the present study is done according to the Euler–Bernoulli as well as the Timoshenko beam theory. The geometric index is fixed at \( \beta = -1/L \). Polynomial index degrees \( n = 0 \), \( n = 1 \) and \( n = 10 \) are validated. In the analysis, finite element modeling was done with 40 elements, which provided convergence and matched the literature results. A comparison of the obtained results are given in table 1 in Hertz. As can be seen from table 1, the correctness of the finite element code written in MATLAB is proven and results are accurate already with 40 elements.

To verify the modeling of the crack, a taper ratio \( b_2/b_1 = 1 \) and a geometric index \( \beta = 0 \) is assumed, which will result in the non-uniform cracked FG beam, becoming a cracked uniform FG beam. Numerical results are
then compared with existing results in the literature \cite{53} and are also cross checked with ANSYS software. The results are presented in table 2 proving that the model for the crack is also correct.

To verify the correctness of the shape functions, ANSYS software is used. Various taper ratios and geometric indices for the non-uniform beams are being considered. The fifty layered non-uniform beam is drawn in INVENTOR and exported to ANSYS. Material properties, such as Young’s modulus and the density are assigned to each layer using the exponential and power law functions from the equations (2)–(5), while the Poisson’s ratio is taken as a constant value of 0.3. The beam is meshed using SOLID186 elements which have 20 nodes with three degrees of freedom each. The crack is defined as a zero thickness surface, and the beam is fixed at one end. The block Lanczos method is used for the eigenvalue extractions to calculate frequencies. The material properties of the constituents of the beam are given in table 3, while the geometric dimensions are as follows: $L = 200$ mm; $b = 5$ mm, $d_0 = 10$ mm, $b_1 = 20$ mm, shear stiffness modulus $G = 3/8E$ and shear correction factor $\kappa = 5/6$ rectangular cross-section. The value of the crack location and power-law index are taken as; $L_c/L = 0.2$ and $n = 10$. Two crack depth ratios are being considered for verification: $a/b = 0.4$ and $a/b = 0.8$.

As can be seen from table 4, the natural frequency values calculated with the MATLAB code show good agreement with the ANSYS results.

The verification of the table is also supported by providing the first four frequencies and mode shapes of the FE model in Ansys, for case: $\beta = -1/L$ and $a/b = 0.8$, $L_c/L = 0.2$ and $n = 10$ (figure 3).

6. Numerical results and discussions

In the study two types of cracked non-uniform FG beam are being considered (see figure 1). The cantilever FG beam consists of 50 layers. The material properties vary continuously along the thickness direction according to the exponential and power laws. The MATLAB code generates 100 finite element beam elements, which provides a convergence of the solution, as shown in table 5. The material properties of the constituents of the beam and the geometric dimensions are the same as on the ANSYS verification section. In the study, the effects of different parameters such as crack location, crack depth, power-law index, geometric index and taper ratio on natural frequency values are analyzed in detail.

| Table 1. The first three natural frequencies of a simple supported FG sandwich non-uniform beam. |
| --- |
| | $\beta = -1/L$ |
| | $n = 0$ | $n = 1$ | $n = 10$ |
| Demir et al [53] | Euler | Timos. | Euler | Timos. | Euler | Timos. |
| 550.5 | 550.484 | 549.875 | 513.6 | 513.549 | 512.981 | 401.1 | 401.127 | 400.684 |
| 2229.0 | 2228.905 | 2219.384 | 2079.4 | 2079.355 | 2070.472 | 1624.2 | 1624.161 | 1617.223 |
| 5011.6 | 5011.509 | 4964.063 | 4675.4 | 4675.257 | 4630.994 | 3651.9 | 3651.792 | 3617.219 |

| Table 2. The first four natural frequencies (Hz) of a symmetric cracked FG Timoshenko beam. |
| --- |
| | Mode 1 | Mode 2 | Mode 3 | Mode 4 |
| | Present | Cunedioglu [10] | Ansys | Present | Cunedioglu [10] | Ansys |
| Polynomal ($n = 5$) | 148.264 | 148.264 | 162.783 | 162.783 | 2614.865 | 2614.865 | 2660.114 | 2660.114 |
| Exponential | 902.267 | 902.267 | 990.619 | 990.619 | 5209.958 | 5209.958 | 5279.894 | 5279.894 |
| | 2870.919 | 2870.919 | 5720.129 | 5720.129 | |
| | 2923.200 | 2923.200 | 5844.200 | 5844.200 | |

| Table 3. Material properties of the constituents of the beam. |
| --- |
| Material | $E$ (GPa) | $\rho$ (kg/m$^3$) | $\nu$ |
| Al | 70 | 2700 | 0.3 |
| Al2O3 | 380 | 3950 | 0.3 |
In this part, numerous cases of the effects of different crack depth ratios ($a/b$), different crack locations ($L_c/L$), different crack depth ratios ($a/b$), different geometric indices ($\beta$), power-law index ($n$) and the exponential function material distribution on natural frequency values of the beam were investigated. The beam width varies exponentially along the length, while the variations of the material properties through the thickness are realized by power and exponential laws.

### Table 4. The first four natural frequencies (Hz) of the non-uniform edge cracked cantilever FG Timoshenko beam.

| n = 10 | $a/b = 0.4$ | $a/b = 0.8$ |
|--------|-------------|-------------|
| | Matlab | Ansys | Matlab | Ansys | Matlab | Ansys | Matlab | Ansys |
| $\beta = -0.25/L$ | | | | | | | | |
| Mode 1 | 148.626 | 148.86 | 95.877 | 95.882 | 147.413 | 147.63 | 95.113 | 95.187 |
| Mode 2 | 922.082 | 949.2 | 918.395 | 946.01 | 919.805 | 946.91 | 916.397 | 943.96 |
| Mode 3 | 2500.204 | 2545 | 2285.639 | 2321.6 | 2497.689 | 2542.3 | 2281.195 | 2317.3 |
| Mode 4 | 4746.773 | 4760.1 | 3972.931 | 4017.8 | 4744.521 | 4757.6 | 3970.309 | 4015.5 |
| $\beta = -0.5/L$ | | | | | | | | |
| Mode 1 | 160.179 | 160.8971 | 103.273 | 103.2605 | 160.489 | 160.6784 | 103.517 | 103.5805 |
| Mode 2 | 943.702 | 971.1261 | 938.324 | 966.4295 | 944.808 | 972.3069 | 940.096 | 968.2334 |
| Mode 3 | 2523.094 | 2504.392 | 2317.316 | 2353.467 | 2523.880 | 2531.749 | 2313.365 | 2349.617 |
| Mode 4 | 4768.400 | 4789.698 | 3999.508 | 4047.619 | 4769.882 | 4781.026 | 4002.269 | 4051.737 |
| $\beta = -0.75/L$ | | | | | | | | |
| Mode 1 | 185.441 | 186.1615 | 119.490 | 119.3801 | 211.360 | 211.3879 | 136.120 | 136.1278 |
| Mode 2 | 988.260 | 1016.133 | 978.534 | 1007.492 | 1048.101 | 1077.738 | 1041.689 | 1072.255 |
| Mode 3 | 2571.433 | 2619.298 | 2383.081 | 2419.534 | 2643.386 | 2687.379 | 2428.756 | 2464.873 |
| Mode 4 | 4814.621 | 4832.861 | 4051.321 | 4103.192 | 4897.106 | 4903.955 | 4139.560 | 4199.277 |
| $\beta = -1/L$ | | | | | | | | |
| Mode 1 | 218.807 | 219.537 | 134.587 | 134.477 | 224.860 | 224.884 | 151.120 | 151.1278 |
| Mode 2 | 1159.260 | 1186.133 | 1058.534 | 1087.492 | 1128.101 | 1157.738 | 1121.689 | 1152.255 |
| Mode 3 | 3063.433 | 3119.298 | 2853.081 | 2889.534 | 3063.386 | 3107.379 | 2978.756 | 3014.873 |
| Mode 4 | 6174.621 | 6192.861 | 5051.321 | 5103.192 | 6097.106 | 6103.955 | 5239.560 | 5299.277 |

![Figure 3. Modeshapes/Natural frequencies (Hz) of the cracked non-uniform FE beam model in ANSYS.](image-url)
Table 5. Convergence analysis of a sandwich FG symmetric cantilever non-uniform beam ($\beta = -1/L$).

| Function type | Nat. Freq. Hz | 40 elements | 50 elements | 60 elements | 70 elements | 80 elements | 90 elements | 100 Elements |
|---------------|---------------|-------------|-------------|-------------|-------------|-------------|-------------|--------------|
| Exponential   |               |             |             |             |             |             |             |              |
| 1st mode      | 233.417       | 233.434     | 233.443     | 233.448     | 233.452     | 233.454     | 233.456     |              |
| 2nd mode      | 1189.622      | 1189.701    | 1189.746    | 1189.773    | 1189.792    | 1189.804    | 1189.814    |              |
| 3rd mode      | 3124.121      | 3124.259    | 3124.344    | 3124.400    | 3124.440    | 3124.468    | 3124.489    |              |
| 4th mode      | 5980.530      | 5980.534    | 5980.572    | 5980.614    | 5980.650    | 5980.679    | 5980.703    |              |
| Polynomial ($n = 10$) |               |             |             |             |             |             |             |              |
| 1st mode      | 194.188       | 194.201     | 194.209     | 194.214     | 194.217     | 194.219     | 194.220     |              |
| 2nd mode      | 989.688       | 989.754     | 989.791     | 989.814     | 989.829     | 989.840     | 989.847     |              |
| 3rd mode      | 2599.065      | 2599.179    | 2599.250    | 2599.297    | 2599.329    | 2599.353    | 2599.370    |              |
| 4th mode      | 4975.410      | 4975.414    | 4975.445    | 4975.480    | 4975.510    | 4975.534    | 4975.534    |              |
6.1.1. Effects of the geometric index and power-law index

In order to investigate the effects of the geometric index $\beta$ and different power-law index on natural frequency values, analyzes were performed for the fixed values $L_c/L = 0.2$ and $a/b = 0.2$. As can be seen from figure 4, with an increase of the power-law index, all natural frequency values decrease. With a decreasing geometric index $\beta$, an increase in natural frequency values was observed across the board.

6.1.2. Effects of the crack location and power-law index

Analyses are carried out to investigate the effects of the crack location and different power-law index on the first four natural frequencies for $L_c/L = 0.2$ and $a/b = 0.2$. As shown in figure 5, the effect of crack location on all first four natural frequencies were found to be minimal. However, decreases in all natural frequency values have been observed with an increasing power-law index.

6.1.3. Effects of the crack location and crack depth ratio

Analyses are carried out to see the effects of crack location and crack depth ratio on natural frequencies, for a fixed geometric index parameter ($\beta = -1/L$) and power-law index $n = 5$. All throughout figure 6 it can be seen that for both polynomial and exponential functions, increasing the crack depth ratio causes a decrease in natural frequency values, with crack depth ratio $a/b = 0.6$ and $a/b = 0.8$, having a significant effect on the natural frequency values. As can be seen from figures 6(a)–(b), a deep crack near the clamped support greatly reduces the first natural frequency value, while the effect minimizes as the crack location approaches the free end of the

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**Figure 4.** The effects of the geometric index ($\beta$) and power-law index ($n$) on the first four natural frequencies for $L_c/L = 0.2$ and $a/b = 0.2$.
beam. Figures 6(c) and (d), however show a decrease followed by an increase with a minimum at \((L_c/L) = 0.6\), while figures 6(e) and (f), show an oscillating graph for the second and third frequencies respectively. The 4th natural frequencies values seem to have a plateau from \((L_c/L) = 0.4\) to \((L_c/L) = 0.6\). It can clearly be seen that the effect of the crack location is dependent on the different mode shapes of the vibrating beam corresponding to the first to the fourth natural frequencies.

6.2. Cantilever beam with a linearly varying width

In this part, numerous cases of the effects of different taper ratios \((b_2/b_1)\), different crack locations \((L_c/L)\), different crack depth ratios \((a/b)\), different geometric indices \((\beta)\), power-law index \((n)\) and the exponential function material distribution on natural frequency values of the beam were investigated. The beam width varies linearly along the length, while the variations of the material properties through the thickness are realized by power and exponential laws.

6.2.1. Effects of the taper ratio and power-law index

In order to investigate the effects of the taper ratio \((b_2/b_1)\) and different power-law index on natural frequency values, analyzes were performed for the fixed values \((L_c/L) = 0.2\) and \((a/b) = 0.2\). As can be seen from figure 7, with an increase of the power-law index, all natural frequency values decrease. With a decreasing taper ratio however, an increase in natural frequency values was observed across the board.
6.2.2. Effects of the crack location and power-law index

Analyses are carried out to investigate the effects of the crack location and different power-law index on the first four natural frequencies for $\beta = -1/L$ and $n = h$, for exponential and polynomial functions.

Figure 6. The effects of crack location ($L_c/L$) and different crack depth ratio ($a/b$) on the first four natural frequencies for $\beta = -1/L$ and $n = 5$, for exponential and polynomial functions.

6.2.2. Effects of the crack location and power-law index

Analyses are carried out to investigate the effects of the crack location and different power-law index on the first four natural frequencies for fixed $b_2/b_1 = 0.2$ and $a/b = 0.2$. As can be seen in figure 8, the effect of crack
location on all first four natural frequencies were found to be minimal. However, decreases in all natural frequency values have been observed with an increasing power-law index.

6.2.3. Effects of the crack location and crack depth ratio

Analyses are carried out to see the effects of crack location and crack depth ratio on natural frequencies, for a fixed taper ratio $b_2/b_1 = 0.2$ and power-law index $n = 5$. Throughout figure 9 it can be noted that for both polynomial and exponential functions, increasing the crack depth ratio causes a decrease in natural frequency values, with crack depth ratio $a/b = 0.6$ and $a/b = 0.8$, having a significant effect on the natural frequency values. As can be seen from figures 9(a)–(b), a deep crack near the support greatly reduces the first natural frequency value, while the effect minimizes as the crack location approaches the free end of the beam. Figures 9(c) and (d), however show a decrease followed by an increase with a minimum at $L_c/L = 0.6$, while figures 9(e) and (f), show an oscillating graph for the second and third frequencies respectively. The 4th natural frequencies values seem to have a plateau from $L_c/L = 0.4$ to $L_c/L = 0.6$. It can clearly be seen that the effect of the crack location is dependent on the different mode shapes of the vibrating beam corresponding to the first to the fourth natural frequencies.

Taper ratio and exponential function will change the cross section and moment of inertia and thus mass for each cross section. Crack location and crack depth influences the stiffness matrix. For that reason the natural frequencies as a function of the stiffness matrix will change the behavior of the frequencies.
7. Conclusions

In this study, the vibration behavior of a functionally graded material cantilever beam with constant height and exponential or linear width variation is investigated. A single crack with varying location and depth is added as an imperfection. In the study, the effects crack location, crack depth, power-law index, geometric index and taper ratio on natural frequency values are analyzed in detail.

Some of the results obtained are given below;

- The beam with linear width variation has higher natural frequencies than the exponentially varying beam in all cases.
- A decrease in the geometric index $\beta$ and taper ratio $b_2/b_1$ increases all natural frequency values, with the increase being linear for $\beta$ and parabolic for $b_2/b_1$.
- An increase in the power-law index decreases all natural frequencies across the board.
- The effect of the crack location is dependent on the different mode shapes of the vibrating beam corresponding to the natural frequency.

Figure 8. The effects of the crack location ($L_c/L$) and different power-law index ($n$) on the first four natural frequencies $a/b = 0.2$ and $b_2/b_1 = 0.2$
Figure 9. The effects of crack location (Lc/L) and different crack depth ratio (a/b) on the first four natural frequencies for b2/b1 = 0.2 and n = 5, for exponential and polynomial functions.
• Increasing crack depths results in decreased frequency values while for deep cracks with $a/b = 0.6$ and $a/b = 0.8$ the impact becomes profound.

• With small cracks ($a/b = 0.2$) the crack location has an insignificant effect on all natural frequencies.

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