Entanglement evolution of bipartite $m \otimes n$-dimensional systems

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Abstract. Entanglement evolution in high dimensional bipartite systems under dissipation is studied through a lower bound of entanglement of formation. Discontinuities for the time derivative of these quantity are found depending on the initial conditions for entangled states. These abrupt changes along the evolution appear as precursors of entanglement sudden death.

1. Introduction
In recent years, the concept of entanglement has become one of the most important resources in the fields of Quantum information and Quantum Computing [1], not only its interesting fundamental properties but rather its potential applications like quantum cryptography [2, 3], quantum teleportation [4], etc. Among the cases mostly studied up to date, the quantum correlations of a system composed by two qubits has shown the most fruitful results and new phenomena have appeared, among others the effect called entanglement sudden death [5, 6]. However, obtaining a closed form for describing the entanglement of a general bipartite system in which the dimensions of both subsystems are arbitrary, has been a very hard task and is still an open problem. When this arbitrary bipartite system is in a pure state, analytical expression to quantify the entanglement has been proposed [7]. For the case when the bipartite system is in a mixed state, only very recently, some criterions have appeared in order to quantify entanglement [8–10]. The expressions obtained in these works are based on two major criterions: (i) the so-called positivity under partial transpose criterion (PPT criterion) [11, 12] and (ii) the realignment criterion [8, 9]. In a recent work [10], these two criterions have been unified to form a general measure to quantify the EOF and an analytical expression for its lower bound has been given. In this work we explore dynamical behavior of entangled states in larger bipartite systems under the action of independent reservoirs. We show that the case of two qutrits may present not only ESD, but also intermediate abrupt changes in the disentanglement dynamics.
2. Entanglement Measures

For a pure $m \otimes n$-dimensional system, the entanglement can be quantified with the measure called tangle [7]. This tangle can be written as

$$\tau_{AB} = 2\nu_A\nu_B \left(1 - \text{tr} \left( \rho_A^2 \right) \right)$$  \hspace{1cm} (1)

where $\rho_A$ is the reduced density operator obtained by tracing out the subsystem $B$. The coefficients $\nu_A$, $\nu_B$ are arbitrary scale factors depending on the dimension of each subsystem $A$ and $B$.

On the other hand, when the system is in an overall mixed state, analytical expressions for the EOF are a very hard task to face on. However, recently, analytical expressions have been obtained for lower bounds of EOF [10]. According to this, the EOF can be written as

$$E(\rho) \geq \begin{cases} 0 & \text{if } \Lambda = 1, \\ H_2 [\gamma(\Lambda)] + [1 - \gamma(\Lambda)] \log_2 (m - 1) & \text{if } \Lambda \in \left[\frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}} \right], \\ \log_2 (m - 1) \log_2 (m) + \log_2 (m) & \text{if } \Lambda \in \left[\frac{4m - 1}{m}, m \right], \end{cases}$$  \hspace{1cm} (2)

where $m$ is the dimension of the first subsystem, $\gamma$ is given by

$$\gamma(\Lambda) = \frac{1}{m^2} \left[ \sqrt{\Lambda} + \sqrt{(m - 1)(m - \Lambda)} \right]^2$$  \hspace{1cm} (3)

with $\Lambda = \max(\|\rho^{TA}\|, \|R(\rho)\|)$ and $H_2(x) = x \log_2(x) - (1 - x) \log_2(1 - x)$, where $\|\cdot\|$ represents the trace norm such that $\|G\| = \text{tr}(GG^\dagger)^\frac{1}{2}$. The matrix $\rho^{TA}$ is the partial transpose with respect to the subsystem $A$, this is, $\rho^{TA}_{i{k,j}l} = \rho_{jk,il}$, and the matrix $R(\rho)$ is defined as $R(\rho)_{i{ij}kl} = \rho_{ik,jl}$.

As we mentioned above, this analytical expression for the EOF combines the PPT criterion with the realignment criterion. The PPT criterion says that $\rho^{TA} \geq 0$ for a separable state [11]. Moreover, $\rho^{TA} \geq 0$ is also sufficient for separability of $2 \otimes 2$ and $2 \otimes 3$ bipartite systems.

On the other hand, the realignment criterion says that a realigned version of $\rho$, denoted $R(\rho)$, must satisfy for a separable state the condition: $\|R(\rho)\| \leq 1$.

Numerical calculations in the physical systems involve in this work, show that $\Lambda$ is a monotone of the EOF. Then, it is possible to quantify entanglement through this quantity, which results to be $\Lambda = m$ when the bipartite state is maximally entangled, and $\Lambda = 1$ when the state is separable. By a numerical calculation we realize that $\|\rho^{TA}\| \geq \|R(\rho)\|$ for all times, so that, we need to concentrate only in $\Lambda(t) = \|\rho^{TA}\|$.

3. Entanglement Evolution

We are mainly concerned in studying the evolution of maximally entangled states, as well as special states corresponding to particular generalizations of the “X”states [6, 13] to higher dimensions. Maximally entangled states for arbitrary dimensions rely on the generalization of the Hadamard gate $H$, which is the discrete quantum fourier transform $F$ and the $XOR_{ij}$ gate, which is $GXR_{ij}$ defined in [14]. In an arbitrary $d$-dimensional quantum system, these states are defined as

$$|\Psi_{\alpha,\beta}\rangle = GXR_{12}(F_1|\alpha\rangle_1|\beta\rangle_2) = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{\frac{2\pi i}{d} \alpha k} |k\rangle_1 |k \otimes \beta\rangle_2$$  \hspace{1cm} (4)

where $\alpha, \beta = 0, 1, \ldots, d$. Using this protocol, we can generate high-dimensional entangled states that will be studied along this work.

Let us consider entangled quantum states of two qutrits, with at most two excitations, in the presence of dissipation at zero temperature. Such situation can be conveniently described by the evolution equation:

$$\dot{\rho} = \sum_{1,2} \frac{\Gamma_i}{2} \left[ 2c_i^\dagger c_i^\dagger \rho - c_i^\dagger c_i^\dagger c_i c_i - \rho c_i c_i^\dagger \right],$$  \hspace{1cm} (5)
Figure 1. Evolution of $\Lambda(t)$ and linear entropy $s(t)$ for the nine maximally entangled states. Solid line correspond to the group (1) of maximally entangled states $\{|\Psi_{\alpha,0}\rangle\}$. Dashed line correspond to the group (2) of maximally entangled states $\{|\Psi_{\alpha,1}\rangle, |\Psi_{\alpha,2}\rangle\}$ with $\alpha = 0, 1, 2$.

where $c_i, c_i^\dagger$ describes annihilation and creation operators for bosonic modes and $\rho$ is a $3 \otimes 3$ density matrix in the computational basis $\{|0\rangle, |1\rangle, |2\rangle\} \otimes \{|0\rangle, |1\rangle, |2\rangle\}$ of both qutrits. As a starting point, we can study the dissipative effects on the entanglement evolution of all possible Bell states that can be defined for two-qutrits. In Fig. 1, the entanglement evolution given by $\Lambda(t)$ and the linear entropy $s(t) = 1 - \text{tr}\rho^2$ for the nine maximally entangled states of two qutrits are shown. From this numerical calculation we realize that dynamics evolution of entanglement is divided in two classes. The nine maximally entangled states can be divided into two groups, depending on the portion of the Hilbert space they initially occupy. The two groups of maximally entangled states appear according to the number of excitations they have initially: (a) the first group containing up to four excitations is composed by states $\{|\Psi_{\alpha,0}\rangle\}$, (b) the second group containing up to three excitations is composed by states $\{|\Psi_{\alpha,1}\rangle\}$, and composed by the states $\{|\Psi_{\alpha,2}\rangle\}$, with $\alpha = 0, 1, 2$. For both cases, we can observe from Fig. 1 that for initially maximally entangled states, entanglement decrease asymptotically.

Let us now study the entanglement evolution for initial mixed states. In order to do this we consider a particular generalization of the “X”states [13] for two qutrits system as follows

$$
\rho(0) = \frac{1}{3} \left( \begin{array}{cccccccc}
1 & 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array} \right),
$$

where $\lambda_i's$ are real parameters ranging from $\lambda_i = 0$ to $\lambda_i = 1$. In the extreme cases, for $\lambda_i's = 0$, we have
Figure 2. Evolution of $\Lambda(t)$ for the initial mixed state of Eq. (6) with $\lambda = 0.1$ as a function of the adimensional time $\Gamma t$.

a separable state whereas for $\lambda_i$’s =1 we have a maximally entangled state $|\Psi_{0,0}\rangle$.

Fig. 2 shows the evolution of $\Lambda(t)$ for $\lambda = 0.1$. As we observe, $\Lambda(t)$ undergoes sudden changes along its evolution exhibiting discontinuous derivatives, finally evolving to a situation where entanglement abruptly dies. As compared with the case of two qubits a reacher dynamical behavior of entanglement appears. From numerical calculations we realize that the abrupt changes of entanglement evolution are dominated by the behavior of a restricted number of eigenvalues of the matrix $\rho^{TA} \cdot (\rho^{TA})^\dagger$, which are given by:

$$
E_1(t) = (\rho_{12,12})^2 + (\rho_{11,22})^2 - 2\rho_{12,12}\rho_{11,22},
$$

$$
E_2(t) = (\rho_{00,11})^2 + (\rho_{01,01})^2 - 2\rho_{00,11}\rho_{01,01},
$$

$$
E_3(t) = (\rho_{00,22})^2 + (\rho_{02,02})^2 - 2\rho_{00,22}\rho_{02,02},
$$

where the density matrix elements $\rho_{ij,kl}$ can be calculated exactly. From Eqs. (7), the times where the abrupt changes occur, can be analytically calculated in terms of the parameter $\lambda$:

$$
t_1 = \ln(2/(2 - \lambda)), t_2 = \ln(1/(1 - \lambda)), t_3 = \ln(1/(1 - \sqrt{\lambda})).
$$

In particular, for the maximally entangled state with $\lambda = 1$, there is sudden change for $t_1 = \ln 2$, and the time of the second and third sudden change, which is the ESD, goes to infinite, showing that the entanglement decays asymptotically. However, we can also understand these abrupt changes in $\Lambda(t)$ by observing the behavior of the eigenvalues of the partial transpose matrix $\rho^{TA}$. In our case only three eigenvalues give us information about these sudden changes and are plotted in Fig. 3. We notice that these eigenvalues change from negative to positive values for specific times which are in agreement with the sudden changes in the entanglement evolution. In other words, the disentanglement rate changes whenever the rank of the partially transposed matrix changes abruptly. We can also associate to each eigenvalue of $\rho^{TA}$ a corresponding entanglement witness operator, such that $\alpha_i(t) = Tr (W_i\rho(t))$ with $i = 1, 2, 3$ and each $W_i$ is given by

$$
W_1 = \frac{1}{2} |21\rangle\langle 21| - |11\rangle\langle 11| - |22\rangle\langle 22| + |12\rangle\langle 12|,
$$

$$
W_2 = \frac{1}{2} |10\rangle\langle 10| - |00\rangle\langle 00| - |11\rangle\langle 11| + |01\rangle\langle 01|,
$$

$$
W_3 = \frac{1}{2} |02\rangle\langle 02| - |00\rangle\langle 00| - |22\rangle\langle 22| + |20\rangle\langle 20|.
$$
At $t = 0$, all three operators can be used to identify entanglement in $\rho$. As time goes by, they consecutively lose this capacity until there is no entanglement left.

4. Summary

In summary, we have studied the evolution of entanglement for high dimensional dissipative quantum systems. By evaluating the entanglement contained in the system using the Chen, Albeverio and Fei measure we have observed outstanding new effects. Quantum correlations undergo abrupt changes as precursors of ESD. The dynamical changes are related to sudden changes in the rank of the partial transpose matrix $\rho^{T_A}$ and the ESD is recovered as a particular case of these sudden dynamical changes.

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