The Reachability Problem for Petri Nets is Not Primitive Recursive

Jérôme Leroux
CNRS & University of Bordeaux
jerome.leroux@labri.fr

Abstract
We present a way to lift up the Tower complexity lower bound of the reachability problem for Petri nets to match the Ackermannian upper bound closing a long standing open problem. We also prove that the reachability problem in dimension 17 is not elementary.

1 Introduction
Vector addition systems with states [7], or equivalently vector addition systems [8], or Petri nets are one of the most popular formal methods for the representation and the analysis of parallel processes [5].

Those equivalent models are acting on counters ranging over the natural numbers. Formally, an implicit infinite countable set of elements called counters is given. A configuration is a function $\rho$ that maps every counter on a natural number in such a way that the set of counters $c$ such that $\rho(c) \neq 0$ is finite. We denote by 0 the configuration $\rho$ such that $\rho(c) = 0$ for every counter $c$. In this paper, we use standard notions from model theory by considering counters as variables, and configurations as valuations of the variables. It means that a configuration is used to replace in an expression $e$ over the counters, each occurrence of a counter $c$ by $\rho(c)$. We denote by $\rho(e)$ the expression we obtain this way. When $\phi$ is a constraint over the counters, we also denote by $\rho(\phi)$ the constraint we obtain by considering $\phi$ as an expression over the counters. We say that $\rho$ satisfies $\phi$ if the constraint $\rho(\phi)$ is true. For instance we say that $\rho$ satisfies $y = (1 + b)x$ if $\rho(y) = (1 + \rho(b))\rho(x)$.

The increment and decrement commands of a counter $c$ are respectively denoted as $\text{inc}(c)$ and $\text{dec}(c)$. We associate with such a command $\text{cmd}$, the binary relation $\xrightarrow{\text{cmd}}$ over the configurations defined by $\alpha \xrightarrow{\text{cmd}} \beta$ if $\alpha(x) = \beta(x)$ for every counter $x \neq c$, and $\beta(c) = \alpha(c) + 1$ if $\text{cmd}$ is an increment, and $\beta(c) = \alpha(c) - 1$ if $\text{cmd}$ is a decrement.

A vector addition system with states (VASS for short) is an automaton $V = (Q, T, q_{\text{in}}, q_{\text{out}})$ where $Q$ is a non empty finite set of elements called states, and $T$ is a finite set of triples $(p, \text{cmd}, q)$ called transitions with $p, q \in Q$ and $\text{cmd}$ is an increment or a decrement command, $q_{\text{in}}, q_{\text{out}} \in Q$ are respectively called the initial and final states. The size of a VASS is defined as $|Q| + |T|$, and its dimension is the cardinal of the set of counters used by $V$, i.e. counters that occurs in a command labeling a transition of $V$. A state-configuration is a pair $(q, \rho)$ denoted as $q(\rho)$ where $q$ is a state and $\rho$ is a configuration. The semantics is defined by introducing the binary relation $\Rightarrow$ over the state-configurations defined by $p(\alpha) \Rightarrow q(\beta)$ if there exists a transition $(p, \text{cmd}, q)$ in $T$ such that $\alpha \xrightarrow{\text{cmd}} \beta$. We denote by $\Rightarrow^*$ the reflexive and transitive closure of $\Rightarrow$, usually called the reachability relation.

*This research has been supported by ANR programme BraVAS (ANR-17-CE40-0028).
The central algorithmic problem for VASS is reachability: given a VASS \( \mathcal{V} \), decide whether \( q_{in}(0) \rightarrow^* q_{out}(0) \). Many important computational problems in logic and complexity reduce or are even equivalent to this problem [20, 6]. After an incomplete proof by Sacerdote and Tenney [19], decidability of the problem was established by Mayr [14, 15], whose proof was then simplified by Kosaraju [9]. Building on the further refinements made by Lambert in the 1990s [10], in 2015, a first complexity upper bound of the reachability problem was provided [12] more than thirty years after the presentation of the algorithm introduced by Mayr [9, 10]. The upper bound given in that paper is “cubic Ackermannian”, i.e. in \( \mathcal{F}_{\omega^3} \) (see [21]). This complexity bound is obtained by analyzing the Mayr algorithm. With a refined algorithm and a new ranking function for proving termination, an Ackermannian complexity upper bound was obtained [13] This means that the reachability problem can be solved in time bounded by \( A(p(n)) \) where \( p \) is a primitive recursive function and where \( A \) is the Ackermann function. This paper also showed that the reachability problem in fixed dimension \( d \) is primitive recursive by bounding the length of executions by \( O(F_{d+4}(n)) \) where \( F_{d+4} \) is a primitive recursive function of the Grzegorczyk hierarchy (see Section 2 for the definition of those functions).

Concerning the complexity lower bound, in 1976, the reachability problem was proved to be ExpSpace-hard [2]. This bound used to be the best one for more than forty years until 2019 when it was improved to a Tower complexity lower bound [3], i.e. a non elementary complexity.

**Contributions.** In this paper we provide an Ackermannian complexity lower bound for the reachability problem for VASS closing the gap with the Ackermannian complexity upper bound solving a 45 year old open problem. Moreover, in fixed dimension \( 4d + 5 \), we prove that the reachability problem is hard for the complexity class \( \mathcal{F}_d \) introduced in [22] associated with the function \( F_d \). As a direct corollary, we derive that the reachability problem is not elementary in dimension 17.

This paper provides the last piece leading to the exact complexity of the reachability problem for VASS. As previously mentioned, it follows a long serie of results. This piece of work is not the most difficult one but it is an important one since it closes a long standing open problem. Technically, the most difficult piece is the notion of \( K \)-amplifiers introduced in the Tower complexity lower bound paper [3] that provides a way to postpone at the end of an execution the tests commands of a Minsky machine (a machine like a VASS but with commands that can test to zero some counters) with counters bounded by \( K \).

In this paper, we provide several gadgets for implementing an Ackermannian amplifier of polynomial size. Recently and independently, other gadgets for implementing such an amplifier have been introduced by Wojciech Czerwiński, and Łukasz Orlikowski [4]. We think that several different solutions to such an important problem is useful not only for the confidence in the claimed result but also for future work. In fact, the reachability problem for VASS extensions is open for almost all natural extensions except for VASS with hierarchal zero tests [1, 18]. Moreover, the best known complexity lower bounds for those models only come from the reachability problem for classical VASS. Finding gadgets that take benefits from the extra power given by the considered extensions is an open problem.

Last but not least, the parameterized complexity of the reachability problem for VASS is still open. In fact, in fixed dimension \( d \), there is a complexity gap between the upper bound \( \mathcal{F}_{d+4} \) and the lower bound \( \mathcal{F}_{d-5} \) introduced in this paper. Moreover, this gap growth with the dimension \( d \). Proving a complexity lower bound of the form \( \mathcal{F}_{d+O(1)} \) based on amplifiers requires new gadgets for implementing an \( F_d(n) \)-amplifiers of elementary size, using only \( d + O(1) \) counters. Actually, the most efficient implementation [1] (with respect to the number of used counters) of such an amplifier is using \( 3d + 2 \) counters [11]. We think that several gadgets for implementing amplifiers should be useful for closing the complexity problem in fixed dimensions.
2 Big Picture

In this section, we recall central ideas used in the Tower complexity lower bound paper \cite{2} that are also used in this paper. We then explain the main idea to lift up the Tower complexity lower bound to an Ackermannian one.

We introduce the sequence \((p_d)_{d\in\mathbb{N}}\) of functions \(p_d : \mathbb{N} \to \mathbb{N}\) defined by \(p_0(n) = n + 1\), and defined by induction on \(d \geq 1\) by \(p_d(n) = p_{d-1}^{p_{d-1}}(n)\). Rather than take the original definition of Ackermann function, we let \(A(n) = F_w(n)\) defined by \(F_w(n) = F_{n+1}(n)\), which behaves like the classical function for our complexity-theoretical purpose.

**Remark 1.** We have \(F_1(n) = 2n + 1\) and \(F_2(n) = 2^{n+1}(n + 1) - 1\) for every \(n \in \mathbb{N}\). The function \(F_3\) behaves like a tower of \(n\) exponential and it is not elementary. Each function \(F_d\) is primitive recursive and \(F_w\) is not primitive recursive.

2.1 Programs

Programs \cite{3} provide a uniform model to express reachability problems for VASS and complexity classes beyond Elementary \cite{21}.

Let us first associate with a counter \(c\), the test command \(c := 0\) and the binary relation \(c \xRightarrow{0}\) over the configurations defined by \(\alpha \xRightarrow{0} \beta\) if \(\alpha(c) = 0\) and \(\beta = \alpha\).

A (general) program \(M\) is defined as an increment/decrement/test command, or inductively as a loop program \texttt{loop} \(M_0\), a serie composition \(M_1; M_2\), or a non-deterministic choice \(M_1\) or \(M_2\) where \(M_0, M_1, M_2\) are programs. The size of a program \(M\) is the number size\((M)\) defined inductively as 1 if \(M\) is a command, \(1 + \text{size}(M_0)\) if \(M = \texttt{loop} \ M_0\), and \(1 + \text{size}(M_1) + \text{size}(M_2)\) if \(M = M_1; M_2\) or \(M = M_1\) or \(M_2\). The dimension of \(M\) is the cardinal of the set of counters used by \(M\). We associate with every program \(M\) the binary relation \(\rightarrow\) over the configurations defined as follows:

\[
\begin{align*}
M \rightarrow &= \left\{ \begin{array}{ll}
\xrightarrow{\text{cmd}} & \text{if } M = \text{cmd} \text{ is a command} \\
\xrightarrow{(M_0)^*} & \text{if } M = \texttt{loop} \ M_0 \\
\xrightarrow{M_1; M_2} & \text{if } M = M_1; M_2 \\
\xrightarrow{M_1 \cup M_2} & \text{if } M = M_1 \text{ or } M_2
\end{array} \right.
\]

Where \((\rightarrow_0)^*\) is the reflexive and transitive closure of \(\rightarrow_0\), and \(\rightarrow_1; \rightarrow_2\) is defined by \(\alpha \rightarrow_1 \rho \rightarrow_2 \beta\) if there exists a configuration \(\rho\) such that \(\alpha \rightarrow_1 \rho\) and \(\rho \rightarrow_2 \beta\). Serie compositions and non-deterministic choices are clearly associative with respect to the relation \(\rightarrow\). In particular given a sequence \(M_1, \ldots, M_k\) of Minsky programs, the relations \(\xrightarrow{M_1; \ldots; M_k}\) and \(\xrightarrow{M_1 \text{ or } \ldots \text{ or } M_k}\) are well defined. We denote by \(M^{(k)}\) the serie composition of \(M\) by itself \(k\) times. We also denote by \(\text{inc}(c_1, \ldots, c_k)\) the program \(\text{inc}(c_1)\ldots; \text{inc}(c_k)\), by \(\text{dec}(c_1, \ldots, c_k)\) the program \(\text{dec}(c_1)\ldots; \text{dec}(c_k)\), and similarly \(c_1, \ldots, c_k \xRightarrow{0} 0\) the program \(c_1 \xRightarrow{0} 0; \ldots; c_k \xRightarrow{0} 0\).

We say that a program is test-free if it does not use any test command. A checking program is a program of the form \(M; c_1, \ldots, c_k \xRightarrow{0} 0\) where \(M\) is a test-free program and \(c_1, \ldots, c_k\) are some counters. The reachability problem for programs asks, given a program \(M\), whether there exists a configuration \(\beta\) such that \(0 \xrightarrow{M} \beta\). Notice that the reachability problem for VASS and for checking programs are equivalent (i.e. inter-reducible).

**Remark 2.** The reachability problem for test-free programs and the state reachability problem for VASS are equivalent (i.e. inter-reducible). This last problem asks, given a VASS \(V\), whether \(q_{\text{in}}(0) \Rightarrow^* q_{\text{out}}(\beta)\) for some configuration \(\beta\). This problem, also called the coverability problem, is strictly simpler than the reachability problem for VASS since it is known to be ExpSpace-complete \cite{2, 17}.
The reachability problem for general programs is undecidable [10] even in dimension 2. However, that problem can be used to define complexity classes beyond Elementary by bounding the range of the counters. More formally, we say that a configuration $\rho$ is $K$-bounded for some $K \in \mathbb{N}$ if $\rho(c) \leq K$ for every counter $c$. Denoting by $\text{Conf}_K$ the set of $K$-bounded configurations, we define the binary relation $\frac{M}{K} \to \leq K$ over $\text{Conf}_K$ where $M$ is a program inductively as follows:

$$
\frac{M}{K} \to \leq K = \begin{cases}
(\text{cmd}) \cap (\text{Conf}_K \times \text{Conf}_K) & \text{if } M = \text{cmd} \text{ is a command} \\
(\frac{M_1}{K} \to \leq K) \cap (\text{Conf}_K \times \text{Conf}_K) & \text{if } M = \text{loop } M_0 \\
\frac{M_1}{K} \to \leq K; \frac{M_2}{K} \to \leq K & \text{if } M = M_1; M_2 \\
\frac{M_1}{K} \cup \frac{M_2}{K} \to \leq K & \text{if } M = M_1 \text{ or } M_2
\end{cases}
$$

Intuitively $\alpha \frac{M}{K} \to \leq K \beta$ for two configurations $\alpha, \beta$ if, and only if, there exists an execution of $M$ such that every visited configuration including $\alpha$ and $\beta$, during the computation is $K$-bounded. The relation $\frac{M}{K} \to \leq K$ is called the $K$-bounded semantics of a program $M$. It provides natural complete problems for several complexity classes beyond Elementary. In fact, following [21], given a function $f : \mathbb{N} \to \mathbb{N}$, the problem that asks, given a 2-dimensional program $M$ of size $n$, whether there exists $\beta$ such that $0 \frac{M}{K} \to \leq K \beta$ with $K = f(n)$ is $F$-complete where:

$$
F = \begin{cases}
\mathbb{F}_3 & \text{if } f(n) = 3(1)^n \\
\mathbb{F}_d & \text{if } f(n) = 1 + F_d(n) \text{ and } d \geq 3 \\
\mathbb{F}_{\omega} & \text{if } f(n) = 1 + F_{\omega}(n)
\end{cases}
$$

Remark 3. The complexity classes $\mathbb{F}_3$ and $\mathbb{F}_{\omega}$ are respectively called the Tower, and the Ackermannian complexity classes. Other complexity classes beyond Elementary can be defined similarly by considering various functions $f$ (see [21] for more details). For instance, the “cubic Ackermannian” complexity class $\mathbb{F}_{\omega,3}$, previously mentioned in the introduction is defined this way.

### 2.2 Amplifiers

The $K$-bounded semantics of a program $M$ can be simulated by a checking program thanks to the so-called $K$-amplifiers, with $K \in \mathbb{N}$. A $K$-amplifier [3] for a triple of counters $(x, y, b)$ is a checking program $A$ such that for any configuration $\beta$, we have $0 \frac{A}{b} \to \beta$ if, and only if, there exists $\ell > 0$ such that $\beta(x, y, b) = (\ell, K\ell, K)$ and $\beta(c) = 0$ for any counter $c \notin \{x, y, b\}$.

Following [3], given a $K$-amplifier $A$ and a general program $M$, we can compute in linear time a checking program $A \triangleright M$ such that for any configuration $\beta$, we have $0 \frac{A \triangleright M}{b} \to \beta$, if, and only if, $0 \frac{A \triangleright M}{b} \to \beta$. It follows that $K$-amplifiers provide a way to postpone at the end of an execution test commands of $K$-bounded general programs. The size of $A \triangleright M$ is $\text{size}(A) + O(\text{size}(M))$. Concerning the dimension of $A \triangleright M$, let us first classify the counters used by $A$. We say that a counter $c$ used by the amplifier $A$ is unsafe (for the simulation), if it belongs to $\{x, y, b\}$ or if it occurs in a test command at the end of $A$, and let us say it is safe if it is not unsafe. Then denoting by $u$ and $s$ respectively the number of unsafe and safe counters of $A$, and by $n$ the dimension of $M$, the dimension of $A \triangleright M$ is equal to:

$$u + \max(s, 2n)$$

The checking program $A \triangleright M$ is obtained as follows. Let us denote by $c_1, \ldots, c_n$ the counters used by $M$. We associate to each counter $c_i$ an auxiliary one $c'_i$ denoted with a prime. By renaming the counters of $A$, we can assume without loss of generality that the unsafe counters
of $A$ are disjoint from \{c_1, c_1', \ldots, c_n, c_n'\}. Moreover, with such a renaming we can additionally assume that the cardinal of the safe counters of $A$ union \{c_1, c_1', \ldots, c_n, c_n'\} is max(s, 2n). We assume that $A$ is the checking program $A': d_1, \ldots, d_m \Leftarrow 0$ where $A'$ is a test-free program. We introduce the test-free program $M'$ obtained from $M$ by replacing each increment command of $c_i$ by $\text{inc}(c_i')$; $\text{dec}(c_i)$, each decrement command of $c_i$ by $\text{dec}(c_i')$; $\text{inc}(c_i)$, and each test command of $c_i$ by the following test-free program:

$$
\text{test}_i = \begin{cases}
\text{loop} \\
\quad \text{dec}(c_i); \text{inc}(c_i'); \text{dec}(y) \\
\quad \text{dec}(c_i'); \text{inc}(c_i); \text{dec}(y) \\
\quad \text{dec}(x) \ (2)
\end{cases}
$$

We also introduce the following test-free program used at the end of $A \triangleright M$ to transfer the value of $c_i'$ in $c_i$ and reset $c_i'$ to zero:

$$
\text{transfer}_i = \begin{cases}
\text{loop} \\
\quad \text{dec}(c_i); \text{dec}(y) \\
\quad \text{dec}(c_i'); \text{inc}(c_i); \text{dec}(y) \\
\quad \text{dec}(x)
\end{cases}
$$

The checking program $A \triangleright M$ is then defined as follows:

$$
A \triangleright M = \begin{align*}
1: & \ A' \\
2: & \ \text{loop} \\
3: & \ \text{dec}(b); \text{inc}(c_1, \ldots, c_n) \\
4: & \ M' \\
5: & \ \text{transfer}_1; \ldots; \text{transfer}_n \\
6: & \ y, d_1, \ldots, d_m \Leftarrow 0
\end{align*}
$$

For every configuration $\beta$, we have $0 \xrightarrow{M} \beta \leq K$ if, and only if, $0 \xrightarrow{A \triangleright M} \beta$. A formal proof for a variant construction of $A \triangleright M$ (that cannot reuse safe counters of $A$ and that introduces several additional test commands) is given in \cite{3}. In the next two paragraphs, we just recall briefly the key ingredients used in the two constructions.

For one direction, assume that $0 \xrightarrow{M} \beta \leq K$ for some configuration $\beta$ and let us consider an execution of $M$ from 0 to $\beta$ such that every visited configuration including $\beta$ is $K$-bounded. Denoting by $m$ the number of times this execution is using a test command, we introduce $\ell = 2m + n$. We consider an execution of $A'$ that leads to a configuration $\rho$ such that $\rho(x, y, b) = (\ell, K\ell, K)$ and $\rho(c) = 0$ for every counter $c \notin \{x, y, b\}$. From $\rho$, we execute line 3 exactly $K$ times in such a way we get a configuration $\alpha$ satisfying $c_i = K$ for every $1 \leq i \leq n$, $b = 0$, $x = \ell$, and $y = K\ell$. From the execution of $M$, we derive an execution of $M'$ from $\alpha$ to a configuration $\delta$ satisfying $\delta(y) = Kn$, $\delta(c') = \beta(c)$, $\delta(c) = K - \beta(c)$. Such an execution is obtained by observing that if the two loops of test$_i$ are executed exactly $K$ times then $x$ is decremented by 2 and $y$ by $2K$. Such an execution is possible since the configurations we obtain when executing test$_i$, satisfy $c_i' = 0$ and $c_i = K$. Finally, we get an execution from $\delta$ to $\beta$ by executing from $\delta$, for each $i \in \{1, \ldots, n\}$, the first and the second loop of transfer$_i$, respectively $K - \beta(c_i)$ and $\beta(c_i)$ times. We derive $0 \xrightarrow{A \triangleright M} \beta$ from such an execution.

For the other direction, assume that $0 \xrightarrow{A \triangleright M} \beta$ for some configuration $\beta$ and let us consider an execution of $A \triangleright M$ witnessing that property. In $A \triangleright M$ the test-free program $A'$ is executed first and the test commands $d_1, \ldots, d_m \Leftarrow 0$ of $A$ are postponed at the end of $A \triangleright M$. Since those tested counters are no longer used in between, the execution of $A'$ can only produce triples
of the form \((\ell, K\ell, K)\) for the valuation of the counters \((x, y, b)\). In fact, any other computation will be blocked at the end of \(A \triangleright M\) when the program will try to execute the postponed test commands \(d_1, \ldots, d_m\) ?= 0. Then notice that line 3 is executed as most \(K\) times since \(b\) is decremented each time the line is executed. It follows that \(\bigwedge_{i=1}^n c_i + c'_i \leq K\) is an invariant of the program since the counter expression \(c_i + c'_i\) is invariant during the execution of \(M'\), and it can only be decreased by the program \textbf{transfer}\(_i\). Notice that \(y \geq Kx\) is also a forward invariant after the execution of \(A'\). It follows that if \(0 \xrightarrow{A \triangleright M} \beta\) then \(\beta\) satisfies \(y \geq Kx\). Since additionally \(\beta(y) = 0\) due to the test command \(y\) ?= 0 at the end of \(A \triangleright M\) we deduce that \(\beta\) satisfies \(y = Kx\). As this equality is backward invariant, we deduce that for every \(i\), the two loops of \textbf{transfer}\(_i\) are executed in total exactly \(K\) times. It follows that line 3 is executed exactly \(K\) times as well. We also deduce that the configurations before and after the execution of \textbf{test}\(_i\) in \(M'\) satisfy \(c_i = K\) and \(c'_i = 0\), and the two loops of that program are also executed exactly \(K\) times. We derive that \(\beta(c) = 0\) for every counter \(c \not\in \{c_1, \ldots, c_n\}\). Moreover, from the execution of \(M'\) we deduce that \(0 \xrightarrow{M' \leq K} \beta\).

It follows that the \(K\)-bounded semantics of programs can be simulated by checking programs of small size as soon as there exists a small \(K\)-amplifier. One can easily define a 3-dimensional \(K\)-amplifier \(A_K\) just by considering the following test-free program:

\[
A_K = \begin{array}{c}
\text{inc}(b)^{(K)} \\
\text{inc}(x); \text{inc}(y)^{(K)} \\
\text{loop} \\
\text{inc}(x); \text{inc}(y)^{(K)}
\end{array}
\]

However, the size of such a program is linear in \(K\). More generally, it can be observed that a test-free \(K\)-amplifier of size \(n\) satisfies \(K \leq O(2^n)\) using a Rackoff argument [17]. It means that small size \(K\)-amplifiers for large \(K\) must use the full power of checking programs, i.e. the test commands at the end.

In \(\mathbb{R}\), a general program \textbf{factorial} such that for any \(K\)-amplifier \(A\), the checking program \(A \triangleright \textbf{factorial}\) is a \(K!\)-amplifier is introduced. Intuitively, \textbf{factorial} is a general program using counters \(x, y, b\) and some additional ones in such a way that for any \(K \in \mathbb{N}\) and any configuration \(\beta\), it satisfies \(0 \xrightarrow{\text{factorial} \leq K} \beta\) if, and only if, there exists \(\ell \in \mathbb{N}\) such that \(\beta(x, y, b) = (\ell, K!\ell, K!\ell, K!)\) and such that \(\beta(c) = 0\) for any counter \(c \not\in \{x, y, b\}\). Based on this observation, in \(\mathbb{R}\) it is shown that the following checking program is a \(K\)-amplifier of size \(O(d)\) computable in time \(O(d)\) with \(K = 3(!)^d\) (where \textbf{factorial} occurs \(d\) times).

\[
(\ldots ((A_3 \triangleright \textbf{factorial}) \triangleright \textbf{factorial}) \ldots) \triangleright \textbf{factorial}
\]

Intuitively, this amplifier is using triples of counters \((x_i, y_i, b_i)\) for \(1 \leq i \leq d\) in a very particular order. In fact, the triple of counters of index \(i + 1\) interact only with the triple of counters of index \(i\).

### 2.3 Ackermannian Amplifiers

By relaxing the way triples are used, we present in Section 6 a family of \((1 + F_d(n))\)-amplifiers \textbf{Ack}_{d,n} of size \(O(n^2d)\) using \(4d + 5\) counters and such that \(2d - 2\) of them are safe. From these amplifiers, called \textbf{Ackermannian amplifiers}, we deduce as a direct corollary the following theorem.

**Theorem 4.** The reachability problem for VASS is \(\mathbb{F}_\omega\)-hard. Moreover, the reachability problem for \(4d + 5\)-dimensional VASS is \(\mathbb{F}_d\)-hard when \(d \geq 3\). In particular the reachability problem for 17-dimensional VASS is not Elementary.
Proof. For the first result, given a 2-dimensional program $M$, let $n = \text{size}(M)$ and just notice that the checking program $\text{Ack}_{k,n+1} \triangleright M$ simulates the $K$-bounded semantics of $M$ with $K = 1 + F_d(n)$. For the second result, given $d \geq 3$ and a 2-dimensional program $M$, let $n = \text{size}(M)$ and just notice that the checking program $\text{Ack}_{d,n} \triangleright M$ simulates the $K$-bounded semantics of $M$ with $K = 1 + F_d(n)$.

Theorem \[4\] is nearly optimal since in \[13\] it is proved that the reachability problem for VASS is in $F_w$, and it is proved that the reachability problem for $d$-dimensional VASS is in $F_{d+4}$.

2.4 Recursion

Functions $F_d$ can be computed with a recursive algorithm using a stack of height $d$, i.e. the stack contains $d$ natural numbers. By encoding such a stack as a vector in $\mathbb{N}^d$, we deduce a way for computing $F_d$ with vectors in $\mathbb{N}^d$. We present such an algorithm in this section.

The correctness of this algorithm will be obtained by observing that the following function $\text{val} : \mathbb{N}^d \to \mathbb{N}$ defined for every vector $v \in \mathbb{N}^d$ as follows is an invariant:

$$\text{val}(v) = F_{d-1}^v \circ \cdots \circ F_0^v[1](0)$$

Let us denote by $\mathbb{1}_{d,i}$ the unit vector of $\mathbb{N}^d$ defined for every $1 \leq j \leq d$ by:

$$\mathbb{1}_{d,i}[j] = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

We introduce the function $\text{evalF}_d : \mathbb{N}^d \to \mathbb{N}^d$ partially defined over the vectors $v \in \mathbb{N}^d$ such that $v[i] > 0$ for some $i \in \{2, \ldots, d\}$ by $\text{evalF}_d(v) = v - \mathbb{1}_{d,i} + (1 + v[1]) \mathbb{1}_{d,i-1}$ where $i$ is the minimal index in $\{2, \ldots, d\}$ such that $v[i] > 0$. One can easily prove \[22\] that if $v$ is in the definition domain of $\text{evalF}_d$, then $\text{val}(v) = \text{val}(\text{evalF}_d(v))$. It follows that $\text{val}$ is an invariant of the function $\text{evalF}_d$.

Since $\text{evalF}_d(v)$ is strictly smaller than $v$ for the lexicographic order, it follows that we can iterate the function $\text{evalF}_d$ on a vector $v$ only a finite number of times (we use the well-foundedness of the lexicographic order). Let us introduce the function $\text{evalF}_d^\text{max} : \mathbb{N}^d \to \mathbb{N}^d$ defined by $\text{evalF}_d^\text{max}(v) = \text{evalF}_d^k(v)$ where $k$ is the maximal number of times (it can be zero) we can apply the function $\text{evalF}_d$ on a vector $v \in \mathbb{N}^d$. Since val is an invariant, we have $\text{val}(v) = \text{val}(\text{evalF}_d^\text{max}(v))$. Additionally, since $\text{evalF}_d^\text{max}(v)$ is a vector of the form $(m, 0, \ldots, 0)$ for some $m \in \mathbb{N}$, we deduce that $\text{val}(\text{evalF}_d^\text{max}(v)) = m$. It follows that the following equality holds (see \[22\] more details):

$$\text{evalF}_d^\text{max}(v) = (F_{d-1}^v \circ \cdots \circ F_0^v[1](0), 0, \ldots, 0)$$

As a direct corollary, we deduce the following lemma.

Lemma 5. We have $\text{evalF}_d^\text{max}(n, 0, \ldots, 0, n + 1) = (F_d(n), 0, \ldots, 0)$ for every $n \geq 0$ and $d \geq 2$.

2.5 Outline of the paper

We introduce in Section \[3\] some additional constructions based on the use of auxiliary counters that will be helpful in the sequel for iterating programs a number of times limited by some counter expressions. In Section \[4\] we introduce the notion of good and bad configurations and show that they can be distinguished thanks to a checking program $\text{proper}_d$. Those configurations are used in Section \[5\] to prove the correctness of a program $\text{evalF}_d$ that implements the function $\text{evalF}_d$. Intuitively, from a bad configuration the program can only produce bad
configurations, and from a good configuration that encodes a vector \( v \in \mathbb{N}^d \) in the valuation of \((B_1, \ldots, B_d)\) for some counter expressions \( B_1, \ldots, B_d \), the execution of \( \text{evalF}_d \) will produce either a bad configuration or a good one such that the valuation of \((B_1, \ldots, B_d)\) is \( \text{evalF}_d(v) \). Finally, in Section 6 we show that by first initializing \((B_1, \ldots, B_d)\) to \((n,0,\ldots,0,1+n)\) and by iterating with a loop the program \( \text{evalF}_d \), and then by executing a variant of the program \( \text{proper}_d \), one can define the \((1+F_d(n))\)-amplifier \( \text{Ack}_{d,n} \).

3 Auxiliary Counters

We assume that every counter \( c \) is implicitly associated with an auxiliary one denoted with a prime as \( c' \), and we denote by \( C \) the counter expression \( c+c' \). Intuitively, auxiliary counters are used to transfer back and forth values between \( c \) and \( c' \) without changing the counter expression \( C \). It will be useful for repeating a program at most \( C \) times as described in the sequel. We can non deterministically change the way \( C \) is split between \( c \) and \( c' \) by introducing the following test-free program:

\[
\text{split } C = \left\{ \begin{array}{ll}
\text{loop } \ (\text{dec}(c); \text{inc}(c')) & \text{or } (\text{dec}(c'); \text{inc}(c)) \\
\text{M} & \\
\text{split } C
\end{array} \right.
\]

Observe that \( \alpha \xrightarrow{\text{split } C} \beta \) if, and only if, \( \alpha(C) = \beta(C) \) and \( \alpha, \beta \) coincide on every counter except on \( c \) and \( c' \). We denote by \( \text{split } C_1, \ldots, C_n \) the program \( \text{split } C_1; \ldots; \text{split } C_n \).

In order to increment, decrement or test the counter expression \( C \), we introduce the programs \( \text{inc}(C) \), \( \text{dec}(C) \), and \( C \?\neq\ 0 \) defined respectively as \( \text{inc}(c) \) or \( \text{inc}(c') \), \( \text{dec}(c) \) or \( \text{dec}(c') \), and \( c \?\neq\ 0; c' \?\neq\ 0 \). We also denote by \( C_1, \ldots, C_k \?\neq\ 0 \) the program \( C_1 ?\neq\ 0; \ldots; C_k ?\neq\ 0 \).

Auxiliary counters provide a way for implementing the following test-free program where \( M \) is any test-free program:

\[
\text{loop at most } C \text{ times } M = \left\{ \begin{array}{ll}
\text{split } C & \\
\text{loop } \ (\text{dec}(c); \text{inc}(c')) & \\
\text{M} & \\
\text{split } C
\end{array} \right.
\]

The last \( \text{split } C \) of the program is used to simplify the statement of the following lemma that proves the correctness of the program \( \text{loop at most } C \text{ times } M \).

**Lemma 6.** Let \( \alpha \) and \( \beta \) be two configurations, and let \( M \) be a test-free program that does not use \( c \) but may use \( c' \) only with increment commands. We have:

\[
\alpha \xrightarrow{\text{loop at most } C \text{ times } M} \beta
\]

if, and only if, there exists \( k \in \{0, \ldots, \alpha(C)\} \) such that:

\[
\alpha \xrightarrow{M^{(k)}; \text{split } C} \beta
\]

**Proof.** Just observe that \( \xrightarrow{\text{split } C; M; \text{split } C} \) is equal to \( \xrightarrow{M; \text{split } C} \), and \( \xrightarrow{\text{inc}(c')} \) commutes with \( \xrightarrow{\text{dec}(c)} \), and \( \xrightarrow{\text{inc}(c')} \). Moreover, every time the loop is executed, the counter \( c \) is decremented since \( M \) does not use \( c \). \( \square \)

In the sequel, we use the following test-free program:

\[
\text{loop at most } 1+C \text{ times } M = \left\{ \begin{array}{ll}
\text{inc}(C) & \\
\text{loop at most } C \text{ times } M & \\
\text{dec}(C)
\end{array} \right.
\]
4 The Good, the Bad, and the Proper

Let us fix a natural number \( d \geq 1 \). We are going to manipulate configurations provided by the counters \( x_0, b, b_1, x_1, \ldots, b_d, x_d \) and their auxiliary counters denoted with a prime. As mentioned in Section 3 we denote by \( X_i, B_i \), and \( B \) the counter expressions \( x_i + x_i', b_i + b_i', \) and \( b + b' \).

We say that a configuration is \((d+1)\)-good if \( X_0 \geq 1 \) and \( B_1 = B \), and inductively for \( j \in \{1, \ldots, d\} \) we say that a configuration is \( j \)-good if it is \((j+1)\)-good and it satisfies \( X_j = (1 + B_j)X_{j-1} \). A configuration is said to be \textit{good} if it is \( 1 \)-good. A configuration is said to be \textit{j-bad} for some \( j \in \{1, \ldots, d\} \) if it is \((j+1)\)-good and it satisfies \( X_j > (1 + B_j)X_{j-1} \). A configuration is said to be \textit{bad} if it is \( j \)-bad from some \( j \). A configuration is said to be \textit{proper} if it is good or bad. In the sequel our programs are only using proper configurations.

**Example 1.** With \( d = 2 \). A configuration satisfying \( X_0 \geq 1, B_1 = B, X_1 = (1 + B_1)X_2, \) and \( X_2 = (1 + B_2)X_3 \) is good. A configuration satisfying \( X_0 \geq 1, B_1 = B, X_2 > (1 + B_2)X_3 \) is \( 2 \)-bad. Notice that such a configuration can satisfy \( X_1 < (1 + B_1)X_2 \) or \( X_1 \geq (1 + B_1)X_2 \). A proper configuration satisfying \( X_0 \geq 1, B_1 = B, X_2 = (1 + B_2)X_3 \) must satisfy \( X_1 = (1 + B_1)X_0 \) (it is \( 1 \)-good) or \( X_1 > (1 + B_1)X_0 \) (it is \( 1 \)-bad).

We can check if a proper configuration is good thanks to a checking program that decreases \( d \) with \( 1 \) each decrement of \( X \) proper configuration satisfying \( X \). We are now ready to define and prove the correctness of the following checking program:

\[
\text{proper}_d = \quad \text{loop} \quad \text{rdec}_{d,0} \\
X_d \neq 0
\]

**Lemma 7.** Let \( \alpha \) and \( \beta \) be two configurations, \( p \in \{0, \ldots, d\} \), and \( k \in \mathbb{N} \). We have:

\[
\alpha \xrightarrow{(\text{rdec}_{d,p})^{(k)}} \beta
\]

if, and only if, there exists a sequence \( \ell_p, \ldots, \ell_d \) such that \( \ell_p = k, \, \ell_i \leq (1 + \alpha(B_i))\ell_{i-1} \) for every \( p < i \leq d \), and satisfying:

\[
\alpha \xrightarrow{\text{dec}(X_p)^{(\ell_p)}; \ldots; \text{dec}(X_d)^{(\ell_d)}; \text{split} \, B_{p+1}; \ldots; B_d} \beta
\]

**Proof.** Simple induction on \( p \) based on Lemma 6 \( \blacksquare \)

We are now ready to define and prove the correctness of the following checking program:

\[
\alpha \xrightarrow{(\text{rdec}_{d,0})^{(k)}} \beta
\]

**Lemma 8.** Given any proper configuration \( \alpha \) and arbitrary \( \beta \), there exists an execution of \( \text{proper}_d \) from \( \alpha \) to \( \beta \) if, and only if, \( \alpha \) is good. Moreover in that case \( \beta(X_i) = 0 \) for every \( 1 \leq i \leq d \).

**Proof.** Observe that if \( \alpha \) is good, we can apply Lemma 7 with \( k = \alpha(X_0) \), and \( \ell_0, \ldots, \ell_d \) defined by \( \ell_i = \alpha(X_i) \) and get an execution of \( \text{proper}_d \). Conversely, assume that there exists an execution of \( \text{proper}_d \) from a proper configuration \( \alpha \) to a configuration \( \beta \). Denoting by \( k \) the
number of times the main loop is executed, we deduce that \( \alpha \frac{\text{rdec}}{\text{dec}} \beta \). From Lemma 7, there exists a sequence \( \ell_0, \ldots, \ell_d \) such that \( \ell_p = k \), \( \ell_i \leq (1 + \alpha(B_i))\ell_{i-1} \) for every \( 1 \leq i \leq d \), and satisfying:

\[
\alpha \frac{\text{dec}(X_0)^{(\ell_0)}; \cdots ; \text{dec}(X_d)^{(\ell_d)}; \text{split} \ B_1, \ldots, B_d} \beta
\]

In particular \( \alpha(X_i) \geq \ell_i \) for every \( 0 \leq i \leq d \). Since the last line of the program was successfully executed, we derive \( \beta(X_d) = 0 \). Let \( i \in \{1, \ldots, d\} \). From \( \beta(X_i) = \alpha(X_i) - \ell_i \) and \( \beta(X_{i-1}) = \alpha(X_{i-1}) - \ell_{i-1} \) we derive the following inequality (just replace \( \beta(X_i) \) and \( \beta(X_{i-1}) \) by \( \alpha(X_i) - \ell_i \) and \( \alpha(X_{i-1}) - \ell_{i-1} \), simplify, and use the fact that \( \ell_i \leq (1 + \alpha(B_i))\ell_{i-1} \)):

\[
\beta(X_i) \geq \alpha(X_i) - (1 + B_i)X_{i-1} + (1 + \alpha(B_i))\beta(X_{i-1})
\]

(1)

In particular if \( \alpha \) is \( i \)-bad for some \( i \in \{1, \ldots, d\} \) then \( \beta(X_i) > 0 \). Assume by contradiction that there exists \( j \) such that \( \beta(X_j) > 0 \) and let \( j \) be the maximal one. We have \( j < d \) since \( \beta(X_d) = 0 \). Configuration \( \alpha \) cannot be \( i \)-bad for some \( i \in \{j+1, \ldots, d\} \) by maximality of \( j \). It follows that \( \alpha \) is \((j+1)\)-good. By replacing \( i \) by \( j+1 \) in equation (1) we get (notice that the first term of the right hand side of (1) is zero since \( \alpha \) is \((j+1)\)-good):

\[
\beta(X_{j+1}) \geq (1 + \alpha(B_{j+1}))\beta(X_j)
\]

In particular since \( \beta(X_j) > 0 \) we get \( \beta(X_{j+1}) > 0 \) and we get a contradiction on the maximality of \( j \). We deduce that \( \beta(X_i) = 0 \) for every \( i \). As previously observed, it implies that \( \alpha \) cannot be bad. Since \( \alpha \) is proper, it is good. \( \square \)

**Remark 9.** The proof of the previous lemma can be simplified by replacing the last line \( X_d \Leftarrow 0 \) in the program \( \text{proper}_d \) by \( X_0, \ldots, X_d \Leftarrow 0 \). We do not consider this simplification in order to be able to safely reuse the counters \( x_i \) and \( x'_i \) for \( 0 \leq i \leq d-1 \) in the construction \( \triangleright \).

### 5 EvalF Program

In this section, we introduce a test-free program \( \text{evalF}_d \) that implements the function evalF\(_d\) introduced in Section 2.4. This program is defined as \( \text{evalF}_d \) or \( \cdots \) or \( \text{evalF}_{d,p} \) where \( \text{evalF}_{d,p} \) with \( p \in \{2, \ldots, d\} \) implements the restriction of evalF\(_d\) on vectors \( v \in \mathbb{N}^d \) such that \( v[p] > 0 \) and such that \( v[i] = 0 \) for every \( 2 \leq i < p \).

Intuitively, let us consider a vector \( v \in \mathbb{N}^d \) such that \( v[p] > 0 \) and such that \( v[i] = 0 \) for every \( 2 \leq i < p \). Assume that \( v \) is encoded in a good configuration \( \alpha \) thanks to the counter expressions \( B_1, \ldots, B_d \) in such a way that \( \alpha(B_1, \ldots, B_d) = v \). By executing the program \( \text{evalF}_{d,p} \) starting from \( \alpha \), we are going to show that under some conditions, we can enforce the program to produce a good configuration \( \beta \) such that \( \beta(B_1, \ldots, B_d) = \text{evalF}_d(v) \) since any other behaviour produces a bad configuration. Moreover, from any bad configuration the program can produce only bad configurations. In order to avoid producing bad configurations the program should ensure that each constraint \( X_i = (1 + B_i)X_{i-1} \) is maintained. In particular, since \( \text{evalF}_{d,p} \) must decrement \( B_p \), it means that \( X_{p-1} \) must be subtracted to \( X_p \). We are going to use that property to enforce the program to perform the maximal number of times the loop at most blocks.

The program \( \text{evalF}_{d,p} \) is described just below. The last line is a macro for \( \text{inc}(b_1, b') \) if \( p = 2 \) and \( \text{inc}(b_{p-1}) \) if \( p \geq 3 \). In fact, since a proper configuration must satisfy \( B = B_1 \), it follows that every increment of \( B_{p-1} \) when \( p = 2 \) must be associated with an increment of \( B \). Notice that this line is using \( b' \) inside a loop at most \( (1 + B) \) times. Even in that case, Lemma 6 can be applied since \( b' \) is only used inside an increment command.
Lemma 10. Let $\alpha$ be a good configuration satisfying $X_0$ is divisible by $2 + B_1$, $B_p > 0$ and $B_{p-1}, \ldots, B_2 = 0$. There exists a good configuration $\beta$ such that $\alpha \xrightarrow{\text{evalF}_{d,p}} \beta$ and such that $\beta(X_0) = \alpha(X_0 \frac{1}{2 + B_1})$ and $\beta(B_1, \ldots, B_d) = \text{evalF}_d(\alpha(B_1, \ldots, B_d))$.

\begin{proof}
We introduce the program $M = \text{dec}(x_p)^{\ell_p}; \ldots; \text{dec}(x_d)^{\ell_d}; \text{split} \ B_{p+1}, \ldots, B_d$ where $\ell_p = 1$ and $\ell_i = (1 + \alpha(B_i))\ell_{i-1}$ for every $p < i \leq d$. Lemma 7 shows that we can assume for this proof that $\text{rdec}_{d,p}$ is replaced by $M$ in the program $\text{evalF}_{d,p}$.

Let us introduce the numbers $m, n, r, s$ defined as follows:

\begin{align*}
m &= \alpha(\frac{X_0}{2 + B_1}) \\
n &= \alpha(1 + B_1)m \\
r &= \alpha(1 + B)n \\
s &= \alpha(1 + B)
\end{align*}

We are going to prove that there exists an execution of the program such that lines 3, 5, 7, 10 are executed respectively $m, n, r, s$ times. The only problem that may prevent doing such an execution is a decrement command applied on a zero counter. To avoid that problem for counters $x_0, \ldots, x_{p-1}$ that are only decremented after the first line $\text{split} \ X_0, \ldots, X_{p-1}$, we execute that first line in such a way that those counters are maximal. More formally, denoting by $\rho$ the configuration we obtain after executing the first line, we can assume that $\rho(x_i) = \alpha(x_i)$ and $\rho(x'_i) = 0$ for every $0 \leq i \leq p - 1$.

Now, in order to prove the existence of an execution up to just before line 8, we just need to check that we can perform every decrement command the right number of times. It means that we need to prove that $\rho(x_0) \geq m + n$ in order to execute $m + n$ times $\text{dec}(x_0)$, $\rho(x_i) \geq n + r$ in order to execute $n + r$ times $\text{dec}(x_i)$ for every $1 \leq i \leq p - 1$, and $\alpha(X_i) \geq (n + r)\ell_i$ for every $p \leq i \leq d$ in order to execute $n + r$ times $\text{dec}(x_i)$. A simple verification shows that all those conditions are fulfilled since $\alpha$ is good and satisfies $B_2, \ldots, B_{p-1} = 0$ (in fact the inequality conditions are equalities).
It follows that we get an execution from $\rho$ to a configuration $\gamma$ just before line $\text{S}$ satisfying $\gamma(B_i) = \alpha(B_i)$ for every $1 \leq i \leq d$, $\gamma(B) = \alpha(B)$ and such that for every $0 \leq i \leq d$, we have:

$$\beta(X_i) = \alpha(X_i) - \begin{cases} 
  n & \text{if } i = 0 \\
  r & \text{if } 1 \leq i \leq p - 2 \\
  0 & \text{if } i = p - 1 \\
  (n + r)\ell_i & \text{if } p \leq i \leq d 
\end{cases}$$

Since $\gamma(B_p) > 0$, notice that the decrement command $\text{dec}(B_p)$ on line $\text{S}$ can be executed. Since line $\text{T}$ can only increments counters, we can execute that loop $s$ times and apply Lemma $\text{R}$ to obtain an execution of the program starting with $\gamma$ just before line $\text{S}$ that leads to a configuration $\beta$ satisfying $\beta(X_i) = \gamma(X_i)$ for every $0 \leq i \leq d$, and such that $\beta(B) = \beta(B_1)$, and for every $1 \leq i \leq d$:

$$\beta(B_i) = \alpha(B_i) + \begin{cases} 
  -1 & \text{if } i = p \\
  s & \text{if } i = p - 1 \\
  0 & \text{otherwise} 
\end{cases}$$

A simple verification shows that $\beta$ satisfies the lemma. \hfill \square

**Corollary 11.** Let $\alpha$ be a good configuration satisfying $X_0$ is divisible by $2 + B_1$ and $B_i > 0$ for some $i \in \{2, \ldots, d\}$. There exists a good configuration $\beta$ such that $\alpha \xrightarrow{\text{evalF}_d} \beta$ and such that $\beta(X_0) = \alpha(\frac{X_0}{2+B_1})$ and $\beta(B_1, \ldots, B_d) = \text{evalF}_d(\alpha(B_1, \ldots, B_d))$.

Now we prove that any other execution behaves the expected way.

**Lemma 12.** Assume that $\alpha \xrightarrow{\text{evalF}_d} \beta$ for some proper configuration $\alpha$ and an arbitrary configuration $\beta$. Then $\beta$ is proper and if $\beta$ is good then $\alpha$ is good and it satisfies $X_0$ is divisible by $2 + B_1$, $B_p > 0$ and $B_{p-1}, \ldots, B_2 = 0$, and moreover $\beta(X_0) = \alpha(\frac{X_0}{2+B_1})$ and $\beta(B_1, \ldots, B_d) = \text{evalF}_d(\alpha(B_1, \ldots, B_d))$.

**Proof.** Let $m, n, p, q$ be the number of times lines $\text{S}$, $\text{T}, \text{R}, \text{U}$ are executed respectively during an execution from a proper configuration $\alpha$ to a configuration $\beta$. Notice that $\alpha$ and $\beta$ satisfies $B_1 = B$ and $X_0 > 0$.

Let us prove some inequalities over $m, n, r, s$. From lemma $\text{R}$ we deduce that $n \leq (1 + \alpha(B_1))m$, $r \leq (1 + \alpha(B))n$, and $s \leq (1 + \alpha(B))$. Since $n + r$ is the number of times the command $\text{dec}(x_i)$ with $i \in \{1, \ldots, p - 1\}$ is executed, it follows that $\alpha(X_i) \geq n + r$ for every $i \in \{1, \ldots, p - 1\}$. Since $m + n$ is the number of times the command $\text{dec}(x_0)$ is executed, it follows that $\alpha(X_0) \geq m + n$.

Notice that we have $\beta(B_p) = \alpha(B_p) - 1$, $\beta(B_{p-1}) = \alpha(B_{p-1}) + s$, and $\beta(B_i) = \alpha(B_i)$ for every $i \in \{1, \ldots, d\} \setminus \{p - 1, p\}$. Moreover, $\beta(X_0) = \alpha(X_0) - n$, $\beta(X_i) = \alpha(X_i) - r$ for every $1 \leq i \leq p - 2$, and $\beta(X_{p-1}) = \alpha(X_{p-1})$. Lemma $\text{S}$ shows that there exists a sequence $\ell_p, \ldots, \ell_d$ such that $\ell_p = n + r$, $\ell_i \leq (1 + \alpha(B_i))\ell_{i-1}$ for every $p < i \leq d$, and satisfying $\beta(X_i) = \alpha(X_i) - \ell_i$ for every $p \leq i \leq d$.

By replacing $\beta(X_i)$ by $\alpha(X_i) - \ell_i$, $\beta(B_i)$ by $\alpha(B_i)$, and $\beta(X_{i-1})$ by $\alpha(X_{i-1}) - \ell_{i-1}$, we get the following equality for every $p \leq i \leq d$:

$$\beta(X_i - (1 + B_i)X_{i-1}) = \alpha(X_i - (1 + B_i)X_{i-1}) + ((1 - \alpha(B_i))\ell_{i-1} - \ell_i) \tag{2}$$

Assume first that there exists $i \in \{p + 1, \ldots, d\}$ such that $\ell_i < (1 + \alpha(B_i))\ell_{i-1}$ and let $i$ be the maximal one. Since $\alpha$ is proper, notice that $\alpha$ is $i$-good, or it is $j$-bad for some $j \geq i$. Equation $\tag{2}$ shows that in the first case $\beta$ is $i$-bad and in the second case $\beta$ is $j$-bad. So, we
can assume that for every $i \in \{p + 1, \ldots, d\}$, we have $\ell_i = (1 + \alpha(B_i))\ell_{i-1}$. Observe that if $\alpha$ is $j$-bad for some $j > p$ then $\beta$ is $j$-bad as well thanks to equation (2). So we can assume that $\alpha$ is $(p + 1)$-good. Equation (refeq:si) shows that $\beta$ is $(p + 1)$-good as well. As $\alpha$ is $(p + 1)$-good and proper, it is $p$-bad or $p$-good. So, $\alpha$ satisfies $X_p - (1 + B_p)X_{p-1} \geq 0$ in any case.

By replacing $\beta(X_p)$ by $\alpha(X_p) - (n + r)$, $\beta(B_p)$ by $\alpha(B_p) - 1$, and $\beta(X_{p-1})$ by $\alpha(X_{p-1})$, we get the following equality:

$$\beta(X_p - (1 + B_p)X_{p-1}) = \alpha(X_p - (1 + B_p)X_{p-1}) + (\alpha(X_{p-1}) - (n + r))$$

(3)

As $\alpha(X_p - (1 + B_p)X_{p-1})$ and $\alpha(X_{p-1}) - (n + r)$ are non negative, we deduce that if one of those two numbers is strictly positive then $\beta$ is $p$-bad. So, we can assume that $\alpha$ is $p$-good and $\alpha(X_{p-1}) = n + r$. It follows that $\beta$ is $p$-good as well.

As $\alpha(X_i) \geq n + r$ for every $1 \leq i \leq p - 2$, and $n + r = \alpha(X_{p-1})$ we deduce that $\alpha$ satisfies $X_i \geq X_{p-1}$ for every $1 \leq i < p - 1$. Assume by contradiction that $\alpha$ is $j$-bad or $\alpha(B_j) > 0$ for some $j \in \{2, \ldots, p - 1\}$ and let us consider the maximal such $j$. Observe that in that case $\alpha$ satisfies $X_j > X_{j-1}$ and $X_i = X_{i-1}$ for every $i \in \{j + 1, \ldots, p - 1\}$. In particular $\alpha$ satisfies $X_{p-1} > X_{j-1}$ and we get a contradiction. So we can assume that $\alpha$ is $2$-good and satisfies $B_i = 0$ for every $2 \leq i \leq p - 1$. Since $\alpha$ is proper and $2$-good, it is either $1$-good or $1$-bad. We deduce that $\alpha$ satisfies $X_1 - (1 + B_1)X_0 \geq 0$ in any case.

By replacing $\alpha(X_1)$ by $n + r$, and $\alpha(B)$ by $\alpha(B_1)$, we get:

$$\alpha(X_1 - (1 + B_1)X_0) = (r - (1 + \alpha(B))n) + (n - (1 + \alpha(B_1))m) + (1 + \alpha(B_1))(m + n - \alpha(X_0))$$

(4)

Since the three terms are non positive integers and the total sum is non negative, it follows that $\alpha$ is $1$-good, $r = (1 + \alpha(B))n$, $n = (1 + \alpha(B_1))m$, and $\alpha(X_0) = m + n$. In particular $\alpha(X_0) = (2 + \alpha(B_1))m$. Since $\beta(X_0) = \alpha(X_0) - n = m + n - n = m$, we derive $\beta(X_0) = \alpha(\frac{X_0}{2 + B_1})$. From $\alpha(X_0) > 0$ we derive $n > 0$.

From $\beta(X_{p-2}) = \alpha(X_{p-2}) - r$ and $\alpha(X_{p-2}) = n + r$, we deduce that $\beta(X_{p-2}) = n$. From $\beta(B_{p-1}) = \alpha(B_{p-1}) + s$ and $\alpha(B_{p-1}) = 0$, we derive $\beta(B_{p-1}) = s$. Moreover, $\beta(X_{p-1}) = \alpha(X_{p-1}) = n + r = (2 + \alpha(B))n$. By replacing $\beta(X_{p-1})$ by $(2 + \alpha(B))n$, $\beta(B_{p-1})$ by $s$, and $\beta(X_{p-2})$ by $n$, we get:

$$\beta(X_{p-1} - (1 + B_{p-1})X_{p-2}) = ((1 + \alpha(B)) - s)n$$

Since $s \leq 1 + \alpha(B)$ we deduce that if the inequality is strict then $\beta$ is $(p - 1)$-bad (recall that $n > 0$). So, we can assume that $s = 1 + \alpha(B)$. In particular $\beta$ is $(p - 1)$-good.

Observe that for every $j \in \{2, \ldots, p - 2\}$ we have:

$$\beta(X_j - (1 + B_j)X_{j-1}) = \alpha(X_j) - r - (\alpha(X_{j-1}) - r) = 0$$

It follows that $\beta$ is $2$-good.

Finally, observe that we have:

$$\beta(X_1 - (1 + B_1)X_0) = n + r - r - (1 + \alpha(B_1))(m + n - n) = 0$$

Thus $\beta$ is good. Now, a simple verification shows that $\beta$ satisfies the lemma.

As a direct corollary, we get the following result.

**Corollary 13.** Assume that $\alpha$ evalF$_d$ $\beta$ for some proper configuration $\alpha$ and an arbitrary configuration $\beta$. Then $\beta$ is proper and if $\beta$ is good then $\alpha$ is good and it satisfies $X_0$ is divisible by $2 + B_1$, $B_i > 0$ for some $i \in \{2, \ldots, d\}$, $\beta(X_0) = \alpha(\frac{X_0}{2 + B_1})$ and $\beta(B_1, \ldots, B_d) = \text{evalF}_d(\alpha(B_1, \ldots, B_d))$. 


6 Ackermannian Amplifier

We introduce in this section a $K$-amplifier for $K = 1 + F_d(n)$ where $d \geq 2$. Intuitively, we are going to use good configurations in such a way that initially ($B_1, \ldots, B_d$) will contain $(n, 0, \ldots, 0, 1 + n)$ and $X_0$ is non deterministically initialized to a positive number, and then we apply some number of times the reduce program $\text{evalF}_d$. Finally, we check that the configuration we obtain is good and that $\text{evalF}_d$ can no longer be executed (i.e. $B_2, \ldots, B_d = 0$).

We first introduce the following test-free program:

\[
\text{init}_{d,n} = \begin{array}{l}
\text{inc}(b_1, b)^{(n)} \\
\text{inc}(b_d)^{(1+n)}; \text{inc}(x_0); \text{inc}(x_1, \ldots, x_{d-1})^{(1+n)}; \text{inc}(x_d)^{(2+n)(1+n)} \\
\text{loop} \\
\text{inc}(b_1, b)^{(n)}; \text{inc}(b_d)^{(1+n)} \\
\text{inc}(x_0); \text{inc}(x_1, \ldots, x_{d-1})^{(1+n)}; \text{inc}(x_d)^{(2+n)(1+n)} \\
\text{loop} \\
\text{evalF}_d
\end{array}
\]

**Lemma 14.** For every $\ell > 0$ there exists a good configuration $\beta$ such that $0 \xrightarrow{\text{init}_{d,n}} \beta$, $\beta(X_0) = \ell$, and $\beta(B_1, \ldots, B_d) = (F_d(n), 0, \ldots, 0)$.

**Proof.** Let $k$ be the maximal number of times the function $\text{evalF}_d$ can be applied on the vector $(n, 0, \ldots, 0, 1 + n)$, and let us prove by induction on $r \in \{0, \ldots, k\}$ that for every $\ell > 0$ there exists a configuration $\beta$ such that $\beta(X_0) = \ell$, $\beta(B_1, \ldots, B_d) = \text{evalF}_d^r(n, 0, \ldots, 0, 1+n)$, and such that $0 \xrightarrow{\text{init}_{d,n}} \beta$. The rank 0 is immediate by iterating the first loop $(\ell - 1)$ times. Assume the rank $r - 1$ proved for some $r \in \{1, \ldots, k\}$. Let $\ell > 0$ and let $v = \text{evalF}_d^{r-1}(n, 0, \ldots, 0, 1+n)$. By induction, there exists a good configuration $\beta$ such that $\beta(X_0) = (2 + v[1])\ell$, $\beta(B_1, \ldots, B_d) = v$, and such that $0 \xrightarrow{\text{init}_{d,n}} \beta$. From Corollary 13 we deduce that there exists a good configuration $\gamma$ such that $\beta \xrightarrow{\text{evalF}_d} \gamma$ and such that $\gamma(X_0) = \ell$ and $\gamma(B_1, \ldots, B_d) = \text{evalF}_d(\beta(B_1, \ldots, B_d)) = \text{evalF}_d(n, 0, \ldots, 0, 1+n)$. Now, just notice that the induction is proved since $0 \xrightarrow{\text{init}_{d,n}} \gamma$. We have proved the claim thanks to Lemma 5.

**Lemma 15.** If $0 \xrightarrow{\text{init}_{d,n}} \beta$ for an arbitrary configuration $\beta$ then $\beta$ is proper and if $\beta$ is good then $\text{val}(\beta(B_1, \ldots, B_d)) = F_d(n)$.

**Proof.** Let $\alpha$ be the configuration we obtain just before executing line 5. Observe that $\alpha$ is good and $\beta(B_1, \ldots, B_d) = (n, 0, \ldots, 0, 1 + n)$. In particular $\text{val}(\beta(B_1, \ldots, B_d)) = F_d(n)$. Since $\text{val}(\beta(B_1, \ldots, B_d))$ is an invariant when $\text{evalF}_d$ produces a good configuration from a good one (see Corollary 13), the lemma is proved.

We are going to simulate with the counters $x$ and $y$ the counter expressions $X_0$ and $X_1$ of $\text{init}_{d,n}$. To do so, we transform the program $\text{init}_{d,n}$ in such a way that every increment command of $x_0$ or $x_0'$ is followed by an increment command of $x$, every decrement command of $x_0$ or $x_0'$ is followed by a decrement command of $x$, and symmetrically for $y$. This way, we get a program that we denote by $\text{init}'_{d,n}$.

We are now ready to introduce the following checking program:
\[ \text{Ack}_{d,n} = \begin{cases} \text{init}'_{d,n} \\
\text{loop} \\
\quad \text{rdec}_{d,0} \\
\text{split} B \\
\text{loop} \\
\quad \text{dec}(B_1) \\
\quad \text{inc}(b) \\
\quad b'?=0 \\
B_1, \ldots, B_d, X_d=?=0 \end{cases} \]

Theorem 16. The checking program \( \text{Ack}_{d,n} \) is a \((1 + F_d(n)) \)-amplifier.

Proof. Let \( K = (1 + F_d(n)) \).

Let us first consider a positive number \( \ell > 0 \). Lemma 14 shows that there exists a good configuration \( \gamma \) such that \( \gamma(X_0) = \ell \), \( \gamma(B_1, \ldots, B_d) = (F_d(n), 0, \ldots, 0) \), and such that \( 0 \xrightarrow{\text{init}'_{d,n}} \gamma \). It follows that \( \gamma(B_1) = K - 1 \) and \( \gamma(B_i) = 0 \) for every \( 2 \leq i \leq d \). Since \( x \) simulates \( X_0 \), from \( \gamma(X_0) = \ell \) we derive \( \gamma(x) = \ell \). Moreover, as \( \gamma \) is good we deduce that \( \gamma \) satisfies \( X_1 = (1 + B_1)X_0 \). Since \( x, y \) simulates \( X_0, X_1 \) and \( \gamma(B) = \gamma(B_1) \), we deduce from \( \gamma(X_1) = \gamma((1 + B_1)X_0) \) that \( \gamma(y) = K\ell \). As \( \gamma \) is good, Lemma 8 shows that there exists an execution of \( \text{proper}_d \) from \( \gamma \) to a configuration \( \delta \) satisfying \( \delta(X_d) = 0 \) and \( \delta(B_i) = \gamma(B_i) \) for every \( 1 \leq i \leq d \). From that execution we deduce that \( \gamma \xrightarrow{\text{loop rdec}_{d,0}} \delta \). Now just observe that by executing the \( \text{split} B \) in such a way that the configuration we obtain afterwards satisfies \( b' = 0 \) and by iterating the last loop \( \delta(B_1) \) times, we get from \( \delta \) a final configuration \( \beta \) satisfying \( \beta(B_i) = 0 \) for every \( 1 \leq i \leq d \), \( \beta(X_d) = 0 \), and \( \beta(b') = 0 \). Moreover, since the counters \( x \) and \( y \) are untouched by the end of the program, we deduce that \( \beta(x, y, b) = (\ell, K\ell, K) \).

Conversely, assume that there exists an execution of \( \text{Ack}_{d,n} \) from the zero configuration 0 to a configuration \( \beta \). We use the same notations as in the previous paragraph, meaning that we denote by \( \gamma \) the configuration we obtain after executing \( \text{init}'_{d,n} \) and we denote by \( \delta \) the configuration we obtain after executing \( \text{loop rdec}_{d,0} \). Lemma 15 shows that \( \gamma \) is proper. Since \( X_0, \ldots, X_d \) are untouched by the end of the program and \( \beta(X_d) = 0 \), we deduce that \( \gamma(X_d) = 0 \).

As \( \gamma \xrightarrow{\text{loop rdec}_{d,0}} \delta \) and \( \gamma \) is proper, Lemma 8 shows that \( \gamma \) is good. From Lemma 15 we deduce that \( \text{val}(\gamma(B_1, \ldots, B_d)) = F_d(n) \). Since \( \gamma(B_1) = \beta(B_1) = 0 \) for every \( 2 \leq i \leq d \), we deduce that in fact \( \gamma(B_1) = F_d(n) = K - 1 \). Since \( \gamma(B_1) = \gamma(B) \) we deduce that \( \gamma(B) = K - 1 \). Since \( \beta(b') = 0 \) we deduce that \( \beta(b) = K \). Let \( \ell = \gamma(x) \). Since \( x \) and \( y \) simulates \( X_0 \) and \( X_1 \), and \( \gamma \) is good, from \( \gamma(X_1) = \gamma((1 + B_1)X_0) \) we get \( \gamma(y) = K\ell \). Since the counters \( x, y \) are untouched by the end of the program, we deduce that \( \beta(x, y, b) = (\ell, K\ell, K) \).

Remark 17. Observe that the counters used by \( \text{Ack}_{d,n} \) are \( x_d, b, b', x, y \), and the counters \( x_{i-1}, x'_{i-1}, b_i, b'_i \) for every \( 1 \leq i \leq d \). In fact, the counter \( x'_d \) is not really used since it is always equals to zero. It follows that the dimension of that program is \( 4d + 5 \). Notice that the counters \( x_i, x'_i \) with \( 0 \leq i \leq d - 1 \) are safe.

7 Conclusion

This paper proves that the reachability problem for VASS is Ackermannian-complete.

Acknowledgements

I gratefully thank Philippe Schnoebelen for helpful comments and encouraging feedback.
References

[1] Rémi Bonnet. The reachability problem for vector addition system with one zero-test. In Filip Murlak and Piotr Sankowski, editors, Mathematical Foundations of Computer Science 2011 - 36th International Symposium, MFCS 2011, Warsaw, Poland, August 22-26, 2011. Proceedings, volume 6907 of Lecture Notes in Computer Science, pages 145–157. Springer, 2011. doi: 10.1007/978-3-642-22993-0_16. URL https://doi.org/10.1007/978-3-642-22993-0_16.

[2] E. Cardoza, R. J. Lipton, and A. R. Meyer. Exponential space complete problems for petri nets and commutative semigroups: Preliminary report. In Proceedings of the 8th Annual ACM Symposium on Theory of Computing, May 3-5, 1976, Hershey, Pennsylvania, USA, pages 50–54. ACM, 1976. doi: 10.1145/800113.803630.

[3] W. Czerwiński, S. Lasota, R. Lazić, J. Leroux, and F. Mazowiecki. The reachability problem for petri nets is not elementary. In Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, STOC 2019, Phoenix, AZ, USA, June 23-26, 2019, pages 24–33. ACM, 2019. doi: 10.1145/3313276.3316369.

[4] Wojciech Czerwinski and Lukasz Orlikowski. Reachability in vector addition systems is ackermann-complete. CoRR, abs/2104.13866, 2021. URL https://arxiv.org/abs/2104.13866.

[5] J. Esparza and M. Nielsen. Decidability issues for petri nets - a survey. Bulletin of the European Association for Theoretical Computer Science, 52:245–262, 1994.

[6] M. H. T. Hack. Decidability questions for Petri nets. PhD thesis, MIT, 1975. URL http://publications.csail.mit.edu/lcs/pubs/pdf/MIT-LCS-TR-161.pdf.

[7] J. E. Hopcroft and J.-J. Pansiot. On the reachability problem for 5-dimensional vector addition systems. Theoretical Computer Science, 8:135–159, 1979.

[8] R. M. Karp and R. E. Miller. Parallel program schemata. J. Comput. Syst. Sci., 3(2):147–195, 1969. doi: 10.1016/S0022-0000(69)80011-5.

[9] S. R. Kosaraju. Decidability of reachability in vector addition systems (preliminary version). In STOC, pages 267–281. ACM, 1982. doi: 10.1145/800070.802201.

[10] J.-L. Lambert. A structure to decide reachability in Petri nets. Theor. Comput. Sci., 99 (1):79–104, 1992. doi: 10.1016/0304-3975(92)90173-D.

[11] Slawomir Lasota. Improved ackermannian lower bound for the vass reachability problem, 2021.

[12] J. Leroux and S. Schmitz. Demystifying reachability in vector addition systems. In 30th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2015, Kyoto, Japan, July 6-10, 2015, pages 56–67. IEEE Computer Society, 2015. doi: 10.1109/LICS.2015.16.

[13] J. Leroux and S. Schmitz. Reachability in vector addition systems is primitive-recursive in fixed dimension. In 34th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2019, Vancouver, BC, Canada, June 24-27, 2019, pages 1–13. IEEE, 2019. doi: 10.1109/LICS.2019.8785796.

[14] E. W. Mayr. An algorithm for the general petri net reachability problem. In Proceedings of the 13th Annual ACM Symposium on Theory of Computing, May 11-13, 1981, Milwaukee, Wisconsin, USA, pages 238–246. ACM, 1981. doi: 10.1145/800076.802477.
[15] E. W. Mayr. An algorithm for the general Petri net reachability problem. *SIAM J. Comput.*, 13(3):441–460, 1984. doi: 10.1137/0213029.

[16] Marvin L. Minsky. *Computation: finite and infinite machines*. Prentice-Hall, Inc., 1967. URL https://dl.acm.org/citation.cfm?id=1095587

[17] C. Rackoff. The covering and boundedness problems for vector addition systems. *Theor. Comput. Sci.*, 6:223–231, 1978. doi: 10.1016/0304-3975(78)90036-1.

[18] Klaus Reinhardt. Reachability in petri nets with inhibitor arcs. *Electronic Notes in Theoretical Computer Science*, 223:239–264, 2008. ISSN 1571-0661. doi: https://doi.org/10.1016/j.entcs.2008.12.042. URL https://www.sciencedirect.com/science/article/pii/S1571066108005057 Proceedings of the Second Workshop on Reachability Problems in Computational Models (RP 2008).

[19] G. S. Sacerdote and R. L. Tenney. The decidability of the reachability problem for vector addition systems (preliminary version). In *Proceedings of the 9th Annual ACM Symposium on Theory of Computing, May 4-6, 1977, Boulder, Colorado, USA*, pages 61–76. ACM, 1977. doi: 10.1145/800105.803396.

[20] S. Schmitz. The complexity of reachability in vector addition systems. *SIGLOG News*, 3(1):4–21, 2016. URL https://dl.acm.org/citation.cfm?id=2893585

[21] Sylvain Schmitz. Complexity hierarchies beyond elementary. *TOCT*, 8(1):3:1–3:36, 2016. URL http://doi.acm.org/10.1145/2858784

[22] Philippe Schnoebelen. Revisiting ackermann-hardness for lossy counter machines and reset petri nets. In Petr Hlinený and Antonín Kucera, editors, *Mathematical Foundations of Computer Science 2010, 35th International Symposium, MFCS 2010, Brno, Czech Republic, August 23-27, 2010. Proceedings*, volume 6281 of *Lecture Notes in Computer Science*, pages 616–628. Springer, 2010. doi: 10.1007/978-3-642-15155-2_54. URL https://doi.org/10.1007/978-3-642-15155-2_54.