Implications of an Ambiguity in J.S. Bell’s Analysis of the Einstein-Podolsky-Rosen Problem

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Abstract

An ambiguity is pointed out in J.S. Bell’s argument that the distinction between quantum mechanics and hidden variable theories cannot be found in the behavior of single-particle beams. Within the context of theories for which states are unambiguously defined it is shown that the question of whether quantum mechanics or a locally realistic theory is valid may indeed be answered by single-particle beam measurements. It is argued that two-particle correlation experiments are required to answer the more fundamental question of whether or not the notion of a state can be unambiguously defined. As a byproduct of the discussion the general form of completely entangled states is deduced.

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J.S. Bell’s 1964 paper [1], in which his celebrated inequality first makes its appearance, proves the inconsistency between quantum mechanics and locally realistic theories, i.e. hidden variable theories obeying the Einstein-Podolsky-Rosen (EPR) postulates [2]. In a preliminary discussion Bell makes the following assertion: “Firstly, there is no difficulty in giving a hidden variable account of spin measurements on a single particle.” To prove this he considers a particle “in a pure spin state labeled by a unit vector” and proceeds to construct a hidden variable model that reproduces the quantum mechanical prediction for the expectation value of the spin in any direction. Bell uses this construction to motivate employing two-particle systems to discover the contradiction between locally realistic theories and quantum mechanics by means of his inequality.

There is, however, an ambiguity in the construction. To see this let us ask what one means by saying that a particle is in a pure state labeled x. The operational criterion is that we be able to predict with certainty that the particle will pass an x-filter (e.g. a Stern-Gerlach filter with magnet orientation indicated by x). There are (at least) two methods for doing this: (i) We can first pass the particles through an x-filter. This procedure is “invasive” in that we must interact with the particle to prepare the state. (ii) We can exploit correlations. Thus suppose we have a two-particle system such that there is equal likelihood for a particle to pass any filter x, but that one can predict with certainty that a particle will pass x if its partner passes a filter $x^u$. Here $x^u$ is related to x by some definite rule, and we refer to the system as a $u$-correlated system. Given such a system one prepares a beam in the state $x$ by admitting only the partners of particles that have passed an $x^u$-filter. Method (ii) has the advantage of being “non-invasive” in that if measurements are space-like separated events and we assume Einstein locality, we can be sure that the state preparation does not disturb the particle.

Since the method of state preparation is not unique, the question must first be answered as to whether the outcome of experiments is sensitive to the method. This is a more fundamental question than what the specific outcome may be since it must be answered before we use the term “state”. We are going to prove the following two theorems below:

**Theorem 1:** Quantum mechanics makes the same predictions for both methods of state preparation.

**Theorem 2:** If a locally realistic theory makes the same predictions for both methods of state preparation it will disagree with the quantum mechanical prediction.

We may draw the following inference from these theorems: For brevity let us say that dynamics is “Markovian” if predictions are unambiguous, i.e. independent of the method
of state preparation. (We choose this term because Markov processes are those in which the future is independent of the past for a known present.) Classical mechanics has this property, and Theorem 1 shows that conventional quantum mechanics has it also. Thus a non-Markovian world is going to be even more counter-intuitive than quantum mechanics. But Theorem 2 now shows that one cannot replace quantum mechanics by a locally realistic theory that is also Markovian unless we can prove that the quantum mechanical predictions for single-particle beams (e.g. Malus’ Law for photons) fails. In other words within the Markovian framework it is possible to rule out locally realistic theories without resorting to sophisticated two-particle correlation experiments. Since Bell’s single-particle hidden variable model reproduces the quantum mechanical result it follows that this model describes a non-Markovian world. We therefore concur with the conclusion drawn by R.Griffiths \[3\] from his consistent history analysis that the counter-intuitive aspect of quantum mechanics is already revealed in the behavior of single particle beams. Thus the two-particle correlation experiments are seen to be addressed to a more profound question, one that applies equally to classical and quantum mechanics, namely whether nature permits the unambiguous definition of a state. Hence we suggest that such experiments should be designed to demonstrate that methods (i) and (ii) give the same result rather than focussing on what results the two methods happen to give individually.

We will use the following notation: We indicate by $P_u(x|y)$ the probability that one of the particles in a $u$-correlated system passes a $y$-filter if its partner is known to have passed an $x^u$-filter. Let $p(x|y)$ denote the probability that a particle passes a $y$-filter if it itself has previously passed an $x$-filter. Then the question of sensitivity to method of state preparation may be formulated as:

$$P_u(x|y) = p(x|y).$$ (1)

We first deduce that quantum mechanics verifies (1). Consider quantum mechanical particles described by vectors in an N-dimensional Hilbert space with a basis $|i\rangle, i = 1, 2, \cdots, N$.

**Lemma 1:** A necessary and sufficient condition that a two particle state $|S\rangle$ exhibit $u$-correlation is that it be of the form:

$$|S\rangle = N^{-1/2} \sum_{i=1}^{N} |i, 1\rangle|i^u, 2\rangle,$$ (2)
in which $|i^n\rangle$ is related to $|i\rangle$ by an anti-unitary transformation.

The proof of necessity is given in the appendix. The proof of sufficiency reveals the role of anti-unitarity and is as follows: First note that $|S\rangle$ is independent of the choice of basis. For if $|i\rangle = \sum_j A_{ij} |j'\rangle$ where prime indicates another basis and $A$ is unitary, then by the anti-unitarity of $u$ we have: $|i^{n'\rangle} = \sum_k A^*_{kj} |j'^{n}\rangle$ whence one verifies that (2) is unchanged if $i$ is replaced by $i'$. One may note that in the Bohm state $^4$ ^5 (the spin-0 state of two spin-1/2 particles) the map $u$ is the anti-unitary time-reversal transformation as may be demonstrated by observing that the correlated magnet orientations operate with reversed currents.

Now let $\pi(x,j) \equiv |x,j\rangle \langle x,j|\), \ j = 1, 2. We then have:

$$P_u(x|y) = \frac{\langle S|\pi(x^u,2)\pi(y,1)|S\rangle}{\langle S|\pi(x^u,2)|S\rangle}. \quad (3)$$

The denominator on the right is $1/N$. Using independence of the choice of basis in (2) we select a basis containing $|y\rangle$. The numerator is then verified to be $|\langle x^u,2|y^u,2\rangle|^2/N = |\langle x^u|y\rangle|^2/N$. But $p(x|y) = |\langle x|y\rangle|^2$ and so we have proved that $P_u(x|y) = p(x|y)$ which is Theorem 1.

We now turn to Theorem 2: We show that if the quantum mechanical prediction for the right side of (1) is assumed, and the left side is computed in a locally realistic theory, then the equation will be violated. According to the familiar EPR argument we must be able to assign values to determinations of filter passage. Thus we are to assume that there is a set $\Lambda$ whose elements $\lambda$ are the values of the hidden-variable and that there are subsets $\Lambda_j(x)$ for all filter labels and $j = 1, 2$ such that particle $j$ passes an $x$ filter if $\lambda \in \Lambda_j(x)$ and otherwise does not. If $\mu(\Lambda)$ is the measure then:

$$P_u(x|y) = \frac{\mu(\Lambda_2(x^u) \cap \Lambda_1(y))}{\mu(\Lambda_2(x^u))}. \quad (4)$$

The perfect correlation between the two particles means that $\Lambda_2(x^u) = \Lambda_1(x)$ up to a set of $\mu$-measure zero so that we may write this as:

$$P_u(x|y) = \frac{\mu(\Lambda_1(x) \cap \Lambda_1(y))}{\mu(\Lambda_1(x))}. \quad (5)$$

Since the quantum mechanical prediction for the right side of (1) is symmetric in the two arguments, it follows that $P_u(x|y)$ can reproduce it only if it is also symmetric. But the
numerator on the right of (5) is symmetric and hence the denominator must be independent of \( x \). Hence it can be absorbed into the numerator by redefining \( \mu \) and we may write:

\[
P_u(x|y) = \mu(\Lambda_1(x) \cap \Lambda_1(y)).
\] (6)

Here we may see the striking effect of method (ii) state determination in constraining the form of \( P_u(x|y) \), i.e. we cannot simply take the argument on the right side to be an arbitrary set \( \Lambda(x, y) \), but must rather take it to be the intersection of sets depending separately on \( x \) and \( y \). That Bell’s construction [1] referred to above is non-Markovian results from the fact that he utilizes a set which is not of this form!

Now suppose that (1) is valid. Then:

\[
\sup_z |P_u(x|z) - P_u(y|z)| = \sup_z |p(x|z) - p(y|z)|
\] (7)

We will prove that if the quantum mechanical expression is used for the right and (6) is used for the left we will have:

**Lemma 2:**

\[
\sup_z |P_u(x|z) - P_u(y|z)| = 1 - P_u(x|y).
\] (9)

To prove (8) note that if \( \pi(z) \equiv |z\rangle\langle z| \) we have \( p(x|y) = Tr(\pi(x)\pi(y)) \), and

\[
\sup_z |Tr(\pi(x)\pi(z)) - Tr(\pi(y)\pi(z))| = \sup_z |\langle z|\pi(x) - \pi(y)|z\rangle|.
\] (10)

But this is just the largest eigenvalue [3] of \( \pi(x) - \pi(y) \). Since the \( \pi \)'s are projectors:

\[
(\pi(x) - \pi(y))^3 = (1 - |\langle x|y\rangle|^2)(\pi(x) - \pi(y))
\] (11)

and one reads off the largest eigenvalue to obtain the assertion. To prove (9) note that in \( |\mu(\Lambda(x) \cap \Lambda(z)) - \mu(\Lambda(y) \cap \Lambda(z))| \) the contribution coming from any overlap of \( \Lambda(x) \) and \( \Lambda(y) \) will cancel. Hence one can compute the supremum as if the sets are disjoint. This occurs when either \( z = x \) or \( z = y \) and has the value given by the right side of (9).

The occurrence of the square-root factor in (8) but not in (9) shows that (7) holds if and only if \( p(x|y) \) is restricted to the values zero or unity. This is true classically but false in quantum mechanics, and we have proved Theorem 2 •
The reader may have noted that the Bell inequality has not appeared in the above analysis. It is concealed in the relationship between (8) and (9) in the following way: One readily checks that the quantities on the left of these two equations are metrics\(^7\), and the Bell inequality is just the triangle inequality associated with the metric of (9). The presence of the square root on the right side of (8) that is missing from (9) produces the violation of that inequality and a myriad of other contradictions as well, e.g. absence of a watch-dog effect. The fact that the mathematical origin of the incompatibility between quantum mechanics and locally realistic theories is to be found in the metric was noted by the author\(^8\). That crucial square-root also turns out to provide interesting insight into the structure of quantum mechanics that will be discussed elsewhere\(^9\).

It is to be noted that Theorem 1 is of interest in its own right. It provides a general method for the construction of perfectly entangled states and has been used by the author\(^10\) in the study of the Bennett-Wiesner communication scheme\(^11\).

Appendix — Proof of necessity in Lemma 1

Let \(|S_x, 1\rangle \equiv \langle x^u, 2|S\rangle, \ |S_x, 2\rangle \equiv \langle x, 1|S\rangle\) (which makes sense notationally because \(|S\rangle\) is a two-particle state). Then \(\langle x, 1|S_x, 1\rangle = \langle x^u, 2|S_x, 2\rangle\). One checks that the perfect correlation condition can be expressed in the form:

\[
\langle S_x, 1|x, 1\rangle\langle x, 1|S_x, 1\rangle = \langle S_x, 1|S_x, 1\rangle \neq 0,
\]

and is equal to the same expression with 1 replaced by 2. Hence one deduces:

\[
|S_x, 1\rangle = \gamma(x)|x, 1\rangle \quad \text{and} \quad |S_x, 2\rangle = \gamma(x)|x^u, 2\rangle,
\]

where \(\gamma(x)\) is a non-vanishing complex number. Then

\[
\gamma(x)\langle y, 1|x, 1\rangle = \langle y, 1|S_x, 1\rangle = \langle x^u, 2|y, 1\rangle = \gamma(y)\langle x^u, 2|y^u, 2\rangle.
\]

Multiplying the left and right members by the complex conjugate \(\gamma(x)^*\) and noting that

\[
\gamma(x)^*\langle x^u, 2|y^u, 2\rangle = \gamma(x)^*\langle y^u, 2|x^u, 2\rangle = \gamma(y)^*\langle y, 1|x, 1\rangle,
\]

it follows that

\[
|\gamma(x)| = |\gamma(y)| \quad \text{if} \quad \langle x|y\rangle \neq 0,
\]
so that $\gamma(x)$ is unimodular up to a constant factor. Hence, by redefining $|x^u, 2\rangle$ to absorb the unimodular factor $\gamma(x)$, it follows from (14) that the map $u$ is anti-unitary. One may then select any basis to express $|S\rangle$ in the form

$$|S\rangle = \sum_{i,j=1}^{N} \alpha_{ij} |i, 1\rangle |j^u, 2\rangle, \quad (17)$$

and use (12) to show that $\alpha_{ij} = \delta_{ij}/\sqrt{N}$ thereby giving (2). $\bullet$

I would like to thank T.Jacobson, A. Shimony, and C.H. Woo for very helpful discussions.
References

[1] J.S. Bell, Physics 1 195 (1964).
[2] A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. 47,777 (1935).
[3] R.B. Griffiths - Amer. J. Phys. 55,11 (1987).
[4] D. Bohm, Quantum Theory. Prentice-Hall, Englewood Cliffs, 1951.
[5] S.J.Freedman and J.F.Clauser, Phys. Rev. Lett. 28 938 (1972).
[6] V. Bargmann, Ann. Math. 59 (2), 1 (1954).
[7] J. Belinfante, J. Math. Phys. 17 No. 3, 285 (1976).
[8] D.I. Fivel, Phys. Rev. Lett. 67, 285 (1991).
[9] D.Fivel, U.of Md. Preprint UMD-PP-94-133
[10] D.Fivel, U.of Md. Preprint UMD-PP-94-131
[11] C.H. Bennett and S.J. Wiesner, Phys. Rev. Let. 69, 2881 (1992).