Isometry groups of formal languages for generalized Levenshtein distances *

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Abstract
This article is a partial answer to the question of which groups can be represented as isometry groups of formal languages for generalized Levenshtein distances. Namely, it is proved that for any language the modulus of the difference between the lengths of its words and the lengths of their images under isometry for an arbitrary generalized Levenshtein distance that satisfies the condition that the weight of the replacement operation is less than twice the weight of the removal operation is bounded above by a constant that depends only on the language itself. From this, in particular, it follows that the isometry groups of formal languages with respect to such metrics always embed into the group $\prod_{n=1}^{\infty} S_n$. We also construct a number of examples showing that this estimate is, in a certain sense, unimprovable.

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1 Introduction
This paper is a partial answer to the question of which groups can be represented as isometry groups of formal languages for generalized Levenshtein distances. Namely, the following theorem is proved:

**Theorem 1.** Let $L$ be an arbitrary formal language, $d$ be an arbitrary generalized Levenshtein distance satisfying the condition that the weight of the replacement operation is less than twice the weight of the removal operation. Then there exists $m \in \mathbb{N}$ such that $|\phi(w)| - |w| \leq m$

Note that for generalized Levenshtein distances with a replacement weight greater than or equal to twice the insertion weight, this statement is not true.

It follows from Theorem 1, in particular, that the isometry group of any formal language with respect to the Levenshtein metric embeds in $\prod_{n=1}^{\infty} S_n$.

A number of examples are also constructed, demonstrating that this estimate is, in a certain sense, unimprovable, moreover, for languages with different growths.

**Definition 1.** **Growth** of $L$ is a function of $n \mapsto |\{w \in L | |w| \leq n\}|$

**Theorem 2.** Let $G_1, G_2, \ldots$ be a countable sequence of arbitrary finite groups, $d$ be an arbitrary generalized Levenshtein distance. Then there exists $n \in \mathbb{N}$ and language $L \subset \{0, 1\}^{24n}$ such that $|L| = n$, $\text{Isom}_d(L) \cong G$ and $d(u, v) \in \{4, 6\}$ for any two distinct words $u, v \in L$.

Here and below, $A^k$ denotes the set of all words of length $k$ over the alphabet $A$.

**Theorem 3.** Let $G_1, G_2, \ldots$ be a countable sequence of arbitrary finite groups, $d$ be an generalized Levenshtein distance. Then there exists a language $L \subset \{0, 1\}^*$ with growth $O(n)$ such that $\text{Isom}_d(L) \cong \prod_{n=1}^{\infty} G_n$.

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Theorem 4. Let \(d\) be an arbitrary generalized Levenshtein distance. Then for any integer \(k \geq 2\), there exists a language \(L\) over an alphabet of \(k\) characters with growth \(\Theta(k\sqrt{n})\) such that \(\text{Isom}_d(L) \cong S_k^\infty \times \Pi_{n=1}^\infty S_{kn}\).

The last fact is interesting because the isometry group obtained by Theorem 1 is maximal for all languages. Also, special attention is paid to regular languages:

Definition 2. Regular language is a formal language that can be obtained from finite languages by applying a finite number of union operations \((U \cup V)\), product \((UV = \{uv | u \in U, v \in V\})\) and Kleene stars \((L^* = \bigcup_{n=0}^\infty L^n)\).

It is proved in the article that their isometry groups are also sufficiently diverse and the given bound for them is unimprovable:

Theorem 5. Let \(G\) and \(H\) be arbitrary finite groups, \(d\) be an arbitrary generalized Levenshtein distance. Then there is a regular language \(L \subset \{0; 1\}^*\) such that \(\text{Isom}_d(L) \cong G \times H^n\).

Theorem 6. Let \(d\) be an arbitrary distance from the family of generalized Levenshtein distances. Then there exists a regular language \(L\) such that \(\text{Isom}_d(L) \cong \Pi_{n=1}^\infty S_{2n}\).

The last fact is interesting because the isometry group obtained by Theorem 1 is maximal for all languages, not necessarily regular ones.

Here and below, \(S_n\) denotes a symmetric group on \(n\) elements, \(C_n\) denotes a cyclic group of order \(n\), \(\Pi\) denotes the Cartesian product of groups.

The isometry groups of finite languages have been studied before. For example, in [3] it is proved that for an arbitrary finite alphabet \(A\), the isometry group \(A^n\) with respect to the Hamming distance is isomorphic to \(S_{|A|^n} \times S_n\).

Another paper on a similar topic is [6], where it is proved that for an arbitrary finite alphabet \(A\) and \(2 \leq k_1 < k_2\) the isometry group of the language \(\bigcup_{k=k_1}^{k_2} A^k\) with respect to the \(\text{internal Levenshtein distance}\) (the minimum number of operations of insertions, deletions and replacements of characters that transform one word into another in such a way that all intermediate words lie in the source language) is isomorphic to \(S_{|A|^n} \times C_2\).

In the same article, the isometry groups of languages, including infinite ones, are studied with respect to the family of generalized Levenshtein distances. Their definitions and main properties will be given in Section 1.

The work consists of 9 sections (including introduction):

- Section 2 formulates the definition of a family of generalized Levenshtein metrics, and also classifies isometry groups of one-character languages.
- Section 3 proves Theorem 1 and constructs a counterexample for the case when the weight of the replacement is greater than or equal to twice the weight of the insertion.
- In Section 4, we carry out the preparatory work necessary for the proof of Theorem 2.
- Section 5 proves Theorem 2.
- Section 6 proves Theorem 3.
- Section 7 provides a proof
- Section 7 proves Theorem 4.
- Section 8 proves Theorem 5.
- Section 9 proves Theorem 6.

2 Generalized Levenshtein distances

Definition 3. Let \(A\) be a finite alphabet and the words \(u, v \in A^*\). Then generalized Levenshtein distance with insertion weight \(\gamma\) and replacement weight \(\theta\) between \(u\) and \(v\) is

\[
\text{lev}_{\gamma, \theta}(u, v) = \min\{\gamma n + \theta m | u \text{ can be translated into } v \text{ by } n \text{ insertions or deletions and } m \text{ replacements}\}
\]

Obviously, for \(\gamma, \theta > 0\) \(\text{lev}_{\gamma, \theta}\) is a distance on \(A^*\).

The most studied special cases of the generalized Levenshtein distance are:
\[ \text{lev}_{1,1} \text{ classical Levenshtein metric} – \text{the minimum number of insertions, deletions, or substitutions required to transform one word into another (first discussed in [4])} \]

\[ \text{lev}_{1,2} \text{ is the minimum number of insertions or deletions required to transform one word into another.} \]

\[ \text{lev}_{n,1} \text{ for } n \to \infty \text{ converges pointwise to the Hamming metric, the minimum number of substitution operations required to transform one word into another.} \]

Also, there is an alternative way to specify the generalized Levenshtein distance by the recursive formula:

**Proposition 1** ([3]).

\[
\text{lev}_{\gamma,\theta}(a, b) = \begin{cases} 
\gamma|a| & |b| = 0 \\
\gamma|b| & |a| = 0 \\
\text{lev}_{\gamma,\theta}(a.\text{tail}, b.\text{tail}) & a.\text{head} = b.\text{head} \\
\min(\theta + \text{lev}_{\gamma,\theta}(a.\text{tail}, b.\text{tail}), \gamma + \text{lev}_{\gamma,\theta}(a.\text{tail}, b), \gamma + \text{lev}_{\gamma,\theta}(b.\text{tail}, a)) & a.\text{head} \neq b.\text{head}
\end{cases}
\]

where \(a.\text{tail}\) is the suffix of \(a\) containing all of its characters except the first one, and \(a.\text{head}\) is the first element of \(a\).

In particular, from this formula, as well as the invariance of the Levenshtein distance under \(\|\text{reflection}\|\_\text{L}\) of words, it is true that \(\text{lev}_{\gamma,\theta}(uxv, uyv) = \text{lev}_{\gamma,\theta}(x, y)\).

The following inequalities also always hold:

- \(\text{lev}_{\gamma,\theta}(u, v) \leq (\theta - \gamma) \min(|u|, |v|) + \gamma \max(|u|, |v|)\)
- \(\text{lev}_{\gamma,\theta}(u, v) \geq \gamma||u| - |v||\), and equality is achieved if and only if the shorter word is contained in the longer one as a subsequence.

**Definition 4.** Let \((M_1, d_1)\) and \((M_2, d_2)\) be metric spaces. We will call a bijection \(\phi : M_1 \to M_2\) homothety if there exists \(t \in \mathbb{R}\) such that for any words \(u, v \in \mathbb{L}\)

\[d_2(\phi(u), \phi(v)) = td_1(u, v)\]

We will call a bijection \(\phi : M_1 \to M_2\) isometry if for any words \(u, v \in \mathbb{L}\)

\[d_2(\phi(u), \phi(v)) = d_1(u, v)\]

The set \(\text{Isom}_d(M)\) of all isometries of the metric space \((M, d)\) into itself forms a composition group. Moreover, the isometry group of a metric space is always isomorphic to the isometry group of its image under homothety.

**Proposition 2.** Let \(d\) be a generalized Levenshtein metric and \(L\) an arbitrary language. Then there exists \(\theta \in (0; 2]\) such that the trivial mapping \((L, d)\) to \((L, \text{lev}_{1,\theta})\) is a homothety.

**Proof.** As \(\theta\) we can take \(\min(\frac{2}{\theta}, 2)\)

We will denote the distance \(\text{lev}_{1,\theta}\) as \(\text{lev}_\theta\), and the isometry group of the language \(L\) with respect to it as \(\text{Isom}_\theta(L)\).

The class of isometry groups of languages over a one-element alphabet is rather small.

**Proposition 3.** Let \(L \subset \{a\}^*, \theta \in (0; 2]\). Then \(\text{Isom}_\theta(L)\) is either trivial or isomorphic to a cyclic group of order 2.

**Proof.** \(l : a^n \mapsto n\) is an isometry of \((\{a\}^*, \text{lev}_\theta)\) onto the metric space \((\mathbb{N}, d(m, n) = |n - m|)\).

From this we can conclude that \(\text{Isom}(L) \cong \text{Isom}(l(L))\).

Suppose \(N \subset \mathbb{N}, |N| \geq 2\) (everything is obvious for \(|N| = 1\)). Let \(n_0, n_1, ...\) be elements of \(N\) sorted in ascending order, \(\phi\) be an isometry from \(N\).

\(\phi(n_0) \in \{n_0, n_{|N|}\}\). Otherwise, if \(\phi(n_0) = n_i \neq n_{|N|}\), then

\[|n_{i+1} - n_{i-1}| = |\phi^{-1}(n_{i+1}) - \phi^{-1}(n_{i-1})| < \]

\[< |\phi^{-1}(n_{i+1}) - n_0| + |n_0 - \phi^{-1}(n_{i-1})| = \]

\[= |(n_{i+1} - n_i) + (n_i - n_{i-1})| = |n_{i+1} - n_{i-1}|\]

It’s impossible.

Let \(\phi(n_0) = n_0\). Let us prove by induction that \(\phi\) is trivial:
Base: For \( N = 0 \) \( \phi(n_0) = n_0 \) (no other items).

Step: suppose that \( \phi(n_i) = n_i \) for all \( i < k \). Then \( n_k \) is the only nearest element to \( n_{k-1} \) not contained in \( \{ n_0, \ldots, n_{k-1} \} \). Therefore, \( \phi(n_k) \) is the only nearest element to \( \phi(n_{k-1}) = n_{k-1} \) not contained in \( \phi(\{ n_0, \ldots, n_{k-1} \}) = \{ n_0, \ldots, n_{k-1} \} \), i.e. \( \phi(n_k) = n_k \).

Let \( \phi(n_0) = n_{|N|} \). Let us prove by induction that \( \phi : n_k \mapsto n_{|N|-k} \).

Base: For \( N = 0 \) \( \phi(n_0) = n_{|N|} \) (no other elements).

Step: suppose that \( \phi(n_i) = n_{|N|-i} \) for all \( i < k \). Then \( n_k \) is the only nearest element to \( n_{k-1} \) not contained in \( \{ n_0, \ldots, n_{k-1} \} \). Therefore, \( \phi(n_k) \) is the only nearest element to \( \phi(n_{k-1}) = n_{|N|-k+1} \) not contained in \( \phi(\{ n_0, \ldots, n_{k-1} \}) = \{ n_{|N|}, \ldots, n_{|N|-k+1} \} \), i.e. \( \phi(n_k) = n_{|N|-k} \).

Thus, there can be no other isometries, except for these two (and the second one is far from always realized). Hence \( Isom(L) \) is either trivial or isomorphic to a cyclic group of order 2.

However, already for the two-element alphabet, the isometry groups are more complicated, which will be demonstrated in the following sections.

3 Proof of Theorem 1

**Lemma 1 (2).** There is no infinite language over a finite alphabet in which no word occurs as a subsequence of another.

We will denote by \( M(L) \) the set of all minimal words in the language \( L \) with respect to the inclusion order as a subsequence.

**Lemma 2.** Let \( L \) be a formal language, \( \theta \in (0; 2) \). Then if \( Isom_\theta(L) \) acts transitively on \( L \), then \( L \) is finite.

**Proof.** Let \( L \) be an infinite language. \( M(L) \) is finite by Lemma 1. Therefore, by the Dirichlet principle, \( L \) has an infinite sublanguage with a unique minimal word. Let’s denote it as \( L_0 \) and the only minimal word in it as \( w_0 \). Language \( \{ w \in L_0 | |w| > \frac{1}{\theta} |w_0| \} \) is also infinite. So, arguing similarly, it also has an infinite sublanguage with a single minimal word. Let’s denote it as \( L_1 \) and the only minimal word in it as \( w_1 \). Language \( \{ w \in L_1 | |w| > 2|w_1| \} \) is also infinite. So, arguing similarly, it also has an infinite sublanguage with a single minimal word. Let’s denote it as \( L_2 \), and the only minimal word in it as \( w_2 \).

Now suppose that \( Isom_\theta(L) \) acts transitively on \( L \). Then there exists \( \phi \in Isom_\theta(L) \) such that \( \phi(w_1) = w_0 \). Then the chain of inequalities is fulfilled:

\[
lev_\theta(w_0, w_1) + lev_\theta(w_1, w_2) = lev_\theta(w_0, w_2) = lev_\theta(\phi(w_0), \phi(w_2)) \leq \max(|\phi(w_0)|, |\phi(w_2)|) + (\theta - 1) \min(|\phi(w_0)|, |\phi(w_2)|) \leq \theta|w_0| + \max(lev_\theta(w_0, \phi(w_0)), lev_\theta(w_0, \phi(w_0))) \leq \theta|w_0| + \max(lev_\theta(w_1, w_0), lev_\theta(w_1, w_2)) + (\theta - 1) \min(lev_\theta(w_0, \phi(w_0)), lev_\theta(w_0, \phi(w_2))) = \theta|w_0| + \max(lev_\theta(w_1, w_0), lev_\theta(w_1, w_2)) + (\theta - 1) \min(lev_\theta(w_1, w_0), lev_\theta(w_1, w_2)) = \theta|w_0| + |w_2| - |w_1| + (\theta - 1)\min(|w_1| - |w_0|) < lev_\theta(w_0, w_1) + lev_\theta(w_1, w_2)
\]

Contradiction.

Let now \( m = \max\{|\phi(v)| - |v| |v \in M(L), \phi \in Isom_\theta(L)\} \) (this set is finite by Lemmas 1 and 2). Let us prove by contradiction that \( |w| - |\phi(w)| \leq m \).

Let \( |w| - |\phi(w)| > m \). Without loss of generality, we assume that \( |w| < |\phi(w)| \). Now let \( u \in M(L) \) be contained in \( w \) as a subsequence.

Then

\[
lev_\theta(w, u) = lev_\theta(\phi(w), \phi(u)) \geq |\phi(w)| - |\phi(u)| > |w| + m - |\phi(u)| \geq |w| + |\phi(u)| - |u| - |\phi(u)| = |w| - |u| = lev_\theta(w, u)
\]

Contradiction.

4
Corollary 1. The isometry group of any formal language $L$ embeds in $\Pi_{n=1}^\infty S_n$.

Proof. Let $n_1, n_2, \ldots$ be the orders of the orbits of the natural action of $\text{Isom}(L)$ on $L$ (the orbits are finite by Lemma 2). Since $\text{Isom}(L)$ acts effectively on $L$, it embeds in $\Pi_{n=1}^\infty S_n \leq \Pi_{n=1}^\infty S_n$. $\square$

Proposition 4. There is a regular language $L \subset \{0,1\}^*$ such that the isometry group $\text{Isom}_2(L) \cong D_\infty$, and it acts transitively on $L$.

Proof. Consider a regular language $L = 1^* \cup 0^*$ and a bijection $\phi : L \to \mathbb{Z}$ given by the formulas

$$\phi(0^n) = n$$
$$\phi(1^n) = -n$$

for all $n \in \mathbb{N}_0$.

Let us show that $\phi$ is an isometry of the spaces $(L, \text{lev}_2)$ and $(\mathbb{Z}, d(x,y) = |x-y|)$. Really

$$\text{lev}_2(0^n,0^m) = |m-n|$$
$$\text{lev}_2(1^n,1^m) = |m-n|$$
$$\text{lev}_2(0^n,1^m) = m+n$$

for all $n, m \in \mathbb{N}_0$.

Moreover, the isometry group of the metric space $(\mathbb{Z}, d(x,y) = |x-y|)$ acts transitively on it and is isomorphic to $D_\infty$. $\square$

4 Word stretching

We will denote the Hamming distance as $h$.

Definition 5. The word stretching is the operation $st : A^* \times A^* \to A^*$ given recursively:

$$st(\Lambda, w_2) = \Lambda$$
$$st(w_1a, w_2) = st(w_1, w_2)w_2a$$

for all $w_1, w_2 \in A^*$, $a \in A$.

Lemma 3. Let the words $w_1, w_2 \in A^*$, where $|w_1| = |w_2|$.

Let $k \in \mathbb{N}$, where $k > h(w_1, w_2)$.
Let the symbols $a, b \in A$, and $a \neq b$.
Let $\theta \in (0;2]$.
Then the equality holds:

$$\text{lev}_0(st(w_1, a^kba^k), st(w_2, a^kba^k)) = h(w_1, w_2)$$

Proof. Let us show that there is a minimal $i\text{path}_{ab}^k$ from $st(w_1, a^kba^k)$ to $st(w_2, a^kba^k)$ that does not contain insertions or deletions.

Indeed, suppose that the shortest $i\text{path}_{ab}^k$ contains a deletion. $k$ deletions cannot contain this $i\text{path}_{ab}^k$ (otherwise it will not be the shortest one). This means that the $i\text{segment}_{ab}^k$ of the word between this deletion and one of the insertions closest to it (on one side or the other) is shifted by a distance less than $k$.

At the same time, at positions that are not multiples of $k+1$ (numbering starts from 1), there are no symbols different from $a$, and at positions comparable to $k+1$ modulo $2k + 2$ there are no symbols different from $b$.

So, if there are $t$ significant symbols on this $i\text{segment}_{ab}^k$, then there are also $t-1$ $i\text{central}_{ab}^k$ $b$ symbols on it that turned out to be shifted. It would take $2(t-1)$ replacement operations to put them in order. This means that the total cost of operations on the interval (including the initial deletion and insertion) will be at least $2(t-1)\theta + 2$. In this case, element-by-element replacement of all significant characters on the segment without any insertions and deletions will cost $t\theta \leq 2(t-1)\theta + 2$.

So, gradually changing the pairs $i\text{insert} / \text{delete}_{ab}^k$ to replace significant characters, we get the desired $i\text{path}_{ab}^k$. $\square$
5 Proof of Theorem 2

Lemma 4. Let $\Gamma(V, E)$ be a finite simple cubic graph. Then there exists a language $\{w_v\}_{v \in V} \subset \{0, 1\}^{[E]}$ such that

$$h(w_u, w_v) = \begin{cases} 
4 & \{u, v\} \in E \\
6 & (u, v) \not\in E 
\end{cases}$$

Proof. Let $E = \{e_0, ..., e_m\}$. Let’s define $i : V \times E \to \{0, 1\}^*$ as follows:

$$i(v, e) = \begin{cases} 
1 & v \in e \\
0 & v \not\in e 
\end{cases}$$

Let’s define the word $w_u = i(v_1)e_0i(v_2)e_1...i(v_iE_i)$.

It is easy to see that the Hamming distances between words turned out to be as required. □

Theorem 7. ([1]) A group is finite if and only if it is isomorphic to the automorphism group of some cubic graph.

Let $G$ be a finite group, $\Gamma(E, V)$ be the cubic graph corresponding to it by Theorem 7, $L_0$ be the language constructed for $\Gamma$ by Lemma 2, $L_1 = st(L_0, 1^701^7)$.

Since $diam(L_1) = 6$, by Lemma 1, $Isom(G, L_1) \cong G$. The length of words in $L_1$ is equal to $16|E| = 24|V| = 24|L_1|$. □

6 Proof of Theorem 3

Let $G_1, G_2, ...$ be a countable sequence of finite groups. $L_1, L_2, ...$ are the corresponding languages constructed by Corollary 1.

We construct the languages $L'_1, L'_2, ...$ using recursion:

$$L'_0 = \{\Lambda\}$$

$$L'_{n+1} = (01)^{l_n+7}L_{n+1}$$

where $l_n$ is the length of words in $L'_n$.

It is easy to see that the isometry group $L'_n$ coincides with the isometry group $L_n$. Moreover, for any words $w_n \in L'_n$ and $w_m \in L'_m$, $m > n$ let $\theta(w_n, w_m) = l(L'_m) - l(L'_n)$. This equality is true because, to get a shorter word from a longer one, it is enough to remove everything except the $(01)^{l(L_n)}$ prefix, and then select the correct character from each 01 block.

It follows from this that for any sequence $\phi_1, \phi_2, ...$ of isometries of languages $L'_1, L'_2, ...$ it is true that $\phi : w_n \mapsto \phi(w_n)$ for all $n \in \mathbb{N}$, $w_n \in L'_n$ is an isometry of $\bigcup_{n=1}^{\infty} L'_n$.

Moreover, $\bigcup_{n=1}^{\infty} L'_n$ has no other isometries. Indeed, $\Lambda$ goes into itself as the only word without neighbors at a distance of 4 or 6. So the length of words in this language (that is, the distance to $\Lambda$) is an invariant.

In other words, $Isom_\theta(\bigcup_{n=1}^{\infty} L'_n) = \Pi_{n=1}^{\infty} L'_n = \Pi_{n=1}^{\infty} L_n = \Pi_{n=1}^{\infty} G_n$.

$\bigcup_{n=1}^{\infty} L'_n$ has growth $O(n)$ since for any $n \in \mathbb{N}$ the length of all words in $L'_n$ is greater $24|L_n|$, which means $|\{w \in L_n|w| \leq k\}| \leq 1 + \frac{k}{24}$. □

7 Proof of Theorem 4

Let $A = \{a_1, ..., a_k\}$ be an arbitrary alphabet. Let’s build languages $L_1, L_2, ...$ recursively:

$$L_0 = \Lambda$$

$$L_n = (a_1...a_k)^{l(k, n-1)}st(A^{k^n}, a_1^{k^{n+1}} a_2 a_1^{k^{n+1}})$$

where $l(k, n)$ is the length of $L_n$.

Note that $l(k, n) = l(k, n-1) + 2k^n(k^{n+1} + 2) = O(k^{2n})$. In this case, $|L_n| = k^{k^n}$. Hence, the language $\bigcup_{n=1}^{\infty} L_n$
has growth \( \sum_{l(k,t) \leq n} k^{l(t)} = \Theta(k^{\sqrt{n}}) \)
Moreover, it is easy to see that the isometry group \( L_n \) is isomorphic to \( S_k^n \times S_2^n \). Moreover, the distance between any words \( u_n \in L_n', w_m \in L_m \), where \( m > n \), is equal to \( l(L_m) - l(L_n) \), because to get a shorter word from a longer one, it is enough to remove everything except the prefix \((a_1...a_k)^l(L_n)\), and then select the correct character from each \( a_1...a_k \).
It follows from this that for any sequence \( \phi_1, \phi_2, ... \) of isometries of languages \( L_1, L_2, ... \) it is true that \( \phi : w_n \mapsto \phi(w_n) \) for all \( n \in \mathbb{N}, w_n \in L_n \) is an isometry of \( \bigcup_{n=1}^{\infty} L_n \).
Moreover, \( \bigcup_{n=1}^{\infty} L_n \) has no other isometries.
Indeed, \( A \) goes into itself as the only word without neighbors at a distance of 1. So the length of words in this language (that is, the distance to \( A \)) is an invariant.
In other words, \( Isom_\theta((\bigcup_{n=1}^{\infty} L_n) = \Pi_{n=1}^{\infty} Isom_\theta(L_n) = S_k^{\infty} \times \Pi_{n=1}^{\infty} S_k^n \).

### 8 Proof of Theorem 5

**Lemma 5.** Let \( L \subset \{0,1\}^n, \theta \in (0;2] \). Then \( Isom_\theta(L((01)^n)^* ) \cong Isom(L)^N \).

**Proof.** Let \( u, v \in L \) and \( p, q \in \mathbb{N} \). Let us show that

\[
lev_\theta(u(01)^pn, v(01)^qn) = \begin{cases} 2|p - q| & p \neq q \\ d(u,v) & p = q \end{cases}
\]

Indeed, in the first case, a longer word is obtained from a shorter one by removing everything superfluous. In the second case, it is enough to perform transformations only on different prefixes.
From this, in particular, it follows that any transformation of the form \( \phi : u(01)^kn \mapsto \psi(k)(u)(01)^kn \), where \( \psi - \) an arbitrary function \( \mathbb{N} \rightarrow Isom_\theta(L) \), is an isometry of \( (L((01)^n)^*, lev_\theta) \).
To verify that there are no other isometries, note that \( u(01)^kn \) has exactly \( 2|L| \) ”neighbors” at a distance of \( 2tn \), for \( t \leq k \) and exactly \( |L| \) for \( t > k \).

Using Theorem 2, we can construct uniform languages \( L_1 \subset \{0,1\}^n \) and \( L_2 \subset \{0,1\}^m \) such that \( Isom_\theta(L_1) \cong G \), while \( Isom_\theta(L_2) \cong H \).
By Lemma 3 \( Isom_\theta(L_2((01)^m)^*) \cong H^N \).
Now consider the language \( L_1((01)^n+m) L_2((01)^m)^* \). It is easy to see that \( lev_\theta(u, (01)^{n+m} v(01)^{mk}) = 2(n+(k+2)m) \).
Hence, for every \( \phi \in Isom_\theta(L_1) \) and \( \psi \in Isom_\theta(L_2((01)^m)^*) \) the map \( \chi \) taking all \( u \in L_1 \) into \( \phi(u) \), and all \( (01)^{n+m} v, \) where \( v \in L_2((01)^m)^* \), is an isometry of \( L_1 \cup (01)^{n+m} L_2((01)^m)^* \).
Moreover, there can be no other isometries, since the word \( v \) from the new language belongs to \( L_1 \) if and only if it has no ”neighbors” at a distance greater than \( n \) but less than \( n + 2m \).

So \( Isom_\theta(L_1 \cup (01)^{n+1} L_2((01)^m)^*) \cong G \times H^N \). Moreover, \( L_1 \cup (01)^{n+1} L_2((01)^m)^* \) is regular by construction.

### 9 Proof of Theorem 6

Consider the language \( L = (01)^*(110)(010)^* \cap \{0,1\}^6)^* \). Let us show that for arbitrary two words \( u, w \in L \) such that \( |u| - |v| = 6, v \) can be obtained from \( u \) with exactly 6 deletions. Note that at positions dividing 3 (we assume the numbering of positions starts from 0), each word can contain only one unit. Let these special units in the words \( u \) and \( v \) be at positions 3i and 3j, respectively. Then, if \( i = j \), it suffices to remove the suffix of length 6 (of the form 010010) from \( u \). Otherwise, you must first remove the subword \( u_{3i} u_{3i+1} u_{3i+2} = 110 \) from \( u \), and then from the subword \( u_{3j} u_{3j+1} u_{3j+2} u_{3j+3} = 0100 \) (numbering in the new order after deletion) remove \( u_{3j} \) and \( u_{3j+2} u_{3j+3} \). Thus we get \( v \).
From the above it follows by induction that for \( u, v \in L \)

\[
lev_\theta(u, v) = \max(||u| - |v||, 2)
\]

Thus we see that the distance between words does not depend on anything other than their length. This means that for any sequence of permutations \( \sigma_t \) of elements \( L \cap A^6 \) the map \( \phi : u \mapsto \sigma_{|u|}(u) \) is an isometry.

At the same time, the absence of other isometries follows from the fact that each word \( u \in L \) has exactly \( \frac{2|u|}{3} \) neighbors at a distance of 6.
That is, since \( |L \cap A^6| = 2i, Isom_\theta(L) \cong \Pi_{n=1}^{\infty} S_2^n \).
10  Thanks

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