A dilaton-pion mass relation

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Recently, Golterman and Shamir presented an effective field theory which is supposed to describe the low-energy physics of the pion and the dilaton in an $SU(N_c)$ gauge theory with $N_f$ Dirac fermions in the fundamental representation. By employing this formulation with a slight but important modification, we derive a relation between the dilaton mass squared $m_{\tau}^2$, with and without the fermion mass $m$, and the pion mass squared $m_{\pi}^2$ to the leading order of the chiral logarithm. This is analogous to a similar relation obtained by Matsuzaki and Yamawaki on the basis of a somewhat different low-energy effective field theory. Our relation reads

$$m_{\tau}^2 = m_{\tau}^2|_{m=0} + KN_f \hat{f}_{\pi}^2 m_{\pi}^2/(2 \hat{f}_{\tau}^2) + O(m_{\pi}^4 \ln m_{\pi}^2)$$

with $K = 9$, where $\hat{f}_{\pi}$ and $\hat{f}_{\tau}$ are decay constants of the pion and the dilaton, respectively. This mass relation differs from the one derived by Matsuzaki and Yamawaki on the points that $K = (3 - \gamma_m)(1 + \gamma_m)$, where $\gamma_m$ is the mass anomalous dimension, and the leading chiral logarithm correction is $O(m_{\pi}^2 \ln m_{\pi}^2)$. For $\gamma_m \sim 1$, the value of the decay constant $\hat{f}_{\tau}$ estimated from our mass relation becomes $\sim 50\%$ larger than $\hat{f}_{\tau}$ estimated from the relation of Matsuzaki and Yamawaki.
1. Introduction

The idea that the spontaneous breaking of a (approximate) dilatational or scale symmetry and the associated (pseudo) Nambu–Goldstone boson (i.e., the dilaton) play a certain role in elementary particle physics dates back to the 1970’s [1–6]. For recent investigations, see for example Refs. [7–14] and references cited therein. This interesting idea has again attracted attention recently as it might provide a natural understanding on the appearance of a flavor-singlet parity-even light meson in the \( N_f = 8 \) SU(3) gauge theory [15, 16].\(^1\) The appearance of such a flavor-singlet light scalar meson is extremely interesting because, combined with the idea of the walking technicolor [20–25], the light scalar meson might be identified with the light Higgs particle. Thus it seems quite interesting if the lightness of the scalar meson can be understood as a consequence of the spontaneous symmetry breaking of a flavor-singlet scalar symmetry: The dilatational symmetry broken by the fermion condensate is a natural candidate.

It is well-recognized, however, that it is not simple to formulate the spontaneous breaking of the dilatation symmetry. The Ward–Takahashi relation associated with the dilatation is almost always intrinsically broken by the trace or conformal anomaly; this implies that one cannot derive the corresponding Nambu–Goldstone theorem. The theories in which the dilatation holds in quantum level, i.e, conformal field theories, do not possess the dynamical mass scale and thus we do not expect the condensate of an order parameter. The notion of the spontaneous breaking of the dilatation and the associated Nambu–Goldstone boson must thus be essentially approximate. If we know a parameter which controls the validity of an approximate symmetry, such as the quark mass for the chiral symmetry in QCD, the notion of the spontaneous symmetry breaking of an approximate symmetry is still quite useful [26–28]. For the dilatation, however, it is not clear at all whether such a useful parameter which controls the magnitude of the trace anomaly exists or not.

Recently, Golterman and Shamir made an interesting proposal on this issue [29]. They take an \( SU(N_c) \) gauge theory with \( N_f \) Dirac fermions in the fundamental representation. If \( N_f \) is within the so-called conformal window, \( N_f(N_c) < N_f < (11/2)N_c \), the theory can be conformal; here, \( N_f(N_c) = 34N_c^3/(13N_c^2 - 3) \) in the two-loop approximation. In Ref. [29], the authors consider confining theories in which \( N_f \) is outside the conformal window \( N_f < N_f(N_c) \) but is very close to the lower boundary of the window, \( N_f \approx N_f(N_c) \). If \( N_f \) is very close to \( N_f(N_c) \), the \( \beta \)-function in the low-energy region at which the chiral symmetry is spontaneously broken might be regarded as very small and the Ward–Takahashi relation associated with the dilatation could be regarded approximately restored; this is the basic idea of Ref. [29]. Further, to introduce a parameter which controls the “closeness” to the window boundary, they consider the Veneziano limit [30] in which \( N_c \to \infty \) while the ratio \( n_f \equiv N_f/N_c \) is kept fixed. Then \( n_f \) may be regarded as a continuous parameter and the difference \( n_f(N_c) - n_f \), where \( n_f \equiv \lim_{N_c \to \infty} N_f(N_c)/N_c \), would be used to parametrize the “smallness” of the dilatational symmetry breaking in quantum theory.

On the basis of the above idea, in Ref. [29], Golterman and Shamir formulated an effective field theory which describes the low-energy physics of the pion and the dilaton in an \( SU(N_c) \)

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\(^1\) Such a flavor-singlet parity-even light state has been observed also in an SU(3) gauge theory with \( N_f = 2 \) Dirac fermions in the symmetric second-rank representation [17]. For recent review on lattice study of many flavor gauge theories, see Refs. [18, 19].
gauge theory with \( N_f \) flavors. It is then interesting to study consequences of the effective theory and compare them with results of the lattice simulation for example, to examine whether the picture of the “spontaneous dilatational symmetry breaking” is physically relevant or not. This is the motivation of the present work.

In the present paper, by employing the formulation of Ref. [29] with an important modification elucidated in Sect. 2.2, we derive a relation between the dilaton mass squared \( m_\tau^2 \), with and without the fermion mass \( m \), and the pion mass squared \( m_\pi^2 \) to the leading order of the chiral logarithm. This relation is analogous to a similar relation obtained by Matsuzaki and Yamawaki in Ref. [31] on the basis of a somewhat different low-energy effective theory. Our relation reads

\[
m_\tau^2 = m_\tau^2 |_{m=0} + K \frac{N_f \hat{f}_\pi^2}{2 \hat{f}_\tau^2} m_\pi^2 + O(m_\pi^4 \ln m_\pi^2),
\]

with \( K = 9 \), where \( \hat{f}_\pi \) and \( \hat{f}_\tau \) are decay constants of the pion and the dilaton, respectively. Our mass relation differs from the one derived by Matsuzaki and Yamawaki on the points that \( K = (3 - \gamma_m)(1 + \gamma_m) \), where \( \gamma_m \) is the mass anomalous dimension, and the leading chiral logarithm correction is \( O(m_\pi^2 \ln m_\pi^2) \). The relation in Ref. [31] has already been utilized to estimate the dilaton decay constant \( \hat{f}_\tau \) from lattice data [15]. For \( \gamma_m \sim 1 \), the value of the decay constant \( \hat{f}_\tau \) estimated from our mass relation becomes \( \sim 50\% \) larger than \( \hat{f}_\tau \) estimated from the relation of Matsuzaki and Yamawaki. We hope that our mass relation will be examined by lattice simulations in the future in the parameter region in which the finite volume effect is well under control.

We basically follow the notation of Ref. [29].

2. Derivation of the mass relation

2.1. Microscopic theory

As Ref. [29], our microscopic theory is an \( SU(N_c) \) gauge theory with \( N_f \) Dirac fermions in the fundamental representation. We assume the dimensional regularization with the spacetime dimension \( D = 4 - 2\epsilon \), which will be especially useful in what follows. Thus we set \( S = \int d^D x \mathcal{L}(x) \), where the Lagrangian density is

\[
\mathcal{L}(x) \equiv \frac{1}{4g_0^2} F_{\mu\nu}^a(x) F_{\mu\nu}^a(x) + \bar{\psi}(x)(\slashed{D} + m_0)\psi(x),
\]

where \( \slashed{D} = \gamma_\mu D_\mu \) and \( D_\mu = \partial_\mu + A_\mu \).

To constrain the possible form of the low-energy effective theory, following Ref. [29] (see also Ref. [32]), we introduce spurious fields, \( \chi(x) \) which is an \( N_f \times N_f \) matrix field and \( \sigma(x) \in \mathbb{R} \), corresponding to the chiral symmetry and the dilatational symmetry, respectively. The action is thus modified as

\[
S = \int d^D x e^{(D-4)\sigma(x)} \left\{ \frac{1}{4g_0^2} F_{\mu\nu}^a(x) F_{\mu\nu}^a(x) + \bar{\psi}(x) \slashed{D} \psi(x) + \bar{\psi}(x) \left[ \chi(x) P_R + \chi(x) P_L \right] \psi(x) \right\},
\]

(2.2)

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so that it is invariant under the $SU(N_f)_L \times SU(N_f)_R$ chiral transformation and the dilatation. The former is given by $(g_L, g_R \in SU(N_f))$

$$\psi(x) \rightarrow (g_R P_R + g_L P_L)\psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x)(P_L g_R^\dagger + P_R g_L^\dagger),$$

$$\chi(x) \rightarrow g_L \chi(x) g_R^\dagger,$$  \hspace{1cm} (2.3)

and other fields are kept intact. The latter is realized by $(\lambda \in \mathbb{R}_+)$

$$A_\mu(x) \rightarrow \lambda A_\mu(\lambda x), \quad \bar{\psi}(x) \rightarrow \lambda^{3/2} \bar{\psi}(\lambda x),$$

$$\psi(x) \rightarrow \lambda^{3/2} \psi(\lambda x), \quad \chi(x) \rightarrow \lambda \chi(\lambda x), \quad \sigma(x) \rightarrow \sigma(\lambda x) + \ln \lambda.$$  \hspace{1cm} (2.4)

These symmetries are of course spurious and, going back to the original theory (2.1) by setting,

$$\sigma(x) = 0, \quad \chi(x) = m_0 \mathbb{1},$$  \hspace{1cm} (2.5)

the symmetries are explicitly broken. Still, these spurious symmetries are quite useful to determine the possible form of the corresponding effective theory.

Now, one of the crucial relations for our argument is

$$\delta \chi(x) S \bigg|_{\sigma(x) = 0, \chi(x) = m_0 \mathbb{1}} = m_0 \bar{\psi}(x) \psi(x),$$  \hspace{1cm} (2.6)

where we have introduced the notation

$$\delta \chi(x) \equiv \text{Re} \chi_{ij}(x) \frac{\delta}{\delta \text{Re} \chi_{ij}(x)} + \text{Im} \chi_{ij}(x) \frac{\delta}{\delta \text{Im} \chi_{ij}(x)}.$$  \hspace{1cm} (2.7)

In terms of the generating functional of connected correlation functions $W$ corresponding to $S$ (2.2), Eq. (2.6) says that

$$\langle m_0 \bar{\psi}(x) \psi(x) \rangle = \delta \chi(x) W \bigg|_{\sigma(x) = 0, \chi(x) = m_0 \mathbb{1}}.$$  \hspace{1cm} (2.8)

In both sides of this expression, we can assume the presence of source fields for gauge invariant operators.

Another basic relation, which can be obtained from a result of Ref. [33], is

$$\frac{\delta}{\delta \sigma(x)} S \bigg|_{\sigma(x) = 0, \chi(x) = m_0 \mathbb{1}} = (D - 4) \mathcal{L}(x) - \partial_{\mu} S_{\mu}(x) - m_0 \bar{\psi}(x) \psi(x),$$  \hspace{1cm} (2.9)

which holds in correlation functions containing gauge invariant operators only. We note that the combination $m_0 \bar{\psi}(x) \psi(x)$ is ultraviolet finite. In this expression, $S_{\mu}(x)$ denotes the
dilatation current, defined by

\[ S_\mu(x) \equiv x_\mu \left[ T_{\mu\nu}(x) + \frac{D-1}{D} \delta_{\mu\nu} \bar{\psi} \left( \frac{1}{2} \overrightarrow{\partial} + m_0 \right) \psi(x) \right] \tag{2.10} \]

from the energy–momentum tensor, whose definition in Ref. [33] is

\[ T_{\mu\nu}(x) \equiv \frac{1}{g_0^2} \left[ F^a_{\mu\nu}(x) F^a_{\nu\sigma}(x) - \frac{1}{4} \delta_{\mu\nu} F^a_{\rho\sigma}(x) F^a_{\rho\sigma}(x) \right] + \frac{1}{4} \bar{\psi}(x) \left( \gamma_\mu \overrightarrow{D}_\nu + \gamma_\nu \overrightarrow{D}_\mu \right) \psi(x) - \delta_{\mu\nu} \bar{\psi}(x) \left( \frac{1}{2} \overrightarrow{\partial} + m_0 \right) \psi(x), \tag{2.11} \]

where

\[ \overrightarrow{D}_\mu \equiv \partial_\mu - A_\mu. \tag{2.12} \]

The last term in Eq. (2.10), which is proportional to the equation of motion of the fermion fields, is added so that \( \partial_\mu S_\mu(x) \) generates the dilatation transformation on the fermion fields with the scaling dimension \((D-1)/2\) through the Ward–Takahashi (WT) relation.

From Eqs. (2.6) and (2.9), assuming the limit \( D \to 4 \), we infer that

\[ \partial_\mu S_\mu(x) = - \left[ \frac{\delta}{\delta \sigma(x)} + \delta_\chi(x) \right] S \bigg|_{\sigma(x)=0, \chi(x)=m_0}, \tag{2.13} \]

or, in terms of the generating functional of the connected correlation functions \( W \),

\[ \langle \partial_\mu S_\mu(x) \rangle = - \left[ \frac{\delta}{\delta \sigma(x)} + \delta_\chi(x) \right] W \bigg|_{\sigma(x)=0, \chi(x)=m_0}. \tag{2.14} \]

This is a fundamental relation for our argument.

### 2.2. Low-energy effective theory

Next, we consider a low-energy effective field theory along the line of reasoning in Ref. [29]. We assume that the low-energy degrees of freedom are represented by the pion field \( \Sigma(x) \in SU(N_f) \) and by the dilaton field \( \tau(x) \in \mathbb{R} \). Under the \( SU(N_f)_L \times SU(N_f)_R \) chiral transformation, these field are transformed as

\[ \Sigma(x) \to g_L \Sigma(x) g_R^\dagger, \quad \tau(x) \to \tau(x), \tag{2.15} \]

and, under the dilatation,

\[ \Sigma(x) \to \Sigma(\lambda x), \quad \tau(x) \to \tau(\lambda x) + \ln \lambda. \tag{2.16} \]
Thus, remembering the transformation laws of the spurion fields in Eqs. (2.3) and (2.4), the most general form of an invariant action to the order $p^2 \sim m$ is given by [29]²

$$\tilde{S} = \int d^D x \left\{ \frac{f_\pi^2}{4} V_\pi(\tau(x) - \sigma(x)) e^{2\tau(x)} \text{tr} \left[ \partial_\mu \Sigma(x)^\dagger \partial_\mu \Sigma(x) \right] 
+ \frac{f_\tau^2}{2} V_\tau(\tau(x) - \sigma(x)) e^{2\tau(x)} \partial_\mu \tau(x) \partial_\mu \tau(x) 
- \frac{f_\pi^2 B_\pi}{2} V_M(\tau(x) - \sigma(x)) e^{y_\tau(x)} \text{tr} \left[ \Sigma(x) + \Sigma(x)^\dagger \right] 
+ f_\tau^2 B_\tau V_d(\tau(x) - \sigma(x)) e^{4\tau(x)} \right\}. \quad (2.17)$$

In this expression, the functions $V_I(\tau)$ ($I = \pi$, $\tau$, $M$, and $d$) cannot be determined from the invariance of the action alone [29]. Here, we have assumed that the action is polynomial in the spurion field $\chi(x)$. Otherwise, the term such as $V_\chi(\tau(x) - \sigma(x)) e^{3\tau(x)} \text{tr} [\chi(x)^\dagger \chi(x)]^{1/2}$ must be taken into account.

In Eq. (2.17), we have multiplied the bare spurious field $\chi(x)$ by the mass renormalization factor $Z_m$, defined by (we set $D = 4 - 2\epsilon$)

$$m \equiv Z_m(g) m_0, \quad g_0^2 \equiv \mu^{2\epsilon} g^2 Z(g), \quad (2.18)$$

where $m$ and $g$ are the renormalized mass and gauge coupling, respectively (for definiteness, we have assumed the minimal subtraction (MS) scheme with the renormalization scale $\mu$), so that $Z_m(\chi(x)) = Z_m m_0 1 = m 1$ becomes a ultraviolet finite quantity. Note that in our renormalization scheme (2.18), the renormalization constant $Z_m$ is independent of the spurious field $\sigma(x)$. Then, for the action (2.17) to be invariant under Eqs. (2.4) and (2.16), the parameter $y$ in the third line of Eq. (2.17) must be

$$y = 3. \quad (2.19)$$

Eq. (2.19) is also required from the equivalence of the effective theory (2.17) and the microscopic theory (2.2). Consider the total divergence of the dilatation current $\partial_\mu S_\mu(x)$ in the effective theory, which must reproduce the relation (2.14) for the generating functional $W$. As computed in Appendix D of Ref. [29], for the action (2.17), the Noether method for the dilatation (2.16) yields

$$\partial_\mu S_\mu(x) \big|_{\sigma(x) = 0, \chi(x) = m_0 1} = \frac{f_\pi^2}{4} V_\pi(\tau(x)) e^{2\tau(x)} \text{tr} \left[ \partial_\mu \Sigma(x)^\dagger \partial_\mu \Sigma(x) \right] + \frac{f_\tau^2}{2} V_\tau(\tau(x)) e^{2\tau(x)} \partial_\mu \tau(x) \partial_\mu \tau(x) 
- \frac{f_\pi^2 B_\pi m}{2} V_M(\tau(x)) e^{y_\tau(x)} \text{tr} \left[ \Sigma(x) + \Sigma(x)^\dagger \right] 
+ f_\tau^2 B_\tau V_d(\tau(x)) e^{4\tau(x)} 
+ (4 - y) \frac{f_\pi^2 B_\pi m}{2} V_M(\tau(x)) e^{y_\tau(x)} \text{tr} \left[ \Sigma(x) + \Sigma(x)^\dagger \right] 
= - \left[ \frac{\delta}{\delta \sigma(x)} \right] \left. \tilde{S} \right|_{\sigma(x) = 0, \chi(x) = m_0 1}, \quad (2.20)$$

²If we require the dilatation invariance in $D$ dimensions, the Lagrangian must be multiplied by the factor $e^{-2\epsilon \tau(x)}$, where $D = 4 - 2\epsilon$. This “evanescent factor” contributes, through ultraviolet divergences, from the one-loop order; its effect on the mass relation is however $O(m_0^4)$ and is higher order in our present treatment. It appears interesting to study a possible effect of this evanescent factor.
where we have used Eq. (2.18). This implies

$$
\left\langle \partial \mu S_\mu(x) \big|_{\sigma(x)=0,\chi(x)=m_0} \right\rangle = - \left[ \frac{\delta}{\delta \sigma(x)} + (4 - y) \delta \chi(x) \right] W \bigg|_{\sigma(x)=0,\chi(x)=m_0} \ , \quad (2.21)
$$

which coincides with our basic relation (2.14) if \( y = 3 \).

Here, we note that in Ref. [29], the parameter \( y \) in the third line of Eq. (2.17) is taken as \( y = 3 - \gamma_m \),

$$
\text{(2.22)}
$$

where \( \gamma_m \) is the mass anomalous dimension, defined by

$$
\gamma_m(g) \equiv - \mu \frac{\partial}{\partial \mu} \ln m \bigg|_{g_0, m_0} = - \mu \frac{\partial}{\partial \mu} g \bigg|_{g_0} \frac{d}{dg} \ln Z_m(g) . \quad (2.23)
$$

The reasoning which leads to Eq. (2.22) is elucidated in detail in a recent reference, Ref. [34]; to in our language, it corresponds to a different renormalization scheme which involves the constant mode of \( \sigma(x) \), \( \sigma_0 \) through the relations,

$$
\tilde{m} \equiv Z_m(\tilde{g})m_0, \quad g_0^2 \equiv e^{-2\epsilon \sigma_0} \mu^2 \tilde{g}^2 Z(\tilde{g}) , \quad (2.24)
$$

where the functions \( Z_m(g) \) and \( Z(g) \) themselves are identical to those in Eq. (2.18). Note that the schemes in Eqs. (2.18) and (2.24) coincide for \( \sigma_0 = 0 \). In this scheme, we have

$$
\frac{\partial}{\partial \sigma_0} \ln Z_m(\tilde{g}(e^{\epsilon \sigma_0} \mu^{-\epsilon} g_0)) \bigg|_{g_0, \mu} = - \frac{\partial}{\partial \mu} \ln Z_m(\tilde{g}(e^{\epsilon \sigma_0} \mu^{-\epsilon} g_0)) \bigg|_{\sigma_0, g_0} = - \frac{\partial}{\partial \tilde{g}} \bigg|_{\sigma_0, g_0} \frac{d}{d\tilde{g}} \ln Z_m(\tilde{g}) = \gamma_m(\tilde{g}) = \gamma_m(g) , \quad (2.25)
$$

where we have noted Eqs. (2.24) and (2.23). This shows that the mass renormalization factors in the above two schemes are related as

$$
Z_m(\tilde{g}) = e^{\gamma_m(g)\sigma_0} Z_m(g) \quad (2.26)
$$

at \( D = 4 \). Thus, if we use this scheme, the third line of Eq. (2.17) would become

$$
- \frac{f_2^2 B_\pi}{2} V_M(\tau(x) - \sigma(x)) e^{y \tau(x)} e^{\epsilon \gamma_m(x)} \left[ Z_m(g) \chi(x)^\dagger \Sigma(x) + \Sigma(x)^\dagger Z_m(g) \chi(x) \right] 
= - \frac{f_2^2 B_\pi}{2} V_M(\tau(x) - \sigma(x)) e^{\epsilon \gamma_m[\tau(x) - \sigma(x)]} e^{y + \gamma_m \tau(x)} 
\times \left[ Z_m(g) \chi(x)^\dagger \Sigma(x) + \Sigma(x)^\dagger Z_m(g) \chi(x) \right] . \quad (2.27)
$$

Here, we have set \( \sigma_0 \to \sigma(x) \) as it would be justified in the lowest order of the derivative expansion. Thus, in this scheme, the invariance of the effective theory under Eqs. (2.4) and (2.16) requires Eq. (2.22). \footnote{It seems difficult, however, to impose the invariance for \( D \neq 4 \) in this scheme, because \( \gamma_m(\tilde{g}) \) depends on \( \sigma(x) \) for \( D \neq 4 \).}
The difference between Eqs. (2.19) and (2.22) has, however, no physical relevance at this stage because, as Eq. (2.27) shows, the factor $e^{\gamma(x)\tau(x)-\sigma(x)}$ that comes from the difference can be absorbed into the definition of the yet undermined function $V_M(\tau(x)-\sigma(x))$. Only when we make a certain choice on $V_M$, the difference between Eqs. (2.19) and (2.22) does matter. Here, following the proposal of Ref. [29], we set

$$V_\pi(\tau) = V_\tau(\tau) = V_M(\tau) = 1, \quad V_d(\tau) = c_0 + c_1 \tau. \quad (2.28)$$

These might be regarded as the leading order approximation in the Veneziano limit with the tuning $N_f \rightarrow N_f^*(N_c)$, where $N_f^*(N_c)$ is the number of flavors at the lower boundary of the conformal window [29]. Then the crucial question is that which of the representations, Eq. (2.17) or Eq. (2.27), is more appropriate for the reasoning which leads to $V_M = 1$. Recalling the basic reasoning in Ref. [29] that the term with the lowest powers of $\sigma(x)$ becomes the leading term in the assumed expansion, we think that the representation (2.17) is rather consistent with the choice $V_M = 1$; Eq. (2.27) has additional dependences on $\sigma(x)$ even for $V_M = 1$. This completes the explanation on our choice Eq. (2.19) with $V_M = 1$.

Our low-energy effective theory is thus given by (with Eq. (2.19))

$$\tilde{S} \bigg|_{\sigma(x)=0, \chi(x)=m_0} = \int d^Dx \left\{ \frac{f_\pi^2}{4} e^{2\tau(x)} \text{tr} \left[ \partial_\mu \Sigma(x)^\dagger \partial_\mu \Sigma(x) \right] + \frac{f_\pi^2}{2} e^{2\tau(x)} \partial_\mu \tau(x) \partial_\mu \tau(x) \right. $$

$$- \frac{f_\pi^2 B_\pi m}{2} e^{\eta(x)} \text{tr} \left[ \Sigma(x) + \Sigma(x)^\dagger \right] $$

$$+ \frac{f_\pi^2 B_\tau e^{4\tau(x)}}{2} \left[ c_0 + c_1 \tau(x) \right] \right\}. \quad (2.29)$$

Here and in what follows, the fact that the low-energy constants $f_\pi$, $f_\tau$, $B_\pi$, and $B_\tau$ are independent of the fermion mass $m$ is crucially important. This follows from the chiral symmetry of the underlying action (2.17). That is, the mass parameter $m$ can arise only through the expectation value of the spurion field $\chi(x)$.

### 2.3. Tree level physics

To read off the tree level physics from Eq. (2.29), we set

$$\Sigma(x) = \exp \left[ \frac{2\pi(x)}{f_\pi} \right], \quad \tilde{\pi}(x) = \tilde{\pi}^A(x)t^A, \quad \text{tr}(t^At^B) = -\frac{1}{2} \delta^{AB}, \quad (2.30)$$

and expand the action to yield

$$\tilde{S} \bigg|_{\sigma(x)=0, \chi(x)=m_0} = \int d^Dx \left\{ -e^{2\tau(x)} \text{tr} \left[ \partial_\mu \tilde{\pi}(x) \partial_\mu \tilde{\pi}(x) + m_\pi^2(\tau(x))\tilde{\pi}(x)\tilde{\pi}(x) \right] \right. $$

$$+ \frac{f_\pi^2}{2} e^{2\tau(x)} \partial_\mu \tau(x) \partial_\mu \tau(x) + V(\tau(x)) \right.$$  

$$- \frac{2}{3} \frac{1}{f_\pi^2} e^{2\tau(x)} \text{tr} \left[ \tilde{\pi}(x)^2 \partial_\mu \tilde{\pi}(x) \partial_\mu \tilde{\pi}(x) - \tilde{\pi}(x) \partial_\mu \tilde{\pi}(x) \tilde{\pi}(x) \partial_\mu \tilde{\pi}(x) \right] $$

$$\left. - \frac{1}{3} \frac{m_\pi^2(\tau(x))}{f_\pi^2} e^{2\tau(x)} \text{tr} \left[ \tilde{\pi}(x)^4 \right] + O(\tilde{\pi}^6) \right\}, \quad (2.31)$$

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where
\[ m_\pi^2(\tau) \equiv 2B_\pi m e^{(y-2)\tau}, \quad (2.32) \]

and
\[ V(\tau) \equiv f_\tau^2 B_\tau e^{4\tau}(c_0 + c_1 \tau) - N_f f_\pi^2 B_\pi m e^{y\tau}. \quad (2.33) \]

The minimum of the dilaton potential (2.33) \( w \) is given by \( V'(w) = 0 \) and
\[
w = v + y \frac{N_f f_\pi^2 B_\pi m}{4 c_1 f_\tau^2 B_\tau} e^{(y-4)w} + O(m^2), \quad (2.34)\]

where \( v \) is the potential minimum at the chiral limit \( m \to 0 \),
\[
v = -\frac{1}{4} - \frac{c_0}{c_1}, \quad (2.35)\]

and we have introduced the “physical” parameters in the chiral limit,
\[
\hat{f}_\pi \equiv f_\pi e^v, \quad \hat{f}_\tau \equiv f_\tau e^v, \quad \hat{B}_\pi \equiv B_\pi e^{(y-2)v}, \quad \hat{B}_\tau \equiv B_\tau e^{2v}. \quad (2.36)\]

Then, from Eq. (2.31), the physical pion mass squared \( m_\pi^2 \) is given by
\[
m_\pi^2 = m_\pi^2(w) \equiv 2B_\pi m e^{(y-2)w}. \quad (2.37)\]

Finally, the dilaton mass squared \( m_\tau^2 \) is given by \( V''(w)/(f_\tau^2 e^{2w}) \) and, by using \( V'(w) = 0 \), we have
\[
m_\tau^2 = 4c_1 B_\tau e^{2w} + y(4 - y) \frac{N_f f_\pi^2 B_\pi m}{2 f_\tau^2} e^{(y-2)w} \nonumber \\
= 4c_1 B_\tau e^{2w} + y(4 - y) \frac{N_f f_\pi^2 B_\pi m^2}{2 f_\tau^2} \nonumber \\
= 4c_1 \hat{B}_\tau + y(6 - y) \frac{N_f f_\pi^2}{2 f_\tau^2} m_\pi^2 + O(m^2), \quad (2.38)\]

where Eqs. (2.34) and (2.37) have been used in the last equality. Since \( 4c_1 \hat{B}_\tau \) is independent of the mass parameter \( m \) as already noted, using Eq. (2.19), we obtain the mass relation (1.1) with \( K = 9 \) in the tree level.

It is instructive to see how the derivation of the (tree-level) mass relation in Ref. [31] can be understood in the context of the present low-energy effective theory. The low-energy effective theory in Ref. [31] corresponds to Eq. (2.29) with the following particular choice of parameters (in our notation),
\[
c_0 = -\frac{1}{16} \frac{m_\phi^2}{B_\tau} + y \frac{N_f f_\pi^2 B_\pi m}{4 f_\tau^2 B_\tau}, \quad c_1 = \frac{1}{4} \frac{m_\phi^2}{B_\tau}, \quad (2.39)\]

where \( m_\phi \) is a mass parameter introduced in Ref. [31] which is supposed to be independent of the fermion mass \( m \); the parameter \( y \) is given by Eq. (2.22). The second term in \( c_0 \)
in Eq. (2.39), which depends on the fermion mass \( m \), arises from the additional term in the action,

\[
\int d^D x \frac{1}{4} y f_\pi^2 \hat{B}_\pi e^{4r(x)} \left\{ N_f \text{tr} \left[ \chi(x) \right] \right\}^{1/2}.
\]  

(2.40)

In this setup, thus \( c_0 \) in Eq. (2.39), and consequently \( v \) in Eq. (2.35) depends on the mass \( m \),

\[
v = -y \frac{N_f f_\pi^2 \hat{B}_\pi m}{f_\pi^2 m_\phi^2},
\]  

(2.41)

and we have to expand also the first term of Eq. (2.38) in the mass \( m \) as

\[
4c_1 \hat{B}_\tau = 4c_1 B_\tau e^{2v} = m_\phi^2 - y N_f f_\pi^2 \frac{2}{f_\pi^2} m_\pi^2 + O(m^2).
\]  

(2.42)

Using this in Eq. (2.38), we have

\[
m_\tau^2 = m_\phi^2 + y(4 - y) N_f f_\pi^2 \frac{2}{f_\pi^2} m_\pi^2 + O(m^2).
\]  

(2.43)

With Eq. (2.22), we have Eq. (1.1) with \( K = (3 - \gamma_m)(1 + \gamma_m) \); this reproduces the tree-level mass relation in Ref. [31].

### 2.4. One-loop chiral logarithmic corrections

Next, we study the one-loop radiative corrections to the mass formula (2.38). We will consider only the leading-order chiral log corrections of the form \( m_\pi^2 \ln m_\pi^2 \) and \( m_\pi^4 \ln m_\pi^2 \) which would surpass \( m_\phi^2 \) and \( m_\pi^2 \) in the chiral limit \( m \to 0 \). Since there is no reason that the dilaton becomes massless as \( m \to 0 \), in what follows we will consider only the radiative corrections due to the pion which becomes massless in the chiral limit.

From Eq. (2.31), by the standard method, we have the one-loop corrections to the effective action as

\[
I^{(1)} = \int d^D x \left\{ \frac{1}{(4\pi)^2} \frac{m_\pi^2}{f_\pi^2} \frac{N_f}{3} \left[ -\frac{1}{\epsilon} + \ln \left( \frac{m_\pi^2}{4\pi} \right) + \gamma - 1 \right] \text{tr} [\partial_\mu \tilde{\pi}(x) \partial_\mu \tilde{\pi}(x)] 
\right. 
\left. + \frac{1}{(4\pi)^2} \frac{m_\pi^2}{f_\pi^2} \left( \frac{N_f}{3} - \frac{1}{N_f} \right) \left[ -\frac{1}{\epsilon} + \ln \left( \frac{m_\pi^2}{4\pi} \right) + \gamma - 1 \right] m_\pi^2 \text{tr} [\tilde{\pi}(x) \tilde{\pi}(x)] 
\right. 
\left. + \frac{1}{(4\pi)^2} m_\pi^2 (N_f^2 - 1) \left\{ \frac{3 - y}{2} \left[ -\frac{1}{\epsilon} + \ln \left( \frac{m_\pi^2}{4\pi} \right) + \gamma \right] + \frac{1}{24} (y^2 - 4y - 8) \right\} 
\right. 
\left\times \partial_\mu \tau(x) \partial_\mu \tau(x) 
\right. 
\left. + \frac{1}{(4\pi)^2} \frac{1}{4} (N_f^2 - 1) \left[ m_\pi^2(\tau) \right]^2 \left\{ -\frac{1}{\epsilon} + \ln \left( \frac{m_\pi^2(\tau)}{4\pi} \right) + \gamma - \frac{3}{2} \right\}, \right. 
\]  

(2.44)

up to terms irrelevant for the corrections to the mass formula (2.38) (the function \( m_\pi^2(\tau) \) is defined in Eq. (2.37)). The ultraviolet divergences in this expression are canceled by

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\(^4\)Recall that we eliminated the possibility of such a term by requiring that the low-energy effective theory is polynomial in the spurions.
appropriate invariant counterterms of $O(p^4)$ or $O(m^2)$ (cf. [27]), such as
\[ e^\sigma(x) \left[ \chi(x)^\dagger \Sigma(x) + \Sigma(x)^\dagger \chi(x) \right] \left[ \partial_\mu \Sigma(x)^\dagger \partial_\mu \Sigma(x) \right], \quad (2.45) \]
\[ e^{2\sigma}(x) \left\{ \left[ \chi(x)^\dagger \Sigma(x) + \Sigma(x)^\dagger \chi(x) \right]^2 \right\}, \quad (2.46) \]
\[ e^\sigma(x) \left[ \chi(x)^\dagger \Sigma(x) + \Sigma(x)^\dagger \chi(x) \right] \partial_\mu \tau(x) \partial_\mu \tau(x), \quad (2.47) \]
\[ e^{2\tau(x)} \left[ \chi(x)^\dagger \chi(x) \right]. \quad (2.48) \]

See also the discussion in Ref. [29]. The resulting finite expression then depends on new low-energy constants, the coefficients of those higher dimensional terms in the action. Still, the coefficients of the logarithmic factors are invariant under this renormalization. Thus, after the renormalization, including only the chiral log corrections, we have the effective action to the one-loop order as ($\Lambda$ is a renormalization scale)
\[ \Gamma = \int d^4x \left\{ -e^{2w'} \left[ 1 - \frac{N_f}{3} L(m_\pi^2) \right] \text{tr} \left[ \partial_\mu \tilde{\pi}(x) \partial_\mu \tilde{\pi}(x) \right] 
\]
\[ - e^{2w'} m_\pi^2 (w') \left[ 1 - \left( \frac{N_f}{3} - \frac{1}{N_f} \right) L(m_\pi^2) \right] \text{tr} \left[ \tilde{\pi}(x) \tilde{\pi}(x) \right] \right. 
\]
\[ + \frac{f_\pi^2}{2} e^{2w'} \left[ 1 + (3 - y)r \frac{N_f^2 - 1}{2N_f} L(m_\pi^2) \right] \partial_\mu \tau(x) \partial_\mu \tau(x) \right. 
\]
\[ + V(w') + \frac{1}{(4\pi)^2} \frac{1}{4} (N_f^2 - 1) m_\pi^4 e^{2(y-2)(w'-w')} \ln \left( \frac{m_\pi^2}{\Lambda^2} \right) + O(m_\pi^4) \right\}, \quad (2.49) \]

where
\[ L(m_\pi^2) \equiv m_\pi^2 \ln \left( \frac{m_\pi^2}{\Lambda^2} \right), \quad r \equiv \frac{2N_f f_\pi^2}{f_\pi^2}. \quad (2.50) \]

$w'$ is the minimum of the dilaton potential which is given by the last line of Eq. (2.49). For $m_\pi \to 0$, we have
\[ w' = w - \frac{1}{8\pi^2} (y - 2) r m_\pi^2 \frac{N_f^2 - 1}{2N_f} L(m_\pi^2) + O(m_\pi^4), \quad (2.51) \]

where $w$ is the minimum of the tree-level potential, Eq. (2.34).

Finally, the dilaton mass is given by the second derivative of the potential with a correction factor arising from the wave function renormalization. Taking also the correction to the pion mass into account, we find
\[ m_\tau^2 = m_\tau^2 \bigg|_{m=0} \left[ 1 - (3 - y) r \frac{N_f^2 - 1}{2N_f} L(m_\pi^2) \right] 
\]
\[ + \frac{y(6 - y)}{4} r m_\pi^2 \left\{ 1 - \left[ (3 - y) r \frac{N_f^2 - 1}{2N_f} + \frac{1}{N_f} \right] L(m_\pi^2) \right\} 
\]
\[ - (y - 2)(5 - y) r m_\pi^2 \frac{N_f^2 - 1}{2N_f} L(m_\pi^2) + O(m_\pi^4). \quad (2.52) \]

Now, we notice that the value (2.19) has a special meaning in view of Eq. (2.52). When $y = 3$, the log correction in the first line of Eq. (2.52) vanishes and the leading log correction
becomes $O(m^4 \ln m^2)$ as presented in Eq. (1.1). This logarithmic correction is certainly subdominant compared with $m^2$ in the sense of the conventional chiral perturbation theory. On the other hand, if $y \neq 3$, then the leading log correction becomes enhanced to $O(m^2 \ln m^2)$ as the first line of Eq. (2.52) which might exceed the tree-level quantity $m^2$ in the second line of Eq. (2.52) in the conventional chiral limit. This inversion of the expansion ordering can happen because of the presence of the another mass scale, the dilaton mass $m^2_{\tau}|_{m=0}$, which is not small in the conventional chiral expansion. Although $m^2_{\tau}|_{m=0} = 4|\hat{B}_{\tau}$ may be regarded as a small quantity in the new expansion scheme of Ref. [29], this inversion of the expansion ordering might be troublesome when the relation is applied to fit to the lattice data for example. The above our observation shows that such situation does not occur. In this way, we have obtained Eq. (1.1) with $K = 9$ including the leading chiral logarithm.

3. Conclusion

In the present paper, from the motivation to examine the validity of the physical picture of the “spontaneous dilatational symmetry breaking” in nearly-conformal $SU(N_c)$ gauge theories with $N_f$ flavors, we derived a relation among the dilaton, the pion, and the fermion masses in the chiral limit. We hope that this mass relation will be tested by lattice simulations in the future. Generalization to theories with fermions in higher dimensional gauge representations and supersymmetric theories seems interesting.

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