Nonequilibrium dynamics of strings in time-dependent plane wave backgrounds

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Abstract

We formulate and study the nonequilibrium dynamics of strings near the singularity of the time-dependent plane wave background in the framework of the Nonequilibrium Thermo Field Dynamics (NETFD). In particular, we construct the Hilbert space of the thermal string oscillators at nonequilibrium and generalize the NETFD to describe the coordinates of the center of mass of the thermal string. The equations of motion of the thermal fields and the Hamiltonian are derived. Due to the time-dependence of the oscillator frequencies, a counterterm is present in the Hamiltonian. This counterterm determines the correlation functions in a perturbative fashion. We compute the two point correlation function of the thermal string at zero order in the power expansion.

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1 Introduction

Understanding the dynamics of strings in as large a number of spacetime manifolds as possible represents an essential step in the development of the string theory and the design of new applications and tests of it. In particular, one expects to obtain important informations concerning the semiclassical properties of the gravitational interaction in the string theory such as the backreaction of the matter to the spacetime background, the cosmological singularity and evolution and the initial conditions of the cosmology. Since there are no methods available to explore systematically the full string physics in arbitrary spacetimes (see [1] for a review of some early results), just a few backgrounds in which the above problems can be addressed consistently within the framework of the available semiclassical methods have been studied thoroughly.

The manifolds on which it is possible to approach the string dynamics from the semiclassical point of view have the property of possessing time-like Killing vectors at least locally. They can be generalized to string backgrounds that are formed by the time-dependent spacetimes and massless fields from the closed sector. From these, the distinguished class of time-dependent plane wave backgrounds have received a considerably attention lately. There are at least three reasons for which the plane waves are interesting for the string theory. In the first place, they can be related to cosmological, string and D-brane backgrounds through the Penrose-Güven limit procedure as shown in [2]. Secondly, the plane waves have the interesting property that they generate conformal backgrounds in the string theory in which the strings can be canonically quantized [3, 4]. And thirdly, the techniques developed to quantize the strings in plane wave backgrounds can be applied to string theory in other backgrounds with similar symmetry structure, for example, to the light-like [5] and the Kasner-like backgrounds [6, 7, 8](see for a recent review [9] and the references therein). The symmetries of the plane wave backgrounds allows one to study the problems related to the cosmological singularity in a simpler framework in which the tools of the perturbative string quantization are available.

In order to fully understand the string dynamics near the cosmological singularities or the singularities of the time-dependent plane waves, it is imperative to investigate the thermal aspects of the string theory in the corresponding time-dependent backgrounds. In the static backgrounds, the thermal effects can be studied under the hypothesis of the local thermal equilibrium. Consequently, the thermodynamical quantities of interest can be derived from the statistical partition function of string. On the other hand, the assumption of the thermal equilibrium in the time-dependent plane wave backgrounds leads to the conclusion that the string has inequivalent canonical representations near the singularity usually taken at zero or infinity time and at an arbitrary finite value of time, respectively [10]. In general, the necessity of different theories for different values of the time parameter signals a phase transition in between the two values. However, that is not displayed by the thermal equilibrium dynamics. Extensive analysis of the temperature under the thermal equilibrium hypothesis in time-dependent plane wave backgrounds has been carried out in [11, 12, 13, 14, 15, 16, 17, 18, 19] where it has been shown that what is the equivalent of the Hagedorn temperature in the Minkowski space-time could be interpreted either as a limiting temperature or a critical temperature or even an oscillating modes dependent temperature.

The results obtained so far by using different techniques of the quantum fields at finite temperature point towards the conclusion that the thermal equilibrium is, at best, a hypothesis valid only locally and asymptotically away from the singularity. The very general situation in the time-dependent backgrounds is that the interaction between the gravitational field and the string involves an exchange of energy at a time dependent rate. As a consequence, the
frequencies of the string modes are time-dependent functions of time and the string cannot maintain its thermal equilibrium. The time-dependence of oscillator frequencies lead to sensible effects in the theory. For example, the correlators of the field theory at finite temperature have the following general form

\[ G(x, y) = G_0(x, y) - i \frac{1}{2} \text{sign}_C(t_x - t_y) \rho(x, y), \]

where \( G_0(x, y) \) is the statistical correlator and \( C \) is a closed real-time contour. At thermal equilibrium, \( G_0(x, y) \sim \rho(x, y) \) while in nonequilibrium \( G_0(x, y) \) is no longer proportional to the spectral function \( \rho(x, y) \). Thus, the heuristic arguments as well as the calculations in the equilibrium field theory suggest that, in general, the string ensembles in time-dependent backgrounds are nonequilibrium system.

There are few attempts in the literature to study the nonequilibrium strings even in the simpler backgrounds, mainly due to the inner difficulties of the nonequilibrium field theory. However, since the strings are solvable by canonical methods in the time-dependent plane wave backgrounds one could look for a formulation of strings in terms of nonequilibrium ensembles of oscillators. To our knowledge, the few works that address this problem are doing so in the framework of one approach, known as the Liouville-von Neumann method [21, 22] (based on the generalization of the Lewis-Riesenfeld invariant theorem [23]) has been used recently to map the time-dependent quantum mechanics to the Renormalization Group flow and was applied to the Penrose-Güven limit of stacks of D-branes and of the Pilch-Warner solution in [24]. The same method was used to analyze the string spectrum in the Penrose-Güven limit of the NS5-brane plane wave background in [25] and to construct the density operator of the type IIB superstring in the singular scale-invariant plane wave background in [26] where it was shown that the Hagedorn behavior of strings is the same as in the flat spacetime. However, the results obtained so far are not complete since they provide a simple description at oscillator level without explaining the nonequilibrium dynamics of string fields and the way in which the thermal string observables or correlations should be calculated from it.

In this paper, we propose a new formulation of the nonequilibrium string dynamics in the time-dependent plane wave backgrounds based on the Nonequilibrium Thermo Field Dynamics (NETFD) [27]. The TFD is a real-time canonical operator formalism in which the statistical averages at thermodynamical equilibrium are calculated not by performing the explicit manipulation of the density operator but rather by averaging the corresponding observables in the thermal vacuum that obeys the thermal state condition that is defined on the representation space of the TFD (also called thermal space [27]). The mathematical structure of the thermal space is that of the direct product of two identical copies of the Hilbert space of the system at zero temperature over the field of temperature dependent functions and it is endowed with an involution denoted by tilde. The physical interpretation of the thermal vectors is that of thermal degrees of freedom expressed in terms of tilde and non-tilde quasi-particles. There is a group of linear involution-invariant and time-invariant maps from the direct product of the tilde and non-tilde Hilbert spaces at zero temperature to thermal spaces whose elements are the thermal Bogoliubov operators that preserve the canonical structure. The thermal group is \( SU(1, 1) \) for bosons and \( SO(2) \) for fermions. The choice of a certain Bogoliubov transformation is equivalent to the choice of the thermal vacuum state defined by the corresponding condensate of pairs of tilde and non-tilde particles. This is so because the Bogoliubov operators entangle in a specific way the states from the tilde and non-tilde spaces at zero temperature to produces thermal states [27]. The equilibrium TFD was applied to the string theory and the string field theory in flat spacetime in [28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40]. More recently, it has been used to calculate the thermodynamics of strings in static backgrounds in

\[ G_0(x, y) \sim \rho(x, y) \]
and to construct the thermal D-branes and calculate their thermodynamical functions in (see for reviews ). In , the equilibrium TFD was generalized to accommodate the topological sector of the strings and D-branes on $R^{1,p} \times T^{d-p-1}$.

The NETFD is a canonical real-time formalism in which the nonequilibrium correlators can be calculated from states and operators that have a simple algebraic structure. The NETFD formalism retains the basic structure of the equilibrium TFD with the thermal space being the direct product of two identical copies of the Hilbert space of the system acted upon by two kinds of operators with and without tilde, respectively. They form the representation space of the operator algebra. The thermal vectors are interpreted as thermal degrees of freedom of the system in nonequilibrium and they can be constructed in the Fock representation of the thermal space from the thermal vacuum which is a solution of the master equation

$$i \frac{\partial}{\partial t} |0(t)\rangle = \hat{H} |0(t)\rangle.$$ (1)

This is the Schroedinger equation with the infinitesimal time evolution generated by the operator

$$\hat{H} = H - \tilde{H},$$ (2)

where $H$ and $\hat{H}$ are the Hamiltonians of the non-tilde and tilde degrees of freedom, respectively. The minus sign is due to the unbounded energy of the tilde excitations and is a characteristic of the thermal processes in the TFD formalism [27]. Also, it shows that in nonequilibrium the thermal vacuum state is generally unstable and time-dependent. Nevertheless, one can still define a set of time-dependent creation and annihilation operators such that the thermal vacuum be canonically defined for any value of the time parameter [27]. Since the time-dependence is carried by the states and the operators, new time-independent operators are introduced by time-dependent Bogoliubov maps in order to define stable quasi-particle states. They allow one to calculate the correlators of the time-dependent thermal operators in the time-independent quasi-particle vacuum with or without interactions [27]. The relativistic generalization of the formalism has been proposed very recently in [65, 66].

This paper is organized as follows. In Section 2 we briefly review the canonical structure of the free string in the time-dependent plane wave background following [4]. After gauging the world-sheet and the target-space symmetries, the string can be reduced to an ensemble of time-dependent non-interacting oscillators in the Brinkmann coordinates. By exploiting this structure, we construct in Section 3 the time-dependent Bogoliubov operators and the time-independent quasi-particle vacuum. In Section 4 we construct the time-dependent correlation functions of the thermal string. We show that they contain the transition matrix between the initial time $t_0 = -\infty$ and the time of the singularity $t_1 = 0$, respectively. In NETFD the transition matrix is defined in terms of the interaction Hamiltonian of the tilde and non-tilde strings and a counterterm that is responsible for the time evolution of the number operators. A double Dyson-Schwinger expansion is possible in the two terms. We compute the correlations at zero order of the expansion. The last section is devoted to conclusions. In order to make the paper self-consistent, some basic facts about the NETFD formalism are collected in the Appendix.

While we have been finishing this paper, we learned of the reference [67] in which similar results to ours are obtained for the complex scalar field in the Minkowski spacetime. The main difference, other than the context, is the time-dependence of the frequencies of the string oscillators in contrast to the constant frequencies of the oscillators of the scalar field from [67].

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2 Strings in plane wave background

In this section we review the canonical quantization of the free closed string in the time-dependent singular plane wave background \[1, 9\] and establish our notations. In what follows, we are going to use the Brinkmann coordinates in which the background belongs to the following general class

\[
ds^2 = 2dx^+ dx^- - \lambda(x^+) \left( x^i dx^+ \right)^2 + \left( dx^i \right)^2,
\]

(3)

\[
\phi = \phi(x^+).
\]

(4)

Here, \(x^\pm\) are the light-cone coordinates in the target-space and \(i, j = 2, 3, \ldots D\). The dilaton compensates in the Ricci tensor the contribution of the metric and ensures that the background is conformally flat. In the case of the singular plane waves, the function \(\lambda(x)\) behaves as \(\lambda(x) \to kx^{-2} + O(x^r)\) at \(x \to 0\), where \(k\) is a real parameter and \(r > 2\). For \(k > 0\) the mass is positive and the dilaton takes the form

\[
\phi(x^+) = \phi_0 - Cx^+ + \frac{(D - 2)k}{2} \ln(x^+),
\]

(5)

where \(C\) is a constant and \(D\) is the spacetime dimension. The string action in curved dilatonic background has the general form \[68\]

\[
S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} \left[ h^{\alpha\beta}(\sigma) g_{\mu\nu}(x) \partial_\alpha x^\mu(\sigma) \partial_\beta x^\nu(\sigma) + \frac{\alpha'}{2} R^{(2)}(\sigma) \phi(\sigma) \right],
\]

(6)

where \(\sigma^\alpha = (\sigma^0, \sigma^1) = (\tau, \sigma)\) are the world-sheet coordinates, \(h^{\alpha\beta}(\sigma)\) is the world-sheet metric, \(x^\mu(\sigma)\) are the string fields, \(g_{\mu\nu}(x)\) is the space-time metric defined by the line element \[4\] with \(\mu, \nu = \pm, 2, 3, \ldots D\), and \(R^{(2)}(\sigma)\) is the scalar curvature of the world-sheet. The symmetries of the action \[10\] are fixed by choosing \(\partial_\alpha h_{\alpha\alpha} = 0\) and the light-cone gauge \(x^+ = \alpha' p^+ \tau\).

The general solution of the equations of motion obtained from \[6\] has the Fourier expansion \[4\]

\[
x^i(\tau, \sigma) = x^i_0(\tau) + i \sqrt{\frac{\alpha'}{2}} \sum_{n > 0} \frac{1}{n} \left[ F_n(\tau) \left( \alpha_n^i e^{2in\sigma} + \beta_n^i e^{-2in\sigma} \right) - F_n^*(\tau) \left( \alpha_n^i e^{-2in\sigma} + \beta_n^i e^{2in\sigma} \right) \right].
\]

(7)

The creation and annihilation operators from \(x^i(\tau, \sigma)\) are normalized as string modes rather than oscillator modes and the following linear superpositions of the Bessel functions have been introduced

\[
F_n(\tau) = \exp(-\frac{i\pi\nu}{2}) \sqrt{n\pi\nu} \left[ J_{\nu-\frac{1}{2}}(2\pi\tau) - i Y_{\nu-\frac{1}{2}}(2\pi\tau) \right],
\]

(8)

\[
G_n(\tau) = \exp(-\frac{i\pi\nu}{2}) \sqrt{n\pi\nu} \left[ J_{\nu+\frac{1}{2}}(2\pi\tau) - i Y_{\nu+\frac{1}{2}}(2\pi\tau) \right],
\]

(9)

where \(\nu = (1/2) \left( 1 + \sqrt{1 - 4k} \right)\). The modes of the string center of mass can be written in terms of time-independent oscillators as follows

\[
x^i_0(\tau) = \left\{ \begin{array}{ll}
\sqrt{\frac{\alpha'}{2(2\nu-1)}} \left[ (\alpha^i_0 + \alpha^i_0) \tau^{1-\nu} - 2i \left( \alpha^i_0 - \alpha^i_0 \right) \tau^\nu \right], & k \neq \frac{1}{4}, \\
\sqrt{\frac{\alpha'}{2}} \left[ (\alpha^i_0 + \alpha^i_0) - 2i \left( \alpha^i_0 - \alpha^i_0 \right) \ln \tau \right], & k = \frac{1}{4},
\end{array} \right.
\]

(10)
The Hamiltonian of string in the light-cone gauge has the following oscillator structure

\[ H = \frac{\alpha'}{2} \sum_{i=2}^{D} \left[ (p_{0}^{i})^2 + \frac{k}{4\alpha'^2\tau^2} (x^{i}_{0})^2 \right] + \frac{1}{2} \sum_{i=2}^{D} \sum_{n=1}^{\infty} \left[ \Omega_n(\tau) \left( \alpha_n^i \alpha_n^i + \beta_n^i \beta_n^i \right) - \Phi_n(\tau) \alpha_n^i \beta_n^i - \Phi*_n(\tau) \alpha_n^i \beta_n^i \right], \tag{11} \]

where the time-dependent coefficients \( \Omega_n(\tau) \) and \( \Phi_n(\tau) \) are given by the following relations

\[ \Omega_n(\tau) = \left( 1 + \frac{\nu}{4\tau^2n^2} \right) |F_n(\tau)|^2 + |G_n(\tau)|^2 - \frac{\nu}{2\Omega_n(\tau)} [F_n(\tau)G_n^*(\tau) + F_n^*(\tau)G_n(\tau)], \tag{12} \]
\[ \Phi_n(\tau) = \left( 1 + \frac{\nu}{4\tau^2n^2} \right) F_n(\tau)^2 + G_n(\tau)^2 - \frac{\nu}{2\Omega_n(\tau)} F_n(\tau)G_n(\tau). \tag{13} \]

From equations (12) and (13), one can see that the string behaves as a collection of self-interacting time-dependent string modes even if it is a free string. The non-diagonal interacting terms do not mix modes of different frequencies. They are generated by the time-dependent couplings \( \Omega_n(\tau) \) and \( \Phi_n(\tau) \) induced by the time-dependent gravitational field. The self-interacting sector of the Hamiltonian should be treated non-perturbatively because \( \Omega_n(\tau) \) and \( \Phi_n(\tau) \) are of the same strength. However, as was shown in [4], the Hamiltonian can be written as a collection of non-interacting time-dependent harmonic oscillators by applying the following linear map

\[ A_0^i(\tau) = u_n(\tau) \alpha_n^i + v_n(\tau) \beta_n^i, \tag{14} \]
\[ A_1^i(\tau) = u_n^*(\tau) \alpha_n^i + v_n(\tau) \beta_n^i, \tag{15} \]
\[ B_0^i(\tau) = u_n(\tau) \beta_n^i + v_n(\tau) \alpha_n^i, \tag{16} \]
\[ B_1^i(\tau) = u_n^*(\tau) \beta_n^i + v_n(\tau) \alpha_n^i, \tag{17} \]

where

\[ u_n(\tau) = \frac{1}{2} e^{2i\omega_n(\tau)\tau} \left[ F_n(\tau) + \frac{i}{2\omega_n(\tau)} \partial_\tau F_n(\tau) \right], \tag{18} \]
\[ v_n(\tau) = \frac{1}{2} e^{-2i\omega_n(\tau)\tau} \left[ -F_n(\tau) + \frac{i}{2\omega_n(\tau)} \partial_\tau F_n(\tau) \right]. \tag{19} \]

The \( A \) and \( B \) oscillators satisfy the canonical commutation relations

\[ [A_n^i(\tau), B^i_\alpha(\tau)] = [A_n^i(\tau), B^i_\alpha(\tau)] = \delta_{n\alpha} \delta^{ij}, \quad [A_n^i(\tau), B^i_\alpha(\tau)] = [A_n^i(\tau), B^i_\alpha(\tau)] = 0. \tag{20} \]

In terms of the \( A \) and \( B \) operators, the light-cone Hamiltonian takes the diagonal form

\[ H = \frac{\alpha'}{2} \sum_{i=2}^{D} \left[ (p_{0}^{i})^2 + \left( \frac{k}{4\alpha'^2\tau^2} \right)^2 (x^{i}_{0})^2 \right] + \frac{1}{2} \sum_{i=2}^{D} \sum_{n=1}^{\infty} \left[ \omega_n(\tau) \left( A^i_0 A^i_1 + B^i_0 B^i_1 \right) \right] + h(\tau), \tag{21} \]

where the time-dependent frequencies are

\[ \omega_n(\tau) = \sqrt{n^2 + \frac{k}{4\tau^2}}. \tag{22} \]
The time-dependent function

\[ h(\tau) = (D - 2) \sum_{n=1}^{\infty} \omega_n(\tau) \]  

plays the rôle of the normal-ordering constant. It is logarithmic divergent and it can be cancelled by a field renormalization of the dilaton for all values of \( \tau \) \[4\]. The string fields \((7)\) can be expanded in terms of \( A \) and \( B \) oscillators as follows

\[
x_i(\tau, \sigma) = x_i^0(\tau) + i \sqrt{\frac{\alpha'}{2}} \sum_{n>0} \frac{1}{\sqrt{n \omega_n(\tau)}} \left\{ e^{-2i \omega_n(\tau) \tau} A_i^n(\tau) e^{2i n \sigma} - e^{2i \omega_n(\tau) \tau} A_i^n(\tau) e^{-2i n \sigma} \right\} + i \sqrt{\frac{\alpha'}{2}} \sum_{n>0} \frac{1}{\sqrt{n \omega_n(\tau)}} \left\{ e^{-2i \omega_n(\tau) \tau} B_i^n(\tau) e^{-2i n \sigma} - e^{2i \omega_n(\tau) \tau} B_i^n(\tau) e^{2i n \sigma} \right\}.
\]

In general, the particle-like states of time-dependent oscillators are well defined only asymptotically \[27\]. In the present case, there are two asymptotic Fock spaces of string oscillators for observers at \( \tau \rightarrow \pm \infty \). Since the oscillator sector is asymptotically the same as that of the free string in the flat space-time, the asymptotic states can be constructed from the light-cone vacua \( |0(\pm \infty)\rangle_{A-B} \) which are eigenstates of \( \alpha_i^n \) and \( \beta_i^n \) operators. At finite \( \tau \), the free string picture is lost due to the self-interactions. However, there are free states that can be obtained by exciting the time-dependent vacuum \( |0(\tau)\rangle_{A-B} \) in the representation defined by the operators \( A \) and \( B \), respectively. The dynamics of the center-of-mass of string does not reduce to the corresponding one in flat space-time, and thus is characteristic to the time-dependent plane wave background.

3 Nonequilibrium dynamics of strings

The fact that the string theory in the time-dependent plane wave backgrounds is solvable in the canonical quantization suggests that the nonequilibrium dynamics could have a canonical formulation, too. There are several methods that can be used to this end. As mentioned in the introduction, we are going to formulate the nonequilibrium dynamics in the NETFD framework because of the simple interpretation that can be given to the states, the operators and the nonequilibrium correlators.

3.1 Thermal string fields

The first step to be taken in order to develop the NETFD method for strings is to define the thermal degrees of freedom. The thermal states belong to the thermal space \( \hat{\mathcal{H}} = \mathcal{H} \otimes \tilde{\mathcal{H}} \) which is the direct product of two identical copies of the Hilbert space of string. On \( \hat{\mathcal{H}} \) act operators of the form \( O \otimes \tilde{1} \) and \( 1 \otimes \tilde{O} \) that can be conveniently tensored with vectors from \( \mathbb{R}^2 \) to form the so called thermal doublets \(27\). We start with the diagonal form of the theory in which the string fields are given by the relation \(24\). The coordinates of the tilde string are obtained by
applying the tilde operation to $x^i(\tau, \sigma)$ (see the Appendix)

$$
\tilde{x}^i(\tau, \sigma) = \tilde{x}_0^i(\tau) - i \sqrt{\frac{\alpha'}{2}} \sum_{n>0} \frac{1}{\sqrt{n \omega_n(\tau)}} \left\{ e^{2 i \omega_n(\tau) \tau} A_n^i(\tau) - e^{-2 i \omega_n(\tau) \tau} \tilde{A}_n^i(\tau) e^{2 i n \sigma} \right\} 
- i \sqrt{\frac{\alpha'}{2}} \sum_{n>0} \frac{1}{\sqrt{n \omega_n(\tau)}} \left\{ e^{2 i \omega_n(\tau) \tau} \tilde{B}_n^i(\tau) - e^{-2 i \omega_n(\tau) \tau} B_n^i(\tau) e^{2 i n \sigma} \right\}.
$$

(25)

It is convenient to absorb the time-dependent exponentials into the canonical operators by the mean of the following change of variables

$$
e^{-2 i \omega_n(\tau) \tau} A_n^i(\tau) = a_n^i(\tau), \quad e^{2 i \omega_n(\tau) \tau} \tilde{A}_n^i(\tau) = \tilde{a}_n^i(\tau),
$$

(26)

with similar redefinition of operators in other sectors. The string thermal doublet, denoted by $\phi^{i_\alpha}(\tau, \sigma)$, can be obtained by organizing the fields from (24) and (25) in to a two dimensional vector field in such a way that the $a-b$ structure be maintained

$$
\phi^{i_\alpha}(\tau, \sigma) = \phi_0^{i_\alpha}(\tau) + \phi_a^{i_\alpha}(\tau, \sigma) + \phi_b^{i_\alpha}(\tau, \sigma)
$$

$$
= \phi_0^{i_\alpha}(\tau) + i \sqrt{\frac{\alpha'}{2}} \sum_{n>0} \frac{1}{\sqrt{n \omega_n(\tau)}} \left\{ a_n^{i_\alpha}(\tau) e^{2 i n \sigma} - (s_3 \tilde{a}_n^{i_\alpha}(\tau)^T) \alpha e^{-2 i n \sigma} \right\}
- i \sqrt{\frac{\alpha'}{2}} \sum_{n>0} \frac{1}{\sqrt{n \omega_n(\tau)}} \left\{ \tilde{b}_n^{i_\alpha}(\tau) e^{-2 i n \sigma} - (s_3 \tilde{b}_n^{i_\alpha}(\tau)^T) \alpha e^{2 i n \sigma} \right\},
$$

(27)

where $\alpha$ is the $\mathbb{R}^2$ index, $\phi_0^{i_\alpha}(\tau)$ stands for the thermal doublet of the center of mass and $s_3$ is the Pauli matrix. Also, we have introduced the standard $\mathbb{R}^2$ thermal doublet

$$
a_n^{i_\alpha}(\tau) = \left( a_n^i(\tau) \right), \quad \tilde{a}_n^{i_\alpha}(\tau) = \left( \tilde{a}_n^i(\tau) \right).
$$

(28)

Similar operators can be introduced in the $b$ - sector. The conjugate of the thermal field given by the relation (27) is obtained by applying the operations defined in the Appendix

$$
\tilde{\phi}^{i_\alpha}(\tau, \sigma) = \tilde{\phi}_0^{i_\alpha}(\tau) + \tilde{\phi}_a^{i_\alpha}(\tau, \sigma) + \tilde{\phi}_b^{i_\alpha}(\tau, \sigma)
$$

$$
= \tilde{\phi}_0^{i_\alpha}(\tau) - i \sqrt{\frac{\alpha'}{2}} \sum_{n>0} \frac{1}{\sqrt{n \omega_n(\tau)}} \left\{ \tilde{a}_n^{i_\alpha}(\tau) e^{-2 i n \sigma} - (a_n^i(\tau)^T s_3) \alpha e^{2 i n \sigma} \right\}
+ i \sqrt{\frac{\alpha'}{2}} \sum_{n>0} \frac{1}{\sqrt{n \omega_n(\tau)}} \left\{ \tilde{b}_n^{i_\alpha}(\tau) e^{2 i n \sigma} - (b_n^i(\tau)^T s_3) \alpha e^{-2 i n \sigma} \right\}.
$$

(29)

Note that the coordinates of the center of mass do not have the standard form of a thermal doublet as given by the NETFD. This can be remedied by organizing the coordinates as follows

$$
x_n^i(\tau) = U(\tau) a_n^i(\tau) + U^*(\tau) \tilde{a}_n^i(\tau),
$$

(30)

where the time-dependent coefficients are given by the following relations

$$
U(\tau) = \begin{cases} 
\sqrt{\frac{\alpha'}{2(2\nu-1)}}^{1-\nu} - i \sqrt{\frac{2\alpha'}{2\nu-1}} \tau^\nu, & k \neq \frac{1}{4} \\
\sqrt{\frac{\alpha'}{2}} - i \sqrt{2\alpha'} \ln \tau, & k = \frac{1}{4}.
\end{cases}
$$

(31)
The coordinates of the center of mass of the tilde string can be obtained by applying the conjugation rules given in the Appendix. Then the thermal fields are generalized to the center of mass by the following relations

\[ \phi_{0}^{i\beta}(\tau) = \left( \frac{x_{0}^{i}(\tau)}{s_{3}^{\lambda}T_{0}(\tau)} \right) = U(\tau)\alpha_{0}^{i\beta} + U^{*}(\tau)\left(s_{3}^{\alpha}T_{0}\right)^{\beta}. \] (32)

The thermal doublets define a time-parametrized family of representations of the thermal field constructed from the thermal vacua \{0(\tau)\}. Any of these representations can be mapped to the time-independent quasi-particle representation by the inverse of the Bogoliubov map (see the Appendix)

\[ a_{n}^{\alpha}(\tau) = B_{a,n}^{-1}(\tau)^{\alpha\beta}\xi_{n}^{\beta}, \quad b_{n}^{\alpha}(\tau) = B_{b,n}^{-1}(\tau)^{\alpha\beta}\chi_{n}^{\beta}. \] (33)

Then the nonequilibrium dynamics is determined by the time variation of the thermal doublets \(a_{n}^{\alpha}(\tau)\) and \(b_{n}^{\alpha}(\tau)\) in the time-independent vacuum \(|0\rangle\) defined by \(\xi_{n}^{\alpha}\) and \(\chi_{n}^{\alpha}\), respectively. The Bogoliubov operators can be different in the \(a\) and \(b\) sectors, respectively, but they are the same for all transversal directions. However, since the two sectors differ only in the orientation of the modes along the space-like direction of the world-sheet, we can take \(B_{a,n}(\tau) = B_{b,n}(\tau) = B_{n}(\tau)\). It follows that the Bogoliubov map of the \(k\)-mode has the form given by the relation \(|0\rangle\) in the linear thermal gauge

\[ \xi_{k}^{\alpha} = \exp \left[ i \int_{\tau_{0}}^{\tau} d\omega_{k}(\lambda) \right] B_{k}^{\alpha\beta}(\tau) a_{k}^{i\beta}(\tau), \] (34)

\[ \tilde{\xi}_{k}^{\alpha} = \tilde{a}_{k}^{i\beta}(\tau) \exp \left[ -i \int_{\tau_{0}}^{\tau} d\omega_{k}(\lambda) \right] \left[B_{k}^{-1}\right]^{\beta\alpha}(\tau), \] (35)

\[ \chi_{k}^{\alpha} = \exp \left[ i \int_{\tau_{0}}^{\tau} d\omega_{k}(\lambda) \right] B_{k}^{\alpha\beta}(\tau) b_{k}^{i\beta}(\tau), \] (36)

\[ \tilde{\chi}_{k}^{\alpha} = \tilde{b}_{k}^{i\beta}(\tau) \exp \left[ -i \int_{\tau_{0}}^{\tau} d\omega_{k}(\lambda) \right] \left[B_{k}^{-1}\right]^{\beta\alpha}(\tau), \] (37)

where the positive exponential represents the pure complex phase function and

\[ n_{k}(\tau)\delta_{k,l} = n_{k}^{0}(\tau)\delta_{k,l} = \left\langle 0(\tau) | a_{k}^{i\dagger}a_{l}^{i} | 0(\tau) \right\rangle. \] (38)

The initial boundary conditions are taken at \(\tau_{0} \rightarrow -\infty\). The time-dependent quasi-particle operators are obtained by multiplying the transformations given in the equations \(34\) - \(37\) with the inverse of the corresponding phase functions to obtain

\[ \xi_{k}^{\alpha}(\tau) = B_{k}^{\alpha\beta}(\tau) a_{k}^{i\beta}(\tau), \] (39)

\[ \tilde{\xi}_{k}^{\alpha}(\tau) = \tilde{a}_{k}^{i\beta}(\tau) \left[B_{k}^{-1}\right]^{\beta\alpha}(\tau), \] (40)

\[ \chi_{k}^{\alpha}(\tau) = B_{k}^{\alpha\beta}(\tau) b_{k}^{i\beta}(\tau), \] (41)

\[ \tilde{\chi}_{k}^{\alpha}(\tau) = \tilde{b}_{k}^{i\beta}(\tau) \left[B_{k}^{-1}\right]^{\beta\alpha}(\tau). \] (42)

The quasi-particle representation is based on the oscillator equation of motion that is satisfied by the string modes \(7\) and by the natural assumption that the system of oscillators evolves according to the Schrödinger equation with the Hamiltonian \(21\) in the \(a - b\) representation.

\footnote{In the case of the complex scalar field discussed in \(67\) this equality does not hold because the left and right moving states have different charges.}
By taking the first derivative of $\xi^i_k \alpha (\tau)$ with respect to $\tau$ one can easily check that $a^{j\beta}_k (\tau)$ satisfies the first order differential equation

$$ \left( i \frac{d}{d\tau} - \omega_k(\tau) \right) a^{j\beta}_k (\tau) + P^{\alpha\beta}_k (\tau) a^{i\beta}_k (\tau) = 0, \quad (43) $$

where

$$ P^{\alpha\beta}_k (\tau) = i \frac{dn_k(\tau)}{d\tau} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \quad (44) $$

The same equations hold for $b^{j\beta}_k (\tau)$ operators. This is a general result of NETFD for the set of time-dependent oscillators \[27\]. From (43) it follows that the time-evolution of the thermal string modes obeys the following equations

$$ i \frac{d}{d\tau} a^{j\alpha}_k (\tau) = \left[ a^{j\alpha}_k (\tau), \hat{H}_{Qa} \right] , \quad (45) $$

$$ i \frac{d}{d\tau} \bar{a}^{j\alpha}_k (\tau) = \left[ \bar{a}^{j\alpha}_k (\tau), \hat{H}_{Qa} \right], \quad (46) $$

where

$$ \hat{H}^a_{Q}(\tau) = \sum_{i=2}^{D} \sum_{k} \left[ \omega_k(\tau) \delta^{\alpha\beta} + P^{\alpha\beta}_k (\tau) \right] \bar{a}^{i\alpha}_k (\tau) a^{i\beta}_k (\tau). \quad (47) $$

The second term in the equation (47) is the thermal counterterm which should be added to the Hamiltonian as a consequence of the time-dependent Bogoliubov map. As can be seen from its definition (44) it is related to the variation of $n_k(\tau)$ from the relation (38). The same relations holds in the $b$ - sector. Since the total Hamiltonian can be written as \[27\]

$$ \hat{H} = \hat{H}_0 + \hat{H}_{int} = \hat{H}_Q + \hat{H}_I, \quad (48) $$

the counterterm $\hat{H}_Q(\tau) = \hat{H}_c m(\tau) + \hat{H}^a_{Q}(\tau) + \hat{H}^b_{Q}(\tau)$ from (48) should be compensated by a term to be added to the interaction term. In the case of the free theory, the interaction Hamiltonian contains only the compensator of the counterterm. Note that the quasi-particle fields, even if stable, cannot be used to define the asymptotic thermal states because the thermal Hamiltonian is unbounded from below \[27\].

### 3.2 Dynamics of thermal fields

Once the time-evolution of each mode of the thermal string, described by the equations (45) and (46), is understood, we can proceed to deriving the equations of motion of the thermal fields. Since we are working in the Hamiltonian formalism, we need to determine the canonical conjugates of $\phi^{i\alpha}(\tau, \sigma)$ and $\bar{\phi}^{i\alpha}(\tau, \sigma)$. The conjugate momenta are defined such that they satisfy the equal-time commutation relations \[27\]

$$ \left[ \phi^{i\alpha}(\tau, \sigma), \pi^{i\beta}(\tau, \sigma') \right] = i \delta^{ij} \delta(\sigma - \sigma') s^{\alpha\beta}, \quad (49) $$

$$ \left[ \bar{\phi}^{i\alpha}(\tau, \sigma), \pi^{i\beta}(\tau, \sigma') \right] = i \delta^{ij} \delta(\sigma - \sigma') \delta^{\alpha\beta}. \quad (50) $$
By using the (45) and (46) in the above commutators we obtain

\[ \pi^{i\alpha}(\tau, \sigma) = \pi_0^{i\alpha}(\tau) + \pi_a^{i\alpha}(\tau, \sigma) + \pi_b^{i\alpha}(\tau, \sigma) \]

\[ = \frac{i}{2\pi} \left[ U^{-1}(\tau) \left( s_3 \bar{a}^i_0 \right)^{\alpha} + (U^{*}(\tau))^{-1} \alpha^{i\alpha}_0 \right] \]

\[ + \frac{i}{\sqrt{2\alpha'}} \sum_{n>0} \sqrt{n\omega_n(\tau)} \left\{ \tilde{a}^i_n(\tau) e^{2in\sigma} + \left( s_3 \bar{a}^i_n(\tau) T^\alpha \right) e^{-2in\sigma} \right\} \]

\[ - \frac{i}{\sqrt{2\alpha'}} \sum_{n>0} \sqrt{n\omega_n(\tau)} \left\{ \tilde{b}^i_n(\tau) e^{-2in\sigma} + \left( s_3 \bar{b}^i_n(\tau) T^\alpha \right) e^{2in\sigma} \right\}. \] (51)

The conjugate momenta of the fields $\tilde{\phi}^{i\alpha}(\tau, \sigma)$ can be obtained by applying the tilde conjugation to $\pi^{i\alpha}(\tau, \sigma)$ and they take the following form

\[ \tilde{\pi}^{i\alpha}(\tau, \sigma) = \pi_0^{i\alpha}(\tau) + \pi_a^{i\alpha}(\tau, \sigma) + \pi_b^{i\alpha}(\tau, \sigma) \]

\[ = \frac{i}{2\pi} \left[ - (U^{*}(\tau))^{-1} \left( \alpha^{i\alpha}_0 T^\alpha \right) s_3 + U^{-1}(\tau) \alpha^{i\alpha}_0 \right] \]

\[ + \frac{i}{\sqrt{2\alpha'}} \sum_{n>0} \sqrt{n\omega_n(\tau)} \left\{ \tilde{a}^i_n(\tau) e^{2in\sigma} + \left( \tilde{a}^i_n(\tau) T^\alpha s_3 \right) e^{-2in\sigma} \right\} \]

\[ - \frac{i}{\sqrt{2\alpha'}} \sum_{n>0} \sqrt{n\omega_n(\tau)} \left\{ \tilde{b}^i_n(\tau) e^{-2in\sigma} + \left( \tilde{b}^i_n(\tau) T^\alpha s_3 \right) e^{2in\sigma} \right\}. \] (52)

The fields obtained in this way satisfy the commutation relations

\[ \left[ \tilde{\phi}^{i\alpha}(\tau, \sigma), \tilde{\pi}^{j\beta}(\tau, \sigma') \right] = i\delta{}^{ij}\delta(\sigma - \sigma')\delta^{\alpha\beta}, \] (53)

which qualifies them for the canonical momenta of the $\tilde{\phi}^{i\alpha}(\tau, \sigma)$ fields.

Let us determine the equations of motion of the oscillator sector of the fields $\phi^{i\alpha}(\tau, \sigma), \tilde{\phi}^{i\alpha}(\tau, \sigma), \pi^{i\alpha}(\tau, \sigma)$ and $\tilde{\pi}^{i\alpha}(\tau, \sigma)$. They can be obtained by deriving the fields with respect to $\tau$ and by using the equations (45) and (46). Before doing the calculation, it is convenient to express the oscillator frequencies in string units through the rescaling

\[ \omega_n(\tau) \rightarrow \frac{n\omega_n(\tau)}{\alpha'}. \] (54)

Lengthy calculations give the following equations of motion for the oscillating modes

\[ \left[ \left( 1 + 2i\partial_\tau n_\nabla \nabla^{-1} T_0 \right)^{\alpha\beta} \partial_\tau + \partial_\tau \nabla \cdot \nabla^{-1} \left( 1 + 2i\partial_\tau n_\nabla \nabla^{-1} T_0 \right)^{\alpha\beta} \right] \phi^{i\beta}_{osc}(\tau, \sigma) = \pi^{i\alpha}_{osc}(\tau, \sigma), \] (55)

\[ \left( 1 - 2i\partial_\tau n_\nabla \nabla^{-1} T_0 \right)^{\alpha\beta} \nabla^2 \phi^{i\beta}_{osc}(\tau, \sigma) + \left[ \partial_\tau - \partial_\tau \nabla \cdot \nabla^{-1} \right] \pi^{i\alpha}_{osc}(\tau, \sigma) = 0. \] (56)

Here, we have introduced the notation $\nabla = \sqrt{-\partial_\tau^2 + \frac{b}{\alpha'}}$. The equations (55) and (56) describe the classical dynamics of the oscillator sector of string fields near the plane wave singularity. Since the equations of motion are the result of the time evolution generated by the classical Hamiltonian, one can use (55), (56) and (53) to integrate the Hamilton equations. The resulting Hamiltonian is composed from two terms

\[ \hat{H}_Q = \hat{H}_{cm} + \hat{H}_{osc}, \] (57)
where $\hat{H}_\text{cm}$ is the Hamiltonian of the center of mass and

$$\hat{H}_\text{osc} = \frac{1}{2} \int d\sigma \sum_{i=2}^{D} \sum_{\alpha > 0} \left[ \pi_{\alpha \alpha}^{i \alpha} \right] (1 + 2i\partial_\tau n_i|\nabla^{-1}T_0)^{\alpha \beta} \pi_{\alpha \alpha}^{i \beta}$$

$$+ \phi^{\alpha}(\tau, \sigma) \left( 1 - 2i\partial_\tau n_i|\nabla^{-1}T_0 \right)^{\alpha \beta} \nabla^2 \phi^{\beta}(\tau, \sigma)$$

$$- \tilde{\pi}_{\alpha \alpha}^{i \alpha} \left( 1 + T_n \right)^{\alpha \beta} \nabla \cdot \nabla^{-1} (1 + 2i\partial_\tau n_i|\nabla^{-1}T_0)^{\beta \gamma} \phi^{\gamma}(\tau, \sigma)$$

$$- \tilde{\phi}^{\alpha}(\tau, \sigma) \partial_\tau \nabla \cdot \nabla^{-1} \pi_{\alpha \alpha}^{i \alpha} \right] ,$$

where

$$T_0 = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & 1 \end{pmatrix} , \quad T_n = \begin{pmatrix} 2i\partial_\tau n_i|\nabla^{-1} & 1 - 2i\partial_\tau n_i|\nabla^{-1} \\ 1 - 2i\partial_\tau n_i|\nabla^{-1} & 1 - 2i\partial_\tau n_i|\nabla^{-1} \end{pmatrix} .$$

The oscillator Hamiltonian can be split in two terms: the free oscillator $\hat{H}_\text{osc}^0$ term and the counterterm $\hat{Q}$ which can be both read off of the right hand side of (58). The Hamiltonian (57) generates the equations of motion (55), (56) and (63). The counterterm $\hat{Q}$ provides the dynamics of the number function $n_i|\nabla|^{\alpha}(\tau)$ from which is derived the time-evolution of each oscillator mode.

### 3.3 Dynamics of center of mass

Note that the derivation of the oscillator terms from (51) and (52) follows from the standard NETFD method. However, the coefficient of the zero mode component $\pi_{0}^{0}(\tau)$ is not given by the general formalism. In order to obtain the correct factor one has to generalize the NETFD in the following way. Firstly, we consider the series representation of the delta-function

$$\delta(\sigma - a) = \frac{1}{2\pi} \sum_{n=-\infty}^{n=\infty} e^{2in(\sigma - a)} .$$

Then one can show that the commutation relations (49) and (50) are reproduced if the coordinates of the center of mass satisfy the following commutators

$$[\phi^{\alpha}_0(\tau), \pi^{\beta}_0(\tau)] = -i \frac{1}{\pi} \delta^{\gamma \delta} s^{\gamma}_3 .$$

From that we can determine the momentum which has the form

$$\pi_{0}^{0}(\tau) = \frac{i}{2\pi} \left[ U^{-1}(\tau) \left( s_3 \alpha_0^i \right)^{\alpha} + (U^*)(\tau)^{-1} \alpha_0^{i \alpha} \right] .$$

This fixes the constant in front of the zero mode term from equation (51). The coefficient of the corresponding term from the equation (52) can be obtained in the same way.

The equations of motion of the center of mass can be obtained by taking the time derivative of the equation (62). After some algebra one obtains the following equation

$$\left[ \partial_\tau + \frac{2\pi}{U^*(\tau)} \left( \frac{\partial_\tau U(\tau)}{2\pi - i} - i\partial_\tau U^*(\tau) \right) \right] \phi^{i \alpha}_0(\tau) = 2\pi U(\tau) \left( \frac{\partial_\tau U(\tau)}{2\pi - i} - i\partial_\tau U^*(\tau) \right) \pi_{0}^{i \alpha} .$$

The equation of motion of the momenta is

$$\left[ \partial_\tau - i \left( \frac{1}{2\pi + i} \frac{\partial_\tau U(\tau)}{U(\tau)} + \frac{1}{2\pi - i} i\partial_\tau U^*(\tau) \right) \right] \pi_{0}^{i \alpha} =$$

$$- i \left[ \frac{1}{2\pi (2\pi + i)} \frac{\partial_\tau U(\tau)}{|U(\tau)|^2} - \frac{1}{2\pi - i} \frac{\partial_\tau U^*(\tau)}{(U^*)^3} \right] \phi^{i \alpha}_0(\tau) .$$

(64)
The equations of motion of $\tilde{\phi}_0^{i\alpha}(\tau)$ and $\tilde{\pi}_0^{i\alpha}(\tau)$ can be obtained in the same way. From the requirement that the equations of motion be the Hamilton equations for the Hamiltonian of the center of mass $\hat{H}_{cm}$, one can calculate the explicit form of it

$$\hat{H}_{cm} = \sum_{i=2}^{D} 2\pi U(\tau) \left( \frac{\partial_{\tau}U(\tau)}{2\pi + i} + \frac{\partial_{\tau}U^*(\tau)}{2\pi - i} \right) \tilde{\pi}_0^{i\alpha} \tilde{\pi}_0^{i\alpha} +$$

$$- i \left( \frac{i}{2\pi(2\pi + i)} - \frac{1}{2\pi - i} \right) \frac{\partial_{\tau}U(\tau)}{U(\tau)} \frac{\partial_{\tau}U^*(\tau)}{U^*(\tau)} \tilde{\phi}_0^{i\alpha}(\tau) \phi_0^{i\alpha}(\tau)$$

$$- i \left( \frac{1}{2\pi + i} \frac{\partial_{\tau}U(\tau)}{U(\tau)} + \frac{1}{2\pi - i} \frac{i\partial_{\tau}U^*(\tau)U(\tau)}{[U^*(\tau)]^2} \right) \tilde{\phi}_0^{i\alpha}(\tau) \pi_0^{i\alpha}$$

$$- i \frac{2\pi}{2\pi - i} \frac{\partial_{\tau}U(\tau)}{U(\tau)} \frac{i\partial_{\tau}U^*(\tau)}{2\pi(2\pi + i)} \tilde{\pi}_0^{i\alpha} \phi_0^{i\alpha} \right]. \quad (65)$$

The total Hamiltonian $\hat{H}_{cm} + \hat{H}_{osc}$ determine the dynamics of the thermal string field in the proximity of the singularity.

4 Thermal string correlations

In order to make predictions about the physical properties of the thermal string, observable quantities must be defined. At equilibrium, the thermodynamical functions and their derivatives provide such observables. In the nonequilibrium, it is more natural to express the observables as transition probabilities for string states at different values of the time parameter. These transition probabilities can be defined in terms of correlations functions. Since we are considering the string at zero order in the Euler number, i.e. on cylinder, and in the $a - b$ representation there are no interactions among string modes, one would naively expect that the correlation numbers be defined by

$$\langle T[\phi^{i_1\alpha_1}(\tau_1)\phi^{i_2\alpha_2}(\tau_2) \cdots \phi^{i_n\alpha_n}(\tau_n)] \rangle. \quad (66)$$

However, the analysis from the section shows that even without interactions, there is a counterterm $\hat{Q}$ that is generated in the $\hat{H}_{osc}$ as a consequence of the the time evolution of the frequencies. In the interacting field theory, $\hat{Q}$ should be cancelled by self-energy interaction in order to leave the Hamiltonian unchanged [27]. Therefore, in order to define the correlation functions one has to work in the interaction picture defined by the interaction Hamiltonian

$$\hat{H}_I = \hat{Q}. \quad (67)$$

In this representation the oscillators evolve according as given in the equations (45) and (46), that is with respect to the free Hamiltonian $\hat{H}_Q$ as defined by the relation (45). Then the correlation functions are defined as the expectation values

$$\left\langle T[\phi^{i_1\alpha_1}(\tau_1)\phi^{i_2\alpha_2}(\tau_2) \cdots \tilde{\phi}^{i_n\alpha_n}(\tau_n)\hat{S}(0, \infty)] \right\rangle, \quad (68)$$

where the $\hat{S}$-operator is defined as usual

$$\hat{S}(0, \infty) = \lim_{\tau_1 \to 0} \lim_{\tau_2 \to \infty} \hat{u}(\tau_1, \tau_2) = T \left[ \exp \left( -i \int_0^\infty d\zeta \hat{H}_I(\zeta) \right) \right]. \quad (69)$$
The thermal vacuum in the interaction picture satisfies the following relation

\[ (0)\, \hat{u}(0, \tau) = \langle 0 \rangle . \]  

(70)

This is the Schwinger-Dyson formalism for the thermal fields in the canonical formulation. The correlations defined by the relation (68) can be calculated by expanding in powers of \( \hat{Q} \)

\[ \left\langle T[\phi^{i_1\alpha_1}(\tau_1)\phi^{i_2\alpha_2}(\tau_2)\cdots\phi^{i_n\alpha_n}(\tau_n)\hat{S}(0, \infty)] \right\rangle = \]

\[ \left\langle T[\phi^{i_1\alpha_1}(\tau_1)\phi^{i_2\alpha_2}(\tau_2)\cdots\phi^{i_n\alpha_n}(\tau_n)] - i\left\langle T[\phi^{i_1\alpha_1}(\tau_1)\phi^{i_2\alpha_2}(\tau_2)\cdots\phi^{i_n\alpha_n}(\tau_n) \int_0^\infty ds \hat{H}_1(s)] \right\rangle \right. \]

\[ + \cdots . \]  

(71)

Note that the expansion is valid for small values of the time derivatives of \( n_{|c>\bar{c}|} \). Thus, the NETFD gives the general method to compute the correlations of the thermal string in a perturbative like fashion.

In order to exemplify this formalism, let us compute the two points correlator at zero order. To this end we consider the first term from (71) in the time-independent vacuum

\[ D^{ij,\alpha\beta}(\tau_1, \tau_2; \sigma_1, \sigma_2) = -i \left\langle T \left[ \phi^{j\alpha}(\tau_1, \sigma_1)\phi^{i\beta}(\tau_2, \sigma_2) \right] \right\rangle \]  

(72)

The fields from the equation (27) act on the direct space \( \hat{\mathcal{H}} \) without mixing the \( a \) and \( b \) sectors. It follows that the Fourier transform of the oscillator propagators has the same form in the two sectors and the cross terms vanish. Thus, there are just three non-vanishing terms in the left-hand side of the relation (72) that correspond to the three distinct Hilbert spaces of the string fields

\[ D^{ij,\alpha\beta}(\tau_1, \tau_2; \sigma_1, \sigma_2) = D_0^{ij,\alpha\beta}(\tau_1, \tau_2) + D_a^{ij,\alpha\beta}(\tau_1, \tau_2; \sigma_1, \sigma_2) + D_b^{ij,\alpha\beta}(\tau_1, \tau_2; \sigma_1, \sigma_2), \]  

(73)

where lower indices denote the center of mass and the corresponding sectors of the string oscillators. By substituting the zero modes from (32) into the above relation, one can show that the propagator of the center of mass has the following form

\[ D_0^{ij,\alpha\beta}(\tau_1, \tau_2) = -\frac{i}{2} \theta(\tau_1 - \tau_2) [U(\tau_1)U(\tau_2) + U(\tau_1)U^*(\tau_2)] (1 + s_3)^{\alpha\beta} \delta^{ij} \]

\[ -\frac{i}{2} \theta(\tau_2 - \tau_1) [-U(\tau_2)U(\tau_1) + U^*(\tau_2)U(\tau_1)] (1 + s_3)^{\alpha\beta} \delta^{ij}. \]  

(74)

The calculation of the propagators of the oscillator sectors \( a \) and \( b \) from the equation (73) is somewhat lengthy but not too difficult. Since the sectors \( a \) and \( b \) can be related by the reflection transformation \( \sigma_1 - \sigma_2 \rightarrow \sigma_2 - \sigma_1 \) one can calculate \( D_a^{ij,\alpha\beta}(\tau_1, \tau_2; \sigma_1, \sigma_2) \) and obtain \( D_b^{ij,\alpha\beta}(\tau_1, \tau_2; \sigma_1, \sigma_2) \) from it by applying the reflection. In order to compute the expectation value from \( D_a^{ij,\alpha\beta}(\tau_1, \tau_2; \sigma_1, \sigma_2) \) we use the Fourier decompositions (27) and (29) and apply the Bogoliubov transformations (34) and (35) to transform the operators \( a(\tau) \) into the time-independent quasi-particle operators \( \xi \) that correspond to the vacuum from the equation (72). After doing the algebra, the result takes the following form

\[ D_a^{ij,\alpha\beta}(\tau_1, \tau_2; \sigma_1, \sigma_2) = -\frac{i\alpha'}{4} \delta^{ij} \sum_{n>0} \left\{ B_{n}(\tau_1) \left[ e^{2i\alpha(\sigma_1-\sigma_2)} D_{a,+}^{ij}(\tau_1, \tau_2; n) + e^{-2i\alpha(\sigma_1-\sigma_2)} D_{a,-}^{ij}(\tau_1, \tau_2; n) \right] B_{n}(\tau_2) \right\}^{\alpha\beta}. \]  

(75)
We have denoted by the indices $+ \text{ and } -$ terms that would correspond to the advanced and retarded propagators, respectively, if the frequency did not depend on time. These terms are given by the following equations

$$D_{a,+}^{\alpha\beta}(\tau_1, \tau_2; n) = \frac{\exp\left[i \int_{\tau_1}^{\tau_2} d\lambda \omega_n(\lambda)\right]}{2n \sqrt{\omega_n(\tau_1) \omega_n(\tau_2)}} \left[\theta(\tau_1 - \tau_2) (\mathbf{1}_2 + s_3) + \theta(\tau_2 - \tau_1) (\mathbf{1}_2 - s_3)\right]^{\alpha\beta},$$

(76)

$$D_{a,-}^{\alpha\beta}(\tau_1, \tau_2; n) = \frac{\exp\left[-i \int_{\tau_1}^{\tau_2} d\lambda \omega_n(\lambda)\right]}{2n \sqrt{\omega_n(\tau_1) \omega_n(\tau_2)}} \left[C_n(\tau_1) \left[\theta(\tau_1 - \tau_2) (\mathbf{1}_2 + s_3)ight] + \theta(\tau_2 - \tau_1) (\mathbf{1}_2 - s_3)\right]^{-1} \omega_n(\tau_2)^{-1} C_{\alpha\beta}^{-1}(\tau_2),$$

(77)

where we have introduced the matrix for each mode $m$

$$C_m(\tau) = \begin{pmatrix} 1 + 2n_m(\tau) & -1 \\ -1 & 0 \end{pmatrix}.$$  

(78)

The Fourier transforms of the nonequilibrium propagators in the $a$ sector given by the relations (76) and (77) are similar to the nonequilibrium propagators of the scalar field in the Minkowski spacetime [65, 66, 67]. As a difference from the scalar field, we note the factor $-\frac{1}{\sqrt{\omega_n(\tau_1) \omega_n(\tau_2)}}$ that is not present in the Minkowski spacetime. This factor is a consequence of the dependence of the field modes $\phi^{\alpha n}_n(\tau_1)$ on $-\frac{1}{\sqrt{n \omega_n(\tau)}}$, more specifically, it is the result of the interaction between the gravitational background and the strings. The nonequilibrium propagators from the $b$ sector can be calculated in exactly the same way.

## 5 Conclusions

In this paper, we have given a canonical method to study the strings near singularity of plane wave backgrounds based on the NETFD. This method allows one to construct and interpret the Hilbert space and to compute the correlation functions of thermal string. The correlation functions of the string oscillators receive corrections from the counterterm $\hat{Q}$ that generates the time evolution of the number parameter $n_1^{\nabla}(\tau)$. Compared with the literature, these results are similar to the ones that are obtained in the case of relativistic scalar field in Minkowski [65, 66, 67]. However, the extra degrees of freedom provided by the center of mass of string do not have any analogue in the thermal field theory. Therefore, in order to compute their thermal behavior, we had to generalize the NETFD by postulating a new commutation relation for the canonically conjugate variables of the center of mass (61) that is consistent with the equal time commutators of the thermal fields given by the relations (49), (50) and (53). In the limit of small time variations of the number parameter, the NETFD provides a perturbative method to compute the correlation function that we have used to determine the two point functions at zero order. Due to the importance of the applications of the nonequilibrium string theory, the development of the method presented in this paper deserves attention. In particular, it is important to calculate the two point functions at first order and the equation that describe the evolution of the thermal correlators. We will report on these topics elsewhere.

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A Nonequilibrium Thermo Field Dynamics

The NETFD is a real-time canonical formalism of thermal quantum field theory. It has been originally developed as an alternative to the nonequilibrium imaginary-time formalisms which presented difficulties in defining the multi-time correlators and the density operators. The NETFD can be derived from the same basic set of axioms as the equilibrium TFD which can be summarized as follows

1. The thermal physical system is described by two sets of commuting (anti-commuting) field operators \( \phi(x) \) and \( \tilde{\phi}(x) \), respectively, that act on the thermal or total Hilbert space which is the direct product of two identical Hilbert spaces \( \hat{\mathcal{H}} = \mathcal{H} \otimes \tilde{\mathcal{H}} \).

2. To operators \( O = O(\phi(x), \phi^\dagger(x)) \) that act on \( \hat{\mathcal{H}} \) are associated operators \( \tilde{O} = O^*(\tilde{\phi}(x), \tilde{\phi}^\dagger(x)) \).

3. The tilde defines an involution on \( \hat{\mathcal{H}} \) that obeys the following rules

\[
\begin{align*}
(c_1 O_1 + c_2 O_2) & = c_1^* \tilde{O}_1 + c_2^* \tilde{O}_2, \\
(O_1 O_2) & = \tilde{O}_1 \tilde{O}_2, \\
(O^\dagger) & = \tilde{O}^\dagger, \\
(\tilde{O}) & = \varepsilon O,
\end{align*}
\]

for all \( c_1, c_2 \in \mathbb{C} \) and all \( O, O_1, O_2 \in \text{End}(\hat{\mathcal{H}}) \). \( \varepsilon = +1(-1) \) for bosonic (fermionic) operators.

4. The vacuum state is invariant under the involution

\[
\langle \tilde{0}(t) | = | 0(t) \rangle, \quad \langle 0(t) | \tilde{0}(t) \rangle = \langle 0(t) |.
\]

5. The time evolution is generator \( \hat{H} \) should satisfy the following condition

\[
\left( i \hat{H} \right) = i \hat{H}.
\]

From this condition one can see that the Hamiltonian is the difference between the two identical Hamiltonians

\[
\hat{H} = H - \tilde{H}
\]

and it is given by the Heisenberg equation for non-tilde operators

\[
i \frac{d}{dt} O(t) = \left[ O, \hat{H} \right].
\]

6. There is a set of operators \( \left\{ \xi, \tilde{\xi}, \xi^\dagger, \tilde{\xi}^\dagger \right\} \) that define the time-independent free quasi-particle representation and have the following action on the vacuum

\[
\xi |0\rangle = \tilde{\xi} |0\rangle = 0, \quad \langle 0 | \xi^\dagger = \langle 0 | \tilde{\xi}^\dagger = 0.
\]

The thermal states are defined by the following thermal state condition

\[
\langle 0 | O(t) = \langle 0 | \tilde{O}^\dagger(t) \quad \text{for bosons},
\]

\[
\langle 0 | O(t) = e^{i\theta} \langle 0 | \tilde{O}^\dagger(t) \quad \text{for fermions},
\]

where \( \theta \) is determined by the tilde conjugation rules.
7. The thermal average of the dynamical observable $O$ is given by
\[
\langle O \rangle = \langle 0 | O | 0 \rangle.
\] (91)

8. For any value of the time variable $t$, there is an invertible map between the total Hilbert space and the time-dependent quasi-particle representation given by a time-dependent Bogoliubov transformation
\[
\begin{pmatrix}
\phi(t) \\
e^{i\theta} \tilde{\phi}(t)
\end{pmatrix} = B^{-1}(t) \begin{pmatrix}
\xi(t) \\
e^{i\theta} \tilde{\xi}(t)
\end{pmatrix}.
\] (92)

9. The ket-vacuum and the bra-vacuum of the time-dependent representation is defined as
\[
\xi(t) |0(t)\rangle = \tilde{\xi}(t) |0(t)\rangle = 0,
\] (93)
\[
\langle 0(t) | \xi^\dagger(t) = \langle 0(t) | \tilde{\xi}^\dagger(t) = 0.
\] (94)

The time-evolution of the time-dependent bra- and ket-vacua is given by the equations
\[
i \frac{\partial}{\partial t} |0(t)\rangle = \hat{H} |0(t)\rangle,
\] (95)
\[
\langle 0(t) | \hat{H} = 0.
\] (96)

10. The stationary thermal states in the Schroedinger representation are defined by the equation
\[
\lim_{t \to \infty} \hat{H} |0(t)\rangle = 0,
\] (97)
assuming that the thermal equilibrium is obtained at $t \to \infty$.

The NETFD is a canonical formalism. Therefore, there is the time-dependent canonical representation defined in terms of the canonical operators $\{a(t), a^\dagger(t), \tilde{a}(t), \tilde{a}^\dagger(t)\}$. The family of time-parametrized mappings between the canonical representation and any time-dependent quasi-particle representation $\{\xi(t), \xi^\dagger(t), \tilde{\xi}(t), \tilde{\xi}^\dagger(t)\}$ is given by the time-dependent invertible Bogoliubov map
\[
B(t) : \left\{ a(t), a^\dagger(t), \tilde{a}(t), \tilde{a}^\dagger(t) \right\} \longrightarrow \left\{ \xi(t), \xi^\dagger(t), \tilde{\xi}(t), \tilde{\xi}^\dagger(t) \right\}.
\] (98)

The quasi-particle representation is defined by the properties of the Bogoliubov map that can satisfy three conditions: i) it can preserve the canonical structure; ii) it can preserve the Hermitian conjugation and iii) it can preserve the tilde conjugation:
\[
B(t) (s_2 \otimes s_3) B^T(t) = s_2 \otimes s_3,
\] (99)
\[
B^*(t) (s_1 \otimes 1) = (s_1 \otimes 1) B(t),
\] (100)
\[
B^*(t) (s_1 \otimes s_1) = (s_1 \otimes s_1) B(t),
\] (101)
where $s_1$, $s_2$ and $s_3$ are the Pauli matrices. The first requirement is made in all cases. The last two ones define non-standard, albeit useful representations of the thermal field theory. It can be shown that the most general form of the non-Hermitian Bogoliubov operator in the doublet representation from equation (92) is
\[
B_k(t) = (1 + \varepsilon n_k(t))^{1/2} e^{\gamma_k(t) s_3} \begin{pmatrix}
1 \\
\varepsilon f^\alpha_k(t) - f^\alpha_k(t)
\end{pmatrix} \theta_k(t),
\] (102)
where \( k \) denotes the canonical mode of the field, \( \varepsilon = +1 (-1) \) for bosons (fermions) and \( \alpha_k \).\( n_k(t) \) and \( \gamma_k(t) \) are parameters of the gauge group of thermal mappings which is \( SU(1,1) \) for bosons and \( SO(2) \) for fermions. The pure complex phase function \( \theta_k(t) \) depends on the energy of the mode and satisfies the first order differential equation

\[
\frac{d}{d\tau} \theta_n(\tau) = \omega_n(\tau) \theta_n(\tau).
\] (103)

The physical interpretation \( n_k(t) \) is that of the number density defined as

\[
n_k(t)\delta(k-l) = \langle 0(t) \left| a_k^\dagger a_l \right| 0(t) \rangle.
\] (104)

The function \( f^{\alpha_k}(t) \) is the statistical distribution

\[
f^{\alpha_k}(t) = \frac{n_k(t)}{1 + \varepsilon n_k(t)}.
\] (105)

The parameters \( \alpha_k \in [0,1] \) lead, in general, to different representations of the thermal field in the Liouville space. The choice of the gauge \( \alpha_k = 1 \) and \( \gamma_k(t) = \ln [1 + \varepsilon n_k(t)] \) renders the Bogoliubov transformations linear \( [60] \)

\[
B_k(t) = \begin{pmatrix}
1 + \varepsilon n_k(t) & -n_k(t) \\
\varepsilon & 1
\end{pmatrix}.
\] (106)
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