FIDELITY OF MÖBIUS MATRICES RELATED WITH LORENTZ BOOSTS

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Abstract. In this article we consider the extended version of a real counterpart of qubit density matrices, called the Möbius matrix, and we see that it is a normalized Lorentz boost. Using the isomorphic gyrogroup structures between the set P of all Lorentz boosts and the Einstein gyrogroup on the open unit ball B of R^n we give an explicit formula of the fidelity for Möbius matrices in terms of Lorentz gamma factors.

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1. Introduction

A qubit density matrix is a 2 × 2 positive semidefinite Hermitian matrix with trace 1. It can be described by a Bloch vector

\[ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \|\mathbf{v}\| \leq 1, \]

where \( \| \cdot \| \) is the Euclidean norm. In details,

\[ \rho_\mathbf{v} = \frac{1}{2} \begin{pmatrix} 1 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 \end{pmatrix} = \frac{1}{2} (v_1 \sigma_x + v_2 \sigma_y + v_3 \sigma_z), \]

where

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

are Pauli matrices. It is known that all qubit pure states are parameterized by the unit sphere, while all qubit mixed states (or invertible density matrices) are parameterized by the open unit ball in \( \mathbb{R}^3 \). In the following we denote the open unit ball in \( \mathbb{R}^n \) as \( B_n \) and consider column vectors \( \mathbf{v} \) in \( \mathbb{R}^n \).

In general, it is difficult to extend the qubit mixed state \( \rho_\mathbf{v} \) to a density matrix that is parametrized by an n-dimensional Bloch vector \( \mathbf{v} = (v_1, v_2, \ldots, v_n) \in B_n \) for \( n > 3 \). On the
other hand, A. A. Ungar has suggested in [7, Section 9.5] the real counterpart of \( \rho_v \) that shares similar properties with \( \rho_v \) and its extended version such as

\[
\mu_{n,v} = \frac{2\gamma_v^2}{(n-3) + 4\gamma_v^2} \left( 1 - \frac{1}{2\gamma_v} \right) \begin{pmatrix}
1 & v^T \\
v & \frac{1}{2\gamma_v} I_n + vv^T
\end{pmatrix}
\]

\[
= \frac{2\gamma_v^2}{(n-3) + 4\gamma_v^2} \begin{pmatrix}
v_1 & v_2 & \cdots & v_n \\
v_2 & v_1v_2 + \frac{1}{2\gamma_v} & \cdots & \vdots \\
v_3 & \vdots & \ddots & \vdots \\
v_n & \vdots & \cdots & v_n + \frac{1}{2\gamma_v} + v_n^2
\end{pmatrix}
\]

where \( I_n \) is the \( n \times n \) identity matrix and \( \gamma_v = \frac{1}{\sqrt{1 - \|v\|^2}} \) is known as the Lorentz gamma factor. This is called a M"{o}bius matrix parameterized by the vector \( v = (v_1, v_2, \ldots, v_n) \in \mathbb{B}_n \).

Although it is not a natural extension of the qubit density matrix, it is meaningful that we explore \( \mu_{n,v} \) as a density matrix in the study of higher-level quantum states. The aim of this paper is to see the fidelity of M"{o}bius matrices, as one of the known measurements.

Lorentz was seeking the transformation under which Maxwell’s equations were invariant when transformed from the ether to a moving frame. In 1905 Henri Poincaré recognized that the transformation has the properties of a mathematical group and named it after Lorentz. Later in the same year Albert Einstein derived the Lorentz transformation under the assumption of the principle of relativity and the constancy of the speed of light in any inertial reference frame. Lorentz transformation of the relativistically admissible vector is currently an important tool in special relativity, since it enables us to study relativistic mechanics in hyperbolic geometry. It also may include a rotation of space, and especially a rotation-free Lorentz transformation is called a Lorentz boost. The Lorentz boost is a positive definite member of the Lorentz group \( O(1, n) \), the group (under composition) of all linear transformations preserving the Lorentz form \( \mathcal{L} \) defined by

\[
\mathcal{L}(\langle s, x_1, \ldots, x_n \rangle, \langle t, y_1, \ldots, y_n \rangle) = -st + \sum_{i=1}^n x_i y_i.
\]

Indeed, the Lorentz boost is a member of the restricted Lorentz group \( SO^+(1, n) \), the identity component of the Lorentz group consisting of all proper orthochronous maps.

In this paper, we can see the interesting result that the M"{o}bius matrix is a normalized Lorentz boost. So the study of M"{o}bius matrices will be associated with the algebraic structure of Lorentz boosts. In Section 2 we review a non-associative algebra structure
(called a gyrogroup) on the set of Lorentz boosts and provide an isomorphism with the Einstein gyrogroup on the open unit ball $B_n$. In Section 3 we show that the Möbius matrix is a normalized Lorentz boost via a diagonalization, and in Section 4 we calculate the fidelity of Möbius matrices and give an explicit formula in terms of Lorentz gamma factors.

2. Gyrogroup for Lorentz boosts

We review first the Einstein’s relativistic sum of admissible velocities of which magnitude is less than the speed of light $c \approx 3 \times 10^5$ km/sec. In our purpose of this article, we assume the speed of light is normalized by the value 1, so that the admissible vectors are in the open unit ball

$$B_n := \{ v \in \mathbb{R}^n : \|v\| < 1 \}.$$ 

Then the relativistic sum of two admissible vectors $u$ and $v$ in $B_n$ is given by

$$u \oplus v = \frac{1}{1 + u^T v} \left\{ u + \frac{1}{\gamma_u} v + \frac{\gamma_u}{1 + \gamma_u (u^T v)} u \right\},$$

where $\gamma_u$ is the well-known Lorentz factor

$$\gamma_u = \frac{1}{\sqrt{1 - \|u\|^2}}.$$  

Definition 2.1. The formula (2.1) defines a binary operation, called the Einstein velocity addition, on the open unit ball $B_n$ of $\mathbb{R}^n$.

Remark 2.2. The Einstein addition $u \oplus v$ of two admissible vectors $u$ and $v$ in $B_n$ may be alternatively obtained by applying the Lorentz boost

$$B(u) = \begin{pmatrix} \gamma_u & \gamma_u u^T \\ \gamma_u u & I + \frac{\gamma_u u^T u}{1 + \gamma_u u^T u} \end{pmatrix}$$

(2.3)

to \( \begin{pmatrix} \gamma_v \\ \gamma_v v \end{pmatrix} \) and obtaining

$$B(u) \begin{pmatrix} \gamma_v \\ \gamma_v v \end{pmatrix} = \begin{pmatrix} \gamma_{u \oplus v} \\ \gamma_{u \oplus v} (u \oplus v) \end{pmatrix},$$

where we use the gamma identity $\gamma_{u \oplus v} = \gamma_u \gamma_v (1 + u^T v)$.

To abstractly analyze Einstein velocity addition in the theory of special relativity, A. A. Ungar has introduced and studied in several papers and books structures that he has called gyrogroups; see [7] and its bibliography. His algebraic axioms are reminiscent of those for a group, but a gyrogroup operation is neither associative nor commutative in general.
**Definition 2.3.** A triple $(G, \oplus, 0)$ is a gyrogroup if the following axioms are satisfied for all $a, b, c \in G$.

1. **(G1)** $0 \oplus a = a \oplus 0 = a$ (existence of identity);
2. **(G2)** $a \oplus (-a) = (-a) \oplus a = 0$ (existence of inverses);
3. **(G3)** There is an automorphism $\text{gyr}[a, b] : G \to G$ for each $a, b \in G$ such that
   
   $$a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a, b]c$$
   (gyroassociativity);
4. **(G4)** $\text{gyr}[0, a] = \text{id}_G$;
5. **(G5)** $\text{gyr}[a \oplus b, b] = \text{gyr}[a, b]$ (loop property).

A gyrogroup $(G, \oplus)$ is gyrocommutative if it satisfies

$$a \oplus b = \text{gyr}[a, b](b \oplus a)$$
(gyrocommutativity).

A gyrogroup is uniquely 2-divisible if for every $b \in G$, there exists a unique $a \in G$ such that $a \oplus a = b$.

The map $\text{gyr}[a, b]$ is called the gyroautomorphism or Thomas gyration generated by $a$ and $b$, which is analogous to the precession map in a loop theory. It has been shown in [6] that gyrocommutative gyrogroups are equivalent to Bruck loops with respect to the same operation. It follows that uniquely 2-divisible gyrocommutative gyrogroups are equivalent to $B$-loops, uniquely 2-divisible Bruck loops. J. Lawson and Y. Lim have recently introduced dyadic symmetric sets in [3] and showed the equivalence with uniquely 2-divisible gyrocommutative gyrogroups. In our purpose of this article we follow the notion of gyrogroups.

A. A. Ungar has shown in [7, Chapter 3] by computer algebra that Einstein addition on the open unit ball $B_n$ is a gyrocommutative gyrogroup operation, and the gyroautomorphisms are orthogonal transformations preserving the Euclidean inner product and the inherited norm. We call $(B_n, \oplus)$ the Einstein (gyrocommutative) gyrogroup, where $\oplus$ is defined by the equation (2.1).

**Remark 2.4.** We note that the Einstein gyrogroup $(B_n, \oplus)$ is uniquely 2-divisible; for any $v \in B_n$ there exists a unique

$$w = \frac{\gamma_v}{1 + \gamma_v} v \in B$$

such that $w \oplus w = v$ (see the equation (6.297) of [7]). We denote it simply by $w := (1/2) \otimes v$, or $v = 2 \otimes w$.

We now see the gyrogroup structure on the set $\mathbb{P}$ of all Lorentz boosts given in the equation (2.3). From the polar decomposition of $B(u)B(v)$ for $u, v \in B_n$ we have the relation

$$B(u \oplus v) = (B(u)B(v)B(u))^1/2,$$
(2.4)
see [2] for more details. Hence, we obtain

**Theorem 2.5.** The Lorentz boost map $B$ is an isomorphism from $(B_n, \oplus, 0)$ to $(\mathbb{P}, \star, I)$, where

$$B(u) \star B(v) = (B(u)B(v)^2B(u))^{1/2}.$$ 

Furthermore, the powers and roots in $(\mathbb{P}, \star)$ agree with those of matrix multiplication.

On the cone $\Omega$ of positive definite Hermitian matrices, the squaring map $D : \Omega \to \Omega$, $D(A) = A^2$ gives us a different algebraic structure on the set $\mathbb{P}$. We note that the squaring map $D$ is a bijection since any positive definite Hermitian matrix has a unique square root in $\Omega$.

**Theorem 2.6.** The composition $D \circ B : (B_n, \oplus) \to (\mathbb{P}, \star)$ is also an isomorphism, where

$$B(u) \star B(v) = B(u)^{1/2} B(v) B(u)^{1/2}.$$ 

**Remark 2.7.** From Theorem 2.5 and Theorem 2.6 we see that both $(\mathbb{P}, \star, I)$ and $(\mathbb{P}, \star, I)$ are uniquely 2-divisible gyrocommutative gyrogroups. Moreover, we have

$$B(2 \otimes v) = B(v)^2, \quad B((1/2) \otimes v) = B(v)^{1/2}$$

for any $v \in B_n$.

### 3. Möbius matrices and Lorentz boosts

First of all, we see the Möbius matrix parameterized by the vector $v \in B_n, n \geq 3$, as an extended version of the real counterpart of qubit density matrices:

$$\mu_{n,v} = \frac{2\gamma^2_v}{(n-3) + 4\gamma^2_v} \left( 1 - \frac{1}{2\gamma_v} v^T \frac{1}{\gamma_v} I_n + vv^T \right).$$  \hspace{1cm} (3.5)$$

This is an $(n+1) \times (n+1)$ symmetric matrix, and we verify a diagonalization of Möbius matrix.

**Theorem 3.1.** For each $v \in B_n$ there exist an orthogonal matrix $O_v$ and a diagonal matrix $D_v$

$$O_v = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{2||v||}} v & \frac{1}{\sqrt{2||v||}} v & u_1 & \cdots & u_{n-1} \end{pmatrix},$$

$$D_v = \frac{1}{(n-3) + 4\gamma^2_v} \begin{pmatrix} \lambda^2 & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda^2} & \cdots & 0 \\ 0 & 0 & \cdots & I_{n-1} \end{pmatrix}.$$
such that \( \mu_{n,v} = O_v^T D_v O_v \), where \( \{u_j : v^T u_j = 0 \text{ for all } j = 1, 2, \ldots, n - 1\} \) is an orthonormal set obtained by the Gram-Schmidt process, and

\[
\lambda = \sqrt{\frac{1 + \|v\|}{1 - \|v\|}} > 1.
\]

**Proof.** Let

\[
A = \frac{(n - 3) + 4\gamma_v^2}{2\gamma_v^2} \mu_{n,v} = \left( \begin{array}{cc}
1 - \frac{1}{2\gamma_v^2} & v^T \\
v & \frac{1}{2\gamma_v^2} I_n + vv^T
\end{array} \right)
\]

It is enough to show that

\[
A = O_v^T \cdot \frac{1}{2\gamma_v^2} \left( \begin{array}{ccc}
\lambda^2 & 0 & 0 \\
0 & \frac{1}{\lambda^2} & 0 \\
0 & 0 & I_{n-1}
\end{array} \right) \cdot O_v.
\]

Indeed,

\[
A \left( \frac{1}{\sqrt{2\|v\|}} v \right) = \left( \frac{1}{\sqrt{2\|v\|}} \left( 1 - \frac{1}{2\gamma_v^2} \right) + \frac{1}{\sqrt{2\|v\|}} v^T v \right)
\]

\[
= \left( \frac{1}{2\sqrt{2\|v\|}} (1 + \|v\|)^2 \right)
\]

\[
= \frac{(1 + \|v\|)^2}{2} \left( \frac{1}{\sqrt{2\|v\|}} v \right)
\]

Here, \( \frac{(1 + \|v\|)^2}{2} = \frac{1}{2\gamma_v^2} \cdot \frac{1 + \|v\|}{1 - \|v\|} = \frac{\lambda^2}{2\gamma_v^2} \). Similarly,

\[
A \left( \frac{-1}{\sqrt{2\|v\|}} v \right) = \frac{(1 - \|v\|)^2}{2} \left( \frac{-1}{\sqrt{2\|v\|}} v \right) = \frac{1}{2\gamma_v^2 \lambda^2} \left( \frac{-1}{\sqrt{2\|v\|}} v \right)
\]

Finally for each \( j = 1, 2, \ldots, n - 1 \)

\[
A \left( \begin{array}{c}
0 \\
u_j
\end{array} \right) = \left( \frac{v^T u_j}{\frac{1}{2\gamma_v^2} I_n + vv^T} u_j \right) = \frac{1}{2\gamma_v^2} \left( \begin{array}{c}
v^T u_j \\
u_j
\end{array} \right)
\]

since \( v^T u_j = 0 \).

\( \square \)

**Remark 3.2.** By Theorem 3.1 we have that the matrix \( \mu_{n,v} \) is positive definite,

\[
\text{tr} \mu_{n,v} = \frac{1}{(n - 3) + 4\gamma_v^2} \left( \lambda^2 + \frac{1}{\lambda^2} + n - 1 \right) = 1,
\]
and
\[ \det \mu_{n, \nu} = \left( \frac{1}{(n - 3) + 4\gamma^2} \right)^{n+1} = \left( \frac{1 - \|\nu\|^2}{(n + 1) - (n - 3)\|\nu\|^2} \right)^{n+1} > 0. \]

We proved the equation (9.85) in [7] and that \( \mu_{n, \nu} \) is an \((n + 1) \times (n + 1)\) real mixed state.

In [2, Theorem 5.6] it has been shown that
\[ \begin{align*}
B(\nu) &= O^T \begin{pmatrix}
\lambda & 0 & 0 \\
0 & \frac{1}{\lambda} & 0 \\
0 & 0 & I_{n-1}
\end{pmatrix} O_v, 
\end{align*} \tag{3.6} \]
where \( O_v \) is the same orthogonal matrix in Theorem 3.1. So we obtain the interesting result that Möbius matrix is the normalized Lorentz boost generated by the vector \( 2 \otimes \nu \), since
\[ \text{tr} B(\nu)^2 = \lambda^2 + \frac{1}{\lambda^2} + n - 1 = 2\gamma^2 (1 + \|\nu\|^2) + n - 1 = (n - 3) + 4\gamma^2 \nu. \]

Proposition 3.3. For each \( \nu \in B_n \),
\[ \mu_{n, \nu} = \frac{1}{\text{tr} B(\nu)^2} B(\nu)^2 = \frac{1}{\text{tr} B(2 \otimes \nu)} B(2 \otimes \nu). \]

4. Fidelity

It has been issued how to measure the distance of quantum states represented by density matrices, i.e., positive semidefinite Hermitian matrices with trace 1. The fidelity is one of crucial measurements although it is actually not a metric for quantum states. On the other hand, it is a measure of the closedness of two quantum states, that is, the fidelity is 1 if and only if two quantum states are identical. Moreover, it does give rise to a useful metric and is able to apply for a variety of research areas in quantum information and computation theory; see [5] and [4, Section 9.2.2].

The fidelity for density matrices \( \rho \) and \( \sigma \) is defined by
\[ F(\rho, \sigma) := \text{tr} \sqrt{\rho^{1/2} \sigma \rho^{1/2}}. \tag{4.7} \]

We review some basic properties of the fidelity.

Lemma 4.1. The following are satisfied for any density matrices \( \rho \) and \( \sigma \).

(i) \( 0 \leq F(\rho, \sigma) \leq 1 \).
(ii) \( F(\rho, \sigma) = 1 \) if and only if \( \rho = \sigma \).
(iii) \( F(\rho, \sigma) = F(\sigma, \rho) \).
(iv) \( F(U\rho U^*, U\sigma U^*) = F(\rho, \sigma) \) for any unitary \( U \).

The property (iv) of Lemma 4.1 is called the invariance under unitary congruence transformation, so that the fidelity is basis-independent.
Remark 4.2. The fidelity $F$ can be quite difficult to calculate, but it takes a simple form for the 2-by-2 density matrices $\rho$ and $\sigma$: see the equation (8.52) in [1],

$$F(\rho, \sigma)^2 = \text{tr}(\rho \sigma) + 2\sqrt{\det(\rho) \det(\sigma)}. \quad (4.8)$$

From the equations (9.64) and (9.68) in [7] we have alternative expression of the fidelity for the 2-by-2 density matrices $\rho_\mu$ and $\rho_\nu$, where $\mu, \nu \in B_3$:

$$F(\rho_\mu, \rho_\nu)^2 = 1 + \frac{\gamma_{\mu \oplus \nu}}{2\gamma_{\mu} \gamma_{\nu}} = \frac{1}{2} \left\{ 1 + u^T v + \sqrt{1 - \|u\|^2} \sqrt{1 - \|v\|^2} \right\}. \quad (4.9)$$

One can verify that two equations (4.8) and (4.9) are the same.

We introduce a normalized Lorentz boost to be a density matrix, especially the Möbius matrix. Let us denote

$$\hat{B}(v) := \frac{1}{\text{tr} B(v)} B(v)$$

for any $v \in B_n$. Indeed, $\hat{B}(v)$ must be a mixed state, a positive definite density matrix, and by Proposition 3.3 we have $\mu_{n,v} = \hat{B}(2 \otimes v)$.

We see useful properties to show the main result.

Lemma 4.3. For any positive semidefinite matrices $A, B$ and $\alpha, \beta > 0$,

$$F(\alpha A, \beta B) = \sqrt{\alpha \beta} F(A, B).$$

Lemma 4.4. For any $u, v \in (B_n, \oplus)$

$$\text{tr}[B(u)B(v)^2B(u)]^{1/2} = 2\gamma_{u \oplus v} + n - 1.$$

Proof. From the equation (3.6) we have

$$\text{tr} B(v) = \lambda + \frac{1}{\lambda} + n - 1 = 2\gamma_v + n - 1.$$

So it is proved by the equation (2.4). \qed

We now see an explicit formula of the fidelity for normalized Lorentz boosts in terms of Lorentz factors.

Theorem 4.5. For any $u, v \in (B_n, \oplus)$

$$F(\hat{B}(u), \hat{B}(v)) = \frac{2\gamma_w + n - 1}{\sqrt{(2\gamma_u + n - 1)(2\gamma_v + n - 1)}},$$

where

$$w = \frac{1}{2} \otimes u \oplus \frac{1}{2} \otimes v.$$
Proof. Let \( u' := (1/2) \otimes u \) and \( v' := (1/2) \otimes v \). Then

\[
F(\hat{B}(u), \hat{B}(v)) = \frac{F(B(u), B(v))}{\sqrt{\text{tr} B(u) \text{tr} B(v)}} = \frac{\text{tr}[B(u')B(v')^2B(u')]}{\sqrt{\text{tr} B(u) \text{tr} B(v)}}
\]

\[
= \frac{2\gamma_{u' \otimes v'} + n - 1}{\sqrt{(2\gamma_u + n - 1)(2\gamma_v + n - 1)}}.
\]

The first equality follows from Lemma 4.3, the second follows from Remark 2.7, and the last follows from Lemma 4.4. \( \square \)

For any \( u, v \in B_n \), in general,

\[
(1/2) \otimes u \oplus (1/2) \otimes v \neq (1/2) \otimes (u \oplus v),
\]

see [7, Chapter 6] for more details. On the other hand, we give an formula of Lorentz factor for \( w = (1/2) \otimes u \oplus (1/2) \otimes v \), so that the fidelity for Möbius matrices can be simply calculated.

**Lemma 4.6.** For any \( u, v \in (B_n, \oplus) \),

\[
\gamma_w = \frac{1}{\sqrt{1 + 2\gamma_u}} \frac{1}{\sqrt{1 + 2\gamma_v}} ((1 + \gamma_u)(1 + \gamma_v) + \gamma_u\gamma_v u^T v),
\]

where \( w = (1/2) \otimes u \oplus (1/2) \otimes v \).

**Proof.** Let \( u' := (1/2) \otimes u \) and \( v' := (1/2) \otimes v \). By Remark 2.4

\[
\gamma_{v'} = \frac{1}{\sqrt{1 - \|v'\|^2}} = \frac{1}{\sqrt{1 - \gamma_v^2/(1 + \gamma_v)^2}} = \frac{1 + \gamma_v}{\sqrt{1 + 2\gamma_v}}.
\]

Applying the gamma identity \( \gamma_{u \oplus v} = \gamma_u \gamma_v (1 + u^T v) \) to \( u' \) and \( v' \), it is proved. \( \square \)

**Remark 4.7.** Directly from the Einstein velocity addition we have

\[
2 \otimes v = v \oplus v = \frac{2\gamma_v^2}{2\gamma_v^2 - 1} v,
\]

so that \( \gamma_{2 \otimes v} = 2\gamma_v^2 - 1 \). Hence, the result of Theorem 4.5 reduces to

\[
F(\mu_{n,u}, \mu_{n,v}) = F(\hat{B}(2 \otimes u), \hat{B}(2 \otimes v)) = \frac{\gamma_{u \otimes v} + n - 1}{\sqrt{(4\gamma_u^2 + n - 3)(4\gamma_v^2 + n - 3)}}.
\]

Especially, if \( n = 3 \),

\[
F(\mu_{3,u}, \mu_{3,v}) = \frac{1 + \gamma_{u \otimes v}}{2\gamma_u \gamma_v} = F(\rho_u, \rho_v)^2.
\]
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