SYMMETRIES OF THE SPACE OF SOLUTIONS TO SPECIAL DOUBLE CONFLUENT HEUN EQUATION OF NEGATIVE INTEGER ORDER AND ITS APPLICATIONS

SERGEY I. TERTYCHNIY

Abstract.
Three linear operators ($\mathcal{L}$-operators) determining automorphisms of the space of solutions to a special double confluent Heun equation (sDCHE) of negative integer order are considered. Their composition rules involving in a natural way the monodromy transformation are given. Introducing eigenfunctions $E_{\{i\}}, E_{\{\}i}$ of one of $\mathcal{L}$-operators ($\mathcal{L}_C$) which also satisfy sDCHE, the four polylocal quadratic functionals playing role of the first integrals of sDCHE are derived. Basing on them, the explicit matrix representations of $\mathcal{L}$-operators and the monodromy operator with respect to the basis constituted by $E_{\{i\}}$ are constructed. The composition rules of $\mathcal{L}$-operators lead to functional equations for the eigenfunctions $E_{\{i\}}$ which can be interpreted as analytic continuations of solutions to sDCHE from the half-plane $\mathbb{R}z > 0$ to their whole domain by means of algebraic transformations. Application of the above results to the theory of the first order non-linear differential equation utilized, in particular, for the modeling of overdamped Josephson junctions in superconductors and closely related to sDCHE is presented. The automorphisms of the set of its solutions induced by $\mathcal{L}$-operators and by the monodromy operator (certain shift of the solution domain in the latter case) represented by algebraic operations are found. Among them, one transformation is involutive and another can be regarded as the square root of the transformation induced by the monodromy transformation.

Let us consider the following linear homogeneous second order differential equation

$$z^2 E'' + ((l + 1)z + \mu(1 - z^2)) E' + (-\mu(l + 1)z + \lambda) E = 0,$$

where $l, \mu \neq 0$, and $\lambda$ are some constant parameters, the argument of the holomorphic function $E = E(z)$ is a point $z$ of the universal cover $\mathcal{C}^*$ of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, the complex plane with zero removed, $z$ denotes result of the canonical projection $\iota : \mathcal{C} \to \mathbb{C}^*$ of $z = \iota z$. The Riemann surface $\mathcal{C}^*$ plays the role of the natural domain of holomorphic solutions to Eq. (1) singular at zero. Its use is necessary for the capturing of some their global properties but most of local considerations, including establishing of fulfillment of Eq. (1), allows to replace the points of $\mathcal{C}^*$ by their projections (complex numbers) living in $\mathbb{C}^*$.

Eq. (1) belongs to the family of so called double confluent Heun equations (1, 2) which name is often abbreviated to DCHE. In Ref. (1, 2) a generic DCHE is given in the following form

$$D^2 v + \alpha(z + z^{-1}) D v + ((\beta_1 + 1/2)\alpha z + \alpha^2/2 - \gamma + (\beta_{-1} - 1/2)\alpha z^{-1}) v = 0,$$

where $v = v(z)$ is the unknown, $D = z d/dz$, and $\alpha, \beta_1, \beta_{-1}, \gamma$ are arbitrary complex constants. A straightforward computation shows that the ansatz

$$v = z^{l/2} e^{-\mu z} E,$$

$$\alpha = \mu, \quad \beta_1 = -(l + 1)/2, \quad \beta_{-1} = (1 - l)/2, \quad \gamma = l^2/4 - \lambda - \mu^2/2$$

leads just to Eq. (1) (up to local interpretation of the argument of unknown). Eq. (1) is therefore equivalent to Eq. (2), provided the constraint

$$\beta_1 + 1/2 = \beta_{-1} - 1/2$$

is imposed. Since Eq. (1) is characterized by the three free constant parameters while there are four such parameters in Eq. (2), Eq. (1) is sometimes named special double confluent Heun equation. The reason why equations of the form (1) were segregated within the general DCHE family will be discussed below.

Supported in part by RFBR grant N 17-01-00192.
In the present notes, we consider a particular case of Eq. (1) which arises when the parameter \( l \) (sometimes named its order) is integer. Moreover, since the case of positive integer \( l \) has been studied elsewhere (see [3]), we claim also of \( l \) to be negative, i.e. assume that

\[
l = -\ell \quad \text{for some } \ell \in \mathbb{N},
\]

(4)

A particular interest to Eq. (1) of integer order is motivated by discovering of additional symmetries of the space \( \Omega \) of its solutions. It is manifested by existence of the three linear operators admitting simple explicit representations which take \( \Omega \) onto itself [4]. Moreover, apart of rather special values of parameters, these operators are invertible and thus act on \( \Omega \) as automorphisms. In conjunction with the monodromy map, these give rise to the group of symmetries of \( \Omega \).

Specifically, let us denote by the symbols \( \mathcal{L}_A, \mathcal{L}_B, \mathcal{L}_C \) the transformations of functions \( H \) holomorphic in \( \mathbb{C}^* \) which are described by the following formulas:

\[
\mathcal{L}_A : E(\zeta) \mapsto \mathcal{L}_A[E](\zeta) = z^2(\ell + 1)\zeta^{\mu(z+1/z)}
\]

\[
\mathcal{L}_B : E(\zeta) \mapsto \mathcal{L}_B[E](\zeta) = 2\omega z^1\zeta^{\mu(z+1/z)}
\]

\[
\mathcal{L}_C : E(\zeta) \mapsto \mathcal{L}_C[E](\zeta) = 2\omega z^1(1-E' - \mu E)
\]

(5), (6), (7)

In them, \( \omega \) is an arbitrary non-zero constant while the symbols \( r, q, s, t, s \) denote certain functions of \( z \) (depending also on the parameters \( \ell, \lambda, \mu \)) which are defined as follows.

(1) Let us introduce the two sequences of the pairs of functions \( \{p_k, q_k\}, \{r_k, s_k\}, k = 0, 1, 2, \ldots, \) of the variable \( z \in \mathbb{C}^* \) by means of the following recurrence schemes:

\[
p_0 = 0, \quad q_0 = 1, \quad r_0 = z^{-2}, \quad s_0 = -\mu;
\]

\[
p_k = (1 - \ell)z p_{k-1} + q_{k-1} + z^2 p'_{k-1}, \quad \text{for } k \in \mathbb{N}
\]

\[
q_k = z^2(-\lambda + (\ell + 1)\mu z)q_{k-1} + \mu (1 - z^2) q_{k-1} + z^2 q'_{k-1}
\]

\[
r_k = 2(k - 2)z r_{k-1} - s_{k-1} - z^2 r'_{k-1},
\]

\[
s_k = z^2(\lambda - (\ell + 1)\mu) r_{k-1} + ((2(k - 1) - (\ell + 1)) z + \mu (z^2 - 1)) s_{k-1} - z^2 s'_{k-1};
\]

(8), (9), (10)

(2) let us pick out their “diagonal” elements for which the pair’s indices are equal to \( \ell \) and assign

\[
p = p_{\ell}, \quad q = q_{\ell}, \quad r = r_{\ell}, \quad s = s_{\ell}.
\]

(11)

Albeit \( r_0 \) is singular at \( z = 0 \), for any greater index \( k \) all the functions \( p_k, q_k, r_k, s_k \) are polynomial in \( z \). In particular, the functions \( p, q, r, s \) are also polynomials and it can be shown that their degrees are equal to \( 2(\ell - 1), 2\ell, 2(\ell - 1), 2\ell, \) respectively [4]; accordingly, the functions \( p = p(z), \ldots, s = s(z) \) are well defined and are holomorphic everywhere in \( \mathbb{C} \). (It is worth noting that they are polynomial in \( \lambda \) and \( \mu \) as well although this dependence is not manifested in our formulas). Thus, in the formulas \( \mathcal{L}_A, \mathcal{L}_B, \mathcal{L}_C \), the symbols \( p, q, r, s \) stand for the known entire functions of \( z = i \), which can be computed in explicit form for every given \( \ell \). As above, the symbols \( E, E' \) also denote some holomorphic functions but their argument is \( \zeta \in \mathbb{C}^* \). Finally, the “aggregates” of the form \( \{ z = \zeta \} \) denote the results of substituting of the expressions \( \zeta \) and \( \zeta \) into the expression \( \{ \cdot \cdot \cdot \} \) considered as a function of \( \zeta \) and \( \zeta \).

Having explained the notations, it is important to note that Eq.s (5)-(7) cannot not still serve themselves exhaustive definitions of certain transformation rules as they should. Their insufficiency lurks in the lack of uniqueness inherent to the lifts of the maps \( A | B | C : z \to -1/z | -z | 1/z \) from the punctured complex plane \( \mathbb{C}^* \) to its universal cover \( \mathbb{C}^* \). In other words, one must additionally clarify what is the actual meaning of the records \(-1/\zeta, -\zeta, 1/\zeta \) encompassed in \( \mathcal{L}_A, \mathcal{L}_B, \mathcal{L}_C \), the point which is simple but yet plays sometimes a non-trivial role.

To be more specific, let us model the Riemann surface \( \mathbb{C}^* \) by the set \( \mathbb{R}_+ \times \mathbb{R} \) of all the pairs \( (\rho, \phi) \) of real numbers (coordinates) of which the first one is strictly positive, \( \rho > 0 \). Then the
projection \( \iota \) of \( \mathbb{C}^* \) to \( \mathbb{C}^* \) is defined by the map \( \mathbb{C}^* \ni z := (\rho, \phi) \mapsto \rho e^{i\phi} = z =: \iota z \in \mathbb{C}^* \). We shall refer to the above construction as the similog model (of \( \mathbb{C}^* \)). It allows to identify simply connected subsets of \( \mathbb{C}^* \) with the corresponding subsets of \( \mathbb{C}^* \). Then functions holomorphic in \( \mathbb{C}^* \) are defined as those which locally coincide with functions holomorphic in \( \mathbb{C}^* \). The algebraic operations in \( \mathbb{C}^* \) must be conveyed through the projection \( \iota \) to the corresponding operations in \( \mathbb{C}^* \).

Let us consider, say, how the map \( B : z \mapsto -z \) can be lifted to \( \mathbb{C}^* \). In frames of the above model, the obvious concordant transformations are \( \hat{-} : (\rho, \phi) \mapsto (\rho, \pi + \phi) \) (in projection to \( \mathbb{C}^* \), the counterclockwise rotation through \( \pi \) which is homotopic to the identical transformation) and \( \hat{-} : (\rho, \phi) \mapsto (\rho, -\pi + \phi) \) (similar but this time clockwise rotation through the same angle, respectively). These lifts of \( B \) are distinct and neither of them is preferable. The projections of the results of their actions to a point, at first, coincide and, at second, differ from the projection of the argument by the opposite sign alone. For example, “reversing sign” of the point \( J := (1, 0) \) projected to \(-1\) we obtain the two distinct points of \( \mathbb{C}^* \), \(-J := (1, \pi)\) and \(-J := (1, -\pi)\), which both project to \( (1, 0) \) (“play the role of”) the number \(-1\).

For the inversion map \( C : z \mapsto 1/z \), the eye-catching transformation which could serve its lift to \( \mathbb{C}^* \), is obviously \( 1/ : (\rho, \phi) \mapsto (1/\rho, -\phi) \). Although it may seem that nothing more is needed, it would actually be insufficient to limit our choice by such an instance alone. Indeed, on \( \mathbb{C}^* \), the map \( C \) possesses the two fixed points, \( z = +1 \) and \( z = -1 \). At the same time, the above lift \( 1/ \) has the only fixed point \( J := (1, 0) \), which projects to \(+1\), obviously. Hence for the sake of completeness we have to introduce yet another lift of the inversion map which would possess a fixed point projected to \(-1\). But we have noted above two points with such projection, being not able to choose a preferable one among them. The corresponding lifts of the inversion map are easily constructed: the map \( \hat{1} / : (\rho, \phi) \mapsto (1/\rho, 2\pi - \phi) \) possesses the only fixed point \( -J \), the only fixed point of the map \( \hat{1} / : (\rho, \phi) \mapsto (1/\rho, -2\pi - \phi) \) is \( -J \). Thus, in total, the three candidates to the role of lift of the inversion map \( C \) to \( \mathbb{C}^* \) have arisen.

In case of the third involutive map \( A : z \mapsto -1/z \) the use of the fixed points is also a convenient trick helping to figure out the “minimal collection” of its lifts. Now, in \( \mathbb{C}^* \), the fixed points are the pure imaginary \( i \) and \(-i\). The points projecting to them which are most close both between themselves and also to the lift of the unit \( J \) (an additional criterion for our selection) are \( i := (1, \pi/2) \) and \(-i := (1, -\pi/2) \). They are the fixed points of the maps \(-\hat{1} / : (\rho, \phi) \mapsto (1/\rho, \pi - \phi) \) and \(-\hat{1} / : (\rho, \phi) \mapsto (1/\rho, -\pi - \phi) \), respectively, which both represent the lifts of \( A \).

It follows from the above examples that although the replacements \( z := -1/z \) \(-z \mid 1/z \) utilized in formulas \( 12 \)-\( 14 \), where they act to arguments of holomorphic functions, can be extended to the case of the domain \( \mathbb{C}^* \), the triplet of the corresponding lifts is in no way unique. To avoid such an uncertainty, we pick the only instance among the \( 2 \times 3 \times 2 \) of opportunities ensured by the above speculation (their amount is actually unlimited) and assume to adhere to the following

**Convention 1.** Let us assume the replacements of the argument \( z \in \mathbb{C}^* \) of the function \( E \) and its derivative \( E' \) involved in the formulas \( 22 \)-\( 24 \) to be realized in accordance with the following rules

\[
A : \quad z := (\rho, \phi) \mapsto (1/\rho, \pi - \phi) =: \hat{-} z, \\
B : \quad z := (\rho, \phi) \mapsto (\rho, \pi + \phi) =: -z, \\
C : \quad z := (\rho, \phi) \mapsto (1/\rho, -\phi) =: 1/z,
\]

respectively.

It is worth noting that the different realizations of lifts of the same transformations arise from ones indicated in Eq.s \( 12 \)-\( 14 \) by means of subtraction/addition of \( 2\pi \) from/to the second \( (\phi) \) component of the “conventional” result. In other words, they can be obtained applying additionally a number of the replacements

\[
M^{\pm 1} : \quad z := (\rho, \phi) \mapsto (\rho, \pm 2\pi + \phi) =: M^{\pm 1} z
\]

which represent the monodromy transformation and its inverse. Application of arbitrarily folded monodromy transformation \( M^k, k \in \mathbb{Z} \), sends any lift of a map on \( \mathbb{C}^* \) to another its lift. There exist
In a more thorough language transformation one as follows rule (15). For functions on $E$ point-wise analytic continuations of functions in $C$ transformation $M$ situated at zero which are passed in the anticlockwise direction (clockwise for inverse monodromy here $M$ in fact) as well as the space of its solutions. More generally, the set of functions holomorphic in $C$ identical map. Eq. (1) is invariant with respect to such operations (which are the linear operators $L$) which will be utilized here.

**Theorem 1.** Transformations $[5]-[7]$ send any solution to Eq. (1) to solution of the same equation.

In other words, these leave the space $\Omega$ of solutions to Eq. (1) invariant.

**Proof outline.** Let us substitute the right-hand side expressions of the formulas $[5]-[7]$ to Eq. (1) and eliminate from the result the higher derivatives $E''$, $E'''$ with the help of the same equation (1). In cases of $L_A$ and $L_B$, the resulting expressions contain also the first and second order derivatives of the polynomials $p,q,r,s$. They are eliminated with the help of the equations

\[
\begin{align*}
\ell^2 p' &= (\mu + (\ell - 1)z)p - q + (-1)^\ell \ell^2 r, \\
q' &= (\lambda - (\ell + 1)\mu z)p + \mu q + (-1)\ell^2 s, \\
\ell^2 q' &= (-1)^{\ell + 1}(\lambda + \mu^2)p + \mu(2(\ell - 1) - \mu z)r - s, \\
\ell^2 r' &= (-1)^{\ell + 1}(\lambda + \mu^2)q + \ell^2(\lambda - (\ell + 1)\mu z)q + ((\ell - 1)z - \mu)s.
\end{align*}
\]

which these functions obey (see [4]). Then, in all the three cases, algebraic simplifications finish with identically zero result.

Leaning on the specific properties of the functions $p,q,r,s$, in every other way, the proof reduces to straightforward computations due to explicit form of the basic formulas $[5]-[7]$. Similar (though somewhat more lengthy) computations allows one to obtain the composition rules of the $L$-operators, provided their domains are restricted to the space $\Omega$ of solutions to Eq. (1). Namely, these read

\[
\begin{align*}
L_A \circ L_A &= -D \cdot \mathbb{I}, \\
L_B \circ L_B &= (2\omega)^2(\lambda + \mu^2)D \cdot \mathcal{M}, \\
L_C \circ L_C &= (2\omega)^2(\lambda + \mu^2) \cdot \mathcal{M}^{-1} \circ \mathcal{M}, \\
L_A \circ L_B &= D \cdot \mathcal{M}^{-1} \circ L_A, \\
L_B \circ L_A &= -D \cdot L_C, \\
L_B \circ L_C &= -(2\omega)^2(\lambda + \mu^2) \cdot \mathcal{M} \circ L_A, \\
L_C \circ L_B &= (2\omega)^2(\lambda + \mu^2) \cdot L_A, \\
L_C \circ L_A &= L_B, \\
L_A \circ L_C &= -\mathcal{M}^{-1} \circ L_B \\
\text{where } D &= z^{2(1-\ell)}(ps - qr).
\end{align*}
\]

Here $\mathcal{M}$ and $\mathcal{M}^{-1}$ denote the monodromy transformation and its inverse which act on holomorphic functions in $\mathbb{C}^*$. We have defined the monodromy transformation $\mathcal{M}$ in action to points of $\mathbb{C}^*$ by the rule (10). For functions on $\mathbb{C}^*$, the monodromy transformation is inferred from it as an ancillary one as follows

\[(\mathcal{M}^{\pm 1}E)(z) = E(\mathcal{M}^{\pm 1}z).\]

In a more thorough language transformation $\mathcal{M} \equiv \mathcal{M}^1$ can also be described as the result of point-wise analytic continuations of $E$ along the arcs projected to full circles in $\mathbb{C}^*$ with centers situated at zero which are passed in the anticlockwise direction (clockwise for inverse monodromy transformation $\mathcal{M}^{-1}$). On lifts of functions holomorphic on $\mathbb{C}^*$ the transformations $\mathcal{M}^{\pm 1}$ act as the identical map. Eq. (11) is invariant with respect to such operations (which are the linear operators in fact) as well as the space of its solutions. More generally, the set of functions holomorphic in $\mathbb{C}^*$ is monodromy invariant.
The monodromy transformation arises in formulas (17)-(25) as a consequence of composition rules for transformations of arguments of the functions $E$ and $E'$ involved in the formulas (5)-(7). The results of such pairwise compositions are described by the following table of rules

| $\circ$ | A   | B   | C   |
|---------|-----|-----|-----|
| A       | $\text{Id}$ | $M^{-1} \circ C$ | $M^{-1} \circ B$ |
| B       | $C$ | $M$ | $M \circ A$ |
| C       | $B$ | $A$ | $\text{Id}$ |

(27)

which is a direct consequence of definitions (12)-(13).

In view of appearance of the monodromy operator, the collection of the composition rules (17)-(25), binding pairwise all the $L$-operators, proves to be non-closed or, better to say, incomplete. Indeed, once the monodromy transformation had there appeared, one must add to them the rules describing compositions of $L$-operators with the very monodromy operator. These take the form of commutation rules and read

$$ \mathcal{M}^{\pm 1} \circ \mathcal{L}_A = \mathcal{L}_A \circ \mathcal{M}^{\mp k}, $$

(28)

$$ \mathcal{M}^{\pm k} \circ \mathcal{L}_B = \mathcal{L}_B \circ \mathcal{M}^{\mp k}, $$

(29)

$$ \mathcal{M}^{\pm k} \circ \mathcal{L}_C = \mathcal{L}_C \circ \mathcal{M}^{\mp k}, \quad \forall k \in \mathbb{Z}. $$

(30)

Thus, in the first and last cases, the commutation inverses the monodromy operator but keeps it unaltered in the second case.

The two comments concerning the proof of Eq.s (17)-(25) are now apt.

At first, as it has been noted, they can be obtained by means of straightforward computation on the base of the formulas (5)-(7). Since they are only valid on the space of solutions to Eq. (1), it is used in calculations for elimination of higher derivatives of the function $E$. However, just like as in the proof of the theorem the crucial role play here the specialties of the functions $p, q, r, s$.

Indeed, expanding the left-hand sides of Eq.s (17)-(25), their derivatives arise. As above, these are eliminated with the help of Eq.s (16). Another effect to the functions $p, q, r, s$ is produced by the replacements of their argument $z$. As a result, the expressions with some $p, q, r, s$ with arguments either $-1/z$ or $-z$ or $1/z$ arise. Uniformity of arguments is achieved by means of the following transformations

$$ p(-z^{-1}) = (-1)^{\ell + 1}z^{-2(\ell - 1)}p(z), $$

$$ q(-z^{-1}) = z^{-2\ell}((-1)^{\ell}p(z) + z^2r(z)), $$

$$ r(-z^{-1}) = z^{-2(\ell - 1)}(\mu z^2p(z) + q(z)), $$

$$ s(-z^{-1}) = -z^{-2\ell}(\mu z^2p(z) + q(z)) + (-1)^{\ell}z^2(\mu z^2r(z) + s(z)); $$

$$ p(-z) = (-1)^{\ell + 1}(\lambda + \mu^2)^{-1}(\mu z^2r(z) + s(z)), $$

$$ q(-z) = \mu z^2p(z) + q(z) + (-1)^{\ell}(\lambda + \mu^2)^{-1}\mu z^2r(z) + s(z)), $$

$$ r(-z) = r(z), $$

$$ s(-z) = (-1)^{\ell + 1}(\lambda + \mu^2)p(z) - \mu z^2r(z); $$

$$ p(z^{-1}) = (\lambda + \mu^2)^{-1}z^{-2(\ell - 1)}(\mu z^2r(z) + s(z)), $$

$$ q(z^{-1}) = (\lambda + \mu^2)^{-1}z^{-2\ell}(\mu z^2r(z) - \mu s(z)), $$

$$ r(z^{-1}) = z^{-2(\ell - 1)}(\mu z^2p(z) + q(z)), $$

$$ s(z^{-1}) = z^{-2\ell}(\mu z^2p(z) - \mu q(z)), $$

(32)

which manifest the characteristic property of the polynomials $p, q, r, s$ [5].

The above relationships suffice for the proving validity of the rules (17)-(25), (28)-(30).

At second, the quantity $D$ defined in (20) as a function of $z$ (a power function times a polynomial) does not actually depend on this indeterminate. This is proven by means of computation of its derivative which turns out to be the identical zero as a consequence of Eq.s (10). $D$ is therefore a
constant or, more exactly, a function (polynomial) of the parameters $\lambda$ and $\mu$ (depending also on $\ell$). Given $p, q, r, s,$ it can be computed substituting into the right-hand side of any numerical value of $z$. The most simple case corresponds to $z = 1$, obviously. In this way, we note that the same substitution to the equalities yields the following reductions

\[ q(1) = r(1) - \mu p(1), \quad s(1) = (\lambda + \mu^2)p(1) - \mu r(1). \]  

(34)

Applying them, we obtain

\[ D = (\lambda + \mu^2)p(1)^2 - r(1)^2. \]  

(35)

Obviously, the properties of the composition rules in the cases $D \neq 0$ and $D = 0$ drastically differ. We may consider the latter of them degenerate as compared to the former and, limiting our consideration to a generic situation, assume to be imposed below the restriction

\[ D \neq 0. \]  

(36)

treating it as a genericity condition.

Similar remark concerns the vanishing of the sum $\lambda + \mu^2$. Apart of affecting the properties of the rules , it would considerably alter the meaning of the equalities . The condition

\[ \lambda + \mu^2 \neq 0 \]  

(37)

is the third restriction, after inequalities $\mu \neq 0$ and , which will be assumed to be fulfilled throughout.

It seems worthwhile to mention here yet another application of the transformations . They can be utilized for modification of the formulas by means of the explicit manifestation of the result of replacements of the variable $z$ which are carried out over their right-hand expressions. In particular, the functions $p, q, r, s$ with modified argument can be expressed in terms of themselves with the “standard” argument $z$. In this way, the following equivalent alternative definitions of the operators $L_A$ and $L_B$ can be obtained:

\[ L_A[E](\gamma) = (-1)^\ell e^{\mu(z+1/z)} \left[ (p(z)(E'(\gamma) - \mu E(\gamma)) + \right. \]  

\[ + (q(z) + \mu z^2 p(z)) E(\gamma) \right], \]  

(38)

\[ L_B[E](\gamma) = 2\omega z^{1-\ell} e^{\mu(z+1/z)} \left[ (q(z) + \mu z^2 p(z)) (E'(\gamma) - \mu E(\gamma)) + \right. \]  

\[ + (\lambda + \mu^2)p(z) E(\gamma) \right]. \]  

(39)

Among $L$-operators, $L_C$ is of particular interest due its inherent relations to Eq. . Inter alia, the following statement holds true.

**Proposition 2.** Let the function $E_\zeta$ holomorphic in $\mathbb{C}^*$ be an eigenfunction of the operator $L_C$ with eigenvalue $\zeta \neq 0$. Then $E_\zeta$ verifies Eq. with some $\lambda \neq -\mu^2$.

**Proof.** The identity

\[ L_C \circ L_C[E](\gamma) = (2\omega)^2(\lambda + \mu^2)E(\gamma) - (2\omega)^2 \text{lhs}(1), \]  

(40)

where lhs(1) stands for the left-hand side expression of Eq. , is established by means of straightforward computation. If $L_C[E_\zeta] = \zeta E_\zeta$ then it converts to the equality $(2\omega)^2 \text{lhs}(1) = ((2\omega)^2(\lambda + \mu^2) - \zeta^2)E$. Obviously, there exists $\lambda$ such that it implies lhs(1) = 0, i.e. Eq. with such $\lambda$ is fulfilled. \[ \square \]

It is worth noting that the only smooth eigenfunction of $L_C$ with zero eigenvalue is obviously the function $E_1(z) = e^{\mu z}$ which verifies Eq. if and only if $\lambda = -\mu^2$. On the other hand, for such $\lambda$, the function

\[ E_2(z) = \exp(\mu z + \varepsilon(z)), \]  

\[ \varepsilon'(z) = z^{\ell-1}e^{-\mu(z-1/z)}/\varepsilon(z), \]  

\[ \varepsilon'(z) = z^{\ell-1}e^{-\mu(z-1/z)}, \]  

(41)

where $\varepsilon(z) = z^{\ell-1}e^{-\mu(z-1/z)}$,
Let us suppose the opposite, i.e. let all the non-trivial solutions to the given Eq. (1) be at the same time eigenfunctions of \( \mathcal{L}_C \), i.e. they obey the equation \( \mathcal{L}_C \) with eigenvalues \( \pm(2\omega)(\lambda + \mu^2)^{1/2}, \) respectively, or one of them is the identically zero function. The latter case takes place if and only if \( E \) is itself an eigenfunction of \( \mathcal{L}_C \); then the corresponding eigenvalue coincides with one of the two ones pointed out above.

The next almost obvious statement is germane to the above assertion:

**Lemma 2.** There exists a solution to Eq. (1) which is not an eigenfunction of the given operator \( \mathcal{L}_C \).

*Proof.* Let us suppose the opposite, i.e. let all the non-trivial solutions to the given Eq. (1) be at the same time eigenfunctions of \( \mathcal{L}_C \). Let, in particular, \( E_1 \) and \( E_2 \) be such solutions which are linear independent. Due to (19) and (37), their eigenvalues \( \zeta_L \) are both non-zero. We have \( \mathcal{L}_C(E_1 + E_2) = \frac{1}{2}(\zeta_1 + \zeta_2)(E_1 + E_2) + \frac{1}{2}(\zeta_1 - \zeta_2)(E_1 - E_2) \). The sum \( E_1 + E_2 \) verifies Eq. (1) and then, in accordance with supposition, \( \mathcal{L}_C(E_1 + E_2) = \zeta(E_1 + E_2) \) for some non-zero constant \( \zeta \). Thus we obtain \( (\zeta_1 + \zeta_2 - 2\zeta)(E_1 + E_2) + (\zeta_1 - \zeta_2)(E_1 - E_2) = 0 \). Since \( E_1 \) and \( E_2 \) are linear independent, both of these coefficients vanish. Accordingly, it holds \( \zeta_1 = \zeta_2 = \zeta \neq 0 \). Now, let us mention that the value of the expression

\[
\frac{1}{z^1} e^{-\mu(\zeta/z)} (E_1'(\zeta) E_2(\zeta) - E_2'(\zeta) E_1(\zeta)),
\]

where, as usual, \( z = \zeta \), does not depend on \( \zeta \). Indeed, straightforward computation using (1) for eliminating of second derivatives shows that its derivative vanishes identically. The constant \( w \) (the appropriately rescaled wronskian of Eq. (1), in fact) is zero if and only if the solutions \( E_1 \) and \( E_2 \) are linear dependent. Under conditions now assumed, it is therefore nonzero. But the solutions \( E_1 \) and \( E_2 \) are at the same time the eigenfunctions of \( \mathcal{L}_C \), i.e. they obey the equation

\[
E'(\zeta) = \mu E(\zeta) + (2\omega)^{-1} \zeta z^{-1} E(1/3).
\]

Using it, the derivatives \( E_1', E_2' \) can be eliminated and definition (11) takes the form

\[
w = -(2\omega)^{-1} \zeta e^{-\mu(1+1/z)} (E_1(\zeta) E_2'(1/3) - E_2(\zeta) E_1'(1/3)).
\]

But in accordance with (11) \( 1/1 = 1 \), and the last factor in parentheses still vanishes at the point \( \zeta = 1 \), at least. Thus, ultimately, we have come to a contradiction. The assumption incompatible with the lemma assertion is therefore false and the latter is proven. \( \square \)

It is also worth noting that the fulfillment of Eq. (15) by eigenfunctions of \( \mathcal{L}_C \) implies the following statement.

---

1. Here we consider the solutions \( E_1, E_2 \) in some vicinity of the unit alone and hence may identify therein their ‘genuine argument’ \( z \) with its projection \( z = \zeta \).

2. Another known exception is the case of existence of a polynomial solution which may arise just if the order is a negative integer, see [6, 7].
Lemma 3. If \( E \) is an eigenfunction the operator of \( \mathcal{L}_C \) with eigenvalue \( \zeta \) then

\[
E'(t) = (\mu + \frac{\zeta}{2\omega})E(t).
\]

(46)

Summarizing the above relationships, we obtain the following statement.

Theorem 3. Let the condition (37) be fulfilled and Eq. (11) be given. Then the solutions \( E_{(+)} \), \( E_{(-)} \) of the Cauchy problems posed at the point \( z = 1 \) with the initial data obeying the restrictions

\[
E'_{(+)}(1) = (\mu \pm (\lambda + \mu^2)^{1/2})E_{(+)}(1).
\]

(47)

are at the same time the eigenfunctions of the operator \( \mathcal{L}_C \) with eigenvalues \( \pm(2\omega)(\lambda + \mu^2)^{1/2} \), respectively. They constitute the basis of the linear space \( \Omega \) of solutions to Eq. (11) and are unique up to multiplications by constant factors. There are no other (i.e. linear independent with both \( E_{(+)} \) and \( E_{(-)} \), separately) eigenfunctions of \( \mathcal{L}_C \) obeying Eq. (11).

Corollary 1.

\[
E_{(+)}(1) = 0.
\]

(48)

Up to now, we considered the parameter \( \omega \) scaling two operators described by Eqs. (6) and (7) as an arbitrary non-zero constant. Assuming here and below the fulfillment of the condition (37), we may fix it by the following constraint

\[(2\omega)^2(\lambda + \mu^2) = 1.\]

(49)

Its advantage is the most simple form of the eigenvalues which the operator \( \mathcal{L}_C \) acquires on the space of solutions to Eq. (11). Indeed, in case of fulfillment of (49) Eqs. (38) take the form

\[
\mathcal{L}_CE_{(+)} = E_{(+)} \quad \text{and} \quad \mathcal{L}_CE_{(-)} = -E_{(-)}.
\]

(50)

We shall derive here one more property of eigenfunctions of the operator \( \mathcal{L}_C \) which follows from the above relationships. Namely, it follows from the constancy of the expression (44) that

\[
z^{1-\mu}(z^{1/2})(E'_{(+)}(3)E_{(+)}(3) - E_{(-)}(3)E_{(+)}(3)) = e^{-2\mu}(E'_{(+)}(1)E_{(+)}(1) - E'_{(-)}(1)E_{(-)}(1)).
\]

Besides, in view of Eq. (14) in which the substitutions \( E \rightarrow E_{(2)} \) and \( \zeta \rightarrow \pm 1 \) were carried out, one has

\[
E'_{(+)}(3)E_{(-)}(3) - E'_{(-)}(3)E_{(+)}(3) = (2\omega)^{-1}(E_{(+)}(3)E_{(-)}(1/3) + E_{(-)}(3)E_{(+)}(1/3)).
\]

As a consequence, we obtain

\[
E_{(+)}(3)E_{(-)}(1/3) + E_{(-)}(3)E_{(+)}(1/3) = 2e^{\mu(z^{1/2}-2)}E_{(+)}(1)E_{(-)}(1) = 0.
\]

(51)

Let us also specialize Eq. (51) to the cases \( z = -1 \) and \( z = i \). One has \( 1/z = -1 \) and \( 1/i = -i \); therefore, the following 3-point relations

\[
E_{(+)}(-1)E_{(-)}(-i) + E_{(-)}(-1)E_{(+)}(-i) = 2e^{-\mu}E_{(+)}(1)E_{(-)}(1),
\]

(52)

\[
E_{(+)}(1)E_{(-)}(-i) + E_{(-)}(1)E_{(+)}(-i) = 2e^{-\mu}E_{(+)}(1)E_{(-)}(1)
\]

(53)

hold true.

In accordance with (59), with respect to the basis of the space of solutions to Eq. (11) constituted by the functions \( E_{(+)} \), \( E_{(-)} \), the operator \( \mathcal{L}_C \) is represented by the diagonal matrix \( \text{diag}(+1, -1) \). It would be useful to possess the corresponding matrix representations of the operators \( \mathcal{L}_A \), \( \mathcal{L}_B \) as well. Their derivation is equivalent to computation the expansions of the four solutions \( \{E_{(+)} \}, \{E_{(-)} \} \) with respect to the above basis. To that end, beginning with Eqs. (38), (39), we eliminate the derivatives \( E' \) with the help of Eq. (45) which in case of fulfillment of Eq. (49) reads

\[
E'_{(+)}(3) = \mu E_{(+)}(3) \pm (2\omega)^{-1}z^{1/2}E_{(+)}(1/3).
\]

(54)
In this way we obtain
\[
\mathcal{L}_A[E_{\{\pm\}}(s)] = e^{\mu(z+1/z)}(\mp (2\omega)^{-1} z^{1-\ell} p(z) \cdot \mathcal{M}^{-1} E_{\{\pm\}}(\tilde{-3})) \\
+ (-1)^\ell (q(z) + \mu z^2 p(z)) \cdot E_{\{\pm\}}(\tilde{-1}/3),
\]
\[
\mathcal{L}_B[E_{\{\pm\}}(s)] = e^{\mu(z+1/z)}((2\omega)^{-1} z^{1-\ell} p(z) \cdot E_{\{\pm\}}(\tilde{-3})) \\
+ (-1)^\ell (q(z) + \mu z^2 p(z)) \cdot \mathcal{M}^{-1} E_{\{\pm\}}(\tilde{-1}/3),
\]
We further utilize the following algebraic identity
\[
e^{\mu(z+1/z)}[(2\omega)^{-1} z^{1-\ell} p(z) A \mp (-1)^\ell (q(z) + \mu z^2 p(z)) B] = \\
e^{2\mu}(4\omega E_1(I) E_2(I))^{-1} \times \\
\left[ (-z^{1-\ell} p(z) A + 2\omega (-1)^\ell (q(z) + \mu z^2 p(z)) B) \cdot \{\{E_1, E_2\}(3, 1/3) + W_{\{\pm\}}[A, \pm B; E_2(3, 1/3)E_1(3)] + W_{\{\pm\}}[A, \pm B; E_1(3, 1/3)] \right]
\]
which is verified by straightforward computation. Here \( z = \iota \), \( E_1 \) and \( E_2 \) stand for some functions holomorphic in \( \mathbb{C}^* \). \( A \) and \( B \) are arbitrary expressions (they cancel out), and the following abbreviated notations are utilized
\[
\{\{E_1, E_2\}(3, 3) = E_1(3) E_2(3) + E_2(3) E_1(3) - 2 e^{\mu(z+1/z-2)} E_1(I) E_2(I),
\]
\[
W_{\{\pm\}}[A, B; E](3, 3) = \pm 2\omega (-1)^\ell (q(1/z) + \mu z^2 p(1/z)) A \cdot E(3) \\
- 2\omega(-1)^\ell (q(z) + \mu z^2 p(z)) B \cdot E(3) \\
\mp z^{\ell-1} p(1/z) B \cdot E(3) + z^{1-\ell} p(z) A \cdot E(3) \\
\mp z^{\ell-1} p(1/z) B \cdot E(3) + z^{1-\ell} p(z) A \cdot E(3),
\]
the last equality taking place in view of Eqs. \( 53 \). If one substitutes into \( 57 \) \( A = \mp \mathcal{M}^{-1} E_{\{\pm\}}(\tilde{-3}) \), \( B = \mp E_{\{\pm\}}(\tilde{-1}/3) \) then its left-hand side coincides with the right-hand side of \( 55 \). Similarly, after substitutions \( A = E_{\{\pm\}}(\tilde{-3}) \) and \( B = \mp \mathcal{M}^{-1} E_{\{\pm\}}(\tilde{-1}/3) \) the left-hand side of \( 57 \) coincides with the right-hand side of \( 50 \). Thus we may replace the right-hand sides of \( 55 \) and \( 56 \) with the corresponding instances of \( 57 \) in which we have also carried out the substitutions \( E_1 = E_{\{\pm\}}, E_2 = E_{\{\mp\}} \). Before the writing down the result, it is important to note that in view of the above selection of the functions \( E_1, E_2 \), the factor \( \{\{E_1, E_2\} \) in \( 57 \) (and hence the whole first summand therein on the right) vanishes due to Eq. \( 51 \). Accordingly, only \( W \)-involving contributions remain and the results read
\[
\mathcal{L}_A[E_{\{\pm\}}(s)] = e^{2\mu}(4\omega E_{\{\mp\}}(I) E_{\{\mp\}}(I))^{-1} \times \\
\left( W_{\{\pm\}}[\mathcal{M}^{-1} E_{\{\pm\}}(\tilde{-3}), \pm E_{\{\pm\}}(\tilde{-1}/3); E_{\{\mp\}}(3, 1/3) \cdot E_{\{\mp\}}(3) + W_{\{\pm\}}[\mathcal{M}^{-1} E_{\{\pm\}}(\tilde{-3}), \pm E_{\{\pm\}}(\tilde{-1}/3); E_{\{\mp\}}(3, 1/3) \cdot E_{\{\mp\}}(3) \right),
\]
\[
\mathcal{L}_B[E_{\{\pm\}}(s)] = e^{2\mu}(4\omega E_{\{\mp\}}(I) E_{\{\mp\}}(I))^{-1} \times \\
\left( W_{\{\pm\}}[E_{\{\pm\}}(\tilde{-3}), \pm \mathcal{M}^{-1} E_{\{\pm\}}(\tilde{-1}/3); E_{\{\mp\}}(3, 1/3) \cdot E_{\{\mp\}}(3) + W_{\{\pm\}}[E_{\{\pm\}}(\tilde{-3}), \pm \mathcal{M}^{-1} E_{\{\pm\}}(\tilde{-1}/3); E_{\{\mp\}}(3, 1/3) \cdot E_{\{\mp\}}(3) \right).
\]
Now, we have to note the following important property which the \( W \)-functional possess.

**Theorem 4.** The values of the functionals \( W_{\{\pm\}} \) and \( W_{\{\mp\}} \) with arguments specified in Eqs. \( 57 \) and \( 58 \) do not depend on the value of \( s \).
The most simple results arise if \( z \) = \( -1 \) and \( 1 \). Hence the condition (36) can also be represented as the union of the two inequalities

\[
\begin{align*}
\text{Proof.} & \quad \text{It reduces to computation of the derivative of the corresponding expressions. In them, the derivatives } E'_\pm(\pm) \text{ are eliminated with the help of Eq. (34), the derivatives } p', r' \text{ are eliminated with the help of Eq}s. (10), \text{ and the functions } p, q, r, s \text{ with the argument } 1/3 \text{ are expressed in terms of the same functions with the argument } 3 \text{ with the help of Eq}s. (35). In all the four cases, the results of these transformations prove to be reducible to identical zero.}
\end{align*}
\]

Thus Eq.s (60), (61) may be interpreted as the decompositions with constant coefficients of their left-hand sides with respect to the basis \{\( E_{(\pm)} \), \( E_{(-)} \). We may bring them to “more explicit” forms by means of fixation of \( z \) in expressions of the coefficients, i.e. in the factors \( W(\pm)[\cdots](3,1/3). \) The most simple results arise if \( z = 1 \). They read

\[
\begin{align*}
\mathcal{L}_A[E_{(-)}(3)] = \frac{2\mu}{4\omega} \left( \frac{D_+}{E_{(+)}(I)}(E_{(+)}(-I) + E_{(+)}(-I)) \cdot E_{(+)}(3) - \frac{D_-}{E_{(-)}(I)}(E_{(+)}(-I) \pm E_{(+)}(-I)) \cdot E_{(-)}(3) \right),
\end{align*}
\]

\[
\begin{align*}
\mathcal{L}_B[E_{(-)}(3)] = \frac{2\mu}{4\omega} \left( \frac{D_+}{E_{(+)}(I)}(E_{(+)}(-I) + E_{(+)}(-I)) \cdot E_{(+)}(3) + \frac{D_-}{E_{(-)}(I)}(E_{(+)}(-I) \pm E_{(+)}(-I)) \cdot E_{(-)}(3) \right),
\end{align*}
\]

where \( D_\pm = p(1) \pm (-1)^{\mu}2\omega r(1). \) Let us note that in view of Eq. (49) and in accordance with expanded definition (35) of \( D \) it holds

\[
D_+D_- = (2\omega)^{-2}D.
\]

Hence the condition (36) can also be represented as the union of the two inequalities

\[
D_+ + 0 \equiv D_-
\]

which remove two algebraic curves from the set of values of the parameters \( \lambda, \mu. \)

A direct consequence of Eq.s (62), (63) are the following explicit \textit{matrix representations} \( \mathcal{L}_A, \mathcal{L}_B \) of the operators \( \mathcal{L}_A, \mathcal{L}_B \) with respect to the basis \{\( E_{(+)} \), \( E_{(-)} \)\}

\[
\begin{align*}
\mathcal{L}_A &= \frac{2\mu}{4\omega} \left( \frac{E_{(+)}(-I) - E_{(+)}(-I) - (E_{(+)}(-I) - E_{(-)}(-I))}{E_{(-)}(-I) + E_{(-)}(-I) - (E_{(-)}(-I) - E_{(-)}(-I))} \right) D^+,
\end{align*}
\]

\[
\begin{align*}
\mathcal{L}_B &= \frac{2\mu}{4\omega} \left( \frac{E_{(+)}(-I) - E_{(+)}(-I) + E_{(+)}(-I) - E_{(-)}(-I)}{E_{(-)}(-I) + E_{(-)}(-I) - E_{(-)}(-I)} \right) D^+,
\end{align*}
\]

where \( D^+ = \text{diag} \left( \frac{D_+}{E_{(+)}(I)}, \frac{D_-}{E_{(-)}(I)} \right). \)

Now let us mention that in accordance with the composition rule (17) the matrix representation of the operator \( \mathcal{L}_A \) has to fulfill the equation \( \mathcal{L}_A^2 = -D \text{diag}(1, 1). \) A straightforward computation with the matrix (67) shows that this is the case if and only if

\[
\frac{D_+}{E_{(+)}(I)}(E_{(+)}(-I) - E_{(+)}(-I)) = \frac{D_-}{E_{(-)}(I)}(E_{(-)}(-I) - E_{(-)}(-I)).
\]

This is yet another relationship constraining the values of the eigenfunctions of the operator \( \mathcal{L}_C, \) this time at the points projected to \(-1 \) and \( 1. \) Applying it, the matrices \( \mathcal{L}_A, \mathcal{L}_B \) can be transformed
to the following representations
\[
L_A = \frac{\varepsilon^{2\mu}}{4\omega} D_+ \begin{pmatrix}
E_{(\varepsilon)}(-\varepsilon I) - E_{(-\varepsilon)}(-\varepsilon I) & -(E_{(\varepsilon)}(-\varepsilon I) + E_{(-\varepsilon)}(-\varepsilon I)) \\
E_{(\varepsilon)}(-\varepsilon I) + E_{(-\varepsilon)}(-\varepsilon I) & -(E_{(\varepsilon)}(-\varepsilon I) - E_{(-\varepsilon)}(-\varepsilon I))
\end{pmatrix},
\]
(71)
\[
L_B = \frac{\varepsilon^{2\mu}}{4\omega} D_+ \begin{pmatrix}
E_{(\varepsilon)}(-\varepsilon I) - E_{(-\varepsilon)}(-\varepsilon I) & E_{(\varepsilon)}(-\varepsilon I) + E_{(-\varepsilon)}(-\varepsilon I) \\
E_{(\varepsilon)}(-\varepsilon I) + E_{(-\varepsilon)}(-\varepsilon I) & E_{(\varepsilon)}(-\varepsilon I) - E_{(-\varepsilon)}(-\varepsilon I)
\end{pmatrix} = L_A \text{diag}(1, -1),
\]
(72)
where \(D_+ = \text{diag} \left( \frac{D_+}{E_{(\varepsilon)}(-\varepsilon I)}, \frac{D_+}{E_{(-\varepsilon)}(-\varepsilon I)} \right) \).
(73)

Straightforward computations taking into account Eq. (52) show that
\[
\det L_A = \mathcal{D}, \quad \det L_B = -\mathcal{D}.
\]
(74)

In view of the requirement (50), the matrices \(L_A, L_B\) are always invertible. The same holds true for the operators \(\mathcal{L}_A, \mathcal{L}_B\). Thus, we have the following statement.

**Theorem 5.** Under restrictions currently assumed, \(\mathcal{L}\)-operators determine linear automorphisms of the space of solutions to (1).

(In case of the operator \(\mathcal{L}_C\), this property follows from definitions and is, in fact, evident.)

The composition rule (14) of \(\mathcal{L}\)-operators has gained us the constraint (70) characterizing certain variations of the eigenfunctions of \(\mathcal{L}_C\). In a similar way, converting the rule (18), which now reads \(\mathcal{L}_B \circ \mathcal{L}_B = DM\), to its matrix form, the matrix \(M = D^{-1}L_B^2\) of the monodromy transformation \(M\) can be derived. Specifically, using the representation (72) and Eq. (70), one obtains
\[
M = e^{2\mu(2E_{(\varepsilon)}(I)E_{(-\varepsilon)}(-I))^{-1} \times \begin{pmatrix}
E_{(\varepsilon)}(-I)E_{(-\varepsilon)}(-I) + E_{(\varepsilon)}(-I)E_{(-\varepsilon)}(-I) - E_{(\varepsilon)}(-I)E_{(-\varepsilon)}(-I) \\
E_{(-\varepsilon)}(-I) + E_{(\varepsilon)}(-I)E_{(-\varepsilon)}(-I) - E_{(\varepsilon)}(-I)E_{(-\varepsilon)}(-I)
\end{pmatrix}}.
\]
(75)

Eq. (52) then implies that
\[
\det M = 1, \quad \text{and, therefore,} \quad M^{-1} = \text{diag}(1, -1) M \text{diag}(1, -1)
\]
(76)
is produced from \(M\) by means of the reversing signs of its antidiagonal elements.

The monodromy transformation enables one to carry out analytic continuation of solutions to Eq. (1) from certain subdomain to the whole domain \(\mathcal{C}^*\) by means of algebraic operations. It suffices to realize such continuation for the solutions \(E_{(\varepsilon)}\) constituting the basis of \(\Omega\). To that end, let us note that if we introduce the two-element column \(E = (E_{(\varepsilon)}, E_{(-\varepsilon)})^T\), then the gist of the matrix (55) is exactly to serve the operator of the transformation
\[
E(M^k) = M^k E(3), \quad k \in \mathbb{Z}.
\]
(77)
As far as one concerns the left-hand side, the map \(E(3) \to E(M^k3)\) can be interpreted as certain “shift” of a subset of the domain where \(E\) is evaluated. Specifically, let the values of \(E(3)\) be known (“have been computed”) everywhere in the subdomain represented in frames of semilog model of \(\mathcal{C}^*\) by the semi-strip \(\mathcal{C}^*_{\delta} := \mathbb{R}_{+} \times (-\pi, \pi)\) which projects bijectively to \(\mathbb{C}^* \setminus \mathbb{R}_{-}\). The map \(3 \to M^k3\) defined by Eq. (115) takes \(\mathcal{C}^*_{\delta}\) to the semi-strip \(\mathcal{C}^*_{\delta} := \mathbb{R}_{+} \times (\pi k - \pi, \pi k + \pi)\). Accordingly, one can determine the value of \(E\) at any point of \(\mathcal{C}^*\) from its value at certain point of \(\mathcal{C}^*_{\delta}\) by means of Eq. (77). Since the union of all \(\mathcal{C}^*_{\delta}\) covers the whole \(\mathcal{C}^*\) but the rays \((\forall \rho > 0, \pi k)\), where \(E\) still can be determined by continuity, its value can be found in this way at any point of the whole domain \(\mathcal{C}^*\) by means of an algebraic transformation.

The making use of the monodromy transformation for continuation of solutions from a subdomain with closure of projection coinciding with the complex plane is not specific for Eq. (1). However, in our case, there exists the opportunity to carry out continuation from a subdomain which is “twice less”. Namely, we shall show that the values on the eigenfunctions \(E_{(\varepsilon)}\) can be determined by means of algebraic operations at any point of their domain from their values in a subdomain projected to the semi-plane \(\mathbb{R}_z > 0\). The corresponding transformations are derived...
from the composition rules \( \mathcal{L} \) in which one \( \mathcal{L} \)-operator is represented by the formula \( \mathcal{E}_0 \) or \( \mathcal{E}_1 \) while another one is taken in a matrix representation, see Eq.s \( 77 \), \( 78 \). They read

\[
E_{(\pm)}(z) = \pm \frac{z^{\pm 1}}{2} e^{\mu(2z^{-1/2})} \times 
\left( (-1)^{\pm} 2\omega z^{\pm 1} \eta(1/z) \right) \left( \frac{E_{(\pm)}(\pm 1)}{D_+ E_{(\mp)}(1)} \cdot E_{(+)}(1/3) + \frac{E_{(\mp)}(\mp 1)}{D_+ E_{(-)}(1)} \cdot E_{(-)}(1/3) \right) + 
\left( \frac{E_{(\pm)}(\pm 1)}{D_- E_{(\mp)}(1)} \cdot E_{(-)}(1/3) + \frac{E_{(\mp)}(\mp 1)}{D_- E_{(-)}(1)} \cdot E_{(+)}(1/3) \right) + 
\left( \frac{E_{(\pm)}(\pm 1)}{D_- E_{(+)}(1)} \cdot E_{(+)}(3) + \frac{E_{(\mp)}(\mp 1)}{D_- E_{(-)}(1)} \cdot E_{(-)}(3) \right),
\]

\[
E_{(\pm)}(\tilde{z}) = \pm \frac{z^{\pm 1}}{2} e^{\mu(2z^{-1/2})} \times 
\left( (-1)^{\pm} 2\omega z^{\pm 1} \eta(1/z) \right) \left( \frac{E_{(\pm)}(\pm 1)}{D_+ E_{(\mp)}(1)} \cdot E_{(+)}(1/3) + \frac{E_{(\mp)}(\mp 1)}{D_+ E_{(-)}(1)} \cdot E_{(-)}(1/3) \right) + 
\left( \frac{E_{(\pm)}(\pm 1)}{D_- E_{(\mp)}(1)} \cdot E_{(-)}(1/3) + \frac{E_{(\mp)}(\mp 1)}{D_- E_{(-)}(1)} \cdot E_{(+)}(1/3) \right) + 
\left( \frac{E_{(\pm)}(\pm 1)}{D_- E_{(+)}(1)} \cdot E_{(+)}(3) + \frac{E_{(\mp)}(\mp 1)}{D_- E_{(-)}(1)} \cdot E_{(-)}(3) \right),
\]

Let us remind the meaning of some notations utilized above. Here \( \mathfrak{z} \) designates an arbitrary point of the Riemann surface \( \mathfrak{C}^* \) and \( z = \mathfrak{z} \) is its projection to \( \mathfrak{C} = \mathfrak{C} \backslash \{0\} \) for which \( \mathfrak{C}^* \) serves the universal cover (each point of the former is lifted to the two-side sequence of points from the latter). The points \( -1, -1 \) and \( -1 \) belonging to \( \mathfrak{C}^* \) both project to \(-z\). They are the boundary points of the non-closed arc projecting bijectively to the “punctured circle” \( S^1_{(-)} = \{ \tilde{z} : \tilde{z} \in \mathfrak{C}, |\tilde{z}| = |z|, \tilde{z} \neq -z \} \), in the middle of which \( -1 \) is situated. The points \( -1 \) and \( -1 \) projecting to \(-1\), represent the particular case of \( -1 \) and \( -1 \) arising if \( \mathfrak{z} = 1 \). The inversion \( z \rightarrow 1/z \) is understood in accordance with the rule \( 14 \). Correspondingly, the “right half” of \( \mathfrak{C}^*_0 \), the subdomain \( \mathfrak{C}^*_0 \) modeled by the half-strip \( \mathbb{R}_+ \times (-\pi/2, \pi/2) \) and projecting to the half-plane \( \mathbb{R} \tilde{z} > 0 \), is invariant with respect to it. It means that, varying \( z \) within \( \mathfrak{C}^*_0 \), the arguments of the functions \( E_{(\pm)} \) in the right-hand sides of Eq.s \( 78 \) and \( 79 \) remain well within the same subdomain. At the same time, the corresponding subdomains, to which the arguments of the left-hand side functions of Eq.s \( 78 \) and \( 79 \) belong, are distinct; in particular, they do not intersect with \( \mathfrak{C}^*_0 \). Indeed, it is easy to see that in case of the equation \( 78 \) the argument of the functions \( E_{(\pm)} \) on the left runs through the subdomain modeled by the half-strip \( \mathbb{R}_+ \times (\pi/2, 3\pi/2) \) while for the left-hand side function in Eq. \( 79 \) the subdomain where it is evaluated is modeled by the half-strip \( \mathbb{R}_+ \times (-3\pi/2, -\pi/2) \), these maps being 1-to-1. Thus, with the help of Eq.s \( 78 \), \( 79 \), starting from \( \mathfrak{C}^*_0 \), one can continue both functions \( E_{(\pm)} \) to the subdomain modeled by the half-strip \( \mathbb{R}_+ \times (-3\pi/2, 3\pi/2) \) which is “three times wider” than the original one. In particular, it embodies \( \mathfrak{C}^*_0 \) and, hence, applying further the monodromy transformation in a way discussed above, the functions \( E_{(\pm)} \) can be continued to their whole domain \( \mathfrak{C}^* \), all the operations involved in such transformations being algebraic.

It is important to note, however, that Eq.s \( 78 \) and \( 79 \), as they stand, can not be considered as the ready-to-be-used tools for continuation of the eigenfunctions \( E_{(\pm)} \) from their subdomain \( \mathfrak{C}^*_0 \). The point is that these contain the quantities \( E_{(\pm)}(1) \) and \( E_{(\pm)}(-1) \) where the functions \( E_{(\pm)} \) are evaluated at the points which belong neither to the subdomain from which the continuation has to be carried out, nor to its boundary. Yet, the difficulty can be settled as follows.

Let us evaluate Eq. \( 78 \) at the point \( z = -1 \) and Eq. \( 79 \) at the point \( z = 1 \). An inspection shows that in such cases all the functions \( E_{(\pm)} \) of arguments depending on \( z \) involved in these equations prove to be evaluated just at these two points. But the both points \( 1 \) and \(-1 \) belong to the boundary of \( \mathfrak{C}^*_0 \). Hence, by continuity, the values of \( E_{(\pm)}(1) \) and \( E_{(\pm)}(-1) \) may be considered to be known. Then the two pairs of the above equations may be treated as the closed inhomogeneous linear system with four unknowns \( E_{(\pm)}(1) \) and \( E_{(\pm)}(-1) \). It proves to be solvable leading to the
The following formulas
\[ E_{(\pm)}(-I) + E_{(\mp)}(-I) = (D_{\pm}E_{(\pm)}(I))^{-1} \times \]
\[ \left( -i^{t+1}(p(-i)E_{(\pm)}^2(i) - (-1)^{t}(p(i)E_{(\mp)}^2(-i)) \right. \]
\[ \left. + (-1)^{t}2\omega(q(i) + q(-i) - \mu(p(i) + p(-i)))E_{(\pm)}(i)E_{(\mp)}(-i) \right) \]
\[ E_{(\pm)}(-I) - E_{(\mp)}(-I) = (D_{\mp}E_{(\mp)}(I))^{-1} \times \]
\[ \left( -i^{t+1}(p(-i)E_{(\mp)}(i)E_{(\pm)}(-i) + (-1)^{t}(p(i)E_{(\pm)}(-i)) \right) \]
\[ + (-1)^{t}2\omega((q(i) + \mu(p(i))E_{(\mp)}(i)E_{(\pm)}(-i) - (q(-i) - \mu(p(-i))E_{(\mp)}(i)E_{(\pm)}(-i))) \right) \]
\[ \tag{80} \]
\[ \tag{81} \]
These allow one to eliminate their left-hand side sums and differences from the right-hand sides of Eqs. (78) and (79), exempting them from presence of quantities unknown in advance. Eqs. (78), (79) allow one to eliminate their left-hand side sums and differences from the right-hand sides of Eqs. (78), (79) which are equivalent to Eq. (82). The transformation connecting the triplets of their parameters reads
\[ \varphi(t) + \sin \varphi(t) = B + A \cos \omega t, \tag{82} \]
in which \( A, B, \omega \) are real constants, \( A \neq 0, \omega > 0 \), and dot denotes derivative with respect to the free real variable \( t \). This equation and its generalizations appear in a number of problems of physics (most notably, in the modeling of Josephson junctions [10, 11, 12]), mechanics [13, 14, 15], dynamical systems theory [16], geometry [17]. On the other hand, Eq. (82) is closely linked to the double confluent Heun equation [8]. More exactly, it is just the family of equations of the form (1) which are equivalent to Eq. (82). The transformation connecting the triplets of their parameters reads
\[ B = \omega l, \quad A = 2\omega \mu, \quad (2\omega)^{-1} = \sqrt{\lambda + \mu^2}. \tag{83} \]
The last equation agrees with condition (49) but, to keep all the parameters real, one has to impose the restriction \( \lambda + \mu^2 > 0 \), more severe than (49). Moreover, \( \lambda \) and \( \mu \) are to be assumed to be real themselves as well.

The transition from Eq. (41) to Eq. (52) is carried out in two steps [9]. First, we replace Eq. (52) with the following Riccati equation
\[ \varphi' + (2i\omega)^{-1}z\varphi^2 - 1 = (lz^{-1} + \mu(1 + z^{-2}))\varphi \tag{84} \]
for holomorphic function \( \varphi = \Phi(\varphi) \). \( \Phi \) is introduced as an analytic continuation of the real analytic function \( e^{i\omega t} \) from (the lift of) the unit circle, i.e. it obeys the equation
\[ \Phi((l^{-1})e^{i\omega t}) = e^{i\omega t}, \quad t \in (-\pi/\omega, \pi/\omega), \tag{85} \]
where \( l^{-1} \) denote the lift of \( \mathbb{C}^+ \setminus \mathbb{R}_- \) to \( \mathbb{C}^* \) such that \( l^{-1}1 = 1 \). We shall need also the function \( \Psi = \Psi(\varphi) \) which is a similar analytic continuation of the exponentiated quadrature \( \int_0^t \cos \varphi(t) \, dt = P(t) \), i.e., obeys the equation
\[ \Psi((l^{-1})e^{i\omega t}) = e^{P(t)} = \exp \int_0^t \cos \varphi(t) \, dt, \quad t \in (-\pi/\omega, \pi/\omega), \tag{86} \]
the corresponding differential equation reading
\[ 2i\omega \Psi' = z^{-1}(\Phi + \Phi^{-1})\Psi. \tag{87} \]
Obviously, at the point \( z = 1 \), it holds
\[ |\Phi(1)| = 1, \quad \Psi(1) = 1. \tag{88} \]
Eq. (S2) is equivalent to Eq. (S1) considered in vicinity of (the lift of) the unit circle with \(-1\) removed for which the initial condition posed at \(s = 1\) obeys the constraint (S8). This relation is ensured by explicit locally invertible transformations of the independent variables and unknown functions.

In the second step, the relations between Eq. (1) and Eq. (S4) are established in the form transformations taking solutions to one of them to solutions to the other equation. More exactly, as long as one concerns Eq. (1), it proves more convenient to consider not arbitrary solutions but the eigenfunctions \(E_{(\pm)}\) of the operator \(L_C\).

In this way, the transformation taking the functions \(E_{(\pm)}\) to solutions to Eq. (1) is as follows

\[
\Phi(s) = -iz^{-\ell} \frac{\cos(\frac{1}{2} \alpha) E_{(\pm)}(3) + i \sin(\frac{1}{2} \alpha) E_{(\pm)}(-1)}{\cos(\frac{1}{2} \alpha) E_{(\pm)}(1/3) - i \sin(\frac{1}{2} \alpha) E_{(\pm)}(-1/3)},
\]

Here \(\alpha\) is an arbitrary real constant. Since in accordance with (S5)

\[
\Phi(1) = e^{i\varphi(0)} = -i \frac{\cos(\frac{1}{2} \alpha) E_{(\pm)}(1) + i \sin(\frac{1}{2} \alpha) E_{(\pm)}(-1)}{\cos(\frac{1}{2} \alpha) E_{(\pm)}(1/3) - i \sin(\frac{1}{2} \alpha) E_{(\pm)}(-1/3)},
\]

where both \(E_{(\pm)}(1)\) are non-zero, it is obvious that, varying \(\alpha\) through the interval \([0, 2\pi]\) (i.e., though a circle/projective line \(P(\mathbb{R})\)), one is able to obtain all solutions to Eq. (S2) (up to uncertainty allowing additions of integer multiples of \(2\pi\)). Let us give also similar relation yielding the correspondingly transformed \(\Psi\):

\[
\Psi(s) = (\cos^2(\frac{1}{2} \alpha) E_{(\pm)}^2(1) + \sin^2(\frac{1}{2} \alpha) E_{(\pm)}^2(-1))^{-1} \times e^{\mu(2-s-1/3)} (\cos(\frac{1}{2} \alpha) E_{(\pm)}(3) + i \sin(\frac{1}{2} \alpha) E_{(\pm)}(-3)) \times (\cos(\frac{1}{2} \alpha) E_{(\pm)}(1/3) - i \sin(\frac{1}{2} \alpha) E_{(\pm)}(-1/3)).
\]

Notice that one has \(\Psi(1) = 1\) for it, the first factor just ensuring such a normalization. Besides, varying independently the normalizations of the functions \(E_{(\pm)}\), i.e., for instance, the values of \(E_{(\pm)}(1) \neq 0\), the solution \(\Phi\) and the function \(\Psi\) vary coherently, this time in a non-trivial way.

The validity of the above transformations is verified by means of the plugging their right-hand sides into the corresponding equations, and elimination of the derivatives \(E'_{(\pm)}\) with the help of Eq.s (S1).

The inverse transformations, taking an arbitrary solution (up to a minor exception, see below) to Eq. (S2) to the eigenfunctions \(E_{(\pm)}\) of the operator \(L_C\), fulfilling Eq. (1) as well, read

\[
E_{(\pm)}(3) = \pm e^{\mu(3+1/3-2)} z^{1/2} \left[ \frac{1 \mp i}{2} \Psi(3)^{1/2} \Phi(3)^{1/2} + \frac{1 \mp i}{\sqrt{2}} \Psi(3/5)^{1/2} \Phi(3/5)^{1/2} \right].
\]

The fulfillment of Eq.s (S4) is verified by straightforward substitution and application of Eq.s (S3), (S7).

Taking into account Eq. (S8), one obtains the equation

\[
E_{(\pm)}(1) = \mp \Psi(1) \sin \frac{1}{2} (\varphi(0) \pm \pi/2),
\]

manifesting the specific normalization of \(E_{(\pm)}\) and \(E_{(-)}\) innate to the representations (11). It reveals a peculiarity of the solutions \(\varphi(t)\) to Eq. (1) solving one of initial data problems with

\[
\varphi(0) = \pi/2 (\text{mod } \pi) = \pm \pi/2 (\text{mod } 2\pi),
\]

implying \(\Phi(1) = \mp i\).

For such a \(\Phi\) the pair of formulas (11) yields only one non-trivial eigenfunction because instead of the second one the identically zero function is produced. It also means that in such cases for one of two choices of the signs the expression brackets in (11) vanishes, i.e. it holds

\[
\Psi(3) \Phi(3) = \Psi(1/3) \Phi(1/3),
\]

The close relation of Eq. (1) and Eq. (S4) outlined above assumes allusion to possible manifestation of symmetries the space of solutions to the former possesses, if any, in properties of the set of solutions to the latter. Specifically, we have noted above a number of maps sending solutions to Eq. (1) to solutions of the same equation. One may expect that there exist associated maps
acting this time to solutions to $\Phi$ and $\Psi$ with similar composition properties. Below we analyze such an opportunity in more details for operators given above in explicit form.

To begin, let us note first that the numerator of the fraction in the right-hand side of $\Phi$ and $\Psi$ is a solution to Eq. (1). The denominator is also a solution but with the argument inversed. If one applies a linear operator taking them to other solutions then the latter can also be used as numerators and denominators in the formula $\Phi$ and $\Psi$, and such a transformation has finally effect of certain modification of the parameter $\alpha$. Replacing finally therein the functions $E_{\{\ell\}}$ by their representations in terms of $\Phi$ and $\Psi$ in accordance with Eq. (91), the closed transformation acting to solutions to $\Phi$ and $\Psi$ and referring to nothing else should result.

A toy example is provided by consideration of the operator $L_C$. Specifically, let us consider $\Phi$ with $\alpha = \pi/4$ and plug in it the expressions of $E_{\{\ell\}}$ in terms of $\Phi$ and $\Psi$ given in Eq. (91). The result is the original solution $\Phi(\alpha)$. However, if we apply on a course of transformation the operator $L_C$ to its eigenfunctions $E_{\{\ell\}}$, i.e. keep $E_{\{\ell\}}$ unchanged but replace $E_{\{\ell\}}$ with $-E_{\{\ell\}}$, then the result becomes equal to $-\Phi(1/3)^{-1}$. This function verify Eq. (94) as long as $\Phi(\alpha)$ does. The transformation

$$\Phi(\alpha) \rightarrow -\Phi(1/3)^{-1}$$

is therefore the map of the set of solutions to Eq. (94) onto itself. In accordance with way of its derivation, it is also the representation of the operator $L_C$ on the latter set. It is also worth noting that, when restricted to the (lift of) unit circle, Eq. (93) converts to a more or less evident transformation

$$\varphi(t) \rightarrow \pi - \varphi(-t)$$

leaving Eq. (94) invariant.

Proceeding next with the less trivial monodromy transformation, it is useful to represent in advance its matrix $M$ (which serves for transformation of solutions to Eq. (1), not to Eq. (94)) in terms solutions to Eq. (94). In such a representation it is equal to

$$M = (-1)^t e^{-P(0)} \sec \varphi(0) \left( \begin{array}{cc} \beta_+ & i(\beta_- + \gamma) \\ i(-\beta_- + \gamma) & \beta_+ \end{array} \right),$$

where $\beta_{\pm} = \frac{1}{2} \left( e^{P(T/2)} \cos \varphi(\frac{T}{2}) \pm e^{-P(T/2)} \cos \varphi(-\frac{T}{2}) \right)$, where $T = 2\pi/\omega$,

$$\gamma = e^{P(T/2)} e^{P(T/2)} \sin \left( \frac{T}{2} \varphi(\frac{T}{2}) - \frac{T}{2} \varphi(-\frac{T}{2}) \right).$$

Utilizing this result, one can obtain the following explicit representation of the coupled monodromy transformations of solutions to Eq.s (94) and (97). They read

$$\mathcal{M} \Phi(\alpha) = \left( e^{P(T/2)} + \cos \left( \frac{T}{2} \varphi(\frac{T}{2}) \right) \right) \Psi(1/3)^{1/2} \Phi(\alpha)^{1/2} \times$$

$$\begin{equation}
\times \left( e^{P(T/2)} + \frac{T}{2} \cos \left( \frac{T}{2} \varphi(\frac{T}{2}) \right) \right) \Psi(1/3)^{1/2} \Phi(\alpha)^{-1/2} \times$$

$$-i e^{P(-T/2)} \sin \left( \frac{T}{2} \varphi(\frac{T}{2}) - \frac{T}{2} \varphi(-\frac{T}{2}) \right) \Psi(1/3)^{1/2} \Phi(\alpha)^{1/2} \times$$

$$\times \left( \Phi(1/3)^{1/2} \Phi(\alpha)^{1/2} - \Phi(1/3)^{-1/2} \Phi(\alpha)^{-1/2} \right)^{-1} \times$$

$$\mathcal{M} \Psi(\alpha) = 2e^{-P(-T/2)} \cos \left( \frac{T}{2} \varphi(\frac{T}{2}) \right) \sec^2 \left( \frac{T}{2} \varphi(\frac{T}{2}) + \frac{T}{2} \varphi(-\frac{T}{2}) \right) \times$$

$$\begin{equation}
\times \left( e^{P(T/2)} + \frac{T}{2} \cos \left( \frac{T}{2} \varphi(\frac{T}{2}) \right) \right) \Psi(1/3)^{1/2} \Phi(\alpha)^{1/2} \times$$

$$+ e^{P(-T/2)} \sin \left( \frac{T}{2} \varphi(\frac{T}{2}) - \frac{T}{2} \varphi(-\frac{T}{2}) \right) \Psi(1/3)^{1/2} \Phi(\alpha)^{-1/2} \times$$

$$-i \cos \left( \frac{T}{2} \varphi(\frac{T}{2}) \right) e^{P(T/2)} e^{P(-T/2)^2} \sin \left( \frac{T}{2} \varphi(\frac{T}{2}) - \frac{T}{2} \varphi(-\frac{T}{2}) \right) \times$$

$$\times \left( \Phi(1/3)^{1/2} \Phi(\alpha)^{1/2} - \Phi(1/3)^{-1/2} \Phi(\alpha)^{-1/2} \right)^{-1}.$$
Their restriction to the lift of the unit circle leads to the transformations of solutions to Eq. 82

\[ e^{i\varphi(t \pm T)} = \left( e^{P(t)/2} \cos\left(\frac{\varphi(t)}{2}\right) + e^{-P(t)/2} \right) \times \]

\[ \pm i e^{P(t)/2} \sin\left(\frac{\varphi(t)}{2}\right) \cdot e^{-P(t)/2} e^{-i\varphi(t)/2} \times \]

\[ \left( e^{P(t)/2} \cos\left(\frac{\varphi(t)}{2}\right) - e^{-P(t)/2} e^{-i\varphi(t)/2} \right) \]

\[ = e^{P(t)/2} \cos\left(\frac{\varphi(t)}{2}\right) \times \]

\[ e^{-P(t)/2} \cos\left(\frac{\varphi(t)}{2}\right) \cdot e^{-P(t)/2} e^{-i\varphi(t)/2} \times \]

\[ \pm i e^{P(t)/2} \sin\left(\frac{\varphi(t)}{2}\right) \cdot e^{-P(t)/2} e^{-i\varphi(t)/2} \times \]

\[ \left( e^{P(t)/2} \cos\left(\frac{\varphi(t)}{2}\right) - e^{-P(t)/2} e^{-i\varphi(t)/2} \right) \]

\[ = e^{P(t)/2} \cos\left(\frac{\varphi(t)}{2}\right) \times \]

\[ e^{-P(t)/2} \cos\left(\frac{\varphi(t)}{2}\right) \cdot e^{-P(t)/2} e^{-i\varphi(t)/2} \times \]

\[ \pm i e^{P(t)/2} \sin\left(\frac{\varphi(t)}{2}\right) \cdot e^{-P(t)/2} e^{-i\varphi(t)/2} \times \]

\[ \left( e^{P(t)/2} \cos\left(\frac{\varphi(t)}{2}\right) - e^{-P(t)/2} e^{-i\varphi(t)/2} \right) \]

\[ = e^{P(t)/2} \cos\left(\frac{\varphi(t)}{2}\right) \times \]

\[ e^{-P(t)/2} \cos\left(\frac{\varphi(t)}{2}\right) \cdot e^{-P(t)/2} e^{-i\varphi(t)/2} \times \]

\[ \pm i e^{P(t)/2} \sin\left(\frac{\varphi(t)}{2}\right) \cdot e^{-P(t)/2} e^{-i\varphi(t)/2} \times \]

\[ \left( e^{P(t)/2} \cos\left(\frac{\varphi(t)}{2}\right) - e^{-P(t)/2} e^{-i\varphi(t)/2} \right) \]

which extend them from the segment of variation of \( t \) of the length equal to the period \( T \) of the right-hand side of Eq. 82 to the both adjacent segments of the same length.

It is worth noting that the usual interpretation of monodromy refers to the behavior of solutions of a differential equation along loops encircling the chosen singular point of this equation (if there are several ones). These loops can be arbitrarily small and even may contract to this point, the monodromy being insensitive to their continuous deformations. In particular, such approach applies in case of the linear equation (11) for which the only singular point (in \( C \)) is zero. Its solutions are regular everywhere (with a natural exception) and the loop can be arbitrarily small as well as arbitrarily large. However, in case of the functions \( \Phi \) and \( \Psi \) of which the former obeys the non-linear equation (84), the monodromy described by the formulas (96), (97) is determined, basically, along the loop coinciding with the unit circle. It also can be deformed but not contracted to zero since the unit disk always contain the singular points of the function \( \Phi \). In accordance with (81) they are associated with roots of certain linear combinations of the functions \( E_{\ell} \). Since the latter verify a linear homogeneous ODE, these roots are simple and hence all the singularities of \( \Phi \), except at zero, are simple poles.

Eqs. 97 and 98 do not involve the parameter \( \ell = -l \) which had been restricted here to positive integers. Hence it can be supposed that such a limitation is superfluous for their validity. It is indeed the case and a straightforward computation shows that the above transformations hold true as long as \( \varphi \) is a solution to Eq. 82 irrespectively to the values of the constant parameters.

On the contrary, the transformation of solutions to Eq. 84 generated by the operators \( \mathcal{L}_A \) and \( \mathcal{L}_B \) can be constructed only if \( \ell \in \mathbb{N} \). They can be represented in the following way.

\[ \Phi_A(z) = \frac{V \Psi^{1/2}(z) \Phi^{1/2}(z) - i U \Psi^{1/2}(1/z) \Phi^{-1/2}(1/z)}{V \Psi^{1/2}(z) \Phi^{-1/2}(z) + i U \Psi^{1/2}(1/z) \Phi^{1/2}(1/z)} \]

\[ \Psi_A(z) = e^{-P(0)} (U^2 + V^2 - 2UV \sin \varphi(0))^{-1} \times \]

\[ (V \Psi^{1/2}(z) \Phi^{1/2}(z) - i U \Psi^{1/2}(1/z) \Phi^{-1/2}(1/z)) \times \]

\[ (V \Psi^{1/2}(z) \Phi^{-1/2}(z) + i U \Psi^{1/2}(1/z) \Phi^{1/2}(1/z)) \]

\[ \Phi_B(z) = \frac{U \Psi^{1/2}(z) \Phi^{1/2}(z) - i V \Psi^{1/2}(1/z) \Phi^{-1/2}(1/z)}{U \Psi^{1/2}(z) \Phi^{-1/2}(z) + i V \Psi^{1/2}(1/z) \Phi^{1/2}(1/z)} \]

\[ \Psi_B(z) = e^{-P(0)} (U^2 + V^2 - 2UV \sin \varphi(0))^{-1} \times \]

\[ (U \Psi^{1/2}(z) \Phi^{1/2}(z) - i V \Psi^{1/2}(1/z) \Phi^{-1/2}(1/z)) \times \]

\[ (U \Psi^{1/2}(z) \Phi^{-1/2}(z) + i V \Psi^{1/2}(1/z) \Phi^{1/2}(1/z)) \]
where \( u_± = e^{i\varphi(T/2)/2} \pm (-1)^i e^{-i\varphi(T/2)/2} \),
\( v_± = (-1)^i e^{\varphi(-T/2)/2} \pm i e^{-i\varphi(-T/2)/2} \),
\( w_± = \cos(\varphi(0)) \pm i \sin(\varphi(0)) \),
\( U = 2e^{P(T/2)/2}(D_+u_+w_+ - iD_-u_-w_-) \),
\( V = D_+w_+(e^{P(T/2)/2}u_+ + e^{P(-T/2)/2}w_+) + iD_-w_-(e^{P(T/2)/2}u_- + e^{P(-T/2)/2}w_-) \). (104)

In a sense, these transformations inherit the composition properties of their prototypes. The corresponding results can be resumed as follows

**Theorem 6.** Let a solution \( \Phi \) to Eq. \( \text{(81)} \) holomorphic in some vicinity of the lift of \( S^1 \) be given and \( \Psi = \Psi(z) \) be a solution to Eq. \( \text{(81)} \), the conditions \( \text{(88)} \) being fulfilled. Then

- Eqs \( \text{(90)}, \text{(97)} \), in which \( \varphi(0), \varphi(\pm \frac{T}{2}), P(0), P(\pm \frac{T}{2}) \) are determined from Eqs \( \text{(85)} \) and \( \text{(86)} \), represent the monodromy transformations of projections of \( \Phi \) and \( \Psi \) for the paths homotopic to \( S^1 \).
- The transformations \( \text{(100)-(103)} \) take these solutions to solutions of the same equations.
  Moreover, when applied twice, the maps \( \text{(100)}, \text{(101)} \) reproduce the original \( \Phi, \Psi \) while two-fold maps \( \text{(102)}, \text{(103)} \) become the above monodromy map.

Thus the transformation represented by Eqs \( \text{(100)}, \text{(101)} \) is involutive and transformation represented by Eqs \( \text{(102)}, \text{(103)} \) can be called a square root of the monodromy transformation.

The final remark concerning implications of symmetries of the space of solutions to Eq. \( \text{(1)} \) for solutions to Eq. \( \text{(82)} \) is as follows. The monodromy transformation in the form \( \text{(75)} \) enabled us to realize the algebraic extension of solutions to Eq. \( \text{(1)} \) from the subdomain with closure of projection coinciding with the complex plane to their whole domains. In terms of solutions to Eq. \( \text{(82)} \) such extension looks like their continuation from the segment \((-T/2, T/2)\) of length \( T = 2\pi/\omega \) to the whole real axis. There are also transformations Eqs \( \text{(76)-\text{(81)}} \) which result in possibility of ‘dissemination’ of solutions to Eq. \( \text{(1)} \) from the half-plane \( \Re z > 0 \). When restricted to the unit circle, this relation means possibility of algebraic extending of solutions to Eq. \( \text{(82)} \) from the segment \((-T/4, T/4)\) of length \( T/2 \).

**Appendix**

When considered on simply connected subsets of \( \mathbb{C}^n \), Eq. \( \text{(1)} \) determines solutions which are single-valued holomorphic functions. In particular, this holds for projections (in fact, restrictions) \( t_+E_{(+)}(z), t_-E_{(-)}(z) \) of the eigenfunctions \( E_{(+)}(z), E_{(-)}(z) \) of the operator \( E_{C} \) to \( \mathbb{C}^n = \mathbb{C}^n \mathbb{R}^m \), the complex plane without zero with negative real axis \( \mathbb{R}^- \) removed. Their one-side limits \( t_+E_{(+)}(x + 0 \cdot i) \) and \( t_-E_{(-)}(x - 0 \cdot i) \) at points \( x \) of \( \mathbb{R}^- \) approached from above \( (x + 0 \cdot i) \) and from below \( (x - 0 \cdot i) \), respectively, do not coincide but are connected by the linear transformation determined by the matrix \( \text{(75)} \) (by definition, the monodromy transformation) which does not depend on \( x \). Notice also that if
\[ E_{(+)}(-)^2 - E_{(+)}(-)^2 = 0 \neq E_{(-)}(-)^2 - E_{(-)}(-)^2 \] (105)
then the non-diagonal elements of \( \text{(75)} \) are non-zero and each function \( E_{(+)} \) or \( E_{(-)} \), as well as their projections \( t_+E_{(+)} \), is taken to a function linearly independent of it. At the same time there exist solutions to Eq. \( \text{(1)} \) for which the same transformation does not engender a distinct solution but reduces to multiplication of the argument by some constant alone. Indeed, let us define the two particular linear combinations \( E_{(+)}^{(M)} \) and \( E_{(-)}^{(M)} \) of \( E_{(+)} \) and \( E_{(-)} \) as follows:
\[ E_{(+)}^{(M)} = (i(E_{(-)}(-)^2 - E_{(-)}(-)^2))^{1/2} E_{(+)} \pm (i(E_{(+)}(-)^2 - E_{(+)}(-)^2))^{1/2} E_{(-)} \]. (106)

A straightforward computation using Eqs \( \text{(77)}, \text{(76)} \), which describe the monodromy transformation \( \mathcal{M} \) of \( E_{(+)} \) in explicit form, shows that
\[ \mathcal{M}E_{(+)}^{(M)} = \Lambda_{\pm} E_{(+)}^{(M)} \], (107)
where the complex (or, perhaps, real) numbers

$$\Lambda_\pm = e^{i\mu} (2E_{(+)}(I)E_{(-)}(I))^{-1/2} \left( E_{(+)}(-I)E_{(-)}(-I) + E_{(+)}(-I)E_{(-)}(-I) \right)$$

are the eigenvalues of the matrix (173) and, at the same time, the eigenvalues of the monodromy operator itself. It is worth noting that

$$\Lambda_+ \cdot \Lambda_- = 1$$

(this follows from definitions (108) and Eq. (102), yet another way of proving leans on the first equation among Eqs. (106)). The inequalities (105) are precisely the condition of non-coincidence of these eigenvalues. If it is fulfilled, the functions $E_{(\pm)}^{(M)}$ are linear independent and constitute the basis of the space of solutions to Eq. (1).

Eq. (107) states in particular that on the edges of $C^*$ contacting $\mathbb{R}_-$ the projections of $E_{(\pm)}^{(M)}$ obey the constraints

$$\iota_{\pm} E_{(\pm)}^{(M)}(x + 0 \cdot i) = \Lambda_+ \cdot \iota_{\pm} E_{(\pm)}^{(M)}(x - 0 \cdot i), \quad x \in \mathbb{R}_-.$$  

(110)

The power functions $z^{(2\pi i)^{-1}\text{Log}\Lambda_\pm}$ possess exactly the same property. This means that for each choice of the signs the one-side limiting values of the product $z^{(2\pi i)^{-1}\text{Log}\Lambda_\pm} \cdot \iota_{\pm} E_{(\pm)}^{(M)}(z)$ on the edges of $C^*$ coincide. Accordingly, they provide its extension to $\mathbb{R}_-$ yielding a single-valued continuous function $G_{(\pm)}(z)$. In view of existence and uniqueness of analytic continuations across $\mathbb{R}_-$ in both directions, this function is holomorphic everywhere in $\mathbb{C}^*$. We have established, therefore, the following result.

**Theorem 7.** Let $E_{(+)}(\mathfrak{s})$ and $E_{(-)}(\mathfrak{s})$ be solutions to Eq. (1) such that $E_{(+)}(I) \neq 0$, and the derivatives $E_{(+)}'(I)$ obey Eqs. (47). Let the constant parameters of Eq. (1) be such that the conditions (105) are fulfilled. Let us define the constants $\gamma_{\pm} = i(2\pi)^{-1}\text{Log}\Lambda_{\pm}$, where the numbers $\Lambda_{\pm}$ defined by the formulas (108) are non-zero in view of Eq. (109) which also implies that $\gamma_+ + \gamma_- = 0$. Then the functions $z^{\gamma_{\pm}} E_{(\pm)}^{(M)}(\mathfrak{s})$, where $z^{\gamma_{\pm}}$ are the lifts of the power functions $z^{\gamma_{\pm}}$ such that $\gamma_{\pm} = 1$, coincide with the lifts (pullbacks) to $\mathbb{C}^*$ of some functions $G_{\pm}(z)$ holomorphic on $\mathbb{C}^*$, i.e.

$$z^{\gamma_{\pm}} E_{(\pm)}^{(M)}(\mathfrak{s}) = G_{\pm}(\iota \mathfrak{s}).$$

(111)

**Corollary 2.** Under conditions assumed by the above theorem, any solution to Eq. (1) is a linear combination of two products of a power function and a function holomorphic on $\mathbb{C}^*$.

In accordance with Laurent theorem, the functions $G_{\pm}(z)$ admit the expansions in Laurent series with center at zero which converge everywhere except at zero. In terms of properties of solutions to Eq. (1) this means the following.

**Corollary 3.** Under conditions assumed by the above theorem, there exists $\gamma \neq 0$ such that there exist the two two-sides sequences $g_{\pm}^{(k)}$, $k \in \mathbb{Z}$, obeying the equations

$$0 = -\mu(k \pm \gamma + 1)g_{\pm}^{(k-1)} + (k \pm \gamma)(k \pm \gamma + 1)g_{\pm}^{(k)} + \mu(k \pm \gamma + 1)g_{\pm}^{(k+1)}$$

(112)

such that the two generalized power series

$$E_{(\pm)}(z) = \sum_{k=-\infty}^{\infty} g_{\pm}^{(k)} z^{k\pm\gamma}$$

(113)

converge everywhere except at zero and satisfy Eq. (1). These are permuted by the operator $\mathcal{L}_C$ (up to some numerical factors).

The use of power series with center at zero as a “template” for solutions to DCHE including Eq. (1) is the wellknown method, see, for example, Eq. (2.454), the next one, and Eq. (2.455) in Ref. [1]. Its curious feature is that, solving Eqs. (112), the series (113) can be constructed for arbitrary $\gamma$. Plugging it further into Eq. (1) and carrying out computations term by term, one
finds that the latter is satisfied. The point however is that such a solution is actually formal, i.e., the series (113) diverges for any non-zero $z$. The only exception (up to some formally distinct but equivalent ones) corresponding to an actual solutions to Eq. (112) is the choice specified by the above theorem. The reasoning leading to it provide us with a simple proof on existence of $\gamma$ making some solution to Eq.s (112) convergent. Another proof based on the Hadamard-Perron theorem had been proposed in Ref. [18].

REFERENCES

[1] D. Schmidt, G. Wolf Double confluent Heun equation, in: Heun's differential equations, Ronveaux (Ed.) Oxford Univ. Press, Oxford, N.Y., (1995), Part C;
[2] S.Yu. Slavyanov, W. Lay. Special Function: A Unified Theory Based on Singularities I, Oxford; New York: Oxford University Press, 2000.
[3] S.I. Tertychnyi Solution space monodromy of a special double confluent Heun equation and its applications, Theoret. and Math. Phys., 201:1, (2019),1426-1441.
[4] V.M. Buchstaber, S.I. Tertychnyi Automorphisms of the solution spaces of special double-confluent Heun equations, Funct. Anal. Appl., 50:3, (2016), 176192.
[5] V.M. Buchstaber, S.I. Tertychnyi Representations of the Klein group determined by the quads of polynomials associated with double confluent Heun equation, Math. Notes, 103:3 (2018), 24-38
[6] S.I. Tertychnyi The modeling of a Josephson junction and Heun polynomials, arXiv:math-ph/0601064 (2006).
[7] V. M. Buchstaber, S. I. Tertychnyi Explicit solution family for the equation of the resistively shunted Josephson junction model, Theoret. and Math. Phys., 176:2 (2013), 965986.
[8] S.I. Tertychnyi Long-term behavior of solutions of the equation $\ddot{\phi} + \sin \phi = f$ with periodic $f$ and the modeling of dynamics of overdamped Josephson junctions, arXiv:math-ph/0512058 (2005), p. 12, footnote 7.
[9] S.I. Tertychnyi The interrelation of the special double confluent Heun equation and the equation of RSJ model of Josephson junction revisited, arXiv:math-ph/1811.03971, (2018)
[10] P. Mangin, R. Kahn Superconductivity. An introduction, Springer (2017).
[11] W.C. Stewart Current-voltage characteristics of Josephson junctions, Appl. Phys. Lett., 12 (1968), 277-280;
[12] D.E. McCumber Effect of ac impedance on dc voltage-current characteristics of superconductor weak-link junctions, J. Appl. Phys., 39 (1968), 3113-3118;
[13] R. Foote Geometry of the Prytz planimeter, Reports Math. Physics, 42 (1998), 249-271;
[14] M. Levi, S. Tabachnikov On bicycle tire tracks geometry, hatchet planimeter, Menzies conjecture and oscillation of unicycle tracks, Experiment. Math., 18:2 (2009), 173-186;
[15] R.L. Foote, M. Levi, S. Tabachnikov Tractrices, Bicycle Tire Tracks, Hatchet Planimeters, and a 100-year-old Conjecture, The Amer. Math. Monthly, 120:3 (2013), 199-216;
[16] J. Guckenheimer, Yu. S. Ilyashenko The duck and the devil: canards on the staircase, Mosc. Math. J., 1:1 (2001), 27–47.
[17] G. Bor, M. Levi, R. Perline, S. Tabachnikov Tire tracks and integrable curve evolution, arXiv:math.DG/1705.06314 (2017), 1-61.
[18] V.M. Buchstaber, A.A. Glutsuk, On monodromy eigenfunctions of Heun equations and boundaries of phase-lock areas in a model of overdamped Josephson effect, Proc. Steklov Inst. Math., 297:1 (2017) 50-89.

Russian Metrological Institute of Technical Physics and Radio Engineering (VNIIFTRI), Mendeleev, 141570, Russia