A basis for the Diagonal Harmonic Alternants
by
A. M. Garsia(†) and M. Zabrocki

Abstract

It will be shown here that there are differential operators $E, F$ and $H = [E, F]$ for each $n \geq 1$, acting on Diagonal Harmonics, yielding that $DH_n$ is a representation of $sl[2]$ (see [3] Chapter 3). Our main effort here is to use $sl[2]$ theory to predict a basis for the Diagonal Harmonic Alternants, $DHA_n$. It can be shown that the irreducible representations $sl[2]$ are all of the form $P, EP, E^2P, \ldots, E^kP$, with $FP = 0$ and $E^{k+1}P = 0$. The polynomial $P$ is known to be called a “String Starter”. From $sl[2]$ theory it follows that $DHA_n$ is a direct sum of strings. Our main result so far is a formula for the number of string starters. A recent paper by Carlsson and Oblomkov (see [2]) constructs a basis for the space of Diagonal Coinvariants by Algebraic Geometrical tools. It would be interesting to see if any our results can be derived from theirs.

Introduction

We set $X_n = x_1, x_2, \ldots, x_n$ and $Y_n = y_1, y_2, \ldots, y_n$, we will be working here with polynomials $P(X_n; Y_n)$ with rational coefficients, that is $P(X_n; Y_n) \in Q[x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n]$.

The diagonal action of $S_n$ is defined by setting for any $\sigma \in S_n$

$$\sigma P(X_n; Y_n) = P(x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma n}; y_{\sigma 1}, y_{\sigma 2}, \ldots, y_{\sigma n}).$$

Another important tool in studying $S_n$ modules that are invariant under the diagonal action is the scalar product

$$\langle P, Q \rangle = L_o P(\partial X_n; \partial Y_n)Q(X_n; Y_n).$$

where the differential operator $P(\partial X_n; \partial Y_n)$ is obtained by the replacements $x_i \rightarrow \partial x_i$ and $y_i \rightarrow \partial y_i$. It is easy to see that we have

$$\langle \sigma P, \sigma Q \rangle = \langle P, Q \rangle \quad \text{(for all } \sigma \in S_n \text{).}$$

In this paper we will study the $S_n$ module $DH_n$ of Diagonal Harmonic polynomials. This module was originally defined as the orthogonal complement, with respect to the scalar product in I.2, of the ideal of polynomials that are invariant under the diagonal action. By a result of Hermann Weyl (see [16]) it follows that $P(X_n; Y_n) \in DH_n$ if and only if

$$\sum_{i=1}^n \partial_{x_i}^p \partial_{y_i}^q P(X_n; Y_n) = 0 \quad \text{(for all } p + q \geq 1 \text{).}$$

This simpler definition makes it obvious that $DH_n$ is invariant under the diagonal action.

It also immediately follows from I.4 that if $P \in DH_n$ then all the bi-homogeneous components of $P$ are in $DH_n$. This implies that we have the direct sum decomposition

$$DH_n = \bigoplus_{0 \leq r+s \leq \binom{n}{2}} \mathcal{H}_{r,s}(DH_n),$$

where $\mathcal{H}_{r,s}(DH_n)$ is the subspace of diagonal Harmonics polynomials which are bi-homogeneous of degree $r$ in the $x's$ and degree $s$ in the $y's$. It then follows that the character resulting from the diagonal action of $S_n$ on $DH_n$ can be written in the form

$$\chi^{DH_n} = \sum_{0 \leq r+s \leq \binom{n}{2}} t^r q^s \chi^{r,s}, \quad \text{(where } \chi^{r,s} \text{ is the character of } \mathcal{H}_{r,s}(DH_n).}$$

The upper bound $\binom{n}{2}$ in I.5 follows from the operator conjecture (proved by Mark Haiman in [13]). $DH_n$ can be obtained by applying differential operators to the Vandermonde determinant $\prod_{1 \leq i < j \leq n}(x_i - x_j)$.

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The Frobenius map \( \mathcal{F} \) (see [3]) considerably simplifies the operation of computing the character of an \( S_n \) representation. Frobenius uses the dimension equality between the class functions of \( S_n \) and the space \( \Lambda^w_n \) of homogeneous symmetric functions of degree \( n \). This given, \( \mathcal{F} \) maps Class Functions onto the power basis by the formula
\[
\mathcal{F}C_\mu = p_\mu/z_\mu.
\]  
where \( C_\mu \) is the sum of all the permutations of cycle structure \( \mu \), \( p_\mu = p_1p_2\cdots p_{l(\mu)} \), where \( l(\mu) \) denotes the length of \( \mu \) and
\[
z_\mu = 1^{a_1}2^{a_2}\cdots n^{a_n}a_1a_2\cdots a_n!
\]  
when \( \mu = 1^{a_1}2^{a_2}\cdots n^{a_n} \mid n \) \( \]  
It follows from I.7 that Young’s irreducible character \( \chi^\lambda \) is given by the identity
\[
\chi^\lambda = \mathcal{F}^{-1}s_\lambda
\]  
If \( M_n \) is an \( S_n \) module with character \( \chi^{M_n} \) then the Frobenius Characteristic of \( M_n \) is the symmetric polynomial \( \mathcal{F} \chi^{M_n} \). It was conjectured (in [6]) and shown by Mark Haiman using tools of Algebraic Geometry (in [12]) that the Frobenius characteristic of \( DH_n \) is the symmetric rational function
\[
\mathcal{F} \chi^{DH_n} = \sum_{\mu \vdash n} T_\mu \tilde{H}_\mu(X;q,t)MB_\mu(q,t)\Pi_\mu(q,t)/w_\mu(q,t).
\]  
To define the ingredients that appear in this formula it is convenient to use Macdonald’s notation [14] yet identify partitions with their French Ferrers diagram. Given a partition \( \mu \) we derive that
\[
\chi^\mu = \mathcal{F}^{-1}s_\mu
\]  
where \( s_\mu \) is the symmetric polynomial \( \left( \sum x^\mu \right)^n \). It was conjectured (in [6]) and proved by Mark Haiman in [12] to be the Frobenius Characteristic of the linear span of derivatives of the alternant that corresponds to the partition \( \mu \).

Our first goal is to construct operators that preserve \( DH_n \). We will prove this property for all the differential operators
\[
a) \quad F_{r,s} = \sum_{i=1}^n \frac{x_i \partial x_i^r y_i^s}{z_i}, \quad \quad b) \quad E_{r,s} = \sum_{i=1}^n \frac{y_i \partial x_i^r y_i^s}{z_i} \quad \text{ (for } r + s \geq 1) \quad .
\]  
We will also prove the relations
\[
a) \quad \left[ F_{p,q}, F_{r,s} \right] = (p-r)F_{p+r-1,q+s},
\]  
\[
b) \quad \left[ F_{p,q}, E_{r,s} \right] = qF_{p+r,q+s-1} - rE_{p+r-1,q+s},
\]  
\[
c) \quad \left[ E_{p,q}, E_{r,s} \right] = (q-s)E_{p+r,q+s-1}.
\]  
Furthermore by setting
\[
a) \quad F = F_{0,1}, \quad \quad b) \quad E = E_{1,0}, \quad \quad c) \quad H = [E,F] = \sum_{i=1}^n \left( y_i \partial y_i - x_i \partial x_i \right)
\]  
we derive that \( DH_n \) is a direct sum of irreducible representations of \( sl[2] \).
Finally, using the results of the $q,t$-Catalan paper (see [8],[9]) we derive that the Hilbert polynomial
\[ c_n(q,t) = \mathcal{F} \chi_{DHA_n} \left| _{s[1^n]} \right. \]  
I.15

of the Diagonal Alternants can be obtained by a purely combinatorial construction. To describe this construction we need further definitions. By a Dyck path in the $n \times n$ lattice square $\mathcal{L}_n$ we mean a path which proceeds by $n$ unit North steps and $n$ unit East steps and goes from $(0,0)$ to $(n,n)$ always remaining weakly above the main diagonal of $\mathcal{L}_n$. This is the straight line that joins $(0,0)$ to $(n,n)$. In the illustration on the right we colored light green all the cells bisected by the main diagonal of $\mathcal{L}_n$. A Dyck path is depicted there in light blue. It is clear that we only need to give the abscissas of the North steps of the Dyck path $D$. Those are the integers that are on the left of the rows of $\mathcal{L}_n$. Thus $D = [0,0,1,1,2,3,3,5]$. Each Dyck path has two statistics which we call $\text{area}(D)$ and $\text{bounce}(D)$. The area statistic is quite simple, its formula is $\text{area}(D) = [0,1,\ldots,n-1] - [0,d_1,\ldots,d_{n-1}]$. This is the number of cells between the Dyck path and the lattice diagonal (the green cells). In our case $\text{area}(D) = 13$. The bounce statistic is the sum of the places where the bounce path hits the main diagonal of $\mathcal{L}_n$. In our display we depicted the bounce path in red using a thinner line. In the general case it starts straight North until it touches the West end of an East step. Then it goes straight East until it touches the diagonal. Then goes straight North until it touches the West end of an East step... alternating straight North and straight East until it reaches $(n,n)$. We place in the possible diagonal touching points the labels $1,2,\ldots,n-1$ as indicated in our display. In our example, $\text{bounce}(D) = 3 + 6$. We must emphasize that the bounce path does not change direction by touching the East end of an East step. In our display that happens in the $5^{th}$ and $8^{th}$ rows.

In particular we obtain
\[ c_n(q,t) = \sum_{D \in \mathcal{D}_n} t^{\text{bounce}(D)} q^{\text{area}(D)} = \sum_{D \in \mathcal{D}_n} t^{\text{area}(D)} q^{\text{dinv}(D)} \]  
I.16

The first identity was conjectured by Jim Haglund the second was conjectured by Mark Haiman. The $\text{dinv}$ statistic has also a purely combinatorial definition. Let $\text{dinv}_a(D) = \sum_{1 \leq i < j \leq n} \chi(u_i = u_j)$ where $u_i$ is the contribution to the area statistic by the $i^{th}$ north step. Similarly we let $\text{dinv}_b(D) = \sum_{1 \leq i < j \leq n} \chi(u_i = u_j+1)$, then set $\text{dinv}(D) = \text{dinv}_a(D) + \text{dinv}_b(D)$. The problem to construct a further statistic that combined with area gives I.16 was stated in [6], these two solutions were discovered quite a few years later.

For $n = 3$ we get
\[ \mathcal{F} DH_3 \left| _{s[1^3]} \right. = \sum_{D \in \mathcal{D}_3} t^{\text{bounce}(D)} q^{\text{area}(D)} = t^3 + t^2 q + tq + tq^2 + q^3. \]  
I.17

This identity reveals that the alternating character $\chi^{[1,1,1]}$ occurs in bi-degrees $(3,0), (2,1), (1,2), (0,3)$ and $(1,1)$. In this particular case these are Frobenius images of the $sl[2]$ strings generated by the following two alternants
\[ a) \quad \Delta_{1,1,1} = \det \left( \begin{array}{ccc} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{array} \right), \quad b) \quad \Delta_{2,1} = \det \left( \begin{array}{ccc} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{array} \right). \]  
I.18

It stands to reason that we should be able to construct a basis for the Diagonal Harmonics Alternants using the operators in I.12, $sl[2]$ theory and the combinatorics of the $q,t$-Catalan.

Mark Haiman’s proof in [13] of the Operator Conjecture implies the following result

**Theorem I.1**

*For any $n \geq 1$ let $m$ be the dimension of $DHA_n$ in bi-degree $(a,b)$, then for this bi-degree, it is always possible to construct $m$ sequences $1 \leq r_1 \leq r_2 \leq \cdots \leq r_m \leq n$ such that the Diagonal Harmonics Alternants $E_{r_1,0}E_{r_2,0}\cdots E_{r_m,0}\Delta_{1^n}$ are linearly independent. This requires that $n \choose 2 - (r_1 + r_2 + \cdots + r_b) = a$*
The only significant results we will prove in this paper, besides introducing $sl[2]$ theory and proving the invariance under the diagonal action of the differential operators in I.12 and proving the properties in I.13, are a formula for the number of starters and an algorithm that gives that number for every $n$.

1. The differential operators.

We start with an auxiliary fact concerning the interaction between multiplication operators and differential operators.

**Proposition 1.1**

For any variable $y$ and integer exponent $q \geq 1$ we have

\[ \partial_y^q y = q \partial_y^{q-1} + y \partial_y^q, \quad \text{where "}$y$" is the multiplication by $y$ operator} \tag{1.1} \]

**Proof**

Suppose that $P(y)$ is a polynomial in $y$. Then for $q = 1$ we get

\[ \partial_y P(y) = P(y) + y \partial_y P(y). \tag{1.2} \]

Thus 1.1 is true for $q = 1$. Proceeding by induction on $q$, suppose that 1.1 is true up to $q - 1$. Then we have

\[
\partial_y^q y P(y) = \partial_y \partial_y^{q-1} y P(y) = \partial_y (q - 1) \partial_y^{q-2} P(y) + \partial_y y \partial_y^{q-1} P(y) = (q - 1) \partial_y^{q-1} P(y) + \partial_y^{q-1} P(y) + y \partial_y^q P(y) = q \partial_y^{q-1} P(y) + y \partial_y^{q-1} P(y).
\]

this proves 1.1.

As an example we will show that

**Theorem 1.1**

The $sl[2]$ operators

\[ a) \quad F = \sum_{i=1}^{n} x_i \partial_{y_i}, \quad b) \quad E = \sum_{i=1}^{n} y_i \partial_{x_i}, \tag{1.4} \]

preserve $DH_n[X_n; Y_n]$.

**Proof**

To this end we will first compute the bracket

\[ [\Pi_{p,q}^n, E] = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \partial_{x_i}^p \partial_{y_j}^q, \ y \partial_{x_j} \right], \quad \text{(where } \Pi_{p,q}^n = \sum_{i=1}^{n} \partial_{x_i}^p \partial_{y_i}^q \text{)}. \tag{1.5} \]

Since for $j \neq i$ the differential and multiplication operators commute, we only need to work with

\[ [\Pi_{p,q}^n, E] = \sum_{i=1}^{n} \left[ \partial_{x_i}^p \partial_{y_i}^q, \ y \partial_{x_i} \right]. \tag{1.6} \]

Using 1.1 for $q \geq 1$ we obtain

\[ \partial_{x_i}^p \partial_{y_i}^q y \partial_{x_i} = \partial_{x_i}^p (q \partial_{y_i}^{q-1} + y \partial_{y_i}^q) \partial_{x_i} = q \partial_{x_i}^{p+1} \partial_{y_i}^{q-1} + y \partial_{x_i}^{p+1} \partial_{y_i}^q. \tag{1.7} \]

We also have

\[ y \partial_{x_i} \partial_{x_i}^p \partial_{y_i}^q = y \partial_{x_i}^{p+1} \partial_{y_i}^q \tag{1.8} \]

so 1.6 becomes

\[ [\Pi_{p,q}^n, E] = q \sum_{i=1}^{n} \partial_{x_i}^{p+1} \partial_{y_i}^{q-1} + \sum_{i=1}^{n} \partial_{x_i}^{p+1} \partial_{y_i}^q - \sum_{i=1}^{n} \partial_{y_i}^{p+1} \partial_{y_i}^q = q \sum_{i=1}^{n} \partial_{x_i}^{p+1} \partial_{y_i}^{q-1}, \tag{1.9} \]

or equivalently

\[ \Pi_{p,q}^n E = E \Pi_{p,q}^n + q \Pi_{p+1,q-1}^n. \tag{1.10} \]

Thus applying $\Pi_{p,q}^n E$ to a diagonal harmonic polynomial $P(X_n; Y_n)$, gives

\[ \Pi_{p,q}^n E P(X_n; Y_n) = E \Pi_{p,q}^n P(X_n; Y_n) + q \Pi_{p+1,q-1}^n P(X_n; Y_n) = 0. \tag{1.11} \]

The case $q = 0$ is trivial. Proving that the operator $E$ preserves diagonal harmonics. Working with $F$ we will reach the same result using analogous steps.
We can use the same idea on the operators

\[ a) \quad F_{r,s} = \sum_{i=1}^{n} x_i \partial_{x_i}^p \partial_{y_i}^q, \quad \quad \quad b) \quad E_{r,s} = \sum_{i=1}^{n} y_i \partial_{x_i}^p \partial_{y_i}^q. \]  

As before we can reduce the calculation to \( j = i \) and work with

\[ \Pi_{p,q}^n F_{r,s} = \sum_{i=1}^{n} \partial_{x_i}^p x_i \partial_{x_i}^q \partial_{y_i}^{q+s} = \sum_{i=1}^{n} (p \partial_{x_i}^{p+r} \partial_{y_i}^{q+s} + x_i \partial_{x_i}^p \partial_{y_i}^{q+s}). \]

Likewise we have

\[ F_{r,s} \Pi_{p,q}^n = \sum_{i=1}^{n} x_i \partial_{x_i}^p \partial_{y_i}^{q+s}. \]

Thus

\[ \Pi_{p,q}^n F_{r,s} = F_{r,s} \Pi_{p,q}^n + p \Pi_{p+r-1,q+s}^n. \]

Now if \( P(X_n; Y_n) \) is Diagonal Harmonic then

\[ \Pi_{p,q}^n F_{r,s} P(X_n; Y_n) = F_{r,s} \Pi_{p,q}^n P(X_n; Y_n) + p \Pi_{p+r-1,q+s}^n P(X_n; Y_n) = 0. \]

Proving that \( F_{r,s} \) preserves Diagonal Harmonics. An analogous argument yields the same result for \( E_{r,s}. \)

**Theorem 1.2**

The following identities hold true for all \( p + q \geq 1 \) and \( r + s \geq 1. \)

\[ a) \quad [F_{p,q}, F_{r,s}] = (p-r)F_{p+r-1,q+s}, \quad \quad \quad b) \quad [F_{p,q}, E_{r,s}] = qF_{p+r-1,q+s} - rE_{p+r-1,q+s}, \quad \quad \quad c) \quad [E_{p,q}, E_{r,s}] = (q-s)E_{p+r-1,q+s-1}. \]

**Proof**

Reducing again to the case \( j = i \) we can write

\[ [F_{p,q}, F_{r,s}] = \sum_{i=1}^{n} [x_i \partial_{x_i}^p \partial_{x_i}^q \partial_{y_i}^s] = \sum_{i=1}^{n} x_i (\partial_{x_i}^p \partial_{y_i}^s) \partial_{x_i}^r \partial_{y_i}^q - \sum_{i=1}^{n} x_i (\partial_{x_i}^r \partial_{y_i}^q) \partial_{x_i}^p \partial_{y_i}^s. \]

Since we have

\[ a) \quad \partial_{x_i}^p x_i = p \partial_{x_i}^{p-1} + x_i \partial_{x_i}^p, \quad \quad \quad b) \quad \partial_{x_i}^r x_i = r \partial_{x_i}^{r-1} + x_i \partial_{x_i}^r. \]

The identity in 1.18 becomes, using 1.19

\[ [F_{p,q}, F_{r,s}] = \sum_{i=1}^{n} x_i (p \partial_{x_i}^{p-1} + x_i \partial_{x_i}^p) \partial_{x_i}^r \partial_{y_i}^q - \sum_{i=1}^{n} x_i (r \partial_{x_i}^{r-1} + x_i \partial_{x_i}^r) \partial_{x_i}^p \partial_{y_i}^s. \]

Now this can be rearranged to

\[ [F_{p,q}, F_{r,s}] = \sum_{i=1}^{n} x_i (p \partial_{x_i}^{p-1} \partial_{x_i}^r \partial_{y_i}^q) - \sum_{i=1}^{n} x_i (r \partial_{x_i}^{r-1} \partial_{x_i}^p \partial_{y_i}^s) + \sum_{i=1}^{n} (x_i^2 - x_i) \partial_{x_i}^{p+r} \partial_{y_i}^{q+s}. \]

From which we derive that

\[ [F_{p,q}, F_{r,s}] = (p-r) \sum_{i=1}^{n} x_i \partial_{x_i}^{p+r-1} \partial_{y_i}^{q+s} = (p-r)F_{p+r-1,q+s}. \]

This proves a) of 1.17.
Next we work on b) of 1.17. Reducing to $j = i$ we can write
\[
[F_{p,q}, E_{r,s}] = \sum_{i=1}^{n} [x_i \partial_{y_i} \partial_{x_i} \partial_{y_i}^p \partial_{x_i}^p] = \sum_{i=1}^{n} x_i (\partial_{y_i} \partial_{x_i}^p \partial_{x_i}^p) - \sum_{i=1}^{n} y_i (\partial_{x_i} \partial_{y_i}^p \partial_{x_i}^p) \quad 1.22
\]
Using 1.19 we get
\[
[F_{p,q}, E_{r,s}] = \sum_{i=1}^{n} x_i (\partial_{y_i}^{-p} + y_i \partial_{y_i}^p) \partial_{x_i}^p \partial_{y_i}^p - \sum_{i=1}^{n} y_i (\partial_{x_i}^{-p} + x_i \partial_{x_i}^p) \partial_{y_i}^p \partial_{x_i}^p = q \sum_{i=1}^{n} \partial_{x_i}^p \partial_{y_i}^p - r \sum_{i=1}^{n} y_i \partial_{x_i}^p \partial_{y_i}^p + \sum_{i=1}^{n} \partial_{x_i} (y_i - x_i \partial_{y_i}^p \partial_{x_i}^p) \partial_{y_i}^p \partial_{x_i}^p \quad 1.23
\]
This proves 1.17 b). The identity in 1.17 c) is proved the same way we proved a).

**Remark 1.1**

Using b) of 1.17 we can prove the result in c) of I.14. In fact, since $E = E_{1,0}$ and $F = F_{0,1}$, setting $p = 0, q = 1, r = 1, s = 0$ in
\[
[F_{p,q}, E_{r,s}] = q F_{p+r,q+s-1} - r E_{p+r-1,q+s},
\]
we obtain
\[
[F_{0,1}, E_{1,0}] = F_{1,0} - E_{0,1},
\]
Using 1.12 this gives that the $\mathfrak{sl}[2]$ operator in I.14
\[
H = [E, F] = \sum_{i=1}^{n} y_i \partial_{y_i} - \sum_{i=1}^{n} x_i \partial_{x_i}, \quad 1.24
\]
on Diagonal Harmonics is none other than the Euler operator in the $y$'s minus the Euler operator in the $x$'s.

**Remark 1.2**

We will make multiple uses of the operators $E_{r,0}$, we must point out that that these operators commute regardless of the values of $r$. This is one of the consequences of the identities in 1.17. In fact, if $q = s$ in c) that will immediately force the commutativity of $E_{p,q}$ and $E_{r,s}$. The analogous result can also be obtained for the differential operators pairs in 1.17 a).

**Proof of Theorem I.1**

For $A = (a_1, a_2, \ldots, a_n)$ with $a_i \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$ set
\[
E^A = E_{1,0}^{a_1} E_{2,0}^{a_2} \cdots E_{n,0}^{a_n}. \quad 1.25
\]
The proof in [13] of the operator conjecture implies that if we set $\mathcal{H}[x_1, x_2, \ldots, x_n] = L_\lambda[\Delta_1^n]$ (the ordinary Harmonics of $S_n$), then
\[
\sum_A E^A \mathcal{H}[x_1, x_2, \ldots, x_n] = \mathcal{H}[x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n], \quad 1.26
\]
the space of Diagonal Harmonics. This implies that if $\{h_1, h_2, \ldots, h_n\}$ is any basis for $\mathcal{H}[x_1, x_2, \ldots, x_n]$ then the elements $\{E^A h_1, E^A h_2, \ldots, E^A h_n\}$ span the Diagonal Harmonics, in particular they span the Diagonal Harmonic Alternants. If the basis is isotypical that is there are as many independent elements generating the irreducible representation character $\chi^\lambda$ as the dimension of this representation. Since the alternating element occurs with multiplicity one and is the Vandermonde in $x_1, x_2, \ldots, x_n$ the basis elements must be all of the form
\[
E^A \Delta_1^n = E_{r_1,0} E_{r_2,0} \cdots E_{r_b,0} \Delta_1^n, \quad 1.27
\]
with $0 \leq r_1 \leq r_2 \leq \cdots \leq r_b$. This proves the theorem.
2. Analysing the computer data for \( n = 5, 6 \).

Although our differential operators are constructing Diagonal Harmonic alternants we will be guided by the Frobenius characteristic of the alternants in \( DH_n \)

\[
c_n(q,t) = \sum_{D \in D_n} t^{\text{bounce}(D)} q^{\text{area}(D)}.
\]

We will do that for \( n = 5 \) and \( n = 6 \).

In the display on the right we have depicted a visual image of this polynomial for \( n = 5 \). For instance the term of \( c_5(q,t) \) that is in the fifth row and fifth column corresponds to the Dyck path \( D \in D_5 \) for which \( t^{\text{bounce}(D)} q^{\text{area}(D)} = 2 t^4 q^4 \). We can easily locate the \( sl[2] \) string in this display. In \( DH_5 \) they are of the form \( P(X_n; Y_n) \to EP(X_n; Y_n) \to \cdots \to E^k P(X_n; Y_n) \), where \( P(X_n; Y_n) \) is killed by \( F \) and is a homogeneous polynomial of bi-degree \((a, b)\). The operator \( E \) diminishes degree in \( x' \)'s by one and increases degree in \( y' \)'s by one, thus the polynomial \( E^i P(X_n; Y_n) \) is homogeneous of bi-degree \((a - i, b + i)\). The final element is of bi-degree \((b, a)\) and is killed by \( E \). The circled entries correspond to alternants indexed by partitions. The one corresponding to \( t^{10} \) is the Vandermonde \( \Delta_{1,5} \). The one corresponding to \( t^6 q^1 \) is \( \Delta_{2,1}, \Delta_{2,1} \) and \( \Delta_{3,2} \) form an \( sl[2] \) string. Our list ends with \( \Delta_{1,5} \) and the Vandermonde \( \Delta_5 \). This display shows that the alternants in \( DH_5 \) factor into a direct sum of 7 of \( sl[2] \) strings. The one of highest total degree is started by \( \Delta_{5,1} \). Let us call it String 1. String 2 starts with bi-degree \((8, 1)\) and ends in bi-degree \((1, 8)\). String 3 starts with bi-degree \((7, 1)\) and ends in bi-degree \((1, 7)\). String 4 starts with bi-degree \((6, 2)\) and ends in bi-degree \((2, 6)\). That accounts for the \( 2^i \)'s in the image of its path. String 5 starts with bi-degree \((6, 1)\) and ends in bi-degree \((1, 6)\). These two ends are \( \Delta_{2,1}, \Delta_{4,1} \). String 6 starts with bi-degree \((6, 2)\) and ends in bi-degree \((2, 6)\). That accounts for all the \( 2^i \)'s in the image of its path. String 7 starts with bi-degree \((4, 2)\) and ends in bi-degree \((2, 4)\).

Our computations using MAPLE yielded the following seven starters:

\[
\Delta_{5,5}, E_{2,0} \Delta_{1,5}, E_{3,0} \Delta_{1,5}, E_{4,0} \Delta_{1,5}, 3E_{2,0} E_{2,0} \Delta_{1,5} + 2EE_{3,0} \Delta_{1,5}, 2E_{2,0} E_{2,0} E_{2,0} \Delta_{1,5} - 3EE E_{4,0} \Delta_{1,5}, E_{4,0} E_{2,0} \Delta_{1,5}.
\]

Notice that for \( n < 7 \) there is at most one starter at any bi-degree. If we find one that starts at bi-degree \((a, b)\) and ends at \((b, a)\) we can safely complete the string in a construction of a basis. However, the result is not a basis that is consistent with Theorem I.1. This is what happens with all the elements of the strings started by the 5th \( \text{th} \) and 6th \( \text{th} \) starters.

It will be good to make a few observations before we work on \( n = 6 \). Firstly, Remark 1.2 tells us that the operators \( E_{r,0} \) commute whatever is the value of \( r \). From Theorem I.1 we know that to obtain an alternant of bi-degree \((a, b)\) in \( DH_n \) using the polynomial \( E_{r_1,0} E_{r_2,0} \cdots E_{r_n,0} \Delta_{1^n} \) we must require the equality

\[
r_1 + r_2 + \cdots + r_n = \binom{n}{2} - a
\]

Due to the commutativity fact, there is no loss in assuming that our sequences are weakly decreasing \((r_1 \geq r_2 \geq \cdots \geq r_n)\) and satisfy 2.2. This is the number of \( b \) parts partitions of \( \binom{n}{2} - a \).

Notice that our display at the row indexed by \( t^5 \) and column indexed by \( q^2 \) reveals that there is only one alternant in \( DH_5 \) with bi-degree \((5, 2)\). Yet the 2 parts partitions of 5 are 32 and 41. Thus when we reduce the two polynomials \( E_{3,0} E_{2,0} \Delta_{1,5} \) and \( E_{4,0} E_{1,0} \Delta_{1,5} \) to have leading monomial with coefficient 1 the resulting alternants must be identical!

This fact shows that we can only use partitions as an upper bound to the number of strings that start at a given bi-degree. However, in our computer experimentation with \( DH A_6 \) we discovered that pairs
that were discarded in $DHA_5$ had also to be discarded as factors in the construction of string starters in $DHA_6$. This suggests that the construction of a basis for $DHA_n$ might demand a recursion on $n$.

These findings suggest that it might be possible to obtain a recursive construction of a basis. The following result gives us a tool for not using factors that have been discarded for $n - 1$ in the construction of basis elements for $n$. For simplicity we will state it in the simplest useful case.

**Theorem 2.1**

Suppose that $C_{n-1}(q,t)|_{t=0q^{b-1}} = k - 1$ and $C_n(q,t)|_{t=0q^b} = k$, then the polynomial

$$P(X_n; Y_n) = E_{r_1,0} E_{r_2,0} \cdots E_{r_b,0} \Delta_1^n$$

cannot be used as a starter in bi-degree $(a, b)$ if

$$a = n - 1 + c - r_b$$

for any discarded pairs of solutions of

$$r_1 + r_2 + \cdots + r_b = \binom{n - 1}{2} - c.$$  

**Proof**

Since our conjecture requires that

$$r_1 + r_2 + \cdots + r_b = 1 + 2 + \cdots + n - 1 - a,$$

this can be rewritten in an inductive way by relating for $n$ what we already obtained for $n - 1$

$$0 = r_1 + r_2 + \cdots + r_{b-1} - \binom{n - 2}{2} - c = n - 1 - a + c - r_b.$$

This proves 2.4. Our task is now to explore $n = 6$.

On the right we have a visual display of a portion of the Frobenius characteristic of the alternants in $DHA_6$. This is the polynomial $c_6(q,t)$, The number of strings start at bi-degree $(a, b)$ is again

$$c_6(q,t)|_{t=0q^{b-1}} - c_6(q,t)|_{t=0q^b} \leq 1$$

Without the 0's the string starters are $(1 \to) \Delta_1^6$, $(2 \to) E_2 \Delta_1^6$, $(3 \to) E_3 \Delta_1^6$, $(4 \to) 11 E_2 E_2 \Delta_1^6 + 4 E_3 E_1 \Delta_1^6$

$(5 \to) E_4 \Delta_1^6$, $(6 \to) 2 E_3 E_2 \Delta_1^6 + E_2^2 E_1 \Delta_1^6$, $(7 \to) 12 E_2 E_2 \Delta_1^6 + 1 E_3^2 E_1 \Delta_1^6 + 4 E_3 E_1 \Delta_1^6 + 5 E_2 E_1 \Delta_1^6$

$(8 \to) E_5 \Delta_1^6$, $(9 \to) 2 E_4 E_2 \Delta_1^6 - 9 E_5 E_1 \Delta_1^6$

$(10 \to) 28 E_3 E_2^2 \Delta_1^6 - 19 E_5 E_1 \Delta_1^6$

$(11 \to) 5 E_3^2 E_2 E_1 \Delta_1^6 + 4 E_3 E_1^2 E_1 \Delta_1^6 - 18 E_3 E_1 \Delta_1^6$

$(12 \to) 3 E_3 E_2 \Delta_1^6 + 2 E_2 E_2 \Delta_1^6$

$(13 \to) 20 E_2 E_2 \Delta_1^6 + 3 E_2 E_2 \Delta_1^6 + 9 E_2 E_2 \Delta_1^6$

$(14 \to) 9 E_2 E_2 \Delta_1^6$

$(15 \to) 5 E_3 \Delta_1^6$, $(16 \to) 5 E_3 \Delta_1^6 - 9 E_2 \Delta_1^6$

$(17 \to) E_2 E_2 \Delta_1^6$.

We can derive an algorithm for obtaining the number of starters for any $n$, that MAPLE permits, from the following calculation. Since every string is of the form

$$P(X_n; Y_n) \rightarrow EP(X_n; Y_n) \rightarrow \cdots \rightarrow E^k P(X_n; Y_n).$$

If $P(X_n; Y_n)$ is an alternant homogeneous of bi-degree $(u, v)$ then the alternant $E^i P(X_n; Y_n)$ is homogeneous of bi-degree $(u - i, v + i)$, with $E^k P(X_n; Y_n)$ homogeneous of bi-degree $(v, u)$. The contribution of this string to the Hilbert series of $DHA_n$ is the polynomial $\sum_{i=0}^{u-v} u^{-i} q^{v+i}$. Note that

$$t^u q^v = (qt)^u q^{v+i},$$

so we can write

$$t^u q^v \left(1 + \left(\frac{q}{t}\right) + \cdots + \left(\frac{q}{t}\right)^{u-v}\right) = (qt)^u \frac{q^{v+i}}{1 - \left(\frac{q}{t}\right)} = (qt)^u \frac{q^{v+i} - \left(\frac{q}{t}\right)^{u-i+1}}{1 - \left(\frac{q}{t}\right)}. $$

$$= (qt)^u \frac{\left(\frac{q}{t}\right)^{v+i} - \left(\frac{q}{t}\right)^{u-i+1}}{1 - \left(\frac{q}{t}\right)} = (qt)^u \frac{\left(\frac{q}{t}\right)^{v+i} - \left(\frac{q}{t}\right)^{u-i+1}}{1 - \left(\frac{q}{t}\right)}.$$
Thus the Hilbert series of $\text{DHA}_n$ is the polynomial
\[
h(q, t) = \frac{\binom{n}{2} \binom{n}{2}}{u \sum \sum b_{u,v} (qt) \frac{\frac{1}{2} q^{u(v+1)} - \frac{1}{4} q^{u+v+1}}{q^{u+v+1} - q^{u+v+1}}} \tag{2.10}
\]
where $b_{u,v}$ is the number of strings that start in bi-degree $(u,v)$. Making the specialization $t \rightarrow q^{-1}$ gives
\[
h(q, q^{-1}) = \frac{\binom{n}{2} \binom{n}{2}}{u \sum \sum b_{u,v} \frac{q^{u(v+1)} - q^{u(v+1)}}{q - q^{-1}}}
\]
or better
\[
(q - q^{-1})h(q, q^{-1}) = \sum \sum b_{u,v} (q^{u(v+1)} - q^{u(v+1)}) \tag{2.11}
\]
We can rewrite the identity in 2.11 in the form
\[
(q - q^{-1})h(q, q^{-1}) = \sum \sum \chi(u - v + 1 = r) (q^{r} - q^{-r}) \sum b_{u,v} \tag{2.12}
\]
If we let $c_r = \sum_{u-v+1=r} b_{u,v}$ then 2.12 becomes
\[
(q - q^{-1})h(q, q^{-1}) = \sum \sum \chi(u - v + 1 = r) (q^{r} - q^{-r}) c_r \tag{2.13}
\]
\[
= \sum_{r=1}^{\binom{n}{2}} (q^{r} - q^{-r}) c_r \sum \sum \chi(r = 1 - u + v) = \sum_{r=1}^{\binom{n}{2}} (q^{r} - q^{-r}) c_r.
\]
Haiman has proved in [12] that $h(q,t)$ is exactly the Garsia-Haiman $q,t$-Catalan. In [6] the specialization $t \rightarrow q^{-1}$ is derived from Macdonald identities to be related to the none other than the $q$-analogue of the number of Dyck paths in the $n \times n$ lattice rectangle, more precisely we have:
\[
q^{\binom{n}{2}} h(q, q^{-1}) = \frac{1}{[n+1]_q} \binom{2n}{n}_q. \tag{2.14}
\]
Thus the left hand side of 2.13 is
\[
(q - q^{-1})q^{\binom{n}{2}} \frac{q - 1}{q^{n+1} - 1} \frac{(q^{n+1} - 1) \cdots (q^{2n} - 1)}{(q^{n} - 1) \cdots (q^{n} - 1)} = q^{-\binom{n}{2} - 1} \frac{(q^{n+2} - 1) \cdots (q^{2n} - 1)}{(q^{3} - 1) \cdots (q^{n} - 1)} \tag{2.15}
\]
and 2.13 becomes
\[
q^{-\binom{n}{2} - 1} \frac{(q^{n+2} - 1) \cdots (q^{2n} - 1)}{(q^{3} - 1) \cdots (q^{n} - 1)} = \sum_{r=1}^{\binom{n}{2}} c_r q^{r} - \sum_{r=1}^{\binom{n}{2}} c_r q^{-r} \tag{2.16}
\]
Since the number of starters is $\sum_{r=1}^{\binom{n}{2}} c_r$ all we need is to sum the positive coefficients of the powers of $q$. Of course this is really only an algorithm, but we can also obtain a formula from 2.16.

To do this we first apply an odd power of the Euler operator $q \partial_q$ to both side of 2.16:
\[
(q \partial_q)^{2k+1} \left( q^{-\binom{n}{2} - 1} \frac{(q^{n+2} - 1) \cdots (q^{2n} - 1)}{(q^{3} - 1) \cdots (q^{n} - 1)} \right) = \sum_{r=1}^{\binom{n}{2}} r^{2k+1} c_r (q^r + q^{-r}) \tag{2.17}
\]
The determinant of the matrix $\|z_{r,k}\|_{r,k=1}^{n(n-1)/2+1}$ factorized into a product of $(z_{r}^2 - z_{2}^2)$ and each is different from zero as long as $z_r \neq z_s$. this proves that the matrix $\|r^{2k+1}\|_{r,k=1}^{n(n-1)/2+1}$ has non zero determinant. Denoting by $\|d_{s,r}\|_{s,r=1}^{n(n-1)/2+1}$ the inverse, it follows that
\[
\sum_{r=1}^{\binom{n}{2}} d_{s,r}(q \partial_q)^{2k+1} \left( q^{-\binom{n}{2} - 1} \frac{(q^{n+2} - 1) \cdots (q^{2n} - 1)}{(q^{3} - 1) \cdots (q^{n} - 1)} \right) = \sum_{r=1}^{\binom{n}{2}} c_r (q^r + q^{-r}) \tag{2.18}
\]
Thus

$$\frac{1}{2} \sum_{r=1}^{\binom{n}{2}} d_{x,r}(q^r)^{2k+1} \left( q^{-\binom{n}{2}} - 1 + \frac{q^{n+2} - 1}{q^3 - 1} \cdots \frac{q^{2n} - 1}{(q^n - 1)} \right) \bigg|_{q=1} = \sum_{r=1}^{\binom{n}{2}} c_r$$

as desired.

Using the algorithm in 2.16, we obtain the following sequence giving the number of string starters for each $n \geq 2$.

$$> \text{seca(25);}$$

$$[1, 2, 3, 7, 17, 44, 120, 344, 1016, 3087, 9604, 30461, 98239, 321447, 1065129, 3568828, 12076553, 41228526, 141874879, 491740681, 1715554813, 6029062946, 21246914682, 75354926331]$$

$$> \text{Nstarters}(60)$$

$$6635868046474817500511616158676$$

The Encyclopedia of Integer Sequences was not aware of the existence of this sequence.

Conjecture in [1] and Theorem 2.1, give us an algorithm for constructing starters, this done a basis for $DHA_n$ is easily constructed. From the display in 2.20, we see that the number of starters for $n = 7$ is 44 and for $n = 8$ is already 120. With some work we may be able to construct the starters for $n = 7$, with even more work we may be able to do it for $n = 8$. However, a paper [1] published in 1998 by Ed Allen yields us a basis for every $n$. To see how we can use [1] we need only to depict an example.

On the right we have the lattice square $L_6$. The green cells are the lattice diagonal. We have depicted in yellow the cells of the english Ferrers diagram of the partition $[4, 2, 1]$. Starting from $(0, 0)$ we have 3 black North steps, followed by 3 red North steps, delimiting a Dyck path $D$ in $L_6$. The 7 yellow cells give the co-area of $D$. We will call the yellow partition the “co-partition” of $D$. In [6] we conjecture that the dimension of $DHA_n$ is given by the number Dyck paths in $L_n$. Accordingly in [1] it is conjectured that the following succession of steps constructs a basis of $DHA_n$. First step, compute the power function expansion of the Schur function indexed by the co-partition for each Dyck path in $L_n$. Second step, replace each $p_r$ by the operator $E_r$. Of course that means $p_r^k$ gets replaced by $E_r$ repeated $k$ times. This done, apply the resulting differential operator to the Vandermonde determinant in $x_1, x_2, \ldots, x_n$. This sequence of steps should yield a basis for $DHA_n$.

Since the power function expansion of a Schur function involves many terms the resulting alternant may not be bi-homogeneous. For instance for the co-partition $[4, 2, 1]$ the power basis expansion of the corresponding Schur function is the sum of the following terms

$$\left[ \frac{1}{12} p_1^7, \frac{1}{12} p_1^2 p_3, \frac{1}{12} p_1^3 p_2, \frac{1}{24} p_1^4 p_4, \frac{1}{24} p_1 p_3^2 p_2, \frac{1}{24} p_1^2 p_3 p_2^2, \frac{1}{24} p_3^3, \frac{1}{24} p_3^2 p_2^2, \frac{1}{24} p_3 p_2^3, \frac{1}{24} p_3^3 p_2, \frac{1}{24} p_3 p_2^3 \right]$$

The next result reinforces Conjecture in [1].

**Theorem 2.2**

*On the validity of the conjecture in [1] it is possible to construct a bi-homogeneous basis for $DHA_n$.***

**Proof**

It suffices to construct such a basis by an algorithm that works for all $n$. A fundamental fact is that $DHA_n$ is a bi-graded vector space. Moreover it is shown in [12] that $DHA_n$ has dimension equal to the number of Dyck paths in $L_n$. The conjecture in [1] starts with the power basis expansion of Schur functions indexed by co-partitions in $L_n$. In the next step we replace each $p_r$ in the resulting expansion by the operator $E_r$ repeated as many times as the exponent of $p_r$. This done we apply the resulting operator to the Vandermonde in $x_1, x_2, \ldots, x_n$. The resulting polynomial will be a bi-homogeneous element of $DHA_n$, for every term in the power function expansion.
To construct a basis we proceed with $0 \leq i \leq \binom{n}{2}$, starting by the diagonal of the string generated by the Vandermonde in $x_1, x_2, \ldots, x_n$ in bi-degree $\binom{n}{2}, 0$ and ending with the Vandermonde in $y_1, y_2, \ldots, y_n$ in bi-degree $0, \binom{n}{2}$. Then process all the bi-degrees in the shorter diagonals as $i$ becomes progressively smaller.

Say that in bi-degree $a,b$ the Frobenius characteristic of $\text{DHA}_n$ tells us that there are $m$ independent bi-homogeneous polynomials. To construct such a set we process from smallest to biggest the co-partition expansions breaking ties in the lex order of the partition sequence. We must eventually find $m$ independent polynomials of bi-degree $a,b$ if the basis conjectured in [1] is valid. This completes our proof.

The display above and on the right we exhibited the Frobenius characteristic of $\text{DHA}_7$. The integers immediately to the left of the display give the degrees in $x'_i$s of the polynomials exhibited in the columns. The integers at the bottom of the display give the degrees in $y'_i$s of the polynomials exhibited in the columns. The smaller integers under the diagonals are the number of independent polynomials that occupy that particular bi-degree. The circles indicate the polynomials indexed by the partitions of $n = 7$. For instance the circle in bi-degree 21,0 is the Vandermonde in the $x'_i$s and the circle in bi-degree 0,21 is the Vandermonde in the $y'_i$s. Since our guide is the Frobenius characteristic of $\text{DHA}_n$, we can see that each of the four polynomials in bi-degrees $(7, 4), (6, 5), (5, 6), (4, 7)$ will be picked up, in the construction of a basis, by our algorithm on the validity of the conjecture in [1] for $n = 7$. We mention this fact since N. Wallach (see [15]) that predicts, for any number $k$ of distinct variables, the polynomials that will occur in any basis. What we are witnessing here, for $k = 2$ and $n = 7$, the validity of a general result of this general result.

**Theorem 2.3**

In $\text{DHA}_n$ there is a basis of $\ker F$ of the form

$$E_2^{a_2} E_3^{a_3} \cdots E_{n-1}^{a_{n-1}} \Delta_1^n + E_1 L_Q[E_1, E_2, \ldots, E_{n-1}] \Delta_1^n$$

where $a_i \in \mathbb{Z}_{\geq 0}$ and $L_Q$ denotes the linear span with coefficients in $\mathbb{Q}$.

**Proof**

It is well known from $sl[2]$ theory that If $W$ is a subspace of $\mathbb{C}[x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n]$ that is invariant under $E$ and $F$ then

$$W = E_1 W \oplus \ker F|_W.$$  

Moreover any irreducible representation of dimension $k + 1$ starts with a polynomial $v_0$ such that

- $a)\ F \ v_0 = 0,$
- $b)\ H \ v_0 = -k \ v_0.$

Then a basis of the representation is

$$v_0, E v_0, E^2 v_0, \ldots, E^k v_0$$

Thus if $f \in \text{DHA}_n$ then

$$f = L_Q[E_1, E_2, \cdots, E_{n-1}]\Delta_1^n + E_1 L_Q[E_1, E_2, \cdots, E_{n-1}]\Delta_1^n.$$  

From the operator theorem it follows that there is a basis of the form

$$L_Q[E_2, E_3, \cdots, E_{n-1}]\Delta_1^n.$$  

If $F f = 0$ then we can assure that

$$f = L_Q[E_2, E_3, \cdots, E_{n-1}]\Delta_1^n + E_1 L_Q[E_1, E_2, \cdots, E_{n-1}]\Delta_1^n.$$  

This proves the Theorem.
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