A Critical Study of Baldelli and Bourdin’s *On the Asymptotic Derivation of Winkler-Type Energies From 3D Elasticity*

Kavinda Jayawardana*

**Abstract**

In our analysis, we show that Baldelli and Bourdin’s work [1] is only valid when describing the behaviour of a film bonded to an elastic pseudo-foundation, where Poisson’s ratios of both bodies are in between $-1$ and 0 or in between 0 and $\frac{1}{2}$ (where both Poisson’s ratios are sufficiently away from 0 and $\frac{1}{2}$), and with the asymptotic condition that is different to what the authors present. We also show that, for all Poisson’s ratios, the authors’ phase diagram is four-dimensional and not two-dimensional. Also, due to the Poisson’s ratio dependence, the asymptotic scalings that the authors present are insufficient to derive their proposed models. Furthermore, the authors’ scaling of the displacement field implies that their method cannot be applicable to films (or strings) with planar loading, unless the normal displacement is zero. Finally, by deriving a Winkler foundation type solution for a plate supported by an elastic pseudo-foundation via the method implied by the authors, we show that the authors’ method cannot be applied to plates due to the structure of the overlying body (i.e. limits of integration of the plate) and the foundation (i.e. planar-stress free condition of the foundation), unless planar displacement field is identically zero. Despite the limitations of the authors’ work, we highlight its strength by showing that, unlike the classical derivation of the Winkler foundation equation, Baldelli and Bourdin’s approach does not violates the volume conversation laws or the governing equations of mathematical elasticity.

**Keywords:** Contact Mechanics, Films, Mathematical Elasticity, Plate

---

*Corresponding author  
*Email address: zcahe58@ucl.ac.uk (Kavinda Jayawardana)
1. Introduction

Consider a situation where two elastic bodies that are bonded together, and for this case, one can easily model this problem with the simple use of the three-dimensional elastic equations. Now, consider the scenario where one of the elastic bodies is very thin in comparison to the other body, and planar in a Euclidean sense. Then the thin body can be approximated by a plate or a film, and such models are frequently used in the field of stretchable and flexible electronics. Applications of such models can be found in the field of conformal displays [2], thin film solar cells [3–6], electronic skins for robots and humans [7] and conformable electronic textiles [8]. For such applications, the degree of the deformation of the electronic body can endure, before its basic functions (i.e. conductivity, transparency or light emission) are adversely affected, is immensely important. However, design and process engineers who are working on the implementation of flexible electronics often lack confidence due to a lack of understanding or a lack of input data for reliable modelling tools [9]. Thus, there is a tremendous amount of research being conducted in the field of academia (Oxford Centre for Nonlinear Partial Differential Equations, Lu Research Group the University of Texas at Austin, Flexible Electronics and Display Center Arizona State University) as well as in the commercial sector (LG Electronics [10, 11], Samsung Group [12–14]).

Elastic foundation models are also used in the study of other mechanical concepts such as the buckling of stiff films bound to compliant substrates under compression [15–17] (which are considered to be important in designing of structural sandwich panels [18]) and in the study of crack patterns in thin films subjected to equi-biaxial residual tensile stress [19] (which are considered to be important in the study of spiral cracks in thin brittle adhesive layers bonding glass plates together, due to environmental interactions with the residual stress state resulting from processing [20]). For such applications, Baldelli and Bourdin [1] analyses the asymptotic behaviour of bonded thin elastic structures (i.e. films and plates) on elastic foundations. The work is presented as the first attempt at providing a rigorous derivation of these heuristic models from three-dimensional elasticity.
However, we show that Baldelli and Bourdin’s work [1] is only valid when describing the behaviour of films bonded to elastic pseudo-foundations, where Poisson’s ratios of the both bodies are in between $-1$ and $0$ or in between $0$ and $\frac{1}{2}$ (where both Poisson’s ratios are sufficiently away from $0$ and $\frac{1}{2}$), and with the asymptotic condition that is different to what the authors present. We also show that, for all Poisson’s ratios, the authors’ phase diagram is four-dimensional, but not two-dimensional as the authors present. Also, due to the Poisson’s ratio dependence, the asymptotic scalings that the authors present are insufficient to derive their proposed models. Furthermore, the authors’ scaling of the displacement field implies that their method cannot be applicable to films (or strings) with planar loading, unless the normal displacement is zero. Finally, by deriving a Winkler foundation type solution for a plate supported by an elastic pseudo-foundation via the method implied by the authors, we show that the authors’ method cannot be applied to plates due to the structure of the overlying body (i.e. limits of integration of the plate) and the foundation (i.e. planar-stress free condition of the foundation), unless planar displacement field is identically zero. Despite the limitations of the authors’ work, we conclude by showing that, unlike the classical derivation of the Winkler foundation equations, Baldelli and Bourdin’s approach does not violates the volume conversation laws or the governing equations of mathematical elasticity.

1.1. Baldelli and Bourdin’s Work

Baldelli and Bourdin [1] perform an asymptotic study to explore the different asymptotic regimes reached in the limit as the thickness of the overlying thin body goes to zero: for varying thickness of the foundation and stiffness ratios. They give a two-dimensional phase diagram to visualise the asymptotically reduced dimension models as a function of two relevant parameters. Two of the major presented results are the identification of the regime of films over in-plane elastic foundations and the identification of the regime of plates over out-of-plane elastic foundations.

Baldelli and Bourdin [1] begin by classifying the study of thin objects on elastic foundations as Winkler foundations and asserting that the derivation of the Winkler foundations equations must be done with rigorous asymptotic analysis, possibly as described by their publication. However, the authors’ assertion is false. Winkler foundation is a very specific mathematical problem, where an elastic body is supported unilaterally on a bed of continuously...
distributed springs with a foundation modulus $K_0$, and where the surface of the foundation is lubricated so that no tangential forces can develop (see section 5.5 of Kikuchi and Oden [21] and section 10.4.1 of Ding et al. [22]). Thus, Winkler foundation type problem is a boundary condition that exists regardless of the elastic properties of the elastic body or the bed of springs (see equation 5.111 of Kikuchi and Oden [21]). Often in the engineering community, Winkler foundation equations are used to describe the behaviour of beams and plates on elastic foundations with infinite depth [23], with complete disregard to understanding why the Winkler foundation equations are applicable to modelling such problems. Thus, the work of the authors may have intended to be used in justifying the use of Winkler foundation equation in modelling such problems.

![Figure 1: 'Film on the bonding layer']

The core idea behind Baldelli and Bourdin [1] is as follows. Consider a thin overlying elastic body (which the authors called the film or the membrane) with a constant thickness $\varepsilon h_f$, bonded to an elastic foundation (which the authors called the bonding layer) with a constant thickness $\varepsilon^{\alpha-1} h_b$, where the displacement of the bottom of the bonding layer is zero, i.e. displacement field of the bonding layer satisfies zero-Dirichlet boundary condition at its lowest boundary (see figure 1). The parameter $\varepsilon$ is considered to be a small constant and $\alpha$ is yet to be determined. In their analysis, the authors assume that there exists a common asymptotic behaviour between the elastic properties (the Young’s modulus and the Poisson’s ratio) of the overlying body and the bonding layer. To be more precise, the authors assume that
both first and second Lamé’s parameters of the overlying body are of the same order, i.e. \( \lambda_f \sim \mu_f \), both first and the second Lamé’s parameters of the bonding layer are of the same order, i.e. \( \lambda_b \sim \mu_b \) (see figure 1 or figure 1 of Baldelli and Bourdin [1]), both Poisson’s ratios of the overlying body and the bonding layer are of the same order, i.e. \( \nu_f \sim \nu_b \) (see hypothesis 2 of Baldelli and Bourdin [1]), and both Poisson’s ratios of the overlying body and the bonding layer are the same sign, i.e. \( \nu_b/\nu_f > 0 \) (see remark 2 of Baldelli and Bourdin [1]). Now, these conditions result in

\[-1 < \nu_f \approx \nu_b < 0 \quad \text{or} \quad 0 < \nu_f \approx \nu_b < \frac{1}{2},\]

given that both Poisson’s ratios are sufficiently away from 0 and \( \frac{1}{2} \). To be more precise, the conditions \( \lambda_f \sim \mu_f \) and \( \lambda_b \sim \mu_b \) imply that \( \lambda_f = c_f \mu_f \) and \( \lambda_b = c_b \mu_b \) for some \( c_f, c_b \sim 1 \) constants, and thus, \( \nu_f = \frac{1}{2}(1 + c_f)^{-1}c_f \) and \( \nu_b = \frac{1}{2}(1 + c_b)^{-1}c_b \). Furthermore, the conditions \( \nu_f \sim \nu_b \) and \( \nu_b/\nu_f > 0 \) imply that

\[-\frac{|c_f|}{2(1 - |c_f|)} = \nu_f \approx \nu_b = -\frac{|c_b|}{2(1 - |c_b|)} \quad \text{or} \quad \frac{|c_f|}{2(1 + |c_f|)} = \nu_f \approx \nu_b = \frac{|c_b|}{2(1 + |c_b|)}.

Thus, one gets the condition \(-1 < \nu_f \approx \nu_b < 0 \), sufficiently away from 0, or the antithetical condition \( 0 < \nu_f \approx \nu_b < \frac{1}{2} \), sufficiently away from 0 and \( \frac{1}{2} \). However, Poisson’s ratio of an object can vary strictly between \(-1 \) and \( \frac{1}{2} \) [24] and different materials have different Poisson’s ratios, and thus, Baldelli and Bourdin’s assertion [1] cannot hold in general. For example, assume that the bonding layer’s Poisson’s ratio is \( \frac{1}{4} \) and the overlying body’s Poisson’s ratio is infinitesimally small, i.e. \( \varepsilon \), and thus, one finds \( \nu_b/\nu_f \sim \varepsilon^{-1} \), which violates the authors’ assumption.

As a result of the restrictive nature of Poisson’s ratios of the authors’ analysis, they assert that all asymptotic behaviour of the overlying bonded body on an elastic bonding layer can be expressed on a two-dimensional phase diagram (see figure 2). However, this cannot hold in general as the phase diagram is four-dimensional due to the four asymptotic scalings

\[ \left\{ \varepsilon^{-2/2} h_b, \frac{\lambda_b}{\mu_f}, \frac{\mu_b}{\mu_f}, \frac{2\nu_f}{(1 - 2\nu_f)} \right\}, \]

5
Figure 2: The phase plane: 'The square-hatched region represents systems behaving as “rigid” bodies, under the assumed scaling hypotheses on the loads. Along the open half line (displayed with a thick solid and dashed stroke) (δ, 0), δ > 0 lay systems whose limit for vanishing thickness leads to a “membrane over in-plane elastic foundation” mode ... The solid segment 0 < γ < 1 (resp. dashed open line γ > 1) is related to systems in which bonding layer is thinner (resp. thicker) than the film, for γ = 1 (black square) their thickness is of the same order of magnitude. All systems within the horizontally hatched region γ > 0, 0 < δ ≤ 1, δ > γ behave, in the vanishing thickness limit, as “plates over out-of-plane elastic foundation”.' [1]. Note that γ = ½(1 + q − α), δ = ½(α + q − 3) and $E_b/E_f \sim \varepsilon^g$.

for all Poisson’s ratios. The only way one may collapse the dimensionality of the phase diagram is by assuming that one is only considering Poisson’s ratios with the very specific values $-1 < \nu_f \approx \nu_b < 0$ or $0 < \nu_f \approx \nu_b < \frac{1}{7}$, given that both Poisson’s ratios are sufficiently away from 0 and $\frac{1}{2}$. 


While describing the rigorous asymptotic analysis, the authors asymptotically rescale the displacement field as $u_\varepsilon = (\varepsilon u^1_\varepsilon, \varepsilon u^2_\varepsilon, u^3_\varepsilon)_E$ (see equation 9 of Baldelli and Bourdin [1]), which is implied by hypothesis 1 of the publication. If one defines the displacement field as described, then the only physical interpretation is that the planar displacement field $(\varepsilon u^1, \varepsilon u^2, 0)$ is infinitesimally small relative to the normal displacement field $(0, 0, u^3)$, and such scaling results in only plate like problems. Thus, the authors’ claim regarding leading order solution for a film bonded to an elastic foundation is possible is in direct conflict with their choice of the scalings for the displacement field.

As an example of their analysis, the authors put forward a model for an overlying film (defined as a membrane) with a very high Young’s modulus (i.e. stiff) bonded to an elastic foundation (see theorem 1 of Baldelli and Bourdin [1]). With rigorous mathematics the authors show that there exists a unique solution in $H^1(\omega)$, where $\omega \subset \mathbb{R}^2$ is the unstrained contact surface between the film and the bonding layer (see section 3.2 of Baldelli and Bourdin [1]). Beneath the authors’ analysis, the method in which the authors use to derive the governing equations is simple. Below, we describe in detail the method used by the authors to derive the energy functional of a film bonded to an elastic foundation. However, we omit the authors’ restrictive scalings of the displacement field (see equation 9 of Baldelli and Bourdin [1]) and Poisson’s ratios (see figure 1 of Baldelli and Bourdin [1]), and the insufficient asymptotic condition $E_f h_f \gg E_b h_b$, where $E_f$ and $E_b$ are the respective Young’s moduli of the film and the bonding layer (see definition of $\delta$ of Baldelli and Bourdin [1]). Note that Einstein’s summation notation is assumed throughout, we regard the indices $i, j, k, l \in \{1, 2, 3\}$ and $\alpha, \beta, \gamma, \delta \in \{1, 2\}$, and the coordinates $(x^1, x^2, x^3) = (x, y, z)$, unless it is strictly stated otherwise.

Consider an overlying film with a Poisson ratio $\nu_f$, a Young’s modulus $E_f$ and a thickness $h_f$, and a bonding layer (i.e. a foundation) with a Poisson ratio $\nu_b$, a Young’s modulus $E_b$ and a thickness $h_b$. Now, define the displacement field of the film by

$$u_f(u) = (u^1(x^1, x^2), u^2(x^1, x^2), 0)$$
and the displacement field of the foundation by

\[ u_b(u) = \left(1 + \frac{1}{h_b}x^3\right)(u^1(x^1, x^2), u^2(x^1, x^2), 0), \tag{1} \]

where \( x^\alpha \in \omega \) and \( x^3 \in (-h_b, 0) \). One can see that the displacement field of the foundation satisfies zero-Dirichlet boundary condition (i.e. \( u_b(u)|_{x^3 = -h_b} = 0 \)), and placement field of the film and the foundation are continuous at the contact region (i.e. \( u_b(u)|_{x^3 = 0} = u_f(u)|_{x^3 = 0} \)). Now, the energy functional of the system can be expressed as

\[
J(u) = \int_0^{h_f} \int_\omega \left[ \frac{1}{2} A_f^{\alpha\beta\gamma\sigma} \epsilon_{\alpha\beta}(u_f(u)) \epsilon_{\gamma\delta}(u_f(u)) - f^\alpha u_\alpha \right] d\omega dx^3 \\
+ \int_{-h_b}^{0} \int_\omega \frac{1}{2} A_b^{ijkl} \epsilon_{ij}(u_b(u)) \epsilon_{kl}(u_b(u)) d\omega dx^3,
\]

where \( \epsilon_{ij}(v) = \frac{1}{2}(\partial_i v_j + \partial_j v_i) \) is the linearised Green-St Venant strain tensor, \((f^1, f^2, 0)\) is an external planar force density field,

\[
A_f^{\alpha\beta\gamma\delta} = \frac{2\mu_f \lambda_f}{(\lambda_f + 2\mu_f)} \delta^{\alpha\beta} \delta^{\gamma\delta} + \mu_f (\delta^{\alpha\gamma} \delta^{\beta\delta} + \delta^{\alpha\delta} \delta^{\beta\gamma})
\]

is the elasticity tensor of the film (or the plate in subsequent analysis),

\[
A_b^{ijkl} = \lambda_b \delta^{ij} \delta^{kl} + \mu_b (\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk})
\]

is the elasticity tensor of the foundation, and were \( \delta^i_j \) is the Kronecker delta. Note that

\[
\lambda_f = \frac{\nu_f E_f}{(1 + \nu_f)(1 - 2\nu_f)} \quad \text{and} \quad \mu_f = \frac{E_f}{2(1 + \nu_f)}
\]

are the first and the second Lamé’s parameters of the film respectively, and

\[
\lambda_b = \frac{\nu_b E_b}{(1 + \nu_b)(1 - 2\nu_b)} \quad \text{and} \quad \mu_b = \frac{E_b}{2(1 + \nu_b)}
\]
are the first and the second Lamé’s parameters of the foundation respectively. Due to Poisson’s ratio dependence, one comes to the conclusion that
\[
\begin{align*}
\lambda_f h_f \sim \mu_b \frac{\text{meas}(\omega; \mathbb{R}^2)}{h_b},
\Lambda_f h_f \gg (\lambda_b + 2\mu_b) h_b
\end{align*}
\]
is the only possible asymptotic scaling that allows any valid leading-order governing equations (i.e. problems that allow traction), where
\[
\Lambda_f = 4\mu_f \frac{\lambda_f + \mu_f}{(\lambda_f + 2\mu_f)} E_f = \frac{E_f}{(1 + \nu_f)(1 - \nu_f)},
\]
and \(\text{meas}(\cdot; \mathbb{R}^n)\) is the standard Lebesgue measure in \(\mathbb{R}^n\). To be more precise, scaling (2) is the asymptotic scaling that allows
\[
\int_0^{h_f} \int_{\omega} \frac{1}{2} A_f^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta}(u_f(u)) \epsilon_{\gamma\delta}(u_f(u)) \, d\omega dx^3 \approx \int_{-h_b}^0 \int_{\omega} A_b^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta}(u_b(u)) \epsilon_{\gamma\delta}(u_b(u)) \, d\omega dx^3 \quad \text{and}
\int_0^{h_f} \int_{\omega} \frac{1}{2} A_f^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta}(u_f(u)) \epsilon_{\gamma\delta}(u_f(u)) \, d\omega dx^3 \gg \int_{-h_b}^0 \int_{\omega} \frac{1}{2} A_b^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta}(u_b(u)) \epsilon_{\gamma\delta}(u_b(u)) \, d\omega dx^3.
\]
To see why relation (3) implies the condition \(\Lambda_f h_f \gg (\lambda_b + 2\mu_b) h_b\), please consult the proof of theorems 3.9-1 and theorem 4.4-1 of Ciarlet [25].

Now, with a little more asymptotic analysis, one can express the leading-order terms of the energy functional of a film bonded to an elastic pseudo-foundation as
\[
J(u) = \frac{1}{2} h_f \int_{\omega} \left[ \frac{2\mu_f \lambda_f}{(\lambda_f + 2\mu_f)} \epsilon_{\alpha}(u) \epsilon_{\beta}(u) + 2\mu_f \epsilon_{\alpha}(u) \epsilon_{\beta}(u) \right] \, d\omega,
\]
If \(\omega \subset \mathbb{R}^2\) is a connected bounded plane with a Lipchitz-continuous boundary \(\partial \omega\), and \(f^\alpha \in L^1(\omega)\), then there exist a unique minimiser \((u^1, u^2) \in H^1(\omega)\)
to equation (4) (see section 1.5 of Ciarlet [26], in particular Korn’s inequality on a surface without boundary conditions, but with $u^3 = 0$). In particular, this unique minimiser is also a critical point in $H^1(\omega)$ (see section 1.5 of Badiale and Serra [27]). Note that $L^k(\cdot)$ are the standard $L^k$-Lebesgue spaces and $H^k(\cdot)$ are the standard $W^{k,2}(\cdot)$-Sobolev spaces (see section 5.2.1 of Evans [28]). Also note that we called the elastic foundation as an elastic pseudo-foundation as now the displacement field of the foundation (1) is grossly oversimplified.

Note that the asymptotic condition $\Lambda_f h_f \sim h_b^{-1} \mu_b \text{meas}(\omega; \mathbb{R}^2)$ implies that arbitrarily increasing the Young’s modulus of the film or arbitrary decreasing the thickness of the film, relative to the foundation (implied by the authors’ asymptotic scaling $E_f h_f \gg E_b h_b$), may not generate the most accurate solution. This is demonstrated numerically by Jayawardana [29] (see section 3.6 and 3.7 of Jayawardana [29]) in curvilinear coordinates (i.e. a shell bonded to an elastic foundation), as the author observe optimal values for the Young’s moduli, the Poisson’s ratios and thickness of each elastic body (and the curvature of the contact region), where the error of planar solution (with respect to the solution implied by standard linear elasticity) attains a minimum.

Baldelli and Bourdin [1] go further with their approach to derive a set of governing equations to describe the behaviour of a stiff plate bonded to an elastic foundation (see theorem 2 of Baldelli and Bourdin [1]). For this, the authors define the energy functional to this problem as

$$J(u) = \int_0^{h_f} \int_\omega \left[ \frac{1}{2} A_f^{\alpha\beta\gamma\sigma} \epsilon_{\alpha\beta}(u_f(u)) \epsilon_{\gamma\delta}(u_f(u)) - f^i u_i \right] d\omega dx^3 + \int_{-h_b}^0 \int_\omega \frac{4 \mu_b (\lambda_b + \mu_b)}{\lambda_b + 2 \mu_b} \epsilon^3_{\beta}(u_b(u)) \epsilon^3_{\beta}(u_b(u)) d\omega dx^3 ,$$

where in the theorem, respective displacement fields of the plate and the foundation are defined as

$$u_f(u) = (u^1(x^1, x^2), u^2(x^1, x^2), u^3(x^1, x^2))$$
$$u_b(u) = (u^1(x^1, x^2), u^2(x^1, x^2), u^3(x^1, x^2)) + (x^3 + h_b) (\partial^1 u_3(x^1, x^2), \partial^2 u_3(x^1, x^2), 0) ,$$
and in the proof of the theorem, respective displacement fields of the plate and the foundation are defined as

\[ u_f(u) = (u^1(x^1, x^2), u^2(x^1, x^2), u^3(x^1, x^2)) - \left( x^3 - \frac{1}{2} h_f \right) \left( \partial^1 u_3(x^1, x^2), \partial^2 u_3(x^1, x^2), 0 \right) \]

and

\[ u_b(u) = (u^1(x^1, x^2), u^2(x^1, x^2), (h_b + x^3) u^3(x^1, x^2)) \]

Unfortunately, the authors’ theorem 2 is erroneous (see equation 18-22 of Baldelli and Bourdin [1]) and their proof theorem 2 (see section 3.3 of Baldelli and Bourdin [1]) is inapplicable to the authors’ theorem. For example, one can clearly see that there are two conflicting definitions for the displacement fields, both definitions of the displacement fields are not continuous at the contact region, i.e. \( u_b(\mathbf{u})|_{x^3=0} \neq u_f(\mathbf{u})|_{x^3=0} \), and both definition of the displacement field of the foundation does not satisfy the zero-Dirichlet boundary condition, i.e. \( u_b(\mathbf{u})|_{x^3=-h_b} \neq 0 \). Also, the authors’ asymptotic scalings from their proof of theorem 2 does not result in equation (5). To be precise, consider a plate bonded to an elastic foundation with a constant thickness, such that the lower surface of the foundation satisfies the zero-Dirichlet boundary condition, where the displacement fields of the plate and the foundation can be express respectively as,

\[ u_f(u) = (u^1(x^1, x^2), u^2(x^1, x^2), u^3(x^1, x^2)) - x^3(\partial^1 u_3(x^1, x^2), \partial^2 u_3(x^1, x^2), 0) \]

and

\[ u_b(u) = \left( 1 + \frac{1}{h_b} x^3 \right) (u^1(x^1, x^2), u^2(x^1, x^2), u^3(x^1, x^2)) \]

Thus, the energy of this two elastic bodies can be expressed as

\[
J(\mathbf{u}) = \int_0^{h_f} \int_\omega \left[ \frac{1}{2} A_f^{\alpha\beta\gamma\sigma} \epsilon_{\alpha\beta}(u_f(\mathbf{u})) \epsilon_{\gamma\sigma}(u_f(\mathbf{u})) - f^i u_i \right] d\omega dx^3 + \int_{-h_b}^0 \int_\omega \left[ \frac{1}{2} A_b^{ijkl} \epsilon_{ij}(u_b(\mathbf{u})) \epsilon_{kl}(u_b(\mathbf{u})) \right] d\omega dx^3 .
\]

However, there exists no asymptotic scaling (what Baldelli and Bourdin present [1] or otherwise) such that the leading order terms of equation (6) would result in equation (5), unless the planar displacement field is an order of magnitude smaller than the normal displacement, i.e. \( u^3 \gg u^\alpha, \forall \alpha \in \{1, 2\} \).
To see this more clearly, integrate equation (6) explicitly in $x^3$ dimension to find

$$J(u) = J_{\text{plate}}(u) + J_{\text{foundation}}(u) ,$$

where

$$J_{\text{plate}}(u) = \int_\omega \left[ \frac{1}{2} A^{\alpha\beta\gamma\sigma}_f \left( h_f \varepsilon_{\alpha\beta}(u) \varepsilon_{\gamma\delta}(u) ight) 
- \frac{1}{2} h_f^2 (\partial_{\alpha\beta} u^3 \varepsilon_{\gamma\delta}(u) + \varepsilon_{\alpha\beta}(u) \partial_{\gamma\delta} u^3) 
+ \frac{1}{3} h_f^3 \partial_{\alpha\beta} u^3 \partial_{\gamma\delta} u^3 \right] d\omega \quad \text{(8)}$$

and

$$J_{\text{foundation}}(u) = \int_\omega \frac{1}{2} \left[ \lambda_b \left( \frac{1}{3} h_b \varepsilon_{\alpha}^\alpha(u) \varepsilon_{\beta}^\beta(u) + u^3 \varepsilon_{\alpha}^\alpha(u) + \frac{1}{h_b} u^3 u_3 \right) 
+ 2 \mu_b \left( \frac{1}{3} h_b \varepsilon_{\alpha\beta}(u) \varepsilon_{\gamma\delta}(u) 
+ \frac{1}{h_b} u^\alpha u_\alpha + u^\alpha \partial_\alpha u^3 + \frac{1}{3} h_b \partial_\alpha u^3 \partial_\alpha u_3 
+ \frac{1}{h_b} u^3 u_3 \right) \right] d\omega .$$

As the reader can see that there exist no asymptotic scaling that one can apply to equation (7) to get a leading order equation of the form (5), unless $u^\alpha = 0, \forall \alpha \in \{1, 2\}$. Note that in standard linear plate theory, if the mid-plane of the plate is located at $x^3 = 0$, then the limits of integration for a plate with thickness $h_f$ are $x^3 \in [-\frac{1}{2} h_f, \frac{1}{2} h_f]$, which results in only the terms $\varepsilon_{\alpha\beta}(u) \varepsilon_{\gamma\delta}(u)$ and $\partial_{\alpha\beta} u^3 \partial_{\gamma\delta} u^3$ in the plate’s energy functional. However, as a result of the authors’ limits of integration, i.e. $x^3 \in [0, h_f]$, we see the product $\varepsilon_{\alpha\beta}(u) \partial_\gamma u^3$ appearing in equation (8).

Now, we derive a solution for a plate supported by an elastic foundation, in accordance with the techniques implied by the proof of theorem 2, but with mathematical rigour. Just as before, we omit the authors’ restrictive scalings of the displacement field (see equation 9 of Baldelli and Bourdin [1]) and Poisson’s ratios (see figure 1 of Baldelli and Bourdin [1]), and the
insufficient asymptotic condition $E_f h_f \gg E_b h_b$ (see definition of $\delta$ of Baldelli and Bourdin [1]). Also, we only include leading order terms for the sake of readability.

Assume that we are considering Winkler foundation type problem, and let $T_j^i(v) = A_{ijkl}^b \epsilon_{kl}(v)$ be the second Piola-Kirchhoff stress tensor and let $v \in \omega \times [-h_b, 0)$ be the displacement field of the foundation. By definition, Winkler foundations cannot admit planar stress (see section 5.5 of Kikuchi and Oden [21]), and thus, implies $T_{j}^{i}(v) = 0$, $\forall \alpha, \beta \in \{1, 2\}$. Now, use these conditions to modify the elasticity tensor and the displacement field of the foundation, i.e. use the conditions $\epsilon_{\alpha}^{\beta}(v) = 0$, $\forall \alpha, \beta \in \{1, 2\}$ and $\epsilon_{\alpha}^{\alpha}(v) = - (\lambda_b + \mu_b)^{-1} \lambda_b \epsilon_{33}^{3}(v)$ to obtain

$$T_{3}^{j}(v(u_b)) = \frac{\lambda_b \mu_b}{(\lambda_b + \mu_b)} \epsilon_{33}^{3}(v(u_b)) \delta_{3}^{j} + 2 \mu_b \epsilon_{3}(v(u_b)),$$

and thus, one gets $T_{3}^{j3}(u_b) = A_{3k}^{jkl} \epsilon_{k3}(u_b)$, where

$$A_{3k}^{jkl} = \frac{\lambda_b \mu_b}{(\lambda_b + \mu_b)} \delta_{ji} \delta_{kl} + \mu_b (\delta_{jk} \delta_{il} + \delta_{jl} \delta_{ik})$$

is the new elasticity tensor of the foundation and $u_b$ is the new the displacement field of the foundation that satisfies the conditions $\epsilon_{3}^{\alpha}(u_b) = 0$, $\forall \alpha, \beta \in \{1, 2\}$. Now, seek a displacement field $u_b$ of the form such that $\epsilon_{3}^{\alpha}(u_b) = 0$, $\epsilon_{33}^{3}(u_b) \neq 0$ and $u_{3}^{b}(x^1, x^2, -h_b) = 0$, and thus, one finds that the displacement field of the foundation can be expressed as

$$u_b(u) = \left(1 + \frac{1}{h_b} x^3\right) (0, 0, u^3(x^1, x^2))$$

where $x^3 \in (0, -h_b)$. Now, assume that there is an overlying plate supported by the foundation. As the displacement field at the contact region must be continuous, one finds that the displacement field of the plate can be expressed as

$$u_f(u) = (-x^3 \partial^1 u_3(x^1, x^2), -x^3 \partial^2 u_3(x^1, x^2), u^3(x^1, x^2))$$

where $x^3 \in [0, h_f)$. Due to Poisson’s ratio dependence, one comes to the conclusion that

$$\{ \Lambda_f h_f^3 \sim \mathcal{K}_b(\text{meas}(\omega; \mathbb{R}^2))^2, \ \Lambda_f h_f^3 \gg h_b \mu_b \text{meas}(\omega; \mathbb{R}^2) \} \quad (9)$$
is the only possible asymptotic scaling that allows any valid governing equations (i.e. that allows Winkler foundation type problems), where

\[ K_b = \frac{\mu_b (3\lambda_b + 2\mu_b)}{h_b (\lambda_b + \mu_b)} \]

\[ = \frac{E_b}{h_b} \]

is the foundation modulus. To be more precise, scaling (9) is the asymptotic scaling that allows

\[ \int_0^{h_f} \int_\omega \frac{1}{2} A^\alpha_{\gamma \delta} \epsilon_{\alpha \beta} (u_f(u)) \epsilon_{\gamma \delta} (u_f(u)) \, d\omega \, dx^3 \approx \int_{-h_b}^0 \int_\omega \frac{1}{2} A^{3333}_{\gamma \beta} \epsilon_{33} (u_b(u)) \epsilon_{33} (u_b(u)) \, d\omega \, dx^3 \]

and

\[ \int_0^{h_f} \int_\omega \frac{1}{2} A^\alpha_{\gamma \delta} \epsilon_{\alpha \beta} (u_f(u)) \epsilon_{\gamma \delta} (u_f(u)) \, d\omega \, dx^3 \gg \int_{-h_b}^0 \int_\omega A^{\alpha 33} \epsilon_{\alpha \beta} (u_b(u)) \epsilon_{\beta 3} (u_b(u)) \, d\omega \, dx^3 . \]

To see why relation (10) implies the condition \( \Lambda_f h_f^3 \sim K_b (\text{meas}(\omega; \mathbb{R}^2))^2 \), please consult the proof of theorems 3.9-1 and theorem 4.4-1 of Ciarlet [25].

Now, with a little more asymptotic analysis, one can express the leading-order terms of the energy functional of a plate supported by an elastic pseudo-foundation as

\[ J(u) = \frac{1}{2} h_f \int_\omega \left[ \frac{1}{3} \Lambda_f h_f^2 \Delta u_3 \Delta u^3 + \left( \frac{K_b}{h_f} u_3 u^3 - 2 f^3 u_3 \right) \right] \, d\omega , \]

where \( \Delta \) is the scalar Laplacian in the Euclidean plane and \( f^3 \) is an external normal force density. As the reader can clearly see that equation (11) is different to equation (11), i.e. there exist clear discrepancies between what Baldelli and Bourdin’s theorem 2 [1] and the method implied by the proof of theorem 2. Note that we used the condition \( \partial_\beta u^3 |_{\partial \omega} = 0, \forall \beta \in \{1, 2\} \), to obtain equation (11), where \( n \) is the unit outward normal to the boundary of the plate, \( \partial \omega \).
Equation (11) is a Winkler foundation type problem for a plate that is supported by a continuous bed of springs with a foundation modulus of $K_b$. Furthermore, if $\omega \subset \mathbb{R}$ is an connected bounded plane with a Lipschitz-continuous boundary $\partial\omega$, $f^3 \in L^1(\omega)$, and $\partial_\beta u^3|_{\partial\omega} = 0$, $\forall \beta \in \{1, 2\}$, in a trace sense, then there exists a unique minimiser $u^3 \in H^2(\omega)$ to equation (11) (see section 1.5 of Ciarlet [26], in particular Korn’s inequality on a surface without boundary conditions, but with $(u^1, u^2) = 0$). In particular, this unique minimiser is also a critical point in $H^2(\omega)$ (see section 1.5 of Badiale and Serra [27]).

Should one try to derive the Winker foundation equation to this problem via the classical approach, Dillard et al.’ work [30] (see equation (3) of Dillard et al. [30]) implies an energy functional of the following form,

$$J_{\text{classical}}(u) = \frac{1}{2} h_f \int_\omega \left[ \frac{1}{12} \Lambda f h_f^2 \Delta u^3 \Delta u^3 + \left( \frac{K_b}{h_f} \right) u^3 u^3 - 2f^3 u^3 \right] d\omega , \quad (12)$$

where the discrepancy with respect to Baldelli and Bourdin’s approach [1] (i.e. the $\frac{1}{2}$ term in equation (11) and the $\frac{1}{12}$ term equation (12)) is due the limits of integration of the plate, i.e. $[-\frac{1}{2} h_f, \frac{1}{2} h_f]$ for Dillard et al. [30] (classical derivation) and $[0, h_f]$ for Baldelli and Bourdin [1]. Now, given that the plate resides in the set $\omega \times [-\frac{1}{2} h_f, \frac{1}{2} h_f]$, if the foundation resides in the set $\omega \times [-h_b, 0]$, then $[-\frac{1}{2} h_f, \frac{1}{2} h_f]$ is a region of overlap (i.e. it violates the volume conservation laws), and if the foundation resides in the set $\omega \times [-h_b, \frac{1}{2} h_f, -\frac{1}{2} h_f]$, then $[-\frac{1}{2} h_f, 0]$ is a region of discontinuity (i.e. it violates the governing equations of mathematical elasticity). However, in Baldelli and Bourdin’s approach [1], the plate resides in the set $\omega \times [0, h_f]$ and the boundary resides in the set $\omega \times [-h_b, 0]$, and thus, no such such regions of overlap or discontinuities. Thus, unlike the classical approach, Baldelli and Bourdin’s approach [1] is consistent with the volume conservation laws the governing equations of mathematical elasticity.

Note that, if one is assuming a plate bonded to an elastic foundation (i.e. not a Winker foundation, and thus, we assume an elasticity tensor of the form $A^{ijkl}_{b}$ in the foundation), and if one is considering normal displacement to be dominant, then one finds an energy functional of the form

$$J(u) = \frac{1}{2} h_f \int_\omega \left[ \frac{1}{3} \Lambda f h_f^2 \Delta u^3 \Delta u^3 + \left( \frac{\lambda_b + 2\mu_b}{h_f h_b} \right) u^3 u^3 - 2f^3 u^3 \right] d\omega ,$$
where the elastic and geometric properties must satisfy the following scaling
\[
\left\{ \Lambda_f h_f^3 \sim (\lambda_b + 2\mu_b)\frac{\text{meas}(\omega; \mathbb{R}^2)^2}{h_b}, \quad \Lambda_f h_f^3 \gg h_b\mu_b\text{meas}(\omega; \mathbb{R}^2) \right\},
\]
which is, again, different to what the authors present.

Figure 3: (a) ‘Cracked lettering at Ecole Polytechnique, Palaiseau, France. A vinyl sticker is bonded to an aluminium substrate and exposed to the sun which causes tensile stresses and subsequent cracking.’ (b) ‘Numerical experiment: nucleation at weak singularities, multiple cracking in the smooth domain, periodic fissuration of slender segments’ [31].

Despite above highlighted flaws, the strength of Baldelli and Bourdin’s [1] work appears lies in the study of overlying bonded films on elastic foundations, which is the subject of study in the PhD thesis of A. A. L. Baldelli [31], where the author use the bonded film model to examine the crack patterns that occurs in thin structures (see page 147 of Baldelli [31]). Baldelli [31] numerically shows that, without any ‘priori’ assumptions on the crack geometry, one can capture complex evolving crack patterns in different asymptotic regimes: parallel, sequential, periodic cracking and possible debonding in a uni-axial traction test as well as the appearance of polygonal crack patterns in a two-dimensional equi-biaxial load, and cracking in a geometrically complex domain. One of the perfect examples of the author’s work is a comparison against a real life crack pattern and the author’s numerical model, where the reader can see in figure 3 the author’s numerical result in figure 3 (b) is almost identical to the real crack pattern observed in figure 3 (a).

[1] http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.402.3946&rep=rep1&type=pdf
2. Conclusions

In conclusion, Baldelli and Bourdin’s [1] work is only valid when describing the behaviour of overlying bonded films on elastic pseudo-foundations (note that the authors’ foundation is not an actual elastic foundation as the displacement field of the foundation is grossly over simplified), where Poisson’s ratios of the both bodies are in between $-1$ and $0$ or in between $0$ and $\frac{1}{2}$ (where both Poisson’s ratios are sufficiently away from $0$ and $\frac{1}{2}$), and with the asymptotic condition $\{E_f h_f \sim h_b^{-1} E_b \text{meas}(\omega; \mathbb{R}^2), E_f h_f \gg E_b h_b\}$ (where the authors assume only the condition $E_f h_f \gg E_b h_b$ is significant). The authors assert that their asymptotic approach is valid for all elastic properties. However, we mathematically proved that the authors’ asymptotic approach is only valid if both Poisson’s ratios are in between $-1$ and $0$ or in between $0$ and $\frac{1}{2}$ (where both Poisson’s ratios are sufficiently away from $0$ and $\frac{1}{2}$). For all Poisson’s ratios, the authors’ phase diagram is four-dimensional, but not two-dimensional as the authors present. Also, due to the Poisson’s ratio dependence, the only scalings that can yield any valid asymptotic solutions are $\{\Lambda_f h_f \sim h_b^{-1} \mu_b \text{meas}(\omega; \mathbb{R}^2), \Lambda_f h_f \gg (\lambda_b + 2\mu_b) h_b\}$ for a film that is bonded to an elastic foundation, and $\{\Lambda_f h_f^3 \sim \mathcal{K}_b(\text{meas}(\omega; \mathbb{R}^2))^2, \Lambda_f h_f^3 \gg h_b \mu_b \text{meas}(\omega; \mathbb{R}^2)\}$ for a plate that is supported by an elastic foundation, but not $E_f h_f \gg E_b h_b$ as the authors present. The authors’ scaling of the displacement field implies that the method cannot be applicable to films (or strings) with planar loading, unless $u^3$ is zero. Finally, by deriving a Winkler foundation type solution for a plate supported by an elastic pseudo-foundation via the method implied by the authors’ proof of theorem 2, we showed that the authors’ method cannot be applied to plates due to the structure of the overlying body (i.e. limits of integration of the plate) and the foundation (i.e. planar-stress free condition of the foundation), unless field $(u^1, u^2)$ is identically zero.

We conclude by noting that the benefit of Baldelli and Bourdin’s work [1] is that, unlike the classical derivation of the Winkler foundation equations, the authors’ approach [1] is consistent with the volume conservation laws the governing equations of mathematical elasticity.

References

[1] A. A. L. Baldelli, B. Bourdin, On the asymptotic derivation of winkle-type energies from 3d elasticity, Journal of Elasticity (2015) 1–27.
[2] S. R. Forrest, The path to ubiquitous and low-cost organic electronic appliances on plastic, Nature 428 (2004) 911–918.

[3] M. C. Choi, Y. Kim, C. S. Ha, Polymers for flexible displays: From material selection to device applications, Progress in Polymer Science 33 (2008) 581–630.

[4] G. Crawford, Flexible flat panel displays, John Wiley & Sons, 2005.

[5] J. Lewis, Material challenge for flexible organic devices, Materials today 9 (2006) 38–45.

[6] M. Pagliaro, R. Ciriminna, G. Palmisano, Flexible solar cells, Chem-SusChem 1 (2008) 880–891.

[7] S. Wagner, S. P. Lacour, J. Jones, I. H. Pai-hui, J. C. Sturm, Z. Li, T. and Suo, Electronic skin: architecture and components, Physica E: Low-dimensional Systems and Nanostructures 25 (2004) 326–334.

[8] E. Bonderover, S. Wagner, A woven inverter circuit for e-textile applications, Electron Device Letters, IEEE 25 (2004) 295–297.

[9] S. Logothetidis, Handbook of Flexible Organic Electronics: Materials, Manufacturing and Applications, Woodhead Publishing Series in Electronic and Optical Materials, Elsevier Science, 2014.

[10] M. S. Yang, Liquid crystal display device and method for manufacturing the same, 1998. US Patent 5,793,460.

[11] M. B. Song, Electronic paper display device, manufacturing method and driving method thereof, 2010. US Patent 7,751,115.

[12] D. U. Jin, J. S. Lee, T. W. Kim, S. G. An, D. Straykhilev, Y. S. Pyo, H. S. Kim, D. B. Lee, Y. G. Mo, H. D. Kim, et al., 65.2: Distinguished paper: World-largest (6.5”) flexible full color top emission amoled display on plastic film and its bending properties, in: SID Symposium Digest of Technical Papers, volume 40, Wiley Online Library, pp. 983–985.

[13] S. H. Kim, Portable display device having an expandable screen, 2001. US Patent 6,262,785.
[14] Y. S. Kim, J. S. Park, Folder-type portable communication device having flexible display unit, 2012. US Patent 8,229,522.

[15] B. Audoly, A. Boudaoud, Buckling of a stiff film bound to a compliant substrate—part i:: Formulation, linear stability of cylindrical patterns, secondary bifurcations, Journal of the Mechanics and Physics of Solids 56 (2008) 2401–2421.

[16] B. Audoly, A. Boudaoud, Buckling of a stiff film bound to a compliant substrate—part ii:: A global scenario for the formation of herringbone pattern, Journal of the Mechanics and Physics of Solids 56 (2008) 2422–2443.

[17] B. Audoly, A. Boudaoud, Buckling of a stiff film bound to a compliant substrate—part iii:: Herringbone solutions at large buckling parameter, Journal of the Mechanics and Physics of Solids 56 (2008) 2444–2458.

[18] H. G. Allen, Analysis and Design of Structural Sandwich Panels: The Commonwealth and International Library: Structures and Solid Body Mechanics Division, Elsevier, 2013.

[19] Z. C. Xia, J. W. Hutchinson, Crack patterns in thin films, Journal of the Mechanics and Physics of Solids 48 (2000) 1107–1131.

[20] D. A. Dillard, J. A. Hinkley, W. S. Johnson, T. L. S. Clair, Spiral tunneling cracks induced by environmental stress cracking in larc™-tpi adhesives, The Journal of Adhesion 44 (1994) 51–67.

[21] N. Kikuchi, J. Oden, Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods, Studies in Applied Mathematics, Society for Industrial and Applied Mathematics, 1988.

[22] H. Ding, W. Chen, L. Zhang, Elasticity of transversely isotropic materials, volume 126, Springer Science & Business Media, 2006.

[23] M. Hetényi, Beams on elastic foundation: theory with applications in the fields of civil and mechanical engineering, University of Michigan, 1971.

[24] P. Howell, G. Kozyreff, J. Ockendon, Applied solid mechanics, 43, Cambridge University Press, 2009.
[25] P. G. Ciarlet, An introduction to differential geometry with applications to elasticity, Journal of Elasticity 78 (2005) 1–215.

[26] P. G. Ciarlet, Theory of Plates, Mathematical Elasticity, Elsevier Science, 1997.

[27] M. Badiale, E. Serra, Semilinear Elliptic Equations for Beginners: Existence Results via the Variational Approach, Springer Science & Business Media, 2010.

[28] L. C. Evans, Partial Differential Equations, Graduate studies in mathematics, American Mathematical Society, 2010.

[29] K. Jayawardana, Mathematical Theory of Shells on Elastic Foundations: An Analysis of Boundary Forms, Constraints, and Applications to Friction and Skin Abrasion, Ph.D. thesis, UCL (University College London), 2016.

[30] D. A. Dillard, B. Mukherjee, P. Karnal, R. C. Batra, J. Frechette, A review of winkler’s foundation and its profound influence on adhesion and soft matter applications, Soft matter 14 (2018) 3669–3683.

[31] A. A. L. Baldelli, On Fracture of Thin Films: a Variational Approach, Ph.D. thesis, Citeseer, 2013.