Geometrical Objects on the First Order Jet Space $J^1(T, \mathbb{R}^5)$ Produced by the Lorenz Atmospheric DEs System

Mircea Neagu
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Abstract
The aim of this paper is to construct natural geometrical objects on the 1-jet space $J^1(T, \mathbb{R}^5)$, where $T \subset \mathbb{R}$, like a non-linear connection, a generalized Cartan connection, together with its d-torsions and d-curvatures, a jet electromagnetic d-field and a jet Yang-Mills energy, starting from the given Lorenz atmospheric DEs system and the pair of Euclidian metrics $\Delta = (1, \delta_{ij})$ on $T \times \mathbb{R}^5$.

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1 Jet Riemann-Lagrange geometry produced by a first order non-linear DEs system

Many authors, like Asanov [1], Saunders [8] and many others, studied the contravariant differential geometry of 1-jet spaces. Going on with the geometrical studies of Asanov and using as a pattern the Lagrangian geometrical ideas developed by Miron and Anastasiei [4], the author of this paper has developed the Riemann-Lagrange geometry of 1-jet spaces [5], which is very suitable for the geometrical study of the solutions of a given DEs or PDEs system, via the least squares variational method proposed by Udrişte and Neagu in [7], [9].

In what follows we present the main jet Riemann-Lagrange geometrical results that, in author opinion, characterize a given non-linear DEs system of order one. In this direction, let $T = [a, b] \subset \mathbb{R}$ be a compact interval of the set of real numbers and let us consider the jet fibre bundle of order one

$$J^1(T, \mathbb{R}^n) \rightarrow T \times \mathbb{R}^n, \quad n \geq 2,$$
whose local coordinates \((t, x^i, x_1^i), i = 1, n\), transform by the rules

\[
\tilde{t} = \tilde{t}(t), \quad \tilde{x}^i = \tilde{x}^i(x^j), \quad \tilde{x}_1^i = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{dt}{dx^j} \cdot x_1^j.
\]

**Remark 1.1** From a physical point of view, the coordinate \(t\) has the physical meaning of relativistic time, the coordinates \((x^i)_{i=1}^n\) represent spatial coordinates and the coordinates \((x_1^i)_{i=1}^n\) have the physical meaning of relativistic velocities.

Let us consider that \(X = \left(X_{(1)}^{(i)}(x^k)\right)\) is an arbitrary d-tensor field on the 1-jet space \(J^1(T, \mathbb{R}^n)\), whose local components transform by the rules

\[
\tilde{X}_{(1)}^{(i)} = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{dt}{dx^j} \cdot X_{(1)}^{(j)}.
\]

It is obvious that the d-tensor field \(X\) produces the jet first order DEs system (jet dynamical system)

\[
x_1^i = X_{(1)}^{(i)}(x^j(t)), \quad \forall i = 1, n,
\]

where \(c(t) = (x^i(t))\) is an unknown curve on \(\mathbb{R}^n\) (i. e., a jet field line of the d-tensor field \(X\)) and we used the notation

\[
x_1^i \neq \dot{x}^i = \frac{dx^i}{dt}, \quad \forall i = 1, n.
\]

Automatically, the jet first order DEs system, together with the pair of Euclidian metrics \(\Delta = (1, \delta_{ij})\) on \(T \times \mathbb{R}^n\), produces the jet least squares Lagrangian function

\[
JLS_{\Delta}^{\text{DEs}} : J^1(T, \mathbb{R}^n) \to \mathbb{R}_+,
\]

expressed by

\[
JLS_{\Delta}^{\text{DEs}}(x^k, x_1^k) = \sum_{i,j=1}^{n} \delta_{ij} \left[x_1^i - X_{(1)}^{(i)}(x)\right] \left[x_1^j - X_{(1)}^{(j)}(x)\right] =
\]

\[
= \sum_{i=1}^{n} \left[x_1^i - X_{(1)}^{(i)}(x)\right]^2,
\]

where \(x = (x^k)_{k=1}^n\). Because the global minimum points of the jet least squares energy action

\[
E_{\Delta}^{\text{DEs}}(c(t)) = \int_{a}^{b} JLS_{\Delta}^{\text{DEs}}(x^k(t), \dot{x}^k(t)) dt
\]

are exactly the solutions of class \(C^2\) of the jet first order DEs system, it follows that we may regard the jet least squares Lagrangian function \(JLS_{\Delta}^{\text{DEs}}\) as a natural geometrical substitut for the DEs system of order one, on the 1-jet space \(J^1(T, \mathbb{R}^n)\).
Remark 1.2 It is important to note that any solution of class $C^2$ of the jet first order DEs system (1.1) verify the second order Euler-Lagrange equations produced by the jet least squares Lagrangian function $JLS_{\Delta}^{DEs}$ (jet geometric dynamics)

$$\frac{\partial [JLS_{\Delta}^{DEs}]}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial [JLS_{\Delta}^{DEs}]}{\partial \dot{x}^i} \right) = 0, \forall i = 1, n. \quad (1.3)$$

Conversely, this statement is not true because there exist solutions for the Euler-Lagrange DEs system (1.3) which are not global minimum points for the jet least squares energy action $H_{\Delta}^{DEs}$, that is which are not solutions for the jet first order DEs system (1.1).

But, a Riemann-Lagrange geometry on $J^1(T, \mathbb{R}^n)$ produced by the jet least squares Lagrangian function $JLS_{\Delta}^{DEs}$, via its Euler-Lagrange equations (1.3), geometry in the sense of non-linear connection, generalized Cartan connection, d-torsions, d-curvatures, jet electromagnetic field and jet Yang-Mills energy, is now completely done in the papers [5], [6] and [7]. For that reason, we introduce the following concept:

Definition 1.3 Any geometrical object on $J^1(T, \mathbb{R}^n)$, which is produced by the jet least squares Lagrangian function $JLS_{\Delta}^{DEs}$, via its second order Euler-Lagrange equations (1.3), is called geometrical object produced by the jet first order DEs system (1.1).

In a such context, we give the following geometrical result, which is proved in [6] and, for the multi-time general case, in [7]. For all details, the reader is invited to consult the book [5].

Theorem 1.4 (i) The canonical non-linear connection on $J^1(T, \mathbb{R}^n)$ produced by the jet first order DEs system (1.1) has the local components

$$\Gamma^{DEs} = \begin{pmatrix} M^{(i)}_{(1)j} \\ N^{(i)}_{(1)j} \end{pmatrix},$$

where

$$M^{(i)}_{(1)j} = 0 \text{ and } N^{(i)}_{(1)j} = -\frac{1}{2} \left[ \frac{\partial X^{(i)}_{(1)}}{\partial x^j} - \frac{\partial X^{(j)}_{(1)}}{\partial x^i} \right], \forall i, j = 1, n.$$ (ii) All adapted components of the canonical generalized Cartan connection $CT^{DEs}$ produced by the jet first order DEs system (1.1) vanish.

(iii) The effective adapted components $R^{(i)}_{(1)jk}$ of the torsion d-tensor $T^{DEs}$ of the canonical generalized Cartan connection $CT^{DEs}$ produced by the jet first order DEs system (1.1) are

$$R^{(i)}_{(1)jk} = -\frac{1}{2} \left[ \frac{\partial^2 X^{(i)}_{(1)}}{\partial x^k \partial x^j} - \frac{\partial^2 X^{(j)}_{(1)}}{\partial x^k \partial x^i} \right], \forall i, j, k = 1, n.$$
(iv) All adapted components of the curvature d-tensor $R_{DEs}^{ij}$ of the canonical generalized Cartan connection $C_{DEs}$ produced by the jet first order DEs system \((1.1)\) vanish.

(v) The geometric electromagnetic distinguished 2-form produced by the jet first order DEs system \((1.1)\) has the expression

$$F_{DEs} = F_{(1)ij} \delta x^i_1 \wedge dx^j,$$

where

$$\delta x^i_1 = dx^i_1 + N^{(i)}_{(1)k} dx^k, \quad \forall i = \overline{1, n},$$

and

$$F_{(1)ij} = \frac{1}{2} \left[ \frac{\partial X^{(i)}_{(1)}}{\partial x^j} - \frac{\partial X^{(j)}_{(1)}}{\partial x^i} \right], \quad \forall i, j = \overline{1, n}.$$  

(vi) The adapted components $F_{(1)ij}$ of the electromagnetic d-form $F_{DEs}$ produced by the jet first order DEs system \((1.1)\) verify the generalized Maxwell equations

$$\sum_{\{i,j,k\}} F_{(1)ij\mid k} = 0,$$

where $\sum_{\{i,j,k\}}$ represents a cyclic sum and

$$F_{(1)ij\mid k} = \frac{\partial F_{(1)ij}}{\partial x^k}$$

has the geometrical meaning of the horizontal local covariant derivative produced by the Berwald linear connection $B\Gamma_0$ on $J^1(T, \mathbb{R}^n)$. For more details, please consult [5].

(vii) The geometric jet Yang-Mills energy produced by the jet first order DEs system \((1.1)\) is defined by the formula

$$E_{YM}^{DEs}(x) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left[ F_{(1)ij} \right]^2.$$  

Remark 1.5 If we use the following matriceal notations

- $J(X_{(1)}) = \left( \frac{\partial X^{(i)}_{(1)}}{\partial x^j} \right)_{i,j=\overline{1, n}}$ - the Jacobian matrix,
- $N_{(1)} = \left( N^{(i)}_{(1)j} \right)_{i,j=\overline{1, n}}$ - the non-linear connection matrix,
- $R_{(1)jk} = \left( R^{(i)}_{(1)jk} \right)_{i,j,k=\overline{1, n}}, \quad \forall k = \overline{1, n},$ - the torsion matrices,
- $F^{(1)} = \left( F_{(1)ij} \right)_{i,j=\overline{1, n}}$ - the electromagnetic matrix.
then the following matricial geometrical relations attached to the jet first order DEs system (1.1) hold good:

1. \( N(1) = -\frac{1}{2} [J(X(1)) - T \cdot J(X(1))] \),

2. \( R_{(1)k} = \frac{\partial}{\partial x^k} [N_{(1)}] , \forall k = 1, n, \)

3. \( F^{(1)} = -N_{(1)}, \)

4. \( EYM^{DEs}(x) = \frac{1}{2} \cdot \text{Trace} [F^{(1)} \cdot T \cdot F^{(1)}] \), that is the jet Yang-Mills energy coincides with the norm of the skew-symmetric electromagnetic matrix \( F^{(1)} \) in the Lie algebra \( o(n) = L(O(n)). \)

In the sequel, we apply the above jet contravariant Riemann-Lagrange geometrical results to the Lorenz five-components atmospheric DEs system introduced by Lorenz [3] and studied, via the Melnikov function method for Hamiltonian systems on Lie groups, by Birtea, Puta, Rațiu and Tudoran [2].

### 2 Jet Riemann-Lagrange geometry produced by the Lorenz simplified model of Rossby gravity wave interaction

The first model equations for the atmosphere are that so called primitive equations. It seems that this model produces wave-like motions on different time scales:

- on the one hand, this model produces the slow motions which have a period of order of days (these slow-waves are called Rossby waves);
- on the other hand, this model produces fast motions which have a period of hours (these fast-waves are called gravity waves).

The question of how to balance these two time scales leads Lorenz [3] to consider a simplified version of the primitive equations model, which is given by the following non-linear system of five differential equations [2]:

\[
\begin{align*}
\frac{dx_1}{dt} &= -x^2 x^3 + \varepsilon x^2 x^5 \\
\frac{dx_2}{dt} &= x^1 x^3 - \varepsilon x^1 x^5 \\
\frac{dx_3}{dt} &= -x^1 x^2 \\
\frac{dx_4}{dt} &= -x^5 \\
\frac{dx_5}{dt} &= x^4 + \varepsilon x^1 x^2,
\end{align*}
\]

(2.1)
where the variables $x^4$, $x^5$ represent the fast gravity wave oscillations and the variables $x^1$, $x^2$, $x^3$ are the slow Rossby wave oscillations, with a parameter $\varepsilon$ which is related to the physical Rossby number.

**Remark 2.1** It is obvious that, from a physical point of view, the Lorenz atmospheric DEs system (2.1) couples the Rossby waves with the gravity waves.

Naturally, the Lorenz atmospheric DEs system (2.1) can be regarded as a non-linear DEs system of order one on the 1-jet space $J^1(T, \mathbb{R}^5)$, which is produced by the d-tensor field $X = \left( X^{(i)}(x) \right)$, where $i = 1, 5$ and

$$ x = (x^1, x^2, x^3, x^4, x^5), $$

having the local components

\begin{align*}
X^{(1)}(x) &= -x^2x^3 + \varepsilon x^2x^5, \\
X^{(2)}(x) &= x^1x^3 - \varepsilon x^1x^5, \\
X^{(3)}(x) &= -x^1x^2, \quad \text{(2.2)} \\
X^{(4)}(x) &= -x^5, \\
X^{(5)}(x) &= x^4 + \varepsilon x^1x^2.
\end{align*}

Consequently, via the Theorem 1.4 we assert that the Riemann-Lagrange geometrical behavior on the 1-jet space $J^1(T, \mathbb{R}^5)$ of the Lorenz atmospheric DEs system (2.1) is described in the following

**Corollary 2.2** (i) The canonical non-linear connection on $J^1(T, \mathbb{R}^5)$ produced by the Lorenz atmospheric DEs system (2.1) has the local components

$$ \tilde{\Gamma} = \left( 0, \tilde{N}^{(i)}_{(1)j} \right), $$

where $\tilde{N}^{(i)}_{(1)j}$ are the entries of the matrix

$$ \tilde{N}_{(1)} = \left( \tilde{N}^{(i)}_{(1)j} \right)_{i,j=1,5} = \begin{pmatrix} 0 & x^3 - \varepsilon x^5 & 0 & 0 & 0 \\
-x^3 + \varepsilon x^5 & 0 & -x^1 & 0 & \varepsilon x^1 \\
0 & x^1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & -\varepsilon x^1 & 0 & -1 & 0 \end{pmatrix}. $$

(ii) All adapted components of the canonical generalized Cartan connection $C\tilde{\Gamma}$ produced by the Lorenz atmospheric DEs system (2.1) vanish.
(iii) All adapted components of the torsion $d$-tensor $\hat{T}$ of the canonical generalized Cartan connection $C\hat{\Gamma}$ produced by the Lorenz atmospheric DEs system (2.1) are zero, except

$$\hat{R}^{(3)}_{(1)21} = -\hat{R}^{(2)}_{(1)31} = 1, \quad \hat{R}^{(5)}_{(1)21} = -\hat{R}^{(2)}_{(1)51} = -\varepsilon,$$

$$\hat{R}^{(2)}_{(1)13} = -\hat{R}^{(1)}_{(1)23} = -1, \quad \hat{R}^{(2)}_{(1)15} = -\hat{R}^{(1)}_{(1)25} = \varepsilon.$$

(iv) All adapted components of the curvature $d$-tensor $\hat{R}$ of the canonical generalized Cartan connection $C\hat{\Gamma}$ produced by the Lorenz atmospheric DEs system (2.1) vanish.

(v) The geometric electromagnetic distinguished 2-form produced by the Lorenz atmospheric DEs system (2.1) has the expression

$$\hat{F} = \hat{F}^{(1)}_{(i)j}\delta x^i \wedge dx^j,$$

where

$$\delta x^i = dx^i + \hat{N}^{(i)}_{(1)k}dx^k, \quad \forall \ i = 1, 5,$$

and the adapted components $\hat{F}^{(1)}_{(i)j}$ are the entries of the matrix

$$\hat{F}^{(1)} = \left(\hat{F}^{(1)}_{(i)j}\right)_{i,j = 1,5} = -\hat{N}^{(1)}.$$

(vi) The jet geometric Yang-Mills energy produced by the Lorenz atmospheric DEs system (2.1) is given by the formula

$$E_{YM}^{Lorenz}(x) = (\varepsilon x^5 - x^3)^2 + (x^1)^2 + (\varepsilon x^1)^2 + 1.$$

Proof. The Lorenz atmospheric DEs system (2.1) is a particular case of the jet first order DEs system (1.1) for $n = 5$ and $X = \left(X^{(i)}_{(1)}(x)\right)_{i = 1,5}$ given by the relations (2.2). In conclusion, applying the Theorem 1.4 together with the Jacobian matrix

$$J \left(X_{(1)}\right) =
\begin{pmatrix}
0 & -x^3 + \varepsilon x^5 & -x^2 & 0 & \varepsilon x^2 \\
x^3 - \varepsilon x^5 & 0 & x^1 & 0 & -\varepsilon x^1 \\
-x^2 & -x^1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
\varepsilon x^2 & \varepsilon x^1 & 0 & 1 & 0
\end{pmatrix},$$

we obtain what we were looking for. \qed

Remark 2.3 Let us remark that, although the jet Yang-Mills electromagnetic energy $E_{YM}^{Lorenz}$ produced by the Lorenz atmospheric DEs system (2.1) depends only by the coordinates $x^1, x^3$ and $x^5$, it still couples the slow Rossby wave oscillations with the fast gravity wave oscillations. However, the coordinates $x^2$ and $x^4$ are missing in the expression of $E_{YM}^{Lorenz}$. There exists a physical interpretation of this fact?
3 Yang-Mills energetical hypersurfaces of constant level produced by the Lorenz atmospheric DEs system

In the preceding Riemann-Lagrange geometrical theory on the 1-jet space $J^1(T, \mathbb{R}^5)$ the Lorenz atmospheric DEs system (2.1) "produces" a jet Yang-Mills energy given by the formula

$$E^{YM}_{\text{Lorenz}}(x) = (1 + \varepsilon^2)(x^1)^2 + (x^3)^2 + \varepsilon^2(x^5)^2 - 2\varepsilon x^3 x^5 + 1,$$

where $x = (x^1, x^2, x^3, x^4, x^5)$. In what follows, let us study the jet Yang-Mills energetical hypersurfaces of constant level produced by the Lorenz atmospheric DEs system (2.1), which are defined by the implicit equations

$$\Sigma^{\text{Lorenz}}_C : (\varepsilon x^5 - x^3)^2 + (1 + \varepsilon^2)(x^1)^2 = C - 1,$$

where $C$ is a constant real number.

Because $\Sigma^{\text{Lorenz}}_C$ is a quadric in the system of axes $Ox^1x^3x^5$ for every $C \in \mathbb{R}$, then, using the reduction to the canonical form of a quadric, we find the following geometrical results:

1. If $C < 1$, then we have $\Sigma^{\text{Lorenz}}_{C<1} = \emptyset$;
2. If $C = 1$, then we have

$$\Sigma^{\text{Lorenz}}_{C=1} : \begin{cases} x^1 = 0 \\ x^3 - \varepsilon x^5 = 0, \end{cases}$$

that is $\Sigma^{\text{Lorenz}}_{C=1}$ is a straight line in the system of axes $Ox^1x^3x^5$;
3. If $C > 1$, then we have

$$\Sigma^{\text{Lorenz}}_{C>1} : (x^3)^2 + (x^5)^2 = \frac{C - 1}{1 + \varepsilon^2},$$

that is $\Sigma^{\text{Lorenz}}_{C>1}$ is a degenerate non-empty quadric in the system of axes $Ox^1x^3x^5$, whose canonical form is

$$\Sigma^{\text{Lorenz}}_{C>1} : (X^3)^2 + (X^5)^2 = \frac{C - 1}{1 + \varepsilon^2},$$

where the rotation of the system of axes $Ox^1x^3x^5$ into the system of axes $OX^1X^3X^5$ is given by the matricial relation

$$\begin{pmatrix} x^1 \\ x^3 \\ x^5 \end{pmatrix} = \frac{1}{\sqrt{1 + \varepsilon^2}} \begin{pmatrix} 0 & \sqrt{1 + \varepsilon^2} & 0 \\ \varepsilon & 0 & 1 \\ 1 & 0 & -\varepsilon \end{pmatrix} \begin{pmatrix} X^1 \\ X^3 \\ X^5 \end{pmatrix}.$$
In conclusion, the degenerate non-empty quadric $\Sigma^{\text{Lorenz}}_{C > 1}$ is in the system of axes $Ox^1x^3x^5$ a slant circular cylinder of radius

$$R = \sqrt{\frac{C - 1}{1 + \varepsilon^2}}$$

having as axis of symmetry the straight line $\Sigma^{\text{Lorenz}}_{C = 1}$.

**Open problem.** There exist real physical interpretations, in the study of the Lorenz atmospheric DEs system (2.1), for the preceding geometrical results?

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**Author’s address:** Mircea NEAGU
University Transilvania of Brasov, Faculty of Mathematics and Informatics
Department of Algebra, Geometry and Differential Equations
B-dul Iuliu Maniu, No. 50, BV 500091, Brasov, Romania

**E-mails:** mircea.neagu@unitbv.ro, mirceaneagu73@yahoo.com

**Website:** [http://www.2collab.com/user:mirceaneagu](http://www.2collab.com/user:mirceaneagu)