Guaranteeing Input Tracking For Constrained Systems: Theory and Application to Demand Response

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Abstract

A method for certifying exact input trackability for constrained discrete time linear systems is introduced in this paper. A signal is assumed to be drawn from a reference set and the system must track this signal with a linear combination of its inputs. Using methods from robust model predictive control, the proposed approach certifies the ability of a system to track any reference drawn from a polytopic set on a finite time horizon by solving a linear program. Optimization over a parameterization of the set of reference signals is discussed, and particular instances of parameterization of this set that result in a convex program are identified, allowing one to find the largest set of trackable signals of some class. Infinite horizon feasibility of the methods proposed is obtained through use of invariant sets, and an implicit description of such an invariant set is proposed. These results are tailored for the application of power consumption tracking for loads, where the operator of the load needs to certify in advance his ability to fulfill some requirement set by the network operator. An example of a building heating system illustrates the results.

1 Introduction

This work proposes a methodology to handle input tracking for constrained discrete-time linear systems. More precisely, restriction of the inputs to a particular subspace of the input space is considered. For example, restricting the power consumption of a multi-input multi-output (MIMO) system to track a particular signal over time, the power consumption being a function of the inputs of this system, falls into our characterization of input tracking. In a nutshell, the method guarantees that, on a finite horizon, any signal drawn from a polyhedral set can be tracked exactly as a function of the inputs of our system, assuming only knowledge of this reference set. It ensures that whatever will be drawn from the tracking set, the system can follow while still satisfying input and state constraints, as well as remain indefinitely feasible after that.

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The method also computes a controller that allows the system to track the reference causally when it is revealed only one step at a time.

This methodology can be viewed as a form of constrained model reduction, where the behavior of a higher-dimensional model is captured by a simpler one, in the sense that the set of trackable references can be described as the feasible input sequences of the smaller system. The representation is however valid only for a finite time length. For example, in the application section, the set of all power profiles that a building can follow is approximated as the set of feasible inputs to a battery, whose parameters can be optimized for best fit. Another way to look at the method is as a disturbance rejection scheme, where the disturbance restricts the inputs to lie into some particular subspace. The control method proposed uses results from the robust MPC literature [1].

To the authors’ knowledge, provision of tracking certifications for constrained systems is a problem that has not been largely addressed. Contributions from the output tracking literature [2, 3] study asymptotic tracking of the output of a reference generator for continuous-time systems in an unconstrained setting. Our problem differs from this classical setting in several regards: first, we consider discrete-time systems and exact tracking of the reference on a finite time horizon. Secondly, the references are not the output of a generator system, but are assumed to be drawn from a convex set. Finally, we consider tracking as a function of the inputs only for constrained systems. A recent work in which guarantees for inexact tracking of a constrained system have been considered is [4], where the problem is tackled by means of robust invariance. To provide infinite horizon guarantees, so-called max-min invariant sets need to be computed. The conservatism in the computation of these sets leads to only inexact tracking guarantees and limits the scalability of the approach. We avoid these issues by ensuring tracking on a finite-time horizon, while still maintaining infinite horizon feasibility.

Related ideas have emerged from more applied works interested in characterizing the flexibility in terms of power consumption of energy systems. Indeed, the envisioned higher need for regulating power on the grid has pushed authors to consider load-side participation through programs such as Demand Response [5], which incentivize loads to modify their power consumption through price or request mechanisms, or to provide ancillary services [6]. [7] considers power envelopes to characterize the power consumption flexibility of SISO systems, while the authors of [?] consider the aggregated representation of Thermostatically Controlled Load (TCL) populations by simple battery models. These approaches are partly subsumed in this work. For example, the proposed method generalizes some concepts of [7] by considering MIMO systems and a more general tracking characterization. Another related work is [8], where the flexibility of multiple buildings is aggregated. Our approach can be combined with the hierarchical scheme proposed there to leverage more flexibility from the buildings. Finally, [9] studies the Demand Response potential of different types of loads considering their characteristics regarding more practical aspects of the implementation and therefore complements this work.

This paper tackles the theoretical questions that arise from the constrained input tracking problem, and make the connection to the building application with an example. The contribution of the paper is three-fold: First, it gives a method for a priori certification of input trackability of a set of references. Secondly, it shows how the tracking set can be optimized in a tractable fashion, for example to derive the largest possible set of references that can be tracked out of a particular class. Lastly, infinite horizon feasibility is discussed and an implicit characterization of an invariant set is proposed.
The paper is organized as follows: Section 2 introduces the general formulation of the input trackability problem for constrained discrete time linear systems. Section 3.1 introduces restrictions on the general problem to propose a method to solve it with convex programming. Section 3.2 introduces an approach to optimize over parameterizations of the reference set and presents a parameterization of the reference set that renders the problem convex. Section 3.4 presents an implicit parametrization of the invariant set. Finally Section 4 illustrates the method on the case of a building HVAC system offering flexibility to a power grid operator.

2 Formulation of the problem

We consider constrained discrete-time linear system of the form:

\[ x_{i+1} = Ax_i + Bu_i \]

where \( x \) and \( u \) are the state and input of the system, which are constrained to lie in the set:

\[ (x, u) \in X \times U = Z \]

The reference signal that the system is required to track at step \( i \) is represented by \( r_i \in \mathbb{R} \). The reference \( r \) is required to be tracked by a linear function of the control input \( u \) so that

\[ g^T u_i = r_i \]  

Remark 1. For simplicity, we consider tracking of a one-dimensional signal. Multidimensional signal tracking is a straightforward extension.

Remark 2. Results also apply to the case where the system is subjected to known exogenous inputs and described by:

\[ x_{i+1} = Ax_i + Bu_i + B_d d_i \]

provided the disturbance \( d \) is assumed to be known in advance. The external input \( d \) can be included in the regular input vector \( u \), with proper modification to system matrices, and constrained in the sequel to be equal to their known or forecasted values. Therefore, the system can be described by the regular constrained system equations which we use for all derivations.

In the following, bold letters denote trajectories over a horizon, e.g. \( u_i = (u^T_0, u^T_1, \ldots, u^T_{i-1})^T \).

For the remainder of the paper, we will consider a fixed horizon \( N \), and so the subscript \( N \) will be dropped when it is clear from the context. The map \( \pi_i(x_i, r_i) : X^{i+1} \times \mathbb{R} \to \mathbb{R}^n \) represents a causal feedback policy to be used at step \( i \) after observing the state sequence \( x_i \), and the reference \( r_i \). \( \phi_i(x_0, r_i, \pi_i) \) denotes the state of the system at step \( i \), if it starts from state \( x_0 \), applies the control policy sequence \( \pi_i \), and receives the reference sequence \( r_i \).

Remark 3. It is assumed that the reference signal is observed at the time it needs to be tracked. However, the above definition and following results can be easily adapted to situations where the system has access to future reference values.
Definition 1. The set $P \subset \mathbb{R}^N$ is input trackable by system (1) in state $x_0$ if there exists a sequence of feedback policies $\pi_\infty$ such that the following conditions hold:

\[
\begin{align*}
\forall r_{N-1} & \in P, \forall i \geq 0, \quad \phi_i(x_0, r_i, \pi_i) \in X, \\
\forall r_{N-1} & \in P, \forall i \geq 0, \quad \pi_i(x_i, r_i) \in U, \\
\forall r_{N-1} & \in P, \forall i \in \{0, \ldots, N-1\}, \quad g^T \pi_i(x_i, r_i) = r_i
\end{align*}
\]

The above definition asserts the ability of the system to track all possible reference sequences that can be drawn from the set $P$ using causal feedback policies. Furthermore, it requires that the system is able to stay feasible indefinitely after tracking the reference.

Given a set $P$, our goal is to find a policy sequence $\pi_\infty$ that satisfies the requirements of Definition 1. It is not tractable to look for a control policy over an infinite horizon. However, by enforcing the state $\phi_N$ to lie in a set $X_f$ which is controlled-invariant for the system (1), it is sufficient to find the first $N$ elements of the policy sequence, described by $\pi_{N-1}$, while inclusion of $\phi_N$ in $X_f$ guarantees the existence of the remaining elements of $\pi_\infty$. This method is a classical technique in Model Predictive Control. Even considering the finite sequence $\pi_{N-1}$, it is difficult to solve this problem due to the infinite dimension of the policy space. The next section presents a finite parametrization of $\pi_{N-1}$.

3 Tracking with affine feedback policies

3.1 Tracking on a finite horizon

As reviewed in [1], affine parameterizations have been introduced due to their nice computational properties (notably the convexity of the set of admissible such policies). Though additive state disturbances are not considered here, we show how to exploit the results in [1] in the following developments.

We consider the case where the set of feasible state and input constraints are polytopic and given by

\[
Z = \{(x, u) \mid Fx + Gu \leq c\}, \quad X_f = \{x \mid Hx \leq p\}. \tag{5}
\]

The state sequence $x$ is fully determined by $u$ and the initial condition $x_0$, so that the dynamics and constraints are given by:

\[
x = Ax_0 + Bu \\
Fx + Gu \leq c
\]

in condensed form with appropriate matrices. Let us consider the following affinely parameterized control policy:

\[
u_i = \sum_{j=0}^{i} M_{i,j} r_j + v_i, \quad \forall i \in \{0, \ldots, N-1\} \tag{7}
\]
Using condensed notation:

\[ u = Mr + v \]  

(8)

where \( M \) is block lower triangular.

Policy (8) is a reference sequence feedback policy. As established in [1], it is equivalent to a state sequence feedback policy, since past references and past states are related in an affine fashion. Therefore the policy (8) is compliant with the input trackability conditions (4).

According to the input trackability conditions (4), the set of admissible affine parameterizations \((M, v)\) can be written as:

\[
\mathcal{F}(x_0, \mathcal{P}) = \left\{ (M, v) \left| \begin{array}{l}
\forall r \in \mathcal{P} \\
\Sigma u \leq \sigma + \Xi x_0 \\
u = Mr + v \\
\Gamma u = r \\
M \in \mathcal{L}T
\end{array} \right. \right\}, \tag{9}
\]

where \( \Sigma := FB + G \), \( \sigma := c \) and \( \Xi := -FA \) and \( \Gamma := I_N \otimes g^T \), with \( I_N \) the identity matrix of size \( N \) and \( \otimes \) the Kronecker product of the matrices. The structural constraint on \( M \) such that it is a lower block triangular matrix is denoted as \( M \in \mathcal{L}T \).

In the sequel, we will simply write \( \mathcal{F}(x_0) \) omitting the dependency on \( \mathcal{P} \).

Let us also define the set of initial states \( x \) for which an admissible policy exists:

\[
\mathcal{X} := \{ x \in \mathbb{R}^n \mid \mathcal{F}(x) \neq \emptyset \} . \tag{10}
\]

**Lemma 1.** Both \( \mathcal{F}(x) \) and \( \mathcal{X} \) are convex.

**Proof.**

\[
\mathcal{F}(x) = \bigcap_{r \in \mathcal{P}} \left\{ (M, v) \left| \begin{array}{l}
\Sigma u \leq \sigma + \Xi x_0 \\
u = Mr + v \\
\Gamma u = r \\
M \in \mathcal{L}T
\end{array} \right. \right\} \tag{11}
\]

Written as such, \( \mathcal{F}(x) \) is clearly the intersection of a family of convex sets and therefore is convex. \( \mathcal{X} \) can be written as the projection on the \( x \) subspace of the set defined as:

\[
\left\{ (M, v, x) \left| \begin{array}{l}
\forall r \in \mathcal{P} \\
\Sigma u \leq \sigma + \Xi x \\
u = Mr + v \\
\Gamma u = r \\
M \in \mathcal{L}T
\end{array} \right. \right\} \tag{12}
\]

This set can itself similarly be written as the intersection of a family of convex sets, and therefore is convex, as is its projection \( \mathcal{X}_N \).

**Remark 4.** The reference set \( \mathcal{P} \) needs not be time-invariant along the horizon, nor does it need to be time-uncorrelated. It does not even need to be convex for the previous lemma to hold.
Lemma 2. If $\mathcal{P}$ is a full-dimensional polyhedral set described by:

$$\mathcal{P} = \{ r \mid S r \leq h \},$$

then

$$\mathcal{F}(x_0) = \\left\{ (M, v) \mid \exists Z \text{ s.t.} \begin{align*}
Z &\geq 0 \\
\Sigma v + Z^T h &\leq \sigma + \Xi x_0 \\
Z^T S &= \Sigma M^T \\
\Gamma M &= I_N, \Gamma v = 0 \\
M &\in LT
\end{align*} \right\}$$

Proof. Notice first that since $\mathcal{P}$ is full-dimensional, the linear equalities $\forall r \in \mathcal{P}$, $\Gamma(Mr + v) = r$ result in $\Gamma M = I_N$ and $\Gamma v = 0$ by balancing both sides of the equation. For the inequality constraints, the universal quantifier can be removed via dualization. One can replace the universal quantifier with a maximization:

$$\forall r \in \mathcal{P}, \Sigma(Mr + v) \leq \sigma + \Xi x_0 \iff \Sigma v + \max_{S r \leq h} \Sigma Mr \leq \sigma + \Xi x_0$$

where the maximization is taken row-wise. Dualizing these maximization problems and introducing the dual variable $Z$ associated to the inequality constraints describing $\mathcal{P}$ in the different maximization problems, the description of the set $\mathcal{F}$ reduces to (14).

Restricting ourselves to polyhedral reference sets and affinely parameterized control policies, we can solve the tracking certification problem described by conditions tractably by solving a single LP.

3.2 Optimizing over the reference set

In general, one is looking for the “largest” set of reference signals that the system can track. Suppose the reference set $\mathcal{P}$ is parameterized with some parameters $\theta \in \Theta \subseteq \mathbb{R}^n$. Let us further assume that for all values of $\theta$, $\mathcal{P}$ is a polyhedral set.

For simplicity, let us redefine the notations of the previous sections as follows: For a particular value of $\theta$, we define:

$$\mathcal{F}_\theta(x_0) := \left\{ (M, v) \mid \forall r \in \mathcal{P}(\theta), \begin{align*}
\Sigma(Mr + v) &\leq \sigma + \Xi x_0 \\
\Gamma(Mr + v) &= r \\
M &\in LT
\end{align*} \right\},$$

$$\mathcal{X}_\theta := \{ x \in \mathbb{R}^n \mid \mathcal{F}_\theta(x) \neq \emptyset \},$$

where $\mathcal{F}_\theta(x_0)$ is the set of all admissible affine disturbance feedback policies and $\mathcal{X}_\theta$ the set of feasible initial states for a particular value of $\theta$. We further define

$$\Theta(x_0) := \{ \theta \mid \mathcal{F}_\theta(x_0) \neq \emptyset \}$$
as the set of all parameters defining the reference set for which there exists an admissible affine policy.

To find the "largest" input trackable reference set, the goal is to solve a problem of the form:

$$\begin{align*}
\text{maximize} & \quad J(\theta) \\
\text{subject to} & \quad \theta \in \Theta(x_0).
\end{align*}$$

(18)

where $J$ captures the size (or value) of the set $P(\theta)$. This problem is in general non-convex and difficult to solve. For example, looking at characterization (14), one sees that a linear parametrization of $S$ and $h$ in $\theta$ results in a problem with bilinear equalities and inequalities. The following subsection presents instances of the problem for which it can be solved efficiently. Essentially we are looking for special cases where the parameterization of the disturbance set results in a convex search space $\Theta_N(x_0)$.

### 3.3 Scaling of a fixed shape polytope

Let us consider the parametrization $P(\theta) = \text{diag}(\theta_1)T + \theta_2 = \{ \text{diag}(\theta_1)r + \theta_2 \mid r \in T \}$ where $T$ is a given polyhedron of dimension $N$, $\theta_1 \in \mathbb{R}^N$ a scaling vector, $\theta_2 \in \mathbb{R}^N$ an offset vector and $\theta = (\theta_1^T, \theta_2^T)^T$. $\text{diag}(\theta_1)$ denotes the diagonal matrix with diagonal $\theta_1$. In the following, we show that we can efficiently optimize over $\theta$. The following lemma is instrumental for this.

**Lemma 3.** If

$$\begin{align*}
A &= \left\{ (M, v) \mid \forall r \in P \quad \Sigma(Mr + v) \leq \sigma \\
& \quad \Gamma v = \lambda, \Gamma M = \Lambda \\
& \quad M \in \mathcal{LT}\right\} \\
B &= \left\{ (M, v) \mid \forall r \in \mathcal{LP} + \lambda \\
& \quad \Sigma(Mr + v) \leq \sigma \\
& \quad \Gamma v = 0, \Gamma M = I_N \\
& \quad M \in \mathcal{LT}\right\}
\end{align*}$$

with $\Lambda$ a diagonal invertible matrix of appropriate dimension, then

$$A = \emptyset \text{ if and only if } B = \emptyset.$$

**Proof.** Suppose $A$ is not empty and $(M, v) \in A$. Then $(M\Lambda^{-1}, v - M\Lambda^{-1}\lambda) \in B$. Indeed, because $\Lambda^{-1}$ is diagonal, we also have $M\Lambda^{-1} \in \mathcal{LT}$. Moreover, $\forall r \in \mathcal{LP} + \lambda$, $\exists \bar{r} \in P : r = \Lambda\bar{r} + \lambda$. Consequently, $\Sigma(M\Lambda^{-1}\bar{r} + v - M\Lambda^{-1}\lambda) = \Sigma(M\Lambda^{-1}(\Lambda\bar{r} + \lambda) + v - M\Lambda^{-1}\lambda) = \Sigma(M\bar{r} + v) \leq \sigma$. The last inequality comes from the definition of $A$. Secondly, $\Gamma M\Lambda^{-1} = \Lambda\Lambda^{-1} = I_N$ and $\Gamma(v - M\Lambda^{-1}\lambda) = \lambda - \Gamma M\Lambda^{-1}\lambda = 0$. These together mean that $(M\Lambda^{-1}, v - M\Lambda^{-1}\lambda) \in B$. Conversely, if $(M, v) \in B$, then $(M\Lambda, v + M\Lambda^{-1}\lambda) \in A$. \hfill $\square$

**Remark 5.** It can be useful to think of $\theta_2$ as the nominal input trajectory of the system and $\theta_1$ as the flexibility around this nominal trajectory.

Considering the parameterization $P(\theta) = \text{diag}(\theta_1)T + \theta_2$, convexity of $\Theta_N(x_0)$ follows.
Lemma 4. If \( \theta \in \mathbb{R}^N_+ \times \mathbb{R}^N \) where \( \mathbb{R}^N_+ \) is the real positive line and \( \mathcal{P}(\theta) = \text{diag}(\theta_1)T + \theta_2 \), then \( \Theta(x_0) \) is convex.

Proof. Following Lemma 3 and removing the universal quantifier over \( \mathcal{P} \) with dualization, we can write the description of \( \Theta(x_0) \) as:

\[
\Theta(x_0) = \left\{ \theta \begin{array}{l}
\exists (M, v, Z) \\
Z \geq 0 \\
\Sigma v + Z^T h \leq \sigma + \Xi x_0 \\
Z^T S = (\Sigma M)^T \\
\Gamma M = \text{diag}(\theta_1), \Gamma v = \theta_2 \\
M \in \mathcal{L}T
\end{array} \right. \tag{19}
\]

\( \Theta(x_0) \) is the projection of a set defined by a family of linear equalities and inequalities, and therefore is convex.

Note that we do not need to explicitly compute the projection to optimize over \( \theta \), since it is implicitly described by the set of equations given above. From a practical point of view, it means that we can optimize over all possible component-wise scaling of a polyhedral disturbance set efficiently. This includes, as a particular case, uniform scalings of polyhedron \( \mathcal{T} \) if we consider matrices \( \Lambda = \mu I_N \). This allows us to find the largest volume reference set of given shape for a given horizon.

We can also consider a cost function over both the control policy and reference set parameterizations \( J(M, v, \theta) \). From Lemma 3 and Lemma 4, we can jointly optimize over the admissible control policies and reference sets in a computationally tractable way, if \( J \) is a convex function. This opens the possibility of re-optimizing the control policies after each step during closed loop operation. In such applications, availability of the initial admissible control policy that guarantees tracking and infinite horizon feasibility, ensures the recursive feasibility of the optimization problem, if the initial tracking requirement is not changed.

3.4 An implicit terminal set

We have shown how to find a control policy that will ensure input trackability as specified by the conditions (4), and how to optimize over the tracking set in a computationally tractable way, assuming that the terminal set \( X_f \) is given. However, finding an explicit description of a controlled invariant set is usually difficult. In this section, we introduce an implicitly defined terminal condition, which ensures infinite horizon feasibility. The method scales well with dimension, as it does not require explicit set calculations.

From (6) and (8), the terminal state \( x_N \) is given by an affine function of the reference \( r \)

\[
x_N = \bar{A}x_0 + \bar{B}Mr + \bar{B}v, \tag{20}
\]

where \( \bar{A} := \begin{bmatrix} 0 & I_n \end{bmatrix} A, \bar{B} := \begin{bmatrix} 0 & I_n \end{bmatrix} B. \)

From the discussion in Section 2, we require that \( x_N \) lies in a controlled invariant set for all values of \( r \). Note that this differs from the standard robust invariance condition since after the horizon of \( N \) steps, there is no further reference to track and
therefore no uncertainty. We follow the idea of [11], by enforcing that \( x_N \) is a feasible steady state of the system for each value of the reference \( r \):

\[
\begin{align*}
x_N &= Ax_N + Bu_N \\
Fx_N + Gu_N &\leq c
\end{align*}
\]

The input at the \( N \)th step \( u_N \) is not specified by the control policy \( u = Mr + v \), so we propose again an affine parametrization

\[
u_N = M_{ss}r + v_{ss}.
\]

Combining (20) - (22) gives the conditions

\[
\begin{align*}
LM + BM_{ss} &= 0 \\
Lv + Bv_{ss} + T x_0 &= 0 \\
FB[Mr + v] + G[M_{ss}r + v_{ss}] &\leq c - FAx_0
\end{align*}
\]

where \( L := [A - I_n] \bar{B} \) and \( T := [A - I_n] \bar{A} \).

**Remark 6.** In some applications, such as building control, it is preferable to keep the system in a periodic steady state, due to periodic nature of the disturbances and constraints. Periodic steady states can be easily incorporated into the definition of the set (23) by representing the periodic system as a lifted version of the original system [1], that describes the state evolution throughout a period of \( N_P \) steps, and modifying the equations (20) - (22).

The conditions (23) ensures that the control policy defined by \((M, v, M_{ss}, v_{ss})\) is able to drive the system to the set of admissible steady states, which is a control invariant set. Therefore we can characterize the admissible control policies by:

\[
\mathcal{H}_\theta(x_0) = \left\{ (M, v, M_N, v_N) \mid \begin{array}{l}
(M, v) \in \mathcal{F}_\theta(x_0) \\
(M, v, M_{ss}, v_{ss}) \text{ satisfies (23)} \\
\forall r \in \mathcal{P}(\theta)
\end{array} \right\}
\]

\( \mathcal{H}_\theta(x_0) \) is a convex set, as it is the intersection of convex sets which are described by conditions (23) and \( \mathcal{F}_\theta \). When \( \mathcal{P}(\theta) \) is a polyehdron, the universal quantifier on \( r \) can be eliminated by dualization as previously explained in section 3.1.

**Remark 7.** Addition of steady state control policy parametrization \((M_{ss}, v_{ss})\) and conditions (23) does not effect the results of Section 3.3 regarding the scaling of the reference set, since an admissible \((\tilde{M}_{ss}, \tilde{v}_{ss})\) for the scaled reference set can be constructed as explained in the proof of Lemma 3.

### 4 Applications

In this section, we present an application of the developed method, in which the flexibility in the power consumption of a building, around a nominal consumption profile, is characterized by means of a simple battery. This is highly desirable for assessing the capabilities of buildings to participate in demand response programs, in which participants are rewarded for their flexibility in consumption.

Consider the following battery model, with constraints on the state-of-charge \( s \), and the rate of charge \( r \).
\[ s_{i+1} = as_i + r_i \]
\[ 0 \leq s_i \leq s_{max} \]
\[ -r_{max} \leq r_i \leq r_{max} \]

(25)

For a finite horizon length, the constrained model (25) serves as a reference generator and describes the polytopic reference set \( \mathcal{T} \), as the set of admissible power input trajectories \( r \) of the battery. In order to assess that the office building can be represented as the given battery model, the building needs to be able to track any reference \( r \in \mathcal{T} \) around a nominal trajectory.

We fix the shape of \( \mathcal{T} \) by setting the battery parameters as: \( a = 1, s_0 = s_{max}/2, s_{max}/r_{max} = 5 \), in order to represent the possible behavior of the reference signal. Given these constraints, the battery is fully described by its maximum power limit \( r_{max} \). By virtue of Lemma 3, we can search for uniformly scaled versions of the reference set \( F(\theta) = \theta_1 \mathcal{T} + \theta_2 \), where \( \theta_2 \) represents the nominal trajectory around which the battery is defined. Note that uniformly scaling the battery by \( \theta_1 \) is equivalent to scaling of \( r_{max} \), and therefore the storage limits and the initial condition of the battery, by the same amount. This means that starting with an initial \( r_{max} \), we can directly optimize over the parameter \( \theta_1 \) to find the optimal power limit \( r_{max}^* = \theta_1^* r_{max} \) and the corresponding battery parameters.

Remark 8. We avoid non-uniform scaling, as it would distort the battery dynamics. However, it can be applied in a setting where the flexibility polytope is a box, i.e. there are no dynamics involved, enabling the building to provide asymmetric up-down flexibility.

We consider a simplified problem, where the external disturbances, such as weather, occupancy and solar radiation, are considered to be perfectly forecast and periodic. In a practical setting, the controller should also be robustified against uncertainty in the external disturbances, but we omit this issue in order to underline the presented methodology. Furthermore, we assume that the thermal power consumption of the building is equal to the electricity consumption of the HVAC system. The building and the external disturbances represent a small office building in summer conditions. The model is obtained with the OpenBuild toolbox [?]. The building consists of three zones and the power input to each zone is considered as a separate input. The total power consumption of the HVAC system of the building is simply the sum of all inputs, therefore following a power consumption reference is an input tracking problem. The HVAC system is required to satisfy the comfort constraints, which are represented as temperature constraints on each zone.

For the cost function of the building, we consider the case where the building is asked for symmetric up - down flexibility and only paid for the power limits. At the beginning of each day, the building finds the optimum battery it can support for a specific activation period, and the corresponding affinely parameterized control policy that guarantees input trackability of all possible battery trajectories. The building then offers the corresponding power limits to the demand response operator, while minimizing its cost for the next day, described as:

\[ J(v, r_{max}) = c_1^T v - c_2^T 1 r_{max} \]

(26)

where 1 is a vector of ones. The linear cost function \( J \) represents the cost of electricity and flexibility reward paid to the building for the active periods. The flexibility reward
is taken to be twice the price of electricity. The extra cost that might arise from the tracking realizations are assumed to be compensated. The activation period covers 10 hours and is between 8:00 and 18:00. The battery is only defined during the activation period and the reference signal $r$ is updated at each step with a discretization of one hour. The building uses predictions for the next day and also aims for a periodic steady state with a period of one day. Thus, the horizon length is $N = 24$, with an additional $N_P = 24$ steps for the periodic steady state.

The control policy resulting from the optimization is simulated with randomly drawn samples from the reference set, as shown in Figure 1. The flexibility captured by the battery ensures a storage capacity of $41.2 \text{ kWh}$ with a power range of $16.5 \text{ kW}$, which represents 36% of the maximum nominal consumption of the HVAC system, which stands at $45.4 \text{ kW}$.

**Figure 1:** Randomly sampled open loop trajectories, where the building uses affinely parameterized control policy found at the initial step. The top three figures represent the evolution of building outputs, inputs, and total power consumption of the HV AC system, while the last two figures show the flexibility signal $r$ and the state-of-charge $s$ described by the battery equations (25). Note that the second day of the simulation represents the periodic steady states computed by the control policy.

### 5 Conclusion

We demonstrate in this paper how to certify in advance that a system can track (as a function of its inputs) references drawn from a reference signal set. The method utilizes
a causal affine reference feedback policy to formulate a convex optimization problem that certifies trackability on a finite horizon window. Use of a terminal invariant set constraint also certifies that the system stays feasible indefinitely after the tracking period. The tracking reference set can be optimized tractably in some cases, allowing one to find the largest dimension-wise scaling of a set that can be tracked. An implicit characterization of a terminal set as the set of all feasible steady states is proposed for this particular setup. Results are illustrated by computing the power consumption flexibility of a building HVAC system.

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