Einstein–Yang–Mills Black Hole Interiors: Serious Problems But Simple Solution

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Recently E. E. Donets, D. V. Gal’tsov, and the author reported the results of numerical and analytical investigation of the SU(2) Einstein–Yang–Mills (EYM) black hole interior solutions (gr-qc/9612067). It was shown that a generic interior solution develops a new type of an infinitely oscillating behavior with exponentially growing amplitude. Numerical data for three sequential oscillations were presented. The numerical integration technique was not discussed. Later P. Breitenlohner, G. Lavrelashvili, and D. Maison confirmed our main results (gr-qc/9703047). But they have made some misleading statements. In particular, they claimed, discussing the oscillations, that “as one performs the numerical integration one quickly runs into serious problems...” so that “it is practically impossible to follow more than one or two of them numerically” because “the numerical integration procedure breaks down” (pp. 3, 12). It is shown here that trivial logarithmic substitutions and integration along the integral curve solve these “serious problems” easily.

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“We”, said Owl, “the customary procedure in such cases is as follows.”
A. A. Milne ;)

Recall the main notation from [1]. We assume the static spherically symmetric magnetic ansatz for the YM potential

\[ A = (W(r) - 1) (T_\phi d\theta - T_\theta d\phi) , \]

\[ (T_\phi, T_\theta) \text{ are spherical projections of the SU(2) generators}, \]

and the following parametrization of the metric:

\[ ds^2 = (\Delta/r^2) \sigma^2 dt^2 - (r^2/\Delta) dr^2 - r^2 d\Omega^2 , \]

where \( \Delta, \sigma \) depend on \( r \). The field equations include a coupled system for \( W, \Delta, \sigma \):

\[ \Delta U'' + FW' = VW/r , \]  
\[ (\Delta/r)' + 2\Delta U'^2 = F , \]

where \( U = W'/r, V = W^2 - 1, F = 1 - V^2/r^2 \), and a decoupled equation for \( \sigma \):

\[ (\ln \sigma)' = 2rU^2 . \]

These equations admit BH solutions in the domain \( r \geq r_h \) for any radius \( r_h \) of the event horizon.

The system (1, 2) was integrated numerically in the region \( 0 < r < r_h \) using an adaptive step size Runge–Kutta method starting at the left vicinity of the event horizon \( r_h \) with one free parameter \( W_h = W(r_h) \) satisfying inequalities \( |W_h| < 1 \) and \( 1 - W_h^2 \leq r_h \), which are the necessary conditions for asymptotic flatness.

One of the main results of our investigation is that a generic EYM BH interior solution develops a new type of an infinitely oscillating behavior with exponentially growing amplitude, during which the ‘falls’ of \( \Delta (\Delta < 0) \) may be approximated as

\[ \Delta(r) = \frac{\Delta(r_k)}{r_k} r \exp \left[ U(r_k)^2 (r^2 - r_k^2) \right] , \quad r_k \geq r , \quad (3) \]

where \( r_k \) is the “starting point” of the \( k \)-th oscillation, \( (U(r_k) r_k)^2 \approx 1 \), and all the regime can be described by a two-dimensional dynamical system (see [1] for the details).

Our investigation of the EYM BH interior solutions reveals, that one meets only two problems during numerical integration of (1, 2).

Problem 1. Very small values of \( r \) as \( r \to 0 \) and \( |\Delta| \) in the neighborhoods of \( \Delta \)'s local maxima, and very large values of \( |U| \) as \( r \to 0 \), and \( |\Delta| \) in the neighborhoods of \( \Delta \)'s local minima.

Problem 2. Intervals of very slow variation of \( r \) but extremely fast variation of \( \Delta \) and maybe \( U \).

Obviously, the first problem can hardly be called a problem because it may be solved by trivial substitutions like \( \ln(1/r^2) \) (or just \( \ln r \)), \( \ln \ln(1/r^2) \), etc., for \( r \); \( \ln(-\Delta) \), \( \ln |\ln(-\Delta)| \), etc., for \( \Delta \) an so on.

The second one is not much more difficult. Let us show how it was solved for obtaining numerical data for three sequential oscillations, presented in [1] (we shall also use figures 6, 7 from [1]).

Let us rewrite (1, 2) in a more “comfortable” way, suggested in [2]:

\[ DW'' + FW' = \frac{1}{x} VW , \]
\[ D' + \left( \alpha W^2 - \frac{1}{2x} \right) D = F , \]

where \( x = (r/r_h)^2, D = 2\Delta/r_h^2, \alpha = 4/r_h^2, V = (W^2 - 1)/2, F = 1 - \alpha V^2/x \), and \( ' \equiv d/dx \). Here \( W' \) plays the
same role as $U$ in (1, 2). In a normal form, needed for the Runge–Kutta method, this system reads as
\begin{align*}
W' &= P, \\
P' &= \frac{1}{xD}V(W + \alpha VP) - \frac{P}{D}, \\
D' &= F - \left(\alpha P^2 - \frac{1}{2x}\right)D.
\end{align*}
(6)

It is clear from (6) that small (absolute) value of denominator $xD$, which takes place in the metric function local maximum neighborhood, may cause numerical problems. It may be unessential for some first maxima (‘max 1’ and ‘max 2’ in Fig. 1), but becomes considerable as oscillations progress. Really, integrating system (6) for $r_h = 2$, $W_h = -0.342072$, one finds out that $x$ practically stops in the third local maximum neighborhood (‘max 3’ in Figs. 2, 3) with $D \approx -10^{-16}$, $W' \approx 10^{18}$ (I mean a PC with 15 digits after the decimal point). This does not allow to pass the third maximum correctly. One of the possible solutions is to desingularize (6) by introducing a parameter, say $t$, in order to have the solution comparable values, and $G = \ln(-D)$ in order to pass not only the third local maximum, but also the third local minimum (see (3) and Fig. 3). Thus, the new system may be written as
\begin{align*}
\begin{bmatrix}
\frac{d}{dt} \\
\frac{d}{dt} \\
\frac{d}{dt}
\end{bmatrix}
\begin{bmatrix}
W \\
P \\
G
\end{bmatrix}
= \\
\begin{bmatrix}
\frac{1}{C}e^{\delta t}P \\
NP - QVW \\
\frac{1}{2} - \alpha e^{\delta t}P^2 - N
\end{bmatrix},
\end{align*}
(8)
where $Q = \exp(-G)$, $H = e^{\delta t} - \alpha V^2$, $N = HQ$, $\delta \equiv d/dl$, and, for example, $dl = (1 + G^2)^{1/2}dt$ for integration along $G$. This system allows to pass the third local maximum and parts A and B of the third oscillation (Figs. 3, 4).
Note that part B in fact does not need to be passed along the integral curve. It is sufficient just to use \( t = \ln x \) (or, e.g., \( t = \ln(1/x) \)) as an independent variable.

At last, one needs to pass part C (Fig. 3) in order to reach (and pass) the fourth local maximum of \( \Delta \). \( \Delta \) vanishes fast at this interval; this forces \( W' \) to increase strongly (\( |U\Delta| \approx |U(r_{\text{max}}^{\text{3}})| \) at this interval \( \Box \)). Therefore one needs to introduce \( Z = \ln W' \). System \( \Box \), rewritten in the corresponding way, solves this problem. (Evidently, this can be done before passing ‘max 3’, but \( \Box \) works a bit better along part A.)

It is also convenient to improve a system, prepared for integration along part C in a way similar to (7). Let us introduce another parameter, say \( p \), such that \( dt = -dp \). Then one obtains

\[
\begin{bmatrix}
\dot{p} \\
\dot{t} \\
\dot{W} \\
\dot{Z} \\
\dot{G}
\end{bmatrix} = \frac{dp}{dt} \begin{bmatrix}
1 \\
E \\
E \exp(t + Z) \\
H - VW \exp(-Z) \\
\frac{1}{2}E - H - \alpha E \exp(t + 2Z)
\end{bmatrix},
\]

where \( E = \exp(G) \). This system works fine both along part C and while passing local maxima (both ‘max 3’ and ‘max 4’).

The next oscillation needs the next “order” of logarithmic substitutions (\( \ln \ln(1/x) \), etc.), but the numerical integration technique remains the same.

Recall, at last, that the investigation, presented in \( \Box \), was restricted to the class of asymptotically flat solutions. If one also studies solutions, which are singular in the exterior region, then a situation, shown in Fig. 3, is rather typical. This makes the claims cited in the abstract even more surprising.

Thus, if we do not discuss restrictions, which are inherent for every numerical method, there are hardly any “serious problems”, which can “break down” the numerical integration of the SU(2) EYM equations inside the event horizon. All the difficulties, connected with the numerical investigation, are typical and can be solved by integration along the integral curve.

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\[\text{FIG. 4. } \Delta \text{ versus } l: \text{ the third local maximum.}\]

\[\text{FIG. 5. This curve represents a solution, singular in the exterior region. One can see 4 first oscillations of } \Delta \text{ and the beginning of the 5th one, which can be easily integrated using the technique, described above.}\]

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