In-medium QCD forces at high temperature

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Y. Akamatsu, arXiv:1209.5068[hep-ph]
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1. INTRODUCTION
Confinement & Deconfinement

• Vacuum & In-Medium Potentials

Singlet channel

\[ V(R) = KR - \frac{4 \alpha_s}{3} + O\left(\frac{1}{M^2}\right) \]

String tension \( K \sim 0.9 \text{GeVfm}^{-1} \)

Mass spectra (c\bar{c}, b\bar{b})

Debye screened potential

\[ V(R) = -\frac{4 \alpha_s}{3} \exp\left(-\omega_D R\right) + O\left(\frac{1}{M^2}\right) \]

Debye mass \( \omega_D \sim gT \) (HTL)

The Schrödinger equation

Existence of bound states (c\bar{c}, b\bar{b})

\( \rightarrow \) \( J/\Psi \) suppression in heavy-ion collisions

What is the in-medium potential?

Matsui & Satz (86)
Quarkonium Suppression

- Sequential melting of bottomonia

\[
\frac{Y(2S)/Y(1S)_{PbPb}}{Y(2S)/Y(1S)_{pp}} = 0.21 \pm 0.07\,\text{(stat)} \pm 0.02\,\text{(syst)}
\]

\[
\frac{Y(3S)/Y(1S)_{PbPb}}{Y(3S)/Y(1S)_{pp}} = 0.06 \pm 0.06\,\text{(stat)} \pm 0.06\,\text{(syst)} < 0.17\,\text{(95\% CL)}
\]
2. IN-MEDIUM QCD FORCES
In-Medium Potential

• Definition

\[ \langle \Psi(t; R) \rangle_T \propto \langle J(t; R) J^\dagger(0; R) \rangle_T \]
\[ \propto \sum_{n,m} |m \langle J^\dagger(0; R) | n \rangle| e^{-\beta E_{n}(R)} \exp[i\{E_{n}(R) - E_{m}(R)\}t] \]
\[ \sim \sum_{\alpha=(n,m) \in \text{lowest peak}} W_{i=0}(\alpha; T) \exp[-iE_{\alpha}(R)t] \]
\[ \sim \exp[-i\{V(R,T) - i\Gamma(R,T)/2\}t] \]

Long time dynamics

In analogy to vacuum case

\[ G(\tau; R, T) = \langle J(-i\tau; R) J^\dagger(0; R) \rangle_T \sim \int D[A] e^{-S(A; T)} \]

(0<\tau<\beta)

Lorentzian fit of \( \sigma(\omega; R, T) \)

Spectral decomposition

\[ \sigma(\omega; R, T) \]

\[ i=0 \quad i=1 \quad \ldots \]

\[ \Gamma(R, T) \]

\[ V(R, T) \]

\[ \omega \]

\[ t, i\tau \]

\[ r \]

\[ R \]
In-Medium Potential

• Complex Potential

\[ \langle \Psi(t; R) \rangle_T \sim \sum_{\alpha \in \text{lowest peak}} W_{i=0}(\alpha; T) \exp[-iE_\alpha(R)t] \]

\[ \sim \exp[-i\{V(R,T) - i\Gamma(R,T)/2\}t] \]

\[ \sim \int D\Theta(s) \exp[-\int_0^t ds \Theta(s)^2/\Gamma(R,T)] \exp[-i\int_0^t ds\{V(R,T) + \Theta(s)\}] \]

Long time dynamics

Lorentzian fit of \(\sigma(\omega;R,T)\)

Suggests stochastic & unitary description

Laine et al (07), Beraudo et al (08), Bramilla et al (10), Rothkopf et al (12).
In-Medium Potential

- **Stochastic Potential**

\[
\Psi(t + \Delta t, R) = \exp \left[ -i \Delta t \left\{ V(R) + \Theta(t, R) \right\} \right] \Psi(t, R), \quad (T \text{ omitted})
\]

\[
\langle \Theta(t, R) \rangle = 0, \quad \langle \Theta(t, R) \Theta(t', R') \rangle = \Gamma(R, R') \delta_{tt'} / \Delta t,
\]

Introduce noise field \( \Theta(t, R) \)

Density matrix: Non-local correlation relevant

\[
\rho(t, R_1, R_2) \equiv \langle \Psi^*(t, R_1) \Psi(t, R_2) \rangle
\]

\[
i \frac{\partial}{\partial t} \Psi(t, R) = \left\{ V(R) - \frac{i}{2} \Gamma(R, R) + \Xi(t, R) \right\} \Psi(t, R),
\]

Imaginary potential = Local correlation

\[
\Xi(t, R) \equiv \Theta(t, R) - \frac{i \Delta t}{2} \left\{ \Theta(t, R)^2 - \langle \Theta(t, R)^2 \rangle \right\}, \quad \langle \Xi(t, R) \rangle = 0
\]

Akamatsu & Rothkopf ('12)
In-Medium Forces

- $M < \infty$

$M = \infty$
- Debye screened force
- Fluctuating force

$M < \infty$
- Drag force
- Langevin dynamics

(Stochastic) Potential force
→ Hamiltonian dynamics

Non-potential force
→ Not Hamiltonian dynamics

How to describe in-medium QCD forces?
3. INFLUENCE FUNCTIONAL OF QCD
Open Quantum System

• Basics

Hilbert space

von Neumann equation

Trace out the environment

Reduced density matrix

Master equation

\[ H_{\text{tot}} = H_{\text{sys}} \otimes H_{\text{env}} \]

\[ i \frac{d}{dt} \hat{\rho}_{\text{tot}}(t) = [\hat{H}_{\text{tot}}, \hat{\rho}_{\text{tot}}(t)] \]

\[ \hat{\rho}_{\text{red}}(t) = \text{Tr}_{\text{env}}[\hat{\rho}_{\text{tot}}(t)] \]

\[ i \frac{d}{dt} \hat{\rho}_{\text{red}}(t) = ? \]

(Markovian limit)

sys = heavy quarks
env = gluon, light quarks
Closed-Time Path

\[ \rho[\varphi_1^{\text{ini}}, \varphi_2^{\text{ini}}] \]

\[ \begin{array}{c}
\text{QCD} \\
\end{array} \]

\[ Z[\eta_1, \eta_2] \sim \int D[\psi_{1,2}, q_{1,2}, A_{1,2}] \rho[\psi_{1}^{\ast}, q_{1}^{\ast}, A_{1}^{\ast}]^{\text{ini}} \times \rho[\psi_2, q_2, A_2]^{\text{ini}} \times \exp \left[ iS[\psi_1] - iS[\psi_2] + i \int \psi_1 \eta_1 - i \int \psi_2 \eta_2 \right] \times \exp \left[ iS[q_1 A_1] - iS[q_2 A_2] + ig \int j_1 A_1 - ig \int j_2 A_2 \right] \]

\[ \rho_{\text{tot}} = \rho_{\text{env}}^\text{eq} \otimes \rho_{\text{sys}} \text{ Factorized initial density matrix} \]

\[ \rightarrow \rho_{\text{tot}}[\psi_{1}^{\ast}, q_{1}^{\ast}, A_{1}^{\ast}]^{\text{ini}}, \psi_2, q_2, A_2]^{\text{ini}} = \rho_{\text{env}}^\text{eq}[q_{1}^{\ast}, A_{1}^{\ast}]^{\text{ini}}, q_2, A_2]^{\text{ini}} \cdot \rho_{\text{sys}}[\psi_{1}^{\ast}, \psi_2^{\ast}]^{\text{ini}} \]

Influence functional Feynman & Vernon (63)

\[ = Z_{qA}[j_1, j_2] \equiv \exp \left[ iS_{\text{FV}}^{\text{FV}}[j_1, j_2] \right] \]

\[ = \exp \left[ - g^2/2 \int j_1 G_A^F j_1 + j_2 G_A^F j_2 - j_1 G_A^> j_2 - j_2 G_A^< j_1 + \int g^3 G_A^{(3)} j j j + g^4 G_A^{(4)} j j j j + \cdots \right] \]
Influence Functional

- Open Quantum System

\[
Z[\eta_1, \eta_2] \sim \int D[\psi_{1,2}] \rho_{\text{sys}}[\psi_1^{*\text{ini}}, \psi_2^{\text{ini}}] \\
\times \exp\left[ iS[\psi_1] - iS[\psi_2] + iS^{\text{FV}}[j_1, j_2] + i\int \psi_1 \eta_1 - i\int \psi_2 \eta_2 \right]
\]

Path integrate until \( s \), with boundary condition \( \psi_1(s) = \psi_1, \psi_2(s) = \psi_2 \)

\[
\rho_{\text{sys}}[\psi_1^{*\text{ini}}, \psi_2^{\text{ini}}] = \rho_{\text{red}}[s, \psi_1^*, \psi_2] = \langle \psi_1^* | \hat{\rho}_{\text{red}}(s) | \psi_2 \rangle
\]
Influence Functional

- **Functional Master Equation**

  \[ \rho_{\text{red}}[t, \psi^*_1, \psi_2] \sim \int_{-\infty}^{t} D[\psi_{1,2}] \rho_{\text{sys}}[\psi^*_{\text{ini}}, \psi_{\text{ini}}] \times \exp \left[ iS[\psi_1] - iS[\psi_2] + iS^{\text{FV}}[j_1, j_2] \right] \]

  - Effective initial wave function
  - Effective action \( S_{1+2} \)
  - \( \rightarrow \) Single time integral
  - Long-time behavior (Markovian limit)
  - Analogy to the Schrödinger wave equation

- **Functional differential equation**

  \[ i \frac{\partial}{\partial t} \rho_{\text{red}}[t, \psi^*_1, \psi_2] = H_{1+2}^{\text{func}}[\psi^*_1, \psi_2] \rho_{\text{red}}[t, \psi^*_1, \psi_2] \]

  - How does this formalism work in perturbation theory in non-relativistic limit?
4. DYNAMICAL EQUATIONS
Density Matrix

• Coherent State

\[ \rho_{\text{red}} \left[ t, Q_{1(c)}^*, \tilde{Q}_{2(c)}^* \right] = \left\langle Q_{1(c)}^* \right| \hat{\rho}_{\text{red}}(t) \left| \tilde{Q}_{2(c)}^* \right\rangle \]

\[ \left\langle Q_{1(c)}^* \right| = \langle \Omega \rangle \exp \left[ - \int_{\tilde{x}} \left\{ \hat{Q} Q_{1(c)}^* + \hat{Q}_c Q_{1(c)}^* \right\} \right] \]

\[ \left| \tilde{Q}_{2(c)}^* \right\rangle = \exp \left[ - \int_{\tilde{x}} \left\{ \tilde{Q}_{2(c)}^* \hat{\Delta} + \tilde{Q}_{2(c)}^* \hat{\Delta}_c \right\} \right] \langle \Omega \rangle \]

Source for HQs

\[ \frac{\delta}{\delta Q_{1(c)}^*(\tilde{x})} \left\langle Q_{1(c)}^* \right| \bigg|_{Q_{1(c)}^* = 0} = \langle \Omega \rangle \hat{\Delta}(\tilde{x}) \]

\[ \frac{\delta}{\delta \tilde{Q}_{2(c)}^*(\tilde{x})} \left| \tilde{Q}_{2(c)}^* \right\rangle \bigg|_{\tilde{Q}_{2(c)}^* = 0} = -\hat{\Delta}_c^+(\tilde{x}) \langle \Omega \rangle \]
Density Matrix

• A few HQs

One HQ

\[ \rho_Q(t, \tilde{x}, \tilde{y}) = \langle \tilde{x} | \hat{\rho}_Q(t) | \tilde{y} \rangle \propto \langle \Omega | \hat{Q}(\tilde{x}) \hat{\rho}_{\text{red}}(t) \hat{Q}^\dagger(\tilde{y}) | \Omega \rangle \]

\[ = -\frac{\delta}{\delta Q_1^*(\tilde{x})} \frac{\delta}{\delta Q_2^*(\tilde{y})} \rho_{\text{red}} \left[ t, Q_{1(c)}^*, \tilde{Q}_{2(c)}^* \right] \bigg|_{Q_{1(c)}^* = \tilde{Q}_{2(c)}^* = 0} \]

Similar for two HQs, ...

\[ \rho_{QQ_c}(t, \tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2), \cdots \]
Master Equation

- Functional Master Equation

\[ i \frac{\partial}{\partial t} \rho_{\text{red}}[t, Q^*_1(c), \tilde{Q}^*_2(c)] = H_1^{\text{func}}[Q^*_1(c), \tilde{Q}^*_2(c)] \rho_{\text{red}}[t, Q^*_1(c), \tilde{Q}^*_2(c)] \]

Functionally differentiated
Color traced

Master equation

\[ i \partial_t \rho_Q(t, \bar{x}, \bar{y}) = \left\{ (a - a^*)M + \left( -\frac{\nabla^2_x - \nabla^2_y}{2M} \right) \right\} \rho_Q(t, \bar{x}, \bar{y}) + C_F \left\{ -iD(\bar{x} - \bar{y}) + \frac{\nabla_x D(\bar{x} - \bar{y})}{4T} \cdot \frac{\nabla_x - \nabla_y}{iM} \right\} \rho_Q(t, \bar{x}, \bar{y}) \]

Reduces to Caldeira-Leggett master equation at \( x \sim y \)
Master Equation

• HQ Number Conservation

\[
\text{Tr} \hat{\rho}_Q(t) = \int \rho_Q(t, \bar{x}, \bar{x})
\]

\[
i \frac{d}{dt} \text{Tr} \hat{\rho}_Q(t) = \int \delta(\bar{x} - \bar{y})(i \partial_t \rho_Q(t, \bar{x}, \bar{y})) = 0
\]

• Ehrenfest Equation

\[
\frac{d}{dt} \langle \bar{x} \rangle = \frac{\langle \bar{p} \rangle}{M},
\]

\[
\frac{d}{dt} \langle \bar{p} \rangle = -\frac{\gamma}{2MT} \langle \bar{p} \rangle,
\]

\[
\gamma = \frac{C_F}{3} \nabla^2 D(x) \bigg|_{x=0} = -\frac{g(T)^2 C_F}{9} \nabla^2 \tilde{G}_{00,aa}^{>} (\omega = 0, x) \bigg|_{x=0}
\]

\[
\frac{d}{dt} \langle E \rangle = -\frac{\gamma}{MT} \left( \langle E \rangle - \frac{3T}{2} \right),
\]

\[
g \frac{g(T)^2 C_F}{9} \int \frac{d^3k}{(2\pi)^3} k^2 \tilde{G}_{00,aa}^{>} (\omega = 0, k)
\]

Moore et al (05,08,09)
Other Results

• Complex Potential

\[ \left\langle \Psi(t; \bar{x}, \bar{y}) \right\rangle_T \propto \left\langle J(t; \bar{x}, \bar{y}) J^\dagger(0; \bar{x}, \bar{y}) \right\rangle_T \]

\[ \propto \frac{\mathcal{O}^2}{\mathcal{O}_1^* (\bar{x}) \mathcal{O}_1^* (\bar{y})} \rho_{\text{red}} \left[ Q_1^*, \widetilde{Q}_2^*, t \right] \]

\( Q_1^*(c) = \widetilde{Q}_2^*(c) = 0 \)

\[ V_{\text{singlet}}(R) = 2(a - 1)M - C_F V(R) = -\frac{g(T)^2 C_F}{4\pi} \left( \omega_D + \frac{e^{-\omega_D R}}{R} + iT \phi(\omega_D R) \right) \]

Laine et al (07), Beraudo et al (08), Brambilla et al (10)
5. SUMMARY & OUTLOOK
• Quantum Dynamics of HQs in Medium
  – Stochastic potential, drag force

• Non-Equilibrium Quantum Field Theory
  – Open quantum system, closed-time path, influence functional
  – Functional master equation, master equation, etc.

• Non-Perturbative Region
  – Model the renormalized effective Hamiltonian
  – Higher-order perturbative analyses (process involving real gluons)
  – Application to phenomenology
BACKUP
In-Medium Potential

- **Definition**

\[ T=0, \quad M=\infty \]

\[ \Psi(t; R) \propto \langle \text{vac} | J(t; R) J^\dagger (0; R) | \text{vac} \rangle \]

\[ = \sum_m \langle m | J^\dagger (0; R) | \text{vac} \rangle^2 \exp \left[ -iE_m (R)t \right] \]

\[ \sim \exp \left[ -iE_{\text{min}} (R)t \right] = e^{-iV(R)t} \]

Long time dynamics

\[ G(\tau; R) = \langle \text{vac} | J(-i\tau; R) J^\dagger (0; R) | \text{vac} \rangle \sim \int D[A] e^{-S(A)} \]

\[ \sim \exp \left[ -E_{\text{min}} (R)\tau \right] = e^{-\tau V(R)} \]

\( V(R) \) from large \( \tau \) behavior
Closed-Time Path

• Basics

\[ \rho[\varphi_1^{\text{ini}}, \varphi_2^{\text{ini}}] \]

Partition function

\[
Z[\eta_1, \eta_2] = \text{Tr} \left( \hat{U}(\infty, -\infty; \eta_1) \hat{\rho} \hat{U}(\infty, -\infty; \eta_2) \right) \\
= \text{Tr} \left( \hat{U}(\infty, -\infty; \eta_2)^\dagger \hat{U}(\infty, -\infty; \eta_1) \hat{\rho} \right) \\
\sim \int D\varphi_{1,2} \rho[\varphi_1^{\text{ini}}, \varphi_2^{\text{ini}}] \exp \left[ iS[\varphi_1] - iS[\varphi_2] + i\int \eta_1 \varphi_1 - i\int \eta_2 \varphi_2 \right]
\]

\[
\Pi_i \frac{\delta}{\delta \eta_i(x_i)} \ln Z[\eta_1, \eta_2] \bigg|_{j_{1,2} = 0} \propto \left\langle T \prod_i \hat{\phi}_i(x_i) \right\rangle_{T,\text{conn}}
\]

\[
\frac{\delta^2}{\delta \eta_1(x_1) \delta \eta_1(x_2)} \ln Z[\eta_1, \eta_2] \bigg|_{j_{1,2} = 0} \propto \left\langle T \hat{\phi}(x_1) \hat{\phi}(x_2) \right\rangle_{T,\text{conn}} = G^F_{\varphi}(x_1, x_2)
\]

\[
\frac{\delta^2}{\delta \eta_1(x_1) \delta \eta_2(x_2)} \ln Z[\eta_1, \eta_2] \bigg|_{j_{1,2} = 0} \propto \left\langle \hat{\phi}(x_2) \hat{\phi}(x_1) \right\rangle_{T,\text{conn}} = G^<_{\varphi}(x_1, x_2)
\]
Approximations

• Leading-Order Perturbation

Influence Functional

\[
\exp \left[ i S^{FV} [j_1, j_2] \right] \\
\approx \exp \left[ - g^2 / 2 \int j_1 G_A^F j_1 + j_2 G_A^{\tilde{F}} j_2 - j_1 G_A^> j_2 - j_2 G_A^< j_1 \right]
\]

Expansion up to 4-Fermi interactions

\[
G_A^F(x_1 - x_2) = \left\langle T \hat{A}(x_1) \hat{A}(x_2) \right\rangle_T, \quad G_A^>(x_1 - x_2) = \left\langle \hat{A}(x_2) \hat{A}(x_1) \right\rangle_T \\
G_A^<(x_1 - x_2) = \left\langle \hat{A}(x_1) \hat{A}(x_2) \right\rangle_T, \quad G_A^{\tilde{F}}(x_1 - x_2) = \left\langle \tilde{T} \hat{A}(x_1) \hat{A}(x_2) \right\rangle_T
\]

Leading-order result by HTL resummed perturbation theory
Approximations

• Heavy Mass Limit

Non-relativistic kinetic term

\[ S[\psi] \approx S^\text{NR}_{\text{kin}}[Q, Q_c] \quad \psi \sim (Q, Q_c^\dagger) \]

\[ S^\text{NR}_{\text{kin}}[Q, Q_c] = Q^\dagger[i\partial_0 - M + \nabla^2/2M]Q \]

\[ + Q_c^\dagger[i\partial_0 + M + \nabla^2/2M]Q_c^\dagger \]

Non-relativistic 4-current (density, current)

\[ j_a^0 = Q^\dagger t^a Q + Q_c t^a Q_c^\dagger \equiv \rho_a \]

\[ \bar{j}_a \approx Q^\dagger \left( \frac{\nabla}{2iM} \right) t^a Q - Q_c \left( \frac{\nabla}{2iM} \right) t^a Q_c^\dagger \]

(quenched)

\[ \nabla Q \sim \sqrt{MT} \cdot Q \]

\[ \leftrightarrow \nabla G \sim (g)T \cdot G \]

Expansion up to

\[ \sim \sqrt{\frac{T}{M}} \]
Approximations

• **Long-Time Behavior**

Time-retardation in interaction

\[
\tilde{G}(\tilde{x} - \tilde{y}, \omega) \approx \tilde{G}(\tilde{x} - \tilde{y},0) + \omega \tilde{G}'(\tilde{x} - \tilde{y},0) \equiv \overline{G}(\tilde{x} - \tilde{y}) + \omega \overline{G}'(\tilde{x} - \tilde{y})
\]

\[
\Leftrightarrow G(x - y) \approx \overline{G}(\tilde{x} - \tilde{y}) \delta(x^0 - y^0) + i \overline{G}'(\tilde{x} - \tilde{y}) \frac{\partial}{\partial (x^0 - y^0)} \delta(x^0 - y^0)
\]

Low frequency expansion

\[
\int j(x)G(x - y) j(y) \approx \int \int_{t \tilde{x} \tilde{y}} \left[ \overline{G}(\tilde{x} - \tilde{y}) j(t, \tilde{x}) j(t, \tilde{y}) - \frac{i}{2} \overline{G}'(\tilde{x} - \tilde{y}) \left\{ \partial_0 j(t, \tilde{x}) j(t, \tilde{y}) - j(t, \tilde{x}) \partial_0 j(t, \tilde{y}) \right\} \right]
\]

Using free equation of motion
Effective Action

- LO pQCD, NR Limit, Slow Dynamics

\[ S_{1+2} \approx S_{\text{kin}}^{\text{NR}} [Q_1, Q_{1c}] - S_{\text{kin}}^{\text{NR}} [Q_2, Q_{2c}] + S_{\text{FV}}^{\text{LONR}} [j_1, j_2] \]

\[ S_{\text{FV}}^{\text{LONR}} [j_1, j_2] = -\frac{1}{2} \int_{t,\tilde{x},\tilde{y}} \left\{ V(\tilde{x} - \tilde{y}) \rho_{1a}(t, \tilde{x}) \rho_{1a}(t, \tilde{y}) - V^*(\tilde{x} - \tilde{y}) \rho_{2a}(t, \tilde{x}) \rho_{2a}(t, \tilde{y}) \right\} \]

\[ + \int_{t,\tilde{x},\tilde{y}} \left\{ iD(\tilde{x} - \tilde{y}) \rho_{1a}(t, \tilde{x}) \rho_{2a}(t, \tilde{y}) - \frac{1}{4} T \nabla D(\tilde{x} - \tilde{y}) \cdot \left( \vec{j}_{1a}(t, \tilde{x}) \rho_{2a}(t, \tilde{y}) + \rho_{1a}(t, \tilde{x}) \vec{j}_{2a}(t, \tilde{y}) \right) \right\} \]

\[ - g^2 \left\{ \overline{G}_{00,ab}^{R}(\tilde{x} - \tilde{y}) + i \overline{G}_{00,ab}^{>}(\tilde{x} - \tilde{y}) \right\} \equiv V(\tilde{x} - \tilde{y}) \delta_{ab} \]

\[ - g^2 \overline{G}_{00,ab}(\tilde{x} - \tilde{y}) \equiv D(\tilde{x} - \tilde{y}) \delta_{ab} = \text{Im} V(\tilde{x} - \tilde{y}) \delta_{ab} \]

Stochastic potential (finite in $M \to \infty$)

Drag force (vanishes in $M \to \infty$)
Hamiltonian Formalism (technical)

• Order of Operators = Time Ordered

\[ \psi_1^*(t_+, \bar{x}) \Leftrightarrow \psi_1(t_-, \bar{x}), \ \psi_2^*(t_-, \bar{x}) \Leftrightarrow \psi_2(t_+, \bar{x}) \]

Kinetic term

Instantaneous interaction

or

Remember the original order

• Change of Variables (canonical transformation)

Make 1 & 2 symmetric

\[ \tilde{\psi}_2 = (\bar{Q}_2, \bar{Q}_2^*) = \psi_2^* = (Q_2^*, Q_{2c}) \]

Determines \( H_{1+2}^{\text{func}} [\psi_1^*, \tilde{\psi}_2^*] \) without ambiguity
Hamiltonian Formalism (technical)

• Variables of Reduced Density Matrix

\[ \rho_{\text{red}}[t, \psi_1^*, \tilde{\psi}_2^*] = \langle \psi_1^* | \hat{\rho}_{\text{red}}(t) | \tilde{\psi}_2^* \rangle \]

Latter is better (explained later)

\[ \Leftrightarrow \rho_{\text{red}}[t, Q_{1(c)}^*, \tilde{Q}_{2(c)}^*] = \langle Q_{1(c)}^* | \hat{\rho}_{\text{red}}(t) | \tilde{Q}_{2(c)}^* \rangle \]

• Renormalization

Convenient to move all the functional differential operators to the right in

\[ i \frac{\partial}{\partial t} \rho_{\text{red}}[t, Q_{1(c)}^*, \tilde{Q}_{2(c)}^*] = H_{1+2}^{\text{func}}[Q_{1(c)}^*, \tilde{Q}_{2(c)}^*] \rho_{\text{red}}[t, Q_{1(c)}^*, \tilde{Q}_{2(c)}^*] \]

In this procedure, divergent contribution from Coulomb potential at the origin appears \( \rightarrow \) needs to be renormalized
Functional Master Equation

- Renormalized Effective Hamiltonian

\[
\hat{H}_{1+2} \approx \int_{\vec{x}} \left[ a M \hat{Q}_1^\dagger \hat{Q}_1 + \hat{Q}_1^\dagger \left( -\nabla^2 / 2M \right) \hat{Q}_1 \right] + \text{(same for } \hat{Q}_{1c} \text{)} \\
+ \int_{\vec{x}} \left[ a^* M \hat{Q}_2^\dagger \hat{Q}_2 + \hat{Q}_2^\dagger \left( -\nabla^2 / 2M \right) \hat{Q}_2 \right] + \text{(same for } \hat{Q}_{2c} \text{)}
\]

\[
= \frac{1}{2} \int_{\vec{x} \vec{y}} \left[ V(\vec{x} - \vec{y}) N\left\{ \hat{j}_1^{a0}(\vec{x}) \hat{j}_1^{a0}(\vec{y}) \right\} - V^*(\vec{x} - \vec{y}) N\left\{ \hat{j}_2^{a0}(\vec{x}) \hat{j}_2^{a0}(\vec{y}) \right\} \\
- 2iD(\vec{x} - \vec{y}) N\left\{ \hat{j}_1^{a0}(\vec{x}) \hat{j}_2^{a0}(\vec{y}) \right\} \\
+ \frac{1}{2T} \vec{\nabla} D(\vec{x} - \vec{y}) \cdot N\left\{ \hat{\tilde{j}}_1^a(\vec{x}) \hat{j}_2^{a0}(\vec{y}) + \hat{j}_1^{a0}(\vec{x}) \hat{\tilde{j}}_2^a(\vec{y}) \right\} \right]
\]

\[
a = 1 + \frac{C_F}{2M} \lim_{r \to 0} V^{(T>0)}(r), \quad V^{(T>0)}(r) \equiv V(r) - V^{(T=0)}(r)
\]
Functional Master Equation

• Schrödinger wave equation

Anti-commutator in functional space

\[ \{ \hat{Q}_1 (\bar{x}), \hat{Q}_1^\dagger (\bar{y}) \} = \{ \hat{Q}_{1c} (\bar{x}), \hat{Q}_{1c}^\dagger (\bar{y}) \} = \delta (\bar{x} - \bar{y}) \iff Q_{1(c)} = \frac{\delta}{\delta \hat{Q}_{1(c)}^*} \]

\[ \{ \hat{Q}_2 (\bar{x}), \hat{Q}_2^\dagger (\bar{y}) \} = \{ \hat{Q}_{2c} (\bar{x}), \hat{Q}_{2c}^\dagger (\bar{y}) \} = -\delta (\bar{x} - \bar{y}) \iff \hat{Q}_{2(c)} = -\frac{\delta}{\delta \hat{Q}_{2(c)}^*} \]

\[ \hat{H}_{1+2} \iff H_{1+2}^{\text{func}} \]

\[ i \frac{\partial}{\partial t} \rho_{\text{red}} [t, Q_{1(c)}, \tilde{Q}_{2(c)}^*] = H_{1+2}^{\text{func}} [Q_{1(c)}^*, \tilde{Q}_{2(c)}^*] \rho_{\text{red}} [t, Q_{1(c)}, \tilde{Q}_{2(c)}^*] \]
Other Results

• Stochastic Dynamics

\( M=\infty \): Stochastic potential

\[
\exp \biggl[ i S_{\text{FV}}^{\text{LONR}} [ j_1, j_2 ] \biggr] = \exp \left[ -i/2 \int_{t, \bar{x}, \bar{y}} \text{Re} V(\bar{x} - \bar{y}) \{ \rho_{1a}(t, \bar{x}) \rho_{1a}(t, \bar{y}) - \rho_{2a}(t, \bar{x}) \rho_{2a}(t, \bar{y}) \} \right] 
\times \left\langle \exp \left[ -i \int_{t, \bar{x}, \bar{y}} \xi_a(t, \bar{x}) \{ \rho_{1a}(t, \bar{x}) - \rho_{2a}(t, \bar{x}) \} \right] \right\rangle_{\xi}
\]

\[
\left\langle \xi_a(t, \bar{x}) \xi_b(s, \bar{y}) \right\rangle = -\delta_{ab} \delta(t-s) D(\bar{x} - \bar{y}) \quad D(x-y): \text{Negative definite}
\]

\( M<\infty \): Drag force

Two complex noises \( c_1, c_2 \)
\( \rightarrow \) Non-hermitian evolution

\[
\rho_Q(t, \bar{x}, \bar{y}) = \left\langle \Psi(t, \bar{x}) \tilde{\Psi}(t, \bar{y}) \right\rangle_{\xi c_1, c_2}
\]

\( \tilde{\Psi}(t, \bar{x}) \not\Rightarrow \Psi^*(t, \bar{x}) \)