Making Byzantine Decentralized Learning Efficient

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Abstract

Decentralized-SGD (D-SGD) distributes heavy learning tasks across multiple machines (a.k.a., nodes), effectively dividing the workload per node by the size of the system. However, a handful of Byzantine (i.e., misbehaving) nodes can jeopardize the entire learning procedure. This vulnerability is further amplified when the system is asynchronous. Although approaches that confer Byzantine resilience to D-SGD have been proposed, these significantly impact the efficiency of the process to the point of even negating the benefit of decentralization. This naturally raises the question: can decentralized learning simultaneously enjoy Byzantine resilience and reduced workload per node?

We answer positively by proposing MoNNA that ensures Byzantine resilience without losing the computational efficiency of D-SGD. Essentially, MoNNA weakens the impact of Byzantine nodes by reducing the variance in local updates using Polyak’s momentum. Then, by establishing coordination between nodes via signed echo broadcast and a nearest-neighbor averaging scheme, we effectively tolerate Byzantine nodes whilst distributing the overhead amongst the non-Byzantine nodes. To demonstrate the correctness of our algorithm, we introduce and analyze a novel Lyapunov function that accounts for the non-Markovian model drift arising from the use of momentum. We also demonstrate the efficiency of MoNNA through experiments on several image classification tasks.

1 Introduction

The increasing complexity of modern-day machine learning (ML) has led to the emergence of distributed learning methodologies [1, 29]. These essentially consist in distributing the expensive task of computing gradients over multiple machines (a.k.a., nodes) [5, 48]. In doing so, the computational workload gets divided by the total number of nodes in the system. Typically, this requires the nodes to coordinate their actions through trusted parameter servers, e.g., by sharing their gradients or models. However, the parameter-server architecture induces a single point of vulnerability: the server could get hacked by malicious entities or simply crash. In that case, the whole learning procedure is compromised [16].

∗Alphabetical order.
The need for central servers can be alleviated by adopting a decentralized peer-to-peer scheme where the nodes coordinate by communicating directly with each other [37]. Specifically, in the decentralized version of SGD (or D-SGD), each node maintains its own local model, which is updated in each iteration in two steps. First, each node performs a local model update using its locally computed gradient. Second, the nodes coordinate by broadcasting their local models and then average all received models. Remarkably, even though the execution of D-SGD does not depend on the presence of any parameter server, the workload per node is still divided by the size of the system [33, 37]. Nevertheless, even this decentralized scheme is vulnerable to faulty or misbehaving machines, a.k.a. Byzantine nodes [46]. The problem of robustness against Byzantine nodes has received significant attention in the parameter sever architecture [2, 3, 10, 14, 18, 26, 43, 44, 49, 55]. Yet, it remains largely understudied in the decentralized setting, especially when the system is not assumed synchronous [15, 51, 12, 52]. For additional discussion on related work, refer to Section 6.

A few prior works addressed the challenge of Byzantine resilience in asynchronous decentralized learning [15, 16, 28]. However, in [28] the loss function is assumed to be strongly convex, which is seldom true in modern-day ML. Furthermore, in [16] convergence is only guaranteed asymptotically. Lastly, the algorithm proposed in [15] inflicts a prohibitively large computational overhead on honest (i.e., non-Byzantine) nodes: each honest node computes $O\left(\frac{1}{\epsilon^2}\right)$ gradients to attain $\epsilon$-stationarity. This nullifies the computational benefit of distributing the learning process. Indeed, an honest node can solve the learning problem more efficiently using only $O\left(\frac{1}{\epsilon}\right)$ gradients by simply running SGD locally (a.k.a., local SGD), without coordination. This naturally raises an important question:

**Can asynchronous decentralized learning simultaneously enjoy Byzantine resilience and reduced computational workload per node?**

**Contributions.** We answer this question positively by introducing MoNNA, a new asynchronous decentralized learning algorithm. In a system of $n$ nodes including $f$ Byzantines, an honest node executing MoNNA computes $O\left(\frac{1+f}{n\epsilon^2}\right)$ gradients to reach $\epsilon$-stationarity, which is comparable to D-SGD when $f \ll n$. Essentially, we grant Byzantine robustness to D-SGD by incorporating three key components.

1. **Local momentum.** For the local model updates, each honest node uses the Polyak’s momentum of its gradients [42]. This reduces the variance of the updates, which proves crucial for tolerating Byzantine information when the nodes exchange their models.

2. **Signed Echo Broadcast (SEB).** To enable coordination amongst honest nodes, key to reducing the workload, we use the SEB communication primitive [9]. This prevents Byzantine nodes from sending mismatching messages, thereby increasing the tolerable fraction of such nodes.

3. **Nearest-Neighbor Averaging (NNA).** Drawing inspiration from robust optimization [25, 26, 28], we propose a new aggregation scheme named NNA. Each honest node filters out the $f$ models it receives that are the farthest from its own, and averages the remainders. The filtering mitigates the impact of Byzantine models, while the averaging reduces the computational overhead.

**Technical challenges.** The convergence analysis of MoNNA reveals challenging mainly for two reasons. First, although momentum is essential in reducing the variance of local updates, it induces a stochastic bias and a non-Markovian drift in honest nodes’ local model updates. This makes existing techniques for analyzing the convergence of SGD with momentum (e.g., see [39, 56])
inapplicable to our decentralized setting. Second, using a non-linear aggregation rule such as NNA in our coordination step further complicates the analysis. In particular, we can no longer rely on previous proof techniques assuming linear aggregation with doubly stochastic matrices, such as in \cite{32,33,35}. To address these challenges, we devise a new proof technique relying on a novel Lyapunov function (formally described in Section 4.1). We believe our technical contribution to be of independent interest to the field of decentralized optimization.

**Empirical evaluation.** We empirically compare MoNNA against D-SGD, local SGD, and the state-of-the-art algorithm for Byzantine resilient asynchronous decentralized ML, RB-TM \cite{15}. MoNNA significantly outperforms RB-TM both in terms of Byzantine robustness and computational efficiency. In fact, under strong attacks \cite{3,4,50}, we observe that RB-TM can be rendered practically ineffective, despite its theoretical guarantees \cite{15}. To be more precise, RB-TM fails to learn on CIFAR-10 \cite{34} even after computing over $10^6$ gradients, while MoNNA reaches over 80% accuracy by using 20 times fewer gradients on the same task. In all our experiments, MoNNA almost matches the performance of D-SGD both in terms of learning accuracy and computational cost. Predictably, our algorithm also outperforms local SGD.

## 2 Preliminaries: Background and Problem Formulation

We consider a peer-to-peer system comprising $n$ nodes, represented by the set of indices $[n] = \{1, \ldots, n\}$. Let $\mathcal{X}$ be the space of data points, and $\mathcal{D}$ be the data generating distribution defined over $\mathcal{X}$. For a given learning model parameterized by vector $\theta \in \mathbb{R}^d$, each data point $x \in \mathcal{X}$ incurs a loss $q(\theta, x)$ where $q : \mathbb{R}^d \times \mathcal{X} \to \mathbb{R}$. Each node $i$ aims to compute a model parameter $\hat{\theta}^{(i)}$ that minimizes the global loss function defined as

$$Q(\theta) := \mathbb{E}_{x \sim \mathcal{D}}[q(\theta, x)]. \quad (1)$$

We assume the point-wise loss function $q$ to be differentiable. Also note that, as the loss function $Q(\theta)$ need not be convex, the above optimization problem is NP-hard in general \cite{7}. Thus, a more reasonable goal is to design a learning algorithm that enables each node to compute a parameter vector that is a critical (or stationary) point of $Q(\theta)$. We formally define this objective below.

**Definition 1.** An algorithm (possibly randomized) is said to guarantee $\epsilon$-stationarity if it enables each node $i \in [n]$ to compute a parameter vector $\hat{\theta}^{(i)}$ such that

$$\mathbb{E} \left[ \| \nabla Q(\hat{\theta}^{(i)}) \|^2 \right] \leq \epsilon,$$

where the expectation $\mathbb{E}[\cdot]$ is taken over the randomness of the algorithm.

### 2.1 Decentralized-SGD (D-SGD)

The aforementioned learning problem can be trivially solved by each node using the SGD algorithm locally (a.k.a., local SGD). However, by coordinating their actions as per the D-SGD algorithm presented below, the nodes can significantly reduce their individual computational workloads \cite{32,37}. In each iteration $t = 1, 2, \ldots$, every node $i$ maintains a local parameter vector $\theta_t^{(i)}$ that is updated using the two steps described below \cite{37}.
1. **Local update step.** Each node $i$ independently samples a data point $x \sim \mathcal{D}$ and computes a stochastic estimate $g^{(i)}_t$ of the true gradient $\nabla Q \left( \theta^{(i)}_t \right)$ as

$$g^{(i)}_t = \nabla q \left( \theta^{(i)}_t, x \right).$$

Then, it partially updates its model as $\theta^{(i)}_{t+1/2} = \theta^{(i)}_t - \gamma g^{(i)}_t$, where $\gamma > 0$ is the learning rate.

2. **Coordination step.** Then, the nodes communicate their partially updated parameter vectors $\{\theta^{(i)}_{t+1/2}, i \in [n]\}$ to each other. Then, each node $i$ computes its new updated model $\theta^{(i)}_{t+1}$ as follows,

$$\theta^{(i)}_{t+1} = \frac{1}{n} \sum_{i=1}^n \theta^{(i)}_{t+1/2}.$$  

**Remark 1.** For simplicity of presentation, we described D-SGD assuming a synchronous system. However, this paper considers a more general setting of asynchrony, detailed below in Section 2.4.

### 2.2 Convergence of Decentralized-SGD

Similarly to SGD, the convergence of D-SGD is usually studied under the following classical assumptions that hold true for many learning problems [6].

**Assumption 1** (Lipschitz smoothness). There exists $L < \infty$ such that for all $\theta_1, \theta_2 \in \mathbb{R}^d$,

$$\|\nabla Q(\theta_1) - \nabla Q(\theta_2)\| \leq L \|\theta_1 - \theta_2\|.$$  

Note that, from (1), we have $\mathbb{E}_{x \sim \mathcal{D}} [\nabla q(\theta, x)] = \nabla Q(\theta)$.

**Assumption 2** (Bounded variance). There exists $\sigma < \infty$ such that for all $\theta \in \mathbb{R}^d$,

$$\mathbb{E}_{x \sim \mathcal{D}} \left[ \|\nabla q(\theta, x) - \nabla Q(\theta)\|^2 \right] \leq \sigma^2.$$  

The update step for each node in the D-SGD algorithm above can be shown equivalent to the centralized mini-batch SGD algorithm with a batch-size of $n$. Thus, thanks to the existing results on the convergence of mini-batch SGD [6], we obtain that if D-SGD is run for $T$ iterations and each node $i$ outputs $\hat{\theta}^{(i)} \sim \mathcal{U}\left\{ \theta^{(i)}_1, \ldots, \theta^{(i)}_T \right\}$, the algorithm achieves $O\left(1/\sqrt{nT}\right)$-stationarity. Recall that each node computes one gradient per iteration. Accordingly, D-SGD guarantees $\epsilon$-stationarity by having each node compute $O\left(1/n^2\epsilon^2\right)$ gradients, which is $n$ times more efficient than local SGD.

**Remark 2.** Note that D-SGD can be shown to have the same computational efficiency even when the system is asynchronous [58].

$1\mathcal{U}\left\{ \theta^{(i)}_1, \ldots, \theta^{(i)}_T \right\}$ denotes the uniform distribution on parameter vectors $\left\{ \theta^{(i)}_1, \ldots, \theta^{(i)}_T \right\}$.
2.3 Byzantine resilience

The above convergence analysis for D-SGD assumes the system to be fault-free. However, this assumption is seldom satisfied in practice where some nodes may exhibit faults, e.g., they might either crash or send incorrect values. To study robustness against such faulty nodes, we consider a scenario where \( f \) out of \( n \) nodes are Byzantine, i.e., they may deviate arbitrarily from the prescribed protocol \([35, 46]\). Our objective remains the same, except that now we only aim for the non-faulty (a.k.a., honest) nodes to compute a stationary point of the global loss function \([25]\).

**Definition 2.** An algorithm is said to guarantee \((f, \epsilon)\)-approximate Byzantine resilient learning if despite the presence of \( f \) Byzantine nodes, it enables each honest node \( i \) to compute \( \hat{\theta}^{(i)} \) such that

\[
\mathbb{E} \left[ \left\| \nabla Q \left( \hat{\theta}^{(i)} \right) \right\|^2 \right] \leq \epsilon,
\]

where the expectation \( \mathbb{E} [\cdot] \) is taken over the randomness of the algorithm.

The above problem can again be trivially solved by each honest node simply using local SGD. However, we are interested in designing a solution that not only ensures Byzantine learning, but also reduces the computational workload for honest nodes (as for D-SGD).

2.4 Asynchrony

In a practical setting, the assumption of synchrony is seldom satisfied. Typically, a real system comprises nodes with heterogeneous computing capabilities that might undergo sudden slowness. A standard way to model such a system is to assume communication delays in the network to be unknown and a priori unbounded \([21]\). Obviously, under asynchrony, a node cannot wait to receive messages from all the other nodes since it can be indefinitely stalled by a single Byzantine node that chooses to never send the expected message. In this paper, we use the asynchronous round-based model proposed by \([8]\), where each node only waits for \( n - f \) messages (including the message from itself). Then, each node is guaranteed to eventually terminate every communication round, and receive at least \( n - 2f \) messages from honest nodes. However, as a consequence of asynchrony, two honest nodes may receive different sets of messages in any communication round. Note that the asynchronous system with a complete communication topology, that we consider, can be modeled by a synchronous system with a dynamic incomplete topology (e.g., see \([32]\)). Therefore, our results on the convergence of MoNNA in Section \([3]\) also hold for certain incomplete topologies under synchrony.

3 MoNNA: Our Algorithm

MoNNA is our new decentralized learning algorithm, enhancing D-SGD with three critical elements, namely Polyak’s momentum, Signed Echo Broadcast (SEB), and Nearest-Neighbor Averaging (NNA). We present these key features and then describe MoNNA in Algorithm \([1]\).

**Polyak’s momentum.** Recall that in each iteration \( t \) of D-SGD, each node \( i \) updates its local parameter vector \( \theta^{(i)}_t \), using a stochastic gradient \( g^{(i)}_t \) defined in \([2]\). We propose to modify this update rule performing local updates using the Polyak’s momentum \([12]\) of the honest gradients instead. To do so, in each iteration \( t \), each honest node \( i \) maintains a momentum vector

\[
m^{(i)}_t := \beta m^{(i)}_{t-1} + (1 - \beta) g^{(i)}_t,
\]

where \( 0 < \beta < 1 \) is a momentum parameter.
This scheme reduces the variance of honest nodes’ local updates, which proves essential to tolerate Byzantine information upon model aggregation. By the SEB primitive, we establish the communication consistency between honest nodes in the coordination step \cite[Section 3.10.4]{9}. This allows us to enhance tolerance to Byzantine nodes from sending mismatching messages to different nodes. SEB only imposes a linear message complexity of $O(n)$ on each honest node, which is identical to that of D-SGD. As the details of SEB are not crucial to the description of MoNNA, we defer them to Appendix A.

**Signed Echo Broadcast (SEB).** We rely on the SEB primitive to establish the communication between honest nodes in the coordination step \cite[Section 3.10.4]{9}. This allows us to enhance MoNNA’s tolerance to Byzantine nodes. Basically, when $n > 3f$, SEB ensures consistency, i.e., it prevents Byzantine nodes from sending mismatching messages to different nodes. SEB only imposes a linear message complexity of $O(n)$ on each honest node, which is identical to that of D-SGD. As the details of SEB are not crucial to the description of MoNNA, we defer them to Appendix A.

**Nearest-Neighbor Averaging (NNA).** We propose NNA, a scheme for aggregating the locally updated parameters exchanged among nodes. Essentially, given a reference vector $z^{(0)}$ and a set of $n - f - 1$ input vectors $Z := \{z^{(1)}, \ldots, z^{(n-f-1)}\}$, NNA outputs the average of the $n - 2f$ nearest neighbors of $z^{(0)}$ in $Z \cup \{z^{(0)}\}$. Specifically, let $\tau$ denote a permutation on $\{1, \ldots, n - f - 1\}$ that sorts in non-decreasing order the vectors in $Z$ based on their distance to $z^{(0)}$. This means that $\|z^{(0)} - z^{(\tau(1))}\| \leq \ldots \leq \|z^{(0)} - z^{(\tau(n-f-1))}\|$. Then, NNA is defined as:

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**Algorithm 1: MoNNA as executed by an honest node $i$**

**Initialization:** First model estimate $\theta_t^{(i)} \in \mathbb{R}^d$, initial momentum $m_0^{(i)} = 0$, momentum coefficient $\beta \in [0, 1)$, total iterations $T$, learning rate $\gamma$, number of coordination rounds $K$, and Byzantine threshold $f$.

In each iteration $t = 1, \ldots, T$, do the following steps:

1. **Local update step:**
   
   a) Update local momentum $m_t^{(i)} = \beta m_{t-1}^{(i)} + (1 - \beta) g_t^{(i)}$ where $g_t^{(i)}$ is defined as in (2).
   
   b) Partially update local model $\hat{\theta}^{(i)}_{t+1/2} := \theta_t^{(i)} - \gamma m_t^{(i)}$.

2. **Coordination step:** Initialize vector $x_0^{(i)} := \theta_{t+1/2}^{(i)}$ and execute the following $K$ coordination rounds.
   
   a) In each coordination round $k = 1, \ldots, K$, do the following
   
   i. Initialize $R_k^{(i)} = \emptyset$.
   
   ii. Send vector $x_{k-1}^{(i)}$ to the other nodes using the Signed Echo Broadcast (SEB) primitive. \textit{(A Byzantine node $j$ may send an arbitrary value for $x_{k-1}^{(j)}$)}
   
   iii. While $|R_k^{(i)}| < n - f - 1$ do:
   
   Upon receiving a vector from node $j$ via SEB, update $R_k^{(i)} = R_k^{(i)} \cup \{j\}$.
   
   iv. Compute $x_k^{(i)} = \text{NNA} \left( x_{k-1}^{(i)}; \{x_{k-1}^{(j)} \mid j \in R_k^{(i)}\} \right)$, as defined in (6).

   b) Update local model $\theta_{t+1}^{(i)} = x_K^{(i)}$.

**Output:** $\hat{\theta}^{(i)} \sim \mathcal{U} \left\{ \theta_1^{(i)}, \ldots, \theta_T^{(i)} \right\}$.

where the parameter $\beta \in [0, 1)$ is the \textit{momentum coefficient}, and $m_0^{(i)} = 0$ by convention. Then, each honest node $i$ partially updates its parameter $\theta_t^{(i)}$ as follows

$$
\theta_{t+1/2}^{(i)} := \theta_t^{(i)} - \gamma m_t^{(i)}.
$$

(5)

This scheme reduces the variance of honest nodes’ local updates, which proves essential to tolerate Byzantine information upon model aggregation.
\[
NNA \left( z^{(0)}; \left\{ z^{(1)}, \ldots, z^{(n-f-1)} \right\} \right) = \frac{\sum_{i=0}^{n-2f-1} z^{(\tau(i))}}{n-2f},
\]

where \( z^{(\tau(0))} = z^{(0)} \) by convention. The rationale for using NNA is threefold. First, by essentially mimicking the averaging scheme on honest nodes, NNA preserves the D-SGD property of dividing the workload of each node by the total number of nodes in the system. Second, although the decentralized framework is much harder to analyse theoretically than the parameter-server setting, it has a hidden advantage. In particular, each honest node has access to a vector that is guaranteed to be honest (i.e., its own). NNA leverages this advantage by making every honest node use its own vector as a pivot to filter out outliers. Finally, NNA is computationally efficient as it can be computed in \( O(nd) \). More precisely, NNA requires \( O(nd) \) computations to evaluate the distances, \( O(n \log n) \) to sort them, and \( O(nd) \) to compute the final average. Therefore, the total complexity is \( O(n(d + \log n)) \), which almost matches the \( O(nd) \) complexity of the simple averaging scheme. Note that in most modern machine learning settings, \( d \) is very much larger than \( n \), making this \( \log(n) \) factor negligible.

**MoNNA.** Similarly to D-SGD, in each learning iteration \( t \), the honest nodes update their local parameter vectors in two steps. First, each honest node \( i \) partially updates its parameter \( \theta_t^{(i)} \) using its momentum vector \( m_t^{(i)} \), as shown in (5). Second, each honest node \( i \) initializes a local vector \( x_{0}^{(i)} = \theta_{t+1/2}^{(i)} \) and engages in \( K \) coordination rounds. In each round \( k \in [K] \), every honest node \( i \) starts by broadcasting \( x_{k-1}^{(i)} \) using the SEB primitive, and waits to receive at least \( n-f-1 \) vectors from other nodes (also via SEB). Upon gathering these vectors, each node updates its model using NNA (6). We detail the overall learning procedure in Algorithm 1.

### 4 Theoretical Analysis

We present here our main theoretical result demonstrating the finite time convergence of Algorithm 1. As a corollary, we derive the result that Algorithm 1 reaches \((f, \epsilon)-Byzantine resilience.\)

#### 4.1 Main convergence result

We first present our main result in Theorem 1 below. Essentially, we analyze Algorithm 1 under assumptions 4 and 2, and upon assuming a sufficiently small learning rate \( \gamma \). In short, when \( n \geq 11f \), i.e., when the fraction of Byzantine players is not too large, it suffices to perform one round per coordination step (i.e., \( K = 1 \)) to obtain convergence. More precisely, let \( Q^* = \min_{\theta \in \mathbb{R}^d} Q(\theta) \) and \( \theta_t := \frac{1}{|\mathcal{H}|} \sum_{i \in \mathcal{H}} \theta_t^{(i)} \), where \( \mathcal{H} \) denotes the set of honest nodes. We get the following.
Theorem 1. Suppose that assumptions 1 and 2 hold true, and that $n \geq 11f$. Let us denote

\[
\alpha := \frac{9.88f}{n-f}, \quad \lambda := \frac{9f}{n-f},
\]

\[
c_1 := \left( \frac{1}{n-f} + 9\lambda + 3\alpha(1 + \alpha) \right) \left( 8 + 27\lambda \right),
\]

\[
c_2 := 8L \left( Q(\overline{\theta}) - Q^* \right), \quad \text{and} \quad c_3 := \frac{18\alpha(1 + \alpha)}{(1 - \alpha)^2}.
\]

Consider Algorithm 1 with $T \geq c_2 \sigma^2 c_1 \max\{1, c_3\}$, $K = 1$, $\gamma = \frac{1}{12L} \sqrt{\frac{c_2}{\sigma^2 c_1}}$, and $\beta = \sqrt{1 - 12\gamma L}$. Then, for each honest node $i \in \mathcal{H}$, we obtain that

\[
\mathbb{E} \left[ \|\nabla Q(\hat{\theta}^{(i)})\|^2 \right] \leq \frac{36\sqrt{\sigma^2 c_1 c_2}}{T^{3/2}} + \left( \frac{c_2 c_3}{24c_1} \right) \frac{n-f}{T}
\]

\[
+ 36 \sqrt{\frac{\sigma^2 c_2}{(n-f)^2 c_1}} \frac{1}{T^{3/2}}.
\]

Proof sketch. We identify two key challenges in the proof. (1) The use of momentum in local model updates leads to stochastic bias (or deviation), resulting in a non-Markovian dynamics of the drift in honest nodes’ models. (2) The aggregation scheme of NNA in the coordination step is non-linear and also non-deterministic with respect to the honest nodes’ inputs. These peculiarities render the existing Lyapunov functions, designed for studying the convergence of standard D-SGD (e.g., see [32, 37]), inapplicable to our algorithm. We thus introduce a novel Lyapunov function

\[
V_t := \mathbb{E} \left[ Q(\overline{\theta}_t) + \frac{1}{4L} \| \overline{m}_t - \nabla Q_t \|^2 \right]
\]

where $\overline{m}_t = \frac{1}{n-f} \sum_{i \in \mathcal{H}} m_t^{(i)}$ and $\nabla Q_t := \frac{1}{n-f} \sum_{i \in \mathcal{H}} \nabla Q(\hat{\theta}_t^{(i)})$. The first term, i.e., $Q(\overline{\theta}_t)$, is a remnant of the traditional Lyapunov function. The second term containing $\| \overline{m}_t - \nabla Q_t \|^2$ is what we introduce to study the impact of deviation and non-Markovian model drift. For studying the growth of $V_t$, we derive four intermediate sub-results. i) The contraction and robust mean estimation properties of our coordination step. ii) A bound on the growth of honest nodes’ model drift. iii) A bound on the growth of momentum deviation. iv) A bound on the growth of $Q(\overline{\theta}_t)$. We combine these results to finally study the growth of $V_t$ and obtain the stated result. \qed

Extension to heterogeneity. When the honest nodes do not have identical data distributions, achieving Byzantine resilience becomes much more challenging [15]. Nevertheless, we also analyze the convergence of our algorithm under data heterogeneity, and obtain an error matching the existing lower bound [31]. The result is deferred to Appendix B for pedagogical reasons.

Applicability to incomplete topologies. We present the convergence of MoNNA in Theorem 1 for the complete communication topology under asynchrony. Interestingly, under synchrony, our result also applies directly to a more general class of graphs, namely arbitrarily dynamic $k$-regular topologies with $k \geq n-f$. Indeed, the communication topology may change
## Byzantine tolerance

Local SGD

\[ O \left( \frac{1}{\epsilon^2} \right) \]

Any \( f < n \)

D-SGD

\[ O \left( \frac{1}{n \epsilon^2} \right) \]

\[ O \left( \frac{1}{\epsilon^2} \right) \]

\( f = 0 \)

MoNNA-11

\[ O \left( \frac{1+f}{n \epsilon^2} \right) \]

\[ O \left( \frac{1+f}{\epsilon^2} \right) \]

\( 11f \leq n \)

MoNNA-5

\[ O \left( \frac{(1+f)^2}{n \epsilon^2} \right) \]

\[ O \left( \frac{(1+f)^2 \log(n)}{\epsilon^2} \right) \]

\( 5f < n \)

### Table 1: Comparison of the number of gradient computations and number of messages sent per node to achieve \((f, \epsilon)\)-approximate Byzantine resilient learning for MoNNA, local SGD, and D-SGD. MoNNA-11 and MoNNA-5 denote MoNNA with \( n \geq 11f \) and \( n > 5f \) respectively.

## Computational workload per node

Note that the convergence rate of MoNNA is inversely proportional to \( \sqrt{n} \). This primarily comes from the fact that NNA averages a subset of the honest vectors, hence dividing the workload per node in the same manner averaging does in D-SGD. However, contrarily to D-SGD, MoNNA operates in the presence of Byzantine nodes, which also makes the rate proportional to \( \sqrt{1+f} \). Essentially, MoNNA reaches \( \epsilon \)-stationarity with a workload per node in \( O \left( \frac{1+f}{n \epsilon^2} \right) \), compared to \( O \left( \frac{1}{n \epsilon^2} \right) \) for D-SGD and \( O \left( \frac{1}{\epsilon^2} \right) \) for local SGD (see Table 1). When Byzantine nodes represent a negligible fraction of the system \( (f \ll n) \), MoNNA benefits from the same reduction in the workload per node as D-SGD, compared to local SGD. This is in sharp contrast with RB-TM [15] that requires \( O \left( \frac{1}{\epsilon^2} \right) \) gradient computations to reach \( \epsilon \)-stationarity, which is several orders of magnitude larger than the \( O \left( \frac{1}{\epsilon^2} \right) \) of local SGD.

## Communication complexity

While signed echo broadcast guarantees consistency, it only requires each honest node to communicate \( O(n) \) times. Furthermore, MoNNA requires only one communication round per iteration (i.e., \( K = 1 \)), achieving a per node message complexity of \( O \left( \frac{1+f}{\epsilon^2} \right) \), almost matching the number of messages sent in D-SGD. On the other hand, [15] performs \( O(n^2 \log(1/\epsilon)) \) communications per node in every iteration, which arguably makes the message complexity of the algorithm significantly larger than that of D-SGD.

## Byzantine resilience

We assume in both Theorem 1 and Corollary 1 that \( n \geq 11f \). Yet, we can still show Byzantine resilience of MoNNA whenever \( n > 5f \) (as summarized in Table 1). A formal statement of this result can be found in Appendix B. However, to obtain Byzantine resilience...
in this case, we need to set $K = \mathcal{O}(\log(n))$, effectively increasing the number of coordination rounds. Furthermore, when $5f < n < 11f$, each honest node needs to compute $\mathcal{O}((1+f)^2/n^{\epsilon})$ gradients to reach $\epsilon$-stationarity, hence inflating the computational workload per node. Designing an efficient decentralized algorithm that tolerates a larger fraction of Byzantines remains an open question.

5 Empirical Evaluation

We compare the performance of MoNNA to D-SGD, local SGD, and the Byzantine resilient decentralized algorithm introduced in [15] under the name RB-TM. We test the robustness properties of MoNNA and RB-TM against three state-of-the-art attacks, whereas D-SGD and local SGD are executed in non-adversarial environments. We consider two classical image classification tasks, as described below. Each experiment is repeated 5 times using seeds 1 to 5 for reproducibility. Our code will be made freely available online.

5.1 Experimental setup

Datasets. We use the MNIST [36] and CIFAR-10 [34] datasets, pre-processed as in [4] and [17]. Refer to Appendix C for more details on pre-processing.

Model architecture and hyperparameters. For MNIST, we consider a convolutional neural network (CNN) with two convolutional layers followed by two fully-connected layers. The model is trained using a learning rate $\gamma = 0.75$ for $T = 600$ iterations. We use a total number of nodes $n = 26$, out of which $f \in \{5, 8\}$ are Byzantine. For CIFAR-10, we use a CNN with four convolutional layers and two fully-connected layers. Furthermore, we set $\gamma = 0.5$ and $T = 2000$ iterations. The experimental setup in this case consists of $n = 16$ nodes, among which $f \in \{3, 5\}$ are Byzantine. A detailed presentation of our models’ architectures can be found in Appendix C. We execute MoNNA with a momentum coefficient $\beta = 0.99$ and $K = 1$ (i.e., one coordination round per iteration), and compare its performance to RB-TM in three adversarial settings.

Byzantine attacks and asynchrony. We consider three state-of-the-art attacks performed by the Byzantine nodes, namely a little is enough (ALIE) [4], fall of empires (FOE) [50], sign-flipping (SF) [3], and label-flipping (LF) [3]. These attacks are explained in detail in Appendix C.3. In order to emulate the ill-effects of asynchrony, we ensure that the Byzantine messages get received first by the honest nodes. Put differently, to construct any honest node $i$’s set of $n - f - 1$ first received messages $\mathcal{R}_k(i)$, we first insert the Byzantines’ messages. We then complete the set by sampling the remaining messages uniformly from the honest vectors.

Evaluation details. To serve as benchmark for MoNNA and RB-TM under attack, we also run D-SGD and local SGD in a Byzantine-free environment (i.e., without faults) and with momentum $\beta = 0.99$. For simplicity, we use the same number of iterations $T$ for all algorithms and compare their learning accuracies and computational workloads per node.

5.2 Experimental results

Figure 1 showcases the performance of MoNNA and RB-TM on both MNIST and CIFAR-10. For space limitations and better readability, we only consider in Figure 1 the FOE and ALIE attacks. The remaining plots of SF and LF are given in Appendix D.1.
The main takeaways of Figure 1 are threefold. First, MoNNA clearly outperforms RB-TM on both MNIST and CIFAR-10 in terms of robustness. The improved resilience properties of our method over RB-TM are particularly striking on the more difficult task of the two, namely CIFAR-10. In fact, the model trained with RB-TM on CIFAR-10 barely reaches 30% in accuracy after 2000 iterations under the FOE attack. Furthermore, ALIE completely annihilates the learning capabilities of RB-TM, which stagnates at 10% accuracy.

Second, the amount of gradients each node has to compute using RB-TM is 20 times (resp. 12 times) larger than the computational workload required per node for MoNNA when learning on CIFAR-10 (resp. MNIST). The inflated computational costs per node associated to RB-TM are explained by the dynamic sampling technique the algorithm implements, whereby the batch-size is gradually augmented across iterations. On the other hand, MoNNA, D-SGD, and local SGD compute the same number of gradients since they all share a constant batch-size $b$, fixed during the entire learning procedure. This highlights the superiority of MoNNA in terms of computational efficiency, as our solution matches the workload complexity of the benchmark methods.

Third, MoNNA consistently outperforms local SGD. When running both algorithms for the same number of iterations $T$, MoNNA achieves higher accuracies than local SGD, even under attack. This observation is especially striking in the case of CIFAR-10. Indeed, local SGD evidently needs more iterations to achieve the same performance as MoNNA, which confirms our theoretical analysis on the reduced computational complexity of our algorithm. Similarly, the performance of our solution almost matches that of D-SGD both in regards to accuracy and workload in all settings presented in Figure 1. This also corroborates our theoretical findings.

6 Conclusion & Discussion

Most prior works addressing Byzantine resilience in decentralized learning consider a synchronous system [22, 23, 24, 25, 27, 41, 47]. However, synchrony is often violated in practice [38]. It is thus of
primary importance to design asynchronous decentralized algorithms. This task however reveals quite challenging [15, 21]. In fact, even in a fully connected topology, addressing asynchronous Byzantine decentralized learning essentially remained an open problem. Most prior solutions either rely upon strong assumptions (e.g., strong convexity [28]), or provide impractical guarantees (e.g., only asymptotic convergence [16] and a prohibitively large computational overhead [15]). Our presented algorithm MoNNA is the first efficient solution to the Byzantine resilience problem in asynchronous decentralized learning. We have demonstrated that the efficiency of our method matches that of D-SGD under the nominal case when $f \ll n$, which is orders of magnitude more efficient than the current state-of-the-art. We have empirically validated the superiority of our solution via experiments on MNIST and CIFAR-10. Furthermore, the upper bound on the convergence error of MoNNA in case of data heterogeneity matches the existing lower bound [31]. We provide additional comparisons between MoNNA and existing solutions below.

**Momentum.** One of the key ingredients of our solution is distributed momentum. While some recent works investigated this scheme for boosting Byzantine resilience in distributed machine learning [17, 20, 30], they only consider the parameter-server architecture assuming a trusted server. Their results do not extend trivially to the peer-to-peer case when no node is assumed a priori trusted. Furthermore, these works do not consider asynchronous communications which is often the case in practice and further complicates the theoretical analysis. To the best of our knowledge, MoNNA is the first algorithm to leverage distributed momentum to obtain tight convergence guarantees for asynchronous decentralized Byzantine learning.

**General communication topologies.** A few prior works that consider larger classes of network topologies include [27, 41, 45, 53, 54]. However, with the exception of [27], all these works assume the optimization problem to be convex, which considerably simplifies their analyses but is rarely true in practice. Moreover, while the algorithm proposed in [27] applies to non-convex settings, it assumes a very small fraction of Byzantine nodes that is orders of magnitude smaller than what can usually be tolerated in a distributed system [10].

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13
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A Signed Echo Broadcast (SEB)

SEB is composed of four rounds of communication. First, in round SEND, each sender node $i$ sends its message $m_i$ to all the other nodes $j$, which contains an identifier and a signature by node $i$. In the context of MoNNA, the message is a vector $x_{k-1}^{(i)}$, with an identifier of the form $(i,t,k)$, where $t$ is the SGD iteration and $k$ is the coordination round. Second, in round ECHO, upon receiving $m_i$, each recipient node $j$ verifies that $m_i$ is the first message with a valid signature from node $i$ and with identifier $(i,t,k)$. If so, node $j$ signs $m_i$ with its private key $pk_j$, thereby obtaining a signature $s_{ij}$ of message $m_i$ which node $j$ sends to node $i$. Otherwise, $m_i$ is ignored. Third, in round FINAL, upon receiving at least $n+2f-1$ valid signatures $s_{ij}$, the sender node $i$ sends the set $S_i$ of received signatures $s_{ij}$ to all other nodes. Fourth and finally, in round ACCEPT, upon receiving the set $S_i$, each recipient node $j$ verifies the signatures of the set, and if they are valid, node $j$ accepts $m_i$ and terminates the protocol. For $n > 3f$, SEB guarantees validity, even under asynchrony, i.e., any honest node’s message is eventually delivered to any other honest nodes. It also guarantees consistency, i.e., two different honest nodes cannot deliver different messages from a Byzantine node [9, Section 3.10.4]. The message complexity of this protocol is linear in the total number of nodes, i.e., $O(n)$.

B Proof of our main Theorem and additional results

We prove a generalized version of Theorem 1 stated below as Theorem 2 for the case when the nodes have heterogeneous data distribution. Specifically, we consider each honest node $i$ to have a data distribution $D_i$, and define its local loss function to be

$$Q^{(i)}(\theta) := \mathbb{E}_{x \sim D_i} [q(\theta, x)]. \quad (7)$$

We however assume the heterogeneity to be bounded. Specifically, drawing inspiration from prior work [13, 19, 31], we assume the following on the average squared distance between the gradients of a pair of honest nodes. Recall that $\mathcal{H}$ denotes the set of honest nodes, and that $|\mathcal{H}| = n - f$.

Assumption 3 (R-bounded heterogeneity). There exists $R < \infty$ such that for all $\theta \in \mathbb{R}^d$,

$$\frac{1}{|\mathcal{H}|^2} \sum_{i,j \in \mathcal{H}} \left\| \nabla Q^{(i)}(\theta) - \nabla Q^{(j)}(\theta) \right\|^2 \leq R^2.$$

Finally, in this particular case, we define the loss function $Q$ to be $Q(\theta) := \frac{1}{|\mathcal{H}|} \sum_{i \in \mathcal{H}} Q^{(i)}(\theta)$.

Remark 3. When the nodes have identical data distribution, i.e., $D_i = D$ for each honest node $i$: (a) the above definition of $Q$ is equivalent to the one given in (1), (b) Assumption 3 is trivially satisfied for $R = 0$, and (c) Theorem 2 below reduces to Theorem 1.
Theorem 2. Suppose that assumptions 1, 2 and 3 hold true, and that $n \geq 11f$. Let us denote

\[
\alpha := \frac{9.88 f}{n - f}, \quad \lambda := \frac{9f}{n - f}, \quad c_1 := \left( \frac{1}{n - f} + 9\lambda + \frac{3\alpha(1 + \alpha)}{(1 - \alpha)^2} (8 + 27\lambda) \right),
\]

\[
c_2 := 8L \left( Q(\bar{\theta}_1) - Q^* \right), \quad \text{and} \quad c_3 := \frac{18\alpha(1 + \alpha)}{(1 - \alpha)^2}.
\]

Consider Algorithm 1 with $K = 1, T \geq \frac{c_2}{\sigma^2 c_1} \max \{1, \frac{c_3}{18}\}, \gamma = \left( \frac{\sqrt{c_2/c_1}}{12L\sigma^2} \right)^{\frac{1}{2}}$, and $\beta = \sqrt{1 - T^2\gamma L}$. Then, for each honest node $i \in \mathcal{H}$, we obtain that

\[
\mathbb{E} \left[ \left\| \nabla Q(\bar{\theta}^{(i)}) \right\|^2 \right] \leq \frac{36\sqrt{\sigma^2 c_1 c_2}}{T^{3/2}} + \frac{c_2 c_3 (n - f)}{48 c_1} \left( 2 + \frac{3R^2}{\sigma^2} \right) \frac{1}{T} + 36 \sqrt{\frac{\sigma^2 c_2}{(n - f)^2 c_1}} \frac{1}{T^{3/2}}
\]

\[+ 9 \left( 4c_3 + 3\lambda (4c_3 + 9) \right) R^2.
\]

We present below the skeleton of our proof of Theorem 2.

B.1 Skeleton of a convergence proof for MoNNA

Our proof comprises 4 key steps, listed as follows.

Step-I: Demonstrating that the coordination step of Algorithm 1 is a contraction operator.

Step-II: Analyzing the parameter drift and the momentum drift.

Step-III: Analyzing the momentum deviation from the true gradient.

Step-IV: Studying the growth of loss function $Q$.

To present the technical details we introduce the following notation.

Notation: We denote by $\mathcal{P}_t$ the history of nodes from steps 1 to $t$. Specifically, we define

\[
\mathcal{P}_t := \{ \theta_1^{(i)}, \ldots, \theta_t^{(i)}; m_1^{(i)}, \ldots, m_t^{(i)}; i = 1, \ldots, n \}.
\]

By convention, $\mathcal{P}_1 = \{ \theta_1^{(i)}; i = 1, \ldots, n \}$. Furthermore, we denote by $\mathbb{E}_t [\cdot] := \mathbb{E} [\cdot | \mathcal{P}_t]$ the conditional expectation given the history $\mathcal{P}_t$, and by $\mathbb{E} [\cdot]$ the total expectation over the randomness of the algorithm $\mathbb{E} [\cdot] := \mathbb{E}_1 [\cdots \mathbb{E}_T [\cdot]]$. We recall that $\mathcal{H}$ denotes the set of honest nodes, and that $|\mathcal{H}| = n - f$. For an arbitrary $t$, we denote by $\Gamma(\star_t)$ the averaged pairwise squared distances between respective honest nodes’ local values, denoted by $\star_t$, i.e., $\Gamma(\star_t) = \frac{1}{(n - f)^2} \sum_{i,j \in \mathcal{H}} \left\| \theta_t^{(i)} - \star_t^{(j)} \right\|^2$. For instance,

\[
\Gamma(\theta_t) = \frac{1}{(n - f)^2} \sum_{i,j \in \mathcal{H}} \left\| \theta_t^{(i)} - \theta_t^{(j)} \right\|^2, \quad \Gamma(m_t) = \frac{1}{(n - f)^2} \sum_{i,j \in \mathcal{H}} \left\| m_t^{(i)} - m_t^{(j)} \right\|^2
\]

and $\Gamma(g_t) = \frac{1}{(n - f)^2} \sum_{i,j \in \mathcal{H}} \left\| g_t^{(i)} - g_t^{(j)} \right\|^2$.

We present below technical summaries of the aforementioned 4 steps.
Step-I: Coordination step with NNA and consistent broadcast

To analyse our proposed coordination step protocol in the presence of $f$ Byzantine nodes, we define the following generic property. Recall that in each iteration $t$, each node $i$ (including the Byzantine) initializes the coordination step with input $x_0^{(i)}$ and obtains $x_K^{(i)}$ at its completion.

**Definition 3 (($\alpha, \lambda$)-reduction).** A coordination step protocol is said to guarantee ($\alpha, \lambda$)-reduction if for any given set of honest nodes’ input vectors $\{x_0^{(i)}, i \in H\}$ the resulting set of output vectors $\{x_K^{(i)}, i \in H\}$ satisfy the following:

$$\Gamma(x_K) \leq \alpha \Gamma(x_0) \quad \text{and} \quad \|\bar{x}_0 - \bar{x}_K\|^2 \leq \lambda \Gamma(x_0)$$

where $\bar{x}_0$ and $\bar{x}_K$ denote the vector averages of $\{x_0^{(i)}, i \in H\}$ and $\{x_K^{(i)}, i \in H\}$, respectively.

If parameter $\alpha < 1$ then the coordination step protocol results in a contraction of honest nodes’ inputs, which is critical for the convergence of the peer-to-peer algorithm. The parameter $\lambda$ quantifies the accuracy of average estimation.

First, we obtain the following properties for our coordination step protocol, i.e., $K$ rounds of NNA with consistent broadcast.

**Lemma 1.** Suppose that $n \geq 11f$. For any $K \geq 1$, the coordination step of Algorithm 1 guarantees ($\alpha, \lambda$)-reduction for

$$\alpha = \left(\frac{9.88f}{n-f}\right)^K \quad \text{and} \quad \lambda = \frac{9f}{n-f} \cdot \min\left\{K, \frac{1}{(1 - \sqrt{\alpha})^2}\right\}.$$

Step-II: Parameter drift and the momentum drift

Second we note that, at any step $t$, neither the momentums $m_t^{(i)}$ nor the parameters $\theta_t^{(i)}$ of the honest nodes are guaranteed to stay close to each other even when the stochastic gradients $g_t^{(i)}$ come from a common gradient oracle. Yet, given our lemmas 1 and 5, we show in Lemma 2 below that the drift both between the honest nodes’ momentums and between their parameters can be controlled by cleverly parametrizing the momentum coefficient $\beta$. Hence, we can guarantee approximate agreement on both the parameters and the momentums of the honest nodes.
Lemma 2. Suppose that assumptions \[ \text{1}, \text{2}, \text{3} \] hold true. Consider Algorithm \[ \text{4} \] with \( \gamma \leq \frac{1-\alpha}{L\sqrt{54\alpha(1+\alpha)}} \). Suppose that the coordination step satisfies \( (\alpha, \lambda) \)-reduction for \( \alpha < 1 \). For each \( t \in [T] \), we obtain that

\[
E[\Gamma(\theta_t)] \leq E(\alpha)\gamma^2 \left( 2\sigma^2 \left( \frac{1-\beta}{1+\beta} \right) + 3R^2 \right),
\]

and

\[
E[\Gamma(m_t)] \leq 6\sigma^2 \left( \frac{1-\beta}{1+\beta} \right) + 9R^2 + 18L^2\gamma^2 E(\alpha) \left( 2\sigma^2 \left( \frac{1-\beta}{1+\beta} \right) + 3R^2 \right),
\]

where

\[
E(\alpha) := \frac{18\alpha(1+\alpha)}{(1-\alpha)^2}.
\]

Step-III: Momentum deviation.

Next, we study the deviation of the average honest momentum \( m_t \) from the average of the true gradients \( \nabla Q_t \), at step \( t \). Let us denote by

\[
\nabla Q_t := \frac{1}{n-f} \sum_{i \in \mathcal{H}} \nabla Q^{(i)}(\theta_t^{(i)}),
\]

the average of the true local gradient vectors at nodes’ local models. We define

\[
\delta_t := m_t - \nabla Q_t.
\]

We now have the following lemma.

Lemma 3. Suppose that assumptions \[ \text{1} \] and \[ \text{3} \] hold true. Consider Algorithm \[ \text{4} \]. For all \( t \in [T] \), we obtain that

\[
E \left[ \|\delta_{t+1}\|^2 \right] \leq \beta^2 \left( 1 + 4L\gamma \right) \left( 1 + \frac{9}{8}L\gamma \right) E \left[ \|\delta_t\|^2 \right] + \frac{3}{4} \beta^2 L\gamma (1 + 4L\gamma) E \left[ \| \nabla Q(\bar{\theta}_t) \|^2 \right]
\]

\[
+ 9\beta^2 L^2 \left( 1 + \frac{1}{4\gamma L} \right) \left( E \left[ \Gamma(\theta_{t+1}) \right] + E \left[ \Gamma(\theta_t) \right] + E \left[ \| \bar{\theta}_{t+1} - \bar{\theta}_{t+1/2} \|^2 \right] \right)
\]

\[
+ \frac{9}{4} \beta^2 L\gamma (1 + 4L\gamma) L^2 E \left[ \Gamma(\theta_t) \right] + \frac{(1-\beta)^2 \sigma^2}{n-f}.
\]

Step-IV: Growth function.

Finally, we analyze the growth of loss function \( Q \) computed at the average parameter of the honest nodes \( \bar{\theta}_t \) along the trajectory of Algorithm \[ \text{1} \]. Let us denote by \( \bar{\theta}_t := \frac{1}{n-f} \sum_{i \in \mathcal{H}} \theta_t^{(i)} \) the average parameter of the honest nodes at step \( t \). Then we obtain the following lemma.
Lemma 4. Suppose that assumptions 1 and 2 hold true. Consider Algorithm 1 with $\gamma \leq 1/L$. For each $t \in [T]$, we obtain that

$$
\mathbb{E} \left[ Q(\overline{\theta}_{t+1}) - Q(\overline{\theta}_t) \right] \leq -\frac{\gamma}{2} \mathbb{E} \left[ \|\nabla Q(\overline{\theta}_t)\|^2 \right] + \frac{3\gamma}{2} \mathbb{E} \left[ \|\delta_t\|^2 \right] + \frac{3}{2\gamma} \mathbb{E} \left[ \|\overline{\theta}_{t+1/2} - \overline{\theta}_{t+1}\|^2 \right] + \frac{3\gamma}{2} L^2 \mathbb{E} \left[ \Gamma'(\theta_t) \right].
$$

This means that Algorithm 1 can actually be treated as DSGD with an additional error term which is proportional to the coupled drift of the momentums and the parameters at each step $t$.

Combining steps I, II, III and IV

To obtain, our final convergence result, as stated in Theorem 2, we combine these elements. Note however that the deviation term in Lemma 4 cannot be readily treated with a standard convergence analysis. To address this issue, we devise a novel Lyapunov function

$$
V_t := \mathbb{E} \left[ Q(\overline{\theta}_t) + \frac{1}{4L} \|\delta_t\|^2 \right].
$$

By analyzing the growth of $V_t$ along the steps of Algorithm 1, we prove Theorem 2 as follows.

B.2 Proof for Theorem 2

Recall that $\mathcal{H}$ denotes the set of honest nodes, and that $|\mathcal{H}| = n - f$. Consider the Lyapunov function $V_t$ defined in (8), i.e.,

$$
V_t := \mathbb{E} \left[ Q(\overline{\theta}_t) + \zeta \|\delta_t\|^2 \right] \quad \text{where} \quad \zeta = \frac{1}{4L}.
$$

Consider an arbitrary $t \in [T]$. From Lemma 4 and Lemma 3 we obtain that

$$
V_{t+1} - V_t = \mathbb{E} \left[ Q(\overline{\theta}_{t+1}) - Q(\overline{\theta}_t) \right] + \zeta \mathbb{E} \left[ \|\delta_{t+1}\|^2 - \|\delta_t\|^2 \right]
$$

$$
\leq -\frac{\gamma}{2} \mathbb{E} \left[ \|\nabla Q(\overline{\theta}_t)\|^2 \right] + \frac{3\gamma}{2} \mathbb{E} \left[ \|\delta_t\|^2 \right] + \frac{3}{2\gamma} \mathbb{E} \left[ \|\overline{\theta}_{t+1/2} - \overline{\theta}_{t+1}\|^2 \right] + \frac{3\gamma}{2} L^2 \mathbb{E} \left[ \Gamma'(\theta_t) \right]
$$

$$
+ \zeta \beta^2 (1 + 4L\gamma) \left( 1 + \frac{9}{8} L\gamma \right) \mathbb{E} \left[ \|\delta_t\|^2 \right] + \frac{3}{4} \zeta \beta^2 L\gamma (1 + 4L\gamma) \mathbb{E} \left[ \|\nabla Q(\overline{\theta}_t)\|^2 \right]
$$

$$
+ 9\zeta \beta^2 L^2 \left( 1 + \frac{1}{4\gamma L} \right) \left( \mathbb{E} \left[ \Gamma'(\theta_{t+1}) \right] + \mathbb{E} \left[ \Gamma'(\theta_t) \right] + \mathbb{E} \left[ \|\overline{\theta}_{t+1} - \overline{\theta}_{t+1/2}\|^2 \right] \right)
$$

$$
+ \frac{9}{4} \zeta \beta^2 L\gamma (1 + 4L\gamma) L^2 \mathbb{E} \left[ \Gamma'(\theta_t) \right] + \zeta (1 - \beta) \frac{\sigma^2}{n - f} - \zeta \mathbb{E} \left[ \|\delta_t\|^2 \right].
$$
Upon re-arranging the terms on the R.H.S. we obtain that
\[ V_{t+1} - V_t \leq -\gamma \left( \frac{1}{2} - \frac{3}{4} \zeta \beta^2 L(1 + 4L\gamma) \right) \mathbb{E} \left[ \| \nabla Q(\tilde{\theta}_t) \|^2 \right] \]
\[ + \left( \frac{3\gamma}{2} + \zeta \beta^2(1 + 4L\gamma) \left( 1 + \frac{9}{8}L\gamma \right) - \zeta \right) \mathbb{E} \left[ \| \delta_t \|^2 \right] \]
\[ + 9\zeta \beta^2 L^2 \left( 1 + \frac{1}{4\gamma L} \right) \left( \mathbb{E} [\Gamma(\theta_{t+1})] + \mathbb{E} [\Gamma(\theta_t)] + \mathbb{E} \left[ \| \tilde{\theta}_{t+1} - \tilde{\theta}_{t+1/2} \|^2 \right] \right) \]
\[ + \frac{9}{4} \zeta \beta^2 L\gamma \left( 1 + 4L\gamma \right) L^2 \mathbb{E} [\Gamma(\theta_t)] + \frac{3\gamma}{2} \frac{\sigma^2}{n - f} \]
\[ + \frac{3\gamma}{2\gamma} \mathbb{E} \left[ \| \tilde{\theta}_{t+1/2} - \tilde{\theta}_{t+1} \|^2 \right] + \frac{3\gamma}{2} L^2 \mathbb{E} [\Gamma(\theta_t)] . \]

We denote,
\[
A := \frac{1}{2} - \frac{3}{4} \zeta \beta^2 L(1 + 4L\gamma), \]
\[
B := \frac{3\gamma}{2} + \zeta \beta^2(1 + 4L\gamma) \left( 1 + \frac{9}{8}L\gamma \right) - \zeta, \quad \text{and} \]
\[
C := 9\zeta \beta^2 L^2 \left( 1 + \frac{1}{4\gamma L} \right) \left( \mathbb{E} [\Gamma(\theta_{t+1})] + \mathbb{E} [\Gamma(\theta_t)] + \mathbb{E} \left[ \| \tilde{\theta}_{t+1} - \tilde{\theta}_{t+1/2} \|^2 \right] \right) \]
\[ + \frac{9}{4} \zeta \beta^2 L\gamma \left( 1 + 4L\gamma \right) L^2 \mathbb{E} [\Gamma(\theta_t)] + \frac{3\gamma}{2\gamma} \mathbb{E} \left[ \| \tilde{\theta}_{t+1/2} - \tilde{\theta}_{t+1} \|^2 \right] + \frac{3\gamma}{2} L^2 \mathbb{E} [\Gamma(\theta_t)] . \]

Substituting from above in (9) we obtain that
\[ V_{t+1} - V_t \leq -\gamma A \mathbb{E} \left[ \| \nabla Q(\tilde{\theta}_t) \|^2 \right] + B \mathbb{E} \left[ \| \delta_t \|^2 \right] + C + \zeta(1 - \beta)^2 \frac{\sigma^2}{(n - f)}. \] (10)

Now, we separately analyse the terms $A$, $B$ and $C$ below by using the following,
\[
\zeta = \frac{1}{4L}, \quad \gamma \leq \frac{1}{12L}, \quad \text{and} \quad 1 - \beta^2 = 12\gamma L. \] (11)

Note that the condition on $\gamma$ above follows from the fact that $\gamma = \frac{1}{12L} \sqrt{\frac{\sigma^2}{c_1 T \sigma^2}}$ where $T \geq \frac{c_2}{\sigma^2 c_1} \max \left\{ 1, \frac{c_3}{48} \right\}$.

**Term A.** Using the facts that $\zeta = 1/4L$, $\gamma \leq 1/12L$ and that $\beta^2 < 1$, we obtain that
\[ A = \frac{1}{2} - \frac{3}{4} \zeta \beta^2 L(1 + 4L\gamma) \geq \frac{1}{2} - \frac{3}{16} \left( 1 + \frac{4}{12} \right) = \frac{1}{4}. \] (12)

**Term B.** As $1 - \beta^2 = 12\gamma L$ and $\beta^2 < 1$, we obtain that
\[ B = \frac{3\gamma}{2} - \zeta(1 - \beta^2) + \zeta \beta^2 \left( \frac{41}{8} L\gamma + \frac{9}{2} L^2 \gamma^2 \right) \leq \frac{3\gamma}{2} - 12\zeta \gamma L + \zeta \left( \frac{41}{8} L\gamma + \frac{9}{2} L^2 \gamma^2 \right). \]

As $\zeta = 1/4L$ and $\gamma \leq 1/12L$, from above we obtain that
\[ B \leq \frac{3\gamma}{2} - 3\gamma + \frac{\gamma}{4} \left( \frac{41}{8} + \frac{9}{2} L\gamma \right) \leq -\frac{3\gamma}{2} + \frac{\gamma}{4} \left( \frac{41}{8} + \frac{9}{24} \right) \leq 0. \] (13)
Term C. Using the facts that $\beta^2 < 1$ and $\zeta = 1/4L$, we obtain that

$$C = 9\zeta \beta^2 L^2 \left(1 + \frac{1}{4\gamma L}\right) \left(\mathbb{E} \left[ \Gamma (\theta_{t+1}) \right] + \mathbb{E} \left[ \Gamma (\theta_t) \right] + \mathbb{E} \left[ \left\| \overline{\theta}_{t+1} - \overline{\theta}_{t+1/2} \right\|^2 \right] \right)$$

$$+ \frac{9}{4} \zeta \beta^2 L \gamma (1 + 4L\gamma) L^2 \mathbb{E} \left[ \left\| \Gamma (\theta_t) \right\|^2 \right] + \frac{3}{2\gamma} \mathbb{E} \left[ \left\| \overline{\theta}_{t+1/2} - \overline{\theta}_{t+1} \right\|^2 \right] + \frac{3\gamma L^2}{2} \mathbb{E} \left[ \Gamma (\theta_t) \right]$$

$$\leq \mathbb{E} \left[ \Gamma (\theta_t) \right] \left( \frac{9L}{16\gamma L} (1 + 4\gamma L) + \frac{9}{16} \gamma L (1 + 4L\gamma) + \frac{3\gamma L^2}{2} \right)$$

$$+ \mathbb{E} \left[ \Gamma (\theta_{t+1}) \right] \frac{9L}{16\gamma L} (1 + 4\gamma L) + \left( \frac{3}{2\gamma} + \frac{9L}{16\gamma L} (1 + 4\gamma L) \right) \mathbb{E} \left[ \left\| \overline{\theta}_{t+1/2} - \overline{\theta}_{t+1} \right\|^2 \right].$$

Using the fact $\gamma \leq 1/12L$ we then have

$$C \leq \mathbb{E} \left[ \Gamma (\theta_t) \right] \left( \frac{9}{16\gamma} \left( \frac{4}{3} \right) + \frac{9}{16} \left( \frac{1}{144\gamma} \right) \left( \frac{4}{3} \right) + \frac{3}{288\gamma} \right)$$

$$+ \mathbb{E} \left[ \Gamma (\theta_{t+1}) \right] \frac{9}{16\gamma} \left( \frac{4}{3} \right) + \left( \frac{3}{2\gamma} + \frac{9}{16\gamma} \left( \frac{4}{3} \right) \right) \mathbb{E} \left[ \left\| \overline{\theta}_{t+1/2} - \overline{\theta}_{t+1} \right\|^2 \right]$$

$$\leq \frac{1}{\gamma} \mathbb{E} \left[ \Gamma (\theta_t) \right] + \frac{1}{\gamma} \mathbb{E} \left[ \Gamma (\theta_{t+1}) \right] + \frac{9}{4\gamma} \mathbb{E} \left[ \left\| \overline{\theta}_{t+1/2} - \overline{\theta}_{t+1} \right\|^2 \right]. \quad (14)$$

From Lemma 1 we have

$$\mathbb{E} \left[ \left\| \overline{\theta}_{t+1/2} - \overline{\theta}_{t+1} \right\|^2 \right] \leq \lambda \mathbb{E} \left[ \Gamma (\theta_{t+1/2}) \right].$$

From Algorithm 1 we have for all $i \in \mathcal{H}$, $\theta_{t+1/2}^{(i)} = \theta_t^{(i)} - \gamma m_t^{(i)}$. Therefore, by definition of $\Gamma (\cdot)$, $\Gamma (\theta_{t+1/2}) \leq 2\Gamma (\theta_t) + 2\gamma^2 \Gamma (m_t)$. Thus, from above we obtain that

$$\mathbb{E} \left[ \left\| \overline{\theta}_{t+1/2} - \overline{\theta}_{t+1} \right\|^2 \right] \leq \lambda \left( 2\mathbb{E} \left[ \Gamma (\theta_t) \right] + 2\gamma^2 \mathbb{E} \left[ \Gamma (m_t) \right] \right)$$

Substituting from above in (14) we obtain that

$$C \leq \frac{1}{\gamma} \left( 1 + \frac{9\lambda}{2} \right) \mathbb{E} \left[ \Gamma (\theta_t) \right] + \frac{1}{\gamma} \mathbb{E} \left[ \Gamma (\theta_{t+1}) \right] + \frac{9\lambda \gamma}{2} \mathbb{E} \left[ \Gamma (m_t) \right].$$

By invoking Lemma 2 we obtain from above that

$$C \leq \left( 2 + \frac{9\lambda}{2} \right) E(\alpha) \gamma \left( 2\sigma^2 \left( \frac{1 - \beta}{1 + \beta} \right) + 3R^2 \right)$$

$$+ \frac{9\lambda \gamma}{2} \left( 6\sigma^2 \left( \frac{1 - \beta}{1 + \beta} \right) + 9R^2 + 18L^2 \gamma^2 E(\alpha) \left( 2\sigma^2 \left( \frac{1 - \beta}{1 + \beta} \right) + 3R^2 \right) \right).$$

Upon re-arranging the terms, and using the facts that $\gamma \leq 1/12L$, we obtain that

$$C \leq \gamma R^2 \left( 6E(\alpha) + \frac{9\lambda}{2} \left( 3E(\alpha) + 9 + \frac{3E(\alpha)}{8} \right) \right) + \gamma \sigma^2 \left( \frac{1 - \beta}{1 + \beta} \right) \left( 4E(\alpha) + \frac{9\lambda}{2} \left( 2E(\alpha) + 6 + \frac{2E(\alpha)}{8} \right) \right)$$

$$\leq \gamma R^2 \left( 6E(\alpha) + \frac{9\lambda}{2} \left( 4E(\alpha) + 9 \right) \right) + \gamma \sigma^2 \left( \frac{1 - \beta}{1 + \beta} \right) \left( 4E(\alpha) + \frac{9\lambda}{2} \left( 3E(\alpha) + 6 \right) \right). \quad (15)$$
Note that
\[
\frac{1 - \beta}{1 + \beta} = \frac{1 - \beta^2}{(1 + \beta)^2} \leq 1 - \beta^2 = 12\gamma L.
\]
Substituting from above in (15) we obtain that
\[
C \leq \gamma R^2 \left( 6E(\alpha) + \frac{9\lambda}{2} (4E(\alpha) + 9) \right) + 12\gamma^2 \sigma^2 L \left( 4E(\alpha) + \frac{9\lambda}{2} (3E(\alpha) + 6) \right).
\]
Combining A, B and C. Substituting from (12) and (13) in (10), and using the fact that \(\zeta = \frac{1}{4L}\), we obtain that
\[
V_{t+1} - V_t \leq -\gamma A \mathbb{E} \left[ \| \nabla Q(\theta_t) \|^2 \right] + B \mathbb{E} \left[ \| \delta_t \|^2 \right] + C + \zeta (1 - \beta) \frac{\sigma^2}{(n - f)}.
\]
Note that, as \(\beta \in (0, 1)\), \(1 - \beta = (1 - \beta^2)/(1 + \beta) \leq 1 - \beta^2\). Using this above we obtain that
\[
V_{t+1} - V_t \leq -\frac{\gamma}{4} \mathbb{E} \left[ \| \nabla Q(\theta_t) \|^2 \right] + C + (1 - \beta^2) \frac{\sigma^2}{4L(n - f)}.
\]
Recall that \(1 - \beta^2 = 12\gamma L\). Therefore,
\[
V_{t+1} - V_t \leq -\frac{\gamma}{4} \mathbb{E} \left[ \| \nabla Q(\theta_t) \|^2 \right] + C + 36\gamma^2 L \frac{\sigma^2}{n - f}.
\]
This implies that
\[
\mathbb{E} \left[ \| \nabla Q(\theta_t) \|^2 \right] \leq (V_t - V_{t+1}) - \frac{4}{\gamma} C + 144\gamma L \frac{\sigma^2}{n - f}.
\]
By taking the average on both sides from \(t = 1\) to \(T\), we obtain that
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \| \nabla Q(\theta_t) \|^2 \right] \leq (V_1 - V_{T+1}) - \frac{4}{\gamma} C + 144\gamma L \frac{\sigma^2}{n - f}.
\]
Analysis on \(V_t\). Recall that \(Q^* = \inf_{\theta} Q(\theta)\). Note that for any \(t\),
\[
V_t - Q^* = \mathbb{E} [Q(\theta_t) - Q^*] + \zeta \mathbb{E} \left[ \| \delta_t \|^2 \right] \geq \mathbb{E} [Q(\theta_t) - Q^*] \geq 0.
\]
Thus, \(V_{T+1} \geq Q^*\). Using this in (18) we obtain that
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \| \nabla Q(\theta_t) \|^2 \right] \leq (V_1 - Q^*) - \frac{4}{\gamma} C + 144\gamma L \frac{\sigma^2}{n - f}.
\]
Recall that
\[
V_1 = Q(\theta_1) + \frac{1}{4L} \mathbb{E} \left[ \| \delta_1 \|^2 \right].
\]
Recall that, by Definition [51] of $\delta_t$, we have $\delta_1 = \overline{m}_1 - \nabla Q_1$. Thus, under Assumption 2 we have
\[
\mathbb{E} \left[ \| \delta_1 \|^2 \right] = \mathbb{E} \left[ \| (1 - \beta) \overline{y}_1 - \nabla Q_1 \|^2 \right] \\
\leq 2(1 - \beta)^2 \mathbb{E} \left[ \| \overline{y}_1 - \nabla Q_1 \|^2 \right] + 2\beta^2 \| \nabla Q_1 \|^2 \\
\leq 2(1 - \beta)^2 \left( \frac{\sigma^2}{n - f} \right) + 2\beta^2 \| \nabla Q_1 \|^2.
\]

Recall that $1 - \beta^2 = 12\gamma L$. Thus, $(1 - \beta)^2 \leq (1 - \beta^2)^2/(1 + \beta)^2 \leq (1 - \beta^2) = 144\gamma^2 L^2$. Substituting this above, and using the fact that $\beta^2 < 1$, we obtain that
\[
\mathbb{E} \left[ \| \delta_1 \|^2 \right] \leq 288\gamma^2 L^2 \left( \frac{\sigma^2}{n - f} \right) + 2 \| \nabla Q (\overline{y}_1) \|^2.
\]

Therefore,
\[
V_1 \leq Q (\overline{y}_1) + \frac{1}{4L} \left( 288\gamma^2 L^2 \left( \frac{\sigma^2}{n - f} \right) + 2 \| \nabla Q (\overline{y}_1) \|^2 \right) = Q (\overline{y}_1) + 72\gamma^2 L \left( \frac{\sigma^2}{n - f} \right) + \frac{1}{2L} \| \nabla Q (\overline{y}_1) \|^2.
\]

Substituting from above in (19) we obtain that
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \| \nabla Q (\overline{y}_t) \|^2 \right] \leq (V_1 - Q^*) \frac{4}{\gamma T} + \frac{4}{\gamma} C + 144\gamma L \frac{\sigma^2}{n - f}. \tag{20}
\]

Substituting $C$ from (16) in above, we obtain that
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \| \nabla Q (\overline{y}_t) \|^2 \right] \leq \frac{4}{\gamma T} \left( Q (\overline{y}_1) - Q^* \right) + \frac{288\gamma L}{T} \left( \frac{\sigma^2}{n - f} \right) + \frac{2}{\gamma L T} \| \nabla Q (\overline{y}_1) \|^2 + \frac{4}{\gamma} C + 144\gamma L \frac{\sigma^2}{n - f}
\]
\[+ 4R^2 \left( 6E(\alpha) + \frac{9\lambda}{2} (4E(\alpha) + 9) \right) + 48\gamma^2 L \left( 4E(\alpha) + \frac{9\lambda}{2} (3E(\alpha) + 6) \right).
\]

Re-arranging the terms, we then have
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \| \nabla Q (\overline{y}_t) \|^2 \right] \leq \frac{4}{\gamma T} \left( Q (\overline{y}_1) - Q^* \right) + \frac{2}{L} \| \nabla Q (\overline{y}_1) \|^2 + \frac{288\gamma L}{T} \left( \frac{\sigma^2}{n - f} \right)
\]
\[+ 48\gamma L \sigma^2 \left( \frac{3}{n - f} + 4E(\alpha) + \frac{9\lambda}{2} (3E(\alpha) + 6) \right)
\]
\[+ 4R^2 \left( 6E(\alpha) + \frac{9\lambda}{2} (4E(\alpha) + 9) \right).
\]

We now define
\[
c_1 := \left( \frac{1}{n - f} + 9\lambda + \frac{3\alpha(1 + \alpha)}{(1 - \alpha)^2} (8 + 27\lambda) \right),
\]

26
\[ c_2 := 8L (Q (\bar{\theta}_1) - Q^*) ,\] and \[ c_3 := E(\alpha) = \frac{18\alpha(1 + \alpha)}{(1 - \alpha)^2}. \]

Note that \(4L (Q (\bar{\theta}_1) - Q^*) + 2 \| \nabla Q (\bar{\theta}_1) \|^2 \leq c_2\) (see Remark 4 below). Thus

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \| \nabla Q (\bar{\theta}_t) \|^2 \right] \leq \frac{c_2}{L \gamma T} + 144c_1 \gamma L \sigma^2 + \frac{288\gamma L}{T} \left( \frac{\sigma^2}{n - f} \right) + 4R^2 \left( 6c_3 + \frac{9\lambda}{2} (4c_3 + 9) \right).
\]

Substituting

\[
\gamma = \frac{1}{12L} \sqrt{\frac{c_2}{c_1 T \sigma^2}},
\]

we obtain that

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \| \nabla Q (\bar{\theta}_t) \|^2 \right] \leq 24 \sqrt{\frac{c_1 c_2 \sigma^2}{T}} + 24 \sqrt{\frac{c_2 \sigma^2}{c_1 T^3}} \left( \frac{1}{n - f} \right) + 4R^2 \left( 6c_3 + \frac{9\lambda}{2} (4c_3 + 9) \right). \tag{21}
\]

Note that for an arbitrary honest node \(i \in \mathcal{H}\), we have

\[
\mathbb{E} \left[ \| \nabla Q (\theta_t^{(i)}) \|^2 \right] \leq \frac{3}{2} \mathbb{E} \left[ \| \nabla Q (\bar{\theta}_t) \|^2 \right] + 3 \mathbb{E} \left[ \| \nabla Q (\bar{\theta}_t) - \nabla Q (\theta_t^{(i)}) \|^2 \right]
\]
\[
\leq \frac{3}{2} \mathbb{E} \left[ \| \nabla Q (\bar{\theta}_t) \|^2 \right] + 3L^2 \mathbb{E} \left[ \| \bar{\theta}_t - \theta_t^{(i)} \|^2 \right],
\]

where the second inequality follows from Assumption \[ \Box \] This implies that

\[
\mathbb{E} \left[ \| \nabla Q (\theta_t^{(i)}) \|^2 \right] \leq \frac{3}{2} \mathbb{E} \left[ \| \nabla Q (\bar{\theta}_t) \|^2 \right] + 3L^2 (n - f) \frac{1}{n - f} \sum_{i \in \mathcal{H}} \mathbb{E} \left[ \| \bar{\theta}_t - \theta_t^{(i)} \|^2 \right]
\]
\[
\leq \frac{3}{2} \mathbb{E} \left[ \| \nabla Q (\bar{\theta}_t) \|^2 \right] + 3L^2 (n - f) \mathbb{E} [\Gamma (\theta_t)].
\]

Using the bound on \( \mathbb{E} [\Gamma (\theta_t)] \) from Lemma \[ \Box \] we obtain that

\[
\mathbb{E} \left[ \| \nabla Q (\theta_t^{(i)}) \|^2 \right] \leq \frac{3}{2} \mathbb{E} \left[ \| \nabla Q (\bar{\theta}_t) \|^2 \right] + 3L^2 c_3 (n - f) \gamma^2 \left( 2\sigma^2 \left( \frac{1 - \beta}{1 + \beta} \right) + 3R^2 \right).
\]

Thus (as \( \frac{1 - \beta}{1 + \beta} \leq 1\)),

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \| \nabla Q (\theta_t^{(i)}) \|^2 \right] \leq \frac{3}{2} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \| \nabla Q (\bar{\theta}_t) \|^2 \right] + 3L^2 c_3 (n - f) \gamma^2 \left( 2\sigma^2 + 3R^2 \right).
\]

Substituting from (21) in the above, and setting \( \gamma = \left( \frac{\sqrt{c_2/c_1}}{12L} \right)^{1/\sqrt{T}} \), we obtain that

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \| \nabla Q (\theta_t^{(i)}) \|^2 \right] \leq 36 \sqrt{\frac{c_1 c_2 \sigma^2}{T}} + 36 \sqrt{\frac{c_2 \sigma^2}{c_1 T^3}} \left( \frac{1}{n - f} \right) + \frac{c_3 (n - f)c_2}{48c_1 T \sigma^2} \left( 2\sigma^2 + 3R^2 \right)
\]
\[
+ 6R^2 \left( 6c_3 + \frac{9\lambda}{2} (4c_3 + 9) \right). \tag{22}
\]

27
As \( \hat{\theta}^{(i)} \sim \mathcal{U}\{\theta_1^{(i)}, \ldots, \theta_T^{(i)}\} \), we get
\[
\mathbb{E} \left[ \| \nabla Q \left( \hat{\theta}^{(i)} \right) \|^2 \right] = \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \| \nabla Q \left( \theta_t^{(i)} \right) \|^2 \right].
\]

Substituting from above in (22) concludes the proof.

**Remark 4.** Note that in the proof, we used the fact that for any \( x \in \mathbb{R}^d \), we have
\[
\| \nabla Q(x) \|^2 \leq 2L(Q(x) - Q^*).
\]

We provided the proof of this fact below.

**Proof.** By the Lipschitzness of \( \nabla Q \), for any \( x, y \in \mathbb{R}^d \), we have
\[
Q(y) \leq Q(x) + \langle \nabla Q(x), y - x \rangle + \frac{L}{2} \| y - x \|^2.
\]

For \( y = x - \frac{1}{L} \nabla Q(x) \), we obtain that
\[
Q \left( x - \frac{1}{L} \nabla Q(x) \right) \leq Q(x) - \frac{1}{L} \| \nabla Q(x) \|^2 + \frac{1}{2L} \| \nabla Q(x) \|^2 = Q(x) - \frac{1}{2L} \| \nabla Q(x) \|^2.
\]

Now note that as \( Q^* \) is the minimum, we have
\[
Q^* \leq Q \left( x - \frac{1}{L} \nabla Q(x) \right) \leq Q(x) - \frac{1}{2L} \| \nabla Q(x) \|^2.
\]

Rearranging the terms, for any \( x \in \mathbb{R}^d \), we have
\[
\| \nabla Q(x) \|^2 \leq 2L(Q(x) - Q^*).
\]

**B.3 Proof of Corollary [1]**

**Proof.** Note that ignoring the higher order terms in the bound of Theorem [2] we have
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \| \nabla Q \left( \theta_t^{(i)} \right) \|^2 \right] \in \mathcal{O} \left( \sqrt{\frac{c_1 c_2 \sigma^2}{T}} + R^2 \left( c_3 + \lambda (c_3 + 1) \right) \right).
\]

Now note also that in Theorem [2] for \( n \geq 11f \), we have \( \alpha \leq 0.988 < 1 \). This implies that \( \frac{1 + \alpha}{(1 - \alpha)^2} \in \mathcal{O}(1) \). Therefore,
\[
c_3 = \frac{18\alpha(1 + \alpha)}{(1 - \alpha)^2} \in \mathcal{O}(\alpha).
\]

We also obtain that
\[
c_1 = \frac{1}{n - f} + 9\lambda + \frac{3\alpha(1 + \alpha)}{(1 - \alpha)^2} (8 + 27\lambda) \in \mathcal{O} \left( \frac{1}{n} + \alpha + \lambda \right).
\]

\[28\]
Finally note that $c_2$ is a function of constants $L, Q(\bar{\theta}_1), Q^*$, and $\|\nabla Q(\bar{\theta}_1)\|$ and thus $c_2 \in \mathcal{O}(1)$. Combining all, then yields

$$
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \|\nabla Q(\theta_t^{(i)})\|^2 \right] \in \mathcal{O} \left( \sqrt{\frac{\sigma^2}{T} \left( \frac{1}{n} + \alpha + \lambda \right) + (\alpha + \lambda)R^2} \right).
$$

Now note that, we have $\alpha = \frac{9.88f}{fn} \in \mathcal{O}(\frac{f}{n})$, and $\lambda = \frac{9f}{n^2} \in \mathcal{O}(\frac{f}{n})$ and for the homogeneous case of Corollary 1, we have $R = 0$. Therefore,

$$
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \|\nabla Q(\theta_t^{(i)})\|^2 \right] \in \mathcal{O} \left( \sqrt{\frac{\sigma^2}{T} \left( \frac{1}{n} + \frac{f}{n} \right)} \right).
$$

This is the desired result for the homogeneous case.

**Remark 5.** Note that for the heterogeneous case ($R \neq 0$), we have

$$
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \|\nabla Q(\theta_t^{(i)})\|^2 \right] \in \mathcal{O} \left( \sqrt{\frac{\sigma^2}{T} \left( \frac{1}{n} + \frac{f}{n} \right) + \frac{R^2}{n}} \right).
$$

This shows that our algorithm converges with the same rate in the heterogeneous case to a neighbourhood of solution and the size of neighbourhood matches the lower-bound of [31].
B.4 Convergence of MoNNA for \( n > 5f \)

Note that theorems 1 and 2 are stated for the case where \( n \geq 11f \). This comes from the fact that we need the number of honest nodes to be sufficiently large to guarantee \((\alpha, \lambda)\)-reduction as stated in Lemma 1. However, by setting \( K \in O(\log(n)) \) we can still guarantee \((\alpha, \lambda)\)-reduction for \( n > 5f \) as stated in the following Lemma. The proof of this Lemma is given in Section B.9.

**Lemma 5.** Suppose that there exists \( \delta > 0 \) such that \( n \geq (5 + \delta)f \). For \( K = \frac{\log(8(n-f))}{2\log(\frac{n}{\delta})} \in O(\log(n)) \), the coordination step of Algorithm 1 guarantees \((\alpha, \lambda)\)-reduction for

\[
\alpha = \frac{2f}{n-f} \leq \frac{1}{2} \quad \text{and} \quad \lambda = \left(\frac{3 + \delta}{\delta}\right)^2 \frac{(8f)^2}{n-f}.
\]

Replacing Lemma 1 by Lemma 5 and following the same steps as the proof of Theorem 1 we can show the following result which is essentially a convergence proof for MoNNA while tolerating a larger fraction of Byzantine nodes \( (n > 5f) \).

**Corollary 2.** Suppose that assumptions 1, 2 and 3 hold true. Suppose also that there exists \( \delta > 0 \) such that \( n \geq (5 + \delta)f \). Denote

\[
\alpha := \frac{2f}{n-f}, \quad \lambda := \left(\frac{3 + \delta}{\delta}\right)^2 \frac{(8f)^2}{n-f}, \quad c_1 := \left(\frac{1}{n-f} + \frac{3\alpha(1+\alpha)}{(1-\alpha)^2} (8 + 27\lambda)\right),
\]

\[
c_2 := 8L \left( Q(\bar{\theta}_i) - Q^* \right), \quad \text{and} \quad c_3 := \frac{18\alpha(1+\alpha)}{(1-\alpha)^2}.
\]

Consider Algorithm 1 with \( K = \frac{\log(8(n-f))}{2\log(\frac{n}{\delta})} \in O(\log(n)) \), \( T \geq \frac{c_2}{\sigma^2 c_1} \max \{1, \frac{c_3}{18}\} \), \( \gamma = \left(\frac{\sqrt{c_2/c_1}}{12L\sigma^2}\right)^{1/4} \), and \( \beta = \sqrt{1-12\gamma L} \). Then, for each honest node \( i \in \mathcal{H} \), we obtain that

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\left[ \left\| \nabla Q\left(\theta_i^{(t)}\right) \right\|^2 \right] \leq \frac{36\sqrt{\sigma^2 c_1 c_2}}{T^{1/2}} + \frac{c_2 c_3 (n-f)}{48 c_1} \left(2 + \frac{3R^2}{\sigma^2}\right) \frac{1}{T} + 36 \sqrt{\frac{\sigma^2 c_2}{(n-f)^2 c_1 T^{1/2}}} \frac{1}{T} + 9 \left(4c_3 + 3\lambda (4c_3 + 9)\right) R^2.
\]

**Remark 6.** Ignoring higher order terms and following the same reasoning as that of proof of Corollary 2, we can show that for \( n > 5f \), MoNNA guarantees \((f, \epsilon)\)-approximate Byzantine resilient learning for

\[
\epsilon \in O\left( \sqrt{\frac{1+f^2}{nT}} \right).
\]
B.5 Proof of Lemma 1

Throughout the proof we make use of the following notation.

**Notation:** Recall that \( \mathcal{R}_k^{(i)} \) is the set of indices received by node \( i \) at coordination round \( k \). Let \( \psi_k^{(i)} : [n - f - 1] \to \mathcal{R}_k^{(i)} \) be a bijection that sorts the elements in \( \mathcal{R}_k^{(i)} \) based on the distance of their corresponding vector to \( x_{k-1}^{(i)} \), i.e.,

\[
\left\| x_{k-1}^{(i)} - x_{k-1}^{(i)} \right\| \leq \ldots \leq \left\| x_{k-1}^{(i)} - x_{k-1}^{(i)} \right\|
\]

We then denote by

\[
S_k^{(i)} := \left\{ \psi_k^{(i)}(j) : j \in [n - 2f - 1] \right\} \cup \{i\},
\]

the set of indices of the vectors selected by the NNA function. From (6), we then have

\[
x_k^{(i)} = \text{NNA} \left( x_{k-1}^{(i)} ; \left\{ x_{k-1}^{(j)} \mid j \in \mathcal{R}_k^{(i)} \right\} \right) = \frac{1}{n - 2f} \sum_{j \in S_k^{(i)}} x_{k-1}^{(j)}.
\]  

We first prove the following few useful lemmas.

**Lemma 6.** For any pair of honest nodes \( p, q \) and coordination round \( k \in [K] \) we obtain that

\[
\left| S_k^{(q)} \setminus S_k^{(p)} \right| = \left| S_k^{(p)} \setminus S_k^{(q)} \right| \leq 2f.
\]  

**Proof.** Consider an arbitrary pair of honest nodes \( p, q \), and coordination round \( k \). By definition of set \( S_k^{(i)} \) for all \( i \in \mathcal{H} \) in (23) we obtain that

\[
\left| S_k^{(q)} \setminus S_k^{(p)} \right| = \left| S_k^{(p)} \setminus S_k^{(q)} \right| \leq n - (n - 2f) = 2f.
\]

\( \square \)

**Lemma 7.** If \( n \geq 11f \) then for each coordination round \( k \in [K] \) we obtain that

\[
\Gamma(x_k) \leq \alpha \Gamma(x_{k-1}) \quad \text{for } \alpha = \frac{9.88f}{n - f}.
\]

**Proof.** Consider two arbitrary honest nodes \( p \) and \( q \) in \( \mathcal{H} \), and an arbitrary \( k \in [K] \). We first introduce some sets that will be used later in the proof. We denote by \( F_p \) the set of Byzantine nodes whose local parameters are selected by \( p \) (using the NNA rule) but not by \( q \) in the \( k \)-th coordination round, i.e.,

\[
F_p := \left\{ i \in [n] \setminus \mathcal{H} \mid i \in S_k^{(p)} \setminus S_k^{(q)} \right\}.
\]

Similarly, \( F_q := \left\{ i \in [n] \setminus \mathcal{H} \mid i \in S_k^{(q)} \setminus S_k^{(p)} \right\} \).

Recall that by Lemma 6 we have \( \left| S_k^{(p)} \setminus S_k^{(q)} \right| \leq 2f \). We consider an arbitrary subset \( H_p \) comprising honest nodes selected by node \( p \) in round \( k \) such that \( |H_p| + |F_p| = 2f \) and \( S_k^{(p)} \setminus S_k^{(q)} \subseteq H_p \), i.e.,

\[
H_p := \left\{ i \in \mathcal{H} \cap S_k^{(p)} \mid |H_p| + |F_p| = 2f, S_k^{(p)} \setminus S_k^{(q)} \subseteq H_p \right\}.
\]  

31
Similarly, $H_q := \{i \in \mathcal{H} \cap S_k^{(q)} \mid |H_q| + |F_q| = 2f, S_k^{(q)} \setminus S_k^{(p)} \subseteq H_q\}$. We let $f_p := |F_p|$ and $f_q := |F_q|$. Note that $f_p + f_q \leq f$. We sort the nodes in $H_p$ based on the distance of their vectors to $x_{k-1}^{(q)}$ (with ties broken arbitrarily). Let $H_p[i]$ denote the $i$-th element in $H_p$ after the sorting. Thus, we have $\|x_{k-1}^{(q)} - x_{k-1}^{(H_p[i])}\| \leq \|x_{k-1}^{(q)} - x_{k-1}^{(H_p[i]+1)}\|$. We do the similar operation on $H_q$.

By definition of NNA (24), we obtain that

$$
\|x_k^{(p)} - x_k^{(q)}\| = \left\| \frac{1}{n - 2f} \sum_{j \in S_k^{(p)}} x_k^{(j)} - \frac{1}{n - 2f} \sum_{j \in S_k^{(q)}} x_k^{(j)} \right\|
$$

$$
= \frac{1}{n - 2f} \left\| \sum_{j \in F_p} x_k^{(j)} + \sum_{j \in H_p} x_k^{(j)} - \sum_{j \in F_q} x_k^{(j)} - \sum_{j \in H_q} x_k^{(j)} \right\|
$$

$$
= \frac{1}{n - 2f} \left\| \left( \sum_{j \in F_p} x_k^{(j)} - \sum_{j \in H_p} x_k^{(H_p[j])} \right) + \left( \sum_{j \in [f+1, 2f-f_p]} x_k^{(H_p[j])} - \sum_{j \in [f_p+1, f]} x_k^{(H_q[j])} \right) \right.
$$

$$
- \left( \sum_{j \in F_q} x_k^{(j)} - \sum_{j \in [f_q]} x_k^{(H_q[j])} \right)
- \left( \sum_{j \in [f+1, 2f-f_q]} x_k^{(H_q[j])} - \sum_{j \in [f_q+1, f]} x_k^{(H_q[j])} \right) \right\|
$$

Therefore,

$$
\|x_k^{(p)} - x_k^{(q)}\| = \frac{1}{n - 2f} \left\| \left( \sum_{j \in F_p} x_k^{(j)} - x_k^{(p)} \right) - \sum_{j \in [f_p]} (x_k^{(H_p[j])} - x_k^{(p)}) \right.
$$

$$
+ \left( \sum_{j \in [f+1, 2f-f_p]} (x_k^{(H_p[j])} - x_k^{(p)}) - \sum_{j \in [f_p+1, f]} (x_k^{(H_q[j])} - x_k^{(p)}) \right) \right.
$$

$$
- \left( \sum_{j \in F_q} x_k^{(j)} - x_k^{(q)} \right)
- \left( \sum_{j \in [f+1, 2f-f_q]} (x_k^{(H_q[j])} - x_k^{(q)}) - \sum_{j \in [f_q+1, f]} (x_k^{(H_q[j])} - x_k^{(q)}) \right) \right\|
$$

32
Using triangle inequality above we obtain that

\[
\| x_k^{(p)} - x_k^{(q)} \| \leq \frac{1}{n-2f} \left[ \left( \sum_{j \in F_p} \| x_{k-1}^{(j)} - x_{k-1}^{(p)} \| + \sum_{j \in [f]} \| x_{k-1}^{(H_p[j])} - x_{k-1}^{(p)} \| \right) 
+ \left( \sum_{j \in [f+1,f+2]} \| x_{k-1}^{(H_p[j])} - x_{k-1}^{(p)} \| + \sum_{j \in [f+1,f]} \| x_{k-1}^{(H_q[j])} - x_{k-1}^{(p)} \| \right) \right]
\]

As the right hand side above is a summation over 4f terms, we obtain that

\[
\| x_k^{(p)} - x_k^{(q)} \| \leq \frac{4f}{(n-2f)^2} \left[ \left( \sum_{j \in F_p} \| x_{k-1}^{(j)} - x_{k-1}^{(p)} \|^2 + \sum_{j \in [f]} \| x_{k-1}^{(H_p[j])} - x_{k-1}^{(p)} \|^2 \right) 
+ \left( \sum_{j \in [f+1,f+2]} \| x_{k-1}^{(H_p[j])} - x_{k-1}^{(p)} \|^2 + \sum_{j \in [f+1,f]} \| x_{k-1}^{(H_q[j])} - x_{k-1}^{(p)} \|^2 \right) \right]
\]

\[
\leq \frac{4f}{(n-2f)^2} \left[ \sum_{j \in F_p} \| x_{k-1}^{(j)} - x_{k-1}^{(p)} \|^2 + \sum_{j \in [f]} \| x_{k-1}^{(H_p[j])} - x_{k-1}^{(p)} \|^2 + \sum_{j \in [f+1,f+2]} \| x_{k-1}^{(H_p[j])} - x_{k-1}^{(p)} \|^2 \right]
\]

Note that \( S_k^{(p)} \) contains at least \( f_p \) Byzantine nodes. Thus there are at most \( n-2f - f_p \) honest nodes in \( S_k^{(p)} \). This implies that there are at least \( f + f_p \) honest nodes that are not selected by node \( p \). We define \( H_p' \) to be a subset of \( f + f_p \) honest nodes not in \( S_k^{(p)} \) that are farthest from \( x_k^{(p)} \). We sort the nodes in \( H_p' \) such that \( \| x_k^{(p)} - x_{k-1}^{(H_p'[i])} \| \leq \| x_k^{(p)} - x_{k-1}^{(H_p'[i+1])} \| \) for \( i = 1, \ldots, f + f_p - 1 \). Note that for each Byzantine node in \( S_k^{(p)} \) there is an honest node in set \( R_k^{(p)} \backslash S_k^{(p)} \). Thus, by definition of \( S_k^{(p)} \) in (23) we obtain that

\[
\sum_{j \in F_p} \| x_{k-1}^{(j)} - x_{k-1}^{(p)} \|^2 \leq \sum_{j \in [f+1,f+f_p]} \| x_{k-1}^{(H_p'[j])} - x_{k-1}^{(p)} \|^2 \] (27)
By definition of $H'_p$, for each $j \in [f]$, $\|x^{(p)}_{k-1} - x^{(H'_q[j])}_{k-1}\| \leq \|x^{(p)}_{k-1} - x^{(H'_p[j])}_{k-1}\|$. Thus,

$$
\sum_{j \in [f]} \|x^{(H'_q[j])}_{k-1} - x^{(p)}_{k-1}\|^2 \leq \sum_{j \in [f]} \|x^{(H'_p[j])}_{k-1} - x^{(p)}_{k-1}\|^2.
$$

(28)

From (27) and (28) we obtain that

$$
\sum_{j \in F_p} \|x^{(j)}_{k-1} - x^{(p)}_{k-1}\|^2 + \sum_{j \in [f]} \|x^{(H'_q[j])}_{k-1} - x^{(p)}_{k-1}\|^2 \leq \sum_{j \in H'_p} \|x^{(j)}_{k-1} - x^{(p)}_{k-1}\|^2.
$$

Therefore, we have

$$
\sum_{j \in F_p} \|x^{(j)}_{k-1} - x^{(p)}_{k-1}\|^2 + \sum_{j \in [f]} \|x^{(H'_q[j])}_{k-1} - x^{(p)}_{k-1}\|^2 + \sum_{j \in [f+1, 2f-f_p]} \|x^{(H'_p[j])}_{k-1} - x^{(p)}_{k-1}\|^2 \leq \sum_{j \in H'_p} \|x^{(j)}_{k-1} - x^{(p)}_{k-1}\|^2.
$$

(29)

Similarly,

$$
\sum_{j \in F_q} \|x^{(j)}_{k-1} - x^{(q)}_{k-1}\|^2 + \sum_{j \in [f]} \|x^{(H'_q[j])}_{k-1} - x^{(q)}_{k-1}\|^2 + \sum_{j \in [f+1, 2f-f_q]} \|x^{(H'_p[j])}_{k-1} - x^{(q)}_{k-1}\|^2 \leq \sum_{j \in H'_q} \|x^{(j)}_{k-1} - x^{(q)}_{k-1}\|^2.
$$

(30)

Substituting from (29) and (30) in (26) we obtain that

$$
\|x^{(p)}_k - x^{(q)}_k\|^2 \leq \frac{4f}{(n-2f)^2} \left[ \sum_{j \in H'_p} \|x^{(j)}_{k-1} - x^{(p)}_{k-1}\|^2 + \sum_{j \in H'_q} \|x^{(j)}_{k-1} - x^{(q)}_{k-1}\|^2 \right].
$$

As the above holds true for an arbitrary pair of honest nodes $p$ and $q$, by averaging over all such possible pairs we obtain that

$$
\frac{1}{(n-f)^2} \sum_{p,q \in H} \|x^{(p)}_k - x^{(q)}_k\|^2 \leq \frac{8f(n-f)}{(n-2f)^2} \frac{1}{(n-f)^2} \sum_{p,q \in H} \|x^{(p)}_{k-1} - x^{(q)}_{k-1}\|^2.
$$

Recall the notation $\Gamma(\cdot)$. The above implies that

$$
\Gamma(x_k) \leq \frac{8f(n-f)}{(n-2f)^2} \Gamma(x_{k-1}).
$$

As $n \geq 11f$, $\frac{(n-f)^2}{(n-2f)^2} \leq \frac{100}{81}$. Using this above proves the lemma, i.e.,

$$
\Gamma(x_k) \leq \frac{800f}{81(n-f)} \Gamma(x_{k-1}) \leq \frac{9.88f}{n-f} \Gamma(x_{k-1}).
$$
Thus, of the NNA operator in (24) we have that the trivially from Lemma 7 for the stated value of $\alpha$. The first condition of Proof. Upon decomposing the right hand side we obtain that

$$x_k^{(i)} - x_{k-1} = \frac{1}{n-2f} \sum_{j \in S_k^{(i)}} x_{k-1}^{(j)} - x_{k-1}^{(i)}$$

Upon decomposing the right hand side we obtain that

$$x_k^{(i)} - x_{k-1} = \left(\frac{1}{n-2f} - \frac{1}{n-\lambda}\right) \sum_{j \in S_k^{(i)} \cap H} (x_{k-1}^{(j)} - x_{k-1}^{(i)}) + \frac{1}{n-2f} \sum_{j \in S_k^{(i)} \setminus H} (x_{k-1}^{(j)} - x_{k-1}^{(i)})$$

Thus,

$$x_k^{(i)} - x_{k-1} = \frac{1}{(n-f)(n-2f)} \left( f \sum_{j \in S_k^{(i)} \cap H} (x_{k-1}^{(j)} - x_{k-1}^{(i)}) + (n-f) \sum_{j \in S_k^{(i)} \setminus H} (x_{k-1}^{(j)} - x_{k-1}^{(i)}) \right) - (n-2f) \sum_{j \in H \setminus S_k^{(i)}} (x_{k-1}^{(j)} - x_{k-1}^{(i)})$$

By taking norm on both sides and then applying the triangle inequality we obtain that

$$\left\| x_k^{(i)} - x_{k-1} \right\| \leq \frac{1}{(n-f)(n-2f)} \left( f \sum_{j \in S_k^{(i)} \cap H} \left\| x_{k-1}^{(j)} - x_{k-1}^{(i)} \right\| \right)$$

$$+ (n-f) \sum_{j \in S_k^{(i)} \setminus H} \left\| x_{k-1}^{(j)} - x_{k-1}^{(i)} \right\| + (n-2f) \sum_{j \in H \setminus S_k^{(i)}} \left\| x_{k-1}^{(j)} - x_{k-1}^{(i)} \right\| \right).$$

Lemma 1. Suppose that $n \geq 11f$. For any $K \geq 1$, the coordination step of Algorithm guarantees $(\alpha, \lambda)$-reduction for

$$\alpha = \left( \frac{9.88f}{n-f} \right)^K \quad \text{and} \quad \lambda = \frac{9f}{n-\lambda} \cdot \min \left\{ K, \frac{1}{(1-\sqrt{\alpha})^2} \right\}.$$
Now let $v := \left| S_k^{(i)} \cap \mathcal{H} \right|$. We then have $v = \left| S_k^{(i)} \right| + |\mathcal{H}| - \left| S_k^{(i)} \cup \mathcal{H} \right| \geq n - 2f + n - f - n = n - 3f$.

Also, $\left| S_k^{(i)} \setminus \mathcal{H} \right| = n - 2f - v$ and $\left| \mathcal{H} \setminus S_k^{(i)} \right| = n - f - v$. There for the number $A(v)$ of items that are added in (31) is

$$A(v) = f + (n - 2f - v)(n - f) + (n - 2f)(n - f - v) = 2(n - 2f)(n - f - v), \quad (32)$$

which is decreasing in $v$. There for the maximum of $A(v)$ is reached for $v = n - 3f$ and we have $A(v) \leq 4f(n - 2f)$. Therefore, (31) yields

$$\left\| x_k^{(i)} - \bar{x}_{k-1} \right\|^2 \leq \frac{4f(n - 2f)}{(n - f)^2(n - 2f)^2} \left( f \sum_{j \in S_k^{(i)} \setminus \mathcal{H}} \left\| x_{k-1}^{(j)} - x_{k-1}^{(i)} \right\|^2 + (n - f) \sum_{j \in S_k^{(i)} \setminus \mathcal{H}} \left\| x_{k-1}^{(j)} - x_{k-1}^{(i)} \right\|^2 \right)$$

$$\leq \frac{4f(n - 2f)}{(n - f)^2(n - 2f)^2} \left( f \sum_{j \in \mathcal{H}} \left\| x_{k-1}^{(j)} - x_{k-1}^{(i)} \right\|^2 + (n - f) \sum_{j \in \mathcal{H}} \left\| x_{k-1}^{(j)} - x_{k-1}^{(i)} \right\|^2 \right)$$

$$\leq \frac{4f(n - 2f)(2n - 2f)}{(n - f)^2(n - 2f)^2} \sum_{j \in \mathcal{H}} \left\| x_{k-1}^{(j)} - x_{k-1}^{(i)} \right\|^2$$

$$= \frac{8f}{(n - f)(n - 2f)} \sum_{j \in \mathcal{H}} \left\| x_{k-1}^{(j)} - x_{k-1}^{(i)} \right\|^2. \quad (33)$$

But now note that

$$\left\| \bar{x}_k - \bar{x}_{k-1} \right\|^2 = \left\| \frac{1}{n - f} \sum_{i \in \mathcal{H}} x_k^{(i)} - \bar{x}_{k-1} \right\|^2$$

$$\leq \frac{1}{n - f} \sum_{i \in \mathcal{H}} \left\| x_k^{(i)} - \bar{x}_{k-1} \right\|^2.$$

Combining above with (33) then yields

$$\left\| \bar{x}_k - \bar{x}_{k-1} \right\|^2 \leq \frac{8f}{n - 2f} \cdot \frac{1}{(n - f)^2} \sum_{i,j \in \mathcal{H}} \left\| x_{k-1}^{(j)} - x_{k-1}^{(i)} \right\|^2.$$

Using the notation $\Gamma(\cdot)$, we then have

$$\left\| \bar{x}_k - \bar{x}_{k-1} \right\|^2 \leq \frac{8f}{n - 2f} \Gamma(x_{k-1}) \leq \frac{8f\alpha^{k-1}}{n - 2f} \Gamma(x_0), \quad (34)$$

36
where in the second inequality we used Lemma 7. Now note that
\[ \| \bar{x}_K - \bar{x}_0 \|^2 = \left\| \sum_{k \in [K]} (\bar{x}_k - \bar{x}_{k-1}) \right\|^2 \]
\[ = \left\langle \sum_{k \in [K]} (\bar{x}_k - \bar{x}_{k-1}), \sum_{k \in [K]} (\bar{x}_k - \bar{x}_{k-1}) \right\rangle \]
\[ = \sum_{k,l \in [K]} \langle \bar{x}_k - \bar{x}_{k-1}, \bar{x}_l - \bar{x}_{l-1} \rangle. \]

By the Cauchy–Schwarz inequality we then have
\[ \| \bar{x}_K - \bar{x}_0 \|^2 \leq \sum_{k,l \in [K]} \sqrt{\| \bar{x}_k - \bar{x}_{k-1} \|^2 \cdot \| \bar{x}_l - \bar{x}_{l-1} \|^2}. \]

Combining this with \ref{eq:34} we obtain that
\[ \| \bar{x}_K - \bar{x}_0 \|^2 \leq \frac{8f}{n-2f} \Gamma (x_0) \sum_{k,l \in [K]} \sqrt{\alpha^{k-1} \alpha^{l-1}} \]
\[ = \frac{8f}{n-2f} \Gamma (x_0) \sum_{k \in [K]} (\sqrt{\alpha})^{k-1} \sum_{l \in [K]} (\sqrt{\alpha})^{l-1}. \]

Now since \( \alpha < 1 \) we have \( \sum_{k \in [K]} (\sqrt{\alpha})^{k-1} \leq K, \) and thus
\[ \| \bar{x}_K - \bar{x}_0 \|^2 \leq \frac{8fK^2}{n-2f} \Gamma (x_0). \] (35)

Moreover, we have
\[ \sum_{k \in [K]} (\sqrt{\alpha})^{k-1} \leq \sum_{k=1}^{\infty} (\sqrt{\alpha})^{k-1} = \frac{1}{1 - \sqrt{\alpha}}, \]
and thus
\[ \| \bar{x}_K - \bar{x}_0 \|^2 \leq \frac{8f}{n-2f} \cdot \frac{1}{(1 - \sqrt{\alpha})^2} \Gamma (x_0). \] (36)

Combining (35) and (36) and noting that \( \frac{8f}{n-2f} \leq \frac{9f}{n-f} \) for \( n \geq 11f \) proves the lemma. \( \square \)
B.6 Proof of Lemma 2

We recall the lemma below.

**Lemma 2.** Suppose that assumptions 1, 2, and 3 hold true. Consider Algorithm 1 with \( \gamma \leq \frac{1 - \alpha}{L \sqrt{34\alpha(1 + \alpha)}} \). Suppose that the coordination step satisfies \((\alpha, \lambda)\)-reduction for \( \alpha < 1 \). For each \( t \in [T] \), we obtain that

\[
E[\Gamma(\theta_t)] \leq E(\alpha) \gamma^2 \left( 2\sigma^2 \frac{1 - \beta}{1 + \beta} + 3R^2 \right),
\]

and

\[
E[\Gamma(m_t)] \leq 6\sigma^2 \left( \frac{1 - \beta}{1 + \beta} \right) + 9R^2 + 18L^2 \gamma^2 E(\alpha) \left( 2\sigma^2 \frac{1 - \beta}{1 + \beta} + 3R^2 \right),
\]

where

\[
E(\alpha) := \frac{18\alpha(1 + \alpha)}{(1 - \alpha)^2}.
\]

**Proof.** Consider an arbitrary step \( t \in [T] \). The proof comprises 3 steps.

**Step i.** In this step, we analyse the growth of \( E[\Gamma(\theta_t)] \). From Algorithm 1 recall that for all \( i \in \mathcal{H} \), we have \( \theta_{t+1/2}^{(i)} = \theta_t^{(i)} - \gamma m_t^{(i)} \). As \((x + y)^2 \leq (1 + c)x^2 + (1 + 1/c)y^2\) for any \( c > 0 \), we obtain for all \( i, j \in \mathcal{H} \) that

\[
E \left[ \left\| \theta_{t+1/2}^{(i)} - \theta_{t+1/2}^{(j)} \right\|^2 \right] \leq E \left[ \left\| \theta_t^{(i)} - \theta_t^{(j)} - \gamma (m_t^{(i)} - m_t^{(j)}) \right\|^2 \right] \leq (1 + c) E \left[ \left\| \theta_t^{(i)} - \theta_t^{(j)} \right\|^2 \right] + \left( 1 + \frac{1}{c} \right) \gamma^2 E \left[ \left\| m_t^{(i)} - m_t^{(j)} \right\|^2 \right].
\]

Thus, by definition of notation \( \Gamma(\ast_t) \), we have

\[
E \left[ \Gamma(\theta_{t+1/2}) \right] \leq (1 + c) E[\Gamma(\theta_t)] + \left( 1 + \frac{1}{c} \right) \gamma^2 E[\Gamma(m_t)]. \tag{37}
\]

Recall that, as shown in Lemma 1, the coordination step of Algorithm 1 satisfies \((\alpha, \lambda)\)-reduction. Thus, for all \( t \), we have \( \Gamma(\theta_{t+1}) \leq \alpha \Gamma(\theta_{t+1/2}) \). Substituting from above we obtain that

\[
E \left[ \Gamma(\theta_{t+1}) \right] \leq (1 + c)\alpha E[\Gamma(\theta_t)] + \left( 1 + \frac{1}{c} \right) \alpha \gamma^2 E[\Gamma(m_t)]. \tag{38}
\]

**Step ii.** In this step, we analyse the growth of \( E[\Gamma(m_t)] \). From the definition of momentum
In (4), we obtain for all $i, j \in \mathcal{H}$ that

$$
\mathbb{E} \left[ \left\| m_t^{(i)} - m_t^{(j)} \right\|^2 \right] = \mathbb{E} \left[ \left\| (1 - \beta) \sum_{s=1}^{t} \beta^{t-s} (g_s^{(i)} - g_s^{(j)}) \right\|^2 \right]
$$

$$
= (1 - \beta)^2 \mathbb{E} \left[ \left\| \sum_{s=1}^{t} \beta^{t-s} (g_s^{(i)} - \nabla Q^{(i)} (\theta_s^{(i)}) + \nabla Q^{(i)} (\theta_s^{(i)})) - \nabla Q^{(j)} (\theta_s^{(j)}) + \nabla Q^{(j)} (\theta_s^{(j)}) - g_s^{(j)}) \right\|^2 \right].
$$

Using the fact that $(x + y + z)^2 \leq 3x^2 + 3y^2 + 3z^2$, from above we obtain that

$$
\mathbb{E} \left[ \left\| m_t^{(i)} - m_t^{(j)} \right\|^2 \right] \leq 3(1 - \beta)^2 \mathbb{E} \left[ \left\| \sum_{s=1}^{t} \beta^{t-s} (g_s^{(i)} - \nabla Q^{(i)} (\theta_s^{(i)})) \right\|^2 \right]
$$

$$
+ 3(1 - \beta)^2 \mathbb{E} \left[ \left\| \sum_{s=1}^{t} \beta^{t-s} (g_s^{(j)} - \nabla Q^{(j)} (\theta_s^{(j)})) \right\|^2 \right]
$$

$$
+ 3(1 - \beta)^2 \mathbb{E} \left[ \left\| \sum_{s=1}^{t} \beta^{t-s} (\nabla Q^{(i)} (\theta_s^{(i)})) - (\nabla Q^{(j)} (\theta_s^{(j)})) \right\|^2 \right].
$$

(39)

Consider an arbitrary $i \in \mathcal{H}$, and denote

$$
A_t := \mathbb{E} \left[ \left\| \sum_{s=1}^{t} \beta^{t-s} (g_s^{(i)} - \nabla Q^{(i)} (\theta_s^{(i)})) \right\|^2 \right].
$$

(40)

Note that

$$
A_t = \mathbb{E} \left[ \left\| \sum_{s=1}^{t} \beta^{t-s} (g_s^{(i)} - \nabla Q^{(i)} (\theta_s^{(i)})) \right\|^2 \right]
$$

$$
= \mathbb{E} \left[ \left\| \sum_{s=1}^{t-1} \beta^{t-s} (g_s^{(i)} - \nabla Q^{(i)} (\theta_s^{(i)})) + (g_t^{(i)} - \nabla Q^{(i)} (\theta_t^{(i)})) \right\|^2 \right].
$$

From above we obtain that

$$
A_t = \mathbb{E} \left[ \left\| \sum_{s=1}^{t-1} \beta^{t-s} (g_s^{(i)} - \nabla Q^{(i)} (\theta_s^{(i)})) \right\|^2 \right] + \mathbb{E} \left[ \left\| g_t^{(i)} - \nabla Q^{(i)} (\theta_t^{(i)}) \right\|^2 \right]
$$

$$
+ \mathbb{E} \left[ \left\langle \sum_{s=1}^{t-1} \beta^{t-s} (g_s^{(i)} - \nabla Q^{(i)} (\theta_s^{(i)})), g_t^{(i)} - \nabla Q^{(i)} (\theta_t^{(i)}) \right\rangle \right].
$$

Recall that in the above, $\mathbb{E} [\cdot] = \mathbb{E}_1 [\ldots \mathbb{E}_t [\cdot]]$. Thus, due to Assumption 2, we have

$$
\mathbb{E} \left[ \left\| g_t^{(i)} - \nabla Q^{(i)} Q (\theta_t^{(i)}) \right\|^2 \right] \leq
$$

39
Using this above we obtain that
\[
A_t \leq \mathbb{E} \left[ \left( \sum_{s=1}^{t-1} \beta^{t-s} \left( g_s^{(i)} - \nabla Q^{(i)} \left( \theta_s^{(i)} \right) \right) \right)^2 + \sigma^2 \right] + \mathbb{E} \left[ \left( \sum_{s=1}^{t-1} \beta^{t-s} \left( g_s^{(i)} - \nabla Q^{(i)} \left( \theta_s^{(i)} \right) \right) , g_t^{(i)} - \nabla Q^{(i)} \left( \theta_t^{(i)} \right) \right) \right] .
\]

Also, by tower rule we have
\[
\mathbb{E} \left[ \left( \sum_{s=1}^{t-1} \beta^{t-s} \left( g_s^{(i)} - \nabla Q^{(i)} \left( \theta_s^{(i)} \right) \right) , g_t^{(i)} - \nabla Q^{(i)} \left( \theta_t^{(i)} \right) \right) \right] = \\
\mathbb{E}_1 \left[ \ldots \mathbb{E}_t \left[ \left( \sum_{s=1}^{t-1} \beta^{t-s} \left( g_s^{(i)} - \nabla Q^{(i)} \left( \theta_s^{(i)} \right) \right) , g_t^{(i)} - \nabla Q^{(i)} \left( \theta_t^{(i)} \right) \right) \right] \right] .
\]

By the definition of conditional expectation \( \mathbb{E}_t \left[ \cdot \right] \), we have
\[
\mathbb{E}_t \left[ \left( \sum_{s=1}^{t-1} \beta^{t-s} \left( g_s^{(i)} - \nabla Q^{(i)} \left( \theta_s^{(i)} \right) \right) , g_t^{(i)} - \nabla Q^{(i)} \left( \theta_t^{(i)} \right) \right) \right] = \\
\left( \sum_{s=1}^{t-1} \beta^{t-s} \left( g_s^{(i)} - \nabla Q^{(i)} \left( \theta_s^{(i)} \right) \right) , \mathbb{E}_t \left[ g_t^{(i)} - \nabla Q^{(i)} \left( \theta_t^{(i)} \right) \right] \right) .
\]

By Assumption 2, we obtain that \( \mathbb{E}_t \left[ g_t^{(i)} - \nabla Q^{(i)} \left( \theta_t^{(i)} \right) \right] = \nabla Q^{(i)} \left( \theta_t^{(i)} \right) - \nabla Q^{(i)} \left( \theta_t^{(i)} \right) = 0 \). Using this above implies that
\[
\mathbb{E} \left[ \left( \sum_{s=1}^{t-1} \beta^{t-s} \left( g_s^{(i)} - \nabla Q^{(i)} \left( \theta_s^{(i)} \right) \right) , g_t^{(i)} - \nabla Q^{(i)} \left( \theta_t^{(i)} \right) \right) \right] = 0.
\]

Substituting from above in (41) we obtain that
\[
A_t \leq \mathbb{E} \left[ \left( \sum_{s=1}^{t-1} \beta^{t-s} \left( g_s^{(i)} - \nabla Q^{(i)} \left( \theta_s^{(i)} \right) \right) \right)^2 + \sigma^2 \right] + \mathbb{E} \left[ \left( \sum_{s=1}^{t-1} \beta^{t-s} \left( g_s^{(i)} - \nabla Q^{(i)} \left( \theta_s^{(i)} \right) \right) , g_t^{(i)} - \nabla Q^{(i)} \left( \theta_t^{(i)} \right) \right) \right] = \\
\beta^2 \mathbb{E} \left[ \left( \sum_{s=1}^{t-1} \beta^{t-1-s} \left( g_s^{(i)} - \nabla Q^{(i)} \left( \theta_s^{(i)} \right) \right) \right)^2 + \sigma^2 \right] + \beta^2 A_{t-1} + \sigma^2.
\]

Note from the definition of \( A_t \) in (40) that, under Assumption 2, \( A_1 \leq \sigma^2 \). Thus, from above we obtain that
\[
A_t := \mathbb{E} \left[ \left( \sum_{s=1}^t \beta^{t-s} \left( g_s^{(i)} - \nabla Q^{(i)} \left( \theta_s^{(i)} \right) \right) \right)^2 \right] \leq \sigma^2 \sum_{s=0}^{t-1} \beta^{2s} \leq \frac{\sigma^2}{1 - \beta^2}.
\]
By Jensen’s inequality, we obtain that
\[
E \left[ \Gamma (m_t) \right] = \frac{1}{(n-f)^2} \sum_{i,j \in \mathcal{H}} E \left[ \left\| m_t^{(i)} - m_t^{(j)} \right\|^2 \right]
\leq 6(1 - \beta)^2 \frac{\sigma^2}{1 - \beta^2} + \frac{3(1 - \beta)^2}{(n-f)^2} \sum_{i,j \in \mathcal{H}} \left[ \sum_{s=1}^t \beta^{t-s} \left( \nabla Q^{(i)} \left( \theta_s^{(i)} \right) - \nabla Q^{(j)} \left( \theta_s^{(j)} \right) \right) \right]^2.
\tag{42}
\]

Let us denote
\[
C_t := \frac{1}{(n-f)^2} \sum_{i,j \in \mathcal{H}} \left[ \sum_{s=1}^t \beta^{t-s} \left( \nabla Q^{(i)} \left( \theta_s^{(i)} \right) - \nabla Q^{(j)} \left( \theta_s^{(j)} \right) \right) \right]^2.
\tag{43}
\]

Now note that
\[
C_t = \frac{1}{(n-f)^2} \sum_{i,j \in \mathcal{H}} \left[ \sum_{s=1}^{t-1} \beta^{t-1-s} \left( \nabla Q^{(i)} \left( \theta_s^{(i)} \right) - \nabla Q^{(j)} \left( \theta_s^{(j)} \right) \right) \right] + \left( \nabla Q^{(i)} \left( \theta_t^{(i)} \right) - \nabla Q^{(j)} \left( \theta_t^{(j)} \right) \right)
\]

By Jensen’s inequality, we obtain that
\[
C_t \leq \frac{1}{(n-f)^2} \sum_{i,j \in \mathcal{H}} \beta \left[ \sum_{s=1}^{t-1} \beta^{t-1-s} \left( \nabla Q^{(i)} \left( \theta_s^{(i)} \right) - \nabla Q^{(j)} \left( \theta_s^{(j)} \right) \right) \right]^2 + \frac{1}{(n-f)^2} \sum_{i,j \in \mathcal{H}} (1 - \beta) \left[ \frac{1}{1 - \beta} \left( \nabla Q^{(i)} \left( \theta_t^{(i)} \right) - \nabla Q^{(j)} \left( \theta_t^{(j)} \right) \right) \right]^2,
\]

By Assumption 1 we have that \( \left\| \nabla Q^{(i)} \left( \theta_t^{(i)} \right) - \nabla Q^{(j)} \left( \theta_t^{(j)} \right) \right\|^2 \leq L^2 \left\| \theta_t^{(i)} - \theta_t^{(j)} \right\|^2 \) and \( \left\| \nabla Q^{(j)} \left( \theta_t^{(j)} \right) - \nabla Q^{(j)} \left( \theta_t^{(j)} \right) \right\| \]

\[
\leq L^2 \left\| \theta_t^{(i)} - \theta_t^{(j)} \right\|^2.
\]

Using this above we obtain that
\[
C_t \leq \beta C_{t-1} + \frac{3L^2}{(n-f)^2(1 - \beta)} \sum_{i,j \in \mathcal{H}} \left( E \left[ \left\| \theta_t^{(i)} - \theta_t^{(j)} \right\|^2 \right] + E \left[ \left\| \theta_t^{(i)} - \bar{\theta}_t \right\|^2 \right] \right) + \frac{3}{(n-f)^2(1 - \beta)} \sum_{i,j \in \mathcal{H}} \left\| \nabla Q^{(i)} \left( \bar{\theta}_t \right) - \nabla Q^{(j)} \left( \bar{\theta}_t \right) \right\|^2.
\]

41
Using Assumption 3 above we obtain that
\[ C_t \leq \beta C_{t-1} + \frac{6L^2}{(n-f)(1-\beta)} \sum_{i \in H} \mathbb{E} \left[ \| \theta_t^{(i)} - \bar{\theta}_t \|^2 \right] + \frac{3R^2}{1-\beta}. \] (44)

Now note that
\[
\frac{1}{n-f} \sum_{i \in H} \mathbb{E} \left[ \| \theta_t^{(i)} - \bar{\theta}_t \|^2 \right] = \frac{1}{n-f} \sum_{i \in H} \mathbb{E} \left[ \left\| \sum_{j \in H} (\theta_t^{(i)} - \theta_t^{(j)}) \right\|^2 \right] \\
\leq \frac{1}{(n-f)^2} \sum_{i,j \in H} \mathbb{E} \left[ \| \theta_t^{(i)} - \theta_t^{(j)} \|^2 \right] = \mathbb{E} \left[ \Gamma (\theta_t) \right].
\]

Combining this with (44), we obtain that
\[ C_t \leq \beta C_{t-1} + \frac{6L^2}{1-\beta} \mathbb{E} \left[ \Gamma (\theta_t) \right] + \frac{3R^2}{1-\beta}. \]

Note that by definition, \( C_0 = 0 \). Thus, from above we obtain that
\[
C_t \leq \frac{6L^2}{1-\beta} \sum_{s=1}^{t} \beta^{t-s} \mathbb{E} \left[ \Gamma (\theta_s) \right] + \frac{3R^2}{1-\beta} \sum_{s=1}^{t} \beta^{t-s} \\
\leq \frac{6L^2}{1-\beta} \sum_{s=1}^{\infty} \beta^{t-s} \mathbb{E} \left[ \Gamma (\theta_s) \right] + \frac{3R^2}{1-\beta} \sum_{s=0}^{\infty} \beta^s \\
= \frac{6L^2}{1-\beta} \sum_{s=1}^{\infty} \beta^{t-s} \mathbb{E} \left[ \Gamma (\theta_s) \right] + \frac{3R^2}{(1-\beta)^2}.
\]

Substituting from above in (42) we obtain that
\[ \mathbb{E} \left[ \Gamma (m_t) \right] \leq 6\sigma^2 \left( \frac{1-\beta}{1+\beta} \right) + 18(1-\beta)L^2 \sum_{s=1}^{t} \beta^{t-s} \mathbb{E} \left[ \Gamma (\theta_s) \right] + 9R^2. \] (45)

Recall from (38) that
\[ \mathbb{E} \left[ \Gamma (\theta_{t+1}) \right] \leq (1+c)\alpha \mathbb{E} \left[ \Gamma (\theta_t) \right] + \left( 1 + \frac{1}{c} \right) \alpha \gamma^2 \mathbb{E} \left[ \Gamma (m_t) \right]. \] (46)

In the next and the final step we use the results derived in (45) and (46) above to conclude the proof.

**Step iii.** Now in (46) we let
\[ c = \frac{1-\alpha}{2\alpha} > 0. \] (47)
Substituting this in (46) we obtain that
\[
\mathbb{E} \left[ \Gamma (\theta_{t+1}) \right] \leq \frac{1 + \alpha}{2} \mathbb{E} [ \Gamma (\theta_t)] + \left( \frac{1 + \alpha}{1 - \alpha} \right) \alpha \gamma^2 \mathbb{E} [\Gamma (m_t)].
\]

Substituting from (45) above we obtain that
\[
\mathbb{E} \left[ \Gamma (\theta_{t+1}) \right] \leq \left( \frac{1 + \alpha}{2} \right) \mathbb{E} [ \Gamma (\theta_t)] \\
+ \frac{\alpha(1 + \alpha)}{1 - \alpha} \gamma^2 \left( 6\sigma^2 \left( \frac{1 - \beta}{1 + \beta} \right) + 9R^2 + 18(1 - \beta) L^2 \sum_{s=1}^{t-1} \beta^{t-s} \mathbb{E} [\Gamma (\theta_s)] \right) \\
= \left( \frac{1 + \alpha}{2} \right) \mathbb{E} [ \Gamma (\theta_t)] + \frac{\alpha(1 + \alpha)}{1 - \alpha} \left( 6\sigma^2 \frac{1 - \beta}{1 + \beta} + 9R^2 \right) \\
+ 18(1 - \beta) \gamma^2 L^2 \left( \frac{\alpha(1 + \alpha)}{1 - \alpha} \right) \sum_{s=1}^{t} \beta^{t-s} \mathbb{E} [\Gamma (\theta_s)].
\]

(48)

As we assume that \( \gamma \leq \frac{1 - \alpha}{L\sqrt{54\alpha(1 + \alpha)}} \), we have
\[
\gamma^2 L^2 \leq \frac{(1 - \alpha)^2}{54\alpha (1 + \alpha)}.
\]

Using the above in (48) we obtain that
\[
\mathbb{E} \left[ \Gamma (\theta_{t+1}) \right] \leq \left( \frac{1 + \alpha}{2} \right) \mathbb{E} [ \Gamma (\theta_t)] \\
+ \frac{\alpha(1 + \alpha)}{1 - \alpha} \gamma^2 \left( \frac{1 - \beta}{1 + \beta} \right) 6\sigma^2 + 9R^2 \right) \\
+ \left( \frac{1 - \alpha}{3} \right) (1 - \beta) \sum_{s=1}^{t} \beta^{t-s} \mathbb{E} [\Gamma (\theta_s)].
\]

For convenience, we denote
\[
D = \frac{\alpha(1 + \alpha)\gamma^2}{1 - \alpha} \left( 6\sigma^2 \frac{1 - \beta}{1 + \beta} + 9R^2 \right).
\]

Thus,
\[
\mathbb{E} \left[ \Gamma (\theta_{t+1}) \right] \leq \left( \frac{1 + \alpha}{2} \right) \mathbb{E} [ \Gamma (\theta_t)] + D + \left( \frac{1 - \alpha}{3} \right) (1 - \beta) \sum_{s=1}^{t} \beta^{t-s} \mathbb{E} [\Gamma (\theta_s)].
\]

(49)

As (49) above holds true for an arbitrary \( t \in [T] \), we reason below by mathematical induction that for all \( t \),
\[
\mathbb{E} [\Gamma (\theta_t)] \leq \frac{6D}{1 - \alpha}.
\]

(50)
First, note that, as \( \theta^{(i)}_1 = \theta^{(j)}_1 \) for all \( i, j \in \mathcal{H} \), the above is trivially true for \( t = 1 \). Second, let us assume that (50) is true for all \( t \in [T] \setminus \{T\} \). Then, from (49) we obtain that

\[
\mathbb{E} \left[ \Gamma \left( \theta_{t+1} \right) \right] \leq \left( \frac{1 + \alpha}{2} \right) \frac{6D}{1 - \alpha} + D + \left( \frac{1 - \alpha}{3} \right) (1 - \beta) \sum_{s=1}^{t} \beta^{t-s} \frac{6D}{1 - \alpha}
\]

\[
= \left( \frac{1 + \alpha}{2} \right) \frac{6D}{1 - \alpha} + D + 2D = \frac{6D}{1 - \alpha}.
\]

Thus, (50) holds true for \( t + 1 \). Therefore, for all \( t \in [T] \), we have

\[
\mathbb{E} \left[ \Gamma \left( \theta_t \right) \right] \leq \frac{\alpha(1 + \alpha)^2}{(1 - \alpha)^2} \left( 36\sigma^2 \frac{1 - \beta}{1 + \beta} + 54R^2 \right)
\]

We now define \( E(\alpha) := \frac{18\alpha(1+\alpha)}{(1-\alpha)^2} \). We then have

\[
\mathbb{E} \left[ \Gamma \left( \theta_t \right) \right] \leq E(\alpha) \gamma^2 \left( 2\sigma^2 \frac{1 - \beta}{1 + \beta} + 3R^2 \right).
\]

Combining this with (45), we then obtain

\[
\mathbb{E} \left[ \Gamma \left( m_t \right) \right] \leq 6\sigma^2 \left( \frac{1 - \beta}{1 + \beta} \right) + 18 (1 - \beta) L^2 \sum_{s=1}^{t} \beta^{t-s} \mathbb{E} \left[ \Gamma \left( \theta_s \right) \right] + 9R^2
\]

\[
\leq 6\sigma^2 \left( \frac{1 - \beta}{1 + \beta} \right) + 9R^2 + 18 (1 - \beta) L^2 \sum_{s=1}^{t} \beta^{t-s} \left( E(\alpha) \gamma^2 \left( 2\sigma^2 \frac{1 - \beta}{1 + \beta} + 3R^2 \right) \right)
\]

\[
\leq 6\sigma^2 \left( \frac{1 - \beta}{1 + \beta} \right) + 9R^2 + 18 (1 - \beta) L^2 \sum_{s=0}^{\infty} \beta^s \left( E(\alpha) \gamma^2 \left( 2\sigma^2 \frac{1 - \beta}{1 + \beta} + 3R^2 \right) \right)
\]

\[
= 6\sigma^2 \left( \frac{1 - \beta}{1 + \beta} \right) + 9R^2 + 18 L^2 \gamma^2 E(\alpha) \left( 2\sigma^2 \frac{1 - \beta}{1 + \beta} + 3R^2 \right)
\]

\( \square \)
B.7 Proof of Lemma \textsuperscript{3}

Recall that
\[ \nabla Q_t := \frac{1}{n-f} \sum_{i \in \mathcal{H}} \nabla Q^{(i)}(\theta^{(i)}_t), \]
and that
\[ \delta_t := m_t - \nabla Q_t. \] (51)

Also, we recall the lemma below.

\begin{lemma}
Suppose that assumptions \textsuperscript{7} and \textsuperscript{2} hold true. Consider Algorithm \textsuperscript{2}. For all \( t \in [T] \), we obtain that
\[ \mathbb{E} \left[ \|\delta_{t+1}\|^2 \right] \leq \beta^2 (1 + 4L\gamma)(1 + \frac{9}{8}L\gamma) \mathbb{E} \left[ \|\delta_t\|^2 \right] + \frac{3}{4} \beta^2 L\gamma (1 + 4L\gamma) \mathbb{E} \left[ \|\nabla Q(\bar{\theta}_t)\|^2 \right] + 9\beta^2 L\left( 1 + \frac{1}{4L\gamma} \right) \left( \mathbb{E} \left[ \Gamma (\theta_{t+1}) \right] + \mathbb{E} \left[ \Gamma (\theta_t) \right] + \mathbb{E} \left[ \|\bar{\theta}_{t+1} - \bar{\theta}_{t+1/2}\|^2 \right] \right) + \frac{9}{4} \beta^2 L\gamma (1 + 4L\gamma) L^2 \mathbb{E} \left[ \Gamma (\theta_t) \right] + \frac{(1 - \beta)^2 \sigma^2}{n - f}. \]
\end{lemma}

\textbf{Proof.}
Consider an arbitrary step \( t \geq 1 \). Recall from Algorithm \textsuperscript{1} that
\[ \bar{m}_{t+1} := \beta \bar{m}_t + (1 - \beta) g_{t+1}. \]
Combining this with (51) we obtain that
\[ \delta_{t+1} = \beta \bar{m}_t + (1 - \beta) \bar{g}_{t+1} - \nabla Q_{t+1}. \]

Adding and subtracting \( \beta \nabla Q_t \) and \( \beta \nabla Q_{t+1} \) on the R.H.S. we then obtain that
\[ \delta_{t+1} = \beta \left( \bar{m}_t - \nabla Q_t \right) + \beta \left( \nabla Q_t - \nabla Q_{t+1} \right) + (1 - \beta) \left( \bar{g}_{t+1} - \nabla Q_{t+1} \right), \]
which yields
\[ \|\delta_{t+1}\|^2 = \beta^2 \|\bar{m}_t - \nabla Q_t\|^2 + \beta^2 \|\nabla Q_t - \nabla Q_{t+1}\|^2 + (1 - \beta)^2 \|\bar{g}_{t+1} - \nabla Q_{t+1}\|^2 + 2\beta^2 \langle \bar{m}_t - \nabla Q_t, \nabla Q_t - \nabla Q_{t+1} \rangle + 2\beta(1 - \beta) \langle \bar{g}_{t+1} - \nabla Q_{t+1}, \bar{m}_t - \nabla Q_t \rangle + 2\beta(1 - \beta) \langle \nabla Q_t - \nabla Q_{t+1}, \bar{g}_{t+1} - \nabla Q_{t+1} \rangle. \]

Applying the conditional expectation \( \mathbb{E}_{t+1} \left[ \cdot \right] \) on both sides, and noting that \( \bar{m}_t, \nabla Q_t, \) and \( \nabla Q_{t+1} \) are deterministic values when given \( \mathcal{P}_{t+1} \), we obtain that
\[ \mathbb{E}_{t+1} \left[ \|\delta_{t+1}\|^2 \right] = \beta^2 \mathbb{E}_{t+1} \left[ \|\delta_t\|^2 \right] + \beta^2 \|\nabla Q_t - \nabla Q_{t+1}\|^2 + (1 - \beta)^2 \mathbb{E}_{t+1} \left[ \|\bar{g}_{t+1} - \nabla Q_{t+1}\|^2 \right] + 2\beta^2 \langle \delta_t, \nabla Q_t - \nabla Q_{t+1} \rangle + 2\beta(1 - \beta) \langle \mathbb{E}_{t+1} \left[ \bar{g}_{t+1} - \nabla Q_{t+1} \right], \bar{m}_t - \nabla Q_t \rangle + 2\beta(1 - \beta) \langle \nabla Q_t - \nabla Q_{t+1}, \mathbb{E}_{t+1} \left[ \bar{g}_{t+1} - \nabla Q_{t+1} \right] \rangle. \]
where (a) uses the facts that the gradient estimations are independent and $\mathbb{E}_{t+1} \left[ g_{t+1}^{(i)} - \nabla Q_{t+1} \right] = 0$. Therefore,

$$\mathbb{E}_{t+1} \left[ \| \delta_{t+1} \|^2 \right] = \beta^2 \| \delta_t \|^2 + \beta^2 \| \nabla Q_t - \nabla Q_{t+1} \|^2 + (1 - \beta)^2 \mathbb{E}_{t+1} \left[ \| g_{t+1} - \nabla Q_{t+1} \|^2 \right] + 2\beta^2 \left( \langle \delta_t, \nabla Q_t - \nabla Q_{t+1} \rangle \right). \tag{52}$$

Note that

$$\begin{align*}
(1 - \beta)^2 \mathbb{E}_{t+1} \left[ \| g_{t+1} - \nabla Q_{t+1} \|^2 \right] &= (1 - \beta)^2 \mathbb{E}_{t+1} \left[ \left\| \frac{1}{n-f} \sum_{i \in \mathcal{H}} \left( g_{t+1}^{(i)} - \nabla Q^{(i)} \left( \theta_{t+1}^{(i)} \right) \right) \right\|^2 \right] \\
&= \frac{(1 - \beta)^2}{(n-f)^2} \sum_{i,j \in \mathcal{H}} \mathbb{E}_{t+1} \left[ \langle g_{t+1}^{(i)} - \nabla Q^{(i)} \left( \theta_{t+1}^{(i)} \right), g_{t+1}^{(j)} - \nabla Q^{(j)} \left( \theta_{t+1}^{(j)} \right) \rangle \right] \\
&= (a) \frac{(1 - \beta)^2}{(n-f)^2} \sum_{i \in \mathcal{H}} \mathbb{E}_{t+1} \left[ \| g_{t+1}^{(i)} - \nabla Q^{(i)} \left( \theta_{t+1}^{(i)} \right) \|^2 \right] \leq (b) \frac{(1 - \beta)^2 \sigma^2}{n-f},
\end{align*} \tag{53}$$

where (a) uses the facts that the gradient estimations are independent and $\mathbb{E}_{t+1} \left[ g_{t+1}^{(i)} - \nabla Q^{(i)} \left( \theta_{t+1}^{(i)} \right) \right] = 0$, and (b) is due to Assumption 2. Substituting from (53) in (52) we obtain that (upon applying Cauchy-Schwartz inequality)

$$\begin{align*}
\mathbb{E}_{t+1} \left[ \| \delta_{t+1} \|^2 \right] &\leq \beta^2 \| \delta_t \|^2 + \beta^2 \| \nabla Q_t - \nabla Q_{t+1} \|^2 + 2\beta^2 \left( \langle \delta_t, \nabla Q_t - \nabla Q_{t+1} \rangle \right) + \frac{(1 - \beta)^2 \sigma^2}{n-f} \\
&\leq \beta^2 \| \delta_t \|^2 + \beta^2 \| \nabla Q_t - \nabla Q_{t+1} \|^2 + 2\beta^2 \| \delta_t \| \| \nabla Q_t - \nabla Q_{t+1} \| + \frac{(1 - \beta)^2 \sigma^2}{n-f}.
\end{align*}$$

Upon taking total expectation on both sides above we obtain that

$$\mathbb{E} \left[ \| \delta_{t+1} \|^2 \right] \leq \beta^2 \mathbb{E} \left[ \| \delta_t \|^2 \right] + \beta^2 \mathbb{E} \left[ \| \nabla Q_t - \nabla Q_{t+1} \|^2 \right] + 2\beta^2 \mathbb{E} \left[ \| \delta_t \| \| \nabla Q_t - \nabla Q_{t+1} \| \right] + \frac{(1 - \beta)^2 \sigma^2}{n-f}.$$ 

As $2xy \leq cx^2 + \frac{y^2}{c'}$ for all $c > 0$, by substituting $c = 4\gamma L$ we obtain that $2 \| \delta_t \| \| \nabla Q_t - \nabla Q_{t+1} \| \leq 4\gamma L \| \delta_t \|^2 + \frac{1}{4\gamma L} \| \nabla Q_t - \nabla Q_{t+1} \|^2$. Using this above we obtain that

$$\mathbb{E} \left[ \| \delta_{t+1} \|^2 \right] \leq \beta^2 (1 + 4\gamma L) \mathbb{E} \left[ \| \delta_t \|^2 \right] + \beta^2 \left( 1 + \frac{1}{4\gamma L} \right) \mathbb{E} \left[ \| \nabla Q_t - \nabla Q_{t+1} \|^2 \right] + \frac{(1 - \beta)^2 \sigma^2}{n-f}. \tag{54}$$

Note that as $\nabla Q_t := \frac{1}{n-f} \sum_{i \in \mathcal{H}} \nabla Q^{(i)} \left( \theta_t^{(i)} \right)$, by Assumption 1 we obtain that

$$\| \nabla Q_t - \nabla Q_{t+1} \| = \frac{1}{n-f} \sum_{i \in \mathcal{H}} \| \nabla Q^{(i)} \left( \theta_t^{(i)} \right) - \nabla Q^{(i)} \left( \theta_{t+1}^{(i)} \right) \| \leq \frac{L}{n-f} \sum_{i \in \mathcal{H}} \| \theta_t^{(i)} - \theta_{t+1}^{(i)} \|.$$

This implies that

$$\| \nabla Q_t - \nabla Q_{t+1} \|^2 \leq L^2 \left( \frac{1}{n-f} \sum_{i \in \mathcal{H}} \| \theta_t^{(i)} - \theta_{t+1}^{(i)} \| \right)^2 \leq \frac{L^2}{n-f} \sum_{i \in \mathcal{H}} \| \theta_t^{(i)} - \theta_{t+1}^{(i)} \|^2. \tag{55}$$
By triangle inequality we obtain that
\[
\| \theta^{(i)}_t - \theta^{(i)}_{t+1} \| \leq \| \theta^{(i)}_t - \bar{\theta} \| + \| \bar{\theta} - \theta^{(i)}_{t+1} \| + \| \theta^{(i)}_{t+1/2} - \theta^{(i)}_t \|.
\]
Recall that \( \bar{\theta}_{t+1/2} = \bar{\theta} - \gamma m_t \). Thus, \( \| \theta^{(i)}_t - \bar{\theta} \| \leq \| \theta^{(i)}_t - \theta^{(i)}_t \| + \gamma \| m_t \| \) and
\[
\| \theta^{(i)}_t - \theta^{(i)}_t \| \leq \| \theta^{(i)}_t - \bar{\theta} \| + \| \bar{\theta} - \theta^{(i)}_{t+1/2} \| + \| \theta^{(i)}_{t+1/2} - \theta^{(i)}_t \| + \gamma \| m_t \|.
\]
As \((x + y)^2 \leq (1 + c)x^2 + (1 + \frac{1}{c})y^2\), taking square on both sides for \(c = 2\) we obtain that
\[
\| \theta^{(i)}_t - \theta^{(i)}_{t+1} \|^2 \leq 3 \left( \| \theta^{(i)}_t - \bar{\theta} \|^2 + \| \bar{\theta} - \theta^{(i)}_{t+1} \|^2 + \| \theta^{(i)}_{t+1/2} - \theta^{(i)}_t \|^2 \right) + \frac{3}{2} \gamma^2 \| m_t \|^2.
\]
And thus
\[
\| \theta^{(i)}_t - \theta^{(i)}_{t+1} \|^2 \leq 9 \| \theta^{(i)}_t - \bar{\theta} \|^2 + 9 \| \bar{\theta} - \theta^{(i)}_{t+1} \|^2 + 9 \| \theta^{(i)}_{t+1/2} - \theta^{(i)}_t \|^2 + \frac{3}{2} \gamma^2 \| m_t \|^2. \tag{56}
\]
Note that for any \(t\), by definition of \( \bar{\theta}_t \) we have
\[
\| \bar{\theta}_t - \theta^{(i)}_t \|^2 = \frac{1}{n-f} \sum_{j \in \mathcal{H}} \left( \theta^{(j)}_t - \theta^{(i)}_t \right)^2 \leq \left( \frac{1}{n-f} \sum_{j \in \mathcal{H}} \theta^{(j)}_t - \theta^{(i)}_t \right)^2 \leq \frac{1}{n-f} \sum_{j \in \mathcal{H}} \left( \theta^{(j)}_t - \theta^{(i)}_t \right)^2.
\]
Substituting from above in (56) we obtain that
\[
\| \theta^{(i)}_t - \theta^{(i)}_{t+1} \|^2 \leq \frac{9}{n-f} \left( \sum_{j \in \mathcal{H}} \| \theta^{(j)}_t - \bar{\theta} \|^2 + \sum_{j \in \mathcal{H}} \| \theta^{(j)} - \theta^{(i)}_t \|^2 \right) + \frac{3}{2} \gamma^2 \| m_t \|^2.
\]
Substituting from above in (55) we obtain that
\[
\| \nabla Q_t - \nabla Q_{t+1} \|^2 \leq \frac{9L^2}{(n-f)^2} \left( \sum_{i,j \in \mathcal{H}} \| \theta^{(j)}_{t+1} - \theta^{(i)}_{t+1} \|^2 + \sum_{i,j \in \mathcal{H}} \| \theta^{(j)}_t - \theta^{(i)}_t \|^2 \right)
\]
\[
+ \frac{9L^2}{n-f} \sum_{i \in \mathcal{H}} \| \bar{\theta}_{t+1} - \bar{\theta}_{t+1/2} \|^2 + \frac{3}{2} \gamma^2 \| m_t \|^2.
\]
Taking total expectation on both sides above, and using the notation \( \Gamma (\ast_t) \), we obtain that
\[
\mathbb{E} \left[ \| \nabla Q_t - \nabla Q_{t+1} \|^2 \right] \leq 9L^2 \left( \mathbb{E} \left[ \Gamma (\theta_{t+1}) \right] + \mathbb{E} \left[ \Gamma (\theta_t) \right] \right) + \frac{3}{2} \gamma^2 \mathbb{E} \left[ \| m_t \|^2 \right].
\]
By definition of \(c_t\) we have \(\| m_t \| \leq \| m_t - \nabla Q_t \| + \| \nabla Q_t \| = \| \delta_t \| + \| \nabla Q_t \| \). Therefore, \(\| m_t \|^2 \leq 3 \| \delta_t \|^2 + \frac{3}{2} \| \nabla Q_t \|^2 \). Using this above we obtain that
\[
\mathbb{E} \left[ \| \nabla Q_t - \nabla Q_{t+1} \|^2 \right] \leq 9L^2 \left( \mathbb{E} \left[ \Gamma (\theta_{t+1}) \right] + \mathbb{E} \left[ \Gamma (\theta_t) \right] + \mathbb{E} \left[ \| \delta_t \|^2 \right] + \frac{3}{2} \mathbb{E} \left[ \| \nabla Q_t \|^2 \right] \right).
\]
Substituting from (57) above in (54) we obtain that

\[
\mathbb{E} \left[ \|\delta_{t+1}\|^2 \right] \leq \beta^2 (1 + 4\gamma L) \mathbb{E} \left[ \|\delta_t\|^2 \right] + \beta^2 \left( 1 + \frac{1}{4\gamma L} \right) \frac{9}{4} L^2 \gamma^2 \left( 2 \mathbb{E} \left[ \|\delta_{t+1}\|^2 \right] + \mathbb{E} \left[ \|\nabla Q_t\|^2 \right] \right) \\
+ \beta^2 \left( 1 + \frac{1}{4\gamma L} \right) 9L^2 \left( \mathbb{E} \left[ \Gamma (\theta_{t+1}) \right] + \mathbb{E} \left[ \Gamma (\theta_t) \right] + \mathbb{E} \left[ \|\bar{\theta}_{t+1} - \bar{\theta}_{t+1/2}\|^2 \right] \right) + \frac{(1 - \beta^2)\sigma^2}{n - f} \\
= \beta^2 (1 + 4L\gamma) (1 + \frac{9}{8} L\gamma) \mathbb{E} \left[ \|\delta_t\|^2 \right] + \frac{9}{16} \beta^2 L\gamma (1 + 4L\gamma) \mathbb{E} \left[ \|\nabla Q_t\|^2 \right] \\
+ \beta^2 \left( 1 + \frac{1}{4\gamma L} \right) 9L^2 \left( \mathbb{E} \left[ \Gamma (\theta_{t+1}) \right] + \mathbb{E} \left[ \Gamma (\theta_t) \right] + \mathbb{E} \left[ \|\bar{\theta}_{t+1} - \bar{\theta}_{t+1/2}\|^2 \right] \right) + \frac{(1 - \beta^2)\sigma^2}{n - f}.
\]

Now note also that \( \|\nabla Q_t\|^2 \leq 4 \|\nabla Q_t - \nabla Q (\bar{\theta}_t)\|^2 + \frac{4}{3} \|\nabla Q (\bar{\theta}_t)\|^2 \); thus

\[
\mathbb{E} \left[ \|\delta_{t+1}\|^2 \right] \leq \beta^2 (1 + 4L\gamma) (1 + \frac{9}{8} L\gamma) \mathbb{E} \left[ \|\delta_t\|^2 \right] + \frac{3}{4} \beta^2 L\gamma (1 + 4L\gamma) \mathbb{E} \left[ \|\nabla Q (\bar{\theta}_t)\|^2 \right] \\
+ \beta^2 \left( 1 + \frac{1}{4\gamma L} \right) 9L^2 \left( \mathbb{E} \left[ \Gamma (\theta_{t+1}) \right] + \mathbb{E} \left[ \Gamma (\theta_t) \right] + \mathbb{E} \left[ \|\bar{\theta}_{t+1} - \bar{\theta}_{t+1/2}\|^2 \right] \right) + \frac{(1 - \beta^2)\sigma^2}{n - f} \\
+ \frac{9}{4} \beta^2 L\gamma (1 + 4L\gamma) \mathbb{E} \left[ \|\nabla Q_t - \nabla Q (\bar{\theta}_t)\|^2 \right].
\] (58)

But now note that

\[
\mathbb{E} \left[ \|\nabla Q_t - \nabla Q (\bar{\theta}_t)\|^2 \right] = \mathbb{E} \left[ \left\| \frac{1}{n - f} \sum_{i \in \mathcal{H}} \nabla Q^{(i)} (\theta_t^{(i)}) - \frac{1}{n - f} \sum_{i \in \mathcal{H}} \nabla Q^{(i)} (\bar{\theta}_t) \right\|^2 \right] \\
= \mathbb{E} \left[ \left\| \frac{1}{n - f} \sum_{i \in \mathcal{H}} \left( \nabla Q^{(i)} (\theta_t^{(i)}) - \nabla Q^{(i)} (\bar{\theta}_t) \right) \right\|^2 \right] \\
\leq \frac{1}{n - f} \sum_{i \in \mathcal{H}} \mathbb{E} \left[ \left\| \nabla Q^{(i)} (\theta_t^{(i)}) - \nabla Q^{(i)} (\bar{\theta}_t) \right\|^2 \right] \\
\leq \frac{L^2}{n - f} \sum_{i \in \mathcal{H}} \mathbb{E} \left[ \left\| \theta_t^{(i)} - \bar{\theta}_t \right\|^2 \right] \leq L^2 \mathbb{E} \left[ \Gamma (\theta_t) \right].
\]

Combining this with (58), we obtain

\[
\mathbb{E} \left[ \|\delta_{t+1}\|^2 \right] \leq \beta^2 (1 + 4L\gamma) (1 + \frac{9}{8} L\gamma) \mathbb{E} \left[ \|\delta_t\|^2 \right] + \frac{3}{4} \beta^2 L\gamma (1 + 4L\gamma) \mathbb{E} \left[ \|\nabla Q (\bar{\theta}_t)\|^2 \right] \\
+ 9\beta^2 L^2 \left( 1 + \frac{1}{4\gamma L} \right) \left( \mathbb{E} \left[ \Gamma (\theta_{t+1}) \right] + \mathbb{E} \left[ \Gamma (\theta_t) \right] + \mathbb{E} \left[ \|\bar{\theta}_{t+1} - \bar{\theta}_{t+1/2}\|^2 \right] \right) \\
+ \frac{9}{4} \beta^2 L\gamma (1 + 4L\gamma) L^2 \mathbb{E} \left[ \Gamma (\theta_t) \right] + \frac{(1 - \beta^2)\sigma^2}{n - f},
\]

which is the lemma.
B.8 Proof of Lemma 4

We recall the lemma below. Also, recall that \( \theta^* := 1/n \sum_{i \in \mathcal{H}} \theta^{(i)} \) and \( Q(\theta) = 1/n \sum_{i \in \mathcal{H}} Q^{(i)}(\theta) \).

Lemma 4. Suppose that assumptions 7 and 3 hold true. Consider Algorithm 1 with \( \gamma \leq 1/L \). For each \( t \in [T] \), we obtain that

\[
\mathbb{E} [Q(\bar{\theta}_{t+1}) - Q(\bar{\theta}_t)] \leq -\frac{\gamma}{2} \mathbb{E} \left[ \| \nabla Q(\bar{\theta}_t) \|^2 \right] + \frac{3 \gamma}{2} \mathbb{E} \left[ \| \delta_t \|^2 \right] + \frac{3}{2 \gamma} \mathbb{E} \left[ \left\| \bar{\theta}_{t+1} - \bar{\theta}_{t+1} \right\| \right] + \frac{3 \gamma}{2} L^2 \mathbb{E} \left[ \Gamma(\theta_t) \right].
\]

Proof. Consider an arbitrary \( t \in [T] \). We define \( G_t := \frac{\bar{\theta}_t - \bar{\theta}_{t+1}}{\gamma} \), the step taken by the average of local models at iteration \( t \). By the smoothness of the loss function (Assumption 4), we have

\[
Q(\bar{\theta}_{t+1}) - Q(\bar{\theta}_t) \leq \langle \bar{\theta}_{t+1} - \bar{\theta}_t, \nabla Q(\bar{\theta}_t) \rangle + \frac{L}{2} \| \bar{\theta}_{t+1} - \bar{\theta}_t \|^2
\]

\[
= -\gamma \langle G_t, \nabla Q(\bar{\theta}_t) \rangle + \frac{L \gamma^2}{2} ||G_t||^2.
\]

Using the fact that \( \gamma \leq 1/L \), we obtain that

\[
Q(\bar{\theta}_{t+1}) - Q(\bar{\theta}_t) \leq -\gamma \langle G_t, \nabla Q(\bar{\theta}_t) \rangle + \frac{\gamma}{2} ||G_t||^2.
\]

Now note that \( \langle x, y \rangle + \|x\|^2 = \|y\|^2 = -\|y\|^2 + \|x-y\|^2 \); thus

\[
Q(\bar{\theta}_{t+1}) - Q(\bar{\theta}_t) \leq -\frac{\gamma}{2} \| \nabla Q(\bar{\theta}_t) \|^2 + \frac{\gamma}{2} \| G_t - \nabla Q(\bar{\theta}_t) \|^2
\]

\[
\leq -\frac{\gamma}{2} \| \nabla Q(\bar{\theta}_t) \|^2 + \frac{\gamma}{2} \left\| \frac{\bar{\theta}_t - \bar{\theta}_{t+1}}{\gamma} - \nabla Q(\bar{\theta}_t) \right\| \|^2
\]

\[
= -\frac{\gamma}{2} \| \nabla Q(\bar{\theta}_t) \|^2 + \frac{\gamma}{2} \left\| \frac{\bar{\theta}_t - \bar{\theta}_{t+1/2} + \bar{\theta}_{t+1/2} - \bar{\theta}_{t+1}}{\gamma} - \nabla Q_t + \nabla Q_t - \nabla Q(\bar{\theta}_t) \right\| \|^2
\]

\[
= -\frac{\gamma}{2} \| \nabla Q(\bar{\theta}_t) \|^2 + \frac{\gamma}{2} \left\| \bar{\theta}_t - \bar{\theta}_{t+1/2} + \bar{\theta}_{t+1/2} - \bar{\theta}_{t+1} \right\| \|^2 - \nabla Q_t + \nabla Q_t - \nabla Q(\bar{\theta}_t) \right\| \|^2
\]

\[
\leq -\frac{\gamma}{2} \| \nabla Q(\bar{\theta}_t) \|^2 + \frac{\gamma}{2} \left\| \bar{\theta}_t - \nabla Q_t \right\| \|^2 + \frac{3}{2 \gamma} \left\| \bar{\theta}_{t+1/2} - \bar{\theta}_{t+1} \right\| \|^2 + \frac{3 \gamma}{2} \left\| \nabla Q_t - \nabla Q(\bar{\theta}_t) \right\| \|^2,
\]

where \( \nabla Q_t = \frac{1}{n-\gamma} \sum_{i \in \mathcal{H}} \nabla Q^{(i)}(\theta^{(i)}) \). Now recall from Lemma 3 that we define \( \delta_t = \bar{m}_t - \nabla Q_t \).

Thus, taking the expectation from both sides of the above, we obtain that

\[
\mathbb{E} [Q(\bar{\theta}_{t+1}) - Q(\bar{\theta}_t)] \leq -\frac{\gamma}{2} \mathbb{E} \left[ \| \nabla Q(\bar{\theta}_t) \|^2 \right] + \frac{3 \gamma}{2} \mathbb{E} \left[ \| \delta_t \|^2 \right] + \frac{3}{2 \gamma} \mathbb{E} \left[ \left\| \bar{\theta}_{t+1/2} - \bar{\theta}_{t+1} \right\| \right] + \frac{3 \gamma}{2} \mathbb{E} \left[ \left\| \nabla Q_t - \nabla Q(\bar{\theta}_t) \right\| \right].
\]
Now note that

\[
\mathbb{E} \left[ \left\| \nabla Q_t - \nabla Q(\bar{\theta}_t) \right\|^2 \right] = \mathbb{E} \left[ \left\| \frac{1}{n-f} \sum_{i \in \mathcal{H}} \nabla Q^{(i)}(\theta_t^{(i)}) - \frac{1}{n-f} \sum_{i \in \mathcal{H}} \nabla Q^{(i)}(\bar{\theta}_t) \right\|^2 \right]
\]

\[
= \mathbb{E} \left[ \left\| \frac{1}{n-f} \sum_{i \in \mathcal{H}} \left( \nabla Q^{(i)}(\theta_t^{(i)}) - \nabla Q^{(i)}(\bar{\theta}_t) \right) \right\|^2 \right]
\]

\[
\leq \frac{1}{n-f} \sum_{i \in \mathcal{H}} \mathbb{E} \left[ \left\| \nabla Q^{(i)}(\theta_t^{(i)}) - \nabla Q^{(i)}(\bar{\theta}_t) \right\|^2 \right]
\]

\[
\leq \frac{L^2}{n-f} \sum_{i \in \mathcal{H}} \mathbb{E} \left[ \left\| \theta_t^{(i)} - \bar{\theta}_t \right\|^2 \right] \leq L^2 \mathbb{E} \left[ \Gamma(\theta_t) \right] .
\]

Combining this with (59), we obtain that

\[
\mathbb{E} \left[ Q(\bar{\theta}_{t+1}) - Q(\bar{\theta}_t) \right] \leq -\frac{\gamma}{2} \mathbb{E} \left[ \left\| \nabla Q(\bar{\theta}_t) \right\|^2 \right] + \frac{3\gamma}{2} \mathbb{E} \left[ \left\| \delta_t \right\|^2 \right]
\]

\[
+ \frac{3}{2\gamma} \mathbb{E} \left[ \left\| \bar{\theta}_{t+1/2} - \bar{\theta}_{t+1/2} \right\|^2 \right] + \frac{3\gamma}{2} L^2 \mathbb{E} \left[ \Gamma(\theta_t) \right] .
\]

which is the lemma. \qed
B.9 Proof of Lemma 5

In this section, we prove that if $n > 5f$, then $K \in \mathcal{O}(\log(n))$ coordination rounds is enough to guarantee $(\alpha, \lambda)$-reduction.

Lemma 8. Consider the coordination step of Algorithm[7] Suppose that there exists $\delta > 0$ such that $n \geq (5 + \delta)f$. For any $k \geq 1$ we have
\[
\max_{i,j \in \mathcal{H}} \| x^{(i)}_k - x^{(j)}_k \| \leq \left( \frac{3}{n - 2f} \right)^k \max_{i,j \in \mathcal{H}} \| x^{(i)}_0 - x^{(j)}_0 \|.
\]

Proof. Consider an arbitrary round $k \in [K]$. Consider two honest nodes $p, q \in \mathcal{H}$. We then have
\[
\| x^{(p)}_k - x^{(q)}_k \| = \left\| \frac{1}{n - 2f} \sum_{j \in S^{(p)}_k} x^{(j)}_{k-1} - \frac{1}{n - 2f} \sum_{j \in S^{(q)}_k} x^{(j)}_{k-1} \right\|
\]
\[
= \frac{1}{n - 2f} \left\| \sum_{j \in S^{(p)}_k \setminus S^{(q)}_k} x^{(j)}_{k-1} - \sum_{j \in S^{(q)}_k \setminus S^{(p)}_k} x^{(j)}_{k-1} \right\|. \tag{60}
\]

Now similar to Lemma[7] we define:
\[
F_p := \left\{ i : i \notin \mathcal{H}, i \in S^{(p)}_k, i \notin S^{(q)}_k \right\},
\]
\[
F_q := \left\{ i : i \notin \mathcal{H}, i \in S^{(q)}_k, i \notin S^{(p)}_k \right\},
\]
\[
H_p := \left\{ i : i \in \mathcal{H}, i \in S^{(p)}_k, i \notin S^{(q)}_k \right\},
\]
\[
H_q := \left\{ i : i \in \mathcal{H}, i \in S^{(q)}_k, i \notin S^{(p)}_k \right\}.
\]

We also define $f_p := |F_p|$ and $f_q := |F_q|$. We also order these four sets such that, e.g., $F_p[i]$ refers to a unique element in set $F_p$. Without loss of generality, we assume $|H_p| \geq f_q$ and $|H_q| \geq f_p$.\footnote{Otherwise we add sufficiently many honest vectors from $S^{\mathcal{P}}_k \cap S^{\mathcal{Q}}_k$ to both $H_p$ and $H_q$ such that $|H_p| \geq f_q$ and $|H_q| \geq f_p$.} Now from (60), we obtain that
\[
(n - 2f) \left\| x^{(p)}_k - x^{(q)}_k \right\| = \left\| \sum_{j \in F_p} x^{(j)}_{k-1} + \sum_{j \in H_p} x^{(j)}_{k-1} - \sum_{j \in F_q} x^{(j)}_{k-1} - \sum_{j \in H_q} x^{(j)}_{k-1} \right\|
\]
\[
= \left\| \sum_{j \in [f_p]} \left( x^{(F_p[j])}_{k-1} - x^{(H_q[j])}_{k-1} \right) - \sum_{j \in [f_q]} \left( x^{(F_q[j])}_{k-1} - x^{(H_p[j])}_{k-1} \right) + \sum_{j \in [f_p] - f_q} \left( x^{(H_p[j] + f_q)}_{k-1} - x^{(H_q[j] + f_q)}_{k-1} \right) \right\|. \tag{60}
\]
By triangle inequality, we then have
\[
(n - 2f) \| x_k^{(p)} - x_k^{(q)} \| \leq \sum_{j \in [f_p]} \| x_{k-1}^{(F_p[j])} - x_{k-1}^{(H_q[j])} \| + \sum_{j \in [f_q]} \| x_{k-1}^{(F_q[j])} - x_{k-1}^{(H_p[j])} \|
\]
\[
+ \sum_{j \in [|H_p| - f_q]} \| x_{k-1}^{(F_p[j])} - x_{k-1}^{(H_q[j])} \|
\]
\[
\leq \sum_{j \in [f_p]} \| x_{k-1}^{(j)} - x_{k-1}^{(p)} \| + \| x_{k-1}^{(p)} - x_{k-1}^{(H_q[j])} \|
\]
\[
+ \sum_{j \in [f_q]} \| x_{k-1}^{(j)} - x_{k-1}^{(q)} \| + \| x_{k-1}^{(q)} - x_{k-1}^{(H_p[j])} \|
\]
\[
+ \sum_{j \in [|H_p| - f_q]} \| x_{k-1}^{(j)} - x_{k-1}^{(H_q[j])} \|
\]
\[
= (2f_p + f_q) + (|H_p| - f_q) \max_{i,j \in H} \| x_{k-1}^{(i)} - x_{k-1}^{(j)} \|.
\]

Now recall from Lemma 6 that \( |S_k^{(q)} \setminus S_k^{(p)}| \leq 2f \). Therefore,
\[
(n - 2f) \| x_k^{(p)} - x_k^{(q)} \| \leq (f_p + f_q + 2f) \max_{i,j \in H} \| x_{k-1}^{(i)} - x_{k-1}^{(j)} \|
\]
\[
\leq 3f \max_{i,j \in H} \| x_{k-1}^{(i)} - x_{k-1}^{(j)} \|.
\]

Equivalently,
\[
\| x_k^{(p)} - x_k^{(q)} \| \leq (f_p + f_q + 2f) \max_{i,j \in H} \| x_{k-1}^{(i)} - x_{k-1}^{(j)} \|
\]
\[
\leq \frac{3f}{n - 2f} \max_{i,j \in H} \| x_{k-1}^{(i)} - x_{k-1}^{(j)} \|.
\]

As the above inequality holds for any choice of \( p \) and \( q \), we obtain that
\[
\max_{i,j \in H} \| x_k^{(i)} - x_k^{(j)} \| \leq \frac{3f}{n - 2f} \max_{i,j \in H} \| x_{k-1}^{(i)} - x_{k-1}^{(j)} \|.
\]
As the above holds for any \( k \geq 1 \), we obtain that
\[
\max_{i,j \in \mathcal{H}} \|x_k^{(i)} - x_k^{(j)}\| \leq \left( \frac{3f}{n - 2f} \right)^k \max_{i,j \in \mathcal{H}} \|x_0^{(i)} - x_0^{(j)}\|.
\]
This is the lemma.

**Lemma 9.** Consider the coordination step of Algorithm 1. Assume there exists \( \delta > 0 \) such that \( n \geq (5 + \delta)f \). For \( K = \frac{\log(8(n-f))}{2 \log(\frac{1}{1 - \delta})} \in \mathcal{O}(\log(n)) \), we have
\[
\Gamma(x_K) \leq \frac{2f}{n-f} \Gamma(x_0).
\]

**Proof.** For \( k = K \) in Lemma 8, we have
\[
\max_{i,j \in \mathcal{H}} \|x_K^{(i)} - x_K^{(j)}\| \leq \left( \frac{3f}{n - 2f} \right)^K \max_{i,j \in \mathcal{H}} \|x_0^{(i)} - x_0^{(j)}\|.
\]
Now squaring both sides, we obtain that
\[
\max_{i,j \in \mathcal{H}} \|x_K^{(i)} - x_K^{(j)}\|^2 \leq \left( \frac{3f}{n - 2f} \right)^{2K} \max_{i,j \in \mathcal{H}} \|x_0^{(i)} - x_0^{(j)}\|^2.
\]
Now as we assume \( n \geq (5 + \delta)f \), we have
\[
\frac{3f}{n - 2f} \leq \frac{3f}{(3 + \delta)f} = \frac{3}{3 + \delta} < 1.
\]
We then have
\[
\max_{i,j \in \mathcal{H}} \|x_K^{(i)} - x_K^{(j)}\|^2 \leq \left( \frac{3}{3 + \delta} \right)^{2K-1} \left( \frac{3f}{n - 2f} \right)^{2K} \max_{i,j \in \mathcal{H}} \|x_0^{(i)} - x_0^{(j)}\|^2.
\]
Now note that as \( n > 5f \), we have \( 4(n - 2f) \geq 3(n - f) \), thus,
\[
\max_{i,j \in \mathcal{H}} \|x_K^{(i)} - x_K^{(j)}\|^2 \leq \left( \frac{3}{3 + \delta} \right)^{2K-1} \left( \frac{4f}{n - f} \right)^{2K} \max_{i,j \in \mathcal{H}} \|x_0^{(i)} - x_0^{(j)}\|^2. \quad (61)
\]
Now note that
\[
\Gamma(x_K) = \frac{1}{(n-f)^2} \sum_{i,j \in \mathcal{H}} \|x_K^{(i)} - x_K^{(j)}\|^2 \\
\leq \frac{1}{(n-f)^2} \sum_{i,j \in \mathcal{H}} \max_{i,j \in \mathcal{H}} \|x_K^{(i)} - x_K^{(j)}\|^2 = \max_{i,j \in \mathcal{H}} \|x_K^{(i)} - x_K^{(j)}\|^2. \quad (62)
\]
Note also that
\[
\max_{i,j \in \mathcal{H}} \|x_0^{(i)} - x_0^{(j)}\|^2 \leq 4 \max_{i \in \mathcal{H}} \|x_0^{(i)} - x_0\|^2 \\
\leq 4 \sum_{i \in \mathcal{H}} \|x_0^{(i)} - x_0\|^2 \\
\leq 4(n-f) \frac{1}{n-f} \sum_{i \in \mathcal{H}} \|x_0^{(i)} - x_0\|^2 = 4(n-f) \Gamma(x_0). \quad (63)
\]
Combining (61), (62), and (63), we then obtain that

$$\Gamma(x_K) \leq \left(\frac{3}{3 + \delta}\right)^{2K-1} \left(\frac{2f}{n-f}\right) 8(n-f)\Gamma(x_0).$$

Setting $K = \left\lceil \frac{\log(8(n-f))}{2\log\left(\frac{3+\delta}{3}\right)} \right\rceil + 1$, we then obtain that

$$\Gamma(x_K) \leq \frac{2f}{n-f}\Gamma(x_0).$$

This is what we wanted.

We recall Lemma 5 below.

**Lemma 5.** Suppose that there exists $\delta > 0$ such that $n \geq (5 + \delta)f$. For $K = \frac{\log(8(n-f))}{2\log\left(\frac{3+\delta}{3}\right)} \in \mathcal{O}(\log(n))$, the coordination step of Algorithm 1 guarantees $(\alpha, \lambda)$-reduction for

$$\alpha = \frac{2f}{n-f} \leq \frac{1}{2} \quad \text{and} \quad \lambda = \left(\frac{3 + \delta}{\delta}\right)^2 \left(\frac{8f}{n-f}\right)^2.$$

**Proof.** The first inequality is already proved by Lemma 3. Here we prove the second inequality. Consider an arbitrary round $k \in [K]$. For an honest node $i \in \mathcal{H}$ we have

$$\left\|x_k^{(i)} - \bar{x}_{k-1}\right\| = \left\|\frac{1}{n-2f} \sum_{j \in S_k^{(i)}} x_{k-1}^{(j)} - \frac{1}{n-f} \sum_{j \in \mathcal{H}} x_{k-1}^{(j)}\right\|$$

$$= \left\|\left(\frac{1}{n-2f} - \frac{1}{n-f}\right) \sum_{j \in S_k^{(i)} \cap \mathcal{H}} x_{k-1}^{(j)} + \frac{1}{n-2f} \sum_{j \in S_k^{(i)} \setminus \mathcal{H}} x_{k-1}^{(j)} - \frac{1}{n-f} \sum_{j \in \mathcal{H} \setminus S_k^{(i)}} x_{k-1}^{(j)}\right\|$$

$$= \frac{1}{(n-f)(n-2f)} \left(f \sum_{j \in S_k^{(i)} \cap \mathcal{H}} (x_{k-1}^{(j)} - x_{k-1}^{(i)}) + (n-f) \sum_{j \in S_k^{(i)} \setminus \mathcal{H}} (x_{k-1}^{(j)} - x_{k-1}^{(i)}) - (n-2f) \sum_{j \in \mathcal{H} \setminus S_k^{(i)}} (x_{k-1}^{(j)} - x_{k-1}^{(i)})\right).$$

By triangle inequality, we then have

$$(n-f)(n-2f) \left\|x_k^{(i)} - \bar{x}_{k-1}\right\| \leq f \sum_{j \in S_k^{(i)} \cap \mathcal{H}} \left\|x_{k-1}^{(j)} - x_{k-1}^{(i)}\right\| + (n-f) \sum_{j \in S_k^{(i)} \setminus \mathcal{H}} \left\|x_{k-1}^{(j)} - x_{k-1}^{(i)}\right\|$$

$$+ (n-2f) \sum_{j \in \mathcal{H} \setminus S_k^{(i)}} \left\|x_{k-1}^{(j)} - x_{k-1}^{(i)}\right\|.$$
we must have $\|x_{k-1}^{(j^*)} - x_{k-1}^{(i)}\| \leq \max_{p,q \in \mathcal{H}} \|x_{k-1}^{(p)} - x_{k-1}^{(q)}\|$. And clearly for any honest node $i^*$ we have $\|x_{k-1}^{(i^*)} - x_{k-1}^{(i)}\| \leq \max_{p,q \in \mathcal{H}} \|x_{k-1}^{(p)} - x_{k-1}^{(q)}\|$. Therefore,

$$\|x_{k-1}^{(i)} - x_{k-1}\| \leq \frac{f}{(n-f)(n-2f)} \left[ |S_k^{(i)} \cap \mathcal{H}| + (n-f) |S_k^{(i)} \setminus \mathcal{H}| + (n-2f) \left| \mathcal{H} \setminus S_k^{(i)} \right| \right] \max_{p,q \in \mathcal{H}} \|x_{k-1}^{(p)} - x_{k-1}^{(q)}\|.$$ 

Now let $v := |S_k^{(i)} \cap \mathcal{H}|$. We then have $v = |S_k^{(i)}| + |\mathcal{H}| - |S_k^{(i)} \cup \mathcal{H}| \geq n - 2f + n - f - n = n - 3f$. Also, $|S_k^{(i)} \setminus \mathcal{H}| = n - 2f - v$ and $|\mathcal{H} \setminus S_k^{(i)}| = n - f - v$. Now we define

$$A(v) := f v + (n - 2f - v)(n - f) + (n - 2f)(n - f - v) = 2(n - 2f)(n - f - v), \quad (64)$$

which is decreasing in $v$. Then the maximum of $A(v)$ is reached for $v = n - 3f$ and we have $A(v) \leq 4f(n - 2f)$. Therefore,

$$\|x_{k-1}^{(i)} - x_{k-1}\| \leq \frac{4f}{n - f} \max_{p,q \in \mathcal{H}} \|x_{k-1}^{(p)} - x_{k-1}^{(q)}\|.$$ 

But now note that

$$\|x_{k-1}^{(i)} - x_{k-1}\| = \left\| \frac{1}{n-f} \sum_{i \in \mathcal{H}} x_{k-1}^{(i)} - x_{k-1}\right\| \\
\leq \frac{1}{n-f} \sum_{i \in \mathcal{H}} \|x_{k-1}^{(i)} - x_{k-1}\| \\
\leq \frac{1}{n-f} \sum_{i \in \mathcal{H}} \frac{4f}{n-f} \max_{p,q \in \mathcal{H}} \|x_{k-1}^{(p)} - x_{k-1}^{(q)}\| \\
= \frac{4f}{n-f} \max_{p,q \in \mathcal{H}} \|x_{k-1}^{(p)} - x_{k-1}^{(q)}\|.$$ 

Now applying Lemma 8, we obtain that

$$\|x_{k-1}^{(i)} - x_{k-1}\| \leq \left( \frac{3f}{n-2f} \right)^{k-1} \frac{4f}{n-f} \max_{p,q \in \mathcal{H}} \|x_0^{(i)} - x_0^{(j)}\|.$$ 

Now as $n \geq (5 + \delta)f$, we obtain that

$$\|x_{k-1}^{(i)} - x_{k-1}\| \leq \left( \frac{3}{3+\delta} \right)^{k-1} \frac{4f}{n-f} \max_{p,q \in \mathcal{H}} \|x_0^{(i)} - x_0^{(j)}\|.$$
Now note that by triangle inequality we have

\[
\|\bar{x}_K - \bar{x}_0\| \leq \sum_{k \in [K]} \|\bar{x}_k - \bar{x}_{k-1}\|
\]

\[
\leq \sum_{k \in [K]} \left(\frac{3}{3 + \delta}\right)^{k-1} \frac{4f}{n - f} \max_{i,j \in \mathcal{H}} \|x_0^{(i)} - x_0^{(j)}\|
\]

\[
\leq \sum_{k=1}^{\infty} \left(\frac{3}{3 + \delta}\right)^{k-1} \frac{4f}{n - f} \max_{i,j \in \mathcal{H}} \|x_0^{(i)} - x_0^{(j)}\|
\]

\[
= \frac{1}{1 - \frac{3}{3 + \delta}} \frac{4f}{n - f} \max_{i,j \in \mathcal{H}} \|x_0^{(i)} - x_0^{(j)}\|
\]

\[
= \frac{3 + \delta}{\delta} \frac{4f}{n - f} \max_{i,j \in \mathcal{H}} \|x_0^{(i)} - x_0^{(j)}\|
\]

Squaring both sides, we then have

\[
\|\bar{x}_K - \bar{x}_0\|^2 \leq \left(\frac{3 + \delta}{\delta}\right)^2 \left(\frac{4f}{n - f}\right)^2 \max_{i,j \in \mathcal{H}} \|x_0^{(i)} - x_0^{(j)}\|^2.
\] (65)

Now note that

\[
\max_{i,j \in \mathcal{H}} \|x_0^{(i)} - x_0^{(j)}\|^2 \leq 4\max_{i \in \mathcal{H}} \|x_0^{(i)} - \bar{x}_0\|^2
\]

\[
\leq 4 \sum_{i \in \mathcal{H}} \|x_0^{(i)} - \bar{x}_0\|^2
\]

\[
\leq 4(n - f) \frac{1}{n - f} \sum_{i \in \mathcal{H}} \|x_0^{(i)} - \bar{x}_0\|^2 = 4(n - f) \Gamma(x_0).
\] (66)

Combining above with (65), we obtain that

\[
\|\bar{x}_K - \bar{x}_0\|^2 \leq \left(\frac{3 + \delta}{\delta}\right)^2 \left(\frac{8f}{n - f}\right)^2 \Gamma(x_0).
\]
C Additional Information on the Experimental Setup

C.1 Dataset Pre-Processing

MNIST receives an input image normalization of mean 0.1307 and standard deviation 0.3081. Furthermore, the images of CIFAR-10 are horizontally flipped, and per channel normalization is also applied with means 0.4914, 0.4822, 0.4465 and standard deviations 0.2023, 0.1994, 0.2010.

C.2 Model Architecture and Detailed Experimental Setup

In order to present the detailed architecture of the models used, we adopt the following compact notation introduced as done, e.g., in [17].

L(#outputs) represents a fully-connected linear layer, R stands for ReLU activation, S stands for log-softmax, C(#channels) represents a fully-connected 2D-convolutional layer (kernel size 5, padding 0, stride 1), M stands for 2D-maxpool (kernel size 2), B stands for batch-normalization, and D represents dropout (with fixed probability 0.25).

The architecture of the models, as well as other details on the experimental setup, are presented in Table 2. Note that CNN stands for convolutional neural network, and NLL refers to the negative log likelihood loss.

| Dataset  | MNIST | CIFAR-10 |
|----------|-------|----------|
| Model type | CNN   | CNN      |
| Model architecture | C(20)-R-M-C(20)-R-M-L(500)-R-L(10)-S | (3,32×32)-C(64)-R-B-C(64)-R-B-M-D-C(128)-R-B-C(128)-R-B-M-D-L(128)-R-D-L(10)-S |
| Loss     | NLL   | NLL      |
| Gradient clipping | 2     | 5        |
| $\ell_2$-regularization | $10^{-4}$ | $10^{-2}$ |
| Learning rate | $\gamma = 0.75$ | $\gamma = 0.5$ |
| Batch-size | $b = 25$ | $b = 50$ |
| Number of nodes | $n = 26$ | $n = 16$ |
| Number of Byzantines | $f = 5, 8$ | $f = 3, 5$ |
| Number of Iterations | $T = 600$ | $T = 2000$ |

Table 2: Detailed experimental setting of Section 5
C.3 Byzantine Attacks

We use four state-of-the-art gradient-based attacks, namely fall of empires (FOE) \cite{50}, a little is enough (ALIE) \cite{4}, sign-flipping (SF) \cite{3}, and label-flipping (LF) \cite{3}. Since these attacks are originally devised on gradients, we first modify them to be executed on parameter vectors. The first three adapted attacks (FoE, ALIE, and SF) rely on the following key notion. Let $a_t$ be the attack vector in iteration $t$ and let $\zeta_t$ be a fixed non-negative real number. In every iteration $t$, all Byzantine nodes broadcast the same vector $\theta_t + \zeta_t a_t$ to all other nodes, where $\theta_t$ is the average of the parameter vectors of the honest nodes in iteration $t$. Each attack follows the general scheme we just described, with the following particularities.

(a) **ALIE.** In this attack, $a_t = -\sigma_t$, where $\sigma_t$ is the opposite vector of the coordinate-wise standard deviation of $\theta_t$. In our experiments on ALIE, $\zeta_t$ is chosen through an extensive grid search. Essentially, in each iteration $t$, we choose the value that results in the worst Byzantine vector, i.e., the vector for which the distance to $\theta_t$ is the largest.

(b) **FOE.** In this attack, $a_t = -\theta_t$. All Byzantine nodes thus send $(1 - \zeta_t)\theta_t$ in iteration $t$. Similarly to ALIE, $\zeta_t$ for FoE is also estimated through grid searching.

(c) **SF.** In this attack, $a_t = -\theta_t$ and $\zeta_t = 2$. All Byzantine nodes thus send $-\theta_t$ in iteration $t$.

Under the LF attack, all Byzantine nodes send the same vector $\hat{\theta}_t$, where $\hat{\theta}_t$ is the average of the parameter vectors of honest nodes computed on flipped labels in iteration $t$. In order to do so, in each iteration $t$, every honest worker also computes a gradient on flipped labels. Since the labels for MNIST and CIFAR-10 are in $\{0, 1, ..., 9\}$, the labels are flipped by computing $l' = 9 - l$ for every training datapoint of the batch, where $l$ is the original label and $l'$ is the flipped/modified label. Each Byzantine node then averages all flipped parameter vectors of honest nodes to get $\hat{\theta}_t$.

C.4 Computing Infrastructure

C.4.1 Software Dependencies:

Python 3.8.10 has been used to run our scripts. Besides the standard libraries associated with Python 3.8.10, our scripts use the following libraries:
| Library     | Version  |
|------------|----------|
| numpy      | 1.19.1   |
| torch      | 1.6.0    |
| torchvision| 0.7.0    |
| pandas     | 1.1.0    |
| matplotlib | 3.0.2    |
| PIL        | 7.2.0    |
| requests   | 2.21.0   |
| urllib3    | 1.24.1   |
| chardet    | 3.0.4    |
| certifi    | 2018.08.24 |
| idna       | 2.6      |
| six        | 1.15.0   |
| pytz       | 2020.1   |
| dateutil   | 2.6.1    |
| pyparsing  | 2.2.0    |
| cycler     | 0.10.0   |
| kiwisolver | 1.0.1    |
| cffi       | 1.13.2   |

Some dependencies are essential, while others are optional (e.g., only used to process the results and produce the plots). Furthermore, our code has been tested on the following OS: Ubuntu 20.04.4 LTS (GNU/Linux 5.4.0-121-generic x86_64).

C.4.2 Hardware Dependencies:

We list below the hardware components used:

- 1 Intel(R) Core(TM) i7-8700K CPU @ 3.70GHz
- 2 Nvidia GeForce GTX 1080 Ti
- 64 GB of RAM
D Additional Experimental Results

D.1 Remaining Plots of Sign-flipping (SF) and Label-flipping (LF) Attacks

Figure 2: Comparison of the learning accuracies and number of gradients computed per node of D-SGD, local SGD, MoNNA, and RB-TM. The Byzantine nodes execute the SF (column 1) and LF (column 2) attacks. First row: MNIST with \( n = 26 \) and \( f = 5 \). Second row: CIFAR-10 with \( n = 16 \) and \( f = 3 \).

We complete the missing results from Figure 1 (in the main paper) by presenting in Figure 2 the cross-accuracies and number of gradients computed per node for D-SGD, local SGD, MoNNA, and RB-TM under the SF and LF attacks. As mentioned in Section 5, we consider the classical ML datasets MNIST and CIFAR-10, and we execute D-SGD and local SGD without faults (i.e., without Byzantine nodes). Similar observations to the one made in Section 5.2 also hold here.

D.2 The Importance of NNA and Momentum in MoNNA

In this section, we report on a set of experiments on CIFAR-10 with \( n = 16 \) nodes and \( f = \frac{n-1}{5} = 3 \) Byzantines. The main goal of these experiments is to evaluate the importance of two key ingredients of MoNNA, namely Polyak’s momentum and NNA. To demonstrate the relevance of our solution in Byzantine settings, we compare its performance (in green) in terms of learning accuracy to that of three other algorithms: (1) MoNNA executed without momentum (i.e., \( \beta = 0 \)), (2) MoTM, a variant of MoNNA executed with Trimmed Mean (TM) \(^3\) instead of NNA, and finally (3) TM, i.e., Algorithm 1 executed without momentum (\( \beta = 0 \)) and with TM instead of NNA. We do not compare the different algorithms in terms of computational workload because they all compute the same total number of gradients per node (100,000 as shown in Figure 1). Our observations are

\(^3\)TM is a Byzantine resilient gradient aggregation scheme from the literature \(^5\).
threefold.

First, it is clear from Figure 3 that momentum plays a crucial role in ensuring the Byzantine resilience of MoNNA in the considered setting. Indeed, although momentum-less MoNNA ($\beta = 0$) is able to learn well against the FoE and SF attacks, ALIE destroys its performance, showcasing a very low cross-accuracy constant at around 10% (equivalent to random guessing) throughout the entire learning. However, MoNNA (our solution, green curve) drastically mitigates these attacks. As shown in Figure 3, the model steadily increases in accuracy to finally reach 80% under all three attacks, almost exactly matching the benchmark accuracy of D-SGD (shown in blue).

Second, we discuss MoTM in purple to show the critical importance of using the NNA scheme when defending against Byzantine nodes. Although more resilient than momentum-less MoNNA ($\beta = 0$), MoTM is still somewhat vulnerable to the different attacks which are still able to hinder the learning to a certain extent. Indeed, even though the accuracy increases, it plateaus at 70%, which is almost 10% less than the accuracy obtained with our solution.

Finally, note that when removing momentum from MoNNA and replacing NNA by TM, all three attacks heavily deteriorate the learning (brown curve). This entire analysis demonstrates the non-triviality of our solution in distributed asynchronous systems.

![Figure 3: Experiments on CIFAR-10 with $n = 16$ nodes and $f = 3$ Byzantines. We compare the learning accuracies of D-SGD, local SGD, MoNNA ($\beta = 0.99$), MoNNA without momentum ($\beta = 0$), MoTM ($\beta = 0.99$), and TM ($\beta = 0$). The Byzantines execute FOE (left), ALIE (right), and SF (bottom).]
Figure 4: Comparison of the learning accuracies and number of gradients computed per node of D-SGD, local SGD, MoNNA, and RB-TM. The Byzantine nodes execute the FOE (row 1, left), ALIE (row 1, right), SF (row 2, left), and LF (row 2, right) attacks. Setting: MNIST with $n = 26$ workers and $f = 8$.

D.3 Stress Test: Setting $f = \frac{n-1}{3}$ on MNIST and CIFAR-10

We report in Figures 4 and 5 on experiments performed on MNIST and CIFAR-10, respectively, in a distributed peer-to-peer system. For MNIST, we have $n = 26$ nodes and $f = \lfloor \frac{n-1}{3} \rfloor = 8$ Byzantines, while $n = 16$ nodes and $f = \frac{n-1}{3} = 5$ Byzantines for CIFAR-10.

These experiments serve as a stress test for MoNNA, i.e., we test the boundaries of our solution by implementing it in an a-priori intolerable Byzantine setting where the number of Byzantines $f$ exceeds the theoretical limit required by MoNNA ($n > 5f$ in B.4). Interestingly, Figure 4 shows that MoNNA remains Byzantine resilient on MNIST when $f = \frac{n-1}{3}$. Indeed, its performance almost matches that of D-SGD in terms of learning accuracy and computational workload per node against all three attacks. However, on a more complicated learning task such as CIFAR-10, we can see from Figure 5 that ALIE deteriorates MoNNA’s performance, while the remaining two attacks prove to be ineffective against our solution. These experiments thus seem to indicate the tightness of our analysis regarding the tolerable fraction of Byzantine nodes for MoNNA. Designing an efficient algorithm that tolerates a larger proportion of Byzantines (i.e., $3f < n \leq 5f$) remains an open question.
Figure 5: Comparison of the learning accuracies and number of gradients computed per node of D-SGD, local SGD, MoNNA, and RB-TM. The Byzantine nodes execute the FOE (row 1, left), ALIE (row 1, right), and SF (row 2) attacks. Setting: CIFAR-10 with $n = 16$ workers and $f = 5$. 