Generation and application of multi-path cat states of light

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\textbf{Abstract.} A new scheme to generate a multi-path cat state of light, in which $N$ photons are propagating in $N$ different paths as a photonic macro-molecule, is proposed. Starting with $N$ indistinguishable single-photon sources and using linear optical circuits, the target state is extracted with high fidelity. If a state preserving quantum non-demolition measurement of photon number can be incorporated into the generation scheme, the fidelity is further improved.

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1. Introduction

A certain type of non-classical light using path-entangled photon number states, such as NOON states \([1]\) and photonic de Broglie waves \([2]\), exhibits a higher order quantum interference effect as well as a first-order classical interference effect. The potential applications of such non-classical states of light in metrology and imaging include the optical interferometry in the Heisenberg limit \([1]–[4]\) and photo lithography beyond the classical diffraction limit \([5, 6]\).

A path-entangled photon number state is generally produced by an optical parametric down converter with a Poissonian photon emission process, a linear optical interference circuit and a single-photon detector for post-selection. Naturally, the overall efficiency of generating a desired photon state is extremely low for practical applications \([7]\).

For optical quantum information processing systems, such as quantum key distribution and linear optical quantum computation, a deterministic single-photon source is required and has been extensively studied in the last decade. Now, indistinguishable single photons with identical frequency and pulse shape can be generated on-demand using a cavity quantum electrodynamics (QED) system with a single semiconductor quantum dot \([8]\), a neutral atom \([9]\) or a trapped ion \([10]\). In fact, the indistinguishable single-photon sources can be monolithically integrated on a chip so that many indistinguishable single photons can be generated simultaneously for massive parallel post-processing \([11]\).

In this paper, we will theoretically study the feasibility of converting such indistinguishable single photons into the path-entangled photon number states, which are useful for quantum lithography mentioned above. In particular, we will propose to generate a novel state, expressed as \((|N00\ldots0\rangle + |0N0\ldots0\rangle + \cdots + |000\ldots0N\rangle)/\sqrt{N}\), using massive parallel indistinguishable single-photon sources and quantum interference circuits. We refer to this state as a multi-path...
cat state of light in this paper. The fidelity of the generated state is increased if the post-selection based on a state preserving quantum non-demolition (QND) measurement of photon number \([12, 13]\) is incorporated. The novel state will be useful for quantum lithography using multiple beams due to the improved spatial resolution and phase sensitivity.

2. A proposed scheme

Bunching behavior of bosonic particles manifests itself, for example, by two indistinguishable single photons colliding at a 50 : 50 beam splitter and is a remarkable quantum interference effect. This particular phenomenon is called a Hong–Ou–Mandel dip \([14]\). The input photons coalesce with each other because of their indistinguishability. This leads to path entanglement at the output, expressed as \((|20\rangle + |02\rangle)/\sqrt{2}\). The probability amplitude for an output state \(|11\rangle\) is cancelled out by destructive interference between the direct and exchange terms of a symmetrized two photon wave function.

Motivated by the above-mentioned phenomenon, we hope to produce a new path-entangled state of light with more indistinguishable single photons. By injecting the indistinguishable single photons as each input to a linear optical device with \(N\) input–\(N\) output ports, we hope to get a novel state

\[
\frac{1}{\sqrt{N}}(|N00\ldots0\rangle + |0N0\ldots0\rangle + \cdots + |000\ldots0\rangle N\rangle),
\]

by a quantum interference effect similar to the one between two identical bosonic particles. Here, we call the above state in equation (1) a ‘multi-path cat state’.

We assume the \(N \times N\) linear optical circuit is described by the interaction Hamiltonian

\[
\hat{H} = \hbar \kappa \sum_{i=1}^{N} \sum_{j \neq i} \hat{a}_i \hat{a}^\dagger_j ,
\]

where \(\kappa\) is a real number coupling constant. A basic scheme of the proposed system is shown in figure 1. In the next section, we will discuss the two cases for \(N = 3\) and \(4\).

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3. Fidelity for extracting a target state

3.1. Unitary time-evolution operator

To analyze the fidelity of the proposed scheme to produce the multi-path cat states of light, we must calculate the unitary time-evolution operator of the interferometer device, which is an optical branching circuit with \( N \) inputs and \( N \) outputs driven by \( N \) indistinguishable single photons and generated by the interaction Hamiltonian (2). Note that the photon number operator \( \hat{N} = \sum_1^4 \hat{a}_i^\dagger \hat{a}_i \) commutes with equation (2). Thus, the dimension of the Hilbert space of which we have to take account is finite. Even for such a truncated Hilbert space, defined by the total number of photons \( N \) and the total number of modes \( N \), the size of the unitary matrix is \( 10 \times 10 \) for \( N = 3 \) and \( 35 \times 35 \) for \( N = 4 \). Instead of directly obtaining such a large unitary matrix by solving the Schrödinger equation, here we solve the Heisenberg equations of motion for the interaction Hamiltonian, equation (2). The output operators are related to the input operators for \( N = 3 \) by the following scattering matrix:

\[
\begin{pmatrix}
\hat{a}_1(t) \\
\hat{a}_2(t) \\
\hat{a}_3(t) \\
\hat{a}_4(t)
\end{pmatrix} =
\begin{pmatrix}
f(t) & h(t) & h(t) \\
h(t) & f(t) & h(t) \\
h(t) & h(t) & f(t)
\end{pmatrix}
\begin{pmatrix}
\hat{a}_1(0) \\
\hat{a}_2(0) \\
\hat{a}_3(0)
\end{pmatrix},
\]

where

\[
f(t) = \frac{3}{4} e^{i\kappa t} + \frac{1}{4} e^{-2i\kappa t}, \tag{4}
\]

\[
h(t) = -\frac{1}{4} e^{i\kappa t} + \frac{1}{2} e^{-2i\kappa t}. \tag{5}
\]

The scattering matrix for the optical branching circuit for \( N = 4 \) can be similarly obtained as follows:

\[
\begin{pmatrix}
\hat{a}_1(t) \\
\hat{a}_2(t) \\
\hat{a}_3(t) \\
\hat{a}_4(t)
\end{pmatrix} =
\begin{pmatrix}
u(t) & v(t) & v(t) & v(t) \\
v(t) & u(t) & v(t) & v(t) \\
v(t) & v(t) & u(t) & v(t) \\
v(t) & v(t) & v(t) & u(t)
\end{pmatrix}
\begin{pmatrix}
\hat{a}_1(0) \\
\hat{a}_2(0) \\
\hat{a}_3(0) \\
\hat{a}_4(0)
\end{pmatrix},
\]

where

\[
u(t) = \frac{3}{4} e^{i\kappa t} + \frac{1}{4} e^{-3i\kappa t}, \tag{7}
\]

\[
v(t) = -\frac{1}{4} e^{i\kappa t} + \frac{1}{2} e^{-3i\kappa t}. \tag{8}
\]

We now turn to the time evolution of the input state in Schrödinger picture. The input states consist of three indistinguishable single photons, expressed as \( |\psi_{in}\rangle = |111\rangle = \hat{a}_1^\dagger(0)\hat{a}_2^\dagger(0)\hat{a}_3^\dagger(0)\langle 000| \). To obtain the output state, we can calculate the unitary time-evolution operator by the following step:

\[
|\psi_{out}\rangle = e^{-i\hat{H}_T/\hbar} |111\rangle
= e^{-i\hat{H}_T/\hbar} \hat{a}_1^\dagger(0)\hat{a}_2^\dagger(0)\hat{a}_3^\dagger(0)|000\rangle
= e^{-i\hat{H}_T/\hbar} \hat{a}_1^\dagger(0)e^{i\hat{H}_T/\hbar} e^{-i\hat{H}_T/\hbar} \hat{a}_2^\dagger(0)e^{i\hat{H}_T/\hbar} e^{-i\hat{H}_T/\hbar} \hat{a}_3^\dagger(0)e^{i\hat{H}_T/\hbar} e^{-i\hat{H}_T/\hbar} |000\rangle
= \hat{a}_1^\dagger(-t)\hat{a}_2^\dagger(-t)\hat{a}_3^\dagger(-t)|000\rangle. \tag{9}
\]
Here, we used the three identity operators, $\hat{I} = e^{i\hat{H}_t/h}e^{-i\hat{H}_t/h}$, and the time evolution of the Heisenberg operators such as $\hat{a}_1(t) = e^{i\hat{H}_t/h}\hat{a}_1(0)e^{-i\hat{H}_t/h}$, $\hat{a}_2(t) = e^{i\hat{H}_t/h}\hat{a}_2(0)e^{-i\hat{H}_t/h}$, $\hat{a}_3(t) = e^{i\hat{H}_t/h}\hat{a}_3(0)e^{-i\hat{H}_t/h}$ and $e^{-i\hat{H}_t/h}|000\rangle = |000\rangle$. Therefore, the output state can be expressed in terms of the $3 \times 3$ scattering matrix and the state $|000\rangle$:

\[
|\psi_{\text{out}}\rangle = \left[ f(t)\hat{a}_1^\dagger(0) + h(t)(\hat{a}_2^\dagger(0) + \hat{a}_3^\dagger(0)) \right] \\
\times \left[ f(t)\hat{a}_2^\dagger(0) + h(t)(\hat{a}_1^\dagger(0) + \hat{a}_3^\dagger(0)) \right] \\
\times \left[ f(t)\hat{a}_3^\dagger(0) + h(t)(\hat{a}_1^\dagger(0) + \hat{a}_2^\dagger(0)) \right]|000\rangle \\
= 3\sqrt{2}f(t)(h(t))^2|\psi_{\text{target}}\rangle \\
+ 2\sqrt{3}[f(t)(h(t))^2 + (f(t))^2h(t) + (h(t))^3]|\varphi\rangle \\
+ [(f(t))^3 + 3f(t)(h(t))^2 + 2(h(t))^3][111],
\]

(10)

where a normalized target state $|\psi_{\text{target}}\rangle$ and garbage state $|\varphi\rangle$ are defined as follows:

\[
|\psi_{\text{target}}\rangle = \frac{1}{3\sqrt{2}}[\hat{a}_1^\dagger(0)^3 + \hat{a}_2^\dagger(0)^3 + \hat{a}_3^\dagger(0)^3]|000\rangle \\
= \frac{1}{\sqrt{3}}(|300\rangle + |030\rangle + |003\rangle),
\]

(11)

\[
|\varphi\rangle = \frac{1}{2\sqrt{3}}[\hat{a}_1^\dagger(0)^2\hat{a}_2^\dagger(0) + \cdots + \hat{a}_2^\dagger(0)\hat{a}_3^\dagger(0)^2]|000\rangle \\
= \frac{1}{\sqrt{6}}(|210\rangle + |120\rangle + |201\rangle + |102\rangle + |021\rangle + |012\rangle).
\]

(12)

We employed the identities $f^*(-t) = f(t)$ and $h^*(-t) = h(t)$ to obtain equation (10) from equation (3). It should be noted that the target state is the multi-path cat state for $N = 3$.

We can similarly obtain the output state for the case using four indistinguishable single photons as input,

\[
|\psi_{\text{out}}\rangle = e^{-i\hat{H}_t/h}|1111\rangle \\
= 4\sqrt{6}u(t)(v(t))^3|\psi_{\text{target}}\rangle \\
+ 6\sqrt{2}[u(t)^2(v(t))^2 + 2u(t)(v(t))^3 + (v(t))^4]|\varphi_1\rangle \\
+ 4\sqrt{3}[u(t)^2(v(t))^2 + u(t)(v(t))^3 + (v(t))^4]|\varphi_2\rangle \\
+ 2\sqrt{3}[u(t)^3v(t) + 2(u(t))^2(v(t))^2 + 5u(t)(v(t))^3 + 4(v(t))^4]|\varphi_3\rangle \\
+ [(u(t))^4 + 6(u(t))^2(v(t))^2 + 8u(t)(v(t))^3 + 9(v(t))^4][1111],
\]

(13)

where we introduce the normalized target state and three garbage states

\[
|\psi_{\text{target}}\rangle = \frac{1}{2}(|0000\rangle + |0400\rangle + |0040\rangle + |0004\rangle),
\]

(14)

\[
|\varphi_1\rangle = \frac{1}{2\sqrt{3}}(|3100\rangle + |3010\rangle + |3001\rangle + |1300\rangle + |0310\rangle + |0301\rangle \\
+ |1030\rangle + |0130\rangle + |0031\rangle + |1003\rangle + |0103\rangle + |0013\rangle).
\]

(15)
Figure 2. Probabilities of finding a target state $|\psi_{\text{target}}\rangle$, garbage state $|\varphi\rangle$ and input state $|111\rangle$ versus interaction parameter $\kappa t$ for $N=3$. The optimal interaction parameter $\kappa t = \frac{2\pi}{9}$ is indicated by a vertical dashed line.

Note that the target state given by equation (14) is the multi-path cat state for $N=4$. We notice that the output states are linear superpositions of orthogonal states, ten different states for $N=3$ and 35 different states for $N=4$, at the arbitrary evolution time $t$.

3.2. Fidelity $F$ versus interaction parameter $\kappa t$

We now discuss the fidelity of extracting the target state $|\psi_{\text{target}}\rangle$, defined as $F = |\langle \psi_{\text{target}} | \psi_{\text{out}} \rangle|^2$. Let us begin by considering the case of $N=3$. We can obtain the probabilities of finding the states $|\psi_{\text{target}}\rangle$, $|\varphi\rangle$ and $|111\rangle$ from equation (10). The numerical result is plotted in figure 2 against the interaction parameter $\kappa t$. The probabilities oscillate with a period of $\kappa t = \frac{2\pi}{3}$. We will focus on the first optimal interaction parameter $\kappa t = \frac{2\pi}{9}$ where the probability for the target state, which is equal to the fidelity $F$, has its maximum value $\frac{2}{3}$. At this optimal interaction parameter, the output state is a superposition of the target state and the input state only, expressed as $|\psi_{\text{out}}\rangle = \sqrt{\frac{2}{3}} |\psi_{\text{target}}\rangle + \sqrt{\frac{1}{3}} |111\rangle$. At this optimal point, the probability for the garbage state $|\varphi\rangle$ is totally suppressed. We will interpret this result in the next section.

We can similarly obtain the fidelity for the case of $N=4$ using equation (13). As shown in figure 3, the output state at the optimal interaction parameter $\pi/4$ is given by $|\psi_{\text{out}}\rangle = \sqrt{\frac{3}{8}} |\psi_{\text{target}}\rangle + \sqrt{\frac{3}{8}} |\varphi_2\rangle + \sqrt{\frac{1}{8}} |1111\rangle$. Thus, the maximum fidelity $F$ is $\frac{3}{8}$. The two garbage states
Figure 3. Probabilities of finding a target state \( |\psi_{\text{target}}\rangle \), garbage state \( |\varphi_1\rangle \), \( |\varphi_2\rangle \), \( |\varphi_3\rangle \) and input state \( |1111\rangle \) versus interaction parameter \( \kappa t \) for \( N = 4 \). The optimal interaction parameter \( \kappa t = \pi/4 \) is indicated by a vertical dashed line.

\( |\varphi_1\rangle \) and \( |\varphi_3\rangle \) are completely suppressed as before but the garbage state \( |\varphi_2\rangle \) is not suppressed even at this optimal interaction parameter.

3.3. Interpretation for destructive interference for garbage states and imperfect destructive interference for input states

In this section, we will discuss the physical interpretation for suppressed probability amplitudes for the garbage states in the output at the optimal interaction parameter, using a complex vector space. To start with, we analyze the simplest case \( N = 2 \), where a garbage state \( |11\rangle \) is suppressed and the desired output state \((|20\rangle + |02\rangle)/\sqrt{2}\) is obtained deterministically. As shown in figure 4(a), the garbage state \( |11\rangle \) consists of two possible paths, direct and exchange terms of equal amplitudes and opposite phases. This leads to the complete destructive interference for the probability amplitude for the garbage state.

Let us next consider the case of \( N = 3 \). One of the components in the garbage state \( |\varphi\rangle \), \( |210\rangle \), has three possible paths as shown in figure 4(b). The three possible paths have vectors with equal amplitudes and mutual phase difference by \( 2\pi/3 \), leading to the complete destructive interference among them. On the other hand, the input state \( |111\rangle \) has six possible paths as shown in figure 4(c). The three possible paths have vectors of equal amplitudes and phases, whereas the remaining three possible paths have vectors with \( 2\pi/3 \) phase shift. Thus, the imperfect destructive interference results in a finite probability amplitude for the input state \( |111\rangle \).

In the case of \( N = 4 \), two garbage states \( |\varphi_1\rangle \) and \( |\varphi_3\rangle \) are completely suppressed by the perfect destructive interference among four different paths and 12 different paths, respectively. On the other hand, the probability amplitudes of finding one garbage state \( |\varphi_2\rangle \) and input state \( |1111\rangle \) remain at a finite value due to the imperfect destructive interference among three possible paths and 24 possible paths, respectively.
3.4. General solution for arbitrary \( N \)

The general solution of the Heisenberg equations of motion for arbitrary input photon number \( N \) is expressed by

\[
\hat{a}_j(t) = f_N(t)\hat{a}_j(0) + h_N(t)\sum_{i \neq j}^N \hat{a}_i(0),
\]

(18)
Figure 5. (a) A state preserving QND measurement scheme that distinguishes the target state from the input state for $N = 3$. When $m$ takes an odd integer number in equation (23), a phase shifter $\pi/3$ must be added at the output port. (b) A scheme of QND detection using a single three-level atom in a cavity.

where

$$f_N(t) = \frac{N - 1}{N} e^{i\kappa t} + \frac{1}{N} e^{-(N-1)i\kappa t}, \quad (19)$$

$$h_N(t) = -\frac{1}{N} e^{i\kappa t} + \frac{1}{N} e^{-(N-1)i\kappa t}. \quad (20)$$

The fidelity of generating the multi-path cat state for arbitrary number $N$ is then expressed as

$$F = N^2 (N - 1)! |f_N(t)|^2 |h_N(t)|^2(N-1). \quad (21)$$

We can estimate that the fidelity reaches its maximum when $|f_N(t)| = |h_N(t)| = 1/\sqrt{N}$ because of the symmetrical form of the target state in equation (1). At this optimal interaction parameter, the fidelity is given by $F = N!/N^{N-1} \sim 1/e^N$ with indistinguishable single particles, whereas the fidelity with distinguishable particles is given by $F = 1/N^{N-1} \sim 1/N^N$. It is now clear that the fidelity improves by a factor of $N!$ due to the indistinguishability in the quantum interference.

4. State preserving QND measurement of photon number and post-selection

4.1. QND measurement of photon number with a cavity QED system

In order to improve the fidelity of generating the multi-path cat state at a cost of decreasing success probability, we can implement a QND detector at the output of the linear coupling circuit as shown in figure 5(a). A conditioned QND measurement of photon number with a finite success probability can be constructed with only linear optical elements [15, 16]. Since the input states into the linear coupling circuit are identical single photons as shown in figure 5(a) and the ancilla qubits required for the linear optics QND detector are also the same identical single photons, this scheme is particularly attractive.

Alternatively, we can implement a deterministic QND detector using an atom–field interaction in a cavity [17, 18]. Let us consider a cavity QED system with a single three-level atom (figure 5(b)). An $N$-photon wave packet couples into a cavity through a partially reflecting mirror, where an off-resonant stimulated Raman transition from $|g\rangle$ to $|e\rangle$ takes place with a help of a strong pump pulse. In the case of a single-photon capturing experiment [17, 18], the
effective pulse area is made equal to $\pi$, so that the incident single photon is absorbed and the atomic state is transferred from the ground state $|g\rangle$ to the excited state $|e\rangle$. In our case of a QND detector for an $N$-photon state, the effective pulse area is set equal to the multiple of $2\pi$, so that the $N$-photon state is reemitted from the cavity after a full Rabi oscillation, leaving the atomic state in $|g\rangle$.

To read out the atomic state after the above atom–field interaction, either $|g\rangle$ or $|e\rangle$, we can resonantly pump the transition between $|e\rangle$ and $|ex\rangle$ in figure 5(b). If the atomic state is in $|g\rangle$, there will be no photon emitted from the system at a transition frequency $\omega_1$ between $|g\rangle$ and $|ex\rangle$. If the atomic state is in $|e\rangle$, there will be one photon emitted at $\omega_1$.

4.2. State preserving QND measurement condition

In this section, we introduce a scheme to improve the fidelity by distinguishing and discarding the garbage and input states by post-selection. For this purpose, we need to distinguish the target state from the garbage and input states without collapsing the target state into one of its constituents such as $|N0000\rangle$, i.e. preserving the form of superposition as expressed in equation (1). Therefore we must use a special, state preserving QND measurement of photon number. Here, we will propose a scheme using the above QND detector of photon number (figure 5(b)). The atom–field interaction can be described by the effective Rabi frequency $g = g_1g_2\sqrt{N_p}/\Delta$ between states $|g\rangle$ and $|e\rangle$. Here, $g_1 = (d_1/\hbar)\sqrt{\hbar\omega_1/2\epsilon_0 V}$ is the coherent Rabi frequency between $|g\rangle$ and $|ex\rangle$, $N = 1$, where $d_1$ is an electric dipole moment, $\omega_1$ is a transition frequency between $|g\rangle$ and $|ex\rangle$, and $V$ is an effective cavity volume. Similarly, $g_2\sqrt{N_p}$ is the coherent Rabi frequency between $|e\rangle$ and $|ex\rangle$, where $N_p$ is the number of pump photons.

First, we will discuss the condition for achieving the state preserving QND measurement satisfying the above requirements for the case of $N = 3$. As we have shown in the previous section, the output state contains the target state consisting of photon number states $|0\rangle$ and $|3\rangle$ and the input state $|111\rangle$ at the optimal interaction parameter. Let us solve the Schrödinger equation under the following initial conditions:

$$|\psi(0)\rangle = |g\rangle_a \otimes \begin{cases} |3\rangle_f \text{ or } |0\rangle_f, & \text{target state,} \\ |1\rangle_f, & \text{input state,} \end{cases}$$

where the subscripts $f$ and $a$ denote a photon state and an atomic state, respectively. When the initial state is $|g\rangle_a|0\rangle_f$, the state does not evolve. In order to satisfy the desired QND measurement condition, we now set an interaction parameter to recover the initial state $|g\rangle_a|3\rangle_f$ so that the photon number is preserved. For the initial state $|g\rangle_a|3\rangle_f$, the final state is given by $|\psi(t)\rangle = \cos(\sqrt{3}gt)|g\rangle_a|3\rangle_f - i\sin(\sqrt{3}gt)|e\rangle_a|2\rangle_f$. Thus, the above condition for QND measurement for the target state with a three-level atom is

$$gt = \frac{1}{\sqrt{3}}m\pi,$$

where $m$ is a positive integer. When $m$ takes an odd integer number in equation (23), there is a $\pi$ phase shift for the state $|g\rangle_a|3\rangle_f$ due to the spinor property of the atom’s spin- $\frac{1}{2}$ system. For this reason, we should impose a phase shifter of $\pi/3$ at the output port to recover the target state $|\psi\rangle$, as shown in figure 5(a).

We hope to obtain the final atomic state $|e\rangle_a$ at the above interaction parameter, given by equation (23), for an input state $|g\rangle_a|1\rangle_f$ to completely distinguish it from the target state. The
solution of the Schrödinger equation for the input state $|g⟩_a|1⟩_f$ is $|ψ(t)⟩ = \cos(gt)|g⟩_a|1⟩_f - i \sin(gt)|e⟩_a|0⟩_f$, from which we can calculate the probability of obtaining the atomic state $|g⟩_a$ at the interaction parameter in equation (23), as follows:

$$P_g = |\cos(gt)|^2 = \cos^2\left(\frac{1}{\sqrt{3}} m \pi \right).$$

(24)

This value can be made arbitrarily close to zero by choosing an appropriate value for $m$, as will be discussed in the next subsection.

Now, we turn to the QND measurement for the case of $N = 4$. In this case, the target state is a coherent superposition of the photon number states $|4⟩$ and $|0⟩$, whereas the remaining garbage state includes the photon number states $|2⟩$ and $|0⟩$ at the optimal interaction parameter. We also have a residual input state $|1111⟩$ to be suppressed. For an initial state $|g⟩_a|4⟩_f$, the requirement for returning to the same initial state is

$$gt = \frac{1}{2} p \pi. \tag{25}$$

Similarly, for an initial state $|g⟩_a|1⟩_f$, the condition for converting to $|e⟩_a|0⟩_f$ is

$$gt = (\frac{1}{2} + q) \pi. \tag{26}$$

The two conditions are simultaneously satisfied if we take an odd number for $p$ and $q = (p - 1)/2$. Under this condition, the input state $|1111⟩$ is completely distinguished and discarded while the target state is untouched.

In order to distinguish and suppress the garbage state $|ϕ⟩_2$, we can use four consecutive QND measurements and reject the wave packet unless the readout is $|g⟩_a|g⟩_a|g⟩_a|g⟩_a$. The probability of obtaining the atomic state $|g⟩_a$ for the initial state $|g⟩_a|2⟩_f$ at the interaction parameter given by equation (25) is

$$P_g = |\cos(\sqrt{2}gt)|^2 = \cos^2\left(\frac{1}{\sqrt{2}} p \pi \right). \tag{27}$$

This value can also be made close to zero by choosing an appropriate odd integer number $p$.

### 4.3. Fidelity and success probability

We will calculate the improved fidelity by the state preserving QND measurement for $N = 3$. As shown in figure 5, we will measure an atomic state, $|e⟩_a$ or $|g⟩_a$, in parallel at each output port. The fidelity after one QND measurement and post-selection with a result $|g⟩_a$ is

$$F = \frac{2}{3} + \frac{1}{3} P_g, \tag{28}$$

while the fidelity after three QND measurements and post-selection with a result $|g⟩_a|g⟩_a|g⟩_a$ is

$$F = \frac{2}{3} + \frac{1}{3} P_g. \tag{29}$$

Table 1 shows the results of different QND measurement schemes.

Next, we will calculate the fidelity for the case of $N = 4$. The fidelity after four QND measurements and post-selection with a result $|g⟩_a|g⟩_a|g⟩_a|g⟩_a$ for this case is

$$F = \frac{3}{8} + \frac{1}{8} P_g. \tag{30}$$
The central lobes of the interference pattern on the screen are shown by dashed lines in figure 5. Application to quantum lithography

For instance, if we take \( p = 5 \) in equation (25), the fidelity is up to \( F = 0.9998 \) at a success probability of \( S = \frac{3}{8} + \frac{3}{8} P_g^2 = 0.3751 \).

In actual QND measurement, the interaction parameter for coherent Rabi oscillation might deviate from the optimal values, such as \( g t = \frac{1}{\sqrt{3}} (m \pi + \delta) \) instead of equation (23). Here, we estimate its impact on the degradation of the fidelity in equation (28) as follows:

\[
\Delta F \leq \left| \frac{2 \sin(2m\pi/\sqrt{3})}{\sqrt{3}(2 + \cos^2(m\pi/\sqrt{3}))^2} \delta \right|.
\]

To suppress \( \Delta F \) within the order of \( 10^{-2} \) for \( m = 1 \), we need to set a relative deviation \( |\delta|/\pi \lesssim 3 \times 10^{-2} \), which is feasible.

5. Application to quantum lithography

Finally in this section, we will demonstrate the usefulness of the multi-path cat states in quantum lithography. For this purpose, we calculate the interference patterns of the state \( |\psi_N^{(N)}\rangle = \left[ \sum_{p=1}^{N} |N\rangle_p (\otimes_{q \neq p} |0\rangle_q) \right]/\sqrt{N} \), from \( N \) slits, where the states are designated by the subscript \( p \) defined at the respective aperture of the slit \( p \). For example, the state is expressed as \( (|3\rangle_1 |0\rangle_2 |0\rangle_3 + |0\rangle_1 |3\rangle_2 |0\rangle_3 + |0\rangle_1 |0\rangle_2 |3\rangle_3) / \sqrt{3} \) for \( N = 3 \). A normalized interference pattern from \( N \) slits is given by

\[
\frac{1}{N} \sum_{p=1}^{N} e^{i(p-1)\Delta \phi} \int_{(N-2p+1)d/2-(\Delta y/2)}^{(N-2p+1)d/2+(\Delta y/2)} e^{ik_y y} dy.
\]

where a finite width of a slit is \( \Delta y \), the slit separation is \( d \), and the distance between the center of the slits and the screen is \( L \). The parameter \( l_p \) measures the distance from a particular spot in the slit \( p \) to the particular spot on the screen (figure 6). A phase shift \( (p-1)\Delta \phi \) is introduced in front of the slit \( p \) so that we can evaluate the phase sensitivity for displacing an exposure pattern in quantum lithography.

When a single photon is incident on the setup which is expressed by \( |\psi_N^{(1)}\rangle = \left[ \sum_{p=1}^{N} |1\rangle_p (\otimes_{q \neq p} |0\rangle_q) \right]/\sqrt{N} \) and a single-photon counting detector is placed on the screen, the wave number \( k \) in equation (31) is given by \( k = 2\pi/\lambda \), where \( \lambda \) is an optical wavelength. The central lobes of the interference pattern on the screen are shown by dashed lines in figure 7. If a multi-path cat state \( |\psi_N^{(N)}\rangle \) is incident on the \( N \) slits and \( N \) photons are simultaneously registered by a coincidence counting detector, then the wave number \( k \) in equation (31) becomes \( k = N \times 2\pi/\lambda \), which corresponds to the photonic de Broglie wavelength \( \lambda_{\text{dB}} = \lambda/N \). Note that

Table 1. Fidelity of extracting the target state by the single and multiple (three) QND measurements and post-selection for \( N = 3 \) with respect to various interaction parameter \( g t \).

| Interaction Parameter \( g t \) | Single QND scheme | Multiple QND scheme |
|-------------------------------|-------------------|----------------------|
| \( \pi/\sqrt{3} \)          | 0.9719            | 0.9937               |
| \( 6\pi/\sqrt{3} \)        |                   | 0.9998               |
| \( 13\pi/\sqrt{3} \)       |                   |                      |

\[ \frac{3}{8} + \frac{3}{8} P_g^2 = 0.3751 \]

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the state for $N = 2$, $(|2⟩_1|0⟩_2 + |0⟩_1|2⟩_2)/\sqrt{2}$, corresponds to the NOON state. The central lobes of the interference pattern for the NOON state and the multi-path cat state are shown by solid lines in figure 7. As $N$ increases, the central lobe concentrates more tightly due to the increased number of slits and the decreased photonic de Broglie wavelength. As shown in figure 7, the multi-path cat state is not only superior to the classical method, but also more suitable for implementing quantum lithography than the NOON state due to the improved spatial resolution.

Next, the numerical results for the shifted interference patterns are plotted in figure 8. It is clear that the required phase modulation for displacing the central lobe by one unit of lobe decreases by a factor of $N$ when the multi-path cat state is used.

6. Conclusion

We have proposed a scheme to generate a multi-path cat state of light, $|ψ⟩ = (|N00⋯0⟩ + |0N0⋯0⟩ + ⋯ + |000⋯0N⟩)/\sqrt{N}$, starting with $N$ indistinguishable single photons and linear optical branching circuits. The fidelity of extracting the target state can be increased
by a state preserving QND measurement of photon number and post-selection. We make a remark on the fidelity that it is enhanced by a factor of $N!$ compared to the scheme using distinguishable particles. However, the fidelity still decreases exponentially as input photon number $N$ increases. We might need to take nonlinearity (attractive interaction among photons) into consideration to get improved fidelity for large $N$ [19]. The novel state is shown to improve the spatial resolution and phase sensitivity in quantum lithography.

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