On the compactness of the (non)radial Sobolev spaces

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Abstract

In this note, we give the affirmative answer of the question in [18], which is a compactness result of the non-radial Sobolev spaces. As an application, we show the existence of an extremal function of the critical Hardy inequality under spherical average zero. Next, we give an improvement of the compactness results of the radial Sobolev spaces in [8]. In Appendix, we give an alternative proof of Hardy type inequalities under spherical average zero.

Keywords: Radial compactness, Spherical average zero, Hardy inequality

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1. Introduction

Let $N \geq 3$ and $2^* = \frac{2N}{N-2}$. The Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is non-compact for any $q \in [2, 2^*]$. However, it is well-known that if we restrict $H^1(\mathbb{R}^N)$ to the radial Sobolev space $H^1_{\text{rad}}(\mathbb{R}^N)$, then the embedding $H^1_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ becomes compact for any $q \in (2, 2^*)$ (Ref. [21]). It is called Strauss's radial compactness. We can deny the possibility of non-compactness with respect to translation invariance thanks to the restriction of $H^1(\mathbb{R}^N)$ to $H^1_{\text{rad}}(\mathbb{R}^N)$ such as the non-compact sequence $\{u_m\}_{m=1}^{\infty} \subset H^1(\mathbb{R}^N)$ of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, where $u_m(x) = u(x + x_m)$ ($x \in \mathbb{R}^N$), $u$ is a smooth function on $\mathbb{R}^N$ and $|x_m| \to \infty$ ($m \to \infty$). Therefore, we have the compactness and the non-compactness results of three embeddings as follows.

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Theorem A. (Strauss’s radial compactness) Let $N \geq 3$ and $2 < q < 2^*$. Then

\[ H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) : \text{non-compact}, \]

\[ H^1_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) : \text{compact}, \]

\[ \left( H^1_{\text{rad}}(\mathbb{R}^N) \right)^\perp \hookrightarrow L^q(\mathbb{R}^N) : \text{non-compact}. \]

Here, \( (H^1_{\text{rad}}(\mathbb{R}^N))^\perp \) is the orthogonal complement of the radial Sobolev space \( H^1_{\text{rad}}(\mathbb{R}^N) \) and the last one in Theorem A follows from Proposition 1 in Appendix. For simplicity, we call \( (H^1_{\text{rad}}(\mathbb{R}^N))^\perp \) the non-radial Sobolev space. Several generalizations of Strauss’s radial compactness have been investigated, see [16, 10, 11]. Note that even if we restrict \( H^1(\mathbb{R}^N) \) to \( H^1_{\text{rad}}(\mathbb{R}^N) \), the embedding \( H^1_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \) is still non-compact for \( q = 2 \) or \( 2^* \). In fact, in the case where \( q = 2 \), there is a non-compact sequence with respect to \( L^2 \)-invariance, which is a vanishing sequence \( \{u_m\}_{m=1}^\infty \subset H^1_{\text{rad}}(\mathbb{R}^N) \), where \( u_m(x) = m^{-\frac{N}{q}} u\left(\frac{x}{m}\right) (x \in \mathbb{R}^N, m \in \mathbb{N}) \). On the other hand, in the case where \( q = 2^* \), there is a non-compact sequence with respect to \( \dot{H}^1 \) and \( L^{2^*} \)-invariance, which is a concentration sequence \( \{u_m\}_{m=1}^\infty \subset H^1_{\text{rad}}(\mathbb{R}^N) \), where \( u_m(x) = m^{\frac{N-2}{2}} u(mx) (x \in \mathbb{R}^N, m \in \mathbb{N}) \). As we can see from Strauss’s radial compactness, we may deny the possibility of non-compactness under some restriction of \( H^1(\mathbb{R}^N) \). In fact, if we restrict \( H^1_{\text{rad}}(\mathbb{R}^N) \) to \( H^1_{\text{rad}}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \), then the embedding \( H^1_{\text{rad}}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \) becomes compact for \( p \in [1, 2) \) and the embedding \( H^1_{\text{rad}}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \) becomes compact for \( p > 2^* \), see [8] Corollary 1. In §3 we extend these results to Lorentz spaces \( L^{p,q}(\mathbb{R}^N) \).

In this note, we give another example of the compactness of the Sobolev embedding by answering the following question in [18]:

Let \( B^a_1 \subset \mathbb{R}^N \) be the unit ball, \( a > 1, q \geq 2 \). Then, is the embedding \( \left( H^1_{0,\text{rad}}(B^a_1) \right)^\perp \hookrightarrow L^q\left( B^a_1; |x|^{-2} (\log \frac{a}{|x|})^{-1-\frac{q}{2}} \right) dx \) compact?

This question comes from a heuristic consideration and calculation via harmonic transplantation which is proposed by Hersch [12]. For a summary of harmonic transplantation, see e.g. [20] §3. First, we give the affirmative answer of the above question as follows.

Theorem 1. Let \( B^a_1 \subset \mathbb{R}^2 \) be the unit ball, \( a > 1 \) and \( q \geq 2 \). Then the embedding \( \left( H^1_{0,\text{rad}}(B^a_1) \right)^\perp \hookrightarrow L^q\left( B^a_1; |x|^{-2} (\log \frac{a}{|x|})^{-1-\frac{q}{2}} \right) dx \) is compact.

By combining Theorem 1 and the non-compactness of the embedding \( H^1_{0,\text{rad}}(B^a_1) \hookrightarrow L^q\left( B^a_1; |x|^{-2} (\log \frac{a}{|x|})^{-1-\frac{q}{2}} \right) dx \), we have the following.
Corollary 1. ("Non-radial compactness") Let $B_1^2 \subset \mathbb{R}^2$ be the unit ball, $a > 1$ and $q \geq 2$. Then

$$H_0^1(B_1^2) \hookrightarrow L^q \left( B_1^2, |x|^{-2} \left( \log \frac{a}{|x|} \right)^{-1-\frac{q}{2}} dx \right) : \text{non-compact},$$

$$H_{0,rad}^1(B_1^2) \hookrightarrow L^q \left( B_1^2, |x|^{-2} \left( \log \frac{a}{|x|} \right)^{-1-\frac{q}{2}} dx \right) : \text{non-compact},$$

$$\left( H_{0,rad}^1(B_1^2) \right)^\perp \hookrightarrow L^q \left( B_1^2, |x|^{-2} \left( \log \frac{a}{|x|} \right)^{-1-\frac{q}{2}} dx \right) : \text{compact}.$$

In Theorem A, the non-compact embedding becomes compact under the restriction of $H^1(\mathbb{R}^N)$ to $H_{rad}^1(\mathbb{R}^N)$, while the non-compact embedding becomes compact under the restriction of $H_0^1(B_1^2)$ to $\left( H_{0,rad}^1(B_1^2) \right)^\perp$ in Corollary I. In this sense, Corollary I implies an opposite phenomenon to Strauss’s radial compactness.

Second, as an application of Theorem I with $q = 2$, we can obtain an extremal of the critical Hardy inequality under spherical average zero:

$$A_a \int_{B_1^2} \frac{|u|^2}{|x|^2 \left( \log \frac{a}{|x|} \right)^2} dx \leq \int_{B_1^2} |\nabla u|^2 dx$$

for any $u \in H_0^1(B_1^2)$ with $\int_0^{2\pi} u(r\theta) d\theta = 0 \ (r \in [0, 1])$. Namely, we can obtain a minimizer of the following minimization problem $A_a$

$$A_a := \inf \left\{ \frac{\int_{B_1^2} |\nabla u|^2 dx}{\int_{B_1^2} |u|^2 \left( \log \frac{a}{|x|} \right)^2 dx} \left| u \in H_0^1(B_1^2) \setminus \{0\}, \int_0^{2\pi} u(r\theta) d\theta = 0 \ (r \in [0, 1]) \right\}$$

$$= \inf \left\{ \frac{\int_{B_1^2} |\nabla u|^2 dx}{\int_{B_1^2} |u|^2 \left( \log \frac{a}{|x|} \right)^2 dx} \left| u \in \left( H_{0,rad}^1(B_1^2) \right)^\perp \setminus \{0\} \right\}$$

$$\geq B_a := \inf \left\{ \frac{\int_{B_1^2} |\nabla u|^2 dx}{\int_{B_1^2} |u|^2 \left( \log \frac{a}{|x|} \right)^2 dx} \left| u \in H_0^1(B_1^2) \setminus \{0\} \right\} = \frac{1}{4}.$$

For the second equality, see e.g. [3]. It is well-known that for any $a \geq 1$, there is no minimizer of $B_a = \frac{1}{4}$, see e.g. [1], [14].

For the Hardy inequality under spherical average zero, see e.g. [4, 9, 3] or Proposition 3 in Appendix.
Corollary 2. Let \(q = \frac{1}{a} = B_a\) for \(a > 1\) and \(A_1 = \frac{1}{4} = B_1\). Moreover, there exists a minimizer of \(A_a\) if and only if \(a > 1\).

Also, in the case where \(q > 2\), we obtain the following corollary.

**Corollary 3.** Let \(q > 2\) and \(a > 1\). Set

\[
D_a := \inf \left\{ \frac{\int_{B_1^2} |\nabla u|^2 \, dx}{\left( \int_{B_1^2} \frac{|u|^q}{|x|^2 \log \frac{|x|}{r}} \, dx \right)^{\frac{2}{q}}} \, \middle| \, u \in H_0^1(B_1^2) \setminus \{0\}, \int_0^{2\pi} u(r\theta) \, d\theta = 0 (t' \in [0, 1]) \right\}.
\]

Then there exists a minimizer of \(D_a (> 0)\).

Since the proof of Corollary 3 is the same as the proof of Corollary 2, we omit the proof of Corollary 3. Set

\[
G_a := \inf \left\{ \frac{\int_{B_1^2} |\nabla u|^2 \, dx}{\left( \int_{B_1^2} \frac{|u|^q}{|x|^2 \log \frac{|x|}{r}} \, dx \right)^{\frac{2}{q}}} \, \middle| \, u \in H_0^1(B_1^2) \setminus \{0\} \right\},
\]

\[
G_{a,\text{rad}} := \inf \left\{ \frac{\int_{B_1^2} |\nabla u|^2 \, dx}{\left( \int_{B_1^2} \frac{|u|^q}{|x|^2 \log \frac{|x|}{r}} \, dx \right)^{\frac{2}{q}}} \, \middle| \, u \in H_0^{1, \text{rad}}(B_1^2) \setminus \{0\} \right\},
\]

\[
D_a = \inf \left\{ \frac{\int_{B_1^2} |\nabla u|^2 \, dx}{\left( \int_{B_1^2} \frac{|u|^q}{|x|^2 \log \frac{|x|}{r}} \, dx \right)^{\frac{2}{q}}} \, \middle| \, u \in \left( H_0^{1, \text{rad}}(B_1^2) \right) \setminus \{0\} \right\}.
\]

It is known that \(G_{a,\text{rad}}\) is not attained for any \(a \in (1, \infty)\), see [13]. The second author [17] showed that there exists \(a_* > 1\) such that \(G_a < G_{a,\text{rad}}\) and \(G_a\) is attained for \(a \in (1, a_*)\), and \(G_a = G_{a,\text{rad}}\) and \(G_a\) is not attained for \(a > a_*\). We can interpret the existence of a minimizer of \(G_a\) for \(a \in (1, a_*)\) intuitively as the embedding \(H_0^{1, \text{rad}}(B_1^2) \hookrightarrow L^q(B_1^2; |x|^{-2} \left( \log \frac{|x|}{r} \right)^{-1 - \frac{2}{q}} \, dx)\) is dominant when \(G_a < G_{a,\text{rad}}\), and the existence of a minimizer of \(G_a\) comes from the effect of the compactness of the embedding shown in Theorem 1.
Remark 1. \((a = 1)\) In the case where \(a = 1\), we have the followings.

\[
H^1_0(B^2) \not\hookrightarrow L^q \left( B^2_1; |x|^{-2} \left( \log \frac{1}{|x|} \right)^{-\frac{1}{2}} \right) dx
\]

\[
H^1_0(\text{rad}(B^2_1)) \not\hookrightarrow L^q \left( B^2_1; |x|^{-2} \left( \log \frac{1}{|x|} \right)^{-\frac{1}{2}} \right) dx
\]

\[
(H^1_0(\text{rad}(B^2_1)))^\perp \not\hookrightarrow L^q \left( B^2_1; |x|^{-2} \left( \log \frac{1}{|x|} \right)^{-\frac{1}{2}} \right) dx
\]

For the first and the second one, see \([13, 17]\). The third one follows from the first and the second one. Therefore, we have \(D_1 = G_1 = 0\).

The minimization problems associated with the Rellich type inequalities under spherical average zero are studied by \([6]\). Also, Hardy type inequalities with another average zero condition, which comes from Neumann problem, are studied by \([7, 19]\).

In the next section, we show Theorem 1, Corollary 1 and Corollary 2. In §3, we show that the embedding \(H^1_{\text{rad}}(\mathbb{R}^N) \cap L^{p,\infty}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)\) is compact for \(p \in [1, 2)\) and the embedding \(H^1_{\text{rad}}(\mathbb{R}^N) \cap L^{p,\infty}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)\) is compact for \(p > 2^*\). Since \(L^p(\mathbb{R}^N) \subsetneq L^{p,\infty}(\mathbb{R}^N)\), it is an improvement of Corollary 1 in \([8]\). In Appendix, we give an alternative proof of Hardy type inequalities under spherical average zero. In \([3]\) p.13-14, the optimality of the constant of the Hardy inequality under spherical average zero was shown on the whole space by using Fourier analysis and Lemma 3.8 in \([23]\). On the other hand, our proof is available not only on the whole space but also on the ball. Furthermore, our proof is self-contained.

**Notation.** \(|A|\) denotes the Lebesgue measure of a set \(A \subset \mathbb{R}^N\) and \(\omega_{N-1}\) denotes an area of the unit sphere in \(\mathbb{R}^N\). \(H^1_0(B^1_N)\) is the completion of \(C^0_0(B^1_N)\) with respect to the norm \(||\nabla(\cdot)||_2\). Throughout this note, if a radial function \(u\) is written as \(u(x) = \tilde{u}(|x|)\) by some function \(\tilde{u} = \tilde{u}(r)\), we write \(u(x) = u(|x|)\) with admitting some ambiguity. Also, we use \(C\) or \(C_i (i \in \mathbb{N})\) as positive constants. If necessary and these constants depend on \(\varepsilon\), they will be denoted by \(C(\varepsilon)\).

2. Compactness of the non-radial Sobolev spaces: Proofs of Theorem 1, Corollary 1 and Corollary 2

First, we show Theorem 1 and Corollary 1.
Proof. (Theorem 1) First, we assume that \( q = 2 \). Let \( \{u_m\}_{m=1}^{\infty} \subset (H^1_{0,\text{rad}}(B_1^2))^\perp \) be a bounded sequence. Since \( (H^1_{0,\text{rad}}(B_1^2))^\perp \) is a closed subspace of the reflexive Banach space \( H^1_{0,\text{rad}}(B_1^2) \), \( (H^1_{0,\text{rad}}(B_1^2))^\perp \) is also the reflexive Banach space. Therefore, there exist \( u \in (H^1_{0,\text{rad}}(B_1^2))^\perp \) and a subsequence \( \{u_{m_j}\}_{j=1}^{\infty} \) (we use \( \{u_m\} \) again) such that \( u_m \rightharpoonup u \) in \( (H^1_{0,\text{rad}}(B_1^2))^\perp \) as \( m \to \infty \). Since \( H^1_{0,\text{rad}}(B_1^2) \hookrightarrow L^2(B_1^2) \) is compact, we have \( u_m \to u \) in \( L^2(B_1^2) \).

Therefore, we have

\[
\int_{B_1^2} \frac{|u_m - u|^q}{|x|^2(\log \frac{a}{|x|})^2} \, dx \leq \left( \int_{B_1^2} \frac{|u_m - u|^2}{|x|^2} \, dx \right)^{q/2} \left( \int_{B_1^2} \frac{|u_m - u|^2}{|x|^2(\log \frac{a}{|x|})^2} \, dx \right)^{1-q/2},
\]

where the second inequality comes from the Hardy inequality under spherical average zero (see e.g. Proposition 2 in §4). Letting \( m \to \infty \) and \( \epsilon \to 0 \), we see that \( u_m \to u \) in \( L^2(\mathbb{R}^2; |x|^{-2}(\log \frac{a}{|x|})^{-2} \, dx) \). Hence the embedding \( (H^1_{0,\text{rad}}(B_1^2))^\perp \hookrightarrow L^2(\mathbb{R}^2; |x|^{-2}(\log \frac{a}{|x|})^{-2} \, dx) \) is compact. Next, we assume that \( q > 2 \). Let \( \{u_m\}_{m=1}^{\infty} \subset (H^1_{0,\text{rad}}(B_1^2))^\perp \) be a bounded sequence and \( u_m \rightharpoonup u \) in \( (H^1_{0,\text{rad}}(B_1^2))^\perp \). From the case where \( q = 2 \), we have \( u_m \to u \) in \( L^2(B_1^2; |x|^{-2}(\log \frac{a}{|x|})^{-2} \, dx) \). Let \( 1 < p < \infty \). By the Hölder inequality, we have

\[
\int_{B_1^2} \frac{|u_m - u|^q}{|x|^2(\log \frac{a}{|x|})^{2+1}} \, dx \leq \left( \int_{B_1^2} \frac{|u_m - u|^2}{|x|^2} \, dx \right)^{q/2} \left( \int_{B_1^2} \frac{|u_m - u|^2}{|x|^2(\log \frac{a}{|x|})^{2}} \, dx \right)^{1-q/2},
\]

where \( r = \frac{p}{p-1} (q - \frac{2}{p}) = q + \frac{q-2}{p-1} > 2 \), \( \bar{r} = \frac{p}{p-1} \left( 1 + \frac{q}{2} - \frac{2}{p} \right) \). Note that \( \bar{r} = \frac{q}{2} + 1 \). By the embedding \( H^1_{0,\text{rad}}(B_1^2) \hookrightarrow L^p(B_1^2; |x|^{-2}(\log \frac{1}{|x|})^{-1+\frac{q}{2}} \, dx) \), we have

\[
\int_{B_1^2} \frac{|u_m - u|^q}{|x|^2(\log \frac{a}{|x|})^{2+1}} \, dx \leq \left( \int_{B_1^2} \frac{|u_m - u|^2}{|x|^2(\log \frac{a}{|x|})^2} \, dx \right)^{q/2} \left( \int_{B_1^2} \frac{|\nabla (u_m - u)|^2}{|x|^2(\log \frac{a}{|x|})^{2}} \, dx \right)^{1-q/2} \to 0,
\]

as \( m \to \infty \). Therefore, \( (H^1_{0,\text{rad}}(B_1^2))^\perp \hookrightarrow L^q(B_1^2; |x|^{-2}(\log \frac{a}{|x|})^{-1+\frac{q}{2}} \, dx) \) is compact for any \( q \geq 2 \).

Proof. (Corollary 1) Although it is already shown by the scaling argument (see e.g. [13,17]), we write the proof here for reader’s convenience. Since \( H^1_{0,\text{rad}}(B_1^2) \subset H^1_{0,\text{rad}}(B_1^2) \), we have...
\(H^1_0(B^2_1),\) it is enough to show the non-compactness of the embedding \(H^1_{0,\text{rad}}(B^2_1) \hookrightarrow L^2\left(B^2_1; |x|^{-\frac{2}{3}} \left(\log \frac{a}{|x|}\right)^{-\frac{1}{2}} dx\right).\) Let \(u \in C^\infty_{c,\text{rad}}(B^2_1) \setminus \{0\}.)\) Consider \(u_\lambda(x) = \lambda^{-\frac{1}{2}} u(\lambda x),\ y = \left(\frac{m}{a}\right)^{1-\frac{1}{2}} x \text{ for } \lambda \leq 1.\) Note that \(\text{supp } u_\lambda \subset B_{a^{-\frac{3}{4}}} \subset B^2_1 \text{ for } \lambda \leq 1.\) Let \(|x| = r, |y| = s.\) Since \(\frac{a}{y} = \frac{a}{x}\) and \(r \frac{dy}{dr} = \lambda s,\) for any \(\lambda \leq 1\) we have
\[
\int_{B^2_1} |\nabla u_\lambda|^2 \, dx = 2 \pi \int_0^{1/\lambda} \left| \frac{dy}{dr}(r) \right|^2 \, r \, dr \\
= 2 \pi \lambda^{-1} \int_0^{1/\lambda} \left| \frac{dy}{dr}(s) \right|^2 \, r \left( \frac{ds}{dr} \right) \, ds = \int_{B^2_1} |\nabla u|^2 \, dy < \infty,
\]
\[
\int_{B^2_1} \frac{|u_\lambda|^q}{|x|^2 (\log \frac{a}{|x|})^{1+q/2}} \, dx = 2 \pi \lambda^{-1} \int_0^{1/\lambda} \frac{|u_\lambda(r)|^q}{(\log \frac{a}{r})^{1+q/2}} \, \frac{dr}{r} \\
= 2 \pi \lambda^{-\frac{1}{2}} \int_0^{1/\lambda} \frac{|u(s)|^q}{(\log \frac{a}{s})^{1+q/2}} \, ds = \int_{B^2_1} \frac{|u|^q}{(\frac{a}{\lambda} \log \frac{a}{s})^{1+q/2}} \, ds
\]

Especially, if we choose \(\lambda = \frac{1}{m} (m \in \mathbb{N}),\) then we see that \(\{u_\lambda\}_{m=1}^\infty \subset H^1_{0,\text{rad}}(B^2_1)\) is a non-compact sequence. In fact, we see that \(u_\lambda \to 0\) in \(H^1_{0,\text{rad}}(B^2_1)\) and \(u_\lambda \nrightarrow 0\) in \(L^q\left(B^2_1; |x|^{-\frac{2}{3}} \left(\log \frac{a}{|x|}\right)^{-\frac{1}{2}} dx\right).\) Therefore, the embedding \(H^1_{0,\text{rad}}(B^2_1) \hookrightarrow L^q\left(B^2_1; |x|^{-\frac{2}{3}} \left(\log \frac{a}{|x|}\right)^{-\frac{1}{2}} dx\right)\) is non-compact. \(\square\)

Next, we show Corollary \(\text{Corollary} 2.\)

**Proof.** (Corollary \(\text{Corollary} 2\)) Consider the following test function.
\[u_\alpha(x) = g_2(\theta) f_\alpha(r) \left(\alpha > \frac{1}{2}\right),\] where \(f_\alpha(r) = \begin{cases} 2(\log 2)^{\alpha} r & \text{if } r \in [0, \frac{1}{2}], \\ (\log \frac{1}{r})^{\alpha} & \text{if } r \in (\frac{1}{2}, 1), \end{cases}\]

and \(g_2 = g_2(\theta)\) satisfies
\[
\int_0^{2\pi} g_2(\theta) \, d\theta = 0, \quad \int_0^{2\pi} |g_2'(\theta)|^2 \, d\theta = \int_0^{2\pi} |g_2(\theta)|^2 \, d\theta.
\]
Then we have

\[ A_1 \leq \int_{B_1^2} |\nabla u_a|^2 \, dx \leq \int_{1/2}^{1} \int_{0}^{2\pi} \alpha \left( \log \frac{1}{r} \right)^{\alpha-1} \frac{1}{r} \, dx 
\int_{1/2}^{1} \left( \frac{1}{r} \right)^{2\alpha-2} |g_2'(\theta)|^2 r^{-1} \, d\theta + C \]

\[ = \alpha^2 + \int_{1/2}^{1} \left( \frac{1}{r} \right)^{2\alpha-2} r^{-1} \, dr \]

\[ = \alpha^2 + \int_{0}^{\log 2} \frac{t^{2\alpha}}{t^{2\alpha-2}} \, dt = \frac{1}{4} + o(1) \quad (\alpha \to \frac{1}{2}) \].

Since \( B_1 = \frac{1}{4} \) is not attained, \( A_1 = \frac{1}{4} \) is not attained.

Assume that \( a > 1 \). Let \( \{u_m\}_{m=1}^{\infty} \subset (H_{0,\text{rad}}(B_1^2))^\perp \) be a minimizing sequence of \( A_a \) satisfying

\[ \int_{B_1^2} |u_m|^2 \, dx = 1, \quad \int_{B_1^2} |\nabla u_m|^2 \, dx = A_a + o(1) \quad (m \to \infty). \]

Since \( \{u_m\} \) is bounded, in the same way as the proof of Theorem 1, we have \( u_m \rightharpoonup u \) in \( (H_{0,\text{rad}}(B_1^2))^\perp \). By Theorem 1 we have

\[ u_m \to u \text{ in } L^2 \left( B_1^2; |x|^{-2} \left( \log \frac{a}{|x|} \right)^{-2} \right), \]

\[ \int_{B_1^2} \frac{|u|^2}{|x|^2 (\log \frac{a}{|x|})^2} \, dx = \lim_{m \to \infty} \int_{B_1^2} \frac{|u_m|^2}{|x|^2 (\log \frac{a}{|x|})^2} \, dx = 1, \]

\[ \int_{B_1^2} |\nabla u|^2 \, dx \leq \liminf_{m \to \infty} \int_{B_1^2} |\nabla u_m|^2 \, dx = A_a. \]

Therefore, \( u \) is a minimizer of \( A_a \). Since \( A_a \) is attained for \( a > 1 \), we see that \( A_a > \frac{1}{4} = B_a \).  \( \square \)
3. Compactness of the radial Sobolev spaces: Improvement of the compactness result in [8]

Let \( N \geq 3, 2^* = \frac{2N}{N-2}, 1 \leq p < \infty \), \( L^{p,q}(\mathbb{R}^N) \) be the Lorentz space which is given by

\[
L^{p,q}(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable} \left\| u \right\|_{p,q} < \infty \right\},
\]

\[
\left\| u \right\|_{p,q} = \begin{cases} \left( \int_0^\infty s^{\frac{1}{q}} u^*(s)^q \frac{ds}{s} \right)^\frac{1}{q} & \text{if } 1 \leq q < \infty, \\ \sup_{s \in (0,\infty)} s^{\frac{1}{q}} u^*(s) & \text{if } q = \infty, \end{cases}
\]

\( u^* : [0, \infty) \to [0, \infty] \) denotes the decreasing rearrangement of \( u \) and \( u^\# : \mathbb{R}^N \to [0, \infty] \) denotes the Schwartz symmetrization of \( u \) which are given by

\[
u^*(t) = \inf \left\{ \lambda > 0 \left| \frac{|x \in \mathbb{R}^N | |u(x)| > \lambda|}{x} \right| \leq t \right\},
\]

\[
u^\#(x) = u^\#(|x|) = \inf \left\{ \lambda > 0 \frac{|x \in \mathbb{R}^N | |u(x)| > \lambda|}{|B^N|} \right\}
\]

(Ref. [2, 15]). Obviously, we have

\[
u^*(t) = u^\#(r), \text{ where } t = |B^N_{|x|}| = \frac{\omega_{N-1} s^N}{N}
\]

The authors [8] obtained the following compactness result in the framework of Lebesgue spaces.

**Theorem B.** ([8] Corollary 1)

(i) \( H^1_{\text{rad}}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \) is compact for \( p \in [1, 2) \).

(ii) \( H^1_{\text{rad}}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \) is compact for \( p > 2^* \).

**Remark 2.** Actually, we do not need the radially symmetry in (i). Namely, \( H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \) is compact for \( p \in [1, 2) \).

In this section, we show the following compactness result in the framework of Lorentz spaces.

**Theorem 2.** Let \( q \in [1, \infty] \). Then the followings hold.

(i) \( H^1_{\text{rad}}(\mathbb{R}^N) \cap L^{p,q}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \) is compact for \( p \in [1, 2) \).

(ii) \( H^1_{\text{rad}}(\mathbb{R}^N) \cap L^{p,q}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \) is compact for \( p > 2^* \).
Remark 3. Since $L^p \subseteq L^{p,q} \subseteq L^{p,\infty}$ for any $q \in (p, \infty)$, Theorem 2 is an improvement of Theorem B.

Remark 4. Set $u_m(x) = m^{\frac{N-2}{p}} u(mx)$ for $x \in \mathbb{R}^N$, $m \in \mathbb{N}$ and $u \in C^\infty_{c, \text{rad}}(\mathbb{R}^N)$. Direct calculation implies that

\[ \|u_m\|_{p,q} = m^{\frac{N-2}{p}} \|u\|_{p,q} \rightarrow \infty \text{ as } m \rightarrow \infty \text{ if } p > 2^*. \]

Therefore, we can deny the possibility of non-compactness with respect to $\dot{H}^1$ and $L^{2^*}$-invariance under the restriction of $H^1(\mathbb{R}^N)$ to $H^1(\mathbb{R}^N) \cap L^{p,q}(\mathbb{R}^N)$ when $p > 2^*$.

Let $G$ be a closed subgroup of the orthogonal group $O(N)$. We call $u(x)$ a $G$ invariant function if $u(gx) = u(x)$ for all $g \in G$ and $x \in \mathbb{R}^N$. Set

\[ H^1_G(\mathbb{R}^N) = \{ u \in H^1(\mathbb{R}^N) \mid u \text{ is } G \text{ invariant} \}. \]

Clearly, we see that $H^1_{\text{rad}}(\mathbb{R}^N) = H^1_{O(N)}(\mathbb{R}^N)$. The following result is a generalization of Theorem A to $H^1_G(\mathbb{R}^N)$.

**Theorem C.** ([16], see also [22] §1.5) Let $N_j \geq 2$, $j = 1, 2, \cdots, k$, $\sum_{j=1}^k N_j = N$ and

\[ G := O(N_1) \times O(N_2) \times \cdots \times O(N_k). \]

Then the embedding $H^1_G(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $p \in (2, 2^*)$.

In the same way, we can generalize Theorem 2 to $H^1_G(\mathbb{R}^N)$.

**Theorem 3.** Let $q \in [1, \infty]$, $N_j \geq 2$, $j = 1, 2, \cdots, k$, $\sum_{j=1}^k N_j = N$ and

\[ G := O(N_1) \times O(N_2) \times \cdots \times O(N_k). \]

Then the followings hold.

(i) $H^1_G(\mathbb{R}^N) \cap L^{p,q}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ is compact for $p \in [1, 2]$.

(ii) $H^1_G(\mathbb{R}^N) \cap L^{p,q}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is compact for $p > 2^*$.

Since $H^1_{\text{rad}}(\mathbb{R}^N) \subset H^1_G(\mathbb{R}^N)$, Theorem 2 follows from Theorem 3. We show Theorem 3 only in the case where $q = \infty$, see Remark 3.
Proof. (Theorem 3) Let \( \{v_m\}_{m=1}^{\infty} \subset H^1_G(\mathbb{R}^N) \cap L^{p,\infty}(\mathbb{R}^N) \) be a bounded sequence and \( v_m \rightharpoonup v \) in \( H^1_G(\mathbb{R}^N) \cap L^{p,\infty}(\mathbb{R}^N) \). Set \( u_m = v_m - v \). Then \( u_m \rightharpoonup 0 \) in \( H^1_G(\mathbb{R}^N) \). From Theorem C, we have \( u_m \to 0 \) in \( L^q(\mathbb{R}^N) \) for \( q \in (2, 2^*) \). By the Pólya-Szegö inequality, we have

\[
||u_m||^2_{H^1(\mathbb{R}^N)} = ||\nabla u_m||^2_{L^2(\mathbb{R}^N)} + ||u_m||^2_{L^2(\mathbb{R}^N)} \leq ||\nabla u_m||^2_{L^2(\mathbb{R}^N)} + ||u_m||^2_{L^2(\mathbb{R}^N)} = ||u_m||^2_{H^1(\mathbb{R}^N)} < \infty
\]

which implies that \( \{u_m\}_{m=1}^{\infty} \) be the bounded sequence in \( H^1_{rad}(\mathbb{R}^N) \). Therefore, there exists \( u \in H^1_{rad}(\mathbb{R}^N) \) such that \( u_m \rightharpoonup u \) in \( H^1_{rad}(\mathbb{R}^N) \). From Theorem A, \( u_m \to u \) in \( L^q(\mathbb{R}^N) \) for \( q \in (2, 2^*) \). Since

\[
0 = \lim_{m \to \infty} ||u_m||_q = \lim_{m \to \infty} ||u_m||_q = ||u||_q,
\]

we have \( u \equiv 0 \). Therefore, \( u_m \rightharpoonup 0 \) in \( H^1_{rad}(\mathbb{R}^N) \).

(i) Let \( p < 2 \). Since \( H^1(B^N_R) \hookrightarrow L^2(B^N_R) \) is compact, we have \( u_m \rightharpoonup u \) in \( L^2(B^N_R) \). Then, we have

\[
\int_{\mathbb{R}^N} |u_m|^2 \, dx =: C_1(m, R) \to 0 \quad \text{as } m \to \infty \text{ for fixed } R > 0.
\]

On the other hand,

\[
\int_{\mathbb{R}^N} |u_m|^2 \, dx = \int_{\mathbb{R}^N} |u_m|^2 \, dx
\]

\[
= \int_{\mathbb{R}^N \setminus B^N_R} |u_m|^2 \, dx + C_1(m, R)
\]

\[
= \omega_{N-1} \int_R^{\infty} |u_m(r)|^2 r^{N-1} \, dr + C_1(m, R)
\]

\[
= \int_R^{\infty} |u_m(r)|^2 \, dt + C_1(m, R)
\]

\[
\leq ||u_m||^2_{L^{p,\infty}} \int_R^{\infty} r^{-\frac{2}{p}} \, dt + C_1(m, R) = CR^{-\frac{2}{p}+1} + C_1(m, R).
\]

Since \( p < 2 \),

\[
\lim_{R \to \infty} \lim_{m \to \infty} \int_{\mathbb{R}^N} |u_m|^2 \, dx = 0
\]

which implies that \( v_m \rightharpoonup v \) in \( L^2(\mathbb{R}^N) \). Hence, the embedding \( H^1_G(\mathbb{R}^N) \cap L^{p,\infty}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \) is compact for \( p \in [1, 2) \).
(ii) Let \( p > 2^* \). First, we recall the following pointwise estimate for any radial function \( f \).

\[
|f(x)| \leq \sqrt{\frac{2}{\omega_{N-1}}} \|f\|_2^{\frac{1}{2}} \|
abla f\|_2^{\frac{1}{2}} |x|^{-\frac{N-1}{2}} \text{ a.e. in } x \in \mathbb{R}^N
\]  

(1)

See e.g. [11] Proposition 6. By (1), we have

\[
\int_{\mathbb{R}^N \setminus B_R^N} |u_m^\#|^{2^*} \, dx \leq C \|u_m^\#\|_{H^1(\mathbb{R}^N)}^{2^*} \int_{\mathbb{R}^N \setminus B_R^N} |x|^{-\frac{N(N-1)}{2N-2}} \, dx \leq CR^{-\frac{N}{N-2}}.
\]  

(2)

By the compactness of the one-dimensional Sobolev embedding \( H^1(\varepsilon, R) \hookrightarrow L^s(\varepsilon, R) \) for any \( s \geq 1 \) and the equivalence of \( H^1(\varepsilon, R); r^{N-1} \, dr \approx H^1(\varepsilon, R) \), we have the compactness of the embedding \( H^1(\varepsilon, R); r^{N-1} \, dr \approx H^1(\varepsilon, R) \) for any \( s \geq 1 \) and for any fixed \( R > \varepsilon > 0 \). Therefore, we have

\[
\int_{B_R^N \setminus B_\varepsilon^N} |u_m^\#|^{2^*} \, dx =: C_2(m, \varepsilon, R) \rightarrow 0 \quad \text{as } m \rightarrow \infty \text{ for fixed } R, \varepsilon > 0.
\]  

(3)

By (2) and (3), we have

\[
\int_{\mathbb{R}^N} |u_m^\#|^{2^*} \, dx = \int_{\mathbb{R}^N} |u_m^\#|^{2^*} \, dx
\]

\[
\leq \int_{B_\varepsilon^N} |u_m^\#|^{2^*} \, dx + C_2(m, \varepsilon, R) + CR^{-\frac{N}{N-2}}
\]

\[
= \int_0^\varepsilon |u_m^\#(t)|^{2^*} \, dt + C_2(m, \varepsilon, R) + CR^{-\frac{N}{N-2}}
\]

\[
\leq ||u_m||_{p, \infty}^{2^*} \int_0^\varepsilon t^{-\frac{N}{p^*}} \, dt + C_2(m, \varepsilon, R) + CR^{-\frac{N}{N-2}}
\]

\[
= C\varepsilon^{-\frac{N}{p^*} + 1} + C_2(m, \varepsilon, R) + CR^{-\frac{N}{N-2}}.
\]

Since \( p > 2^* \),

\[
\lim_{\varepsilon \rightarrow 0} \lim_{R \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{\mathbb{R}^N} |u_m^\#|^{2^*} \, dx = 0
\]

which implies that \( v_m \rightarrow v \) in \( L^{2^*}(\mathbb{R}^N) \). Hence, the embedding \( H^1_G(\mathbb{R}^N) \cap L^{p, \infty}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \) is compact for \( p > 2^* \). \qed
In the end of this section, we give remarks about Hölder inequalities and interpolation inequalities, and give another proof of Theorem B. Theorem B was shown in [3] by using Theorem A and the Hölder inequality: \( \|fg\| \leq \|f\|_q \|g\|_p \), where \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \). Especially, they used the following interpolation inequalities by setting \( f = |u|^\lambda \), \( g = |u|^{1-\lambda} \).

\[
\begin{align*}
(i) \quad \|u\|_2 & \leq \|u\|_q^\lambda \|u\|_p^{1-\lambda} \quad \left( p < 2 < q < 2^* , \lambda = \frac{q(2-p)}{2(q-p)} \right) \\
(ii) \quad \|u\|_{2^*} & \leq \|u\|_q^\lambda \|u\|_p^{1-\lambda} \quad \left( 2 < q < 2^* < p , \lambda = \frac{q(2^*-p)}{2(q-p)} \right)
\end{align*}
\]

In fact, if we assume that \( u_m \rightharpoonup u \) in \( H^{1,1}_\text{rad}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \), then we have \( u_m \to u \) in \( L^q(\mathbb{R}^N) \) from Theorem A. By using the interpolation inequalities above, we obtain \( u_m \to u \) in \( L^2(\mathbb{R}^N) \) in the case (i) and \( u_m \to u \) in \( L^{2^*}(\mathbb{R}^N) \) in the case (ii). Therefore, (i) and (ii) in Theorem B hold. However, in the weak-Lebesgue spaces \( L^{p,\infty}(\mathbb{R}^N) \), the Hölder inequality does not hold in general. In fact, the following will be proved

\[ \forall C > 0 \text{ s.t. } \|fg\|_2 \leq C\|f\|_q \|g\|_{p,\infty} \left( \forall f \in L^q(\mathbb{R}^N), \forall g \in L^{p,\infty}(\mathbb{R}^N) \right), \quad (4) \]

where \( p < 2 < q < 2^* \). Indeed, if we consider

\[ f(x) = |x|^{-\alpha} I_{B(0)}(x), \quad g(x) = |x|^{-\frac{N}{q}}, \quad \alpha = \frac{N}{q} - \varepsilon, \quad \varepsilon > 0, \]

then

\[ \|f\|_q = \left( \int_{B(0)} |x|^{-\alpha q} \, dx \right)^{\frac{1}{q}} = \left( \int_{B(0)} |x|^{N+\varepsilon q} \, dx \right)^{\frac{1}{q}} = C\varepsilon^{-\frac{1}{q}}, \]

\[ \|g\|_{p,\infty} < C, \]

\[ \|fg\|_2 = \left( \int_{B(0)} |x|^{-2\alpha - \frac{N}{p}} \, dx \right)^{\frac{1}{2}} = \left( \int_{B(0)} |x|^{-N+2\varepsilon} \, dx \right)^{\frac{1}{2}} = C\varepsilon^{-\frac{1}{2}}. \]

Therefore,

\[ C\varepsilon^{-\frac{1}{2}} = \|fg\|_2 \leq C\|f\|_q \|g\|_{p,\infty} = C\varepsilon^{-\frac{1}{q}} \]

which implies (4) when we take \( \varepsilon \to 0 \). However, in spite of (4), we can show the following interpolation inequalities from Proposition below.

\[
\begin{align*}
(i) \quad \|u\|_2 & \leq C\|u\|_q^\lambda \|u\|_{p,\infty}^{1-\lambda} \quad \left( p < 2 < q < 2^* , \lambda \in (0,1) \right) \\
(ii) \quad \|u\|_{2^*} & \leq C\|u\|_q^\lambda \|u\|_{p,\infty}^{1-\lambda} \quad \left( 2 < q < 2^* < p , \lambda \in (0,1) \right)
\end{align*}
\]
As a consequence, thanks to Proposition 1, we can show Theorem 3 in the same way as the proof of Theorem B.

**Proposition 1.** Let $1 \leq p < q < r \leq \infty$. Then the interpolation inequality

$$
\|u\|_q \leq D \|u\|_{p,\infty}^\lambda \|u\|_{r,\infty}^{1-\lambda},
$$

holds for any $u \in L^{p,\infty}(\mathbb{R}^N) \cap L^{r,\infty}(\mathbb{R}^N)$, where $D = \left(\frac{q(r-p)}{(r-q)(q-p)}\right)^{\frac{1}{2}}$ and $\lambda = \frac{q(r-p)}{(r-q)(q-p)}$.

**Proof.** For any $s > 0$, we have

$$
\|u\|_q^s = \int_0^s |u^*(t)|^q \, dt + \int_s^\infty |u^*(t)|^q \, dt \leq \|u\|_{r,\infty}^q \int_0^s r^{-\frac{q}{r}} \, dt + \|u\|_{p,\infty}^q \int_s^\infty r^{-\frac{q}{r}} \, dt = As^a + Bs^{-b},
$$

where

$$
A = \frac{r}{r-q} \|u\|_{r,\infty}^q, \quad a = \frac{r-q}{r}, \quad B = \frac{p}{q-p} \|u\|_{p,\infty}^q, \quad b = \frac{q-p}{p}.
$$

Note that the function $f(s) = As^a + Bs^{-b}$ attains its minimum at $s = \left(\frac{BB}{AA}\right)^{\frac{1}{a+b}}$. Since

$$
\left(\frac{BB}{AA}\right)^{\frac{1}{a+b}} = \left(\frac{\|u\|_{p,\infty}}{\|u\|_{r,\infty}}\right)^{\frac{q(r-p)}{(r-q)(q-p)}},
$$

we have

$$
\|u\|_q^s \leq \frac{r}{r-q} \|u\|_{r,\infty}^q \left(\frac{\|u\|_{p,\infty}}{\|u\|_{r,\infty}}\right)^{\frac{q(r-p)}{(r-q)(q-p)}} + \frac{p}{q-p} \|u\|_{p,\infty}^q \left(\frac{\|u\|_{p,\infty}}{\|u\|_{r,\infty}}\right)^{-\frac{q(r-p)}{(r-q)(q-p)}} = D^s \|u\|_{p,\infty}^\lambda \|u\|_{r,\infty}^{1-\lambda}.
$$

**4. Appendix**

First, we give a proof of two Hardy type inequalities under spherical average zero on the ball.

**Proposition 2.** For any $u \in C_0^\infty(B_1^N)$ with $\int_{S^{N-1}} u(r\omega) \, dS_\omega = 0 \ (\forall r \geq 0)$, the following inequalities hold.

$$
\frac{N^2}{4} \int_{B_1^N} \frac{|u|^2}{|x|^2} \, dx \leq \int_{B_1^N} |\nabla u|^2 \, dx \tag{5}
$$

$$
\frac{5}{4} \int_{B_1^N} \frac{|u|^2}{|x|^2 (\log \frac{r}{|x|})^2} \, dx \leq \int_{B_1^N} |\nabla u|^2 \, dx \tag{6}
$$

Moreover, the best constant $\frac{N^2}{4}$ in (5) is not attained.
Proof. We use the polar coordinate:

\[ x = r \omega (r = |x|, \omega \in \mathbb{S}^{N-1}), \quad \nabla u(x) = \left( \frac{\partial u}{\partial r} (r \omega) \right) \omega + \frac{1}{r^2} \nabla_{\mathbb{S}^{N-1}} u(r \omega) \]

From the classical Hardy inequality:

\[ \left( \frac{N - 2}{2} \right)^2 \int_{B^N_1} \frac{|u|^2}{|x|^2} \, dx < \int_{B^N_1} \left| \nabla u \cdot \frac{x}{|x|} \right|^2 \, dx \quad (\forall u \in C_0^\infty (B^N_1)) \quad (7) \]

and the Poincaré inequality on the sphere \( \mathbb{S}^{N-1} \):

\[ (N - 1) \int_{\mathbb{S}^{N-1}} |g(\omega)|^2 \, dS_\omega \leq \int_{B^N_1} \left| \nabla_{\mathbb{S}^{N-1}} g \right|^2 \, dx \quad (\forall g \in C^\infty (\mathbb{S}^{N-1}), \int_{\mathbb{S}^{N-1}} g \, dS_\omega = 0), \]

for any \( u \in C^\infty_c (B^N_1) \) with \( \int_{\mathbb{S}^{N-1}} u(r \omega) \, dS_\omega = 0 \) (\( \forall r \geq 0 \)) we have

\[
\int_{B^N_1} |\nabla u|^2 \, dx = \int_{B^N_1} \left| \nabla u \cdot \frac{x}{|x|} \right|^2 \, dx + \int_0^1 \int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}} u(r \omega)|^2 \, r^{N-3} \, dr \, dS_\omega \\
> \left( \frac{N - 2}{2} \right)^2 \int_{B^N_1} \frac{|u|^2}{|x|^2} \, dx + (N - 1) \int_0^1 \int_{\mathbb{S}^{N-1}} |u(r \omega)|^2 \, r^{N-3} \, dr \, dS_\omega \\
= \frac{N^2}{4} \int_{B^N_1} \frac{|u|^2}{|x|^2} \, dx
\]

which implies the inequality (5). It is enough to show the optimality of the constant \( \frac{N^2}{4} \) in (5). Consider the following test function.

\[ u_a(x) = g_2(\omega) f_a(r) \quad (a > 0), \quad \text{where} \quad f_a(r) = \begin{cases} \frac{r^a}{\beta} & \text{if } r \in [0, \frac{1}{2}], \\
\text{smooth} & \text{if } r \in \left(\frac{1}{2}, 1\right), \\
0 & \text{if } r \in [1, \infty) \end{cases} \]

and \( g_2 \) is the second eigenfunction of the Laplace-Beltrami operator \( -\Delta_{\mathbb{S}^{N-1}} \) which satisfies

\[
\int_{\mathbb{S}^{N-1}} g_2(\omega) \, dS_\omega = 0, \quad \int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}} g_2(\omega)|^2 \, dS_\omega = (N - 1) \int_{\mathbb{S}^{N-1}} |g_2(\omega)|^2 \, dS_\omega.
\]
Then, we have
\[
\frac{\int_{B_1^N} |\nabla u|^2}{\int_{B_1^N \setminus \{0\}} \frac{|u|^2}{|x|^2}} dx \leq \frac{1}{2} \int_0^{1/2} \int_{S_{N-1}} \frac{|u|^2}{|x|^2} dS + \frac{1}{2} \int_{1/2}^1 \int_{S_{N-1}} |\nabla u|^2 \frac{|x|^2}{|x|^2} dS + C
\]
\[
= \left( a^2 + N - 1 \right) + \frac{C}{2} \int_0^{1/2} \int_{S_{N-1}} r^2 |\nabla g_2|^2 dS + C_1
\]
\[
= \left( a^2 + N - 1 \right) + R(a, N).
\]
Since \( R(a, N) \to 0 \) as \( a \to 0 \), we can obtain the optimality of the constant \( 1 = \frac{N^2}{4} \) when \( N = 2 \). In the case where \( N \geq 3 \), consider the following test function.

\[ v_m(x) = g_2(x) h_m(r), \text{ where } h_m(r) = \begin{cases} 
0 & \text{if } r \in [0, \frac{1}{2m}), \\
2m \left( (m^2 - 1) \left( r - \frac{1}{2m} \right) \right) & \text{if } r \in \left( \frac{1}{2m}, \frac{1}{m} \right), \\
r \frac{N^2}{4} - 1 & \text{if } r \in \left[ \frac{1}{m}, 1 \right].
\end{cases} \]

Then we have
\[
\frac{\int_{B_1^N} |\nabla v_m|^2}{\int_{B_1^N \setminus \{0\}} \frac{|v_m|^2}{|x|^2}} dx \leq \frac{1}{2} \int_1 \int_{S_{N-1}} \frac{(N-2) \frac{N^2}{4} - 1}{|x|^2} dS + C_1
\]
\[
= \left( \frac{(N-2)}{4} + N - 1 \right) \log m + C_1
\]
\[
= \frac{N^2}{4} + o(1) \quad (m \to \infty)
\]
which implies the optimality of the constant \( \frac{N^2}{4} \) when \( N \geq 3 \).

On the other hand, if we use the critical Hardy inequality:
\[
\frac{1}{4} \int_{B_1^N} \frac{|u|^2}{|x|^2} dS < \int_{B_1^N} |\nabla u \cdot \frac{x}{|x|}|^2 dS \quad (\forall u \in C_0^\infty(B_1^N))
\]
instead of the classical Hardy inequality, for \( u \in C_0^\infty(B_1^N) \) with \( \int_{S_{N-1}} u(r\omega) dS = \cdots \)
which implies the inequality (6). \[\square\]

Next, we give a simple proof of the Hardy inequality under spherical average zero on the whole space. For another proof, see e.g. \[3\].

**Proposition 3.** For any \( u \in C_0^\infty(\mathbb{R}^N) \) with \( \int_{S^{N-1}} u(r\omega) dS_\omega = 0 \) (\( \forall r \geq 0 \)), the inequality

\[
\frac{N^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx
\]

holds. Moreover, the best constant \( \frac{N^2}{4} \) is not attained.

**Proof.** We show only the optimality of the constant \( \frac{N^2}{4} \). Consider the following test function.

\[
u_a(x) = g_2(\omega) f_a(r), \quad \text{where } f_a(r) = \begin{cases} r & \text{if } r \in [0, 1], \\ r^{-a} & \text{if } r \in (1, \infty) \end{cases}
\]

and \( g_2 \) is given by the proof of Proposition 2. Then we have

\[
\frac{\int_{\mathbb{R}^N} |\nabla \nu_a|^2 \, dx}{\int_{\mathbb{R}^N} \frac{|\nu_a|^2}{|x|^2} \, dx} \leq \frac{\int_0^1 r^{N-1} \, dr + \int_1^{\infty} |ar^{-a-1}|^2 r^{N-1} \, dr + (N-1) \left[ \int_0^1 r^{N-1} + \int_1^{\infty} r^{-2a+N-3} \, dr \right]}{\int_1^{\infty} r^{-2a+N-3} \, dr}
\]

\[
= \left( a^2 + N - 1 \right) + \frac{1}{\int_1^{\infty} r^{-2a+N-3} \, dr}
\]

\[
= \left( a^2 + N - 1 \right) + R(a, N).
\]

Since \( R(a, N) \to 0 \) as \( a \to \frac{N-2}{2} \), we see that the constant \( \frac{N^2}{4} \) is optimal. \[ \square \]
Finally, we show the following result to show Theorem A in Introduction.

**Proposition 4.** Let $X$ be a Hilbert space, $Y$ be a Banach space, $A \subset X$ be a closed subspace and $X = A \oplus A^\perp$. If the embedding $X \hookrightarrow Y$ is non-compact and $A \hookrightarrow Y$ is compact, then $A^\perp \hookrightarrow Y$ is non-compact.

**Proof.** Assume that $A^\perp \hookrightarrow Y$ is compact. For any bounded sequence $\{u_m\}_{m=1}^\infty \subset X$, there exist $\{v_m\}_{m=1}^\infty \subset A$ and $\{w_m\}_{m=1}^\infty \subset A^\perp$ such that $u_m = v_m + w_m$, $(v_m, w_m)_X = 0$ for any $m \in \mathbb{N}$. Since $X$ is reflexive and $X \hookrightarrow Y$, there exist $u \in X$ and a subsequence $\{u_m\}_{j=1}^\infty \subset X$ such that $u_m \rightharpoonup u$ in $X$. Moreover, since both sequences $\{v_m\}_{m=1}^\infty \subset A$, $\{w_m\}_{m=1}^\infty \subset A^\perp$ are also bounded and any closed subspace in reflexive Banach space is also reflexive (see e.g. [5] Proposition 3.20), we have $v_m \rightharpoonup v$ in $A$ and $w_m \rightharpoonup w$ in $A^\perp$. Since both $A \hookrightarrow Y$ and $A^\perp \hookrightarrow Y$ are compact, we have $v_m \rightarrow v$ and $w_m \rightarrow w$ in $Y$. Therefore, we see that $u_m \rightarrow v + w$ in $Y$. Thanks to the uniqueness of the weak limit, we have $u = v + w$ which implies that $u_m \rightarrow u$ in $Y$. This contradicts the non-compactness of the embedding $X \hookrightarrow Y$. Hence, the embedding $A^\perp \hookrightarrow Y$ is non-compact. □

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