Nearest neighbor spacing distribution of prime numbers
and quantum chaos

Marek Wolf
Cardinal Stefan Wyszyński University, Faculty of Mathematics and Natural Sciences. College of Sciences
ul. Więceikiego 1/3, PL-01-938 Warsaw, Poland, e-mail: m.wolf@uksw.edu.pl

We give heuristic arguments and computer results to support the hypothesis that, after appropriate rescaling, the statistics of spacings between adjacent prime numbers follows the Poisson distribution. The scaling transformation removes the oscillations in the NNSD of primes. These oscillations have the very profound period of length six. We also calculate the spectral rigidity \( \Delta_3 \) for prime numbers by two methods. After suitable averaging one of these methods gives the Poisson dependence \( \Delta_3(L) = L/15 \).

I. INTRODUCTION

The primes numbers often provided a toy model for some physical ideas in the past. For example in [1] the multifractal formalism was applied to prime numbers, in [2] the appropriately defined Lyapunov exponents for the distribution of primes were calculated numerically. In the paper [3] it was shown that the distribution of prime numbers displays the \( 1/f \) noise, while in [4] the noise \( 1/f^2 \) was found in the difference between the prime-number counting \( \pi(x) \) function and Riemann’s function \( R(x) \). In [5] and [6] random walks on primes numbers were defined. In [7] an attempt to construct the dynamical model for prime numbers was taken and computable information content as well as entropy information of the set of prime numbers were calculated.

The prime numbers can be regarded as eigenvalues of some quantum hamiltonian. The problem of construction of a simple one–dimensional Hamiltonian whose spectrum coincides with the set of primes was considered in [8], [9], [10], see also review [11]. Then it is natural to investigate the spacings between prime numbers, i.e. in physical language the nearest neighbor spacing distribution (NNSD). Several authors have undertaken a study of this problem in the past, see [12], [13], [14]. Below we will treat prime numbers as the energy levels and we will apply methods used to describe statistical properties of discrete spectra. Let the quantum system possess the discrete spectrum \( E_1, E_2, \ldots \) and let \( N(E) = \sum_n \Theta(E - E_n) \) (\( \Theta \) is a unit step function) denote the function counting the number of energy levels smaller than \( E \). Usually spectral staircase \( N(E) \) can be split into the “smooth” \( \overline{N}(E) \) and fluctuating (oscillating) \( \overline{N}(E) \) parts. For example, for a large class of differential operators on \( d \) dimensional bounded manifold \( \Omega \subset \mathbb{R}^d \) the Weyl’s law

\[
\overline{N}(E) \sim \frac{\mathrm{vol}(\Omega)}{(2\pi)^d} E^{d/2},
\]

holds, see e.g. [15, Ch.1].

Given the spectrum \( E_1, E_2, \ldots \) the statistics of normalized and dimensionless (“unfolded”) spectrum, see e.g. [16, Sect.4.7]) gaps between two consecutive energy levels \( s_n = (E_{n+1} - E_n)/\overline{D}(E) \), where \( \overline{D}(E) \) is the mean distance between energy levels up to \( E \), was extensively studied in the past. For general systems \( E_{n+1} - E_n \) are arbitrary real numbers and histogram of the level spacings \( s_n \) is built. It is well known, that level–spacings distributions of quantum systems can be grouped into a few universality classes connected with the symmetry properties of the hamiltonians: Poisson distribution (i.e. \( e^{-x} \)) for systems with underlying regular classical dynamics, Gaussian orthogonal ensemble (GOE, also called the Wigner–Dyson distribution) — hamiltonians invariant under time reversal, Gaussian unitary ensemble (GUE) — not invariant under time reversal and Gaussian symplectic ensemble (GSE) for half-spin systems with time reversal symmetry. There are many reviews on these topics, we cite here [17], [16], [18].

There is some confusion regarding the proper statistics of the gaps between consecutive primes: in [12] it was claimed that NNSD of primes follows GOE distribution, while in [13, 14], the possibilities of GOE, Poisson and exotic Berry-Robnik [19] distribution were investigated. Liboff and Wong have obtained Wigner distribution and level repulsion for NNSD of primes by artificially including the gaps 0 (no degeneracy — all primes are different) and 1, see [12, p.3113]. The gap 1 appears only once between 2 and 3 and should be skipped in the wake of infinity of primes. There is a very often reproduced figure showing some typical spectra (see [17, Fig. 1.2], [18, Fig.3], [20, Fig. I.8], [21, front figure], [22, p. 32]): random levels with no correlations (Poisson series), sequence of prime numbers, resonance levels of erbium 166 nucleus, the energies a free particle in the Sinai billiard, nontrivial zeros of the Riemann zeta function. In [17, p. 10] it is stated that “case of prime numbers... are far from either regularly spaced uniform series or the completely random Poisson series with no correlations”.

It is the purpose of this paper “to settle once and for ever” that NSDD of primes follows the Poisson distribution. The next Section II is devoted to this problem. In [23] M.V. Berry has calculated spectral rigidity \( \Delta_3 \) for zeros of the Riemann zeta function and in Sect.III we will study spectral rigidity for prime numbers.

II. NNSD FOR PRIME NUMBERS

In the case of primes numbers all gaps \( d_n = p_{n+1} - p_n \) (except the first pair of primes \( p_1 = 2, p_2 = 3 \)) are even integers 2, 4, 6, ... . These spacings are dimensionless and we will not perform unfolding for time being (see next Section) — the usual (17) unfolding obscures analysis of the oscillations present in the NNSD between original...
primes. Let $\tau_d(x)$ denote a number of pairs of consecutive primes smaller than a given bound $x$ and separated by $d$:

$$\tau_d(x) = \sharp\{p_n, p_{n+1} < x, \text{ with } p_{n+1} - p_n = d\}. \quad (2)$$

For odd $d = 2k + 1$ we supplement this definition by putting $\tau_{2k+1}(x) = 0$.

In 1922 G. H. Hardy and J.E. Littlewood in the famous paper [24] have proposed 15 conjectures. The conjecture B of their paper states that there are infinitely many prime pairs $(p, p')$, where $p' = p + d$, for every even $d$. If $\pi_d(x)$ denotes the number of prime pairs differing by $d$ and less than $x$, then

$$\pi_d(x) \sim C_2 \prod_{p|d} \frac{p-1}{p-2} \frac{x}{\ln^2(x)}. \quad (3)$$

Here $C_2 \equiv 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = 1.32032\ldots$ is called the “twins constant”.

In the middle of 2013 the major step towards the proof of the conjecture B was made: Yitang Zhang has submitted to Annals of Mathematics the paper in which he proved unconditionally that $\liminf_{n \to \infty} (p_{n+1} - p_n) < 7 \times 10^7$, see e.g. [25]. Very soon this bound was lowered many times by mathematicians and present record is $\liminf_{n \to \infty} (p_{n+1} - p_n) \leq 600$ and was obtained by J. Maynard [26].

The conjecture B of G. H. Hardy and J.E. Littlewood gives the number of pairs of primes not necessarily consecutive and we would like to stress that in (2) $\tau_d(x)$ denotes number of pairs of consecutive primes $p_n, p_{n+1}$ with difference $p_{n+1} - p_n = d$. The pairs of primes separated by $d = 2$ and $d = 4$ are special as they always have to be consecutive primes (with the exception of the pair $(3,7)$ containing 5 in the middle): in the triple of integers $2k+1, 2k+3, 2k+5$ the middle $2k+3$ has to be divisible by 3 if $2k+1, 2k+5$ are prime (in particular not divisible by 3). For $d = 6$ (and larger $d$) we have $\pi_6(x) > \tau_6(x)$, for example $(5, 7, 11), (7, 11, 13), (11, 13, 17), \ldots$. From the conjecture B of G. H. Hardy and J.E. Littlewood [24] it follows that the number of gaps $d = 2$ (“twins”) is approximately equal to the number of gaps $d = 4$ (“cousins”): $\pi_2(x) \equiv \pi_2(x) \approx \pi_4(x) \equiv \pi_4(x)$, see also [6]. For $d \geq 6$ in

![FIG. 1: Plots of $\tau_d(x)$ for $x = 2^{24}, 2^{26}, \ldots, 2^{46}, 2^{48}$. The histogram step widths are 2; because $\tau_2(x) \approx \tau_4(x)$, therefore the visible step for $d = 2, 4$ has width 4. In red exponential fits $a(x)e^{-db(x)}$ are plotted. In the inset the plots of $\tau_d(x)/P(d)$ are shown.](image-url)
Here \( \pi(x) = \sum_n \Theta(n, p_n) \) denotes the number of primes up to \( x \) and by the Prime Number Theorem (PNT) is very well approximated by the logarithmic integral

\[
\pi(x) \sim \text{Li}(x) \equiv \int_2^x \frac{du}{\ln(u)}.\]

Integration by parts gives the asymptotic expansion which should be cut at the term \( n_0 = [\ln(x)] \):

\[
\text{Li}(x) = \frac{x}{\ln(x)} + \frac{x}{\ln^2(x)} + \frac{2!x}{\ln^3(x)} + \frac{3!x}{\ln^4(x)} + \cdots. \tag{5}
\]

There is a series giving \( \text{Li}(x) \) for all \( x > 2 \) and quickly convergent which has \( n! \) in denominator and \( \ln^n(x) \) in nominator instead of opposite order in (5) (see [28, Sect. 5.1])

\[
\text{Li}(x) = \gamma + \ln(x) + \sum_{n=1}^{\infty} \frac{\ln^n(x)}{n \cdot n!} \quad \text{for} \quad x > 1, \tag{6}
\]

Here \( \gamma = 0.577216\ldots \) is the Euler-Mascheroni constant.

Putting in (4) \( \pi(x) \sim x/\ln(x) \) the compact formula expressing \( \tau_d(x) \) by explicitly known functions

\[
\tau_d(x) \sim C_2 \frac{x^2}{\ln^2(x)} \prod_{p | d, p > 2} \frac{p-1}{p} e^{-d/\ln(x)} \quad \text{for} \quad d \geq 6, \tag{7}
\]

is obtained. Comparing it with the original Hardy–Littlewood conjecture (3) we obtain that the number \( \tau_d(x) \) of successive primes \( (p_{n+1}, p_n) \) smaller than \( x \) and of the difference \( d \) \((= p_{n+1} - p_n)\) is diminished by the factor \( \exp(-d/\ln(x)) \) in comparison with the number of all pairs of primes \( (p, p') \) apart in the distance \( d = p' - p \):

\[
\tau_d(x) \sim \pi_d(x)e^{-d/\ln(x)} \quad \text{for} \quad d \geq 6. \tag{8}
\]

The expression (7) for \( \tau_d(x) \) was proved (in slightly different form required by the precision of the formulation of the theorem) under the assumption of the conjecture B of Hardy–Littlewood by D. A. Goldston and A. H. Lefoan [29] in 2012.

During over a seven months long run of the computer program we have collected the values of \( \tau_d(x) \) up to \( x = 2^{48} \approx 2.8147 \times 10^{14} \). The data representing the function \( \tau_d(x) \) were stored at values of \( x \) forming the geometrical progression with the ratio 2 at \( x = 2^{15}, 2^{16}, \ldots, 2^{47}, 2^{48} \). Such a choice of the intermediate thresholds as powers of 2 was determined by the employed computer program in which the primes were coded as bits. The data is available for downloading from http://pracownicy.uksw.edu.pl/mwolf/gapstau.zip.

The resulting curves are plotted in Fig.1. Characteristic oscillating pattern of points is caused by the product

\[
P(d) \equiv \prod_{p | d, p > 2} \frac{p-1}{p} \quad \text{appearing in (4), see inset in Fig. 1. This product for the first time appeared in the paper of Hardy and Littlewood [24] and it has local maxima for } d \text{ equal to the products of consecutive primes ("primorials", i.e. factorials over primes } 2 \cdot 3 \cdot 5 \cdot \ldots \cdot p_n \equiv p_n(2). \text{ Clearly visible in Fig. 1 are oscillations of the period } 6 = 2 \times 3 \text{ with overimposed higher harmonics } 30 = 2 \times 3 \times 5 \text{ and } 210 = 2 \times 3 \times 5 \times 7, \text{ i.e. when } P(d) \text{ has local maxima } P(6) = 2, \ P(30) = 8/3 = 2.666\ldots \ P(210) = 16/5 = 3.2 \text{ (local minima are 1 and they correspond to } d = 2^n). \text{ We have performed the discrete Fourier Transform of } P(d), \text{ i.e. we calculated numerically}

\[
\tilde{P} \left( \frac{n}{2M} \right) = \sum_{k=0}^{M-1} P(2k) e^{2\pi kn/M}, \tag{10}
\]

where \( n = 0, 1, 2, \ldots, M - 1 \) and \( n/2M \) plays the role of discrete frequency. Having \( \tilde{P}(f) \) we can calculate the power spectrum density \( S(f) = |P(f)|^2 \). The large value of \( S(f) \) at some frequency \( f \) means that the dependence of \( P(d) \) on \( d \) has some harmonic component of the period \( T = 1/f \). Thus in the Fig.2 we have plotted \( S(f) \) versus \( 1/f = d \) to show main periods 5, 6 = 2 \times 3, 10 = 2 \times 5, 14 = 2 \times 7, 30 = 2 \times 3 \times 5 \ldots \) of \( P(d) \). These oscillations are the reason why the Poisson distribution was not attributed to NNSD of primes in the past: e.g. \( P(2) = P(4) = 1 \) while \( P(6) = 2 \) and the plot should be made with logarithmic scale on the \( y \) axis to suppress these oscillations.

In [31] E. Bombieri and H. Davenport have proved that:

\[
\sum_{k=1}^{n} \prod_{p \parallel k, p>2} \frac{p-1}{p-2} = \prod_{p>2} \frac{n}{p-2} - \frac{1}{(p-1)^2} + O(\ln^2(n)); \tag{11}
\]

i.e. in the limit \( n \rightarrow \infty \) the number \( 2/C_2 \) is the arithmetical average of the product \( \prod_{p | k} \frac{p-1}{p-2} \). The main period of oscillations is 6 hence we can write:

\[
P(d) = \prod_{p | d, p>2} \frac{p-1}{p-2} \approx \alpha + \beta \cos \left( \frac{2\pi d}{6} \right). \tag{12}
\]

The numerical value of \( \alpha \) is equal to \( 2/C_2 \) to reproduce the average value of \( P(d) \) in (11). It can be explained by taking into account that \( \cos(2\pi 2k/6) = 1 \)
while \( \cos(2\pi(2k+2)/6) = \cos(2\pi(2k+4)/6) = -\frac{1}{2} \) and hence by the equation

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left( \alpha + \beta \cos \left( \frac{2\pi k}{6} \right) \right) = \alpha \quad (13)
\]

the value of the parameter \( \beta \) does not contribute to the average of r.h.s. of (12). Thus from (11) we have \( \alpha = 2/C_2 \approx 1.5147801281 \). Requiring, that the combination \( \alpha + \beta \cos(2\pi d/6) \) for \( d = 6 \) takes the value 2 times larger than for \( d = 2 \) and \( d = 4 \): \( \alpha + \beta = 2(\alpha - \beta/2) \) gives \( \beta = \alpha/2 \approx 0.75739 \). Fitting of the parameters \( \alpha \) and \( \beta \) can be done also numerically by standard General Linear Least Squares, see e.g. [32]. We have used the procedure fit from [32] with 2500000 numbers of points: for \( d = 2, 4, 6, \ldots, 5000000 \). The output of the computer run was: \( \alpha = 1.51478 \approx 2/C_2, \quad \beta = 0.75471 \approx 1/C_2 \). Hence we propose the compact formula (see inset in Fig. 2):

\[
P(d) = \prod_{p|d, p > 2} \frac{p-1}{p-2} \approx \frac{1}{C_2} \left( 2 + \cos \left( \frac{2\pi d}{6} \right) \right), \quad (14)
\]

which allows to substitute for \( P(d) \) an expression more amenable for algebraic manipulations. Such an approximation may be relevant for calculations of correlations functions for zeros of the Riemann zeta function, where sums involving product \( P(d) \) appear very often [33]. It turns out, that \( \cos(2\pi d/6) \) takes for even \( d \) only two values: \(-1/2\) for \( d = 6k+2 \) and \( 6k+4 \), and 1 for \( d = 6k \). Because \( d \) and \( d^2 \) have the same prime divisors it follows that \( P(d^2) = P(d) \). The same relation is also obeyed by the approximation (14) because \((6k+2)^2 = 6k' + 4 \) and \((6k+4)^2 = 6k'' + 4 \) and the square of the \( d = 6k \) is obviously again a number of the same form.

The smallest gap between adjacent primes is 2 (twin primes), while the maximal gap \( G(x) = \max_{p_n < x} (p_n - p_{n-1}) \) grows with \( x \). We can obtain the formula for \( G(x) \) from (4) assuming that the largest gap up to \( x \) between two consecutive “levels” \( p_{n+1} - p_n \) appears only once: \( \tau_G(x) = 1 \). Skipping the oscillating term \( P(d) \), which is very often close to 1, we get for \( G(x) \) the following estimation expressed directly by \( \pi(x) \):

\[
G(x) \sim \frac{x}{\pi(x)} \left( 2\ln(\pi(x)) - \ln(x) + c \right), \quad (15)
\]

where \( c = \ln(C_2) = 0.2778769 \ldots \). Substituting here the PNT in the form \( \pi(x) \sim x/\ln(x) \) gives the Cramer’s conjecture [34] \( G(x) \sim \ln^2(x) \) in the limit of large \( x \). The maximal gaps \( G(x) \) are scattered chaotically, the largest currently known gap of 1476 follows the prime 1425172824437699411, see [30]. The comparison of the above formula with real data is presented in Fig. 3.

We finish this section recalling the result of P. Gallagher [35]. He proved, assuming the special generalization of the \( n \)-tuple conjecture of Hardy–Littlewood (3), that the fraction of intervals which contain exactly \( k \) primes follows a Poisson distribution. More precisely he proved, that the number \( P_k(h, N) \) of such \( n < N \) that the interval \( (n, n + h) \) contains exactly \( k \) primes is asymptotically for \( N \to \infty \) given by

\[
P_k(h, N) \sim N \frac{\lambda^k e^{-\lambda}}{k!},
\]

where \( \lambda \sim h/\ln(N) \) is a parameter of the Poisson distribution. In [36] E. Kowalski has generalized the Gallagher theorem to other families of primes. In particular the numbers of twins, primes of the form \( m^2 + 1 \) or Sophie Germain primes (i.e. primes \( p \) with \( 2p + 1 \) also prime) in short intervals are asymptotically Poisson distributed.
III. UNFOLDED PRIMES

For energy spectrum $E_1, E_2, \ldots$ one usually performs unfolding to focus on fluctuations around the smooth part of staircase and simultaneously to pass to the dimensionless variables $e_1, e_2, \ldots$ via the definition:

$$e_n = \overline{N}(E_n).$$  \hspace{1cm} (16)

Then the average spacing between two consecutive $e_n, e_{n+1}$ is equal to 1 and this procedure removes the individual properties of a system. Although primes are dimensionless we can perform the unfolding using the definition

$$r_n = \text{Li}(p_n).$$  \hspace{1cm} (17)

Then the unfolded spacings are $D_n = r_{n+1} - r_n$, writing $p_{n+1} = p_n + d_n$ (d$_n$ are “pure” spacings, not unfolded) and using $\text{Li}(x) \sim x/\ln(x)$ we obtain

$$D_n \approx \frac{d_n}{\ln(p_n) + d_n/p_n}$$  \hspace{1cm} (18)

and for large $p_n$ it goes into $D_n = d_n/\ln(p_n)$. In other words we can say, that the unfolded gaps (level spacings) between very large consecutive primes are $D_n = (p_{n+1} - p_n)/\ln(p_n)$. Because the average distance between primes $(p_{n-1}, p_n)$ is $\ln(p_n)$ we have from (18) for large $p_n$ that the average spacing between two consecutive $(r_n, r_{n+1})$ is equal to 1, as it should be for unfolded variables. The values of $D_n$ are arbitrary real numbers, while $d_n$ assume only even values. For example, for twin primes $p_{n+1} = p_n + 2$ the gap $d = 2$ will be mapped into $D_n \approx 2/\ln(p_n)$ with explicit dependence on $p_n$ and it goes to zero with increasing $p_n$ (if there are infinity of twins, as it is widely believed). On the other side the maximal gap up to $2^{34}$ is $G(2^{34}) = 382$ and it appears at $p_{486570087} = 10726904659 \approx \text{Li}(p_n)$ for primes up to $2^{34} = 1.72 \ldots \times 10^{10}$. Three widths of bins are used: $\Delta D = 0.1$. $\Delta D = 0.001$ and $\Delta D = 16.54/28000$. In black is shown the plot for the unfolding defined by eq. (19).

![Diagram](image)

FIG. 4: The plot of histograms of unfolded spacings $D_n = r_{n+1} - r_n$ where $r_n = \text{Li}(p_n)$ for primes up to $2^{34} = 1.72 \ldots \times 10^{10}$. Three widths of bins are used: $\Delta D = 0.1$. $\Delta D = 0.001$ and $\Delta D = 16.54/28000$. In black is shown the plot for the unfolding defined by eq. (19).

$p_{486570087} = 10726904659$ we get that the maximal value of $D$ is $382/\ln(10726904659) = 16.54 \ldots$ and the size of bin should be $16.54/28000 \approx 0.00059$. In Fig. 4 red line presents the plot for this choice of the bin size, the blue line is for roughly ten times larger division $\Delta D = 0.005$ while green plot presents the histogram of prime pairs with $D$ divided into bins of the size $\Delta D = 10^{-1}$. These plots can be normalized by dividing all values by the maximal value present in the histogram for a given bin size.

The explicit form of the equation (4) allows us to define the unfolding in the following way: Let us define the rescaled quantities:

$$T_d(x) = \frac{x\tau_d(x)}{C_2 P(d)\pi^2(x)}, \quad D(x, d) = \frac{d\pi(x)}{x}. \hspace{1cm} (19)$$

The product $P(d)$ in the denominator of the first formula removes the oscillations and gives the analog of the histogram free of size bin ambiguity. The second equation defines the proper unfolding for prime numbers. Because $x/\pi(x) \approx \ln(x)$ is the mean distance between two consecutive primes $d \approx \ln(x)$ up to $x$, we see that $D(x, d)$ corresponds to the distances between “unfolded” primes — normalized spacing between two consecutive primes is $D(x, d) \approx d/\ln(x)$ and hence the mean value of $D(x, d)$ is simply 1. For large $x$ the quantity $D(x, d)$ agrees with expression (18) for large $p_n$: $D(p_n, d_n) \approx d_n/\ln(p_n) = D_n$ and hence values of $D(x, d) \in [2/\ln(x), \ln(x)]$. From the conjecture (4) we expect that for each $x$ the points $(D(x, d), T_d(x))$, $d = 2, 4, \ldots, G(x)$ should coincide — the function $\tau_d(x)$ displays scaling in the physical terminology. In Fig. 5 we have plotted the points $(D(x, d), T_d(x))$ for $x = 2^{28}, 2^{38}, 2^{48}$, and indeed we affirm the tendency of all these curves to collapse into the universal one. To make this plot we have used ex-
act values of \( \pi(x) \), not any of the approximate formulas like \( Li(x) \): from the definition of \( \tau_d(x) \) it follows that 
\[
\pi(x) = \sum_d \tau_d(x) + 1
\]
and it allowed us to calculate from \( \tau_d(x) \) precise values of \( \pi(x) \) for \( x = 2^{28} \cdot 3^{38} \cdot 5^{48} \). If we denote \( u = D(x, d) \) then all these scaled functions should exhibit the pure exponential decrease \( e^{-u} \): Poisson distribution shown in red in Fig. 5. We have determined by the least square method slope \( s(x) \) and prefactor \( a(x) \) of the fits \( a(x) e^{-s(x)L} \) to the linear parts of plots of \( (D(x, d), \ln(T_d(x))) \). The results are presented in Fig. 6. The slope very slowly tend to 1: for over 6 orders of \( x \) (\( \tau \)) values of \( ax \) parameters are calculated and put equal to zero, what gives the very well 
\[
\text{by the least square method, i.e. the partial derivatives}
\]

Finally let us remark that there is no repulsion of small gaps between primes: usually for GOE or GUE there is a prohibition of small gaps between energy levels (in fact the number of gaps with \( s = 0 \) is equal to zero), but for our case the smallest gap corresponds to twins and it is believed that there is infinity of them. From (4) it follows that the number of twins and cousins is roughly a half of the number of primes separated by \( d = 6 \). In fact for all plots of \( \tau_d(x) \) in Fig. 1 \( d = 6 \) is the highest point — i.e. it is most often occurring gap. However in Fig. 1 local spikes appear at multiplicities of \( 30 = 2 \cdot 3 \cdot 5 \cdot 7 \), where the product \( P(d) \) has local maxima. As \( x \) increases the slopes of plots of \( \tau_d(x) \) decrease and at some value around \( x \approx 10^{36} \) the peak at \( d = 30 \) will be greater than that at \( d = 6 \). At much larger \( x \approx 10^{428} \) the spike at 210 will take over \( d = 30 \). It leads to the so called problem of champions, i.e. most occurring gap between consecutive primes, see [37]. Thus primes are repelled in a very special way: the most often occurring gaps are products of consecutive primes, but they become the “champions” at extremely large values of \( x \). For the unfolded according to eq. (18) gaps \( D_n \) (or eq. (19) and quantities \( D \) as well) there is no repelling: the most common value of \( D_n \) is \( 2 \pi(x)/x \approx 2/\ln(x) \) and it tends to zero with increasing \( x \) — behavior typical for the Poisson distribution.

Similar unfolding procedure has been used in dynamical systems e.g. in the stadium billiard were the existence of strong oscillations due to bouncing ball orbits strongly influence the spectral statistics \( \Delta_3 \) and to get a good agreement with the Gaussian Orthogonal Ensemble (GOE) predictions one has to perform unfolding which includes explicitly the contribution of the bouncing ball periodic orbits (see [38]).

It is a common belief that the Poisson NNSD of the quantum energy levels is linked with integrable systems with more than one degree of freedom. In [39] P. Crehan has shown that for any sequence of energy levels obeying a certain growth law \( (|E_n| < e^{an+b} \), for some \( a \in \mathbb{R}^+, \ b \in \mathbb{R} \), there are infinitely many classically integrable Hamiltonians for which the corresponding quantum spectrum coincides with this sequence. Because from PNT it follows, that the \( n \)-th prime \( p_n \) grows like \( p_n \sim n \ln(n) \) the results of Crehan’s paper can be applied and there exist classically integrable hamiltonians whose spectrum coincides with prime numbers, see also [11].

IV. SPECTRAL RIGIDITY OF PRIME NUMBERS

In [40] several statistical measures to describe fluctuations in the energy levels \( \{E_n\} \) of complex systems were introduced. One which attracted much attention is the spectral rigidity \( \Delta_3 \). The spectral rigidity for arbitrary system with spectral staircase \( N(E) \) is defined as the averaged mean square deviation of the best local fit straight line \( ax + b \) to the \( N(E) \) on the interval \( (x, x + L) \):

\[
\Delta_3(x; L) = \frac{1}{L} \left\langle \min_{a, b} \int_0^L (N(x + \epsilon) - ax - b)^2 \, d\epsilon \right\rangle
\]

The averaging procedure \( \langle \cdot \rangle \) depends on the specific problem, e.g. for random matrices it is the mean value from an ensemble of generated matrices or average over a set of atomic nuclei in real experiments, see e.g. [41]; sometimes average over the initial point \( x \) is applied. There are in general two ways of performing the operation \( \min_{a, b} \), see the discussion in [40]. One can calculate partial derivatives of r.h.s. of (20) with respect to \( a \) and \( b \), equate them to zero, solve for \( a \) and \( b \) and substitute solutions back to r.h.s. what leads to the double integrals, see e.g. [42, Appendix II]. We will present here the procedure for calculating \( \Delta_3 \) in this way devised by O. Bohigas and M.-J. Giannoni in [43] and [20]. First the energies are unfolded \( E_N \rightarrow e_n \) using the smooth part \( \bar{N}(E_n) \) of the staircase function, see eq. (16). If the sequence of unfolded levels \( e_1, e_2, \ldots, e_n \) falls in the interval \( (x, x + L) \) the following explicit formula for \( \Delta_3(x; L) \) is obtained:

\[
\Delta_3(x; L) = \frac{n^2}{16} - \frac{1}{L} \left( \sum_{k=1}^n e_k \right)^2 + \frac{3n}{2L^2} \sum_{k=1}^n e_k^2 - \frac{3}{L^2} \left( \sum_{k=1}^n e_k^2 \right) + \frac{1}{L} \sum_{k=1}^n (n - 2k + 1) e_k,
\]

where \( e_k = e_k - (x + L/2) \). In the second approach the parameters \( a \) and \( b \) are obtained by fitting the straight line \( ax + b \) to the set of points \( (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \) by the least square method, i.e. the partial derivatives of \( \sum_{k=1}^n (y_k - ax_k - b)^2 \) with respect to \( a \) and \( b \) are calculated and put equal to zero, what gives the very well known expressions:

\[
a = \frac{n \sum_{k=1}^n x_k y_k - \sum_{k=1}^n x_k \sum_{k=1}^n y_k}{n \sum_{k=1}^n x_k^2 - (\sum_{k=1}^n x_k)^2}
\]
and 10
mula for ∆
contribution from trivial zeros
−

The spectral rigidity obtained in this second way we will
define by (20) with
\[ \zeta(x) \]

2

π

us of all primes are needed and we have used primes
N
instead of
3
In the case of ∆
ρ
The sum over
N
sufficient for calculation of ∆


\[ \sum_{k=1}^{n} (y_k - ax_k) \]

In the case of ∆₃(x; L) we have \( x_k = E_k, y_k = N(E_k) \).
The spectral rigidity obtained in this second way we will
distinguish from (21) by apostrophe ∆′

3

\[ \Delta'_3(x; L) \]

Spectral rigidity for primes we define by (20) with
\[ \pi(x) \]

\[ \tau(x) \]

\[ \mu(n) \]

\[ \mu(n) = \begin{cases} 
1 & \text{for } n = 1 \\
0 & \text{when } p^2 | n \\
(-1)^r & \text{when } n = p_1 p_2 \ldots p_r.
\end{cases} \]

The sum over \( \rho \) runs over nontrivial zeros of the Riemann \( \zeta(s) \) function \( \zeta(\rho) = 0 \) and the last integral contains contribution from trivial zeros \( -2m \) of zeta: \( \zeta(-2m) = 0, \ m = 1, 2, 3, \ldots \). If the Riemann Hypothesis is true then for all nontrivial zeros \( \Re(\rho) = \frac{1}{2} \) and the contribution to the sum over \( k \) in (22) is dominated by the first term, what leads to the following approximation to \( \pi(x) \):

\[ R(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \ln(x^k). \]  

FIG. 5: Plots of \( (\mathcal{D}(x,d), \tau_d(x)) \), \( d = 2, 4, \ldots \) for \( x = 2^{28}, 2^{38}, 2^{48} \) and in red the plot of \( e^{-u} \). Only the points with \( \tau_d(x) > 1000 \) were plotted to avoid fluctuations at large \( D(x,d) \) due to small values of \( \tau_d(x) \) for large \( d \).

FIG. 6: Plot of slopes \( s(x) \) and prefactors \( a(x) \) in
the dependence \( a(x)e^{-s(x)} \) obtained from fitting it to
\[ (\mathcal{D}(d,x), \ln(\tau_d(x))) \] for \( x = 2^{28}, 2^{38}, \ldots, 2^{48} \).
The difference \( \pi(x) - R(x) \) changes the sign already at \( x \) as low as \( x \in [2, 100] \), see e.g. tables obtained by T. R. Nicely in [30] and up to \( 10^{14} \) there are over 50 millions of sign change of \( \pi(x) - R(x) \) [49], however on average the behavior of both differences \( \pi(x) - \text{Li}(x) \) and \( \pi(x) - R(x) \) seems to be the same [50]. The above function \( R(x) \) can be obtained, without the need of calculating the logarithmical integral \( \text{Li}(x) \), from the series obtained by J.P. Gram, see e.g. [51, p.51]:

\[
R(x) = 1 + \sum_{m=1}^{\infty} \frac{\ln^m(x)}{m!n!\zeta(m+1)}
\tag{24}
\]

Hence we have made the unfolding of primes according to the rule

\[
r_n = R(p_n).
\tag{25}
\]

At this point let us remark that from (6) and (24) we see that because \( \zeta(m) \to 1 \) for \( m \to \infty \) very quickly (e.g. \( \zeta(4) = \pi^4/90 = 1.082323 \ldots \), \( \zeta(6) = \pi^6/945 = 1.017343 \ldots \) for large \( x \) the functions \( \text{Li}(x) \) and \( R(x) \) should differ by roughly \( \ln(x) \) and this quantity can be discarded in comparison with values of series involving powers of \( \ln(x) \) present in (6) and (24). Indeed, from (23) it follows using the first term from asymptotic expansion (5) that for large \( x \) the approximate relation \( R(x)/\text{Li}(x) = 1 - 1/\sqrt{x} \) holds. Thus for large \( x \) the particular form of unfolding (17) or (25) should be irrelevant, despite the fact that \( \pi(x) - \text{Li}(x) \) changes the sign first time somewhere in the vicinity of \( x = 10^{316} \) while \( \pi(x) - R(x) \) changes the sign already for \( x \) between 10 and 20, see tables of Nicely [30].

We will present the plots of \( \Delta_3(x; L) \) for three values of \( x \): \( 10^8, 10^9 \), and \( 10^{10} \). The values of primes for which the unfolded variables begin to fall into the intervals \( (10^8, 10^9 + L), (10^9, 10^{10} + L), (10^{10}, 10^{10} + L) \), are accordingly \( 2038076627, 22801797631, 252097715777 \) for probabilistic primes:

\[
R(2038076627) = 10^8 + 1.8496 \ldots, R(22801797631) = 10^9 + 2.3178 \ldots, R(252097715777) = 10^{10} + 0.0024 \ldots
\]

As there seems to be no clear relation between the values of \( L \) in comparison with chosen \( x \) we have used the wide range of values of \( L \): we have calculated from (21) spectral rigidity for values \( L = 2^7 = 128, \ldots, 2^{26} = 67108864 \). The results are presented in Fig. 7. It is well known that for stationary Poisson ensemble \( \Delta_3(x; L) = L/15 \), see e.g. [40, eq.(61)] or [42, Appendix II, and on the Fig. 7 this theoretical prediction is plotted in blue. The obtained plots of \( \Delta_3(x; L) \) seem to tend to the line \( L/15 \) with increasing \( x \). For primes there is no natural averaging procedure present in the definition (20) and in Fig. 7 prominent fluctuations are seen. To simulate the averaging we have performed the following “Monte Carlo” experiment for \( x = 10^{10} \). From the PNT in the form \( \pi(k) \sim k/\ln(k) \) it follows that the chance that randomly chosen large integer \( k \) should be a prime is \( 1/\ln(k) \). Such a probabilistic model for primes was created by H. Cramer in the 1930’s [34]. We have started to test if a given natural \( k \) number is the probabilistic “artificial” prime from the first \( k_0 \) for which \( R(k_0) > 10^{10} \), i.e. for \( k_0 = 252097715777 \) for which \( R(k_0) = 10^{10} + 0.0024 \ldots \). The natural number \( k > k_0 \) (even the even numbers were allowed — when even numbers are skipped the probability of odd number \( k \) to be a “prime” should be \( 2/\ln(k) \)) was accepted to be a “probabilistic” prime if \( 1/\ln(k) \) was
larger than the uniformly generated from the interval \((0, 1)\) random number \textit{random}: \(\text{random} < 1/\ln(k)\). For such a “prime” \(k\) the unfolding was performed using the equation \(r'_k = R(k)\). The random drawing of “primes” was continued until the unfolded “prime” was larger than \(x + L\) for \(L = 128, \ldots, 2^{26}\). For the set of such generated unfolded quantities in the intervals \((x, x + L)\) the “artificial” spectral rigidity \(\Delta_3^{(p)}(x; L)\) was calculated using (21). The result of this procedure is plotted in green in Fig. 8 and there are fluctuations seen resembling those present in Fig. 7 for “true” primes. But now we can generate many independent sets of the artificial probabilistic primes. We have repeated this procedure 100 times and the averaged over all these samples spectral rigidity \(\Delta_3^{(p)}(x; L)\) is presented in Fig. 8 in black. Now the fluctuations disappeared and the obtained plot follows perfectly the predicted dependence \(L/15\). This allows us to claim that the spectral rigidity for prime numbers unfolded via the Riemann function \(R(x)\) is the same as for Poisson statistics (we have checked that the same result is obtained for unfolding with \(\text{Li}(x)\) as in eq. (17)). Let us mention that usually saturation of \(\Delta_3\) is observed in physical systems, i.e. after the initial dependence resembling \(L/15\) spectral rigidity stops to increase and is constant for large \(L\), see e.g. [23] or [52]. However our system is infinite and there is no departure from straight line \(L/15\).

Next we will present spectral rigidity for second method of minimizing the r.h.s of (20) over \(a, b\), namely determination of \(a, b\) by the least square method. In the case of primes numbers, for large \(x\), the smooth part of staircase \(\pi(x)\) given by \(x/\ln(x)\) is almost linear in the interval \((x, x + L)\) as the denominator changes from \(\ln(x)\) to \(\ln(x + L) = \ln(x) + L/x + \ldots\) what for \(x \gg L \gg 1\) again is \(\ln(x)\). There are a few websites [30] offering the tables of values of \(\pi(x)\) (as well as other number theoretic functions). In these data files the values of \(\pi(x)\) are tabulated with different step size of \(x\), the best resolution is at the A. V. Kulsha’s page: the file pi.txt of the size 421MB contains counts of \(\pi(x)\) with a step of \(10^9\) from \(x = 10^9\) to \(x = 2.5 \times 10^{16}\). Now we will give the formula for calculating the integral appearing in the definition of \(\Delta_3^{(p)}(x; L)\):

\[
\mathcal{I}(x; L) = \int_0^L (\pi(x + \epsilon) - a\epsilon - b)^2 d\epsilon 
\]

appropriate for our data. Let us assume, that the values of \(\pi(x)\) in the integral (26) are known with the resolution \(h\): \(y_k = \pi(x_k + kh)\); hence we assume that \(\pi(x)\) is constant on the intervals \((kh, (k + 1)h)\) (in fact \(\pi(x)\) is constant only between two consecutive primes). We regard this sampling of \(\pi(x)\) with different steps \(h\) as the averaging procedure hidden in the angle bracket in (20) — taking values of \(\pi(x)\) at all consecutive \(h\) primes would introduce fluctuations. The combination \(\pi(x + \epsilon) - a\epsilon - b\) is the linear function on the intervals \((kh, (k + 1)h)\) and we can write (we assume here that \(L\) is the integer multiple of \(h\)):

\[
\mathcal{I}(x; L) = \int_0^L (\pi(x + \epsilon) - a\epsilon - b)^2 d\epsilon = \\
\sum_{k=0}^{L/h-1} \int_{kh}^{(k+1)h} (y_k - a\epsilon - b)^2 d\epsilon.
\]

Performing elementary integration we obtain:

\[
\Delta_3^{(p)}(x; L) = b^2 + abL + \frac{a^2L^2}{3} + \frac{1}{L} \sum_{k=0}^{L/h-1} y_k(y_k - 2b)h - ay_k(2k + 1)h^2
\]

in these figures: the constant in \(L\) values of \(\Delta_3^{(p)}\) depending on \(h\) and the collapse of plots of \(\Delta_3^{(p)}\) for all \(h\) when the increase of \(\Delta_3^{(p)}\) with \(L\) begins. It seems that to get rid of dependence on \(h\) the sufficiently large number \(L/h\) of terms in the formula (27) has to be summed up. The inspection of data shows, that to have the independence of \(\Delta_3^{(p)}\) on \(h\) a few thousands of terms in the sum in (27) are sufficient (for largest \(L\) there are millions of terms in this sum, see plots in royal red in Fig. 10 and 11). It is possible to find heuristically the values of the constant in \(L\) parts of \(\Delta_3^{(p)}\). To find the analytical expressions for \(a\) and \(b\) we consider the smooth part of \(\pi(x)\) given by \((x + \epsilon)/\ln(x + \epsilon)\) and the straight line \(ae + b\) obtained by best fitting to the values of \((x + kh)/\ln(x + kh)\). The experiments show, that the fits cross \((x + \epsilon)/\ln(x + \epsilon)\) on the interval \(\epsilon \in (0, L)\) roughly at \(\epsilon = L/4\) and \(\epsilon = 3L/4\), see Fig. 9, thus from \((x + L/4)/\ln(x + L/4) = aL/4 + b\)

It should be noted, that parameters \(a\) and \(b\) in eq. (27) obtained from fitting \(ae + b\) to points \(\pi(x + \epsilon), 0 \leq \epsilon \leq L\), by least-square method are functions of \(L\) and \(x\), see below (29).

The value of \(\Delta_3^{(p)}(x; L)\) given by (27) should not depend on \(h\). To test this presumption we have calculated \(\Delta_3^{(p)}(x; L)\) for \(x = 10^{15}\) and \(x = 10^{16}\) and for \(h_1 = 10^{9}, h_2 = 2 \times 10^{10}, h_3 = 4 \times 10^{9}\). We have chosen the following sequence of values of the length of intervals \(L = 16h_1 = 1.6 \times 10^{10}, 32h_1 = 3.2 \times 10^{10}, \ldots, 2^{22}h_1 = 8.388608 \times 10^{15}\) for both \(x_1, x_2\) and additionally \(L = 1.5 \times 10^{16}\) for \(x_2 = 10^{16}\). It means that the number of terms in the sum in (27) was \(2^3, 2^4, \ldots, 2^{22} = 4194304\) for \(h_2\) and \(2^2, 2^4, \ldots, 2^{21} = 2097152\) for \(h_3\) respectively.

For each \(L\) the parameters \(a\) and \(b\) were fitted by the least-square method to the points \((x_k = x + kh, y_k = \pi(x + kh)), k = 0, 1, \ldots, L/h - 1\). In Figures 10 and 11 we present the results. Two types of behaviors are seen
and \((x + 3L/4)/\ln(x + 3L/4) = a3L/4 + b\) we get

\[
a = \frac{2}{L} \left( \frac{x + 3L/4}{\ln(x + 3L/4)} - \frac{x + L/4}{\ln(x + L/4)} \right) = \frac{1}{\ln(x)} - \frac{L}{2x\ln^2(x)} + \text{terms} \frac{1}{x^2} \ \text{or higher} \tag{28}
\]

\[
b = \frac{x + L/4}{\ln(x + L/4)} - aL/4 = \frac{x}{\ln(x)} - \frac{L}{4\ln^2(x)} + \text{terms} \frac{1}{x} \ \text{or higher} \tag{29}
\]

Using \(y_k = (x + kh)/\ln(x + kh) \approx (x + kh)/\ln(x) -(kh)^2/2x\ln^2(x)\) we obtain in (27) sums over \(k\) which can be calculated exactly and retaining the leading terms gives:

\[
\Delta'_3(x; L) = \frac{\hbar^2}{3\ln^2(x)} - \frac{\hbar L}{4\ln^3(x)} + \left( \text{terms} \frac{1}{x} \ \text{or higher} \right) \tag{30}
\]

Because \(\Delta'_3(x; L) > 0\) we have from above \(\hbar^2/3\ln^2(x) > hL/4\ln^3(x)\), i.e. \(L < 4h\ln(x)/3\), what for \(x = 10^{13}\) gives \(L < 40h\). Surprisingly the first term in (30), not depending on \(L\) but being the function of \(x\), gives the expression

\[
\Delta'_3(x; L; h) = \frac{\hbar^2}{3\ln^2(x)} + \ldots \tag{31}
\]

which works very well even for \(L = 1024h\) for \(x_1 = 10^{13}\) and \(L = 8192h\) for \(x_2 = 10^{16}\), as it is seen in Figures 10 and 11, where the predicted values \(\hbar^2/3\ln^2(x)\) are plotted by dashed lines together with the plots of \(\Delta'_3(x; L; h)\) obtained from (27). In fact this agreement is astonishing: e.g. all \(\Delta'_3(10^{16}; L; h_1)\) for initial 11 values of \(L\) have first three digits the same: \(2.455 \ldots \times 10^{14}\) while (31) predicts \(2.45588 \ldots \times 10^{14}\). In Fig. 10 we were able to make the plot for \(L\) up to almost \(10^9x_1\), while in Fig. 11 the largest \(L\) is smaller than \(x_2\), thus we expect bending of \(\Delta'_3(x_2; L; h)\) for larger \(L\), similar to the behavior of \(\Delta'_3(x_1; L; h)\) on Fig. 10. In the plots of \(\Delta'_3\) we see the crossover at value \(L^*\) above which the steeper increase of spectral rigidities begins and this dependence is \(L^*\), with \(\gamma \approx 3.1\). Heuristically existence of this crossover can be justified by the following reasoning: for moderate values of \(L\) the straight line \(ae + b\) approximates \(\pi(x + \epsilon)\) quite well leading to the small values of the integral \(\int_x^{x+L} (\pi(x + \epsilon) - ae - b)^2 \, dx\), while for larger \(L\) the discrepancy between \(\pi(x + \epsilon)\) and the straight line increases leading to larger values of \(\Delta_3\). The spectral rigidity calculated in second way displays different behavior than \(\Delta_3(x; L)\) obtained in the first manner. Let us remark at this point that the proof of \(\Delta_3(x; L) = L/15\) for the Poisson ensemble was obtained in [42, Appendix II] only for the first method of minimalization over \(a\) and \(b\) in (20).

\section{Conclusions}

In this paper we have treated prime numbers as energy levels and we applied the physical methods used to study spectra of quantum systems to the description of distribution of prime numbers. We presented large numerical data (up to \(x = 2.814 \ldots \times 10^{14}\)) in support of the formula (4) for NNSD between consecutive primes. It was also possible to obtain analytical formula (15) for the Poisson ensemble was obtained in [42, Appendix II] only for the first method of minimalization over \(a\) and \(b\).

\[\text{FIG. 9: The illustration of the experimental fact that the straight line best fitting } (x + \epsilon)/\ln(x + \epsilon) \text{ on the interval } \epsilon \in (0, L) \text{ crosses it at } \epsilon = L/4 \text{ and } \epsilon = 3L/4.\]
of the form $4k^2 + 1$; in the latter case the “energy levels” are the values of $k$ for which $4k^2 + 1$ is prime.

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