Berry Phase Generation and Measurement in a Single Trapped Ion

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In this work, we propose a new design of an ion trap which can enable us to generate state specific Berry phase in a single trapped ion. Such a design will enable us to study the physics at the boundary of abelian and non-abelian symmetries and can also have significant impact in quantum computation.

I. INTRODUCTION

Single trapped ion provides the cleanest quantum system available in nature. In the past decade or more, trapped ions has been used in a variety of scenarios, from developing atomic clocks in the optical domain to implementing protocols required for quantum computation. It also provides a test bed for measurement of various fundamental properties of quantum mechanics and quantum field theories, such as parity non-conservation and measurements on limits of electron electric dipole moment as well as time variation of fundamental constants of nature.

Geometric phases in general and Berry phase in particular has been studied for a variety of physical systems and conditions. The theoretical treatment of geometric phases has been generalized for composite systems having internal structure particularly in the light of quantum optics. This has practical importance from the view point of systemic effects associated with rotating electric and magnetic fields in precision experiments. However, higher order multipole effects has not been considered except for a few cases like the Nuclear Quadrupole Resonances in a rotating frame where frequency shift in the NQR spectrum due to Berry phase has been observed. In this work, we put forward a proposal for the measurement of Berry phases and Berry phase generated energy shifts using single trapped ion. The ion trap experiment will not only provide observable energy shift but also a clean and controlled measurement to understand the geometric phase in Abelian and non-Abelian cases. To the best of our knowledge, this has so far not been explored either theoretically or experimentally. In a way, this proposal allows one to simulate a situation when a quantum system changes from one symmetry to another symmetry. In either regime the Berry phases may be used to implement various noise tolerant quantum gates, which form the heart of quantum computation, without the complexity of using lasers.

II. BERRY PHASE AND PHASE DEPENDENT ENERGY SHIFT

A. Introduction to Berry Phase

Berry phase is a phase acquired by the eigenstates of a Hamiltonian, which is changing with time implicitly in an adiabatic fashion in a parameter space so that the system remains in the same eigenstate during the entire evolution, i.e. the time dependence does not lead to a transition in the system. For adiabatic condition to hold, time scale of the change must be less than any other relevant time scale of the system. The Schrödinger equation for the adiabatic evolution of a system can be written as

$$H(\mathcal{R}(t))\psi_n(\mathcal{R}(t)) = E_n(\mathcal{R}(t))\psi_n(\mathcal{R}(t)),$$

where the time dependence comes through the parameter $R(t)$. Under such time evolution, the total wave-function is given by

$$|\psi_n(\mathcal{R}(t))\rangle = e^{\frac{\text{i}}{\hbar} \int_0^t E_n(\mathcal{R}(t'))dt} e^{i\gamma_n(t)} |n(\mathcal{R}(t))\rangle ,$$

where $\gamma_n(t)$ is the Berry phase acquired in time $t$ by the state $|n(\mathcal{R}(t))\rangle$ eigenstate and $E_n(\mathcal{R}(t))$ is the corresponding eigenvalue. For a periodic time evolution of the Hamiltonian (which is the most practical of experimentally viable scenarios), the Berry phase acquired by the $n$th eigenstate is given by

$$\gamma_n^0 = \int \langle \psi(\mathcal{R}) | \nabla_{\mathcal{R}} |\psi(\mathcal{R})\rangle \cdot d\mathcal{R},$$
where the integration is carried over the path traversed in the parameter space.

B. Phase Dependent Energy Shift

The phase of the wavefunction in Eq. [2] in the linear approximation of variation of Berry Phase with time due to adiabatic change can be re-written as

$$\frac{i}{\hbar} \int_0^t (E_n(R(t)) + \frac{\hbar \gamma_n}{T}) dt,$$

where $\gamma_n$ is the Berry Phase acquired in one cycle with a time period $T$.

As can be seen from Eq. [3] the eigenvalue of the system gets modified by $\frac{\hbar \gamma_0}{T}$. Thus in presence of a periodically evolving time dependent Hamiltonian, adiabatic in nature, the energy eigenvalue of the Hamiltonian gets modified depending on the acquired phase and the time period of the cyclical evolution.

III. PROPOSED EXPERIMENTAL SYSTEM

In this work we propose a concrete experiment where we can observe such phase dependent energy shifts using a single trapped ion. The trap is modified to impose a cyclic Hamiltonian on the ion and we can observe the resultant shift in the electronic levels of the trapped ion. The interaction by which we propose to impose such a time dependent Hamiltonian is the interaction of the quadrupole moment of some chosen electronic levels with a time dependent electric field gradient, provided by a modified trap geometry. To the best of our knowledge, the only experimental observation of similar splitting and shift has been by R. Tycko who used a single crystal of KClO$_3$, where the crystal field gradient interacted with the nuclear quadrupole moment and the time dependence has been incorporated by mechanical rotation of the crystal. The proposed experiment using a single trapped ion however is fundamentally different in two ways. First, the electronic state of the ion allows the manipulation of the Hamiltonian by light field. Second, the presence or absence of static magnetic field allows the system to change between Abelian and non-Abelian geometry. This transition from one regime to another has so far not been studied either experimentally or theoretically.

A. Calculation of Berry Phase

For the generation of a Berry phase, we need a time dependent field gradient and a system with a quadrupole moment. In this case the time dependent field gradient is provided by the modified trap design.

The quadrupole moment can be defined as

$$Q_{ij} = c \left( \frac{1}{2} S_i S_j - S_j S_i - \frac{1}{3} S^2 \right)$$

where $S_k$ corresponds to the $k^{th}$ component of the spin and $c$ is a numeric constant. The Hamiltonian of the interaction is given as

$$H = \frac{1}{6} Q_{ij} \frac{\partial E_i}{\partial x_j},$$

where $\frac{\partial E_i}{\partial x_j}$ is the $ij$th component of the electric field gradient tensor. Now, we first move into the frame of the field gradient and define the $\hat{z}'$-axis along the field gradient tensor. In that case, the Hamiltonian becomes

$$H = \alpha (S_z^2 - \frac{1}{3} S^2),$$

where the pre-factor $\alpha$ contains the value of the electric field gradient and the quadrupole moment of the state under study. Assuming a spin $\frac{3}{2}$ state, which is relevant, as the experimental system we are interested in is the $5d_{5/2}$ state of a trapped Ba$^+$ ion, the eigenvalues for this Hamiltonian are $\alpha \hbar^2$ and $-\alpha \hbar^2$ with each of them being doubly degenerate due to Kramers degeneracy. $\frac{1}{2} >$ and $\frac{1}{2} >$ are the first set of doubly degenerate eigenstates and $\frac{1}{2} >$ and $\frac{1}{2} >$ consists of the second set. The value of alpha will depend on the value of the quadrupole moment of the $5d_{5/2}$ of Ba$^+$ ion [17], as well as the magnitude of the components of electric field gradient.

Now to transform back to the laboratory frame, we apply the Wigner D-matrices. Transformation to the laboratory frame consists of a rotation of $-\phi$ along the $z$-axis and then $-\theta$ along the rotated $y$ axis and the third rotation being a null rotation. The time dependence comes from $\phi = \omega t$. Applying the Wigner matrices and using Eq. [3] we obtain the following Berry phases for the eigenstates of the Hamiltonian

$$\gamma_4 = -3\pi (\cos \theta - 1)$$
$$\gamma_3 = -\pi ((4 - 3 \cos^2 \theta)\frac{1}{2} - 1)$$
$$\gamma_2 = \pi ((4 - 3 \cos^2 \theta)\frac{1}{2} - 1)$$
$$\gamma_1 = 3\pi (\cos \theta - 1),$$

where 4, 3, 2 and 1 signify the states $|3/2\rangle$, $|1/2\rangle$, $|-1/2\rangle$ and $|-3/2\rangle$ respectively. The constants of integration are so chosen as to ensure zero phase for $\theta = 0$. Thus incorporating the phase dependent energy shifts, we obtain the energies of the eigenstates as
information processing.

potential applicable to perform fault tolerant quantum

depends on the geometry of the rotation axis w.r.t. lab z-axis

rotation of the Hamiltonian. The Berry phase itself

phase of each of the states as well as the frequency of

rise to a field gradient whose principal axes are tilted

with respect to the trap symmetry axes as shown in Fig

V\frac{\partial V}{\partial z} = \frac{1}{2}V_0 z'^2.

Thus the eigenvectors split depending on the Berry

phase of each of the states as well as the frequency of

rotation of the Hamiltonian. The Berry phase itself

is independent of any external field value but only

depends on the geometry of the rotation axis with

respect to the quantization axis. This is not surprising

since this phase is a purely geometric in nature and is

potentially applicable to perform fault tolerant quantum

information processing.

B. Application of Rotating Field Gradient

The proposal is based on the fact that a potential
tilted with respect to the trap axis ($\frac{1}{2}V_0 z'^2$) can give
rise to a field gradient whose principal axes are tilted

with respect to the trap symmetry axes as shown in Fig

However it has been shown by our calculation and also

in Ref [17] that the main contributing component is the

$\frac{\partial V}{\partial z}$ component. It can be proved by taking the potential

$V = \frac{1}{2}V_0 x^2 + \frac{1}{2}V_0 y^2 - \frac{1}{2}V_0 z^2$ and calculating the Hamil-
	onian of quadrupole interaction. Hence for simplicity,

we consider only the $\frac{\partial V}{\partial z}$ component of the field gradient

matrix to be non-zero with the other components to be

zero. The proof of the fact that a tilted potential can
give rise to a tilted field gradient will be obvious by cal-
culating the field gradient matrix from a potential

representing it in the trap basis, i.e. $R^T E_z' R$ with Euler

angles $\phi, \theta$ and $0$.

Thus the design goal is to have electrode geometry

which can produce a parabolic potential rotating about

the linear trap axis. It can be established using a four

rod structure for the end caps in a linear paul trap and

applying RF voltages on diagonally opposite end cap

rods across the body. As is demonstrated in figure [3]

by connecting body diagonally opposite end cap rods

e.g. $A$ and $3$ and applying suitably phase shifted RF

voltage on each of the four pair of end cap rods ($\frac{\pi}{2}$), it

is possible to rotate a parabolic potential about the trap

axis [18], leading to rotation of the field gradient about

the axis of symmetry.

Here we will show that such a distorted trap indeed

leads to a tilted potential. We simulate the trap

potential, given its geometry and the rotating time
dependent potential, using Simion 7.0 and plot the three

components of the electric field for a position, off axis

from the trap center (exactly at the trap center the

field is zero). Also we plot the electric field as obtained

comparing, Fig. 4 and Fig. 5.

As can be seen there are certain differences between

the two figures. They arise out of the fact that even

though $\frac{1}{2}V_0 z'^2$ assumes a $x'$ and $y'$ symmetry, the

actual trap does not have so and hence it can lead
to the difference that is observed. The value of $\theta$ has

FIG. 1: Principle axis of field gradient relative to laboratory

axis. $\phi = \omega t$ is the time dependent parameter leading to

rotation of the principle axis w.r.t. lab z-axis

FIG. 2: The tilted potential $\frac{1}{2}V_0 z'^2$ and resultant field gradi-

ten $E_{zz}'$

\[
E_1 = \alpha \hbar^2 + \frac{\hbar 3 \pi \cos \theta - 1}{T}
\]
\[
E_2 = -\alpha \hbar^2 + \frac{\hbar \pi ((4 - 3 \cos^2 \theta) \frac{\pi}{2} - 1)}{T}
\]
\[
E_3 = -\alpha \hbar^2 - \frac{\hbar 3 \pi \cos \theta - 1}{T}
\]

Thus the eigenvectors split depending on the Berry

phase of each of the states as well as the frequency of

rotation of the Hamiltonian. The Berry phase itself

is independent of any external field value but only

depends on the geometry of the rotation axis with

FIG. 3: (i) Trap design and the resultant field gradient axis. 

(ii) Cross-sectional view and phase of applied RF.

\[
E_1 = \alpha \hbar^2 - \frac{\hbar 3 \pi (\cos \theta - 1)}{T}
\]

\[
E_2 = -\alpha \hbar^2 - \frac{\hbar \pi ((4 - 3 \cos^2 \theta) \frac{\pi}{2} - 1)}{T}
\]

\[
E_3 = -\alpha \hbar^2 + \frac{\hbar 3 \pi (\cos \theta - 1)}{T}
\]

\[
E_4 = \alpha \hbar^2 - \frac{\hbar 3 \pi (\cos \theta - 1)}{T}
\]

\[
E_4 = \alpha \hbar^2 + \frac{\hbar \pi ((4 - 3 \cos^2 \theta) \frac{\pi}{2} - 1)}{T}
\]
end caps are increased, the parabolic nature of the potentials also increase. However, the berry phase and the resultant energy difference is independent of the magnitude of the field gradient and hence it should not affect the splitting of the levels.

The frequency of rotation however has to be very slow. This is because, to maintain the adiabatic condition, the precession frequency has to be much smaller than the level splitting caused by the Hamiltonian itself. As an example, for Ca ion, the splitting is of the order of 150 Hz for an electric field gradient of about 50V/mm$^2$ [21]. Since the quadrupole moment of Ba$^+$ is almost double that of Ca$^+$ [17], hence we can assume that given similar gradient magnitudes, the splitting will be of the order of 300 Hz and the rotation frequency should be much less than that. For such low frequencies, it will not affect the motion of ions in the trap as the relevant trap frequencies are of the order of MHz.

IV. APPLICATIONS

From the point of view of Abelian and Non-abelian physics, this system can provide insights into transition from abelian to non-abelian situations. For example, when abelian situation is there, that is the states are degenerate, the phase acquired by the $|\frac{1}{2}>$ and $|-\frac{1}{2}>$ sub-states of the $5d_{\frac{3}{2}}$ state are $\pm \pi(4 - 3 \cos^2 \theta)$ [21]. Since the geometric nature of the acquired phases make this system quite attractive for quantum information processing.

V. CONCLUSION

In this article we have shown that an ion trap with a modified geometry can be used to generate observable splitting which are purely geometric in nature, on metastable D-states of ions. The advantage of using the electric quadrupole moment is that it will lead to selective splitting, occurring only for states which have quadrupole moments. The geometric nature of the acquired phases as well as the state specificity of the phases make this system quite attractive for quantum information processing. This type of shifts can be considered as possible systematic in precision experiments dealing with rotating fields and their gradients. As a conclusion the proposed experiment is the first direct attempt to test quantum Physics in an interface of two symmetries namely, Abelian and non-Abelian.
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