The Sup Connective in IMALL:
A Categorical Semantics

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Abstract
We explore a proof language for intuitionistic multiplicative additive linear logic, incorporating the sup connective that introduces additive pairs with a probabilistic elimination, and sum and scalar products within the proof-terms. We provide an abstract characterization of the language, revealing that any symmetric monoidal closed category with biproducts and a monomorphism from the semiring of scalars to the semiring $\text{Hom}(I, I)$ is suitable for the job. Leveraging the binary biproducts, we define a weighted codiagonal map at the heart of the sup connective.

Keywords: Probabilistic setting, Linear logic, Categorical model.

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1. Introduction

1.1. Historical origins
In the quest for a logic for quantum computing, the non-cloning principle \cite{30} is one of the challenges to tackle. This principle states that it is impossible to create an identical copy of an arbitrary unknown quantum state. This is a consequence of the linearity of the quantum mechanics operators, which is a fundamental principle of quantum computing. However,
the first step into considering this linearity is to have a language where that linearity can be expressed. With this aim, calculi with sums and scalar product in the proof-terms has been used for quantum computing and algebraic lambda-calculi in many occasions [2–6, 10–14, 28, 29, 31]. The idea is that if \( t \) and \( u \) are proofs of the same proposition \( A \), then \( t+u \) and \( s\cdot t \) are also proofs of \( A \), with \( s \) in some set of scalars. Most of these works consider a call-by-value strategy for the reduction of the terms, which forces a kind of linearity by considering the reduction rules \( t(u+v) \rightarrow tu+tv \) and \( t(s\cdot u) \rightarrow s\cdot tu \), when \( u \) and \( v \) are values.

In [16] the approach to have linearity in the proof-language is different. There is no need to define a reduction strategy. Instead, the logic considered is Intuitionistic Multiplicative Additive Linear Logic (IMALL), and in the proof language, there is one proof of the proposition \( 1 \) (the multiplicative truth) as elements of a semiring of scalars \( S \). Then, the proofs of \( \&_{i=1}^{n}1 \) (for any parentheses) are in one-to-one correspondence with the elements of \( S^n \). In such a calculus any closed proof \( t \) of the proposition \( A \rightarrow o B \) is proved to be linear in the syntactic sense. That is, the proof \( t(u+v) \) is proof-equivalent to \( tu+tv \) and \( t(s\cdot u) \) is proof-equivalent to \( s\cdot tu \). Moreover, any \( S \)-homomorphism \( S^n \xrightarrow{f} S^m \) has a representation in a proof-term of the proposition \( \&_{i=1}^{n}1 \rightarrow \&_{i=1}^{m}1 \).

The proof language in question is the \( LS \)-calculus. It is a proof language for IMALL, which contains sums and scalar products as proof constructors, but whose probable formulae are nothing more—and nothing less—than the tautologies of IMALL.

A second challenge of a proof-language for quantum computing is the non-determinism of the measurement. In [9], non-determinism has been treated as a new connective in Intuitionistic Propositional Logic, in a Natural Deduction presentation. This connective, \( \odot \) (read as “sup” for superposition), is introduced to express the superposition of data and, more importantly, the measurement operation. The connective sup arises from the observation that a superposition behaves as a conjunction, where both propositions are true (and so, its proof is the pair of proofs), but also, when measured, it behaves as a disjunction, where only one proposition will be recovered in a non-deterministic process. The \( \odot^S \)-calculus contains the sup connective, and also sums and scalar products. While not enforcing linearity (thus allowing cloning), it allows encoding a basic quantum lambda calculus.

The sup connective then has the introductions and eliminations of con-
junction, but it goes further by also including one extra elimination rule, that of the disjunction. This elimination is, in fact, derivable in Natural Deduction when sup is replaced with a conjunction, but the derivation is not unique, thus enabling non-determinism. The $\odot^S$-calculus shows that superposition and measurement can be represented by this new connective.

In [16], alongside the $L^S$-calculus, the $L\odot^S$-calculus is also considered, which incorporates the sup connective within the linear setting. This distinguishes it from other approaches to non-deterministic and probabilistic linear calculus, such as PCF$^R$ [21], where the non-deterministic reduction arises from terms like $t_1$ or $t_2$ with $t_1$ and $t_2$ of the same type. In contrast, the $L\odot^S$-calculus introduces the non-deterministic reduction as a pair destructor: $\pi_1^\odot$ and $\pi_2^\odot$ serve as deterministic pair destructors, while $\delta^\odot$ is non-deterministic. That is, $\delta^\odot([t_1, t_2], x.s_1, y.s_2)$ reduces to either $(t_1/x)s_1$ or $(t_2/x)s_2$. Consequently, the non-deterministic behaviour is explicit in its elimination and is not triggered by an introduction term. This approach also allows for a choice among elements of different types.

In the present paper, our aim is to provide an abstract categorical characterization for a proof-language of IMALL with $\odot$. IMALL with $\odot$ is essentially IMALL, as the sup connective can be regarded as an additive conjunction, with an extra rule that is derivable by more than one deduction tree—resulting in non-determinism. Further technical details are presented in Remark 2.5, following the presentation of the deduction rules.

PCF$^R$ [21] not only addresses the non-determinism, with its or constructor, but also the probabilistic choice, with the $\bullet$ constructor. Hence, $(p \bullet t_1)$ or $(q \bullet t_2)$ expresses the probabilistic choice between $t_1$ and $t_2$, with probabilities $p$ and $q$ respectively. In fact, it is slightly more general than a probabilistic choice since the scalars belong to the continuous semiring $\mathbb{R}$. In the case of $\mathcal{R} = \mathbb{R}^{\geq 0}$, it is a proper probabilistic calculus. We will refer to this as “generalised probabilistic choice”.

We generalise the $L\odot^S$-calculus to the $L\odot^{Sp}$-calculus, where instead of considering the non-deterministic destructor $\delta^\odot$, we employ a (generalised) probabilistic destructor $\delta_{pq}^\odot$, with $p$ and $q$ scalars in the semiring $\mathcal{S}$ summing to one. What PCF$^R$ expresses as $(p \bullet t_1)$ or $(q \bullet t_2)$ can be written in the $L\odot^{Sp}$-calculus as $\delta_{pq}^\odot([t_1, t_2], x.x, y.y)$. This expression is interpreted similarly if we choose their category of “weighted relations” [21] as a concrete example of our abstract categorical construction. Nonetheless, we can also write the term $(p \bullet t_1) + (q \bullet t_2)$, which carries the same interpretation but does not have a probabilistic reduction. Instead, it represents a linear combination of terms.
that enables us to express linear functions (matrices) and vectors. Thus, the $\mathcal{L} \odot \text{Sp}$-calculus uses the sums and scalar product provided by its model not only to express probabilistic reductions but also to denote sums and scalar products within the proof language.

1.2. Modelling the sup connective

Introducing a (generalised) probabilistic operator to a linear language is not straightforward. We begin the informal analysis of this section with the concrete category $\text{SM}_S$ of semimodules over the semiring $S$ and linear maps, as a means to aid intuition. Such a category is one of the concrete construction examples we will use throughout the paper.

Our interpretation does not use the Powerset Monad, as it is usual to express non-deterministic effects [23], since this monad is not valid in this category. Such approach would consist in using the Cartesian product of the several non-deterministic paths, and gather them together into a set. The problem is that the needed map $A \times A \xrightarrow{\xi} \mathcal{P} A$ defined by $\xi(a_1, a_2) = \{a_1, a_2\}$ is not linear and so it is not in the category.

Our approach is instead inspired by the density matrix quantum formalism (see, for example, [24, Section 2.4]), wherein we consider the linear combination of results as a representation of a probability distribution. Let $t$ be a term reducing with probability $p$ to $t_1$ and a probability of $q$ to $t_2$, with $p + q = 1$. We interpret $t$ as $\nabla_{pq}(t_1, t_2) = p \bullet_A t_1 + q \bullet_A t_2$, where, if $\hat{p}$ is the mapping that multiplies its argument by $p$, then $\nabla_{pq}$ is defined as $[\hat{p}, \hat{q}]$, that is

$$
\begin{array}{c}
A \xrightarrow{i_1} A + A \leftarrow A \xrightarrow{i_2} A \\
\downarrow \nabla_{pq} \downarrow \\
A
\end{array}
$$

This approach is closed to that used for PCF$^R$ in [21].

In an abstract categorical setting, this means that we need at least a category with biproducts, so we can interpret $\nabla_{pq}$ as $\nabla \circ (\hat{p} \oplus \hat{q})$, where $\hat{p}$ and $\hat{q}$ are appropriated maps $A \to A$. For those scalar maps, we consider the category to also be monoidal, so we can count with the semiring of scalars $\text{Hom}(I, I)$ [20], where $I$ is the tensor unit. Then, we define a monomorphism $\langle \cdot \rangle : S \to \text{Hom}(I, I)$, which ensures that if two proof-terms are interpreted by the same map, then they are somehow equivalent.
1.3. Related works

The probabilistic choice in linear logic has been studied in many settings.

**Compact closed categories.** In [1], the authors proposed a categorical semantics of quantum protocols using symmetric monoidal closed categories with biproducts, which are also compact. The compactness property provides a notion of dagger, which gives a natural definition of measurements in terms of the *Born rule* in quantum mechanics. Thus, the main difference between our presentation for a model of IMALL+⊙ and their presentation for a model of quantum protocols is their reliance on a dagger operator and their use of the compactness property for this purpose. Remark 3.15 illustrates that some properties would be significantly easier to prove if the category were compact closed. However, assuming compactness would limit the generality of the results.

**Probabilistic coherent spaces.** In [8], based on an idea from Girard [19], the authors proposed a model of linear logic using probabilistic coherence spaces, interpreting types through continuous domains. Morphisms in the associated category are Scott-continuous. Additionally, they provide a probabilistic interpretation of terms, extending PCF with a probabilistic choice construction which selects a natural number from a probability distribution. They show the denotational semantics of closed terms in their base type as sub-probability distributions.

**Cones.** In [26], the author employed the concept of normed cones to provide an interpretation for the probabilities inherent in quantum programming. An abstract cone is analogous to an *R*-vector space, except that scalars are drawn from the set of non-negative real numbers. This idea has been further developed in [17], and then proved to be a model of intuitionistic linear logic in [18]. In addition, it is proved [7] that this model is a conservative extension of the probabilistic coherent spaces.

**Weighted relational models.** In [21], the authors proposed a model of PCF∗—that is, PCF with a probabilistic choice operator—based on the category of weighted relations. The first main difference with our approach is that they have a probabilistic choice operator, while we have a probabilistic pair destructor, as mentioned in the previous sections. The second difference is that they use a concrete model in the category of matrices over a continuous semiring, while we use an abstract categorical model. They also consider a fixed-point operator, which is outside the scope of this paper.
1.4. Contents of the paper

In Section 2, we introduce the $\mathcal{L} \otimes^{Sp}$-calculus, detailing its grammars, deduction and reduction rules, and how to use it to encode matrices and vectors. Then, we state its correctness properties.

In Section 3, we introduce the categorical construction together with some specific maps, such as $\nabla_{pq}$ and $\hat{p}$, which are fundamental to interpreting the language.

Section 4 is dedicated to providing the denotational semantics of the $\mathcal{L} \otimes^{Sp}$-calculus within the category just defined, and establishing its soundness and adequacy proofs.

Finally, in Section 5, we offer some concluding remarks.

2. The $\mathcal{L} \otimes^{Sp}$-calculus

2.1. Grammars

**Definition 2.1** (Propositions of the $\mathcal{L} \otimes^{Sp}$-logic). The propositions of the $\mathcal{L} \otimes^{Sp}$-logic are those of IMALL with $\otimes$.

$$A = 1 \mid A \otimes A \mid A \rightarrow A$$

**multimultiplicative**

$$\mid \top \mid 0 \mid A \& A \mid A \oplus A \mid A \otimes A$$

**additive**

**Remark 2.2.** In intuitionistic linear logic there is no multiplicative falsehood ($\bot$), multiplicative disjunction ($\forall$), nor additive implication ($\Rightarrow$).

**Definition 2.3** (Proof-terms of the $\mathcal{L} \otimes^{Sp}$-calculus). The proof-terms of the $\mathcal{L} \otimes^{Sp}$-calculus are those produced by the following grammar, where $x \in \text{Vars}$, an infinite set of variables, $\mathcal{S}$ is a fixed semiring, $s, p, q \in \mathcal{S}$, and $p + s q = 1_s$.

| introductions | eliminations | connective |
|---------------|--------------|------------|
| $t = x \mid t + t \mid s \bullet t$ | $| s \star \mid \delta_1(t, t)$ | $1$ |
| $\mid \lambda x.t \mid tt$ | $\mid \delta_\otimes(t, x y.t)$ | $\otimes$ |
| $\mid t \otimes t$ | $\mid \delta_\circ(t)$ | $\circ$ |
| $\mid \langle \rangle$ | $\mid \pi_1(t) \mid \pi_2(t)$ | $\&$ |
| $\mid \langle t, t \rangle$ | $\mid \delta_{\otimes}(t, x t, y t)$ | $\oplus$ |
| $\mid \text{inl}(t) \mid \text{inr}(t)$ | $\mid \pi_1^\otimes(t) \mid \pi_2^\otimes(t) \mid \delta_{\otimes}^{pq}(t, x t, y t)$ | $\otimes$ |
The substitution of \(x\) by \(u\) in \(t\) is written \((u/x)t\).

**Definition 2.4 (Proof-term context).** We let \(K\) be a proof-term with a distinguished variable \([\cdot]\). We write \(K[t]\) for \((t/[\cdot])K\), that is, the substitution of \([\cdot]\) by \(t\) in \(K\).

### 2.2. Deduction rules

The deduction rules are given in Figure 1. They include the standard rules of IMALL, plus the extra rules for \(\oplus\), \(\cdot\), and \(\odot\).

**Remark 2.5.** Rules \(\odot_i\), \(\odot_{e1}\), and \(\odot_{e2}\) coincide with \(\&_i\), \(\&_{e1}\), and \(\&_{e2}\). If we use those rules instead, the extra rule \(\odot_e\) could be derivable in IMALL as follows:

\[
\frac{\Gamma \vdash A \& B \quad \Delta \vdash C}{\Gamma, \Delta \vdash C} \quad \text{\(\&_{e1}\)}
\]

or, similarly

\[
\frac{\Gamma \vdash A \& B \quad \Delta \vdash C}{\Gamma, \Delta \vdash C} \quad \text{\(\&_{e2}\)}
\]

The goal of having \(\odot\) instead of just these two derivations is that these two have a deterministic cut-elimination, while \(\odot_e\) makes a non-deterministic choice between the two.

### 2.3. Reduction rules

The reduction rules define a relation between two proof-terms and a scalar in \(S\) (in the particular case of \(S = \mathbb{R}^\geq\), it can be seen as a probabilistic reduction relation). The first group of rules, that we call “beta group” and are presented in Figure 2, are standard, except for those corresponding to the term \(\delta_\odot^{pg}\).

**Remark 2.6.** Continuing with Remark 2.5, if we consider instead of \([t_1, t_2]\), the term \(\langle t_1, t_2 \rangle\), the rule \((\delta_\odot^\ell)\) would be equivalent to

\[
\delta_\ominus (\text{inl}(\pi_1\langle t_1, t_2 \rangle), x.u, y.v) \rightarrow_p (t_1/x)u,
\]

and the rule \((\delta_\odot^r)\) to

\[
\delta_\ominus (\text{inr}(\pi_2\langle t_1, t_2 \rangle), x.u, y.v) \rightarrow_q (t_2/y)v.
\]
The deduction rules \(\oplus\) and \(\bullet(s)\) allows building proofs that cannot be reduced because the introduction rule of some connective and its elimination rule are separated by an interstitial rule. For example,

\[
\begin{array}{c}
\Gamma \vdash A \\
\Xi_1
\end{array} \quad \begin{array}{c}
\Gamma \vdash A \\
\Xi_2
\end{array} \quad \begin{array}{c}
\Gamma \vdash A \\
\Xi_3
\end{array} \quad \begin{array}{c}
\Gamma \vdash A \\
\Xi_4
\end{array} \quad \begin{array}{c}
\Gamma, \Delta \vdash C \\
\Xi_5
\end{array}
\]

Reducing such a proof, sometimes called a commuting cut, requires reduction rules to commute the rule sum either with the elimination rule below or with

\[
\begin{array}{c}
\Gamma \vdash C \\
\Xi_6
\end{array}
\]

\[\Gamma \vdash C \]
Figure 2: The beta group of reduction rules of the $\mathcal{L}_p$-calculus.

As the commutation with the introduction rules above is not always possible, for example in the proof

\[
\frac{\Gamma \vdash A \quad \pi_1}{\Gamma \vdash A \oplus B \oplus_1} \quad \frac{\Gamma \vdash B \quad \pi_2}{\Gamma \vdash A \oplus B \oplus_2}
\]

the commutation with the elimination rule below is often preferred. However, in the $\mathcal{L}_p$-calculus, the commutation of the interstitial rules with the introduction rules is chosen, rather than with the elimination rules, whenever it is possible, that is for all connectives except the disjunction and the tensor. For example, the proof

\[
\frac{\pi_3}{\Gamma \vdash A \& B \&_i} \quad \frac{\pi_4}{\Gamma \vdash A \& B \&_i}
\]

\[
\frac{\pi_1 \quad \pi_2}{\Gamma \vdash A \& B \& \oplus \&_i} \quad \frac{\pi_3 \quad \pi_4}{\Gamma \vdash A \& B \& \oplus \&_i}
\]
Figure 3: The commutation group of reduction rules of the $\mathcal{L}_{\odot}^{\mathcal{S}_{\oplus}}$-calculus.

reduces to

$$
\begin{array}{cccc}
\pi_1 & \pi_3 & \pi_2 & \pi_4 \\
\Gamma \vdash A & \Gamma \vdash A & \Gamma \vdash B & \Gamma \vdash B \\
\hline
\Gamma \vdash A \& B & \Gamma \vdash B & \&_i.
\end{array}
$$

Such a choice of commutation yields a stronger introduction property for the considered connective (Theorem 2.10): Most connectives have as closed normal forms, introductions, rather than linear combinations of those. The reduction rules corresponding to these commutations are presented in Figure 3.

2.4. Correctness

The safety properties (subject reduction, confluence, strong normalization, and introduction) have been established in [16] (with the exception for
confluence, which has been proved for the fragment of the calculus without \( \odot \).

**Theorem 2.7** (Subject reduction [16, Theorem 2.2]). If \( \Gamma \vdash t : A \) and \( t \rightarrow_p u \), then \( \Gamma \vdash u : A \).  

**Theorem 2.8** (Confluence [16, Theorem 2.3]). The \( L^{\odot_{Sp}} \)-calculus is confluent if we exclude the rules \( \delta^l_{\odot} \) and \( \delta^r_{\odot} \).

**Theorem 2.9** (Strong normalization [16, Corollary 2.29]). The \( L^{\odot_{Sp}} \)-calculus is strongly normalizing.

**Theorem 2.10** (Introduction [16, Theorem 2.30]). Let \( \vdash t : A \) and \( t \) irreducible.

- If \( A = 1 \), then \( t = \ast \).
- If \( A = B \otimes C \), then \( t = u \otimes v \), \( u \cdot v \), or \( s \cdot u \).
- If \( A = B \rightarrow C \), then \( t = \lambda x.u \).
- If \( A = \top \), then \( t = \langle \rangle \).
- \( A \) cannot be equal to \( \odot \).
- If \( A = B \& C \), then \( t = \langle u, v \rangle \).
- If \( A = B \oplus C \), then \( t = \text{inl}(l) \), \( t = \text{inr}(r) \), \( u \cdot v \), or \( s \cdot u \).
- If \( A = B \odot C \), then \( t = [u, v] \).  

2.5. Vectors and matrices

In this section we replicate some results of [16] for the \( L^S \)-calculus, that is, the fragment of \( L^{\odot_{Sp}} \)-calculus without \( \odot^3 \). These results show that the \( L^{\odot_{Sp}} \)-calculus can be used to encode vectors and matrices, and, moreover, that the sum and scalar product in the syntax represent the sum and scalar product of the elements of a semimodule, and that all the abstractions that we can construct with these symbols are homomorphisms. Please, refer to that paper for a comprehensive treatment of the subject.

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3In fact, we can just remove \( \delta^r_{\odot^3} \), since \( \odot \), without rule \( \odot_e \) from Figure 1 become a second additive conjunction where all the results are still valid.
The set of semimodule propositions $\mathcal{V}$ is inductively defined as follows: $1 \in \mathcal{V}$, and if $A$ and $B$ are in $\mathcal{V}$, then so is $A \& B$. To each proposition $A \in \mathcal{V}$, we associate a positive natural number $d(A)$, which is the number of occurrences of the symbol $1$ in $A$: $d(1) = 1$ and $d(B \& C) = d(B) + d(C)$.

If $A \in \mathcal{V}$ and $d(A) = n$, then the closed irreducible proofs of $A$ and the elements of the semimodule $S^n$ are in one-to-one correspondence: to each closed irreducible proof $t$ of $A$, we associate an element $t$ of $S^n$ and to each element $u$ of $S^n$, we associate a closed irreducible proof $\overline{u}^A$ of $A$.

**Definition 2.11** (One-to-one correspondence [16, Definition 3.6]). Let $A \in \mathcal{V}$ with $d(A) = n$. To each closed irreducible proof $t$ of $A$, we associate an element $t$ of $S^n$ as follows.

- If $A = 1$, then $t = (a \star)$. We let $\overline{t}^A = (a \star)$.
- If $A = A_1 \& A_2$, then $t = (u, v)$. We let $\overline{t}$ be $\overline{t} = (\overline{u}^A, \overline{v}^A)$, where we use the block notation with the convention that if $u = \left(\begin{array}{c} 1 \\ 2 \end{array}\right)$ and $v = \left(\begin{array}{c} 3 \\ 4 \end{array}\right)$, then $\overline{(u)^A} = \left(\begin{array}{c} 1 \\ 3 \end{array}\right)$ and not $\overline{(\overline{u}^A)} = \left(\begin{array}{c} 1 \\ 4 \end{array}\right)$.

To each element $u$ of $S^n$, we associate a closed irreducible proof $\overline{u}^A$ of $A$.

- If $n = 1$, then $u = (a \star)$. We let $\overline{u}^A = (a \star)$.
- If $n > 1$, then $A = A_1 \& A_2$, let $n_1$ and $n_2$ be the dimensions of $A_1$ and $A_2$. Let $u_1$ and $u_2$ be the two blocks of $u$ of $n_1$ and $n_2$ rows, so $u = (\overline{u_1}^A, \overline{u_2}^A)$. We let $\overline{u}^A = (\overline{u_1}^A, \overline{u_2}^A)$.

We extend the definition of $t$ to any closed proof of $A$, $t$ is by definition $t'$ where $t'$ is the irreducible form of $t$.

**Theorem 2.12** (Matrices [16, Theorem 3.10]). Let $A, B \in \mathcal{V}$ with $d(A) = m$ and $d(B) = n$ and let $M$ be a matrix with $m$ columns and $n$ rows, then there exists a closed proof $t$ of $A \rightarrow B$ such that, for all the elements $u$ of $S^m$, we have $\overline{u}^A = Mu$.

The proof of [16, Theorem 3.10] gives the explicit proof-terms representing the matrices. For these details, please refer to that paper.

**Theorem 2.13** (Linearity [16, Corollary 4.12]). Let $A$ and $B$ be propositions, $t$ a closed proof of $A \rightarrow B$ and $u_1$ and $u_2$ be closed proofs of $A$. 

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• If $B \in V$, we have 
\[(t(u_1 \bullet u_2))_\downarrow = (tu_1 \bullet tu_2)_\downarrow \quad \text{and} \quad (t(a \bullet u_1))_\downarrow = (a \bullet tu_1)_\downarrow \]
where $(t)_\downarrow$ is the normal form of $t$.

• In the general case, we have 
\[t(u_1 + u_2) \equiv tu_1 + tu_2 \quad \text{and} \quad t(a \bullet u_1) \equiv a \bullet tu_1 \]
where $\equiv$ is the computational equivalence.

The next corollary is the converse of Theorem 2.12.

**Corollary 2.14** (Linearity [16, Corollary 4.13]). Let $A, B \in V$, such that $d(A) = m$ and $d(B) = n$, and $t$ be a closed proof of $A \rightarrow B$. Then the function $S^m \xrightarrow{f} S^n$, defined as $f(u) = t u^A$ is linear. 

Finally, we can prove that the sum and scalar product in the syntax represent the sum and scalar product of the elements of a semimodule.

**Theorem 2.15** (Syntactic sum and scalar multiplication [16, Lemmas 3.7 and 3.8]). Let $A \in V$, and $u$ and $v$ be two closed proofs of $A$. Then, $u + v = u + v$ and $a \bullet u = au$. 

3. The categorical construction

3.1. Some properties of categories with biproducts

**Definition 3.1.** A semiadditive category is a category with an Abelian monoid structure on each of its morphism sets such that compositions are bilinear and the unit of the monoid is an absorbing element. That is,

• each set $\text{Hom}(A, B)$ of morphisms has a monoid structure;

• $f(g + h) = fg + fh$ for every $B \xrightarrow{f} C$, $A \xrightarrow{g} B$, and $A \xrightarrow{h} B$;

• $(g + h)f = gf + hf$ for every $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, and $B \xrightarrow{h} C$; and

• the unit of each monoid satisfies $0_{BB}f = f0_{AA} = 0_{AB}$ for every $A \xrightarrow{f} B$. 

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**Definition 3.2.** In a category with biproduct, we can define the following operation between maps.

\[
\begin{array}{ccc}
A & \xrightarrow{f+g} & B \\
\downarrow & & \uparrow \\
A \oplus A & \xrightarrow{f \oplus g} & B \oplus B,
\end{array}
\]

for \(A \xrightarrow{f} B\) and \(A \xrightarrow{g} B\), where \(\Delta = (\text{id}, \text{id})\) and \(\nabla = [\text{id}, \text{id}]\).

**Theorem 3.3** (Semiadditive structure [22, Proposition 18.4]). A category with a biproduct has a unique semiadditive structure in the sense of Definition 3.1, where the sum of maps is given by Definition 3.2, and the unit of each monoid is given by the map \(0_{AB}\) defined as \(A \xrightarrow{1} 0 \xrightarrow{1} B\) (where the zero object 0 is due to the biproducts).

**Corollary 3.4** (Semiring). In a category with biproduct, each \(\text{Hom}(A, A)\) of morphisms is a semiring with \(+\) given by Definition 3.2 as additive operation, \(\circ\) as product operator, where \(0_{AA}\) and \(\text{id}_A\) are the units of the addition and product respectively.

**Proof.** Straightforward. \(\square\)

### 3.2. The category \(C_S\)

**Definition 3.5** (The category \(C_S\)). Let \(S\) be a fixed semiring. The category \(C_S\) is a symmetric monoidal closed category with biproduct where there exists a monomorphism from the semiring \(S\) to the semiring \(\text{Hom}(I, I)\), being \(I\) the unit object.

**Notation 3.6.** We write

\([A \rightarrow B]\) for the internal hom between \(A\) and \(B\),

\(\otimes\) for the tensor product,

\(\oplus\) for the biproduct,

\(I\) for the unit object.

The usual coherence maps are noted as follows.

\[
\begin{align*}
A \otimes B & \xrightarrow{\sigma_{A,B}} B \otimes A, \\
A \otimes (B \otimes C) & \xrightarrow{\alpha_{A,B,C}} (A \otimes B) \otimes C, \\
I \otimes A & \xrightarrow{\lambda_A} A, \\
A \otimes I & \xrightarrow{\rho_A} A.
\end{align*}
\]
The usual maps for the biproduct are noted as follows.

\[ A \oplus B \xrightarrow{\pi_1} A, \quad A \oplus B \xrightarrow{\pi_2} B, \quad A \xrightarrow{i_1} A \oplus B, \quad A \xrightarrow{i_2} A \oplus B. \]

Finally, we note \( S \xrightarrow{\cdot} \text{Hom}(I, I) \) the monomorphism.

**Example 3.7.** The following are examples of categories with the properties asked by Definition 3.5.

1. The category \((\text{Rel}, \times, \{\star\}, \cup)\), where objects are sets, arrows are relations, the tensor is the Cartesian product, and the biproduct is the disjoint union, under the condition that \( S = \{\star\} \), otherwise the map from \( S \) to \( \text{Hom}(\{\star\}, \{\star\}) \) would not be injective.
2. The category \((\text{SM}_S, \otimes, S, \oplus)\), where objects are semimodules over the semiring \( S \), arrows are semimodule homomorphisms, the tensor is the semimodules tensor, and the biproduct is Cartesian product. The map \( \langle s \rangle \mapsto s' \mapsto s \cdot s' \).
   Our first model for the \( \mathcal{L}^S \)-calculus has been given in this category in a previous draft [15].
3. The category \((\text{CPM}, \otimes, 1, \times)\), where objects are the lists of natural numbers, arrows are matrices over the continuous semiring \( R \), the composition is the matrix product, the tensor is the tensor of vector spaces, and the biproduct is the Cartesian product. In this category \( I = 1 \) and \( \text{Hom}(I, I) \simeq R_{\geq 0} \), so any monomorphism from \( S \) to \( R_{\geq 0} \) is enough. For the \( \mathcal{L}_{CP}^{R_{\geq 0}} \)-calculus, we can take the identity.
   This category has been defined in [27] and used to model quantum computing in [25].
4. The category \((\mathcal{R}^\Pi, \times, \{\star\}, \oplus)\), where objects are sets, arrows are matrices over the continuous semiring \( \mathcal{R} \), the composition is the matrix product, the tensor is the Cartesian product, and the biproduct is the disjoint union. In this category we have \( I = \{\star\} \) and \( \text{Hom}(I, I) \simeq \mathcal{R} \), so any monomorphism from \( S \) to \( \mathcal{R} \) is enough. For the \( \mathcal{L}_{\otimes}^{\mathcal{R}_{\geq 0}} \)-calculus, we can take the identity.
   This category has been defined and used to model PCF\(^R \), a probabilistic extension of PCF, in [21].

Notice that the category \((\text{Pcoh}, \otimes, (\{\star\}, [0, 1]))\) where objects are probabilistic coherent spaces and arrows are given by matrices, used in [8] to model a probabilistic extension of PCF is not an example of our construction since it
does not have a biproduct. Indeed, the interpretation of $1$ is $(\{\star\}, [0, 1])$ and so both $1 \& 1$ and $1 \oplus 1$ have the same web $\{0, 1\}$ but $P(1 \& 1) = [0, 1] \times [0, 1]$ whereas $P(1 \oplus 1) = \{(\alpha, \beta) \in [0, 1] \times [0, 1] : \alpha + \beta \leq 1\}$.

**Definition 3.8.** A semiadditive functor is a functor preserving the monoid structure on each hom.

**Lemma 3.9.** Let $F : \mathcal{C}_S \to \mathcal{C}_S$ be a semiadditive functor. Then, $F(A) \oplus F(B) \cong F(A \oplus B)$.

**Proof.** Given in Appendix B.

**Corollary 3.10 (Distributions).** In monoidal closed categories with biproduct, there exists the following natural transformations.

1. $(A \oplus B) \otimes C \xrightarrow{d} (A \otimes C) \oplus (B \otimes C)$ with $d = (\pi_1 \otimes \text{id}_C, \pi_2 \otimes \text{id}_C)$.
2. $(A \otimes C) \oplus (B \otimes C) \xrightarrow{d^{-1}} (A \oplus B) \otimes C$ with $d^{-1} = [i_1 \otimes \text{id}, i_2 \otimes \text{id}]$.
3. $[A \to B \oplus C] \xrightarrow{\gamma} [A \to B] \oplus [A \to C]$ with $\gamma = ([A \to \pi_1], [A \to \pi_2])$.
4. $[A \to B] \oplus [A \to C] \xrightarrow{\gamma^{-1}} [A \to B \oplus C]$ with $\gamma^{-1} = [[A \to i_1], [A \to i_2]]$.

**Proof.** Direct consequence of Lemma 3.9.

3.3. The map $\hat{s}$

**Lemma 3.11 (Scalar map).** Let $I \xrightarrow{s} I$. The map $A \xrightarrow{\hat{s}_A} A$ defined by $\hat{s}_A = \rho_A \circ (\text{id} \otimes s) \circ \rho_A^{-1}$, is a natural transformation.

**Proof.** Given in Appendix C.

**Lemma 3.12 (Some properties of the scalar map).**

1. $\hat{s}_I = s$.
2. $\hat{s}_{A \otimes B} = \hat{s}_A \otimes \text{id}_B$.
3. $\hat{s}_{A \oplus B} = \hat{s}_A \oplus \hat{s}_B$.

**Proof.** Given in Appendix D.

Property 2 of Lemma 3.12 can be rephrased to $F(\hat{s}) = \hat{s}$, in the particular case of $F$ being the functor $- \otimes B$. If we change the functor to be $[A \to -]$, the property, which would be stated as $[A \to \hat{s}] = \hat{s}$ is more subtle to prove. We do this in Lemma 3.14, but on its proof we need to use the map $\tau$ associated with the adjunction between the tensor product and the hom, and its naturality with respect to $I$ (Lemma 3.13).
Lemma 3.13 (The map $\tau$). The following map in the arrows of $C_S$ is a natural transformation with respect to $I$.

$$\tau = [A \rightarrow B] \otimes I \xrightarrow{\varphi_{A,[A \rightarrow B] \otimes I,B \otimes I}(\varepsilon \otimes \text{id})} [A \rightarrow B \otimes I],$$

where $\varphi_{A,[A \rightarrow B] \otimes I,B \otimes I}$ is the map given by the adjunction

$$\Hom(X \otimes Y, Z) \xleftarrow{\varphi_{X,Y,Z}} \Hom(Y, [X \rightarrow Z]),$$

by taking $X = A$, $Y = [A \rightarrow B] \otimes I$, and $Z = B \otimes I$.

Proof. Given in Appendix E. \qed

Lemma 3.14. $[A \rightarrow \hat{s}_B] = \hat{s}_{[A \rightarrow B]}$.

Proof. Consequence of the commutation of the following diagram.

\[
\begin{array}{ccc}
[A \rightarrow B] & \xrightarrow{[A \rightarrow \rho]} & [A \rightarrow B \otimes I] \\
\downarrow{\rho} & \xrightarrow{(*)} & \downarrow{[A \rightarrow \text{id} \otimes s]} \\
[A \rightarrow B] \otimes I & \xrightarrow{\text{id} \otimes s} & [A \rightarrow B \otimes I] \\
\downarrow{\tau} & \xrightarrow{(**)} & \downarrow{[A \rightarrow \rho^{-1}]} \\
[A \rightarrow B] \otimes I & \xrightarrow{\rho^{-1}} & [A \rightarrow B]
\end{array}
\]

The commutation of the diagram $(*)$ is proved by an equivalent diagram, obtained through the adjunction of Lemma 3.13, taking $X = A$, $Y = [A \rightarrow B]$, and $Z = B \otimes I$. Beware, these are not the same variables taken in the
The resulting diagram is as follows.

\[
\begin{array}{ccc}
A \otimes [A \to B] & \xrightarrow{\cdot \id \otimes [A \to B]} & A \otimes [A \to B \otimes I] \\
\xrightarrow{\varepsilon} & B \otimes I \\
\xrightarrow{\rho} & A \otimes [A \to B \otimes I] \otimes \id \tau & \xrightarrow{\cdot \id \otimes [A \to B \otimes I]} & A \otimes [A \to B \otimes I]
\end{array}
\]

The commutation of the diagram (**) is also proved by an equivalent diagram, obtained through the adjunction of Lemma 3.13, taking this time the same variables as in the definition of \(\tau\): \(X = A, Y = [A \to B] \otimes I,\) and \(Z = B \otimes I.\)

**Remark 3.15.** In order to prove the Lemma 3.14 we needed the natural transformation \(\tau\) coming from the adjunction given by the fact that the category is assumed to be closed. Notice, however, that the property is almost trivial in the case of compact categories.
Also, if $F = [A \to -]$ were a monoidal functor, the property could have been easily proven by the following diagram.

\[
\begin{array}{ccc}
F(A) & \xrightarrow{F(\rho)} & F(A \otimes I) \\
\downarrow \text{(Monoidality axiom)} & & \downarrow \text{(Naturality of } m) \downarrow \\
F(A) \otimes F(I) & \xrightarrow{\text{id} \otimes F(s)} & F(A) \otimes F(I) \\
\downarrow \text{(Monoidality axiom)} & & \downarrow \text{(Naturality of } m) \downarrow \\
F(A) \otimes I & \xrightarrow{\text{id} \otimes s} & F(A) \otimes I
\end{array}
\]

3.4. The map $\nabla_{pq}$

The map $\nabla_{pq}$ is the key map mentioned in the introduction. Its related map $\Delta_{pq}$ is not needed for the interpretation, instead, we need the usual diagonal map $\Delta$, since $\nabla_{pq}$, for some particular $(p, q)$, are left inverses of $\Delta$, as shown by Lemma 3.24.

**Lemma 3.16 (Weighted codiagonal).** The map $A \oplus A \xrightarrow{\nabla_{pq}} A$ defined by $\nabla_{pq} = [\hat{p}, \hat{q}]$ is a natural transformation.

**Proof.** Given in Appendix F. \qed

**Lemma 3.17.** Let $F$ be a semiadditive functor such that $F(\hat{s}) = \hat{s}$. Then,

$$\nabla_{pq} \circ \langle F(\pi_1), F(\pi_2) \rangle = F(\nabla_{pq}).$$

**Proof.** We must show that $F(\nabla_{pq}) = \nabla_{pq} \circ \langle F(\pi_1), F(\pi_2) \rangle = F(\nabla_{pq})$. We show equivalently (cf. Appendix B), that $F(\nabla_{pq}) \circ [F(i_1), F(i_2)] = \nabla_{pq}$.

$$F(\nabla_{pq}) \circ [F(i_1), F(i_2)] = F([\hat{p}, \hat{q}] \circ [F(i_1), F(i_2)])$$

$$= [F([\hat{p}, \hat{q}] \circ i_1), F([\hat{p}, \hat{q}] \circ i_2)]$$

$$= [F(\hat{p}) \circ i_1, F(\hat{q}) \circ i_2]$$

$$= [\hat{p}, \hat{q}]$$

$$= \nabla_{pq} \quad \Box$$

**Corollary 3.18.**
1. $\nabla_{pq} \circ d = \nabla_{pq} \otimes B$.
2. $\nabla_{pq} \circ \gamma = [A \to \nabla_{pq}]$.
3. $\nabla \circ d = \nabla \otimes B$.
4. $\nabla \circ \gamma = [A \to \nabla]$.

Where $d$ and $\gamma$ are the distribution maps of Corollary 3.10.

Proof.

- Items 1 and 2: It is straightforward to check that both $- \otimes B$ and $[A \to -]$ are semiadditive functors. Thus, by Lemmas 3.12.2 and 3.14, these functors meet the conditions of Lemma 3.17, which concludes the proof.

- Items 3 and 4: These are particular cases of Items 1 and 2, respectively, since $\nabla = [\text{id}, \text{id}] = [\hat{\text{id}}, \hat{\text{id}}] = \nabla_{\text{id}, \text{id}}$.

Analogously to Lemma 3.17, we can state and prove the following lemma for $\Delta$, with a similar corollary to Corollary 3.18. Remark that in Lemma 3.19 we do not need the hypothesis $F(\hat{s}) = \hat{s}$, since we only need to make use of the trivial property $F(\text{id}) = \text{id}$.

Lemma 3.19. Let $F$ be a semiadditive functor. Then,

$$[F(i_1), F(i_2)] \circ \Delta = F(\Delta).$$

Proof. We must show that $F(\Delta) = [F(i_1), F(i_2)] \circ \Delta$. We show equivalently (cf. Appendix B), that $\langle F(\pi_1), F(\pi_2) \rangle \circ F(\Delta) = \Delta$.

$$\langle F(\pi_1), F(\pi_2) \rangle \circ F(\Delta) = \langle F(\pi_1), F(\pi_2) \rangle \circ F(\langle \text{id}, \text{id} \rangle)$$
$$= \langle F(\pi_1) \circ F(\langle \text{id}, \text{id} \rangle), F(\pi_2) \circ F(\langle \text{id}, \text{id} \rangle) \rangle$$
$$= \langle F(\pi_1 \langle \text{id}, \text{id} \rangle), F(\pi_2 \langle \text{id}, \text{id} \rangle) \rangle$$
$$= \langle F(\text{id}), F(\text{id}) \rangle$$
$$= \langle \text{id}, \text{id} \rangle$$
$$= \Delta$$

Corollary 3.20.

1. $d^{-1} \circ \Delta = \Delta \otimes \text{id}$.
2. $\gamma^{-1} \circ \Delta = [A \to \Delta]$. 

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Where \( d \) and \( \gamma \) are the distribution maps of Corollary 3.10.

Proof. It is straightforward to check that both \(- \otimes B\) and \([ A \to -] \) are semi-additive functors. Thus, we conclude by Lemma 3.19.

The usual extension of \( \Delta \) and \( \nabla \) to more general objects is also valid for \( \nabla_{pq} \). The next lemma shows this, for \( \Delta \) and \( \nabla_{pq} \), which are the only cases we need.

Lemma 3.21.

1. \( (\nabla_{pq} \oplus \nabla_{pq}) \circ (\text{id} \oplus \sigma \oplus \text{id}) = \nabla_{pq} \).
2. \( (\text{id} \oplus \sigma \oplus \text{id}) \circ (\Delta \oplus \Delta) = \Delta \).

Proof. Given in Appendix G.

3.5. The set \( W \)

Definition 3.22. \( W = \{(p, q) \in \text{Hom}(I, I) \times \text{Hom}(I, I) : p + q = \text{id}_I\} \).

Example 3.23.

1. In the category \( \text{Rel} \), \( W = \{ (\emptyset, \text{id}), (\text{id}, \emptyset), (\text{id}, \text{id}) \} \), where \( \emptyset \) is the empty relation.
   Indeed, there are only two elements in \( \text{Hom}(I, I) \), which are \( \emptyset \) and \( \text{id} \), and we can check that for \( s_1, s_2 \in \{\emptyset, \text{id}\} \), the equation \( \nabla \circ (s_1 \oplus s_2) \circ \Delta = \text{id} \) is non-valid in the case \( (\emptyset, \emptyset) \), and it is valid in the other cases.
   First, notice that \( I \oplus I = \{T, F\} \) with \( T = (\ast, 0) \) and \( F = (\ast, 1) \). Thus, \( I \Delta I \oplus I \) is the relation \( \{ (\ast, T), (\ast, F) \} \). In the same way, \( I \Delta I \nabla I \) is the relation \( \{ (T, \ast), (F, \ast) \} \).
   Now, we can analyse the four cases:
   - Let \( s_1 = s_2 = \emptyset \). In this case \( \nabla \circ (\emptyset \oplus \emptyset) \circ \Delta = \nabla \circ \emptyset \circ \Delta = \emptyset \neq \text{id} \).
   - Let \( s_1 = \emptyset, s_2 = \text{id} \). In this case,
     \[
     \nabla \circ (\emptyset \oplus \text{id}) \circ \Delta
     = \{ (T, \ast), (F, \ast) \} \circ \{ (F, F) \} \circ \{ (\ast, T), (\ast, F) \}
     = \{ (T, \ast), (F, \ast) \} \circ \{ (F, F) \}
     = \{ (\ast, \ast) \}
     = \text{id}.
     \]
• Let \( s_1 = \text{id}, \ s_2 = \emptyset \) Analogous to the previous case.
• Let \( s_1 = s_2 = \text{id} \). In this case,
  \[
  \nabla \circ (\text{id} \oplus \text{id}) \circ \Delta
  = \{(T, \star), (F, \star)\} \circ \{(T, T), (F, F)\} \circ \{(\star, T), (\star, F)\}
  = \{(T, \star), (F, \star)\}
  = \{(\star, \star)\}
  = \text{id}.
\]

2. In the category \( \text{SM}_S \), \( \mathcal{W} = \{(p, q) \in \mathcal{S}^2 : p + S q = 1_S\} \).
3. In the category \( \text{CPM} \), \( \mathcal{W} = \{(f, g) \in \mathcal{C}^2 : f, g \in \mathbb{R}_{\geq 0} \text{ and } f + g = 1\} \), that is, the stochastic vectors in \( \mathcal{C}^2 \).
4. In the category \( \mathcal{R}_\Pi \), \( \mathcal{W} = \{(p, q) \in \mathcal{R}^2 : p + \mathbb{R} q = 1_\mathbb{R}\} \).

The relation between \( \Delta \) and \( \nabla_{pq} \) has become evident in Section 3.4. However, the next lemma goes a bit further, showing that in the particular cases of \( (p, q) \in \mathcal{W} \), \( \nabla_{pq} \) are the left inverses of \( \Delta \).

**Lemma 3.24.** If \( (p, q) \in \mathcal{W} \), then \( \nabla_{pq} \circ \Delta = \text{id}_A \).

**Proof.** Consequence of the commutation of the following diagram.

\[\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \oplus A \xrightarrow{\rho \oplus \rho} (A \otimes I) \oplus (A \otimes I) \\
& & \downarrow \rho' \\
A & & (p, q) \in \mathcal{W} \\
& & \downarrow \rho \\
A \otimes I & \xrightarrow{id} & (A \otimes I) \oplus (A \otimes I) \\
& & \downarrow \rho \\
A & \xrightarrow{\nabla} & A \oplus A \xrightarrow{\rho^{-1} \oplus \rho^{-1}} (A \otimes I) \oplus (A \otimes I) \\
\end{array}\]
Remark 3.25. In the categories $\text{SM}_{\mathbb{R}^+}$ and $((\mathbb{R}^+)^\Pi$, where $W = \{(p, q) : p + q = 1\}$, we have $\nabla_{pq}(a_1, a_2) = p.a_1 + q.a_2$, that is, a probability distribution. In this particular case, Lemma 3.24 simply states that $p.a + (1 - p).a = a$.

3.6. The map $\delta$

In Lemma 3.26 we introduce a natural transformation $\delta$, which shares some similarity with the map $d$ in its interaction with the map $\nabla_{pq}$. The property from Corollary 3.18 has an analogy with $\delta$ when $(p, q) \in W$, as shown by Lemma 3.27. However, its proof does not use Lemma 3.17, since the functor $F = - \oplus B$ does not satisfy the hypothesis $F(\hat{s}) = \hat{s}$ needed by Lemma 3.17. The same analogy applies to $\Delta$ with Corollary 3.20, as shown by Lemma 3.28.

Lemma 3.26. The map $(A \oplus B) \oplus C \xrightarrow{\delta} (A \oplus C) \oplus (B \oplus C)$ defined by

$$(A \oplus B) \oplus C \xrightarrow{id \oplus \Delta} (A \oplus B) \oplus (C \oplus C) \xrightarrow{id \oplus \sigma \oplus id} (A \oplus C) \oplus (B \oplus C),$$

is a natural transformation.

Proof. The maps $\sigma$ and $\Delta$ are natural, thus, $\delta$ is natural. $\square$

Lemma 3.27. If $(p, q) \in W$, then $\nabla_{pq} \circ \delta = \nabla_{pq} \oplus \text{id}$.

Proof. Given in Appendix H. $\square$

Lemma 3.28. $\Delta = \delta \circ (\Delta \oplus \text{id})$.

Proof. Given in Appendix I. $\square$

The next property is about the iteration of two weighted codiagonal maps.

Lemma 3.29. If $(p, q) \in W$, then $\nabla_{p'q'} \circ (\nabla_{pq} \oplus \text{id}) = \nabla_{pq} \circ (\nabla_{p'q'} \oplus \nabla_{p'q'}) \circ \delta$.

Proof. Consequence of the commutation of the following diagram.

\[
\begin{array}{cccccccc}
(A \oplus A) \oplus A & \xrightarrow{id \oplus \Delta} & (A \oplus A) \oplus (A \oplus A) & \xrightarrow{id \oplus \sigma \oplus id} & (A \oplus A) \oplus (A \oplus A) \\
\downarrow \nabla_{pq} \oplus \text{id} & & \downarrow \nabla_{pq} \oplus \nabla_{pq} & & \downarrow \nabla_{pq} \oplus \nabla_{pq} \\
A \oplus A & \xrightarrow{\nabla_{p'q'}} & A & \xleftarrow{\nabla_{pq}} & A \oplus A
\end{array}
\]

(Lemma 3.24) (Lemma 3.21) (Lemma 3.16) $\square$
4. Denotational semantics

4.1. Definitions and properties

In this section, we give an interpretation of the $\mathcal{L} \otimes S^n$-calculus in the category $C_S$. The interpretation of types and contexts are standard, interpreting the $\otimes$ as the biproduct.

**Definition 4.1** (Interpretation of propositions). We consider the following interpretation of propositions in the objects of $C_S$.

\[
\begin{align*}
[1] &= I \\
[A \otimes B] &= [A] \otimes [B] \\
[A \rightarrow B] &= [[A] \rightarrow [B]] \\
[\top] &= 0 \\
[A \& B] &= [A] \oplus [B] \\
[A \oplus B] &= [A] \oplus [B] \\
[A \odot B] &= [A] \oplus [B]
\end{align*}
\]

**Definition 4.2** (Interpretation of contexts).

\[
\begin{align*}
[\emptyset] &= I \\
[\Gamma, x : A] &= [\Gamma] \otimes [A]
\end{align*}
\]

**Definition 4.3** (Interpretation of deduction rules). We consider the following interpretation of proof-terms in the arrows of $C_S$, where $(s)$ is the interpretation of the scalar $s$ in $\text{Hom}(I, I)$ (see Definition 3.5).

Since the deduction system is syntax directed (cf. Figure 1), we give instead an interpretation for each deduction rule.

\[
\begin{align*}
\Gamma \vdash t : A & \quad \text{(ax)} \\
\Gamma \vdash t : A & \quad \text{(s)} \\
\Gamma \vdash t : A & \quad \text{(\otimes)} \\
\Gamma \vdash t : A & \quad \text{(\circ)} \\
\Gamma \vdash t : A & \quad \text{(\odot)} \\
\Gamma \vdash t : A & \quad \text{(\circlearrowright)} \\
\Gamma \vdash t : A & \quad \text{(\odot)}
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash t : A & \quad \text{(ax)} \\
\Gamma \vdash t : A & \quad \text{(s)} \\
\Gamma \vdash t : A & \quad \text{(\otimes)} \\
\Gamma \vdash t : A & \quad \text{(\circ)} \\
\Gamma \vdash t : A & \quad \text{(\odot)} \\
\Gamma \vdash t : A & \quad \text{(\circlearrowright)} \\
\Gamma \vdash t : A & \quad \text{(\odot)}
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash t : A & \quad \text{(ax)} \\
\Gamma \vdash t : A & \quad \text{(s)} \\
\Gamma \vdash t : A & \quad \text{(\otimes)} \\
\Gamma \vdash t : A & \quad \text{(\circ)} \\
\Gamma \vdash t : A & \quad \text{(\odot)} \\
\Gamma \vdash t : A & \quad \text{(\circlearrowright)} \\
\Gamma \vdash t : A & \quad \text{(\odot)}
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash t : A & \quad \text{(ax)} \\
\Gamma \vdash t : A & \quad \text{(s)} \\
\Gamma \vdash t : A & \quad \text{(\otimes)} \\
\Gamma \vdash t : A & \quad \text{(\circ)} \\
\Gamma \vdash t : A & \quad \text{(\odot)} \\
\Gamma \vdash t : A & \quad \text{(\circlearrowright)} \\
\Gamma \vdash t : A & \quad \text{(\odot)}
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash t : A & \quad \text{(ax)} \\
\Gamma \vdash t : A & \quad \text{(s)} \\
\Gamma \vdash t : A & \quad \text{(\otimes)} \\
\Gamma \vdash t : A & \quad \text{(\circ)} \\
\Gamma \vdash t : A & \quad \text{(\odot)} \\
\Gamma \vdash t : A & \quad \text{(\circlearrowright)} \\
\Gamma \vdash t : A & \quad \text{(\odot)}
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash t : A & \quad \text{(ax)} \\
\Gamma \vdash t : A & \quad \text{(s)} \\
\Gamma \vdash t : A & \quad \text{(\otimes)} \\
\Gamma \vdash t : A & \quad \text{(\circ)} \\
\Gamma \vdash t : A & \quad \text{(\odot)} \\
\Gamma \vdash t : A & \quad \text{(\circlearrowright)} \\
\Gamma \vdash t : A & \quad \text{(\odot)}
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash t : A & \quad \text{(ax)} \\
\Gamma \vdash t : A & \quad \text{(s)} \\
\Gamma \vdash t : A & \quad \text{(\otimes)} \\
\Gamma \vdash t : A & \quad \text{(\circ)} \\
\Gamma \vdash t : A & \quad \text{(\odot)} \\
\Gamma \vdash t : A & \quad \text{(\circlearrowright)} \\
\Gamma \vdash t : A & \quad \text{(\odot)}
\end{align*}
\]
\[
\begin{align*}
\Gamma, x : A \vdash t : B & \quad \vdash \eta_{\eta_1} \quad \vdash [A] \to [\Gamma] \otimes [A] \quad \vdash [\eta_{\eta_1}] \to [\Gamma] \to [B] \\
\Gamma \vdash \lambda x.t : A \to B & \quad \vdash \lambda \Delta \vdash [\Delta] \to [A] \\
\Gamma, \Delta \vdash t : A \to B, \Delta \vdash u : A & \quad \vdash [\Delta] \to [B] \\
\Gamma, \Delta \vdash tu : B & \quad \vdash [\Delta] \otimes [A] \to [B] \\
\end{align*}
\]
Remark 4.4. The interpretation of deduction rules is mostly standard, having \( \circ_e, \circ_{e_1}, \) and \( \circ_{e_2} \) interpreted as if they were the rules for the additive conjunction \&_{i}, \&_{e_1}, \) and \( \&_{e_2} \) respectively. However, even if the rule \( \circ_e \) looks quite similar to the elimination of the additive disjunction, \( \oplus_e \), its interpretation has a slight but important difference: instead of applying the mediating arrow of the coproduct \([A] \oplus [B] \xrightarrow{[u,v]} [C]\), which is equivalent to
\[
[A] \oplus [B] \xrightarrow{u \oplus v} [C] \oplus [C] \xrightarrow{\nabla} [C],
\]
we use
\[
[A] \oplus [B] \xrightarrow{u \oplus v} [C] \oplus [C] \xrightarrow{\nabla pq} [C].
\]

4.2. Soundness

Our interpretation is sound (Theorem 4.6) with respect to reduction.

Lemma 4.5 (Substitution). If \( \Gamma, x : A \vdash t : B \) and \( \Delta \vdash v : A \), then
\[
[\Gamma \vdash (v/x)t : B] = [\Gamma, x : A \vdash t : B] \circ (id \otimes [\Gamma \vdash v : A]).
\]
That is, the following diagram commutes.

\[
\begin{array}{ccc}
[\Gamma] \otimes [\Delta] & \xrightarrow{(v/x)t} & [B] \\
\downarrow{id \otimes v} & & \\
[\Gamma] \otimes [A] & \xrightarrow{1} & [B]
\end{array}
\]

Proof. By induction on \( t \). Given in Appendix J.

Theorem 4.6 (Soundness). Let \( \Gamma \vdash t : A \).

- If \( t \xrightarrow{\delta^\ell} r \), by any rule but \( (\delta^\ell_e) \) and \( (\delta^r_e) \), then
  \[
  [\Gamma \vdash t : A] = [\Gamma \vdash r : A].
  \]

- If \( t \xrightarrow{p} r_1 \) by rule \( (\delta^\ell_e) \) and \( t \xrightarrow{q} r_2 \) by rule \( (\delta^r_e) \), then
  \[
  [\Gamma \vdash t : A] = \nabla_{(p,q)} \circ ( [\Gamma \vdash r_1 : A] \oplus [\Gamma \vdash r_2 : A]) \circ \Delta,
  \]
  that is,
  \[
  \begin{array}{ccc}
  [\Gamma] & \xrightarrow{\nabla_{(p,q)}} & [A] \\
  \downarrow{\Delta} & & \\
  [\Gamma] \oplus [\Gamma] & \xrightarrow{r_1 \oplus r_2} & [A] \oplus [A].
  \end{array}
  \]

Proof. By induction on the relation \( \xrightarrow{p} \), using the properties proven in the previous section. The full details are given in Appendix K.
4.3. Adequacy

As usual, our interpretation is not complete with respect to the reduction relation because we do not consider eta-rules. For example, \( \vdash \lambda x.\langle \pi_1 x, \pi_2 x \rangle : A \& B \rightarrow A \& B \) has the same interpretation as \( \vdash \lambda x.x : A \& B \rightarrow A \& B \), but one does not reduce to the other. Thus, we can only expect it to be adequate with respect to an observational equivalence.

In addition, we have chosen to not distinguish in the semantics certain situations. For example, let \( S = \mathbb{R}^2, t = \delta_{\odot}^2 ([1_2 \cdot \star, 1_2 \cdot \star], x.x, y.y) \) and \( u = \delta_{\odot}^2 ([3_4 \cdot \star, 1_4 \cdot \star], x.x, y.y) \). Then \( t \) reduces with probability 1 to \( 1_2 \cdot \star \), while \( u \) reduces with equal probability to \( 3_4 \cdot \star \) and to \( 1_4 \cdot \star \). However,

\[
\llbracket \vdash t : 1 \rrbracket = \frac{1}{2} \cdot \frac{\hat{1}}{2} + \frac{1}{2} \cdot \frac{\hat{1}}{2} = \frac{\hat{1}}{2} = \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{4} = \llbracket \vdash u : 1 \rrbracket
\]

Indeed, we have drawn inspiration from the density matrices formalism in quantum physics. In this formalism, the mixed state produced by the equal distribution of pure states \( |0\rangle \) and \( |1\rangle \), and the mixed state produced by the equal distribution of pure states \( |+\rangle \) and \( |-\rangle \), are indistinguishable.

Thus, the adequacy result (Theorem 4.16) is established in relation to a “mixed computational equivalence” (Definition 4.14), equating terms as \( t \) and \( u \) from the previous example.

**Definition 4.7** (Elimination context). An elimination context is a typed proof-term context (cf. Definition 2.4) produced by the following grammar.

\[
K := \cdot | \delta(K, xy.t) | Kt | \pi_1(K) | \pi_2(K) | \delta_{\otimes}(K, x.t, y.u) | \pi_1^{\otimes}(K) | \pi_2^{\otimes}(K),
\]

where in \( \delta_{\otimes}(K, xy.t) \), and \( \delta_{\otimes}(K, x.t, y.u) \), the proposition proved by \( K \) is strictly bigger than that proved by \( t \) and \( u \).

To distinguish between two programs, we can require that these programs, when placed in any elimination context of a certain type, produce the same outputs. In particular, that type must admit more than one closed value, which is the case with \( 1 \), when there are more than one element in \( S \), as all the proof-terms \( s_\star \) are proofs of \( 1 \). This renders our adequacy result unsuitable for the category \textbf{Rel}, which serves as a model of the \textbf{L}_\otimes^{Sp}-calculus only in the case of \( S = \{ \star \} \) (see Example 3.7.1). However, the case \( S = \{ \star \} \) is a degenerate case, as all the rules from Figure 3 become trivial, and we may be able to find a simpler model for that particular case than the one presented in this paper.

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**Definition 4.8.** A basic proposition $\tau$ is either $1$ or $\top$.

**Definition 4.9.** Let $P = [t_0, \ldots, t_n]$ be a list of terms. We write $t \rightarrow_P^p v$ if

$$t = t_0 \rightarrow_{p_1} t_1 \rightarrow_{p_2} \cdots \rightarrow_{p_n} t_n = v,$$

where $n \geq 0$ and $\prod_{i=1}^n p_i = p$.

That is, the product of the probabilities of all the reductions along the path, give us the probability of the path.

**Notation 4.10.** We write $P_v$ for the list $P$ if $v$ is the last element of the list.

**Definition 4.11 (Probabilistic computational equivalence).** Let $\vdash t : A$ and $\vdash u : A$. We say that $t$ and $u$ are probabilistically equivalent, notation $t \sim u$, if for every elimination context $[.] : A \vdash K : \tau$ such that $\tau$ is a basic proposition, we have that $\forall P, K[t] \rightarrow_P^p v$ iff $K[u] \rightarrow_P^p v$.

**Definition 4.12 (Multiset of probability distribution of values of a term).** The multiset of probability distributions of values of a term $t$ is the following multiset of terms,

$$\mathcal{M}_t = \{p \cdot v : t \rightarrow_P^p v\}.$$

**Notation 4.13.** We write $\sum_{t \in T} t$ for the term produced by the constructor $\oplus$ with the terms of the set $T$ taken in a lexicographical order, and associating the parenthesis to the left. For example, let $T = \{t_1, t_2, t_3\}$ with $t_1 < t_2 < t_3$, then,

$$\sum_{t \in T} t = (t_1 \oplus t_2) \oplus t_3.$$

**Definition 4.14 (Mixed computational equivalence).** Let $\vdash t : A$ and $\vdash u : A$. We say that $t$ and $u$ are mixed computational equivalent, notation $t \approx u$, if

$$\sum_{t' \in \mathcal{M}_t} t' \sim \sum_{u' \in \mathcal{M}_u} u'.$$

In order to prove adequacy (Theorem 4.16), we need the following alternative formulation of soundness.

**Corollary 4.15 (Soundness).** $\llbracket \vdash t : A \rrbracket = \llbracket \vdash \sum_{t' \in \mathcal{M}_t} t' : A \rrbracket$.

**Proof.** Without lost of generality, we make the following assumptions.
• In order to represent the reductions, we make the following modification: for each reduction rule of the form \( t \to_1 r \) we add a new reduction new rule of the form \( t \to_0 S r \). It is easy to see that Theorem 4.6 still stands, since \( f = \nabla \circ (\text{id}_I \times 0_I) \circ (f \times f) \circ \Delta = \nabla_{\{1_S\} \cup \{0_S\}} \circ f \times f \circ \Delta \).

• In addition, to make the analysis easier, we also consider that the reduction tree for a term have all its leaves at the same level, by simply continue reducing the values to themselves until all the branches had reached its values. This does not alter the analysis, because the interpretation of a term is the same to its reduct when it reduces with “probability” \( 1_S \).

• Finally, we use this notation: The first reducts of \( t \) are the terms \( r_0 \) and \( r_1 \). The next level is as follows. The reducts of \( r_0 \) are \( r_{00} \) and \( r_{01} \) and the reducts of \( r_1 \) are \( r_{10} \) and \( r_{11} \). The next level will add one more bit, and so on. The scalars associated to each reduction follows the same pattern.

Thus, the term \( r_{b_1 \ldots b_n} \) is the one reached by the following path
\[
\begin{align*}
  t & \rightarrow p_{b_1} \ r_{b_1} \rightarrow p_{b_1 b_2} \ r_{b_1 b_2} \rightarrow p_{b_1 b_2 b_3} \cdots \rightarrow p_{b_1 \ldots b_n} \ r_{b_1 \ldots b_n}.
\end{align*}
\]

By Theorem 4.6, we know that
\[
\begin{align*}
  [\cdot] t : A = \nabla_{p_0 \ldots p_1} \circ ([\cdot] r_0 : A) \times [\cdot] r_1 : A) \circ \Delta.
\end{align*}
\]

Using the same Theorem 4.6, we also have
\[
\begin{align*}
  [\cdot] r_0 : A &= \nabla_{p_{00} \ldots p_{01}} \circ ([\cdot] r_{00} : A) \times [\cdot] r_{01} : A) \circ \Delta, \text{ and} \\
  [\cdot] r_1 : A &= \nabla_{p_{10} \ldots p_{11}} \circ ([\cdot] r_{10} : A) \times [\cdot] r_{11} : A) \circ \Delta.
\end{align*}
\]

Let \( \overline{p_{b_1 \ldots b_n}} = p_{b_1} p_{b_1 b_2} \ldots p_{b_1 \ldots b_n} \). Then, we have
\[
\begin{align*}
  \left[ \sum_{t' \in \mathcal{M}_t} t' : A \right] &= \left[ [\cdot] p_{0 \ldots 0} \bullet r_{0 \ldots 0} + \cdots + [\overline{p_{1 \ldots 1}}] \bullet r_{1 \ldots 1} : A \right] \\
  &= \nabla_n \circ ([\overline{p_{0 \ldots 0}}] \times \cdots \times [\overline{p_{1 \ldots 1}}]) \circ ([\cdot] r_{0 \ldots 0} : A) \times [\cdot] r_{1 \ldots 1} : A) \circ \overline{\Delta_n}.
\end{align*}
\]
where
\[ \Delta_1 = \Delta \]
\[ \Delta_{n+1} = \Delta^{2^n} \circ \Delta_n \]
\[ \nabla_1 = \nabla \]
\[ \nabla_{n+1} = \nabla^{2^n} \circ \nabla_n. \]

We must prove that (1) = (2).

From equation (1), we have:

\[
\begin{align*}
\Gamma(\{p_0\} \circ \{p_1\}) & \circ \left( \Gamma(\{r_0\} \times \{r_1\}) \circ \Delta \right) \\
& = \nabla_{\{p_0\} \circ \{p_1\}} \circ \left( \Gamma(\{r_0\} \times \{r_1\}) \circ \Delta \right) \\
& = \nabla_{\{p_0\} \circ \{p_1\}} \circ \left( \left( \nabla_{\{p_0\}} \circ \left( \Gamma(\{r_0\} \times \{r_1\}) \circ \Delta \right) \right) \times \\
& \quad \left( \nabla_{\{p_1\}} \circ \left( \Gamma(\{r_0\} \times \{r_1\}) \circ \Delta \right) \right) \circ \Delta \\
& = \nabla_1 \circ \left( \Gamma(\{p_0\}) \times \Gamma(\{p_1\}) \right) \circ \\
& \quad \left( \left( \nabla_{\{p_0\}} \circ \left( \Gamma(\{r_0\}) \times \Gamma(\{r_1\}) \right) \circ \Delta \right) \times \\
& \quad \left( \nabla_{\{p_1\}} \circ \left( \Gamma(\{r_0\}) \times \Gamma(\{r_1\}) \right) \circ \Delta \right) \right) \circ \Delta_1 \\
& \quad (\ast) = \nabla_1 \circ \left( \Gamma(\{p_0\}) \times \Gamma(\{p_1\}) \right) \circ \\
& \quad \left( \left( \nabla_0 \circ \left( \Gamma(\{r_0\}) \times \Gamma(\{r_1\}) \right) \circ \nabla_1 \right) \times \\
& \quad \left( \nabla_1 \circ \left( \Gamma(\{r_0\}) \times \Gamma(\{r_1\}) \right) \circ \nabla_1 \right) \right) \circ \Delta_1
\end{align*}
\]

Where the equality (\ast) is justified by the following commuting diagram.

\[
\begin{array}{cccccc}
S & \xrightarrow{\nabla_1} & (S \times S) & \xrightarrow{(\Gamma_0 \times \Gamma_1) \times (\Gamma_0 \times \Gamma_1)} & (A \times A) \times (A \times A) \\
\downarrow \nabla_1 \times \nabla_1 & \downarrow \Gamma_0 \times \Gamma_1 \circ (\Gamma_0 \times \Gamma_1) & \downarrow \nabla_1 \times \nabla_1 & \downarrow \nabla_1 \times \nabla_1 & \downarrow \nabla_1 \times \nabla_1 & \downarrow \nabla_1 \times \nabla_1 \\
A \times A & \xleftarrow{\Gamma_0 \times \Gamma_1 \circ (\Gamma_0 \times \Gamma_1)} & (A \times A) \times (A \times A) & \xrightarrow{\nabla_1 \times \nabla_1} & A \times A & \xrightarrow{\nabla_1 \times \nabla_1} & A \times A
\end{array}
\]

where
\[
\begin{align*}
f & = (\Gamma_0 \circ \{p_0\} \times \Gamma_1 \circ \{p_0\}) \times (\Gamma_0 \circ \{p_1\} \times \Gamma_1 \circ \{p_1\}) \\
g & = (\Gamma_0 \circ \{p_0\} \times \Gamma_0 \circ \{p_0\} \circ \{p_1\} \circ \{p_1\}) \times (\Gamma_1 \circ \{p_1\} \times \Gamma_1 \circ \{p_1\} \circ \{p_1\} \circ \{p_1\})
\end{align*}
\]

\[ \square \]
Theorem 4.16 (Adequacy of \( \approx \)). If \([\vdash t : A] = [\vdash u : A]\) then \( t \approx u \).

Proof. To prove \( t \approx u \) we need to prove that
\[
\sum_{t' \in M_t} t' \sim \sum_{u' \in M_u} u'.
\]
That is, for every elimination context \([\cdot]: A \vdash K : \tau\), we have
\[
\forall P, \quad K[\sum_{t' \in M_t} t'] \rightarrow^P_P v \iff K[\sum_{u' \in M_u} u'] \rightarrow^P_P v.
\]
Or, since the only elimination context \([\cdot]: \tau \vdash K' : \tau\) is \([\cdot]\),
\[
K[\sum_{t' \in M_t} t'] \sim K[\sum_{u' \in M_u} u'].
\] (3)

We proceed by induction on \( K \).

• Let \( K = [\cdot] \), we have two cases.
  
  - If \( A = 1 \), then by Theorem 2.10, \( M_t = \{p_i \cdot s_i.\ast\}_i \in I \) and \( M_u = \{p'_j \cdot s'_j.\ast\}_j \in J \). Then,
    \[
    \sum_{t' \in M_t} t' \rightarrow^{1_s^*} \big( \sum_i p_i \cdot s_i \big) \ast \ast \ast \text{ and } \sum_{u' \in M_u} u' \rightarrow^{1_s^*} \big( \sum_j p'_j \cdot s'_j \big) \ast \ast \ast .
    \]
    Then, using Corollary 4.15, we have
    \[
    \langle \sum_i p_i \cdot s_i \rangle = \vdash (\sum_i p_i \cdot s_i) \ast 1 = [\vdash t : 1] = [\vdash u : 1] = \langle \sum_j p'_j \cdot s'_j \rangle .
    \]
    Therefore, since \( \langle \cdot \rangle \) is a monomorphism, \( (\sum_i p_i \cdot s_i) \ast = (\sum_j p'_j \cdot s'_j) \ast \ast \ast \), thus \( t \approx u \).
    
  - If \( A = \top \), then by Theorem 2.10, \( t \rightarrow^{1_s^*} \langle \rangle \) and \( u \rightarrow^{1_s^*} \langle \rangle \), and thus, \( t \sim u \) and so \( t \approx u \).

• Let \( K \neq [\cdot] \). By Corollary 4.15, we have
\[
\begin{align*}
\langle \sum_{t' \in M_t} t' : A \rangle & = [\vdash t : A] = [\vdash u : A] = [\vdash \sum_{u' \in M_u} u' : A].
\end{align*}
\]
Hence, by composition,

\[
\Gamma \vdash K \left[ \sum_{t' \in M_t} t' \right] : \tau = \Gamma \vdash K \left[ \sum_{u' \in M_u} u' \right] : \tau.
\]

Since \( \tau \) is smaller than \( A \), otherwise \( K \) would have been \([],\)[1], we can apply the induction hypothesis to conclude that

\[
K \left[ \sum_{t' \in M_t} t' \right] \approx K \left[ \sum_{u' \in M_u} u' \right].
\]

(4)

Since any reduction path started from \( K \left[ \sum_{t' \in M_t} t' \right] \) or from \( K \left[ \sum_{u' \in M_u} u' \right] \) use only reductions \( \rightarrow_{1_S} \), Equation (4) implies \( 1_S \bullet K \left[ \sum_{t' \in M_t} t' \right] \sim 1_S \bullet K \left[ \sum_{u' \in M_u} u' \right] \), which implies the Equation (3).

\[\Box\]

5. Conclusion

In this paper, we have presented a categorical characterization for the proof language \( L_{\odot}^{Sp} \)-calculus, an extension of IMALL with the generalised probabilistic connective \( \odot \). We have shown that the essential structure of a symmetric monoidal closed category with biproducts suffices for modelling the \( L_{\odot}^{Sp} \)-calculus, when there exists a monomorphism from the semiring of scalars to the semiring \( \text{Hom}(I, I) \). A key element in our approach was the abstract definition of the weighted codiagonal map, which underpins the representation of generalised probabilities. We established soundness and adequacy proofs for this model.

In particular, Corollary 4.15 gives a summary of the approach: the interpretation of a term is the same as the interpretation of the weighted linear combination of the values it achieves. The map \( \nabla_{pq} \) gives us an abstract representation for this.

Our work offers an alternative approach to existing models relying on probabilistic coherence spaces, cones, or compactness requirements. Also generalises the model \( R^\Pi \) of PCF\(^R\) given in [21] in two ways: we give a categorical characterization, and we give a language where the sums and scalar product not only serve as a way to represent probabilities but also as a way to represent vectors and matrices. Furthermore, the categorical model for \( L_{\odot}^{Sp} \)-calculus paves the way for future investigations into the connections between linear logic, verifiable quantum algorithms, and the development of probabilistic programming languages.

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References

[1] Abramsky, S., Coecke, B., 2004. A categorical semantics of quantum protocols, in: Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, 2004., pp. 415–425.

[2] Altenkirch, T., Grattage, J., 2005. A functional quantum programming language, in: Proceedings of the 20th Annual IEEE Symposium on Logic in Computer Science (LICS), IEEE. pp. 249–258.

[3] Arrighi, P., Díaz-Caro, A., 2012. A System F accounting for scalars. Logical Methods in Computer Science 8(1:11).

[4] Arrighi, P., Díaz-Caro, A., Valiron, B., 2017. The vectorial lambda-calculus. Information and Computation 254, 105–139.

[5] Arrighi, P., Dowek, G., 2017. Lineal: A linear-algebraic lambda-calculus. Logical Methods in Computer Science 13.

[6] Assaf, A., Díaz-Caro, A., Perdrix, S., Tasson, C., Valiron, B., 2014. Call-by-value, call-by-name and the vectorial behaviour of the algebraic $\lambda$-calculus. Logical Methods in Computer Science 10(4:8).

[7] Crubillé, R., 2018. Probabilistic stable functions on discrete cones are power series, in: Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2018), ACM. pp. 275–284.

[8] Danos, V., Ehrhard, T., 2011. Probabilistic coherence spaces as a model of higher-order probabilistic computation. Information and Computation 209, 966–991.

[9] Díaz-Caro, A., Dowek, G., 2023. A new connective in natural deduction, and its application to quantum computing. Theoretical Computer Science 957, 113840.
[10] Díaz-Caro, A., Dowek, G., Rinaldi, J.P., 2019a. Two linearities for quantum computing in the lambda calculus. BioSystems 186, 104012. Post-proceedings of TPNC 2017.

[11] Díaz-Caro, A., Guillermo, M., Miquel, A., Valiron, B., 2019b. Realizability in the unitary sphere, in: Proceedings of the 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2019), pp. 1–13.

[12] Díaz-Caro, A., Malherbe, O., 2019. A concrete categorical semantics for Lambda-S, in: Accattoli, B., Olarte, C. (Eds.), Proceedings of the 13th Workshop on Logical and Semantic Frameworks with Applications (LSFA 2018), Elsevier. pp. 83–100.

[13] Díaz-Caro, A., Malherbe, O., 2020. A categorical construction for the computational definition of vector spaces. Applied Categorical Structures 28, 807–844.

[14] Díaz-Caro, A., Malherbe, O., 2022. Quantum control in the unitary sphere: Lambda-$S_1$ and its categorical model. Logical Methods in Computer Science 18, 32:1–32:36.

[15] Díaz-Caro, A., Malherbe, O., 2022. Semimodules and the (syntactically-)
linear lambda calculus. arXiv:2205.02142v2.

[16] Díaz-Caro, A., Dowek, G., 2024. A linear linear lambda-calculus. Mathematical Structures in Computer Science (accepted for publication). Preprint at arXiv:2201.11221.

[17] Ehrhard, T., Pagani, M., Tasson, C., 2017. Measurable cones and stable, measurable functions: a model for probabilistic higher-order programming, in: Proceedings of the 44th ACM SIGPLAN Symposium on Principles of Programming Languages (POPL 2017), ACM. pp. 1–28.

[18] Ehrnard, T., 2020. Cones as a model of intuitionistic linear logic, in: Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2020), ACM. pp. 370–383.

[19] Girard, J.Y., 2004. Between Logic and Quantic: a Tract. Cambridge University Press, Cambridge. chapter 10. London Mathematical Society Lecture Note Series, pp. 346–381.
[20] Kelly, G.M., Laplaza, M.L., 1980. Coherence for compact closed categories. Journal of Pure and Applied Algebra 19, 193–213.

[21] Laird, J., Manzonetto, G., McCusker, G., Pagani, M., 2013. Weighted relational models of typed lambda-calculi, in: Proceedings of the 28rd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2013), ACM. pp. 301–310.

[22] Mitchell, B., 1965. Theory of categories. Academic Press.

[23] Moggi, E., 1991. Notions of computation and monads. Information and Computation 93, 55–92.

[24] Nielsen, M., Chuang, I., 2010. Quantum Computation and Quantum Information. 10th years anniversary ed., Cambridge University Press.

[25] Pagani, M., Selinger, P., Valiron, B., 2014. Applying quantitative semantics to higher-order quantum computing, in: Proceedings of the 41st Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL 2014), ACM. pp. 647–658.

[26] Selinger, P., 2004a. Toward a semantics for higher-order quantum computation, in: Selinger, P. (Ed.), 2nd International Workshop on Quantum Programming Languages (QPL 2004), Turku Centre for Computer Science. pp. 127–143.

[27] Selinger, P., 2004b. Towards a quantum programming language. Mathematical Structures in Computer Science 14, 527–586.

[28] Selinger, P., Valiron, B., 2006. A lambda calculus for quantum computation with classical control. Mathematical Structures in Computer Science 16, 527–552.

[29] Vaux, L., 2009. The algebraic lambda calculus. Mathematical Structures in Computer Science 19, 1029–1059.

[30] Wootters, W., Zurek, W.H., 1982. A single quantum cannot be cloned. Nature 1982.

[31] Zorzi, M., 2016. On quantum lambda calculi: a foundational perspective. Mathematical Structures in Computer Science 26, 1107–1195.
Appendix A. Proof of Theorem 2.10

Theorem 2.10 (Introduction). Let $\vdash t : A$ and $t$ irreducible.

- If $A = 1$, then $t = \star$.
- If $A = B \otimes C$, then $t = u \otimes v$.
- If $A = B \rightarrow C$, then $t = \lambda x.u$.
- If $A = \top$, then $t = \langle \rangle$.
- $A$ cannot be equal to $0$.
- If $A = B \& C$, then $t = \langle u, v \rangle$.
- If $A = B \oplus C$, then $t = \text{inl}(l)$, $t = \text{inr}(r)$.
- If $A = B \odot C$, then $t = [u, v]$.

Proof. By induction on $t$. If $t$ is one of $\star$, $u \otimes v$, $\lambda x.u$, $\langle \rangle$, $\langle u, v \rangle$, $\text{inl}(l)$, $\text{inr}(r)$, or $[u, v]$, then we are done.

- $t$ cannot be a variable or $\delta_0(u)$ since it is closed.
- Let $t = \delta_1(u, v)$, then $\vdash u : 1$. Thus, by the induction hypothesis, $u = \star$, but then $\delta_1(u, v)$ is reducible, which is absurd.
- Let $t = \delta_\odot(u, xy.v)$, then $\vdash u : A \odot B$. Thus, by the induction hypothesis, $u = u_1 \odot u_2$, but then $\delta_\odot(u, xy.v)$ is reducible, which is absurd.
- Let $t = uv$, then $\vdash u : B \rightarrow A$ and $\vdash v : B$. Thus, by the induction hypothesis, $u = \lambda x.s$, but then $uv$ is reducible, which is absurd.
- Let $t = \pi_1(u)$, then $\vdash u : A \& B$. Thus, by the induction hypothesis, $u = \langle s_1, s_2 \rangle$, but then $\pi_1(u)$ is reducible, which is absurd.
- Let $t = \pi_2(u)$. This case is analogous to the previous one.
- Let $t = \delta_\oplus(u, x.s_1, y.s_2)$, then $\vdash u : B \oplus C$. Thus, by the induction hypothesis, $u = \text{inl}(r)$ or $u = \text{inr}(r)$, but then $\delta_\oplus(u, x.s_1, y.s_2)$ is reducible, which is absurd.
Appendix B. Proof of Lemma 3.9

Lemma 3.9. Let \( F \) a semiadditive functor in \( C_S \). Then, \( F(A) \oplus F(B) \cong F(A \oplus B) \).

Proof. Consider the following arrows.

\[
F(A \oplus B) \xrightarrow{f} F(A) \oplus F(B) \text{ given by } f = \langle F(\pi_1), F(\pi_2) \rangle \\
F(A) \oplus F(B) \xrightarrow{f^{-1}} F(A \oplus B) \text{ given by } f^{-1} = [F(i_1), F(i_2)]
\]

- First we check that \( f^{-1} \circ f = \text{id} \)

\[
f^{-1} \circ f = f^{-1} \circ \text{id} \circ f \\
= f^{-1} \circ (i_1 \circ \pi_1 \oplus i_2 \circ \pi_2) \circ \langle F(\pi_1), F(\pi_2) \rangle \\
= f^{-1} \circ (i_1 \circ \pi_1 \circ \langle F(\pi_1), F(\pi_2) \rangle \oplus i_2 \circ \pi_2 \circ \langle F(\pi_1), F(\pi_2) \rangle) \\
= f^{-1} \circ (i_1 \circ F(\pi_1) \oplus i_2 \circ F(\pi_2)) \\
= f^{-1} \circ i_1 \circ F(\pi_1) \oplus f^{-1} \circ i_2 \circ F(\pi_2) \\
= [F(i_1), F(i_2)] \circ i_1 \circ F(\pi_1) \oplus [F(i_1), F(i_2)] \circ i_2 \circ F(\pi_2) \\
= F(i_1) \circ F(\pi_1) \oplus F(i_2) \circ F(\pi_2) \\
= F(i_1 \circ \pi_1) \oplus F(i_2 \circ \pi_2) \\
= F(\text{id}) \\
= \text{id}
\]

- To check \( f \circ f^{-1} = \text{id} \) we check instead

\[
\begin{cases}
    f \circ f^{-1} \circ i_1 = i_1 \\
    f \circ f^{-1} \circ i_2 = i_2
\end{cases}
\]
For the first equation, we have

\[
\begin{align*}
f \circ f^{-1} \circ i_1 &= \langle F(\pi_1), F(\pi_2) \rangle \circ [F(i_1), F(i_2)] \circ i_1 \\
&= \langle F(\pi_1), F(\pi_2) \rangle \circ F(i_1) \\
&= \langle F(\pi_1) \circ F(i_1), F(\pi_2) \circ F(i_1) \rangle \\
&= \langle F(\pi_1 \circ i_1), F(\pi_2 \circ i_1) \rangle \\
&= \langle F(id), F(0) \rangle \\
&= \langle id, 0 \rangle
\end{align*}
\]

\[\ast\) = i_1\]

Where the equality \((\ast)\) is given by the fact that \(\pi_1 \circ i_1 = id = \pi_1 \circ \langle id, 0 \rangle\) and \(\pi_2 \circ i_1 = 0 = \pi_2 \circ \langle id, 0 \rangle\).

The second equation is analogous.

- We check that \(f\) is a natural transformation.

\[
\begin{array}{ccc}
F(A \oplus B) & \longrightarrow & F(A) \oplus F(B) \\
\downarrow F(g \oplus h) & & \downarrow F(g) \oplus F(h) \\
F(C \oplus D) & \longrightarrow & F(C) \oplus F(D)
\end{array}
\]

\[
\begin{align*}
f \circ F(g \oplus h) &= \langle F(\pi_1), F(\pi_2) \rangle \circ F(g \oplus h) \\
&= \langle F(\pi_1 \circ (g \oplus h)), F(\pi_2 \circ (g \oplus h)) \rangle \\
&= \langle F(g \circ \pi_1), F(h \circ \pi_2) \rangle \\
&= \langle F(g) \circ F(\pi_1), F(h) \circ F(\pi_2) \rangle \\
&= \langle F(g) \circ \pi_1 \circ (F(\pi_1), F(\pi_2)), F(h) \circ \pi_2 \circ (F(\pi_1), F(\pi_2)) \rangle \\
&= \langle F(g) \circ \pi_1 \circ f, F(h) \circ \pi_2 \circ f \rangle \\
&= \langle F(g) \circ \pi_1, F(h) \circ \pi_2 \rangle \circ f \\
&= (F(g) \oplus F(h)) \circ f
\end{align*}
\]

- Finally, \(f^{-1}\) is also a natural transformation, since it is the inverse of a natural transformation. \(\Box\)
Appendix C. Proof of Lemma 3.11

Lemma 3.11 (Scalar). Let $I \to s \to I$. The map $A \xrightarrow{s_A} A$ defined by $s_A = \rho_A \circ (\text{id} \otimes s) \circ \rho_A^{-1}$, is a natural transformation.

Proof. Consequence of the commutation of the following diagram.

\[
\begin{array}{ccccccccc}
A & \xrightarrow{\rho} & A \otimes I & \xrightarrow{\text{id} \otimes s} & A \otimes I & \xrightarrow{\rho^{-1}} & A \\
\downarrow & & \downarrow & & \downarrow & & \\
B & \xrightarrow{\rho} & B \otimes I & \xrightarrow{\text{id} \otimes s} & B \otimes I & \xrightarrow{\rho^{-1}} & B
\end{array}
\]

Proof. Consequence of the commutation of the following diagram.

\[
\begin{array}{ccccccccc}
I & \xrightarrow{\rho} & I \otimes I & \xrightarrow{\text{id} \otimes s} & I \otimes I & \xrightarrow{\rho^{-1}} & I \\
\downarrow & & \downarrow & & \downarrow & & \\
I & \xrightarrow{s} & I \otimes I & \xrightarrow{\rho^{-1}} & I \otimes I & \xrightarrow{\rho^{-1}} & I
\end{array}
\]

Appendix D. Proof of Lemma 3.12

Lemma 3.12 (Properties of the scalar map).

1. $\hat{s}_I = s$.
2. $\hat{s}_{A \otimes B} = \hat{s}_A \otimes \text{id}_B$.
3. $\hat{s}_{A \oplus B} = \hat{s}_A \oplus \hat{s}_B$.

Proof.

1. Consequence of the commutation of the following diagram.

\[
\begin{array}{ccccccccc}
A \otimes B & \xrightarrow{\rho_A \otimes B} & A \otimes I \otimes I & \xrightarrow{\text{id} \otimes \text{id} \otimes s} & A \otimes I \otimes I
\end{array}
\]

2. Consequence of the commutation of the following diagram.

\[
\begin{array}{ccccccccc}
A \otimes B \otimes I & \xrightarrow{\rho_{A \otimes B}} & A \otimes I \otimes B & \xrightarrow{\rho_{A \otimes B}^{-1}} & A \otimes I \otimes B
\end{array}
\]
3. Consequence of the commutation of the following diagram.

\[
\begin{array}{ccc}
A \oplus B & \xrightarrow{\rho_{A \oplus B}} & A \oplus B \\
\downarrow & & \downarrow \\
(A \oplus B) \otimes I & \xrightarrow{\text{id} \otimes s} & (A \oplus B) \otimes I \\
\downarrow & & \downarrow \\
(A \otimes I) \oplus (B \otimes I) & \xrightarrow{(\text{id} \otimes s) \oplus (\text{id} \otimes s)} & (A \otimes I) \oplus (B \otimes I)
\end{array}
\]

Where the commutation of the diagram \((*)\) is proved as follows.

\[
d \circ \rho_{A \oplus B} = (\pi_1 \otimes \text{id}, \pi_2 \otimes \text{id}) \circ \rho_{A \oplus B} = (\pi_1 \otimes \text{id}) \circ \rho_{A \oplus B}, (\pi_2 \otimes \text{id}) \circ \rho_{A \oplus B}
\]

(Naturality of \(\rho\)) = \langle \rho_A \circ \pi_1, \rho_B \circ \pi_2 \rangle

= \rho_A \oplus \rho_B

The commutation of the diagram \((**)\) is a direct consequence of the commutation of the diagram \((*)\). Indeed, since \(d \circ \rho_{A \oplus B} = \rho_A \oplus \rho_B\), we have \((\rho_A^{-1} \oplus \rho_B^{-1}) \circ d \circ \rho_{A \oplus B} = \text{id}\), thus, \((\rho_A^{-1} \oplus \rho_B^{-1}) \circ d = \rho_{A \oplus B}^{-1}\).

\[\square\]

Appendix E. Proof of Lemma 3.13

Lemma 3.13 (The map \(\tau\)). The following map in the arrows of \(C_S\) is a natural transformation with respect to \(I\).

\[
\tau = [A \to B] \otimes I \xrightarrow{\varphi_{A,[A \to B] \otimes I,B \otimes I}(\varepsilon \otimes \text{id})} [A \to B \otimes I]
\]

where \(\varphi_{A,[A \to B] \otimes I,B \otimes I}\) is the map given by the adjunction

\[
\text{Hom}(X \otimes Y, Z) \cong \text{Hom}(Y, [X \to Z])
\]

by taking \(X = A, Y = [A \to B] \otimes I,\) and \(Z = B \otimes I\).

Proof. We need to prove the commutation of the following diagram.

\[
\begin{array}{ccc}
[A \to B] \otimes I & \xrightarrow{\tau} & [A \to B \otimes I] \\
\downarrow & \downarrow & \downarrow \\
[A \to B] \otimes I & \xrightarrow{\tau} & [A \to B \otimes I]
\end{array}
\]
Since \( \varphi \) is natural, the following diagram commutes.

\[
\begin{array}{ccc}
\text{Hom}(A \otimes [A \to B] \otimes I, B \otimes I) & \xrightarrow{\varphi} & \text{Hom}([A \to B] \otimes I, [A \to B \otimes I]) \\
\text{Hom}(id, id \otimes s) & & \text{Hom}(id, [A \to id \otimes s]) \\
\downarrow & & \downarrow \\
\text{Hom}(A \otimes [A \to B] \otimes I, B \otimes I) & \xrightarrow{\varphi} & \text{Hom}([A \to B] \otimes I, [A \to B \otimes I])
\end{array}
\]

Thus, \([A \to id \otimes s] \circ \tau = [A \to id \otimes s] \circ \varphi(\varepsilon \otimes id) = \varphi(\text{Hom}(id, id \otimes s)(\varepsilon \otimes id))\).

Therefore, we must prove that

\[
\varphi(\text{Hom}(id, id \otimes s)(\varepsilon \otimes id)) = \varphi(\varepsilon \otimes id) \circ (id \otimes s) = \tau \circ (id \otimes s)
\]

We prove instead that

\[
\text{Hom}(id, id \otimes s)(\varepsilon \otimes id) = \varphi^{-1}(\tau \circ (id \otimes s))
\]

where \( \varphi_{X,Y,Z}^{-1}(g) = X \otimes Y \xrightarrow{id \otimes g} X \otimes [X \to Z] \xrightarrow{\varepsilon} Z \).

We have

\[
\text{Hom}(id, id \otimes s)(\varepsilon \otimes id) = (id \otimes s) \circ (\varepsilon \otimes id)
\]

\[
= \varepsilon \otimes s
\]

\[
(\ast) = \varepsilon \circ ((id \otimes \tau) \circ (id \otimes id \otimes s))
\]

\[
= \varepsilon \circ (id \otimes (\tau \circ (id \otimes s)))
\]

\[
= \varphi^{-1}(\tau \circ (id \otimes s))
\]

where the equality \((\ast)\) is justified by the commutation of the following diagram.

\[
\begin{array}{ccc}
A \otimes [A \to B] \otimes I & \xrightarrow{\varepsilon \otimes s} & B \otimes I \\
\downarrow & & \downarrow \\
A \otimes [A \to B] \otimes I & \xrightarrow{id \otimes \tau} & A \otimes [A \to B \otimes I]
\end{array}
\]

\[
\text{(Functoriality of } \otimes)\]

\[
(\varepsilon \otimes id = \varphi^{-1}(\tau))
\]

\[
\square
\]

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Appendix F. Proof of Lemma 3.16

Lemma 3.16 (Weighted codiagonal). The map $A \oplus A \xrightarrow{\nabla_{pq}} A$ defined by $\nabla_{pq} = [\hat{p}, \hat{q}]$ is a natural transformation.

Proof. Consequence of the commutation of the following diagram.

\[
\begin{array}{ccc}
A \oplus A & \xrightarrow{\nabla_{pq}} & A \\
\downarrow & & \downarrow \\
B \oplus B & \xrightarrow{\nabla_{pq}} & B
\end{array}
\]

We have

\[
f \circ \nabla_{pq} = f \circ [\hat{p}, \hat{q}] = [f \circ \hat{p}, f \circ \hat{q}]
\]

(Lemma 3.11) $[\hat{p} \circ f, \hat{q} \circ f]$

$=[[\hat{p}, \hat{q}] \circ i_1 \circ f, [\hat{p}, \hat{q}] \circ i_2 \circ f]$

$=[\hat{p}, \hat{q}] \circ [i_1 \circ f, i_2 \circ f]$

$= \nabla_{pq} \circ (f \oplus f)$

Appendix G. Proof of Lemma 3.21

Lemma 3.21.

1. $(\nabla_{pq} \oplus \nabla_{pq}) \circ (\text{id} \oplus \sigma \oplus \text{id}) = \nabla_{pq}$.
2. $(\text{id} \oplus \sigma \oplus \text{id}) \circ (\Delta \oplus \Delta) = \Delta$

Proof.

1. Consequence of the commutation of the following diagram.

\[
\begin{array}{ccc}
(A \oplus A) \oplus (B \oplus B) & \xleftarrow{\text{id} \oplus \sigma \oplus \text{id}} & (A \oplus B) \oplus (A \oplus B) \\
& \xrightarrow{\nabla_{pq} \oplus \nabla_{pq}} & \\
& \xleftarrow{\nabla_{pq}} & A \oplus B
\end{array}
\]

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We have

\[(\nabla_{pq} \oplus \nabla_{pq}) \circ (\id \oplus \sigma \oplus \id) = [\hat{p}_A, \hat{q}_A] \oplus [\hat{p}_B, \hat{q}_B] \circ (\id \oplus \sigma \oplus \id)\]
\[= (\nabla \circ (\hat{p}_A \oplus \hat{q}_A)) \oplus (\nabla \circ (\hat{p}_B \oplus \hat{q}_B)) \circ (\id \oplus \sigma \oplus \id)\]
\[= \nabla \circ (\hat{p}_A \oplus \hat{q}_A \oplus \hat{p}_B \oplus \hat{q}_B) \circ (\id \oplus \sigma \oplus \id)\]
\[= \nabla \circ (\hat{p}_A \oplus \hat{p}_B \oplus \hat{q}_A \oplus \hat{q}_B)\]

(Lemma 3.12) = \nabla \circ (\hat{p}_{A \oplus B} \oplus \hat{q}_{A \oplus B})
\[= [\hat{p}_{A \oplus B}, \hat{q}_{A \oplus B}]\]
\[= \nabla_{pq}\]

2. Consequence of the commutation of the following diagram

\[\begin{array}{ccc}
A \oplus B & \xrightarrow{\Delta \oplus \Delta} & (A \oplus A) \oplus (B \oplus B) \\
\downarrow \Delta & & \downarrow \id \oplus \sigma \oplus \id \\
(A \oplus B) \oplus (A \oplus B) & & (A \oplus B) \oplus (A \oplus B)
\end{array}\]

To check \((\id \oplus \sigma \oplus \id) \circ (\Delta \oplus \Delta) = \Delta\), we check instead

\[\left\{\begin{array}{l}
\pi_1 \circ (\id \oplus \sigma \oplus \id) \circ (\Delta \oplus \Delta) = \pi_1 \circ \Delta \\
\pi_2 \circ (\id \oplus \sigma \oplus \id) \circ (\Delta \oplus \Delta) = \pi_2 \circ \Delta
\end{array}\right.\]

We have

\[\pi_{A \oplus B, A \oplus B}^1 \circ (\id \oplus \sigma \oplus \id) \circ (\Delta \oplus \Delta)\]
\[(*) = (\pi_{A, A}^1 \oplus \pi_{B, B}^1) \circ (\Delta \oplus \Delta)\]
\[= (\pi_{A, A}^1 \circ \Delta) \oplus (\pi_{B, B}^1 \circ \Delta)\]
\[= \id_A \oplus \id_A\]
\[= \id_{A \oplus B}\]
\[= \pi_{A \oplus B, A \oplus B}^1 \circ \Delta\]

where the equality \((*)\) is justified as follows, using the fact that

\[f_A \oplus g_B = (f_A \circ \pi_{A \oplus B, A \oplus B}^1, g_B \circ \pi_{A \oplus B}^2)\] (G.1)
\[
\pi_{A\oplus B, A\oplus B}^1 \circ (\text{id}_A \oplus (\sigma_{A\oplus B}^1 \oplus \text{id}_B)) = \pi_{A\oplus B, A\oplus B}^1 \circ (\text{id}_A \oplus (\pi_{A, B}^2, \pi_{A, B}^1) \oplus \text{id}_B)
\]

\[= \pi_{A\oplus B, A\oplus B}^1 \circ (\langle \text{id}_A \circ \pi_{A, A\oplus B}^1 \circ \langle \pi_{A, B}^2, \pi_{A, B}^1 \rangle \circ \pi_{A, A\oplus B}^2 \rangle \oplus \text{id}_B)
\]

\[= \pi_{A\oplus B, A\oplus B}^1 \circ (\langle \pi_{A, A\oplus B}^1 \circ \langle \pi_{A, B}^2, \pi_{A, B}^1 \rangle \circ \pi_{A, A\oplus B}^2 \rangle \oplus \text{id}_B)
\]

\[= \pi_{A\oplus B, A\oplus B}^1 \circ \langle \pi_{A, A\oplus B}^2 \circ \pi_{A, A\oplus B}^1 \circ \langle \pi_{A, B}^2, \pi_{A, B}^1 \rangle \circ \pi_{A, A\oplus B}^2 \rangle, \pi_{A, A\oplus B}^2 \rangle
\]

\[= \pi_{A\oplus B, A\oplus B}^1 \circ \langle \pi_{A, A\oplus B}^2 \circ \pi_{A, A\oplus B}^1 \circ \langle \pi_{A, B}^2, \pi_{A, B}^1 \rangle \circ \pi_{A, A\oplus B}^2 \rangle, \pi_{A, A\oplus B}^2 \rangle
\]

The case with \(\pi^2\) is analogous. \(\square\)

**Appendix H. Proof of Lemma 3.27**

**Lemma 3.27.** If \((p, q) \in W\), then \(\nabla_{pq} \circ \delta = \nabla_{pq} \oplus \text{id} \).

**Proof.** Consequence of the commutation of the following diagram.
Appendix I. Proof of Lemma 3.28

Lemma 3.28. \( \Delta = \delta \circ (\Delta \oplus \text{id}) \).

Proof. Consequence of the commutation of the following diagram.

\[
\begin{array}{ccc}
A \oplus B & \rightarrow & (A \oplus B) \oplus (A \oplus B) \\
\downarrow \Delta \oplus \text{id} & & \downarrow \delta \\
(A \oplus A) \oplus B & & \\
\end{array}
\]

We have

\[
\delta \circ (\Delta \oplus \text{id}) = (\text{id} \oplus \sigma \oplus \text{id}) \circ (\text{id} \oplus \Delta) \circ (\Delta \oplus \text{id})
\]

\[
= (\text{id} \oplus \sigma \oplus \text{id}) \circ (\Delta \oplus \Delta)
\]

(Lemma 3.21) \( \Delta \) \( \Box \)

Appendix J. Proof of Lemma 4.5

Lemma 4.5 (Substitution). If \( \Gamma, x : A \vdash t : B \) and \( \Delta \vdash v : A \), then

\[
[\Gamma \vdash (v/x)t : B] = [\Gamma, x : A \vdash t : B] \circ (\text{id} \otimes [\Gamma \vdash v : A]).
\]

That is, the following diagram commutes.

\[
\begin{array}{ccc}
[\Gamma] \otimes [\Delta] & \rightarrow & [B] \\
\downarrow \text{id} \otimes v & & \downarrow 1 \\
[\Gamma] \otimes [A] & & \\
\end{array}
\]

Proof. By induction on \( t \). To avoid cumbersome notation, we write \( A \) instead of \([A]\).

- Let \( t = x \). Then, \( \Gamma = \emptyset \), and \( A = B \). Then, the commuting diagram is the following.

\[
\begin{array}{ccc}
I \otimes \Delta & \equiv & A \\
\downarrow \text{id} \otimes v & & \downarrow \text{id} \\
I \otimes A & = & A
\end{array}
\]

- The case \( t = y \neq x \) is no possible since \( \Gamma, x : A \neq y : B \).
• Let $s \star$, it is not possible since $\Gamma, x : A \neq \emptyset$.

• Let $t = r \oplus u$. Then, the commuting diagram is the following.

\[
\begin{array}{c}
\Gamma \otimes \Delta \\
\downarrow (\text{Nat. of } \Delta) \downarrow (\text{IH and funct. of } \otimes) \\
(\Gamma \otimes \Delta) \oplus (\Gamma \otimes \Delta) \\
\downarrow (\text{id} \otimes v) \oplus (\text{id} \otimes v) \\
(\Gamma \otimes A) \oplus (\Gamma \otimes A) \\
\downarrow \Delta \\
\Gamma \otimes A \\
\end{array}
\]

\[
\begin{array}{c}
\Gamma \otimes \Delta \\
\downarrow (\text{Nat. of } \Delta) \downarrow (\text{IH and funct. of } \otimes) \\
(\Gamma \otimes \Delta) \oplus (\Gamma \otimes \Delta) \\
\downarrow (\text{id} \otimes v) \oplus (\text{id} \otimes v) \\
(\Gamma \otimes A) \oplus (\Gamma \otimes A) \\
\downarrow \Delta \\
\Gamma \otimes A \\
\end{array}
\]

\[
\begin{array}{c}
\Gamma \otimes \Delta \\
\downarrow (\text{Nat. of } \Delta) \downarrow (\text{IH and funct. of } \otimes) \\
(\Gamma \otimes \Delta) \oplus (\Gamma \otimes \Delta) \\
\downarrow (\text{id} \otimes v) \oplus (\text{id} \otimes v) \\
(\Gamma \otimes A) \oplus (\Gamma \otimes A) \\
\downarrow \Delta \\
\Gamma \otimes A \\
\end{array}
\]

• Let $t = s \bullet r$. Then, the commuting diagram is the following.

\[
\begin{array}{c}
\Gamma \otimes \Delta \\
\downarrow (\text{Nat. of } \Delta) \downarrow (\text{IH}) \\
(\Gamma \otimes \Delta) \oplus (\Gamma \otimes \Delta) \\
\downarrow (\text{id} \otimes v) \oplus (\text{id} \otimes v) \\
(\Gamma \otimes A) \oplus (\Gamma \otimes A) \\
\downarrow \Delta \\
\Gamma \otimes A \\
\end{array}
\]

• Let $t = \delta_1(r, u)$, so $\Gamma = \Gamma_1, \Gamma_2$.

  – Let $x \in FV(r)$, then, the commuting diagram is the following.

\[
\begin{array}{c}
\Gamma_1 \otimes \Delta \otimes \Gamma_2 \\
\downarrow (\text{IH} \text{ and funct. of } \otimes) \downarrow (\text{IH} \text{ and funct. of } \otimes) \\
I \otimes A \\
\downarrow \delta_1(v \otimes x, r, u) \\
\Gamma_1 \otimes A \otimes \Gamma_2 \\
\end{array}
\]
Let \( x \in FV(u) \), then, the commuting diagram is the following.

\[
\begin{aligned}
\Gamma_1 \otimes \Gamma_2 \otimes \Delta & \quad \xrightarrow{\delta_i(r,(v/x)u)} \quad A \\
& \quad \downarrow \text{(IH \ and \ funct. \ of \ \otimes)} \quad \downarrow \text{(Def.)} \\
\Gamma_1 \otimes \Gamma_2 \otimes A & \quad \xrightarrow{\delta_1(r,u)} \quad I \otimes A \\
& \quad \downarrow \text{id \otimes v} \quad \downarrow \text{id}_{r \otimes A} \\
\Gamma_1 \otimes \Gamma_2 & \quad \xrightarrow{\text{I \ and \ functoriality \ of \ \otimes}} \quad B_1 \otimes B_2 \\
& \quad \downarrow \text{id \otimes v \otimes id} \quad \downarrow \text{id} \otimes r \otimes \delta_0(u) \\
\end{aligned}
\]

Let \( t = r \otimes u \).

– Let \( x \in FV(u) \), so \( \Gamma = \Gamma_1, \Gamma_2 \). Then, the commuting diagram is the following.

\[
\begin{aligned}
\Gamma_1 \otimes \Delta \otimes \Gamma_2 & \quad \xrightarrow{(v/x)t_1 \otimes t_2} \quad B_1 \otimes B_2 \\
& \quad \downarrow \text{id \otimes v \otimes id} \quad \downarrow \text{t_1 \otimes t_2} \\
\Gamma_1 \otimes A \otimes \Gamma_2 & \quad \xrightarrow{\text{IH \ and \ functoriality \ of \ \otimes}} \quad B_1 \otimes B_2 \\
& \quad \downarrow \text{id} \otimes r \otimes \delta_0(u) \\
\end{aligned}
\]

Let \( x \in FV(r) \). This case is analogous to the previous one.

• Let \( t = \delta_\otimes(r, yz.u) \). Then \( \Gamma = \Gamma_1, \Gamma_2 \).

– Let \( x \in FV(r) \). Then, the commuting diagram is the following.

\[
\begin{aligned}
\Gamma_1 \otimes \Gamma_2 \otimes \Delta & \quad \xrightarrow{\delta_\otimes((v/x)r,yz,u)} \quad B \\
& \quad \downarrow \text{id \otimes (v/x)r} \quad \downarrow \text{id} \otimes r \otimes \delta_0(u) \\
\Gamma_1 \otimes \Gamma_2 \otimes A & \quad \xrightarrow{\text{IH \ and \ funct. \ of \ \otimes}} \quad B \\
& \quad \downarrow \text{id} \otimes \delta_0(r) \otimes \delta_0(u) \\
\end{aligned}
\]
Let \( x \in FV(u) \). Then, the commuting diagram is the following

\[
\begin{array}{c}
\Gamma_1 \otimes \Gamma_2 \otimes \Delta \\
\downarrow \delta_{\otimes}(r, yz, (v/x)u) \\
\otimes \downarrow (v/x)u \\
C \otimes D \otimes \Gamma_2 \otimes \Delta \quad (\text{HI}) \\
\downarrow \delta_{\otimes}(yz, v) \\
C \otimes D \otimes \Gamma_2 \otimes A \\
\downarrow \delta_{\otimes}(y, x) \\
\Gamma_1 \otimes \Gamma_2 \otimes A
\end{array}
\]

- Let \( t = \lambda y. r \), so \( B = C \rightarrow D \). Then, the commuting diagram is the following.

\[
\begin{array}{c}
\Gamma \otimes \Delta \\
\downarrow \eta_C \\
\otimes \downarrow (v/x)(\lambda y. r) \\
C \rightarrow \Gamma \otimes \Delta \otimes C \\
\downarrow (v/x)(\lambda y. r) \\
C \rightarrow \Gamma \otimes A \otimes C \\
\downarrow \eta_C \\
\Gamma \otimes A
\end{array}
\]

- Let \( t = ru \), so \( \Gamma = \Gamma_1, \Gamma_2 \).

  - Let \( x \in FV(r) \), so \( \Gamma_1 \vdash u : C \) and \( \Gamma_2, x : A \vdash r : C \rightarrow B \). Then,
the commuting diagram is the following

\[
\begin{array}{c}
\Gamma_1 \otimes \Gamma_2 \otimes \Delta \\
\downarrow \text{(IH and funct. of } \otimes) \\
\Gamma_1 \otimes \Gamma_2 \otimes A
\end{array}
\]

\[
\begin{array}{c}
\Gamma \otimes [C \to \check{A}u] \\
\downarrow \text{ru} \\
\Gamma \otimes A
\end{array}
\]

- Let \( x \in FV(u) \), so \( \Gamma_1, x : A \vdash u : C \) and \( \Gamma_2 \vdash r : C \to A \). Then, the commuting diagram is the following.

\[
\begin{array}{c}
\Gamma_1 \otimes \Delta \otimes \Gamma_2 \\
\downarrow \text{(IH and funct. of } \otimes) \\
\Gamma_1 \otimes A \otimes \Gamma_2
\end{array}
\]

\[
\begin{array}{c}
C \otimes [C \to B] \\
\downarrow \text{ru} \\
0
\end{array}
\]

- Let \( t = () \). Then, the commuting diagram is the following.

\[
\begin{array}{c}
\Gamma \otimes \Delta \\
\downarrow \text{ru} \\
\Gamma \otimes A
\end{array}
\]

- Let \( t = \delta_0(r) \).

- Let \( x \in FV(r) \), so \( \Gamma = \Gamma_1, \Gamma_2 \) and \( \Gamma_1, x : A \vdash r : \circ \). Then, the commuting diagram is the following.

\[
\begin{array}{c}
\circ \otimes \Gamma_2 \\
\downarrow \text{ru} \\
\Gamma_1 \otimes A \otimes \Gamma_2
\end{array}
\]
Let \( x \notin FV(r) \), so \( \Gamma = \Gamma_1, \Gamma_2 \) and \( \Gamma_1 \vdash r : \circ \). Then, the commuting diagram is the following:

\[
\begin{array}{cccccc}
\Gamma_1 \otimes \Gamma_2 \otimes \Delta & \xrightarrow{(v/x)\delta_0(r)=\delta_0(r)} & B \\
\downarrow r \otimes \text{id} & & & & \downarrow B(r) \\
0 \otimes \Gamma_2 \otimes \Delta & \xleftarrow{0} & (0 \text{ arrow}) \\
\downarrow \text{id} \otimes v & & & & \downarrow \text{Def.} \\
0 \otimes \Gamma_2 \otimes A & \xleftarrow{\text{Def.}} & \text{Functoriality of } \otimes \\
\downarrow r \otimes \text{id} & & & & \downarrow \text{Def.} \\
\Gamma_1 \otimes \Gamma_2 \otimes A & \xleftarrow{\text{Def.}} & \text{Functoriality of } \otimes
\end{array}
\]

- Let \( t = \langle r, u \rangle \), so \( B = C \odot D \). Then, the commuting diagram is the following:

\[
\begin{array}{cccccc}
\Gamma \otimes \Delta & \xrightarrow{(v/x)(r,u)} & C \oplus D \\
\downarrow \Delta & & & & \downarrow (v/x)r \oplus (v/x)u \\
(\Gamma \otimes \Delta) \oplus (\Gamma \otimes \Delta) & \xleftarrow{\text{Def.}} & (\Gamma \otimes A) \oplus (\Gamma \otimes A) \\
\downarrow (id \otimes v) \oplus (id \otimes v) & & & & \downarrow (\pi_1, v) \\
(\Gamma \otimes A) \oplus (\Gamma \otimes A) & \xleftarrow{\text{Def.}} & (\Gamma \otimes A) \\
\downarrow \Delta & & & & \downarrow (\pi_1, v) \\
\Gamma \otimes A & \xleftarrow{\text{Def.}} & \text{Naturality of } \Delta
\end{array}
\]

- Let \( t = \pi_1(r) \). Then, the commuting diagram is the following:

\[
\begin{array}{cccccc}
\Gamma \otimes \Delta & \xrightarrow{(v/x)\pi_1(r)} & B \\
\downarrow (v/x) & & & & \downarrow (\pi_1, v) \\
B \oplus C & \xleftarrow{\text{Def.}} & \text{Functoriality of } \oplus \\
\downarrow \pi_1 & & & & \downarrow \pi_1(r) \\
\Gamma \otimes A & \xleftarrow{\text{Def.}} & \pi_1
\end{array}
\]

50
• Let $t = \pi_2(r)$. This case is analogous to the previous case.

• Let $t = \text{inl}(r)$, so $B = C + D$. Then, the commuting diagram is the following

\[ \Gamma \otimes \Delta \xrightarrow{(v/x)\text{inl}(r)} C \oplus D \]

\[ \downarrow \text{(Def.)} \]

\[ \Gamma \otimes A \]

\[ \Gamma \otimes A \otimes \Gamma_2 \]

\[ \Gamma_1 \otimes \Delta \otimes \Gamma_2 \xrightarrow{(v/x)\delta_\oplus(r,y,s_1,z,s_2)} B \]

\[ \downarrow \text{Def.} \]

\[ \Gamma_1 \otimes \Delta \otimes \Gamma_2 \]

\[ \Gamma_1 \otimes A \otimes \Gamma_2 \]

• Let $t = \text{inr}(r)$. This case is analogous to the previous case.

• Let $t = \delta_\oplus(r, y.s_1, z.s_2)$. Then, $\Gamma = \Gamma_1 \land \Gamma_2$.

  – Let $x \in \text{FV}(r)$, so $\Gamma_1, x : A \vdash r : C \lor D$, $y : C, \Gamma_2 \vdash s_1 : B$, and $z : D, \Gamma_2 \vdash s_2 : B$. Then, the commuting diagram is the following

\[ \Gamma_1 \otimes \Delta \otimes \Gamma_2 \xrightarrow{(v/x)\delta_\oplus(r,y,s_1,z,s_2)} B \]

\[ \downarrow \text{Def.} \]

\[ \Gamma_1 \otimes \Delta \otimes \Gamma_2 \]

\[ \Gamma_1 \otimes A \otimes \Gamma_2 \]

  – Let $x \in \text{FV}(s_1) \cup \text{FV}(s_2)$, so $\Gamma_1 \vdash r : C \lor D$, $y : C, \Gamma_2, x : A \vdash s_1 : B$, and $z : D, \Gamma_2, x : A \vdash s_2 : B$. Then, the commuting diagram is
Let \( t = [r, u] \). Analogous to case \( t = \langle r, u \rangle \).

Let \( t = \pi_1^\odot(r) \). Analogous to case \( t = \pi_1(r) \).

Let \( t = \pi_2^\odot(r) \). Analogous to case \( t = \pi_2(r) \).

Let \( t = \delta_{pq}^\odot(r, y.s_1, z.s_2) \). Then, \( \Gamma = \Gamma_1, \Gamma_2 \).

- Let \( x \in FV(r) \), so \( \Gamma_1, x : A \vdash r : C \odot D, y : C, \Gamma_2 \vdash s_1 : B \), and \( z : D, \Gamma_2 \vdash s_2 : B \). Then, the commuting diagram is the following

- Let \( x \in FV(s_1) \cup FV(s_2) \), so \( \Gamma_1 \vdash r : C \odot D, y : C, \Gamma_2, x : A \vdash s_1 : B \), and \( z : D, \Gamma_2, x : A \vdash s_2 : B \). Then, the commuting diagram is
the following

\[(C \oplus D) \otimes \Gamma_2 \otimes \Delta \quad \rightarrow \quad (C \otimes \Gamma_2 \otimes \Delta) \oplus (D \otimes \Gamma_2 \otimes \Delta)\]

\[(C \otimes \Gamma_2 \otimes \Delta) \oplus (D \otimes \Gamma_2 \otimes \Delta) \quad \rightarrow \quad \Gamma_1 \otimes \Gamma_2 \otimes \Delta \quad \rightarrow \quad (C \oplus D) \otimes \Gamma_2 \otimes A\]

\[(C \oplus \Gamma_2 \otimes A) \oplus (D \otimes \Gamma_2 \otimes A) \quad \rightarrow \quad \Gamma_1 \otimes \Gamma_2 \otimes A \quad \rightarrow \quad B \oplus B\]

Appendix K. Proof of Theorem 4.6

Theorem 4.6 (Soundness). Let \(\Gamma \vdash t : A\).

- If \(t \rightarrow_{1,s} r\), by any rule but \((\delta^l)\) and \((\delta^r)\), then
  \[[\Gamma \vdash t : A] = [\Gamma \vdash r : A]\]

- If \(t \rightarrow_p r_1\) by rule \((\delta^l)\) and \(t \rightarrow_q r_2\) by rule \((\delta^r)\), then.
  \[[\Gamma \vdash t : A] = \nabla_{(p,q)} \circ ([\Gamma \vdash r_1 : A] \oplus [\Gamma \vdash r_2 : A]) \circ \Delta\]

That is,

\[
\begin{array}{c}
\Gamma \\
\downarrow \Delta \\
\end{array} \quad t \quad \begin{array}{c}
\Delta \\
\downarrow \nabla_{(p,q)} \\
\end{array} \\
\begin{array}{c}
[\Gamma] \\
\end{array} \quad \rightarrow \quad [A] \\
\begin{array}{c}
\end{array} \quad \rightarrow \quad [A] \oplus [A] \quad \begin{array}{c}
\end{array}
\]

Proof. By induction on the relation \(\rightarrow_p\), using implicitly the coherence maps when needed. To avoid cumbersome notation, we write \(A\) instead of \([A]\).

Basic cases:
\[ \frac{\Gamma \vdash t : A}{\Gamma \vdash \delta_1(s \cdot t) : A} \quad \rightarrow_{1s} \quad \frac{\Gamma \vdash s \cdot t : A}{\Gamma \vdash r} \]

\[ \delta_1(s \cdot t) \] (Def.)

\[ \Gamma \xrightarrow{\lambda^{-1}} I \otimes \Gamma \xrightarrow{\hat{s} \otimes t} I \otimes A \xrightarrow{\lambda} A \] (Funct. of \( \otimes \))

\[ \lambda^{-1} \]

\[ \{ \text{(Naturality of } \lambda^{-1} \text{)}\} \]

\[ I \otimes A \]

\[ \{ \text{(Naturality of } \lambda \text{)}\} \]

\[ \lambda^{-1} \]

\[ \hat{s} \]

\[ s \cdot t \]

\[ \{ \text{Def.}\} \]

\[ \{ \text{Lemma 4.5}\} \]

\[ 54 \]

\[ \Gamma, x : A, y : B \vdash r : C \]

\[ \Delta_1 \vdash t : A \quad \Delta_2 \vdash u : B \]

\[ \Gamma, \Delta_1, \Delta_2 \vdash \delta_\otimes(t \otimes u, x y.r) : C \]

\[ \rightarrow_{1s} \quad \Gamma_1, \Gamma_2, \Delta \vdash (t/x, u/y)r : C \]

\[ \{ \text{Lemma 4.5}\} \]

\[ (t/x, u/y)r \]

\[ \{ \text{Funct. of } \otimes \} \]

\[ \{ \text{Def.}\} \]

\[ \{ \text{Funct. of } \otimes \} \]

\[ \{ \text{Lemma 4.5}\} \]

\[ \delta_\otimes(t \otimes u, x y.r) \]
\( \Gamma, x : A \vdash u : B \)

- \( \Gamma \vdash \lambda x.t : A \rightarrow B \quad \Delta \vdash u : A \)

\[ \rightarrow \delta \quad \Gamma, \Delta \vdash (\lambda x.t)u : B \]

\( \Gamma \otimes \Delta \xrightarrow{\eta \otimes u} [B \rightarrow \Gamma \otimes B] \otimes B \quad \ldots \quad [B \rightarrow \iota] \otimes \text{id} \xrightarrow{\epsilon} [B \rightarrow A] \otimes B \)

\( \Gamma \otimes B \quad \text{(Lemma 4.5)} \)

\( \Gamma \vdash t : A \quad \Gamma \vdash u : B \)

- \( \Gamma \vdash \langle t, u \rangle : A \otimes B \quad \rightarrow \delta \quad \Gamma \vdash \pi_1(t, u) : A \)

\( \pi_1 \circ \Delta = \text{id}_\Gamma \)

\( \Gamma \vdash t : A \quad \Gamma \vdash u : B \)

- \( \Gamma \vdash \langle t, u \rangle : A \otimes B \quad \rightarrow \delta \quad \Gamma \vdash \pi_2(t, u) : B \)

Analogous to the previous case.

\( \Gamma \vdash \text{inl}(t) : A \uplus B \quad \Delta, x : A \vdash u : C \quad \Delta, y : B \vdash v : C \)

\[ \rightarrow \delta \quad \Gamma, \Delta \vdash \delta_B(\text{inl}(t), x.u, y.v) : C \]

\[ \rightarrow \delta \quad \Gamma, \Delta \vdash (t/x)u : C \]
The commutation of the diagram (*) is justified as follows.

\[ d \circ (i_1 \otimes \text{id}) = \langle \pi_1 \otimes \text{id}, \pi_2 \otimes \text{id} \rangle \circ (i_1 \otimes \text{id}) \]
\[ = \langle (\pi_1 \otimes \text{id}) \circ (i_1 \otimes \text{id}), (\pi_2 \otimes \text{id}) \circ (i_1 \otimes \text{id}) \rangle \]
\[ = \langle (\pi_1 \circ i_1) \otimes \text{id}, (\pi_2 \circ i_1) \otimes \text{id} \rangle \]
\[ = \langle \text{id} \otimes \text{id}, 0 \otimes \text{id} \rangle \]
\[ = \langle \text{id}, 0 \rangle \]
\[ = i_1 \]

\[
\begin{array}{c}
\Gamma \vdash t : B \\
\Gamma \vdash \text{inr} : A \oplus B \quad \Delta, x : A \vdash u : C \quad \Delta, y : B \vdash v : C \\
\Gamma, \Delta \vdash \delta_{\oplus}(\text{inr}(t), x.u, y.v) : C \\
\rightarrow_{1_S} \quad \Gamma, \Delta \vdash (t/y)v : C
\end{array}
\]

Analogous to the previous case.

\[
\begin{array}{c}
\Gamma \vdash t : A \quad \Gamma \vdash u : B \\
\Gamma \vdash [t, u] : A \otimes B \\
\rightarrow_{1_S} \quad \Gamma \vdash t : A
\end{array}
\]

Since the interpretation of the derivations are the same to the case of \( \pi_1(t, u) \), this case is analogous to that one.

\[
\begin{array}{c}
\Gamma \vdash t : A \quad \Gamma \vdash u : B \\
\Gamma \vdash [t, u] : A \otimes B \\
\rightarrow_{1_S} \quad \Gamma \vdash u : B
\end{array}
\]

Analogous to the previous case.
\[
\begin{align*}
\Gamma \vdash t_1 : A & \quad \Gamma \vdash t_2 : B \\
\Gamma \vdash [t_1, t_2] : A \otimes B & \quad x : A, \Delta \vdash u : C \\
& \quad y : B, \Delta \vdash v : C \\
\Gamma, \Delta \vdash \delta^\Delta_{\otimes}([t_1, t_2], x, u, y, v) & : C
\end{align*}
\]

\[
\begin{align*}
\Delta & \quad \Delta \\
\Delta \otimes \text{id} & \quad \Delta \\
\text{(Corollary 3.20)} & \quad \text{(Lemma 3.9)} \\
\text{(Def.)} \quad C & \quad C
\end{align*}
\]

Since \(\dashv\) is a homomorphism, we have

\[
\llbracket \vdash (s_1 \# s_2).\star : 1 \rrbracket = \llbracket s_1 \# s_2 \rrbracket = \llbracket s_1 \rrbracket + \llbracket s_2 \rrbracket = \llbracket s_1.\star \# s_2.\star : 1 \rrbracket
\]

\[
\begin{align*}
\Delta & \vdash t : B \otimes C & \Delta & \vdash u : B \otimes C \\
\Gamma & \vdash \delta_{\otimes}(t, x, y, v) : A & \Gamma, \Delta & \vdash \delta_{\otimes}(u, x, y, v) : A
\end{align*}
\]
\[ \Gamma \otimes \Delta \] (Def.)
\[ \Gamma \otimes (\Delta \oplus \Delta) \] (Corollary 3.20)
\[ (\Gamma \otimes \Delta) \oplus (\Gamma \otimes \Delta) \] (Lemma 3.9)
\[ (\Gamma \otimes B \otimes C) \oplus (\Gamma \otimes B \otimes C) \] (Corollary 3.18)
\[ (\Gamma \otimes B \otimes C) \] (Naturality of \( \nabla \))
\[ A \oplus A \] (Def.)

- \( \frac{\Gamma, x : A \vdash t : B \quad \Gamma, x : A \vdash u : B}{\Gamma \vdash \lambda x.t + \lambda x.u : A \rightarrow B} \)

- \( \frac{\Gamma, x : A \vdash t : B \quad \Gamma, x : A \vdash u : B}{\Gamma, x : A \vdash t \oplus u : B \quad \rightarrow_{1s}^{\delta_{\oplus}(t, u, x, v)} \frac{\Gamma, x : A \vdash \lambda x.(t \oplus u) : A \rightarrow B}{\Gamma \vdash \lambda x.t \oplus \lambda x.u : A \rightarrow B} \)
\[
\begin{align*}
\Gamma \vdash \langle \rangle : \top & \quad \Gamma \vdash \langle \rangle : \top \\
\Gamma \vdash \langle \rangle \oplus \langle \rangle : \top & \quad \Gamma \vdash \langle \rangle : \top
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash \langle \rangle : \top & \quad \Gamma \vdash \langle \rangle : \top \\
\Gamma \vdash \langle \rangle \oplus \langle \rangle : \top & \quad \Gamma \vdash \langle \rangle : \top
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash t_1 : A \quad \Gamma \vdash t_2 : B & \quad \Gamma \vdash u_1 : A \quad \Gamma \vdash u_2 : B \\
\Gamma \vdash \langle t_1, t_2 \rangle : A \& B & \quad \Gamma \vdash \langle u_1, u_2 \rangle : A \& B \\
\Gamma \vdash \langle t_1, t_2 \rangle \oplus \langle u_1, u_2 \rangle : A \& B
\end{align*}
\]
\[ \Gamma \vdash t_1 : A \quad \Gamma \vdash u_1 : A \quad \Gamma \vdash t_2 : A \quad \Gamma \vdash u_2 : A \]

\[ \Gamma \vdash (t_1 \oplus u_1, t_2 \oplus u_2) : A \& B \]

\[ \Gamma \vdash t_1 : B \oplus C \quad \Gamma \vdash u : B \oplus C \]

\[ x : B, \Delta \vdash v : A \quad y : C, \Delta \vdash w : A \]

\[ \Delta \vdash \delta_{\oplus}(t \oplus u, x.v, y.w) : A \]

where

\[ D_1 = \begin{array}{c}
\Gamma \vdash t : B \oplus C \\
x : B, \Delta \vdash v : A \\
y : C, \Delta \vdash w : A
\end{array} \]

\[ D_2 = \begin{array}{c}
\Gamma \vdash u : B \oplus C \\
x : B, \Delta \vdash v : A \\
y : C, \Delta \vdash w : A
\end{array} \]
This case is analogous to that of pairs.

By Lemma 3.12, we have that for any \( I \xrightarrow{s} I \), \( \hat{s} = s \). In addition, we have that \( \langle \cdot \rangle \) is a homomorphism. Thus,

\[
\{\langle s_1 \cdot s_2 \rangle, \ast : 1 \} = \langle s_1 \rangle \circ \langle s_2 \rangle = \langle \langle s_1 \rangle \circ \langle s_2 \rangle \rangle = \{\langle s_1 \cdot s_2 \rangle, \ast : 1 \}
\]

\[
\Gamma, x : B, y : C \vdash v : A \quad \Delta \vdash t : B \otimes C \quad \Delta \vdash \delta \otimes (s \cdot t, x.y.v) : A
\]
$\Gamma, x : B, y : C \vdash v : A \quad \Delta \vdash t : B \otimes C$

$\Gamma, \Delta \vdash \delta_\otimes(t, xy.v) : A$

$\Gamma, \Delta \vdash s \bullet \delta_\otimes(t, xy.v) : A$

$\Gamma \otimes \Delta$

$\Gamma \otimes B \otimes C$

$\Gamma \otimes B \otimes C$

$A$

$s \bullet \delta_\otimes(t, xy.v)$

$\Gamma, x : A \vdash t : B$

$\Gamma \vdash \lambda x.t : A \rightarrow B$

$\Gamma \vdash s \bullet \lambda x.t : A \rightarrow B$

$\Gamma \vdash s \bullet \lambda x.t : A \rightarrow B$

$\Gamma, x : A \vdash t : B$

$\Gamma \vdash \lambda x.s \bullet t : B$

$\Gamma \vdash \lambda x.s \bullet t : B$

$\Gamma \vdash \langle \rangle : \top$

$\Gamma \vdash s \bullet \langle \rangle : \top$

$\langle \rangle$

$\langle \rangle$

$\Gamma$
\[
\begin{align*}
\Gamma \vdash t : A & \quad \Gamma \vdash u : B \\
\Gamma \vdash s \cdot \langle t, u \rangle : A \& B \\ 
\Gamma \vdash t : A & \quad \Gamma \vdash u : B \\
\Gamma \vdash s \cdot t : A & \quad \Gamma \vdash s \cdot u : B
\end{align*}
\]
\[ \Gamma \vdash t : A, \Gamma \vdash u : B \quad \Gamma \vdash [t,u] : A \odot B \]

This case is analogous to that of pairs.

**Inductive cases:** The cases where the reduction is deterministic (that is, any case but those due to rules \( \delta^t_0 \) and \( \delta^t_0 \)), are trivial by composition. Therefore, the only interesting case is that of the non-deterministic rules.

The interesting case corresponds to the reductions

\[
\begin{align*}
t &\rightarrow_p r_1 \\
K'[t] &\rightarrow_p K'[r_1] \\
t &\rightarrow_q r_2 \\
K'[t] &\rightarrow_q K'[r_2]
\end{align*}
\]

We proceed by induction on the shape of \( K'[] \).

- If \( K[] = [] \), then this is the basic case of the non-deterministic rules.

- If \( K[] = K'[] + u \). Let

\[
\begin{align*}
f_1 &= [[\Gamma \vdash K'[r_1] : A] \\
g &= [[\Gamma \vdash u : A]
\end{align*}
\]

Then,

\[
\begin{align*}
[[\Gamma \vdash K'[t] : A] &= [[\Gamma \vdash K'[t] + u : A] \\
&= \nabla \circ ([[\Gamma \vdash K'[t] : A] \oplus g) \circ \Delta \\
(by \, IH) &= \nabla \circ ((\nabla \circ (f_1 \oplus f_2) \circ \Delta) \oplus (g)) \circ \Delta \\
(*) &= \nabla_{(p|q)} \circ ((\nabla \circ (f_1 \oplus g) \circ \Delta) \oplus (\nabla \circ (f_2 \oplus g) \circ \Delta)) \circ \Delta \\
&= \nabla_{(p|q)} \circ ([[\Gamma \vdash K'[r_1] : A] \oplus [[\Gamma \vdash K'[r_2] : A]) \circ \Delta
\end{align*}
\]
Where the equality (*) is justified by the following commuting diagram.

\[
\begin{array}{c}
\Gamma \\ \Delta \\
\downarrow \\
\Delta \oplus \Delta \\
\downarrow \\
\Gamma \oplus \Gamma \\
\oplus \\
\downarrow \\
(\Gamma \oplus \Gamma) \oplus (\Gamma \oplus \Gamma) \\
\oplus \\
\downarrow \\
(f_1 \oplus g) \oplus (f_2 \oplus g) \\
\uparrow \\
\Phi \oplus \Phi \\
\oplus \\
\downarrow \\
(A \oplus A) \oplus (A \oplus A) \\
\oplus \\
\downarrow \\
\Phi \oplus \Phi \\
\oplus \\
\downarrow \\
A \oplus A \\
\end{array}
\]

\[
\begin{array}{c}
\naturality \text{ of } \Delta \\
\end{array}
\]

(Lemma 3.28)

(Lemma 3.26)

(Lemma 3.27)

(Lemma 3.16)

Then,

\[
\begin{array}{c}
\Gamma \vdash K'[t] : A \\
= \Gamma \vdash s \bullet K'[t] \\
= \hat{s} \circ \Gamma \vdash K'[t] : A \\
= \hat{s} \circ (\hat{s} \circ f_1) \circ \Delta \\
= \hat{s} \circ ((\Gamma \vdash s \bullet K'[r_1] : A) \oplus (\Gamma \vdash s \bullet K'[r_2] : A)) \circ \Delta \\
= \hat{s} \circ ((\Gamma \vdash K[r_1] : A) \oplus (\Gamma \vdash K[r_2] : A)) \circ \Delta \\
\end{array}
\]

• If $K[] = u \bullet K'[\cdot]$. This case is analogous to the case $K'[\cdot] \bullet s$.

• If $K[] = s \bullet K'[\cdot]$. Let

\[
f_1 = \llbracket \Gamma \vdash K'[r_1] : A \rrbracket \\
f_2 = \llbracket \Gamma \vdash K'[r_2] : A \rrbracket
\]

Then,

\[
\begin{array}{c}
\Gamma \vdash K'[t] : A \\
= \Gamma \vdash s \bullet K'[t] \\
= \hat{s} \circ \Gamma \vdash K'[t] : A \\
= \hat{s} \circ (\hat{s} \circ f_1) \circ \Delta \\
= \hat{s} \circ ((\Gamma \vdash s \bullet K'[r_1] : A) \oplus (\Gamma \vdash s \bullet K'[r_2] : A)) \circ \Delta \\
= \hat{s} \circ ((\Gamma \vdash K[r_1] : A) \oplus (\Gamma \vdash K[r_2] : A)) \circ \Delta
\end{array}
\]
Where the equality (*) is justified by the following commuting diagram.

\[
\begin{array}{cccccc}
\Gamma \oplus \Gamma & \xrightarrow{\Delta} & \Gamma \\
\downarrow & & \\
A \oplus A & \xrightarrow{\nabla_{(0,0)}} & A \\
\downarrow & & \\
A \oplus A & \xrightarrow{\nabla_{(0,0)}} & A
\end{array}
\]

\[\begin{array}{c}
\text{(Lemma 3.16)}
\end{array}\]

\[\mathbf{\Box}\]

- If \(K[] = \delta_1(K'[\cdot], u)\). Then \(\Gamma = \Gamma_1, \Gamma_2\). Let

\[f_1 = \llbracket \Gamma_1 \vdash K'[r_1] : 1 \rrbracket \quad f_2 = \llbracket \Gamma_1 \vdash K'[r_2] : 1 \rrbracket \quad g = \llbracket \Gamma_2 \vdash u : A \rrbracket\]

Then,

\[
\begin{align*}
\llbracket \Gamma \vdash K[t] : A \rrbracket &= \llbracket \Gamma \vdash \delta_1(K'[t], u) : A \rrbracket \\
&= \llbracket \Gamma_1, \Gamma_2 \vdash \delta_1(K'[t], u) : A \rrbracket \\
&= \lambda \circ (\llbracket \Gamma \vdash K'[t] : 1 \rrbracket \otimes g) \\
(\text{by IH}) &= \lambda \circ ((\nabla_{(p,q)} \circ (f_1 \oplus f_2) \circ \Delta) \otimes g) \\
(\text{(*)}) &= \nabla_{(p,q)} \circ ((\lambda \circ (f_1 \otimes g)) \oplus (\lambda \circ (f_2 \otimes g))) \circ \Delta \\
&= \nabla_{(p,q)} \circ (\llbracket \Gamma \vdash K'[r_1], u : A \rrbracket \\
&\quad \oplus \llbracket \Gamma \vdash K'[r_2], u : A \rrbracket) \circ \Delta \\
&= \nabla_{(p,q)} \circ (\llbracket \Gamma \vdash K[r_1] : A \rrbracket \oplus \llbracket \Gamma \vdash K[r_2] : A \rrbracket) \circ \Delta
\end{align*}
\]
Where the equality (*) is justified by the following commuting diagram.

\[
\begin{array}{cccccc}
\Gamma_1 \otimes \Gamma_2 & \xrightarrow{\Delta} & (\Gamma_1 \otimes \Gamma_2) \oplus (\Gamma_1 \otimes \Gamma_2) & \xrightarrow{(f_1 \otimes g) \oplus (f_2 \otimes g)} & (I \otimes A) \oplus (I \otimes A) \\
\downarrow & & \downarrow & & \downarrow \\
(\Gamma_1 \oplus \Gamma_1) \otimes \Gamma_2 & \xrightarrow{\text{(Lemma 3.9)}} & (I \oplus I) \otimes A & \xrightarrow{\text{(Lemma 3.18)}} & A \oplus A \\
\downarrow & & \downarrow & & \downarrow \\
(I \otimes A) & \xrightarrow{\text{(Corollary 3.20)}} & (I \otimes \Gamma_2) \otimes \text{id} & \xrightarrow{\lambda \otimes \lambda} & A \\
\end{array}
\]

- If \( K[] = \delta_1(u, K'[\cdot]) \). Then \( \Gamma = \Gamma_1, \Gamma_2 \). Let

\[
f_1 = \Gamma_2 \vdash K'[r_1] : A \\
g = \Gamma_1 \vdash u : 1
\]

Then,

\[
\begin{aligned}
[\Gamma \vdash K[t] : A] & = [\Gamma \vdash \delta_1(u, K'[t]) : A] \\
& = [\Gamma_1, \Gamma_2 \vdash \delta_1(u, K'[t]) : A] \\
& = \lambda \circ (g \otimes [\Gamma_2 \vdash K'[t] : A]) \\
& \text{(by IH)} = \lambda \circ (g \otimes (\nabla_{[\cdot][\cdot]}(f_1 \oplus f_2) \circ \Delta)) \\
& \text{(*)} = \nabla_{[\cdot][\cdot]}(\lambda \circ (g \otimes f_1)) \oplus (\lambda \circ (g \otimes f_2)) \circ \Delta \\
& = \nabla_{[\cdot][\cdot]}(\lambda \circ (\Gamma \vdash \delta_1(u, K'[r_1]) : A) \\
& \quad \oplus [\Gamma \vdash \delta_1(u, K'[r_2]) : A]) \circ \Delta \\
& = \nabla_{[\cdot][\cdot]}(\Gamma \vdash K[r_1] : A) \oplus [\Gamma \vdash K[r_2] : A]) \circ \Delta
\end{aligned}
\]
Where the equality (*) is justified by the following commuting diagram.

\[
\begin{array}{cccccccc}
\Gamma_1 \otimes \Gamma_2 & \xrightarrow{\Delta} & (\Gamma_1 \otimes \Gamma_2) \oplus (\Gamma_1 \otimes \Gamma_2) \\
\downarrow \text{id} \otimes \Delta & & \downarrow \text{(Corollary 3.20)} & & \downarrow \text{(Lemma 3.9)} \\
\Gamma_1 \otimes (\Gamma_2 \oplus \Gamma_2) & \xrightarrow{\delta_r} & (I \otimes A) \oplus (I \otimes A) \\
\downarrow g \otimes (f_1 \oplus f_2) & & \downarrow \lambda \oplus \lambda & & \downarrow \lambda \\
I \otimes (A \oplus A) & \xrightarrow{\delta_r} & A \oplus A \\
\downarrow \text{id} \otimes \nabla_{\langle p_{\langle q \rangle}, \langle q \rangle \rangle} & & \downarrow \nabla_{\langle p \rangle} \circ \nabla_{\langle q \rangle} & & \downarrow \nabla_{\langle p \rangle} \\
I \otimes A & \xrightarrow{\lambda} & A \\
\end{array}
\]

- If \( K'[] = K'[] \otimes u \). Then \( A = B \otimes C \) and \( \Gamma = \Gamma_1, \Gamma_2 \). Let
  \[
  f_1 = \llbracket \Gamma_1 \vdash K'[r_1] : B \rrbracket \quad f_2 = \llbracket \Gamma_2 \vdash K'[r_2] : B \rrbracket \\
g = \llbracket \Gamma_2 \vdash u : C \rrbracket
  \]

Then,

\[
\llbracket \Gamma \vdash K[t] : A \rrbracket = \llbracket \Gamma_1, \Gamma_2 \vdash K'[t] \otimes u : B \otimes C \rrbracket \\
= \llbracket \Gamma_1 \vdash K'[t] : B \rrbracket \otimes g \\
(\text{by IH}) = (\nabla_{\langle p \rangle} \circ (f_1 \oplus f_2) \circ \Delta) \otimes g \\
(*) = \nabla_{\langle p \rangle} \circ ((f_1 \otimes g) \oplus (f_2 \otimes g)) \circ \Delta \\
= \nabla_{\langle p \rangle} \circ (\llbracket \Gamma_1, \Gamma_2 \vdash K'[r_1] \otimes u : B \otimes C \rrbracket \\
\quad \oplus \llbracket \Gamma_1, \Gamma_2 \vdash K'[r_2] \otimes u : B \otimes C \rrbracket) \circ \Delta \\
= \nabla_{\langle p \rangle} \circ (\llbracket \Gamma \vdash K[r_1] : A \rrbracket \oplus \llbracket \Gamma \vdash K[r_2] : A \rrbracket) \circ \Delta
\]
Where the equality \((*)\) is justified by the following commuting diagram.

\[
\begin{array}{c}
\triangleleft \Delta \rightarrow \triangleleft \Gamma_1 \otimes \Gamma_2 \rightarrow \triangleleft \Gamma_1 \otimes \Gamma_2 \oplus \Gamma_1 \otimes \Gamma_2 \oplus (B \otimes C) \oplus (B \otimes C)
\end{array}
\]

\[\begin{array}{c}
\begin{array}{c}
\Delta \circ \text{id} \\
\delta \\
(f_1 \otimes g) \oplus (f_2 \otimes g)
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\delta \\
\nabla_{\langle \rho \rangle \langle \zeta \rangle} \circ \text{id}
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\nabla_{\langle \rho \rangle \langle \zeta \rangle} \circ \text{id} \circ (f_1 \oplus f_2) \circ \Delta
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\nabla_{\langle \rho \rangle \langle \zeta \rangle} \circ ((g \circ (\text{id} \otimes f_1)) \oplus (g \circ (\text{id} \otimes f_2))) \circ \Delta
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\nabla_{\langle \rho \rangle \langle \zeta \rangle} \circ ([\Gamma \vdash K''[r_1]: A] \oplus [\Gamma \vdash K''[r_2]: A]) \circ \Delta
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\nabla_{\langle \rho \rangle \langle \zeta \rangle} \circ ((g \circ (\text{id} \otimes f_1)) \oplus (g \circ (\text{id} \otimes f_2))) \circ \Delta
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\nabla_{\langle \rho \rangle \langle \zeta \rangle} \circ (\Gamma \vdash K''[r_1]: A) \oplus (\Gamma \vdash K''[r_2]: A) \oplus (\Gamma \vdash K''[r_2]: A)
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\nabla_{\langle \rho \rangle \langle \zeta \rangle} \circ (\Gamma \vdash K''[r_1]: A) \oplus (\Gamma \vdash K''[r_2]: A) \circ \Delta
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\nabla_{\langle \rho \rangle \langle \zeta \rangle} \circ (\Gamma \vdash K''[r_1]: A) \oplus (\Gamma \vdash K''[r_2]: A) \circ \Delta
\end{array}
\end{array}\]

Where the equality \((*)\) is justified by the following commuting diagram.

- If \(K[] = u \otimes K''[]\). This case is analogous to the case \(K''[] \otimes u\).
- If \(K[] = \delta_{\otimes}(K''[], xy.u\). Then \(\Gamma = \Gamma_1, \Gamma_2\). Let

\[
f_1 = [\Gamma_2 \vdash K''[r_1]: B \otimes C] \quad f_2 = [\Gamma_2 \vdash K''[r_2]: B \otimes C]
\]

\[
g = [\Gamma_1, x : B, y : C \vdash u : A]
\]

Then,

\[
[\Gamma \vdash [\Gamma \vdash K[t]: A] = [\Gamma_1, \Gamma_2 \vdash \delta_{\otimes}(K''[t], xy.u): A]
\]

\[\begin{array}{c}
\begin{array}{c}
g \circ (\text{id} \otimes [\Gamma_2 \vdash K''[t]: B \otimes C])
\end{array}
\end{array}\]

(by IH) \[\begin{array}{c}
\begin{array}{c}
g \circ (\text{id} \otimes (\nabla_{\langle \rho \rangle \langle \zeta \rangle} \circ (f_1 \oplus f_2) \circ \Delta))
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\nabla_{\langle \rho \rangle \langle \zeta \rangle} \circ ((g \circ (\text{id} \otimes f_1)) \oplus (g \circ (\text{id} \otimes f_2))) \circ \Delta
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\nabla_{\langle \rho \rangle \langle \zeta \rangle} \circ (\Gamma \vdash K''[r_1]: A) \oplus (\Gamma \vdash K''[r_2]: A) \circ \Delta
\end{array}
\end{array}\]

Where the equality \((*)\) is justified by the following commuting diagram.
\( \Gamma_1 \otimes \Gamma_2 \xrightarrow{id \otimes \Delta} \Gamma_1 \otimes (\Gamma_2 \oplus \Gamma_2) \)

(Corollary 3.20)

\( \Delta \)

\( \Gamma_1 \otimes (\Gamma_2 \oplus \Gamma_2) \xrightarrow{id \otimes (f_1 \oplus f_2)} \Gamma_1 \otimes ((B \otimes C) \oplus (B \otimes C)) \)

(Lemma 3.9)

\( \Gamma_1 \otimes ((B \otimes C) \oplus (B \otimes C)) \xrightarrow{\text{Lemma 3.17}} \Gamma_1 \otimes B \otimes C \)

\( \Gamma_1 \otimes B \otimes C \xrightarrow{\nabla_{\psi}[q]} \Gamma_1 \otimes C \)

(Lemma 3.16)

\( C \oplus C \xrightarrow{\nabla_{\psi}[q]} C \)

- If \( K[] = \delta_\otimes(u, xy.K'[\[]) \). Then \( \Gamma = \Gamma_1, \Gamma_2 \). Let

\[
\begin{align*}
f_1 &= \llbracket \Gamma_1, x : B, y : C \vdash K'[r_1] : A \rrbracket \quad f_2 = \llbracket \Gamma_1, x : B, y : C \vdash K'[r_2] : A \rrbracket \\
g &= \llbracket \Gamma_2 \vdash u : B \otimes C \rrbracket
\end{align*}
\]

\[
\begin{align*}
\llbracket \Gamma \vdash K[t] : A \rrbracket &= \llbracket \Gamma_1, \Gamma_2 \vdash \delta_\otimes(u, xy.K'[t]) : A \rrbracket \\
&= \llbracket \Gamma_1, x : B, y : C \vdash K'[t] : A \rrbracket \circ (id \otimes g) \\
&\text{(by IH)} = \nabla_{\psi}[q] \circ (f_1 \oplus f_2) \circ \Delta \circ (id \otimes g) \\
&\quad \text{(*)} = \nabla_{\psi}[q] \circ (f_1 \oplus f_2) \circ ((id \otimes g) \oplus (id \otimes g)) \circ \Delta \\
&= \nabla_{\psi}[q] \circ ((f_1 \circ (id \otimes g)) \oplus (f_2 \circ (id \otimes g))) \circ \Delta \\
&= \nabla_{\psi}[q] \circ (\llbracket \Gamma_1, \Gamma_2 \vdash \delta_\otimes(u, xy.K'[r_1]) : A \rrbracket \\
&\quad \oplus \llbracket \Gamma_1, \Gamma_2 \vdash \delta_\otimes(u, xy.K'[r_2]) : A \rrbracket) \circ \Delta \\
&= \nabla_{\psi}[q] \circ (\llbracket \Gamma \vdash K[r_1] : A \rrbracket \oplus \llbracket \Gamma \vdash K[r_2] : A \rrbracket) \circ \Delta
\end{align*}
\]
Where the equality (*) is justified by the following commuting diagram.

\[
\begin{array}{ccc}
\Gamma_1 \otimes \Gamma_2 & \xrightarrow{id \otimes g} & \Gamma_1 \otimes B \otimes C \\
\downarrow \Delta & & \downarrow \Delta \\
(\Gamma_1 \otimes \Gamma_2) \oplus (\Gamma_1 \otimes \Gamma_2) & \xrightarrow{(id \otimes g) \oplus (id \otimes g)} & (\Gamma_1 \otimes B \otimes C) \oplus (\Gamma_1 \otimes B \otimes C) \\
& \downarrow f_1 \oplus f_2 & \\
A & \xrightarrow{\nabla_{\langle p,q \rangle}} & A \oplus A
\end{array}
\]

- Let \( K[] = \lambda x.K'[\cdot] \). Then \( A = B \rightarrow C \). Let

\[
f_1 = [\Gamma, x : B \vdash K'[r_1] : C] \quad f_2 = [\Gamma, x : B \vdash K'[r_2] : C]
\]

Then,

\[
[\Gamma \vdash K[t] : A] = [\Gamma \vdash \lambda x.K'[t] : B \rightarrow C] \\
= [B \rightarrow [\Gamma, x : B \vdash K'[t] : C]] \circ \eta^B \\
(by \ IH) = [B \rightarrow \nabla_{\langle p,q \rangle} \circ (f_1 \oplus f_2) \circ \Delta] \circ \eta^B \\
(*) = \nabla_{\langle p,q \rangle} \circ ([B \rightarrow f_1] \circ \eta^B \oplus [B \rightarrow f_2] \circ \eta^B) \circ \Delta \\
= \nabla_{\langle p,q \rangle} \circ ([\Gamma \vdash \lambda x.K'[r_1] : B \rightarrow C] \\
\quad \oplus [\Gamma \vdash \lambda x.K'[r_2] : B \rightarrow C]) \circ \Delta \\
= \nabla_{\langle p,q \rangle} \circ (\Gamma \vdash K[r_1] : A \oplus [\Gamma \vdash K[r_2] : A]) \circ \Delta
\]
Where the equality (*) is justified by the following commuting diagram.

\[
\begin{array}{ccc}
\Gamma & \overset{\eta^B}{\longrightarrow} & [B \to \Gamma \otimes B] \\
\downarrow & & \downarrow \\
\Gamma \oplus \Gamma & \overset{\eta^B \oplus \eta^B}{\longrightarrow} & [B \to (\Gamma \otimes B) \oplus (\Gamma \otimes B)] \\
\downarrow & & \downarrow \\
[B \to \Gamma \otimes B] \oplus [B \to \Gamma \otimes B] & \overset{[B \to f_1 \oplus f_2]}{\longrightarrow} & [B \to C \oplus C] \\
\downarrow & & \downarrow \\
[B \to C] \oplus [B \to C] & \overset{\nabla_{\oplus(\otimes)}}{\longrightarrow} & [B \to C] \\
\end{array}
\]

- If \( K[] = K'[u] \). Then \( \Gamma = \Gamma_1, \Gamma_2 \). Let

\[
\begin{align*}
  f_1 &= \lbrack \Gamma_1 \vdash K'[r_1] : B \to A \rbrack \\
  f_2 &= \lbrack \Gamma_1 \vdash K'[r_2] : B \to A \rbrack \\
  g &= \lbrack \Gamma_2 \vdash u : B \rbrack
\end{align*}
\]

Then,

\[
\begin{align*}
\lbrack \Gamma \vdash K[t] : A \rbrack &= \lbrack \Gamma \vdash K'[t]u : A \rbrack \\
&= \varepsilon \circ (\lbrack \Gamma_1 \vdash K'[t] : B \to A \rbrack \otimes g) \\
&= \varepsilon \circ ((\nabla_{(\otimes(\oplus))} \circ (f_1 \oplus f_2) \circ \Delta) \otimes g) \\
&= \nabla_{(\otimes(\oplus))} \circ ((\varepsilon \circ (f_1 \otimes g)) \oplus (\varepsilon \circ (f_2 \otimes g))) \circ \Delta \\
&= \nabla_{(\otimes(\oplus))} \circ (\lbrack \Gamma \vdash K[r_1]u : A \rbrack + \lbrack \Gamma \vdash K'[r_2]u : A \rbrack) \circ \Delta \\
&= \nabla_{(\otimes(\oplus))} \circ (\lbrack \Gamma \vdash K[r_1] : A \rbrack + \lbrack \Gamma \vdash K'[r_2] : A \rbrack) \circ \Delta
\end{align*}
\]
Where the equality (*) is justified by the following commuting diagram.

\[
\begin{array}{cccccccc}
\Gamma_1 \otimes \Gamma_2 & \xrightarrow{\Delta} & \Delta & \xrightarrow{\Delta \otimes \text{id}} & (\Gamma_1 \otimes \Gamma_2) \oplus (\Gamma_1 \otimes \Gamma_2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(\Gamma_1 \oplus \Gamma_1) \otimes \Gamma_2 & \xrightarrow{(\Gamma_1 \oplus \Gamma_1) \otimes \Delta} & (\Gamma_1 \oplus \Gamma_1) \otimes \Gamma_2 & \xrightarrow{(f_1 \otimes g) \oplus (f_2 \otimes g)} & ([B \to A] \otimes B) \oplus ([B \to A] \otimes B) \\
\downarrow & & \downarrow & & \downarrow \\
([B \to A] \oplus [B \to A]) \otimes B & \xrightarrow{\nabla_{[B \to A] \otimes \text{id}}} & ([B \to A] \oplus [B \to A]) \otimes B & \xrightarrow{\nabla_{[B \to A] \otimes \text{id}}} & A \oplus (\Delta \otimes \text{id}) \\
\downarrow & & \downarrow & & \downarrow \\
[B \to A] \otimes B & \xrightarrow{\epsilon} & \epsilon & \xrightarrow{\epsilon} & A
\end{array}
\]

- If \( K[] = uK'[] \). Then \( \Gamma = \Gamma_1, \Gamma_2 \). Let

\[
\begin{align*}
f_1 &= \llbracket \Gamma_2 \vdash K'[r_1]: B \rrbracket \\
g &= \llbracket \Gamma_1 \vdash u: B \rightarrow A \rrbracket
\end{align*}
\]

Then,

\[
\llbracket \Gamma \vdash K[t]: A \rrbracket = \llbracket \Gamma \vdash uK'[t]: A \rrbracket
= \epsilon \circ (g \otimes \llbracket \Gamma_2 \vdash K'[t]: B \rrbracket)
(by \text{IH}) = \epsilon \circ (g \otimes (\nabla_{[B \to A]} \circ (f_1 \oplus f_2) \circ \Delta))
(*) = \nabla_{[B \to A]} \circ ((\epsilon \circ (g \otimes f_1)) \oplus (\epsilon \circ (g \otimes f_2))) \circ \Delta
= \nabla_{[B \to A]} \circ ([\Gamma \vdash uK'[r_1]: A] \oplus [\Gamma \vdash uK'[r_2]: A]) \circ \Delta
= \nabla_{[B \to A]} \circ ([\Gamma \vdash K[r_1]: A] \oplus [\Gamma \vdash K[r_2]: A]) \circ \Delta
\]
Where the equality (*) is justified by the following commuting diagram.

\[
\begin{array}{ccc}
\Gamma_1 \otimes \Gamma_2 & \xrightarrow{\Delta} & (\Gamma_1 \otimes \Gamma_2) \oplus (\Gamma_1 \otimes \Gamma_2) \\
\downarrow & & \downarrow \\
\Gamma_1 \otimes (\Gamma_2 \oplus \Gamma_2) & \xrightarrow{(\sigma \oplus \sigma) \circ \sigma} & (\Gamma_1 \otimes \Gamma_2) \oplus ([B \to A] \otimes B) \oplus ([B \to A] \otimes B) \\
\downarrow & & \downarrow \\
\downarrow \\
\end{array}
\]

(Corollary 3.20) (Lemma 3.9) (Lemma 3.17) (Lemma 3.16)

\[
\begin{array}{ccc}
[B \to A] \otimes (B \oplus B) & \xrightarrow{\epsilon} & A \\
\downarrow & & \downarrow \\
\end{array}
\]

\[
\begin{array}{ccc}
[\Gamma] = \delta_0(K') & \xrightarrow{f_1 = [\Gamma_1 \vdash K'[r_2] : \circ] \quad f_2 = [\Gamma_1 \vdash K'[r_2] : \circ]} & [\Gamma] = \delta_0(K') \\
\end{array}
\]

Then,

\[
\begin{array}{l}
[\Gamma] = \delta_0(K'[t]) : A \\
= 0 \circ (\Gamma_1 \vdash K'[t] : \circ) \times \text{id} \\
= 0 \circ (\nabla_{[\Gamma]}(q) \circ (f_1 \oplus f_2) \circ \Delta) \times \text{id} \\
(\star) = \nabla_{[\Gamma]}(q) \circ ((0 \circ (f_1 \times \text{id})) \oplus (0 \circ (f_2 \times \text{id}))) \circ \Delta \\
= \nabla_{[\Gamma]}(q) \circ ([\Gamma] \oplus \delta_0(K'[r_1]) : A) \oplus [\Gamma] \oplus \delta_0(K'[r_2]) : A \circ \Delta \\
= \nabla_{[\Gamma]}(q) \circ ([\Gamma] \oplus K[r_1] : A) \oplus [\Gamma] \oplus K[r_2] : A \circ \Delta
\end{array}
\]

Where the equality (*) is justified by the following commuting diagram.

\[
\begin{array}{ccc}
\Gamma_1 \otimes \Gamma_2 & \xrightarrow{\Delta \circ \text{id}} & (\Gamma_1 \otimes \Gamma_2) \oplus (\Gamma_1 \otimes \Gamma_2) \\
\downarrow & & \downarrow \\
(\Gamma_1 \oplus \Gamma_1) \otimes \Gamma_2 & \xrightarrow{(0 \circ f_1 = 0)} & (0 \otimes \Gamma_2) \oplus (0 \otimes \Gamma_2) \\
\downarrow & & \downarrow \\
(0 \otimes 0) \otimes \Gamma_2 & \xrightarrow{0 \circ 0 = 0} & A \oplus A \\
\downarrow & & \downarrow \\
0 \otimes A & \xrightarrow{0} & A \\
\end{array}
\]

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• If $K[] = \langle K'[\cdot], u \rangle$, then $A = B \& C$. Let

\[
\begin{align*}
    f_1 &= [\Gamma \vdash K'[r_1] : B] & f_2 &= [\Gamma \vdash K'[r_2] : B] \\
    g &= [\Gamma \vdash u : C]
\end{align*}
\]

Then,

\[
\begin{align*}
    [\Gamma \vdash K[t] : A] &= [\Gamma \vdash [K'[t], u] : B \& C] \\
    &= ([\Gamma \vdash K'[t] : B] \oplus g) \circ \Delta \\
    \text{(by IH)} &= ((\nabla_{(b)}(q) \circ (f_1 \oplus f_2) \circ \Delta) \oplus g) \circ \Delta \\
    (*) &= \nabla_{(b)}(q) \circ (((f_1 \oplus g) \circ \Delta) \oplus ((f_2 \oplus g) \circ \Delta)) \circ \Delta \\
    &= \nabla_{(b)}(q) \circ ([\Gamma \vdash [K'[r_1], u] : B \& C] \\
    &= \oplus [\Gamma \vdash [K'[r_2], u] : B \& C]) \circ \Delta \\
    &= \nabla_{(b)}(q) \circ ([\Gamma \vdash K[r_1] : A] \oplus [\Gamma \vdash K[r_2] : A]) \circ \Delta
\end{align*}
\]

Where the equality (*) is justified by the following commuting diagram.

\[
\begin{array}{cccccc}
\Gamma & \Delta & \rightarrow & \Gamma \oplus \Gamma & \Delta \oplus \Delta & \rightarrow & (\Gamma \oplus \Gamma) \oplus \Gamma \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\oplus \Delta & \rightarrow & (\Gamma \oplus \Gamma) \oplus (\Gamma \oplus \Gamma) & \rightarrow & (B \oplus B) \oplus C & \rightarrow & (B \oplus C)
\end{array}
\]

(Lemma 3.28)

\[
\begin{align*}
    & \delta \downarrow \\
    & \rightarrow \\
    & (f_1 \oplus f_2) \oplus g \\
    \end{align*}
\]

(Lemma 3.26)

\[
\begin{align*}
    & \|f_1 \oplus g\| + (f_2 \oplus g) \\
    \end{align*}
\]

(Lemma 3.27)

\[
\begin{align*}
    & \nabla_{(b)}(q) \oplus \text{id} \\
    \end{align*}
\]

\[
\begin{align*}
    & \rightarrow \\
    & \rightarrow \\
    & \rightarrow
\end{align*}
\]

• If $K[] = \langle s, K'[\cdot] \rangle$. This case is analogous to the case $\langle K'[\cdot], s \rangle$.

• If $K[] = \pi_1(K'[\cdot])$. Let

\[
\begin{align*}
    f_1 &= [\Gamma \vdash K'[r_1] : A \& B] & f_2 &= [\Gamma \vdash K'[r_2] : A \& B]
\end{align*}
\]

Then,

\[
\begin{align*}
    [\Gamma \vdash K[t] : A] &= [\Gamma \vdash \pi_1(K'[t]) : A] \\
    &= \pi_1 \circ [\Gamma \vdash K[t] : A \& B] \\
    \text{(by IH)} &= \pi_1 \circ \nabla_{(b)}(q) \circ (f_1 \oplus f_2) \circ \Delta \\
    (*) &= \nabla_{(b)}(q) \circ ((\pi_1 \circ f_1) \oplus (\pi_1 \circ f_2)) \circ \Delta \\
    &= \nabla_{(b)}(q) \circ (((\pi_1 \circ f_1) \oplus (\pi_1 \circ f_2)) \circ \Delta) \\
    &= \nabla_{(b)}(q) \circ ([\Gamma \vdash \pi_1(K'[r_1]) : A] \oplus [\Gamma \vdash \pi_1(K'[r_2]) : A]) \circ \Delta \\
    &= \nabla_{(b)}(q) \circ ([\Gamma \vdash K[r_1] : A] \oplus [\Gamma \vdash K[r_2] : A]) \circ \Delta
\end{align*}
\]
Where the equality (*) is justified by the following commuting diagram.

\[
\Gamma \xrightarrow{\Delta} \Gamma \oplus \Gamma \xrightarrow{f_1 \oplus f_2} (A \oplus B) \oplus (A \oplus B) \xrightarrow{\pi_1 \oplus \pi_1} A \oplus A
\]

\[
\begin{array}{c}
\n \\downarrow \n \\n \\n \n\end{array}
\]

(Lemma 3.16)

\[
\begin{array}{c}
\n \\downarrow \n \\n \\n \n\end{array}
\]

\[
\begin{array}{c}
\n \\n \\n \\n \n\end{array}
\]

\[\pi_1\]

\[
\begin{array}{c}
\n \\n \\n \\n \n\end{array}
\]

\[
\begin{array}{c}
\n \\n \\n \\n \n\end{array}
\]

\[\Delta\]

• If \(K[] = \pi_2(K')[[]]\). This case is analogous to the case \(\pi_1(K')[[]]\).

• If \(K[] = \text{inl}(K')[[]]\), then \(A = B \oplus C\). Let

\[
f_1 = \llbracket \Gamma \vdash K'[r_1] : B \rrbracket, \quad f_2 = \llbracket \Gamma \vdash K'[r_2] : B \rrbracket
\]

Then,

\[
\llbracket \Gamma \vdash K[t] : A \rrbracket = \llbracket \Gamma \vdash \text{inl}(K'[t]) : B \oplus C \rrbracket
\]

\[
= i_1 \circ \llbracket \Gamma \vdash K'[t] : B \rrbracket
\]

(by IH) = \(i_1 \circ \nabla_{(p)(q)} \circ (f_1 \oplus f_2) \circ \Delta\)

(*) = \(\nabla_{(p)(q)} \circ (i_1 \oplus i_1) \circ (f_1 \oplus f_2) \circ \Delta\)

\[
= \nabla_{(p)(q)} \circ (i_1 \circ f_1) \oplus (i_1 \circ f_2) \circ \Delta
\]

\[
= \nabla_{(p)(q)} \circ (\llbracket \Gamma \vdash i_1(K'[r_1]) : B \rrbracket \oplus \llbracket \Gamma \vdash i_1(K'[r_2]) : B \rrbracket) \circ \Delta
\]

\[
= \nabla_{(p)(q)} \circ (\llbracket \Gamma \vdash K[r_1] : A \rrbracket \oplus \llbracket \Gamma \vdash K[r_2] : A \rrbracket) \circ \Delta
\]

Where the equality (*) is justified by the following commuting diagram.

\[
\begin{array}{c}
\Gamma \oplus \Gamma \xleftarrow{\Delta} \Gamma
\end{array}
\]

\[
\begin{array}{c}
\n \\downarrow \n \\n \\n \n\end{array}
\]

\[
\begin{array}{c}
\n \\n \\n \\n \n\end{array}
\]

\[\pi_1\]

\[
\begin{array}{c}
\n \\n \\n \\n \n\end{array}
\]

\[\Delta\]

• If \(K[] = \text{inr}(K')[[]]\). This case is analogous to the case \(\text{inl}(K')[[]]\).

• If \(K[] = \delta \oplus (K'[[]], x.u_1, y.u_2)\). Then \(\Gamma = \Gamma_1, \Gamma_2\). Let

\[
f_1 = \llbracket \Gamma_1 \vdash K'[r_1] : B \oplus C \rrbracket, \quad f_2 = \llbracket \Gamma_1 \vdash K'[r_2] : B \oplus C \rrbracket
\]

\[
g_1 = \llbracket x : B, \Gamma_2 \vdash u_1 : A \rrbracket, \quad g_2 = \llbracket y : C, \Gamma_2 \vdash u_2 : A \rrbracket
\]

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Then,

\[
[\Gamma \vdash K[t] : A] = [\Gamma_1, \Gamma_2 \vdash \delta_\oplus(K'[t], x.u_1, y.u_2) : A]
\]

\[
= [g_1, g_2] \circ d \circ ([\Gamma \vdash K'[t] : B \oplus C] \otimes \text{id})
\]

(by IH)

\[
(*) = \nabla_{[(p,q)]} \circ (([g_1, g_2] \circ d \circ (f_1 \oplus f_2) \circ \Delta) \otimes \text{id})
\]

\[
= \nabla_{[(p,q)]} \circ ([\Gamma_1, \Gamma_2 \vdash \delta_\oplus(K'[r_1], x.u_1, y.u_2) : A] \oplus [\Gamma_1, \Gamma_2 \vdash \delta_\oplus(K'[r_2], x.u_1, y.u_2) : A]) \circ \Delta
\]

Where the equality (*) is justified by the following commuting diagram.

\[
\begin{array}{cccc}
\Gamma_1 \otimes \Gamma_2 & \xrightarrow{\Delta \otimes \text{id}} & (\Gamma_1 \otimes \Gamma_2) \oplus (\Gamma_1 \otimes \Gamma_2) \\
\downarrow & & \downarrow \\
(\Gamma_1 \oplus \Gamma_1) \otimes \Gamma_2 & \xrightarrow{d} & ((B \oplus C) \otimes \Gamma_2) \oplus ((B \oplus C) \otimes \Gamma_2) \\
\downarrow & & \downarrow \\
((B \oplus C) \oplus (B \oplus C)) \otimes \Gamma_2 & \xrightarrow{d} & (B \oplus \Gamma_2 \oplus C \otimes \Gamma_2) \oplus (B \oplus \Gamma_2 \oplus C \otimes \Gamma_2) \\
\downarrow & & \downarrow \\
(B \oplus C) \otimes \Gamma_2 & \xrightarrow{\nabla_{[(p,q)]} \otimes \text{id}} & A \oplus A \\
\downarrow & & \downarrow \\
B \oplus \Gamma_2 \oplus C \otimes \Gamma_2 & \xrightarrow{[g_1,g_2]} & A
\end{array}
\]

- If \(K[] = \delta_\oplus(v, x.K'[], y.u_2)\). Then \(\Gamma = \Gamma_1, \Gamma_2\). Let

\[
\begin{align*}
f_1 &= [x : B, \Gamma_2 \vdash K'[r_1] : A] \\
g &= [\Gamma_1 \vdash v : B \oplus C] \\
f_2 &= [x : B, \Gamma_2 \vdash K'[r_2] : A] \\
h &= [y : C, \Gamma_2 \vdash u_2 : A]
\end{align*}
\]
Then,

\[ [\Gamma \vdash K[t] : A] = [[\Gamma_1, \Gamma_2 \vdash \delta_{\oplus}(v, x.K'[t], y.u_2) : A]] \]
\[ = [[x : B, \Gamma_2 \vdash K'[t] : A], h] \circ d \circ (g \otimes \text{id}) \]

(by IH) \[ = [\nabla_{\langle p \rangle \langle q \rangle} \circ (f_1 \oplus f_2) \circ \Delta, h] \circ d \circ (g \otimes \text{id}) \]

\[ = \nabla_{\langle p \rangle \langle q \rangle} \circ (\langle f_1, h \rangle \circ d \circ (g \otimes \text{id})) \oplus (\langle f_2, h \rangle \circ d \circ (g \otimes \text{id})) \circ \Delta \]

Where the equality (*) is justified by the following commuting diagram.

\[ \begin{array}{ccc}
\Gamma_1 \otimes \Gamma_2 & \xrightarrow{\Delta} & (\Gamma_1 \otimes \Gamma_2) \oplus (\Gamma_1 \otimes \Gamma_2) \\
\downarrow & & \downarrow \\
(I \oplus I) \otimes \Gamma_2 & \xrightarrow{\Delta} & ((I \oplus I) \otimes \Gamma_2) \oplus ((I \oplus I) \otimes \Gamma_2) \\
\downarrow & & \downarrow \\
I \otimes \Gamma_2 \oplus I \otimes \Gamma_2 & \xrightarrow{\Delta} & (I \otimes \Gamma_2 \oplus I \otimes \Gamma_2) \oplus (I \otimes \Gamma_2 \oplus I \otimes \Gamma_2) \\
\downarrow & & \downarrow \\
\Gamma_2 \oplus \Gamma_2 & \xrightarrow{\Delta} & (\Gamma_2 \oplus \Gamma_2) \oplus (\Gamma_2 \oplus \Gamma_2) \\
\downarrow & & \downarrow \\
[A] & \xrightarrow{\Delta} & A \oplus A
\end{array} \]
The commutation of the diagram (***) is justified as follows.

\[
[\nabla_{[p]}(q) \odot (f_1 \oplus f_2) \odot \Delta, h] = [\nabla_{[p]}(q) \odot (f_1 \oplus f_2) \odot \Delta, \text{id} \circ h]
\]

\[= \nabla_{[p]}(q) \odot ((f_1 \oplus f_2) \odot \Delta, \Delta \circ h)\]

\[= \nabla_{[p]}(q) \odot \nabla \odot ((f_1 \oplus f_2) \odot \Delta) \oplus (\Delta \circ h)\]

(Naturality of \(\Delta\))

\[= \nabla_{[p]}(q) \odot \nabla \odot ((f_1 \oplus f_2) \odot \Delta) \oplus ((h \oplus h) \circ \Delta)\]

\[= \nabla_{[p]}(q) \odot \nabla \odot (f_1 \oplus f_2 \oplus h \oplus h) \circ (\Delta \oplus \Delta)\]

(***)

\[= (\nabla \oplus \nabla) \odot (f_1 \oplus h \oplus f_2 \oplus h) \circ \Delta\]

\[= ((\nabla \odot (f_1 \oplus h)) \oplus (\nabla \odot (f_2 \oplus h))) \circ \Delta\]

\[= \nabla_{[p]}(q) \odot ([f_1, h] \oplus [f_2, h]) \circ \Delta\]

Where the equality (***') is justified by the following commuting diagram, using the fact that \(\nabla_{11} = \nabla\).

\[\begin{array}{ccc}
\Gamma \oplus \Gamma & \xrightarrow{\Delta \oplus \Delta} & \Gamma \oplus \Gamma \oplus \Gamma \\
\Delta & \downarrow & \\
\Gamma \oplus \Gamma \oplus \Gamma \oplus \Gamma & \xrightarrow{\text{id} \oplus \sigma \oplus \text{id}} & A \oplus A \oplus A \oplus A \\
f_1 \oplus h \oplus f_2 \oplus h & \downarrow & \\
A \oplus A \oplus A \oplus A & \xrightarrow{\nabla \oplus \nabla} & A \oplus A
\end{array}\]

(Lemma 3.21)

(Naturality of \(\sigma\))

(Lemma 3.21)

• If \(K[] = \delta_{[\oplus]}(v, x, u_1, y, K'[]]\) This case is analogous to the case \(\delta_{[\oplus]}(v, x, K'[], y, u_2).\)

• If \(K[] = [K', s]\), then \(A = B \odot C\). This case is identical to the case \(\langle K', s \rangle\).

• If \(K[] = [s, K']\). This case is identical to the case \(\langle s, K' \rangle\).

• If \(K[] = \pi_1(K')\). This case is identical to the case \(\pi_1(K')\).

• If \(K[] = \pi_2(K')\). This case is identical to the case \(\pi_2(K')\).

• If \(K[] = \delta_{[\otimes q]}(K', x, u_1, y, u_2)\).
Then $\Gamma = \Gamma_1, \Gamma_2$. Let

$$f_1 = \llbracket \Gamma_1 \vdash K'[r_1] : B \otimes C \rrbracket \quad f_2 = \llbracket \Gamma_1 \vdash K'[r_2] : B \otimes C \rrbracket$$

$$g_1 = \llbracket x : B, \Gamma_2 \vdash u_1 : A \rrbracket \quad g_2 = \llbracket y : C, \Gamma_2 \vdash u_2 : A \rrbracket$$

Then,

$$\llbracket \Gamma \vdash K[t] : A \rrbracket = \llbracket \Gamma_1, \Gamma_2 \vdash \delta_{\otimes}^{p'q'}(K'[t], x.u_1, y.u_2) : A \rrbracket$$

(by IH) = $\nabla_{[p']\{q'_1\}} \circ (g_1 \oplus g_2) \circ d \circ (\llbracket \Gamma \vdash K'[t] : B \oplus C \rrbracket \otimes \text{id})$

($\ast$) = $\nabla_{[p']\{q'_1\}} \circ ((\nabla_{[p']\{q'_2\}} \circ (g_1 \oplus g_2) \circ d \circ (f_1 \otimes \text{id}))$

$$\oplus (\nabla_{[p']\{q'_2\}} \circ (g_1 \oplus g_2) \circ d \circ (f_2 \otimes \text{id})) \circ \Delta$$

$$= \nabla_{[p']\{q'_1\}} \circ (\llbracket \Gamma_1, \Gamma_2 \vdash \delta_{\otimes}^{p'q'}(K'[r_1], x.u_1, y.u_2) : A \rrbracket$$

$$\oplus \llbracket \Gamma_1, \Gamma_2 \vdash \delta_{\otimes}^{p'q'}(K'[r_2], x.u_1, y.u_2) : A \rrbracket \rrbracket \circ \Delta$$

Where the equality ($\ast$) is justified by the following commuting diagram.

![Diagram](image_url)
• If \( K[] = \delta_{\circ}^{q'} q' (s, x.K'[s], y.u_2). \)

Then \( \Gamma = \Gamma_1, \Gamma_2. \) Let

\[
f_1 = [x : B, \Gamma_2 \vdash K'[r_1] : A] \\
g = [\Gamma_1 \vdash v : B \otimes C] \\
f_2 = [x : B, \Gamma_2 \vdash K'[r_2] : A] \\
h = [y : C, \Gamma_2 \vdash u_2 : A]
\]

\[
[\Gamma \vdash K[t] : A] = \left[ \Gamma_1, \Gamma_2 \vdash \delta^{q'}_{\circ} (v, x.K'[t], y.u_2) : A \right]
\]

(by IH) \[= \nabla_{[p']\{q'\}} \circ \left( (\left[ [x : B, \Gamma_2 \vdash K'[t] : A] \right] \oplus h) \circ d \circ (g \otimes \text{id}) \right) \]

\[= \nabla_{[p']\{q'\}} \circ \left( (\left( \nabla_{(p')\{q'\}} \circ (f_1 \oplus f_2) \circ \Delta \right) \oplus h) \circ d \circ (g \otimes \text{id}) \right) \circ \Delta \]

(\((*)\) = \[= \nabla_{[p']\{q'\}} \circ \left( \left[ \left[ \Gamma_1, \Gamma_2 \vdash \delta^{q'}_{\circ} (v, x.K'[r_1], y.u_2) : A \right] \right] \oplus \left[ \Gamma_1, \Gamma_2 \vdash \delta^{q'}_{\circ} (v, x.K'[r_2], y.u_2) : A \right] \right) \circ \Delta \]

Where the equality \((*)\) is justified by the following commuting diagram.

\[
\begin{array}{c}
\Gamma_1 \otimes \Gamma_2 \xrightarrow{\Delta} (\Gamma_1 \otimes \Gamma_2) \oplus (\Gamma_1 \otimes \Gamma_2) \\
\downarrow \text{\text{(Naturality of \(\Delta\))}} \quad \downarrow \text{\text{(Naturality of \(\Delta\))}} \\
(B \oplus C) \otimes \Gamma_2 \xrightarrow{\Delta} ((B \oplus C) \otimes \Gamma_2) \oplus ((B \oplus C) \otimes \Gamma_2) \\
\downarrow d \quad \downarrow d \oplus d \\
(B \otimes \Gamma_2) \oplus (C \otimes \Gamma_2) \xrightarrow{\Delta \oplus \text{id}} (B \otimes \Gamma_2 \oplus C \otimes \Gamma_2) \oplus (B \otimes \Gamma_2 \oplus C \otimes \Gamma_2) \\
\downarrow \Delta \oplus \text{id} \quad \downarrow (f_1 \oplus h) \oplus (f_2 \oplus h) \\
(B \otimes \Gamma_2 \oplus B \otimes \Gamma_2) \oplus (C \otimes \Gamma_2) \xrightarrow{\delta} (A \oplus A) \oplus (A \oplus A) \\
\downarrow (f_1 \oplus f_2) \oplus h \quad \nabla_{[p']\{q'\}} \oplus \nabla_{[p']\{q'\}} \\
(A \oplus A) \oplus A \xrightarrow{\gamma} A \oplus A \\
\nabla_{[p']\{q'\}} \oplus \text{id} \quad \nabla_{[p']\{q'\}} \oplus \text{id} \\
A \oplus A \xrightarrow{\delta} A
\end{array}
\]
\[ K[] = \delta^{p'q'}_{\odot}(v, x.u_1, y.K'[]). \] This is analogous to the case \[ \delta^{p'q'}_{\odot}(v, x.K'[], y.u_2). \]