A comparison of numerical solvers for the delay eigenvalue problem of coupled oscillators

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Abstract. In this work, we conduct a comparison for various delay differential equations’ solvers to determine the strength and weakness points of each one. The comparison is clarified through a numerical example in which we consider the coupled harmonic oscillators. We obtain the eigenvalues of the delayed coupled harmonic oscillators system by solving the governing equations by all considered methods. However, the methods of concern in this paper are as follows: the Taylor series expansion, Rekasius’s substitution, self-consistent approach, and the Krylov method with Chebyshev interpolation. The main aspects that we investigate in this paper are the stability, robustness, and numerical efficiency. We found that the Krylov method with Chebyshev interpolation performs better than all the other considered ones.

1. Introduction

Commonly, ordinary differential equations (ODEs) are used to describe the dynamics of physical processes assuming that the interactions are instantaneous, and hence the behavior is determined solely by the present state. This is certainly valid for the cases where the transient behavior is slow compared to the duration of interaction passage. However, for fast dynamics, considering the delay and the past states in mathematical modeling can provide a more reliable description [1]. The applications in which the delay parameter is inevitable while modeling physical phenomena are countless and cover practically almost all fields of science [2–5]. By imposing causality, the resulted governing equations in such cases are systems of delay differential equations (DDEs). Thus, developing numerically efficient solvers for DDEs is a time needed task in science and engineering [6]. However, working with time-delay dynamical systems is challenging since such systems are infinite-dimensional, and the dynamics are significantly influenced by instability and chaos [7, 8]. The complexity of dealing with DDEs escalates when delays are time-dependent, or more precisely, state-dependent. In this paper, the considered systems are autonomous and have one constant delay.

Let \( \tau \in \mathbb{R}^+ \) be the delay of the system, and \( X \) denotes the state space of continuous functions \( C([-\tau, 0], \mathbb{R}^n) \). The dynamical equation of the considered system can be written as:

\[
\dot{y}(t) = f(y_t), \quad y \in \mathbb{R}^n,
\]
for \( t \in \mathbb{R}, f : C \rightarrow \mathbb{R}^n \), and \( y_t \in X \), which is defined as

\[
y_t(\theta) := y(t + \theta), \quad \theta \in [-\tau, 0],
\]

The solution of Eq. (1) in a compact representation can be

\[
y(t) = \begin{cases} 
\psi(0) + \int_\mu^t f(y_s) \, ds, & t \geq \mu, \\
\psi(t), & t \in [-\tau_{\text{max}}, \mu].
\end{cases}
\]

Solving Eq. (3) by ordinary substitution (i.e., \( Ae^\lambda t \)) yields transcendental equations that do not possess a closed-form solution.

To solve DDEs, one must apply numerical methods (which are mostly applicable for ODEs [9]) such as Taylor series approximation [10], Krylov method with efficient interpolation [11, 12], iterative self-consistent methods, and the Rekasius’s substitution [6]. The Taylor series method is a straightforward approach that turns the transcendental equations into algebraic equations and hence converts DDEs to ODEs. The stability characteristics of this method are considered by Insperger [10], which discusses some restrictions; nevertheless, it is widely used. The second method (i.e., Krylov method) relies on the fact that any linear delay differential equation can be equivalently represented by a linear infinite-dimensional ordinary differential operator [11, 12]. The infinite-dimensional is then reduced by proper time-space interpolation. For Rekasius’s substitution method, an extensive stability analysis is conducted by Ebenbauer and Allgower [6] that illustrate its validity for a single time-delay. In reality, numerical methods that have been proposed and used to tackle the problem of DDEs are many; yet, one can conclude that the ones mentioned previously are the most popular and consistent methods.

In this paper, we compare the computational accuracy and efficiency of four numerical methods to study coupled harmonic oscillators with delayed interaction. The considered numerical methods are as follows: the Taylor series approximation, Krylov approach with Chebyshev interpolation, standard self-consistent method, and Rekasius’s substitution. The mathematical formulation, assumptions of each approach, and the imposed conditions are addressed. For the sake of comparison, the aspects examined in this article are numerical stability, robustness, and numerical efficiency. Since the stability properties for each method cannot be covered entirely in this article, they will be touched briefly.

As aforementioned, the comparison focuses on applying these methods to the delay eigenvalue problem (DEP) associated with the coupled harmonic oscillators (CHOs) model owing to its significance in the interpretation of various physics phenomena [13]. The proposed model of delayed CHOis is describing a linear autonomous system in which the delay is a single constant parameter, and thereby the model aligns with the prime purpose of analyzing such systems in this article. Ultimately, the paper concludes with multiple remarks that highlight the strength and weakness points in each method of concern, then recommend which one is proven to be more efficient in solving the DEP of the delayed CHOIs model. The comparison shall not consider the rigorous mathematical reasoning behinds each method since that beyond the scope of the work. Furthermore, the algorithms stemmed from each approach aiming to solve the DEP via computer software are not discussed in this article.

2. The delayed coupled harmonic oscillators

The conventional CHOIs model involves Hooke coupling in the governing equations for the dynamics of two masses \((m_1, m_2)\) coupled together by a spring with a coupling constant \((\kappa)\) and each mass is connected to a fixed point by another two springs with spring constants \((k_1, k_2)\) as illustrated in Figure 1. Now, the delay \((\tau)\) comes into play in the interaction between the two
Figure 1. Two coupled harmonic oscillators with spring constants $k_1, k_2$ and masses $m_1, m_2$. The coupling between the two masses is represented by $\kappa$. The oscillation is driven by an external force $F(t)$, which acts upon $m_1$ to initiate the motion.

masses, i.e., in the coupling term. The system allows energy exchange between the two masses with loss rates $(\gamma_1, \gamma_2)$. The displacements from equilibrium are expressed by $x_1(t)$, and $x_2(t)$. We consider solving the homogeneous system such that the system is triggered initially. In other words, $F(t)$ is assumed to be zero, $x_1(t) \neq 0$, and $x_2(t) = 0$. The resulted equations of motion has $f(\phi)$ (as defined in Eq. 1) become

$$f(\phi) = M\phi(0) + L\phi(\theta), \quad (4)$$

where

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\kappa + k_1}{m_1} & -\gamma_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{\kappa + k_2}{m_2} & -\gamma_2 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\kappa}{m_1} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\kappa}{m_2} & 0 & 0 & 0 \end{pmatrix}, \quad M \in \mathbb{R}^{4 \times 4}, \quad L \in \mathbb{R}^{4 \times 4}. \quad (5)$$

Re-writing Eq. 1 & Eq. 3, respectively in terms of the given definition of $f(\phi)$ in Eq. 4 and system’s parameters in Eq. 5 results in the following

$$\dot{y}(t) = My(t) + Ly(t + \theta), \quad \theta \in [-\tau, 0], \quad y = \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix} \in \mathbb{R}^4. \quad (6)$$

$$y(t) = \begin{cases} \psi(0) + \int_\mu^t My(s) + Ly(s + \theta) \, ds, & t \geq \mu \\ \psi(t), & t \in [-\tau, \mu]. \end{cases}, \quad \mu \in \mathbb{R}. \quad (7)$$

By Eq. 7, the formulation of the model is complete, and the next subsections investigate the different considered methods to solve the problem.

2.1. Taylor series expansion
As known by the Taylor series, an infinitely continuous function can be expanded as an infinite sum of function’s derivatives at a single point. Clearly, as the order of expansion increases, the accuracy of approximation improves as well. Practically, Taylor’s expansion is a numerical
method used frequently. For the model under investigation, the delayed term about \( \theta = 0 \) can be approximated by Taylor’s expansion as

\[
y(t + \theta) = \sum_{n=0}^{\infty} \frac{y^{(n)}(t)}{n!} \theta^n = y(t) + \theta \dot{y}(t) + \frac{1}{2} \theta^2 \ddot{y}(t) + ... \tag{8}
\]

since \( \theta \in [-\tau, 0] \) (i.e., negative value) where \( \tau \in \mathbb{R}^+ \), substituting a specific delay leads to an alternating series. Consequently, a higher order of expansion results in instability in the system of ODEs, which can be detected by the Routh-Hurwitz stability criterion \[10\]. The method is performing well in terms of numerical efficiency in solving differential-algebraic equations \[14\], which can infer its low computational cost in solving DDEs as well. As the value of the delay increases, the accuracy decreases to the extent that the approximated system might show different qualitative characteristics and becomes inapplicable to capture the dynamic of the real system; however, if the delayed part of \( f(\phi) \) is tiny compared to the instantaneous part, then there is a possibility that large values of delay can be handled \[10\]. Thus, the method should be used in particular applications to analyze the asymptotic behavior of systems.

2.2. Rekasius’s substitution

Similar to the Taylor series method, Rekasius’s substitution aims to transform the transcendental equations to polynomial equations, thereby obtaining a simpler model that can be solved relatively easier \[6\]. Inspecting the stability requires a lower bound determination for the maximum possible tolerated delay in the system. An efficient technique applied to the stability analysis of linear systems with the Rekasius’s substitution is the sum of squares \[6\]. Assuming \( y(t) = \sum_k y_k(t) \) such that \( y_k(t) = c_k \exp(\lambda_k t) v_k \), and \( M + L \) yields a stable system for \( \theta = 0 \), i.e., Hurwitz criterion, the characteristic equation of (6) is

\[
det(\lambda_k I - M - L \exp(\lambda_k \theta)) = 0, \tag{9}
\]

where \( n \rightarrow \infty \). The term causing the problem in solving (9) is \( \exp(\lambda_k \theta) \), which can be substituted with Rekasius’s substitution as follows

\[
\exp(\lambda_k \theta) = \frac{1 - T_k \lambda_k}{1 + T_k \lambda_k}, \quad T_k = -\frac{1}{\lambda_k} \tanh \frac{\lambda_k \theta}{2}, \quad T \geq 0. \tag{10}
\]

Substituting Eq. 10 in Eq. 9 yields

\[
det((\lambda_k I - M)(1 - T_k \lambda_k) - L(1 + T_k \lambda_k)) = 0. \tag{11}
\]

Different approaches can be applied to find the roots of Eq. 11, for instance, Routh-Hurwitz, and the sum of squares. The former is exact but computationally hard; yet, the latter is more efficient and well-studied by Ebenbauer and Allgower \[6\].

2.3. Self-consistent method

This method is iterative. We start by assuming that the solution can take the following form

\[
y(t) = V \exp(\Lambda t)c, \tag{12}
\]

such that \( V = [v_1, v_2, ..., v_n] \in \mathbb{R}^{4 \times n} \) combining the eigenvectors, \( \Lambda = \text{diag}[\lambda_1, \lambda_2, ..., \lambda_n] \in \mathbb{R}^{n \times n} \) for the eigenvalues, and \( c^T = [c_1, c_2, ..., c_n] \in \mathbb{R}^{n \times 1} \) represents the constants’ vector. The remaining terms after substituting (12) into (6) and simplifying the resultant expression are

\[
C_m = M + L \exp(C_m \theta), \quad C_m = V \Lambda V^{-1}. \tag{13}
\]
As can be seen from Eq. 13, the term $C_m$ appears in the R.H.S as well as in the L.H.S, which explains the naming of the method. The solution can be obtained by assuming a start value $C_{m_0}$, then carrying out the iteration process until $\|C_{m_k} - C_{m_{k-1}}\| < \epsilon$, where $\epsilon \to 0$. Since the process is sequential, obtaining better accuracy requires more iterations. Although the convergence is guaranteed using this approach, it is computationally inefficient.

2.4. Krylov method
Spectral discretization of an infinite-dimensional operator is a traditional treatment in dealing with the DEP of the DDEs. The discretization is applied by many [11,15,16] as an attempt to solve DDEs. In the Krylov method, the main concept of the method is to utilize the fact that any linear delay differential equation is equivalently representable by a linear infinite-dimensional ordinary differential operator. This infinite dimensionality is reduced to a finite one by efficient Chebyshev interpolation [11,12]. Interpolation generates errors that can be then corrected by post-processing. The formulation of the method is equivalent to the one shown in Eq. 1, in which they define $f(\phi)$ presented in Eq. 4 as an infinite-dimensional linear operator.

3. Numerical example
We start by assuming that the two masses ($m_1, m_2$), spring constants ($k_1, k_2$), and loss rates ($\gamma_1, \gamma_2$) depicted in Figure 1 are the same. In other words, the two oscillators in the system are identical. Yet, the application can be readily extended to the non-identical case. The values of the parameters used in this demonstration are selected carefully to ensure that the system is physically stable. In our analysis, we assign the following values to the system's parameters: 1 to $m_1, m_2, k_1,$ and $k_2$; 0.01 to $\kappa, \gamma_1$, and $\gamma_2$. The main goal is to find the eigenvalues of the system by the considered methods. Capturing the dynamics very well with reasonable implementation complexity leads to expand the Taylor series to the 40th order of expansion. Similarly, in Rekasius's substitution, and Self-consistent methods, we used 5000 iterations to reach the minimum possible value of error. Finally, to have comparable results with the Taylor series method, 40 Chebyshev nodes are used for the interpolation implemented for the Krylov method.

The change in the real part of the two fundamental eigenfrequencies associated with the conventional CHO's model with the variation in correspondence to the delay is demonstrated in Figure 2. The results are intriguing since there is a critical point in which the eigenfrequencies intersects and swap, which definitely affects the exchange of energy between the two oscillators. Despite the fact that eigenfrequencies are changing by varying the delay, the stability is achieved

![Figure 2](image-url)

**Figure 2.** The real part of the two fundamental eigenfrequencies ($\sigma$) versus the delay ($\tau$) using (a) Taylor series expansion, (b) Rekasius’s substitution, (c) Self-consistent, and (d) Krylov methods.
Figure 3. The imaginary part of the two fundamental eigenfrequencies ($\omega$) versus the delay ($\tau$) using (a) Taylor series expansion, (b) Rekasius’s substitution, (c) Self-consistent, and (d) Krylov methods.

since their real part never exceed zero. By examining the system’s stability, one can infer that it is independent of $\tau$ and highly dependent on the coupling $\kappa$.

Moreover, the change in the imaginary part subject to the delay is considered in the analysis as well. As illustrated in Figure 3, the behavior of the imaginary part exhibits some crossing of the eigenfrequencies. According to [13], anti-crossing behavior is directly proportional to the coupling constant, which is fixed in our analysis. Hence, the prime delay impact on the system’s eigenfrequencies is enforcing them to meet at a certain point. It is important to note that the imaginary part has an identical image that is produced by the reflection over the $x$-axis, which is natural since our system has four first-order differential equations.

4. Comparison study

Although Figure 2 and Figure 3 cannot tell the differences among DDEs’ solvers clearly since the behavior of the eigenfrequencies is almost identical, some methods have more appealing features than the others. For numerical stability, as aforementioned, the physical parameters are chosen such that the system is physically stable. Hence, the real part of the eigenvalues stemmed from solving the equations of the model should lie in the left half-plane so that the system is a stable one. Concerning the robustness, it is here assessed by the number and accuracy of the calculated eigenpairs associated with the system. In other words, it is after the effective capturing of the system’s dynamics, which can be evaluated by the number of the resulted eigenvalues and the description of the real physical behavior. The last aspect of concern is the numerical efficiency, which refers to the speed of calculations as well as the memory requirements. For instance, the Krylov method performs better in terms of stability compared with other approaches, as shown in Table 1. As the Krylov method depends on the accuracy of interpolation, using Chebyshev interpolation guarantees improving accuracy with an increasing number of used nodes. As a result, the resulted solutions are more stable, especially with an increasing number of nodes. In contrast, the Taylor series expansion is an “alternating” series due to the delay. According to the Routh-Hurwitz criteria, this can result in numerical instability. The stability analysis for Rekasius’s substitution is excessively discussed in [6], where the technique of the sum of squares is deployed as an efficient alternative to the Routh-Hurwitz criteria. The same analysis and conclusions apply to the self-consistent method.

As for robustness, the Rekasius’s substitution and the self-consistent methods are iterative approaches starting from some initial guess. The obvious guesses are the four eigenpairs for the system without considering the delay. Consequently, they result in only four eigenpairs, which is not enough to capture the full dynamics since the delay converts the system into an infinite-dimensional one. Therefore, a higher number of eigenpairs are needed, which can be
realized by both the Taylor series expansion and the Krylov method. Yet, the latter is more accurate and hence more robust.

The last considered aspect is numerical efficiency. Since both the Rekasius’s substitution and the self-consistent methods are iterative approaches, they require more time to be executed. That is, they are not as fast as the Taylor series expansion. The reason for the high numerical efficiency of the latter method is mainly the high sparsity of the resulted matrices, which are almost diagonal as well. Predictably, the Krylov method has relatively lower numerical efficiency compared with the Taylor series expansion owing to its robustness, the post-processing time, and the less sparsity of the resulted matrices.

Thus, if the accuracy is more vital than the numerical efficiency, then the Krylov method is the best choice. Conversely, if there is a limitation in the computational resources, then either Rekasius’s substitution or self-consistent methods can perform well in that case. The stability issue that the Taylor series expansion suffers from places it in the last position except for very small $\tau$.

5. Conclusions
In conclusion, investigating DDEs is a developing field that attracts more attention due to its wide range of applications in science and engineering. The fact that addressing the existence of the delay in a large class of models is an inevitable action contributes to enhancing the numerical methods that can solve DEPs. We consider in this paper four different DDEs’ solvers, namely, the Taylor series expansion, Rekasius’s substitution, self-consistent, and Krylov methods. A numerical example of a delayed system is solved completely by all methods under examination. To summarize the main results, the challenge that DEP introduces is the infinite-dimensional property of the system, which is tackled numerically by different interpolations, such as Chebyshev interpolation in the Krylov method and Taylor series expansion, to mention a few. As the number of discretization points increases, the accuracy of capturing the real dynamics of the system increases as well. For a big delay, the Taylor series fails to approximate DDEs and should not be used to solve the DEP unless the coefficients of the delayed terms are small enough. The Rekasius’s substitution and self-consistent methods have common properties and can be employed interchangeably with moderate performance. The approach that shows more reliability in describing the delayed CHOs is the Krylov method with Chebyshev interpolation; nevertheless, it is computationally more expensive than the other three methods.

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