$L^p$-Solutions and Comparison Results for Lévy Driven BSDEs in a Monotonic, General Growth Setting

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Abstract

We present a unified approach to $L^p$-solutions ($p > 1$) of backward stochastic differential equations (BSDEs) driven by Lévy processes in a general setting and prove existence, uniqueness and comparison results. The generator functions obey a time-dependent extended monotonicity condition in the $y$-variable and have general growth in $y$. Within this setting, the results generalize those of Royer [23], Yin and Mao [29], Yao [28], Kruse and Popier [14] and Geiss and Steinicke [11].

Keywords: Backward stochastic differential equation; Lévy process; $L^p$-solutions

Mathematics Subject Classification: 60H10

1 Introduction

The existence and uniqueness of $L^p$-solutions to a backward stochastic differential equation (BSDE) driven by a Lévy process has been investigated in various specific settings. In this paper we both unify and simplify various settings for a general BSDE framework and further relax the assumptions for guaranteeing unique $L^p$-solutions, $p > 1$, to a BSDE with terminal condition $\xi$ and generator $f$ that satisfies a monotonicity condition. An $L^p$-solution is a triplet of adapted processes $(Y, Z, U)$ from suitable $L^p$-spaces (defined in section 2) which satisfies a.s.

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_{[t,T] \times \mathbb{R}^d \setminus \{0\}} U_s(x)\tilde{N}(ds, dx),$$

for each $t \in [0, T]$, where $W$ is a Brownian Motion, $\tilde{N}$ is a compensated Poisson random measure independent of $W$. The BSDE (1) itself will be denoted by $(\xi, f)$.

1.1 Related Works

For nonlinear BSDEs $(\xi, f)$ driven by Brownian motion, existence and uniqueness results were first systematically studied by Pardoux and Peng [22] with $(\omega, y, z) \mapsto f(\omega, y, z)$ Lipschitz in $(z, y)$ and $\xi$ square integrable. The importance of BSDEs in mathematical finance and stochastic optimal control was further elaborated by various works e.g. by El Karoui et al. [8] who considered Lipschitz generators, $L^p$-solutions and Malliavin derivatives of BSDEs in the Brownian setting. The ambition to

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weaken the assumptions on $f$ and $\xi$ to still guarantee a unique solution gave birth to a large number of contributions, where – in the case of a generator with Lipschitz dependence on the $z$-variable – at least a few should be mentioned herein: Pardoux [21] and Briand and Carmona [2] considered monotonic generators w.r.t. $y$ with different growth conditions. Mao [16] used the Bihari-LaSalle inequality to generalize the growth condition. Briand et al. [3] proved existence and uniqueness of a solution in the case where the generator may have a general growth in the $y$-variable and $f(\cdot, 0, 0), \xi$ belong to $L^p$ for some $p \geq 1$. Generalizing the driving randomness, Tang and Li [27] and many other papers studied BSDEs including jumps by a Poisson random measure independent of the Brownian Motion.

Treating BSDEs in the case of quadratic growth in the $z$-variable, a considerable amount of articles was published in the recent years starting from the seminal paper of Kobylanski [13] in 2000 to recent papers using BMO methods such as [4] in the Brownian case or also comparison theorems like in [10] who consider an additional Poisson random measure as driving noise. We skip detailed comments in the direction of quadratic growth BSDEs as we will not consider this setting in our article.

Recent and most relevant for the present paper are the results by Kruse and Popier [14] considering $L^p$-solutions for BSDEs under a monotonicity condition driven by Brownian motion, a Poisson random measure and an additional martingale term. They included the case of random time horizons. Yao [28] studied $L^p$-solutions to BSDEs with a finite activity Lévy process for $1 < p < 2$ and used a generalization for the monotonicity assumption similar to the one of [16] and also used in Sow [25]. Generalizing the $L^p$-assumptions for the monotonic generator setting, in [9] the existence (and uniqueness in [5]) of a solution was proven for a scalar linearly growing BSDE when the terminal value $\xi$ admitted integrability of $|\xi| \exp\left(\mu \sqrt{2 \log(1 + |\xi|)}\right)$ for a parameter $\mu > \mu_0$, for some critical value $\mu_0 > 0$. Also a counterexample shows that the preceding integrability is not sufficient to guarantee the existence of the solution. See also [9] for the critical case $\mu = \mu_0$.

1.2 Main Contribution

In [15], Kruse and Popier designed function spaces such that their results of [14] extend to $1 < p < 2$. In this article, we show that the BSDEs’ solutions for $1 < p < 2$ are even contained in the usual $L^p$ spaces as defined for $p \geq 2$. An additional martingale term $M$ orthogonal to $W$ and $\tilde{N}$ as used by Kruse and Popier [14] could also be added to our setting as an extension, as the careful analysis in their paper shows how the bracket process $[M]$ has to be treated in an a priori estimate. Nonetheless we decided to omit the martingale term to avoid more technicalities in this paper. The paper of Geiss and Steinicke [11], placed in a 1-dimensional $L^2$-setting, requires a linear growth condition on the generator and needs approximation results for the comparison theorem, while the present setting allows for general growth, uses a simpler approximation technique for the comparison theorem avoiding deep lying measurability results and, for $p \geq 2$, only requires comparison of the generators on the solution processes.

In contrast to [3], [14], et. al., this article uses the more general monotonicity condition with a non-decreasing, concave function $\rho$ to relax the generator’s dependence on $y$ (see also Mao [16]). This includes e.g. continuities of the type as $y \mapsto -y \log(|y|)$ possesses at $y = 0$. Within this setting, similar a priori estimates still hold true in order to guarantee existence and uniqueness of an $L^p$-solution, $p \geq 2$ to a BSDE. Also the results of Yao [28] are extended in the sense that we do not restrict the jump process to require a finite Lévy measure. Hence, we close several gaps in the theoretical understanding of solutions to BSDEs driven by a Lévy process, for the class of generators which are Lipschitz in the $z$- and $u$-variables.
Our proofs for existence and uniqueness are inspired by [3] along with [14] and [8]. In that spirit, before starting the main proofs, we obtain useful a priori estimates for the solution processes. For the comparison theorem we enhance ideas and simplify proofs from [11], that already generalized the comparison result from [23].

1.3 Paper Structure

This paper is organized in the following way: First we establish the setting in section 2 and state the assumptions and the main theorem (section 3). After developing a priori estimates in section 4 we finally prove existence and uniqueness of \( L^p \)-solutions for \( p > 1 \) in section 5 and end up with the comparison results for \( p \geq 2 \) and \( 1 < p < 2 \) in section 6.

2 Setting

Throughout the paper, we will use the following setting: In dimension \( d \geq 1 \), let \(| \cdot |\) denote the Euclidean distance. For \( x, y \in \mathbb{R}^d \) we write \( \langle x, y \rangle = \sum_{i=1}^d x_i y_i \), and for \( z \in \mathbb{R}^{d \times k} \) with full column rank, we denote \( |z|^2 = \text{trace}(zz^*) \). The operations \( \min(a, b) \) and \( \max(a, b) \) will be denoted by \( a \land b \) and \( a \lor b \).

Let \( X = (X_t)_{t \in [0,T]} \) be a càdlàg Lévy process with values in \( \mathbb{R}^d \) on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with Lévy measure \( \nu \). By \((\mathcal{F}_t)_{t \in [0,T]}\) we will denote the augmented natural filtration of \( X \) and assume that \( \mathcal{F} = \mathcal{F}_T \). Equations or inequalities for objects on these spaces are considered up to \( \mathbb{P} \)-null sets. Conditional expectations \( \mathbb{E}[\cdot | \mathcal{F}_t] \) will be denoted by \( \mathbb{E}_t \).

The Lévy-Itô decomposition of \( X \) can be written as

\[
X_t = at + \Sigma W_t + \int_{[0,t] \times \{|x| \leq 1\}} x\tilde{N}(ds, dx) + \int_{[0,t] \times \{|x| > 1\}} xN(ds, dx),
\]

where \( a \in \mathbb{R}^d \), \( \Sigma \in \mathbb{R}^{d \times k} \) with full column rank, \( W \) is a \( k \)-dimensional standard Brownian motion and \( N \) (\( \tilde{N} \)) is the (compensated) Poisson random measure corresponding to \( X \). For the general theory of Lévy processes, we refer to [1] or [24]. This setting can be adapted to a pure jump process, if one sets \( \Sigma = 0 \) and omits the stochastic integrals with respect to \( W \) in the BSDE. Generalizing the above setting slightly, \( \mathcal{F} \) can be assumed to be generated by a \( k \)-dimensional Brownian Motion \( W \) and an independent (from \( W \)) compensated Poisson random measure \( \tilde{N} \) on \( \mathbb{R}^\ell \setminus \{0\} \) for some \( \ell \geq 1 \), the assumption of a driving process \( X \) can in principle be omitted. For convenience however, we will stick to the setting emerging from a driving Lévy process \( X \).

2.1 Notation

Let \( 0 < p \leq \infty \).

- We use the notation \((L^p, \| \cdot \|_p) := (L^p(\Omega, \mathcal{F}, \mathbb{P}), \| \cdot \|_{L^p})\) for the space of all \( \mathcal{F} \)-measurable functions \( g : \Omega \rightarrow \mathbb{R}^d \) with

\[
\|g\|_{L^p} := \left( \int_\Omega |g|^p d\mathbb{P} \right)^{1/p} < \infty \text{ if } p < \infty , \text{ and } \|g\|_{L^\infty} := \text{esssup}_{\omega \in \Omega} |g(\omega)| < \infty.
\]
• Let $\mathcal{S}^p$ denote the space of all $(\mathcal{F}_t)_{t\in[0,T]}$-progressively measurable and càdlàg processes $Y : \Omega \times [0, T] \to \mathbb{R}^d$ such that

$$\|Y\|_{\mathcal{S}^p} := \left\| \sup_{0 \leq t \leq T} |Y_t| \right\|_p < \infty.$$  

• We define $L^p(W)$ as the space of all progressively measurable processes $Z : \Omega \times [0, T] \to \mathbb{R}^{d \times k}$ such that

$$\|Z\|_{L^p(W)} := \left\| \left( \int_0^T |Z_s|^2 \, ds \right)^{\frac{1}{2}} \right\|_p < \infty.$$  

• Let $\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}$. We define $L^p(\tilde{N})$ as the space of all random fields $U : \Omega \times [0, T] \times \mathbb{R}_0^d \to \mathbb{R}^d$ which are measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_0^d)$ (where $\mathcal{P}$ denotes the predictable $\sigma$-algebra on $\Omega \times [0, T]$ generated by the left-continuous $(\mathcal{F}_t)_{t\in[0,T]}$-adapted processes, and $\mathcal{B}$ is the Borel-$\sigma$-algebra) such that

$$\|U\|_{L^p(\tilde{N})} := \left\| \left( \int_0^T \int_{\mathbb{R}_0^d} |U_s(x)|^2 \nu(dx) ds \right)^{\frac{1}{2}} \right\|_p < \infty.$$  

• $L^2(\nu) := L^2(\mathbb{R}_0^d, \mathcal{B}(\mathbb{R}_0^d), \nu), \| \cdot \| := \| \cdot \|_{L^2(\nu)}.$  

• $L^p([0,T]) := L^p([0,T], \mathcal{B}([0,T]), \lambda)$, where $\lambda$ is the Lebesgue measure on $[0,T]$.  

• $L_{\text{loc}}(W)$ denotes the space of $\mathbb{R}^{d \times k}$-valued progressively measurable processes, such that for every $t > 0$,

$$\int_0^t |Z_s|^2 ds < \infty, \quad \mathbb{P}\text{-a.s.}$$  

• $L_{\text{loc}}(\tilde{N})$ denotes the space of $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_0^d)$-measurable random fields $U : \Omega \times [0, T] \times \mathbb{R}_0^d \to \mathbb{R}^d$, such that for every $t > 0$,

$$\int_0^t \int_{\mathbb{R}_0^d} (|U_s(x)|^2 \vee |U_s(x)|) \nu(dx) ds < \infty, \quad \mathbb{P}\text{-a.s.}$$  

• With a slight abuse of notation we define

$$L^p(\Omega; L^1([0,T])) := \left\{ F : \Omega \times [0, T] \to \mathbb{R} : F \text{ is } \mathcal{F} \otimes \mathcal{B}([0,T])\text{-measurable, } \left\| \int_0^T |F(\omega,t)| dt \right\|_p < \infty \right\}.$$  

For $F \in L^p(\Omega; L^1([0,T]))$ we define

$$I_F(\omega) := \int_0^T F(\omega,t) dt \quad \text{and} \quad K_F(\omega,s) := \begin{cases} \frac{F(\omega,s)}{I_F(\omega)}, & \text{if } I_F(\omega) \neq 0 \\ 0, & \text{if } I_F(\omega) = 0. \end{cases} \quad (3)$$
We consider the terminal condition $\xi$ to be an $\mathcal{F}_T$-measurable random variable and the generator to be a random function $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \times L^2(\nu) \rightarrow \mathbb{R}^d$.

**Definition 2.1.** An $L_{\text{loc}}$-solution to a BSDE $(\xi, f)$ with terminal condition $\xi$ and generator $f$ is a triplet $(Y, Z, U) \in L_{\text{loc}}(W) \times L_{\text{loc}}(W) \times L_{\text{loc}}(\tilde{N})$, adapted to $(\mathcal{F}_t)_{t \in [0, T]}$, which satisfies for all $t \in [0, T]$,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_{[t, T] \times \mathbb{R}^d} U_s(x) \tilde{N}(ds, dx), \quad \mathbb{P}\text{-a.s.}$$

**Definition 2.2.** An $L^p$-solution to a BSDE $(\xi, f)$ with terminal condition $\xi$ and generator $f$ is an $L_{\text{loc}}$-solution $(Y, Z, U)$ to the BSDE $(\xi, f)$ which satisfies

$$(Y, Z, U) \in S^p \times L^p(W) \times L^p(\tilde{N}).$$

### 2.2 Lévy process with finite measure

The driving Lévy process, given by its Lévy-Itô-decomposition (2) will be approximated for $n \geq 1$ by

$$X^n_t = at + \Sigma W_t + \int_{[0, t] \times \{|x| > 1\}} x N(ds, dx) + \int_{[0, t] \times \{1/n \leq |x| \leq 1\}} x \tilde{N}(ds, dx).$$

The process $X^n$ has a finite Lévy measure. Note furthermore, that the compensated Poisson random measure associated to $X^n$ can be expressed as $\tilde{N}^n = \chi_{\{1/n \leq |x| \}} \tilde{N}$, where $\chi_A$ denotes the indicator function of a set $A$. Let

$$\mathcal{F}^0 := \{\Omega, \emptyset\} \lor \mathcal{N},$$

$$\mathcal{F}^n := \sigma(X^n) \lor \mathcal{N}, \quad n \geq 1,$$

where $\mathcal{N}$ stands for the null sets of $\mathcal{F}$. Denote by $\mathbb{E}_n$ the conditional expectation $\mathbb{E}[\cdot | \mathcal{F}^n]$.

### 3 Main Theorem

With this setting in mind, we now state the main theorem based on the following assumptions, with a slight distinction for $p \geq 2$ and $p < 2$, which turns out to be quite natural for the proofs. Instead of a Lipschitz condition, we require the weaker conditions $(A3_{\geq 2})$ and respectively $(A3_{< 2})$ referred to as one-sided Lipschitz or monotonicity condition for the generator $f$.

#### 3.1 Assumptions

(A1) For all $(y, z, u) \in \mathbb{R}^d \times \mathbb{R}^{d \times k} \times L^2(\nu) : (\omega, s) \mapsto f(\omega, s, y, z, u)$ is progressively measurable and the process $f_0 = (f(t, 0, 0, 0))_{t \in [0, T]}$ is in $L^p(\Omega; L^1([0, T]))$.

(A2) For all $r > 0$ there are nonnegative, progressively measurable processes $\Phi, \psi_r$ with

$$\left\| \int_0^T \Phi(\cdot, s)^2 ds \right\|_{\infty} < \infty$$
and \( \psi_r \in L^1(\Omega \times [0, T]) \) such that for all \((z, u) \in \mathbb{R}^{d \times k} \times L^2(\nu)\),

\[
\sup_{|y| \leq r} |f(t, y, z, u) - f_0(t)| \leq \psi_r(t) + \Phi(t)(|z| + \|u\|), \quad \mathbb{P} \otimes \lambda\text{-a.e.}
\]

**(A3\_\geq 2)** For \( p \geq 2 \):

For \( \lambda \)-almost all \( s \), the mapping \((y, z, u) \mapsto f(s, y, z, u)\) is \( \mathbb{P}\)-a.s. continuous. Moreover, there is a nonnegative function \( \alpha \in L^1([0, T]) \) and progressively measurable processes \( \mu, \beta \) with \( \int_0^T (\mu(\omega, s) + \beta(\omega, s)^2) \, ds < \infty \), \( \mathbb{P}\)-a.s. such that for all \((y, z, u)\), \((y', z', u')\) \( \in \mathbb{R}^d \times \mathbb{R}^{d \times k} \times L^2(\nu)\),

\[
|y - y'|^p - (y - y', f(t, y, z, u) - f(t, y', z', u'))^p \\
\leq \alpha(t)|y - y'|^p - \mu(t)|y - y'|^p + \beta(t)|y - y'|^p - (|\beta_1(t)||z - z'| + |\beta_2(t)||u - u'|)|, \quad (4)
\]

\( \mathbb{P} \otimes \lambda\text{-a.e.}, \) for a nondecreasing, continuous and concave function \( \rho \) from \([0, \infty[\) to itself, satisfying \( \rho(0) = 0 \), \( \lim_{x \to 0} \frac{\rho(x^2)}{x} = 0 \) and \( \int_0^1 \frac{1}{\rho(x^2)} \, dx = \infty \), for some \( \epsilon > 0 \).

**(A3\_< 2)** For \( 0 < p < 2 \):

For \( \lambda \)-almost all \( s \), the mapping \((y, z, u) \mapsto f(s, y, z, u)\) is \( \mathbb{P}\)-a.s. continuous. Moreover, there is a nonnegative function \( \alpha \in L^1([0, T]) \), \( C > 0 \) and progressively measurable processes \( \mu, \beta_1, \beta_2 \) with \( \int_0^T (\mu(\omega, s) + \beta_1(\omega, s)^2 + \beta_2(\omega, s)^q) \, ds < C \), \( \mathbb{P}\)-a.s. for some \( q > 2 \) such that for all \((y, z, u)\), \((y', z', u')\) \( \in \mathbb{R}^d \times \mathbb{R}^{d \times k} \times L^2(\nu)\), \( y \neq y' \)

\[
|y - y'|^p - (y - y', f(t, y, z, u) - f(t, y', z', u'))^p \\
\leq \alpha(t)\rho(|y - y'|^p) + \mu(t)|y - y'|^p + |y - y'|^p - (\beta_1(t)|z - z'| + \beta_2(t)||u - u'||), \quad (5)
\]

\( \mathbb{P} \otimes \lambda\text{-a.e.}, \) for a nondecreasing, continuous and concave function \( \rho \) from \([0, \infty[\) to itself, satisfying \( \rho(0) = 0 \), \( \lim_{x \to 0} \frac{\rho(x^p)}{x} = 0 \) and \( \int_0^1 \frac{1}{\rho(x^2)} \, dx = \infty \).

**Remark 3.1.**

(i) The limit assumptions \( \lim_{x \to 0} \frac{\rho(x^2)}{x} = 0 \) together with \( (4) \) or \( \lim_{x \to 0} \frac{\rho(x^p)}{x} = 0 \) together with \( (5) \) already imply that the generator \( f \) is Lipschitz in \( z, u \). Moreover \( \beta \) (and analogously for \( \beta_1 + \beta_2 \)) can take the role of \( \Phi \) in \( (A2) \). Nonetheless, for convenience in the proofs, we will still use the generic function \( \Phi \).

(ii) The \( \rho \)-function appearing in the right hand sides of \( (A3\_\geq 2) \) and \( (A3\_< 2) \) admits the following inequalities, which play important roles in the proofs:

(a) \( \alpha(t)\rho(|y|^2)|y|^{p-2} \leq \alpha(t)\rho(|y|^p) + \alpha(t)\rho(1)|y|^p \), \quad for \( p \geq 2 \),

(b) \( \alpha(t)\rho(|y|^p)|y|^{2p} \leq \alpha(t)\rho(|y|^2) + \alpha(t)\rho(1)|y|^2 \), \quad for \( 0 < p < 2 \).

**Proof.** For \( (ii) \) we see that, if \( |y| < 1 \), then \( |y|^{p-2} < 1 \) and by the concavity of \( \rho \),

\[
\rho(|y|^2)|y|^{p-2} \leq \rho(|y|^2)|y|^{p-2} = \rho(|y|^p).
\]

For \( |y| \geq 1 \) we have by the concavity of \( \rho \),

\[
\rho(|y|^2)|y|^{p-2} \leq \rho(1)|y|^2|y|^{p-2} = \rho(1)|y|^p.
\]

The case \( 0 < p \leq 2 \) is similar. \( \square \)
Remark 3.2.

(i) In (A3₂ < 2), if \( \beta_2 \) is deterministic, we could impose the weaker condition \( q = 2 \) as described later in Remark 4.7.

(ii) The following example is constructed in order to demonstrate the possibilities in this setting for \( d = 1, p > 1 \). All the involved expressions are chosen to exploit the assumptions on the coefficients which may be time-dependent, unbounded and some even random. The generator’s dependence on \( y \) is not Lipschitz (not even one-sided Lipschitz) and of super-linear growth:

\[
f(\omega, t, y, z, u) = -\frac{1}{\sqrt{t}} y \log(|y|) - \mu(\omega, t) \left(y^3 + |y|^3\right) + \beta_1(\omega, t)(z + \sin(z) \cos(y)) + \beta_2(\omega, t) \int_{\mathbb{R}_0} (\arctan(y\kappa(x) u(x)) + u(x)) \kappa(x) \nu(dx) + f_0(t),
\]

where

- \( \mu \) is given by \( \mu(\omega, t) = \sum_{n=1}^{\infty} \frac{1}{n^2 \sqrt{t-t_n(\omega)}} \), with \( (t_n(\omega))_{n \geq 1} \) being a numeration of the jumps of the trajectory \( t \mapsto X_t(\omega) \) of the Lévy process and \( \mu(t, \omega) = 0 \) if \( t \mapsto X_t(\omega) \) has no jumps,

- \( \beta_1(\omega, t) = \begin{cases} \chi_{[T/2, T]}(t) \vphantom{\sqrt{t-W_{T/2}(\omega)}} & \text{when defined,} \\ 0 & \text{else,} \end{cases} \)

- \( \beta_2(\omega, t) = \begin{cases} \chi_{[T/3, T]}(t) \vphantom{\sqrt{t-W_{T/2}(\omega)}} & \text{when defined,} \\ \left| \log \left( \frac{|\omega|}{|W_{T/3}(\omega)|} \right) \right| & \text{else,} \end{cases} \)

- \( \kappa(x) = 1 \wedge |x| \),

- \( f_0(\omega, t) = \begin{cases} \int_0^t s \exp \left( \frac{W_s^2}{2t} \right) ds \vphantom{\frac{\varepsilon}{\beta}} & \text{when defined,} \\ 0 & \text{else.} \end{cases} \)

3.2 Main Theorem

**Theorem 3.3 (Existence and Uniqueness).** Assume that the terminal condition \( \xi \) is in \( L^p \) and the generator \( f \) satisfies \([A 1],[A 2],[A 3_\geq 2]\) for \( p \geq 2 \) or \([A 1],[A 2],[A 3_\leq 2]\) for \( 1 < p < 2 \), then there exists a unique \( L^p \)-solution to the BSDE \((\xi, f)\).

We will prove this theorem in section 5 after presenting necessary a priori estimates in the next section.

4 A Priori Estimates and Stability

Throughout the next sections, recall that \( f_0(t) = f(t, 0, 0, 0) \), and that \( I_{|f_0|} \) and \( K_{|f_0|} \) are defined as in (3).
Remark 4.1. For the results in this section, instead of \([A_{3<2}]\), it suffices to require the weaker condition

\[
|y|^{p-2} \langle y, f(t, y, z, u) \rangle 
\leq \alpha(t)|y|^{p-2} \rho(|y|^2) + \mu(t)|y|^p + \beta(t)|y|^{p-1}(|z| + \|u\|) + |y|^{p-1}|f_0(t)|,
\]

and instead of \([A_{3<2}]\)

\[
|y|^{p-2} \langle y, f(t, y, z, u) \rangle 
\leq \alpha(t)\rho(|y|^p) + \mu(t)|y|^p + |y|^{p-1} (\beta_1(t)|z| + \beta_2(t)\|u\|) + |y|^{p-1}|f_0(t)|,
\]

\(\mathbb{P} \otimes \lambda\text{-a.e. for all } (y, z, u) \in \mathbb{R}^d \times \mathbb{R}^{d \times k} \times L^2(\nu)\).

Lemma 4.2. If a sequence of random variables \((V_n)_{n \in \mathbb{N}}\) in \(L^p\) satisfies \(\lim_{n \to \infty} \mathbb{E}|V_n|^p = 0\), then for a function \(\rho\) as in the assumptions, we have

\[
\lim_{n \to \infty} \mathbb{E} \left[ \rho \left( |V_n|^2 \right)^{\frac{p}{2}} \right] = 0.
\]

Proof. This follows from the continuity of \(\rho\), \(\rho(0) = 0\) and the uniform integrability of \((|V_n|^p)_{n \geq 1}\),

\[
\rho \left( |V_n|^2 \right)^{\frac{p}{2}} \leq (a + b|V_n|^2)^{\frac{p}{2}} \leq 2^{\frac{p}{2} - 1} \left( a^{\frac{p}{2}} + b^{\frac{p}{2}} |V_n|^p \right),
\]

since \(\rho(x) \leq a + bx\) for some \(a, b > 0\) and the above inequality shows that also \((\rho(|V_n|^2)^{\frac{p}{2}})_{n \geq 1}\) is a uniformly integrable sequence. \(\square\)

The following two propositions show that the norms of the \(Z\) and \(U\) processes can be controlled by expressions in \(Y\) and \(f_0\). Note that the bounds in Proposition 4.3 and Proposition 4.4 differ slightly, so that the application of Proposition 4.3 in section 5 needs the assertion of Lemma 4.2.

Proposition 4.3. Let \(p \geq 2\) and let \((Y, Z, U)\) be an \(L_{\text{loc}}\)-solution to the BSDE \((\xi, f)\). If \(\xi \in L^p\), \(Y \in S^p\) and \([A_1]\) and \([A_{3<2}]\) are satisfied, then \((Y, Z, U)\) is an \(L^p\)-solution.

More precisely, there is a constant \(C > 0\) depending on \(p, T, \alpha, \mu, \beta\) such that for all \(t \in [0, T]\),

\[
\mathbb{E} \left[ \left( \int_t^T |Z_s|^2 ds \right)^{\frac{p}{2}} \right] + \mathbb{E} \left[ \left( \int_t^T \|U_s\|^2 ds \right)^{\frac{p}{2}} \right] 
\leq C \left( \mathbb{E} \left[ \sup_{s \in [t,T]} |Y_s|^p \right] + \mathbb{E} \left[ \rho \left( \sup_{s \in [t,T]} |Y_s|^2 \right)^{\frac{p}{2}} \right] + \mathbb{E} \left[ \left( \int_t^T |f_0(s)|ds \right)^{p} \right] \right).
\]

Proof. This proof generalizes the arguments in [3] Lemma 3.1.

Step 1:
For \(t \in [0, T]\) and \(n \geq 1\) define the stopping times

\[
\tau_n := \inf \left\{ s \in [t, T] : \int_t^s |Z_s|^2 ds \geq n \right\} \wedge \inf \left\{ s \in [t, T] : \int_t^s \|U_s\|^2 ds \geq n \right\}.
\]
W e continue our estimate (with another constant $c$)

$$\begin{align*}
|Y_t|^2 + \int_t^{\tau_n} |Z_s|^2 ds + \int_t^{\tau_n} ||U_s||^2 ds = |Y_{\tau_n}|^2 &+ 2 \int_t^{\tau_n} \langle Y_s, f(s, Y_s, Z_s, U_s) \rangle ds \\
- 2 \int_t^{\tau_n} \langle Y_s, Z_s dW_s \rangle &- \int_{[t, \tau_n] \times \mathbb{R}^d} \left( |Y_s - U_s(x)|^2 - |Y_s - x|^2 \right) \tilde{N}(ds, dx),
\end{align*}$$

from which we infer by $[A.3.2]$ that

$$\begin{align*}
\int_t^{\tau_n} |Z_s|^2 ds + \int_t^{\tau_n} ||U_s||^2 ds &\leq |Y_{\tau_n}|^2 + 2 \int_t^{\tau_n} (\alpha(s)\rho(|Y_s|^2) + \mu(s)|Y_s|^2) ds \\
+ \int_t^{\tau_n} \beta(s)|Y_s|(|Z_s| + ||U_s||) ds &+ 2 \int_t^{\tau_n} |Y_s||f_0(s)| ds \\
+ 2 \int_t^{\tau_n} \langle Y_s, Z_s dW_s \rangle &+ \int_{[t, \tau_n] \times \mathbb{R}^d} \left( |Y_s - U_s(x)|^2 - |Y_s - x|^2 \right) \tilde{N}(ds, dx) \bigg].
\end{align*}$$

Taking the power $\frac{2}{p}$, we find a constant $c_0 > 0$ such that

$$\begin{align*}
\left[ \int_t^{\tau_n} |Z_s|^2 ds \right]^\frac{2}{p} + \left[ \int_t^{\tau_n} ||U_s||^2 ds \right]^\frac{2}{p} &\leq c_0 \left[ |Y_{\tau_n}|^p + \left[ \int_t^{\tau_n} (\alpha(s)\rho(|Y_s|^2) + \mu(s)|Y_s|^2) \right] ds \right]^\frac{2}{p} \\
+ \left[ \int_t^{\tau_n} \beta(s)|Y_s|(|Z_s| + ||U_s||) ds \right]^\frac{2}{p} &+ \left[ \int_t^{\tau_n} |Y_s||f_0(s)| ds \right]^\frac{2}{p} \\
+ \left[ \int_t^{\tau_n} \langle Y_s, Z_s dW_s \rangle \right]^\frac{2}{p} &+ \int_{[t, \tau_n] \times \mathbb{R}^d} \left( |Y_s - U_s(x)|^2 - |Y_s - x|^2 \right) \tilde{N}(ds, dx)^\frac{2}{p}. \tag{8}
\end{align*}$$

We continue our estimate (with another constant $c_1 > 0$)

$$\begin{align*}
\left[ \int_t^{\tau_n} |Z_s|^2 ds \right]^\frac{2}{p} &+ \left[ \int_t^{\tau_n} ||U_s||^2 ds \right]^\frac{2}{p} \\
\leq c_1 \left[ \sup_{s \in [t, T]} |Y_s|^p \right] &+ \left[ \int_t^{T} \alpha(s) ds \right]^\frac{2}{p} \rho \left( \sup_{s \in [t, T]} |Y_s|^2 \right) + \left[ \int_t^{T} \mu(s) ds \right]^\frac{2}{p} \sup_{s \in [t, T]} |Y_s|^p \\
+ \left[ \int_t^{\tau_n} \beta(s)|Y_s|(|Z_s| + ||U_s||) ds \right]^\frac{2}{p} &+ \left[ \int_t^{\tau_n} |f_0(s)| ds \right]^\frac{2}{p} \sup_{s \in [t, T]} |Y_s|^\frac{2}{p} \\
+ \left[ \int_t^{\tau_n} \langle Y_s, Z_s dW_s \rangle \right]^\frac{2}{p} &+ \int_{[t, \tau_n] \times \mathbb{R}^d} \left( |Y_s - U_s(x)|^2 - |Y_s - x|^2 \right) \tilde{N}(ds, dx)^\frac{2}{p}. \tag{8}
\end{align*}$$

To estimate the above further, we have to split up the range of values of $p \geq 2$.

**Case 1: $2 \leq p \leq 4$**

We use the following inequality given e.g. in [13] Theorem 3.2, which states that for a local martingale $M$, given by $M(t) = \int_{[0, t] \times \mathbb{R}^d} g_s(x) \tilde{N}(ds, dx), t \in [0, T]$, there exists $c_2 > 0$ such that the following inequality holds for $p' \in [0, 2]$:

$$\mathbb{E} \sup_{t \in [0, T]} |M_t|^{p'} \leq c_2 \mathbb{E} \left[ \left( \int_0^T \int_{\mathbb{R}^d} |g_s|^2 \nu(dx) ds \right)^{\frac{p'}{2}} \right].$$
Here, we will apply this inequality for $p' = p/2$ to the martingale

$$s \mapsto \int_{[t, s \wedge \tau_n] \times \mathbb{R}^d} \left( |Y_{s-} + U_s(x)|^2 - |Y_{s-}|^2 \right) \, N(ds, dx).$$

Note that we can estimate the square of the above integrand $\mathbb{P} \otimes \lambda \otimes \nu$-a.e. by

$$\left( |Y_{s-} + U_s(x)| + |Y_{s-}| \right)^2 \left( |Y_{s-} + U_s(x)| - |Y_{s-}| \right)^2 \leq 16 \sup_{r \in [t, T]} |Y_r|^2 |U_s(x)|^2,$$

(9)

since for all $s \in [t, T]$ we can bound the jump sizes $|U_s(x)|$ by $2 \sup_{r \in [t, T]} |Y_r|$. $\mathbb{P} \otimes \lambda \otimes \nu$-a.e. (see [20 Corollary 1]) and since $|Y_{s-} + U_s(x)| - |Y_{s-}| \leq |U_s(x)|$.

We take suprema and expectations to get a constant $c_3 > 0$ such that

$$\mathbb{E} \left[ \int_t^{\tau_n} |Z_s|^2 ds \right]^{\frac{p}{2}} + \mathbb{E} \left[ \int_t^{\tau_n} \|U_s\|^2 ds \right]^{\frac{p}{2}} \leq c_3 \left( \mathbb{E} \left[ \sup_{s \in [t, T]} |Y_s|^p \right] + \mathbb{E} \left[ \rho \left( \sup_{s \in [t, T]} |Y_s|^2 \right)^{\frac{p}{2}} \right] \right) + \mathbb{E} \left[ \int_t^{\tau_n} \beta(s)|Y_s| |Z_s| + \|U_s\| ds \right]^{\frac{p}{2}} + \mathbb{E} \left[ \left( \int_t^{\tau_n} |f_0(s)| ds \right)^{\frac{p}{2}} \sup_{s \in [t, T]} |Y_s|^2 \right] + \mathbb{E} \left[ \int_t^{\tau_n} |Y_s|^2 |Z_s|^2 ds \right]^{\frac{p}{2}} + \mathbb{E} \left[ \int_t^{\tau_n} \sup_{r \in [t, T]} |Y_r|^2 \|U_s\|^2 ds \right]^{\frac{p}{2}}.

Young’s inequality (see [Theorem A.1] in the Appendix) now gives us for an arbitrary $R > 0$,

$$\mathbb{E} \left[ \int_t^{\tau_n} |Z_s|^2 ds \right]^{\frac{p}{2}} + \mathbb{E} \left[ \int_t^{\tau_n} \|U_s\|^2 ds \right]^{\frac{p}{2}} \leq c_3 \left( \mathbb{E} \left[ \sup_{s \in [t, T]} |Y_s|^p \right] + \mathbb{E} \left[ \rho \left( \sup_{s \in [t, T]} |Y_s|^2 \right)^{\frac{p}{2}} \right] \right) + \mathbb{E} \left[ \int_t^{\tau_n} \frac{R}{2} \beta(s)^2 |Y_s|^2 ds \right]^{\frac{p}{2}} + \frac{1}{(2R)^\frac{p}{2}} \left( \mathbb{E} \left[ \int_t^{\tau_n} |Z_s|^2 ds \right]^{\frac{p}{2}} + \mathbb{E} \left[ \int_t^{\tau_n} \|U_s\|^2 ds \right]^{\frac{p}{2}} \right) + \frac{1}{2} \mathbb{E} \left[ \int_t^{\tau_n} |f_0(s)| ds \right]^{p} + \frac{1}{2} \sup_{s \in [t, T]} |Y_s|^p + 2 \left( \frac{R}{2} \right)^\frac{p}{2} \mathbb{E} \sup_{s \in [t, T]} |Y_s|^p + \frac{1}{(2R)^\frac{p}{2}} \left( \mathbb{E} \left[ \int_t^{\tau_n} |Z_s|^2 ds \right]^{\frac{p}{2}} + \mathbb{E} \left[ \int_t^{\tau_n} \|U_s\|^2 ds \right]^{\frac{p}{2}} \right).

Choosing now $R$ such that $\frac{2c_3}{(2R)^\frac{p}{2}} < 1$ yields a constant $C > 0$ such that

$$\mathbb{E} \left[ \int_t^{\tau_n} |Z_s|^2 ds \right]^{\frac{p}{2}} + \mathbb{E} \left[ \int_t^{\tau_n} \|U_s\|^2 ds \right]^{\frac{p}{2}} \leq C \left( \mathbb{E} \left[ \sup_{s \in [t, T]} |Y_s|^p \right] + \mathbb{E} \left[ \rho \left( \sup_{s \in [t, T]} |Y_s|^2 \right)^{\frac{p}{2}} \right] + \mathbb{E} \left[ \int_t^{T} |f_0(s)| ds \right]^{p} \right).
Taking the limit for \( n \to \infty \) shows the assertion for \( 2 \leq p \leq 4 \).

**Case 2: \( p > 4 \)**

We start from (5) following the same lines of the previous case. In this case the only difference is: [18] Theorem 3.2 states that for a local martingale \( M \), given by \( M(t) = \int_{[0,t] \times \mathbb{R}^d} g_s(x) \tilde{N}(ds, dx) \), \( t \in [0, T] \) there exists \( c_4 > 0 \) such that the following inequality holds for all \( p' \geq 2 \):

\[
\mathbb{E} \sup_{t \in [0,T]} |M_t|^{p'} \leq c_4 \mathbb{E} \left( \int_0^T \int_{\mathbb{R}^d} |g_s|^2 \nu(dx)ds \right)^{\frac{p'}{2}} + \int_0^T \int_{\mathbb{R}^d} |g_s|^{p'} \nu(dx)ds.
\] (10)

For \( p' = \frac{p}{2} \), we apply this inequality to the local martingale

\[
s \mapsto \int_{[t,s \wedge \tau_n] \times \mathbb{R}^d} \left( |Y_{s-} + U_s(x)|^2 - |Y_{s-}|^2 \right) \tilde{N}(ds, dx).
\]

The first summand of (10) can be treated as in case 1. We focus on the second term which equals

\[
\int_t^T \int_{\mathbb{R}^d} (|Y_{s-} + U_s(x)|^2 - |Y_{s-}|^2) \frac{2}{p} \nu(dx)ds
\]

\[
= \int_t^T \int_{\mathbb{R}^d} (|Y_{s-} + U_s(x)|^2 - |Y_{s-}|^2)^{\frac{p-2}{2}} (|Y_{s-} + U_s(x)|^2 - |Y_{s-}|^2)^{\frac{2}{p}} \nu(dx)ds.
\] (11)

We can bound the integrands (as explained in (9)) by

\[
(|Y_{s-} + U_s(x)| + |Y_{s-}|) (|Y_{s-} + U_s(x)| - |Y_{s-}|) \leq 16 \sup_{r \in [t,T]} |Y_r|^2,
\]

and

\[
(|Y_{s-} + U_s(x)| + |Y_{s-}|) (|Y_{s-} + U_s(x)| - |Y_{s-}|) \leq 4 \sup_{r \in [t,T]} |Y_r||U_s(x)|.
\]

Hence we find a constant \( c_5 > 0 \), such that (11) is smaller than

\[
c_5 \int_t^T \int_{\mathbb{R}^d} \sup_{r \in [t,T]} |Y_r|^{p-2} |U_s(x)|^2 \nu(dx)ds
\]

By Young’s inequality for the conjugate couple \( \left( \frac{p}{2}, \frac{p}{p-2} \right) \) there exists \( R_1 > 0 \) with

\[
\int_t^T \int_{\mathbb{R}^d} \sup_{r \in [t,T]} |Y_r|^{p-2} |U_s(x)|^2 \nu(dx)ds = \sup_{r \in [t,T]} |Y_r|^{p-2} \int_t^T \int_{\mathbb{R}^d} |U_s(x)|^2 \nu(dx)ds
\]

\[
\leq \left( \frac{p-2}{p} R_1^{\frac{p}{p-2}} \sup_{r \in [t,T]} |Y_r|^p + \frac{2}{p R_1^p} \left( \int_t^T \int_{\mathbb{R}^d} |U_s(x)|^2 \nu(dx)ds \right)^{\frac{p}{2}} \right).
\]

From here, similar steps as in case 1 conclude the proof. \(\square\)

**Proposition 4.4.** Let \( 0 < p < 2 \) and let \( (Y, Z, U) \) be an \( L_{loc}^p \)-solution to the BSDE \((\xi, f)\). If \( \xi \in L^p \), \( Y \in S^p \) and \([A1]\) and \([A3_{<2}]\) are satisfied, then \( (Y, Z, U) \) is an \( L^p \)-solution.
More precisely, there is a constant $C$ depending on $p, T, \alpha, \rho, 1, \mu, \beta_1, \beta_2$ such that for all $t \in [0, T]$,

$$
\mathbb{E} \left[ \int_t^T |Z_s|^2 ds \right]^\frac{p}{2} + \mathbb{E} \left[ \int_t^T \|U_s\|^2 ds \right]^\frac{p}{2} 
\leq C \left( \mathbb{E} \left[ \sup_{s \in [t, T]} |Y_s|^p \right] + \mathbb{E} \left[ \int_t^T \alpha(s) \rho \left( |Y_s|^p \right) ds \right] + \mathbb{E} \left[ \int_t^T |f_0(s)|^\gamma ds \right]^\frac{p}{\gamma} \right).
$$

The assertion holds true even if $q = 2$ in (A3$<$2$>$) since we do not use a higher integrability condition in the proof.

**Proof.** As in the proof before, for $t \in [0, T]$ and $n \geq 1$ we define the stopping times

$$
\tau_n := \inf \left\{ s \in [t, T] : \int_t^s |Z_s|^2 ds \geq n \right\} \wedge \inf \left\{ s \in [t, T] : \int_t^s \|U_s\|^2 ds \geq n \right\}.
$$

Itô’s formula implies

$$
|Y_t|^2 + \int_t^\tau_n |Z_s|^2 ds + \int_t^\tau_n \|U_s\|^2 ds = |Y_{\tau_n}|^2 + 2 \int_t^\tau_n \langle Y_s, f(s, Y_s, Z_s, U_s) \rangle ds 
- 2 \int_t^\tau_n \langle Y_s, Z_s dW_s \rangle - \int_{[t, \tau_n] \times \mathbb{R}^d} \left( |Y_s - + U_s(x)|^2 - |Y_s - |^2 \right) \tilde{N}(ds, dx),
$$

from which we infer by (A3$<$2$>$) and Remark 4.1 putting $\beta_1 + \beta_2 =: \beta$,

$$
\int_t^\tau_n |Z_s|^2 ds + \int_t^\tau_n \|U_s\|^2 ds \leq |Y_{\tau_n}|^2 + 2 \int_t^\tau_n \left( \alpha(s) \rho (|Y_s|^p) |Y_s|^{2-p} + \mu(s) |Y_s|^2 \right) ds 
+ \int_t^\tau_n \beta(s) |Y_s|(|Z_s| + \|U_s\|) ds + 2 \int_t^\tau_n |Y_s| f_0(s) ds 
+ 2 \left| \int_t^\tau_n \langle Y_s, Z_s dW_s \rangle \right| + \left| \int_{[t, \tau_n] \times \mathbb{R}^d} \left( |Y_s - + U_s(x)|^2 - |Y_s - |^2 \right) \tilde{N}(ds, dx) \right|.
$$

Taking the power $\frac{2}{p}$, we find a constant $c_0 > 0$ such that

$$
\left[ \int_t^\tau_n |Z_s|^2 ds \right]^\frac{p}{2} + \left[ \int_t^\tau_n \|U_s\|^2 ds \right]^\frac{p}{2} 
\leq c_0 \left( |Y_{\tau_n}|^p + \sup_{s \in [t, T]} \|Y_s\|^{2-p} \int_t^\tau_n \alpha(s) \rho (|Y_s|^p) ds + \sup_{s \in [t, T]} |Y_s|^2 \int_t^\tau_n \mu(s) ds \right) \left[ \int_t^\tau_n |Y_s| f_0(s) ds \right]^\frac{p}{2} 
+ \left[ \int_t^\tau_n \beta(s) |Y_s|(|Z_s| + \|U_s\|) ds \right]^\frac{p}{2} 
+ \left| \int_t^\tau_n \langle Y_s, Z_s dW_s \rangle \right|^{\frac{p}{2}} + \left| \int_{[t, \tau_n] \times \mathbb{R}^d} \left( |Y_s - + U_s(x)|^2 - |Y_s - |^2 \right) \tilde{N}(ds, dx) \right|^{\frac{p}{2}}.
$$
We estimate further with \( c_1 > 0 \)
\[
\left[ \int_t^{r_n} |Z_s|^2 ds \right]^\frac{p}{2} + \left[ \int_t^{r_n} |U_s|^2 ds \right]^\frac{p}{2} \leq c_1 \left( |Y_{r_n}|^p + \sup_{s \in [t,T]} |Y_s|^{\frac{(2-p)p}{2}} \right) \left[ \int_t^{r_n} \alpha(s)\rho(|Y_s|^p) ds \right]^\frac{p}{2} \\
+ \sup_{s \in [t,T]} |Y_s|^p \left[ \int_t^{r_n} \mu(s) ds \right]^\frac{p}{2} + \left[ \int_t^{r_n} \beta(s)|Y_s|(|Z_s| + |U_s|) ds \right]^\frac{p}{2} + \left[ \int_t^{r_n} |Y_s|f_0(s) ds \right]^\frac{p}{2} \\
+ \left| \int_t^{r_n} (Y_s, Z_s dW_s) \right|^\frac{p}{2} + \int_{[t,r_n] \times \mathbb{R}^d} \left( |Y_{s-} + U_s(x)|^2 - |Y_{s-}|^2 \right) \tilde{N}(ds,dx)^\frac{p}{2}.
\]

With Young’s inequality for \( \left( \frac{2}{p}, \frac{2}{2-p} \right) \) and a new constant \( c_2 > 0 \) we get,
\[
\left[ \int_t^{r_n} |Z_s|^2 ds \right]^\frac{p}{2} + \left[ \int_t^{r_n} |U_s|^2 ds \right]^\frac{p}{2} \leq c_2 \left( |Y_{r_n}|^p + \sup_{s \in [t,T]} |Y_s|^p + \int_t^{r_n} \alpha(s)\rho(|Y_s|^p) ds \right) \\
+ \left[ \int_t^{r_n} \beta(s)|Y_s|(|Z_s| + |U_s|) ds \right]^\frac{p}{2} + \left[ \int_t^{r_n} |Y_s|f_0(s) ds \right]^\frac{p}{2} \\
+ \left| \int_t^{r_n} (Y_s, Z_s dW_s) \right|^\frac{p}{2} + \int_{[t,r_n] \times \mathbb{R}^d} \left( |Y_{s-} + U_s(x)|^2 - |Y_{s-}|^2 \right) \tilde{N}(ds,dx)^\frac{p}{2}.
\]

From here on the proof can be concluded similar to case 1 of Proposition 4.3.

From the proposition above, we now know how to bound \( Z \) and \( U \) in terms of \( Y \) and \( f_0 \). For the core of the existence proof later we need to control the \( Y \) part of the solution triplet by a bound depending only on \( \xi \) and \( f \), which we will show in the sequel.

**Proposition 4.5.** Let \( p \geq 2 \) and let \((Y, Z, U)\) be an \( L^p \)-solution to the BSDE \((\xi, f)\). If \( \xi \in L^p \) and \([AT]\) and \([A_{3,2}]\) are satisfied, then there exists a function \( h : [0, \infty) \to [0, \infty] \) with \( h(x) \to 0 \) as \( x \to 0 \) such that
\[
\|Y\|_{S_p}^p \leq h \left( \mathbb{E}[|\xi|^p + \mathbb{E}[f_{0}]^p] \right),
\]
where \( h \) depends on \( p, T, \rho, \alpha, \beta, \mu \).

**Proof.**

**Step 1:**
Let \( \Psi(y) := |y|^p \) and \( \eta = (\eta_t)_{0 \leq t \leq T} \in L^{\infty}(\Omega; L^1([0, T])) \) be a progressively measurable, continuous process, which we will determine later. Itô’s formula (see also [14] Proposition 2)) for \( t \in [0, T] \) implies
\[
e^{\int_0^t \eta(s) ds} |Y_t|^p + \int_t^T e^{\int_s^t \eta(r) dr} \left[ \eta(s)|Y_s|^p + \frac{1}{2} \text{trace}(D^2\Psi(Y_s)Z_sZ_s^*) \right] ds + P(t) \\
= e^{\int_0^t \eta(s) ds} |\xi|^p + \int_t^T e^{\int_s^t \eta(r) dr} \left[ P(Y_s|Y_s|^{p-2}, f(s, Y_s, Z_s, U_s)) ds + M(t) \right], \tag{12}
\]
where \( D^2\Psi \) denotes the Hessian matrix of \( \Psi \),
\[
P(t) = \int_t^T \int_{\mathbb{R}^d} e^{\int_0^s \eta(r) dr} \left[ |Y_{s-} + U(s, x)|^p - |Y_{s-}|^p - \langle U(s, x), pY_s|Y_s|^{p-2} \rangle \right] \nu(dx) ds
\]
and

\[
M(t) = - \int_t^T e^{\int_0^t \eta(s) ds} \langle p(Y_s | Y_{s-2}, Z_s dW_s) \\
- \int_t^T \int_{\mathbb{R}^d} e^{\int_0^t \eta(s) ds} \left[ |Y_{s-} + U(s, x)|^p - |Y_{s-}|^p \right] \hat{N}(ds, dx).
\]

By the argument in [14] we can use the estimates \( \text{trace}(D^2 \Psi(y) zz^*) \geq p|y|^{p-2}z^2 \) and

\[P(t) \geq p(1 - p)3^{1-p} \int_t^T e^{\int_0^t \eta(s) ds} |Y_{s-}|^{p-2} \|U_s\|^2 ds,
\]

leading to

\[
e^{\int_0^t \eta(s) ds} |Y_t|^p + \int_t^T e^{\int_0^t \eta(s) ds} \left[ \eta(s) |Y_s|^p + \frac{1}{2}p(p - 1)|Y_s|^{p-2} |Z_s|^2 \right] ds
\]
\[
+ p(1 - p)3^{1-p} \int_t^T e^{\int_0^t \eta(s) ds} |Y_s|^{p-2} \|U_s\|^2 ds
\]
\[
\leq e^{\int_0^t \eta(s) ds} |\xi|^p + \int_t^T e^{\int_0^t \eta(s) ds} \langle p(Y_s | Y_{s-2}, f(s, Y_s, Z_s, U_s)) ds + M(t).
\]

Using (A3.2) and Young’s inequality for arbitrary \( R_z, R_u > 0 \), we obtain with \( I_{[f_0]}, K_{[f_0]} \) from (3), \( c_z = \frac{1}{2}p(1 - p) \) and \( c_u = p(1 - p)3^{1-p} \),

\[
e^{\int_0^t \eta(s) ds} |Y_t|^p + \int_t^T e^{\int_0^t \eta(s) ds} \left( \eta(s) |Y_s|^p + c_z |Y_s|^{p-2} |Z_s|^2 + c_u |Y_s|^{p-2} \|U_s\|^2 \right) ds
\]
\[
\leq e^{\int_0^t \eta(s) ds} |\xi|^p + \int_t^T e^{\int_0^t \eta(s) ds} \left( \alpha(s)\rho(|Y_s|^p) + \left( \alpha(s)\rho(1) + \mu(s) + \frac{R_z + R_u}{2} \beta(s)^2 \right) |Y_s|^p \right) ds
\]
\[
+ \int_t^T e^{\int_0^t \eta(s) ds} \langle p(Y_s | \eta(s)) \left( \frac{|Z_s|^2}{2R_z} + \frac{\|U_s\|^2}{2R_u} \right) ds + \int_t^T e^{\int_0^t \eta(s) ds} \langle p(Y_s | \eta(s)) |K_{[f_0]}(s) ds
\]
\[
+ \int_t^T e^{\int_0^t \eta(s) ds} \langle f_0(s) | |f_0(s)|^{p-1} ds + M(t).
\]

We set \( R_z = p/c_z, R_u = p/c_u \) and \( \eta = p(\alpha \rho(1) + \mu + \beta^2(R_z + R_u)/2) + (p - 1)K_{[f_0]} \) leading to

\[
e^{\int_0^t \eta(s) ds} |Y_t|^p + \int_t^T e^{\int_0^t \eta(s) ds} \left( \frac{c_z}{2} |Y_s|^{p-2} |Z_s|^2 + \frac{c_u}{2} |Y_s|^{p-2} \|U_s\|^2 \right) ds
\]
\[
\leq e^{\int_0^t \eta(s) ds} |\xi|^p + \int_t^T e^{\int_0^t \eta(s) ds} \rho(|Y_s|^p) ds + \int_t^T e^{\int_0^t \eta(s) ds} |f_0(s)|^{p-1} ds + M(t).
\]

Now, we omit \( e^{\int_t^0 \eta(s) ds} |Y_t|^p \) and take expectations,

\[
\mathbb{E} \int_t^T e^{\int_0^t \eta(s) ds} \left( \frac{c_z}{2} |Y_s|^{p-2} |Z_s|^2 + \frac{c_u}{2} |Y_s|^{p-2} \|U_s\|^2 \right) ds
\]
\[
\leq \mathbb{E} e^{\int_t^0 \eta(s) ds} |\xi|^p + \mathbb{E} \int_t^T e^{\int_0^t \eta(s) ds} \rho(|Y_s|^p) ds + \mathbb{E} I_{[f_0]}^{p-1} \int_t^T e^{\int_0^t \eta(s) ds} |f_0(s)| ds.
\]
Hence, we find a constant $c_0 > 0$, to end the step with
\[
\mathbb{E} \int_t^T (|Y_s|^{p-2}|Z_s|^2 + |Y_s|^{p-2}|U_s|^2) \, ds \leq c_0 \left( \mathbb{E} |\xi|^p + \mathbb{E} \int_t^T \alpha(s) \rho(|Y_s|^p) \, ds + \mathbb{E} I_{[f_0]}^p \right), \quad (14)
\]

**Step 2:**
We start again from (13). Now we set $R_z = p/(2c_2)$, $R_u = p/(2c_2)$ and $\eta = p(\alpha(s)\rho(1) + \mu + \beta^2(R_z + R_u)/2) + (p - 1)K_{[f_0]}$. By the choice of a suitable constant $c_1 > 0$
\[
e^0_\eta(s) d|Y_t|^p \leq e^0_\eta(s) |\xi|^p + \int_t^T e^0_\eta(\eta(r)) \rho(s) \rho(|Y_s|^p) \, ds + c_1 I_{[f_0]}^p + M(t).
\]
We now take suprema,
\[
\sup_{s \in [t,T]} e^0_\eta(s) d|Y_s|^p \leq e^0_\eta(s) |\xi|^p + \int_t^T e^0_\eta(\eta(r)) \rho(s) \rho(|Y_s|^p) \, ds + c_1 I_{[f_0]}^p + \sup_{s \in [t,T]} |M(s)|,
\]
and estimate the expectation of the suprema in the next step.

**Step 3:**
Let us decompose $M(t) = -A(t, T) - B(t, T)$ with
\[
A(s', t') = \int_{s'}^{t'} e^0_\eta(\eta(r)) \rho(s) \rho(|Y_s - |Y_s|^{p-2}, Z_s dW_s),
\]
\[
B(s', t') = \int_{s'}^{t'} \int_{\mathbb{R}_d} e^0_\eta(\eta(r)) \rho(|Y_s - U(s, x)|^p - |Y_s - |^p) \tilde{N}(ds, dx)
\]
for $0 \leq s' \leq t' \leq T$. From the Burkholder-Davis-Gundy inequality ([12] Theorem 10.36) we get $c_2 > 0$ and from Young’s inequality for arbitrary $R > 0$, we have
\[
\mathbb{E} \sup_{s \in [t,T]} |A(t, s)| \leq c_2 \mathbb{E} \left( \int_t^T (e^{0_\eta(\eta(r))} \rho(s) |Y_s|^{p-2}|Z_s|^2) \, ds \right)^{1/2}
\]
\[
\leq c_2 p \mathbb{E} \sup_{s \in [t,T]} |Y_s|^{p/2} \left( \int_t^T e^{0_\eta(\eta(r))} \rho(s) |Y_s|^{p-2}|Z_s|^2 \, ds \right)^{1/2}
\]
\[
\leq c_3 \mathbb{E} \left( \frac{1}{R} \sup_{s \in [t,T]} |Y_s|^{p} + R \int_t^T |Y_s|^{p-2}|Z_s|^2 \, ds \right),
\]
for another constant $c_3 > 0$. We use inequality [18] Theorem 3.2] again to get $c_4 > 0$ such that
\[
\mathbb{E} \sup_{s \in [t,T]} |B(t, s)| \leq c_4 \mathbb{E} \left( \int_t^T \int_{\mathbb{R}_d} e^{0_\eta(\eta(r))} \rho(|Y_s - U(s, x)|^p - |Y_s - |^p) \nu(dx) \, ds \right)^{1/2}.
\]
By the mean value theorem, we have for $p > 1$ and $b > a \in \mathbb{R}$, that $b^p - a^p \leq pb^{p-1}(b - a)$. 

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Appendix) finishes the proof.

where we also used the concavity of \( \rho \)

We choose \( R \) such that

\[
E \sup_{s \in [t,T]} |B(t, s)| \leq c_5 E \left( \frac{1}{R} \sup_{s \in [t,T]} |Y_s|^p + R \int_t^T |Y_s|^{p-2} \|U_s\|^2 ds \right). 
\]

Step 4:

With the last step’s results we continue from (15) to get a constant \( D > 0 \) satisfying

\[
E \sup_{s \in [t,T]} e^{\int_0^s \eta(\tau) d\tau} |Y_s|^p \leq D \left( 1 + E \sup_{s \in [t,T]} |Y_s|^p \right) + \frac{D}{R} E \sup_{s \in [t,T]} |Y_s|^p.
\]

We apply inequality (14) yielding

\[
E \sup_{s \in [t,T]} e^{\int_0^s \eta(\tau) d\tau} |Y_s|^p \leq D_1 \left( E |\xi|^p + \sup_{r \in [s,T]} |Y_r|^p \right) + \frac{D}{R} E \sup_{s \in [t,T]} |Y_s|^p,
\]

where we also used the concavity of \( \rho \). Now, the Bihari-LaSalle inequality (see Theorem A.2 in the Appendix) finishes the proof.

Proposition 4.6. Let \( 1 < p < 2 \) and let \((Y, Z, U)\) be an \( L^p \)-solution to the BSDE \((\xi, f)\). If \( \xi \in L^p \) and \((A1)\) and \((A3_{<2})\) are satisfied, then there exists a function \( h: [0, \infty[ \to [0, \infty[ \) with \( h(x) \to 0 \) as \( x \to 0 \) such that

\[
\|Y\|_{L^p(W)} + \|Z\|_{L^p(W)} + \|U\|_{L^p(\mathcal{N})} \leq h \left( E |\xi|^p + E I_{[0\uparrow]} \right),
\]

where \( h \) depends on \( p, T, \rho, \alpha, \beta_1, \beta_2, \mu \).

Proof. Step 1:

We begin this proof similarly to the case \( p \geq 2 \): Let \( \eta \) be a progressively measurable, continuous process in \( L^\infty(\Omega; L^1([0, T])) \), which we will determine later. As carried out in detail in [14, Proposition 3], Itô’s formula, applied to the smooth function \( x \mapsto (|x|^2 + \varepsilon)^{\frac{p}{2}} \) and taking the limit \( \varepsilon \to 0 \) implies that for \( c_0 = \frac{\beta_1(p-1)}{2} \) and \( t \in [0, T] \),

\[
\begin{align*}
e^{\int_0^t \eta(\tau) d\tau} &|Y_t|^p + \int_t^T e^{\int_0^s \eta(\tau) d\tau} \left[ \eta(s) |Y_s|^p + c_0 |Y_s|^{p-2} |Z_s|^2 \chi_{\{Y_s \neq 0\}} \right] ds + P(t) \\
\leq M(t) + e^{\int_0^T \eta(\tau) d\tau} |\xi|^p + \int_t^T e^{\int_0^s \eta(\tau) d\tau} P(Y_s |Y_s|^{p-2}, f(s, Y_s, Z_s, U_s)) ds,
\end{align*}
\]

Since \( |Y_s + U_s(x)| \vee |Y_s - | \leq 3 \sup_{s \in [t,T]} |Y_s| \), \( \mathbb{P} \otimes \lambda \otimes \nu \)-a.e., we obtain

\[
E \sup_{s \in [t,T]} |B(t, s)| \leq c_4 E \left( \int_t^T \int_{\mathbb{R}^d} e^{2f_0^{\lambda} \eta(\tau) d\tau} \left( p3^{p-1} \sup_{s \in [t,T]} |Y_s|^{p-1} |U_s(x)| \right)^2 \nu(dx) ds \right)^{\frac{1}{2}}.
\]
where

\[ P(t) = \int_t^T \int_{\mathbb{R}^d} e^{\int_t^s \eta(r) dr} \left[ |Y_s - U_s(x)|^p - |Y_s - |p - \langle U_s(x), pY_s - |Y_s - |p-2 \rangle \right] \nu(dx) ds \]

and

\[ M(t) = - \int_t^T e^{\int_t^s \eta(r) dr} p(Y_s - |Y_s - |p-2, Z_s dW_s) \]
\[ - \int_t^T \int_{\mathbb{R}^d} e^{\int_t^s \eta(r) dr} \left[ |Y_s - U_s(x)|^p - |Y_s - |p \right] \tilde{N}(ds, dx). \]

By the argument in [14, Proposition 3] we can use the estimate

\[ P(t) \geq c_0 \int_t^T e^{\int_t^s \eta(r) dr} \int_{\mathbb{R}^d} \left( |Y_s - | \vee |Y_s - U_s(x)| \right)^{p-2} |U_s(x)|^2 \chi(|Y_s - | \vee |Y_s - U_s(x)|) \neq 0) \nu(dx) ds \]

leading to

\[ e^{\int_t^s \eta(r) dr} |Y_s|^p + \int_t^T e^{\int_t^s \eta(r) dr} \left[ \eta(s) |Y_s|^p + c_0 |Y_s|^{p-2} |Z_s|^2 \chi(Y_s \neq 0) \right] ds \]
\[ + c_0 \int_t^T e^{\int_t^s \eta(r) dr} \int_{\mathbb{R}^d} \left( |Y_s - | \vee |Y_s - U_s(x)| \right)^{p-2} |U_s(x)|^2 \chi(|Y_s - | \vee |Y_s - U_s(x)|) \neq 0) \nu(dx) ds \]
\[ \leq M(t) + e^{\int_t^T \eta(r) dr} |\xi|^p + \int_t^T e^{\int_t^s \eta(r) dr} p(Y_s |Y_s|^{p-2}, f(s, Y_s, Z_s, U_s)) ds. \]

Using (A3_2) and Young’s inequality, we obtain for an arbitrary \( R_z > 0 \),

\[ e^{\int_t^s \eta(s) ds} |Y_t|^p + \int_t^T e^{\int_t^s \eta(r) dr} \left( \eta(s) |Y_s|^p + c_0 |Y_s|^{p-2} |Z_s|^2 \chi(Y_s \neq 0) \right) ds \]
\[ + c_0 \int_t^T e^{\int_t^s \eta(r) dr} \int_{\mathbb{R}^d} \left( |Y_s - | \vee |Y_s - U_s(x)| \right)^{p-2} |U_s(x)|^2 \chi(|Y_s - | \vee |Y_s - U_s(x)|) \neq 0) \nu(dx) ds \]
\[ \leq e^{\int_t^s \eta(s) ds} |\xi|^p + \int_t^T e^{\int_t^s \eta(r) dr} p(\alpha(s) \rho(|Y_s|^p) + \left( \mu(s) + \frac{R_z \beta_1(s)^2}{2} + (p-1) K_{\|f_0\|}(s) \right) |Y_s|^p) ds \]
\[ + \frac{p}{2R_z} \int_t^T e^{\int_t^s \eta(r) dr} \left( |Y_s|^{p-2} |Z_s| \chi(Y_s \neq 0) ds + \int_t^T e^{\int_t^s \eta(r) dr} \rho(\beta_2(s) |Y_s|^{p-1} \|U_s\| ds \]
\[ + \int_t^T e^{\int_t^s \eta(r) dr} |f_0(s)|^{p-1} ds + M(t). \] (17)

We choose \( R_z = \frac{\mu}{\mu + R_z \beta_1} \) and \( \eta = p \left( \mu + \frac{R_z \beta_1^2}{2} + (p-1) K_{\|f_0\|} \right) \), take expectations and omit the first term to arrive at

\[ \frac{c_0}{2} \mathbb{E} \int_t^T e^{\int_t^s \eta(r) dr} |Y_s|^p \chi(Y_s \neq 0) ds \]
\[ + c_0 \mathbb{E} \int_t^T e^{\int_t^s \eta(r) dr} \int_{\mathbb{R}^d} \left( |Y_s - | \vee |Y_s - U_s(x)| \right)^{p-2} |U_s(x)|^2 \chi(|Y_s - | \vee |Y_s - U_s(x)|) \neq 0) \nu(dx) ds \]
\[ \leq \mathbb{E} e^{\int_t^T \eta(s) ds} |\xi|^p + \mathbb{E} \int_t^T e^{\int_t^s \eta(r) dr} p\alpha(s) \rho(|Y_s|^p) ds \]
\[ + \mathbb{E} \int_t^T e^{\int_t^s \eta(r) dr} \rho(\beta_2(s) |Y_s|^{p-1} \|U_s\| ds + \mathbb{E} \int_t^T e^{\int_t^s \eta(r) dr} |f_0(s)|^{p-1} ds, \]
Proposition 4.5, (15) - (16) which yields constants the suprema of the stochastic integrals appearing in (17) by similar means as in step 3 of the proof of several terms differently and using the integrability assumptions on

\[ \left| Y_{s-} \right| \leq c \sup_{s \in [t,T]} \left| Y_s \right| \]

Step 2:

In this step we leave the argumentation lines of Kruse and Popier [14,15] and Briand et al, estimating several terms differently and using the integrability assumptions on \( \beta_2 \). We start from estimating the suprema of the stochastic integrals appearing in (17) by similar means as in step 3 of the proof of Proposition 4.5 [15] - (16) which yields constants \( c, c_1 > 0 \), such that for an arbitrary \( R > 0 \) we get

\[
\sup_{s \in [t,T]} |M(s)| \leq \sup_{s \in [t,T]} |A(t,s)| + \sup_{s \in [t,T]} |B(t,s)|
\]

\[
\leq c_1 \left( \frac{1}{R} \mathbb{E} \sup_{s \in [t,T]} |Y_s|^p + R \mathbb{E} \int_t^T |Y_s|^{p-2} |Z_s|^2 \chi_{\{Y_s \neq 0\}} ds \right.
\]

\[
+ E \left[ \mathbb{E} \int_t^T \int_{\mathbb{R}_0^d} \left( |Y_{s-} + U_s(x)| \lor |Y_{s-}| \right)^{p-1} |U_s(x)|^2 \chi_{\{Y_{s-} \lor U_s(x) \neq 0\}} \nu(dx) ds \right]^{\frac{2}{p}}
\]

\[
\leq c_1 \left( \frac{1}{R} \mathbb{E} \sup_{s \in [t,T]} |Y_s|^p + R \mathbb{E} \int_t^T |Y_s|^{p-2} |Z_s|^2 \chi_{\{Y_s \neq 0\}} ds \right.
\]

\[
+ \mathbb{E} \int_t^T \int_{\mathbb{R}_0^d} \left( |Y_{s-} + U_s(x)| \lor |Y_{s-}| \right)^{p-2} |U_s(x)|^2 \chi_{\{Y_{s-} \lor U_s(x) \neq 0\}} \nu(dx) ds \right)
\]

where again we used Young's inequality as well as that \( |Y_{s-} + U_s(x)| \lor |Y_{s-}| \leq 3 \sup_{s \in [t,T]} |Y(s)|, \)

\( \mathbb{P} \otimes \lambda \otimes \nu \)-a.e.

Therefore, taking suprema of the left and right hand side in (17), omitting the integral terms on the left hand side, we come to

\[
\mathbb{E} \sup_{s \in [t,T]} e^{\int_0^s \eta(\tau) d\tau} |Y_s|^p
\]

\[
\leq \mathbb{E} e^{\int_0^T \eta(s) ds} |\xi|^p + \mathbb{E} \int_t^T e^{\int_0^s \eta(\tau) d\tau} p \alpha(s) \rho(|Y_s|^p) ds
\]

\[
+ \mathbb{E} \int_t^T e^{\int_0^s \eta(\tau) d\tau} |Y_s|^{p-1} \beta_2(s) \|U_s\| ds + \mathbb{E} \int_t^T e^{\int_0^s \eta(\tau) d\tau} |f_0(s)|^{p-1} ds
\]

\[
+ \frac{c_1}{R} \sup_{s \in [t,T]} \mathbb{E} e^{\int_0^s \eta(\tau) d\tau} |Y_s|^p + c_1 R \left( \mathbb{E} \int_t^T |Y_s|^{p-2} |Z_s|^2 \chi_{\{Y_s \neq 0\}} ds \right.
\]

\[
+ \mathbb{E} \int_t^T \int_{\mathbb{R}_0^d} \left( |Y_{s-} \lor Y_{s-} + U_s(x)| \right)^{p-2} |U_s(x)|^2 \chi_{\{Y_{s-} \lor Y_{s-} + U_s(x) \neq 0\}} \nu(dx) ds \right).
\]
Now inequality (18) can be plugged in for the last parentheses to estimate, for another constant $D > 0$,

$$
\mathbb{E} \sup_{s \in [t, T]} |Y_s|^p \leq D \left( (1 + R) \mathbb{E} |\xi|^p + (1 + R) \mathbb{E} \int_t^T \alpha(s) \rho(|Y_s|^p) ds \right.
$$

$$
+ (1 + R) \mathbb{E} \int_t^T \beta_2(s) |Y_s|^{p-1} |U_s| ds + (1 + R) \mathbb{E} \int_t^T |f_0(s)| P_{[f_0]} ds + \frac{1}{R} \mathbb{E} \sup_{s \in [t, T]} |Y_s|^p \right).
$$

(19)

We focus on the term $\mathbb{E} \int_t^T \beta_2(s) |Y_s|^{p-1} |U_s| ds$, which we estimate by

$$
\mathbb{E} \sup_{s \in [t, T]} |Y_s|^{p-1} \int_t^T \beta_2(s) ||U_s|| ds
$$

$$
\leq \frac{c_2}{R_1} \mathbb{E} \sup_{s \in [t, T]} |Y_s|^p + c_2 R_1^{p-1} \mathbb{E} \left[ \left( \int_t^T \beta_2(s) ds \right)^\frac{p}{2} \left( \int_t^T ||U_s||^2 ds \right)^\frac{1}{2} \right],
$$

(20)

for $R_1 > 0$ and a constant $c_2 > 0$ coming from Young’s inequality for the couple $(p, \frac{p}{p-1})$. By the Cauchy-Schwarz inequality,

$$
\mathbb{E} \sup_{s \in [t, T]} |Y_s|^{p-1} \int_t^T \beta_2(s) ||U_s|| ds
$$

$$
\leq \frac{c_2}{R_1} \mathbb{E} \sup_{s \in [t, T]} |Y_s|^p + 2 c_2 R_1^{p-1} \mathbb{E} \left[ \left( \int_t^T \beta_2(s) ds \right)^\frac{p}{2} \left( \int_t^T ||U_s||^2 ds \right)^\frac{1}{2} \right],
$$

and using the additional integrability of $\beta_2$ with a power $q > 2$ (for the case $\beta_2 \in L^2([0, T])$ see Remark 4.7 after the proof, here we treat a non-deterministic $\beta_2$ where higher integrability is needed in the sequel), Hölder’s inequality gives us

$$
\mathbb{E} \sup_{s \in [t, T]} |Y_s|^{p-1} \int_t^T \beta_2(s) ||U_s|| ds
$$

$$
\leq \frac{c_2}{R_1} \mathbb{E} \sup_{s \in [t, T]} |Y_s|^p + 2 c_2 R_1^{p-1} \mathbb{E} \left[ \left( \int_t^T \beta_2(s) ds \right)^\frac{p}{q} \left( \int_t^T ||U_s||^2 ds \right)^\frac{q}{2} \right].
$$

Now, by the boundedness of $\int_0^T \beta_2(s)^q ds$, and applying Proposition 4.4, we get a constant $D_1 > 0$ such that

$$
\mathbb{E} \sup_{s \in [t, T]} |Y_s|^{p-1} \int_t^T \beta_2(s) ||U_s|| ds
$$

$$
\leq \frac{c_2}{R_1} \mathbb{E} \sup_{s \in [t, T]} |Y_s|^p + D_1 R_1^{p-1} (T - t) \frac{p}{q} \left( \mathbb{E} \sup_{s \in [t, T]} |Y_s|^p + \int_t^T \alpha(s) \rho(|Y(s)|^p) ds + \mathbb{E} P_{[f_0]} \right).
$$

(21)
Inserting (21) to estimate inequality (19), we get
\[
\mathbb{E} \sup_{s \in [t,T]} |Y_s|^p \leq D \left( (1 + R)(1 + D_1R_1^{p-1}T^{-\eta/q})\mathbb{E}|\xi|^p + (1 + R) \left( \mathbb{E} \int_t^T \alpha(s)\rho(|Y_s|^p)ds + (1 + R)D_1R_1^{p-1}T^{-\eta/q} \mathbb{E}I^p_{[f_0]} \right) + \left( \frac{1}{R} + \frac{(1 + R)c_2}{R_1} + (1 + R)D_1R_1^{p-1}(T-t)^{-\eta/q} \right) \sup_{s \in [t,T]} |Y_s|^p \right).
\]

Now, choose \( R \) such that \( \frac{D}{R} < \frac{1}{2} \), afterwards choose \( R_1 \) such that \( \frac{D(1+R)c_2}{R_1} < \frac{1}{8} \). Now, our goal for the next step is to divide \([0, T]\) into small parts in order to make the third term containing \((T-t)\) small too.

**Step 3:**

From here on, let the time interval \([0, T]\) be partitioned into \( 0 = t_0 < t_1 < \ldots < t_n = T \), such that for all \( 1 \leq i \leq n \), \( D(1 + R)D_1R_1^{p-1}(t_i - t_{i-1})^{-\eta/q} \leq \frac{1}{8} \). Thus, on the interval \([t_{n-1}, T]\), we come to a constant \( D_2 > 0 \) such that
\[
\mathbb{E} \sup_{s \in [t,T]} |Y_s|^p \leq D_2 \left( \mathbb{E}|\xi|^p + \int_t^T \alpha(s)\rho(\mathbb{E}|Y_s|^p)ds + \mathbb{E}I^p_{[f_0]} \right).
\]

Now, the Bihari-LaSalle inequality (Theorem A.2) shows that there is a function \( h_n \) such that
\[
\mathbb{E} \sup_{s \in [t_{n-1}, T]} |Y_s|^p \leq h_n(\mathbb{E}|\xi|^p + \mathbb{E}I^p_{[f_0]}).
\]

Performing the same steps as above for the interval \([t_{n-2}, t_{n-1}]\), we find a function \( h_{n-1} \) such that
\[
\mathbb{E} \sup_{s \in [t_{n-2}, t_{n-1}]} |Y_s|^p \leq h_{n-1}(\mathbb{E}|Y_{t_{n-1}}|^p + \mathbb{E}I^p_{[f_0]}) \leq h_{n-1}(h_n(\mathbb{E}|\xi|^p + \mathbb{E}I^p_{[f_0]}) + \mathbb{E}I^p_{[f_0]}).
\]

Iterating the procedure backwards in time, we end up with functions \( h_1, \ldots, h_n \), accumulating to a function \( h \), such that
\[
\mathbb{E} \sup_{s \in [0,T]} |Y_s|^p \leq \tilde{h}(\mathbb{E}|\xi|^p + \mathbb{E}I^p_{[f_0]}).
\]

The bound for \( \|Z\|_{L^p(W)}^p + \|U\|_{L^p(\widetilde{N})}^p \) then follows from Proposition 4.4, concluding the proof.

**Remark 4.7.** In step 3, if \( \beta_2 \) is deterministic, we could impose the weaker condition, namely \( \beta_2 \) being only square-integrable (instead of in \( L^q \), for some \( q > 2 \)). Then we do not need to apply Hölder’s inequality to (20) in order to choose the division of \([0, T]\) such that \( D_1(t_i - t_{i-1})^{-\eta/q} \) is small. Instead we choose the partition such that the \( \int_{t_{i-1}}^{t_i} \beta_2(s)^2ds \) become sufficiently small.

With the technique from the two a priori estimates above in hand, we can now prove another key part for the existence proof: boundedness stability of the \( Y \) process, meaning that the solution process \( Y \) stays bounded, when the data \((\xi, f)\) has boundedness properties:
Proposition 4.8. Let \( p > 1 \) and \((Y, Z, U)\) be an \( L^p \)-solution to the BSDE \((\xi, f)\).
If \( \xi, I_{[f_0]} \in L^\infty \) and

(i) for \( p \geq 2, (A 1) \) and \( (A 3 \geq 2) \) hold,

(ii) or for \( 1 < p < 2 \), \( (A 1) \) and \( (A 3 < 2) \) hold,

then there exists a constant \( C > 0 \) such that for all \( t \in [0, T] \),

\[ |Y_t|^p \leq C, \quad \mathbb{P}\text{-a.s.} \]

Here \( C \) depends on \( \|\xi\|_\infty, \|I_{[f_0]}\|_\infty, \mu, p, T, \rho, \alpha, \beta \) for \( p \geq 2 \). If \( p < 2 \), \( C \) depends on the same variables but \( \beta_1, \beta_2, q \) instead of \( \beta \).

Proof. We copy the proofs of Proposition 4.5 and Proposition 4.6 for the mutual cases \( 1 < p < 2 \) and \( 2 \leq p \), replacing the operator \( \mathbb{E} \) by \( \mathbb{E}[\cdot | F_t] \) considering the BSDEs on \([t, T]\), which leads to the estimates \( \mathbb{E}\left[\sup_{s \in [t, T]} |Y_s|^p | F_t\right] < C \) for all \( t \in [0, T] \). The assertion now follows from the monotonicity of the conditional expectation.

5 Proof of the Main Theorem 3.3

The proof basically follows the one in Briand et al. [3, Theorem 4.2]. For convenience of the reader, we give a detailed proof adapted to our more general setting. We consider only the case \( 1 < p < 2 \) as the case \( p \geq 2 \) is similar but easier.

Step 1: Uniqueness
Assume we have another solution \((Y', Z', U')\). Then Proposition 4.6 applied to the BSDE \((0, g)\) with \( g(t, y, z, u) = f(t, y+Y', z+Z', u+U') - f(Y', Z', U') \) implies \((Y - Y', Z - Z', U - U') = (0, 0, 0)\).

Step 2:
In this step, we construct a first approximating sequence of generators for \( f \) and show several estimates for the solution processes. Assume that \( \xi, I_{[f_0]} \in L^\infty \). As \( (A 3 < 2) \) is satisfied, the condition is also satisfied for the changed parameter \( \mu' = \rho(1)\alpha + \mu \). We take the constant \( C \) appearing in Proposition 4.8 and choose an \( r > C \).

Take a smooth real function \( \theta_r \) such that \( 0 \leq \theta_r \leq 1, \theta_r(y) = 1 \) for \( |y| \leq r \) and \( \theta_r(y) = 0 \) for \( |y| \geq r + 1 \) and define

\[ h_n(t, y, z, u) := \theta_r(y) \left( f(t, y, c_n(z), \tilde{c}_n(u)) - f_0(t) \right) \frac{n}{\psi_{r+1}(t) n} + f_0(t). \]

Here, \( c_n, \tilde{c}_n \) are the projections \( x \mapsto nx/(|x| \lor n) \) onto the closed unit balls of radius \( n \), respectively in \( \mathbb{R}^{d \times k} \) and \( L^2(\nu) \).

These generators \( h_n \) satisfy the following properties for all \( n \in \mathbb{N} \):

(A i) Condition \( (A 1) \) is satisfied.

(A ii) By \( (A 2) \)

\[ |h_n(t, y, z, u)| \leq n + |f_0(t)| + \Phi(t)(|z| + |u|). \]
(A iii) By [A 2] and [A3≤2] with \( \beta = \beta_1 + \beta_2 \), and \( C_r \) denoting the Lipschitz constant of \( \theta_r \), it holds that

\[
\langle y - y', h_n(t, y, z, u) - h_n(t, y', z', u') \rangle = \theta_r(y) \frac{n}{\psi_{r+1}(t) \vee n} \langle y - y', f(t, y, c_n(z), \tilde{c}_n(u)) - f(t, y', c_n(z'), \tilde{c}_n(u')) \rangle \\
+ \frac{n}{\psi_{r+1}(t) \vee n} (\theta_r(y) - \theta_r(y')) \langle y - y', f(t, y', c_n(z'), \tilde{c}_n(u')) - f_0(t) \rangle \\
\leq \frac{n}{\psi_{r+1}(t) \vee n} \left( \alpha(t) \rho(y - y'|p) \frac{|y - y'|^p}{|y - y'|^{p-2}} + \mu(s) |y - y'|^2 + \beta(t) |z - z'| + ||u - u'|| \right) \\
+ C_r (2\Phi(s) + 1) n |y - y'|^2 \\
\leq \alpha(t) \rho(y - y'|^2) + (\rho(1) \alpha(s) + \mu(s) + C_r (2\Phi(s) + 1) n) |y - y'|^2 \\
+ \beta(t) |z - z'| + ||u - u'||,
\]

where we used Remark 3.1(ii)(b).

(A iv) By [A3≤2] again with \( \beta = \beta_1 + \beta_2 \) we have,

\[
y h_n(t, y, z, u) \leq |y| f_0(t) + \alpha(t) \rho(y| p^2) + (\rho(1) \alpha(t) + \mu(t)) |y|^2 + \beta(t) |y| (|z| + ||u||).
\]

Properties [A i][A iii] imply that for \( \mu_r(s) := (\rho(1) \alpha(s) + \mu(s) + C_r (2\Phi(s) + 1) n) \) the generator \( g_n := e^{\int_0^T \mu_r(s) ds} (h_n - \mu_r \cdot y) \) satisfies assumptions (A 1)-(A 3) of Theorem 3.1 in [11] (or rather a straightforward adaptation to \( d \) dimensions of it) and admits a unique solution of BSDE \( (e^{\int_0^T \mu_r(s) ds} \xi, g_n) \). Thus, by the transformation \( (Y, Z, U) := e^{-\int_0^T \mu_r(s) ds} (Y, Z, U) \), one gets that also \( (\xi, h_n) \) has a unique solution in \( (Y^n, Z^n, U^n) \).

Moreover, by property [A iv] and assertion 6 of Remark 4.1 we are able to apply Proposition 4.8 to get that \( ||Y^n||_\infty \leq r \). Since \( Y^n \) is bounded by \( r \), we get that \( (Y^n, Z^n, U^n) \) is also a solution to the BSDE \( (\xi, f_n) \), with

\[
f_n(t, y, z, u) := (f(t, y, c_n(z), \tilde{c}_n(u)) - f_0(t)) \frac{n}{\psi_{r+1}(t) \vee n} + f_0(t).
\]

Comparing the solutions \( (Y^n, Z^n, U^n) \) and \( (Y^m, Z^m, U^m) \) for \( m \geq n \), we use the standard methods from [12]-[13], for the differences

\[
(\Delta Y, \Delta Z, \Delta U) := (Y^m, Z^m, U^m) - (Y^n, Z^n, U^n).
\]

In this procedure, we replace the use of the monotonicity condition [A3≥2] in Proposition 4.6 by

\[
|\Delta Y_s|^{p-2} \langle \Delta Y_s, f_m(s, Y^m_s, Z^m_s, U^m_s) - f_n(s, Y^n_s, Z^n_s, U^n_s) \rangle \\
= |\Delta Y_s|^{p-2} \langle \Delta Y_s, f_m(s, Y^m_s, Z^m_s, U^m_s) - f_m(s, Y^n_s, Z^n_s, U^n_s) \rangle \\
+ |\Delta Y_s|^{p-2} \langle \Delta Y_s, f_m(s, Y^m_s, Z^m_s, U^m_s) - f_n(s, Y^n_s, Z^n_s, U^n_s) \rangle \\
\leq \alpha(s) \rho(|\Delta Y_s|^p) + \left( \rho(1) \alpha(s) + \mu(s) + \frac{R_2 + R_a}{2} \beta(s)^2 \right) |\Delta Y_s|^p \\
+ \frac{1}{2} \left( |\Delta Z_s|^2 + \frac{|\Delta U_s|^2}{2R_a} \right) \\
\leq \alpha(s) \rho(|\Delta Y_s|^p) + \left( \rho(1) \alpha(s) + \mu(s) + \frac{R_2 + R_a}{2} \beta(s)^2 \right) |\Delta Y_s|^p \\
+ \frac{1}{2} \left( |\Delta Z_s|^2 + \frac{|\Delta U_s|^2}{2R_a} \right),
\]

for all \( s \in [0, T] \) and \( t \in [s, T] \).

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such that the same steps of the proof of Proposition 4.6 can be conducted to get a function \( h \) with

\[
\|\Delta Y\|_{L^p(W)}^p + \|\Delta Z\|_{L^p(\tilde{N})}^p + \|\Delta U\|_{L^p(\tilde{N})}^p
\leq h \left( \mathbb{E} \int_0^T |\Delta Y_s|^{p-2} \left| (\Delta Y_s, f_m(s, Y^n_s, Z^n_s, U^n_s) - f_n(s, Y^n_s, Z^n_s, U^n_s)) \right| ds \right)
\]

(in the case for \([A3_{\geq 2}]\) we use the steps from Proposition 4.5 and Proposition 4.3). So \( \|\Delta Y\|_{L^p(W)}^p + \|\Delta Z\|_{L^p(\tilde{N})}^p + \|\Delta U\|_{L^p(\tilde{N})}^p \) tends to zero if

\[
\mathbb{E} \int_0^T |\Delta Y_s|^{p-2} \left| (\Delta Y_s, f_m(s, Y^n_s, Z^n_s, U^n_s) - f_n(s, Y^n_s, Z^n_s, U^n_s)) \right| ds
\]
does, which we will show next (in the case of \([A3_{\geq 2}]\) this follows from Proposition 4.5 together with assertion 7 of Remark 4.1).

Since \( |Y^m_t|, |Y^n_t| \leq r \), we estimate

\[
\mathbb{E} \int_0^T |\Delta Y_s|^{p-2} \left| (\Delta Y_s, f_m(s, Y^n_s, Z^n_s, U^n_s) - f_n(s, Y^n_s, Z^n_s, U^n_s)) \right| ds
\leq (2r)^{p-1} \mathbb{E} \int_0^T \left| f_m(s, Y^n_s, Z^n_s, U^n_s) - f_n(s, Y^n_s, Z^n_s, U^n_s) \right| ds. \tag{22}
\]

Because of the definition of \( f_m, f_n \) and since \( m \geq n \), the integrand is zero if \( |Z_s| \leq n, |U_s| \leq n \) and \( \psi_{r+1}(s) \leq n \) at the same time and bounded by

\[
2\Phi(s) \left( |Z^n_s| + \|U^n_s\| \right) \chi_{\{|Z^n_s|+\|U^n_s\|>n\}} + 2\Phi(s) \left( |Z^n_s| + \|U^n_s\| \right) \chi_{\{|\psi_{r+1}(s)|>n\}}
\]
\[+ 2\psi_{r+1}(s) \chi_{\{|\psi_{r+1}(s)|>n\}} + 2\psi_{r+1}(s) \chi_{\{|Z^n_s|+\|U^n_s\|>n\}} \tag{23}
\]
on otherwise.

To show convergence of the integral of \((23)\), we use the uniform integrability of the families \( \Phi(|Z^n| + \|U^n\|) \) with respect to the measure \( \mathbb{P} \otimes \lambda \), which follows from

\[
\mathbb{E} \int_0^T \Phi(s)(|Z^n_s| + \|U^n_s\|) ds \leq \mathbb{E} \int_0^T \left( \Phi(s)^2 + |Z^n_s|^2 + \|U^n_s\|^2 \right) ds
\]
\[\leq \left\| \int_0^T \Phi(s)^2 ds \right\|_{\infty} + r',
\]
since by Proposition 4.6 and \([A_{iv}]\) there is \( r' > 0 \) such that \( \|Z^n\|_{L^2(W)}^2 + \|U^n\|_{L^2(\tilde{N})}^2 < r' \). Therefore, as \((23)\) (as sequence in \( n \)) is uniformly integrable w.r.t. \( \mathbb{P} \otimes \lambda \), dominating the sequence \( \left( |f_m(s, Y^n_s, Z^n_s, U^n_s) - f_n(s, Y^n_s, Z^n_s, U^n_s)| \right)_{n \geq 0} \), which approaches zero pointwisely, also \((22)\) tends to zero as \( m \to n \to \infty \). Hence, also \( |\Delta Y|_{L^p(W)}^p + |\Delta Z|_{L^p(\tilde{N})}^p + |\Delta U|_{L^p(\tilde{N})}^p \) tend to zero, showing that the \((Y^n, Z^n, U^n)\) form a Cauchy sequence in \( S^p \times L^p(W) \times L^p(\tilde{N}) \) and converge to an element \((Y, Z, U)\).

**Step 3:**

We now show that the \((Y, Z, U)\) satisfies the BSDE with \((\xi, f)\) from step 2. The stochastic integral
terms of the BSDEs \((\xi, f_n)\) with solution \((Y^n, Z^n, U^n)\) converge to the corresponding terms of the BSDE \((\xi, f)\) also in probability. It is left to show that, at least for a subsequence,

\[
\int_t^T f_n(s, Y^n_s, Z^n_s, U^n_s)ds \to \int_t^T f(s, Y_s, Z_s, U_s)ds, \quad \mathbb{P}\text{-a.s.}
\]

For an appropriate subsequence all other terms of the BSDEs converge almost surely. W.l.o.g, this subsequence is assumed to be the original one. Hence, we know that there is a random variable \(V_t\) such that

\[
\int_t^T f_n(s, Y^n_s, Z^n_s, U^n_s)ds \to V_t, \quad \mathbb{P}\text{-a.s.}
\]

We take expectations and split up the integral into

\[
\delta^{(1)} := \mathbb{E} \int_t^T (f_n(s, Y^n_s, Z^n_s, U^n_s) - f(s, Y^n_s, Z^n_s, U^n_s)) ds
\]

and

\[
\delta^{(2)} := \mathbb{E} \int_t^T (f(s, Y^n_s, Z^n_s, U^n_s) - f(s, Y_s, Z_s, U_s)) ds.
\]

By the same argument as for inequality (23) above,

\[
|\delta^{(1)}| \leq \mathbb{E} \int_t^T \left[ 2\Phi(s) (|Z^n_s| + \|U^n_s\|) \chi_{\{|Z^n_s| + \|U^n_s\| > n\}} + 2\Phi(s) (|Z^n_s| + \|U^n_s\|) \chi_{\{\psi_{r+1}(s) > n\}} + 2\psi_{r+1}(s) \chi_{\{\psi_{r+1}(s) > n\}} + 2\psi_{r+1}(s) \chi_{\{|Z^n_s| + \|U^n_s\| > n\}} \right] ds,
\]

which converges to zero. Now, for \(\delta^{(2)}\), we know that \((Y^n, Z^n, U^n) \to (Y, Z, U)\) in the measure \(\mathbb{P} \otimes \lambda\). Thus also \(f(s, Y^n_s, Z^n_s, U^n_s) \to f(s, Y_s, Z_s, U_s)\) in \(\mathbb{P} \otimes \lambda\). Since now

\[
|f(s, Y^n_s, Z^n_s, U^n_s)| + |f(s, Y_s, Z_s, U_s)| \\
\leq \psi_{r+1}(s) + \Phi(s) (|Z^n_s| + |Z_s| + \|U^n_s\| + \|U_s\|),
\]

and \((\psi_{r+1} + \Phi(|Z^n| + |Z| + \|U^n\| + \|U\|))_{n \geq 1}\) is uniformly integrable with respect to \(\mathbb{P} \otimes \lambda\), it follows that also \(|\delta^{(2)}| \to 0\).

Thus,

\[
\mathbb{E} \int_t^T f_n(s, Y^n_s, Z^n_s, U^n_s)ds \to \mathbb{E} \int_t^T f(s, Y_s, Z_s, U_s)ds,
\]

and extracting a subsequence \((n_t)_{t \geq 1}\) satisfying \(\mathbb{P}\text{-a.s.}\)

\[
\int_t^T f_{n_t}(s, Y^n_{n_t}, Z^n_{n_t}, U^n_{n_t})ds \to \int_t^T f(s, Y_s, Z_s, U_s)ds,
\]

shows that \(V_t = \int_t^T f(s, Y_s, Z_s, U_s)ds\). So \((Y, Z, U)\) satisfies the BSDE \((\xi, f)\).

**Step 4:**

We now approximate a general \(\xi \in L^p\) by \(c_n(\xi)\) and the generator \(f\) by

\[
f^n(t, y, z, u) := f(t, y, z, u) - f^0(t) + c_n(f^0(t)).
\]
A solution \((Y^n, Z^n, U^n)\) to \((c_n(\xi), f^n)\) exists due to the last step. Now we get, for \(m \geq n\), denoting differences again by

\[
(\Delta Y, \Delta Z, \Delta U) := (Y^m, Z^m, U^m) - (Y^n, Z^n, U^n)
\]

via Proposition 4.6 (we use the generator \(g_{m,n}(t,y,z,u) := f^m(t,y + Y^n, z + Z^n, u + U^n) - f^n(Y^n, Z^n, U^n)\)):

\[
\|\Delta Y\|_{L^p}^p + \|\Delta Z\|_{L^2(W)}^2 + \|\Delta U\|_{L^2(N)}^2 \\
\leq h \left( \mathbb{E}|c_n(\xi) - c_n(\xi)|^p + \mathbb{E} \left( \int_0^T |f^m(s, Y^n_s, Z^n_s, U^n_s) - f^n(Y^n_s, Z^n_s, U^n_s)| ds \right)^p \right) \\
= h \left( \mathbb{E}|c_n(\xi) - c_n(\xi)|^p + \mathbb{E} \left( \int_0^T |c_m(f_0(t)) - c_n(f_0(t))| ds \right)^p \right).
\]

As \(n \to \infty\), the latter term tends to zero showing convergence of the sequence \((Y^n, Z^n, U^n)\) to \((Y, Z, U)\) in \(S^p \times L^p(W) \times L^p(N)\). Again it remains to check that

\[
\int_t^T f^n(s, Y^n_s, Z^n_s, U^n_s) ds \to \int_t^T f(s, Y_s, Z_s, U_s) ds, \quad \mathbb{P}\text{-a.s.},
\]

at least for a subsequence. The first difference, after splitting up again, is

\[
\int_t^T (f^n(s, Y^n_s, Z^n_s, U^n_s) - f(s, Y^n_s, Z^n_s, U^n_s)) ds = \int_t^T (c_n(f_0(t)) - f_0(t)) ds,
\]

which tends to zero as \(n \to \infty\).

The second difference is more complicated:

Here, we extract a subsequence \((n_l)_{l \geq 1}\) such that

\[
\sup_{t \in [0,T]} |Y_t - Y_{n_l}^m| \to 0, \quad \mathbb{P}\text{-a.s.,} \quad (Z^{n_l}, U^{n_l}) \to (Z, U), \quad \lambda\text{-a.e.,} \quad \mathbb{P}\text{-a.s.,}
\]

and

\[
\left( \int_0^T |Z^{n_l}|^2 ds, \int_0^T |U^{n_l}|^2 ds \right) \to \left( \int_0^T |Z_s|^2 ds, \int_0^T |U_s|^2 ds \right), \quad \mathbb{P}\text{-a.s.,}
\]

which is possible due to the convergence of \((Y^n, Z^n, U^n)\) in \(S^p \times L^p(W) \times L^p(N)\). Severini-Egorov’s theorem now permits for a given \(\varepsilon > 0\) the existence of a set \(\Omega_\varepsilon, \mathbb{P}(\Omega_\varepsilon) > 1 - \varepsilon\) such that there is a number \(N_\varepsilon > 0\) with

\[
\sup_{t \in [0,T]} |Y_l(t) - Y_{n_l}^m(t)| < 1 \quad \text{for all } l > N_\varepsilon \text{ and } \omega \in \Omega_\varepsilon,
\]

and the convergences above persist on this set. For given \(r > 0\) on \(\Omega_\varepsilon^r := \Omega_\varepsilon \cap \left\{ \sup_{t \in [0,T]} |Y_t| \leq r \right\}\), we have for \(l > N_\varepsilon\),

\[
\int_0^T |f(s, Y^{n_l}_s, Z^{n_l}_s, U^{n_l}_s) - f(s, Y_s, Z_s, U_s)| ds \\
\leq \int_0^T (2\psi_{r+1}(s) + \Phi(s) (|Z^{n_l}_s| + |Z_s| + \|U^{n_l}_s\| + \|U_s\|)) ds < C < \infty,
\]
where $C$ may still depend on $\omega \in \Omega_\varepsilon$ and for such $\omega$, the family $(s \mapsto (|Z^{n_i}_s(\omega)|^2 + \|U^{n_i}_s(\omega)\|^2)_{t \geq 0}$ is uniformly integrable with respect to $\lambda$. Thus, since $f(s, Y^{n_i}_s, Z^{n_i}_s, U^{n_i}_s) \to f(s, Y_s, Z_s, U_s)$, for $\lambda$-a.a. $s \in [0, T]$ on $\Omega_\varepsilon$, dominated convergence yields

$$
\lim_{l \to \infty} \int_t^T f(s, Y^{n_i}_s, Z^{n_i}_s, U^{n_i}_s) ds = \int_t^T f(s, Y_s, Z_s, U_s) ds
$$

for $l \to \infty$ on $\Omega_\varepsilon$ for all $\varepsilon > 0$, $r > 0$. The last identity now even holds on $\Omega^0 := \bigcup_{r > 0} \bigcup_{q \geq 1} \Omega_{1/q}$ which is an almost sure event. So, the limit of $\int_t^T f^n(s, Y^n_s, Z^n_s, U^n_s) ds$ is uniquely determined as $\int_t^T f(s, Y_s, Z_s, U_s) ds$. Hence, $(Y, Z, U)$ is a solution to BSDE $(\xi, f)$.

\[\Box\]

## 6 Comparison Results

We switch to dimension $d = 1$ and set $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$ for the following comparison results, generalizing those in [11] to the case of generators that do not have linear growth in the $y$-variable and to an $L^p$-setting for $p > 1$. Moreover, in contrast to [11], our proof does not depend on approximation theorems for BSDEs that demand deep measurability results.

**Theorem 6.1 (Comparison, $p \geq 2$).** Let $p, p' \geq 2$ and $(Y, Z, U)$ be the $L^p$-solution to $(\xi, f)$ and $(Y', Z', U')$ be the $L^{p'}$-solution to $(\xi', f')$. Furthermore let $f$ and $f'$ satisfy $(A1)$ and $(A3 \geq 2)$ for the according $p, p'$. If the following assumptions hold

(i) $\xi \leq \xi'$, $\mathbb{P}$-a.s.,

(ii) $f(s, Y'_s, Z'_s, U'_s) \leq f'(s, Y'_s, Z'_s, U'_s)$, for $\mathbb{P} \otimes \lambda$-a.a. $(\omega, s) \in \Omega \times [0, T]$ and

(A7) for all $u, u' \in L^2(\nu)$ with $u' \geq u$

$$f(s, y, z, u) - f(s, y, z, u') \leq \int_{\mathbb{R}_0} (u'(x) - u(x))\nu(dx), \quad \mathbb{P} \otimes \lambda$-a.e.,

(24)

then for all $t \in [0, T]$, we have $\mathbb{P}$-a.s.

$$Y_t \leq Y'_t.$$

The same assertion follows from an equivalent formulation for $f'$, requiring $f(s, Y_s, Z_s, U_s) \leq f'(s, Y'_s, Z'_s, U'_s)$ and (24) being satisfied for $f'$.

**Proof.** The basic idea for this proof was inspired by the one of Theorem 8.3 in [7] and is an extension and simplification of the one in [11].

**Step 1:**

First, note that $(Y, Z, U), (Y', Z', U')$ are solutions in $S^2 \times L^2(W) \times L^2(\tilde{N})$.

We use the conditional expectation $\mathbb{E}_n$ (see subsection 2.2) on the BSDEs $(\xi, f)$ and $(\xi', f')$ to get (for the BSDE $(\xi, f)$)

$$
\mathbb{E}_n Y_t = \mathbb{E}_n \xi + \int_t^T \mathbb{E}_n f(s, Y_s, Z_s, U_s) ds - \int_t^T \mathbb{E}_n Z_s dW_s
$$

$$- \int_{[t, T] \times \mathbb{R}_0} \chi_{\{1/n \leq |x|\}} \mathbb{E}_n U_s(x) \tilde{N}(ds, dx).
$$
Note that here all the processes \((\mathbb{E}_n Z_t)_{t \in [0,T]}, (\mathbb{E}_n U_t(x))_{t \in [0,T], x \in \mathbb{R}_0}, (\mathbb{E}_n f(t, Y_t, Z_t, U_t))_{t \in [0,T]}\) are considered to be progressively measurable processes that equal the conditional expectation \(\mathbb{P}\)-a.s. for almost all \(t \in [0, T]\) (or \(\lambda \otimes \nu\) almost every \((t,x) \in [0, T] \times \mathbb{R}_0\) for the \(U\)-process). In the case of \(Y, (\mathbb{E}_n Y_t)_{t \in [0,T]}\), denotes a progressively measurable version of this process. For bounded or nonnegative processes, the construction of such processes can be achieved by using optional projections with parameters (see [19] for optional projections with parameters, [11] for the mentioned construction).

In the present case, we are confronted with merely integrable processes: \((\mathbb{E}_n Y_t, Z_t, U_t)\) denotes a progressively measurable version of this process. For bounded or nonnegative \(\int_{[0,t]} \mathbb{E}_n (f(s, Y_s, Z_s, U_s)) ds, (\mathbb{E}_n Y_t, Z_t, U_t)\) integrable and hence also \(f(s, Y_s, Z_s, U_s) \in L^1(W)\). The construction of a progressively measurable version for the processes at hand can be found in Lemma A.4 and Remark A.5 in the Appendix.

Moreover, assume for the rest of the proof that the coefficient \(\mu\) of \(f\) is zero: If this was not the case, we could use the transformed variables \((\tilde{Y}_t, \tilde{Z}_t, \tilde{U}_t) := e^{\int_0^t \mu(s) ds}(Y_t, Z_t, U_t)\) and \((\tilde{Y}', \tilde{Z}', \tilde{U}') := e^{\int_0^t \mu(s) ds}(Y'_t, Z'_t, U'_t)\).

**Step 2:**
We use Tanaka-Meyer’s formula (cf. [6] Section 2.11)) to see that for \(\eta := 18\beta^2\),

\[
e^{\int_0^t \eta(s) ds}(\mathbb{E}_n Y_t - \mathbb{E}_n Y'_t)_+^2 = e^{\int_0^T \eta(s) ds}(\mathbb{E}_n \xi - \mathbb{E}_n \xi')_+^2 + M(t)
+ \int_t^T e^{\int_0^t \eta(r) dr} \mathcal{X}(\mathbb{E}_n Y_s - \mathbb{E}_n Y'_s)_+^2 \times \frac{2(\mathbb{E}_n Y_s - \mathbb{E}_n Y'_s)_+ + \mathbb{E}_n(f(s, Y_s, Z_s, U_s) - f'(s, Y'_s, Z'_s, U'_s))}{\mathbb{E}_n Z_s - \mathbb{E}_n Z'_s}^2 - \eta(s) |\mathbb{E}_n Y_s - \mathbb{E}_n Y'_s|^2
- \int_{\{1/n \leq |x|\}} ((\mathbb{E}_n Y_s - \mathbb{E}_n Y'_s + \mathbb{E}_n U_s(x) - \mathbb{E}_n U'_s(x))_+^2 - (\mathbb{E}_n Y_s - \mathbb{E}_n Y'_s)_+^2
- 2(\mathbb{E}_n U_s(x) - \mathbb{E}_n U'_s(x))(\mathbb{E}_n Y_s - \mathbb{E}_n Y'_s)_+ \nu(dx)) ds.
\]

Here, \(M(t)\) is a stochastic integral term with zero expectation which follows from \(Y, Y' \in S^2\). Moreover, we used that on the set \(\{\Delta^n Y_s \geq 0\}\) (where \(\Delta^n Y := \mathbb{E}_n Y - \mathbb{E}_n Y'\)) we have \((Y_s - Y'_s)_+ = Y_s - Y'_s\). Taking means and denoting the differences by \(\Delta^n \xi := \mathbb{E}_n \xi - \mathbb{E}_n \xi', \Delta^n Z := \mathbb{E}_n Z - \mathbb{E}_n Z'\) and \(\Delta^n U := \mathbb{E}_n U - \mathbb{E}_n U'\) leads us to

\[
e^{\int_0^t \eta(s) ds}(\Delta^n Y_t)_+^2 = e^{\int_0^t \eta(s) ds}(\Delta^n \xi)_+^2
+ \mathbb{E} \int_t^T e^{\int_0^t \eta(r) dr} \mathcal{X}(\Delta^n Y_s)_+^2 \times \frac{2(\Delta^n Y_s)_+ + \mathbb{E}_n(f(s, Y_s, Z_s, U_s) - f'(s, Y'_s, Z'_s, U'_s))}{\Delta^n Z_s}^2 - \eta(s) |\Delta^n Y_s|^2
- \int_{\{1/n \leq |x|\}} ((\Delta^n Y_s + \Delta^n U_s(x))_+^2 - (\Delta^n Y_s)_+^2 - 2(\Delta^n U_s(x))(\Delta^n Y_s)_+ \nu(dx)) ds,
\]

We split up the set \(\{1/n \leq |x|\}\) into

\(B_n(s) = B_n(\omega, s) = \{1/n \leq |x|\} \cap \{\Delta^n U_s(x) \geq -\Delta^n Y_s\}\) and its complement \(B_n^c(s)\).
Taking into account that \( \xi \leq \xi' \Rightarrow \mathbb{E}_n \xi \leq \mathbb{E}_n \xi' \), we estimate
\[
\mathbb{E}_f\int_0^t \eta(ds) \langle \Delta^n Y_t \rangle_+^2 \\
\leq \mathbb{E} \int_t^T e^{-\int_0^t \eta(r)dr} \chi(\Delta^n Y_s > 0) \\
\times \left[ 2(\Delta^n Y_s)_{+} \mathbb{E}_n (f(s, Y_s', Z_s', U_s') - (\Delta^s Z_s')_+^2 - \eta(s)|\Delta^s Y_s|^2 \\
- \int_{B_n(s)} \langle \Delta^n U_s(x) \rangle_+^2 \nu(dx) + \int_{B_n(s)} \langle (\Delta^n Y_s)_+^2 + 2(\Delta^n U_s(x)) \rangle_+ \nu(dx) \right] ds. \quad (25)
\]

We focus on \( (\Delta^n Y_s)_+ \mathbb{E}_n (f(s, Y_s, Z_s, U_s) - f'(s, \Theta_s')) \). By abbreviating \( \Theta := (Y, Z, U) \), \( \Theta' := (Y', Z', U') \) and by the assumption \( f(s, \Theta) \leq f'(s, \Theta') \) we get,
\[
(\Delta^n Y_s)_+ \mathbb{E}_n (f(s, \Theta_s) - f'(s, \Theta'_s)) = (\Delta^n Y_s)_+ \mathbb{E}_n (f(s, \Theta_s) - f(s, \Theta'_s) + f(s, \Theta'_s) - f'(s, \Theta'_s)) \\
\leq (\Delta^n Y_s)_+ \mathbb{E}_n (f(s, \Theta_s) - f(s, \Theta'_s)). \quad (26)
\]

Since [A3.22] implies the Lipschitz property in the \( u \) and \( z \)-variables, we infer, inserting and subtracting the same terms,
\[
(\Delta^n Y_s)_+ \mathbb{E}_n (f(s, Y_s, Z_s, U_s) - f(s, Y'_s, Z'_s, U'_s)) \\
= (\Delta^n Y_s)_+ \mathbb{E}_n (f(s, Y_s, Z_s, U_s) - f(s, Y'_s, Z'_s, U'_s) + (f(s, Y_s, \mathbb{E}_n Z_s, \mathbb{E}_n U_s) - f(s, Y'_s, \mathbb{E}_n Z'_s, \mathbb{E}_n U'_s)) \\
- \left( f(s, Y_s, \mathbb{E}_n Z_s, \mathbb{E}_n U_s) - f(s, Y'_s, \mathbb{E}_n Z'_s, \mathbb{E}_n U'_s) \right)) \\
\leq (\Delta^n Y_s)_+ \mathbb{E}_n (f(s, Y_s, \mathbb{E}_n Z_s, \mathbb{E}_n U_s) - f(s, Y'_s, \mathbb{E}_n Z'_s, \mathbb{E}_n U'_s)) \\
+ (\Delta^n Y_s)_+ \beta(s) \left( |Z_s - \mathbb{E}_n Z_s| + |Z'_s - \mathbb{E}_n Z'_s| + \|U_s - \mathbb{E}_n U_s\| + \|U'_s - \mathbb{E}_n U'_s\| \right). \quad (27)
\]

We estimate, inserting and subtracting terms again, then using [24],
\[
(\Delta^n Y_s)_+ \mathbb{E}_n (f(s, Y_s, \mathbb{E}_n Z_s, \mathbb{E}_n U_s) - f(s, Y'_s, \mathbb{E}_n Z'_s, \mathbb{E}_n U'_s)) \\
\leq (\Delta^n Y_s)_+ \mathbb{E}_n \left( f(s, Y_s, \mathbb{E}_n Z_s, \mathbb{E}_n U_s \chi_{B_n(s)} + \mathbb{E}_n U_s \chi_{\overline{B}_n(s)}) \\
- f(s, Y'_s, \mathbb{E}_n Z'_s, \mathbb{E}_n U'_s \chi_{B_n(s)} + \mathbb{E}_n U'_s \chi_{\overline{B}_n(s)}) \right) \\
+ (\Delta^n Y_s)_+ \mathbb{E}_n \left( f(s, Y_s, \mathbb{E}_n Z_s, \mathbb{E}_n U'_s \chi_{B_n(s)} + \mathbb{E}_n U'_s \chi_{\overline{B}_n(s)}) - f(s, Y'_s, \mathbb{E}_n Z'_s, \mathbb{E}_n U'_s \chi_{B_n(s)} + \mathbb{E}_n U'_s \chi_{\overline{B}_n(s)}) \right) \\
\leq (\Delta^n Y_s)_+ \mathbb{E}_n \left( f(s, Y_s, \mathbb{E}_n Z_s, \mathbb{E}_n U'_s \chi_{B_n(s)} + \mathbb{E}_n U'_s \chi_{\overline{B}_n(s)}) \\
- f(s, Y'_s, \mathbb{E}_n Z'_s, \mathbb{E}_n U'_s \chi_{B_n(s)} + \mathbb{E}_n U'_s \chi_{\overline{B}_n(s)}) \right) \\
- \int_{B_n(s)} (\Delta^s Y_s)_{+} \Delta^s U_s \nu(dx). \quad (28)
\]

Next we apply Jensen’s inequality in two dimensions for the product of positive random variables and
also \([A_{3} \geq 2]\) and Young’s inequality to arrive at
\[
(\Delta^n Y_{s})_{+} \mathbb{E}_{n} \left( f(s, Y_{s}, E_{n} Z_{s}, E_{n} U_{s} \chi_{B_{n}(s)} + E_{n} U_{s} \chi_{B_{n}'(s)}) \right.

- f(s, Y'_{s}, E_{n} Z'_{s}, E_{n} U'_{s} \chi_{B_{n}(s)} + E_{n} U_{s} \chi_{B_{n}'(s)})

\leq \mathbb{E}_{n} \left[ \Delta Y_{s} \left( f(s, Y_{s}, E_{n} Z_{s}, E_{n} U_{s} \chi_{B_{n}(s)} + E_{n} U_{s} \chi_{B_{n}'(s)}) \right. \right.

- f(s, Y'_{s}, E_{n} Z'_{s}, E_{n} U'_{s} \chi_{B_{n}(s)} + E_{n} U_{s} \chi_{B_{n}'(s)})

\leq \mathbb{E}_{n} \alpha(s) \rho((\Delta Y_{s})^2_{+}) + 2\mathbb{E}_{n} \beta(s)^2(\Delta Y_{s})^2_{+} + \frac{|\Delta^n Z_{s}|^2}{4} + \mathbb{E}_{n} \|\Delta^n U_{s} \chi_{B_{n}(s)}\|^2.
\] (29)

Taking together inequalities (26), (27), (28) and (29), we get with Young’s inequality again
\[
(\Delta^n Y_{s})_{+} \mathbb{E}_{n} \left( f(s, Y_{s}, Z_{s}, U_{s} - f'(s, Y'_{s}, Z'_{s}, U'_{s}) \right)

\leq \mathbb{E}_{n} \alpha(s) \rho((\Delta Y_{s})^2_{+}) + 2\mathbb{E}_{n} \beta(s)^2(\Delta Y_{s})^2_{+} + \frac{|\Delta^n Z_{s}|^2}{4} + \mathbb{E}_{n} \|\Delta^n U_{s} \chi_{B_{n}(s)}\|^2

- \int_{B_{n}'(s)} (\Delta^n Y_{s})_{+} \Delta^n U_{s} \nu(dx)

+ 4\beta(s)^2(\Delta^n Y_{s})^2_{+} + \frac{1}{4} (|Z_{s} - E_{n} Z_{s}|^2 + |Z'_{s} - E_{n} Z'_{s}|^2 + \|U_{s} - E_{n} U_{s}\|^2 + \|U'_{s} - E_{n} U'_{s}\|^2).
\]

Therefore, (25) evolves to
\[
\mathbb{E} \int_{0}^{T} \eta(s) ds (\Delta^n Y_{s})^2_{+}

\leq \mathbb{E} \int_{0}^{T} e^{\int_{0}^{s} \eta(t) dt} \chi_{\{\Delta^n Y_{s} > 0\}} \left[ 2\mathbb{E}_{n} \alpha(s) \rho((\Delta Y_{s})^2_{+}) + 4\mathbb{E}_{n} \beta(s)^2(\Delta Y_{s})^2_{+} + \frac{|\Delta^n Z_{s}|^2}{2} 

+ \frac{\|\Delta^n U_{s} \chi_{B_{n}(s)}\|^2}{2} - \int_{B_{n}'(s)} 2(\Delta^n Y_{s})_{+} \Delta^n U_{s}(x) \nu(dx) + 8\beta(s)^2 (\Delta^n Y_{s})^2_{+}

+ \frac{1}{2} (|Z_{s} - E_{n} Z_{s}|^2 + |Z'_{s} - E_{n} Z'_{s}|^2 + \|U_{s} - E_{n} U_{s}\|^2 + \|U'_{s} - E_{n} U'_{s}\|^2) - |\Delta^n Z_{s}|^2

- \eta(s)|\Delta^n Y_{s}|^2 - \int_{B_{n}(s)} |\Delta^n U_{s}(x)|^2 \nu(dx) + \int_{B_{n}'(s)} ((\Delta^n Y_{s})^2_{+} + 2(\Delta^n Y_{s})_{+} (\Delta^n U_{s}(x))) \nu(dx) \right] ds.
\]

We cancel out terms and end this step with the estimate
\[
\mathbb{E} \int_{0}^{T} e^{\int_{0}^{s} \eta(t) dt} \chi_{\{\Delta^n Y_{s} > 0\}} \alpha(s) \rho((\Delta Y_{s})^2_{+}) ds =: \delta_{\rho}
\]
and
\[
\mathbb{E} \int_0^T e^{\int_0^t \eta(\tau) \, d\tau} \chi_{\{\Delta^n Y_s > 0\}} \beta(s)^2 (\Delta Y_s)^2_+ \, ds =: \delta_y.
\]
are positive numbers. All other cases would simplify the proof. Since \(\mathbb{E}_n Y_s \to Y_s\) a.s. for all \(s\), dominated convergence shows that also
\[
\mathbb{E} \int_0^T e^{\int_0^t \eta(\tau) \, d\tau} \left| \chi_{\{\Delta^n Y_s > 0\}} \alpha(s) \rho((\Delta Y_s)^2_+) - \chi_{\{\Delta^n Y_s > 0\}} \alpha(s) \rho((\Delta^n Y_s)^2_+) \right| \, ds,
\]
and
\[
\mathbb{E} \int_0^T e^{\int_0^t \eta(\tau) \, d\tau} \left| \chi_{\{\Delta Y_s > 0\}} \alpha(s) \rho((\Delta Y_s)^2_+) - \chi_{\{\Delta Y_s > 0\}} \alpha(s) \rho((\Delta Y_s)^2_+) \right| \, ds
\]
converge to zero. For domination we use
\[
\mathbb{E} \int_0^T e^{\int_0^t \eta(\tau) \, d\tau} \alpha(s) \left( \rho((\Delta Y_s)^2_+) + \rho \left( \sup_{n \geq 0} \mathbb{E}_n (\Delta Y_s)^2_+ \right) \right) \, ds
\leq e^{C \|\alpha\|_{L^1([0,T])}} (1 + b \|Y\|_{S^2})^2,
\]
and
\[
\mathbb{E} \int_0^T e^{\int_0^t \eta(\tau) \, d\tau} \left| \chi_{\{\Delta^n Y_s > 0\}} \alpha(s) \rho((\Delta Y_s)^2_+) - \chi_{\{\Delta Y_s > 0\}} \alpha(s) \rho((\Delta Y_s)^2_+) \right| \, ds < \delta_y - \frac{1}{m},
\]
where we applied that \(\int_0^T \eta(s) \, ds < C\) a.s., Doob’s martingale inequality and that there is \(b > 0\) such that for \(x \geq 0: \rho(x) \leq 1 + bx\).

For \(m \geq 0\) with \(\delta_y - \frac{1}{m} > 0\) let us now choose \(N_m \in \mathbb{N}\) large enough, such that for \(n \geq N_m\):
\[
\mathbb{E} \int_0^T e^{\int_0^t \eta(\tau) \, d\tau} \left| \chi_{\{\Delta^n Y_s > 0\}} \alpha(s) \rho((\Delta Y_s)^2_+) - \chi_{\{\Delta^n Y_s > 0\}} \alpha(s) \rho((\Delta Y_s)^2_+) \right| \, ds < \delta_y - \frac{1}{m},
\]
and
\[
\mathbb{E} \int_0^T e^{\int_0^t \eta(\tau) \, d\tau} \chi_{\{\Delta^n Y_s > 0\}} \alpha(s) \rho((\Delta Y_s)^2_+) \, ds \geq \delta_y - \frac{1}{m}.
\]
For such an \(n\) we get that
\[
\mathbb{E} \int_0^T e^{\int_0^t \eta(\tau) \, d\tau} \chi_{\{\Delta^n Y_s > 0\}} \alpha(s) \rho((\Delta Y_s)^2_+) \, ds
\leq 2 \mathbb{E} \int_0^T e^{\int_0^t \eta(\tau) \, d\tau} \chi_{\{\Delta^n Y_s > 0\}} \alpha(s) \rho((\Delta^n Y_s)^2_+) \, ds.
\]
In the same way, one can choose \(m, N_m \in \mathbb{N}\) also large enough such that for all \(n \geq N_m\),
\[
\mathbb{E} \int_0^T e^{\int_0^t \eta(\tau) \, d\tau} \chi_{\{\Delta^n Y_s > 0\}} \beta(s)^2 (\Delta Y_s)^2_+ \, ds
\leq 2 \mathbb{E} \int_0^T e^{\int_0^t \eta(\tau) \, d\tau} \chi_{\{\Delta^n Y_s > 0\}} \beta(s)^2 (\Delta^n Y_s)^2_+ \, ds.
\]
(31)
Similarly, by martingale convergence $\mathbb{E}_n Z_s \to Z_s$ and a domination argument, we can conclude that for $n \geq N_m$ ($N_m$ may have to be rechosen large enough),

\[
\mathbb{E} \int_0^T e^{\int_0^t \eta(r)dr} \chi_{\{\Delta^n Y_s > 0\}} |Z_s - \mathbb{E}_n Z_s|^2 ds \leq \mathbb{E} \int_0^T e^{\int_0^t \eta(r)dr} \chi_{\{\Delta^n Y_s > 0\}} \beta(s)^2 |\Delta^n Y_s|^2 ds
\]

and

\[
\mathbb{E} \int_0^T e^{\int_0^t \eta(r)dr} \chi_{\{\Delta^n Y_s > 0\}} ||U_s - \mathbb{E}_n U_s||^2 ds \leq \mathbb{E} \int_0^T e^{\int_0^t \eta(r)dr} \chi_{\{\Delta^n Y_s > 0\}} \beta(s)^2 |\Delta^n Y_s|^2 ds, \tag{32}
\]

since the left hand sides tend to zero, while the right hand sides converge to $\delta_y$. The same estimates hold for $Z'$ and $U'$ as well.

Hence, applying (31) and (32) to (30) yields

\[
\mathbb{E} e^{\int_0^t \eta(s)ds} (\Delta^n Y_t)_+^2 \leq \mathbb{E} \int_t^T e^{\int_0^s \eta(r)dr} \chi_{\{\Delta^n Y_s > 0\}} \left[4\alpha(s) \rho((\Delta^n Y_s)_+^2) + 18\beta(s)^2 (\Delta^n Y_s)_+^2 - \eta(s) |\Delta^n Y_s|^2 + \int_0^t (\Delta^n Y_s)_+^2 \nu(dx) \right] ds. \tag{33}
\]

**Step 4:**

Bounding $\int_{B^n_t} (\Delta^n Y_s)_+^2 \nu(dx)$ by $\nu(\{1/n \leq |x|\} (\Delta^n Y_s)_+^2$ in (33), leads us to

\[
\mathbb{E} e^{\int_0^t \eta(s)ds} (\Delta^n Y_t)_+^2 \leq \mathbb{E} \int_t^T e^{\int_0^s \eta(r)dr} \chi_{\{\Delta^n Y_s > 0\}} \left[4\alpha(s) \rho((\Delta^n Y_s)_+^2) + (\nu(\{1/n \leq |x|\}) + 18\beta(s)^2 (\Delta^n Y_s)_+^2 - \eta(s) |\Delta^n Y_s|^2 \right] ds.
\]

It remains, recalling that $\eta = 18\beta^2$,

\[
\mathbb{E} e^{\int_0^t \eta(s)ds} (\Delta^n Y_t)_+^2 \leq \mathbb{E} \int_t^T e^{\int_0^s \eta(r)dr} \left[4\alpha(s) \rho((\Delta^n Y_s)_+^2) + \nu(\{1/n \leq |x|\} \right] ds
\]

\[
\leq \mathbb{E} \int_t^T e^{\int_0^s \eta(r)dr} \left[4\alpha(s) \rho + \nu(\{1/n \leq |x|\} \right] ds.
\]

The term $e^{\int_0^T \eta(r)dr}$ is $\mathbb{P}$-a.s. bounded by a constant $C > 0$. Thus, by the concavity of $\rho_n := \rho + \nu(\{1/n \leq |x|\} \) id, which satisfies the same assumptions as $\rho$, we arrive at

\[
\mathbb{E} (\Delta^n Y_t)_+^2 \leq \mathbb{E} e^{\int_0^t \eta(s)ds} (\Delta^n Y_t)_+^2 \leq \int_t^T 4C (\alpha(s) \rho + \nu(\{1/n \leq |x|\} \) id) (\Delta^n Y_s)_+^2 ds.
\]

Then, the Bihari-LaSalle inequality (Theorem A.2) shows that $\mathbb{E} (\Delta^n Y_t)_+^2 = 0$ for all $t \in [0, T]$.

**Step 5:**

Steps 1-4 granted that $\mathbb{E}_n Y \leq \mathbb{E}_n Y'$ for $n$ greater than a certain value. The convergence of the sequences to the solutions $Y$ and $Y'$ of $(\xi, f)$ and $(\xi', f')$, respectively in $L^2(W)$ shows $Y \leq Y'$, and the theorem is proven. \[\square\]
In the following [Theorem 6.2] we state a version of the above theorem for the case $1 < p < 2$. The difference to [Theorem 6.1] is that here we cannot compare the generators on the solution only. If one wants to keep the comparison of the generators on the solution but accepts a slightly stronger condition than $\text{(A3)}$ given as $(\text{H}_{\text{comp}})$ in [14],

\[(A\gamma') \quad f(s, y, z, u) - f(s, y, z, u') \leq \int_{\mathbb{R}_0}(u(x) - u'(x))\gamma_t(x)\nu(dx), \quad \mathbb{P} \otimes \lambda\text{-a.e.}
\]

for a predictable process $\gamma = \gamma^{y, z, u, u'}$, such that $-1 \leq \gamma_t(x)$ and $|\gamma_t(u)| \leq \vartheta(u)$, where $\vartheta \in L^2(\nu)$, then the proof of [14, Proposition 4] can also be conducted for generators satisfying the conditions $(A3_{\geq 2})$ or $(A3_{< 2})$.

**Theorem 6.2** (Comparison, $p > 1$). Let $p, p' > 1$ and $(Y, Z, U)$ be the $L^p$-solution to $(\xi, f)$ and $(Y', Z', U')$ be the $L^{p'}$-solution to $(\xi', f')$. Furthermore let $f$ and $f'$ satisfy $(A1)$ and $(A3_{\geq 2})$ or $(A3_{< 2})$ for the according $p, p'$. If the following assumptions hold

(i) $\xi \leq \xi', \mathbb{P}$-a.s.,

(ii) $f(s, y, z, u) \leq f'(s, y, z, u)$, for all $(y, z, u) \in \mathbb{R} \times \mathbb{R} \times L^2(\nu)$, for $\mathbb{P} \otimes \lambda$-a.a. $(\omega, s) \in \Omega \times [0, T]$ and

$(A\gamma)$ for all $u, u' \in L^2(\nu)$ with $u' \geq u$

\[f(s, y, z, u) - f(s, y, z, u') \leq \int_{\mathbb{R}_0}(u'(x) - u(x))\nu(dx), \quad \mathbb{P} \otimes \lambda\text{-a.e., (34)}\]

then for all $t \in [0, T]$, we have $\mathbb{P}$-a.s.,

$Y_t \leq Y'_t$.

The same assertion follows from an equivalent formulation for $f'$, requiring that $(34)$ hold for $f'$.

**Proof.** We approximate the generators and terminal conditions by $\xi_n := \frac{n}{n \geq 0} \xi$, $\xi'_n := \frac{n}{n \geq 0} \xi'$ and

\[f_n(t, y, z, u) := \frac{n}{|f(t, y, z, u)|} f(t, y, z, u), \quad f'_n(t, y, z, u) := \frac{n}{|f'(t, y, z, u)|} f'(t, y, z, u).
\]

This procedure preserves order relations. Furthermore, the generators $f_n, f'_n$ satisfy $(A1)$ $(A3_{\geq 2})$ or $(A3_{< 2})$ with respect to their mutual coefficients. Also $(34)$ remains satisfied for $f_n$.

Thus, the solutions $Y_n$ and $Y'_n$ of all these equations satisfy $Y_n \leq Y'_n$ since (by the boundedness of $\xi_n, \xi'_n, f_n, f'_n$ and since [Remark 3.1] (b) implies $(A3_{\geq 2})$) for $p = 2$ for the $f_n, f'_n$) they are also $L^2$-solutions. Convergence of those solutions to $Y$ in $S^p$ and $Y'$ in $S^{p'}$ follows from [Proposition 4.5] and since $\mathbb{E} \left( \int_0^T |f(Y_s, Z_s, U_s)ds \right)^p < \infty$, which can easily be derived since

$$\mathbb{E} \left( \int_0^T f(Y_s, Z_s, U_s)ds \right)^p < \infty \quad \text{and} \quad \mathbb{E} \left( \int_0^T -f(Y_s, Z_s, U_s)ds \right)^p < \infty.$$ 

Then, for all $t$,

$Y_{n,t} \to Y_t, \quad Y'_{n,t} \to Y'_t$

in probability, $Y_t \leq Y'_t, \mathbb{P}$-a.s. follows. \qed
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A Appendix

A.1 Inequalities

Theorem A.1 (Young’s inequality). For $R > 0$ and $a, b \in \mathbb{R}$, we can bound the product

$$ab = aR^{\frac{1}{p}}R^{\frac{1}{q}}b \leq \frac{a^p}{pR} + \frac{b^q}{qR}$$

with $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

The Bihari-LaSalle inequality. For the Bihari-LaSalle inequality we refer to [17, pp. 45-46]. Here we formulate a backward version of it which has been applied in [29]. The proof is analogous to that in [17].

Theorem A.2. Let $c > 0$. Assume that $\rho : [0, \infty] \rightarrow [0, \infty]$ is a continuous and nondecreasing function such that $\rho(x) > 0$ for all $x > 0$. Let $K$ be a nonnegative, integrable Borel function on $[0, T]$, and $y$ a nonnegative, bounded Borel function on $[0, T]$, such that

$$y(t) \leq c + \int_{t}^{T} K(s)\rho(y(s))ds.$$ 

Then for $z(t) := c + \int_{t}^{T} K(s)\rho(y(s))ds$ it holds that

$$y(t) \leq G^{-1}\left(G(c) + \int_{t}^{T} K(s)ds\right)$$

for all $t \in [0, T]$ such that $G(c) + \int_{t}^{T} K(s)ds \in \text{dom}(G^{-1})$. Here

$$G(x) := \int_{1}^{x} \frac{dr}{\rho(r)},$$

and $G^{-1}$ is the inverse function of $G$.

Furthermore, if for an $\epsilon > 0$, $\int_{0}^{\epsilon} \frac{dr}{\rho(r)} = \infty$, then

$$y(t) = 0$$

for all $t \in [0, T]$.

Remark A.3. As a special case we have Gronwall’s inequality: If $\rho(r) = r$ for $r \in [0, \infty]$, we get

$$y(t) \leq ce^{\int_{t}^{T} K(s)ds}.$$
A.2 Construction of progressively measurable versions

We will use the next Lemma for the construction of progressively measurable versions of conditional expectations necessary for the proof of Theorem 6.1. To prepare it, we need the following definitions:

Let $D[0, T]$ denote the space of càdlàg functions on $[0, T]$ endowed with the σ-algebra generated by the projection maps $p_t: D[0, T] \to \mathbb{R}, x \mapsto x_t$. The measure we consider on this sigma field is the pushforward measure $\mathbb{P}_X$ of the Lévy process $X$, given by the trajectory mapping $X: \Omega \to D[0, T], \omega \mapsto X(\omega)$.

Note, that for each $t \in [0, T]$ the map, that ’stops’ a trajectory at time $t$,

$$(\cdot)^t: D[0, T] \mapsto D[0, T], x \mapsto x^t = (s \mapsto x_s \chi_{[0,t]}(s) + x_t \chi_{[t,T]}(s))$$

is measurable.

Recalling that $\mathcal{P}$ denotes the predictable σ-algebra according to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$, we have the following result:

**Lemma A.4.** Let $(V, \mathcal{V}, \mu)$ be a measure space and $K: \Omega \times [0, T] \times V \to \mathbb{R}$ be a $\mathcal{P} \otimes \mathcal{V}$-measurable process such that for $\mu$-almost all $x \in V$, 

$$\mathbb{E} \int_0^T |K(s, x)| ds < \infty.$$ 

Moreover, let $K$ be represented by a measurable functional $F^K: D[0, T] \times [0, T] \times V \to \mathbb{R}$, such that $\mathbb{P}$-a.s.

$$K(\cdot, \cdot) = F^K(X, \cdot, \cdot)$$

(for details on this representation we refer to [26]). Let $X^n$ be the ’cut-off’ Lévy-process from Subsection 2.2.

We assert that the process $(G(t, x))_{(t,x) \in [0,T] \times V}$, given by

$$G(t, x) = \begin{cases} \mathbb{E} F^K(h + X - X^n, t, x)|_{h=X^n}, & \text{whenever the expectation exists and is finite}, \\ 0, & \text{else}, \end{cases}$$

is $\mathcal{P} \otimes \mathcal{V}$-measurable, and

$$G(\omega, t, x) = \mathbb{E}_n K(\omega, t, x), \text{P-a.s. for } \lambda \otimes \mu-a.a. \ (t, x) \in [0, T] \times V.$$ 

**Proof.** Since $F^K$ is measurable, by concatenation the mapping

$$D[0, T] \times D[0, T] \times [0, T] \times V \to \mathbb{R}, (h, k, t, x) \mapsto F^K(h + k, t, x)$$

is measurable too. Since $X^n$ and $X - X^n$ are independent, we get, denoting the pushforward measures of those processes by $\mathbb{P}_{X^n}, \mathbb{P}_{X - X^n}$, that

$$\int_0^T \int_{D[0, T]} \mathbb{E} F^K(h + X - X^n, t, x) d\mathbb{P}_{X^n}(h) dt = \int_0^T \int_{D[0, T]} \mathbb{E} F^K(h + k, t, x) d\mathbb{P}_{X - X^n}(k) d\mathbb{P}_{X^n}(h) dt = \int_0^T \mathbb{E} F^K(X, t, x) dt < \infty, \ \mu-a.e.$$
since $F^K(X, t, x)$ equals $K(t, x)$ $\mathbb{P}$-a.s. By Fubini's theorem, the map
\[
(h, t, x) \mapsto \left\{ \begin{array}{ll}
\mathbb{E}F^K(h + X - X^n, t, x) |_{h = X^n}, & \text{whenever the expectation exists and is finite,} \\
0, & \text{else},
\end{array} \right.
\]
is measurable, and the first case applies for $\mathbb{P}_{X^n} \otimes \lambda \otimes \mu$-almost all $(h, t, x) \in D[0, T] \times [0, T] \times V$. The predictability of the process $G$ can be seen as follows: For a simple predictable process $\tilde{K}$ of the form $\tilde{K} = \sum_{k=1}^{N} \alpha_k((X)^{t_{k-1}}, \cdot \chi[t_{k-1}, t_k])$, where $N \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots < t_N = T$, and $\alpha_k: D[0, T] \times V \to \mathbb{R}$, measurable, with $\int_{D[0, T]} \alpha_k(h, x) d\mathbb{P}(h) < \infty$, $\mu$-a.e., the process $\tilde{G}$, given by
\[
\tilde{G}(t, x) = \left\{ \begin{array}{ll}
\mathbb{E} \sum_{k=1}^{N} \alpha_k(h^{t_{k-1}} + (X - X^n)^{t_{k-1}}, \cdot \chi[t_{k-1}, t_k]) |_{h = X^n}, & \text{whenever the expectation exists and is finite,} \\
0, & \text{else},
\end{array} \right.
\]
is again a simple, predictable process. As the process $K$ can be written as (pointwise) limit of simple predictable processes $(\tilde{K}_k)_{k \geq 1}$ of the form $\tilde{K}$ above, and the integrability assumptions of $K$ admit the use of the dominated convergence theorem, the limit of the respective simple predictable processes $(\tilde{G}_k)_{k \geq 1}$, defined by the procedure (35), is $G$, which then is also measurable w.r.t. $\mathcal{P} \otimes \mathcal{V}$.

It is left to show that $\lambda \otimes \mu$-a.e. $G$ equals $\mathbb{E}_n K$ a.s. To that end, let $A \in \mathcal{F}^n$ and take $(t, x)$ such that $\mathbb{E}F^K(X, t, x)$ exists and is finite (that are $\lambda \otimes \mu$-almost all). Integration over $A$ yields
\[
\int_A K(t, x) d\mathbb{P} = \int_A F^K(X, t, x) d\mathbb{P} = \mathbb{E}_{\chi_A} F^K(X, t, x) = \mathbb{E}_{\chi_A} F^K(X^n + X - X^n, t, x).
\]
Using the independence of $X^n$ and $X - X^n$, and representing the function $\chi_A$ by a functional $\tilde{\chi}_A: D[0, T] \to \mathbb{R}$, such that $\chi_A = \tilde{\chi}_A(X^n)$, $\mathbb{P}$-a.s., we continue with
\[
\mathbb{E}_{\chi_A} F^K(X^n + X - X^n, t, x) = \int_{D[0, T]} \int_{D[0, T]} \tilde{\chi}_A(h) F^K(h + k, t, x) d\mathbb{P}X - X^n(k) d\mathbb{P}X^n(h)
\]
\[
= \int_{D[0, T]} \tilde{\chi}_A(h) \int_{D[0, T]} F^K(h + k, t, x) d\mathbb{P}X - X^n(k) d\mathbb{P}X^n(h)
\]
\[
= \mathbb{E} \left[ \chi_A \int_{D[0, T]} F^K(h + k, t, x) d\mathbb{P}X - X^n(k) \bigg|_{h = X^n} \right] = \int_A \left[ \mathbb{E} F^K(h + X - X^n, t, x) \bigg|_{h = X^n} \right] d\mathbb{P}
\]
\[
= \int_A G(t, x) d\mathbb{P}.
\]
This means that for $\lambda \otimes \mu$-a.e. $(t, x) \in [0, T] \times V$, we end up with
\[
\int_A K(t, x) d\mathbb{P} = \int_A G(t, x) d\mathbb{P},
\]
which shows that $\mathbb{E}_n K(t, x) = G(t, x)$, $\mathbb{P}$-a.s., proving the assertion. \hfill \Box

**Remark A.5.** A similar proof as above can be done if one considers progressively measurable sets instead of predictable ones. If the process $K$ does not depend on an additional parameter set $V$ and is such that $\mathbb{E} |K(t)| < \infty$, for all $t$, then again very similar arguments as in the above Lemma can be used to find that there is a progressively measurable process $G$ with $\mathbb{E}_n K(t) = G(t)$ for all $t \in [0, T]$ (this case is the one needed for a progressively measurable version of $(\mathbb{E}_n Y_t)_{t \in [0, T]}$ in the proof of Theorem 6.7).
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