Polyakov Loops in 2D QCD

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Abstract

We discuss SU(N) gluodynamics at finite temperature and on a spatial circle. We show that the effective action for the Polyakov Loop operator is a one dimensional gauged SU(N) principle chiral model with variables in the loop space and loop algebra of the gauge group. We find that the quantum states can be characterized by a discrete \( \theta \)-angle which appears with a particular 1-dimensional topological term in the effective action. We present an explicit computation of the partition function and obtain the spectrum of the model, together with its dependence on the discrete theta angle. We also present explicit formulae for 2-point correlators of Polyakov loop operators and an algorithm for computing all N-point correlators.

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Yang-Mills theory in 1+1-dimensions is a prototypical example of a topological field theory. The classical theory is trivial and the quantum theory has no propagating degrees of freedom. It nevertheless has interesting features, particularly in the large $N$ limit or when it is defined on a multiply connected spacetime \cite{1, 2} where its partition function depends on the area as well as topological invariants. On a sphere it exhibits a large $N$ phase transition for some critical value of $eA$ where $e$ is the coupling and $A$ is the area of the sphere \cite{3}. Wilson loops can be computed in both the strong and weak coupling phases and exhibit interesting behaviors \cite{4, 5}. It has been shown that the strong coupling expansion can be rewritten as a lower-dimensional string theory \cite{6}. Furthermore, when defined on Riemann surfaces with non-zero genus it is known to have degrees of freedom related to the gauge group holonomy on the homology cycles of the surface \cite{1, 2}. In particular, on cylindrical spacetime, it can be solved explicitly and has quantum mechanical degrees of freedom corresponding to the eigenvalues of the Wilson loop operators which wind around the compact spatial direction \cite{7} - \cite{10}. In that case, topology also affects the gauge fixing, \cite{11}. Gauge invariant two-dimensional models are also intimately connected with one-dimensional integrable quantum systems \cite{12, 13}.

In this paper we shall examine Yang-Mills theory on a torus. It corresponds to finite temperature 1+1-dimensional gluodynamics on a spatial circle. We show that, in any 1+1-dimensional gauge theory, correlators of Polyakov loop operators, which are the Wilson loop operators that wind around the periodic time, can be computed as correlators in a particular 1-dimensional gauged principle chiral model. In either pure Yang-Mills theory, or in Yang-Mills theory coupled to matter which is in the adjoint representation of the gauge group, the effective field theory has a global symmetry which transforms the Polyakov loop operator by elements of the center of the gauge group. For the case of pure Yang-Mills theory, we find the effective field theory explicitly. We also find a solution of the effective theory and use it to give an explicit computation of the partition function and the two-point correlators of Polyakov loop operators as well as an algorithm for computing all correlators of Polyakov loop operators.

A few years ago, Polyakov \cite{14} and Susskind \cite{15} suggested that the question of confinement in quantum gluodynamics at finite temperature is intimately related to the realization of a global symmetry involving the center of the gauge group. This symmetry governs the expectation value of the
Polyakov loop operator

\[ P(x) = \text{tr} \mathcal{P} \exp \left( i \int_0^{1/T} d\tau A_0(x, \tau) \right) \]  

(1)

which measures the gauge group holonomy in the periodic Matsubara time, 

\[ < P(x_1) \ldots P(x_m) P^\dagger(y_1) \ldots P^\dagger(y_n) > = \]

\[ = \frac{\int dA_\mu e^{-\int_0^{1/T} d\tau d\sigma \mathcal{F}_{\mu \nu}(\tau, \sigma)/2} P(x_1) \ldots P(x_m) P^\dagger(y_1) \ldots P^\dagger(y_n)}{\int dA_\mu e^{-\int_0^{1/T} d\tau d\sigma \mathcal{F}_{\mu \nu}(\tau, \sigma)/2}} \]  

(2)

(The Euclidean path integral has periodic boundary conditions, \( A_\mu(1/T, \vec{x}) = A_\mu(0, \vec{x}) \) and \( T \) is the temperature. For a discussion of the path integral formulation of finite-temperature gauge theory, see [16].) The action and measure in the path integral (2) are symmetric under gauge transformations

\[ A_\mu(\tau, x) \rightarrow g^\dagger(\tau, x) A_\mu(\tau, x) g(\tau, x) + i g^\dagger(\tau, x) \nabla_\mu g(\tau, x) \]  

(3)

where \( g(\tau, x) \) is a gauge group element (here we consider SU(N)) which is periodic up to an element of the center of the gauge group, \( Z(N) \),

\[ g(1/T, x) = g(0, x)e^{2\pi in/N} \]  

(4)

The path-ordered exponential transforms as

\[ \text{tr} \mathcal{P} e^{i \int_0^{1/T} d\tau A_0(x, \tau)} \rightarrow \text{tr} g^\dagger(1/T, x) \mathcal{P} e^{i \int_0^{1/T} d\tau A_0(x, \tau)} g(0, x) = e^{2\pi in/N} \text{tr} \mathcal{P} e^{i \int_0^{1/T} d\tau A_0(x, \tau)} \]  

(5)

The expectation value (2) is interpreted as the free energy of the system in the presence of an array of external fundamental representation quark sources at positions \( x_1, \ldots, x_m \) and anti-quarks at \( y_1, \ldots, y_n \). The realization of the \( Z(N) \) symmetry is related to confinement - it is represented faithfully in a confined phase and it is spontaneously broken in a de-confined phase. In 3+1-dimensional chromodynamics a first order phase transition between confining and deconfining phases is expected at some critical temperature. The domain walls and \( Z(N) \) bubbles between different \( Z_N \) vacua that should appear in the de-confined, broken \( Z(N) \) phase, have been extensively investigated [17, 18].

On the other hand, 1+1-dimensional gluodynamics is confining and the global \( Z(N) \) symmetry is unbroken at all temperatures. There, the correlators of Polyakov loop operators are known to be related to the correlators in a generalized Sutherland model [12].
The Hamiltonian of 1+1-dimensional gluo-dynamics is

\[ H = \frac{e^2}{2} \int_0^L dx \sum_a E^a(x)^2 \]  

(6)

with nonvanishing commutator

\[ [A^a(x), E^b(y)] = i \delta^{ab} \delta(x-y) \]  

(7)

gauge constraint

\[ G^a(x) \equiv \nabla E^a(x) + f^{abc} A^b(x) E^c(x) \sim 0 \]  

(8)

and periodic boundary conditions, \( E^a(L) = E^a(0), A^a(L) = A^a(0) \). In Eqs. (6,7,8) \( a, b = 1 \ldots N^2 - 1 \), the traceless hermitean generators \( T^a \) are normalized so that \( \text{tr} T^a T^b = \frac{1}{2} \delta^{ab} \), the structure constants are defined by \([T^a, T^b] = i f^{abc} T^c\) and we define the \( N \times N \) matrices \( E(x) \equiv E^a(x) T^a, A(x) \equiv A^a(x) T^a \).

Since \( A(x) \) and \( E(x) \) transform under the adjoint action of the gauge group, their periodic boundary conditions are preserved by time independent gauge transformations which are periodic up to an element of the center of the group, \( g_n(L) = g_n(0) e^{2\pi i n/N} \). \( G_\chi = \int_0^L dx G^a(x) \chi^a(x) \) commutes with the Hamiltonian and generates time independent, periodic gauge transforms

\[ e^{iG_\chi} E(x) e^{-iG_\chi} = g_\chi(x) E(x) g_\chi^\dagger(0) \]  

(9)

\[ e^{iG_\chi} A(x) e^{-iG_\chi} = g_\chi(x) A(x) g_\chi^\dagger(0) + i g_\chi^\dagger(0) \nabla g_\chi(0) \]  

(10)

where \( g_\chi(0) = 1 - i \chi(0) + \ldots \) and where \( g_\chi(0) = g_\chi(L) \). The constraint \( G_\chi \) indicates that the physical states are invariant under all strictly periodic gauge transforms. The coset group of all gauge transformations modulo periodic ones is isomorphic to the center of the gauge group, \( Z_N \). The physical states must transform under an irreducible unitary representation of the coset. Representations of \( Z_N \) are one dimensional and are characterized by a discrete angle.\footnote{The number of connected components of the group of periodic gauge transformations, which is the loop group based on \( SU(N)/Z_N \), is

\[ \Pi_0(\text{loops on } SU(N)/Z_N) = \Pi_1(SU(N)/Z_N) = \Pi_0(Z_N) = Z_N \]  

(11)

The group \( Z_N \) has \( N \) 1-dimensional irreducible representations, labelled by an integer \( m = 0, 1, \ldots, N-1 \) and where the elements are \( U(n; \theta_m) = e^{i\theta_m n} \) and \( \theta_m = 2\pi m/N \).}
This theta-angle will also characterize the quantization of Yang-Mills theory with coupling to any adjoint representation matter fields. Some related issues have been pointed out in [19]. Also, for adjoint QCD on a hyper-torus in d spatial dimensions, there are d angles related to periodicity of gauge transformations as well as perhaps others related to higher dimensional winding numbers. In either case, if one introduces matter fields which are in the fundamental representation, their boundary conditions reduce the gauge symmetry to gauge transformations with a fixed periodicity, i.e. one of the degenerate states, and there is no theta-angle.

In the Schrödinger picture, where states are wavefunctionals $\Psi[A]$, and the electric field is realized as $E^a(x) \equiv \frac{1}{i} \delta A^a(x)$, the physical state condition is

$$\left(\frac{1}{i} \nabla \frac{\delta}{\delta A^a(x)} + f^{abc} A^b(x) \frac{\delta}{\delta A^c(x)}\right) \Psi_{\text{phys}}[A] = 0$$

(12)

The wavefunctionals of physical states therefore transform as

$$\Psi_{\text{phys}}[g^\dagger_n A g_n + i g^\dagger_n \nabla g_n; \theta] = e^{i\theta} \Psi[A; \theta]$$

(13)

The thermodynamic partition function is obtained by taking a trace of the Boltzmann factor, $e^{-H/T}$, over the physical states. This is most conveniently expressed as a trace using eigenstates of $A(x)$ and a projection onto gauge invariant theta-states,

$$Z = \frac{1}{\text{VOL} G} \sum_n \int [dg_n] e^{-i\theta n} \int_x dA^a(x) \langle A|e^{-H/T}|g^\dagger_n A g_n + i g^\dagger_n \nabla g_n \rangle$$

(14)

The integration over gauge transforms projects onto physical states with angle $\theta$. This integral can be expressed in terms of the standard Euclidean path integral (see [17]) which is used to take expectation values of observables as in (2) by first considering the integral with twisted boundary conditions,

$$Z = \frac{1}{\text{VOL} G} \sum_n \int [dg_n] e^{-i\theta n} \prod_{x,\tau} dA^a(x, \tau) e^{-\int_0^{1/T} d\tau dx \frac{1}{2} \text{tr} \dot{A}^2(x, \tau)}$$

(15)

$$A(1/T, x) = g^\dagger_n(x) A(0, x) g_n(x) + i g^\dagger_n(x) \nabla g_n(x)$$

(16)

The boundary conditions can be made periodic by performing a non-periodic gauge transformation.

$$A(\tau, x) \rightarrow g^\dagger_n(\tau, x) A(\tau, x) g_n(\tau, x) + i g^\dagger_n(\tau, x) \nabla g_n(\tau, x)$$

(17)
where \( g_n(1/T, x) = g_n(0, x)g_n(x) \). The result is the standard form of the path integral in (2). The loop operator is identified as

\[
P e^{i \int_0^{1/T} d\tau A_0(\tau, x)} \equiv g_n(x)
\]

and the trace of the Polyakov loop operator is equal to the trace of the gauge group valued variable in (14).

\[
P(x) = \text{tr}g_n(x)
\]  

The integrand in (14),

\[
e^{-i\theta n} < A|e^{-H/T}|g_n^\dagger Ag_n + ig_n^\dagger \nabla g_n > \equiv e^{-S_{\text{eff}}[A, g_n]} \frac{1}{N} \text{tr} \left( g_n(L)g_n^\dagger(0) \right)^{-\theta N/2\pi}
\]

is a functional of \( A \) and \( g_n \) which has the symmetry properties

\[
S_{\text{eff}}[A, g_n] = S_{\text{eff}}[u^\dagger Au + iu^\dagger \nabla u, u^{-1}g_nu]
\]

\[
S_{\text{eff}}[A, g_nz] = S_{\text{eff}}[A, g_n]
\]

The variables \( A \) and \( g_n \) take values in the loop algebra and loop group of the gauge group, respectively. This is a 1-dimensional gauged principle chiral model. The theta-term in (20) is a topological term in the sense that it is insensitive to local variations of \( g_n(x) \) and depends only on its holonomy on the circle. (22) implies that the effective action is invariant under the \( Z_N \) symmetry \( g_n \to g_nz \). The expectation values of Polyakov loop operators are given by the correlators of traces of the group elements

\[
< P(x_1) \ldots P^\dagger(y_k) > = 
\sum_n \int dA(x)[dg_n(x)]e^{-S_{\text{eff}}[A, g_n]} \frac{1}{N} \text{tr} \left( g_n(L)g_n^\dagger(0) \right)^{-\theta N/2\pi} \text{tr}g_n(x_1) \ldots \text{tr}g_n^\dagger(y_k)
\]

\[
\sum_n \int dA(x)[dg_n(x)]e^{-S_{\text{eff}}[A, g_n]} \frac{1}{N} \text{tr} \left( g_n(L)g_n^\dagger(0) \right)^{-\theta N/2\pi}
\]

The equation (20) defines an \( d \)-dimensional effective action for the Polyakov loop operator in any \( d+1 \)-dimensional gauge theory coupled to matter fields where there is a trace over the matter degrees of freedom on the right hand side of the formula. In all cases, the resulting effective action has the gauge invariance property in (21). When the matter is in the adjoint representation, or any other representation which is invariant under transformations
in the center of the gauge group, the effective action also has the global $Z_N$ symmetry in (22). In the present paper, we shall concentrate on finding the effective action for pure 1+1-dimensional Yang-Mills theory.

By evaluating the matrix element in (14),

$$<A| \exp \left( -\frac{e^2}{2T} \sum_a \int_0^L dx \frac{-\delta^2}{\delta A^a(x)^2} \right) |g_n^\dagger A g_n + ig_n^\dagger \nabla g_n >$$  \hspace{1cm} (24)

$$= \exp \left( -\frac{T}{e^2} \int_0^L dx \left( \frac{1}{2} \text{tr} (A - g_n^\dagger A g_n - ig_n^\dagger \nabla g_n)^2 \right) \right)$$  \hspace{1cm} (25)

(when defined using zeta function regularization, the constant factor which should appear on the right-hand-side is one) we present the partition function as

$$Z[\theta, T, L] = \frac{1}{\text{VOL}G} \sum_n \int \prod_{x,a} dA^a(x) [dg_n(x)] \exp \left( -\frac{T}{e^2} \int_0^L dx \left( \frac{1}{2} \text{tr} (Dg_n) \right) \right) \text{tr} \left( g_n(L)g_n^\dagger(0) \right)^{-\theta N/2\pi}$$  \hspace{1cm} (26)

where $Dg_n = \nabla g_n + [A, g_n]$. The integral in (26) can be done by using gauge symmetry to diagonalize the unitary matrix $g_n = Ug_n^DU^{-1}$ where $g_n^D(x) = \text{diagonal}(e^{i\phi_1^n(x)}, e^{i\phi_2^n(x)}, \ldots, e^{i\phi_N^n(x)})$. The $\phi$ variables are angles $\phi \in [0, 2\pi]$. The measure in the integral has the form

$$[dg_n(x)] = \prod_{x \in [0,L]} \prod_{\alpha} d\phi_\alpha^n(x) \prod_{\alpha<\beta} \frac{1}{2} \sin^2(\phi_\alpha^n(x) - \phi_\beta^n(x)) \delta \left( \sum_\alpha \phi_\alpha^n(x) \right) [dU(x)]$$  \hspace{1cm} (27)

and the action is $\frac{T}{e^2} \int_0^L dx \left( \sum_\alpha \nabla \phi_\alpha^n \nabla \phi_\alpha^n + \sum_{\alpha, \beta} |A_{\alpha\beta}|^2 e^{i\phi_\alpha^n} - e^{i\phi_\beta^n} \right)^2$. The integral over $A_{\alpha\beta}$, where $\alpha \neq \beta$, cancels the Vandermonde determinant in the integration measure. The integration over diagonal components of $A$ yields an infinite factor which compensates the infinite normalization of the plane-wave state which was used to take the trace in (14). The partition function is an integral over the eigenvalue variables,

$$Z = \frac{1}{\text{VOL}G} \sum_{n=0}^{N-1} e^{-\theta n} \int \prod_{x} \prod_{\alpha=1}^{N-1} d\phi_\alpha^n(x) \exp \left[ -\frac{T}{e^2} \int_0^L dx \sum_{\alpha=1}^{N-1} \nabla \phi_\alpha^n \left( \nabla \phi_\alpha^n + \sum_{\beta=1}^{N-1} \nabla \phi_\beta^n \right) \right]$$  \hspace{1cm} (28)
where the integration variables have the boundary condition \( \phi^\alpha_n(L) = \phi^\alpha_n(0) + \theta n + 2\pi p_\alpha \), \( \theta = 2\pi m/N \) and \( p \in \mathbb{Z} \) and the integration measure integrates \( \phi(x) \), at each point \( x \), over the range \([0, 2\pi)\) and sums over the integers \( p_\alpha \). The integral is invariant under the field translation symmetry

\[
\phi^\alpha_n(x) \to \phi^\alpha_n(x) + c^\alpha
\]  

where \( c^\alpha \) is a constant. Note that this is not a symmetry of the action or the integration measure in the path integral (26) separately but only appears after the integration over \( A \). This symmetry can be used to extend the limits on the integration over \( \phi(x) \) at each \( x \) to infinity. One can untwist the boundary conditions and take into account the periodicity of the \( \phi^\alpha_n(x) \) variables, by changing the functional integration variables to \( \phi^\alpha(x) \), by means of

\[
\phi^\alpha_n(x) = \frac{2\pi(n + Np_\alpha)x}{NL} + \phi^\alpha(x) ,
\]

with \( p_\alpha \in \mathbb{Z} \). In (30) \( \phi^\alpha(x) \) is real and periodic, \( \phi^\alpha(L) = \phi^\alpha(0) \). The partition function can be written as the product of an analytic part \( Z_a \) containing the integral over the periodic fields \( \phi(x) \) and a topological part \( Z_t \) containing the summation over integers \( p_\alpha \), \( Z = Z_aZ_t \). \( Z_a \) can be computed by means of a zeta function regularization. Expanding \( \phi^\alpha(x) \) in modes

\[
\phi^\alpha(x) = \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{\infty} a_k e^{i2\pi kx/L}
\]

the functional measure is defined as \( \prod_x d\phi(x) \equiv \prod_k da_k \). The integration over the zero mode produces an infinite, irrelevant, temperature independent factor proportional to the volume of the gauge group. The functional integral in \( Z_a \) is proportional to the determinant of the laplacian,

\[
Z_a = \prod_{k \neq 0} \left( \det \frac{N^2}{i} \Omega k^2 \right)^{-\frac{1}{2}} = \det \left( \frac{N^2}{i} \Omega \right)^{-\xi(0)} e^{(N-1)\xi'(0)} = \sqrt{N} \left( \frac{2T}{e^2L} \right)^{\frac{N-1}{2}}.
\]

where \( \Omega \) is the \((N-1) \times (N-1)\) matrix

\[
\Omega = i - \frac{4\pi T}{e^2LN^2} \begin{pmatrix} 2 & 1 & \ldots & 1 \\ 1 & 2 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 2 \end{pmatrix} \equiv ig_{ij}.
\]
and we have used zeta-function regularization with
\[ \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^{2s}} \] (34)
and \( \zeta(0) = -1/2 \) and \( \zeta'(0) = -(\log 2\pi)/2 \).

\( Z_t \) is given by a finite sum over \( n \) and an infinite sum over all the \( p_\alpha \).

Rewriting
\[ \exp \left( -\frac{i2\pi mn}{N} \right) = \exp \left[ i2\pi m \sum_{\alpha=1}^{N-1} (n + Np_\alpha) \right] , \] (35)
the finite sum over \( n \) and the \( N - 1 \) infinite sums over the \( p_\alpha \) become just \( N - 1 \) infinite sums over the integers \( n_\alpha = n + Np_\alpha \). The \( n_\alpha \) in fact span \( \mathbb{Z} \) when \( 0 \leq n \leq N - 1 \) and \( p \in \mathbb{Z} \). \( Z_t \) then reads
\[ Z_t = \sum_{\vec{n} = -\infty}^{+\infty} \exp \left( 2\pi i\vec{n} \cdot \vec{z} + \pi i\vec{n} \Omega \vec{n} \right) \equiv \theta (\vec{z}, \Omega) . \] (36)

Here we introduced the \( \theta \)-function of several variables according to the conventions of Ref. [20], \( \vec{n}, \vec{z} \) are thought of as \( N - 1 \)-dimensional column vectors, \( \vec{n} = (n_1, n_2, \ldots, n_{N-1}), \vec{z} = (m/N, m/N, \ldots, m/N) \).

In order to obtain the energy spectrum of the theory we can now apply the generalized Poisson resummation formula
\[ \sum_{\vec{n} = -\infty}^{+\infty} \exp \left( -\pi g_{ij} m_i m_j - 2\pi i m_i a_i \right) = \frac{1}{\sqrt{\det g_{ij}}} \sum_{\vec{n} = -\infty}^{+\infty} \exp \left[ -\pi g_{ij} (n_i - a_i)(n_j - a_j) \right] . \] (37)
where \( g_{ij} \) is the inverse of \( g_{ij} \)
\[ g_{ij} = \frac{e^2 LN}{4\pi T} \begin{pmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & N-1 \end{pmatrix} . \] (38)
and \( \det g_{ij} = (4\pi T/e^2LN^2)^{(N-1)}N \). The \( \sqrt{\det g_{ij}} \) cancels the temperature dependence of \( Z_a \) so that the partition function reads
\[ Z = \sum_{\vec{n} = -\infty}^{+\infty} \exp \left\{ -\frac{e^2L}{4T} \left[ \sum_{\alpha=1}^{N-1} (Nn_\alpha + m)^2 - \frac{1}{N} \left( \sum_{\alpha=1}^{N-1} (Nn_\alpha + m) \right)^2 \right] \right\} . \] (39)
From (39) one can deduce the energy spectrum of the theory for each $m$-sector. The result is energy levels which are characterized by $N-1$ integers $\vec{n}$ and

$$E(\vec{n}) = \frac{e^2 L}{4} \left[ \sum_{\alpha=1}^{N-1} (Nn_\alpha + m)^2 - \frac{1}{N} \left( \sum_{\alpha=1}^{N-1} (Nn_\alpha + m) \right)^2 \right]$$

(40)

which agrees with the one obtained, in a different context, in Refs. [7, 8, 9] with the small difference that one finds that each energy eigenvalue belongs to a specific topological sector, labelled by $m$. For example for $SU(2)$ in Refs. [7, 8, 9] was found $E_n = e^2 L n^2 / 8$ and here we see that the eigenvalues corresponding to even integers belong to the $m = 0$ sector ($E_n = e^2 L (2n)^2 / 8$) whereas those corresponding to odd integers ($E_n = e^2 L (2n+1)^2 / 8$) belong to the $m = 1$ sector.

The calculation of correlators of Polyakov loop operators straightforward. The path integral is easily performed and the sums on $n$ and the $p_\alpha$ can be treated as before.

Correlators are nonvanishing only if they are invariant under the transformations by the center of the group, $Z(N)$, in (22). This occurs when the number of $g$’s in a correlator is equal to the number of $g^\dagger$’s, modulo $N$. Using Eqs.(37), the 2-point correlator $P_m^{(2)}(x, y) = \langle tr \{ g(x) \} tr \{ g^\dagger(y) \} \rangle_m / N^2$ can be presented as

$$P_m^{(2)}(x, y) = \frac{1}{Z N^2} \sum_{\vec{n} = -\infty}^{+\infty} \exp \left[ -\pi g^{ij} \left( n_i + \frac{m}{N} - \frac{x - y}{NL} \right) \left( n_j + \frac{m}{N} - \frac{x - y}{NL} \right) \right]$$

$$+ \sum_{\alpha=1}^{N-1} \exp \left[ -\pi g^{ij} \left( n_i + \frac{m}{N} - \frac{x - y}{NL} \delta_{i\alpha} \right) \left( n_j + \frac{m}{N} - \frac{x - y}{NL} \delta_{j\alpha} \right) \right] \right\} G(x, y) , \ \ (41)$$

where

$$G(x, y) = \exp \left[ -\frac{Le^2 (N-1)}{4TN} \left( \frac{|x-y|}{L} - \frac{(x-y)^2}{L^2} \right) \right] . \ \ \ \ \ \ \ (42)$$

For $SU(2)$

$$P_m^{(2)}(x, y) = \frac{1}{2Z} \sum_{n=-\infty}^{+\infty} \exp \left\{ -\frac{e^2 L}{8T} \left[ (2n + m)^2 - (4n + 2m) \frac{x-y}{L} \right] + \frac{x-y}{L} \right\} . \ \ \ \ \ \ \ (43)$$

Any other correlator of Polyakov loop operators can be analogously obtained.
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