A Relation between Coherent States and Generalized Bell States

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Abstract

In the first half we show an interesting relation between coherent states and the Bell states in the case of spin 1/2, which was suggested by Fivel.

In the latter half we treat generalized coherent states and try to generalize this relation to get several generalized Bell states.

Our method is based on a geometry and our task may give a hint to open a deep relation between a coherence and an entanglement.

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1 Introduction

The recent progress of quantum information theory including quantum computer, quantum cryptography and quantum teleportation is marvelous enough. The coherence and entanglement play an essential role in quantum information theory. See the papers in [1] or [4].

In [2] Bell considered the so-called Bell states to test the EPR problem (“paradox”) and proposed the famous inequality, see [3] or [4]. The Bell states are typical examples of the entanglement. Interestingly enough they have been used in the field of quantum teleportation. They are in the case of spin 1/2. Of course we can consider states with general spin $j$. We call them generalized Bell states.

On the other hand coherent states are fundamental tools in quantum optics and they are of course entangled. See [5]. Coherent states (generalized coherent states) are related with unitary representations of compact or non-compact Lie groups such as $U(n)$ or $U(n-1,1)$, see [6].

What is a relation between coherent states and Bell states or generalized Bell states? We would like to construct a mathematical theory between them. In [7] Fivel defined the generalized Bell states as the integral of tensor product of generalized coherent state and its “twisted” one. We redefine Fivel’s one to be more calculable and perform several integrals. Then we recover the Bell states and, moreover, get Bell states with general spin and more. In a certain sense the states of Fivel are overcomplete expression of Bell states or generalized Bell states.

By the way we are now developing Holonomic Quantum Computation, [14]–[21]. One of our aim of this study is to apply the idea of generalized Bell states to it. But we have a trouble. The Fivel’s states are not defined for coherent states based on non-compact Lie group such as $U(n-1,1)$. This point is unsatisfactory to us. Therefore we need to extend our method more widely.
2 Review on General Theory

We make a review of [7] within our necessity. Let $G$ be a compact linear Lie group (for example $G = U(n)$) and consider a coherent representation of $G$ whose parameter space is a compact complex manifold $S = G/H$, where $H$ is a subgroup of $G$. For example $G = U(n)$ and $H = U(k) \times U(n-k)$, then $S = U(n)/U(k) \times U(n-k) \cong G_k(C^n)$, a complex Grassmann manifold. See in detail [3] or [10]. Let $Z$ be a local coordinate and $|Z\rangle$ a generalized coherent state in some representation space $V \cong \mathbb{C}^K$ for some big $K \in \mathbb{N}$). Then we have from the definition the measure $d\mu(Z, Z^\dagger)$ that satisfies the resolution of unity

$$\int_S d\mu(Z, Z^\dagger) |Z\rangle\langle Z| = 1_V \quad \text{and} \quad \int_S d\mu(Z, Z^\dagger) = \dim V . \quad (1)$$

Next we define an anti-automorphism $b : S \rightarrow S$. We call $Z \rightarrow Z^b$ an anti-automorphism if and only if

(i) $Z \rightarrow Z^b$ induces an automorphism of $S$, \hfill (2)

(ii) $b$ is an anti-map, namely $\langle Z^b|W^\gamma \rangle = \langle W|Z\rangle$. \hfill (3)

Now let us define the generalized Bell state [7] :

**Definition** (Fivel) The generalized Bell state is defined as

$$|\rangle\rangle_B = \frac{1}{\sqrt{\dim V}} \int_S d\mu(Z, Z^\dagger) |Z\rangle \otimes |Z^b\rangle . \quad (4)$$

Then we have

$$\langle \langle B||B\rangle \rangle = \frac{1}{\dim V} \int_S \int_S d\mu(Z, Z^\dagger)d\mu(W, W^\dagger)(\langle Z| \otimes \langle Z^b|)(|W\rangle \otimes |W^\gamma\rangle)$$

$$= \frac{1}{\dim V} \int_S \int_S d\mu(Z, Z^\dagger)d\mu(W, W^\dagger)(Z|W^\gamma \rangle \langle Z^b|W^\gamma\rangle)$$

$$= \frac{1}{\dim V} \int_S \int_S d\mu(Z, Z^\dagger)d\mu(W, W^\dagger)(Z|W^\gamma \rangle \langle W|Z\rangle)$$

$$= \frac{1}{\dim V} \int_S d\mu(Z, Z^\dagger)\langle Z|Z\rangle = \frac{1}{\dim V} \int_S d\mu(Z, Z^\dagger) = 1,$$

where we have used (1) and (3).
Therefore we can get several generalized Bell states as choosing several anti-automorphisms. In the next section we will show that these states just coincide with the famous Bell states \[2\] in the case of spin \(\frac{1}{2}\):

\[
\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), \quad \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle), \\
\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle), \quad \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) .
\] (5)

Next we make a review of complex projective spaces, \[11\], \[9\] and \[12\]. For \(N \in \mathbb{N}\) the complex projective space \(\mathbb{C}P^N\) is defined as follows: For \(\zeta, \mu \in \mathbb{C}^{N+1} - \{0\}\) \(\zeta\) is equivalent to \(\mu\) \((\zeta \sim \mu)\) if and only if \(\zeta = \lambda \mu\) for some \(\lambda \in \mathbb{C} - \{0\}\). We show its equivalence relation class as \([\zeta]\) and set \(\mathbb{C}P^N \equiv \mathbb{C}^{N+1} - \{0\} / \sim\). When \(\zeta = (\zeta_0, \zeta_1, \cdots, \zeta_N)\) we write usually as \([\zeta] = [\zeta_0 : \zeta_1 : \cdots : \zeta_N]\). Then it is well–known that \(\mathbb{C}P^N\) has \(N+1\) local charts, namely

\[
\mathbb{C}P^N = \bigcup_{j=0}^{N} U_j , \quad U_j = \{[\zeta_0 : \cdots : \zeta_j : \cdots : \zeta_N] | \zeta_j \neq 0\} .
\] (6)

Since

\[
(\zeta_0, \cdots, \zeta_j, \cdots, \zeta_N) = \zeta_j \left( \frac{\zeta_0}{\zeta_j}, \cdots, \frac{\zeta_{j-1}}{\zeta_j}, 1, \frac{\zeta_{j+1}}{\zeta_j}, \cdots, \frac{\zeta_N}{\zeta_j} \right),
\]

we have the local coordinate on \(U_j\)

\[
\left( \frac{\zeta_0}{\zeta_j}, \cdots, \frac{\zeta_{j-1}}{\zeta_j}, \frac{\zeta_{j+1}}{\zeta_j}, \cdots, \frac{\zeta_N}{\zeta_j} \right) .
\] (7)

But the above definition of \(\mathbb{C}P^N\) is not handy, so we use the well–known expression by projections

\[
\mathbb{C}P^N \cong G_1(\mathbb{C}^{N+1}) = \{P \in M(N+1; \mathbb{C}) | P^2 = P, \ P^\dagger = P \text{ and } \text{tr}P = 1\} .
\] (8)

and this correspondence

\[
[\zeta_0 : \zeta_1 : \cdots : \zeta_N] \iff \frac{1}{|\zeta_0|^2 + |\zeta_1|^2 + \cdots + |\zeta_N|^2} \begin{pmatrix}
|\zeta_0|^2 & \zeta_0 \bar{\zeta}_1 & \cdots & \zeta_0 \bar{\zeta}_N \\
\zeta_1 \bar{\zeta}_0 & |\zeta_1|^2 & \cdots & \zeta_1 \bar{\zeta}_N \\
\vdots & \vdots & \ddots & \vdots \\
\zeta_N \bar{\zeta}_0 & \zeta_N \bar{\zeta}_1 & \cdots & |\zeta_N|^2
\end{pmatrix} \equiv P .
\] (9)
If we set

\[
|\zeta\rangle = \frac{1}{\sqrt{\sum_{j=0}^{N} |\zeta_j|^2}} \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \vdots \\ \zeta_N \end{pmatrix}, \quad (10)
\]

then we can write the right hand side of (9) as

\[
P = |\zeta\rangle \langle \zeta| \quad \text{and} \quad \langle \zeta| \langle \zeta \rangle = 1. \quad (11)
\]

For example on \(U_1\)

\[
(z_1, z_2, \ldots, z_N) = \left(\frac{\zeta_1}{\zeta_0}, \frac{\zeta_2}{\zeta_0}, \ldots, \frac{\zeta_N}{\zeta_0}\right),
\]

we have

\[
P(z_1, \ldots, z_N) = \frac{1}{1 + \sum_{j=1}^{N} |z_j|^2} \begin{pmatrix}
1 & \bar{z}_1 & \cdots & \bar{z}_N \\
\bar{z}_1 & |z_1|^2 & \cdots & z_1 \bar{z}_N \\
\vdots & \vdots & \ddots & \vdots \\
z_N & z_N \bar{z}_1 & \cdots & |z_N|^2
\end{pmatrix}
\]

\[
= |(z_1, z_2, \ldots, z_N)\rangle \langle (z_1, z_2, \ldots, z_N)|, \quad (12)
\]

where

\[
|(z_1, z_2, \ldots, z_N)\rangle = \frac{1}{\sqrt{1 + \sum_{j=1}^{N} |z_j|^2}} \begin{pmatrix} 1 \\ z_1 \\ \vdots \\ z_N \end{pmatrix}. \quad (13)
\]

For the latter use let us give a more detail description for the cases \(N = 1\) and 2.

(a) \(N = 1\) :

\[
P(z) = \frac{1}{1 + |z|^2} \begin{pmatrix} 1 & \bar{z} \\ z & |z|^2 \end{pmatrix} = |z\rangle \langle z|,
\]
\[ \begin{aligned}
\text{where } |z\rangle &= \frac{1}{\sqrt{1 + |z|^2}} \begin{pmatrix} 1 \\ z \end{pmatrix}, \quad z = \frac{\zeta_1}{\zeta_0}, \quad \text{on } U_1, \\
P(w) &= \frac{1}{|w|^2 + 1} \begin{pmatrix} |w|^2 & w \\ \bar{w} & 1 \end{pmatrix} = |w\rangle\langle w|,
\end{aligned} \]

\[ \begin{aligned}
\text{where } |w\rangle &= \frac{1}{\sqrt{|w|^2 + 1}} \begin{pmatrix} w \\ 1 \end{pmatrix}, \quad w = \frac{\zeta_0}{\zeta_1}, \quad \text{on } U_2.
\end{aligned} \]

(b) \( N = 2 \):

\[ \begin{aligned}
P(z_1, z_2) &= \frac{1}{1 + |z_1|^2 + |z_2|^2} \begin{pmatrix} 1 & \bar{z}_1 & \bar{z}_2 \\ z_1 & |z_1|^2 & \bar{z}_1z_2 \\ z_2 & \bar{z}_2z_1 & |z_2|^2 \end{pmatrix} = |(z_1, z_2)\rangle\langle (z_1, z_2)|,
\end{aligned} \]

\[ \begin{aligned}
\text{where } |(z_1, z_2)\rangle &= \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}} \begin{pmatrix} 1 \\ z_1 \\ z_2 \end{pmatrix}, \quad (z_1, z_2) = \left( \frac{\zeta_1}{\zeta_0}, \frac{\zeta_2}{\zeta_0} \right), \quad \text{on } U_1,
\end{aligned} \]

\[ \begin{aligned}
P(w_1, w_2) &= \frac{1}{|w_1|^2 + 1 + |w_2|^2} \begin{pmatrix} |w_1|^2 & w_1 & w_1\bar{w}_2 \\ \bar{w}_1 & 1 & \bar{w}_2 \\ w_2\bar{w}_1 & w_2 & |w_2|^2 \end{pmatrix} = |(w_1, w_2)\rangle\langle (w_1, w_2)|,
\end{aligned} \]

\[ \begin{aligned}
\text{where } |(w_1, w_2)\rangle &= \frac{1}{\sqrt{|w_1|^2 + 1 + |w_2|^2}} \begin{pmatrix} w_1 \\ 1 \\ w_2 \end{pmatrix}, \quad (w_1, w_2) = \left( \frac{\zeta_0}{\zeta_1}, \frac{\zeta_2}{\zeta_1} \right), \quad \text{on } U_4.
\end{aligned} \]

\[ \begin{aligned}
P(v_1, v_2) &= \frac{1}{|v_1|^2 + |v_2|^2 + 1} \begin{pmatrix} |v_1|^2 & v_1 & v_1\bar{v}_2 \\ v_2\bar{v}_1 & |v_2|^2 & v_2 \\ \bar{v}_1 & \bar{v}_2 & 1 \end{pmatrix} = |(v_1, v_2)\rangle\langle (v_1, v_2)|,
\end{aligned} \]

\[ \begin{aligned}
\text{where } |(v_1, v_2)\rangle &= \frac{1}{\sqrt{|v_1|^2 + |v_2|^2 + 1}} \begin{pmatrix} v_1 \\ v_2 \\ 1 \end{pmatrix}, \quad (v_1, v_2) = \left( \frac{\zeta_0}{\zeta_2}, \frac{\zeta_1}{\zeta_2} \right), \quad \text{on } U_3.
\end{aligned} \]
3 Bell States Revisited

In this section we show that (4) coincides with the Bell states (5) by choosing antiautomorphism suitably.

First let us recall the spin $j$-representation of Lie algebra $su(2)$ from [8]. This is a coherent representation of $su(2)$ based on complex manifold $CP^1$ in our terminology. The algebra of $\{J_+, J_-, J_3\}$ reads

$$[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = 2J_3 ,$$

(19)

where $J_\pm = \frac{1}{2}(J_1 \pm iJ_2)$ and actions of $\{J_+, J_-, J_3\}$ on a representation space $V (\cong \mathbb{C}^{2j+1})$

$$J_+|j, m\rangle = \sqrt{(j - m)(j + m + 1)}|j, m + 1\rangle, \quad J_-|j, m\rangle = \sqrt{(j - m + 1)(j + m)}|j, m - 1\rangle, \quad J_3|j, m\rangle = m|j, m\rangle ,$$

(20)

where $-j \leq m \leq j$. We note

$$1_j = \sum_{m=-j}^{j} |j, m\rangle \langle j, m| \quad \text{and} \quad \langle j, m| j, n\rangle = \delta_{mn} .$$

(21)

Then the coherent state $|z\rangle$ ($z \in \mathbb{C} \subset CP^1$) is defined as

$$|z\rangle = \frac{1}{(1 + |z|^2)^{j}} \sum_{k=0}^{2j} \sqrt{2j}C_k z^k |j, -j + k\rangle$$

(22)

and this satisfies the resolution of unity [11]

$$\int_{\mathbb{C}} d\mu(z, \bar{z}) |z\rangle \langle z| = \sum_{m=-j}^{j} |j, m\rangle \langle j, m| = 1_j \quad \text{and} \quad \int_{\mathbb{C}} d\mu(z, \bar{z}) = 2j + 1 ,$$

(23)

where the measure $d\mu(z, \bar{z})$ is

$$d\mu(z, \bar{z}) = \frac{2j + 1}{\pi} \frac{[d^2z]}{(1 + |z|^2)^2} .$$

(24)

We note that this measure is invariant under the transform $z \rightarrow 1/z$, so this one is defined on $CP^1$ not $\mathbb{C}$. In the following we set for simplicity

$$|j, -j + k\rangle = |k\rangle \quad \text{for} \quad 0 \leq k \leq 2j .$$

(25)
For example \(|\frac{1}{2}, -\frac{1}{2}\rangle = |0\rangle\) and \(|\frac{1}{2}, \frac{1}{2}\rangle = |1\rangle\) in the case of spin \(\frac{1}{2}\). In this case we consider the following four anti-automorphisms (2) and (3):

\[
\begin{align*}
(1) \quad z^b &= \bar{z} \\
(2) \quad z^b &= -\bar{z} \\
(3) \quad z^b &= \frac{1}{\bar{z}} \\
(4) \quad z^b &= \frac{-1}{\bar{z}}.
\end{align*}
\]

Then it is easy to see from (14) and (15) we have easily

\[
\frac{d\mu}{\pi} = \int_C \frac{[d^2z]}{(1 + |z|^2)^2} \frac{1}{1 + |z|^2} = 1, \\
\frac{2}{\pi} \int_C \frac{[d^2z]}{(1 + |z|^2)^2} \frac{z}{1 + |z|^2} = 0,
\]

we have easily

\[
\begin{align*}
(1) \quad ||B|| &= \frac{1}{\sqrt{2}} \int_C d\mu(z, \bar{z})|z\rangle \otimes |\bar{z}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), \\
(2) \quad ||B|| &= \frac{1}{\sqrt{2}} \int_C d\mu(z, \bar{z})|z\rangle \otimes |\bar{z}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle), \\
(3) \quad ||B|| &= \frac{1}{\sqrt{2}} \int_C d\mu(z, \bar{z})|z\rangle \otimes |1/\bar{z}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |0\rangle \otimes |0\rangle), \\
(4) \quad ||B|| &= \frac{1}{\sqrt{2}} \int_C d\mu(z, \bar{z})|z\rangle \otimes |-1/\bar{z}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |0\rangle \otimes |0\rangle),
\end{align*}
\]

where \(d\mu(z, \bar{z}) = \frac{2}{\pi} \frac{[d^2z]}{(1 + |z|^2)^2}\). We note that our calculation is based on the following two matrices:

\[
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Lemma 1

\[
\begin{align*}
(1) \quad |z^b\rangle &= |\bar{z}\rangle = \frac{1}{\sqrt{1 + |z|^2}}(|0\rangle + \bar{z}|1\rangle), \\
(2) \quad |z^b\rangle &= |-\bar{z}\rangle = \frac{1}{\sqrt{1 + |z|^2}}(|0\rangle - \bar{z}|1\rangle), \\
(3) \quad |z^b\rangle &= |1/\bar{z}\rangle = \frac{1}{\sqrt{1 + |z|^2}}(|\bar{z}|0\rangle + |1\rangle), \\
(4) \quad |z^b\rangle &= |-1/\bar{z}\rangle = \frac{1}{\sqrt{1 + |z|^2}}(|\bar{z}|0\rangle - |1\rangle).
\end{align*}
\]

Here we have identified \(|0\rangle = \frac{1}{\sqrt{2}} |1\rangle\) and \(|1\rangle = \frac{1}{\sqrt{2}} |0\rangle\). Then making use of elementary facts

\[
\frac{2}{\pi} \int_C \frac{[d^2z]}{(1 + |z|^2)^2} \frac{1}{1 + |z|^2} = 1, \\
\frac{2}{\pi} \int_C \frac{[d^2z]}{(1 + |z|^2)^2} \frac{z}{1 + |z|^2} = 0,
\]

we have easily

Proposition 2

\[
\begin{align*}
(1) \quad ||B|| &= \frac{1}{\sqrt{2}} \int_C d\mu(z, \bar{z})|z\rangle \otimes |\bar{z}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), \\
(2) \quad ||B|| &= \frac{1}{\sqrt{2}} \int_C d\mu(z, \bar{z})|z\rangle \otimes |\bar{z}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle), \\
(3) \quad ||B|| &= \frac{1}{\sqrt{2}} \int_C d\mu(z, \bar{z})|z\rangle \otimes |1/\bar{z}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |0\rangle \otimes |0\rangle), \\
(4) \quad ||B|| &= \frac{1}{\sqrt{2}} \int_C d\mu(z, \bar{z})|z\rangle \otimes |-1/\bar{z}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |0\rangle \otimes |0\rangle),
\end{align*}
\]
We just recovered the Bell states (5). The author does not know whether this result has been known or not.

Since we consider the case of spin \( j \), we write
\[
|z\rangle_j = \frac{1}{\sqrt{2^j + 1}} \sum_{k=0}^{2j} \sqrt{2j C_k} z^k |k\rangle
\]
(36)
to emphasize the dependence of spin \( j \). From Proposition 2 it is very natural to define Bell states with spin \( j \) as follows because the parameter space is the same \( CP^1 \):

**Definition 3 (Bell states with spin \( j \))**

\[
\begin{align*}
(1) & \quad \langle B \rangle = \frac{1}{\sqrt{2^j + 1}} \int_C d\mu(z, \bar{z}) |z\rangle_j \otimes |\bar{z}\rangle_j, \\
(2) & \quad \langle B \rangle = \frac{1}{\sqrt{2^j + 1}} \int_C d\mu(z, \bar{z}) |z\rangle_j \otimes |-\bar{z}\rangle_j, \\
(3) & \quad \langle B \rangle = \frac{1}{\sqrt{2^j + 1}} \int_C d\mu(z, \bar{z}) |z\rangle_j \otimes |1/\bar{z}\rangle_j, \\
(4) & \quad \langle B \rangle = \frac{1}{\sqrt{2^j + 1}} \int_C d\mu(z, \bar{z}) |z\rangle_j \otimes |-1/\bar{z}\rangle_j,
\end{align*}
\]
(37) - (40)
where \( d\mu(z, \bar{z}) = \frac{2^{j+1}}{\pi} \frac{d^2z}{(1 + |z|^2)^2} \). Let us calculate \(|\bar{z}\rangle_j, |-\bar{z}\rangle_j, |1/\bar{z}\rangle_j \) and \(|-1/\bar{z}\rangle_j \). It is easy to see

**Lemma 4**

\[
\begin{align*}
(1) & \quad |\bar{z}\rangle_j = \frac{1}{\sqrt{2^j + 1}} \sum_{k=0}^{2j} \sqrt{2j C_k} \bar{z}^k |k\rangle, \\
(2) & \quad |-\bar{z}\rangle_j = \frac{1}{\sqrt{2^j + 1}} \sum_{k=0}^{2j} \sqrt{2j C_k} (-1)^k \bar{z}^k |k\rangle, \\
(3) & \quad |1/\bar{z}\rangle_j = \frac{1}{\sqrt{2^j + 1}} \sum_{k=0}^{2j} \sqrt{2j C_k} \bar{z}^k |2j - k\rangle, \\
(4) & \quad |-1/\bar{z}\rangle_j = \frac{1}{\sqrt{2^j + 1}} \sum_{k=0}^{2j} \sqrt{2j C_k} (-1)^k \bar{z}^k |2j - k\rangle.
\end{align*}
\]
(41) - (44)
From this lemma and the elementary facts
\[
\frac{2^j + 1}{\pi} \int_C \frac{[d^2z]}{(1 + |z|^2)^2} |z|^{2k} = \frac{1}{2j C_k} \quad \text{for} \quad 0 \leq k \leq 2j,
\]
8
we can give explicit forms to the Bell states with spin $j$ in Definition 3:

**Proposition 5**

(1) $\left\langle |B\rangle\right\rangle = \frac{1}{\sqrt{2j + 1}} \sum_{k=0}^{2j} |k\rangle \otimes |k\rangle,$

(2) $\left\langle |B\rangle\right\rangle = \frac{1}{\sqrt{2j + 1}} \sum_{k=0}^{2j} (-1)^k |k\rangle \otimes |k\rangle,$

(3) $\left\langle |B\rangle\right\rangle = \frac{1}{\sqrt{2j + 1}} \sum_{k=0}^{2j} |k\rangle \otimes |2j - k\rangle,$

(4) $\left\langle |B\rangle\right\rangle = \frac{1}{\sqrt{2j + 1}} \sum_{k=0}^{2j} (-1)^k |k\rangle \otimes |2j - k\rangle.$

A comment is in order. We list the above result once more for the case $j = 1$:

(1) $\frac{1}{\sqrt{3}} (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + |2\rangle \otimes |2\rangle),$

(2) $\frac{1}{\sqrt{3}} (|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle + |2\rangle \otimes |2\rangle),$

(3) $\frac{1}{\sqrt{3}} (|0\rangle \otimes |2\rangle + |1\rangle \otimes |1\rangle + |2\rangle \otimes |0\rangle),$

(4) $\frac{1}{\sqrt{3}} (|0\rangle \otimes |2\rangle - |1\rangle \otimes |1\rangle + |2\rangle \otimes |0\rangle).$

It is easy to see that they are not linearly independent. Namely this case is very special (peculiar).

4 Generalized Bell States

In this section we generalize the result in the preceding section, namely we treat the coherent states of $u(n + 1)$ based on $\mathbb{C}P^n$ (see [3]) and calculate generalized Bell states ([4] for several anti-automorphisms like [26]). But to avoid complicated calculations we consider the case $n = 2$ and $Q = 1$ only, because it is easy to conjecture the corresponding result in general case from this special case.
Let \(\{0, 1, 2\}\) be a basis of the representation space \(V (\cong \mathbb{C}^3)\). Namely
\[
\sum_{j=0}^{2} |j\rangle \langle j| = 1_{Q=1} \quad \text{and} \quad \langle i|j\rangle = \delta_{ij}.
\]

A coherent state \(|(z_1, z_2)\rangle\) for \((z_1, z_2) \in \mathbb{C}^2\) is defined as
\[
|(z_1, z_2)\rangle = \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}}(|0\rangle + z_1|1\rangle + z_2|2\rangle)
\]
and the measure
\[
d\mu(Z, Z^\dagger) = \frac{6}{\pi^2} \frac{[d^2z_1][d^2z_2]}{(1 + |z_1|^2 + |z_2|^2)^3}.
\]

Then we have
\[
\int_{\mathbb{C}^2} d\mu(Z, Z^\dagger)| (z_1, z_2)\rangle \langle (z_1, z_2)| = \sum_{j=0}^{2} |j\rangle \langle j| = 1_{Q=1} \quad \text{and} \quad \int_{\mathbb{C}^2} d\mu(Z, Z^\dagger) = 3.
\]

Let \(\omega\) be an element in \(\mathbb{C}\) satisfying \(\omega^3 = 1\). Then \(1 + \omega + \omega^2 = 0\) and \(\bar{\omega} = \omega^2\). Here we consider the following nine anti-automorphisms (2) and (3):

(a-1, 2, 3) \((z_1, z_2)^b = (\bar{z}_1, \bar{z}_2), \quad (z_1, z_2)^b = (\omega \bar{z}_1, \omega^2 \bar{z}_2), \quad (z_1, z_2)^b = (\omega^2 \bar{z}_1, \omega \bar{z}_2)\),

(b-1, 2, 3) \((z_1, z_2)^b = (\frac{1}{z_2}, \frac{z}{z_2}), \quad (z_1, z_2)^b = (\omega \frac{1}{z_2}, \omega^2 \frac{z}{z_2}), \quad (z_1, z_2)^b = (\omega^2 \frac{1}{z_2}, \omega \frac{z}{z_2})\),

(c-1, 2, 3) \((z_1, z_2)^b = (\frac{\bar{z}_2}{\bar{z}_1}, \frac{1}{\bar{z}_1}), \quad (z_1, z_2)^b = (\omega \frac{\bar{z}_2}{\bar{z}_1}, \omega^2 \frac{1}{\bar{z}_1}), \quad (z_1, z_2)^b = (\omega^2 \frac{\bar{z}_2}{\bar{z}_1}, \omega \frac{1}{\bar{z}_1})\).

A note is in order. We can of course choose another anti-automorphisms instead of the above ones.

Then it is easy to see from (2), (17) and (18).

**Lemma 6**

(a-1) \(|(z_1, z_2)^b\rangle = |(\bar{z}_1, \bar{z}_2)\rangle = \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}}(|0\rangle + \bar{z}_1|1\rangle + \bar{z}_2|2\rangle),

(a-2) \(|(z_1, z_2)^b\rangle = |(\omega \bar{z}_1, \omega^2 \bar{z}_2)\rangle = \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}}(|0\rangle + \omega \bar{z}_1|1\rangle + \omega^2 \bar{z}_2|2\rangle),

(a-3) \(|(z_1, z_2)^b\rangle = |(\omega^2 \bar{z}_1, \omega \bar{z}_2)\rangle = \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}}(|0\rangle + \omega^2 \bar{z}_1|1\rangle + \omega \bar{z}_2|2\rangle),

(b-1) \(|(z_1, z_2)^b\rangle = |(1/\bar{z}_2, \bar{z}_1/\bar{z}_2)\rangle = \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}}(\bar{z}_2|0\rangle + |1\rangle + \bar{z}_1|2\rangle),

10
From this lemma and the elementary facts

\[(b-2) \left| (z_1, z_2) \right|^2 = \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}} (\omega^2 \bar{z}_2 |0 \rangle + | 1 \rangle + \omega \bar{z}_1 |2 \rangle), \]

\[(b-3) \left| (z_1, z_2) \right|^2 = \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}} (\omega \bar{z}_2 |0 \rangle + | 1 \rangle + \omega^2 \bar{z}_1 |2 \rangle), \]

\[(c-1) \left| (z_1, z_2) \right|^2 = \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}} (\bar{z}_1 |0 \rangle + \bar{z}_2 |1 \rangle + |2 \rangle), \]

\[(c-2) \left| (z_1, z_2) \right|^2 = \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}} (\omega \bar{z}_1 |0 \rangle + \omega^2 \bar{z}_2 |1 \rangle + |2 \rangle), \]

\[(c-3) \left| (z_1, z_2) \right|^2 = \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}} (\omega^2 \bar{z}_1 |0 \rangle + \omega \bar{z}_2 |1 \rangle + |2 \rangle). \]

From this lemma and the elementary facts

\[
\begin{align*}
\frac{6}{\pi^2} \int_{C^2} \frac{[d^2z_1][d^2z_2]}{(1 + |z_1|^2 + |z_2|^2)^3} & \frac{1}{1 + |z_1|^2 + |z_2|^2} = \frac{6}{\pi^2} \int_{C^2} \frac{[d^2z_1][d^2z_2]}{(1 + |z_1|^2 + |z_2|^2)^3} \frac{|z_1|^2}{1 + |z_1|^2 + |z_2|^2} = \\
\frac{6}{\pi^2} \int_{C^2} \frac{[d^2z_1][d^2z_2]}{(1 + |z_1|^2 + |z_2|^2)^3} & \frac{|z_2|^2}{1 + |z_1|^2 + |z_2|^2} = 1,
\end{align*}
\]

we have easily

**Proposition 7** (generalized Bell states)

\[(a-1) \mid |B \rangle = \frac{1}{\sqrt{3}} (|0 \rangle \otimes |0 \rangle + |1 \rangle \otimes |1 \rangle + |2 \rangle \otimes |2 \rangle), \]

\[(a-2) \mid |B \rangle = \frac{1}{\sqrt{3}} (|0 \rangle \otimes |0 \rangle + \omega |1 \rangle \otimes |1 \rangle + \omega^2 |2 \rangle \otimes |2 \rangle), \]

\[(a-3) \mid |B \rangle = \frac{1}{\sqrt{3}} (|0 \rangle \otimes |0 \rangle + \omega^2 |1 \rangle \otimes |1 \rangle + \omega |2 \rangle \otimes |2 \rangle), \]

\[(b-1) \mid |B \rangle = \frac{1}{\sqrt{3}} (|0 \rangle \otimes |1 \rangle + |1 \rangle \otimes |2 \rangle + |2 \rangle \otimes |0 \rangle), \]

\[(b-2) \mid |B \rangle = \frac{1}{\sqrt{3}} (|0 \rangle \otimes |1 \rangle + \omega |1 \rangle \otimes |2 \rangle + \omega^2 |2 \rangle \otimes |0 \rangle), \]

\[(b-3) \mid |B \rangle = \frac{1}{\sqrt{3}} (|0 \rangle \otimes |1 \rangle + \omega^2 |1 \rangle \otimes |2 \rangle + \omega |2 \rangle \otimes |0 \rangle), \]

\[(c-1) \mid |B \rangle = \frac{1}{\sqrt{3}} (|0 \rangle \otimes |2 \rangle + |1 \rangle \otimes |0 \rangle + |2 \rangle \otimes |1 \rangle), \]

\[(c-2) \mid |B \rangle = \frac{1}{\sqrt{3}} (|0 \rangle \otimes |2 \rangle + \omega |1 \rangle \otimes |0 \rangle + \omega^2 |2 \rangle \otimes |1 \rangle), \]

\[(c-3) \mid |B \rangle = \frac{1}{\sqrt{3}} (|0 \rangle \otimes |2 \rangle + \omega^2 |1 \rangle \otimes |0 \rangle + \omega |2 \rangle \otimes |1 \rangle). \]
This is our main result. We note that our calculation is deeply based on the following two matrices:

\[ A = \begin{pmatrix} 1 & \omega \\ \omega & \omega^2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \] (73)

From Proposition 7 it is very easy to conjecture the explicit forms of generalized Bell states, [13].

5 Discussion

In this paper we calculated the generalized Bell states defined by Fivel, which are defined as the integral of tensor product of generalized coherent states based on \( \mathbb{C}P^n \) and their “twisted” ones due to anti–automorphisms on \( \mathbb{C}P^n \). The generalization to Grassmann manifolds \( G_k(\mathbb{C}^{N+1}) \) is under consideration, [13].

But unfortunately the generalized Bell states by Fivel are not defined for coherent states based on non–compact complex manifolds such as the Poincare disk or Siegel domains. We need a drastic change of his definition.

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Appendix A Anti–Automorphisms on \( G_1(\mathbb{C}^{N+1}) \)

Let us give global descriptions of anti–automorphisms in \( N = 1 : \{ (26), \text{Lemma 1} \} \) and \( N = 2 : \{ (52), \text{Lemma 6} \} \) for the readers who are not familiar with this field.

(a) \( N = 1 : \quad \phi : G_1(\mathbb{C}^2) \rightarrow G_1(\mathbb{C}^2) \)
Since
\[ P(z) = \frac{1}{1 + |z|^2} \begin{pmatrix} 1 & \bar{z} \\ z & |z|^2 \end{pmatrix} \]
we have
\[ P(\bar{z}) = \frac{1}{1 + |z|^2} \begin{pmatrix} 1 & \bar{z} \\ \bar{z} & |z|^2 \end{pmatrix}, \quad P(-\bar{z}) = \frac{1}{1 + |z|^2} \begin{pmatrix} 1 & -z \\ -\bar{z} & |z|^2 \end{pmatrix}, \]
\[ P(1/\bar{z}) = \frac{1}{1 + |\bar{z}|^2} \begin{pmatrix} |\bar{z}|^2 & \bar{z} \\ \bar{z} & 1 \end{pmatrix}, \quad P(-1/\bar{z}) = \frac{1}{1 + |\bar{z}|^2} \begin{pmatrix} |\bar{z}|^2 & -\bar{z} \\ -\bar{z} & 1 \end{pmatrix}, \]
so that it is easy to see
\[ (1) \quad \flat : P \rightarrow \bar{P} \quad (2) \quad \flat : P \rightarrow \sigma_3 \bar{P} \sigma_3 \]
\[ (3) \quad \flat : P \rightarrow \sigma_1 P \sigma_1 \quad (4) \quad \flat : P \rightarrow \sigma_1 \sigma_3 P \sigma_3 \sigma_1 \quad (74) \]
with \(\{\sigma_3, \sigma_1\}\) in (35). At this moment we know that anti-automorphisms are independent of choices of local coordinates.

(a) \(N = 2\) : \(\flat : G_1(C^3) \rightarrow G_1(C^3)\)

Since
\[ P(z_1, z_2) = \frac{1}{1 + |z_1|^2 + |z_2|^2} \begin{pmatrix} 1 & \bar{z}_1 & \bar{z}_2 \\ z_1 & |z_1|^2 & z_1 \bar{z}_2 \\ z_2 & z_2 \bar{z}_1 & |z_2|^2 \end{pmatrix}, \]
\[ \bar{P}(\bar{z}_1, \bar{z}_2) = \frac{1}{1 + |\bar{z}_1|^2 + |\bar{z}_2|^2} \begin{pmatrix} 1 & \bar{z}_1 & \bar{z}_2 \\ \bar{z}_1 & |\bar{z}_1|^2 & \bar{z}_1 \bar{z}_2 \\ \bar{z}_2 & \bar{z}_2 \bar{z}_1 & |\bar{z}_2|^2 \end{pmatrix} = P(z_1, z_2) \]
we have after some calculations
\[ (a-1) \quad \flat : P \rightarrow \bar{P} \quad (a-2) \quad \flat : P \rightarrow A \bar{P} A \quad (a-3) \quad \flat : P \rightarrow A^2 \bar{P} A^2 \]
\[ (b-1) \quad \flat : P \rightarrow B \bar{P} B \quad (b-2) \quad \flat : P \rightarrow B A \bar{P} A B \quad (b-3) \quad \flat : P \rightarrow B A^2 \bar{P} A^2 B \]
\[ (c-1) \quad \flat : P \rightarrow B^2 \bar{P} B^2 \quad (c-2) \quad \flat : P \rightarrow B^2 A \bar{P} A B^2 \quad (c-3) \quad \flat : P \rightarrow B^2 A^2 \bar{P} A^2 B^2 \quad (75) \]
with \( \{A, B\} \) in (73). We leave the proof to the readers. At this moment anti-automorphisms are independent of choices of local coordinates.

The generalization of the above result to arbitrary \( N \) is very easy, [13].

**Appendix B  Walsh–Hadamard Transformation**

Let us make a brief comment on the Walsh–Hadamard transformation. See [4] and [12] about it.

The Walsh–Hadamard transformation \( W \) defined by

\[
W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in U(2)
\]

(76)

plays a very important role in quantum computation, see [12]. This transformation is characterized as

\[
\begin{pmatrix} 1 \\ 1 \end{pmatrix} = W \begin{pmatrix} 1 \\ -1 \end{pmatrix} W^{-1}.
\]

(77)

Let us take an analogy. If we define \( \tilde{W} \) as

\[
\tilde{W} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \in U(3)
\]

(78)

then we have

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} = \tilde{W} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} \tilde{W}^{-1}.
\]

(79)

Therefore it may be possible for us to call \( \tilde{W} \) generalized Walsh–Hadamard transformation.

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