A Stratonovich SDE with irregular coefficients:
Girsanov’s example revisited

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Abstract

In this paper we study the Stratonovich stochastic differential equation $dX = |X|^\alpha \circ dB$, $\alpha \in (-1,1)$, which has been introduced by Cherstvy et al. [New Journal of Physics 15:083039 (2013)] in the context of analysis of anomalous diffusions in heterogeneous media. We determine its strong solutions, which are homogeneous strong Markov processes spending zero time in 0: for $\alpha \in (0,1)$, these solutions have the form

$$X_t^\theta = ((1 - \alpha)B_t^\theta + (X_0)^{1-\alpha})^{1/(1-\alpha)},$$

where $B^\theta$ is the $\theta$-skew Brownian motion driven by $B$, $\theta \in [-1,1]$, and $(x)^\gamma = |x|^\gamma \text{sign} x$; for $\alpha \in (-1,0]$, only the case $\theta = 0$ is possible. The central part of the paper consists in the proof of the existence of a quadratic covariation $\langle f(B^\theta), B \rangle$ for a locally square integrable function $f$ and is based on the time-reversion technique for Markovian diffusions.

Keywords: Stratonovich integral, Girsanov’s example, non-uniqueness, singular stochastic differential equation, skew Brownian motion, time-reversion, generalized Itô’s formula, local time.

MSC 2010 subject classification: Primary 60H10; secondary 60J55, 60J60.

1 Introduction

Girsanov [1962] considered the stochastic differential equation

$$Y_t = Y_0 + \int_0^t |Y_s|^\alpha dB_s, \quad t \geq 0,$$  \hspace{1cm} (1.1)

driven by a standard Brownian motion $B$ as an example of an SDE with non-unique solution. In particular, it was shown that for $\alpha \in (0,1/2)$, equation (1.1) has infinitely many continuous strong Markov (weak) solutions as well as non-homogeneous Markov solutions; non-Markovian solutions can also be constructed.

Since then, equation (1.1) serves as a benchmark example of various peculiar effects which come to light when one weakens the standard regularity assumptions on the coefficients of an SDE.

The proof of weak uniqueness for $\alpha \geq 1/2$ and non-uniqueness for $\alpha \in (0,1/2)$ with the help of random time change was given by (McKean, 1969, §3.10b) whereas a construction of an uncountable set of weak solutions can be found in (Engelbert and Schmidt, 1983, Example 3.3). The existence and uniqueness of unique strong solutions for $\alpha \in [1/2,1]$ was established by (Zvonkin, 1974, Theorem 4).

Furthermore for $\alpha \in (0,1/2)$, it was shown in (Engelbert and Schmidt, 1983, Theorem 5.2) that for every initial value $Y_0 \in \mathbb{R}$, there is a weak solution to (1.1) that spends zero time at 0 (the so-called fundamental solution) and the law of such a solution is unique. Path-wise uniqueness among those solutions to (1.1) that spend zero time at 0, and existence of a strong solution was proven by (Bass et al., 2007, Theorem 1.2).
An analogue of (1.2) with the Stratonovich integral

\[ X_t = X_0 + \int_0^t |X_s|^{\alpha} \circ dB_s \tag{1.2} \]

was recently introduced in the physical literature by Chertvy et al. (2013) under the name heterogeneous diffusion process. The authors studied the autocorrelation function of this process analytically and sub- and super-diffusive behaviour with the help of numerical simulations.

In this paper we will further investigate equation (1.2) with \( \alpha \in (-1, 1) \). It turns out that equation (1.2) has properties quite different from its Itô counterpart. Let us first make some observations about it. The only problematic point of the diffusion coefficient \( \sigma(x) = |x|^\alpha \) is \( x = 0 \). For \( \alpha \in (0, 1) \), the Lipschitz continuity fails at this point, and for \( \alpha \in (-1, 0) \) even the continuity and boundedness. However one can easily solve (1.2) locally for initial points \( X_0 \neq 0 \).

Indeed, assume for definiteness that \( X_0 > 0 \). For any \( \varepsilon > 0 \), using the properties of the Stratonovich integral, we see that the process given by

\[ X_t^\varepsilon = \left(1 - \alpha\right)B_t + \left[X_0^\varepsilon\right]^{\alpha} \cdot \text{sign} \left(X_0^\varepsilon\right), \tag{1.3} \]

solves equation (1.2) until the time \( \tau_\varepsilon = \inf\{t \geq 0 : X_t = \varepsilon\} \). Moreover, the solution is unique until \( \tau_\varepsilon \). Consequently, the formula (1.3) defines a unique strong solution until the time \( \tau_0 = \inf\{t \geq 0 : X_0^\theta = 0\} \) when the process hits zero. It is clear that extending the solution by zero value beyond \( \tau_0 \) gives a strong solution. However, the uniqueness fails: as it is shown in Section 3, the formula (1.3) defines a strong solution (called a benchmark solution), if we understand the left-hand side as a signed power function, i.e.

\[ \left(x^{\frac{1}{1-\alpha}}\right) = |x|^{\frac{1}{1-\alpha}} \cdot \text{sign} \left(x\right). \]

In Section 3 we also give some non-Markov solutions of (1.2).

The next question is whether, as for the Itô equation, a uniqueness is true within the class of processes spending a zero time at 0. For \( \alpha \in (0, 1) \), this question is answered negatively: in Theorem 1.5 we show that equation (1.2) has also ‘skew’ solutions

\[ X_t^\theta = \left(1 - \alpha\right)B_t^\theta + \left[X_0^\theta\right]^{1-\alpha} \cdot \text{sign} \left(X_0^\theta\right), \]

where for \( \theta \in [-1, 1] \), \( B_t^\theta \) is the skew Brownian motion, which solves the stochastic differential equation \( B_t^\theta = B_t + \theta L_t(B^\theta) \), \( L \) being the symmetric local time in 0. Moreover, we show all solutions, which are homogeneous strong Markov processes and which spend zero time at 0, have this form.

In Section 5 we propose an explanation for huge diversity of strong solutions to (1.2) and discuss further questions regarding the equation.

## 2 Preliminaries and conventions

Throughout the article, we work on a stochastic basis \( (\Omega, \mathcal{F}, \mathbb{P}) \), i.e. a complete probability space with a filtration \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \) satisfying the standard assumptions. The process \( B = (B_t)_{t \geq 0} \) is a standard continuous Brownian motion on this stochastic basis.

First we briefly recall definitions related to stochastic integration. More details may be found in Protter (2004).

The main mode of convergence considered here is the uniform convergence on compacts in probability (the u.c.p. convergence for short): a sequence \( X^n = (X^n_t)_{t \geq 0}, n \geq 1 \), of stochastic processes converges to \( X = (X_t)_{t \geq 0} \) in u.c.p. if for any \( t \geq 0 \)

\[ \sup_{s \in [0,t]} |X^n_s - X_s| \xrightarrow{\mathbb{P}} 0, n \to \infty. \]

Let a sequence of partitions \( D_n = \{0 = t^n_0 < t^n_1 < t^n_2 < \cdots\} = \{0 = t_0 < t_1 < t_2 < \cdots\} \) of \( [0, \infty) \) be such that for each \( t \geq 0 \) the number of points in each interval \( [0, t] \) is finite, and \( \|D_n\| := \sup_{k \geq 1} |t^n_k - t^n_{k-1}| \to 0 \) as \( n \to \infty \). A continuous process \( X \) is said to be of finite quadratic variation if there for every \( t \geq 0 \) the limit in u.c.p.

\[ \lim_{n \to \infty} \sum_{t_k \in D_n, t_k < t} (X_{t_{k+1}} - X_{t_k})^2 = [X]_t \]
exists, which is called the quadratic variation of $X$. Similarly, the quadratic covariation of two continuous processes $X$ and $Y$ is defined as a limit in u.c.p.

$$
\lim_{n \to \infty} \sum_{t_k \in D_n, t_k < t} (X_{t_{k+1}} - X_{t_k})(Y_{t_{k+1}} - Y_{t_k}) = [X,Y]_t.
$$

When $X$ and $Y$ are semimartingales, the quadratic variations $[X]$, $[Y]$ and the quadratic covariation $[X,Y]$ exist, moreover, they have bounded variation on any finite interval.

Further, we define the Itô (forward) integral as a limit in u.c.p.

$$
\int_0^t X_s \, dY_s = \lim_{n \to \infty} \sum_{t_k \in D_n, t_k < t} X_{t_k} (Y_{t_{k+1}} - Y_{t_k})
$$

and the Stratonovich (symmetric) integral as limit in u.c.p.

$$
\int_0^t X_s \circ dY_s = \int_0^t X_s \, dY_s + \frac{1}{2} [X,Y]_t
$$

and

$$
= \lim_{n \to \infty} \sum_{t_k \in D_n, t_k < t} \frac{1}{2} (X_{t_{k+1}} + X_{t_k})(Y_{t_{k+1}} - Y_{t_k}),
$$

provided that both the Itô integral and the quadratic variation exist. Again, when both $X$ and $Y$ are continuous semimartingales, both integrals exists, and the convergence holds in u.c.p. There is an alternative approach to Stratonovich stochastic integration, developed in Russo and Vallois (1993, 1995, 2000), which allows integration with respect to non-semimartingales and non-Markov processes like fractional Brownian motion.

In this paper, we define

$$
\text{sign } x = \begin{cases} 
-1, & x < 0, \\
0, & x = 0, \\
1, & x > 0, 
\end{cases}
$$

and for any $\alpha \in \mathbb{R}$ we set

$$
|x|^\alpha = \begin{cases} 
|x|^\alpha, & x \neq 0, \\
0, & x = 0.
\end{cases}
$$

With this notation, for example, for $\alpha = 0$ we have $|x|^0 = \mathbb{1}(x \neq 0)$. We also denote

$$
(x)^\alpha = |x|^\alpha \text{ sign } x.
$$

## 3 Benchmark solution

Now we turn to equation (1.2). The concept of strong solution is defined in a standard manner.

**Definition 3.1.** A strong solution to (1.2) is a continuous stochastic process $X$ adapted to the augmented natural filtration of $B$ and such that for each $t \geq 0$, the Stratonovich integral $\int_0^t |X_s|^\alpha \circ dB_s$ exist, and (1.2) holds almost surely.

Define the benchmark solution to equation (1.2) by

$$
X_t^0 = \left( (1 - \alpha)B_t + (X_0)^{1-\alpha} \right)^\frac{1}{\alpha}.
$$

The following change of variable result will be crucial for proving that it solves the equation (1.2).

**Theorem 3.2 (Föllmer et al., 1995, Theorem 4.1)).** Let $F$ be absolutely continuous with locally square integrable derivative $f$. Then

$$
F(B_t) = F(B_0) + \int_0^t f(B_s) \, dB_s + [f(B),B]_t.
$$
**Theorem 3.3.** For $\alpha \in (-1, 1)$ and $X_0 \in \mathbb{R}$, the process $X^0$ given by (3.1) is a strong solution to (1.2).

*Proof.* For each $X_0 \in \mathbb{R}$, we note that the function

$$F(x) = \left((1 - \alpha)x - (X_0)^{1-\alpha}\right)^{\frac{1}{1-\alpha}}$$

is absolutely continuous for $\alpha \in (-1, 1)$ and its derivative

$$f(x) = \left|1 - \alpha\right|x - (X_0)^{1-\alpha}\right|^{\frac{1}{1-\alpha}} = |F(x)|^\alpha$$

is locally square integrable, so by Theorem 3.2 the process $X^0 = F(B)$ satisfies (1.2).

As explained in the introduction, the benchmark solution is a unique strong solution until the time

$$\tau_0 = \inf\{t \geq 0: X^0_t = 0\} = \inf\left\{t \geq 0: B_t = -\frac{(X_0)^{1-\alpha}}{1-\alpha}\right\}.$$  

(3.2)

when it hits 0. However, the uniqueness fails after this time. Namely, with the help of Theorem 3.2 one can easily construct other strong solutions. One example is the solution killed at 0.

**Theorem 3.4.** For $\alpha \in (-1, 1)$ and $X_0 \in \mathbb{R}$, the process

$$X'_t = \left((1 - \alpha)B_t + (X_0)^{1-\alpha}\right)^{\frac{1}{1-\alpha}} \mathbb{I}_{t \leq \tau_0},$$

where $\tau_0$ is given by (3.2), is a strong solution to (1.2).

Both $X^b$ and $X'$ possess the strong Markov property, thanks to that of $B$. One can also construct an uncountable family of non-Markov solutions. Namely, for any $-A \leq 0 \leq B$, $A, B > 0$, set

$$F_{A,B}(x) = \begin{cases} 
|x + A|^{\frac{1}{1-\alpha}}, & x < -A, \\
0, & -A \leq x \leq B, \\
|x - B|^{\frac{1}{1-\alpha}}, & x > B.
\end{cases}$$

and

$$X'^{A,B}_t = F_{A,B}((1 - \alpha)B_t + (X_0)^{1-\alpha})$$

which equals to zero as long as $(1 - \alpha)B_t + (X_0)^{1-\alpha} \in [-A, B]$.

**Theorem 3.5.** For $\alpha \in (-1, 1)$ and $X_0 \in \mathbb{R}$, the process $X^{A,B}$ is a strong solution to (1.2).

### 4 Solutions spending zero time at 0

The property of spending zero time at 0 is known to be crucial to guarantee uniqueness, see e.g., Beck (1973) for the deterministic differential equations and Bass et al (2007); Aryasova and Pilipenko (2011) for the stochastic case. We will need also a concept of weak solution to (1.2).

**Definition 4.1.** A weak solution of (1.2) is a pair $(\tilde{X}, \tilde{B})$ of adapted continuous processes on a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ such that

1. $\tilde{B}$ is a standard Brownian motion on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}});
2. for any $t \geq 0$, the Itô integral $\int_0^t |\tilde{X}_s|^\alpha \, d\tilde{B}_s$ and the quadratic covariation $[|\tilde{X}|^\alpha, \tilde{B}]_t$ exist;
3. for any $t \geq 0$,

$$\tilde{X}_t = X_0 + \int_0^t |\tilde{X}_s|^\alpha \, d\tilde{B}_s,$$

holds $\mathbb{P}$-a.s.
Definition 4.2. A process $X$ is said to spend zero time at 0 if for each $t \geq 0$
\[
\int_0^t 1_{\{0\}}(X_s) \, ds = 0 \quad \text{P-a.s.}
\]

In order to define solutions to \eqref{1.2}, different from the benchmark solution \eqref{3.1} and spending zero time at 0, recall the notion of skew Brownian motion. For $\theta \in [-1, 1]$, let $B^\theta$ be the skew Brownian motion being a unique solution of the SDE
\[
B^\theta_t = B_t + \theta L_t(B^\theta), \quad (4.1)
\]
where $L(B^\theta)$ is the symmetric local time of $B^\theta$ in 0, see \cite{HarrisonShepp1981} and \cite{Lejay2006} (Section 5). Roughly speaking, the process $B^\theta$ behaves like a standard Brownian motion outside of zero. At zero its decisions to evolve in the positive or negative directions independently of anything else (the strong Markov property) with the “flipping” probabilities $\beta_\pm = \frac{1 \mp \alpha}{1 - \alpha}$. For $\theta = 0$, $B^0 \equiv B$, and for $\theta = \pm 1$, the solution to the equation \eqref{4.1} can be written explicitly, namely
\[
B^1 = (B^1_t = B_t - \min_{s \leq t} B_s)_{t \geq 0} \overset{d}{=} (|B_t|)_{t \geq 0},
\]
\[
B^{-1} = (B^{-1}_t = B_t - \max_{s \leq t} B_s)_{t \geq 0} \overset{d}{=} (-|B_t|)_{t \geq 0}.
\]

A complete account on the properties of the the Brownian motion can be found in \cite{Lejay2006}.

First we describe the law of the absolute value of weak solution that spends zero time at 0.

Theorem 4.3. Let $\alpha \in (-1, 1)$, and let $X$ be a weak solution of \eqref{1.2} started at $X_0$ such that $X$ spends zero time at 0. Then the law of the process $Z = \left(\frac{1}{1 - \alpha} |X_t|^{1 - \alpha}\right)_{t \geq 0}$ coincides with the law of a reflected Brownian motion started at $\frac{1}{1 - \alpha} |X_0|^{1 - \alpha}$.

Having the law of $|X|$ in hand we will describe all possible laws of the solution $X$ itself. Essentially we look for a process which behaves as a Brownian motion outside of 0 and spends zero time at 0. It is the skew Brownian motion $B^\theta$ which comes to mind first as an example of a process different from $B$ and $|B|$ which satisfies these conditions.

The skew Brownian motion is a homogeneous strong Markov process however it is not the unique process whose absolute value is distributed like a reflected Brownian motion. Indeed, on can construct the so-called variably skewed Brownian motion with a variable skewness parameter $\theta$: $\mathbb{R} \to (-1, 1)$ as a solution to the SDE
\[
B^{\theta}\_t = B_t + \Theta(L_t(B^\theta)), \quad t \geq 0,
\]
where $\Theta(x) = \int_0^x \theta(y) \, dy$. This is a Markov process with $|B^\theta| \overset{d}{=} |B|$ (see \cite{Barlowetal2000} Lemma 2.1)); however, if $\theta$ is non-constant, $B^\theta$ is not homogeneous Markov.

To exclude these cases we restrict ourselves to the case of homogeneous strong Markov solutions.

Theorem 4.4. Let $\alpha \in (-1, 1)$, and let $X$ be a weak solution of \eqref{1.2} such that $X$ is a strong Markov process spending zero time at 0. Then there is $\theta \in [-1, 1]$ such that
\[
X \overset{d}{=} X^\theta = \left(1 - \alpha B^\theta + (X_0)^{1 - \alpha}\right)^{\frac{1}{1 - \alpha}}
\]
for a $\theta$-skew Brownian motion $B^\theta$.

For $\theta = 1$, the skew Brownian motion $B^1$ coincides in law with the reflected Brownian motion, and hence for each $X_0 \geq 0$ the weak solution $X^1$ in non-negative.

\cite{AryasovaPilipenko2011} studied non-negative solutions of a singular SDE written in the weak form. By \cite{AryasovaPilipenko2011} Theorem 1) there exists a strong solution to equation \eqref{1.2} (in the weak form) with initial condition $X_0 \geq 0$ spending zero time at the point 0 and the strong uniqueness holds in the class of solutions spending zero time at 0. Of course it is equal to the solution $X^1$ which can be determined explicitly as
\[
X^1_t = \left(1 - \alpha \left(\min_{s \leq t} B_s\right) + X_0^{1 - \alpha}\right)^{\frac{1}{1 - \alpha}}, \quad t \geq 0,
\]
see Section 10 for details.

Finally we show the existence of strong solutions different from (5.1).

**Theorem 4.5.** 1. Let $\alpha \in (0, 1)$ and $\theta \in [-1, 1]$. Then

$$X_t^\theta = ((1 - \alpha)B_t^\theta + (X_0)^{1-\alpha})^{\frac{1}{1-\alpha}}$$

is a strong solution of (1.2) which is a homogeneous strong Markov process spending zero time at $0$.

Moreover, $X_t^\theta$ is the unique strong solution of (1.2) which is a homogeneous strong Markov process spending zero time at $0$ and such that

$$\mathbb{P}(X_t^\theta \geq 0 \mid X_0 = 0) = \beta_+ = \frac{1 + \theta}{2}, \ t > 0.$$  

2. Let $\alpha \in (-1, 0]$. Then $X_t^\theta = ((1 - \alpha)B_t + (X_0)^{1-\alpha})^{\frac{1}{1-\alpha}}$ is the unique strong solution of (1.2) which is a homogeneous strong Markov process spending zero time at $0$.

The crucial part of the proof of Theorem 4.5 is the existence of the quadratic variation $[|X^\theta|^{\alpha}, B]$ which follows from the following Theorem which is interesting on its own.

**Theorem 4.6.** Let $f \in L^2_{	ext{loc}}(\mathbb{R}^2, \mathbb{R})$ and let the $\theta$-skew Brownian motion $B^\theta$, $\theta \in (-1, 1)$, be the unique strong solution of the SDE (4.1). Then the quadratic variation

$$[f(B^\theta, B)]_t = \lim_{n \to \infty} \sum_{t_k \in B_n, t_k < t} (f(B_{t_k}^\theta, B_{t_k}) - f(B_{t_{k-1}}^\theta, B_{t_{k-1}}))(B_{t_k} - B_{t_{k-1}})$$

exists as a limit in u.c.p. Moreover, let $\{f_n\}_{n \geq 1}$ be a sequence of continuous functions such that for each compact $K \subset \mathbb{R}^2$

$$\lim_{n \to \infty} \int_K |f_n(x, y) - f(x, y)|^2 \, dx \, dy = 0. \quad (4.2)$$

Then

$$[f_n(B^\theta, B), B]_t \to [f(B^\theta, B), B]_t$$

in u.c.p.

The proof of this Theorem combines the approach by Föllmer et al. [1995] with the time reversal technique from Haussmann and Pardoux [1985, 1986]. In the case $\theta \in \{-1, 1\}$, where the time reversal is no longer available, we give a direct proof of Theorem 4.5.

5 **On the relation between the Stratonovich and Itô equations**

The Stratonovich SDE (1.2) can be formally written as an Itô SDE with singular coefficients

$$X_t = X_0 + \int_0^t |X_s|^{\alpha} \, dB_s + \frac{\alpha}{2} \int_0^t (X_s)^{2\alpha - 1} \, ds. \quad (5.1)$$

Let us check whether the process $X_t^\theta$ satisfies this equation. For definiteness, we set $X_0 = 0$.

In order to be able to substitute $X_t^\theta$ into (5.1) we have to guarantee that the both summands of the SDE (5.1) are well defined. Hence, for the existence of the Itô integral we need

$$\int_0^t |X_s|^{2\alpha} \, ds = \mathbb{E} \int_0^t |W_s|^{2\alpha} \, ds < \infty \quad \text{a.s.,} \ (5.2)$$

and for the existence of the drift term we need

$$\int_0^t |X_s|^{2\alpha - 1} \, ds = \mathbb{E} \int_0^t |W_s|^{2\alpha - 1} \, ds < \infty \quad \text{a.s.} \quad (5.3)$$
The Engelbert–Schmidt zero-one law ([Engelbert and Schmidt, 1981, Theorem 1]) implies that for a Borel \( \Phi: \mathbb{R} \to [0, +\infty] \)
\[
P\left( \int_0^t \Phi(W_s) \, ds < \infty, \quad \forall t \geq 0 \right) = 1 \iff \Phi \in L^1_{\text{loc}}(\mathbb{R})
\]
and hence (5.2) is satisfied for all \( \alpha > -1 \) and the drift term (5.3) exists for \( \alpha > 0 \). This indicates that \( X^\theta \) is a solution of (5.1) for \( \theta \in [-1, 1] \) and \( \alpha \in (0, 1) \).

To extend the existence result to \( \alpha \in (-1, 0] \), we will consider the drift term in the principal value sense:

\[
\text{v.p.} \int_0^t (W_s)^{\frac{\alpha+1}{2}} \, ds := \lim_{\varepsilon \to 0} \int_0^t (W_s)^{\frac{\alpha+1}{2}} \cdot I(|W_s| > \varepsilon) \, ds.
\] (5.4)

The principal value definition is intrinsically based on the symmetry of the Brownian motion and the asymmetry of the integrand and hence excludes the cases \( \theta \neq 0 \). Necessary and sufficient conditions for the existence of Brownian principal value integrals are given in ([Cherny, 2001, Theorem 3.1, p. 352]). In particular, the integral (5.4) is finite if and only if \( \alpha > -1 \).

This yields that for \( \alpha \in (-1, 0] \), \( X^0 \) is the solution of the Itô SDE

\[
X_t = X_0 + \int_0^t |X_s|^\alpha \, dB_s + \frac{\alpha}{2} \cdot \text{v.p.} \int_0^t (X_s)^{2\alpha-1} \, ds.
\] (5.5)

In their book, [Cherny and Engelbert, 2005] consider singular SDE in the sense of existence of the absolute integrals (5.2) and (5.3). Since they also assume that the diffusion coefficient does not vanish at 0, it follows from Chapter 5, that for \( \alpha \leq 0 \) the SDE (5.1) has a unique solution for \( X_0 \neq 0 \) which cannot be extended after it hits 0, and it has no solution starting at 0. Since we always assume that the diffusion coefficient and the drift are zero at 0, our solution should stick to 0 after hitting it. This behaviour seems to contradict the fact that \( X \) is a solution to the Stratonovich equation for \( \alpha \in (-1, 1) \) due to Theorem 3.2. This contradiction is resolved by taking into account the fact that for \( \alpha \in (-1, 0] \) the noise-induced drift has to be understood in the principal value sense (5.4).

6 Proof of Theorem 4.3

We use the following characterization of the reflected Brownian motion, see [Varadhan, 2011].

**Proposition 6.1.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a filtered probability space. A continuous stochastic process \( Z \) is a reflected Brownian motion started at \( x \) if and only if

1. \( Z_0 = x \) a.s.;

2. \( Z \) behaves locally like a Brownian motion on \((0, \infty)\), i.e. for any bounded smooth function \( f \) vanishing in some neighbourhood of 0, the process

\[
f(Z_t) - f(x) - \frac{1}{2} \int_0^t f''(Z_s) \, ds
\]

is a martingale;

3. \( E \int_0^\infty \mathbb{I}_{(0)}(Z_s) \, ds = 0. \)

For the proof of Theorem 4.3 we will need a variant of the change of variables formula for functions vanishing in a neighbourhood of the irregular point of the SDE.
Lemma 6.2. Let \( \varphi \in C^1(\mathbb{R}\setminus\{0\}) \) and let \((X, B)\) be a weak solution of the SDE
\[
X_t = x + \int_0^t \varphi(X_s) \circ dB_s := x + \int_0^t \varphi(X_s) dB_s + \frac{1}{2} \langle \varphi(X), B \rangle_t.
\]
Then for any \( g \in C^2(\mathbb{R}, \mathbb{R}) \) which vanishes in a neighbourhood of zero we have
\[
g(X_t) = g(X_0) + \int_0^t g'(X_s) \varphi(X_s) dB_s + \frac{1}{2} \int_0^t \left( g''(X_s) \varphi^2(X_s) + g'(X_s) \varphi'(X_s) \right) ds
\quad (6.1)
\]
The proof of this Lemma essentially follows the lines of the proof of the classical Itô formula for Itô processes and is postponed to Appendix ??.

Eventually, we prove Theorem 7.1. Let \((X, B)\) be a weak solution of the SDE (1.2) spending zero time at 0. We consider the process
\[
Z_t = \frac{1}{1-\alpha} |X_t|^{1-\alpha}, \quad t \geq 0.
\]
which starts at \(Z_0 = \frac{1}{1-\alpha} |X_0|^{1-\alpha}\) and also spends zero time at 0.

Let \( f \in C^2_b(\mathbb{R}_+, \mathbb{R}_+) \) and let \( f \) be zero in a neighbourhood of 0. The function \( g(x) = f\left( \frac{1}{1-\alpha} |x|^{1-\alpha} \right) \) is also twice continuously differentiable and bounded, and is zero in a neighbourhood of 0, and
\[
\begin{align*}
g'(x) &= f'(z)(x)^{-\alpha}, \\
g''(x) &= f''(z)|x|^{-2\alpha} - \alpha f'(z)|x|^{-\alpha-1}, \quad z = \frac{1}{1-\alpha} |x|^{1-\alpha}.
\end{align*}
\]
Then Lemma 6.2 immediately yields
\[
f(Z_t) = g(X_t) = g(X_0) + \int_0^t g'(X_s)|X|^{\alpha} dB_s
\]
\[
+ \frac{1}{2} \int_0^t \left( g''(X_s)|X_s|^{2\alpha} + \alpha g'(X_s)(X_s)^{2\alpha-1} \right) ds
\]
\[
= f(Z_0) + \int_0^t f'(Z_s) \operatorname{sign} X_s dB_s + \frac{1}{2} \int_0^t f''(Z_s) ds,
\]
so that the process
\[
t \mapsto f(Z_t) - f(Z_0) - \frac{1}{2} \int_0^t f''(Z_s) ds \quad (6.2)
\]
is a martingale.

7 Proof of the Theorem 4.4

Let \( X \) be a solution of the SDE (1.2) spending zero time at 0. Then
\[
\frac{1}{1-\alpha} |X_t|^{1-\alpha} \overset{d}{=} |W - \frac{X_0}{1-\alpha}|,
\]
i.e. is a reflected Brownian motion starting at \( \frac{X_0}{1-\alpha} \). The statement of the Theorem will follow from the following Proposition.

Proposition 7.1. Let \( Y \) be a continuous one-dimensional strong Markov process starting at \( y \in \mathbb{R} \) such that \( \|Y\| \overset{d}{=} |W - y| \), \( W \) being a standard Brownian motion. Then there is \( \theta \in [-1, 1] \) such that \( Y \overset{d}{=} B^\theta \), where \( B^\theta \) is the \( \theta \)-skew Brownian motion starting at \( y \).

Proof. Since for any \( \theta \in [-1, 1] \), \( Y \overset{d}{=} W + y \overset{d}{=} B^\theta \) before the first hitting time of 0, it is sufficient to consider the case of the initial starting point \( y = 0 \).
Denote for \( a < 0 < b \)
\[
\tau_{(a,b)} = \inf\{t \geq 0: Y_t \notin (a,b)\}
\]
and show that the probability
\[
p_+(\varepsilon) = P_0(Y_{\tau_{(-\varepsilon, \varepsilon)}} = \varepsilon), \quad \varepsilon > 0,
\]
does not depend on \( \varepsilon \).

Indeed, if \( p_+(\varepsilon) = 0 \) or \( p_+(\varepsilon) = 1 \) for all \( \varepsilon > 0 \), then the statement holds true.

Assume that there is \( \varepsilon > 0 \) such that \( p_+(\varepsilon) = \beta_+ \in (0, 1) \).

Let \( 0 < \varepsilon < \varepsilon' \), then
\[
p_+(\varepsilon') = P_0\left(Y_{\tau_{(-\varepsilon', \varepsilon')}} = \varepsilon' \mid Y_{\tau_{(-\varepsilon, \varepsilon)}} = \varepsilon\right) p_+(\varepsilon) \\
+ P_0\left(Y_{\tau_{(-\varepsilon', \varepsilon')}} = \varepsilon' \mid Y_{\tau_{(-\varepsilon, \varepsilon)}} = -\varepsilon\right)(1 - p_+(\varepsilon)) \\
= P_\varepsilon(Y_{\tau_{(-\varepsilon', \varepsilon')}} = \varepsilon') p_+(\varepsilon) + P_{-\varepsilon}(Y_{\tau_{(-\varepsilon', \varepsilon')}} = \varepsilon')(1 - p_+(\varepsilon)).
\]

Note that
\[
P_\varepsilon(Y_{\tau_{(-\varepsilon', \varepsilon')}} = \varepsilon') = P_\varepsilon(Y_{\tau_{(0, \varepsilon')}} = \varepsilon') + P_\varepsilon(Y_{\tau_{(0, \varepsilon')}} = 0)p_+(\varepsilon') \\
= \frac{\varepsilon}{\varepsilon'} + \left(1 - \frac{\varepsilon}{\varepsilon'}\right) p_+(\varepsilon').
\]

Analogously
\[
P_{-\varepsilon}(Y_{\tau_{(-\varepsilon', \varepsilon')}} = \varepsilon') = P_{-\varepsilon}(Y_{\tau_{(-\varepsilon', 0)}} = 0)p_+(\varepsilon') = \left(1 - \frac{\varepsilon}{\varepsilon'}\right) p_+(\varepsilon').
\]

Hence we obtain that
\[
p_+(\varepsilon') = \frac{\varepsilon}{\varepsilon'} p_+(\varepsilon) + \left(1 - \frac{\varepsilon}{\varepsilon'}\right) p_+(\varepsilon') p_+(\varepsilon) + \left(1 - \frac{\varepsilon}{\varepsilon'}\right) p_+(\varepsilon')(1 - p_+(\varepsilon)) = p_+(\varepsilon) = \beta_+.
\]

Let now \( 0 < \varepsilon' < \varepsilon \). Due to the continuity of the paths of \( Y \), \( p_+(\varepsilon') > 0 \), so repeating the previous argument with \( \varepsilon \) and \( \varepsilon' \) interchanged we eventually obtain that \( p_+(\varepsilon) = \beta_+ \) for all \( \varepsilon > 0 \).

Since \( Y \) is a continuous strong Markov process its law is uniquely determined by the Dynkin characteristic operator
\[
\mathfrak{A} f(x) := \lim_{U \downarrow x} \frac{E_x f(Y_{\tau(U)}) - f(x)}{E_x \tau(U)},
\]
where \( U \) is a bounded open interval containing \( x \), see [Dynkin 1965, Chapter 5 §3].

Choosing \( U = U_{\varepsilon} = (x - \varepsilon, x + \varepsilon) \), a straightforward calculation yields that for \( x \neq 0 \) and \( f \) being twice continuously differentiable at \( x \)
\[
\mathfrak{A} f(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2} \left( f(x + \varepsilon) + f(x - \varepsilon) - 2f(x) \right) = \frac{1}{2} f''(x).
\]

For \( x = 0 \) the limit
\[
\mathfrak{A} f(0) = \lim_{\varepsilon \downarrow 0} \frac{\beta_+ f(\varepsilon) + (1 - \beta_+) f(-\varepsilon) - f(0)}{\varepsilon^2} = \frac{1}{2} \left( \beta_+ f''(0+) + (1 - \beta_+) f''(0-) \right)
\]
exists for any continuous \( f \) such that \( \beta_+ f'(0+) = (1 - \beta_+) f'(0-) \) and \( f''(0+) \) and \( f''(0-) \) exist and \( f''(0+) = f''(0-) \).

Hence \( \mathfrak{A} \) coincides with the generator of the \( \theta \)-skew Brownian motion with \( \theta = 2\beta_+ - 1 \) (see [Lejay 2006]).

8 Proof of Theorem 4.6

Define
\[
\sigma(x) = \frac{2}{1 + \theta \text{sign} x} \quad \text{and} \quad \beta(x) = \frac{1}{\sigma(x)}.
\]
Let $Y^\theta$ be the unique strong solution of the SDE

$$Y^\theta_t = u + \int_0^t \sigma(Y^\theta_s) \, dB_s, \quad u \in \mathbb{R},$$

(8.1)

and consider the two-dimensional Markov process $(Y^\theta, B)$ with the law $P_{u,w} := \text{Law}(Y^\theta, B)|Y_0 = u, B_0 = w).$

The skew Brownian motion with parameter $\theta$, starting from $w_0 \in \mathbb{R}$ and driven by a Brownian motion $B$ is the unique strong solution to the following stochastic differential equation

$$B^\theta_t = w_0 + (B_t - w) + \theta L_t(B^\theta),$$

(8.2)

where $L_t(B^\theta)$ denotes the symmetric local time of $B^\theta$ at zero. The values $\beta_+ = \frac{1+\theta}{2}$ and $\beta_- = \frac{1-\theta}{2}$ can be interpreted as the probabilities of positive and negative values of the skew Brownian motion $B^\theta$ started at $w_0 = 0$.

Further, define the functions

$$r(x) = \frac{x}{\sigma(x)} = x\beta(x) = \begin{cases} \frac{1+\theta}{2}x, & x \geq 0, \\ \frac{1-\theta}{2}x, & x < 0, \end{cases} \quad s(x) = x\sigma(x) = \frac{x}{\beta(x)},$$

then $s(r(x)) = x$. The application of the Itô–Tanaka formula (compare with [Lejay 2006, Section 5.2]) yields

$$r(Y^\theta_t) = r(u) + B_t - w + \frac{\theta}{2}L_t(Y^\theta_t)$$

$$= r(u) + B_t - w + \theta L_t(B) = r(u) - w_0 + B^\theta_t.$$ (8.3)

In the following lemma we will use the functional dependence (8.3) of the processes $(Y^\theta, B)$ and $(B^\theta, L(B^\theta))$ to determine the marginal density of the pair $(Y^\theta_t, B_t)$.

**Lemma 8.1.** For $\theta \in (-1,1) \setminus \{0\}$, $t > 0$, the joint distribution of $Y^\theta_t$ and $B_t$ given $Y^\theta_0 = u, B_0 = w$ is

$$P_{u,w}(Y^\theta_t \in dy, B_t \in dz) = \frac{2\beta^2(y)}{\theta^2 \sqrt{2\pi t^3}} \left(2y\beta^2(y) - \kappa u - z + w\right)$$

$$\times \exp \left(-\frac{1}{2\theta^2 t} \left(2y\beta^2(y) - \kappa u - z + w\right)^2\right) \, dz \, dy,$$

(8.4)

where $\kappa = \frac{1}{2}(1-\theta^2)$, if $\theta^{-1}(r(y) - r(u) - z + w) > 0$, and

$$P_{u,w}(Y^\theta_t \in dy, B_t = w + r(y) - r(u))$$

$$= \frac{\beta(u)}{|\theta| \sqrt{2\pi t^3}} \left(e^{-\frac{(r(y) - r(u))^2}{2t}} - e^{-\frac{(r(y) + r(u))^2}{2t}}\right) \cdot 1_{|y| > 0} \, dy.$$ (8.5)

In particular, the joint density of $(Y^\theta_t, B_t)$ provided that $Y_0 = u = 0, B_0 = w = 0$ is

$$p(t, y, z) = \frac{2\beta^2(y)}{\theta^2 \sqrt{2\pi t^3}} \left(2y\beta^2(y) - z\right) \exp \left(-\frac{1}{2\theta^2 t} \left(2y\beta^2(y) - z\right)^2\right) \cdot 1_{|y| > 0, (r(y) - z) > 0}.$$ (8.6)

**Proof.** First we determine the joint distribution of $B^\theta_t$ and $L_t(B^\theta)$, $t > 0$. We note that the pair $(|B^\theta|, L(B^\theta))$ has the same distribution as $(|B|, L(B))$ whose marginal law can be easily recalculated from the joint distribution of $(|B_t|, L(B)|$ given in [Borodin and Salminen 2002, formulae 3.1.3.8 (1) and (2), p. 335], namely for $w \in \mathbb{R}, t > 0$,

$$\text{P}\left(|B_t| \in db, L_t(B) \in dl \bigg| |B_0| = |w|\right) = \frac{2|w| + b + l}{\sqrt{2\pi t^3}} e^{-\frac{|w| + b + l}{2t}} db \, dl,$$ for $b, l > 0$.

$$\text{P}\left(|B_t| \in db, L_t(B) = 0 \bigg| |B_0| = |w|\right) = \frac{1}{\sqrt{2\pi t^3}} \left(e^{-\frac{|w|}{2t}} - e^{-\frac{|w|}{2t}}\right) db.$$
Then we note that before hitting zero (i.e. on the event $L_t(B^\theta) = 0$), the skew Brownian motion $B^\theta_t$ coincides in law with $B_t$, which leads to

$$\mathbf{P}\left(B^\theta_t \in db, L_t(B) = 0 \mid B_0^\theta = w_0\right) = \frac{1}{\sqrt{2\pi t^3}} \left( e^{-\frac{(b-w_0)^2}{2t}} - e^{-\frac{(b+w_0)^2}{2t}} \right) 1_{|w_0| > 0} db. \quad (8.7)$$

On the other hand, after having hit zero (i.e. on the event $L_t(B^\theta) > 0$), the skew Brownian motion $B^\theta_t$ chooses its sign with probabilities $\beta_\pm = \frac{1}{1+\theta}$ independently of anything else, and hence $B^\theta_t \overset{d}{=} |B_t| \cdot Z$, $Z$ being an independent Bernoulli random variable with $\mathbf{P}(Z = \pm 1) = \beta_\pm$. This immediately leads to the following formula for the density:

$$\mathbf{P}\left(B^\theta_t \in db, L_t(B^\theta) \in dl \mid B_0^\theta = w_0\right) = 2\beta_+ \cdot \mathbf{1}_{b > 0} \frac{l + |w_0| + |b|}{\sqrt{2\pi t^3}} e^{-\frac{(l+|w_0|+|b|)^2}{2t}} dz \, dl$$

$$+ 2\beta_- \cdot \mathbf{1}_{b < 0} \frac{l + |w_0| + |b|}{\sqrt{2\pi t^3}} e^{-\frac{(l+|w_0|+|b|)^2}{2t}} dz \, dl$$

$$= 2\beta(b) \cdot \frac{l + |w_0| + |b|}{\sqrt{2\pi t^3}} e^{-\frac{(l+|w_0|+|b|)^2}{2t}} dz \, dl, \quad b \in \mathbb{R}, \, l > 0, \quad (8.8)$$

where in the last equality we redefined the value of the density at $b = 0$ for convenience. The formula $(8.8)$ can be also found in Appuhamillage et al. [2011].

Recall now that

$$B^\theta_t = r(Y^\theta_t) - r(u) + w_0,$$

$$L_t(B^\theta) = \frac{1}{\theta} \left( r(Y^\theta_t) - r(u) - B_t + w \right).$$

The initial condition of $u = Y^\theta_0$ given, let us fix $w_0 = r(u)$, so that $B^\theta_t = r(Y^\theta_t)$ for $t \geq 0$. Then the change of variables $b = b(y, z), \, l = l(y, z)$,

$$b = r(y), \quad l = \frac{1}{\theta} \left( r(y) - r(u) - z + w \right), \quad (8.9)$$

yields

$$l + |w_0| + |b| = \frac{1}{\theta} \left( r(y) - r(u) - z + w \right) + |r(u)| + |r(y)|$$

$$= r(y) + \theta r(y) - r(u) + \theta |r(u)| - z + w$$

$$= r(y)(1 + \theta \text{sign } y) - r(u)(1 - \theta \text{sign } u) - z + w$$

$$= \frac{2y\beta^2(y)}{\theta} - \frac{u(1 - \theta^2)}{2\theta} - z - w$$

$$= \frac{\theta}{\theta} \text{det } J \quad (8.10)$$

where we made use of the relation $|r(y)| = r(y) \text{sign } y$. For $y, u \neq 0$, the Jacobian for the change of variables $(y, z) \rightarrow (b, l)$ given by $(8.9)$ and its determinant equal

$$J = \begin{pmatrix} \beta(y) & 0 \\ \beta(y) & -1/\theta \end{pmatrix}, \quad |\text{det } J| = \frac{\beta(y)}{|\theta|},$$

whence, noting that $\sigma(b) = \sigma(y)$, we get $(8.3)$. Similarly, we have $(8.5)$. The remaining formula $(8.6)$ follows by plugging in $w = 0$. \hfill \Box

From now on we assume without loss of generality that $\theta \in (0, 1)$. Let all the processes under consideration will be started at zero, $u = w = w_0 = 0$, so that $Y^\theta = B^\theta/\beta(B^\theta)$.

Note that for any $t > 0$

$$r(Y^\theta_t) - B_t = B^\theta_t - B_t = \theta L_t(B^\theta) > 0,$$

$$2Y^\theta_t \beta^2(Y^\theta_t) - B_t = 2\beta(B^\theta_t)B^\theta_t - B_t = (1 + \theta \text{sign } B^\theta_t)B^\theta_t - B_t$$

$$= B^\theta_t - B_t + \theta B^\theta_t \text{sign } B^\theta_t \text{sign } B^\theta_t \theta L_t(B^\theta) + \theta |B^\theta_t| > 0. \quad (8.11)$$
Our aim now is to prove a generalized Itô formula for $B^\theta$, in the spirit of Föllmer et al. (1995). Towards this end, on a fixed time interval $[0, T]$ we first establish a stochastic differential equation for the time-reversed pair $(\bar{Y}^\theta_t, B_t) = (\bar{Y}^\theta_{T-t}, B_{T-t}).$ We follow the method developed by Haussmann and Pardoux (1985, 1986) for Markovian diffusions. For $y \neq 0$ and $z < r(y), s \in [0, T]$ define the functions

\[
\bar{b}^\theta(s, y, z) = \frac{1}{p(T - s, y, z)} \left( \sigma^2(y) \frac{\partial^2 p}{\partial y^2}(T - s, y, z) + \sigma(y) \frac{\partial p}{\partial y}(T - s, y, z) \right),
\]

\[
\bar{b}^\sigma(s, y, z) = \frac{1}{p(T - s, y, z)} \left( \sigma(y) \frac{\partial^2 p}{\partial y}(T - s, y, z) + \frac{\partial p}{\partial z}(T - s, y, z) \right) = \frac{\bar{b}^\theta(s, y, z)}{\sigma(y)},
\]

and set $\bar{b}^\theta(s, 0, z) = \bar{b}^\sigma(s, 0, z) = \bar{b}^{\theta}(T, y, z) = \bar{b}^{\sigma}(T, y, z) = 0.$ Noting that

\[
\frac{\partial^2 p}{\partial z^2}(T - s, y, z) = \frac{2\beta^2(y)}{\theta^2(2\pi(T - s))^2} \exp \left( - \frac{(2y\beta^2(y) - z)^2}{2\theta^2(T - s)} \right) \left( 1 + \frac{(2y\beta^2(y) - z)^2}{\theta^2(T - s)} \right), z < r(y),
\]

and $\frac{\partial^2}{\partial z^2}(T - s, y, z) = -2\beta^2(y)\frac{\partial^2 p}{\partial y}(T - s, y, z),$ we get

\[
\bar{b}^\theta(s, y, z) = \frac{2\beta(y) - 1}{\beta(y)} \left( \frac{1}{2y\beta^2(y) - z} - \frac{2y\beta^2(y) - z}{\theta^2(T - s)} \right)
\]

\[
= \frac{\theta \sign y}{\beta(y)} \left( \frac{1}{2y\beta^2(y) - z} - \frac{2y\beta^2(y) - z}{\theta^2(T - s)} \right), \quad z < r(y).
\]

Proposition 8.2. Let for $\theta \in (-1, 1) \setminus \{0\}$, $B^\theta$ be a solution of (8.12) started at 0. Then for any $T > 0$, $(\bar{Y}^\theta_t, B_t) = (\bar{Y}^\theta_{T-t}, B_{T-t})$ is a weak solution to the stochastic differential equation

\[
\bar{Y}^\theta_t = Y^\theta_T + \int_0^t \bar{b}^\theta(s, \bar{Y}^\theta_s, B_s) \, ds + \int_0^t \sigma(\bar{Y}^\theta_s) \, dW_s,
\]

\[
\bar{B}_t = B_T + \int_0^t \bar{b}^\sigma(s, Y^\theta_s, B_s) \, ds + W_t, \quad t \in [0, T],
\]

$W$ being a standard Brownian motion.

Remark 8.3. a) Thanks to (8.11), the coefficients of (8.13) are well defined.

b) Equation (8.13) is very degenerate, and $\bar{Y}^\theta$ and $\bar{B}$ evolve proportionally whenever $\bar{Y}^\theta_t \neq 0.$ This, however, will not hinder our analysis.

Proof. As above, we continue to assume without loss of generality that $\theta \in (0, 1)$ and $T = 1$. We need to show that $(\bar{Y}^\theta_t, \bar{B}_t)$ is a solution to the martingale problem with the generator

\[
\bar{L}_t f(y, z) = \left( \bar{b}^\theta(t, y, z) \frac{\partial}{\partial y} + \bar{b}^\sigma(t, y, z) \frac{\partial}{\partial y} + \frac{\sigma(y)^2}{2} \frac{\partial^2}{\partial y^2} + \sigma(y) \frac{\partial^2}{\partial y \partial z} + \frac{1}{2} \frac{\partial^2}{\partial z^2} \right) f(y, z).
\]

Thanks to (8.11), it is enough to establish

\[
\mathbb{E} \left[ \left( f(\bar{Y}^\theta_t, \bar{B}_t) - f(\bar{Y}^\theta_s, \bar{B}_s) - \int_s^t \bar{L}_u f(\bar{Y}^\theta_u, \bar{B}_u) \, du \right) \cdot g(\bar{Y}^\theta_s, \bar{B}_s) \right] = 0
\]

for any $0 \leq s < t < 1$ and functions $f, g \in C^2(\mathbb{R}^2)$ having compact support inside the domain $D := \{(y, z) \in \mathbb{R}^2; r(y) - z > 0\}$. Equivalently,

\[
\mathbb{E} \left[ \left( f(Y^\theta_t, B_t) - f(Y^\theta_s, B_s) + \int_s^t \bar{L}_{T-u} f(Y^\theta_u, B_u) \, du \right) \cdot g(Y^\theta_t, B_t) \right] = 0
\]

for any fixed $0 < s < t \leq 1$. Define for $(y, z) \in \mathbb{R}^2$

\[
v(s, y, z) = \mathbb{E} \left[ g(Y^\theta_t, B_t) \right] (Y^\theta_s, B_s) = (y, z).
\]
It is proved in Appendix A that \( v \) solves the partial differential equation
\[
\left( \frac{\partial}{\partial t} + \mathcal{L} \right) v(s, y, z) = 0,
\]
where
\[
\mathcal{L} f(y, z) = \left( \frac{\sigma(y)^2}{2} \frac{\partial^2}{\partial y^2} + \sigma(y) \cdot \frac{\partial^2}{\partial y \partial z} + \frac{1}{2} \frac{\partial^2}{\partial z^2} \right) f(y, z).
\]

Denote by \( \langle \cdot, \cdot \rangle \) the scalar product in \( L^2(\mathbb{R}^2) \). Write
\[
E[f(Y_t^\theta, B_t)g(Y_t^\theta, B_t)] - E[f(Y_s^\theta, B_s)g(Y_s^\theta, B_s)]
= \int_s^t \left( \frac{\partial}{\partial u} p(u), v(u) \right) \, du + \int_s^t \left( \frac{\partial}{\partial u} v(u), p(u) \right) \, du
= \int_s^t \left( f \frac{\partial}{\partial u} p(u), v(u) \right) \, du - \int_s^t \left( f p(u), \mathcal{L} v(u) \right) \, du.
\]

Using (A.2) from the Appendix A we get
\[
\langle f p(u), \mathcal{L} v(u) \rangle = \langle v(u), \mathcal{L} (f p(u)) \rangle.
\]

Write
\[
p(u, y, z) = \frac{2\beta^2(y)}{\theta} \varphi_u'(\frac{z - 2y\beta^2(y)}{\theta}) 1_{z < r(y)}, \tag{8.14}
\]
where \( \varphi_t(x) = \frac{1}{\sqrt{2\pi} e^{-\frac{x^2}{2}}} \) is the standard Gaussian density. Now we have
\[
\mathcal{L} (f p(u))(y, z) = f(y, z) \cdot \mathcal{L} p(u, y, z) + p(u, y, z) \cdot \mathcal{L} f(y, z)
+ \left( \sigma(y) \cdot \frac{\partial}{\partial y} f(y, z) + \frac{\partial}{\partial z} f(y, z) \right) \left( \sigma(y) \cdot \frac{\partial}{\partial y} p(u, y, z) + \frac{\partial}{\partial z} p(u, y, z) \right)
\]
From (8.14), for \( y \neq 0, z < r(y) \),
\[
\mathcal{L} p(u, y, z) = \frac{4\beta^4(y)}{\theta^3} - \frac{4\beta^3(y)}{\theta^3} + \frac{\beta^2(y)}{\theta^3} \cdot \varphi_u''(\frac{z - 2y\beta^2(y)}{\theta})
= \frac{\beta^2(y)}{\theta^3} \cdot (2\beta(y) - 1)^2 \cdot \varphi_u''(\frac{z - 2y\beta^2(y)}{\theta}) = \frac{\beta^2(y)}{\theta} \cdot \varphi_u''(\frac{z - 2y\beta^2(y)}{\theta}).
\]

On the other hand, since \( \frac{\partial}{\partial u} \varphi_t = \frac{1}{\sqrt{2\pi} e^{-\frac{x^2}{2}}} \), we get
\[
\mathcal{L} p(u, y, z) = \frac{\partial}{\partial u} p(u, y, z).
\]

Further, denote
\[
h(u, y, z) = \sigma(y) \cdot \frac{\partial}{\partial y} p(u, y, z) + \frac{\partial}{\partial z} p(u, y, z).
\]

Then we have
\[
\langle v(u), \mathcal{L} (f p(u)) \rangle = \langle v(u), f \frac{\partial}{\partial u} p(u) \rangle + \langle v(u), p(u) \mathcal{L} f \rangle + \langle v(u), h(u) \left( \sigma(y) \cdot \frac{\partial}{\partial y} f + \frac{\partial}{\partial z} f \right) \rangle.
\]

Observe that
\[
\langle v(u), p(u) \mathcal{L} f \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v(u, y, z) p(u, y, z) \mathcal{L} f(y, z) \, dz \, dy
= E[v(u, Y_t^\theta, B_t) \mathcal{L} f(Y_t^\theta, B_t)] = E[E[g(Y_t^\theta, B_t) | Y_u^\theta, B_u] \mathcal{L} f(Y_u^\theta, B_u)]]
= E[g(Y_t^\theta, B_t) \mathcal{L} f(Y_u^\theta, B_u)]
\]
and similarly
\[
\left\langle v(u), h(u) \cdot \left( \sigma(y) \cdot \frac{\partial}{\partial y} f + \frac{\partial}{\partial z} f \right) \right\rangle = \left\langle v(u), p(u) \cdot \frac{h(u)}{p(u)} \cdot \left( \sigma(y) \cdot \frac{\partial}{\partial y} f + \frac{\partial}{\partial z} f \right) \right\rangle
\]
\[
= \left\langle v(u), p(u) \left( \tilde{b}^x(T - u) \frac{\partial}{\partial y} f + \tilde{b}^z(T - u) \frac{\partial}{\partial z} f \right) \right\rangle
\]
\[
= \mathbb{E} \left[ g(Y^0_t, B_t) \left( \tilde{b}^x(T - u, Y_u, B_u) \frac{\partial}{\partial y} f(Y^0_{u}, B_u) + \tilde{b}^z(T - u, Y_u, B_u) \frac{\partial}{\partial z} f(Y^0_{u}, B_u) \right) \right],
\]
Collecting everything,
\[
\mathbb{E}[f(Y^0_t, B_t)g(Y^0_t, B_t)] - \mathbb{E}[f(Y^0_s, B_s)g(Y^0_t, B_t)] = -\mathbb{E}\left[ g(Y^0_t, B_t) \int_s^t \mathcal{L}f(Y^0_s, B_s) \, ds \right]
\]
\[
- \mathbb{E}\left[ g(Y^0_t, B_t) \int_s^t \left( \tilde{b}^x(T - u, Y_u^0, B_u) \frac{\partial}{\partial y} f(Y^0_{u}, B_u) + \tilde{b}^z(T - u, Y_u^0, B_u) \frac{\partial}{\partial z} f(Y^0_{u}, B_u) \right) \, du \right]
\]
\[
= -\mathbb{E}\left[ g(Y^0_t, B_t) \int_s^t \mathcal{L}_{T-u} f(Y^0_s, B_s) \, du \right],
\]
as required.

Consider a sequence of partitions $D_n$ of the form $0 = t_0^n < t_1^n < \cdots < t_n^n = T$ with $|D_n| = \max_{1 \leq k \leq n} (t_{k+1}^n - t_k^n) \to 0$, $n \to \infty$ (we will often omit the superscript $n$).

**Proof of Theorem 4.6.** Note that $f(B^0_t, B_t) = f(r(Y^0_t), B_t) = g(Y^0_t, B_t)$ with $g \in L^2_{\text{loc}}(\mathbb{R}^2)$, so it suffices to establish a similar statement for $Y^0$. The rest of proof goes similarly to Föllmer et al. (1995).

First note that by usual localization argument, we can assume that $g \in L^2(\mathbb{R}^2)$. Also for a continuous function $h$, the quadratic variation
\[
[h(Y^0), B]_t = \lim_{M \to \infty} \sum_{k=1}^{n} \left( h(Y^0_{t_{k-1}}, B_{t_{k-1}}) - h(Y^0_{t_k}, B_{t_{k-1}}) \right) (B_{t_k} - B_{t_{k-1}})
\]
extists as a limit in u.c.p. Indeed, since $B$ is a semimartingale, and $h(Y^0, B)$ is an adapted continuous process, then by Protter (2004, Theorem 21, p. 64),
\[
\lim_{t_k \in D_n, t_{k-1} \leq t} \sum_{k=1}^{n} h(Y^0_{t_{k-1}}, B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}}) = \int_0^t h(s, B_s) \, dB_s
\]
in u.c.p. The time-reversed process is also a semimartingale, so arguing as in Föllmer et al. (1997), we have the convergence to the backward integral
\[
\lim_{t_k \in D_n, t_{k-1} \leq t} \sum_{k=1}^{n} h(Y^0_{t_k}, B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}}) = \int_0^t h(s, B_s) \, dB_s
\]
in u.c.p. Therefore, we obtain
\[
[h(Y^0, B), B]_t = \int_0^t h(Y^0_s, B_s) \, dB_s - \int_0^t h(Y^0_s, B_s) \, dB_s.
\]
Fix some $T > 0$ and let now $\{h_m\}_{m \geq 1}$ be a sequence of continuous functions, such that $h_m \to g$ in $L^2(\mathbb{R}^2)$ as $m \to \infty$. Denote
\[
I_m(t) := \int_0^t \left( h_m(Y^0_s, B_s) - g(Y^0_s, B_s) \right) \, dB_s,
\]
\[
S_{n,m}(t) := \sum_{t_k \in D_n, t_{k-1} \leq t} \left( h_m(Y^0_{t_{k-1}}, B_{t_{k-1}}) - g(Y^0_{t_{k-1}}, B_{t_{k-1}}) \right) (B_{t_k} - B_{t_{k-1}}).
\]
It is easy to see that the density \( p(t, y, z) \) of \( (Y^\theta_t, B_t) \) satisfies \( p(t, y, z) \leq \frac{C}{\sqrt{t}} \) for some \( C > 0 \), so we can estimate, using the Doob inequality that

\[
E \sup_{t \in [0,T]} I^2_m(t) \leq C \int_0^T E \left[ (h_m(Y^\theta_t, B_t) - g(Y^\theta_t, B_t))^2 \right] dt \\
= \int_0^T \int_\infty^{-\infty} \int_\infty^{-\infty} (h_m(y, z) - g(y, z))^2 p(t, y, z) \, dy \, dz \, ds \\
\leq C \|h_m - g\|_{L^2(\mathbb{R}^2)}^2 \cdot \int_0^T \frac{dt}{\sqrt{t}} \leq C \|h_m - g\|_{L^2(\mathbb{R}^2)}^2.
\]

Similarly,

\[
E \sup_{t \in [0,T]} S^2_{n,m}(t) = E \sum_{t_k \in D_n} (h_m(Y^\theta_{t_{k-1}}, B_{t_{k-1}}) - g(Y^\theta_{t_{k-1}}, B_{t_{k-1}}))^2 (t_k - t_{k-1}) \\
\leq C \sum_{t_k \in D_n, k \geq 1} \frac{t_k - t_{k-1}}{\sqrt{t_{k-1}}} \|h_m - g\|_{L^2(\mathbb{R}^2)}^2,
\]

whence

\[
\limsup_{n \to \infty} E \sup_{t \in [0,T]} S^2_{n,m}(t) \leq C \|h_m - g\|_{L^2(\mathbb{R}^2)}^2.
\]

As a result, we get \( \sup_{t \in [0,T]} |I_m(t)| \xrightarrow{P} 0, \ m \to \infty \), and for any \( \varepsilon > 0 \)

\[
\lim_{m \to \infty} \limsup_{n \to \infty} P \left( \sup_{t \in [0,T]} |S_{n,m}(t)| > \varepsilon \right) = 0.
\]

Hence, using that

\[
\sum_{t_k \in D, t_k \leq t} h_m(Y^\theta_{t_{k-1}}, B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}}) \to \int_0^t h_m(Y^\theta_s, B_s) \, dB_s, \ n \to \infty,
\]

uniformly on \([0, T]\) in probability, we get that

\[
\sum_{t_k \in D, t_k \leq t} g(Y^\theta_{t_{k-1}}, B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}}) \to \int_0^t g(Y^\theta_s, B_s) \, dB_s, \ n \to \infty,
\]

uniformly on \([0, T]\) in probability.

Further, recall that the time-reversed process \((\tilde{Y}^\theta_t, \tilde{B}_t) = (Y^\theta_{T-t}, B_{T-t})\) satisfies \((8.13)\) in the weak sense. As far as the convergence in probability is concerned, we can safely assume that \((\tilde{Y}^\theta, \tilde{B})\) satisfies \((8.13)\) with some Brownian motion \( W \). Then we can write

\[
\int_0^T g(Y^\theta_s, B_s) \, dB_s = \int_{T-t}^T g(\tilde{Y}^\theta_s, \tilde{B}_s) \, dB_s \\
= \int_{T-t}^T g(\tilde{Y}^\theta_s, \tilde{B}_s) \, dB_s + \int_{T-t}^T g(\tilde{Y}^\theta_s, \tilde{B}_s) \tilde{b}^\theta(s, Y^\theta_s, \tilde{B}_s) \, ds.
\]

Arguing as above, we have

\[
\sum_{t_k \in D, t_k \leq t} g(Y^\theta_{t_k}, B_{t_k})(W_{T-t_k} - W_{T-t_{k-1}}) \\
= \sum_{t_k \in D, t_k \leq t} g(\tilde{Y}^\theta_{T-t_k}, \tilde{B}_{T-t_k})(W_{T-t_k} - W_{T-t_{k-1}}) \to \int_{T-t}^T g(\tilde{Y}^\theta_s, \tilde{B}_s) \, dW_s, \ n \to \infty.
\]
uniformly on $[0,T]$ in probability. It remains to show that
\[
\sum_{t_k \in D, t_k < t} g(Y^\theta_{t_k}, B_{t_k}) \int_{t_{k-1}}^{t_k} b^\theta(T - s, Y^\theta_s, B_s) \, ds
\]
\[
= \sum_{t_k \in D, t_k < t} g(Y^\theta_{T-t_k}, \bar{B}_{T-t_k}) \int_{T-t_{k-1}}^{T-t_k} b^\theta(s, Y^\theta_s, \bar{B}_s) \, ds
\]
\[
\rightarrow \int_{T-t}^T g(Y^\theta_{T-s}, \bar{B}_{T-s}) b^\theta(s, Y^\theta_s, \bar{B}_s) \, ds = \int_0^t g(Y^\theta_s, B_s) b^\theta(T - s, Y^\theta_s, B_s) \, ds, \ \ n \to \infty,
\]
uniformly on $[0,T]$ in probability.

We will first establish an estimate for $\int_0^T |b^\theta(T - s, Y^\theta_s, B_s) h(Y^\theta_s, B_s)| \, ds$ with $h \in L^2(\mathbb{R}^2)$. From (8.15), using the Cauchy–Schwarz inequality, we have
\[
\int_0^T \mathbb{E}|h(Y^\theta_s, B_s) b^\theta(s, Y^\theta_s, B_s)| \, ds
\leq C \int_0^T \mathbb{E}|h(Y^\theta_s, B_s)| \left( \frac{1}{2Y^\theta_s \beta^2(Y^\theta_s) - B_s} + \frac{2Y^\theta_s \beta^2(Y^\theta_s) - B_s}{s} \right) \, ds
\leq C \int_0^T \left[ \mathbb{E}|h(Y^\theta_s, B_s)|^2 \cdot \mathbb{E} \left( \frac{1}{(2Y^\theta_s \beta^2(Y^\theta_s) - B_s)^2} + \frac{(2Y^\theta_s \beta^2(Y^\theta_s) - B_s)^2}{s^2} \right) \right]^{\frac{1}{2}} \, ds.
\]
As before, from the estimate $p(s, Y_s, B_s) \leq \frac{C}{\sqrt{s}}$ it follows that
\[
\mathbb{E}|h(Y_s, B_s)|^2 \leq \frac{C}{\sqrt{s}} \|h\|^2_{L^2(\mathbb{R}^2)}.
\]
Further, using (8.11), (8.7) and (8.8) with $u_0 = 0$, we get
\[
\mathbb{E} \left( \frac{1}{2Y^\theta_s \beta^2(Y^\theta_s) - B_s} \right)^2 = \mathbb{E} \left( \frac{1}{\beta^2(B^\theta_s + L_s(B^\theta))} \right)^2
\leq \frac{1}{\sqrt{2\pi s^3}} \int_0^\infty \int_{-\infty}^\infty \frac{1}{s^2} e^{-\frac{(l+b)^2}{2s}} \, db \, dl
\leq \frac{C}{\sqrt{s^3}} \int_0^\infty \frac{1}{l^2} e^{-\frac{(l+b)^2}{2s}} \, db \, dl \leq \frac{C}{\sqrt{s^3}} \int_0^\infty e^{-\frac{z^2}{s^2}} \, dz \leq C.
\]
Similarly,
\[
\mathbb{E} \left( 2Y^\theta_s \beta^2(Y^\theta_s) - B_s \right)^2 \leq \frac{C}{\sqrt{s^3}} \int_0^\infty \int_0^\infty (l+b)^3 e^{-\frac{(l+b)^2}{2s}} \, db \, dl \leq \frac{C}{\sqrt{s^3}} \int_0^\infty z^4 e^{-\frac{z^2}{s^2}} \, dz \leq C.
\]
Therefore,
\[
\int_0^T \mathbb{E}|h(Y^\theta_s, B_s) b^\theta(s, Y^\theta_s, B_s)| \, ds
\leq C\|h\|_{L^2(\mathbb{R}^2)} \int_0^T s^{-\frac{1}{4}} ds \leq C\|h\|_{L^2(\mathbb{R}^2)}.
\]
(8.16)

Using similar estimates, we get
\[
\sum_{t_k \in D, t_k < t} \int_{t_{k-1}}^{t_k} \mathbb{E}|h(Y^\theta_{t_k}, B_{t_k}) b^\theta(T - s, Y^\theta_s, B_s)| \, ds
\leq \sum_{t_k \in D, t_k < t} \left( \mathbb{E}|h(Y^\theta_{t_k}, B_{t_k})|^2 \cdot \mathbb{E}|b^\theta(T - s, Y^\theta_s, B_s)|^2 \right)^{\frac{1}{2}} \, ds
\leq C\|h\|_{L^2(\mathbb{R}^2)} \sum_{t_k \in D, t_k < t} t_k^{1/4} s^{-1/2} \, ds \leq C\|h\|_{L^2(\mathbb{R}^2)} \int_0^T s^{-3/4} \, ds \leq C\|h\|_{L^2(\mathbb{R}^2)}.
\]
(8.17)
If \( h \in C(\mathbb{R}^2) \), then
\[
\delta_n = \max_{t_k \in D^n} \sup_{s \in [t_{k-1}, t_k]} |h(Y^\theta_{t_k}, B_{t_k}) - h(Y^\theta_s, B_s)| \to 0, \quad n \to \infty,
\]
almost surely and we can estimate
\[
\left| \sum_{t_k \in D, t_k < t} h(Y^\theta_{t_k}, B_{t_k}) \int_{t_{k-1}}^{t_k} b^\theta(T - s, Y^\theta_s, B_s) \, ds - \int_0^t h(Y^\theta_s, B_s)b^\theta(T - s, Y^\theta_s, B_s) \, ds \right|
\leq \delta_n \int_0^T |b^\theta(T - s, Y^\theta_s, B_s)| \, ds.
\]
Similarly to the calculations above,
\[
\int_0^T \mathbb{E}[b^\theta(T - s, Y^\theta_s, B_s)] \, ds \leq \int_0^T \left( \mathbb{E}[b^\theta(T - s, Y^\theta_s, B_s)]^2 \right)^{\frac{1}{2}} \, ds \leq C \int_0^T s^{-\frac{1}{2}} \, ds \leq C,
\]
so that \( \int_0^T |b^\theta(T - s, Y^\theta_s, B_s)| \, ds \) is bounded in probability. Therefore, for \( h \in C(\mathbb{R}^2) \),
\[
\sum_{t_k \in D, t_k < t} h(Y^\theta_{t_k}, B_{t_k}) \int_{t_{k-1}}^{t_k} b^\theta(T - s, Y^\theta_s, B_s) \, ds \to \int_0^t h(Y^\theta_s, B_s)b^\theta(T - s, Y^\theta_s, B_s) \, ds, \quad n \to \infty,
\]
uniformly on \([0, T]\) in probability. Hence, taking, as before, a sequence \( h_m \in C(\mathbb{R}^2) \) converging to \( g \) in \( L^2(\mathbb{R}^2) \) and using (8.16) and (8.17), we arrive at (8.15). Combined with our previous findings, this leads to
\[
\sum_{t_k \in D, t_k < t} \left( f(B^\theta_{t_k}, B_{t_k}) - f(B^\theta_{t_{k-1}}, B_{t_{k-1}}) \right)(B_{t_k} - B_{t_{k-1}})
\to \int_0^t f(B^\theta_s, B_s) \, dB_s - \int_0^t f(B^\theta_s, B_s) \, dB_s, \quad n \to \infty,
\]
uniformly on \([0, T]\) in probability. Since \( T > 0 \) is arbitrary, this means precisely that the desired u.c.p. convergence holds.

9 Proof of Theorem [4.5], \( \theta \in (-1, 1) \)

For definiteness we set \( X_0 = 0 \).

1. For \( \theta = 0 \) and \( \alpha \in (-1, 1) \), the statement follows directly from the Itô formula proven in [Föllmer et al. 1995].

2. Let \( \theta \neq 0 \) and \( \alpha \in (0, 1) \). If \( h \in C^1(\mathbb{R}) \), \( h(0) = 0 \), \( H(x) = \int_0^x h(y) \, dy \), then by the usual Itô formula for semimartingales (see, e.g., Protter 2004, Theorem II.32), we have
\[
H(B^\theta_t) = H(0) + \int_0^t \left( h(B^\theta_s) \, dB_s + \theta \int_0^t h(B^\theta_s) \, dL(B^\theta_s) + \frac{1}{2}[h(B^\theta), B]_t + \frac{\theta}{2}[h(B^\theta), L(B^\theta)]_t \right),
\]
where the decomposition of quadratic variation into the sum holds true since both \([h(B^\theta), B^\theta]_t]\) and \([h(B^\theta), B]_t]\) exist as u.p.c. limits. Furthermore since \( h(0) = 0 \), the quadratic variation \([h(B^\theta), L(B^\theta)]_t]\) and the integral w.r.t. \( L(B^\theta) \) vanish a.s., so that we obtain the equality
\[
H(B^\theta_t) = H(0) + \int_0^t h(B^\theta_s) \, dB_s + \frac{1}{2}[h(B^\theta), B]_t.
\]

Taking a sequence \( \{h_n\} \) of \( C^1 \)-functions such that, \( h_n(0) = 0 \), \( h_n(x) = |(1 - \alpha)x|^n \) for \( |x| \geq 1 \) and \( \sup_{x \in [0,1]} |h_n(x) - (1 - \alpha)|x|^n| \to 0 \) we utilize the Itô isometry and Theorem [14.5] to get the desired result.

3. It is left to show that for \( \theta \neq 0 \) and \( \alpha \in (-1, 0) \), \( X^\theta \) is not a solution of the SDE.
3.1. Let \( \alpha = 0 \). Clearly,
\[
\int_0^t \mathbb{1}(B^\theta_s \neq 0) \, dB_s = B_t \quad \text{a.s.}
\]
However
\[
[\mathbb{1}(B^\theta \neq 0), B] \equiv 0 \quad \text{a.s.}
\]
since \( h(x) = \mathbb{1}(x \neq 0) \) can be approximated by \( h_n(x) \equiv 1 \) in \( L^2(\mathbb{R}) \), and \([1, B] \equiv 0 \). Hence,
\[
\int_0^t \mathbb{1}(B^\theta_s \neq 0) \, dB_s = B_t \neq X^\theta_t = B_t + \theta L_t(B^\theta).
\]

3.2. For \( \alpha \in (-1, 0) \), the Stratonovich integral w.r.t. \( B \) is well defined as the sum
\[
\int_0^t |B^\theta_s|^{2\alpha} \circ dB_s = \int_0^t |B^\theta_s|^{2\alpha} \, dB_s + \frac{1}{2} [\mathbb{1}(B^\theta), B]_t
\]
Choosing again a sequence of \( C^1 \)-functions \( \{h_n\} \) such that
\[
\begin{align*}
&h_n(x) \equiv |(1-\alpha)x|^{\frac{\alpha}{1-\alpha}}, \quad |x| \geq 1, \\
&\|h_n(\cdot) - |(1-\alpha)\cdot|^{\frac{\alpha}{1-\alpha}}\|_{L^2((-1,1))} \to 0,
\end{align*}
\]
we obtain that
\[
H_n(x) = \int_0^x h_n(y) \, dy \to H(x) = (1-\alpha)x|^{\frac{\alpha}{1-\alpha}}
\]
uniformly on \( \mathbb{R} \), so that we can apply the standard Itô formula to obtain
\[
\begin{align*}
H_n(B^\theta_t) &= H_n(0) + \int_0^t h_n(B^\theta_s) \, dB^\theta_s + \frac{1}{2} [h_n(B^\theta), B^\theta]_t \\
&= H_n(0) + \int_0^t h_n(B^\theta_s) \, dB_s + \theta \int_0^t h_n(B^\theta_s) \, dL_s(B^\theta)
\end{align*}
\]
However it is easy to see e.g. by the monotone convergence (if we choose \( h_n \) monotonically increasing)
\[
\lim_{n \to \infty} \int_0^t h_n(B^\theta_s) \, dL_s = \int_0^t |(1-\alpha)B^\theta_s|^{\frac{\alpha}{1-\alpha}} \circ dB_s \\
:= (1-\alpha)^{\frac{\alpha}{1-\alpha}} \lim_{n \to \infty} \sum_{t_{k-1} < t} \frac{|B^\theta_{t_k}|^{\frac{\alpha}{1-\alpha}} + |B^\theta_{t_{k-1}}|^{\frac{\alpha}{1-\alpha}}}{2} (L_{t_k} - L_{t_{k-1}}) = +\infty,
\]
so that the SDE (12) is not satisfied unless \( \theta = 0 \).

Note that the Riemann-Stieltjes integral w.r.t. \( L \) does not exist since the points of increase of \( L \) coincide with the points of discontinuity of \( |B^\theta|^{\frac{\alpha}{1-\alpha}} \).

10 Proof of Theorem 4.5 for \( \theta = \pm 1 \)

For simplicity and without loss of generality, assume that \( \theta = 1 \) and \( X_0 = 0 \). Denote \( m_t = \min_{0 \leq s \leq t} B_s \) the running minimum of the Brownian motion \( B \).

The process \( B^1_t = B_t - m_t \) is the unique strong solution to SDE
\[
B^1_t = B_t + L_t(B^1_t). \tag{10.1}
\]

**Theorem 10.1.** Let \( \alpha \in (0, 1) \). The process
\[
X^1_t = X_t = (1-\alpha)(B_t - m_t)^{\frac{\alpha}{1-\alpha}}
\]
is a non-negative strong solution to (12).
Proof. For fixed $t > 0$, take a sequence of partitions $0 = t_0^n < t_1^n < \cdots < t_n^n = t$ of $[0, t]$ (as usual, we omit the superscript $n$). Define $\tau_k = \min\{s \geq t_{k-1}: X_t = 0\} \wedge t_k$, $k = 1, \ldots, n$.

On each interval $[t_{k-1}, \tau_k]$, the unique solution is given by

$$X_t = \left( (1 - \alpha)(B_t - B_{t_{k-1}}) + (X_{t_{k-1}})^{1-\alpha} \right)^{\frac{1}{1-\alpha}}$$

$$= \left( (1 - \alpha)(B_t - B_{t_{k-1}}) + (1 - \alpha)(B_{t_{k-1}} - m_{t_{k-1}}) \right)^{\frac{1}{1-\alpha}}$$

$$= \left( (1 - \alpha)(B_t - m_t) \right)^{\frac{1}{1-\alpha}}, \quad t \in [t_{k-1}, \tau_k],$$

where we used that $m_t = m_{t_{k-1}}$ for $t \in [t_{k-1}, \tau_k]$ since $m_t$ changes only when $B$ hits its minimum, equivalently, when $X$ hits zero.

Let us write the telescopic sum

$$X_t = \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}}) + \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})$$

and show that the first sum converges to zero whereas the terms of the second sum satisfy the SDE.

We start by estimating the first sum. Denote $I_k(\omega) = \{k: \tau_k < t_k\}$ the set of indices $k$ such that $X(\omega)$ has a zero in $[t_{k-1}, t_k)$. Then for any $a < \frac{1}{2}$, using the Hölder continuity of $B$, we have

$$\left| \sum_{k=1}^{n} (X_{t_k} - X_{\tau_k}) \right| \leq \sum_{k \in I} \left| X_{t_k} - X_{\tau_k} \right| \leq C(a, \omega) \sum_{k \in I} |t_k - t_{k-1}|^{a-\alpha}.$$  \hspace{1cm} (10.2)

It is well known that $B - m$ has the same distribution as $|B|$. So the limit in law of the sum above is the $\frac{a}{1-\alpha}$-dimensional Hausdorff measure of the set of zeros of $B$. It is well known that the dimension of this set is $1/2$, so choosing $a$ such that $\frac{a}{1-\alpha} > \frac{1}{2}$, we get

$$\sum_{k=1}^{n} (X_{t_k} - X_{\tau_k}) \to 0, \quad n \to \infty,$$  \hspace{1cm} (10.3)

in law, and hence, in probability.

For the second sum, for any $k = 1, \ldots, n$, we apply the Itô formula from Föllmer et al. (1995) to find that

$$X_{t_k} - X_{t_{k-1}} = \int_{t_{k-1}}^{t_k} |X_s|^\alpha \circ dB_s = \int_{t_{k-1}}^{t_k} |X_s|^\alpha dB_s + \frac{1}{2}[|X|^\alpha, B]_{\tau_k, t_{k-1}}.$$  \hspace{1cm} (10.4)

Therefore,

$$\sum_{k=1}^{n} (X_{\tau_k} - X_{t_{k-1}}) = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} |X_s|^\alpha dB_s.$$  \hspace{1cm} (10.5)

It remains to prove that

$$\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} |X_s|^\alpha dB_s = \sum_{k=1}^{n} \left( \int_{\tau_k}^{t_k} |X_s|^\alpha dB_s + \frac{1}{2}[|X|^\alpha, B]_{t_{k-1}, \tau_k} \right) \to 0, \quad n \to \infty,$$

in probability. The Itô integral part is easy to estimate:

$$\mathbb{E} \left( \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} |X_s|^\alpha dB_s \right)^2 = \int_0^t \mathbb{E} \left( \sum_{k=1}^{n} |X_{s}\|^\alpha_{(\tau_k, t_k)}(s) \right)^2 ds \leq C \cdot \mathbb{E} \left( |\tau_k|^{2a(1-\alpha)} \right) \to 0, \quad n \to \infty;$$

here $\delta_n = \max_{k=1,\ldots,n} (t_k - t_{k-1})$ and $w_B(\delta) = \sup_{0 \leq u < s \leq t, \delta \in [t]} |B_s - B_u|$ is the modulus of continuity of $B$. For the remaining part use the well known fact that $(B - m, B)$ has the same distribution as $(|B|, |B| - l)$, where $l$ is the symmetric local time of $B$ at zero. That said,

$$\sum_{k=1}^{n} [|X|^\alpha, B]_{t_{k-1}, \tau_k} \to \sum_{k=1}^{n} [|B|^{\alpha}, |B| - l]_{t_{k}, \sigma_k},$$

where $\sigma_k$ is the symmetric local time of $B$ at zero. That said,
where $\sigma_k = \inf\{t \geq t_{k-1}: B_t = 0\}$. Clearly,
\[
\begin{aligned}
\sum_{k=1}^{n} [\{B\}_{t_k}, |B| - l]_{t_k, \sigma_k} &= \sum_{k=1}^{n} [\{B\}_{\tau_k}, |B|]_{t_k, \sigma_k} - \sum_{k=1}^{n} [\{B\}_{\tau_k}, l]_{t_k, \sigma_k} \\
&= \sum_{k=1}^{n} [\{B\}_{\tau_k}, |B|]_{t_k, \sigma_k} - \sum_{k=1}^{n} [\{B\}_{\tau_k}, l]_{t_k, \sigma_k}.
\end{aligned}
\]

Since $l$ has bounded variation and $|B|_{\tau_k}$ is small when $l$ increases, it is easy to see that the second term is zero. On the other hand, by Itô’s formula,
\[
\sum_{k=1}^{n} [\{B\}_{\tau_k}, |B|]_{t_k, \sigma_k} = 2 \sum_{k=1}^{n} \left(1 - \alpha \right) (|B_{t_k}|_{\tau_k} - |B_{\sigma_k}|_{\tau_k}) - \int_{\tau_k}^{t_k} (B_s)_{\tau_k} dB_s.
\]
As above, we have for any $\frac{1-\alpha}{2} < a < \frac{1}{2}$ that
\[
\left| \sum_{k=1}^{n} (|B_{t_k}|_{\tau_k} - |B_{\sigma_k}|_{\tau_k}) \right| \leq \sum_{k=1}^{n} \left| |B_{t_k}|_{\tau_k} - |B_{\sigma_k}|_{\tau_k} \right| \\
\leq C(a, \omega) \sum_{k=1}^{n} |t_k - t_{k-1}|_{\tau_k} \to 0, \quad n \to \infty.
\]
On the other hand, similarly to (10.4),
\[
E \left( \sum_{k=1}^{n} \int_{\sigma_k}^{t_k} (B_s)_{\tau_k} dB_s \right)^2 \to 0, \quad n \to \infty.
\]
The proof is now complete. \hfill $\square$

A  Partial differential equation for $v(s, x, w)$

Let $\theta \in (0, 1)$. Consider for fixed $t > 0$, $g \in C^\infty(\mathbb{R}^2)$ with support inside $D = \{(y, z) \in \mathbb{R}^2: r(y) - z > 0\}$
\[
v(s, x, w) = E[g(Y_t, B_t)|Y_s = x, B_s = w], \quad s \in [0, t], \quad x, w \in \mathbb{R}.
\]
Denoting $\varphi_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ and taking into account that $\psi_t(x) = -\varphi_t'(x) = \frac{1}{t} \varphi_t(x)$ we employ (8.3) and (8.4) to obtain
\[
v(s, x, w) = \frac{2}{\theta} \int_{-\infty}^{\infty} \beta(y) \left( \frac{\varphi_t(y)}{\sqrt{\theta}} \right) dz dy \\
+ \frac{1}{\theta} \int_{y: y > 0} g(y, r(y) - r(x) + w) \cdot \left( \varphi_t - \varphi_t' \right) dy.
\]
The cases $x > 0$ and $x < 0$ can be treated similarly, so we will consider only the former. Denote for brevity
\[
\beta_+ = \frac{1 + \theta}{2}, \quad g_1(y, z) = \frac{\partial}{\partial z} g(y, z), \quad g_2(y, z) = \frac{\partial^2}{\partial z^2} g(y, z), \quad h(x, y, z, w) = \frac{2y\beta^2(y) - \kappa x - z + w}{\theta}.
\]
Also note that by (8.10), substituting $z = r(x) - r(y) + w$ into $h(x, y, z, w)$ gives $|r(x)| + |r(y)| = r(x) + |r(y)|$.

Then
\[
\frac{\partial}{\partial x} v(s, x, w) = -\frac{2 \beta_+}{\theta} \int_{-\infty}^{\infty} \beta^2(y) \cdot g(y, r(y) - r(x) + w) \cdot \psi_{t-s}(r(x) + |r(y)|) dy \\
- \frac{2 \kappa}{\theta^2} \int_{-\infty}^{\infty} \beta^2(y) \int_{-\infty}^{r(y) - r(x) + w} g(y, z) \cdot \psi'_{t-s}(h(x, y, z, w)) dz dy \\
- \beta_+^2 \int_{0}^{\infty} g_1(y, r(y) - r(x) + w) \cdot \left( \varphi_{t-s} - \varphi_{t-s}' \right) dy + \beta_+^2 \int_{0}^{\infty} g(y, r(y) - r(x) + w) \cdot \left( \varphi'_{t-s} - \varphi_{t-s}' \right) dy,
\]
Further,

\[ \frac{\partial}{\partial w} v(s, x, w) = \frac{2}{\theta} \int_{-\infty}^{\infty} \beta^2(y) \cdot g(y, r(y) - r(x) + w) \cdot \psi_{1-s}(r(x) + |y|) \, dy + \frac{2}{\theta^2} \int_{-\infty}^{\infty} \beta^2(y) \int_{-\infty}^{r(y) - r(x) + w} g(y, z) \cdot \psi'_{1-s}(h(x, y, z, w)) \, dz \, dy + \beta_+ \left[ g_1(y, r(x) - r(y)) - \varphi_{1-s}(r(x) + r(y)) \right] \, dy. \]

Further,

\[ \frac{\partial^2}{\partial x^2} v(s, x, w) = \frac{2\beta^2}{\theta} \int_{-\infty}^{\infty} \beta^2(y) \cdot g_1(y, r(x) - r(x) + w) \cdot \psi_{1-s}(r(x) + |y|) \, dy - \frac{2\beta^2}{\theta} \int_{-\infty}^{\infty} \beta^2(y) \cdot g(y, r(y) - r(x) + w) \cdot \psi_{1-s}(r(x) + |y|) \, dy + \frac{2\kappa\beta_+}{\theta^2} \int_{-\infty}^{\infty} \beta^2(y) \cdot g(y, r(y) - r(x) + w) \cdot \psi''_{1-s}(r(x) + |y|) \, dy + \frac{2\kappa^2}{\theta^3} \int_{-\infty}^{\infty} \beta^2(y) \int_{-\infty}^{r(y) - r(x) + w} g(y, z) \cdot \psi''_{1-s}(h(x, y, z, w)) \, dz \, dy + \beta_+ \left[ g_2(y, r(y) - r(x) + w) \cdot \left[ \varphi_{1-s}(r(x) - r(y)) - \varphi_{1-s}(r(x) + r(y)) \right] \right] \, dy - \frac{2\beta^3}{\theta} \int_{-\infty}^{\infty} \beta^2(y) \cdot g_1(y, r(x) - r(x) + w) \cdot \psi_{1-s}(r(x) + |y|) \, dy + \beta_+ \left[ g_1(y, r(x) - r(y)) - \varphi_{1-s}(r(x) + r(y)) \right] \, dy - 2\beta^3 \int_{-\infty}^{\infty} g(y, r(y) - r(x) + w) \cdot \left[ \varphi_{1-s}(r(x) - r(y)) - \varphi_{1-s}(r(x) + r(y)) \right] \, dy + \beta_+ \left[ g(y, r(y) - r(x) + w) \cdot \left[ \varphi''_{1-s}(r(x) - r(y)) - \varphi''_{1-s}(r(x) + r(y)) \right] \right] \, dy, \]

\[ \frac{\partial^2}{\partial x \partial w} v(s, x, w) = -\frac{2\beta^2}{\theta} \int_{-\infty}^{\infty} \beta^2(y) \cdot g_1(y, r(x) - r(x) + w) \cdot \psi_{1-s}(r(x) + |y|) \, dy - \frac{2\kappa}{\theta^2} \int_{-\infty}^{\infty} \beta^2(y) \cdot g(y, r(y) - r(x) + w) \cdot \psi_{1-s}(r(x) + |y|) \, dy - \frac{2\kappa}{\theta^2} \int_{-\infty}^{\infty} \beta^2(y) \int_{-\infty}^{r(y) - r(x) + w} g(y, z) \cdot \psi''_{1-s}(h(x, y, z, w)) \, dz \, dy - \frac{2\kappa^2}{\theta^3} \int_{-\infty}^{\infty} \beta^2(y) \int_{-\infty}^{r(y) - r(x) + w} g(y, z) \cdot \psi''_{1-s}(h(x, y, z, w)) \, dz \, dy - \beta^2_+ \int_{-\infty}^{\infty} g_1(y, r(y) - r(x) + w) \cdot \left[ \varphi''_{1-s}(r(x) - r(y)) - \varphi''_{1-s}(r(x) + r(y)) \right] \, dy - \beta^2_+ \int_{-\infty}^{\infty} g_2(y, r(y) - r(x) + w) \cdot \left[ \varphi_{1-s}(r(x) - r(y)) - \varphi_{1-s}(r(x) + r(y)) \right] \, dy, \]

\[ \frac{\partial^2}{\partial w^2} v(s, x, w) = \frac{2}{\theta} \int_{-\infty}^{\infty} \beta^2(y) g_1(y, r(y) - r(x) + w) \cdot \psi_{1-s}(r(x) + |y|) \, dy + \frac{2}{\theta^2} \int_{-\infty}^{\infty} \beta^2(y) \cdot g(y, r(y) - r(x) + w) \cdot \psi_{1-s}(r(x) + |y|) \, dy + \frac{2}{\theta^2} \int_{-\infty}^{\infty} \beta^2(y) \int_{-\infty}^{r(y) - r(x) + w} g(y, z) \cdot \psi''_{1-s}(h(x, y, z, w)) \, dz \, dy + \beta_+ \left[ g_2(y, r(y) - r(x) + w) \cdot \left[ \varphi_{1-s}(r(x) - r(y)) - \varphi_{1-s}(r(x) + r(y)) \right] \right] \, dy. \]
Therefore, taking into account that \( \frac{g}{\beta_x} = \frac{1-\theta^2}{1-\theta^2} = 1 - \theta \), we get

\[
\left( \frac{1}{2\beta_x^2} \frac{\partial^2}{\partial x^2} + \frac{1}{\beta_x} \cdot \frac{\partial^2}{\partial x \partial w} + \frac{1}{2} \cdot \frac{\partial^2}{\partial w^2} \right) v(s, x, w)
\]

\[
= \left( -\frac{1}{\theta} + \frac{\kappa}{\beta_x \theta^2} - \frac{2\kappa}{\beta_x \theta^2} + \frac{1}{\theta^2} \right) \int_{-\infty}^{\infty} \beta^2(y) \cdot g(y, r(y) - r(x) + w) \cdot \psi_{l-s}(r(x) + |r(y)|) \, dy
\]

\[
+ \left( \frac{\kappa^2}{\beta_x^2 \theta^2} - \frac{2\kappa}{\beta_x \theta^2} + \frac{1}{\theta^2} \right) \int_{-\infty}^{\infty} \beta^2(y) \int_{-\infty}^{r(y) - r(x) + w} g(y, z) \cdot \psi_{l-s}(h(x, y, z, w)) \, dz \, dy
\]

\[
+ \frac{\beta_x}{2} \int_{0}^{\infty} g(y, r(y) - r(x) + w) \cdot \left[ \varphi''_{l-s}(r(x) - r(y)) - \varphi''_{l-s}(r(x) + r(y)) \right] \, dy
\]

\[
= \left( -\frac{1}{\theta} - \frac{(1-\theta)^2}{\theta^3} + \frac{1}{\theta} \right) \int_{-\infty}^{\infty} \beta^2(y) \int_{-\infty}^{r(y) - r(x) + w} g(y, z) \cdot \psi_{l-s}(h(x, y, z, w)) \, dz \, dy
\]

\[
+ \frac{\beta_x}{2} \int_{0}^{\infty} g(y, r(y) - r(x) + w) \cdot \left[ \varphi''_{l-s}(r(x) - r(y)) - \varphi''_{l-s}(r(x) + r(y)) \right] \, dy
\]

\[
= \frac{1}{\theta} \int_{-\infty}^{\infty} \beta^2(y) \int_{-\infty}^{r(y) - r(x) + w} g(y, z) \cdot \psi_{l-s}(h(x, y, z, w)) \, dz \, dy
\]

\[
+ \frac{\beta_x}{2} \int_{0}^{\infty} g(y, r(y) - r(x) + w) \cdot \left[ \varphi''_{l-s}(r(x) - r(y)) - \varphi''_{l-s}(r(x) + r(y)) \right] \, dy.
\]

On the other hand,

\[
\frac{\partial}{\partial s} v(s, x, w) = \frac{2}{\theta} \int_{-\infty}^{\infty} \int_{-\infty}^{r(y) - r(x) + w} g(y, z) \frac{\partial}{\partial s} \psi_{l-s}(h(x, y, z, w)) \, dz \, dy
\]

\[
+ \frac{\beta_x}{2} \int_{0}^{\infty} g(y, r(y) - r(x) + w) \frac{\partial}{\partial s} \left[ \varphi_{l-s}(r(x) - r(y)) - \varphi_{l-s}(r(x) + r(y)) \right] \, dy.
\]

Taking into account that \( \frac{\partial}{\partial t} \varphi_t(x) = \frac{1}{\theta} \varphi''_t(x) \) and \( \frac{\partial}{\partial t} \psi_t(x) = \frac{1}{\theta} \psi''_t(x) \), we arrive at the desired equation

\[
\left( \frac{\partial}{\partial s} + L \right) v(s, x, w) = 0, \quad x > 0.
\]

Dealing similarly with the case \( x < 0 \), we get

\[
\left( \frac{\partial}{\partial s} + L \right) v(s, x, w) = 0, \quad x \neq 0, \quad w \in \mathbb{R},
\]

where

\[
Lf(x, w) = \left( \frac{\sigma^2(x)}{2} \cdot \frac{\partial^2}{\partial x^2} + \sigma(x) \cdot \frac{\partial^2}{\partial x \partial w} + \frac{1}{2} \cdot \frac{\partial^2}{\partial w^2} \right) f(x, w).
\]

Further, let \( f \in C^\infty(\mathbb{R}^2) \) have bounded support inside \( \{(x, w): w < r(x)\} \), and

\[
p(s, x, w) = \frac{2\beta^2(x)}{\theta} \psi_\lambda \left( \frac{2x\beta^2(x) - w}{\theta} \right)
\]

as given by \([8.6]\). Denoting \( \beta_- = \frac{1-\theta}{2} \) and using integration by parts, we write

\[
\int_{-\infty}^{\infty} \beta^2(x) \cdot p(s, x, w) \cdot f(x, w) \cdot \frac{\partial^2}{\partial x^2} v(s, x, w) \, dx
\]

\[
= f(0, w) \left[ \beta^2_+ \cdot p(s, 0-, x) \cdot \frac{\partial}{\partial x} v(s, 0-, w) - \beta^2_+ \cdot p(s, 0+, w) \cdot \frac{\partial}{\partial x} v(s, 0+, w) \right]
\]

\[
- \int_{-\infty}^{\infty} \beta^2(x) \cdot \frac{\partial}{\partial x} v(s, x, w) \cdot \frac{\partial}{\partial x} \left( p(s, x, w) f(x, w) \right) \, dx.
\]
We have

$$\frac{\partial^2}{\partial x^2} \cdot p(s,0-,x) \frac{\partial}{\partial x} v(s,0-,w) - \frac{\partial^2}{\partial x^2} \cdot p(s,0+,w) \frac{\partial}{\partial x} v(s,0+,w)$$

$$= \frac{2}{\theta} \psi_t \left( \frac{-u}{\theta} \right) \cdot \left[ 2 \int_{-\infty}^{\infty} \beta^2(y) \cdot g(y,r(y) + w) \cdot \psi_{s-y}(r(y)) \, dy \right]$$

$$+ \beta^2 \int_{-\infty}^{0} g(y,r(y) + w) \cdot \left[ \varphi_{r-s}(-r(y)) - \varphi_{r-s}(r(y)) \right] \, dy$$

$$- \beta^2 \int_{0}^{\infty} g(y,r(y) + w) \cdot \left[ \varphi_{r-s}(-r(y)) - \varphi_{r-s}(r(y)) \right] \, dy = 0,$$

since $\varphi_t = \psi_t$ and $\psi_t(-x) = -\psi_t(x)$. Further, using integration by parts again and noting that $v$ is continuous, we get

$$\int_{-\infty}^{\infty} \beta^2(x) \cdot \frac{\partial}{\partial x} v(s,x,w) \cdot \frac{\partial}{\partial x} \left( p(s,x,w) f(x,w) \right) \, dx$$

$$= v(s,0,w) \cdot \frac{\partial}{\partial x} f(0,w) \cdot \left( \beta^2 \cdot p(s,0-,w) - \beta^2 \cdot p(s,0+,w) \right)$$

$$+ v(s,0,w) \cdot f(0,w) \left( \beta^2 \cdot \frac{\partial}{\partial x} p(s,0-,w) - \beta^2 \cdot \frac{\partial}{\partial x} p(s,0+,w) \right)$$

$$- \int_{-\infty}^{\infty} \beta(x)^2 \cdot v(s,x,w) \frac{\partial^2}{\partial x^2} \left( p(s,x,w) f(x,w) \right) \, dx$$

$$= -4v(s,0,w) \cdot f(0,w) \cdot \psi_t^t \left( \frac{w}{\theta} \right) - \int_{-\infty}^{\infty} \beta^2(x) \cdot v(s,x,w) \cdot \frac{\partial^2}{\partial x^2} \left( p(s,x,w) f(x,w) \right) \, dx.$$

Integrating with respect to $w$ then leads to

$$\left\langle \beta^2 \frac{\partial^2}{\partial x \partial w} v(s), fp(s) \right\rangle = \left\langle v(s), \beta^2 \frac{\partial^2}{\partial x \partial w} (fp(s)) \right\rangle + 4 \int_{-\infty}^{\infty} v(s,0,w) f(0,w) \psi_t^t \left( \frac{w}{\theta} \right) \, dw.$$

Similarly, integrating by parts with respect to $w$,

$$\left\langle \beta \frac{\partial^2}{\partial x \partial w} v(s), fp(s) \right\rangle = -\left\langle \beta \frac{\partial}{\partial x} v(s), \frac{\partial}{\partial w} (fp(s)) \right\rangle.$$

and integrating by parts with respect to $x$,

$$\int_{-\infty}^{\infty} \beta(x) \cdot \frac{\partial}{\partial w} \left( p(s,x,w) f(x,w) \right) \frac{\partial}{\partial x} v(s,x,w) \, dx$$

$$= v(s,0,w) f(0,w) \left( \beta \frac{\partial}{\partial w} p(s,0-,x) - \beta \frac{\partial}{\partial w} p(s,0+,w) \right)$$

$$- \int_{-\infty}^{\infty} \beta(x) \cdot v(s,x,w) \frac{\partial^2}{\partial x \partial w} \left( p(s,x,w) f(x,w) \right) - 2v(s,0,w) f(0,w) \psi_t^t \left( \frac{w}{\theta} \right),$$

so

$$\left\langle \beta \frac{\partial^2}{\partial x \partial w} v(s), fp(s) \right\rangle = \left\langle v(s), \beta \frac{\partial^2}{\partial x \partial w} (fp(s)) \right\rangle - 2 \int_{-\infty}^{\infty} v(s,0,w) f(0,w) \psi_t^t \left( \frac{w}{\theta} \right) \, dw.$$

Finally, integrating by parts with respect to $w$ twice,

$$\left\langle \frac{\partial^2}{\partial w^2} v(s), fp(s) \right\rangle = \left\langle v(s), \frac{\partial^2}{\partial w^2} (fp(s)) \right\rangle.$$

Summing everything, we get, with $L$ given by (A.1), that

$$\left\langle L v(s), fp(s) \right\rangle = \left\langle v(s), L (fp(s)) \right\rangle.$$

(A.2)
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