HYDRODYNAMIC LIMIT OF THE KINETIC THERMOMECHANICAL CUCKER-SMALE MODEL IN A STRONG LOCAL ALIGNMENT REGIME

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Abstract. We present a hydrodynamic limit from the kinetic thermomechanical Cucker-Smale (TCS) model to the hydrodynamic Cucker-Smale (CS) model in a strong local alignment regime. For this, we first provide a global existence of weak solution, and flocking dynamics for classical solution to the kinetic TCS model with local alignment force. Then we consider one-parameter family of well-prepared initial data to the kinetic TCS model in which the temperature tends to common constant value determined by initial datum, as singular parameter $\varepsilon$ tends to zero. In a strong local alignment regime, the limit model is the hydrodynamic CS model in [8]. To verify this hydrodynamic limit rigorously, we adopt the technique introduced in [5] which combines the relative entropy method together with the 2-Wasserstein metric.

1. Introduction. Collective dynamics such as flocking often appears in our nature and human society. After Viscek’s seminal work in [24], the research on the flocking behavior of a complex system has received lots of attention in distinct disciplines such as biology, control theory and physics in last two decades. In the past, several mechanical models were proposed and studied from theoretical and numerical viewpoints. Among them, our main interest in this paper lies on the thermomechanical Cucker-Smale (TCS) model with temperature as an internal variable. As its name suggests, the TCS model is one of active particle models for the CS particles [4] incorporating temperature variable together with position and momentum variables.

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In [12], the TCS model was derived based on a heuristic argument from the multi-temperature fluid model via the space homogeneity assumption, Galilean invariance and entropy principle. In a series of papers by the first author and his collaborators, the TCS model has been extensively studied from various points of view, e.g., its emergent collective behavior [7, 9, 12], kinetic description and mean-field limit [7], hydrodynamic description [8], singular interaction kernels [1], coupled with other various kinds of fluids [2, 3]. For notational simplicity, we set

\[ z := (x, v, \theta), \quad dz := dx dv d\theta, \quad \omega := T_z^d \times \mathbb{R}_v, \quad \Omega := T_z^d \times \mathbb{R}_v \times \mathbb{R}_\theta^+. \]

Let \( f = f(t, z) \) be a one-particle distribution function of TCS particles at state space \( z \) at time \( t \). Then, the spatial-temperature evolution of \( f \) is governed by the kinetic TCS model [7]:

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F[f]f) + \partial_\theta (G[f]f) &= 0, \quad (t, z) \in \mathbb{R}_+ \times \Omega, \\
F[f](t, z) &= \kappa_v \int_\Omega \phi(|x - x_*|) \left( \frac{v_*}{\theta_*} - \frac{v}{\theta} \right) f(t, z_*) dz_*, \\
G[f](t, z) &= \kappa_\theta \int_\Omega \zeta(|x - x_*|) \left( \frac{1}{\theta_*} - \frac{1}{\theta} \right) f(t, z_*) dz_*.
\end{align*}
\] (1)

Here, \(| \cdot |\) is the standard distance in \( T_z^d = \mathbb{R}^d / \mathbb{Z}^d \) defined as

\[ |x_1 - x_2| := \inf_{(z_1, z_2) \in [x_1] \times [x_2]} |z_1 - z_2|_{\mathbb{R}^d}, \]

and \( \kappa_v \) and \( \kappa_\theta \) are velocity and temperature coupling strengths, respectively and real-valued functions \( \phi \) and \( \zeta \) are assumed to be Lipschitz continuous, positive and bounded:

\[ \phi, \zeta \in \text{Lip}(\mathbb{R}_+, \mathbb{R}_+), \quad 0 < \phi \leq \phi_M, \quad 0 < \zeta \leq \zeta_M. \]

Note that our aim is to derive a macroscopic description rigorously for the kinetic equation (1) in a “strong local alignment regime”. For the local alignment force for velocity variable [5, 17], we add the term \( \frac{1}{\varepsilon}L_F[f](t, x, v) \) in the left-hand side of (1.1):

\[ u(t, x) := \frac{\int_{\mathbb{R}^d \times \mathbb{R}_+} v f d v d \theta}{\int_{\mathbb{R}^d \times \mathbb{R}_+} f d v d \theta}, \quad L_F[f](t, x, v) := u(t, x) - v. \] (2)

In [19], the local alignment \( L_F[f] \) was first appeared in the study of singular limit of a normalized non-symmetric alignment introduced by Motsch and Tadmor [21]. For the local alignment of temperature variables, we choose a well-prepared initial data asymptotically concentrated on a common constant temperature (see Section 3 for details).

The purpose of this paper is to derive a singular limit of the following rescaled equation in a strong local alignment regime (\( \varepsilon \ll 1 \)): for \( (t, z) \in \mathbb{R}_+ \times \Omega, \)

\[
\begin{align*}
\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \nabla_v \cdot (F[f^\varepsilon]f^\varepsilon) + \partial_\theta (G[f^\varepsilon]f^\varepsilon) + \frac{1}{\varepsilon} \nabla_v \cdot (L_F[f^\varepsilon]f^\varepsilon) &= 0, \\
f^\varepsilon(0, z) &=: f_0^\varepsilon(z), \quad \int_\Omega f_0^\varepsilon dz = 1, \quad \int_\Omega v f_0^\varepsilon dz = 0, \quad \int_\Omega \theta f_0^\varepsilon(z) dz =: \theta_c.
\end{align*}
\] (3)

Moreover, we are interested in the setting where particle’s temperature fields are close to equilibrium temperature \( \theta_c \) pointwise. Note that the convergence rates of velocity and temperature fields toward equilibria are proportional to the coupling strengths (see [9, 10, 11, 13] for TCS and CS models). In this situation, under the
limit process \((\varepsilon \to 0)\), we can expect that \(f^\varepsilon\) converges, in some weak sense, to a mono-kinetic distribution:

\[
\rho(x,t) \otimes \delta_{v=u(x,t)} \otimes \delta_{\theta=\theta_v}.
\]

Here, the macroscopic variables \(\rho\) and \(u\) satisfy the pressureless Euler equations with a nonlocal flocking dissipation:

\[
\begin{aligned}
\partial_t \rho + \nabla_x \cdot \left( \rho u \right) &= 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{T}^d, \\
\partial_t \left( \rho u \right) + \nabla_x \cdot \left( \rho u \otimes u \right) &= \frac{K_u}{\theta_k} \int_{\mathbb{T}^d} \phi(|x-x_*|)(u(x_*) - u(x))\rho(x)\rho(x_*) \, dx_* , \\
\rho(0,x) &= \rho_0(x), \quad u(0,x) := u_0(x), \quad \int_{\mathbb{T}^d} \rho_0 \, dx = 1.
\end{aligned}
\]

One of main difficulties in the passage from (3) to (4) is due to the singularity of the mono-kinetic distribution. In particular, the pressureless Euler equations (4) has a convex entropy \(\eta(\rho,\rho u) = \rho \frac{|u|^2}{2}\), which is not strictly convex with respect to \(\rho\). Hence the relative entropy method in \([16, 17, 20, 23]\) with the associated relative entropy made from the entropy \(\eta\) is not enough to control the convergence of the nonlocal alignment term. In \([5]\), the authors combine the relative entropy method together with the 2-Wasserstein distance to overcome this difficulty. More precisely, they estimate an \(L^2\)-distance of characteristic curves generated by vector fields \(u^\varepsilon\) and \(u\), which control the 2-Wasserstein distance between densities, and then combine the estimates for the relative entropy and the \(L^2\)-distance of characteristics to close the arguments. We will take an advantage of the method in \([5]\) to justify the singular limit from (3) to (4). For other recent results on this kinds of singular limit leading to the Dirac mass, we refer to \([14, 15, 16]\). On the other hand, there is a recent work \([22]\) for the derivation of the continuity equation supplemented by an integral constraint from the Cucker-Smale model with singular communication weight via scaling method.

The main results of this paper are three-fold. First, in Theorem 3.1, we present a global existence of a weak solution to (3). To do this, we use a weighted energy estimate and fixed point argument as in \([18]\): for each \(\delta, \lambda > 0\) and \(p_0 \in \left(1, \frac{d+2}{d+1}\right)\), define

\[
T_{\delta,\lambda} : L^{p_0}((0,T) \times \mathbb{T}^d) \to L^{p_0}((0,T) \times \mathbb{T}^d), \quad T_{\delta,\lambda}(\bar{u}) := \frac{\int_{\mathbb{R}^d \times \mathbb{R}_+} v f \, dv \, d\theta}{\delta + \int_{\mathbb{R}^d \times \mathbb{R}_+} f \, dv \, d\theta},
\]

where \(f\) is a solution of

\[
\begin{aligned}
\partial_t f + v \cdot \nabla_x f + \nabla_x \cdot \left( F[f] f + f(\chi_\lambda(\bar{u}) - v) \right) + \partial_\theta(G[f]f) &= 0, \\
f(0,x,v,\theta) &= f_0(x,v,\theta).
\end{aligned}
\]

Here, \(\chi_\lambda(u)\) is a truncation function defined as \(\chi_\lambda(u) := u 1_{|u| \leq \lambda}\). For each \(\delta\) and \(\lambda\), we find a fixed point \(u_{\delta,\lambda}\) of \(T_{\delta,\lambda}\) and take a limiting process to find a weak solution of (3).

Second, in Theorem 3.2, we provide an exponential decay estimate of projected temperature and velocity support. We consider several characteristics to derive Grönwall-type inequalities on the diameter of temperature and velocity supports.

Third, in Theorem 3.3, we provide a rigorous hydrodynamic limit from (3) to (4) for some well-prepared one parameter family of initial data. We use a relative entropy method to show the desired weak convergence, when the entropy \(\eta\) is defined
by

$$\eta(\rho, \rho u) = \frac{1}{2}\rho|u|^2.$$  

However, since $\eta$ is not strictly convex in $\rho$, we also need to estimate an $L^2$-distance of characteristics generated by $u^x$ and $u$ to control the 2-Wasserstein distance, and conclude the desired result. See [5] for details.

The rest of the paper is organized as follows. In Section 2, we recall the thermo-mechanical Cucker-Smale model and its formal hydrodynamic model, and present the relative entropy method and $a$ priori estimates for the kinetic TCS model (3) to be crucially used later. In Section 3, we briefly summarize our main results. In Section 4, we present detailed proofs for Theorem 3.1 and Theorem 3.2. In Section 5, we also provide a proof of Theorem 3.3. Finally, Section 6 is devoted to the brief summary of our main results and discussion on some unresolved issues for a future work.

**Notation:** For notational simplicity, we use the handy notation $f^\varepsilon = f^\varepsilon(t, x, v, \theta)$ and $f^\varepsilon_* = f^\varepsilon(t, x_*, v_*, \theta_*)$ for kinetic density functions. Finally, the constant $C$ stands for generic constant, which might be different from line to line.

2. **Preliminaries.** In this section, we briefly review the TCS model and its hydrodynamic counterpart, and then discuss the relative entropy method and provide some $a$ priori estimates on the projected velocity and temperature supports.

2.1. **The TCS model.** The TCS model was first introduced in [12] as a one of generalization for the well-known CS flocking model. More specifically, let $(x_i, v_i, \theta_i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+$ be a position, velocity and temperature of the $i$-th TCS particle. Then, due to the Galilean invariance and entropy principle, the dynamics of $(x_i, v_i, \theta_i)$ in the reference frame of center of mass can be written as follows:

$$\begin{aligned}
\frac{dx_i}{dt} &= v_i, \quad t > 0, \quad i = 1, 2, \ldots, N, \\
\frac{dv_i}{dt} &= \frac{\kappa_v}{N} \sum_{j=1}^{N} \phi(|x_i - x_j|) \left( \frac{v_j}{\theta_j} - \frac{v_i}{\theta_i} \right), \\
\frac{d\theta_i}{dt} &= \frac{\kappa_\theta}{N} \sum_{j=1}^{N} \zeta(|x_i - x_j|) \left( \frac{1}{\theta_i} - \frac{1}{\theta_j} \right), \\
(x_0(0), v_0(0), \theta_0(0)) &= (x_i^0, v_i^0, \theta_i^0), \quad \sum_{i=1}^{N} x_i^0 = 0, \quad \sum_{i=1}^{N} v_i^0 = 0, \quad \theta_i^0 > 0,
\end{aligned}$$

(6)

where small diffusion velocity assumption is employed (see [12] for details). Note that in order to guarantee the global existence of the TCS model (6), a temperature term $\theta_i$ appearing in the denominators on the right hand side of (6) should be away from zero for all time. Fortunately, thanks to the structure of temperature interaction (6)3, minimal and maximal temperatures are monotonic in the sense that for all $t > 0$,

$$\min_{1 \leq i \leq N} \theta_i^0 \leq \min_{1 \leq i \leq N} \theta_i(t) \leq \max_{1 \leq i \leq N} \theta_i(t) \leq \max_{1 \leq i \leq N} \theta_i^0.$$

Therefore, if the initial minimum temperature is strictly positive, the global well-posedness of TCS model follows from the standard Cauchy-Lipschitz theory. When all initial temperatures $\theta_i^0$ take the same constant temperature $\theta_e$, then the
temperature dynamics vanishes and the TCS model (6) becomes the CS flocking model:

\[
\frac{dx_i}{dt} = v_i, \quad t > 0, \quad i = 1, 2, \ldots, N,
\]

\[
\frac{dv_i}{dt} = \frac{\kappa}{N} \sum_{j=1}^{N} \tilde{\phi}(|x_i - x_j|)(v_j - v_i), \quad \tilde{\phi} := \frac{\phi}{\theta_e}.
\]

If initial diameter of velocity and temperature are sufficiently small, the temperatures are converge to same constant temperature \(\theta_e\) exponentially fast. See [7, 9] for detailed sufficient frameworks.

Moreover, the antisymmetric structure of (6) also gives the following nice properties for \(\theta_i\) as in Proposition 2.1. First, we define total momentum \(M_1(t)\), internal energy \(E(t)\) and entropy \(S(t)\) as follows:

\[
M_1(t) = \sum_{i=1}^{N} v_i(t), \quad E(t) = \sum_{i=1}^{N} \theta_i(t), \quad S(t) = \sum_{i=1}^{N} \log \theta_i(t), \quad t \geq 0.
\]

**Proposition 1** ([9]). Let \(\{(x_i, v_i, \theta_i)\}\) be the solution to (6). Then, the following assertions hold.

1. The total momentum and internal energy are conserved:

\[
M_1(t) = M_1(0), \quad E(t) = E(0), \quad t \geq 0.
\]

2. The total entropy is nondecreasing:

\[
\frac{dS}{dt}(t) = \frac{\kappa}{2N} \sum_{i,j=1}^{N} \zeta(|x_i - x_j|) \left| \frac{1}{\theta_i} - \frac{1}{\theta_j} \right|^2 \geq 0, \quad t > 0.
\]

**Proof.** We refer to [9] for a detailed proof.

On the other hand, when the number of the TCS particles is large enough, its microscopic dynamics (6) can be effectively approximated by the corresponding kinetic equation. More precisely, if \(f = f(t, x, v, \theta)\) denotes a probability density function of the TCS ensemble, its dynamics is governed by the kinetic TCS equation (1). The mean-field limit from the TCS model (6) to the kinetic TCS equation (1) can be rigorously justified, as for the CS model [11] and in fact its uniform-in-time mean-field limit was also done in [7] under the mono-cluster flocking setting.

### 2.2. Pressureless Euler equations with nonlocal alignments

Next, we briefly explain how to derive hydrodynamic equations from the kinetic equation (1) at the formal level. In [8], the authors derived a pressureless Euler equations with a nonlocal alignment force by using mono-kinetic ansatz for \(f\) in (3):

\[
f(t, x, v, \theta) = \rho(x, t) \otimes \delta_{v=u(t,x)} \otimes \delta_{\theta=e(t,x)}.
\]

In the sequel, we suppress \(t\) dependence in \(\rho, u\) and \(e\) for notational simplicity:

\[
\rho(x) := \rho(t, x), \quad u(x) := u(t, x), \quad e(x) := e(t, x).
\]
Then, the resulting hydrodynamic model reads as
\[ \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \]
\[ \partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) = \kappa_v \int_{T^d} \phi(|x - x_*|) \rho(x) (\frac{u(x)}{e(x)} - \frac{u(x)}{e(x_*)}) \, dx_*, \quad (7) \]
\[ \partial_t (\rho e) + \nabla_x \cdot (\rho e u) = \kappa_\theta \int_{T^d} \zeta(|x - x_*|) \rho(x) \rho(x_*) \left( \frac{1}{e(x)} - \frac{1}{e(x_*)} \right) \, dx_. \]

If we assume that the temperature field is homogeneous in space, i.e., \( e(x) \equiv e_* \), then the energy equation (7) is degenerate, and the above system is clearly reduced to the hydrodynamic CS equation (4).

2.3. Relative entropy method. In this subsection, we review the relative entropy method to be used for the hydrodynamic limit in a series of papers [5, 16, 17, 20, 23]. We set
\[ m := \rho u, \quad U := \begin{pmatrix} \rho \\ m \end{pmatrix}, \quad A(U) := \begin{pmatrix} m \\ m \otimes m \end{pmatrix}, \]
\[ F(U) := \begin{pmatrix} \frac{\rho}{2} \int_{T^d} \phi(|x - x_*|) \rho(x) (u(x) - u(x_*) \, dx_*, \end{pmatrix}. \]

Then, system (4) can be written as a vector form:
\[ \partial_t U + \nabla_x \cdot A(U) = F(U). \quad (8) \]

As noted in [5, 16], system (8) admits a convex entropy \( \eta \):
\[ \eta(U) := \frac{|m|^2}{2\rho}. \]

Then, we consider the relative entropy and relative flux as
\[ \eta(V|U) := \eta(V) - \eta(U) - D\eta(U) \cdot (V - U), \]
\[ A(V|U) := A(V) - A(U) - DA(U) \cdot (V - U), \quad (9) \]

where the matrix \( DA(U) \cdot (V - U) \) is defined as
\[ (DA(U) \cdot (V - U))_{ij} = \sum_{k=1}^{d+1} \partial_{u_k} A_{ij}(U_k - U_k), \quad 1 \leq i \leq d + 1, \quad 1 \leq j \leq d. \]

Moreover, the relative entropy between two macroscopic observables \( V = (q, qw)^T \) and \( U = (\rho, \rho u)^T \) is given by
\[ \eta(V|U) = \frac{q}{2} |w|^2 - \frac{\rho}{2} |u|^2 + \frac{|u|^2}{2} (q - \rho) - u \cdot (qw - \rho u) = \frac{q}{2} |u - w|^2. \quad (10) \]

The next proposition provides a key ingredient to establish a hydrodynamic limit through the relative entropy method.

**Proposition 2** ([16, 20]). Let \( U \) be a strong solution to system (8), and \( V \) be any smooth function. Then, we have
\[ \frac{d}{dt} \int_{T^d} \eta(V|U) \, dx = \frac{d}{dt} \int_{T^d} \eta(V) \, dx - \int_{T^d} \nabla_x (D\eta(U)) : A(V|U) \, dx \\
- \int_{T^d} D\eta(U) \cdot [\partial_t V + \text{div}_x A(V) - F(V)] \, dx \\
- \int_{T^d} [D^2 \eta(U) F(U) (V - U) + D\eta(U) F(V)] \, dx. \]
Proof. For a proof, we refer to [16, 20].

2.4. A priori estimates. In this subsection, we study several basic estimates on the kinetic TCS equation (3). For a given \((x, v, \theta)\) at time \(t\), we define tr-characteristic curves

\[
x^c(s) := x^c(s; t, x, v, \theta), \quad v^c(s) := v^c(s; t, x, v, \theta), \quad \theta^c(s) := \theta^c(s; t, x, v, \theta)
\]
as a solution of the following ODE system:

\[
\begin{aligned}
\frac{d}{ds} x^c(s) &= v^c(s), \quad s > 0, \\
\frac{d}{ds} v^c(s) &= \kappa_v \int _\Omega \phi(|x^c(s) - x_s|) \left( \frac{v^c_s - v^c(s)}{\theta^c(s)} \right) f^c_s \, dz_s + \frac{1}{\varepsilon} (u^c(x^c(s), s) - v^c(s)), \\
\frac{d}{ds} \theta^c(s) &= \kappa_\theta \int _\Omega \zeta(|x^c(s) - x_s|) \left( \frac{1}{\theta^c(s)} - \frac{1}{\theta_s} \right) f^c_s \, dz_s,
\end{aligned}
\]

\((x^c(t), v^c(t), \theta^c(t)) = (x, v, \theta).\) \hspace{1cm} (11)

In next lemma, we estimate velocity and temperature supports of the solution \(f^c\) to (3). For notational convenience, we denote the \(v\)- and \(\theta\)-support of \(f^c\) as

\[
\Omega_v(f^c(t)) := \text{supp}_v f^c(t, \cdot, \cdot, \cdot), \quad \Omega_\theta(f^c(t)) := \text{supp}_\theta f^c(t, \cdot, \cdot, \cdot).
\]

Lemma 2.1. For a positive constant \(T \in [0, \infty)\), let \(f^c\) be a classical solution of \((3)\) on the time interval \([0, T]\).

1. Suppose that the temperature support of the initial datum \(f^c_0\) is contained in an \(\varepsilon\)-independent compact set \([\theta_m, \theta_M]\):

\[
\Omega_\theta(f^c(0)) \subset [\theta^c_m, \theta^c_M] \subset [\theta_m, \theta_M].
\]

Then, the temperature support of \(f\) is non-increasing in time:

\[
\Omega_\theta(f^c(s)) \subset \Omega_\theta(f^c(t)), \quad 0 \leq s \leq t \leq T.
\]

2. In addition, if the initial velocity support of \(f^c_0\) is contained in an \(\varepsilon\)-independent compact set \(\Omega_v\):

\[
\Omega_v(f^c(0)) \subset \Omega_v,
\]

then the velocity support of \(f^c\) is bounded as:

\[
\sup _{v \in \Omega_v(f^c(t))} |v| \leq R(t), \quad 0 \leq t \leq T,
\]

where

\[
R(t) := \left( \sup _{v \in \Omega_v(f^c(0))} |v| \right) \exp \left( \frac{2 \kappa_v \phi_M}{\theta_m} t \right).
\]

Proof. (1) Let \((x^c_1(s), v^c_1(s), \theta^c_1(s))\) be a characteristics along which the temperature support has maximum value:

\[
\theta^c_1(s) = \sup _{\theta \in \Omega_\theta(f^c(s))} \theta.
\]

Then, it follows from \((11)_3\) that we have

\[
\frac{d\theta^c_1(s)}{ds} = \kappa_\theta \int _\Omega \zeta(|x^c_1(s) - x_s|) \left( \frac{1}{\theta^c_1(s)} - \frac{1}{\theta_s} \right) f^c_s \, dz_s \leq 0,
\]

where the last inequality comes from the maximality of \(\theta^c_1(s)\). Therefore, the maximum temperature decreases along the time. By the same argument, we can see
that the minimum temperature increases. This implies the first part of the lemma. In particular, one has
\[ \Omega_\theta(f^\varepsilon(s)) \subset \Omega_\theta(f^\varepsilon(0)) \subset [\theta_m, \theta_M]. \]

(2) Let \((x_2^\varepsilon(s), v_2^\varepsilon(s), \theta_2^\varepsilon(s))\) be characteristic curves such that the velocity support has the maximum value:
\[ |v_2^\varepsilon(s)| = \max_{v \in \Omega_\theta(f^\varepsilon(s))} |v|. \]

Then, we take an inner product \((11)_2\) with \(v_2^\varepsilon(s)\) to obtain
\[ \frac{1}{2} \frac{d}{ds} |v_2^\varepsilon(s)|^2 \]
\[ = \kappa_v \int_\Omega \phi(|x_2^\varepsilon(s) - x_\ast|) v_2^\varepsilon(s) \cdot \left( \frac{v_\ast}{\theta_\ast} - \frac{v_2^\varepsilon(s)}{\theta_2^\varepsilon(s)} \right) f^\varepsilon dz_\ast + \frac{1}{\varepsilon} (u^\varepsilon(x_2^\varepsilon(s), s) - v_2^\varepsilon(s)) \cdot v_2^\varepsilon(s). \]
\[ \leq \kappa_v \int_\Omega \phi(|x_2^\varepsilon(s) - x_\ast|) v_2^\varepsilon(s) \cdot \left( \frac{v_\ast}{\theta_\ast} - \frac{v_2^\varepsilon(s)}{\theta_2^\varepsilon(s)} \right) f^\varepsilon dz_\ast, \]

where the last inequality is due to the maximality of \(v_2^\varepsilon(s)\):
\[ |u^\varepsilon(x_2^\varepsilon(s), s)| = \frac{\int_{\mathbb{R}^d \times \mathbb{R}_+} v f^\varepsilon(s, x_2^\varepsilon(s), v, \theta) dv d\theta}{\int_{\mathbb{R}^d \times \mathbb{R}_+} f^\varepsilon(s, x_2^\varepsilon(s), v, \theta) dv d\theta} \leq |v_2^\varepsilon(s)|. \]

Therefore,
\[ \frac{1}{2} \frac{d}{ds} |v_2^\varepsilon(s)|^2 \leq \kappa_v \int_\Omega \phi(|x_2^\varepsilon(s) - x_\ast|) v_2^\varepsilon(s) \cdot \left( \frac{v_\ast}{\theta_\ast} - \frac{v_2^\varepsilon(s)}{\theta_2^\varepsilon(s)} \right) f^\varepsilon dz_\ast \]
\[ \leq \kappa_v \phi_M |v_2^\varepsilon(s)| \int_\Omega \left( \frac{v_\ast}{\theta_\ast} + \left| \frac{v_2^\varepsilon(s)}{\theta_2^\varepsilon(s)} \right| \right) f^\varepsilon dz_\ast \]
\[ \leq \frac{2 \kappa_v \phi_M |v_2^\varepsilon(s)|^2}{\theta_m}, \]

where the last inequality comes from the maximality of \(v_2^\varepsilon(s)\) and the temperature support of \(f^\varepsilon\). Then, we use the Grönwall lemma to obtain
\[ |v_2^\varepsilon(t)| \leq \left( \sup_{v \in \Omega_\theta(f^\varepsilon(0))} |v| \right) \exp \left( \frac{2 \kappa_v \phi_M}{\theta_m} t \right) =: R(t), \quad (12) \]
which implies the second part of the lemma.

\( \square \)

3. Description of main results. In this section, we briefly present the three main results of this paper. First, we prove the existence of a weak solution to (3) (Theorem 3.1), and we also provide the flocking estimate of a solution to (3) (Theorem 3.2). Then, we justify the hydrodynamic limit of kinetic TCS equation (3) to Euler-type equations (4) under several assumptions on the initial data (Theorem 3.3).

3.1. Weak solutions and flocking dynamics. In this subsection, we present a global existence of weak solutions to (3) and asymptotic flocking estimate for strong solutions. Now, we state our first main result on the global existence of weak solutions as follows.
Theorem 3.1. For any $\varepsilon > 0$, suppose that initial datum $f^0_\varepsilon$ satisfies

(i) $f^0_\varepsilon \geq 0$, $f^0_\varepsilon \in L^1 L^\infty(\Omega)$, $|v|^2 f^0_\varepsilon \in L^1(\Omega)$,

(ii) $f^0_\varepsilon$ has compact temperature support $[\theta_m, \theta_M]$.

Then, there exists a weak solution $f^\varepsilon \geq 0$ of (3) satisfying the following properties:

1. $f^\varepsilon$ satisfies the regularity and integrability:

$$f^\varepsilon \in C(0, T; L^1(\Omega)) \cap L^\infty([0, T) \times \Omega), \quad |v|^2 f^\varepsilon \in L^\infty(0, T; L^1(\Omega)),$$

2. $f^\varepsilon$ satisfies weak formulation: for any $\varphi \in C^\infty_c([0, T) \times \Omega)$, we have

$$\int_0^t \int_\Omega f^\varepsilon \left( \partial_t \varphi + v \cdot \nabla_x \varphi + F[f^\varepsilon] \cdot \nabla_v \varphi + G[f^\varepsilon] \partial_\theta \varphi + \frac{1}{\varepsilon} (u^\varepsilon - v) \cdot \nabla_v \varphi \right) dz \, ds + \int_\Omega f^\varepsilon_0 \varphi(0, \cdot) \, dz = 0.$$

Proof. The proof will be given in Section 4.1. \qed

Next, we present our second main result on the flocking dynamics of (3). Our second main result deals with exponential contraction of temperature support and flocking estimate.

Theorem 3.2. Let $f^\varepsilon$ be a classical solution to (3) with compact and strictly positive initial temperature and velocity support:

$$\Omega_0(f^\varepsilon(0)) \subset [\theta_m, \theta_M] \subset \mathbb{R}_+, \quad \text{diam}(\Omega_0(f(0))) < \infty.$$

Then, the temperature and velocity support of $f$ decays exponentially:

$$\Omega_0(f^\varepsilon(t)) \subset [\theta_e - r_1(t), \theta_e + r_1(t)], \quad \Omega_0(f^\varepsilon(t)) \subset \tilde{B}_{r_2(t)}(0),$$

where

$$r_1(t) := e^{-\frac{\kappa \theta_m t}{\varepsilon}}, \quad \zeta_m := \min_{x \in \mathbb{T}^d} \{ \zeta(|x|) \}, \quad \phi_m := \min_{x \in \mathbb{T}^d} \{ \phi(|x|) \},$$

$$r_2(t) := \text{diam}(\Omega_0(f(0))) e^{-\frac{\kappa \theta_m t}{\varepsilon}} + C \left( e^{-\frac{\kappa \varepsilon \phi M \delta}{\varepsilon \delta M}} - e^{-\frac{\kappa \varepsilon \phi M \delta}{\varepsilon \delta M}} \right), \quad C > 0.$$

3.2. Hydrodynamic limit. For the hydrodynamic limit, we consider one-parameter family of well-prepared initial data $f^0_\varepsilon$: for all $\varepsilon > 0$,

- $(A_1)$: $\left| \int_{\Omega} \frac{|v|^2}{2} f^0_\varepsilon dz - \int_{\mathbb{T}^d} \frac{|u_0|^2}{2} \rho_0 dx \right| \leq O(\varepsilon)$,

- $(A_2)$: $\| \rho^\varepsilon_0 - \rho_0 \|_{L^1(\mathbb{T}^d)} + \| u^\varepsilon_0 - u_0 \|_{L^\infty(\mathbb{T}^d)} \leq O(\varepsilon)$,

- $(A_3)$: $\| \Omega_0(f^\varepsilon(0)) \| = \theta^\varepsilon_M - \theta^\varepsilon_m = O(\varepsilon)$, $\int_{\Omega} \theta^\varepsilon f^\varepsilon dz =: \theta_e$.

Note that the assumption $(A_3)$ implies that the temperature support of $f^0_\varepsilon$ tends to $\theta_e$ pointwisely as $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} \theta^\varepsilon_M = \lim_{\varepsilon \to 0} \theta^\varepsilon_m = \theta_e.$$

On the other hand, for the hydrodynamic limit in the previous articles [5, 16, 20, 23], we need the smoothness of solutions to the limit system (4). We here present a result for the existence of strong solutions to the system (4), which was proved in [6].
Proposition 3 ([6]). Suppose that initial data satisfy
\[ \rho_0 > 0 \quad \text{in} \quad \mathbb{T}^d \quad \text{and} \quad (\rho_0, u_0) \in H^s(\mathbb{T}^d) \times H^{s+1}(\mathbb{T}^d) \quad \text{for} \quad s > \frac{d}{2} + 1. \] (16)
Then, there exists \( T_* > 0 \) such that (4) has a unique smooth solution \((\rho, u)\) satisfying
\[ \rho \in C^0([0, T_*]; H^s(\mathbb{T}^d)) \cap C^1([0, T_*]; H^{s-1}(\mathbb{T}^d)), \]
\[ u \in C^0([0, T_*]; H^{s+1}(\mathbb{T}^d)) \cap C^1([0, T_*]; H^s(\mathbb{T}^d)). \] (17)
Moreover, if the initial data satisfying (16) is suitably small, we obtain a global well-posedness of smooth solutions.

To present our last theorem, we briefly introduce the \( p \)-Wasserstein distance \( W_p \) on \( \mathbb{R}^d \times \mathbb{R}^d \). For \( p \geq 1 \), the \( p \)-Wasserstein distance between two probability measures \( \mu_1 \) and \( \mu_2 \) on \( \mathbb{R}^d \) is defined by
\[ W^P_p(\mu_1, \mu_2) := \inf_{\nu \in \mathcal{A}(\mu_1, \mu_2)} \int_{\mathbb{R}^d} \|x - y\|^p \nu(dx, dy), \]
where \( \mathcal{A}(\mu_1, \mu_2) \) denotes the set of all probability measures \( \nu \) on \( \mathbb{R}^{2d} \) with marginals \( \mu_1 \) and \( \mu_2 \), i.e.,
\[ \pi_{1#} \nu = \mu_1, \quad \pi_{2#} \nu = \mu_2, \]
where \( \pi_1 : (x, y) \mapsto x \) and \( \pi_2 : (x, y) \mapsto y \) are the natural projections from \( \mathbb{R}^d \times \mathbb{R}^d \) to \( \mathbb{R}^d \), and \( \pi_{#} \nu \) denotes the push forward of \( \nu \) through a map \( \pi \), i.e., \( \pi_{#} \nu(B) := \nu(\pi^{-1}(B)) \) for any Borel set \( B \). This definition can be similarly extended to measures on the torus \( \mathbb{T}^d \) with the understanding that \( \|x - y\| \) denotes the distance on the torus.

We are now ready to present a result on hydrodynamic limit from the kinetic TCS equation (3) to the hydrodynamic CS model (4).

Theorem 3.3. Suppose that one-parameter family of initial data \( \{f_0^\varepsilon\}_{\varepsilon > 0} \) satisfy (13) and (A1) – (A3), let \( f^\varepsilon \) be the corresponding weak solution to (3) as a consequence of Theorem 3.1, and \((\rho, u)\) be a local smooth solution to (4) up to the time \( T_* \). Then, the following assertions hold:

1. There exists a positive constant \( C \) (depending on \( T_* \)) such that, for all \( t \leq T_* \),
\[ \int_{\mathbb{T}^d} \rho^\varepsilon |u^\varepsilon - u|^2(t) \, dx + W_2^2(\rho^\varepsilon(t), \rho(t)) \leq C\varepsilon, \] (18)
where \( \rho^\varepsilon \) and \( u^\varepsilon \) are defined as
\[ \rho^\varepsilon := \int_{\mathbb{R}^d \times \mathbb{R}_+} f^\varepsilon \, dv \, d\theta, \quad \rho^\varepsilon |u^\varepsilon := \int_{\mathbb{R}^d \times \mathbb{R}_+} v f^\varepsilon \, dv \, d\theta, \]
and \( W_2 \) denotes the 2-Wasserstein distance.

2. \( f^\varepsilon \) converges to a mono-kinetic distribution weakly:
\[ f^\varepsilon \rightharpoonup \rho(t, x) \otimes \delta_{u=\rho(t,x)} \otimes \delta_{\theta=0}, \quad \text{in} \quad \mathcal{M}((0, T_*) \times \Omega), \] (19)
where \( \mathcal{M}((0, T_*)) \times \Omega) \) is the space of nonnegative Radon measures on \((0, T_*) \times \Omega).
4. Existence of weak solutions and flocking dynamics. In this section, we provide proofs of Theorem 3.1 and Theorem 3.2.

4.1. A global existence of a weak solution. The proof follows the same strategy as in [18], which deals with the existence of weak solution for the original Cucker-Smale flocking equation with local alignment force and diffusion. Thus, we briefly provide a main part of proof for the existence of weak solutions to (3). Fix \( \varepsilon = 1 \) for simplicity. We consider the Cauchy problem to the following regularized equation of (3):

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F[f] f + f(\chi(\lambda \bar{u}) - v)) + \partial_\theta (G[f] f) &= 0, \\
(0, x, v, \theta) &= f_0(x, v, \theta).
\end{align*}
\]

Here, \( \chi = \chi(\lambda) \) denotes the truncation function defined as \( \chi(\lambda) := u1_{|u| \leq \lambda} \), and \( u_\delta \) is defined as

\[
u_\delta := \frac{\int_{\mathbb{R}^d \times \mathbb{R}^+} v f dv \, d\theta}{\delta + \int_{\mathbb{R}^d \times \mathbb{R}^+} f dv \, d\theta} = \frac{\rho_0}{\delta + \rho_0}.
\]

To prove the existence of weak solution to (20), we consider a map \( T \) as

\[
T : L^{p_0}((0, T) \times \mathbb{T}^d) \rightarrow L^{p_0}((0, T) \times \mathbb{T}^d), \quad \bar{u} \mapsto T(\bar{u}) := \frac{\int_{\mathbb{R}^d \times \mathbb{R}^+} v f dv \, d\theta}{\delta + \int_{\mathbb{R}^d \times \mathbb{R}^+} f dv \, d\theta},
\]

where \( p_0 \in \left(1, \frac{d+2}{d+1}\right) \) is the same choice as in [18] and \( f \) is a weak solution of

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F[f] f + f(\chi(\lambda \bar{u}) - v)) + \partial_\theta (G[f] f) &= 0, \\
(0, x, v, \theta) &= f_0(x, v, \theta).
\end{align*}
\]

Then, the existence of weak solution to (20) will be made using Schauder’s fixed point theorem for \( T \). In the sequel, we study the well-definedness and compactness of the mapping \( T \) one by one.

First, we define energy functional:

\[
E(t) := \int_{\Omega} \frac{1}{2} f dz.
\]

Then, the well-definedness of \( T \) will be shown in the sequel.

Lemma 4.1. Suppose that initial datum \( f_0 \) satisfies

\[
f_0 \geq 0, \quad f_0 \in (L^1 \cap L^\infty)(\Omega), \quad |v|^2 f_0 \in L^1(\Omega),
\]

\( f_0 \) has a compact temperature support \([\theta_m, \theta_M]\).

For any \( \bar{u} \in L^p((0, T) \times \mathbb{T}^d) \) with \( p \in [1, \infty] \), let \( f \) be the solution to (21) with fixed \( \lambda, \delta, T > 0 \). Then, there exists a positive constant \( C \) such that

\[
\|f\|_{L^\infty(0, T; L^p(\Omega))} \leq e^{CT} \|f_0\|_{L^p(\Omega)} \quad \text{and} \quad E(t) \leq \left(\frac{\lambda^2}{2C} + E(0)\right) e^{CT}.
\]

Proof. (i) \( (L^\infty\text{-bound estimate}) \): Direct calculation implies that for a fixed \( p < \infty \),

\[
\frac{d}{dt} \int_{\Omega} f^p dz = p \int_{\Omega} f^{p-1} (-v \cdot \nabla_x f - \nabla_v \cdot (F[f] f + f(\chi(\lambda \bar{u}) - v)) - \partial_\theta (G[f] f)) dz
\]

\[
= -(p - 1) \int_{\Omega} f^{p} (\nabla_v \cdot F[f] + \partial_\theta G[f]) dz + (p - 1) \int_{\Omega} f^{p} dz.
\]
Since \( f \) is compactly supported with respect to \( \theta \) by the same argument in Lemma 2.1, we have
\[
f^p|\nabla_v \cdot F[f]| \leq f^p d\kappa_v \int_{\Omega} \phi(|x - x^*|) \frac{1}{\theta} f |dz| \leq \frac{d\kappa_v \phi_M}{\theta_m} f^p,
\]
\[
f^p|\partial_\theta G[f]| \leq f^p \frac{\kappa_\theta}{\theta^2} \int_{\Omega} \zeta(|x - x^*|) f |dz| \leq \frac{\kappa_\theta \zeta_M}{\theta_m^2} f^p.
\]
Therefore, Grönwall inequality implies that
\[
\|f\|_{L^\infty(0,T;L^p(\Omega))} \leq e^{CT_{\text{p}^{-1}}} \|f_0\|_{L^p(\Omega)}.
\]
Now, we take \( p \to \infty \) to get the \( L^\infty \)-bound of \( f \).

(ii) (Energy estimate): For the second assertion, we take a derivative to (22) to obtain
\[
\frac{dE}{dt} = \int_{\Omega} v \cdot F[f] f + f(\chi_\lambda(\bar{u}) - v) \cdot v.
\]
On the other hand, we have the following estimates:
\[
\int_{\Omega} v \cdot F[f] f |dz| \leq \kappa_v \phi_M \left( \int_{\Omega} |v| v |dz| f |dz| + \int_{\Omega} |v| f |dz| f |dz| \right)
\leq \frac{\kappa_v \phi_M}{\theta_m} \left( \left( \int_{\Omega} |v| f |dz| \right)^2 + \int_{\Omega} |v|^2 f |dz| \right)
\leq C \int_{\Omega} |v|^2 f |dz| \leq CE(t),
\]
and
\[
\int_{\Omega} f(\chi_\lambda(\bar{u}) \cdot v |dz| \leq \frac{1}{2} \int_{\Omega} f |\chi_\lambda(\bar{u})|^2 |dz| + \frac{1}{2} \int_{\Omega} f |v|^2 |dz|.
\]
These estimates imply
\[
\frac{dE}{dt} \leq \frac{1}{2} \int_{\Omega} f |\chi_\lambda(\bar{u})|^2 |dz| + CE(t) \leq \frac{\lambda^2}{2} + CE(t), \tag{24}
\]
and the Grönwall lemma yields the desired estimate.

Following [18, Lemma 2.4], the estimates in Lemma 4.1 implies \( \|\rho u\|_{L^\infty(0,T;L^p(\Omega))} \) is finite, and therefore the operator \( T \) is well-defined.

Then, the existence of weak solution to (20) can be done as follows.

**Proposition 4.** Suppose that initial datum \( f_0 \) satisfies
\[
f_0 \geq 0, \quad f_0 \in (L^1 \cap L^\infty)(\Omega), \quad |v|^2 f_0 \in L^1(\Omega),
\]
\[f_0 \text{ has a compact temperature support } [\theta_m, \theta_M].\]

Then, for any \( \delta, \lambda, T > 0 \) and \( p \in [1, \infty] \), there exists a solution \( f \) to (20) and a generic positive constant \( C \) independent of \( \lambda \) and \( \delta \) such that
\[
\|f\|_{L^\infty(0,T;L^p(\Omega))} \leq e^{CT} \|f_0\|_{L^p(\Omega)} \quad \text{and} \quad E(t) \leq CE(0).
\]

**Proof.** For the compactness of the operator \( T \), we use the velocity averaging lemma [18, Proposition 2.5]. To this end, it is enough to show that
\[
\tilde{f} := \int_{\mathbb{R}_+} f \, d\theta
\]
is bounded in $L^\infty(0,T; L^p(\omega))$ for any $1 < p < \infty$, and

$$G := \tilde{f}(\chi_\lambda(\bar{u}) - v) + \int_{\mathbb{R}_+} fF[f] \, d\theta$$

is bounded in $L^\infty(0,T; L^q(\omega))$ for any $1 < q < 2$. Then, it follows from (21) that $\tilde{f}$ and $G$ satisfy

$$\partial_t \tilde{f} + v \cdot \nabla_x \tilde{f} + \nabla_v \cdot G = 0$$

(see [18, Lemma 3.4] and [18, Lemma 3.5]). Using Hölder’s inequality and Lemma 4.1 together with the compactness of temperature support as in Lemma 4.1, we have

$$\int_{\omega} \tilde{f}^p \, dx \, dv = \int_{\omega} \left( \int_{\mathbb{R}_+} f \, d\theta \right)^p \, dx \, dv \leq C \int_{\omega} \int_{\mathbb{R}_+} f \, d\theta \, dx \, dv \leq C \int_{\omega} f_0^p \, dz,$$

and thus

$$\|G\|_{L^q(\omega)}$$

$$\leq C\|\tilde{f}\|_{L^q_{\mathbb{R}_+}}^q + C\|\tilde{f}v\|_{L^q_{\mathbb{R}_+}}^q + C\int_{\omega} \left( \int_{\mathbb{R}_+} f f_s(\|v_s\| + \|v\|) \, dz \, d\theta \right)^q \, dx \, dv$$

$$\leq C\|\tilde{f}\|_{L^q_{\mathbb{R}_+}}^q + C\|\tilde{f}v\|_{L^q_{\mathbb{R}_+}}^q + C\int_{\omega} \left( \tilde{f} \int_{\omega} f_s |v_s| \, dx \, dv \right)^q \, dx \, dv$$

$$\leq C\|\tilde{f}\|_{L^q_{\mathbb{R}_+}}^q + C\|\tilde{f}\|_{L^q_{\mathbb{R}_+}}^q \left( \int_{\mathbb{T}^d \times \mathbb{R}_+} \tilde{f} |v|^2 \, dx \, d\theta \right)^{\frac{q}{2}} + C\|\tilde{f}\|_{L^q_{\mathbb{R}_+}}^q \|\tilde{f}v\|_{L^1_{\mathbb{R}_+}}^q \frac{\pi}{\mathbf{F}_{\mathbb{R}_+}(\omega)}$$

$$\leq C \langle f_0, E(0), T \rangle.$$

This concludes that the operator $T$ is a compact operator. Then, by Schauder’s fixed point theorem, there exists a fixed point $u \in L^{p_0}((0, T) \times \mathbb{R}^d)$. Moreover, since

$$\int_{\Omega} f(\chi_\lambda(u_\delta))^2 \, dz \leq \int_{\mathbb{R}^d} \rho |u|^2 \, dx \leq \int_{\Omega} f|v|^2 \, dz,$$

(24) implies the uniform boundedness of $E(t)$:

$$\frac{dE}{dt} \leq CE(t),$$

where $C$ does not depend on $\delta$ or $\lambda$. 

We have shown the existence of the weak solution $f = f_{\delta, \lambda}$ to (20). The remaining part is for passing to the limit $\lambda \to \infty$ and $\delta \to 0$. However, this limiting process is almost the same as in [18, Section 4], we omit the remaining proof and complete the sketch of proof of the existence of weak solution of (3).

4.2. Proof of Theorem 3.2. In this subsection, we provide our second main result on the flocking dynamics for the kinetic TCS equation with local alignment (3), for a fixed $\varepsilon = 1$. We first obtain temperature flocking estimate and then using this, we show the velocity flocking. Now, we multiply (3) by $\theta$ or $v$ and integrate over state space to obtain

$$\frac{d}{dt} \int_{\Omega} \theta f \, dz = 0, \quad \frac{d}{dt} \int_{\Omega} v f \, dz = 0.$$
This implies that the averaged temperature and velocity are conserved along the evolution, and therefore, if flocking happens, temperature should converges to the average of initial temperature \( \theta_e \), and velocity should converges to 0:

\[
\theta_e = \int_\Omega \theta f_0 \, dz, \quad 0 = \int_\Omega v f_0 \, dz.
\]

The proof of Theorem 3.2 depends on the following two lemmas. First, we show that the diameter of temperature support decays exponentially by using the monotonicity property of its diameter proved in Lemma 2.1, which yields the first part of Theorem 3.2.

**Lemma 4.2.** Let \( f = f(t, x, v, \theta) \) be a classical solution to (3) with compact and strictly positive initial temperature support. Then, the diameter of the temperature support of \( f \) decays exponentially:

\[
\text{diam}(\Omega_\theta(f(t))) \leq \text{diam}(\Omega_\theta(f(0))) e^{-\frac{\kappa \zeta_m t}{\theta_1}} , \quad t \geq 0,
\]

where \( \zeta_m := \min_{x \in \mathbb{T}^d} \zeta(|x|) \).

**Proof.** We basically follow the proof of Lemma 2.1. First, we consider the characteristic curves \( (x_1(s), v_1(s), \theta_1(s)) \) which give us the maximal temperature. Then,

\[
\frac{d(\theta_1(s) - \theta_e)}{ds} = \kappa \theta \int_\Omega \zeta(|x_1(s) - x_*|) \left( \frac{1}{\theta_1(s)} - \frac{1}{\theta_*} \right) f_* \, dz_*
\]

\[
= \kappa \theta \int_\Omega \zeta(|x_1(s) - x_*|) \left( \frac{\theta_* - \theta_1(s)}{\theta_1(s)\theta_*} \right) f_* \, dz_*
\]

\[
\leq \kappa \theta \int_\Omega \frac{\zeta_m}{\theta_1} (\theta_* - \theta_1(s)) f_* \, dz_* = \frac{\kappa \theta \zeta_m}{\theta_1} (\theta_e - \theta_1(s)).
\]

This yields exponential convergence of maximal temperature:

\[
(\theta_1(t) - \theta_e) \leq (\theta_1 - \theta_e) e^{-\frac{\kappa \zeta_m t}{\theta_1}} \quad (25)
\]

Similarly, we use the same argument as above to get

\[
(\theta_e - \theta_2(t)) \leq (\theta_e - \theta_m) e^{-\frac{\kappa \zeta_m t}{\theta_1}} ,
\]

for the characteristic curves \( (x_2(s), v_2(s), \theta_2(s)) \) which give us the minimal temperature. We add two estimates (25) and (26) to obtain desired estimate.

In the following lemma, we estimate the diameter of velocity support \( \Omega_v(f(t)) \).

**Lemma 4.3.** Let \( f = f(t, x, v, \theta) \) be a classical solution to (3) with compact and strictly positive initial temperature support. Then, there exists a constant \( C > 0 \) such that the diameter of the velocity support of \( f \) decays exponentially:

\[
\text{diam}(\Omega_v(f(t))) \leq \text{diam}(\Omega_v(f(0))) e^{-\frac{\kappa \phi_m t}{\theta_1}} + C \left( e^{-\frac{\kappa \zeta_m t}{\theta_1}} - e^{-\frac{\kappa \phi_m t}{\theta_1}} \right) , \quad t \geq 0,
\]

where \( \phi_m := \min_{x \in \mathbb{T}^d} \phi(|x|) \) and \( \zeta_m := \min_{x \in \mathbb{T}^d} \zeta(|x|) \).

**Proof.** In order to estimate the size of diameter, we again choose two characteristic curves \( (x_1(s), v_1(s), \theta_1(s)) \) and \( (x_2(s), v_2(s), \theta_2(s)) \) such that

\[
|v_1(s) - v_2(s)| = \sup_{v, v_* \in \Omega_v(f(s))} |v - v_*|.
\]
Here, the boundedness of velocity support obtained in Lemma 2.1 (2) guarantees the existence of such characteristic curves. Then, we estimate $\frac{1}{2} \frac{|v_1 - v_2|^2}{ds}$ as

$$\frac{1}{2} \frac{|v_1 - v_2|^2}{ds} = (v_1 - v_2) \cdot \left[ \kappa_v \int_{\Omega} \phi(|x_1 - x_*|) \left( \frac{v_s}{\theta_s} - \frac{v_1}{\theta_1} \right) f_* dz_* + (u(x_1, s) - v_1) \right]$$

$$- \kappa_v \int_{\Omega} \phi(|x_2 - x_*|) \left( \frac{v_s}{\theta_s} - \frac{v_2}{\theta_2} \right) f_* dz_* - (u(x_2, s) - v_2) \right]$$

$$= (v_1 - v_2) \cdot (u(x_1, s) - u(x_2, s)) - |v_1 - v_2|^2$$

$$+ \kappa_v (v_1 - v_2) \cdot \int_{\Omega} \left[ \phi(|x_1 - x_*|) \left( \frac{v_s}{\theta_s} - \frac{v_1}{\theta_1} \right) - \phi(|x_2 - x_*|) \left( \frac{v_s}{\theta_s} - \frac{v_2}{\theta_2} \right) \right] f_* dz_*$$

$$=: J_1 + J_2 + J_3.$$  

- (Estimate of $J_1 + J_2$): Note that for any $v \in \Omega_v(f(s))$, the maximality of $|v_1 - v_2|$ implies

$$(v_1 - v_2) \cdot (v_1 - v) \leq |v_1 - v_2|^2; \quad \text{hence} \quad (v_1 - v_2) \cdot v_2 \leq (v_1 - v_2) \cdot v. \quad (27)$$

Similarly, we obtain

$$(v_1 - v_2) \cdot (v - v_2) \leq |v_1 - v_2|^2; \quad \text{hence} \quad (v_1 - v_2) \cdot v \leq (v_1 - v_2) \cdot v_1. \quad (28)$$

Therefore, these estimate yields

$$(v_1 - v_2) \cdot v_2 \leq (v_1 - v_2) \cdot u(x_1, s) = \frac{\int_{\Omega} (v_1 - v_2) \cdot v f(x_1, v, \theta) dv d\theta}{\int_{\Omega} f(x_1, v, \theta) dv d\theta} \leq (v_1 - v_2) \cdot v_1,$$

and

$$(v_1 - v_2) \cdot v_2 \leq (v_1 - v_2) \cdot u(x_2, s) \leq (v_1 - v_2) \cdot v_1.$$  

Therefore, we have

$$\|v_1(s) - v_2(s)) \cdot (u(x_1(s), s) - u(x_2(s), s))\| \leq |v_1(s) - v_2(s)|^2,$$

and consequently, we have $J_1 + J_2 \leq 0$.  

- (Estimate of $J_3$): To estimate $J_3$, we define a (parametrized) nonnegative measure $\Phi(z; z)$ as follows: for any fixed $z \in \Omega$,

$$d\Phi(z; z) := \phi(|x - x_*|) f(z) dz_* + \left( \phi_M - \int_{\Omega} \phi(|x - \tilde{x}|) f(\tilde{z}) d\tilde{z} \right) \delta_z(dz_*),$$

where $\delta_z$ denotes the Dirac measure concentrated at $z$. Then, we have

$$\int_{\Omega} d\Phi(z; z) = \phi_M. \quad (29)$$

Since $\phi_m \leq \phi(|x|) \leq \phi_M$, for all $x \in \mathbb{T}^d$, we find that

$$d\Phi(z; z) - \phi_m f(z) dz_* \geq 0. \quad (30)$$

Moreover, since for any $z$,

$$\int_{\{f > 0\}} \left( \frac{v_s}{\theta_s} - \frac{v}{\theta} \right) \delta_z(dz_*) = 0,$$

we have

$$\int_{\{f > 0\}} \left( \frac{v_s}{\theta_s} - \frac{v}{\theta} \right) d\Phi(z; z) = \int_{\{f > 0\}} \phi(|x - x_*|) \left( \frac{v_s}{\theta_s} - \frac{v}{\theta} \right) f_* dz_* \quad (31).$$
Furthermore, using (27), (28) and (30), we have
\[ J_3 = \kappa_v (v_1 - v_2) \cdot \int_{\{t > 0\}} \left[ \frac{v_s}{\theta_s} - \frac{v_1}{\theta_1} \right] d\Phi(z_s; z_1) - \left( \frac{v_s}{\theta_s} - \frac{v_2}{\theta_2} \right) d\Phi(z_s; z_2) \]
\[ = \kappa_v (v_1 - v_2) \cdot \left[ \phi_M \left( \frac{v_2}{\theta_2} - \frac{v_1}{\theta_1} \right) + \int_{\{t > 0\}} \frac{v_s}{\theta_s} d\Phi(z_s; z_1) - \Phi(z_s; z_2) \right] \]
\[ = \kappa_v (v_1 - v_2) \cdot \left[ \phi_M \left( \frac{v_2}{\theta_2} - \frac{v_1}{\theta_1} \right) + \int_{\{t > 0\}} \frac{v_s}{\theta_s} (d\Phi(z_s; z_1) - \phi_m f(z_s) dz_s) \right. \]
\[ \quad - \left. \int_{\{t > 0\}} \frac{v_s}{\theta_s} (d\Phi(z_s; z_2) - \phi_m f(z_s) dz_s) \right]. \]

Furthermore, using (27), (28) and (30), we have
\[ J_3 \leq \kappa_v (v_1 - v_2) \cdot \left[ \phi_M \left( \frac{v_2}{\theta_2} - \frac{v_1}{\theta_1} \right) + v_1 \int_{\{t > 0\}} \frac{(d\Phi(z_s; z_1) - \phi_m f(z_s) dz_s)}{\theta_s} \right. \]
\[ \quad - v_2 \int_{\{t > 0\}} \frac{(d\Phi(z_s; z_2) - \phi_m f(z_s) dz_s)}{\theta_s} \left. \right] \]
\[ \leq \kappa_v (v_1 - v_2) \cdot \left[ \phi_M \left( \frac{v_2}{\theta_2} - \frac{1}{\theta_1} \right) \right. \]
\[ \quad - \left. \kappa_v |v_1 - v_2|^2 \int_{\Omega} \frac{\phi_m}{\theta_s} f_s dz_s \right] \]
\[ \leq \kappa_v (v_1 - v_2) \cdot \left[ \phi_M \left( \frac{v_2}{\theta_2} - \frac{1}{\theta_1} \right) \right. \]
\[ \quad \left. + v_1 \int_{\Omega} \left( \frac{1}{\theta_s} - \frac{1}{\theta_1} \right) d\Phi(z_s; z_1) \right] \]
\[ \quad - \kappa_v \phi_m |v_1 - v_2|^2. \]

On the other hand, by the estimate in Lemma 4.2, we have
\[ \left| \frac{1}{\theta_i} - \frac{1}{\theta_s} \right| \leq \frac{(\theta_i^0 - \theta_m^0)}{\theta_m^2} e^{-\frac{s \phi_m}{\sigma_m t}}, \quad i = 1, 2. \]

Thus, we obtain
\[ J_3 \leq -\frac{\kappa_v \phi_m}{\theta_M} |v_1 - v_2|^2 + \frac{2 \kappa_v \phi_M (\theta_i^0 - \theta_m^0)}{\theta_m^2} |v_1 - v_2|^2 e^{-\frac{s \phi_m}{\sigma_m t}}. \]

Here, we used
\[ |v_i| = |v_i - \int_{\Omega} \frac{v f dz}{f}| \leq \int_{\Omega} |v_i - v| f dz \leq |v_1 - v_2|. \]

Therefore, we collect all the estimates for \( J_i \) to obtain
\[ \frac{d|v_1(s) - v_2(s)|}{ds} \leq -\frac{\kappa_v \phi_m}{\theta_M} |v_1 - v_2|^2 + \frac{2 \kappa_v \phi_M (\theta_i^0 - \theta_m^0)}{\theta_m^2} |v_1 - v_2|^2 e^{-\frac{s \phi_m}{\sigma_m t}}, \]
which directly implies
\[ |v_1(t) - v_2(t)| \leq |v_1(0) - v_2(0)| e^{-\frac{s \phi_m t}{\sigma_m}} + C \left( e^{-\frac{s \phi_m t}{\sigma_m}} - e^{-\frac{s \phi_m t}{\sigma_m}} \right). \]
4.3. Proof of Theorem 3.2. Now, recall that the average temperature and velocity are conserved:
\[
\int_{\Omega} \theta f \, dz = \int_{\Omega_y(f(t))} \theta f \, dz = \theta_e, \quad \int_{\Omega} v f \, dz = \int_{\Omega_y(f(t))} v f \, dz = 0, \quad t \geq 0.
\]
Then, we can easily check that the Theorem 3.2 holds, since each support \(\Omega_y(f(t))\) \((y = \theta, v)\) is contained in a closed ball \(\overline{B}_r(c)\) centered at \(c\) with radius \(r\), where
\[
c := \int_{\Omega_y(f(t))} y f \, dz, \quad r := \text{diam} \,(\Omega_y(f(t))).
\]

5. From the kinetic TCS equation to hydrodynamic equations. In this section, we provide the proof of Theorem 3.3 by relaxing structural hypotheses in [5, Section 4], and showing that the kinetic equation (3) satisfies the relaxed hypothesis.

5.1. Structural hypotheses. Let \(f^\varepsilon\) be a solution to a given kinetic equation (3) scaled with \(\varepsilon > 0\) corresponding to initial data \(f^\varepsilon_0\), and let \(U^\varepsilon\) and \(U^\varepsilon_0\) be the corresponding hydrodynamic observables associated with \(f^\varepsilon\) and \(f^\varepsilon_0\), respectively.

Let \(U\) be a smooth solution to the limiting balance laws (as a limit system of (3) as \(\varepsilon \to 0\)):
\[
\begin{cases}
\partial_t U + \text{div}_x A(U) = F(U), & x \in \mathbb{T}^d, \quad t > 0, \\
U|_{t=0} = U_0.
\end{cases}
\] (32)

For a given \(T > 0\), suppose that there exists a constant \(C > 0\) (only depending on \(T\) and the regularity of \(U\)) such that for all \(t \in (0, T)\), the following hypotheses hold:

- **(H1)**: There exists a functional \(F\) such that \(\int_{\mathbb{T}^d} F(f^\varepsilon)(t) \, dx \geq 0\) and
  \[
  \int_{\mathbb{T}^d} F(f^\varepsilon)(t) \, dx + \frac{1}{\varepsilon} \int_0^t D_1(f^\varepsilon)(s) \, ds + \int_0^t D_2(f^\varepsilon)(s) \, ds \leq \int_{\mathbb{T}^d} F(f^\varepsilon_0) \, dx + C\varepsilon,
  \]
  where \(D_1, D_2 \geq 0\) are nonnegative functionals.

- **(H2)**: The initial data \(f_0\) and \(U_0\) satisfy the following estimates:
  \[
  \int_{\mathbb{T}^d} \eta(U^\varepsilon_0|U_0) \, dx \leq C\varepsilon, \quad \int_{\mathbb{T}^d} (F(f^\varepsilon_0) - \eta(U^\varepsilon_0)) \, dx \leq C\varepsilon, \quad \int_{\mathbb{T}^d} F(f^\varepsilon_0) \, dx \leq C.
  \]

- **(H3)**: The balance laws have a convex entropy \(\eta\), and \(F\) in \((H1)\) satisfies the minimization property:
  \[
  \eta(U^\varepsilon) \leq F(f^\varepsilon).
  \]

- **(H4)**: The relative entropy in (9) satisfies
  \[
  \left| \int_{\mathbb{T}^d} \nabla_x(D\eta(U)) : A(U^\varepsilon|U) \, dx \right| \leq C \int_{\mathbb{T}^d} \eta(U^\varepsilon|U) \, dx,
  \]
  \[
  \left| \int_0^t \int_{\mathbb{T}^d} D\eta(U) \cdot [\partial_t U^\varepsilon + \text{div}_x A(U^\varepsilon) - F(U^\varepsilon)] \, dx \, ds \right| \leq C \int_0^t D_1(f^\varepsilon)(s) \, ds + C\varepsilon.
  \]
(H5): Let \( \rho^{\varepsilon} \) be the hydrodynamic variable corresponding to \( f^{\varepsilon} \) as the local mass, and \( \rho \) be the corresponding variable for the balance laws (32). Then, we have
\[
- \int_{T_0}^{T} \left[ D^2 \eta(U) F(U) (U^{\varepsilon} - U) + D \eta(U) F'(U^{\varepsilon}) \right] dx \\
\leq D_2(f^{\varepsilon}) + CW^2_2(\rho^{\varepsilon}, \rho) + C \int_{T_0}^{T} \eta(U^{\varepsilon}|U) dx,
\]
\[
W^2_2(\rho^{\varepsilon}, \rho)(t) \leq C \int_0^t \int_{T_0}^{T} \eta(U^{\varepsilon}|U) dx ds + C \varepsilon.
\]

Remark 2. 1. Compared to the hypotheses in [5, Section 4], the only modified parts are (H1) and (H4), which are obviously relaxed version of the previous ones. The nonnegative functionals \( D_1, D_2 \) in (H1) play an important role in controlling the right-hand sides in (H4) and (H5).

2. (H2) is related to a kind of well-prepared initial data. The hypotheses (H1)-(H4) provide a basic structure in applying the relative entropy method to hydrodynamic limits as in previous results (for example, [16, 17, 20]). On the other hand, the hypothesis (H5) provides a crucial connection between the relative entropy and Wasserstein distance.

Under the above hypotheses, we have the following proposition.

Proposition 5. Suppose that the hypotheses (H1)-(H5) hold. Then, for a given \( T > 0 \), there exists a constant \( C_T > 0 \) such that
\[
\int_{T_0}^{T} \eta(U^{\varepsilon}|U)(t) dx + W^2_2(\rho^{\varepsilon}, \rho)(t) \leq C_T \varepsilon, \quad t \leq T.
\]

Proof. Although the proof is almost the same as that of [5, Lemma 4.2], we present its proof in Appendix A for reader’s convenience.

Therefore, once we show that the hypotheses (H1)-(H5) are verified by our equations (3) and (8), the first part (18) of Theorem 3.3 follows from Proposition 5 and (10).

5.2. The first part of Theorem 3.3. In this part, we check whether our setting satisfies structural hypotheses (H1)-(H5) described in previous subsection one by one.

- Step A (Verification of (H3)): Consider the following functional
\[
\mathcal{F}(f)(x, t) := \int_{R^d \times R_+} \frac{|v|^2}{2} f dv d\theta.
\]
Note that for the macroscopic observables
\[
\rho^{\varepsilon} = \int_{R^d \times R_+} f^{\varepsilon} dv d\theta \quad \text{and} \quad \rho^{\varepsilon} u^{\varepsilon} = \int_{R^d \times R_+} v f^{\varepsilon} dv d\theta
\]
constructed from the solution \( f^{\varepsilon} \) to (3), the functional \( \mathcal{F}(f^{\varepsilon}) \) can be regarded as a microscopic quantity of the entropy \( \eta \) of the limit system (4). Indeed, by Hölder’s inequality as in [5, 16], we have the minimization property:
\[
\eta(\rho^{\varepsilon}, \rho^{\varepsilon} u^{\varepsilon}) = \rho^{\varepsilon} \frac{|u^{\varepsilon}|^2}{2} = \frac{\left( \int_{R^d \times R_+} v f^{\varepsilon} dv d\theta \right)^2}{2 \int_{R^d \times R_+} f^{\varepsilon} dv d\theta} \leq \frac{1}{2} \int_{R^d \times R_+} |v|^2 f dv d\theta = \mathcal{F}(f^{\varepsilon}).
\]
Therefore, \((\mathcal{H}3)\) was verified.

- Step B (Verification of \((\mathcal{H}1)\)): Note that we do not expect that \(\mathcal{F}(f^\varepsilon)\) is dissipative, because \(\mathcal{F}(f^\varepsilon)\) does not reflect the dynamics of temperature. Instead, we will take advantage of the assumption \((A_3)\) in Section 3.2 to control some terms related to the temperature. Then, we verify \((\mathcal{H}1)\) as follows.

Lemma 5.1. Let \(f^\varepsilon\) be the solution to (3) with initial data \(f^\varepsilon_0\) satisfying \((A_3)\). Then, for a given \(T > 0\), there exists \(C > 0\) such that

\[
\int_{\mathcal{T}_t} \mathcal{F}(f^\varepsilon)(t) dx + \frac{1}{\varepsilon} \int_0^t D_1(f^\varepsilon)(s) ds + \int_0^t D_2(f^\varepsilon)(s) ds \leq \int_{\mathcal{T}_t} \mathcal{F}(f^\varepsilon_0) dx + C\varepsilon,
\]

where the production functionals \(D_1\) and \(D_2\) are given by the following relations:

\[
D_1(f^\varepsilon) := \int_\Omega |u^\varepsilon - v|^2 f^\varepsilon dz,
\]

\[
D_2(f^\varepsilon) := \frac{\kappa_v}{2\theta_0^2} \int_{\mathcal{T}_2} \phi(|x - x_*|) \rho^\varepsilon(x) \rho^\varepsilon(x_*) |u^\varepsilon(x) - u^\varepsilon(x_*)|^2 dx dx_*.
\]

Proof. We multiply \(\frac{|v|^2}{2}\) to (3) and integrate it over \(\Omega\) to obtain

\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} |v|^2 f^\varepsilon dz
\]

\[
= \int_{\Omega} v \cdot \mathcal{F}[f^\varepsilon] f^\varepsilon dz + \frac{1}{\varepsilon} \int_{\Omega} v \cdot (u^\varepsilon - v) f^\varepsilon dz \tag{33}
\]

\[
= \kappa_v \int_{\Omega \times \Omega} \phi(|x - x_*|) v \cdot \left( \frac{v_* - v}{\theta_*} \right) f^\varepsilon f^\varepsilon_* dz dx_* + \frac{1}{\varepsilon} \int_{\Omega} v \cdot (u^\varepsilon - v) f^\varepsilon dz
\]

\[
=: \mathcal{I}_{31} + \mathcal{I}_{32}.
\]

- Case A (Estimate of \(\mathcal{I}_{31}\)): We use Lemma 2.1 to estimate

\[
\mathcal{I}_{31} = \frac{\kappa_v}{2} \int_{\Omega \times \Omega} \phi(|x - x_*|)(v - v_*) \cdot \left( \frac{v_* - v}{\theta_*} \right) f^\varepsilon f^\varepsilon_* dz dx_*
\]

\[
= \frac{\kappa_v}{2} \int_{\Omega \times \Omega} \phi(|x - x_*|)(v - v_*) \cdot \left( \frac{v_* - v}{\theta_*} \right) f^\varepsilon f^\varepsilon_* dz dx_*
\]

\[
+ \frac{\kappa_v}{2} \int_{\Omega \times \Omega} \phi(|x - x_*|)(v - v_*) \cdot \left( \frac{1}{\theta_*} - \frac{1}{\theta} \right) f^\varepsilon f^\varepsilon_* dz dx_*
\]

\[
\leq -\frac{\kappa_v}{2} \int_{\Omega \times \Omega} \phi(|x - x_*|) \left| \frac{v - v_*}{\theta_*} \right|^2 f^\varepsilon f^\varepsilon_* dz dx_*
\]

\[
+ \frac{\kappa_v}{2} \int_{\Omega \times \Omega} \phi(|x - x_*|)(v - v_*) \cdot \left| \frac{\theta - \theta_*}{\theta_* \theta} \right| f^\varepsilon f^\varepsilon_* dz dx_*
\]

\[
\leq -\frac{\kappa_v}{2} \int_{\Omega \times \Omega} \phi(|x - x_*|) \left| \frac{v - v_*}{\theta_*} \right|^2 f^\varepsilon f^\varepsilon_* dz dx_*
\]

\[
=: \mathcal{I}_{311} + \frac{\kappa_v R^2 \phi_M (\theta^M_* - \theta^m)}{(\theta^m_*)^2} \mathcal{I}_{312}.
\]
We use Lemma 2.1 and \((A_3)\) to get
\[
\mathcal{I}_{311} = -\frac{\kappa_v}{2} \int_{\Omega} \phi(|x - x_*|) \frac{|v - v_*|^2}{\theta_e} f^e f_* dz_* \\
- \frac{\kappa_v}{2} \int_{\Omega} \phi(|x - x_*|) \frac{|v - v_*|^2}{\theta_e} f^e dz_* \\
\leq - \frac{\kappa_v}{2\theta_e} \int_{\Omega} \phi(|x - x_*|) |v - v_*|^2 f^e f_* dz_* + C\varepsilon,
\]
and
\[
\mathcal{I}_{313} \leq C\varepsilon.
\]

For \(\mathcal{I}_{313}\), we follow the proof of [5, Lemma 5.1] to find
\[
\mathcal{I}_{313} \leq -\frac{\kappa_v}{2\theta_e} \int_{\mathbb{R}^d} \phi(|x - x_*|) \rho^e(x) \rho^e(x_*) |u^e(x) - u^e(x_*)|^2 dx dx_*.
\]

- Case B (Estimate of \(\mathcal{I}_{32}\)): We use definition of \(u\) to get
\[
\mathcal{I}_{32} = -\frac{1}{\varepsilon} \int_{\Omega} |u^e - v|^2 f^e dz.
\]

Finally in (33), we combine estimates in Case A and Case B to get
\[
\frac{d}{dt} \int_{\mathbb{T}^d} \mathcal{F}(f^e) dx + \frac{1}{\varepsilon} \mathcal{D}_1(f^e) + \mathcal{D}_2(f^e) \leq C\varepsilon.
\]

This provides the desired estimate.

Now, it remains to show that the hypotheses \((H2)\) and \((H4)-(H5)\) are satisfied. However, those follow directly from the results in [5, Section 5.1]. More precisely, since we consider the same functionals \(\mathcal{F}, \eta\), and initial assumptions \((A_1)-(A_3)\) as in [5], we obtain \((H2)\). In addition, since the functional \(D_4\) in Lemma 5.1 and the balance laws (8) are the same ones as in [5], the hypotheses \((H4)_1\) and \((H5)\) are also satisfied. For the remaining hypothesis \((H4)_2\), we find that (3) formally yields
\[
\partial_t U^e + \nabla_x \cdot A(U^e) = F(U^e) + \nabla_x \left( \left[ \int_{\mathbb{R}^d \times \mathbb{R}_+} (u^e - v) \otimes (u^e - v) f^e dv d\theta \right] \right) \\
+ \left( \kappa_v \int_{\mathbb{T}^d \times \mathbb{R}^2} \phi(|x - x_*|) \left( v_* \left( \frac{1}{\theta_*} - \frac{1}{\theta_e} \right) - \left( \frac{1}{\theta} - \frac{1}{\theta_e} \right) \right) f^e f_* dz_* dv d\theta \right).
\]

More rigorously, we take \(\varphi = D_p \eta(U) + v \cdot D_p \eta(U)\) as a test function in (15) to get
\[
\int_0^t \int_{\mathbb{T}^d} D\eta(U) \cdot \left[ \partial_t U^e + \text{div}_x A(U^e) - F(U^e) \right] dx ds \\
= - \int_0^t \int_{\mathbb{T}^d} \nabla_x D_p \eta(U) : \left[ \int_{\mathbb{R}^d \times \mathbb{R}_+} (u^e - v) \otimes (u^e - v) f^e dv d\theta \\
+ \int_{\mathbb{T}^d \times \mathbb{R}^2} \phi(|x - x_*|) \left( v_* \left( \frac{1}{\theta_*} - \frac{1}{\theta_e} \right) - \left( \frac{1}{\theta} - \frac{1}{\theta_e} \right) \right) f^e f_* dz_* dv d\theta \right] dx ds.
\]
Then, we use Lemma 2.1, (A₃) and recall $D_P \eta(U) = u$ to obtain
\[
\left| \int_0^t \int_{\mathbb{T}^d} D\eta(U) \cdot \left[ \partial_t U^\varepsilon + \text{div}_x A(U^\varepsilon) - F(U^\varepsilon) \right] dx ds \right|
\leq \| \nabla x u \|_{L^\infty((0,T) \times \mathbb{T}^d)} \int_0^t D_1(f^\varepsilon)(s) ds + C\varepsilon,
\]
which verifies (H4)₂.

5.3. **The second part of Theorem 3.3.** First of all, Lemma 5.1 and (A₁) imply
\[
\int_0^T \int f^\varepsilon(|v - u|) dv ds \leq C\varepsilon.
\]
This and (18) yield
\[
\int_0^T \int f^\varepsilon(|v - u| + |u^\varepsilon - u|) dv ds \leq C(1 + T_*)\varepsilon. \tag{34}
\]
Then, for any $\psi \in C^1_0((0, T_*) \times \Omega)$, we have
\[
\int_0^{T_*} \int_\Omega \psi(x,v,\theta) f^\varepsilon dv ds - \int_0^{T_*} \int_{\mathbb{T}^d} \psi(x,v,\theta) \rho \delta_u (dv) \delta_\theta (d\theta) dx ds
= \int_0^{T_*} \int_\Omega \psi(x,v,\theta) f^\varepsilon dv ds - \int_0^{T_*} \int_{\mathbb{T}^d} \psi(x,u,\theta^\varepsilon) \rho dx ds
= \mathcal{I}_{41} + \mathcal{I}_{42}.
\]
- **Case A (Estimate of $\mathcal{I}_{41}$):** We use the assumption (A₃) and (34) to get
\[
\mathcal{I}_{41} \leq \| \nabla \psi \|_{L^\infty} \int_0^{T_*} \int_\Omega \left( |\theta - \theta^\varepsilon| + |v - u| \right) f^\varepsilon dv ds
= C(T_*) \left( \varepsilon + \int_0^{T_*} \int_\Omega |v - u| f^\varepsilon dv ds \right)
\leq C(T_*) \left( \varepsilon + \int_0^{T_*} \int_{|v - u| \leq \sqrt{\varepsilon}} |v - u| f^\varepsilon dv ds + \int_0^{T_*} \int_{|v - u| > \sqrt{\varepsilon}} |v - u| f^\varepsilon dv ds \right)
\leq C(T_*) \left( \varepsilon + \sqrt{\varepsilon} + \frac{1}{\sqrt{\varepsilon}} \int_0^{T_*} \int_{|v - u| > \sqrt{\varepsilon}} |v - u|^2 f^\varepsilon dv ds \right) \leq C(T_*) \sqrt{\varepsilon}.
\]
- **Case B (Estimate of $\mathcal{I}_{42}$):** We again use Kantorovich-Rubinstein theorem and (18) to obtain
\[
\mathcal{I}_{42} \leq T_* \| \psi \|_{W^{1,\infty}(\rho^\varepsilon, \rho)} \leq C(T_*) W_2(\rho^\varepsilon, \rho) \leq C(T_*) \varepsilon.
\]
Now, by letting $\varepsilon \downarrow 0^+$, we obtain the desired weak convergence, and completes the second part of Theorem 3.3.
6. Conclusion. In this paper, we rigorously derived the hydrodynamic CS model from the modified kinetic TCS equation with a local strong alignment force. For the local alignment of velocity and temperature, we used the local alignment force and conditions that the initial temperature support is concentrated to the constant initial average temperature. If we relax the initial condition on temperature, it is expected that the asymptotic limit system will be the hydrodynamic TCS model equipped with an energy balance law rather than the hydrodynamic CS model. However, due to lack of a suitable free energy, we are not able to take a hydrodynamic limit in full generality to recover the energy balance law in the asymptotic limit. Thus, we leave this interesting problem as a future work.

Appendix A. Proof of Proposition 5. In this appendix, we present a proof Proposition 5. First of all, we use Proposition 2 to see

\[
\int_{T^d} \eta(U^\varepsilon | U) \, dx \leq I_{51} + I_{52} + I_{53} + I_{54} + I_{55},
\]

where

\[
I_{51} := \int_{T^d} \eta(U_0^\varepsilon | U_0) \, dx, \quad I_{52} := \int_{T^d} \left( \eta(U^\varepsilon)(t) - \eta(U_0^\varepsilon) \right) \, dx,
\]

\[
I_{53} := -\int_{0}^{t} \int_{T^d} \nabla_x (D\eta(U)) : A(U^\varepsilon | U) \, dx \, ds,
\]

\[
I_{54} := -\int_{0}^{t} \int_{T^d} D\eta(U) \cdot [\partial_t U^\varepsilon + \text{div}_x A(U^\varepsilon) - F(U^\varepsilon)] \, dx \, ds,
\]

\[
I_{55} := -\int_{0}^{t} \int_{T^d} \left[ D^2 \eta(U) F(U)(U^\varepsilon - U) + D\eta(U) F(U^\varepsilon) \right] \, dx \, ds.
\]

Below, we estimate \( I_{51}, i = 1, \ldots, 5 \) one by one.

- (Estimate of \( I_{51} \)): By \((H2)\), we have

\[
I_{51} \leq C \varepsilon.
\]

- (Estimate of \( I_{52} \)): We decompose \( I_{52} \) as

\[
I_{52} = \int_{T^d} (\eta(U^\varepsilon)(t) - F(f^\varepsilon)(t)) \, dx + \int_{T^d} (F(f^\varepsilon)(t) - F(f_0^\varepsilon)) \, dx + \int_{T^d} (F(f_0^\varepsilon) - \eta(U_0^\varepsilon)) \, dx.
\]

\[
= I_{52}^1 + I_{52}^2 + I_{52}^3.
\]

- (Estimate of \( I_{52}^1 \)): By \((H3)\), we have

\[
I_{52}^1 \leq 0.
\]

- (Estimate of \( I_{52}^2 \)): By \((H1)\), we have

\[
I_{52}^2 \leq -\int_{0}^{t} D_2(f^\varepsilon) \, ds + C \varepsilon.
\]

It follows from \((H5)\) that

\[
I_{52}^3 + I_{55} \leq C \varepsilon + C \int_{0}^{t} W_2^2(\rho^\varepsilon, \rho) \, ds + C \int_{T^d} \eta(U^\varepsilon | U) \, dx \, ds.
\]

- (Estimate of \( I_{52}^3 \)): By \((H2)\), we have

\[
I_{52}^3 \leq C \varepsilon.
\]
It follows from (H1) that
\[ \mathcal{I}_{53} \leq C \int_0^t \int T_4 \eta(U^*)|U| dx ds. \]
Since (H1) and (H2) imply
\[ \int_0^t D_1(f^*)(s) ds \leq C \varepsilon, \]
it follow from (H4)2 that
\[ \mathcal{I}_{54} \leq C \varepsilon. \]
Therefore, we have
\[ \int T_4 \eta(U^*)|U| dx \leq C \varepsilon + C \int_0^t \left[ \int T_4 \eta(U^*)|U| dx + W_2^2(\rho^e, \rho) \right] ds. \]
Hence, we combine it with (H5), and use Grönnwall’s inequality to obtain the desired result.

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