Lightlike foliations of semi-Riemannian manifolds

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Abstract. Using screen distributions and lightlike transversal vector bundles we develop a theory of degenerate foliations of semi-Riemannian manifolds. We build lightlike foliations of a semi-Riemannian manifold by suspension of a group homomorphism \( \varphi : \pi_1(B, x_0) \to \text{Isom}(T) \). We compute the basic cohomology groups of the flow determined by a lightlike Killing vector field on a complete semi-Riemannian manifold. We prove a lightlike analog to Rummler's formula and the transversal divergence theorem of F. Kamber et al., [4].

1. Introduction

A lightlike foliation \( \mathcal{F} \) of a semi-Riemannian manifold \( M \) is a foliation each of whose leaves is a lightlike submanifold on \( M \), so that the restriction of the ambient metric to the tangent bundle \( T(\mathcal{F}) \) is degenerate. Therefore \( \text{Rad} \ T(\mathcal{F}) := T(\mathcal{F}) \cap T(\mathcal{F})^\perp \neq 0 \) and one may not develop a satisfactory theory (a geometry of the second fundamental form of \( \mathcal{F} \) in \( M \)) by a mere imitation of the theory of foliations of Riemannian manifolds, cf. e.g. [9], p. 62-73. Indeed the very basics (existence of bundle-like metrics for Riemannian foliations, building adapted connections in the normal bundle, etc.) depend upon the availability of a natural isomorphism \( \sigma : \nu(\mathcal{F}) \approx T(\mathcal{F})^\perp \) whose existence follows from the nondegeneracy of \( T(\mathcal{F}) \). We solve this problem on the lines of [1] (which deals with the case of a single lightlike submanifold) by using the technique of screen distributions and the corresponding lightlike

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transversal bundles. Precisely we build a vector bundle \( tr(TF) \rightarrow M \) (depending on a choice of complements - the so called \textit{screen distributions} - to Rad \( TF \) in \( T(F) \) and \( T(F)\perp \) respectively) playing the role of \( T(F)\perp \) in the theory of foliations with nondegenerate leaves, i.e. \( T(M) = T(F) \oplus tr(TF) \), whose key property is that its lightlike part \( ltr(TF) \) is \textit{not} orthogonal to the radical distribution. The particular cases we study are those of lightlike foliations defined by suspension, flows of lightlike Killing vector fields, and foliations by level sets of lightlike functions i.e. smooth functions on a semi-Riemannian manifold whose gradient is null. By a result of Y. Kamishima, \cite{Y}, a Lorentz spherical manifold \( M \) admits no timelike or lightlike Killing vector fields (and if \( M \) is compact and 3-dimensional there are no spacelike Killing vector fields as well). As an application of our theory, given a complete 3-dimensional Lorentz manifold we may weaken the hypothesis in \cite{Y} by assuming that Isom(\( M \)) = O(4, 1) and that \( M \) has the real homology of a pseudosphere \( S^3_1(\rho) \) proving however a less precise result: such \( M \) admits no \textit{complemented} lightlike Killing vector fields (cf. Corollary \ref{corollary}). More general, given a lightlike Killing vector field and the corresponding flow \( F \) on a complete semi-Riemannian manifold we build a long exact sequence of cohomology groups

\[
H^1_B(F) \rightarrow H^k(M, \mathbb{R}) \rightarrow H^{k-1}_B(F) \xrightarrow{\Delta} H^{k+1}_B(F) \rightarrow \cdots
\]

allowing one to compute the basic cohomology of the flow when the de Rham cohomology of \( M \) is known (e.g. when \( M \sim S^n_\rho(\rho) \) i.e. \( M \) is a real homology pseudosphere, cf. Corollary \ref{corollary}). In the spirit of H. Rummler, \cite{H}, and F. Kamber et al., \cite{FK}, we obtain lightlike analogs to Rummler’s formula (cf. also \cite{H}, p. 66) and to the transversal divergence theorem, though only on foliated semi-Riemannian manifolds without boundary, while the problem of producing a foliated analog of the result by B. Ünal, \cite{U}, is left open.

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\section{Screen distributions and transversal bundles}

Let \( E \rightarrow M \) be a real vector bundle of rank \( m \) \((m \geq 2)\) over a \( C^\infty \) manifold \( M \). In this paper by a \textit{(bundle) metric} in \( E \) we intend a \( C^\infty \) section \( g : x \in M \mapsto g_x \in E_x^* \otimes_{\mathbb{R}} E_x^* \) in \( E^* \otimes E^* \) such that \( g_x \)}
is symmetric and has constant index $\text{ind}(g_x) = \sigma$, for any $x \in M$. If each $g_x$ is nondegenerate and $1 \leq \sigma \leq m - 1$ (respectively if each $g_x$ is positive definite) then $g$ is a semi-Riemannian metric (respectively a Riemannian metric) in $E$. An arbitrary metric $g$ in $E$ is therefore allowed to be degenerate i.e. $(\text{Rad } E)_x \neq (0)$ for some $x \in M$ where $$(\text{Rad } E)_x = \{ v \in E_x : g_x(v, w) = 0, \ w \in E_x \}, \ x \in M.$$ Nevertheless we assume in most cases that $\text{Rad } E$ is a subbundle of $E$ of rank $r$ with $1 \leq r \leq m$ and then refer to $g$ as a $r$-lightlike metric while $\text{Rad } E$ is the radical bundle of $(E, g)$.

Let $\mathcal{F}$ be a codimension $q$ foliation of a real $n$-dimensional manifold $M$. Let $\nu(\mathcal{F}) = T(M)/T(\mathcal{F})$ be the transverse bundle and $\Pi : T(M) \to \nu(\mathcal{F})$ the projection. Let $g$ be a $r$-lightlike metric in $T(\mathcal{F})$ where $1 \leq r \leq \min\{m, q\}$ and $m = n - q$. Then $(\mathcal{F}, g)$ is a tangentially lightlike foliation of $M$ and $\text{Rad } T(\mathcal{F})$ is its tangential radical distribution. It is with this sort of foliations that the present paper is mainly concerned. If $M$ is a $n$-dimensional semi-Riemannian manifold of index $1 \leq s \leq n - 1$ and the metric $g$ above is induced in $T(\mathcal{F})$ by the ambient metric then each leaf of $\mathcal{F}$ is a degenerate or lightlike submanifold of $M$ (cf. $\Pi$, p. 140). We also adopt the terminology in Table 1.

Similarly if $g_Q$ is a $\rho$-lightlike ($1 \leq \rho \leq q$) metric in $Q = \nu(\mathcal{F})$ such that $\tilde{\nabla}_X g_Q = 0$ for any $X \in T(\mathcal{F})$ then $(\mathcal{F}, g_Q)$ is a transversally lightlike foliation. Here $\tilde{\nabla}$ denotes the Bott connection of $(M, \mathcal{F})$ i.e. $\tilde{\nabla}_X s = \Pi[X, Y]$ for any $C^\infty$ section $s$ in $Q$ and any $Y \in T(M)$ such that $\Pi(Y) = s$.

Let $\mathcal{F}$ be a tangentially lightlike foliation of the semi-Riemannian manifold $(M, g)$. We set $$T(\mathcal{F})^\perp = \{ V \in T(M) : g(V, X) = 0, \ X \in T(\mathcal{F}) \}.$$ Let $S(T\mathcal{F})$ and $S(T\mathcal{F}^\perp)$ be complements to the tangential radical distribution in $T(\mathcal{F})$ and $T(\mathcal{F})^\perp$, respectively. Then

(1) $$T(\mathcal{F}) = S(T\mathcal{F}) \oplus \text{Rad } T\mathcal{F},$$

(2) $$T(\mathcal{F})^\perp = S(T\mathcal{F}^\perp) \oplus \text{Rad } T\mathcal{F},$$

and (by Proposition 2.1 in $\Pi$, p. 5) both $S(T\mathcal{F})$ and $S(T\mathcal{F}^\perp)$ are nondegenerate. Consequently

(3) $$T(M) = S(T\mathcal{F}) \oplus S(T\mathcal{F})^\perp.$$ If $T(\mathcal{F})$ were nondegenerate then the ambient semi-Riemannian metric $g$ would induce a bundle metric in $\nu(\mathcal{F})$ by the natural isomorphism $\nu(\mathcal{F}) \approx T(\mathcal{F})^\perp$. As to the study of tangentially lightlike foliations
we circumvent the difficulties (arising from the failure to decompose $T(M) = T(\mathcal{F}) \oplus T(\mathcal{F})^\perp$) by using lightlike transversal bundles (as in the theory of lightlike submanifolds, cf. \[1\], p. 139-148).

| $\mathcal{F}$       | $r$                        |
|---------------------|----------------------------|
| (I) $r$ - lightlike | $1 \leq r < \min\{m, q\}$ |
| (II) co - isotropic | $1 \leq r = q < m$        |
| (III) isotropic     | $1 \leq r = m < q$        |
| (IV) totally lightlike | $1 \leq r = m = q$   |

Table 1. Classification of tangentially lightlike foliations of semi-Riemannian manifolds according to the rank of their tangential radical distribution.

Let us start by noticing that

(4) \[ S(T\mathcal{F})^\perp \supseteq S(T\mathcal{F}^\perp). \]

Indeed if $X \in S(T\mathcal{F}^\perp) \subseteq T(\mathcal{F})^\perp$ then $X$ is orthogonal to $T(\mathcal{F}) \supseteq S(T\mathcal{F})$ hence $X \in S(T\mathcal{F})^\perp$. Next, since $S(T\mathcal{F}^\perp)$ is nondegenerate

(5) \[ S(T\mathcal{F})^\perp = S(T\mathcal{F}^\perp) \oplus S(T\mathcal{F}^\perp)^\perp. \]

We shall need the following

**Lemma 1.** Let $\{\xi_1, \cdots, \xi_r\}$ be a local frame of $\text{Rad} T\mathcal{F}$ defined on the open set $U \subseteq M$. There exist $N_i \in \Gamma^\infty(U, S(T\mathcal{F}^\perp))$, $1 \leq i \leq r$, such that $g(N_i, \xi_j) = \delta_{ij}$ and $g(N_i, N_j) = 0$.

**Proof.** Note first that

(6) \[ \text{Rad} T\mathcal{F} \subseteq S(T\mathcal{F}^\perp)^\perp. \]

Indeed if $X \in \text{Rad} T\mathcal{F}$ then $X$ is orthogonal on $T(\mathcal{F})^\perp \supseteq S(T\mathcal{F}^\perp)$ hence $X \in S(T\mathcal{F}^\perp)^\perp$. Next we choose a complement $E$ to $\text{Rad} T\mathcal{F}$ so that

(7) \[ S(T\mathcal{F}^\perp)^\perp = (\text{Rad} T\mathcal{F}) \oplus E. \]

Consequently $\dim \mathbb{R} E_x = r$ for any $x \in M$. Let then $\{V_1, \cdots, V_r\}$ be a local frame of $E$ on $U$. One may look for the $N_i$’s in the form

$N_i = A^k_i \xi_k + B^k_i V_k$

for some $C^\infty$ functions $A^k_i, B^k_i : U \rightarrow \mathbb{R}$ with the requirement

\[ \delta_{ij} = g(N_i, \xi_j) = B^k_i g_{jk}. \]
where \( g_{j k} = g(\xi_j, V_k) \). Let us set \( G = \det[g_{j k}] \). We claim that \( G(x) \neq 0 \) for any \( x \in U \). The proof is by contradiction. If \( G(x_0) = 0 \) for some \( x_0 \in U \) then there is \( v = (v^1, \ldots, v^r) \in \mathbb{R}^r \setminus \{0\} \) such that
\[
g_{j k}(x_0)v^j = 0, \quad 1 \leq k \leq r.
\]
Let us set \( w = v^j \xi_{j, x_0} \in (\text{Rad } T\mathcal{F})_{x_0} \subset S(T\mathcal{F}^\perp) \). Then (by \( \mathbf{8} \)) \( g_{x_0}(w, V_{k, x_0}) = 0 \). Also \( g_{x_0}(w, \xi_{k, x_0}) = 0 \) by the very definition of \( w \). Then \( w \) sits in \( S(T\mathcal{F}^\perp)_{x_0} \) \( \setminus \{0\} \) and (by \( \mathbf{7} \)) it is perpendicular on \( S(T\mathcal{F}^\perp)_{x_0} \) i.e. \( S(T\mathcal{F}^\perp)_{x_0} \) is degenerate, a contradiction. Therefore it is legitimate to consider \( [g^{j k}] := [g_{j k}]^{-1} \). Then \( B^k_i = g^{k i} \) and the requirement \( g(N_i, N_j) = 0 \) yields
\[
A^j_i + A^j_j + g^{k i}g^{j k}g(V_k, V_\ell) = 0
\]
and we may choose \( A^j_i := -\frac{1}{2}g^{k i}g^{j k}g(V_k, V_\ell) \). Lemma \( \mathbf{1} \) is proved. In particular (with the notations of Lemma \( \mathbf{1} \)) \( \{\xi_1, \ldots, \xi_r, N_1, \ldots, N_r\} \) is a local frame of \( S(T\mathcal{F}^\perp) \) on \( U \). Let us set
\[
\text{ltr}(T\mathcal{F})_x = \sum_{i=1}^r \mathbb{R}N_{i, x}, \quad x \in U.
\]

**Lemma 2.** The definition of the bundle \( \text{ltr}(T\mathcal{F})_x \) doesn’t depend upon the choice of local frames \( \{\xi_j\} \) of \( T\mathcal{F} \) and \( \{V_k\} \) of \( E \) at \( x \). Moreover \( \text{ltr}(T\mathcal{F}) = \bigcup_{x \in M} \text{ltr}(T\mathcal{F})_x \) is a vector bundle over \( M \) and
\[
S(T\mathcal{F}^\perp) = (\text{Rad } T\mathcal{F}) \oplus \text{ltr}(T\mathcal{F}).
\]

The proof of Lemma \( \mathbf{2} \) is imitative of that of Theorem 1.4 in \( \mathbf{1} \), p. 147, and is omitted. We call \( ltr(T\mathcal{F}) \to M \) a *lightlike transversal* vector bundle with respect to the pair \( (S(T\mathcal{F}), S(T\mathcal{F}^\perp)) \). Also
\[
\text{tr}(T\mathcal{F}) := \text{ltr}(T\mathcal{F}) \oplus S(T\mathcal{F}^\perp)
\]
is a *transversal* vector bundle. Then (by \( \mathbf{3} \), \( \mathbf{5} \) and \( \mathbf{9} \))
\[
T(M) = S(T\mathcal{F}) \oplus S(T\mathcal{F}^\perp) \oplus (\text{Rad } T\mathcal{F}) \oplus \text{ltr}(T\mathcal{F})
\]
hence
\[
T(M) = T(\mathcal{F}) \oplus \text{tr}(T\mathcal{F}).
\]
Let \( \sigma : \nu(\mathcal{F}) \to \text{tr}(T\mathcal{F}) \) be the bundle isomorphism given by
\[
\sigma(s) = \text{tra}(Y), \quad \Pi(Y) = s, \quad Y \in T(M),
\]
where \( \text{tra} : T(M) \to \text{tr}(T\mathcal{F}) \) is the natural projection associated to the decomposition \( \mathbf{11} \). Let us set
\[
g_{\text{tra}}(s, r) = g(\sigma(s), \sigma(r)), \quad s, r \in \nu(\mathcal{F}).
\]
If $g_{\text{tra}}$ is holonomy invariant, i.e. $\mathcal{L}_X g_{\text{tra}} = 0$ for any $X \in T(\mathcal{F})$, then $g$ is said to be bundle-like. Here $\mathcal{L}_X$ denotes the Lie derivative in the direction $X$. Let $Q = \nu(\mathcal{F})$ for simplicity. One expects $g_{\text{tra}}$ to be degenerate, as well. Indeed, if we set

$$\text{Rad } Q = \{ s \in Q : g_{\text{tra}}(s, r) = 0, \ r \in Q \}$$

then we have

**Proposition 1.** Let $\mathcal{F}$ be a lightlike foliation of the semi-Riemannian manifold $(M, g)$ and $\text{ltr}(T\mathcal{F}) \to M$ a lightlike transversal vector bundle associated with the screen distributions $S(T\mathcal{F})$ and $S(T\mathcal{F}^\perp)$. Then

$$\sigma(\text{Rad } Q) = \text{ltr}(T\mathcal{F}).$$

**Proof.** Let $N \in \text{ltr}(T\mathcal{F})$ and $r \in Q$. As $N$ is orthogonal to $\text{tr}(T\mathcal{F})$

$$g_{\text{tra}}(\sigma^{-1}(N), r) = g(N, \sigma(r)) = 0$$

it follows that $\sigma^{-1}(N) \in \text{Rad } Q$. For the opposite inclusion let $s \in \text{Rad } Q$ and $Z \in \text{tr}(T\mathcal{F})$. If we set $r = \sigma^{-1}(Z) \in Q$ then

$$0 = g_{\text{tra}}(s, r) = g(\sigma(s), Z).$$

We have $s = \Pi(Y)$ for some $Y \in T(M)$. As a consequence of (10)-(11) $Y = X + N + V$ for some $X \in T(\mathcal{F})$, $N \in \text{ltr}(T\mathcal{F})$ and $V \in S(T\mathcal{F}^\perp)$. Then $\sigma(s) = N + V$. Let $W \in S(T\mathcal{F}^\perp)$. Applying (13) for $Z = W$ gives $g(V, W) = 0$ and then $V = 0$ since $S(T\mathcal{F}^\perp)$ is nondegenerate. It remains that $\sigma(s) = N \in \text{Rad } Q$ and Proposition 1 is proved.

By (10) and Proposition 1 a canonical choice of screen distribution in $Q$ is $S(Q) := \sigma^{-1}S(T\mathcal{F}^\perp)$ so that

$$Q = S(Q) \oplus \text{Rad } Q.$$ 

Let $\nabla^g$ be the Levi-Civita connection of $(M, g)$. We also consider

$$\nabla_X s = \begin{cases} \hat{\nabla}_X s, & X \in T(\mathcal{F}), \\ \Pi \nabla_X^2 \sigma(s), & X \in \text{ltr}(T\mathcal{F}). \end{cases}$$

One checks easily that

**Proposition 2.** $\nabla$ is a connection in $Q$ and $T_{\nabla} = 0$, where $T_{\nabla}(Y, Z) := \nabla_Y \Pi Z - \nabla_Z \Pi Y - \Pi[Y, Z]$ for any $Y, Z \in T(M)$. Moreover $g$ is bundle-like if and only if $\nabla g_{\text{tra}} = 0$. 
3. Lightlike foliations defined by suspension

Let \((N, h)\) be a semi-Riemannian manifold and \(j : B \hookrightarrow N\) a \(m\)-dimensional connected lightlike submanifold i.e. \((\text{Rad } TB)_x = \{X \in T_x(B) : g_{B.x}(X, Y) = 0, \ Y \in T_x(B)\} \ (x \in B)\) has constant dimension \(1 \leq \rho \leq \min\{m, \ell\} \ (\ell = \dim_\mathbb{R} N - m)\). Here \(g_B = j^*h\). Let \(\hat{B}\) be the universal covering manifold of \(B\) and \(\hat{p} : \hat{B} \to B\) the projection. We set \(g_{\hat{B}} = \hat{p}^*g_B\). Next let us consider a \(q\)-dimensional connected semi-Riemannian manifold \((T, g_T)\) and the warped product \(\hat{M} = T \times_f \hat{B}\) i.e. \(\hat{M}\) is the product manifold \(T \times \hat{B}\) endowed with the \((0, 2)\)-tensor field \(\hat{g} = p_1^* g_T + (f \circ p_1)^2 \ p_2^* g_{\hat{B}}\) where \(f \) (the warping function) is a \(C^\infty\) function \(f : T \to (0, +\infty)\). Also \(p_1 : \hat{M} \to T\) and \(p_2 : \hat{M} \to \hat{B}\) are the natural projections. Our notion of warped product generalizes slightly that in [2], p. 204, as \(\hat{g}\) is not a semi-Riemannian metric \((\hat{M}, \hat{g})\) has a nontrivial radical distribution). Let \(\hat{F}\) be the foliation of \(\hat{M}\) whose leaves are the fibres of \(p_1\) i.e. \(T(\hat{F}) = \text{Ker}(dp_1)\) and \(\hat{M}/\hat{F} = \{\{y\} \times \hat{B} : y \in T\}\). The same symbol \(\hat{g}\) denotes the induced metric in \(T(\hat{F})\).

**Lemma 3.** \(\hat{g}\) is a \(\rho\)-lightlike metric in \(T(\hat{F})\).

**Proof.** We set as customary

\[
(\text{Rad } T(\hat{F}))_{(y, \hat{x})} = \{X \in T(\hat{F})_{(y, \hat{x})} : \hat{g}_{(y, \hat{x})}(X, Y) = 0, \ Y \in T(\hat{F})_{(y, \hat{x})}\}
\]

for any \(y \in T\) and \(\hat{x} \in \hat{B}\). If \(\alpha_y : \hat{B} \to \hat{M}\) is the canonical injection \(\alpha_y(\hat{x}) = (y, \hat{x})\) then

\[
(\text{Rad } T(\hat{F}))_{(y, \hat{x})} = (d_{\hat{x}}\alpha_y)(\text{Rad } T\hat{B})_{\hat{x}}.
\]

Indeed

\[
(\text{Rad } T(\hat{F}))_{(y, \hat{x})} = \{X \in \text{Ker}(d_{(y, \hat{x})} p_1) : (p_1^* g_T + (f \circ p_1)^2 \ p_2^* g_{\hat{B}})_{(y, \hat{x})}(X, Y) = 0, \ Y \in \text{Ker}(d_{(y, \hat{x})} p_1)\} =
\]

\[
\{X \in \text{Ker}(d_{(y, \hat{x})} p_1) : g_{\hat{B}, \hat{x}}((d_{(y, \hat{x})} p_2)X, (d_{(y, \hat{x})} p_2)Y) = 0, \ Y \in \text{Ker}(d_{(y, \hat{x})} p_1)\}.
\]

Note that \(\text{Ker}(d_{(y, \hat{x})} p_1) = (d_{\hat{x}}\alpha_y)T_{\hat{x}}(\hat{B})\). Let \(X = (d_{\hat{x}}\alpha_y)v\) with \(v \in T_{\hat{x}}(\hat{B})\). As \(p_2 \circ \alpha_y = 1\) (the identical transformation of \(\hat{B}\)) it follows that \((d_{(y, \hat{x})} p_2)X = v\). We conclude that

\[
(\text{Rad } T(\hat{F}))_{(y, \hat{x})} = \{(d_{\hat{x}}\alpha_y)v : g_{\hat{B}, \hat{x}}(v, w) = 0, \ w \in T_{\hat{x}}(\hat{B})\}
\]

and (14) is proved. \(\square\)
Let $\varphi : \pi_1(B, x_0) \to \text{Diff}(T)$ be a homomorphism of the fundamental group of $B$ with base point $x_0 \in B$ into the group of all $C^\infty$ diffeomorphisms of $T$ in itself. Also we think of $\tilde{B}$ as the set of all homotopy classes of paths issuing at $x_0$. Let $\tilde{M} \times \pi_1(B, x_0) \to \tilde{M}$ be the natural action given by

$$R_{[\gamma]}(y, \tilde{x}) = (\varphi([\gamma]^{-1})(y), \tilde{x} \cdot [\gamma]), \quad (y, \tilde{x}) \in \tilde{M}, \ [\gamma] \in \pi_1(B, x_0),$$

and let $M = \tilde{M}/\pi_1(B, x_0)$ be the quotient space. Let $\mathcal{F}$ be the projection of $\tilde{\mathcal{F}}$ on $M$ i.e. the foliation of $M$ whose leaves are the projection of the leaves of $\tilde{\mathcal{F}}$

$$M/\mathcal{F} = \{\pi(L) : L \in \tilde{\mathcal{F}}\}$$

where $\pi : \tilde{M} \to M$ denotes the natural projection. Infinitesimally $T(\mathcal{F})_{\pi(y, \tilde{x})} = (d_{(y, \tilde{x})}\pi)(T(\tilde{\mathcal{F}})_{(y, \tilde{x})})$. We say $\mathcal{F}$ is the foliation of $M$ defined by suspension of the homomorphism $\varphi$. Let us consider the map $p : M \to B$ given by $p(\pi(y, \tilde{x})) = \tilde{\rho}(\tilde{x})$. Then $p : M \to B$ is a fibre bundle with standard fibre $T$ and structure group $G = \varphi(\pi_1(B, x_0))$. See also [6], p. 29. As well known (cf. e.g. [1], p. 28) $M$ admits a natural $C^\infty$ manifold structure such that $\pi : \tilde{M} \to M$ is an étale mapping i.e. $\text{Ker}(d_{(y, \tilde{x})}\pi) = (0)$ for any $(y, \tilde{x}) \in \tilde{M}$.

**Lemma 4.** If $G \subset \text{Isom}(T)$ and $f$ is $G$-invariant then $\hat{g}$ is $\pi_1(B, x_0)$-invariant. In particular there is a metric $g$ in $T(M)$ such that $\pi^*g = \hat{g}$.

Here $\text{Isom}(T)$ denotes the group of isometries of the semi-Riemannian manifold $(T, g_T)$. **Proof of Lemma 4** Let $[\gamma] \in \pi_1(B, x_0)$ and $u, v \in T_{(y, \tilde{x})}(M)$. Then

$$\hat{g}_{R_{[\gamma]}(y, \tilde{x})}(A, B) = g_{T, \varphi([\gamma]^{-1})(y)}((d_{R_{[\gamma]}(y, \tilde{x})}p_1)A, (d_{R_{[\gamma]}(y, \tilde{x})}p_1)B) +$$

$$+ f(\varphi([\gamma]^{-1})(y))^2 g_{B, \tilde{x}-[\gamma]}((d_{R_{[\gamma]}(y, \tilde{x})}p_2)A, (d_{R_{[\gamma]}(y, \tilde{x})}p_2)B)$$

where $A = (d_{(y, \tilde{x})}R_{[\gamma]})u$ and $B = (d_{(y, \tilde{x})}R_{[\gamma]})v$. Note that

$$p_1 \circ R_{[\gamma]} = \varphi([\gamma]^{-1}) \circ p_1, \quad p_2 \circ R_{[\gamma]} = D_{[\gamma]} \circ p_2,$$

where $D_{[\gamma]} : \tilde{B} \to \tilde{B}$ is the deck transformation $D_{[\gamma]}(\tilde{x}) = \tilde{x} \cdot [\gamma]$. Then

$$(R_{[\gamma]}^*\hat{g})_{(y, \tilde{x})}(u, v) = (\varphi([\gamma]^{-1})^* g_T)_y((d_{(y, \tilde{x})}p_1)u, (d_{(y, \tilde{x})}p_1)v) +$$

$$+ f(\varphi([\gamma]^{-1})(y))^2 (D_{[\gamma]}^*\hat{g})_{\tilde{x}}((d_{(y, \tilde{x})}p_2)u, (d_{(y, \tilde{x})}p_2)v) = \hat{g}_{(y, \tilde{x})}(u, v).$$

To prove the second statement in Lemma 4 let $p \in M$ and $X, Y \in T_p(M)$. Then $p = \pi(y, \tilde{x})$ and $X = (d_{(y, \tilde{x})}\pi)u$, $Y = (d_{(y, \tilde{x})}\pi)v$ for some $y \in T$, $\tilde{x} \in \tilde{B}$ and $u, v \in T_{(y, \tilde{x})}(\tilde{M})$. We set

$$g_p(X, Y) := \hat{g}_{(y, \tilde{x})}(u, v).$$
We only need to check that the definition doesn’t depend upon the choice of representatives. If \((y', \bar{x}') \in \pi^{-1}(p)\) then \((y', \bar{x}') = R_{[\gamma]}(y, \bar{x})\) for some \([\gamma] \in \pi_1(B, x_0)\). If \(u', v' \in T_{(y', \bar{x}')} (M)\) are other representatives of \(X\) and \(Y\) then \(\pi = \pi \circ R_{[\gamma]}\) yields \(u' = (d_{(y, \bar{x})} R_{[\gamma]}) u \in \text{Ker}(d_{(y, \bar{x})} \pi) = \{0\}\) i.e. \(u' = (d_{(y, \bar{x})} R_{[\gamma]}) u\) and similarly \(v' = (d_{(y, \bar{x})} R_{[\gamma]}) v\). Finally (by Lemma 4) \(\hat{g}_{(y', \bar{x}')}(u', v') = (R_{[\gamma]}^\ast \hat{g})_{(y, \bar{x})}(u, v) = \hat{g}_{(y, \bar{x})}(u, v)\) i.e. \(g_p(X, Y)\) is well defined. \(\square\)

The same symbol \(g\) denotes the induced metric in \(T(F)\).

**Proposition 3.** Let \(T\) be a connected semi-Riemannian manifold, \(j : B \hookrightarrow N\) a connected lightlike submanifold of the semi-Riemannian manifold \((N, h)\) such that the induced metric \(j^\ast h\) is \(\rho\)-lightlike, and \(\hat{B}\) the universal covering manifold of \(B\). Let \(\varphi : \pi_1(B, x_0) \to \text{Isom}(T)\) be a group homomorphism and let \(M = T \times_f \hat{B}\) be a warped product with a \(G\)-invariant warping function \(f : T \to (0, +\infty)\) where \(G = \varphi(\pi_1(B, x_0))\). Let \(M = \hat{M}/\pi_1(B, x_0)\). Then the foliation \(\mathcal{F}\) of \(M\) defined by suspension of the homomorphism \(\varphi\) is tangentially lightlike and \(g\) is a \(\rho\)-lightlike metric in \(T(F)\).

**Proof.** The tangential radical distribution is \((\text{Rad} \ T \mathcal{F})_{\pi(y, \bar{x})} = \{X \in T(F)_{\pi(y, \bar{x})} : g_{\pi(y, \bar{x})}(X, Y) = 0, \ Y \in T(F)_{\pi(y, \bar{x})}\}\) = \(\{(d_{(y, \bar{x})} \pi) u : (\pi^\ast g)_{(y, \bar{x})}(u, v) = 0, \ u, v \in T(\hat{F})_{(y, \bar{x})}\}\) = \((d_{(y, \bar{x})} \pi)(\text{Rad} \ T \hat{F})_{(y, \bar{x})} = (d_{(y, \bar{x})} \pi)(d_{\bar{x}} \alpha_y)(\text{Rad} \ T \hat{B})_{\bar{x}}\) (by (14)). Next \(\hat{p}^\ast g_B = g_{\hat{B}}\) yields

\[(15) \quad (d_{\bar{x}} \hat{p})(\text{Rad} \ T \hat{B})_{\bar{x}} = (\text{Rad} \ T B)_{\hat{p}(\bar{x})}\]

for any \(\bar{x} \in \hat{B}\). Our previous calculation, the identity (15) and the commutativity of the diagram

\[
\begin{array}{ccc}
\hat{B} & \overset{\alpha_\bar{y}}{\longrightarrow} & \hat{M} \\
\hat{p} & \downarrow & \downarrow \pi \\
B & \overset{p}{\longleftarrow} & M
\end{array}
\]

imply that

\[(16) \quad (d_{\pi(y, \bar{x})} p)(\text{Rad} \ T \mathcal{F})_{\pi(y, \bar{x})} = (\text{Rad} \ T B)_{\hat{p}(\bar{x})}.\]

Cf. again [6], p. 29, the fibres of \(p\) are connected total transversals of \((M, \mathcal{F})\). In particular

\[T_{\pi(y, \bar{x})}(M) = T(F)_{\pi(y, \bar{x})} \oplus \text{Ker}(d_{\pi(y, \bar{x})} p), \quad \pi(y, \bar{x}) \in p^{-1}(\hat{p}(\bar{x})).\]

Hence the restriction of \(d_{\pi(y, \bar{x})} p\) to \((\text{Rad} \ T \mathcal{F})_{\pi(y, \bar{x})}\) is a \(\mathbb{R}\)-linear isomorphism.
4. Lightlike Killing vector fields

For each lightlike foliation $\mathcal{F}$ of the semi-Riemannian manifold $M$ we denote by $\Omega^k_B(\mathcal{F})$ the space of all basic differential $k$-forms on $(M, \mathcal{F})$ i.e. if $\omega \in \Omega^k_B(\mathcal{F})$ then $X \cdot \omega = 0$ and $X \cdot d\omega = 0$ for any $X \in T(\mathcal{F})$. In particular $\Omega^0_B(\mathcal{F})$ is the space of all basic functions ($f \in C^\infty(M)$ is basic if $X(f) = 0$ for any $X \in T(M)$). Let $H^k_B(\mathcal{F}) := H^k(\Omega^k_B(\mathcal{F}))$, $k \geq 0$, be the corresponding cohomology groups (that is the basic cohomology of $(M, \mathcal{F})$). By standard foliation theory $H^0_B(\mathcal{F}) = \mathbb{R}$ and there is a natural injection $H^1_B(\mathcal{F}) \hookrightarrow H^1(M, \mathbb{R})$ (cf. e.g. [9], p. 119).

Let $(M, g)$ be a geodesically complete semi-Riemannian manifold and $\xi$ a lightlike Killing vector field on $M$. Then (cf. e.g. [7], p. 254) $\xi$ is complete. Let $H$ be the global 1-parameter group of global transformations of $M$ obtained by integrating $\xi$. Let $G = \overline{H}$ be the closure of $H$ in $\text{Isom}(M, g)$. We assume from now on that $G$ is compact. For instance, if $(M, g)$ is a Lorentz manifold ($s = 1$) and $\text{Isom}(M, g) = O(n + 1, 1)$ then the closure of any lightlike 1-parameter subgroup is compact, cf. Lemma 3.1 in [3], p. 584. Let $\Omega^k(M)$ be the de Rham algebra of $M$ and $\Omega^k(M)^G$ the subalgebra of all $G$-invariant differential forms i.e. if $\omega \in \Omega^k(M)^G$ then $L_\xi \omega = 0$. Let $\mathcal{F}$ be the codimension $q = n - 1$ lightlike foliation of $M$ such that $T(\mathcal{F}) = \mathbb{R}\xi$. It is immediate that

\textbf{Proposition 4.} Either $\mathcal{F}$ is isotropic or $M$ is a Lorentz surface ($n = 2$, $s = 1$) and $\mathcal{F}$ is totally lightlike.

Next, note that

(17) $\Omega^k_B(\mathcal{F}) \subset \Omega^k(M)^G$, $k \geq 0$.

Let $i_\xi : \Omega^k(M) \to \Omega^{k-1}(M)$ be the interior product with $\xi$ i.e. $i_\xi \omega = \xi \cdot \omega$ for any $\omega \in \Omega^k(M)$. Then

(18) $i_\xi \Omega^k(M)^G \subset \Omega^{k-1}_B(\mathcal{F})$, $k \geq 1$.

Indeed, let $\omega \in \Omega^k(M)^G$ and $\eta = i_\xi \omega$. Then $i_\xi \eta = i_\xi^2 \omega = 0$ and

$i_\xi d\eta = i_\xi L_\xi \omega - i_\xi^2 d\omega = 0$

by Cartan’s formula. Our main purpose in the present section is to establish

\textbf{Theorem 1.}

Let $\xi$ be a lightlike Killing vector field on the complete semi-Riemannian manifold $(M, g)$ and $\mathcal{F}$ the 1-dimensional foliation tangent to $\xi$. Let $G$ be the closure in $\text{Isom}(M, g)$ of the 1-parameter group generated by $\xi$. 
Assume that there is a globally defined $G$-invariant vector field $V \neq 0$ on $M$ such that

\begin{equation}
S(T\mathcal{F}^\perp) = (\text{Rad } T\mathcal{F}) \oplus \mathbb{R}V.
\end{equation}

If $G$ is compact then for any $k \geq 1$ there is a linear map $\Delta : H^k_B(\mathcal{F}) \to H^{k+1}_B(\mathcal{F})$ such that

\begin{equation}
H^k_B(\mathcal{F}) \xrightarrow{i_\xi} H^k(M, \mathbb{R}) \xrightarrow{(i_\xi)_*} H^k_B(\mathcal{F}) \xrightarrow{\Delta} H^{k+1}_B(\mathcal{F}) \to \cdots
\end{equation}

is a long exact sequence, where $j : \Omega^k_B(\mathcal{F}) \to \Omega^k(M)^G$ is the inclusion. In particular, if $M$ is compact then $\dim \mathbb{R} H^k_B(\mathcal{F}) < \infty$.

**Proof.** The map $i_\xi : \Omega^k(M)^G \to \Omega^{k-1}_B(\mathcal{F})$ is surjective. Indeed, let $V \in S(T\mathcal{F}^\perp)$ as in Theorem 1. Then (by the proof of Lemma 1) $g(\xi, V) \neq 0$ everywhere on $M$. Let us set

$$N = \frac{1}{g(\xi, V)} \{ V - \frac{g(V, V)}{2g(\xi, V)} \xi \}$$

so that $g(\xi, N) = 1$ and $g(N, N) = 0$. Since

$$\xi(g(\xi, V)) = (\mathcal{L}_\xi g)(\xi, V) + g(\xi, \mathcal{L}_\xi V) = 0,$$

$$\xi(g(V, V)) = (\mathcal{L}_\xi g)(V, V) + 2g(\mathcal{L}_\xi V, V) = 0,$$

it follows that

\begin{equation}
\mathcal{L}_\xi N = 0.
\end{equation}

Let us consider the 1-form $\alpha \in \Omega^1(M)$ given by $\alpha(X) = g(X, N)$ for any $X \in T(M)$. Note that $\alpha$ is $G$-invariant. Indeed (by 21)

$$(\mathcal{L}_\xi \alpha)X = \xi(g(X, N)) - g(\mathcal{L}_\xi X, N) = (\mathcal{L}_\xi g)(X, N) = 0$$

for any $X \in T(M)$. Consequently, for any $\omega \in \Omega^{k-1}_B(\mathcal{F})$ the $k$-form $\alpha \wedge \omega$ is $G$-invariant. Finally

$$i_\xi(\alpha \wedge \omega) = (i_\xi \alpha)\omega = \omega$$

so that $i_\xi$ is on-to, as claimed. The next step is to observe that

\begin{equation}
0 \to \Omega^k_B(\mathcal{F}) \to \Omega^k(M)^G \xrightarrow{i_\xi} \Omega^{k-1}_B(\mathcal{F}) \to 0
\end{equation}

is a short exact sequence. By 17 and the first part of the proof of Theorem 1 one only needs to check exactness at the middle term. If $\omega \in \Omega^k_B(\mathcal{F})$ then $i_\xi \omega = 0$ because $\omega$ is a basic form. Viceversa, let $\omega \in \text{Ker}(i_\xi) \subseteq \Omega^k(M)^G$. Then $i_\xi \omega = 0$ and $\mathcal{L}_\xi \omega = 0$ hence $\omega \in \Omega^k_B(\mathcal{F})$.

Let us consider the map

$$\Delta : H^{k-1}_B(\mathcal{F}) \to H^{k+1}_B(\mathcal{F})$$
given by \( \Delta[\omega] = [d\alpha \wedge \omega] \) for any \( \omega \in \Omega^{k-1}_B(F) \) with \( d\omega = 0 \). As \( i_\xi \omega = 0 \) and \( i_\xi \alpha = 1 \) one has

\[
i_\xi(d\alpha \wedge \omega) = (i_\xi d\alpha) \wedge \omega = (\mathcal{L}_\xi \alpha - d i_\xi \alpha) \wedge \omega = 0,
\]

\[
\mathcal{L}_\xi(d\alpha \wedge \omega) = (\mathcal{L}_\xi d\alpha) \wedge \omega = (d i_\xi d\alpha) \wedge \omega = 0,
\]

hence \( d\alpha \wedge \omega \in \Omega^{k+1}_B(F) \). Also the form \( d\alpha \wedge \omega \) is closed, so that its cohomology class \( \text{mod} \ d \Omega^k_B(F) \) is well defined. One checks easily that the definition of \( \Delta[\omega] \) doesn’t depend upon the choice of representative in \([\omega]\). At this point one may use the sequence (22) and the map \( \Delta \) to build the sequence

\[
H^k_B(F) \to H^k(\Omega^*^G(M)) \to H^{k-1}_B(F) \to H^{k+1}_B(F) \to \cdots
\]

which yields (21) as the compactness of \( G \) implies \( H^k(\Omega^*^G(M)) \approx H^k(M, \mathbb{R}) \) (cf. e.g. [2], p. 151). Since (22) is already exact we need to check exactness in (20) only at the terms of the form \( H^{k-1}_B(F) \). For any \( \omega \in \Omega^k(\Omega^*^G(M)) \) with \( d\omega = 0 \) we have

\[
\Delta(i_\xi)_*[\omega] = [d\alpha \wedge i_\xi \omega] = [d(\alpha \wedge i_\xi \omega) + \alpha \wedge di_\xi \omega] =
\]

\[
= [\alpha \wedge (\mathcal{L}_\xi \omega - i_\xi d\omega)] = 0.
\]

Viceversa, if \([\eta] \in \text{Ker}(\Delta)\) then \( \eta \in \Omega^{k-1}_B(F) \) and \( d\eta = 0 \) and

\[
d\alpha \wedge \eta = d\beta
\]

for some \( \beta \in \Omega^{k}_B(F) \). Then \( \omega := \alpha \wedge \eta - \beta \) is a closed \( G \)-invariant form and \( i_\xi \beta = 0 \) yields \((i_\xi)_*[\omega] = [\eta]\). Theorem \( \textbf{1} \) is proved. \( \Box \)

With the notations above a lightlike Killing vector field \( \xi \) is said to be \textit{complemented} if there exist nowhere zero globally defined \( G \)-invariant vector fields \( W \in \mathfrak{S}(T\mathcal{F}^+) \) and \( N \in \text{ltr}(T\mathcal{F}) \) such that 1) \([W, N] = 0\), 2) \( W \) is spacelike and \( g(\xi, N) = 1 \), and 3) for any \( f \in \Omega^0_B(F) \) there are \( a, b \in \Omega^0_B(F) \) such that \( N(b) - W(a) = f \).

\textbf{Proposition 5.} Let \( M \) be a 3-dimensional semi-Riemannian manifold and \( \xi \) a complemented Killing vector field on \( M \). Then \( H^2_B(F) = 0 \).

\textit{Proof.} We may assume without loss of generality that \( g(W, W) = 1 \). Otherwise we set \( W' = g(W, W)^{-1/2}W \) and observe that \( \mathcal{L}_\xi g = 0 \) and \( \mathcal{L}_\xi W = 0 \) yield \( \mathcal{L}_\xi W' = 0 \). Let us set

\[
\lambda(X) = g(X, N), \quad \mu(X) = g(X, \xi), \quad \eta(X) = g(X, W),
\]

for any \( X \in TM \). Then any \( \Omega \in \Omega^2_B(F) \) is given by \( \Omega = f \mu \wedge \eta \) for some \( f \in \Omega^0_B(F) \). Indeed \( \xi \mid \Omega = 0 \) implies that \( \Omega^2_B(F) \) is spanned by \( \mu \wedge \eta \). To see that the coefficient is a basic function one must compute \( \xi \mid d\Omega \). Note that (by \( \mu(\xi) = 0 \))

\[
2(\xi \mid d\mu)X = \xi(\mu(X)) - X(\mu(\xi)) - \mu([\xi, X]) =
\]
\[ = \xi(g(X, \xi)) - g(\mathcal{L}_\xi X, \xi) = (\mathcal{L}_\xi g)(X, \xi) = 0 \]

for any \( X \in T(M) \). Therefore \( \mu \in \Omega^1_B(\mathcal{F}) \). Similarly (by \( \eta(\xi) = 0 \))

\[ 2(\xi \left\lbrack d\eta \right\rbrack X = \xi(\eta(X)) - X(\eta(\xi)) - \eta([\xi, X]) = \]

\[ = \xi(g(X, W)) - g(\mathcal{L}_\xi X, W) = (\mathcal{L}_\xi g)(X, W) + g(X, \mathcal{L}_\xi W) = 0 \]

so that \( \eta \in \Omega^1_B(\mathcal{F}) \). Finally the identities

\[ \xi \left\lbrack (df \wedge \mu \wedge \eta \right\rbrack = \frac{1}{3} \xi(f) \mu \wedge \eta, \]

\[ \xi \left\lbrack (d\mu \wedge \eta \right\rbrack = \frac{2}{3} (\xi \left\lbrack d\mu \right\rbrack \wedge \eta = 0, \xi \left\lbrack \mu \wedge d\eta \right\rbrack = \frac{2}{3} \mu \wedge (\xi \left\lbrack d\eta \right\rbrack = 0, \]

together with \( \xi \left\lbrack d\Omega = 0 \right\rbrack \) yield \( \xi(f) = 0 \).

Next note that

\[ (23) \]

\[ d\mu = 0, \quad d\eta = 0. \]

Indeed \( \xi \left\lbrack d\mu \right\rbrack = 0 \) yields \( d\mu = h \mu \wedge \eta \) for some \( h \in C^\infty(M) \). On the other hand

\[ h = 2(d\mu)(N, W) = \]

\[ = N(\mu(W)) - W(\mu(N)) - \mu([N, W]) = \]

\[ = g(\xi, [W, N]) = 0. \]

The proof that \( d\eta = 0 \) is similar. Let now \( \omega \in \Omega^1_B(\mathcal{F}) \). Then \( \xi \left\lbrack \omega = 0 \right\rbrack \) yields \( \omega = a\mu + b\eta \) for some \( a, b \in C^\infty(M) \). By \( (23) \)

\[ d\omega = da \wedge \mu + db \wedge \eta = \]

\[ = (\xi(a) \lambda + W(a)\eta) \wedge \mu + (\xi(b) \lambda + N(b)\mu) \wedge \eta = \]

\[ = \xi(a) \lambda \wedge \mu + \xi(b) \lambda \wedge \eta + (N(b) - W(a)) \mu \wedge \eta \]

and \( 0 = 2 \xi \left\lbrack d\omega = \xi(a) \mu + \xi(b) \eta \right\rbrack \) shows that both \( a \) and \( b \) are basic functions. Finally

\[ H^2_B(\mathcal{F}) = \frac{\text{Ker}(d : \Omega^2_B(\mathcal{F}) \to \cdot)}{\text{d}\Omega^1_B(\mathcal{F})} = \]

\[ = \frac{\{ f \mu \wedge \eta : f \in \Omega^0_B(\mathcal{F}) \}}{\{(N(b) - W(a)) \mu \wedge \eta : a, b \in \Omega^0_B(\mathcal{F}) \}} = 0. \]

Proposition \( \text{[5]} \) is proved.

**Corollary 1.** Let \( (M, g) \) be a complete 3-dimensional Lorentz manifold with \( \text{Isom}(M, g) = O(4, 1) \). If \( M \sim S^3_1(r) \) i.e. \( M \) is a real homology pseudosphere \( S^3_1(r) \) then \( M \) admits no complemented lightlike Killing vector field.
Here $S^n_\nu(r)$ is the pseudosphere i.e. $S^n_\nu(r) = \{ x \in \mathbb{R}^{n+1} : -\sum_{j=1}^{\nu} x_j^2 + \sum_{j=\nu+1}^{n+1} x_j^2 = r^2 \} \ (r > 0)$. As well known

$$H^j(S^n_\nu(r), \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } j \in \{0, n-\nu\}, \\ 0 & \text{otherwise.} \end{cases}$$

The proof of Corollary 1 is by contradiction. Let $\xi$ be a complemented lightlike Killing vector field on $M$ and let $H$ be the 1-parameter group of transformations generated by $\xi$. Its closure $G = \overline{H} \subset O(4, 1)$ is compact (by Lemma 3.1 in [5]) hence $G$ is a torus. The proof of Theorem 1 relies only on the existence of $N \in S(TF^\perp)$ such that $g(\xi, N) = 1$, $g(N, N) = 0$ and $L_\xi N = 0$ so that we obtain the long exact cohomology sequence (20). In particular

$$H^1(M, \mathbb{R}) \to H^0_B(F) \to H^2_B(F) \to H^2(M, \mathbb{R}) \to H^1_B(F)$$

is exact. Since $M$ is assumed to have the real cohomology of $S^3_\nu(r)$ one has $H^0(M, \mathbb{R}) = H^2(M, \mathbb{R}) = \mathbb{R}$ and $H^1(M, \mathbb{R}) = 0$. Thus $H^1_B(F) = 0$ and (24) yields the exact sequence

$$0 \to \mathbb{R} \to H^2_B(F) \to \mathbb{R} \to 0,$$

in contradiction with Proposition 5. Corollary 1 is proved. Under the same assumptions as those of Theorem 1 (with $M$ not necessarily compact) one also has

**Corollary 2.** 1) If $M \sim S^{\nu+1}_\nu(r)$ then for any $\ell \geq 1$

$$H^{2\ell}_B(F) = \begin{cases} \mathbb{R} & \text{if } \Delta(1) \neq 0, \\ 0 & \text{if } \Delta(1) = 0, \end{cases} \quad H^{2\ell-1}_B(F) = \begin{cases} \mathbb{R} & \text{if } H^1_B(F) \neq 0, \\ 0 & \text{if } H^1_B(F) = 0, \end{cases}$$

where $\Delta : \mathbb{R} \to H^1_B(F)$ is the map $\Delta(c) = c[\alpha], c \in \mathbb{R}$. 2) If $M \sim S^{\nu+2}_\nu(r)$ then for any $\ell \geq 0$

$$H^{2\ell+1}_B(F) = 0,$$

$$H^{2\ell}_B(F) = \begin{cases} \mathbb{R} & \text{if } \exists f \in C^\infty(M) \text{ with } \xi(f) = 1, \\ 2 - \text{dimensional} & \text{otherwise.} \end{cases}$$

3) Assume that $M \sim S^{\nu+p}_\nu(r)$ for some fixed $3 \leq p \leq n - 1$. Then

$$H^j_B(F) = \begin{cases} \mathbb{R} & \text{if } j = \text{even}, \\ 0 & \text{if } j = \text{odd}, \end{cases} \quad 0 \leq j \leq p - 1,$$

$$H^{p+2j}_B(F) \approx H^p_B(F), \quad 1 \leq j \leq \left[ \frac{n-p-1}{2} \right],$$

$$H^{p+2j+1}_B(F) \approx H^{p+1}_B(F), \quad 1 \leq j \leq \left[ \frac{n-p}{2} \right] - 1.$$
Moreover one has either that i) \( j_* H^p(F) = 0 \) and then \( H^p_B(F) \approx H^p(F) \), or that ii) \( j_* H^p_B(F) = \mathbb{R} \) and then either \( H^p_B(F) = 0 \) and \( H^p_B(F) = \mathbb{R} \) or \( H^p_B(F) \neq 0 \) and \( \dim \mathbb{R} H_B(F) = 2 \). Finally either iii) \( (i_\xi)_* (1) = 0 \) and then \( H^{p+1}_B(F) \approx H^{p-1}_B(F) \), or iv) \( (i_\xi)_* (1) \neq 0 \) and then \( \dim \mathbb{R} H^{p+1}_B(F) = \dim \mathbb{R} H^{p-1}_B(F) - 1 \).

Here \([a] \in \mathbb{Z}\) is the integer part of \( a \in \mathbb{R} \). A similar result holds when \( M \) is a real homology pseudohyperbolic space. **Proof of Corollary 2** Statement 1) in Corollary 2 is a straightforward consequence of (20) and \( H^j(M) = \mathbb{R} \) if \( n = \nu + j \ (1 \leq j \leq n-1) \) and \( H^j(M) = 0 \) otherwise. Statement 2) requires again (20) and the following

**Lemma 5.** If there is \( f \in C^\infty(M) \) such that \( \xi(f) = 1 \) then \( H^2_B(F) = \mathbb{R} \). Otherwise \( \dim \mathbb{R} H^2_B(F) = 2 \).

**Proof.** Let \( f \in C^\infty(M) \) such that \( \xi(f) = 1 \) and let us set \( \omega := \alpha - df \). Then \( \xi | \omega = 0 \). Also \( d\omega = do \) so that \( \xi | d\omega = 0 \). Hence \( \omega \in \Omega^1_B(F) \) i.e. \( \omega \) is a basic form and then \( \Delta(1) = [do] = [d\omega] = 0 \in H^2_B(F) \). Note that \( H^2_B(F) \hookrightarrow H^1(M) = 0 \). Then \( \Delta(R) = 0 \) and the exactness of

\[
\mathbb{R} \xrightarrow{\Delta} H^2_B(F) \xrightarrow{j} \mathbb{R} \xrightarrow{\Delta} 0
\]

imply that \( j_* : H^2_B(F) \approx \mathbb{R} \). Otherwise \( \Delta(R) \neq 0 \) and \( H^2_B(F)/\Delta(R) \approx \mathbb{R} \) so that Lemma 5 is proved. Finally, statement 3) in Corollary 2 is implied by (20) and the fact that all cohomology groups of \( M \) vanish except for \( H^p(M) \approx \mathbb{R} \).

5. **Rummler’s formula for lightlike foliations with trivial radical distribution**

Let \((F, S(TF), S(TF^\perp))\) be a \( r \)-lightlike foliation of the semi-Riemannian manifold \( M \), together with a choice of screen distributions. We say that \( F \) is **tangentially** (respectively **transversally**) screen oriented if \( S(TF) \) is oriented (respectively if \( S(TF^\perp) \) is oriented). As both screen distributions are nondegenerate we may consider the local orthonormal frames \( \{X_a : 1 \leq a \leq m - r\} \) in \( S(TF) \) and \( \{W_\alpha : 1 \leq \alpha \leq q - r\} \) in \( S(TF^\perp) \), defined on the open set \( U \subseteq M \). Then \( g(X_a, X_b) = \epsilon_\alpha \delta_{ab} \) and \( g(W_\alpha, W_\beta) = \epsilon_\alpha \delta_{\alpha\beta} \) (where \( \epsilon_j^2 = 1 \)). Moreover we set \( \omega^a(X) = g(X, X_a) \) and \( \eta^a(X) = g(X, W_\alpha) \) for any \( X \in T(M) \). If \( F \) is tangentially screen oriented then the local \((m-r)\)-forms \( \omega^1 \wedge \cdots \wedge \omega^{m-r} \) glue up to give a (globally defined) \((m-r)\)-form \( \chi_{S(TF)} \) on \( M \).

We shall need a lightlike counterpart of Rummler’s formula, cf. H. Rummler, [8] (or identity (6.17) in [9], p. 66). To this end one ought to build a lightlike analog of the characteristic form \( \chi_F \) of a tangentially
oriented foliation $\mathcal{F}$ of a Riemannian manifold, cf. [9], p. 65-66. We do this under the additional assumption that
\[ \text{Rad} \, T \mathcal{F} \approx M \times \mathbb{R}^r \]
(the trivial bundle). If this is the case, let \{\(\xi_i : 1 \leq i \leq r\)\} be a globally defined frame (fixed through the remainder of this section) of \(\text{Rad} \, T \mathcal{F}\) and let \(\{N_i : 1 \leq i \leq r\}\) be the lightlike vector fields furnished by Lemma 1. The construction of \(N_i\) depends on a choice of complement \(E\) to \(\text{Rad} \, T \mathcal{F}\) in \(S(T \mathcal{F}^\perp)^\perp\) as in (7) and on the choice of a local frame \(\{V_i : 1 \leq i \leq r\}\) of \(E\) on \(U \subseteq M\), so that \(a \text{ priori}\) the \(N_i\)’s are but locally defined. Nevertheless if \(E\) is fixed as well and \(\{V'_i : 1 \leq i \leq r\}\) is another local frame of \(E\) on \(U' \subseteq M\) then \(N'_i = N_i\) on \(U \cap U'\). Therefore the proof of Lemma 1 yields a globally defined system of vector fields \(N_i \in S(T \mathcal{F}^\perp)^\perp\) such that
\[ g(\xi_i, N_j) = \delta_{ij} \] and
\[ g(N_i, N_j) = 0. \]
We set \(\lambda_i = g(N_i, X)\) and \(\mu_i = g(\xi_i, X)\) for any \(X \in T(M)\). Moreover let \(\chi_{\mathcal{F}}\) be the \(m\)-form on \(M\) given by
\[ \chi_{\mathcal{F}} = \lambda^1 \wedge \cdots \wedge \lambda^r \wedge \chi_{S(T \mathcal{F})}. \]
We emphasize on a number of elementary properties of the local frame \(\{X_a, \xi_i, W_\beta, N_i\}\). First \(g(N_i, N_j) = 0\) may be written
\[ \lambda^i(N_j) = 0, \quad 1 \leq i, j \leq r. \]
Next
\[ W_\beta \in S(T \mathcal{F}^\perp)^\perp \subseteq T(\mathcal{F})^\perp \subset T(\mathcal{F}) \supset S(T \mathcal{F}) \ni X_a \]
hence \(g(X_a, W_\beta) = 0\) i.e.
\[ \omega^a(W_\beta) = 0, \quad 1 \leq a \leq m-r, \quad 1 \leq \beta \leq q-r. \]
Moreover
\[ N_i \in \text{ltr}(T \mathcal{F}) \subset S(T \mathcal{F}^\perp)^\perp \subset S(T \mathcal{F}^\perp)^\perp \ni W_\beta \]
hence \(g(N_i, W_\beta) = 0\) i.e.
\[ \lambda^i(W_\beta) = 0, \quad 1 \leq i \leq r, \quad 1 \leq \beta \leq q-r. \]
Also
\[ N_i \in \text{ltr}(T \mathcal{F}) \subset S(T \mathcal{F}^\perp)^\perp \subset S(T \mathcal{F}^\perp)^\perp \ni X_a \]
so that \(g(X_a, N_j) = 0\) i.e.
\[ \omega^a(N_j) = 0, \quad 1 \leq a \leq m-r, \quad 1 \leq j \leq r. \]
Then (25)-(28) imply that
\[ tr(T \mathcal{F}) | \lambda^i = 0, \quad tr(T \mathcal{F}) | \omega^a = 0. \]
In particular \(tr(T \mathcal{F}) | \chi_{\mathcal{F}} = 0\) and
\[ \chi_{\mathcal{F}}(\xi_1, \cdots, \xi_r, X_1, \cdots, X_{m-r}) = 1/m!. \]
Let \( \tan : T(M) \to T(F) \) be the natural projection associated with the decomposition (11). If \( Z \in \Gamma^\infty(tr(TF)) \) then
\[
(\mathcal{L}_Z \chi_F)(Y_1, \ldots, Y_m) = Z(\chi_F(Y_1, \ldots, Y_m)) - \\
- \sum_{j=1}^m \chi_F(Y_1, \ldots, \tan[Z, Y_j], \ldots, Y_m)
\]
for any \( Y_j \in T(F) \). We wish to evaluate this identity at \( Y_i = \xi_i \) and \( Y_{a+r} = X_a \). The first term on the right hand side vanishes. By (29) one has \( \tan(X) = \lambda^i(X)\xi_i + \epsilon^a \omega^a(X)X_a \) for any \( X \in T(M) \) (where \( \epsilon^a = \epsilon_a \)). Hence

**Proposition 6.** Let \((F, S(TF), S(TF^\perp))\) be a \( r \)-lightlike foliation of codimension \( q \) of a \( n \)-dimensional semi-Riemannian manifold \((M, g)\) such that \( 1 \leq r \leq m - 1 \) and \( 1 \leq r \leq q - 1 \) where \( m = n - q \). Let us assume that \( \text{Rad } TF \) is trivial and let \( \xi = (\xi_1, \ldots, \xi_r) \) be a global frame of \( \text{Rad } TF \). If we define \( \kappa = \kappa(\xi, E) \) by setting
\[
\kappa(X) = 0, \quad X \in T(F),
\]
\[
\kappa(Z) = \lambda^i([Z, \xi_i]) + \epsilon^a \omega^a([Z, X_a]), \quad Z \in \text{tr}(TF),
\]
then \( \kappa \in \Omega^1(M) \) i.e. \( \kappa \) is globally defined and its definition doesn’t depend upon the choice of local orthonormal frames \( \{X_a\} \subset S(TF) \) and \( \{W_a\} \subset S(TF^\perp) \). If \( F \) is tangentially screen oriented then
\[
(30) \quad \mathcal{L}_Z \chi_F + \kappa(Z) \chi_F = 0
\]
on \( T(F) \otimes \cdots \otimes T(F) \) for any \( Z \in \text{tr}(TF) \).

The identity (30) is the lightlike analog to the Rummler formula we were seeking for while \( \kappa \) is formally similar to the mean curvature form of a Riemannian foliation (cf. e.g. [9], p. 67). Let \( h \) be the second fundamental form of the foliation \( F \) i.e.
\[
h(X, Y) = \Pi \nabla^g_X Y, \quad X, Y \in T(F).
\]
See also (6.1) in [9], p. 62. Let \( h_{S(TF)} \) be the restriction of \( h \) to \( S(TF) \otimes S(TF) \) and \( \tau_{S(TF)} \) the trace of \( h_{S(TF)} \) with respect to \( g \) i.e.
\[
\tau_{S(TF)} = \text{trace}_g h_{S(TF)}
\]
(locally \( \tau_{S(TF)} = \sum_{a=1}^{m-r} \epsilon_a h(X_a, X_a) \)). As it turns out, in the case of lightlike foliations of semi-Riemannian manifolds neither \( \tau_{S(TF)} \) is the mean curvature vector of the distribution \( S(TF) \) in \((M, g)\) nor \( \kappa \) equals \( \tau_{S(TF)} \) (but rather \( \tau_{S(TF)} \) plus extra terms whose geometric meaning is rather obscure). Indeed (as \( \nabla^g \) is torsion-free and \( \nabla^g g = 0 \))
\[
\kappa(Z) = \lambda^i([Z, \xi_i]) + \epsilon^a g(\nabla^g_Z X_a, X_a) - \epsilon^a g(\nabla_X_a Z, X_a) =
\]
that is
\[ \kappa(Z) = \lambda^i([Z, \xi_i]) + \epsilon^a g(Z, \nabla_{X_a}^q X_a) \]

(31)

\[ = \lambda^i([Z, \xi_i]) + g(Z, H_{S(TF)}) \]

where \( H_{S(TF)} \) is the mean curvature vector of \( S(TF) \) in \((M, g)\) i.e.

\[ H_{S(TF)} = \text{trace}_g B_{S(TF)}, \]

\[ B_{S(TF)}(X, Y) = (\nabla^q X^i Y^i)_{S(TF)^\perp}, \quad X, Y \in S(TF), \]

and \( V_{S(TF)^\perp} \) is the \( S(TF)^\perp \)-component of \( V \in T(M) \) with respect to the direct sum decomposition \( \mathbb{E} \). Indeed, as

\[ S(TF)^\perp = \text{tr}(TF) \oplus \text{Rad} TF \]

\( S(TF)^\perp \) is locally the span of \( \{N_i, W_{\alpha}, \xi_i\} \) hence

\[ g(H_{S(TF)}, Z) = \sum_{a=1}^{m-r} \epsilon^a g(\nabla^q_{X_a} X_a, Z) \]

and (31) is proved. The identity (31) also shows that \( \kappa \) is indeed globally defined. On the other hand, if \( s := \Pi(Z) \) then

\[ g_{\text{tra}}(s, \tau_{S(TF)}) = \sum_a \epsilon^a g(Z, \text{tra}(\nabla^q_{X_a} X_a)) \]

so that

\[ g(Z, H_{S(TF)}) = g_{\text{tra}}(s, \tau_{S(TF)}) + \epsilon^a g(\text{ltr}(Z), \nabla^q_{X_a} X_a) \]

where \( \text{ltr} : tr(TF) \to \text{ltr}(TF) \) is the projection associated to the decomposition \( \mathbb{E} \).

We shall need the multiplicative filtration \( \{F^r \Omega^k : r \geq 0, k \geq r - 1\} \) of the de Rham complex \( \Omega^\bullet(M) \) as devised by F. Kamber & P. Tondeur, \[3\] (cf. also \[9\], p. 120). That is

\[ F^r \Omega^k = \{ \omega \in \Omega^k(M) : i_{X_1} \cdots i_{X_{k-r+1}} \omega = 0, \quad X_j \in T(\mathcal{F}) \}. \]

An useful reformulation of (30) is \( \mathcal{L}_{X} \chi_{\mathcal{F}} + \kappa(Z) \chi_{\mathcal{F}} \in F^1 \Omega^m \) or equivalently (as \( T(\mathcal{F}) \upharpoonright \kappa = 0 \))

\[ d\chi_{\mathcal{F}} + (m + 1) \kappa \wedge \chi_{\mathcal{F}} \in F^2 \Omega^{m+1}. \]

The way \( \kappa \) depends upon \((\xi, E)\) is described by the following

**Corollary 3.** Let \( \mathcal{F} \) be a \( r \)-lightlike foliation of a semi-Riemannian manifold with \( \text{Rad} TF \) trivial. Let \( \xi' = (\xi'_1, \cdots, \xi'_r) \) be another global frame of \( \text{Rad} TF \) and let \( E' \to M \) be another complement to \( \text{Rad} TF \) in \( S(TF)^\perp \). Then there exists a \( C^\infty \) function \( f : M \to \text{GL}(r, \mathbb{R}) \) such that

\[ \kappa(\xi', E') = \kappa(\xi, E) + \text{trace}(f^{-1} df). \]

(33)
In particular if $\mathcal{F}$ is a 1-lightlike foliation and there is $(\xi, E)$ such that $d\kappa(\xi, E) = 0$ then the de Rham cohomology class $[\kappa(\xi, E)] \in H^1(M, \mathbb{R})$ doesn’t depend upon the choice of $(\xi, E)$. For instance let $M$ be complete and let $\xi$ be a lightlike Killing vector field on $M$. Let $\mathcal{F}$ be the 1-lightlike foliation of $M$ such that $T(\mathcal{F}) = \mathbb{R}\xi$. Assume that there exist $G$-invariant globally defined nowhere zero vector fields $V \in S(T\mathcal{F}^+) \perp$ and $W \in S(T\mathcal{F}^+) \perp$ such that (33) holds. Then $\kappa(\xi, RV)$ is closed.

Proof. Let $\{V_i' : 1 \leq i \leq r\}$ be a local frame of $E'$, defined on an open set $U' \subseteq M$ such that $U \cap U' \neq \emptyset$. Then $V'_i = a'_i \xi + b'_i V_j$ for some $C^\infty$ functions $a'_i$, $b'_i : U \cap U' \to \mathbb{R}$. Let us set $g'_{ij} = g(\xi', V'_j)$ and (according to the proof of Lemma 11) \([g'^{ij}] = [g'_{ij}]^{-1}\). Let $f = [f_j'] : M \to GL(r, \mathbb{R})$ such that $\xi' = f\xi$. Then

\[
(34) \quad g'^{ij} = b'_i f_j^k g^{k\ell}.
\]

Let $N'_j$ be given by

\[
N'_j = -\frac{1}{2} g'^{ki} g'^{ij} g(V'_k, V'_\ell) \xi'_i + g'^{kj} V'_k, \quad 1 \leq j \leq r.
\]

A calculation based on (34) shows that

\[
(35) \quad N'_j = \sum_{i=1}^{r} (f^{-1})^j_i N_i + \frac{1}{2} \{a'_k g'^{kj} - a'_k g'^{k\ell} (f^{-1})^j_\ell f_i\} \xi_i.
\]

Consequently

\[
\lambda^i([Z, \xi'_j]) = \lambda^i([Z, \xi_i]) + (f^{-1})^j_i Z(f_j')
\]

yielding (33). When $r = 1$ the identity (33) becomes

\[
\kappa(\xi', E') = \kappa(\xi, E) + d \log |f|.
\]

Q.e.d.

6. THE TRANSVERSAL DIVERGENCE THEOREM

Assume from now on that $\mathcal{F}$ is transversally screen oriented. Then the local $(q - r)$-forms $\eta^1 \wedge \cdots \wedge \eta^{q-r}$ glue up to a (globally defined) $(q - r)$-form $\nu_{S(T\mathcal{F}^+)}$ on $M$. Let $\nu_{\mathcal{F}}$ be the $q$-form on $M$ given by $\nu_{\mathcal{F}} = \mu^1 \wedge \cdots \wedge \mu^r \wedge \nu_{S(T\mathcal{F}^+)}$. Then $\omega := \nu_{\mathcal{F}} \wedge \chi_{\mathcal{F}}$ is a volume form on $M$. We denote by $V(\mathcal{F})$ the set of all infinitesimal automorphisms of $\mathcal{F}$ i.e. $Y \in V(\mathcal{F})$ is a vector field on $M$ such that $[X, Y] \in T(\mathcal{F})$ for any $X \in T(\mathcal{F})$. The transversal divergence operator is the map $\text{div}_B : V(\mathcal{F}) \to C^\infty(M)$ given by

\[
\mathcal{L}_Y \nu_{\mathcal{F}} = \text{div}_B(Y) \nu_{\mathcal{F}}, \quad Y \in V(\mathcal{F}).
\]

Cf. e.g. [2], p. 126. One checks easily that
Lemma 6. Assume that 1) the complement $E$ in $\mathcal{L}$ and the local frame 
$\{V_i : 1 \leq i \leq r\}$ may be chosen such that $N_i \in V(\mathcal{F})$, 1 $\leq i \leq r$, and
2) there is a transversal screen distribution $S(T\mathcal{F}^\perp)$ admitting a local
orthonormal frame $\{W_\alpha : 1 \leq \alpha \leq q-r\}$ such that each $W_\alpha$ is a local
infinitesimal automorphism of $\mathcal{F}$. Then $\nu_{\mathcal{F}}$ is holonomy invariant. In
particular $\text{div}_B X = 0$ for any $X \in T(\mathcal{F})$ and $\text{div}_B Y \in \Omega^0_B(\mathcal{F})$ for any
$Y \in V(\mathcal{F})$.

We shall prove the following lightlike analog of a result by F. Kamber
& P. Tondeur & G. Toth, [11]

Theorem 2. Let $\mathcal{F}$ be a tangentially and transversally screen oriented
lightlike foliation of the semi-Riemannian manifold $(M, g)$. Assume
that the radical distribution of $\mathcal{F}$ is trivial (i.e. $\text{Rad} T\mathcal{F} \cong M \times \mathbb{R}$) and
there is a transverse bundle $\text{tr}(T\mathcal{F}) = \text{ltr}(T\mathcal{F}) \oplus S(T\mathcal{F}^\perp)$ with $\text{ltr}(T\mathcal{F})$
and $S(T\mathcal{F}^\perp)$ as in Lemma 6. If $\partial M = \emptyset$ then

$$
\int_M \text{div}_B Y \nu_{\mathcal{F}} \wedge \chi_{\mathcal{F}} = (-1)^q (m+1) \int_M (i_Y \kappa) \nu_{\mathcal{F}} \wedge \chi_{\mathcal{F}}
$$

for any compactly supported $Y \in V(\mathcal{F})$.

When $M$ is a manifold-with-boundary ($\partial M \neq \emptyset$) the problem of
producing a foliated analog to the semi-Riemannian divergence theorem
(cf. B. Unal, [10]) is left open.

Proof of Theorem 2. Let $Y \in V(\mathcal{F})$ and $Z := \text{tra}(Y)$. A calculation
shows that $(i_Z d \nu_{\mathcal{F}}) \wedge \chi_{\mathcal{F}} = 0$. Then we have (by Lemma 6)

$$(\text{div}_B Y) \omega = (\text{div}_B Z) \nu_{\mathcal{F}} \wedge \chi_{\mathcal{F}} = (\mathcal{L}_Z \nu_{\mathcal{F}}) \wedge \chi_{\mathcal{F}} =$$

$$= \{(d i_Z + i_Z d) \nu_{\mathcal{F}}\} \wedge \chi_{\mathcal{F}} = (d i_Z \nu_{\mathcal{F}}) \wedge \chi_{\mathcal{F}} =$$

$$= d((i_Z \nu_{\mathcal{F}}) \wedge \chi_{\mathcal{F}}) + (-1)^q (i_Z \nu_{\mathcal{F}}) \wedge d\chi_{\mathcal{F}}.$$

If we set $\varphi := d\chi_{\mathcal{F}} + (m+1) \kappa \wedge \chi_{\mathcal{F}}$ then (as $T(\mathcal{F}) \not\mid \nu_{\mathcal{F}} = 0$)

$$(\text{div}_B Y) \omega = d((i_Y \nu_{\mathcal{F}}) \wedge \chi_{\mathcal{F}}) + (-1)^q (i_Y \nu_{\mathcal{F}}) \wedge (\varphi - (m+1) \kappa \wedge \chi_{\mathcal{F}}).$$

Note that $\varphi \in F^2 \Omega^{m+1}$ (by (32)). Let $(U, x^1, \ldots, x^m, y^1, \ldots, y^q)$ be a
foliated chart. Since $T(\mathcal{F}) \not\mid i_Y \nu_{\mathcal{F}} = 0$ it follows that $i_Y \nu_{\mathcal{F}}$ is a sum of
monomials of the form $dy^1 \wedge \cdots \wedge dy^{q-1}$ with $C^\infty$ coefficients. Also $\varphi$
is a sum of monomials each of which contains at least a monomial of
the form $dy^i \wedge dy^j$. Hence

$$(i_Y \nu_{\mathcal{F}}) \wedge \varphi \in F^{q+2} \Omega^m = 0$$

and (37) becomes

$$(\text{div}_B Y) \omega = d((i_Y \nu_{\mathcal{F}}) \wedge \chi_{\mathcal{F}}) + (-1)^{q+1} (m+1) (i_Y \nu_{\mathcal{F}}) \wedge \kappa \wedge \chi_{\mathcal{F}}.$$
Next $T(F) | \kappa = 0$ yields $\kappa \wedge \nu_F = 0$ and then $(i_Y \kappa) \wedge \nu_F - \kappa \wedge i_Y \nu_F = 0$ so that
\[(38) \quad (\text{div}_B Y) \omega = d ((i_Y \nu_F) \wedge \chi f) + (-1)^q (m + 1) (i_Y \kappa) \omega.\]
Finally one integrates \((38)\) over \(M\) and uses the Stokes theorem.

### 7. Lightlike functions

Let \((M, g)\) be a \(n\)-dimensional semi-Riemannian manifold. A \(C^\infty\) function \(f : M \to \mathbb{R}\) is said to be lightlike if \(\nabla f\) is null i.e. \(\text{Crit}(f) = \emptyset\) and \(g(\nabla f, \nabla f) = 0\). For instance if \(M = \mathbb{R}^n_s\) \((1 \leq s \leq n - 1)\) then a smooth function \(f : \mathbb{R}^n_s \to \mathbb{R}\) is lightlike if \((f x_1(x), \ldots, f x_n(x)) \neq 0\) at any \(x \in \mathbb{R}^n\) and
\[
\sum_{j=1}^s (f x_j)^2 - \sum_{j=s+1}^n (f x_j)^2 = 0,
\]
where \(f x_j = \partial f / \partial x_j\). Let \(f : M \to \mathbb{R}\) be a lightlike function and let \(F\) be the foliation by level sets of \(f\) so that \(T(F) = \{X \in T(M) : X(f) = 0\}\).

Then \(\dim \mathbb{R} T(F) = 1\). Consequently \(T(F) = \mathbb{R} \nabla f\). Taking into account the classification in Table 1

**Proposition 7.** For any lightlike function on a semi-Riemannian manifold the corresponding foliation by level sets is co-isotropic.

Indeed \(\text{Rad} T(F) = T(F) \cap T(F)^\perp = \mathbb{R} \nabla f\). We wish to apply the results in Section 1 to foliations by level sets of lightlike functions. Therefore we choose a screen distribution \(S(T(F))\) such that
\[(39) \quad T(F) = S(T(F)) \oplus \mathbb{R} \nabla f\]
(and \(S(T(F)^\perp) = 0\)). As \(S(T(F))\) is nondegenerate \(T(M) = S(T(F)) \oplus S(T(F)^\perp)\). In particular \(\dim \mathbb{R} S(T(F)^\perp) = 2\). Let \(E \to M\) be a real line bundle such that
\[(40) \quad S(T(F)^\perp) = (\mathbb{R} \nabla f) \oplus E.\]
Given \(V \in \Gamma^\infty(U, E)\) such that \(V_x \neq 0\) for any \(x \in M\) we set
\[(41) \quad N = \frac{1}{V(f)} \{V - \frac{1}{2} g(V, V) \nabla f\}\]
hence
\[(42) \quad g(N, N) = 0, \quad N(f) = 1.\]
Clearly if \(N'\) is similarly built in terms of a nowhere zero \(V' \in \Gamma^\infty(U', E)\) then \(V' = \lambda V\) for some \(C^\infty\) function \(\lambda : U \cap U' \to \mathbb{R}\) hence \(N = N'\).
on \( U \cap U' \). This furnishes a globally defined \( C^\infty \) section \( N \in S(TF) \perp \) possessing the properties (12). Then \( tr(TF) = \mathbb{R}N \) is a choice of transversal bundle and in particular

\[
T(M) = T(F) \oplus \mathbb{R}N.
\]

**Proposition 8.** Let \( f : M \to \mathbb{R} \) be a lightlike function on the semi-Riemannian manifold \((M, g)\). Let \( F \) be the foliation of \( M \) by level sets of \( f \). Then the isomorphism \( \sigma : \nu(F) \approx tr(TF) \) is given by

\[
\sigma(s) = Y(f)N, \quad X, Y \in T(M).
\]

Consequently the second fundamental form \( h \) of \( F \) in \((M, g)\) is given by

\[
\sigma h(X, Y) = -\text{Hess}_f(X, Y)N, \quad X, Y \in T(F).
\]

Finally \( \kappa \in \Omega^1(M) \) is given by \( T(F) \big| \kappa = 0 \) and

\[
2\kappa(N) = (\mathcal{L}_\xi g)(N, N) - \frac{1}{V(f)} \{\text{trace}_g(\mathcal{L}_Vg)_{S(TF)} - \frac{g(V, V)}{V(f)}[\Box f - 2\text{Hess}_f(\xi, N)]\}
\]
on \( U \subseteq M \), where \( \Box \) is the Laplace-Beltrami operator of \((M, g)\), while \( \xi \) and \( \text{Hess}_f \) are the gradient and Hessian of \( f \).

For instance

**Corollary 4.** Let \( f : \mathbb{R}^n_s \to \mathbb{R} \) be the linear function

\[
f(x_1, \ldots, x_n) = \sqrt{\frac{n-s}{s}} \sum_{i=1}^{s} x_i + \sum_{j=s+1}^{n} x_j
\]

and let \( F \) be the corresponding foliation of \( \mathbb{R}^n_s \) by affine hyperplanes.

Let \( S(TF) \) be the span of \( \{X_a : 1 \leq a \leq n-2\} \) where

\[
X_a = \begin{cases} 
\partial_a - \sqrt{\frac{n-s}{s}} \partial_n, & \text{if } 1 \leq a \leq s-1, \\
\partial_{a+1} - \partial_n, & \text{if } s \leq a \leq n-2.
\end{cases}
\]

Then \( S(TF) \) is a screen distribution i.e. (39) holds. Also if

\[
V = -\sqrt{\frac{n-s}{s}} \sum_{i=1}^{s-1} \partial_i + \sum_{j=s}^{n} \partial_j
\]
then \( E = \mathbb{R}V \) is a complement to \( \text{Rad} TF \) in \( S(TF) \perp \). Finally \( V \) and \( \nabla f \) are Killing vector fields on \( \mathbb{R}^n_s \) and consequently \( h = 0 \) and \( \kappa = 0 \).

Here \( \partial_i \) is short for \( \partial/\partial x_i \). **Proof of Proposition** It suffices to compute \( \kappa(N) \). On one hand

\[
g([N, \xi], N) = \frac{1}{2}(\mathcal{L}_\xi g)(N, N).
\]
On the other hand
\[ \Box f = \text{trace}_g \text{Hess}_f, \]
\[ \text{Hess}_f(X, Y) = X(Y(f)) - (\nabla^g_X Y)(f), \quad X, Y \in T(M), \]
and

**Lemma 7.** Let \((M, g)\) be a \(n\)-dimensional semi-Riemannian manifold of index \(1 \leq s \leq n - 1\). Let \(\{X_a : 1 \leq a \leq n - 2\}\) be a local orthonormal (i.e. \(g(X_a, X_b) = \epsilon_a \delta_{ab}, \epsilon_a^2 = 1\)) frame of \(S(T\mathcal{F})\) defined on the open set \(U \subseteq M\). If \(f : M \to \mathbb{R}\) is a lightlike function and \(\epsilon \in \{\pm 1\}\) then \(\{X_1, \cdots, X_{n-1}, \xi(\epsilon/2) N\}\) is a local orthonormal frame of \(T(M)\) on \(U\). In particular a screen distribution \(S(T\mathcal{F})\) has index \(\text{ind}(S(T\mathcal{F}), g) = s - 1\).

Consequently
\[ -\epsilon^a (\nabla^g_{X_a} X_a)(f) = \Box f - 2 \text{Hess}_f(\xi, N) \]
hence (by (41) and \(V \in S(T\mathcal{F})^\perp\))
\[ g(N, H_{S(T\mathcal{F})}) = \frac{\epsilon}{V(f)} \{ g(V, \nabla^g_{X_a} X_a) - \frac{g(V, V)}{2V(f)} (\nabla^g_{X_a} X_a)(f) \} = - \frac{1}{2V(f)} \text{trace}_g (\mathcal{L}_V g)_{S(T\mathcal{F})} + \frac{g(V, V)}{2V(f)^2} [\Box f - 2 \text{Hess}_f(\xi, N)] \]
and (41) is proved. Let us look at the example (45). The tangent bundle \(T(\mathcal{F})\) is the span of
\[ \{ \partial_i - \sqrt{\frac{n-s}{s}} \partial_n, \partial_j - \partial_n : 1 \leq i \leq s, \quad s + 1 \leq j \leq n - 1 \}. \]
Moreover \(\xi = -\sqrt{\frac{n-s}{s}} \sum_{i=1}^s \partial_i + \sum_{j=s+1}^n \partial_j\) hence \(\xi \in T(\mathcal{F})\) and \(\{X_a, \xi : 1 \leq a \leq n - 2\}\) are linearly independent everywhere on \(M\), where the \(X_a\)'s are given by (46). Therefore \(S(T\mathcal{F})\) is indeed a screen distribution and \(S(T\mathcal{F})^\perp\) is the span of
\[ \{ \partial_s, -\sqrt{\frac{n-s}{s}} \sum_{i=1}^{s-1} \partial_i + \sum_{j=s+1}^n \partial_j \}. \]
Consequently \(V \in S(T\mathcal{F})^\perp\) where \(V\) is given by (47). Next \(\{\xi, V\}\) are independent so that (40) holds. For any \(C^\infty\) function \(f : \mathbb{R}^n \to \mathbb{R}\) one has \(\mathcal{L}_\xi g = 2 \text{Hess}_f\). Finally a calculation shows that
\[ N = \frac{1}{2} \sqrt{\frac{s}{n-s}} \left( - \sum_{i=1}^{s-1} \partial_i + \partial_s + \sqrt{\frac{s}{n-s}} \sum_{j=s+1}^n \partial_j \right) \]
hence (by (44)) \(\kappa(N) = 0\).
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