Antithetic variates in higher dimensions

Sebastian del Baño Rollin\textsuperscript{1} and Joan-Andreu Lázaro-Cami\textsuperscript{2}

\textsuperscript{1}Centre de Recerca Matemàtica. Apartat 50, E-08193 Bellaterra
Barcelona, Spain.
\textsuperscript{2}Imperial College London. Department of Mathematics.
180 Queen’s Gate, SW7 2AZ, London, UK.

Abstract

We introduce the concept of multidimensional antithetic as the absolute minimum of the covariance function defined on the orthogonal group by $A \mapsto \text{Cov}(f(\xi), f(A\xi))$ where $\xi$ is a standard $N$-dimensional normal random variable and $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is an almost everywhere differentiable function. The antithetic matrix is designed to optimise the calculation of $E[f(\xi)]$ in a Monte Carlo simulation. We present an iterative annealing algorithm that dynamically incorporates the estimation of the antithetic matrix within the Monte Carlo calculation.

Keywords: Antithetic variates, Monte Carlo method, Robbins-Monro algorithms, simulated annealing.

1 Introduction

The valuation of financial derivatives is based on the resolution of a parabolic partial differential equation defined by the chosen dynamics for the underlying assets subject to boundary conditions defined by the product (see [MJ03]). These equations are rarely solvable explicitly and a numerical method has to be chosen. The standard methods of choice in the industry are resolution on grids and Monte Carlo. The applicability of the Monte Carlo method is a consequence of the Feynman Kac theorem which solves a parabolic PDE in terms of an expectation. In fact, in many of the more complex equity products with a large dimensionality, the grid method is not efficient and Monte Carlo is \textit{de facto} the only pricing method. In this approach the price can be written as an expectation

$$ m = E[f(\xi)], \quad (1.1) $$

where $\xi \sim \mathcal{N}(0, \text{Id}_N)$ is an $N$-dimensional standard normal random variable describing a random path of the underlying assets and $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is a measurable function representing the payoff of the derivative contract. The Monte Carlo method is essentially a transcription of the strong law of large numbers which claims that $m$ can be approximated by

$$ \frac{1}{n} \sum_{i=1}^{n} f(\xi_i), \quad (1.2) $$

\textsuperscript{1}e-mail: SebastiandB@crm.cat
\textsuperscript{2}e-mail: j.lazaro-cami@imperial.ac.uk

1
where $\xi_1, \ldots, \xi_n$ are independent simulations of the random variable $\xi$.

The main drawback of Monte Carlo methods is that they are usually computationally demanding, often putting great strains on the capability of a trading operation to properly monitor its risks and manage complex positions. Therefore, it is of great value to design methods to improve the performance of the Monte Carlo calculation. It is the aim of this article to present such a method.

In order to speed up the Monte Carlo method we can seek to decrease the simulation error. A measure of this error is given by $\sigma/\sqrt{n}$ for a large enough number of simulations $n$ \cite{HH64}, where $
abla = \sqrt{\text{Var}[f(\xi)]}$ is the standard deviation of $f(\xi)$. Like the expectation $m$, $\sigma$ is usually unknown and needs to be estimated.

Variance reduction techniques are methods that reduce this error by replacing $f(\xi)$ by a different random variable which has the same expectation but a smaller variance. Hopefully, this will ensure a faster convergence of the Monte Carlo method. Antithetic variates is one such method.

Antithetic variates appear for the first time in the seminal work of Hammersley, Morton, and Mauldon \cite{HM56, HM56a}. The crucial idea of this procedure is to recycle the simulations of $\xi$ as samples of $-\xi$, which has the same distribution as $\xi$. Therefore, we can approximate $m$ in (1.1) by

$$\frac{1}{n} \sum_{i=1}^{n} \left( \frac{f(\xi_i) + f(-\xi_i)}{2} \right).$$

The trivial equality

$$\text{Var}[f(\xi) + f(-\xi)] = 2 \text{Var}[f(\xi)] + 2 \text{Cov}(f(\xi), f(-\xi))$$

shows that if the dependence between $f(\xi)$ and $f(-\xi)$ is such that $\text{Cov}(f(\xi), f(-\xi)) < 0$, then the accuracy of (1.3) will be greater than that of the crude Monte Carlo (1.2).

Antithetic variates were developed in the case $\xi$ is one dimensional. For higher dimensional normal variables, practitioners have often used antithetics by changing the sign of some components of the random normal vector in a more or less haphazard manner. In fact, there is a larger underlying group of symmetries since, for any orthogonal matrix $A \in O(N)$, $A\xi$ is again a standard normal vector. We can therefore extend the approach of Hammersley et al. to higher dimensions by replacing the role of $-\xi$ above by $A\xi$ and approximating $m$ as

$$\frac{1}{n} \sum_{i=1}^{n} \left( \frac{f(\xi_i) + f(A\xi_i)}{2} \right).$$

An antithetic method corresponds to the choice of $A \in O(N)$ and the optimal antithetic is the matrix that minimizes the covariance function

$$A \mapsto \text{Cov}(f(\xi), f(A\xi)).$$

Note that, $O(N)$ being a compact group, this function will always have an absolute minimum. The purpose of our work is to propose an algorithm to locate this optimal antithetic matrix $A^* \in O(N)$, provided the covariance is negative for some value $A \in O(N)$. The present paper is the first step of a program whose aim is to describe optimal antithetics to price some popular complex derivatives such as baskets, cliquets, Himalaya options and the like.

Finally, note that solving

$$\min_{A \in O(N)} \text{Cov}(f(\xi), f(A\xi))$$

is equivalent to solving the simpler problem

$$\min_{A \in O(N)} \text{E}[f(\xi) f(A\xi)].$$

(1.5)
and that both are well posed. Indeed, since $O(N)$ is compact, there exists a $A^* \in O(N)$ such that $\min_{A \in O(N)} E[f(\xi) f(A\xi)] = E[f(\xi) f(A^*\xi)]$ provided that $f$ is continuous. More restrictively, we will assume throughout this paper that $f$ is everywhere continuous and continuously differentiable except on a set of zero (Lebesgue) measure.

1.1 Contents

The paper is structured as follows:

In Section 2 we recall some results about simulated annealing algorithms. These algorithms are designed to find the global minima of a given function defined on $\mathbb{R}^r$ by stochastically perturbing a gradient based algorithm which, by itself, would normally only converge to a local minimum.

Section 3 contains a detailed discussion on the exponential covering map from the Lie algebra $\mathfrak{so}(N)$ to $O(N)$. The coordinates induced by the exponential map seem to be the best suited for the optimization problem at hand.

In Section 4, we introduce a novel iterative simulated annealing algorithm adapted to the state space $O(N)$. This algorithm provides a sequence $\{A_k\}_{k \in \mathbb{N}} \subset O(N)$ of random variables converging in probability to the global minimum $A^*$ of (1.5). The efficiency and performance of the algorithm are checked in Section 5, where we use it to find the optimal antithetic for an Asian call option and a covariance swap.

Finally, in Section 6, we define a dynamical antithetic technique where we use $\{A_k\}_{k \in \mathbb{N}} \subset O(N)$ to estimate (1.1). More concretely, we define the sequence

$$S_n := \frac{1}{2n} \sum_{k=1}^{n} (f(\xi_k) + f(A_k\xi_k))$$

and we prove that (1.6) converges almost surely to $m = E[f(\xi)]$, and that $\sqrt{n}(S_n - m)$ converges in law to a normal variable $N(0, \sigma^2)$ with variance

$$\sigma^2 = \frac{1}{2} (\text{Var}[f(\xi)] + \text{Cov}(f(\xi), f(A^*\xi))).$$

That is, the estimated error of the Monte Carlo method (1.6) was reduced as much as it was possible using antithetics.

1.2 Notation

Throughout this article, $(\Omega, \mathcal{F}, P)$ will denote the underlying probability space, where $\mathcal{F}$ is a $\sigma$-algebra and $P : \mathcal{F} \rightarrow [0, 1]$ is a probability measure. The space of all $\mathbb{R}^N$-valued random variables will be denoted by $L^0_{\mathbb{R}^N}(\Omega, P)$. A standard Gaussian vector $\xi : \Omega \rightarrow \mathbb{R}^N$ will be a random variable with a probability density function given by

$$\rho(x) = \frac{1}{(2\pi)^{N/2}} e^{-\frac{1}{2} ||x||^2}$$

with respect to the Lebesgue measure $\lambda$. We will write $\xi \sim \mathcal{N}(0, \text{Id}_N)$ to mean $\xi$ is a standard Gaussian vector. On the other hand, $C^1_\lambda(\mathbb{R}^N)$ will denote the set of real functions $f : \mathbb{R}^N \rightarrow \mathbb{R}$ which are continuously differentiable except on a set of zero Lebesgue measure.
2 Simulated annealing algorithms

In this section we review some of the existing methods to deal with the problem of locating absolute minima for a function of the form

$$\mathcal{U} : \mathbb{R}^r \rightarrow \mathbb{R},$$

$$x \mapsto E[U(x, \xi)],$$

where $\xi \sim \mathcal{N}(0, \text{Id}_N)$ is a Gaussian random vector and $U : \mathbb{R}^r \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a measurable function continuously differentiable with respect to the first $r$ entries. Actually, in our original problem, $U$ is defined on the compact Lie group $O(N)$ rather than $\mathbb{R}^r$; this additional feature will be dealt with later.

The absolute minima of the function $\mathcal{U}$ above will be zeroes of its gradient field $\nabla \mathcal{U}$ which can, under regularity conditions, be calculated as $E[\nabla_x U(x, \xi)]$. Hopefully, one may expect to solve $\nabla \mathcal{U}(x) = 0$ by some numerical scheme such as the gradient method. However, the equation $\nabla \mathcal{U}(x) = E[\nabla_x U(x, \xi)] = 0$ is defined through an expectation. This means that, in general, the gradient $\nabla \mathcal{U}$ needs to be evaluated by a Monte Carlo simulation, which can be too expensive. Recall that we want to solve (1.5) in order to improve the efficiency of a (crude) Monte Carlo. Therefore, calculating the gradient by Monte Carlo estimation in order to improve our original (crude) Monte Carlo makes no sense. The solution is worse than the problem. In order to overcome this difficulty, we try to work directly with $\nabla_x U(x, \xi)$.

2.1 Robbins-Monro algorithms

Let $F(. , \xi) : \mathbb{R}^r \rightarrow \mathbb{R}^r$ be a measurable vector field depending on a Gaussian vector $\xi \in \mathcal{N}(0, \text{Id}_N)$. **Robbins-Monro algorithms** ([MR51]) are designed to solve an equation of the form $E[F(x, \xi)] = 0$ by means of a scheme algorithm such as

$$x_n = x_{n-1} + \gamma_n F(x_{n-1}, \xi_n),$$

where $\{\gamma_n\}_{n \in \mathbb{N}}$ is a non-negative sequence of real numbers and $\{\xi_n\}_{n \in \mathbb{N}}$ are independent Gaussian vectors. Observe that, in our particular case, $F = -\nabla_x U$. We will set $\overline{F}(x) = E[F(x, \xi)]$.

**Theorem 1** ([D97, Theorem 1.4.26]) Assume that $\overline{F}(x)$ has a zero $x^*$. Then the sequence defined by (2.1) converges almost surely to $x^*$ for almost all initial conditions $x_0$ provided that

A1. $\langle x - x^*, \overline{F}(x) \rangle < 0$ for any $x \in \mathbb{R}^r$.

A2. $\sum_{n \in \mathbb{N}} \gamma_n = \infty$ and $\sum_{n \in \mathbb{N}} \gamma_n^2 < \infty$.

A3. $E \left[ \|F(x_{n-1}, \xi_n)\|^2 \mid F_{n-1} \right] \leq K \left( 1 + \|\xi_n\|^2 \right)$ a.s. for some constant $K > 0$ where we set $F_{n-1}$ to be the $\sigma$-algebra generated by the random variables $\{\xi_k \mid k \leq n - 1\}$.

Here $\langle \cdot , \cdot \rangle$ and $\| \cdot \|$ denote the Euclidean scalar product and norm respectively.

**Remark 2** An example of a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ verifying the conditions of A2 is $\gamma_n = \frac{c}{n}$ for some constant $c > 0$. 

Among the hypotheses stated in Theorem A1, A1 is the most problematic. It means, on the one hand, that \( x^* \) is a unique zero of the vector field \( F \), and on the other, that \( \|x - x^*\| \) is a Lyapunov function for \( F \) so that \( x^* \) is an asymptotically stable equilibrium point. When \( F = -\nabla U \), this condition is sometimes hidden behind the requirement that \( U(x) \) is convex and has a unique minimum. Unfortunately, both conditions are too restrictive. In fact, by Morse theory (see [M63]), the number of critical points of a function defined on \( O(N) \) such as our original covariance function has to be at least the dimension of the total rational homology space of \( O(N) \), which is \( 2^{\lfloor N/2 \rfloor} + 1 \) (see [MT91] Corollary III.3.15).

There are variants of the Robbins-Monro algorithms that are guaranteed to converge to critical points of \( F \) which are actual minima of \( U \). However, there is no certainty that these will be absolute minima. In the next subsection, we will discuss simulated annealing methods which will converge to global minima.

Robbins-Monro algorithms combined with importance sampling methods have also been successfully used in the context of derivative pricing to reduce the variance of Monte Carlo simulations (see [A04], [A03], [DV98], and [PS02]). For example, if the price of a derivative is \( E[f(\xi)] \) with \( \xi \sim N(0, \text{Id}_N) \) as in (1.1), then a simple change of variables shows that

\[
E[f(\xi)] = E\left[f(x + \xi)e^{-x^2\xi^2/2}\right].
\]

However, the variance of the variable under the second expectation operator now depends on \( x \). So we will achieve faster convergence by choosing \( x \) that minimises \( \text{Var}[f(x + \xi)e^{-x^2\xi^2/2}] \) or, equivalently,

\[
\overline{U}(x) = E[f^2(\xi)e^{-x^2\xi^2/2}].
\]

(2.2)

If turns out this function is convex and thus has a unique minimum. The Robbins-Monro algorithm (suitable modified with a truncation method to ensure condition A3) can then be applied to \( F(x, \xi) = \nabla_x [f^2(\xi)e^{-x^2\xi^2/2}] \) to yield the optimal value of \( x \in \mathbb{R}^r \) (A04, A03). Note that in our problem the function to be minimised is not a measure change, and that it does not have the growth properties at infinity as a function of \( x \) of (2.2).

### 2.2 Simulated annealing algorithms

As we have mentioned, our minimisation problem will give rise to multiple critical points and thus is not suited to the Robbins-Monro scheme. A method that has been devised to deal with multiple minima is Simulated Annealing. This is a technique inspired in the metallurgy where a metal alloy is heated to pull it out of an equilibrium state, a local minimum of the energy, and then slowly cooled to allow atoms to diffuse into the lowest energy state where the system has some optimal physical property. The analogue of this heat injection in the Robbins-Monro algorithm (2.1) is the addition of an extra source of randomness that hits the approximating sequence out of local minima. In their most general form, simulated annealing algorithms are written as

\[
x_{n+1} = x_{n-1} + \gamma_n F(x_{n-1}, \xi_n) + b_n \zeta_n,
\]

(2.3)

where \( \{b_n\}_{n \in \mathbb{N}} \subset \mathbb{R} \) is a real sequence, the annealing temperature scheme, and \( \{\zeta_n\}_{n \in \mathbb{N}} \) a second sequence of i.i.d Gaussian random vectors independent from the \( \{\xi_n\}_{n \in \mathbb{N}} \). In order to study their convergence, first of all, we need to introduce some notation.

Let \( B_{x,y} \) denote the set of continuous paths \( \varphi : [0,1] \to \mathbb{R}^r \) starting at \( x \in \mathbb{R}^r \) and ending at \( y \in \mathbb{R}^r \). Let \( U(x, \xi) \) be as above and set \( F(x, \xi) = -\nabla_x U(x, \xi) \) and \( \overline{U}(x) = E[U(x, \xi)] \). Recall that
our intention is to locate absolute minima of $\overline{U}$. Let $S$ be the set where the absolute minimum is achieved

$$S = \{x \in \mathbb{R}^r \mid \overline{U}(x) = \min_{y \in \mathbb{R}^r} \overline{U}(y)\}$$

and $S = \{x \in \mathbb{R}^r \mid \nabla \overline{U}(x) = 0\}$ the set of critical points of $\overline{U}$ which we will assume has finitely many connected components. For any $l > 0$, let $B(S, l)$ be the set of points which are at a distance less than $l$ from $S$. Define

$$I(x, y) = \inf_{\varphi \in B_{x,y}} 2 \left( \max_{0 \leq t \leq 1} \overline{U}(\varphi(t)) - \overline{U}(x) \right) \text{ and } d^* = \max_{x \in S} \min_{y \in S} I(x, y). \quad (2.4)$$

Observe that $d^*$ is a measure of how oscillatory the objective function $\overline{U}$ is.

The next theorem provides sufficient conditions for guaranteeing the convergence of simulated annealing algorithms. It is extracted from [FGQ97, Theorem 5.2].

**Theorem 3** Let $\{r_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ and $\{h_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ be the sequences of real numbers defined by $r_n = 1/n^\gamma$, $0 < \gamma < 1$, and $h_n = d/(1-\gamma) \ln n$, where $d > d^*$. For $\{\xi_n\}_{n \in \mathbb{N}}$, $\{\zeta_n\}_{n \in \mathbb{N}}$ two independent sequences of i.i.d Gaussian random vectors define an iteration scheme by

$$x_n = x_{n-1} - r_n \nabla x_{U}(x_{n-1}, \xi_n) + \sqrt{r_n h_n} \zeta_n. \quad (2.5)$$

Assume the function $F(x) = -E[\nabla x U(x, \xi)]$ satisfies the following properties:

**B1.** $\limsup_{\|x\| \to \infty} \frac{\|F(x)\|}{\|x\|} \leq M_1 < \infty$,

**B2.** $\limsup_{\|x\| \to \infty} \frac{\|F(x)\|}{\|x\|^2} \leq -c < 0$, and

**B3.** $\limsup_{\|x\| \to \infty} \frac{E[\|F(x, \xi) - F(x)\|^2]}{\|x\|^2} \leq M_2 < \infty$

for some positive constants $M_1$, $M_2$, and $c$. Then, for any $l > 0$,

$$P(\{x_n \in B(S, l)\}) \to 1 \text{ as } n \to \infty \quad (2.6)$$

uniformly for any initial condition $x_0$ in an arbitrary compact set.

**Remark 4**

1. If the absolute minimum $x^*$ of $\overline{U}$ is unique, then (2.6) implies that the sequence $\{x_n\}_{n \in \mathbb{N}}$ converge in probability to $x^*$.

2. In fact, Fang et al. have proved a more general version of Theorem 3 for a wider spectrum of sequences $\{r_n\}_{n \in \mathbb{N}}$ and $\{h_n\}_{n \in \mathbb{N}}$ (see [FGQ97] for details). For example, they prove that Theorem 3 holds when $r_n = b/n$ and $h_n = d/(n \ln(n))$, $b > 0$, $d > 0$, and $n \in \mathbb{N}$. Simulated annealing algorithms with coefficients $-b/n$ and $\sqrt{b d/(n \ln(n))}$ have already been considered in [GM91, GM93] in the particular case $\nabla x U(x_{n-1}, \xi_n) = \nabla x V(x_{n-1}) + \xi_n$ with $V : \mathbb{R}^r \to \mathbb{R}$ a deterministic function.
3. Hypotheses **B1**, **B2**, and **B3** are weaker than the corresponding hypotheses **A1**, **A2**, and **A3** in the Robbins-Monro algorithm. This is because the former only impose restrictions at infinity. For example, if we know that the absolute minimum of the function $\bar{U}$ lies in a bounded domain, we can modify $\bar{U}$ far away from that domain so that **B1**, **B2**, and **B3** automatically hold and $\{x_n\}_{n \in \mathbb{N}}$ in (2.3) converges. If we are certain that the minimum is within the ball $B(x^*, R)$ of radius $R$ centered at some $x^* \in \mathbb{R}^r$, then we can add a penalty function appropriately on a narrow spherical shell around $\|x\| = R$ such that $-\nabla x \bar{U}$ strongly points towards $B(x^*, R)$.

4. As far as the rate of convergence of (2.5) is concerned, it is also proved in [FGQ97] that there exists a constant $\delta > 0$ such that, for any $\varepsilon > 0$,

$$P \{ \{x_n \in B(\mathbb{S}, l)\} \} \geq 1 - \varepsilon \text{ if } n > \exp \left( -\frac{d}{(1 - \gamma)\delta} \ln(2\varepsilon) \right).$$

### 2.3 Continuous simulated annealing

Simulated annealing is often presented in the literature in its continuous version. The stochastically perturbed iterative algorithm is then replaced by a stochastic differential equation with drift $-\nabla x \bar{U}$ the negative of the gradient of the function $\bar{U} \in C^1(\mathbb{R}^r)$ we wish to minimise, and diffusion coefficient decaying to zero with time as in the annealing method. Unfortunately, as it will become clearer later on, this procedure will only work for deterministic functions; that is, functions which do not depend on a random variable as is indeed our case. However, we wish to review time-continuous simulated annealing for the benefit of a more complete exposition. Most of the results quoted here are extracted from [BHT08] where Baudoin, Hairer, and Teichmann study the Ornstein-Uhlenbeck process on a compact Lie group and its properties to design efficient simulated annealing schemes. This recent paper improves considerably the efficiency of simulated annealing techniques developed so far (see [HKS89]). The reader is also encouraged to check with [B08 Chapter 5] for a more comprehensive approach.

Let $G$ be a compact Lie group and let $\mathcal{L} = \frac{1}{2} \sum_{i=1}^{d} V_i \circ V_i$ be a second order differential operator acting on $L^2(G, \mu_G)$, where $\mu_G$ is the Haar measure and, for any $i = 1, \ldots, d$, $V_i \in \mathfrak{g}$ is a left invariant vector field. We assume that Hörmander’s hypoelliptic condition holds, i.e., that the Lie sub-algebra generated by $\{V_1, \ldots, V_d\}$ coincides with $\mathfrak{g}$. In this context, the **carré du champ operator** $\Gamma$ is defined as

$$\Gamma(g, f) = \mathcal{L}(fg) - f \mathcal{L}(g) - g \mathcal{L}(f), \quad f, g \in L^2(G, \mu_G).$$

Let $\bar{U} \in C^1(G)$ be a differentiable function and let $\varepsilon \in (0, 1]$ be the annealing temperature parameter. Assume that the following integral is finite

$$Z_\varepsilon := \int_G e^{-\bar{U}(g)/\varepsilon^2} d\mu_G(g) < \infty.$$ 

Then we can define the **Gibbs measure** $\mu_\varepsilon$ by $\mu_\varepsilon(B) := \frac{1}{Z_\varepsilon} \int_B e^{-\bar{U}(g)/\varepsilon^2} d\mu_G(g), \quad B \in \mathbb{B}(G)$. Intuitively this measure concentrates on the minima of $\bar{U}$ as the temperature $\varepsilon$ falls to zero. The Gibbs measure is invariant under the differential operator $\mathcal{L}_\varepsilon = \varepsilon^2 \mathcal{L} - \frac{1}{2} \Gamma(\bar{U}, \cdot)$, which is the **infinitesimal generator** of the stochastic differential equation

$$dX^g_t = V_0(X^g_t) dt + \varepsilon \sum_{i=1}^{d} V_i(X^g_t) \delta B^i_t$$

(2.7)
where $V_0 = -\frac{1}{2} \Gamma (\overline{U}, \cdot) \in \mathfrak{g}(G)$ represents the negative gradient vector of the function $\overline{U}$. This is the continuous version of the simulated annealing process. Note that if the temperature $\varepsilon$ is set to zero, we are left with an ordinary differential equation where the carré du champ $\Gamma (\overline{U}, \cdot)$ plays the rôle of the ordinary gradient. Its flow, hopefully, will evolve to the closest minimum of $\overline{U}$. The addition of appropriately selected annealing schedule function $\varepsilon (t)$ will ensure we move to an absolute minimum.

In [BHT08] the following remarkable result is proved:

**Theorem 5** Assume the annealing heat function is given by

$$\varepsilon (t) = \frac{c}{\sqrt{\ln (R + t)}}$$

for positive constants $c, R > 0$. Then, under mild conditions on $\overline{U} \in C^1 (G)$,

$$P \left( \{ X_0^g \in B_\delta \} \right) \leq M \sqrt{\mu (B_\delta)} (2.8)$$

where $M > 0$, $B_\delta := \{ g \in G \mid \overline{U} (g) > \overline{U}_0 + \delta \}$, and $\overline{U}_0$ is the absolute minimum of $\overline{U}$. In particular, provided that there exists only one element $g_0 \in G$ such that $\overline{U} (g_0) = \overline{U}_0$, (2.8) implies that

$$\lim_{t \to \infty} E \left[ f (X_0^g) \right] = f (g_0)$$

for any continuous bounded test function $f \in C (G)$.

In other words, in order to find the global minima of $\overline{U}$ we can simulate numerically the stochastic differential equation (2.7) and approximate its invariant measure $\mu (t)$ which, as time goes by, concentrates around $g_0 \in G$.

Unfortunately, our original problem does not fit directly in this framework. The drift vector field $V_0 = \frac{1}{2} \Gamma (\overline{U}, \cdot)$ in (2.7) is assumed to be deterministic, it cannot depend on an independent Gaussian noise, as happens in our case: according to (1.5), $U = f (\xi) f (A_\xi)$, where $\xi \sim N (0, \text{Id}_N)$, $A \in O (N)$, and $f \in C^1 (\mathbb{R}^N)$. Removing this stochastic dependence would imply working with $\overline{U} (A) = E [ f (\xi) f (A_\xi) ]$. As we already argued, it is computationally inconvenient to take the expectation at this point.

### 3 Lie group methods

We have so far reviewed the simulated annealing algorithms available to globally minimize functions defined on $\mathbb{R}^r$ through an expectation. However, our optimization problem (1.5) does not take place on an Euclidean space but on a Lie group, namely, the orthogonal group $O (N)$. Therefore, we need to adapt the results of Section 2 to $O (N)$ by taking suitable coordinates. As it is customary on Lie groups, we will take local coordinates by means of a local diffeomorphism from the Lie algebra, $\varphi : \mathfrak{o} (N) \to O (N)$, which is a Euclidean space. In this section we are going consider two different choices of $\varphi$: the Cayley transform and the exponential map. These are two of the most used coordinate patches in the design of numerical integrators for ordinary differential equations on Lie groups ([IMPZ05]). This section aims at giving explicit expressions for the gradient of a smooth function $F : O (N) \to \mathbb{R}$ when composed with the local coordinates $\varphi$. Such a gradient will be employed later in our simulated annealing algorithm.

Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. The tangent bundle $\tau_G : TG \to G$ is trivial meaning that it is isomorphic to the product $G \times \mathfrak{g}$. The identification $TG = G \times \mathfrak{g}$ can be carried out by
means of an isomorphism \( \rho : TG \to G \times \mathfrak{g} \) induced by right translations. If \( R_g : G \to G \) denotes the map defined for \( g \in G \) by \( R_g(h) = hg \), then \( \rho(v) = (g, T_g R_g^{-1}(v)) \), where \( g = \tau_G(v) \). We refer to this trivialisation as the **space coordinates** on the tangent bundle.

Given a smooth function \( F : G \to \mathbb{R} \), the tangent map \( TF : TG \to T\mathbb{R} \) it induces will be noted \( \mathcal{T}F \) when written in space coordinates. That is, \( \mathcal{T}F : G \times \mathfrak{g} \to \mathbb{R}^2 \)

\[
(g, X) \mapsto (F(g), T_g F \circ T_g R_g(X)).
\]

Equivalently, if \( \varphi : \mathfrak{g} \to G \) is a local diffeomorphism from a neighborhood of \( 0 \in \mathfrak{g} \), \( \mathcal{T}\varphi \) is defined by \( \mathcal{T}\varphi : \mathfrak{g} \times \mathfrak{g} \to G \times \mathfrak{g} \) stands for the tangent map \( T\varphi : \mathfrak{g} \times \mathfrak{g} \to TG \) in space coordinates, i.e., \( T_X \mathcal{T}\varphi = T_{\varphi(X)} \mathcal{T}\varphi \circ T_X \varphi \), \( X \in \mathfrak{g} \). It can be immediately checked that

\[
T (F \circ \varphi) = \mathcal{T}F \circ T \varphi.
\]

In order to minimise a function on \( G \) or, via a coordinate chart, on \( \mathfrak{g} \), it will be useful to have a notion of gradient field. Assume therefore that we are given an arbitrary metric \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{g} \). We will describe the gradient of a function \( F : G \to \mathbb{R} \) in space coordinates. For \( X, Y \in \mathfrak{g} \), the gradient of \( F \circ \varphi : \mathfrak{g} \to \mathbb{R} \) satisfies

\[
\langle \nabla (F \circ \varphi)(X), Y \rangle = d(F \circ \varphi)(X)(Y) = pr_2 \circ T_X (F \circ \varphi)(Y)
= pr_2 \circ T_{\varphi(X)} \mathcal{T}\varphi \circ T_X \varphi(Y)
\]

where \( pr_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) denotes the projection onto the second factor. Therefore, if \( \{Y_i\}_{i=1, \ldots , \dim(\mathfrak{g})} \) is an orthonormal basis, then

\[
\nabla (F \circ \varphi)(X) = \sum_{i=1}^{\dim(\mathfrak{g})} (pr_2 \circ T_{\varphi(X)} \mathcal{T}\varphi \circ T_X \varphi(Y_i)) Y_i.
\]

**Example 6 (Canonical coordinates of the first kind)** Let \( \varphi : \mathfrak{g} \to G \) be \( \varphi(X) := \exp (X) g \) for some \( g \in G \). The exponential map of a Lie group is a local diffeomorphism from a neighborhood of \( 0 \in \mathfrak{g} \) onto a neighborhood of \( g \in G \). Using these coordinates, it can be checked that

\[
T_X \mathcal{T}\varphi = \exp (ad_X) - \Id = \sum_{j \geq 0} \frac{1}{(j+1)!} ad_X \circ [\, \cdot \, ]^j \circ ad_X
\]

(see [IMPZ05, L98]). Canonical coordinates of the first kind are convenient for nilpotent Lie algebras because then the series (3.2) becomes a finite sum. Unfortunately, \( \mathfrak{so}(N) \) is not nilpotent. However, these coordinates are useful for \( SU(3) \) because, in this particular case, the exponential can be easily computed by means of the **Rodrigues formula**. Indeed, if

\[
X = \begin{pmatrix}
0 & -a & b \\
 a & 0 & -c \\
-b & c & 0
\end{pmatrix} \in \mathfrak{so}(3),
\]

one can prove that

\[
\exp (X) = \Id + \frac{\sin(\sigma)}{\sigma} X + \frac{1 - \cos(\sigma)}{\sigma^2} X^2,
\]

where \( \sigma = \sqrt{a^2 + b^2 + c^2} \), and

\[
T_X \exp = \Id + \frac{1 - \cos(\sigma)}{\sigma} X + \frac{\sigma - \sin(\sigma)}{\sigma^3} X^2.
\]

For \( n \geq 3 \) several methods have been devised to calculate the exponential map by other means than truncation of the series \( \sum_{n \geq 1} \frac{1}{n!} X^n \), which often leads to numerical error (see [BLP05, GX02]).
4 Higher dimensional antithetic variates

We now present an annealing type method to find the optimal antithetic matrix $A^* \in O(N)$ as defined in the introduction. Recall that an optimal antithetic is an absolute minimum of the function $A \mapsto \text{Cov}(f(\xi), f(A\xi))$, where $f \in C^1_b(\mathbb{R}^N)$ is the payout function depending on a path described by a normal vector $\xi \sim N(0, \text{Id}_N)$. This is equivalent to minimising $U(A) = \mathbb{E}[f(\xi)f(A\xi)]$. In the notation of previous sections we write $U(A, \xi) = f(\xi)f(A\xi)$.

The plan is to coordinate $SO(N)$ by means of the exponential map $\exp(Y)$, $Y \in \mathfrak{so}(N)$, introduce an iterative algorithm, and study its convergence in the light of Theorem 3. Unfortunately, $U(\exp(Y), \xi)$ have not the appropriate behaviour at infinity as specified by hypothesis $B_2$ in Theorem 3. Indeed, let $\langle X, Y \rangle := \frac{1}{2} \text{trace}(XY^\top)$ be the standard Euclidean product on $\mathfrak{so}(N)$ and let $\| \cdot \|$ its associated norm, $Y, X \in \mathfrak{so}(N)$. If $F = -\nabla \mathbb{E}[f(\xi)f(\exp(Y)\xi)]$ denotes the gradient of $U \circ \exp$ with respect to an orthonormal basis of $\mathfrak{so}(N)$ then, since $U$ is bounded,

$$\lim_{\|Y\| \to \infty} \frac{\langle F(Y), Y \rangle}{\|Y\|^2} = 0,$$

which is not strictly negative as $B_2$ requires. In order that our algorithm satisfies this and the rest of hypotheses, we will therefore modify $U(\exp(Y))$ far away from a bounded set with a penalty function.

Before stating the main results of this section, we need an auxiliary lemma. Recall that the rank of a Lie group is the dimension of a maximal torus and that $SO(N)$ has rank $\lfloor N/2 \rfloor$, where $\lfloor \cdot \rfloor$ stands for the integer part of a real number. We define $R := \lfloor N/2 \rfloor$ for the sake of a simpler notation.

**Lemma 7** The compact ball $B(0, \pi \sqrt{R}) \subset \mathfrak{so}(N)$ surjects onto $SO(N)$ by the exponential map.

**Proof.** It is a well know fact that any matrix in $SO(N)$ is conjugated to an element in the maximal torus of matrices with $\lfloor N/2 \rfloor$ diagonal blocks of the form

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \theta \in [-\pi, \pi],$$

(see [BD85, Chapter IV Theorem 1.6]). It is immediate to see that these rotations can be written as $\exp\left(\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}\right)$. Since conjugation commutes with exponentiation, we conclude that, if

$$K := \left\{ \text{diag}(S_1, \ldots, S_R) \in \mathfrak{so}(N) \mid S_i = \begin{pmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{pmatrix}, \quad \theta_i \in [-\pi, \pi], \quad i = 1, \ldots, R \right\},$$

then the compact set $\left\{ AkA^\top \mid A \in SO(n), \ k \in K \right\} \subset \mathfrak{so}(N)$ surjects onto $SO(n)$ by the exponential map. The lemma follows because, for any $A \in SO(n)$ and any $k \in K$, we have

$$\| AkA^\top \|^2 = \frac{1}{2} \text{trace}(Akk^\top A^\top) = \frac{1}{2} \text{trace}(kk^\top) \leq R\pi^2$$

As stated above, we will use a penalty function to ensure that our algorithm converges. We use the notation $1_B(Y)$ for the characteristic function of a set $B \subseteq \mathfrak{so}(N)$. 
Theorem 9 Let \( E_{ij} \in \mathfrak{s}(N) \) be the matrix with +1 in the \((i,j)\)th position, −1 in the \((j,i)\)th position and zero elsewhere. The set \( \{E_{ij}\}_{0<i<j\leq N} \) is a basis of \( \mathfrak{s}(N) \), orthogonal with respect to the aforementioned standard scalar product.

Definition 8 Let \( P(Y) \) be the function defined on \( \mathfrak{s}(N) \) by

\[
P(Y) = \|Y\|^2 \cdot 1_{B(0, \pi \sqrt{R})^c}(Y)
\]

The algorithm we present in the following theorem requires some notation. Let \( E_{ij} \in \mathfrak{s}(N) \) be the matrix with +1 in the \((i,j)\)th position, −1 in the \((j,i)\)th position and zero elsewhere. The set \( \{E_{ij}\}_{0<i<j\leq N} \) is a basis of \( \mathfrak{s}(N) \), orthogonal with respect to the aforementioned standard scalar product.

Theorem 9 Let \( \{\xi_n\}_{n \in \mathbb{N}} \) and \( \{\zeta_n\}_{n \in \mathbb{N}} \) be sequences of independent standard normal \( N \)-dimensional vectors, \( A_0 \in O(N) \) and let \( \varphi \) be the exponential function centered at \( A_0 \), \( \varphi(Y) = \exp(Y)A_0 \). Set \( Y_0 = 0 \) and \( Z_0 = 0 \). Then the sequence \( \{A_n\}_{n \in \mathbb{N}} \) defined by

\[
\begin{align*}
Z_n &= \sum_{i<j} (f(\xi_n)\nabla f|_{A_{n-1}\xi_n} T_{Y_{n-1}} \varphi^{ac}(E_{ij}) A_{n-1}\xi_n) E_{ij} \in \mathfrak{s}(N) \quad (4.1a) \\
Y_n &= Y_{n-1} + \frac{1}{n} \left( Z_n + 2Y_{n-1} \cdot 1_{B(0, \pi \sqrt{R})^c}(Y_{n-1}) \right) + \frac{d}{n^\gamma (1-\gamma) \ln n} \zeta_n \quad (4.1b) \\
A_n &= \exp(Y_n)A_0 \quad (4.1c)
\end{align*}
\]

converges in probability to the optimal antithetic \( A^* \) (or the antithetic locus if there are multiple minima) in the connected component of \( SO(N) \) containing \( A_0 \). Here \( d > 0 \) is the same as in Theorem 3.

Proof. The proof consists in showing that the iterative process \( \{Y_n\}_{n \in \mathbb{N}} \) is the simulated annealing algorithm described in Theorem 3 applied to the function \( U(\xi, \exp(Y)) + P(Y), Y \in \mathfrak{s}(N) \).

Since \( U(\xi, \exp(Y)) + P(Y) \) coincides with \( U(\xi, \exp(Y)) \) on \( B(0, \pi \sqrt{R}) \) and is strictly greater outside this ball, the minima of these functions will coincide in \( B(0, \pi \sqrt{R}) \). Thanks to Lemma 7, the exponential of the absolute minima of \( U(\xi, \exp(Y)) + P(Y) \) yield the optimal antithetics.

The Robbins-Monro gradient term in the annealing procedure is

\[
F(\xi_n, Y_n-1) = -\nabla_Y U(\xi_n, \exp(Y)) + P(Y) |_{Y=Y_n-1}. \quad (4.2)
\]

In order to calculate it, let \( \varphi(Y) = \exp(Y)A_0 \) and write \( U_\xi(A) := U(\xi, A), A \in O(n) \). The tangent map \( T_A U_\xi : T_A O(N) \to T_{U_\xi(A)} \mathbb{R} \cong \mathbb{R} \) can be described in space coordinates on \( X \in \mathfrak{s}(N) \) by using the chain rule in the following manner

\[
T_A U_\xi(X) = T_A U_\xi (T_{Id} R_A (X)) = T_{Id} (U_\xi \circ R_A)(X) =
\]

\[
= \frac{d}{dt} \bigg|_{t=0} f(\xi) f(\exp(Xt)A\xi) = f(\xi) \nabla f|_{A\xi} X A\xi,
\]

where \( \nabla f|_A \) is the row vector \((\partial f/\partial \xi^1, \ldots, \partial f/\partial \xi^N)\). Now, let \( \{E_{ij}\}_{0<i<j\leq N} \) be the orthonormal basis of \( \mathfrak{s}(N) \) above-mentioned. By Section 3 Equation 3.1, the gradient \( \nabla_Y (U_\xi \circ \varphi) \) in the first part of (4.2) is built from the composition of \( T_Y U_\xi^{ac} \) with \( T_Y \varphi^{ac} \). Explicitly,

\[
\nabla_Y (U_\xi \circ \varphi) = \sum_{i<j} (f(\xi)\nabla f|_{A\xi} T_Y \varphi^{ac}(E_{ij}) A\xi) E_{ij} \in \mathfrak{s}(N), \quad A = \exp(Y)A_0.
\]

On the other hand, the second part of (4.2) is simply minus the gradient of the penalty function \( P(Y), -2Y \cdot 1_{B(0, \pi \sqrt{R})^c}(Y) \). Thus, the Robbins-Monro part in the simulated annealing algorithm...
is
\[-\left(\sum_{i<j} (f(\xi)\nabla f)^T A\xi T Y^\varphi^{sc}(E_{ij}) A\xi E_{ij}\right) - 2Y \cdot 1_{B(0,\pi\sqrt{R})}^v(Y)\]
evaluated at \(Y = Y_{n-1}\), \(A = A_{n-1}\), and \(\xi = \xi_n\). This yields the iterative scheme described in the statement.

To prove convergence we have to show that hypothesis \(B_1\), \(B_2\) and \(B_3\) in Theorem 3 hold. Since \(U(\xi, A)\) is bounded the only part of the objective function that will contribute in the limits in those conditions will be the penalty function \(P(Y)\). That is, we can replace \(F(Y)\) by \(-\nabla\|Y\|^2 = -2Y^\top\). \(B_1\), \(B_2\) and \(B_3\) follow easily in this case.

Remark 10

1. The choice of initial matrix, \(A_0\), is rather arbitrary unless one has additional information on the behaviour of the function \(f\).

However a couple of observations can be made to aid an informed decision. The first is that \(O(N)\) has two connected components, \(SO(N)\) and \(SO(N)^-\), defined as the preimages of \(+1\) and \(-1\) by the continuous map \(\det : O(N) \to \{+1, -1\}\). The algorithm we will define will stay in the connected component that contains \(A_0\).

The maximum of the covariance function \(\text{Cov}(f(\xi), f(A\xi))\) is achieved at \(A = \text{Id} \in SO(N)\), where it values \(\text{Var}[f(\xi)]\). Assuming the covariance function is continuous, it will be positive around the identity matrix \(\text{Id}\). Therefore, it seems advisable to choose the initial matrix in the other connected component \(SO(N)^-\). It might be of interest to explore whether the minimum of the covariance function always achieved in \(SO(N)^-\).

2. The convergence of algorithm (4.1) depends on the constants \(d > 0\). The parameter \(d\) controls the heat injection in the annealing scheme. As mentioned in Theorem 3, \(d\) must be larger than \(d^*\) in Equation (2.4), which measures the oscillatory nature of the function. In our case, given that the objective function \(\mathcal{F}(Y)\) is essentially the covariance, it can be seen that \(d^* \leq 4\text{Var}[f(\xi)]\). Since \(\text{Var}[f(\xi)]\) is a number that any user of Monte Carlo will be familiar with, we can estimate the magnitude of a valid choice of \(d\).

3. As stated the algorithm applies to functions \(f\) which are differentiable. Many of the functions used in pricing derivatives are non differentiable along a hypersurface. In fact the algorithm will work provided the set of non-differentiable points has measure zero a condition that will hold in most real life scenarios.

5 Examples

In this section, we are going to test the performance of our algorithm (4.1) in finding the optimal antithetic for a couple of elementary derivatives, an (arithmetic) Asian option and a covariance swap. Both products will be priced under the Black-Scholes model. This means that, under the risk-neutral probability, the price of an asset \(S_t\) is expressed as
\[S_t = S_0 e^{(r-d-\nu^2/2)t + \nu B_t},\]  
where \(r\) stands for the continuous interest rate, \(d\) for the continuous dividend yield, and \(\nu\) for the volatility, all assumed constant; \(B_t\) denotes a standard Brownian motion, i.e., \(B_t - B_s \sim N(0, t-s)\).
t > s. On the other hand, if \( f_t(S_1, ..., S_k) \) is the payoff function at time \( t \) of a given contract depending on \( k \) assets, the price of such contract at time \( t \) is given by

\[
e^{-rt} E \left[ f_t(S_1, ..., S_k) \right].
\] (5.2)

We do not intend to find the right price of such products in a realistic market, where more sophisticated models may be required, but simply implement algorithm (4.1) and show its efficiency and usefulness in standard Monte Carlo simulations. We hope to explore antithetics under other asset models in a future work.

5.1 Asian option

An (arithmetic) Asian call option with strike price \( K \) is a derivative contract based on an asset price \( S_t \) whose payoff at the expiry time \( T \) is given by

\[
\max \left( 0, \frac{1}{N} \sum_{i=1}^{N} S_{t_i} - K \right),
\]

where \( 0 \leq t_1 \leq ... \leq t_N = T \) is a sequence of times set in the contract at which \( S_t \) is observed. Consequently, according to (5.2), the price of an Asian call option is

\[
e^{-rT} E \left[ \max \left( 0, \frac{1}{N} \sum_{i=1}^{N} S_{t_i} - K \right) \right].
\] (5.3)

In order to estimate this expectation through a Monte Carlo simulation, we replace each \( S_{t_i} \) with the random variable

\[
S_{t_i} = S_{t_{i-1}} e^{(r-\nu^2/2)(t_i-t_{i-1})+\nu\sqrt{(t_i-t_{i-1})}\xi}, \quad \xi \sim \mathcal{N}(0,1),
\]

so that we can compute the sequence of prices iteratively. Therefore, according to our picture, the expectation in (5.3) can be rewritten as \( E[f(\xi)] \), where

\[
f(x_1, ..., x_N) = \max \left( 0, \frac{1}{N} \sum_{i=1}^{N} \prod_{j=1}^{i} e^{(r-\nu^2/2)(t_j-t_{j-1})+\nu\sqrt{(t_j-t_{j-1})}x_j} - K \right) \quad \text{and} \quad \xi \sim \mathcal{N}(0, \text{Id}_N).
\]

In the previous expression, we implicitly assumed \( t_0 = 0 \).

In this particular example, we are going to find the optimal antithetic for an Asian option averaged over the asset prices on the first of each month along one year. For the sake of simplicity, we suppose that all months have the same number of days. That is, the expiry time \( T = 1 \) equals one year and \( t_i = i/12 \) years, \( i = 1, ..., 12 \). More explicitly, our antithetic matrix will be of dimension 12, which means that our optimizing problem (1.5) takes place on a space of dimension \( \dim(\mathfrak{so}(12)) = 66 \). We will suppose that the asset price at the initial date is \( S_0 = \£100 \) and that the strike price equals \( \£100 \) as well. Furthermore, we set \( r = 2.83\% \) and \( \nu = 10.36\% \).

The results obtained are summarized in Table 1. The first row refers to a crude Monte Carlo estimation of one Asian option according to the data above-mentioned. The second and the third rows contain, respectively, information on the Monte Carlo estimation of (5.2) using the antithetics procedure (4.1) when the matrix \( A \) is minus the identity \( -\text{Id}_{12} \) and the antithetic \( A^* \) provided by algorithm (4.1) with \( \gamma = 1/2 \) and initial condition \( A_0 = -\text{Id}_{12} \). For any estimation, we specify
the price of the option, the estimated variance $\sigma^2 = \text{Var}[e^{-rT} f(\xi)]$ of the method, the error $\sigma/\sqrt{n}$ associated to that estimation, and the time (in seconds) our computer (a laptop with a 1.6 MHz CPU) invested in those computations.

| Price (£) | Variance $\sigma^2$ | MC Error $\sigma/\sqrt{n}$ | Time (s) |
|-----------|---------------------|-----------------------------|----------|
| Crude Monte Carlo | 3.15 | 17.25 | 0.01 | 8.75 |
| Antithetic $-\text{Id}_{12}$ | 3.15 | 3.50 | 0.01 | 3.41 |
| Antithetic $A^*$ | 3.15 | 3.60 | 0.01 | 3.54 |

Table 1: Monte Carlo simulations for an Asian option

First of all, we observe that the use of antithetics dramatically reduces the variance a crude Monte Carlo simulation by, roughly speaking, 5, which means, once an error threshold is fixed, we need less than half of the time required in a crude Monte Carlo to price the option with the same accuracy. However, we can see that the use of antithetics in the original Hammersley sense, that is, taking $-\text{Id}_{12}$, slightly beats the antithetic provided by algorithm (4.1). This is because, in this particular example, $-\text{Id}_{12}$ is one of the points where the covariance function $\text{Cov}(f(\xi), f(A\xi))$ reaches its minimum value. If algorithm (4.1) does not provide an antithetic so performing is because we can only run it for a finite time while its convergence to the set $S = \min_{A \in \mathbb{R}^{12}} \text{Cov}(f(\xi), f(A\xi))$ is assured as $t \to \infty$. However, despite this fact and that $A^*$ is obtained by a numeric method, which by definition intrinsically carries some degree of approximation, the result is satisfactory and we conclude that $A^*$ is very close to an absolute minimum.

The main drawback of algorithm (4.1) is its speed of convergence. Since at each step we have to compute, on the one hand, the exponential of a skew-symmetric matrix (of dimension 12) and, on the other, the tangent map $T \exp^{sc}$ to write down the gradient (4.1a), the algorithm is very time-consuming. For example, the antithetic $A^*$ used in the simulations of Table 1 is obtained after 10000 iterations, which our 1.6MHz computer carried out in approximately forty minutes, nothing comparable with the times spent in the Monte Carlo simulations (see Table 1). Therefore, finding the optimal antithetic $A^*$ before a crude Monte Carlo seems advisable if $A^*$ may be reused in several simulations. We hope to improve the efficiency of the algorithm in a future work.

### 5.2 Covariance swap

The next example aims at showing that the optimal antithetic $A^*$ of some daily traded products can be very different from the antithetic $-\text{Id}$ which, moreover, turns out to be completely useless.

A **covariance swap** depending on two assets $S_1$ and $S_2$ is a contract that, at the expiry date $T$, pays

$$\frac{1}{N} \sum_{i=1}^{N} \left( \frac{(S_1)_{t_i}}{(S_1)_{t_{i-1}}} - 1 \right) \left( \frac{(S_2)_{t_i}}{(S_2)_{t_{i-1}}} - 1 \right)$$

where, as in the case of the Asian option, $0 = t_0 \leq t_1 \leq ... \leq t_N = T$ is a sequence of times fixed in the contract. This quantity can be negative, which means that the holder of the contract, instead of receiving any money, must pay that quantity. It is called covariance swap because, roughly speaking, it measures the realised covariance between the returns of the assets $S_1$ and $S_2$. Sometimes, the returns are alternatively expressed as $\ln \left( \frac{S_{t_i}}{S_{t_{i-1}}} \right)$.

In the Black-Scholes model, where the assets are uncorrelated, the payoff (5.4) is very small (but not zero though, because the spot prices are only observed on a finite number of dates and
not continuously). In order to increase the price of \((5.4)\), so that we avoid working with too many decimal places, we take the maximum between \((5.4)\) and 0,

$$\max \left( 0, \frac{1}{N} \sum_{i=1}^{N} \left( \frac{(S_1)_{t_i}}{(S_1)_{t_{i-1}}} - 1 \right) \left( \frac{(S_2)_{t_i}}{(S_2)_{t_{i-1}}} - 1 \right) \right). \quad (5.5)$$

We insist, however, that this is not the way covariance swaps are traded in real markets, where they are mainly used as a hedge against possible unwanted correlation effects between assets or indices.

We are going to fix the details of our simulation. Let \(\nu_1 = 17.36\%\) and \(\nu_2 = 10.12\%\) be the volatilities of \(S_1\) and \(S_2\). Suppose that the asset \(S_2\) pays a constant dividend yield \(d = 2.03\%\) and that the constant free-risk interest rate is again \(r = 2.83\%\). Hence the assets \(S_1\) and \(S_2\) follow the log-normal processes:

\[
(S_1)_t = (S_1)_0 e^{(r - \nu_1^2/2)t + \nu_1 B^{(1)}_t}, \quad (S_2)_t = (S_2)_0 e^{(r - d - \nu_2^2/2)t + \nu_2 B^{(2)}_t},
\]

where \(B^{(1)}_t\) and \(B^{(2)}_t\) are two independent Brownian motions. As in the previous example, the price of \((5.5)\) can be written as \(e^{-rT} \mathbb{E}[f(\xi)]\). Explicitly,

\[
f(x_1, ..., x_{2N}) = \max \left( 0, \frac{1}{N} \sum_{j=1}^{N} \left( e^{(r - \nu_1^2/2)(t_j - t_{j-1}) + \nu_1 \sqrt{(t_j - t_{j-1})x_{2j}} - 1} \right) \cdot \left( e^{(r - d - \nu_2^2/2)(t_j - t_{j-1}) + \nu_2 \sqrt{(t_j - t_{j-1})x_{2j+1}} - 1} \right) \right) \quad (5.6)
\]

and \(\xi \sim N(0, \text{Id}_{2N})\) is a Gaussian vector whose odd components \(\xi^{2j+1}, \ j = 1, ..., N\), are used to generate the evolution of \(S_1\) while the even components \(\xi^{2j}\) go with \(S_2\). It is clear from \((5.6)\) that the values of \(S_1\) and \(S_2\) at time \(t = 0\) play no role in the final price of the covariance swap option, so we can take them equal to 1.

Table 2 summarises the prices of 100 covariance swaps estimated by Monte Carlo. As for the Asian option before studied, the asset prices \(S_{t_i}\) are read on the first of every month along one year, where all months are supposed to have 30 days, which means that the expiry time \(T = 1\) equals one year and \(t_i = i/12\) years, \(i = 1, ..., 12\). Our (optimal) antithetic matrix will be of dimension 24, so that our optimizing problem \((1.5)\) takes place on a space of dimension \(\dim(\text{so(24)}) = 276\). As in Table 1, the first row is used to provide a crude Monte Carlo, the second one to the antithetics procedure \((1.4)\) when the matrix \(A\) is minus the identity \(\text{Id}_{24}\), and the third one to the antithetic \(A^*\) provided by algorithm \((1.4)\) with \(\gamma = 1/2\) and initial condition \(A_0 = -\text{Id}_{24}\). Again, we specify the price of the option, the estimated variance \(\sigma^2 = \mathbb{V}ar[e^{-rT} f(\xi)]\) of the method, the error \(\sigma/\sqrt{n}\) associated to that estimation, and the time (in seconds) spent in those simulations.

|                      | Price (£) | Variance \(\sigma^2\) | MC Error (\(\sigma/\sqrt{n}\)) | Time (s) |
|----------------------|-----------|------------------------|-------------------------------|----------|
| Crude Monte Carlo    | 0.198     | 0.081                  | 0.001                         | 7.28     |
| Antithetic \(-\text{Id}_{24}\) | 0.197     | 0.081                  | 0.001                         | 15.72    |
| Antithetic \(A^*\)  | 0.198     | 0.029                  | 0.001                         | 5.02     |

Table 2: Monte Carlo simulations for an option on a covariance swap

We can observe that now the antithetic \(A = -\text{Id}_{24}\) does not improve the efficiency of the crude Monte Carlo. Indeed, it is easy to check that, in this particular example, \(\text{Cov}(f(\xi), f(-\xi)) = \mathbb{V}ar[f(\xi)], \xi \sim N(0, \text{Id}_{24})\), so that the variance of \((1.4)\) is the same of the original (crude) Monte Carlo simulations.
del Baño Rollín and Lázaro-Camí: Antithetic variates in higher dimensions

Carlo. However, since at each step we now evaluate the payoff function \( f \) twice, we need twice as much time. In conclusion, the antithetic \( A = -\text{Id}_{24} \) is useless.

On the contrary, the antithetic \( A^* \) reduces the variance 2.8 times which, in turn, means that the computational time is reduced, approximately, by 1.45. Unfortunately, unlike \( -\text{Id}_{24} \), \( A^* \) is an orthogonal dense matrix that has no trivial interpretation. And what is worse, as the dimensionality of the optimisation problem grows, so does the time needed to obtain a good approximation of \( A^* \).

For example, in the location of the skew-symmetric matrix \( Y^* \in \mathfrak{so}(24) \) such that \( \exp(Y^*) = A^* \), a 1.6MHz laptop invested more than an hour in carrying out 100 000 iterations of (4.1). Although there are certainly more performing computers, this time is huge compared with the times of the simulations we want to improve (see Table 2).

6 A dynamically optimized Monte Carlo

Section 4 provides an algorithm to locate the optimal antithetic \( A^* \). Recall that the use of this matrix \( A^* \) will improve the calculation of \( m = \mathbb{E}[f(\xi)] \) as a Monte Carlo average in the form

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{f(\xi_k) + f(A^*\xi_k)}{2}.
\]

In practice, as we saw in Section 5, it seems too expensive to locate \( A^* \) and then use it in the Monte Carlo calculation. It would be better to use the antithetics dynamically, meaning that we use as approximating sum

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{f(\xi_k) + f(A_{k-1}\xi_k)}{2}. \tag{6.1}
\]

In other words, we calculate the optimal antithetic at the same time as we estimate \( \mathbb{E}[f(\xi)] \).

It is worth emphasizing that the random samples \( \{\xi_n\}_{n \in \mathbb{N}} \) used in the location of \( A^* \) via the simulated annealing (4.1) can be also recycled in (6.1) to approximate \( m \). Did we do that, the results in this section would continue being valid. However, the expressions would become more complex but with no additional content. Therefore, in order to illustrate that antithetic variates can be used dynamically, we are only going to study the estimations of \( m \) obtained from (6.1).

Furthermore, for the benefit of a simpler notation, we will abbreviate

\[
g(A_{k-1}, \xi_k) := \frac{1}{2} (f(\xi_k) + f(A_{k-1}\xi_k)).
\]

The random variables \( \{g(A_{k-1}, \xi_k)\}_{k \in \mathbb{N}} \) are neither independent nor equally distributed because, for any \( k \in \mathbb{N} \), \( A_k \) is not deterministic but depends on the previous random variables \( \{\xi_i, \zeta_i\}_{i=1, \ldots, k} \). Consequently, we cannot invoke the Strong Law of Large Numbers to guarantee that (6.1) converges to \( m \) almost surely. In other words, (6.1) is different from computing \( m \) as

\[
\frac{1}{n} \sum_{k=1}^{n} \frac{f(\xi_k) + f(A^*\xi_k)}{2}
\]

for the optimal value \( A^* \), where \( A^* \) is supposed to be deterministic and fixed. Moreover, the dependence of \( \{g(A_{k-1}, \xi_k)\}_{k \in \mathbb{N}} \) might cause the appearance of some positive correlations which might spoil the efficiency of our method. Nevertheless, our situation is not that bad. Indeed, if we define

\[
S_n := \frac{1}{n} \sum_{k=1}^{n} g(A_{k-1}, \xi_k),
\]
we are going to prove in Theorem 11 that, on the one hand, \(S_n\) converges to the expectation \(m = \text{E}[f(\xi)]\) almost surely as \(n \to \infty\) and, on the other, that \(\sqrt{n}(S_n - m)\) converges in law to a normal variable \(N(0, \sigma^2_n)\) with variance

\[
\sigma^2_n = \text{Var}[g(A^*, \xi)] = \frac{1}{2} (\text{Var}[f(\xi)] + \text{Cov}(f(\xi), f(A^*\xi))) ,
\]

\(\xi \sim N(0, \text{Id}_N)\). It is worth noticing that \(\sigma^2_n = \text{Var}[g(A^*, \xi)]\) is the smallest possible variance we can get using antithetic variates.

Finally, in order to estimate \(\sigma^2_n\) and thus obtain empirical confidence intervals for \(m\), we prove in Theorem 12 that

\[
\frac{1}{n} \sum_{k=1}^{n} g^2(A_{k-1}, \xi_k) - S_n^2 \xrightarrow{\text{a.s.}} \sigma^2_n \text{ in probability.}
\]

The following results are inspired by [A04, A03].

**Theorem 11** Let \(A^* \in O(N)\) and suppose that there exists a sequence \(\{A_k\}_{k \in \mathbb{N}}\) of random variables taking values in \(O(N)\) such that \(A_k \to A^*\) in probability as \(k \to \infty\). Let \(\{\xi_k\}_{k \in \mathbb{N}}\) and \(\{\zeta_i\}_{i \in \mathbb{N}}\) be a couple of sequences of i.i.d Gaussian vectors mutually independent. Let \(\mathcal{F}_k = \sigma(\xi_i, \zeta_i \mid 1 \leq i \leq k)\) be the \(\sigma\)-algebra generated by \(\{\xi_i, \zeta_i\}_{i=1,...,k}\) and suppose that \(A_k\) is \(\mathcal{F}_k\)-measurable. Furthermore, if \(\xi \sim N(0, \text{Id}_N)\) denote an independent Gaussian vector, assume that \(\text{E}[g(A_{k-1}, \xi)^2] < \infty\) for any \(k \in \mathbb{N}\) and that the map \(A \mapsto \text{E}[g(A, \xi)^2]\) is continuous at \(A^*\). Then,

\[
\frac{1}{n} \sum_{k=1}^{n} g(A_{k-1}, \xi_k) \xrightarrow{a.s.} m.
\]

**Proof.** Define \(M_0 = 0\) and

\[
M_n = \sum_{k=1}^{n} (g(A_{k-1}, \xi_k) - m), \quad n \geq 1.
\]

Let \(\mathcal{F}_n\) be the \(\sigma\)-algebra generated by \(\{\xi_i, \zeta_i\}_{i=1,...,n}\). Since \(A_{n-1}\) is \(\mathcal{F}_{n-1}\)-measurable and \(\xi_n\) is independent of \(\mathcal{F}_{n-1}\), we have

\[
\text{E}[g(A_{n-1}, \xi_n) \mid \mathcal{F}_{n-1}] = \text{E}[g(A, \xi_n)]_{A=A_{n-1}} = m
\]

and \(\{M_n\}_{n \in \mathbb{N}}\) is a martingale with respect to the filtration \(\{\mathcal{F}_n\}_{n \in \mathbb{N}}\). As we imposed that \(\text{E}[(g(A_{n-1}, \xi_n))^2] < \infty\) for any \(n \in \mathbb{N}\), it is not difficult to see that \(\{M_n\}_{n \in \mathbb{N}}\) is a square integrable martingale. On the other hand, its angle bracket is given by

\[
\langle M \rangle_n = \sum_{k=1}^{n} \text{E}[(\Delta M_k)^2 \mid \mathcal{F}_{k-1}] = \sum_{k=1}^{n} \left( \text{E}[g^2(A_{k-1}, \xi_k) \mid \mathcal{F}_{k-1}] - m^2 \right)
\]

\[
= \sum_{k=1}^{n} \left( \text{E}[g^2(A, \xi_k)]_{A=A_{k-1}} - m^2 \right).
\]

Now, since \(A_k \to A^*\) in probability as \(k \to \infty\) and \(A \mapsto \text{E}[g^2(A, \xi_k)]\) is continuous at \(A^*\), \(\text{E}[g^2(A, \xi_k)]_{A=A_{k-1}}\) also converges in probability to \(\text{E}[g^2(A, \xi)]_{A=A^*}\) by Lemma 13. Applying Lemma 13, we see that

\[
\frac{\langle M \rangle_n}{n} \xrightarrow{n \to \infty} \text{E}[g^2(A, \xi)]_{A=A^*} - m^2 = \text{Var}[g(A^*, \xi)]
\]

in probability. Therefore, by Theorem 15 (i), we conclude that \(\frac{1}{n} M_n \xrightarrow{n \to \infty} 0\) a.s.
Theorem 12 (Central Limit Theorem) With the same notation as in Theorem 11, assume now that $E[|g(A_{k-1}, \xi)|^2] < \infty$ for any $k \in \mathbb{N}$ and that $A \mapsto E[|g(A, \xi)|^p]$ is continuous at $A^*$, $1 \leq p \leq 4$.

(i) The following convergence in law holds:

$$\sqrt{n} (S_n - m) \xrightarrow{n \to \infty} \mathcal{N}(0, \sigma^2_n),$$

where $\sigma^2_n := \text{Var}[g(A^*, \xi)]$.

(ii) Let $\sigma_n^2 := \frac{1}{n} \sum_{k=1}^{n} g^2(A_{k-1}, \xi_k) - S_n^2$. Then $\sigma_n^2 \xrightarrow{n \to \infty} \sigma^2_*$ in probability.

Proof.

(i) First of all, observe that

$$E \left[ |g(A_{k-1}, \xi_k) - m|^4 \mid \mathcal{F}_{k-1} \right] = \frac{E [g^4(A_{k-1}, \xi_k) \mid \mathcal{F}_{k-1}] - 3m^4}{-4m E [g^3(A_{k-1}, \xi_k) \mid \mathcal{F}_{k-1}] + 6m^2 E [g^2(A_{k-1}, \xi_k) \mid \mathcal{F}_{k-1}]},$$

where $\sigma^2 := \text{Var}[g(A^*, \xi)]$.

Since $A \mapsto E[|g(A_{k-1}, \xi)|^p]$ is continuous and $A_k \to A$ in probability as $k \to \infty$, by Lemma 13 we have that $E \left[ |g(A_{k-1}, \xi_k) - m|^4 \mid \mathcal{F}_{k-1} \right]$ converges in probability to a positive random variable $L \in L^4_{\text{eq}}(\Omega, \mathbb{P})$ as $k \to \infty$. Thus, by Lemma 13,

$$\frac{1}{n} \sum_{k=1}^{n} E \left[ |g(A_{k-1}, \xi_k) - m|^4 \mid \mathcal{F}_{k-1} \right] \xrightarrow{n \to \infty} L,$$

in probability.

For any $a > 0$, define

$$F_n := \frac{1}{n} \sum_{k=1}^{n} E \left[ |g(A_{k-1}, \xi_k) - m|^2 \mathbf{1}[|g(A_{k-1}, \xi_k) - m| > a] \mid \mathcal{F}_{k-1} \right].$$

It can be easily checked that

$$F_n(a) \leq \frac{1}{na^2} \sum_{k=1}^{n} E \left[ |g(A_{k-1}, \xi_k) - m|^4 \mid \mathcal{F}_{k-1} \right],$$

so that, taking the limit superior of the sequence $\{F_n(a)\}_{n \in \mathbb{N}}$ in probability, we obtain that $\limsup_{n \to \infty} F_n(a) \leq a^{-2}L$. If now $a = \varepsilon \sqrt{n}$ with $\varepsilon > 0$ fixed, we have

$$\limsup_{n \to \infty} F_n(\varepsilon \sqrt{n}) = 0$$

in probability and the Lindberg’s condition holds. Therefore, $\sqrt{n} (S_n - m) \xrightarrow{n \to \infty} \mathcal{N}(0, \sigma^2_*)$ in law by Theorem 11 (ii).
Lemma 13

(ii) Essentially, we only need to show that \( \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left[ g^2(A, \xi_k) \right] \big|_{A=A_{k-1}} \) and \( \frac{1}{n} \sum_{k=1}^{n} g^2(A_{k-1}, \xi_k) \) have the same limit a.s.. We are going to do so by mimicking the proof of Theorem 11 and quoting Theorem 15.

Define \( M_0 = 0 \) and

\[
M_n := \sum_{k=1}^{n} \left( g^2(A_{k-1}, \xi_k) - \mathbb{E} \left[ g^2(A, \xi_k) \right] \big|_{A=A_{k-1}} \right), \quad n \geq 1.
\]

It is not difficult to check that \( \{M_n\}_{n \in \mathbb{N}} \) is a square integrable martingale with respect to \( \{\mathcal{F}_n\}_{n \in \mathbb{N}}, \mathcal{F}_n = \sigma(\xi_i, \xi_i \mid i = 1, \ldots, n) \), whose angle bracket is given by

\[
\langle M \rangle_n = \sum_{k=1}^{n} \left( \mathbb{E} \left[ g^4(A_{k-1}, \xi_k) \mid \mathcal{F}_{k-1} \right] - \mathbb{E} \left[ g^2(A, \xi_k) \right]^2 \big|_{A=A_{k-1}} \right).
\]

By the continuity of \( A \mapsto \mathbb{E} \left[ |g(A, \xi)|^p \right] \), \( 1 \leq p \leq 4 \), and Lemma 13 we have

\[
\frac{1}{n} \langle M \rangle_n \longrightarrow \mathbb{E} \left[ g^4(A, \xi) \right] \big|_{A=A^*} - \mathbb{E} \left[ g^2(A, \xi) \right]^2 \big|_{A=A^*} = \text{Var} \left[ g^2(A^*, \xi) \right]
\]

in probability. Then Theorem 15 (i) claims that

\[
\frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left[ g^2(A, \xi_k) \right] \big|_{A=A_{k-1}} - \frac{1}{n} \sum_{k=1}^{n} g^2(A_{k-1}, \xi_k) \longrightarrow 0 \text{ a.s.}
\]

Since \( \mathbb{E} \left[ g^2(A, \xi_k) \right] \big|_{A=A_{k-1}} = \mathbb{E} \left[ g^2(A, \xi) \right] \big|_{A=A_{k-1}} \) for any arbitrary Gaussian vector \( \xi \), we conclude that \( \frac{1}{n} \sum_{k=1}^{n} g^2(A_{k-1}, \xi_k) \) and \( \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left[ g^2(A, \xi) \right] \big|_{A=A_{k-1}} \) have the same limit almost surely.

Finally, as \( \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left[ g^2(A, \xi_k) \right] \big|_{A=A_{k-1}} \) and \( S_n^2 \) converge in probability to \( \mathbb{E} \left[ g^2(A^*, \xi) \right] \) and \( \mathbb{E} \left[ g(A^*, \xi) \right]^2 \) as \( n \to \infty \) respectively, we obtain that

\[
\sigma_n^2 = \frac{1}{n} \sum_{k=1}^{n} g^2(A_{k-1}, \xi_k) - S_n^2 \longrightarrow \mathbb{E} \left[ g^2(A^*, \xi) \right] - \mathbb{E} \left[ g(A^*, \xi) \right]^2 = \text{Var} \left[ g(A^*, \xi) \right]
\]

in probability.

\[
\]

\section{Appendix}

In this appendix we recall some auxiliary results. The first one is a probabilistic version of the so-called Cesaro Lemma and reads

\begin{lemma}
Let \( \{\xi_n\}_{n \in \mathbb{N}} \subset L^0_{\mathbb{F}}(\Omega, \mathcal{P}) \) be a sequence of random variables and let \( \xi \in L^0_{\mathbb{F}}(\Omega, \mathcal{P}) \). If \( \xi_n \rightarrow \xi \) in probability as \( n \to \infty \), then

\[
\frac{1}{n} \sum_{k=1}^{n} \xi_k \longrightarrow \xi \text{ in probability as well.}
\]

\end{lemma}
**Proof.** Let $\zeta, \eta \in L^0_{\mathbb{R}^r} (\Omega, P)$. The function $d(\zeta, \eta) = \text{E}[\min(1, ||\zeta - \eta||)]$ is a distance in $L^0_{\mathbb{R}^r} (\Omega, P)$ compatible with convergence in probability. That is, a sequence $\{\eta_n\}_{n \in \mathbb{N}} \subseteq L^0_{\mathbb{R}^r} (\Omega, P)$ converges in probability to $\eta \in L^0_{\mathbb{R}^r} (\Omega, P)$ if and only if $d(\eta_n, \eta) \to 0$ as $n \to \infty$.

Let $\varepsilon > 0$ and let $n_0 \in \mathbb{N}$ be large enough so that $d(\xi_n, \xi) \leq \varepsilon / 2$ for any $n > n_0$. We can choose $n_1 > n_0$ so that

$$\frac{1}{n_1} \sum_{k=1}^{n_0} d(\xi_k, \xi) \leq \frac{\varepsilon}{2}.$$ 

Then, for any $n > n_1$ we have

$$d\left(\frac{1}{n} \sum_{k=1}^{n} \xi_k, \xi\right) \leq \frac{1}{n} \sum_{k=1}^{n} d(\xi_k, \xi) = \frac{1}{n} \sum_{k=1}^{n_0} d(\xi_k, \xi) + \frac{1}{n} \sum_{k=n_0+1}^{n_1} d(\xi_k, \xi) \leq \frac{1}{n_1} \sum_{k=1}^{n_0} d(\xi_k, \xi) + \frac{\varepsilon(n - n_0)}{2n} \leq \varepsilon.$$

Consequently, $\frac{1}{n} \sum_{k=1}^{n} \xi_k$ converges to $\xi$ in probability. $\blacksquare$

The next lemma is rather elementary and well known. However, we state and prove it for the benefit of a clearer exposition.

**Lemma 14** Let $X$ be a topological space. Let $\{x_n\}_{n \in \mathbb{N}} \subseteq L^0_{\mathbb{R}} (\Omega, P)$ be a sequence of random variables and $x \in X$. Let $f : X \to \mathbb{R}$ be a function continuous at $x$. If $x_n \to x$ in probability as $n \to \infty$, then $f(x_n) \to f(x)$ in probability as well.

**Proof.** Let $\varepsilon > 0$. By the continuity of $f$, there exists an open neighborhood $V_x$ around $x$ such that $|f(y) - f(x)| \leq \varepsilon$ for any $y \in V_x$. Then,

$$\{\omega \in \Omega : |f(x_n) - f(x)| > \varepsilon\} \subseteq \{\omega \in \Omega : x_n \notin V_x\}.$$

Since $P(\{\omega \in \Omega : x_n \notin V_x\}) \to 0$ as $n \to \infty$ because $\{x_n\}_{n \in \mathbb{N}}$ converges in probability to $x \in \mathbb{R}$, we see that

$$P(\{\omega \in \Omega : |f(x_n) - f(x)| > \varepsilon\}) \to 0$$

as well, and $\{f(x_n)\}_{n \in \mathbb{N}}$ converges in probability to $f(x)$. $\blacksquare$

Unlike the previous results, the following theorem is much deeper. Roughly speaking, it is a powerful tool to prove general versions of the Law of Large Numbers and the Central Limit Theorem for sequences of random variables which are neither independent nor equally distributed. Its proof can be found in [D97] Corollary 2.1.10.

**Theorem 15** Let $\{M_n\}_{n \in \mathbb{N}}$ be a real, square-integrable martingale adapted to a filtration $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$. Let $(M)_{\omega}$ denote its angle bracket process. Let $\{a_n\}_{n \in \mathbb{N}}$ be an increasing sequence such that $a_n \to \infty$ as $n \to \infty$.

(i) If $\frac{M_n}{a_n} \to \sigma^2 > 0$ in probability, then $\frac{M_n}{a_n} \to 0$ a.s.

(ii) If Lindberg’s condition holds, that is,

$$\frac{1}{a_n} \sum_{k=1}^{n} \text{E}[|M_k - M_{k-1}|^2 1_{\{||M_k - M_{k-1}|| \geq \varepsilon \sqrt{n}\}} \mid \mathcal{F}_{k-1}] \to 0$$

in probability, then $\frac{M_n}{\sqrt{a_n}} \to \mathcal{N}(0, \sigma^2)$ in law.
Acknowledgements. The authors would like to thank Juan-Pablo Ortega for his enlightening comments and suggestions. They also acknowledge support from Centre de Recerca Matemàtica. Sebastian del Baño Rollin is partially supported by Project MTM2006-01351, Ministerio de Educación y Ciencia.

References

[A03] Arouna, B. Robbins-Monro algorithms and variance reduction in Finance. The Journal of Computational Finance 7 (2), pp. 1245-1255, 2003.

[A04] Arouna, B. Adaptative Monte Carlo method, a variation reduction technique. Monte Carlo Methods and Appl. 10 (1), pp. 1-24, 2004.

[BHT08] Baudoin, F., Hairer, M., and Teichmann, J. Ornstein-Uhlenbeck processes on Lie groups. Journal of Functional Analysis 255 (4), pp. 877-890, 2008.

[B08] Bayer, C. Selected Topics in Numerics of Stochastic Differential Equations. PhD Thesis. Technische Universität Wien, 2008.

[BD85] Bröcker, T. and tom Dieck, T. Representations of Compact Lie Groups. Graduate texts in mathematics 98. Springer-Verlag.

[BLP05] Del Buono, N., Lopez, L., and Peluso, R. Computation of the exponential of large sparse skew-symmetric matrices. SIAM J. Sci. Comput. 27 (1), pp. 278–293, 2005.

[D97] Duflo, M. Random Iterative Models. Springer, 1997.

[DV98] Dufresne, D. and Vázquez-Abad, F. Accelerated simulation for pricing Asian options. Proceedings of the Winter Simulation Conference (Washington, DC) pp. 1493-1500, 1998.

[E98] Engø, K. On the construction of geometric integrators in the RKMK class. Technical Report No 158, Dept Comp. Sc., University of Bergen, 1998.

[FGQ97] Fang, H., Gong, G., and Quian, M. Annealing of iterative stochastic schemes. SIAM J. Control Optim. 35 (6), pp. 1886–1907, 1997.

[FS02] Fu, M. C. and Su, Y. Optimal importance sampling in securities pricing. Journal of Computational Finance 5 (4), pp. 27-50, 2002.

[GX02] Gallier, J. and Xu, D. Computing exponentials of skew-symmetric matrices and logarithms of orthogonal matrices. International Journal of Robotics and Automation, 17 (4), 2002.

[GM93] Gelfand, S. B. and Mitter, K. Metropolis-type annealing algorithms for global optimization in $\mathbb{R}^d$. SIAM J. Control. Optim. 31, pp. 111-131, 1993.

[GM91] Gelfand, S. B. and Mitter, K. Recursive stochastic algorithms for global optimization in $\mathbb{R}^d$. SIAM J. Control. Optim. 29, pp. 999-1018, 1991.

[GV96] Golub, G. H. and Van Loan, C. F. Matrix Computations. Third edition. The Johns Hopkins University Press, 1996.

[HH64] Hammersley, J. M. and Handscomb, D. C. Monte Carlo Methods. Fletcher & Son Ltd. 1964.
[HM56a] Hammersley, J. M. and Mauldon, J. G. General principles of antithetic variates. Proc. Cambridge Philosophical Society 52, pp. 476-481, 1956.

[HM56] Hammersley, J. M. and Morton, K. W. A new Monte Carlo technique: antithetic variates. Proc. Cambridge Philosophical Society 52, pp. 449-475, 1956.

[HKS89] Holley, R. A., Kusuoka, S., and Stroock, D. W. Asymptotics of the spectral gap with applications to the theory of simulated annealing. J. Funct. Anal. 83 (2), pp. 333–347, 1989.

[IMPZ05] Iserles, A., Munthe-Kaas, H. Z., and Nørsett, S. P., Zanna, A. Lie group methods. Acta Numerica 9, pp. 215-365, 2000.

[MJ03] Mark Joshi, The Concepts and Practice of Mathematical Finance. Cambridge University Press, 2003.

[K87] Kushner, H. J. Asymptotic global behaviour for stochastic approximation and diffusions with slowly decreasing noise effects: global minimization via Monte Carlo. SIAM J. Appl. Math. 47 (1), pp. 169-185, 1987.

[MR51] Monro, S. and Robbins, H. A stochastic approximation method. Ann. Math. Stat. 22, pp. 400-407, 1951.

[M63] J. Milnor, Morse Theory, Annals of Mathematic Studies 51, Princeton University Press, 1963.

[MT91] Mimura, M. and Toda, H. Topology of Lie Groups I and II. Translations of mathematical monographs, 91. American Mathematical Society, 1991.