The $N=2$ super $W_4$ algebra and its associated generalized KdV hierarchies

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Abstract

We construct the $N=2$ super $W_4$ algebra as a certain reduction of the second Gel’fand-Dikii bracket on the dual of the Lie superalgebra of $N=1$ super pseudo-differential operators. The algebra is put in manifestly $N=2$ supersymmetric form in terms of three $N=2$ superfields $\Phi_i(X)$, with $\Phi_1$ being the $N=2$ energy momentum tensor and $\Phi_2$ and $\Phi_3$ being conformal spin 2 and 3 superfields respectively. A search for integrable hierarchies of the generalized KdV variety with this algebra as Hamiltonian structure gives three solutions, exactly the same number as for the $W_2$ (super KdV) and $W_3$ (super Boussinesq) cases.
1 Introduction

The study of conformal field theories has been important to string theory [1], critical statistical mechanics in two dimensions [2] and two dimensional quantum gravity [3]. The role of the Virasoro algebra and its extensions has been crucial in this study. Of particular interest in recent times is the the class of extended conformal algebras known as $W$ algebras. Not long after its introduction to conformal field theory [4], it was realized [5] that its classical version had been studied earlier in the context of integrable systems [6, 7]. What have since become known as the classical $W_n$ algebras turn out to provide the second Hamiltonian structure of the generalized KdV hierarchies.

Knowledge of the generalized KdV hierarchies can be expected to give insight into the structure of the $W$ algebras. An early work in this direction is [8] in which essential use is made of the Miura transformation to free fields. The quantization of the generalized KdV hierarchies [9], which can be regarded as integrable classical field theories, provides information on perturbations of conformal field theories [10] and, in turn, off-critical statistical mechanics. The generalized KdV hierarchies are themselves of major interest in the matrix models of quantum gravity [3].

The situation with supersymmetric $W$ algebras and supersymmetric generalized KdV hierarchies is not as clear at present. It is probably fair to say that the $N = 2$ theories are currently better understood. These are also of particular interest to string theory, and will be our sole concern here. It is known that there exist three $N = 2$ supersymmetric KdV hierarchies [11, 12]. The construction of $N = 2$ classical super $W$ algebras was recently proposed [13, 14, 15]. Three supersymmetric Boussinesq equations were found [16] to be associated with the $N = 2$ super $W_3$ algebra. In this paper we construct the $N = 2$ super $W_4$ algebra and argue for the existence of three integrable supersymmetric hierarchies with it as Hamiltonian structure. Based on this and on [12] and [16] we conjecture that there are three generalized supersymmetric KdV hierarchies associated with each $N = 2$ super $W_n$ algebra.

2 The classical $N = 2$ super-$W_4$ algebra

The construction of the classical $N = 2$ super $W_n$ algebras in terms of $N = 1$ superfields was proposed independently in [13, 14] and [15]. The $N = 2$ super $W_3$ was explicitly worked out in the latter two papers.

Let us first make our notation clear. We will be working in both (1|1) and (1|2) superspace. Consider thus the superspace (1|N) with coordinates $X = (x, \theta_1, \ldots, \theta_N)$,
and covariant superderivative \( D_i = \partial/\partial \theta_i + \theta_i \partial \) obeying \( D_i^2 = \partial \equiv \partial_x \) and \( D_i D_j = -D_j D_i \) for \( i \neq j \). The Berezin integral of a super function \( f(X) \) is defined as \( \int_B f(X) \equiv \int f(X)dX \) with \( dX = d\theta_X \cdots d\theta_1 dx \). We use the convention \( f \theta_i d\theta_j = \delta_{ij} \) and \( \int d\theta_j = 0 \). The Grassmann parity of a super function \( f(X) \) is denoted by \( |f| \), and takes the value 0 (resp. 1) if \( f \) is even (resp. odd).

We next introduce several objects required to do variational calculus in superspace. Given a super differential operator (SDO) \( L \), we define its adjoint \( L^* \) with respect to the inner product \((f, g) \equiv \int_B fg\) through the relation

\[
(Lf, g) = (-1)^{|L||f|} (f, L^*g),
\]

which consequently has the properties (i) \((L^*)^* = L\) and (ii) \((PQ)^* = (-1)^{|P||Q|} Q^* P^*\).

Given a functional \( F[\Phi] = \int_B f[\Phi] \) of the superfields \( \Phi(X) = (\Phi_1(X), \Phi_2(X), \ldots) \), the variational derivative \( \delta F/\delta \Phi_j \) is defined through the relation

\[
\int_B \Gamma \frac{\delta F}{\delta \Phi_j} = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} F[\ldots, \Phi_j + \epsilon \Gamma, \ldots].
\]

In \( N = 1 \) superspace, it is given explicitly by

\[
\frac{\delta F}{\delta \Phi_j} = \sum_{k=0}^{\infty} (-1)^{|\Phi_j|k(k+1)/2} D^k \frac{\partial f}{\partial (D^k \Phi_j)},
\]

whereas in \( N = 2 \) superspace the corresponding expression is

\[
\frac{\delta F}{\delta \Phi_j} = \sum_{k=0}^{\infty} (-1)^k \partial^k \left( \frac{\partial}{\partial (\partial^k \Phi_j)} - (-1)^{|\Phi_j|} \sum_{i=1}^{2} D_i \frac{\partial}{\partial (\partial^k D_i \Phi_j)} \right) + D_1 D_2 \frac{\partial}{\partial (\partial^k D_1 D_2 \Phi_j)} f.
\]

It was proposed in [13, 14, 15] that the construction of the \( N = 2 \) super \( W_n \) algebra be performed in \((1|1)\) superspace. Given a set of \( N = 1 \) superfields \( u_i(X) \), let \( G \) be the infinite dimensional Lie superalgebra of super pseudo-differential operators \((S\Psi DO's)\), which are formal Laurent series in \( D^{-1} \equiv D\partial^{-1} \) with coefficients which are differential polynomials in \( u_i(X) \). Multiplication in the algebra is defined through the usual Leibniz rule for superdifferentiation augmented by the rules \( \partial \partial^{-1} = \partial^{-1} \partial = 1 \) and

\[
\partial^k f = f \partial^k + \sum_{i=1}^{\infty} \binom{k}{i} (\partial^i f) \partial^{k-i},
\]

for any integer \( k \). With the super residue \( sresP \) of a \( S\Psi DO \) \( P = \sum p_i D^i \) being \( p_{-1} \), the Adler super trace \( StrAB \equiv \int_B sresAB \) is well defined [17], with the property \( Str(AB) = (-1)^{|A||B|} Str(BA) \). The Adler super trace defines a nondegenerate
supersymmetric bilinear invariant form $\langle \cdot , \cdot \rangle$ on $G$ via

$$\langle A, B \rangle = \text{Str}(AB),$$

allowing the identification of $G$ with its dual $G^\ast$.

The Lie superalgebra $G$ splits into a direct sum of the subalgebra $G_+^*$ of SDO’s and the subalgebra $G_-^*$ of “integrational operators”, with $G_\pm^*$ being dual to $G_\mp$. The projection of an element $Y \in G$ into $G_\pm$ is denoted by $Y_\pm$. We concentrate on the subspace $G_n$ of (homogeneous Grassmann parity) SDO’s of the form

$$L = D^n + u_{n-1}D^{n-1} + \cdots + u_0. \quad (1)$$

It was shown in [18] that the map $J : G_\ast^* \to G_n$ defined by

$$J(Y) = L(Y L)_+ - (LY)_+ L \quad \text{(2)}$$
defines a Poisson bracket (the “second” Gel’fand-Dikii bracket) on the space of functionals of $u_i(X)$ through

$$\{ F, G \} = \langle J(dF), dG \rangle, \quad (3)$$

with $dF$ being the gradient of $F$ with respect to the bilinear form $\langle \cdot , \cdot \rangle$:

$$\langle A, dF \rangle = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} F[L + \epsilon A].$$

An analogous Poisson bracket structure can in fact be associated to any Lie superalgebra admitting a trace form and a “unitary Yang-Baxter operator $R$”. For details see [19]. “Coordinatizing” $L$ and $A$ by (1) and $A = \sum_{k=1}^{n-1} A_k D^k$ respectively, $dF$ can be shown to be given by

$$dF = (-1)^{|F| + |L| + 1} \sum_{k=0}^{n-1} (-1)^k D^{-k-1} \frac{\delta F}{\delta u_k}. \quad (4)$$

The $N = 2$ super $W_m$ algebra is the Poisson bracket algebra (3) induced on the subspace $G_{2m-1}^{(0)}$ of $G_{2m-1}$ of SDO’s with vanishing coefficient of $D^{2m-2}$. Actually, what has been proved so far [13] is that this reduction contains the $N = 2$ super Virasoro algebra as a subalgebra. It remains a conjecture that $(m - 2)$ other $N = 2$ primary fields can be constructed out of the $u_i(X)$, and that the Poisson brackets amongst the primary fields close on themselves and their derivatives. This was shown [14, 15] to be true for $m = 3$. We will show that it is also true for $m = 4$. To obtain the Poisson bracket on the subspace $G_{2m-1}^{(0)}$, one can either use the Dirac procedure or, equivalently [13], the following: define the gradient $dF$ of a functional
on \( G_{2m-1}^{(0)} \) as \([4]\) (with \( n = 2m - 1 \)) but with \( \delta F / \delta u_{2m} \) replaced by \( Y_{2m-2} \) such that 
\[ sres [L, dF] = 0. \]

We now concentrate on the construction of the \( m = 4 \) case. The two \( N = 1 \) superfields which generate the classical \( N = 2 \) Virasoro subalgebra of \( N = 2 \) super \( W_4 \) are given by \([13]\)

\[
J(X) = u_5, \quad T(X) = u_4 - \frac{1}{2}(Du_5). \tag{5}
\]

According to the general theory \([13]\), four \( N = 1 \) primary fields \( W_k(X) \) of conformal spins \( k = 2, \frac{5}{2}, 3 \) and \( \frac{7}{2} \) organized into the \( N = 2 \) supermultiplets \((W_2(X), W_{5/2}(X))\) and \((W_3(X), W_{7/2}(X))\) can be constructed out of the \( u_i(X) \) and their derivatives. To determine \( W_k(X) \), we look for the most general combination of the \( u_i(X) \) of spin \( k \) (or “degree” \( k \) - the degree of \( \partial \) and \( D \) being 1 and 1/2 respectively) such that its Poisson bracket with \( T(X) \) is given by

\[
\{W_k(X), T(X')\} = \left( (\partial W_k(X)) + kW_k(X)\partial - (-1)^{|W_k|} \frac{1}{2}(DW_k(X))D \right) \Delta(X - X'), \tag{7}
\]

where \( \Delta(X - X') = (\theta - \theta')\delta(x - x') \) is the \( N = 1 \) supersymmetric delta function. The Poisson bracket is calculated from \([3]\) and, although straightforward, is long and tedious. A program in Mathematica \([20]\) was written to handle the calculation. To check that the program gives the desired result, the following alternative way to determine \( W_k(X) \) was also employed.

Primary fields of a given conformal spin can also be constructed by the requirement that the \( SDO \) \([1]\) (with \( u_{n-1} = 0 \)) is a super covariant operator. That is, we require \( L \) to transform as

\[
L = (D\bar{\theta})^{n+1} \bar{L}(D\bar{\theta})^{n+1}. \tag{8}
\]

under superconformal transformations \([2]\) \( X \rightarrow \bar{X}(X) = \{\bar{x}(x, \theta), \bar{\theta}(x, \theta)\} \) (with \( D = (D\bar{\theta})\bar{D} \)). Primary fields of conformal spin \( k \) transform under such superconformal transformations as \( W_k = \tilde{W}_k(D\bar{\theta})^{2k} \).

First the transformation properties of the \( u_i \) fields are determined from those of \( L \). Substituting \( (\bar{D})^j = ((D\theta)^{-1}D)^j \) into \( \bar{L} \), expanding and equating coefficients of \( D \) yields the superconformal transformation properties of the \( u_i \) functions

\[
\begin{align*}
u_5 &= \tilde{u}_5(D\bar{\theta})^2 \\
u_4 &= \tilde{u}_4(D\bar{\theta})^3 + \tilde{u}_5(D\bar{\theta})(D^2\bar{\theta}) + 6S \\
u_3 &= \tilde{u}_3(D\bar{\theta})^4 - 2\tilde{u}_4(D\bar{\theta})^2(D^2\bar{\theta}) + 2\tilde{u}_5(D\bar{\theta})(D^3\bar{\theta}) + 4DS \\
u_2 &= \tilde{u}_2(D\bar{\theta})^5 + 2\tilde{u}_3(D\bar{\theta})^3(D^2\bar{\theta}) + 4\tilde{u}_4(D\bar{\theta})^2(D^3\bar{\theta})
\end{align*}
\]
\[ u_1 = \tilde{u}_1 (D\tilde{\theta})^6 - \tilde{u}_2 (D\tilde{\theta})^4 (D^2 \tilde{\theta}) + 2 \tilde{u}_3 (D\tilde{\theta})^3 (D^3 \tilde{\theta}) \]
\[ - \tilde{u}_4 ((D\tilde{\theta})^2 (D^4 \tilde{\theta}) + 2(D\tilde{\theta}) (D^2 \tilde{\theta})(D^3 \tilde{\theta})) \]
\[ + \tilde{u}_5 ((D^2 \tilde{\theta}) (D^4 \tilde{\theta}) + 2(D\tilde{\theta})(D^5 \tilde{\theta}) - 2(D^3 \tilde{\theta})^2) + 2D^3 S \]
\[ u_0 = \tilde{u}_0 (D\tilde{\theta})^7 + 3\tilde{u}_1 (D\tilde{\theta})^5 (D^2 \tilde{\theta}) + 3\tilde{u}_2 (D\tilde{\theta})^4 (D^3 \tilde{\theta}) \]
\[ + 3\tilde{u}_3 (D\tilde{\theta})^3 (D^4 \tilde{\theta}) + 3\tilde{u}_4 ((D\tilde{\theta})^2 (D^5 \tilde{\theta}) - (D\tilde{\theta})(D^2 \tilde{\theta})(D^4 \tilde{\theta})) \]
\[ + 3\tilde{u}_5 ((D\tilde{\theta})(D^6 \tilde{\theta}) - (D^3 \tilde{\theta})(D^4 \tilde{\theta}) - (D^2 \tilde{\theta})(D^5 \tilde{\theta})) + 3D^4 S + 9SDS. \] (9)

The first term in each equation is the appropriate transformation for a primary field of 1, \( \frac{3}{2} \), 2, \( \frac{5}{2} \), 3, \( \frac{7}{2} \) while the remaining terms involve more complicated terms of the same degree showing that the \( u_i \) are not primary fields, and also include the super-Schwartzian of the superanalytic map

\[ S = \frac{D^4 \tilde{\theta}}{D\tilde{\theta}} - 2\frac{(D^3 \tilde{\theta})(D^2 \tilde{\theta})}{(D\tilde{\theta})^2}, \] (10)

and its superderivatives. The super-Schwartzian terms signify the presence of a central term in the corresponding operator product expansion and in fact they prove essential in enabling the construction of primary fields for the higher conformal spins. We consider the most general combinations of the \( u_i \)’s and their derivatives consistent with the desired degree, including products of \( u_i \)’s, and using the explicit transformation properties \( \Box \) we determine the coefficients required to obtain primary fields. This method, which was used in \( \Box \) for the case of \( N = 2 \) super \( W_3 \), confirms the form of the primary fields deduced by implementing the superconformal transformations using the Poisson bracket \( \Box \).

The primary fields \( W_k(X) \) were found to be

\[ W_2(X) = u_3 - \frac{2}{3}(Du_4) - \frac{2}{3}(Du_5) + \alpha(u_5)^2, \] (11)
\[ W_{5/2}(X) = u_2 - \frac{1}{2}(Du_3) - (\partial u_4) + \frac{1}{3}(\partial Du_5) - \frac{1}{15}u_5 T \] (12)
\[ W_3(X) = u_1 - \frac{1}{9}(Du_2) - \frac{2}{9}(Du_3) + \frac{1}{5}(\partial Du_4) + \frac{1}{10}(\partial^2 u_5) + \beta u_5 u_3 + \gamma u^2_5 + \frac{2}{15}(Du_5)u_4 + (\frac{1}{3} - \frac{2}{3} \beta) u_5(\partial u_5) - (\frac{1}{15} + \frac{2}{3} \gamma) u_5(Du_4) \] (13)
\[ W_{7/2}(X) = u_0 - \frac{1}{4}(Du_1) - \frac{1}{3}(Du_2) + \frac{1}{6}(\partial Du_3) + \frac{1}{10}(\partial^2 u_4) \]
\[ - \frac{1}{20}(\partial^2 Du_5) + \delta u_5(Du_3) + \epsilon u_5^2 u_4 + \lambda u_5(Du_5) - \frac{1}{20}(Du_4)u_4 \]
\[ - \frac{3}{50}u_3 u_4 + \frac{12}{25}(Du_5) u_4 - \frac{1}{2} u_5^2 (Du_5) + \frac{1}{25} (3 - 20 \lambda - 40 \delta) u_5 u_2 \]
\[ + \frac{1}{15} (-3 + 20 \lambda + 30 \delta) u_5(\partial u_4) \]
\[ + \frac{1}{120} (-33 + 220 \lambda - 360 \epsilon + 360 \delta) u_5(\partial Du_5) \]
\[ + \frac{1}{60} (21 - 205 \lambda + 270 \epsilon - 330 \delta) (\partial u_5)(Du_5) \] (14)
Note the presence of six undetermined constants in the above. In order that 
\((W_2(X), W_{5/2}(X))\) and \((W_3(X), W_{7/2}(X))\) form \(N = 2\) supermultiplets, we require 
\(W_2(X)\) and \(W_3(X)\) to have the following Poisson brackets with the \(U(1)\) current \(J(X)\):

\[
\{W_2(X), J(X')\} = -2W_{5/2}\Delta(X - X') \\
\{W_3(X), J(X')\} = -2W_{7/2}\Delta(X - X').
\]

Together with the corresponding brackets between \(W_{5/2}(X)\) and \(J(X)\) and \(W_{7/2}(X)\) 
and \(J(X)\) implied by the above and by the Jacobi identity, we find that the constants 
\(\alpha - \lambda\) are fixed uniquely:

\[
\alpha = -\frac{11}{36}, \quad \beta = -\frac{3}{10}, \quad \gamma = \frac{23}{360}, \quad \delta = \frac{3}{20}, \quad \epsilon = \frac{23}{160}, \quad \lambda = \frac{3}{20}
\]

In accordance with expectations, we find that the Poisson brackets amongst the 
primary fields \(W_k(X)\) with these values of \(\alpha - \lambda\) close on themselves and their 
derivatives. We can thus say that we have constructed the classical \(N = 2\) super 
\(W_4\) algebra.

There is no point in displaying the Poisson brackets amongst the remaining 
\(N = 1\) primary superfields. Apart from the fact that it would take more than 10 
pages, there is a more natural and compact formalism - namely the \(N = 2\) formalism. 
In fact it would be preferable for the algebra to be constructed in \(N = 2\) superspace 
from scratch, presumably as a reduction of the Gel’fand-Dikii algebra on the dual 
of the Lie algebra of \(N = 2\) \(S\Phi D\)O’s. How to carry this out is however an open 
problem. Let us define the \(N = 2\) superfields

\[
\Phi_1(x, \theta_1, \theta_2) = \alpha_1 T(x, \theta_2) + \alpha_2 J(x, \theta_2), \quad (15) \\
\Phi_2(x, \theta_1, \theta_2) = \beta_1 W_{5/2}(x, \theta_2) + \beta_2 W_2(\theta_2, x), \quad (16) \\
\Phi_3(x, \theta_1, \theta_2) = \gamma_1 W_{7/2}(x, \theta_2) + \gamma_2 W_3(\theta_2, x) \quad (17)
\]

with conformal dimension 1, 2 and 3 respectively. Without loss of generality we 
choose \(\alpha_1 = 1\). We find that the bracket \(\{\Phi_1(X), \Phi_1(X')\}\) closes on \(\Phi_1\) and its 
derivatives if and only if \(\alpha_2 = \pm \frac{i}{2}\). We choose the positive sign. We then find 
that \(\{\Phi_2(X), \Phi_1(X')\}\) and \(\{\Phi_3(X), \Phi_1(X')\}\) close on \(\Phi_i\) and derivatives if and only 
if \(\beta_2 = \frac{1}{2} \beta_1\) and \(\gamma_2 = \frac{1}{2} \gamma_1\). In that case, all the other brackets close on \(\Phi_i\) and 
derivatives as well. We find the choice \(\beta_1 = i\) and \(\gamma_1 = 1\) convenient. In this case, 
all the coefficients on the right hand side of the brackets are real.

The Poisson brackets amongst the \(\Phi_i\) are given by

\[
\{\Phi_1(X), \Phi_1(X')\} = \left(3\partial D_1 D_2 - \Phi_1 \partial + \frac{1}{2}(D_1 \Phi_1) D_i - (\partial \Phi_1) \right) \Delta(X - X') \quad (18)
\]
\[ \{ \Phi_2(X), \Phi_1(X') \} = \left( -2\Phi_2 \partial + \frac{x}{2}(D_i \Phi_2)D_i - (\partial \Phi_2) \right) \Delta(X - X') \]  
\[ \{ \Phi_3(X), \Phi_1(X') \} = \left( -3\Phi_3 \partial + \frac{x}{2}(D_i \Phi_3)D_i - (\partial \Phi_3) \right) \Delta(X - X') \]  
\[ \{ \Phi_2(X), \Phi_2(X') \} = (\mathcal{O}_2 - \mathcal{O}_2') \Delta(X - X') \]  
\[ \{ \Phi_3(X), \Phi_2(X') \} = \mathcal{D}_{32} \Delta(X - X') \]  
\[ \{ \Phi_3(X), \Phi_3(X') \} = (\mathcal{O}_3 - \mathcal{O}_3') \Delta(X - X') \]  
\[ \{ \Phi_2(X), \Phi_2(X') \} = (\mathcal{O}_2 - \mathcal{O}_2') \Delta(X - X') \]  

where \( \Delta(X - X') \equiv (\theta_1 - \theta_1')(\theta_2 - \theta_2')\delta(x - x') \) is the \( N = 2 \) supersymmetric delta function and \( \mathcal{O}_2, \mathcal{O}_3 \) and \( \mathcal{D}_{32} \) are displayed in the Appendix. The fields on the right-hand sides of (18) - (24) are evaluated at the point \( X \). The Poisson bracket (18) defines the classical \( N = 2 \) super Virasoro algebra. The brackets (19) and (20) simply state that \( \Phi_k \) is an \( N = 2 \) primary field with conformal dimension \( k \), for \( k = 2, 3 \). We summarize the Poisson brackets (18) - (23) amongst the primary fields \( \Phi_k \) in the form

\[ \{ \Phi_i(X), \Phi_j(X') \} = \mathcal{D}_{ij} \Delta(X - X'). \]  

The matrix-valued Hamiltonian operator \( \mathcal{D} \) is anti-selfadjoint: \( \mathcal{D}_{ij}^* = -\mathcal{D}_{ji} \), reflecting the anti-(super)symmetry of the Poisson brackets. The Poisson bracket can be extended to arbitrary functionals \( \mathcal{F}[\Phi] \) and \( \mathcal{G}[\Phi] \) via

\[ \{ \mathcal{F}, \mathcal{G} \} = \int_B \mathcal{D}_{ij} \frac{\delta \mathcal{F}}{\delta \Phi_j} \frac{\delta \mathcal{G}}{\delta \Phi_i}. \]  

### 3 The associated integrable hierarchies

Gel’fand-Dikii brackets are intimately connected with integrable Hamiltonian systems. In fact they first arose out of a study of generalized KdV hierarchies \[3\]. One therefore expects that the \( N = 2 \) super \( W_3 \) algebra constructed in the last section can also be constructed out of a Lax operator for an integrable hierarchy. A first step towards making such a connection concrete is to look for the integrable hierarchy (or hierarchies) associated with the algebra - namely an infinite set of Hamiltonian functionals mutually in involution with respect to the Poisson bracket (24). Our approach is constructive and follows that of \[12\]. We will show the existence of three families of Hamiltonian functionals in involution, assuming only space-translation invariance. Naturally the extent of these families as constructed is finite, but we conjecture that they constitute the first few members of each of three infinite families of Hamiltonian functionals in involution.

Let us first note that there is an \( S_2 \) symmetry in the Poisson bracket algebra, which is also present in the \( N = 2 \) super \( W_2 \) (Virasoro) case \[11\] and in the \( N = 2 \) super \( W_3 \) case \[16\]. For the permutation (12) define the map \( \Pi_{(12)} \) acting on the
superfields by \( \Pi_{(12)} : \Phi_1 \mapsto -\Phi_1, \Phi_2 \mapsto \Phi_2, \Phi_3 \mapsto -\Phi_3, D_1 \mapsto D_2, D_2 \mapsto D_1, dX \mapsto -dX \). Then \( \Pi_{(12)} \) together with the identity map \( \Pi_e \) forms a representation of the symmetric group \( S_2 \). We say that a superfield \( f \) is \( \Pi \)-even if \( \Pi_{(12)} f = f \), \( \Pi \)-odd if \( \Pi_{(12)} f = -f \) and \( \Pi \)-definite if it is either \( \Pi \)-even or \( \Pi \)-odd. The symmetric or \( \Pi \)-even (anti-symmetric or \( \Pi \)-odd) part of a superfield \( f \) is given by \( f_s = P f \) (\( f_a = Q f \)) where

\[
P = \sum_{x \in S_2} \Pi_x \\
Q = \sum_{x \in S_2} \delta_x \Pi_x
\]

with \( \delta_x \) being the parity of the permutation \( x \), are respectively the symmetrizer and anti-symmetrizer for the group. One can check that the Poisson bracket \( \{24\} \) respects the \( S_2 \) symmetry. By this we mean the following: Firstly, note that \( \Pi_{(12)} (\delta/\delta \Phi_j) = (-1)^j \delta/\delta \Phi_j \) for \( j = 1, 2, 3 \). Note also that \( \Pi_{(12)} (D_{ij}) = -(-1)^{i+j} D_{ij} \). Therefore we have \( \Pi_{(12)} \{ \mathcal{F}, \mathcal{G} \} = \{ \Pi_{(12)}(\mathcal{F}), \Pi_{(12)}(\mathcal{G}) \} \). In particular, this means that if \( \{ \mathcal{F}, \mathcal{G} \} = 0 \) and that \( \mathcal{F} \) is \( \Pi \)-definite, then \( \mathcal{F} \) is in involution with both \( Q(\mathcal{G}) \) and \( P(\mathcal{G}) \). A similar argument holds for the degree - if one of the Hamiltonian functionals of the hierarchy is of homogeneous degree, then all other Hamiltonians can be taken to be of homogeneous degree.

We now note the importance of the Hamiltonian functional \( H_1^+ = \int_B \Phi_1 \). Its Poisson bracket with any Hamiltonian functional \( H_k[\Phi] = \int_B h_k[\Phi], \) for which there is no explicit \( X \) dependence, is given by

\[
\{ H_1^+, H_k \} = \int_B (D_{11}(1)) \frac{\delta H_k}{\delta \Phi_i} \\
= -\int_B (\partial \Phi_i) \frac{\delta H_k}{\delta \Phi_i} \\
= -\frac{d}{d\epsilon} \bigg|_{\epsilon=0} H_k[\Phi(x+\epsilon)] \\
= 0.
\]

Put another way, the Hamiltonian vector field corresponding to \( H_1 \) is the generator of translations in \( x \), and as long as we require the flows generated by the \( H_k \) which form the hierarchy to be invariant under \( x \)-translations, as is the case for the generalized KdV hierarchies, \( H_1^+ \) must belong to the hierarchy. Since \( H_1^+ \) is \( \Pi \)-definite and of homogeneous degree, we have the result that all the members \( H_k \) are also \( \Pi \)-definite and of homogeneous degree.

To write down the most general \( \Pi \)-even (resp. odd) \( H_n = \int_B h_n \) of a particular degree, we construct all the independent \( \Pi \)-odd (resp. even) densities \( h_n \). Two
densities \( f \) and \( g \) are dependent if there exists non-trivial \( \alpha \) and \( \beta \) such that \( \int_B (\alpha f + \beta g) = 0 \) or, equivalently, if \( \alpha \frac{\delta}{\delta \Phi_j} \int_B f + \beta \frac{\delta}{\delta \Phi_j} \int_B g = 0 \) for all \( j \). A program written in Mathematica [20] was used to generate the \( \mathcal{H}_n^\pm \)'s, where the subscripts indicate the degrees and superscripts the \( \Pi \)-parities.

To determine when two Hamiltonian functionals are in involution, we need a further tool. We define the Fréchet derivative \( \mathcal{D}_K = (\mathcal{D}_K^1, \mathcal{D}_K^2, \ldots) \) of a functional \( K[\Phi] \) to be such that its component \( \mathcal{D}_K^j \) is given by

\[
\mathcal{D}_K^j(\Gamma) = (-1)^{\|\Gamma\|(|\Phi|}\Phi_j)} \frac{d}{d\epsilon} \bigg|_{\epsilon=0} K[\ldots, \Phi_j + \epsilon \Gamma, \ldots],
\]

for any \( \Gamma[\Phi] \). The Fréchet derivative has the properties

\[
\mathcal{D}_j^{(P_m D_1^n D_2^p \Phi_j)} = (-1)^{|\Phi_j| n + p} P_m D_1^n D_2^p
\]

\[
\mathcal{D}_j^{fg} = (-1)^{|f| |\Phi_j|} f \mathcal{D}_g + (-1)^{|g| |\Phi_j| + |f|} g \mathcal{D}_f.
\]

In fact it has an intimate connection with the variational derivative. Namely, one can show that if \( \mathcal{F} = \int_B f \) then

\[
\frac{\delta \mathcal{F}}{\delta \Phi_j} = \mathcal{D}_j^*(1),
\]

(26)

\[
\frac{\delta}{\delta \Phi_j} \int_B f g = \mathcal{D}_j^*(g) + (-1)^{|f| |\Phi_j|} \mathcal{D}_g^*(f).
\]

(27)

We use (27) to determine when two functionals are in involution. Namely, \( \{\mathcal{H}_n, \mathcal{H}_m\} = 0 \) if and only if

\[
\sum_{k=1}^2 \left( \left( \mathcal{D}_k^j \right)^* \left( \frac{\delta \mathcal{H}_m}{\delta \Phi_k} \right) + (-1)^{|S_k| |\Phi_j|} \left( \mathcal{D}_k^j \right)^* \left( S_k \right) \right) = 0
\]

(28)

for \( j = 1, 2 \), with \( S_k \equiv D_{ik} \frac{\delta}{\delta \Phi_j} \mathcal{H}_n \). A program written in Mathematica [20] was used to determine if this criterion is satisfied. The reason for working with Fréchet derivatives rather than with integrals is because we find integration to be more difficult to implement on the computer.

After a systematic search, we find the following three\(^1\) systems of Hamiltonian functionals mutually in involution:

\[
\mathcal{H}_2^- = \int_B P \left( a_1 \Phi_2 + a_2 \Phi_1^2 \right)
\]

(29)

\[
\mathcal{H}_3^+ = \int_B Q \left( b_1 \Phi_3 + b_2 \Phi_1^3 + b_3 \Phi_1 \Phi_2 + b_4 \Phi_1 (D_1 D_2 \Phi_1) \right)
\]

(30)

\(^1\) We explicitly ignore the three solutions corresponding to the \( N = 2 \) super KdV hierarchies \[1, 12\] whose second Hamiltonian structure is the \( N = 2 \) Virasoro subalgebra of \( N = 2 \) super \( W_4 \).
\( \mathcal{H}_1^- = \int_B P \left( c_1 \Phi_2^1 + c_2 \Phi_1^4 + c_3 \Phi_1 \Phi_3 + c_4 \Phi_1 (D_1 D_2 \Phi_2) + c_5 \Phi_1 (\partial^2 \Phi_1) \right) + c_6 \Phi_1 \Phi_2 + c_7 \Phi_1^2 (D_1 D_2 \Phi_1) \) \tag{31}

\( \mathcal{H}_3^+ = \int_B Q \left( d_1 \Phi^5 + d_2 \Phi_2 \Phi_3 + d_3 \Phi_2 (D_1 D_2 \Phi_2) + d_4 \Phi_2 (\partial^2 \Phi_1) + d_5 \Phi_2 \Phi_1 \right) + d_6 \Phi_3 (D_1 D_2 \Phi_1) + d_7 \Phi_1 (\partial^2 D_1 D_2 \Phi_1) + d_8 \Phi_1^2 \Phi_3 + d_9 \Phi_1^2 (D_1 D_2 \Phi_2) \)
\[ + d_{10} \Phi_1^2 (\partial^2 \Phi_1) + d_{11} \Phi_1^3 \Phi_2 + d_{12} \Phi_1^3 (D_1 D_2 \Phi_1) + d_{13} \Phi_1 (D_1 D_2 \Phi_1)^2 \]
\[ + d_{14} \Phi_1 \Phi_2 (D_1 D_2 \Phi_1) \] \tag{32}

\( \mathcal{H}_6^- = \int_B P \left( e_1 \Phi_2^2 + e_2 \Phi_3^2 + e_3 \Phi_1^6 + e_4 (D_1 D_2 \Phi_2)^2 + e_5 (D_1 D_2 \Phi_1)^3 + e_6 (\partial D_1 D_2 \Phi_1)^2 \right) + e_7 \Phi_2 (D_1 D_2 \Phi_3) + e_8 \Phi_2 (\partial^2 D_1 D_2 \Phi_1) + e_9 \Phi_1^2 \Phi_2 + e_{10} \Phi_1^2 (D_1 D_2 \Phi_1) \)
\[ + e_{11} \Phi_3 (\partial^2 \Phi_1) + e_{12} \Phi_1^2 (D_1 D_2 \Phi_3) + e_{13} \Phi_1^2 (D_1 D_2 \Phi_1)^2 + e_{14} \Phi_1^2 (\partial \Phi_1)^2 \]
\[ + e_{15} \Phi_1^2 (\partial^2 \Phi_2) + e_{16} \Phi_1^2 (\partial^2 D_1 D_2 \Phi_1) + e_{17} \Phi_1^3 \Phi_3 + e_{18} \Phi_1^3 (D_1 D_2 \Phi_2) + e_{19} \Phi_1^4 \Phi_2 \]
\[ + e_{20} \Phi_1^4 (D_1 D_2 \Phi_1) + e_{21} \Phi_2 (D_1 D_2 \Phi_1)^2 + e_{22} \Phi_2 (\partial \Phi_1)^2 + e_{23} \Phi_1 \Phi_2 \Phi_3 \]
\[ + e_{24} \Phi_1 \Phi_2 (D_1 D_2 \Phi_1) + e_{25} \Phi_2 (D_1 \Phi_1) (\partial D_2 \Phi_1) \]
\[ + e_{26} \Phi_2 (D_1 \Phi_1) (\partial D_1 \Phi_1) + e_{27} \Phi_1 \Phi_3 (D_1 D_2 \Phi) + e_{28} \Phi_1^2 \Phi_2 (D_1 D_2 \Phi_1) \). \tag{33}

The values of the coefficients such that the involution property is satisfied are given in tables (1) - (4). From this, we postulate the existence of three integrable hierarchies. The flow equations

\[
\frac{\partial \Phi_i}{\partial t} = \mathcal{D}_j \delta \mathcal{H}_j^- / \delta \Phi_j \]
\tag{34}

with \((a_1, a_2) = (1, \frac{7}{3}), (1, -\frac{2}{3}), (1, -\frac{5}{3})\) are the three \(N = 2\) supersymmetric generalized KdV equations associated with \(N = 2\) super \(W_4\). For two of the hierarchies, non-trivial \(\mathcal{H}_n\) exist for every positive integer degree \(n\), whereas for the third hierarchy \(\mathcal{H}_4\) is missing for \(k = 1\) and, we suspect, for every integer \(k\). We have checked by hand that \(\mathcal{H}_3^+\) is indeed a conservation law for the equation (34). The existence of so many conservation laws in involution can be regarded as very strong evidence that three integrable hierarchies exist.

| \(a_2/a_1\) | \(b_1\) | \(b_2\) | \(b_3\) | \(b_4\) |
|---|---|---|---|---|
| \(\frac{7}{3}\) | \(-10\) | \(20\) | \(1\) | \(5\) |
| \(-\frac{2}{3}\) | 20 | \(-\frac{5}{3}\) | 1 | \(\frac{5}{3}\) |
| \(-\frac{5}{3}\) | \(\frac{10}{3}\) | \(-\frac{10}{3}\) | 1 | \(-\frac{5}{3}\) |

Table 1. Values of \(a_i\) and \(b_i\) such that \(\{\mathcal{H}_2^-, \mathcal{H}_3^+\} = 0\).

| \(a_2/a_1\) | \(c_1\) | \(c_2\) | \(c_3\) | \(c_4\) | \(c_5\) | \(c_6\) | \(c_7\) |
|---|---|---|---|---|---|---|---|
| \(\frac{7}{3}\) | \(-\frac{1}{3}\) | \(-\frac{40}{48}\) | 1 | \(-\frac{8}{15}\) | \(\frac{7}{18}\) | \(-\frac{71}{180}\) | \(-\frac{77}{210}\) |
| \(-\frac{5}{3}\) | \(-\frac{1}{3}\) | \(-\frac{40}{48}\) | 1 | \(\frac{7}{18}\) | \(\frac{49}{180}\) | \(-\frac{77}{210}\) |
Table 2. Values of $a_i$ and $c_i$ such that $\{ \mathcal{H}_2^-, \mathcal{H}_4^- \} = 0$.

| $a_2/a_1$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ | $d_7$ | $d_8$ | $d_9$ |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\frac{7}{9}$ | $-\frac{46}{105}$ | 1 | $-\frac{9}{20}$ | $\frac{3}{2}$ | $-\frac{7}{10}$ | $\frac{1}{3}$ | 1 | $\frac{25}{77}$ | $-\frac{2}{3}$ |
| $-\frac{5}{9}$ | $\frac{88}{230}$ | 1 | $\frac{7}{40}$ | $\frac{1}{2}$ | $\frac{2}{15}$ | $-\frac{5}{8}$ | $-\frac{7}{18}$ | $-\frac{5}{9}$ | $-\frac{17}{72}$ |
| $-\frac{2}{9}$ | $-\frac{37}{12960}$ | 1 | $\frac{7}{40}$ | $\frac{1}{12}$ | $-\frac{3}{40}$ | $-\frac{2}{3}$ | $\frac{7}{72}$ | $\frac{5}{18}$ | $-\frac{13}{44}$ |

Table 3. Values of $s_i$ and $d_i$ such that $\{ \mathcal{H}_2^+, \mathcal{H}_5^+ \} = 0$.

| $a_2/a_1$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\frac{7}{9}$ | 1 | 6 | $\frac{397}{729}$ | $\frac{207}{70}$ | $\frac{170}{24}$ | $\frac{15}{7}$ | $-\frac{72}{5}$ | $-\frac{12}{5}$ | $\frac{1151}{150}$ | $\frac{11}{2}$ |
| $-\frac{2}{9}$ | 1 | 24 | $-\frac{8}{729}$ | $-\frac{21}{25}$ | $-\frac{14}{27}$ | $\frac{4}{3}$ | $\frac{72}{5}$ | $-4$ | $\frac{22}{75}$ | $-\frac{7}{2}$ |
| $-\frac{5}{9}$ | 0 | 1 | $\frac{1}{81}$ | $\frac{9}{100}$ | $\frac{1}{12}$ | $\frac{1}{4}$ | $\frac{3}{5}$ | $\frac{3}{10}$ | $\frac{9}{100}$ | 0 |

Table 4. Values of $a_i$ and $e_i$ such that $\{ \mathcal{H}_2^-, \mathcal{H}_6^- \} = 0$.

| $a_2/a_1$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ | $e_{16}$ | $e_{17}$ | $e_{18}$ | $e_{19}$ | $e_{20}$ |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\frac{7}{9}$ | 30 | $-\frac{23}{2}$ | $\frac{841}{108}$ | $\frac{827}{36}$ | $-\frac{33}{2}$ | $-\frac{23}{2}$ | $-20$ | $\frac{893}{180}$ | $\frac{139}{27}$ | $1807$ |
| $-\frac{2}{9}$ | 0 | $-8$ | $-\frac{8}{729}$ | $\frac{22}{27}$ | $\frac{2}{3}$ | $\frac{2}{9}$ | 0 | $-\frac{16}{45}$ | $\frac{4}{27}$ | $\frac{4}{81}$ |
| $-\frac{5}{9}$ | 1 | $-\frac{7}{12}$ | $\frac{25}{37}$ | $\frac{59}{72}$ | $\frac{2}{3}$ | $-\frac{5}{12}$ | $-\frac{2}{9}$ | $-\frac{1}{8}$ | $-\frac{1}{18}$ | $\frac{11}{72}$ |

| $a_2/a_1$ | $e_{21}$ | $e_{22}$ | $e_{23}$ | $e_{24}$ | $e_{25}$ | $e_{26}$ | $e_{27}$ | $e_{28}$ |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\frac{7}{9}$ | 191 | 112 | $-\frac{102}{5}$ | $\frac{237}{25}$ | 0 | $-\frac{64}{5}$ | $-47$ | $\frac{4021}{180}$ |
| $-\frac{5}{9}$ | 92 | 4 | $-\frac{48}{5}$ | $\frac{3}{25}$ | 0 | $-\frac{28}{5}$ | 16 | $-\frac{32}{45}$ |
| $-\frac{2}{9}$ | $-3$ | $-\frac{9}{20}$ | $\frac{3}{5}$ | $\frac{9}{50}$ | 0 | $\frac{3}{10}$ | $-\frac{1}{2}$ | $-\frac{13}{40}$ |

4 Conclusion

In this paper, we have explicitly constructed the classical $N = 2$ super $W_4$ algebra. From a successful search for families of Hamiltonian functionals in involution, we have argued for the existence of three integrable hierarchies (invariant under space translations) for which the classical $N = 2$ super $W_4$ algebra furnish the ("second") Hamiltonian structure. We have not proved integrability of these hierarchies, of
course. We expect this to be a difficult task, given that even in the case of the
$N = 2$ super Virasoro ($W_2$) algebra, integrability of one of the hierarchies [12] has
yet to be proven. Nevertheless, we think that the strong evidence for the existence
of three integrable hierarchies in each of the cases of the $N = 2$ $W_2$, $W_3$ [16]
and $W_4$ algebras suggest that this is the case for all the $N = 2$ super $W_n$ algebras of
[13, 14, 15].

**Acknowledgements**

C.M.Y. thanks I.N.McArthur for helpful conversations. The authors acknowledge
support of the Australian Research Council.

**Appendix**

In this appendix, we present the explicit form for the nonlinear terms of the matrix
Hamiltonian operator $\mathcal{D}_{ij}$ defining the Poisson bracket [14]. The diagonal terms
$\mathcal{D}_{22}$ and $\mathcal{D}_{33}$ are written in manifestly anti-selfadjoint form for compactness. This
form, $\mathcal{D}_{ii} = \mathcal{O}_i - \mathcal{O}_i^*$, is of course non-unique.

\[
\mathcal{O}_2 = -\frac{5}{4} \partial^2 D_1 D_2 - \frac{10}{9} \Phi_1 \partial^3 + \frac{5}{4} (D_i \Phi_1) \partial^2 D_i - \frac{10}{9} (\partial \Phi_1) \partial^2 + \frac{10}{9} (D_1 D_2 \Phi_1) \partial D_1 D_2
\]

\[
- \frac{5}{25} \Phi_1 \partial D_1 D_2 - \frac{7}{3} \Phi_2 \partial D_1 D_2 + \frac{10}{9} (\partial D_i \Phi_1) \partial D_i + \frac{115}{34} \Phi_1 (D_1 D_2 \Phi_1) \partial
\]

\[
- \frac{7}{15} \epsilon^{ij} (D_i \Phi_2) \partial D_j - \frac{60}{34} \epsilon^{ij} \Phi_1 (D_i \Phi_1) \partial D_j - \frac{7}{15} (\partial \Phi_2) D_1 D_2 - \frac{25}{15} \Phi_1 (\partial \Phi_1) D_1 D_2
\]

\[
- \frac{40}{81} \Phi_1 \Phi_2 \partial - \frac{10}{81} \Phi_1^2 \partial - \frac{5}{12} (D_1 D_2 \Phi_1) (D_i \Phi_1) D_i + \frac{5}{81} \Phi_1^2 (D_i \Phi_1) D_i
\]

\[
- 7 \Phi_1 (D_i \Phi_2) D_i + 7 \Phi_2 (D_i \Phi_1) D_i - \frac{5}{108} \epsilon^{ij} (\partial \Phi_1) (D_i \Phi_1) D_j
\]

\[
+ 6 \Phi_3 \partial - (D_i \Phi_3) D_i
\]

\[
\mathcal{D}_{32} = -\frac{2}{5} (\partial D_1 D_2 \Phi_3) + \frac{3}{25} (\partial^3 \Phi_2) + \frac{3}{8} \Phi_2 (\partial \Phi_2) - \frac{8}{25} \Phi_2 (\partial D_1 D_2 \Phi_1)
\]

\[
- \frac{38}{35} \Phi_1 (\partial \Phi_3) - \frac{1}{25} \Phi_1 (\partial D_1 D_2 \Phi_2) + \frac{2}{25} \Phi_1^2 (\partial \Phi_2) - \frac{3}{25} (D_1 D_2 \Phi_2) (\partial \Phi_1)
\]

\[
- \frac{1}{5} (D_1 D_2 \Phi_3) \partial - \frac{8}{25} (D_1 D_2 \Phi_1) (\partial \Phi_2) + \frac{24}{25} (\partial^2 \Phi_2) \partial + \frac{7}{25} \epsilon^{ij} (D_i \Phi_2) (\partial D_j \Phi_1)
\]

\[
- \frac{7}{5} (D_i \Phi_1) (D_i \Phi_3) + \frac{9}{20} \epsilon^{ij} (D_i \Phi_1) (\partial D_j \Phi_2) - \frac{4}{5} \epsilon^{ij} (\partial D_i \Phi_3) D_j
\]

\[
- \frac{9}{105} (\partial^2 D_1 D_2 \Phi_2) D_i + \frac{16}{45} \Phi_1 \Phi_2 (\partial \Phi_1) - \frac{46}{25} \Phi_2 (D_1 D_2 \Phi_1) \partial
\]

\[
- \frac{3}{25} \Phi_2 (D_i \Phi_2) D_i - \frac{1}{25} \epsilon^{ij} \Phi_2 (\partial D_j \Phi_1) D_j - \frac{44}{45} \Phi_1 \Phi_3 \partial
\]

\[
+ 2 \Phi_3 (D_i \Phi_1) D_i - \frac{8}{25} \Phi_1 (D_1 D_2 \Phi_2) \partial - \frac{8}{25} \Phi_1 (D_i \Phi_3) D_i
\]

\[
+ \frac{3}{105} \epsilon^{ij} \Phi_1 (\partial D_i \Phi_2) D_j + \frac{14}{75} \Phi_2 \Phi_2 \partial - \frac{1}{80} \Phi_1^2 (D_i \Phi_2) D_i
\]

\[
+ \frac{3}{80} (D_1 D_2 \Phi_2) (D_i \Phi_1) D_i + \frac{3}{20} (D_1 D_2 \Phi_1) (D_i \Phi_2) D_i - \frac{3}{200} \epsilon^{ij} (\partial \Phi_2) (D_i \Phi_1) D_j
\]

\[
+ \frac{1}{10} (\partial \Phi_2) \partial^2 - \frac{28}{15} (\partial \Phi_3) D_1 D_2 + \frac{3}{50} \epsilon^{ij} (\partial \Phi_1) (D_i \Phi_2) D_j
\]

\[
- \frac{3}{50} (\partial D_1 D_2 \Phi_2) D_1 D_2 + \frac{87}{200} (D_1 \Phi_2) (D_2 \Phi_1) \partial + \frac{87}{200} (D_1 \Phi_1) (D_2 \Phi_2) \partial
\]
\[ O_3 = -\frac{1}{810} \Phi_1^2 \partial D_j - \frac{4}{25} (\partial^2 \Phi_3) \partial D_i - \frac{51}{300} (\partial^2 D_1 D_2 \Phi_2) \partial D_i - \frac{17}{80} (\partial^4 \Phi_1) \partial D_i - \frac{1}{50} (\partial^2 D_1 \Phi_3) D_i \]

\[ + \frac{1}{25} \Phi_2 (\partial \Phi_1) D_1 D_2 - \frac{3}{50} \epsilon^{ij} \Phi_2 (D_i D_j) \partial D_i + \frac{2}{5} \Phi_2 \partial^3 - \frac{28}{5} \Phi_3 \partial D_1 D_2 \]

\[ + \frac{1}{25} \Phi_1 (\partial \Phi_2) D_1 D_2 + \frac{9}{100} \epsilon^{ij} \Phi_1 (D_i \Phi_2) \partial D_j - \frac{9}{50} (D_1 D_2 \Phi_2) \partial D_1 D_2 \]

\[ - \frac{1}{40} (D_i \Phi_2) (D_i \Phi_3) D_1 D_2 - \frac{9}{20} (D_i \Phi_2) \partial^2 D_i + \frac{3}{25} \Phi_1 \Phi_2 \partial D_1 D_2 \]
\[ - \frac{2}{5} (\partial \Phi_1)^2 \partial D_1 D_2 + \frac{21}{200} (\partial^2 \Phi_2) \partial D_1 D_2 + \frac{26}{100} \epsilon^{ij} (\partial^2 \Phi_1) (D_i \Phi_1) \partial D_j \\
+ \frac{1}{5} (\partial^2 \Phi_1) \partial^3 - \frac{1}{20} (\partial^2 D_1 D_2 \Phi_1) \partial D_1 D_2 - \frac{17}{1000} \epsilon^{ij} (D_i \Phi_2) (\partial D_j \Phi_1) D_1 D_2 \\
+ \frac{7}{100} (D_i \Phi_3) \partial^2 D_i + \frac{9}{125} \epsilon^{ij} (D_i \Phi_1) (\partial D_j \Phi_2) D_1 D_2 + \frac{9}{100} (D_i \Phi_1) (\partial^2 D_i \Phi_1) D_1 D_2 \\
+ \frac{21}{250} \epsilon^{ij} (\partial D_i \Phi_2) \partial^2 D_j - \frac{1}{5} (\partial^2 D_i \Phi_1) \partial^2 D_i - \frac{7}{120} (D_i \Phi_3) (D_i \Phi_1) D_1 D_2 \\
+ \frac{168}{9000} \epsilon_i^j \Phi_1 (\partial \Phi_1) D_1 D_2 + \frac{857}{9000} \epsilon^{ij} \Phi_1 (D_i \Phi_1) \partial D_j - \frac{14}{125} \Phi_2 \Phi_1 \partial^3 \\
- \frac{28}{1125} \Phi_2 \Phi_1 (\partial D_1 D_2) + \frac{56}{1125} \Phi_2 (D_1 D_2 \Phi_1) \partial D_1 D_2 + \frac{40}{220} \Phi_1 (D_i \Phi_1) \partial^2 D_i \\
- \frac{1}{90} \Phi_3 \Phi_1 \partial D_1 D_2 - \frac{167}{1080} \Phi_1 (D_1 D_2 \Phi_1) (\partial \Phi_1) D_1 D_2 \\
- \frac{167}{1080} \epsilon_i^j \Phi_1 (D_i D_2 \Phi_1) (D_i \Phi_1) \partial D_j + \frac{11}{35} \Phi_1 (D_1 D_2 \Phi_1) \partial^3 \\
+ \frac{7}{120} \Phi_1 (\partial \Phi_1) (D_i \Phi_1) \partial D_i - \frac{2}{5} \Phi_1 (\partial^2 \Phi_1) \partial D_1 D_2 - \frac{1}{12} \Phi_1 (D_i \Phi_2) (D_i \Phi_1) D_1 D_2 \\
- \frac{7}{125} \Phi_1 (D_i \Phi_2) \partial^2 D_i + \frac{21}{125} \Phi_1 (D_1 D_2 \Phi_1) \partial D_1 D_2 + \frac{1}{48} \Phi_1 (D_i \Phi_1) \partial^2 D_i \\
- \frac{13}{90} (D_1 D_2 \Phi_1) (D_1 \Phi_1) \partial^2 D_i - \frac{1}{210} (\partial \Phi_1) (D_i \Phi_1) (D_2 \Phi_1) D_1 D_2 \\
- \frac{7}{50} \epsilon_i^j (D_i \Phi_1) (D_2 \Phi_1) \partial D_1 D_2 + \frac{1}{210} (D_i \Phi_1) (\partial D_i \Phi_1) \partial D_1 D_2 \\
- \frac{7}{50} \Phi_2 \partial^3 D_1 D_2 - \frac{5}{7} \Phi_1 (\partial \Phi_1) \partial^2 D_1 D_2 + \frac{1}{96} \epsilon_i^j \Phi_1 (D_1 D_2 \Phi_1) \partial^2 D_j \\
- \frac{1}{120} \Phi_1 (D_i \Phi_1) (D_2 \Phi_1) \partial D_1 D_2 + \frac{19}{144} \epsilon_i^j \Phi_1 (D_i \Phi_1) \partial^3 D_j - \frac{1}{12} \Phi_1 \partial^3 D_1 D_2 \\
- \frac{1}{30} \Phi_1 \partial^3 + \frac{11}{12} (D_1 D_2 \Phi_1) \partial^3 D_1 D_2 + \frac{19}{16} (D_1 \Phi_1) \partial^3 D_1 + \frac{4}{30} \partial^3 D_1 D_2 \]
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