GLOBAL WELL-POSEDNESS OF LOGARITHMIC KELLER-SEGEL TYPE SYSTEMS

JAEWOOK AHN, KYUNGKEUN KANG, AND JIHOON LEE

ABSTRACT. We consider a class of logarithmic Keller-Segel type systems modeling the spatio-temporal behavior of either chemotactic cells or criminal activities in spatial dimensions two and higher. Under certain assumptions on parameter values and given functions, the existence of classical solutions is established globally in time, provided that initial data are sufficiently regular. In particular, we enlarge the range of chemotaxis sensitivity \( \chi \), compared to known results, in case that spatial dimensions are between two and eight. In addition, we provide new type of small initial data to obtain global classical solution, which is also applicable to the urban crime model. We discuss long-time asymptotic behaviors of solutions as well.

1. Introduction

The formation of high-density clusters has been observed in the movement of chemotactic cells [11], in the dynamics of self-gravitating particles [16], and in the time-evolution of residential burglary data [22]. The study of spatio-temporal dynamics of such clusters is important since it can be used to gain insight on how to enhance or suppress the formation of clusters. In the real world, for example, it can help suppress the formation of criminal hotspots observed in social problems.

One may use mathematical tools to analyze the spatio-temporal dynamics of clusters. In this paper, we deal with the cross-diffusive system which describes either the movement of chemotactic cells [11] or the propagation of criminal activities [23]:

\[
\begin{aligned}
\partial_t u &= \Delta u - \chi \nabla \cdot (u \nabla \log v) - \sigma uv + \varphi, \quad x \in \Omega, \ t > 0, \\
\partial_t v &= \Delta v - v + uv^\lambda + \psi, \quad x \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0, \\
u(0) &= u_0(x), \ v(x, 0) = v_0(x), \quad x \in \Omega.
\end{aligned}
\]  

(1.1)

Here, \( \Omega \subset \mathbb{R}^d, d \geq 2 \), is a smooth and bounded domain, \( \nu \) is the unit outward normal vector to \( \partial \Omega \), \( \chi, \sigma \) and \( \lambda \) are given non-negative parameters, \( \varphi \) and \( \psi \) are given non-negative functions, and \( u \) and \( v \) are unknowns.

The system (1.1) represents urban crime model (we refer it to (UC) in the sequel), which is a generalized version of logarithmic Keller-Segel model. Namely, if \( \sigma = \varphi = \psi = \lambda = 0 \), (UC) becomes

\[
\partial_t u = \Delta u - \chi \nabla \cdot (u \nabla \log v), \quad \partial_t v = \Delta v - v + u.
\]

From now on, we call it (KS), unless any confusion is to be expected.

In the following, we briefly introduce (KS), (UC) and related works. Our main interests are their global classical solvability and long-time asymptotics.

The system (KS) describes the biased movement of chemotactic cells toward a higher chemoattractant concentration. The unknown functions \( u \) and \( v \) denote the density of chemotactic cells, and the chemoattractant concentration, respectively. As to global solvability in

2010 Mathematics Subject Classification. 35Q92, 35Q91, 35K57.

Key words and phrases. global well-posedness, logarithmic Keller-Segel, urban crime.
(KS), various thresholds of \( \chi \) have been introduced. Winkler [27] obtained the global classical solvability for
\[
\chi < \sqrt{\frac{2}{d}}, \quad d \geq 2,
\]
and later on, Lankeit [13] relaxed the condition (1.2) for \( d = 2 \) as \( \chi < \chi_0 \approx 1.015 \). In the case of the parabolic-elliptic counterpart, Nagai-Senba [15] proved the existence of finite time blowup radial solutions for \( \chi > \frac{2d}{d-2}, \quad d \geq 3 \), and the existence of global radial classical solutions for \( \chi < \frac{2}{d^2}, \quad d \geq 3 \). Furthermore, Fujie-Winkler-Yokota [9] established global classical solvability for \( \chi < \frac{2}{d}, \quad d > 2 \), and later on, this threshold for \( d = 2 \) was enlarged to infinity by Fujie-Senba [7] (see also [8]). For a generalized solution concept, we refer to [2, 3, 4, 14, 24]. As to long-time asymptotics of (KS), Winkler-Yokota [30] obtained the asymptotic stability of constant steady states under (1.2) and the smallness of the domain size \(|\Omega|\). For a generalized solution concept, we refer to [2, 3, 4, 14, 24]. As to long-time asymptotics of (KS), Winkler-Yokota [30] obtained the asymptotic stability of constant steady states under (1.2) and the smallness of the domain size \(|\Omega|\), and later on, Ahn [1] removed out the restriction on the domain size by assuming \( \chi \leq \frac{1}{2} \) and the convexity of \( \Omega \). In the case of the parabolic-elliptic counterpart, qualitative properties of solution such as eventual regularity and asymptotic behavior can be found in [2].

The system (1.1) with \( \sigma > 0 \) and \( \lambda = 0 \) or 1 represents the model (UC), where \( u \) denotes the density of criminals, \( v \) denotes the attractiveness value, and given functions \( \varphi \) and \( \psi \) denote the density of additional criminals, and the source of attractiveness, respectively. Taking into account two effects, the broken window effect and the repeat near-repeat effect, the spatio-temporal dynamic of criminal occurrences is modeled in (UC). For more information on modeling, see [21-23]. As to global solvability in (UC), Rodríguez [17] obtained the global classical solvability for \((\chi, d, \lambda) = (1, 2, 0)\), and Freitag [5] obtained the existence of global classical solutions for \( \chi < \frac{2}{3}, \quad d \geq 2, \quad \lambda = 1 \) (see also [20]). Later on, Rodríguez-Winkler [19] established the global classical solvability for arbitrary \( \chi > 0 \) and \( d = \lambda = 1 \) (see also [26] for the case that \( \varphi, \psi \) are constants). Recently, Tao-Winkler [25] obtained the global classical solvability for arbitrary \( \chi > 0 \) and \((d, \lambda) = (2, 1)\) under the smallness conditions on \( \|u_0\|_{L^2(\Omega)}, \|\nabla \sqrt{u_0}\|_{L^2(\Omega)}, \|\varphi\|_{L^\infty(0, \infty; L^2(\Omega))}, \|\nabla \sqrt{\psi}\|_{L^\infty(0, \infty; L^2(\Omega))}\). For a generalized solution concept, Winkler [29] obtained the global existence of radial renormalized solution for \( \chi > 0, \quad d = 2, \quad \lambda = 1 \). As to long-time asymptotics of (UC), Shen-Li [21] obtained the asymptotic stability of constant steady states for \( \chi < \frac{2}{3}, \quad d \geq 2, \quad \lambda = 1 \) under the assumption that \( \varphi \geq 0 \) and \( \psi > 0 \) are spatial-temporal constants with certain smallness. Moreover, the long-time convergence results \((u, v) \to (0, v_\infty)\) in an appropriate sense have been obtained in Rodríguez-Winkler [19], Winkler [29], and Tao-Winkler [25], where \( v_\infty \) denotes the solution to Neumann problem \(-\Delta v_\infty + v_\infty = \psi_\infty\) with \( \psi_\infty(x) = \lim_{t \to \infty} \psi(x, t)\).

As we mentioned above, global existence results for (KS) or (UC) with general \( \chi > 0 \) are available only for certain generalized solution concepts [14, 24, 29], or restricted to either \( d = 1, 2, 3 \) or \( d = 2 \) with small data [25]. We also note that the conditions on \( \chi \) for global classical solvability in (1.1) such as (1.2) were made when certain energy estimates were obtained. For example, in [5, 20, 27], the condition (1.2) is used to control \( \int_\Omega u^p v^{-q} \) with some \( p > \frac{d}{2} \).

In this paper, we develop a new approach different from energy methods. Our main tool is the maximum principle, which is motivated and modified from those used in Yang-Chen-Liu [31, Theorem 2.3] and Winkler [28]. More precisely, after transforming the system (1.1) into the single equation for \( z = \frac{u^{\sigma}}{v^{1-\lambda}} + (\lambda + \frac{d}{2})|\nabla \log v|^2 \),
\[
\partial_t z - \Delta z - 2\nabla z \cdot \nabla \log v + F(u, v, \varphi, \psi) = G(u, v, \varphi, \psi)
\]
with
\[
F(u, v, \psi) = (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right)^2 + \sigma v \left( \frac{u}{v^{1-\lambda}} \right) + (1 - \lambda)(\lambda + \chi) \left( \frac{u}{v^{1-\lambda}} \right) |\nabla \log v|^2
\]
Theorem 1. Let $\chi$ be a unique non-negative smooth solution $(u,v)$ with $\chi < \chi_{d,\lambda}$ defined in (1.3) (Theorem 1). In particular, our global solvability result covers (KS) with $\chi = \chi_{2,0} = 2$, $d = 2$ for large data. In the case of (UC) with $\chi = 2 > 0 = \chi_{2,1}$, $d = 2$, $\lambda = 1$, our global solvability result requires $\sigma > 0$ and additional smallness conditions on $u_0$, $\nabla \log v_0$, $\varphi$, and $\nabla \sqrt{\psi}$.

The first goal of this paper is the global well-posedness result for large data. The initial data $u_0$, $v_0$, and given functions $\varphi$, $\psi$ of (1.1) are supposed to satisfy the following:

**Assumption 1.** $u_0, v_0, \varphi,$ and $\psi$ are all non-negative and $(u_0, v_0) \in \left(C^{2+\alpha}(\overline{\Omega})\right)^2$ for some $\alpha \in (0, 1)$, $\min_{\overline{\Omega}} v_0 > 0$,

$$
\frac{\partial u_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = 0, \quad x \in \partial \Omega,
$$

$$
\varphi \in C^1(\overline{\Omega} \times [0, \infty)) \cap L^\infty(\Omega \times (0, \infty)), \quad \psi \in C^2(\overline{\Omega} \times [0, \infty)) \cap L^\infty((0, \infty); W^{1,\infty}(\Omega)).
$$

Although $\lambda = 0$ or 1 is the most interesting case, we consider $\lambda \in [0, 1]$ to see how $\lambda$ affects the threshold of $\chi$. For convenience, we denote

$$
\chi_{d,\lambda} := \frac{2}{d} \left(1 - \lambda + \sqrt{2d\lambda(1 - \lambda) + (1 - \lambda)^2}\right).
$$

Our first main result, which is for large data and $\lambda \in [0, 1)$, reads as follows:

**Theorem 1.** Let $\Omega$ be a smooth, bounded and convex domain of $\mathbb{R}^d$, $d \geq 2$. Suppose that $(u_0, v_0, \varphi, \psi)$ satisfies Assumption 1 and let $\sigma > 0$, $0 \leq \lambda < 1$. If $\chi \leq \chi_{d,\lambda}$, then (1.1) possess a unique non-negative smooth solution $(u, v)$ globally in time in the class

$$
(u, v) \in \left(C^{2+\alpha,1+\frac{2}{d}}(\overline{\Omega} \times [0, \infty))\right)^2.
$$

Moreover, if we further assume that $\chi < \chi_{d,\lambda}$ and one of the followings:

- $\sigma > 0$, $\inf_{t \geq 0} \int_{\Omega} \psi(\cdot, t) > 0$,
- $(\varphi(x, t), \psi(x, t)) = (0, b(t))$ for all $x \in \Omega$, $t > 0$,

then there exists a constant $C > 0$ independent of $t$ such that

$$
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|\nabla \log v(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t > 0.
$$

**Remark 1.** According to Theorem 1 (KS) with $\chi \leq \chi_{d,0} = 1$ we possess a unique non-negative smooth solution $(u, v)$ globally in time in the class (1.4). In particular, if the strict inequality $\chi < \chi_{d,0}$ holds, then there exists a constant $C > 0$ satisfying (1.5). Thus, in the case that $2 \leq d \leq 8$, Theorem 1 improves the previous results of Winkler [27] and Lankeit [13] since $\chi_{d,0} = \frac{d}{2} \geq \sqrt{\frac{2}{d}}$ and $\chi_{2,0} = 2 > 1.016$. Theorem 1 is also an improvement of the result in [11] for (UC) with $(\lambda, d) = (0, 2)$. 

$$(2\lambda + \chi)|D^2 \log v|^2 + (1 - \lambda) \left(\frac{u}{v^{1-\chi}}\right)\frac{\psi}{v} + (2\lambda + \chi)\left(\frac{\phi}{v^{1-\chi}}\right)\nabla \log v|^2,$$

and

$$
G(u, v, \varphi, \psi) = -\chi \left(\frac{u}{v^{1-\chi}}\right) \Delta \log v + (1 - \lambda) \left(\frac{u}{v^{1-\chi}}\right) + \frac{\varphi}{v^{1-\chi}} + \frac{(2\lambda + \chi)}{2v} \nabla \log v \cdot \nabla \psi,
$$

we control the right-hand side $G(u, v, \varphi, \psi)$ by using the terms in $F(u, v, \psi)$, and apply the maximum principle argument. Here, the challenging term in $G(u, v, \varphi, \psi)$ is

$$
-\chi \left(\frac{u}{v^{1-\chi}}\right) \Delta \log v
$$

but as we will show in (3.8)–(3.9), we can control this term if $\chi \leq \chi_{d,\lambda}$ with $\chi_{d,\lambda}$ defined in (1.3).
From now on, we assume that \( \psi \) is strictly positive and satisfies
\[
0 < \eta := \inf_{x \in \Omega, s \geq 0} \psi(x, s) \leq \min_{\Omega} v_0.
\]
(1.7)
The second goal of this paper is to establish the global well-posedness result for (1.1) with \( \sigma > 0 \) under certain smallness conditions. More precisely, if either \( \chi \) is large or \( \sigma \eta \) is small, then smallness of given data implies global existence of classical solutions. In case that \( \sigma \eta \) is large compared to \( \chi \), it is not necessary that given data are small, but they are required to be bounded by certain numbers. This seems to be due to damping effects of parameters \( \sigma \) and \( \eta \).

We first state the case \( \lambda = 1 \).

**Theorem 2.** Let \( \Omega \) be a smooth, bounded and convex domain of \( \mathbb{R}^d \), \( d \geq 2 \), and let \( \sigma > 0 \), \( \lambda = 1 \). Suppose that \( (u_0, v_0, \varphi, \psi) \) satisfies Assumption 7 and (1.7). We set
\[
\delta = \min \left\{ \frac{1}{2}, \frac{\sigma \eta (2 + \chi)}{d \chi^2} \right\}, \quad \mu = \max \{ \sigma^{-1}, \| v_0 \|_{L^\infty(\Omega)}, \| \psi \|_{L^\infty(\Omega \times (0, \infty))} \}.
\]
Assume further that \( (u_0, v_0, \varphi, \psi) \) satisfies
\[
\left\| u_0 + (1 + \frac{\lambda}{2}) \nabla \log v_0 \right\|_{L^\infty(\Omega)} < \delta,
\]
and
\[
\left\| \varphi + \frac{4 + 2 \chi}{\eta} | \nabla \sqrt{\psi} |^2 \right\|_{L^\infty(\Omega \times (0, \infty))} < \frac{\eta \delta}{2 \mu}.
\]
Then, (1.1) possess a unique non-negative smooth solution \( (u, v) \) globally in time in the class \( (1.4) \). Moreover, there exists a constant \( C > 0 \) independent of \( t \) satisfying (1.6).

**Remark 2.** For \( d \geq 2 \), \( \sigma > 0 \), and \( \lambda = 1 \), a quadruple \( (u_0, v_0, \varphi, \psi) \) satisfies (1.7)–(1.9) if it is sufficiently close to a constant vector \((0, a, 0, b)\) with \( a \geq b > 0 \).

The proof of Theorem 2 is also valid with some modifications to the case \( \lambda \in (0, 1) \), which seems to be of independent interest, because it shows how \( \lambda \) affects values of \( \delta, \mu, \sigma, \) and \( \eta \). Since the global well-posedness for \( \chi \leq \chi_{d, \lambda} \) with general large data is resolved by Theorem 1 we only treat the case \( \chi > \chi_{d, \lambda} \).

**Theorem 3.** Let \( \Omega \) be a smooth, bounded and convex domain of \( \mathbb{R}^d \), \( d \geq 2 \), and let \( \sigma > 0 \), \( 0 \leq \lambda < 1 \), and \( \chi > \chi_{d, \lambda} \) with \( \chi_{d, \lambda} \) defined in (1.3). Suppose that \( (u_0, v_0, \varphi, \psi) \) satisfies Assumption 7 and (1.7). We set
\[
\delta_0 = \min \left\{ \frac{1}{2}, \frac{\sigma \eta (8 \lambda + 4 \chi)}{d \chi^2} - (1 - \lambda) \right\},
\]
\[
\mu_0 = \max \{2 \sigma^{-1}, \| v_0 \|_{L^\infty(\Omega)}, \| \psi \|_{L^\infty(\Omega \times (0, \infty))} \}.
\]
Assume further that \( (u_0, v_0, \varphi, \psi) \) satisfies
\[
\sigma \eta \geq 4(1 - \lambda),
\]
and
\[
\left\| u_0 + \left( \lambda + \frac{\lambda}{2} \right) \nabla \log v_0 \right\|_{L^\infty(\Omega)} < \delta_0,
\]
(1.10)
and
\[
\left\| \varphi + \frac{4 + 2 \chi}{\eta} | \nabla \sqrt{\psi} |^2 \right\|_{L^\infty(\Omega \times (0, \infty))} < \frac{\eta \delta_0}{2 \mu_0}.
\]
(1.11)
Then, (1.1) possess a unique non-negative smooth solution \( (u, v) \) globally in time in the class \( (1.4) \). Moreover, there exists a constant \( C > 0 \) independent of \( t \) satisfying (1.6).

Finally, we state the long time behavior result for \( \sigma > 0 \).
Theorem 4. Let $\Omega$ be a smooth, bounded and convex domain of $\mathbb{R}^d$, $d \geq 2$, and let $\sigma > 0$, $0 \leq \lambda \leq 1$. Suppose that $(u, v)$ is a unique global classical solution to (1.1) satisfying (1.6). If

$$
\eta_0 \leq \min_{\Omega} v(\cdot, t) \quad \text{for all } t > 0,
$$

where $\eta_0$ is a positive constant independent of $t$, and

$$
\int_0^\infty \int_\Omega \varphi(x, t) dx dt + \int_0^\infty \int_\Omega |\psi(x, t) - \psi_\infty(x)|^2 dx dt < \infty
$$

for some $\psi_\infty \in C(\overline{\Omega})$ with $\min_{\Omega} \psi_\infty > 0$, then $u$ and $v$ satisfy

$$
\|u(\cdot, t)\|_{L^\infty(\Omega)} \to 0, \quad \|v(\cdot, t) - v_\infty\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty,
$$

where $v_\infty$ denotes the solution for

$$
\begin{cases}
  v_\infty - \Delta v_\infty = \psi_\infty, & x \in \Omega, \\
  \frac{\partial v_\infty}{\partial n} = 0, & x \in \partial \Omega.
\end{cases}
$$

(1.14)

Remark 3. We remark that the property (1.6) holds for solutions in Theorem 2, Theorem 3 and Theorem 4 with $\chi < \chi_0 \lambda$ and (1.5). Thus, such solutions satisfy asymptotics in Theorem 4 under the assumptions (1.12)–(1.13).

The remaining part of this paper is organized as follows: in Section 2 we deal with a local well-posedness result involving lower estimate for $v$; in Section 3 we prove the global well-posedness for large data(Theorem 1); in Section 4 we prove the global well-posedness for small data(Theorem 2 and Theorem 3); in Section 5 we prove a long time asymptotics(Theorem 4).

Throughout this paper, $C$ will denote a generic constant that may change from line to line.

2. LOCAL WELL-POSEDNESS

In this section, the local well-posedness for (1.1) is obtained. Note that (1.1) has singular structure $\frac{1}{v}$ in the drift term $\nabla \cdot (u \nabla v)$ but due to the positivity of $v_0$ in Assumption 1 such a singularity does not occur in finite time unless blow-up occurs.

Lemma 1. Let $\Omega$ be a smooth, bounded and convex domain of $\mathbb{R}^d$, $d \geq 2$. Suppose that $(u_0, v_0, \varphi, \psi)$ satisfies Assumption 1 and let $\sigma \geq 0$, $0 \leq \lambda \leq 1$, and $q > d$. Then, there exists the maximal time of existence, $T_{\text{max}} \leq \infty$, such that a unique non-negative solution $(u, v)$ of (1.1) exists and satisfies

$$
u, v \in \left( C^{2+\alpha, \frac{\lambda}{2} \Omega \times [0, T_{\text{max}}]} \right)^2, e^{-t} \min_{\Omega} v_0 \leq \min_{\Omega} v(\cdot, t) \quad \text{for } t < T_{\text{max}}
$$

and

$$
\begin{array}{ll}
\text{either } T_{\text{max}} = \infty \quad \text{or} \quad & \lim_{t \to T_{\text{max}}} \left( \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)} \right) = \infty.
\end{array}
$$

(2.1)

Note that except for the regularity near $t = 0$, $u, v \in \left( C^{2+\alpha, \frac{\lambda}{2} \Omega \times [0, T_{\text{max}}]} \right)^2$, Lemma 1 with $\sigma = \lambda = 1$ was obtained in [5] Lemma 1–Lemma 2. Since $u_0, v_0 \in \left( C^{2+\alpha}(\Omega) \right)^2$, we have $u, v \in \left( C^{2+\alpha, \frac{\lambda}{2} \Omega \times [0, T_{\text{max}}]} \right)^2$ by Schauder’s estimate(see e.g., [12]). We omit the proof of Lemma 1 since its generalization to other cases is rather straightforward(see also [18, 27]).

The temporal bound (2.1) can be replaced by uniform one if we further assume that

$$
\inf_{t \geq 0} \int_\Omega \psi(\cdot, t) > 0 \quad \text{or} \quad \sigma = 0 \leq \lambda < 1, \quad \|u_0\|_{L^1(\Omega)} > 0.
$$

(2.3)
Lemma 2. Let the same assumptions as in Lemma 1 be satisfied. Let \((u,v)\) be the solution to (1.1) given by Lemma 1. If (2.3) holds, then there exists a positive constant \(\eta_1\) independent of \(t\) such that
\[
\eta_1 \leq \min_{\Omega} v(\cdot,t) \quad \text{for all } t < T_{\text{max}}. \tag{2.4}
\]
In particular, if \(\inf_{x \in \Omega, s \geq 0} \psi(x,s) > 0\), then there exists a positive constant
\[
\eta_2 = \min\{\min_{\Omega} v_0, \inf_{x \in \Omega, s \geq 0} \psi(x,s)\}
\]
and denote \(\text{diam } \Omega := \max_{x,y \in \Omega} |x-y|\). As in the proof of [6] Lemma 2.2, we use the representation formula of \(v\) and \(e^{t(\Delta-1)}v_0 \geq 0\) to obtain that
\[
v(t) = e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}(uv^\lambda + \psi)(s)ds \geq \int_0^t e^{(t-s)(\Delta-1)}(uv^\lambda + \psi)(s)ds
\]

\[
\geq \int_0^t \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} e^{-\frac{(\text{diam } \Omega)^2}{4(t-s)}+(t-s)} \left( \int_{\Omega} uv^\lambda(x,s) + \psi(x,s)dx \right) ds. \tag{2.6}
\]

Now, using (2.6), we show (2.4). We treat two cases in (2.3) separately.
\bullet \inf_{t \geq 0} \int_{\Omega} \psi(\cdot,t) > 0\) case.

The right-hand side of (2.6) has a lower bound
\[
\int_0^t \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} e^{-\frac{(\text{diam } \Omega)^2}{4(t-s)}+(t-s)} \left( \int_{\Omega} uv^\lambda(x,s) + \psi(x,s)dx \right) ds \geq \int_0^t \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} e^{-\frac{(\text{diam } \Omega)^2}{4(t-s)}+(t-s)} \left( \int_{\Omega} \psi(x,s)dx \right) ds
\]

\[
\geq \left( \inf_{s \geq 0} \int_{\Omega} \psi(\cdot,s) \right) \int_0^t \frac{1}{(4\pi r)^{\frac{d}{2}}} e^{-\frac{(\text{diam } \Omega)^2}{4r}+r} dr,
\]
which is a monotone increasing function of \(t\) with initial value zero. Combining it with the decreasing lower bound in (2.1), we have a desired bound (2.4).

\bullet \sigma = 0 \leq \lambda < 1, \|u_0\|_{L^1(\Omega)} > 0\) case.

Integrating \(u\)-equation over \(\Omega\), we first note that \(\int_{\Omega} u(\cdot,t) \geq \int_{\Omega} u_0\) for \(t < T_{\text{max}}\). We compute a lower bound of the right-hand side of (2.6) as
\[
\int_0^t \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} e^{-\frac{(\text{diam } \Omega)^2}{4(t-s)}+(t-s)} \left( \int_{\Omega} uv^\lambda(x,s) + \psi(x,s)dx \right) ds
\]

\[
\geq \int_0^t \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} e^{-\frac{(\text{diam } \Omega)^2}{4(t-s)}+(t-s)} \left( \int_{\Omega} uv^\lambda(x,s)dx \right) ds
\]

\[
\geq \int_0^t \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} e^{-\frac{(\text{diam } \Omega)^2}{4(t-s)}+(t-s)} \left( \int_{\Omega} u(x,s)dx \right) (\min_{\Omega} v(\cdot,s))^\lambda ds
\]

\[
\geq \int_0^t \frac{1}{(4\pi r)^{\frac{d}{2}}} e^{-\frac{(\text{diam } \Omega)^2}{4r}+r} dr \left( \int_{\Omega} u_0\right) (\inf_{s \leq t} \min_{\Omega} v(\cdot,s))^\lambda.
\]
By (2.6), it follows that
\[ \min_{\Omega} v(\cdot, t) \geq \int_0^t \frac{1}{(4\pi r)^{d/2}} e^{-\frac{(\text{diam } \Omega)^2}{4r} + r} \left( \int_{\Omega} u_0 \right) (\inf_{s \leq t} \min_{\Omega} v(\cdot, s))^\lambda. \] (2.7)
Note that there exists a unique \( \tau > 0 \) satisfying \( e^{-\tau} \min_{\Omega} v_0 = f(\tau) \), where
\[ f(t) := \left[ \int_0^t \frac{1}{(4\pi r)^{d/2}} e^{-\frac{(\text{diam } \Omega)^2}{4r} + r} \left( \int_{\Omega} u_0 \right) (\inf_{s \leq t} \min_{\Omega} v(\cdot, s)^\lambda \right], \quad t > 0. \]
Indeed, \( e^{-t} \min_{\Omega} v_0 \) is the monotone decreasing function of \( t \) approaching zero for large \( t \), and \( f(t) \) is the monotone increasing function of \( t \) with initial value zero. Note also from (2.1) that \( \min_{\Omega} v(\cdot, t) \geq e^{-\tau} \min_{\Omega} v_0 \) for all \( t \leq \tau \). Now, let \( \tilde{\eta} \in (0, e^{-\tau} \min_{\Omega} v_0) \) and suppose that
\[ \min_{\Omega} v(\cdot, t_1) = \tilde{\eta} \]
for the first time \( t_1 > \tau \). By (2.7), we have
\[ \min_{\Omega} v(\cdot, t_1) \geq \int_0^{t_1} \frac{1}{(4\pi r)^{d/2}} e^{-\frac{(\text{diam } \Omega)^2}{4r} + r} \left( \int_{\Omega} u_0 \right) (\inf_{s \leq t_1} \min_{\Omega} v(\cdot, s))^\lambda 
= \int_0^{t_1} \frac{1}{(4\pi r)^{d/2}} e^{-\frac{(\text{diam } \Omega)^2}{4r} + r} \left( \int_{\Omega} u_0 \right) (\min_{\Omega} v(\cdot, t_1))^\lambda 
\geq \int_0^{\tau} \frac{1}{(4\pi r)^{d/2}} e^{-\frac{(\text{diam } \Omega)^2}{4r} + r} \left( \int_{\Omega} u_0 \right) (\min_{\Omega} v(\cdot, t_1))^\lambda, \]
which leads to contradiction to \( \tilde{\eta} \in (0, e^{-\tau} \min_{\Omega} v_0) \). Therefore, \( \min_{\Omega} v(\cdot, t) \geq e^{-\tau} \min_{\Omega} v_0 \) for all \( t > 0 \) and thus, (2.4) is obtained.

Next, (2.5) is a direct consequence of the maximum principle applied to \( v \)-equation with \( \inf_{x \in \Omega, s \geq 0} \psi(x, s) > 0 \). This completes the proof. \( \square \)

3. Global well-posedness for large data

In this section, we prove Theorem 1. Using the maximum principle argument, we obtain an estimate for \( \|u - \chi\|_{L^\infty} \) and \( \nabla \log v \).

**Proposition 1.** Let \( \Omega \) be a smooth, bounded and convex domain of \( \mathbb{R}^d \), \( d \geq 2 \), and let \( \sigma \geq 0 \), \( 0 \leq \lambda < 1 \). Suppose that \((u_0, v_0, \varphi, \psi)\) satisfies Assumption 1 and \((u, v)\) is a unique solution to (1.1) given by Lemma 1. Then, there exists a constants \( C > 0 \) independent of \( t \) such that we have the following:

(i) If \( \chi \leq \chi_{d, \lambda} \), then
\[ \|\frac{u}{v_1 - \lambda}(\cdot, t)\|_{L^\infty(\Omega)} + \|\nabla \log v(\cdot, t)\|_{L^\infty(\Omega)} \leq Ce^{2t} \quad \text{for } t < T_{\text{max}}, \] (3.1)

(ii) If \( \chi < \chi_{d, \lambda}, \sigma > 0 \), and \( \inf_{s \geq 0} \int_\Omega \psi(\cdot, s) > 0 \), then
\[ \|\frac{u}{v_1 - \lambda}(\cdot, t)\|_{L^\infty(\Omega)} + \|\nabla \log v(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C, \] (3.2)
(iii) If \( \chi < \chi_{d,\lambda} \) and \((\varphi(x,s), \psi(x,s)) = (0, b(s))\) for all \(x \in \Omega\), \(s > 0\), then
\[
\|\frac{u}{v^{1-\lambda}}(\cdot, t)\|_{L^\infty(\Omega)} + \|\nabla \log v(\cdot, t)\|_{L^\infty(\Omega)} \leq C. \tag{3.3}
\]

Proof. From (1.1), we obtain
\[
\begin{align*}
\partial_t \left( \frac{u}{v^{1-\lambda}} \right) - \Delta \left( \frac{u}{v^{1-\lambda}} \right) &+ (1 - \lambda) (\lambda + \chi) \left( \frac{u}{v^{1-\lambda}} \right) \Delta \log v^2 \\
+ (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right)^2 &+ \sigma v \left( \frac{u}{v^{1-\lambda}} \right) + (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right) \frac{\psi}{v} \\
= (2 - 2\lambda - \chi) \nabla \left( \frac{u}{v^{1-\lambda}} \right) \cdot \nabla \log v - \chi \left( \frac{u}{v^{1-\lambda}} \right) \Delta \log v + (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right) + \frac{\varphi}{v^{1-\lambda}}. \tag{3.4}
\end{align*}
\]
From (1.1), we derive
\[
\begin{align*}
\partial_t |\nabla \log v|^2 - \Delta |\nabla \log v|^2 &+ 2|D^2 \log v|^2 + 2 \frac{\psi}{v} |\nabla \log v|^2 \\
= 2 \nabla |\nabla \log v|^2 \cdot \nabla \log v + 2 \nabla \left( \frac{u}{v^{1-\lambda}} \right) \cdot \nabla \log v + 2 \frac{\psi}{v} \nabla \log v \cdot \nabla \psi. \tag{3.5}
\end{align*}
\]
Denote \( z := \frac{u}{v^{1-\lambda}} + \theta |\nabla \log v|^2 \), where \( \theta = \lambda + \frac{\chi}{2} \). Note that
\[
\frac{\partial z}{\partial \nu} \leq 0 \quad \text{on} \quad \partial \Omega, \tag{3.6}
\]
which is due to \( \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \) and the convexity of \( \Omega \). Adding (3.4) and (3.5) \( \times \theta \), we have
\[
\begin{align*}
\partial_t z - \Delta z - 2 \nabla z \cdot \nabla \log v &+ (1 - \lambda) (\lambda + \chi) \left( \frac{u}{v^{1-\lambda}} \right) \Delta \log v^2 + (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right)^2 \\
+ \sigma v \left( \frac{u}{v^{1-\lambda}} \right) &+ (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right) \frac{\psi}{v} + 2 \theta |D^2 \log v|^2 + 2 \theta \frac{\psi}{v} |\nabla \log v|^2 \\
= - \chi \left( \frac{u}{v^{1-\lambda}} \right) \Delta \log v + (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right) + \frac{\varphi}{v^{1-\lambda}} + 2 \theta \frac{\psi}{v} \nabla \log v \cdot \nabla \psi. \tag{3.7}
\end{align*}
\]
Using Young’s inequality and
\[
\nabla f \cdot \nabla \Delta f = \frac{1}{2} \Delta |\nabla f|^2 - |D^2 f|^2 \quad \text{and} \quad |\Delta f| \leq \sqrt{d} |D^2 f| \quad \text{for} \quad f \in C^2(\Omega),
\]
we compute the first term on the right hand side of (3.7) as
\[
- \chi \left( \frac{u}{v^{1-\lambda}} \right) \Delta \log v \leq 2 \theta |D^2 \log v|^2 + (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right)^2 - \left[ (1 - \lambda) - \frac{d \chi^2}{8 \lambda + 4 \chi} \right] \left( \frac{u}{v^{1-\lambda}} \right)^2. \tag{3.8}
\]
Now, we consider three cases, (3.3)–(3.5), separately. The definition of \( \chi_{d,\lambda} \) gives
\[
\begin{align*}
(1 - \lambda) - \frac{d \chi^2}{8 \lambda + 4 \chi} &= 0 \quad \text{if} \quad \chi = \chi_{d,\lambda}, \\
(1 - \lambda) - \frac{d \chi^2}{8 \lambda + 4 \chi} &> 0 \quad \text{if} \quad \chi < \chi_{d,\lambda}. \tag{3.9}
\end{align*}
\]
\[\bullet \ (i) \ \chi \leq \chi_{d,\lambda}.\]
Using Young’s inequality, we compute
\[
\frac{2 \theta}{v} \nabla \log v \cdot \nabla \psi \leq (1 - \lambda) \theta |\nabla \log v|^2 + \frac{\theta}{1 - \lambda} |\nabla \psi|^2 \frac{1}{v^2}.\]
By (3.7)–(3.9), Assumption 1 and (2.1), it follows that
\[ \partial_t z - \Delta z - 2\nabla z \cdot \nabla \log v \leq (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right) + (1 - \lambda)^2 |\nabla \log v|^2 + \frac{\varphi}{v^{1-\lambda}} + \frac{\theta}{1 - \lambda} |\nabla \psi|^2 \frac{1}{v^2}, \]
\[ \leq (1 - \lambda) z + \frac{\|\varphi\|_{L^\infty(\Omega \times (0, \infty))}}{v^{1-\lambda}} + \frac{\theta}{1 - \lambda} \frac{1}{v^2} \left|\nabla \psi\right|^2 L^\infty(\Omega \times (0, \infty)) e^{(1-\lambda)t}, \]
\[ \leq (1 - \lambda) z + C_1 e^{2t}, \]
where \( C_1 \) is a positive constant independent of \( t \).

Since \( \partial_t [z e^{-(1-\lambda)t}] - \Delta [z e^{-(1-\lambda)t}] - 2\nabla [z e^{-(1-\lambda)t}] \cdot \nabla \log v \leq C_1 e^{(1+\lambda)t} \),
we have for \( Z = z e^{-(1-\lambda)t} - \frac{C_1}{1+\lambda} e^{(1+\lambda)t} \) that
\[ \partial_t Z - \Delta Z - 2\nabla Z \cdot \nabla \log v \leq 0. \]

As \( \frac{\partial Z}{\partial t} \leq 0 \) on \( \partial \Omega \), applying the maximum principle to \( Z \)-equation, we have
\[ \|Z(\cdot, t)\|_{L^\infty(\Omega)} \leq \|Z(\cdot, 0)\|_{L^\infty(\Omega)} \quad \text{for } t < T_{\text{max}}. \]
Thus, (3.11) can be deduced.

- (ii) \( \chi < \chi_{d, \lambda}, \sigma > 0, \text{ and } \inf_{\theta \geq 0} \int_\Omega \psi(\cdot, s) > 0. \)

From (3.7)–(3.9), we observe that
\[ \partial_t z - \Delta z - 2\nabla z \cdot \nabla \log v \]
\[ + (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right) |\nabla \log v|^2 + \left( (1 - \lambda) - \frac{d\chi^2}{8\lambda + 4\chi} \right) \left( \frac{u}{v^{1-\lambda}} \right)^2 \]
\[ + \sigma v \left( \frac{u}{v^{1-\lambda}} \right) + (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right) \frac{\psi}{v} + 2\theta \frac{\psi}{v} |\nabla \log v|^2 \]
\[ \leq (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right) + \frac{\varphi}{v^{1-\lambda}} + \frac{2\theta}{v} |\nabla \log v| |\nabla \psi|, \]
which yields
\[ \partial_t z - \Delta z - 2\nabla z \cdot \nabla \log v \]
\[ + \left[ (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right) |\nabla \log v|^2 + \left( (1 - \lambda) - \frac{d\chi^2}{8\lambda + 4\chi} \right) \left( \frac{u}{v^{1-\lambda}} \right)^2 \right] + \sigma v - (1 - \lambda) \]
\[ \leq \frac{\varphi}{v^{1-\lambda}} + \frac{2\theta}{v} |\nabla \log v| |\nabla \psi|. \]
Adding (1.1) \( \times \frac{\varphi}{2} \) to the above inequality and introducing \( Z := z + \frac{\varphi}{2} v \), we have
\[ \partial_t Z - \Delta Z - 2\nabla Z \cdot \nabla \log v \]
\[ + \left[ (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right) |\nabla \log v|^2 + \left( (1 - \lambda) - \frac{d\chi^2}{8\lambda + 4\chi} \right) \left( \frac{u}{v^{1-\lambda}} \right)^2 \right] + \sigma v - (1 - \lambda) \]
\[ \leq \frac{\varphi}{v^{1-\lambda}} + \frac{2\theta}{v} |\nabla \log v| \cdot |\nabla \psi| - \sigma v |\nabla \log v|^2 - \frac{\sigma}{2} v + \frac{\varphi}{2}. \]
Using Young’s inequality, we compute
\[ \frac{2\theta}{v} |\nabla \log v| \cdot |\nabla \psi| \leq \frac{\sigma}{2} v |\nabla \log v|^2 + \frac{4\theta^2}{2\sigma v^3} |\nabla \psi|^2 \]
and using (3.10), we can find a positive number $\varepsilon$ depending only on $\lambda$, $\chi$, and $d$ such that

$$\varepsilon \mathcal{Z} \leq (1 - \lambda) (\lambda + \chi) |\nabla \log v|^2 + [(1 - \lambda) - \frac{d\chi^2}{8\lambda + 4\chi}] \left( \frac{u}{v^{1-\lambda}} \right) + \frac{\sigma}{2} v.$$  

It follows that

$$\partial_t \mathcal{Z} - \Delta \mathcal{Z} - 2 \nabla \mathcal{Z} \cdot \nabla v + [\varepsilon \mathcal{Z} - (1 - \lambda)] \left( \frac{u}{v^{1-\lambda}} \right)$$

$$\leq \frac{\varphi^{\varepsilon - 2\sigma}}{v^{1-\lambda}} + \frac{4\theta^2}{2\sigma v^{\beta}} |\nabla \psi|^2 - \frac{\sigma}{2} u |\nabla \log v|^2 - \frac{\sigma}{2} v + \frac{\sigma}{2} \psi.$$  

Note that $v \geq \eta_1 > 0$ by (2.4) and $\inf_{s \geq 0} \int_{\Omega} \psi(\cdot, s) > 0$. Using this lower bound and Assumption $I$ we compute the right-hand side as

$$\frac{\varphi^{\varepsilon - 2\sigma}}{v^{1-\lambda}} + \frac{4\theta^2}{2\sigma v^{\beta}} |\nabla \psi|^2 - \frac{\sigma}{2} u |\nabla \log v|^2 - \frac{\sigma}{2} v + \frac{\sigma}{2} \psi$$

$$\leq \frac{\|\varphi\|_{L^{\infty}(\Omega \times (0, \infty))}}{\eta_1^{1-\lambda}} + \frac{4\theta^2}{2\sigma \eta_1} \|\nabla \psi\|^2_{L^{\infty}(\Omega \times (0, \infty))} - \frac{\sigma}{2} \eta_1 |\nabla \log v|^2 - \frac{\sigma}{2} v + \frac{\sigma}{2} \|\psi\|_{L^{\infty}(\Omega \times (0, \infty))}$$

$$= C_2 - \frac{\sigma}{2} \eta_1 |\nabla \log v|^2 - \frac{\sigma}{2} v,$$

where $C_2$ is a positive constant independent of $t$. We observe that

$$\partial_t \mathcal{Z} - \Delta \mathcal{Z} - 2 \nabla \mathcal{Z} \cdot \nabla v \leq C_2 - \left\{ [\varepsilon \mathcal{Z} - (1 - \lambda)] \left( \frac{u}{v^{1-\lambda}} \right) + \frac{\sigma}{2} \eta_1 |\nabla \log v|^2 + \frac{\sigma}{2} v \right\}.$$  

(3.11)

Now, suppose that there exists $x_M \in \Omega^\varepsilon$ such that

$$\mathcal{Z}(x_M, t_M) = 2 \max \left\{ \|Z_0\|_{L^{\infty}(\Omega)}, \frac{2 - \lambda}{\varepsilon}, C_2, C_2 \frac{2\theta}{\sigma \eta_1} \right\}$$

(3.12)

for the first time $t_M > 0$. Note that $t_M < T_{\max}$ by (2.2). Note also that the right-hand side of (3.11) is negative at $(x, t) = (x_M, t_M)$. Indeed,

$$C_2 - \left\{ [\varepsilon \mathcal{Z} - (1 - \lambda)] \left( \frac{u}{v^{1-\lambda}} \right) + \frac{\sigma}{2} \eta_1 |\nabla \log v|^2 + \frac{\sigma}{2} v \right\}$$

$$\leq C_2 - \left\{ \left( \frac{u}{v^{1-\lambda}} \right) + \frac{\sigma}{2} \eta_1 |\nabla \log v|^2 + \frac{\sigma}{2} v \right\}$$

$$\leq \left\{ \begin{array}{ll}
C_2 - \left\{ \left( \frac{u}{v^{1-\lambda}} \right) + \theta |\nabla \log v|^2 + \frac{\sigma}{2} v \right\} = C_2 - \mathcal{Z}, & \text{if } \theta \leq \frac{\sigma}{2} \eta_1, \\
C_2 - \left\{ \frac{\varphi}{\theta} \left( \frac{u}{v^{1-\lambda}} \right) + \frac{\sigma}{2} \eta_1 |\nabla \log v|^2 + \frac{\sigma}{2} \eta_1 \frac{\sigma}{\theta} v \right\} = C_2 - \frac{\sigma \eta_1}{\theta} \mathcal{Z}, & \text{if } \theta > \frac{\sigma}{2} \eta_1,
\end{array} \right.$$  

which is strictly negative for $(x, t) = (x_M, t_M)$. If $x_M$ is an interior point of $\Omega$, then

$$\partial_t \mathcal{Z} - \Delta \mathcal{Z} - 2 \nabla \mathcal{Z} \cdot \nabla \log v \geq 0$$

for $(x, t) = (x_M, t_M)$

and thus, this with (3.11) leads to the contradiction. Let $x_M \in \partial \Omega$. Then, by Hopf’s Lemma type argument, $\partial \mathcal{Z} / \partial \nu$ is strictly positive at $(x, t) = (x_M, t_M)$ but this leads to the contradiction because $\partial \mathcal{Z} / \partial \nu \leq 0$ on $\partial \Omega$ for $t < T_{\max}$. Therefore, there is no such $(x_M, t_M) \in \Omega \times (0, T_{\max})$ satisfying (3.12). Thus,

$$\|\mathcal{Z}(\cdot, t)\|_{L^{\infty}(\Omega)} < 2 \max \left\{ \|Z_0\|_{L^{\infty}(\Omega)}, \frac{2 - \lambda}{\varepsilon}, C_2, C_2 \frac{2\theta}{\sigma \eta_1} \right\}, \quad t < T_{\max},$$

namely, (3.2) is obtained.
As in (3.10), by (3.7)–(3.8), we have
\[
\frac{\partial_t z}{\partial t} - \Delta z - 2\nabla z \cdot \nabla \log v + (1 - \lambda)(\lambda + \chi) \left( \frac{u}{v^{1-\lambda}} \right)^2 |\nabla \log v|^2 + \left[ (1 - \lambda) - \frac{d\chi^2}{8\lambda + 4\chi} \right] \left( \frac{u}{v^{1-\lambda}} \right)^2 \\
+ \sigma v \left( \frac{u}{v^{1-\lambda}} \right) + (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right) \frac{\psi}{v} + 2\theta \frac{\psi}{v} |\nabla \log v|^2 \\
\leq (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right)
\]
and thus,
\[
\frac{\partial_t z}{\partial t} - \Delta z - 2\nabla z \cdot \nabla \log v + \left[ (1 - \lambda)(\lambda + \chi)|\nabla \log v|^2 + \left( (1 - \lambda) - \frac{d\chi^2}{8\lambda + 4\chi} \right) \left( \frac{u}{v^{1-\lambda}} \right)^2 \right] \leq 0.
\]
Note from (3.9) that there exists a positive number \( \varepsilon \) depending only on \( \lambda, \chi, \) and \( d \) such that
\[
\frac{\partial_t z}{\partial t} - \Delta z - 2\nabla z \cdot \nabla \log v + \left[ \varepsilon z - (1 - \lambda) \right] \left( \frac{u}{v^{1-\lambda}} \right) \leq 0.
\]
By the maximum principle, we have
\[
\|z(\cdot, t)\|_{L^\infty(\Omega)} \leq \max \left\{ \|z_0\|_{L^\infty(\Omega)}, \frac{1 - \lambda}{\varepsilon} \right\}, \quad t < T_{\max}.
\]
Therefore, (3.3) is obtained. This completes the proof. \( \square \)

To obtain Theorem 1, we prepare two bounds of \( u \) in the following two lemmas. First, we estimate the temporal bound of \( u \) for \( \chi \leq \chi_{d, \lambda} \).

**Lemma 3.** Let the same assumptions as in Proposition 1 be satisfied. Suppose that \( \chi \leq \chi_{d, \lambda} \) and \( T < T_{\max} \). Then, there exists a positive number \( C \) independent of \( T \) such that
\[
\sup_{t \leq T} \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq Ce^{CT}, \quad (3.13)
\]

**Proof.** Integrating the first equation of (1.1) over \( \Omega \) and using a direct computation, we have
\[
\frac{d}{dt} \int_\Omega u = -\sigma \int_\Omega uv + \int_\Omega \varphi \leq \int_\Omega \varphi \leq \|\varphi\|_{L^\infty(\Omega \times (0, \infty))}, \quad \frac{\partial_t z}{\partial t} - \Delta z - 2\nabla z \cdot \nabla \log v + \left[ (1 - \lambda)(\lambda + \chi)|\nabla \log v|^2 + \left( (1 - \lambda) - \frac{d\chi^2}{8\lambda + 4\chi} \right) \left( \frac{u}{v^{1-\lambda}} \right)^2 \right] \leq 0.
\]
which entails by integrating with respect to time that
\[
\|u(\cdot, t)\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)} + \|\varphi\|_{L^\infty(\Omega \times (0, \infty))} t, \quad t < T_{\max}. \quad (3.14)
\]
Using the representation formula of \( u \) and \( -\sigma uv \leq 0 \), we note that
\[
u(t) \leq e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} [-\chi \nabla \cdot (u \nabla \log v) + \varphi](s) ds, \quad t < T_{\max}.
\]
By the smoothing estimate for Neumann heat semigroup \( e^{t\Delta} \) and
\[
\|e^{t\Delta} f\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)} \quad \text{for } f \in L^\infty(\Omega), \quad (3.15)
\]
we have
\[
\|u(t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} + Ce^{CT} \int_0^t \left( 1 + (t - s)^{-\frac{\delta}{2} - \frac{\gamma}{2}} \right)\|u \nabla \log v(s)\|_{L^2d(\Omega)} + \|\varphi(s)\|_{L^\infty(\Omega)} ds.
\]
Using Hölder’s inequality, \(3.11\), \(3.14\), and a direct computation, we further compute the integral on the right-hand side as

\[
\int_0^t (1 + (t - s)^{-\frac{2}{3}} \frac{d}{2})\| u \nabla \log v(s) \|_{L^2(\Omega)} + \| \varphi(s) \|_{L^\infty(\Omega)} ds
\]

\[
\leq \int_0^t (1 + (t - s)^{-\frac{2}{3}})\| u(s) \|_{L^1(\Omega)}^{\frac{1}{3}}\| u(s) \|_{L^\infty(\Omega)}^{\frac{2}{3}}\| \nabla \log v(s) \|_{L^\infty(\Omega)} ds + \| \varphi \|_{L^\infty(\Omega \times (0, \infty))} t
\]

\[
\leq C \int_0^t (1 + (t - s)^{-\frac{2}{3}})(\| u_0 \|_{L^1(\Omega)} + \| \Omega \|_{L^\infty(\Omega \times (0, \infty))})^{\frac{1}{3}} e^{\frac{1}{2} s} ds sup_{s \leq t} \| u(s) \|_{L^\infty(\Omega)}^{\frac{1}{3}} + Ct
\]

\[
\leq C (1 + t^{1 + \frac{1}{2} p}) e^t sup_{s \leq t} \| u(s) \|_{L^\infty(\Omega)}^{\frac{1}{3}} + Ct,
\]

where \(C\) is a positive constant independent of \(t\). Combining above estimates and taking supremum over \(0 \leq t \leq T\), we have

\[
\sup_{t \leq T} \| u(t) \|_{L^\infty(\Omega)} \leq C e^{2T} \sup_{t \leq T} \| u(t) \|_{L^\infty(\Omega)}^{\frac{1}{3}} + C (1 + T).
\]

This yields by Young’s inequality and \(1 + T \leq e^T\) that \(3.13\). This completes the proof. \(\square\)

Next, we obtain the uniform bound of \(u\) for \(\chi < \chi_{d,l}\) when \(1.5\) holds.

**Lemma 4.** Let the same assumptions as in Proposition 1 be satisfied. Assume that \(\chi < \chi_{d,l}\) and \(1.5\) holds. Then, there exists a positive constant \(C\) independent of \(T < T_{\max}\) satisfying

\[
\sup_{t \leq T} \| u(t) \|_{L^\infty(\Omega)} \leq C.
\]

**Proof.** Let \(\chi < \chi_{d,l}\). In the case where \(\sigma > 0\) and \(\inf_{t \geq 0} \int _{\Omega} \psi(\cdot, t) > 0\), it follows by \(3.2\) in Proposition 1 that

\[
\| u \|_{L^\infty((0, T_{\max}); L^\infty(\Omega))} \leq \frac{\| u \|_{L^\infty((0, T_{\max}); L^\infty(\Omega))}}{t^{1-\chi}} \leq C.
\]

Next, in the case of \((\varphi(x, t), \psi(x, t)) = (0, b(t))\) for all \(x \in \Omega, t > 0\), we note that

\[
\partial_t u + u - \Delta u = -\chi \nabla \cdot (u \nabla \log v) - \sigma uv + \varphi + u \leq -\chi \nabla \cdot (u \nabla \log v) + u
\]

and thus,

\[
u(t) \leq e^{t(\Delta - 1)}u_0 + \int_0^t e^{(t-s)(\Delta - 1)}[-\chi \nabla \cdot (u \nabla \log v) + u](s) ds, \quad t < T_{\max}.
\]
By the smoothing estimate for $e^{\Delta}$, (3.3), (3.14) with $\varphi = 0$, (3.15), and Hölder’s inequality we obtain

$$\|u(t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} + C \int_0^t e^{-(t-s)} \left(1 + (t-s)^{-\frac{d}{2} + \frac{d}{4}}\right) \|\nabla \log v(s)\|_{L^{2d}(\Omega)} ds$$

$$+ C \int_0^t e^{-(t-s)} \left(1 + (t-s)^{-\frac{d}{2} + \frac{d}{4}}\right) \|u(s)\|_{L^{2d}(\Omega)} ds$$

$$\leq C + C \int_0^t e^{-(t-s)} \left(1 + (t-s)^{-\frac{d}{2}}\right) \|u(s)\|_{L^1(\Omega)}^{\frac{1}{2}} \|\nabla \log v(s)\|_{L^\infty(\Omega)} ds$$

$$+ C \int_0^t e^{-(t-s)} \left(1 + (t-s)^{-\frac{d}{2}}\right) \|u(s)\|_{L^1(\Omega)} \|v(s)\|_{L^\infty(\Omega)} \|\nabla \log v(s)\|_{L^\infty(\Omega)} ds$$

$$\leq C + C \sup_{s \leq t} \|u(s)\|_{L^1(\Omega)} \|v(s)\|_{L^\infty(\Omega)} ds$$

Taking supremum over $0 \leq t \leq T$ and using Young’s inequality, we have the desired uniform bound. This completes the proof. \qed

We are ready to prove Theorem 1.

**Proof of Theorem 1.** We first obtain $T_{\max} = \infty$ for $\chi \leq \chi_{d,\lambda}$. Suppose not, i.e. $T_{\max} < \infty$. We show that for $q > d$, there exists a positive constant $C$ independent of $T < T_{\max}$ such that

$$\sup_{t \leq T} \|v(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C e^{CT}. \tag{3.16}$$

Let $t \leq T < T_{\max}$. We begin with recalling the representation formula

$$v(t) = e^{t(\Delta - 1)}v_0 + \int_0^t e^{(t-s)(\Delta - 1)} [u v^\lambda + \psi](s) ds. \tag{3.17}$$

Using the smoothing estimate for $e^{\Delta}$ and Hölder’s inequality, we compute

$$\|v(t)\|_{W^{1,q}(\Omega)} \leq C \|v_0\|_{W^{1,q}(\Omega)} + C \int_0^t e^{-(t-s)} \left(1 + (t-s)^{-\frac{d}{2}}\right) \|uv^\lambda(s)\|_{L^\infty(\Omega)} + \|\psi(s)\|_{L^\infty(\Omega)} ds$$

$$\leq C + C \int_0^t e^{-(t-s)} \left(1 + (t-s)^{-\frac{d}{2}}\right) \|u(s)\|_{L^\infty(\Omega)} \|v(s)\|_{L^\infty(\Omega)} + \|\psi\|_{L^\infty(\Omega \times (0,\infty))} ds.$$

Using the embedding relation $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$, (3.13), and a direct computation, we further compute the integral on the right hand side as

$$\int_0^t e^{-(t-s)} \left(1 + (t-s)^{-\frac{d}{2}}\right) \|u(s)\|_{L^\infty(\Omega)} \|v(s)\|_{L^\infty(\Omega)} ds$$

$$\leq C e^{CT} \sup_{s \leq t} \|v(s)\|_{W^{1,q}(\Omega)} + C,$$

where $C$ is a positive constant independent of $t$. Combining above estimates, taking supremum over $0 \leq t \leq T$, and using Young’s inequality, we have (3.16). This leads to the contradiction to the fact, due to the blow-up criterion (2.2) and Lemma 3, that $T_{\max} < \infty$. Thus, $T_{\max} = \infty.$
Next, we obtain \((1.6)\) under \(\chi < \chi_{d,\lambda}\) and \((1.5)\). Thanks to Lemma 4 and \((3.2)\)–\((3.3)\), it is sufficient to show that \(v\) has a uniform bound. When \(\sigma > 0\) and \(\inf_{t \geq 0} \psi(\cdot, t) > 0\) are satisfied, the uniform bound of \(v\) is obtained in \((3.2)\). In the case where \((\varphi(x, t), \psi(x, t)) = (0, b(t))\) for all \(x \in \Omega, t > 0\), using \((3.15)\), \((3.17)\), and Lemma 4, we compute

\[
\|v(t)\|_{L^\infty(\Omega)} \leq \|v_0\|_{L^\infty(\Omega)} + C \int_0^t e^{-(t-s)} \|uv^\lambda(s)\|_{L^\infty(\Omega)} + \|\psi(s)\|_{L^\infty(\Omega)} ds
\]

\[
\leq C + C \int_0^t e^{-(t-s)} \|u(s)\|_{L^\infty(\Omega)} \|v(s)\|_{L^\infty(\Omega)}^\lambda + \|\psi\|_{L^\infty(\Omega \times (0, \infty))} ds
\]

\[
\leq C + C \int_0^t e^{-(t-s)} ds \sup_{s \leq t} \|v(s)\|_{L^\infty(\Omega)}^\lambda \leq C + C \sup_{s \leq t} \|v(s)\|_{L^\infty(\Omega)}^\lambda,
\]

where \(C\) is a positive constant independent of \(t\). Then, taking supremum over time interval and using Young’s inequality, we can conclude that \(v\) is uniformly bounded. This completes the proof.

\[
\square
\]

4. Global well-posedness for small data

In this section, we prove Theorem 2 and Theorem 3. We prepare the lower and upper estimates for \(v\), when \(u \leq \frac{1}{2}\) and \(\min v_0 \geq \inf_{x \in \Omega, s \geq 0} \psi(x, s) > 0\). \(v\) is bounded above and bounded below away from zero.

**Lemma 5.** Let \(\Omega\) be a smooth, bounded and convex domain of \(\mathbb{R}^d, d \geq 2\). Suppose that \((u_0, v_0, \varphi, \psi)\) satisfies Assumption 4 and \(\sigma \geq 0, 0 \leq \lambda \leq 1\). Let \((u, v)\) be the solution to \((1.1)\) given by Lemma 7. If

\[
\|u\|_{L^\infty(0, T_{\max}; L^\infty(\Omega))} \leq \frac{1}{2}
\]

and

\[
\min_{\Omega} v_0 \geq \inf_{x \in \Omega, s \geq 0} \psi(x, s) > 0,
\]

then

\[
\inf_{x \in \Omega, s \geq 0} \psi(x, s) \leq v(x, t) \leq 2 \max\{\|v_0\|_{L^\infty(\Omega)}, \|\psi\|_{L^\infty(\Omega \times (0, \infty))}\}, \quad x \in \Omega, t < T_{\max}.
\]

**Proof.** Let \(t < T_{\max}\). Using the representation formula for \(\partial_t v + (\frac{1}{2} - \Delta) v = uv - \frac{1}{2} v + \psi\), and \(\|u\|_{L^\infty(0, T_{\max}; L^\infty(\Omega))} \leq \frac{1}{2}\), we compute

\[
v(t) = e^{t(\Delta - \frac{1}{2})} v_0 + \int_0^t e^{(t-s)(\Delta - \frac{1}{2})} (uv - \frac{1}{2} v + \psi)(s) ds
\]

\[
\leq e^{t(\Delta - \frac{1}{2})} v_0 + \int_0^t e^{(t-s)(\Delta - \frac{1}{2})} \psi(s) ds.
\]

Using \((3.15)\), it follows that

\[
\|v(t)\|_{L^\infty} \leq \|v_0\|_{L^\infty} e^{-\frac{t}{2}} + 2(1 - e^{-\frac{t}{2}}) \|\psi\|_{L^\infty(\Omega \times (0, \infty))}
\]

and thus, a desired upper bound for \(v\) is obtained. The desired lower bound for \(v\) is a consequence of \((2.5)\) in Lemma 2. This completes the proof.

\[
\square
\]
We are ready to prove Theorem 2 and Theorem 3. Again, we use the maximum principle. **Proof of Theorem 2.** Let \( t < T_{\text{max}} \) and \( z = u + \theta |\nabla \log v|^2 \), where \( \theta = 1 + \frac{\delta}{\bar{\delta}} \). We recall from (3.6)–(3.7) that \( \frac{\partial z}{\partial \nu} \leq 0 \) on \( \partial \Omega \), and

\[
\partial_t z - \Delta z - 2\nabla z \cdot \nabla \log v + \sigma v u + 2\theta |D^2 \log v|^2 + 2\theta \frac{\psi}{v} |\nabla \log v|^2 = -\chi u \Delta \log v + \varphi + \frac{2\theta}{v} \nabla \log v \cdot \nabla \psi.
\]

Using the pointwise estimate \( |\Delta f| \leq \sqrt{d} |D^2 f| \) for \( f \in C^2(\Omega) \) and Young’s inequality, we can compute

\[
-\chi u \Delta \log v \leq 2\theta |D^2 \log v|^2 + \frac{d\chi^2}{8\theta} u^2,
\]

and

\[
\frac{2\theta}{v} \nabla \log v \cdot \nabla \psi \leq \frac{\psi}{v} |\nabla \log v|^2 + \theta \frac{\nabla |\psi|^2}{\psi v}.
\]

Thus, we have

\[
\partial_t z - \Delta z - 2\nabla z \cdot \nabla \log v + \sigma v u + \frac{\psi}{v} |\nabla \log v|^2 \leq \frac{d\chi^2}{8\theta} u^2 + \varphi + \theta \frac{\nabla \psi|^2}{\psi v},
\]

and which entails by \( v \geq \eta, \psi \geq \eta \), and \( u \leq z \) that

\[
\partial_t z - \Delta z - 2\nabla z \cdot \nabla \log v + \eta \left( \sigma - \frac{d\chi^2}{8\theta \eta} \right) u + \frac{\eta}{v} \theta |\nabla \log v|^2 \leq \varphi + \theta \frac{\nabla \psi|^2}{\psi v}.
\]

Note that \( z_0 < \delta = \min \left\{ 1, \frac{8\eta}{\eta \sigma} \right\} \) by (1.8). Now, suppose that there exists \( x_\delta \in \Omega \) such that \( z(x_\delta, t_\delta) = \delta \) for the first time \( t_\delta < T_{\text{max}} \). Using the definition of \( \delta \) and Lemma 5, we have for \((x, t) = (x_\delta, t_\delta)\) that

\[
\eta \left( \sigma - \frac{d\chi^2}{8\theta \eta} \right) u + \frac{\eta}{v} \theta |\nabla \log v|^2 \\
\geq \frac{\eta}{2} \sigma u + \frac{\eta}{2} \max \{ \|v_0\|_{L^\infty(\Omega)}, \|\psi\|_{L^\infty(\Omega \times (0, \infty))} \} \theta |\nabla \log v|^2 \\
\geq \frac{\eta}{2} \min \left\{ \sigma, \frac{1}{\max \{ \|v_0\|_{L^\infty(\Omega)}, \|\psi\|_{L^\infty(\Omega \times (0, \infty))} \} } \right\} z \\
= \frac{\eta}{2} \max \{ \sigma^{-1}, \|v_0\|_{L^\infty(\Omega)}, \|\psi\|_{L^\infty(\Omega \times (0, \infty))} \} \delta
\]

which yields for \((x, t) = (x_\delta, t_\delta)\) that

\[
\partial_t z - \Delta z - 2\nabla z \cdot \nabla \log v \\
\leq \varphi + \frac{4\theta}{\eta} |\nabla \sqrt{\psi}|^2 - \frac{\eta}{2} \max \{ \sigma^{-1}, \|v_0\|_{L^\infty(\Omega)}, \|\psi\|_{L^\infty(\Omega \times (0, \infty))} \} \delta
\]

Note that by (1.9), the right-hand side is strictly negative. If \( x_\delta \) is an interior point of \( \Omega \), then the left-hand side of (4.3) is nonnegative. Thus, \( x_\delta \) is not an interior point of \( \Omega \). Let \( x_\delta \in \partial \Omega \). Then, by Hopf’s lemma type argument, \( \frac{\partial z}{\partial \nu} > 0 \) at \((x, t) = (x_\delta, t_\delta)\) but again, this leads to the contradiction since \( \frac{\partial z}{\partial \nu} \leq 0 \) on \( \partial \Omega \). Therefore, \( z < \delta \) for \( t < T_{\text{max}} \). Since \( u \leq z \), \( v \) has a uniform bound by Lemma 5 and \( \nabla v = v |\nabla \log v| \leq v \frac{\delta}{2} \) also has a uniform bound. Then, by (2.2) and \( \|\nabla v\|_{L^p(\Omega)} \leq C \|\nabla v\|_{L^\infty(\Omega)}, \) we obtain \( T_{\text{max}} = \infty \) and (1.6). This completes the proof.
Proof of Theorem 3. The proof is similar to the proof of Theorem 2. Let $t < T_{\text{max}}$ and $z = \frac{u}{v^{1-\lambda}} + \theta |\nabla \log v|^2$, where $\theta = \lambda + \frac{1}{2}$. Again, we recall from (3.6)–(3.7) that $\frac{\partial u}{\partial t} \leq 0$ on $\partial \Omega$, and

$$
\partial_t z - \Delta z - 2\nabla z \cdot \nabla \log v + (1 - \lambda)(\lambda + \chi) \left( \frac{u}{v^{1-\lambda}} \right) |\nabla \log v|^2 + (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right)^2
$$

$$
+ \sigma v \left( \frac{u}{v^{1-\lambda}} \right) + (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right) \frac{\psi}{v} + 2\theta |D^2 \log v|^2 + 2\theta \frac{\psi}{v} |\nabla \log v|^2
$$

$$
= -\chi \left( \frac{u}{v^{1-\lambda}} \right) \Delta \log v + (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right) + \frac{\varphi}{v^{1-\lambda}} + 2\theta \frac{\nabla \log v \cdot \nabla \psi}{v^2}.
$$

By (4.1)–(4.2), we have

$$
\partial_t z - \Delta z - 2\nabla z \cdot \nabla \log v + (1 - \lambda)(\lambda + \chi) \left( \frac{u}{v^{1-\lambda}} \right) |\nabla \log v|^2
$$

$$
+ \sigma v \left( \frac{u}{v^{1-\lambda}} \right) + (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right) \frac{\psi}{v} + \theta \frac{\psi}{v} |\nabla \log v|^2
$$

$$
\leq \left( \frac{d \chi^2}{8\theta} - (1 - \lambda) \right) \left( \frac{u}{v^{1-\lambda}} \right)^2 + (1 - \lambda) \left( \frac{u}{v^{1-\lambda}} \right) + \frac{\varphi}{v^{1-\lambda}} + \theta \frac{|\nabla \psi|^2}{v^2},
$$

which entails by $v \geq \eta, \psi \geq \eta$ and $\frac{u}{\eta^{1-\lambda}} \leq z$ that

$$
\partial_t z - \Delta z - 2\nabla z \cdot \nabla \log v
$$

$$
+ \left[ \sigma \eta - (1 - \lambda) - \left( \frac{d \chi^2}{8\theta} - (1 - \lambda) \right) \right] \left( \frac{u}{v^{1-\lambda}} \right) + \theta \frac{\eta}{v^2} |\nabla \log v|^2
$$

$$
\leq \frac{\varphi}{\eta^{1-\lambda}} + \frac{4\theta}{\eta} |\nabla \psi|^2.
$$

Now, note from (1.10) that

$$
z_0 < \delta_0 = \min \left\{ \frac{1}{2}, \frac{\sigma \eta}{2} \left( \frac{d \chi^2}{8\theta} - (1 - \lambda) \right)^{-1} \right\}.
$$

As in the proof of Theorem 2, we show $z < \delta_0$ for $t < T_{\text{max}}$. Suppose that there exists $(x_\delta, t_\delta) \in \Omega \times (0, T_{\text{max}})$ such that $z(x_\delta, t_\delta) = \delta_0$ for the first time. Using the definition of $\delta_0$, $\sigma \eta \geq 4(1 - \lambda)$, and Lemma 5, we have for $(x, t) = (x_\delta, t_\delta)$ that

$$
\left[ \sigma \eta - (1 - \lambda) - \left( \frac{d \chi^2}{8\theta} - (1 - \lambda) \right) \right] \left( \frac{u}{v^{1-\lambda}} \right) + \theta \frac{\eta}{v^2} |\nabla \log v|^2
$$

$$
\geq \left[ \sigma \eta - (1 - \lambda) - \frac{\sigma \eta}{2} \right] \left( \frac{u}{v^{1-\lambda}} \right) + \theta \frac{\eta}{v^2} |\nabla \log v|^2
$$

$$
\geq \frac{\eta \sigma}{4} \left( \frac{u}{v^{1-\lambda}} \right) + \frac{\eta}{2 \max \left\{ \|v_0\|_{L^\infty(\Omega)}, \|\psi\|_{L^\infty(\Omega \times (0, \infty))} \right\}} \theta |\nabla \log v|^2
$$

$$
\geq \frac{\eta}{2} \min \left\{ \frac{\sigma}{2}, \frac{1}{\max \left\{ \|v_0\|_{L^\infty(\Omega)}, \|\psi\|_{L^\infty(\Omega \times (0, \infty))} \right\}} \right\} \frac{\delta_0}{2 \max \{2\sigma^{-1}, \|v_0\|_{L^\infty(\Omega)}, \|\psi\|_{L^\infty(\Omega \times (0, \infty))} \}},
$$

which implies for $(x, t) = (x_\delta, t_\delta)$ that

$$
\partial_t z - \Delta z - 2\nabla z \cdot \nabla \log v
$$

$$
\leq \frac{\varphi}{\eta^{1-\lambda}} + \frac{4\theta}{\eta} |\nabla \psi|^2 - \frac{\eta}{2} \frac{\delta_0}{2 \max \{2\sigma^{-1}, \|v_0\|_{L^\infty(\Omega)}, \|\psi\|_{L^\infty(\ Omega \times (0, \infty))}, \frac{\delta_0}{2 \max \{2\sigma^{-1}, \|v_0\|_{L^\infty(\Omega)}, \|\psi\|_{L^\infty(\Omega \times (0, \infty))}}\).
Then, the right-hand side is negative by (1.11). The remaining part of the proof is very similar to that of Theorem 2, and thus the details are omitted. This completes the proof.

5. Long-time asymptotics

In this section, we prove the long time behavior result, Theorem 4. To this end, first, we estimate fractional norms of $u$ and $v$ and using them, we compute $L^\infty$-norms of $u$ and $v$. Below, $A$ denotes the sectorial realization of $-\Delta + 1$ in $L^r(\Omega)$ with $1 < r < \infty$ under homogeneous Neumann boundary condition, and $A^\beta$ with $\beta \in (0,1)$ denotes the fractional power of $A$ (see, e.g. [10] Section 1.4).

**Lemma 6.** Let $\Omega$ be a smooth, bounded and convex domain of $\mathbb{R}^d$, $d \geq 2$, and $\sigma > 0$, $0 \leq \lambda \leq 1$. Suppose that $(u, v)$ is a unique global classical solution to (1.1) satisfying (1.6). Then, for $r > d$ and $\frac{d}{2r} < \beta < \frac{1}{2}$, there exist a positive constant $C$ independent of $t$ such that

$$\|A^\beta u(\cdot, t)\|_{L^r(\Omega)} + \|A^\beta v(\cdot, t)\|_{L^r(\Omega)} \leq C \text{ for all } t > 0. \quad (5.1)$$

Moreover, there exist $\gamma \in (0, 1)$ and a positive constant $C$ independent of $t$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C\|u(\cdot, t)\|_{L^r(\Omega)^{\gamma/(1 - \gamma)}}^{1/(1 - \gamma)} \leq C \text{ for all } t > 0, \quad (5.2)$$

$$\|v(\cdot, t) - v_\infty\|_{L^\infty(\Omega)} \leq C\|v(\cdot, t) - v_\infty\|_{L^2(\Omega)^{2\gamma/(1 - \gamma)}}^{1/(1 - \gamma)} \leq C \text{ for all } t > 0, \quad (5.3)$$

where $v_\infty$ denotes the solution for (1.14).

**Proof.** First, we show (5.1). Let $r > d \geq 2$ and $\frac{d}{2r} < \beta < \frac{1}{2}$. Using the representation formula of $u$, we compute

$$\|A^\beta u(t)\|_{L^r(\Omega)} \leq \|A^\beta e^{t(\Delta - 1)}u_0\|_{L^r(\Omega)} + \int_0^t \|A^\beta e^{(t-s)(\Delta - 1)}(-\chi \nabla \cdot (u\nabla \log v) + u - \sigma uv + \varphi)(s)\|_{L^r(\Omega)} ds$$

$$\leq C\|u_0\|_{W^{1,r}(\Omega)} + C\int_0^t e^{-(t-s)} \left(1 + (t-s)^{\beta - \frac{3}{2}}\right) \|u\nabla \log v(s)\|_{L^r(\Omega)} ds$$

$$+ C\int_0^t e^{-(t-s)} \left(1 + (t-s)^{\beta}\right) \left(\|u(s)\|_{L^r(\Omega)} + \|uv(s)\|_{L^r(\Omega)} + \|\varphi(s)\|_{L^r(\Omega)}\right) ds$$

$$\leq C + C\|u\nabla \log v\|_{L^\infty(\Omega \times (0,\infty))} \int_0^t e^{-(t-s)} \left(1 + (t-s)^{\beta - \frac{3}{2}}\right) ds$$

$$+ C \left(\|u\|_{L^\infty(\Omega \times (0,\infty))} + \|uv\|_{L^\infty(\Omega \times (0,\infty))} + \|\varphi\|_{L^\infty(\Omega \times (0,\infty))}\right) \int_0^t e^{-(t-s)} \left(1 + (t-s)^{\beta}\right) ds \leq C,$$
where $C > 0$ is a constant independent of $t$. Similarly, it follows from the representation formula of $v$
that
\[
\|A^\beta v(t)\|_{L^r(\Omega)} \leq \|A^\beta e^{t(\Delta-1)} v_0\|_{L^r(\Omega)} + \int_0^t \|A^\beta e^{(t-s)(\Delta-1)} (uv^\lambda + \psi) (s)\|_{L^r(\Omega)} ds.
\]
Thus, (5.2) is obtained. Next, we note from (1.14) and the standard elliptic regularity theory
interpolation inequality as above, due to
\[
D u = \gamma, 0 \text{ is a constant independent of } C,
\]
we have
\[
\|u\|_{L^\infty(\Omega)} \leq C\|A^{\beta(1-\gamma)} u\|_{L^r(\Omega)} \leq C\|A^\beta u\|_{L^r(\Omega)}^{1-\gamma} \|u\|_{L^r(\Omega)}^{\gamma}.
\]
It follows by (5.11) and the interpolation inequality $\|u\|_{L^r(\Omega)} \leq \|u\|_{L^1(\Omega)}^{1/(1-1/\gamma)} \|u\|_{L^\infty(\Omega)}^{1/(1-1/\gamma)}$ that
\[
\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{L^1(\Omega)}^{\gamma}.
\]
Thus, (5.2) is obtained. Next, we note from (1.13) and the standard elliptic regularity theory that $v_\infty \in W^{2,p}(\Omega)$ for any finite $p > 1$. Using the same embedding relation and the fractional interpolation inequality as above, due to
\[
\|v(\cdot, t) - v_\infty\|_{L^r(\Omega)} \leq \|v(\cdot, t) - v_\infty\|_{L^2(\Omega)}^{\gamma} \|v(\cdot, t) - v_\infty\|_{L^\infty(\Omega)}^{1-\gamma},
\]
we have
\[
\|v(\cdot, t) - v_\infty\|_{L^\infty(\Omega)} \leq C\|A^{-\gamma} (v(\cdot, t) - v_\infty)\|_{L^r(\Omega)} \leq C\|A^{\beta(1-\gamma)} (v(\cdot, t) - v_\infty)\|_{L^r(\Omega)} \leq C\|A^\beta (v(\cdot, t) - v_\infty)\|_{L^r(\Omega)}^{1-\gamma} \|v(\cdot, t) - v_\infty\|_{L^r(\Omega)}^{\gamma} \leq C\|v(\cdot, t) - v_\infty\|_{L^2(\Omega)}^{\gamma} \|v(\cdot, t) - v_\infty\|_{L^\infty(\Omega)}^{1-\gamma}
\]
and thus,
\[
\|v(\cdot, t) - v_\infty\|_{L^\infty(\Omega)} \leq C\|v(\cdot, t) - v_\infty\|_{L^2(\Omega)}^{\gamma},
\]
where $C > 0$ is a constant independent of $t$. Namely, (5.3) is obtained. This completes the proof.

Now, we are ready to prove Theorem 4. Due to (1.13) and $-uv$ term in $u$-equation, $(u, v)$ converges to $(0, v_\infty)$ as times goes to infinity.

Proof of Theorem 4. First, we show the convergence $u \to 0$. Integrating the first equation of (1.1) over $\Omega$ yields
\[
\frac{d}{dt} \int_\Omega u + \sigma \int_\Omega uv = \int_\Omega \varphi.
\]
Using $v \geq \eta_0$, we have
\[ \frac{d}{dt} \int_{\Omega} u + \eta_0 \int_{\Omega} u \leq \int_{\Omega} \varphi, \]
and thus,
\[ \frac{d}{dt} \left( e^{\eta_0 t} \int_{\Omega} u \right) \leq e^{\eta_0 t} \int_{\Omega} \varphi. \]
Integrating this with respect to the temporal variable over $(\tau, 2\tau)$ for $\tau > 0$ yields
\[ e^{\eta_0 2\tau} \int_{\Omega} u(\cdot, 2\tau) \leq e^{\eta_0 \tau} \int_{\Omega} u(\cdot, \tau) + \int_{\tau}^{2\tau} e^{\eta_0 \sigma t} \int_{\Omega} \varphi(\cdot, t) dt \]
\[ \leq e^{\eta_0 \tau} \int_{\Omega} u(\cdot, \tau) + e^{\eta_0 \sigma t} \int_{\tau}^{2\tau} \int_{\Omega} \varphi(\cdot, t) dt \]
Then,
\[ \int_{\Omega} u(\cdot, 2\tau) \leq e^{-\eta_0 \tau} \int_{\Omega} u(\cdot, \tau) + \int_{\tau}^{\infty} \int_{\Omega} \varphi(\cdot, t) dt. \]
By (1.6) and the spatio-temporal $L^1$-bound of $\varphi$, we have that for any given $\varepsilon_0 > 0$, there exists $\tau > 0$ satisfying
\[ \int_{\Omega} u(\cdot, t) \leq \varepsilon_0 \quad \text{for all } t \geq \tau. \]
Hence, $\int_{\Omega} u$ converges to 0 as time goes to infinity and thus, by (5.2), we obtain the desired convergence
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty. \] (5.4)

Next, we show the convergence $v \to v_\infty$. Note from (1.14) and the standard elliptic regularity theory that $v_\infty \in W^{2,p}(\Omega)$ for any finite $p > 1$. Using (1.11) and (1.14), we can derive
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} |v(\cdot, t) - v_\infty|^2 + \int_{\Omega} |\nabla (v(\cdot, t) - v_\infty)|^2 + \int_{\Omega} |v(\cdot, t) - v_\infty|^2 \]
\[ = \int_{\Omega} (v(\cdot, t) - v_\infty)u v^\lambda + \int_{\Omega} (v(\cdot, t) - v_\infty)(\psi - \psi_\infty) \]
which entails by Young's inequality that
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} |v(\cdot, t) - v_\infty|^2 + \frac{1}{2} \int_{\Omega} |v(\cdot, t) - v_\infty|^2 \leq \int_{\Omega} u^2 v^{2\lambda} + \int_{\Omega} |\psi - \psi_\infty|^2. \]
By (1.6) and a direct computation, we obtain
\[ \frac{d}{dt} \left( e^t \int_{\Omega} |v(\cdot, t) - v_\infty|^2 \right) \leq C e^t \|u(\cdot, t)\|_{L^\infty}^2 + 2 e^t \int_{\Omega} |\psi - \psi_\infty|^2 \]
for some constant $C > 0$ independent of $t$. Integrating this with respect to the temporal variable over $(\tau, 2\tau)$ with $\tau > 0$ yields
\[ e^{2\tau} \int_{\Omega} |v(\cdot, 2\tau) - v_\infty|^2 \]
\[ \leq e^\tau \int_{\Omega} |v(\cdot, \tau) - v_\infty|^2 + C(e^{2\tau} - e^\tau) \sup_{\tau \leq \tau \leq 2\tau} \|u(\cdot, t)\|_{L^\infty}^2 + 2 e^{2\tau} \int_{\tau}^{2\tau} \int_{\Omega} |\psi - \psi_\infty|^2 \]
and therefore,
\[ \int_{\Omega} |v(\cdot, 2\tau) - v_\infty|^2 \]
\[ \leq e^{-\tau} \int_{\Omega} |v(\cdot, \tau) - v_\infty|^2 + C \sup_{\tau \geq \tau} \|u(\cdot, t)\|_{L^\infty}^2 + 2 \int_{\tau}^{\infty} \int_{\Omega} |\psi - \psi_\infty|^2. \]
Due to (1.6), (1.13), and (5.4), for any given $\varepsilon_0 > 0$, there exists $\tau > 0$ satisfying

$$\int_{\Omega} |v(\cdot, t) - v_\infty|^2 \leq \varepsilon_0 \quad \text{for all } t \geq \tau.$$ 

Hence, $L^2(\Omega)$-norm of $v - v_\infty$ approaches 0 as time tends to infinity and thus, due to (5.3), we obtain the desired convergence

$$\|v(\cdot, t) - v_\infty\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty.$$ 

This completes the proof.

\[\square\]

ACKNOWLEDGEMENT

J. Ahn is supported by the Dongguk University Research Fund of 2020. K. Kang is partially supported by NRF-2019R1A2C1084685 and NRF-2015R1A5A1009350. J. Lee is supported by Samsung Science and Technology Foundation under Project Number SSTF-BA1701-05.

REFERENCES

[1] J. Ahn, Global well-posedness and asymptotic stabilization for chemotaxis system with signal-dependent sensitivity, Journal of Differential Equations 266 (2019) 6866–6904, http://dx.doi.org/10.1016/j.jde.2018.11.015
[2] J. Ahn, K. Kang, J. Lee, Eventual smoothness and stabilization of global weak solutions in parabolic-elliptic chemotaxis systems with logarithmic sensitivity, Nonlinear Analysis: Real World Applications 49 (2019) 312–330, http://dx.doi.org/10.1016/j.nonrwa.2019.03.012
[3] P. Biler, Global solutions to some parabolic–elliptic systems of chemotaxis, Advances in Mathematical Sciences and Applications 9 (1999) 347–359.
[4] T. Black, Global generalized solutions to a parabolic–elliptic Keller–Segel system with singular sensitivity, Discrete & Continuous Dynamical Systems - S 13 (2020) 119–137, http://dx.doi.org/10.3934/dcdss.2020007
[5] M. Freitag, Global solutions to a higher-dimensional system related to crime modeling, Mathematical Methods in the Applied Sciences 41 (2018) 6326–6335, http://dx.doi.org/10.1002/mma.5141
[6] K. Fujie, Boundedness in a fully parabolic chemotaxis system with singular sensitivity, Journal of Mathematical Analysis and Applications 424 (2015) 675–684, http://dx.doi.org/10.1016/j.jmaa.2014.11.045
[7] K. Fujie, T. Senba, Global existence and boundedness in a parabolic-elliptic Keller-Segel system with general sensitivity, Discrete & Continuous Dynamical Systems - B 21 (2016) 81–102, http://dx.doi.org/10.3934/dcdsb.2016.21.81
[8] K. Fujie, T. Senba, A sufficient condition of sensitivity functions for boundedness of solutions to a parabolic–parabolic chemotaxis system, Nonlinearity 13 (2018) 1639–1672, http://dx.doi.org/10.1088/1361-6544/aac2bf
[9] K. Fujie, M. Winkler, T. Yokota, Boundedness of solutions to parabolic–elliptic Keller–Segel systems with signal-dependent sensitivity, Mathematical Methods in the Applied Sciences 38 (2015) 1212–1224, http://dx.doi.org/10.1002/mma.3149.
[10] D. Henry, Geometric theory of semilinear parabolic equations, Lecture Notes in Mathematics 840 (1981)
[11] E. F. Keller, L.A. Segel, Model for chemotaxis, Journal of Theoretical Biology 30 (1971) 225–234, http://dx.doi.org/10.1016/0022-5193(71)90050-6
[12] O. Ladyzhenskaya, V. Solonnikov, N. Ural’ceva, Linear and Quasilinear Equations of Parabolic Type, Transl. Math. Monogr. 23, AMS, Providence, RI, (1988)
[13] J. Lankeit, A new approach toward boundedness in a two-dimensional parabolic chemotaxis system with singular sensitivity, Mathematical Methods in the Applied Sciences 39 (2016) 394–404, http://dx.doi.org/10.1002/mma.3489
[14] J. Lankeit, M. Winkler, A generalized solution concept for the Keller–Segel system with logarithmic sensitivity: global solvability for large nonradial data, NoDEA Nonlinear Differential Equations Appl. 24 (2017) 24–49, http://dx.doi.org/10.1007/s00030-017-0472-8
[15] T. Nagai, T. Senba, Global existence and blow-up of radial solutions to a parabolic–elliptic system of chemotaxis, Advances in Mathematical Sciences and Applications 8 (1998) 145–156.
[16] T. Padmanabhan, Statistical mechanics of gravitating systems, Physics Reports 188 (1990) 285–362, http://dx.doi.org/10.1016/0370-1573(90)90051-3
[17] N. Rodríguez, On the global well-posedness theory for a class of PDE models for criminal activity, Physica D: Nonlinear Phenomena 260 (2013) 191–200, http://dx.doi.org/10.1016/j.physd.2012.08.003

[18] N. Rodríguez, A. Bertozzi, Local existence and uniqueness of solutions to a PDE model for criminal behavior, Mathematical Models and Methods in the Applied Sciences 20 (2010) 1425–1457, http://dx.doi.org/10.1142/s0218202510004696

[19] N. Rodríguez, M. Winkler, On the global existence and qualitative behavior of one-dimensional solutions to a model for urban crime, ArXiv (2019)

[20] J. Shen, B. Li, Mathematical analysis of a continuous version of statistical model for criminal behavior, Mathematical Methods in the Applied Sciences 43 (2019) 409–426, http://dx.doi.org/10.1002/mma.5898

[21] M. B. Short, A. L. Bertozzi, P. J. Brantingham, Nonlinear patterns in urban crime: hotspots, bifurcations, and suppression, SIAM Journal on Applied Dynamical Systems 9 (2010) 462–483, http://dx.doi.org/10.1137/090759069

[22] M. B. Short, P. J. Brantingham, A. L. Bertozzi, G. E. Tita, Dissipation and displacement of hotspots in reaction-diffusion models of crime, Proceedings of the National Academy of Sciences 107 (2010) 3961–3965, http://dx.doi.org/10.1073/pnas.0910921107

[23] M. B. Short, M. R. D’Orsogna, V. B. Pasour, G. E. Tita, P. J. Brantingham, A. L. Bertozzi, L. B. Chayes, A statistical model of criminal behavior, Mathematical Models and Methods in the Applied Sciences 18 (2008) 1249–1267, http://dx.doi.org/10.1142/s0218202508003029

[24] C. Stinner, M. Winkler, Global weak solutions in a chemotaxis system with large singular sensitivity, Nonlinear Analysis: Real World Applications 12 (2011) 3727–3740, http://dx.doi.org/10.1016/j.nonrwa.2011.07.006

[25] Y. Tao, M. Winkler, Global smooth solutions in a two-dimensional cross-diffusion system modeling propagation of urban crime, ArXiv (2020)

[26] Q. Wang, D. Wang, Y. Feng, Global well-posedness and uniform boundedness of urban crime models: One-dimensional case, Journal of Differential Equations 269 (2020) 6216–6235, http://dx.doi.org/10.1016/j.jde.2020.04.035

[27] M. Winkler, Global solutions in a fully parabolic chemotaxis system with singular sensitivity, Mathematical Methods in the Applied Sciences 34 (2011) 176–190, http://dx.doi.org/10.1002/mma.1346.

[28] M. Winkler, Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening, Journal of Differential Equations 257 (2014) 1056–1077, http://dx.doi.org/10.1016/j.jde.2014.04.023

[29] M. Winkler, Global solvability and stabilization in a two-dimensional cross-diffusion system modeling urban crime propagation. Annales de l’Institut Henri Poincaré C, Analyse non linéaire 36 (2019) 1747–1790, http://dx.doi.org/10.1016/j.anihpc.2019.02.004

[30] M. Winkler, T. Yokota, Stabilization in the logarithmic Keller–Segel system, Nonlinear Analysis 170 (2018) 123–141, http://dx.doi.org/10.1016/j.na.2018.01.002

[31] Y. Yang, H. Chen, W. Liu, On existence of global solutions and blow-up to a system of reaction-diffusion equations modelling chemotaxis, SIAM Journal on Mathematical Analysis 33 (2001) 763–785, http://dx.doi.org/10.1137/s0036141000337796

Department of Mathematics, Dongguk University, Seoul 04620, Republic of Korea

Email address: jaewookahn@dgu.ac.kr

School of Mathematics & Computing(Mathematics), Yonsei University, Seoul 03722, Republic of Korea

Email address: kkang@yonsei.ac.kr

Department of Mathematics, Chung-Ang University, Seoul 06974, Republic of Korea

Email address: jhleepde@cau.ac.kr