MULTIVARIABLE LUBIN-TATE $(\varphi, \Gamma)$-MODULES
AND FILTERED $\varphi$-MODULES

by

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Abstract. — We define some rings of power series in several variables, that are attached
to a Lubin-Tate formal module. We then give some examples of $(\varphi, \Gamma)$-modules over those
rings. They are the global sections of some reflexive sheaves on the $p$-adic open unit polydisk,
that are constructed from a filtered $\varphi$-module using a modification process. We prove that
we obtain every crystalline $(\varphi, \Gamma)$-module over those rings in this way.

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Introduction

Let $F$ be the unramified extension of $\mathbb{Q}_p$ of degree $h$ and let $q = p^h$ so that the residue
field of $\mathcal{O}_F$ is $\mathbb{F}_q$. We fix an embedding $F \subset \mathbb{Q}_p$ so that if $\sigma : F \to F$ denotes the
absolute Frobenius map, which lifts $x \mapsto x^p$ on $\mathbb{F}_q$, then the $h$ embeddings of $F$ into $\overline{\mathbb{Q}}_p$

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are given by $\text{Id}, \sigma, \ldots, \sigma^{h-1}$. The symbol $\varphi_q$ denotes a $\sigma^h$-semilinear Frobenius map. If $K$ is a subfield of $\overline{\mathbb{Q}}_p$, then let $G_K = \text{Gal}(\overline{\mathbb{Q}}_p/K)$.

The goal of this article is to present a first attempt to construct some “multivariable Lubin-Tate $(\varphi, \Gamma)$-modules”, that is some $(\varphi_q, \Gamma_F)$-modules over rings of power series in $h$ variables, on which $\Gamma_F = \mathcal{O}_F^\times$ acts by a formula arising from a Lubin-Tate formal $\mathcal{O}_F$-module. A construction of such $(\varphi_q, \Gamma_F)$-modules, but “in one variable”, was carried out by Kisin and Ren in [KR09]: they prove that in certain cases, the $(\varphi_q, \Gamma_F)$-modules arising from Fontaine’s standard construction of [Fon90] are overconvergent. In order to do so, Kisin and Ren adapt the construction of $(\varphi, \Gamma)$-modules attached to filtered $(\varphi, N)$-modules given in [Ber08b] to their setting, which allows them to attach a $(\varphi_q, \Gamma_F)$-module in one variable to a filtered $\varphi_q$-module. They then point out in the introduction of [KR09] that “it seems likely that in order to obtain a classification valid for any crystalline $G_K$-representation one needs to consider higher dimensional subrings of $W(\text{Fr} R)$, constructed using the periods of all the conjugates of [the Lubin-Tate group]”.

The motivation for these computations is the hope that we can construct some representations of the Borel subgroup of $\text{GL}_2(F)$, for example using the recipe given by Colmez in [Col10], that would shed some light on the $p$-adic local Langlands correspondence for $\text{GL}_2(F)$ (see [Bre10]). Theorems A, B and C below are a very first step in this direction, but remain insufficient. In particular, the “$p$-adic Fourier theory” of Schneider and Teitelbaum (see [ST01]) will very likely play an important role in the sequel.

We now describe our results in more detail. Let $\text{LT}_h$ be the Lubin-Tate formal $\mathcal{O}_F$-module for which multiplication by $p$ is given by $[p](T) = pT + T^q$. We denote by $[a](T)$ the element of $\mathcal{O}_F[T]$ that gives the action of $a \in \mathcal{O}_F$ on $\text{LT}_h$. We consider two rings $\mathcal{R}^+(Y)$ and $\mathcal{R}(Y)$ of power series in the $h$ variables $Y_0, \ldots, Y_{h-1}$, with coefficients in $F$. The ring $\mathcal{R}^+(Y)$ is the ring of power series that converge on the open unit polydisk, and $\mathcal{R}(Y)$ is the Robba ring that corresponds to it, by adapting Schneider’s construction given in the appendix of [Záb12]. The action of the group $\mathcal{O}_F^\times$ on those rings is given by the formula $a(Y_j) = [\sigma^j(a)](Y_j)$, and the Frobenius map $\varphi_q$ is given by $\varphi_q(Y_j) = [p](Y_j)$.

The construction of $p$-adic periods for Lubin-Tate groups gives rise to a map $\mathcal{R}^+(Y) \to \tilde{B}_{\text{rig}}^+$, where $\tilde{B}_{\text{rig}}^+$ is the Fréchet completion of $\tilde{B}^+ = W(\tilde{E}^+)[1/p]$, and we prove (corollary 3.7) that this map is in fact injective (remark: if $\tilde{R}^+(Y)$ denotes the completion of the perfection of $\mathcal{R}^+(Y)$, then the map above extends to a map $\tilde{R}^+(Y) \to \tilde{B}_{\text{rig}}^+$ but note that, by the theory of the field of norms of [FW79] and [Win83], this latter map is not injective anymore if $h \geq 2$. This has prevented us from studying étale $\varphi_q$-modules using Kedlaya’s methods, so such considerations are absent from this article).
Let $D$ be a finite dimensional $F$-vector space, endowed with an $F$-linear Frobenius map $\varphi_q : D \to D$, and an action of $G_F$ on $D$ that factors through $\Gamma_F$ and commutes with $\varphi_q$. For each $0 \leq j \leq h - 1$, let $\text{Fil}_j^\ast$ be a filtration on $D$ that is stable under $\Gamma_F$.

For example, if $V$ is an $F$-linear crystalline representation of $G_F$ of dimension $d$, then $D_{\text{cris}}(V)$ is a free $F \otimes_{Q_p} F$-module of rank $d$, and we have

$$D_{\text{cris}}(V) = D \oplus \varphi(D) \oplus \cdots \oplus \varphi^{h-1}(D),$$

according to the decomposition of $F \otimes_{Q_p} F$ as $\prod_{\sigma : F \to F} F$. Each $\varphi^j(D)$ has the filtration induced from $D_{\text{cris}}(V)$, and we set $\text{Fil}_j^D = \varphi^{-j}(\text{Fil}^D_{\text{cris}}(V))$.

The composite of the map $\mathcal{R}^{+}(Y) \to \hat{B}_{\text{rig}}^{+}$ with the map $\varphi^{-k} : \hat{B}_{\text{rig}}^{+} \to \hat{B}_{\text{rig}}^{+}$ gives rise to a map $\iota_k : \mathcal{R}^{+}(Y) \to \hat{B}_{\text{rig}}^{+}$. Let $\log_{LT}(T)$ be the logarithm of $LT_h$, and let $\lambda_j = \log_{LT}(Y_j)/Y_j$ and $\lambda = \prod_{j=0}^{h-1} \lambda_j$ (note that the image of $\prod_{j=0}^{h-1} \log_{LT}(Y_j)$ in $\hat{B}_{\text{rig}}^{+}$ is some $Q_p$-multiple of $t = \log(1 + X)$, so that $\lambda$ is an analogue of $t/X$). Define

$$M^{+}(D) = \{ y \in \mathcal{R}^{+}(Y)[1/\lambda] \otimes_F D, \ \iota_k(y) \in \text{Fil}_{-k}^D(B_{\text{dr}} \otimes_{p^{-k}} D) \text{ for all } k \geq h \}.$$

The ring $\mathcal{R}^{+}(Y)$ is a Fréchet-Stein algebra in the sense of [ST03], and we therefore have the notion of coadmissible $\mathcal{R}^{+}(Y)$-modules, which are the global sections of coherent sheaves on the open unit polydisk.

**Theorem A.** — The module $M^{+}(D)$ is a reflexive coadmissible $\mathcal{R}^{+}(Y)$-module, for all $0 \leq j \leq h - 1$, $M^{+}(D)[\lambda_j/\lambda]$ is a free $\mathcal{R}^{+}(Y)[\lambda_j/\lambda]$-module of rank $d$, and we have $M^{+}(D) = \cap_{j=0}^{h-1} M^{+}(D)[\lambda_j/\lambda]$.

The definition of $M^{+}(D)$ is analogous to the one given in [Ber08b], [KR09] and similar articles. When $h = 1$, the proof of theorem A relies on the fact that $M^{+}(D)$ can be seen as a vector bundle on the open unit disk. Our proof of theorem A relies on the one dimensional case, and on the interpretation of $M^{+}(D)$ as the global sections of a coherent sheaf on the open unit polydisk.

**Remark.** — If $h \leq 2$, then $\mathcal{R}^{+}(Y)$ is of dimension $\leq 2$ and one can then prove that $M^{+}(D)$, being reflexive, is actually free of rank $d$ (see remark 5.7). If $h \geq 3$, I do not know whether $M^{+}(D)$ is free of rank $d$ in general, nor even if it is finitely generated.

Let $M(D) = \mathcal{R}(Y) \otimes_{\mathcal{R}^{+}(Y)} M^{+}(D)$, so that $M(D)$ is a $(\varphi_q, \Gamma_F)$-module over the multivariable Robba ring $\mathcal{R}(Y)$ (see definition 6.4).

**Theorem B.** — If $V$ is an $F$-linear crystalline representation of $G_F$, and if $D$ arises from $D_{\text{cris}}(V)$ as above, then there is a natural map $\hat{B}_{\text{rig}}^{+} \otimes_{\mathcal{R}(Y)} M(D) \to \hat{B}_{\text{rig}}^{+} \otimes_{F} V$, and this map is an isomorphism.
If \( M \) is a \((\varphi_q, \Gamma_F)\)-module over \( R(Y) \), then we set \( D_{\text{cris}}(M) = (R(Y)[1/t] \otimes_{R(Y)} M)^{\Gamma_F} \), and we say that \( M \) is crystalline if \( M[\lambda_j/\lambda] \) is a free \( R(Y)[\lambda_j/\lambda] \)-module of some rank \( d \) for all \( j \), \( M = \bigcap_{j=1}^{h-1} M[\lambda_j/\lambda] \), and \( \dim D_{\text{cris}}(M) = d \). For example, if \( D \) is a filtered \( \varphi_q \)-module with \( h \) filtrations \( \text{Fil}_h^* \) as above, on which the action of \( G_F \) is trivial, then \( M(D) \) is a crystalline \((\varphi_q, \Gamma_F)\)-module.

**Theorem C.** — The functors \( M \mapsto D_{\text{cris}}(M) \) and \( D \mapsto M(D) \), between the category of crystalline \((\varphi_q, \Gamma_F)\)-modules over \( R(Y) \) and the category of \( \varphi_q \)-modules with \( h \) filtrations, are mutually inverse.

Note that if \( h = 1 \), then the \((\varphi, \Gamma)\)-modules that we construct are the classical cyclotomic ones, and theorems A, B and C are well-known.

We now give a short description of the contents of this article: in \( \S \) we give some reminders about the \( p \)-adic periods of Lubin-Tate formal \( O_F \)-modules. In \( \S 2 \), we define the various rings of power series that we use, and establish some of their properties. In \( \S 3 \) we embed those rings in the usual rings of \( p \)-adic periods. In \( \S 4 \) we briefly survey Kisin and Ren’s construction and explain why \((\varphi_q, \Gamma_F)\)-modules over rings of power series in several variables are needed. In \( \S 5 \) we attach such objects to filtered \( \varphi_q \)-modules and prove theorem A. In \( \S 6 \) we prove theorem B. In \( \S 7 \) we study crystalline \((\varphi_q, \Gamma_F)\)-modules and prove theorem C.

## 1. Periods of Lubin-Tate formal groups

Let \( LT_h \) be the Lubin-Tate formal \( O_F \)-module for which multiplication by \( p \) is given by \([p](T) = pT + T^q\). We denote by \([a](T)\) the element of \( O_F[[T]] \) that gives the action of \( a \in O_F \) on \( LT_h \) and by \( S(T, U) = T \oplus U \) the element of \( O_F[[T, U]] \) that gives addition.

Let \( \pi_0 = 0 \) and for each \( n \geq 1 \), let \( \pi_n \in \overline{Q}_p \) be such that \([p](\pi_n) = \pi_{n-1} \), with \( \pi_1 \neq 0 \). We have \( \text{val}_p(\pi_n) = 1/q^{n-1}(q-1) \) if \( n \geq 1 \). Let \( F_n = F(\pi_n) \) and \( F_\infty = \cup_{n \geq 1} F_n \). Recall that \( \text{Gal}(F_\infty/F) \simeq O_F^\times \) and that the maximal abelian extension of \( F \) is \( F_\infty \cdot F^{\text{unr}} \). Denote by \( H_F \) the group \( \text{Gal}(\overline{Q}_p/F_\infty) \), by \( \Gamma_F \) the group \( \text{Gal}(F_\infty/F) \) and by \( \chi_{LT} \) the isomorphism \( \chi_{LT} : \Gamma_F \rightarrow O_F^\times \). In the sequel, we sometimes directly identify \( \Gamma_F \) with \( O_F^\times \), that is we drop \( \chi_{LT} \) from the notation to make the formulas less cumbersome.

Let \( \widehat{E}^+ = \lim_{\xleftarrow{n \to \infty}} O_{C_p}/p \) and \( \widehat{A}^+ = W(\widehat{E}^+) \) denote Fontaine’s rings of periods (see \cite{Fon94}). Note that we take the limit with respect to the maps \( x \mapsto x^n \), which does not change the rings. Let \( \varphi_q : \widehat{A}^+ \to \widehat{A}^+ \) be given by \( \varphi_q = \varphi^h \). Recall that in §9.2 of \cite{Col02}, Colmez has constructed a map \( \{\cdot\} : \widehat{E}^+ \to \widehat{A}^+ \) having the following property: if \( x \in \widehat{E}^+ \), then \( \{x\} \) is the unique element of \( \widehat{A}^+ \) that lifts \( x \) and satisfies \( \varphi_q(\{x\}) = [p](\{x\}) \).
Let \( \theta : \tilde{A}^+ \to \mathcal{O}_{C_p} \) denote Fontaine’s map (see [Fon94]). If \( x = (x_0, x_1, \ldots) \), then
\[
\theta(\{x\}) = \lim_{n \to \infty} [p^n] (\tilde{x}_n),
\]
where \( \tilde{x}_n \in \mathcal{O}_{C_p} \) is any lift of \( x_n \).

Let \( u = \{(\pi_0, \pi_1, \ldots) \} \in \tilde{A}^+ \), so that \( g(u) = [\chi_{LT}(g)](u) \) if \( g \in G_F \).

Let \( \log_{LT}(T) \in F[T] \) denote the Lubin-Tate logarithm map, which converges on the open unit disk and satisfies \( \log_{LT}([a](T)) = a \cdot \log_{LT}(T) \) if \( a \in \mathcal{O}_F \). Recall (see §9.3 of [Co02]) that \( \log_{LT}(u) \) converges in \( \mathfrak{B}_{rig}^+ \) to an element \( t_F \) which satisfies \( g(t_F) = \chi_{LT}(g) \cdot t_F \).

Let \( Q_k(T) \) be the minimal polynomial of \( \pi_k \) over \( F \). We have \( Q_0(T) = T \), \( Q_1(T) = p + T^q - 1 \) and \( Q_{k+1}(T) = Q_k([p](T)) \) if \( k \geq 1 \). Note that
\[
\log_{LT}(T) = T \cdot \prod_{k \geq 1} \frac{Q_k(T)}{p}.
\]
Indeed, \( \log_{LT}(T) = \lim_{k \to \infty} p^{-k} \cdot [p^k](T) \) (§9.3 of [Co02]) and \( [p^k](T) = Q_0(T) \cdots Q_k(T) \).

Let \( \exp_{LT}(T) \) denote the inverse of \( \log_{LT}(T) \). We have \( \exp_{LT}(T) = \sum_{k=1}^{\infty} e_k T^k \) with \( v_p(e_k) \geq -k/(q - 1) \). For example, \( \log_{G_m}(T) = \log(1 + T) \) and \( \exp_{G_m}(T) = \exp(T) - 1 \).

**Remark 1.1.** — Our special choice of \( \{p\}(T) = pT + T^q \) is the simplest. Since \( \{p\}(T) \) belongs to \( \mathbb{Z}_p[T] \), the series \( Q_k(T) \), \( \log_{LT}(T) \) and \( \exp_{LT}(T) \) all have coefficients in \( Q_p \). It also implies that \( [\sigma(a)](T) = \sigma([a](T)) \), since \( [a](T) = \exp_{LT}(a \cdot \log_{LT}(T)) \).

**Lemma 1.2.** — If \( z \in \mathfrak{m}_{C_p} \), then
\[
\frac{[1 + a](z) - z}{a} = \log_{LT}(z) \cdot \frac{dS}{dU}(z, 0) + O(a),
\]
as \( a \to 0 \) in \( \mathcal{O}_F \).

**Proof.** — We are looking at the limit of \( (S(z, [a](z)) - z)/a \) as \( a \to 0 \). If \( a \) is small enough, then \( [a](z) = \exp_{LT}(a \cdot \log_{LT}(z)) = a \cdot \log_{LT}(z) + O(a^2) \), which implies the lemma.

** 2. Rings of multivariable power series **

We consider power series in the \( h \) variables \( Y_0, \ldots, Y_{\hat{h} - 1} \). If \( Y^m = Y_0^{m_0} \cdots Y_{\hat{h} - 1}^{m_{\hat{h} - 1}} \) is a monomial, then its weight is \( w(m) = m_0 + pm_1 + \cdots + p^{(h-1)m_{\hat{h} - 1}} \). If \( J \) is a subinterval of \([0; +\infty)\) and if \( J = \{j_1, \ldots, j_k\} \) is a subset of \( \{0, \ldots, h - 1\} \), then (adapting Appendix A of [Záb12] to our situation) we define \( R^I(\{Y_j\}_{j \in J}) \) to be the ring of power series
\[
f(Y_{j_1}, \ldots, Y_{j_k}) = \sum_{m_1, \ldots, m_k \in \mathbb{Z}} a_{m_1 \ldots m_k} Y_{j_1}^{m_1} \cdots Y_{j_k}^{m_k},
\]
such that \( \text{val}_p(a_m) + w(m)/r \to +\infty \) for all \( r \in I \). In other words, \( f(Y) \) is required to converge on the polyannulus \( \{ (Y_0, \ldots, Y_{\hat{h} - 1}) \) such that \( |Y_0| = p^{-1/r}, \ldots, |Y_{\hat{h} - 1}| = p^{-h^{-1}/r} \)
for all $r \in I$. We then define $W(f(Y), r) = \inf_{m \in \mathbb{Z}}(\text{val}_p(a_m) + w(m)/r)$ and, if $I$ is closed, $W(f(Y), I) = \inf_{r \in I} W(f(Y), r)$.

We let $\mathcal{R}^+({\{Y_j\}_{j \in J}}) = \mathcal{R}^{[0; +\infty]}({\{Y_j\}_{j \in J}})$ be the ring of holomorphic functions on the open unit polydisk corresponding to $J$. The Robba ring $\mathcal{R}({\{Y_j\}_{j \in J}})$ is defined as $\mathcal{R}({\{Y_j\}_{j \in J}}) = \bigcup_{r \geq 0} \mathcal{R}^{[r; +\infty]}({\{Y_j\}_{j \in J}})$. In order to lighten the notation, we write $\mathcal{R}^I(Y)$ and $\mathcal{R}(Y)$ instead of $\mathcal{R}^I(Y_0, \ldots, Y_{h-1})$, $\mathcal{R}^+(Y_0, \ldots, Y_{h-1})$ and $\mathcal{R}(Y_0, \ldots, Y_{h-1})$.

The rings $\mathcal{R}^I({\{Y_j\}_{j \in J}})$ are endowed with an $F$-linear action of $\Gamma_F$, given by the formula $a(Y_j) = [\sigma^j(a)](Y_j)$. There is also an $F$-linear Frobenius map $\varphi_q : \mathcal{R}^I({\{Y_j\}_{j \in J}}) \to \mathcal{R}'^I({\{Y_j\}_{j \in J}})$, given by $Y_j \mapsto [p](Y_j)$, for appropriate $I$ and $I'$.

On the ring $\mathcal{R}'^I(Y)$, we can define in addition an absolute $\sigma$-semilinear Frobenius map $\varphi$ by $Y_j \mapsto Y_{j+1}$ for $0 \leq j \leq h-2$ and $Y_{h-1} \mapsto [p](Y_0)$. This map $\varphi$ has the property that $\varphi^h = \varphi_q$, and it also commutes with $\Gamma_F$.

Let $t_i = \log_{LT}(Y_i)$. Since $a(Y_i) = [\sigma^i(a)](Y_i)$ if $a \in \Gamma_F$, we have $a(t_i) = \sigma^i(a) \cdot t_i$ so that $g(t_0 \cdots t_{h-1}) = N_{F/Q_p}(\chi_{LT}(g)) \cdot t_0 \cdots t_{h-1} = \chi_{\text{cyc}}(g) \cdot t_0 \cdots t_{h-1}$ if $g \in G_F$ as well as $\varphi(t_0 \cdots t_{h-1}) = p \cdot t_0 \cdots t_{h-1}$. The element $t_0 \cdots t_{h-1}$ therefore behaves like a $Q_p$-multiple of the “usual” $t$ of $p$-adic Hodge theory (see proposition 3.3 for a more precise statement).

The following two propositions are variations on the “Weierstrass division theorem”.

**Proposition 2.1.** — Let $I = [0; s]$ or $[0; s[ \leq [0; s[$ and let $P(T) \in O_F[T]$ be a monic polynomial of degree $d$ whose nonleading coefficients are all divisible by $p$. If $f \in \mathcal{R}^I({\{Y_j\}_{j \in J}})$, then there exists $g \in \mathcal{R}^I({\{Y_j\}_{j \in J}})$ and $f_0, \ldots, f_{d-1} \in \mathcal{R}^I({\{Y_j\}_{j \in J \setminus \{i\}}})$ such that

$$f = f_0 + f_1Y_i + \cdots + f_{d-1}Y_i^{d-1} + g \cdot P(Y_i).$$

**Proof.** — If $I = [0; s]$ is closed, then this is a straightforward consequence of the Weierstrass division theorem. Since $g$ and the $f_i$s are uniquely determined, the result extends to the case when $I = [0; s[$. □

**Proposition 2.2.** — Let $I = [s; s]$ and let $P(T) \in O_F[T]$ be a monic polynomial of degree $d$, all of whose roots are of valuation $-1/s$. If $f \in \mathcal{R}^I({\{Y_j\}_{j \in J}})$, then there exists $g \in \mathcal{R}^I({\{Y_j\}_{j \in J}})$ and $f_0, \ldots, f_{d-1} \in \mathcal{R}^I({\{Y_j\}_{j \in J \setminus \{i\}}})$ such that

$$f = f_0 + f_1Y_i + \cdots + f_{d-1}Y_i^{d-1} + g \cdot P(Y_i).$$

**Proof.** — The polynomial $Q(T) = P(1/T)T^d/P(0)$ is monic and all its roots are of valuation $1/s$. Write $f = f^+ + f^-$ where $f^+$ contains positive powers of $Y_i$ and $f^-$ contains negative powers of $Y_i$. One may Weierstrass divide $f^+$ by $P(Y_i)$ and $f^-$ by $Q(1/Y_i)$, which implies the proposition. □
Lemma 2.3. — The action of $\Gamma_F$ on $R^I(Y)$ is locally $\mathbb{Q}_p$-analytic, and we have

$$[1 + a](f(Y)) = f(Y) + \sum_{j=0}^{h-1} \sigma^j(a) \cdot \log_{LT}(Y_j) \cdot \frac{dS}{dY_j}(Y, 0) \cdot \frac{df}{dY_j}(Y) + O(a^2).$$

Proof. — The above formula follows from the fact that $[1 + a](Y_j) = Y_j \oplus [a](Y_j) = Y_j \oplus (\sigma^j(a) \cdot \log_{LT}(Y_j) + O(a^2)).$ 

Proposition 2.4. — Let $\rho = (\rho_1, \ldots, \rho_{h-1})$ and let $R^\rho_{F_k}(T_1, \ldots, T_{h-1})$ denote the ring of Laurent series converging for $|T_i| = \rho_i$, with coefficients in $F_k$. If the $z_i \in \mathfrak{m}_{F_{\infty}}$ are such that $\log_{LT}(z_i) \neq 0$, $|z_i| = \rho_i$ and $g(z_i) = [\sigma^i(g)](z_i)$ for $g \in \mathcal{O}_F^\times$, then the map $R^\rho_{F_k}(T_1, \ldots, T_{h-1}) \to \mathbb{C}_p$ given by evaluating at $(z_1, \ldots, z_{h-1})$ is injective.

Proof. — Suppose that $f(z_1, \ldots, z_{h-1}) = 0$ for some $f \in R^\rho_{F_k}(T_1, \ldots, T_{h-1})$. If $g \in \Gamma_{F_k}$, then $f(g(z_1), \ldots, g(z_{h-1})) = 0$. If $g = 1 + a$ with a small, then lemma 1[2] provides us with $h-1$ elements $y_1, \ldots, y_{h-1}$ of $F_{\infty}$ such that $g(z_i) = z_i + \sigma^i(a) \cdot y_i + O(a^2)$. Since $y_i = \log_{LT}(z_i) \cdot \frac{dS}{dU}(z_i, 0)$ and $dS/dU$ is a unit and $\log_{LT}(z_i) \neq 0$, the elements $y_1, \ldots, y_{h-1}$ are all nonzero.

If $f \neq 0$ and $m$ is the smallest index for which $f$ has a nonzero partial derivative of order $m$ at $(z_1, \ldots, z_{h-1})$ and if we expand $f(g(z_1), \ldots, g(z_{h-1}))$ around $(z_1, \ldots, z_{h-1})$ (which generalizes lemma 2[3]), then we get

$$\sum_{j_1 + \ldots + j_{h-1} = m} (\sigma^1(a)y_1)^{j_1} \ldots (\sigma^{h-1}(a)y_{h-1})^{j_{h-1}} \frac{d^m f}{dT_1^{j_1} \ldots dT_{h-1}^{j_{h-1}}}(z_1, \ldots, z_{h-1}) + O(a^{m+1}).$$

Since $f(g(z_1), \ldots, g(z_{h-1})) = 0$, the above linear combination is a homogeneous polynomial, of degree $m$ in $h - 1$ variables and coefficients in $F_\infty$, that is identically zero on $(\sigma^1(a), \ldots, \sigma^{h-1}(a))$. The shortest nonzero polynomial that is identically zero on $(\sigma^1(a), \ldots, \sigma^{h-1}(a))$ can be taken to have coefficients in $F$ and Artin’s theorem on the algebraic independence of characters implies that it is equal to zero. Since all the $y_i$’s are nonzero, all the partial derivatives of order $m$ of $f$ are zero, so that finally $f = 0.$

3. Embeddings in $B_{dR}$

We now explain how to embed the rings of power series of the previous section in the usual rings of $p$-adic periods. Let $\tilde{B}^I$ be the ring defined in §2.1 of [Ber02]. This ring is complete with respect to the valuation $V(\cdot, I)$ (denoted by $V_I(\cdot)$ in §2.1 of ibid.). Recall that if $x = \sum_{k \geq 0} p^k \delta[k] \in \tilde{A}^+$, then $V(x, r) = \inf_k (\text{val}_E(x_k) + kr p/(p-1))$. Set $r_F = p^{h-1} \cdot q/(q-1) \cdot (p-1)/p$ (for example, $r_{\mathbb{Q}_p} = 1$ and if $h > 1$, then $r_F < p^{h-1}$).
Proposition 3.1. — If \( r \geq r_F \) and \( m \in \mathbb{Z} \), then \( V(\varphi^j(u)m, r) = m \cdot p^j \cdot q/(q - 1) \) for \( 0 \leq j \leq h - 1 \).

Proof. — Recall that \( u = \{ \pi \} \) where \( \pi = (\pi_0, \pi_1, \ldots) \) with \( \text{val}_p(\pi_n) = 1/q^n(q - 1) \) for \( n \geq 1 \), so that \( \text{val}_E(\pi) = q/(q - 1) \). We have \( \varphi^j(u) = [\pi^{p^j}] + \sum_{k \geq 1} p^k [u_{k,j}] \) where \( \text{val}_E(u_{k,j}) > 0 \), so that if \( r \geq r_F \), then \( \varphi^j(u)/[\pi^{p^j}] \) is a unit of \( \mathbb{A}^{1,r} \) and the proposition follows. \( \square \)

Note that a better estimate on the \( \text{val}_E(u_{k,j}) \) would allow us to take a smaller value for \( r_F \). Let \( s_n = p^{n-h}(q - 1) \) and let \( r_n = p^{n-1}(p - 1) \) (so that \( s_n \cdot q/(q - 1) = r_n \cdot p/(p - 1) \)).

Proposition 3.2. — If \( n \geq h \), and if \( f(Y) \in \mathcal{R}^{[s_n:s_n]}(Y) \), then \( f(u, \ldots, \varphi^{h-1}(u)) \) converges in \( \tilde{\mathbb{B}}_{[r_n:r_n]} \).

Proof. — If \( f(Y) = \sum_{m \in \mathbb{Z}^h} a_m Y^m \in \mathcal{R}^{[s_n:s_n]}(Y) \), then \( \text{val}_p(a_m) + (m)/p^{n-h}(q - 1) \rightarrow +\infty \). If \( n \geq h \), then \( r_n > r_F \) so that \( V(\varphi^j(u)m, r) = m_j \cdot p^j \cdot q/(q - 1) \) for \( 0 \leq j \leq h - 1 \) by proposition 3.1 and then

\[
V(a_{m_0,\ldots,m_{h-1}} u^{m_0} \ldots \varphi^{h-1}(u)^{m_{h-1}}, r) \rightarrow +\infty.
\]

The series \( f(u, \ldots, \varphi^{h-1}(u)) \) therefore converges in \( \tilde{\mathbb{B}}_{[r_n:r_n]} \). \( \square \)

Corollary 3.3. — If \( n \geq h \), and if \( f(Y) \in \mathcal{R}^{[0:s_n]}(Y) \), then \( f(u, \ldots, \varphi^{h-1}(u)) \) converges in \( \tilde{\mathbb{B}}_{[0:r_n]} \). If \( f(Y) \in \mathcal{R}^+(Y) \), then \( f(u, \ldots, \varphi^{h-1}(u)) \) converges in \( \tilde{\mathbb{B}}^+_\text{rig} \).

Proof. — If \( f \in \mathcal{R}^{[0:s_n]}(Y) \), then each term of the series \( f(u, \ldots, \varphi^{h-1}(u)) \) belongs to \( \tilde{\mathbb{B}}^+ \) so that it converges in \( \tilde{\mathbb{B}}_{[0:r_n]} \) by the maximum modulus principle (corollary 2.20 of [Ber02]). The second assertion follows by passing to the limit. \( \square \)

The image of \( \log_{\text{LT}}(Y_0) \cdots \log_{\text{LT}}(Y_{h-1}) \) in \( \tilde{\mathbb{B}}_{\text{rig}}^+ \subset \mathbb{B}_{\text{rig}}^+ \) is \( a \cdot t \) with \( a \in \mathbb{Q}_p \), as we have seen above. We henceforth denote by \( t \) the element of \( \mathcal{R}^+(Y) \) whose image in \( \tilde{\mathbb{B}}_{\text{rig}}^+ \) is \( t \), that is \( t = \log_{\text{LT}}(Y_0) \cdots \log_{\text{LT}}(Y_{h-1})/a \). In the following proposition, we determine the valuation of \( a \) (this is not used in the rest of this article).

Proposition 3.4. — In the ring \( \mathbb{B}_{\text{rig}}^+ \), the product \( \log_{\text{LT}}(u) \cdots \log_{\text{LT}}(\varphi^{h-1}(u)) \) belongs to \( p^{h-1} \cdot \mathbb{Z}_p^+ \cdot t \), where \( t \) is the usual \( t \) of \( p \)-adic Hodge theory.

Proof. — We have seen that \( \log_{\text{LT}}(u) \cdots \log_{\text{LT}}(\varphi^{h-1}(u)) = a \cdot t \) with \( a \in \mathbb{Q}_p \), and we now compute \( \text{val}_p(a) \). We have \( \log_{\text{LT}}(u) = u \cdot \prod_{k \geq 1} Q_k(u)/p \) and likewise, if \( \pi = [\varepsilon] - 1 \), then \( t = \pi \cdot \prod_{k \geq 1} Q_k^{\text{cyq}}(\pi)/p \). This implies that \( \theta(t/\log_{\text{LT}}(u)) = \theta(\pi/u) \). Since both \( \pi/\varphi^{-1}(\pi) \) and \( u/\varphi_q^{-1}(u) \) are generators of \( \ker(\theta) \) in \( \tilde{\mathbb{A}}^+ \), we have \( \text{val}_p(\theta(t/\log_{\text{LT}}(u))) = 1/(p - 1) - 1/(q - 1) \). On the other hand, \( \text{val}_p(\theta \circ \varphi^j(u)) = \text{val}_p(\lim_{n \to \infty} [p^n](\pi^{p_k^n})) = 1 + p^j/(q - 1) \).
Corollary 3.7. — Let $\nu_n : \mathcal{R}^{[\sigma, \tau]}(Y) \to \mathbf{B}_{\text{dr}}^+$ be the compositum of the map defined above, with the map $\varphi^{-1} : \mathbf{B}^{[\sigma, \tau]} \to \mathbf{B}^{[\tau, \tau]}$ and the map $\mathbf{B}^{[\tau, \tau]} \subset \mathbf{B}_{\text{dr}}^+$ defined in §2.2 of [Ber02].

It follows from the definition as well as the formulas for $\varphi$ and the action of $\Gamma_F$ on $\mathcal{R}^I(Y)$ that $\nu_{n+1}(\varphi(f)) = \nu_n(f)$ when applicable, and that $g(\nu_n(f)) = \nu_n(g(f))$ if $g \in G_F$. Since $\nu_n(t) = t^{n-1}$, we can extend $\nu_n$ to $\nu_n : \mathcal{R}^{[\sigma, \tau]}(Y)[1/t] \to \mathbf{B}_{\text{dr}}$.

Theorem 3.6. — If $n \geq h$, if $f \in \mathcal{R}^{[\sigma, \tau]}(Y)$, and if $n = hk + i$ with $0 \leq i \leq h - 1$, then we have $\nu_n(f) \in \text{Fil}^1 \mathbf{B}_{\text{dr}}^+$ if and only if $f \in Q_k(Y_i) \cdot \mathcal{R}^{[\sigma, \tau]}(Y)$.

Proof. — Recall that $u = \{(\pi_0, \pi_1, \ldots)\} \in \mathbb{A}_+$. If $m \geq 1$ and $u_m = \theta(\varphi^{-m}(u)) \in \hat{F}_\infty$, then $g(u_m) = [\sigma^{-m}(g)](u_m)$. Note that if $m = h\ell$, then $u_m = \theta(\varphi^{-1}(u)) = \pi_{\ell}$. The theorem is equivalent to the assertion that $f^{\sigma^{-n}}(u_n, \ldots, u_{n-h+1}) = 0$ in $C_p$ if and only if $f \in Q_k(Y_i) \cdot \mathcal{R}^{[\sigma, \tau]}(Y)$. We have $u_{n-i} = \pi_k$ so that if $f$ belongs to $Q_k(Y_i) \cdot \mathcal{R}^{[\sigma, \tau]}(Y)$, then $f^{\sigma^{-n}}(u_n, \ldots, u_{n-h+1}) = 0$.

Since $Q_k(T)$ is a monic polynomial of degree $d = q^{k-1}(q - 1)$, whose nonleading coefficients are divisible by $p$, we can use proposition 2.2 to write $f^{\sigma^{-n}} = f_0 + Y_if_1 + \cdots + Y_i^{d-1}f_{d-1} + Q_k(Y_i)r$ with $f_i$ a power series in the $Y_j$’s with $j \neq i$. Proposition 2.4 applied to $f_0 + \pi_kf_1 + \cdots + \pi_k^{d-1}f_{d-1}$, with the $T_j$’s a suitable permutation of the $Y_j$’s, shows that $f_0 + \pi_kf_1 + \cdots + \pi_k^{d-1}f_{d-1} = 0$. Therefore, $f = Q_k(Y_i)r^{\sigma^n}$, which proves the theorem. □

Corollary 3.7. — If $n \geq h$, then the map $\nu_n : \mathcal{R}^{[\sigma, \tau]}(Y) \to \mathbf{B}_{\text{dr}}^+$ is injective. If $n \in \mathbb{Z}$, then the map $\nu_n : \mathcal{R}^+(Y) \to \mathbf{B}_{\text{dr}}^+$ is injective.

Proof. — The first assertion follows from theorem 3.6. The second follows from that, and from the fact that $\nu_{n+1}(\varphi(f)) = \nu_n(f)$ for the other $n$. □

4. $(\varphi_q, \Gamma_F)$-modules in one variable

Before constructing $(\varphi_q, \Gamma_F)$-modules over $\mathcal{R}(Y)$, we review Kisin and Ren’s construction of $(\varphi_q, \Gamma_F)$-modules in one variable and explain why we need rings in several variables.

Let $Y_0$ be the variable of [2] and let $\mathcal{E}(Y_0)$ be Fontaine’s field of [Fon90] with coefficients in $F$, that is $\mathcal{E}(Y_0) = \mathcal{O}_C(Y_0)[1/p]$ where $\mathcal{O}_C(Y_0)$ is the $p$-adic completion of $\mathcal{O}_F[Y_0][1/Y_0]$. We let $\mathcal{E}^I(Y_0)$ and $\mathcal{R}(Y_0)$ denote the corresponding overconvergent and Robba rings. If $I$ is a subinterval of $[0; +\infty]$, then we denote as above by $\mathcal{R}^I(Y_0)$ the set of power series $f(Y_0) = \sum_{m \in \mathbb{Z}} a_m Y_0^m$ that belong to $\mathcal{R}^I(Y_0, \ldots, Y_{h-1})$ via the natural inclusion.
If $K/F$ is a finite extension, then by the theory of the field of norms (see [FW79] and [Win83]), there corresponds to it a finite extension $\mathcal{E}_K(Y_0)$ of $\mathcal{E}(Y_0)$, of degree $[K_\infty : F_\infty]$. A $(\varphi_q, \Gamma_K)$-module over $\mathcal{E}_K(Y_0)$ is a finite dimensional $\mathcal{E}_K(Y_0)$-vector space $D$, along with a semilinear $\varphi_q$ and a compatible action of $\Gamma_K$. We say that $D$ is étale if $D = \mathcal{E}_K(Y_0) \otimes_{\mathcal{O}_K(Y_0)} D_0$ where $D_0$ is a $(\varphi_q, \Gamma_K)$-module over $\mathcal{O}_K(Y_0)$. By specializing the constructions of [Fon90], Kisin and Ren prove the following theorem in their paper (theorem 1.6 of [KR09]).

**Theorem 4.1.** — The functors
\[ V \mapsto (\hat{\mathcal{E}}(Y_0)^{unr} \otimes_F V)^{H_K} \text{ and } D \mapsto (\hat{\mathcal{E}}(Y_0)^{unr} \otimes_{\mathcal{E}_K(Y_0)} D)^{\varphi_q = 1} \]
give rise to mutually inverse equivalences of categories between the category of $F$-linear representations of $G_K$ and the category of étale $(\varphi_q, \Gamma_K)$-modules over $\mathcal{E}_K(Y_0)$.

We say that an $F$-linear representation of $G_K$ is $F$-analytic if it is Hodge-Tate with weights 0 (i.e. $C_p$-admissible) at all embeddings $\tau \neq \text{Id}$. Kisin and Ren then go on to show that if $K \subset F_\infty$, and if $V$ is a crystalline representation of $G_K$, that is $F$-analytic, then the $(\varphi_q, \Gamma_K)$-module attached to $V$ is overconvergent (see §3.3 of ibid.).

Assume from now on that $K \subset F_\infty$, so that $\mathcal{E}_K(Y_0) = \mathcal{E}(Y_0)$. If $D$ is a $(\varphi_q, \Gamma_K)$-module over $\mathcal{R}(Y_0)$, and if $g \in \Gamma_K$ is close enough to 1, then by standard arguments (see §4.1 of [Ber02] or §2.1 of [KR09]), the series $\log(g) = \log(1 + (g - 1))$ gives rise to a differential operator $\nabla_g : D \rightarrow D$. The map $\text{Lie} \Gamma_F \rightarrow \text{End}(D)$ arising from $v \mapsto \nabla_{\exp(v)}$ is $Q_p$-linear, and we say that $D$ is $F$-analytic if this map is $F$-linear (see §2.1 of [KR09] and §1.3 of [FX12]). This is equivalent to the requirement that $\nabla_j = 0$ on $D$ for $1 \leq j \leq h - 1$, where $\nabla_j$ is the partial derivative in the direction $\sigma^j$.

**Theorem 4.2.** — If $V$ is an overconvergent $F$-linear representation of $G_K$, and if $D(V) = \mathcal{R}(Y_0) \otimes_{\mathcal{E}(Y_0)} D^+(V)$, then $D(V)$ is $F$-analytic if and only if $V$ is $F$-analytic.

**Proof.** — Choose $1 \leq j \leq h - 1$, and take $n \gg 0$ such that $n = -j \mod h$. By proposition 3.2 we have a map $\theta \circ \varphi^{-n} : \mathcal{R}^{[s_n, \infty]}(Y_0) \rightarrow B^+_{dR} \rightarrow C_p$, giving rise to an isomorphism
\[ C_p \otimes_{\mathcal{R}^{[s_n, \infty]}(Y_0)} D^{[s_n, \infty]}(V) \rightarrow C_p \otimes_{\mathcal{F}}^{\sigma^j} V. \]

We first prove that if $D(V)$ is $F$-analytic, then $V$ is $C_p$-admissible at the embedding $\sigma^j$. Let $\hat{\mathcal{F}}_\infty^{(j)}$ denote the field of locally $\sigma^j$-analytic vectors of $\hat{\mathcal{F}}_\infty$ for the action of $\Gamma_K$. Note that $\theta \circ \varphi^{-n}(\mathcal{R}^{[s_n, \infty]}(Y_0)) \subset \hat{\mathcal{F}}_\infty^{(j)}$. Let $D^{(j)}_{\text{Sen}}(V)$ be the $\hat{\mathcal{F}}_\infty^{(j)}$-vector space
\[ D^{(j)}_{\text{Sen}}(V) = \hat{\mathcal{F}}_\infty^{(j)} \otimes_{\theta \circ \varphi^{-n}(\mathcal{R}^{[s_n, \infty]}(Y_0))} \theta \circ \varphi^{-n}(D^{[s_n, \infty]}(V)). \]

It is of dimension $d$, its image in $(C_p \otimes_{\mathcal{F}}^{\sigma^j} V)^{H_F}$ generates $C_p \otimes_{\mathcal{F}}^{\sigma^j} V$, and its elements are all locally $\sigma^j$-analytic vectors of $(C_p \otimes_{\mathcal{F}}^{\sigma^j} V)^{H_F}$ because $D(V)$ is $F$-analytic and $\varphi^{-n} \circ \nabla_j =
\[ \nabla_0 \circ \varphi^{-n}. \] If \( y \in D_{\text{Sen}}(V) \), then \( (g(y) - y)/(\sigma^j \circ \chi_{LT}(g) - 1) \) has a limit as \( g \to 1 \), and we call \( \nabla_j(y) \) this limit. We then have \( g(y) = \exp(\log_p(\sigma^j \circ \chi_{LT}(g)) \cdot \nabla_j)(y) \) if \( g \in \Gamma_K \) is close to 1.

Recall that there exists \( a_j \in C_p \) such that \( \log_p(\sigma^j \circ \chi_{LT}(g)) = g(a_j) - a_j \). For example, one can take \( a_j = \log_p(\theta \circ \iota_0(t_j)) \). The element \( a_j \) then belongs to \( \hat{F}^j_{\infty} \) for obvious reasons and satisfies \( \nabla_j(a_j) = 1 \). Take \( y \in D_{\text{Sen}}(V) \), and choose \( a_{j,0} \in F_\infty \) such that \( |a_j - a_{j,0}|_p \) is small enough. The series

\[ C(y) = \sum_{k \geq 0} (-1)^k \frac{(a_j - a_{j,0})^k}{k!} \nabla_j^k(y) \]

then converges for the topology of \( D_{\text{Sen}}(V) \) (the technical details concerning convergence in such spaces of locally analytic vectors can be found in [BC]) and a short computation shows that \( \nabla_j(C(y)) = 0 \), so that \( C(y) \in \left( C_p \otimes_{F}^j V \right)^{G_{F_n}} \) for some \( n = n(y) \gg 0 \). In addition, \( n(y) = n(\nabla_j^k(y)) \) for \( k \geq 0 \), the series for \( C(\nabla_j^k(y)) \) also converges for the topology of \( D_{\text{Sen}}(V) \), and \( y = \sum_{k \geq 0} (a_j - a_{j,0})^k/k! \cdot C(\nabla_j^k(y)) \).

If \( y_1, \ldots, y_d \) is a basis of \( D_{\text{Sen}}(V) \), and if \( n \geq \max n(y_i) \), then the above computations show that the elements \( y_i \) belong to \( \hat{F}^j_{\infty} \otimes_{F_n} \left( C_p \otimes_{F}^j V \right)^{G_{F_n}} \), so that \( \left( C_p \otimes_{F}^j V \right)^{G_{F_n}} \) generates \( \left( C_p \otimes_{F}^j V \right)^H_F \). This implies that \( V \) is \( C_p \)-admissible at the embedding \( \sigma^j \). This is true for all \( 1 \leq j \leq h - 1 \), and therefore \( V \) is \( F \)-analytic.

We now prove that if \( V \) is \( C_p \)-admissible at the embedding \( \sigma^j \), then \( \nabla_j = 0 \) on \( D(V) \). Choose \( n = hm - j \) with \( m \gg 0 \). Since \( j \not\equiv 0 \) mod \( h \), the map \( \theta \circ \varphi^{-n} : \mathcal{R}^{[s_n: s_n]}(Y_0) \to C_p \) is injective by Theorem 3.6. This implies that the map

\[ D^{\left[s_n: s_n\right]}(V) \to C_p \otimes_{\mathcal{R}^{[s_n: s_n]}(Y_0)} D^{\left[s_n: s_n\right]}(V) \]

is injective, and hence the map \( D^{\left[s_n: s_n\right]}(V) \to C_p \otimes_{F}^j V \) is also injective. Therefore, we have an injection \( D^{\left[s_n: s_n\right]}(V) \to (C_p \otimes_{F}^j V)^{H_{F}} \) where \( (C_p \otimes_{F}^j V)^{H_{F}} \) denotes the set of locally \( Q_p \)-analytic vectors of \( (C_p \otimes_{F}^j V)^{H_{F}} \). If \( V \) is \( C_p \)-admissible at the embedding \( \sigma^j \), then \( (C_p \otimes_{F}^j V)^{H_{F}} = (\hat{F}^\infty_{\infty})^d \). One of the main results of [BC] is that \( \nabla_0 = 0 \) on \( \hat{F}^\infty_{\infty} \) (it is shown in [BC] that, in a suitable sense, \( \hat{F}^\infty_{\infty} \) is generated by \( F_\infty \) and the elements \( a_1, \ldots, a_h \)). This implies that \( \nabla_j = 0 \) on \( D^{\left[s_n: s_n\right]}(V) \), since \( \varphi^{-n} \circ \nabla_j = \nabla_0 \circ \varphi^{-n} \).

Note that an analogous argument for the proof of the implication “\( D(V) \) is \( F \)-analytic implies \( V \) is \( F \)-analytic” in certain cases was given by Bingyong Xie.

**Corollary 4.3.** — If \( V \) is an absolutely irreducible \( F \)-linear overconvergent representation of \( G_K \), then there exists a character \( \delta \) of \( \Gamma_K \) such that \( V \otimes \delta \) is \( F \)-analytic.

**Proof.** — We give a sketch of the proof. Choose some \( g \in \Gamma_K \) such that \( \log_p(\chi_{LT}(g)) \neq 0 \), and let \( \nabla = \log(g)/\log_p(\chi_{LT}(g)) \). Choose \( r > 0 \) large enough and \( s \geq qr^r \). If \( a \in \mathcal{O}_F \),
and if $\text{val}_p(a) \geq n$ for $n = n(r, s)$ large enough, then the series $\exp(a \cdot \nabla)$ converges to an operator on $D^{[r|s]}(V)$. This way, we can define a twisted action of $\Gamma_{K_n}$ on $D^{[r|s]}(V)$, by the formula $h \ast x = \exp(\log_p(\chi_{LT}(h)) \cdot \nabla)(x)$. This action is now $F$-analytic by construction.

Since $s \geq qr$, the modules $D^{[q^m\cdot p^m|s]}(V)$ for $m \geq 0$ are glued together by $\varphi_q$ and this way, we get a new action of $\Gamma_{K_n}$ on $D(V)$. Since $\varphi_q$ is unchanged, this new $(\varphi_q, \Gamma_{K_n})$-module is étale, and therefore corresponds to a representation $W$ of $G_{K_n}$. This representation $W$ is $F$-analytic by theorem 4.2 and its restriction to $H_F$ is isomorphic to $V$.

The space $\text{Hom}(V, \text{ind}_{G_{K_n}}^{G_K} W)^{H_F}$ is nonempty, and is a finite dimensional representation of $\Gamma_K$. Since $\Gamma_K$ is abelian, we find (possibly extending scalars) a character $\delta$ of $\Gamma_K$ and a nonzero $f \in \text{Hom}(V, \text{ind}_{G_{K_n}}^{G_K} W)^{H_F}$ such that $h(f) = \delta(h) \cdot f$ if $h \in G_K$. This $f$ gives rise to a nonzero $G_K$-equivariant map $V \otimes \delta \rightarrow \text{ind}_{G_{K_n}}^{G_K} W$. Since $\text{ind}_{G_{K_n}}^{G_K} W$ is $F$-analytic and $V$ is absolutely irreducible, the corollary follows. \hfill $\Box$

Corollary 4.3 (as well as theorem 0.6 of [FX12]) suggests that if we want to attach overconvergent $(\varphi_q, \Gamma_K)$-modules to all $F$-linear representations of $G_K$, then we need to go beyond the objects in only one variable. We finish with a conjecture that seems reasonable enough, since it holds for crystalline representations by the work of Kisin and Ren (see also theorem 0.3 of [FX12]).

**Conjecture 4.4.** — If $V$ is $F$-analytic, then it is overconvergent.

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## 5. Construction of $\mathcal{R}^+(Y)$-modules

We now explain how to construct some $\mathcal{R}^+(Y)$-modules $M^+(D)$ that are attached to some filtered $\varphi_q$-modules $D$. Let $D$ be a finite dimensional $F$-vector space, endowed with an $F$-linear Frobenius map $\varphi_q : D \rightarrow D$, and an action of $G_F$ on $D$ that factors through $\Gamma_F$ and commutes with $\varphi_q$.

If $n \in \mathbb{Z}$, let $B_{\text{dr}} \otimes_{F}^{\sigma^n} D$ denote the tensor product of $B_{\text{dr}}$ and $D$ above $F$, where $F$ maps to $B_{\text{dr}}$ via $\sigma^n$. We then have $b \otimes a \cdot d = \sigma^n(a) \cdot b \otimes d$. Note that $B_{\text{dr}} \otimes_{F}^{\sigma^n} D$ only depends on $n$ mod $h$. For each $0 \leq j \leq h - 1$, let $\text{Fil}_j^*$ be a filtration on $D$ that is stable under $\Gamma_F$, and define $W_{\text{dr}}^{+,j}(D) = \text{Fil}_j^0(B_{\text{dr}} \otimes_{F}^{\sigma^j} D)$ so that $W_{\text{dr}}^{+,j}$ is a $G_F$-stable $B_{\text{dr}}^*$-lattice of $B_{\text{dr}} \otimes_{F}^{\sigma^j} D$.

**Example 5.1.** — If $V$ is an $F$-linear crystalline representation of $G_F$ of dimension $d$, then $D_{\text{cris}}(V)$ is a free $F \otimes_{\mathbb{Q}_p} F$-module of rank $d$ and we have

$$D_{\text{cris}}(V) = D \oplus \varphi(D) \oplus \cdots \oplus \varphi^{h-1}(D),$$

according to the decomposition of $F \otimes_{\mathbb{Q}_p} F$ as $\prod_{\gamma:F \rightarrow F} F$. Each $\varphi^j(D)$ comes with the filtration induced from $D_{\text{cris}}(V)$, and we set $\text{Fil}_j^k D = \varphi^{-j}(\text{Fil}_j^k D_{\text{cris}}(V) \cap \varphi^j(D)).$
We now briefly recall some definitions from [ST03]. The ring $\mathcal{R}^+(Y)$ is a Fréchet-Stein algebra; indeed, its topology is defined by the valuations $\{W(\cdot, [0; s_n])\}_{n \in S}$, where $S$ is any unbounded set of integers, and the ring $\mathcal{R}^{[0; s_n]}(Y)$ is noetherian and flat over $\mathcal{R}^{[0; s_n]}(Y)$ if $m \geq n \in S$. Recall that a coherent sheaf is then a family $\{M^{[0; s_n]}\}_{n \in S}$ of finitely generated $\mathcal{R}^{[0; s_n]}(Y)$-modules, such that $\mathcal{R}^{[0; s_n]}(Y) \otimes \mathcal{R}^{[0; s_m]}(Y) M^{[0; s_m]} = M^{[0; s_n]}$ for all $m \geq n \in S$. A $\mathcal{R}^+(Y)$-module $M$ is said to be coadmissible if $M$ is the set of global sections of a coherent sheaf $\{M^{[0; s_n]}\}_{n \in S}$. We say that $M$ is a reflexive coadmissible $\mathcal{R}^+(Y)$-module if each $M^{[0; s_n]}$ is a reflexive $\mathcal{R}^{[0; s_n]}(Y)$-module. By lemma 8.4 of [ST03], this is the same as requiring that $M$ itself be a reflexive $\mathcal{R}^+(Y)$-module.

Let $\lambda_j = \log_{\text{GT}}(Y_j)/Y_j$ and $\lambda = \lambda_0 \cdots \lambda_{h-1}$, so that for any $n \in \mathbb{Z}$, $t$ is a $\mathbb{Q}_p$-multiple of $t_n(\lambda \cdot Y_0 \cdots Y_{h-1})$. Let $f_j = \lambda/\lambda_j$, so that if $k \neq j \mod h$, then $t_k(f_j)$ is a unit of $\mathcal{B}_d^{+\text{dr}}$.

If $y = \sum_i y_i \otimes d_i \in \mathcal{R}^+(Y)[1/\lambda] \otimes_F D$, let $t_k(y) = \sum_i t_k(y_i) \otimes d_i \in \mathcal{B}_d^{+\text{dr}} \otimes_F^{-k} D$.

**Definition 5.2.** — Let $M^+(D)$ be the set of $y \in \mathcal{R}^+(Y)[1/\lambda] \otimes_F D$ that satisfy $\nu_k(y) \in W_{\text{dr}}^{+,-k}(D)$ for all $k \geq h$.

**Theorem 5.3.** — If $D$ is a $\varphi_q$-module with an action of $\Gamma_F$ and $h$ filtrations, then

1. $M^+(D)$ is a reflexive coadmissible $\mathcal{R}^+(Y)$-module;
2. the $\mathcal{R}^+(Y)[1/f_j]$-module $M^+(D)[1/f_j]$ is free of rank $d$ for $0 \leq j \leq h - 1$;
3. $M^+(D) = \bigcap_{j=0}^h M^+(D)[1/f_j]$.

In the remainder of this section, we prove theorem 5.3. We now establish some preliminary results. Let $S = \{hm + (h - 1) \text{ where } m \geq 1\}$, and take $n \in S$. Recall that on the ring $\mathcal{R}^{[0; s_n]}(Y)$, the map $\nu_k$ is defined for $h \leq k \leq n$. Let

$$M(D)^{[0; s_n]} = \{y \in \mathcal{R}^{[0; s_n]}(Y)[1/\lambda] \otimes_F D, \nu_k(y) \in W_{\text{dr}}^{+,-k}(D) \text{ for all } h \leq k \leq n\}.$$

For $0 \leq j \leq h - 1$, recall that $\mathcal{R}^+(Y_j)$ is a ring of power series in one variable. Let

$$N_j^{[0; s_n]} = \{y \in \mathcal{R}^{[0; s_n]}(Y_j)[1/\lambda_j] \otimes_F D, \varphi^{-k}_q \varphi^{-j}_q(y) \in W_{\text{dr}}^{+,-j}(D) \text{ for all } 1 \leq k \leq m\},$$

$$N_j^+ = \{y \in \mathcal{R}^+(Y_j)[1/\lambda_j] \otimes_F D, \varphi^{-k}_q \varphi^{-j}_q(y) \in W_{\text{dr}}^{+,-j}(D) \text{ for all } k \geq 1\}.$$

**Proposition 5.4.** — The $\mathcal{R}^+(Y_j)$-module $N_j^+$ is free of rank $d$, for all $n$ we have $N_j^{[0; s_n]} = \mathcal{R}^{[0; s_n]}(Y_j) \otimes_{\mathcal{R}^+(Y_j)} N_j^+$, and the map $\mathcal{B}_d^{+\text{dr}} \otimes_{\mathcal{R}^+(Y_j)} N_j^+ \to W_{\text{dr}}^{+, -j}(D)$ is an isomorphism for all $k \geq 1$.

**Proof.** — Since there is only one variable, the proof is a standard argument, analogous to the one which one can find in §II.1 of [Ber08b] or §2.2 of [KR09].

Let $M_j^{[0; s_n]} = \mathcal{R}^{[0; s_n]}(Y)[1/f_j] \otimes_{\mathcal{R}^{[0; s_n]}(Y_j)} N_j^{[0; s_n]}$, where $f_j = \lambda/\lambda_j$.

**Proposition 5.5.** — We have $M(D)^{[0; s_n]}[1/f_j] = M_j^{[0; s_n]}$ and $M(D)^{[0; s_n]} = \bigcap_j M_j^{[0; s_n]}$. 

Proof. — In the sequel, we use the fact that $Q_1(Y_j) \cdots Q_m(Y_j)$ and $\lambda_j$ generate the same ideal of $R^{[0:s_n]}(Y_j)$ (recall that $n = hm + (h-1)$). Let $a$ and $b$ be two integers such that

$$t^a \cdot \mathcal{B}^+_{dr} \otimes F^\ell D \subset W^+_{dr}(D) \subset t^{-b} \cdot \mathcal{B}^+_{dr} \otimes F^\ell D,$$

for all $j$. We then have $M(D)^{[0:s_n]} \subset \lambda^{-b} \cdot R^{[0:s_n]}(Y) \otimes F D$ by theorem 3.6.

We have $\varphi^{-(hk+j)}(R^{[0:s_n]}(Y)[1/f_j]) \subset \mathcal{B}^+_{dr}$ for all $1 \leq k \leq m$ so that if $y \in M_j^{[0:s_n]}$, then $\varphi^{-(hk+j)}(y) \in W^+_{dr}(D)$ for all $1 \leq k \leq m$. On the other hand, if $y \in M_j^{[0:s_n]}$, then $y \in \lambda^{-c} \cdot \mathcal{R}^{[0:s_n]}(Y) \otimes F D$ for some $c \geq 0$, so that $f_j^{a+c}y \in M(D)^{[0:s_n]}$. This implies that $M_j^{[0:s_n]} \subset M(D)^{[0:s_n][1/f_j]}$.

We now prove that $M(D)^{[0:s_n]} \subset M_j^{[0:s_n]}$. Choose $y \in M(D)^{[0:s_n]}$. Since

$$M(D)^{[0:s_n]} \subset \lambda^{-b} \cdot \mathcal{R}^{[0:s_n]}(Y) \otimes F D,$$

we can write $y = \lambda^{-b} \sum k z_k \otimes d_k$. By Weierstrass dividing (proposition 2.1) the $z_k$’s by the polynomial $(Q_1(Y_j) \cdots Q_m(Y_j))^{a+b}$, we can write $y = (Q_1(Y_j) \cdots Q_m(Y_j))^{a+b}z + y_0$ with $y_0 \in \mathcal{R}^{[0:s_n]}(Y)[1/\lambda] \otimes \mathcal{R}^{[0:s_n]}(Y) N_j^{[0:s_n]}$.

Note that $(Q_1(Y_j) \cdots Q_m(Y_j))^{a+b}z \in M_j^{[0:s_n]}$ because $t^a \mathcal{B}^+_{dr} \otimes F^\ell D \subset W^+_{dr}(D)$, so that $(Q_1(Y_j) \cdots Q_m(Y_j))^{a+b}z \in M_j^{[0:s_n]}$. We also have $\varphi_q^{\ell} \varphi_{-j}^\ell y_0 \in W^+_{dr}(D)$ for all $1 \leq \ell \leq m$. By proposition 5.4, the map

$$\mathcal{B}^+_{dr} \otimes \mathcal{R}^{[0:s_n]}(Y) N_j^{[0:s_n]} \to W^+_{dr}(D)$$

is an isomorphism; this implies that $\varphi_q^{\ell} \varphi_{-j}^\ell \mathcal{B}^+_{dr}$ for all $1 \leq \ell \leq m$. Theorem 3.6 now implies that $a_k$ has no pole at any of the roots of $Q_1(Y_j), \ldots, Q_m(Y_j)$, so that we have $a_k \in \mathcal{R}^{[0:s_n]}(Y)[1/f_j]$. This implies that $y_0 \in M_j^{[0:s_n]}$, and therefore also $y$. This proves that $M(D)^{[0:s_n]} \subset M_j^{[0:s_n]}$ and therefore $M(D)^{[0:s_n][1/f_j]} = M_j^{[0:s_n]}$.

If $x \in \cap_j M_j^{[0:s_n]}$, and if $k = j$ mod $h$ with $0 \leq j \leq h-1$, then the fact that $x \in M(D)^{[0:s_n][1/f_j]} = \mathcal{R}^{[0:s_n]}(Y)[1/f_j] \otimes \mathcal{R}^{[0:s_n]}(Y) N_j^{[0:s_n]}$ implies that $t_k(x) \in W^+_{dr}(D)$. This is true for all $h \leq k \leq n$, so that $x \in M(D)^{[0:s_n]}$ and this proves the second assertion.

Lemma 5.6. — We have $M^+(D)[1/f_j] = \mathcal{R}^+(Y)[1/f_j] \otimes \mathcal{R}^+(Y_j) N_j^+$.

Proof. — By combining propositions 5.4 and 5.5, we find that

$$M(D)^{[0:s_n][1/f_j]} = \mathcal{R}^{[0:s_n]}(Y)[1/f_j] \otimes \mathcal{R}^+(Y_j) N_j^+.$$

Since $M(D)^+ = \cap_j M(D)^{[0:s_n]}$, we have $M(D)^+[1/f_j] \subset \cap_j M(D)^{[0:s_n][1/f_j]}$. We also have $\mathcal{R}^+(Y)[1/f_j] \otimes \mathcal{R}^+(Y_j) N_j^+ \subset M^+(D)[1/f_j]$, and those two inclusions imply that $M^+(D)[1/f_j] = \mathcal{R}^+(Y)[1/f_j] \otimes \mathcal{R}^+(Y_j) N_j^+$. 


Proof of theorem 5.3 — We first prove that the family \( \{ M(D)^{[0; s_n]} \}_{n \in S} \) is a coherent sheaf. Take \( n \geq m \in S \). We have

\[
R^{[0; s_m]}(Y) \otimes_{R^{[0; s_n]}(Y)} M(D)^{[0; s_n]} = R^{[0; s_m]}(Y) \otimes_{R^{[0; s_n]}(Y)} ( \bigcap_j R^{[0; s_n]}(Y)[1/f_j] \otimes_{R^{[0; s_n]}(Y)} N_j^{[0; s_n]} ) = \bigcap_j R^{[0; s_m]}(Y)[1/f_j] \otimes_{R^{[0; s_n]}(Y)} N_j^{[0; s_n]} = M(D)^{[0; s_n]}.
\]

This implies that the family \( \{ M(D)^{[0; s_n]} \}_{n \in S} \) is a coherent sheaf. It is clear that its global sections are precisely \( M^+(D) \). By proposition 5.5 we have \( M(D)^{[0; s_n]} = \bigcap_j M(D)^{[0; s_n][1/f_j]} \) where each \( M(D)^{[0; s_n][1/f_j]} \) is free of rank \( d \) over \( R(Y)^{[0; s_n][1/f_j]} \). The fact that \( M(D)^{[0; s_n]} \) is reflexive now follows from proposition 6 of VII.4.2 of [Bou61], and this proves (1).

By combining proposition 5.3 and lemma 5.6 we get item (2) of the theorem. Suppose now that \( x \in \bigcap_j M^+(D)[1/f_j] \). If \( k = j \) mod \( h \) with \( 0 \leq j \leq h - 1 \), then the fact that \( x \in M^+(D)[1/f_j] = R^+(Y)[1/f_j] \otimes_{R^+(Y)} N_j^+ \) implies that \( \iota_k(x) \in W_{dR}^{-1, -k}(D) \). This being true for all \( k \geq h \), we have \( x \in M^+(D) \) and this proves item (3) of the theorem. \( \square \)

Remark 5.7. — If \( h \leq 2 \), then the ring \( R^{[0; s_n]}(Y) \) is of dimension \( \leq 2 \), and reflexive \( R^{[0; s_n]}(Y) \)-modules are therefore projective. By Lütkebohmert’s theorem (see [Lüt77], corollary on page 128), the \( R^{[0; s_n]}(Y) \)-module \( M(D)^{[0; s_n]} \) is then free of rank \( d \). The system \( \{ M(D)^{[0; s_n]} \}_{n \in S} \) then forms a vector bundle over the open unit polydisk. By combining proposition 2 on page 87 of [Gru68] (note that “\( A_m \)” is defined at the bottom of page 82 of loc. cit.), and the main theorem of [Bar81], we get that \( M^+(D) \) is free of rank \( d \) over \( R^+(Y) \). If \( h \geq 3 \), I do not know whether this still holds.

6. Properties of \( M^+(D) \)

We now prove that \( M(D) = R(Y) \otimes_{R^+(Y)} M^+(D) \) is a \( (\varphi_q, \Gamma_F) \)-module over \( R(Y) \), and that if \( D \) arises from a crystalline representation \( V \), then \( M^+(D) \) and \( V \) are naturally related. It is clear from the definition that \( M^+(D) \) is stable under the action of \( \Gamma_F \). We also have \( \lambda^a \cdot R^+(Y) \otimes_F D \subset M^+(D) \) for some \( a \geq 0 \), so that

\[
R^+(Y)[1/\lambda] \otimes_{R^+(Y)} M^+(D) = R^+(Y)[1/\lambda] \otimes_F D.
\]

Say that the module \( D \) with \( h \) filtrations is effective if \( \text{Fil}^0_j(D) = D \) for \( 0 \leq j \leq h - 1 \). Recall that \( n = hm + (h - 1) \) with \( m \geq 1 \).

Lemma 6.1. — If \( D \) is effective, then the \( R^+(Y_j) \)-module \( N_j^+ \) is stable under \( \varphi_q \), and \( N_j^+/\varphi_q(N_j^+) \) is killed by \( Q_1(Y_j)^{a_j} \) if \( a_j \geq 0 \) is such that \( \text{Fil}^{a_j+1}D = \{0\} \).

Proof. — This concerns the construction in one variable, so the proof is standard. See for example §2.2 of [KR09]. \( \square \)
**Proposition 6.2.** — If $D$ is effective, then the $\mathcal{R}^+(Y)$-module $M^+(D)$ is stable under the Frobenius map $\varphi_q$, and $M^+(D)/\varphi_q^*(M^+(D))$ is killed by $Q_1(Y_0)^{a_0} \cdots Q_1(Y_{h-1})^{a_{h-1}}$.

**Proof.** — By (2) of theorem 5.3, we have $M^+(D) = \cap_j M^+(D)[1/f_j]$ and by lemma 5.6, $M^+(D)[1/f_j] = \mathcal{R}^+(Y)[1/f_j] \otimes_{\mathcal{R}^+(Y_j)} N^+_j$. Lemma 6.1 implies that $N^+_j$ is stable under $\varphi_q$, and so the same is true of $M^+(D)[1/f_j]$ and hence $M^+(D)$.

If $x \in M^+(D)$, then $x \in M^+(D)[1/f_j] = \mathcal{R}^+(Y)[1/f_j] \otimes_{\mathcal{R}^+(Y_j)} N^+_j$. Note however that at each $k = i \not\equiv j \mod h$, the coefficients of $x$ can have a pole of order at most $a_i$ since $\text{Fil}^{a_i+1}D = \{0\}$. This implies the more precise estimate

$$M^+(D) \subset \prod_{i \neq j} \lambda_i^{-a_i} \cdot \mathcal{R}^+(Y) \otimes_{\mathcal{R}^+(Y_j)} N^+_j.$$

The $\varphi_q(\mathcal{R}^+(Y))$-module $\mathcal{R}^+(Y)$ is free of rank $q^h$, with basis $\{Y^\ell, \ell \in \{0, \ldots, q-1\}^h\}$. We therefore have

$$Q_1(Y_0)^{a_0} \cdots Q_1(Y_{h-1})^{a_{h-1}} \cdot x \in \prod_{i \neq j} (\lambda_i/Q_1(Y_i))^{-a_i} \cdot \mathcal{R}^+(Y) \otimes_{\mathcal{R}^+(Y_j)} Q_1(Y_j)^{a_j} \cdot N^+_j \subseteq \oplus \mathbb{Q}^\ell . \varphi_q(\mathcal{R}^+(Y)[1/f_j] \otimes_{\mathcal{R}^+(Y_j)} N^+_j).$$

This implies that

$$Q_1(Y_0)^{a_0} \cdots Q_1(Y_{h-1})^{a_{h-1}} \cdot x \in \cap_j \oplus \mathbb{Q}^\ell . \varphi_q(M^+(D)[1/f_j]) = \varphi_q^*(M^+(D)),$$

which proves the second claim. 

**Remark 6.3.** — Instead of working with a $D$ where the filtrations are defined on $D$, we could have asked for the filtrations to be defined on $F_n \otimes_F D$ for some $n \geq 1$. The construction and properties of $M^+(D)$ are then basically unchanged, but the annihilator of $M^+(D)/\varphi_q^*(M^+(D))$ is possibly more complicated than in proposition 6.2. This applies in particular to representations of $G_F$ that become crystalline when restricted to $G_{F_n}$ for some $n \geq 1$.

**Definition 6.4.** — A $(\varphi_q, \Gamma_F)$-module over $\mathcal{R}(Y)$ is a $\mathcal{R}(Y)$-module $M$ that is of the form $M = \mathcal{R}(Y) \otimes_{\mathcal{R}[\varphi_q; \infty]} M[\varphi_q; \infty]$ where $M[\varphi_q; \infty]$ is a coadmissible $\mathcal{R}[\varphi_q; \infty](Y)$-module, endowed with a semilinear Frobenius map $\varphi_q: M[\varphi_q; \infty] \rightarrow M[q\varphi_q; \infty]$, such that $\varphi_q^*(M[\varphi_q; \infty]) = M[q\varphi_q; \infty]$, and a continuous and compatible action of $\Gamma_F$.

**Remark 6.5.** — In the definition above, it would seem natural to impose some additional condition on $M$, such as “torsion-free”. All the $(\varphi_q, \Gamma_F)$-modules over $\mathcal{R}(Y)$ that are constructed in this article are actually reflexive. The definition above should be considered provisional, until we have a better idea of which objects we want to exclude. Note that in the absence of flatness, tensor products may behave badly.
If $D$ is a $\varphi_q$-module with an action of $\Gamma_F$ and $h$ filtrations and if $\ell \in \mathbb{Z}$, let $D(\ell)$ denote the same $\varphi_q$-module with an action of $\Gamma_F$, but with $\text{Fil}^\ell_j(D(\ell)) = (\text{Fil}^{\ell - \ell r_j}((\ell))$. Note that $D(\ell)$ is effective if $\ell \gg 0$.

**Lemma 6.6.** — We have $M(D(\ell)) = \lambda^{-\ell} \cdot M(D)$.

**Proof.** — The fact that $M^+(D(\ell)) = \lambda^{-\ell} \cdot M^+(D)$ follows from the definition.

**Theorem 6.7.** — If $D$ is a $\varphi_q$-module with an action of $\Gamma_F$ and $h$ filtrations as above, then $\mathcal{R}(Y) \otimes_{\mathcal{R}^+(Y)}^\text{rig} M^+(D)$ is a $(\varphi_q, \Gamma_F)$-module over $\mathcal{R}(Y)$.

**Proof.** — If $D$ is effective, then this follows from theorem 5.3 and proposition 6.2. If $D$ is not effective, then $D(\ell)$ is effective if $\ell \gg 0$, and the theorem follows from the effective case and lemma 6.6.

**Remark 6.8.** — In [KR09], Kisin and Ren construct some $(\varphi_q, \Gamma_F)$-modules $M^+_\text{KR}(D)$ in one variable, over the ring $\mathcal{R}^+(Z_0)$, from the data of a $D$ like ours for which the filtration $\text{Fil}^j_+$ is trivial for $j \neq 0$. For those $D$, we have $M^+(D) = \mathcal{R}^+(Y) \otimes_{\mathcal{R}^+(Y)} M^+_\text{KR}(D)$.

More generally, our construction shows that $M^+(D)$ comes by extension of scalars from a $(\varphi_q, \Gamma_F)$-module in as many variables as there are nontrivial filtrations among the $\text{Fil}^j_+$.

**Proposition 6.9.** — If $n = \text{hk} + j \geq h$, then the map

$$B^+_{\text{dr}} \otimes_{\iota_n(\mathcal{R}^+(Y))} \iota_n(M^+(D)) \to \text{Fil}^0_j(B^+_{\text{dr}} \otimes \varphi^{-j} D)$$

is an isomorphism.

**Proof.** — We have $B^+_{\text{dr}} \otimes_{\iota_n(\mathcal{R}^+(Y))} \iota_n(N^+_j) = \text{Fil}^0_j(B^+_{\text{dr}} \otimes \varphi^{-j} D)$, for example by lemma II.1.5 of [Ber08b]. If $x \in \mathcal{R}^+(Y) \otimes_{\mathcal{R}^+(Y)} N^+_j$, then there exists $c \gg 0$ such that $f^j \chi x \in M^+(D)$, and the lemma results from the fact that $\iota_n(f^j)$ is a unit of $B^+_{\text{dr}}$.

Suppose now that $D$ comes from an $F$-linear crystalline representation $V$ of $G_F$ as in example 5.1. In this case, $\text{Fil}^0_j(B^+_{\text{dr}} \otimes \varphi^j D) = B^+_{\text{dr}} \otimes \varphi^j V$. Moreover, one recovers $V$ from $D$ by the formula:

$$V = \{ y \in (\mathcal{B}^+_{\text{rig}}[1/t] \otimes_D F)^{\varphi^j = 1}, \quad y \in \text{Fil}^0_j(B^+_{\text{dr}} \otimes \varphi^j D) \} \text{ for all } 0 \leq j \leq h - 1 \}.$$

Recall that we have constructed in 3 an injective map $\mathcal{R}^+(Y) \to \mathcal{B}^+_{\text{rig}}$. This way we get a map

$$\mathcal{B}^+_{\text{rig}} \otimes_{\mathcal{R}^+(Y)} M^+(D) \to \mathcal{B}^+_{\text{rig}}[1/t] \otimes_D F \to \mathcal{B}^+_{\text{rig}}[1/t] \otimes_F V.$$

Let $\mathcal{B}^+_{\text{rig}}$ be the rings defined in §2.3 [Ber02]. Recall that $n(r)$ is the smallest $n$ such that $r \leq p^{n-1}(p-1)$. We have the following lemma.

**Lemma 6.10.** — If $y \in \mathcal{B}^+_{\text{rig}}[1/t]$ satisfies $\varphi^n(y) \in B^+_{\text{dr}}$ for all $n \geq n(r)$, then $y \in \mathcal{B}^+_{\text{rig}}$. 

Proof. — See lemma 1.1 of [Ber09] and the proof of proposition 3.2 in ibid. \qed

**Theorem 6.11.** — If $D$ comes from a crystalline representation $V$, and if $r \geq p^{h-1}(p-1)$, then the map above gives rise to an isomorphism

$$
\widehat{B}_{\text{rig}}^{l,r} \otimes_{\mathcal{R}^+ (Y)} M^+(D) \to \widehat{B}_{\text{rig}}^{l,r} \otimes_{\mathcal{F}} V.
$$

**Proof.** — We first check that the image of the map above belongs to $\widehat{B}_{\text{rig}}^{l,r} \otimes_{\mathcal{F}} V$. If $y \in \widehat{B}_{\text{rig}}^{l,r} \otimes_{\mathcal{R}^+ (Y)} M^+(D)$, then its image is in $\widehat{B}_{\text{rig}}^{l,r} \otimes_{\mathcal{F}} V$ and satisfies $\varphi^{-n}(y) \in B_{dR} \otimes_{\mathcal{F}} \varphi^{-n} V$ for all $n \geq n(r)$, so the assertion follows from lemma 6.10.

We now prove that $\widehat{B}_{\text{rig}}^{l,r} \otimes_{\mathcal{R}^+ (Y)} M^+(D)$ is a free $\widehat{B}_{\text{rig}}^{l,r}$-module of rank $d$. By (2) of theorem 2.9.6 of [Ked05], $M^+(D)[1/f_j]$ is a free $\mathcal{R}^+(Y)[1/f_j]$-module of rank $d$, and therefore $\widehat{B}_{\text{rig}}^{l,r} \otimes_{\mathcal{R}^+ (Y)} M^+(D)$ is a free $\widehat{B}_{\text{rig}}^{l,r}$-module of rank $d$ for all $j$. The ring $\widehat{B}_{\text{rig}}^{l,r}$ is a Bézout ring by theorem 2.9.6 of [Ked05], and the elements $f_0, \ldots, f_{h-1}$ have no common factor. They therefore generate the unit ideal of $\widehat{B}_{\text{rig}}^{l,r}$, and $\widehat{B}_{\text{rig}}^{l,r} \otimes_{\mathcal{R}^+ (Y)} M^+(D)$ is projective of rank $d$ by theorem 1 of II.5.2 of [Bou61]. Since $\widehat{B}_{\text{rig}}^{l,r}$ is a Bézout ring, $\widehat{B}_{\text{rig}}^{l,r} \otimes_{\mathcal{R}^+ (Y)} M^+(D)$ is free of rank $d$. By proposition 6.9 the map

$$
B_{dR}^{+} \otimes_{\tau_n(\widehat{B}_{\text{rig}}^{l,r})} \iota_n(\widehat{B}_{\text{rig}}^{l,r} \otimes_{\mathcal{R}^+ (Y)} M^+(D)) \to B_{dR}^{+} \otimes_{\mathcal{F}} \varphi^{-n} V
$$

is an isomorphism if $n \geq n(r)$. The two $\widehat{B}_{\text{rig}}^{l,r}$-modules $\widehat{B}_{\text{rig}}^{l,r} \otimes_{\mathcal{R}^+ (Y)} M^+(D)$ and $\widehat{B}_{\text{rig}}^{l,r} \otimes_{\mathcal{F}} V$ therefore have the same localizations at all $n \geq n(r)$, and are both stable under $G_F$, so that they are equal by the same argument as in the proof of lemma 2.2.2 of [Ber08a] (the idea is to take determinants, so that one is reduced to showing that if $x \in \widehat{B}_{\text{rig}}^{l,r}$ generates an ideal stable under $G_F$, and has the property that $\iota_n(x)$ is a unit of $B_{dR}^{+}$ for all $n \geq n(r)$, then $x$ is a unit of $\widehat{B}_{\text{rig}}^{l,r}$). \qed

**Remark 6.12.** — If $D$ comes from a crystalline representation $V$, and if $n \geq 0$, then there is likewise an isomorphism $\widehat{B}_{\text{rig}}^{l,r} \otimes_{\mathcal{R}^+ (Y)} M^+(D) \to \widehat{B}_{\text{rig}}^{l,r} \otimes_{\mathcal{F}} \varphi^{-n} V$ for $r \gg 0$.

### 7. Crystalline $(\varphi_q, \Gamma_F)$-modules

Let $M$ be a $(\varphi_q, \Gamma_F)$-module over $\mathcal{R}(Y)$. In this section, we define what it means for $M$ to be crystalline, and we prove that every crystalline $(\varphi_q, \Gamma_F)$-module $M$ is of the form $M = M(D)$, where $D$ is a filtered $\varphi_q$-module on which the action of $G_F$ is trivial. The results are similar to those of [Ber08b], which deals with the cyclotomic case.

**Lemma 7.1.** — We have $\operatorname{Frac}(\mathcal{R}(Y))^F = F$.

**Proof.** — If $x \in \operatorname{Frac}(\mathcal{R}(Y))^F$, then we can write $x = a/b$ with $a, b \in \mathcal{R}^{[s_n; s_n]}(Y)$ for some $n \gg 0$. By proposition 3.2 the series $a(u, \ldots, \varphi^{h-1}(u))$ and $b(u, \ldots, \varphi^{h-1}(u))$ converge
in \( \mathcal{B}^{[r_n,r_{n+1}]} \). We can therefore see \( \varphi^{-n}(a) \) and \( \varphi^{-n}(b) \) as elements of \( \mathcal{B}^{\text{dR}} \), which satisfy \( \varphi^{-n}(a)/\varphi^{-n}(b) \in \mathcal{B}^{G_F} \). The lemma now follows from the fact that \( \mathcal{B}^{G_F} = F \). \( \square \)

If \( M \) is a \((\varphi,q,\Gamma_F)\)-module over \( \mathcal{R}(Y) \), then let \( D_{\text{cris}}(M) = (\mathcal{R}(Y)[1/t] \otimes_{\mathcal{R}(Y)} M)^{F_F} \).

**Corollary 7.2.** — If \( M \) is a \((\varphi,q,\Gamma_F)\)-module over \( \mathcal{R}(Y) \), then \( \dim D_{\text{cris}}(M) \leq \dim \text{Frac}(\mathcal{R}(Y)) \otimes_{\mathcal{R}(Y)} M \).

**Proof.** — By a standard argument, lemma 7.1 implies that the map \( \text{Frac}(\mathcal{R}(Y)) \otimes_{\mathcal{R}} D_{\text{cris}}(V) \rightarrow \text{Frac}(\mathcal{R}(Y)) \otimes_{\mathcal{R}(Y)} M \)

is injective. \( \square \)

**Definition 7.3.** — We say that a \((\varphi,q,\Gamma_F)\)-module \( M \) over \( \mathcal{R}(Y) \) is crystalline if

1. for some \( s \), \( M^{[s;+\infty]}[1/f_j] \) is a free \( \mathcal{R}(Y)^{[s;+\infty]}[1/f_j] \)-module of finite rank \( d \);
2. \( M^{[s;+\infty]} = \bigcap_{j=0}^{n+1} M^{[s;+\infty]}[1/f_j] \);
3. we have \( \dim D_{\text{cris}}(M) = d \).

For example, if \( D \) is a filtered \( \varphi \)-module on which the action of \( G_F \) is trivial, then \( M(D) \) is a crystalline \((\varphi,q,\Gamma_F)\)-module.

**Proposition 7.4.** — If \( f \in \mathcal{R}^{[s;+\infty]}(Y) \) generates an ideal of \( \mathcal{R}^{[s;+\infty]}(Y) \) that is stable under \( \Gamma_F \), then \( f = u \cdot \prod_{j=0}^{n-1} \prod_{a \geq n(s)} (Q_{a}(Y_{j})/p)^{a_{n,j}} \) where \( u \) is a unit and \( a_{n,j} \in \mathbb{Z}_{\geq 0} \).

**Proof.** — Recall that a power series \( f \in \mathcal{R}^{I}(Y) \) is a unit if and only if it has no zero in the corresponding domain of convergence (by the nullstellensatz, see §7.1.2 of [BGR84]).

Let \( I = [s;u] \) be a closed subinterval of \([s;+\infty[\), so that \( f \in \mathcal{R}^{I}(Y) \), and let \( z = (z_0,z_1,\ldots,z_{h-1}) \) be a point such that \( f(z) = 0 \). Let \( J \) be the set of indices \( j \) such that \( z_j \) is not a torsion point of \( \text{LT}_h \) and let \( f_j \in \mathcal{R}^{I}_{F_k}(\{Y_j\}_{j \in J}) \) be the power series that results from evaluation of the \( Y_m \) at \( z_m \) for all the \( z_m \) that are torsion points of \( \text{LT}_h \) (here \( k \) is large enough so that all those \( z_m \) belong to \( F_k \)). The ideal of \( \mathcal{R}^{I}_{F_k}(\{Y_j\}_{j \in J}) \) generated by the power series \( f_j \) is stable under \( 1+p^k\mathcal{O}_F \), so that the set of its zeroes is stable under the action of \( 1+p^k\mathcal{O}_F \). Furthermore, \( f_j \) has a zero none of whose coordinates are torsion points of \( \text{LT}_h \). The same argument as in the proof of proposition 2.4 shows that \( f_j = 0 \).

If we denote by \( Z_I(f) \) the zero set of \( f \in \mathcal{R}^{I}(Y) \), then the preceding argument shows that \( Z_I(f) \) is the union of finitely many components of the form \( Z_0 \times \cdots Z_{h-1} \) where for each \( j \), either \( Z_j \) is a torsion point of \( \text{LT}_h \) or \( Z_j = Z_I(\{0\}) \). For reasons of dimension, each of these components has precisely one \( Z_j \) which is a torsion point, the remaining \( h-1 \) being \( Z_I(\{0\}) \). This implies that in \( \mathcal{R}^{I}(Y) \), \( f \) is the product of finitely many \( Q_n(Y_j) \) by a unit.
The proposition now follows by a standard infinite factorisation argument, by writing \([s; +\infty] = \bigcup_{u \geq s} [s; u]\).

**Corollary 7.5.** — If \(M\) is a crystalline \((\varphi_q, \Gamma_F)\)-module over \(\mathcal{R}(Y)\), then the map

\[
\mathcal{R}(Y)[1/t] \otimes_F D_{\text{cris}}(M) \to \mathcal{R}(Y)[1/t] \otimes_{\mathcal{R}(Y)} M
\]

is an isomorphism.

**Proof.** — The map is injective by lemma 7.4 and its determinant generates an ideal of \(\mathcal{R}(Y)[1/t]\) that is stable under \(\Gamma_F\). Proposition 7.4 implies that this ideal is the unit ideal of \(\mathcal{R}(Y)[1/t]\), and therefore that the map is an isomorphism.

We now consider filtrations on \(D_{\text{cris}}(M)\).

**Lemma 7.6.** — Let \(D\) be an \(F\)-vector space with a trivial action of \(G_F\), and let \(W\) be a \(B_{\text{dR}}^+\)-lattice of \(B_{\text{dR}} \otimes_F D\) that is stable under \(G_F\). If we set \(\text{Fil}^i D = D \cap t^i \cdot W\), then \(W = \text{Fil}^0(B_{\text{dR}} \otimes_F D)\).

**Proof.** — Let \(e_1, \ldots, e_d\) be a basis of \(D\) adapted to its filtration, with \(e_i \in \text{Fil}^{h_i} \setminus \text{Fil}^{h_i+1} D\). We then have \(\text{Fil}^0(B_{\text{dR}} \otimes_F D) = \bigoplus_{i=1}^d B_{\text{dR}}^+ \cdot t^{-h_i} e_i\). By definition, we have \(t^{-h_i} e_i \in W\), so that \(\text{Fil}^0(B_{\text{dR}} \otimes_F D) \subset W\). We now prove the reverse inclusion.

If \(w \in W\), then we can write \(w = a_1 t^{-h_1} e_1 + \cdots + a_d t^{-h_d} e_d\) with \(a_i \in B_{\text{dR}}\) and we need to prove that \(a_i \in B_{\text{dR}}^+\) for all \(i\). If this is not the case, then there exists \(n \geq 1\) such that if we set \(b_i = t^n a_i\), then we have \(b_1 t^{-h_1} e_1 + \cdots + b_d t^{-h_d} e_d \in t \cdot W\), with \(b_i \in (B_{\text{dR}}^+)^n\) for at least one \(i\). Consider the shortest such relation; in particular, \(b_i \in (B_{\text{dR}}^+)\) for all \(i\) for which \(b_i \neq 0\), and we can assume that \(b_i = 1\) for at least one \(i\). If \(g \in G_F\), then applying \(1 - \chi_{\text{cyc}}(g)^{h_i} g\) to the relation yields a shorter relation. This implies that \((1 - \chi_{\text{cyc}}(g)^{h_i-h_j} g)(b_j) \in t B_{\text{dR}}^+\) for all \(g \in G_F\) and all \(1 \leq j \leq d\). Since \(H^0(G_F, C_p) = F\) and \(H^0(G_F, C_p(h)) = \{0\}\) if \(h \neq 0\), we have \(b_j \in F + t B_{\text{dR}}^+\) if \(h_i = h_j\) and \(b_j \in t B_{\text{dR}}^+\) otherwise. The relation above therefore reduces to an \(F\)-linear combination of those \(e_j\) for which \(h_j = h_i\), belonging to \(D \cap t^{h_i+1} W = \text{Fil}^{h_i+1} D\), and is hence trivial. This proves that \(W \subset \text{Fil}^0(B_{\text{dR}} \otimes_F D)\).

**Definition 7.7.** — Let \(M\) be a crystalline \((\varphi_q, \Gamma_F)\)-module over \(\mathcal{R}(Y)\), which we can write as \(M = \mathcal{R}(Y) \otimes_{\mathcal{R}[s, +\infty]|Y)} M^{s; +\infty}\) for some \(s\) large enough. For \(m \gg 0\) and \(j = 0, \ldots, h - 1\) and \(n = hm - j\), define

\[
\text{Fil}^j(D_{\text{cris}}(M)) = D_{\text{cris}}(M) \cap t^j \cdot (B_{\text{dR}}^+ \otimes_{\mathcal{R}[s, +\infty]|Y)} M^{s; +\infty}).
\]

**Proposition 7.8.** — The definition of \(\text{Fil}^j(D_{\text{cris}}(M))\) does not depend on \(m \gg 0\), and we have \(\text{Fil}^0(B_{\text{dR}} \otimes_{\mathcal{R}[s, +\infty]|Y)} D_{\text{cris}}(M)) = B_{\text{dR}}^+ \otimes_{\mathcal{R}[s, +\infty]|Y)} M^{s; +\infty}\).
Proof. — If \( s \) is large enough, then \( M^{[s;+\infty]} = \varphi_q^s(M^{[s;+\infty]}) \) so that

\[
B^+_{dR} \otimes_{\mathcal{R}_{[s;+\infty]}(Y)} M^{[s;+\infty]} = B^+_{dR} \otimes_{\mathcal{R}_{[s;+\infty]}(Y)} \varphi_q(M^{[s;+\infty]}) = B^+_{dR} \otimes_{\mathcal{R}_{[s;+\infty]}(Y)} M^{[s;+\infty]},
\]

which implies the first statement. The second statement follows from lemma\ref{lem:7.6} applied to \( W = B^+_{dR} \otimes_{\mathcal{R}_{[s;+\infty]}(Y)} M^{[s;+\infty]} \).

**Theorem 7.9.** — The functors \( M \mapsto D_{\text{cris}}(M) \) and \( D \mapsto M(D) \), between the category of crystalline \((\varphi, \Gamma_F)\)-modules over \( \mathcal{R}(Y) \) and the category of \( \varphi_q \)-modules with \( h \) filtrations, are mutually inverse.

Proof. — If \( D \) is a \( \varphi_q \)-module with \( h \) filtrations, then it is clear that \( D_{\text{cris}}(M(D)) = D \) as \( \varphi_q \)-modules. The fact that \( \text{Fil}_j^h(D) = D \cap t^j \cdot \text{Fil}_j^h(B_{dR} \otimes_{F^+} D) \) follows from taking a basis of \( D \) adapted to \( \text{Fil}_j^h \) and

\[
\text{Fil}_j^h(B_{dR} \otimes_{F^+} D) = B^+_{dR} \otimes_{\mathcal{R}_{[s;+\infty]}(Y)} M^{[s;+\infty]}(D) = \text{Fil}_j^h(B_{dR} \otimes_{F^+} D_{\text{cris}}(M(D))
\]

by propositions\ref{lem:6.9} and\ref{lem:7.8}, so that the filtrations on \( D \) and \( D_{\text{cris}}(M) \) are the same.

We now check that if \( M \) is a crystalline \((\varphi_q, \Gamma_F)\)-module over \( \mathcal{R}(Y) \) and \( D = D_{\text{cris}}(M) \) with the filtration given in definition\ref{def:7.1} then \( M = M(D) \). Corollary\ref{cor:7.4} says that we have \( \mathcal{R}(Y)[1/t] \otimes_F D = \mathcal{R}(Y)[1/t] \otimes_{\mathcal{R}(Y)} M \). The theorem now follows from proposition\ref{lem:7.8} and the fact that if \( y \in \mathcal{R}(Y)[1/t] \otimes_{\mathcal{R}(Y)} M \), then \( y \in M \) if and only if \( y \in B^+_{dR} \otimes_{\mathcal{R}_{[s;+\infty]}(Y)} M \) for all \( n \gg 0 \) by theorem\ref{thm:3.6} and items (1) and (2) of definition\ref{def:7.3}.

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