The $r$-mean curvature and rigidity of compact hypersurfaces in the Euclidean space

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Abstract
In this paper, we characterize round spheres in the Euclidean space under some suitable conditions on the $r$-mean curvature.

Keywords Isometric immersions · Higher-order mean curvature · Self-shrinkers · $\lambda$-hypersurfaces · Minkowski integral formulas

Mathematics Subject Classification Primary 53C42 · Secondary 53C24 · 53C44

1 Introduction

Let $x: M^n \to \mathbb{R}^{n+1}$ be an isometric immersion of an orientable Riemannian manifold $M^n$ in the Euclidean space $\mathbb{R}^{n+1}$. Denote by $A$ the second fundamental form of the hypersurface with respect to an unit normal vector field $N$ globally defined on $M$. The $r$-mean curvature of $M$ is defined by

$$H_r = \left(\begin{array}{c} n \\ r \end{array}\right)^{-1} S_r,$$

where $S_r$ is the $r$-elementary symmetric function of the eigenvalues of $A$, for $r = 1, \ldots, n$, and $S_0 = 1$.

The Newton transformations $P_r$ related to the immersion $x$ are the linear maps defined recursively by $P_0 = I$ and $P_r = S_r I - A P_{r-1}$, when $1 \leq r \leq n$. Associated with each $P_r$, we have the linear differential operators of second order $L_r$ given by

$$L_r(u) = \text{tr}(P_r \nabla^2 u).$$

In the right-hand side of this equation, $\nabla^2 u$ stands for the Hessian operator of a smooth function $u$ on $M$. It is well known that the operators $L_r$ are elliptic if and only if the corresponding Newton transformations $P_r$ are positively defined. Moreover,
\[ L_r(u) = \text{div}(P_r(\nabla u)) , \]

where \( \nabla u \) is the gradient of \( u \) and \( \text{div} \) denotes the divergent operator of a vector field on \( M \). Classically, the operators \( P_r \) and \( L_r \) come from the variational aspects related to the problem of minimizing certain \( r \)-area functionals of the immersion \( x \). For more details, see the seminal paper of Rosenberg [14].

A still challenging topic in differential geometry is the rigidity of hypersurfaces \( M^n \) in the Euclidean space under some natural conditions on the topology and under suitable analytical assumptions on \( H_r \) or \( L_r \), for some \( r = 0, \ldots, n \). In this context, it is expected to prove that the immersion is in fact totally umbilical. We point out that there exists a vast literature on this subject, exploring the cases where \( x \) is an embedding or just an immersion. For instance, the celebrated work of Aleksandrov [1] claims that a closed (compact and with empty boundary) embedded hypersurface with constant mean curvature is a round sphere. This result was generalized by Ros [13] in the case that some higher-order mean curvature is constant. In the case that \( x \) is an immersion and \( M \) is a topological 2-sphere in \( \mathbb{R}^3 \) with constant mean curvature, the also celebrated work of Hopf [9] shows that \( M \) a round sphere. It is well known that for any integer \( g \geq 1 \) there are constant mean curvature surfaces with genus \( g \) in \( \mathbb{R}^3 \) (see [17] for \( g = 1 \) and [11, 12] for \( g \geq 2 \)). In higher dimensions, Hsiang, Teng and Yu [10] proved the existence of topological spheres in \( \mathbb{R}^{2n} \) with constant mean curvature that are not round.

The aim of this work is to present new characterizations of the Euclidean sphere in terms of the behavior of the \( r \)-mean curvature \( H_r \) when \( x \) is an immersion.

To state our first results, we denote by \( \rho \) the support function of \( x \), that is, \( \rho : M \to \mathbb{R}, \rho = \langle x, N \rangle \). Geometrically, \( \rho(p) \) is the distance with sign from the origin \( 0 \in \mathbb{R}^{n+1} \) to the hyperplane tangent to \( x(M) \) at \( x(p) \). Assuming that \( \rho \) is non-negative, Deshmukh [7] proved that the mean curvature \( H \) of \( M \) is a solution to the Poisson equation \( \Delta u = 1 + H_1 \rho \) if and only if, \( M \) is isometric to a round sphere. Our first theorem extends this result for the operators \( L_r \).

**Theorem 1.1** Let \( x : M^n \to \mathbb{R}^{n+1} \) be a closed hypersurface with non-negative support function such that operator \( P_r \) is positively definite. Then, the mean curvature \( H_1 \) satisfies the equation \( L_r u = 1 + H_1 \rho \) if and only if \( M \) is isometric to a round sphere.

Assuming that \( H_r \) is constant, we obtain that

**Theorem 1.2** Let \( x : M^n \to \mathbb{R}^{n+1} \) be a closed hypersurface with non-negative support function. Assume that, for some \( 1 \leq r \leq n-2 \), the operator \( P_r \) is positively defined and \( H_r \) is constant. Then, the mean curvature \( H_{r+1} \) satisfies the equation \( L_r u = H_r + H_{r+1} \rho \) if and only if \( M \) is isometric to a round sphere.

The positivity of the operator \( P_r \) is a natural analytical condition, which is automatically verified when \( r = 0 \). It is an interesting problem to prove Theorem 1.2 when \( H_r \) is not constant.

The techniques used to prove the theorems above can be applied to self-shrinkers of the Euclidean space. We recall that \( M^n \subset \mathbb{R}^{n+1} \) is a self-shrinker if the equation is satisfied

\[ H = -\frac{\rho}{2}, \quad (1) \]

where \( H \) is the non-normalized mean curvature of \( M \).
Self-shrinkers form an important class of solutions for the mean curvature flow and stand out in the study of the so-called type I singularities. See, for example, Colding and Minicozzi [5].

Some basic examples of self-shrinkers are hyperplanes passing through the origin, minimal cones, round spheres $\mathbb{S}^n(\sqrt{2n})$ and cylinders $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$, for $k = 1, ..., n - 1$.

In [8], Guo obtained some gap theorems for closed self-shrinkers and concluded that if the scalar curvature of such hypersurfaces is constant, then they are isometric to a round sphere. In the following, we present a direct and more general result.

**Theorem 1.3** Let $x : M^n \to \mathbb{R}^{n+1}$ be a closed self-shrinker with $H_{r+1}$ constant for some $1 \leq r \leq n - 1$. Then, $M = \mathbb{S}^n(\sqrt{2n})$.

As a natural extension of self-shrinkers, we say that $M$ is a $\lambda$-hypersurface if the following equation is satisfied

$$H = -\frac{\rho}{2} + \lambda, \tag{2}$$

where $\lambda$ is a constant. For example, the sphere $\mathbb{S}^n(r)$ with radius $r$ is a $\lambda$-hypersurface in $\mathbb{R}^{n+1}$ for $\lambda = n/r - r/2$.

This concept was introduced by Cheng and Wei in [4] where they studied mean curvature flow that preserves a weighted volume. The authors show, among other facts, that a compact $\lambda$-hypersurface is isometric to a round sphere if $H - \lambda \geq 0$ and $\lambda(A - |A|^2) \geq 0$, where $|A|^2 = \sum_i h_{ij}^2$ is the square of the norm of the second fundamental form and $f = \sum_i h_{ij} h_{jk} h_{ki}$.

Applying the same approach as in the proof of Theorem 1.3, we obtain a simple proof of the following theorem due to Ross [15].

**Theorem 1.4** Let $x : M^n \to \mathbb{R}^{n+1}$ be a closed $\lambda$-hypersurface with $H \geq \lambda$. If $|A|^2 \leq 1/2$, then $M$ is a round sphere.

## 2 Preliminaries

In order to obtain our rigidity results, we need the following propositions, which, besides being important in themselves, have several other applications in problems involving higher-order mean curvatures of hypersurfaces. We emphasize that such propositions are valid in space forms.

**Proposition 2.1** Let $x : M^n \to \mathbb{R}^{n+1}$ be an orientable hypersurface of the Euclidean space. If $\rho : M \to \mathbb{R}$ is the support function of $x$, then

$$L_{\rho}(\rho) = -(r + 1)S_{r+1} - (S_1 S_{r+1} - (r + 2)S_{r+2})\rho - \langle \nabla S_{r+1}, x^T \rangle, \tag{3}$$

where $x^T$ indicates the component of $x$ tangent to $M$.

**Proof** See Alencar and Colares [2], page 209. \qed
Corollary 2.2 Let $x : M^n \to \mathbb{R}^{n+1}$ be an orientable hypersurface of the Euclidean space. If $\rho : M \to \mathbb{R}$ is the support function of $x$, then
\[
\Delta \rho = -nH_1 - |A|^2 \rho - n\langle \nabla H_1, x^T \rangle.
\] (4)

Proof Take $r = 0$ in equation (3) and use the identity $|A|^2 + 2S_2 = S_1^2$. \qed

The so-called Garding and Newton inequalities are used to prove the following result:

Proposition 2.3 Let $x : M^n \to \mathbb{R}^{n+1}$ be a closed orientable hypersurface. If $H_{r+1}$ is positive on $M$, then for every $i$, with $1 \leq i \leq r$, we have:

(a) Each $H_i$ is positive.
(b) $H_i H_{i+2} - H_{i+1} \geq 0$.

Moreover, equality in (b) occurs for some $i$ if, and only if $M$ is a round sphere.

Proof See Silva et al. [6], page 297. \qed

In the next result, we present the classical Minkowski integral formula. For the sake of completeness, we present a concise demonstration following ideas ofAlias and Malacarne in [3].

Proposition 2.4 Let $x : M^n \to \mathbb{R}^{n+1}$ be a closed hypersurface. Then, for every $r$, with $0 \leq r \leq n - 1$, we have
\[
\int_M (H_r + H_{r+1})dM = 0.
\]

Proof Consider the function $g : M^n \to \mathbb{R}$ defined by $g = (1/2)|x|^2$. We know that $\nabla g = x^T$, where $x^T = x - \rho N$. Then, for each tangent vector field $X$ on $M$ we have
\[
(\nabla^2 g)(X) = \nabla_X (\nabla g) = X + \rho A(X).
\]

Therefore,
\[
L_r(g) = tr(P_r \nabla^2 g) = tr(P_r) + tr(AP_r) \rho = c_r H_r + c_r H_{r+1} \rho = c_r (H_r + H_{r+1} \rho),
\]
with $c_r = (n - r) \binom{n}{r}$ and the traces are determined in [14], page 13. By the divergence theorem, it follows that
\[
\int_M (H_r + H_{r+1})dM = 0,
\]
finalizing the proof. \qed

To conclude this section, we present two identities that will be useful for our purposes. First, a directly computation yields
\[
\Delta |x|^2 = 2n(1 + H_1 \rho).
\] (5)
The next identity is a consequence of the divergence theorem.

$$\int_M u L_r(v) dM = - \int_M \langle P_r(\nabla u), \nabla v \rangle dM,$$

whenever $u$ and $v$ are smooth functions on $M$.

### 3 Proof of Theorems

In this section, we present the proofs of our theorems. For the reader’s convenience, we state the theorems again.

**Theorem 3.1** Let $x : M^n \to \mathbb{R}^{n+1}$ be a closed hypersurface with non-negative support function such that operator $P_r$ is positively definite. Then, the mean curvature $H_1$ satisfies the equation $L_r u = 1 + H_1 \rho$ if and only if $M$ is isometric to a round sphere.

**Proof** If $H_1$ is a solution to the equation, we have $H_1 L_r H_1 = H_1 + H_1^2 \rho$. So, applying identity (6) we get

$$-n \int_M \langle P_r(\nabla H_1), \nabla H_1 \rangle dM = n \int_M H_1 dM + n \int_M H_1^2 \rho dM.$$ 

On the other hand, using formula (4)

$$n \int_M H_1 dM = - \int_M |A|^2 \rho dM + \frac{n}{2} \int_M H_1 \Delta |x|^2 dM.$$ 

From (5) and the hypothesis about $H_1$, we rewrite this last equality as

$$n \int_M H_1 dM = - \int_M |A|^2 \rho dM - n^2 \int_M \langle P_r(\nabla H_1), \nabla H_1 \rangle dM.$$ 

Therefore,

$$(n^2 - n) \int_M \langle P_r(\nabla H_1), \nabla H_1 \rangle dM + \int_M (|A|^2 - nH_1^2) \rho dM = 0.$$ 

Since $P_r$ is positively definite and $\rho \geq 0$, it follows that $H_1$ and $\rho$ are constant. Furthermore, we conclude that $M$ is totally umbilical and therefore a round sphere. \hfill \Box

We now recall the following algebraic inequality related to $r$th mean curvature. For each $1 \leq r \leq n-1$, it holds

$$H_r^2 \geq H_{r-1} H_{r+1},$$

and equality occurs only at umbilical points of $M$. See, for example, Steele [16], page 178.

**Theorem 3.2** Let $x : M^n \to \mathbb{R}^{n+1}$ be a closed hypersurface with non-negative support function. Assume that, for some $1 \leq r \leq n-2$, the operator $P_r$ is positively defined and $H_r$ is constant. Then, the mean curvature $H_{r+1}$ satisfies the equation $L_r u = H_r + H_{r+1} \rho$ if and only if $M$ is isometric to a round sphere.
Proof As before, if \( H_{r+1} \) is a solution to that equation, we get

\[- \int_M \langle P_r(\nabla H_{r+1}), \nabla H_{r+1} \rangle dM = \int_M H_r H_{r+1} dM + \int_M H_{r+1}^2 \rho dM.\]

Since \( H_{r+1}^2 \rho \geq H_r H_{r+2} \rho \), and using our hypotheses on \( P_r, H_r \) and the Minkowski formula, we obtain

\[0 \geq - \int_M \langle P_r(\nabla H_{r+1}), \nabla H_{r+1} \rangle dM \geq H_r \int_M (H_{r+1} + H_{r+2} \rho) dM = 0,\]

It follows that \( H_{r+1} \) is constant and so all inequalities above are equalities. It means that \( H_r^2 = H_{r-1} H_{r+1} \) on \( M \), and we conclude that \( M \) is a round sphere. \( \square \)

Now we prove our theorems on self-shrinkers.

Theorem 3.3 Let \( x : M^n \to \mathbb{R}^{n+1} \) be a closed self-shrinker with \( H_{r+1} \) constant for some \( 1 \leq r \leq n-1 \). Then, \( M = \mathbb{S}^n(\sqrt{2n}) \).

Proof Since \( H_{r+1} \) is constant, we obtain by integrating identity (3)

\[0 = -(r+1) \binom{n}{r+1} \int_M H_{r+1} dM - \int_M \left[ n \binom{n}{r+1} H_1 H_{r+1} - (r+2) \binom{n}{r+2} H_{r+2} \right] \rho dM = -(r+1) \binom{n}{r+1} \int_M H_{r+1} dM - \int_M \left[ n \binom{n}{r+1} H_1 H_{r+1} - (n-(r+1)) \binom{n}{r+1} H_{r+2} \right] \rho dM\]

and organizing the terms,

\[-(r+1) \int_M (H_{r+1} + H_{r+2} \rho) dM - n \int_M (H_1 H_{r+1} - H_{r+2}) \rho dM = 0.\]

Therefore, from Proposition 2.4 and by the equation of a self-shrinker we have,

\[\int_M (H_1 H_{r+1} - H_{r+2}) H dM = 0.\]

Choosing the orientation such that \( H_{r+2} \rho > 0 \), we conclude by Proposition 2.3 that \( M \) is totally umbilical. Therefore, \( M = \mathbb{S}^n(\sqrt{2n}) \). \( \square \)

Finally, we will show that

Theorem 3.4 Let \( x : M^n \to \mathbb{R}^{n+1} \) be a closed \( \lambda \)-hypersurface with \( H \geq \lambda \). If \( |A|^2 \leq 1/2 \), then \( M \) is a round sphere.

Proof First, let us consider the case \( \lambda \leq 0 \). Since \( \rho = 2(\lambda - H) \leq 0 \), we can use identity (4) to obtain
It follows from the strong maximum principle that $\rho$ is constant and thus $H$ is also constant. Now we can use Minkowski formula and identity (4) to conclude that $n|A|^2 = H^2$, and so $M$ is totally umbilical.

Now, let us assume that $\lambda \geq 0$. Since $H = nH_1$ and $\rho = 2(\lambda - H)$, we use identity (4) to get

$$0 = \int_M HdM + \int_M 2(\lambda - H)|A|^2 dM + \frac{1}{2} \int_M \langle \nabla H, \nabla |x|^2 \rangle dM$$

$$= \int_M HdM + \int_M 2(\lambda - H)|A|^2 dM - \frac{1}{2} \int_M H|\nabla x|^2 dM$$

$$= \int_M HdM + \int_M 2(\lambda - H)|A|^2 dM - \int_M (n + 2(\lambda - H)H) dM,$$

where we use formula (5) in the last equality. Organizing the terms, we obtain

$$\int_M [(n-1)H + 2(H - \lambda)(|A|^2 - H^2)] dM = 0. \tag{7}$$

Now, using that $H^2 \leq n|A|^2 \leq n/2$ we conclude that

$$(n-1)H + 2(H - \lambda)(|A|^2 - H^2) \geq (n-1)H - 2(H - \lambda)(n-1)|A|^2$$

$$\geq (n-1)H - (n-1)(H - \lambda)$$

$$= (n-1)\lambda \geq 0.$$ 

In view of identity (7), we conclude that all inequalities above are in fact identities. In particular, $M$ is a round sphere. $\square$

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