DLMP-based Coordination Procedure for Decentralized Demand Response under Distribution Network Constraints

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Abstract—Load aggregators are independent private entities whose goal is to optimize energy consumption flexibilities offered by multiple residential consumers. Although aggregators optimize their decisions in a decentralized way, they are indirectly linked together if their respective consumers belong to the same distribution grid. Thus, a major issue for the distribution system operator (DSO), responsible of the good operation of the distribution network, is to ensure that the decentralized decisions taken do not violate the grid constraints and do not increase the global system costs. The problem is also challenging from the information point of view as the network state and characteristics usually constitute confidential information for the DSO. To address this issue, we propose a decentralized coordination procedure between the DSO and multiple aggregators. The procedure, based on distribution locational marginal prices (DLMP), preserves the decentralized structure of information and decisions, and lead to a feasible and optimal solution for the aggregators and the DSO.

Index Terms—Decentralized Systems, Demand Response, Load Aggregator, Distribution Locational Marginal Prices, AC Optimal Power Flow.

INTRODUCTION

The management of electricity consumption flexibilities, or Demand Response [1], offered by new usages such as electric vehicles and smart appliances, is considered as a key component of modern electricity systems to increase the share of renewable energy production, reduce carbon emissions and ensure the grid stability and resilience. In this context, aggregators are new actors of the electric system, whose role is to aggregate a large number of individually negligible consumption flexibilities offered by residential or small consumers, and valuate these flexibilities on the demand response market [2] or as a service offered to the system operator. In liberal electricity markets as in Europe or in the United States, several Load Aggregators (LAs) can compete on the same distribution network. There is a hierarchical decisions structure, as explained in [2], from the distribution system operator (DSO) in charge of the network, which interacts with the present LAs, each LA interacting at the lower level with a subset of affiliated end-consumers. This decentralized system involves multiple actors that are meant to engage in decentralized decisions. However, the decisions of LAs and of the DSO are all linked by the physical constraints of the underlying network (line capacities, voltage limits, etc): if each LA manages its flexibilities ignoring these constraints, the resulting flows could jeopardize the stability of the network or, from the mathematical point of view, be infeasible.

In addition to the multiplicity of actors, the information asymmetry is also a key issue in the coordination: the network physical parameters (topology, line resistances and capacities, etc.) are often considered as confidential by the DSO and not revealed to third parties, while, on the other hand, LAs might have privacy considerations regarding the flexibilities provided by consumers.

The objective of the present paper is to provide a decentralized coordination mechanism for LAs, based on the computation of Distribution Locational Marginal Prices (DLMPs) obtained from a conic relaxation of the Alternative Current Optimal Power Flow (ACOPF) problem. Considering an AC model of the network enables to take into account not only capacity constraints but also voltage and angle constraints, which are limiting in practice in distribution networks, as stated above. The underlying idea of the proposed mechanism is to use DLMPs, centrally computed by the DSO, as price incentives for LAs to manage their flexibilities. Through such a procedure, the decentralized structure as well as the asymmetry and privacy of information are preserved.

Related Works. Recent works have shown the existence of tight conic relaxations of the ACOPF problem, with some cases of exact relaxations exact in particular for radial networks. The works [3, 4] consider an SDP relaxation of ACOPF and show its exactness. The works [5, 6] consider different Second Order Cone Programming (SOCP) relaxations of the OPF problem and show that the SOCP relaxations can be exact under some additional assumptions. The authors in [6] showed that, in tree networks, the branch flow SOCP relaxation is exact whenever the SDP model [3] is exact. As SOCP is computationally simpler than SDP [9, 10], we focus in this paper on the SOCP branch flow model of [6].

The idea of using DLMPs to coordinate LAs and electric vehicles charging in a decentralized fashion appears in [11]. Huang et al [12] propose to use a quadratic model to implement the decentralized procedure using DLMPs, to coordinate LAs of electric vehicles or heat pumps. Both [11] and [12] consider a direct current (DC) power flow model, and DLMPs are the Lagrangian multipliers associated to the capacity constraint for each time period in the DSO optimization problem. Although DC power flow models are much easier to solve than ACOPF problems, a DC model does not consider all existing constraints in distribution networks such as voltage limits.
Papavasiliiou [13] considers two formulations of the ACOPF in distribution networks, based on an implicit function formulation and on the SOCP relaxation [14], and derives expressions and properties of the DLMPs in each formulation. The SOCP formulation adopted in this paper, is the most interesting from a computational point of view.

Liu et al. [15] consider solving the original (nonconvex) ACOPF problem for the DSO to obtain DLMPs sent to multiple LAs to compute flexible consumption profiles satisfying consumers constraints. However, because of its nonconvexity [6], the original ACOPF problem can be very hard to solve in practice, and only a local optimum may be found. The objective functions considered by LAs in [15] are linear, one cannot ensure, as noted in [12], that the combination of the individual LAs solutions obtained through this decentralized procedure does not necessarily correspond to the optimal solution computed by the DSO.

Lin et al. [16] propose a coordination mechanism between transmission and distribution level, considering an AC branch flow model for distribution grids. Their method relies on the exchange of information (boundary variables and lower bounds) between transmission and distribution levels.

Bai et al. [17] propose to solve the ACOPF, considering different types of distributed energy resources (DERs) as well as feeder reconfiguration and on load tap changers, involving discrete decision variables. The authors propose to solve in a first step the problem considering the SOCP relaxation defined in [6] with the discrete variables, to determine the optimal values of these variables. In a second step, the discrete variables are fixed and a linearization of the SOCP problem around the optimal value is solved to obtain DLMPs, which are then transmitted to the DERs.

The framework considered in this paper differs from the above mentioned works in at least two points. First, the procedures proposed in the existing literature are not decentralized as, to compute the DLMPs by solving the OPF, the authors consider that the DSO has access to all the information from the DERs (capacities, state of charge for batteries, power bounds for LAs). A solution could be to impose to all the DERs (capacities, state of charge for batteries, power bounds for LAs). A solution could be to impose to all the DERs (capacities, state of charge for batteries, power bounds for LAs) to provide the necessary information to the DSO in bounds for LAs). A solution could be to impose to all the DERs (capacities, state of charge for batteries, power bounds for LAs).

A. Branch Flow Model and SOCP relaxation

We consider a distribution network represented by a set of nodes \( \mathcal{N} \triangleq \{1, \ldots, N\} \) and given as a graph \( \mathcal{G} \triangleq (\mathcal{N}, \mathcal{E}) \), where \( \mathcal{N}_+ \triangleq \mathcal{N} \cup \{0\} \) with 0 denoting the root node and corresponding to the feeder node (link with transportation network).

Distribution networks are usually designed such that there is no cycle in the electricity lines. Thus, the graph \( \mathcal{G} \) is a tree.

Each node \( n \in \mathcal{N} \) corresponds to an individual household, a group of households or a commercial building linked to the distribution network. We assume that the operation of the distribution network and the management of flexibilities is done on a common time horizon \( T \) (e.g. a day), given as a finite set of discrete time periods:

\[
T \triangleq \{1, \ldots, T\}.
\]

A. Branch Flow Model and SOCP relaxation

We use the branch flow model and the SOCP relaxation introduced in [14] and also considered in [13].

Following [13], we use \( p_n, q_n \) to denote active and reactive power consumption at node \( n \); thus \( p_n < 0 \) means that there is production at bus \( n \). At the root (feeder) node \( n = 0 \), we expect that some power will be produced (or bought from the market) and that there is no consumption: we therefore assume \( p_{0,t} \leq 0 \). Variable \( \varepsilon_n \) stands for the squared voltage magnitude at bus \( n \). Variables \( f_n, g_n \) and \( t_n \) denote the active and reactive power flows and the squared current magnitude...
on the line from \( n \) to the unique ancestor of node \( n \), denoted by \( n_a \). The resistance and reactance on this line are denoted by \( R_n \) and \( X_n \), while the shunt conductance and susceptance at node \( n \) are denoted by \( G_n \) and \( B_n \). The power flow magnitude limit (line capacity) on line \((n, n_a)\) is denoted by \( S_n \).

We obtain the set of branch flow equations:

\[
\begin{align*}
v_{n,t} - 2(R_n f_{n,t} + X_n g_{n,t}) + \ell_{n,t} (R_n^2 + X_n^2) = v_{n_a,t}, & \quad \forall n \in N \quad (1a) \\
f_{n,t} - \sum_{m,m_a=n} (f_{m,t} - \ell_{m,t} R_{m,t}) + p_{n,t} + G_{n,t} v_{n,t} = 0, & \quad \forall n \in N_+ \quad (1b) \\
g_{n,t} + \sum_{m,m_a=n} (g_{m,t} - \ell_{m,t} X_{m,t}) + q_{n,t} - B_{n,t} v_{n,t} = 0, & \quad \forall n \in N_+ \quad (1c) \\
|f_{n,t}^2 + g_{n,t}^2| \leq v_{n,t} \ell_{n,t}, & \quad \forall n \in N \quad (1d) \\
|f_{n,t}^2 + g_{n,t}^2| \leq S_n^2, & \quad \forall n \in N \quad (1e) \\
(f_{n,t} + R_n \ell_{n,t})^2 + (g_{n,t} - X_n \ell_{n,t})^2 \leq S_n^2, & \quad \forall n \in N \quad (1f)
\end{align*}
\]

and we further denote by \( \beta_{n,t}, \lambda_{n,t}, \mu_{n,t} \in \mathbb{R} \) the Lagrangian multipliers associated respectively to \((1a), (1b), (1c)\), and by \( \gamma_{n,t}, \eta_{n,t}, \theta_{n,t}, \sigma_{n,t}, \tau_{n,t} \geq 0 \) the ones associated to constraints \((1d), (1e), (1f)\). In \((1f)\), the equality \( f_{n,t}^2 + g_{n,t}^2 = v_{n,t} \ell_{n,t} \) defining the current magnitude \( \ell_{n,t} \) in the branch flow model, is relaxed as the inequality \((1d)\), which is a cone constraint (see [6] for more details).

**B. Electricity Load Aggregators**

We consider that a set \( A \) of several Load Aggregators (LAs) coexist on the distribution network. Each LA \( a \in A \) manages a subset \( \mathcal{N}_a \) of the nodes in the network, such that \( \bigcup_a \mathcal{N}_a \) forms a partition of \( N \) (each node is affiliated with one LA).

The active and reactive power at each node is not fixed but flexible: the LA \( a \in A \) manages the flexible net power consumption \((p_{n,t}, q_{n,t})\) of individual consumers at \( n \) for each node \( n \in \mathcal{N}_a \), w.r.t. individual constraints. We consider the possible presence of local renewable energy sources (e.g., photovoltaic panels) that can also be managed by the LA, such that \( p_{n,t} \) is composed of a production part \( p_{n,t}^P \geq 0 \) and a consumption part \( p_{n,t}^C \geq 0 \), as:

\[
\forall n \in \mathcal{N}_a, \forall t \in T, \quad p_{n,t} = p_{n,t}^C - p_{n,t}^P.
\]

The constraints on the consumption of each node \( n \) are described through a global energy demand \( E_n \) over the \( T \) time periods, as well as lower and upper bounds for each time period, that is:

\[
\begin{align*}
\sum_{t \in T} p_{n,t}^C & \geq E_n \quad (2b) \\
p_{n,t}^C & \leq p_{n,t}^P \leq P_{n,t}^F \quad (2c)
\end{align*}
\]

Constraints \((2b)\) give a simple model for deferrable loads such as electric vehicles and water heaters, which has been widely used \([18, 23]\).

In practice, each LA can aggregate the flexibilities offered by individual end-consumers to fit in the model \((2b)\). The same idea of aggregate set of constraints in a particular form is formulated for instance in \([24]\), where the authors propose an aggregation procedure considering zonotopic sets instead of the simplex structure \((2b)\).

We consider that active and reactive consumptions are correlated by a given ratio (depending on the type of appliances) as:

\[
q_{n,t}^F = \tau_{n} p_{n,t}^F.
\]

Each LA also controls the power \((p_{n,t}^P)_{n \in N_a, t} \) produced by distributed energy resources (DERs) across its affiliated nodes \( N_a \). The DERs \([17]\) we consider are local and renewable energy (either photovoltaic or wind power) installed in households. We consider that the power \( p_{n,t}^P \) produced at period \( t \) is upper bounded by an intermittent available power \( P_{n,t}^F \) (which depends on the DERs capacities and the wind or sun power), and can be adjusted within \([0, P_{n,t}^F]\). The associated reactive power \( q_{n,t}^F \) can be adjusted through the control of smart inverters \([25]\), which gives:

\[
0 \leq p_{n,t}^P \leq P_{n,t}^F \quad (2e)
\]

We consider that DERs are used at cost zero by the LA operating them: incentives and costs can be distributed by the LA to its affiliated households in a separate process. The study of interactions between an LA and its affiliated consumers is beyond the scope of this paper.

We define the cost function of LA \( a \) as \( \phi_a(\cdot) \) and assume that it depends on the active consumption profiles \((p_n)_{n \in N_a}\).

The cost function \( \phi_a \) is determined by exogenous parameters (e.g. from day-ahead electricity market prices) or by some incentives from the system operator. Thus, the local problem \( P_a \) of LA \( a \) is formulated as:

\[
\begin{align*}
\min_{x_a=(p_n,q_n)_{n \in N_a}} & \quad \phi_a((p_n)_{n \in N_a}) \\
\text{s.t.} & \quad (2)
\end{align*}
\]

where \( x_a = (p_n,q_n)_{n \in N_a} \) denote the variables of LA \( a \in A \) that are controlled locally by \( a \).

**C. Distribution System Operator**

The Distribution System Operator (DSO) is a central, independent entity in charge of the operation of the distribution network, which is able to impose (price) incentives to LAs.

In most of the electric systems, the DSO also bears the costs of the network losses, by buying the necessary energy at the day-ahead market price. The corresponding fees are usually recovered by taxes to utilities collected by the DSO.

We assume the existence of underlying energy price functions \((c_t(\cdot))_{t \in T}\) modeling, for instance, an electricity market or some production costs. Following \([12]\), we assume that each energy price function is affine and increasing, i.e.,
c_l(x) \triangleq \alpha_l + \beta_l x \text{ for some } \alpha_l, \beta_l \geq 0. \text{ We consider different possible objective functions for the DSO:}

i) \text{ minimization of the social cost of LAs:}

\[ \text{SC}(p) \triangleq \sum_{a \in A} \phi_a((p_{n,t})_{n,t}) = \sum_{t \in T} c_t \left( \sum_{n \in N} p_{n,t} \right) ; \quad (3a) \]

ii) \text{ minimization of the total active power injected at the substation node } 0 \text{ into the grid and bought at the DA market price:}

\[ \phi_{\text{m}}(p_0) \triangleq \sum_{t \in T} c_t(p_{0,t}^0) = \sum_{t \in T} c_t(-p_{0,t}) ; \quad (3b) \]

iii) \text{ minimization of active network losses:}

\[ \phi_{\text{loss}}(\ell) \triangleq \sum_{t \in T} \sum_{n \in N} R_n \ell_{n,t} . \quad (3c) \]

The objectives \(3a\) and \(3b\) are different in general: because of network losses, we will have \(p_{0,t} \neq \sum_n p_{n,t}\).

From the multi-agents point of view, the objectives concern different entities: the cost \(\phi_0\) in \(3a\) concerns the LA operator, while the costs \(3b\) and \(3c\) make more sense at the level of the DSO. It is thus relevant to differentiate both cases by the following notation:

- \(x_A \triangleq (x_a)_{a \in A} \text{ denotes the LAs variables as defined above, while } \phi_A(x_A) \triangleq \sum_a \phi_a(x_a) \text{ denotes the LAs part in the cost function;}
- x_0 \triangleq (p_0, q_0, v, \ell, f, g) \text{ denotes the DSO variables, while } \phi_0(x_0) \triangleq \phi_{\text{m}}(p_0) + \alpha_{\text{loss}}\phi_{\text{loss}}(\ell) \text{ denotes the DSO part in the cost function.}

In what follows, we will assume that the DSO objective function \(\Phi\) will be either \(\phi_A\) (as the minimization of social cost) or \(\phi_0\), or a linear combination of the three objectives \(3\):

\[ \Phi(x) \triangleq \phi_0(x_0) + \phi_A(x_A) . \]

In the decentralized framework adopted of this paper, the DSO considers the local variables \(x_A \triangleq (p_n, q_n)_{n \in N} \) as fixed parameters, as those variables are managed by the LAs. Thus, the DSO faces the following optimization problem:

\[
\min_{x_0=\{p_0, q_0, v, \ell, f, g\}} \Phi(x) \quad \text{s.t. } (\mathcal{P}_0(x_A))
\]

Despite he does not control the local variables \(x_A\), the DSO is interested in finding the best operating solution for the whole system, that is, obtaining an optimal solution of the global centralized problem:

\[
\min_{x_0, x_A} \Phi(x) \quad \text{s.t. } (\mathcal{P}^*)
\]

From now on, we assume that problem \(\mathcal{P}^*\) has a solution:

**Assumption 1.** There exists \((x_0, x_A)\) satisfying \(\mathcal{P} \text{ - } 2\) or, equivalently, problem \(\mathcal{P}^*\) is feasible.

In what follows, we recall that the SOC relaxation \(\mathcal{P}^*\) can be exact: a solution of this problem will, in many cases, give an optimal solution of the original OPF problem (i.e. the same problem \(\mathcal{P}^*\) with constraint \(\mathcal{P}\) written as an equality).

**II. Exactness of SOC Relaxation**

In [4, 10], the authors show that, in a radial network, the SOC relaxation \(\mathcal{P}^*\) of the nonlinear OPF problem is exact under some specific assumptions. The first result stated below is a straightforward extension of [4, Thm. 1] to the multi-time periods OPF problem \(\mathcal{P}^*\).

**Theorem 1.** Suppose that the objective function \(\Phi\) is convex, strictly increasing in \(t\), independent of \(f, g, p\) plus one of the following:

- \(\Phi\) is nonincreasing in \(v^c, q^c > 0\), and upper bounds \(\mathcal{A}_n, \mathcal{A}_m\) on \(v^c, q^c\) are not binding (that is, \(\mathcal{P}_{n,t} = \infty, Q_{n,t} = \infty\));
- \(\Phi\) is nondecreasing in \(p^c, q^c > 0\), and lower bounds \(4\) on \(p^c, q^c\) are not binding;

then the SOC relaxation \(\mathcal{P}^*\) given above is exact.

The authors in [10] also consider hypotheses on the voltage magnitude constraints that should not be binding on a strong sense. Thm. 2 below is an immediate extension of [10, Thm.1] to multi-time periods.

**Theorem 2.** Let \(\mathcal{P}_n = \{n, n_1, \ldots, n_l\} \subset N_+\) denotes the unique path from \(n\) to root node \(0\). If \(\Phi(\cdot)\) is strictly increasing in \(p_0\) and if:

- there is no shunt capacitances and admittances \((\forall n \in N, B_n = G_n = 0)\);
- for any optimal \(x = (p, q, f, g, v, \ell) \in \text{sol}(\mathcal{P}^*)\), the linearized flow solutions (considering \(\ell_{n,t} = 0 \in \mathcal{P}\)):

\[
\hat{\ell}_{n,t}(p_t) \triangleq \sum_{m:n \in P_m} p_{m,t}, \quad \tilde{v}_{n,t}(q_t) \triangleq \sum_{m:n \in P_m} g_{m,t} ;
\]

\[
\hat{v}_{n,t}(x) \triangleq V_0 + 2 \sum_{m:n \in P_m} R_m \hat{\ell}_m(p_t) + X_m \tilde{g}_m(q_t) ,
\]

verifies:

- for any \(t \in T\), any path \((n_1, \ldots, n_l)\) to a leaf bus \(l \in N\), and any \((1 \leq s \leq k \leq l)\), we have:

\[
A_{n_{s-1},t} \cdots A_{n_{k-1},t} u_{n_k} > 0 \text{ with } u_n \triangleq \left(\frac{R_n}{X_n}\right)
\]

and \(A_{n,t} \triangleq I - \frac{2}{\sum_{n \in N} \left(\frac{R_n}{X_n}\right)^2} \left[\left[f_n(p)\right]^+ + \left[g_n(q)\right]^+\right]\) then the SOC relaxation \(\mathcal{P}^*\) given above is exact.

It has been verified in [10] that the conditions of Thm. 2 are verified in many standard networks. More importantly, the conditions of Thms 1 and 2 are sufficient conditions, but are not necessary: as shown in the example of Sec. V, but also in [10], in general the SOC relaxation \(\mathcal{P}^*\) is exact even if conditions of Thms 1 and 2 do not hold.

The objective of this paper is not to improve those results of exactness, but to rely on the SOC relaxation of the branch flow model to design an efficient coordination procedure between the DSO and LAs while considering the decentralized information structure. Thus, for the remaining of the paper, we make the following assumption:

**Assumption 2.** For the instances considered, problem \(\mathcal{P}^*\) is an exact relaxation of the actual ACOPF problem.

Assumption 2 is easy to verify a posteriori for a solution: it suffices to check that \(\mathcal{P}\) is an equality.
III. DISTRIBUTION LOCATIONAL MARGINAL PRICES

The idea of using DLMPs has been considered in the literature in the last five years [12, 17, 13]. In addition to being mathematically funded, DLMPs provide a decentralized tool that has shown its efficiency for more than a decade at the transmission level.

In a context of an important level of local renewable energy, DLMPs for the reactive power part [16] can also be a valuable tool to improve the stability and power quality of the distribution grid. In some cases, the reactive power can be adapted locally, for instance through smart inverters associated to renewable sources. Thus, in this paper, we will consider DLMPs for both the active power $p_n$ and the reactive power $q_n$ for node $n$, as defined in Sec. I.

Besides, advances in conic optimization to solve the OPF problem (SOCP [22] and SDP [26] relaxations) make the idea of relying on DLMPs for decentralized coordination more relevant. Indeed, by solving the SOCP problem (22) or $\mathcal{P}(x,\lambda)$ representing an instance of OPF with standard interior point methods [3, Ch. 11], the active (resp. reactive) DLMPs are directly obtained by the system operator as the dual solutions $(\lambda_n)_{n \in N}$ (resp. $(\mu_n)_{n \in N}$), and can be transmitted to the LAs as price incentives. This is the basis of the decentralized coordination methods proposed in Sec. IV.

The SOCP formulation (22) enables to decompose the DLMPs $(\lambda_n, \mu_n)$ and give a mathematical interpretation of those prices. The following proposition completes [13, Prop. 3.2] where the author studies different interpretations of DLMPs in the SOCP formulation (22) of the ACOPF, as well as in two other ACOPF formulations.

**Proposition 1.** The DLMPs $(\lambda_n, \mu_n)$ at node $n$ can be expressed as a linear combination of the DLMPs of the ancestor node $n$, in addition to some dual quantities related to the line capacity constraints (1c), (1d) and the voltage definition constraints (1a), (1b):

\[
\lambda_n = \lambda_n + 2f_n \eta_n + 2f_n \eta^+_n + 2(f_n - R_n \lambda_n) \eta^-_n + 2\beta_n R_n, \\
\mu_n = \mu_n + 2g_n \gamma_n + 2g_n \eta^+_n + 2(g_n - X_n \lambda_n) \eta^-_n + 2\beta_n X_n.
\]

**Proof.** The conditions above are part of the KKT conditions of optimality, obtained by taking the partial derivative of the Lagrangian function of (22) w.r.t. $(f_n, x_n)$ and $(g_n, x_n)$. We refer to [13, Sec. II.B] for expression of the KKT conditions.

Prop. 1 does not provide a closed form of the DLMPs, as we cannot obtain an explicit expression of Lagrangian multipliers $\gamma_n, \beta_n$. The interpretation of multipliers $\gamma_n, \beta_n$ is not straightforward, as noticed in [13], where the author states that the DLMPs $\lambda_n$ can be expressed as nonlinear functions of $\lambda_n, \mu_n, \mu_n, \eta^+_n$, and $\eta^-_n$, although these functions are implicit. However, Prop. 1 shows that the DLMPs at a given node are linearly linked to the DLMPs at the parent’s node. Making some simplifying assumptions, we can actually state a stronger result, given in Prop. 2 below:

**Proposition 2.** If reactances and shunt reactances are neglected at node $n$, i.e. $R_n = X_n = G_n = B_n = 0$ and if the line $(n, m)$ is not saturated at time $t$, i.e. the inequalities (1c), (1d) are strict, then the DLMPs at $n$ at $t$ and at the parent node $m$ are equal:

\[
\lambda_{n,t} = \lambda_{m,t}, \quad \mu_{n,t} = \mu_{m,t}.
\]

**Proof.** As capacity constraints are not binding, we get from the complementarity conditions that $\eta^+_n = \eta^-_n = 0$. From the KKT condition obtained by taking the derivative of the Lagrangian w.r.t. $\ell_n, t$ and as $R_n = X_n = G_n = B_n = 0$, we get:

\[
0 = \gamma_{n,t} + \nu_{n,t}.
\]

As $\nu_{n,t} > 0$, we necessarily have $\gamma_{n,t} = 0$. Thus, simplifying the equalities stated in Prop. 1 gives exactly (4).

Prop. 2 has further consequences: if the conditions hold for all nodes $n \in N$, then all DLMPs are equal to the DLMPs at the root node $\lambda_0, \mu_0$. In particular, for each $n \in N$, $\lambda_n = \nabla p_n \phi_n(p_n)$ i.e. the DLMPs are all equal to the root node marginal production cost, which is what we expect to obtain. If resistances are nonzero and capacity constraints are saturated, the DLMPs will deviate from this value to account for the costs of losses and congestion effects.

IV. DECENTRALIZED COORDINATION METHODS

The idea behind the proposed coordination algorithm is to consider the OPF problem in a decentralized framework, where different parts of the set of decision variables are managed by different agents:

- each LA $a \in A$ decides of local consumption variables:

\[
x_a \triangleq (p_{n,t}, q_{n,t})_{n \in N, t \in T}
\]

related to her affiliated nodes (and the corresponding consumption/production variables);
- the DSO is responsible of the operation of the network, that is, of the variables:

\[
x_0 \triangleq (p_0, q_0, v, \ell, f, g)
\]

Besides, a decentralized coordination mechanism is also relevant to address the partial information held by each agent. Typically, the network characteristics (topology, capacities, etc.) related to constraints (1) are considered as confidential information by the DSO and shall not be revealed to other actors of the system. On the other hand, consumption and production constraints can constitute private information for electric consumers and, thus, should not be revealed by an aggregator to other actors or network operator.

We can reformulate problem (22) formally to consider the decomposition between the DSO and LAs, as:

\[
\begin{align*}
\min \Phi(x) \\
A_0 x_0 + Bx_A &= b \\
x_0 &\in X_0 \\
x_A &\in X_A,
\end{align*}
\]

where $x \triangleq (x_0, x_A)$ and $x_A \triangleq (x_A)_{a \in A}$ denotes the LAs variables, $X_0$ (resp. $X_A$) denotes the convex feasible set for $x_0$ (resp. $x_A$) defined by constraints (1) (resp. (2)).
Constraint (7b) refers to the coupling constraints between LAs and DSO variables (15). It is assumed that matrix $B$ is block diagonal $B = \text{diag}(B_a)_{a \in A}$, where $B_a \in \mathcal{M}_{k_a,n_a}(\mathbb{R})$. We denote by $k = \sum_{a} k_a$ the dimension of the LAs variables, thus $A_0 \in \mathcal{M}_{k,n_0}(\mathbb{R})$. Through this Sec. IV we use the notation $\lambda \in \mathbb{R}^k$ to denote the complete vector of Lagrangian multipliers associated to (7b).

The decentralized coordination procedure that we propose in this paper is the following:

**Procedure 1** DLMP-based Coordination Procedure

1. in a first computation step, an optimization decomposition method is run to obtain consumption decisions $x_a^*$ for each LA $a \in A$ and Lagrangian multipliers $\lambda^*$ for the DSO, corresponding to an optimal primal-dual solution $(x_0^*, x_A^*, \lambda^*)$ of (7).
2. in a second realization step, each LA $a$ realizes the profile $x_a^*$ history and receives the price incentives corresponding to the DLMPs $\lambda^*$, imposed by the DSO. The DSO operates the network optimally from the solution $x_0^*$.

As $(x_0^*, x_A^*, \lambda^*)$ constitutes an optimal solution of (7), it is optimal for each LA $a$ to realize the profile $x_a^*$ under the incentives $\lambda^*$. From the point of view of mechanism design [27, Part III], this shows that the proposed procedure is incentive-compatible. The essential step in this procedure is to rely on a decomposition method that enables to obtain $x_A^*$ and $\lambda^*$ while respecting the decentralized structure of decisions and information: the DSO does not have access to $X_A$ and is responsible of variables $x_0$, while each LA $a$ does not have access to $X_a$ or $X_a^*$ for $a \neq a$, and is responsible of variables $x_a$. In the remaining of this Sec. IV we propose three relevant decomposition methods to be used in this framework.

A. Dual Decomposition

Considering $x_0$ and $x_A$ satisfying (7c) and (7d), the Lagrangian function associated to (7) is defined as:

$$L(x_0, x_A, \lambda) \triangleq \Phi(x) + \lambda^T (A_0 x_0 + B x_A - b) ,$$

which is a basic ingredient of the following method.

The dual decomposition method [28] relies on the consideration of the subproblems:

$$\min_{x_a \in X_a} \phi_a(x_a) + \lambda^T_B B_a x_a , \quad (\mathcal{P}_a^b(\lambda))$$

$$\min_{x \in X_0} \phi_0(x_0) + \lambda^T A_0 x_0 , \quad (\mathcal{P}_0(\lambda))$$

A dual ascent enables to optimize (7) in a distributed way, by considering the following Algo. 2

**Algo. 2 Decomposition through Dual Ascent**

**Require:** $\lambda^{(0)}$, stopping criterion, steps $(\alpha_k)_k$

1. $k \leftarrow 0$
2. while stopping criterion not true do
3. for each LA $a \in A$ do
4. [LA $a$] receive $\lambda^{(k)}(\alpha_k)$ from DSO
5. [LA $a$] $x_a^{(k+1)} \in \text{sol}(\mathcal{P}_a^b(\lambda^{(k)}))$
6. end for
7. [DSO] $x_0^{(k+1)} \in \text{sol}(\mathcal{P}_0(\lambda^{(k)}))$
8. [DSO] $\lambda^{(k+1)} = \lambda^{(k)} + \alpha_k (A_0 x_0^{(k+1)} + B x_A^{(k+1)} - b)$
9. $k \leftarrow k + 1$
10. end while

If the sequence $(\alpha_k)_k$ is chosen such that $\sum_{k} \alpha_k = +\infty$ while $\sum_{k} \alpha_k^2 < +\infty$, and $\Phi$ satisfies some strict convexity assumption, Algo. 2 converges to a solution of (7) [28]. [29] Ch. 6.

B. Auxiliary Problem Principle

This method relies on the augmented subproblems:

$$\min_{x_a \in X_a} \phi_a(x_a) + \lambda^T_B B_a x_a + \frac{1}{\alpha_k} \|x_a - y_a\|^2 , \quad (\mathcal{P}_a^b(\lambda, y_a))$$

$$\min_{x \in X_0} \phi_0(x_0) + \lambda^T A_0 x_0 + \frac{1}{\alpha_k} \|x_0 - y_0\|^2 . \quad (\mathcal{P}_0(\lambda, y_0))$$

The only difference of this method with Algo. 2 is that we consider $(\mathcal{P}_a^b(\lambda, y_a))$ (resp. $(\mathcal{P}_0(\lambda, y_0))$) instead of $(\mathcal{P}_a^b(\lambda))$ (resp. $(\mathcal{P}_0^b(\lambda))$). This method has been proved to converge under some strong monotonicity assumptions, as a special case of auxiliary problem principle [30]. Indeed, one iteration of the obtained algorithm can be written as:

$$(\text{AP}^k) \min_{y \in \mathcal{Y}} \alpha_k \langle \nabla F(y^{(k)}), y \rangle + \|y - y^{(k)}\|^2 , \quad (9)$$

which corresponds to an iteration of [30] Algorithm 2.1] as in the differentiable case $\nabla F(y) = \lambda(y)^T B$, exactly to $(\mathcal{P}_2^b(\lambda^{(k)}))$ defined in $(\mathcal{P}_2^b(\lambda))$.

**Theorem 3**. [30] **Theorem 2.1** Assume that:

- $F$ is convex, differentiable with $L_F$-Lipschitz gradients;
- the functions $K^b$ have $L_K$-Lipschitz gradients on $\mathcal{Y}$ with $L_K \leq L_K$ a positive constant (in our case $K^b(y) = \|y - y^{(k)}\|^2$ are also $1$-strongly convex);
- there exists $\alpha, \beta > 0$ such that $\alpha k \in [\alpha, 2/(L_F + \beta)]$.

Then the sequence $F(y^{(k)})$ is strictly decreasing (unless it is equal to $F^*$) and, if $y^*$ is unique, $y^{(k)}$ converges to $y^*$.

If moreover $F$ is strongly convex with constant $\alpha_F$ then $y^{(k)}$ converges to the unique $y^*$ and

$$\forall k, \|y^{(k+1)} - y^*\| \leq \frac{1}{\alpha_F} \langle L_F^{(k)} \rangle \|y^{(k+1)} - y^{(k)}\| .$$

C. Primal-Dual Gauss-Seidel Iterations

In Algo. 3 below, we propose a new method which relies on a Lagrangian relaxation of the coupling constraint (15), as the dual decomposition methods above. The main difference is that we consider the original instances of $(\mathcal{P}_0^b(\lambda_A))$ instead of relaxed problem. For that, we rely on the resolution of the dual problem of OPF problem $(\mathcal{P}_0(\lambda_A))$ to compute the
DLMPs $\lambda$, while computing the associated LAs decisions $x_A$ afterwards. Using the Lagrangian function $\mathcal{L}$ defined in (8), problem (7) can be written, with $\lambda^\top x_0 \in \lambda_A$:

$$\inf_{x \in X} \sup_{\lambda \in \mathbb{R}^k} \mathcal{L}(x, \lambda),$$

(10)

and the dual problem:

$$\inf_{\lambda \in \mathbb{R}^k} \sup_{x \in X} \mathcal{L}(x, \lambda).$$

(11)

Let us consider the (partial) dual function $\psi_0$ defined as:

$$\psi_0(\lambda, x_A) \triangleq \min_{x_0 \in X_0} \{ \Phi(x_0, x_A) + \lambda^\top (A_0x_0 + Bx_A - b) \}$$

$$= \min_{x_0 \in X_0} \{ \phi_0(x_0) + \lambda^\top A_0x_0 \} + \phi_0(x_A) + \lambda^\top (Bx_A - b).$$

Because $X_0$ is a compact subset, $\psi_0$ is well defined. Using the notation of (7), the problem $(\mathcal{P}_0(x_A))$ can be reformulated as:

$$\min_{x_0 \in X_0} \phi_0(x_0)$$

$$\text{s.t. } A_0x_0 = b - Bx_A$$

(\lambda)

where $\lambda$ is the Lagrangian multiplier associated to the equality constraint, such that $\psi_0$ is the dual function of problem $(\mathcal{P}_0(x_A))$ (translated by $\phi_0(x_A)$). Moreover, we have:

**Proposition 3.** $\psi_0$ is concave in $\lambda$ and convex in $x_A$.

**Proof.** The function $\psi_0$ is the sum of the convex function $x_A \mapsto \phi_0(x_A)$ and of an affine function of $x_A$, thus it is convex in $x_A$. As a minimum of concave functions of $\lambda$ (because affine), it is concave in $\lambda$.

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**Algo. 3 Dual Alternating Decomposition**

**Require:** $\lambda^{(0)}$, stopping criterion

1: $k \leftarrow 0$

2: while stopping criterion not true do

3: \[ \text{[LA $a$]} \quad x_A^{(k+1)} \in \arg\min_{x_A \in X_A} \phi_0(x_A) + \lambda^{(k)}^\top Bx_A \]

4: if $(\mathcal{P}_0(x_A))$ is feasible then

5: \[ \text{[DSO]} \quad \lambda^{(k+1)} \in \arg\max_{\lambda \in \mathbb{R}^k} \psi_0(\lambda, x_A^{(k+1)}) \]

6: else

7: $\lambda^{(k+1)}$ infeasibility certificate of $(\mathcal{P}_0(x_A))$

8: end if

9: $k \leftarrow k + 1$

10: end while

---

We rely on the following assumption to prove the validity of Algo. 3.

**Assumption 3.** A constraint qualification (e.g. Slater’s condition) holds for (7), such that strong duality holds [9, ch. 5]:

$$\min_{x \in X} \max_{\lambda \in \mathbb{R}^k} \mathcal{L}(x, \lambda) = \max_{\lambda \in \mathbb{R}^k} \min_{x \in X} \mathcal{L}(x, \lambda).$$

(12)

The idea behind Algo. 3 is that the primal problem $(\mathcal{P}_0(x_A))$ is not always feasible (depending on the value of $x_A$), but the dual problem is always feasible, although it can be unbounded. In this case, one will obtain a certificate of infeasibility of $(\mathcal{P}_0(x_A))$, that is, a Lagrangian multiplier $\lambda^{(k)}$ giving the direction of an unbounded ray for the dual problem. The multiplier $\lambda^{(k)}$ can still be used as DLMPs to update $x_A$, for the next iteration. We have the following formal result:

**Proposition 4.** In case of convergence, Algo. 3 provides a DLMP-based decentralized coordination method which enables to optimize DERs while satisfying network constraints, giving an optimal solution of $(\mathcal{P}_0)$.

**Proof.** The algorithm is decentralized as $\phi_A(x_A) = \sum_{a \in A} \phi_a(x_a)$ is separable and $B = \text{diag}(B_a)_{a \in A}$ is block-diagonal, so that Line 3 can be executed in a decentralized manner by each LA $a \in A$, while the update of $\lambda$ (Line 4 to Line 8) is executed independently by the DSO.

Let $x_A^\ast, \lambda^\ast$ be a fixed point of Algo. 3. Then $(x_A^\ast, \lambda^\ast)$ satisfy the KKT conditions of $(\mathcal{P}_0)$. Considering a primal solution $x_0^\ast$ for $(\mathcal{P}_0(x_A))$ associated to $\lambda^\ast$, then $(x_0^\ast, x_A^\ast, \lambda^\ast)$ satisfies the KKT conditions of $(\mathcal{P}_0(x_A))$. The union of the two sets of conditions gives exactly the KKT conditions of problem $(\mathcal{P}_0)$, which shows that $(x_0^\ast, x_A^\ast, \lambda^\ast) \in \text{solutions of } (\mathcal{P}_0)$.

An advantage of the proposed Algo. 3 is that we solve the real problems on both sides: LAs simply face the price incentives given by the DLMPs $\lambda$ in their local optimization problem, while the DSO computes an exact solution of the OPF problem.

To help understand the convergence conditions of Algo. 3 it is relevant to consider a zero-sum game [31] interpretation:

**Proposition 5.** Consider the zero-sum game on $\psi_0$ where the first player minimizes $\psi_0$ on $x_A \in X_A$, while the second player maximizes $\psi_0$ on $\lambda \in \mathbb{R}^k$. Suppose that the game has a value and let $(x_A^\ast, \lambda^\ast)$ denote a saddle (equilibrium) point. Then, there exists a primal solution $x_0^\ast$ of $(\mathcal{P}_0(x_A))$ and $(x_0^\ast, x_A^\ast)$ defines a solution to central problem $(\mathcal{P}_0)$.\[ \square \]

**Proof.** Because the game has a value, the dual problem $\max_{\lambda \in \mathbb{R}^k} \psi_0(\lambda, x_A^\ast)$ is bounded, and thus there is a solution $x_0^\ast$ to the primal problem $(\mathcal{P}_0(x_A))$, such that $A_0x_0^\ast + BxA^\ast - b = 0$ (otherwise the dual problem cannot be bounded). We get:

$$\Phi(x_0^\ast, x_A^\ast) + 0 = \mathcal{L}(x_0^\ast, x_A^\ast, \lambda^\ast)$$

$$= \max_{\lambda \in \mathbb{R}^k} \min_{x_A \in X_A} \psi_0(\lambda, x_A)$$

$$= \max_{\lambda \in \mathbb{R}^k} \min_{x_0 \in X_0, x_A \in X_A} \mathcal{L}(x_0, x_A, \lambda)$$

$$= \max_{x_0 \in X_0, x_A \in X_A} \min_{\lambda \in \mathbb{R}^k} \mathcal{L}(x_0, x_A, \lambda)$$

$$= \min_{x_0 \in X_0, x_A \in X_A} \max_{\lambda} \Phi(x_0, x_A, \lambda)$$

(13)

where (13) follows from Assumption 3.\[ \square \]

**Remark 1.** The interpretation based on a zero-sum game goes further, as Algo. 3 actually implements the so-called best response dynamics (BRD), for which Hofbauer and Sorin [31] proved the convergence in zero-sum games with compact strategy sets. This compactness assumption does not hold here as $\lambda$ lies in $\mathbb{R}^k$. This could help in proving the convergence of Algo. 3 and gives interesting directions for further work.
TABLE I: Parameters for the 15 buses network [13]

| n | R_n | X_n | S_n | E[p_{n,t}] | E[q_{n,t}] | B_n 10^3 |
|---|-----|-----|-----|-----------|-----------|---------|
| 0 | 0   | 0   | 0   | 0         | 0         | 0       |
| 1 | 0.001 | 0.12 | 2   | 0.7936    | 0.1855    | 1.1     |
| 2 | 0.0883 | 0.1262 | 0.256 | 0       | 0       | 2.8     |
| 3 | 0.1384 | 0.1978 | 0.256 | 0.0201    | 0.0084    | 2.4     |
| 4 | 0.0191 | 0.0273 | 0.256 | 0.0173    | 0.0043    | 0.4     |
| 5 | 0.0175 | 0.0251 | 0.256 | 0.0291    | 0.0073    | 0.8     |
| 6 | 0.0482 | 0.0689 | 0.256 | 0.0219    | 0.0055    | 0.6     |
| 7 | 0.0523 | 0.0747 | 0.256 | -0.1969   | 0.0019    | 0.6     |
| 8 | 0.0407 | 0.0582 | 0.256 | 0.0235    | 0.0059    | 1.2     |
| 9 | 0.01 | 0.0143 | 0.256 | 0.0229    | 0.0142    | 0.4     |
| 10 | 0.0241 | 0.0345 | 0.256 | 0.0217    | 0.0065    | 0.4     |
| 11 | 0.0103 | 0.0148 | 0.256 | 0.0132    | 0.0033    | 0.1     |
| 12 | 0.001 | 0.12 | 1   | 0.6219    | 0.1291    | 0.1     |
| 13 | 0.1559 | 0.1119 | 0.204 | 0.0014    | 0.0008    | 0.2     |
| 14 | 0.0953 | 0.0684 | 0.204 | 0.0224    | 0.0083    | 0.1     |

V. NUMERICAL EXAMPLE

In this example, we consider the 15 buses network proposed by Papavassiliou [13], but with flexible active and reactive loads instead of fixed ones, and we consider a time set \( T = \{0, 1\} \), of 2 time periods. The network structure can be observed on Fig. 2.

For each bus \( n \), the parameters \( R_n, X_n, S_n, B_n \) are those of [13], given in Tab. I while \( \{p_{n,t}, q_{n,t}, \tau_n^c, \tau_n^r\} \) are generated as follows. For each \( t \in T \), and with \( \tilde{p}_n, \tilde{q}_n \) denoting the fixed active and reactive load values considered in [13]:

- \( p_{n,t} \) is chosen randomly as \( p_{n,t} \sim U([0, \tilde{p}_n]) \), where \( U(I) \) denotes the uniform distribution on \( I \);
- \( q_{n,t} \) is chosen randomly as \( q_{n,t} \sim U([0, \tilde{q}_n]) \);
- \( E_n \) is chosen as \( E_n \sim U([\sum p_{n,t}, \sum q_{n,t}]) \);
- \( \tau_n^c \) is fixed as \( \tau_n^c = \frac{\tilde{q}_n}{\tilde{p}_n} \).

Following [13], we consider that only the bus 11 has a renewable production, with \( \sum p_{n,t} \sim U([0, 0.6]) \) and \( p^p_0 \sim U([0, 0.6]) \) (the renewable production is fully active). The bounds \( [V^m_n, V^M_n] \) are taken to be 0.81 and 1.21 for each \( n \in N \), while \( V_0 = 1.0 \).

We consider the objective \( \Phi(x) = \phi_0(x_0) = \sum_{t \in T} c_t(p_{0,t}) \), with cost functions chosen as follows: time period 0 has an expensive price given by \( c_0 : p \mapsto p + p^2 \), while time period 1 has a cheaper price given by \( c_1 : p \mapsto p^2 \).

The SOCP problem \( \{A\} \) is solved with the CvxOpt Python library [32] in 0.53s on a laptop with a processor of 2.6GHz. The solutions obtained are detailed in Tab. II while Fig. 2 shows the active flows directions and the saturated lines.

The optimal production for node 11 is \( p^p_{11,0} = 0.185 \) and \( p^p_{11,1} = 0.194 \), while the optimal costs obtained are \( c_0(p_{0,0}) = 0.873 \) and \( c_1(p_{0,1}) = 1.299 \). Several comments are to be made.

First, one can observe that, even if the example does not satisfy the theoretical assumptions of Thm. I or Thm. II, the SOCP relaxation is exact and gives the optimal solution of the original OPF problem.

Second, because of the existence of network losses, the optimal centralized production costs \( \phi_0(x_0) \) obtained are higher than the social costs, whose values are \( SC_0(\sum p_{n,0}) = 0.843 \) for \( t = 0 \) and \( SC_1(\sum p_{n,1}) = 1.287 \) for \( t = 1 \).

TABLE II: Results of the OPF for the 15-bus network for time periods \( t=0 \) (above) and \( t=1 \) (below).
Last, the solutions show that the active (and reactive) DLMPs obtained for each time period are closed to the DLMPs at the root node \((\lambda_0, \mu_0)\), except in two cases:

- for the branch composed of nodes 8, 7, 9, 10, 11, the active DLMPs are close to 0.0 on all time periods. This is due to the fact that the renewable production of node 11 at cost 0 and the negative load of node 7, which together fully compensate the demand on this branch. The energy left is exported to the remaining nodes of the network up to the saturation of the line \((3, 8)\). This saturation creates an important difference of the DLMPs between one part of the network and the other. The DLMPs here give incentives for the nodes to either decrease production or increase consumption as much as possible, in order to release the saturation of line \((3, 8)\).

- On the contrary, the active DLMPs on the branch composed of nodes \((12, 13, 14)\) on time period \(t = 1\) are twice as large as the other nodes. Again, this is explained by the saturation of line \((0, 12)\). Because of the cheaper price on time period 1, LAs are encouraged to move the flexible demand onto this period, which results in the congestion of line \((0, 12)\). Here, the DLMPs counterbalance this price difference and give incentives to decrease consumption on \(t = 1\) for nodes \((12, 13, 14)\) in order to stay within the line capacity.

CONCLUSION

We proposed a coordination procedure for aggregators operating on a distribution network, which respects the decentralized structure of decisions and information. This procedure enables to compute decentralized decisions which satisfy AC network constraints and which lead to an optimal grid operation from the system operator point of view.

Several directions of research are interesting for extending this work. First, one could further analyze the theoretical convergence conditions of the proposed Algo.\[2\] Second, one could study the tractability and the limits of the proposed procedure for large-scale instances, considering networks with a large number of nodes and/or a large set of time periods. Last, an interesting but complex extension to the proposed model would be to consider a strategic and game-theoretic framework in which aggregators are subject to the DSO incentives, but are also price-makers on an electricity market.

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