Partially locally rotationally symmetric perfect fluid cosmologies

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Abstract

We show that there are no new consistent cosmological perfect fluid solutions when in an open neighbourhood \( U \) of an event the fluid kinematical variables and the electric and magnetic Weyl curvature are all assumed rotationally symmetric about a common spatial axis, specialising the Weyl curvature tensor to algebraic Petrov type D. The consistent solutions of this kind are either locally rotationally symmetric, or are subcases of the Szekeres dust models. Parts of our results require the assumption of a barotropic equation of state. Additionally we demonstrate that local rotational symmetry of perfect fluid cosmologies follows from rotational symmetry of the Riemann curvature tensor and of its covariant derivatives only up to second order, thus strengthening a previous result.

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1 Introduction

Our long-term aim is to determine all perfect fluid cosmologies which cannot be invariantly defined by the existence of a unique eigentetrad for the fluid shear tensor \( \sigma \). This consideration is of relevance to the equivalence problem of spacetimes [2, 17] and its application to relativistic cosmology. The fluid shear plays a central role in the dynamics of a generic cosmology [8]. If it vanishes, the resultant consistent models are highly special and well-studied for perfect fluid matter sources [5]. In general, \( \sigma \) has two distinct eigenvalues, which may serve to invariantly classify cosmologies on the basis of the related eigentetrad. When these eigenvalues coincide (degenerate), but with \( \sigma \neq 0 \), the \( \sigma \)-eigentetrad is no longer uniquely defined and the fluid shear tensor thus picks out a distinct spatial direction. It is the determination of all cosmological perfect fluid solutions exhibiting this feature that interests us in the long run.

Key symmetries of spacetimes are their continuous isotropies, and cosmologies are either isotropic (and then have a Robertson–Walker metric), locally rotationally symmetric (‘LRS’) (and then are all known up to the form of their metric), or are anisotropic (see, e.g., [6] and [10] for a discussion of these cases). In the case of LRS cosmologies, there is at each event (relative to the family of fundamental observers) precisely one preferred spatial direction, and all physical properties and observations are invariant under rotation about this direction. It follows that these spacetimes are invariant under multiply transitive groups of isometries [6]. In the case of LRS cosmologies, there is at each event (relative to the family of fundamental observers) precisely one preferred spatial direction, and all physical properties and observations are invariant under rotation about this direction. It follows that these spacetimes are invariant under multiply transitive groups of isometries [6]. The question that is interesting from both the physical point of view, and in terms of determining the equivalence of cosmologies, is how weak we can make the assumptions of rotational symmetry and still determine explicitly the family of cosmological spacetimes involved; in
other words, how few physical and geometrical quantities we can make rotationally symmetric, where rotationally symmetric means the quantity concerned is either isotropic, or invariant under arbitrary rotations about a preferred spatial axis.

Central to the equivalence problem formalism are the components of the spacetime Riemann curvature tensor $\mathbf{R}$ and its covariant derivatives $\nabla \ldots \nabla \mathbf{R}$ in a standard tetrad. In \textsuperscript{3} it was shown that a spacetime will be LRS if all tensors algebraically defined by the spacetime Ricci curvature tensor and their covariant derivatives up to third order are rotationally symmetric (note that the fluid 4-velocity field $\mathbf{u}$ is algebraically determined by the Ricci curvature tensor, through the Einstein field equations). The ultimate aim is to weaken this assumption by considering perfect fluid cosmologies in which only the fluid shear tensor $\mathbf{\sigma}$ (and not its covariant derivatives) has this symmetry. All cosmologies that do not satisfy this restriction can be invariantly defined through tensor components relative to the unique $\mathbf{\sigma}$–eigentetrad. In the present work we set off to consider a subcase of this more general project in the context of an approach that is applicable to investigating systematically all cases with rotationally symmetric $\mathbf{\sigma}$. In detail, we consider perfect fluid cosmologies in which all fluid kinematical and Weyl curvature variables are rotationally symmetric about the same spatial axis (the Ricci curvature tensor components are automatically so, because of the perfect fluid assumption), thus restricting the cosmologies to algebraic Petrov type D and simpler cases. We make no similar assumption about the covariant derivatives of these variables. We find all perfect fluid cosmologies satisfying this restriction.

### 1.1 Assumptions

We describe a spacetime by a pseudo-Riemannian manifold $\mathcal{M}$ with a rank two symmetric metric tensor field $\mathbf{g}$. It is convenient to decompose the Riemann curvature tensor $\mathbf{R}$ into the completely tracefree Weyl conformal curvature tensor $C_{abcd}$, the Ricci curvature tensor $R_{ab} := R^c_{acb} = R_{(ab)}$ and the Ricci curvature scalar $R := g^{ab}R_{ab}$ according to

$$R_{abcd} = C_{abcd} + R_{a[c}g_{d]b} - R_{[c}g_{d]a} - \frac{1}{4}Rg_{a[c}g_{d]b} .$$

We will mean by a cosmology a spacetime which satisfies all of the following requirements \textsuperscript{2} [6, 7, 10]:

- It is a self-consistent solution of the Einstein field equations (EFE), relating matter source fields which are represented by an energy–momentum–stress tensor $T_{ab}$ to the Ricci curvature tensor $R_{ab}$ and its trace $R$ as$^2$

$$R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab} .$$

- It is filled with matter energy of some sort which we can represent as a perfect fluid with fundamental unit 4-velocity field $\mathbf{u}$ ($u_a u^a = -1$). The energy–momentum–stress tensor $T_{ab}$ is then given by

$$T_{ab} = \mu u_a u_b + p h_{ab} = T_{(ab)} ,$$

where $h_{ab} := g_{ab} + u_a u_b$ denotes the tensor projecting into the rest 3-spaces of observers moving with 4-velocity $\mathbf{u}$, and $\mu := T_{ab}u^a u^b$ and $p := \frac{1}{3}T_{ab} h^{ab}$, respectively, are the total energy density and isotropic pressure measured in these rest 3-spaces.

- Additionally we assume

$$(\mu + p) > 0 ,$$

so that by \textsuperscript{3}, together with \textsuperscript{2}, $\mathbf{u}$ uniquely defines the timelike eigenvector of $R_{ab}$.

- Observations have shown that the Universe is expanding. We will take this to mean that the (isotropic) expansion of $\mathbf{u}$, described by the fluid expansion scalar $\Theta$, is positive:

$$\Theta > 0 .$$

\textsuperscript{1}When referring to tetrad components in the spacetime, we use Latin indices from the first half of the alphabet: $a, b, c \in \{0, 1, 2, 3\}$; in the spatial sections we use Greek letters: $\alpha, \beta, \gamma \in \{1, 2, 3\}$. For the local coordinate spacetime description we use Latin indices from the second half of the alphabet: $i, j, k \in \{0, 1, 2, 3\}$. Round brackets denote symmetrised indices and square brackets denote skew-symmetrised indices.

\textsuperscript{2}Throughout this work we will employ geometrised units characterised by $c = 1 = 8\pi G/c^2$, and we set the cosmological constant $\Lambda$ equal to zero (the latter implies no loss of generality, as $\Lambda$ can be effectively included by suitably redefining $T_{ab}$).
The Weyl conformal curvature tensor may be decomposed relative to $\mathbf{u}$ into its symmetric tracefree ‘electric’ and ‘magnetic’ tensor parts, $\mathbf{E}$ and $\mathbf{H}$, respectively, according to \[ E_{ab} := C_{cdef} h^c_{\ a} u^e h^d_{\ b} u^f = E_{(ab)} \] \[ H_{ab} := (-\frac{1}{2} \eta_{cegh} C^{gh}_{\ ab}) h^c_{\ a} u^e h^d_{\ b} u^f = H_{(ab)} \] with the completely skew spacetime permutation tensor $\eta_{abcd}$ specified by $\eta_{abcd} = \eta_{(abcd)}$ and $\eta_{0123} = 1$, $\eta_{0123} = -1$. The second Bianchi identities differentially relate components of the Riemann curvature tensor:
\[ \nabla_{[a} R_{bc]de} = 0 \begin{equation} \end{equation} \]
As well as entailing the matter equations of motion $\nabla_b T^{ab} = 0 = \nabla_b (R^{ab} - \frac{1}{2} R g^{ab})$, this relation provides evolution and constraint equations for $\mathbf{E}$ and $\mathbf{H}$ when employing the $1 + 3$ decompositions \[ \begin{equation} \end{equation} \] and \[ \begin{equation} \end{equation} \].

1.2 The problem: symmetry assumptions

The question we answer here is: ‘Under what conditions are all of the fluid acceleration, vorticity and shear and the spacetime electric and magnetic Weyl curvatures either rotationally symmetric about the same spatial axis or isotropic, but the spacetime itself is not?’\footnote{\textsuperscript{3}} A dynamical tensor field is rotationally symmetric if there is a degeneracy in that tensor quantity in terms of its eigenvalues, but it is not isotropic (not all eigenvalues are zero). The justification for this terminology is based on the fact that a spacetime is LRS if all tensor quantities, as well as their covariant derivatives, are either isotropic or rotationally symmetric about the same spatial axis, with at least one not being isotropic \[ \begin{equation} \end{equation} \]. Our ultimate aim is to be able to uniquely classify all perfect fluid cosmologies which have degenerate fluid shear eigenvalues. To this end, we employ the methods developed in \[ \begin{equation} \end{equation} \] which introduce as a reference framework an orthonormal tetrad (‘ONT’) field \{ $\mathbf{e}_a$ \} where the timelike member $\mathbf{e}_0$ is identified with the fluid 4-velocity field $\mathbf{u}$ (‘$1 + 3$ decomposition’). For the spatial rotation coefficients we use the notation suggested in \[ \begin{equation} \end{equation} \] and utilised to great effect by Wainwright and collaborators (see \[ \begin{equation} \end{equation} \] for a survey). We have thus written out for a perfect fluid matter source the evolution and constraint equations\footnote{When $\mathbf{e}_0 = \mathbf{u}$ has non-vanishing vorticity this terminology is doubtful because then the local rest 3-spaces of neighbouring fundamental observers do not mesh to form everywhere spacelike 3-surfaces.} for the tetrad commutation functions $\gamma$ as well as for $\mathbf{E}$ and $\mathbf{H}$, which are obtained from the Jacobi, Ricci and second Bianchi identities, and specialised to a rotationally symmetric fluid shear tensor $\sigma$. These are given in full in appendix \[ \begin{equation} \end{equation} \].

2 PLRS perfect fluid cosmologies

The effect of restricting the geometry of a spacetime is generally expressed by setting certain geometrically defined tensor components to zero. The evolution equations for these quantities now become new constraint equations. These constraint equations must be preserved along the fluid flow lines $\mathbf{u}$ to be consistent. In general this does not happen without further constraints on the dynamics. Of course, the test for preservation then has to be repeated for these new constraints. We list all perfect fluid cosmologies that provide solutions to the EFE which have the tensor fields $\sigma$, $\mathbf{E}$ and $\mathbf{H}$ as well as the vector fields $\hat{\mathbf{u}}$ and $\omega$, but not their covariant derivatives, all rotationally symmetric about the same spatial axis. In our investigation we find that all consistent cosmological solutions to the EFE satisfying this criterion, bar one — the Szekeres dust solutions —, are in fact LRS spacetimes in the definition of Ellis \[ \begin{equation} \end{equation} \] and Stewart and Ellis \[ \begin{equation} \end{equation} \]. We shall generally take $\sigma \neq 0$, as for $\sigma = 0$ we will see that the presently considered types of cosmologies typically specialise to the well known spatially homogeneous and isotropic Friedmann–Lemaître–Robertson–Walker (‘FLRW’) cases.

Definition 2.1 We assume that in an open neighbourhood $\mathcal{U}$ of an event on an expanding perfect fluid spacetime manifold $\mathcal{M}$
\begin{equation}
\begin{aligned}
(\sigma_{22} - \sigma_{33}) &= \sigma_{12} = \sigma_{23} = \sigma_{31} = 0 , \\
(E_{22} - E_{33}) &= E_{12} = E_{23} = E_{31} = 0 ,
\end{aligned}
\end{equation}
That is to say, we have set all of \( \sigma \), \( E \) and \( H \) simultaneously rotationally symmetric about the same spatial axis. In addition, the vectors \( \dot{u} \) and \( \omega \) are aligned with this axis of rotational symmetry. We call all cosmological spacetimes in which there exists a tetrad \( \{ e_\alpha \} \) with respect to which these restrictions are satisfied partially locally rotationally symmetric (‘PLRS’) cosmologies. Those PLRS cosmologies that are not LRS will be referred to as strictly PLRS.

Remarks:

(i) Using a \( \sigma \)–eigentetrad \( \{ e_\alpha \} \) here is equivalent to using an \( E \)– or \( H \)–eigentetrad.

(ii) If either of the vector quantities \( \dot{u} \) or \( \omega \) do not point in the invariant spatial direction defined by the degenerate tensor quantities, then they define an additional invariant spatial direction. The tetrad \( \{ e_\alpha \} \) may then be invariantly defined by, e.g., identifying another of its members with this new direction. And thus, for the situations we want to start from to be PLRS, we must have that \( \dot{u} = \omega = 0 \); that is, \( \dot{u} \parallel e_1 \). Similarly, we must have \( \omega_2 = \omega_3 = 0 \), corresponding to \( \omega \parallel e_1 \). If these conditions did not hold, this would exclude for non-zero \( \dot{u} \) and \( \omega \) the possibility of the spacetime being LRS and thus compromise the presently proposed notion of partial symmetry.

(iii) The present setup arbitrarily adapts to the spatial \( e_1 \)–axis. However, this is only a matter of convention and by a cyclic permutation of indices \( 1 \to 2 \to 3 \to 1 \) one can easily adapt to any of the other spatial axes as well.

(iv) Our assumptions reduce the perfect fluid cosmologies we consider in this article to algebraic Petrov type D; a classification scheme for solutions with this property based on the Newman–Penrose formalism has been suggested by Wainwright in [24].

With Definition 2.1 in place, the momentum conservation equations for all PLRS cosmologies reduce in \( U \) to (77) – (99). Moreover, combining the \( H \)–constraint equations (108) with (109) and (110) with (111), we find that all PLRS cosmologies require in \( U \) that

\[
(n_{22} - n_{33}) (\sigma_{11}^2 + 4 \omega_1^2) = n_{23} (\sigma_{11}^2 + 4 \omega_1^2) = 0;
\]

which means, as we assume that \( \sigma \omega \neq 0 \), that

\[
(n_{22} - n_{33}) = n_{23} = 0.
\]

The related evolution equations obtained from (77) and (78) as well as (70) and (71), respectively, now reduce in \( U \) to constraints on the \( e_2 \)– and \( e_3 \)–gradients of the Fermi-rotation variables \( \Omega_2 \) and \( \Omega_3 \); namely

\[
\begin{align*}
(e_2 + 2n_{31}) (\Omega_2) - (e_3 - 2n_{12}) (\Omega_3) & = 0 \\
(e_2 + 2n_{31}) (\Omega_3) + (e_3 - 2n_{12}) (\Omega_2) & = 0.
\end{align*}
\]

We proceed to check the consistency of the cosmological PLRS subcase of the EFE by mainly computing the time evolution of all new constraints. In the process of doing this we will fix the remaining tetrad freedom and thus invariantly classify the solutions. For each of the cases we consider below we will check the transformation behaviour of the commutation functions and other tensor quantities and use this to fix the freedom conveniently.

We now turn to the issue of choice of spatial triad \( \{ e_\alpha \} \). As \( e_1 \) is presently a uniquely defined vector, \( e_1 \) is fixed. This will thus have an invariantly defined direction, say \( X \). For any given tetrad choice, the components of this fixed direction in the spatial \( e_2 \)– and \( e_3 \)–directions are \( X \cdot e_2 \) and \( X \cdot e_3 \), respectively.

We have the freedom to set one of these components (or any other quantity which does not behave like a scalar under a rotation) to zero in \( U \) by rotating the spatial triad \( \{ e_\alpha \} \) in the \( e_2 / e_3 \)–plane. Alternatively,

\footnote{This restriction on \( n_{\alpha\beta} \) suggests that for PLRS cosmologies according to Definition 2.1 no consistent solutions to the EFE exist that contain gravitational radiation; it is the transverse (with respect to the spatial \( e_1 \)–direction) components \( (n_{22} - n_{33}) \), \( n_{23} \), \( (\sigma_{22} - \sigma_{33}) \) and \( \sigma_{23} \) which typically form the connection characteristic eigenfields associated with the local null cones (cf. [14]). If we relax the PLRS restrictions so that \( E \) is not rotationally symmetric, we find that this feature still holds.}
as can be seen from the transformation property of $\Omega_1$, we may choose to set $\Omega_1 = 0$ in $U$ which fixes the $e_3$-gradient of the rotation angle $\varphi$; and then for example we choose either $n_{22}$ or $n_{33}$ in the 3-space by fixing the $e_1$-gradient of $\varphi$.

For ease of notation we will use

$$\Theta_\alpha := \frac{1}{3} \Theta + \sigma_{\alpha\alpha} \quad \text{(no summation)}.$$  

\subsection{General perfect fluid}

This is the most general case to be considered here where we will take throughout that $\dot{u} \neq 0$ and $\omega \neq 0$.

For $\omega = 0$ see section 2.2, for $\dot{u} = 0$ refer to section 2.3. We assume a perfect fluid, but leave the equation of state unspecified.

\subsubsection{Constraint analysis}

We proceed by fully fixing the tetrad freedom by choosing $e_3$ orthogonal to the projection of the fixed vector $X$ in the $e_2/e_3$-plane. Hence, we rotate the spatial triad such that

$$\Omega_2 = e_3 \cdot \dot{e}_1 = -e_1 \cdot \dot{e}_3 = 0. \quad (11)$$

Under the assumptions of PLRS symmetry (7), which imply (8), and with the above tetrad choice, the $E$-equations (54) and (55) yield the new constraints

$$e_2(H_{11}) = 0 \quad (12)$$
$$\quad (a_2 - n_{31}) H_{11} = 0, \quad (13)$$

while from the $\dot{H}$-equations (53) and (54), the new constraints

$$e_2(\mu) = 3 (a_2 - n_{31}) E_{11} \quad (14)$$
$$e_2(E_{11}) = \frac{1}{3} e_2(\mu) \quad (15)$$
arise. The algebraic condition (13) suggests that we distinguish between two subcases according to

A] $(a_2 - n_{31}) = 0$ or B] $H_{11} = 0$.

A] $(a_2 - n_{31}) = 0$: So now $(a_2 + n_{31}) = 2 a_2$. From the $\dot{\omega}$-equation (104) we see that $e_2(\dot{u}_1) = 0$ and from (14) we have $e_2(\mu) = 0$. The $\dot{\sigma}$-equation (102) then shows that also $\Omega_3 \sigma_{11} = 0$ must hold, providing a split into further subcases according to

A1] $\Omega_3 = 0$ or A2] $\sigma_{11} = 0$.

A1] $\Omega_3 = 0$: This has the implications from (103) that $e_3(\dot{u}_1) = 0$, which implies from (105) that $(a_3 + n_{12}) = 0$, as we assumed $\dot{u}_1 \neq 0$. We note that the following $e_2$- and $e_3$-gradients of certain quantities must vanish: from (12) and (57) we get that $e_2(H_{11}) = e_3(H_{11}) = 0$ and from (13) - (16) we find that $e_2(E_{11}) = e_3(E_{11}) = 0$ and $e_2(\mu) = e_3(\mu) = 0$. Then (72) and (73) reduce to

$$e_2(\Theta_1) = e_3(\Theta_1) = 0. \quad (16)$$

Now we check the propagation property of the vanishing $e_2$- and $e_3$-gradients of the energy density $\mu$. We use the commutator relations (59) and (60) operating on $\mu$ and the energy conservation equation (55) to show that $e_2(\mu) = e_3(\mu) = 0$ if $(\mu + p) e_2(\Theta) = (\mu + p) e_3(\Theta) = 0$, since the momentum conservation equations (18) and (90) must hold. Thus, if we want to stick to purely cosmological solutions as we defined them in the introduction, we must have $e_2(\Theta) = e_3(\Theta) = 0$. Now this means that $e_2(\omega_1) = e_3(\omega_1) = 0$, which we obtain from the Ricci identities (113) and (115), suitably combined with (17). This result is crucial because now we can show that the solutions contained in the present PLRS subclass are not

\footnote{See [14].}
cosmological ones. To do so we find the commutator most useful. We first apply this relation to \( \omega_1 \), yielding
\[
2 \Theta_2 \omega_1 - a_1 n_{11} = 0 ,
\]
where we have used the \( \dot{\omega} \)–equation, and the constraint on \( e_1(\omega_1) \) given by \( \text{(106)} \). We then apply the commutator to the energy density \( \mu \), using \( \text{(64)} \), and find that
\[
2 \Theta_2 \omega_1 (\mu + p) - n_{11} e_1(\mu) = 0 .
\]
Finally we apply the commutator to the electric Weyl curvature component \( E_{11} \) and substitute from \( \text{(79)} \), \( \text{(126)} \) and \( \text{(18)} \) to get
\[
2 \Theta_2 \omega_1 [(\mu + p) - 3 E_{11}] + 3 a_1 n_{11} E_{11} = 0 .
\]
We now use \( \text{(17)} \) in \( \text{(19)} \) to get
\[
\Theta_2 \omega_1 (\mu + p) = 0 \Rightarrow \Theta_2 = 0 ,
\]
violating the premise that our cosmology be an expanding one. We conclude that there exist no cosmologically viable solutions in the present PLRS subclass.

**A2** \( \sigma_{11} = 0 \): From the \( \sigma \)– and \( \dot{\sigma} \)–equations \( \text{(103)} \) and \( \text{(105)} \) we must have
\[
(a_3 + n_{12}) \dot{u}_1 - \Omega_3 \omega_1 = 0 ,
\]
and from the second Bianchi identities \( \text{(86)} \) and \( \text{(87)} \),
\[
(a_3 + n_{12}) H_{11} + \Omega_3 E_{11} = 0 .
\]
Combining these two equations, we get that either \( (a_3 + n_{12}) = \Omega_3 = 0 \) which would be a subcase of **A1** and is thus non-cosmological, or
\[
\dot{u}_1 E_{11} + \omega_1 H_{11} = 0 .
\]
An important algebraic relation is given by the \( H \)–constraint \( \text{(107)} \); that is
\[
2 (\dot{u}_1 + a_1) \omega_1 + H_{11} = 0 .
\]
The tetrad choice employed has the effect of generally eliminating the gradients in the \( e_2 \)–direction of important scalars. In particular, from \( \text{(114)} \) and \( \text{(115)} \) we get
\[
e_2(\omega_1) = 0 = e_3(\Theta) .
\]
Now, from \( \text{(104)} \) we get that
\[
e_2(\dot{u}_1) = 0 ,
\]
and from \( \text{(30)} \)
\[
e_2(H_{11}) = 0 .
\]
From \( \text{(23)} \), suitably combined with \( \text{(24)} \), we get
\[
e_2(E_{11}) = e_2(\mu) = 0 .
\]
If we now take the \( e_2 \)–gradient of equation \( \text{(24)} \) and substitute from the equations \( \text{(25)} \), \( \text{(26)} \) and \( \text{(27)} \), we get the useful result
\[
e_2(a_1) = 0 .
\]
A relation for \( e_3(\omega_1) \) is provided by \( \text{(112)} \) and \( \text{(113)} \):
\[
e_3(\omega_1) + (a_3 + n_{12}) \omega_1 = 0 .
\]
We get relations involving the \( e_3 \)-gradients of both \( E_{11} \) and \( \mu \) from combining (25) and (30) suitably, thus yielding
\[
\begin{align*}
e_3(E_{11}) - \frac{1}{3} e_3(\mu) - 3 \Omega_3 H_{11} &= 0, \\
e_3(\mu) - 3(a_3 + n_{12}) E_{11} + 3 \Omega_3 H_{11} &= 0.
\end{align*}
\] (31) (32)
The commutators provide vital information here. We find from (31) acting on \( \omega_1 \) that
\[- \frac{2}{3} \Theta \omega_1 + a_1 n_{11} = 2 a_2 (a_3 + n_{12}),
\] (33)
where we have used the \( \omega \)-equation (23), the constraints on the gradients of \( \omega_1 \) provided by (106), (25) and (30), and then substituted from (116) with (117) appropriately. We find from the commutator (31) acting on the magnetic Weyl curvature component \( H_{11} \) that
\[3 H_{11} (-\frac{2}{3} \Theta \omega_1 + a_1 n_{11}) - n_1 (\mu + p) \omega_1 - 12 a_2 E_{11} \Omega_3 = 0,
\]
using a relation for \( e_3(H_{11}) \) provided by (37), noting the constraint (27) on \( e_2(H_{11}) \), the constraint on \( e_1(H_{11}) \) given by (23), and then using the \( \mathbf{H} \)-equation (38). We also needed the first part of (23) and (10). Combining the above with (33) we get the useful result
\[n_1 (\mu + p) \omega_1 = -18 a_2 \Omega_3 E_{11},
\] (34)
where we have also used (23). We now take the commutator (31) operating on \( \omega_1 \) and substitute from (25), (29), (27), the vorticity constraint equation (106), and (30) into the resultant expression and we find that \( n_{33} (a_3 + n_{12}) \omega_1 = 0 \). Now if \( a_3 + n_{12} = 0 \), then from (21) we must have \( \Omega_3 = 0 \), and thus this would be a class already dealt with in section A2; those were non-cosmological. So we conclude that \( n_{33} = 0 \). We now take the \( e_3 \)-gradient of (23) and get the key result
\[\dot{\omega}_1 e_3(\mu) = 0 \Rightarrow e_3(\mu) = 0
\] (35)
by using, in addition to (23), equations (22), (30) - (32), (105) and (37). We proceed to check the consistency of (31) and (32) in the combined form \( e_3(E_{11}) - 3(a_3 + n_{12}) E_{11} = 0 \). We propagate this along \( u \) and find
\[\frac{3}{2} n_{11} [ (a_3 + n_{12}) H_{11} - \Omega_3 E_{11} ] = 0,
\] (36)
where we have used the \( \mathbf{E} \)-equation (34), the constraints on \( e_3(E_{11}) \) and \( e_2(\mu) \) given respectively by (28) and (29), the constraints on \( e_3(H_{11}) \) given by (37), and the Jacobi identity (73) together with (114). The evolution along \( u \) of the \( e_3 \)-gradient of \( E_{11} \) is obtained by applying the commutator (10) to \( E_{11} \) and using the equations (23) with (21). Now we may rewrite (36) by using (23), obtaining \( n_{11} (a_3 + n_{12}) H_{11} = 0 \). We show that this means that \( n_{11} = 0 \). If not, then \( (a_3 + n_{12}) H_{11} = 0 \) which means that \( \Omega_3 E_{11} = 0 \) (from (23)), which then contradicts (34).

We have seen here that \( n_{11} = 0 \) is required. But now, again from (34), we must have \( a_2 \Omega_3 E_{11} = 0 \). So either \( a_2 \) or \( \Omega_3 E_{11} \) vanishes. We consider these possibilities below.

- If \( \Omega_3 E_{11} = 0 \), then from (22) we require \( (a_3 + n_{12}) H_{11} = 0 \). If now \( H_{11} = 0 \), then (82) tells us that these solutions are not cosmological since they require in \( \mathcal{U} \) \( (\mu + p) \omega_1 = 0 \Rightarrow (\mu + p) = 0 \). And if \( (a_3 + n_{12}) = 0 \), then from (33) we get in \( \mathcal{U} \) \( \Theta \omega_1 = 0 \Rightarrow \Theta = 0 \), and we are clearly in the non-cosmological realm again.

- If, on the other hand, \( a_2 = 0 \), then again from (33) we must have in \( \mathcal{U} \) \( \Theta \omega_1 = 0 \Rightarrow \Theta = 0 \). So none of the solutions here are of relevance to us.

B] \( H_{11} = 0 \): Immediately we see from (22) that \( (\mu + p) + 3 E_{11} = 0 \). Also, from (38), we get that \( n_{11} E_{11} = 0 \). Now if \( E_{11} = 0 \), then \( (\mu + p) = 0 \) which is not allowed. So \( n_{11} = 0 \). We also get from (93) that \( \Omega_3 E_{11} = 0 \), and once again we deduce that since \( E_{11} = 0 \Rightarrow (\mu + p) = 0 \), it follows that we must have \( \Theta_3 = 0 \). Moreover \( e_3(\dot{u}_1) = 0 \) from (103); and \( e_2(\dot{u}_1) = 0 \) from (102). This tells us from (104) that \( (a_2 - n_{31}) = 0 \) and from (105) that \( (a_3 + n_{12}) = 0 \). And now from (111) we get that \( \Theta_1 \omega_1 = 0 \Rightarrow \Theta_1 = 0 \). We get from (23) and (30) that \( e_2(E_{11}) = e_2(\mu) = 0 \) and from (29) and (30) that \( e_3(E_{11}) = e_3(\mu) = 0 \). If we now take the above two relations for the \( e_2 \)– and \( e_3 \)-gradients of \( \mu \) and substitute them into the commutator (31) applied to \( \mu \), using (64), we get \( \Theta \omega_1 (\mu + p) = 0 \), and so in \( \mathcal{U} \)
\[\Theta (\mu + p) = 0;
\] (37)
in other words, there are no solutions in this PLRS subclass that are of cosmological interest.
2.1.2 Summary

We conclude that there exist no rotating and accelerating perfect fluid cosmologies which are PLRS according to Definition 2.1.

2.2 Irrotational accelerating perfect fluid

These models have \( \omega = 0 \). We assume that \( \mathbf{u} \neq 0 \). An immediate implication for the commutation functions \( n_{a \beta} \), in addition to (8), is that \( n_{11} = 0 \), from (69). Now from the \( \mathbf{H} \)-constraint (107) it follows that \( H_{11} = 0 \), reducing the \( \mathbf{H} \)-equation (88) to a trivial statement. A useful point of departure is provided by the \( \mathbf{E} \)-equations (84) and (86). That is, \( \Omega_2 E_{11} = \Omega_3 E_{11} = 0 \). A brief argument below will show that this means that in \( \mathcal{U} \)

\[
\Omega_2 = \Omega_3 = 0 .
\] (38)

- The argument goes as follows: if \( E_{11} = 0 \), then from (74) we have \( (\mu + p) \sigma_{11} = 0 \Rightarrow \sigma_{11} = 0 \). We find that the following gradients vanish: \( e_1(\mu) = 0 = e_1(\Theta) \) from (120) and (118), respectively. Now from the commutator relation (58) acting on \( \mu \) we get \( \Theta (\mu + p) = 0 \); that is, this is a non-cosmological subcase. In this little argument, we have employed the assumption that the matter fluid has a barotropic equation of state, \( p = p(\mu) \).

So we must have (88) holding, which then implies from (102) and (103) that

\[
e_2(\dot{u}_1) = e_3(\dot{u}_1) = 0 .
\] (39)

In turn the effect of the above is that \( (a_2 - n_{31}) = (a_3 + n_{12}) = 0 \) which derives from (104) and (107). So we may write \( (a_2 + n_{31}) = 2a_2 \) and \( (a_3 - n_{12}) = 2a_3 \).

2.2.1 Tetrad choice and constraint analysis

From the above we see that we are free to employ along \( \mathbf{u} \) a Fermi-propagated \( \sigma \)-eigentriad \( \{ e_a \} \) by setting \( \Omega_1 = e_0(\varphi) \) via a spatial rotation by an angle \( \varphi \) in the \( e_2 / e_3 \)-plane so that \( \Omega_1' = 0 \). We note the following consistency conditions:

\[
\begin{align*}
  n_{11} &= 0 \quad \text{(from (76))} \\
  (a_2 - n_{31}) &= 0 \quad \text{(from (72) and (112))} \\
  (a_3 + n_{12}) &= 0 \quad \text{(from (73) and (114)).} 
\end{align*}
\]

We can further use the tetrad freedom on a 3-surface \( x^0 = c^0 \) to set \( n_{33} = 0 \). This condition is then preserved as we can see from (78). We can also set \( a_3 = 0 \) on a 2-surface \( x^0 = c^0, x^1 = c^1 \). This way we can do with impurity since firstly \( e_0(a_3) = -\Theta a_3 \sigma_{13} \) from (75) and (113). Secondly \( e_1(a_3) = a_1 a_3 \) from (118) and (119). We conclude \( a_3 = 0 \) everywhere. We may summarise: the only non-zero commutation functions are \( \dot{u}_1, \Theta_1, \Theta_2, a_1 \) and \( a_2 \). Of these quantities, only \( a_2 \) does not have its \( e_2^{-} \) and \( e_3^{-} \)-gradients vanishing; as can be seen from (93), (122) and (114) and (115). We may now proceed to use the remaining tetrad freedom to set \( e_4(a_2) = 0 \) on the line \( x^0 = c^0, x^1 = c^1, x^2 = c^2 \). This we can do by observing that the \( e_0^{-}, e_1^{-} \) and \( e_2^{-} \)-derivatives of this quantity are conserved respectively by applying the commutators (90), (112) and (113) to \( a_2 \). We also need (74) and (113), and (120) and (122) to see this. Finally we set \( e_2(a_2) = 0 \) at an event \( x^1 = c^1 \). This can be done because firstly, the \( e_0^{-} \)-derivative of \( e_2(a_2) \) is driven by a multiple of \( e_2(a_2) \) (from (58) operating on \( a_2 \) and \( \mathbf{J} \)); secondly, the \( e_1^{-} \)-derivative of \( e_2(a_2) \) is driven by a multiple of \( e_2(a_2) \) (from \( \mathbf{K} \)) operating on \( a_2 \)). Thirdly, the \( e_2^{-} \)-derivative of \( e_2(a_2) \) is driven by a multiple of \( e_2(a_2) \) (from taking \( e_2 \) of (122)); lastly, the \( e_3^{-} \)-derivative of \( e_2(a_2) \) vanishes (from (113)). The remaining commutation functions have their gradients in the \( e_2 / e_3 \)-plane vanishing, and the only equations remaining constrain quantities in the invariant spatial \( e_1^{-} \)-direction (from (112) and (120)). The now algebraic relation (122) determines \( E_{11} \) in terms of \( \mu \), say. The curvature variables which remain non-zero in \( \mathcal{U} \) are \( p, \mu \), and \( E_{11} \). We find that all of these variables have their gradients vanishing in the \( e_2 / e_3 \)-plane. This is obvious from (83) and (99), and (93) combined with (94), as well as (94) combined with (96). The remaining constraints on the various \( e_1^{-} \)-gradients are given by (17) and (26). The consistent solutions in \( \mathcal{U} \) we obtain are the LRS class II solutions of Stewart and Ellis [21].

2.2.2 Summary

Having assumed in this subsection a barotropic equation of state, \( p = p(\mu) \), we conclude that the only irrotational accelerating perfect fluid cosmologies that are PLRS according to Definition 2.1 are the expanding solutions in LRS class II of Stewart and Ellis [21] (see also [13] and [19]).

2.3 Rotating dust

These models have \( p = 0 \). From part of the twice-contracted Bianchi identities (momentum conservation equations), \( e_\alpha(p) = 0 \Rightarrow \omega^\alpha = 0 \). We here assume that \( \omega \neq 0 \). Since we are dealing with dust, we note that now our tetrad choice is that of Ellis in \([7]\). These models have\(^\text{6}\) operational accelerating perfect fluid cosmologies that are PLRS according to Definition 2.1.

2.3.1 Tetrad choice and constraint analysis

We are free to propagate the spatial triad \( \{e_\alpha\} \) along \( u \) as anti-rotating by choosing \( (\omega_1 + \Omega_1) = e_0(\varphi) \), i.e., \( (\omega_1 + \Omega_1)' = 0 \). We can further use the tetrad freedom in \( \mathcal{U} \) to set \( n_{33} = 0 \) by choosing \( e_1(\varphi) \) such that \( e_1(\varphi) = -n_{33} \). This quantity is conserved, as we can see from (78). But now we may not as yet proceed with a further tetrad specification as in \([7]\), where \( (a_3 - n_{12}) = 0 \) on a 2-surface \( x^0 = c^0, x^1 = c^1 \), because this would constrain the geometry. In particular, it actually requires \( e_2(\omega_1) = 0 \) for \( (a_3 - n_{12}) = 0 \) to hold, using (75) and (115). So we proceed by leaving the tetrad freedom unfixed for now and see what the implications are from consistency checks. From setting \( \sigma \) to be rotationally symmetric we do not get any immediate constraints. But from setting \( E \) to be rotationally symmetric we get new constraints. Specifically we get

\[
(a_2 - n_{31}) H_{11} = 0 \quad \text{(from the } \dot{\mathcal{E}} \text{-equation (84) combined with (131))}
\]

\[
(a_3 + n_{12}) H_{11} = 0 \quad \text{(from the } \dot{\mathcal{E}} \text{-equation (86) combined with (131))}.
\]

So naturally here we have a split into

\[
\text{A] } H_{11} = 0 \quad \text{or} \quad \text{B] } (a_2 - n_{31}) = (a_3 + n_{12}) = 0
\]

\[
\text{A] } H_{11} = 0: \quad \text{For } H_{11} \text{ to vanish, we must have from the } \dot{\mathcal{E}} \text{-equation (88) that } n_{11} E_{11} = 0.
\]

\[
\text{• A brief argument now shows that we are not interested in } E_{11} = 0. \text{ It goes as follows: if } E_{11} = 0, \text{ then from (74) we have } \mu \sigma_{11} = 0 \Rightarrow \sigma_{11} = 0. \text{ For consistency we now require from the } \dot{\sigma} \text{-equations (65) and (66) that } \omega_1 = 0; \text{ that is, this case is dealt with elsewhere. In fact this leads to FLRW solutions as we have already noted.}
\]

Then we conclude that \( E_{11} \neq 0 \) must hold and hence \( n_{11} = 0 \). This has the immediate implication from the \( \mathcal{H} \)-constraint equation (107) that \( a_1 = 0 \). Now we get from the second Bianchi identities the following new constraints

\[
3 (a_2 - n_{31}) E_{11} - e_2(\mu) = 0 \quad \text{(from (93) and (14))} \quad (40)
\]

\[
3 (a_3 + n_{12}) E_{11} - e_3(\mu) = 0 \quad \text{(from (15) and (66)).} \quad (41)
\]

We propagate (12) along \( u \) twice, using the necessary evolution equations. We use (72), (73), and (13) operating on \( \mu \) with (64), to get

\[
\omega_1 (a_3 + n_{12}) (3 E_{11} - \mu) + 2 \mu e_3(\omega_1) = 0.
\]

We now need (12), (72), (73), (64), and (10) operating on \( \omega_1 \), to get for consistency of (12) that

\[
- \omega_1 (a_2 - n_{31}) (3 E_{11} - \mu) + 4 \mu e_2(\omega_1) = 0.
\]

\(^6\)Described in his Theorem 3.1.
We propagate (11) along \( u \) twice, using the necessary evolution equations. These are (73), (79), and (80) operating on \( \mu \) with (64), which thus yields

\[
\omega_1 (a_2 - n_{31}) (3 E_{11} - \mu) + 2 \mu e_2 (\omega_1) = 0 .
\]  
(44)

We now need (69), (72), (79), (64), and (59) operating on \( \omega_1 \) to get for consistency of (43) that

\[
\omega_1 (a_3 + n_{12}) (3 E_{11} - \mu) - 4 \mu e_3 (\omega_1) = 0 .
\]  
(45)

We form linear combinations of the above four constraints to facilitate our task at this point.

\[
\begin{align*}
\mu e_3 (\omega_1) &= 0 \quad \text{(from } (12) - (45) \text{)} \quad \Rightarrow \quad e_3 (\omega_1) = 0 \\
\mu e_2 (\omega_1) &= 0 \quad \text{(from } (13) + (44) \text{)} \quad \Rightarrow \quad e_2 (\omega_1) = 0 \\
(a_3 + n_{12}) (3 E_{11} - \mu) &= 0 \quad \text{(from } 2 \times (12) + (13) \text{)} \\
(a_2 - n_{31}) (3 E_{11} - \mu) &= 0 \quad \text{(from } (13) - 2 \times (13) \text{)}.
\end{align*}
\]

- A brief argument now shows that \( 3 E_{11} - \mu = 0 \) is not applicable. It goes as follows: we note that \( e_2 (\omega_1) = e_3 (\omega_1) = 0 \). So from the commutator (51) acting on \( \omega_1 \) and incorporating the \( \omega \)-equation (93) into this, we get that \( \Theta_2 = 0 \). The consistency of this requires from (56) that

\[
3 E_{11} - \mu + 3 \omega_1^2 = 0,
\]

and if \( 3 E_{11} - \mu = 0 \), it must necessarily follow that \( \omega_1 = 0 \), which we excluded.

So we must conclude that \( (a_2 - n_{31}) = (a_3 + n_{12}) = 0 \). This is severely restrictive. We get from the commutator (51) acting on \( \omega_1 \) that \( \Theta_2 = 0 \), using the \( \omega \)-equation (93). And now from (116) we must also have \( \Theta_1 = 0 \). These last two results, in particular, imply that in \( \mathcal{U} \)

\[
\Theta = 0 ,
\]
(46)

which means that spacetimes in this PLRS subclass are non-cosmological.

B] \( (a_2 - n_{31}) = (a_3 + n_{12}) = 0 \): Firstly we note from (112) and (114) that

\[
e_2 (\Theta_1) = e_3 (\Theta_1) = 0 .
\]
(47)

The critical constraints are obtained from the second Bianchi identities: (93) combined with (94), and (95) combined with (96) once again. They read, respectively,

\[
\begin{align*}
e_2 (E_{11}) &= e_2 (\mu) = 0 \\
e_3 (E_{11}) &= e_3 (\mu) = 0 .
\end{align*}
\]

We proceed to check preservation along \( u \) of \( e_2 (\mu) = e_3 (\mu) = 0 \) by using evolution equations obtained from the commutator relations (59) and (60) when acting on \( \mu \). We also require from this the relations given by (12), (112), (114), (113) and (115). We get that

\[
\begin{align*}
e_0 (e_2 (\mu)) &= \left( \frac{1}{2} \sigma_{11} - \frac{3}{2} \Theta \right) e_2 (\mu) - 2 e_3 (\omega_1) \quad \Rightarrow \quad e_3 (\omega_1) = 0 \\
e_0 (e_3 (\mu)) &= \left( \frac{1}{2} \sigma_{11} - \frac{3}{2} \Theta \right) e_3 (\mu) + 2 e_2 (\omega_1) \quad \Rightarrow \quad e_2 (\omega_1) = 0 .
\end{align*}
\]

Taking these results and putting them back into (112), (114), (113) and (115), and recalling (47), we have \( e_2 (\Theta_1) = e_3 (\Theta_1) = e_2 (\Theta_2) = e_3 (\Theta_2) = 0 \). We note now from (130) and (131) that it is apparent that the gradients of \( H_{11} \) are also degenerate in this fashion: \( e_2 (H_{11}) = e_3 (H_{11}) = 0 \). We check preservation along \( u \) of \( e_2 (E_{11}) = e_3 (E_{11}) = 0 \) by using (124) and the commutators (54) and (60) when acting on \( E_{11} \). The results are that

\[
\begin{align*}
e_0 (e_2 (E_{11})) &= - \frac{3}{2} H_{11} e_2 (n_{11}) \\
e_0 (e_3 (E_{11})) &= - \frac{3}{2} H_{11} e_3 (n_{11}) .
\end{align*}
\]

If now \( H_{11} = 0 \), we have a case already dealt with in A]. So we must therefore conclude that \( e_2 (n_{11}) = e_3 (n_{11}) = 0 \). Now here again crucial algebraic constraints are obtained from the commutator (61). Acting on \( \omega_1 \) we get

\[
\Theta_2 \omega_1 = a_1 n_{11}
\]
(48)
from the $\dot{\omega}$–equation (69) and the $e_1(\omega_1)$–constraint (106). Acting on $n_{11}$ we get
\[ n_{11} (\sigma_{11} \omega_1 + a_1 n_{11}) = 0 \] (49)
from the evolution equation for $n_{11}$ (76) and the $e_1(n_{11})$–constraint (116). We show that we do not get any relevant solutions here. We start with equation (49).

• If we first show that $n_{11} = 0$ leads to trivial solutions. If $n_{11} = 0$, then from (116) we must have $\Theta_1 = 0$ and then from (48) this means that $\Theta_2 = 0$ and so this cannot be cosmological because consequently in $U$
\[ \Theta = 0 \] (50)

• If, on the other hand, $n_{11} \neq 0$, and instead we have from (49) that $\sigma_{11} \omega_1 + a_1 n_{11} = 0$, then we immediately get from (48) that $\Theta \omega_1 = 0$; that is, in $U$
\[ \Theta = 0 \] (51)

So these too are not cosmological solutions.

2.3.2 Summary
We conclude that there are no rotating dust cosmologies which are PLRS according to Definition 2.1.

2.4 Irrotational dust
We shall now consider irrotational dust cosmologies under the PLRS restrictions of Definition 2.1: that is to say, cosmologies where $\omega = 0$ and $p = 0 \Rightarrow \dot{u} = 0$. We can show that the only consistent irrotational dust solutions of the EFE which have $\sigma$ and both $E$ and $H$ rotationally symmetric in the same plane are known solutions.

For example, the case $\sigma = 0$ leads to the FLRW dust solutions [7]. Proceeding, we find from (103) and (102) that in $U$
\[ \Omega_2 = \Omega_3 = 0 \] (52)

2.4.1 Tetrad choice and constraint analysis
We note that for the models we are interested in here, we again choose the $\sigma$–eigentriad $\{ e_\alpha \}$ to be Fermi-propagated along $u$ by setting $\Omega_1 = e_0(\varphi)$ via a spatial rotation by an angle $\varphi$ in the $e_2/e_3$–plane so that $\Omega'_1 = 0$. We may also choose $n'_{33} = 0 \Rightarrow e_1(\varphi) = -n_{33}$, since now the evolution equation (78) for $n_{33}$ becomes involutive. We will show that the present PLRS subclass recovers the well known Szekeres dust solutions [23]. From the $E$–equation (84) we find $e_2(H_{11}) = \frac{3}{2} (a_2 - n_{31}) H_{11}$, while (85) gives $e_3(H_{11}) = 0$, leading to the more useful algebraic result
\[ (a_2 - n_{31}) H_{11} = 0 \] (52)

Then from (86) we find $e_3(H_{11}) + \frac{3}{2} (a_3 + n_{12}) H_{11} = 0$, while (87) gives $e_3(H_{11}) = 0$, now yielding the algebraic result
\[ (a_3 + n_{12}) H_{11} = 0 \] (53)

Now the commutator (61) is helpful at this point. If we operate on $H_{11}$, we find that $n_{11} e_1(H_{11}) = 0$, which subdivides the class into

\[ A] \ n_{11} = 0 \ \ \text{or} \ \ B] \ e_1(H_{11}) = 0 \]

We may, with relative ease generalise this result to the situation where only $\sigma$ and $H$ are rotationally symmetric.
A] $n_{11} = 0$: It is easy to see from (76) that presently $n_{11} = 0$ is conserved along $u$. Hence, the $H$-constraint (107) gives $H_{11} = 0$, which the $H$-equation (88) preserves; thus $H = 0$. Now we look at (124), which reads
\[ e_2(E_{11}) = \frac{1}{3} e_2(\mu), \tag{54} \]
and we use (73), (113), (64), and the commutator (59) acting on $E_{11}$ and $\mu$, to check the time evolution property of this last relation:
\[ e_0(2 e_2(E_{11}) - \frac{1}{3} e_2(\mu)) = -4 \Theta_2 \left[ e_2(E_{11}) - \frac{1}{3} e_2(\mu) \right]. \]
This is solved by $e_2(E_{11}) = \frac{1}{3} e_2(\mu)$, and we may now use (127) with the above to get $e_2(E_{11}) = \frac{3}{2} (a_2 - n_{31}) E_{11} - \frac{1}{6} e_2(\mu)$, which, in this manner, shows the consistency with (63). Now we look at (129), which reads
\[ e_3(E_{11}) = \frac{1}{3} e_3(\mu), \tag{55} \]
and we use (74), (115), (64), and the commutator (60) acting on $E_{11}$ and $\mu$, to check the time evolution property of this last relation:
\[ e_0(3 e_3(E_{11}) - \frac{1}{3} e_3(\mu)) = -4 \Theta_2 \left[ e_3(E_{11}) - \frac{1}{3} e_3(\mu) \right]. \]
This is solved by $e_3(E_{11}) = \frac{1}{3} e_3(\mu)$, and we may now use (128) with the above to get $e_3(E_{11}) = \frac{3}{2} (a_3 + n_{12}) E_{11} - \frac{1}{6} e_3(\mu)$, which, in this manner, shows the consistency with (63). We have from (121) and (113) that along $u$, and, by substitution into the relevant commutators, that everywhere $e_2(a_1) = e_3(a_1) = 0$. We may at this point use some of the remaining freedom in the commutators to (on a 2-surface $x^0 = c^0$, $x^1 = c^1$) set $e_2(a_2 + n_{31}) = e_2(a_3 - n_{12})$. This can be done because firstly
\[ e_0(3(a_2 + n_{31}) - e_2(a_3 - n_{12})) = -2 \Theta_2 \left[ e_3(a_2 + n_{31}) - e_2(a_3 - n_{12}) \right], \]
using the commutator (60) on $(a_2 + n_{31})$ and the commutator (65) on $(a_4 - n_{12})$. Secondly,
\[ e_1(3(a_2 + n_{31}) - e_2(a_3 - n_{12})) = 2 a_1 \left[ e_3(a_2 + n_{31}) - e_2(a_3 - n_{12}) \right]. \]

B] $e_1(H_{11}) = 0$: Immediately we see from the $e_1(H_{11})$-constraint (124) that $a_1 H_{11} = 0$, which provides us with the subdivision

B1] $a_1 = 0$ or B2] $H_{11} = 0$.

B1] $a_1 = 0$: We still have (122) and (123) holding, i.e., $(a_2 - n_{31}) H_{11} = (a_3 + n_{12}) H_{11} = 0$. Now if $H_{11} = 0$, then we are dealing with B2]. So instead here we must have $(a_2 - n_{31}) = (a_3 + n_{12}) = 0$. We may use the tetrad freedom to set $(a_3 - n_{12}) = 0$. From (122) and (113) we have $(a_4 - n_{12}) = 0$ without any further new constraints. And now, from (118), we have $(a_3 - n_{12}) = 0$ without any further new constraints. So this choice is allowed. Thus we may state that
\[ a_3 = n_{12} = 0 \quad \text{and} \quad (a_2 + n_{31}) = 2 a_2. \]

Now from (93) combined with (94), (95) combined with (16), the commutator (61) operating on $\mu$, and then using (31), we get that $e_n(\mu) = e_n(E_{11}) = 0$. Also, from (112) and (113) we have $e_2(\Theta_1) = e_2(\Theta_2) = 0 \Rightarrow e_2(\Theta) = e_2(\sigma_{11}) = 0$, while from (114) and (117) follows $e_3(\Theta_1) = e_3(\Theta_2) = 0 \Rightarrow e_3(\Theta) = e_3(\sigma_{11}) = 0$, which then, from the commutator (60) acting on $\Theta$ and $\sigma_{11}$, gives $e_n(\Theta) = e_n(\sigma_{11}) = 0$. Moreover, it is clear from (64) and (65) that $e_n(H_{11}) = 0$. The constraints (116), (111) and (112) imply $e_n(n_{31}) = 0$. 
By applying the commutators (58), (59) and (60) on the variables \( f \in \{ \mu, E_{11}, H_{11}, \Theta, \sigma_{11}, n_{11} \} \), and utilising their respective evolution equations (123), (124), (86), (124) and (74), we can show that \( \mathbf{e}_a(f) = 0 \). Now from (120) we have \( \mathbf{e}_1(a_2) = 0 \), which allows us to use the remaining freedom to set \( \mathbf{e}_3(a_2) = 0 \). We can do this because

\[
\mathbf{e}_0(\mathbf{e}_3(a_2)) = -2 \Theta_2 \mathbf{e}_3(a_2) \quad \text{from (58) on } a_2, (74) \text{ and (117)},
\]

\[
\mathbf{e}_1(\mathbf{e}_3(a_2)) = 0 \quad \text{from (52) on } a_2 \text{ and (123)},
\]

and

\[
\mathbf{e}_2(\mathbf{e}_3(a_2)) = 6 \mathbf{a}_2 \mathbf{e}_3(a_2) \quad \text{from (61) on } a_2, (122) \text{ and (20)}.
\]

Finally we set \( \mathbf{e}_2(a_2) = 0 \) at an event \( x^i = c^i \). This can be done because firstly, the \( \mathbf{e}_0 \)-derivative of \( \mathbf{e}_2(a_2) \) is given by a multiple of \( \mathbf{e}_2(a_2) \) (from (58) operating on \( a_2 \) and (74)); secondly, the \( \mathbf{e}_1 \)-derivative of \( \mathbf{e}_2(a_2) \) vanishes (from (52) operating on \( a_2 \) and (123)); thirdly, the \( \mathbf{e}_i \)-derivative of \( \mathbf{e}_2(a_2) \) is given by a multiple of \( \mathbf{e}_2(a_2) \) (from taking \( \mathbf{e}_2 \) of (122)); lastly, the \( \mathbf{e}_i \)-derivative of \( \mathbf{e}_2(a_2) \) vanishes (from (61) operating on \( a_2 \)).

The remaining non-zero commutation functions and curvature variables they are coupled to in \( \mathcal{U} \) are \( \mu, \Theta_1, \Theta_2, a_2, n_{11}, E_{11} \text{ and } H_{11} \). The remaining constraints (107), (123) and (122) are all algebraic relations. Thus the presently obtained consistent solutions constitute the subclass of spatially homogeneous cosmologies within the dust LRS class II of Ellis (18).

**B2|** \( H_{11} = 0 \): From the \( \dot{H} \) equation (88) it follows that \( n_{11} \sigma_{11} = 0 \). Now if \( n_{11} = 0 \) then we are dealing with case A1, and if in \( \mathcal{U} \) \( \sigma_{11} = 0 \), these are the FLRW dust solutions and are well known.

### 2.4.2 Summary

We conclude that the only non-trivial irrotational dust cosmologies that are PLRS according to Definition 2.3 are known. These are the Szekeres dust cosmologies (23), which are strictly PLRS, and Ellis’ dust spacetimes in LRS class II (18). Moreover, this last study demonstrates that local rotational symmetry of an irrotational dust cosmology results if, in an open neighbourhood \( \mathcal{U} \) of an event of \( \mathcal{M} \), all tensors algebraically determined from the Riemann curvature tensor and all its covariant derivatives up to only second order are rotationally symmetric, generalising a corresponding result given in (2). This property may be traced back to the fact that a rotationally symmetric covariant derivative \( \nabla_1 \mathbf{e}_1 \), i.e.

\[
\Gamma_{211} = \mathbf{e}_2 \cdot \nabla_1 \mathbf{e}_1 = 0 = \mathbf{e}_3 \cdot \nabla_1 \mathbf{e}_1 = \Gamma_{311},
\]

corresponds by (22) and (58) to

\[
0 = (a_2 - n_{31}) = (a_3 + n_{12}),
\]

eliminating the Szekeres dust models from the above consistent set and reducing to the LRS solutions. Note that these conditions also apply when \( p \neq 0 \Leftrightarrow \dot{u}_1 \neq 0 \) (cf. subsection 2.2).

### 3 Conclusion

The only consistent solutions for the general class of perfect fluid cosmologies with all tensors rotationally symmetric about the same spatial axis are known solutions. All of these solutions are either LRS or they belong to the Szekeres class of dust cosmologies. Thus our work may be viewed as a form of classification scheme for inhomogenous cosmologies which incorporates the Szekeres dust solutions. This may be contrasted with the scheme developed by Szafroń and Collins, where restrictions placed on submanifolds achieve this result (3, 22, 1). A main result of the present work is that for all spatially inhomogeneous perfect fluid cosmologies that are PLRS according to Definition 2.3 we have in \( \mathcal{U} \)

\[
\dot{H} = 0.
\]

This generalises similar results obtained for the Szekeres dust cosmologies in (3), and for perfect fluid cosmologies in LRS class II in (13). It has also been demonstrated that perfect fluid cosmologies are LRS if tensors and their covariant derivatives up to second order are rotationally symmetric, generalising a similar result obtained in (18).
There are no PLRS perfect fluid cosmologies which are rotating. Also, the cases which have vanishing fluid shear are fairly trivial: they are either not cosmologies at all in our understanding or they are of the simple FLRW kind. This last result requires the assumption of a barotropic equation of state $p = p(\mu)$ in the irrotational accelerating perfect fluid case (although it is probably possible to relax this requirement).

We have seen that the only strictly PLRS solutions are the Szekeres dust cosmologies. In later work we intend to relax the requirements of Definition 2.1 to see what partial symmetry features result. One possibility is to keep $\sigma$ and $E$ rotationally symmetric about the same spatial axis, but leave $H$ unrestricted such that a uniquely defined tetrad $\{e_a\}$ does exist. An example of a spatially inhomogeneous perfect fluid cosmology with coinciding eigenvalues of $\sigma$ and non-zero $E$ and $H$ is provided by the diagonal Abelian $G_2$ solutions with radiation equation of state by Senovilla [20].

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Appendix

A Evolution and constraint equations

Restrictions imposed: All 1 + 3 ONT equations we list in the following are written out for cosmologies with a perfect fluid matter source that are subject to the specific restrictions that

- a fluid-comoving 1 + 3 ONT $\{e_a\} = \{u, e_\alpha\}$ is chosen which diagonalises $\sigma$,
- $\sigma$ is rotationally symmetric in the $e_2/e_3$–plane; meaning, $\sigma_{22} = \sigma_{33}$ $(-\frac{1}{2}\sigma_{11})$,
- $\omega_2 = \omega_3 = 0$, that is, $\omega \parallel e_1$,
- $\dot{u}_2 = \dot{u}_3 = 0$; equivalently, $\dot{u} \parallel e_1$.

For brevity we use the variables $\Theta_1$ and $\Theta_2$ in some sets of equations while we use $\sigma_{11}$ and $\Theta$ in other areas where the degeneracy in the eigenvalues of $\sigma$ allows for some simplification. The relations

$$\sigma_{11} = \frac{2}{3}(\Theta_1 - \Theta_2) \quad \text{and} \quad \Theta = \Theta_1 + 2\Theta_2$$

yield the transformation from the former to the latter set of variables.

The 1 + 3 ONT equations for perfect fluid cosmologies in fluid-comoving description without the specialisation stated can be accessed online at the URL [12].

A.1 Rotational tetrad freedom

For coinciding eigenvalues of the fluid shear tensor $\sigma$ such that $\sigma_{22} = \sigma_{33}$, its eigentetrad $\{e_a\}$ is defined only up to spatial rotations $\Lambda = \Lambda(x')$ in the $e_2/e_3$–plane. These may be represented by

$$\Lambda^{-1}_{a'}{}^a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \varphi & \sin \varphi \\ 0 & 0 & -\sin \varphi & \cos \varphi \end{pmatrix},$$

(57)

with, in general, $e_0(\varphi) \neq 0 \neq e_1(\varphi)$. The members of the $\sigma$–eigentetrad then transform according to

$$e_{a'} \rightarrow \Lambda^{-1}_{a'}{}^a e_a.$$
A.2 Commutators

In explicit form the commutators read

\[
\begin{align*}
[\mathbf{e}_0, \mathbf{e}_1] (f) &= \dot{u}_1 \mathbf{e}_0(f) - \Theta_1 \mathbf{e}_1(f) - \Omega_3 \mathbf{e}_2(f) + \mathbf{e}_3(f) + \Omega_2 \mathbf{e}_4(f) \\
[\mathbf{e}_0, \mathbf{e}_2] (f) &= \Omega_3 \mathbf{e}_1(f) - \Theta_2 \mathbf{e}_2(f) - (\omega_1 + \Omega_1) \mathbf{e}_3(f) \\
[\mathbf{e}_0, \mathbf{e}_3] (f) &= -\Omega_2 \mathbf{e}_1(f) + (\omega_1 + \Omega_1) \mathbf{e}_2(f) - \Theta_2 \mathbf{e}_3(f) \\
[\mathbf{e}_2, \mathbf{e}_3] (f) &= 2 \omega_1 \mathbf{e}_1(f) + n_{13} \mathbf{e}_1(f) - (a_3 - n_{12}) \mathbf{e}_2(f) + (a_2 + n_{31}) \mathbf{e}_3(f) \\
[\mathbf{e}_3, \mathbf{e}_1] (f) &= (a_3 + n_{12}) \mathbf{e}_1(f) + n_{22} \mathbf{e}_2(f) - (a_1 - n_{23}) \mathbf{e}_3(f) \\
[\mathbf{e}_1, \mathbf{e}_2] (f) &= -(a_2 - n_{31}) \mathbf{e}_1(f) + (a_1 + n_{23}) \mathbf{e}_2(f) + n_{33} \mathbf{e}_3(f).
\end{align*}
\]

A.3 Evolution equations

A.3.1 Energy density evolution

Conservation of energy requires

\[ e_0(\mu) = -\Theta (\mu + p) . \] (64)

A.3.2 Fluid shear and expansion evolution in terms of E

\[
\begin{align*}
R_{0101} &= - (e_0 + \Theta_1) (\Theta_1) + (e_1 + \dot{u}_1) (\dot{u}_1) = E_{11} + \frac{1}{6} (\mu + 3p) \\
R_{0202} &= - (e_0 + \Theta_2) (\Theta_2) + \omega_1^2 - (a_1 + n_{23}) \dot{u}_1 = E_{22} + \frac{1}{6} (\mu + 3p) \\
R_{0303} &= - (e_0 + \Theta_3) (\Theta_3) + \omega_2^2 - (a_1 - n_{23}) \dot{u}_1 = E_{33} + \frac{1}{6} (\mu + 3p) .
\end{align*}
\] (65) (66) (67)

A.3.3 Fluid expansion evolution

The expansion evolution is obtained by writing out \( R_{0}^{\alpha} \alpha_0 \):

\[ e_0(\Theta) = -\frac{1}{3} \Theta^2 + (e_1 + \dot{u}_1 - 2a_1) (\dot{u}_1) - \frac{3}{2} \sigma_{11}^2 + 2 \omega_1^2 - \frac{1}{2} (\mu + 3p) . \] (68)

A.3.4 Fluid vorticity and spatial commutation function evolution

\[
\begin{align*}
(0)_{012} & \quad e_0(\omega_1) = -2 \Theta_2 \omega_1 - \frac{1}{2} n_{11} \dot{w}_1 \\
(0)_{23} & \quad e_0(a_1 + n_{23}) = -\Theta_1 (a_1 + n_{23}) - (e_1 + \dot{u}_1) (\dot{u}_1) + (e_2 - a_2 + n_{31}) (\Omega_3) \\
& \quad + (\omega_1 + \Omega_1) (n_{22} - n_{33}) + \Omega_2 (a_3 - n_{12}) \\
(3)_{013} & \quad e_0(a_1 - n_{23}) = -\Theta_1 (a_1 - n_{23}) - (e_1 + \dot{u}_1) (\dot{u}_1) - (e_3 - a_3 - n_{12}) (\Omega_2) \\
& \quad - (\omega_1 + \Omega_1) (n_{22} - n_{33}) - \Omega_3 (a_2 + n_{31}) \\
(1)_{012} & \quad e_0(a_2 - n_{31}) = -\Theta_2 (a_2 - n_{31}) - e_2 (\dot{u}_1) - (e_1 + \dot{u}_1 - a_1 - n_{23}) (\Omega_3) \\
& \quad - (\omega_1 + \Omega_1) (a_3 + n_{12}) - \Omega_2 (n_{33} - n_{11}) \\
(1)_{013} & \quad e_0(a_3 + n_{12}) = -\Theta_2 (a_3 + n_{12}) - e_2 (\dot{u}_1) + (e_1 + \dot{u}_1 - a_1 + n_{23}) (\Omega_2) \\
& \quad + (\omega_1 + \Omega_1) (a_2 - n_{31}) + \Omega_3 (n_{11} - n_{22}) \\
(3)_{023} & \quad e_0(a_2 + n_{31}) = -(e_2 + a_2 + n_{31}) (\Theta_2) + (e_3 - a_3 + n_{12}) (\omega_1 + \Omega_1) \\
& \quad + \Omega_3 (a_1 - n_{23}) + \Omega_2 (n_{33} - n_{11}) \\
(2)_{023} & \quad e_0(a_3 - n_{12}) = - (e_3 + a_3 - n_{12}) (\Theta_2) - (e_2 - a_2 - n_{31}) (\omega_1 + \Omega_1) \\
& \quad - \Omega_2 (a_1 + n_{23}) - \Omega_1 (n_{11} - n_{22}) \\
(1)_{023} & \quad e_0(n_{11}) = -\left(\frac{4}{3} \Theta - 2 \sigma_{11}\right) n_{11} - (e_2 - 2n_{31}) (\Omega_2) - (e_3 + 2n_{12}) (\Omega_3) \\
(1)_{013} & \quad e_0(n_{22}) = -\Theta_1 n_{22} - (e_1 + \dot{u}_1 + 2n_{23}) (\omega_1 + \Omega_1) - (e_3 - 2n_{12}) (\Omega_3) \\
(3)_{012} & \quad e_0(n_{33}) = -\Theta_1 n_{33} - (e_1 + \dot{u}_1 - 2n_{23}) (\omega_1 + \Omega_1) - (e_2 + 2n_{31}) (\Omega_2) .
\end{align*}
\] (69) (70) (71) (72) (73) (74) (75) (76) (77) (78)
A.3.5 Evolution equations for E and H

\[(023|23)\]
\[
(e_0 + 3\Theta_2)(E_{11}) = (e_2 - a_2 - n_{31})(H_{31}) - (e_3 - a_3 + n_{12})(H_{12})
- \frac{1}{2}(\mu + p)\sigma_{11} + 2\Omega_2 E_{31} - 2\Omega_3 E_{12}
- \frac{3}{2} a_{11} H_{11} + \frac{3}{2}(n_{22} - n_{33})(H_{22} - H_{33}) + 2n_{31} H_{23}
\]
\[(031|31)\]
\[
(e_0 + \Theta)(E_{22}) = -(e_1 + 2u_1 - a_1 + n_{23})(H_{23}) + (e_3 - a_3 - n_{12})(H_{12})
+ \frac{1}{2}(\mu + p)\sigma_{11} + \frac{3}{2}\sigma_{11} E_{33} + 2\Omega_3 E_{12} - (\omega_1 + 2\Omega_1) E_{23}
- \frac{3}{2} n_{22} H_{22} + \frac{1}{2}(n_{33} - n_{11})(H_{33} - H_{11}) + 2n_{31} H_{31}
\]
\[(012|12)\]
\[
(e_0 + \Theta)(E_{33}) = (e_1 + 2u_1 - a_1 - n_{23})(H_{23}) - (e_2 - a_2 + n_{31})(H_{11})
+ \frac{1}{2}(\mu + p)\sigma_{11} + \frac{3}{2}\sigma_{11} E_{22} - 2\Omega_2 E_{31} + (\omega_1 + 2\Omega_1) E_{23}
- \frac{3}{2} n_{33} H_{33} + \frac{1}{2}(n_{11} - n_{22})(H_{11} - H_{22}) + 2n_{12} H_{12}
\]
\[(031|31)\]
\[
(e_0 + \Theta + \frac{3}{2}\sigma_{11})(E_{23}) = -(e_1 + u_1 - a_1 + n_{23})(H_{23}) + (e_3 - a_3 - 2n_{12})(H_{11})
+ \frac{1}{2}(\mu + p)\omega_1 - \omega_1(E_{33} - E_{11}) - \Omega_1(E_{33} - E_{22}) - \Omega_2 E_{12} + \Omega_3 E_{31}
- \frac{1}{2}(3n_{33} - n_{11} - n_{22}) H_{23} - (a_2 + n_{31}) H_{12} - (a_1 + n_{23}) H_{11} + u_1 H_{22}
\]
\[(012|23)\]
\[
(e_0 + \Theta)(E_{31}) = (e_1 + u_1 - 2a_1 - 2n_{23})(H_{12}) - (e_2 - a_2 + n_{31})(H_{11})
- \Omega_2(E_{11} - E_{33}) - \Omega_3 E_{23} + (2\omega_1 + \Omega_1) E_{12}
- \frac{1}{2}(3n_{33} - n_{11} - n_{22}) H_{31} - (a_2 + n_{12}) H_{23} - (a_2 - n_{31}) H_{12}
\]
\[(023|12)\]
\[
(e_0 + 3\Theta_2)(E_{33}) = (e_2 - a_2 - n_{31})(H_{33}) - (e_3 - a_3 + 2n_{12})(H_{23})
- \Omega_2(E_{11} - E_{33}) + (\Omega_1 - \omega_1) E_{12} - \Omega_3 E_{23}
- \frac{1}{2}(3n_{11} - n_{22} + n_{33}) H_{31} + (u_1 + a_1 - n_{23}) H_{12} + (a_2 + n_{12}) H_{22}
\]
\[(031|23)\]
\[
(e_0 + \Theta)(E_{12}) = -(e_1 + u_1 - 2a_1 + 2n_{23})(H_{11}) + (e_3 - a_3 - n_{12})(H_{11})
- \Omega_3(E_{22} - E_{11}) + \Omega_2 E_{23} - (2\omega_1 + \Omega_1) E_{31}
- \frac{1}{2}(3n_{22} - n_{33} + n_{11}) H_{12} + (a_2 - n_{31}) H_{23} + (a_3 + n_{12}) H_{31}
\]
\[(023|31)\]
\[
(e_0 + 3\Theta_2)(E_{12}) = (e_2 - 2a_2 - 2n_{31})(H_{23}) - (e_3 - a_3 + n_{12})(H_{22})
- \Omega_3(E_{22} - E_{11}) + (\omega_1 - \Omega_1) E_{31} + \Omega_2 E_{23}
- \frac{1}{2}(3n_{11} + n_{22} - n_{33}) H_{12} - (\omega_1 + a_1 + n_{23}) H_{31} - (a_3 - n_{12}) H_{33}
\]
\[(023|01)\]
\[
(e_0 + 3\Theta_2)(H_{11}) = -(e_2 - a_2 - n_{31})(E_{31}) + (e_3 - a_3 + n_{12})(E_{12})
+ \frac{1}{2} n_{11} E_{11} + \frac{3}{2}(n_{22} - n_{33})(E_{22} - E_{33}) - 2n_{23} E_{23} + 2\Omega_2 H_{31} - 2\Omega_3 H_{12}
\]
\[(031|02)\]
\[
(e_0 + \Theta)(H_{22}) = (e_1 + 2u_1 - a_1 + n_{23})(E_{23}) - (e_3 - a_3 - n_{12})(E_{12})
+ \frac{3}{2}\sigma_{11} H_{33} + \frac{3}{2} n_{22} E_{22} - \frac{1}{2}(n_{33} - n_{11})(E_{33} - E_{11})
- 2n_{31} E_{31} - (2\omega_1 + \omega_1) H_{23} + 2\Omega_3 H_{12}
\]
\[(012|03)\]
\[
(e_0 + \Theta)(H_{33}) = -(e_1 + 2u_1 - a_1 - n_{23})(E_{23}) + (e_2 - a_2 + n_{31})(E_{31})
+ \frac{3}{2}\sigma_{11} H_{22} + \frac{3}{2} n_{33} E_{33} - \frac{1}{2}(n_{11} - n_{22})(E_{11} - E_{22})
- 2n_{12} E_{12} + (2\Omega_1 + \omega_1) H_{23} - 2\Omega_2 H_{31}
\]
\[(012|02)\]
\[
(e_0 + \Theta + \frac{3}{2}\sigma_{11})(H_{23}) = -(e_1 + u_1 - a_1 - n_{23})(E_{22}) + (e_2 - 2a_2 + 2n_{31})(E_{12})
+ \omega_1(H_{22} - H_{11}) - \Omega_1(H_{33} - H_{22}) + \Omega_3 H_{31} - \Omega_2 H_{12} - \frac{1}{2} e_1(\mu)
+ \frac{1}{2}(3n_{33} - n_{11} + n_{22}) E_{23} - (a_3 - n_{12}) E_{31} - (a_1 + n_{23}) E_{11} + u_1 E_{33}
\]
A EVOLUTION AND CONSTRAINT EQUATIONS

We contract the second Bianchi identity (6) twice and find that for PLRS cosmologies according to Definition 2.1

\[ (e_0 + \Theta + \frac{3}{2} \sigma_{11})(H_{23}) = (e_1 + \dot{u}_1 - a_1 + n_{23})(E_{33}) - (e_3 - 2a_3 - 2n_{12})(E_{31}) \]

\[ -\omega_1(H_{33} - H_{11}) - \Omega_1(H_{33} - H_{22}) - \Omega_3 H_{12} + \Omega_3 H_{31} + \frac{1}{6} e_1(\mu) + \frac{1}{3}(3n_{22} + n_{33} - n_{11})E_{23} + (a_2 + n_{31})E_{12} + (a_1 - n_{23})E_{11} - \dot{u}_1 E_{22} \]

(92)

\[ (e_0 + \Theta)(H_{31}) = -(e_1 + \dot{u}_1 - 2a_1 - 2n_{23})(E_{12}) + (e_2 - a_2 + n_{31})(E_{11}) - \Omega_2 H_{11} - \Omega_3 H_{33} + (2\omega_1 + \Omega_1)H_{12} + \frac{1}{6} e_2(\mu) + \frac{1}{2}(3n_{33} + n_{11} - n_{22})E_{31} + (a_3 + n_{12})E_{23} + (a_2 - n_{31})E_{22} \]

(93)

\[ (e_0 + 3\Theta_2)(H_{31}) = -(e_2 - a_2 - n_{31})(E_{33}) + (e_3 - 3a_3 + 2n_{12})(E_{23}) - \Omega_2 H_{11} - \Omega_3 H_{33} + (2\omega_1 + \Omega_1)H_{12} - \frac{1}{6} e_2(\mu) + \frac{1}{2}(3n_{11} - n_{22} + n_{33})E_{31} - (\dot{u}_1 + a_1 - n_{23})E_{12} - (a_2 + n_{31})E_{22} \]

(94)

\[ (e_0 + \Theta)(H_{12}) = (e_1 + \dot{u}_1 - 2a_1 + 2n_{23})(E_{31}) - (e_3 - 3a_3 - n_{12})(E_{11}) - \Omega_3 H_{22} - H_{11} + \Omega_2 H_{33} - (2\omega_1 + \Omega_1)H_{31} - \frac{1}{6} e_3(\mu) + \frac{1}{2}(3n_{22} - n_{33} + n_{11})E_{12} - (a_2 - n_{31})E_{23} - (a_3 + n_{12})E_{33} \]

(95)

\[ (e_0 + 3\Theta_2)(H_{12}) = -(e_2 - 2a_2 - 2n_{31})(E_{23}) + (e_3 - a_3 + n_{12})(E_{22}) - \Omega_3 H_{22} - H_{11} + \Omega_2 H_{33} + (2\omega_1 - \Omega_1)H_{31} + \frac{1}{6} e_3(\mu) + \frac{1}{2}(3n_{11} + n_{22} - n_{33})E_{12} + (\dot{u}_1 + a_1 + n_{23})E_{31} + (a_3 - n_{12})E_{33} . \]

(96)

A.4 Constraint equations

A.4.1 Momentum conservation constraints

We contract the second Bianchi identity (6) twice and find that for PLRS cosmologies according to Definition 2.1

\[ e_1(p) + (\mu + p) \dot{u}_1 = 0 \]

(97)

\[ e_2(p) = 0 \]

(98)

\[ e_3(p) = 0 . \]

(99)

A.4.2 Fluid vorticity, spatial commutation function and fluid shear constraints

\[ R_{0202} - R_{0303} = (E_{22} - E_{33}) = -2n_{23} \dot{u}_1 \]

(100)

\[ R_{0203} - (0_{023}) = E_{23} = \frac{1}{2}(n_{22} - n_{33}) \dot{u}_1 \]

(101)

\[ R_{0102} = E_{12} = e_2(\dot{u}_1) + \Omega_2 \omega_1 + \frac{3}{2} \Omega_3 \sigma_{11} \]

(102)

\[ R_{0103} = E_{31} = e_3(\dot{u}_1) + \Omega_3 \omega_1 - \frac{3}{2} \Omega_2 \sigma_{11} \]

(103)

\[ (0_{012}) \]

0 = \frac{1}{2} e_2(\dot{u}_1) - \frac{1}{2} (a_2 - n_{31}) \dot{u}_1 + \Omega_2 \omega_1 \]

(104)

\[ (0_{013}) \]

0 = \frac{1}{2} e_3(\dot{u}_1) - \frac{1}{2} (a_3 + n_{12}) \dot{u}_1 + \Omega_3 \omega_1 \]

(105)

\[ (0_{123}) \]

0 = (e_1 - \dot{u}_1 - 2a_1)(\omega_1) \]

(106)

\[ R_{0123} = \frac{3}{2} n_{11} \sigma_{11} + 2(\dot{u}_1 + a_1) \omega_1 = -H_{11} \]

(107)

\[ R_{0231} = -(e_1 - a_1 + n_{23}) \omega_1 - \frac{3}{2} n_{11} \sigma_{11} - \frac{3}{4} (n_{22} - n_{33}) \sigma_{11} = -H_{22} \]

(108)

\[ R_{0312} = -(e_1 - a_1 - n_{23}) \omega_1 - \frac{3}{2} n_{11} \sigma_{11} + \frac{3}{4} (n_{22} - n_{33}) \sigma_{11} = -H_{33} \]

(109)

\[ R_{0331} = e_1(\Theta_2) + \frac{3}{2} (a_1 - n_{23}) \sigma_{11} + \frac{3}{4} (n_{11} + n_{22} - n_{33}) \omega_1 = -H_{23} \]

(110)

\[ R_{0212} = -e_1(\Theta_2) - \frac{3}{2} (a_1 + n_{23}) \sigma_{11} + \frac{3}{4} (n_{22} - n_{33} - n_{11}) \omega_1 = -H_{23} \]

(111)

\[ R_{0112} = e_2(\Theta_1) - \frac{3}{2} (a_2 - n_{31}) \sigma_{11} + (a_3 + n_{12}) \omega_1 = -H_{31} \]

(112)

\[ R_{0323} = -e_2(\Theta_2) + e_3(\omega_1) = -H_{31} \]

(113)

\[ R_{0131} = -e_3(\Theta_1) + \frac{3}{2} (a_3 + n_{12}) \sigma_{11} + (a_2 - n_{31}) \omega_1 = -H_{12} \]

(114)

\[ R_{0223} = e_3(\Theta_2) + e_2(\omega_1) = -H_{12} \]

(115)
\[
\begin{align*}
R_{3112} &= - \Theta_1 \omega_1 \\
R_{2312} &= - E_{31} \\
R_{3331} &= - E_{12} \\
R_{3323} &= - E_{11} + \frac{1}{3} \mu \\
R_{3113} &= - E_{22} + \frac{1}{3} \mu \\
R_{1212} &= - E_{33} + \frac{1}{3} \mu .
\end{align*}
\]

**A.4.3 Generalised Friedmann equation**

This is obtained by writing out \(R^{\alpha\beta}_{\alpha\beta}\) (which corresponds to summing (122), (123) and (124)).

\[
\begin{align*}
4 e_1(a_1) + 4 e_2(a_2) + 4 e_3(a_3) - 6 a_1^2 - 6 a_2^2 - 6 a_3^2 \\
- 2 n_{22}^2 - 2 n_{11}^2 - 2 n_{12}^2 - \frac{1}{2} n_{11}^2 - \frac{1}{2} n_{22}^2 - 3 n_{23}^2 + n_{11} (n_{22} - n_{33}) \\
+ \frac{2}{3} \theta^2 - \frac{2}{3} \sigma_{11}^2 + 2 \omega_1^2 - 4 \Omega_1 \omega_1 = 2 \mu.
\end{align*}
\]
\begin{align}
- (\mu + p)\omega_1 + (n_{22} - n_{33})H_{23} - (H_{22} - H_{33})n_{23} - 3\omega_1 E_{11} \\
(e_2 - 3a_2)(H_{22}) + (e_3 - 3a_3 + n_{12})(H_{23}) + (e_1 - 3a_1 - n_{23})(H_{12}) = \\
(n_{33} - n_{11})H_{31} - (H_{33} - H_{11})n_{31} - 3\omega_1 E_{12} + \frac{3\sigma_{11}}{2} E_{31} \\
(e_3 - 3a_3)(H_{33}) + (e_1 - 3a_1 + n_{23})(H_{31}) + (e_2 - 3a_2 - n_{31})(H_{23}) = \\
(n_{11} - n_{22})H_{12} - (H_{11} - H_{22})n_{12} - 3\omega_1 E_{31} - \frac{3\sigma_{11}}{2} E_{12}.
\end{align}

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