COSMIC CENSORSHIP OF SMOOTH STRUCTURES

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Abstract. It is observed that on many 4-manifolds there is a unique smooth structure underlying a globally hyperbolic Lorentz metric. For instance, every contractible smooth 4-manifold admitting a globally hyperbolic Lorentz metric is diffeomorphic to the standard \( \mathbb{R}^4 \). Similarly, a smooth 4-manifold homeomorphic to the product of a closed oriented 3-manifold \( N \) and \( \mathbb{R} \) and admitting a globally hyperbolic Lorentz metric is in fact diffeomorphic to \( N \times \mathbb{R} \). Thus one may speak of a censorship imposed by the global hyperbolicity assumption on the possible smooth structures on \((3 + 1)\)-dimensional spacetimes.

Introduction. One form of the strong cosmic censorship hypothesis proposed by Penrose asserts that ‘physically relevant’ spacetimes should be globally hyperbolic (see [14]). The purpose of this note is to point out that global hyperbolicity imposes strong restrictions on the differential topology of the spacetime. The starting point of all our considerations will be the smooth splitting theorem for globally hyperbolic spacetimes established by Bernal and Sánchez [2, 3]. All manifolds will be assumed Hausdorff and paracompact, since Hausdorff spacetimes are necessarily paracompact by [6, pp. 1743–1744].

The first result is valid in all dimensions but seems to be particularly interesting for \((3 + 1)\)-dimensional spacetimes. In that case, the argument makes essential use of the three-dimensional Poincaré conjecture proved by Perelman [17, 18, 19].

Theorem A. Let \( (X, g) \) be a globally hyperbolic \((n + 1)\)-dimensional spacetime. Suppose that \( X \) is contractible. Then \( X \) is diffeomorphic to the standard \( \mathbb{R}^{n+1} \).

For every \( n \geq 3 \), there exist uncountably many contractible smooth \( n \)-manifolds that are not homeomorphic to \( \mathbb{R}^n \) (see [12], [5] and [3]). In dimension four, in addition to that there are uncountably many smooth four-manifolds that are homeomorphic but not diffeomorphic to \( \mathbb{R}^4 \) (the so-called exotic \( \mathbb{R}^4 \)'s, see [3] and [22]). The theorem shows that none of those carry globally hyperbolic Lorentz metrics.

The topological argument used to prove Theorem A in the \((3 + 1)\)-dimensional case was first applied in the context of Lorentz geometry by Newman and Clarke [15]. They showed that a globally hyperbolic spacetime which is diffeomorphic to \( \mathbb{R}^4 \) can have any contractible 3-manifold as its Cauchy surface, see Remark 2.3.

Global hyperbolicity singles out ‘standard’ smooth structures on another large class of \((3 + 1)\)-dimensional spacetimes as well. The following result is based on Perelman’s geometrization theorem for 3-manifolds and the work of Turaev [24].

Theorem B. Let \( (X, g) \) be a globally hyperbolic \((3 + 1)\)-dimensional spacetime. Suppose that \( X \) is homeomorphic to the product of a closed oriented 3-manifold \( N \) and \( \mathbb{R} \). Then \( X \) is diffeomorphic to \( N \times \mathbb{R} \), where \( N \) and \( \mathbb{R} \) have their unique smooth structures.

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In fact, we do not know an example of a topological 4-manifold admitting two non-diffeomorphic smooth structures each of which is the smooth structure of a globally hyperbolic spacetime. However, such manifolds exist in higher dimensions (for instance, $S^7 \times \mathbb{R}$). To show that 4-dimensional examples do not exist, one would need to prove Theorem 3 for a 3-manifold $N$ that may be non-compact or non-orientable.

1. **Globally hyperbolic spacetimes.** A *spacetime* is a time-oriented connected Lorentz manifold $(X, g)$. The Lorentz metric $g$ and the time-orientation define a distribution of future hemicones in $TX$. A piecewise-smooth curve in $X$ is called *future-pointing* if its tangent vectors lie in the future hemicones. For two points $x, y \in X$, we write $x \leq y$ if either $x = y$ or there exists a future-pointing curve connecting $x$ to $y$. A spacetime is called *causal* if $\leq$ defines a partial order on it, that is, if there are no closed non-trivial future-pointing curves.

A spacetime $(X, g)$ is globally hyperbolic if it is causal and the ‘causal segments’ $I_{x,y} = \{ z \in X \mid x \leq z \leq y \}$ are compact for all $x, y \in X$. (This definition is equivalent to the classical one [10, §6.6] by [4, Theorem 3.2].)

A *Cauchy surface* in a spacetime is a subset such that every endless future-pointing curve meets it exactly once. It is a classical fact [10, pp. 211–212] that a spacetime is globally hyperbolic if and only if it contains a Cauchy surface. It has long been conjectured (and sometimes tacitly assumed) that Cauchy surfaces can be chosen to be smooth and spacelike and that a globally hyperbolic spacetime must be diffeomorphic to the product of its Cauchy surface with $\mathbb{R}$; this was finally proved by Bernal and Sánchez in 2003.

**Theorem 1.1** (Bernal–Sánchez [2, 3]). For a globally hyperbolic $(n+1)$-dimensional spacetime $(X, g)$, there exist an $n$-dimensional smooth manifold $M$ and a diffeomorphism $h : M \times \mathbb{R} \to X$ such that

a) $h(M \times \{t\})$ is a smooth spacelike Cauchy surface for all $t \in \mathbb{R}$;

b) $h(\{x\} \times \mathbb{R})$ is a future-pointing timelike curve for all $x \in M$.

Note that it follows by projecting along the timelike $t$-direction that *all* smooth spacelike Cauchy surfaces in $(X, g)$ are diffeomorphic to the same manifold $M$.

2. **Proof of Theorem A.** Suppose that $(X, g)$ is globally hyperbolic and $X$ is contractible. By Theorem 1.1, we know that $X$ is diffeomorphic to the product $M \times \mathbb{R}$ for a smooth $n$-manifold $M$. Since $X$ is contractible, it follows that $M$ is also contractible (as it is homotopy equivalent to $X$). Thus, it remains to invoke the following result.

**Proposition 2.1** (McMillan [11, 13], Stallings [21]). Suppose that $M$ is a contractible smooth $n$-manifold. Then $M \times \mathbb{R}$ is diffeomorphic to $\mathbb{R}^{n+1}$.

**Proof.** The proof splits into three cases according to the dimension of $M$.

1. $\dim M \leq 2$.

The result is obvious because the only contractible manifolds of dimension $\leq 2$ are $\mathbb{R}$ and $\mathbb{R}^2$.

2. $\dim M = 3$ (cf. [13, p. 55]).

We outline McMillan’s argument [11, 13] trying to give precise references for each step. For an introduction to the relevant topological methods, the reader may
consult the book by Rourke and Sanderson [20]. McMillan [11, Theorem 1] proved that if the three-dimensional Poincaré conjecture holds true, then $M$ can be exhausted by compact subsets PL-homeomorphic to handlebodies with handles of index one. It follows by an engulfing argument [11, Proof of Theorem 2, p. 513] that $M \times \mathbb{R}$ is the union of compact subsets $B_n \subset M \times \mathbb{R}$ such that $B_n \subset \text{Int} B_{n+1}$ and each $B_n$ is PL-homeomorphic to the 4-ball. McMillan and Zeeman observed [13, Lemma 4] that this implies that $M \times \mathbb{R}$ is PL-homeomorphic to $\mathbb{R}^4$. However, if a smooth manifold is PL-homeomorphic to $\mathbb{R}^n$, then it is diffeomorphic to $\mathbb{R}^n$ by a result of Munkres [14, Corollary 6.6]. Since the Poincaré conjecture is now known to be true because of Perelman’s work [17, 18, 19], the result follows.

3. $\dim M \geq 4$.

This is a special case of a result of Stallings [21, Corollary 5.3].

**Remark 2.2** (The rôle of the Poincaré conjecture). The three-dimensional Poincaré conjecture enters the preceding argument in the case $n = 3$ through the proof of [11, Theorem 1]. It is used there in the form of the following statement: A null-homotopic embedded 2-sphere in a three-manifold bounds a 3-ball. The assertion that such a sphere bounds a homotopy ball is classical and ‘elementary’ (see e.g. [1, Proposition 3.10]); the Poincaré conjecture ensures that the only homotopy 3-ball is the usual one.

**Remark 2.3** (Standard spacetimes vs non-standard Cauchy surfaces). Following Newman and Clarke [15], let us show that although the underlying manifolds of contractible globally hyperbolic spacetimes are standard, their Cauchy surfaces can be completely arbitrary: For every contractible smooth $n$-manifold $M$, there exists a globally hyperbolic spacetime of the form $(\mathbb{R}^{n+1}, g)$ with Cauchy surface diffeomorphic to $M$. Indeed, take any complete Riemann metric $\bar{g}$ on $M$, then $(M \times \mathbb{R}, \bar{g} \oplus -dt^2)$ is a globally hyperbolic spacetime. By Proposition 2.1 the manifold $M \times \mathbb{R}$ is diffeomorphic to $\mathbb{R}^{n+1}$.

3. **Proof of Theorem** [3]. The manifold $X$ is diffeomorphic to $M \times \mathbb{R}$ for some 3-manifold $M$ by Theorem [17, 18]. We shall prove that $M$ is in fact homeomorphic to $N$. Since homeomorphic 3-manifolds are diffeomorphic [14, Theorem 3.6], it will follow that the smooth 4-manifolds $X \cong M \times \mathbb{R}$ and $N \times \mathbb{R}$ are diffeomorphic.

Note first that

$$H_3(M, \mathbb{Z}) = H_3(M \times \mathbb{R}, \mathbb{Z}) = H_3(X, \mathbb{Z}) = H_3(N \times \mathbb{R}, \mathbb{Z}) = H_3(N, \mathbb{Z}) = \mathbb{Z}$$

and hence $M$ is closed and orientable.

Turaev [24, Theorem 1.4, p. 293] proved that two orientable closed geometric 3-manifolds are homeomorphic if and only if they are topologically $h$-cobordant. In [24, p. 291] geometric 3-manifolds were defined as connected sums of Seifert fibred, hyperbolic, and Haken manifolds. It is now known by the work of Perelman [17, 18] that a non-Haken (and hence atoroidal) irreducible orientable closed 3-manifold is either Seifert fibred (which includes all spherical 3-manifolds [23, p. 248]) or hyperbolic, see e.g. [1, Theorem 1.1.6]. Thus, all closed orientable 3-manifolds are geometric in the sense of [24].

It remains to construct a topological $h$-cobordism between $N$ and $M$. Let $\psi : M \times \mathbb{R} \to N \times \mathbb{R}$ be a homeomorphism. Since $\psi(M \times \{0\})$ is compact, it is contained in $N \times (-\infty, T)$ for some $T \gg 0$. Reversing the $\mathbb{R}$-factor in $M \times \mathbb{R}$ if necessary, we
may assume that $N \times \{T\} \subset \psi(M \times (0, +\infty))$. Set
\[ W = N \times (-\infty, T] \cap \psi(M \times [0, +\infty)) \subset N \times \mathbb{R}. \]
This is a compact topological manifold with boundary that defines a topological cobordism between $M \xrightarrow{\psi} \psi(M \times \{0\})$ and $N = N \times \{T\}$. By the definition of an $h$-cobordism, we have to check now that the inclusions of the boundary components into $W$ are homotopy equivalences.

Let $r_M : M \times \mathbb{R} \to M \times [0, +\infty)$ and $r_N : N \times \mathbb{R} \to N \times (-\infty, T]$ be the obvious strong deformation retractions. Then
\[ r_N \circ \psi \circ r_M \circ \psi^{-1} : N \times \mathbb{R} \to W \]
is a strong deformation retraction. Hence, the inclusion $W \hookrightarrow N \times \mathbb{R}$ is a homotopy equivalence. Since the inclusions $N \times \{T\} \hookrightarrow N \times \mathbb{R}$ and $\psi(M \times \{0\}) \hookrightarrow N \times \mathbb{R}$ are also homotopy equivalences, it follows that $W$ is a topological $h$-cobordism indeed, which completes the proof of Theorem B.

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