GENERALIZED MULTI-HUMP WAVE SOLUTIONS OF KDV-KDV SYSTEM OF BOUSSINESQ EQUATIONS

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(Communicated by Jerry Bona)

Abstract. The KdV-KdV system of Boussinesq equations belongs to the class of Boussinesq equations modeling two-way propagation of small-amplitude long waves on the surface of an ideal fluid. It has been numerically shown that this system possesses solutions with two humps which tend to a periodic solution with much smaller amplitude at infinity (called generalized two-hump wave solutions). This paper presents the first rigorous proof. The traveling form of this system can be formulated into a dynamical system with dimension 4. The classical dynamical system approach provides the existence of a solution with an exponentially decaying part and an oscillatory part (small-amplitude periodic solution) at positive infinity, which has a single hump at the origin and is reversible near negative infinity if some free constants, such as the amplitude and the phase shift of the periodic solution, are activated. This eventually yields a generalized two-hump wave solution. The method here can be applied to obtain generalized 2k-hump wave solutions for any positive integer k.

1. Introduction. We consider in this paper the following coupled system of two nonlinear dispersive wave equations in one space dimension

\[ \eta_t + u_\xi + (\eta u)_\xi + \frac{1}{6} u_{\xi\xi\xi} = 0, \]
\[ u_t + \eta_\xi + uu_\xi + \frac{1}{6} \eta_{\xi\xi\xi} = 0, \]

which is called the KdV-KdV system in [3, 5] because the dispersive terms are third-order spatial derivatives. This system is originally derived in [3] and later [1, 21] from Boussinesq systems

\[ \eta_t + u_\xi + (\eta u)_\xi + au_{\xi\xi\xi} - b\eta_{\xi\xi\xi} = 0, \]
\[ u_t + \eta_\xi + uu_\xi + cu_\xi\xi\xi - du_{\xi\xi\xi} = 0 \]

for appropriate constants a, b, c and d. The system (1) has been proved to be a valid approximation to the full two-dimensional Euler equations for small-amplitude long
waves with two-way propagation at the surface of an ideal fluid in a long rectangular channel with a flat bottom under the force of gravity (for example, see [3]). The variables here are dimensionless and scaled. The independent variables $\xi$ and $t$ represent the position and the elapsed time along the channel respectively whilst $\eta(\xi, t)$ is the deviation of the free surface from its rest position and $u(\xi, t)$ is the horizontal velocity.

Since the full Euler equations are much more complex in both laboratory studies and applications, the system (1) has been extensively studied by numerical and analytic techniques in many literatures to explore the properties of the full Euler equations, and then a lot of interesting results have been obtained such as well-posedness problems [3, 4, 7, 16, 19, 22, 23], Hamiltonian structures [4], infinitely many nonlocal conserved quantities [20], the blowup phenomenon [15, 24], the energy preserving integration [12], PT-invariant solutions [13], periodic solutions with variable coefficients [26], exact solitary wave solutions [8], collision experiments of solitary wave solutions [2, 4, 21] and numerical existence of radiating solitary wave solutions [6]. Most of properties of (1) can also be found in the review [11].

Here, we focus on the traveling wave solutions of the system (1) with the form

$$\eta(t, \xi) = \eta(x), \quad u(t, \xi) = u(x)$$

for $x = \xi - ct$ where $c$ is the speed. Note that if $(\eta(x), u(x))^T$ is a solution of (1) for some $c > 0$, then $(\eta(x), -u(x))^T$ is also a solution propagating with speed $-c$, i.e. to the left, as $t$ increases. Thus, we assume $c > 0$. It is easy to check that $c = 1$ is a critical point near which the rear parts of some eigenvalues change from zero to nonzero (see (8)). Hence we consider $c$ as a parameter $c = 1 + \mu$ for small $\mu$.

Let $a_1 = \eta, a_2 = \dot{\eta}, a_3 = u$ and $a_4 = \dot{u}$ where the dot stands for the derivative with respect to $x$. Substituting (2) into (1) one obtains, after integrating once and taking the integration constants equal to zero, the system of nonlinear ordinary differential equations

$$\begin{align*}
\dot{a}_1 &= a_2, \\
\dot{a}_2 &= -6a_1 + 6(1 + \mu)a_3 - 3a_3^2, \\
\dot{a}_3 &= a_4, \\
\dot{a}_4 &= 6(1 + \mu)a_1 - 6a_3 - 6a_1a_3.
\end{align*}$$

Now, the solitary wave solutions of (1) correspond to the homoclinic solutions of (3). It is clear that the origin of (3) is a saddle-center equilibrium for $\mu > 0$, that is, the linear operator around the origin has a positive eigenvalue, a negative eigenvalue and a pair of purely imaginary eigenvalues (see (8) or Figure 3). Since the stable and unstable manifolds are both one dimensional, it is very hard to prove that the stable and unstable manifolds will intersect to form the classical homoclinic solutions (see (1) of Figure 1). [5] numerically and theoretically verified that the system (3) actually possesses a generalized single-hump homoclinic solution by the standard Galerkin-finite element method and the theory of Lombardi [17] respectively, i.e., homoclinic solution has a main hump which does not tend to zero but to a small-amplitude periodic solution at infinity (see (2) of Figure 1). Moreover, [5] numerically obtained a generalized two-hump homoclinic solution for $\mu = 0.2$ (see Figure 6 in [5] or Figure 2 here). The heights of the two humps are same and are about 0.44, and the distance of two humps is around 30. This implies that two humps are far apart. Does the system (3) really have a generalized two-hump homoclinic solution with same heights? It is still an interesting open problem.
Intuitively, if two generalized single-hump homoclinic solutions are far away, they will not interact with each other so that we can paste these two solutions together and obtain a new solution with two humps. In this paper we will give a first rigorous proof. The basic idea is based on [5, 10, 17]. Note that the system (3) has a generalized single-hump homoclinic solution [5], denoted by \((\eta_s, \dot{\eta_s}, u_s, \dot{u}_s)^T(x)\). We just look at the \(\eta\)-component \(\eta_s(x)\) (the similar idea can be applied to \(u_s(x)\)), which is even in \(x\) and exponentially tends to a periodic solution with small amplitude \(I\) as \(x \to \pm \infty\). The hump of \(\eta_s(x)\) is at the origin. In order to get two-hump solutions, we have to break the evenness of the solution \(\eta_s(x)\). We choose different amplitudes for the periodic solution in \(\eta_s(x)\), that is, the amplitudes are taken as \(I^+\) for \(x > 0\) and \(I^-\) for \(x < 0\) respectively. The difference of \(I^+ - I^-\) is nonzero and regarded as a new small parameter. Since the amplitudes are different, the modified solution, say \(\eta^*_s(x)\), exists only on \([-\tau, +\infty)\), not on \((-\infty, +\infty)\) where \(\tau > 0\) is large and can be chosen as we need. We might find a point \(x = -x_0\) for large \(x_0 \in (0, \tau]\) where the derivative \((\eta^*_s)'(-x_0) = 0\). The reversibility of the system implies that \(\eta^*_s(x)\) can be smoothly extended on \((-\infty, +\infty)\) so that it has two humps at \(x = 0\) and \(x = -2x_0\). Therefore, the two-hump solutions are obtained and then the existence of the generalized two-hump wave solution with same heights observed in [5] is proved rigorously. The similar idea may be used to construct \(2^k\)-hump solutions for any positive integer \(k\). We would like to mention that our constructive method is different from other methods such as in [9, 18, 25] since we can provide more details of the obtained solutions such as the heights and the distance of two humps, and the small amplitude of the periodic solution, which play important roles in the applications. The main result can be summarized as follows.

\[ \text{Theorem 1.1. Let a positive integer } n \text{ and two constants } I_0 > 0 \text{ and } \frac{3}{4} \leq \tilde{I} \leq 2n + \frac{1}{2} \text{ be given. Then for each } 0 < \mu < \mu_0 \text{ and } I^+ = I_0\mu^{n+1/2} \text{ with some small } \mu_0 > 0, \]

![Figure 1](image1.png)  \(1\) Homoclinic solutions.  \(2\) Generalized homoclinic solutions.

![Figure 2](image2.png)  Generalized two-hump homoclinic solutions.
there exist two real numbers $a_0$ and $x_0 > 0$ satisfying
\[
\left| a_0 - \frac{9216}{c_1 I_0} \right| \leq M\mu^{1/4}, \quad \mu^{2n+2+i}e^{2\sqrt{n\mu} x_0} = c_1
\]
with $c_1$ near 1 only dependent on $\mu$, and a traveling solution $(\eta(x; x_0), u(x; x_0))^T$ of (1) with $x = \xi - (1 + \mu)t$, where
\[
\begin{align*}
\eta(x; x_0) &= \begin{cases}
2\mu \text{sech}^2 \left( \frac{\sqrt{\mu}}{2}(x - x_0) \right) + \eta_\epsilon(x - x_0) \eta_0^+ (x - x_0 - \theta^+(I^+); I^+) + \eta_\epsilon(x - x_0) & \text{for } x > x_0, \\
2\mu \text{sech}^2 \left( \frac{\sqrt{\mu}}{2}(x - x_0) \right) + \eta_\epsilon(x - x_0) \eta_0^+ (x - x_0 + \theta^+(I^+); I^-) + \eta_\epsilon(x - x_0) & \text{for } 0 \leq x \leq x_0,
\end{cases}
\end{align*}
\]
\[
\begin{align*}
u(x; x_0) &= \begin{cases}
2\mu \text{sech}^2 \left( \frac{\sqrt{\mu}}{2}(x - x_0) \right) + u_\epsilon(x - x_0) \eta_0^+ (x - x_0 - \theta^+(I^+); I^+) + u_\epsilon(x - x_0) & \text{for } x > x_0, \\
2\mu \text{sech}^2 \left( \frac{\sqrt{\mu}}{2}(x - x_0) \right) + u_\epsilon(x - x_0) \eta_0^+ (x - x_0 + \theta^+(I^+); I^-) + u_\epsilon(x - x_0) & \text{for } 0 \leq x \leq x_0,
\end{cases}
\end{align*}
\]
$\eta_\epsilon(x - x_0)$, $u_\epsilon(x - x_0)$, and satisfy the estimates
\[
\begin{align*}
|\eta_\epsilon(x; I^+, \theta^+(I^+))| + |u_\epsilon(x; I^+, \theta^+(I^+))| & \leq M\mu^2 e^{-\left(\frac{1}{\sqrt{4\mu}} - 1\right)|x|}, \\
|\eta_\epsilon(x; I)| - \frac{2}{\lambda_0} \mu I \cos \left((\lambda_0 + r_1)x\right) & \leq M\mu I^2, \\
|u_\epsilon(x; I)| + \frac{2}{\lambda_0} \mu I \cos \left((\lambda_0 + r_1)x\right) & \leq M\mu I^2, \quad |r_1| \leq M\mu I^2,
\end{align*}
\]
with $\lambda_0 = \sqrt{12 + 6\mu}$. Here, the remainders $\eta_\epsilon(x - x_0)$ and $u_\epsilon(x - x_0)$ are smooth functions decaying exponentially at $+\infty$ and satisfy the estimates
\[
|\eta_\epsilon(x - x_0)| + |u_\epsilon(x - x_0)| \leq M\mu^{n+7/4} \quad \text{for } x \in [-x_0, x_0 - 2],
\]
And, $M > 0$ is a constant independent of $\mu$ and $\sigma(x)$ is a smooth even cutoff function with $\sigma(x) = 0$ for $|x| \leq 1$ and $\sigma(x) = 1$ for $|x| \geq 2$.

**Remark 1.** (1) The deviation $\eta(x; x_0)$ of the free surface obtained in Theorem 1.1 is even in $x$ and has two humps with same height $2\mu$ at $\pm x_0$ respectively (see Figure 2), which has small oscillations of amplitude at infinity. The amplitude of the oscillations at $x$ near zero is of order $O(\mu I^+) = O(\mu^{n+3/2})$. The distance between two humps is $2x_0 = O(\mu^{-1/2} \ln \mu)$, This is exactly the one observed in [5].

(2) If $I^- = I^+$, it means that the difference $I^+ - I^-$ is zero and then $x_0$ is infinity so that this solution becomes a generalized solitary wave solution.
The methods here can be applied to other types of Boussinesq equations and other systems, such as Bona-Smith system which has a generalized two-hump wave solutions with same heights numerically given in [2].

This paper is organized as follows. Section 2 reformulates the system (3) into a new dynamical system with the eigenvectors of the linear operator in (3) as a basis for the space $\mathbb{R}^4$. After doing the appropriate scaling and the change of variables, the dominant system is presented and its homoclinic solution $H(x)$ is obtained. In Section 3, the periodic solution $X_p(x)$ of the whole system is given with the Fourier series technique and the amplitude $I > 0$ is chosen as a parameter. In order for two-hump solutions, we intentionally choose $I^+$ for $x > 0$ and $I^-$ for $x < 0$ respectively to break the reversibility. In particular, the difference $I^+ - I^-$ is considered as an additional free constant. Section 4 writes the whole solution as the summation of $H(x)$, $X_p(x)$ and the unknown perturbation term $Z(x)$, and transforms the problem of the existence of generalized two-hump homoclinic solutions into the one for an integral equation with respect to $Z(x)$ such that the fixed point theorem can be applied. The appropriate Banach spaces are constructed in Section 5 and $Z(x)$ is decomposed into two parts: the reversible part $Z_r(x)$ and the remainder part $Z_e(x)$. Section 6 yields the estimates and the existence proof of $Z_r(x)$ on $[0, +\infty)$ and $Z_e(x)$ on $[-2, +\infty)$ by applying the fixed point theorem, while the existence of $Z(x)$ on $[-\tau, +\infty)$ is given in Section 7 for some large $\tau > 0$. In Section 8, the obtained solution $Z(x)$ on $[-\tau, +\infty)$ is smoothly extended on $(-\infty, +\infty)$ if some free constants such as $I^+ - I^-$ are adjusted, which gives the generalized two-hump homoclinic solution of (3). This finishes the proof of Theorem 1.1. Appendices present the proofs of some lemmas left in the previous sections.

Throughout this paper, $M$ denotes a generic positive constant and $B = O(C)$ means that $|B| \leq M|C|$.

2. Formulations. Symbolically, the system (3) can be written as

$$\dot{U} = LU + N(U),$$

where $U = (a_1, a_2, a_3, a_4)^T$,

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -6 & 0 & 6(1 + \mu) & 0 \\ 0 & 0 & 0 & 1 \\ 6(1 + \mu) & 0 & -6 & 0 \end{pmatrix}, \quad N(U) = \begin{pmatrix} 0 \\ -3a_3^2 \\ 0 \\ -6a_1a_3 \end{pmatrix}.$$

The system (6) is reversible with a reverser $S$ defined by

$$S(a_1, a_2, a_3, a_4) = (a_1, -a_2, a_3, -a_4),$$

that is, $SU(-x)$ is also a solution whenever $U(x)$ is. A solution $U(x)$ is reversible if $SU(-x) = U(x)$. This implies that $a_1(x)$ and $a_3(x)$ are even functions, and $a_2(x)$ and $a_4(x)$ are odd functions.

It is easily seen that the eigenvalues of $L$ are

$$\lambda_1 = \sqrt{6\mu}, \quad \lambda_2 = -\sqrt{6\mu}, \quad \lambda_3 = i\lambda_0, \quad \lambda_4 = -i\lambda_0,$$

where $\lambda_0 = \sqrt{12 + 6\mu}$. See Figure 3. In what follows, we assume that $\mu > 0$ (in this case, the origin is a saddle-center equilibrium). Then the corresponding eigenvectors
are
\[ U_1 = \begin{pmatrix} \frac{1}{\lambda_1} \\ 1 \\ \frac{1}{\lambda_1} \\ 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} -\frac{1}{\lambda_1} \\ 1 \\ -\frac{1}{\lambda_1} \\ 1 \end{pmatrix}, \quad U_3 = \begin{pmatrix} -\frac{i}{\lambda_0} \\ 1 \\ -\frac{i}{\lambda_0} \\ 1 \end{pmatrix}, \quad U_4 = \begin{pmatrix} -\frac{i}{\lambda_0} \\ -1 \\ -\frac{i}{\lambda_0} \\ 1 \end{pmatrix}, \]
which satisfy
\[ SU_1 = -U_2, \quad SU_2 = -U_1, \quad SU_3 = -U_4, \quad SU_4 = -U_3. \tag{9} \]

Note that the solutions of the real system (6) can be expressed in terms of the above eigenvectors, that is,
\[ U = \tilde{A}U_1 + \tilde{B}U_2 + (\tilde{v}_1 - i\tilde{v}_2)U_3 + (\tilde{v}_1 + i\tilde{v}_2)U_4, \tag{10} \]
where \( \tilde{A}, \tilde{B}, \tilde{v}_1 \) and \( \tilde{v}_2 \) are real functions of \( x \). Thus, the system (6) is equivalent to the following real system
\[
\begin{align*}
\dot{\tilde{A}} &= \sqrt{6\mu} \tilde{A} + \frac{1}{8\lambda_0^2 \mu} \left( 4\sqrt{6\mu} \lambda_0 (\tilde{A} - \tilde{B}) \tilde{v}_2 + 24\mu \tilde{v}_2^2 - 3\lambda_0^2 (\tilde{A} - \tilde{B})^2 \right), \\
\dot{\tilde{B}} &= -\sqrt{6\mu} \tilde{B} + \frac{1}{8\lambda_0^2 \mu} \left( 4\sqrt{6\mu} \lambda_0 (\tilde{A} - \tilde{B}) \tilde{v}_2 + 24\mu \tilde{v}_2^2 - 3\lambda_0^2 (\tilde{A} - \tilde{B})^2 \right), \\
\dot{\tilde{v}}_1 &= \lambda_0 \tilde{v}_2 + \frac{1}{8\lambda_0^2 \mu} \left( -4\sqrt{6\mu} \lambda_0 (\tilde{A} - \tilde{B}) \tilde{v}_2 - \lambda_0^2 (\tilde{A} - \tilde{B})^2 + 72\mu \tilde{v}_2^2 \right), \\
\dot{\tilde{v}}_2 &= -\lambda_0 \tilde{v}_1. 
\end{align*} \tag{11} \]
Furthermore, we let \( \hat{A} = \frac{\tilde{A} + i\tilde{B}}{2} \) and \( \hat{B} = \frac{\tilde{A} - i\tilde{B}}{2} \). Then (11) is changed into
\[
\begin{align*}
\dot{\hat{A}} &= \sqrt{6\mu} \hat{A} + \frac{1}{4\lambda_0^2 \mu} \left( -3\lambda_0^2 \hat{A}^2 + 24\mu \tilde{v}_2^2 + 4\sqrt{6\mu} \lambda_0 \hat{A} \tilde{v}_2 \right), \\
\dot{\hat{B}} &= \sqrt{6\mu} \hat{B} + \frac{1}{4\lambda_0^2 \mu} \left( -3\lambda_0^2 \hat{B}^2 + 24\mu \tilde{v}_2^2 + 4\sqrt{6\mu} \lambda_0 \hat{B} \tilde{v}_2 \right), \\
\dot{\tilde{v}}_1 &= \lambda_0 \tilde{v}_2 + \frac{9}{\lambda_0^2 \mu} \tilde{v}_2^2 - \frac{\sqrt{3}}{2\mu \lambda_0} \hat{A} \tilde{v}_2 - \frac{1}{8\mu} \hat{A}^2, \\
\dot{\tilde{v}}_2 &= -\lambda_0 \tilde{v}_1. 
\end{align*} \tag{12} \]
Do the scaling and suppose that
\[ \tilde{A} = \mu^{3/2} \hat{A}, \quad \tilde{B} = \mu^{3/2} \hat{B}, \quad \tilde{v}_1 = \mu \tilde{v}_2, \quad \tilde{v}_2 = \mu \tilde{v}_1, \]
which gives from (12)

\[ \begin{align*}
A &= \sqrt{6\mu} B, \\
\dot{B} &= \sqrt{6\mu A} - \frac{3\sqrt{\mu}}{4} A^2 + \frac{\sqrt{\mu}}{\lambda_0} w_1 (6w_1 + \sqrt{6}\lambda_0 A), \\
\dot{w}_1 &= -\lambda_0 w_2, \\
\dot{w}_2 &= \lambda_0 w_1 - \frac{\mu}{8} A^2 + \frac{\mu}{2\lambda_0} w_1 (18w_1 - \sqrt{6}\lambda_0 A),
\end{align*} \]

or, equivalently

\[ \dot{X} = F(\mu, X) + \mathcal{N}(\mu, X), \] (13)

where \( X = (A, B, w_1, w_2)^T \), and

\[ \begin{align*}
F(\mu, X) &= \begin{pmatrix}
\sqrt{6\mu} B \\
\sqrt{6\mu A} - \frac{3\sqrt{\mu}}{4} A^2 \\
-\lambda_0 w_2 \\
\lambda_0 w_1
\end{pmatrix}, \\
\mathcal{N}(\mu, X) &= \begin{pmatrix}
0 \\
\frac{\sqrt{\mu}}{\lambda_0} w_1 (6w_1 + \sqrt{6}\lambda_0 A) \\
0 \\
-\frac{\mu}{8} A^2 + \frac{\mu}{2\lambda_0} w_1 (18w_1 - \sqrt{6}\lambda_0 A)
\end{pmatrix}.
\] (14)

Clearly, the existence problem of generalized two-hump wave solutions of (1) is transformed into one of generalized two-hump homoclinic solutions of (14), and \( \eta(x), u(x) \) have the following forms

\[ \eta(x) = \frac{1}{\sqrt{6}} \mu A(x) + \frac{2}{\lambda_0} \mu w_1(x), \quad u(x) = \frac{1}{\sqrt{6}} \mu A(x) - \frac{2}{\lambda_0} \mu w_1(x). \] (15)

In what follows, we pay our attention on the system (13).

It is obvious that from (7) and (9) the reverser \( S \) is now given by

\[ S(A, B, w_1, w_2) = (A, -B, w_1, -w_2). \] (16)

The dominant system of (13)

\[ \dot{X} = F(\mu, X) \]

has a homoclinic solution

\[ H(x) = (H_A(x), H_B(x), 0, 0)^T, \] (17)

which satisfies

\[ SH(-x) = H(x), \quad |H(x)| \leq Me^{-\sqrt{6}|x|} \quad \text{for} \quad x \in \mathbb{R}, \] (18)

where

\[ \begin{align*}
H_A(x) &= 2\sqrt{6} \sech^2 \left( \frac{\sqrt{3}\mu}{\sqrt{2}} x \right), \\
H_B(x) &= -2\sqrt{6} \sech^2 \left( \frac{\sqrt{3}\mu}{\sqrt{2}} x \right) \tanh \left( \frac{\sqrt{3}\mu}{\sqrt{2}} x \right).
\end{align*} \] (19)

In the rest of this paper, we will use this homoclinic solution \( H(x) \) to construct a generalized two-hump homoclinic solution.
3. Periodic solutions. Since the generalized two-hump homoclinic solution to be proved exponentially approaches a periodic solution at infinity, this section applies the Fourier series to look for the periodic solutions of the system (13) with period $2\pi/(\lambda_0 + r_1)$.

Assume

$$\tilde{x} = (\lambda_0 + r_1)x,$$  \hspace{1cm} (20)

where $r_1$ is a small real constant to be determined later. The system (13) can be transformed into

$$A' = \frac{\sqrt{6}\mu}{\lambda_0 + r_1}B,$$

$$B' = \frac{1}{\lambda_0 + r_1} \left( \sqrt{6}\mu A - \frac{3\sqrt{\mu}}{4} A^2 + \frac{\sqrt{\mu}}{\lambda_0^2} w_1(6w_1 + \sqrt{6}\lambda_0 A) \right),$$

$$w'_1 = -\frac{\lambda_0}{\lambda_0 + r_1}w_2,$$

$$w'_2 = \frac{\lambda_0}{\lambda_0 + r_1}w_1 + \frac{1}{\lambda_0 + r_1} \left( -\frac{\mu}{8} A^2 + \frac{\mu}{2\lambda_0^2} w_1(18w_1 - \sqrt{6}\lambda_0 A) \right),$$  \hspace{1cm} (21)

where the prime stands for the derivative with respect to $\tilde{x}$.

Denote $H^m_p(0,2\pi)$ by the space of periodic functions of $\tilde{x}$ with period $2\pi$ such that their derivatives up to order $m$ are in $L^2(0,2\pi)$, whose norm is denoted by $\|\cdot\|_m$. The reversible periodic solutions of (21) in $(H^m_p(0,2\pi))^4$ can be expressed as

$$A(\tilde{x}) = \sum_{n=0}^{\infty} A_n \cos(n\tilde{x}), \quad B(\tilde{x}) = \sum_{n=1}^{\infty} B_n \sin(n\tilde{x}), \quad w_2(\tilde{x}) = \sum_{n=1}^{\infty} w_{2n} \sin(n\tilde{x}),$$

$$w_1(x) = I \cos(\tilde{x}) + \bar{w}_1(\tilde{x}) = I \cos \tilde{x} + \sum_{n=0, n \neq 1}^{\infty} w_{1n} \cos(n\tilde{x}),$$  \hspace{1cm} (22)

where $I > 0$ is a free real constant.

Plugging (22) into (21), setting each Fourier coefficient equal zero, and applying the fixed point theorem with suitable estimates, one can solve for $A(\tilde{x}), B(\tilde{x}), w_1(\tilde{x})$ and $w_2(\tilde{x})$ in $(H^m_p(0,2\pi))^4$ and the real constant $r_1$ as smooth functions of $(\mu, I)$ for small $(\mu, I)$ (more details can be seen in [14]), which can be written as

$$(A, B, w_1, w_2)(\tilde{x}; \mu, I) = (A, B, w_1, w_2)(\tilde{x}; \mu, I), \quad r_1 = r_1(\mu, I).$$

Moreover, using the fact that the right sides of (21) are polynomials of degree 2,

$$\|A\|_m + \|B\|_m \leq MI^2, \quad \|w_1\|_m + \|w_2\|_m \leq MI, \quad |r_1(\mu, I)| \leq M\mu I^2$$

for $I \in (0, I_1]$ and $\mu \in (0, \mu_1]$ where the positive constants $I_1$ and $\mu_1$ are small.

Due to (20), we denote this periodic solution by

$$X_p(x) = (A_p, B_p, w_{1p}, w_{2p})^T(x) = (A, B, w_1, w_2)^T((\lambda_0 + r_1)x; \mu, I),$$

which satisfies

$$\|X_p[j]\|_m \leq MI^2, \quad \|X_p[k]\|_m \leq MI$$  \hspace{1cm} (23)

for $j = 1, 2$ and $k = 3, 4$ and any positive integer $m$, where $X_p[j]$ denotes the $j$-th component of $X_p$. The Sobolev embedding theorem implies that (23) are also valid.
under the $C^m_B(\mathbb{R})$-norm, where $C^m_B(\mathbb{R})$ is the space of continuously differentiable functions up to order $m$ with a supremum norm.

A direct calculation shows that the dominant terms of the periodic solution $X_p(x)$ for small $(\mu, I)$ can be expressed as

$$A_p(x) = -\frac{\sqrt{6}}{2\lambda_0^3} I^2 - \frac{6\mu I^2}{4\lambda_0^3 + 6\lambda_0^2\mu} \cos (2(\lambda_0 + r_1)x) + O(\mu I^3),$$

$$B_p(x) = \frac{3\sqrt{\mu}}{2\lambda_0^3 + 3\lambda_0\mu} I^2 \sin (2(\lambda_0 + r_1)x) + O(\sqrt{\mu} I^3),$$

$$w_{1p}(x) = I \cos ((\lambda_0 + r_1)x) - \frac{9\mu I^2}{2\lambda_0^3} + \frac{3\mu I^2}{2\lambda_0^3} \cos (2(\lambda_0 + r_1)x) + O(\mu I^3),$$

$$w_{2p}(x) = I \sin ((\lambda_0 + r_1)x) + \frac{3\mu I^2}{\lambda_0} \sin (2(\lambda_0 + r_1)x) + O(\mu I^3),$$

where

$$r_1 = \frac{3\mu I^2}{4\lambda_0^3} - \frac{135\mu^2 I^2}{4\lambda_0^5} + \frac{9\mu^2 I^2}{8\lambda_0(2\lambda_0^4 + 3\lambda_0^2\mu)} + O(\mu I^3).$$

In the following sections, we will prove the existence of a generalized two-hump homoclinic solution exponentially approaching the periodic solution $X_p^+(x - \theta^+)$ with amplitude $I^+ > 0$ and phase shift $\theta^+$ as $x \to +\infty$ and another periodic solution $X_p^-(x - \theta^-)$ with amplitude $I^- > 0$ and phase shift $\theta^-$ on the left side of the homoclinic solution $H(x)$ in (17). The reason to choose different periodic solutions is to break the reversibility. Otherwise, we obtain only one-hump solution. The difference of $I^+ - I^-$ will be used as a new parameter (see (69) and Lemma 8.1).

It is also essential to introduce the phase shifts $\theta^\pm$ such that a matching condition (see (57)) between the decaying part and the oscillatory part at infinity holds. The purpose to take the phase shift $\theta^- = -\theta^+$ for $x < 0$ instead of $\theta^+$ for $x > 0$ is just to simplify the calculations later and only the amplitude of the periodic solution is changed. These two periodic solutions can be denoted by

$$X_p^\pm(x - \theta^\pm) = (A_p^\pm, B_p^\pm, w_{1p}^\pm, w_{2p}^\pm)^T(x - \theta^\pm),$$

where the plus sign is taken for $x > 0$ and the minus sign is used for $x < 0$, respectively.

Then, according to (24) and (25), we have the following lemma, which will be used later.

**Lemma 3.1.** For $x \geq 0$ and $\theta^- = -\theta^+$, if $\mu \in (0, \mu_1]$ and $I^+, I^- \in (0, I_1]$, we have

$$A_p^+(x - \theta^+) - A_p^-(x + \theta^+) = -\frac{\sqrt{6}}{\lambda_0} I^+(I^+ - I^-)$$

$$- \frac{3\sqrt{6}}{2\lambda_0^3 + 3\lambda_0\mu} \mu I^+(I^+ - I^-) \cos (2(\lambda_0 + r_1^+)(x - \theta^+))$$

$$+ O(\mu(I^+)^2|I^+ - I^-|) + O(|I^+ - I^-|^2),$$

$$B_p^+(x - \theta^+) + B_p^-(x + \theta^+)$$

$$= \frac{6}{2\lambda_0^3 + 3\lambda_0\mu} \sqrt{\mu} I^+(I^+ - I^-) \sin (2(\lambda_0 + r_1^+)(x - \theta^+))$$

$$+ O(\sqrt{\mu}(I^+)^2|I^+ - I^-|) + O(\sqrt{\mu}|I^+ - I^-|^2),$$

$$w_{1p}^+(x - \theta^+) - w_{1p}^-(x + \theta^+) = (I^+ - I^-) \cos ((\lambda_0 + r_1^+)(x - \theta^+))$$

$$+ O(\mu(I^+)^2|I^+ - I^-|) + O(\mu|I^+ - I^-|^2).$$
4. Integral equations. This section transforms the existence problem of two-
hump solutions for (13) into the one for an integral equation so that the fixed
point theorem can be applied.

Define smooth even cutoff functions $\sigma(x)$ and $\tilde{\sigma}(x)$ as follows

$$
\sigma(x) = \begin{cases}
1 & \text{for } |x| \geq 2, \\
0 & \text{for } |x| \leq 1,
\end{cases}
$$

and

$$
\tilde{\sigma}(x) = 1 - \sigma(x).
$$

Since the dominant term $H(x)$ in (17) and the small-amplitude periodic solutions
$
X_p^\pm(x - \theta^\pm)
\$ are obtained, the solution of (13) near $H(x)$ can be assumed to have a form

$$
X(x; \mu, \theta^\pm, I^\pm) = H(x) + Z(x) + \sigma(x)X_p^\pm(x - \theta^\pm),
$$

(29)

where $Z(x) = (A, B, w_1, w_2)^T(x)$, (Here $(A, B, w_1, w_2)^T$ is now denoted for $Z$, not
for $X$ in the previous sections since no confusion appears), is a perturbation term to be determined and exponentially decays to 0 as $x \to +\infty$. Substituting (29) into

(13) yields the equation of $Z(x)$

$$
\dot{Z}(x) = \mathcal{L}(x)Z + \mathcal{F}(x, Z, X_p^\pm),
$$

(30)

where $\mathcal{L}(x) = dF[\mu, H]$, $d$ means taking the Fréchet derivative, and

$$
\mathcal{F}(x, Z, X_p^\pm) = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)^T(x),
$$

$$
\mathcal{F}_1(x) = -\sigma'(x)A_p^\pm(x - \theta^\pm),
$$

$$
\mathcal{F}_2(x) = h_2(x) + g_2(x),
$$

$$
h_2(x) = -\sigma'(x)B_p^\pm(x - \theta^\pm),
$$

$$
g_2(x) = -\frac{3}{4}\sqrt{\mu}\left(2\sigma HA_p^\pm + (A + \sigma A_p^\pm)^2 - \sigma(A_p^\pm)^2\right)
$$

$$
\quad + \frac{6}{\lambda_0}\sqrt{\mu}\left((w_1 + \sigma w_{1p}^\pm)^2 - \sigma(w_{1p}^\pm)^2\right)
$$

$$
\quad + \frac{\sqrt{6}}{\lambda_0}\sqrt{\mu}\left(H_A(w_1 + \sigma w_{1p}^\pm) + A(w_1 + \sigma w_{1p}^\pm)\right)
$$

$$
\mathcal{F}_3(x) = -\sigma'(x)w_{1p}^\pm(x - \theta^\pm),
$$

$$
\mathcal{F}_4(x) = \mathcal{F}_4(x) + g_4(x),
$$

$$
h_4(x) = -\sigma'(x)w_{2p}^\pm(x - \theta^\pm),
$$

$$
g_4(x) = -\frac{3}{4}\sqrt{\mu}\left(2\sigma HA_p^\pm + (A + \sigma A_p^\pm)^2 - \sigma(A_p^\pm)^2\right)
$$

$$
\quad + \frac{6}{\lambda_0}\sqrt{\mu}\left((w_1 + \sigma w_{1p}^\pm)^2 - \sigma(w_{1p}^\pm)^2\right)
$$

$$
\quad + \frac{\sqrt{6}}{\lambda_0}\sqrt{\mu}\left(H_A(w_1 + \sigma w_{1p}^\pm) + A(w_1 + \sigma w_{1p}^\pm)\right).
The adjoint system of \( (32) \) also has four linearly independent solutions

\[
g_1(x) = -\frac{1}{8} \mu \left( H_A^2 + A^2 + 2H_A A + 2H_A \sigma A_{p}^{\pm} + 2\sigma A_{p}^{\pm} + (\sigma^2 - \sigma)(A_{p}^{\pm})^2 \right)
- \frac{\sqrt{6}}{2\lambda_0} \mu \left( H_A(w_1 + \sigma w_{1p}^{\pm}) + A(w_1 + \sigma w_{1p}^{\pm}) \right)
+ \sigma A_{p}^{\pm} (w_1 + \sigma w_{1p}^{\pm}) - \sigma w_{1p}^{\pm} A_{p}^{\pm}
+ \frac{9}{\lambda_0^2} \mu \left( (w_1 + \sigma w_{1p}^{\pm})^2 - \sigma (w_{1p}^{\pm})^2 \right).
\]

The linear system of \( (30) \)

\[
\dot{Z}(x) = \mathcal{L}(x) Z
\]

has four linearly independent solutions

\[
s_1(x) = \left( 2 \sech^2 \left( \frac{\sqrt{3} \mu}{\sqrt{2}} x \right) \tanh \left( \frac{\sqrt{3} \mu}{\sqrt{2}} x \right), \right.
\]
\[
\left. (2 - \cosh(\sqrt{6}\mu x)) \sech^4 \left( \frac{\sqrt{3} \mu}{\sqrt{2}} x \right), 0, 0 \right)^T,
\]
\[
u_1(x) = \left( \frac{1}{9} \left( 6\sqrt{6} + \sqrt{6} \cosh(\sqrt{6}\mu x) \right)
- 15 \sech^2 \left( \frac{\sqrt{3} \mu}{\sqrt{2}} x \right) \left( \sqrt{6} - 3\sqrt{\mu} x \tanh \left( \frac{\sqrt{3} \mu}{\sqrt{2}} x \right) \right), \right.
\]
\[
\left. \frac{1}{24\sqrt{6}} \sech^4 \left( \frac{\sqrt{3} \mu}{\sqrt{2}} x \right) \left( -60 \sqrt{6} \mu x (-2 + \cosh(\sqrt{6}\mu x)) \right.ight.
\[
\left. + 185 \sinh(\sqrt{6}\mu x) + 4 \sinh(2\sqrt{6}\mu x) + \sinh(3\sqrt{6}\mu x)) / 0, 0 \right)^T,
\]
\[
u_2 = \left( 0, 0, \cos(\lambda_0 x), \sin(\lambda_0 x) \right)^T, \quad \nu_3 = \left( 0, 0, \sin(\lambda_0 x), -\cos(\lambda_0 x) \right)^T,
\]

which satisfy for \( x \in \mathbb{R} \)

\[
|s_1(x)| \leq M e^{-\sqrt{6}\mu|x|}, \quad |u_1(x)| \leq M e^{\sqrt{6}\mu|x|}, \quad |u_2(x)| + |u_3(x)| \leq M.
\]

The adjoint system of \( (32) \) also has four linearly independent solutions

\[
s_1^*(x) = \left( \frac{1}{128} \sech^4 \left( \frac{\sqrt{3} \mu}{\sqrt{2}} x \right) \left( -60 \sqrt{6} \mu x (-2 + \cosh(\sqrt{6}\mu x)) \right)
+ 185 \sinh(\sqrt{6}\mu x) + 4 \sinh(2\sqrt{6}\mu x) + \sinh(3\sqrt{6}\mu x)) / 0, 0 \right)^T,
\]
\[
u_1^*(x) = \left( \frac{3\sqrt{3}}{8\sqrt{2}} (-2 + \cosh(\sqrt{6}\mu x)) \sech^4 \left( \frac{\sqrt{3} \mu}{\sqrt{2}} x \right), \right.
\]
\[
3\sqrt{6} \cosh^3(\sqrt{6}\mu x) \sinh^4 \left( \frac{\sqrt{3} \mu}{\sqrt{2}} x \right), 0, 0 \right)^T,
\]
\[ u_2^* = \begin{pmatrix} 0, 0, \cos(\lambda_0 x), \sin(\lambda_0 x) \end{pmatrix}^T, \quad u_3^* = \begin{pmatrix} 0, 0, \sin(\lambda_0 x), -\cos(\lambda_0 x) \end{pmatrix}^T, \tag{35} \]

which satisfy for \( x \in \mathbb{R} \)

\[ |s_1^*(x)| \leq Me^{\sqrt{\nu}|x|}, \quad |u_1^*(x)| \leq Me^{-\sqrt{\nu}|x|}, \quad |u_2^*(x)| + |u_3^*(x)| \leq M, \tag{36} \]

and

\[ \langle s_1(x), s_1^*(x) \rangle = 1, \quad \langle s_1(x), u_k^*(x) \rangle = 0, \quad \langle u_k(x), s_1^*(x) \rangle = 0, \]
\[ \langle u_k(x), u_k^*(x) \rangle = 1, \quad \langle u_k(x), u_j^*(x) \rangle = 0, \quad k, j = 1, 2, 3, \quad j \neq k, \]

where \( \langle \cdot, \cdot \rangle \) denotes the Euclidean inner product on \( \mathbb{R}^4 \). The solution of (30) can be formally written as

\[ Z(x) = \int_0^x \langle F(s, Z, X_p^+), s_1^*(s) \rangle ds s_1(x) - \sum_{k=1}^3 \int_x^\infty \langle F(s, Z, X_p^+), u_k^*(s) \rangle ds u_k(x) \]
\[ \triangleq E[Z](x) = (E_1, E_2, E_3, E_4)^T[Z](x) \triangleq (A, B, W_1, W_2)^T(x). \tag{37} \]

Thus, the existence problem of \( Z(x) \) is transformed into the fixed point problem of the operator \( E \) in (37).

5. Function spaces. In order to use the fixed point theorem for the operator \( E \) in (37), we define three function spaces as follows

\[ \mathcal{H}_e = \left\{ f \in C^{2n}(-\infty, +\infty) \mid f(x) \text{ is even}, \right\} \]
\[ \|f\|_e = \sum_{k=0}^{2n} \mu^{-k/2} \left( \sup_{x \geq 0} |f^{(k)}(x)| e^{\nu x} \right) < +\infty, \]

\[ \mathcal{H}_d = \left\{ f \in C^{2n}(-\infty, +\infty) \mid f(x) \text{ is odd}, \right\} \]
\[ \|f\|_d = \sum_{k=0}^{2n} \mu^{-k/2} \left( \sup_{x \leq 0} |f^{(k)}(x)| e^{\nu x} \right) < +\infty, \]

\[ \mathcal{H}_r = \left\{ f \in C^{2n}([-\tau, +\infty)) \mid \|f\|_r = \sum_{k=0}^{2n} \left( \sup_{x \in [-\tau, +\infty)} |f^{(k)}(x)| e^{\nu x} \right) < +\infty \right\}, \]

where \( \tau > 0 \) is a large constant, \( n \) is a positive integer,

\[ \nu = \sqrt{6\mu - \mu^\alpha}, \quad \frac{1}{2} < \alpha < 1, \tag{38} \]

and \( \alpha \) is a constant. Here the constants \( \tau \) and \( \alpha \) will be fixed later. From the definition of \( \mathcal{H}_r \), we see that \( f \in \mathcal{H}_r \) may be exponentially large near negative infinity for large \( \tau \). In order to control this bad property, we will adjust the difference \( I^+ - I^- \) (see (94)).

Due to the reversibility of the system (13), we decompose the solution \( Z(x) \) accordingly, and let \( \mathcal{H}_1 = \mathcal{H}_e \oplus \mathcal{H}_r \) and \( \mathcal{H}_2 = \mathcal{H}_d \oplus \mathcal{H}_r \). For \( f \in \mathcal{H}_1 \), it means that \( f \in C^{2n}([-\tau, +\infty)) \) and can be decomposed as

\[ f(x) = f_e(x) + f_{er}(x), \tag{39} \]

where

\[ f_e(x) = f(|x|) - f_{e0}(x), \]
\[ f_{e0}(x) = (1 - \sigma(x)) |x| \left( f'(0) + \frac{f^{(3)}(0)}{3!} x^2 + \cdots + \frac{f^{(2n-1)}(0)}{(2n-1)!} x^{2n-2} \right), \]

\[ f_{er}(x) = f(x) - f_e(x) = (1 - \mathbb{H}(x)) (f(x) - f(-x)) + f_{e0}(x), \tag{40} \]

and \( f_e \in \mathcal{H}_e, f_r \in \mathcal{H}_r \). The even cutoff function \( \sigma(x) \) is defined in (27), and the Heaviside function \( \mathbb{H}(x) \) is given as follows

\[ \mathbb{H}(x) = \begin{cases} 1 & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases} \]

Similarly, for \( f \in \mathcal{H}_2 \), it means that \( f \in C^{2n}([-\tau, +\infty)) \) and can be written as

\[ f(x) = f_d(x) + f_{dr}(x), \tag{41} \]

where

\[ f_d(x) = \begin{cases} f(x) - f_{d0}(x) & \text{for } x > 0, \\ -f(|x|) - f_{d0}(x) & \text{for } x \leq 0, \end{cases} \]

\[ f_{d0}(x) = (1 - \sigma(x)) \left( f(0) + \frac{f^{(2)}(0)}{2!} x^2 + \cdots + \frac{f^{(2n)}(0)}{(2n)!} x^{2n} \right), \]

\[ f_{dr}(x) = f(x) - f_d(x) = (1 - \mathbb{H}(x)) (f(x) + f(-x)) + f_{d0}(x), \]

\[ \tilde{f}_{d0}(x) = \begin{cases} f_{d0}(x) & \text{for } x \geq 0, \\ -f_{d0}(x) & \text{for } x < 0, \end{cases} \tag{42} \]

and \( f_d \in \mathcal{H}_d, f_{dr} \in \mathcal{H}_r \). It is easy to check that the above decompositions are unique and all functions \( f_e(x), f_d(x), f_{er}(x) \) and \( f_{dr}(x) \) are smooth.

Note that \( f_{er}(x) = f_{dr}(x) = 0 \) for \( x \geq 2 \) and \( f_{er}(x) = f(x) - f(-x) \) and \( f_{dr}(x) = f(x) + f(-x) \) for \( x \leq -2 \). Before we apply the fixed point theorem for the operator \( \mathcal{E} \) in (37), we separate the interval \([-\tau, +\infty)\) into two parts: \( x \in [-2, +\infty) \) and \( x \in [-\tau, -2] \). Correspondingly, \( \mathcal{H}_r \) restricted on \([-2, +\infty)\), i.e., equivalently on \([-2, 2]\) (or on \([-\tau, -2]\)) is denoted by \( \mathcal{H}_{r0} \) (or \( \mathcal{H}_{r1} \)). More precisely,

\[ \mathcal{H}_{r0} = \{ f \in C^{2n}([-2, 2]) \} \tag{43} \]

with a norm \( \| \cdot \|_0 \) given by

\[ \| f \|_0 = \sum_{k=0}^{2n} \sup_{|x| \leq 2} |f^{(k)}(x)| \approx \sum_{k=0}^{2n} \left( \sup_{|x| \leq 2} |f^{(k)}(x)| e^{\nu|x|} \right) < +\infty, \tag{44} \]

and

\[ \mathcal{H}_{r1} = \{ f \in C^{2n}([-\tau, -2]) \} \tag{45} \]

with a norm \( \| \cdot \|_1 \) given by

\[ \| f \|_1 = \sum_{k=0}^{2n} \left( \sup_{x \in [-\tau, -2]} |f^{(k)}(x)| e^{\nu|x|} \right) < +\infty. \tag{46} \]

Let \( \mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_1 \times \mathcal{H}_2 \). For \( Z = (A, B, w_1, w_2)^T \in \mathcal{H} \), we symbolically write \( Z \) as

\[ Z(x) = Z_e(x) + Z_r(x), \tag{47} \]

where

\[ Z_e(x) = (A_e(x), B_d(x), w_1e(x), w_2d(x))^T, \]
which norms are defined by
\[ \|Z\|_e = \|A_e\|_e + \|B_d\|_d + \|w_{1e}\|_e + \|w_{2d}\|_d, \]
\[ \|Z\|_0 = \|A_{er}\|_0 + \|B_{dr}\|_0 + \|w_{1er}\|_0 + \|w_{2dr}\|_0 \]  
\tag{49}
for \( Z_r \) only well defined with \( |x| \leq 2 \) (in this case \( \mathcal{H} \) is denoted by \( \mathcal{H}^0 \)) and
\[ \|Z\|_1 = \|A_{er}\|_1 + \|B_{dr}\|_1 + \|w_{1er}\|_1 + \|w_{2dr}\|_1 \]  
\tag{50}
for \( Z_r \) only well defined with \( x \in [-\tau, -2] \) (in this case, \( \mathcal{H}^1 \) stands for \( \mathcal{H} \)). If no confusion arises, we sometimes use \( (A_r, B_r, w_{1r}, w_{2r}) \) for \( (A_{er}, B_{dr}, w_{1er}, w_{2dr}) \).

6. Existence proof of solutions for \( x \in [-2, +\infty) \). This section gives the estimates of the operator \( \mathcal{E} \) in (37) for \( Z_e \) with \( x \in [0, +\infty) \) and \( Z_r \) with \( x \in [-2, +\infty) \) since \( Z_e \) is reversible. Note that \( Z_r(x) = 0 \) for \( x \geq 2 \) due to (40), (42) and (48). Thus, \( \mathcal{H}^0 \) with the norm \( \| \cdot \|_0 \) will be used for \( Z_r \). See (43), (44) and (49).

From the definitions \( f_{er}(x) \) and \( f_{dr}(x) \) in Section 5, one obtains that
\[ A_r(x) = \begin{cases} 0, & x \in [2, +\infty), \\ A_{e0}(x), & x \in [0, 2], \\ A(x) - A(-x) + A_{e0}(x), & x \in [-2, 0], \\ A(x) - A(-x), & x \in [-\tau, -2], \end{cases} \]  
\tag{51}
\[ B_r(x) = \begin{cases} 0, & x \in [2, +\infty), \\ B_{d0}(x), & x \in [0, 2], \\ B(x) + B(-x) - B_{d0}(x), & x \in [-2, 0], \\ B(x) + B(-x), & x \in [-\tau, -2], \end{cases} \]  
\tag{52}
\[ w_{1r}(x) = \begin{cases} 0, & x \in [2, +\infty), \\ w_{1e0}(x), & x \in [0, 2], \\ w_1(x) - w_1(-x) + w_{1e0}(x), & x \in [-2, 0], \\ w_1(x) - w_1(-x), & x \in [-\tau, -2], \end{cases} \]  
\tag{53}
\[ w_{2r}(x) = \begin{cases} 0, & x \in [2, +\infty), \\ w_{2e0}(x), & x \in [0, 2], \\ w_2(x) + w_2(-x) - w_{2e0}(x), & x \in [-2, 0], \\ w_2(x) + w_2(-x), & x \in [-\tau, -2]. \end{cases} \]  
\tag{54}

A direct calculation shows that from (31) and (37) for \( x \leq -2 \)
\[ \mathcal{E}_{1r}[Z] = \mathcal{A}(|x|) - \mathcal{A}(-|x|) = \int_0^{[x]} \left( (\mathcal{F}_1(t) + \mathcal{F}_1(-t)) s_1^*[1](t) \\ + (\mathcal{F}_2(t) - \mathcal{F}_2(-t)) s_1[2]^*[t] \right) dt \ s_1[1](|x|) \\
+ \int_0^{[x]} \left( (\mathcal{F}_1(t) + \mathcal{F}_1(-t)) u_1^*[1](t) \\ + (\mathcal{F}_2(t) - \mathcal{F}_2(-t)) u_1[2]^*[t] \right) dt \ u_1[1](|x|), \]  
\tag{55}
\( \mathcal{E}_{3r}[Z] = \mathcal{B}(|x|) + \mathcal{B}(-|x|) = \int_{0}^{x} \left( (\mathcal{F}_1(t) + \mathcal{F}_1(-t))s_1[1](t) ight. \\
\left. + (\mathcal{F}_2(t) - \mathcal{F}_2(-t))s_1[2]^*(t) \right) dt \sin(s_1[2](|x|)) \\
\left. + \int_{0}^{x} \left( (\mathcal{F}_1(t) + \mathcal{F}_1(-t))u_1[1]^*(t) ight) dt \sin(s_1[2](|x|)) \right), \\
(56) \\
\mathcal{E}_{4r}[Z] = W_1(|x|) - W_1(-|x|) \\
= -2 \int_{0}^{x} \left( \mathcal{F}_3(t) \sin(\lambda_0 t) - \mathcal{F}_4(t) \cos(\lambda_0 t) \right) dt \sin(\lambda_0 |x|) \\
\left. + \int_{0}^{x} \left( \mathcal{F}_3(t) + \mathcal{F}_3(-t) \right) \cos(\lambda_0 (|x| - t)) \\
- (\mathcal{F}_4(t) - \mathcal{F}_4(-t)) \sin(\lambda_0 (|x| - t)) \right) dt, \\
\mathcal{E}_{4r}[Z] = W_2(|x|) + W_2(-|x|) \\
= 2 \int_{0}^{x} \left( \mathcal{F}_3(t) \sin(\lambda_0 t) - \mathcal{F}_4(t) \cos(\lambda_0 t) \right) dt \cos(\lambda_0 |x|) \\
\left. + \int_{0}^{x} \left( \mathcal{F}_3(t) + \mathcal{F}_3(-t) \right) \sin(\lambda_0 (|x| - t)) \\
+ (\mathcal{F}_4(t) - \mathcal{F}_4(-t)) \cos(\lambda_0 (|x| - t)) \right) dt.

Notice that \( \mathcal{F}_3(t) + \mathcal{F}_3(-t) \) and \( \mathcal{F}_4(t) - \mathcal{F}_4(-t) \) will have a factor \( I^+ - I^- \), and we will use \( I^+ - I^- \) to control the term \( e^{2\sqrt{\mu}r} \) (see (94) and (95)). Hence, we assume
\[
\int_{0}^{x} \left( \mathcal{F}_3(t) \sin(\lambda_0 t) - \mathcal{F}_4(t) \cos(\lambda_0 t) \right) dt = 0, \\
(57)
\]
which implies
\[
\mathcal{E}_{3r}[Z] = W_1(|x|) - W_1(-|x|) = \int_{0}^{x} \left( \mathcal{F}_3(t) + \mathcal{F}_3(-t) \right) \cos(\lambda_0 (|x| - t)) \\
- (\mathcal{F}_4(t) - \mathcal{F}_4(-t)) \sin(\lambda_0 (|x| - t)) \right) dt, \\
(58) \\
\mathcal{E}_{4r}[Z] = W_2(|x|) + W_2(-|x|) = \int_{0}^{x} \left( \mathcal{F}_3(t) + \mathcal{F}_3(-t) \right) \sin(\lambda_0 (|x| - t)) \\
+ (\mathcal{F}_4(t) - \mathcal{F}_4(-t)) \cos(\lambda_0 (|x| - t)) \right) dt. \\
(59)
\]

Before we can proceed further, we solve the equation (57) for \( \theta^+ \). The plus sign in \( \mathcal{F}_3 \) and \( \mathcal{F}_4 \) is taken since the integral is from zero to positive infinity. For the sake of simplicity, we in advance assume that
\[
I^+ = O(\mu^{n+1/2}), \quad I^- = O(\mu^{n+1/2}). \\
(60)
\]

Lemma 6.1. Suppose that (60) is valid and
\[
\|A_r\|_0 + \|B_r\|_0 + \|w_1r\|_0 + \|w_2r\|_0 \leq M\mu^n,
\]
for $Z = (A, B, w_1, w_2)^T \in \mathcal{H}^0$. The equation (67) is equivalent to
\[
\theta^+ = \bar{\Phi}(\theta^+, \mu, I^+) = \mu^{1/8} \Phi(\theta^+, \mu, I^+),
\]
where $\Phi(\theta^+, \mu, I^+)$ is differentiable with respect to its arguments. Moreover, it and its derivatives with respect to $\theta^+$ are uniformly bounded for small $(\theta^+, \mu, I^+)$. The proof is given in Appendix 1. Take a closed interval $\bar{B}_1(0) \subset \mathbb{R}$ with a length $r = O(\mu^{1/16})$. It is obvious that $\hat{\Phi}$ is a contraction map on $\bar{B}_1(0)$ for small $(\mu, I^+)$. The fixed point theorem yields that $\hat{\Phi}$ has a unique solution $\theta^+ = \theta^+(\mu, I^+)$ satisfying
\[
|\theta^+| \leq M \mu^{1/8}.
\]

Now we estimate $\mathcal{E}[Z]$ in (37).

**Lemma 6.2.** Under the assumption (60), if $\|Z_e\|_e + \|\hat{Z}_e\|_e + \|Z_r\|_0 + \|\hat{Z}_r\|_0 \leq M_0$ for $Z = (A, B, w_1, w_2)^T$, $\hat{Z} = (\bar{A}, \bar{B}, \bar{w}_1, \bar{w}_2)^T \in \mathcal{H}^0$, where $M_0$ is some positive constant, then the mapping $\mathcal{E}$ in (37) satisfies
\[
\|\mathcal{E}[Z]\|_e + \|\mathcal{E}_d[Z]\|_d \leq M \left( \mu^{1/2-\alpha-n} I^+ + \sqrt{\mu} + \mu^{1/2-\alpha-n} I^+ \|A_e\|_e + \|A_e\|^2_e \right.
\]
\[
+ \sqrt{\mu}(\|A_e\|_e + \|w_{1e}\|_e + \mu^{1/2-n} \|A_r\|_0 + \mu^{1/2-n} \|w_{1r}\|_0) ,
\]
\[
\|\mathcal{E}[Z]\|_e + \|\mathcal{E}_d[Z]\|_d \leq M \sqrt{\mu} \left( 1 + \mu^{1/2-n} I^+ \right) \|A_e\|_e + \|A_e\|^2_e
\]
\[
+ \mu^{1/2-n} \|A_r\|_0 + \mu^{1/2-n} \|w_{1r}\|_0),
\]
\[
\|\mathcal{E}[Z] - \mathcal{E}_d[Z]\|_e + \|\mathcal{E}_d[Z] - \mathcal{E}_d[\hat{Z}]\|_d \leq M \left( \mu^{1/2-\alpha-n} I^+ \|A_e - \hat{A}_e\|_e \right.
\]
\[
+ \sqrt{\mu}(\|A_e - \hat{A}_e\|_e + \|w_{1e} - \hat{w}_{1e}\|_e) + (\|A_e\|_e + \|\hat{A}_e\|_e) \|A_e - \hat{A}_e\|_e
\]
\[
+ \mu^{1/2-n} \|A_r - \hat{A}_r\|_0 + \mu^{1/2-n} \|w_{1r} - \hat{w}_{1r}\|_0),
\]
\[
\|\mathcal{E}[Z] - \mathcal{E}[\hat{Z}]\|_e + \|\mathcal{E}[\hat{Z}] - \mathcal{E}_d[\hat{Z}]\|_d \leq M \sqrt{\mu} \left( \|A_e - \hat{A}_e\|_e + \|w_{1e} - \hat{w}_{1e}\|_e
\]
\[
+ \mu^{1/2-n} \|A_r - \hat{A}_r\|_0 + \mu^{1/2-n} \|w_{1r} - \hat{w}_{1r}\|_0),
\]
and for $x \geq -2$
\[
\|\mathcal{E}[Z]\|_0 + \|\mathcal{E}_d[Z]\|_0 \leq M \sqrt{\mu} \left( I^+ - I^- \right) + \|A_r\|_0 + \|w_{1r}\|_0),
\]
\[
\|\mathcal{E}[Z]\|_0 + \|\mathcal{E}_d[Z]\|_0 \leq M \sqrt{\mu} \left( \mu^{1/2-\alpha} I^+ + \mu^{1/2} I^- - I^- \right) + \mu^{n+1/2}
\]
\[
+ \mu^n \|A_e\|^2_e + \mu^{n+1/2} (\|A_e\|_e + \|w_{1e}\|_e) + \sqrt{\mu} \|A_r\|_0 + \sqrt{\mu} \|w_{1r}\|_0),
\]
\[
\|\mathcal{E}[Z] - \mathcal{E}[\hat{Z}]\|_0 + \|\mathcal{E}[\hat{Z}] - \mathcal{E}_d[\hat{Z}]\|_0 \leq M \sqrt{\mu} \left( \|A_r - \hat{A}_r\|_0 + \|w_{1r} - \hat{w}_{1r}\|_0 \right),
\]
\[
\|\mathcal{E}[Z] - \mathcal{E}_d[Z] + \|\mathcal{E}_d[Z] - \mathcal{E}_d[\hat{Z}]\|_d \leq M \sqrt{\mu} \left( \mu^n (\|A_e\|_e + \|\hat{A}_e\|_e) \|A_e - \hat{A}_e\|_e
\]
\[
+ \mu^{n+1/2} (\|A_e - \hat{A}_e\|_e + \|w_{1e} - \hat{w}_{1e}\|_e) + \sqrt{\mu} \|A_r - \hat{A}_r\|_0
\]
\[
+ \sqrt{\mu} \|w_{1r} - \hat{w}_{1r}\|_0 \right).
The proof of this lemma is presented in Appendix 2. Note that $\mu^{1/2-\alpha-n}I^\pm$ in (61) and $\mu^{-1/2-n}I^\pm$ in (62) arise, which means that the assumption $I^\pm = O(\mu^{n+1/2})$ in (60) is reasonable.

Assume that for $x \geq -2$

\[ Z_r(x) = \mu^{n+1/2}\tilde{Z}_r(x), \quad \tilde{Z}_r(x) = \mu^{n+1/2}\tilde{\tilde{Z}}_r(x), \]

and

\[ I^+ = I_0\mu^{n+1/2}, \quad I^+ - I^- = a_0\mu^{n+1/2+i}, \quad \tilde{i} \geq \frac{1}{4}, \quad \alpha = \frac{3}{4}, \] (69)

where $I_0$ and $i$ are positive constants, and the constant $a_0$ will be fixed later. Clearly, (38) and (60) are satisfied. Now consider $\mathcal{H}^0$ as a product space of $(Z, \tilde{Z}_r)$ and choose a closed ball $\bar{B}_2(0)$ in $\mathcal{H}^0$ such that for $\bar{Z} = (Z, \tilde{Z}_r)$ in $\bar{B}_2(0)$

\[ \|Z_c\| + \|\tilde{Z}_r\|_0 \leq M\mu^{1/8}. \]

Correspondingly, let $\hat{\mathcal{E}}[\bar{Z}] = (\hat{\mathcal{E}}_c[\bar{Z}], \hat{\mathcal{E}}_r[\bar{Z}])$ where

\[ \hat{\mathcal{E}}_c[\bar{Z}] = \mathcal{E}_c[Z], \quad \hat{\mathcal{E}}_r[\bar{Z}] = \mu^{-n-1/2}\mathcal{E}_r[Z]. \]

According to Lemma 6.2, we obtain that for $\bar{Z}, \tilde{\bar{Z}} \in \bar{B}_2(0)$

\[ \|\hat{\mathcal{E}}_c[\bar{Z}]\|_e + \|\hat{\mathcal{E}}_r[\bar{Z}]\|_e \leq M\mu^{1/4}, \quad \|\hat{\mathcal{E}}_c[\tilde{Z}]\|_e + \|\hat{\mathcal{E}}_r[\tilde{Z}]\|_e \leq M\sqrt{\mu}, \]

\[ \|\hat{\mathcal{E}}_c[\bar{Z}]\|_0 + \|\hat{\mathcal{E}}_r[\bar{Z}]\|_0 \leq M\mu^{5/8}, \quad \|\hat{\mathcal{E}}_c[\tilde{Z}]\|_0 + \|\hat{\mathcal{E}}_r[\tilde{Z}]\|_0 \leq M\mu^{1/4}, \]

\[ \|\hat{\mathcal{E}}_c[\bar{Z}] - \hat{\mathcal{E}}_c[\tilde{Z}]\|_e + \|\hat{\mathcal{E}}_r[\bar{Z}] - \hat{\mathcal{E}}_r[\tilde{Z}]\|_0 \leq M\mu^{1/8}\left( \|Z_c - \tilde{Z}_c\|_e + \|\tilde{Z}_r - \tilde{\tilde{Z}}_r\|_0 \right), \] (70)

which implies that $\hat{\mathcal{E}}$ is a contraction mapping on $\bar{B}_2(0)$ for small $\mu > 0$. Therefore, $\hat{\mathcal{E}}$ has a unique fixed point $\bar{Z} \in \mathcal{H}^0$ for $x \in [-2, \infty)$. This shows that (37) has a unique solution $Z \in \mathcal{H}^0$ for $x \in [-2, \infty)$ satisfying

\[ \|A_c\|_e + \|B_c\|_d \leq M\mu^{1/4}, \quad \|w_{1c}\|_e + \|w_{2d}\|_d \leq M\sqrt{\mu}, \]

\[ \|A_r\|_0 + \|B_r\|_0 \leq M\mu^{n+9/8}, \quad \|w_{1r}\|_0 + \|B_{2r}\|_0 \leq M\mu^{n+3/4}. \] (71)

The more exact estimates for $A_c(x)$ and $w_{1c}(x)$ will be needed later.

**Lemma 6.3.** Under the assumption (69), we have for small $\mu > 0$

\[ A_c(x) = \frac{3\sqrt{6}}{\lambda_0^2} \mu \cosh \left( \sqrt{6\mu} x \right) \sech^4 \left( \frac{\sqrt{3\mu}}{\sqrt{2}} x \right) + O(\mu^{3/2}\mu^{-|x|}), \]

\[ w_{1c}(x) = \frac{\mu}{8\lambda_0} H^2_A(x) + O(\mu^{3/2}\mu^{-|x|}), \] (72)

and then

\[ \|A_c\|_e + \|w_{1c}\|_e \leq M\mu. \] (73)

**Proof.** A direct calculation shows that by (30), (69) and (71)

\[ \ddot{A}_c = 6\mu A_c - \frac{3\sqrt{6}}{2} \mu H A_c - \frac{3\sqrt{6}}{4} \mu A_c^2 \]

\[ + \frac{6\sqrt{6}}{\lambda_0^2} \mu w_{1e} + \frac{6\mu}{\lambda_0} (H_A w_{1e} + A_c w_{1e}) + f_1(x), \]

\[ \ddot{w}_{1e} = -\lambda_0^2 w_{1e} + \frac{\lambda_0}{8} (H^2_A + 2H_A A_c + A_c^2) + \frac{\sqrt{6}}{2} \mu (H_A w_{1e} + A_c w_{1e}) + f_2(x), \]
where
\[ f_1(x) = O(\mu^{5/2}e^{-\nu|x|}), \quad f_2(x) = O(\mu^{3/2}e^{-\nu|x|}). \]

Notice that \( w_{1c}(x) \) is even. It is straightforward to see that
\[ w_{1c}(x) = c_0 \cos(\lambda_0 x) + \frac{1}{\lambda_0} \int_0^x \sin(\lambda_0 (x-t)) \left( \frac{\lambda_0}{8} \mu (H_A^2(t) + 2H_A(t)A_c(t) + A_c^2(t)) \right. \right.
\[ + \frac{\sqrt{6}}{2} \mu (H_A(t)w_{1c}(t) + A_c(t)w_{1c}(t)) + f_2(t) \left. \right) dt \]  
(74)
Since \( w_{1c}(x) \) exponentially tends to 0 as \( x \to \infty \), we have with integration by parts
\[ c_0 = \frac{1}{\lambda_0} \int_0^\infty \sin(\lambda_0 t) \left( \frac{\lambda_0}{8} \mu (H_A^2(t) + 2H_A(t)A_c(t) + A_c^2(t)) \right. \right.
\[ + \frac{\sqrt{6}}{2} \mu (H_A(t)w_{1c}(t) + A_c(t)w_{1c}(t)) + f_2(t) \left. \right) dt \]

\[ = \frac{\mu}{\lambda_0} \left( \frac{\lambda_0}{8} (H_A^2(0) + 2H_A(0)A_c(0) + A_c^2(0)) + \frac{\sqrt{6}}{2} (H_A(0)w_{1c}(0) + A_c(0)w_{1c}(0)) \right) \]
\[ + \frac{\mu}{\lambda_0} \int_0^\infty \cos(\lambda_0 t) \left( \frac{\lambda_0}{8} (H_A^2(t) + 2H_A(t)A_c(t) + A_c^2(t)) \right. \right.
\[ + \frac{\sqrt{6}}{2} (H_A(t)w_{1c}(t) + A_c(t)w_{1c}(t)) + \frac{1}{\mu} f_2(t) \left. \right) dt + O(\mu^{3/2}) \]

\[ = \frac{\mu}{\lambda_0^2} \left( \frac{\lambda_0}{8} (H_A^2(0) + 2H_A(0)A_c(0) + A_c^2(0)) + \frac{\sqrt{6}}{2} (H_A(0)w_{1c}(0) + A_c(0)w_{1c}(0)) \right) \]
\[ + \frac{\mu}{\lambda_0} \int_0^\infty \sin(\lambda_0 t) \left( \frac{\lambda_0}{8} (H_A^2(t) + 2H_A(t)A_c(t) + A_c^2(t)) \right. \right.
\[ + \frac{\sqrt{6}}{2} (H_A(t)w_{1c}(t) + A_c(t)w_{1c}(t)) + \frac{1}{\mu} f_2(t) \left. \right)'' dt + O(\mu^{3/2}) \]
\[ = \frac{\mu}{\lambda_0^2} \left( \frac{\lambda_0}{8} (H_A^2(x) + 2H_A(x)A_c(x) + A_c^2(x)) + \frac{\sqrt{6}}{2} (H_A(x)w_{1c}(x)) \right. \right.
\[ + A_c(x)w_{1c}(x)) \left. \right) \]
\[ - \frac{\mu}{\lambda_0^2} \int_0^x \sin(\lambda_0 (x-t)) \left( \frac{\lambda_0}{8} (H_A^2(t) + 2H_A(t)A_c(t) + A_c^2(t)) \right. \right.
\[ + \frac{\sqrt{6}}{2} (H_A(t)w_{1c}(t) + A_c(t)w_{1c}(t)) + \frac{1}{\mu} f_2(t) \left. \right)'' dt + O(\mu^{3/2}e^{-\nu|x|}) \]
\[ = \frac{\mu}{8\lambda_0} \left( H_A^2(x) + 2H_A(x)A_c(x) \right) + O(\mu^{3/2}e^{-\nu|x|}), \]

which implies that \( ||w_{1c}||_c \leq M_\mu \). Using this result with a similar method, we obtain
\[ A_c(x) = \frac{3\sqrt{6}}{\lambda_0^2} \mu \cosh \left( \sqrt{6\mu x} \right) \text{sech}^4 \left( \frac{\sqrt{3\mu}}{\sqrt{2}} x \right) + O(\mu^{3/2}e^{-\nu|x|}), \]
which further implies that

\[ w_{1e}(x) = \frac{\mu}{8\lambda_0} H_2^2(x) + O(\mu^{3/2} e^{-\nu|x|}). \]

The proof is completed. \(\square\)

7. **Existence proof of solution** \(Z_r(x)\) on \([-\tau, -2]\). In this section, we prove the existence of \(Z_r(x) = (A_r, B_r, w_{1r}, w_{2r})^T(x)\) for \(x \in [-\tau, -2]\) and the space \(H^1\) with the norm \(\|\cdot\|_1\) will be used (see (50)). For the simplicity of notations, let

\[
\begin{align*}
X_1(x) &= A(|x|) - A(-|x|), \\
X_2(x) &= B(|x|) + B(-|x|), \\
X_3(x) &= w_1(|x|) - w_1(-|x|), \\
X_4(x) &= w_2(|x|) + w_2(-|x|)
\end{align*}
\]

for \(x \in [-\tau, 0]\), which means that from (51)-(54)

\[
A_r(x) = -X_1(x), \quad B_r(x) = X_2(x), \quad w_{1r}(x) = -X_3(x), \quad w_{2r}(x) = X_4(x)
\]

for \(x \in [-\tau, -2]\). In the following, we concentrate only on the equations (55), (56), (58) and (59). From these integrals, we will see that \(X_j(x) (j = 1, 2, 3, 4)\) should be well defined on \([-\tau, \tau]\), or equivalently on \([-\tau, 0]\) because of the symmetry. Hence we first prove the existence of \(X_j(x) (j = 1, 2, 3, 4)\) for \(x \in [-\tau, 0]\) and then restrict them on \(x \in [-\tau, -2]\) to get the existence of \(Z_r(x)\). Meanwhile, we will use the following space with the norm \(\|\cdot\|_2\)

\[
H_2 = \left\{ f \in C^{2n}([\tau, 0]) \mid \|f\|_2 = \sum_{k=0}^{2n} \left( \sup_{x \in [-\tau, 0]} |f^{(k)}(x)| e^{-\nu|x|} \right) < \infty \right\}.
\]

When we extend \(Z_r(x)\) on \((-\infty, +\infty)\) in Section 8, the dominant terms of \(B(|x|) + B(-|x|)\) near \(x = -\tau\) are needed (see (103)), which requires that more vital terms of \(W_1(|x|) - W_1(-|x|)\) should be computed. To implement these, we define for \(x \in [-\tau, 0]\)

\[
\begin{align*}
\tilde{X}_1(x) &= w_1(|x|) - w_1(-|x|) + \sigma(x) \left( w_{1p}^+(|x| - \theta^+) - w_{1p}^-(|x| + \theta^+) \right), \\
\tilde{X}_2(x) &= W_1(|x|) - W_1(-|x|) + \sigma(x) \left( w_{2p}^+(|x| - \theta^+) - w_{2p}^-(-|x| + \theta^+) \right), \\
\tilde{X}_3(x) &= w_2(|x|) + w_2(-|x|) + \sigma(x) \left( w_{3p}^+(|x| - \theta^+) + w_{3p}^-(-|x| + \theta^+) \right), \\
\tilde{X}_4(x) &= W_2(|x|) + W_2(-|x|) + \sigma(x) \left( w_{4p}^+(|x| - \theta^+) + w_{4p}^-(-|x| + \theta^+) \right),
\end{align*}
\]

and first pay our attention on \(W_1(|x|) - W_1(-|x|)\) and \(W_2(|x|) + W_2(-|x|)\).

7.1. **Estimate** \(W_1(|x|) - W_1(-|x|)\) and \(W_2(|x|) + W_2(-|x|)\) for \(x \in [-\tau, 0]\). By Lemma 3.1 and (31), it is obvious that for \(x \in [-\tau, 0]\)

\[
\begin{align*}
\mathcal{F}_3(|x|) &= F_3(|x|) = -\sigma'(|x|) \left( (I^+ - I^-) \cos (\lambda_0 + r_1^+)(|x| - \theta^+) \right) \\
&\quad - \frac{9}{\lambda_0} \mu I^+(I^+ - I^-) + \frac{3}{\lambda_0} \mu I^+(I^+ - I^-) \cos (2(\lambda_0 + r_1^+)(|x| - \theta^+)) \\
&\quad + |\sigma'(|x|)| O(\mu |I^+|^2 I^+ - I^-) + |\sigma'(|x|)| O(\mu |I^+|^2 I^- - I^2),
\end{align*}
\]

\[
\begin{align*}
h_4(|x|) &= h_4(-|x|) = -\sigma'(|x|) \left( (I^+ - I^-) \sin (\lambda_0 + r_1^+)(|x| - \theta^+) \right) \\
&\quad + \frac{6}{\lambda_0} \mu I^+(I^+ - I^-) \sin (2(\lambda_0 + r_1^+)(|x| - \theta^+)) \\
&\quad + |\sigma'(|x|)| O(\mu |I^+|^2 I^+ - I^-) + |\sigma'(|x|)| O(\mu |I^+|^2 I^- - I^2),
\end{align*}
\]
\[ g_4(|x|) - g_4(-|x|) = -\frac{1}{4}\mu H_A(x)X_1(x) - \frac{\sqrt{6}}{2\lambda_0}\mu H_A(x)\tilde{X}_3(x) \]

\[-\frac{1}{4}\mu H_A(x)\sigma(x) \left( A_\mu^+(|x| - \theta^+) - A_\mu^-(|x| + \theta^+) \right) \]

\[-\frac{9}{\lambda_0}\mu \sigma(x) \left( (\omega_\mu^+)^2(|x| - \theta^+) - (\omega_\mu^-)^2(|x| + \theta^+) \right) + \tilde{g}_{40} \]

\[ = -\frac{1}{4}\mu H_A(x)X_1(x) - \frac{\sqrt{6}}{2\lambda_0}\mu H_A(x)\tilde{X}_3(x) + \frac{\sqrt{6}}{4}\mu I^+(I^+ - I^-)\sigma(x)H_A(x) \]

\[ + \frac{3\sqrt{6}}{4(2\lambda_0^3 + 3\lambda_0^2\mu)}\mu^2 I^+(I^+ - I^-)\sigma(x)H_A(x) \cos\left(2(\lambda_0 + r_1^+)(|x| - \theta^+)\right) \]

\[ - \frac{18}{\lambda_0}\mu I^+(I^+ - I^-)\sigma(x) \cos^2\left((\lambda_0 + r_1^+)(|x| - \theta^+)\right) + \tilde{g}_{40}. \] (85)

where the even functions like \( \frac{1}{2}\mu H_A^2(x) \) in \( g_4(x) \) do not appear and

\[ |\tilde{g}_{40}| + |\tilde{g}_{40}| \leq M_\mu \left( |I^+ - I^-|^2 + (I^\pm)^2 |I^+ - I^-| \right) \]

\[ + (\|A_\mu\|e + \|A_\nu\|e + \|w_{1e}\|e + \|w_{1e}\|e + \|w_{1r}\|e + \|w_{1r}\|e + \|w_{1r}\|e + \|w_{1r}\|e) \|I^\pm|I^+ - I^-| \]

\[ + e^{-\epsilon|x|}(\|A_\mu\|e + \|A_\nu\|e + \|w_{1e}\|e + \|w_{1e}\|e + \|w_{1r}\|e + \|w_{1r}\|e) \|I^\pm|I^+ - I^-| \]

\[ + e^{\epsilon|x|} \|X_1\|_2 + \|\tilde{X}_3\|_2 \right) + e^{2\epsilon|x|} \|X_1\|_2^2 + \|\tilde{X}_3\|_2^2 \right). \]

Thus, according to (58), (76) and (80) together with (25), we have with integration by parts

\[ \tilde{X}_3(x) = \sigma(x)\left( w_{1p}^+|x| - \theta^+) - w_{1p}^-(-|x| + \theta^+)) + W_1(|x|) - W_1(-|x|) \]

\[ = \sigma(x)\left( (I^+ - I^-) \cos\left((\lambda_0 + r_1^+)(|x| - \theta^+)\right) - \frac{9}{\lambda_0}\mu I^+(I^+ - I^-) \]

\[ + \frac{3}{\lambda_0}\mu I^+(I^+ - I^-) \cos\left((\lambda_0 + r_1^+)(|x| - \theta^+)\right) \]

\[ + O(\mu(I^\pm)^2 |I^+ - I^-|) + O(\mu |I^+ - I^-|^2) \right) \]

\[ + \int_0^{\frac{|x|}{2}} -\sigma'(s)\left( (I^+ - I^-) \cos\left((\lambda_0 + r_1^+)(s - \theta^+)\right) - \frac{9}{\lambda_0}\mu I^+(I^+ - I^-) \]

\[ + \frac{3}{\lambda_0}\mu I^+(I^+ - I^-) \cos\left((\lambda_0 + r_1^+)(s - \theta^+)\right) \]

\[ + O(\mu(I^\pm)^2 |I^+ - I^-|) + O(\mu |I^+ - I^-|^2) \right) \cos(\lambda_0(|x| - s)) ds \]

\[ - \int_0^{\frac{|x|}{2}} \left( -\sigma'(s)\left( (I^+ - I^-) \sin\left((\lambda_0 + r_1^+)(s - \theta^+)\right) \]

\[ + \frac{6}{\lambda_0}\mu I^+(I^+ - I^-) \sin\left((\lambda_0 + r_1^+)(s - \theta^+)\right) \]

\[ + O(\mu(I^\pm)^2 |I^+ - I^-|) + O(\mu |I^+ - I^-|^2) \right) \]

\[ - \frac{1}{4}\mu H_A(s)X_1(s) - \frac{\sqrt{6}}{2\lambda_0}\mu H_A(s)\tilde{X}_3(s) + \frac{\sqrt{6}}{4}\mu I^+(I^+ - I^-)\sigma(s)H_A(s) \]

\[ + \frac{3\sqrt{6}}{4(2\lambda_0^3 + 3\lambda_0^2\mu)}\mu^2 I^+(I^+ - I^-)\sigma(s)H_A(s) \cos\left(2(\lambda_0 + r_1^+)(s - \theta^+)\right) \]
This implies that derived that $\nu$

According to (71), the definitions of $x$

Similarly, from (59) it is obtained that

\[
\int_{0}^{\lambda_0}(x) \cos \left(2(\lambda_0 + r_1^+)(s - \theta^+)\right) \right) \sin \left(\lambda_0(|x| - s)\right) ds
\]

\[
- \frac{3}{\lambda_0^2} \mu I^+(I^+ - I^-) \sigma(s) \cos \left(2(\lambda_0 + r_1^+)(s - \theta^+)\right) \right) \sin \left(\lambda_0(|x| - s)\right) ds
\]

\[+ \frac{3}{4(2\lambda_0^2 + 3\lambda_0 \mu)} \mu^2 I^+(I^+ - I^-) \sigma(s) H_A(s) \cos \left(2(\lambda_0 + r_1^+)(s - \theta^+)\right) \right) \sin \left(\lambda_0(|x| - s)\right) ds
\]

\[+ \mathcal{X}_3, \quad (86)
\]

where

\[
|\mathcal{X}_3| \leq M \sqrt{\mu} \left(\sqrt{|x|}|I^+ - I^-|^2 + \sqrt{|x|}(I^\pm|I^+ - I^-|)
\]

\[+ \sqrt{|x|} \left(\|A_c\|_e + \|A_r\|_0 + \|w_{1c}\|_e + \|w_{1r}\|_0\right) \left(\|X_1\|_2 + \|\mathcal{X}_3\|_2\)
\]

\[+ \sqrt{|x|} \left(\|A_c\|_e + \|A_r\|_0 + \|w_{1c}\|_e + \|w_{1r}\|_0\right) \left(I^\pm|I^+ - I^-|\right)
\]

\[+ e^x|I^\pm| \left(\|X_1\|_2 + \|\mathcal{X}_3\|_2\) + \sqrt{|x|} \left(\|X_1\|_2 + \|\mathcal{X}_3\|_2\) \right).
\]

This implies that

\[
\|\mathcal{X}_3\|_2 \leq M \sqrt{\mu} \left(\sqrt{|x|} \left(\|A_c\|_e + \|A_r\|_0 + \|w_{1c}\|_e + \|w_{1r}\|_0\right) \left(\|X_1\|_2 + \|\mathcal{X}_3\|_2\)
\]

\[+ \sqrt{|x|} \left(\|A_c\|_e + \|A_r\|_0 + \|w_{1c}\|_e + \|w_{1r}\|_0\right) \left(I^\pm|I^+ - I^-|\right)
\]

\[+ e^x|I^\pm| \left(\|X_1\|_2 + \|\mathcal{X}_3\|_2\) \right).
\]

According to (71), the definitions of $\nu$ in (38) and $\alpha$ in (69), and Lemma 6.3, it is derived that

\[
\|\mathcal{X}_3\|_2 \leq M \sqrt{\mu} \left(\mu^{-1/4} \left(\|X_1\|_2 + \|\mathcal{X}_3\|_2\) + I^\pm|I^+ - I^-| + |I^+ - I^-|^2
\]

\[+ \sqrt{|x|} \left(\|X_1\|_2 + \|\mathcal{X}_3\|_2\) \right).
\]

Similarly, from (59) it is obtained that for $x \in [-\tau, 0]$

\[
\int_{0}^{\lambda_0}(x) \cos \left(2(\lambda_0 + r_1^+)(s - \theta^+)\right) \right) \sin \left(\lambda_0(|x| - s)\right) ds
\]

\[
- \frac{3}{\lambda_0^2} \mu I^+(I^+ - I^-) \sigma(s) \cos \left(2(\lambda_0 + r_1^+)(s - \theta^+)\right) \right) \sin \left(\lambda_0(|x| - s)\right) ds
\]

\[+ \frac{3}{4(2\lambda_0^2 + 3\lambda_0 \mu)} \mu^2 I^+(I^+ - I^-) \sigma(s) H_A(s) \cos \left(2(\lambda_0 + r_1^+)(s - \theta^+)\right) \right) \sin \left(\lambda_0(|x| - s)\right) ds
\]
\[ + O(\mu(I^\pm)^2|I^+-I^-|) + O(\mu|I^+-I^-|^2) \sin(\lambda_0(|x|-s)) \, ds \]
\[ + \int_0^{|x|} \left( -\sigma'(s) \left( (I^+-I^-) \sin((\lambda_0 + r^+_1)(s-\theta^+)) + 6 \frac{\mu}{\lambda_0} I^+(I^+-I^-) \sin(2(\lambda_0 + r^+_1)(s-\theta^+)) + O(\mu(I^\pm)^2|I^+-I^-|) + O(\mu|I^+-I^-|^2) \right) \right. \]
\[ - \frac{1}{4} \mu H_A(s) X_1(s) - \frac{\sqrt{6}}{2\lambda_0} \mu H_A(s) \tilde{X}_3(s) + \frac{\sqrt{6}}{4\lambda_0} \mu I^+(I^+-I^-) \sigma(s) H_A(s) \]
\[ + \frac{3\sqrt{6}}{4(2\lambda_0^3 + 3\lambda_0^2 \mu)} \mu^2 I^+(I^+-I^-) \sigma(s) H_A(s) \cos(2(\lambda_0 + r^+_1)(s-\theta^+)) \]
\[ - \frac{18}{\lambda_0^2} \mu I^+(I^+-I^-) \sigma(s) \cos^2((\lambda_0 + r^+_1)(s-\theta^+)) \cos(\lambda_0(|x|-s)) \, ds \right) \]
\[ + \tilde{X}_{40}, \]}

where
\[ |\tilde{X}_{40}| \leq M |X_{30}|. \]

Hence, according to (81)
\[ \tilde{X}_1(x) = \sigma(x) \left( w_{2\sigma}(|x|-\theta^+) + w_{2\sigma}(-|x|+\theta^+) \right) + W_2(|x|) + W_2(-|x|) \]
\[ = \frac{6}{\lambda_0^2} \mu I^+(I^+-I^-) \sigma(x) \sin(2(\lambda_0 + r^+_1)(|x|-\theta^+)) \]
\[ + \int_0^{|x|} \left( -\sigma'(s) \left( -\frac{9}{\lambda_0^2} \mu I^+(I^+-I^-) \sin((\lambda_0(|x|-s)) \right. \]
\[ + \frac{3}{\lambda_0^3} \mu I^+(I^+-I^-) \cos(2(\lambda_0 + r^+_1)(s-\theta^+)) \sin(\lambda_0(|x|-s)) \right) \]
\[ + \frac{6}{\lambda_0^3} \mu I^+(I^+-I^-) \sin(2(\lambda_0 + r^+_1)(s-\theta^+)) \cos(\lambda_0(|x|-s)) \right) \]
\[ - \frac{1}{4} \mu H_A(s) X_1(s) \cos(\lambda_0(|x|-s)) - \frac{\sqrt{6}}{2\lambda_0} \mu H_A(s) \tilde{X}_3(s) \cos(\lambda_0(|x|-s)) \]
\[ + \frac{\sqrt{6}}{4\lambda_0^2} \mu I^+(I^+-I^-) \sigma(s) H_A(s) \cos(\lambda_0(|x|-s)) \]
\[ + \frac{3\sqrt{6}}{4(2\lambda_0^3 + 3\lambda_0^2 \mu)} \mu^2 I^+(I^+-I^-) \sigma(s) H_A(s) \cos(2(\lambda_0 + r^+_1)(s-\theta^+)) \]
\[ \times \cos(\lambda_0(|x|-s)) \right) \]
\[ - \frac{18}{\lambda_0^2} \mu I^+(I^+-I^-) \sigma(s) \cos^2((\lambda_0 + r^+_1)(s-\theta^+)) \cos(\lambda_0(|x|-s)) \right) \, ds \right) \]
\[ + \tilde{X}_{40}, \]

where
\[ |\tilde{X}_{40}| + |\tilde{X}_{40}| \leq M |X_{30}|, \]

which yields
\[ ||\tilde{X}_4|| \leq M ||\tilde{X}_3|| + \sqrt{\mu} \left( \mu^{-1/4} ||X_1||_2 + ||\tilde{X}_3||_2 + I^\pm |I^+-I^-| + |I^+-I^-|^2 \right) \]
where $j$ It is straightforward to check that for $h \in F^2$, using Lemma 3.1 and (31), we get

\[
F_1(|x|) + F_1(-|x|) = -\sigma'(|x|) \left( -\frac{\sqrt{6}}{\lambda_0^2} I^+(I^+ - I^-) - \frac{3\sqrt{6}}{2\lambda_0^2 + 3\mu_0^2}\mu I^+(I^+ - I^-) \cos \left( 2(\lambda_0 + r_1^+)(|x| - \theta^+) \right) + O(\mu(I^+)^2|I^+ - I^-|) + O(|I^+ - I^-|^2) \right),
\]

\[
h_2(|x|) - h_2(-|x|) = -\sigma'(|x|) \left( \frac{6}{2\lambda_0^2 + 3\mu_0^2}\sqrt{\mu} I^+(I^+ - I^-) \sin \left( 2(\lambda_0 + r_1^+)(|x| - \theta^+) \right) + O(\sqrt{\mu}|I^+ - I^-|^2) \right),
\]

\[
g_2(|x|) - g_2(-|x|) = -\frac{3}{2} \sqrt{\mu}\sigma(x)H_A(x) \left( -\frac{\sqrt{6}}{\lambda_0^2} I^+(I^+ - I^-) - \frac{3\sqrt{6}}{2\lambda_0^2 + 3\mu_0^2}\mu I^+(I^+ - I^-) \cos \left( 2(\lambda_0 + r_1^+)(|x| - \theta^+) \right) + O(\mu(I^+)^2|I^+ - I^-|) + O(|I^+ - I^-|^2) \right),
\]

\[
+ \frac{12}{\lambda_0^2 + 3\mu_0^2}\sqrt{\mu}\sigma(x)\hat{X}_3(x) - \frac{12}{\lambda_0^2 + 3\mu_0^2}\sqrt{\mu} I^+(I^+ - I^-) \cos \left( (\lambda_0 + r_1^+)(|x| - \theta^+) \right) + \mu_1(x)X_1(x)
\]

\[
+ \frac{\sqrt{6}}{\lambda_0^2 + 3\mu_0^2}\sqrt{\mu} \sigma(x)w_1(x) \left( -\frac{\sqrt{6}}{\lambda_0^2} I^+(I^+ - I^-) \sin \left( 2(\lambda_0 + r_1^+)(|x| - \theta^+) \right) + O(\mu(I^+)^2|I^+ - I^-|) + O(|I^+ - I^-|^2) \right),
\]

\[
\tilde{g}_{20}(x),
\]

(90)

where

\[
|\tilde{g}_{20}(x)| \leq M\sqrt{\mu} \left( \tilde{\sigma}(x)(|A_r|_0 + |w_1|_0)(|X_1|_2 + |\hat{X}_3|_2) + (I^+)^2|I^+ - I^-| + |I^+ - I^-|^2 + e^{2\mu|x|}(|X_1|_2^2 + |\hat{X}_3|_2^2) + e^{\mu|x|}I^+(|X_1|_2 + |\hat{X}_3|_2) + \tilde{\sigma}(x)(|A_r|_0 + |w_1|_0)I^+|I^+ - I^-| \right).
\]

It is straightforward to check that for $j = 1, 2$

\[
\int_0^{|x|} (F_1(t) + F_1(-t))s^*_1[1](t)dt s_1[j](|x|)
\]

(89)
\[
\int_0^{[x]} \left( F_1(t) + F_1(-t) \right) u_1^*[1](t) dt u_1[j]([x]) \quad \leq M e^{\sqrt{\mu}|x|}(I^\pm |I^+ - I^-| + |I^+ - I^-|^2),
\]
\[
\int_0^{[x]} (h_2(t) - h_2(-t)) s_1^*[2](t) ds_1[j]([x]) + \int_0^{[x]} (h_2(t) - h_2(-t)) s_1^*[2](t) dt u_1[j]([x]) \quad \leq M e^{\sqrt{\mu}|x|}\sqrt{\mu}(I^\pm |I^+ - I^-| + |I^+ - I^-|^2),
\]
\[
\int_0^{[x]} (g_2(t) - g_2(-t)) u_1^*[2](t) dt u_1[j]([x]) \quad \leq M e^{\sqrt{\mu}|x|}(I^\pm |I^+ - I^-| + |I^+ - I^-|^2)
\quad + \|\tilde{X}_3\|_2 + (||A_e||_e + ||A_r||_0 + ||w_{1e}||_e + ||w_{1r}||_0)\|X_1\|_2
\quad + e^{2\nu|x|}(||X_1||_2^2 + \|\tilde{X}_3\|_2^2) + e^{\nu|x|}I^\pm (\|X_1\|_2 + \|\tilde{X}_3\|_2),
\]
\[
\int_0^{[x]} (g_2(t) - g_2(-t)) u_1^*[2](t) dt u_1[j]([x]) \quad \leq M e^{\sqrt{6\mu}|x|}(I^\pm |I^+ - I^-| + |I^+ - I^-|^2)
\quad + |I^+ - I^-|^2 + \|\tilde{X}_3\|_2 + (||A_e||_e + ||A_r||_0 + ||w_{1e}||_e + ||w_{1r}||_0)\|X_1\|_2
\quad + e^{(2\nu-\sqrt{\mu})|x|}(\|X_1\|^2 + \|\tilde{X}_3\|^2_2) + \frac{\sqrt{\mu}}{\sqrt{6\mu} - \nu} I^\pm (\|X_1\|_2 + \|\tilde{X}_3\|_2).
\]

where (34) and (36) are used. Applying (55), (56) and the above inequalities yields
\[
\|A(\cdot) - A(-\cdot)\|_2 + \|B(\cdot) + B(-\cdot)\|_2 \leq M \left(e^{(\sqrt{6\mu} - \nu)\tau}I^\pm |I^+ - I^-| + e^{(\sqrt{6\mu} - \nu)\tau}||X_3||_2 + e^{2\nu\tau}(\|X_1\|^2 + \|\tilde{X}_3\|^2_2) + \frac{\sqrt{\mu}}{\sqrt{6\mu} - \nu} I^\pm (\|X_1\|_2 + \|\tilde{X}_3\|_2)\right).
\]

(91)

From (79) and (80), we know the relationship between \(\tilde{X}_3(x)\) and \(\tilde{X}_3(x)\). Thus, using (69) and (71) and replacing \(\|X_3\|_2\) in the term \(e^{(\sqrt{6\mu} - \nu)\tau}||X_3||_2^2\) of (91) by the estimate \(\|\tilde{X}_3\|_2\) in (87), we have
\[
\|A(\cdot) - A(-\cdot)\|_2 + \|B(\cdot) + B(-\cdot)\|_2 \leq M \left(e^{(\sqrt{6\mu} - \nu)\tau}I^\pm |I^+ - I^-| + e^{(\sqrt{6\mu} - \nu)\tau}||X_3||_2 + e^{(\sqrt{6\mu} - \nu)\tau}(\|X_1\|^2 + \|\tilde{X}_3\|^2_2) + \frac{\sqrt{\mu}}{\sqrt{6\mu} - \nu} I^\pm (\|X_1\|^2 + \|\tilde{X}_3\|^2). \right)
\]

(92)

7.3. Existence of \(Z_r(x)\) for \(x \in [-\tau, -2]\). For \(x \in [-\tau, 0]\), we consider the space \(\mathcal{H}_2\) with the norm \(\|\cdot\|_2\) in (78) and let
\[
\mathcal{Z}(x) = (X_1(x), X_2(x), \tilde{X}_3(x), \tilde{X}_4(x))^T
\]
and \(\Xi[\mathcal{Z}](x) = (\Xi_1, \Xi_2, \Xi_3, \Xi_4)^T\mathcal{Z}(x)\) where
\[
\Xi_1[\mathcal{Z}] = A[\mathcal{Z}](|x|) - A[\mathcal{Z}](|x|), \quad \Xi_2[\mathcal{Z}] = B[\mathcal{Z}](|x|) + B[\mathcal{Z}](|x|),
\]
\[
\Xi_3[\mathcal{Z}] = \tilde{X}_3[\mathcal{Z}](x), \quad \Xi_4[\mathcal{Z}] = \tilde{X}_4[\mathcal{Z}](x).
\]
This means that (55), (56), (58) and (59) can be changed into
\[ Z = \Xi[Z], \tag{93} \]

The assumption (69) indicates that \( I^\pm|I^+ - I^- = O(\mu^{2n+1+i}) \). To control the
term \( e^{2\sqrt{6\mu}\tau} \) (see (92)), we choose
\[ \mu I^\pm|I^+ - I^-| e^{2\sqrt{6\mu}\tau} = O(1), \tag{94} \]
that is,
\[ \mu^{2n+2+i} e^{2\sqrt{6\mu}\tau} = c_0 \geq 10, \tag{95} \]
where \( c_0 \) is a constant. Thus, we have
\[ \tau = \frac{1}{2\sqrt{6\mu}} \left( \ln c_0 - (2n + 2 + i) \ln \mu \right) \to +\infty, \quad \text{as} \ \mu \to 0^+, \]
and by (38) and \( \alpha = 3/4 \)
\[ (\sqrt{6\mu} - \nu) \tau = \mu^{3/4} \tau \to 0, \quad \text{as} \ \mu \to 0^+. \]
From (89) and (92), it is easy to obtain the following lemma.

**Lemma 7.1.** Under the assumptions (69) and (95), if \( \|Z_1\|_2 + \|Z_2\|_2 \leq M_1 \) for
some positive constant \( M_1 \) and \( Z_1, Z_2 \in \mathcal{H}_2 \), then
\[ \|\Xi_1[Z_1]\|_2 + \|\Xi_2[Z_1]\|_2 \leq M \left( I^\pm|I^+ - I^-| + \mu^{1/4}(\|X_1\|_2 + \|\tilde{X}_3\|_2) \right) \]
\[ + \mu^{-(2n+2+i)/2}(\|X_1\|_2^2 + \|\tilde{X}_3\|_2^2)), \]
\[ \|\Xi_3[Z_1]\|_2 + \|\Xi_4[Z_1]\|_2 \leq M \sqrt{\mu} \left( I^\pm|I^+ - I^-| + \mu^{-1/4}(\|X_1\|_2 + \|\tilde{X}_3\|_2) \right) \]
\[ + \mu^{-(2n+2+i)/2}(\|X_1\|_2^2 + \|\tilde{X}_3\|_2^2)), \]
\[ \|\Xi[Z_1] - \Xi[Z_2]\|_2 \leq M \left( \mu^{1/4} + \mu^{-(2n+2+i)/2}(\|Z_1\|_2 + \|Z_2\|_2) \right) \|Z_1 - Z_2\|_2, \]
where \( X_1 \) and \( \tilde{X}_3 \) are the first and third components of \( Z_1 \) respectively.

Take a closed set \( \mathcal{B}_3(0) \) in \( \mathcal{H}_2 \) such that for \( Z \in \mathcal{B}_3(0) \)
\[ \|Z\|_2 \leq M \mu^{-1/4} I^\pm|I^+ - I^-| = O(\mu^{2n+3/4+i}). \tag{96} \]
Lemma 7.1 shows that \( \Xi \) is a contraction mapping on \( \mathcal{B}_3(0) \) for small \( \mu > 0 \). Therefore, (93) has a unique solution \( Z \in \mathcal{B}_3(0) \), which is smooth with respect to its
arguments and satisfies (96) together with the more accurate estimates
\[ \|A(\cdot) - A(-\cdot)\|_2 + \|B(\cdot) + B(-\cdot)\|_2 \leq M I^\pm|I^+ - I^-|, \]
\[ \|\tilde{X}_3\|_2 + \|\tilde{X}_4\|_2 \leq M^{1/4} I^\pm|I^+ - I^-|. \tag{97} \]
From the relationship between \( Z \) and \( Z_r \) (see (76), (77), (79) and (81)), it is obvi-
ous that \( Z_r(x) = Z(x) \) if \( x \) is restricted on \([-\tau, -2] \). Hence, the existence and
smoothness of \( Z_r(x) \) for \( x \in [-\tau, -2] \) are obtained. Moreover, \( Z_r(x) \) satisfies
\[ \|A_r\|_1 + \|B_r\|_1 \leq M I^\pm|I^+ - I^-|, \quad \|w_1 r\|_1 + \|w_2 r\|_1 \leq M |I^+ - I^-|, \tag{98} \]
where the norm \( \|\cdot\|_1 \) is defined in (46). Thus, (37) has a unique smooth solution
\( Z(x) = Z_r(x) + Z_r(x) \) for \( x \in [-\tau, \infty) \).

In order to smoothly extend the obtained solution \( Z(x) \) on \((-\infty, +\infty) \), we need
more details about \( A(|x|) - A(|-x|) \) and \( B(|x|) + B(|-x|) \). Using the above results,
we have the following lemma.
Lemma 7.2. Under the assumptions (69) and (95), for \( x \in [-\tau, 0] \),

\[
A(|x|) - A(-|x|) = \frac{\sqrt{6}}{\lambda_0^2} I^+(I^+ - I^-) \int_0^{|x|} \sigma'(s)s_1^*[1](s)dsu_1[1](|x|) \\
+ \frac{\sqrt{6}}{\lambda_0} I^+(I^+ - I^-) \int_0^{|x|} \sigma'(s)u_1^*[1](s)dsu_1[1](|x|) \\
+ \frac{3\sqrt{6}}{2\lambda_0^3} \sqrt{\mu}I^+(I^+ - I^-) \int_0^{|x|} \sigma(s)H_A(s)s_1^*[2](s)dsu_1[1](|x|) \\
+ \frac{3\sqrt{6}}{2\lambda_0^3} \sqrt{\mu}I^+(I^+ - I^-) \int_0^{|x|} \sigma(s)H_A(s)u_1^*[2](s)dsu_1[1](|x|) \\
- \frac{12}{\lambda_0^2} \sqrt{\mu}I^+(I^+ - I^-) \int_0^{|x|} \sigma(s)\cos^2 \left( (\lambda_0 + r_1^+) (s - \theta^+) \right) s_1^*[2](s)dsu_1[1](|x|) \\
- \frac{12}{\lambda_0^2} \sqrt{\mu}I^+(I^+ - I^-) \int_0^{|x|} \sigma(s)\cos^2 \left( (\lambda_0 + r_1^+) (s - \theta^+) \right) u_1^*[2](s)dsu_1[1](|x|) \\
+ O(\mu^{1/4}I^+I^+ - I^-|\epsilon\sqrt{\mu}|x|),
\]

and

\[
B(|x|) + B(-|x|) = -\frac{9}{4\lambda_0^2} \mu I^+(I^+ - I^-)u_1[2](|x|) + O(\mu^{5/4}I^+I^+ - I^-|\epsilon\sqrt{\mu}|x|) \\
+ O(\sqrt{\mu}I^+I^+ - I^-|\epsilon\sqrt{\mu}|x) + O(I^+I^+ - I^-). \tag{100}
\]

The proof is given in Appendix 3.

8. Existence of two-hump solutions. In previous sections, we have proved the existence of the solution \( X(x; \mu, \theta^\pm, I^\pm) \) of (13) for \( x \in [-\tau, +\infty) \), which has a hump at the origin. See (17) and (29). The reversibility and the translational invariance indicate that \( X(x - x_0; \mu, \theta^\pm, I^\pm) \) for \( x - x_0 \geq -\tau \) and \( SX(-x - x_0; \mu, \theta^\pm, I^\pm) \) for \( -x - x_0 \geq -\tau \) are also solutions of (13) with any \( x_0 \in \mathbb{R} \). To have a reversible solution with two humps for (13), we just glue these two solutions together at \( x = -x_0 \) for some large \( x_0 \in (0, \tau] \). This is equivalent to find \( x_0 \) so that the following equation

\[
(I - S)X(-x_0; \mu, \theta^\pm, I^\pm) = 0 \tag{101}
\]

holds where \( I \) is an identity mapping. If this \( x_0 \) can be found, we define

\[
X_1(x; \mu, \theta^\pm, I^\pm) = \begin{cases} 
X(x - x_0; \mu, \theta^\pm, I^\pm) & \text{for } x \geq 0, \\
SX(-x - x_0; \mu, \theta^\pm, I^\pm) & \text{for } x \leq 0.
\end{cases} \tag{102}
\]

Then \( X_1(x; \mu, \theta^\pm, I^\pm) \) is a reversible smooth solution of (13) with two humps.

The definition of the reverser \( S \) given in (16) implies that (101) is equivalent to

\[
X_1[2](-x_0; \mu, \theta^\pm, I^\pm) = 0; \quad X_1[4](-x_0; \mu, \theta^\pm, I^\pm) = 0,
\]

or

\[
H_B(-x_0) + B(-x_0) + B_p^\prime(-x_0 + \theta^+) = 0, \\
w_2(-x_0) + w_2^\prime(-x_0 + \theta^+) = 0,
\]

where the cutoff function \( \sigma(-x_0) = 1 \) is used. Since \( -x_0 \) is near negative, the decompositions of \( B(x) \) and \( w_2(x) \) yield
Thus, we have
\[- H_B(x_0) - B_d(x_0) + B(-x_0) + B(x_0) + B_p(-x_0 + \theta^+) = 0, \tag{103}\]
\[- w_{2d}(x_0) + w_2(-x_0) + w_2(x_0) + w_{2p}(-x_0 + \theta^+) = 0. \tag{104}\]

Hence, two constants are needed to solve the above two equations. We will choose these two free constants \(x_0\) and \(a_0\) defined in \((69)\).

Let
\[x_0 = \theta^+ + \frac{2\pi n_0 + \beta}{\lambda_0 + r_1}, \quad |\beta| \leq \pi, \tag{105}\]
and
\[\mu^{2n+2+\tilde{l}}e^{2\sqrt{6}t_0x_0} = c_1. \tag{106}\]

Then
\[
\ln c_1 = -\frac{4\pi \sqrt{6}i}{\lambda_0 + r_1} \left( -\frac{\lambda_0 + r_1^-}{4\pi \sqrt{6}i} \left( (2n + 2 + \tilde{l}) \ln \mu + 2\sqrt{6}i \theta^+ + \frac{2\sqrt{6}i \beta}{\lambda_0 + r_1} \right) - n_0 \right).
\]
Choose
\[n_0 = \left[ -\frac{\lambda_0 + r_1^-}{4\pi \sqrt{6}i} \left( (2n + 2 + \tilde{l}) \ln \mu + 2\sqrt{6}i \theta^+ + \frac{2\sqrt{6}i \beta}{\lambda_0 + r_1} \right) \right] + \infty, \quad \text{as} \quad \mu \to 0^+,
\]
where \([x]\) denotes the largest integer less than or equal to \(x\), which shows that \(n_0\) is a large positive integer and
\[
\ln c_1 = -\frac{4\pi \sqrt{6}i}{\lambda_0 + r_1} \left( -\frac{\lambda_0 + r_1^-}{4\pi \sqrt{6}i} \left( (2n + 2 + \tilde{l}) \ln \mu + 2\sqrt{6}i \theta^+ + \frac{2\sqrt{6}i \beta}{\lambda_0 + r_1} \right) \right)
- \left[ -\frac{\lambda_0 + r_1^-}{4\pi \sqrt{6}i} \left( (2n + 2 + \tilde{l}) \ln \mu + 2\sqrt{6}i \theta^+ + \frac{2\sqrt{6}i \beta}{\lambda_0 + r_1} \right) \right]
\to 0, \quad \text{as} \quad \mu \to 0^+.
\]

Thus, we have \(c_1 \to 1\) as \(\mu \to 0^+\), which implies that \(x_0 \in (0, \tau)\) by \((95)\). It is straightforward to check that
\[
x_0 = \frac{1}{2\sqrt{6}i} \left( \ln c_1 - (2n + 2 + \tilde{l}) \ln \mu \right), \quad e^{x_0} \leq M\mu^{-(2n+2+\tilde{l})/2},
\]
\[e^{(\sqrt{6} - \mu) x_0} = e^{\mu^{3/4} x_0} \leq M, \tag{107}\]

Finally, we are ready to solve \((103)\) and \((104)\) for \(\beta\) and \(a_0\), and have the following lemma.

**Lemma 8.1.** Suppose that \((69)\) hold and
\[\frac{3}{4} \leq \tilde{l} \leq 2n + \frac{1}{2}. \tag{108}\]
Then for small \(\mu > 0\) the equations \((103)\) and \((104)\) are equivalent to
\[\Upsilon = \Phi(\Upsilon; \mu) = \mu^{1/4} \Phi_1(\Upsilon; \mu) \tag{109}\]
where \(\Upsilon = (\beta, \tilde{a}_0)^T, \tilde{a}_0 = a_0 - \frac{3216}{c_1 t_0}, \Phi(\Upsilon; \mu)\) is differentiable with respect to its arguments, and \(\Phi_1(\Upsilon; \mu)\) and its derivatives with respect to \(\Upsilon\) are uniformly bounded for small bounded \((\Upsilon; \mu)\).
shows that the condition for \( \tilde{l} \) in (69) is satisfied. Take a closed ball \( \bar{B}_2(0) \) in \( \mathbb{R}^2 \) with a radius \( r_2 = O(\mu^{1/8}) \). Then, \( \Phi \) is a contraction mapping on \( \bar{B}_4(0) \) for small \( \mu > 0 \). The fixed point theorem yields that \( \Phi \) has a unique fixed point \( \Upsilon \) satisfying

\[
|\beta| + |\alpha_0| \leq M\mu^{1/4}.
\]

Hence, (103) and (104) hold by choosing \((x_0, a_0)\) obtained. The uniqueness of the solution for an initial value problem implies that \( X_1(x; \mu, \theta^\pm, I^\pm) \) defined in (102) is a smooth solution of the system (13). It is obvious that \( SX_1(-x; \mu, \theta^\pm, I^\pm) = X_1(x; \mu, \theta^\pm, I^\pm) \), that is, the solution \( X_1(x; \mu, \theta^\pm, I^\pm) \) is a reversible solution of (13). Clearly, \( X_1(x; \mu, \theta^\pm, I^\pm) \) exponentially approaches periodic solutions with small amplitudes at infinity and has two humps at \( x = \pm x_0 \) respectively. The distance between two humps is

\[
2x_0 = \frac{1}{\sqrt{6}\mu} \left( \ln c_1 - (2n + 2 + \tilde{l}) \ln \mu \right) = O(\mu^{-1/2} \ln \mu),
\]

which becomes very large if \( \mu \) is small. According to the relationships among the original system (1), the system (3) and the system (13), it is straightforward to see that

\[
\begin{align*}
\eta(x) &= \frac{1}{\sqrt{6}} \mu \left( H_A(x) + \sigma(x) A_p^\pm(x - \theta^\pm) + A(x) \right) \\
&\quad + \frac{2}{\lambda_0} \mu \left( \sigma(x) w_1^\pm(x - \theta^\pm) + w_1(x) \right), \\
\end{align*}
\]

\[
\begin{align*}
u(x) &= \frac{1}{\sqrt{6}} \mu \left( H_A(x) + \sigma(x) A_p^\pm(x - \theta^\pm) + A(x) \right) \\
&\quad - \frac{2}{\lambda_0} \mu \left( \sigma(x) w_1^\pm(x - \theta^\pm) + w_1(x) \right),
\end{align*}
\]

which, together with (15) and (102), yields the proof of Theorem 1.1.

**Remark 2.** With a similar idea, we may prove the existence of a generalized \( 2^m \)-hump wave solution for any positive integer \( m \).

**Proof of Lemma 8.1.** It is obvious that

\[
u_1[2](x_0) = \frac{1}{3\sqrt{6}} e^{\sqrt{6}\mu x_0} + O(x_0 e^{-\sqrt{6}\mu x_0}),
\]

\[
\begin{align*}
\sech^2 \left( \sqrt{\frac{3\mu}{2}} x_0 \right) \tanh \left( \sqrt{\frac{3\mu}{2}} x_0 \right) &= 4 e^{-\sqrt{6}\mu x_0} + O(e^{-2\sqrt{6}\mu x_0}).
\end{align*}
\]

(110)

The assumption (108) implies that \( \tilde{l} \geq 3/4 \). According to Lemma 7.2, we have

\[
B(x_0) + B(-x_0) = -\frac{9}{4\lambda_0} \mu I^+(I^+ - I-) u_1[2](x_0) + P_1
\]

\[
= -\frac{3}{4\sqrt{6}\lambda_0} \mu I^+(I^+ - I-) e^{\sqrt{6}\mu x_0} + P_2,
\]

and by (88)

\[
w_2(x_0) + w_2(-x_0) = W_2(x_0) + W_2(-x_0)
\]

\[
= -(I^+ - I^-) \sin \left( \lambda_0(x_0 - \theta^+) \right) + P_3,
\]

(112)
where
\[ |P_1| + |P_2| \leq M \left( \mu^{5/4}I^\pm |I^+ - I^-|e^{\sqrt{\mu}x_0} + I^\pm |I^- - I^+| \right) , \]
\[ |P_3| \leq M\mu^{1/4}|I^+ - I^-| . \] (113)

From (24) and (71), one obtains
\[ |B_d(x_0)| + |B_{1\tau}(-x_0 + \theta^+)| \leq M\mu^{1/4}e^{-\nu x_0} + M\sqrt{\mu}(I^\pm)^2 . \]

Therefore, the equation (103) is changed to
\[ 0 = 2\sqrt{6}\text{sech}^2\left( \frac{\sqrt{3}\mu}{\sqrt{2}}x_0 \right) \tanh\left( \frac{\sqrt{3}\mu}{\sqrt{2}}x_0 \right) - \frac{3}{4\sqrt{6}\lambda_0}\mu I^+(I^+ - I^-)e^{\sqrt{\mu}x_0} + P_4 \]
\[ = 8\sqrt{6}e^{-\sqrt{\mu}x_0} \frac{1}{192\sqrt{6}}\mu I^+(I^+ - I^-)e^{\sqrt{\mu}x_0} + P_5 , \]
where (19), (110) and the definition of \( \lambda_0 \) below (8) are used, and
\[ |P_4| + |P_5| \leq M \left( e^{-2\sqrt{\mu}x_0} + \mu^{5/4}I^\pm |I^+ - I^-|e^{\sqrt{\mu}x_0} + I^\pm |I^- - I^+| + \mu^{1/4}e^{-\nu x_0} + \sqrt{\mu}(I^\pm)^2 \right) . \]

Since \( I^+ = I_0\mu^{n+1/2} \) and \( I^+ - I^- = a_0\mu^{n+1/2+i} \) (see (69)), we have from (106)-(108)
\[ 0 = 8\sqrt{6} - \frac{1}{192\sqrt{6}}c_1I_0a_0 + e^{\sqrt{\mu}x_0}P_5 , \]
or
\[ 0 = 8\sqrt{6} - \frac{1}{192\sqrt{6}}c_1I_0a_0 + e^{\sqrt{\mu}x_0}P_5 , \] (114)

where \( \tilde{a}_0 = a_0 - \frac{9216}{c_1I_0} \), and
\[ |P_6| \leq Me^{\sqrt{\mu}x_0}|P_5| \leq M\mu^{1/4} , \quad \phi_2(\beta, \tilde{a}_0; \mu) \leq M . \]

Again Lemma 6.3, (69), (107), (112) and (113) yield
\[ |w_{2d}(x_0)| \leq Me^{-\nu x_0} \leq M\mu^{n+2+I/2} , \]
\[ |w_2(x_0) + w_2(-x_0)| \leq M|I^+ - I^-| \leq M\mu^{n+1/2+i} . \]

Thus, the equation (104) can be transformed into by (24) and (105)
\[ 0 = -I^- \sin(\beta) + P_7 , \]
or
\[ 0 = -I^- \sin(\beta) + P_7 , \] (115)

where \( P_8 = \arcsin(P_7/I^-) \) and
\[ |P_7| \leq M \left( \mu^{2n+2} + \mu^{n+2+I/2} + \mu^{n+1/2+i} \right) , \]
\[ |P_8| \leq M \left( \mu^{n+3/2} + \mu^{3/2+i/2} + \mu^{I} \right) \leq M\mu^{1/4} , \]
\[ |\phi_1(\beta, \tilde{a}_0; \mu)| \leq M . \]

It is also straightforward to check that \( \phi_1 \) and \( \phi_2 \) are differentiable with respect to their arguments and their derivatives with respect to \( \beta \) and \( \tilde{a}_0 \) are uniformly bounded for small bounded \((\beta, \tilde{a}_0, \mu)\). Hence, (114) and (115) give (109), which completes the proof. \( \square \)
9. Appendices.

9.1. Appendix 1.

Proof of Lemma 6.1. The definition of \( \mathcal{H}_1 \) (or \( \mathcal{H}_2 \)) in Section 5 shows that for \( f \in \mathcal{H}_1 \) (or \( f \in \mathcal{H}_2 \)), \( f_e(x) \) (or \( f_d(x) \)) decays with a rate \( e^{-\nu|x|} \), and \( f_{er}(x) \) (or \( f_{dr}(x) \)) is zero for \( x \geq 2 \) (we choose \( \bar{\sigma}(x) \) defined in (28) to describe this property) and the factor \( \mu^{j/2} \) will come out when \( \frac{d^j f(x)}{dx^j} \) is estimated, that is,

\[
\left| \frac{d^j f(x)}{dx^j} \right| \leq M(\mu^{j/2}e^{-\nu|x|}\|f_e\|_e + \bar{\sigma}(x)\|f_r\|_0), \quad x \geq -2,
\]

where the space \( \mathcal{H}_{r0} \) in (43) is be used.

Using the fact that

\[
|\sigma'(x)| + |\sigma''(x) - \sigma(x)| \leq M\bar{\sigma}(x) \quad \text{for} \quad x \in \mathbb{R},
\]

we have from (31)

\[
|g_{4}(x)| \leq M\mu \left( e^{-2\sqrt{\mu}|x|} + e^{-\sqrt{\mu}|x|} I^\pm + e^{-\sqrt{\mu}|x|} I^\pm \|A_e\|_e \right.
\]
\[
+ e^{-\sqrt{\mu}|x|} \left( \|w_{1e}\|_e + e^{-2\nu|x|} \|A_e\|_e^2 + e^{-2\nu|x|} \|w_{1e}\|_e + e^{-\nu|x|} I^\pm \|A_e\|_e \right)
\]
\[
+ e^{-\nu|x|} I^\pm \|w_{1e}\|_e + \bar{\sigma}(x)\|A_r\|_0 + \bar{\sigma}(x)\|w_{1r}\|_0 + \bar{\sigma}(x)(I^\pm)^2 \right),
\]

\[
|g_{4}^{(j)}(x)| \leq M\mu \left( \mu^{j/2}e^{-2\sqrt{\mu}|x|} + e^{-\sqrt{\mu}|x|} I^\pm + \mu^{j/2}e^{-\sqrt{\mu}|x|} I^\pm \|A_e\|_e \right.
\]
\[
+ \mu^{j/2}e^{-\sqrt{\mu}|x|} \left( \|w_{1e}\|_e + \mu^{j/2}e^{-2\nu|x|} \|A_e\|_e^2 + \mu^{j/2}e^{-2\nu|x|} \|w_{1e}\|_e^2 + \bar{\sigma}(x)\|A_r\|_0 + \bar{\sigma}(x)\|w_{1r}\|_0 + \bar{\sigma}(x)(I^\pm)^2 \right),
\]

for \( x \geq -2 \) and \( j = 1, \ldots, 2n \).

Since the integral in (57) is from zero to positive infinity, the plus sign in \( \mathcal{F}_3 \) and \( \mathcal{F}_4 \) is used. Write \( g_4(x) = g_{4e}(x) + g_{4r}(x) \) where \( g_{4e}(x) \) is even in \( x \) and exponentially tends to zero as \( x \to +\infty \), and \( g_{4r}(x) \) is the remainder. Here, \( g_{4e}(x) \) mainly consists of the terms including \( A_e(x) \) and \( w_{1e}(x) \).

Lemma 9.1. For an even (or odd) function \( f \in C^{2n}(-\infty, +\infty) \), if \( f(x) \) and its derivatives \( f^{(j)}(x) \), \( j = 1, 2, \ldots, 2n \) exponentially decay to zero as \( x \to +\infty \), then

\[
\int_0^\infty f(t) \cos(\lambda_0 t) dt = \frac{(-1)^n}{\lambda_0^{2n}} \int_0^\infty f^{(2n)}(t) \cos(\lambda_0 t) dt
\]
\[
\left( \text{or} \int_0^\infty f(t) \sin(\lambda_0 t) dt = \frac{(-1)^n}{\lambda_0^{2n}} \int_0^\infty f^{(2n)}(t) \sin(\lambda_0 t) dt \right).
\]

The proof is straightforward with integration by parts. Using this lemma, it is obtained that

\[
\left| \int_0^\infty g_{4e}(x) \cos(\lambda_0 x) dx \right| = \left| \frac{(-1)^n}{\lambda_0^{2n}} \int_0^\infty g_{4e}^{(2n)}(x) \cos(\lambda_0 x) dx \right|
\]
\[
\leq M\mu^{n+1/2} \left( \mu^{k/2} + \|A_e\|_e + \|w_{1e}\|_e \right) \quad (117)
\]
for any positive integer \( k \) where the term \( \mu^{k/2} \) only corresponds to the term \( H_2(x) \) in \( g_4(x) \). Similarly, with \( \left| \int_0^\infty \tilde{\sigma}(x)dx \right| \leq M \) we have
\[
\left| \int_0^\infty g_4(x) \cos(\lambda_0 x)dx \right| \leq M \sqrt{\mu} \left( I^+ + \sqrt{\mu} \| A_r \|_0 + \sqrt{\mu} \| w_{1r} \|_0 \right),
\]
which yields, together with (117)
\[
\left| \int_0^\infty g_4(x) \cos(\lambda_0 x)dx \right| \leq M \sqrt{\mu} \left( \mu^{k/2+n} + \mu^n \| A_e \|_0 + \mu^n \| w_{1e} \|_0 + I^+ \right)
+ \sqrt{\mu} \| A_r \|_0 + \sqrt{\mu} \| w_{1r} \|_0).
\] (118)

By (24) and (31), we have
\[
\mathcal{F}_3(x) = -\sigma'(x)I^+ \cos ((\lambda_0 + r_1^+)(x-\theta^+)) + |\sigma'(x)|O(\mu(I^+)^2),
\]
\[
\mathcal{F}_4(x) = -\sigma'(x)I^+ \sin ((\lambda_0 + r_1^+)(x-\theta^+)) + g_4(x) + |\sigma'(x)|O(\mu(I^+)^2).
\] (119)

Using the definition of \( r_1^+ \) in (25), it is deduced that
\[
I^+ \int_0^\infty \left( -\sigma'(t) \cos ((\lambda_0 + r_1^+)(t-\theta^+)) \right) \sin(\lambda_0 t)
+ \sigma'(t) \sin ((\lambda_0 + r_1^+)(t-\theta^+)) \cos(\lambda_0 t) dt
= I^+ \int_0^\infty \sigma'(t) \sin (-\lambda_0 \theta^+ + r_1^+(t-\theta^+)) dt
= I^+ \int_0^\infty \sigma'(t) \sin(-\lambda_0 \theta^+) dt + O(\mu(I^+)^3)
= I^+ \sin(-\lambda_0 \theta^+) + O(\mu(I^+)^3). \] (120)

Hence, by (118)-(120), the equation (57) is changed into
\[
\theta^+ = \tilde{\Phi}(\theta^+, \mu, I^+) = (I^+)^{-1} \Phi_0(\theta^+, \mu, I^+) = \mu^{1/8} \Phi(\theta^+, \mu, I^+),
\]
where
\[
|\Phi_0(\theta^+, \mu, I^+)| \leq M \sqrt{\mu} \left( \mu^{k/2+n} + \mu^n \| A_e \|_0 + \mu^n \| w_{1e} \|_0 + I^+ \right)
+ \sqrt{\mu} \| A_r \|_0 + \sqrt{\mu} \| w_{1r} \|_0)
\leq M \mu^{n+5/8}.
\]

Similarly, one can check that \( \Phi(\theta^+, \mu, I^+) \) is differentiable with respect to its arguments, and its derivative with respect to \( \theta^+ \) is uniformly bounded for small bounded \( (\theta^+, \mu, I^+) \). The proof is completed. \( \square \)

9.2. Appendix 2.

Proof of Lemma 6.2. Since we consider the case \( x \geq -2 \), the space \( \mathcal{H}^0 \) with the norm \( \| \cdot \|_0 \) in (49) for \( Z_r(x) \) will be used. By (24), (25) and (31), using a similar method for (116), we have for \( x \geq -2 \) and any positive integer \( k \),
\[
|\mathcal{F}_1(x)| + |\mathcal{F}_1^{(k)}(x)| \leq M \tilde{\sigma}(x)(I^\pm)^2, \quad |\mathcal{F}_3(x)| + |\mathcal{F}_3^{(k)}(x)| \leq M \tilde{\sigma}(x)I^\pm, \quad (121)
|\mathcal{F}_2(x)| \leq M \sqrt{\mu} \left( \tilde{\sigma}(x)(I^\pm)^2 + \tilde{\sigma}(x) \| A_r \|_0 + \tilde{\sigma}(x) \| w_{1r} \|_0 + e^{-\sqrt{\mu}|x|} I^\pm \right)
\]
\[ + e^{-((\sqrt{\theta}+\nu)|\xi|)w_{1e}e} + e^{-2\nu|\xi|}||A_{e}\|^{2} + e^{-|\nu|}||w_{1e}||^{2} \]
\[ + e^{-\nu|\xi|}I_{\pm}||A_{e}|| + e^{-\nu|\xi|}I_{\pm}||w_{1e}||^{2} \),
\[ (122) \]
\[ |F_{2}(x)| \leq M\sqrt{\mu}\left(\bar{\sigma}(x)(I)^{2} + \bar{\sigma}(x)||A_{e}||_{0} + \bar{\sigma}(x)||w_{1r}||_{0} + e^{-\sqrt{\theta}|\xi|}I_{\pm} \]
\[ + \mu^{k/2}e^{-((\sqrt{\theta}+\nu)|\xi|)w_{1e}e} + \mu^{k/2}e^{-2\nu|\xi|}||w_{1e}||^{2} \]
\[ + \mu^{k/2}e^{-2\nu|\xi|}||A_{e}||^{2} + e^{-\nu|\xi|}I_{\pm}||A_{e}|| + e^{-\nu|\xi|}I_{\pm}||w_{1e}||^{2} \),
\[ (123) \]
\[ |F_{4}(x)| \leq M\mu\left(\mu^{-1}\bar{\sigma}(x)I_{\pm} + e^{-2\nu|\xi|} + e^{-\sqrt{\theta}|\xi|}I_{\pm} + e^{-((\sqrt{\theta}+\nu)|\xi|)w_{1e}e} \]
\[ + e^{-\nu|\xi|}I_{\pm}||A_{e}|| + e^{-\nu|\xi|}I_{\pm}||w_{1e}||_{0} + \bar{\sigma}(x)||A_{e}||_{0} + \bar{\sigma}(x)||w_{1r}||_{0} \),
\[ (124) \]
\[ |F_{4}^{(k)}(x)| \leq M\mu\left(\mu^{-1}\bar{\sigma}(x)I_{\pm} + e^{-2\nu|\xi|} + e^{-\sqrt{\theta}|\xi|}I_{\pm} \]
\[ + \mu^{k/2}e^{-((\sqrt{\theta}+\nu)|\xi|)w_{1e}e} + \mu^{k/2}e^{-2\nu|\xi|}||w_{1e}||^{2} \]
\[ + \mu^{k/2}e^{-2\nu|\xi|}||A_{e}||^{2} + e^{-\nu|\xi|}I_{\pm}||A_{e}|| + e^{-\nu|\xi|}I_{\pm}||w_{1e}||_{0} \),
\[ (125) \]
where \( \bar{\sigma}(x) \) is defined in (28). Now we divide the proof into four steps.

**Step 1.** Estimates for \( E_{1e}(x) \) and \( E_{2d}(x) \).

(40) and (42) show that we should estimate \( E_{1}(x), E_{2}(x), E_{1r0}(x) \) and \( E_{2d0}(x) \) and their derivatives respectively.

For \( x \geq -2 \) and \( m = 1, 2 \), by (34), (36), (37) and (60), it is obtained that

\[ |E_{m}(x)|e^{\nu|\xi|} \leq M \int_{0}^{\sqrt{\theta}|\xi|} \left( |F_{1}(s)| + |F_{2}(s)| \right)e^{\sqrt{\theta}s}ds e^{-\sqrt{\theta}x} \]
\[ + M \int_{\sqrt{\theta}|\xi|}^{\infty} \left( |F_{1}(s)| + |F_{2}(s)| \right)e^{-\sqrt{\theta}s}ds e^{\sqrt{\theta}x} \]
\[ \leq M \left( \frac{\sqrt{\theta}}{\sqrt{6\mu} - \nu} I_{\pm} ||A_{e}||_{0} + \sqrt{\theta} ||w_{1r}||_{0} + ||A_{e}||^{2} + ||w_{1e}||^{2} \right) \]
\[ + \frac{\sqrt{\theta}}{\sqrt{6\mu} - \nu} I_{\pm} ||A_{e}|| + \frac{\sqrt{\theta}}{\sqrt{6\mu} - \nu} I_{\pm} ||w_{1e}||^{2} \),
\[ (126) \]
and

\[ \mu^{-k/2}|E_{m}^{(k)}(x)|e^{\nu|\xi|} \]
\[ \leq M\mu^{-k/2}e^{\nu|\xi|} \left( \int_{0}^{\sqrt{\theta}|\xi|} \left( F_{1}(s)s_{1}^{[1]}(s) + F_{2}(s)s_{1}^{[2]}(s) \right)ds \right) \]
\[ + \sum_{p=0}^{k-1} \left( F_{1}(x)s_{1}^{[1]}(x) + F_{2}(x)s_{1}^{[2]}(x) \right)^{(k-1-p)} (s_{1}[m]^{(p)}(x) \]
\[ + \sum_{p=0}^{k-1} \left( F_{1}(x)s_{1}^{[1]}(x) + F_{2}(x)s_{1}^{[2]}(x) \right)^{(k-1-p)} (s_{1}[m]^{(p)}(x) \]
Thus, using (51), (52) and (128), we have for

\[ x \]

where the even function will disappear in

\[ \theta \]

we have that for

\[ \theta \]

appears together with the periodic solution

\[ \theta \]

that the (2\( m \))th derivative of any smooth even (odd) function is zero, such that no term including \( I^{\pm} \) arises. Using the fact that the \( (2k + 1) \)-th (or \( 2k \)-th) derivative of any smooth even (odd) function is zero, we have for \( m = 0, \cdots, n - 1 \) and \( j = 0, 1, \cdots, n \),

\[ |\mathcal{E}^{(2m+1)}_{1}(0)| + |\mathcal{E}^{(2j)}_{2}(0)| \leq M \sqrt{\|A_{r}\|_{0} + \|w_{1r}\|_{0}}, \tag{128} \]

which yield that

\[ \mu^{-k/2}e^{\nu|x|}(|\mathcal{E}^{(k)}_{1}(0)| + |\mathcal{E}^{(k)}_{2}(0)|) \leq M \mu^{1/2-k/2}(\|A_{r}\|_{0} + \|w_{1r}\|_{0}), \tag{129} \]

where \( k = 0, 1, \cdots, 2n \). Thus, by the definition of \( \nu \) in (38) and the assumption (60), together with (126)-(129), we obtain that for \( x \geq -2 \),

\[ \|\mathcal{E}_{1}\| + \|\mathcal{E}_{2}\|_{d} \leq M \left( \mu^{1/2-\alpha-n} I^{\pm} + \mu^{1/2-\alpha-n} I^{\pm} \right) + \|A_{r}\|_{0} + \|w_{1r}\|_{0} \leq M \mu^{1/2-n}(\|A_{r}\|_{0} + \|w_{1r}\|_{0}) \leq M \mu^{1/2-n}(\|A_{r}\|_{0} + \|w_{1r}\|_{0}). \tag{130} \]

Before we can use the the fixed point theorem, we have to replace the term \( \|w_{1r}\|_{c} \) in the above inequality with the estimate of \( \mathcal{E}_{3e} \).

**Step 2.** Estimates for \( \mathcal{E}_{1r}(x) \) and \( \mathcal{E}_{2r}(x) \) for \( x \in [-2, 2] \).

According to (51) and (52), we first look at \( \mathcal{E}_{1}(|x|) - \mathcal{E}_{1}(-|x|) \) and \( \mathcal{E}_{2}(|x|) + \mathcal{E}_{2}(-|x|) \) given by (55) and (56). From Lemma 3.1 and (31), it is obtained that for \( x \in [-2, 2] \) and \( j = 0, 1, \cdots, 2n \),

\[ \left| \left( \mathcal{F}_{1}(|x|) + \mathcal{F}_{1}(-|x|) \right) \right|^{(j)} \leq M I^{\pm} I^{+} - I^{-}, \]

\[ \left| \left( \mathcal{F}_{2}(|x|) - \mathcal{F}_{2}(-|x|) \right) \right|^{(j)} \leq M \sqrt{\mu} \left( I^{+} - I^{-} \right) + \|A_{r}\|_{0} + \|w_{1r}\|_{0}, \]

where the even function will disappear in \( \left( \mathcal{F}_{2}(|x|) - \mathcal{F}_{2}(-|x|) \right) \). By (55) and (56), we have that for \( x \in [-2, 2] \)

\[ \|\mathcal{E}_{1}(\cdot) + \mathcal{E}_{1}(\cdot)\|_{0} + \|\mathcal{E}_{2}(\cdot) + \mathcal{E}_{2}(\cdot)\|_{0} \leq M \sqrt{\mu} \left( I^{+} - I^{-} \right) + \|A_{r}\|_{0} + \|w_{1r}\|_{0}. \]

Thus, using (51), (52) and (128), we have for \( x \in [-2, 2] \)

\[ \|\mathcal{E}_{1r}\|_{0} + \|\mathcal{E}_{2r}\|_{0} \leq M \sqrt{\mu} \left( I^{+} - I^{-} \right) + \|A_{r}\|_{0} + \|w_{1r}\|_{0}. \]
which gives the inequality (65).

**Step 3.** Estimates for $E_{3e}(x)$ and $E_{4d}(x)$.

From (37), it is easy to show that

$$E_3(x) = -\int_{x}^{\infty} F_3(t) \cos(\lambda_0(x-t)) - F_4(t) \sin(\lambda_0(x-t)) dt$$

$$= \int_{0}^{\infty} F_3(x-\tau) \cos(\lambda_0 \tau) - F_4(x-\tau) \sin(\lambda_0 \tau) dt,$$

$$E_4(x) = -\int_{x}^{\infty} F_3(t) \cos(\lambda_0(x-t)) + F_4(t) \cos(\lambda_0(x-t)) dt,$$

$$= \int_{0}^{\infty} F_3(x-\tau) \sin(\lambda_0 \tau) + F_4(x-\tau) \cos(\lambda_0 \tau) dt,$$

and then for $k = 1, \cdots, 2n$

$$E_{3i}^{(k)}(x) = -\int_{x}^{\infty} F_{3i}^{(k)}(t) \cos(\lambda_0(x-t)) - F_{4i}^{(k)}(t) \sin(\lambda_0(x-t)) dt,$$

$$E_{4i}^{(k)}(x) = -\int_{x}^{\infty} F_{3i}^{(k)}(t) \sin(\lambda_0(x-t)) + F_{4i}^{(k)}(t) \cos(\lambda_0(x-t)) dt. \quad \text{(131)}$$

Like what we did in Step 1, by (121), (124) and (125), we have for $x \geq -2$ and $k = 1, \cdots, 2n$,

$$|E_{3i}(x)| e^{\rho|x|} \leq M \left( I^\pm + \sqrt{\mu} + \sqrt{\mu} \|A_c\|_e + \sqrt{\mu} \|w_{1e}\|_e + \mu \|A_r\|_0 + \mu \|w_{1r}\|_0 \right), \quad \text{(133)}$$

$$\mu^{-k/2} |E_{3i}^{(k)}(x)| e^{\rho|x|} \leq M \sqrt{\mu} \left( \mu^{-1/2-k/2} I^\pm + 1 + \|A_c\|_e + \|w_{1e}\|_e \right.$$

$$\left. + \mu^{1/2-k/2} \|A_r\|_0 + \mu^{1/2-k/2} \|w_{1r}\|_0 \right). \quad \text{(134)}$$

Now we focus on the derivatives of the functions $E_3(x)$ and $E_4(x)$ at $x = 0$. Using the properties of the cutoff function $\sigma(x)$ and integration by parts together with (31) and (57), it is straightforward to obtain

$$\int_{0}^{\infty} F_{3i}^{(2j+1)}(t) \cos(\lambda_0 t) + F_{4i}^{(2j+1)}(t) \sin(\lambda_0 t) dt$$

$$= (-1)^j \lambda_0^{2j+1} \int_{0}^{\infty} F_3(t) \sin(\lambda_0 t) - F_4(t) \cos(\lambda_0 t) dt$$

$$= (-1)^j \lambda_0^{2j+1} \int_{0}^{\infty} g_4(t) \cos(\lambda_0 t) dt$$

with $j = 0, 1, \cdots, n-1$, which yields by (131)

$$E_{3i}^{(2j+1)}(0) = (-1)^j \lambda_0^{2j+1} \int_{0}^{\infty} g_4(t) \cos(\lambda_0 t) dt - \int_{0}^{\infty} g_{4i}^{(2j+1)}(t) \sin(\lambda_0 t) dt.$$ 

Thus, according to (118) and Lemma 9.1

$$|E_{3i}^{(2j+1)}(0)| \leq M \left( \int_{0}^{\infty} g_4(t) \cos(\lambda_0 t) dt \right) + \int_{0}^{\infty} g_{4i}^{(2j+1)}(t) \sin(\lambda_0 t) dt$$

$$\leq M \sqrt{\mu} \left( \mu^{l/2} + I^\pm + \mu^l \|A_c\|_e + \mu^n \|w_{1e}\|_e + \sqrt{\mu} \|A_r\|_0 + \sqrt{\mu} \|w_{1r}\|_0 \right),$$

where $l$ is any positive integer, which gives

$$|E_{3e0}(x)| \leq M \sqrt{\mu} \left( \mu^{l/2} + I^\pm + \mu^l \|A_c\|_e + \mu^n \|w_{1e}\|_e$$

$$+ \sqrt{\mu} \|A_r\|_0 + \sqrt{\mu} \|w_{1r}\|_0 \right) \quad \text{(135)}$$
\begin{equation}
\sqrt{\mu} \| A_r \|_0 + \sqrt{\mu} \| w_{1r} \|_0 \). \tag{135}
\end{equation}

Hence, with (133) and (134),
\begin{equation}
\| E_{3e} \|_e \leq M \sqrt{\mu} \left( 1 + \mu^{-1/2} | I^\pm + \| A_e \|_e + \| w_{1e} \|_e \\
+ \mu^{1/2-\eta} \| A_r \|_0 + \mu^{1/2-\eta} \| w_{1r} \|_0 \right). \tag{136}
\end{equation}

Similarly, it is obtained that
\begin{equation}
\| E_{4d} \|_d \leq M \sqrt{\mu} \left( 1 + \mu^{-1/2} | I^\pm + \| A_e \|_e + \| w_{1e} \|_e \\
+ \mu^{1/2-\eta} \| A_r \|_0 + \mu^{1/2-\eta} \| w_{1r} \|_0 \right). \tag{137}
\end{equation}

Therefore, the inequality (62) is proved. Plugging (136) into (130) with replacing \| w_{1e} \|_e by \| E_{3e} \|_e gives the inequality (61).

**Step 4.** Estimates for \( E_{3r} (x) \) and \( E_{4r} (x) \) for \( x \in [-2, 2] \).

For \( x \in [-2, 2] \) and \( j = 0, 1, \cdots, 2n \), it is similarly obtained that by (31)
\begin{align*}
\left| (F_3(x) + F_3(-|x|))^{(j)} \right| & \leq M | I^+ - I^- |, \\
\left| (F_4(x) - F_4(-|x|))^{(j)} \right| & \leq M \mu \left( |I^+ - I^-| + \| A_r \|_0 + \| w_{1r} \|_0 \right),
\end{align*}
from (58), which yields that
\begin{equation}
\| E_3 (\cdot) - E_3 (-\cdot) \|_0 \leq M \mu \left( |I^+ - I^-| + \| A_r \|_0 + \| w_{1r} \|_0 \right).
\end{equation}

Thus, we have due to (135)
\begin{equation}
\| E_{3r} \|_0 \leq M \sqrt{\mu} \left( |I^\pm + \mu^{-1/2} | I^+ - I^- | + \mu^n (\| A_e \|_e + \| w_{1e} \|_e) \\
+ \sqrt{\mu} \| A_r \|_0 + \sqrt{\mu} \| w_{1r} \|_0 \right), \tag{138}
\end{equation}
where the fact that \( l \) is arbitrary is used. Similarly, we have
\begin{equation}
\| E_{4r} \|_0 \leq M \sqrt{\mu} \left( |I^\pm + \mu^{-1/2} | I^+ - I^- | + \mu^n (\| A_e \|_e + \| w_{1e} \|_e) \\
+ \sqrt{\mu} \| A_r \|_0 + \sqrt{\mu} \| w_{1r} \|_0 \right). \tag{139}
\end{equation}

Substituting (61) and (62) into (138) and (139) by replacing \| A_e \|_e and \| w_{1e} \|_e with \| E_{1e} \|_e, \| E_{3e} \|_e respectively yields the inequality (66).

The rest of the inequalities can be similarly obtained. The proof is completed. \( \Box \)

**9.3. Appendix 3.**

**Proof of Lemma 7.2.** Since the existence of \( Z(x) \) for \( x \in [-\tau, \infty) \) has been proved, we will use the following results
\begin{align*}
A(|x|) - A(-|x|) = A(|x|) - A(-|x|), \\
B(|x|) + B(-|x|) = B(|x|) + B(-|x|), \quad \tilde{X}_k (x) = \tilde{X}_k (x)
\end{align*}
for \( k = 3, 4 \).

We first pay our attention on \( A(|x|) - A(-|x|) \) which will be substituted into \( B(|x|) + B(-|x|) \). (55), (90), (97) and Lemmas 3.1 and 6.3 yield
\begin{equation}
A(|x|) - A(-|x|) = \int_0^{\infty} \left( (F_1(s) + F_1(-s)) s^k [1](s)
\end{equation}
\[
\begin{align*}
&+ \int_0^{|x|} \left( -\sigma'(s)(A^+_p(s - \theta^+) - A^-_p(-s + \theta^+))u^*_1[1](s) \\
&\quad - \sigma'(s)(B^+_p(s - \theta^+) + B^-_p(-s + \theta^+))u^*_1[2](s) \\
&\quad + \left( -\frac{3}{2} \sqrt{\mu\sigma(s)}H_A(s) \right) \left( -\sqrt{\frac{6}{\lambda^0}} I^+(I^+ - I^-) \\
&\quad - \frac{3\sqrt{6}}{2\lambda^0 + 3\mu\lambda^0} \mu I^+(I^+ - I^-) \cos \left( 2(\lambda^0 + r^+_1)(s - \theta^+) \right) \right) \\
&\quad - \frac{3}{2} \sqrt{\mu A_c(s)} X_1(s) - \frac{3}{2} \sqrt{\mu\sigma(s)}A_c(s) \left( -\sqrt{\frac{6}{\lambda^0}} I^+(I^+ - I^-) \\
&\quad - \frac{3\sqrt{6}}{2\lambda^0 + 3\mu\lambda^0} \mu I^+(I^+ - I^-) \cos \left( 2(\lambda^0 + r^+_1)(s - \theta^+) \right) \right) \\
&\quad + \frac{12}{\lambda^0} \sqrt{\mu w_{1c}(s)} \tilde{X}_3(s) - \frac{12}{\lambda^0} \sqrt{\mu I^+(I^+ - I^-)\sigma(s)\cos^2 \left( (\lambda^0 + r^+_1)(s - \theta^+) \right) } \\
&\quad + \frac{\sqrt{6}}{\lambda^0} \sqrt{\mu H_A(s)} \tilde{X}_3(s) + \frac{\sqrt{6}}{\lambda^0} \sqrt{\mu A_c(s)} \tilde{X}_3(s) + \frac{\sqrt{6}}{\lambda^0} \sqrt{\mu \omega_{1c}(s)} X_1(s) \\
&\quad + \frac{\sqrt{6}}{\lambda^0} \sqrt{\mu\sigma(s)} w_{1c}(s) \left( -\sqrt{\frac{6}{\lambda^0}} I^+(I^+ - I^-) \\
&\quad - \frac{3\sqrt{6}}{2\lambda^0 + 3\mu\lambda^0} \mu I^+(I^+ - I^-) \cos \left( 2(\lambda^0 + r^+_1)(s - \theta^+) \right) \right) \\
&\quad + \tilde{g}_{20}(s) u^*_1[2](s) \right) ds u[1](|x|)
\end{align*}
\]

which gives (99).

Now we look for the dominant term of \( B(|x|) + B(-|x|) \). From (56) and (90), together with integration by parts if needed, one has

\[
B(|x|) + B(-|x|) = \int_0^{|x|} \left( (F_1(s) + F_1(-s))s^*_1[1](s)
\right)
\]

\[ \theta(u) \]
\[
\begin{align*}
&+ (\mathcal{F}_2(s) - \mathcal{F}_2(-s))s_1^2[2](t)\big|ds_1[2](|x|) \\
&+ \int_0^{|x|} \left( -\sigma'(s)(A^+_p(s - \theta^+) - A^-_p(-s + \theta^+))u^*_1[1](s) \\
&- \sigma'(s)(B^+_p(s - \theta^+) + B^-_p(-s + \theta^+))u^*_1[2](s) \\
&+ \left( \frac{3\sqrt{6}}{2\lambda_0^4 + 3\mu\lambda_0^2}\mu I^+(I^+ - I^-) \cos(2(\lambda_0 + r^+_1)(s - \theta^+)) \right) \\
&- \frac{12}{\lambda_0^4}\sqrt{\mu}w_{1e}(s)\tilde{X}_3(s) - \frac{12}{\lambda_0^4}\sqrt{\mu}I^+(I^+ - I^-)\sigma(s)\cos^2(2(\lambda_0 + r^+_1)(s - \theta^+)) \\
&+ \frac{\sqrt{6}}{\lambda_0^4}\sqrt{\mu}H_A(s)\tilde{X}_3(s) + \frac{\sqrt{6}}{\lambda_0^4}\sqrt{\mu}A_e(s)\tilde{X}_3(s) + \frac{\sqrt{6}}{\lambda_0^4}\sqrt{\mu}\omega_{1e}(s)X_1(s) \\
&- \frac{6}{\lambda_0^4}\sqrt{\mu}I^+(I^+ - I^-)\sigma(s)w_{1e}(s)u^*_1[2](s)\big|dsu_1[2](|x|) \\
&+ O(\mu^{5/4}\lambda^+|I^+ - I^-|e^{\sqrt{\mu}|x|}) + O(|I^+ - I^-|^2e^{\sqrt{\mu}|x|}) \\
&+ O(I^+|I^+ - I^-|). 
\end{align*}
\]

In the following, we analyze the right side of (141) step by step.

**Step 1.** Terms without \(X_1(s)\) and \(\tilde{X}_3(s)\).
Using Lemma 3.1 and the fact that
\[ u_1^*[1](0) = \frac{-3\sqrt{3}}{8\sqrt{2}}, \quad u_1^*[2](0) = 0, \quad \int_0^\infty u_1^*[2](s)ds = \frac{3}{8\sqrt{2}}, \]
\[ \int_0^\infty H_A(s)u_1^*[2](s)ds = \frac{3\sqrt{3}}{4\sqrt{2}}, \]
one can check that
\[
\int_0^{|x|} -\sigma'(s) \left( A_p^+(s - \theta^+) - A_p^-(s + \theta^+) \right) u^*_1[1](s)ds \\
+ \int_0^{|x|} \frac{3\sqrt{6}}{2\lambda_0} \sqrt{\mu I^+(I^+-I^-)} \sigma(s)H_A(s)u^*_1[2](s)ds \\
= \frac{\sqrt{6}}{\lambda_0^2} I^+(I^+-I^-) \int_0^{|x|} \sigma'(s)u_1^*[1](s)ds \\
+ \frac{3\sqrt{6}}{2\lambda_0} \sqrt{\mu I^+(I^+-I^-)} \int_0^{|x|} \sigma(s)H_A(s)u^*_1[2](s)ds \\
+ \frac{3\sqrt{6}}{2\lambda_0^3 + 3\lambda_0^2 \mu} \mu I^+(I^+-I^-) \int_0^{|x|} \sigma'(s)u_1^*[1](s) \cos \left( 2(\lambda_0 + r_t^+) (s - \theta^+) \right) ds \\
+ O(\mu(I^+)^2 |I^+-I^-|) + O(|I^+-I^-|^2) \\
= \frac{\sqrt{6}}{\lambda_0^2} I^+(I^+-I^-)\sigma(x)u^*_1[|x|] - \frac{\sqrt{6}}{\lambda_0^2} I^+(I^+-I^-) \int_0^{|x|} \sigma(s)(u^*_1[1])'(s)ds \\
+ \frac{3\sqrt{6}}{2\lambda_0} \sqrt{\mu I^+(I^+-I^-)} \int_0^\infty H_A(s)u_1^*[2](s)ds \\
- \frac{3\sqrt{6}}{2\lambda_0^2} \sqrt{\mu I^+(I^+-I^-)} \int_0^{|x|} H_A(s)u^*_1[2](s)ds \\
- \frac{3\sqrt{6}}{2\lambda_0^2} \sqrt{\mu I^+(I^+-I^-)} \int_0^{|x|} (1 - \sigma(s))H_A(s)u^*_1[2](s)ds \\
+ \frac{3\sqrt{6}}{2\lambda_0^2 + 3\lambda_0^2 \mu} \mu I^+(I^+-I^-) \int_0^{|x|} \sigma'(s)u_1^*[1](s) \cos \left( 2(\lambda_0 + r_t^+) (s - \theta^+) \right) ds \\
+ O(\mu(I^+)^2 |I^+-I^-|) + O(|I^+-I^-|^2) \\
= -\frac{\sqrt{6}}{\lambda_0^2} I^+(I^+-I^-) \int_0^\infty (u_1^*[1])'(s)ds \\
+ \frac{3\sqrt{6}}{2\lambda_0^2} \sqrt{\mu I^+(I^+-I^-)} \int_0^\infty H_A(s)u_1^*[2](s)ds \\
+ \frac{\sqrt{6}}{\lambda_0^2} I^+(I^+-I^-) \int_0^{|x|} (1 - \sigma(s))(u_1^*[1])'(s)ds \\
- \frac{3\sqrt{6}}{2\lambda_0^2} \sqrt{\mu I^+(I^+-I^-)} \int_0^{|x|} (1 - \sigma(s))H_A(s)u^*_1[2](s)ds \\
+ \frac{3\sqrt{6}}{2\lambda_0^2 + 3\lambda_0^2 \mu} \mu I^+(I^+-I^-) \int_0^{|x|} \sigma'(s)u_1^*[1](s) \cos \left( 2(\lambda_0 + r_t^+) (s - \theta^+) \right) ds \\
+ O(e^{-\sqrt{6} \mu |x|} |I^+-I^-|) + O(\mu(I^+)^2 |I^+-I^-|) + O(|I^+-I^-|^2)
\[
\begin{align*}
\int_0^{|x|} & \frac{\sqrt{6}}{\lambda_0^2} I^+(I^+ - I^-) u_1^*[1](s) + \frac{3\sqrt{6}}{2\lambda_0^2} I^+(I^+ - I^-) \frac{3\sqrt{3}}{4\sqrt{2}} \\
& + \frac{\sqrt{6}}{\lambda_0^2} I^+(I^+ - I^-) \int_0^{[x]} (1 - \sigma(s))(u_1^*[1])'(s)ds \\
& - \frac{3\sqrt{6}}{2\lambda_0^2} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{[x]} (1 - \sigma(s)) H_A(s) u_1^*[2](s)ds \\
& + \frac{3\sqrt{6}}{2\lambda_0^2 + 3\lambda_0^2 \mu} I^+(I^+ - I^-) \int_0^{[x]} \sigma'(s) u_1^*[1](s) \cos\left(2(\lambda_0 + r_1^+)(s - \theta^+)\right)ds \\
& + O(e^{-\sqrt{6\mu}|x||I^+ - I^-|}) + O(\mu(I^+)^2|I^+ - I^-|) + O(|I^+ - I^-|^2) \\
& = \frac{9}{4\lambda_0^2} I^+(I^+ - I^-) + \frac{6}{\lambda_0^2} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{[x]} (1 - \sigma(s))(u_1^*[1])'(s)ds \\
& - \frac{3\sqrt{6}}{2\lambda_0^2} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{[x]} (1 - \sigma(s)) H_A(s) u_1^*[2](s)ds \\
& + \frac{3\sqrt{6}}{2\lambda_0^2 + 3\lambda_0^2 \mu} I^+(I^+ - I^-) \int_0^{[x]} \sigma'(s) u_1^*[1](s) \cos\left(2(\lambda_0 + r_1^+)(s - \theta^+)\right)ds \\
& + O(e^{-\sqrt{6\mu}|x||I^+ - I^-|}) + O(\mu(I^+)^2|I^+ - I^-|) + O(|I^+ - I^-|^2),
\end{align*}
\]

and
\[
\int_0^{[x]} \frac{12}{\lambda_0^2} \sqrt{\mu} I^+(I^+ - I^-) \sigma(s) \cos^2\left((\lambda_0 + r_1^+)(s - \theta^+)\right) u_1^*[2](s)ds
\]
\[
= -\frac{6}{\lambda_0^2} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{[x]} \sigma(s) u_1^*[2](s)ds \\
- \frac{6}{\lambda_0^2} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{[x]} \sigma(s) \cos\left(2(\lambda_0 + r_1^+)(s - \theta^+)\right) u_1^*[2](s)ds \\
= -\frac{6}{\lambda_0^2} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{\infty} u_1^*[2](s)ds + \frac{6}{\lambda_0^2} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{[x]} u_1^*[2](s)ds \\
+ \frac{6}{\lambda_0^2} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{[x]} (1 - \sigma(s)) u_1^*[2](s)ds \\
- \frac{3}{\lambda_0^2 (\lambda_0 + r_1^+)} \sqrt{\mu} I^+(I^+ - I^-) \sigma(|x|) \sin\left(2(\lambda_0 + r_1^+)(|x| - \theta^+)\right) u_1^*[2](|x|) \\
+ \frac{3}{\lambda_0^2 (\lambda_0 + r_1^+)} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{[x]} \sin\left(2(\lambda_0 + r_1^+)(s - \theta^+)\right) (\sigma(s) u_1^*[2](s))'ds \\
= -\frac{9}{4\lambda_0^2} I^+(I^+ - I^-) + \frac{6}{\lambda_0^2} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{[x]} (1 - \sigma(s)) u_1^*[2](s)ds \\
+ \frac{3}{\lambda_0^2 (\lambda_0 + r_1^+)} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{[x]} \sin\left(2(\lambda_0 + r_1^+)(s - \theta^+)\right) (\sigma(s) u_1^*[2](s))'ds \\
+ O(e^{-\sqrt{6\mu}|x||I^+ - I^-|}),
\]

which implies
\[
\int_0^{[x]} -\sigma'(s) \left(A_p^+(s - \theta^+) - A_p^-(s + \theta^+)\right) u_1^*[1](s)ds
\]
\[
+ \int_0^{|x|} \frac{3\sqrt{6}}{2\lambda_0} \sqrt{\mu} I^+(I^+ - I^-) \sigma(s) H_A(s) u_1^*[2](s) ds \\
+ \int_0^{|x|} \frac{12}{\lambda_0^2} \mu (I^+ - I^-)^2 \sigma^2 ((\lambda_0 + r_1^+)(s - \theta^+)) u_1^*[2](s) ds \\
= \frac{3\sqrt{6}}{2\lambda_0^2 + 3\lambda_0^2 \mu} \mu I^+(I^+ - I^-) \int_0^{|x|} \sigma'(s) u_1^*[1](s) \cos (2(\lambda_0 + r_1^+)(s - \theta^+)) ds \\
+ \frac{3}{\lambda_0^2(\lambda_0 + r_1^+)} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{|x|} \sin (2(\lambda_0 + r_1^+)(s - \theta^+)) (\sigma(s) u_1^*[2](s))' ds \\
+ O(e^{-\sqrt{\mu|\lambda|}|I^+ - I^-|}) + O(\mu(I^+)^2|I^+ - I^-|) + O(|I^+ - I^-|^2),
\]
where we use the fact that
\[
(u_1^*[1])'(s) - \frac{3}{2} \sqrt{\mu} H_A(s) u_1^*[2](s) + \sqrt{\mu} u_1^*[2](s) = 0.
\]
It is easy to see that
\[
\int_0^{|x|} -\sigma'(s) \left( B_p^+(s - \theta^+) + B_p^-(s - \theta^+) \right) u_1^*[2](s) ds \\
= -\frac{6}{2\lambda_0^2 + 3\lambda_0 \mu} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{|x|} \sigma'(s) u_1^*[2](s) \sin (2(\lambda_0 + r_1^+)(s - \theta^+)) ds \\
+ O(\sqrt{\mu} I^+(I^+ - I^-) + O(I^+ - I^-)^2).
\]
Hence, using \( r_1^+ = O(\mu(I^+)^2) \) and \((u_1^*[2])'(x) = -\sqrt{\mu} u_1^*[1](x)\) with integration by parts,
\[
\int_0^{|x|} -\sigma'(s) \left( A_p^+(s - \theta^+) - A_p^-(s - \theta^+) \right) u_1^*[1](s) ds \\
+ \int_0^{|x|} -\sigma'(s) \left( B_p^+(s - \theta^+) + B_p^-(s - \theta^+) \right) u_1^*[2](s) ds \\
+ \frac{3\sqrt{6}}{2\lambda_0^2} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{|x|} \sigma(s) H_A(s) u_1^*[2](s) ds \\
- \frac{12}{\lambda_0^2} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{|x|} \sigma(s) \cos^2 ((\lambda_0 + r_1^+)(s - \theta^+)) u_1^*[2](s) ds \\
= \frac{3\sqrt{6}}{2\lambda_0^2 + 3\lambda_0^2 \mu} \mu I^+(I^+ - I^-) \int_0^{|x|} \sigma'(s) u_1^*[1](s) \cos (2(\lambda_0 + r_1^+)(s - \theta^+)) ds \\
+ \frac{3}{\lambda_0^2(\lambda_0 + r_1^+)} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{|x|} \sigma(s) (u_1^*[2])'(s) \sin (2(\lambda_0 + r_1^+)(s - \theta^+)) ds \\
+ O(e^{-\sqrt{\mu|\lambda|} |I^+ - I^-|}) + O(\mu^{3/2} I^+|I^+ - I^-|) + O(|I^+ - I^-|^2) \\
= -\frac{3\sqrt{6}}{2\lambda_0^2} \mu I^+(I^+ - I^-) \int_0^{|x|} \sigma(s) (u_1^*[1])'(s) \cos (2(\lambda_0 + r_1^+)(s - \theta^+)) ds \\
+ O(e^{-\sqrt{\mu|\lambda|} |I^+ - I^-|}) + O(\mu^{3/2} I^+|I^+ - I^-|) + O(|I^+ - I^-|^2) \\
= \frac{3\sqrt{6}}{4\lambda_0^2(\lambda_0 + r_1^+)} \mu I^+(I^+ - I^-) \int_0^{|x|} (\sigma(s) (u_1^*[1])'(s))' \sin (2(\lambda_0 + r_1^+)(s - \theta^+)) ds \\
+ O(e^{-\sqrt{\mu|\lambda|} |I^+ - I^-|}) + O(\mu^{3/2} I^+|I^+ - I^-|) + O(|I^+ - I^-|^2)
By Lemma 6.3, it follows that

$$- \frac{6}{\lambda_0^2} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{\vert x \vert} \sigma(s) w_1 e(s) u_1^*[2](s) ds$$

$$= - \frac{3}{4 \lambda_0^3} \mu^{3/2} I^+(I^+ - I^-) \int_0^{\vert x \vert} H_A^2(s) \sigma(s) u_1^*[2](s) ds + O(\mu^{3/2} I^\pm |I^+ - I^-|)$$

$$= - \frac{3}{4 \lambda_0^3} \mu^{3/2} I^+(I^+ - I^-) \int_0^{\infty} H_A^2(s) u_1^*[2](s) ds$$

$$+ O(e^{-\sqrt{\mu} \vert x \vert} I^\pm |I^+ - I^-|) + O(\mu^{3/2} I^\pm |I^+ - I^-|)$$

$$= - \frac{9}{4 \lambda_0^3} \mu^{3/2} I^+(I^+ - I^-) + O(e^{-\sqrt{\mu} \vert x \vert} I^\pm |I^+ - I^-|) + O(\mu^{3/2} I^\pm |I^+ - I^-|),$$

and

$$\frac{3 \sqrt{\mu}}{2 \lambda_0^2} \sqrt{\mu} I^+(I^+ - I^-) \int_0^{\vert x \vert} \sigma(s) A_e(s) u_1^*[2](s) ds$$

$$= \frac{27}{4 \lambda_0^3} \mu^{3/2} I^+(I^+ - I^-) \int_0^{\infty} \cosh(\sqrt{\mu} s) \sech^2 \left( \frac{\sqrt{3} \mu}{\sqrt{2}} \right) u_1^*[2](s) ds$$

$$+ O(e^{-\sqrt{\mu} \vert x \vert} I^\pm |I^+ - I^-|) + O(\mu^{3/2} I^\pm |I^+ - I^-|)$$

$$= \frac{27}{4 \lambda_0^3} \mu^{3/2} I^+(I^+ - I^-) + O(e^{-\sqrt{\mu} \vert x \vert} I^\pm |I^+ - I^-|) + O(\mu^{3/2} I^\pm |I^+ - I^-|).$$

The above calculations imply that (141) is changed into

$$B(|x|) + B(-|x|) = \frac{9}{2 \lambda_0^3} \mu^{3/2} I^+(I^+ - I^-) u_1[2](|x|)$$

$$- \frac{3 \sqrt{\mu}}{2 \lambda_0^2} \int_0^{\vert x \vert} A_e(s) X_1(s) u_1^*[2](s) ds u_1[2](|x|)$$

$$+ \frac{\sqrt{6}}{\lambda_0^3} \sqrt{\mu} \int_0^{\vert x \vert} H_A(s) \tilde{X}_3(s) u_1^*[2](s) ds u_1[2](|x|)$$

$$+ \frac{\sqrt{6}}{\lambda_0^3} \sqrt{\mu} \int_0^{\vert x \vert} \sigma(s) X_1(s) u_1^*[2](s) ds u_1[2](|x|)$$

$$+ O(\mu^{3/2} I^\pm |I^+ - I^-| e^{\sqrt{\mu} \vert x \vert}) + O(|I^+ - I^-|^2 e^{\sqrt{\mu} \vert x \vert})$$

$$+ O(\sqrt{\mu} |I^+ - I^-|).$$  \hfill (142)

**Step 2.** Terms only including \(X_1(s)\).

Notice that \(\omega_{1e}(x)\) is given in Lemma 6.3 and the expression of \(X_1(x) = A(|x|) - A(-|x|)\) is obtained in (140). Before we plug them into (142), we simplify some involved terms.

It is easy to check that with integration by parts and \(s_1[1](0) = 0\)

$$\int_0^{\vert x \vert} H_A^2(s) u_1^*[2](s) s_1[1](s) \int_0^{\infty} \sigma'(\tau) s_1[1](\tau) d\tau ds$$

$$= \int_0^{\vert x \vert} H_A^2(s) u_1^*[2](s) s_1[1](s) \sigma(s) s_1[1](s) ds$$
\[
- \int_0^{|x|} H_\lambda^2(s) u_s^*[2](s) s_1[1](s) \int_0^s \sigma(\tau)(s_1^*[1])'(\tau) d\tau ds \\
= \int_0^{|x|} H_\lambda^2(s) u_s^*[2](s) s_1[1](s) s_1^*[1](s) ds \\
- \int_0^{|x|} H_\lambda^2(s) u_s^*[2](s) s_1[1](s) \int_0^s (s_1^*[1])'(\tau) d\tau ds \\
- \int_0^{|x|} H_\lambda^2(s) u_s^*[2](s) s_1[1](s)(1 - \sigma(s)) s_1^*[1](s) ds \\
+ \int_0^{|x|} H_\lambda^2(s) u_s^*[2](s) s_1[1](s) \int_0^s (1 - \sigma(s))(s_1^*[1])'(\tau) d\tau ds \\
= - \int_0^{|x|} H_\lambda^2(s) u_s^*[2](s) s_1[1](s)(1 - \sigma(s)) s_1^*[1](s) ds \\
+ \int_0^{|x|} H_\lambda^2(s) u_s^*[2](s) s_1[1](s) \int_0^s (1 - \sigma(s))(s_1^*[1])'(\tau) d\tau ds \\
= O(1), \\
\int_0^{|x|} H_\lambda^2(s) u_s^*[2](s) u_s[1](s) \int_0^s \sigma'(x) u_s^*[1](\tau) d\tau ds \\
= u_s^*[1](0) \int_0^\infty H_\lambda^2(s) u_s^*[2](s) u_s[1](s) ds \\
- u_s^*[1](0) \int_x^\infty H_\lambda^2(s) u_s^*[2](s) u_s[1](s) ds + O(1) \\
= - \frac{2(-5 + \ln 64)}{7\sqrt{\mu}} + O(1) + O(\mu^{-1/2} e^{-\sqrt{\mu} |x|}), \\
\int_0^{|x|} H_\lambda^2(s) u_s^*[2](s) s_1[1](s) \int_0^s \sigma(x) H_\lambda(\tau) s_1^*[2](\tau) d\tau ds \\
= \int_0^\infty H_\lambda^2(s) u_s^*[2](s) s_1[1](s) \int_0^s H_\lambda(\tau) s_1^*[2](\tau) d\tau ds \\
+ O(\mu^{1/2}) + O(\mu^{-1/2} e^{-\sqrt{\mu} |x|}) \\
= \frac{4(1091 + 48 \ln 2)}{2145\mu} + O(\mu^{1/2}) + O(\mu^{-1/2} e^{-\sqrt{\mu} |x|}), \\
\int_0^{|x|} H_\lambda^2(s) u_s^*[2](s) u_s[1](s) \int_0^s \sigma(x) H_\lambda(\tau) u_s^*[2](\tau) d\tau ds \\
= \int_0^\infty H_\lambda^2(s) u_s^*[2](s) u_s[1](s) \int_0^s H_\lambda(\tau) u_s^*[2](\tau) d\tau ds \\
+ O(\mu^{1/2}) + O(\mu^{-1/2} e^{-\sqrt{\mu} |x|}) \\
= \frac{24(-1 + \ln 512)}{143\mu} + O(\mu^{-1/2}) + O(\mu^{-1/2} e^{-\sqrt{\mu} |x|}), \\
\int_0^{|x|} H_\lambda^2(s) u_s^*[2](s) s_1[1](s) \int_0^s \sigma(x) \cos^2 \left((\lambda_0 + r^+_1)(\tau - \theta^+)\right) s_1^*[2](\tau) d\tau ds \\
= \frac{1}{2} \int_0^{|x|} H_\lambda^2(s) u_s^*[2](s) s_1[1](s) \int_0^s \sigma(x) s_1^*[2](\tau) d\tau ds
\[ \frac{1}{2} \int_0^{[x]} H_A(s) u_1^*[2](s) s_1[1](s) \int_0^{\theta} \sigma(\tau) \cos \left(2(\lambda_0 + r_1^+)(\tau - \theta^+)\right) s_1^*[2](\tau) d\tau ds \]
\[ = \frac{1}{2} \int_0^{[x]} H_A(s) u_1^*[2](s) s_1[1](s) \int_0^{\theta} s_1^*[2](\tau) d\tau ds + O(\mu^{-1/2}) + O(\mu^{-1} e^{-\sqrt{\mu}|x|}) \]
\[ = \frac{229 - 48 \ln 2}{385} + O(\mu^{-1/2}) + O(\mu^{-1} e^{-\sqrt{\mu}|x|}), \]
\[ \int_0^{[x]} H_A(s) u_1^*[2](s) s_1[1](s) \int_0^{\theta} \sigma(\tau) \cos \left(2(\lambda_0 + r_1^+)(\tau - \theta^+)\right) u_1^*[2](\tau) d\tau ds \]
\[ = \frac{1}{2} \int_0^{[x]} H_A(s) u_1^*[2](s) s_1[1](s) \int_0^{\theta} u_1^*[2](\tau) d\tau ds + O(\mu^{-1/2}) + O(\mu^{-1} e^{-\sqrt{\mu}|x|}) \]
\[ = \frac{\sqrt{3}(1 + 24 \ln 2)}{154 \sqrt{2\mu}} + O(\mu^{-1/2}) + O(\mu^{-1} e^{-\sqrt{\mu}|x|}). \]

Applying (140), Lemma 6.3 and the above results yield
\[ \frac{\sqrt{6}}{\lambda_0} \sqrt{\mu} \int_0^{[x]} \omega(s) u_1^*[2](s) X_1(s) ds = \frac{9}{4\lambda_0} \mu I^+(I^+ - I^-) + O(\mu^{5/4} I^\pm |I^+ - I^-|) + O(\sqrt{\mu} I^\pm |I^+ - I^-| e^{-\sqrt{\mu}|x|}), \]
which also implies that
\[ \frac{\sqrt{6}}{4\lambda_0} \mu^{3/2} \int_0^{[x]} H_A(s) u_1^*[2](s) s_1[1](s) ds = \frac{9}{2\lambda_0} \mu I^+(I^+ - I^-) + O(\mu^{5/4} I^\pm |I^+ - I^-|) + O(\sqrt{\mu} I^\pm |I^+ - I^-| e^{-\sqrt{\mu}|x|}). \] (143)

It is similarly derived that
\[ - \frac{3}{2} \sqrt{\mu} \int_0^{[x]} A(s) u_1^*[2](s) X_1(s) ds = - \frac{9}{\lambda_0} \mu I^+(I^+ - I^-) + O(\mu^{5/4} I^\pm |I^+ - I^-|) + O(\sqrt{\mu} I^\pm |I^+ - I^-| e^{-\sqrt{\mu}|x|}). \]

**Step 3.** Terms only including $\tilde{X}_3(s)$.

From (86) and (143), a similar computation yields, together with changing the order of integration and integration by parts if needed,
\[ \frac{\sqrt{6}}{\lambda_0} \sqrt{\mu} \int_0^{[x]} H_A(s) u_1^*[2](s) \tilde{X}_3(s) ds \]
\[ = \frac{\sqrt{6}}{\lambda_0} \sqrt{\mu} \int_0^{[x]} H_A(s) u_1^*[2](s) \left( - \frac{9}{\lambda_0} \mu I^+(I^+ - I^-) \sigma(s) + \frac{3}{\lambda_0} \mu I^+(I^+ - I^-) \sigma(s) \cos \left(2(\lambda_0 + r_1^+)(s - \theta^+)\right) \right. \]
\[ + \frac{9}{\lambda_0} \mu I^+(I^+ - I^-) \int_0^{s} \sigma'(\tau) \cos \left(\lambda_0(s - \tau)\right) d\tau \]
\[ + \frac{3}{\lambda_0} \mu I^+(I^+ - I^-) \int_0^{s} \sigma'(\tau) \cos \left(2(\lambda_0 + r_1^+)(s - \theta^+)\right) \cos \left(\lambda_0(s - \tau)\right) d\tau \]
\[ - \int_0^{s} \left( - \frac{6}{\lambda_0} \mu I^+(I^+ - I^-) \sigma'(\tau) \sin \left(2(\lambda_0 + r_1^+)(s - \theta^+)\right) \right. \]
\[ \left. - \frac{3}{\lambda_0} \mu I^+(I^+ - I^-) \sigma'(\tau) \cos \left(\lambda_0(s - \tau)\right) \right) d\tau \]
\[\begin{align*}
- \frac{1}{4} \mu H_A(\tau) X_1(\tau) & \quad - \frac{\sqrt{6}}{2\lambda_0} \mu H_A(\tau) \dot{X}_3(\tau) + \frac{\sqrt{6}}{4\lambda_0^2} \mu I^+(I^+ - I^-) \sigma(\tau) H_A(\tau) \\
& \quad + \frac{3\sqrt{6}}{4(2\lambda_0^2 + 3\lambda_0^2\mu)} \mu^2 I^+(I^+ - I^-) \sigma(\tau) H_A(\tau) \cos \left(2(\lambda_0 + r^+_1)(\tau - \theta^+)\right) \\
& \quad - \frac{18}{\lambda_0^2} \mu I^+(I^+ - I^-) \sigma(\tau) \cos^2 \left((\lambda_0 + r^+_1)(\tau - \theta^+)\right) \sin \left(\lambda_0(s - \tau)\right) d\tau \\
& \quad + \dot{X}_{30} \right) ds \\
& = - \frac{9\sqrt{6}}{\lambda_0^2} \mu^{3/2} I^+(I^+ - I) \int_0^{|x|} H_A(s) u^+_1[2](s) \sigma(s) ds \\
& \quad - \frac{3}{2\lambda_0^2} \mu^{3/2} I^+(I^+ - I) \int_0^{|x|} H_A(s) u^+_1[2](s) \int_0^s H_A(\tau) \sigma(\tau) \sin \left(\lambda_0(s - \tau)\right) d\tau ds \\
& \quad + \frac{18\sqrt{6}}{\lambda_0^2} \mu^{3/2} I^+(I^+ - I) \int_0^{|x|} H_A(s) u^+_1[2](s) \\
& \quad \times \int_0^s \sigma(\tau) \cos^2 \left((\lambda_0 + r^+_1)(\tau - \theta^+)\right) \sin \left(\lambda_0(s - \tau)\right) d\tau ds \\
& \quad + \frac{\sqrt{6}}{4\lambda_0^2} \mu^{3/2} \int_0^{|x|} H_A(s) u^+_1[2](s) \int_0^s H_A(\tau) X_1(\tau) \sin \left(\lambda_0(s - \tau)\right) d\tau ds \\
& \quad + \frac{3}{\lambda_0^2} \mu^{3/2} \int_0^{|x|} H_A(s) u^+_1[2](s) \int_0^s H_A(\tau) \dot{X}_3(\tau) \sin \left(\lambda_0(s - \tau)\right) d\tau ds \\
& \quad + O(\mu^{5/4} I^+|I^+ - I^-|) + O(\sqrt{\mu} |I^+ - I^-|^2) + O(I^+|I^+ - I^-| e^{-\sqrt{6\mu}|x|}) \\
& = - \frac{9\sqrt{6}}{\lambda_0^2} \mu^{3/2} I^+(I^+ - I) \int_0^{|x|} H_A(s) u^+_1[2](s) \sigma(s) ds \\
& \quad - \frac{3}{2\lambda_0^2} \mu^{3/2} I^+(I^+ - I) \int_0^{|x|} H^2_A(s) \sigma(s) u^+_1[2](s) ds \\
& \quad + \frac{9\sqrt{6}}{\lambda_0^2} \mu^{3/2} I^+(I^+ - I) \int_0^{|x|} H_A(s) u^+_1[2](s) \sigma(s) ds \\
& \quad + \frac{\sqrt{6}}{4\lambda_0^2} \mu^{3/2} \int_0^{|x|} H^2_A(s) u^+_1[2](s) X_1(s) ds \\
& \quad + \frac{3}{2\lambda_0^2} \mu^{3/2} \int_0^{|x|} H^2_A(s) u^+_1[2](s) \dot{X}_3(s) ds \\
& \quad + O(\mu^{5/4} I^+|I^+ - I^-|) + O(\sqrt{\mu} |I^+ - I^-|^2) + O(I^+|I^+ - I^-| e^{-\sqrt{6\mu}|x|}) \\
& = - \frac{3}{2\lambda_0^2} \mu^{3/2} I^+(I^+ - I) \int_0^\infty H^2_A(s) u^+_1[2](s) ds \\
& \quad + \frac{\sqrt{6}}{4\lambda_0^2} \mu^{3/2} \int_0^\infty H^2_A(s) u^+_1[2](s) X_1(s) ds \\
& \quad + O(\mu^{5/4} I^+|I^+ - I^-|) + O(\sqrt{\mu} |I^+ - I^-|^2) + O(I^+|I^+ - I^-| e^{-\sqrt{6\mu}|x|}) \\
& = O(\mu^{5/4} I^+|I^+ - I^-|) + O(\sqrt{\mu} |I^+ - I^-|^2) + O(I^+|I^+ - I^-| e^{-\sqrt{6\mu}|x|}).
\end{align*}\]
From the above steps, using (142), we have
\[ B(|x|) + B(-|x|) = -\frac{9}{4X_0} |I^+(I^+ - I^-)u_1[2](|x|) + O(\mu^{5/4} |I^+ - I^-| e^{\sqrt{\mu} |x|})
+ O(\sqrt{\mu} |I^+ - I^-|^2 e^{\sqrt{\mu} |x|}) + O(I^\pm |I^+ - I^-|). \]
This yields (100). The proof is completed.

Acknowledgments. The author thanks the anonymous referees for valuable comments and suggestions. This research is supported by the National Natural Science Foundation of China (No. 11771197), the Guangdong Natural Science Foundation of China (No. 2017A030313030), and the Scientific Research Funds of Huaqiao University.

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Received for publication October 2017.

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