STABLY ISOMORPHIC DUAL OPERATOR ALGEBRAS

G.K. ELEFTHERAKIS AND V.I. PAULSEN

Abstract. We prove that two unital dual operator algebras $A, B$ are stably isomorphic if and only if they are $\Delta$-equivalent [7], if and only if they have completely isometric normal representations $\alpha, \beta$ on Hilbert spaces $H, K$ respectively and there exists a ternary ring of operators $M \subseteq B(H,K)$ such that $\alpha(A) = [M^*\beta(B)M]^{-w^*}$ and $\beta(B) = [M\alpha(A)M^*]^{-w^*}$.

1. Introduction

Two dual operator algebras [12] $A, B$ are called stably isomorphic if there exists a cardinal $I$ such that the algebras $M_I(A), M_I(B)$ of matrices indexed by $I$, whose finite submatrices have uniformly bounded norms, are algebraically isomorphic through an isomorphism which is completely isometric and $w^*$-(bi)continuous. In the special case of $W^*$-algebras [1], this happens if and only if $A$ and $B$ are Morita equivalent in the sense of Rieffel [11]. A proof of this fact for separably acting von Neumann algebras can be found in [12] and the general case is in [1].

In [6, 7] two new equivalence relations between dual operator algebras were defined:

Definition 1.1. [6] Let $A, B$ be $w^*$-closed algebras acting on Hilbert spaces $H$ and $K$, respectively. If there exists a ternary ring of operators (TRO) $M \subseteq B(H,K)$, i.e. a subspace satisfying $MM^*M \subseteq M$, such that $A = [M^*BM]^{-w^*}$ and $B = [MAM^*]^{-w^*}$ we write $A \overset{M}{\sim} B$. We say that the algebras $A, B$ are TRO equivalent if there exists a TRO $M$ such that $A \overset{M}{\sim} B$.

If $A$ is a dual operator algebra, then we call a completely contractive, $w^*$-continuous homomorphism $\alpha : A \rightarrow B(H)$ where $H$ is a Hilbert space, a normal representation of $A$.

Key words and phrases. Morita equivalence, stable isomorphism, ternary ring.

This project is cofunded by European Social Fund and National Resources - (EPEAEK II) "Pythagoras II" grant No. 70/3/7997.
In [7] the notion of $\Delta$-equivalence of two unital dual operator algebras $A, B$ was defined in terms of equivalence of two appropriate categories. In the present paper, we will adopt the following definition of $\Delta$-equivalence.

**Definition 1.2.** Two unital dual operator algebras $A, B$ are called $\Delta$-equivalent if they have completely isometric normal representations $\alpha, \beta$ such that the algebras $\alpha(A), \beta(B)$ are TRO equivalent.

**Remark 1.1.** The conclusion of the present paper (Theorem 3.2) was used in [7, Theorem 1.3]. It was proved in that theorem that definition 1.2 is in fact equivalent to the one given in [7, Definition 1.4]: there, two unital dual operator algebras $A$ and $B$ are called $\Delta$-equivalent if there exists an equivalence functor between their categories of normal representations which intertwines not only the representations of the algebras but also their restrictions to the diagonals.

Two completely isometrically and $w^*$-continuously isomorphic unital dual operator algebras are not necessarily TRO equivalent, but they are $\Delta$-equivalent. Also two $W^*$-algebras are Morita equivalent in the sense of Rieffel if and only if they are $\Delta$-equivalent [7]. In this work we are going to prove that two unital dual operator algebras are $\Delta$-equivalent if and only if they are stably isomorphic.

We explain now why two stably isomorphic unital dual operator algebras are $\Delta$-equivalent. We need first to present some definitions and results, see for example [1]. If $I$ is a cardinal and $X$ is a dual operator space, we denote by $\Omega_I(X)$ the linear space of all matrices with entries in $X$. If $x \in \Omega_I(X)$ and $r$ is a finite subset of $I$ we write $x^r = (x_{ij})_{i,j \in r}$. We define

$$\|x\| = \sup_{r \subset I, \text{finite}} \|x^r\|$$ and $M_I(X) = \{x \in \Omega_I(X), \|x\| < +\infty\}$.

This space is a dual operator space. If $X$ is a dual operator algebra then $M_I(X)$ is also a dual operator algebra. In case $X$ is a $w^*$-closed subspace of $B(H, K)$ for some Hilbert spaces $H, K$ we naturally identify $M_I(X)$ as a subspace of $B(H^I, K^I)$ where $H^I$ (resp.$K^I$) is the direct sum of $I$ copies of $H$ (resp.$K$). We denote the $w^*$-closed subspace of $B(H^I, K)$ consisting of bounded operators of the form

$$H^I \to K : (\xi_i)_{i \in I} \to \sum_i x_i(\xi_i)$$

for $\{x_i : i \in I\} \subset X$ by $R^w_I(X)$ and the $w^*$-closed subspace of $B(H, K^I)$ consisting of bounded operators of the form

$$H \to K^I : \xi \to (x_i(\xi))_{i \in I}$$
for \( \{ x_i : i \in I \} \subset X \) by \( C^\pi_f(X) \). Observe that if \( X \) is a \( w^* \)-closed TRO then the spaces \( R^\pi_f(X), C^\pi_f(X) \) are \( w^* \)-closed TRO’s.

Suppose now that the unital dual operator algebras \( A_0, B_0 \) are stably isomorphic for a cardinal \( I \). By [9] there exist completely isometric normal representations of \( A_0, B_0 \) whose images we denote by \( A, B \), respectively. Observe that the algebras \( A, M_I(A) \) are TRO equivalent, indeed, \( A \simeq M_I(A) \), where \( M = C^\pi_f(\Delta(A)) \), and \( \Delta(A) = A \cap A^* \) is the diagonal of \( A \). Similarly the algebras \( B, M_I(B) \) are TRO equivalent. Since \( \Delta \)-equivalence is an equivalence relation preserved by normal completely isometric homomorphisms we conclude that the initial algebras are \( \Delta \)-equivalent.

The purpose of this paper is to prove the converse: \( \Delta \)-equivalent algebras are stably isomorphic. Since every completely isometric normal homomorphism \( A \to B \) for dual operator algebras naturally “extends” to a completely isometric normal homomorphism \( M_I(A) \to M_I(B) \) for every cardinal \( I \) [4], it suffices to show that the TRO equivalent algebras are stably isomorphic.

2. Generated bimodules.

In this section we prove that if \( A \) (resp. \( B \)) is a \( w^* \)-closed subalgebra of \( B(H) \) (resp. \( B(K) \)) for a Hilbert space \( H \) (\( K \)) and \( \mathcal{M} \subset B(H, K) \) is a TRO such that \( A \cong M_I(B) \), then there exist bimodules \( X, Y \) over these algebras, i.e., \( AXB \subset X, BYA \subset Y \), which are generated by \( \mathcal{M} \), such that \( A \cong X \otimes_B Y \) and \( B \cong Y \otimes_A X \) as dual spaces, where \( X \otimes_B Y \) (\( Y \otimes_A X \)) is an appropriate quotient of the normal Haagerup tensor product \( X \otimes^h Y \) (\( Y \otimes^h X \)) [5].

We start with some definitions and symbols. If \( \Omega \) is a Banach space we denote by \( \Omega^* \) its dual. If \( X, Y, Z \) are linear spaces, \( n \in \mathbb{N} \) and \( \sigma : X \to Y \) is a linear map we denote again by \( \sigma \) the map \( M_n(X) \to M_n(Y) : (x_{ij}) \to (\sigma(x_{ij})) \). If \( \phi : X \times Y \to Z \) is a bilinear map and \( n, p \in \mathbb{N} \) we denote again by \( \phi \) the map \( M_{n,p}(X) \times M_{p,n}(Y) \to M_n(Z) : ((x_{ij}), (y_{kj})) \to \left( \sum_{k=1}^{p} \phi(x_{ik}, y_{kj}) \right)_{ij} \). If \( X, Y \) are operator spaces we denote by \( CB(X, Y) \) the space of completely bounded maps from \( X \) to \( Y \) with the completely bounded norm. If \( Z \) is another operator space, a bilinear map \( \phi : X \times Y \to Z \) is called completely bounded [10] if there exists \( c > 0 \) such that \( \| \phi(x, y) \| \leq c \| x \| \| y \| \) for all \( x \in M_{n,p}(X), y \in M_{p,n}(Y), n, p \in \mathbb{N} \). The least such \( c \) is the completely bounded norm of \( \phi \) and it is denoted by \( \| \phi \|_{cb} \). We write

\[
CB(X \times Y, Z) = \{ \phi : X \times Y \to Z, \ \phi \text{ is \ completely \ bounded} \}.
\]
This is an operator space under the identification
\[ M_n(CB(X \times Y, Z)) = CB(X \times Y, M_n(Z)) \]
for all \( n \in \mathbb{N} \).

We denote the Haagerup tensor product of \( X, Y \) by \( X \otimes Y \). The map
\[ CB(X \times Y, Z) \rightarrow CB(X \otimes Y, Z) : \omega \mapsto \tilde{\omega} \]
given by \( \tilde{\omega}(x \otimes y) = \omega(x, y) \) for all \( x \in X, y \in Y \) is a complete isometry. If \( X, Y \) are dual operator spaces we denote by \( CB^\sigma(X, Y) \) the space of completely bounded \( w^* \)-continuous maps. If \( Z \) is another dual operator space a bilinear map \( \phi : X \times Y \rightarrow Z \) is called \textbf{normal} if it is separately \( w^* \)-continuous. We denote by \( CB^\sigma(X \times Y, Z) \) the space of completely bounded normal bilinear maps.

We now recall the normal Haagerup tensor product [5]. In the rest of this section we fix dual operator spaces \( X, Y \) and the map
\[ \pi : CB(X \times Y, \mathbb{C}) \rightarrow CB(X \otimes Y, \mathbb{C}) = (X \otimes Y)^* \]
given by \( \pi(\omega) = \tilde{\omega}, \tilde{\omega}(x \otimes y) = \omega(x, y) \). We denote by \( \Omega_1 \) the space \( \pi(CB^\sigma(X \times Y, \mathbb{C})) \) and by \( X \otimes Y \) the dual of \( \Omega_1 \). This space is the \( w^* \)-closed span of its elementary tensors \( x \otimes y, x \in X, y \in Y \) and it has the following property: For all dual operator spaces \( Z \) there exists a complete onto isometry
\[ J : CB^\sigma(X \times Y, Z) \rightarrow CB^\sigma(X \otimes Y, Z) : \phi \mapsto \phi^\sigma \]
where \( \phi^\sigma(x \otimes y) = \phi(x, y) \).

We now fix a dual operator algebra \( B \) such that \( X \) is a right \( B \)-module and \( Y \) is left \( B \)-module and the maps
\[ X \times B \rightarrow X : (x, b) \mapsto xb, \quad B \times Y \rightarrow Y : (b, y) \mapsto by \]
are complete contractions and normal bilinear maps. A bilinear map \( \omega : X \times Y \rightarrow Z \) is called \textbf{\( B \)-balanced} if \( \omega(xb, y) = \omega(x, by) \) for all \( x \in X, b \in B, y \in Y \). For every dual operator space \( Z \) we define the space
\[ CB^{\sigma B}(X \times Y, Z) = \{ \omega \in CB^\sigma(X \times Y, Z) : \omega \text{ is } B\text{-balanced} \}. \]
We denote by \( \Omega_2 \) the space \( \pi(CB^{\sigma B}(X \times Y, \mathbb{C})) \). Observe that \( \Omega_2 \) is a closed subspace of \( \Omega_1 \subset (X \otimes Y)^* \). Also we define the space
\[ N = [xb \otimes y - x \otimes by : x \in X, b \in B, y \in Y]^{-w^*} \subset X \sigmah \otimes Y. \]
We denote by \( X \otimes_B Y \) the space \( (X \sigmah \otimes Y)/N \) and we use the symbol \( x \otimes_B y \) for \( x \otimes y + N, x \in X, y \in Y \).
Proposition 2.1. The spaces $X^\sigma_{\epsilon_B} \otimes B Y$ and $\Omega^*_2$ are completely isometric and $w^*$-homeomorphic.

Proof. The adjoint map $\theta : X^\sigma \otimes Y \to \Omega^*_2$ of the inclusion $\Omega_2 \hookrightarrow \Omega_1$ is a complete quotient map and $w^*$-continuous. Check now that $N = Ker(\theta)$. □

Proposition 2.2. If $Z$ is a dual operator space and $\phi \in CB^\sigma(X \times Y, Z)$ then there exists a $w^*$-continuous and completely bounded map $\phi_{B\sigma h} : X^\sigma_{\epsilon_B} \otimes Y \to Z$ such that $\phi_{B\sigma h}(x \otimes_B y) = \phi(x, y)$ for all $x \in X$, $y \in Y$. In fact the map $CB^\sigma(X \times Y, Z) \to CB^\sigma(X^\sigma_{\epsilon_B} \otimes Y, Z) : \phi \to \phi_{B\sigma h}$ is a complete isometry, onto.

Proof. Suppose that $Z_*$ is the operator space predual of $Z$. For every $\omega \in Z_*, \omega \circ \phi \in \Omega_2$. So we can define a map $\phi_* : Z_* \to \Omega_2 : \phi_*(\omega) = \omega \circ \phi$. We denote by $\phi_{B\sigma h}$ the adjoint map of $\phi_*$. So that $\phi_{B\sigma h} \in CB(\Omega^*_2, Z) = CB(X^\sigma_{\epsilon_B} \otimes Y, Z)$ by Proposition 2.1. For every $x \in X, y \in Y, \omega \in Z_*$ we have $\langle \phi_{B\sigma h}(x \otimes_B y), \omega \rangle = \langle \phi(x, y), \omega \rangle$ so $\phi_{B\sigma h}(x \otimes_B y) = \phi(x, y)$.

Let $i : \Omega_2 \to \Omega_1$ denote the inclusion map so that $q = i^* : \Omega^*_1 \to \Omega^*_2$ is a $w^*$-continuous complete quotient map. The map of composition with $q$ gives a completely isometric inclusion, $q^* : CB^\sigma(\Omega^*_2, Z) \to CB^\sigma(\Omega^*_1, Z)$.

By Proposition 2.1 we may identify $\Omega^*_2 = X^\sigma_{\epsilon_B} \otimes Y$ and also we have $\Omega^*_1 = X^\sigma \otimes Y$ by definition. Thus, modulo these identifications, we have that $q^* : CB^\sigma(X^\sigma_{\epsilon_B} \otimes Y, Z) \to CB^\sigma(X^\sigma \otimes Y, Z)$ is a $w^*$-continuous complete isometry.

We also have that $CB^{B\sigma}(X \times Y, Z) \subseteq CB^\sigma(X \times Y, Z)$ is a subspace endowed with the same matrix norms. Thus, $J : CB^{B\sigma}(X \times Y, Z) \to CB^\sigma(X^\sigma \otimes Y, Z)$ is also a completely isometric inclusion.

Now observe that $J(\phi) = q^*(\phi_{B\sigma h})$, so that $\phi \to \phi_{B\sigma h}$ is a complete isometry and $J(CB^{B\sigma}(X \times Y, Z)) \subseteq q^*(CB^\sigma(X^\sigma_{\epsilon_B} \otimes Y, Z))$.

It remains to show that the map is onto so that the above inclusion is an equality of sets. To see that $\phi \to \phi_{B\sigma h}$ is onto $CB^\sigma(X^\sigma_{\epsilon_B} \otimes Y, Z)$, let $\widetilde{\psi} \in CB^\sigma(X^\sigma_{\epsilon_B} \otimes Y, Z)$ and $\theta : X^\sigma \otimes Y \to X^\sigma_{\epsilon_B} \otimes Y : x \otimes y \to x \otimes_B y$ be the map in Proposition 2.1. Since $\widetilde{\psi} \circ \theta \in CB^\sigma(X^\sigma \otimes Y, Z)$ the map $\widetilde{\psi} : X \times Y \to Z$ given by $\psi(x, y) = \widetilde{\psi} \circ \theta(x \otimes_B y) = \widetilde{\psi}(x \otimes_B y)$ belongs to the space $CB^\sigma(X \times Y, Z)$. We have to prove that $\widetilde{\psi}$ is balanced.
If \( \omega \in Z_* \) then \( \omega \circ \tilde{\psi} \) belongs to the predual of \( X \otimes_B Y \). So there exists \( \chi \in CB^{sa}(X \times Y, \mathbb{C}) \) such that \( \chi(x, y) = \omega(\psi(x, y)) \) for all \( x \in X, y \in Y \). Now for every \( x \in X, y \in Y, b \in B \) we have
\[
\omega(\psi(xb, y)) = \chi(xb, y) = \chi(x, by) = \omega(\psi(x, by)).
\]
The functional \( \omega \) is arbitrary in \( Z_* \) so \( \psi(xb, y) = \psi(x, by) \). We have proved that the map \( CB^{sa}(X \times Y, Z) \to CB^{sa}(X \otimes_B Y, Z) : \phi \to \phi_{B\sigma h} \) is an onto. \( \Box \)

Suppose now that \( H, K \) are Hilbert spaces, \( A \) and \( B \) are unital \( w^* \)-closed subalgebras of \( B(K) \) and \( B(H) \) respectively and \( \mathcal{M} \subset B(K, H) \) is a \( w^* \)-closed TRO such that \( A \not\sim B \).

**Definition 2.1.** The spaces \([\mathcal{AM}^*]^{-w^*}, [\mathcal{MA}]^{-w^*}\) are called the \( \mathcal{M} \)-generated \( A - B \) bimodules.

In what follows we assume that \( X = [\mathcal{AM}^*]^{-w^*}, Y = [\mathcal{MA}]^{-w^*} \). We can check that
\[
X = [\mathcal{M}^*B]^{-w^*}, Y = [BM]^{-w^*},
\]
(2.1) \( AXB \subset X, BYA \subset Y, A = [XY]^{-w^*}, B = [YX]^{-w^*} \).

Let \( a \in A \). We define a map
\[
CB^{sa}(X \times Y, \mathbb{C}) \to CB^{sa}(X \times Y, \mathbb{C}) : \omega \to \omega_a,
\]
by \( \omega_a(x, y) = \omega(x, ya) \). This map is continuous. The adjoint map \( \pi_a : X \otimes_B Y \to X \otimes_B Y \) satisfies \( \pi_a(x \otimes_B y) = x \otimes_B (ya) \). For every \( z \in X \otimes_B Y \) we define \( za = \pi_a(z) \). Observe that if \( \left( \sum_{i=1}^{k_j} x_i^j \otimes_B y_i^j \right)_j \) is a net such that \( z = w^* - \lim_j \sum_{i=1}^{k_j} x_i^j \otimes_B y_i^j \) then \( za = w^* - \lim_j \sum_{i=1}^{k_j} x_i^j \otimes_B (y_i^j a) \).

**Lemma 2.3.** Let \( z \in X \otimes_B Y \). If \( (a_\lambda)_\lambda \subset A \) is a net such that \( a_\lambda \stackrel{w^*}{\to} a \) then \( za_\lambda \stackrel{w^*}{\to} za \).

**Proof.** Choose \( \omega \in Ball(CB^{sa}(X \times Y, \mathbb{C})) \). From the normal version of the Christensen, Sinclair, Paulsen, Smith theorem, see for example Theorem 5.1 in [5], there exist a Hilbert space \( H \) and normal completely contractive maps \( \phi_1 : X \to B(H, \mathbb{C}), \phi_2 : Y \to B(C, H) \) such that \( \omega(x, y) = \phi_1(x) \phi_2(y) \). Observe that the bilinear map \( Y \times A \to B(C, H) : (y, a) \to \phi_2(ya) \) is completely contractive and normal. So by the same theorem there exist a Hilbert space \( K \) and complete contractions \( \phi_3 : A \to B(C, K), \phi_4 : Y \to B(K, H) \).
such that $\phi_2(ya) = \phi_1(y)\phi_2(a)$ for all $y \in Y, a \in A$. The bilinear map $X \times Y \to B(K, \mathbb{C}) : (x, y) \to \phi_1(x)\phi_4(y)$ is normal and a complete contraction. So there exists a completely contractive $w^*$-continuous map $\pi : X \hat{\otimes} Y \to B(K, \mathbb{C})$ such that $\pi(x \otimes y) = \phi_1(x)\phi_4(y)$. Now the map

$$
\tau(\omega) : (X \hat{\otimes} Y) \times A \to \mathbb{C} : \tau(\omega)(z, a) = \pi(z)\phi_3(a)
$$

is normal, completely contractive and satisfies

$$
\tau(\omega)(x \otimes y, a) = \pi(x \otimes y)\phi_3(a)
$$

for all $x \in X, y \in Y, a \in A$. The conclusion is that we can define a contraction

$$
\tau : CB^\sigma(X \times Y, \mathbb{C}) \to CB^\sigma(X \hat{\otimes} Y \times A, \mathbb{C}) : \omega \to \tau(\omega)
$$

which has adjoint map $\sigma : (X \hat{\otimes} Y) \hat{\otimes} A \to X \hat{\otimes} Y$ satisfying $\sigma((x \otimes y) \otimes a) = x \otimes (ya)$. We recall from Proposition 2.1 the map

$$
\theta : X \hat{\otimes} Y \to X \hat{\otimes}_B Y : \theta(x \otimes y) = x \otimes_B y.
$$

Choose arbitrary $z \in X \hat{\otimes}_B Y$ and $z_0 \in X \hat{\otimes}_h Y$ such that $\theta(z_0) = z$. If $\left(\sum_{i=1}^{k_j} x_i^j \otimes y_i^j\right)_j$ is a net such that $z_0 = \omega^* - \lim \sum_{i=1}^{k_j} x_i^j \otimes y_i^j$ then for all $a \in A$

$$
\theta \circ \sigma(z_0 \otimes a) = \theta \circ \sigma \left( \lim_j \left( \left( \sum_{i=1}^{k_j} x_i^j \otimes y_i^j \right) \otimes a \right) \right) = \lim_j \sum_{i=1}^{k_j} \theta(x_i^{j} \otimes (y_i^j a)) = \lim_j \sum_{i=1}^{k_j} x_i^{j} \otimes_B (y_i^j a) = za.
$$

If $(a_\lambda)_\lambda \subset A$ is a net such that $a_\lambda \overset{w^*}{\to} a$ then $z_0 \otimes a_\lambda \overset{w^*}{\to} z_0 \otimes a$ in $(X \otimes Y) \otimes A$. Since $\theta \circ \sigma$ is $w^*$-continuous we have $\theta \circ \sigma(z_0 \otimes a_\lambda) \overset{w^*}{\to} \theta \circ \sigma(z_0 \otimes a)$ or equivalently $za_\lambda \overset{w^*}{\to} za$. 

**Theorem 2.4.** $A \cong X \hat{\otimes}_B Y$ and $B \cong Y \hat{\otimes}_A X$ completely isometrically and $w^*$-homeomorphically.

**Proof.** The map $X \times Y \to A : (x, y) \to xy$ is normal, completely contractive and $B$-balanced. So by Proposition 2.2 it defines a completely contractive and $w^*$-continuous map

$$
\pi : X \hat{\otimes}_B Y \to A : \pi(x \otimes_B y) = xy.
$$
We shall show that $\pi$ is a complete isometry. Since $A = [XY]^{-w^*}$, it will follow from the Krein Smulian theorem that $\pi$ is onto $A$.

Let $z = (z_{ij}) \in M_n(\sigma h(X \otimes_B Y))$. It suffices to show that $\|z\| \leq \|\pi(z)\|$. Since $X \sigma h \otimes_B Y = (CB^{B\sigma}(X \times Y, \mathbb{C}))^*$ given $\epsilon > 0$ there exist $m \in \mathbb{N}$ and $(\omega_{kl}) \in Ball(M_m(CB^{B\sigma}(X \times Y, \mathbb{C})))$ such that

$$\sum \omega_{kl}(z_{ij}v_s^*v_s) = 0$$

so

$$\sum \omega_{kl}(z_{ij}v_s^*v_s) = 0$$

for all $k, l, i, j$. It follows that there exist partial isometries $\{v_i : i \in I\} \subset \mathcal{M}$ such that $\|z\| - \epsilon \leq \|((\omega_{kl}(z_{ij}v_s^*v_s))_{ij})_{kl}\|$.

Since $X \sigma h \otimes_B Y$ is the $w^*$-closure of the space $(X \otimes Y)/N$, see Proposition 2.1 there exists a net $(z_\lambda)_\lambda \subset M_n(X \otimes Y/N)$ such that $z_\lambda \xrightarrow{w^*} z$. If $z_\lambda = (z_{ij}(\lambda))_{ij}$ for all $\lambda$ we have $z_{ij}(\lambda) \xrightarrow{w^*} z_{ij}$, hence $\sum_{s=1}^r \omega_{kl}(z_{ij}(\lambda)v_s^*v_s) \to \sum_{s=1}^r \omega_{kl}(z_{ij}v_s^*v_s)$ for all $i, j, k, l$. It follows that there exists $\lambda_0$ such that

$$\|z\| - \epsilon \leq \left\|\left(\sum_{s=1}^r \omega_{kl}(z_{ij}(\lambda)v_s^*v_s)\right)_{ij}\right\|_{kl} \text{ for all } \lambda \geq \lambda_0.$$

Fix $i, j, \lambda$ and suppose that $z_{ij}(\lambda) = \sum_{p=1}^t x_p \otimes_B y_p$, then $\omega_{kl}(z_{ij}(\lambda)v_s^*v_s) = \sum_{p=1}^t \omega_{kl}(x_p, y_p v_s^*v_s)$ for all $k, l, s$. Since $y_p v_s^* \in Y \subset B$ and $\omega_{kl}$ is $B$-balanced we have

$$\omega_{kl}(z_{ij}(\lambda)v_s^*v_s) = \sum_{p=1}^t \omega_{kl}(x_p y_p v_s^*, v_s) = \omega_{kl}(\pi(z_{ij}(\lambda))v_s^*, v_s).$$

So we take the inequality

$$\|z\| - \epsilon \leq \left\|\left(\sum_{s=1}^r \omega_{kl}(\pi(z_{ij}(\lambda))v_s^*, v_s)\right)_{ij}\right\|_{kl} \text{ for all } \lambda \geq \lambda_0.$$
Let \( v \) since \( \pi \) we have
\[
\| z \| - \epsilon \leq \left\| \left( \sum_{s=1}^{r} \omega_{kl}(\pi(z_{ij}) v_{s}^{*}, v_{s}) \right)_{ij} \right\|_{mn}.
\]
Let \( v = (v_1, ..., v_r)^t \) and
\[
x = (\pi(z_{ij}))_{ij} \begin{bmatrix} v^* & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \end{bmatrix} \in M_{n,nr}(X), \quad y = \begin{bmatrix} v \\ 0 & \cdots & 0 \\ v \end{bmatrix} \in M_{nr,n}(Y).
\]
The above inequality can be written in the following form
\[
\| z \| - \epsilon \leq \| (\omega_{kl}(x,y))_{kl} \|_{mn}.
\]
Since
\[
\| (\omega_{kl}) \|_{m} = \| (\omega_{kl}) : X \times Y \to M_m \|_{cb} \leq 1
\]
we have
\[
\| z \| - \epsilon \leq \| x \| \| y \| \leq \| (\pi(z_{ij}))_{ij} \| \| v^* \| \| v \| \leq \| \pi(z) \|.
\]
Since \( \epsilon > 0 \) is arbitrary we obtain \( \| z \| \leq \| \pi(z) \| \). This completes the proof of \( A \cong X \overset{sh}{\otimes}_B Y \). Similarly we can prove \( B \cong Y \overset{sh}{\otimes}_A X \) \( \square \)

3. THE MAIN THEOREM

In this section we shall prove that two unital dual operator algebras are \( \Delta \)-equivalent if and only if they are stably isomorphic. As we noted in section 1 it suffices to show that TRO equivalent algebras are stably isomorphic. Thus in what follows, we fix unital \( w^* \)-closed algebras \( A, B \) acting on Hilbert spaces \( H, K \) respectively and a \( w^* \)-closed TRO \( \mathcal{M} \) such that \( A \overset{\mathcal{M}}{\cong} B \). Let \( X = [A \mathcal{M}^*]^{-w^*}, Y = [\mathcal{M} A]^{-w^*} \) be the \( \mathcal{M} \)-generated \( A - B \) bimodules which satisfy (2.1). We give the following definition (see the analogous definition in [2]). If \( U_i \subset B(L, H), V_i \subset B(H, L), i = 1, 2 \) are spaces such that \( U_i V_i \subset A, i = 1, 2 \) a pair of maps \( \sigma : U_1 \to U_2, \pi : V_1 \to V_2 \) is called \textbf{A-inner product preserving} if \( \sigma(x)\pi(y) = xy \) for all \( x \in U_1, y \in V_1 \).

Lemma 3.1. There exist a cardinal \( I \) and completely isometric, \( w^* \)-continuous, onto, \( A \)-module maps \( \sigma : R^w_i(X) \to R^w_i(A), \pi : C^w_i(Y) \to C^w_i(A) \) such that the pair \( (\sigma, \pi) \) is \( A \)-inner product preserving.

Proof. From Lemma 8.5.23 in [1] there exist partial isometries \( \{m_i : i \in I\} \subset \mathcal{M} \) with mutually orthogonal initial spaces and \( \{n_j : j \in J\} \subset \mathcal{M} \) with mutually orthogonal final spaces such that \( \sum_{i \in I} \oplus m_i^* m_i = I_H, \sum_{j \in J} \oplus n_j^* n_j = I_K \).
By introducing sufficiently many 0 partial isometries to each set, we may assume that $T^2 = I = J$. We denote by $m$ the column $(m_i)_{i \in I} \in C^w_I(M)$. We have $m^*m = I_H$ and we denote by $p$ the projection $mm^* \in M_I(B)$.

In what follows if $U_n \subset B(H_n, K)$ are $w^*$-closed subspaces, $H_n, K$ Hilbert spaces, $n \in \mathbb{N}$, we denote by $U_1 \oplus_r U_2 \oplus_r \ldots$ the $w^*$-closed subspace of $B(\sum_n \oplus H_n, K)$ generated by the bounded operators of the form $(u_1, u_2, \ldots), u_n \in U_n, n \in \mathbb{N}$. Also if $V_n \subset B(K, H_n)$ are $w^*$-closed subspaces, $n \in \mathbb{N}$ we denote by $V_1 \oplus_c V_2 \oplus_c \ldots$ the $w^*$-closed subspace of $B(K, \sum_n \oplus H_n)$ generated by the bounded operators of the form $(v_1, v_2, \ldots), v_n \in V_n, n \in \mathbb{N}$. If $(x_i)_{i \in I} \in R^w_I(R^w_I(X))$ where $x_i \in R^w_I(X)$ then $x_i m \in A$ and so we can define the maps

\[ \tau_1 : R^w_I(R^w_I(X)) \to R^w_I(A) \oplus_r R^w_I(R^w_I(X)p^\perp), \]

\[ \tau_1((x_i)_{i \in I}) = ((x_i m_i)_{i \in I}, (x_i p^\perp)_{i \in I}), \quad x_i \in R^w_I(X) \]

and

\[ \tau_2 : C^w_I(C^w_I(Y)) \to C^w_I(A) \oplus_c C^w_I(p^\perp C^w_I(Y)), \]

\[ \tau_2((y_i)_{i \in I}) = ((m^* y_i)_{i \in I}, (p^\perp y_i)_{i \in I})^t, \quad y_i \in C^w_I(Y). \]

This pair of maps is $A$-inner product preserving: if $x \in R^w_I(R^w_I(X)), y \in C^w_I(C^w_I(Y))$ then

\[ \tau_1(x) \tau_2(y) = (xm, xp^\perp)(m^* y, p^\perp y)^t = xmm^* y + x p^\perp y = xy + xp^\perp y = xy. \]

These maps are onto because every $a \in A$ may be written $a = (am^*)m$ with $am^* \in R^w_I(X)$ and also $a = m^* (ma)$ with $ma \in C^w_I(Y)$ and they are clearly $w^*$-continuous $A$-module maps. Also they are complete isometries. We check this fact for $\tau_1$ and $n = 2$ : If $x = (x_{ij}) \in M_2(R^w_I(R^w_I(X)))$ we have

\[ \|\tau_1(x)\|^2 = \left\| \begin{bmatrix} x_{11}m & x_{11}p^\perp & x_{12}m & x_{12}p^\perp \\ x_{21}m & x_{21}p^\perp & x_{22}m & x_{22}p^\perp \end{bmatrix} \right\|^2 \]

\[ = \left\| \begin{bmatrix} x_{11}m & x_{12}m & x_{11}p^\perp & x_{12}p^\perp \\ x_{21}m & x_{22}m & x_{21}p^\perp & x_{22}p^\perp \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} x & 0 \\ 0 & m \end{bmatrix} \right\|^2 \]

\[ = \|xx^*\| = \|x\|^2. \]

We use the symbol $\infty$ for the $\aleph_0$ cardinal. The following spaces are isomorphic as $A$-modules and as dual operator spaces:

\[ R^w_{I\infty}(R^w_I(R^w_I(X))) \cong R^w_I(A) \oplus_r R^w_I(R^w_I(X)p^\perp) \oplus_r R^w_I(A) \oplus_r \ldots \cong R^w_I(A) \oplus_r R^w_{\infty}(R^w_I(R^w_I(X))) \]
and

$$C^w_\infty(C^w_I(C^w_I(Y))) \cong C^w_I(A) \oplus C^w_I(p^\perp C^w_I(Y,X)) \oplus C^w_I(A) \oplus \ldots$$

$$\cong C^w_I(A) \oplus C^w_\infty(C^w_I(C^w_I(Y)))$$

Since $I^2 = I$ it follows that $\infty I = I$ so we have

$$R^w_I(X) \cong R^w_\infty(R^w_I(R^w_I(X)))$$

and

$$C^w_I(Y) \cong C^w_\infty(C^w_I(C^w_I(Y))).$$

We conclude that there exist completely isometric, $w^*$-continuous, $A$-module bijections

$$\lambda_1 : R^w_I(X) \to R^w_I(A) \oplus R^w_I(X)$$

and

$$\lambda_2 : C^w_I(Y) \to C^w_I(A) \oplus C^w_I(Y).$$

We can choose $\lambda_1, \lambda_2$ to be $A$-inner product preserving. Similarly working with the partial isometries $\{n_j : j \in I\}$ (see the beginning of the proof) we obtain a pair $(\nu_1, \nu_2)$ of $A$-inner product preserving, completely isometric, $w^*$-continuous $A$-module bijections:

$$\nu_1 : R^w_I(A) \oplus R^w_I(X) \to R^w_I(A)$$

and

$$\nu_2 : C^w_I(A) \oplus C^w_I(Y) \to C^w_I(A).$$

The maps

$$\sigma = \nu_1 \circ \lambda_1 : R^w_I(X) \to R^w_I(A)$$

and

$$\pi = \nu_2 \circ \lambda_2 : C^w_I(Y) \to C^w_I(A)$$

satisfy our requirements.

**Theorem 3.2.** Two unital dual operator algebras are $\Delta$-equivalent if and only if they are stably isomorphic.

**Proof.** It suffices to show that if the algebras, $A$ and $B$, are TRO-equivalent, then they are stably isomorphic. Let $I, \sigma, \pi$ be as in Lemma 3.1. Observe that $A C^w_I(M) \cong M_I(B)$ and the $C^w_I(M)$-generated $A - M_I(B)$ bimodules (see definition 2.1) are the spaces $R^w_I(X)$ and $C^w_I(Y)$. So by Theorem 2.3 the map

$$\psi_1 : C^w_I(Y) \overset{\sigma_h}{\otimes}_A R^w_I(X) \to M_I(B) : \psi_1(y \otimes_A x) = yx$$

is a completely isometric, $w^*$-continuous bijection. For the same reason the map

$$\psi_2 : C^w_I(A) \overset{\sigma_h}{\otimes}_A R^w_I(A) \to M_I(A) : \psi_2(a \otimes_A c) = ac$$

is a completely isometric, $w^*$-continuous bijection. The map

$$C^w_I(Y) \times R^w_I(X) \to C^w_I(A) \overset{\sigma_h}{\otimes}_A R^w_I(A) : (y, x) \to \pi(y) \otimes_A \sigma(x)$$

is completely contractive, separately $w^*$-continuous and $A$-balanced. So by Proposition 2.2 there exists a completely contractive $w^*$-continuous map

$$C^w_I(Y) \overset{\sigma_h}{\otimes}_A R^w_I(X) \to C^w_I(A) \overset{\sigma_h}{\otimes}_A R^w_I(A) : y \otimes_A x \to \pi(y) \otimes_A \sigma(x).$$
We denote this map by $\pi \otimes \sigma$. Similarly we can define a complete contraction $\pi^{-1} \otimes \sigma^{-1} : C^w_I(A) \otimes_A \sigmah I R^w_I(A) \to C^w_I(Y) \otimes_A \sigmah I R^w_I(X)$. Since $\pi^{-1} \otimes \sigma^{-1}$ is the inverse of $\pi \otimes \sigma$ we conclude that $\pi \otimes \sigma$ is a complete isometry. It follows that the map 
\[
\gamma = \psi_2 \circ (\pi \otimes \sigma) \circ \psi_1^{-1} : M_I(B) \to M_I(A)
\]
is a completely isometric, $w^*$-continuous bijection. It remains to check that it is an algebraic homomorphism. Since $M_I(B) = [C^w_I(Y)R^w_I(X)]^{-w^*}$ it suffices to show that $\gamma(y_1 x_1 \cdot y_2 x_2) = \gamma(y_1 x_1) \cdot \gamma(y_2 x_2)$ for all $x_1, x_2 \in R^w_I(X), y_1, y_2 \in C^w_I(Y)$. Indeed, 
\[
\begin{align*}
\gamma(y_1 x_1 \cdot y_2 x_2) &= \psi_2 \circ (\pi \otimes \sigma) \circ \psi_1^{-1}(y_1 x_1 y_2 \cdot x_2) = \psi_2(\psi_1(y_1 x_1 y_2) \otimes_A \sigma(x_2)) \\
&= \psi_2(\psi_1(y_1 x_1 y_2) \otimes_A \sigma(x_2)) = (x_1 y_2 \in A \text{ and } \pi \text{ is a } A\text{-module map}) \\
&= \pi(y_1 x_1 y_2) \sigma(x_2) \\
&= \pi(y_1) x_1 \sigma(x_1) \sigma(x_2) = \psi_2(\pi(y_1) \otimes_A \sigma(x_1)) \cdot \psi_2(\pi(y_2) \otimes_A \sigma(x_2)) \\
&= \psi_2(\pi(y_1) x_1 \sigma(x_1)) \cdot \psi_2(\pi(y_1) \cdot \pi(\pi \otimes \sigma)(y_2 \otimes_A x_2) \\
&= \psi_2(\pi(y_1) \cdot \pi \otimes \sigma)(y_1 x_1) \cdot \psi_2(\pi(y_2) \otimes_A \sigma(x_2)) \\
&= \gamma(y_1 x_1) \cdot \gamma(y_2 x_2)
\end{align*}
\]

\[\square\]

**Remark 3.3.** When the unital dual operator algebras $A, B$ have completely isometric normal representations $\alpha, \beta$ on separable, Hilbert spaces such that $\alpha(A)$ and $\beta(B)$ are TRO equivalent, then the proof of the above theorem shows that $M_\infty(A)$ and $M_\infty(B)$ are completely isometrically isomorphic, i.e., the index set $I$ may be taken to be countable.

### 4. Stably isomorphic CSL algebras.

In this section we assume that all Hilbert spaces are separable. A set of projections on a Hilbert space is called a **lattice** if it contains the zero and identity operators and is closed under arbitrary suprema and infima. If $A$ is a subalgebra of $B(H)$ for some Hilbert space $H$, the set 
\[
\text{Lat}(A) = \{l \in pr(B(H)) : l^\perp Al = 0\}
\]
is a lattice. Dually if $\mathcal{L}$ is a lattice the space 
\[
\text{Alg}(\mathcal{L}) = \{a \in B(H) : l^\perp al = 0 \ \forall \ l \in \mathcal{L}\}
\]
is an algebra. A commutative subspace lattice (CSL) is a projection lattice $\mathcal{L}$ whose elements commute; the algebra Alg($\mathcal{L}$) is called a **CSL algebra**.
Let \( \mathcal{L} \) be a CSL and \( l \in \mathcal{L} \). We denote by \( l \downarrow \) the projection \( \lor \{ r \in \mathcal{L} : r < l \} \). Whenever \( l < l \) we call the projection \( l - l \) an atom of \( \mathcal{L} \). If the CSL \( \mathcal{L} \) has no atoms we say that it is a continuous CSL. If the atoms span the identity operator we say that \( \mathcal{L} \) is a totally atomic CSL.

If \( \mathcal{L}_1, \mathcal{L}_2 \) are CSL’s, \( \phi : \mathcal{L}_1 \to \mathcal{L}_2 \) is a lattice isomorphism (a bijection which preserves order) and \( p \) (resp. \( q \)) is the span of the atoms of \( \mathcal{L}_1 \) (resp. of \( \mathcal{L}_2 \)) there exists a well defined lattice isomorphism \( \mathcal{L}_1|_p \to \mathcal{L}_2|_q : l|_p \to \phi(l)|_q \) (Lemma 5.3 in [6].) Observe that the CSL’s \( \mathcal{L}_1|_p^\perp \), \( \mathcal{L}_2|_q^\perp \) are continuous. But it is not always true that \( \phi \) induces a lattice isomorphism from \( \mathcal{L}_1|_p^\perp \) onto \( \mathcal{L}_1|_q^\perp \). In [3, 7.19] there exists an example of isomorphic nests \( \mathcal{L}_1, \mathcal{L}_2 \) such that \( p^\perp = 0 \) and \( q^\perp \neq 0 \).

This motivates the following definition:

**Definition 4.1.** [6] Let \( \mathcal{L}_1, \mathcal{L}_2 \) be CSL’s, \( \phi : \mathcal{L}_1 \to \mathcal{L}_2 \) be a lattice isomorphism, \( p \) the span of the atoms of \( \mathcal{L}_1 \) and \( q \) the span of the atoms of \( \mathcal{L}_2 \). We say that \( \phi \) respects continuity if there exists a lattice isomorphism \( \mathcal{L}_1|_p^\perp \to \mathcal{L}_2|_q^\perp \) such that \( l|_p^\perp \to \phi(l)|_q^\perp \) for every \( l \in \mathcal{L}_1 \).

The following was proved in [6] (Theorem 5.7).

**Theorem 4.1.** Let \( \mathcal{L}_1, \mathcal{L}_2 \) be separably acting CSL’s. The algebras \( \text{Alg}(\mathcal{L}_1), \text{Alg}(\mathcal{L}_2) \) are TRO equivalent if and only if there exists a lattice isomorphism \( \phi : \mathcal{L}_1 \to \mathcal{L}_2 \) which respects continuity.

Also we recall Theorem 3.2 in [8].

**Theorem 4.2.** Two CSL algebras are \( \Delta \)-equivalent if and only if they are TRO equivalent.

Combining Theorems 4.1, 4.2 with Theorem 3.2 we obtain the following:

**Theorem 4.3.** Two CSL algebras, acting on separable Hilbert spaces, are stably isomorphic if and only if there exists a lattice isomorphism which respects continuity.

**Remark 4.4.** In fact, since the CSL algebras, say \( \text{Alg}(\mathcal{L}_i), i = 1, 2 \) are acting on separable Hilbert spaces, we have that if there exists a lattice isomorphism between \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) that respects continuity, then \( M_\infty(\text{Alg}(\mathcal{L}_1)) \) and \( M_\infty(\text{Alg}(\mathcal{L}_2)) \) are completely isometrically isomorphic.

A consequence of this theorem is that two separably acting CSL algebras with continuous or totally atomic lattices are stably isomorphic if and only if they have isomorphic lattices.
REFERENCES

[1] D.P. Blecher, C. Le Merdy, Operator algebras and their modules, *London Mathematical Society Monographs*, 2004.
[2] D.P. Blecher, P.S. Muhly, V.I. Paulsen, Categories of operator modules-Morita equivalence and projective modules, *Memoirs of the A.M.S.* 143 (2000) No 681.
[3] Kenneth R. Davidson, *Nest algebras*, volume 191 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1988. Triangular forms for operator algebras on Hilbert space.
[4] E.G. Effros and Z-J Ruan, Operator spaces, London Mathematical Society Monographs, New series 23, The Clarendon Press, Oxford University Press, New York, 2000.
[5] E.G. Effros and Z-J Ruan, Operator space tensor products and Hopf convolution algebras, *J. Operator Theory* 50(2003), 131-156.
[6] G.K. Eleftherakis, TRO equivalent algebras, preprint, ArXiv:math.OA/0607488
[7] G.K. Eleftherakis, A Morita type equivalence for dual operator algebras, *Journal of Pure and Applied Algebra* (to appear), ArXiv:math.OA/0607489v4
[8] G.K. Eleftherakis, Morita type equivalences and reflexive algebras, Arxiv:math.OA/0709.0600.
[9] Christian Le Merdy, An operator space characterization of dual operator algebras, *American Journal of Mathematics*, 121 (1999), 55-63.
[10] V.I. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge Studies in Advanced Math. 78, Cambridge University Press, Cambridge, 2002.
[11] M.A. Rieffel, Morita equivalence for $C^*$-algebras and $W^*$-algebras, *Journal of Pure and Applied Algebra* 5: 51-96, 1974.
[12] Zhong-Jin Ruan, Type decomposition and the Rectangular AFD property for $W^*$-TRO’s, *Canad. J. Math.*, 56(2004), no 4, 843-870.

Dept. of Mathematics, University of Athens, Athens, Greece
E-mail address: gelefth@math.uoa.gr

Dept. of Mathematics, University of Houston, Houston, TX, 77204
E-mail address: vern@math.uh.edu