Security Games with Decision and Observation Errors

Kien C. Nguyen, Tansu Alpcan, and Tamer Başar

Abstract—We study two-player security games which can be viewed as sequences of nonzero-sum matrix games played by an Attacker and a Defender. The evolution of the game is based on a stochastic fictitious play process. Players do not have access to each other’s payoff matrix. Each has to observe the other’s actions up to present and plays the action generated based on the best response to these observations. However, when the game is played over a communication network, there are several practical issues that need to be taken into account: First, the players may make random decision errors from time to time. Second, the players’ observations of each other’s previous actions may be incorrect. The players will try to compensate for these errors based on the information they have. We examine convergence property of the game in such scenarios, and establish convergence to the equilibrium point under some mild assumptions when both players are restricted to two actions.

I. INTRODUCTION

Game theory has recently been used as an effective tool to model and solve many security problems in computer and communication networks. In a noncooperative matrix game between an Attacker and a Defender, if the payoff matrices are assumed to be known to both players, each player can compute the set of Nash equilibria of the game and play one of these strategies to maximize her expected gain (or minimize its expected loss). However, in practice, the players do not necessarily have full knowledge of each other’s payoff function. If the game is repeated, a mechanism called fictitious play (FP) can be used for each player to learn her opponent’s motivations. In a FP process, each player observes all the actions and makes estimates of the mixed strategy of her opponent. At each stage, she updates this estimate and plays the pure strategy that is the best response (or generated based on the best response) to the current estimate of the other’s mixed strategy. It can be seen that in a FP process, if one person plays a fixed strategy (either of the pure or mixed type), the other person’s strategy will converge to the best response to this fixed strategy. Furthermore, it has been shown that, for many classes of games, such a FP process will finally render both players playing the Nash equilibrium.

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1The problem of each player choosing a Nash equilibrium out of multiple Nash equilibria is not discussed within the scope of this paper.

In this paper, we examine a two-player game, where an Attacker (denoted as player 1 or \(P_1\)) and a Defender (denoted as player 2 or \(P_2\)) participate in a discrete-time repeated nonzero-sum matrix game. In a general setting, the Attacker has \(m\) possible actions and the Defender has \(n\) possible actions to choose from. When such a security game is played between two automated systems over a network, in order to have a good model, we have to take into account several practical issues. First, the players may make random decision errors from time to time. Instead of playing an action \(a_i^t\) that is the output of the best-response computation, player \(i\) may play another action \(a_i^{t'}\) with some probability (which is typically small for functional systems). Second, the observation that each player makes on her opponent’s actions may also be incorrect, which will definitely affect her own responding actions. There are many factors giving rise to these problems: The non-ideality of electronic and software systems, the uncertain and noisy characteristic of observation data, and the erroneous nature of the channels on which commands and observations are communicated, to name a few.

It is these scenarios that we aim to address in this paper. We examine convergence of players’ strategies in the FP process with decision and observation errors. If these strategies do converge, we quantify the new Nash equilibrium and thus estimate how these decision and observation errors affect the learning process and the equilibrium of the game.

Security games have been examined extensively in a number of papers, see for example, [1]–[4]. The work in [5] employs the framework of Bayesian games to address the intrusion detection problem in wireless ad hoc networks. In [6], the author examines the intrusion detection problem in heterogenous networks as a nonzero-sum static game. The work in [7] addresses this problem using the framework of zero-sum stochastic games [8]. In [9], we develop a network model based on linear influence networks that allows us to take into consideration the correlation among the nodes in terms of both security assets and vulnerabilities.

Relevant literature on fictitious play can be found in [10]–[16]. For two-player zero-sum classical FP, the convergence proof was obtained for arbitrary numbers of actions for each player \((m \times n)\) [10]. For nonzero-sum games, the proofs for two-player FP have been found for the case where one player is restricted to two actions (See [12] for classical FP and [13] for stochastic FP). In [19], we address the classical FP and stochastic FP with imperfect observations for the case where each player is restricted to two actions.

Our contributions in this paper are as follows. First, we formulate the repeated security games where players make
random decision errors as a fictitious play process. We discuss the convergence of such games in the general case with arbitrary numbers of actions for each player. We then establish the convergence property for several classes of games with decision errors where both players are restricted to two actions. Second, we examine the fictitious play process where the players’ observations are imperfect and the players try to compensate for the observation errors. We again establish the convergence property for the case where both players are restricted to two actions. We point out a number of scenarios that can be considered as special cases of this result.

In Section II, we introduce some background and notation adopted from [13], [14]. The analysis for the stochastic FP with decision errors is presented in Section III. In Section IV, we address the FP with observation errors. Finally, some concluding remarks end the paper.

II. BACKGROUND

A. Static games

We present an overview of some concepts for static security games, where player $P_i$ has $m$ and player $P_j$ has $n$ possible actions. In equations written for the generic player $P_i$, $i = 1, 2$, we use $k$ to denote $m$ or $n$. Denote by $p_1 \in \Delta(m)$ and $p_2 \in \Delta(n)$ a pair of mixed strategies for $P_1$ and $P_2$, respectively, where $\Delta(k)$ is the simplex in $\mathbb{R}^k$, i.e.,

$$\Delta(k) \equiv \left\{ s \in \mathbb{R}^k | s_j \geq 0, j = 1, \ldots, k, \sum_{j=1}^k s_j = 1 \right\}. \quad (1)$$

The utility function of $P_i$, $U_i(p_i, p_{-i})$, is given by $^2$

$$U_i(p_i, p_{-i}) = \sum_{a_i} M_{a_i} p_{a_i} + \beta_i H(p_i), \quad (2)$$

where $M_{a_i}$ is the payoff matrix of $P_i$, $i = 1, 2$, and $H : Int(\Delta(k)) \rightarrow \mathbb{R}$ is the entropy function of the probability vector $p_i$; $H(p_i) = -p_i^T \log(p_i)$ (Note that $M_{a_i}$ is of dimension $m \times n$ and $M_{a_i} n \times m$). The weighted entropy $\beta_i H(p_i)$ with $\beta_i \geq 0$ is introduced to boost mixed strategies. In a security game, $\beta_i$ represents how much player $i$ wants to randomize its actions, and thus is not necessarily known to the other player. Also, for $\beta_1 = \beta_2 = 0$ (referred to as classical FP), the best response mapping can be set-valued, while it has a unique value when $\beta_i \geq 0$ (referred to as stochastic FP) [4], [14]. For a static game, each player selects an integer action $a_i$ according to the mixed strategy $p_i$. The (instant) payoff for player $P_i$ is $u_i^T M_{a_i} v_{a_{-i}} + \beta_i H(p_i)$, where we use $u_i, a_i = 1, \ldots, k$, to indicate the $j$th vertex of the simplex $\Delta(k)$ (For example, when $k = 2$, $v_1 = [1 \ 0]^T$ for the first action, and $v_2 = [0 \ 1]^T$ for the second action). For a pair of mixed strategies $(p_1, p_2)$, the utility functions are given by the expected payoffs:

$$U_i(p_i, p_{-i}) = E \left[ v_{a_{-i}}^T M_{a_i} v_{a_{-i}} + \beta_i H(p_i) \right]. \quad (3)$$

Now, the best response mappings $\beta_1 : \Delta(n) \rightarrow \Delta(m)$ and $\beta_2 : \Delta(m) \rightarrow \Delta(n)$ are defined as:

$$\beta_i(p_{-i}) = \arg \max_{p_i \in \Delta(k)} U_i(p_i, p_{-i}). \quad (4)$$

If $\beta_i > 0$, from (4), the best response is unique as mentioned earlier, and is given by the soft-max function:

$$\beta_i(p_{-i}) = \sigma \left( \frac{M_{p_{-i}}}{\beta_i} \right), \quad (5)$$

where the soft-max function $\sigma : \mathbb{R}^k \rightarrow \text{Interior}(\Delta(k))$ is defined as

$$\sigma(x) = \frac{e^{x_j}}{\sum_{j=1}^k e^{x_j}}, j = 1, \ldots, k. \quad (6)$$

Note that $(\sigma(x))_j > 0$, and thus the range of the soft-max function is just the interior of the simplex.

Finally, a (mixed strategy) Nash equilibrium is defined to be a pair $(p^*_1, p^*_2) \in \Delta(m) \times \Delta(n)$ such that for all $p_1 \in \Delta(m)$ and $p_2 \in \Delta(n)$

$$U_i(p_i, p^*_{-i}) \leq U_i(p^*_i, p^*_{-i}). \quad (7)$$

We can also write a Nash equilibrium $(p^*_1, p^*_2)$ as the fixed point of the best response mappings:

$$p^*_1 = \beta_i(p^*_{-i}). \quad (8)$$

B. Fictitious play

1) Discrete-Time Fictitious Play: From the static game described in Subsection II-A, we define discrete-time FP as follows. Suppose that the game is repeated at times $k \in \{0, 1, 2, \ldots\}$. The empirical frequency $q_i(k)$ of player $P_i$ is given by

$$q_i(k + 1) = \frac{1}{k+1} \sum_{j=0}^k v_{a_i(j)} \quad (9)$$

Using induction, we can prove the following recursive relation:

$$q_i(k + 1) = \frac{k}{k+1} q_i(k) + \frac{1}{k+1} v_{a_i(k)} \quad (10)$$

At time $k$, player $P_i$ picks the best response to the empirical frequency of the opponent’s actions:

$$p_i(k) = \beta_i(q_{-i}(k)). \quad (11)$$

2) Continuous-Time Fictitious Play: From the equations of discrete-time FP (9), (10), the continuous-time version of the iteration can be stated as follows ([13], [14], also see [15], [19] for the derivation):

$$\dot{p}_i(t) = \beta_i(p_{-i}(t)) - p_i(t), \quad i = 1, 2. \quad (12)$$

C. Algorithms

We present in this subsection two algorithms for discrete-time stochastic FP. Algorithm II-C.1, derived from [13], [14], [19], is used for the case when players’ observations are considered to be perfect or when they have no estimates of observation errors. Algorithm II-C.2, a generalized version of the one in [19], is used for players who have estimates of observation errors and want to compensate for these errors.
1) **Stochastic FP with perfect observations:** In stochastic FP, at time \( k \), player \( i, i = 1, 2 \), carries out the following steps:

1. Update the empirical frequency of the opponent using (10).
2. Compute the best response \( \beta_i(q_{-i}(k)) \) using (5). (Note that the result is always a completely mixed strategy.)
3. Generate an action \( a_i(k) \) using the mixed strategy from step (2), \( a_i(k) = \text{rand}[\beta_i(q_{-i}(k))] \), where we use \( \text{rand} \) to denote the randomizer function that gives \( a_i(k) \) such that the expectation \( E[a_i(k)] = \beta_i(q_{-i}(k)) \).

2) **Stochastic FP with imperfect observations:** At time \( k \), player \( i, i = 1, 2 \), carries out the following steps:

1. Update the observed frequency of the opponent \( \overline{q}_{-i} \) using (10).
2. Compute the estimated frequency
   \[
   q_{-i} = f_{-i}(\overline{q}_{-i}).
   \] (13)
3. Compute the best response \( \beta_i(q_{-i}(k)) \) using (5). (Note that the result is always a completely mixed strategy.)
4. Generate an action \( a_i(k) \) using the mixed strategy from step (3), \( a_i(k) = \text{rand}[\beta_i(q_{-i}(k))] \).

**D. A convergence result for \( m = n = 2 \) with perfect observations**

We restate the following theorem from [13], [19], for the general case where the coefficients of the entropy terms for the players (\( \tau_1 \) and \( \tau_2 \)) are not necessarily equal (Cf. Equation (2)). This theorem in [13] is stated for \( \tau_1 = \tau_2 \), however, one can always scale the payoff matrices to get the general case.

**Theorem 1:** (A variant of Theorem 3.2 [13] for general \( \tau_1, \tau_2 > 0 \)) Consider a two-player two-action fictitious play process with \( (L^T M_1 L)(L^T M_2 L) \neq 0 \), where \( M_i \) are the payoff matrices of \( P_i, i = 1, 2 \), and \( L := (1, -1)^T \). The solutions of continuous-time FP (12) satisfy
\[
\lim_{t \to \infty} (p_1(t) - \beta_1(p_2(t))) = 0 \quad (14)
\]
\[
\lim_{t \to \infty} (p_2(t) - \beta_2(p_1(t))) = 0, \quad (15)
\]
where \( \beta_i(p_{-i}, i = 1, 2 \), are given in (5).

**III. SECURITY GAMES WITH DECISION ERRORS**

In this section, we consider the situations where players are not totally rational or the channels carrying commands are error prone. Specifically, \( P_i \) makes decision errors with probabilities \( \alpha_{ij} \)’s where \( \alpha_{ij}, i, j = 1 \ldots m \), is the probability that \( P_i \) intends to play action \( i \) but ends up playing action \( j \), \( \alpha_{ij} \geq 0, \sum_{j=1}^m \alpha_{ij} = 1, i = 1 \ldots m \). Similarly, \( P_2 \)’s decision error probabilities are given by \( \epsilon_{ij}, \epsilon_{ij} \geq 0, \sum_{j=1}^m \epsilon_{ij} = 1, i = 1 \ldots n \). This is called “trembling hand” problem in the game theory literature (See for example, Reference [17], Subsection 3.5.5). The decision error matrices \( D_1 \) and \( D_2 \) are given below.
\[
D_1 = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \ldots & \alpha_{1m} \\
\alpha_{21} & \alpha_{22} & \ldots & \alpha_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mm}
\end{pmatrix}, \quad (16)
\]
\[
D_2 = \begin{pmatrix}
\epsilon_{11} & \epsilon_{12} & \ldots & \epsilon_{1n} \\
\epsilon_{21} & \epsilon_{22} & \ldots & \epsilon_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\epsilon_{n1} & \epsilon_{n2} & \ldots & \epsilon_{nn}
\end{pmatrix}. \quad (17)
\]

When \( m = n = 2 \), the decision error matrices can be written as:
\[
D_1 = \begin{pmatrix}
1 - \alpha & \gamma \\
\alpha & 1 - \gamma
\end{pmatrix}, \quad D_2 = \begin{pmatrix}
1 - \epsilon & \mu \\
\epsilon & 1 - \mu
\end{pmatrix}. \quad (18)
\]

The decision errors of each player in this case are illustrated in Figure 1. In what follows, we state two standard results in digital communications. The proofs are similar to those for the case \( m = n = 2 \) in [19].

**Proposition 1:** Consider the two-player discrete-time fictitious play with decision errors where the error probabilities are given in Equations (16) and (17). Let \( \bar{\alpha}_{ij}, i, j = 1 \ldots m \), and \( \bar{\epsilon}_{ij}, i, j = 1 \ldots n \), be the empirical decision error frequencies of \( P_1 \) and \( P_2 \), respectively. If decision errors are assumed to be independent from stage to stage, it holds that
\[
\lim_{k \to \infty} \text{a.s. } \bar{\alpha}_{ij} = \alpha_{ij}, i, j = 1 \ldots m,
\]
\[
\lim_{k \to \infty} \text{a.s. } \bar{\epsilon}_{ij} = \epsilon_{ij}, i, j = 1 \ldots n. \quad (19)
\]
where we use \( \text{a.s.} \) to denote almost sure convergence.

**Proposition 2:** Consider a two-player discrete-time fictitious play with decision errors where the error probabilities are given in Equations (16) and (17). Let \( \overline{q}_i \) be the empirical frequency of player \( i \)’s real actions and \( q_i \) be the frequency of player \( i \)’s intended actions (generated from the best response at each stage). If decision errors are assumed to be independent from stage to stage, it holds that
\[
\lim_{k \to \infty} \text{a.s. } \overline{q}_i = D_i(\lim_{k \to \infty} \text{a.s. } q_i), i = 1, 2, \quad (20)
\]
where \( D_i \) are the decision error matrices given in Equations (16) and (17).

**A. If the players know their own decision error probabilities**

We first consider the case where the players both have complete information about the decision error matrices \( D_i \), \( i = 1, 2 \). If they both also know the payoff matrices \( M_i, i = 1, 2 \), then each can compute and play one of the Nash equilibria right from beginning. The problem then can be considered as a stochastic version of the trembling hand problem. Specifically, suppose that each player still wants to randomize their empirical frequency \( p_i \) (instead of the frequency of their intended actions, or intended frequency, \( p_i \)) by including an entropy term in their utility function, we have that
\[
U_i(p_i, p_{-i}) = p_i^T \tilde{M}_i p_{-i} + \tau_i H(D_i p_i), i = 1, 2, \quad (21)
\]
where \( p_i \)'s are intended frequencies, \( M_1 = D_1^T M_1 D_2 \) and \( M_2 = D_2^T M_2 D_1 \) (These are the payoff matrices resulted from decision errors using the results in Propositions 1 and 2, see for example [17] for derivation). Using \( \overline{p}_i := D_i p_i, \ i = 1, 2 \), the utility functions now can be written as

\[
U_i(p_i, p_{-i}) = \overline{p}_i^T M_i \overline{p}_{-i} + \tau_i H(\overline{p}_i), \quad i = 1, 2. \tag{22}
\]

The game is thus reduced to the one without decision errors and the Nash Equilibrium of the static game is known from Subsection II-A to satisfy:

\[
\overline{p}_i = \beta_i(\overline{p}_{-i}), \quad i = 1, 2, \tag{23}
\]

or equivalently (with the assumption that \( D_i \)'s are invertible):

\[
p_i^* = (D_i)^{-1} \beta_i(D_{-i} p_{-i}^*), \quad i = 1, 2. \tag{24}
\]

The best response is now given as

\[
p_i = (D_i)^{-1} \beta_i(\overline{p}_{-i}) = (D_i)^{-1} \beta_i(\overline{p}_{i-1}) \tag{25}
\]

In the corresponding FP process (the "trembling hand stochastic FP"), as each player \( P_i \) can observe her opponent’s empirical frequency \( \overline{p}_{-i} \), she does not need to know \( D_{-i} \) to compute the best response. We thus state below a convergence result for the FP process with decision errors for the case \( m = n = 2 \).

**Proposition 3:** Consider a two-player two-action fictitious play process where players make decision errors with invertible decision error matrices \( D_1 \) and \( D_2 \), respectively. Suppose that at each step, each player calculates the best response taking into account their own decision errors using Equation (25). If \( (L^T M_1 L)(L^T M_2 L) \neq 0, \ L := (1, -1)^T \), the solutions of the continuous-time FP process with decision errors will satisfy

\[
\lim_{t \to \infty} p_1(t) = D_1^{-1} \beta_1(\overline{p}_{-1}) \tag{26}
\]

\[
\lim_{t \to \infty} p_2(t) = D_2^{-1} \beta_2(\overline{p}_{-2}) \tag{27}
\]

where \( \sigma(.) \) is the soft-max function defined in (6).

**Proof:** The proof can be obtained using Theorem 1 and the fact \( \overline{p}_i := D_i p_i, \ i = 1, 2. \)

It thus can be seen that with knowledge of their own decision errors, players can completely precompensate for these errors and the equilibrium empirical frequencies remain the same as those of the original game without decision errors.

**B. If the players are unaware of all the decision error probabilities**

We consider in this subsection a two-player fictitious play process with decision errors where the decision error probabilities are not known to both players. Each player plays the regular stochastic FP Algorithm II-C.1. We are interested in whether or not the FP process will converge, and when it does, what the equilibrium will be. We first examine the general case with arbitrary \( m, n \), and then the special case where \( m = n = 2 \). We first use Proposition 2 and the same arguments as in the proof of Theorem 3 [19] to approximate the discrete-time FP with the continuous-time version. At time step \( k \), as each player \( P_i \) generates her action \( v_{ai}(k) \) based on the best response to her opponent’s empirical frequency \( \overline{p}_{-i} \), the expectation of \( v_{ai}(k) \), \( i = 1, 2 \), will be given by

\[
E[v_{ai}(k)] = D_1 \beta_1(\overline{p}_2(k)), \quad E[v_{ai}(k)] = D_2 \beta_2(\overline{p}_1(k)),
\]

where \( D_1 \) and \( D_2 \) account for decision errors. The mean dynamic of the empirical frequencies then can be written as follows

\[
\overline{p}_1(k+1) = \frac{k}{k+1} \overline{p}_1(k) + \frac{1}{k+1} D_1 \beta_1(\overline{p}_2(k)), \quad \overline{p}_2(k+1) = \frac{k}{k+1} \overline{p}_2(k) + \frac{1}{k+1} D_2 \beta_2(\overline{p}_1(k)). \tag{28}
\]

From the mean dynamic, we can derive the continuous-time approximation (See [20] for the derivation):

\[
\dot{\overline{p}}_1(t) = D_1 \beta_1(\overline{p}_2(t)) - \overline{p}_1(t), \quad \dot{\overline{p}}_2(t) = D_2 \beta_2(\overline{p}_1(t)) - \overline{p}_2(t). \tag{29}
\]

It can be seen that a pair of mixed strategies \( (p_1^*, p_2^*) \) that satisfies

\[
\overline{p}_1(t) = D_1 \beta_1(\overline{p}_2(t)), \quad \overline{p}_2(t) = D_2 \beta_2(\overline{p}_1(t)),
\]

will be an equilibrium point of the dynamics (28). For some results on the stability of the equilibrium point in the continuous-time system and the discrete-time system for general values of \( m, n \), we refer to [20]. When \( m = n = 2 \), it turns out the point \( (\overline{p}_1, \overline{p}_2) \) is globally stable for the continuous-time system under some mild assumptions. We thus state the following theorem for this special case.

**Theorem 2:** Consider a two-player two-action fictitious play process where players make decision errors with decision error matrices \( D_1 \) and \( D_2 \), respectively. Suppose that the players are unaware of all the decision error probabilities
and use the regular stochastic FP algorithm II-C.1. If $D_i$, $i = 1, 2$, are invertible and $(LT M_1 D_2 L)(LT M_2 D_1 L) \neq 0$, the solutions of continuous-time FP process with decision errors (28) will satisfy

$$
\lim_{t \to \infty} p_1(t) = D_1 \sigma \left( \frac{M_1 \lim_{t \to \infty} p_2(t)}{\tau_1} \right),
\lim_{t \to \infty} p_2(t) = D_2 \sigma \left( \frac{M_2 \lim_{t \to \infty} p_1(t)}{\tau_2} \right),
$$

(29)

where $\sigma(.)$ is the soft-max function defined in (6).

**Proof:** The proof, some remarks, and a numerical example can be found in [20].

**IV. SECURITY GAMES WITH OBSERVATION ERRORS**

In [19], we study the effect of observation errors on convergence to the NE in a $2 \times 2$ FP process. We also prove that if each player has a correct estimate of error probabilities of observations, they can reverse the effect of the channel to obtain the NE of the original static game. In this section, we present a generalized version of these results. Consider a two-player fictitious play game with imperfect observations where the error channels are given in Equations (30) and (31).

$$
C_1 = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \ldots & \alpha_{1m} \\
\alpha_{21} & \alpha_{22} & \ldots & \alpha_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mm}
\end{pmatrix},
$$

(30)

$$
C_2 = \begin{pmatrix}
\epsilon_{11} & \epsilon_{12} & \ldots & \epsilon_{1n} \\
\epsilon_{21} & \epsilon_{22} & \ldots & \epsilon_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\epsilon_{m1} & \epsilon_{m2} & \ldots & \epsilon_{mn}
\end{pmatrix},
$$

(31)

where $\alpha_{ij}, i, j = 1 \ldots m$ is the probability that $P_1$’s action $i$ is erroneously observed as action $j$, $\alpha_{ij} \geq 0$, $\sum_{j=1}^{m} \alpha_{ij} = 1$, $i = 1 \ldots m$, and $\epsilon_{ij}, i, j = 1 \ldots n$ is the probability that $P_2$’s action $i$ is erroneously observed as action $j$, $\epsilon_{ij} \geq 0$, $\sum_{j=1}^{n} \epsilon_{ij} = 1$, $i = 1 \ldots n$. Suppose that the players have their estimates of the error probabilities as follows:

$$
\overline{C}_1 = \begin{pmatrix}
\overline{\alpha}_{11} & \overline{\alpha}_{12} & \ldots & \overline{\alpha}_{1m} \\
\overline{\alpha}_{21} & \overline{\alpha}_{22} & \ldots & \overline{\alpha}_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\overline{\alpha}_{m1} & \overline{\alpha}_{m2} & \ldots & \overline{\alpha}_{mm}
\end{pmatrix},
$$

(32)

$$
\overline{C}_2 = \begin{pmatrix}
\overline{\epsilon}_{11} & \overline{\epsilon}_{12} & \ldots & \overline{\epsilon}_{1n} \\
\overline{\epsilon}_{21} & \overline{\epsilon}_{22} & \ldots & \overline{\epsilon}_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\overline{\epsilon}_{m1} & \overline{\epsilon}_{m2} & \ldots & \overline{\epsilon}_{mn}
\end{pmatrix},
$$

(33)

where $\overline{\alpha}_{ij} \geq 0$, $\sum_{j=1}^{m} \overline{\alpha}_{ij} = 1$, $i = 1 \ldots m$, and $\overline{\epsilon}_{ij} \geq 0$, $\sum_{j=1}^{n} \overline{\epsilon}_{ij} = 1$, $i = 1 \ldots n$. We first restate Propositions 1 and 2 in the context of repeated games with imperfect observations.

**Proposition 4:** Consider the two-player discrete-time fictitious play with imperfect observations where error probabilities are given in Equations (30) and (31). Let $\overline{\alpha}_{ij}, i, j = 1 \ldots m$, and $\overline{\epsilon}_{ij}, i, j = 1 \ldots n$, be the empirical error frequencies of observations on $P_1$’s and $P_2$’s actions, respectively. If channel errors are assumed to be independent from stage to stage, it holds that

$$
\lim_{k \to \infty} a.s. \overline{\alpha}_{ij} = \alpha_{ij}, \ i,j = 1 \ldots m,
\lim_{k \to \infty} a.s. \overline{\epsilon}_{ij} = \epsilon_{ij}, \ i,j = 1 \ldots n.
$$

(34)

where we use $\lim$ a.s. to denote almost sure convergence.

**Proposition 5:** Consider the two-player discrete-time fictitious play with imperfect observations where error probabilities are given in Equations (30) and (31). Let $\overline{q}_i$ be the observed frequency and $q_i$ be the empirical frequency of player $i$. If channel errors are assumed to be independent from stage to stage, it holds that

$$
\lim_{k \to \infty} a.s. \overline{q}_i = C_i \left( \lim_{k \to \infty} a.s. q_i \right), \ i = 1, 2,
$$

(35)

where $C_i$ are the channel error matrices given in Equations (30) and (31).

If both players have their estimates of the error probabilities as in Equations (32) and (33), they can play the stochastic FP algorithm given in II-C.2 with $f_i(\overline{\theta}_{-i}) = (\overline{C}_i)^{-1}\overline{q}_{-i}$ to compensate for observation errors (Using the results in Propositions 4 and 5). Again we can use the same procedure as in Subsection III-B to approximate the discrete-time FP with the continuous-time version.

$$
q_1(k+1) = \frac{k}{k+1} q_1(k) + \frac{1}{k+1} \sigma \left( \frac{M_1(\overline{C}_2)^{-1}C_2 q_2(k)}{\tau_1} \right),
$$

$$
q_2(k+1) = \frac{k}{k+1} q_2(k) + \frac{1}{k+1} \sigma \left( \frac{M_2(\overline{C}_1)^{-1}C_1 q_1(k)}{\tau_2} \right).
$$

The continuous-time approximation is given by:

$$
p_1(t) = \sigma \left( \frac{M_1(\overline{C}_2)^{-1}C_2 p_2(t)}{\tau_1} \right) - p_1(t),
$$

$$
p_2(t) = \sigma \left( \frac{M_2(\overline{C}_1)^{-1}C_1 p_1(t)}{\tau_2} \right) - p_2(t).
$$

(36)
It can be seen that a pair of mixed strategies \((q_1^*, q_2^*)\) that satisfies
\[
\begin{align*}
p_1^*(t) &= \sigma \left( \frac{M_1(C_2)^{-1}C_2p_2^*(t)}{\tau_1} \right), \\
p_2^*(t) &= \sigma \left( \frac{M_2(C_1)^{-1}C_1p_1^*(t)}{\tau_2} \right),
\end{align*}
\]
will be an equilibrium point of the dynamics (36). For some results on the stability of the equilibrium point in the continuous-time system and the discrete-time system for general values of \(m\) and \(n\), we refer to [20]. When \(m = n = 2\), again the point \((p_1^*, p_2^*)\) is globally stable for the continuous-time system under some mild assumptions. We have the following theorem.

**Theorem 3:** Consider a two-player two-action fictitious play game with imperfect observations where the error channels are given in Figure 2 and Equation (37).
\[
C_1 = \left( \begin{array}{cc} 1 - \alpha & \gamma \\ \alpha & 1 - \gamma \end{array} \right), \quad C_2 = \left( \begin{array}{cc} 1 - \epsilon & \mu \\ \epsilon & 1 - \mu \end{array} \right)
\]
Suppose that the players have their estimates of the error probabilities as follows:
\[
\overline{C}_1 = \left( \begin{array}{cc} 1 - \overline{\alpha} & \overline{\gamma} \\ \overline{\alpha} & 1 - \overline{\gamma} \end{array} \right), \quad \overline{C}_2 = \left( \begin{array}{cc} 1 - \overline{\epsilon} & \overline{\mu} \\ \overline{\epsilon} & 1 - \overline{\mu} \end{array} \right)
\]
The players then play the stochastic FP given in II-C.2. If \((L^T M_1(C_2)^{-1}C_2L)(L^T M_2(C_1)^{-1}C_1L) \neq 0\), the solutions of continuous-time FP with imperfect observations (12) will satisfy
\[
\begin{align*}
limit_{t \to \infty} p_1(t) &= \sigma \left( \frac{M_1(C_2)^{-1}C_2 \lim_{t \to \infty} p_2(t)}{\tau_1} \right), \\
limit_{t \to \infty} p_2(t) &= \sigma \left( \frac{M_2(C_1)^{-1}C_1 \lim_{t \to \infty} p_1(t)}{\tau_2} \right).
\end{align*}
\]
where \(\sigma(\cdot)\) is the soft-max function defined in (6).

*Proof:* The proof, some remarks, and a numerical example can be found in [20].

V. CONCLUSION

In this paper, we have introduced and discussed some repeated security game models that take into account players' decision errors and observation errors. Each player does not have access to her opponent's payoff matrix and thus has to learn this through the fictitious play process. However, in a practical setting, each player is expected to make random decision errors from time to time and also has to respond to imperfectly observed actions of the other player. We have studied the convergence property of such games and, if the FP process does converge, quantified the new equilibrium. Such analyses will help provide guidelines for players to maximize their gain or minimize their loss in a nonideal environment.

We normally start from the mean dynamics of the discrete-time version of a game, proceed to continuous-time approximation and then analyze convergence of this continuous-time version. Although the convergence of the continuous-time fictitious play does not guarantee the almost sure convergence of the discrete-time counterpart, it does provide the necessary limiting results for the discrete-time version.

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