Abstract. In this article, we investigate the orbit configuration spaces of some equivariant closed manifolds over simple convex polytopes in toric topology, such as small covers, quasi-toric manifolds and (real) moment-angle manifolds; especially for the cases of small covers and quasi-toric manifolds. These kinds of orbit configuration spaces are all non-free and noncompact, but still built via simple convex polytopes. We obtain an explicit formula of Euler characteristic for orbit configuration spaces of small covers and quasi-toric manifolds in terms of the $h$-vector of a simple convex polytope. As a by-product of our method, we also obtain a formula of Euler characteristic for the classical configuration space, which generalizes the Félix-Thomas formula. In addition, we also study the homotopy type of such orbit configuration spaces. In particular, we determine an equivariant strong deformation retract of the orbit configuration space of 2 distinct orbit-points in a small cover or a quasi-toric manifold, which turns out that we are able to further study the Betti numbers and (equivariant) cohomology of such an orbit configuration space.

1. Introduction

Let $G$ be a topological group and let $M$ be a $G$-space. The (ordered) orbit configuration space $F_G(M,n)$ is defined by

$$F_G(M,n) = \{(x_1, \ldots, x_n) \in M^n \mid G(x_i) \cap G(x_j) = \emptyset \text{ for } i \neq j\}$$

with subspace topology, where $n \geq 2$ and $G(x)$ denotes the orbit at $x$. In the case where $G$ acts trivially on $M$, the space $F_G(M,n)$ is the classical configuration space denoted by $F(M,n)$.

The notion of configuration space had been introduced in physics in 1940s [N, vT] concerning the topology of configurations with various study on this important object since then. In mathematics, configuration spaces were first introduced by Fadell and Neuwirth [FN] in 1962 with various applications [Ar, BCWW, Bir, Bo, Co, T, V]. Since 1990s, the notion of configurations was introduced in robotics community to study safe-control problems of robots. Mathematically the topological robotics was recently established by Ghrist and Farber [Fa, Gh], where the topology of configuration spaces on graphs plays an important role in this new created area. The orbit configuration spaces with labels provide combinatorial models for equivariant...
loop spaces \([\mathbb{X}]\). Moreover orbit configuration space is an analogy to fiber-type arrangements \([\text{Coh}]\). The fundamental groups of orbit configuration spaces enrich the theory of braids \([\text{CKX}, \text{CX}]\).

If the group \(G\) acts properly discontinuously on a manifold \(M\), there are various fibrations available related to \(F_G(M,n)\) \([\mathbb{X}]\). Thus the standard methods of spectral sequences in algebraic topology can be used for studying the cohomology of \(F_G(M,n)\). In particular, the cohomology of \(F_{\mathbb{Z}}(S^k,n)\) has been determined in \([\text{FZ}, \mathbb{X}]\), where \(\mathbb{Z}\) acts (freely) on \(S^k\) by antipodal map. However the determination of the homotopy type or cohomology of \(F_G(M,n)\) become much harder provided that the group \(G\) does not act freely on a manifold \(M\) because of lacking effective tools from algebraic topology. For instance, the classical Fadell-Neuwirth fibration \([\text{FN}]\) fail in non-free cases in general.

In 1991, Davis and Januszkiewicz \([\text{DJ}]\) introduced four classes of particularly nicely behaving manifolds over simple convex polytopes–small covers, quasi-toric manifolds and (real) moment-angle manifolds, which have become important objects in toric topology. Note that (real) moment-angle manifolds were named by Buchstaber and Panov \([\text{BP}]\) later when they studied the topology of (real) moment-angle manifolds as submanifolds in polydisks. A quasi-toric manifold (resp. small cover), as the topological version of a compact non-singular toric variety (resp. real toric variety), is a smooth closed manifold \(M\) of dimension \(2m\) (resp. dimension \(m\)) with a locally standard action of torus \(T^m\) (resp. real torus \(\mathbb{Z}_2^m\)) such that its orbit space is a simple convex \(m\)-polytope \(P\). A (real) moment-angle manifold can directly be constructed from a simple convex polytope \(P\) such that it admits an action of real torus or torus with \(P\) as its orbit space. There are strong links between topology and geometry of these equivariant manifolds and combinatorics of polytopes. In this article, we put these equivariant manifolds into the framework of orbit configuration spaces, especially for the cases of small covers and quasi-toric manifolds. In other words, we pay our much more attention on non-free orbit configuration space \(F_{G_d^m}(M,n)\) for a \(dm\)-dimensional \(G_d^m\)-manifold \(\pi_d : M \rightarrow P\) over a simple convex \(m\)-polytope \(P\), \(d = 1, 2\), where \(M\) is a small cover and \(G_1^m = \mathbb{Z}_2^n\) when \(d = 1\), and a quasi-toric manifold and \(G_2^m = T^m\) when \(d = 2\). We still expect that there is an essential connection between topology and geometry of \(F_{G_d^m}(M,n)\) and combinatorics of \(P\).

Our first result is an explicit formula for the Euler characteristic of \(F_{G_d^m}(M,n)\) in terms of the \(h\)-vector \((h_0, h_1, \ldots, h_m)\) of \(P\), and in particular, \(\chi(F_{T^m}(M,n)) = \chi(F(M,n))\) if \(d = 2\). Let \(h_p(t) = h_0 + h_1 t + \cdots + h_m t^m\) be a polynomial in \(\mathbb{Z}[t]\). Then our result is stated as follows.

**Theorem 1.1.** Let \(\pi_d : M \rightarrow P\) be a \(dm\)-dimensional \(G_d^m\)-manifold over a simple convex \(m\)-polytope \(P\) where \(d = 1, 2\). Then the Euler characteristic of \(F_{G_d^m}(M,n)\) is

\[
\chi(F_{G_d^m}(M,n)) = \begin{cases} 
(-1)^{mn} \sum_{I = (n_1, \ldots, n_s)} \mathcal{C}_I \prod_{k=1}^{s} h_p(1 - 2^{n_k}) & \text{if } d = 1 \\
\chi(F(M,n)) = \sum_{I = (n_1, \ldots, n_s)} \mathcal{C}_I h_p(1)^s & \text{if } d = 2 
\end{cases}
\]

where \(I = (n_1, \ldots, n_s)\) runs over all partitions of \(n\), \(\mathcal{C}_I = \frac{n! (-1)^{n-s} r_1 r_2 \cdots r_s}{r_1! r_2! \cdots r_s! n_1 n_2 \cdots n_s}\) and \(r_k\) is the time number that \(n_k\) appears in \(I\).
Our method for proving this theorem is to investigate the combinatorial structure on \( F_{G^n_d}(M, n) \). As a consequence of this method, we can also give a formula for the Euler characteristic of a non-equivariant configuration space \( F(M, n) \) in terms of a polynomial of \( \chi(M) \).

**Theorem 1.2.** Let \( M \) be a compact triangulated homology \( m \)-manifold. Then

\[
\chi(F(M, n)) = (-1)^{mn} \prod_{k=0}^{n-1} (\chi(M) - k) = (-1)^{mn} n! \left( \frac{\chi(M)}{n} \right).
\]

**Remark 1.** The above formula can be rewritten as

\[
1 + \sum_{n=1}^{\infty} \frac{(-1)^{mn} \chi(F(M, n))}{n!} t^n = 1 + \sum_{n=1}^{\infty} \left( \frac{\chi(M)}{n} \right) t^n = (1 + t)^{\chi(M)}.
\]

Let \( m \) be even. Then we obtain the Félix-Thomas formula \([FT, \text{Theorem B}]\). Hence Theorem 1.2 generalizes the Félix-Thomas formula.

Next we shall be concerned with the homotopy type of \( F_{G^n_d}(M, n) \) for \( m \geq 1 \).

We first consider the case \( m = 1 \). In this case, we obtain

**Theorem 1.4.** Let \( \pi_d : M \to P \) be a \( d \)-dimensional \( G^n_d \)-manifold over \( P \). Then, when \( d = 1 \), \( F_{G^n_d}(M, n) \) has the same homotopy type as \( n! 2^{n-2} \) points, and when \( d = 2 \), \( F_{S^1}(M, n) \) has the same homotopy type as a disjoint union of \( n! \) copies of \( T^{n-2} \).

**Remark 2.** The classical configuration spaces on the circle is related to hyperbolic Dehn fillings \([YNK]\). The orbit version might give some additional information.

For general \( M \) and \( n \), the spaces \( F_{G^n_d}(M, n) \) can be expressed as an intersection of the subspaces of \( M \times n \) which are homeomorphic to \( M \times (n-2) \times F_{G^n_d}(M, 2) \) under coordinate permutations (see Proposition 2.1). Thus the study on \( F_{G^n_d}(M, 2) \) is the first step for the general cases. In this article we focus on this case to give an experimental investigation about the homotopy type of \( F_{G^n_d}(M, 2) \), where the combinatorial methods successfully overcome the technical difficulties in this case. The spaces \( F_{G^n_d}(M, n) \) for general \( n \) will be explored in our subsequent work. By the reconstruction of small covers and quasi-toric manifolds, we are able to determine an equivariant strong deformation retract of \( F_{G^n_d}(M, 2) \) in terms of the combinatorial data from \( P \) via \( \pi_d \). The result is stated as follows.

**Theorem 1.4.** Let \( \pi_d : M \to P \) be a \( d \)-dimensional \( G^n_d \)-manifold over a simple convex polytope \( P \). Then there is an equivariant strong deformation retraction of \( F_{G^n_d}(M, 2) \) onto

\[
X_d(M, 2) = \bigcup_{F_1, F_2 \in \mathcal{F}(P)} (\pi_d^{-1})^\times 2(F_1 \times F_2)
\]

where \( \mathcal{F}(P) \) is the set of all faces of \( P \).

The equivariant strong deformation retract in Theorem 1.4 plays an important role on further studying the algebraic topology of \( F_{G^n_d}(M, 2) \). As a result, we obtain that

**Theorem 1.5.** Given a simple convex polytope \( P \), assume that \( \pi_d : M \to P \) is a \( d \)-dimensional \( G^n_d \)-manifold over \( P \). Then

\[
b_{2d}(F_{T^m}(M, 2)) = b_{d}^2(F_{G^n_d}(M, 2)),
\]
which only depends upon the combinatorial structure of \( P \), where \( b_i^{2^m}(F_{2^m}(M, 2)) \) is the \( i \)-th mod 2 Betti number of \( F_{2^m}(M, 2) \), and \( b_i(F_{T^m}(M, 2)) \) is the \( i \)-th Betti number of \( F_{T^m}(M, 2) \). In particular, the homology of \( F_{T^m}(M, 2) \) vanishes in odd dimensions and is free abelian in even dimensions.

As a further application of Theorem 1.4, the equivariant cohomology of the orbit configuration spaces \( F_{T^m}(M, 2) \) can be also determined as follows:

**Theorem 1.6.** Let \( \pi_2 : M \to P \) be a 2\( m \)-dimensional quasi-toric manifold over a simple convex polytope \( P \). Then the Leray–Serre spectral sequence of the fibration

\[
E^2_{m, n} \to \pi_2^*(H^m(M, \mathbb{Z})).
\]

with \( F_{T^m}(M, 2) \) collapses at the \( E_2 \) term, that is, \( E_\infty = E_2 \).

Furthermore, as a consequence of Theorems 1.4 and the Mayer–Vietoris spectral sequence, we can also determine:

1. the integral homology of \( F_{G_m^m}(M, 2) \) and
2. the (mod 2) homology of \( F_{G_m^m}(M, 2) \) for \( M \) to be a \( G_m^m \)-manifold over an \( m \)-simplex \( \Delta^m \).

The article is organized as follows. In section 2 we give a brief review on the notions of small covers, quasi-toric manifolds and (real) moment-angle manifolds and investigate basic constructions and properties of their orbit configuration spaces. We calculate the Euler characteristic of the orbit configuration spaces for small covers and quasi-toric manifolds in section 3 where Theorem 1.1 is Theorem 3.1 and the proof of Theorem 1.2 is given in subsection 3.3. In section 4 we study the homotopy type of \( F_{G_m^m}(M, n) \) and \( F_{G_m^m}(M, 2) \) with giving the proofs of Theorems 1.3 and 1.4. As an application of Theorem 1.4, we study the Betti numbers and (equivariant) cohomology of \( F_{G_m^m}(M, 2) \) and prove Theorems 1.5 and 1.6 in section 5. In section 6 we compute the integral homology of \( F_{G_m^m}(M, 2) \) and the (mod 2) homology of \( F_{G_m^m}(M, 2) \) for the \( G_m^m \)-manifold \( M \) over an \( m \)-simplex. The Mayer–Vietoris spectral sequence will be one of major tools for our computations and so we give a review on the Mayer-Vietoris spectral sequence in section 7 as an appendix.

2. \( G_m^m \)-manifolds and (real) moment-angle manifolds over simple convex polytopes and their orbit configuration spaces

2.1. \( G_m^m \)-manifolds and (real) moment-angle manifolds over simple convex polytopes. Following [DJ], let \( P \) be a simple convex \( m \)-polytope, and let \( G_m^m \) be the 2-torus \( \mathbb{Z}_{2^m}^m \) of rank \( m \) if \( d = 1 \), and the torus \( T^m \) of rank \( m \) if \( d = 2 \). A \( dm \)-dimensional \( G_m^m \)-manifold over \( P \), \( \pi_d : M \to P \), is a smooth closed \( dm \)-dimensional manifold \( M \) with a locally standard \( G_m^m \)-action such that the orbit space is \( P \). A \( G_m^m \)-manifold \( \pi_d : M \to P \) is called a small cover if \( d = 1 \) and a quasi-toric manifold if \( d = 2 \). We know from [DJ] that each \( G_m^m \)-manifold \( \pi : M \to P \) determines a characteristic function \( \lambda_d \) on \( P \), defined by mapping all facets (i.e., (\( m - 1 \))-dimensional faces) of \( P \) to nonzero elements of \( R_d^m \) such that \( m \) facets meeting at each vertex are mapped to a basis of \( R_d^m \) where \( R_d = \begin{cases} \mathbb{Z}_2 & \text{if } d = 1 \\ \mathbb{Z} & \text{if } d = 2. \end{cases} \)

Conversely, the pair \((P, \lambda_d)\) can be reconstructed to the \( M \) as follows: first \( \lambda_d \) gives
exists a (real) moment-angle manifold on $P \times G^m_d$

$$\lambda_d: (x, g) \sim \lambda_d (y, h) \iff \begin{cases} x = y, g = h & \text{if } x \in \text{int}(P) \\ x = y, g^{-1}h \in G_F & \text{if } x \in \text{int}F \subset \partial P \end{cases}$$

then the quotient space $P \times G^m_d / \sim \lambda_d$ is equivariantly homeomorphic to the $M$, where $G_F$ is explained as follows: for each point $x \in \partial P$, there exists a unique face $F$ of $P$ such that $x$ is in its relative interior. If $\dim F = k$, then there are $m - k$ facets, say $F_1, \ldots, F_{m-k}$, such that $F = F_1 \cap \cdots \cap F_{m-k}$, and furthermore, $\lambda_d(F_1), \ldots, \lambda_d(F_{m-k})$ determine a subgroup of rank $m - k$ in $G^m_d$, denoted by $G_F$. This reconstruction of $M$ tells us that any topological invariant of $\pi_d: M \to P$ can be determined by $(P, \lambda_d)$. Davis and Januszkiewicz showed that $\pi_d: M \to P$ has a beautiful algebraic topological invariant in terms of $(P, \lambda_d)$. For example, the equivariant cohomology with $R_d$ coefficients of $\pi_d: M \to P$ is homotopy-equivalent to the Stanley–Reisner face ring of $P$, and the mod 2 Betti numbers $(b_0, b_1, \ldots, b_m)$ of $M$ for $d = 1$ and the Betti numbers $(b_0, b_2, \ldots, b_{2m})$ of $M$ for $d = 2$ agree with the $h$-vector $(h_0, h_1, \ldots, h_m)$ of $P$.

In addition, associated with a simple convex $m$-polytope $P$ with $h$ facets $F_1, \ldots, F_h$, Davis and Januszkiewicz also introduced a $G^h_d$-manifold $Z_{P,d}$ of dimension $(d - 1)h + m$ over $P$ as follows: first define a map $\theta_d: \{F_1, \ldots, F_h\} \to R^h_d$ by mapping $F_i \mapsto e_i$, where $\{e_1, \ldots, e_h\}$ is the standard basis of $R^h_d$, and then use $\theta_d$ to give an equivalence relation $\sim \theta_d$ on $P \times G^h_d$ as in (2.1), so that the required $G^h_d$-manifold $Z_{P,d}$ is just the quotient $P \times G^h_d / \sim \theta_d$ with a natural $G^h_d$-action having orbit space as $P$. Later on, Buchstaber and Panov [BP] further studied the topology of $Z_{P,d}$ as a submanifold in the polydisk $(D^d)^h$, and named it a real moment-angle manifold. Davis and Januszkiewicz showed in [DJ] that the real moment-angle manifold and a moment-angle manifold are often denoted by $\mathbb{R}Z_{P} = Z_{P,d}$ and $Z_{P}$, respectively.

As pointed out in [DJ Nonexample 1.22], given a simple convex $m$-polytope $P$ with $m > 3$, there may not exist any $G^m_d$-manifold over $P$. However, there always exists a (real) moment-angle manifold over $P$.

When $P$ admits a characteristic function $\lambda_d$ (so there is a $G^m_d$-manifold $M^{dm}$ over $P$ reconstructed by $(P, \lambda_d)$), regarding $R^h_d$ as a free module generated by $\{F_1, \ldots, F_h\}$, the map $\lambda_d$ may linearly extend to a surjection $\lambda_d: R^h_d \to R^m_d$. Then the kernel of $\lambda_d$ determines a subgroup $H$ of rank $h - m$ of $G^h_d$, which can freely act on $Z_{P,d}$ such that the quotient manifold $Z_{P,d}/H$ is exactly equivariantly homeomorphic to the $G^m_d$-manifold $M^{dm}$. Thus, the natural projection $\rho_d: Z_{P,d} \to M^{dm}$ is a fibration with fiber $G^{h-m}_d$. Davis and Januszkiewicz showed in [DJ] that the Borel constructions $EG^h_d \times G^h_d Z_{P,d}$ and $EG^m_d \times G^m_d M^{dm}$ are homotopy-equivalent. For more details of these equivariant manifolds above with many interesting developments and applications, e.g., see [DJ, BP, BBCG, CL, CMS, CPS, IFM, LT, LY, M, MS, U].

**2.2. Basic constructions and properties of orbit configuration spaces of $G^m_d$-manifolds and (real) moment-angle manifolds.** Let $\pi_d: M \to P$ be a $dm$-dimensional $G^m_d$-manifold over a simple convex polytope $P$. Then the product $\pi_{d}^{\times n}: M^{\times n} \to P^{\times n}$ is also a $dmn$-dimensional $(G^m_d)^{\times n}$-manifold over a simple convex polytope $P^{\times n}$.
Now let $\tilde{\Delta}(P^{xn})$ be the weak diagonal of $P^{xn}$, i.e., $\tilde{\Delta}(P^{xn}) = \bigcup_{1 \leq i < j \leq n} \Delta_{i,j}(P^{xn})$ where $\Delta_{i,j}(P^{xn}) = \{(p_1, p_2, \ldots, p_n) \in P^{xn} | p_i = p_j\}$. By definition, we have that $F(P, n) = P^{xn} - \tilde{\Delta}(P^{xn})$. By the constructions of $M$, we see that $F_G^d(M, n)$ is the pullback from $\pi_d^{xn} : M^{xn} \to P^{xn}$ via the inclusion $F(P, n) \hookrightarrow P^{xn}$. So there is the following commutative diagram:

$$
\begin{array}{ccc}
F_G^d(M, n) & \longrightarrow & M^{xn} \\
\pi_d^{xn} \downarrow & & \downarrow \pi_d^{xn} \\
F(P, n) & \longrightarrow & P^{xn}.
\end{array}
$$

Furthermore, we obtain that $F_G^d(M, n) \subset M^{xn}$ is a non-free orbit configuration space, and admits an action of $(G^m_d)^{xn}$ such that the orbit space is exactly $F(P, n)$.

**Proposition 2.1.** Let $\pi_d : M \to P$ be a $dn$-dimensional $G^m_d$-manifold over a simple convex polytope $P$. Then

$$
F_G^d(M, n) = \bigcap_{1 \leq i < j \leq n} (M^{xn} - (\pi_d^{xn})^{-1}(\Delta_{i,j}(P^{xn}))).
$$

**Proof.** The required result follows by using the De Morgan formula. \qed

**Remark 3.** In Proposition 2.1, each $M^{xn} - (\pi_d^{xn})^{-1}(\Delta_{i,j}(P^{xn}))$ is homeomorphic to $M^{(n-2)} \times F_G^d(M, 2)$.

**Remark 4.** Let $M \to P$ be a quasi-toric manifold over $P$. As shown in [DJ Corlary 1.9], there is a conjugation involution $\tau$ on $M$ such that its fixed point set $M^\tau$ is exactly a small cover over $P$. This means that there is still an involution on $F_{T^n}(M, n)$ such that its fixed point set is $F_{\mathbb{Z}_2^n}(M^\tau, n)$.

Let $p_d : Z_{P,d} \to P$ be the (real) moment-angle manifold over $P$ with $h$ facets. Similarly, we see from the constructions of $Z_{P,d}$ that $F_{G^h_d}(Z_{P,d}, n)$ admits an action of $(G^h_d)^{xn}$ and is the pullback from $p_d^{xn} : Z_{P,d}^{xn} \to P^{xn}$ via the inclusion $F(P, n) \hookrightarrow P^{xn}$, so there is a commutative diagram

$$
\begin{array}{ccc}
F_{G^h_d}(Z_{P,d}, n) & \longrightarrow & Z_{P,d}^{xn} \\
\pi_d^{xn} \downarrow & & \downarrow p_d^{xn} \\
F(P, n) & \longrightarrow & P^{xn}.
\end{array}
$$

Thus we have that

$$
F_{G^h_d}(Z_{P,d}, n) = \bigcap_{1 \leq i < j \leq n} (Z_{P,d}^{xn} - (p_d^{xn})^{-1}(\Delta_{i,j}(P^{xn}))).
$$

If we assume that there exists a $G^m_d$-manifold $\pi_d : M \to P$ over $P$, then we know that $Z_{P,d}$ is a principal $G^h_d$-$m$-bundle over $M$, denoted by $\rho_d : Z_{P,d} \to M$. Then we have that $p_d = \pi_d \circ \rho_d$. Furthermore, we have the following commutative
diagram:
\[
\begin{array}{ccc}
F_{G_d^h}(Z_{P,d},n) & \longrightarrow & Z_P^x \times Z_d^x \\
\downarrow \pi_d^x & & \downarrow \rho_d^x \\
F_{G_d}(M,n) & \longrightarrow & M_P^x \times M_d^x \\
\downarrow \pi_d^x & & \downarrow \pi_d^x \\
F(P,n) & \longrightarrow & P^x.
\end{array}
\]

It is not difficult to see that \(F_{G_d^h}^x : F_{G_d^h}(Z_{P,d},n) \longrightarrow F_{G_d^h}(M,n)\) is a fibration with fiber \((G_{d-m}^h)^x\). In the same way as in [DJ, 4.1] and [BP, Proposition 6.34], we have the following homotopy-equivalent Borel constructions
\[
E(G_d^m)^x \times (G_d^m)^x E_{F_d^h}(Z_{P,d},n) \simeq E(G_d^m)^x \times (G_d^m)^x F_{G_d^h}(M,n).
\]
Thus we conclude that

**Proposition 2.2.** Given a simple convex \(m\)-polytope \(P\) with \(h\) facets, assume that \(\pi_d : M \rightarrow P\) is a \(G_{d}^m\)-manifold over \(P\). Let \(p_d : Z_{P,d} \rightarrow P\) be the (real) moment-angle manifold over \(P\). Then the equivariant cohomologies of \(F_{G_d^h}(M,n)\) and \(F_{G_d^h}(Z_{P,d},n)\) are isomorphic, i.e.,
\[
H^*_{(G_d^m)^x}(F_{G_d^h}(M,n)) \cong H^*_{(G_d^m)^x}(F_{G_d^h}(Z_{P,d},n)).
\]

3. **Euler characteristic of \(F_{G_d^h}(M,n)\)**

The objective of this section is to calculate the Euler characteristic \(\chi(F_{G_d^h}(M,n))\) for a \(dm\)-dimensional \(G_{d}^m\)-manifold \(\pi_d : M \rightarrow P\).

3.1. **Euler characteristic of union–the inclusion-exclusion principle.** Suppose that \(X_1, \ldots, X_N\) are CW-complexes such that their all possible non-empty intersections are subcomplexes of \(X_1 \cup \cdots \cup X_N\). Let \(\Delta^N\) be the abstract simplex on vertex set \([N] = \{1, \ldots, N\}\), i.e., \(\Delta^N = 2^{[N]}\) (the power set of \([N]\)). For each \(a \in 2^{[N]}\), set
\[
X_a = \begin{cases} 
\bigcap_{i \in a} X_i & \text{if } a \neq \emptyset \\
\bigcup_{i=1}^N X_i & \text{if } a = \emptyset.
\end{cases}
\]
Since each pair \((X_i, X_j)\) is an excisive couple of \(X_i \cup X_j\), we have the following well-known formula for euler characteristics.

**Proposition 3.1** (Inclusion-exclusion principle).
\[
\chi(X_{\emptyset}) = \sum_{a \in 2^{[N]} \atop a \neq \emptyset} (-1)^{|a|-1} \chi(X_a).
\]

3.2. **The \(h\)-polynomial and the cell-vector of \(P\).** Let \(P\) be a simple convex \(m\)-polytope. The \(f\)-vector of \(P\) is an integer vector \((f_0, f_1, \ldots, f_{m-1})\), where \(f_i\) is the number of faces of \(P\) of codimension \(i+1\) (i.e., of dimension \(m-i-1\)). Then the \(h\)-vector of \(P\) is the integer vector \((h_0, h_1, \ldots, h_m)\) defined from the following equation
\[
(1.1) \quad h_0 + h_1 t + \cdots + h_m t^m = (t-1)^m + f_0 (t-1)^{m-1} + \cdots + f_{m-2} (t-1) + f_{m-1}.
\]
The \(f\)-vector and the \(h\)-vector determine each other by Equation (3.1).
Furthermore, by Equation (3.1) the $h$-polynomial of $P$. Given a finite CW-complex $X$ of dimension $l$, the cell–vector $c(X)$ of $X$ is the integer vector $(c_0, c_1, \ldots, c_l)$ where $c_i$ denotes the number of all $i$-cells in $X$. Each simple convex $m$-polytope $P$ has a natural cell decomposition such that the interior $\text{int} F$ of an $i$-face $F$ of $P$ is an $i$-cell. Thus,

$$c(P) = (f_{m-1}, f_{m-2}, \ldots, f_1, f_0, 1)$$

where $f(P) = (f_0, f_1, \ldots, f_{m-1})$ is the $f$-vector of $P$.

For an arbitrary positive integer $\ell$, by $\Delta(P^{\times \ell})$ we denote the strong diagonal of $P^{\times \ell}$, i.e.,

$$\Delta(P^{\times \ell}) = \{(p, p, \ldots, p) \in P^{\times \ell} | p \in P\}.$$

**Lemma 3.1.** Let $\pi_d : M \rightarrow P$ be a $dn$-dimensional $G_d^m$-manifold over a simple convex $m$-polytope $P$. Then for a positive integer $\ell$, the Euler characteristic of $(\pi_1^x)^{-1}(\Delta(P^{\times \ell}))$ is

$$\chi((\pi_1^x)^{-1}(\Delta(P^{\times \ell}))) = \begin{cases} h_P(1 - 2^d) & \text{if } d = 1 \\ h_P(1) = \chi(M) & \text{if } d = 2. \end{cases}$$

**Proof.** Fix the cell decomposition of $P$ as above such that its cell–vector $c(P) = (f_{m-1}, f_{m-2}, \ldots, f_1, f_0, 1)$. Let $F$ be a face of dimension $i$. By [DJ, Lemma 1.3], we know that $\pi_d^{-1}(F)$ is still a $di$-dimensional $G_d^m$-manifold over $F$, and in particular, $\pi_d^{-1}(\text{int} F) = G_d^m \times \text{int} F$. When $d = 1$, $\pi_1^{-1}(\text{int} F)$ is the disjoint union of $2^i$ copies of $\text{int} F$. Since the strong diagonal $\Delta(P^{\times \ell})$ is combinatorially equivalent to $F$, $(\pi_1^x)^{-1}(\Delta((\text{int} F)^{\times \ell}))$ is the disjoint union of $2^i$ copies of $\Delta((\text{int} F)^{\times \ell})$. Thus, the cell–vector of $(\pi_1^x)^{-1}(\Delta(P^{\times \ell}))$ is

$$(f_{m-1}, 2^i f_{m-2}, \ldots, 2^i f_{m-i-1}, 2^i f_{m-i}, \ldots, 2^{i-2} f_{m-1}, f_0, 2^i f_{m-1}).$$

Furthermore, by Equation (3.1)

$$\chi((\pi_1^x)^{-1}(\Delta(P^{\times \ell}))) = f_{m-1} - 2^i f_{m-2} + \cdots + (-1)^i 2^i f_{m-i-1} + \cdots + (-1)^{m-2} 2^{i-2} f_{m-i} + (-1)^m 2^i f_0 = h_P(1 - 2^i).$$

When $d = 2$, we see easily that $(\pi_2^x)^{-1}(\Delta((\text{int} F)^{\times \ell})) = T^{\ell} \times \Delta((\text{int} F)^{\times \ell})$. Now, for $i > 0$, we give a cell decomposition for each circle $S^1$ in $T^{\ell}$, with one 0-cell and one 1-cell. Then $(\pi_2^x)^{-1}(\Delta((\text{int} F)^{\times \ell}))$ contains $\binom{\ell}{i} k$ cells of dimension $(i + k)$ for $0 \leq k \leq i\ell$. Thus, all $i$-cells of $P$ contribute $\binom{\ell}{i} f_{m-i-1}$ cells of dimension $(i + k)$ in $(\pi_2^x)^{-1}(\Delta(P^{\times \ell}))$ where $0 \leq k \leq i\ell$. Since $\sum_{k=0}^{i\ell} (-1)^k \binom{i\ell}{k} = 0$ for every $i > 0$, by a direct calculation we have

$$\chi((\pi_2^x)^{-1}(\Delta(P^{\times \ell}))) = f_{m-1} = h_P(1) = \chi(M)$$

as desired. \qed

### 3.3. Subgraphs of $\mathcal{K}_n$ and partitions of $n$ and $[n]$

Let $\mathcal{K}_n$ be the complete graph of degree $n - 1$, which contains $n$ vertices and $\binom{n}{2}$ edges. We label $n$ vertices of $\mathcal{K}_n$ by $1, ..., n$ respectively, and $\binom{n}{2}$ edges by pairs $(i, j), 1 \leq i < j \leq n$, respectively. Thus we may identify $\mathcal{K}_n$ with the 1-skeleton of the abstract $(n - 1)$-simplex $\Delta^n = 2^n$ on vertex set $[n] = \{1, ..., n\}$. 

Definition 3.1. A subgraph $\Gamma$ of $K_n$ is said to be **vertex-full** if the vertex set of $\Gamma$ is $[n]$.

By $\text{VF}(K_n)$ we denote the set of all vertex-full subgraphs of $K_n$.

Lemma 3.2. There is a one-to-one correspondence between all subsets of the power set $2^{[n]}$ and all vertex-full subgraphs of $\text{VF}(K_n)$, where $[[n]] = \{(i, j) | 1 \leq i < j \leq n\}$.

**Proof.** Each vertex-full subgraph $\Gamma$ of $K_n$ uniquely determines a subset $E(\Gamma)$ of $2^{[n]}$, where $E(\Gamma)$ denotes the set of all edges of $\Gamma$. Note that the discrete subgraph $[n]$ of $K_n$ corresponds to the empty set $\emptyset$ of $2^{[n]}$. Conversely, let the empty set $\emptyset$ of $2^{[n]}$ correspond to the discrete subgraph $[n]$ of $K_n$. Each nonempty subset of $2^{[n]}$ determines a unique subgraph $\Gamma$ of $K_n$. If the vertex set of $\Gamma$ does not cover $[n]$, then we can add those missing vertices as one-point subgraphs to $\Gamma$ to give the required vertex-full subgraph of $K_n$.

Given a vertex-full subgraph $\Gamma$ in $\text{VF}(K_n)$, define

$$\Delta_\Gamma(P \times n) = \begin{cases} \bigcap_{(i, j) \in E(\Gamma)} \Delta_{i,j}(P \times n) & \text{if } E(\Gamma) \neq \emptyset \\
 \Delta(P \times |V(\Gamma)|) & \text{if } E(\Gamma) = \emptyset \end{cases}$$

where $E(\Gamma)$ denotes the set of all edges of $\Gamma$. Generally, $\Gamma$ may not be connected. By $C(\Gamma)$ we denote the set of all connected subgraphs of $\Gamma$. Write $C(\Gamma) = \{\Gamma_1, ..., \Gamma_s\}$. Then $\Gamma = \biguplus_{k=1}^s \Gamma_k$ (a disjoint union of $\Gamma_1, ..., \Gamma_s$).

Lemma 3.3. Let $\Gamma$ be a vertex-full subgraph in $\text{VF}(K_n)$ with $C(\Gamma) = \{\Gamma_1, ..., \Gamma_s\}$. Then

$$\Delta_\Gamma(P \times n) = \prod_{k=1}^s \Delta(P \times |V(\Gamma_k)|)$$

where $V(\Gamma_k)$ denotes the vertex set of $\Gamma_k$.

**Proof.** Obviously, if $\Gamma = [n]$, then the required equality holds. Suppose that $\Gamma \neq [n]$. For each component $\Gamma_k$ of $\Gamma$, $\bigcap_{(i, j) \in E(\Gamma_k)} \Delta_{i,j}(P \times n)$ is combinatorially equivalent to $P^{\times n - |V(\Gamma_k)|} \times \Delta(P \times |V(\Gamma_k)|)$. Thus

$$\Delta_\Gamma(P \times n) = \bigcap_{k=1}^s \Delta_{i,j}(P \times n) = \prod_{k=1}^s \Delta(P \times |V(\Gamma_k)|)$$

as desired.

Recall that a **partition** of $n$ is an unordered sequence $(n_1, ..., n_s)$ of positive integers with sum $n$, and a **partition** of $[n]$ is an unordered sequence of nonempty subsets of $[n]$ which are pairwise disjoint and whose union is $[n]$. Clearly, every vertex-full subgraph $\Gamma = \biguplus_{k=1}^s \Gamma_k$ of $\text{VF}(K_n)$ gives a partition $([V(\Gamma_1)], ..., [V(\Gamma_s)])$ of $n$, denoted by $n(\Gamma)$. In addition, each vertex-full subgraph of $\text{VF}(K_n)$ also determines a partition $(V(\Gamma_1), ..., V(\Gamma_s))$ of $[n]$.

Lemma 3.4. Let $I = (n_1, ..., n_s)$ be a partition of $n$. Then the number of those combinatorially equivalent vertex-full subgraphs $\Gamma$ with $n(\Gamma) = I$ of $\text{VF}(K_n)$ is

$$\frac{n!}{n_1! \cdots n_s! r_1! \cdots r_s!}$$

where $r_i$ denotes the time number that $n_i$ appears in $I$. 

Proof. Obviously, those combinatorially equivalent vertex-full subgraphs $\Gamma$ with $n(\Gamma) = I$ of $VF(\mathcal{K}_n)$ bijectively correspond to those partitions $(a_1, \ldots, a_s)$ with $|a_k| = n_k$ of $[n]$. The desired number then follows from an easy argument. $\square$

### 3.4. Calculation of Euler characteristic.

Now let us calculate $\chi(F_{G_d^m}(M, n))$ for a $G_d^m$-manifold $\pi_d : M \rightarrow P$ over a simple convex polytope $P$.

**Definition 3.2.** Let $I = (n_1, \ldots, n_s)$ be a partition of $n$. Define

$$C_I = \sum_{\Gamma \in \mathcal{VF}(\mathcal{K}_n) \atop n(\Gamma) = I} (-1)^{|E(\Gamma)|}.$$

**Theorem 3.1.** Let $\pi_d : M \rightarrow P$ be a $dm$-dimensional $G_d^m$-manifold over a simple convex polytope. Then

$$\chi(F_{G_d^m}(M, n)) = \begin{cases} (-1)^{mn} \sum_{I=(n_1, \ldots, n_s)} C_I \prod_{k=1}^s h_P(1 - 2^{n_k}) & \text{if } d = 1 \\ \chi(F(M, n)) & \text{if } d = 2 \end{cases}$$

where $I = (n_1, \ldots, n_s)$ runs over all partitions of $n$.

**Proof.** First, we calculate $\chi((\pi_d^\times)^{-1}(\Delta(P^\times)))$ by Proposition 2.1. Since $[n][n]$ is combinatorially equivalent to $2^{|N|}$ where $N = \binom{n}{2}$, by Proposition 3.1 and Lemmas 3.4, 3.5, we have that

$$\chi((\pi_d^\times)^{-1}(\Delta(P^\times))) = \sum_{r = \Pi_{k=1}^s \Gamma_k \in \mathcal{VF}(\mathcal{K}_n) \atop E(\Gamma) \neq \emptyset} (-1)^{|E(\Gamma)| - 1} \chi((\pi_d^\times)^{-1}(\Delta(P^\times)))$$

$$= \sum_{r = \Pi_{k=1}^s \Gamma_k \in \mathcal{VF}(\mathcal{K}_n) \atop E(\Gamma) \neq \emptyset} (-1)^{|E(\Gamma)| - 1} \chi((\pi_d^\times)^{-1}(\Delta(P^\times)))$$

$$= \sum_{r = \Pi_{k=1}^s \Gamma_k \in \mathcal{VF}(\mathcal{K}_n) \atop E(\Gamma) \neq \emptyset} (-1)^{|E(\Gamma)| - 1} \prod_{k=1}^s \chi((\pi_d^\times)^{V(\Gamma_k)})^{-1}(\Delta(P^\times)^{V(\Gamma_k)})$$

$$= \sum_{r = \Pi_{k=1}^s \Gamma_k \in \mathcal{VF}(\mathcal{K}_n) \atop E(\Gamma) \neq \emptyset} (-1)^{|E(\Gamma)| - 1} \prod_{k=1}^s \chi((\pi_d^\times)^{V(\Gamma_k)})^{-1}(\Delta(P^\times)^{V(\Gamma_k)}))$$

Since each vertex-full subgraph $\Gamma = \bigcup_{k=1}^s \Gamma_k$ of $\mathcal{K}_n$ corresponds to a unique partition $n(\Gamma) = (|V(\Gamma_1)|, \ldots, |V(\Gamma_s)|)$ of $n$, we further have that

$$\chi((\pi_d^\times)^{-1}(\Delta(P^\times))) = \begin{cases} - \sum_{I=(n_1, \ldots, n_s) \atop \Gamma(\Gamma) \neq (1, \ldots, 1)} C_I \prod_{k=1}^s h_P(1 - 2^{n_k}) & \text{if } d = 1 \\ - \sum_{I=(n_1, \ldots, n_s) \atop \Gamma(\Gamma) \neq (1, \ldots, 1)} C_I \chi(M)^s & \text{if } d = 2 \end{cases}$$

where $I$ runs over those partitions except for $(1, \ldots, 1)$ of $n$. A direct calculation gives that $C_{(1, \ldots, 1)} = 1$, so

$$\chi(M^\times) = \begin{cases} C_{(1, \ldots, 1)} h_P(-1)^n & \text{if } d = 1 \\ C_{(1, \ldots, 1)} h_P(1)^n = C_{(1, \ldots, 1)} \chi(M)^n & \text{if } d = 2 \end{cases}$$

by Lemma 3.4.

Next, by Lefschetz duality theorem and Proposition 2.1 we conclude that...
\text{Proof of Theorem 1.2.}\]

By using the pigeonhole principle, since \( k < n \)

Thus, using the proof method of Theorem 3.1, we can obtain the following formula for more general non-equivariant configuration spaces.

\[ \chi(F(M, n)) = (-1)^{mn} \sum_{I} \chi(M)^{s} \]

where \( I = (n_1, \ldots, n_s) \) runs over all partitions of \( n \).

In Theorem 3.2 if we further write

\[ \chi(F(M, n)) = (-1)^{mn} \sum_{I} \chi(M)^{s} = (-1)^{mn} \sum_{I} \left( \sum_{s=1}^{n} \chi(M)^{s} \right) \]

then we see that \( \chi(F(M, n)) \) is actually a polynomial (with \( \mathbb{Z} \) coefficients) of \( \chi(M) \) of degree \( n \). By \( g(t) \) we denote this polynomial in \( \mathbb{Z}[t] \). We first complete the proof of Theorem 1.2.

\text{Proof of Theorem 1.2.}\] It suffices to show that \( g(t) = (-1)^{mn} \prod_{k=0}^{n-1} (t - k) \). For \( 0 \leq k < n \), choose \( M \) as a set consisting of \( k \) points. Then \( M \) is a 0-dimensional manifold if \( 0 < k < n \), and a empty set (or \(-1\)-dimensional manifold) if \( k = 0 \). Thus

\[ \chi(M) = \begin{cases} 
  k & \text{if } 0 < k < n \\
  0 & \text{if } k = 0. 
\end{cases} \]

By using the pigeonhole principle, since \( k < n \), we see that

\[ F(M, n) = \{(x_1, \cdots, x_n) \in M^{\times n} | x_i \neq x_j \text{ for } i \neq j \} \]

must be empty, so \( \chi(F(M, n)) = 0 \). This implies that \( g(k) = 0 \) for \( 0 \leq k < n \), and thus each \( k \) is a root of \( g(t) \). Furthermore, we can write \( g(t) = (-1)^{mn} e \prod_{k=0}^{n-1} (t - k) \).
where \( c \) is a constant number. Since \( C_{(1,\ldots,1)} = 1 \), we conclude that \( c \) must be 1. This completes the proof. □

**Corollary 3.3.**

\[ C_{(n)} = (-1)^{n-1}(n-1)!. \]

**Proof.** This can be obtained by comparing the coefficients of \( \chi(M) \) on both sides of the following equality

\[
\sum_{s=1}^{n} \left( \sum_{I=(n_1,\ldots,n_s)} C_I \chi(M)^s \right) = \prod_{k=0}^{n-1} (\chi(M) - k).
\]

□

Finally, to complete the proof of Theorem 1.1, it remains to determine the number \( C_I \) for every partition \( I \) of \( n \).

**Proposition 3.2.** Let \( I = (n_1,\ldots,n_s) \) be a partition of \( n \). Then

\[ C_I = \frac{n!(-1)^{n-s}}{r_1!r_2!\cdots r_s!n_1n_2\cdots n_s} \]

where \( r_k \) denotes the time number that \( n_k \) appears in \( I \).

**Proof.** Let \( A_I \) denote the set of those vertex-full subgraphs \( \Gamma \) with \( n(\Gamma) = I \) of \( \text{VF}(K_n) \), all of which are not combinatorially equivalent to each other. By Lemma 3.4 and Corollary 3.3 we have that

\[
C_I = \sum_{\Gamma \in \text{VF}(K_n) \atop n(\Gamma) = I} (-1)^{|E(\Gamma)|} \chi(M)^s \prod_{k=1}^{s} (\chi(M) - k).
\]

\[
= \frac{n!}{n_1!\cdots n_s!r_1!\cdots r_s!} \sum_{\Gamma_k \subseteq A_I \atop |V(\Gamma_k)| = n_k} \prod_{k=1}^{s} (-1)^{|E(\Gamma_k)|}
\]

\[
= \frac{n!}{n_1!\cdots n_s!r_1!\cdots r_s!} \prod_{k=1}^{s} (-1)^{|E(\Gamma_k)|}
\]

\[
= \frac{n!}{n_1!\cdots n_s!r_1!\cdots r_s!} \prod_{k=1}^{s} C_{(n_k)}
\]

\[
= \frac{n!}{n_1!\cdots n_s!r_1!\cdots r_s!} \prod_{k=1}^{s} (-1)^{n_k-1}(n_k - 1)!
\]

\[
= \frac{n!(-1)^{n-s}}{r_1!r_2!\cdots r_s!n_1n_2\cdots n_s}
\]

as desired. □

**Example 3.1.** By the formula of Theorem 1.1 we have that

\[ \chi(F_{Z^2}(M, 2)) = h_P(-1)^2 - h_P(-3) \]

and

\[ \chi(F_{Z^2}(M, 3)) = (-1)^{3m}(h_P(-1)^3 - 3h_P(-1)h_P(-3) + 2h_P(-7)) \]

where \( M \) is a small cover over \( P \).
Corollary 3.4. Given a simple convex $m$-polytope $P$ with $h$ facets, assume that there exists a small cover over $P$. Then

$$\chi(F_{22m}(Z_{P,1}, n)) = (-1)^{mn}2^{(h-m)n} \sum_{I=\{n_1, \ldots, n_s\}} \frac{n!(-1)^{n-s}}{r_1!r_2!\cdots r_s!n_1n_2\cdots n_s} \prod_{k=1}^{s} h_P(1-2^{n_k})$$

where $I = (n_1, \ldots, n_s)$ runs over all partitions of $n$, and $r_k$ is the time number that $n_k$ appears in $I$.

Proof. Let $M$ be a small cover over $P$. Then $F_{22m}(Z_{P,1}, n)$ is a principal $(Z_{2}^{h-m})^{\times n}$-bundle over $F_{22m}(M, n)$. By [AP, p. 86, (1.5)(d)] we have that

$$\chi(F_{22m}(Z_{P,1}, n)) = |(Z_{2}^{h-m})^{\times n}| \chi(F_{22m}(M, n)).$$

Moreover, the required result follows from Theorem 1.1. \qed

Remark 5. It should be interesting to give an explicit formula of $\chi(F_{22m}(Z_{P,1}, n))$ without the existence assumption of a small cover over $P$ in Corollary 3.4.

Proposition 3.3. Let $P$ be a simple convex polytope with $h$ facets. Then

$$\chi(F_{T^h}(Z_{P,2}, n)) = 0.$$

Proof. We know from [BP, Proposition 7.29] that the diagonal circle subgroup of $T^h$ acts freely on $Z_{P,2}$, so the diagonal circle subgroup of $(T^h)^{\times n}$ also acts freely on $F_{T^h}(Z_{P,2}, n)$. Therefore, $F_{T^h}(Z_{P,2}, n)$ admits a principal $S^1$-bundle structure, which induces that $\chi(F_{T^h}(Z_{P,2}, n)) = 0$ by [AP, p. 86, (1.5)(c)]. \qed

Remark 6. Buchstaber and Panov [BP] expanded the construction of $Z_{P,d}$ over a simple convex polytope $P$ to the case of general simplicial complex $K$. The resulting space denoted by $Z_{K,d}$ is not a manifold in general, called the (real) moment-angle complex. When $d = 2$, $Z_{K,2}$ still admits a principal $S^1$-bundle structure, so the Euler characteristic of its orbit configuration space is zero. It should also be interesting to give an explicit formula for the Euler characteristic of the orbit configuration space of $Z_{K,1}$ in terms of the combinatorial data of $K$. In addition, it was showed in [CL, U] that the Halperin-Carlsson conjecture holds for $Z_{K,d}$ with the restriction free action. Naturally, we wish to know whether this is also true for the orbit configuration space of $Z_{K,d}$.

4. Homotopy type of $F_{G_{d}^{m}}(M, n)$ for $m = 1$ or $n = 2$

Throughout the following, assume that $\pi_{d} : M \to P$ is a $dm$-dimensional $G_{d}^{m}$-manifold over a simple convex $m$-polytope $P$. It is easy to see that $F(P, n)$ is disconnected if $m = 1$ and path-connected if $m > 1$ since $\Delta(P^{\times 2})$ is combinatorially equivalent to $P$.

4.1. Homotopy type of $F_{G_{d}^{1}}(M, n)$.

Theorem 4.1. Let $\pi_{d} : M \to P$ be a $d$-dimensional $G_{d}^{1}$-manifold over $P$. Then, when $d = 1$, $F_{G_{d}^{1}}(M, n)$ has the same homotopy type as $n!2^{n-2}$ points, and when $d = 2$, $F_{G_{d}^{1}}(M, n)$ has the same homotopy type as a disjoint union of $n!$ copies of $T^{n-2}$.
Proof. It is well-known that when \( d = 1 \), \( M \) is a circle \( S^1 \) with a reflection fixing two isolated points such that the orbit polytope \( P \) is a 1-dimensional simplex, and when \( d = 2 \), \( M \) is a 2-sphere \( S^2 \) with a rotation action of \( S^1 \), fixing two isolated points, such that the orbit polytope \( P \) is also a 1-dimensional simplex. Since a 1-simplex is homeomorphic to the interval \([0, 1] \), we may identify \( P \) with the cube \([0, 1]^n \). It is easy to see that each point \( x = (x_1, ..., x_n) \in F(P, n) \subset [0, 1]^n \) determines a unique permutation \((\sigma(1), ..., \sigma(n))\) of \([n]\) such that \( x_i < x_j \) as long as \( \sigma(i) < \sigma(j) \), where \( \sigma \in S_n \), and \( S_n \) is the symmetric group on \([n]\). Define a homotopy \( H : F(P, n) \times [0, 1] \to F(P, n) \) by

\[
((x_1, ..., x_n), t) \mapsto \left( (1-t)x_1 + t \frac{\sigma(1) - 1}{n-1}, ..., (1-t)x_n + t \frac{\sigma(n) - 1}{n-1} \right).
\]

An easy argument shows that this homotopy \( H \) is a deformation retraction of \( F(P, n) \) onto \( n! \) points in \( \mathcal{A} = \left\{ (\frac{\sigma(1)-1}{n}, ..., \frac{\sigma(n)-1}{n}) | \sigma \in S_n \right\} \subset F(P, n) \). This means that \( F(P, n) \) contains \( n! \) connected components \( C_\sigma \), \( \sigma \in S_n \), each of which may continuously collapse to a point in \( \mathcal{A} \). For each \( \sigma \), since \( 0, 1 \in \left\{ \frac{\sigma(1)-1}{n}, ..., \frac{\sigma(n)-1}{n} \right\} \) and \( \left\{ \frac{\sigma(1)-1}{n}, ..., \frac{\sigma(n)-1}{n} \right\} - \{0, 1\} \) is in the open interval \((0, 1)\), \( C_\sigma \) has also the deformation retract \( R_\sigma \) that is homeomorphic to an \((n-2)\)-dimensional open ball \( B_{\sigma} \) in \( F(P, n) \), and is contained in \( \partial P \times \partial P \). Therefore, each \( R_\sigma \) can be chosen in the interior of an \((n-2)\)-face in \( P \times P \), so by [13] Lemma 4.1,

\[
(\pi^P_d)^{-1}(R_\sigma) = G^{n-2}_d \times R_\sigma = \begin{cases} \mathbb{Z}^{n-2}_d \times R_\sigma & \text{if } d = 1 \\ T^{n-2} \times R_\sigma & \text{if } d = 2 \end{cases}
\]

which is homotopic to \( 2^{n-2} \) points if \( d = 1 \) and \( T^{n-2} \) if \( d = 2 \). This completes the proof. \( \square \)

4.2. An equivariant strong deformation retract of \( F_{G^P_n}(M, 2) \) with \( m > 1 \).

Let \( F(P) \) denote the set of all faces of \( P \).

Lemma 4.1. There is a strong deformation retraction \( H : F(P, 2) \times [0, 1] \to F(P, 2) \) of \( F(P, 2) \) onto

\[
\mathcal{A}(P, 2) = \bigcup_{F_1, F_2 \in F(P)} F_1 \times F_2.
\]

Proof. First, we note that \( F(P, 2) = P \times P - \Delta(P \times P) \) is path-connected since \( m > 1 \). Since \( \Delta(P \times P) \) always contains the interior points of \( P \times P \), we have that \( F(P, 2) \) can continuously collapse onto \( \partial(P \times P) - \Delta(P \times P) \). Since \( \partial(P \times P) = \partial P \times P \cup P \times \partial P \), in a similar way, it is easy to see that both \( \partial P \times P \) and \( P \times \partial P \) can further continuously collapse onto \( \partial P \times \partial P - \Delta(\partial P \times \partial P) \). Now we see that \( \partial P \times \partial P - \Delta(\partial P \times \partial P) \) is the union of subsets of the following forms

\[
F \times F - \Delta(F \times F), F \times F' - \Delta(\partial P \times \partial P), F \times F''
\]

where \( F, F', F'' \) are facets of \( P \) with \( F \cap F' \neq \emptyset \) and \( F \cap F'' = \emptyset \). In particular, we further see that \( F \times F' - \Delta(\partial P \times \partial P) \) can be continuously shrunk to the union of subsets of the following forms

\[
(F \cap F') \times (F \cap F') - \Delta((F \cap F')^2), Q \times (F \cap F') - \Delta(\partial P \times \partial P),
\]

\[
(F \cap F') \times Q' - \Delta(\partial P \times \partial P), Q_1 \times Q_1
\]

where \( Q \) and \( Q_1 \) are facets of \( F \), \( Q' \) and \( Q_1' \) are facets of \( F' \), such that \( Q \times (F \cap F') \neq \emptyset \), \( (F \cap F') \times Q' \neq \emptyset \), and \( Q_1 \cap Q_1' = \emptyset \). We continuous the above process to
Proof. Since $M = P \times G_d^m / \sim_{\lambda_d}$, we have that $F_{G_d^m}(M, 2) = F(P, 2) \times G_d^{2m} / \sim_{\lambda_d \times \lambda_d}$.

For the strong deformation retraction $H : F(P, 2) \times [0, 1] \to F(P, 2)$ in Lemma 4.1, it may naturally be lift to a strong deformation retraction

$$H_1 : F(P, 2) \times G_d^{2m} \times [0, 1] \to F(P, 2) \times G_d^{2m}$$

by mapping $(a, g, t)$ to $(H(a, t), g)$. Now assume that two points $(a, g')$ of $F(P, 2) \times G_d^{2m}$ satisfy $(a, g) \sim_{\lambda_d \times \lambda_d} (a, g')$.

Claim A. $H_1(a, g, t) = (H(a, t), g) \sim_{\lambda_d \times \lambda_d} (H(a, t), g') = H_1(a, g', t)$.

If $a \in \text{int}(P \times 2)$, then, by the construction of $M$, $g = g'$. It follows that $H_1(a, g, t) = H_1(a, g', t)$, so $(H(a, t), g) \sim_{\lambda_d \times \lambda_d} (H(a, t), g')$ regardless of whether $H(a, t)$ belongs to $\text{int}(P \times 2)$ or not.

If $a \in \partial(P \times 2)$, then $a$ belongs to $F \times P$ or $P \times F'$ where $F$ and $F'$ are facets of $P$. Without the loss of generality, we merely consider the case of $a \in F \times P$ in the following argument. By Lemma 4.1, we see that $\sigma(t) = (H(a, t), g)$ is a path from $H(a, 0)$ to $H(a, 1)$, and there exists a sequence of faces in $F \times P$

$$F \times P \supseteq Q_1 \times Q_1' \supseteq \cdots \supseteq Q_{l-1} \times Q_{l-1}' \supseteq Q_l \times Q_l'$$

with $Q_i \cap Q_i' \neq \emptyset$ for $i = 1, \ldots, l - 1$ and $Q_l \cap Q_l' = \emptyset$, such that $\sigma(t)$ continuously runs from $\sigma(0) = H(a, 0) = a \in \text{int}(Q_1 \times Q_1')$ to $\sigma(1) = H(a, 1) \in \text{int}(Q_l \times Q_l')$ through

$$\text{int}(Q_1 \times Q_1') \supseteq \cdots \supseteq \text{int}(Q_{l-1} \times Q_{l-1}') \supseteq \text{int}(Q_l \times Q_l').$$

Thus, by the definition of $\sim_{\lambda_d}$, we have that $g^{-1}g' \in G_{Q_1} \times G_{Q_1'}$. On the other hand, by the construction of $M$, we have the following sequence of subgroups of $G_d^{2m}$

$$G_{Q_1} \times G_{Q_1'} < \cdots < G_{Q_{l-1}} \times G_{Q_{l-1}'} < G_{Q_l} \times G_{Q_l'}$$

where $G_Q$ is the subgroup of $G_d^m$, determined by $Q$ and the characteristic function of $P$ (see subsection 2.1). This means that $g^{-1}g' \in G_{Q_i} \times Q_i'$ for all $1 \leq i \leq l$. Thus, $(H(a, t), g) \sim_{\lambda_d \times \lambda_d} (H(a, t), g')$ by the definition of $\sim_{\lambda_d}$.

Moreover, we conclude by Claim A that $H_1$ descends to an equivariant strong deformation retraction of $F_{G_d^m}(M, 2)$ onto $X_d(M, 2)$. □
Remark 8. By the intersection property of all \((\pi_2^{-1})^2(F_1 \times F_2), F_1, F_2 \in \mathcal{F}(P)\) with \(F_1 \cap F_2 = \emptyset\), in the way as shown in section 7, \(X_d(M, 2)\) can determine a simplicial complex \(K\) such that \(K\) is exactly the dual cell decomposition of \(\mathcal{A}(P, 2)\).

**Corollary 4.3.** Let \(\pi_2 : M \rightarrow P\) be a \(2m\)-dimensional quasi-toric manifold over a simple convex polytope \(P\). Then \(F_{T_m}(M, 2)\) is simply connected.

**Proof.** We know from [DJ, 1.10 and Theorem 3.1] that for any two faces \(F_1\) and \(F_2\) of \(P\), \((\pi_2^{-1})^2(F_1 \times F_2)\) is a quasi-toric manifold over \(F_1 \times F_2\), and each quasi-toric manifold has no odd-dimensional cells. Thus, \(X_2(M, 2)\) is simply connected, so is \(F_{T_m}(M, 2)\) by Theorem 4.2. \(\square\)

4.3. **Examples.** Now let us look at the case \(m = 2\). In this case, \(P\) is a polygon.

1. When \(P\) is a 3-polygon, we have that \(F_{Z_2}(M, 2)\) has the homotopy type of the following 1-dimensional simplicial complex

![1-dimensional simplicial complex](image1)

and \(F_{T_2}(M, 2)\) has the homotopy type of a 2-dimensional simplicial complex produced by replacing six circles of the above complex by six 2-spheres.

2. When \(P\) is a 4-polygon, we have that \(F_{Z_2}(M, 2)\) has the homotopy type of the following 2-dimensional simplicial complex

![2-dimensional simplicial complex](image2)

and \(F_{T_2}(M, 2)\) has the homotopy type of a 4-dimensional simplicial complex produced by replacing four tori of the above complex by four copies of \(S^2 \times S^2\).
(3) When \( P \) is a 5-polygon, we have that \( F_{\mathbb{Z}_2}(M, 2) \) has the homotopy type of

The resulting space is obtained by gluing same colored circles together

and \( F_{\mathbb{T}^2}(M, 2) \) has the homotopy type of a 4-dimensional simplicial complex produced by replacing all tori and circles of the above complex by \( S^2 \times S^2 \) and \( S^2 \) respectively.

5. Betti numbers and (equivariant) cohomology of \( F_{G^2_d}(M, 2) \)

Throughout the following, assume that \( \pi_d : M \rightarrow P \) is a \( dm \)-dimensional \( G^m_d \)-manifold over a simple convex polytope \( P \). By Theorem 4.2 let \( \mathcal{H} : F_{G^m_d}(F, 2) \times [0, 1] \rightarrow F_{G^m_d}(F, 2) \) be the equivariant strong deformation retraction of \( \tilde{F}_{G^m_d}(F, 2) \) onto \( X_d(M, 2) \).

**Lemma 5.1.** The equivariant cohomologies of \( F_{G^m_d}(F, 2) \) onto \( X_d(M, 2) \) are isomorphic, i.e.,

\[
H^*_G(F_{G^m_d}(F, 2)) \cong H^*_G(X_d(M, 2)).
\]

**Proof.** Consider the following equivariant lifting of \( \mathcal{H} \)

\[
\tilde{\mathcal{H}} : EG^m_d \times F_{G^m_d}(F, 2) \times [0, 1] \rightarrow EG^m_d \times F_{G^m_d}(F, 2)
\]
by mapping \((x, y, t)\) to \((x, \mathcal{H}(y, t))\). This lifting \(\mathcal{H}\) descends to a deformation retraction

\[
EG_d^{2m} \times G_d^{2m} F_G^2(F, 2) \times [0, 1] \to EG_d^{2m} \times G_d^{2m} F_G^2(F, 2)
\]

of \(EG_d^{2m} \times G_d^{2m} F_G^2(F, 2)\) onto \(EG_d^{2m} \times G_d^{2m} X_d(M, 2)\), which induces the required result.

Now let us look at the cell structure of \(X_d(M, 2)\).

**Lemma 5.2.** \(X_d(M, 2)\) has a perfect cell structure with respect to \(R_d\) coefficients in the sense of Morse theory, i.e., the closure of each cell of \(X_d(M, 2)\) is a (pseudo) manifold, where \(R_d\) is \(Z_2\) if \(d = 1\), and \(Z\) if \(d = 2\).

**Proof.** As shown in \([DJ, \text{Theorem 3.1}]\), each quasi-toric manifold (or small cover) has a perfect cell structure with respect to \(Z\) (or \(Z_2\)) coefficients in the sense of Morse theory. From the construction of \(X_d(M, 2)\) in Theorem \([5.2]\), we see that all possible intersections of submanifolds of \(\{(\pi_d^{-1}) \times^2 (F_1 \times F_2) | F_1, F_2 \in F(P) \text{ with } F_1 \cap F_2 = \emptyset\}\) in \(X_d(M, 2)\) are also small covers if \(d = 1\) and quasi-toric manifolds if \(d = 2\), so the lemma follows from this.

**Remark 9.** Lemma \([5.2]\) implies that \(Z_2^{2m}\) acts trivially on \(H^*(F_{2^{2m}}(M, 2); Z_2)\).

5.1. The homology of \(F_{T^m}(M, 2)\). Now let us observe the homology of \(F_{T^m}(M, 2)\) for a \(2m\)-dimensional quasi-toric manifold \(\pi_2 : M \to P\).

**Proposition 5.1.** Let \(\pi_2 : M \to P\) be a \(2m\)-dimensional quasi-toric manifold over a simple convex polytope \(P\). Then the homology of \(F_{T^m}(M, 2)\) vanishes in odd dimensions and is free abelian in even dimensions.

**Proof.** By Theorem \([4.2]\) it suffices to consider the homology of \(X_2(M, 2)\). \([DJ, \text{Theorem 3.1}]\) tells us that all cells of each quasi-toric manifold are of even dimension, so \(X_2(M, 2)\) has only even-dimensional cells. Then the required result follows from Lemma \([5.2]\).

We see that the dual cell decomposition \(K\) of \(A(P, 2)\) as a polyhedron is a simplicial complex, and it indicates the intersection property of submanifolds of \(\{(\pi_d^{-1}) \times^2 (F_1 \times F_2) | F_1, F_2 \in F(P) \text{ with } F_1 \cap F_2 = \emptyset\}\) in \(X_d(M, 2)\) for a \(G_d^{2m}\)-manifold \(\pi_d : M \to P\). Then we know from section \([4]\) that \(X_d(M, 2)\) with \(K\) together can be associated to the Mayer–Vietoris spectral sequence \(E^1_{p,q}(K), ..., E_{p,q}^\infty(K)\). The following result is a consequence of Proposition \([5.1]\) and Theorem \([5.1]\).

**Corollary 5.1.** Let \(\pi_2 : M \to P\) be a \(2m\)-dimensional quasi-toric manifold over a simple convex polytope \(P\). Then the associated Mayer–Vietoris spectral sequence of \(X_2(M, 2)\) collapses at the \(E^2_{p,q}\) term. Moreover,

\[
H_i(F_{T^m}(M, 2)) = \bigoplus_{p+q=i} E^2_{p,q}(K).
\]

5.2. Relation between Betti numbers of \(F_{T^m}(M, 2)\) and mod 2 Betti numbers of \(F_{2^{2m}}(M, 2)\). We know from \([DJ, \text{Theorem 3.1}]\) that the Betti numbers (resp. mod 2 Betti numbers) of a quasi-toric manifold (resp. a small cover) \(M \to P\) only depends upon the combinatorics (more precisely, the \(h\)-vector) of \(P\). Lemma \([5.2]\) tells us that \(X_d(P, 2)\) has still a perfect cell structure in the sense of Morse theory, and in particular, this perfect cell structure only depends upon the structure of
$A(P, 2) = \bigcup_{F_1, F_2 \in \mathcal{F}(P)} F_1 \times F_2$. In other words, whichever $d = 1$ or 2, the number of all cells at any dimension in the perfect cell decomposition of $X_d(M, 2)$ is completely determined by the $h$-vectors of those polytopes $F_1 \times F_2$ with $F_1 \cap F_2 = \emptyset$, $F_1, F_2 \in \mathcal{F}(P)$. Therefore, with Lemma 5.2 and Proposition 5.1 together, we conclude that

**Theorem 5.2.** Given a simple convex polytope $P$, assume that $\pi_d : M \to P$ is a $d$-dimensional $G^m_d$-manifold over $P$. Then

$$b_2(F_{T^m}(M, 2)) = b^d_2(F_{Z^m_2}(M, 2))$$

and both only depend upon the combinatorial structure of $P$.

5.3 Equivariant cohomology of $F_{G^m_d}(M, 2)$. Let $\phi_d : EG^d_m \times G^m_d \to F_{G^m_d}(M, 2) \to BG^d_m$ be the fibration with fiber $F_{G^m_d}(M, 2)$.

**Theorem 5.3.** Let $\pi_d : M \to P$ be a $d$-dimensional $G^m_d$-manifold over a simple convex polytope $P$. Then the Leray–Serre spectral sequence (with $R_d$ coefficients) of the fibration $\pi_d$ collapses at the $E_2$ term (i.e., $E^{pq}_2 = E^{pq}_2$) if $d = 2$, and has the property $E^{pq}_2 = H^*(BZ^m_2; \mathbb{Z}) \otimes H^q(F_{Z^m_2}(M, 2); \mathbb{Z})$ if $d = 1$.

**Proof.** We see from the proof of Lemma 5.1 that the fibration $\pi_d$ is homotopic to the fibration $\tilde{\pi}_d : EG^d_m \times G^m_d X_d(M, 2) \to BG^d_m$ with fiber $X_d(M, 2)$, so it suffices to consider the fibration $\pi_d$. When $d = 2$, all the differentials in the spectral sequence are trivial since both $BT^2_m$ and $X_2(M, 2)$ have only even-dimensional cells. Thus, in this case, $E^{pq}_2 = E^{pq}_2$. When $d = 1$, we have that the fundamental group $\pi_1(BZ^m_2) \cong \mathbb{Z}^m_2$, which acts trivially on $H^*(X_1(M, 2); \mathbb{Z})$ by Remark 14. Thus we have that $E^{pq}_2 = H^p(BZ^m_2; \mathbb{Z}) \otimes H^q(F_{Z^m_2}(M, 2); \mathbb{Z})$. $\Box$

**Corollary 5.4.** Let $\pi_2 : M \to P$ be a $2$-dimensional quasi-toric manifold over a simple convex polytope $P$. Then $H^2_1(F_{T^m}(M, 2)) = H^*(F_{T^m}(M, 2)) \otimes H^*(BT^2_m)$ is a free $H^*(BT^2_m)$-module, and the inclusion of fiber $F_{T^m}(M, 2) \hookrightarrow ET^2_m \times T^2_m F_{T^m}(M, 2)$ induces an epimorphism.

6. Calculation of the (mod 2) homology of $F_{G^m_d}(M, 2)$ and $F_{G^m_d}(M, 2)$ for $P$ to be an $m$-simplex

In this section, using Theorem 5.2 and the Mayer–Vietoris spectral sequence we calculate the (mod 2) homology of $F_{G^m_d}(M, 2)$ and $F_{G^m_d}(M, 2)$ for $P$ to be an $m$-simplex. By Theorem 1.5 this is equivalent to determining the (mod 2) Betti numbers. Our results are stated as follows:

**Proposition 6.1.** Let $\pi_d : M \to P$ be a $d$-dimensional $G^m_d$-manifold over a polygon $P$ with $\ell$ vertices. When $\ell = 3$, all nonzero Betti numbers of $F_{Z^m_2}(M, 2)$ (resp. $F_{T^2}(M, 2)$) are $(b_0, b_1) = (1, 7)$ (resp. $(b_0, b_2) = (1, 7)$); when $\ell > 3$, all nonzero Betti numbers of $F_{Z^m_2}(M, 2)$ (resp. $F_{T^2}(M, 2)$) are $(b_0, b_1, b_2) = (1, 2\ell + 1, \ell(\ell - 3))$ (resp. $(b_0, b_2, b_3) = (1, 2\ell + 1, \ell(\ell - 3))$). In particular, the non-vanishing homology of $F_{Z^m_2}(M, 2)$ is free abelian.

**Remark 10.** As we have seen in Proposition 6.1, we actually determine the integral homology of $F_{Z^m_2}(M, 2)$. However, unlike 2-dimensional small covers, the non-vanishing homology of $F_{Z^m_2}(M, 2)$ has no torsion. This can also be seen from the special examples in subsection 4.3.
Proposition 6.2. Let \( \pi_d : M \rightarrow \Delta^m \) be a \( dm \)-dimensional \( G_m^d \)-manifold over an \( m \)-simplex \( \Delta^m \). When \( d = 1 \), all nonzero mod 2 Betti numbers of \( F_{Z_2^m}(M, 2) \) are

\[
(b_0^{Z_2}, b_1^{Z_2}, ..., b_{m-2}^{Z_2}, b_{m-1}^{Z_2}) = (1, 2, ..., m - 1, \frac{3^{m+1} + 2m - 3}{4}).
\]

When \( d = 2 \), all nonzero Betti numbers of \( F_{T^m}(M, 2) \) are

\[
(b_0, b_2, ..., b_{2m-4}, b_{2m-2}) = (1, 2, ..., m - 1, \frac{3^{m+1} + 2m - 3}{4}).
\]

In order to show Propositions 6.1 and 6.2, by Theorems 1.5 and 4.2 we need merely consider the case of \( d = 1 \) and calculate the (mod 2) Betti numbers of \( X_1(M, 2) \) in the following discussion.

6.1. The integral homology of \( F_{Z_2^m}(M, 2) \) for 2-dimensional small covers.

Let \( P \) be a \( \ell \)-polygon with facets \( F_1, ..., F_\ell \) and vertices \( v_1, ..., v_\ell \), as shown in the following diagram:

By Theorem 4.2, \( F_{Z_2^m}(M, 2) \) is homotopic to

\[
X(\ell) = \begin{cases} 
\bigcup_{F_i \cap F_j = \emptyset} (\pi_1^{-1})^\times 2(F_i \times F_j) & \text{if } \ell > 3 \\
\bigcup_{v_i \cap F_j = \emptyset} (\pi_1^{-1})^\times 2(v_i \times F_j) \bigcup \bigcup_{F_i \cap v_j = \emptyset} (\pi_1^{-1})^\times 2(F_i \times v_j) & \text{if } \ell = 3.
\end{cases}
\]

Now let us determine the simplicial complex \( K(\ell) \) dual to \( \mathcal{A}(P, 2) \).

When \( \ell = 3, 4 \), it is easy to see that \( K(3) \) is a 6-polygon with vertices \( v_1 \times F_3, v_2 \times F_1, v_3 \times F_2, F_1 \times v_2, F_2 \times v_3, F_3 \times v_1 \), and \( K(4) \) is a 4-polygon with four vertices \( F_1 \times F_3, F_2 \times F_4, F_3 \times F_1, F_4 \times F_2 \). When \( \ell = 5 \), \( K(5) \) is a 2-dimensional simplicial complex with 10 vertices. Actually \( K(5) \) is exactly an annulus as shown
in the following diagram:

```
In general, when \( \ell > 5 \), \( K(\ell) \) is a 3-dimensional simplicial complex with vertex set 
\( \{F_i \times F_j | F_i \cap F_j = \emptyset\} \) such that there are 3-dimensional simplices of the form

\[
\{F_i \times F_j, F_{i+1} \times F_j, F_i \times F_{j+1}, F_{i+1} \times F_{j+1}\}
\]

where \( F_{i+1} \) will be \( F_1 \) if \( i = \ell \), and \( F_{j+1} \) will be \( F_1 \) if \( j = \ell \), and some additional 
2-dimensional simplices of the form

\[
\{F_i \times F_{i+2}, F_i \times F_{i+3}, F_{i+1} \times F_{i+3}\} \text{ or } \{F_{i+2} \times F_i, F_{i+3} \times F_i, F_{i+3} \times F_{i+1}\}
\]

where \( F_{i+2} \) will be \( F_1, F_2 \) if \( i = \ell - 1, \ell \) and \( F_{i+3} \) will be \( F_1, F_2, F_3 \) if \( i = \ell - 2, \ell - 1, \ell \). 
Obviously, \( K(\ell) \) contains \( \ell(\ell - 3) \) vertices. We also know from Remark\[7\] that \( K(\ell) \) 
is homotopic to a circle.

Next, for the convenience of calculation, let us choose a locally nice subcomplex 
\( L(\ell) \) of \( K(\ell) \) (for the notion of a locally nice subcomplex, see Defintion\[7.1\]). When 
\( \ell \leq 5 \), take \( L(\ell) = K(\ell) \). When \( \ell \geq 6 \), we take \( L(\ell) \) in such a way that \( L(\ell) \) contains 
\( \emptyset \) and all vertices of \( K(\ell) \), and \( 2\ell(\ell - 4) \) 2-dimensional simplices of the following forms

\[
\{F_i \times F_j, F_{i+1} \times F_j, F_{i+1} \times F_{j+1}\} \text{ and } \{F_i \times F_j, F_i \times F_{j+1}, F_{i+1} \times F_{j+1}\}
\]

In this case, it is easy to check that \( L(\ell) \) is exactly an annulus, and it has \( \ell(3\ell - 11) \) 
1-simplices. For example, when \( \ell = 6 \), \( L(6) \) is a 2-dimensional simplicial complex
as shown in the following picture:

With the above arguments together, we have

**Lemma 6.1.** When \( \ell \leq 4 \), \( L(3) \) is a 6-polygon and \( L(4) \) is a 4-polygon. When \( \ell \geq 5 \), \( L(\ell) \) is a triangulation of an annulus with \( \ell(\ell - 3) \) vertices, \( \ell(3\ell - 11) \) 1-simplices and \( 2\ell(\ell - 4) \) 2-simplices.

Now, to complete the proof of Proposition 6.1, it suffices to show the following result.

**Proposition 6.3.** \( X(3) \) is a 1-dimensional connected CW complex with Betti numbers \((b_0, b_1) = (1, 7)\). When \( \ell \geq 4 \), \( X(\ell) \) is a 2-dimensional connected CW complex with Betti numbers \((b_0, b_1, b_2) = (1, 2\ell + 1, \ell(\ell - 3))\).

**Proof.** Each vertex of \( L(\ell) \) is of the form \( F_i \times F_j \) if \( \ell > 3 \), and of the form \( v_i \times F_j \) or \( F_i \times v_j \) if \( \ell = 3 \). Since each \( \pi_i^{-1}(F_i) \) is a circle and \( \pi_1^{-1}(v_i) \) is a point, we have that \( (\pi_1^{-1})^2(F_i \times F_j) \) is a torus, and \( (\pi_1^{-1})^2(v_i \times F_j) \) (or \( (\pi_1^{-1})^2(F_i \times v_j) \)) is also a circle. Also, for all \( \ell \geq 3 \), \( L(\ell) \) is connected. Thus, \( X(3) \) is a 1-dimensional connected CW complex and when \( \ell \geq 4 \), \( X(\ell) \) is a 2-dimensional connected CW complex.

Now we first have by Remark [11] and Lemma [6.1] that \( E^2_{p,0}(L(\ell)) = H_p(S^1) \), so \( E^2_{p,0}(L(\ell)) = 0 \) if \( p > 1 \) and \( E^2_{1,0}(L(\ell)) \cong E^2_{1,0}(L(\ell)) \cong \mathbb{Z} \).

If \( \ell = 3 \), it is easy to see that \( E^1_{p,q}(L(3)) = 0 \) for \( p > 0 \) and \( q > 1 \), and \( E^1_{0,1}(L(3)) \cong \mathbb{Z}^6 \). So, we have that

\[
E^2_{p,q}(L(3)) = \begin{cases} 
0 & \text{if either } p > 1 \text{ and } q = 0 \text{ or } p > 0 \text{ and } q > 1 \\
\mathbb{Z} & \text{if } p \leq 1 \text{ and } q = 0 \\
\mathbb{Z}^6 & \text{if } p = 0 \text{ and } q = 1 
\end{cases}
\]
so \( E^\infty(L(3)) = E^2(L(3)) \).

If \( \ell > 3 \), then we have that \( E^2_{p,q}(L(\ell)) = 0 \) for either \( p > 1 \) and \( q > 2 \) or \( p > 0 \) and \( q = 2 \). By a direct calculation, we obtain that \( E^2_{0,2}(L(\ell)) \cong \mathbb{Z}^{\ell(\ell-3)} \), \( E^1_{0,1}(L(\ell)) \cong \mathbb{Z}^{2\ell(\ell-3)} \) and \( E^1_{1,1}(L(\ell)) \cong \mathbb{Z}^{2\ell(\ell-4)} \). When \( \ell = 4 \), \( \mathbb{Z}^{2\ell(\ell-4)} \) means the trivial group 0, and so \( E^2_{1,1}(L(4)) = 0 \) and \( E^2_{0,1}(L(4)) \cong \mathbb{Z}^4 \).

When \( \ell > 4 \), it is easy to check that

\[
0 \longrightarrow E^1_{1,1}(L(\ell)) \longrightarrow E^0_{0,1}(L(\ell)) \longrightarrow 0
\]

is isomorphic to the simplicial complex given by the disjoint union of \( 2\ell \) copies of a segment, so \( E^2_{1,1}(L(\ell)) = 0 \) and \( E^2_{0,1}(L(\ell)) \cong \mathbb{Z}^{2\ell} \). Thus, if \( \ell > 3 \),

\[
E^2_{p,q}(L(\ell)) = \begin{cases} 
0 & \text{if either } p > 1 \text{ and } q = 0 \text{ or } p > 0 \text{ and } q > 2 \\
\mathbb{Z} & \text{if } p \leq 1 \text{ and } q = 0 \\
\mathbb{Z}^{2\ell} & \text{if } p = 0 \text{ and } q = 1 \\
\mathbb{Z}^{\ell(\ell-3)} & \text{if } p = 0 \text{ and } q = 2
\end{cases}
\]

so \( E^\infty(L(\ell)) = E^2(L(\ell)) \).

Furthermore, we may conclude the desired Betti numbers for \( X(\ell) \) by Theorem 6.3.

\[ \square \]

6.2. The case in which \( P \) is an \( m \)-simplex \( \Delta^m \) with \( m > 1 \). Let \( sd(Bd(\Delta^m)) \) be the barycentric subdivision of the boundary complex of \( \Delta^m \), which is an \((m-1)\)-dimensional simplicial complex. Each simplex of \( sd(Bd(\Delta^m)) \) will be expressed as the form \( \sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_k \), where \( \sigma_i \) is a face of \( Bd(\Delta^m) \) (so \( 0 \leq \dim \sigma_i \leq m-1 \)), and is understood as a vertex of \( sd(Bd(\Delta^m)) \). Let \( i, j \) be non-negative integers with \( i + j + 1 \leq m \). By \( K^m_{i,j} \) we denote the subcomplex of \( sd(Bd(\Delta^m)) \), formed by those simplices \( \{ \sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_k | \dim \sigma_1 \geq i, \dim \sigma_k \leq m - j - 1 \} \). It is easy to check that \( K^m_{i,j} \) has the following properties:

- \( K^m_{i,j} \) is \((m - i - j - 1)\)-dimensional and connected.
- For \( i' \leq i \) and \( j' \leq j \), \( K^m_{i,j} \subset K^m_{i',j'} \).
- \( K^0_{0,0} \) is the barycentric subdivision of the \((m - j - 1)\)-dimensional skeleton of \( Bd(\Delta^m) \). In particular, \( K^m_{0,0} = Bd(\Delta^m) \).

**Lemma 6.2.** \( K^m_{i,j} \) is combinatorially equivalent to \( K^m_{i',i} \).

**Proof.** This follows by mapping each vertex \( \sigma \) of \( K^m_{i',i} \) to the vertex \( \overline{\sigma} \) of \( K^m_{i,j} \) and each simplex \( \sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_k \) to \( \overline{\sigma_1} \subset \cdots \subset \overline{\sigma'_k} \subset \overline{\sigma} \), where \( \overline{\sigma} \) means the complement of \( \sigma \) in the boundary complex \( Bd(\Delta^m) \) of \( \Delta^m \), i.e., \( \overline{\sigma} \) is the face of \( \Delta^m \), determined by those vertices which are not contained in \( \sigma \).

**Lemma 6.3.** When \( m = i + j + 1 \), \( H_r(K^m_{i,j}) = 0 \) if \( r \neq 0 \), and when \( m > i + j + 1 \), \( H_r(K^m_{i,j}) = 0 \) if \( r \neq 0 \) and \( r \neq m - i - j - 1 \).

**Proof.** When \( m = i + j + 1 \), \( K^m_{i,j} \) is a 0-dimensional complex, so \( H_r(K^m_{i,j}) = 0 \) if \( r \neq 0 \).

Now suppose that \( m > i + j + 1 \). Given a simplex \( \sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_k \) in \( K^m_{i,j} \), if \( \dim \sigma_k < m - j - 1 \), then it belongs to \( K^m_{i,j+1} \). If \( \dim \sigma_k = m - j - 1 \), then it is easy to see that this simplex \( \sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_k \) belongs to the following subcomplex of \( K^m_{i,j} \)

\[
\bigcup_{\sigma_i \in K^m_{i,j} \atop \dim \sigma_i = m - j - 1} St(\sigma_i, K^m_{i,j}).
\]
Also, $Lk(\sigma_k, K^n_{m,j}) = \{ \sigma_1 \subset \sigma_2 \cdots \subset \sigma_s | \dim \sigma_1 \geq i, \sigma_s \subset \sigma_k \text{ and } \sigma_s \neq \sigma_k \}$, and obviously it is the subcomplex $K^n_{i,0} - j - 1$ of $sd(Bd(\sigma_k))$. Thus, we conclude that

\[(6.1) \quad K^m_{i,j} = K^m_{i,j+1} \bigcup \bigcup_{\dim \sigma_l = m-j-1} St(\sigma_l, K^m_{i,j})\]

and for $\sigma_l \in K^m_{i,j}$ with $\dim \sigma_l = m-j-1$

\[(6.2) \quad St(\sigma_l, K^m_{i,j}) \bigcap K^m_{i,j+1} = Lk(\sigma_l, K^m_{i,j}) = K^m_{i,0} - j - 1.\]

Claim B. For $m > i + j + 1$, $H_s(K^m_{i,j}, K^m_{i,j+1}) = 0$ if $s \neq m - i - j - 1$.

Using the axiom of excision and (6.1)–(6.2), we see that

\[H_s(K^m_{i,j}, K^m_{i,j+1}) \cong \bigoplus_{\sigma_l \in K^m_{i,j}} H_s(St(\sigma_l, K^m_{i,j}), Lk(\sigma_l, K^m_{i,j})).\]

For $s > 1$, since

\[H_s(St(\sigma_l, K^m_{i,j}), Lk(\sigma_l, K^m_{i,j})) = H_{s-1}(Lk(\sigma_l, K^m_{i,j})) = H_{s-1}(K^m_{i,0} - j - 1)\]

and $K^m_{i,0} - j - 1$ has the same homology as the $(m - i - j - 2)$-skeleton of $Bd(\sigma_i)$, it follows that Claim B holds.

We note that $K^m_{i,0}$ is just $sd(Bd(\Delta^m))$, so $H_r(K^m_{i,0}) = 0$ if $r \neq 0$ and $r \neq m - 1$. Consider the long exact sequence

\[\cdots \rightarrow H_k(K^m_{i,j+1}) \rightarrow H_k(K^m_{i,j}) \rightarrow H_k(K^m_{i,j}, K^m_{i,j+1}) \rightarrow H_{k-1}(K^m_{i,j+1}) \rightarrow \cdots.\]

Moreover, using an induction and Claim B, we may easily obtain the required result. \(\square\)

Now let $M$ be a small cover over an $m$-dimensional simplex $\Delta^m$. Then by Theorem 4.2, $F_{Z^2}(M, 2)$ is homotopic to

\[X_1(M, 2) = \bigcup_{\sigma_i \in F(\Delta^m)} (\pi_1^{-1})^2(\sigma_i \times \overline{\sigma_i}).\]

Obviously, $X_1(M, 2)$ is an $(m - 1)$-dimensional CW complex. In order to apply Mayer-Vietoris spectral sequence, we choose a locally nice complex $L$ in such a way that the vertex set of $L$ consists of all $(\pi_1^{-1})^2(\sigma_i \times \overline{\sigma_i})$ where $\sigma_i \in F(\Delta^m)$, and each simplex of $L$ is of the form

\[\{(\pi_1^{-1})^2(\sigma_i \times \overline{\sigma_i}), \cdots, (\pi_1^{-1})^2(\sigma_{t'} \times \overline{\sigma_{t'}})\}\]

with $\sigma_i \subset \cdots \subset \sigma_{t'}$. If we map $(\pi_1^{-1})^2(\sigma_i \times \overline{\sigma_i})$ to $\sigma_i$, we see that $L$ is combinatorially equivalent to $sd(Bd(\Delta^m))$. With this understood, we will identify $L$ with $sd(Bd(\Delta^m))$.

Proof of Proposition 6.2. Given a simplex $a$ of the form $\sigma_{i_1} \subset \cdots \subset \sigma_{i_t}$ in $L$, it is easy to see that

\[X_a = (\pi_1^{-1})^2(\sigma_{i_1} \times \overline{\sigma_{i_1}}) \cap \cdots \cap (\pi_1^{-1})^2(\sigma_{i_t} \times \overline{\sigma_{i_t}}) = (\pi_1^{-1})^2(\sigma_{i_1} \times \overline{\sigma_{i_1}}) \cap \cdots \cap (\pi_1^{-1})^2(\sigma_{i_t} \times \overline{\sigma_{i_t}}).\]

It is well-known that $M$ is homeomorphic to $\mathbb{R}P^m$ and for each face $\sigma$ of $\Delta^m$, $\pi_1^{-1}(\sigma)$ is homeomorphic to $\mathbb{R}P^{\dim \sigma}$. Thus, $X_a$ is actually homeomorphic to $\mathbb{R}P^s \times \mathbb{R}P^t$ where $s = \dim \sigma_{i_1}$ and $t = \dim \sigma_{i_t}$. Moreover,

\[H_k(X_a; \mathbb{Z}_2) \cong H_k(\mathbb{R}P^s \times \mathbb{R}P^t; \mathbb{Z}_2) = \sum_{i+j=k} H_i(\mathbb{R}P^s; \mathbb{Z}_2) \otimes H_j(\mathbb{R}P^t; \mathbb{Z}_2).\]
is generated by $\beta^i \otimes \gamma^j$ with $i + j = k$, where $\beta^i$ and $\gamma^j$ are generators of $H_i(\mathbb{R}P^s; \mathbb{Z}_2)$ and $H_j(\mathbb{R}P^t; \mathbb{Z}_2)$. To emphasize $\beta^i \otimes \gamma^j$ as an element of $H_k(X_a; \mathbb{Z}_2)$, we shall denote it by $(\beta^i \otimes \gamma^j)_a$.

Define a map

$$f : E^1_{*,q}(L) = \bigoplus_{a \in L} H_q(X_a; \mathbb{Z}_2) \to \bigoplus_{i+j=q} \mathcal{C}_*(K^n_{i,j}; \mathbb{Z}_2)$$

by mapping $(\beta^i \otimes \gamma^j)_a$ to $a \in K^n_{i,j}$, where $\mathcal{C}_*(K^n_{i,j}; \mathbb{Z}_2)$ is the chain complex of $K^n_{i,j}$ with $\mathbb{Z}_2$ coefficients. Then $f$ is a chain map since the boundary operator on $E^1_{*,q}(L)$ agrees with the boundary operator on $\bigoplus_{i+j=q} \mathcal{C}_*(K^n_{i,j}; \mathbb{Z}_2)$ by Remark 11. It is easy to check that $f$ is a bijection, so $f$ is actually a chain isomorphism. Thus, we obtain that

$$(6.3) \quad E^2_{p,q}(L) \cong \bigoplus_{i+j=q} H_p(K^n_{i,j}; \mathbb{Z}_2).$$

Note that if $q = 0$, then $K^n_{0,0} = \text{sd}(\text{Bd}((\Delta^m)_a)) = L$, so $E^2_{0,0}(L) \cong H_p(S^{m-1}; \mathbb{Z}_2)$. This is also shown in Remark 11 since each $X_a$ is connected.

Now by Lemma 6.3 and (6.3), we have that $E^2_{p,q}(L) = 0$ if $p + q \neq m - 1$ and $p \neq 0$. Also it is easy to see that $\dim E^2_{0,q}(L) = q + 1$ if $q \leq m - 1$.

Combining the above arguments, we have that if $p + q \neq m - 1$, then $E^\infty_{p,q}(L) = E^2_{p,q}(L)$, so $\dim H_k(X_1(M, 2); \mathbb{Z}_2) = k + 1$ for $k < m - 1$.

Now by Hopf trace theorem, we have that

$$\sum_{k=0}^{m-1} (-1)^k \dim H_k(X_1(M, 2); \mathbb{Z}_2) = \sum_{k=0}^{m-1} (-1)^k \dim D_k(X_1(M, 2); \mathbb{Z}_2)$$

where $D_k(X_1(M, 2); \mathbb{Z}_2)$ is the $k$-dimensional cellular chain group of $X_1(M, 2)$. By the construction of small covers, it is easy to check that

$$\dim D_k(X_1(M, 2); \mathbb{Z}_2) = 2^k \sum_{i+j=k} \binom{m+1}{i+1} \binom{m-i}{j+1}$$

so

$$\dim H_{m-1}(X_1(M, 2); \mathbb{Z}_2)$$

$$= \sum_{k=0}^{m-1} (-1)^k \dim D_k(X_1(M, 2); \mathbb{Z}_2) - \sum_{k=0}^{m-2} (-1)^k \dim H_k(X_1(M, 2); \mathbb{Z}_2)$$

$$= \sum_{k=0}^{m-1} (-2)^k \sum_{i+j=k} \binom{m+1}{i+1} \binom{m-i}{j+1} - \sum_{k=0}^{m-2} (-1)^k (k + 1)$$

$$= \frac{3^{m+1} + 2m - 3}{4}.$$ 

This completes the proof. \(\square\)

7. Appendix—Mayer-Vietoris spectral sequence

Suppose that $X$ is a CW-complex with all cells indexed by $J$, and $X_1, ..., X_N$ are subcomplexes of $X$ such that $\bigcup X_i = X$ and all possible intersections of $X_1, ..., X_N$ are subcomplexes of $X$. Associated with $X$, we may define an abstract simplicial
complex $K$ (including empty set) with vertices $1, ..., N$ (or $X_1, ..., X_N$) as follows: if $X_{i_1} \cap \cdots \cap X_{i_r} \neq \emptyset$, then $\{i_1, ..., i_r\} \in K$. For each $a \in K$, we define

$$X_a = \begin{cases} \bigcap_{i \in a} X_i & \text{if } a \neq \emptyset \\ X & \text{if } a = \emptyset. \end{cases}$$

Set $D^K_{p,q}(X) = \bigoplus_{a \in K, |a| = p+1} D_q(X_a)$ where $D_q(X_a) = \{D_q(X_a)\}$ is the cellular chain complex of $X_a$. Then we shall see that $D^K_{*,*}(X)$ has a natural double complex structure.

Let $e_a$ be a cell of $X$ in $\{e_\alpha | \alpha \in J\}$. Define $K(e_a) = \{a \in K | e_a \subset X_a\}$. Obviously, $K(e_a)$ is a subcomplex determined by some simplex of $K$, so it is acyclic. If $a \in K(e_a)$, then $e_a$ would be a generator of $D_{\dim e_a}(X_a)$, denoted by $e_{a,a}$. Furthermore, we may write each cellular chain of $D^K_{p,q}(X) = \bigoplus_{a \in K, |a| = p+1} D_q(X_a)$ as

$$\sum_{\alpha \in J(q)} \sum_{a \in K(e_a) \atop |a| = p+1} k_{\alpha,a} e_{\alpha,a}$$

where $k_{\alpha,a} \in \mathbb{Z}$ and $J(q)$ means that for $a \in J(q)$, $\dim e_a = q$.

Let $c = \sum_{a \in K(e_a)} \lambda_a a$ be a chain in the simplicial chain complex $C_*(K(e_a))$ of $K(e_a)$. Define $e_{\alpha,c} = \sum_{a \in K(e_a)} \lambda_a e_{\alpha,a}$. Then it is easy to check that

**Lemma 7.1.** $e_{\alpha,c} = 0$ if and only if $c = 0$.

Now two differentials on $D^K_{*,*}(X)$ are defined as follows: One is $\partial_1 : D^K_{p,q}(X) \to D^K_{p,q-1}(X)$ given by

$$\partial_1 \left( \sum_{\alpha \in J(q)} \sum_{a \in K(e_a) \atop |a| = p+1} k_{\alpha,a} e_{\alpha,a} \right) = \sum_{\alpha \in J(q)} \sum_{a \in K(e_a) \atop |a| = p+1} k_{\alpha,a} \partial(e_{\alpha,a})$$

which is induced by the boundary homomorphism $\partial$ of $D_q(X_a)$, and the other one is $\partial_2 : D^K_{p,q}(X) \to D^K_{p-1,q}(X)$ given by

$$\partial_2 \left( \sum_{\alpha \in J(q)} \sum_{a \in K(e_a) \atop |a| = p+1} k_{\alpha,a} e_{\alpha,a} \right) = \sum_{\alpha \in J(q)} \sum_{a \in K(e_a) \atop |a| = p+1} k_{\alpha,a} e_{\alpha,\partial' a}$$

which is induced by the boundary homomorphism $\partial'$ of the simplicial chain complex $C_*(K(e_a))$. Note that for the empty set $\emptyset \in K$, $\partial' \emptyset = 0$.

It is easy to check that $\partial_1 \partial_2 = \partial_2 \partial_1$. Thus we have the following commutative diagram:
Furthermore, we have that for each $\alpha$

$$
\partial_1 \leftarrow D^K_{1,0}(X) \leftarrow D^K_{-1,1}(X) \leftarrow \cdots \leftarrow D^K_{1,q}(X) \leftarrow \cdots \\
\partial_2 \uparrow \quad \partial_2 \uparrow \quad \partial_2 \uparrow \quad \partial_2 \uparrow
$$

for each $p$ such that $\Box$ is also a boundary chain, as desired.

$C$ is a cycle in $e^{0}_{0}$.

Now, let us look at the structure of this double complex $(D^K_{*,*}(X), \partial_1, \partial_2)$.

**Proposition 7.1.** Every column of the above diagram is exact, i.e., for each $q$,

$$
0 \xleftarrow{\partial_2} D^K_{-1,q}(X) \xrightarrow{\partial_1} D^K_{0,q}(X) \xleftarrow{\partial_2} \cdots \xrightarrow{\partial_2} D^K_{p,q}(X) \xleftarrow{\partial_2} \cdots
$$

is exact.

**Proof.** Suppose that $\sum_{\alpha \in J(q)} \sum_{a \in K(e_\alpha)} k_{a,a} e_{\alpha,a}$ is a cycle in $D^K_{p,q}(X)$. Then

$$
\partial_2 \left( \sum_{\alpha \in J(q)} \sum_{a \in K(e_\alpha)} k_{a,a} e_{\alpha,a} \right) = 0.
$$

Furthermore, we have that for each $\alpha \in J(q)$, $\partial_1 \left( \sum_{a \in K(e_\alpha)} k_{a,a} a \right) = 0$. By Lemma 7.1, we obtain that for each $\alpha \in J(q)$, $\partial_1 \left( \sum_{a \in K(e_\alpha)} k_{a,a} a \right) = 0$, so $\sum_{a \in K(e_\alpha)} k_{a,a} a$ is a cycle in $C_p(K(e_\alpha))$. Since $K(e_\alpha)$ is acyclic, there exists a chain $c_\alpha$ in $C_{p+1}(K(e_\alpha))$ such that $\partial'(c_\alpha) = \sum_{a \in K(e_\alpha)} k_{a,a} a$, so $\sum_{a \in K(e_\alpha)} k_{a,a} e_{\alpha,a} = e_{\alpha,1} \sum_{a \in K(e_\alpha)} k_{a,a} a = e_{\alpha,1} \partial'(c_\alpha)$. Therefore, we conclude that

$$
\sum_{\alpha \in J(q)} \sum_{a \in K(e_\alpha)} k_{a,a} e_{\alpha,a} = \partial_2 \left( \sum_{\alpha \in J(q)} e_{\alpha,c_\alpha} \right)
$$

is also a boundary chain, as desired. \qed

Define $E^1_{p,q}(K)$ as the $q$-th homology group of the $p$-th row in the above diagram for $p \geq 0$, and when $p < 0$, let $E^1_{p,q}(K) = 0$. Let $\text{Im}^K_{p,q} \partial_1 = \text{Im}(D^K_{p,q}(X) \xrightarrow{\partial_1} D^K_{p-1,q}(X))$. Then we have the induced chain complex:

$$
0 \leftarrow \text{Im}^K_{p,0} \partial_1 \leftarrow \text{Im}^K_{p,1} \partial_2 \leftarrow \text{Im}^K_{p,2} \partial_2 \leftarrow \cdots \leftarrow \text{Im}^K_{p,q} \partial_2 \leftarrow \cdots
$$
Let \( A_{p,q}(K) \) be the \( q \)-th homology group of this chain complex. Note that when \( p < 0 \), let \( A_{p,q}(K) = A_{0,p+q}(K) \).

Because every column in the above diagram is exact, we have the following short exact sequence:

\[
0 \longrightarrow \text{Im}^{K}_{p+1,*} \partial_{2} \longrightarrow D^{K}_{p,*}(X) \longrightarrow \text{Im}^{K}_{p,*} \partial_{2} \longrightarrow 0
\]

Furthermore, we may obtain the following long exact sequence:

\[
\cdots \overset{i}{\rightarrow} A_{p+1,q}(K) \overset{j}{\rightarrow} E^{1}_{p,q}(K) \overset{k}{\rightarrow} A_{p,q}(K) \overset{i}{\rightarrow} A_{p+1,q-1}(K) \overset{j}{\rightarrow} E^{1}_{p,q-1}(K) \overset{k}{\rightarrow} \cdots
\]

Then we have the exact couple \( A(K) \quad \xrightarrow{j} \quad A(K) \), and the spectral sequence \( E^{1}_{p,q}(K), E^{2}_{p,q}(K), \ldots, E^{\infty}_{p,q}(K) \).

By the theory of spectral sequence, we have that

**Theorem 7.1.**

\[
H_{i}(X) = \bigoplus_{p+q=i} E^{\infty}_{p,q}(K)
\]

**Remark 11.** We can explicitly write out the differential \( j \circ k \) of the chain complex

\[
\cdots \overset{j \circ k}{\rightarrow} E^{1}_{p,q}(K) \overset{j \circ k}{\rightarrow} E^{1}_{p-1,q}(K) \overset{j \circ k}{\rightarrow} \cdots \overset{j \circ k}{\rightarrow} E^{1}_{0,q}(K) \rightarrow 0.
\]

At first, for \( a \in K \), given an element \( \beta_{a} \in H_{*}(X_{a}) \). Let \( b \subset a \). Then we define \( \beta_{a,b} \) as the image of \( \beta_{a} \) under the map \( H_{*}(X_{a}) \rightarrow H_{*}(X_{b}) \) induced by the nature imbedding \( X_{a} \hookrightarrow X_{b} \). Now if \( c = \sum_{i} \lambda_{i} b_{i} \) is a chain of the simplicial chain complex of \( K \) where \( b_{i} \subset a \), define \( \beta_{a,c} \) as \( \sum_{i} \lambda_{i} \beta_{a,b_{i}} \). Next, by the definition of \( E(K) \), we know that \( E^{1}_{p,q}(K) = \bigoplus_{|a|=p+1} H_{q}(X_{a}) \), so we can write its element as \( \sum_{a \in K \ | \ |a|=p+1} \beta_{a} \).

Moreover, we see easily that

\[
j \circ k \left( \sum_{a \in K \ | \ |a|=p+1} \beta_{a} \right) = \sum_{a \in K \ | \ |a|=p+1} \beta_{a,b}.\]

In particular, when \( q = 0 \) and every \( X_{a} \) is connected, the chain complex

\[
\cdots \overset{j \circ k}{\rightarrow} E^{1}_{p,0}(K) \overset{j \circ k}{\rightarrow} E^{1}_{p-1,0}(K) \overset{j \circ k}{\rightarrow} \cdots \overset{j \circ k}{\rightarrow} E^{1}_{0,0}(K) \rightarrow 0
\]

is isomorphic to the simplicial chain complex of \( K \setminus \{\emptyset\} \). Thus, \( E^{2}_{p,0}(K) \) is isomorphic to \( H_{p}(K \setminus \{\emptyset\}) \). Note that since we have assumed that \( \emptyset \in K \), \( H_{p}(K) \) is isomorphic to the reduced homology \( \tilde{H}_{p}(K \setminus \{\emptyset\}) \).

Generally, \( K \) may have a very complicated structure. This will lead to a difficulty for calculating the spectral sequence \( E^{1}_{p,q}(K), E^{2}_{p,q}(K), \ldots, E^{\infty}_{p,q}(K) \) induced by the double complex \( D^{K}_{p,*}(X) \). For the purpose of our application, we shall choose a suitable subcomplex of \( K \), so that this may give a simpler calculation.

**Definition 7.1.** A subcomplex \( L \) of \( K \) is said to be **locally nice** if \( L \) satisfies the following properties:

- \( L \) contains all vertices of \( K \) and the empty set \( \emptyset \).
- For each cell \( e_{a} \) of \( X \), \( L(e_{a}) = \{ a \in L | e_{a} \subset X_{a} \} \) is acyclic.
Now let \( L \) be a locally nice subcomplex of \( K \). Similarly, we can define a double complex \( D_{p,q}(X) = \{ D_{p,q}(X) \} \), where \( D_{p,q}(X) = \bigoplus_{a \in L, |a| = p+1} D_q(X_a) \). Then we see that for each \( q \),

\[
0 \leftarrow D_{p,q}(X) \leftarrow D_{p,q}(X) \leftarrow \cdots \leftarrow D_{p,q}(X) \leftarrow \cdots
\]

is still exact since \( L(e_a) = \{ a \in L | e_a \subset X_a \} \) is acyclic for each cell \( e_a \) of \( X \). Thus, we can induce a corresponding spectral sequence \( E_{p,q}^1(L), E_{p,q}^2(L), \ldots, E_{p,q}^\infty(L) \) such that

\[
H_i(X) = \bigoplus_{p+q=i} E_{p,q}^\infty(L).
\]

It should be pointed out that there is also a cohomological version of the above argument. Namely, we can obtain a spectral sequence \( E_{p,q}^1(L), E_{p,q}^2(L), \ldots, E_{p,q}^\infty(L) \) from \( (X, L) \) such that

\[
H^i(X) = \bigoplus_{p+q=i} E_{p,q}^\infty(L).
\]

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