ON SOME HAMILTONIAN STRUCTURES OF COUPLED PAINLEVÉ II SYSTEMS IN DIMENSION FOUR

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Abstract. We find and study a two-parameter family of coupled Painlevé II systems in dimension four with affine Weyl group symmetry of several types.

0. Introduction

In this paper, we study coupled Painlevé II systems in dimension four
\[
\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w}, \quad \frac{dw}{dt} = -\frac{\partial H}{\partial z}
\]
with the following Hamiltonian
\[
H = H_{II}(x, y, t; \alpha_1) + H_{II}(z, w, t; \alpha_2) + ayw.
\]
Here \(x, y, z\) and \(w\) denote unknown complex variables and \(\alpha_1, \alpha_2\) are complex parameters and \(a\) is a coupling constant. The symbol \(H_{II}(q, p, t; \alpha)\) denotes the Hamiltonian of the second-order Painlevé II systems given by
\[
H_{II}(x, y, t; \alpha) = a_1 x^2y + a_2 y^2/2 + a_3 ty + a_1 \alpha x.
\]
Here \(\alpha\) is a parameter and \(a_i\) \((i = 1, 2, 3)\) are nonzero parameters which can be fixed arbitrarily.

Moreover, the system (1) becomes again a polynomial Hamiltonian system in each coordinate system \((x_i, y_i, z_i, w_i)\) \((i = 1, 2)\):
\[
(x_1, y_1, z_1, w_1) = (1/x, -(xy + \alpha_1)x, z, w), \quad (x_2, y_2, z_2, w_2) = (x, y, 1/z, -(zw + \alpha_2)z).
\]
Each coordinate system contains a three-parameter family of meromorphic solutions of the system (1).

Moreover, the system (1) is invariant under the following birational and symplectic transformations:
\[
S_1 : (x, y, z, w; \alpha_1, \alpha_2) \rightarrow (x + \frac{\alpha_1}{y}, y, z, w; -\alpha_1, \alpha_2),
\]
\[
S_2 : (x, y, z, w; \alpha_1, \alpha_2) \rightarrow (x, y, z + \frac{\alpha_2}{w}, w; \alpha_1, -\alpha_2).
\]

In the case of \(a = 3/4\), in 1999, M. Noumi discovered coupled Painlevé II systems with affine Weyl group symmetry of type \(C_2^{(1)}\) (see [2]). This system is explicitly
written as

\[
\begin{align*}
\frac{dx}{dt} &= 2x^2 + y/4 + 3w/4 - t, \\
\frac{dy}{dt} &= -4xy - \alpha_1, \\
\frac{dz}{dt} &= 2z^2 + w/4 + 3y/4 - t, \\
\frac{dw}{dt} &= -4zw - \alpha_2
\end{align*}
\]

with the Hamiltonian

\[H = 2x^2y + y^2/8 - ty + \alpha_1x + 2z^2w + w^2/8 - tw + \alpha_2z + 3yw/4.\]

In the case of \(\alpha = 1\), in 2005, the author found a 2-parameter family of coupled Painlevé II systems with affine Weyl group symmetry of type \(C_2^{(1)}\) (see [3]). This system is explicitly written as

\[
\begin{align*}
\frac{dx}{dt} &= -x^2 + y + w - t/2, \\
\frac{dy}{dt} &= 2xy + \alpha_1, \\
\frac{dz}{dt} &= -z^2 + y + w - t/2, \\
\frac{dw}{dt} &= 2zw + \alpha_2
\end{align*}
\]

with the Hamiltonian

\[H = -x^2y + y^2/2 - ty/2 - \alpha_1x - z^2w + w^2/2 - tw/2 - \alpha_2z + yw.\]

In this paper, we study the case of \(\alpha = -3\). This system is explicitly given by

\[
\begin{align*}
\frac{dx}{dt} &= -2x^2 + 4y - 3w - 2t, \\
\frac{dy}{dt} &= 4xy + 2\alpha_2, \\
\frac{dz}{dt} &= z^2 + 2w - 3y + t, \\
\frac{dw}{dt} &= -2zw - \alpha_3
\end{align*}
\]

with the Hamiltonian

\[H = -2x^2y + 2y^2 - 2ty - 2\alpha_2x + z^2w + w^2 + tw + \alpha_3z - 3yw.\]

This paper is organized as follows. In Section 1, we study symmetry of the system (7). In Section 2, we will study holomorphy condition of the system (7). In Section 3, we study some Hamiltonian structures of the system (7).

1. Symmetry of the system (7)

In this section, we study symmetry of the system (7).
Theorem 1.1. The system (7) is invariant under the following birational and symplectic transformations:

\begin{align}
    s_1 &: (x, y, z, w, t; \alpha_1, \alpha_2, \alpha_3) \rightarrow (x + \frac{\alpha_2}{y}, y, z, w, t; \alpha_1 + 2\alpha_2, -\alpha_2, \alpha_3), \\
    s_2 &: (x, y, z, w, t; \alpha_1, \alpha_2, \alpha_3) \rightarrow (x, y + \frac{\alpha_3}{w}, w, t; \alpha_1 + \alpha_3, \alpha_2, -\alpha_3), \\
    s_3 &: (x, y, z, w, t; \alpha_1, \alpha_2, \alpha_3) \rightarrow \\
    &\left(\frac{x(y - x^2 - w - t) - (x + z)w + \alpha_1}{y - x^2 - w - t}, \right. \\
    &\left. \frac{y - x^2 - w + \frac{(x(y - x^2 - w - t) - (x + z)w + \alpha_1)^2}{(y - x^2 - w - t)^2}}{y - x^2 - w - t}, \right. \\
    &\left. \frac{(x + z)(-1 - (x + z)w + 2\alpha_1 + 2\alpha_2)}{y - x^2 - w - t}, \right. \\
    &\left. \frac{y - x^2 - w - t - x(y - x^2 - w - t) - (x + z)w + \alpha_1}{x + z}, \right. \\
    &\left. \frac{(x + z)(-1 - (x + z)w + 2\alpha_1 + 2\alpha_2)}{y - x^2 - w - t}, t; -\alpha_1 - \alpha_3, 2\alpha_1 + \alpha_2 + \alpha_3, \alpha_3). \right)
\end{align}

We note that the parameters \(\alpha_1, \alpha_2, \alpha_3\) satisfy the following relation:

\[2\alpha_1 + 2\alpha_2 + \alpha_3 = 1.\]

Theorem 1.1 can be checked by a direct calculation.

Corollary 1.1. The transformations described in Theorem 1.1 satisfy the following relations:

\[s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^2 = 1.\]

Moreover, the transformation \(s_3 s_1\) acts on parameters \((\alpha_1, \alpha_2, \alpha_3)\) as follows:

\[s_3 s_1 : (\alpha_1, \alpha_2, \alpha_3) \rightarrow (\alpha_1 + 1, \alpha_2 - 1, \alpha_3).\]

Proposition 1.1. The system (7) has the following invariant cycles:

| codimension | invariant cycle | parameter’s relation |
|-------------|----------------|----------------------|
| 1           | \(f_1 := y\)   | \(\alpha_2 = 0\)    |
| 1           | \(f_2 := w\)   | \(\alpha_3 = 0\)    |
| 2           | \(f_3^{(1)} := y - x^2 - w - t, f_3^{(2)} := x + z\) | \(\alpha_1 = 0\) |

2. Holomorphy of the system (7)

If we look for a polynomial Hamiltonian system which admits the symmetry given in Theorem 1.1, we have to consider huge polynomials in variables \(x, y, z, w, t, \alpha_1, \alpha_2, \alpha_3\). Instead of the symmetry, we consider the holomorphy of the system (7). In the holomorphy requirement, we only need to consider polynomials in \(x, y, z, w\). This reduces the number of unknown coefficients drastically (see [4]).

Theorem 2.1. Let us consider a polynomial Hamiltonian system with Hamiltonian \(H \in \mathbb{C}(t)[x, y, z, w]\). We assume that

\[(A1)\deg(H) = 5\] with respect to \(x, y, z, w\).
(A2) This system becomes again a polynomial Hamiltonian system in each coordinate system \((x_i, y_i, z_i, w_i) (i = 1, 2, 3)\):

\[
x_1 = \frac{1}{x}, \quad y_1 = -x(xy + \alpha_2), \quad z_1 = z, \quad w_1 = w,
\]

\[
x_2 = x, \quad y_2 = y, \quad z_2 = \frac{1}{z}, \quad w_2 = -(zw + \alpha_3)z,
\]

\[
x_3 = \frac{1}{x}, \quad y_3 = -(y - x^2 - w - t)x - (x + z)w + \alpha_1)x, \quad z_3 = \frac{-w}{x}, \quad w_3 = x(x + z).
\]

Then such a system coincides with the system (7).

By solving a linear problem, Theorem 2.1 can be checked by a direct calculation.

Remark 2.1. Each coordinate system given in Theorem 2.1 contains a 3-parameter family of meromorphic solutions of (7) (see [5]).

Theorem 2.2. On each affine open set \((x_j, y_j, z_j, w_j) \in U_j \times B\) in Theorem 2.1, each Hamiltonian \(H_j\) on \(U_j \times B\) is expressed as a polynomial in \(x_j, y_j, z_j, w_j, t,\) and satisfies the following condition:

\[
dx \wedge dy + dz \wedge dw - dH \wedge dt = dx_j \wedge dy_j + dz_j \wedge dw_j - dH_j \wedge dt\quad (j = 1, 2),
\]

\[
dx \wedge dy + dz \wedge dw - d(H + x) \wedge dt = dx_3 \wedge dy_3 + dz_3 \wedge dw_3 - dH_3 \wedge dt.
\]

3. Hamiltonian structures

There is the following rational and symplectic transformations other than the transformations given in Theorems 1.1 and 2.1.

Theorem 3.1. The system (7) is equivalent to the following Hamiltonian systems:

by using the rational and symplectic transformation \(\varphi_1\)

\[
\varphi_1 : (x, y, z, w) \rightarrow (x - \frac{(x + z)w - \alpha_1}{-t - w - x^2 + y}, -t - w - x^2 + y, -w(-t - w - x^2 + y), \frac{x + z}{-t - w - x^2 + y}),
\]

the system (7) is transformed to

\[
\begin{align*}
\frac{dx}{dt} &= 2x^2 + 4y + 2t - zw^2 + \alpha_3w, \\
\frac{dy}{dt} &= -4xy + 2zw + 2\alpha_1, \\
\frac{dz}{dt} &= -2xz - 2yzw + \alpha_3y, \\
\frac{dw}{dt} &= yw^2 + 2xw + 1
\end{align*}
\]

with the Hamiltonian

\[
H = 2x^2y + 2y^2 + 2ty - yzw^2 + \alpha_3yw - 2xzw - 2\alpha_1x - z,
\]

and by using the symplectic transformation \(\varphi_2\)

\[
\varphi_2 : (x, y, z, w) \rightarrow (x, -t - w - x^2 + y, -w, x + z),
\]
the system (7) is transformed to
\[
\begin{align*}
\frac{dx}{dt} &= 2x^2 + 4y - z + 2t, \\
\frac{dy}{dt} &= -4xy - 2zw - 2\alpha_1, \\
\frac{dz}{dt} &= -2zw + 2xz + \alpha_3, \\
\frac{dw}{dt} &= w^2 - 2xw + y
\end{align*}
\]
with the Hamiltonian
\[
H = 2x^2y + 2y^2 - yz + 2ty + 2\alpha_1x + 2xz + \alpha_3w - zw^2.
\] (15)

If \(\alpha_1 = 0\), the system (14) has a particular solution \(y = w = 0\). Here \((x, z)\) satisfy
\[
\begin{align*}
\frac{dx}{dt} &= 2x^2 - z + 2t, \\
\frac{dz}{dt} &= 2xz + \alpha_3.
\end{align*}
\] (16)

Theorem 3.1 can be checked by a direct calculation.

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