Extending vector hysteresis operators

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Abstract. In some recent papers we studied how to extend to $BV$ a hysteresis operator defined on Lipschitzian inputs, preserving suitable continuity properties. More precisely we considered the so called strict metric defined by means of the essential variation. This approach may have some drawbacks from the physical point of view, therefore in the present paper we show how to extend a general hysteresis operator with respect to a notion of convergence which takes into account of the pointwise variation rather than the essential variation.

1. Introduction
In two recent papers ([1, 2]) we studied how to extend to $BV$ a hysteresis operator defined on Lipschitzian inputs on the interval $[0,T]$, preserving suitable continuity properties. More precisely we considered the so called strict metric defined by

$$d_s(u,v) := \|u - v\|_{L^1} + |V_e(u,[0,T]) - V_e(v,[0,T])|. \quad (1.1)$$

Here $V_e(u,[0,T])$ denotes the essential variation, which is defined by

$$V_e(u,[0,T]) := \inf \{V_p(w,[0,T]) : w = u \text{ a.e. in } [0,T]\},$$

where $V_p(w,[0,T])$ is the pointwise variation:

$$V_p(w,[0,T]) := \sup \sum_{j=1}^{m} |w(t_j) - w(t_{j-1})|,$$

the supremum being taken over all subdivisions $0 = t_0 < t_1 < \cdots < t_m = T$ of the interval $[0,T]$. The strict metric defined in (1.1) is natural if we consider the problem of approximating a $BV$ function by convolution: indeed if $u \in BV$ and $u_\varepsilon$ is its convolution, then $u_\varepsilon$ converges to $u$ with respect to the strict metric. Moreover two functions differing on a set of Lebesgue measure zero have the same essential variation, thus this is the natural notion of variation in the framework of Lebesgue equivalence classes of integrable functions. The essential variation is considered also in other works on hysteresis (see, e.g., [3, 4]).

However this approach may have some drawback if we consider a situation like the following one. Let us consider the hysteresis play operator which can be described by means of the following simple mechanical model: a cylinder of length $2r$ which can move along the $y$-axis when driven by a piston. The position of the piston at the time $t$ is denoted by the coordinate...
u(t), whereas \( y(t) \) denotes the position of a point of the cylinder, for instance its center. The operator \( u \mapsto y \) is the relation defining the play operator \( y := P(u) \). Assume that the initial position of the cylinder is \( y(0) = 0 \) and the movement \( u(t) \) of the piston is given by

\[
 u(t) = \begin{cases} 
 0 & \text{if } t < T/2 \\
 2r & \text{if } t = T/2 \\
 0 & \text{if } t > T/2 
\end{cases}
\]

i.e. at the time \( T/2 \) the input \( u \) moves instantaneously to the position \( 2r \) and comes back to 0. From the point of view of Lebesgue measure the input \( u \) is equivalent to the constant function \( v(t) \equiv 0 \), indeed the singleton \( T/2 \) has measure zero. But this description is non physical if we interpret the term “instantaneously” as a movement that is so fast that it cannot be measured by the available measure instruments: we expect that after the time \( T/2 \) the output of the play operator is \( y(t) = r \), whereas \( P(v) \equiv 0 \) is the output associated to \( v(t) \equiv 0 \).

This example suggests that we should consider the pointwise variation rather than the essential variation. The aim of this note is to prove an extension theorem for a general hysteresis operator which is continuous with respect the convergence of the pointwise variations, more precisely we consider the following notion of convergence:

\[
 u_n \to u \quad \text{if and only if} \quad \begin{cases} 
 u_n(t) \to u(t) & \forall t \in [0, T] \\
 V_p(u_n, [0, T]) \to V_p(u, [0, T]) 
\end{cases} 
\]

We will prove our results in the framework of vector hysteresis operators acting between curves in a Hilbert spaces. We use a procedure which is analogous to the one used in [2], but we have to modify the proofs in order to treat with the pointwise variation rather than with the essential variation. In particular we also need an approximation property of \( BV \) with respect to the convergence defined in (1.2).

Let us remark that the problem of extending a hysteresis operator to spaces of non regular functions was dealt by several authors: see, e.g., the monographs [5, 3, 6, 7], the papers [4, 8, 9, 10] and the references therein. In most of these references the extension was obtained for particular cases of hysteresis operators or for scalar hysteresis operators. Our result holds for a general vector hysteresis operators having suitable continuity properties and applies to a large variety of particular cases occurring in the literature.

2. Preliminaries

In the sequel \( \mathbb{N} \) denotes the set of strictly positive integers and \( T > 0 \). We say that a real function \( f : [0, T] \to \mathbb{R} \) is increasing if \( (f(s) - f(t))(s - t) \geq 0 \) for every \( s, t \in [0, T] \).

We recall that \( \chi_S \) denotes the characteristic function of a set \( S \): \( \chi_S(t) = 1 \) if \( t \in S \) and \( \chi_S(t) = 0 \) if \( t \notin S \). The interior of a subset \( S \subseteq \mathbb{R} \) is denoted by \( S \). Throughout the paper we assume that \( \mathcal{H} \) is a real Hilbert space endowed with the scalar product \( \langle \cdot, \cdot \rangle \) and we set \( \|u\|_\mathcal{H} := \langle u, u \rangle^{1/2}, u \in \mathcal{H} \). The Lebesgue measure on \( \mathbb{R} \) is denoted by \( \mathcal{L} \), and by \( L^1(0, T; \mathcal{H}) \) we indicate the space of Lebesgue \( \mathcal{H} \)-valued integrable function defined on \( [0, T] \) (cf., e.g., [11, Part Three]). If \( f : [0, T] \to \mathcal{H} \) the \( \|f\|_\mathcal{H} := \sup_{t \in [0, T]} \|f(t)\|_\mathcal{H} \), by Cont(\( f \)) we denote the set of continuity points of \( f \), whereas Discont(\( f \)) := \( [0, T] \setminus \text{Cont}(f) \). As usual we set \( \text{Lip}(f) := \sup_{t \neq s} \|f(t) - f(s)\|_\mathcal{H}/|t - s| \) and \( \text{Lip}([0, T]; \mathcal{H}) := \{f : [0, T] \to \mathcal{H} : \text{Lip}(f) < \infty\} \).

By the symbol \( W^{1,p}(0, T; \mathcal{H}) \) we denote the Sobolev spaces of \( \mathcal{H} \)-valued functions, \( p \in [1, \infty) \). It is well known that \( W^{1,\infty}(0, T; \mathcal{H}) \cap C([0, T]; \mathcal{H}) = \text{Lip}([0, T]; \mathcal{H}) \) (see, e.g., the appendix of [12]).

**Definition 2.1.** If \( J \) is a subinterval of \( [0, T] \), the symbol \( St(J; \mathcal{H}) \) denotes the set of \( \mathcal{H} \)-valued step maps on \( J \), i.e. maps \( f : J \to \mathcal{H} \) such that \( J \) can be partitioned into a finite number
of (possibly degenerate) intervals \(J_1, \ldots, J_m\) and \(f\) is constant on each \(J_j\) for \(j = 1, \ldots, m\). A function \(f : J \to \mathcal{H}\) is called \emph{regulated on} \(J\) if at each point \(t \in \mathbb{J}\) there exist the one sided limits \(f(t^-)\) and \(f(t^+))\) in \(\mathcal{H}\) and if there exist \(f(\inf J)\) and \(f(\sup J^-)\) in \(\mathcal{H}\). We denote by \(\text{Reg}(J; \mathcal{H})\) the set of regulated maps on \(J\).

It is well known that \(\mathcal{S}(0, T; \mathcal{H})\) is dense in \(\text{Reg}([0, T]; \mathcal{H})\) with respect to the supremum norm \(\| \cdot \|_{\infty}\). In particular every step function is Lebesgue measurable and \(\text{Discont}(f)\) is a countable set (see, e.g., next Proposition 3.1). We also remark that if \(f \in \mathcal{S}(J, \mathcal{H})\) and \(J\) is compact, then \(f\) is bounded.

**Definition 2.2.** We recall that a \emph{subdivision} of a nondegenerate subinterval \(J \subseteq [0, T]\) is a family \((s_j)_{j=0}^m, m \in \mathbb{N}\), with the property that \(s_0 < \cdots < s_m\) and \(s_j \in J\) for \(j = 0, \ldots, m\). The set of all subdivisions of \(J\) is indicated by \(\mathcal{S}(J)\). If \(f : J \to \mathcal{H}\) and \(s = (s_j)_{j=0}^m \in \mathcal{S}(J)\), the \emph{variation of \(f\) with respect to \(s\)} is defined by

\[
V(f, s) := \sum_{j=1}^{m} \| f(s_j) - f(s_{j-1}) \|_{\mathcal{H}}.
\]

If \(J\) is nondegenerate the \emph{pointwise variation of} \(u\) \emph{on} \(J\) is defined by

\[
V_p(f, J) := \sup\{ V(f, s) : s \in \mathcal{S}(J) \},
\]

otherwise we set \(V_p(f, J) = 0\). We define \(BV_p([0, T]; \mathcal{H}) := \{ f : [0, T] \to \mathcal{H} : V_p(f, [0, T]) < \infty \}\), the set of functions with pointwise bounded variation on \([0, T]\).

It is well known and clear from the definition that the pointwise variation is lower semicontinuous with respect to the pointwise convergence, i.e. \(V_p(f, J) \leq \liminf_{n \to \infty} V_p(f_n, J)\) whenever \(f_n(t) \to f(t)\) for every \(t \in J\). If \(f : [0, T] \to \mathcal{H}\), \(V_p(f, [0, T]) < \infty\), and \(t, t_0 \in [0, T]\), the inequality \(\| f(t) \|_{\mathcal{H}} \leq \| f(t_0) \|_{\mathcal{H}} + V_p(f, [0, T])\) yields the boundedness of \(f\). Moreover it is well known that \(f \in \text{Reg}([0, T]; \mathcal{H})\), therefore we can define the maps \(f_-, f_+ : [0, T] \to \mathcal{H}\) by setting

\[
f_-(t) := f(t^-), \quad f_+(t) := f(t^+), \quad t \in [0, T], \quad (2.1)
\]

with the convention that \(f(0^-) := f(0)\) and \(f(T^+) := f(T)\). It is easy to check that \(V_p(f_+, [0, T]) = V_p(f, [0, T]) - \| f(0^+) - f(0) \|_{\mathcal{H}}\) and \(V_p(f_-, [0, T]) = V_p(f, [0, T]) - \| f(T^-) - f(T) \|_{\mathcal{H}}\).

Let us observe that if \(g_1, g_2 : [0, T] \to \mathcal{H}\) are two functions in the same \(L^1\)-equivalence class and \(V_p(g_j, [0, T]) < \infty, j = 1, 2\), then every \(t \in [0, T]\) is a left Lebesgue point of \(g_j\), hence

\[
(g_1)_-(t) = \lim_{h \to 0^+} \frac{1}{h} \int_{t-h}^{t} g_1(s) \, ds = \lim_{h \to 0^+} \frac{1}{h} \int_{t-h}^{t} g_2(s) \, ds = (g_2)_-(t) \quad \forall t \in [0, T],
\]

In the same manner we see that \((g_1)_+ = (g_2)_+\) on \([0, T]\). This remark allows us to formulate the following

**Definition 2.3.** Let \(f : [0, T] \to \mathcal{H}\) be given. If there is no \(L^1\)-representative \(g\) of \(f\) such that \(V_p(g, [0, T]) < \infty\), we set \(V_e(f, [0, T]) := \infty\). Otherwise if \(g : [0, T] \to \mathcal{H}\) is such that \(f = g\) \(L^1\)-a.e. in \([0, T]\) and \(V_p(g, [0, T]) < \infty\), we set

\[
V_e(f, [0, T]) := V_p(g_-, [0, T]) + \| f(T^-) - f(T) \|_{\mathcal{H}} = V_p(g_+, [0, T]) + \| f(0^+) - f(0) \|_{\mathcal{H}}, \quad (2.2)
\]

where \(g_\pm\) is defined in (2.1). The real extended number \(V_e(f, [0, T])\) is called the \emph{essential variation of} \(f\). The space of maps of bounded essential variation on \([0, T]\) is denoted by \(BV_e([0, T]; \mathcal{H})\).
Let us recall that $W^{1,p}(0,T;\mathcal{H}) \subseteq BV_p([0,T];\mathcal{H})$ and $V_p(f, [0,T]) = \int_0^T \|f'(t)||\mathcal{H} \, dt$ for every $f \in W^{1,p}(0,T;\mathcal{H})$, where $f'$ denotes the distributional derivative of $f$ (see the appendix of [12]).

**Remark 2.1.** Observe that in the previous definition the essential variation of $f$ takes into account of the values of $f$ in 0 and $T$, therefore the points 0 and $T$ have a privileged role in comparison with the singletons of $[0,T]$. In this way our definition coincide with the one in [4]. One could also define the essential variation neglecting the values $f(0)$ and $f(T)$ and setting $V_p(f, [0,T]) := V_p(g, [0,T])$ where $g = f \in L^1$-a.e. and has pointwise bounded variation. In this way we obtain that $V_p(f, [0,T]) = \inf \{ V_p(g, [0,T]) : f = g \in L^1$-a.e. in $[0,T]\}$, as we wrote in the Introduction in order to make the presentation clearer. In [2, Section 4.4] it is shown the the first approach can be reduced to the second one, and vice versa.

Let us now recall the notion of arc length (see [13, Section 2.5.16]). If $u \in BV_p([0,T];\mathcal{H})$ we define $\ell_u : [0,T] \longrightarrow [0,T]$ by

$$\ell_u(t) := \begin{cases} \frac{T}{V_p(u,[0,T])} V_p(u,[0,t]) & \text{if } V_p(u,[0,T]) \neq 0, \\ 0 & \text{if } V_p(u,[0,T]) = 0, \end{cases} \quad t \in [0,T].$$ \hfill (2.3)

The function $\ell_u$ is increasing, $\ell_u(0) = 0$ and $\ell_u(T) = T$. Moreover $\text{Discont}(\ell_u) = \text{Discont}(u)$ and

$$[0,T] \setminus \ell_u([0,T]) = \bigcup_{t \in \text{Discont}(u)} [\ell_u(t-), \ell_u(t+)] \setminus \{\ell_u(t)\}.$$

Moreover $\ell_u$ is Lipschitz continuous if $u \in Lip([0,T];\mathcal{H})$. If $t_1 < t_2$ we have $\|u(t_1) - u(t_2)\|_H \leq V_p(u,[t_1,t_2]) - V_p(u,[0,t_2]) - V_p(u,[0,t_1])$ therefore

$$\|u(t_1) - u(t_2)\|_H \leq \frac{V_p(u,[0,T])}{T} |\ell_u(t_1) - \ell_u(t_2)| \quad \forall t_1, t_2 \in [0,T].$$ \hfill (2.4)

This inequality yields that $u(\ell_u^{-1}(\sigma))$ is a singleton for every $\sigma \in \ell_u([0,T])$, therefore there is a unique function $U : \ell_u([0,T]) \longrightarrow \mathcal{H}$ such that $U \circ \ell = u$. From (2.4) it also follows that $\text{Lip}(U) \leq V_p(u,[0,T])/T$. In order to extend $U$ to all of $[0,T]$ we define $\tilde{u} : [0,T] \longrightarrow \mathcal{H}$ by setting

$$\tilde{u}(\sigma) := (1 - \lambda)u(t-) + \lambda u(t) \quad \text{if } \sigma = (1 - \lambda)\ell_u(t-) + \lambda\ell_u(t), \quad t \in [0,T], \quad \lambda \in [0,1],$$

$$\tilde{u}(\sigma) := (1 - \lambda)u(t) + \lambda u(t+) \quad \text{if } \sigma = (1 - \lambda)\ell_u(t) + \lambda\ell_u(t+), \quad t \in [0,T], \quad \lambda \in [0,1].$$

It is clear that $\tilde{u}$ extends $U$ and that $\text{Lip}(\tilde{u}) = \text{Lip}(U)$. The function $\tilde{u}$ may be regarded as a kind of reparametrization of $u$ by the normalized arc length. We summarize the previous discussions in the following

**Proposition 2.1.** If $u \in BV_p([0,T];\mathcal{H})$ and $\ell_u : [0,T] \longrightarrow [0,T]$ is its "normalized" arc length defined by (2.3), then there exists a unique function $\tilde{u} \in Lip([0,T];\mathcal{H})$ such that

$$u = \tilde{u} \circ \ell_u \quad \forall t \in [0,T],$$ \hfill (2.5)

$\tilde{u}$ is affine on $[\ell_u(t-), \ell_u(t)]$ \quad $\forall t \in \text{Discont}(u),$ \hfill (2.6)

$\tilde{u}$ is affine on $[\ell_u(t), \ell_u(t+)]$ \quad $\forall t \in \text{Discont}(u).$ \hfill (2.7)

Moreover $\text{Lip}(\tilde{u}) \leq V_p(u,[0,T])/T$. 


3. Various kind of convergences and approximations

In this section we discuss some kind of convergences and approximations in the spaces $BV_p([0, T]; H)$ and $BV_v([0, T]; H)$.

**Definition 3.1.** Let us consider $u_n, u : [0, T] \to H$. We say that:

(i) $u_n \to u$ strictly if $\|u_n - u\|_{L^1(0,T; H)} + |V_\varepsilon(u_n, [0, T]) - V_\varepsilon(u, [0, T])| \to 0 \text{ as } n \to \infty$,

(ii) $u_n \to u$ uniformly strictly if $\|u_n - u\|_\infty + |V_p(u_n, [0, T]) - V_p(u, [0, T])| \to 0 \text{ as } n \to \infty$,

whenever these convergences makes sense.

Let us start recalling a classical argument showing how to approximate a function $u \in \text{Reg}([0, T]; H)$ by a sequence of step functions (cf. [14, Chapter 2, Section 1] and [4, Proposition 1.3]). We define iteratively a sequence of subdivisions of $[0, T]$. If $n \in \mathbb{N}$, since $u$ is regulated, for every $t \in [0, T]$ there exists an interval $J_n(t) = [a_n(t), b_n(t)]$ such that $t \in J_n(t)$, $L^1(J_n(t)) < 1/n$ and

\[
\begin{align*}
    s & \in [a_n(t), t] \cap [0, T] \implies \|f(t) - f(s)\|_H < 1/4n, \\
    s & \in [t, b_n(t)] \cap [0, T] \implies \|f(s) - f(t)\|_H < 1/4n.
\end{align*}
\]

(3.1)

(3.2)

Since $[0, T]$ is compact, we find a subdivision $s_n = (t^n_0, t^n_1, \ldots, t^n_{m_n})$ of $[0, T]$ such that

\[
[0, T] \subseteq \bigcup_{j=1}^{m_n} [a_n(t^n_j), b_n(t^n_j)].
\]

(3.3)

It is easy to see that we can assume that $a_n(t^n_{j-1}) \leq a_n(t^n_j)$ and $b_n(t^n_{j-1}) \leq b_n(t^n_j)$ and we can restrict the intervals $[a_n(t^n_j), b_n(t^n_j)]$ in such a way they are mutually disjoint (hence $b_n(t^n_{j-1}) = a_n(t^n_j)$). In the next step for $n + 1$ we can assume that the subdivision $s_{n+1}$ contains all the point of the previous one: $\{t^n_0, \ldots, t^n_{m_n}\} \subseteq \{t^{n+1}_0, \ldots, t^{n+1}_{m_{n+1}}\}$. It also follows that

\[
\text{osc}_{[0,T]}(t) := \lim sup_{r \to 0} \sup_{|s-t| \leq r} \|f(s) - f(t)\|_H < 1/n \quad \forall t \in [0, T] \setminus \{t^n_0, \ldots, t^n_{m_n}\},
\]

therefore

\[
\text{Discont}(u) \subseteq \bigcup_{n \in \mathbb{N}} \{t^n_0, t^n_1, \ldots, t^n_{m_n}\}.
\]

Now we can define for every $n \in \mathbb{N}$ the step function $v_n$ by setting for every $t \in [0, T]$

\[
v_n(t) := \begin{cases} 
    u(t^-) & \text{if } t \in [a_n(t^n_j), t^n_j), \\
    u(t_j) & \text{if } t = t^n_j, \\
    u(t^+ & \text{if } t \in (t^n_j, b_n(t^n_j)].
\end{cases}
\]

(3.4)

It is clear that $V_p(v_n, [0, T]) \leq V_p(u, [0, T])$ and $\|v_n - u\|_\infty < 1/n$. Therefore we infer the following

**Proposition 3.1.** For every $u \in \text{Reg}([0, T]; H)$ there exists a sequence $v_n \in \text{St}([0, T]; H)$ such that $v_n \to u$ uniformly and $V_p(v_n, [0, T]) \to V_p(u, [0, T])$ as $n \to \infty$.

Indeed by semicontinuity with respect of the pointwise convergence we have that $V_p(u, [0, T]) \leq \lim inf_{n \to \infty} V_p(v_n, [0, T]) \leq \lim sup_{n \to \infty} V_p(v_n, [0, T]) \leq V_p(u, [0, T]).$

Thus we have seen that $\text{St}([0, T]; H)$ is dense in $\text{Reg}([0, T]; H)$ with respect to the uniform strict convergence. In order to obtain a more regular approximation we need a modification of
the previous argument. We partition every $J_n(t)$ in the following way: we consider $\rho_n(t)$ such that $0 < \rho_n(t) < \min\{(t - a_n(t))/3, (b_n(t) - t)/3\}$ and we set
\[
A_n^1(t) := [t - \rho_n(t), t], \quad A_n^2(t) := [a_n(t), t - \rho_n(t)],
\]
\[
B_n^1(t) := [t, t + \rho_n(t)], \quad B_n^2(t) := [t + \rho_n(t), b_n(t) - \rho_n(t)], \quad B_n^3(t) := [b_n(t) - \rho_n(t), b_n(t)],
\]
hence
\[
J_n(t) = A_n^2(t) \cup A_n^1(t) \cup \{t\} \cup B_n^1(t) \cup B_n^2(t) \cup B_n^3(t).
\]
It is enough to consider the unique continuous function $u_n : [0, T] \rightarrow \mathcal{H}$ such that
\[
\begin{align*}
&u_n(t) = v_n(t) \text{ if } t \in A_n^2(t^n_j) \cup B_n^2(t^n_j) \\
&u_n(t) = v(t^n_j) \text{ if } t = t^n_j \\
&u_n \text{ is affine on } A_n^1(t^n_j) \\
&u_n \text{ is affine on } B_n^1(t^n_j) \\
&u_n \text{ is affine on } B_n^3(t^n_j) \quad (\text{only for } j \neq m_n)
\end{align*}
\]
instead if $t \in B_n^3(t^n_m)$ we define $u_n(t) := v_n(t)$. It is clear that $u_n \in \text{Lip}([0, T]; \mathcal{H})$ and we still have that $V_p(u_n, [0, T]) \leq V_p(u, [0, T])$. Of course we lose uniform convergence, but the pointwise convergence still holds, indeed we have the following

**Proposition 3.2.** If $u \in \text{Reg}([0, T]; \mathcal{H})$ then there exists a sequence $u_n \in \text{Lip}([0, T]; \mathcal{H})$ such that $u_n(t) \rightarrow u(t)$ for every $t \in [0, T]$ and $V_p(u_n, [0, T]) \rightarrow V_p(u, [0, T])$ as $n \rightarrow \infty$. In particular $u_n \rightarrow u$ in $L^1(0, T; \mathcal{H})$ as $n \rightarrow \infty$.

**Proof.** Fix $t \in [0, T]$. We first prove that $u_n(t) \rightarrow u(t)$. If $t = t^n_j$ for some $n$ and some $j$, then by construction $u_n(t)$ is definitely equal to $u(t)$ and we are done. Thus let us consider the case when $t \neq t^n_j$ for every $n$ and every $j$. In particular $t \in \text{Cont}(u)$. If for every $n$ there exists $j_n$ such that $t \in A_n^2(t^n_{j_n}) \cup B_n^2(t^n_{j_n})$ for every $n$ then $\|u_n(t) - u(t)\|_{\mathcal{H}} < 1/n$ for every $n$ and we are done. If instead for every $n$ there exists $j_n \in \{1, \ldots, m_n-1\}$ such that $t \in B_n^3(t^n_{j_n})$, then, as $u_n(t)$ belongs to the line segment $[u(t^n_{j_n}+), u(t^n_{j_n}+1)]$ and $b_n(t^n_j) = a_n(t^n_{j_n+1})$,
\[
\|u_n(t) - v_n(t)\|_{\mathcal{H}} = \|u_n(t) - u(t^n_{j_n}+\|_{\mathcal{H}} \leq \|u(t^n_{j_n}+\) - u(t^n_{j_n}+1\)\|_{\mathcal{H}} \\
\leq \|u(t^n_{j_n}+\) - u(b(t^n_{j_n}))\|_{\mathcal{H}} + \|u(a(t^n_{j_n+1})) - u(t^n_{j_n+1}\)\|_{\mathcal{H}} < 1/2n,
\]
thus also in this case we obtain the desired convergence.

Now let us consider the case when there is a sequence $j_n \in \{1, \ldots, m_n\}$ such that $t \in A_n^1(t^n_{j_n}) \cup B_n^1(t^n_{j_n})$. We can assume that $t^n_{j_n} \rightarrow \tilde{t} \in [0, T]$, at least for a subsequence which we do not relabel. Therefore, as $\rho_n(s) \rightarrow 0$ for every $s$, we infer that $t = \tilde{t}$, so that $\tilde{t} \neq t^n_j$ for every $n$ and every $j$ and $t \in \text{Cont}(u)$. It follows that $u(t^n_{j_n}+1) - u(t^n_{j_n})$ and $u(t^n_{j_n}+1) - u(t^n_{j_n})$ as $n \rightarrow \infty$. Since $u_n(t)$ belongs to the line segment $[u(t^n_{j_n}+), u(t^n_{j_n}+1)]$ for every $n$, we deduce that $u_n(t) \rightarrow u(t)$. All the other cases can be reduced to the previous ones by considering subsequences.

The convergence of the variation is proved as above. Concerning the $L^1$-convergence, we observe that $\|u_n\|_{\infty} \leq \|u_n(0)\|_{\mathcal{H}} + V_p(u_n, [0, T])$, therefore the sequence $\|u_n\|_{\infty}$ is bounded and the dominated convergence theorem applies. \(\square\)

In the papers [1, 2] we considered the problem of extending a hysteresis operators in a continuous way with respect to the strict convergence. In order to consider the pointwise variation one could modify the notion of strict convergence by declaring that $u_n \rightarrow u$ if and
only if \( \|u_n - u\|_{L^1([0,T];\mathcal{H})} \to 0 \) and \( V_p(u_n,\mathcal{H}) \to V_p(u,\mathcal{H}) \) as \( n \to \infty \). This definition has a drawback: the limits are not unique, for instance if \( u_n \) is defined by \( u_n := \chi(T/2) \) if \( n \) is odd, \( u_n := -\chi(T/2) \) if \( n \) is even, then \( u_n \) is not pointwise convergent, but has several limits in the previous sense, for instance any function of the kind \( \chi_{t_0} \) with \( t_0 \in [0,T] \).

Since we consider the pointwise variation, we do distinguish functions which differ on a set of Lebesgue measure zero. Therefore a suitable convergence we may consider is the one we introduce in the following

**Definition 3.2.** We say that \( u_n : [0,T] \to \mathcal{H} \) is pointwise strictly convergent to \( u \) if \( V_p(u_n,\mathcal{H}) \to V_p(u,\mathcal{H}) \) and \( u_n(t) \to u(t) \) for every \( t \in [0,T] \) as \( n \to \infty \).

Observe that this convergence is not induced by a metric, but the limits are unique. The previous discussion shows that \( \text{Lip}([0,T];\mathcal{H}) \) is dense in \( BV_p([0,T];\mathcal{H}) \) with respect to this convergence. Let us conclude this section with the following

**Lemma 3.1.** Assume that \( u, u_n \in BV_p([0,T];\mathcal{H}) \cap C([0,T];\mathcal{H}) \) for every \( n \) and \( u_n \to u \) pointwise strictly on \( [0,T] \). Then \( u_n \to u \) uniformly on \( [0,T] \).

**Proof.** First of all let us observe that \( \|u_n\|_{\infty} \leq \|u_0\|_{\infty} + V_p(u_n,\mathcal{H}) \), therefore the sequence \( \|u_n\|_{\infty} \) is bounded and by the dominated convergence theorem we find that \( u_n \to u \) in \( L^1([0,T];\mathcal{H}) \). Since \( u \) and \( u_n \) are continuous, the pointwise and essential variations coincide, therefore we have that \( u_n \to u \) strictly on \( [0,T] \). Thus we can apply [2, Corollary 4.2] and infer the uniform convergence. \( \square \)

**4. Extension of hysteresis operators**

In this section we show how to extend a hysteresis operator \( R : \text{Lip}([0,T];\mathcal{H}) \to BV_p([0,T];\mathcal{H}) \cap C([0,T];\mathcal{H}) \) which is continuous with respect to the pointwise strict convergence.

**Lemma 4.1.** Assume that \( u, u_n \in BV_p([0,T];\mathcal{H}) \) and let \( \ell_u \) and \( \ell_{u_n} \) be their renormalized arc lengths defined by Proposition 2.1. If \( u_n \to u \) pointwise strictly on \( [0,T] \) then \( \ell_{u_n}(t) \to \ell_u(t) \) for every \( t \in [0,T] \).

**Proof.** Fix \( t \in [0,T] \). By the lower semicontinuity we have that \( V_p(u,c,d) \leq \liminf_{n \to \infty} V_p(u_n,c,d) \) for every \( c,d \in [0,T] \), \( c \leq d \). On the other hand, exploiting again the lower semicontinuity and the convergence \( V_p(u_n,\mathcal{H}) \to V_p(u,\mathcal{H}) \), we infer that

\[
\limsup_{n \to \infty} V_p(u_n,\mathcal{H}) = \limsup_{n \to \infty} (V_p(u_n,\mathcal{H}) - V_p(u_n,t,\mathcal{H})) \\
\leq V_p(u,\mathcal{H}) - \liminf_{n \to \infty} V_p(u_n,t,\mathcal{H}) \\
\leq V_p(u,\mathcal{H}) - V_p(u_n,t,\mathcal{H}) = V_p(u,\mathcal{H}),
\]

thus we have proved that \( V_p(u_n,\mathcal{H}) \to V_p(u,\mathcal{H}) \) as \( n \to \infty \). Therefore \( \lim_{n \to \infty} \ell_{u_n}(t) = \ell_u(t) \). \( \square \)

Now we need to prove a generalization of [2, Lemma 4.1].

**Lemma 4.2.** Let \( v \in C([0,T];\mathcal{H}) \) be such that \( V_p(v,\mathcal{H}) < \infty \) and let \( \beta : [0,T] \to [0,T] \) be an increasing function satisfying \( \beta(0) = 0 \) and \( \beta(T) = T \). Moreover assume that

\[
V_p(v,\mathcal{H}) = \|v(\beta)\|_{\mathcal{H}} \quad \forall \beta \in \text{Discont}(\beta), \quad (4.1)
\]

\[
V_p(v,\mathcal{H}) = \|v(\beta(t))\|_{\mathcal{H}} \quad \forall t \in \text{Discont}(\beta). \quad (4.2)
\]

Then \( V_p(v \circ \beta,\mathcal{H}) = V_p(v,\mathcal{H}) \).
Proof. We prove the lemma when $\beta(0) = \beta(0^+)$ and $\beta(T^-) = \beta(T)$, the other cases needing very slight modifications. The inequality $V_p(v \circ \beta, [0, T]) \leq V_p(v, [0, T])$ is obvious, hence $V_p(v, [0, T])$ is an upper bound for $\{\sum_{j=1}^{n} \|v(\beta(t_j)) - v(\beta(t_{j-1}))\|_H : n \in \mathbb{N}, 0 = t_0 \leq \cdots \leq t_n = T\}$. Let $\varepsilon > 0$ be arbitrarily fixed. There exists a subdivision $(t_j)_{j=0}^{n}$ of $[0, T]$ such that

$$V_p(v, [0, T]) < \sum_{j=1}^{n} \|v(t_j) - v(t_{j-1})\|_H + \varepsilon/2. \quad (4.3)$$

For every $\sigma \in \text{Discont}(\beta)$ there is a possibly empty subset $E_\sigma \subseteq \{t_j\}$ contained in $[\beta(\sigma^-), \beta(\sigma^+)]$. Adding the points $\beta(\sigma^-), \beta(\sigma), \beta(\sigma^+)$ to $E_\sigma$, the sum in (4.3) can only increase (of course it can happen that $\beta(\sigma)$ is one of the points $\beta(\sigma^-), \beta(\sigma^+)$). Moreover, thanks to the assumptions (4.1)-(4.2) we can also replace $E_\sigma$ by $\{\beta(\sigma^-), \beta(\sigma), \beta(\sigma^+)\}$ without affecting such a sum. Therefore we can assume that (4.3) holds for a subdivision $(t_j)$ such that

$$(t_j)_{j=0}^{n} = \{s_0^1, \ldots, s_{k_1}^1\} \cup \{\beta(\sigma_1^-), \beta(\sigma_1), \beta(\sigma_1^+)\}$$

$$\cup \{s_0^2, \ldots, s_{k_2}^2\} \cup \{\beta(\sigma_2^-), \beta(\sigma_2), \beta(\sigma_2^+)\} \cup \cdots$$

$$\cup \{s_0^m, \ldots, s_{k_m}^m\} \cup \{\beta(\sigma_m^-), \beta(\sigma_m), \beta(\sigma_m^+)\} \cup \{s_{k_m+1}^m, \ldots, s_{k_{m+1}}^{m+1}\}$$

where $T = s_{k_{m+1}}^{m+1}$ and

$$\sigma_i \in \text{Discont}(\beta) \quad \forall i = 1, \ldots, m;$$

$$\{s_0^i, \ldots, s_{k_i}^i\} \subseteq [\beta([0, T]) \quad \forall i = 1, \ldots, m + 1,$$

$$s_{h-1}^i < s_h^i < \beta(\sigma_i) < s_{h+1}^i < s_{h+1}^i \quad \forall i = 1, \ldots, m, \forall h = 0, \ldots, k_i.$$

Hence, selecting

$$\tau_h^i \in \beta^{-1}(s_h^i) \quad i = 1, \ldots, m + 1, \quad h = 0, \ldots, k_i,$$

we can write

$$\sum_{j=1}^{n} \|v(t_j) - v(t_{j-1})\|_H = \sum_{i=1}^{m} \left( \sum_{h=1}^{k_i} \|v(\beta(\tau_h^i)) - v(\beta(\tau_{h-1}^i))\|_H ight.$$  

$$+ \|v(\beta(\sigma_i^-)) - v(\beta(\tau_{k_i}^i))\|_H + \|v(\beta(\sigma_i)) - v(\beta(\tau_{h}^i))\|_H$$

$$+ \|v(\beta(\sigma_i^+)) - v(\beta(\sigma_i))\|_H) \right)$$

$$+ \sum_{h=1}^{k_{m+1}} \|v(\beta(\tau_{h}^{m+1})) - v(\beta(\tau_{h-1}^{m+1}))\|_H.$$

The continuity of $v$ yields that for every $i = 1, \ldots, m$ there exist $\tilde{\sigma}_i, \tilde{\sigma}_{i+1}$ very near $\sigma_i$ such that

$$\tilde{\sigma}_i < \sigma_i < \tilde{\sigma}_{i+1}, \quad \|v(\beta(\tilde{\sigma}_i)) - v(\beta(\sigma_i^-))\|_H < \varepsilon/(4m), \quad \text{and} \quad \|v(\beta(\sigma_i^+)) - v(\beta(\tilde{\sigma}_{i+1}))\|_H < \varepsilon/(4m),$$

so that

$$\sum_{j=1}^{n} \|v(t_j) - v(t_{j-1})\|_H \leq \sum_{i=1}^{m} \left( \sum_{h=1}^{k_i} \|v(\beta(\tau_h^i)) - v(\beta(\tau_{h-1}^i))\|_H ight.$$  

$$+ \|v(\beta(\tilde{\sigma}_i)) - v(\beta(\tau_{k_i}^i))\|_H + \|v(\beta(\sigma_i)) - v(\beta(\tilde{\sigma}_{i+1}))\|_H$$

$$+ \|v(\beta(\sigma_i^+)) - v(\beta(\sigma_i))\|_H + \|v(\beta(\tau_{h}^{i+1})) - v(\beta(\sigma_i^+))\|_H + \varepsilon/m)$$

$$+ \sum_{h=1}^{k_{m+1}} \|v(\beta(\tau_{h}^{m+1})) - v(\beta(\tau_{h-1}^{m+1}))\|_H.$$
That is, we have found a subdivision \((\theta_j)_{j=0}^n\) such that \(V_p(v, [0, T]) < \sum_{j=1}^n \|v(\beta(\theta_j)) - v(\beta(\theta_{j-1}))\|_H + \varepsilon\), and the lemma is proved.

**Corollary 4.1.** If \(u \in BV_p([0, T]; H)\) and \(\tilde{u}\) is the reparametrization of \(u\) defined in Proposition 2.1, then \(V_p(u, [0, T]) = V_p(\tilde{u}, [0, T])\).

Let us recall the following lemma (see [2, Lemma 4.7]).

**Lemma 4.3.** Let \(c, d \in \mathbb{R}\) be such that \(c < d\) and assume \(x, y \in H\). Then the affine map \(w : [c, d] \rightarrow H\) defined by \(w(t) := x + t(y - x)/(d - c)\) is the only minimizer of the functional \(v \mapsto \|v'\|^2_{L^2([c, d]; H)}\) in the set \(\{v \in Lip([c, d]; H) : v(c) = x, v(d) = y\}\).

Now we can prove the following proposition connecting the convergences of \(u_n\) and \(\tilde{u}_n\). The previous preparatory lemmas enable us to exploit the argument of [2, Proposition 4.10], but in the present situation the convergence of \(\ell_{u_n}\) holds everywhere in \([0, T]\), rather than almost everywhere. In the proof we will use the notions of weak and weak star convergence, denoted respectively by \(\rightharpoonup\) and \(\rightharpoonup^*\) (see, e.g., [15, Chapter 3]).

**Proposition 4.1.** Assume that \(u, u_n \in BV_p([0, T]; H)\) for every \(n \in \mathbb{N}\) and \(u_n \rightarrow u\) pointwise strictly on \([0, T]\). Let \(\tilde{u}\) and \(\tilde{u}_n\) be their reparametrizations defined defined by Proposition 2.1. Then

\[
\tilde{u}_n \rightarrow \tilde{u} \quad \text{in } W^{1,p}(0, T; H) \quad \forall p \in [1, +\infty].
\]

In particular \(\tilde{u}_n \rightarrow \tilde{u}\) pointwise strictly on \([0, T]\).

**Proof.** Now observe that \(\tilde{u}(0) = u(0)\) and \(\tilde{u}_n(0) = u_n(0)\), therefore

\[
\tilde{u}_n(0) \rightarrow \tilde{u}(0) \quad \text{in } H
\]

as \(n \rightarrow \infty\), since \(u_n\) is pointwise convergent to \(u\). We also have

\[
\|\tilde{u}_n(\sigma)\|_H \leq \|\tilde{u}_n(0)\|_H + \int_0^T \|\tilde{u}_n'(\tau)\|_H d\tau
\]

\[
= \|\tilde{u}_n(0)\|_H + V_p(\tilde{u}_n, [0, T])
\]

\[
= \|\tilde{u}_n(0)\|_H + V_p(u_n, [0, T]) \quad \forall \sigma \in [0, T]
\]

and, by Proposition 2.1 and Corollary 4.1,

\[
\|\tilde{u}_n\|_{L^\infty([0, T]; H)} \leq \frac{V_p(\tilde{u}_n, [0, T])}{T} = \frac{V_p(u_n, [0, T])}{T}.
\]

Hence (4.5)-(4.7) imply that \((\tilde{u}_n)\) is bounded in \(W^{1,p}(0, T; H)\) for every \(p \in [1, \infty]\). Hence there exists \(\tilde{u} \in Lip([0, T]; H)\) such that, at least for a subsequence which we do not relabel,

\[
\tilde{u}_n \rightharpoonup^* \tilde{u} \quad \text{in } W^{1,p}(0, T; H) \quad \forall p \in [1, \infty].
\]

This convergence, together with (4.5) implies that \(\tilde{u}_n(0) \rightarrow \tilde{u}(0) = \tilde{u}(0)\) in \(H\), from which we infer that

\[
\tilde{u}_n(\sigma) \rightarrow \tilde{u}(\sigma) \quad \forall \sigma \in [0, T],
\]

indeed for every \(x \in H\)

\[
\langle \tilde{u}_n(\sigma) - \tilde{u}(\sigma), x \rangle = \langle \tilde{u}_n(0) - \tilde{u}(0), x \rangle + \int_0^\sigma \langle \tilde{u}_n'(\tau) - \tilde{u}'(\tau), x \rangle d\tau \rightarrow 0.
\]
Now we prove that \( \tilde{u}_n \rightarrow \tilde{u} \) in \( W^{1,p}(0,T;\mathcal{H}) \). For every \( n \in \mathbb{N} \) we have that, using also the Hölder inequality, \[
abla \tilde{u}_n' \in L^p(0,T;\mathcal{H}) \] for every \( p \in [1, +\infty] \). For every \( n \in \mathbb{N} \) we have that \[
abla \tilde{u}_n' \rightarrow \nabla \tilde{u}' \] in \( L^p(0,T;\mathcal{H}) \) as \( n \rightarrow \infty \), hence, as \( L^p(0,T;\mathcal{H}) \) is uniformly convex for \( p \in [1, +\infty] \), thanks to [15, Proposition III.30] we have that (4.12)–(4.13) imply that \[
abla \tilde{u}_n' \rightarrow \nabla \tilde{u}' \] in \( L^p(0,T;\mathcal{H}) \) for every \( p \in [1, +\infty] \), and we are done.
Now we are in position to extend a general rate independent operator \( R : \text{Lip}([0, T]; \mathcal{H}) \rightarrow BV_p([0, T]; \mathcal{H}) \cap C([0, T]; \mathcal{H}) \), i.e. an operator such that
\[
R(u \circ \phi) = R(u) \circ \phi \quad \forall u \in \text{Lip}([0, T]; \mathcal{H})
\]
whenever \( \phi : [0, T] \rightarrow [0, T] \) is a Lipschitz continuous increasing map such that \( \phi([0, T]) = [0, T] \). If \( u \in BV_p([0, T]; \mathcal{H}) \) then by Proposition 3.2 there exists a sequence \( u_n \in \text{Lip}([0, T]; \mathcal{H}) \) such that \( \lim_{n \to \infty} V_p(u_n, [0, T]) = V_p(u, [0, T]) \) and \( u_n \) is pointwise convergent to \( u \). Thanks to the rate independence we find that
\[
R(u_n) = R(\tilde{u}_n \circ \ell_{u_n}) = R(\tilde{u}_n) \circ \ell_{u_n} \quad \forall n \in \mathbb{N}.
\]

If we assume that \( R \) is continuous with respect to the pointwise strict convergence, then \( R(\tilde{u}_n) \rightarrow R(\tilde{u}) \) pointwise strictly because by Proposition 4.1 we know that \( \tilde{u}_n \rightarrow \tilde{u} \) pointwise strictly on \([0, T]\). Moreover \( R(\tilde{u}) \) and \( R(\tilde{u}_n) \) are continuous functions, therefore by Lemma 3.1 it follows that the convergence of \( R(\tilde{u}_n) \) is uniform. On the other hand, thanks to Lemma 4.1 \( \ell_{u_n} \) is pointwise convergent to \( \ell_u \), therefore
\[
\lim_{n \to \infty} R(u_n)(t) = (R(\tilde{u}) \circ \ell_u)(t) \quad \forall t \in [0, T].
\]
The continuity of \( R \) we infer that
\[
\lim_{n \to \infty} V_p(R(u_n), [0, T]) = V_p(R(\tilde{u}), [0, T]).
\]

However the pointwise variation of \( R(\tilde{u}) \) is not the same of \( R(\tilde{u}) \circ \ell_u \), because \( \ell_u \) is not continuous. A sufficient condition for the variations to be the same is given by Lemma 4.2: if
\[
\begin{align*}
V_p(R(\tilde{u}), [\ell_u(t-), \ell_u(t)]) &= \|R(\tilde{u})(\ell_u(t-)) - R(\tilde{u})(\ell_u(t))\|_\mathcal{H} \quad \forall t \in \text{Discont}(\ell_u), \\
V_p(R(\tilde{u}), [\ell_u(t+), \ell_u(t)]) &= \|R(\tilde{u})(\ell_u(t+)) - R(\tilde{u})(\ell_u(t))\|_\mathcal{H} \quad \forall t \in \text{Discont}(\ell_u),
\end{align*}
\]
then \( V_p(R(\tilde{u}), [0, T]) = V_p(R(\tilde{u}) \circ \ell_u, [0, T]) \). Indeed these conditions holds true by virtue of (2.6)–(2.7). Summarizing we have proved the following

**Theorem 4.1.** Assume that \( R : \text{Lip}([0, T]; \mathcal{H}) \rightarrow BV_p([0, T]; \mathcal{H}) \cap C([0, T]; \mathcal{H}) \) is a rate independent operators which is continuous with respect to the pointwise strict convergence. Define \( \bar{R} : BV_p([0, T]; \mathcal{H}) \rightarrow BV_p([0, T]; \mathcal{H}) \) by
\[
\bar{R}(u) := R(\tilde{u}) \circ \ell_u, \quad u \in BV_p([0, T]; \mathcal{H}),
\]
where \( \tilde{u} \) is the reparametrization defined by Proposition 2.1. Then \( \bar{R} \) is the unique continuous extension of \( R \) in the following sense: if \( u_n \rightarrow u \) pointwise strictly on \([0, T]\) then \( \bar{R}(u_n)(t) \rightarrow \bar{R}(u)(t) \) for every \( t \in [0, T] \). Moreover if we assume that
\[
V_p(v, [c, d]) = ||v(d) - v(c)||_\mathcal{H} \quad \implies V_p(R(v), [c, d]) = ||R(v)(d) - R(v)(c)||_\mathcal{H}
\]
whenever \( 0 \leq c < d \leq T \), then \( \bar{R} \) is continuous with respect to pointwise strict convergence.
In [2] we called \textit{locally isotone} an operator satisfying (4.21). The argument of [2, Proposition 4.2] shows that condition (4.21) is also necessary in order to obtain that $\mathcal{R}$ is continuous with respect to pointwise strict convergence. If $\mathcal{H} = \mathbb{R}$ this condition is a slight generalization of the well known local monotonicity.

In the scalar case $\mathcal{H} = \mathbb{R}$ the hypotheses of Theorem 4.1 are satisfied by most of the concrete hysteresis operators occurring in applications (see, e.g., [1, Section 5] and [16, Section 5]). In the multidimensional case the condition (4.21) is very restrictive: for instance the vector play operator satisfies it if and only if its characteristic is a vector subspace or is the intersection of two parallel half spaces (cf. [2]). However the vector play operator is continuous with respect to the pointwise strict convergence on $\text{Lip}([0,T];\mathcal{H})$ ([2, Theorem 3.3]), therefore it can be extended in a weaker sense to $BV_p$ by means of formula (4.20).

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