Abstract—In this work, we address optimization problems where the objective function is a nonlinear function of an expected value, i.e., compositional stochastic strongly convex programs. We consider the case where the decision variable is not vector-valued but instead belongs to a reproducing Kernel Hilbert Space (RKHS), motivated by risk-aware formulations of supervised learning and Markov Decision Processes defined over continuous spaces. We develop the first memory-efficient stochastic algorithm for this setting, which we call Compositional Online Learning with Kernels (COLK). COLK, at its core a two time-scale stochastic approximation method, addresses the fact that (i) compositions of expected value problems cannot be addressed by classical stochastic gradient due to the presence of the inner expectation; and (ii) the RKHS-induced parameterization has complexity which is proportional to the iteration index which is mitigated through greedily constructed subspace projections. We establish almost sure convergence of COLK with attenuating step-sizes, and linear convergence in mean to a neighborhood with constant step-sizes, as well as the fact that its complexity is at-worst finite. The experiments with robust formulations of supervised learning demonstrate that COLK reliably converges, attains consistent performance across training runs, and thus overcomes overfitting.

I. INTRODUCTION

In this work, we focus on compositional stochastic programming, a setting where the objective function is an expectation over a set of random convex functions, each of which depends on the expected value of a different random convex function. This problem setting has received recent attention in operations research [2], [3] and machine learning [4] due to its ability to gracefully address technicalities that arise in the theory of Markov Decision Problems (MDPs) [5] and bias-variance issues in supervised learning [6]. Our goal is to solve this class of problems when the decision variable is not vector-valued, as in [2], but is instead itself a function. This setting arises intrinsically when addressing MDPs defined over the continuous state and action spaces [7] or when accounting for risk [8] in supervised learning with nonlinear interpolators [9].

The theory of optimization in function space began with variational calculus [10] and Hamilton’s Principle [11]. However, in modern applications, we require solutions to such problems in situations where classical methods no longer apply. Two different issues arise: (1) how to evaluate the expectation (integral) and (2) how to parameterize the function so that tractable updates may be obtained. Setting aside (1) for now, to address (2), i.e., to handle the intractability of general functional optimization, one must restrict the function we seek to not only yield a computationally tractable formulation, but also one be rich enough to address common experimental settings. In learning theory, for instance, we typically restrict the function to be a neural network [12] or a nonparametric basis expansion in terms of data [13], whereas in control systems, polynomial interpolation [14] and kriging [15] are popular. In this work, we address the case where the function class is a Reproducing Kernel Hilbert Space (RKHS), motivated by a recently developed memory-efficient parameterization of a function that is infinite dimensional [16]. This approach subsumes polynomial interpolation [14], avoids the memory explosion associated with large sample-size kriging [15], and preserves convexity, thus avoiding convergence to poor stationary points rampant in neural network training [17].

With the function class specified, we turn to discuss how to solve the associated functional stochastic program: doing so requires iterative stochastic methods [18], [19], since deterministic approaches [20] require computing gradients that depend on infinitely many realizations of a random variable, thus exhibiting prohibitive complexity. Unfortunately, standard stochastic gradient descent (SGD) is inappropriate to the compositional setting, because, for a single stochastic descent direction, one requires the evaluation of an additional inner expectation, an observation that was popularized in reinforcement learning as the “double sampling problem” [5].

To ameliorate this issue, we develop a functional nonparametric extension of stochastic quasi-gradient (SQG) method [2], [21], which uses two time-scale stochastic approximation: one uses a quasi-stationary estimate of the inner expectation, whereas the other executes stochastic descent [22]. However, the choice of $H$ as an RKHS, and sequential application of the Representer Theorem [23], means the function parameterization grows with the iteration index [9], and thus becomes untenable for expected value problems. In short, there exists no affordable memory method to solve compositional stochastic programs over an RKHS. Thus, our main contributions are to:

- extend SQG to RKHS, whose parameterization we compress with matching pursuit [24] (Sec. III). We tailor the compression to the step-size to ensure valid descent [16], [25]. We call this method Compositional Online Learning with Kernels (COLK).
- establish that COLK converges almost surely to the optimal function with decreasing learning rates and compression budget (Theorem 1).
- guarantee that the algorithm converges to a neighborhood whose radius depends on step-sizes and problem constants when used with constant learning rates and compression budget (Theorem 3), which occurs at a linear rate (Theorem 2). Further, the worst-case complexity of the function sequence is finite (Theorem 4).
- experimentally (Sec. V) validate this method on a problem instantiation defined by robust supervised learning. Doing so yields nonlinear statistical models whose bias and variance is

A.S. Bedi and A. Koppel contributed equally to this work. They both are with the U.S. Army Research Laboratory, Adelphi, MD, USA. (e-mail: amrit0714@gmail.com, akoppel@seas.upenn.edu). K. Rajawat is with the Department of Electrical Engineering, Indian Institute of Technology Kanpur, Kanpur 208016, India (e-mail: ketan@iitk.ac.in). A part of this work is presented in American Control Conference (ACC), Philadelphia, USA, 2019 [1].
small, first on a synthetic data regression outliers which has a heavier tailed distribution, i.e., more outliers are present, and then on benchmark data: lidar [26]. We observe that COLK yields consistently accurate performance across training realizations, meaning that it does not overfit, in contrast to other methods that cannot minimize risk functionals [6].

II. COMPOSITIONAL STOCHASTIC PROGRAMMING IN RKHS

In this work, we focus on solving functional optimization problems whose objective is a nonlinear function of an expected value. More broadly, the objective function is a composition of two functions, each of which is an expected value over a set of functions parameterized by a pair of random variables. More specifically, there are two sets of random variables \( \{ \xi_i \} \subset \mathbb{R}^p \) and \( \{ \theta_i \} \subset \mathbb{R}^p \). In general both the random variables are allowed to be dependent, but for the ease of analysis and understanding, we assume that \( \xi, \theta \in \mathbb{R}^p \) and \( \xi, \theta \) are independent of each other. Considering these random pairs, the cost takes the form \( J(f) := \langle L(f), \theta \rangle \), where \( \mathbb{E}_\theta [K_\xi(f(\xi))] \) is a map \( H : \mathbb{R}^m \to \mathbb{R}^m \) that is an expectation over a set of random functions \( K_\xi(f(\xi)) \) of \( \mathbb{R}^m \). Similarly, \( L(u) = \mathbb{E}_\theta [\phi_\theta(u)] \) is a map \( L : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R} \) that is an expected value over a random collection variable. Further, \( H \) is a function space to be subsequently specified. In this work, we focus on the functional compositional stochastic program:

\[
\min_{f \in H} \mathbb{E}_\theta \left[ \phi_\theta \left( \mathbb{E}_\xi \left[ K_\xi(f(\xi)) \right] \right) \right] + \frac{1}{2} \| f \|^2_H , \tag{1}
\]

where we assume that \( J(f) \) is convex with respect to function \( f \) and add a Tikhonov regularizer \( \frac{1}{2} \| f \|^2_H \) to ensure strong convexity [27], defining the regularized loss

\[
f^* = \arg\min_{f \in H} R(f) := J(f) + \frac{1}{2} \| f \|^2_H . \tag{2}
\]

The feasible set \( H \) of \( \theta \), the domain of \( \mathcal{H} \), and hence \( J \), is not Euclidean space \( \mathbb{R}^p \), as in [2], but instead is a Hilbert space equipped with a unique distance-like kernel function, \( \kappa : \mathcal{U} \times \mathcal{U} \to \mathbb{R} \), such that:

\[
\begin{align*}
(i) & \quad \langle f, \kappa(\cdot, \cdot) \rangle_H = f(\cdot) \quad \text{for all } u \in \mathcal{U}, \\
(ii) & \quad H = \text{span} \{ \kappa(\cdot, \cdot) \} \quad \text{for all } u \in \mathcal{U}.
\end{align*} \tag{3}
\]

where \( \langle \cdot, \cdot \rangle_H \) denotes the Hilbert inner product for \( H \) and \( \mathcal{U} := \Xi \cup \Theta \) denotes the union of data domains \( \Xi \) and \( \Theta \), whose elements \( \xi \) and \( \theta \) are random variables \( \xi \) and \( \theta \). We further assume that the kernel is positive semi-definite, i.e., \( \kappa(u, u') \geq 0 \) for all \( u, u' \in \mathcal{U} \) so that it is a Mercer kernel. Function spaces with this structure are called reproducing kernel Hilbert spaces (RKHS) [28].

In [3], property (i) is called the reproducing property of the kernel and comes from the Riesz Representation Theorem [29]. Replacing \( f \) by \( \phi(\cdot) \) in (i) yields the expression \( \langle \phi(\cdot), \kappa(\cdot, \cdot) \rangle_H = \phi(\cdot) \), which is why \( \kappa \) is called “reproducing.” This property provides a practical means by which to access a nonlinear transformation of the input space \( \mathcal{U} \). Specifically, denote by \( \phi(\cdot) \) a nonlinear map of the feature space that assigns to each \( u \) to the kernel function \( \kappa(\cdot, u) \). Then the reproducing property of the kernel allows us to write the inner product of the image of distinct feature vectors \( u \) and \( u' \) under the map \( \phi \) in terms of kernel evaluations only: \( \langle \phi(u), \phi(u') \rangle_H = \kappa(u, u') \). This is commonly referred to as the kernel trick, and it provides a principled method for function estimation.

Moreover, property (ii) states that any function \( f \in \mathcal{H} \) may be written as a linear combination of kernel evaluations. For kernelized and regularized empirical risk minimization (i.e., the sample average approximation of \( \theta \) for some fixed \( N \) realizations of \( \xi \) and \( \theta \)), the Representer Theorem [23, 28] establishes that the optimal \( f \) in function class \( \mathcal{H} \) may be written as an expansion of kernel evaluations only at elements of the training set as

\[
f(u) = \sum_{n=1}^{N} w_n \kappa(\xi_n, u) . \tag{4}
\]

where \( w = [w_1, \ldots, w_N]^T \in \mathbb{R}^N \) denotes the weight vector. The upper summand index \( N \) in (4) is henceforth referred to as the model order. Common choices \( \kappa \) include the polynomial kernel and the radial basis kernel, i.e., \( \kappa(u, u') = (u' u + b)^\xi \) and \( \kappa(u, u') = \exp \left\{ -\frac{\| u - u' \|^2}{2\sigma^2} \right\} \), respectively, where \( u, u' \in \mathcal{U} \).

Then, one may use the Representer Theorem [4] to transform the sample average approximation of (1) over all of \( H \) into the parametric problem of two \( N \)-dimensional weight vector \( w \). However, as \( N \to \infty \), the function representation becomes infinite as well. In this work, we seek to find functions that are close-to-optimal solutions to (1) but also admit a finite-memory representation. Before turning to develop an algorithmic tool which does so, we note that the problem setting (1) arises in diverse applications. Here we mention two, the first of which is the focus of this work.

**Example 1 (Robust Supervised Learning)** Consider a random pair \( (x, y) \in \mathcal{X} \times \mathcal{Y} \), realizations of which are training examples \((x_n, y_n)\), and \( \mathcal{X} \subset \mathbb{R}^p \), the \( p \)-dimensional Euclidean space. In comparison to formulation in (1), we have \( \theta = \xi = x \) and \( \theta = y = y \) for this example which represents the corresponding target values. In the case of classification with \( C \) classes, \( \mathcal{Y} = \{1, \ldots, C\} \), whereas in the case of regression \( \mathcal{Y} \subset \mathbb{R}^d \). In supervised learning, we learn an estimator \( f(x) \) according to its ability to minimize a loss function \( I : H \times \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \) averaged over data:

\[
f^* = \arg\min_{f \in H} \mathbb{E}[I(f(x_0), y_0)] . \tag{5}
\]

where we define \( L(f) = \mathbb{E}[I(f(x_0), y_0)] \), and ignore the regularizer for the moment. The loss \( l \) quantifies the merit of the estimator \( f(x) \) with respect to its target \( y \). However, as is well known in statistics [27], solving (5) is really only an approximation of the Bayes optimal estimator

\[
\hat{y}^* = \arg\min_{\hat{y} \in \mathcal{Y}^X} \mathbb{E}[I(\hat{y}(x_0), y_0)] , \tag{6}
\]

where \( \mathcal{Y}^X \) denotes the space of all functions \( \hat{y} : \mathcal{X} \to \mathcal{Y} \) that map data \( x \) to target variables \( y \). Suppose we try to minimize the loss (5) and obtain some estimate \( \hat{f} \). Then the performance
Given as objective evaluation, which practically may be interpreted as a regularizer back into (12). Solutions of (12), as compared with estimation or approximation error. Note that we have added the optimal supervised cost to \[30\] following eqn. (13) and Sections 3.1 - 3.2 further details.

In supervised learning, typically we try to make the model bias small as the number of data points goes to infinity, resigning ourselves to the fact that the variance is an intrinsic penalty we suffer for selecting a particular modeling hypothesis: the function class to which \( f \) belongs. However, authors in operations research [8] and applied probability [31] have proposed to optimize both the expected loss over all data plus a measure of the dispersion of the estimate with respect to its target variable as a way of accounting for the unknown approximation error of the modeling hypothesis in \([7]\). Many measures of dispersion are possible, but one which yields a convex formulation is the semivariance.

\[
\tilde{\text{Var}}[l(f(x_0), y_\theta)] = \mathbb{E} \left\{ \left( l(f(x_0), y_\theta) - \mathbb{E}[l(f(x_\xi), y_\xi)] \right)_+^2 \right\},
\]

(8)

where \( a_+ = \max(a, 0) \) denotes the positive projection. Note that when we omit the positive projection, \([5]\) reduces to the variance of the instantaneous loss \( l(f(x_0), y_\theta) \). To see that variance is a composite function observe that when \( \theta \) and \( \xi \) are independent, we have that

\[
\mathcal{L}_\theta(u) = (u^1 - u^2)^2 \quad \mathcal{L}(u) = \mathbb{E} \left[ (u^1 - u^2)^2 \right] \quad (9)
\]

\[
[\mathcal{H}(f)]_1 = l(y_\theta - f(\theta)) \quad [\mathcal{H}_\xi](f)_1 = l(y_\xi - f(\theta)) \quad (10)
\]

\[
[\mathcal{H}_\xi](f)_2 = l(y_\xi - f(\xi)) \quad [\mathcal{H}_\xi\xi](f)_2 = \mathbb{E} \left[ l(y_\xi - f(\xi)) \right] \quad (11)
\]

where note that \( u_1 \) and \( u_2 \) are also random quantities so the first expectation cannot be dropped.

However, the subtraction of the second moment of \( l(f(x_\theta), y_\theta) \) without positive projection makes the problem non-convex. Thus, using positive projections yields semivariance rather than true variance \([8]\). With the measure of dispersion in \((8)\), one robust formulation of supervised learning over an RKHS \( \mathcal{H} \) which tries to account for the approximation error is

\[
f^* = \arg \min_{f \in \mathcal{H}} E[l(f(x_0), y_\theta)] + \eta \tilde{\text{Var}}[l(f(x_0), y_\theta)] + \frac{\lambda}{2} \| f \|_H^2.
\]

(12)

where \( \eta \) is a scaling parameter that tunes the importance of estimation or approximation error. Note that we have added the regularizer back into \((12)\). Solutions of \((12)\), as compared with \([5]\), are better attuned to data points associated with high variance objective evaluation, which practically may be interpreted as outliers or in the case of classification, situations where training examples possess characteristics corresponding to multiple classes. Due to the fact that our analysis requires Lipschitz gradients (Section \([4]\), numerically approximate the positive projection in \([8]\) with the softmax in Section \([7]\) – see \([20]\). An alternative risk measure popular in finance is the conditional value-at-risk (CVaR) \([32]\), which quantifies the loss function at different quantiles of its distribution.

**Example 2 (Policy Evaluation in Continuous Markov Decision Problems)** Another instantiation of \((1)\) is the task of policy evaluation in a continuous Markov Decision Problem (MDP) \([33]\), which is increasingly relevant to emerging technologies such as robotics \([34]\), power systems \([35]\), and others. A MDP is a quintuple \((X, A, P, r, \gamma)\), where \( P \) is the action-dependent transition probability of the process: when the agent starts in state \( x_t \in X \subset \mathbb{R}^p \) at time \( t \) and takes an action \( a_t \in A \), a transition to next state \( y_t \in X \) is distributed according to \( y_t \sim P(\cdot | x_t, a_t) \). After the agent transitions to a particular \( y_t \), the MDP provides to it an instantaneous reward \( r(x_t, a_t, y_t) \), where the reward function is a map \( r : X \times A \times X \to \mathbb{R} \).

In policy evaluation, control decisions \( a_t \) are chosen according to a stochastic stationary policy \( \pi : X \to \rho(A) \), where \( \rho(A) \) denotes the set of probability distributions over \( A \), and one seeks to compute the value of a policy when starting in state \( x \), quantified by the discounted expected sum of rewards, or value function \( V^\pi(x) \):

\[
V^\pi(x) = \mathbb{E}_y \left[ \sum_{i=0}^{\infty} \gamma^i r(x_i, a_i, y_i) \mid x_0 = x, \{a_i = \pi(x_i)\}_{i=0}^\infty \right].
\]

(13)

For a single trajectory through the state space \( X, y_t = x_{t+1} \). The discount factor \( \gamma \in (0, 1) \) determines the agent’s farsightedness. From the definition of the value function in \((13)\), one may derive the Bellman evaluation equation \([33]\):

\[
V^\pi(x) = \int_X \left[ r(x, \pi(x), y) + \gamma V(y) \right] P(dy \mid x, \pi(x)) \quad \text{for all } x \in X,
\]

(14)

The functional fixed point problem \((14)\) defined by Bellman’s equation may be reformulated as a nested stochastic program. To do so, rewrite the integral as an expectation, subtract the value function \( V^\pi(x) \) that satisfies the fixed point relation from both sides, and then pull it inside the expectation. Then, to solve \((14)\) in an initialization-independent manner, integrate out \( x \), the starting point of the trajectory defining the value function \((13)\), as well as policy \( \pi(x) \), to obtain the compositional problem

\[
V^\pi = \arg \min_{V \in B(X)} J(V)
\]

\[
:= \arg \min_{V \in B(X)} \mathbb{E}_{x, \pi(x)} \left\{ \frac{1}{2} \left( \mathbb{E}_y [r(x, \pi(x), y) + \gamma V(y) - V(x)] \mid x, \pi(x) \right)^2 \right\},
\]

(15)

However, since it is intractable to optimize over all bounded functions \( B(X) \), one may restrict focus to an RKHS \( \mathcal{H} \). This hypothesis, however, requires the introduction of regularization. Assuming that the Bellman fixed point \( V^\pi \) is a continuous function, the RKHS approximation may be made close to the true \( V^\pi \) when used with a universal kernel \([36]\) – see \([25]\). We note that the problem formulation in \((15)\) is slight generalization of the problem formulation in \((12)\). In particular, observe that the inner
random variable \( y \) in (15) depends on the outer random variables \( x \) and \( \sigma(x) \), necessitating the use of the conditional expectation operator. However, this generalization can be readily handled as detailed in [2] and the subsequent analysis considers the special case in (12) for the sake of brevity.

With the problem setting clarified, we shift focus in subsequent sections to developing an iterative numerical method to solve (1).

### III. Algorithm Development

Now we turn to solving the stochastic compositional convex optimization (1) over the RKHS \( \mathcal{H} \). We focus on the development of stochastic approximation methods such that we may minimize \( R \) over \( \mathcal{H} \) with only sequentially revealed independent and identically distributed realizations of \( \Theta \) and \( \xi \), but no knowledge of their underlying probability distributions. Related ideas are developed for the specialized objective of Example 2 in [2] and in the vector-valued case in [2]. The fundamental building blocks of our proposed algorithm are a functional generalization of the stochastic quasi-gradient method (Section III-A) operating in tandem with low-dimensional subspace projections that are greedily constructed using matching pursuit (Section III-B), which we detail next.

#### A. Functional Stochastic Quasi-gradient Descent

Note that the functional gradient of the objective function \( J(f) \) is given by

\[
\mathbb{E} \left[ \langle \nabla J f \hat{\mathcal{H}} f(\xi) \rangle, \nabla \Theta \mathbb{E} [\hat{\mathcal{H}} f(\xi)] \right]
\]

where \( \nabla \Theta \mathbb{E} [\hat{\mathcal{H}} f(\xi)] \) is defined as \[ \mathbb{E} \left[ \nabla \Theta f(\xi) \right] = \frac{\partial \mathbb{E} f(\xi)}{\partial \Theta} \]. Observe that the stochastic version of the functional gradient is obtained by dropping the outer expectation as follows

\[
\langle \nabla J f \hat{\mathcal{H}} f(\xi), \nabla \Theta \mathbb{E} [\hat{\mathcal{H}} f(\xi)] \rangle.
\]

However, the stochastic gradient (17) at a specific random variable \( \xi \), \( \Theta \) is not available due to the expectation involved in the argument of \( \mathbb{E} \Theta f(\xi) \). This issue precludes use of vanilla stochastic gradient method for solving (1).

Thus, we propose using a two time-scale stochastic approximation strategy called stochastic quasi-gradient method [21], [27]. This algorithm operates by defining a scalar sequence \( \hat{g}_t \) that tracks the sequence of instantaneous functions \( \hat{\mathcal{H}} f(\xi) \) evaluated at \( \xi_t \) using the equation

\[
\hat{g}_{t+1} = (1 - \beta_t) \hat{g}_t + \beta_t \hat{\mathcal{H}} f(\xi_t)
\]

with the intent of estimating the expectation \( \mathbb{E} [\hat{\mathcal{H}} f(\xi)] \). In (18), \( \beta_t \) is a scalar learning rate chosen from the unit interval (0,1) which may be either diminishing or constant. Then, we define a function sequence \( f_t \in \mathcal{H} \) initialized as null \( f_0 = 0 \), that we sequentially update using stochastic quasi-gradient descent:

\[
f_{t+1} = (1 - \lambda_t) f_t - \lambda_t \langle \nabla \Theta f(\xi_t), \nabla \Theta \Theta (\hat{g}_{t+1}) \rangle,
\]

where \( \lambda_t \) is a step-size parameter chosen as diminishing or constant. Typically, we require that \( \lambda_t < \beta_t \). Further note that the term \( \langle \nabla \Theta f(\xi_t) \rangle \) is a function in \( \mathcal{H} \), and thus infinite dimensional. However, by applying the chain rule and the reproducing property of the kernel ("the kernel trick") stated in (5), we obtain

\[
\langle \nabla f \hat{\mathcal{H}} f(\xi_t), \nabla \Theta \Theta (\hat{g}_{t+1}) \rangle
\]

where the functional gradient of the objective function ("the kernel trick") stated in (5), we obtain

\[
= \sum_{i=1}^m \nabla f \hat{\mathcal{H}} f(\xi_t) \left( \frac{\partial \Theta (\hat{g}_{t+1})}{\partial \Theta (\hat{g}_{t+1})} \right) \bigg|_{\Theta = \Theta (\hat{g}_{t+1})}
\]

\[
= \sum_{i=1}^m \frac{\partial \Theta (\hat{g}_{t+1})}{\partial \Theta (\hat{g}_{t+1})} \bigg|_{\Theta = \Theta (\hat{g}_{t+1})} \langle \nabla \Theta f(\xi_t), \nabla \Theta \Theta (\hat{g}_{t+1}) \rangle
\]

\[
= \langle \nabla \Theta f(\xi_t), \nabla \Theta \Theta (\hat{g}_{t+1}) \rangle.
\]

In (22), we have used the vector inner product notation to denote the summation in (21). Note that the kernel function \( k(\xi, \cdot) \) in (21) is common and therefore outside the inner product in (22). Utilizing this notation, the function update equation of (19) may be written as

\[
f_{t+1} = (1 - \lambda_t) f_t - \lambda_t \langle \nabla \Theta f(\xi_t), \nabla \Theta \Theta (\hat{g}_{t+1}) \rangle k(\xi_t, \cdot).
\]

Observe that in (23), the vector \( h'(f(\xi_t)) \) associates with the sample point \( \xi_t \) evaluated by the kernel, whereas \( \Theta (\hat{g}_{t+1}) \) associates with the tracking parameter \( \Theta (\hat{g}_{t+1}) \). Moreover, the function sequence in (23) belongs to a RKHS defined over the kernel \( k(\xi, \cdot) \). Specifically, using the fact that \( f_0 = 0 \in \mathcal{H} \), one may obtain through induction that the function \( f_t \) at time \( t \) admits an expansion in terms of kernel evaluations of past data realizations \( \xi_n \) and scalar weights \( w_n \) for \( n < t \):

\[
f_t(u) = \sum_{n=1}^{t-1} w_n k(\xi_n, u) = w_t^T k_{U_t}(u).
\]

where we define weight vector \( w_t := [w_1, \ldots, w_{t-1}] \), kernel dictionary \( U_t := [\xi_1, \ldots, \xi_{t-1}] \), and the empirical kernel map \( k_{U_t}(u) := [k(\xi_1, u), \ldots, k(\xi_{t-1}, u)] \). Thus, performing the stochastic quasi-gradient iteration in the RKHS amounts to the following parametric updates on the coefficient vector \( w \) and kernel dictionary \( U_t \):

\[
U_{t+1} = \left[ U_t, \xi_t \right],
\]

\[
w_{t+1} = \left[ (1 - \lambda_t) w_t - \lambda_t h'(f(\xi_t), \Theta (\hat{g}_{t+1})) \right].
\]

In (25), observe the kernel dictionary parameterizing function \( f_t \) is a matrix \( U_t \in \mathcal{H}^{p \times (t-1)} \) which stacks past realizations of random variable \( \xi \), and the coefficient vector \( w_t \in \mathcal{H}^{t \times 1} \) as the associated scalars in the kernel expansion (24) which are updated according to (25). Observe that the function update of (25) implies that the complexity of \( f_t \) is \( O(t) \), due to the fact that the number of columns in \( U_t \), or model order \( M_t \), is \( (t-1) \), and thus is unsuitable for settings where the total number of data samples is not finite, or are arriving sequentially and repeatedly. This is an inherent challenge of extending [2] to optimizing over nonlinear functions that belong to RKHSs. To address this, we consider projections of (23) onto low-dimensional subspaces, inspired by (16), which we detail in the following subsection.

#### B. Subspace Projections for Complexity Control

In this subsection, we turn to address the untenable growth of the function representational complexity discussed in the previous
section, namely, that the model order is $M_t = (t - 1)$, and grows without bound with the iteration index $t$. To do so, we adopt the idea of bias-inducing proximal projections onto low-dimensional subspaces developed in [16].

Specifically, we construct an approximate sequence of functions by orthogonally projecting functional stochastic gradient updates onto subspaces $H_D \subseteq H$ that consist only of functions that can be represented using some dictionary $D = \{d_1, \ldots, d_M\} \in \mathbb{R}^{p \times M}$, i.e., $H_D = \{f : \mathbb{R} \rightarrow \mathbb{R} | f = \sum_{m=1}^{M} w_m \kappa(d_{m}, \cdot) = \mathbb{W} \kappa(\cdot)\}$. Here we define $d_n \in \mathbb{R}^p$ as a model point which stacks exemplar realizations of $\xi$, i.e., $d_n = \xi_n$. Further define $\kappa(\cdot) = \{\kappa(d_1, \cdot) \ldots \kappa(d_M, \cdot)\}$, and $K_{D,D}$ as the resulting kernel matrix from this dictionary. We will enforce sparsity in function representation by selecting dictionaries $D$ such that $M_t \ll t$.

Note that the function update written in (19) may be rewritten as follows

$$\tilde{f}_{t+1} = \arg\min_{f \in H}\left\|f - (1 - \lambda a_t) f_t - a_t \langle \mathcal{H}_t^D(f(\xi)), \mathcal{C}_t^D(g_{t+1}) \rangle \kappa(\xi, \cdot) \right\|_H$$

where the first equality in (26) comes by ignoring the constant terms which vanish when we compute the derivative with respect to $f$. The second equality comes from the fact that $f_{t+1}$ can be represented by using the points available in dictionary $U_{t+1}$. Observe that (26) expresses $f_{t+1}$ as the orthogonal projection of the update

$$(1 - \lambda a_t) f_t - a_t \langle \mathcal{H}_t^D(f(\xi)), \mathcal{C}_t^D(g_{t+1}) \rangle \kappa(\xi, \cdot)$$

onto the subspace defined by dictionary $U_{t+1}$. If we replace $U_{t+1}$ by some other dictionary $D_{t+1}$, we obtain the projection:

$$f_{t+1} = \arg\min_{f \in H_{D_{t+1}}} \left\|f - (1 - \lambda a_t) f_t - a_t \langle \mathcal{H}_t^D(f(\xi)), \mathcal{C}_t^D(g_{t+1}) \rangle \kappa(\xi, \cdot) \right\|_H$$

$$= P_{H_{D_{t+1}}} \left[\left(1 - \lambda a_t\right) f_t - a_t \langle \mathcal{H}_t^D(f(\xi)), \mathcal{C}_t^D(g_{t+1}) \rangle \kappa(\xi, \cdot)\right].$$

We propose to replace (26) with the proximately projected iteration (27) using some other dictionary $D$, $D = D_{t+1}$, which is extracted from the data points observed thus far, at each iteration, and is of dimension $p \times M_{t+1}$, with $M_{t+1} \ll t$. As a result, we shall generate a function sequence $f_t$ that differs from the functional stochastic quasi-gradient method presented in Section III-A. The function $f_{t+1}$ is parameterized by dictionary $D_{t+1}$ and weight vector $w_{t+1}$. We denote columns of $D_{t+1}$ as $d_n$ for $n = 1, \ldots, M_{t+1}$, where the time index is dropped for notational clarity but may be inferred from the context.

**Coefficient update** The update (27), for a fixed dictionary $D_{t+1} \in \mathbb{R}^{p \times M_{t+1}}$, may be expressed in terms of the parameter space of coefficients only. To do so, first define the stochastic gradient update, given function $f_t$ parameterized by dictionary $D_t$ and coefficients $w_t$, as

$$\tilde{f}_{t+1} = (1 - \lambda a_t) f_t - a_t \langle \mathcal{H}_t^D(f(\xi)), \mathcal{C}_t^D(g_{t+1}) \rangle \kappa(\xi, \cdot).$$

This update may be represented using dictionary and weight vector

$$\tilde{D}_{t+1} = [D_t, \xi_t], \quad \tilde{w}_{t+1} = [\{1 - \lambda a_t\} w_t, -a_t \langle \mathcal{H}_t^D(f(\xi)), \mathcal{C}_t^D(g_{t+1}) \rangle].$$

Observe that since two data points are added to dictionary at each instant, $D_{t+1}$ has $M = M_t + 1$ columns, and the length of $\tilde{w}_{t+1}$ is $1$. Given that the projection of $\tilde{f}_{t+1}$ onto the stochastic subspace $H_{D_{t+1}}$ for a fixed dictionary $D_{t+1}$, the stochastic projection in (27) amounts to a least-squares problem on the coefficient vector. To see this, make use of the Representer Theorem to rewrite (27) in terms of kernel expansions, and that the coefficient vector is the only free parameter to write as in [16]. In (27), the first equality comes from expanding the square, and the second comes from defining the cross-kernel matrix $K_{D_{t+1}, D_{t+1}}$ whose $(n,m)^{th}$ entry is given by $\kappa(d_n, d_m)$. The outer kernel matrices $K_{D_{t+1}, D_{t+1}}$ and $K_{D_{t+1}, D_{t+1}}$ are similarly defined. Note that $M_{t+1}$ is the number of columns in $D_{t+1}$, while $M = M_t + 1$ is the number of columns in $D_{t+1}$ [cf. (29)]. The explicit solution of (27) may be obtained by noting that the last term is a constant independent of $w$, and thus by computing gradients and solving for $w_{t+1}$ we obtain

$$w_{t+1} = K_{D_{t+1}, D_{t+1}}^{-1} K_{D_{t+1}, D_{t+1}} w_{t+1},$$

which is the required coefficient update step. Given that the projection of $\tilde{f}_{t+1}$ onto the stochastic subspace $H_{D_{t+1}}$, for a fixed dictionary $D_{t+1}$, amounts to a simple least-squares multiplication, we now detail how the kernel dictionary $D_{t+1}$ is selected from past data $\{\xi_n\}_{n \in \mathbb{N}}$.

**Dictionary Update** The selection procedure for the kernel dictionary $D_{t+1}$ is based upon greedy sparse approximation, a topic studied extensively in the compressive sensing community [38]. The function

$$f_{t+1} = (1 - \lambda a_t) f_t - a_t \langle \mathcal{H}_t^D(f(\xi)), \mathcal{C}_t^D(g_{t+1}) \rangle \kappa(\xi, \cdot),$$

defined by stochastic quasi-gradient method without projection is parameterized by dictionary $\tilde{D}_{t+1}$ [cf. (29)], whose model order is $M = M_t + 1$. We form $D_{t+1}$ by selecting a subset of $M_{t+1}$ columns from $D_{t+1}$ that are best for approximating $f_{t+1}$ in terms of error with respect to the Hilbert norm. As previously noted, numerous approaches are possible for seeking a sparse representation. We make use of kernel orthogonal matching pursuit (KOMP) [39] with allowed error tolerance $\epsilon_t$ to find a kernel dictionary matrix $D_{t+1}$ based on the one which adds the latest sample point $\xi_t$. This choice, inspired by [16], is due to the fact that we can tune its stopping criterion to guarantee a decrement property in expectation, and further guarantee the complexity of the function representation remains finite – see Section LV for details.

We now describe the variant of KOMP we propose using, called Destructive KOMP with Pre-Fitting (see [39], Section 2.3), which is summarized in Algorithm 1. This flavor of KOMP takes as an input a candidate function $\tilde{f}$ of model order $M$ parameterized by its kernel dictionary $\tilde{D} \in \mathbb{R}^{p \times M}$ and coefficient vector $\tilde{w} \in \mathbb{R}^M$. The method then seeks to approximate $\tilde{f}$ by a parsimonious function $f \in H$ with a lower model order. Initially, this sparse approximation is the original function $f = \tilde{f}$ so that its dictionary is initialized with that of the original function $D = \tilde{D}$, with corresponding coefficients $w = \tilde{w}$. Then, the algorithm
Algorithm 1 Destructive Kernel Orthogonal Matching Pursuit (KOMP)

Require: function $\tilde{f}$ defined by dict. $\tilde{D} \in \mathbb{R}^{p \times M}$, coeffs. $\tilde{w} \in \mathbb{R}^{M/2}$, approx. budget $\epsilon_t > 0$

initialize $f = \tilde{f}$, dictionary $D = \tilde{D}$ with indices $I$, model order $M = M$, coeffs. $w = \tilde{w}$.

while candidate dictionary is non-empty $I \neq \emptyset$ do

for $j = 1, \ldots, M$ do

Find minimal approximation error with dictionary element $d_j$ removed

$$y_j = \min_{w \in \mathbb{R}^M} \| \tilde{f}() - \sum_{k \in I \setminus \{j\}} w_k \kappa(d_k, \cdot) \|_H.$$ 

end for

Find dictionary index minimizing approximation error:

$$j^* = \arg \min_{j \in I} y_j$$

if minimal approximation error exceeds threshold $y_{j^*} > \epsilon_t$ stop
else

Prune dictionary $D \leftarrow D \setminus \{j^*\}$, remove both the columns associated with index $j^*$

Revise set $I \leftarrow I \setminus \{j^*\}$ and model order $M \leftarrow M - 1$.

Update weights $w$ defined by current dictionary $D$

$$w = \arg \min_{w \in \mathbb{R}^M} \| \tilde{f}() - w^T \kappa_d(\cdot) \|_H$$

end while

return $f, D, w$ of complexity $M \leq \hat{M}$ s.t. $\| f - \tilde{f} \|_H \leq \epsilon_t$

Algorithm 2 Compositional Online Learning with Kernels (COLK)

Require: $\{\theta_i, \xi_i, a_i, \beta_i, \epsilon_i\}_{i=0,1,2,\ldots}$
initialize $f_0() = 0$, $D_0 = \emptyset$, $w_0 = \emptyset$, i.e. initial dictionary, coefficient vectors are empty

for $t = 0, 1, 2, \ldots$ do

Update auxiliary variable $g_{t+1}$ according to

$$g_{t+1} = (1 - \beta_t) g_t + \beta_t \kappa(f(\xi_t))$$

Update function with a stochastic quasi-gradient step

$$\hat{f}_{t+1} = (1 - \lambda a_t) f_t - \lambda a_t \kappa'(f(\xi_t), \ell'_{\theta_t}(g_{t+1})) \kappa(\xi_t, \cdot)$$

Revise parameterization: dictionary and weights

$$\tilde{D}_{t+1} = [D_t, \xi_t]$$

$$\tilde{w}_{t+1} = [(1 - a_t) w_t - a_t \kappa'(f(\xi_t), \ell'_{\theta_t}(g_{t+1}))]$$

Compress the function representation using Algorithm 1

$$(f_{t+1}, D_{t+1}, w_{t+1}) = \text{KOMP}(\hat{f}_{t+1}, \tilde{D}_{t+1}, \tilde{w}_{t+1}, \epsilon_t)$$

end for

Sequentially removes dictionary elements from dictionary $\tilde{D}$, yielding a sparse approximation $f$ of $\tilde{f}$, until the error threshold $\| f - \tilde{f} \|_H \leq \epsilon_t$ is violated, in which case it terminates.

At each stage of KOMP, a pair of dictionary element associated with index $j$ of $D$ is selected to be removed which contributes the least to the Hilbert-norm approximation error

$$\min_{f \in \mathbb{H}_{D_{\setminus\{j\}}}} \| f - \tilde{f} \|_H$$

of the original function $\tilde{f}$, when dictionary $D$ is used. Since at each stage the kernel dictionary is fixed, this amounts to a computation involving weights $w \in \mathbb{R}^{M-1}$ only: that is, the error of removing dictionary point $d_j$ is computed for each $j$ as

$$y_j = \min_{w \in \mathbb{R}^{M-1}} \| \tilde{f}() - \sum_{k \in I \setminus \{j\}} w_k \kappa(d_k, \cdot) \|_H.$$ 

We use the notation $w_{I \setminus \{j\}}$ to denote the entries of $w \in \mathbb{R}^M$ restricted to the sub-vector associated with indices $I \setminus \{j\}$. Then, we define the dictionary element which contributes the least to the approximation error as

$$j^* = \arg \min_{j \in I \setminus \{j\}} y_j.$$ 

If the error associated with removing this kernel dictionary element exceeds the given approximation budget $y_{j^*} > \epsilon_t$, the algorithm terminates. Otherwise, this dictionary elements associated with $d_{j^*}$ are removed, the weights $w$ are revised based on the pruned dictionary as

$$w = \arg \min_{w \in \mathbb{R}^{M}} \| \tilde{f}() - w^T \kappa_d(\cdot) \|_H,$$

and the process repeats as long as the current function approximation is defined by a nonempty dictionary.

With Algorithm 1 stated, we may summarize the key steps of the proposed method in Algorithm 2 for solving (1) while maintaining a finite model order, thus breaking the “curse of kernelization” for compositional stochastic programming over an RKHS. The method, Compositional Online Learning with Kernels (COLK), executes the stochastic subspace projection of a functional stochastic quasi-gradient step onto sparse subspaces $\mathcal{H}_{D_{t+1}}$ stated in (27). The initial function is set to null $f_0 = 0$, meaning that it has empty kernel dictionary $D_0 = \emptyset$ and coefficient vector $w_0 = \emptyset$. The notation $[]$ is used to denote the empty matrix or vector respective size $p \times 0$ or $0$. Then, at each step, given an independent training example $\xi_t$ and step-size $\eta_t$, we update the auxiliary variable $g_{t+1} = (1 - \beta_t) g_t + \beta_t \kappa(f(\xi_t))$. Then, this updated scalar is used to compute the unconstrained functional stochastic gradient iterate

$$\hat{f}_{t+1} = (1 - \lambda a_t) f_t - \lambda a_t \kappa'(f(\xi_t), \ell'_{\theta_t}(g_{t+1})) \kappa(\xi_t, \cdot),$$

which admits the parametric representation $\tilde{D}_{t+1}$ and $\tilde{w}_{t+1}$ as stated in (29). These parameters are then fed into KOMP with approximation budget $\epsilon_t$, such that

$$(f_{t+1}, D_{t+1}, w_{t+1}) = \text{KOMP}(\hat{f}_{t+1}, \tilde{D}_{t+1}, \tilde{w}_{t+1}, \epsilon_t).$$

IV. CONVERGENCE ANALYSIS

This section establishes the convergence and finite-memory properties of Algorithm 2 which is characterized in terms of the behavior of the function iterates $f_t$ with respect to the minimizer of (1). This quantity is shown to go to null in almost sure (a.s.) sense when learning rates and compression-induced error attenuate, and to a small constant-radius neighborhood depending on step-sizes when chosen as constant. To proceed with the analysis, a few quantities are first defined to simplify the exposition. Firstly, define the functional stochastic quasi-gradient of the objective function (cf. (2))

$$\hat{f}_f R(f_t, g_{t+1}, \xi_t, \theta_t) = \langle \kappa'(f(\xi_t), \ell'_{\theta_t}(g_{t+1})) \kappa(\xi_t, \cdot) + \lambda f_t, \xi_t \rangle,$$

We assume that the output of Algorithm 1 has bounded Hilbert norm, which may be explicitly enforced, for instance, by thresholding the weights if they climb above some large threshold value.
Proposition 1

Given independent identical realizations \((\xi_t, \theta_t)\) of the two associated random variables \((\xi, \theta)\), for all \(t\) it holds that

\[
\|\hat{\nabla}_f R(f_t, g_{t+1}; \xi_t, \theta_t) - \hat{\nabla}_f R(f_t, g_{t+1}; \xi_t, \theta_t)\|_H \leq \varepsilon_t, 
\]

where \(\alpha_t > 0\) denotes the step-size and \(\varepsilon_t > 0\) is the compression parameter of Algorithm 1.

Next we present an intermediate lemma which is vital to establishing a stochastic descent relationship, and hence convergence.

**Lemma 1** For the given algorithm history \(F_t\) at time \(t\), under the Assumptions [2] - [4] consider the sequence of iterates \(f_t\) generated by Algorithm 2. Then:

i) The conditional expectation of the Hilbert-norm difference of the next function estimate \(f_{t+1}\) and current iteration \(f_t\) \(\alpha_t > 0\) satisfies

\[
\mathbb{E} \left[ \|f_{t+1} - f_t\|_H^2 \right| F_t] \leq 4\alpha_t^2 U^2(G^2_H G^2_\varepsilon + \lambda^2 K^2) + 2\varepsilon_t^2. 
\]

ii) The conditional expectation of the Hilbert-norm difference of the next function estimate \(f_{t+1}\) and optimal function \(f^*\) satisfies

\[
\mathbb{E} \left[ \left\| f_{t+1} - f^* \right\|_H^2 \right| F_t] \leq \left( 1 + L_c U^2 \frac{\sigma_f^2}{\beta_t} \right) \| f_t - f^* \|_H^2 + 2\varepsilon_t \| f_t - f^* \|_H + 2\alpha_t \| R(f_t) - R(f^*) \|_H^2 + \alpha_t^2 \sigma_f^2 + L_c U^2 \beta_t \mathbb{E} \left[ \left\| \delta_{t-1} - g_{t+1} \right\|_H^2 \right| F_t]. 
\]

iii) Considering the definition of \(\delta_t\) provided in Assumption 2, the evolution of the auxiliary sequence \(g_t\) with respect to \(\delta_t\) satisfies

\[
\mathbb{E} \left[ \| g_{t+1} - \delta_t \|_H^2 \right| F_t] \leq (1 - \beta_t) (\| g_t - \delta_{t-1} \|_H^2 + \frac{2\alpha_t}{\beta_t} (\| f_t - f_{t-1} \|_H^2 + 2\beta_t \sigma_f^2). 
\]

The proof of this lemma is provided in Appendix VIII in the supplementary material. The relationships stated in Lemma 1 under Assumptions 1 - 4 allow us to obtain almost sure convergence.

**Theorem 1** Consider the sequence \(g_t\) [cf. (32)] and \(f_t\) [cf. (53)] as stated in Algorithm 2. Assume the regularizer is positive \(\lambda > 0\), Assumptions [2] - [4] hold, and the step-size conditions hold:

\[
\sum_{i=1}^{\infty} \alpha_i = \infty, \sum_{i=1}^{\infty} \beta_i = \infty, \sum_{i=1}^{\infty} \alpha_i^2 + \beta_i^2 + \frac{\alpha_i^2}{\beta_i} < \infty, \epsilon_i = \alpha_i^2,
\]

then \(f_t \to f^*\) defined by (2) with probability 1.
Proof of Theorem 7. To establish Theorem 1 we first state a result defining the limiting behavior of coupled decreasing stochastic processes, namely, the Coupled Supermartingale Theorem (41) (Lemma 6).

Lemma 2 (Coupled Supermartingale Theorem (41) (Lemma 6)) Let \( \{ \phi_t \}, \{ \zeta_t \}, \{ u_t \}, \{ \tilde{u}_t \}, \{ \eta_t \}, \{ \chi_t \}, \{ \varepsilon_t \}, \{ \mu_t \}, \{ v_t \} \) be sequences of nonnegative random variables such that

\[
\mathbb{E}[\phi_{t+1}] \leq (1 + \eta_t) \phi_t - u_t + c \chi_t \zeta_t + \mu_t, \quad (48)
\]

\[
\mathbb{E}[\zeta_{t+1}] \leq (1 - \chi_t) \zeta_t - \tilde{u}_t + \varepsilon_t \phi_t + v_t, \quad (49)
\]

where \( G_t = \{ \phi_t, \zeta_t, u_t, \tilde{u}_t, \eta_t, \chi_t, \varepsilon_t, \mu_t, v_t \} \) is the filtration, and \( c > 0 \) is a scalar. Suppose the following summability conditions hold:

\[
\sum_{k=0}^{\infty} \eta_k < \infty, \quad \sum_{k=0}^{\infty} \varepsilon_k < \infty, \quad \sum_{k=0}^{\infty} \mu_k < \infty, \quad \sum_{k=0}^{\infty} v_k < \infty, \quad \text{almost surely.} \quad (50)
\]

Then \( \phi_t \) and \( \zeta_t \) converge almost surely to two respective nonnegative random variables, and we may conclude that

\[
\sum_{k=0}^{\infty} u_k < \infty, \quad \sum_{k=0}^{\infty} \tilde{u}_k < \infty, \quad \sum_{k=0}^{\infty} \chi_k \zeta_k < \infty, \quad \text{almost surely.} \quad (51)
\]

The limiting behavior of coupled stochastic processes in Lemma 2 may be connected with Algorithm 2 through the expressions in Lemma 1.

Begin by using the approximation budget \( e_t = a_{t}^2 \) and \( ||f_t - f^*||_H \leq 2K \) (which comes from our specification that the output of KOMP has bounded RKHS norm) into (102) to write

\[
\mathbb{E} \left[ ||f_{t+1} - f^*||_H^2 | F_t \right] \leq \left( 1 + L_c U^2 \frac{a_{t}^2}{\beta_t} G_h^2 \right) ||f_t - f^*||_H^2 - 2a_t \left[ R(f_t) - R(f^*) \right] + \frac{a_{t}^2}{2} \sigma_h^2 + 4K \nonumber \\
+ L_c U^2 \beta_t \mathbb{E} \left[ \left| \delta_t - g_{t+1} \right| \right] ||f_t - f^*||_H \right), \quad (52)
\]

and then substitute (46) regarding the evolution of \( g_t \), with respect to its conditional expectation into (52) to obtain:

\[
\mathbb{E} \left[ ||f_{t+1} - f^*||_H^2 | F_t \right] \leq \left( 1 + L_c U^2 \frac{a_{t}^2}{\beta_t} G_h^2 \right) ||f_t - f^*||_H^2 - 2a_t \left[ R(f_t) - R(f^*) \right] + \frac{a_{t}^2}{2} \sigma_h^2 + 4K \nonumber \\
+ L_c U^2 \beta_t \left| \delta_t - g_{t+1} \right| ||f_t - f^*||_H + 2L_c U^2 \beta_t^2 \sigma_h^2. \quad (53)
\]

Assume that \( \beta_t \in (0,1) \) for all \( t \), so that the right-hand side of (53) may be simplified to

\[
\mathbb{E} \left[ ||f_{t+1} - f^*||_H^2 | F_t \right] \leq \left( 1 + L_c U^2 \frac{a_{t}^2}{\beta_t} G_h^2 \right) ||f_t - f^*||_H^2 - 2a_t \left[ R(f_t) - R(f^*) \right] + \frac{a_{t}^2}{2} \sigma_h^2 + 4K \nonumber \\
+ L_c U^2 \beta_t \left| \delta_t - g_{t+1} \right| ||f_t - f^*||_H + 2L_c U^2 \beta_t^2 \sigma_h^2. \quad (54)
\]

Now, to apply the result of Supermartingale Lemma 2 ([48], [54], [49]) and (46)

\[
\phi_t = ||f_t - f^*||_H^2, \quad \eta_t = L_c U^2 \frac{a_{t}^2}{\beta_t} G_h^2, \quad u_t = 2a_t \left[ R(f_t) - R(f^*) \right], \nonumber \\
\mu_t = \frac{a_{t}^2}{2} \sigma_h^2 + 4K + L_c U^2 \beta_t \left| \delta_t - g_{t+1} \right| + 2L_c U^2 \beta_t^2 \sigma_h^2, \nonumber \\
c = L_c U^2 \beta_t \zeta_t = ||g_t - \delta_{t-1}||^2, \quad \chi_t = \beta_t, \quad (55)
\]

where \( u_t \geq 0 \) by the definition of the optimal objective \( R(f^*) \). Further to show the summability of the sequence \( \mu_t \), note that

\[
\sum_{t=1}^{\infty} \mu_t = (\sigma_h^2 + 4K) \sum_{t=1}^{\infty} \frac{a_{t}^2}{2} + L_c U^2 \sum_{t=1}^{\infty} ||f_t - f_{t-1}||_H^2 \nonumber \\
+ 2L_c U^2 \beta_t^2 \sum_{t=1}^{\infty} \sigma_h^2. \quad (56)
\]

The sum in (56) is clearly bounded owing to the fact that \( \sum_{t} a_{t}^2 < \infty, \sum_{t} \beta_t^2 < \infty, \) and \( \sum_{t} ||f_t - f_{t-1}||_H^2 < \infty \). The boundedness of the sum \( \sum_{t} ||f_t - f_{t-1}||_H^2 < \infty \) is formalized in the following proposition.

Proposition 2 Under the Assumptions 1 - 4, the sequence of function iterates \( \{ f_t \} \) as stated in Algorithm 2 is such that \( \sum_{t=1}^{\infty} ||f_t - f_{t-1}||_H^2 < \infty \) holds with probability 1.

Proof: Note the result in (44), by considering \( e_t = a_{t}^2 \) we have

\[
\mathbb{E} \left[ ||f_{t+1} - f^*||_H^2 | F_t \right] \leq J a_t^2, \quad (57)
\]

where \( J = (4U^2(G_h^2 G^2 + \lambda K^2) + 2a_{t}^2) \) is a constant. Take the total expectation on both sides of (57) and calculate the sum from \( t = 1 \) to \( \infty \), we get

\[
\sum_{t=1}^{\infty} \mathbb{E} \left[ ||f_{t+1} - f_t||_H^2 \right] \leq \sum_{t=1}^{\infty} \mathbb{E} \left[ \delta_t \right] < \infty, \quad (58)
\]

which holds from the fact that \( \sum_{t} a_{t}^2 \) is finite. To prove that the instantaneous sequence \( \sum_{t=1}^{\infty} ||f_{t+1} - f_t||_H^2 \) is finite, we pull the expectation (integral) outside the limit by using the use of the Monotone Convergence Theorem 2 to obtain

\[
\lim_{n \to \infty} \sum_{t=1}^{n} \mathbb{E} \left[ ||f_{t+1} - f_t||_H^2 \right] = \mathbb{E} \left[ \lim_{n \to \infty} \sum_{t=1}^{n} ||f_{t+1} - f_t||_H^2 \right]. \quad (59)
\]

Using the preceding expression together with (58) and the fact that a positive random variable with finite expected value is almost surely finite, we obtain that \( \sum_{t=1}^{\infty} ||f_{t+1} - f_t||_H^2 \) is finite as \( n \to \infty \) with probability 1.

Now, we proceed by noting that (46) is related to (49) via the identifications:

\[
\tilde{u}_t = 0, \quad \varepsilon_t = 0, \quad \varepsilon_t = \frac{L_c U^2}{\beta_t} ||f_t - f_{t-1}||_H^2 + 2\beta_t a_{t}^2 \quad (60)
\]

with \( \zeta_t = ||g_t - \delta_{t-1}||^2 \) and \( \chi_t = \beta_t \). The summability of \( \varepsilon_t \) follows from the same logic as provided in Corollary 2 with the fact that \( \sum_{t=1}^{\infty} \frac{a_{t}^2}{\beta_t} < \infty \).
Together with the conditions on the step-size sequences $\alpha_i$ and $\beta_i$ \cite{17}, the summability conditions \cite{20} of Coupled Supermartingale Theorem [Lemma\cite{2}] are satisfied. Thus we may allow us to conclude that $\phi_i = \|f_i - f^*\|_H^2$ and $\xi_i = \|g_i - \delta_{i-1}\|_2^2$ converge to two nonnegative random variables with probability 1, and that:

$$\sum_i \alpha_i |R(f_i) - R(f^*)| < \infty \text{, } \sum_i \beta_i \|g_{i+1} - \delta_i\|^2 < \infty \text{, a.s.} \quad (61)$$

The non-summability of the step-size sequences $\alpha_i$ and $\beta_i$ \cite{27} allows us to conclude that:

$$\liminf_{i \to \infty} R(f_i) = R(f^*) \text{, } \liminf_{i \to \infty} \|g_{i+1} - \delta_i\|^2 = 0 \text{, a.s.} \quad (62)$$

Therefore, the sequences $\|f_i - f^*\|_H^2$ and $\|g_{i+1} - \delta_i\|^2$ converge to two nonnegative random variables with probability 1. It remains to show that $f_i$ converges almost surely to a random point in the set of optimal solutions $\mathcal{H}^*$. The rest of this proof is analogous to \cite{2}, but is repeated in Appendix \cite{X} in the supplementary material for completeness.

This result establishes that the function sequence generated by Algorithm \cite{3} converge to the minimizer of (1) asymptotically almost surely. This result generalizes \cite{2} to the case where the optimization variable is a nonparametric function that depends on the random variable sequence $\theta_i, \xi_i$. This result requires step-sizes to go to null at specific rates \cite{27}, one example of which is $\alpha_i = \Theta(r^{-3/4+\epsilon/2})$, $\beta_i = \Theta(r^{-1+\epsilon/2})$, $\epsilon_i = \Theta(\alpha_i^2) = \Theta(r^{-3/2+\epsilon})$, where $\epsilon > 0$ is a small positive constant to make sure that the sums $\sum \alpha_i$ and $\sum \beta_i$ are finite. In general to satisfy the requirements in (27), we require that $\alpha_i = \Theta(r^{p_a}), \beta_i = \Theta(r^{-p_b})$ with $p_a \in (3/4, 1)$ and $p_b \in (1/2, 2p_a - 1)$.

Observe, however, that Theorem \cite{1} requires that the compression budget to diminish with the iteration index, and thus as $t \to \infty$ we have that $\epsilon_i \to 0$. Unfortunately, this means that the price of exact convergence in the RKHS is possibly infinite complexity of $f_i$ in the limit. To avoid this scenario, and maintain control of the function complexity, we next consider the case that both step-size and compression budget parameters are held constant. To obtain these results, we first require the following Lemma \cite{3} whose proof is in Appendix \cite{X} in the supplementary material.

**Lemma 3** Consider a constant step-size $\alpha_i = \alpha$ and $\beta_i = \beta$ with compression budget $\epsilon_i = \epsilon = Ca^2$, the sequence of $f_i$ satisfies:

$$\mathbb{E} \left[ \|f_{i+1} - f^*\|_H^2 \right] \\ \leq (1 - \lambda_0) \mathbb{E} \left[ \|f_i - f^*\|_H^2 \right] + \alpha^2 (\sigma_f^2 + 4CK) + 2\beta^2 \sigma_\delta^2 \quad (63)$$

$$+ \frac{2Lr \epsilon}{\beta} \left[ 4\alpha^2 U^2 (G_h^2 \sigma^2_e + (L_r U^2 G_h^2 \alpha / \beta + \lambda_0)^2 K^2) + 2C^2 a_4^2 \right],$$

where $\lambda = L_r U^2 G_h^2 \alpha / \beta + \lambda_0$ for $\lambda_0 < 1$.

Using Lemma \cite{3} we may derive two related convergence results for Algorithm \cite{2} when used with constant step-sizes, specifically, Theorem \cite{2} and Theorem \cite{3}.

**Theorem 2** Consider a constant step-size $\alpha_i = \alpha$, $\beta_i = \beta$ such that $0 < \alpha \leq \beta < 1$, and constant compression budget $\epsilon_i = \epsilon$ with regularizer $\lambda_0 < 1$. Then the function sub-optimality of Algorithm \cite{2} $\mathbb{E} \left[ \|f_i - f^*\|_H^2 \right]$ converges linearly to an error bound, i.e.:

$$\mathbb{E} \left[ \|f_i - f^*\|_H^2 \right] \leq (1 - \lambda_0)^i \mathbb{E} \left[ \|f_0 - f^*\|_H^2 \right] + \Theta \left( \alpha^2 + \frac{\alpha^2}{\beta} \left[ 1 + \frac{\alpha^2}{\beta} \frac{\alpha^2}{\beta} \right] \right) \quad (64)$$

**Proof of Theorem 2** From the statement of Lemma \cite{3} \cite{6}, the sequence $\mathbb{E} \left[ \|f_{i+1} - f^*\|_H^2 \right]$ is upper bounded in term of previous value $\mathbb{E} \left[ \|f_i - f^*\|_H^2 \right]$ and an error terms as follows:

$$\mathbb{E} \left[ \|f_{i+1} - f^*\|_H^2 \right] \leq (1 - \lambda_0) \mathbb{E} \left[ \|f_i - f^*\|_H^2 \right] + \mathcal{Z}(\alpha, \beta), \quad (65)$$

where

$$\mathcal{Z}(\alpha, \beta) = a^2 (\sigma_f^2 + 4CK) + 2\beta^2 \sigma_\delta^2 + \frac{2Lr \epsilon}{\beta} \left[ 4\alpha^2 U^2 (G_h^2 \sigma^2_e + (L_r U^2 G_h^2 \alpha / \beta + \lambda_0)^2 K^2) \right]$$

with $\lambda_0 < 1$. Note that

$$\mathcal{Z}(\alpha, \beta) = \Theta \left( \alpha^2 + \frac{\alpha^2}{\beta} \left[ 1 + \frac{\alpha^2}{\beta} \right] \right) \quad (66)$$

From the recursive relation on (65), we can write for $t$

$$\mathbb{E} \left[ \|f_{i+1} - f^*\|_H^2 \right] \leq (1 - \lambda_0)^i \mathbb{E} \left[ \|f_0 - f^*\|_H^2 \right] + \mathcal{Z}(\alpha, \beta)$$. \quad (67)

Now substitute the upper bound in (67) into (65), we get

$$\mathbb{E} \left[ \|f_{i+1} - f^*\|_H^2 \right] \leq (1 - \lambda_0)^2 \mathbb{E} \left[ \|f_i - f^*\|_H^2 \right] + \mathcal{Z}(\alpha, \beta)(1 + (1 - \lambda_0))$$. \quad (68)

By repeating the steps in (67), we can write

$$\mathbb{E} \left[ \|f_{i+1} - f^*\|_H^2 \right] \leq (1 - \lambda_0)^{i+1} \mathbb{E} \left[ \|f_0 - f^*\|_H^2 \right] + \mathcal{Z}(\alpha, \beta) \sum_{n=0}^{i} (1 - \lambda_0)^n \quad (69)$$

Replacing $t+1$ by $t$ and calculating the sum on right hand side of (69), we get

$$\mathbb{E} \left[ \|f_{i+1} - f^*\|_H^2 \right] \leq (1 - \lambda_0)^i \mathbb{E} \left[ \|f_0 - f^*\|_H^2 \right] + \frac{\mathcal{Z}(\alpha, \beta)}{\lambda_0} (1 - (1 - \lambda_0)^i) \quad (70)$$

Note that the term $[1 - (1 - \lambda_0)^i] < 1$ and utilizing this inequality in (70) and equality in (66), we obtain the result claimed in Theorem \cite{2} \cite{6} describes the non-asymptotic behavior of the mean-square distance $\mathbb{E} \left[ \|f_i - f^*\|_H^2 \right]$. This is comparable to well-known finite sample analysis of stochastic gradient algorithms, which linearly converge to a $\Theta(\alpha)$ neighborhood of the optimal value for step-size $\alpha$. This result reduces to a comparable statement when $\alpha = \beta$. Note that we cannot take the limit on both sides of (64) and obtain the corresponding asymptotic behavior in limit as $t \to \infty$ due to the fact that a priori the limit of $\mathbb{E} \left[ \|f_i - f^*\|_H^2 \right]$ may not exist. Therefore, we must separately establish its limiting behavior, as is done next as Theorem \cite{3}. For this result, for simplicity, we set the learning rates $\alpha, \beta$ to be equal.
Theorem 3 Consider a constant step size $\alpha = \alpha$ and $\beta = \beta$ and constant compression budget $c_\epsilon = \epsilon$, and with sufficiently large regularization, i.e.

$$0 < \beta < 1 , \alpha = \beta , c = Ca^2 , \lambda = G_f \frac{a}{\beta} + \lambda_0 ,$$

(71)

where $C > 0$ is a scalar, and $0 < \lambda_0 < 1$. Then, with the step sizes $0 < \alpha \leq \beta < 1$ and under Assumptions 3-6 the optimality gap of the sequence $\|f_i - f^*\|_{H_l}$ converges mean to a neighborhood:

$$\liminf_{i \to \infty} \mathbb{E} \left[ \|f_{i+1} - f^*\|_{H_l}^2 \right] = \mathcal{O}(\alpha).$$

(72)

Proof of Theorem 3 To establish the convergence to neighborhood, we will provide the proof by contradiction. From the statement of Lemma 3, it holds that

$$W : = a^2(\sigma_0^2 + 4CC) + 2\beta^2 \sigma_0^2$$

(73)

$$+ \frac{2Lf}{\beta} \left[ 4a^2 U^2(G_h^2G_r^2 + (L_r U^2 G_h^2 \alpha/\beta + \lambda_0)^2 K^2) + 2C^2 \sigma_0^2 \right].$$

Let us consider the hypothesis that

$$\liminf_{i \to \infty} \mathbb{E} \left[ \|f_{i+1} - f^*\|_{H_l}^2 \right] > \frac{W}{\lambda_0}.$$ 

(74)

The above condition implies that there exists some time index $t < \infty$ and some $\delta > 0$ such that

$$\mathbb{E} \left[ \|f_{i+1} - f^*\|_{H_l}^2 \right] > \frac{W}{\lambda_0} + \delta.$$ 

(75)

for all $t \geq t_0$. After rearranging the terms, we get

$$\lambda_0 \mathbb{E} \left[ \|f_{i+1} - f^*\|_{H_l}^2 \right] - \lambda_0 \delta > W.$$ 

(76)

Substituting the bound in (76) into (73), we get

$$\mathbb{E} \left[ \|f_{i+1} - f^*\|_{H_l}^2 \right] \leq (1 - \lambda_0) \mathbb{E} \left[ \|f_i - f^*\|_{H_l}^2 \right] + W$$

$$\leq \mathbb{E} \left[ \|f_i - f^*\|_{H_l}^2 \right] - \lambda_0 \delta$$

(77)

$$\leq \mathbb{E} \left[ \|f_i - f^*\|_{H_l}^2 \right]$$

where we have used that fact that $-\lambda_0 \delta \leq 0$ and canceled the common factor of $\lambda_0 \mathbb{E} \left[ \|f_i - f^*\|_{H_l}^2 \right]$ from the right-hand side. Note that, we start with the hypothesis that $\liminf_i \mathbb{E} \left[ \|f_i - f^*\|_{H_l}^2 \right] > W/\lambda_0$ but still the sequence $\mathbb{E} \left[ \|f_i - f^*\|_{H_l}^2 \right]$ is decreasing monotonically to zero. This is the contradiction to our hypothesis which implies that our hypothesis is false, and hence

$$\liminf_{i \to \infty} \mathbb{E} \left[ \|f_i - f^*\|_{H_l}^2 \right] = W$$

$$= \mathcal{O} \left( a^2 + \beta^2 + \frac{a^2}{\beta} \left[ 1 + a^2 + \frac{a}{\beta} + \frac{a^2}{\beta^2} \right] \right).$$

(78)

When $a = \beta$, the posynomial of the learning rates in (78) simplifies to $\mathcal{O}(a + a^2 + a^3)$. Since, $\beta \in (0, 1)$, we have $\mathcal{O}(a + a^2 + a^3) = \mathcal{O}(a)$, which is as stated in (72) (Theorem 3).

This result establishes that for a constant learning rate algorithm $\alpha = \alpha, \beta_i = \beta$ and fixed compression budget $c = Ca^2$, the sequence of functions $f_i$ converges in expectation to the neighborhood of optimal value $f^*$. The primary advantage of using constant learning rates, aside from their experimental simplicity, is that we may formally establish that the resulting function’s model order always remains finite.

Model Order Control To mitigate the complexity of functional stochastic approximation algorithms in RKHS, we proposed using sparse projections in Sec. III-B. This projection indeed limits the complexity of the function representation, as is formalized in next. This result is an extension of Theorem 3 in [16]. To proceed, an extra assumption related to the behavior of the inner and outer instantaneous functions defined in (1) is required, which we state here.

Assumption 5 (Gradient boundedness) The instantaneous gradient of both the inner function $\ell_0^\star(u)$ and outer function $\mathcal{H}^\star_{\epsilon}(f(\xi))$ are bounded as

$$|\ell_0^\star(u)| \leq C_{\epsilon} \quad \text{and} \quad |\mathcal{H}^\star_{\epsilon}(f(\xi))| \leq C_{\epsilon}. \quad (79)$$

It is remarked that this assumption is not required for rest of the analysis performed in this paper. But the assumption is a standard one and automatically satisfied for most smooth convex functions when data domains are compact. With this assumption stated, we may proceed to the proof of Theorem 4 provided in supplementary material XI-A.

Theorem 4 Consider a constant step size sequence $\alpha = \alpha, \beta_i = \beta$, with finite compression budget $c = Ca^2$, and regularization parameter as $\lambda = G_f^2 \frac{a}{\beta} + \lambda_0 = \mathcal{O}(\alpha/\beta + 1)$. Let $M_i$ denote the model order of the function iterate $f_i$, i.e. number of columns in current dictionary $D_i$. then under Assumptions 3-5 there exists a finite upper bound $M^{\infty}$ on the model order such that $M_i \leq M^{\infty}$ for all $t \geq 0$. This result states that the model order of the limiting function $m^{\infty} = \lim_{i \to \infty} f_i$ is finite only.

Theorem 4 ensures that the number of data points in the kernel representation of $f_i$ generated from Algorithm 2 is finite in limit. The condition for online sparsification performed by KOMP algorithm boils down to the condition for packing number of the kernelized feature space $\phi(\mathbf{x})$ as described in [138]. The packing of kernelized feature space is inversely proportional to the radius $\frac{C_0}{C \epsilon C_0}$. As this radius increases, the packing number reduces, meaning the model order required to cover feature space decreases. This radius depends upon the parsimony constant $C$, which scales the compression budget. A larger radius may be attained by choosing a larger parsimony constant $C$, meaning that fewer points are required to cover the data domain, and thus yields a lower model order. Pragmatically, this finite model order result means that the required memory to store the kernel dictionary will be under control.

V. EXPERIMENTS

To show the efficacy of the proposed algorithm, we consider a problem of nonlinear regression (filtering) over a $p$-dimensional parameter space. We have again have two sets of random variables $(x, x' \in \mathcal{X} \subset \mathbb{R}^p$ but now the target variables are real valued $y, y' \in \mathbb{R}$. The merit criterion of model fitness for a given training example $(x_n, y_n)$ is the humble square loss:

$$\ell(f(x_n), y_n) = (f(x_n) - y_n)^2 \quad (80)$$
However, due to the bias-variance tradeoff in Section 1, we do not want to only minimize the expectation of \((80)\) plus a regularizer \(\|f\|_H^2\) over all data \((x, y)\), but also some surrogate \([31]\) for the approximation error over data \((x', y')\). Due to the fact that many probability distributions may be completely characterized by their moments \([42]\), Chapter 3, a reasonable choice for the risk is to choose the dispersion measure as all \(p\)-th order central moments,

\[
D[\ell(f(x), y)] = \sum_{p=2}^{P} E_{x,y} \left\{ (\ell(f(x), y) - E_{x', y'}[\ell(f(x'), y')])^p \right\},
\]

which are just deviations raised to the \(p\)-th power \([8]\). However, since the computational overhead scales with \(P\), we truncate the upper summand index in \((81)\) to \(P = 4\). It is remarked that the dispersion measure in \((81)\) is non-convex which is used for the experimental purposes which corresponds to the variance, skewness, and kurtosis of the loss distribution. Note that \((81)\) may be convexified through a positive projection of \((\ell(f(x), y) - E_{x', y'}[\ell(f(x'), y')])\), in which case the standard deviation becomes a semi-deviation, as do its higher-order analogues \([43]\). However, for simplicity, we omit the positive projection in experiments. Next, we apply the proposed algorithm to solve the nonlinear regression problem which results in the following updates.

\[
g_{i+1} = (1 - \beta_i) g_i + \beta_i \ell(f_i(x_i^t) - y_i^t)^2.
\]

![Graphs showing experimental behavior on a regression on a synthetic data set, without and with training outliers. The presence of outliers does not break the learning stability, and test accuracy remains comparable, at the cost of increased complexity. Here COLK minimizes bias, variance, and third and fourth-order deviations.](image)

\[
\tilde{f}_{i+1} = (1 - \lambda a_i) f_i - a \left\{ 2(f_i(x_i) - y_i) \kappa(x_i, \cdot) - \sum_{p=2}^{P} m(p, g_{i+1}) [2(f_i(x_i) - y_i) \kappa(x_i, \cdot) - 2(f_i(x'_i) - y'_i) \kappa(x'_i, \cdot)] \right\}
\]

where \(m(p, g_{i+1}) := [\rho(f_i(x_i) - y_i)^2 - g_{i+1}]^{p-1}\). Note that the gradient bound of the outer function is not bounded in the above mentioned problem, but the gradient of the projected version after applying KOMP will surely be bounded.

We evaluate COLK on synthetic and real data sets whose distributions are skewed or heavy-tailed, and compare its test accuracy against existing benchmarks that minimize only bias. Firstly, we evaluate performance on synthetic data regression outliers which has a heavier tailed distribution, i.e., more outliers are present. We inquire as to which methods overfit versus learn successfully: COLK (Algorithm 2), or methods such as BSGD \([44]\), NPBSG \([45]\), POLK \([16]\).

We generate 20 different sets from the same data distribution and then run both POLK and COLK to learn a regression function. To generate the synthetic dataset regression outliers, we used the function \(y = 2x + 3\sin(6x)\) as the original function and target \(y\)'s observed after adding a zero mean Gaussian noise to \(2x + 3\sin(6x)\). First we generate 60000 samples of the data, and then select 20% as the test data set. From the remaining 48000 samples, we select 50% at random to generate 20 different training sets. We run COLK over these training set with the following parameter selections: a Gaussian kernel with bandwidth \(\sigma = .06\), step-size parameters \(a = 0.02\), \(\beta = 0.01\),
\( \epsilon = K \alpha^2 \) with parsimony constant \( K = 5 \), variance coefficient \( \eta = 0.1 \), and mini-batch size of 1. Similarity, for POLK we use \( \alpha = 0.5 \) and \( \epsilon = K \alpha^2 \) with parsimony constant \( K = 0.09 \). We fix the kernel type and bandwidth across the different methods, and the parameters that define comparator algorithms are hand-tuned to optimize performance with the restriction that their model complexity is comparable to each other. We run these algorithms for different realizations of training data and evaluate their test accuracy as well as its standard deviation.

Before summarizing these results, we present an example sample path of Algorithm 1 for this experiment for the regression outliers data, with and without outliers, in Fig. 1. Specifically, Fig. 1a shows that the mean plus variance of the loss function is minimized as the number of samples processed increases. The time-series of the test-set error of COLK is given in Fig. 1b which converges as the training samples increases. In Fig. 1c we plot the model order of the function sequence defined by COLK, and observe it stabilizes over time regardless of the presence of outliers. These preliminary results validate the convergence results established in Section IV. The advantage of minimizing the bias as well as variance is depicted in Fig. 1d which plots the learned function for POLK and COLK for two training sets. It can be observed that how POLK learning varies from one training set to other while COLK is robust to this change.

We now discuss our experimental results in terms of how COLK compares to existing techniques that only fit to the mean loss. We run COLK as well as the others for 20 total training runs and report the average test error and the standard deviation in the box and whisker plot given in Fig. 2. The box represents average test error and whisker represents the standard deviation of the corresponding estimate. Observe that COLK yields the lowest error as well as the lowest standard deviation, meaning it yields inferences that are both low bias and low variance. To check the proposed algorithm for real data, we consider the performance of filtering laser scans to interpolate range to a target via the lidar data \([26]\) (with added outliers) with the results shown in Fig. 3. It is clear from the figure that the proposed algorithm is robust to outliers in the data.

**VI. CONCLUSION**

In this work, we addressed compositional stochastic programming in Reproducing Kernel Hilbert Space by developing a functional generalization of the stochastic quasi-gradient method operating in parallel with greedy subspace projections. This method, Compositional Online Learning with Kernels (COLK), converges both under attenuating and constant learning rates, and yields memory-efficient parameterizations. Our particular motivation for this problem class comes from formulations of supervised learning which accounts for error variance through coherent risk, as well as function approximation in Markov Decision Problems over continuous spaces. We experimentally observed both with synthetic and benchmark data that COLK applied to robust supervised learning overcomes the problem of overfitting: by accounting for error variance through coherent risk, we observed consistent performance across training runs.

In future work, we hope to investigate the use of hierarchical kernels for larger parameter problems in vision or acoustics, and thus design deep learners that do not overfit. Moreover, modifying the underlying stochastic optimization methods to include, for instance, momentum, may improve the convergence rate or more effectively mitigate the error caused by the lossy compression required to keep the model order under control.

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This result establishes that norm of the difference between the projected and unprojected stochastic quasi-gradient is of the order of the ratio of compression budget \( \epsilon_t \) and step size \( a_t \).

**Proof 1** Consider the square-Hilbert-norm difference of \( \hat{\nabla}_f R(f_t, g_{t+1}; \xi_t, \theta_t) \) and \( \hat{\nabla}_f R(f_t, g_{t+1}; \xi_t, \theta_t) \) defined in (34) and (35), respectively,

\[
\| \hat{\nabla}_f R(f_t, g_{t+1}; \xi_t, \theta_t) - \hat{\nabla}_f R(f_t, g_{t+1}; \xi_t, \theta_t) \|_H^2
\]

Multiply and divide \( \hat{\nabla}_f R(f_t, g_{t+1}; \xi_t, \theta_t) \), the last term, by \( a_t \), and reorder terms to write

\[
\left\| \frac{f_t - a_t \hat{\nabla}_f R(f_t, g_{t+1}; \xi_t, \theta_t)}{a_t} \right\|_H^2
\]

Next, using the upper bounds as per Assumption 2 [cf. (39)], we get the final equation given by \( \| f_{t+1} - f_t \|_H \leq \epsilon_t \).

**VIII. PROOF OF LEMMA 1**

There are three results presented in the statement of Lemma 1. A separate proof for each statement is provided below.

**Proof of Lemma 1**: At the current time instant \( t \), consider the Hilbert-norm difference between the next function iterate \( f_{t+1} \) and current estimate \( f_t \) using the definition of \( f_{t+1} \) in (36), i.e.,

\[
\| f_{t+1} - f_t \|_H^2 = a_t^2 \| \hat{\nabla}_f R(f_t, g_{t+1}; \xi_t, \theta_t) \|_H^2
\]

Subtract and apply the inequality \((a + b)^2 \leq 2a^2 + 2b^2\) which holds for any \( a, b \). Now, we apply the result stated in Proposition 1 to the second term on the right hand side of (36). After performing this, taking the conditional expectation on the filtration \( F_t \) yields

\[
E[\| f_{t+1} - f_t \|_H^2 | F_t] = 2a_t^2 E[\| \hat{\nabla}_f R(f_t, g_{t+1}; \xi_t, \theta_t) \|_H^2 | F_t] + 2\epsilon_t^2.
\]

Next, utilize the definition of the stochastic quasi-gradient \( \hat{\nabla}_f R(f_t, g_{t+1}; \xi_t, \theta_t) \) on the first line of (36) and again using \((a + b)^2 \leq 2(a^2 + b^2)\), we get

\[
E[\| f_{t+1} - f_t \|_H^2 | F_t] \leq 4a_t^2 E\left\{ \| a_t \hat{\nabla}_f R(f_t, g_{t+1}; \xi_t, \theta_t) \|_H^2 \right\} + 2\epsilon_t^2.
\]

Applying the Cauchy-Schwartz inequality to the first term on right hand side of (38) yields

\[
E[\| f_{t+1} - f_t \|_H^2 | F_t] \leq 4a_t^2 E\left\{ \| a_t \hat{\nabla}_f R(f_t, g_{t+1}; \xi_t, \theta_t) \|_H^2 \right\} + 4a_t^2 \epsilon_t^2 + 2\epsilon_t^2.
\]

The second inequality in (39) is obtained from the Law of total expectation and using Assumption 1 [cf. (38)] which implies that \( \| \kappa(\xi_t, \cdot) \|_H^2 \leq U^2 \). Next, using the upper bounds as per Assumption 2 [cf. (39)], we get the final equation

\[
E[\| f_{t+1} - f_t \|_H^2 | F_t] \leq 4a_t^2 U^2 (G_g^2 \cdot G_\theta^2 + \lambda^2 K^2) + 2\epsilon_t^2.
\]
Proof of Lemma \[\text{Lemma 1}\]: This proof is a generalization of Lemma 3 in Appendix G.2 in the Supplementary Material of [2] to a function-valued stochastic quasi-gradient step combined with bias induced by the sparse subspace projections $P_{H_{U+1}}[\cdot]$ in (27). Let us consider the square-Hilbert norm distance of $f_{t+1}$ from the optimal $f^*$, i.e.,

$$\|f_{t+1} - f^*\|^2_H = \|f_t - a_t \nabla f R(f_t, g_{t+1}; \xi_t, \theta_t) - f^*\|^2_H$$

$$= \|f_t - f^*\|^2_H - 2a_t \langle \nabla f R(f_t, g_{t+1}; \xi_t, \theta_t); f_t - f^* \rangle_H + a_t^2 \|\nabla f R(f_t, g_{t+1}; \xi_t, \theta_t)\|^2_H,$$

(91)

where we utilized the reformulation of the function update defined in (36) for the first equality, and expand the square in the second. Now, adding and subtracting $\nabla f R(f_t, g_{t+1}; \xi_t, \theta_t)$ (which is the (un-projected) functional stochastic quasi-gradient (34)) to first term in the inner product on right-hand side of (91) yields

$$\|f_{t+1} - f^*\|^2_H = \|f_t - f^*\|^2_H - 2a_t \langle \nabla f R(f_t, g_{t+1}; \xi_t, \theta_t); f_t - f^* \rangle_H + 2a_t \|\nabla f R(f_t, g_{t+1}; \xi_t, \theta_t)\|^2_H$$

$$+ 2a_t \|\nabla f R(f_t, g_{t+1}; \xi_t, \theta_t)\|^2_H.$$

(92)

Applying the Cauchy-Schwartz inequality to the third inner product term on the right-hand side of (92) and then utilizing the upper bound developed in Proposition 1 we get

$$\|f_{t+1} - f^*\|^2_H = \|f_t - f^*\|^2_H - 2a_t \langle \nabla f R(f_t, g_{t+1}; \xi_t, \theta_t); f_t - f^* \rangle_H$$

$$+ 2a_t \|\nabla f R(f_t, g_{t+1}; \xi_t, \theta_t)\|^2_H + 2\epsilon_t \|f_t - f^*\|_H$$

$$+ \|\nabla f R(f_t, g_{t+1}; \xi_t, \theta_t)\|^2_H.$$

(93)

Now, as defined in Lemma 1 let $\delta_t := \delta_\xi(f(\xi_t))$ with $\delta_t := \mathbb{E} \left[ \delta_t | \theta_t \right]$. Add and subtract $\nabla f R(f_t, \delta_t; \xi_t, \theta_t)$, which is nothing but the stochastic quasi-gradient evaluated at $(f_t, \delta_t)$ rather than $(f_t, g_{t+1})$, inside the inner-product term on the right-hand side of (93), to obtain

$$\|f_{t+1} - f^*\|^2_H = \|f_t - f^*\|^2_H - 2a_t \langle \nabla f R(f_t, \delta_t; \xi_t, \theta_t); f_t - f^* \rangle_H$$

$$+ 2a_t \|\nabla f R(f_t, \delta_t; \xi_t, \theta_t)\|^2_H + 2\epsilon_t \|f_t - f^*\|_H$$

$$+ \|\nabla f R(f_t, \delta_t; \xi_t, \theta_t)\|^2_H.$$

(94)

where we substitute in the definitions of $\nabla f R(f_t, \delta_t; \xi_t, \theta_t)$ and $\nabla f R(f_t, g_{t+1}; \xi_t, \theta_t)$ [cf. (43), (44), respectively] in (24), and cancel out the common regularization term $\lambda f_t$. To proceed further, let us define the directional error term related to stochastic quasi-gradient and the stochastic gradient as

$$v_t = 2a_t \langle \langle \delta_\xi(f(\xi_t)), (\ell'(\delta_t) - \ell'(g_{t+1})); \xi_t, \theta_t \rangle \rangle \|\xi(\xi_t), \cdot \rangle, f_t - f^* \rangle_H.$$

(95)

From here, compute the conditional expectation on the algorithm history $F_t$:

$$\mathbb{E} \left[ \|f_{t+1} - f^*\|^2_H \bigg| F_t \right] = \|f_t - f^*\|^2_H - 2a_t \langle \nabla f R(f_t, \delta_t; \xi_t, \theta_t); f_t - f^* \rangle_H$$

$$+ 2\epsilon_t \|f_t - f^*\|_H + \|\nabla f R(f_t, \delta_t; \xi_t, \theta_t)\|^2_H.$$

(96)

Utilizing the fact that the compositional objective $R(f)$ defined in (1) is convex with respect to $f$ and utilizing the first order convexity condition, we have

$$\langle \mathbb{E} \left[ \nabla f R(f_t, \delta_t; \xi_t, \theta_t) \bigg| F_t \right], f_t - f^* \rangle_H \geq R(f_t) - R(f^*) \geq R(f_t) - R(f^*) \geq 0.$$

(97)

Using the inequality in (77) and Assumption 2 [cf. (40)] into (96), we get

$$\mathbb{E} \left[ \|f_{t+1} - f^*\|^2_H \bigg| F_t \right] = \|f_t - f^*\|^2_H - 2a_t \left( R(f_t) - R(f^*) \right)$$

$$+ 2\epsilon_t \|f_t - f^*\|_H + \|\nabla f R(f_t, g_{t+1}; \xi_t, \theta_t)\|^2_H.$$

(98)

It remains to analyze $v_t$, the directional error associated with using stochastic quasi-gradients rather than stochastic gradients. Proceed by applying the Cauchy-Schwartz inequality to (95), which allows us to write

$$v_t \leq 2a_t \|\nabla \xi(\xi_t)\|_H \|\ell'(\delta_t) - \ell'(g_{t+1})\|_H \|\xi(\xi_t), \cdot \|_H \|f_t - f^*\|_H$$

$$\leq 2a_t \|L_{\xi} \| \|\nabla \xi(\xi_t)\| \|\delta_t - g_{t+1}\| \|f_t - f^*\|_H.$$

(99)
where the second inequality in (99) uses Assumptions 1 [cf. (38)] and 3 [cf. (42)]. Consider Peter-Paul’s inequality $2ab \leq \rho a^2 + b^2 / \rho$ for $\rho, a, b > 0$, which we apply to (99) with \( a = \| \delta_i - g_{i+1} \| \), \( b = \alpha_i(\| k_{x_i} (f(\xi_i))) \| f_i - f^* \| H \), and $\rho = \beta_i$ so that (99) becomes

\[
v_i \leq L_r U^2 \left[ \beta_i \| \delta_i - g_{i+1} \|^2 + L_r U^2 \frac{\alpha_i^2}{\beta_i} \| k_{x_i} (f(\xi_i))) \| f_i - f^* \| H \right].
\]  

(100)

The conditional mean of $v_i$ [cf. (95)], using (100), is then

\[
\mathbb{E} \left[ v_i \mid F_i \right] \leq L_r U^2 \beta_i \mathbb{E} \left[ \| \delta_i - g_{i+1} \|^2 \mid F_i \right] + L_r U^2 \frac{\alpha_i^2}{\beta_i} \mathbb{E} \left[ \| k_{x_i} (f(\xi_i))) \| f_i - f^* \| H \mid F_i \right] \quad \text{(101)}
\]

\[ \leq L_r U^2 \beta_i \mathbb{E} \left[ \| \delta_i - g_{i+1} \|^2 \mid F_i \right] + L_r U^2 \frac{\alpha_i^2}{\beta_i} \Gamma^2 \| f_i - f^* \| H , \]

where we have used the Assumption 1 [cf. (38)]. Now, substitute (101) into the right-hand side of (98) and gather like terms:

\[
\mathbb{E} \left[ \| f_{i+1} - f^* \|_H \mid F_i \right] \leq \left( 1 + L_r U^2 \frac{\alpha_i^2}{\beta_i} \Gamma^2 \right) \| f_i - f^* \|_H + 2 \epsilon_i \| f_i - f^* \|_H
\]

\[ - 2 \alpha_i \mathbb{E} \left[ R(f_i) - R(f^*) \right] + \alpha_i^2 \sigma^2 + L_r U^2 \beta_i \mathbb{E} \left[ \| \delta_i - g_{i+1} \|^2 \mid F_i \right] . \quad \text{(102)}
\]

which is as stated in Lemma 1(ii).

---

**Proof of Lemma 1(ii):** This proof is an adaptation of Lemma 2 in Appendix G.1 in the Supplementary Material of [2] to the recursively averaged sequence $g_i$ defined in (15). Start by defining the vector quantity $e_i$ as the difference of conditional expected value $\delta_i$ of the inner function and $\delta_{i-1}$ scaled by the forgetting factor $\beta_i$, i.e. $e_i = (1 - \beta_i)(\delta_i - \delta_{i-1})$. Then we consider the difference of the evolution of the auxiliary variable $g_{i+1}$ with respect to the conditional mean $\delta_i$, plus the difference of its expected value

\[
g_{i+1} - \delta_i + e_i = (1 - \beta_i) g_i + \beta_i \delta_i - (1 - \beta_i) \delta_{i-1} + \beta_i \delta_i = (1 - \beta_i) (g_i - \delta_{i-1}) + \beta_i (\delta_i - \delta_{i-1}).
\]  

(103)

where we have used the definition of $g_{i+1}$ in (15), the fact that $\delta_i = [(1 - \beta_i) \delta_i + \beta_i \delta_{i-1}]$, and the definition of $e_i$ on the first line of (103). In the second line of (103), we have collected the like terms with respect to factors $(1 - \beta_i)$ and $\beta_i$ and cancel the redundant $\delta_i$ term. Now, square the expression (103) on both sides, we get

\[
\| g_{i+1} - \delta_i + e_i \|^2 = \| (1 - \beta_i) (g_i - \delta_{i-1}) + \beta_i (\delta_i - \delta_{i-1}) \|^2
\]

\[ = (1 - \beta_i)^2 \| g_i - \delta_{i-1} \|^2 + \beta_i^2 \| \delta_i - \delta_{i-1} \|^2 + 2(1 - \beta_i) \beta_i (g_i - \delta_{i-1}, \delta_i - \delta_{i-1}) . \quad \text{(104)}
\]

Now, computing the conditional expectation of (104) on the filtration $F_i$ results in

\[
\mathbb{E}[\| g_{i+1} - \delta_i + e_i \|^2 \mid F_i] = (1 - \beta_i)^2 \mathbb{E}[\| g_i - \delta_{i-1} \|^2 \mid F_i] + \beta_i^2 \mathbb{E}[\| \delta_i - \delta_{i-1} \|^2 \mid F_i]
\]

\[ + 2(1 - \beta_i) \beta_i \mathbb{E}[\langle g_i - \delta_{i-1}, \delta_i - \delta_{i-1} \rangle \mid F_i] . \quad \text{(105)}
\]

Now using the assumption of finite variance of $\delta_i$ (cf. Assumption 2 in (41)), we get

\[
\mathbb{E}[\| g_{i+1} - \delta_i + e_i \|^2 \mid F_i] \leq (1 - \beta_i)^2 \| g_i - \delta_{i-1} \|^2 + \beta_i^2 \sigma^2 .
\]  

(106)

Consider the inequality $\| a + b \|^2 \leq (1 + \rho) \| a \|^2 + (1 + 1/\rho) \| b \|^2$ which holds for any $\rho > 0$: set $a = g_{i+1} - \delta_i + e_i$, $b = -e_i$, and $\rho = \beta_i$. Substituting into (106), we get

\[
\| g_{i+1} - \delta_i \|^2 \leq (1 + \beta_i) \| g_{i+1} - \delta_i + e_i \|^2 + \left( 1 + \frac{1}{\beta_i} \right) \| e_i \|^2 .
\]  

(107)

Now, take the conditional expectation on both sides in (107) and then using the expression from (106), we will get a multiplicative factor of $(1 + \beta_i)$ on the right-hand side of (106) plus the error term $(1 + 1/\beta_i) \| e_i \|^2$, yielding

\[
\mathbb{E}[\| g_{i+1} - \delta_i \|^2 \mid F_i] \leq (1 + \beta_i)((1 - \beta_i) \| g_i - \delta_{i-1} \|^2 + \beta_i^2 \sigma^2) + \left( 1 + \frac{1}{\beta_i} \right) \| e_i \|^2 .
\]  

(108)

Note that $(1 - \beta_i)^2(1 - \beta_i) \leq (1 - \beta_i)$ and applying this to the first term in (108). Similarly, we it holds that $(1 + \beta_i) \beta_i^2 \leq 2 \beta_i^2$ and using this result for the second (since $\beta_i \in (0, 1)$) term of (108) to simplify it as

\[
\mathbb{E}[\| g_{i+1} - \delta_i \|^2 \mid F_i] = (1 - \beta_i) \| g_i - \delta_{i-1} \|^2 + 2 \beta_i^2 \sigma^2 + \left( 1 + \frac{1}{\beta_i} \right) \| e_i \|^2 .
\]  

(109)
After this, it remains to handle the error term $e_t$, which is given by

$$\|e_t\| = (1 - \beta_t)\|D_t - D_{t-1}\| \leq (1 - \beta_t)L \|f_t - f_{t-1}\|_H,$$

where we have applied the Lipschitz continuity of the inner function as mentioned in Assumption 4 [cf. (5)]. Utilizing this upper bound of $\|e_t\|$ into it (109), and using the inequality $(1 - \beta_t)/\beta_t \leq 1/\beta_t$, we get the expression in (46).

IX. CONVERGENCE TO OPTIMAL SET IN THEOREM [1]

Let $\Omega_{f,*}$ be the collection of sample paths such that $\Omega_{f,*} = \{\omega : \lim_{t\to\infty} \|f_t(\omega) - f^*\| = 0\}$, where $\omega$ represents the random variable at which function is evaluated. We just established above that $P(\Omega_{f,*}) = 1$ for any $f^* \in H$. Now the remaining task is to prove that any limiting value function of the sequence $f_t$ is optimal, for which we need to establish that $\cap_{f \in H} \Omega_{f,*}$ is measurable and $P(\cap_{f \in H} \Omega_{f,*}) = 1$.

Since $R$ is a convex function, the set of minimizers of $R$, denoted as $H^* \subset H$, is separable, and has a countably dense subset $H^*_Q$. Thus the probability of divergence for some $f^* \in H^*_Q$ is the probability of a union of countably many sets, each having null probability. Therefore, we may write

$$P\left(\bigcap_{H^*_Q} \Omega_{f,*}\right) = 1 - P\left(\bigcup_{H^*_Q} \Omega_{f,*}\right) \geq 1 - \sum_{f^* \in H^*_Q} P\left(\Omega_{f,*}\right) = 1$$

by simple application of De Morgan’s Law and Boole’s inequality. Then consider any $\bar{f} \in H^*$ which is the limit of a sequence of optimal value functions $\{\bar{f}_k\}_{k=1}^{\infty} \subset H^*$. We can prove that $\|\bar{f}_t(\omega) - \bar{f}\|$ is convergent provided that $\|\bar{f}_t(\omega) - \bar{f}_k\|$ is convergent for all $k$. Note that

$$\|\bar{f}_t(\omega) - \bar{f}_k\|_H - \|\bar{f}_k - \bar{f}\|_H \leq \|f_t(\omega) - \bar{f}\|_H \leq \|f_t(\omega) - \bar{f}_k\| + \|\bar{f}_k - \bar{f}\|_H$$

Then, since $\|f_t(\omega) - \bar{f}_k\|_H$ has a limit, we can take $t \to \infty$ in (112) which yields:

$$\lim_{t \to \infty} \|f_t(\omega) - \bar{f}_k\|_H - \|\bar{f}_k - \bar{f}\|_H \leq \lim_{t \to \infty} \inf_{k \to \infty} \|f_t(\omega) - \bar{f}\|_H$$

which, by subtracting $\lim_{t \to \infty} \|f_t(\omega) - \bar{f}\|_H$ from both sides in (113), cancelling the common limit $\lim_{t \to \infty} \|f_t(\omega) - \bar{f}_k\|_H$, and combining terms, allows us to write

$$\lim_{t \to \infty} \|f_t(\omega) - \bar{f}\|_H - \lim_{t \to \infty} \inf_{k \to \infty} \|f_t(\omega) - \bar{f}_k\|_H \leq \|\bar{f}_k - \bar{f}\|_H$$

Take $k \to \infty$ in (114), for which $\|\bar{f}_t - \bar{f}\|_H \to 0$, yielding

$$\lim_{t \to \infty} \|f_t(\omega) - \bar{f}\|_H = \lim_{t \to \infty} \inf_{k \to \infty} \|f_t(\omega) - \bar{f}_k\|_H,$$

and therefore $\|f_t(\omega) - \bar{f}\|_H$ has a limit, so $\omega \in \Omega_{f,*}$, and therefore $\cap_{H^*_Q} \Omega_{f,*} \subset \Omega_{\bar{f}}$. Consequently, $P\left(\cap_{H^*_Q} \Omega_{f,*}\right) = 1$. As a result, we have $(\cap_{H^*_Q} \Omega_{f,*})^c \subset (\cap_{H^*_Q} \Omega_{f,*})^c$, both of which are measurable and have null probability: $P\left((\cap_{H^*_Q} \Omega_{f,*})^c\right) \leq P\left((\cap_{H^*_Q} \Omega_{f,*})^c\right) = 0$. So therefore $(\cap_{H^*_Q} \Omega_{f,*})$ is measurable and occurs with probability 1. Put another way, $f_t - \bar{f}\|_H$ is convergent for all $f^* \in H^*$ with probability 1.

Now, we can use this fact together with (62), namely, $\lim_{t \to \infty} R(f_t) = R(f^*)$, to establish that $f_t$ converges to the minimizer of $R(f)$ almost surely. To do so, let $f^* \in H^*$ the set of optimizers of $R$. Since $\|f_t(\omega) - f^*\|_H$ converges, it is bounded. Then, $f_t(\omega)$ must have a limit point $\bar{f}$ being an optimal solution, $R(\bar{f}) = R^*$ with $\bar{f} \in H^*$, by the continuity of $R$. Since $\omega \in \cap_{H^*} \Omega_{f,*} \subset \Omega_{\bar{f}}$, $R(\|f_t(\omega) - \bar{f}\|_H)$ is a convergent sequence whose limit is null. Thus, $\|f_t(\omega) - \bar{f}\|_H \to 0$, so $f_t(\omega) \to \bar{f}$ on this sample path. $\bar{f}$ is a random variable dependent on the sample path, parameterized by $\omega$. The set of all such sample paths has probability 1, so that $f_t$ converges to a random point of the set of solutions $H^*$ of (2).

X. PROOF OF LEMMA [3]

Before analyzing the mean convergence behavior of the distance from optimal $E[\|f_{t+1} - f^*\|_H^2]$, consider the following temporal difference between $g_{t+1}$ and $\delta_t$ as stated in (113) and compute total expectation given by

$$E[\|g_{t+1} - \delta_t\|^2] \leq (1 - \beta_t)E[\|g_t - \delta_{t-1}\|^2] + \frac{L_t}{\beta_t}E[\|f_t - f_{t-1}\|_H^2] + 2\beta^2\delta^2,$$
where we have used $\beta_t = \beta$ in (116). Next, substitute the upper bound on the total expectation of $\|f_t - f_{t-1}\|_H^2$ as described in (110) into (116), with constant step-size $a_t = a$ and compression budget $c_t = \epsilon$ to obtain

\[
E \left[ \|g_{t+1} - \bar{\delta}_t\|^2 \right] \leq \frac{(1 - \beta)E}{\|g_t - \bar{\delta}_{t-1}\|^2} + \frac{2L_c}{\beta} \left[ 4a^2U^2(G_h^2G_c^2 + \lambda^2K^2) + 2\epsilon^2 \right] + 2\beta^2\sigma^2_{\delta}. \tag{117}
\]

It is interesting to observe that (117) relates $E \left[ \|g_{t+1} - \bar{\delta}_t\|^2 \right]$ to its previous iterate value. Utilizing this recursion, we can write the following by replacing $t + 1$ by $t$

\[
E \left[ \|g_t - \bar{\delta}_{t-1}\|^2 \right] \leq (1 - \beta)^2E \left[ \|g_{t-1} - \bar{\delta}_{t-2}\|^2 \right] + [1 + (1 - \beta)^2] \left( \frac{2L_c}{\beta} \left[ 4a^2U^2(G_h^2G_c^2 + \lambda^2K^2) + 2\epsilon^2 \right] + 2\beta^2\sigma^2_{\delta} \right). \tag{118}
\]

Substituting (118) into the right-hand side of (117) yields

\[
E \left[ \|g_{t+1} - \bar{\delta}_t\|^2 \right] \leq (1 - \beta)^{t+1}\|g_0 - \bar{\delta}_{-1}\|^2 + \sum_{i=0}^{t-1} (1 - \beta)^i \left( \frac{2L_c}{\beta} \left[ 4a^2U^2(G_h^2G_c^2 + \lambda^2K^2) + 2\epsilon^2 \right] + 2\beta^2\sigma^2_{\delta} \right). \tag{120}
\]

The first term in (120) vanishes due to the initialization $g_0 = 0$ and the convention $\bar{\delta}_1 = 0$. Moreover, the second term represents a geometric series and sum can be evaluated using $\sum_{i=0}^{t-1} (1 - \beta)^i = 1 - (1 - \beta)^t/\beta$ provided $\beta < 1$. In this geometric sum expression, since the numerator is strictly less than unit, we can further simplify (120) to

\[
E \left[ \|g_{t+1} - \bar{\delta}_t\|^2 \right] \leq \frac{2L_c}{\beta^2} \left[ 4a^2U^2(G_h^2G_c^2 + \lambda^2K^2) + 2\epsilon^2 \right] + 2\beta\sigma^2_{\delta} = O \left( \frac{\epsilon^2 + \lambda^2}{\beta^2} + \beta \right). \tag{121}
\]

After establishing this auxiliary sequence order in terms of the order of step sizes $a$ and $\beta$, we shift our focus again to the sub-optimality gap $\|f_t - f^*\|_H$ in expectation. Before proceeding, note that the Hilbert-norm regularizer $(\lambda/2)\|f\|_H^2$ in (2) makes the objective $R(f)$ strongly convex, i.e.

\[
\frac{\lambda}{2}\|f_t - f^*\|_H^2 \leq R(f_t) - R(f^*). \tag{122}
\]

Using the inequality in (122) along with $a_t = \alpha, \beta_t = \beta, \epsilon_t = \epsilon$ into the expression of Lemma (111), we obtain

\[
E \left[ \|f_{t+1} - f^*\|_H \mid F_t \right] \leq \left( 1 + L_cU^2\frac{\alpha^2}{\beta}G_h^2 - \alpha \lambda \right) \|f_t - f^*\|_H^2 + 2\epsilon\|f_t - f^*\|_H + a^2\sigma^2_{\bar{\delta}} + L_cU^2\beta E \left[ \|g_{t+1} - \bar{\delta}_t\|_H^2 \mid F_t \right]. \tag{123}
\]

Taking the total expectation of (123), with compression budget $\epsilon = Ca^2$ for some arbitrary constant $C > 0$, and the fact that $\|f_t - f^*\|_H \leq 2K$, and applying (122) to the last term on the right-hand side of the preceding expression to obtain:

\[
E \left[ \|f_{t+1} - f^*\|_H^2 \right] \leq \left( 1 + L_cU^2\frac{\alpha^2}{\beta}G_h^2 - \alpha \lambda \right) E \left[ \|f_t - f^*\|_H^2 \right] + a^2\sigma^2_{\bar{\delta}} + 4CK \|f_t - f^*\|_H^2 + \frac{2L_c}{\beta} \left[ 4a^2U^2(G_h^2G_c^2 + \lambda^2K^2) + 2\epsilon^2 \right]. \tag{124}
\]

To get the final result in (63), note that we can substitute the regularizer selection $\lambda = L_cU^2G_h^2a/\beta + \lambda_0$ for $\lambda_0 < 1$ into (124).

**XI. PRELIMINARIES FOR THEOREM 4**

Before discussing the finiteness of the model order, we discuss a lemma which helps us to relate the stopping criterion of specification in KOMP to the Hilbert subspace.

**Lemma 4** Let us define the distance of an arbitrary random feature vector $(\xi)$ calculated as $\phi(\xi) = \kappa(\xi, \cdot)$ to $\mathcal{H}_D = \text{span}(\kappa(D_n, \cdot))_{n=1}^M$, the subspace of the Hilbert space spanned by a dictionary $D$ of size $M$, as

\[
dist(\kappa(\xi, \cdot), \mathcal{H}_D) = \min_{f \in \mathcal{H}_D} \|\kappa(\xi, \cdot) - v^T\kappa(\cdot)\|_H. \tag{125}
\]
This set distance simplifies to following least-squares projection when $D \in \mathbb{R}^{P \times M}$ is fixed
\[
dist(\kappa(\xi, \cdot), H_D) = \left\| \kappa(\xi, \cdot) - [K_{D,D}^{-1} \kappa_D(\xi)]^T \kappa_D(\cdot) \right\|_H. \tag{126}
\]

**Proof 2** We can write the distance to a Hilbert space $H_D$ as follows
\[
dist(\kappa(\xi, \cdot), H_D) = \min_{f \in H_D} \| \kappa(\xi, \cdot) - f^T \kappa_D(\cdot) \|_H = \min_{v \in \mathbb{R}^M} \| \kappa(\xi, \cdot) - v^T \kappa_D(\cdot) \|_H, \tag{127}
\]
which is obtained since the dictionary $D$ is fixed and the only free parameter left to optimize is $v$. Now similarly to (26) - (31), we can obtain an optimal weight vector $\tilde{v}^* = K_{d,D}^{-1} \kappa_D(\theta, \xi)$ and then substitute back into (127), we get
\[
dist(\kappa(\xi, \cdot), H_D) = \left\| \kappa(\xi, \cdot) - [K_{D,D}^{-1} \kappa_D(\xi)]^T \kappa_D(\cdot) \right\|_H. \tag{128}
\]

**A. Proof of Theorem 2**

**Proof 3** Consider two arbitrary time instants $t$ and $t + 1$, at which $f_t$ and $f_{t+1}$ are the iterates generated by Algorithm 2 with corresponding model order denoted by $M_t$ and $M_{t+1}$, respectively. We consider a constant step size algorithm with fixed approximation budget $\epsilon = Cu^2$ for some constant $K > 0$. For the sake of analysis, suppose that the model order at $t + 1$ is $M_{t+1} \leq M_t$ which means that model order does not grow as we go to iterate $t + 1$ from $t$. Note that the model order remains the same from $t$ to $t + 1$ if the error introduced by the removal of recently appended data points $(\xi, \gamma)$ to dictionary $D_{t+1} = [D_t; \xi]$ [cf. (29)] is less than the approximation budget $\epsilon$. In other words, the model order does not grow if the stopping criteria of KOMP (Algorithm 1), stated as $\min_{j=1,\ldots,M_t} \gamma_j > \epsilon$, is not satisfied. This leads us to the conclusion that the model order remains the same from $t$ to $t + 1$ if
\[
\min_{j=1,\ldots,M_t+1} \gamma_j \leq \epsilon. \tag{129}
\]

Further observe that the left hand side of (129) is a lower bound for the approximation error $\gamma_{M_{t+1}}$ at $t + 1$ because of the minimization over all $j = 1 \cdots (M_t + 1)$. This states that the error $\gamma_{M_{t+1}}$ introduced by removing the recently appended pair $(\xi, \gamma)$ is such that $\min_{j=1,\ldots,M_t+1} \gamma_j \leq \gamma_{M_{t+1}}$. Therefore, the model order does not grow if $\gamma_{M_{t+1}} \leq \epsilon$ holds because then (129) is satisfied. Let us analyze the error term $\gamma_{M_{t+1}}$ as follows. The definition of $\gamma_{M_{t+1}}$ with the substitution of $f_{t+1}$ in (28) with the notation
\[
V_t' := \langle h_t^{(t)}(f(\xi_t)), \ell_{\theta_t}(\xi_{t+1}) \rangle
\]
allows us to write
\[
\gamma_{M_{t+1}} = \min_{u \in \mathbb{R}^{M_t}} \left\| (1 - \alpha \lambda) f_t - a V_t' \kappa(\xi_t, \cdot) - \sum_{k \in I \setminus \{M_t+1\}} u_k^{(t)} \kappa(\xi_t, \cdot) \kappa(d_k, \cdot) \right\|_H \tag{130}
\]
where we denote the $k^{th}$ data column of $\mathbb{D}_t$ as $d_k$. To minimize the error, consider the square of the term inside the minimization term and then expanding the square, we get
\[
\left\| (1 - \alpha \lambda) \sum_{k \in I \setminus \{M_t+1\}} u_k^{(t)} \kappa(\xi_t, \cdot) \kappa(d_k, \cdot) - a V_t' \kappa(\xi_t, \cdot) \kappa(\xi_t, \cdot) - \sum_{k \in I \setminus \{M_t+1\}} u_k^{(t)} \kappa(\xi_t, \cdot) \kappa(d_k, \cdot) \right\|_H^2 \tag{131}
\]
\[
= (1 - \alpha \lambda)^2 \kappa^T \mathbb{K}_{D_t} \kappa + a^2 V_t'^2 \kappa(\xi_t, \xi_t) + u^T \mathbb{K}_{D_t} u - 2(1 - \alpha \lambda) \eta u^T \kappa(\xi_t, \xi_t) - 2(1 - \alpha \lambda)^2 \kappa^T \mathbb{K}_{D_t} \kappa \mathbb{K}_{D_t} u. \]

To obtain the value $\tilde{u}^*$ which minimizes the above expression, we calculate the gradient of above expression with respect to $u$ and set it to zero. Following the similar logic to that of (26) - (31), we get the following result
\[
\tilde{u}^* = (1 - \alpha \lambda) \kappa - a V_t' \mathbb{K}_{D_t}^{-1} \kappa(\xi_t) \tag{132}
\]
Next, utilizing the optimal value $\tilde{u}^*$ from (132) into the expression in (130) along with the short-hand notation $f_t(\cdot) = \mathbb{K}_d(\cdot)$ and $\sum_k u_k^{(t)} \kappa(d_k, \cdot) = u^T \mathbb{K}_d(\cdot)$, we get
\[
\left\| (1 - \alpha \lambda) \kappa^T \mathbb{K}_d(\cdot) - a V_t' \kappa(\xi_t, \cdot) \kappa(\xi_t, \cdot) - u^T \kappa(\xi_t, \cdot) \right\|_H \tag{133}
\]
\[
= \left\| (1 - \alpha \lambda) \kappa^T \mathbb{K}_d(\cdot) - a V_t' \kappa(\xi_t, \cdot) \kappa(\xi_t, \cdot) - u^T \kappa(\xi_t, \cdot) \right\|_H \tag{134}
\]
\[
= \left\| (1 - \alpha \lambda) \kappa^T \mathbb{K}_d(\cdot) - a V_t' \kappa(\xi_t, \cdot) \kappa(\xi_t, \cdot) - [(1 - \alpha \lambda) \kappa - a V_t' \mathbb{K}_{D_t}^{-1} \kappa(\xi_t)] \kappa(\xi_t, \cdot) \right\|_H \tag{135}
\]
Further simplifying the above expression by cancelling the similar terms \((1 - \alpha)w^T \kappa_{D_i}(\cdot)\) and taking the common term \(a|V_t|\) outside the norm as

\[
\left\| -aV_t \kappa(\xi_i, \cdot) \kappa(\xi_i, \cdot) + aV_t [K_{D_i,D_i}^{-1} \kappa_D(\xi_i)]^T \kappa_{D_i}(\cdot) \right\|_H
= a|V_t| \left\| \kappa(\xi_i, \cdot) \kappa(\xi_i, \cdot) - [K_{D_i,D_i}^{-1} \kappa_D(\xi_i)]^T \kappa_{D_i}(\cdot) \right\|_H .
\]

(134)

It is remarked that the norm expression in the right hand side of (134) describes the distance to the subspace \(\mathcal{H}_{D_i}\) as described in (126) and defined in Lemma 4 with a scaling factor of \(a|V_t|\). The right hand side of (134) can be written as

\[
\alpha|V_t| \left\| \kappa(\xi_i, \cdot) \kappa(\xi_i, \cdot) - [K_{D_i,D_i}^{-1} \kappa_D(\xi_i)]^T \kappa_{D_i}(\cdot) \right\|_H = \alpha|V_t| \text{dist}(\kappa(\xi_i, \cdot) \kappa(\xi_i, \cdot), \mathcal{H}_{D_i}) .
\]

(135)

where the result in (126) on the right hand side of (135) to replace the Hilbert-norm term. Observe that when the stopping criteria of KOMP is violated, (129) holds and thus \(\gamma_{M+1} \leq \varepsilon\). Therefore, we have that the right-hand side of (135) will be upper-bounded by \(\varepsilon\), and we can write the following inequality for \(\varepsilon = C\alpha^2\) as

\[
\frac{\text{dist}(\kappa(\xi_i, \cdot) \kappa(\xi_i, \cdot), \mathcal{H}_{D_i})}{|V_t|} \leq \frac{C\alpha}{|V_t|} .
\]

(136)

The error associated with the model order \(M_{t+1}\) is denoted by \(\gamma_{M_{t+1}}\). Observe that if (136) holds, then \(\gamma_{M_{t+1}} \leq \varepsilon\) holds, but since \(\gamma_{M_{t+1}} \geq \min_{j} \gamma_j\), we may conclude that (129) is satisfied. Consequently the model order at the subsequent step does not grow \(M_{t+1} \leq M_t\) whenever (136) is valid.

Now, consider the contrapositive of the preceding expressions. Observe that the model order growth condition \((M_{t+1} = M_t + 1)\) implies that

\[
\frac{\text{dist}(\kappa(\xi_i, \cdot) \kappa(\xi_i, \cdot), \mathcal{H}_{D_i})}{|V_t|} \geq \frac{C\alpha}{|V_t|} .
\]

(137)

holds. This condition establishes the fact that every time a new data \(\xi\) is appended to kernel dictionary, then the associated product kernel is guaranteed to be at least a distance of \(\frac{C\alpha}{|V_t|}\) from every other kernel function in the current model.

Now utilizing the Cauchy Schwartz inequality and Assumption 5 we get

\[
|V_t| \leq \|f_{\xi}^{(t)}(g_{t+1})(\xi)\|_{\mathcal{H}_d} \|f_{\xi}(\xi)\|_{\mathcal{H}_d} \leq C_{F} C_{H} .
\]

This upper bound implies that \(1/|V_t| \geq 1/(C_{F} C_{H})\), therefore we can lower bound the right hand side in (137) as follows

\[
\frac{C\alpha}{|V_t|} \geq \frac{C\alpha}{C_{F} C_{H}} .
\]

(138)

Therefore, the KOMP stopping criterion is violated for the newest point whenever distinct dictionary points \(\mathbf{d}_k\) and \(\mathbf{d}_j\) for \(j, k \in \{1, \ldots, M_t\}\), satisfy the condition \(\|\phi(\mathbf{d}_j) - \phi(\mathbf{d}_k)\|_2 > \frac{C\alpha}{C_{F} C_{H}}\). Next, we proceed in a similar manner to that of Theorem 3.1 in [46]. Note that since the space \(\mathcal{U}\) is compact and \(\kappa\) is continuous, the range \(\phi(\mathcal{U})\) (where \(\phi(\mathbf{u}) = \kappa(\mathbf{u}, \cdot)\) for \(\mathbf{u} \in \mathcal{U}\)) of the kernel transformation of feature space \(\mathcal{U}\) is compact. This allows us to conclude that the number of balls of radius \(\chi\) (here, \(\chi = \frac{C\alpha}{C_{F} C_{H}}\) required to completely cover the set \(\phi(\mathcal{U})\) is finite (see, e.g., [47]). Thus, for some finite \(M_{\infty}\), if \(M_t = M_{\infty}\), the condition in (137) is satisfied, which implies (129) is true for all \(t\). This leads us to the conclusion that that \(M_t \leq M_{\infty}\) for all \(t\).