Compactly supported solution of the time-fractional porous medium equation on the half-line
Łukasz Płociniczak, Mateusz Świtała

Abstract
In this work we prove that the time-fractional porous medium equation on the half-line with Dirichlet boundary condition has a unique compactly supported solution. The approach we make is based on a transformation of the fractional integro-differential equation into a nonlinear Volterra integral equation. Then, the shooting method is applied in order to facilitate the analysis of the free-boundary problem. We further show that there exists an exactly one choice of initial conditions for which the solution has a zero which guarantees the no-flux condition. Then, our previous considerations imply the unique solution of the original problem.

Keywords: fractional derivative, porous medium equation, compact support, existence, uniqueness

1 Introduction
In the last few decades an interest in the fractional calculus increased significantly. This can be seen, for instance, in a profound development of theoretical aspects of this branch and its utility as a powerful tool in modelling many physical phenomena. A good example is anomalous diffusion, which has been deeply explored recently [9]. Many experiments have been performed and their results indicate either sub- or superdiffusive character of various processes. For instance, apart from its emergence in biomechanical transport [8, 10] and condensed matter physics [11], the anomalous diffusion is also present in percolation of some porous media [3, 7, 18].

For us, the relevant physical experiment is based on a moisture imbibition in certain materials. For some specimens water diffuses at a slower pace than in the classical situation and the usual porous medium equation is not adequate to model this correctly [3, 2]. To deal with this problem an approach based on the modelling of the waiting times distribution has been proposed and a time-fractional porous medium equation has been introduced to successfully to describe the subdissusive version of the process [4, 12, 20]. In our previous works [15, 13, 14] we stated the main assumptions of the model and proved a number of its mathematical properties.

By \( u = u(x, t) \) we denote the (nondimensional) moisture concentration at a point \( x \geq 0 \) and time \( t \geq 0 \). We consider the following nonlocal PDE

\[
\partial_t^\alpha u = (u^m u_x)_x, \quad 0 < \alpha < 1, \quad m > 1, \tag{1}
\]

where the temporal derivative is of the Riemann-Liouville type

\[
\partial_t^\alpha u(x, t) = \frac{1}{\Gamma(1 - \alpha)} \frac{\partial}{\partial t} \int_0^t (t - s)^{-\alpha} u(x, s)ds. \tag{2}
\]

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In [14] the Reader can find the derivation of the above equation as a consequence of the trapping phenomenon. The initial-boundary conditions are as follows
\[
\begin{align*}
    u(x, 0) &= 0, \quad u(0, t) = 1, \quad x > 0, \quad t > 0,
\end{align*}
\]
which models a one-dimensional semi-infinite medium with the interface kept in a contact with water. Since the equation [11] is degenerate at \( u = 0 \) it is natural to expect that its solution has a compact support for at least some values of \( m \) (as in the classical case [1]). This has a straightforward physical meaning, namely the wetting front propagates at a finite speed.

In the previous work [16] we have proved that (1) possess a unique weak compactly supported solution in the class of sufficiently almost everywhere smooth functions which have a zero at some point. This paper presents a result that weakens those assumptions and leaves only the requirement of physically motivated boundedness.

2 Main Result

In this work we will consider the equation of anomalous diffusion in the self-similar form (for more details see [13, 14]). The transformation leading to it can be commenced by putting \( \eta = xt^{-\alpha/2} \) for some \( 0 < \alpha \leq 1 \) and denoting \( u(x, t) = U(xt^{-\alpha/2}) \). In the previous work [16] we proved the existence and uniqueness of the weak compactly supported solution of the resulting transformed problem
\[
(U^m U')' = \left[ (1 - \alpha) - \frac{a}{2} \eta \frac{d}{d\eta} \right] I^{0,1-\alpha}_{\frac{1}{2}} U(\eta), \quad U(0) = 1, \quad U(\infty) = 0, \quad m > 1, \quad \beta > 0,
\]
under the a-priori assumption that this solution has a zero, i.e. \( U(\eta^*) = 0 \) for some \( \eta^* > 0 \). Here, the Erdélyi-Kober operator \( I^{a,b,c} \) (see [5, 6, 19]), is defined by
\[
I^{a,b,c}_{\alpha} U(\eta) = \frac{1}{\Gamma(b)} \int_0^1 (1 - s)^{b-1} s^a U(s^{1/\alpha}\eta) ds.
\]
The main result of this work is to prove that such \( \eta^* \) exists under the assumption that \( U \) is bounded. To ensure the uniqueness of \( \eta^* \) the auxiliary condition needs to be posed, namely
\[
\lim_{\eta \to \eta^*} -U(\eta)^m U'(\eta) = 0.
\]
Physically, this is simply the no-flux requirement through the wetting front and can be derived from the equation alone (see [16]).

In what follows we will consider \( \alpha \) and \( m \) to be a fixed constants. To prove our statement we will consider the following auxiliary problem
\[
(U^m U')' = \left[ (1 - \alpha) - \frac{a}{2} \eta \frac{d}{d\eta} \right] I^{0,1-\alpha}_{\frac{1}{2}} U(\eta), \quad U(0) = 1, \quad U'(0) = -\beta, \quad m > 1,
\]
where \( \beta \) is a positive constant. Our technique is based on the fact that the above integro-differential problem can be transformed into an Volterra equation.

**Proposition 1.** If \( U(\eta) \) is a continuous solution of
\[
U(\eta)^{m+1} = 1 + (m + 1) \left[ -\beta \eta + \int_0^\eta ((1 - \frac{a}{2})\eta - z) I^{0,1-\alpha}_{\frac{1}{2}} U(z) \right] d\eta,
\]
then it is twice-differentiable and is a solution of (7).
Proof. Assume that \( U(\eta) \) is a solution of \([8]\). The right-hand side of \([8]\) is \( C^2 \) since \( I_{\frac{0}{\alpha}}^{1-\alpha}U \) is continuous. Then, the left-hand side is also \( C^2 \) since \( m > 1 \). Differentiating twice the equation \([8]\) we get \([7]\). Thus \( U(\eta) \) is also a solution of \([7]\).

Firstly we will prove that the solution of \([8]\) exists. To do this we need to introduce the function space in which we will operate

\[ X := \{ U \in C[0, \infty] : 0 \leq U \leq 1 \}, \tag{9} \]

with the uniform norm, i.e. \( \| U \| = \sup_{0 \leq t \leq \infty} |U(t)| \). Next, we introduce the function space \( M \),

\[ M := \{ U \in C[0, \infty] : U(0) = 1, 0 \leq U \leq 1 \}, \tag{10} \]

which is subspace of \( X \) and in which the solution of \([8]\) will be sought. It is easy to see that \( X \) is a Banach space (\( X \) is subspace of \( B[0, \infty] \), the space of bounded functions, and is closed). For the same reason \( M \) is also a Banach space. Moreover, we can make the following simple observation.

**Proposition 2.** The subspace \( M \subset X \) is bounded and convex.

Proof. Let \( \gamma \in (0, 1) \), \( u, v \in M \) and introduce the function \( w(x) = \gamma u(x) + (1 - \gamma)v(x) \). From definition of \( M \) we know that \( u(0) = 1 \), \( v(0) = 1 \) and \( 0 \leq u \leq 1 \), \( 0 \leq v \leq 1 \). From properties continuous functions the \( w \) is also a continuous function. Next, if \( u(0) = 1 \) and \( v(0) = 1 \) then \( w(0) = \alpha u(0) + (1 - \beta) u(0) = \gamma + (1 - \gamma) = 1 \). And the last properties to show is the boundedness

\[ \gamma \cdot 0 + (1 - \gamma) \cdot 0 = 0 \leq w = \gamma u + (1 - \gamma)v \leq \gamma + 1 - \gamma = 1. \tag{11} \]

Hence, \( w \in M \) which implies that \( M \) is convex.

In order to state the main result we have to construct an appropriate integral operator and show that it possesses a fixed point. First, define the auxiliary operator \( S_\beta : M \to M \) (the well-definiteness will be settled in the following lemma)

\[ S_\beta(Y) = 1 + (m + 1) \left[ -\beta \eta + \int_0^\eta ((1 - \frac{\eta}{2}) - z) I_{\frac{\eta}{\alpha}}^{1-\alpha} Y(z)^{1/(1+m)} \, dz \right], \tag{12} \]

then the equation \([8]\) is equivalent to the fixed-point problem for \( S \)

\[ S_\beta(Y) = Y, \tag{13} \]

what can be seen by the substitution \( Y = U^{m+1} \). Now, we can prove several important properties of \( S \).

**Lemma 1.** Assume that

\[ \beta \geq \frac{2 - \alpha}{\sqrt{2\Gamma(2-\alpha)(m+1)}} =: \beta_0, \tag{14} \]

then for \( Y \in M \) the following holds.

1. (Existence of a zero) There exists \( \eta^*(\beta) \) such that \( S_\beta(Y)(\eta^*(\beta)) = 0 \). Moreover,

\[ \eta_2(\beta) \leq \eta^*(\beta) \leq \eta_1(\beta), \tag{15} \]

where

\[ \eta_1(\beta) := \frac{4 \beta \Gamma(2-\alpha)}{(2-\alpha)^2} - \frac{2 \sqrt{\beta^2 (m+1) \Gamma(2-\alpha)^2} - (2-\alpha) \Gamma(3-\alpha)}{(2-\alpha)^2 \sqrt{m+1}}, \]

\[ \eta_2(\beta) := \frac{2 \sqrt{\Gamma(2-\alpha) (\alpha^2 + 2 \beta^2 (m+1) \Gamma(2-\alpha))}}{\alpha^2 \sqrt{m+1}} - \frac{4 \beta \Gamma(2-\alpha)}{\alpha^2}. \tag{16} \]

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2. (Estimates) We have the following estimates

\[ g_2(\eta, \beta) \leq S_\beta(Y)(\eta) \leq g_1(\eta, \beta), \]  

where

\[
\begin{align*}
g_1(\eta, \beta) &:= 1 + (m + 1) \left( -\beta \eta + \frac{(2 - \alpha)^2 \eta^2}{\Gamma(2 - \alpha)} \right), \\
g_2(\eta, \beta) &:= 1 + (m + 1) \left( -\beta \eta - \frac{\alpha^2 \eta^2}{\Gamma(2 - \alpha)} \right). 
\end{align*}
\]  

(17)

(18)

3. (Range) The operator \( S_\beta \) maps \( M \) into itself.

Proof. Let us take \( Y \in M \) and check when the integral operator \( S \) is well-defined. From the definition of the space \( M \) we can write an inequality for the Erdélyi-Kober operator

\[
0 \leq I_0^{0,1-\alpha} U(\eta) = \frac{1}{\Gamma(1-\alpha)} \int_0^1 (1-s)^{-\alpha} Y(s^{-\frac{\alpha}{2}} \eta)^{1/(1+m)} ds \leq \frac{1}{\Gamma(2-\alpha)}, \]  

(19)

which we will use in further parts of the proof. The function \( S_\beta(Y)(\eta) \) can be estimated from the above by

\[
S_\beta(Y)(\eta) \leq 1 + (m + 1) \left[ -\beta \eta + \frac{1}{\Gamma(2-\alpha)} \int_0^1 \left( \left(1 - \frac{\alpha}{2} \eta - z \right) dz \right)^{(1-\frac{\alpha}{2})} \right],
\]  

(20)

what can be integrated to yield \( S_\beta(Y)(\eta) \leq g_1(\eta, \beta) \). On the other hand, in the same way we can find the lower limit of \( S_\beta(Y)(\eta) \)

\[
S_\beta(Y)(\eta) \geq 1 + (m + 1) \left[ -\beta \eta - \frac{1}{\Gamma(2-\alpha)} \int_0^\eta \left( \left(1 - \frac{\alpha}{2} \eta - z \right) dz \right)^{(1-\frac{\alpha}{2})} \right],
\]  

(21)

which implies that \( S_\beta(Y)(\eta) \geq g_2(\eta, \beta) \).

Now, notice that \( g_1 \) is decreasing in the interval \([0, \eta^*(\beta)]\) and attains its maximal value equal to 1 at \( \eta = 0 \). Hence, we get that \( S_\beta(Y)(\eta) \leq 1 \). Also, from the definition we immediately have \( S_\beta(Y)(0) = 1 \) and thus \( S_\beta(Y) \in M \).

We see that both, \( g_1 \) and \( g_2 \) are quadratic functions so it is straightforward (but tedious) to find their positivity intervals. We get that \( g_1(\eta, \beta) \geq 0 \) for \( \eta \leq \eta_1(\beta) \) and the function \( g_2(\eta, \beta) \geq 0 \) for \( \eta \leq \eta_2(\beta) \). Further, we can see that \( \eta_2 \) is always greater than zero. The square-root in \( \eta_1(\beta) \) is real only when

\[
\beta \geq \frac{2 - \alpha}{\sqrt{2\Gamma(2-\alpha)(m + 1)}},
\]  

(22)

what is satisfied by the assumption. Hence, due to the continuity of \( g_1 \), \( g_2 \) and \( S_\beta(Y)(\eta) \) we can conclude the existence of at least one \( \eta^*(\beta) \) which satisfies inequalities.

\[
\eta_2(\beta) \leq \eta^*(\beta) \leq \eta_1(\beta).
\]  

(23)

This concludes the proof. \( \square \)

We can now see that for the appropriate values of \( \beta \) the operator \( S_\beta(Y) \) is well-defined on \( M \). We can now state the main existence result.
Theorem 1. For $\beta \geq \beta_0$ the equation \cite{8} has at least one nonnegative compactly supported solution $Y \in M$. Moreover, $\text{supp } Y = [0, \eta^* (\beta)]$.

Proof. It is convenient to introduce yet another operator which maps $M$ into itself

$$A_\beta(Y) = \begin{cases} 1 + (m + 1) \left[ -\beta \eta + \int_0^1 ((1 - \frac{\alpha}{2}) \eta - z) I_{-\frac{2}{\alpha}}^{0,1-\alpha} Y(z) \frac{1}{(1+m)} \, dz \right], & \text{for } \eta \leq \eta^* (\beta), \\ 0, & \text{for } \eta > \eta^* (\beta), \end{cases}$$

where $\eta^*$ is the smallest argument that satisfies the equation below (Lemma 1 assures its well-definiteness)

$$1 + (m + 1) \left[ -\beta \eta^* (\beta) + \int_0^{\eta^* (\beta)} ((1 - \frac{\alpha}{2}) \eta^* (\beta) - z) I_{-\frac{2}{\alpha}}^{0,1-\alpha} Y(z) \frac{1}{(1+m)} \, dz \right] = 0. \tag{25}$$

Our original problem is thus reduced to showing the existence of a solution of the following

$$A_\beta(Y) = Y, \tag{26}$$

where operator $A$ is defined above.

From Proposition \cite{2} we have that $M$ is bounded, closed and convex. Moreover, $M \subset X$, i.e. $M$ is a subspace of a Banach space. The kernel of integral operator $A$ is continuous, hence using the Theorem 3.4 from \cite{17} we conclude that the operator $A$ is completely continuous. Conclusively, we can use Schauder’s theorem \cite{17} \cite{21}, which says that equation \cite{26} has a solution. The form of the compact support comes from the definition of $A_\alpha$. \hfill \qed

As for now we know that \cite{8} has a bounded solution which possesses a zero provided that $\beta$ is large enough. It is also true that inside the interval of the admissible $\beta$ there exists exactly one value if of such for which the no-flux condition \cite{6} is satisfied.

Theorem 2. There exists a value of $\beta$ (and hence $\eta^* (\beta)$), for which the solution of \cite{8} satisfies \cite{6}.

Proof. Assume that $U \in M$ and differentiate the equation \cite{8} to get

$$U(\eta^* (\beta))^m U'(\eta^* (\beta)) = -\beta + \left(1 - \frac{\alpha}{2}\right) \int_0^{\eta^* (\beta)} I_{-\frac{2}{\alpha}}^{0,1-\alpha} U(z) \, dz - \frac{\alpha}{2} \eta^* (\beta) I_{-\frac{2}{\alpha}}^{0,1-\alpha} U(\eta^* (\beta)). \tag{27}$$

Since $U(\eta) = 0$ for $\eta \geq \eta^* (\beta)$ we have

$$\frac{\alpha}{2} \eta^* (\beta) I_{-\frac{2}{\alpha}}^{0,1-\alpha} U(\eta^* (\beta)) = 0, \tag{28}$$

which, along with the former formula, implies

$$U(\eta^* (\beta))^m U'(\eta^* (\beta)) = -\beta + \left(1 - \frac{\alpha}{2}\right) \int_0^{\eta^* (\beta)} I_{-\frac{2}{\alpha}}^{0,1-\alpha} U(z) \, dz. \tag{29}$$

Now, we estimate the magnitude of $U(\eta^* (\beta))^m U'(\eta^* (\beta))$. Using the boundedness of the function $U$ we have

$$U(\eta^* (\beta))^m U'(\eta^* (\beta)) \leq -\beta + \left(1 - \frac{\alpha}{2}\right) \frac{1}{\Gamma(2-\alpha)} \eta_1 (\beta) =: f_+(\beta). \tag{30}$$
On the other hand, we can write

\[
U(\eta^*(\beta))^m U'(\eta^*(\beta)) \geq -\beta + \frac{1 - \frac{\alpha^2}{2}}{\Gamma(2 - \alpha)} \int_0^{\eta_2(\beta)} \left(1 + (m + 1) \left(-\beta z - \frac{\alpha^2}{\Gamma(2 - \alpha)} z^2 \right) \right) \frac{1}{1+m} \, dz
\]  

\[=: f_-(\beta). \]  

(31)

The above estimates can be written as

\[f_-(\beta) \leq U(\eta^*(\beta))^m U'(\eta^*(\beta)) \leq f_+(\beta). \]  

(32)

Next, we compute the limits \( \lim_{\beta \to \infty} f_\pm(\beta) \). It is easy to see that

\[\lim_{\beta \to \infty} \eta_{1,2}(\beta) = 0, \]  

(33)

and hence \( \lim_{\beta \to \infty} f_\pm(\beta) = -\infty \). Moreover,

\[f_+(\beta_0(m)) = \frac{\alpha}{\sqrt{2} \sqrt{(m + 1)\Gamma(2 - \alpha)}} > 0, \]  

(34)

We further see that \( \eta_2(0) > 0 \) so consequently \( f_-(0) > 0 \). Functions \( f_\pm \) are \( \beta \)-decreasing and continuous. Therefore, since \( f_\pm(\beta) \) change their sign when \( \beta \) increases from the Darboux Theorem there exists a \( \beta^* \) (and hence \( \eta^* \)) such that the condition \( U(\eta^*)^m U'(\eta^*) = 0 \) is satisfied.

Equation (8) has thus a unique solution that belongs to \( M \), satisfies (6) and has a zero. Now, we can invoke our previous results [16] to conclude that (4) has a unique bounded solution.

**Collorary 1** ([16], Corollary 1). Let \( U \) be a bounded weak solution of (4) such that \( 0 \leq U(\eta) \leq 1 \). Then,

- it is compactly supported with supp \( U = [0, \eta^*] \) for a unique \( \eta^* > 0 \),
- it is unique,
- it is twice differentiable in the neighbourhood of \( \eta \) such that \( U(\eta) > 0 \),
- it is monotone decreasing.

3 Conclusion

In this paper we proved that the solution of (4) has a compact support. We proceeded by reducing our problem to an auxiliary equation to which a shooting method was applied. One of the most important steps was to transform the fractional integro-differential equation into a nonlinear Volterra integral and choose an appropriate function space, where the solution of (7) was sought. To find a unique \( \eta^* \) we posed the no-flux condition (6) which arises from both physical and mathematical reasons. Based on our current results and theorems provided in [16], we can say that solution of (4) exist, is unique compactly supported decreasing function. This result rigorously confirms the obvious physical observation and, additionally, gives some useful estimates on the wetting front.

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