Integrability of the odd eight-vertex model with symmetric weights

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Abstract

In this paper we investigate the integrability properties of a two-state vertex model on the square lattice whose microstates at a vertex always have an odd number of incoming or outcoming arrows. This model was named the odd eight-vertex model by Wu and Kunz (2004 J. Stat. Phys. 116 67) to distinguish it from the well-known eight-vertex model possessing an even number of arrow orientations at each vertex. When the energy weights are invariant under arrow inversion we show that the integrable manifold of the odd eight-vertex model coincides with that of the even eight-vertex model. The form of the $R$-matrix for the odd eight-vertex model is however not the same as that of the respective Lax operator. Altogether we find that these eight-vertex models give rise to a generic sheaf of $R$-matrices satisfying the Yang–Baxter equations resembling intertwiner relations associated to equidimensional representations.

Keywords: eight-vertex models, Yang–Baxter equations, integrability

1. Introduction

The name vertex model is used to denote a lattice model in which the statistical configurations sit on each line connecting a pair of nearest neighbor sites of the lattice. This type of model first emerged in the discussion by Pauling of the residual entropy of ice [1] and afterwards in the study of certain phase transitions exhibited by hydrogen-bonded crystals [2]. In these situations the microstates are characterized by two possible positions of the hydrogens commonly represented by incoming and outcoming arrows placed along the lattice links [3]. It turns out that on the square lattice we have 16 vertex possibilities and these configurations can be organized in terms of two distinct families of eight-vertex states according to the even or odd number arrow orientations at a vertex. When the number of in and out arrows are even we have the standard eight-vertex model whose vertices are shown in figure 1. The corresponding configurations with an odd number of arrows are exhibited in figure 2.
The even eight-vertex model is known to have an integrable manifold when the weights are invariant under inversion of all arrows at each vertex [4]. In this situation the energy weights are symmetric,

\[ w_1 = w_2 = a, \quad w_3 = w_4 = b, \quad w_5 = w_6 = c, \quad w_7 = w_8 = d. \]  

(1)

In this case Baxter [4] found that there exists a family of commuting transfer matrices provided the weights are lying on the following intersection of quadrics,

\[ \frac{ab - cd}{ab + cd} = \Gamma \quad \text{and} \quad \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)} = \Delta, \]

(2)

where \( \Gamma \) and \( \Delta \) are free parameters.

A natural question to be asked is whether or not one can also build commuting transfer matrices for the odd eight-vertex model on the subspace of symmetric weights, namely

\[ v_1 = v_2 = a, \quad v_3 = v_4 = b, \quad v_5 = v_6 = c, \quad v_7 = v_8 = d, \]

(3)

and what is the respective algebraic variety constraining the weights \( a, b, c \) and \( d \). At this point we use bold letters to distinguish the weights of the symmetric odd eight-vertex model from those of the even eight-vertex model.

In this paper we shall argue that both even and odd symmetric eight-vertex models are integrable when their weights are lying on the same algebraic curve (2). Altogether they give rise to a sheaf of \( R \)-matrices satisfying the Yang–Baxter equations with a structure similar to the intertwiner relations of two distinct representations of a given algebra. The common invariance here is the symmetry of the weights upon the inversion of the arrows at a vertex.

We start the next section recalling a result by Wu and Kunz [5] who have shown that the partition function of the odd eight-vertex model can be expressed as the partition function of an alternating bipartite even eight-vertex model. This equivalence can be reversed and the partition function of the even eight-vertex model can also be written as that of a staggered odd eight-vertex model. These mappings for symmetric weights suggest that both even and odd eight-vertex may be integrable on the same algebraic manifold. This is elaborated on in section 3 where we show that the transfer matrices of the even and odd eight-vertex models with symmetric weights are in fact related. We then study the Yang–Baxter equation for the symmetric odd eight-vertex uncovering the structure of the respective \( R \)-matrix which turns
out to be distinct from that of the Lax operator. This guided us to propose the form of four types of $R$-matrices and to show that they satisfy a set of Yang–Baxter relations on the same elliptic curve. For the reader’s convenience this result has been stated in equations (19)-(21). In section 4 we comment on the possibility of integrability in the case of asymmetric weights.

2. Staggered eight-vertex models

In the staggered eight-vertex model there are different vertex weights for the two possible sublattices of a $2N \times 2N$ square lattice. This is schematized in figure 3 in which the sublattices are represented by the symbols $X$ and $Y$. For the sublattice $X$ we assign the weights $\omega_X$ while for the sublattice $Y$ the weights are $\omega_Y$.

Let us denote by $Z_{8ev}^{stag}(w_X;w_Y)$ the partition function of the staggered model built using the weights configurations of the even eight-vertex model. From the work by Wu and Kunz [5] one concludes that the partition of the odd eight-vertex model can be rewritten as follows

$$Z_{8od}(v_1, \ldots, v_8) = Z_{8ev}^{stag} \left( \begin{array}{c} v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 \\ X \end{array} ; \begin{array}{c} v_3, v_4, v_1, v_2, v_8, v_7, v_6, v_5 \\ Y \end{array} \right),$$

where the weights on the right-hand side of equation (4) are organized as in figure 1 with the identification $w_i = v_i$.

This equivalence was used to compute the free-energy of the odd eight-vertex model when the weights satisfy the free-fermion condition [5],

$$v_1v_2 + v_3v_4 - v_5v_6 - v_7v_8 = 0,$$

but the possibility of building commuting transfer matrices for finite lattices either within the restriction (5) or for any other manifold has not yet been studied in the literature.

The same reasoning devised by Wu and Kunz [5] can be used to relate the partition function of the even eight-vertex model with that of an alternating odd eight-vertex model. In other words, one can also establish the following correspondence,

$$Z_{8ev}(w_1, \ldots, w_8) = Z_{8od}^{stag} \left( \begin{array}{c} w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8 \\ X \end{array} ; \begin{array}{c} w_3, w_4, w_1, w_2, w_6, w_7, w_5 \\ Y \end{array} \right),$$

where now the right-hand side weights in equation (6) are read off according to figure 2 with $v_i = w_i$.

The above relationships become simpler when the weights of both models are symmetric. Taking into account the identifications (1,3) we obtain,

$$Z_{8od}(a, \ldots, d) = Z_{8ev}^{stag} \left( \begin{array}{c} a, b, c, d \\ X \end{array} ; \begin{array}{c} b, a, d, c \\ Y \end{array} \right),$$

as well as,

$$Z_{8ev}(a, \ldots, d) = Z_{8od}^{stag} \left( \begin{array}{c} a, b, c, d \\ X \end{array} ; \begin{array}{c} b, a, d, c \\ Y \end{array} \right).$$

1 Here we also add that the partition function of the odd eight-vertex model for periodic boundary conditions on $N \times N$ square lattice is always zero when the number of sites $N$ is odd.
We now note that the integrable manifold (2) of the symmetric even eight-vertex is invariant by the weights exchange \( a \leftrightarrow b \) and \( c \leftrightarrow d \). This fact together with the equivalences (7) and (8) suggest that the odd eight-vertex model with symmetric weights may also have commuting transfer matrices on the manifold (2). In the next section we show that this is indeed the case and discuss the respective Yang–Baxter equation. We finally remark that recently other mappings among the even and odd eight-vertex have been investigated by Assis [6]. They appear useful to related models with generic statistical weights but for the subspace of symmetric configuration stringent conditions among the weights are required. In particular, a mapping between the symmetric even eight-vertex and the odd eight-vertex model has not been reported in the work [6].

3. Yang–Baxter relations

In two spatial dimensions the method of commuting transfer matrices provides an efficient device to uncover integrable lattice systems of statistical mechanics [4]. For a recent review on the integrability of different families of lattice models see for example [7]. In the case of vertex models the transfer matrix can be built by tensoring a number of local operators \( L_{\alpha,\beta}(\omega) \) which acts on the product of two spaces associated with the horizontal (\( \alpha \)) and vertical (\( \beta \)) edge statistical configurations. Here the symbol \( \omega \) represents the set of energy weights of the vertex model. Assuming periodic boundary conditions on a square lattice of size \( N \times N \) the transfer matrix can be represented by the following trace,

\[
T(\omega) = \text{Tr}_0 [L_{01}(\omega)L_{02}(\omega) \ldots L_{0N}(\omega)],
\]

where the index 0 denotes the space associated to the horizontal degrees of freedom.

In our specific case the horizontal and vertical spaces are two-dimensional and the corresponding Lax operators can therefore be represented in terms of \( 4 \times 4 \) matrices. The form of this matrix for the symmetric even eight-vertex model is,
\[ L^{(\text{ev})}(a, b, c, d) = \begin{bmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{bmatrix}, \]

while for the symmetric odd eight-vertex model one has,

\[ L^{(\text{od})}(a, b, c, d) = \begin{bmatrix} 0 & a & d & 0 \\ b & 0 & 0 & c \\ c & 0 & 0 & b \\ 0 & d & a & 0 \end{bmatrix}. \]

It turns out that these operators can be related to each other by a weight independent local transformation given by,

\[ L^{(\text{od})}(a, b, c, d) = (\sigma^x \otimes \sigma^x) L^{(\text{ev})}(a, b, c, d) (\sigma^x \otimes I_2), \]

where \( \sigma^x \) is the standard Pauli matrix and \( I_2 \) refers to the \( 2 \times 2 \) identity matrix. Here we recall that symmetry relations among vertex models with distinct statistical weights have been discussed long before in the literature, see for instance \([8, 9]\). They are usually referred to as a gauge or twist transformation.

Thanks to the correspondence (12) we, from now on, can drop the use of bold letters to emphasize that the action of the matrices \( \sigma^x \) on the auxiliary space are cancelled out and as a result we obtain,

\[ T^{(\text{od})}(a, \ldots, d) = (\sigma^x_1 \otimes \sigma^x_2 \otimes \cdots \otimes \sigma^x_N) T^{(\text{ev})}(a, \ldots, d), \]

where \( \sigma^x_j \) acts as \( \sigma^x \) on the \( j \)-th space and as an identity otherwise. In addition, the symmetry under the inversion of arrow orientations implies the following commutation relations,

\[ [T^{(\text{ev})}(a, \ldots, d), \sigma^x_1 \otimes \sigma^x_2 \otimes \cdots \otimes \sigma^x_N] = [T^{(\text{od})}(a, \ldots, d), \sigma^x_1 \otimes \sigma^x_2 \otimes \cdots \otimes \sigma^x_N] = 0. \]

An immediate consequence of these results is that both the even and odd eight-vertex models with symmetric weights have in fact commuting transfer matrices when the weights are sited on the algebraic manifold (2). The next natural step is to uncover the integrable manifold directly from the study of the Yang–Baxter equation \([4]\) for the symmetric odd eight-vertex model. To this end we need to find the form of the \( R \)-matrix satisfying the following relation,

\[ R_{12}(\omega', \omega'') L^{(\text{od})}_{13}(a', \ldots, d') L^{(\text{od})}_{23}(a'', \ldots, d'') = L^{(\text{od})}_{23}(a'', \ldots, d'') L^{(\text{od})}_{13}(a', \ldots, d') R_{12}(\omega', \omega''). \]

In order to determine the \( R \)-matrix we first take two distinct numerical points \( a', \ldots, d' \) and \( a'', \ldots, d'' \) on the curve (2). We then solve numerically the Yang–Baxter equation (15) for a general \( 16 \times 16 \) \( R \)-matrix and conclude that many matrix elements vanish. The form of the \( R \)-matrix is found to be similar to that of the even eight-vertex model.
\[
\mathbf{R}(\omega', \omega'') = \begin{bmatrix}
\omega_1 & 0 & 0 & \omega_4 \\
0 & \omega_2 & \omega_3 & 0 \\
0 & \omega_3 & \omega_2 & 0 \\
\omega_4 & 0 & 0 & \omega_1
\end{bmatrix},
\]
and therefore the R-matrix and the Lax operator of the odd eight-vertex model have distinct structures.

We next substitute the Lax operator (11) and the R-matrix proposal (16) in the Yang–Baxter equation and as a result we obtain six independent functional relations. Their explicit expressions are,

\[
\begin{align*}
\omega_4 c' b'' + \omega_1 a' c'' - \omega_2 a' d'' - \omega_3 c' d'' &= 0, \\
\omega_1 d' a'' + \omega_4 b' d'' - \omega_2 c' a'' - \omega_3 a' d'' &= 0, \\
\omega_4 d' a'' + \omega_1 b' c'' - \omega_3 d' b'' - \omega_2 b' c'' &= 0, \\
\omega_1 c' b'' + \omega_4 c' c'' - \omega_2 d' b'' - \omega_3 c' d'' &= 0, \\
\omega_1 a' b'' + \omega_4 c' d'' - \omega_3 b' a'' - \omega_2 d' d'' &= 0, \\
\omega_2 a' a'' + \omega_3 c' d'' - \omega_4 b' b'' - \omega_3 d' c'' &= 0.
\end{align*}
\]

At this point we observe that such functional equations are similar to those associated with the even eight-vertex model [4]. They become exactly the same relations once we identify the entries of the R-matrix with the weights as follows,

\[
\begin{align*}
\omega_1 &\rightarrow c, \quad \omega_2 \rightarrow d, \quad \omega_3 \rightarrow a, \quad \omega_4 \rightarrow b,
\end{align*}
\]
and the invariants (2) can therefore be derived along the lines already discussed by Baxter [4].

The above result suggests that the Yang–Baxter structure sitting on the curve (2) is richer than that solely associated to the even eight-vertex model. In order to see this feature in a concise way we first recall the weights parameterization introduced by Baxter [4] for the even eight-vertex model,

\[
\begin{align*}
a(\mu) &= -i \Theta(\lambda) H \left[ \frac{1}{2} (\lambda - \mu) \right] \Theta \left[ \frac{1}{2} (\lambda + \mu) \right], \\
b(\mu) &= -i \Theta(\lambda) \Theta \left[ \frac{1}{2} (\lambda - \mu) \right] H \left[ \frac{1}{2} (\lambda + \mu) \right], \\
c(\mu) &= -i H(\lambda) \Theta \left[ \frac{1}{2} (\lambda - \mu) \right] \Theta \left[ \frac{1}{2} (\lambda + \mu) \right], \\
d(\mu) &= i H(\lambda) H \left[ \frac{1}{2} (\lambda - \mu) \right] H \left[ \frac{1}{2} (\lambda + \mu) \right],
\end{align*}
\]

where \( \mu \) represents the curve spectral variable, \( \lambda \) is a free parameter and \( H(\mu) \) and \( \Theta(\mu) \) are theta functions of modulus \( k \) as defined in [4]. For the sake of completeness these functions have been presented in the appendix.

We next define a family of R-matrices as follows².

² Note that the index exchange ev ↔ od is equivalent to the weights replacements \( a(\mu) \leftrightarrow c(\mu) \) and \( b(\mu) \leftrightarrow d(\mu) \).
and as result we find that these $R$-matrices fulfill the following set of functional relations,

$$R^{(\alpha_1,\alpha_2)}_{(\mu_1\mu_2)} R^{(\alpha_1,\alpha_3)}_{(\mu_1\mu_2)} = R^{(\alpha_2,\alpha_3)}_{(\mu_2\mu_1)} R^{(\alpha_1,\alpha_3)}_{(\mu_1\mu_2)} R^{(\alpha_1,\alpha_2)}_{(\mu_1\mu_2)},$$  \hspace{1cm} (21)

where the upper indices $\alpha_i = \text{ev}, \text{od}$ giving rise to family of Yang–Baxter relations.

We note that the choices $\alpha_1 = \alpha_2 = \alpha_3 = \text{ev}$ and $\alpha_1 = \alpha_2 = \text{od}, \alpha_3 = \text{ev}$ correspond to the Yang–Baxter equations associated to the even and odd eight-vertex models, respectively. The remaining relations can be verified by using theta function properties nowadays implemented in computer algebra systems such as Mathematica or Maple. Alternatively, it is possible to show that all eight branches of equation (21) lead to the same functional relations defined by equations (17) and (18). In the latter case it is helpful to use the following analogs of the operator identity (12),

$$R^{(\text{od,od})}(\mu) = (\sigma^x \otimes I_2) R^{(\text{ev,od})}(\mu) (I_2 \otimes \sigma^x),$$

$$R^{(\text{ev,od})}(\mu) = (I_2 \otimes \sigma^x) R^{(\text{ev,od})}(\mu) (\sigma^x \otimes \sigma^y),$$

$$R^{(\text{od,ev})}(\mu) = (\sigma^y \otimes \sigma^x) R^{(\text{ev,ev})}(\mu) (\sigma^y \otimes I_2).$$  \hspace{1cm} (22)

We finally observe that the above scenario is similar to that of solutions of the Yang–Baxter equation associated to different representations of a given group symmetry having the same dimension.

4. Concluding remarks

The even eight-vertex model is known to be solvable in the thermodynamic limit when the weights satisfy the free-fermion condition [10]. This means that an exact expression for the infinity volume free-energy can be written provided that,

$$w_1 w_2 + w_3 w_4 - w_5 w_6 - w_7 w_8 = 0.$$  \hspace{1cm} (23)

However, in order to have commuting transfer matrices for finite lattice sizes it is necessary to add other algebraic constraints. This was first pointed out by Krinsky [11] when the weights satisfy the condition $w_6 = w_3$ and $w_8 = w_7$ and later on reconfirmed by other authors, see for instance [12–15]. It is possible to adapt Krinsky’s work for eight arbitrary weights and as a result one finds the need of three extra restrictions on the weights. They are given by,

$$\frac{w_6 w_8}{w_5 w_7} = \Delta_1, \quad \frac{w_1 w_4 + w_2 w_3}{w_5 w_7} = \Delta_2, \quad \frac{w_1^2 + w_2^2 - w_2^2 - w_3^2}{w_5 w_7} = \Delta_3,$$  \hspace{1cm} (24)

where $\Delta_1, \Delta_2$ and $\Delta_3$ are free parameters.

At this point we recall that Wu and Kunz have shown that the free-energy of the odd eight-vertex model can also be computed in the thermodynamic limit when the weights fulfill the
free-fermion condition (5). This is because the equivalent staggered even eight-vertex model also has weights on the free-fermion condition and the infinite volume free-energy can again be computed by means of a Pfaffian solution [16]. Motivated by this fact we attempt to check whether or not it is possible to construct commuting transfer matrices for the odd eight-vertex model under the free-fermion condition (5). By explicit computation of transfer matrices commutators for two sites we conclude that if such a manifold exists it is certainly not given by the constraints (24) when the weight \( w_i \) is replaced by \( v_i \). This however does not exclude the existence of another type of integrable manifold for the asymmetric odd eight-vertex model with weights satisfying the free-fermion condition. Otherwise the free-fermion odd eight-vertex model will be an example in which the free-energy can be exactly computed in the continuum without the existence of commuting transfer matrices for finite lattice sizes. This issue appears to be worthy of further investigation.

We finally note that the mapping (6) suggests that the staggered odd eight-vertex model may have commuting transfer matrices when the weights are lying on the manifolds (23) and (24) with \( w_i = v_i \). In this case the partition function can be written as a trace of a power of the product of two different operators, namely

\[
Z^{stag}_{8od}(v_1, \ldots, v_8) = \text{Tr} \left[ (T_1(v_1, \ldots, v_8)T_2(v_1, \ldots, v_8))^N \right],
\]

where the corresponding transfer matrices are built by alternating two types of Lax operators,

\[
T_1(v_1, \ldots, v_8) = T_{01} \left[ L_{01}(v_1, \ldots, v_8)L_{02}(v_1, \ldots, v_8) \ldots L_{02N-1}(v_1, \ldots, v_8)L_{02N}(v_1, \ldots, v_8) \right],
\]

\[
T_2(v_1, \ldots, v_8) = T_{01} \left[ L_{01}(v_1, \ldots, v_8)L_{02}(v_1, \ldots, v_8) \ldots L_{02N-1}(v_1, \ldots, v_8)L_{02N}(v_1, \ldots, v_8) \right].
\]

(26)

The Lax operators encode the staggered weights \( \omega_X \) and \( \omega_Y \) as stated in the right-hand side of the correspondence (6) with \( w_i = v_i \). Their explicit matrices representation are,

\[
L(v_1, \ldots, v_8) = \begin{bmatrix}
0 & v_1 & 0 \\
v_3 & 0 & v_6 \\
v_5 & 0 & v_4 \\
v_8 & v_2 & 0
\end{bmatrix}, \quad L(v_1, \ldots, v_8) = \begin{bmatrix}
0 & v_3 & v_6 & 0 \\
v_1 & 0 & 0 & v_7 \\
v_8 & 0 & 0 & v_2 \\
v_5 & v_4 & 0 & 0
\end{bmatrix}.
\]

(27)

We have verified for \( N = 1, 2 \) that the operator product \( T_1(v_1, \ldots, v_8)T_2(v_1, \ldots, v_8) \) commutes for two distinct sets of random numerical values for the weights sited on the manifolds (23) and (24) with \( w_i = v_i \). It would be of interest to confirm this fact both algebraically and for arbitrary \( N \) considering the respective Yang–Baxter equation for the tensor product of the two types of Lax operators (27). Here we stress that the transfer matrices \( T_1(v_1, \ldots, v_8) \) and \( T_2(v_1, \ldots, v_8) \) do not have vanishing commutators for distinct weights on the manifolds (23) and (24). This appears to be an example in which individual transfer matrices may not commute but the product of two of them can lead us to a family of commuting operators.

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Appendix. Theta functions

The theta functions are functions of two variables which can be chosen to be the spectral variable $\mu$ and the so-called nome parameter $q$. For instance, following Baxter [4] the theta functions used in this paper are,

\begin{align}
H(\mu) &= 2q^{1/4} \sin \left[ \frac{\pi \mu}{2K} \right] \prod_{j=1}^{\infty} \left( 1 - 2q^{2j} \cos \left[ \frac{\pi \mu}{K} \right] + q^{4j} \right) (1 - q^{2j}), \tag{A.1} \\
\Theta(\mu) &= \prod_{j=1}^{\infty} \left( 1 - 2q^{(2j-1)} \cos \left[ \frac{\pi \mu}{K} \right] + q^{4j-2} \right) (1 - q^{2j}) \tag{A.2}
\end{align}

where $K$ is one of the half-period magnitudes which is given by,

\begin{equation}
K = \frac{\pi}{2} \prod_{j=1}^{\infty} \left[ \left( 1 + q^{2j-1} \right) \left( 1 - q^{2j-1} \right) \right]^2.
\tag{A.3}
\end{equation}

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