Nonstandard entropy production in the standard map

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March 23, 2002

Abstract

We investigate the time evolution of the entropy for a paradigmatic conservative dynamical system, the standard map, for different values of its controlling parameter a. When the phase space is sufficiently “chaotic” (i.e., for large |a|), we reproduce previous results. For small values of |a|, when the phase space becomes an intricate structure with the coexistence of chaotic and regular regions, an anomalous regime emerges. We characterize this anomalous regime with the generalized nonextensive entropy, and we observe that for values of a approaching zero, it lasts for an increasingly large time. This scenario displays a striking analogy with recent observations made in isolated classical long-range N-body Hamiltonians, where, for a large class of initial conditions, a metastable state (whose duration diverges with 1/N → 0) is observed before it crosses over to the usual, Boltzmann-Gibbs regime.

PACS numbers: 05.20.-y, 05.45.-a, 05.70.Ce

In his critical remarks about the domain of validity of Boltzmann principle, Einstein stressed [1] that the basis of statistical mechanics lies on dynamics. During the past century an impressive amount of work has been made to clarify this connection, but certainly this crucial point is far from being well understood. In this paper, following recent observations on dissipative maps, we analyze numerically the entropy production of a far-from-equilibrium conservative dynamical system, and we exhibit a new scenario where anomalous effects arise, and that may serve as a basis for extending the usual statistical-mechanical formalism.

In low-dimensional dynamical systems, both dissipative and conservative, intensive effort has been done in analyzing the connection between the property of mixing of the system and the evolution of the statistical entropy (see, e.g., [2, 3, 4, 5, 6, 7, 8] and references therein). Particularly, an interesting observation was made in [8], where it was found a simple connection between the Kolmogorov-Sinai entropy rate (the one that stems from the properties of mixing of the system), and the statistical entropy (functional of a probability...
distribution). It was shown in fact that, partitioning the phase space in \( W \) cells and starting from \( M > > 1 \) points within one cell (details are given later), the time dependence of the usual, Boltzmann-Gibbs (BG), statistical entropy

\[ S_1 = - \sum_{i=1}^{W} p_i \ln p_i, \quad (1) \]

includes a linear stage whose slope \( K_1 \equiv \lim_{t \to \infty} \lim_{W \to \infty} \lim_{M \to \infty} S_1(t) \) coincides with the Kolmogorov-Sinai entropy rate, calculated for instance via Pesin equality using the Lyapunov exponents. Nevertheless, in the context of one-dimensional dissipative systems (e.g., the logistic map), it was found in \( [7] \) that at the edge of chaos the usual entropy \( (1) \) fails to exhibit with nonvanishing slope such a behavior (which is of course expected since the Lyapunov exponent vanishes). Instead, the nonextensive entropy \( [9] \) (for a recent review, see \( [10] \))

\[ S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1} \quad (q \in \mathbb{R}), \quad (2) \]

succeeds for a specific value of \( q < 1 \) (\( q^* \approx 0.2445 \) for the logistic map). We remind that the entropic form \( (2) \) reduces to \( (1) \) in the limit \( q \to 1 \).

In Hamiltonian dynamics, one of the most studied models is the standard map (see, for instance, \( [11] \)), also referred to as the kicked-rotator model:

\[ \begin{align*}
  x_{t+1} &= y_t + \frac{a}{2\pi} \sin(2\pi x_t) + x_t \pmod{1}, \\
  y_{t+1} &= y_t + \frac{a}{2\pi} \sin(2\pi x_t) \pmod{1},
\end{align*} \quad (3) \]

where \( a \in \mathbb{R} \) \((t = 0, 1, 2, \ldots)\). The system is integrable when \( a = 0 \), while, for large-enough values of \( a \) (typically \(|a| \geq 7\)), it is strongly chaotic\( ^1 \) in Fig. 1 we illustrate the phase portrait of the map \( (3) \) for different values of \( a \). It can be easily verified that, for all values of \( a \), this two-dimensional map is conservative; it has a pair of \((x, y)\)-dependent Lyapunov exponents, which only differ in sign (simplectic structure) and globally vanish when \( a \) vanishes. Inspired by the results obtained in \( [7, 8] \), in this letter we study, for the standard map, the time evolution of both the extensive \((q = 1)\) and the nonextensive \((q \neq 1)\) statistical entropies, focusing our attention on small values of the parameter \( a \), i.e., on those situations where the border between the chaotic and the regular regions has a relevant influence. For smaller and smaller values of \(|a|\) (say below 7), one encounters in fact an increasingly rich fractal-like structure, characteristic of a large class of Hamiltonian systems. Typically, chains of regular islands corresponding to elliptic points in resonance condition begin to appear, and between each couple of elliptic points of a chain there is a hyperbolic point, thus forming another chain of hyperbolic points responsible for the chaotic areas. Around each island there is another chain of islands of higher order and, once again, another chain of hyperbolic points between the islands (see, for instance, \( [13] \) for details). This structure is usually referred to as “islands-around-islands”.

\( ^{1} \)We are not interested here in the accelerator-mode islands that appear at various critical values of \( a \) (see, for instance, \( [14] \) and references therein).
Figure 1: Phase portrait of the standard map for typical values of $a$. $M = 20 \times 20$ orbits (black dots) were started with a uniform distribution in the unit square and traced for $0 \leq t \leq 200$.

If we think that in some sense the islands-around-islands structure plays, for a Hamiltonian system, the role that edge-of-chaos plays for a dissipative system, we verify that for the standard map its relative area of influence increases when $a$ approaches 0.

To study numerically the time evolution of the entropy, we introduce a coarse-graining partition of the mapping phase space by dividing it in $W$ cells of equal size, and we set many copies of the system ($M$ points) in a far-from-equilibrium situation putting all the $M$ points randomly or uniformly distributed inside a single cell. The occupation number $M_i$ of each cell $i$ ($\sum_{i=1}^{W} M_i = M$) provides a probability distribution $p_i(t) \equiv M_i(t)/M$, hence an entropy value $S_q(t)$ ($\forall q$). Using then the dynamic equations (3), at each step the points spread in the phase space causing the entropy value to change. In order to obtain a consistent definition of the probabilities $\{p_i\}$ for all $t$ we are analyzing, we impose the constraint $M \gg W_{\text{max}}$, where $W_{\text{max}} \leq W$ is the maximum number of cells occupied. Finally, a global quantity over the whole $(x,y)$ phase space is obtained averaging many histories. Each history is characterized by the choice of the cell with which we start at $t = 0$ (see [14] for a preliminary exhibition of this phenomenon). These cells are randomly chosen in the unit square. We stress here that in this way we are implementing a Gibbsian-like viewpoint in the sense that we obtain an ensemble description of how the system evolves towards equilibrium. Given $a$, the result of this analysis is then a single curve of the entropy versus time, for each entropic form $S_q$. The initial
Figure 2: Schematic representation of $S_q(t)$ for different values of $q$, in the case of a strongly chaotic phase space. Only for $q = 1$ we have, in the limit $W \to \infty$, a finite entropy production $K_1$ ($K_q = 0$ for $q > 1$ and $K_q \to \infty$ for $q < 1$).

part of the curve characterizes the entropy production when the system is very far from equilibrium. Then comes an intermediate regime whose duration increases with $W$. Finally, the entropy starts approaching its equilibrium, where the entropy saturates at a value equal or below $(W^{1-q} - 1)/(1-q)$ (value of $S_q$ at equiprobability). With the aim of mathematically characterizing, for fixed $a$, the properties of the entropy production in the limit $W \to \infty$, we perform the calculation for increasingly large values of $W$. The curves asymptotically approach the $W \to \infty$ curve (see Fig. 2). Numerically, the crucial point is to execute many times the calculation with large $W$ satisfying the constraint $M >> W_{\text{max}}$. This exhausts the memory capacity of a single, standard computer if we wish to attain $W$ larger than say $10^6$; also, a lot of machine-time is required. To overcome these problems, we implemented a distributed computing technique [15] which allows for the simultaneous utilization of many computers linked within a network.

Let us start with $a = 7$, where the phase space is characterized by a connected chaotic sea (Fig. 1(a)). In this case there is just one value of the entropic index ($q = 1$) that yields a linear time evolution of the entropy $S_q(t)$, after a possible transient that depends on the details of the region where the initial data are set, and before the effects of saturation. After the transient and before saturation, $S_q(t)$ is a convex function for $q < 1$ and a concave function for $q > 1$. We schematize this analysis in Fig. 2, neglecting the initial transient. Moreover, as said before, during the linear stage, the slope of the BG entropy $S_1(t)$ coincides with the positive Lyapunov exponent.

If we now pass to $a = 5$, the phase space presents two large (and infinitely many small) islands and a connected chaotic sea (Fig. 1(b)). Depending on where we set the initial data, we may now face a strongly chaotic, or a regular, or even an islands-around-islands region. Our averaging procedure does not privilege any of these regions and allows the predominant ones to emerge. The key point is to perform many different histories so that the average stabilizes on a definite curve. The resulting entropy curves reflect two different phenomena:
Figure 3: $S_q(t)$ for $a = 5$ and different values of $q$ (1000 histories were averaged). Full (empty) circles correspond to $M/10 = W = 708 \times 708$ ($M/10 = W = 224 \times 224$). The lines are guides to the eye. The slope of $S_1(t)$ between $t_1 = 2$ and $t_2 = 6$ is $K_1(5) \simeq 0.97$. 
Figure 4: $S_1(t)$ for different values of $a \leq 2$ (5000 to 7000 histories were averaged). Full circles correspond to $M = W = 1000 \times 1000$ for $a = 2, 1.5, 1.1, 1.05, 1.1$ ($M = W = 5000 \times 5000$ for $a = 0.65$); empty circles correspond to $M = W = 448 \times 448$ for $a = 2, 1.5, 1.1$ ($M = W = 2236 \times 2236$ for $a = 0.65$). Inset: Slopes of $S_1(t)$ ($K_1(a)$) and of $S_{0.3}(t)$ ($K_{0.3} \simeq 0.71 + 0.30 a^{3.6}$) in their linear regimes (see also Fig. 5). The lines are guides to the eye.

the sensitivity to the initial conditions of each area and the relative extension of that area with respect to the whole phase space (unit square). In spite of the presence of large islands, the results we obtain in this case qualitatively coincide with Fig. 2; the only difference with the case $a = 7$ is that the slope of the linear stage of $S_1(t)$, $K_1(a)$, is smaller, consistently with the observation that the positive Lyapunov exponent is approaching 0. In Fig. 3 we present the actual numerical results for $a = 5$; our estimation of $K_1(5) \simeq 0.97$ is slightly smaller from that found in [8] because here we average over the entire phase space, including the islands.

What happens if we further reduce $a$, approaching situations similar to Fig. 1(c)? Let us first concentrate on the BG entropy $S_1(t)$. Fig. 4 displays that, for $a$ approaching zero, the part which for large values of $a$ was linear starts bending downwards. Then, after a certain characteristic time (noted $t_{cross}(a)$), linearity is obtained once again. If we now change $q$, we find that only for a special value $q^* \simeq 0.3$, $S_{q^*}(t)$ grows linearly for $t \leq t_{cross}(a)$. In Fig. 5 we plot $S_{0.3}(t)$ for $a$ approaching zero. For small values of $a$ and for $t \leq t_{cross}$, the dynamical system appears to be trapped in islands-around-islands regions, so that the usual entropy, $S_1$, is forced to grow slowly. For $t$ above $t_{cross}$, the normal dynamical behavior is restored; in other words, there is a crossover from an “anomalous” regime, where $S_{q^*}$ grows linearly, to the “usual” regime, where, in accordance with the observation in [8], it is $S_1$ that grows linearly (before
Figure 5: $S_{0.3}(t)$ for: $a = 1.5$ (a); $a = 1.1$ (b); $a = 0.9$ (c); $a = 0.7$ (d); $a = 0.65$ (e) (5000 to 7000 histories were averaged). Full circles correspond to $M = W = 1000 \times 1000$ for (a), (b), (c), $M = W = 5000 \times 5000$ for (d), (e); empty circles correspond to $M = W = 448 \times 448$ for (a), (b), (c), $M = W = 2236 \times 2236$ for (d), (e).

Inset: $t_{\text{cross}}(a)$ defined as the intersection of the linear part (before it starts bending) and a standard extrapolation of the bended part of the curves $S_{0.3}(t)$. Notice that $t_{\text{cross}}(a)$ diverges for $a \to 0$. The lines are guides to the eye. Our results suggest $\lim_{t \to \infty} \lim_{a \to 0} \lim_{W \to \infty} \lim_{M \to \infty} \frac{S_{0.3}(t)}{t} \approx 0.71$ for $q = q^* \approx 0.3$, whereas this limit vanishes (diverges) for $q > q^*$ ($q < q^*$).

Summarizing, we have studied the production of entropy for a well known low-dimensional conservative map controlled by the parameter $a$. It appears that, for fixed $a \neq 0$ and in the limit $t \to \infty$ (to be in all cases taken after the $W \to \infty$ limit), it is the standard, BG entropy $S_1$, which is associated with a finite entropy production. However, during the time interval preceding this extreme limit, an increasingly large interval emerges for which it is $S_{q^*}$, with $q^* \approx 0.3$, which is associated with a finite entropy production. Consistently, our results strongly suggest that, in the $\lim_{t \to \infty} \lim_{a \to 0}$ ordering, the only linear regime in fact observed is that corresponding to $q^* \approx 0.3$, whereas in the $\lim_{a \to 0} \lim_{t \to \infty}$ ordering the linear regime observed corresponds to $q = 1$. This fact provides a very appealing scenario for (meta) equilibrium thermostatistics in long-range Hamiltonians such as that described in [17, 18]. For these systems, a longstanding metastable state can exist (preceding the BG one) whose duration diverges with $N \to \infty$, and whose distribution of velocities is not Maxwellian,
Figure 6: Schematic representation of $S_q(t)$ for different values of $q$, in the limit $W \to \infty$, when there is a crossover between two different regimes.

but rather a power-law. The role played by $a$ in our present simple map and that played by $1/N$ in such long-range-interacting many-body systems, appear to be very similar, thus providing a dynamical basis for nonextensive statistical mechanics [9].

1 Acknowledgments

We acknowledge E.G.D. Cohen for stressing our attention on Einstein’s 1910 paper. We have benefitted from partial support by PRONEX, CNPq, CAPES and FAPERJ (Brazilian agencies).

References

[1] A. Einstein, Annalen der Physik 33, 1275 (1910) [ “Usually $W$ is put equal to the number of complexions... In order to calculate $W$, one needs a complete (molecular-mechanical) theory of the system under consideration. Therefore it is dubious whether the Boltzmann principle has any meaning without a complete molecular-mechanical theory or some other theory which describes the elementary processes. $S = \frac{k}{N} \log W + \text{const.}$ seems without content, from a phenomenological point of view, without giving in addition such an Elementartheorie.” (Translation: Abraham Pais, Subtle is the Lord..., Oxford University Press, 1982)]

[2] C. Tsallis, A.R. Plastino and W.-M. Zheng, Chaos, Solitons and Fractals 8, 885 (1997). See also P. Grassberger and M. Scheunert, J. Stat. Phys. 26 697 (1981), T. Schneider, A. Politi and D. Wurtz, Z. Phys. B 66 469 (1987), G. Anania and A. Politi, Europhys. Lett. 7 119 (1988) and H. Hata, T. Horita and H. Mori, Progr. Theor. Phys. 82 897 (1989).

[3] U.M.S. Costa, M.L. Lyra, A.R. Plastino and C. Tsallis, Phys. Rev. E 56, 245 (1997).
[4] M.L. Lyra, C. Tsallis, Phys. Rev. Lett. 80, 53 (1998).
[5] U. Tirnakli, C. Tsallis and M.L. Lyra, Eur. Phys. J. B 10, 309 (1999).
[6] F.A.B.F. de Moura, U. Tirnakli and M.L. Lyra, Phys. Rev. E 62, 6361 (2000).
[7] V. Latora, M. Baranger, A. Rapisarda and C. Tsallis, Phys. Lett. A 273, 97 (2000); U. Tirnakli, G.F.J. Ananos and C. Tsallis, Phys. Lett. A 289, 51 (2001). See also J. Yang and P. Grigolini, Phys Lett. A 263, 323 (1999).
[8] V. Latora and M. Baranger, Phys. Rev. Lett. 82, 520 (1999).
[9] C. Tsallis, J. Stat. Phys 52, 479 (1988); E.M.F. Curado and C. Tsallis, J. Phys. A 24, L69 (1991) [Corrigenda: 24, 3187 (1991) and 25, 1019 (1992)]; C. Tsallis, R.S. Mendes and A.R. Plastino, Physica A 261, 534 (1998).
[10] C. Tsallis, in Nonextensive Statistical Mechanics and Its Applications, eds. S. Abe and Y. Okamoto, Lecture Notes in Physics 560, 3 (Springer, Berlin, 2001); C. Tsallis, E.P. Borges and F. Baldovin, in Non Extensive Statistical Mechanics and Physical Applications, eds G. Kaniadakis, M. Lissia and A. Rapisarda, Physica A 305, 1 (2002); See http://tsallis.cat.cbpf.br/biblio.htm for full bibliography.
[11] M. Tabor, Chaos and Integrability in Nonlinear Dynamics (Wiley, New York, 1989), Sec. 4.2.e.
[12] G.M. Zaslavsky and B.A. Niyazov, Phys. Rep. 283, 73 (1997). C.
[13] M. Henon, in Chaotic Behavior of Deterministic Systems, eds. G. Ioos, R.H.G. Helleman and R. Stora (North-Holland, Amsterdam, 1983).
[14] F. Baldovin, Physica A 305, 124 (2002) .
[15] F. Baldovin and B. Schulze, in progress.
[16] F. Baldovin and C. Tsallis, in progress.
[17] V. Latora, A. Rapisarda and C. Tsallis, Phys. Rev. E 64, 056134 (2001).
[18] A. Campa, A. Giansanti and D. Moroni, Physica A 305, 137 (2002).