FLOW OF THE VISCOUS-ELASTIC LIQUID IN THE
NON-HOMOGENEOUS TUBE

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ABSTRACT. A problem on propagation of waves in deformable shells with flowing liquid is very urgent in connection with wide use of liquid transportation systems in living organisms and technology. It is necessary to consider shell motion equations for influence of moving liquid in cavity on the dynamics of a shell by solving such kind problems.

Nowadays a totality of such problems is a widely developed field of hydrodynamics. However, a number of peculiarities connected with taking into account viscous-elastic properties of the liquid and inhomogeneity of the shell material generates considerable mathematical difficulties connected with integration of boundary value problems with variable coefficients.

In the paper we consider wave flow of the liquid enclosed in deformable tube. The used mathematical model is described by the equation of motion of incompressible viscous elastic liquid combined with equation of continuity and dynamics equation for a tube inhomogeneous in length. It is accepted that the tube is cylindrical, semi-infinite and rigidly fastened to the environment. At the infinity the tube is homogeneous. As a final result, the problem is reduced to the solution of Volterra type integral equation that is solved by sequential approximations method. Pulsating pressure is given at the end of the tube to determine the desired hydrodynamic functions.

KEYWORDS. viscous-elastic liquid, non-homogeneous tube

Statement and mathematical ground of hydroelasticity problem is considered that admits to describe small amplitude wave propagation in elastic tube nonhomogeneous in length (with modulus of elasticity $E = E(x)$ and density $\rho_m = \rho_m(x)$), containing liquid. The basis of liquid’s model is the accounting of its viscous-elastic properties. Here, one-dimensional linearized equations are used.

1. Given a rectilinear semi-infinite cylindrical thin-shelled tube of constant thickness $h$ and radius $R$ whose material is subjected to Hooke’s law. In the considered case the system of hydroelasticity equations is of the form [1,2]

$$\frac{\partial u(x, t)}{\partial x} + \frac{2}{R} \frac{\partial w(x, t)}{\partial t} = 0;$$

(1)
\[
\rho f \frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x} \left\{ \sigma(x,t) - p(x,t) \right\}; \quad (2)
\]

\[
\prod_{j=1}^{r} \left\{ \sigma(x,t) + \lambda_j \frac{\partial \sigma(x,t)}{\partial t} \right\} = 2\eta \prod_{j=1}^{s} \left\{ \frac{\partial u(x,t)}{\partial x} + \theta_j \frac{\partial^2 u(x,t)}{\partial x \partial t} \right\}; \quad (3)
\]

\[
p(x,t) - \frac{h}{\pi R^2} E(x) w(x,t) = h \rho_m(x) \frac{\partial^2 w(x,t)}{\partial t^2}. \quad (4)
\]

Rheological relation (3) describes sufficiently well the flow of liquid continua containing long “high-molecular” compounds. It should be noted that there exist two classes of variants of model (3). The media possessing instantaneous elasticity for which \( r = s + 1 \) belong to the first class. The models detecting viscous behavior at instantaneous loading belong to the second class. For them \( r = s \).

At the equations (1) - (4) \( u(x,t) \) is longitudinal speed of flow of liquid, \( w(x,t) \) is radial displacement of tube’s walls, \( p(x,t) \) is pressure, \( \sigma(x,t) \) is “viscous-elastic” stress, \( \rho_f \) is liquid’s density, \( \eta \) is its dynamic viscosity coefficient, the quantities \( \lambda_j \) and \( \theta_j \) form relaxation and retardation spectra, respectively.

Not losing generality, we represent the functions \( E(x) \) and \( \rho_m \) by means of the equalities \( E(x) = E_\infty g_1(x) \) and \( \rho_m = \rho_m\infty g_2(x) \) and accept that the functions \( g_1(x) \) and \( g_2(x) \) are twice differentiable. We’ll also assume that the tube is homogeneous at infinity. Hence we have:

\[
\lim_{x \to \infty} g_1(x) = \lim_{x \to \infty} g_2(x) = 1. \quad (5)
\]

At the same time we assume

\[
\lim_{x \to \infty} g_1'(x) = 0; \quad \lim_{x \to \infty} g_2'(x) = 0; \quad \lim_{x \to \infty} g_1''(x) = 0; \quad \lim_{x \to \infty} g_2''(x) = 0, \quad (6)
\]

where here and later on the primes mean differentiation with respect to \( x \).

Transforming the system (1)-(4) by formula

\[
Q(x,t) = \pi R^2 u(x,t)
\]

we get the following expance system of equation

\[
\frac{\partial Q(x,t)}{\partial x} + 2\pi R \frac{\partial w(x,t)}{\partial x} = 0, \quad (7)
\]

\[
\rho_j \frac{\partial Q(x,t)}{\partial t} = \pi R^2 \frac{\partial}{\partial x} \left\{ \sigma(x,t) - p(x,t) \right\}, \quad (8)
\]

\[
\pi R^2 \prod_{j=1}^{r} \left\{ \sigma(x,t) + \lambda_j \frac{\partial \sigma(x,t)}{\partial t} \right\} = 2\eta \prod_{j=1}^{s} \left\{ \frac{\partial Q(x,t)}{\partial x} + \theta_j \frac{\partial^2 Q(x,t)}{\partial t \partial x} \right\}, \quad (9)
\]
\[
p(x, t) - \frac{h E_\infty}{R^2} g_1(x) w(x, t) = \rho_{m\infty} h g_2(x) \frac{\partial^2 w(x, t)}{\partial t^2}.
\]  

(10)

We separate variables and look for the solution of system (1) - (4) in the following form:

\[
Q(x, t) = Q_1(x) \exp(i\omega t), \quad w(x, t) = w_1(x) \exp(i\omega t), \quad \sigma(x, t) = \sigma_1(x) \exp(i\omega t), \quad p(x, t) = p_1(x) \exp(i\omega t),
\]

(11)

that allows to reduce the initial system of equations to the system of differential equations of the form

\[
Q_1^1(x) + 2\pi R \omega w_1(x) = 0,
\]

(12)

\[
\rho_f i\omega Q_1(x) = \pi R^2 (\sigma_1^1(x) - p_1^1(x)),
\]

(13)

\[
p_1(x) = \left\{ \frac{h E_\infty}{R^2} g_1(x) - h \omega^2 \rho_{m\infty} g_2(x) \right\} w_1(x),
\]

(14)

\[
\sigma_1(x) = \frac{2\eta}{\rho R^2 \alpha} Q_1^1(x),
\]

(15)

where in

\[
\prod_{j=1}^{r} (1 + i\omega \lambda_j) = a = a_0 + i\alpha, \quad \prod_{j=1}^{s} (1 + i\omega \theta_j) = b = b_0 + ib_1
\]

is accepted, and \(\omega\) is the given angular frequency. Further we introduce the denotation

\[
G(x) = \frac{2\eta}{\rho_f} b a - \frac{\omega_0^2}{\omega^2} g_1(x) + \frac{R h \rho_{m\infty}}{2\rho_f} g_2(x), \quad \left( \omega_0 = \frac{h E_\infty}{2R \rho_f} \right),
\]

(16)

whence

\[
G'(x) = \frac{R h \rho_{m\infty}}{2\rho_f} g_2'(x) - \frac{\omega_0^2}{\omega^2} g_1'(x).
\]

(17)

Combining equations (12-15), after some transformations we get the following equation for the function \(u_1(x)\)

\[
G(x) u_1''(x) + G'(x) u_1'(x) - u_1(x) = 0.
\]

(18)

We use the Liouville substitution

\[
y(x) = Q_1(x) \exp \left\{ \frac{1}{2} \int \frac{G'(x)}{G(x)} dx \right\} = Q_1(x) \sqrt{|G(x)|}
\]

(19)

and reduce equation (18) to the form
where the invariant \( I(x) \) is determined by the relation

\[
I(x) = \frac{1}{4} \left\{ \frac{G'(x)}{G(x)} \right\}^2 - \frac{1}{2} \frac{G''(x)}{G(x)} - \frac{1}{G(x)}.
\]

Using relations (5) and (6) we establish the following limit equalities:

\[
\lim_{x \to \infty} G(x) = \frac{2\eta}{\rho_f \omega^2} \frac{b}{ia} - \frac{c_0^2}{\omega^2} + \frac{R \rho \rho_m}{2\rho_f}, \quad \lim_{x \to \infty} G'(x) = 0.
\]

Hence we follow (21) and get the expression

\[
\lim_{x \to \infty} I(x) = \delta^2 = \frac{k_0}{k_0^2 + k_1^2} - i \frac{k_1}{k_0^2 + k_1^2},
\]

wherein

\[
k_0 = \frac{2\eta}{\rho_f \omega^2} \frac{a_1 b_0 - a_0 b_1}{a_0^2 + a_1^2} + \frac{c_0^2}{\omega^2} + \frac{R \rho \rho_m}{2\rho_f},
\]

\[
k_1 = \frac{2\eta}{\rho_f \omega^2} \frac{a_0 b_0 - a_1 b_1}{a_0^2 + a_1^2}.
\]

Now, using denotation

\[
q(x) = 1 - \frac{I(x)}{\delta^2},
\]

we reduce equation (20) to the form

\[
y''(x) + \delta^2 y(x) = \delta^2 q(x) y(x).
\]

We’ll use a root at which \( Im \delta < 0 \) later on, and on the potential \( q(x) \) we impose the integrability condition

\[
\int_{0}^{\infty} |q(x)| \, dx < +\infty.
\]

In order to construct the solutions, equation (24) should be completed with the following boundary conditions

\[
y(0) = y_0, \quad y \to 0 \text{ as } x \to \infty.
\]

The quantity \( y_0 \) depends on the functional regime of the system, and the second condition (26) provides the boundedness of the desired solution. As a result, the solution of the hydroelasticity problem is reduced to a singular boundary value problem of Sturm-Liouville type (24) and (26) under the condition (25).
2. In equation (24) considering $\delta^2 q(x)$ as an external source and applying the method of arbitrary constants variation we can reduce the solution of the problem to the equivalent integral equation [3]

$$y(x,-\delta) = C e^{-i\delta x} + \delta \int_{x}^{\infty} \sin \delta (\xi - x) q(\xi) y(\xi, -\delta) d\xi,$$  \hspace{1cm} (27)

Here the constant $C$ is determined as

$$C = \frac{y_0}{f(0, -\delta)},$$

and

$$y = \frac{y_0}{f(0, -\delta)} f(x, -\delta),$$

where by [4] a new function $f(x, -\delta)$ is determined from the solution of the integral equation

$$f(x, -\delta) = e^{-i\delta x} + \delta \int_{x}^{\infty} \sin \delta (\xi - x) q(\xi) f(\xi, -\delta) d\xi.$$  \hspace{1cm} (28)

Equation (28) is a Volterra type equation and is solved by the sequential approximations method:

$$f(x, -\delta) = \sum_{n=0}^{\infty} \delta^n f_n (x, -\delta).$$  \hspace{1cm} (29)

Therein we have the following recurrent relations

$$f_0 (x, -\delta) = e^{-i\delta x},$$

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$$f_n (x, -\delta) = \delta \int_{x}^{\infty} \sin \delta (\xi - x) q(\xi) f_{n-1} (\xi, -\delta) d\xi \quad (n = 1, 2, ...).$$  \hspace{1cm} (30)

By inequality (25) from uniform convergences of sequential approximations by the Weierstrass test we can establish that the unique solution of integral equation (28) is determined by relation (29). We can directly establish that this solution solution of input equation (24) as well.

Further, it is easy to determine the functions from systems (12)-(15) and write $u_1$, $w_1$, $p_1$, $\sigma_1$

$$Q(x, t) = \frac{\pi R^2 y(x)}{\sqrt{|G(x)|}} y_0 e^{i\omega t} F(x);$$  \hspace{1cm} (31)

$$w(x, t) = -\frac{R y_0}{2\omega i} e^{i\omega t} F'(x);$$  \hspace{1cm} (32)
\[ \sigma (x, t) = 2\eta y_0 \frac{b}{a} e^{i\omega t} F'(x); \quad (33) \]

\[ p(x, t) = y_0 e^{i\omega t} \left[ \frac{hE_\infty i}{2R\omega} g_1(x) - \frac{R\rho m_\infty i}{2} g_2(x) \right] F'(x), \quad (34) \]

where

\[ F(x) = \frac{1}{\sqrt{|G(x)|}} \frac{f(x, -\delta)}{f_0(0, -\delta)}. \]

To determine the quantity \( y_0 \) at the end of the tube \((x = 0)\) we give the pulsating pressure

\[ p(0, t) = p_0 \exp(i\omega t). \]

Using equality (34) this circumstance admits immediately to determine the quantity \( y_0 \). It is of the form:

\[ y_0 = -i \frac{p_0}{\left\{ \frac{hE_\infty i}{2R\omega} g_1(0) - \frac{R\rho m_\infty}{2} g_2(0) \right\}} F'(0). \]

Thus, the solution of the problem is complete. In conclusion note that series (29) in combination with relations (30) gives constructive representation of the desired solution and real parts (31)-(34) represent physical quantity.

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