Killing Correspondence in Finsler spaces

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Abstract

The present paper deals with the Killing correspondence between some Finsler spaces. We consider a Finsler space equipped with a $\beta$-change of metric and study the Killing correspondence between the original Finsler space and the Finsler space equipped with $\beta$-change of metric. We obtain necessary and sufficient condition for a vector field Killing in the original Finsler space to be Killing in the Finsler space equipped with $\beta$-change of metric. Certain consequences of such result are also discussed.

Keywords: Finsler spaces, $\beta$-change, Killing vector field

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1 Introduction

As a matter of investigation, it is important to observe how properties of a Finsler space change under a change in the metric. Several geometers from different parts of the globe have been working in this direction for the last 2-3 decades. M.S. Knebelman [1], S.Golab [2] and M.Hashiguchi [3] studied conformal change of Finsler metrics. Park and Lee [4] discussed various Randers changes of Finsler spaces with $(\alpha, \beta)$-metrics of Douglas type. M. Matsumoto [6] and T. Aikou [5] studied and investigated several properties of projective change and projective Randers change. In 1984, C. Shibata [8] studied $\beta$-change of Finsler metrics and discussed certain invariant tensors under such a change.
Killing equations play important role in the study of a Finsler space which undergoes a change in the metric. In fact, they give an equivalent characterization for the transformations to preserve distances. In 1979, Singh, et. al. [7] studied a Randers space $F^n\left(M, L(x, y) = (g_{i j}(x) y^i y^j)^{1/2} + b_i(x) y^i\right)$, $n \geq 2$ which undergoes a change $L(x, y) \mapsto L^*(x, y) = L^2(x, y) + (\alpha_i(x)y^i)^2$. They discussed Killing correspondence of the spaces $F^n(M, L)$ and $F^*n(M, L^*)$.

In the present paper, we consider a general Finsler space $F^n(M, L)$ which undergoes a $\beta$-change, that is $L(x, y) \mapsto \bar{L}(x, y) = f(L, \beta)$, where $\beta(x, y) = b_i(x)y^i$ is a 1-form. We study Killing correspondence of the Finsler spaces $F^n(M, L)$ and $\bar{F}^n(M, \bar{L})$. For the notations and terminology, we refer the reader to the books [9] and [10], and the paper [8] by Shibata.

The paper is organized as follows. In section 2, we give some preliminaries which are used in the discussion of subsequent sections. Section 3 deals with Killing correspondence of $F^n(M, L)$ and $F^n(M, L)$, where $L(x, y) = f(L, \beta)$. In section 4, we give conclusion to the results obtained in the paper and discuss future possible work to be done in this direction.

## 2 Preliminaries

Let $F^n(M, L)$, $n \geq 2$ be an $n$-dimensional Finsler space. Suppose that the metric function $L(x, y)$ undergoes a change $L(x, y) \mapsto \bar{L}(x, y) = f(L, \beta)$, where $\beta(x, y) = b_i(x)y^i$ is a 1-form and the new space is $\bar{F}^n(M, \bar{L})$. This change of metric is called a $\beta$-change (see [8] and [9]).

The angular metric tensor $\bar{h}_{ij}$ of the space $\bar{F}^n$ is given by [8]

$$\bar{h}_{ij} = ph_{ij} + q_0 m_i m_j, \tag{2.1}$$

where

$$\begin{align*}
p &= f f_1/L, \quad q_0 = f f_22, \quad m_i = b_i - \beta y^i/L^2, \\
f_1 &= \partial f/\partial L, \quad f_2 = \partial f/\partial \beta, \tag{2.2}
\end{align*}$$
$h_{ij}$ being the angular metric tensor of $F^n$. The fundamental metric tensor $\bar{g}_{ij}$ and its inverse $\bar{g}^{ij}$ of $\bar{F}^n$ are expressed as

\begin{equation}
\bar{g}_{ij} = p g_{ij} + p_0 b_i b_j + p_{-1} (b_i y_j + b_j y_i) + p_{-2} y_i y_j,
\end{equation}

\begin{equation}
\bar{g}^{ij} = \frac{g^{ij}}{p - s b^i b^j - s_{-1} (b^i y^j + b^j y^i) - s_{-2} y^i y^j},
\end{equation}

where

\begin{equation}
\left\{ \begin{array}{l}
p_0 = q_0 + f_2^2, \\
q_{-1} = f f_{12}/L, \quad p_{-1} = q_{-1} + p f_2/f, \quad q_{-2} = f (f_{11} - f_{1}/L) / L^2, \\
p_{-2} = q_{-2} + p^2 / f^2, \\
b^i = g^{ij} b_j, \quad b^2 = g^{ij} b_i b_j, \quad s_0 = \bar{L} q_0 / (\tau p L^2), \\
s_1 = p_{-1} \bar{L}^2 / (\tau p L^2), \quad s_{-2} = p_{-1} (\nu p L^2 - b^2 \bar{L}^2) / (\tau p L^2 \beta), \\
\tau = \bar{L}^2 (p + \nu q_0) / L^2, \quad \nu = b^2 - \beta^2 / L^2,
\end{array} \right.
\end{equation}

$g_{ij}$ and $g^{ij}$ respectively being the metric tensor and inverse metric tensor of $F^n$. The Cartan tensor $\bar{C}_{ijk}$ and the associate Cartan tensor $\bar{C}^h_{ij}$ of $\bar{F}^n$ are given by the following expressions:

\begin{equation}
\bar{C}_{ijk} = p C_{ijk} + \frac{1}{2} p_{-1} \mathcal{S}_{(ijk)} \{ h_{ij} m_k \} + \frac{1}{2} p_{02} m_i m_j m_k,
\end{equation}

\begin{equation}
\bar{C}^h_{ij} = C^h_{ij} - V^h_{ij},
\end{equation}

where

\begin{equation}
V^h_{ij} = Q^h (p C_{imj} b^m - p_{-1} m_i m_j) - \frac{1}{p} (m^h - \nu Q^h)(p_{02} m_i m_j + p_{-2} h_{ij}) / 2 \\
- p_{-1} (h^h_i m_j + h^h_j m_i) / (2p),
\end{equation}

\begin{equation}
Q^h = s_0 b^h + s_{-1} y^h, \quad h^h_i = g^{hr} h_{ir}, \quad m^h = g^{hr} m_r, \quad p_{02} = \partial p_0 / \partial \beta,
\end{equation}

$\mathcal{S}_{(ijk)}$ denote the cyclic sum with respect to the indices $i$, $j$ and $k$; $C_{ijk}$ and $C^h_{ij}$ respectively being the Cartan tensor and associate Cartan tensor of $F^n$. 


The spray coefficients \( \bar{G}^i \) of \( \bar{F}^n \) in terms of the spray coefficients \( G^i \) of \( F^n \) are expressed as

\[
(2.10) \quad \bar{G}^i = G^i + D^i,
\]

where

\[
D^i = \left( \frac{q}{p} \right) F^i_0 + \left( \frac{p}{2} \right) E_0 - 2qF(r)b^i(s^{-1} y^i + s_0 b^i)/2,
\]

\[
F^i_j = g^{ir}F_{rj}, \quad E_{jk} = \left( \frac{1}{2} \right)(b_{j|k} + b_{k|j}), \quad F_{jk} = \left( \frac{1}{2} \right)(b_{j|k} - b_{k|j}),
\]

the symbol ‘\( \| \)’ denote the \( h \)-covariant derivative with respect to the Cartan connection \( CT \) and the lower index ‘\( 0 \)’ (except in \( s_0 \)) denote the contraction by \( y^i \).

The relation between the coefficients \( \bar{N}^i_j \) of Cartan nonlinear connection in \( \bar{F}^n \) and the coefficients \( N^i_j \) of the corresponding Cartan nonlinear connection in \( F^n \) is given by

\[
(2.11) \quad \bar{N}^i_j = N^i_j + D^i_j,
\]

where

\[
(2.12) \quad D^i_j = \dot{\partial}_j D^i, \quad \dot{\partial}_j \equiv \partial/\partial y^j.
\]

The coefficients \( \bar{F}^i_{jk} \) of Cartan connection \( CT \) in \( \bar{F}^n \) and the coefficients \( F^i_{jk} \) of the corresponding Cartan connection \( CT \) in \( F^n \) are related as

\[
(2.13) \quad \bar{F}^i_{jk} = F^i_{jk} + D^i_{jk},
\]

where

\[
D^i_{jk} = \left( \frac{1}{p} \right) g^{is} - Q^i b^s - y^s(s^{-1} b^i + s y^i)
\]

\[
( B_{sj}b_{0|k} + B_{sk}b_{0|j} - B_{kj}b_{0|s} + F_{sj}Q_k + F_{sk}Q_j + E_{kj}Q_s + pC_{jkr}D^r_s
\]

\[
+ V_{jkr}D^r_s - pC_{skm}D^m_j - V_{sjm}D^m_k - pC_{sjm}D^m_k - V_{skm}D^m_j ) ;
\]

\[
B_{kj} = 2\dot{\partial}_j Q_k.
\]

The difference tensor \( D^i_{jk} \) satisfies the following properties:

(i) \( D^i_{j0} = B^i_{j0} = D^i_j \),  
(ii) \( D^i_{00} = 2D^i \),  
where \( B^i_{jk} = \dot{\partial}_k D^i_j \).
3 Killing Correspondence of $F^n$ and $\bar{F}^n$

Let us consider an infinitesimal transformation

\begin{equation}
\begin{aligned}
\dot{x}^i &= x^i + \epsilon v^i(x), \\
\end{aligned}
\end{equation}

where $\epsilon$ is an infinitesimal constant and $v^i(x)$ is a contravariant vector field.

The vector field $v^i(x)$ is said to be a Killing vector field in $F^n$ if the metric tensor of the Finsler space with respect to the infinitesimal transformation (3.1) is Lie invariant, that is

\begin{equation}
\mathfrak{L}_v g_{ij} = 0,
\end{equation}

$\mathfrak{L}_v$ being the operator of Lie differentiation. Equivalently, the vector field $v^i(x)$ is Killing in $F^n$ if

\begin{equation}
v_{ij} + v_{ji} + 2C_{i}^{h} v_{h|0} = 0,
\end{equation}

where $v_{i} = g_{i\ell} v^\ell$.

Now, we prove the following result which gives a necessary and sufficient condition for a Killing vector field in $F^n$ to be Killing in $\bar{F}^n$:

**Theorem 3.1.** A Killing vector field $v^i(x)$ in $F^n$ is Killing in $\bar{F}^n$ if and only if

\begin{equation}
V_{ij}^{h} v_{h|0} + C_{rji} v^l D_{i}^{l} + C_{rji} v^l D_{j}^{l} + v_{r} D_{ij}^{r} + \bar{C}_{ij}^{h} (2C_{rhi} v^l D_{r}^{l} + v_{r} D_{h}^{r}) = 0,
\end{equation}

where $\bar{C}_{ij}^{h}$ is the associate Cartan tensor of $\bar{F}^n$.

**Proof.** Assume that $v^i(x)$ is Killing in $F^n$. Then (3.3) is satisfied. By definition, the $h$-covariant derivatives of $v_{i}$ with respect to $C\bar{T}$ and $C\Gamma$ are respectively given as

\begin{equation}
\begin{aligned}
(a) \quad v_{i||j} &= \partial_{j} v_{i} - (\dot{\partial}_{r} v_{i}) G_{j}^{r} - v_{r} F_{ij}^{r}, \\
(b) \quad v_{ij} &= \partial_{j} v_{i} - (\dot{\partial}_{r} v_{i}) G_{j}^{r} - v_{r} F_{ij}^{r},
\end{aligned}
\end{equation}

where $\partial_{j} \equiv \partial/\partial x^{j}$ and '||' denote the $h$-covariant differentiation with respect to $C\bar{T}$. Equation (3.5)(a), by virtue of (2.10), (2.13) and (3.5)(b), takes the form

\begin{equation}
v_{i||j} = v_{ij} - 2C_{rji} v^l D_{j}^{l} - v_{r} D_{ij}^{r}.
\end{equation}
Now, from (3.6), we have

\[ v_{i|j} + v_{j|i} + 2\dd C_{ij}^h v_h|0 = v_{i|j} + v_{j|i} + 2\dd C_{ij}^h v_h|0 - 2 C_{r|i} v^j D_j^r - 2 C_{r|j} v^l D_l^r - 2 v_r D_{ij}^r - 2 \dd C_{ij}^h (2 C_{rhl} v_l D_r^j + v_r D_{hi}^r). \]

Using (2.7) in (3.7) and applying (3.3), we get

\[ v_{i|j} + v_{j|i} + 2\dd C_{ij}^h v_h|0 = -2 V_{ij}^h v_h|0 - 2 C_{r|i} v^j D_j^r - 2 C_{r|j} v^l D_l^r - 2 v_r D_{ij}^r - 2 \dd C_{ij}^h (2 C_{rhl} v_l D_r^j + v_r D_{hi}^r). \]

Proof completes with the observation that \( v^i(x) \) is Killing in \( F^n \) if and only if \( v_{i|j} + v_{j|i} + 2\dd C_{ij}^h v_h|0 = 0 \), that is, if and only if (3.4) holds.

If a vector field \( v^i(x) \) is Killing in \( F^n \) and \( \dd F^n \), then from Theorem 3.1, (3.4) holds, which on transvection by \( y^i \) yields

\[ 2 C_{rjl} v^j D_l^r + v_r D_j^r = 0. \]

Equation (3.4), in view of (3.9), enables us to state the following:

**Corollary 3.1.** If a vector field \( v^i(x) \) is Killing in \( F^n \) and \( \dd F^n \), then

\[ V_{ij}^h v_h|0 + C_{r|j} v^j D_i^r + C_{r|i} v^l D_l^r + v_r D_{ij}^r = 0. \]

As another important consequence of Theorem 3.1 we have the following:

**Corollary 3.2.** If a vector field \( v^i(x) \) is Killing in \( F^n \) and \( \dd F^n \), then the vector \( v_i(x, y) \) is orthogonal to the vector \( D^i(x, y) \).

**Proof.** As \( v^i(x) \) is Killing in \( F^n \) and \( \dd F^n \), (3.4) holds, which on transvection by \( y^i \) gives (3.9). Again transvecting (3.9) by \( y^j \), it follows that \( v_r D^r = 0 \). This proves the result.

## 4 Discussion and Conclusion

We proved Theorem 3.1 as the main result and as its consequences we obtained Corollary 3.1 and Corollary 3.2. Since the Killing equation (3.2) is a necessary and
sufficient condition for the transformation (3.1) to be a motion in $F^n$ (vide [10]), the condition (3.4) obtained in Theorem 3.1 may be taken as the necessary and sufficient condition for the vector field $v^i(x)$, generating a motion in $F^n$, to generate a motion in $\bar{F}^n$ as well. Also, since every motion is an affine motion and every affine motion is a projective motion (vide [11]-[13]), it is clear that vector field $v^i(x)$, generating an affine motion (respectively projective motion) in $F^n$, generates an affine motion (respectively projective motion) in $\bar{F}^n$ if condition (3.4) holds. The main result and its consequences, obtained in the paper, may be further utilized to link various transformations in $F^n$ with corresponding transformations in $\bar{F}^n$.

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