Continuity equation in LlogL for the 2D Euler equations under the enstrophy measure

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Abstract

The 2D Euler equations with random initial condition has been investigat es by S. Albeverio and A.-B. Cruzeiro in [1] and other authors. Here we prove existence of solutions for the associated continuity equation in Hilbert spaces, in a quite general class with LlogL densities with respect to the enstrophy measure.

1 Introduction

We consider the 2D Euler equations on the torus $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$, formulated in terms of the vorticity $\omega$

$$\partial_t \omega + u \cdot \nabla \omega = 0$$

(1)

where $u$ is the velocity, divergence free vector field such that $\omega = \partial_2 u_1 - \partial_1 u_2$. We consider this equation in the following abstract Wiener space structure. We set $H = L^2 (T^2)$ with scalar product $\langle \cdot, \cdot \rangle_H$ and norm $\|\cdot\|_H$. Given $\delta > 0$, we consider the negative order Sobolev space $B := H^{-1-\delta} (T^2)$, its dual $B^* = H^{1+\delta} (T^2)$, and we write $\langle \cdot, \cdot \rangle$ for the dual pairing between elements of $B$ and $B^*$. More generally, we shall use the notation $\langle \cdot, \cdot \rangle$ also for the dual pairing between elements of $C^\infty (T^2)'$ and $C^\infty (T^2)$; in all cases $\langle \cdot, \cdot \rangle$ reduces to $\langle \cdot, \cdot \rangle_H$ when both elements are in $H$. Let $\mu$ be the so called ”enstrophy measure”, the centered Gaussian measure on $B$ (in fact it is supported on $H^{-1-} (T^2) \cap_{\delta > 0} H^{-1-\delta} (T^2)$; but not on $H^{-1} (T^2)$) such that

$$\int_B \langle \omega, \phi \rangle \langle \omega, \psi \rangle \mu (d\omega) = \langle \phi, \psi \rangle_H$$

for all $\phi, \psi \in C^\infty (T^2)$. Equation (1) has been investigated in this framework and it has been proved that, with a suitable interpretation of the nonlinear term of the equation, it
has a (possibly non unique) solution for \( \mu \)-almost every initial condition in \( B \). Moreover, on a suitable probability space \((\Xi, \mathcal{F}, \mathbb{P})\), there exists a stationary process with continuous trajectories in \( B \), with marginal law \( \mu \) at every time \( t \) (in this sense we could say that \( \mu \) is invariant for equation (1); see also the infinitesimal invariance [2]), whose trajectories are solutions of equation (1) in that suitable specified sense. These results have been proved first by Albeverio and Cruzeiro in [1] and proved with a different concept of solution (used below) in [12].

We want to study the continuity equation, associated to equation (1), for a density \( \rho_t (\omega) \) with respect to \( \mu \). Let us introduce the notation

\[
b (\omega) = -u (\omega) \cdot \nabla \omega
\]

for the drift in equation (1), where we stress by writing \( u (\omega) \) the fact that \( u \) depends on \( \omega \). The precise meaning of \( b (\omega) \) is a nontrivial problem discussed below; for the time being, let us take it as an heuristic notation. Let \( \mathcal{F}C^1_{b,T} \) be the set of all functionals \( F : [0,T] \times C^\infty (\mathbb{T}^2) \to \mathbb{R} \) of the form \( F (t, \omega) = \sum_{i=1}^m \tilde{f}_i (\langle \omega, \phi_1 \rangle, ..., \langle \omega, \phi_n \rangle) g_i (t) \), with \( \phi_1, ..., \phi_n \in C^\infty (\mathbb{T}^2) \), \( \tilde{f}_i \in C^1_b (\mathbb{R}^n) \), \( g_i \in C^1 ([0,T]) \) with \( g_i (T) = 0 \). The weak form of the continuity equation is

\[
\int_0^T \int_B (\partial_t F (t, \omega) + \langle b (\omega), DF (t, \omega) \rangle) \rho_t (d\omega) \, dt = -\int_B F (0, \omega) \rho_0 (\omega) \mu (d\omega). \tag{2}
\]

The most critical term, which requires a careful definition, is \( \langle b (\omega), DF (t, \omega) \rangle \). Let us discuss this issue.

When \( F (t, \omega) = \sum_{i=1}^m \tilde{f}_i (\langle \omega, \phi_1 \rangle, ..., \langle \omega, \phi_n \rangle) g_i (t) \) as above, given any element \( \eta \in C^\infty (\mathbb{T}^2) \) the limit

\[
\lim_{\epsilon \to 0} \epsilon^{-1} (F (t, \omega + \epsilon \eta) - F (t, \omega))
\]

exists for every \( (t, \omega) \in [0,T] \times C^\infty (\mathbb{T}^2) \) and it is equal to

\[
\sum_{i=1}^m \sum_{j=1}^n \partial_j \tilde{f}_i (\langle \omega_t, \phi_1 \rangle, ..., \langle \omega_t, \phi_n \rangle) g_i (t) \langle \eta, \phi_j \rangle.
\]

Assume we have defined \( \langle b (\omega), \phi \rangle \) when \( \omega \) is a typical element under \( \mu \) and \( \phi \in C^\infty (\mathbb{T}^2) \). Then we set

\[
\langle b (\omega), DF (t, \omega) \rangle := \sum_{i=1}^m \sum_{j=1}^n \partial_j \tilde{f}_i (\langle \omega_t, \phi_1 \rangle, ..., \langle \omega_t, \phi_n \rangle) g_i (t) \langle b (\omega), \phi_j \rangle. \tag{3}
\]

To complete the meaning of \( \langle b (\omega), DF (t, \omega) \rangle \) we thus have to give a meaning to \( \langle b (\omega), \phi \rangle \) for every \( \phi \in C^\infty (\mathbb{T}^2) \). Formally

\[
\langle b (\omega), \phi \rangle = -\langle u (\omega) \cdot \nabla \omega, \phi \rangle.
\]
In Theorem 7 of Section 2 we shall define (for each \( \phi \in C_\infty (T^2) \)) a random variable \( \omega \mapsto \langle b(\omega), \phi \rangle \) on the space \((B, B, \mu)\) (\(B\) being the Borel \(\sigma\)-field on \(B\)). With this definition, identity (3) provides a rigorous definition of the measurable map \( (\omega, t) \mapsto \langle b(\omega), DF(t, \omega) \rangle \), with certain integrability properties in \( \omega \) coming from the results of Section 2.

**Remark 1** To help the intuition, let us heuristically write equation (2) in the form
\[
\partial_t \rho_t + \text{div}_\mu (\rho_t b) = 0 \tag{4}
\]
with initial condition \( \rho_0(\omega) \), where \( \text{div}_\mu (v) \), when defined, for a vector field \( v \) on \( B \), is (heuristically) defined by the identity
\[
\int_B F(\omega) \text{div}_\mu (v(\omega)) \mu(d\omega) = -\int_B \langle v(\omega), DF(\omega) \rangle \mu(d\omega) \tag{5}
\]
for all \( F \in FC^1_b \), where \( FC^1_b \) is defined as \( FC^1_{b,T} \) but without the time-dependent components \( g_i \).

In [12] it is proved that the random variable \( \omega \mapsto \langle b(\omega), \phi \rangle \) on \((B, B, \mu)\) has all finite moments; here we improve the result and show that it is exponentially integrable: given \( \phi \in C_\infty (T^2) \), it holds
\[
\int_B e^{\epsilon|\langle b(\omega), \phi \rangle|} \mu(d\omega) < \infty \tag{6}
\]
for some \( \epsilon > 0 \), which depends only on \( \|\phi\|_\infty \); see Theorem 8 in Section 2 below.

This exponential integrability is a key ingredient to extend, to the 2D Euler equations, the result of the authors [7] for abstract equations in Hilbert spaces (in that work the measure \( \mu \) is not necessarily Gaussian, but the nonlinearity is bounded). Indeed, we aim to prove existence in the class of densities \( \rho_t(\omega) \) such that
\[
\sup_{t \in [0,T]} \int_B \rho_t(\omega) \log \rho_t(\omega) \mu(d\omega) < \infty. \tag{7}
\]
Since \( ab \leq e^{a} + e^{-1}b (\log e^{-1}b - 1) \), if \( \rho_t(\omega) \) satisfies (7) and property (6) is proved, then
\[
\int_B \langle b(\omega), DF(t, \omega) \rangle \rho_t(\omega) \mu(d\omega)
\]
is well defined. With these preliminaries we can give the following definition.

**Definition 2** Given a measurable function \( \rho_0 : B \to [0, \infty) \) such that \( \int_B \rho_0(\omega) \log \rho_0(\omega) \mu(d\omega) < \infty \), we say that a measurable function \( \rho : [0,T] \times B \to [0, \infty) \) is a solution of equation (4) of class \( L\log L \) if property (7) is satisfied and identity (2) holds for every \( F \in FC^1_{b,T} \).

Our main result, proved in Section 3 is:

**Theorem 3** If
\[
\int_B \rho_0(\omega) \log \rho_0(\omega) \mu(d\omega) < \infty
\]
then there exists a solution of equation (4) of class \( L\log L \).
2 Definition and properties of $\langle b (\omega) , \phi \rangle$

We denote by $\{ e_n \}$ the complete orthonormal system in $L^2 (T^2 ; \mathbb{C})$ given by $e_n (x) = e^{2 \pi i n \cdot x}$, $n \in \mathbb{Z}^2$. As already said in the Introduction, given a distribution $\omega \in C^\infty (T^2)'$ and a test function $\phi \in C^\infty (T^2)$, we denoted by $\langle \omega , \phi \rangle$ the duality between $\omega$ and $\phi$ (namely $\omega (\phi)$), and we use the same symbol for the inner product of $L^2 (T^2)$. We set $\hat{\omega} (n) = \langle \omega , e_n \rangle$, $n \in \mathbb{Z}^2$ and we define, for each $s \in \mathbb{R}$, the space $H^s (T^2)$ as the space of all distributions $\omega \in C^\infty (T^2)'$ such that $
 \| \omega \|^2_{H^s} := \sum_{n \in \mathbb{Z}^2} \left( 1 + |n|^2 \right)^s |\hat{\omega} (n)|^2 < \infty.$

We use similar definitions and notations for the space $H^s (T^2 , \mathbb{C})$ of complex valued functions.

We want to define, for every $\phi \in C^\infty (T^2)$, the random variable $\langle b (\omega) , \phi \rangle = - \langle u (\omega) \cdot \nabla \omega , \phi \rangle = - \int_{T^2} u (\omega) (x) \cdot \nabla \omega (x) \phi (x) \, dx$ $= \int_{T^2} \omega (x) u (\omega) (x) \cdot \nabla \phi (x) \, dx$ where we have used integration by parts and the condition $\text{div } u = 0$ (the computation is heuristic, or it holds for smooth periodic functions; we are still looking for a meaningful definition). Recall that $u$ is divergence free and associated to $\omega$ by $\omega = \partial_2 u_1 - \partial_1 u_2$. This relation can be inverted using the so called Biot-Savart law: $u (x) = \int_{T^2} K (x - y) \omega (y) \, dy$ where $K (x, y)$ is the Biot-Savart kernel; in full space it is given by $K (x - y) = \frac{1}{2\pi |x - y|^2}$; on the torus its form is less simple but we still have $K$ smooth for $x \neq y$, $K (y - x) = -K (x - y)$, $|K (x - y)| \leq \frac{C}{|x - y|}$ for small values of $|x - y|$. See for instance [14] for details.

The difficulty in the definition of $\langle b (\omega) , \phi \rangle$ is that $\omega$ is of class $H^{-1-\delta} (T^2)$ and $u$ of class $H^{-\delta} (T^2)$, so we need to multiply distributions. The following remark recalls a trick used in several works on measure-valued solutions of 2D Euler equations, like [9], [10], [13], [14], [15].

**Remark 4** If $\omega$ is sufficiently smooth and periodic, using Biot-Savart law we can write $\langle b (\omega) , \phi \rangle = \int_{T^2} \int_{T^2} \omega (x) \omega (y) K (x - y) \cdot \nabla \phi (x) \, dx \, dy.$

4
Since the double integral, when we rename $x$ by $y$ and $y$ by $x$, is the same (the renaming doesn’t affect the value), and $K(x - y) = -K(y - x)$, we get

$$\langle b(\omega), \phi \rangle = \int_{T^2} \int_{T^2} \omega(x) \omega(y) H_\phi(x, y) \, dx \, dy$$

where

$$H_\phi(x, y) := \frac{1}{2} K(x - y) : \left( \nabla \phi(x) - \nabla \phi(y) \right).$$

The advantage of this symmetrization is that $H_\phi$ (opposite to $K(x - y) \cdot \nabla \phi(x)$) is a bounded function. It is smooth outside the diagonal $x = y$, discontinuous on the diagonal; more precisely, we can write

$$H_\phi(x, y) = \frac{1}{2\pi} \left\langle D^2 \phi(x) \frac{x - y}{|x - y|}, \frac{x - y}{|x - y|} \right\rangle + R_\phi(x, y) \quad (8)$$

where $R_\phi(x, y)$ is Lipschitz continuous, with

$$|R_\phi(x, y)| \leq C |x - y|.$$

To summarize, when $\omega$ is sufficiently smooth and periodic, we have

$$\langle b(\omega), \phi \rangle = \langle \omega \otimes \omega, H_\phi \rangle_{L^2(T^2 \times T^2)}$$

where $\omega \otimes \omega : T^2 \times T^2 \to \mathbb{R}$ is defined as $(\omega \otimes \omega)(x, y) = \omega(x) \omega(y)$.

**Remark 5** The previous expression is meaningful when $\omega$ is a measure, since $H_\phi$ is Borel bounded. When $\omega$ is only a distribution, of class $H^{-1-\delta}(T^2)$, one can define $\omega \otimes \omega$ as the unique element of $H^{-2-2\delta}(T^2 \times T^2)$ such that

$$\langle \omega \otimes \omega, f \rangle = \langle \omega, \phi \rangle \langle \omega, \psi \rangle$$

for every smooth $f : T^2 \times T^2 \to \mathbb{R}$ of the form $f(x, y) = \phi(x) \psi(y)$, where the dual pairing $\langle \omega \otimes \omega, f \rangle$ is on $T^2 \times T^2$. But $H_\phi$ is not of class $H^{2+2\delta}(T^2 \times T^2)$, hence there is no simple deterministic meaning for $\langle \omega \otimes \omega, H_\phi \rangle$ when $\omega \in H^{-1-\delta}(T^2)$. It is here that probability will play the essential role.

In [12] the following result has been proved. As remarked above, when $f \in H^{2+2\delta}(T^2 \times T^2)$, $\langle \omega \otimes \omega, f \rangle$ is well defined for all $\omega \in H^{-1-\delta}(T^2)$, hence for a.e. $\omega$ with respect to the Entropy measure $\mu$.

**Lemma 6** Assume $f \in H^{2+\epsilon}(T^2 \times T^2)$ for some $\epsilon > 0$. One has

$$\int_B |\langle \omega \otimes \omega, f \rangle|^p \, d\omega \leq \frac{(2p)!}{2^{2p}p!} \|f\|_\infty^p$$
for every positive integer $p \geq 2$,
\[ \int_B \langle \omega \otimes \omega, f \rangle \mu(d\omega) = \int_{T^2} f(x,x) \, dx \]
and, when $f$ is also symmetric,
\[ \int_B \left| \langle \omega \otimes \omega, f \rangle - \int_{T^2} f(x,x) \, dx \right|^2 \mu(d\omega) = 2 \int_{T^2} \int_{T^2} f(x,y)^2 \, dx \, dy. \]

The consequence proved in [12] is:

**Theorem 7** Let $\omega : \Xi \to C^\infty(T^2)$ be a white noise and $\phi \in C^\infty(T^2)$ be given. Assume that $H^\phi_n \in H^{2+}(T^2 \times T^2)$ are symmetric and approximate $H_\phi$ in the following sense:

\[ \lim_{n \to \infty} \int_{T^2} \int_{T^2} (H^\phi_n - H_\phi)^2(x,y) \, dx \, dy = 0 \]
\[ \lim_{n \to \infty} \int_{T^2} H^\phi_n(x,x) \, dx = 0. \]

Then the sequence of r.v.'s $\langle \omega \otimes \omega, H^\phi_n \rangle$ is a Cauchy sequence in mean square. We denote by

\[ \langle b(\omega), \phi \rangle = \langle \omega \otimes \omega, H_\phi \rangle \]
its limit. Moreover, the limit is the same if $H^\phi_n$ is replaced by $\widetilde{H}^\phi_n$ with the same properties and such that $\lim_{n \to \infty} \int \int (H^\phi_n - \widetilde{H}^\phi_n)^2(x,y) \, dx \, dy = 0$.

A simple example of functions $H^\phi_n$ with these properties is given in [12]. In addition to these fact, here we prove exponential integrability, see (6).

**Theorem 8** Given a bounded measurable $f$ with $\|f\|_\infty \leq 1$, we have

\[ \int_B e^{\epsilon|\omega \otimes \omega, f|} \mu(d\omega) < \infty \]
for all $\epsilon < \frac{1}{2}$.

**Proof.**

\[ \mathbb{E} \left[ e^{\epsilon|\langle \omega \otimes \omega, f \rangle|} \right] = \sum_{p=0}^{\infty} \frac{e^{p\epsilon} \mathbb{E} \left[ |\langle \omega \otimes \omega, f \rangle|^p \right]}{p!} \leq \sum_{p=0}^{\infty} \left( \frac{\epsilon}{2} \right)^p (2p)! \frac{(2p)!}{p!}. \]

This series converges for $\epsilon < \frac{1}{2}$ because (using ratio test)

\[ \frac{\left( \frac{\epsilon}{2} \right)^p (2p+2)!}{(2p+1)! (p+1)!} = \frac{\epsilon (2p+2) (2p+1)}{2} \frac{(2p)!}{(p+1) (p+1)} \to 2\epsilon. \]
3 Proof of Theorem 3

3.1 Approximate problem

Recall from the Introduction that $\delta > 0$ is fixed and we set $B = H^{-1-\delta} (\mathbb{T}^2)$, $H = L^2 (\mathbb{T}^2)$; recall also from Section 2 that we write $e_n (x) = e^{2\pi i n \cdot x}$, $x \in \mathbb{T}^2$, $n \in \mathbb{Z}^2$, that is a complete orthonormal system in $H^C := L^2 (\mathbb{T}^2; \mathbb{C})$. Given $N \in \mathbb{N}$, let $H^C_N$ be the span of $e_n$ for $|n|_\infty \leq N$, $|n|_\infty := \max (|n_1|, |n_2|)$ for $n = (n_1, n_2)$; it is a subspace of $H^C$. Let $H_N$ be the subspace of $H^C_N$ made of real-valued elements; it is a subspace of $H$ and is characterized by the following property: $\omega = \sum_{|n|_\infty \leq N} \omega_n e_n$ is in $H_N$ if and only if $\overline{\omega_n} = \omega_{-n}$, for all $n$ such that $|n|_\infty \leq N$.

Let $\pi_N$ be the orthogonal projection of $H$ onto $H_N$. It is given by $\pi_N \omega = \sum_{|n|_\infty \leq N} \langle \omega, e_n \rangle_H e_n$, for all $\omega \in H$. We extend $\pi_N$ to an operator on $B$ by setting

$$
\pi_N : B \to H_N,
$$
$$
\pi_N \omega = \sum_{|n|_\infty \leq N} \langle \omega, e_n \rangle e_n,
$$

where now $\langle \omega, e_n \rangle$ is the dual pairing. We may introduce the Dirichlet kernel

$$
\theta_N (x_1, x_2) = \sum_{n_1=-N}^{N} \sum_{n_2=-N}^{N} e^{2\pi i (n_1 x_1 + n_2 x_2)} = \sum_{|n|_\infty \leq N} e^{2\pi i n \cdot x}
$$

for $x = (x_1, x_2) \in \mathbb{T}^2$, and check that

$$
\pi_N \omega = \theta_N * \omega.
$$

We define the operator $b_N : B \to H_N$ as

$$
b_N (\omega) = -\pi_N (u (\pi_N \omega) \cdot \nabla \pi_N \omega), \quad \omega \in B
$$

where $u (\pi_N \omega)$ denotes the result of Biot-Savart law applied to $\pi_N \omega$,

$$
u (\pi_N \omega) (x) := \int_{\mathbb{T}^2} K (x - y) (\pi_N \omega) (y) \, dy.
$$

The operator $b_N$ has the following properties. We denote by $\text{div} \, b_N (\omega)$ the function

$$
\text{div} b_N (\omega) = \sum_{|n|_\infty \leq N} \partial_n \langle b_N (\omega), e_n \rangle_H
$$
where, when defined, $\partial_n F(\omega) = \lim_{\epsilon \to 0} \epsilon^{-1} (F(\omega + \epsilon e_n) - F(\omega))$, for a function $F$ defined on $B$. We say that $\text{div} b_N(\omega)$ exists if $\partial_n \langle b_N(\omega), e_n \rangle_H$ exists for all $|n|_{\infty} \leq N$. Moreover, we set

$$\text{div}_\mu b_N(\omega) := \text{div} b_N(\omega) - \langle \omega, b_N(\omega) \rangle$$

where $\langle \omega, b_N(\omega) \rangle$ is the dual pairing. It is easy to check that this definition is coherent with the general one (5) given in the Introduction.

**Lemma 9** The divergence $\text{div} b_N(\omega)$ exists for all $\omega \in B$ and

$$\text{div} b_N(\omega) = 0$$

$$\langle \omega, b_N(\omega) \rangle = 0$$

and thus

$$\text{div}_\mu b_N(\omega) = 0.$$

**Proof.** **Step 1:** A basic identity is

$$\langle \omega, b_N(\omega) \rangle = 0$$

for all $\omega \in B$, where as usual $\langle ., . \rangle$ denotes dual pairing. This identity holds because

$$\langle \omega, \pi_N (u(\pi_N \omega) \cdot \nabla \pi_N \omega) \rangle = \langle \pi_N \omega, u(\pi_N \omega) \cdot \nabla \pi_N \omega \rangle_H = 0$$

where the first equality can be checked by writing $\omega = \sum \langle \omega, e_n \rangle e_n$ (the series converges in $B$), and the second equality is true because

$$\langle v \cdot \nabla f, f \rangle = \frac{1}{2} \int_{\mathbb{T}^2} v(x) \cdot \nabla f^2 (x) \, dx = -\frac{1}{2} \int_{\mathbb{T}^2} \text{div} v(x) f^2 (x) \, dx = 0$$

for all sufficiently smooth divergence free vector field $v$ (we take $v = u(\pi_N \omega)$ that is a smooth divergence free vector field) and all sufficiently smooth functions $f$ (we take $f = \pi_N \omega$).

**Step 2:** Recall that $u(e_n)(x)$ is periodic, divergence free, and such that $\nabla \perp \cdot u(e_n) = e_n$ (it is also given by the Biot-Savart law $u(e_n)(x) := \int_{\mathbb{T}^2} K(x-y) e_n(y) \, dy$). Then we have

$$u(e_n)(x) \cdot \nabla e_n(x) = 0$$

for every $n \in \mathbb{Z}^2$. Indeed,

$$u(e_n)(x) \cdot \nabla e_n(x) = 2\pi i (u(e_n)(x) \cdot n) e_n(x)$$

and this is zero because $u(e_n)(x) \cdot n = 0$. To prove the latter property, it is necessary to understand the shape of $u(e_n)(x)$. It is

$$u(e_n)(x) = \frac{n^\perp}{|n|^2} e_n(x)$$
(which implies \( u(e_n)(x) \cdot n = 0 \) because \( n^\perp \cdot n = 0 \)). Indeed, this function \( u \) is periodic, divergence free (one has \( \text{div} \ u(e_n)(x) = \frac{n^\perp}{|n|} e_n(x) \cdot n = 0 \)) and \( \nabla^\perp \cdot u(e_n)(x) = \frac{n^\perp}{|n|} e_n(x) \cdot n = e_n(x) \).

**Step 3:** Finally we can prove that \( \text{div} b_N(\omega) = 0 \). It is

\[
\text{div} b_N(\omega) = -\sum_{|n| \leq N} \partial_n \langle \pi_N (u(\pi_N \omega) \cdot \nabla \pi_N \omega), e_n \rangle_H.
\]

We have

\[
\partial_n \langle \pi_N (u(\pi_N \omega) \cdot \nabla \pi_N \omega), e_n \rangle_H = \partial_n \langle u(\pi_N \omega) \cdot \nabla \pi_N \omega, e_n \rangle_H = -\partial_n \langle \pi_N \omega, u(\pi_N \omega) \cdot \nabla e_n \rangle_H
\]

(we have used integration by parts and \( \text{div} u(\pi_N \omega) = 0 \) in the last identity)

\[
= -\partial_n \left( \sum_{|n'| \leq N} \langle \omega, e_{n'} \rangle e_{n'}, \sum_{|n''| \leq N} \langle \omega, e_{n''} \rangle u(e_{n''}) \cdot \nabla e_n \right)_H
\]

\[
= -\langle e_n, u(\pi_N \omega) \cdot \nabla e_n \rangle_H - \langle \pi_N \omega, u(e_n) \cdot \nabla e_n \rangle_H.
\]

The first term is zero by the same general rule recalled in Step 1. The second term is zero by Step 2. Therefore \( \text{div} b_N(\omega) = 0 \).

Consider the finite dimensional ordinary differential equation in the space \( H_N \) defined as

\[
\frac{d\omega_i^N}{dt} = b_N(\omega_i^N), \quad \omega_i^N \in H_N.
\]  

(10)

The function \( b_N \), in \( H_N \), is differentiable, bounded with bounded derivative on bounded sets. Hence, for every \( \omega_0^N \in H_N \), there is a unique local solution \( \omega_i^N(\omega_0^N) \) of equation (10) and the flow map \( \omega_0^N \mapsto \omega_i^N(\omega_0^N) \), where defined, is continuously differentiable, invertible with continuously differentiable inverse. The solution is global because of the energy estimate

\[
\frac{d}{dt} \|\omega_i^N\|_H^2 = 2 \langle b_N(\omega_i^N), \omega_i^N \rangle_H = 0
\]

which implies \( \sup_{t \in [0,T]} \|\omega_i^N\|_H^2 \leq \|\omega_0^N\|_H^2 \) on any interval \([0,T]\) of local existence; the property \( \langle b_N(\omega_i^N), \omega_i^N \rangle_H = 0 \) holds by Lemma 9. We denote by \( \Phi_i^N : H_N \to H_N \) the global flow defined as \( \Phi_i^N(\omega_0^N) = \omega_i^N(\omega_0^N) \).
Denote by $\mu^N(\omega)$ the image measure, on $H_N$, of $\mu(\omega)$ under the projection $\pi_N$. This measure is invariant under the flow $\Phi^N_t$, because $\text{div}_\mu b_N(\omega) = 0$: for every smooth $F: H_N \to [0, \infty)$, bounded with bounded derivatives,

$$\int_{H_N} \langle b_N(\omega), DF(\omega) \rangle_{H_N} \mu^N(\omega) d\omega = \int_B \langle b_N(\omega), DF(\pi_N(\omega)) \rangle_H \mu(\omega)$$

$$= - \int_B F(\pi_N(\omega)) \text{div}_\mu b_N(\omega) \mu(\omega) d\omega = 0.$$

### 3.2 Continuity equation for the approximate problem

Given a measurable function $\rho^N_0: H_N \to [0, \infty)$, with $\int_B \rho^N_0(\pi_N(\omega)) \mu(\omega) < \infty$, consider the measure $\rho^N_0(\pi_N(\omega)) \mu^N(\omega)$ and its push forward under the flow map $\Phi^N_1$; denote it by $\nu^N_t$. By definition, for bounded measurable $F: H_N \to [0, \infty)$,

$$\int_{H_N} F(\omega) \nu^N_t(\omega) d\omega = \int_{H_N} F(\omega) \rho^N_0(\omega) \mu^N(\omega) d\omega.$$

From the invariance of $\mu^N$ under the flow $\Phi^N_t$, we have

$$\int_{H_N} F(\omega) \nu^N_t(\omega) d\omega = \int_{H_N} F(\omega) \rho^N_0(\omega) \left(\Phi^N_1(\omega)^{-1}\right) \mu^N(\omega)$$

hence

$$\nu^N_t(\omega) = \rho^N_t(\pi_N(\omega)) \mu^N(\omega)$$

where

$$\rho^N_t(\omega) = \rho^N_0 \left(\Phi^N_1(\omega)^{-1}\right), \quad \omega \in H_N. \quad (11)$$

We have partially proved the following statement.

**Lemma 10** Consider equation (10) in $H_N$, with the associated flow $\Phi^N_t$. Given at time zero a measure of the form $\rho^N_0(\pi_N(\omega)) \mu^N(\omega)$ with $\int_B \rho^N_0(\pi_N(\omega)) \mu(\omega) < \infty$, its push forward at time $t$, under the flow map $\Phi^N_t$, is a measure of the form $\rho^N_t(\pi_N(\omega)) \mu^N(\omega)$, with $\int_B \rho^N_t(\pi_N(\omega)) \mu(\omega) < \infty$. If in addition $\int_B \rho^N_0(\pi_N(\omega)) \log \rho^N_0(\pi_N(\omega)) \mu(\omega) < \infty$, the same is true at time $t$ and

$$\int_B \rho^N_t(\pi_N(\omega)) \log \rho^N_t(\pi_N(\omega)) \mu(\omega) d\omega = \int_B \rho^N_0(\pi_N(\omega)) \log \rho^N_0(\pi_N(\omega)) \mu(\omega) d\omega. \quad (12)$$

If in addition $\rho^N_0$ is bounded, then $\rho^N_t \leq \left\| \rho^N_0 \right\|_{\infty}$. Finally, $\rho^N_t$ satisfies the continuity equation

$$\int_0^T \int_B \left( \partial_t F(t, \omega) + \langle DF(t, \omega), b_N(\omega) \rangle_H \right) \rho^N_t(\pi_N(\omega)) \mu(\omega) dt = - \int_B F(0, \omega) \rho^N_0(\pi_N(\omega)) \mu(\omega) d\omega$$

for all $F \in \mathcal{F}_{C^1_{b,T}}$ of the form $\tilde{F}(\omega) = \sum_{i=1}^m \tilde{f}_i(\omega, e_n), |n|_{\infty} \leq N, g_i(t)$.
Proof. The integrability of $\rho_t^N$ comes from the invariance of $\mu^N$ under $\Phi_t^N$, as well as the \text{LlogL} property; let us check this latter one. Using (11) we have

$$
\int_B \rho_t^N (\pi_N \omega) \log \rho_t^N (\pi_N \omega) \mu (d\omega) = \int_{H_N} \rho_t^N (\omega) \log \rho_t^N (\omega) \mu^N (d\omega)
$$

$$
= \int_{H_N} \rho_0^N \left( (\Phi_t^N)^{-1} (\omega) \right) \log \rho_0^N \left( (\Phi_t^N)^{-1} (\omega) \right) \mu^N (d\omega)
$$

$$
= \int_{H_N} \rho_0^N (\omega) \log \rho_0^N (\omega) \mu^N (d\omega)
$$

$$
= \int_B \rho_0^N (\pi_N \omega) \log \rho_0^N (\pi_N \omega) \mu (d\omega).
$$

When $\rho_0^N$ is bounded, we have

$$
\rho_t^N (\omega) = \rho_0^N \left( (\Phi_t^N)^{-1} (\omega) \right) \leq \| \rho_0^N \|_\infty.
$$

Finally, form the chain rule applied to $F \left( t, \Phi_t^N (\omega) \right)$, $\omega \in H_N$, we get the weak form of the continuity equation.

Remark 11 We may construct $\rho_t^N$ and prove (12) also by the following procedure, closer to [7]. We study the transport equation in $H_N$

$$
\partial_t \rho_t^N + \langle b_N, D \rho_t^N \rangle_H = 0
$$

with initial condition $\rho_0^N$, which has the solution (11) by the method of characteristics. Its weak form reduces to (13) because (for $F$ like those of the Lemma)

$$
\int_{H_N} F (t, \omega) \langle b_N (\omega), D \rho_t^N (\omega) \rangle_H \mu^N (d\omega)
$$

$$
= \int_B F (t, \omega) \langle b_N (\omega), D \rho_t^N (\pi_N \omega) \rangle_H \mu (d\omega)
$$

$$
= - \int_B \langle DF (t, \omega), b_N (\omega) \rangle_H \rho_t^N (\pi_N \omega) \mu (d\omega)
$$

where we have used the property $\text{div}_\mu b_N (\omega) = 0$. Finally, to prove (12) as in [7], we compute

$$
\frac{d}{dt} \int_{H_N} \rho_t^N (\log \rho_t^N - 1) d\mu^N
$$

$$
= \int_{H_N} \log \rho_t^N \partial_t \rho_t^N d\mu^N = - \int_{H_N} \log \rho_t^N \langle b_N, D \rho_t^N \rangle d\mu^N
$$

$$
= - \int_{H_N} \langle b_N, D \left[ \rho_t^N (\log \rho_t^N - 1) \right] \rangle d\mu^N
$$

$$
= \int_{H_N} \left[ \rho_t^N (\log \rho_t^N - 1) \right] \text{div}_\mu b_N d\mu^N = 0.
$$
3.3 Construction of a solution to the limit problem

3.3.1 First case: bounded $\rho_0$

Consider first the case when $\rho_0$ is a bounded measurable function on $B$. Define the sequence of equibounded functions $\rho_0^N$ on $H_N$ by setting $\rho_0^N (\pi_N \omega) = \rho_0 (\pi_N \omega)$. For each one of them, consider the associated function $\rho_t^N (\pi_N \omega)$ given by Lemma 10. There is a subsequence, still denoted for simplicity by $\rho_t^N (\pi_N \omega)$ which converges to some function $\rho_t$ weak* in $L^\infty ([0,T] \times B)$; moreover we have (12) which implies (see [7] for similar computations)

$$\int_B \rho_t (\omega) \log \rho_t (\omega) \mu (d\omega) \leq \int_B \rho_0 (\omega) \log \rho_0 (\omega) \mu (d\omega).$$

Finally we have to prove that $\rho_t$ satisfies the weak formulation. We have to pass to the limit in (13). The only problem is the term

$$\int_0^T \int_B \langle b_N (\omega), DF (t, \omega) \rangle_H \rho_t^N (\pi_N \omega) \mu (d\omega) dt.$$

We add and subtract the term

$$\int_0^T \int_B \langle b (\omega), DF (t, \omega) \rangle \rho_t^N (\pi_N \omega) \mu (d\omega) dt$$

and use integrability of $\langle b (\omega), DHF (t, \omega) \rangle$ and weak* convergence of $\rho_t^N (\pi_N \omega)$ to $\rho_t (\omega)$ to pass to the limit in one addend. It remains to prove that

$$\lim_{k \to \infty} \int_0^T \int_B (\langle b_N (\omega), DF (t, \omega) \rangle_H - \langle b (\omega), DF (t, \omega) \rangle) \rho_t^N (\pi_N \omega) \mu (d\omega) dt = 0.$$

Keeping in mind again the weak* convergence of $\rho_t^N (\pi_N \omega)$, it is sufficient to prove that $\int_B \langle b_N (\omega), DHF (t, \omega) \rangle_H$ converges strongly to $\langle b (\omega), DHF (t, \omega) \rangle$ in $L^1 (0,T; L^1 (B, \mu))$. Due to the form of $F$, it is sufficient to prove the following claim: given $\phi \in C^\infty (\Omega^2)$,

$$\lim_{k \to \infty} \int_B |\langle b_N (\omega), \phi \rangle_H - \langle b (\omega), \phi \rangle| \mu (d\omega) = 0.$$

The remainder of this subsection is devoted to the proof of this claim. It is not restrictive to assume that $\phi \in H_{N_0}$ for some $N_0$. Hence, for $N$ large enough so that $\pi_N \phi = \phi$, we have

$$\langle b_N (\omega), \phi \rangle_H = - \langle \pi_N (u (\pi_N \omega) \cdot \nabla \pi_N \omega), \phi \rangle_H$$

$$= - \langle u (\pi_N \omega) \cdot \nabla \pi_N \omega, \phi \rangle_H$$

$$= \langle \pi_N \omega, u (\pi_N \omega) \cdot \nabla \phi \rangle_H$$

$$= \langle (\pi_N \omega) \otimes (\pi_N \omega), H \phi \rangle$$.
where the last identity is proved as in Remark 4. We have
\[ \langle (\pi_N^* \omega) \otimes (\pi_N^* \omega), H_\phi \rangle = \langle \omega \otimes \omega, (H_\phi)_N \rangle \]
where
\[ (H_\phi)_N (x,y) = \sum_{|n| \leq N} \sum_{|n'| \leq N} e_n(x) e_{n'}(y) \int_{T^2} \int_{T^2} e_{n'}(y') e_n(x') H_\phi(x',y') \, dx'dy'. \]

Therefore, our aim is to prove that, given \( \phi \in C^\infty (\mathbb{T}^2) \),
\[ \lim_{k \to \infty} \int_B |\langle \omega \otimes \omega, (H_\phi)_N - H_\phi \rangle| \, \mu (d\omega) = 0. \]

Thanks to Lemma 6 and Theorem 7, with a simple argument on Cauchy sequences one can see that it is sufficient to prove that \( (H_\phi)_N \to H_\phi \) in \( L^2 (\mathbb{T}^2 \times \mathbb{T}^2) \) and
\[ \int_{T^2} (H_\phi)_N(x,x) \, dx \to 0. \]
(14)

From the theory of Fourier series, \( (H_\phi)_N \to H_\phi \) in \( L^2 (\mathbb{T}^2 \times \mathbb{T}^2) \). The limit property (14) requires more work. The result is included in the next lemma, which completes the proof that \( \rho_t \) is a weak solution, in the case when \( \rho_0 \) is bounded.

**Lemma 12** i) The Dirichlet kernel \( \theta_N(x_1, x_2) = \theta_N(x_2, x_1) \)
\[ \theta_N(-x_1, x_2) = \theta_N(x_1, x_2). \]

ii) If a kernel \( \theta_N(x), x \in \mathbb{T}^2 \), has these two properties, the the kernel \( W_N = \theta_N * \theta_N \) has the same properties.

iii) It follows that, for any symmetric matrix \( S \),
\[ \int_{T^2} W_N(x) \left< S \frac{x}{|x|}, \frac{x}{|x|} \right> \, dx = 0. \]

iv) It follows also that
\[ \lim_{N \to \infty} \int_{T^2} \int_{T^2} W_N(x-y) H_\phi(x,y) \, dx \, dy = 0. \]

In the case when \( \theta_N \) is the Dirichlet kernel, this property is the limit property (14).
Proof. Property (i) is obvious. The proof of (ii) is elementary, but we give the computations for completeness:

\[ W_N(x_1, x_2) = \int_{T^2} \theta_N(x_1 - y_1, x_2 - y_2) \theta_N(y_1, y_2) \, dy_1 dy_2 \]
\[ = \int_{T^2} \theta_N(x_2 - y_2, x_1 - y_1) \theta_N(y_2, y_1) \, dy_1 dy_2 \]
\[ = W_N(x_2, x_1) \]

\[ W_N(-x_1, x_2) = \int_{T^2} \theta_N(-x_1 - y_1, x_2 - y_2) \theta_N(y_1, y_2) \, dy_1 dy_2 \]
\[ = \int_{T^2} \theta_N(x_1 + y_1, x_2 - y_2) \theta_N(y_1, y_2) \, dy_1 dy_2 \]
\[ = \int_{T^2} \theta_N(x_1 - y_1, x_2 - y_2) \theta_N(-y_1, y_2) \, dy_1 dy_2 \]
\[ = \int_{T^2} \theta_N(x_1 - y_1, x_2 - y_2) \theta_N(y_1, y_2) \, dy_1 dy_2 \]
\[ = W_N(x_1, x_2) \]

Let us prove (iii). We can write

\[ \langle S \frac{x}{|x|}, \frac{x}{|x|} \rangle = (S_{11} + S_{22}) \frac{x_1 x_2}{|x|^2} + S_{12} \frac{x_2^2 - x_1^2}{|x|^2}. \]

Let us show that the integrals corresponding to each one of the two terms vanish. We have

\[ \int_{T^2} W_N(x) \frac{x_1 x_2}{|x|^2} dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} W_N(x) \frac{x_1 x_2}{|x|^2} dx_1 dx_2 \]

The integration in the second quadrant,

\[ \int_{0}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} W_N(x) \frac{x_1 x_2}{|x|^2} dx_1 dx_2 \]

cancels with the integration in the first quadrant,

\[ \int_{0}^{\frac{1}{2}} \int_{0}^{\frac{1}{2}} W_N(x) \frac{x_1 x_2}{|x|^2} dx_1 dx_2 \]

because of property \( W_N(-x_1, x_2) = W_N(x_1, x_2) \) (point (ii)); similarly for the integrations in the other quadrants. So \( \int_{T^2} W_N(x) \frac{x_1 x_2}{|x|^2} dx = 0 \). For the other integral, just by renaming the variables we have

\[ \int_{T^2} W_N(x_1, x_2) \frac{x_1^2}{|x|^2} dx_1 dx_2 = \int_{T^2} W_N(x_2, x_1) \frac{x_2^2}{|x|^2} dx_2 dx_1 \]
and then, using \( W_N(x_1, x_2) = W_N(x_2, x_1) \) (point (ii))
\[
= \int_{T^2} W_N(x_1, x_2) \frac{x_2^2}{|x|^2} dx_1 dx_2
\]

hence \( \int_{T^2} W_N(x) \frac{x^2-x_1^2}{|x|^2} dx = 0 \). We have proved (iii).

Finally, the limit in (iv) is a consequence of the decomposition \((8)\). Indeed,
\[
\int_{T^2} \int_{T^2} W_N(x - y) \left( D^2 \phi(x) \frac{x - y}{|x - y|}, \frac{(x - y)^\perp}{|x - y|} \right) dx dy
\]
\[
= \int_{T^2} \left( \int_{T^2} W_N(z) \left( D^2 \phi(x) \frac{z}{|z|}, \frac{z^\perp}{|z|} \right) dz \right) dx = 0
\]
by (iii), and
\[
\lim_{N \to \infty} \int_{T^2} \int_{T^2} W_N(x - y) R_\phi(x, y) dx dy = 0
\]
because \( R_\phi(x, y) \) is Lipschitz continuous with \( |R_\phi(x, y)| \leq C |x - y| \). To complete the proof of the claims of part (iv), let us check that, when \( \theta_N \) is the Dirichlet kernel, the property stated in (iv) coincides with the limit property \((14)\). We have
\[
\int_{T^2} (H_\phi)_N(x, x) dx = \sum_{|n'|_\infty \leq N} \sum_{|n|_\infty \leq N} \int_{T^2} \int_{T^2} e^{2\pi in' \cdot (x-x')} e^{2\pi in \cdot (x-y')} H_\phi(x', y') dy' dx' dx
\]
\[
= \int_{T^2} \int_{T^2} \left( \sum_{|n'|_\infty \leq N} \sum_{|n|_\infty \leq N} \int_{T^2} e^{2\pi in' \cdot (x-x')} e^{2\pi in \cdot (x-y')} dx \right) H_\phi(x', y') dy' dx'
\]
\[
= \int_{T^2} \int_{T^2} W_N(x' - y') H_\phi(x', y') dy' dx'.
\]
The proof is complete. \(\blacksquare\)

### 3.3.2 General case: \( \rho_0 \) of class \( L \log L \)

Assume now that \( \rho_0 \) satisfies only the assumptions of the main theorem. Define \( \rho_0^n = \rho_0 \wedge n \).

For each \( n \), apply the result of the first case and construct a weak solution \( \rho_t^n \), which fulfills in particular
\[
\int_B \rho_t^n(\omega) \log \rho_t^n(\omega) \mu(d\omega) \leq \int_B \rho_0^n(\omega) \log \rho_0^n(\omega) \mu(d\omega) \leq \int_B \rho_0(\omega) \log \rho_0(\omega) \mu(d\omega).
\]

From this inequality we deduce the existence of a subsequence, still denoted for simplicity by \( \rho_t^n(\omega) \) which converges to some function \( \rho_t \) weak* in \( L^1(0, T; L^1(B, \mu)) \), which satisfies
property (7), and moreover, from the duality of Orlicz spaces, such that

$$\int_0^T \int_B G(t, \omega) \rho^n_t (\pi_N \omega) \mu(d\omega) \, dt \to \int_0^T \int_B G(t, \omega) \rho_t (\omega) \mu(d\omega) \, dt$$

for all $G$ such that, for some $\epsilon > 0$,

$$\sup_{t \in [0,T]} \int_B e^{\epsilon G(t, \omega)} \mu(d\omega) < \infty. \quad (15)$$

Due to these fact, in order to prove that $\rho_t$ satisfies the weak formulation of the continuity equation, we have only to prove that

$$\int_0^T \int_B \langle b(\omega), DF(t, \omega) \rangle \rho^n_t (\omega) \mu(d\omega) \, dt \to \int_0^T \int_B \langle b(\omega), DF(t, \omega) \rangle \rho_t (\omega) \mu(d\omega) \, dt.$$

Since $G(t, \omega) := \langle b(\omega), DF(t, \omega) \rangle$ has property (15) by Theorem 8, this is true, and the proof is complete.

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