CANONICAL SUBSHEAVES OF TORSIONFREE SEMISTABLE SHEAVES

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Abstract. Let $F$ be a torsionfree semistable coherent sheaf on a polarized normal projective variety defined over an algebraically closed field. We prove that $F$ has a unique maximal locally free subsheaf $V$ such that $F/V$ is torsionfree and $V$ also admits a filtration of subbundles for which each successive quotient is a stable vector bundle whose slope is $\mu(F)$. We also prove that $F$ has a unique maximal reflexive subsheaf $W$ such that $F/W$ is torsionfree and $W$ admits a filtration of subsheaves for which each successive quotient is a stable reflexive sheaf whose slope is $\mu(F)$. We show that these canonical subsheaves behave well with respect to the pullback operation by étale Galois covering maps. Given a separable finite surjective map $\phi : Y \to X$ between normal projective varieties, we give a criterion for the induced homomorphism of étale fundamental groups $\phi_* : \pi_{et}^1(Y) \to \pi_{et}^1(X)$ to be surjective. The criterion in question is expressed in terms of the above mentioned unique maximal locally free subsheaf associated to the direct image $\phi_* \mathcal{O}_Y$.

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1. INTRODUCTION

Let $X$ be an irreducible normal projective variety, defined over an algebraically closed field. Fix a very ample line bundle on $X$ to define (semi)stable sheaves. Here (semi)stability would always refer to $\mu$-(semi)stability.

Let $E$ be a reflexive semistable sheaf on $X$. Then $E$ admits a unique maximal polystable subsheaf $E_1 \subset E$ such that $\mu(E) = \mu(E_1)$ and $E/E_1$ is torsionfree. It may be mentioned that a similar result holds also for Gieseker semistable sheaves. However, if $E$ is just a torsionfree semistable sheaf on $X$, then a similar subsheaf $E_1$ does not exist in general (such an example is given in Section 3.1).

We prove the following (see Proposition 3.3 and Proposition 3.5):

**Proposition 1.1.** Let $F$ be a torsionfree semistable sheaf on $X$.

(a) Assume that $F$ contains a polystable reflexive subsheaf $F'$ with $\mu(F') = \mu(F)$. Then there is a unique reflexive subsheaf $V \subset F$ satisfying the following three conditions:

1. $V$ is polystable with $\mu(V) = \mu(F)$,
2. $F/V$ is torsionfree, and
3. $V$ is maximal among all reflexive subsheaves of $F$ satisfying the above two conditions.

(b) Assume that $F$ contains a polystable locally free subsheaf $F''$ with $\mu(F'') = \mu(F)$. Then there is a unique locally free subsheaf $W \subset F$ satisfying the following three conditions:

1. $W$ is polystable with $\mu(W) = \mu(F)$,
2. $F/W$ is torsionfree, and
3. $W$ is maximal among all locally free subsheaves of $F$ satisfying the above two conditions.

In part (a) of Proposition 1.1 if $F$ does not contain any polystable reflexive subsheaf $F'$ with $\mu(F') = \mu(F)$, then we set $V = 0$. In part (b) of Proposition 1.1 if $F$ does not contain any polystable locally free subsheaf $F''$ with $\mu(F'') = \mu(F)$, then we set $W = 0$.

A semistable sheaf $F$ is called a pseudo-stable sheaf if $F$ admits a filtration of subsheaves

$$0 = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq F_n = F$$

such that $F_i/F_{i-1}$ is a stable reflexive sheaf with $\mu(F_i/F_{i-1}) = \mu(F)$ for all $1 \leq i \leq n$. If $F$ and all $F_i/F_{i-1}$ are locally free, then a pseudo-stable sheaf is called a pseudo-stable bundle.

An iterative application of Proposition 1.1 gives the following (see Theorem 4.1 and Theorem 4.3):

**Theorem 1.2.** Let $F$ be a torsionfree semistable sheaf on $X$. 
(a) Assume that \( F \) contains a polystable reflexive subsheaf \( F' \) with \( \mu(F') = \mu(F) \). Then there is a unique pseudo-stable subsheaf \( V \subset F \) satisfying the following three conditions:
(1) \( \mu(V) = \mu(F) \),
(2) \( F/V \) is torsionfree, and
(3) \( V \) is maximal among all pseudo-stable subsheaves of \( F \) satisfying the above two conditions.

(b) Assume that \( F \) contains a polystable locally free subsheaf \( F'' \) with \( \mu(F'') = \mu(F) \). Then there is a unique coherent subsheaf \( W \subset F \) satisfying the following four conditions:
(1) \( \mu(W) = \mu(F) \),
(2) \( W \) is a pseudo-stable bundle,
(3) \( F/W \) is torsionfree, and
(4) \( W \) is maximal among all subsheaves of \( F \) satisfying the above three conditions. In particular, \( W \) is locally free.

As before, in part (a) of Theorem 1.2 if \( F \) does not contain any polystable reflexive subsheaf \( F' \) with \( \mu(F') = \mu(F) \), then we set \( V = 0 \). In part (b) of Theorem 1.2 if \( F \) does not contain any polystable locally free subsheaf \( F'' \) with \( \mu(F'') = \mu(F) \), then we set \( W = 0 \).

In Section 5.1 we show that the canonical subsheaves in Theorem 1.2 behave well with respect to the pullback operation by étale Galois covering maps; see Proposition 5.1 and Proposition 5.2.

Let \( X \) and \( Y \) be irreducible normal projective varieties over an algebraically closed field \( k \), and let

\[ \phi : Y \rightarrow X \]

be a separable finite surjective map. Then \( \mu_{\text{max}}(\phi_*\mathcal{O}_Y) = 0 \). Let \( F \subset \phi_*\mathcal{O}_Y \) be the first nonzero term of the Harder–Narasimhan filtration of \( \phi_*\mathcal{O}_Y \). Let

\[ W \subset F \quad (1.1) \]

be the unique locally free pseudo-stable bundle given by the second part of Theorem 1.2. Then we have \( \mathcal{O}_X \subset W \).

We prove the following (see Proposition 6.3):

**Proposition 1.3.** The homomorphism of étale fundamental groups induced by \( \phi \)

\[ \phi_* : \pi_1^{\text{et}}(Y) \rightarrow \pi_1^{\text{et}}(X) \]

is surjective if and only if \( W = \mathcal{O}_X \), where \( W \) is the subsheaf in (1.1).

When \( \dim X = 1 \) (equivalently, \( \dim Y = 1 \)), Proposition 1.3 was proved in [BP]. We note that \( F \) coincides with \( W \) in (1.1) when \( \dim X = 1 \).


2. Pseudo-stable sheaves

Let $k$ be an algebraically closed field. Let $X$ be an irreducible normal projective variety defined over $k$. Fix a very ample line bundle $L$ on $X$; using it, define the degree

$$\deg(F) \in \mathbb{Z}$$

of any torsionfree coherent $F$ sheaf on $X$ [HL, p. 13–14, Definition 1.2.11]. The slope of $F$, which is denoted by $\mu(F)$, is defined to be

$$\mu(F) := \frac{\deg(F)}{\text{rank}(F)}.$$

We recall that $F$ is called stable (respectively, semistable) if

$$\mu(V) < \mu(F) \quad \text{(respectively, } \mu(V) \leq \mu(F))$$

for all coherent subsheaves $V \subset F$ with $0 < \text{rank}(V) < \text{rank}(F)$ [HL p. 14, Definition 1.2.11]. Also, $F$ is called polystable if

- $F$ is semistable, and
- $F$ is a direct sum of stable sheaves.

Following [BS], we define:

**Definition 2.1.**

1. A semistable sheaf $F$ is called a pseudo-stable sheaf if $F$ admits a filtration of subsheaves

$$0 = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq F_n = F$$

such that $F_i/F_{i-1}$ is a stable reflexive sheaf with $\mu(F_i/F_{i-1}) = \mu(F)$ for every $1 \leq i \leq n$.

2. A semistable vector bundle $F$ is called a pseudo-stable bundle if $F$ admits a filtration of subbundles

$$0 = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq F_n = F$$

such that $F_i/F_{i-1}$ is a stable vector bundle with $\mu(F_i/F_{i-1}) = \mu(F)$ for every $1 \leq i \leq n$.

Note that a polystable reflexive sheaf is a pseudo-stable sheaf, and a polystable vector bundle is a pseudo-stable bundle.

**Remark 2.2.** Since any polystable sheaf is a direct sum of stable sheaves of same slope, and any polystable vector bundle is a direct sum of stable vector bundles of same slope, we conclude that a semistable sheaf $F$ is pseudo-stable if it admits a filtration of subsheaves

$$0 = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq F_n = F$$

such that $F_i/F_{i-1}$ is a polystable reflexive sheaf with $\mu(F_i/F_{i-1}) = \mu(F)$ for all $1 \leq i \leq n$. Similarly, a semistable vector bundle $F$ is a pseudo-stable bundle if $F$ admits a filtration of subbundles

$$0 = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq F_n = F$$
such that $F_i/F_{i-1}$ is a polystable vector bundle with $\mu(F_i/F_{i-1}) = \mu(F)$ for all $1 \leq i \leq n$.

**Lemma 2.3.** Let $F$ be a pseudo-stable sheaf on $X$. Then $F$ is reflexive.

*Proof.* Let

$$0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$$

be a short exact sequence of coherent sheaves on $X$ such that both $A$ and $B$ are reflexive. Then it can be shown that $E$ is reflexive. To prove this, consider the commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & A \\
\downarrow \hat{\iota} & & \downarrow \iota \\
0 & \rightarrow & A^{**}
\end{array}
\begin{array}{ccc}
& & E \\
& & \downarrow b \\
E^{**} & \rightarrow & B^{**}
\end{array}
\begin{array}{ccc}
\rightarrow & & B \\
\downarrow l' & & \downarrow l \\
\rightarrow & & 0
\end{array}
$$

Since $A$ and $B$ are reflexive it follows that $\hat{\iota}$ and $\iota'$ are isomorphisms. The homomorphism $b^{**}$ is surjective because both $b$ and $\iota'$ are surjective. This and the fact that both $\hat{\iota}$ and $\iota'$ are isomorphisms together imply that $\iota$ is an isomorphism. Consequently, the coherent sheaf $E$ is reflexive.

The sheaf $F$ admits a filtration of subsheaves

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = F$$

such that $F_i/F_{i-1}$ is reflexive for all $1 \leq i \leq n$. Since $F_1$ and $F_2/F_1$ are reflexive, the above observation implies that $F_2$ is reflexive. Now we inductively observe that if $F_i$ is reflexive, then $F_{i+1}$ is reflexive. Therefore, $F$ is reflexive. $\square$

### 3. Maximal Polystable Subsheaves of Semistable Sheaves

We first recall a proposition from [BDL].

**Proposition 3.1** ([BDL p. 1034, Proposition 3.1]). Let $F$ be a reflexive semistable sheaf on $X$. Then there is a unique coherent subsheaf $V \subset F$ satisfying the following three conditions:

1. $V$ is polystable with $\mu(V) = \mu(F)$,
2. $F/V$ is torsionfree, and
3. $V$ is maximal among all subsheaves of $F$ satisfying the above two conditions.

A few clarifications on Proposition 3.1 are in order.

**Remark 3.2.**

1. Although the base field in Proposition 3.1 of [BDL] is $\mathbb{C}$, it is straight-forward to check that the proof of Proposition 3.1 in [BDL] gives Proposition 3.1.
2. Now we use the notation of [BDL p. 1034, Proposition 3.1]. For the subsheaf $E' \subset E$ in [BDL Proposition 3.1], the quotient $E/E'$ is actually torsionfree because $E'$ is a maximal polystable subsheaf, as asserted in [BDL Proposition 3.1]. Note that if $E/E'$ has torsion, then the inverse image in $E$ of the torsion subsheaf.
(\(E/E'\))_{\text{torsion}} \subset E/E', under the quotient map \(E \rightarrow E/E'\), is a polystable subsheaf containing \(E'\).

(3) A similar result for Gieseker semistable sheaves is known (see [HL, p. 23, Lemma 1.5.5]).

In Proposition 3.1 the assumption that \(F\) is reflexive is essential, as shown by the following example.

3.1. An example. Take \((X, L)\) with \(\dim X \geq 2\) such that there are two line bundle \(A\) and \(B\) on \(X\) satisfying the following two conditions:

- \(A \neq B\), and
- \(\text{degree}(A) = \text{degree}(B)\).

Fix a point \(x \in X\), and take a line \(S \subset (A \oplus B)_x = A_x \oplus B_x\) such that \(S \neq A_x\) and \(S \neq B_x\).

Consider the composition of homomorphisms

\[
A \oplus B \rightarrow (A \oplus B)_x \rightarrow (A \oplus B)_x/S;
\]

both \((A \oplus B)_x\) and \(S\) are torsion sheaves supported at \(x\). Let \(F\) denote the kernel of this composition of homomorphisms. We list some properties of \(F\):

- \(F\) is torsionfree as it is a subsheaf of \(A \oplus B\).
- \(F\) is not reflexive, because \(F^{**} = A \oplus B\); here the condition that \(\dim X \geq 2\) is used.
- \(\mu(F) = \mu(A \oplus B) = \mu(A) = \mu(B)\).
- \(F/(F \cap A) = B\) and \(F/(F \cap B) = A\).
- \(F\) is semistable because \(A \oplus B\) is so.
- \(F\) is not polystable. This is because the short exact sequence

\[
0 \rightarrow F \cap A \rightarrow F \rightarrow B \rightarrow 0
\]

does not split.

Therefore, if we set \(V = F \cap A\) or \(V = F \cap B\), then the following three conditions are valid:

1. \(V\) is polystable with \(\mu(V) = \mu(F)\),
2. \(F/V\) is torsionfree, and
3. \(V\) is maximal among all subsheaves satisfying the above two conditions.

Hence Proposition 3.1 fails for \(F\).

3.2. A unique maximal reflexive subsheaf. Although Proposition 3.1 fails for general non-reflexive torsionfree sheaves, the following variation of it holds.

**Proposition 3.3.** Let \(F\) be a torsionfree semistable sheaf on \(X\). Assume that \(F\) contains a polystable reflexive subsheaf \(F'\) with \(\mu(F') = \mu(F)\). Then there is a unique coherent subsheaf \(V \subset F\) satisfying the following four conditions:
(1) $V$ is reflexive,
(2) $V$ is polystable with $\mu(V) = \mu(F),$
(3) $F/V$ is torsionfree, and
(4) $V$ is maximal among all subsheaves of $F$ satisfying the above three conditions.

**Proof.** The coherent sheaf $F^{**}$ is reflexive and semistable; also, we have $\mu(F^{**}) = \mu(F)$. Apply Proposition 3.1 to $F^{**}$. Let

$$W \subset F^{**}$$

be the unique polystable subsheaf satisfying the three conditions in Proposition 3.1.

Let $\mu_0 := \mu(W) = \mu(F)$. Since $W$ is polystable, it is a direct sum of stable sheaves of slope $\mu_0$. Let $C$ denote the space of all coherent subsheaves $E \subset W$ satisfying the following two conditions:

1. $E$ is a direct summand of $W$, meaning there is a coherent subsheaf $E' \subset W$ such that the natural homomorphism $E \oplus E' \rightarrow W$ is an isomorphism.
2. $E$ is indecomposable, meaning if $E = E_1 \oplus E_2$, then either $E_1 = 0$ or $E_2 = 0$.

Note that any $E \in C$ is stable with $\mu(E) = \mu_0$. Moreover, $E$ is reflexive, because it is direct summand of the reflexive sheaf $W$. Since $W$ is polystable, for any subset $C' \subset C$, the coherent subsheaf of $W$ generated by all subsheaves $E \in C'$ is a direct summand of $W$. Let

$$C_F \subset C$$

be the subset consisting of all subsheaves $E \in C$ such that $E \subset F$. The space $C_F$ is nonempty, because $F$ contains a polystable reflexive subsheaf $F'$ with $\mu(F') = \mu(F)$.

Let

$$V \subset W$$

be the coherent subsheaf of $W$ generated by all subsheaves $E \in C_F$, where $C_F$ is defined in (3.1). We will show that $V$ defined in (3.2) satisfies all the conditions in the proposition.

Recall from Proposition 3.1 that $W$ is reflexive and polystable with $\mu(W) = \mu_0$. Since $V$ in (3.2) is a direct summand of $W$, we conclude that $V$ is reflexive and polystable with $\mu(V) = \mu_0$.

We note that $F^{**}/V$ fits in the short exact sequence

$$0 \rightarrow W/V \rightarrow F^{**}/V \rightarrow F^{**}/W \rightarrow 0.$$ 

Since both $F^{**}/W$ and $W/V$ are torsionfree, it follows that $F^{**}/V$ is also torsionfree. Hence the subsheaf $F/V \subset F^{**}/V$ is torsionfree.

Let $\widetilde{V} \subset F$ be a subsheaf satisfying the following three conditions:

1. $\widetilde{V}$ is reflexive,
2. $\widetilde{V}$ is polystable with $\mu(\widetilde{V}) = \mu(F)$, and
3. $F/\widetilde{V}$ is torsionfree.

Then from the properties of $W$ we know that $\widetilde{V} \subset W$. From this it follows that $\widetilde{V} \subset V$. This proves the uniqueness of $V$ satisfying the four conditions in the proposition.  \qed
Remark 3.4. In Proposition 3.3, if $F$ does not contain any polystable reflexive subsheaf $F'$ with $\mu(F') = \mu(F)$, then we set the maximal subsheaf $V$ in Proposition 3.3 to be the zero subsheaf $0 \subset F$. To explain this convention, consider the sheaf $F$ in Section 3.1. Note that $F$ does not contain any reflexive subsheaf $F'$ with $\mu(F') = \mu(F)$. Hence, by this convention, the subsheaf $V$ of $F$ given by Proposition 3.3 is the zero subsheaf $0 \subset F$.

3.3. A unique maximal locally free subsheaf. Proposition 3.3 has the following variation.

**Proposition 3.5.** Let $F$ be a torsionfree semistable sheaf on $X$. Assume that $F$ contains a polystable locally free subsheaf $F''$ with $\mu(F'') = \mu(F)$. Then there is a unique coherent subsheaf $V^f \subset F$ satisfying the following four conditions:

1. $V^f$ is locally free,
2. $V^f$ is polystable with $\mu(V^f) = \mu(F)$,
3. $F/V^f$ is torsionfree, and
4. $V^f$ is maximal among all subsheaves of $F$ satisfying the above three conditions.

**Proof.** The proof is very similar to the proof of Proposition 3.3. Take $W \subset F^{**}$ as in the proof of Proposition 3.3. Consider $C_F$ defined in (3.1). Let

$$C_F^f \subset C_F$$

be the subset consisting of all subsheaves $E \in C_F$ such that $E$ is locally free. We note that the set $C_F^f$ is nonempty, because $F$ contains a polystable locally free subsheaf $F''$ with $\mu(F'') = \mu(F)$.

Let

$$V^f \subset W$$

be the coherent subsheaf of $W$ generated by all subsheaves $E \in C_F^f$, where $C_F^f$ is defined in (3.3).

We will describe an alternative construction of the subsheaf $V^f$ in (3.4).

A theorem of Atiyah says that any coherent sheaf $E$ on $X$ can be expressed as a direct sum of indecomposable sheaves, and if

$$E = \bigoplus_{i=1}^{m_1} E_i^1 = \bigoplus_{i=1}^{m_2} E_i^2$$

are two decompositions of $E$ into direct sum of indecomposable sheaves, then

- $m_1 = m_2$, and
- there is a permutation $\sigma$ of $\{1, \cdots, m_1\}$ such that $E_i^1$ is isomorphic to $E_{\sigma(i)}^2$ for all $1 \leq i \leq m_1$.

(See [At] p. 315, Theorem 2(i)].) From this theorem of Atiyah it follows immediately that any coherent sheaf $E$ on $X$ can be expressed as

$$E = E^f \oplus E', \quad (3.5)$$
where $\mathcal{E}'$ is locally free, and every indecomposable component of $\mathcal{E}'$ is not locally free. Furthermore, the decomposition of $\mathcal{E}$ in (3.3) is unique if

- there is no nonzero homomorphism from $\mathcal{E}'$ to $\mathcal{E}'$, and
- there is no nonzero homomorphism from $\mathcal{E}'$ to $\mathcal{E}'$.

Note that this condition is satisfied for $W$; this is because any nonzero homomorphism between two stable reflexive sheaves of same slope is an isomorphism.

Let $W^f \subset W$ be the (unique) maximal locally free component of $W$. So $W$ is a direct sum of stable vector bundles of slope $\mu_0$. The subsheaf $V^f$ in (3.4) is generated by all direct summands of $W^f$ that are contained in $F$.

Since $V^f$ is a direct summand of $W^f$, it follows that $V^f$ is locally free. It clearly satisfies all the conditions in the proposition, and it is unique. □

Remark 3.6. In Proposition 3.5, if $F$ does not contain any polystable locally free subsheaf $F''$ with $\mu(F'') = \mu(F)$, then we set the maximal subsheaf $V^f$ in Proposition 3.5 to be the zero subsheaf $0 \subset F$. For the sheaf $F$ in Section 3.1, the subsheaf given by Proposition 3.5 is the zero subsheaf $0 \subset F$.

4. Maximal pseudo-stable subsheaves

Theorem 4.1. Let $F$ be a torsionfree semistable sheaf on $X$. Assume that $F$ contains a polystable reflexive subsheaf $F'$ with $\mu(F') = \mu(F)$. Then there is a unique pseudo-stable subsheaf $V \subset F$ satisfying the following three conditions:

1. $\mu(V) = \mu(F)$,
2. $F/V$ is torsionfree, and
3. $V$ is maximal among all pseudo-stable subsheaves of $F$ satisfying the above two conditions.

Proof. The idea is to apply Proposition 3.3 iteratively. Let $V_1 \subset F$ be the subsheaf given by Proposition 3.3. Assume that $V_1 \neq 0$. Let

$$q_1 : F \longrightarrow F/V_1$$

be the quotient map. We note that $F/V_1$ is torsionfree and semistable; moreover, we have $\mu(F/V_1) = \mu(F)$, if $F/V_1 \neq 0$. Let

$$V_1' \subset F/V_1$$

be the subsheaf given by Proposition 3.3. Define

$$V_2 := q_1^{-1}(V_1') \subset F,$$

where $q_1$ is the projection in (4.1). So $F/V_2 = (F/V_1)/V_1'$ is torsionfree and semistable; moreover, we have $\mu(F/V_2) = \mu(F)$, if $F/V_2 \neq 0$. 


Proceeding step-by-step, this way we obtain a filtration of subsheaves
\[ 0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n = V \subset F \tag{4.2} \]
such that \( V_i/V_{i-1} \) is the subsheaf of \( F/V_{i-1} \) given by Proposition \[3.3\] for every \( 0 \leq i \leq n \), and subsheaf of \( F/V_n \) given by Proposition \[3.3\] is the zero subsheaf.

From Proposition \[3.3\] it follows immediately that

- \( V \) is pseudo-stable (see Remark \[2.2\]),
- \( \mu(V) = \mu(F) \), and
- \( F/V \) is torsionfree.

We need to show that \( V \) is the unique maximal one among all subsheaves of \( F \) satisfying these three conditions.

Let
\[ W \subset F \]
be a coherent subsheaf such that

- \( W \) is pseudo-stable,
- \( \mu(W) = \mu(F) \), and
- \( F/W \) is torsionfree.

Let
\[ 0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{m-1} \subset W_m = W \]
be a filtration of \( W \) such that \( W_i/W_{i-1} \) is a stable reflexive sheaf with \( \mu(W_i/W_{i-1}) = \mu(W) \) for all \( 1 \leq i \leq m \). Then it follows immediately that \( W_i \subset V_i \), where \( V_i \) is the subsheaf in \( (4.2) \). For the same reason, we have \( W_i \subset V_i \) for all \( i \) (see \( (4.2) \)). This proves that \( V \) is the unique maximal one among all subsheaves of \( F \) satisfying the three conditions. \( \square \)

**Remark 4.2.** In Theorem \[4.1\] if \( F \) does not contain any polystable reflexive subsheaf \( F' \) with \( \mu(F') = \mu(F) \), then we set the maximal subsheaf \( V \) in Theorem \[4.1\] to be the zero subsheaf \( 0 \subset F \).

**Theorem 4.3.** Let \( F \) be a torsionfree semistable sheaf on \( X \). Assume that \( F \) contains a polystable locally free subsheaf \( F'' \) with \( \mu(F'') = \mu(F) \). Then there is a unique coherent subsheaf \( V \subset F \) satisfying the following four conditions:

1. \( \mu(V) = \mu(F) \),
2. \( V \) is a pseudo-stable bundle,
3. \( F/V \) is torsionfree, and
4. \( V \) is maximal among all subsheaves of \( F \) satisfying the above three conditions.

In particular, \( V \) is locally free.

**Proof.** We apply Proposition \[3.5\] iteratively, just as Proposition \[3.3\] was applied iteratively in the proof of Theorem \[4.1\]. Apart from that, the proof is identical to the proof of Theorem \[4.1\] we omit the details. \( \square \)
**Remark 4.4.** In Theorem 4.3 if $F$ does not contain any polystable locally free subsheaf $F''$ with $\mu(F'') = \mu(F)$, then we set the maximal subsheaf $V$ in Theorem 4.3 to be the zero subsheaf $0 \subset F$.

5. **Galois coverings and descent**

5.1. **Galois covering.** As before, $X$ is an irreducible normal projective variety over $k$, equipped with an ample line bundle $L$. Let $Y$ be an irreducible projective variety and

$$\phi : Y \longrightarrow X$$ (5.1)

an étale Galois covering. The Galois group $\text{Gal}(\phi)$ will be denoted by $\Gamma$.

Note that the line bundle $\phi^*L$ on $Y$ is ample. The degree of sheaves on $Y$ will be defined using $\phi^*L$. For any torsionfree coherent sheaf $E$ on $X$,

$$\text{degree}(\phi^*E) = (\#\Gamma) \cdot \text{degree}(E).$$ (5.2)

Let $E$ be a semistable torsionfree sheaf on $X$. Then it can be shown that $\phi^*E$ is semistable. To prove this, assume that $\phi^*E$ is not semistable. Let $E' \subset \phi^*E$ be the maximal semistable subsheaf, namely the first term of the Harder–Narasimhan filtration of $\phi^*E$ (see [HL, p. 16, Theorem 1.3.4]). From the uniqueness of $E'$ it follows immediately that the natural action of the Galois group $\Gamma$ on $\phi^*E$ preserves $E'$. Therefore, there is a unique subsheaf

$$E_1 \subset E$$

such that $\phi^*E_1 = E'$. Since $E'$ contradicts the semistability condition for $\phi^*E$, using (5.2) we conclude that $E_1$ contradicts the semistability condition for $E$. But $E$ is semistable. This proves that $\phi^*E$ is semistable.

5.2. **An example.** Let

$$V \subset E \quad \text{and} \quad W \subset \phi^*E$$

be the polystable subsheaves given by Proposition 3.1. In general, $W \neq \phi^*V$. Such an example is given below.

Take $\Gamma$ such that the $\Gamma$–module $k[\Gamma]$ is not completely reducible; this requires the characteristic of $k$ to be positive. Set $E = \phi_\ast \mathcal{O}_Y$. We note that $\phi_\ast \mathcal{O}_Y$ is the vector bundle associated to the principal $\Gamma$–bundle $Y \stackrel{\phi}{\longrightarrow} X$ for the $\Gamma$–module $k[\Gamma]$. Then $\phi^*E$ is the trivial vector bundle $Y \times k[\Gamma] \longrightarrow Y$ with fiber $k[\Gamma]$. Hence the polystable subsheaf

$$W \subset \phi^*E$$

given by Proposition 3.1 is $\phi^*E$ itself. But $E$ is not polystable (though it is semistable) because the $\Gamma$–module $k[\Gamma]$ is not completely reducible. Therefore, the polystable subsheaf

$$V \subset E$$

given by Proposition 3.1 is a proper subsheaf of $E$. In particular, we have $W \neq \phi^*V$. 

5.3. **Descent of the reflexive subsheaf.** We continue with the set-up of Section 5.1.

**Proposition 5.1.** Let $F$ be a semistable torsionfree sheaf on $X$. Let 
$V \subset F$ and $W \subset \phi^*F$
be the pseudo-stable subsheaves given by Theorem 4.1. Then
$W = \phi^*V$
as subsheaves on $\phi^*F$.

**Proof.** We trace the construction of the subsheaf in Proposition 3.3. First note that we have
$\phi^*(F^{**}) = (\phi^*F)^{**}$, because $\phi$ is étale. Therefore, the Galois group $\Gamma$ has a natural
action on $(\phi^*F)^{**}$. Since the polystable subsheaf $B' \subset (\phi^*F)^{**}$
given by Proposition 3.1 is unique, the action of $\Gamma$ on $(\phi^*F)^{**}$ preserves $B'$. Consequently,
the polystable subsheaf $B \subset \phi^*F$
given by Proposition 3.3 is preserved by the action of $\Gamma$ on $\phi^*F$. Hence there is a unique
coherent subsheaf $A \subset F$
such that $\phi^*A = B$ as subsheaves on $\phi^*F$.

The sheaf $A$ is semistable, because for a subsheaf $A_1 \subset A$ contradicting the semistability condition for $A$, the subsheaf $\phi^*A_1 \subset \phi^*A = B$ contradicts the semistability condition for $B$. We will prove that $A$ is pseudo-stable.

To prove that $A$ is pseudo-stable, first note that $A$ is reflexive, because $\phi$ is étale and $B = \phi^*A$ is reflexive. Let
$0 \rightarrow S \rightarrow A \rightarrow Q \rightarrow 0$
be a short exact sequence such that

- $\mu(S) = \mu(A)$, and
- $Q$ is torsionfree.

Since $A$ is semistable, and $\mu(S) = \mu(A)$, it follows that both $S$ and $Q$ are semistable and also we have $\mu(Q) = \mu(A)$.

To prove that $A$ is pseudo-stable, it suffices to show that $Q$ is reflexive.

As noted in Section 5.1, the semistability of $S$ implies the semistability of $\phi^*S$. We also have $\mu(\phi^*S) = \mu(\phi^*A)$, because $\mu(S) = \mu(A)$. On the other hand $\phi^*A = B$ is polystable. These together imply that $\phi^*S$ is a direct summand of $\phi^*A$. Fix a subsheaf
$S' \subset \phi^*A$
such that the natural homomorphism $S' \oplus \phi^*S \rightarrow \phi^*A$ is an isomorphism. Since $\phi^*A$ is reflexive, it follows that $S'$ is also reflexive. But $S' = \phi^*Q$. Therefore, we conclude that $Q$ is reflexive.
As noted before, this proves that $A$ is pseudo-stable.

Now following the iterative construction in Theorem 4.1 it is straightforward to deduce that

$$W \subset \phi^* V.$$  \hfill (5.3)

To complete the proof we need to show that

$$\phi^* V \subset W.$$  \hfill (5.4)

We note that to prove (5.4) it suffices to show the following:

*Let $E$ be a reflexive stable sheaf on $X$. Then $\phi^* E$ is pseudo-stable.*

To prove the above statement, let $E$ be a reflexive stable sheaf on $X$. So $\phi^* E$ is reflexive and semistable (shown in Section 5.1). Let

$$E_1 \subset \phi^* E$$

be the polystable subsheaf given by Proposition 3.1. As observed earlier, from the uniqueness of $E_1$ it follows immediately that the natural action of $\Gamma$ on $\phi^* E$ preserves $E_1$. Let

$$E' \subset E$$

be the unique subsheaf such that

$$E_1 = \phi^* E'$$

as subsheaves of $\phi^* E$.

As noted before, $E'$ is semistable, because $\phi^* E$ is so; also, we have $\mu(E') = \mu(E)$, because $\mu(\phi^* E') = \mu(\phi^* E)$. Furthermore, $E'$ is reflexive because $\phi^* E'$ is so. Since $E$ is polystable, these together imply that $E'$ is a direct summand of $E$. Since $E$ is reflexive, and $E/E'$ is a direct summand of $E$, we conclude that $E/E'$ is also reflexive. Consequently, the pullback

$$\phi^*(E/E') = (\phi^* E)/E_1$$

is also reflexive. Using this it is straightforward to deduce that $\phi^* E$ is pseudo-stable. Indeed, in the above argument, substitute $E/E'$ in place of $E$, and iterate.

Hence the inclusion in (5.4) holds. The proposition follows from (5.3) and (5.4). \hfill $\Box$

### 5.4. Descent of the locally free subsheaf.

**Proposition 5.2.** Let $F$ be a semistable torsionfree sheaf on $X$. Let

$$V \subset F \text{ and } W \subset \phi^* F$$

be the pseudo-stable bundles given by Theorem 4.3. Then

$$W = \phi^* V$$

as subsheaves on $\phi^* F$.

**Proof.** The proof is very similar to the proof of Proposition 5.1. The construction of the subsheaf in Proposition 3.5 needs to traced instead of the construction in Proposition 3.3. We omit the details. \hfill $\Box$
6. Direct image of structure sheaf

6.1. The case of group quotients. Let $Z$ be an irreducible normal projective variety over $k$ such that a finite (reduced) group $\Gamma$ is acting faithfully on it. Then the quotient $X := Z/\Gamma$ is also an irreducible normal projective variety, and the quotient map
\[ \varphi : Z \longrightarrow Z/\Gamma =: X \] (6.1)
is separable. Note that we are not assuming that the action of $\Gamma$ is free; so the map $\varphi$ need not be étale. We fix an ample line bundle $L$ on $X$, and we equip $Z$ with $\varphi^*L$; so we have the notion of degree of sheaves on both $X$ and $Z$.

Consider the direct image $\varphi_*O_Z$; it is reflexive because $X$ and $Z$ are normal. We have
\[ (\varphi^*\varphi_*O_Z)/\text{Torsion} \subset O_Z \otimes_k k[\Gamma]. \] (6.2)
Indeed, over the open subset $U \subset Z$ where $\varphi$ is flat, we have
\[ \hat{\varphi}^*\hat{\varphi}_*O_U \subset O_U \otimes_k k[\Gamma], \]
where $\hat{\varphi} = \varphi|_U$. Since the codimension of $Z \setminus U \subset Z$ is at least two, and $Z$ is normal, this inclusion map over $U$ extends to an inclusion map as in (6.2). From (6.2) it follows that $\mu_{\text{max}}((\varphi^*\varphi_*O_Z)/\text{Torsion}) \leq 0$, and hence
\[ \mu_{\text{max}}(\varphi_*O_Z) \leq 0. \]
On the other hand, we have $O_X \subset \varphi_*O_Z$. Combining these it follows that
\[ \mu_{\text{max}}(\varphi_*O_Z) = 0. \] (6.3)
Let
\[ F_1 \subset \varphi_*O_Z \] (6.4)
be the maximal semistable subsheaf of degree zero (see (6.3)); in other words, $F_1$ is the first nonzero term of the Harder–Narasimhan filtration of $\varphi_*O_Z$. We note that $F_1$ is reflexive.

Let
\[ W_1 \subset F_1 \] (6.5)
be the unique maximal locally free pseudo-stable bundle given by Theorem 4.3 for the sheaf $F_1$ in (6.4). We note that Theorem 4.3 is applicable because $O_X \subset F_1$.

Lemma 6.1. The pullback $\varphi^*W_1$ of the sheaf $W_1$ in (6.5) by the map in (6.1) is a trivial bundle on $Z$.

Proof. We know that $\varphi^*W_1$ is locally free of degree zero; let $r$ be its rank. Using the inclusion map in (6.2) we have
\[ \bigwedge^r \varphi^*W_1 \subset \bigwedge^r (O_Z \otimes_k k[\Gamma]) = O_Z \otimes_k \bigwedge^r k[\Gamma]. \] (6.6)
Since $\bigwedge^r \varphi^*W_1$ is a line bundle of degree zero, any nonzero homomorphism $\bigwedge^r \varphi^*W_1 \longrightarrow O_Z$ is an isomorphism. So the subsheaf $\bigwedge^r \varphi^*W_1$ in (6.6) is a subbundle. From this it follows that $\varphi^*W_1$ is a subbundle of $O_Z \otimes_k k[\Gamma]$. 
Any subbundle of degree zero of the trivial bundle is trivial. This proves that $\varphi^*W_1$ is trivial. □

6.2. The general case. Let $X$ and $Y$ be irreducible normal projective varieties over $k$, and let

$$\phi : Y \longrightarrow X \quad (6.7)$$

be a separable finite surjective map. Let

$$\varphi : Z \longrightarrow X \quad (6.8)$$

be the normal Galois closure of $\phi$. So there is a commutative diagram

$$\begin{array}{ccc}
Z & \xrightarrow{\phi} & X \\
\downarrow & & \parallel \\
Y & \xrightarrow{\phi} & X
\end{array}$$

We fix an ample line bundle $L$ on $X$, and we equip both $Y$ and $Z$ with its pullback.

We have

$$\phi_*\mathcal{O}_Y \subset \varphi_*\mathcal{O}_Z. \quad (6.9)$$

Hence from (6.3) it follows that

$$\mu_{\text{max}}(\phi_*\mathcal{O}_Y) = 0;$$

recall that $\mathcal{O}_X \subset \phi_*\mathcal{O}_Y$.

Let

$$F \subset \phi_*\mathcal{O}_Y \quad (6.10)$$

be the maximal semistable subsheaf of degree zero (equivalently, it is the first nonzero term of the Harder–Narasimhan filtration); let

$$W \subset F \quad (6.11)$$

be the unique maximal locally free pseudo-stable bundle given by Theorem 4.3 for the sheaf $F$ in (6.10). As before, Theorem 4.3 is applicable because $\mathcal{O}_X \subset F$.

The algebra structure on $\mathcal{O}_Y$ produces an algebra structure on $\phi_*\mathcal{O}_Y$. Let

$$m : (\phi_*\mathcal{O}_Y) \otimes (\phi_*\mathcal{O}_Y) \longrightarrow \phi_*\mathcal{O}_Y \quad (6.12)$$

be the corresponding multiplication map.

**Proposition 6.2.** The subsheaf $W \subset \phi_*\mathcal{O}_Y$ (see (6.11), (6.10)) satisfies the condition

$$m(W \otimes W) \subset W,$$

where $m$ is the map in (6.12).

**Proof.** From the inclusion in (6.9) it follows that $F \subset F_1$ (defined in (6.10) and (6.4)), and hence we have

$$W \subset W_1$$

(defined in (6.11) and (6.5)). This implies that

$$\varphi^*W \subset \varphi^*W_1.$$
Now, $\varphi^*W_1$ is trivial by Lemma 6.1 and $\varphi^*W$ is a locally free subsheaf of it of degree zero. Therefore, using the argument in the proof of Lemma 6.1 we conclude that $\varphi^*W$ is a trivial subbundle of $\varphi^*W_1$.

Since $\varphi^*W$ is trivial, it follows that $(\varphi^*W) \otimes (\varphi^*W) = \varphi^*(W \otimes W)$ is trivial. This implies that $W \otimes W$ is semistable. Also, $\text{degree}(W \otimes W) = 0$, because $\text{degree}(W) = 0$. From these it follows that

$$m(W \otimes W) \subset F$$

(see (6.11) and (6.12) $F$ and $m$).

Any torsionfree quotient, of degree zero, of a trivial vector bundle is also a trivial vector bundle; this follows using the argument in Lemma 6.1. From this, and the characterization of $W$ in Theorem 4.3, we have

$$m(W \otimes W) \subset W.$$  \hfill $\square$

**Proposition 6.3.** Take $\phi$ as in (6.7). Then the induced homomorphism of étale fundamental groups

$$\phi_* : \pi_1^{\text{et}}(Y) \longrightarrow \pi_1^{\text{et}}(X)$$

is surjective if and only if $W = \mathcal{O}_X$, where $W$ is defined in (6.11).

**Proof.** First assume that the homomorphism induced by $\phi$

$$\phi_* : \pi_1^{\text{et}}(Y) \longrightarrow \pi_1^{\text{et}}(X)$$

(6.13)

is not surjective. Since $\phi$ is a finite surjective map,

$$\phi_*(\pi_1^{\text{et}}(Y)) \subset \pi_1^{\text{et}}(X)$$

is a subgroup of finite index. Let

$$\phi' : Y' \longrightarrow X$$

be the étale covering corresponding to this finite index subgroup $\phi_*(\pi_1^{\text{et}}(Y))$. So there is a morphism

$$\phi'' : Y \longrightarrow Y'$$

such that $\phi = \phi' \circ \phi''$.

We will now show that

$$\text{degree}(\phi'_*\mathcal{O}_{Y'}) = 0. \tag{6.14}$$

First, (6.14) holds when $\dim X = 1$ [Ha, p. 306, Ch. IV, Ex. 2.6(d)]. If $\dim X \geq 2$, take any pair $(C, \delta)$, where $C$ is an irreducible smooth projective curve over $k$ and $\delta : C \longrightarrow X$ is a morphism. Since (6.14) holds for curves, it follows that $\text{degree}(\delta^*\phi'_*\mathcal{O}_{Y'}) = 0$. As this holds for all pairs $(C, \delta)$ of the above type we conclude that (6.14) holds.

It may be mentioned that norm $\text{Norm}(\phi'_*\mathcal{O}_{Y'})$ of $\phi'_*\mathcal{O}_{Y'}$ is trivial, because $\phi'$ is étale. Since $\text{Norm}(\phi'_*\mathcal{O}_{Y'})^{\otimes 2} = (\det \phi'_*\mathcal{O}_{Y'})^{\otimes 2}$, the line bundle $(\det \phi'_*\mathcal{O}_{Y'})^{\otimes 2}$ is trivial. However, for our purpose (6.14) suffices.

Let

$$\varphi_1 : Z' \longrightarrow X$$
be the normal Galois closure of $\phi'$. Since (6.14) holds, from the proof of Proposition 6.2 it follows that $\varphi'_1\phi'_*\mathcal{O}_{Y'}$ is trivial. Hence

$$\phi'_*\mathcal{O}_{Y'} \subset W.$$ 

But $\text{rank}(\phi'_*\mathcal{O}_{Y'}) = \deg(\phi') > 1$, because $\phi_*$ in (6.13) is not surjective. This implies that $\text{rank}(W) \geq \text{rank}(\phi'_*\mathcal{O}_{Y'}) > 1$, and hence $W \neq \mathcal{O}_X$.

To prove the converse assume that the homomorphism $\phi_*$ in (6.13) is surjective.

Recall that $\mathcal{O}_X \subset W$. Assume that $W \neq \mathcal{O}_X$. Hence we have $\text{rank}(W) \geq 2$.

The spectrum of the subalgebra bundle $W \subset \varphi_*\mathcal{O}_Y$ in Proposition 6.2 produces a covering (it need not be étale)

$$\phi' : Y' \longrightarrow X.$$ 

(6.15)

In the proof of Proposition 6.2 we saw that $\varphi^*W$ is trivial, where $\varphi$ is the map in (6.8). This implies that the pullback of the covering $\phi'$ in (6.15) to $Z$ in (6.8) is trivial. Consequently, the map $\phi'$ is étale.

The inclusion map of $W$ in $\phi_*\mathcal{O}_Y$ (see (6.10) and (6.11)) produces a covering

$$\phi'' : Y \longrightarrow Y'$$

such that $\phi = \phi' \circ \phi''$, where $\phi'$ is the map in (6.15).

But we have $\deg(\phi') = \text{rank}(W) \geq 2$. Hence $\phi_*$ in (6.13) is not surjective. Since this contradicts the hypothesis, we conclude that $W = \mathcal{O}_X$. This completes the proof. □

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