COASTING COSMOLOGIES WITH

TIME DEPENDENT COSMOLOGICAL CONSTANT

LUIS O. PIMENTEL* and LUZ M. DIAZ-RIVERA
Departamento de Física, Universidad Autónoma Metropolitana-Iztapalapa,
A. P. 55-534, CP 09340, México D. F., México

Received (28.IV.1998)
Revised (16.VI.1998)

The effect of a time dependent cosmological constant is considered in a family of scalar-
tensor theories. Friedmann- Robertson- Walker cosmological models for vacuum and
perfect fluid matter are found. They have a linear expansion factor, the so called coast-
ing cosmology, the gravitational "constant" decreases inversely with time; that is these
models satisfy the Dirac Hypotheses. The cosmological "constant" decreases inversely
with the square of time, therefore we can have a very small value for it at present time.

1. Introduction

The renewed interest in the scalar- tensor theories of gravitation is caused by
two main factors: First, most of the unified theories, including super-string theories
contain a scalar field (dilaton, size of the extra compact space in Kaluza-Klein
theories) which play a similar role to the scalar field of the scalar-tensor theories.
Secondly, the new scenario of extended inflation which solves the fine tuning problem
of the old, new and chaotic inflation has a scalar field that slows the expansion rate
of the universe, from exponential to polynomial, allowing the completion of the
phase transition from the de Sitter phase to a radiation dominated universe, the
graceful exit problem.

In this work we want to consider a family of scalar- tensor theories with a
potential that is equivalent to a time dependent cosmological constant. Recently
several authors\textsuperscript{1–12} have considered the cosmological consequences of a time varying
cosmological constant. Most of them introduce the time dependence in an ad hoc
manner. In this work we consider an equivalent problem in a the well known general
scalar-tensor theory of gravity where the time dependence can occur in a natural
way, without any new assumption or modification of the theory.

2. Field Equations

\*E-mail: lopr@xanum.uam.mx
We start our discussion with the action for the most general scalar-tensor theory of gravitation:

\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} [\phi R - \phi^{-1} \omega g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 2\phi \lambda(\phi)] + S_{NG}, \quad (1) \]

where \( g = \det (g_{\mu\nu}) \) G is Newton’s constant, \( S_{NG} \) is the action for the nongravitational matter. We shall use the signature \((-+,+,+,-)\). The arbitrary functions \( \omega(\phi) \) and \( \lambda(\phi) \) distinguish the different scalar-tensor theories of gravitation; \( \lambda(\phi) \) is a potential function and plays the role of a cosmological constant, \( \omega(\phi) \) is the coupling function of the particular theory. General relativity is the limit of this theory when \( |\omega| \to \infty \) and \( \lambda(\phi) \to 0 \); and as it is well known, the solar system experiments imply that \( |\omega| \geq 500 \).

The explicit field equations are

\[ G_{\mu\nu} = \frac{8\pi T_{\mu\nu}}{\phi} + \lambda(\phi) g_{\mu\nu} + \omega \phi^{-2}(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\lambda}\phi_{,\lambda}) + \phi^{-1}(\phi_{;\mu\nu} - g_{\mu\nu} \phi), \quad (2) \]

\[ \square \phi + \frac{2\phi^2 d\lambda/d\phi - 2\phi \lambda(\phi)}{3 + 2\omega(\phi)} = \frac{1}{3 + 2\omega(\phi)} \left( 8\pi T - \frac{d\omega}{d\phi} \phi_{,\mu}\phi_{,\mu} \right), \quad (2c) \]

where \( G_{\mu\nu} \) is the Einstein tensor. The last equation can be substituted by

\[ \square \phi + \frac{2\phi^2 d\lambda/d\phi - 2\phi \lambda(\phi)}{3 + 2\omega(\phi)} = \frac{1}{3 + 2\omega(\phi)} \left( 8\pi T - \frac{d\omega}{d\phi} \phi_{,\mu}\phi_{,\mu} \right), \quad (2c) \]

where \( T = T^\mu_{\mu} \) is the trace of the stress-energy matter tensor. The divergenceless condition of the stress-energy matter tensor is satisfied if the field equation (3) is satisfied (as is shown in appendix), therefore we shall consider equations (2) and (2c) as our field equations.

In what follows we shall assume that \( \omega(\phi) = \omega_0 = \text{constant} \), \( \lambda(\phi) = c \phi^m \), \( \phi = \phi_1 t^q \), with \( c, m \) and \( q \) constants (for recent results when \( \omega \) is variable see 14). The field equations for this choice of \( \omega \) and \( \lambda \) and with a perfect fluid for the matter content in the isotropic and homogeneous line element will be considered,

\[ ds^2 = dt^2 - a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (4) \]

The field equations are

\[ 3(\dot{a}/a)^2 + 3k/a^2 - c \phi^m = \frac{8\pi \rho}{\phi} + (\omega/2)(\dot{\phi}/\phi)^2 - 3(\ddot{a}/a)(\dot{\phi}/\phi), \quad (5) \]

\[ -2(\ddot{a}/a) - (\dot{a}/a)^2 - k/a^2 + c \phi^m = \frac{8\pi p}{\phi} + (\omega/2)(\dddot{\phi}/\phi)^2 + \dddot{\phi}/\phi + 2(\dddot{a}/a)(\dot{\phi}/\phi), \quad (6) \]

\[ [\dddot{\phi}/\phi + 3(\dot{a}/a)(\dot{\phi}/\phi)]B = 2c(1 - m)\phi^m + \frac{8\pi (\rho - 3p)}{\phi}, \quad (7) \]
where \( B = 3 + 2\omega \). In the next sections we show some exact solutions for these equations when the fluid is a barotropic one, \( p = \epsilon \rho \), including the vacuum case.

### 3. The vacuum solutions

In the case where we neglect all the nongravitational matter we have obtained the following solutions.

#### 3.1. \( k \neq 0 \)

\[
a = a_1 t, \quad a_1 = \pm \sqrt{\frac{k m^2}{2 + 2m + 2\omega - m^2}},
\]

\[
\phi = \phi_1 t^{-2/m}, \quad c = \frac{2(3 + 2\omega)}{\phi_1^m m^2},
\]

\[
\lambda = \lambda_1 t^2, \quad \lambda_1 = \frac{2(3 + 2\omega)}{m^2}
\]

where \( \phi_1 \) and \( m \) are arbitrary constants. In order to have real \( a_1 \), the values of \( \omega \) are restricted in the following way,

\[
\omega > \frac{m^2}{2} - m - 1, \text{ for } k = 1,
\]

\[
\omega < \frac{m^2}{2} - m - 1, \text{ for } k = -1.
\]

The value of \( \phi_1 \) can be related to present day observations if we recall that relation of \( \phi \) at the present time \( t_0 \),

\[
\phi_0 = \frac{G_0^{-1} 4 + 2\omega}{3 + 2\omega},
\]

and the definition of the Hubble constant,

\[
H_0 = \left( \frac{\dot{a}}{a} \right)_0 = \frac{1}{t_0},
\]

(\( t_0 \) is the age of the universe), we obtain the value of the constant \( \phi_1 \),

\[
\phi_1 = \frac{1}{H_0^{2/m} G_0 3 + 2\omega}.
\]

#### 3.2. \( k = 0 \) solutions

\[
a = a_1 t,
\]
\[ \phi = \phi_1 t^{1+\sqrt{3+2\omega}} \tag{16} \]

\[ m = 1 \pm \sqrt{3+2\omega}, \quad c = \frac{2(m-1)^2}{m^2\phi_1^m} \tag{17} \]

\[ \lambda = \frac{\lambda_1}{t^2}, \quad \lambda_1 = \frac{3+2\omega}{\omega+2\pm \sqrt{3+2\omega}} \tag{18} \]

Here \( a_1 \) and \( \phi_1 \) are arbitrary constants. The value of \( \phi_1 \) for this case is determined in terms of present day values of \( G \) and the Hubble parameter, as above, to be

\[ \phi_1 = \frac{1}{H_0^2/(1+\sqrt{3+2\omega})} \frac{4+2\omega}{G_0 3+2\omega} \tag{19} \]

4. Barotropic equation of state

Assuming the equation of state \( p = \epsilon \rho \) we have from the conservation equation \( \rho = s/a^{3(\epsilon+1)} \); substituting into the field equations we obtain the following solution

\[ a = a_1 t, \quad \phi = \phi_1 t^{-(1+3\epsilon)}, \]

\[ \rho = \frac{s}{a^{3(1+\epsilon)}}, \]

\[ \lambda = \frac{\lambda_1}{t^2}, \quad \lambda_1 = \frac{a_1^2 \left[ \omega(1+5\epsilon+3\epsilon^2-9\epsilon^3) + 2(3\epsilon+1) \right] + 2k(1+3\epsilon)}{2(1+\epsilon)a_1^2} \tag{20} \]

where

\[ c = \frac{a_1^{1+3\epsilon} \phi_1}{8\pi(1+\epsilon)} \{ 2k - a_1^2 [ \omega(9\epsilon^2+6\epsilon+1) + 9\epsilon^2+12\epsilon+1] \}, \]

\[ s = \frac{2}{1+3\epsilon} \]

\[ m = \frac{2}{1+3\epsilon} \tag{21} \]

and

\[ \phi_1 = \frac{1}{H_0^2/(1+3\epsilon)} \frac{4+2\omega}{G_0 3+2\omega} \tag{22} \]

The above solution is not valid for \( \epsilon = -1 \), that is for the equation of state that corresponds to the quantum vacuum. In the next section we consider this particular case.

5. Vacuum fluid
Here we want to consider the case when $p = -\rho$, that is the well known equation of state for the quantum vacuum. This case means that in addition to the contribution of the gravitational theory to the cosmological constant we have some other contribution(s) from the vacuum expectation value of some quantum fields. Therefore the effective cosmological constant is $\Lambda_{\text{eff}} = \frac{8\pi \rho}{\phi} + c\phi^m$.

For $\epsilon = -1$ we have obtained the following solution,

$$a = a_1 t, \quad \phi = \phi_1 t^2, \quad \rho = \rho_0 = \text{const.}, \quad \lambda = \frac{\lambda_1}{t^2}$$

with

$$a_1 = \sqrt{\frac{k}{2\omega - 1}}, \quad c = \phi_1 (6 + 4\omega) - 8\pi \rho_0, \quad \lambda_1 = 2\left[\frac{\phi_1 (2\omega + 3)}{\phi_1} - 4\pi \rho_0\right], \quad m = -1.$$ (24)

The effective cosmological constant is

$$\Lambda_{\text{eff}} = \frac{8\pi \rho}{\phi} + c\phi^m = \frac{2(3 + 2\omega)}{t^2}$$ (25)

We can see now that in this model regardless of how large is the contribution to the cosmological constant of the vacuum energy of quantum fields the gravitational contribution reduce the value of the effective cosmological constant to the present value of $\simeq 4\omega H_0^2$.

In this case $\phi_1$ can also be related to the present value of the gravitational constant,

$$\phi_1 = \frac{H_0^2}{G_0} \frac{4 + 2\omega}{3 + 2\omega}.$$ (26)

6. Final remarks

In this work we have presented a family of solutions to the general scalar-tensor theory of gravity with a potential which plays the role of a time dependent cosmological constant.

The time dependence of the cosmological constant in each obtained solutions has the form $\lambda \sim t^{-2}$. This dependence occurs in a natural way when we use a field potential $\phi \sim t^q$ and we limited to ourself to $a \approx t$ that guarantees a coasting period in which the solutions are true. This type of solutions in general relativity were studied some time ago\textsuperscript{15,16}, where the time dependence of $\lambda$ is chosen ad hoc. The resulting age of the universe, $t_0 = 1/H_0$, is not in conflict with the observational determination\textsuperscript{17,18,19}. Some other astrophysical consequences of the models, like nucleosynthesis,\textsuperscript{20} remain to be explored.

7. Appendix

In this appendix we show that the divergenceless condition is satisfied if the field equation (3) is satisfied.
We start our demonstration with the expression of the stress-energy matter tensor, which can be written from Eq. (2) as

\[ 8\pi T^\mu_\nu = \phi (R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R) - \frac{\omega(\phi)}{\phi} (\phi^{\mu_\nu} - \frac{1}{2} \delta^\mu_\nu \phi, \lambda \phi^\lambda) - (\phi^{\mu_\nu} - \frac{1}{2} \delta^\mu_\nu \phi, \lambda \phi^\lambda) - \lambda(\phi) \delta^\mu_\nu \phi, \lambda \phi^\lambda, \]  

(27)

taking the divergence of this last equation and rearranging terms, we get

\[ 0 = -\phi^\mu_\nu \delta^\mu_\nu R - \frac{\omega(\phi)}{\phi} \phi^\mu_\nu \phi, \lambda \phi^\lambda + \frac{d\omega(\phi)}{d\phi} \frac{1}{2\phi^\mu_\nu} \phi^\mu_\nu \phi^\lambda - \frac{\omega(\phi)}{\phi} \phi^\mu_\nu \phi^{\mu_\nu}, \]

(28)

simplifying this last equation:

\[ 0 = -\phi^\mu_\nu \frac{1}{2} \phi^\mu_\nu R - \frac{\omega(\phi)}{2\phi^\mu_\nu} \phi^\mu_\nu \phi, \lambda \phi^\lambda + \frac{d\omega(\phi)}{d\phi} \frac{1}{2\phi^\mu_\nu} \phi^\mu_\nu \phi^\lambda - \frac{\omega(\phi)}{\phi} \phi^\mu_\nu \phi^{\mu_\nu} - \phi^\mu_\nu \phi^\mu_\nu + (\phi \phi)._\nu, \]

(29)

Taking into account the identities \( \phi^{\mu_\nu} R^\mu_\nu = (\phi^{\mu_\nu}) - (\phi \phi)._\nu \), in the previous equation, we have

\[ 0 = -\frac{\omega(\phi)}{\phi} \phi^\mu_\nu \left[ \phi + \frac{1}{2} \phi, \lambda \phi^\lambda \frac{d}{d\phi} \ln \left( \frac{\omega(\phi)}{\phi} \right) + \frac{1}{2} \phi \phi \left[ R + 2 \frac{d}{d\phi}(\phi \phi \lambda(\phi)) \right] \right] \]

(30)

This is the field equation (3) times the factor \( -\frac{\omega(\phi)}{\phi} \phi^\mu_\nu \), therefore if this field equation is satisfied, then the divergenceless condition is satisfied too and it is enough to use equations (2) and (2c) as the field equations of our system from which we obtain Equations (5), (6) and (7).

8. Acknowledgment

This work was partially supported by CONACyT GRANT 1861-E9212.
References

1. S. Weinberg, Rev. Mod. Phys. 61, 1 (1989)
2. Y. J. Ng, Int. J. Mod. Phys. 1, 145 (1992)
3. J. C. Carvalho, J.A.S. Lima and I. Waga, Phys. Rev. D46, 2404 (1992)
4. V. Silveira and I. Waga, Phys. Rev. D50, 4890 (1994)
5. J. Matyjasek, Phys. Rev. D51, 4154 (1995)
6. D. Kalligas, P. Wesson and C. W. F. Everitt, Gen. Rel. Grav. 24, 351 (1992)
7. A. Beesham, Gen. Rel. Grav. 26, 159 (1994)
8. D. Kalligas, P. Wesson and C. W. F. Everitt, Gen. Rel. Grav. 27, 645 (1995)
9. S. Capozziello and R. de Ritis, A time dependent cosmological “constant”. astro-ph/9605070
10. J. W. Moffat, Phys. Lett. B357, 526 (1995).
11. H. Haber, C. Kane and T. Sterling, Nucl. Phys. 493 (1979)
12. M. Özer, M. O. Taha, Mod. Phys. Lett. A13, 571-580 (1998)
13. C. M. Will, Theory and experiment in gravitational physics, Cambridge University Press, Cambridge (1981)
14. J. D. Barrow, P. Parsons, Phys. Rev. D55, 1906-1930 (1997)
15. E. W. Kolb, ApJ 344, 543 (1989)
16. D. S. Lemons and W. Peter, Astron. Astrophys., 265, 373 (1992).
17. C. H. Lineweaver, D. Barbosa, A. Blanchard, J. G. Bartlett, Astron. Astrophys., 322, 365 (1997)
18. M. Tegmark, ApJ Lett 464, L35 (1996)
19. B. Chaboyer, P. Demarque, P. Kerman and L. M. Kraus, the age of globular clusters in light of Hipparcos: resolving the age problem. astro-ph/9706128
20. M. Birkel and S. Sarkar, Phys. Lett. B408, 59-56 (1997)