Symbol calculus on a projective space

Naoya MIYAZAKI
Department of Mathematics, Keio University,
Yokohama, 223-8521, JAPAN

Abstract: In this article, we introduce symbol calculus on a projective scheme. Using holomorphic Poisson structures, we construct deformations of ring structures for structure sheaves on projective spaces $\mathbb{CP}^n$'s.

Mathematics Subject Classification (2000): Primary 58B32; Secondary 53C28, 53D55

Keywords: deformation theory, structure sheaf, sheaf-cohomology, twistor theory, quantization, etc.

Acknowledgements: The author would like to dedicate the present article to Professor Akira Yoshioka on his 60th birthday. This research is partially supported by JSPS Grant-in-Aid for Scientific Research, and the academic fund of Keio University.

1 Introduction

The terminology “symbol calculus” is used in Fourier analysis and pseudo-differential operator in order to study partial differential equations. Especially, symbol calculus of pseudo-differential operator with respect to elliptic operator gives fruitful contribution to the index theorem, i.e. an important role is played by pseudo-differential operator in the index theory for elliptic operators appearing geometry, where pseudo-differential operators are used to extend the class of possible deformation of an elliptic operator which has essential topological data of the base manifold.

In this article, using holomorphic Poisson structures, we consider symbol calculus on a projective scheme. Especially, we give concrete examples using $\mathbb{CP}^n$. As mentioned above, structure sheaves on algebraic varieties have important and essential feature which plays crucial role to analyze their fundamental properties with respect to the base varieties. Using a holomorphic Poisson structure on the base variety, we construct symbol calculus on the structure sheaf. We here state the main theorems of the present article.

**Theorem 1.1** Assume that $Z = [z_0 : z_1 : \ldots : z_n]$ is the homogeneous coordinate system of $\mathbb{CP}^n$, and $\Lambda = \sum_{\alpha,\beta} \partial_{\bar{z}_0} \Lambda^{\alpha,\beta} \partial_{\bar{z}_0}$ defines a holomorphic skew-symmetric biderivation\(^1\) of order zero acting on the structure sheaf $\mathcal{O}_{\mathbb{CP}^n}$ satisfying the Jacobi rule\(^2\) and an assumption below:

\[
\left( \partial_{\bar{z}_{a_1}} \Lambda^{\alpha_1,\beta_1} \partial_{\bar{z}_{b_1}} \right) \cdots \left( \partial_{\bar{z}_{a_k}} \Lambda^{\alpha_k,\beta_k} \partial_{\bar{z}_{b_k}} \right) = \partial_{\bar{z}_{a_1} \cdots a_k} \Lambda^{\alpha_1,\beta_1} \cdots \Lambda^{\alpha_k,\beta_k} \partial_{\bar{z}_{b_1} \cdots b_k}, \quad (1)
\]

\(^1\)We use Einstein’s convention unless confusing.

\(^2\)The biderivation used here might be called an “algebraic Poisson structure.” I am not sure.
where
\[ f(Z) \partial_{Z_{\alpha_1} \ldots \alpha_k} \quad \text{(resp.} \partial_{Z_{\beta_1} \ldots \beta_k} g(Z) \text{)} \]
means
\[ \partial_{Z_{\alpha_1}} \partial_{Z_{\alpha_2}} \ldots \partial_{Z_{\alpha_k}} f(Z) \quad \text{(resp.} \partial_{Z_{\beta_1}} \partial_{Z_{\beta_2}} \ldots \partial_{Z_{\beta_k}} g(Z) \text{)} \]

Then, for any point "p" and germs \([f(Z)]_\mu, [g(Z)]_\mu\) of the stalk \(O_{\mathbb{CP}^n, p}[\mu] \) we have
\[
f(Z) \# g(Z) := \sum_{k=0}^{\infty} \frac{1}{k!} \mu^2 \Lambda_{\alpha_1} \beta_1 \Lambda_{\alpha_2} \beta_2 \ldots \Lambda_{\alpha_k} \beta_k \partial_{Z_{\alpha_1}} \partial_{Z_{\alpha_2}} \ldots \partial_{Z_{\alpha_k}} f(Z) \partial_{Z_{\beta_1}} \partial_{Z_{\beta_2}} \ldots \partial_{Z_{\beta_k}} g(Z) \]
defines a non-commutative and associative ring structure, where \(\mu\) is a formal parameter.

We also have

**Theorem 1.2** Under the same assumptions and notations of Theorem 1.1, the product \(\#\) induces globally defined non-commutative, associative product on the sheaf-cohomology space \(\sum_{k=0}^{\infty} H^0(\mathbb{CP}^n, O_{\mathbb{CP}^n}(k)) \) where \(\mu\) can be specialized a scalar (for example \(\mu = 1\)).

Using the product \(\#\), we also have

**Theorem 1.3** Suppose that the same assumptions of Theorem 1.1. Let \(A[Z] := ZA^T Z\) be a quadratic form with homogeneous degree 2. Then we have
\[
e^\frac{1}{\mu} A[Z] = \det -1/2 \left( e^{\Lambda A} + e^{-\Lambda A} \right) e^{1/\mu \left( \Lambda^T \tan(\sqrt{\text{det}(\Lambda A)}) \right)} \sum_{k=0}^{\infty} H^0(\mathbb{CP}^n, O_{\mathbb{CP}^n}(k)[\mu, \mu^{-1}])
\]

(3)

2 Outlines of Proofs of Theorems 1.1, 1.2 and 1.3

In this section we give outline of proofs of our main theorems (cf. [18]). Before giving proofs, we recall fundamentals of projective scheme. In fact, we need a slight modification of the standard theory of scheme.

Let \(S = \bigoplus_{n=0}^{\infty} S_n\) be a graded commutative ring. Then, \(S_0\) is obviously commutative and \(S\) is an \(S_0\)-algebra. It is well-known that the homogeneous ideal \(S_+ := \bigoplus_{n=1}^{\infty} S_n\) is called the irrelevant ideal. And the following is well-known:

**Proposition 2.1** A graded commutative ring \(S\) is noetherian if and only if \(S_0\) is noetherian and \(S\) is finitely generated by \(S_1\) as an \(S_0\)-algebra.

It is also well-known that a projective scheme
\[ \text{Proj}(S) := \{ \mathfrak{p} : \text{a homogeneous prime ideal} \mid \neg (S_+ \subset \mathfrak{p}) \} \]
admits the canonical scheme structure in the following way: Set
\[ D_+(f) := \{ \mathfrak{p} \in \text{Proj}(S) \mid \neg (f \in \mathfrak{p}) \} \]

3Here \([\mathfrak{g}]\) denotes either \([\mu, \mu^{-1}]\) or \([\mu, \mu^{-1}]\), and we have to choose carefully in context.
for any homogeneous element $f \in S_d$ with degree $d$, then the family $\{D_+(f)\}_{f \in S_d}, d \in \mathbb{Z}_{\geq 0}$ forms a basis of open sets. Hence it gives the canonical topology $\mathcal{O}_{\text{Proj}(S)}$ (that is, the Zariski topology) for $\text{Proj}(S)$. Note that $-\alpha(f \in p)$ means $f(p) \neq 0$, intuitively.

We also set

$$\Gamma(D_+(f), \mathcal{O}_{\text{Proj}(S)}) := \left\{ \frac{g}{f^m} \mid g \in S_m, \quad m \geq 0 \right\},$$

$$\mathcal{O}_{\text{Proj}(S)} : \mathcal{O}_{\text{Proj}(S)} \ni D_+(f) \mapsto \Gamma(D_+(f), \mathcal{O}_{\text{Proj}(S)}) \in \text{Mod.}$$

The functor above is well-known as the structure sheaf. We remark that when $g \in S_0$, we easily see that $g/1 = fg/f$ ($fg \in S_d$). Hence we may consider $g/f^m$ ($m \geq 1$) instead of $g/f^m$ ($m \geq 0$). We obtain that R.H.S. of $4$ is a part of degree 0 of localization $S_f$ of $S$ by a product closed set $\{f^t\}_{t = 0, 1, 2, \ldots}$. We denote it by $(S_f)_0$ or $S(f)_0$. Strictly speaking, for any homogeneous element $f$ with $\deg(f) = d$,

$$(S_f)_0 := S(f) := \left\{ \frac{g}{f^m} \mid g \in S_{md}, \quad m \geq 0 \right\}.$$  

Hence we proved that

**Proposition 2.2** As for $D_+(f)$,

$$(D_+(f), \mathcal{O}_{\text{Proj}(S)}|_{D_+(f)}) \cong \text{Spec}(S(f)).$$

Thus, $\text{Proj}(S)$ is obtained by gluing of affine schemes. It indicates that $(\mathcal{O}_{\text{Proj}(S)}, \mathcal{O}_{\text{Proj}(S)})$ is a scheme in the genuine sense.

Next we consider cohomology of quasi-coherent sheaf over $\text{Proj}(S)$. Assume that a graded ring $S$ is generated by $S_1$ as an $S_0$-algebra. For instance

$S = R[z_0, z_1, \ldots, z_n], \quad S_0 = R, \quad S_1 = \{a \in S \mid \deg(a) = 1\}$.

As for a quasi-coherent sheaf $\mathcal{F}$, we set

$$\mathcal{F}(m)[[x]] := \mathcal{F} \otimes_{\mathcal{O}_{\text{Proj}(S)}} \mathcal{O}_{\text{Proj}(S)}(m)[[x]],$$

and define

$$\Gamma_*(\mathcal{F}) := \oplus_{m \in \mathbb{Z}} \Gamma(X, \mathcal{F}(m))[[x]], \quad \deg(a) := m, \quad (\forall a \in \Gamma(X, \mathcal{F}(m))).$$

Then we see that $\Gamma_*(\mathcal{F}[[x]])$ is a graded $\Gamma(\mathcal{O}_{\text{Proj}(S)}[[x]])$-module. For any element $f \in S_d$, we set $a_0(f) := a/1$. Then it is well-known that

**Proposition 2.3** The map $a_0$ obtained above defines a homomorphism

$$a_0(f) : S_d[[x]] \ni a \mapsto a/1 \in S(d)_0 = \Gamma(D_+(f), \mathcal{O}_{\text{Proj}(S)}(d)[[x]]).$$

A family $\{a_0(f)\}_f$ homogeneous induces a module homomorphism

$$a_d : S_d[[x]] \rightarrow \Gamma(\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)}(d)[[x]]).$$

Hence, using the module homomorphisms $\{a_d\}$, a graded ring homomorphism

$$\alpha := \oplus_{n=0}^\infty a_d : S = \oplus_{n=0}^\infty S_d[[x]] \rightarrow \Gamma(\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)}[[x]])$$

can be defined for any quasi-coherent sheaf $\mathcal{F}$. Thus, $\Gamma(\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)}[[x]])$ admits a graded $S[[x]]$-module structure.

---

A sheaf $\mathcal{F}$ is quasi-coherent if and only if there is a pre-sheaf exact sequence $\mathcal{O}_{\text{Proj}(S)}^0 \rightarrow \mathcal{O}_{\text{Proj}(S)}^n \rightarrow \mathcal{F} \rightarrow 0$. A sheaf $\mathcal{F}$ is coherent if and only if there is a pre-sheaf exact sequence $\mathcal{O}_{\text{Proj}(S)}^n \rightarrow \mathcal{F} \rightarrow 0$ ($n \in \mathbb{N}$).
Definition 2.4 We denote the pair \((\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)}[[z]])\) by \(\text{Proj}(S)[[z]]\).

We are also interested in \(\Gamma_{\ast}(\mathcal{F}[[z]])(f)\). As for any element \(f \in S_d\), and \(x \in \Gamma(\text{Proj}(S), \mathcal{F}[[z]])\), we see \(x/f^n \in \Gamma_{\ast}(\text{Proj}(S), \mathcal{F}[[z]])(f)\). We denote the restriction of \(x\) to \(D_{\ast}(f)\) by \(x|_{D_{\ast}(f)}\). Then matching the degrees of \(x|_{D_{\ast}(f)}\) and \((a_d(f)|_{D_{\ast}(f)})^n\) we get \(x|_{D_{\ast}(f)}/(a_d(f)|_{D_{\ast}(f)})^n\).

\[
\beta(f) : \Gamma_{\ast}(\text{Proj}(S), \mathcal{F}[[z]])(f) \ni \frac{x}{f^m} \mapsto \frac{x|_{D_{\ast}(f)}}{(a_d(f)|_{D_{\ast}(f)})^m} \in \Gamma(D_{\ast}(f), \mathcal{F}[[z]]). \tag{10}
\]

As similarly for \(\alpha\), for any homogeneous element \(g \in S\), we obtain a diagram:

\[
\begin{array}{ccc}
\Gamma_{\ast}(\mathcal{F}[[z]])(f) & \rightarrow & \Gamma(D_{\ast}(f), \mathcal{F}[[z]]) \\
\downarrow \hspace{2cm} \downarrow \hspace{2cm} \downarrow \\
\Gamma_{\ast}(\mathcal{F}[[z]])(fg) & \rightarrow & \Gamma(D_{\ast}(fg), \mathcal{F}[[z]])
\end{array}
\]

By a similar argument as above, we define an \(\text{Proj}(S)[[z]]\)-module homomorphism

\[
\beta_{\mathcal{F}} : \Gamma_{\ast}(\mathcal{F}[[z]]) \rightarrow \mathcal{F}[[z]]. \tag{11}
\]

Proposition 2.5 Assume that a graded ring \(S\) is generated by \(S_1 = \{f_1, f_2, \ldots, f_\ell\}\) (\(\forall \ell \in \mathbb{Z}_{\geq 0}\)) as \(S_0\)-algebra. Then we see

(i) If \(S\) is a domain then the map \(\alpha\) induced in \([\mathcal{F}[[z]]\] is injective.

(ii) If \((f_\ell)_{\ell=1,2,\ldots,\ell}\) are all prime ideals, then the map \(\alpha\) is an isomorphism.

(iii) When \(S = k[z_0, z_1, z_2, \ldots, z_n]\), then the map \(\alpha\) is an isomorphism.

We also have the following:

Proposition 2.6 Assume that \(\mathcal{F}[[z]]\) is a quasi-coherent sheaf \(\text{Proj}(S)[[z]]\)-module. Then the homomorphism \(\beta_{\mathcal{F}}\) induced in \((\mathcal{F}[[z]]\) is an isomorphism. Furthermore, via \((\mathcal{F}[[z]\), we see

\[
\begin{align*}
H^0(\mathbb{P}^n_{k[z_0, z_1, \ldots, z_n]}[\mathbb{Z}]) & := H^0(\text{Proj}(k[z_0, z_1, z_2, \ldots, z_n], \mathcal{O}_{\text{Proj}(k[z_0, z_1, z_2, \ldots, z_n]}(m)[[z]]) \\
& = \begin{cases} 
0 & (\text{if } m < 0), \\
\{k[z_0, z_1, z_2, \ldots, z_n]|_m & (\text{o.w. } m \geq 0). \tag{12}
\end{cases}
\end{align*}
\]

Definition 2.7 Assume that \(\Lambda\) is a holomorphic skew-biderivation satisfying Jacobi rule. Then, \(\#\) is called symbol calculus on \(\text{Proj}(S)\) if \((\text{Proj}(S)[[z]], \#)\) has an associative algebra sheaf structure such that

\[
f(Z)\#g(Z) = f \circ g + \left(\frac{\ell}{2}\right) \Lambda^{\alpha\beta} \partial_{\alpha} f(Z) \partial_{\beta} g(Z) + \cdots.
\]
Proof of Theorem 1.1  Under the assumption (1), it is easy to check
\[
\sum_{k=0}^{\infty} \frac{1}{k!} \left( \mu^2 \right)^k \Lambda_{\alpha_1}^{\beta_1} \partial_{Z_1} \Lambda_{\alpha_2}^{\beta_2} \partial_{Z_2} \cdots \Lambda_{\alpha_k}^{\beta_k} \partial_{Z_k} g(Z)
\]
Then the right hand side of (13) coincides with the asymptotic expansion formula for product of the Weyl type pseudo-differential operators. Thus, it completes the proof.

Proof of Theorem 1.2  As seen in the previous argument, as for sheaf cohomology of projective space, we obtain that
\[
\sum_{k=0}^{\infty} H_0(\mathbb{CP}^n, O_{\mathbb{CP}^n}(k)) \cong \sum_{k=0}^{\infty} \mathbb{C}[Z]_k,
\]
where \( \mathbb{C}[Z]_k \) stands for the space of homogeneous polynomials of degree \( k \in \mathbb{Z}_{\geq 0} \).

Then a direct computation using (13) shows that \( \mu \) can be specialized a scalar.

Proof of Theorem 1.3  We would like to compute exponentials having the following form
\[
f(Z) = g(t) e^{\mu Q[Z](t)}
\]
with respect to \# for quadratic polynomials under a quite general setting.

Let \( Z = [z^1 : \ldots : z^n] \), \( A[Z] := ZA[Z] \), where \( A \in Sym(n, \mathbb{C}) \), i.e. \( A \) is an \( n \times n \)-complex symmetric matrix. In order to compute the exponential \( F(t) := e^{\frac{1}{\mu} A[Z]} \) with respect to the Weyl type product formula, we treat the following evolution equation:
\[
\partial_t F = \frac{1}{\mu} A[Z] \# F,
\]
with an initial condition
\[
F_0 = e^{\frac{1}{\mu} B[Z]},
\]
where \( B \in Sym(n, \mathbb{C}) \).

As seen above, our setting is rather different from the situations considered in the article [12] and in [21, 22, 24]. See also [25]. However, to compute exponentials, we can use similar methods employed in the articles above, as will be seen below:

Under the assumption \( F(t) = g \cdot e^{\frac{1}{\mu} Q[Z]} \) \( (g = g(t), Q = Q(t)) \), we would like to find a solution of the equations (15) and (16).

Direct computations give
\[
\text{L.H.S. of (15)} = g' e^{\frac{1}{\mu} Q[Z]} + g \frac{1}{\mu} Q'[Z] e^{\frac{1}{\mu} Q[Z]},
\]
\[
\text{R.H.S. of (15)} = \frac{1}{\mu} A[Z] \# F
\]
\[
= \frac{1}{\mu} A[Z] \cdot F + \frac{i\hbar}{2} A^{i_1 j_1} \partial_{z_1} \frac{1}{\mu} A[Z] \cdot \partial_{z_2} F
\]
\[
- \frac{i\hbar^2}{2A} A^{i_1 j_1} A^{i_2 j_2} \partial_{z_1} \partial_{z_2} \frac{1}{\mu} A[Z] \partial_{j_1 j_2} F
\]
(17)

Quillen’s method employing the Cayley transform is very useful to compute superconnection character forms and supertrace of Dirac-Laplacian heat kernels (cf. [3]).
where $A = (A_{ij})$, $\Lambda = (\Lambda_{ij})$ and $Q = (Q_{ij})$. Comparing the coefficient of $\mu^{-1}$ gives
\[
Q'[Z] = A[Z] - 2tAAQ[Z] - Q\Lambda A\Lambda Q[Z].
\] (18)

Applying $\Lambda$ by left and setting $q := \Lambda Q$ and $a := \Lambda A$, we easily obtain
\[
\Lambda Q' = \Lambda A + \Lambda Q\Lambda - \Lambda A A\Lambda Q - \Lambda Q\Lambda A\Lambda Q
= (1 + \Lambda Q)A(1 - \Lambda Q)
= (1 + q)a(1 - q).
\] (19)

As to the coefficient of $\mu^0$, we have
\[
g' = \frac{1}{2} \Lambda_{i_1 j_1} \Lambda_{i_2 j_2} A_{i_1 i_2} gQ_{j_1 j_2}
= -\frac{1}{2} \text{tr}(aq) \cdot g,
\] (20)

where “tr” means the trace. Thus we obtain

**Proposition 2.8** The equation (15) is rewritten by
\[
\partial_t q = (1 + q)a(1 - q),
\] (21)
\[
\partial_t g = -\frac{1}{2} \text{tr}(aq) \cdot g.
\] (22)

In order to solve the equations (21) and (22), we now recall the “Cayley transform.”

**Proposition 2.9** Set
\[
C(X) := \frac{1 - X}{1 + X}
\] (23)

if $\det (1 + X) \neq 0$. Then

1. $X \in sp\Lambda(n, \mathbb{R}) \iff \Lambda X \in Sym(n, \mathbb{R})$,
   and then $C(X) \in Sp\Lambda(n, \mathbb{R})$, where
   
   $Sp\Lambda(n, \mathbb{R}) := \{g \in GL(n, \mathbb{R}) | g\Lambda g = \Lambda\}$,
   $sp\Lambda(n, \mathbb{R}) := \text{Lie}(Sp\Lambda(n, \mathbb{R}))$.

2. $C^{-1}(g) = \frac{1 - g}{1 + g}$, (the “inverse Cayley transform”).
3. $e^{2\sqrt{-1} \tan(a)} = c(-\sqrt{-1} \tan(a))$.
4. $\log a = 2\sqrt{-1} \arctan(\sqrt{-1}C^{-1}(g))$.
5. $\partial_t q = (1 + q)a(1 - q) \iff \partial_t C(q) = -2aC(q)$.

Solving the above equation 5 in Proposition 2.9, we have
\[
C(q) = e^{-2at}C(b),
\]
where $b = AB$ and then
\[
q = C^{-1}\left(e^{-2at} \cdot C(b)\right) = C^{-1}\left(C(-\sqrt{-1} \tan(\sqrt{-1}at)) \cdot C(b)\right).
\]

Hence, according to the inverse Cayley transform, we can get $Q$ in the following way.
Proposition 2.10

\[ Q = -\Lambda \cdot C^{-1} \left( C(-\sqrt{-1}\tan(\sqrt{-1}\Lambda t)) \cdot C(\Lambda B) \right). \] (24)

Next we compute the amplitude coefficient part \( g \). Solving

\[ g' = -\frac{1}{2} tr(aq) \cdot g \] (25)
gives

Proposition 2.11

\[ g = \det \left\{ \frac{\frac{e^{\Lambda t}(1 + b) + e^{-\Lambda t}(1 - b)}{2}}{2} \right\}. \] (26)

Setting \( t = 1 \), \( a = \Lambda A \) and \( b = 0 \), we get

Theorem 2.12

\[ e_{\mu}^{\frac{1}{2}A[Z]} = \det \left( \frac{e^{\Lambda t} + e^{-\Lambda t}}{2} \right) \cdot e_{\mu}^{\frac{1}{2}(\sqrt{-1}\tan(\sqrt{-1}\Lambda t))[Z]} \] (27)

As usual, using the Čech resolution, we can compute the sheaf-cohomology \( \sum_{k=0}^{\infty} H^0(\mathbb{C}P^n, \mathcal{O}_{\mathbb{C}P^n}[\mu, \mu^{-1}]) \). Combining it with Theorems 2.12, we see

\[ e_{\mu}^{\frac{1}{2}A[Z]} \in \sum_{k=0}^{\infty} H^0(\mathbb{C}P^n, \mathcal{O}_{\mathbb{C}P^n}[\mu, \mu^{-1}]). \]

This completes the proof of Theorem 1.13 \( \square \)

3 Remarks

Consider the following diagram:

\[ (x^{\alpha, \dot{\alpha}}), [\pi_1 : \pi_2] \in M := \mathbb{C}^4 \times \mathbb{C}P^1 \]

\[ \Pi_1 \rightarrow \] (\( [z_1 : \ldots : z_4] \) \in \mathbb{C}P^3)

\[ \rightarrow (x^{\alpha, \dot{\alpha}}) \in \mathbb{C}^4 \]

\[ \Pi_2 \]
where $x^{\alpha,\dot{\alpha}}$ are even variables, we set
\[
(x^{\alpha}, \dot{x}^\alpha) := (x^1, \dot{x}^1, x^2, \dot{x}^2),
\[
([z_1 : \ldots : z_4]) := ([x^{\alpha,1} \pi_\alpha : x^{\alpha,2} \pi_\alpha : \pi_1 : \pi_2]).
\]

Here we use Einstein’s convention (we will often omit $\sum$ unless there is a danger of confusion). We call $([z_1 : \ldots : z_4])$ the homogeneous coordinate system of $\mathbb{CP}^3$.

1. The relations
\[
[z^{\dot{\alpha}}, z^{\dot{\beta}}] = \hbar D^{\alpha,\dot{\beta}} \pi_\alpha \pi_\beta,
\]
where $z^1 := z_1$, $z^2 := z_2$, give a globally defined non-commutative associative product $\#$ on $\mathbb{CP}^3$, where $D^{\alpha,\dot{\beta}}$ is a skew symmetric matrix.

2. Let $A[Z]$ be a homogeneous polynomial of $z^1 = x^{\alpha,1} \pi_\alpha$, $z^2 = x^{\alpha,2} \pi_\alpha$ with degree 2. Then a star exponential function $e^{\frac{\hbar}{\pi} A[Z]}$ gives a “function” on $\mathbb{CP}^3$.

More precisely,

**Theorem 3.1** Assume that $\Lambda := \hat{\Lambda}$ and $A[Z]$ a homogeneous polynomial of $z^1 = x^{\alpha,1} \pi_\alpha$, $z^2 = x^{\alpha,2} \pi_\alpha$ with degree 2. Then a star exponential function $e^{\frac{\hbar}{\pi} A[Z]}$ gives a cohomology class of $\mathbb{CP}^3$ with coefficients in the sheaf $\sum_{k=0}^\infty \mathcal{O}_{\mathbb{CP}^3}(k)$.

\[
M := \mathbb{C}^4|^{4N} \times \mathbb{CP}^1
\]

\[
\Pi
\]

\[
\mathbb{C}^4|^{2N} \times \mathbb{CP}^1
\]

\[
\Pi_1 \quad \Pi_2
\]

\[
\mathbb{CP}^{3|N} \quad \mathbb{C}^4|^{2N}
\]

where $\Pi$ denotes the chiral projection, we can consider non-anti-commutative deformation of super twistor space.

In order to give a brief explanation, we recall the definition of super twistor manifold ([10, 30, 31]).

**Definition 3.2** $(3|N)$-dimensional complex super manifold $Z$ is said to be a super twistor space if the following conditions (1) – (3) are satisfied.

---

6Here $[,]$ denotes the commutator bracket.
(1) $p : Z \to \mathbb{CP}^1$ is a holomorphic fiber bundle.

(2) $Z$ has a family of holomorphic sections of $p$ whose normal bundle is isomorphic to $\mathcal{O}_{\mathbb{CP}^1}(1) \oplus \mathcal{O}_{\mathbb{CP}^1}(1) \oplus \mathbb{C}^N \oplus \Pi \mathcal{O}_{\mathbb{CP}^1}(1)$.

(3) $Z$ has an anti-holomorphic involution $\sigma$ being compatible with (1), (2) and $\sigma$ has no fixed point.

We define $\mathbb{CP}^3|_{\alpha'} = (\mathbb{CP}^3|_{\mathcal{O}_{\mathbb{CP}^3}})_{\alpha'}$.

Let $f(z, \xi; \alpha')$ be a local section defined in the following manner:

$$f(z, \xi; \alpha') = \sum_{k=0}^{N} \sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_k \leq N} f_{k_1 \cdots k_k}(z) \xi_1^{k_1} \xi_2^{k_2} \cdots \xi_k^{k_k}$$ (29)

where $f_{k_1 \cdots k_k}(z)$ is a homogeneous element of $z = [z_1 : z_2 : z_3 : z_4]$ with homogeneous degree $(-k)$ on $\mathbb{CP}^3$. Then we can introduce a structure sheaf $\mathcal{O}_{\mathbb{CP}^3|_{\alpha'}}$ whose local section is given by $f(z, \xi; \alpha')$.

Under these notations, we can introduce a ringed space denoted by $\mathbb{CP}^3|_{\alpha'} = (\mathbb{CP}^3, \mathcal{O}_{\mathbb{CP}^3|_{\alpha'}})$. We shall call it non-anti-commutative complex projective super space.

As for the non-anti-commutative deformed product $\ast_{\alpha'}$ associated with the non-anti-commutative complex projective super space, we have commutation relations of local coordinate functions:

(1) Let $(z_1, z_2, \pi_1, \pi_2|\xi_1, \ldots, \xi_N)$ be a local coordinate system of $\mathcal{P}|_{\alpha'}$ where $\mathcal{P}|_{\alpha'}$ denotes the non-anti-commutative open super twistor space. Then

$$\{\xi_1, \xi_1\}_{\ast_{\alpha'}} = \alpha' C^{\alpha, \beta} \pi_\alpha \pi_\beta, \quad (0 \text{ o.w.})$$

(2) A local coordinate system $(z_1, z_2, \pi_1, \pi_2|\xi_1, \ldots, \xi_N)$ of $\mathbb{CP}^3|_{\alpha'}$ satisfies

$$\{\xi_1, \xi_1\}_{\ast_{\alpha'}} = \alpha' C^{\alpha, \beta} \pi_\alpha \pi_\beta, \quad (0 \text{ o.w.})$$

Here we do not explain more the notion and notations which appeared above and do not give the proof of them. For details, see [30].

References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, Deformation theory and quantization I, Ann. of Phys. 111 (1978), 61-110.

[2] B. V. Fedosov, A simple geometrical construction of deformation quantization, Jour. Diff. Geom. 40 (1994), 213-238.

[3] K. Gomi and Y. Terashima, Chern-Weil construction for twisted K-theory, preprint.

[4] Y. Homma and T. Tate, private communications.

[5] L. Hörmander, Fourier integral operators. I, Acta Mathematica 127, (1971), 79-183.

[6] L. Hörmander, The Weyl calculus of general pseudodifferential operators, Commun. Pure. Appl. Math. 32 (1979), 359-443.
[7] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer-Verlag, 1989

[8] L. Hörmander, *The Analysis of Linear Partial Differential Operators III*, Springer-Verlag, 1984

[9] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. 66, (2003), 157-216.

[10] C. LeBrun, Y. S. Poon and R. O. Wells, Jr., *Projective embedding of complex supermanifolds*, Commun. Math. Phys. 126 (1990) 433-452.

[11] Y. Maeda, N. Miyazaki, H. Omori and A. Yoshioka, *Star exponential functions as two-valued elements*, Progr. Math. 232 (2005), 483-492, Birkhäuser.

[12] J. M. Maillard, *Star exponential for any ordering of the elements of the inhomogeneous symplectic Lie algebra*, Jour. Math. Phys. vol.45, no. 2, (2004), 785-794.

[13] N. Miyazaki, *On a certain class of oscillatory integral transformations which determine canonical graphs*, Japanese Jour. Math. 24, no.1 (1998) 61-81.

[14] N. Miyazaki, *Formal deformation quantization and the Index theorem*, unpublished, (2003), in Japanese.

[15] N. Miyazaki, *Lifts of symplectic diffeomorphisms of a Weyl algebra bundle with Fedosov connection*, International Journal of Geometric Method in Modern Physics, Vol. 4, No. 4 (2007) 533-546.

[16] N. Miyazaki, *A Lie Group Structure for Automorphisms of a Contact Weyl Manifold*, Progr. Math. 252 (2007), 25-44, Birkhäuser.

[17] N. Miyazaki, *Remarks on deformation quantization*, Kyoto University RIMS Kokyuroku 1692, Geometric Mechanics, (2010), 1-16.

[18] N. Miyazaki, *Deformation of structure sheaf on a projective scheme*, in preparation.

[19] H. Moriyoshi, private communications.

[20] G. J. Murphy, *$C^*$-algebras and operator theory*, Academic Press, (1990)

[21] H. Omori, *Physics in Mathematics*, The Univ. Tokyo Press, (2004), in Japanese.

[22] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Star exponential functions for quadratic forms and polar elements*, Contemporary Mathematics. 315 (Amer. Math. Soc.), (2002), 35-38.

[23] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Convergent star product on Fréchet-Poisson algebras of Heisenberg type*, Contemporary Mathematics 434 (Amer. Math. Soc.), (2007), 99-123.

[24] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *A new nonformal noncommutative calculus: Associativity and finite part regularization*, Astérisque 321 (2008), 267-297.

[25] D. Quillen, *Superconnection character forms and the Cayley transform*, Topology, 27 (2), (1988), 211-238.

[26] H. Sato, private communications.

[27] M. A. Shubin, *Pseudodifferential Operators and Spectral Theory*, Springer-Verlag.

[28] D. Sternheimer, *Deformation quantization twenty years after*, AIP Conf. Proc. 453 (1998), 107-145.
[29] T. Suzuki, private communications.

[30] T. Taniguchi and N. Miyazaki, On non(anti)commutative super twistor spaces, International Journal of Geometric Method in Modern Physics Vol. 7, No. 4 (2010) 655-668.

[31] R. S. Ward and R. O. Wells, Jr., Twistor geometry and field theory, Cambridge Monographs on Math. Phys. 1990.

[32] N. Woodhouse, Geometric quantization, Clarendon Press, (1980), Oxford.

[33] A. Yoshioka, Contact Weyl manifold over a symplectic manifold, in “Lie groups, Geometric structures and Differential equations”, Adv. Stud. Pure Math. 37(2002), 459-493.