GEOMETRIC PROPERTIES OF DIRICHLET FORMS UNDER ORDER ISOMORPHISMS

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ABSTRACT. We study pairs of Dirichlet forms related by an intertwining order isomorphisms between the associated $L^2$-spaces. We consider the measurable, the topological and the geometric setting respectively. In the measurable setting, we deal with arbitrary (irreducible) Dirichlet forms and show that any intertwining order isomorphism is necessarily unitary (up to a constant). In the topological setting we deal with quasi-regular forms and show that any intertwining order isomorphism induces a quasi-homeomorphism between the underlying spaces. In the geometric setting we deal with both regular Dirichlet forms as well as resistance forms and essentially show that the geometry defined by these forms is preserved by intertwining order isomorphisms. In particular, we prove in the strongly local regular case that intertwining order isomorphisms induce isometries with respect to the intrinsic metrics between the underlying spaces under fairly mild assumptions. This applies to a wide variety of metric measure spaces including $\text{RCD}(K,N)$-spaces, complete weighted Riemannian manifolds and complete quantum graphs. In the non-local regular case our results cover in particular graphs as well as fractional Laplacians as arising in the treatment of $\alpha$-stable Lévy processes. For resistance forms we show that intertwining order isomorphisms are isometries with respect to the resistance metrics.

Our results can be understood as saying that diffusion always determines the Hilbert space, and – under natural compatibility assumptions – the topology and the geometry respectively. As special instances they cover earlier results for manifolds and graphs.

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INTRODUCTION

There is a strong interplay between geometry of a manifold, spectral theory of its Laplace-Beltrami operator and stochastic properties of the associated Brownian motion. Clearly, the geometry determines both the spectral theory and the stochastic properties and a vast literature is devoted to this topic.

On the fundamental level also the converse is of interest. Indeed, a famous question of Kac asks whether the spectral theory of the Laplacian determines the geometry. This question was originally asked for the two dimensional setting \[Kac66\]
and has triggered a substantial research over the time. Starting with Milnor's counterexample in 16 dimensions [Mil64] over Sunada's general method [Sun85] it took quite while till a negative answer was given in two dimensions by Gordon, Webb and Wolpert in [GWW92]. In a similar spirit one may ask whether the diffusion determines the geometry. This question was brought up by Arendt in [Are02] and a positive answer was given there for domains in Euclidean space satisfying a mild regularity assumption. Later a positive answer was given for manifolds by Arendt, Biegert, ter Elst in [ABtE12], see also [AtE] for a short proof in the compact case and [Are01] for related material.

Now, Brownian motion and, more generally, symmetric Markov processes are not only a basic object of study on manifolds, but can rather be considered on arbitrary measure spaces. A convenient analytic framework to describe this is given by Dirichlet spaces, i.e., measure spaces together with a Dirichlet form. More specifically, each Dirichlet space comes naturally with a Laplace type operator as well as a symmetric Markov process and, conversely, any symmetric Markov process induces a Dirichlet form on the underlying space. Prominent instances of Dirichlet spaces are metric measure spaces with the Cheeger energy, fractals as well as both discrete and metric graphs. Likewise fractional Laplacians on subsets of Euclidean space and the associated $\alpha$-stable Lévy processes give rise to Dirichlet spaces. In all these cases the underlying space is not only a measure space but carries further geometric structure, and various aspects of the interplay between this geometric structure, spectral theory of the associated Laplacians and stochastic properties of the corresponding diffusion process has received ample attention, see e.g. the recent articles [AGS14, AGS15, BBCK09, Che99, FG16, FLS16, Gig15, Kig12, KZ12, KSZ14] or the survey collection [EKKST08, FL14] and references therein.

Given this situation it is very natural to ask the question whether diffusion determines the geometry for other Dirichlet spaces than manifolds.

In the case of discrete graphs a positive partial answer to this question has recently been given in [KLSW15] using the special ingredients available in that situation.

In the present article we consider the question for arbitrary Dirichlet spaces. As mentioned above, such Dirichlet spaces do not necessarily come with a specific geometry. Instead the underlying space may only be a topological space or even just a measure space. For this reason we deal with the question successively on the level of measure theory and Hilbert spaces, on the level of topology and on the level of geometry. Our main results can be summarized as giving positive answers to the (correspondingly modified) question on each of these levels.

We will next be more specific and discuss the contents of the article in further detail.

In line with the mentioned works [Are02, AtE, ABtE12] our point of view is that two Markovian semigroups on (possibly) different spaces are naturally equivalent if they are intertwined by an order isomorphism between the corresponding $L^2$-spaces. Accordingly, our main thrust is to find features of the underlying spaces which are stable under existence of such intertwining order isomorphisms. In particular, a precise version of the idea that diffusion determines the geometry is then that equivalence of semigroups over (suitable) metric spaces entails that there exists an isometric bijection between the underlying spaces. Similarly, a precise version of the idea that diffusion determines the topology is then that equivalence of semigroups on (suitable) topological spaces entails that there exists a homeomorphism between these spaces.

We begin our investigations in the next section with a discussion of some background on order isomorphism between $L^2$-spaces. By a well-known Lamperti type result such an order isomorphism is composed of a measurable map with measurable
a.e. inverse between the underlying spaces, called \textit{transformation}, and a so-called \textit{scaling function}, which is an almost everywhere strictly positive measurable function on the range space (Proposition 1.1). This easily allows to compute their adjoints as well (Lemma 1.2). Given these basic results the bulk of the article is concerned with investigating finer features of the transformation underlying an order isomorphism between the associated $L^2$-spaces. As mentioned already, this is done on three levels:

We first consider a measurable setting and deal with arbitrary (irreducible) Dirichlet forms on measure spaces. We show that any intertwining order isomorphism is unitary up to an overall constant (Theorem 2.3). Hence, diffusion determines the Hilbert space structure. This also gives that existence of an intertwining order isomorphism is stronger than unitary equivalence. On the structural level this can be seen as explanation why diffusion – unlike spectral theory – can determine the geometry. We also show that the scaling function underlying an order isomorphism is necessarily excessive and that it is therefore constant whenever the Dirichlet spaces are recurrent. In the transient case however, a non-constant scaling can not be excluded in the generality of our setting. All this is contained in Section 2.

We then turn to a topological setting in Section 3. Here, we deal with Dirichlet forms on topological spaces. Of course, in order to obtain meaningful results we have to assume some type of compatibility between the Dirichlet form and the underlying topology. This leads us to consider quasi-regular Dirichlet forms. These provide the most general setting ensuring such a compatibility. Here, we show that the transformation underlying an intertwining order isomorphism provides a (quasi)-homeomorphism of the underlying space (Theorem 3.11). In this sense diffusion (quasi)-determines the topology. This provides an optimal result in the given setting. Indeed, in general, one cannot expect the underlying spaces to be homeomorphic, as can be seen from the example of a Euclidean ball and a punctured ball with same radius and dimension, which are indistinguishable for the Brownian motion.

This result and its proof can be seen as the heart of our article. Loosely speaking the proof requires the passage from measure theory to topology. In order to achieve this, we have to overcome a major obstacle, which was not present in any of the earlier investigations: Specifically, we have to deal with the fact that there is no well-defined pointwise evaluation of functions available in this generality. In the case of manifolds this did not play a role as one can always restrict attention to smooth functions, which allow for pointwise evaluation. In the case of graphs this did not play a role as the underlying space is discrete anyway. In order to tackle this obstacle we develop some structure theory centered around capacity and nests for quasi-regular forms. This may be useful in other contexts as well.

We complement the results of Section 3.11 by a study of the Beurling-Deny decomposition for quasi-regular form in Section 4. In particular, we show that there is no interaction between jump and strongly local part for intertwined forms (Theorem 4.2). This insight is completely new as the mentioned earlier results only dealt with situations in which the decomposition is trivial (i.e. has only one term).

Let us emphasize that Section 3 and Section 4 deal with quasi-regular Dirichlet forms in full generality thereby covering also infinite dimensional cases.

Finally, we turn to geometry in the three final sections of the article. Here, again, for meaningful results we have to assume some type of compatibility between the geometry and the Dirichlet form. In order to achieve this we rely on intrinsic geometry coming about with any regular Dirichlet form on a locally compact space together with some mild additional ‘smoothness’ assumptions. These assumptions are general enough to cover all common settings of geometric analysis. Specifically, we proceed as follows:
For strongly local regular Dirichlet forms intrinsic geometry has been developed via the concept of the intrinsic metric in a seminal work by Sturm [Stu94]. Our main result (Theorem 5.8) shows that the transformation underlying the intertwining order isomorphism provides an isometric homeomorphism between the closures of the underlying spaces (where all metric concepts are defined with respect to the intrinsic metrics). Hence, geometry is determined by diffusion in this case. This result considerably extends the corresponding results on manifolds of [Are02, AtE, AEtE12]. In fact, the assumptions are satisfied for large classes of metric measure spaces including RCD($K,N$)-spaces, complete weighted Riemannian manifolds and complete quantum graphs. Details are discussed in Section 5.

For arbitrary regular Dirichlet forms a framework of intrinsic metrics has recently been presented by Frank, Lenz and Wingert in [FLW14]. A new phenomenon featured in this theory is that there are in general several non-compatible intrinsic metrics available whereas in the strongly local case the intrinsic metric discussed in [Stu94] can be seen as the maximal intrinsic metric in the sense of [FLW14]. Our main result in this context takes care of this multitude of intrinsic metrics by invoking the assumption of recurrence. It states that after possible removal of closed sets with zero capacity, the transformation underlying the order isomorphism induces a bijection between the set of intrinsic metrics (Theorem 6.4). So, here again, the geometry (as given by the family of intrinsic metrics) is determined by the diffusion. As a consequence we obtain a bijection between the sets of intrinsic metrics on the whole spaces whenever points have positive capacity. This result is new even for graphs. Theorem 6.4 also applies to the Dirichlet forms associated to $\alpha$-stable Levy processes thereby giving the first non-local examples outside of the graph setting of diffusion determining the geometry. All of this is discussed in Section 6.

Finally, there are resistance forms. Such forms play a crucial role in the study of fractals. The class of resistance forms is not disjoint from the class of strongly local forms or the class of regular Dirichlet forms. However, in most prominent cases of resistance forms intrinsic metrics as given above are not useful as they are trivial [Hin05]. To overcome this obstacle we use that any resistance form comes with a metric, the resistance metric, which captures the geometry. In this context our result says that the transformation underlying the order isomorphism induces an isometric homeomorphism with respect to the resistance metrics (Theorem 7.3). It is the first result of this form for fractals and covers all the usual models. This is discussed in Section 7.

As far as methods are concerned, the considerations in Section 5, Section 6 and Section 7 can all be seen as building on the method given in the proof of Theorem 3.11 by additionally using the tools at hand in the corresponding specific situations. In terms of results these sections provide a rather complete treatment of diffusion determining the geometry. As a consequence we not only recover the previously known results but can deal with a wealth of new situations.

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1. ORDER ISOMORPHISMS BETWEEN $L^p$-SPACES

In this section we collect some basic results about order isomorphisms between $L^p$-spaces. All the statements given are essentially well-known and many hold in more general settings, but we restrict our attention to the situation we will need later in the article.
A measurable space is called standard Borel space if it is isomorphic to a Polish (i.e. separable complete metric) space with its Borel σ-algebra. We also say that a measure space is standard Borel if the underlying measurable space is. Throughout the section we denote by \((X_i, B_i, m_i), i \in \{1, 2\}\), σ-finite standard Borel spaces.

All equalities and inequalities in \(L^p\) are to be understood as equalities and inequalities almost everywhere (later we will have to distinguish between almost everywhere and quasi everywhere). As usual we will often write a.e. for almost everywhere in the context of functions on measure spaces.

A linear map \(U : L^p(X_1, m_1) \rightarrow L^p(X_2, m_2)\) is called positivity preserving if \(f \geq 0\) implies \(Uf \geq 0\). An invertible positivity preserving linear map with positivity preserving inverse is called order isomorphism. It is a standard result that positivity preserving operators are continuous (see e.g. [AB85], Theorem 4.3).

The structure of order isomorphisms between \(L^p\)-spaces is characterized by the following Banach-Lamperti-type theorem (see e.g. [Wei84], Proposition 5.1).

**Proposition 1.1** (Order isomorphism as weighted composition operator). If \(p \in [1, \infty)\) and \(U : L^p(X_1, m_1) \rightarrow L^p(X_2, m_2)\) is an order isomorphism, then there exist a measurable map \(h : X_2 \rightarrow (0, \infty)\) and a measurable map \(\tau : X_2 \rightarrow X_1\) with measurable a.e. inverse such that

\[
Uf = h \cdot (f \circ \tau)
\]

for all \(f \in L^p(X_1, m_1)\). The maps \(h\) and \(\tau\) are unique up to equality almost everywhere.

The maps \(h\) and \(U\) associated with an order isomorphism according to the previous proposition are the main players in the present article. We call \(h\) the scaling and \(\tau\) the transformation associated with \(U\).

Given the scaling and transformation associated with an order isomorphism, it is easy to calculate its adjoint. Here and in the following we denote by \(\varphi_#\mu\) the pushforward of the measure \(\mu\) along the map \(\varphi\).

**Lemma 1.2** (Adjoint of an order isomorphism). Let \(p \in [1, \infty)\), \(q\) the dual exponent, and let \(U : L^p(X_1, m_1) \rightarrow L^p(X_2, m_2)\) be an order isomorphism with associated scaling \(h\) and transformation \(\tau\). Then \(\tau_#m_2\) and \(m_1\) are mutually absolutely continuous. Moreover, the adjoint of \(U\) is given by

\[
U^* : L^q(X_2, m_2) \rightarrow L^q(X_1, m_1), U^*g = \frac{d(\tau_#m_2)}{dm_1}(hg) \circ \tau^{-1}.
\]

Furthermore, for \(p = 2\) one has

\[
U^*Uf = \frac{d(\tau_#m_2)}{dm_1}(h \circ \tau^{-1})^2f \quad \text{and} \quad UU^*g = \frac{dm_2}{d(\tau_#^{-1}m_1)}h^2g
\]

for all \(f \in L^2(X_1, m_1), g \in L^2(X_2, m_2)\).

**Proof.** Let \(A \in X_1\) be measurable with \(m_1(A) = 0\). Then \(1_A = 0\) in \(L^p(X_1, m_1)\), hence \(U1_A = 0\) in \(L^p(X_2, m_2)\). Thus, as \(h > 0\) a.e. we find

\[
\tau_#m_2(A) = \int_{X_2} 1_A \circ \tau \, dm_2 = \int_{X_2} \frac{1}{h} \cdot U1_A \, dm_2 = 0.
\]

The converse direction works analogously by invoking \(U\) instead of \(U^{-1}\).
Now we can compute the adjoint. For all \( f \in L^p(X_1, m_1) \), \( g \in L^p(X_2, m_2) \) we have
\[
\langle Uf, g \rangle = \int_{X_2} gh \cdot (f \circ \tau) \, dm_2
= \int_{X_1} (gh) \circ \tau^{-1} f \, d(\tau_#m_2)
= \int_{X_1} f \frac{d(\tau_#m_2)}{dm_1}(gh) \circ \tau^{-1} \, dm_1.
\]
Thus, \( \frac{d(\tau_#m_2)}{dm_1}(hg) \circ \tau^{-1} \) induces a continuous linear functional on \( L^p(X_1, m_1) \) and must therefore be an element of \( L^q \). Furthermore, the above computation shows the formula for \( U^* \). The formulae for \( UU^* \) and \( U^*U \) follow easily. \( \square \)

2. ORDER ISOMORPHISMS INTERTWining Markovian semigroups

In the previous section we have discussed the basic structure of order isomorphism. In this section we investigate the structure of order isomorphisms that intertwine Markovian semigroups. In particular, we show that intertwining on \( L^p \) implies intertwining on \( L^2 \) and that intertwining operators on \( L^2 \) between irreducible Markovian semigroups are necessarily unitary up to a constant (Theorem 2.3) and provide a strong rigidity result for intertwined semigroups in the recurrent case (Corollary 2.9).

Along the way we will need (and recall) various pieces of the theory of Markovian semigroups and Dirichlet forms. For background and references we refer to the standard textbooks such as [FOT94], more results on (not necessarily bounded) intertwiners of Markovian semigroups and probabilistic interpretations can be found in [PSZ17].

Throughout this section, \((X_1, \mathcal{B}_1, m_1), (X_2, \mathcal{B}_2, m_2)\) and \((X, \mathcal{B}, m)\) denote \(\sigma\)-finite standard Borel spaces.

If \(E_1, E_2\) are Banach spaces and \(S_1, S_2\) are (not necessarily bounded) operators on \(E_1\) and \(E_2\) respectively, an invertible operator \(U: E_1 \to E_2\) is said to intertwine \(S_1\) and \(S_2\) if
\[
UD(S_1) = D(S_2) \quad \text{and} \quad US_1f = S_2Uf
\]
for all \(f \in D(S_1)\). Two families \((S_\alpha^{(1)})_{\alpha \in A}\) and \((S_\alpha^{(2)})_{\alpha \in A}\) are said to be intertwined by \(U\) if \(U\) intertwines \(S_\alpha^{(1)}\) and \(S_\alpha^{(2)}\) for all \(\alpha \in A\).

For later use we note that two strongly continuous symmetric contraction semigroups are intertwined by \(U\) if and only if their generators are (cf. [KLSW15], Appendix A).

Denote by \(L_+(X, m)\) the space of all equivalence classes of measurable functions \(X \to [0, \infty]\). A positivity preserving operator \(S: L^p(X_1, m_1) \to L^p(X_2, m_2)\) can be uniquely extended to a map \(\tilde{S}: L_+(X_1, m_1) \to L_+(X_2, m_2)\) that satisfies \(\tilde{S}f_n \nearrow \tilde{S}f\) if \(f_n \nearrow f\). For some properties of this extension see [Kaj17], Proposition 1. Unless it is ambiguous, we will often omit the tilde and simply use the same symbol for the given operator and this extension.

If \((T_i)\) are positivity preserving operators on \(L^p(X_i, m_i)\), \(i \in \{1, 2\}\), which are intertwined by an order isomorphism \(U: L^p(X_1, m_1) \to L^p(X_2, m_2)\), it is easily verified that \(\tilde{U}T_1 = T_2U\).

From now on we will deal with Markovian semigroups, that is, strongly continuous symmetric contraction semigroups \((T_i)\) of positivity preserving operators on \(L^2\) such that \(T_1 \leq 1\) for all \(t \geq 0\).

For a Markovian semigroup \((T_i)\), the restriction \((T_i|_{L^p(X)}}) \) extends continuously to \(L^p\) for all \(p \in [1, \infty)\). We will denote this extension also by \((T_i)\). It is compatible
with the extension $\hat{T}_t$, i.e., $T_tf = \hat{T}_tf$ for all $f \in L^p_+$, $t > 0$. Moreover, the semigroups on $L^p$ and $L^q$ are mutually adjoint for dual exponents $p, q$.

**Lemma 2.1.** Let $(T^{(i)}_t)$, $i \in \{1, 2\}$, Markovian semigroups on $L^2(X_i, m_i)$, let $p \in [1, \infty)$ and let $q$ be the dual exponent. If $U: L^p(X_1, m_1) \rightarrow L^p(X_2, m_2)$ is an order isomorphism intertwining $(T^{(1)}_t)$ and $(T^{(2)}_t)$, its adjoint $U^*: L^q(X_2, m_2) \rightarrow L^q(X_1, m_1)$ is an order isomorphism intertwining $(T^{(2)}_t)$ and $(T^{(1)}_t)$.

**Proof.** Denote by $\langle \cdot, \cdot \rangle$ the dual pairing of $L^p$ and $L^q$. For $f \in L^q$ we have $f \geq 0$ if and only if $\langle f, g \rangle \geq 0$ for all $g \in L^p$.

For $f \in L^q(X_1, m_1)$ we have

$$\langle U^*f, g \rangle = \langle f, Ug \rangle \geq 0$$

for all $g \in L^p(X_1, m_1)$. Thus, $U^*f \geq 0$. The same argument holds for the inverse $(U^*)^{-1} = (U^{-1})^*$ so that $U^*$ is an order isomorphism. The intertwining property follows by taking adjoints and using the fact that the semigroups on $L^p$ and $L^q$ are adjoint. □

A measurable subset $A$ of $X$ is called *invariant* under the Markovian semigroup $(T_t)$ if $\hat{T}_t \mathbb{1}_A \subseteq \mathbb{1}_A$ for all $t \geq 0$ (for various characterizations of invariance see [Sch04]). If every invariant set is either a null set or the complement of a null set, $(T_t)$ is called *irreducible* (or *ergodic*).

If $U$ is an order isomorphism intertwining Markovian semigroups $(T^{(1)}_t)$ and $(T^{(2)}_t)$ and $\tau$ is the associated transformation, it is easy to see that a measurable set $A \subset X_2$ is invariant under $(T^{(2)}_t)$ if and only if $\tau(A)$ is invariant under $(T^{(1)}_t)$. Consequently, $(T^{(1)}_t)$ is irreducible if and only if $(T^{(2)}_t)$ is irreducible.

By a slight abuse of notation, we say that a measurable function $f$ is almost everywhere constant if there exists $\alpha \in \mathbb{R}$ such that $f = \alpha$ a.e.

**Lemma 2.2.** Let $(T_t)$ be an irreducible Markovian semigroup and let $\varphi: X \rightarrow [0, \infty)$ be measurable. If $\hat{T}_t(\varphi f) = \varphi \hat{T}_tf$ for all $f \in L_+(X, m)$ and $t \geq 0$, then $\varphi$ is almost everywhere constant.

**Proof.** Let $\lambda \geq 0$ and define $A(\lambda) = \{x \in X \mid \varphi(x) \leq \lambda\}$. We will show that $A(\lambda)$ is invariant under $T$.

By the commutation relation for $\varphi$ and $\hat{T}_t$ we have

$$\varphi \hat{T}_t \mathbb{1}_{A(\lambda)} = \hat{T}_t(\varphi \mathbb{1}_{A(\lambda)}) \leq \lambda \hat{T}_t \mathbb{1}_{A(\lambda)},$$

hence $(\varphi - \lambda) \hat{T}_t \mathbb{1}_{A(\lambda)} \leq 0$ for all $t \geq 0$. For $y \in A(\lambda)^c$ this implies

$$0 \geq (\varphi(y) - \lambda)(\hat{T}_t \mathbb{1}_{A(\lambda)})(y),$$

so $\hat{T}_t \mathbb{1}_{A(\lambda)}(y) \leq 0$ for all $t \geq 0$. On the other hand, $0 \leq \mathbb{1}_{A(\lambda)} \leq 1$ implies $\hat{T}_t \mathbb{1}_{A(\lambda)} \leq 1$.

Put together, we can conclude $\hat{T}_t \mathbb{1}_{A(\lambda)} \leq \mathbb{1}_{A(\lambda)}$ for all $t \geq 0$, that is, $A(\lambda)$ is invariant.

Now obviously,

$$A(\lambda) \subset A(\gamma) \text{ for } \lambda \leq \gamma,$$

$$X = \bigcup_{\lambda \geq 0} A(\lambda) \text{ and } \bigcap_{\lambda \geq 0} A(\lambda)^c = \emptyset.$$

Thus, since $(T_t)$ is irreducible, there is a (unique) $\beta \geq 0$ such that $m(A(\lambda)) = 0$ for $\lambda < \beta$ and $m(A(\lambda)^c) = 0$ for $\lambda > \beta$. It follows easily that $\varphi = \beta$ almost everywhere. □

**Remark.** The commutation property given in the lemma gives in fact a characterization of irreducibility, as can be seen from the characterization of invariant sets in [Sch04], Theorems 6 and 7.
Theorem 2.3 (Main properties of intertwining order isomorphisms). Let $p \in [1, \infty)$ and let $U : L^p(X_1, m_1) \rightarrow L^p(X_2, m_2)$ be an order isomorphism intertwining irreducible Markovian semigroups $(T^{(1)}_t)$ and $(T^{(2)}_t)$. Then there is a constant $\beta > 0$ such that $UU^* = \beta$, $U^*U = \beta$ and

$$\tau_\#(h^2m_2) = \beta m_1$$

for the transformation $\tau$ associated with $U$. Moreover, if $p \neq 2$, $h$ and $h^{-1}$ are bounded, and for every $r \in [1, \infty]$ the restriction $U|_{L^p \cap L^r}$ extends to an order isomorphism $U^{(r)} : L^r(X_1, m_1) \rightarrow L^r(X_2, m_2)$ intertwining $(T^{(1)}_t)$ and $(T^{(2)}_t)$ with the same scaling and transformation as $U$.

Proof. From Lemma 2.2 it follows easily that for $f \in L_+(X_1, m_1)$ we have

$$\hat{U}^*\hat{U}f = \varphi f$$

with $\varphi = \frac{d(\tau_\#m_2)}{dm_1}(h \circ \tau^{-1})^2$.

Denote by $q$ the dual exponent of $p$. Since $U$ is an order isomorphism intertwining $(T^{(1)}_t)$ and $(T^{(2)}_t)$ on $L^p$ and $U^*$ is an order isomorphism intertwining $(T^{(2)}_t)$ and $(T^{(1)}_t)$ on $L^q$, we have $\hat{U}T^{(1)}_t = \hat{T}^{(2)}_t\hat{U}$ and $U^*\hat{T}^{(2)}_t = \hat{T}^{(1)}_tU^*$. This gives

$$\hat{U}^*\hat{U}T^{(1)}_t = \hat{T}^{(1)}_t\hat{U}^*\hat{U}$$

Putting the last two displayed inequalities together we find

$$\varphi T^{(1)}_t = T^{(2)}_t\varphi$$

for all $t \geq 0$. By Lemma 2.2 there exists then a $\beta > 0$ such that $\varphi = \beta$ almost everywhere, that is, $\beta m_1 = \tau_\#(h^2m_2)$. This implies

$$\|Uf\|_{L^p(X_2, m_2)} = \int_{X_2} h^p|f \circ \tau|^p dm_2$$

$$= \beta \int_{X_2} h^{p-2}|f \circ \tau|^p d(\tau^{-1}_#m_1)$$

$$= \beta \int_{X_1} (h \circ \tau^{-1})^{p-2}|f|^p dm_1.$$ 

Since $U$ is bounded, the multiplication operator $M_{(h \circ \tau^{-1})^{p-2}}$ is bounded on $L^p(X_1, m_1)$ and so $h^{p-2}$ is bounded. The same argument for $U^{-1}$ yields that $h^{2-p}$ is also bounded. Hence, if $p \neq 2$, both $h$ and $h^{-1}$ are bounded.

Now it is easily seen that $U|_{L^p \cap L^r}$ is bounded for any $r \in [1, \infty]$ and that the extension to $L^r(X_1, m_1)$ is given by

$$U^{(r)} : L^r(X_1, m_1) \rightarrow L^r(X_2, m_2), U^{(r)}f = h \cdot (f \circ \tau).$$

In particular, $U^{(r)} = \hat{U}|_{L^r}$ and so $\hat{U}T^{(1)}_t = \hat{T}^{(2)}_t\hat{U}$ implies that $U^{(r)}T^{(1)}_t = T^{(2)}_tU^{(r)}$. \hfill $\square$

As a direct consequence of Theorem 2.3 and the equality $\|UU^*\| = \|U\|^2 = \|U^*U\|$ for $U : L^2(X_1, m_1) \rightarrow L^2(X_2, m_2)$, we get the following result in the case $p = 2$.

Corollary 2.4 (Intertwining order isomorphisms on $L^2$ are (almost) unitary). Let $U : L^2(X_1, m_1) \rightarrow L^2(X_2, m_2)$ be an order isomorphism intertwining the irreducible Markovian semigroups $(T^{(1)}_t)$ and $(T^{(2)}_t)$. Then $\frac{1}{\|U\|}U$ is unitary and

$$\tau_\#(h^2m_2) = \|U\|^2m_1.$$ 

Remark. The previous theorem justifies why we restrict our attention to intertwining on $L^2$. It shows that intertwining by order isomorphisms of irreducible semigroups on any $L^p$-space yields intertwining by order isomorphisms on $L^2$. Moreover, the corollary then shows that intertwining order isomorphisms on $L^2$ are actually (almost) unitary.
In the following sections we will primarily deal with the associated quadratic forms instead of the semigroup itself. A strongly continuous symmetric contraction semigroup on \( L^2 \) is Markovian if and only if the associated closed, densely defined quadratic form \( Q \) is a Dirichlet form, that is, \( u \wedge 1 \in D(Q) \) and \( Q(u \wedge 1) \leq Q(u) \) for all \( u \in D(Q) \) (where \( f \wedge g \) denotes the minimum of \( f \) and \( g \)).

In general, the Dirichlet form does not transform nicely under order isomorphisms intertwining the associated Markovian semigroup since it is not clear how \( U \) behaves with respect to the Hilbert space structure on \( L^2 \). However, in the irreducible case the situation is much better as the intertwining operator is (almost) unitary.

**Corollary 2.5.** For \( i \in \{1, 2\} \) let \((T_i^{(1)})\) be irreducible Markovian semigroups on \( L^2(X_i, m_i) \), \( Q_i \) the corresponding Dirichlet forms, and \( U \) an order isomorphism intertwining \((T_i^{(1)})\) and \((T_i^{(2)})\). Then \( U D(Q_1) = D(Q_2) \) and

\[
Q_2(U f, U g) = \|U\|^2 Q_1(f, g)
\]

for all \( f, g \in D(Q_1) \).

**Proof.** The operator \( V := \frac{1}{\|U\|^2} U \) clearly intertwines the semigroups and is unitary by the preceding corollary. Given this the statement of the corollary follows by standard arguments. Here, are the details: The associated Dirichlet form is derived from the semigroup via

\[
D(Q_i) = \{ f \in L^2(X_i, m_i) \mid \lim_{t \to 0} \frac{1}{t} \langle f - T_i^{(1)} f, f \rangle \text{ exists} \},
\]

\[
Q_i(f, g) = \lim_{t \to 0} \frac{1}{t} \langle f - T_i^{(1)} f, g \rangle.
\]

Hence, for all \( f, g \in L^2(X_1, m_1) \) we have

\[
\frac{1}{t} \langle V f - T_i^{(2)} V f, V g \rangle = \frac{1}{t} \langle V f - T_i^{(1)} f, V g \rangle = \frac{1}{t} \langle f - T_i^{(1)} f, g \rangle.
\]

In particular, \( f \in D(Q_1) \) if and only if \( V f \in D(Q_2) \), and \( Q_2(V f, V g) = Q_1(f, g) \) for \( f, g \in D(Q_1) \) and the desired statements follow. \( \square \)

It turns out that the scaling \( h \) belongs to a special class of functions and this can be used to show that it must be constant under an additional assumption of recurrence. Details are discussed in the remaining part of this section.

We first introduce the relevant class of functions.

**Definition 2.6.** Let \((T_i)\) be a Markovian semigroup on \( L^2(X, m) \). A function \( u \in L_+(X, m) \) is called \((T_i)\)-excessive if \( T_i u \leq u \) for all \( t \geq 0 \).

**Lemma 2.7** (\( h \) as excessive function). Let \((T_i^{(1)})\) be Markovian semigroups on \( L^2(X_i, m_i) \), \( i \in \{1, 2\} \), and \( U \) an order isomorphism intertwining \((T_i^{(1)})\) and \((T_i^{(2)})\). Then the associated scaling \( h \) is \((T_i^{(2)})\)-excessive.

**Proof.** Since \( \tilde{T}_i^{(1)} \leq 1 \), we have

\[
\tilde{T}_i^{(2)} h = \tilde{T}_i^{(2)} \tilde{U} 1 = \tilde{U} \tilde{T}_i^{(1)} 1 \leq \tilde{U} 1 = h
\]

for all \( t \geq 0 \). \( \square \)

We now turn to the additional assumption on the semigroup. Let \((T_i)\) be a Markovian semigroup on \( L^2(X, m) \). For \( f \in L_+^2(X, m) \) define the integral \( S_N f = \int_0^N T_t f \, dt \) in the Bochner sense. Then \( (S_N f) \) is a monotone increasing sequence in \( L_+^2(X, m) \) and therefore there exists a \( G f \in L_+(X, m) \) such that \( S_N f \nearrow G f \). The operator \( G \) is positivity preserving and therefore extends to \( G : L_+(X, m) \to L_+(X, m) \). A Markovian semigroup \((T_i)\) is called transient if \( G f < \infty \) almost...
everywhere for some $f \in L^4(X,m)$ with $f > 0$ a.e. It is called recurrent if $m(\{0 < Gf < \infty\}) = 0$ for all $f \in L^1_+(X,m)$. A Dirichlet form is called irreducible (resp. transient, recurrent) if the associated Markovian semigroup is irreducible (resp. transient, recurrent). Irreducibility of $Q$ is equivalent to the following property (see [FOT94], Theorem 1.6.1): If $A$ is a measurable subset of $X$ with $1_A f \in D(Q)$ and

$$Q(f) = Q(1_A f) + Q(1_{A^c} f)$$

for all $f \in D(Q)$, then $m(A) = 0$ or $m(A^c) = 0$. Recurrence of $Q$ is equivalent to $1 \in D(Q)_+$ and $Q(1) = 0$ (see [FOT94], Theorem 1.6.3).

By a standard result, an irreducible Markovian semigroup is either recurrent or transient (see e.g. [FOT94], Lemma 1.6.4). In this case, recurrence and transience can be characterized by a Liouville-type property, namely the (non-) existence of non-constant excessive functions. A convenient formulation for this is given in the following result (see [Kaj17], Theorem 1).

**Lemma 2.8.** Let $(T_t)$ be an irreducible Markovian semigroup. Then $(T_t)$ is recurrent if and only if every $(T_t)$-excessive function is a.e. constant.

Putting Lemma 2.7 and Lemma 2.8 together we obtain the following rigidity statement for recurrent semigroups. For the special case of graphs this result is already known and in fact one of the main achievements of [KLSW15].

**Corollary 2.9.** Let $(T_t^{(i)})$, $i \in \{1,2\}$, be irreducible, recurrent Markovian semigroups on $L^2(X_i,m_i)$. If $U : L^2(X_1,m_1) \to L^2(X_2,m_2)$ is an order isomorphism intertwining $(T_t^{(1)})$ and $(T_t^{(2)})$, then the associated scaling $h$ is a.e. constant and, in particular, there exists $\alpha > 0$ with

$$\tau^h_{\tau} m_2 = \alpha m_1.$$

**Remark.** As the preceding considerations show the scaling function is constant in the recurrent situation. Thus, it may be worthwhile to point out that in our setting in general non-trivial scaling can not be avoided. To see this we consider an arbitrary irreducible Markovian semigroup $T_t$ on $L^2(X,m)$ admitting an non-trivial excessive function $h$. Then it is not hard to see that $h$ must be strictly positive and that the semigroup $T_t^{(2)} := M_h^\tau T_t M_h$ on $L^2(X,h^2m)$ is also Markovian, where $M_h$ denotes the operator by multiplication with $g$. Now, clearly the semigroups $T_t$ and $T_t^{(2)}$ are intertwined by $U : L^2(X,m) \to L^2(X,h^2m), U f = \frac{1}{h} f$.

**Remark.** The considerations of this section can easily be carried over to other families of Markovian operators such as semigroups of Markovian operators over the natural numbers.

### 3. Regularity properties

In this section we study regularity properties of the scaling $h$ and transformation $\tau$ when the Dirichlet forms are not defined merely on measure spaces, but on topological spaces. For that purpose we need some compatibility of the Dirichlet form and the underlying topology. We achieve this by working with quasi-regular forms. Indeed, for our purposes the setting of quasi-regular forms is not more involved than the – maybe more common – framework of regular Dirichlet forms. At the same time it offers the advantage that we can deal with topological spaces without local compactness features. In this setting there exist natural replacements of the concepts of continuity and homeomorphism viz quasi-continuity and quasi-homeomorphisms. The main result of this section then shows that the transformation associated with an intertwining order isomorphism is a quasi-homeomorphism (Theorem 3.11). Along
the way of proving this result we have to develop the theory of general nests for the analytic capacity. This may well be of use in other situations as well.

First we recall some basic notions from the potential theory of Dirichlet forms. For a comprehensive treatment see [FOT94], Chapter 2, and [MR92], Chapter III. Since there are several slightly different definitions for various objects of relevance in potential theory in the literature, we give an (almost) comprehensive list of definitions and comment on subtleties.

Throughout this section let \( X, X_1, X_2 \) be Polish spaces and let \( m, m_1, m_2 \) be \( \sigma \)-finite Borel measure of full support on \( X, X_1, X_2 \). In particular, this assumption ensures that the arising measure spaces are standard Borel spaces.

Let \( Q \) be a Dirichlet form on \( X \). The form norm \( \| \cdot \|_Q \) on \( D(Q) \) is given by

\[
\| f \|_Q = (Q(f) + \| f \|_2^2)^{1/2}.
\]

For \( \varphi \in L^2(X, m) \) with \( \varphi > 0 \) a.e. let \( \psi = (L + 1)^{-1}\varphi \). The capacity is defined as

\[
\text{cap}_\psi(O) = \inf \{ \| f \|_Q^2 \mid f \mathbb{1}_O \geq \psi \mathbb{1}_O \text{ a.e.} \}
\]

for \( O \subset X \) open, and by

\[
\text{cap}_\psi(E) = \inf \{ \text{cap}_\psi(O) \mid O \supset E \text{ open} \}
\]

for arbitrary \( E \subset X \). Since \( \psi \) is nonnegative and belongs to \( D(Q) \), the capacity is always well-defined. We omit the index \( \psi \) whenever the choice does not matter.

An ascending sequence \( (G_k)_{k \in \mathbb{N}} \) of subsets of \( X \) is called a nest if

\[
\lim_{k \to \infty} \text{cap}(G_k) = 0.
\]

From the subadditivity of the capacity (see [MR92], Theorem 2.8) it follows that for nests \( (F_k) \) and \( (G_k) \) the refinement \( (F_k \cap G_k) \) is also a nest. Notice that we do not demand the sets \( G_k \) to be closed as is usually done. Nevertheless, by the definition of cap and its monotonicity, for each nest \( (G_k) \) there exists a nest of closed sets \( (F_k) \) with \( F_k \subset G_k \).

Remark. So far only nests of closed sets seem to have been considered in the setting of Dirichlet forms on topological spaces. In the context of Dirichlet forms on measure spaces rater general nests have already been studied in [AH05, Sch16]. However, let us stress that our definition of a nest does not coincide with the ones given there. This is due to the fact that we are in a topological setting and work with the topological (analytic) capacity instead of the measure theoretic one.

For a subset \( G \) of \( X \) let

\[
D(Q)_G = \{ f \in D(Q) \mid \text{there exists } F \subset G \text{ closed with } f \mathbb{1}_{F^c} = 0 \text{ a.e.} \}.
\]

Hence, a function \( f \in D(Q) \) belongs to \( D(Q)_G \) if and only if its measure theoretic support, i.e. the support of the measure \( f \mu \), is contained in \( G \).

The following lemma characterizes nests in terms of the density of functions vanishing outside the nest. It is a slight extension of [MR92], Theorem 2.11, which only treats nests of closed sets.

**Lemma 3.1.** An ascending sequence \( (G_k) \) of subsets of \( X \) is a nest if and only if \( \bigcup_{k \in \mathbb{N}} D(Q)_{G_k} \) is dense in \( D(Q) \) with respect to \( \| \cdot \|_Q \).

**Proof.** For a nest of closed sets \( (F_k) \) the density of \( \bigcup_{k \in \mathbb{N}} D(Q)_{F_k} \) in \( D(Q) \) follows from [MR92], Theorem 2.11. As remarked above, for an arbitrary nest \( (G_k) \) there exists a nest of closed sets \( F_k \) with \( F_k \subset G_k \). This inclusion implies

\[
\bigcup_{k \in \mathbb{N}} D(Q)_{F_k} \subset \bigcup_{k \in \mathbb{N}} D(Q)_{G_k}.
\]
and the desired density follows from the statement for $\bigcup_{k \in \mathbb{N}} D(Q)_{F_k}$.

Let $(G_k)$ be an ascending sequence of subsets of $X$ such that $\bigcup_{k \in \mathbb{N}} D(Q)_{G_k}$ is dense in $D(Q)$ with respect to $\| \cdot \|_Q$ and let $\psi \in D(Q)$ be the function that appears in the definition of cap. There then exists a sequence $\psi_n \in \bigcup_{k \in \mathbb{N}} D(Q)_{G_k}$ with $\psi_n \to \psi$ with respect to $\| \cdot \|_Q$. According to our definition of $D(Q)_{G_k}$, for each $n$ there is $k_n \in \mathbb{N}$ and a closed set $F_{k_n} \subset G_{k_n}$ with $\psi_n1_{F_{k_n}} = 0$. Moreover, it follows from Lemma 2.72 that

$$\text{cap}(O) = \inf \{ \| \psi - f \|_Q^2 \mid f \mathbb{1}_O = 0 \text{ a.e.} \}$$

for every open $O \subset X$. With this observation, the choice of $(\psi_n)$ implies

$$\lim_{k \to \infty} \text{cap}_\psi(G_k^c) = \inf_k \text{cap}_\psi(G_k^c) \leq \inf_n \text{cap}_\psi(F_{k_n}^c) \leq \inf_n \| \psi - \psi_n \|_Q^2 = 0,$$

showing that $(G_k)$ is a nest.

A set $N \subset X$ is called polar or $Q$-exceptional if there is a nest $(G_k)_{k \in \mathbb{N}}$ such that $N \subset \bigcap_k G_k^c$. Then, it is easy to see that a set $N \subset X$ is polar if and only if $\text{cap}(N) = 0$. A pointwise property is said to hold quasi-everywhere (q.e. for short) if it holds for all points outside a polar set.

For a nest $(G_k)$ let

$$C(\{G_k\}) = \{ u : X \to \mathbb{R} \mid u|_{G_k} \text{ continuous for all } k \in \mathbb{N} \}.$$ 

A function $f : X \to \mathbb{R}$ is called quasi-continuous if there is a nest $(G_k)$ such that $f \in C(\{G_k\})$. If an a.e. defined function $f$ has a quasi-continuous version, we write $\tilde{f}$ for such a version (which is a.e. and q.e. unique). Note that we also used a tilde to indicate extensions of positive operators to $L^1(X, m)$. We believe that no confusion should arise from this conflict in notation.

Let $\hat{X}$ be a Polish space, $\hat{m}$ a $\sigma$-finite Borel measure on $\hat{X}$ and $\hat{Q}$ a Dirichlet form on $L^2(\hat{X}, \hat{m})$. A map $\Phi : X \to \hat{X}$ is called quasi-homeomorphism if there are nests $(G_k)$ in $X$, $(\hat{G}_k)_{k \in \mathbb{N}}$ in $\hat{X}$ of closed sets such that $\Phi : G_k \to \hat{G}_k$ is a homeomorphism for all $k \in \mathbb{N}$.

A Dirichlet form $Q$ on a locally compact space Polish space $X$ is called regular if $C_c(X) \cap D(Q)$ is dense in $C_c(X)$ with respect to the supremum norm and in $D(Q)$ with respect to the form norm. (Here, $C_c(X)$ denotes the set of continuous functions on $X$ with compact support). A generalization to our setting of – not necessarily locally compact – Polish spaces $X$ is given by quasi-regular Dirichlet forms. Here, a Dirichlet form $Q$ is called quasi-regular if

- there exists a nest of compact sets,
- there exists a dense subset of $(D(Q), \| \cdot \|_Q)$ whose elements have quasi-continuous versions,
- there exist $f_n \in D(Q)$, $n \in \mathbb{N}$, with quasi-continuous versions $\tilde{f}_n$ and a polar set $N$ such that $\{ f_n \mid n \in \mathbb{N} \}$ separates points of $X \setminus N$.

The connection between regular and quasi-regular forms is given by the following characterization (see [CMR94], Theorem 3.7): A Dirichlet form $Q$ is quasi-regular if and only if $Q$ is locally compact Polish space $\hat{X}$, a Radon measure $\hat{m}$ of full support on $\hat{X}$, a regular Dirichlet form $\hat{Q}$ on $L^2(\hat{X}, \hat{m})$ and a quasi-homeomorphism $\Phi : X \to \hat{X}$ such that $\Phi_* m = \hat{m}$ and $Q(f \circ \Phi) = \hat{Q}(f)$ for all $f \in L^2(\hat{X}, \hat{m})$.

A set $F \subset X$ is called quasi-open (resp. quasi-closed) if there exists a nest $(F_k)_{k \in \mathbb{N}}$ of closed sets such that $F \cap F_k$ is open (resp. closed) in $F_k$.

**Lemma 3.2.** Let $Q$ be a quasi-regular Dirichlet form and $f : X \to \mathbb{R}$ quasi-continuous. Then preimages of open (resp. closed) sets under $f$ are quasi-open (resp. quasi-closed).
Proof. Let \((F_k)\) be a nest of closed sets such that \(f \in C(\{F_k\})\), and let \(A \subset \mathbb{R}\) be open (resp. closed). Then \(f^{-1}(A) \cap F_k\) is open (resp. closed) in \(F_k\) as the preimage of an open (resp. closed) set under a continuous map. Thus, \(f^{-1}(A)\) is quasi-open (resp. quasi-closed).

\[Q.E.D.\]

**Lemma 3.3.** Let \(\Phi : X \to \hat{X}\) be a quasi-homeomorphism that maps nests to nests. Then images of quasi-open (resp. quasi-closed) sets are quasi-open (resp. quasi-closed).

**Proof.** Let \(\Phi : X \to \hat{X}\) be a quasi-homeomorphism and \(A \subset X\) quasi-open. Let \((F_k)\) be a nest of closed sets in \(X\) such that \(\Phi(F_k)\) is closed and \(\Phi : F_k \to \Phi(F_k)\) is a homeomorphism for all \(k \in \mathbb{N}\), and let \((A_k)\) be a nest of closed subsets of \(X\) such that \(A \cap A_k\) is open in \(A_k\) for all \(k \in \mathbb{N}\).

Then \((A_k \cap F_k)\) is a nest, and so is \((\Phi(A_k \cap F_k))\) by assumption. Since \(\Phi\) is a homeomorphism on \(F_k\) and \(\Phi(F_k)\) is closed, the set \(\Phi(A_k \cap F_k)\) is closed. Moreover, \(\Phi(A) \cap \Phi(A_k \cap F_k) = \Phi(A \cap A_k \cap F_k)\) is open in \(\Phi(A_k \cap F_k)\) as the image of an open set under a homeomorphism.

Of course, the proof for quasi-closed sets works analogously. \(Q.E.D.\)

Let \((G_k)\) be a nest. We say that \(f : X \to \mathbb{R}\) is in the local space of \((G_k)\) if for all \(k \in \mathbb{N}\) there exists an \(f_k \in D(Q)\) with \(f|_{G_k} = f_k|_{G_k}\) a.e. We write \(D_{\text{loc}}(\{G_k\})\) for the space of all functions in the local space of \((G_k)\) and

\[D(Q)_{\text{loc}}^* = \bigcup_{n=1}^{\infty} D_{\text{loc}}(\{G_k\}),\]

where the union is taken over all nests \((G_k)\) of quasi-open sets. This definition of the local space is taken from [Kuw98], Section 4.

**Lemma 3.4.** Let \(Q\) be a quasi-regular Dirichlet form, \(f \in D(Q)_{\text{loc}}^*\) and \(D \subset D(Q)\) a dense subspace. Then \(f\) has a quasi-continuous version \(\tilde{f}\), and there exists a sequence \((f_n)\) in \(D\) such that \(f_n \to \tilde{f}\) q.e.

**Proof.** That \(f\) has a quasi-continuous version is the content of [Kuw98], Lemma 4.1. Hence there is a nest \((G_k)\) of quasi-open subsets and a sequence \((g_k)\) in \(D(Q)\) such that \(f|_{G_k} = g_k|_{G_k}\) q.e. Let \(G'_k = \{x \in G_k \mid \tilde{f}(x) = g_k(x)\}\). Since \(D\) is dense in \(D(Q)\), it follows from [MR92], Proposition III.3.5 that for every \(n \in \mathbb{N}\) there is a closed set \(F_n \subset X\) with \(\text{cap}(F_n) < 2^{-n}\) and an \(f_n \in D\) such that

\[\lim_{n \to \infty} \sup_{x \in F_n} |\tilde{f}(x) - g_n(x)| = 0.\]

Let \(F'_k = \bigcap_{n=k}^{\infty} F_n\) for \(k \in \mathbb{N}\). Then \((F'_k)\) is ascending and

\[\text{cap}(X \setminus F'_k) = \text{cap}\left(\bigcup_{n=k}^{\infty} F_n\right) \leq \sum_{n=k}^{\infty} 2^{-n} \to 0, \ k \to \infty.\]

Hence \((F'_k \cap G'_k)\) is a nest. For all \(x \in F'_k \cap G'_k\) and \(n \geq k\) we have

\[|\tilde{f}(x) - f_n(x)| \leq \sup_{y \in F'_k \cap G'_n} |\tilde{f}(y) - f_n(y)| = \sup_{y \in F'_k \cap G'_n} |\tilde{f}(y) - g_n(y)| \to 0, \ n \to \infty.\]

Since \(X \setminus \bigcup_k (F'_k \cap G'_k)\) is polar, \((f_n)\) converges q.e. to \(\tilde{f}\). \(Q.E.D.\)

The following lemma is the technical key to establishing regularity properties for excessive functions.

**Lemma 3.5** (Main tool). Let \((T_t)\) be a Markovian semigroup and \(Q\) the associated Dirichlet form. If \(h \in L_+(X, m)\) is \((T_t)\)-excessive and \(f \in D(Q)\), then \(f \land h, (f - h)_+ \in D(Q)\) and \(Q(f \land h) \leq Q(f), Q((f - h)_+) \leq 4Q(f)\).
Proof. The statement about $f \wedge h$ is the content of [Kaj17], Proposition 4. As for the statement about $(f-h)_+$, note that $(f-h)_+ = f-f \wedge h$. Thus
\[
Q((f-h)_+) = Q(f-f \wedge h) \leq 2Q(f) + 2Q(f \wedge h) \leq 4Q(f).
\]
\[\square\]

Remark. With a proof along the lines of [Sch10], Lemma 2.50 and Theorem 2.57, one can show $Q((f-h)_+) \leq Q(f)$, but the weaker estimate from the lemma is sufficient for our purposes.

Lemma 3.6. Let $(T_t)$ be a Markovian semigroup and assume that the associated Dirichlet form $Q$ is quasi-regular. If $h$ is $(T_t)$-excessive, then there is a nest $(G_k)$ of quasi-open sets such that $h \land M \in D_{loc}\{G_k\}$ for all $M \geq 0$.

Proof. By [Kuw98], Theorem 4.1, there is a nest $(G_k)$ of quasi-open sets and a sequence $(f_k) \in D(Q)$ with $f_k|_{G_k} = 1$ a.e. for all $k \in \mathbb{N}$. According to Lemma 3.5, $(Mf_k \land h) \in D(Q)$. Now $(Mf_k \land h)|_{G_k} = (h \land M)|_{G_k}$, hence $h \land M \in D_{loc}\{G_k\}$.

Proposition 3.7 (Excessive functions contained in local space). Let $(T_t)$ be a Markovian semigroup and assume that the associated Dirichlet form $Q$ is quasi-regular. If $h$ is $(T_t)$-excessive, then $h \in D(Q)^{\star}_{loc}$.

Proof. By Lemma 3.6 and [Kuw98], Lemma 4.1 there is a quasi-continuous version $\hat{h} \land n$ of $h \land n$ for all $n \in \mathbb{N}$. We first show that $(H_n) = \{(h \land n) \leq n\}$ is a nest.

The sequence $(H_n)$ consists of quasi-closed sets, see Lemma 3.2. Therefore, there exists a nest of closed sets $(F_n)$ such that for each $n$ the set $H_n \cap F_n$ is closed. By Lemma 3.1 and since $(F_n)$ is a nest, it suffices to prove that $\bigcup_{n \in \mathbb{N}} D(Q)_{H_n \cap F_n}$ is dense in $\bigcup_{n \in \mathbb{N}} D(Q)_{F_n}$.

To this end, let $k \in \mathbb{N}$ and $f \in D(Q)_{F_k}$ with $f \geq 0$. Since $h$ is $(T_t)$-excessive, so is $h \land n$. Let $f_n := (f \land n - \frac{h}{n} \land n)_+$. Obviously, $f_n = 0$ a.e. on $H_n$, and $|f_n| \leq |f| = 0$ a.e. on $F_k$. For $n^2 \geq k$ this shows $f_n = 0$ a.e. on $(H_n \cap F_n)^c$. By Lemma 3.5 we have $f_n \in D(Q)$, and $H_n \cap F_n$ is closed. Therefore, $f_n \in \bigcup_{k \in \mathbb{N}} D(Q)_{H_k \cap F_k}$ for all $n \in \mathbb{N}$.

According to Lemma 3.5, $f_n \in D(Q)$ and $Q(f_n) \leq 4Q(f \land n) \leq 4Q(f)$. Thus, every subsequence of $(f_n)$ has a weakly convergent subsequence in $(D(Q), \| \cdot \|_Q)$. Since $f_n \to f$ in $L^2$, the limit is $f$. Hence $f_n \to f$ weakly in $(D(Q), \| \cdot \|_Q)$. Weak closures and strong closures of convex sets agree in Hilbert spaces. Therefore, $f$ belongs to the closure of $\bigcup_{n \in \mathbb{N}} D(Q)_{H_n \cap F_n}$ in $(D(Q), \| \cdot \|_Q)$ and we arrive at the conclusion that $(H_n)$ is a nest.

By Lemma 3.2 the set $G_n = \{(h \land n) < n + 1\}$ is quasi-open. Since $H_n \subset G_n$, $(G_n)$ is a nest. Let $(\hat{G}_n)$ be a nest as in Lemma 3.6. Then the refinement $(G_n \cap \hat{G}_n)$ is also a nest. For $n \in \mathbb{N}$ let $f_n \in D(Q)$ such that $f_n|_{\hat{G}_n} = h \land (n + 1)|_{\hat{G}_n}$ a.e. By the definition of $G_n$ we have $(h \land (n + 1))_{\hat{G}_n} = h|_{\hat{G}_n}$ a.e. Thus, $f_n|_{G_n \cap \hat{G}_n} = h|_{\hat{G}_n}$ a.e. and we arrive at $h \in D_{loc}\{G_k \cap \hat{G}_k\}$. Since $G_n$ and $\hat{G}_n$ are quasi-open, so is their intersection and we obtain $h \in D(Q)^{\star}_{loc}$.

Remark. We were not able to find the result above in the literature. For a strongly local regular Dirichlet form it is proven in [Stu94], Lemma 3, that locally bounded excessive function belong to the local space (with respect to a nest of compact sets).

Lemma 3.8. Let $Q_i$ be irreducible quasi-regular Dirichlet forms on $L^2(X_i, m_i), i \in \{1, 2\}$, and let $U : L^2(X_1, m_1) \rightarrow L^2(X_2, m_2)$ be an order isomorphism intertwining the associated semigroups. Denote by $\tau$ the associated transformation. Then there is a nest $(G_k)_{k \in \mathbb{N}}$ of quasi-open subsets of $X_1$ such that $f \circ \tau \in D(Q_2)$ for all $f \in \bigcup_k D(Q_1)_{G_k} \cap L^\infty(X_1, m_1)$.

Proof. Denote by $H$ the scaling of $U^{-1}$. We have $H \circ \tau = 1/h$, where $h$ is the scaling of $U$. By Lemma 2.7 and Proposition 3.7 it satisfies $H \in D(Q_1)^{\star}_{loc}$. By [Kuw98],
Theorem 4.1. There is a nest \((G_k)_{k \in \mathbb{N}}\) of quasi-open subsets of \(X_1\) and a sequence \((H_k)_{k \in \mathbb{N}}\) in \(D(Q_1) \cap L^\infty(X_1, m_1)\) such that \(H|_{G_k} = H_k|_{G_k}\) a.e. for all \(k \in \mathbb{N}\).

Let \(f \in D(Q_1) \cap L^\infty(X_1, m_1)\) with \(f|_{G_k'} = 0\) a.e. Then
\[
f \circ \tau = h(Hf) \circ \tau = h(H_kf) \circ \tau\quad \text{a.e.}
\]

Since \(D(Q_1) \cap L^\infty(X_1, m_1)\) is an algebra (see [FOT94], Theorem 1.4.2), we have \(H_kf \in D(Q_1)\) and therefore \(f \circ \tau = U(H_kf) \in D(Q_2)\) by Corollary 2.5.

Lemma 3.9. Let \(Q_i\) be irreducible quasi-regular Dirichlet forms on \(L^2(X_i, m_i)\), \(i \in \{1, 2\}\), and let \(U : L^2(X_1, m_1) \to L^2(X_2, m_2)\) an order isomorphism intertwining the associated semigroups. Denote by \(h\) and \(\tau\) the associated scaling and transformation. Then \(\{h = 0\}\) is polar, \(f \circ \tau\) has a quasi-continuous version and \(\tilde{U}f = \hat{h} \cdot f \circ \tau\) q.e. for all \(f \in D(Q_1)\).

Proof. Let \((G_k)\) be a nest as in Lemma 3.8 and let \(f \in D(Q_1)_{G_k} \cap L^\infty(X_1, m_1)\) for some \(k \in \mathbb{N}\). Then \(f \circ \tau \in D(Q_2)\), hence it has a quasi-continuous version \(f \circ \tau\).

Since both \(\tilde{U}f\) and \(\hat{h} \cdot f \circ \tau\) are versions of \(Uf\), they coincide q.e. (see [FOT94], Lemma 2.1.4). In particular, \(\tilde{U}f = 0\) q.e. on \(\{h = 0\}\).

By [MR92], Proposition III.3.5 there is a subsequence \((f_{n_j})\) such that \(Uf_{n_j} \to \tilde{g}\) q.e. Thus \(\tilde{g} = 0\) q.e. on \(\{h = 0\}\) for all \(g \in D(Q_2)\). Since \(Q_2\) is quasi-regular, there is a countable collection \(\{g_n | n \in \mathbb{N}\} \subset D(Q_2)\) and a polar set \(N \subset X_2\) such that quasi-continuous versions \(\{g_n | n \in \mathbb{N}\}\) separate the points of \(X \setminus N\). Moreover, by the subadditivity of the capacity there is another polar set \(N'\) such that \(\tilde{g}_n = 0\) on \(\{h = 0\} \setminus N'\) for all \(n \in \mathbb{N}\). In particular, \(\{g_n | n \in \mathbb{N}\}\) does not separate the points of \(\{h = 0\} \setminus N'\) and so \(\{h = 0\}\) must be polar.

In order to prove the main theorem of this section, we need to recall the following regularity property for nests. A closed set \(F \subset X\) is called regular if its measure theoretic support satisfies \(\supp(1_{F \cap m}) = F\). A nest of closed sets \((F_k)\) is regular if for all \(k \in \mathbb{N}\) the set \(F_k\) is regular. The main merit of working with regular nests is that for such nests the concepts of quasi-everywhere and almost-everywhere are compatible in the following sense: For a regular nest \((F_k)\) a function \(f \in C(F_k)\) satisfies \(f \geq 0\) a.e. if and only if \(f(x) \geq 0\) for all \(x \in \bigcup_k F_k\), see [FOT94], Theorem 2.1.2. Note that for any nest of closed sets \((F_k)\) the sequence \((\supp(1_{F_k} m))\) is a regular nest, see [FOT94], Lemma 2.1.3. Thus, for most purposes nests of closed sets can be assumed to be regular.

We begin with a lemma, which is a variant of [FOT94], Theorem 2.1.2.

Lemma 3.10. Let \(Q_i\) be quasi-regular Dirichlet forms on \(L^2(X_i, m_i)\), \(i \in \{1, 2\}\). If \(\Phi, \Phi' : X_2 \to X_1\) are quasi-homeomorphisms such that \(\Phi = \Phi'\) a.e., then \(\Phi = \Phi'\) q.e.

Proof. Let \((G_k), (G'_k)\) be nests such that \(\Phi|_{G_k}\) and \(\Phi'|_{G'_k}\) are continuous. Otherwise restricting to \(\supp(1_{G_k} m_2)\) and \(\supp(1_{G'_k} m_2)\) respectively, we can assume that the nests are regular. Then \((G_k \cap G'_k)\) is also a regular nest, and thus \(\Phi = \Phi'\) a.e. implies \(\Phi|_{G_k \cap G'_k} = \Phi'|_{G_k \cap G'_k}\). Therefore, \(\Phi = \Phi'\) q.e.

Theorem 3.11 (Regularity of the transformation). Let \(Q_i\) be irreducible quasi-regular Dirichlet forms on \(L^2(X_i, m_i)\), \(i \in \{1, 2\}\), and let \(U : L^2(X_1, m_1) \to L^2(X_2, m_2)\) be an order isomorphism intertwining the associated semigroups. Then
the transformation associated with \(U\) has a version \(\tilde{\tau}\) that is a quasi-homeomorphism. Up to equality q.e. this version is unique.

Proof. Since every quasi-regular Dirichlet form is quasi-homeomorphic to a regular Dirichlet form and compositions of quasi-homeomorphisms are quasi-homeomorphisms, it suffices to prove the assertion in the regular case.

Since \(X_1\) is assumed to be metrizable and separable and \(Q_1\) is regular, there is a countable dense subalgebra \(D \subset C_0(X_1) \cap D(Q_1)\) that is uniformly dense in \(C_0(X_1)\). Since \(UD \subset D(Q_2)\) and \(Q_2\) is regular, there is a regular nest \((F_k)\) of closed subsets of \(X_2\) such that for every \(f \in D\) there is a version \(\tilde{U}f\) of \(Uf\) such that \(\tilde{U}f \in C(\{F_k\})\), and \(h\) has a version \(\tilde{h}\) such that \(\tilde{h} \in C(\{F_k\})\), see [FOT94], Theorem 2.1.2. Since \(\{\tilde{h} = 0\}\) is polar by Lemma 3.9, we may additionally assume that \(\tilde{h} > 0\) on \(\bigcup_k F_k\). Refining by a regular nest of compact sets, we can moreover assume the \((F_k)\) to be compact.

Since \(|Uf| \leq \text{h}\|f\|_{\infty}\) and \((Uf)(Ug) = hU(fg)\) a.e. for all \(f, g \in D\), the regularity of the nest \((F_k)\) implies \(|\tilde{U}f| \leq \text{h}\|f\|_{\infty}\) and \((\tilde{U}f)(\tilde{U}g) = \tilde{h}\tilde{U}(fg)\) on \(\bigcup_k F_k\) for all \(f, g \in D\). Thus, for \(y \in F_k\) the map

\[
D \rightarrow \mathbb{R}, \; f \mapsto \frac{1}{h(y)} \tilde{U}f(y)
\]

extends continuously to a multiplicative linear map \(\chi_y\) on \(C_0(X_1)\). By Gelfand-Naimark theory, there exists a unique \(\tilde{\tau}(y) \in X_1\) such that \(\chi_y(f) = f(\tilde{\tau}(y))\) for all \(f \in C_0(X_1)\).

We prove that \(\tilde{\tau}\) is continuous on \(F_k\). For if not, there exists a \(y \in F_k\), a neighborhood \(V\) of \(\tilde{\tau}(y)\) and a sequence \((y_n)\) such that \(y_n \rightarrow y\) and \(\tilde{\tau}(y_n) \notin V\). Let \(f \in D\) with \(f(\tilde{\tau}(y)) > 0\) and \(\text{supp} f \subset V\). Then

\[
\tilde{h}(y)f(\tilde{\tau}(y)) = \tilde{U}f(y) = \lim_{n \rightarrow \infty} \tilde{U}f(y_n) = \tilde{h}(y_n) f(\tilde{\tau}(y_n)) = 0,
\]

a contradiction to \(\tilde{h}(y) > 0\).

It is easy to see that the construction is consistent for different \(k\) so that \(\tilde{U}f = \tilde{h} \cdot (f \circ \tilde{\tau})\) on \(\bigcup_k F_k\) for all \(f \in D\). Since \(m_2(X \setminus \bigcup_k F_k) = 0\) and \(h > 0\) on \(\bigcup_k F_k\), we obtain \(f \circ \tilde{\tau} = f \circ \tau\) a.e. for all \(f \in D\) and so \(\tilde{\tau} = \tau\) a.e.

Since \(F_k\) is compact and \(\tilde{\tau}\) is continuous on \(F_k\), the image \(\tilde{\tau}(F_k)\) is compact. The measure \(\tilde{\tau}_#m_2 = \tau_#m_2\) is equivalent to \(m_1\) and so \(\tilde{\tau}(F_k)\) is regular. We prove that \((\tilde{\tau}(F_k))\) is a nest by showing \(UD(Q_1)_{\tilde{\tau}(F_k)} = D(Q_2)_{F_k}\). Since both \(F_k\) and \(\tilde{\tau}(F_k)\) are closed, Corollary 2.5 implies that it suffices to show \((Uf)(1_{X_1 \setminus F_k}) = 0\) a.e. if and only if \(f 1_{X_1 \setminus \tilde{\tau}(F_k)} = 0\) a.e. This however is a consequence of Corollary 2.4 and the fact that \(\tau = \tilde{\tau}\) a.e.

The transformation associated with \(U^{-1}\) is given by \(\tau^{-1}\). An application of the above arguments to \(U^{-1}\) yields a regular nest of compact sets \(\{F'_k\}\) in \(X_1\) and an \(m_1\)-version \(\tau^{-1}\) of \(\tau^{-1}\) that is continuous on \(F'_k\) for all \(k \in \mathbb{N}\).

We let \(G_k := F_k \cap \tilde{\tau}^{-1}(F'_k) = F_k \cap \tilde{\tau}^{-1}(F'_k \cap \tilde{\tau}(F_k))\). Since \(\tilde{\tau}\) is continuous on \(F_k\) and \(F_k\) is compact, \(G_k\) is compact. Moreover, as a refinement of two regular nests \((\tilde{\tau}(G_k)) = (F'_k \cap \tilde{\tau}(F_k))\) is a regular nest. With the same arguments as for proving that \((\tilde{\tau}(F_k))\) is a regular nest, it follows that \(G_k\) is a regular nest.

Next we prove that \(\tilde{\tau}|_{G_k}\) is injective. We chose \(G_k\) such that the restriction of the composition \(\tau^{-1} \circ \tilde{\tau}\) is continuous. Moreover, since \(\tilde{\tau}\) is an \(m_2\)-version of \(\tau\), \(\tau^{-1}\) is an \(m_1\)-version of \(\tau^{-1}\) and since \(\tau^{-1}_#m_1\) and \(m_2\) are equivalent, we have \(\tau^{-1} \circ \tilde{\tau} = \tau^{-1} \circ \tau = \text{id}_{X_2}\) a.e. Thus, the regularity of \(G_k\) and the continuity of \(\tau^{-1} \circ \tilde{\tau}\) implies \(\tau^{-1} \circ \tilde{\tau}|_{G_k} = \text{id}_{X_2}\) proving the injectivity of \(\tilde{\tau}|_{G_k}\).
Since $\tilde{\tau}|_{G_k}$ is continuous, $G_k$ is compact and $\tilde{\tau}|_{G_k} : G_k \to \tilde{\tau}(G_k)$ is bijective, it follows that $\tilde{\tau}|_{G_k} : G_k \to \tilde{\tau}(G_k)$ is a homeomorphism. As we have seen above ($G_k$) and ($\tilde{\tau}(G_k)$) are nests and so $\tilde{\tau}$ is a quasi-homeomorphism.

Since the transformation $\tau$ is determined up to equality a.e., the q.e. uniqueness of $\tilde{\tau}$ follows from Lemma 3.10.

In the situation of the previous theorem, we call $\tilde{\tau}$ simply the quasi-homeomorphism associated with $U$. As result of the theorem, it is uniquely determined up to equality q.e.

Our final aim in this section it to show that intertwining order isomorphism preserve not only the form domain but also the local spaces. We will need the following technical feature of $\tilde{\tau}$, $\tilde{\tau}^{-1}$.

**Proposition 3.12.** Let $Q_i$ be irreducible quasi-regular Dirichlet forms on $L^2(X_i, m_i)$, $i \in \{1, 2\}$, and let $U : L^2(X_1, m_1) \to L^2(X_2, m_2)$ an order isomorphism intertwining the associated semigroups and let $\tilde{\tau}$ be the quasi-homeomorphism associated with $U$. Then $\tilde{\tau}, \tilde{\tau}^{-1}$ map nests to nests. Moreover, if the elements of the nest are quasi-open, then so are the elements of its image under $\tilde{\tau}, \tilde{\tau}^{-1}$.

**Proof.** Let $(F_k)$ be a nest of closed subsets of $X_1$. By definition there are nests $(A_k), (B_k)$ of closed subsets of $X_1, X_2$ such that $\tilde{\tau} : B_k \to A_k$ is a homeomorphism. Then $(A_k \cap F_k)$ is a nest in $X_1$, $\tilde{\tau}^{-1}(A_k \cap F_k)$ is closed in $X_2$ and $D(Q_2)_{\tilde{\tau}^{-1}}(A_k \cap F_k) = U D(Q_1)_{A_k \cap F_k}$ (cf. proof of Theorem 3.11). Hence $(\tilde{\tau}^{-1}(A_k \cap F_k))$ is a nest in $X_2$ and so is $(\tilde{\tau}^{-1}(F_k))$. If $(G_k)$ is a nest of not necessarily closed subsets of $X_1$, there is a nest $(F_k)$ of closed sets such that $F_k \subset G_k$ by definition. Thus $\tilde{\tau}^{-1}$ maps nests to nests (and of course the same holds for $\tilde{\tau}$).

The last statement follows from the already shown part and Lemma 3.13.

The following lemma extends Corollary 2.5.

**Lemma 3.13** (Order isomorphisms preserve local space). Let $Q_i$ be irreducible quasi-regular Dirichlet forms on $L^2(X_i, m_i)$, $i \in \{1, 2\}$, and let $U : L^2(X_1, m_1) \to L^2(X_2, m_2)$ an order isomorphism intertwining the associated semigroups. Denote by $h$ and $\tilde{\tau}$ the associated scaling and quasi-homeomorphism. Let $f \in D(Q_1)_{\loc}$ be arbitrary. Then, both $h \cdot (f \circ \tau)$ and $f \circ \tau$ belong to $D(Q_2)_{\loc}$. Moreover, $f$ and $f \circ \tau$ have quasi-continuous versions that are related by $f \circ \tau = \tilde{f} \circ \tilde{\tau}$ quasi everywhere.

**Proof.** We first show $h \cdot (f \circ \tau) \in D(Q_2)_{\loc}$ for $f \in D(Q_1)_{\loc}$.

Let now $(G_n)$ be a nest of quasi-open sets and $(f_n)$ a sequence in $D(Q_1)$ such that $f_n|_{G_n} = f|_{G_n}$ a.e. Then

$$h \cdot (f \circ \tau)|_{\tilde{\tau}^{-1}(G_n)} = h \cdot (f_n \circ \tau)|_{\tilde{\tau}^{-1}(G_n)} = U f_n|_{\tilde{\tau}^{-1}(G_n)}.$$

As the image of $(G_n)$ under $\tilde{\tau}^{-1}$ is a quasi-open nest by the previous proposition, we arrive at $h \cdot (f \circ \tau) \in D(Q_2)_{\loc}$ as desired.

We now turn to proving the remaining statements of (b). Here, we improve the arguments from Lemma 3.13 with what we have already shown. Let $H$ be the scaling $U^{-1}$ and let $f \in D(Q_1)_{\loc}$. Since $D(Q_1)_{\loc}$ is an algebra by [Kuw98], Theorem 4.1, and [FOT94], Theorem 1.4.2, we have $H f \in D(Q_1)_{\loc}$. By what we have already shown this gives $f \circ \tau = h \cdot (H f) \circ \tau \in D(Q_2)_{\loc}$.

According to [Kuw98], Lemma 4.1, both $f$ and $f \circ \tau$ have quasi-continuous versions. Since $\tau = \tilde{\tau}$ a.e., it follows that $\tilde{f} \circ \tilde{\tau} = \tilde{f} \circ \tilde{\tau}$ a.e. and since $\tilde{\tau}$ is a quasi-homeomorphism that maps nets to nests, $f \circ \tilde{\tau}$ is quasi-continuous. Therefore, $f \circ \tilde{\tau} = \tilde{f} \circ \tilde{\tau}$ by [FOT94], Theorem 2.1.2.
4. The Beurling-Deny decomposition

In this section we show how the Beurling-Deny decomposition of a quasi-regular Dirichlet form transforms under an order isomorphism (Theorem 4.2).

Let $Q$ be a quasi-regular Dirichlet form on $L^2(X, m)$. Let $\Gamma$ be a measure on $X$ that charges no polar sets. A quasi-closed set $F \subset X$ is called quasi-support of $\Gamma$ if $\Gamma(F^c) = 0$ and for every quasi-closed set $\tilde{F}$ with $\Gamma(\tilde{F}^c) = 0$ the set $F \setminus \tilde{F}$ is polar.

The quasi-support of $\Gamma$ is determined up to a polar set, and we write $\text{supp}_Q(\Gamma)$ whenever the choice does not matter. The quasi-support of a measurable function $f : X \to \mathbb{R}$ is defined as the quasi-support of $|f| m$ and we denote it by $\text{supp}_Q(f)$.

Let $\Delta_X = \{(x, x) \mid x \in X\}$. The form $Q$ can be decomposed as

$$Q(f, g) = Q^{(c)}(f, g) + \int_{X \times X \setminus \Delta_X} (\tilde{f}(x) - \tilde{f}(y))(\tilde{g}(x) - \tilde{g}(y)) \, dJ(x, y) + \int_X \tilde{f}(x)\tilde{g}(x) \, dk(x)$$

for all $f, g \in D(Q)$, where

- $Q^{(c)}$ is a positive symmetric bilinear form such that $Q^{(c)}(f, g) = 0$ for all $f, g \in D(Q)$ such that $f$ is constant on a quasi-open set containing $\text{supp}_Q(g)$,
- $J$ is a $\sigma$-finite symmetric measure on $(X \times X) \setminus \Delta_X$ such that $J((N \times X) \setminus \Delta_X) = 0$ for all polar $N \subset X$,
- $k$ is a $\sigma$-finite measure on $X$ such that $k(N) = 0$ for all polar $N \subset X$.

Moreover, $Q^{(c)}$, $J$ and $k$ are unique among all maps with the properties listed above (see [DMS97], Theorem 1.2, or [Kuw98], Theorem 5.1).

For $f \in D(Q) \cap L^\infty(X, m)$ the local part of the energy measure $\Gamma^{(c)}(f)$ is the measure defined by the identity

$$\int_X \tilde{\varphi} d\Gamma^{(c)}(f) = Q^{(c)}(\varphi f, f) - \frac{1}{2} Q^{(c)}(\varphi, f^2)$$

for $\varphi \in D(Q) \cap L^\infty(X, m)$, see [Kuw98], Theorem 5.2. The local part of $Q$ then satisfies

$$Q^{(c)}(f) = \int_X d\Gamma^{(c)}(f)$$

for all $f \in D(Q) \cap L^\infty(X, m)$. By polarization $\Gamma^{(c)}(\cdot)$ can be extended to a measure valued bilinear form $\Gamma^{(c)}(\cdot, \cdot)$ on $D(Q) \cap L^\infty(X, m)$. It satisfies the following product rule (see [Kuw98], Lemma 5.2):

$$\Gamma^{(c)}(fg, h) = \tilde{f}\Gamma^{(c)}(g, h) + \tilde{g}\Gamma^{(c)}(f, h)$$

for all $f, g, h \in D(Q) \cap L^\infty(X, m)$. Moreover, due to the locality property of $Q^{(c)}$, the energy measure satisfies $1_G \Gamma^{(c)}(f) = 0$ whenever $f$ is constant a.e. on the quasi-open set $G$, see [Kuw98], Lemma 5.1. Thus $\Gamma^{(c)}$ can be extended to $D(Q)^{\ast}$ via

$$1_{G_n} \Gamma^{(c)}(f) = 1_{G_n} \Gamma^{(c)}(f_n),$$

where $(G_n)$ is a nest of quasi-open sets and $f_n \in D(Q)$ such that $1_{G_n} f_n = 1_{G_n} f$ a.e. This extension still satisfies the Leibniz rule (see [Kuw98], Lemma 5.3 for details).

Conversely, the product rule for the energy measure implies the strong locality of the form. This result is essentially well-known, see e.g. [Stu94]. We state (and prove) it here in a form suitable for the proof of the main theorem of this section.

**Lemma 4.1.** Let $Q$ be a quasi-regular Dirichlet form on $L^2(X, m)$. Denote by $M(X)$ the signed measures on $X$ endowed with the total variation norm and let

$$\tilde{\Gamma} : D(Q) \times D(Q) \to M(X)$$
be a positive, symmetric and bilinear map such that \( \hat{\Gamma}(f,f)(X) \leq Q(f) \) and \( \hat{\Gamma}(f,f) \) changes no polar set for all \( f \in D(Q) \). Let \( \hat{Q}(f,g) = \hat{\Gamma}(f,g)(X) \) for \( f, g \in D(Q) \). If \( D \subset D(Q) \) is a dense subspace and \( \hat{\Gamma} \) satisfies the product rule

\[
\hat{\Gamma}(f,g,h) = \hat{f}\hat{\Gamma}(g,h) + \hat{g}\hat{\Gamma}(f,h)
\]

for all \( f, g, h \in D \cap L^\infty(X,m) \), then \( \hat{Q} \) is strongly local in the sense that

\[
\hat{Q}(f,g) = 0
\]

whenever there exists an open set \( V \) such that \( \text{supp}_Q g \subset V \) and \( f \) is constant a.e. on \( V \).

**Proof.** First we prove that the product rule holds indeed for all \( f, g, h \in D(Q) \cap L^\infty(X,m) \). Let \( (f_k), (g_k), (h_k) \) be sequences in \( D \cap L^\infty(X,m) \) with \( f_k \to f \), \( g_k \to g \), \( h_k \to h \) in \( D(Q) \). We may assume additionally that \( \|f_k\|_\infty \leq \|f\|_\infty \) for all \( k \in \mathbb{N} \), \( f_k \to f \) q.e. and the same for \( (g_k), (h_k) \).

By [FOT94], Theorem 1.4.2, we have \( f_k g_k \to fg \) in \( D(Q) \). Since \( \hat{\Gamma} \) is bounded, we have \( \hat{\Gamma}(f_k g_k, h_k) \to \hat{\Gamma}(fg,h) \) in total variation norm. On the other hand, the product rule implies

\[
\hat{\Gamma}(f_k g_k, h_k) = \hat{f}_k \hat{\Gamma}(g_k, h_k) + \hat{g}_k \hat{\Gamma}(f_k, h_k).
\]

Moreover,

\[
\|\hat{f}_k \hat{\Gamma}(g_k, h_k) - \hat{f}_k \hat{\Gamma}(g_k, h_k)\| \leq \|\hat{f}_k - \hat{f}\| \hat{\Gamma}(g_k, h_k) + \|\hat{f}_k \hat{\Gamma}(g_k, h_k) - \hat{\Gamma}(g_k, h_k)\|
\]

\[
\leq \int_X |\hat{f}_k - \hat{f}| d\hat{\Gamma}(g_k, h_k) + \|f\|_\infty \|\hat{\Gamma}(g_k, h_k) - \hat{\Gamma}(g_k, h_k)\|.
\]

The first summand converges to 0 by Lebesgue’s theorem, while the convergence of the second summand follows once again from the boundedness of \( \hat{\Gamma} \). The same argument can be applied to \( \hat{\Gamma}(f_k, h_k) \) so that we arrive at

\[
\hat{\Gamma}(fg,h) = \lim_{k \to \infty} (\hat{f}_k \hat{\Gamma}(g_k, h_k) + \hat{g}_k \hat{\Gamma}(f_k, h_k)) = \hat{f}\hat{\Gamma}(g,h) + \hat{g}\hat{\Gamma}(f,h)
\]

as desired.

From the product rule we can infer \( \hat{\Gamma}(f,f)(G) = 0 \) for every quasi-opens set \( G \) and every \( f \in D(Q) \) such that \( f \) is constant a.e. on \( G \). Indeed, this follows from [FOT94], Corollary 3.2.1, by the transfer principle as noted in [Kuw98], Lemma 5.2. Notice that while these results are formulated for the strongly local energy measure, the proofs only use the assumptions of the present lemma.

Now, if \( V \subset X \) is open, \( \text{supp}_Q g \subset V \) and \( f \) is constant a.e. on \( V \), then

\[
|\hat{\Gamma}(f,g)(X)| \leq |\hat{\Gamma}(f,g)(V)| + |\hat{\Gamma}(f,g)(\text{supp}_Q g)^c|.
\]

As \( f \) is constant a.e. on \( V \) and \( g = 0 \) a.e. on \( (\text{supp}_Q g)^c \), \( |\hat{\Gamma}(f,g)(X)| = 0 \) follows from the Cauchy-Schwarz inequality for \( \hat{\Gamma} \), which also carries over easily to this setting. Thus, \( \hat{Q}(f,g) = 0 \).

For \( f, \varphi \in D(Q) \cap L^\infty(X,m) \) we define the truncated form

\[
Q_{\varphi}(f) = Q(\varphi f) - Q(\varphi f^2, \varphi).
\]

From the product rule it follows that

\[
Q_{\varphi}(f) = \int_X \varphi^2 d\Gamma(c)(f) + \int_{(X \times X) \setminus \Delta_X} \varphi(x) \varphi(y) (\hat{f}(x) - \hat{f}(y))^2 dJ(x,y).
\]
Theorem 4.2 (Transformation Beurling-Deny decomposition). Let $Q_i$ be irreducible quasi-regular Dirichlet forms on $L^2(X_i, m_i)$, $i \in \{1, 2\}$, and let $U : L^2(X_1, m_1) \to L^2(X_2, m_2)$ an order isomorphism intertwining the associated semigroups. Denote by $h$ and $\tilde{\tau}$ the associated scaling and quasi-homeomorphism. Then, the following holds:

- The energy measures $\Gamma_1^{(c)}$ and $\Gamma_2^{(c)}$ are related by
  \[
  \|U\|^2 \Gamma_1^{(c)}(f) = \tilde{\tau}_#(h^2 \Gamma_2^{(c)}(f \circ \tilde{\tau}))
  \]
  for all $f \in D(Q_1)_{\text{loc}}$.

- The jump measures $J_1$ and $J_2$ satisfy
  \[
  \|U\|^2 J_1 = (\tilde{\tau} \times \tilde{\tau})_#((h \otimes h)J_2).
  \]

Proof. We can assume without loss of generality that $\|U\| = 1$. By Lemma 3.8 there is a nest $(G_k)_{k \in \mathbb{N}}$ of quasi-open subsets of $X_1$ such that $f \circ \tau \in D(Q_2) \cap L^\infty(X_2, m_2)$ for all $f \in \bigcup_k D(Q_1)_{G_k} \cap L^\infty(X_1, m_1)$.

Since $h \in D(Q_2)_{\text{loc}}$, by [Kuw98], Theorem 4.1 there exists a nest of quasi-open sets $(G'_k)$ such that $h \in L^\infty(G'_k)$ for each $k \in \mathbb{N}$. The maps $\tilde{\tau}, \tilde{\tau}^{-1}$ are quasi-homeomorphisms that map nests to nests, see Proposition 3.12. Therefore, they also map quasi-open sets to quasi-open sets, see Lemma 3.8. It follows from these properties that we can assume $h \in L^\infty(\tilde{\tau}^{-1}(G_k))$ for all $k \in \mathbb{N}$.

Let $f, \varphi \in \bigcup_k D(Q_1)_{G_k} \cap L^\infty(X_1, m_1)$. Using the intertwining property we obtain

\[
Q_{1,\varphi}(f) = Q_1(\varphi f) - Q_1(\varphi f^2, \varphi) = Q_2(U(\varphi f)) - Q_2(U(\varphi f^2), U\varphi) = Q_2((U\varphi)(f \circ \tau)) - Q_2((U\varphi)(f^2 \circ \tau), U\varphi) = Q_{2, U\varphi}(f \circ \tau).
\]

Thus,

\[
\int \varphi^2 d\Gamma_1^{(c)}(f) + \int \varphi(x)\varphi(y)|\tilde{f}(x) - \tilde{f}(y)|^2 dJ_1(x, y) = \int (\tilde{U}\varphi)^2 d\Gamma_2^{(c)}(f \circ \tau) + \int \tilde{U}\varphi(x)\tilde{U}\varphi(y)|\tilde{f}(\tilde{\tau}(x)) - \tilde{f}(\tilde{\tau}(y))|^2 dJ_2(x, y).
\]

By Lemma 3.3 there is a sequence $(\psi_n)$ in $\bigcup_k D(Q_1)_{F_k}$ such that $\psi_n \to 1$ q.e. Define $(\varphi_n)$ recursively by $\varphi_1 = (\psi_1 \wedge 1) \lor 0$, $\varphi_{n+1} = (\psi_n \wedge 1) \lor \varphi_n$. Then $\varphi_n \in \bigcup_k D(Q_1)_{F_k} \cap L^\infty(X_1, m_1)$ and $\varphi_n \not\rightarrow 1$ q.e.

Moreover, since $\tilde{U}\varphi_n = h \cdot \varphi_n \circ \tau = \tilde{h} \cdot (\varphi_n \circ \tilde{\tau})$ q.e., see Lemma 3.9 and Lemma 3.13 and since $\tilde{\tau}$ maps nests to nests, we also have $\tilde{U}\varphi_n \not\rightarrow h$ q.e.

An application of the monotone convergence theorem and the identity $\tilde{f} \circ \tilde{\tau} = \tilde{f} \circ \tilde{\tau}$, see Lemma 3.13, gives

\[
\int d\Gamma_1^{(c)}(f) + \int |\tilde{f}(x) - \tilde{f}(y)|^2 dJ_1(x, y) = \lim_{n \to \infty} \left( \int \varphi_n^2 d\Gamma_1^{(c)}(f) + \int \varphi_n(x)\varphi_n(y)|\tilde{f}(x) - \tilde{f}(y)|^2 dJ_1(x, y) \right)
\]

\[
= \lim_{n \to \infty} Q_{1, \varphi_n}(f)
\]

\[
= \lim_{n \to \infty} Q_{2, \varphi_n}(f)
\]

\[
= \lim_{n \to \infty} \left( \int \tilde{U}\varphi_n^2 d\Gamma_2^{(c)}(f \circ \tau) + \int \tilde{U}\varphi_n(x)\tilde{U}\varphi_n(y)|\tilde{f}(\tilde{\tau}(x)) - \tilde{f}(\tilde{\tau}(y))|^2 dJ_2(x, y) \right)
\]

\[
= \int \tilde{h}^2 d\Gamma_2^{(c)}(f \circ \tau) + \int \tilde{h}(x)\tilde{h}(y)|\tilde{f}(\tilde{\tau}(x)) - \tilde{f}(\tilde{\tau}(y))|^2 dJ_2(x, y).
\]
Let
\[ \hat{\Gamma}_1^c(f, g) = \hat{\tau}_\#(\hat{h}^2 \, d\Gamma_2^c(f \circ \tau, g \circ \tau)) \]
and \( \hat{Q}_1^c(f, g) = \hat{\Gamma}_1^c(f, g)(X_1) \) for \( f, g \in \bigcup_k D(Q_1)_F \cap L^\infty(X_1, m_1) \). Then \( \hat{\Gamma}_1^c \) is a positive symmetric bilinear map that satisfies
\[
0 \leq \hat{\Gamma}_1^c(f)(X) \leq Q_1(f) - \int \hat{h}(x) \hat{h}(y) |\hat{f}(\hat{\tau}(x)) - \hat{f}(\hat{\tau}(y))|^2 \, dJ_2(x, y) \leq Q_1(f)
\]
for all \( f \in \bigcup_k D(Q_1)_F \cap L^\infty(X_1, m_1) \). Hence \( \hat{\Gamma}_1^c \) (and thus \( \hat{Q}_1^c \)) extends continuously to \( D(Q_1) \) and
\[
Q_1^c(f) + \int |\hat{f}(x) - \hat{f}(y)|^2 \, dJ_1(x, y) + \int |\hat{f}(x)|^2 \, dk_1(x) = \hat{Q}_1^c(f) + \int |\hat{f}(x) - \hat{f}(y)|^2 \, d((\hat{\tau} \times \hat{\tau})_\#((h \otimes h) \circ J_2))(x, y) + \int |\hat{f}(x)|^2 \, dk_1(x)
\]
for all \( f \in D(Q_1) \).

Since \( \tau^{-1} \) maps polar sets to polar sets, the measure \( \hat{\Gamma}_1^c(f) \) does not charge polar sets for all \( f \in D(Q_1) \). Moreover, the product rule for \( \Gamma_2^c \) implies immediately the product rule for \( \hat{\Gamma}_1^c \) on \( \bigcup_k D(Q_1)_F \cap L^\infty(X_1, m_1) \). Thus, \( \hat{Q}_1^c \) is strongly local by Lemma 4.1.

Therefore the uniqueness of the Beurling-Deny decomposition yields
\[
\hat{Q}_1^c(f) = Q_1^c(f)
\]
\[
J_1 = (\hat{\tau} \times \hat{\tau})_\#((h \otimes h) \circ J_2).
\]

Finally, it is not hard to see that the energy measure of \( \hat{Q}_1^c \) is given by
\[
\hat{\Gamma}_1^c(f) = \hat{\tau}_\#(\hat{h}^2 \Gamma_2^c(f \circ \tau))
\]
for \( f \in \bigcup_k D(Q_1)_F \cap L^\infty(X_1, m_1) \). The formula for arbitrary \( f \in D(Q_1) \) follows by localization.

Remark. 
- In the special situation of graphs the transformation of the jump parts (which are the only parts present in that situation) is already known (see [KLSW13], Theorem 3.6).
- The transformation of the killing parts can be deduced from the formulas for the jump and strongly local part. However, it should be noted that the killing measure of \( Q_2 \) may depend not only on the killing measure of \( Q_1 \), but also on the jump and strongly local part.

5. Strongly local Dirichlet forms

In this section we consider strongly local Dirichlet forms satisfying some additional regularity assumptions. In this situation we show that the transformation \( \tau \) has a version that is an isometry with respect to the intrinsic metrics (Theorem 5.8). As a technical byproduct we prove an alternative formula for the intrinsic metric of regular strongly local Dirichlet forms (Proposition 5.6), which may be of independent interest.

A quasi-regular Dirichlet form \( Q \) is called strongly local if the jump measure \( J \) and the killing measure \( k \) in the Beurling-Deny decomposition vanish. For simplicity’s sake, we then write \( \Gamma \) for the local energy measure \( \Gamma^c \). In the proofs of this section we will freely use properties of \( \Gamma \) such as extension to local spaces and product rule discussed in Section 4.
For a strongly local regular Dirichlet form $Q$, the intrinsic metric $d_Q$ (see [BM95, Stu94]) on $X$ is defined by

$$d_Q(x,y) = \sup\{|f(x) - f(y)|: f \in D(Q)_{\text{loc}} \cap C(X), \Gamma(f) \leq m\},$$

where $D(Q)_{\text{loc}}$ is the set of all functions $f: X \to \mathbb{R}$ such that for every open, relatively compact $G \subset X$ there exists $f_G \in D(Q)$ such that $f \mathbb{1}_G = f_G \mathbb{1}_G$.

Notice that despite its name containing the word metric, $d_Q$ may attain the value $0$ off the diagonal and may be infinite for some points.

We say that a strongly local Dirichlet form $Q$ is strictly local if $d_Q$ is a complete metric that induces the original topology on $X$. We say that $Q$ satisfies the Sobolev-to-Lipschitz property if every $f \in D(Q)$ with $\Gamma(f) \leq m$ has a $1$-Lipschitz version.

**Example 5.1.** If $Q$ is the Dirichlet energy with Dirichlet boundary conditions on a Riemannian manifold $(M, g)$, it is folklore that $d_Q$ coincides with the geodesic metric induced by $g$, see e.g. Proposition 2.2 of [ABtE12]. Thus, $Q$ is strictly local if and only if $(M, g)$ is (geodesically) complete. Moreover, $Q$ satisfies the Sobolev-to-Dirichlet property by the converse to Rademacher’s theorem. For an $\mathbb{R}^n$ version of this property, which can be transferred to manifolds by localization, see e.g. [Hei01].

**Example 5.2.** Dirichlet forms of Riemannian energy measure spaces (see [AGS15], Def. 3.16) are strictly local and satisfy the Sobolev-to-Lipschitz property. In particular, this applies to the Cheeger energy on RCD$(K, \infty)$ spaces in the sense of [AGS14]. If a metric measure space satisfies the stronger RCD$(K, N)$ condition for finite $N$, then it is also locally compact and the Cheeger energy is a regular Dirichlet form.

**Example 5.3.** Dirichlet forms on metric graphs (with Kirchhoff boundary conditions) are strictly local and satisfy the Sobolev-to-Lipschitz property, see e.g. [LSS, Hae15].

In the course of the following two lemmas we will see that the Sobolev-to-Lipschitz property can be suitably localized.

**Lemma 5.4.** Let $Q$ be a quasi-regular strongly local Dirichlet form on $L^2(X, m)$ and $f \in D(Q)_{\text{loc}}^* \cap L^{\infty}(X, m)$ with $\int d\Gamma(f) < \infty$. Then $fg \in D(Q)$ for all $g \in D(Q) \cap L^{\infty}(X, m)$.

**Proof.** Let $(G_k)$ be a nest of quasi-open subsets of $X$ such that $f \in D_{\text{loc}}(\bigcup_k G_k)$. Let $g_n$ be a sequence in $\bigcup_k D(Q)_{G_k} \cap L^{\infty}(X, m)$ such that $g_n \|g\|_2$ and $\|g_n\|_\infty \leq \|g\|_\infty$. By definition, for all $n \in \mathbb{N}$ there exists $f_n \in D(Q) \cap L^{\infty}(X, m)$ with $\|f_n\|_\infty \leq \|f\|_\infty$ such that $f = f_n$ on a quasi-open set containing the quasi-support of $g_n$. Then, $fg_n = f_n g_n \in D(Q) \cap L^{\infty}(X, m)$ (as $D(Q) \cap L^{\infty}(X, m)$ is an algebra) and $fg_n \to fg$ in $L^2(X, m)$.

The lower semicontinuity of $Q$, its locality and the product-rule for the energy measure imply

$$Q(fg) \leq \liminf_{n \to \infty} Q(fg_n)$$

$$= \liminf_{n \to \infty} Q(f_n g_n)$$

$$\leq 2 \liminf_{n \to \infty} \left( \int f_n^2 d\Gamma(g_n) + \int g_n^2 d\Gamma(f_n) \right)$$

$$= 2 \liminf_{n \to \infty} \left( \int f_n^2 d\Gamma(g_n) + \int g_n^2 d\Gamma(f) \right)$$

$$\leq 2 \left( \|f\|_\infty^2 Q(g) + \|g\|_\infty^2 \int d\Gamma(f) \right).$$

Hence $fg \in D(Q)$. \hfill $\square$
Remark. The space of functions that appeared in the previous lemma satisfies
\[ \{ f \in D(Q)^{\bullet}_{\text{loc}} \cap L^\infty(X, m) \mid \int d\Gamma(f) < \infty \} = D(Q)^{\text{ref}} \cap L^\infty(X, m), \]
where \( D(Q)^{\text{ref}} \) is the so-called reflected Dirichlet space of \( Q \) (see [Kuw02], Definition 4.1). It is known that \( D(Q) \cap L^\infty(X, m) \) is an (multiplicative) ideal in \( D(Q)^{\text{ref}} \cap L^\infty(X, m) \), c.f. [Kuw02], Lemma 4.2. We included a proof for the convenience of the reader.

Lemma 5.5. Let \( Q \) be a strongly local regular Dirichlet form on \( L^2(X, m) \) with the Sobolev-to-Dirichlet property. Assume that \( d_Q \) induces the original topology on \( X \). Then, any \( f \in D(Q)^{\bullet}_{\text{loc}} \cap L^\infty(X, m) \) with \( \Gamma(f) \leq L^2 m \) has a continuous version.

Proof. Let \( (G_k) \) be an ascending sequence of open, relatively compact subsets of \( X \) with \( \bigcup_k G_k = X \). Let \( H_k = \{ x \in G_k \mid d(x, G_k^c) > \frac{1}{k} \} \) and \( \varphi_k = (1 - kd_Q(x, H_k))_+ \). By Lemma 1.9 of [Sm94] (see Lemma 1 of [Sm95] as well), the function \( \varphi_k \) then belongs to \( D(Q)_{\text{loc}} \) and satisfies \( \Gamma(\varphi_k) \leq k^2 m \). Moreover, we clearly have \( \text{supp} \varphi_k \subset G_k \).

Hence \( \varphi_k \) belongs to \( D(Q) \cap L^\infty(X, m) \) as any function from \( D(Q)_{\text{loc}} \) with compact support belongs to \( D(Q) \) by Theorem 3.5 of [FLW14].

By the product rule and Cauchy-Schwarz inequality we have \( f \varphi_k \in D(Q)^{\bullet}_{\text{loc}} \cap L^\infty(X, m) \) and
\[ \Gamma(f \varphi_k) = \tilde{f}^2 \Gamma(\varphi_k) + \varphi_k^2 \Gamma(f) + 2 \tilde{f} \varphi_k \Gamma(f, \varphi_k) \leq 2(k^2 \| f \|_2^2 + L^2) m. \]
Moreover, the locality of \( \Gamma \) implies \( \mathbf{1}_{G_{k+1}} \Gamma(f \varphi_k) = 0 \) and so \( d \Gamma(f \varphi_k) < \infty \). Since \( G_k \) is relatively compact and \( Q \) regular, we can pick \( \psi_k \in D(Q) \cap L^\infty(X, m) \) such that \( \psi_k \mathbf{1}_{G_k^c} = \mathbf{1}_{G_k} \). Lemma 5.4 implies \( f \varphi_k = f \varphi_k \psi_k \in D(Q) \).

By the Sobolev-to-Lipschitz property there is a Lipschitz version \( \tilde{f}_k \) of \( f \varphi_k \). Since \( \tilde{f}_{k+1} \mathbf{1}_{G_{k+1}} = f_k \mathbf{1}_{G_k} \) pointwise, the limit \( f(x) = \lim_{k \to \infty} \tilde{f}_k(x) \) exists, is continuous and coincides with \( f \) a.e. on \( \bigcup_k G_k = X \).

The following lemma may well be of independent interest. It shows that in the definition of \( d_Q \) the space \( D(Q)^{\bullet}_{\text{loc}} \) can be replaced by the larger \( D(Q)^{\bullet}_{\text{loc}} \) if \( Q \) is regular.

Lemma 5.6. Let \( Q \) be a strongly local regular Dirichlet form on \( L^2(X, m) \). If \( f \in C_c(X) \cap D(Q)^{\bullet}_{\text{loc}} \) with \( \Gamma(f) \leq m \), then \( f \in D(Q)^{\bullet}_{\text{loc}} \). In particular, the intrinsic metric \( d_Q \) can be expressed as
\[ d_Q(x, y) = \sup \{|f(x) - f(y)| : f \in C_b(X) \cap D(Q)^{\bullet}_{\text{loc}}, \Gamma(f) \leq m \}. \]

Proof. Let \( G \subset X \) be relatively compact and open. Since \( Q \) is regular, there exists \( \varphi \in C_c(X) \cap D(Q) \) such that \( \mathbf{1}_G \leq \varphi \leq 1 \). By the product rule and Cauchy-Schwarz inequality, we have \( \Gamma(f \varphi) \leq 2 \tilde{f}^2 \Gamma(\varphi) + 2 \varphi \Gamma(f) \). Since \( f \) is locally bounded, the support of \( \Gamma(\varphi) \) is compact and \( \Gamma(f) \) is bounded, it follows that \( f d\Gamma(f \varphi) < \infty \). Now we infer from Lemma 5.4 that \( f \varphi^2 \in D(Q) \), and clearly, \( f \varphi^2 |_G = f |_G \).

That one can restrict the optimization problem for \( d_Q \) to bounded functions follows by a standard cut-off argument.

Proposition 5.7. Let \( Q_i, i \in \{1, 2\} \), be regular irreducible local Dirichlet forms on \( L^2(X_i, m_i) \) with the Sobolev-to-Lipschitz property, and assume that each \( d_{Q_i} \) induces the original topology on \( X_i \). Let \( U : L^2(X_1, m_1) \to L^2(X_2, m_2) \) be an order isomorphism intertwining the associated semigroups with associated transformation \( \tau \).

Then there exist closed polar sets \( N_i \subset X_i, i \in \{1, 2\} \), and a version \( \tilde{\tau} \) of \( \tau \) such that
\[ \tilde{\tau} : X_2 \setminus N_2 \to X_1 \setminus N_1 \]
is a homeomorphism.
Proof. The proof is similar to that of Theorem 3.11. Denote by $\mathcal{L}$ the Lipschitz functions on $X_1$ with compact support. By Theorem 4.9 of [FLW14] we have $\mathcal{L} \subset D(Q_{1})_{\text{loc}}$ and $\Gamma_1(f) \leq L^2 m_1$ whenever $f \in \mathcal{L}$ is $L$-Lipschitz. As functions from $D(Q_{1})_{\text{loc}}$ with compact support belong to $D(Q)$ by Theorem 3.5 of [FLW14], we then also find $\mathcal{L} \subset D(Q_{1})$.

By Corollary 2.4 and Theorem 1.2 we have $f \circ \tau \in D(Q_{2})_{\text{loc}}$ and $\Gamma_{2}(f \circ \tau) \leq L m_2$ for all $f \in \mathcal{L}$. By Lemma 5.5, $f \circ \tau$ has a continuous version $\tilde{f} \circ \tau$ for all $f \in \mathcal{L}$.

Let $N_2 = \{ y \in Y \mid \tilde{f} \circ \tau(y) = 0 \text{ for all } f \in \mathcal{L} \}$. Then $N_2$ is closed as the intersection of closed sets. It is polar by the same arguments as in Lemma 3.9.

By the Stone-Weierstraß theorem, $\mathcal{L}$ is dense in $C_0(X_1)$. Just as in the proof of Theorem 3.11 we get a map $\tilde{\tau}: X_2 \setminus N_2 \rightarrow X_1$ such that $\tilde{f} \circ \tilde{\tau} = f \circ \tau$ for all $f \in \mathcal{L}$. Continuity follows similarly since for every $x \in X_1$ the bump function $(1 - \frac{1}{r}d(x, \cdot))_+$ is in $\mathcal{L}$ for all $r > 0$. An application of the same arguments to $U^{-1}$ with its transformation $\tau^{-1}$ yields the claim.

**Theorem 5.8** (Isometric transformation). Let $Q_i$, $i \in \{1, 2\}$, be strongly local regular irreducible Dirichlet forms on $L^2(X_i, m_i)$ with the Sobolev-to-Lipschitz property. Assume that each $d_{Q_i}$ induces the original topology on $X_i$ and denote by $\overline{X_i}$ the completion of $X_i$ w.r.t. $d_{Q_i}$.

If $U: L^2(X_1, m_1) \rightarrow L^2(X_2, m_2)$ is an order isomorphism intertwining $Q_1$ and $Q_2$, then the associated transformation $\tau$ has a version $\tilde{\tau}$ that extends to an isometry from $X_2$ onto $X_1$. In particular, if $Q_1$ and $Q_2$ are strictly local, then $(X_1, d_{Q_1})$ and $(X_2, d_{Q_2})$ are isometric.

Proof. For $f \in C_b(X_1) \cap D(Q_{1})_{\text{loc}}$ with $\Gamma_1(f) \leq m_1$ we have $f \circ \tau \in D(Q_{2})_{\text{loc}} \cap L^\infty(X_2, m_2)$ by Lemma 3.13 and $\Gamma_2(f \circ \tau) \leq m_2$ by Corollary 2.4 and Theorem 4.2. Now, choose $\tilde{\tau}$ according to Proposition 5.7. Then, $f \circ \tilde{\tau}$ is continuous on $X_2 \setminus N_2$ by Proposition 5.7. Furthermore, $f \circ \tilde{\tau}$ has a continuous extension to $X_2$ by Lemma 5.5.

Using Lemma 5.6 we see that for all $x, y \in X_2 \setminus N_2$ we have

$$d_{Q_1}(\tilde{\tau}(x), \tilde{\tau}(y)) = \sup \{|f(\tilde{\tau}(x)) - f(\tilde{\tau}(y))| : f \in C_b(X_1) \cap D(Q_{1})_{\text{loc}}, \Gamma_1(f) \leq m_1\} \leq \sup \{|g(x) - g(y)| : g \in C_b(X_2) \cap D(Q_{2})_{\text{loc}}, \Gamma_2(g) \leq m_2\} = d_{Q_2}(x, y).$$

The converse inequality follows by exchanging the roles of $Q_1$ and $Q_2$.

The sets $N_i$ from Proposition 5.7 are in particular null sets, hence $X_i \setminus N_i$ is dense in $\overline{X_i}$ since $m_i$ has full support. Thus $\tilde{\tau}$ extends to an isometry from $\overline{X_2}$ onto $\overline{X_1}$.

**Remark.**

- The previous theorem can be seen as an extension of the main result of [ABtE12] (which deals with manifolds) to strongly local regular Dirichlet forms. In fact, even in the smooth setting, our result breaks new ground as it covers weighted Riemannian manifolds. Let us note, however, that in order to obtain an isometry $X_2 \rightarrow X_1$ (instead of their completions), our assumptions are slightly more restrictive than those from [ABtE12]. We assume completeness of the space, whereas in the mentioned work only some form of regularity of the boundary of the metric completion is assumed. Our result should also hold true in this more general setting with the regularity condition of the metric boundary adapted to Dirichlet forms.

- It is clear that some form of regularity of the space is needed in order to obtain an isometry on the underlying spaces and not only on completions. Indeed, for a given regular Dirichlet form one can remove a polar set from the underlying space and consider its trace on the remaining part. By the
polarity of the set the generated semigroups coincide but the underlying spaces do not.

6. Non-local Dirichlet forms and recurrence

In this section we study intertwining of general (i.e. not necessarily local) regular Dirichlet forms. In this context metric considerations are also possible due to the recently developed theory of intrinsic metric for general regular Dirichlet forms \cite{FLW14}. The considerations of \cite{FLW14} do not provide one intrinsic metric but rather a whole family of intrinsic metrics. The strongly local case is then distinguished by there being one largest intrinsic metric (referred to as ‘the’ intrinsic metric in the previous section). For general regular Dirichlet forms the intrinsic metrics will in general not be compatible and we will have to deal with the whole family. As main result in this section we establish a bijective correspondence between the families of intrinsic metrics for intertwined Dirichlet forms (Theorem \ref{thm:intr}) under suitable conditions.

We say that a subset \( F \) of \( L^0(X,m) \) is a core of bump functions if every \( f \in F \) has a continuous representative and \( F \cap C_0(X) \) is dense in \( C_0(X) \).

**Lemma 6.1.** Let \( Q_i \) be recurrent, irreducible, quasi-regular Dirichlet forms on \( L^2(X_i,m_i), i \in \{1,2\} \), and let \( U \): \( L^2(X_1,m_1) \to L^2(X_2,m_2) \) be an order isomorphism intertwining the associated semigroups. Assume that there exist cores of bump functions \( F_i \subset L^2(X_i,m_i), i \in \{1,2\} \), such that \( UF_1 = F_2 \). Then, there are closed, polar sets \( N_i \subset X_i, i \in \{1,2\} \), and a version \( \tilde{\tau} \) of \( \tau \) such that

\[
\tilde{\tau}: X_2 \setminus N_2 \to X_1 \setminus N_1
\]

is a homeomorphism.

**Proof.** By Corollary \ref{cor:scale}, the scaling \( h \) is constant. Since \( UF_1 = F_2 \subset C(X_2) \), we have \( f \circ \tau \in C(X_2) \) for all \( f \in F_1 \cap C_0(X_1) \). From here on we can proceed exactly as in Proposition \ref{prop:isom}.

The condition on the existence of suitable cores of bump functions in the proposition above does not only depend on the Dirichlet forms, but also on the order isomorphism. However, for several classes of Dirichlet forms, this condition is satisfied for all intertwining order isomorphism.

**Example 6.2.** If \( (X_i,b_i,c_i,m_i), i \in \{1,2\} \), are connected weighted graphs and \( Q_i \) the associated Dirichlet forms with Dirichlet boundary conditions (see \cite{KL12}), then \( F_i := D(Q_i) \) is a core of bump functions since \( C_c(X_i) \subset D(Q_i) \subset C(X_i) \). By definition, any order isomorphism intertwining \( Q_1 \) and \( Q_2 \) maps \( F_1 \) to \( F_2 \).

**Example 6.3.** Let \( (M_i,g_i) \) be Riemannian manifolds, \( \alpha \in (0,1] \), and \( Q_i \) the Dirichlet form generated by the fractional Laplacian \( (-\Delta_i)^{\alpha/2} \) on \( M_i \). The space \( F_i = \bigcap_{k=1}^{\infty} D((-\Delta_i)^{k\alpha/2}) \) is a core of bump functions by the Sobolev embedding theorem. Moreover, any order isomorphism intertwining \( Q_1 \) and \( Q_2 \) maps \( F_1 \) to \( F_2 \).

Let \( Q \) be a regular Dirichlet form on \( L^2(X,m) \). For an open subset \( V \) of \( X \) define the Dirichlet form \( Q^{(V)} \) on \( L^2(V,m) \) as the closure of the restriction of \( Q \) to \( D(Q) \cap C_c(V) \). If \( X \setminus V \) is polar, then

\[
Q(f) = \int_V d\mathcal{L}^{(c)}(f) + \int_{(V \times V) \setminus \Delta_V} (f(x) - f(y))^2 dJ(x,y) + \int_V f^2 \,dk
\]

for all \( f \in D(Q^{(V)}) \cap C_c(V) \). From the uniqueness of the Beurling-Deny decomposition we can infer that the strongly local energy measure, jump measure and killing measure for \( Q^{(V)} \) are given by the restrictions the measures for \( Q \) to \( V \) resp. \( (V \times V) \setminus \Delta_V \).
We now turn to intrinsic metrics in this context following the theory developed in [FLWT14]. As mentioned already, when dealing with non-local Dirichlet forms, there is not a distinguished intrinsic metric, but a family of intrinsic metrics. Let us briefly recall the relevant definitions.

Recall that the local form domain $D(Q)_{\text{loc}}$ of a regular Dirichlet form is the set of all functions $f: X \to \mathbb{R}$ such that for every open, relatively compact $G \subset X$ there exists $f_G \in D(Q)$ such that $f \mathbb{1}_G = f_G \mathbb{1}_G$.

Let $J$ be the jump measure of $Q$ in the Beurling-Deny decomposition. The space $D(Q)_{\text{loc}}^*$ is the set of all functions $f \in D(Q)_{\text{loc}}$ such that

$$\int_{(K \times X) \setminus \Delta_X} (\tilde{f}(x) - \tilde{f}(y))^2 \, dJ(x, y) < \infty$$

for all compact $K \subset X$.

For $f \in D(Q)_{\text{loc}}$ the Radon measure $\Gamma^b(f)$ is defined by

$$\Gamma^b(f)(E) = \int_{(E \times X) \setminus \Delta_X} (\tilde{u}(x) - \tilde{u}(y))^2 \, dJ(x, y)$$

for $E \subset X$ measurable.

A pseudo-metric $d: X \times X \to [0, \infty]$ is called intrinsic metric for $Q$ if there are Radon measure $m^b$ and $m^c$ with $m^b + m^c \leq m$ such that for all $A \subset X$ and all $M > 0$ the function $d_A = d(\cdot, A)$ satisfies

- $d_A \wedge M \in D(Q)_{\text{loc}}^* \cap C(X)$,
- $\Gamma^b(d_A \wedge T) \leq m^b$,
- $\Gamma^c(d_A \wedge T) \leq m^c$.

The set of all intrinsic metrics for $Q$ is denoted by $\mathcal{I}(Q)$.

If $Q$ is strongly intrinsic, then the metric $d_Q$ discussed in Section 5 is an intrinsic metric in the sense of the definition above, and every intrinsic metric for $Q$ is pointwise dominated by $d_Q$ (see [FLWT14], Theorem 6.1).

**Theorem 6.4.** Let $Q_i$, $i \in \{1, 2\}$. Let $U: L^2(X_1, m_1) \to L^2(X_2, m_2)$ be an order isomorphism intertwining the associated semigroups and denote by $\tau$ the associated transformation. Assume that there exist cores of bump functions $F_i \subset L^2(X_i, m_i)$ such that $UF_1 = F_2$. Let $N_i \subset X_i$ be closed polar sets, $V_i = X_i \setminus N_i$, and $\tilde{\tau}$ a version of $\tau$ such that $\tilde{\tau}: V_2 \to V_1$ is a homeomorphism. Then

$$\Phi: \mathcal{I}(Q_1^{(V_1)}) \to \mathcal{I}(Q_2^{(V_2)}), \quad d \mapsto d(\tilde{\tau}(\cdot), \tilde{\tau}(\cdot))$$

is a bijection.

**Proof.** It suffices to show that $\Phi$ maps intrinsic metrics to intrinsic metrics. Let $d_1$ be an intrinsic metric for $Q_1^{(V_1)}$ and let $m_1^b, m_1^c$ be Radon measures on $X_1$ such that $m_1^b + m_1^c \leq m_1$ and $\Gamma_1^b((d_1)_A \wedge M) \leq m_1^b$, $\Gamma_1^c((d_1)_A \wedge M) \leq m_1^c$ for all $A \subset U_1$ and $M > 0$.

Let $d_2 = \Phi(d_1)$. Since $Q_1, Q_2$ are recurrent, there is an $\alpha > 0$ with $h = \alpha$ a.e. It follows easily from Theorem 4.2 that $(d_2)_{\tilde{\tau}^{-1}(A) \wedge M} = ((d_1)_A \wedge M) \circ \tilde{\tau} \in D(Q_2)_{\text{loc}}^*$ for all $A \subset U_1$ and $M > 0$. Moreover, the fact that $\tilde{\tau}: U_2 \to U_1$ is a homeomorphism guarantees $(d_2)_A \wedge T \in C(X_2)$.

Let $m_2^b = \alpha^2 m_1^b, m_2^c = m_1^c$. An application of Theorem 4.2 gives $m_2^b + m_2^c \leq m_2$ and $\Gamma_2^b((d_2)_{\tilde{\tau}^{-1}(A) \wedge M}) \leq m_2^b$, $\Gamma_2^c((d_2)_{\tilde{\tau}^{-1}(A) \wedge M}) \leq m_2^c$ for all $A \subset V_1$ and $M > 0$. Hence $d_2$ is an intrinsic metric.

**Remark.** When points have positive capacity, the sets $V_1, V_2$ coincide with $X_1, X_2$. Consequently, in this case $\tilde{\tau}$ induces a bijection between the sets of intrinsic metrics for the original Dirichlet forms. This result is new even for the case of graphs treated in [KLSW15].
7. Resistance forms

In this section we study Dirichlet forms induced by resistance forms. The theory of such forms goes back to Kigami and plays a major role in analysis and probability theory on fractals \cite{Kig01,Kig12}. Resistance forms naturally come with a metric viz. the resistance metric and this metric is a fundamental tool in their study. As main result of this section we show that the transformation associated with an intertwining order isomorphism is an isometry (up to an overall factor) with respect to the resistance metrics (Theorem \ref{lem:resistance_isomorphism}).

A resistance form on a set $X$ is a pair $(\mathcal{E}, \mathcal{F})$ consisting of a subspace $\mathcal{F} \subset \mathbb{R}^X$ and a positive quadratic form $\mathcal{E}$ on $\mathcal{F}$ such that the following conditions hold:

- The constant functions are contained in $\mathcal{F}$ and $\mathcal{E}(f) = 0$ if and only if $f$ is constant.
- The quotient $\mathcal{F}/\mathbb{R}1$ is a Hilbert space with norm $\mathcal{E}^{1/2}$.
- If $V \subset X$ is finite and $g: V \rightarrow \mathbb{R}$, there is a function $f \in \mathcal{F}$ such that $f|_V = g$.
- For all $x, y \in X$, the distance

$$R(x, y) = \sup\{|f(x) - f(y)|^2 \mid f \in \mathcal{F}, \mathcal{E}(f) \leq 1\}$$

is finite.

- If $f \in \mathcal{F}$, then $\bar{f} := (f \vee 0) \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(\bar{f}) \leq \mathcal{E}(f)$.

It is easily seen that $R$ is a metric on $X$, the so-called resistance metric associated with $(\mathcal{E}, \mathcal{F})$. We will endow $X$ with the topology induced by $R$.

The resistance form $(\mathcal{E}, \mathcal{F})$ is called regular if $X$ is locally compact and $\mathcal{F} \cap C_c(X)$ is dense in $C_0(X)$. By \cite{Kig12}, Theorem 6.3, this condition is equivalent to the following:

If $K \subset X$ is compact and $U \subset X$ open such that $K \subset U$ and $\bar{U}$ is compact, then there is a function $\varphi \in \mathcal{F}$ such that $\varphi|_K = 1$, im $\varphi \subset [0, 1]$ and supp $\varphi \subset \bar{U}$.

Given a Radon measure $m$ of full support on $(X, R)$, the restriction of $\mathcal{E}$ to $\mathcal{F} \cap L^2(X, m)$ is closed in $L^2(X, m)$ (cf. \cite{Kig01}, 2.4.1).

We call $Q$ a Dirichlet form associated with $(\mathcal{E}, \mathcal{F})$ if $C_c(X) \cap \mathcal{F} \subset D(Q) \subset \mathcal{F} \cap L^2(X, m)$ and $Q$ is a restriction of $\mathcal{E}$. If $(\mathcal{E}, \mathcal{F})$ is regular, the closure of the restriction of $\mathcal{E}$ to $C_c(X) \cap \mathcal{F}$ is a regular Dirichlet form.

**Lemma 7.1.** Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on $X$ and $m$ a finite Radon measure on $X$. Then $\mathcal{E}$ restricted to $\mathcal{F} \cap L^2(X, m)$ is an irreducible, recurrent Dirichlet form.

**Proof.** Denote by $Q$ the restriction of $\mathcal{E}$ to $\mathcal{F} \cap L^2(X, m)$. We have $1 \in \mathcal{F} \cap L^2(X, m) = D(Q) \subset D(Q)_e$ and $Q(1) = \mathcal{E}(1) = 0$. Thus, $Q$ is recurrent.

If $A \subset X$ is a $Q$-invariant set, then $1_A \in D(Q) \subset \mathcal{F}$ and $Q(1_A) = Q(1) - Q(1_{A^c}) \leq 0$, hence $1_A = 0$ or $1_A = 1$. Thus, $Q$ is irreducible.

**Lemma 7.2.** Let $(\mathcal{E}, \mathcal{F})$ be a regular resistance form. Then the resistance metric is given by

$$R(x, y) = \sup\{|f(x) - f(y)|^2 \mid f \in \mathcal{F} \cap C_b(X), \mathcal{E}(f) \leq 1\}, \ x, y \in X.$$ 

**Proof.** Let $x, y \in X$. For $\epsilon > 0$ choose $g \in \mathcal{F}$ such that $\mathcal{E}(g) \leq 1$ and $|g(x) - g(y)|^2 \geq R(x, y) - \epsilon$. Assume without loss of generality that $g(x) \geq g(y)$. Let $f = (g \wedge g(x)) \vee g(y)$. Then $f \in \mathcal{F} \cap C_b(X), \mathcal{E}(f) \leq \mathcal{E}(g) \leq 1$ and

$$|f(x) - f(y)|^2 = |g(x) - g(y)|^2 \geq R(x, y) - \epsilon.$$ 

Since $\epsilon$ was arbitrary, the assertion follows. \hfill \qed
Here comes the main result of this section. It essentially shows that intertwining order isomorphisms between resistance forms are induced by isometries w.r.t. resistance metric between the spaces.

**Theorem 7.3** (*U* is induced from an isometry). Let \((\mathcal{E}_i, \mathcal{F}_i)\) be regular resistance forms on \(X_i, i \in \{1, 2\}\), \(m_i\) finite Borel measures of full support on \((X_i, R_i)\) and \(Q_i\) irreducible, recurrent Dirichlet forms associated with \((\mathcal{E}_i, \mathcal{F}_i)\). If \(U: L^2(X_1, m_1) \rightarrow L^2(X_2, m_2)\) is an order isomorphism intertwining \(Q_1\) and \(Q_2\), then there is a version \(\tilde{\tau}\) of the associated transformation \(\tau\), which is a homeomorphism, and a constant \(\alpha > 0\) such that \(h = \alpha\) a.e. and

\[
\|U\|^2 R_1(\tilde{\tau}(y), \tilde{\tau}(z)) = \alpha^2 R_2(y, z)
\]

for all \(y, z \in X_2\).

**Proof.** Corollary 2.4 implies that there is a constant \(\alpha > 0\) such that \(h = \alpha\) a.e.

Since \(m_i\) are finite, we have \(\mathcal{F}_i \cap C_b(X_i) = D(Q_i) \cap C_b(X_i)\). Moreover, by [Kig12, Proposition 9.13], \(Q_i\)-quasi-continuous functions are continuous. Hence \(\tau\) has a version \(\tilde{\tau}\) that is a homeomorphism. Furthermore, \(U(D(Q_1) \cap C_b(X_1)) = D(Q_2) \cap C_b(X_2)\).

If \(f \in D(Q_1)\), it follows that

\[
Q_2(\alpha f \circ \tau) = Q_2(U f) = \|U\|^2 Q_1(f).
\]

Thus, \(Q_1(f) \leq 1\) if and only if \(\alpha^2 Q_2(f \circ \tau) \leq \|U\|^2\). Therefore,

\[
R_1(\tilde{\tau}(y), \tilde{\tau}(z)) = \sup\{|f(\tilde{\tau}(y)) - f(\tilde{\tau}(z))|^2 \mid f \in D(Q_1) \cap C_b(X_1), Q_1(f) \leq 1\}
\]

\[
= \sup\{|g(y) - g(z)|^2 \mid g \in D(Q_2) \cap C_b(X_2), \alpha^2 Q_2(g) \leq \|U\|^2\}
\]

\[
= \frac{\alpha^2}{\|U\|^2} R_2(y, z). \quad \square
\]

As a consequence of the previous considerations we can treat typical examples of resistance forms as arising in the study of fractals. This is the content of the next corollary.

**Corollary 7.4** (Typical example). Let \((\mathcal{E}_i, \mathcal{F}_i)\) be resistance forms on \(X_i, i \in \{1, 2\}\), such that \((X_i, R_i)\) are compact, \(m_i\) probability measures on \(X_i\) and \(Q_i\) Dirichlet forms associated with \(\mathcal{E}_i\). If \(U: L^2(X_1, m_1) \rightarrow L^2(X_2, m_2)\) is an order isomorphism intertwining \(Q_1\) and \(Q_2\), then \(\tilde{\tau}\) is a surjective isometry with respect to the resistance metrics \(R_1, R_2\).

**Proof.** By Lemma 7.3 the Dirichlet forms \(Q_1\) and \(Q_2\) are irreducible and recurrent. It suffices to show that \(\alpha = \|U\|\) in Theorem 7.3. By Corollary 2.4 we have \(\alpha^2 \tau_# m_2 = \|U\|^2 m_1\). Since \(m_1(X_1) = 1 = m_2(X_2)\), the assertion follows. \(\square\)

**Remark.** Inspection of the proof of the corollary shows that the compactness assumption on the \(X_i\) is not necessary. Also, it is not necessary that the \(m_i\) are probability measures. It suffices that they are finite measures with \(m_1(X_1) = m_2(X_2)\).

**Remark.** The set of resistance forms is not disjoint from the sets of strictly local forms. So, for strictly local resistance forms one could also try and use the intrinsic metrics discussed above. However, in typical situations these intrinsic metrics will be zero and, hence, do not capture any geometry of the underlying set [Hin05].

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