INTEGRABLE SYSTEMS AND ISOMONODROMY DEFORMATIONS

RICHARD BEALS AND DAVID H. SATTINGER
Yale University and University of Minnesota

ABSTRACT. We analyze in detail three classes of isomondromy deformation problems associated with integrable systems. The first two are related to the scaling invariance of the $n \times n$ AKNS hierarchies and the Gel’fand-Dikii hierarchies. The third arises in string theory as the representation of the Heisenberg group by $[(L^{k/n})_+, L] = I$ where $L$ is an $n^{th}$ order scalar differential operator. The monodromy data is constructed in each case; the inverse monodromy problem is solved as a Riemann-Hilbert problem; and a simple proof of the Painlevé property is given for the general case.

Introduction
1. Overdetermined systems and isomonodromy equations
2. The forward monodromy problem at $z = \infty$
3. The forward monodromy problem at $z = 0$
4. The inverse problem and the Painlevé property
5. Rational solutions
6. Bäcklund transformations
7. Scaling, self-similarity, and construction of isomonodromy deformations
8. Gel’fand-Dikii equations and isomonodromy
9. Isomonodromy deformations and string equations

Introduction

It has been known since work of Ablowitz and Segur [AS1] that there is an intimate relationship between equations like KdV which are integrable by the inverse scattering method and Painlevé equations; see also [AS2]. It was observed by Flaschka and Newell [FN] that just as KdV gives an isospectral flow for the Schrödinger operator, Painlevé equations are monodromy preserving flows for linear systems with irregular singular points. Certain of these problems have been investigated in detail, for $2 \times 2$ systems and second order scalar problems: [FN], [FZ], [IN], [JMU], [JM1], [JM2].

Research of the authors was supported by National Science Foundation grants DMS-8916968 and DMS-9123844
In this paper we discuss general isomonodromy problems associated to \( n \times n \) systems and higher order equations. The corresponding isomonodromy equations are generally of order greater than 2, so they are not the classical Painlevé transcendents. However they do have the Painlevé property, in fact the stronger property that any solution has a single-valued meromorphic extension to the entire plane; cf. [Ma], [Mi]. We hope, among other things, to simplify and clarify the treatment of isomonodromy deformations and their relation to Riemann-Hilbert problems, on the principle that the more general the case the less reliance on special features.

The paper is organized as follows. In §1 we describe the formal connection between matrix isomonodromy equations and overdetermined \( n \times n \) systems in two variables. This connection is made rigorous in the next three sections, which describe the forward problem at the singular points \( z = \infty \) and \( z = 0 \), connect it to a Riemann-Hilbert problem, and prove the Painlevé property. A few examples of equations and systems which occur in this context are: the Painlevé II equation

\[
4(xu)_x + u_{xxx} - 6u^2u_x = 0;
\]

the system of three equations of order one

\[
(xu_i)_x = a_i u + b_i u_j u_k, \quad \{i, j, k\} = \{1, 2, 3\};
\]

the system of two equations of order two with cubic nonlinearity

\[
(xu_1)_x + \frac{1}{2}u_{1xx} - u_1 u_2 = 0 = (xu_2)_x - \frac{1}{2}u_{2xx} + u_2^2 u_1;
\]

and the system of two equations of order two with quadratic nonlinearity

\[
(xu_1)_x + \frac{i}{\sqrt{3}}u_{1xx} + 2u_2 u_{2x} = 0 = (xu_2)_x - \frac{i}{\sqrt{3}}u_{2xx} + 2u_1 u_{1x}.
\]

In §5 the rational solutions of the isomonodromy equations are constructed by solving finite linear systems. This extends results of Airault [Ai], who found Bäcklund transformations giving rational solutions of some Painlevé equations. We develop the gauge theory of Bäcklund transforms in §6. The gauge transformations take the wave functions for one solution to those of a new solution and thus transform solutions to solutions.

Isomonodromy deformations arise from integrable systems in two ways. Some can be obtained as self-similar solutions of the given nonlinear evolution equations; see [AS2]. This construction is given in §7 for isospectral deformations of \( n \times n \) first-order operators.
(AKNS-ZS systems); each of the examples above is of this type. In §7 we also treat the isospectral deformations of an \( n \)-th order scalar differential operator (the Gel’fand-Dikii hierarchy). Examples include the equation

\[
(xu)_x + u + \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x = 0
\]

which corresponds to self-similar solutions of the KdV equation and the system

\[
(xu_1)_x = u_{1xx} - 2u_{0x}, \quad (xu_0)_x + u_0 = \frac{2}{3}u_{1xx} + \frac{2}{3}u_1u_{1x} - u_{0xx}
\]

which corresponds to self-similar solutions of the Boussinesq system. In all these cases the Lax pair for the integrable system can be rescaled to obtain a Lax pair for the corresponding isomonodromy deformation problem. This was done in detail in the \( 2 \times 2 \) case by Flaschka and Newell [FN] for the modified KdV equation and its associated isomonodromy problem, the Painlevé II equation, as well as the sine-Gordon equation and the Painlevé III equation (cf. also Its and Novokshenov [IN]). In section 8 we show that all self-similar solutions of Gel’fand-Dikii flows are in fact solutions of isomonodromy equations, and we treat the direct and inverse problems for these equations. The results are analogous to those of §§2, 3, 4.

A second class of isomonodromy problems was obtained by M. Douglas [Do] in recent two-dimensional theories of quantum gravity. These problems are obtained by replacing \( \dot{L} \) by \( \hbar I \) in the Lax equation \( \dot{L} = [ (L^k/n)_+ , L ] \) for the Gel’fand-Dikii flows. This device leads to a representation of the Heisenberg group and to a class of isomonodromy problems different from those obtained by the scaling invariance. Two of the simplest examples are

\[
\frac{1}{4}u_{xxx} + \frac{3}{2}uu_x + \hbar = 0;
\]

\[
\frac{1}{16}u^{(5)} + \frac{5}{8}uu_{xxx} + \frac{5}{4}u_xu_{xx} + \frac{15}{8}u^2u_x + \hbar = 0.
\]

This class of isomonodromy problems was treated by Moore [Mo1], [Mo2]. We thank him for useful discussions of the topic. In §9 we focus on some aspects of the mathematical analysis of these problems and give a different, more complete and self-contained treatment.

1. Overdetermined systems and isomonodromy equations

In this section we outline the formal connection between certain overdetermined linear \( n \times n \) systems and monodromy preserving equations.

Let \( J \) and \( \mu \) be diagonal matrices belonging to the space \( M_d(\mathbb{C}) \) of \( d \times d \) complex matrices; \textit{we assume that each has distinct diagonal entries and has trace zero}. Suppose
that \( q = q(x) \) is an off-diagonal matrix-valued function defined on some real interval or some connected complex domain \( I \). Consider the overdetermined system for a matrix-valued function \( \psi(x, z) \) of real or complex \( x \) and complex \( z \), \( \psi \in SL(n, \mathbb{C}), n \geq 2 \):

\[
(1.1) \quad \frac{\partial \psi}{\partial x} = [zJ + q(x)]\psi;
\]
\[
(1.2) \quad z\frac{\partial \psi}{\partial z} = A(x, z)\psi = \sum_{j=0}^{n} z^j A_j(x)\psi, \quad A_n \equiv \mu.
\]

The compatibility condition (“zero curvature condition”) for these equations is

\[
(1.3) \quad \left[ \frac{\partial}{\partial x} - (zJ + q), z \frac{\partial}{\partial z} - A(x, z) \right] = 0.
\]

Set

\[
(1.4) \quad A_j = F_{n-j}, \ n > 1; \quad A_1 = F_{n-1} + xJ; \quad A_0 = F_n + xq(x).
\]

Then the zero-curvature condition (1.3) is equivalent to the sequence of conditions

\[
(1.5) \quad F_0 = \mu, \quad [J, F_{j+1}] = \left[ \frac{d}{dx} - q, F_j \right], \quad 0 \leq j < n,
\]

together with the equation

\[
(1.6) \quad \left[ \frac{d}{dx} - q, xq + F_n \right] = 0.
\]

The conditions (1.4) do not determine the \( F_j \) uniquely but, as we shall indicate, there is a determination for which the \( F_j \) are polynomials in \( q \) and its derivatives. Then (1.6) is a nonlinear differential equation for \( q \), and we show that it is an isomonodromy equation for the system of differential equations in the complex domain (1.2): under the flow (1.6), the monodromy of (1.2) is invariant.

**Examples:** 1. The simplest nontrivial example is obtained with \( n = 2, k = 3, \mu = J = \text{diag}(1, -1) \), and

\[
q(x) = u(x) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

Then

\[
F_1 = q, \quad F_2 = \frac{1}{2} \begin{bmatrix} -u^2 & u_x \\ -u_x & u^2 \end{bmatrix}, \quad F_3 = \frac{1}{4} \begin{bmatrix} 0 & u_{xx} - 2u^3 \\ u_{xx} - 2u^3 & 0 \end{bmatrix}.
\]
and equation (1.6) becomes the equation satisfied by self-similar solutions of the mKdV equation, namely the Painlevé II equation
\[ 4(xu)_x + u_{xxx} - 6u^2u_x = 0. \]

See §§7, 8 for a full discussion of self-similarity.

2. Although we assume generally that \( n \) is at least two, certain cases with \( n = 1 \) can be treated, e.g. when \( J \) and \( \mu \) are both real. If we assume this and assume that \( q + q^t = 0 \), then the \( 3 \times 3 \) case with \( n = 1 \) and \( k = 1 \) leads to a system of equations of the form
\[
(xu_i)_x + a_i u + b_i u_j u_k = 0, \quad \{i, j, k\} = \{1, 2, 3\},
\]
where the \( a_i \)'s and \( b_i \)'s are constants. These equations correspond to self-similar solutions of the three-wave interaction equation. Some more examples are sketched at the end of this section.

We suppose from now on that \( q \) is a smooth function on the domain \( I \subset \mathbb{R} \) and that it is off-diagonal:
\[
q(x)_{jj} = 0, \quad 1 \leq j \leq d.
\]

Let
\[
\psi(x, z) = m(x, z)e^{\Phi(x, z)}, \quad \Phi(x, z) = \frac{1}{n}z^n \mu + xzJ.
\]

The equations (1.1), (1.2) for \( \psi \) are equivalent to
\[
(1.8) \quad \frac{\partial m}{\partial x} = [zJ, m] + qm, \\
(1.9) \quad z \frac{\partial m}{\partial z} = A(x, z)m - m(z^n \mu + xzJ).
\]

By a formal solution of (1.8) we mean a formal power series in \( z^{-1} \),
\[
m(x, z) = \sum_{j=0}^{\infty} z^{-j} f_j(x), \quad f_j \in C^\infty(I; M_d(\mathbb{C})), \quad f_0 = 1
\]
which satisfies (1.8) formally; this means that for each \( N \)
\[
\left( \frac{\partial}{\partial x} - z\text{ad}J - q \right) \sum_{j=0}^{N} z^{-j} f_j(x) = O(z^{-N}).
\]
Equivalently,

\[
\frac{df_j}{dx} - qf_j = [J, f_{j+1}].
\]

These relations determine the off-diagonal part of \( f_{j+1} \) from \( f_j \) by inverting \( \text{ad} J \). At the next step they determine the diagonal part of \( f_{j+1} \), up to a constant matrix, in terms of the off-diagonal part of \( f_{j+1} \) since the diagonal part of \([ J, f_{j+2} ]\) vanishes and the diagonal part of \( qf_{j+1} \) involves only the off-diagonal part of \( f_{j+1} \). Thus these relations are solvable recursively, so formal solutions exist. Note in particular that \( q = f_1J - Jf_1 \) and that the diagonal part of \( f_1 \) may be taken to be 0. For later use we need to go one step further in this discussion. Let us fix a point \( x_0 \) and choose the unique formal solution with the property that \( f_j(x_0) \) is off-diagonal for all \( j > 0 \). Then one can prove by induction that each Taylor coefficient at \( x_0 \) of each \( f_j \) is given by a universal polynomial in the Taylor coefficients at \( x_0 \) of the entries of \( q \). Moreover, each of these polynomials has constant term 0. In particular, \( f_{j+1}(x_0) \) is a polynomial in the \( j \)-jet of \( q \) at \( x_0 \).

We define formal solutions of (1.9) in an analogous manner.

**Theorem 1.1.** Let \( m = \sum_{j=0}^{\infty} z^{-j} f_j(x) \) be a formal solution of (1.8). The formal series

\[
F = mm^{-1} = \sum_{j=0}^{\infty} z^{-j} F_j(x)
\]

is independent of the choice of \( m \). The coefficients \( F_j = F_{j,\mu} \) are traceless polynomials in \( q \) and its derivatives:

\[
F_j(x) = P_j(q(x), q'(x), \ldots, q^{j-1}(x)).
\]

Moreover, the \( F_j \) satisfy the conditions (1.5).

If \( m \) is also a formal asymptotic solution of (1.2), then the relation between the coefficients \( A_j \) and \( F_j \) is given by (1.4) and \( q \) satisfies the isomonodromy equation (1.6).

**Proof.** Suppose that \( m_1 \) and \( m_2 \) are two formal asymptotic solutions; then \( m_2 \) has a formal inverse and \( g = m_1^{-1} m_2 \) is a formal asymptotic solution of the equation \( dg/dx = [zJ, g] \). The corresponding relations are

\[
[J, g_{j+1}] = \frac{dg_j}{dx},
\]

and since \( g_0 = 1 \) it follows recursively that each \( g_j \) is diagonal and constant. Therefore \( m_2 \) is a formal power series product \( m_2 = m_1 g \) with \( g \) diagonal and so \( m_2 mm_2^{-1} = m_1 mm_1^{-1} \).
It is clear from the preceding that $m \mu m^{-1}$ is independent of the choice of formal solution $m$. The assertion about the coefficients of $m \mu m^{-1}$ may be derived by localizing and using a result of [Sa], but we give here another proof which adapts more readily for use in §8. Fix any $x_0$ in the domain of $q$, and let $m$ be the formal solution whose diagonal part at $x_0$ is 1. From the remarks above we conclude that the coefficients of $m \mu m^{-1}$ at $x_0$ are given by universal polynomials in the entries of $q$ and their derivatives at $x_0$. But this fact is independent of the choice of $m$ and of $x_0$.

If $m$ is a formal solution of (1.8), then $F = m \mu m^{-1}$ is readily seen to be a formal solution of

\[(1.13) \quad \left[ \frac{d}{dx} - zJ - q, F \right] = 0.\]

Therefore its coefficients satisfy the recursion relations corresponding to (1.13), which are (1.5).

Finally, suppose that $m$ is also a formal solution of (1.9). Then $z m x$ has no constant term, so we conclude that (as formal power series in $z^{-1}$)

\[A(x, z) = m(x, z) [z^n + xzJ] m(x, z)^{-1} + O(z^{-1})\]

\[= z^n F + xzJ + x f_1 J - xJ f_1 + O(z^{-1}) = z^n F + xzJ + xq + O(z^{-1}).\]

This is equivalent to (1.4). Now (1.3) can also be obtained by equating terms in formal power series expansions, and we deduce the isomonodromy equation (1.6). □

Let us note explicitly that

\[F_0 = \mu, \quad F_1 = (\text{ad} J)^{-1}[\mu, q], \]

\[F_{j+1} = \left( \frac{d}{dx} \right)^j (\text{ad} J)^{-j}[\mu, q] + \{ \text{terms of order } \prec j \}.\]

We assume from now on that the $F_j$ are those in Theorem 1.2.

**Corollary 1.2.** The isomonodromy equation (1.6) is an algebraic differential equation for the matrix function $q$.

**Further examples:** 3. Here we take $n = 2$, $J = \mu = \text{diag}(1, -1)$, and $k = 2$, with $q$ a general off-diagonal matrix,

\[q = \begin{bmatrix} 0 & u_1 \\ u_2 & 0 \end{bmatrix}.\]
Some computation shows that $F_1 = q$, $F_2 = \frac{1}{2}qJq + \frac{1}{2}Jq_x$, and equation (1.6) becomes

$$(xq)_x + \frac{1}{2}Jq_{xx} - q^2Jq = 0.$$ Explicitly,

$$(xu_1)_x + \frac{1}{2}u_{1xx} - u_1^2u_2 = 0 = (xu_2)_x - \frac{1}{2}u_{2xx} + u_2^2u_1.$$ If we replace $J$ by $iJ$ the formula is the same and is compatible with the reductions $u_2 = \pm \bar{u}_1$, in which case the solutions are exactly the self-similar solutions of the cubic nonlinear Schrödinger equations.

4. Finally we take $n = 3$, $J = \text{diag}(\alpha, \alpha^2, 1)$, where $\alpha = e^{2\pi i/3}$ is a primitive cube root of 1, $\mu = J^2$, and $k = 2$. We assume that $q$ has the form $q = u_1 \Pi + u_2 \Pi^2$, where $u_1$ and $u_2$ are scalar and $\Pi$ is the permutation matrix

$$\Pi = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$ Then $F_1 = [J^2, r]$ and $F_2 = Jr_x + r_xJ + q^2 - 2u_1u_21$, where

$$r = (1 - \alpha)^{-1}u_1J^2\Pi + (1 - \alpha^2)^{-1}u_2J\Pi^2.$$ Equation (1.6) becomes $(xq)_x + Jr_{xx} + r_{xx} + (q^2)_x - 2(u_1u_2)_x1$, or explicitly the coupled quadratic nonlinear system

$$(xu_1)_x + \frac{i}{\sqrt{3}}u_{1xx} + 2u_2u_{2x} = 0 = (xu_2)_x - \frac{i}{\sqrt{3}}u_{2xx} + 2u_1u_{1x}.$$

2. The forward monodromy problem at $z = \infty$

In this section we continue to assume that $q$ is a smooth, off-diagonal matrix function on a real interval or a connected complex domain $I$ and we assume it satisfies the algebraic differential equation (1.6). (Consequently $q$ is analytic in $I$.)

Using a partial sum of a formal solution of the $x$-equation as a gauge transformation (or parametrix) we show that the $z$-equation has actual solutions which are well-behaved in specified sectors and have a special form. Moreover, the classical Stokes matrices, which link the solutions from different sectors, are independent of $x$. (It is classical to use a formal solution of the $z$-equation as a gauge transformation to obtain a tractable integral equation for the transformed solution; cf. Chapter 5 of [CL]. The special feature here is that the phase function which occurs in the $z$-asymptotics of the solutions has a very
special form, and this is a consequence of the fact that the formal solutions are also formal solutions of the \( x \)-equation.)

Let the polynomial part of the formal series \( z^n F \) be denoted

\[
[z^n F]_+ = \sum_{j=0}^{n} z^{n-j} F_j.
\]

We assume that \( q \) satisfies the isomonodromy equation (1.6):

\[
\left[ \frac{d}{dx} - q, xq + F_n \right] = 0.
\]

In view of the recursion relations for the \( F_j \), an alternative form is

\[
(xq)_x + [J, F_{n+1}] = 0.
\]

It is convenient to introduce a condensed notation for the differential operators which recur throughout:

\[
D_x \equiv \frac{\partial}{\partial x} - (zJ + q); \quad D_z \equiv z \frac{\partial}{\partial z} - [z^n F]_+ - x(zJ + q)
\]

Then the conditions (1.5) are satisfied and (1.6) is the remaining requirement for the commutator condition

\[(2.1) \quad [D_x, D_z] = 0.\]

We shall show that (1.2) has certain normalized solutions with asymptotic expansions in sectors of the \( z \)-plane. To describe these sectors, we set

\[
\Sigma(j, k) = \{ z \in \mathbb{C} : \text{Re} z^n (\mu_j - \mu_k) = 0 \}, \quad 1 \leq j < k \leq d; \quad \Sigma = \bigcup_{j<k} \Sigma(j, k).
\]

Each \( \Sigma(j, k) \) is a union of \( n \) lines through the origin, with angle \( \pi/n \) between successive rays. The distinguished open sectors \( \{ \Omega_{\nu} \} \) which we consider are described as follows. Each sector is bounded by two rays from \( \Sigma \), and its interior contains exactly one ray from each \( \Sigma(j, k) \). We take the leading edge of a sector to be the boundary ray with greater argument and we number sectors in order of increasing argument of leading edge, from some starting point.
Theorem 2.1. Suppose that \( q \) satisfies the isomonodromy equation (1.6). Then for each \( x \) and each sector \( \Omega_\nu \) there is a unique solution of
\[
(2.2) \quad z \frac{\partial \psi}{\partial z} = ([z^n F]_+ + xz J + xq) \psi, \quad z \in \Omega_\nu
\]
which has the form \( \psi\nu = m_\nu e^\Phi \) as in (1.7), where the \( m_\nu \) have the (same) asymptotic expansion
\[
(2.3) \quad m_\nu \sim \sum_{j=0}^\infty z^{-j} f_j(x) \quad \text{as} \quad z \to \infty,
\]
with \( f_0 = 1 \).

The expansion (2.3) is valid uniformly as \( z \) tends to infinity in any closed subsector of \( \Omega_\nu \cup \{0\} \) and as \( x \) varies in any compact set.

There are constant matrices (Stokes matrices) \( S_\nu \) such that the relation between successive solutions is
\[
(2.4) \quad \psi_{\nu+1}(x, z) = \psi_\nu(x, z) S_\nu, \quad z \in \Omega_\nu \cap \Omega_{\nu+1}.
\]

Finally, the \( \psi_\nu \) satisfy
\[
(2.5) \quad \det \psi_\nu = 1; \quad \frac{\partial \psi_\nu}{\partial x} = (z J + q) \psi_\nu.
\]

The rest of this section is devoted to the proof of Theorem 2.1. We begin with a gauge transformation to convert (2.2) to a more tractable form for large \( z \). Let \( m = \sum z^{-j} f_j \) be a formal solution of (1.8) with \( f_0 = 1 \) and set
\[
f(x, z) = \sum_{j=0}^n z^{-j} f_j(x).
\]
Then \( f \mu f^{-1} = F + O(z^{-n-1}) \), while (1.8) implies that \( (z J + q) f = zf J + O(z^{-1}) \), so
\[
(2.6) \quad ([z^n F]_+ + xz J + xq) f = f (z^n \mu + xz J) + O(z^{-1}).
\]

If we look for a solution of (2.2) in the form \( \psi = f \hat{\psi} \), then (2.2) is equivalent to
\[
\frac{\partial \hat{\psi}}{\partial z} = (z^{n-1} \mu + xJ) \hat{\psi} + r(x, z) \hat{\psi}
\]
with \( r(x, z) = O(z^{-2}) \). Setting \( \hat{\psi} = \hat{m} e^\Phi \), we convert this to the integral equation
\[
(2.7) \quad \hat{m} = 1 + \int^z_{\infty} e^{\Phi(z) - \Phi(\zeta)} r(\zeta) \hat{m}(\zeta) e^{-\Phi(\zeta) + \Phi(\zeta)} d\zeta,
\]
where we have suppressed the dependence on \( x \). Note that (2.7) is a matrix equation and the path of integration may differ entry by entry.
Lemma 2.2. In each sector $\Omega_\nu$, the paths of integration in (2.7) may be chosen so that there is a solution for large $z$ in each closed subsector, with an asymptotic expansion valid uniformly in the smaller sector.

Proof. Let 

$$
\Sigma_{xz}(j, k) = \{ \zeta \in \mathbb{C} : \text{Re}[\Phi(x, z) - \Phi(x, \zeta)]_{jj} = \text{Re}[\Phi(x, z) - \Phi(x, \zeta)]_{kk} \}, \quad j < k.
$$

This set consists of $n$ regular curves, one of which goes through $z$. For $z$ not in $\Sigma = \bigcup \Sigma(j, k)$ the curve through $z$ is asymptotic to the nearest two rays of $\Sigma(j, k)$. It follows that for each $z \in \Omega_\nu$ we may choose the path of integration for the $(j, k)$ and $(k, j)$ entries in (2.7) to lie along this branch and to be asymptotic to the ray of $\Sigma(j, k)$ which lies in the interior of $\Omega_\nu$. The effect is that conjugation by $e^{\Phi(z) - \Phi(\zeta)}$ multiplies each entry by a bounded exponential.

If $z$ lies in a closed subsector of $\Omega_\nu$, the associated paths of integration lie in $\{ |\zeta| \geq R \}$ when $|z|$ is large. Since $r = O(z^{-2})$, it follows that eventually the integral equation (2.7) with our choice of contour has a unique solution obtained by successive approximations. By a standard argument, this solution has an asymptotic expansion which is valid uniformly in each closed subsector. □

Note that the solutions so obtained are holomorphic, and thus extend to the entire sector; indeed they extend as functions on $\mathbb{C} \setminus 0$ (multi-valued in general).

Lemma 2.3. Suppose $\tilde{\psi}_\nu$ is a second solution of (2.2) in $\Omega_\nu$, for fixed $x$, with the property that the limit of $\tilde{\psi}_\nu e^{-\Phi}$ as $z$ tends to infinity along any ray in $S_\nu$ is 1. Then $\tilde{\psi}_\nu = \psi_\nu$.

Proof. There is a constant matrix $S$ such that $\tilde{\psi}_\nu = \psi_\nu S$. From the asymptotic expansions it follows that the conjugation 

$$
e^{\Phi(x, z)} Se^{-\Phi(x, z)} = [\tilde{\psi}_\nu e^{-\Phi}]^{-1} [\psi_\nu e^{-\Phi}]
$$

converges to 1 as $z$ tends to $\infty$ along any ray in $\Omega_\nu$. Then the diagonal entries of $S$ are 1. By boundedness, 

$$
S_{jk} = 0 \quad \text{if} \quad \text{Re} z^n (\mu_j - \mu_k) > 0
$$

along such a ray. By assumption, $\Omega_\nu$ contains such rays for each $j \neq k$, so $S = 1$. □

Lemma 2.4. The functions $\psi_\nu$ satisfy (2.5).

Proof. The commutativity property (2.1) implies that the function 

$$
\tilde{\psi}_\nu = [\frac{\partial}{\partial x} - zJ - q] \psi_\nu
$$
is also a solution of (2.2), so $D_x \tilde{\psi}_\nu = \psi_\nu T(x)$ for some matrix-valued function $T$. The
construction of $\psi_\nu$ shows that the corresponding $m_\nu$ is differentiable with respect to $x$
and the asymptotic expansion of $m_\nu$ can be differentiated term by term with respect to $x$. Then

$$m_\nu^{-1} \left( \frac{\partial}{\partial x} - z \text{ad} J - q \right) m_\nu = e^\Phi T e^{-\Phi}. \tag{2.8}$$

The left side is bounded as $z \to \infty$ so, as in the proof of Lemma 2.3, it follows that the
matrix $T$ must be diagonal. Thus the right side of (2.8) is independent of $z$. On the other
hand, a check of the asymptotics of the left side of (2.8) shows that the leading term for
large $z$ is $[m_\nu, J] - q$, where $m_\nu$ is the coefficient of $z^{-1}$ in the asymptotic expansion
of $m_\nu$. This term is off-diagonal since $q$ is off-diagonal. (This is the first time we have
used the assumption that $q$ is off-diagonal.) Thus both sides of (2.8) must vanish, hence
$D_x \psi_\nu = 0$. We have proved in the process that

$$q(x) = -[J, m_\nu], \quad \text{where} \quad m_\nu = 1 + z^{-1}m_{\nu 1} + O(z^{-2}). \tag{2.9}$$

Finally, the equation (1.8) for $m_\nu$ implies that $\det m_\nu$ is constant, hence is identically 1.
We have assumed that $\Phi$ is traceless, so $\psi_\nu = m_\nu e^\Phi$ also has determinant 1. □

**Lemma 2.5.** There are constant matrices $S_\nu$ such that for all $x$ in $I$

$$\psi_{\nu + 1}(x, z) = \psi_\nu(x, z) S_\nu, \quad z \in \Omega_{\nu + 1} \cap \Omega_\nu. \tag{2.10}$$

**Proof.** This is clear from the fact that in the common domain of definition, both functions
satisfy linear first order systems of differential equations in $x$ and in $z$.

The preceding lemma makes possible the final step in the proof of Theorem 2.1.

**Corollary 2.6.** The coefficients of the asymptotic series for $m_\nu$ do not depend on the
sector.

**Proof.** By Lemma 2.3, 

$$m_\nu = m_{\nu + 1} e^\Phi S_\nu^{-1} e^{-\Phi}$$

on any ray in $\Omega_{\nu + 1} \cap \Omega_\nu$, and our arguments show that the only non-zero entries of $S_\nu$ must
remain bounded under the conjugation by $e^\Phi$ as $z$ tends to infinity on the ray. Therefore
the off-diagonal entries decay exponentially as $z \to \infty$. This implies that the asymptotic
expansions of $m_\nu$ and $m_{\nu + 1}$ are the same.
3. The forward monodromy problem at \( z = 0 \)

To complete the analysis of the \( z \)-equation one must examine the behavior at the regular singular point \( z = 0 \).

We continue to assume that \( q \) is a solution of the isomonodromy equation (1.6) on a domain \( I \). We take the sectors \( \Omega_{\nu} \) and the matrix functions \( \psi_{\nu}, m_{\nu}, \Phi \) as in \( \S 2 \), and we assume that there are \( N \) sectors numbered cyclically: \( \Omega_N = \Omega_0 \).

**Theorem 3.1.** There are invertible constant matrices \( C_{\nu} \) and a matrix function \( U(z) \), holomorphic and invertible in \( \mathbb{C} \setminus 0 \), such that

\[
m_{\nu}(x,z)^{\Phi(x,z)} C_{\nu}^{-1} U(z)^{-1} = m_{\nu}(x,z)^{\Phi(x,z)} U_{\nu}^{-1}
\]

is regular at \( z = 0 \) for every \( x \in I \) and is independent of \( \nu \).

**Lemma 3.2.** There is a fundamental solution of \( D_x \psi = 0, D_z \psi = 0 \) which has the form

\[
\psi(x,z) = f(x,z)V(z)z^B, \quad z \in \mathbb{C} \setminus 0,
\]

with \( f(x,\cdot) \) entire, \( V \) entire, \( B \) constant, and \( \det(Vz^B) \equiv 1 \).

**Proof.** Choose \( x_0 \in I \) and let \( f(x,z) \) be the unique solution of \( D_x f = 0 \) which satisfies the initial condition \( f(x_0,z) = 1 \); it is entire as a function of \( z \). Since \([D_x,D_z] = 0\), it follows that \( D_x D_z f = D_z D_x f = 0 \), so \( D_z f \) is also in the kernel of \( D_x \). Therefore \( D_z f = f(x,z)C(z) \) for some (entire) function \( C \). Note that \( \det f \) is constant with respect to \( x \), hence is identically \( 1 \). We look for \( \psi \) in the form \( \psi(x,z) = f(x,z)U(z) \). The necessary and sufficient condition for \( D_z(fU) = 0 \) is

\[
z \frac{dU}{dz} = -CU.
\]

From the equation \( D_z f = fC \) and the condition \( f(x_0,z) = 1 \) we find that \( C \) is a polynomial in \( z \) and \( -C(0) = x_0q(x_0) + F_n(x_0) \). Equation (3.2) has a regular singular point at the origin, so it has a fundamental solution \( U(z) = V(z)z^B \), where \( V \) is entire and \( B \) is constant \([\text{CL, Ch. 4, Theorem 4.2}]\). Now \( \det U \) is constant, so we may choose \( \det U \equiv 1 \). \( \square \)

**Proof of Theorem 3.1.** Each fundamental solution \( \psi_{\nu} = m_{\nu}^{\Phi} \) differs from the solution constructed in Lemma 3.2 by right multiplication by an invertible constant matrix \( C_{\nu} \). \( \square \)

Generically one has more information about the fundamental solution at \( z = 0 \) than is given above. We begin with a general remark about the constant term \( A_0(x) = xq(x) + F_n(x) \).
Lemma 3.3. The isomonodromy equation (1.6) implies that the matrices $xq + F_n$ are similar for all values of $x$.

Proof. Let $a(x)$ be a non-singular solution of $da/dx = qa$. Then

$$
\frac{d}{dx} \left[ a^{-1}(xq + F_n)a \right] = a^{-1} \left[ \frac{d}{dx} - q, xq + F_n \right] a = 0
$$

so $a^{-1}(xq + F_n)a$ is constant. □

Corollary 3.4. If $q$ and its derivatives of order less than $n$ have limit zero anywhere, then $xq + F_n$ vanishes identically.

Definition. The matrix function $q$ is proper if $D_z \psi = 0$ has a fundamental solution of the form

$$
(3.3) \quad \psi(x_0, z) = w(x_0, z) z^{A_0(x_0)}, \quad w(x_0, \cdot) \text{ entire, } w(x_0, 0) = 1.
$$

for some $x_0 \in I$.

Remarks. This is a condition on the $(n - 1)$-jet of $q$ at $x_0$. By the proof of Lemma 3.2 the condition is in fact independent of $x_0$, and applies to all $x \in I$.

In particular, $q$ is proper if for some $x_0$ no two eigenvalues of $A(x_0)$ differ by a non-zero integer (CL, Theorem 4.4.1); hence generically, $q$ is proper.

When the eigenvalues of $A_0(x_0)$ differ by an integer, there is still a solution at the origin of the form $wz^B$, with $w$ entire ([CL], Ch. 4, Theorem 4.2); but generically, $w$ is degenerate at the origin, and $B$ need no longer be similar to $A_0(x_0)$. ($B$ is similar to $A_0(x_0)$ if $w$ is invertible at the origin.)

These two situations are simply illustrated by the Painlevé II equation.

Example: Half-Integer Solutions of Painlevé II. As in §1, we take

$$
J = \mu = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad q = u\sigma = u \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};
$$

$$
F_1 = q, \quad F_2 = \frac{1}{2} \begin{bmatrix} -u^2 & u_x \\ -u_x & u^2 \end{bmatrix}, \quad F_3 = \frac{1}{4} \begin{bmatrix} 0 & u_{xx} - 2u^3 \\ u_{xx} - 2u^3 & 0 \end{bmatrix}.
$$

As noted earlier the isomonodromy equation is

(PII) \quad \frac{1}{4}u_{xx} - \frac{1}{2}u^3 + xu = \nu
where \( \nu \) is a constant; and \( A_0(x) \equiv \nu \sigma \) is constant.

Let us consider the case \( \nu = 1/2 \) and look for a solution of the form \( wz^{\sigma/2} \). We pick an initial point \( x_0 \) and let
\[
 u(x_0) = \alpha, \quad u'(x_0) = \beta.
\]
The equation \( D_z(wz^{\sigma/2}) = 0 \) is
\[
 z \frac{d}{dz}(wz^{\sigma/2}) = zwz^{\sigma/2} + wz^{\sigma/2}\sigma = (A_0 + A_1 z + \ldots)wz^{\sigma/2}.
\]
Substituting a power series expansion \( w = w_0 + w_1 z + \ldots \) into this equation, we get the recursion relations
\[
[w_0, \sigma] = 0, \quad w_1 + \frac{1}{2}[w_1, \sigma] = A_1 w_0, \quad \ldots,
\]
where
\[
A_1 = F_2 + xq = \begin{bmatrix} x_0 - \alpha^2/2 & \beta/2 \\ -\beta/2 & -x_0 + \alpha^2/2 \end{bmatrix}.
\]
From the first equation, \( w_0 \) commutes with \( \sigma \); and since the equations are invariant under right multiplication by a constant matrix, we may factor out \( w_0 \) on the right. Hence without loss of generality we may take \( w_0 = 1 \). Taking
\[
w_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]
and turning to the second recursion relations, we find \( a + d = b + c = 0 \) and the constraint
(C) \[ \beta + \alpha^2 = 2x_0. \]
Since this constraint is independent of the initial point \( x_0 \) we get the additional equation
\[
u' + u^2 = 2x.
\]
It is easily seen that any solution of this equation is also a solution of (PII). Moreover, setting \( u = \chi'/\chi \) one finds that \( \chi \) satisfies the Airy equation
\[
\chi'' - 2x\chi = 0.
\]
These special solutions were originally discovered by Gambier [Ga]; other special half-integer solutions were constructed by Airault [Ai], using Bäcklund transformations.
The constraint $C$ on the 1-jet of the solution at $x_0$ shows that the proper solutions of Painlevé II for $\nu = 1/2$ form a submanifold of codimension 1, and so are non-generic. The same considerations apply in general to solutions of the isomonodromy equations when the eigenvalues of $A_0$ differ by integers. The monodromy data for the general case is discussed in Theorem 4.1.

It is important to note that $q$ need not be proper. The condition that it be proper is a non-trivial condition on the $(n-1)$–jet of $q$, which determines the operator $D_z$; see Chapter 4 of [CL].

**Theorem 3.5.** Suppose that $q$ is proper. Then there are unique constant matrices $B_\nu$ such that the functions $\psi_\nu(x,z)z^{-B_\nu}$ are entire functions of $z$, invertible at $z = 0$. Each $B_\nu$ is similar to $A_0(x_0)$ and therefore has trace 0. Moreover

$$B_{\nu+1} = S^{-1}_\nu B_\nu S_\nu; \quad \exp(2\pi iB_\nu) = S_\nu S_{\nu+1} \cdots S_{\nu-1}. \quad (3.4)$$

**Proof.** Combining the assumption that $q$ is proper with the construction in Lemma 3.3, we obtain a fundamental solution

$$\psi(x,z) = w(x,z)z^{A_0(x_0)}$$

with $w(x,.)$ entire and $w(x_0,0) = 1$. Then there are constant matrices $C_\nu$ such that

$$\psi_\nu = \psi C_\nu = wC_\nu z^{B_\nu}, \quad B_\nu = C^{-1}_\nu A_0(x_0)C_\nu. \quad (3.5)$$

The first relation in (3.4) follows from (2.4). The second relation follows from the fact that the analytic continuations of $\psi_\nu$ and $z^{B_\nu}$ around the origin are $\psi_\nu S_\nu S_{\nu+1} \cdots S_{\nu-1}$ and $z^{B_\nu} \exp(2\pi iB_\nu)$ respectively.

If $wz^B = \tilde{w}z^\tilde{B}$ with $w$ and $\tilde{w}$ regular and invertible at the origin, then $z^{\tilde{B}} z^{-B}$ is regular at the origin and it follows that $\tilde{B} = B$. Thus the $B_\nu$ are unique.
4. The inverse problem and the Painlevé property

The inverse problem, to determine the solution of the isomonodromy equation from its monodromy data, can be formulated as a Riemann-Hilbert problem. This idea was introduced in the context of evolution equations by Shabat and utilized by Flaschka and Newell in $2 \times 2$ cases. We give the general formulation in this section. We also show that every solution of a Riemann-Hilbert problem of this type is associated to a solution of an isomonodromy equation (1.6).

It has been shown by Malgrange [Ma] and Miwa [Mi] that equations like those considered here have the Painlevé property. We sketch here an argument based on the Riemann-Hilbert problem.

Suppose again that $q$ is a solution of the isomonodromy equation.

**Theorem 4.1.** The function $q$ is uniquely determined by its Stokes matrices $S_\nu$, the function $\Phi(x,z) = \frac{1}{n} z^n \mu + xzJ$, and the functions $U_\nu$ of Theorem 3.1.

If $q$ is proper, then $q$ is uniquely determined by the Stokes matrices $S_\nu$, the function $\Phi$, and the exponents $B_\nu$ of Theorem 3.5.

**Proof.** Let the functions $\psi_\nu$, $m_\nu$ be as defined previously. Let $\Sigma_\nu$ be a ray in the intersection $\Omega_\nu \cap \Omega_{\nu+1}$, let $\Gamma = \{|z| = 1\}$ be the unit circle, and let

$$K = \Gamma \cup \left( \bigcup_\nu \Sigma_\nu \cap \{ |z| > 1 \} \right).$$

Fix $x \in I$ and define a function $M$ on $\mathbb{C} \setminus K$ by

$$M(z) = \psi_\nu(x,z) U_\nu(z)^{-1}, \quad \text{for} \quad |z| < 1 \quad (4.1)$$

$$M(z) = m_\nu(x,z) \quad \text{for} \quad |z| > 1, \quad z \text{ lying between } \Sigma_{\nu-1} \text{ and } \Sigma_\nu. \quad (4.2)$$

(Recall that by Theorem 3.1 the value given by (4.1) is independent of $\nu$.) The matrix function $M$ has the properties:

$$M \text{ is holomorphic and invertible on } \mathbb{C} \setminus K; \quad (4.3)$$

$$M \text{ has limit } 1 \text{ as } z \to \infty; \quad (4.4)$$

$$M \text{ is continuous up to the boundary from each component.} \quad (4.5)$$

Let $M_\Gamma$ and $M_\nu$ denote the boundary values of $M$ on the circle from the unit disc and on the boundary of the region outside the circle between the rays $\Sigma_{\nu-1}$ and $\Sigma_\nu$, respectively. These boundary values are linked on their common domains of definition by

$$M_\nu(z) = M_\Gamma(z) U_\nu e^{-\Phi(x,z)}; \quad M_{\nu+1}(z) = M_\nu e^{\Phi(x,z)} S_\nu e^{-\Phi(x,z)}. \quad (4.6)$$
Uniqueness in the general case will follow if we show that $M$ is uniquely determined by the Riemann–Hilbert problem (4.3)–(4.6). But this is standard: if $\tilde{M}$ is a second solution of (4.3)-(4.6), then $\tilde{M}M^{-1}$ is piecewise holomorphic and continuous, with value 1 at $\infty$, so $\tilde{M}M^{-1} \equiv 1$.

Suppose now that $q$ is proper. Then we redefine $M$ on the unit disc by

$$M(z) = \psi_\nu(x, z)z^{-B_\nu} \text{ if } |z| < 1.$$  

Then $M$ has the previous properties, with (4.6) modified to

$$M_\nu(z) = M_\Gamma(z)z^{B_\nu}e^{-\Phi(x, z)}; \quad M_{\nu+1}(z) = M_\nu e^{\Phi(x, z)}S_\nu e^{-\Phi(x, z)}.$$  

As before, this $M$ is uniquely determined by the conditions (4.3)-(4.5) and the boundary relations (4.8).  

Our principal result concerning the isomonodromy equation is the following.

**Theorem 4.2.** Any solution of the isomonodromy equation (1.6) on an interval or connected domain has a single-valued meromorphic extension to all $x \in \mathbb{C}$. In particular (1.6) has the Painlevé property: the only movable singularities of its solutions are poles.

**Proof.** The Riemann–Hilbert problem (4.3) to (4.6) has data which depend holomorphically on $x$ through the function $\Phi(x, z) = \Phi(0, z) + xzJ$.

Since the ray $\Sigma_\nu$ lies in the sector $\Omega_\nu$, the off-diagonal elements of $e^{\Phi}S_\nu e^{-\Phi}$ decay at infinity like $\exp(-\epsilon|z|^n)$, which more than offsets any growth from $\exp(xzJ)$ and its derivatives with respect to $x$.

The modified Riemann–Hilbert problem can be written as a singular integral equation with parameter $x$, which we write symbolically as

$$M = 1 + C_x M,$$

where $C_x$ also depends on the data $S, U$. The operator $Id - C_x$ depends holomorphically on $x$ in the entire plane and is invertible for all $x$ in the original domain $I$ of $q$. It is shown in [BC] that $Id - C_x$ is Fredholm with index zero; in fact it is shown that inverting $Id - C_x$ can be accomplished (if at all) in two steps: inverting a small perturbation of the identity, followed by solving a finite system of linear equations which depend on $x$. In the present case both the perturbation of the identity and the system of linear equations depend holomorphically on $x$, and it follows that the solution is meromorphic with respect...
to $x$ in the entire $x$–plane. Therefore the associated potential $q$ is also meromorphic in the entire $x$–plane. (In [BC] the sufficient conditions include compatibility conditions for the data at the intersections points of its domain of definition; these conditions are shown in [BDT] to be a consequence of a product condition on Taylor expansions of the data at the intersection points. In the present case the data is assumed to have come from the direct problem, so the conditions are automatically satisfied.) □

The next result closes the circle, insofar as proper solutions are concerned. Recall the following facts about the Stokes matrices and the matrices $B_\nu$:

\begin{align}
(4.10) & \quad e^\Phi S_\nu e^{-\Phi} \text{ is bounded as } z \to \infty, \quad z \in \Omega_\nu, \quad (S_\nu)_{jj} = 1; \\
(4.11) & \quad \text{tr}(B_\nu) = 0 \quad \exp(2\pi i B_\nu) = S_\nu S_{\nu+1} \cdots S_{\nu-1}.
\end{align}

Theorem 4.3. Suppose matrices $S_\nu$, $B_\nu$ satisfy the conditions (4.10), (4.11). If the associated Riemann-Hilbert problem (4.3)-(4.5), (4.8) has a solution $M(x_0, z)$ for some value $x_0$ of $x$, then it has a solution $M(x, z)$ except for a discrete set of values of $x \in \mathbb{C}$. For $|z| > 1$, $z \notin \bigcup \Sigma_\nu$, the function $\psi(x, z) = M(x, z)e^\Phi$ satisfies the system of equations

\begin{align}
(4.12) & \quad D_x \psi = 0, \quad D_z \psi = 0
\end{align}

where the operators $D_x = \partial/\partial x - zJ - q$ and $D_z = z\partial/\partial z - [z^n F]_+ - xzJ - xq$ are defined by a unique off-diagonal matrix function $q$. The function $q$ satisfies the isomonodromy equation and is proper.

Proof. Because of the properties of the Riemann-Hilbert problem as indicated in the previous proof, solvability at $x = x_0$ implies solvability except in a discrete subset of $\mathbb{C}$. The determinant satisfies a Riemann-Hilbert problem which forces $\det M \equiv 1$; in fact since $\text{tr}(\Phi) = \text{tr}(B_\nu) = 0$, it follows that $\det M$ is continuous across $K$ and hence entire in $z$, and (4.4) implies that $\det M$ is 1 at infinity. Thus $M$ is invertible. The function

\[
\left[ \left( \frac{\partial}{\partial x} - z \text{ad } J \right) M \right] M^{-1}
\]

is piecewise holomorphic, continuous, and bounded, hence has value $q(x)$ independent of $z$. Therefore $\psi = Me^\Phi$ satisfies the first of the equations (4.12), for this choice of $q$. Moreover $M(x, \cdot)$ has an asymptotic expansion $\sum z^{-j} f_j(x)$ as $z \to \infty$ and $q = -[J, f_1]$ is off-diagonal. On the other hand, the function

\[
\left[ \left( z \frac{\partial}{\partial z} - z^n \text{ad } \mu - xz \text{ad } J \right) M \right] M^{-1}
\]
is piecewise holomorphic, continuous, and $O(z^{n-1})$ as $z \to \infty$, hence is a polynomial of degree less than $n$ in $z$. Therefore $\psi$ satisfies an equation

$$z \frac{\partial}{\partial z} \psi = A(x, z) \psi = \sum_{j=0}^{n} z^j A_j(x), \quad A_n = \mu.$$  

As in the proof of Theorem 1.1, we conclude from the asymptotic expansion of $M$ at infinity that

$$A(x, z) = [z^n F]_+ + xzJ + xq.$$  

Thus $\psi$ satisfies the equations $D_x \psi = 0$ and $D_z \psi = 0$, so $q$ satisfies the isomonodromy equation.

Finally, we must show that $q$ is proper. Let $\psi_{\nu}$ be the restriction of $\psi$ to $\Omega_{\nu} \cap \{ |z| > 1 \}$ and extend $\psi_{\nu}$ to the unit disc by setting $\psi_{\nu} = Mz^{B_{\nu}}$ for $|z| < 1$. Then $\psi_{\nu}$ is continuous across the circular arc, because of (4.8), and is therefore holomorphic. Since $M$ is regular at the origin, $\psi_{\nu}$ has the desired form and $q$ is proper. □

5. Rational solutions

In this section we investigate rational solutions of the isomonodromy equations. These are the analogues of the reflectionless potentials in inverse scattering theory; indeed there is a close connection between the Stokes matrices $S_{\nu}$ being trivial ($\equiv 111$), the normalized wave functions $m_{\nu}$ being rational in $z$, and $q$ being rational in $x$.

**Theorem 5.1.** Suppose that $q$ satisfies the isomonodromy equation (1.6). The associated system (1.8), (1.9) has a solution $m$ which is rational in $z$ and equals $111$ at $z = \infty$ if and only if each Stokes matrix $S_{\nu}$ equals $1$.

If also $q$ is proper, then $q$ and $m$ are rational functions of $x$.

**Proof.** If (1.8), (1.9) has a solution $m$ which is rational in $z$ and equals $1$ at $\infty$, then by uniqueness $m_{\nu} = m$, all $\nu$, and so each $S_{\nu} = 1$. Conversely, suppose each $S_{\nu} = 1$. Then the $m_{\nu}$ coincide and so $m = m_{\nu}$ is regular on the Riemann sphere minus the origin. It follows that in the representation (3.1), $m = f(x, z)V(z)z^B$ with $fV$ entire, the factor $z^B$ is single-valued. Therefore $\exp(2\pi i B) = 1$, and therefore $B$ is diagonalizable, with integer eigenvalues. This shows that the origin is a pole for $m$, hence $m$ is rational.

Now suppose that $q$ is proper, i.e. that $B$ has trace zero. Suppose that $m$ has a pole of order $r$ at $z = 0$. Then the smallest eigenvalue of $B$ is $-r$. Denote the dimension of the eigenspace corresponding to the eigenvalue $s$ by $d_s$. Let $(v_1, v_2, \ldots, v_{d_s})$ be a basis of
eigenvectors of $B$ with eigenvalues $s = \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d = -r$. For $1 - r \leq k \leq s = \alpha_1$ define a $d \times (d_k + \ldots + d_s)$ matrix $V_k$ by taking the columns to be those $v_j$ with $\alpha_j \geq k$. Let

$$e^{\Phi(x,z)} = \sum_{j=0}^{\infty} z^j C_j(x),$$

where the matrix-valued function $C_j$ is a polynomial of degree $j$ in $x$. Then the condition

(5.1) \[ m e^{\Phi} z^{-B} = [1 + z^{-1} f_1 + \cdots + z^{-r} f_r] e^{\Phi} z^{-B} \text{ is regular at } z = 0 \]

is equivalent to the sequence of equations

(5.2) \[ f_r V_{1-r} = 0, \]

\[ [f_r C_1 + f_{r-1}] V_{2-r} = 0, \]

\[ \ldots \]

\[ [f_r C_{r-1} + \cdots + f_1] V_0 = 0, \]

\[ [f_r C_r + \cdots + f_1 C_1] V_1 = -V_1, \]

\[ \ldots \]

\[ [f_r C_{s+r-1} + \cdots + f_1 C_s] V_s = -V_s. \]

The first equation in (5.2) represents $d(d_{1-r} + d_{2-r} + \ldots)$ linear equations for the $d^2$ entries of $f_r$. Altogether the number of linear equations represented by (5.2) is

$$d \left[ (d_{1-r} + d_{2-r} + \ldots) + (d_{2-r} + d_{3-r} + \ldots) + (d_{3-r} + \ldots) + \ldots \right]$$

\[ = d \left[ (1 - r)d_{1-r} + (2 - r)d_{2-r} + \ldots \right] + d r \left[ d_{1-r} + d_{2-1} + \ldots \right] \]

\[ = d [\text{tr}(B) + rd_{-r}] + d r [d - d_{-r}] = rd^2. \]

Thus (5.2) is a set of $rd^2$ equations for the $rd^2$ entries of $f_j$. As noted above, the coefficients in these equations are polynomials in $x$. To prove $m$ is rational in $x$ we only need to show that the system has a unique solution. By assumption it has one solution. As noted, (5.2) is equivalent to (5.1). Note also that

$$\lim_{z \to \infty} \det(f e^{\Phi} z^{-B}) = 1,$$

so (5.1) implies that $f e^{\Phi} z^{-B}$ is invertible for all $z$. Consequently, if $\tilde{f}$ is a second solution of (5.1) then $\tilde{f} f^{-1}$ is rational, entire, and $1$ at $\infty$, hence identically $1$. Thus the solution is unique and $m$ is rational in $x$. Now the potential $q = -[J, f_1]$, so $q$ is also rational.

Conversely, the equations (5.2) allow construction of rational solutions of the isomonodromy equation.
Theorem 5.2. Suppose that $B$ is a constant matrix with trace zero and minimum eigenvalue $-r$, and that $\exp(2\pi iB) = 1$. Suppose that the equations (5.2) have a solution. Then $q = -[J, f_1]$ is rational and is a proper solution of the isomonodromy equation having normalized wave function $m = 1 + z^{-1} f_1 + \cdots + z^{-r} f_r$. Both $m$ and $q$ are rational functions of $x$.

Proof. In view of the previous proof, all we need to show is that $m = 1 + z^{-1} f_1 + \cdots + z^{-r} f_r$ satisfies equations of the form (1.8), (1.9). Let $\psi = me^\Phi = wz^B$. By assumption, $w$ is entire and invertible everywhere. Consider the function

$$\left[\left(\frac{d}{dx} - zJ\right)\psi\right] \psi^{-1} = \frac{\partial w}{\partial x} w^{-1} - zJ.$$ 

which is entire. The behavior at $z = \infty$ is obtained by rewriting the function as

$$\left[\left(\frac{d}{dx} - z \text{ad}J\right)m\right] m^{-1} = -[J, f_1] + O(z^{-1}).$$

Thus by Liouville's theorem the function is independent of $z$ and is identically $q(x) = -[J, f_1]$, and $m$ satisfies (1.8) with this choice of $q$. Similarly, consider the function

$$A(x, z) = \left[z \frac{\partial \psi}{\partial z}\right] \psi^{-1} = \left[z \frac{\partial}{\partial z} \left(wz^B\right)\right] z^{-B} w^{-1} = \left[z \frac{\partial w}{\partial z}\right] w^{-1} + B.$$ 

It is regular at $z = 0$ and its behavior at $z = \infty$ is seen by rewriting it in terms of $m$:

$$\left[z \frac{\partial}{\partial z} \left(me^\Phi\right)\right] e^{-\Phi} m^{-1} = \left[z \frac{\partial m}{\partial z}\right] m^{-1} + m(z^n \mu + xzJ) m^{-1} = z^n \mu + O(z^{n-1}).$$

Thus $A$ is a polynomial in $z$ with leading term $z^n \mu$, and $m$ satisfies (1.9). This completes the proof.

We turn now to a closer inspection of the system (5.2), particularly the behavior for large $x$. Note that the coefficients $C_j$ of the expansion of $e^\Phi$ have the form

$$C_j(x) = \frac{1}{j!} x^j J^j + O(x^{j-n}).$$

Rescale the problem for large $x$ by setting $x^j f_j = \tilde{f}_j$. Then in view of (5.3) the equations
(5.2) have the form

\[
\tilde{f}_r V_{1-r} = O(x^{-n}),
\]

\[
[f_r J + \tilde{f}_{r-1}] V_{2-r} = 0(x^{-n}),
\]

\[
\ldots
\]

\[
\frac{1}{(r-1)!} f_r J^{r-1} + \cdots + \tilde{f}_1 V_0 = 0(x^{-n}),
\]

\[
\frac{1}{r!} f_r J^r + \cdots + \tilde{f}_1 V_1 = -V_1 + 0(x^{-n}),
\]

\[
\ldots
\]

\[
\frac{1}{(s+r-1)!} f_r J^{s+r-1} + \cdots + \frac{1}{s!} \tilde{f}_1 J^s] V_s = -V_s + 0(x^{-n}).
\]

The key to the behavior for large \( x \) is thus the \( rd \times rd \) matrix

\[
\begin{pmatrix}
V_{1-r} & JV_{2-r} & \ldots & \frac{1}{(r-1)!} J^{r-1} V_0 & \ldots & \frac{1}{(s+r-1)!} J^{s+r-1} V_s \\
0 & V_{2-r} & \ldots & \frac{1}{(r-2)!} J^{r-2} V_0 & \ldots & \frac{1}{(s+r-2)!} J^{s+r-2} V_s \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \ldots & \ldots & V_0 & \ldots & \frac{1}{s!} J^s V_s
\end{pmatrix}
\]

(5.6)

Since the \( V_j \) are constructed from eigenvectors of \( B \), invertibility of (5.6) is a condition on the matrices \( J, B \) alone.

**Theorem 5.3.** Suppose that \( e^{2\pi i B} = 1 \), the least eigenvalue of \( B \) is \(-r\), and the matrices \( J \) and \( B \) are such that the matrix (5.6) is invertible. Then the equations (5.2) are solvable for all large \( x \) and the associated potential \( q \) is rational; moreover \( q(x) = x^{-1} q_0 + O(x^{-1-n}) \), where the constant matrix \( q_0 \) is similar to \( B \).

**Proof.** The preceding discussion establishes that invertibility of (5.6) implies solvability of (5.2) for large \( x \). It is clear from (5.6) that the \( f_j \) are of the form \( x^{-j} \tilde{f}_j = x^{-j} f_{j0} + O(x^{-j-n}) \) with \( f_{j0} \) constant. Therefore \( q = -[J, f_1] \) has the form asserted above. Finally, it follows by induction that \( F_j \) of Theorem 1.1 is \( O(x^{-j}) \). Now \( B \) is similar to \( F_n + xq = q_0 + O(x^{-n}) \) for every \( x \), so \( B \sim q_0 \).

**Example.** As a simple example to illustrate the construction of rational solutions we consider the AKNS hierarchy; cf. also [FN]. Here

\[
J = \mu = \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} 0 & u \\ v & 0 \end{bmatrix}.
\]
From the recursion relations (1.5) we find

\[ F_0 = \sigma_3, \quad F_1 = q, \quad F_2 = \frac{1}{2} \begin{bmatrix} -uv & u_x \\ -v_x & uv \end{bmatrix} \]

and the isomonodromy deformation equations (1.6) at order 2 are

\[ (xu)_x + \frac{1}{2} u_{xx} - u^2 v = 0, \quad (xv)_x + \frac{1}{2} v_{xx} + uv^2 = 0. \]

Under the symmetry reduction \( u = \pm v \) these equations reduce to the single equation

\[ (xu)_x + \frac{1}{2} u_{xx} \pm u^3 = 0. \]

For the case \( r = 1 \) equations (5.3) are simply

\[ f_1 v_1 = 0, \quad (1 + f_1 C_1) v_1 = 0, \]

where \( C_1 = x \sigma_3 \), and \( v_1 \) is the eigenvector of \( B \) with eigenvalue 1. Since we do not know \( B \) a priori, let us take \( v = [1, \ h]^t \) where \( h \) is to be determined. We find

\[ f_1 = \frac{1}{2x} \begin{bmatrix} -1 & h^{-1} \\ -h & 1 \end{bmatrix}, \quad m = 1 + z^{-1} f_1, \quad q = -[\sigma_3, f_1] = -\frac{1}{x} \begin{bmatrix} 0 & h^{-1} \\ h & 0 \end{bmatrix}. \]

It is easily verified that \( \det (1 + f_1 z^{-1}) = 1 \) for all \( h \). Moreover this result is independent of \( n \), the order of the equation in the hierarchy. Thus for all \( h \) we get a solution with

(5.8)

\[ u = -\frac{1}{hx}, \quad v = -\frac{h}{x}, \quad h \neq 0. \]

The construction of rational solutions for \( r = 2 \) is somewhat more involved. Under the symmetry reduction \( u = v \), we have \( \sigma m(x, z) \sigma^{-1} = m(x, -z) \), where

\[ \sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \]

hence

\[ f_j = \begin{bmatrix} a_j & (-1)^j b_j \\ b_j & (-1)^j a_j \end{bmatrix}. \]

For \( n = 2, \)

\[ C_1 = x \sigma_3, \quad C_2 = \frac{x^2}{2} 1 + \sigma_3, \quad C_3 = \frac{x^3}{3!} \sigma_3 + x 1. \]
In the case $r = 2$, and $h = 1$ one finds

$$q = -[\sigma_3, f_1] = \left( \frac{3(x^2 + 1)(x - 1)}{x(x^4 + 3)} - \frac{1}{x} \right) \sigma.$$ 

It is also interesting to consider the symmetry reduction $v = \epsilon \bar{u}$ where $\epsilon = \pm 1$. Replacing $t$ by $it$ in the time evolution equation $q_t = [\frac{\partial}{\partial x} - q, F_2]$, one obtains the nonlinear Schrödinger equation

$$i u_t = \frac{1}{2} u_{xx} - \epsilon |u|^2 u.$$ 

There are solitons only in the case $\epsilon = -1$, called the self-focussing case. This equation is invariant under the scaling $u(x, t) \rightarrow \lambda u(\lambda x, \lambda^2 t)$ and the associated similarity solutions take the form

$$u(x, t) = \frac{1}{\sqrt{t}} \phi(\xi), \quad \xi = \frac{x}{\sqrt{t}}$$

where $\phi$ satisfies the ordinary differential equation

$$i(\xi \phi)' + 2\epsilon |\phi|^2 \phi - \phi'' = 0.$$ 

We can construct “rational solutions” of this equation for real $x$, although they cannot be continued into the complex $x$ plane. For example, taking $h = i$ in (5.8) we obtain

$$u = \frac{i}{x}, \quad v = -\frac{i}{x} = \bar{u}$$

for real $x$. Hence we obtain rational solutions of (5.10) for real $x$ in the defocussing case.

In the self-focussing case, $xq + F_2$ is skew-Hermitian, so always has imaginary eigenvalues. Therefore the solutions $wz^{A_0}$ have non-trivial monodromy in a neighborhood of the origin, and solutions of (1.8), (1.9) for $m$ rational in $z$ do not exist. Thus the theory of the present section does not apply.

6. Bäcklund transformations

In this section we adapt a procedure which goes back at least to Moutard [Mt] for transforming a pair consisting of a linear differential operator and its eigenfunctions to a new such pair. Here the idea is to make a gauge transformation of normalized solutions $\psi$ associated to a given problem. We look for a gauge transformation of the form

$$\tilde{\psi}(x, z) = G(x, z)\psi(x, z) = [1 - z^{-1}a(x)]\psi(x, z).$$
We assume that $\psi$ is invertible and satisfies (1.1), (1.2), where $q$ satisfies the isomonodromy equation. Then $\tilde{\psi}$ satisfies

\begin{equation}
\frac{\partial \tilde{\psi}}{\partial x} = [G(Jz + q)G^{-1} - G_xG^{-1}] \tilde{\psi},
\end{equation}

\begin{equation}
z \frac{\partial \tilde{\psi}}{\partial z} = [GA(z,x)G^{-1} - zG_zG^{-1}] \tilde{\psi} = \tilde{A}(x,z) \tilde{\psi}.
\end{equation}

In order for $6.2$ to take the form $\partial \tilde{\psi}/\partial x = (Jz + \tilde{q}) \tilde{\psi}$ it is necessary and sufficient that

$\tilde{q} = q + [J,a]$, $\frac{da}{dx} = [q + Ja, a].$

This matrix Riccati equation for $a$ can be linearized by setting $a = bcb^{-1}$ with $c$ constant; then the equation for $a$ is satisfied provided $db/dx = qb + Jbc$. In particular, there is a solution for each choice of $c$ and each choice of (invertible) initial value $b(x_0)$. Note that $c^2 = 0$ implies $a^2 \equiv 0$.

The next problem is to ensure that $\tilde{A}(x, \cdot)$ in (6.3) is a polynomial with leading term $z^n\mu$.

**Theorem 6.1.** Suppose $a$ satisfies (6.4), and $a^2 \equiv 0$. Suppose that $\tilde{A}(x_0, \cdot)$ is a polynomial in $z$. Then $\tilde{A}(x, \cdot)$ is a polynomial in $z$ for each $x$ for which $A$ is defined, and $\tilde{q}$ satisfies the isomonodromy equation.

**Proof.** The assumption $a^2 = 0$ implies that $G^{-1} = 1 + z^{-1}a$, so

$\tilde{A}(x,z) = \left[1 - z^{-1}a(x)\right]A(x,z)\left[1 + z^{-1}a(x)\right] - z^{-1}a(x).$

The zero-curvature condition (1.3) is preserved by gauge transformations, so

$\left[ \frac{\partial}{\partial x} - zJ - \tilde{q}, \tilde{A} \right] = zJ.$

Equation (6.5) shows that $\tilde{A}$ has highest term $z^n\mu$ and (6.6) shows that if $A(x, \cdot)$ is regular at $z = 0$ for one value of $x$, then this is true for all values of $x$. To show that (6.6) amounts to the isomonodromy equation for $\tilde{q}$, we must relate $\tilde{A}$ to the coefficients of the expansion of $\tilde{m}_\mu \tilde{m}^{-1}$, where $\tilde{m} = \psi \Phi$. The required relation (1.4) follows from the second part of Theorem 1.1: let $\psi_\nu = m_\nu \Phi$ be the normalized wave function for the sector $\Omega_\nu$. Then $\tilde{m}_\nu = Gm_\nu$ has an asymptotic expansion as $z \to \infty$ in $\Omega_\nu$ with leading term $1$, as required. Thus $\tilde{q}$ satisfies the isomonodromy equation.

We now consider the case of proper $q$, i.e. $q$ for which the fundamental solutions have the form (3.3) at $z = 0$. 
Theorem 6.2. Suppose that \( q \) is proper and that \( xq + F_n \) is diagonalisable. Then either \( q \) satisfies an algebraic differential equation of degree \( n - 1 \), or there is a nontrivial Bäcklund transformation \( q \to \tilde{q} \) of the form (6.4) such that the eigenfunctions transform by (6.1) and such that \( \tilde{q} \) satisfies the isomonodromy equation. Moreover, \( \tilde{q} \) is also proper and \( x\tilde{q} + \tilde{F}_n \) is diagonalizable.

Proof. Fix \( x_0 \) and write matrices with respect to a basis for which \( A_0(x_0) \) is diagonal. Choose a solution of the form (3.3), so \( w(x_0,0) = 1 \). Then \( \det w \equiv 1 \). Write \( w(x_0,z) = \sum z^k w_k \). Suppose that for some \( j \neq k \), \( (w_1)_{kj} \neq 0 \), and take

\[
(6.7) \quad a_{jk}w_{1,jk} = 1; \quad \text{all other entries of } a_{rs}(x_0) = 0.
\]

Then a simple calculation shows that the \( jj \)-entry of \( G(x_0,z)w(x_0,z) \) vanishes at \( z = 0 \), and all entries except the \( jk \)-entry are regular at \( z = 0 \). Therefore

\[
G(x_0,z)w(x_0,z) = \tilde{w}(x_0,z)z^C,
\]

with \( \tilde{w} \) entire, \( \det \tilde{w} \equiv 1 \), \( C \) diagonal, \( C = e_{jj} - e_{kk} \). It follows that

\[
\tilde{A}(x_0,\cdot) = \left[ z \frac{\partial}{\partial z} \tilde{w} + \tilde{w}B \right] \tilde{w}^{-1}, \quad B = C + A_0(x_0).
\]

Thus \( \tilde{A}(x_0,\cdot) \) is regular at \( z = 0 \). We can prolong \( a \) by (6.4) and deduce the result from the preceding theorem. Clearly \( \tilde{q} \) is regular at \( x_0 \) and hence for all \( x \).

The obstruction to this construction at \( x = x_0 \) is the vanishing of the off-diagonal entries of \( w_1(x_0) \), so \( [A_0, w_1(x_0)] = 0 \). The equation for \( w \) is

\[
z \frac{dw}{dz} = A(z, w)w - wB.
\]

Substitution of the power series for \( w \) into this equation yields \( w_1 = [A_0, w_1] + A_1 \); hence one would have \( [A_0, A_1] = 0 \) at \( x_0 \). Thus the obstruction is the vanishing of this commutator at every point, which is the algebraic ODE for \( q \) of order \( n - 1 \):

\[
[F_n + xq, F_{n-1} + xJ] = 0.
\]

A consequence of the argument just given is that the matrix \( a \) for the transformation in Theorem 6.2 can be constructed algebraically, bypassing the differential equation in (6.4), by looking at \( w(x, \cdot) \) in a basis which diagonalizes \( A_0(x) \). Note that \( x\tilde{q} + \tilde{F}_n \) is similar to
$B$ and hence diagonalizable. Relative to $xq + F_n$, one eigenvalue has been increased by 1 and a second eigenvalue has been decreased by 1.

**Example.** We illustrate the method to construct integer solutions of the Painlevé II equation by Bäcklund transformations (cf. also Airault [Ai]); we use the notation in §3.

The Bäcklund transformation from the zero solution to the solution for $r = 1$ coincides with the reduced wave function $m$. In fact, (1.8) is equivalent to the intertwining relation

$$m\left(\frac{\partial}{\partial x} - zJ\right) = \left(\frac{\partial}{\partial x} - zJ - q\right)m.$$

We work in a basis in which $q$ is diagonal. Note that

$$P^{-1}\sigma P = \sigma_3, \quad P^{-1}\sigma_3P = -\sigma, \quad P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Thus in this basis $J = -\sigma$.

We look for a gauge transformation of the form (6.1) with $a(x) = v(x)T$. In order that $a$ satisfy (6.4) we must have

$$-[\sigma T, T] = \lambda T, \quad T^2 = 0$$

for some scalar $\lambda$. These equations are satisfied by the choices $T = N_{\pm}$:

$$N_+ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad N_- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The Riccati equation (6.4) is

$$v_x N_{\pm} = v^2 [ -\sigma N_{\pm}, N_{\pm} ] = v^2 N_{\pm},$$

hence

$$v_x = v^2.$$

We use (6.7) to get the initial condition at an arbitrary point $x_0$. Since we are gauging from the trivial solution, $w = \exp\{ -(xz + z^3/3)\sigma \}$. Therefore, $w_1(x_0) = -x_0\sigma$, while $a(x_0) = v(x_0)N_{\pm}$. In either case, (6.7) implies

$$v(x_0) = -\frac{1}{x_0}.$$
Since \(x_0\) was arbitrary (other than \(x_0 = 0\)) we see that \(v(x) \equiv -1/x\), and this function satisfies (6.8). Thus, as remarked above, \(v\) is fully determined by the algebraic condition (6.7).

The corresponding potentials and wave functions are

\[ q_\pm = [J, vN_\pm] = \pm \frac{1}{x} \sigma_3, \quad m_\pm = I + \frac{1}{xz} N_\pm. \]

Let us take the case \(q = q_+\) and construct the gauge transformation from \(r = 1\) to \(r = 2\). We begin by determining \(w_1\). The wave function for \(q_+\) is

\[ m_+ e^\Phi = \begin{bmatrix} 1 \quad 0 \\ (xz)^{-1} \quad 1 \end{bmatrix} e^{-(xz+z^3/3)\sigma}. \]

Expanding the exponential to third order terms in \(z\) we find

\[ m_+ e^\Phi = \left\{ \begin{bmatrix} 0 & -x \\ x^{-1} & 0 \end{bmatrix} + z \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix} + \cdots \right\} \begin{bmatrix} z^{-1} & 0 \\ 0 & z \end{bmatrix} = \begin{bmatrix} 1 + z \begin{bmatrix} 0 & x \\ -w & 0 \end{bmatrix} + \cdots \right\} \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 0 & -x \\ x^{-1} & 0 \end{bmatrix}. \]

where

\[ w = \frac{x^3 - 1}{3x}. \]

Therefore

\[ w_1 = \begin{bmatrix} 0 & x \\ -w & 0 \end{bmatrix} \]

in (6.7). Then with \(k = 2, j = 1\) we take

\[ v(x) = -\frac{1}{w} = \frac{3x^2}{1-x^3}. \]

This choice indeed satisfies the Riccati equation (6.4), which in this case is

\[ v_x = \frac{2v}{x} + v^2. \]

The potential and wave function for \(r = 2\) is

\[ q = -\left( \frac{6x^2}{1-x^3} + \frac{2}{x} \right), \quad m = \begin{bmatrix} 1 & -\frac{3x^2}{(1-x^3)z} \\ 0 & \frac{1}{(xz)^{-1} 1} \end{bmatrix}. \]

The gauge transformation \(G\) is the first factor of \(m\).
7. Scaling, self-similarity, and construction of isomonodromy deformations

The isomonodromy deformations (1.1) and (1.2) can be derived from the Lax pairs for integrable systems by a scaling invariance. This was done for the mKdV equation by Flaschka and Newell [FN], working with the $2 \times 2$ AKNS system. For the general $d \times d$ system (1.1) and a given constant matrix $\mu$, the $n$-th equation in the associated hierarchy of equations is

\begin{equation}
\frac{\partial q}{\partial t} = [J, F_{q,n+1}] = \left[ \frac{\partial}{\partial x} - q, F_{q,n} \right].
\end{equation}

Here $F_{q,k}$ is the coefficient of $z^{-k}$ in the formal series of Theorem 1.1. This equation is associated to the operator

\[ D_{t,q} = \frac{\partial}{\partial t} - [z^n F_q]_+, \]

where $[z^n F_q]_+$ denotes the polynomial part of $z^n F_q$. In fact (7.1) is equivalent to the zero-curvature condition

\begin{equation}
[D_{x,q}, D_{t,q}] = 0,
\end{equation}

which is the compatibility condition for the overdetermined system

\begin{equation}
D_{x,q} \psi = 0, \quad D_{t,q} \psi = 0,
\end{equation}

where

\[ D_{x,q} = \frac{\partial}{\partial x} - zJ - q. \]

A solution $q = q(x,t)$ of (7.1) is said to be self-similar if

\begin{equation}
q(x,t) = q_\lambda(x,t) \equiv \lambda^{-1} q(\lambda^{-1} x, \lambda^{-n} t), \quad \lambda > 0.
\end{equation}

Note that such a function is uniquely determined by its values at fixed $t$, say at $t = 1/n$.

Define dilations $T_\lambda$, $\lambda > 0$, acting on functions of $(x, t, z)$ by

\begin{equation}
T_\lambda f(x, t, z) = f(\lambda x, \lambda^n t, \lambda^{-1} z).
\end{equation}

Let $q$ be self-similar; then

\begin{equation}
D_{x,q} T_\lambda = \lambda T_\lambda D_{x,q}, \quad D_{t,q} = \lambda^n T_\lambda D_{t,q}.
\end{equation}
The first of these operator identities is immediate from (7.4). For the second, let \( m_0(x,z) \) be a formal solution of \( D_{x,q}m = -zmJ \) at \( t = 1/n \) and extend \( m_0 \) as a function of \( t \) by \( m(x,t,z) = m_0((nt)^{-1/n}x,(nt)^{1/n}z) \). Then \( m \) is invariant under the dilations, so the first identity in (7.6) shows that \( m \) is a formal solution for each fixed \( t \). It follows that \( F_q = m_0m^{-1} \) is dilation invariant, which implies the second identity in (7.6). A consequence is that the evolution equations (7.1) are invariant under dilation.

The basic observation is that self-similar solutions of the evolution equation (7.1) correspond to solutions of (1.6) and conversely.

**Theorem 7.1.** If \( q(x,t) \) is a self-similar solution of (7.1), then \( q(x,1/n) \) is a solution of the isomonodromy equation

\[
\left[ \frac{d}{dx} - q, xq + F_n \right] = 0. \tag{7.7}
\]

Conversely, if \( q_0(x) \) is a solution of (7.7), then \( q(x,t) = (nt)^{-1/n}q_0((nt)^{-1/n}x) \) is a self-similar solution of (7.1) with \( q(x,1/n) = q_0(x) \).

**Proof.** Suppose \( q \) is a self-similar solution of (7.1). Differentiating (7.4) with respect to \( \lambda \) at \( \lambda = 1 \), we obtain the Euler equation

\[
x \frac{\partial q}{\partial x} + nt \frac{\partial q}{\partial t} + q = 0. \tag{7.8}
\]

At \( t = 1/n \) this identity becomes \( q_t = -(xq)_x \). Substituting this last identity in (7.1), we obtain (7.7).

Conversely suppose that \( q_0(x) \) satisfies (7.7). Let \( q \) be the self-similar extension. By the argument leading to the second identity in (7.6), \( F_q \) is the invariant extension of \( F \). Consequently we only need to check (7.1) or (7.2) when \( t = 1/n \), where it is implied by (7.7), (7.8).

It is worth noting that the scaling argument carries over to the wave functions. Scaling invariance \( T_\lambda \psi = \psi \) is equivalent to the condition that \( \psi(x,1/n,z) \) satisfy (1.2). In fact scaling invariance implies the Euler equation

\[
x \frac{\partial \psi}{\partial x} + nt \frac{\partial \psi}{\partial t} - z \frac{\partial \psi}{\partial z} = 0, \tag{7.10}
\]

and, at \( t = 1/n \), (7.10) and (7.3) imply (1.1). Conversely, (1.1) and (7.10) for a function \( \psi_0(x,z) \) imply that the self-similar extension \( \psi(x,t,z) \equiv \psi_0((nt)^{-1/n}x,(nt)^{1/n}z) \) satisfies (7.5).
Similar considerations apply to the Gel’fand-Dikii flows

\[(7.11) \quad \dot{L} = [[L^{k/n}_+, L], L], \quad L = L_n = D^n + \sum_{j=0}^{n-2} u_j(x,t)D^j, \quad D = \frac{d}{dx},\]

where \( k \in \mathbb{N} \) is not divisible by \( n \) and \( [L^{k/n}_+] \) denotes the differential part of the pseudo-differential fractional power. The coefficients of \( (L^{k/n})_+ \) are polynomials in the \( u_j \)'s and their derivatives, with no constant terms. The simplest examples are

\[
L_2 = D^2 + u, \quad [L^{3/2}_2]_+ = D^3 + \frac{3}{2} u D + \frac{3}{4} u_x; \\
L_3 = D^3 + u_1 D + u_0, \quad [L^{2/3}_3]_+ = D^2 + \frac{2}{3} u_1.
\]

These lead to the KdV and Boussinesq equations respectively:

\[
\begin{align*}
u_t &= [[L^{3/2}_2]_+, L_2] = \frac{1}{4} u_{xxx} + \frac{3}{2} u u_x; \\
u_{1t} D + u_{0t} &= [[L^{2/3}_3]_+, L_3] = (2u_{0x} - u_{1xx})D + (u_{0xx} - \frac{2}{3} u_{1xxx} - \frac{2}{3} u_1 u_{1x}).
\end{align*}
\]

The next equation in the KdV hierarchy,

\[
u_t = \frac{1}{16} D^5 u + \frac{5}{8} u u_{xxx} + \frac{5}{4} u_x u_{xx} + \frac{15}{8} u^2 u_x,
\]

comes from

\[
[L^{5/2}_2]_+ = D^5 + \frac{5}{2} u D^3 + \frac{15}{4} u_x D^2 + \frac{5}{8} (5u_{xx} + 3u^2) D + \frac{15}{16} (u_{xxx} + 2uu_x).
\]

A solution \( L \) of (7.11) is said to be self-similar if

\[(7.12) \quad u_j(x,t) = \lambda^{n-j} u_j(\lambda x, \lambda^k t), \quad \lambda > 0, \quad j = 0, 1, \ldots, n-2.\]

**Examples.** The simplest cases are the self-similar KdV and Boussinesq solutions which satisfy, respectively:

\[
(xu)_x + u + \frac{1}{4} u_{xxx} + \frac{3}{2} u u_x = 0;
\]

\[
(xu_1)_x = u_{1xx} - 2u_{0x}, \quad (xu_0)_x + u_0 = \frac{2}{3} u_{1xx} + \frac{2}{3} u_1 u_{1x} - u_{0xx}.
\]

The next equation in the KdV hierarchy leads to

\[
(xu)_x + u + \frac{1}{16} D^5 u + \frac{5}{8} u u_{xxx} + \frac{5}{4} u_x u_{xx} + \frac{15}{8} u^2 u_x = 0.
\]
Note that (7.11) is the compatibility condition for the system of two scalar equations

\begin{equation}
Lv(x, t, z) = z^n v(x, t, z), \quad \frac{\partial v}{\partial t} = [L^{k/n}]_+ v.
\end{equation}

The self-similarity condition (7.12) is equivalent to a pair of operator identities analogous to (7.6):

\begin{equation}
[L - z^n I] T_\lambda = \lambda^n T_\lambda [L - z^n I], \quad (L^{k/n})_+ T_\lambda = \lambda^k T_\lambda (L^{k/n})_+,
\end{equation}

where

\begin{equation}
T_\lambda v(x, t, z) = v(\lambda x, \lambda^k t, \lambda^{-1} z).
\end{equation}

The self-similarity condition is compatible with the scaling condition

\begin{equation}
v(\lambda x, \lambda^k t, \lambda^{-1} z) = v(x, t, z), \quad \lambda > 0,
\end{equation}
on solutions of (7.15).

We study the scalar equation \(Lv = z^n v\) by using the standard procedure to convert it to a first-order system. Let

\[\psi = (v, \frac{\partial v}{\partial x}, \ldots, \frac{\partial^{n-1} v}{\partial x^{n-1}})^t.\]

Then \(Lv = z^n v\) is equivalent to

\begin{equation}
\frac{\partial \psi}{\partial x} = (J_z + q)\psi
\end{equation}

where

\begin{equation}
J_z = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
& & \ddots & & \\
0 & 0 & 0 & \ldots & 1 \\
z^n & 0 & 0 & \ldots & 0
\end{bmatrix}, \quad -q = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
& & \ddots & & \\
u_0 & u_1 & u_2 & \ldots & 0
\end{bmatrix}.
\end{equation}

The commutator \([(L^{k/n})_+, L]\) is a differential operator of order \(n - 2\):

\begin{equation}
[(L^{k/n})_+, L] = \sum_{j=0}^{n-2} w_j \left(\frac{d}{dx}\right)^j.
\end{equation}
If \( L \) is an operator and \( v \) is a function, then the derivatives of \( (L^{k/n})v \) with respect to \( x \) can be expressed in terms of the derivatives of \( D^jv \) of order less than \( n \), i.e. the entries of \( \psi \). This leads to an identity
\[
( (L^{k/n})v, D(L^{k/n})v, \ldots, D^{n-1}(L^{k/n})v )^t = G_k(x, z) \psi
\]
where the \( n \times n \) matrix \( G_k = G_{n,k} \) is a polynomial in \( z^n \) and also in the \( u_j \) and their derivatives of order less than \( k \); (7.20) holds for all such solutions \( v \).

**Examples.** The two simplest cases are
\[
G_{2,3} = J_z^3 + \frac{1}{4} \begin{bmatrix}
-2u_x & 2u \\
-2z u - 2u^2 - u_{xx} & u_x
\end{bmatrix};
\]
\[
G_{3,2} = J_z^2 + \frac{1}{3} \begin{bmatrix}
2u_1 & 0 & 0 \\
-3u_0 + 2u_{1x} & -u_1 & 0 \\
2u_{1xx} - u_{0x} & -3u_0 + u_{1x} & -u_1
\end{bmatrix}.
\]

We differentiate (7.20) with respect to \( x \) to find that
\[
\left[ \frac{\partial}{\partial x} - J_z - q, G_k \right] = r
\]
where \( -r \) is the matrix corresponding to (7.19), i.e.
\[
-r = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & & & & \\
w_0 & w_1 & w_2 & \cdots & 0
\end{bmatrix}.
\]

It follows that the Gel’fand-Dikii equation (7.11) is the same as the matrix equation
\[
q_t = r.
\]

With this preparation we turn to the self-similar solutions of the Gel’fand-Dikii equation. The self-similarity condition (7.12) is equivalent to
\[
q(\lambda x, \lambda^k t) = \lambda^{-1} d(\lambda)^{-1} q(x, t) d(\lambda),
\]
where \( d(\lambda) = \text{diag}(1, \lambda, \lambda^2, \ldots, \lambda^{n-1}) \).

If a scalar function \( v \) is invariant under the dilations (7.15), the corresponding column vector \( \psi \) transforms by
\[
\psi(x, t, z) = d(\lambda) \psi(\lambda x, \lambda^k t, \lambda^{-1} z).
\]

Let \( P \) denote the diagonal matrix
\[
P = \frac{d}{d\lambda} [d(\lambda)]_{\lambda=1} = \text{diag}(0, 1, 2, \ldots, n-1).
\]

The Euler equation equivalent to (7.24) is
\[
x \frac{\partial q}{\partial x} + kt \frac{\partial q}{\partial t} + [P, q] + q = 0.
\]

The following result is analogous to Theorem 7.1 and can be proved in the same way.
Theorem 7.2. If the matrix $q(x,t)$ corresponds to a self-similar solution of the Gel’fand-Dikii equation (7.11), then $q(x,1/k)$ is a solution of the system of algebraic ordinary differential equations

$$ (xq)_x + [P, q] + r = 0. $$

Conversely, the self-similar extension of a solution $q_0(x)$ of (7.28) corresponds to a solution of (7.11).

Examples. The simplest cases are the self-similar KdV and Boussinesq solutions which satisfy, respectively:

$$ (xu)_x + u + \frac{1}{4}u_{xxx} + \frac{3}{2}u_x = 0; $$

$$ (xu_1)_x = u_{1xx} - 2u_{0x}, \quad (xu_0)_x + u_0 = \frac{2}{3}u_{1xx} + \frac{2}{5}u_1 u_{1x} - u_{0xx}. $$

Proposition 7.3. Equation (7.28) is equivalent to the commutator condition

$$ [\frac{\partial}{\partial x} - J_z - q, z \frac{\partial}{\partial z} - x(J_z + q) - P - G_k] = 0 $$

Proof. This is a straightforward calculation using (7.22) and the identity

$$ [P, J_z] - \frac{\partial}{\partial z} (J_z) + J_z = 0. $$

We show in the next section that (7.29) is the equation for a monodromy-preserving flow.

8. Gel’fand-Dikii equations and isomonodromy

The equation (7.29) which characterizes self-similar solutions of the Gel’fand-Dikii equation (7.11), is the compatibility condition for the system

$$ \frac{\partial \psi}{\partial x} = [J_z + q] \psi, $$

$$ z \frac{\partial \psi}{\partial z} = [xJ_z + xq + P + G_k] \psi. $$

The basic result for the forward monodromy problem is the following analogue of results of §§2, 3.
Theorem 8.1. The system (8.2) has a regular singular point at $z = 0$; the Stokes matrices for the irregular singular point at $z = \infty$ are preserved under the flow (7.28).

The fact that the origin is a regular singular point for the system (8.2) is a consequence of the fact that $G_k$ is a polynomial in $z$ (in fact a polynomial in $z^n$). To prove that the Stokes matrices are constant we must analyze the behavior of solutions of (8.2) as $z$ tends to infinity. The argument is similar to that in §2, and needs some preparation.

Note first that in the trivial case $q = 0$ there is a fundamental solution of (8.1) having the form $\psi = \Lambda z e^{xzJ} = d(z) \Lambda e^{xzJ}$ where

$$\Lambda = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \alpha_1^{-1} & \alpha_2^{-1} & \ldots & \alpha_n^{-1} \end{bmatrix}, \quad J = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n),$$

and the $\alpha_j$ are the $n$–th roots of unity; to fix the choice we let $\alpha_j = \exp(2\pi ij/n)$. Moreover

$$\tilde{q} \equiv \Lambda^{-1} q \Lambda = O(z^{-1}), \quad \tilde{r} \equiv \Lambda^{-1} r \Lambda = O(z^{-1}).$$

It is natural to look for solutions to (8.2) in the form $\psi = \Lambda z e^{xzJ}$, so (8.1) and (8.2) become

$$\frac{\partial m}{\partial x} = [zJ, m] + \tilde{q}m,$$

$$z \frac{\partial m}{\partial z} = [xzJ + x\tilde{q} + \tilde{G}] m - m[(zJ)^k + xzJ], \quad \tilde{G} = \Lambda^{-1} G \Lambda.$$

Equation (8.6) has formal solutions as in §1: $m = \sum_{j=0}^{\infty} z^{-j} f_j$ with $f_0 = 1$.

Example. With $n = 2$ and with $x_0$ fixed, the unique formal solution with diagonal part $\equiv 1$ at $x = x_0$ is

$$m(x_0, z) = 1 + \left(\frac{1}{4} z^{-2} u + \frac{1}{16} z^{-4} [u_{xx} + 2u^2]\right) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$+ \left(\frac{1}{8} z^{-3} u_x + \frac{1}{32} z^{-5} [u_{xxx} + 6uu_x]\right) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + O(z^{-6}).$$

For later use we note a symmetry property of (8.6). Let $\Pi$ be the permutation matrix

$$\Pi = J_1 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & \ddots & \ddots & \ddots & \ddots \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$
Then simple calculations show

\[(8.8) \quad \Pi (zJ) \Pi^{-1} = (\alpha z)J, \quad \Pi \tilde{q}_z \Pi^{-1} = \tilde{q}_{az}, \quad \Pi \tilde{r}_z \Pi^{-1} = \tilde{r}_{az}, \quad \Pi \Lambda_z \Pi^{-1} = \Lambda_{az}.\]

Note that \(\tilde{G}_k = \Lambda^{-1}_z G_k \Lambda_z\) is a rational function of \(z\) with leading term \(z^k J^k\) as \(z\) tends to infinity.

**Example.** With \(n = 2\) and \(k = 3\),

\[
\tilde{G}_3 = z^3 J + \frac{1}{2} zu \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \frac{1}{4} u_x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{8} z^{-1}(u_{xx} + 2u^2) \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.
\]

The next step in our analysis of the behavior of solutions for large \(z\) involves understanding \(\tilde{G}_k\).

**Lemma 8.2.** If \(\hat{m}\) is a formal solution of \((8.6)\), then the product \(\hat{m} (zJ)^k \hat{m}^{-1}\) differs from \(\tilde{G}_k\) only in terms of negative degree in \(z\).

**Proof.** Equations \((7.21)\) and \((8.6)\) imply that

\[
\left[ \frac{\partial}{\partial x} - zJ, \hat{m}^{-1} \tilde{G}_k \hat{m} \right] = \hat{m}^{-1} \tilde{r} \hat{m} = O(z^{-1}).
\]

We write

\[(8.9) \quad \hat{m}^{-1} \tilde{G}_k \hat{m} = \sum_{j=0}^{\infty} z^{k-j} g_j, \quad \hat{m}^{-1} \tilde{r} \hat{m} = \sum_{j=0}^{\infty} z^{-j} h_j,
\]

and take \(g_j = h_j = 0\) for \(j < 0\), so that the equation above is equivalent to

\[(8.10) \quad [J, g_{j+1}] = (g_j)_x - h_{j-k}, \quad \text{all } j.
\]

It follows from the relations \((8.10)\) and the identity \(g_0 = J^k\) that \(g_j\) is diagonal and constant for \(j \leq k\). Note that this fact does not depend on \((8.2)\); it is true for arbitrary smooth \(q\).

As in §1, the fact that the diagonal part of \(\hat{m}^{-1} \tilde{G}_k \hat{m}\) is independent of the choice of formal solution \(\hat{m}\) implies that the coefficients of the diagonal part are polynomials in the entries of \(q\) and their derivatives, and that all but the top order term \((zJ)^k\) are polynomials with constant term zero. The only such polynomial which is constant for arbitrary \(q\) is the zero polynomial. Therefore \(g_j = 0\) for \(0 < j \leq k\) and the assertion is proved.

As in §2 we may use a partial sum of a formal solution to regularize our problem at infinity. Let \(f = \sum_{j=0}^{k} z^{-j} f_j\) be a partial sum of \(\hat{m}\); we have

\[
f^{-1} \tilde{G}_k f = z^k J^k + O(z^{-1}).
\]
We look for a solution of (8.2) in the form $\psi = \Lambda_z f \hat{\psi}$. Note that
\begin{equation}
z \frac{\partial \psi}{\partial z} - P \psi = \Lambda_z \frac{\partial}{\partial z}(\Lambda_z^{-1} \psi)
\end{equation}
so the equation for $\hat{\psi}$ is
\begin{equation}
z \frac{\partial \hat{\psi}}{\partial z} = [-zf^{-1} f_z + x f^{-1}(zJ + \bar{q})f + f^{-1}\bar{G}_k f] \hat{\psi} = [z^k J^k + xzJ + r(x, z)] \hat{\psi}, \quad r(x, z) = O(z^{-1}).
\end{equation}
From this point the analysis is essentially the same as in §2. Equation (8.2) has unique solutions of the form
\begin{equation}
\psi_\nu = \Lambda_z m_\nu e^{\Phi}, \quad \Phi(x, z) = \frac{1}{k} z^k J^k + xzJ,
\end{equation}
where $m_\nu$ has an asymptotic expansion with leading coefficient 1, valid in a sector $\Omega_\nu$. As in §2 these functions also satisfy (8.1) and therefore the Stokes matrices which relate the $\psi_\nu$ are constant. This completes our sketch of the proof of Theorem 8.1.

**Remark.** The symmetry property (8.8) and the uniqueness of the solutions (8.13) allow us to conclude that these solutions have the $n$-fold symmetry which can be stated roughly as
\[ m(x, \alpha z) = \Pi m(x, z) \Pi^{-1}. \]
To state it more precisely we need to investigate the set
\[ \Sigma = \{ z : \text{Re}(z\alpha_j)^k = \text{Re}(z\alpha_i)^k, \text{ some } \alpha_j^k \neq \alpha_i^k \}. \]
Let $p$ be the smallest positive integer such that $\alpha^{pk} = 1$, i.e. such that $n$ divides $pk$. Then
\[ \Sigma = \{ z : z^k \in i\mathbb{R} \} \text{ if } p = 2; \quad \Sigma = \{ z : z^{2pk} \in \mathbb{R}_+ \} \text{ if } p > 2. \]
We define regions $\Omega_\nu$ as in §2, bounded by various of the rays of the set $\Sigma$. It follows from the form of $\Sigma$ that $\Omega_{\nu+1}$ is obtained by rotation of $\Omega_\nu$ through an angle $\pi/pk$, ($\pi/k$ if $p = 2$) and the precise form of the symmetry is
\begin{equation}
m_{\nu+s}(x, \alpha z) = \Pi m_\nu(x, z) \Pi^{-1}, \quad \begin{cases} s = 2kp/n & \text{if } p > 2 \\ s = 2k/n & \text{if } p = 2. \end{cases}
\end{equation}
The Stokes matrices satisfy the corresponding symmetry

\[ S_{\nu+s} = \Pi S_{\nu} \Pi^{-1}. \]  

as well as the standard constraints

\[ \text{diag}(S_{\nu}) = 1, \quad e^\Phi S_{\nu} e^{-\Phi} \text{ is bounded as } z \to \infty, \quad z \in \Omega_{\nu}, \quad z \text{ near } \Sigma_{\nu}, \]

where \( \Sigma_{\nu} \) is any ray in \( \Omega_{\nu} \cap \Omega_{\nu+1} \); it will be convenient to choose \( \Sigma_{\nu} \) to bisect this sector.

The forward monodromy problem at \( z = 0 \) is the same as in §3, but we must take the symmetry into account. For convenience we consider only the generic case, when there is a traceless constant matrix \( A \) and a solution of \( \psi \) of (8.1), (8.2) with the property that \( \psi(x,z)z^{-A} \) is an entire function of \( z \), invertible for each \( z \). If so, then there are traceless constant matrices \( B_{\nu} \) such that \( \psi_{\nu} z^{-B_{\nu}} \) are entire, and

\[ B_{\nu+1} = S_{\nu}^{-1} B_{\nu} S_{\nu}; \quad \exp(2\pi i B_{\nu}) = S_{\nu} S_{\nu+1} \cdots S_{\nu-1}, \quad B_{\nu+s} = \Pi B_{\nu} \Pi^{-1}. \]

**Theorem 8.3.** In the generic case, if two solutions of the isomonodromy equation (7.29) have a common domain and the same monodromy data \( \{S_{\nu}, B_{\nu}\} \), then they are identical.

**Proof.** Suppose that \( q \) and \( q' \) are two solutions of (7.29) on a common domain, having the same monodromy data. Let \( \psi_{\nu} \) and \( \psi'_{\nu} \) be the corresponding normalized solutions of (8.1), (8.2). Let \( m = \psi_{\nu} e^{-\Phi} \) for \( |z| > 1 \), \( z \) between \( \Sigma_{\nu-1} \) and \( \Sigma_{\nu} \) and \( m = \psi_{\nu} z^{-B_{\nu}} e^{-\Phi} \) on \( \{|z| < 1\} \). Define \( m' \) analogously. Each of \( m \) and \( m' \) is a solution of a matrix Riemann-Hilbert factorization problem; a vector version of this problem is described in more detail below. It follows that \( m^{-1}m' \) is continuous and piecewise holomorphic, hence entire. From the asymptotics of (8.13) we know that \( \Lambda_z^{-1}m \sim 1 \) at infinity, and the same for \( m' \), so \( Q = m^{-1}m' \) is a polynomial in \( z \) and

\[ d(z)^{-1} Q d(z) = 1 + O(z^{-1}). \]

This last fact implies that \( Q - 1 \) is strictly lower triangular. Therefore \( m' = Qm \) has the same first row as \( m \). But \( m \) and \( m' \) are each determined by the first row \( M \); in fact the \( j \)-th row is \( D^{j+1} (Me^\Phi) e^{-\Phi} \). Thus \( \psi = \psi' \) and \( q = [D \psi \ J_z \psi] \psi^{-1} = q' \). \( \square \)

In order to formulate the inverse problem, in which matrices \( \{S_{\nu}, B_{\nu}\} \) are given, we must give a precise description of the vector Riemann-Hilbert problem alluded to above. Let \( M \) be the first row of \( \psi_{\nu} e^{-\Phi} \) for \( |z| > 1 \), \( z \) between \( \Sigma_{\nu-1} \) and \( \Sigma_{\nu} \). On \( \{|z| < 1\} \)
let \( M \) be the first row of \( \psi \nu z^{-B\nu} e^{-\Phi} = \Lambda z m_\nu \); this is independent of \( \nu \). Because of the asymptotics of \( m_\nu \) and the relations given by the Stokes matrices, one can see that the row vector function \( M \) has the properties

\begin{align}
(8.18) & \quad M \text{ has limit } (1, 1, \ldots, 1) \text{ as } z \to \infty; \\
(8.19) & \quad M \text{ is bounded and holomorphic where defined}; \\
(8.20) & \quad M \text{ is continuous up to the boundary from each component}; \\
\end{align}

Moreover the boundary values \( M_\nu \), the limit on the boundary of the region outside the unit circle between the rays \( \Sigma_{\nu-1} \) and \( \Sigma_\nu \), and \( M_\Gamma \), the limit on the unit circle from the disc, satisfy

\begin{align}
(8.21) & \quad M_{\nu+1} = M_\nu e^\Phi S_\nu e^{-\Phi}; \quad M_\nu = M_\Gamma e^\Phi z B_\nu e^{-\Phi}, \\
\end{align}

while the row vector \( M \) itself has the symmetry

\begin{align}
M(x, \alpha z) = M(x, z) \Pi^{-1}, \quad |z| > 1, \quad z \notin \bigcup \Sigma_\nu. \\
(8.22)
\end{align}

Conversely, suppose matrices \( \{S_\nu, B_\nu\} \) are given. Generically the Riemann-Hilbert problem (8.18)-(8.22) has a unique solution, say for \( x \) in some domain.

**Theorem 8.4.** Suppose that \( S_\nu \) and \( B_\nu \) are constant matrices which satisfy the conditions (8.15)-(8.17), and \( \text{tr} B_\nu = 0 \). Suppose the Riemann-Hilbert problem (8.18)-(8.22) has a unique solution \( M(x, \cdot) \), for \( x \) in some domain. Then there is a unique \( n \)-th order operator

\[
L = D^n + \sum_{j=0}^{n-2} u_j(x) D^j, \quad D = \frac{d}{dx}
\]

such that the row vector \( v = M e^\Phi \) satisfies the equation \( Lv = z^n v \). The system corresponding to \( L \) satisfies the isomonodromy equation (7.29).

**Proof.** The vector function \( M \) has an asymptotic expansion in powers of \( z^{-1} \) as \( z \to \infty \),

\begin{align}
M(x, z) \sim \sum_{j=0}^{\infty} z^{-j} a_j(x) \quad (8.23)
\end{align}

and the symmetry (8.22) implies that the coefficients have the form

\begin{align}
a_j(x) = b_j(x)(1, 1, \ldots, 1)J^{-j}, \quad (8.24)
\end{align}
where the $b_j$’s are scalars. The expansion can be differentiated term by term with respect to $x$.

Next we construct the sequence of vectors

$$M^{(j)} = D^j (Me^\Phi) e^{-\Phi} = (D^j v) e^{-\Phi}, \quad j \geq 0. \tag{8.25}$$

Note that

$$M^{(j)} = (1, 1, \ldots, 1)(zJ)^j + O(z^{j-1}). \tag{8.26}$$

In particular $M^{(n)} - z^n M$ is $O(z^{n-1})$. We use the symmetries (8.24) and their analogues for the $M^{(j)}$’s to conclude that there is a unique scalar function $u_{n-1}(x)$ such that

$$M^{(n)} - z^n M + u_{n-1}M^{(n-1)} = O(z^{n-2}). \tag{8.27}$$

Continuing, we find unique functions $u_{n-2}, \ldots, u_0$ such that

$$M^{(n)} - z^n M + \sum_{j=0}^{n-1} u_j M^{(j)} = 0(z^{-1}). \tag{8.28}$$

Now the $M^{(j)}$’s and $z^n M$ are solutions of the Riemann-Hilbert problem (8.19)-(8.22). Therefore the left side of (8.28) is a solution, with limit 0 as $z$ tends to $\infty$. By our uniqueness assumption for solutions of (8.18)-(8.22) the left side of (8.28) must vanish. It follows that the row vector $v = Me^\Phi$ satisfies the equation

$$Lv \equiv [D^n + \sum_{j=0}^{n-1} u_j D^j] v = z^n v. \tag{8.29}$$

Our next task is to show that $u_{n-1} \equiv 0$, so that $L$ has the desired form.

We may reduce to a first-order system as before and conclude that $v$ is the first row of a solution to a system of the form (8.1). The $j$-th row of the matrix $\psi$ is precisely $M^{(j-1)} e^\Phi$. Note that the trace of the matrix $q$ is $-u_{n-1}$. It follows that the determinant $\Delta = \det(\psi) = \det(\psi e^{-\Phi})$ satisfies

$$\Delta_x \equiv -u_{n-1} \Delta. \tag{8.30}$$

On the other hand, $\Delta$ is a solution of a scalar Riemann-Hilbert problem with trivial multiplicative jumps. Therefore $\Delta$ is an entire function. The asymptotic behavior is the
same as that of $\det \Lambda_z$, so $\triangle$ is a polynomial with leading coefficient $\det \Lambda_z$. Since this coefficient does not depend on $x$, (8.30) implies $u_{n-1} \equiv 0$.

Similarly, $zM_z - xM^{(1)} + M[(zJ)^k + xzJ] - M^{(k)}$ is a solution of (8.19)-(8.22) which is $O(z^{k-1})$ and we may use the symmetries and (8.26) to deduce that

$$zM_z - xM^{(1)} + M[(zJ)^k + xzJ] = M^{(k)} + a_{k-1}M^{(k-1)} + O(z^{k-1}).$$

As before, we conclude that $v$ satisfies an equation of the form $zv_z = xv_x + L\# v$, where $L\#$ is a differential operator in the $x$-variable, of order $k$. This equation is equivalent to the first-order system

$$z\psi_z = x(J_z + q)\psi + P\psi + G\psi,$$

where the matrix $G(x, z)$ is characterized by the condition analogous to (7.20):

(8.31) $(L\# v, D L\# v, \ldots, D^{n-1} L\# v)^t = G\psi.$

Equation (7.29) is the compatibility condition for (8.31) and (8.31), provided we may replace $L\#$ by $(L^{k/n})_+$. We may make this replacement if and only if $L\# v = (L^{k/n})_+ v$, and this is true if the matrices $G$ and $G_k$ have the same first row.

We write $\psi = \Lambda_z m e^\Phi$ and note that $m$ has an expansion at $\infty$ with leading term $1$. Moreover, $m$ satisfies

$$xm_x = [xzJ, m] + x\tilde{q}m; \quad zm_z = [xzJ, m] + [x\tilde{q} + \tilde{G}]m - m(zJ)^k,$$

where $\tilde{G} = \Lambda^{-1}_z G \Lambda_z$. It follows from this and Lemma 8.2 that

(8.32) $\tilde{G} = m(zJ)^k m^{-1} + xm_x - zm_z = m(zJ)^k m^{-1} + O(z^{-1}) = \tilde{G}_k + O(z^{-1}).$

Therefore

$$d(z)^{-1}[G - G_k]d(z) = \Lambda[\tilde{G} - \tilde{G}_k]\Lambda^{-1} = O(z^{-1})$$

which implies that $G - G_k$ is strictly lower triangular. In particular $G$ and $G_k$ have the same first row. This in turn implies that $L\# v = (L^{k/n})_+ v$, so we may replace $L\#$ by $(L^{k/n})_+$. □

9. Isomonodromy deformations and string equations

In this section we consider a second classes of (systems) of ordinary differential equations associated to the Gel’fand-Dikii hierarchy (7.11) and show that each is an isomonodromy
deformation for a first order system. This class was obtained by M. Douglas [Do] in connection with theories of quantum gravity. The Gel’fand-Dikii flow itself is replaced by the equation

\[(9.1) \quad [(L^{k/n})_+ , L] = \hbar I\]

where \(I\) is the identity operator. For convenience of notation we consider \(\hbar = 1\). The case \(n = 2, k = 3\) and the case \(n = 3, k = 2\) each lead to a form of the PI equation; for \(n = 2, k = 3\) the equation is

\[\frac{1}{4} u_{xxx} + \frac{3}{2} uu_x + 1 = 0;\]

for \(n = 3, k = 2\) the constants are different. The case \(n = 2, k = 5\) is

\[\frac{1}{16} D^5 u + \frac{5}{8} uu_{xxx} + \frac{5}{4} u_x u_{xx} + \frac{15}{8} u^2 u_x + 1 = 0.\]

**Remark.** The development in this section applies equally to the case when the operator \((L^{k/n})_+\) is replaced by a finite linear combination \(\sum_k a_k (L^{k/n})_+\) with arbitrary constants \(a_k\). The phase function \(\Phi\) which is considered below must be changed accordingly.

Note that (9.1) can be written as a commutator equation

\[(9.2) \quad [L - z^n I , \frac{\partial}{\partial z} - nz^{n-1}(L^{k/n})_+] = 0,\]

which is the compatibility condition for the system of two scalar equations

\[(9.3) \quad Lv = z^n v, \quad n v_z = nz^n (L^{k/n})_+ v.\]

Reduction to a first order system as in §8 gives a matrix version of (9.2):

\[(9.4) \quad \begin{bmatrix} \frac{\partial}{\partial x} - (J_z + q) \end{bmatrix}, z \frac{\partial}{\partial z} - nz^n G_k \right] = 0.\]

In turn, (9.4) is the compatibility condition for the system

\[(9.5) \quad \frac{\partial \psi}{\partial x} = [J_z + q] \psi,\]

\[(9.6) \quad z \frac{\partial \psi}{\partial z} = nz^n G_k \psi.\]

We sketch a proof of the following analogue of Theorem 8.1.
Theorem 9.1. The system (9.6) has a regular point at \( z = 0 \); the Stokes matrices for the irregular singular point at \( z = \infty \) are preserved under the flow (9.4).

Since \( G_k \) is a polynomial in \( z^n \) it follows that the origin is a regular point for (9.6). The second assertion is a consequence of Lemma 9.3 below. We look for exact solutions of (9.6) having the form \( \psi = \Lambda z f \hat{\psi} \), where \( f = \sum_{j=0}^{n+k+1} z^{-j} m_j \). Then as before there are solutions \( \psi_\nu \) of this form which have the appropriate asymptotic expansions in sectors \( \Omega_\nu \), and which are related by constant Stokes matrices \( S_\nu \).

Thus we look for solutions of (9.5), (9.6) having the form \( \psi = \Lambda z m e^\Phi \) for some diagonal matrix-valued function \( \Phi \). The equations become

\[
D_x m \equiv m_x - zJm - q_z m = -m \Phi_x \\
D_z m \equiv zm_z - nz^n \tilde{G}_km + \Lambda_z^{-1} P \Lambda_z m = -zm \Phi_z.
\]

Lemma 9.2. Suppose \( \hat{m} \) is a formal solution of (9.7). Then the diagonal part of the coefficient \( g_j \) of \( z^{-j} \) in the formal series \( \hat{m}^{-1}r_z \hat{m} \) has the form \( c_j J^{-j} \), where \( c_j \) is a scalar function which is a polynomial without constant term in the entries of \( q \) and their derivatives. Moreover, \( c_j = 0 \) if \( j \) is divisible by \( n \).

The proof is postponed.

Lemma 9.3. Suppose (9.1) is satisfied. Then there is a unique diagonal matrix-valued function

\[
\Phi(x, z) = \frac{n}{n+k} (zJ)^{n+k} + \sum_{j=1}^{n-1} b_j (zJ)^j + xzJ - \frac{1}{2}(n-1) \log z 1,
\]

where the \( b_j \)'s are scalar constants, and a unique formal power series in \( z^{-1} \), \( m = \sum_{j=0}^{\infty} z^{-j} m_j(x) \) such that \( m_0 = 1 \) and \( m \) is a formal solution of the equations (9.7), (9.8).

Proof. We begin by refining Lemma 8.2. Let \( \hat{m} \) be a formal solution of (9.5) and use the notation of (8.9).

We already know from the proof of Lemma 8.2 that \( g_1 = \cdots = g_k = 0 \). It follows from Lemma 9.2 and the recursion relations (8.10) that the diagonal part of \( g_{k+j} \) has derivative \( c_j J^{-j} \) for \( j > 0 \). The diagonal part of \( g_{k+j} \) itself is a polynomial without constant term in the entries of \( q \) and their derivatives, so we can write

\[
(g_{k+j})^{\text{diag}} = \frac{n-j}{n} b_{n-j} J^{-j} = \frac{n-j}{n} b_{n-j} J^{n-j}, \quad j > 0,
\]
where \( b_j \) is scalar polynomial in the entries of \( q \) and their derivatives, without constant term.

Equation (9.1), which we now invoke, implies that \( w_0 = -1 \) and \( w_j = 0, \ j > 0 \). Therefore \( h_j = 0 \) for \( j < n - 1 \) and

\[
(h_{n-1})^{\text{diag}} = -\frac{1}{n}w_0J = \frac{1}{n}J.
\]

We use the recursion relations (8.10) again and conclude that \( g_{k+j} \) is diagonal and is constant for \( 0 < j < n - 1 \). Thus the scalar \( b_j \) is constant, i.e. this expression in \( q \) and its derivatives is invariant under the \( x \)-flow (9.1).

At the next step (8.10) implies

\[
g_{k+n} = \frac{1}{n}(xJ + b_1J), \quad b_1 \text{ constant, scalar;}
\]

\[
(g_{k+n})_x = [J, g_{k+n+1}] + h_n.
\]

By Lemma 9.2, \( h_n \) is off-diagonal and we find that the diagonal part of \( g_{k+n} \) is constant. This fact is independent of (9.1) and holds for arbitrary \( q \), which again allows us to conclude that the diagonal part of \( g_{k+n} \) vanishes in all cases.

Note that the diagonal part of \( \Lambda_z^{-1}P\Lambda_z = \Lambda^{-1}PA \) is \( \frac{1}{2}(n-1) \mathbf{1} \). The preceding argument shows that

\[
(9.10)
\]

\[
\hat{m}^{-1}[nz^n\hat{G}_k - \Lambda_z^{-1}P\Lambda_z]\hat{m}
= nz^{n+k}J^k + \sum_{j=1}^{n-1} j b_j (zJ)^j + xzJ - \frac{1}{2}(n-1) \mathbf{1} + E
= z \frac{\partial}{\partial z} [\sum_{j=1}^{n+k} j b_j (zJ)^j + xzJ - \frac{1}{2}(n-1) \log z \mathbf{1}] + E
= z \frac{\partial}{\partial z} [\Phi(x, z)] + E
\]

where the \( b_j \) are scalar constants and \( \Phi \) is defined by (9.9). The remainder term \( E \) is a formal series with terms of non-positive degree in \( z \), whose term of degree 0 in \( z \) is off-diagonal.

Finally we look for a solution to (9.8) in the form \( m = ph \) where \( p \) is the sum of the terms of degree \( \geq -n - k \) in the formal solution \( \hat{m} \). An analogue of (9.10) holds with \( p \) in place of \( m \), so (9.8) is equivalent to

\[
zh_z = [\Phi_z, h] - Fh, \quad F = O(z^{-1})
\]
where the off-diagonal part of $F$ is $O(z^{-2})$. The proof of Theorem 2.1 in [CL, Ch. 5] shows that this equation has a formal solution $h = 1 + O(z^{-1})$.

We return to unfinished business.

Proof of Lemma 9.2. The assertion is independent of the choice of formal solution, so we may take $\hat{m}$ to be the unique formal solution normalized so $\hat{m}(x_0, z) \equiv 1$. Because of uniqueness and the symmetry properties (8.8), we can conclude that $\Pi \hat{m}(x, z) \Pi^{-1} \equiv \hat{m}(x, \alpha z)$. With this and another use of (8.8) we find that

$$\Pi \hat{m}(x, z)^{-1} r_z \hat{m}(x, z) \Pi^{-1} = \hat{m}(x, \alpha z)^{-1} r_{\alpha z} \hat{m}(x, \alpha z).$$

Therefore the coefficients in the expansion $\hat{m}^{-1} r_z \hat{m} = \sum z^{-j} h_j$ satisfy $\Pi h_j \Pi^{-1} = \alpha^{-j} h_j$. This in turn implies that

$$\Pi (J^j h_j) \Pi^{-1} = (\Pi J^j \Pi^{-1})(\Pi h_j \Pi^{-1}) = (\alpha^j J^j)(\alpha^{-j} h_j) = J^j h_j.$$ 

It follows that the diagonal part of $J^j h_j$ commutes with $\Pi$ and therefore is a scalar $c_j$.

We noted earlier that since the diagonal part of $h_j$ is independent of the choice of $\hat{m}$, its entries are polynomials in the entries of $q$ and their derivatives.

When $j$ is divisible by $n$, the diagonal part of $h_j$ has trace $n c_j$, so to complete the proof it is enough to show that the trace is zero. But

$$\text{tr}(\hat{m}^{-1} r_z \hat{m}) = \text{tr}(r_z) = \text{tr}(\Lambda^{-1}_z r \Lambda_z) = \text{tr}(r) \equiv 0.$$ 

Examples. 1. In view of the previous calculations for the case $n = 2$, $k = 3$, we find that in general

$$\hat{m}^{-1}(2z \tilde{G}_3) \hat{m} = 2z^4 J^3 + \frac{1}{8} \begin{bmatrix} 2u_{xx} + 6u^2 & z^{-1}(u_{xxx} + 6uu_x) \\ z^{-1}(u_{xxx} + 6uu_x) & -2u_{xx} - 6u^2 \end{bmatrix}. $$

Equation (9.1) in this case is $u_{xxx} + 6uu_x = -4$, so $u_{xx} + 3u^2 = -4(x + b)$ for some constant $b$. Thus the general formula reduces to

$$\hat{m}^{-1}(2z \tilde{G}_3) \hat{m} = 2z^4 J^5 + (x + b)J - \frac{1}{2} z^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It follows that for this case

$$\Phi(x, z) = \frac{2}{5}(zJ)^5 + (x + b)zJ - \frac{1}{2} \log z \mathbf{1}.$$
2. For the case \( n = 3, k = 2 \), in general

\[
[\hat{m}^{-1}(3z^2\hat{G}_2)\hat{m}]^{\text{diag}} = 3z^4 J^5 + (u_{1xx} - 2u_0)zJ^2 + (\frac{2}{3}u_{1xx} + \frac{1}{3}u_1^2 - u_{0x})J - z^{-1}1.
\]

Equation (9.1) is equivalent to: \( u_{1xx} - 2u_0 \equiv 2b_2 \) and \( \frac{2}{3}u_{1xx} + \frac{1}{3}u_1^2 - u_{0x} = b_1 + x \) for some constants \( b_2, b_1 \). Therefore

\[
\Phi(x, z) = \frac{2}{3}(zJ)^5 + b_2(zJ)^2 + (b_1 + x)zJ - \log z 1.
\]

**Remark.** The set of solutions of (9.1) is invariant under translation, so one can always get rid of the constant \( b_1 \) in \( \Phi \) by translating.

**Remark.** As we noted earlier, the origin is a regular point for the equation (9.6), which is satisfied by \( \psi = \Lambda z me^\Phi \). Therefore \( me^\Phi \) has trivial monodromy. It follows that the product of the Stokes matrices is \((-1)^{n-1}1\):

(9.11)

\[
S_1S_2\cdots S_0 = (-1)^{n-1}1.
\]

The following is the analogue of Theorem 8.3, and is proved in the same way.

**Theorem 9.4.** If two solutions of the isomonodromy equation (9.4) have a common domain and the same monodromy data \( \{S_\nu\} \), then they are identical.

The direct problem is similar to, but somewhat simpler than, the direct problem treated in §8. The set \( \Sigma \) has the form

(9.12)

\[
\Sigma = \{z : z^{n+k} \in i\mathbb{R}\} \text{ if } p = 2; \quad \Sigma = \{z : z^{2pn+2pk} \in \mathbb{R}_+\} \text{ if } p > 2,
\]

where again \( p \) is the smallest integer such that \( n \) divides \( pk \). The symmetry condition (8.8) and its consequences carry over to the present situation, with \( s = 2p(n+k)/n \) if \( p > 2 \) or \( s = 2(n+k)/n \) if \( p = 2 \).

Again we associate a Riemann-Hilbert problem. Let \( \psi_\nu = \Lambda z m_\nu e^\Phi \) with \( m_\nu = 1 + 0(z^{-1}) \) in \( \Omega_\nu \) and let \( M \) be the first row of \( m_\nu \) for \( |z| > 1 \), \( z \) between \( \Sigma_{\nu-1} \) and \( \Sigma_\nu \); on the unit disc let \( M \) be the first row of \( z^{(n+1)/2} \psi_0 e^{-\Phi} \). Then \( M \) satisfies (8.18)-(8.20), and (8.22), and its boundary values satisfy

(9.13)

\[
M_{\nu+1} = M_\nu e^\Phi S_\nu e^{-\Phi}, \quad |z| > 1;
\]

(9.14)

\[
M_{\nu+1} = z^{-(n+1)/2} M_\Gamma e^\Phi S_0S_1\cdots S_\nu e^{-\Phi}, \quad |z| = 1.
\]

For the inverse problem, suppose that the \( S_\nu \) and \( \Phi \) are given. Generically the Riemann-Hilbert problem (8.18)-(8.20), (9.13), (9.14) has a unique solution \( M(x, \cdot) \) for \( x \) in some domain.
Theorem 9.5. Suppose that the constant matrices $S_\nu$ satisfy (8.15), (8.16), and (9.11) and that the matrix $\Phi(x,z)$ has the form (9.9). Suppose that the Riemann-Hilbert problem (8.18)-(8.20), (9.13), (9.14) has a unique solution for $x$ in some domain. Then there is a unique $n$-th order operator $L = D^n + \sum_{j=0}^{n-2} u_j D^j$ such that $v = Me^\Phi$ satisfies the equations $Lv = z^n v$, $zv_z = nz^n (L^{k/n})_+ v$. In particular, the isomonodromy equation (9.1) is satisfied.

**Proof.** Arguing exactly as in the proof of Theorem 8.4, we examine the expressions

$$M^{(m)} - z^n M, \quad zM_z + M(z\Phi_z) - nz^n M^{(k)}$$

and show that $v = Me^\Phi$ satisfies an equation of the correct form $Lv = z^n v$ as well as a second equation of the form $zv_z = nz^n L^# v$. Here $L^#$ is an operator of order $k$. The corresponding first order systems are

$$\psi_x = [J_z + q] \psi, \quad z\psi_z = nz^n G \psi,$$

with $G$ determined by (8.31). As before we write $\psi = \Lambda_z me^\Phi$, so $m = 1 + O(z^{-1})$ and

$$zm_z = [nz^n \tilde{G} - \Lambda^{-1}_z PA \Lambda_z] m - m(z\Phi_z) = [nz^n \tilde{G} - \Lambda^{-1} PA] m - m(z\Phi_z).$$

Because of the form of $\Phi$ and the asymptotic behavior of $m$, this equation implies

$$\tilde{G} = m(zJ)^k m^{-1} + O(z^{-1}) = \tilde{G}_k + O(z^{-1}).$$

As in §8 we conclude from this fact that $G$ and $G_k$ have the same first row and therefore that we may replace $L^#$ by $(L^{k/n})_+$. 

**References**

[AS1] M. J. Ablowitz and H. Segur, *Exact linearization of a Painlevé transcendent*, Phys. Rev. Lett. **38** (1977), 1103-1106.

[AS2] M. J. Ablowitz and H. Segur, “Solitons and the Inverse Scattering Transform”, SIAM Studies in Applied Mathematics, Philadelphia 1981.

[Ai] H. Airault, *Rational Solutions of Painlevé Equations*, Studies in Applied Mathematics **61** (1979), 31-53.

[BC] R. Beals and R. R. Coifman, *Scattering and inverse scattering for first order systems*, Comm. Pure Appl. Math. **87** (1984), 39-90.
[CL] E. A. Coddington and N. Levinson, “Theory of Ordinary Differential Equations”, McGraw-Hill, New York 1955.

[Do] M. Douglas, Strings in less than one dimension and the generalized KdV hierarchies, Phys. Lett. 238B (1990), 176-180.

[FN] H. Flaschka and A.C. Newell, Monodromy and Spectrum Preserving Deformations. I, Comm. in Math. Phys. 76 (1980), 65-116.

[FZ] A. S. Fokas and X. Zhou, Integrability of Painlevé transcendents, to appear.

[Ga] E. Gambier, Sur les équations différentielles du second ordre et du premier degré dont l’intégrale générale est à points critiques fixes, Acta Math. 33 (1910), 1-55.

[IN] A. R. Its and V. Y. Novokshenov, “The Isomonodromic Deformation Method in the Theory of Painlevé Equations,” Lecture Notes in Mathematics no. 1191, Springer Verlag, Heidelberg 1986.

[JMU] M. Jimbo, T. Miwa, and K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, Physica D 2 (1981), 306-352.

[JM1] M. Jimbo and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II, Physica D 2 (1981), 407-448.

[JM2] M. Jimbo and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. III, Physica D 4 (1983), 26-46.

[Ma] B. Malgrange, “Équations Différentielles à Coefficients Polynomiaux,” Birkhaüser, Boston 1991.

[Mi] T. Miwa, Painlevé property of monodromy preserving equations and the analyticity of the $\tau$ function, Publ. R.I.M.S. Kyoto University 17 (1981), 703-721.

[Mo1] G. Moore, Geometry of the string equations, Comm. Math. Phys. 133 (1990), 261-304.

[Mo2] G. Moore, Matrix models of 2D gravity and isomonodromic deformation, Progress Theor. Phys., Suppl. no. 102 (1990), 255-285.

[Mt] T. F. Moutard, Note sur les équations différentielles linéaires du second ordre,
C. R. Acad. Sci. Paris 80 (1876), 729.

[Sa] D. H. Sattinger, *Hamiltonian hierarchies on semisimple Lie algebras*, Studies in Applied Math. 72 (1985), 65-86.

[Zu] V. Zurkowski, “Scattering for first order linear systems on the line and Bäcklund transformations,” Ph.D. Thesis, University of Minnesota - Minneapolis 1987.