Finite Temperature Deconfining Transition
in the BRST Formalism

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Abstract

We present a toy model study of the high temperature deconfining transition in Yang-Mills theory as a breakdown of the confinement condition proposed by Kugo and Ojima. Our toy model is a kind of topological field theory obtained from the Yang-Mills theory by taking the limit of vanishing gauge coupling constant $g_{YM} \to 0$, and therefore the gauge field $A_\mu$ is constrained to the pure-gauge configuration $A_\mu = g^\dagger \partial_\mu g$. At zero temperature this model has been known to satisfy the confinement condition of Kugo and Ojima which requires the absence of the massless Nambu-Goldstone-like mode coupled to the BRST-exact color current. In the finite temperature case based on the real-time formalism, our model in 3+1 dimensions is reduced, by the Parisi-Sourlas mechanism, to the “sum” of chiral models in 1+1 dimensions with various boundary conditions of the group element $g(t, x)$ at the ends of the time contour. We analyze the effective potential of the $SU(2)$ model and find that the deconfining transition in fact occurs due to the contribution of the sectors with non-periodic boundary conditions.

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1 Introduction

Deconfining transition in high temperature QCD has been a subject of active researches for both theoretical and phenomenological interests. This transition may be stated as follows: below a certain critical temperature $\beta_c^{-1}$ we cannot observe color non-singlet excitations, while above $\beta_c^{-1}$ colored excitations such as quarks are allowed as physical ones. Widely adopted as a criterion of (de)confinement at finite temperature is the expectation value $\langle P \rangle_\beta$ of the Polyakov loop (thermal Wilson loop) operator $P(x) = \text{tr} \exp \left( \int_0^\beta d\tau A_4(\tau, x) \right)$ which measures the free-energy of a quark put in the system as an external source [1, 2]. However, the Polyakov loop cannot be used as a criterion of confinement in a system with dynamical (quantized) quark fields. Moreover, $\langle P \rangle_\beta$ tells nothing about the (de)confinement of other color carrying fields, for example, the gluon. In this paper we shall study the deconfining transition on the basis of another confinement criterion proposed by Kugo and Ojima (KO) [3]. In contrast with the Polyakov loop, the confinement mechanism of KO treats directly the confinement of quantized colored fields (particles) of any kind.

The KO confinement criterion is based on the BRST quantized Yang-Mills theory [3] described by the lagrangian

$$\mathcal{L}_{YM} = \frac{1}{2g^2_{YM}} \text{tr} F^2_{\mu\nu} + \mathcal{L}_{\text{matter}} - i\delta_B G,$$

where $\delta_B$ is the BRST transformation defined as usual by

$$\delta_B A_\mu = D_\mu c \equiv \partial_\mu c + [A_\mu, c], \quad \delta_B c = -\frac{1}{2}\{c, c\}, \quad \delta_B \bar{c} = iB, \quad \delta_B B = 0. \quad (1.2)$$

The last term of eq. (1.1) gives the gauge-fixing and the corresponding ghost terms. The key quantity in the KO confinement mechanism at zero temperature is the BRST-exact conserved color current $N_\mu$,

$$N_\mu = -i\delta_B K_\mu = \{Q_B, K_\mu\}, \quad (1.3)$$

where $Q_B$ is the BRST charge, and $K_\mu$ is obtained from $\mathcal{L}_{YM}$ of eq. (1.1) by making a local gauge transformation $\delta_\epsilon$, $\delta_\epsilon A_\mu = D_\mu \epsilon$ and $\delta_\epsilon \phi = [\phi, \epsilon]$ for $\phi = c, \bar{c}, B$, as

$$\delta_\epsilon \mathcal{L}_{YM} = -i\delta_\epsilon \delta_B G = -i\delta_B K^a_\mu \cdot \partial_\mu \epsilon^a. \quad (1.4)$$

* We restrict the gauge group to $SU(N)$. The field variables $\phi = A_\mu, c, \bar{c}, B$ are Lie algebra valued and are expressed as $\phi = \sum_{a=1}^{N^2-1} \phi^at^a$ in terms of Hermitian fields $\phi^a$ and the (anti-Hermitian) basis $t^a$ with the normalization $\text{tr}(t^at^b) = -(1/2)\delta^{ab}$. 

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Note that $\delta_L \mathcal{L}_{YM}$ has its contribution only from the gauge-term $\delta_B G$, and the last expression is because we have assumed that the gauge-fixing function $G$ preserves the global color rotation symmetry. In the Feynman-type gauge with $G = \text{tr} \left[ \partial_\mu A_\mu - \alpha B \right]$, we have $K_\mu = D_\mu \bar{c}$. The ordinary Noether color current $J_\mu^a = \bar{q} \gamma_\mu t^a q + \ldots$ containing the matter fields is related to $N_\mu$ by $J_\mu = N_\mu - (1/g_{YM}^2) \partial^\nu F_{\nu\mu}$ using the equation of motion. Therefore both $N_\mu$ and $J_\mu$ generate the same global color rotation.

The KO confinement condition requires that the BRST-exact color current $N_\mu$ (1.3) contains no (Nambu-Goldstone-like) massless one-particle component. If this condition is satisfied, then the integration $\int d^3x N_0$ has a well-defined meaning and hence the color charge $Q^a$ can be written in a BRST-exact form, $Q^a = \{ Q_B, \int d^3x K^a_0 \}$, which implies color confinement in the sense that any color non-singlet asymptotic state is necessarily BRST unphysical and unobservable.†

In order to extend the KO confinement mechanism to the finite temperature case, we shall recapitulate the elements of finite temperature gauge theory. First, in statistical mechanics of gauge theories in the BRST formalism, the statistical average must be taken only over physical states. Let $\mathcal{P}_{\text{phys}}$ be the projection operator to the subspace of physical states $\mathcal{H}_{\text{phys}} = \text{Ker} Q_B / \text{Im} Q_B$. Then we have the following useful identity [5] for the thermal expectation value $\langle O \rangle_\beta$ of a BRST invariant operator $O$ satisfying $[Q_B, O] = 0$:

$$\langle O \rangle_\beta \equiv \text{Tr} \left( \mathcal{P}_{\text{phys}} e^{-\beta H} O \right) / Z(\beta) = \text{Tr} \left( e^{-\beta H + i \pi N_{gh}} O \right) / Z(\beta), \quad (1.5)$$

where $N_{gh}$ is the ghost number operator, and Tr means the trace operation over all (physical as well as unphysical) states:

$$\text{Tr} O \equiv \sum_{k,l} \langle k | O | l \rangle \eta^{-1}_{kl} \left( \eta_{kl} = \langle k | l \rangle, \quad \sum_l \eta_{kl} \eta^{-1}_{lm} = \delta_{km} \right). \quad (1.6)$$

The partition function $Z(\beta)$ itself is also given by $Z(\beta) \equiv \text{Tr} \left( \mathcal{P}_{\text{phys}} e^{-\beta H} \right) = \text{Tr} \left( e^{-\beta H + i \pi N_{gh}} \right)$. Eq. (1.3) is a consequence of the formula $1 = \mathcal{P}_{\text{phys}} + \{ Q_B, \mathcal{P}_{\text{phys}} \}$ where 1 is the identity operator. We adopt the last expression of eq. (1.5) with statistical weight $e^{-\beta H + i \pi N_{gh}}$ as the definition of $\langle O \rangle_\beta$ even when $O$ is not a BRST invariant quantity.

A framework of finite temperature field theory which is suitable for discussing the KO mechanism is the real-time formalism where we can treat fields with ordinary time variable $t (-\infty < t < \infty)$. There are two formulations of the real-time formalism. One is the path-integral formalism [6], and the other is the operator formalism called thermo field dynamics

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† Detailed explanation of the KO confinement mechanism at zero temperature is found in the original paper [3] (see also ref. [4] for a brief explanation).
The path-integral formalism is convenient for concrete calculations, while TFD is necessary to generalize the KO confinement mechanism, which has been formulated in the BRST operator formalism, to the finite temperature case. In this paper we shall use both of the two formalisms regarding them as equivalent. We use the same symbol for both an operator and the corresponding path-integration variable.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{The time contour $C$.}
\end{figure}

First the path-integral formalism applied to a gauge theory is as follows. We consider thermal Green’s functions generated by

$$Z[j] = \text{Tr} \left\{ e^{-\beta H + i\pi N_{gh} T_C} \left[ \exp i \int_C j \phi \right] \right\}, \quad (1.7)$$

where we have used the abbreviation $\int_C A \equiv \int_C d\tau \int d^3 x A(\tau, x)$ with the contour $C$ in the complex time-plane depicted in fig. 1. and $T_C$ denotes the ordering along the contour $C$. $\phi$ and $j$ represents a generic field operator in the system and the corresponding source. The generating functional $Z[j]$ (1.7) has a path-integral expression [3]:

$$Z[j] = \int_{\text{periodic}} \mathcal{D}\phi \exp \left\{ i \int_C (\mathcal{L}_{YM}(\phi) + j \phi) \right\}, \quad (1.8)$$

where the functional integration $\mathcal{D}\phi \equiv DA_\mu \mathcal{D}c \mathcal{D}\tau \mathcal{D}B$ should be done with periodic boundary condition $\phi(-T - i\beta) = \phi(-T)$ for all the fields $\phi = A_\mu, c, \tau, B$. In particular, the effect of the factor $e^{i\pi N_{gh}}$ in eq. (1.7) is to turn the boundary condition of the fermionic fields $c$ and $\bar{c}$ to the periodic one. Since the contour $C$ of fig. 1 contains the horizontal (real-time) segments $C_1$ and $C_2$ of infinite length (we take the limit $T \to \infty$), we can consider thermal Green’s functions.
with ordinary real time arguments in contrast to the imaginary-time formalism where the contour $C$ is simply a straight vertical line $[0, -i\beta]$.

Thermo field dynamics (TFD) \cite{7, 8} is the operator formalism which reproduces the thermal average of eq. (1.5) as an expectation value with respect to “temperature dependent vacuum” $|0(\beta)\rangle$:

$$\langle \mathcal{O} \rangle_{\beta} = \langle 0(\beta) | \mathcal{O} | 0(\beta) \rangle.$$  \hspace{1cm} (1.9)

Application of TFD to gauge theories is given in ref. \cite{9}. Corresponding to two horizontal segments $C_1$ and $C_2$ of the contour $C$ in fig. \cite{1}, we have to double the fields in TFD as compared to the theory at zero temperature. The fields and states corresponding to $C_1$ is denoted as before by $\phi(t, \mathbf{x})$ and $|k\rangle$, and those corresponding to $C_2$ is denoted with tilde; $\tilde{\phi}(t, \mathbf{x})$ and $\tilde{|k\rangle}$. Then $|0(\beta)\rangle$ is given as

$$|0(\beta)\rangle = Z(\beta)^{-1/2} \sum_{k,l} \exp \left( -\frac{1}{2} \beta H + \frac{i\pi}{2} N_{gh} \right) |k\rangle \otimes \tilde{|l\rangle} \eta_{kl}^{-1}.$$  \hspace{1cm} (1.10)

One can easily see that eq. (1.9) holds for this $|0(\beta)\rangle$. The lagrangian $\tilde{\mathcal{L}}_{YM}$ describing TFD is

$$\tilde{\mathcal{L}}_{YM} = \mathcal{L}_{YM} - \mathcal{L}_{YM},$$  \hspace{1cm} (1.11)

where $\tilde{\mathcal{L}}_{YM}$ is the lagrangian for the tilde fields $\tilde{\phi}$. The physical states in TFD are specified by $Q_B |\text{phys}\rangle = 0$ using the BRST charge of the whole system $Q_B \equiv Q_B - \tilde{Q}_B$. See refs. \cite{9} for precise definitions.

Now we are ready to generalize the KO (de)confinement criterion to the finite temperature Yang-Mills theory. In the finite temperature case based on TFD, the relevant BRST-exact color current has contributions from both the non-tilde and tilde fields (cf. eq. (1.11)):

$$\mathbb{N}_\mu = N_\mu - \tilde{N}_\mu = \{Q_B, K_\mu \},$$  \hspace{1cm} (1.12)

where $\mathbb{N}_\mu$ is not invariant under a separate color rotation on non-tilde or tilde fields generated by $N_\mu$ or $\tilde{N}_\mu$. Using this $\mathbb{N}_\mu$, the KO confinement mechanism is generalized to the finite temperature Yang-Mills system as follows. \textit{If the BRST-exact color current $\mathbb{N}_\mu$ (1.12) contains no massless one-particle component, it implies color confinement: the system contains no colored excitations (quasi-particles) as physical ones.} \footnote{\textup{We are assuming that the asymptotic field like analysis applies also to TFD.}}

In order to study the deconfining transition, we have to first prepare a situation where the confinement is realized at zero temperature ($\beta = \infty$). Although the KO confinement...
criterion has not been shown to hold in real QCD at $\beta = \infty$, we have a toy model of four-dimensional Yang-Mills system where the confinement condition of KO is known to be satisfied at $\beta = \infty$. This toy model is obtained from the real Yang-Mills system (1.1) by taking the limit of vanishing gauge coupling constant; $g_{YM} \to 0$. In this limit, the field strength term, $(1/g_{YM}^2)\text{tr}F_{\mu\nu}^2$, of eq. (1.1) forces the gauge field $A_{\mu}$ constrained to the pure-gauge configuration, $A_{\mu}(x) = g^\dagger(x)\partial_{\mu}g(x)$, and hence the system is reduced to something like a topological field theory described by the BRST-exact lagrangian $-i\delta_B G[A_{\mu} = g^\dagger\partial_{\mu}g]$ \[^{[11]}\]. We call this toy model the pure-gauge model.

Although the pure-gauge model contains no physical degrees of freedom, the KO confinement criterion is still a non-trivial dynamical problem. In naive perturbation theory of both the real Yang-Mills theory and the pure-gauge model at zero temperature, we have a massless pole coupled to $N_{\mu}$. This is revealed by the Green’s function,

$$\int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot x} \langle TN_{\mu}^a(x) A_{\nu}^b(0) \rangle_{\beta = \infty} \sim \delta^{ab} p_{\mu} p_{\nu}/p^2.$$  

(1.13)

However, in the case of pure-gauge model with a special gauge called $OSp(4/2)$ symmetric gauge, we can show the massless pole in (1.13) is in fact missing. This is because the pure-gauge model in 3+1 dimensions is shown to be “equivalent” by the Parisi-Sourlas mechanism \[^{[12]}\] to the chiral model in 1+1 dimensions where the vacuum is realized in disordered phase with a mass gap \[^{[13]}\]. It is expected that, in the real Yang-Mills theory also, the KO confinement condition (at $\beta = \infty$) holds due to large gauge field fluctuation in the direction of local gauge transformation.\[^{[4]}\]

Having completed the preparation, let us turn to the explanation of our analysis of finite temperature case carried out in this paper. Recalling that the pure-gauge model is obtained as the $g_{YM} \to 0$ limit of Yang-Mills theory (1.1), we consider the two-point function

$$\lim_{g_{YM} \to 0} \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot x} \langle TN_{\mu}^a(x) A_{\nu}^b(0) \rangle_{\beta},$$  

(1.14)

in the limit $g_{YM} \to 0$. (In eq. (1.14), $T$ denotes the time-ordering in TFD and the $T_C$-ordering in the path-integral formalism.) We know that (1.14) is free from massless poles at zero temperature $\beta = \infty$, and study whether a massless pole is generated at high temperature $\beta < \beta_c$ with some critical $\beta_c$. For this purpose we observe that in the path-integral formalism (1.8) the $A_{\mu}$-integration is reduced in the $g_{YM} \to 0$ limit to the $g$-integration ($A_{\mu} = g^\dagger\partial_{\mu}g$)

\[^{\S}\] In fact, the color confinement by the KO mechanism is interpretable as a consequence of the *restoration* of local gauge symmetry with transformation parameter $\epsilon(x) \sim a_{\mu}x^\mu$ ($a_{\mu}$: const.) and hence $\delta_\epsilon A_{\mu} \sim a_{\mu}$ \[^{[14]}\].

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and, in addition, that, although \( A_\mu = g^\dagger \partial_\mu g \) has to satisfy the periodic boundary condition \( A_\mu(-T - i\beta, x) = A_\mu(-T, x) \), the group element \( g(t, x) \) need not be strictly periodic. In fact, the boundary conditions using the constant \( SU(N) \) left-transformations, \( g(-T - i\beta, x) = h \cdot g(-T, x) \), are allowed ones which respect the periodicity of \( g^\dagger \partial_\mu g \). Adopting the \( OSp(4/2) \) symmetric gauge of ref. [11] we find that the two-point function (1.14) in 3+1 dimensions is equal to the average of the Green’s function \( \int \frac{d^2p}{(2\pi)^2} e^{ip \cdot x} \langle T A^a_\mu(x) A^b_\nu(0) \rangle_\beta \) over the boundary conditions of \( g(t, x) \) in the 1+1 dimensional chiral model. This implies that color confinement by the KO mechanism breaks down if the corresponding chiral models in 1+1 dimensions are realized in the ordered Nambu-Goldstone phase for a finite range of the boundary conditions.

Since we do not have a systematic non-perturbative methods to analyze finite temperature chiral model in the real-time formalism with unusual boundary conditions, we shall carry out the analysis of the effective potential of the \( O(4) \) non-linear \( \sigma \)-model (which is equivalent to the \( SU(2) \) chiral model) obtained by the large-\( N \) like method. Our analysis shows that there is indeed a desired deconfining transition at \( 1/\beta \sim m \) (\( m \): mass gap at \( \beta = \infty \)). The transition occurs because the infrared singularity which realized the disordered phase at zero temperature is weakened by the non-periodic boundary conditions.

The rest of this paper is organized as follows. In Sec. 2, we introduce the finite temperature pure-gauge model in the real-time formalism as the \( g_{YM} \to 0 \) limit of the Yang-Mills theory, and explain the Parisi-Sourlas reduction to the two-dimensional chiral model. In Sec. 3, we carry out the analysis of the \( O(4) \) non-linear \( \sigma \)-model with generalized boundary conditions. The final section (Sec. 4) is devoted to summary and discussion.

## 2 The pure-gauge model

As stated in the Introduction we shall consider the \( g_{YM} \to 0 \) limit in finite temperature Yang-Mills system in the real-time formalism described by the path-integral (1.8). In this limit the gauge field \( A_\mu \) is restricted to the pure-gauge configuration,

\[
A_\mu(x) = g^\dagger(x) \partial_\mu g(x) \quad \text{with} \quad g(x) \in SU(N), \tag{2.1}
\]
due to the \( (1/g_{YM}^2) \text{tr} F_{\mu\nu}^2 \) term in \( L_{YM} \) and, and the system is reduced to the pure-gauge model (PGM) with dynamical variables \((g, c, \bar{c}, B)\) just as in the zero-temperature case [10, 11]. What is particular to the finite temperature case is the boundary condition of \( g(t, x) \). Although \( A_\mu = g^\dagger \partial_\mu g \) has to satisfy the periodic boundary condition,

\[
A_\mu(-T - i\beta, x) = A_\mu(-T, x), \tag{2.2}
\]
in eq. (1.8), the group element \( g(t,x) \) need not be strictly periodic. Boundary conditions related by a constant \( SU(N) \) left-transformation,

\[
B_h : g(-T - i\beta, x) = h \cdot g(-T, x) \quad (h \in SU(N)),
\]

are allowed ones which respect the periodicity of \( A_\mu = g^\dagger_\mu g \).

Therefore it is natural to assume that in the limit \( g_{YM} \to 0 \) the \( A_\mu \)-integration with periodic boundary condition is reduced to the \( g \)-integration using all the boundary conditions \( B_h \); namely we have to integrate over the boundary condition parameter \( h \):

\[
\int \mathcal{D} A_\mu \quad \to \quad \int dh \int_{B_h} \mathcal{D} g ,
\]

where \( \int \mathcal{D} \) denotes the Haar measure of \( SU(N) \). Then the expectation value of an operator \( \mathcal{O} \) is reduced in the \( g_{YM} \to 0 \) limit to

\[
\langle \mathcal{O} \rangle_\beta \to \int dh \int_{B_h} \mathcal{D} g \mathcal{D} c \mathcal{D} \bar{c} \mathcal{D} B \exp \left( i \int_C \mathcal{L}_{PGM} \right) / \int dh \mathcal{Z}_h ,
\]

where \( \mathcal{L}_{PGM} \) is the lagrangian of PGM,

\[
\mathcal{L}_{PGM} = -i\delta_B G|_{A_\mu = g^\dagger_\mu g} ,
\]

and \( \mathcal{Z}_h \) is the partition function of the PGM with the boundary condition \( B_h \). The BRST transformation on \( g \) is given by \( \delta_B g = gc \). The partition function \( \mathcal{Z}_h \) is shown to be equal to 1 for any \( h \) by reversing the manipulation of eq. (1.5):

\[
\mathcal{Z}_h = \text{Tr} \left( U_h e^{-\beta H + i\pi N h} \right) = \text{Tr} \left( U_h \mathcal{P}_{phys} e^{-\beta H} \right) = 1,
\]

where \( U_h \) is the operator of the left-transformation by \( h \), \( U_h^{-1} g U_h = h \cdot g \). In eq. (2.7) use has been made of the property \( [Q_B, U_h] = 0 \), and the facts that the vacuum \( |0\rangle \) is the only physical state in the PGM (at zero temperature) and that \( U_h |0\rangle = |0\rangle \) for any \( h \). Therefore the expectation value \( \langle \mathcal{O} \rangle_\beta \) in the limit \( g_{YM} \to 0 \) is expressed as the average over \( h \) of the expectation values \( \langle \mathcal{O} \rangle_{\beta,h} \) in the PGM with the boundary condition \( B_h \):

\[
\langle \mathcal{O} \rangle_\beta \to \int dh \frac{1}{\mathcal{Z}_h} \int_{B_h} \mathcal{D} g \mathcal{D} c \mathcal{D} \bar{c} \mathcal{D} B \exp \left( i \int_C \mathcal{L}_{PGM} \right) \equiv \int dh \langle \mathcal{O} \rangle_{\beta,h} .
\]

The integrations over \( (c, \bar{c}, B) \) should be done using periodic boundary condition.

\footnote{We consider for simplicity a system without quark fields. The quark fields do not decouple from the \((g, c, \bar{c}, B)\) system if we impose a non-periodic boundary condition on \( g \).}
The PGM is still not easy to analyze for a general gauge-fixing function \( G[A_\mu, c, \bar{c}, B] \). Fortunately the matter becomes remarkably simple if we adopt what is called the \( OS\!(4/2) \) symmetric gauge \([15]\) given by the gauge-fixing function \( G_{OS}\!p \):

\[
G_{OS\!p} = \frac{2}{\lambda} \delta_A \left\{ \text{tr} \left( A^2_\mu + 2ic\bar{c} \right) \right\},
\]

where \( \lambda \) is the gauge parameter and \( \delta_A \) is the anti-BRST transformation:

\[
\delta_A g = -g\bar{c}, \quad \delta_A \bar{c} = \frac{1}{2}\{\bar{c}, c\}, \quad \delta_A c = -i\overline{B}, \quad \delta_A \overline{B} = 0, \quad \left( \overline{B} \equiv i\{c, \bar{c}\} - B \right).
\]

This is because the action of the PGM is written in a manifestly \( OS\!(4/2) \) symmetric form by introducing the superspace \((x_\mu, \theta, \bar{\theta})\) with Grassmann-odd coordinates \( \theta \) and \( \bar{\theta} \) \([11]\):

\[
S_{OS\!p} = \frac{2i}{\lambda} \int d^4 x \, \delta_A \delta_B \left\{ \text{tr} \left( A^2_\mu + 2ic\bar{c} \right) \right\} = \frac{2i}{\lambda} \int d^4 x \int d\theta d\bar{\theta} \text{tr} \left( \eta^{MN} \partial_M G'^{\dagger} \partial_N G \right),
\]

where the superfield \( G(x, \theta, \bar{\theta}) \) is defined by

\[
G(x, \theta, \bar{\theta}) = \left( 1 + \theta \delta_B + \theta \delta_A + \bar{\theta} \theta \delta_A \delta_B \right) g(x)
= g + \bar{\theta} gc - \theta g\bar{c} + \bar{\theta} \theta g \left( iB + \bar{c} \big\bar{c} \right),
\]

and the superspace metric \( \eta_{MN} \) \((M, N = 0, 1, 2, 3, \theta, \bar{\theta})\) is given as

\[
\eta_{\mu\nu} = \text{diag} \left( 1, -1, -1, -1 \right), \quad \eta_{\theta\bar{\theta}} = -\eta_{\bar{\theta}\theta} = i, \quad \text{others} = 0.
\]

\( OS\!(4/2) \) is the rotation in the superspace \((x_\mu, \theta, \bar{\theta})\) which leaves the metric \( \eta^{MN} \) invariant. The BRST and anti-BRST transformation, \( \delta_B \) and \( \delta_A \), correspond to the translation of \( \bar{\theta} \) and \( \theta \), respectively.

Then, the Parisi-Sourlas dimensional reduction mechanism \([12]\) tells that the PGM of eq. (2.11) in four dimensions is “equivalent” to the chiral model in two dimensions described by the action

\[
S_{\text{chiral}} = -\frac{4\pi}{\lambda} \int d^2 x \, \text{tr} \left( \partial^\mu g'^{\dagger} \partial_\mu g \right).
\]

The equivalence holds also in the present case of the finite temperature system in the real-time formalism with twisted boundary conditions \( B_h \) \([2,3]\) (the proof of ref. \([16]\) which does not rely on perturbation theory applies to the present case): two spatial coordinates \((x_2, x_3)\) and two Grassmann-odd coordinates \((\theta, \bar{\theta})\) in the 3+1 dimensional PGM cancel to leave the 1+1 dimensional chiral model with the same boundary condition. The equivalence implies in particular that \([11]\)

\[
\int d^4 x e^{ip\cdot x} \langle T \bar{N}_\mu^a(x) A^b_\mu(0) \rangle_{\beta, h}^{3+1} = -2\pi \int d^2 x e^{ip\cdot x} \langle T \bar{A}_\mu^a(x) A^b_\mu(0) \rangle_{\beta, h}^{1+1},
\]

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where the LHS (RHS) is the Green’s function in the 3+1 dimensional PGM (1+1 dimensional chiral model) with common $\beta$ and the boundary condition $B_h$. In eq. (2.13) the four-momentum $p_\mu$ and the indices $\mu$ and $\nu$ on the LHS should have components only in the two dimensional part $\mu, \nu = 0, 1$ of the RHS. The reason why $N_\mu$ on the LHS of eq. (2.15) is converted to $A_\mu \equiv A_\mu - \tilde{A}_\mu (\tilde{A}_\mu = \tilde{g}\partial_\mu \tilde{g})$ on the RHS is the relation $N_\mu = -i\delta A_\delta B (A_\mu)$. Note that $A_\mu$ is the Noether current of the $SU(N)$ $R$ symmetry in the chiral model.

From eqs. (2.8) and (2.15), we see that the confinement condition of KO fails if the chiral $SU(N)_R$ current $\overline{A}_\mu$ in the 1+1 dimensional chiral model contains a massless mode for $h$ in a finite range of $SU(N)$. At zero temperature ($\beta = \infty$), the chiral model in 1+1 dimensions is realized in the disordered phase with a mass gap (the system is insensitive to the boundary condition $B_h$ when $\beta = \infty$) and hence the KO confinement condition holds \cite{11}. We would like therefore to know whether the chiral model with a given boundary condition $B_h$ undergoes a phase transition to an ordered phase having a massless Nambu-Goldstone mode coupled to $\overline{A}_\mu$.

3 Analysis of the $SU(2)$ model

Since we do not have a systematic non-perturbative technique to analyze the $SU(N)$ chiral model in the real-time formalism with various boundary conditions $B_h$, we shall carry out the following approximate analysis to the $SU(2)$ chiral model. Note that the $SU(2)$ chiral model is equivalent to the $O(4)$ non-linear $\sigma$-model via the expression $g = \varphi_0 1 + i \sum_{a=1}^{3} \varphi_a \sigma^a (\varphi_0^2 + \varphi_a^2 = 1)$. Rescaling $\varphi_i$ and introducing the multiplier field $\alpha(x)$, let us consider the following $O(4)$ non-linear $\sigma$-model system:

$$L_{O(4)} = \frac{1}{2} \partial^\mu \varphi \cdot \partial_\mu \varphi - \frac{1}{2} \alpha \left( \varphi^2 - \frac{1}{\lambda_0} \right),$$  \hspace{1cm} (3.1)

where the 4 component field $\varphi = (\varphi_i) = (\varphi_0, \ldots, \varphi_3)$ is free from the constraint, and $\lambda_0$ is the bare coupling constant. The boundary condition $B_h$ (2.3) for the $SU(2)$ element $h = \exp(\eta_a \sigma^a / i)$ reads in terms of $\varphi$ as

$$B_h : \varphi_i (-T - i\beta, x) = \left( e^{\eta_a T^a} \right)_{ij} \varphi_j (-T, x),$$

where the $4 \times 4$ matrix $T^a$ ($a = 1, 2, 3$) satisfies the property

$$[T^a, T^b] = 2\epsilon^{abc} T^c, \quad \{T^a, T^b\} = -2\delta^{ab} 1, \hspace{1cm} (3.3)$$
and therefore we have

\[ e^{\eta_a T_a} = \cos \eta \mathbf{1} + \sin \eta T_\eta, \quad (T_\eta)^2 = -1, \quad \left( \eta \equiv \left( \eta^2 \right)^{1/2}, \quad T_\eta \equiv \eta_a T_a / \eta \right). \tag{3.4} \]

The range of \( \eta \) we should consider is \( 0 \leq \eta < 2\pi \).

Our analysis to the model (3.1) is to carry out the large-\( N \) like calculation of the effective potential to determine the phase of the model. Namely, we consider the effective potential which is the sum of the tree part and the one-loop (trace-log) term coming from the \( \varphi \)-integration. Since the large \( N \) expansion is a valid non-perturbative technique for the (ordinary) \( O(N) \) non-linear \( \sigma \)-model, it is expected that our analysis here will give a qualitatively correct result for the present \( O(4) \) model with a generalized boundary condition.

Our question is whether the system (3.1) with the boundary condition (3.2) undergoes a phase transition at \( \beta = \beta_c(\eta) \) from the disordered symmetric phase in the large \( \beta \) region to an ordered phase with massless Nambu-Goldstone modes coupled to the Noether current \( \overline{A}_\mu \) (recall eq. (2.13)). If at some \( \beta \) the \( O(4) \) model is in the Nambu-Goldstone phase for a finite interval of the boundary condition parameter \( \eta_a \), it implies that the KO confinement condition breaks down in the original vanishing \( g_{YM} \) limit of the Yang-Mills theory.

It is straightforward to apply the path-integral formalism of ref. [6] to the \( O(4) \) model of eq. (3.1) with unusual boundary conditions \( B_h \) (3.2). We need the Green’s function \( D_\beta(x - y)_{ij} \) (with the \( O(4) \) indices \( i, j \)) on the contour of fig. 1 satisfying

\[ (-\Box_C - m^2) D_\beta(x - y)_{ij} = \delta_C(x - y) \delta_{ij}, \tag{3.5} \]

and the boundary condition

\[ D_\beta(\tau - i\beta, x)_{ij} = \left( e^{\eta_a T_a} \right)_{ik} D_\beta(\tau, x)_{kj}. \tag{3.6} \]

In eq. (3.5) we have \( \Box_C \equiv (\partial/\partial \tau)^2 - (\partial/\partial x)^2 \) and \( \delta_C \) is the contour \( \delta \)-function [1]. \( D_\beta \) is obtained in the form

\[ D_\beta(x - y) = D_\beta^\geq(x - y) \theta_C(\tau_x - \tau_y) + D_\beta^\leq(x - y) \theta_C(\tau_y - \tau_x), \tag{3.7} \]

with \( D_\beta^\geq \) given (in Fourier-transformed form with respect to the spatial variable) by

\[ D_\beta^\geq(\tau; k_1) = \frac{1}{2i \omega f(\omega)} \left\{ \left[ (1 - e^{-\beta \omega} \cos \eta) \mathbf{1} \mp e^{-\beta \omega} \sin \eta T_\eta \right] e^{i \omega \tau} + \left[ (\cos \eta - e^{-\beta \omega}) \mathbf{1} \pm \sin \eta T_\eta \right] e^{-\beta \omega \pm i \omega \tau} \right\}, \tag{3.8} \]

We do not know whether we can regard the present analysis as the \( N = 4 \) case of the large \( N \) expansion of an \( O(N) \) model since the boundary condition (3.2) is particular to \( N = 4 \) and it breaks explicitly the \( O(3)_L \) symmetry.
where $\omega = |k_1|$ and
\[
f(\omega) \equiv 1 - 2e^{-\beta \omega} \cos \eta + e^{-2\beta \omega}.
\] (3.9)

For any boundary condition $B_h$, the generating functional $Z[j]$ defined by the contour $C$ of fig. 1,
\[
Z[j] = \int_{B_h} D\varphi \int_{\text{periodic}} D\alpha \exp \left\{ i \int_C \left( L_{O(4)} + j \cdot \varphi + J_\alpha \right) \right\},
\] (3.10)

factorizes in the limit $T \to \infty$ to the $C_1C_2$ and $C_3C_4$ parts,
\[
Z[j] = Z[j; C_1C_2] Z[j; C_3C_4].
\] (3.11)

We are interested in $Z[j; C_1C_2]$ with infinite real-time segments. It is expressed as
\[
Z[j; C_1C_2] = \int \prod_{A=1,2} D\varphi_A D\alpha_A \exp \left\{ i \int_{-\infty}^{\infty} dt \int dx \left( \frac{1}{2} \varphi_A \cdot (D^{-1})^{AB} \varphi_B - \frac{1}{2} \alpha_1 \left( \varphi_A^2 - \frac{1}{\lambda} \right) + \frac{1}{2} \alpha_2 \left( \varphi_A^2 - \frac{1}{\lambda} \right) + j_A \cdot \varphi_A + J_A \alpha_A \right) \right\},
\] (3.12)

where we have defined
\[
j_1(t, x) = j(t, x), \quad j_2(t, x) = -j \left( t - \frac{i\beta}{2}, x \right),
\] (3.13)

and similarly for $J_A$. Note that $(\varphi_1, \varphi_2)$ corresponds to $(\varphi, \tilde{\varphi})$ in TFD.

The finite-temperature propagator $D_{AB}^\beta (k_0, k_1)$ is given in momentum space as
\[
D_{AB}^\beta (k_0, k_1) = \begin{pmatrix}
C & S_- \\
S_+ & C
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & \frac{1}{k^2 + i\epsilon}
\end{pmatrix}
\begin{pmatrix}
0 & -1 \\
\frac{1}{k^2 - i\epsilon} & 0
\end{pmatrix}
\begin{pmatrix}
C & S_- \\
S_+ & C
\end{pmatrix}

= \begin{pmatrix}
\frac{1}{k^2 + i\epsilon} - 2\pi i\delta(k^2)S_+ S_- & -2\pi i\delta(k^2)CS_- \\
-2\pi i\delta(k^2)CS_+ & \frac{1}{k^2 - i\epsilon} - 2\pi i\delta(k^2)S_+ S_-
\end{pmatrix},
\] (3.14)

where the $4 \times 4$ matrices $C$ and $S_\pm$ are expressed in the form $a\mathbf{1} + bT_\eta$ and should satisfy
\[
C^2 - S_+ S_- = \mathbf{1},
\]
\[
S_+ S_- = \frac{e^{-\beta |k_0|}}{f(|k_0|)} \left\{ \left( \cos \eta - e^{-\beta |k_0|} \right) \mathbf{1} - \epsilon(k_0) \sin \eta T_\eta \right\},
\] (3.15)
\[
CS_\pm = \pm \frac{e^{-\beta |k_0|}}{f(|k_0|)} \left\{ \left( 1 - e^{\mp \beta |k_0|} \cos \eta \right) \mathbf{1} \mp e^{\mp \beta |k_0|} \sin \eta T_\eta \right\} e^{\mp 1/2 \beta |k_0|} \epsilon(k_0),
\]
with $\epsilon(x) \equiv \text{sign}(x)$. We present here only the explicit expression of $C$:

$$C = \frac{1}{\sqrt{2f(|k_0|)}} \left( u(|k_0|) \ 1 - \frac{e^{-\beta x} \epsilon(k_0)}{u(|k_0|)} \sin \eta \ T_\eta \right),$$

(3.16)

where $u(x) \equiv \left(1 - e^{-\beta x} \cos \eta + \sqrt{f(x)} \right)^{1/2}$. Note that, in the particular cases of $\eta = 0$ and $\eta = \pi$, the propagator (3.14) reduces to the familiar propagator \[6\] for bosons and fermions, respectively. The inverse propagator $\left(D_{\beta}^{-1}\right)^{AB}$ appearing in eq. (3.12) is given by

$$\left(D_{\beta}^{-1}\right)^{AB}(k_0, k_1) = \begin{pmatrix} C & -S_- \\ -S_+ & C \end{pmatrix} \begin{pmatrix} k^2 + i\epsilon & 0 \\ 0 & -k^2 + i\epsilon \end{pmatrix} \begin{pmatrix} C & -S_- \\ -S_+ & C \end{pmatrix}$$

$$= \begin{pmatrix} k^2 + i(C^2 + S_+ S_-) \epsilon & -2iCS_- \epsilon \\ -2iCS_+ \epsilon & -k^2 + i(C^2 + S_+ S_-) \epsilon \end{pmatrix}.$$

(3.17)

Although the boundary conditions $B_h$ breaks explicitly the $O(3)_L$ symmetry except the cases $\eta = 0$ and $\pi$, the effect of the breaking in the inverse propagator appears only at the $i\epsilon$ parts.

As stated at the beginning of this section, we consider the effective potential $V_\beta(\varphi_A, \alpha_A)$ which is the sum of the tree part and the trace-log term coming from the $\varphi$-integration:

$$V_\beta(\varphi_A, \alpha_A) = \frac{1}{2} \alpha_1 \left( \varphi_1^2 - \frac{1}{\lambda_0} \right) - \frac{1}{2} \alpha_2 \left( \varphi_2^2 - \frac{1}{\lambda_0} \right)$$

$$- \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \text{tr} \ln \left[ D_{\beta}^{-1} - \begin{pmatrix} \alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix} \right].$$

(3.18)

The phase of the theory is determined by the stationary condition of $V_\beta(\varphi_A, \alpha_A)$ with respect to $\varphi_A$ and $\alpha_A$ ($A = 1, 2$). The stationary points exist in the subspace $\alpha_1 = \alpha_2$ ($V_\beta$ (3.18) is singular when $\alpha_1 \neq \alpha_2$ \[3\]). Therefore, the conditions to determine the groundstate are

$$\frac{\partial V_\beta}{\partial \varphi_A} \bigg|_{\alpha_1 = \alpha_2 = \alpha} = \alpha \varphi_A = 0,$$

(3.19)

$$(-)^{A+1} \frac{\partial V_\beta}{\partial \alpha_A} \bigg|_{\alpha_1 = \alpha_2 = \alpha} = \frac{1}{2} \left( \varphi_1^2 - \frac{1}{\lambda_0} \right) + \frac{N}{8\pi} \int_{-\Lambda}^{\Lambda} \frac{dk}{\sqrt{k^2 + \alpha}} 1 - e^{-2\beta \sqrt{k^2 + \alpha}} f(\sqrt{k^2 + \alpha}) = 0,$$

(3.20)

where $N = \text{tr}1 = 4$ in the present case, and we have introduced the cut-off $\Lambda$ for the $k$ (spatial momentum) integration. The first condition (3.19) tells that we have either $\alpha = 0$ or $\varphi_A = 0$, and from the second condition (3.20) we see that $\varphi_1^2 = \varphi_2^2$. The second condition (3.20) is rewritten using the mass gap $m$ at zero temperature instead of $\lambda_0$ and $\Lambda$. Note that $m$ is determined by eq. (3.20) with $\beta = \infty$, $\alpha = m^2$ and $\varphi_A = 0$ as

$$\frac{2\pi}{N\lambda_0} = \int_{-\Lambda}^{\Lambda} \frac{dk}{\sqrt{k^2 + m^2}} = \ln\left(\frac{2\Lambda}{m}\right),$$

(3.21)
which implies a familiar formula \( m = 2\Lambda \exp \left( -\frac{2\pi}{N\lambda_0} \right) \). Using eq. (3.21) we have

\[
(-)^{A+1} \frac{\partial}{\partial \alpha_A} V_\beta \bigg|_{\alpha_1=\alpha_2=\alpha} = \frac{1}{2} \varphi_A^2 + \frac{N}{4\pi} K_\beta(\alpha) = 0, 
\]

(3.22)

with \( K_\beta(\alpha) \) defined by

\[
K_\beta(\alpha) \equiv \int_0^\infty dk \left( \frac{1}{\sqrt{k^2 + \alpha}} - \frac{1}{\sqrt{k^2 + m^2}} \right). 
\]

(3.23)

The property of our \( O(4) \) model is qualitatively different between the periodic boundary condition (\( \eta = 0 \)) and the twisted one (\( 0 < \eta < 2\pi \)). We shall discuss the two cases separately below.

### Periodic boundary condition

It is easily seen that eq. (3.22) with \( \eta = 0 \) has no solution with \( \alpha = 0 \) for any \( \beta \) since \( K_\beta(\alpha = 0) \) is positive and infrared divergent (note that \( f(x) = (1 - e^{-x})^2 \) when \( \eta = 0 \)). Therefore we have always \( \varphi_A = 0 \). The expectation value of \( \alpha \), which is the \((\text{mass})^2\) of the \( \varphi \) particles, is determined by eq. (3.22) with \( \eta = 0 \) and \( \varphi_A^2 = 0 \), and it is a monotonically increasing function of the temperature \( 1/\beta \). Therefore, the system with periodic boundary condition is in the disordered phase for any \( \beta \).

### Twisted boundary condition

In this case, there is a critical temperature \( \beta_c(\eta) \) determined by the condition \( K_{\beta=\beta_c(\eta)}(0) = 0 \) or explicitly,

\[
\ln \left( \frac{m\beta_c(\eta)}{2} \right) = \int_0^\infty dx \ln x \cdot \frac{d}{dx} \left( \frac{\cosh(x/2) \sinh(x/2)}{\sinh^2(x/2) + \sin^2(\eta/2)} \right). 
\]

(3.24)

The critical temperature \( \beta_c(\eta) \) as a function of the boundary condition parameter \( \eta \) (\( 0 \leq \eta < 2\pi \)) is depicted in fig. 2 in units of \( \beta_c(\pi) = \pi e^{-\gamma}/m \) (\( \beta_c \) for the anti-periodic boundary condition [17, 18]). As seen from fig. 2, \( \beta_c(\eta) \) is a monotonically increasing function of \( \sin(\eta/2) \) and vanishes only when \( \eta = 0 \). The meaning of \( \beta_c(\eta) \) is that \( K_{\beta}(0) > 0 \) (\(< 0 \)) when \( \beta > \beta_c \) (\( \beta < \beta_c \)) (see fig. 2). The two phases separated by \( \beta_c(\eta) \) are as follows:

i) Low temperature region \( \beta > \beta_c(\eta) \): Eq. (3.22) has no solution of the type \( (\varphi_A \neq 0, \alpha = 0) \) since we have \( K_{\beta}(\alpha = 0) > 0 \) when \( \beta > \beta_c \) (see fig. 2). Therefore, the system is in the disordered phase. The non-vanishing value of \( \alpha \) determined by eq. (3.22) with \( \varphi_A = 0 \), i.e., the intercept of the curve of fig. 2 with the \( \alpha \)-axis, gives the \((\text{mass})^2\) of the \( \varphi \) excitation.

The property of our \( O(4) \) model is qualitatively different between the periodic boundary condition (\( \eta = 0 \)) and the twisted one (\( 0 < \eta < 2\pi \)). We shall discuss the two cases separately below.
Figure 2: The critical temperature $\beta_c(\eta)$ in units of $\beta_c(\pi)$.

ii) High temperature region $\beta < \beta_c(\eta)$: Eq. (3.22) has no solution of the type $(\varphi_A = 0, \alpha > 0)$ since $K_\beta(\alpha)$, which is a monotonically decreasing function of $\alpha$, is negative definite when $\alpha > 0$ (see fig. 3). Therefore the system is realized in the ordered phase with $(\varphi_A \neq 0, \alpha = 0)$. The Noether current of $SU(2)_R$, $A^a_\mu = (A^a_\mu)_{A=1} - (A^a_\mu)_{A=2}$ ($A^a_\mu = \epsilon^{abc}\varphi_0\partial_\mu\varphi_c + \varphi_0\partial_\mu\varphi_a$), is coupled to the Nambu-Goldstone mode irrespectively of the direction of the expectation value $\varphi_A$.

The above results are summarized as follows. When $\beta > \beta_c(\pi)$, our $O(4)$ model is in the disordered phase for all the boundary conditions $B_h$. When $\beta < \beta_c(\pi)$, the models with $\eta$ in the range $\sin(\beta_c^{-1}(\beta)) < \sin(\eta/2) \leq 1$ ($\beta_c^{-1}$ is the inverse function of $\beta_c$) are in the Nambu-Goldstone phase, while the models in the other range of boundary conditions are in the disordered phase. The range of the boundary conditions corresponding to the ordered phase increases as we increase the temperature $1/\beta$. Translating this result back to the original $SU(2)$ PGM in 3+1 dimensions obtained as the vanishing $g_{\text{YM}}$ limit of Yang-Mills theory, the KO confinement condition is satisfied at low temperature $\beta > \beta_c(\pi)$, however it breaks down at high temperature $\beta < \beta_c(\pi)$ due to the contribution of the twisted boundary condition sectors in the average (2.8) with $O = \nabla_\mu A_\mu$. The Green’s function (1.14) develops a massless pole when $\beta < \beta_c(\pi)$.

Some comments are in order. First, we should comment on the compatibility of our result

** The $k$-integration in eq. (3.23) can be continued to the $\alpha < 0$ region, and eq. (3.22) with $\varphi_A = 0$ has a negative $\alpha$ solution even when $\beta < \beta_c$. However, we do not adopt this solution since it implies that the $\varphi$ excitation is tachyonic.
(i.e., that the Nambu-Goldstone phase is realized when $\beta < \beta_c(\eta)$ for the twisted sectors) with Coleman’s theorem which forbids the Nambu-Goldstone bosons in two dimensions. The origin of the absence of the Nambu-Goldstone bosons is that the (ordinary) massless scalar propagator in 1+1 dimensions

$$\int \frac{d^2k}{(2\pi)^2} \frac{e^{ikx}}{k^2 + i\epsilon},$$

(3.25)
does not exist because of infrared divergence. In the present case of the $O(4)$ model with a twisted boundary condition, the massless propagator in momentum space (see eq. (3.14)) takes in the infrared $k_\mu \sim 0$ the following form:

$$D^{AB}_\beta(k) \sim \left( \mathcal{P}\left(\frac{1}{k^2}\right) + i\pi \epsilon(k_0)\delta(k^2) \cot\left(\frac{\eta}{2}\right) T_\eta \right) \frac{i\pi}{1 - \cot\left(\frac{\eta}{2}\right) T_\eta} \epsilon(k_0)\delta(k^2) \left( i + \frac{\eta}{2} T_\eta \right) \epsilon(n_0)\delta(k^2) - \mathcal{P}\left(\frac{1}{k^2}\right) + i\pi \epsilon(k_0)\delta(k^2) \cot\left(\frac{\eta}{2}\right) T_\eta,$$

(3.26)

where $\mathcal{P}$ denotes the principal part. Coleman’s theorem is evaded since the Fourier transform of the RHS of eq. (3.26) does exist:

$$\int \frac{d^2k}{(2\pi)^2} \left( \mathcal{P}\left(\frac{1}{k^2}\right) + i\pi \epsilon(k_0)\delta(k^2) \right) e^{ikx} = \left( -\frac{1}{2} \theta(x^2), \frac{1}{4} \epsilon(x_0)\theta(x^2) \right),$$

(3.27)

where $\theta(x)$ is the step function, $\theta(x) = (\epsilon(x) + 1)/2$. An intuitive reason why the deconfining transition occurs in our model is that the infrared singularity (in the perturbative ordered
phase) which caused the disordered phase at zero temperature is weakened by the twisted boundary conditions. The effect of the boundary condition becomes stronger as we raise the temperature and hence triggers the transition to the ordered phase.

Our second comment is on the imaginary-time formalism. In this paper we have employed the real-time formalism of finite temperature field theory since our interest is in the KO confinement condition, which needs continuous four-momentum and cannot be discussed in the imaginary-time formalism. Forgetting this fact for the moment, let us consider what happens if we adopt the imaginary-time formalism defined by the straight vertical time contour \([0, -i\beta]\) (cf. fig. 1) in the above analysis of the \(O(4)\) non-linear \(\sigma\)-model. Then, for the exactly periodic sector with \(\eta = 0\) we get the same conclusion that \(\langle \alpha \rangle_\beta > 0\) and \(\langle \varphi \rangle_\beta = 0\) for all \(\beta\). For the twisted boundary condition sector, however, we get a completely different result from the real-time formalism: the disordered phase with \(\langle \varphi \rangle_\beta = 0\) is realized for all \(\beta\). This is because in the imaginary-time formalism with a twisted boundary condition, \(\varphi\) has no zero-mode to develop an expectation value. In particular, expanding \(\varphi\) into modes,

\[
\varphi(\tau, x) = \sum_{n=-\infty}^{\infty} \varphi_n(x)e^{i(2n+1)\pi \tau / \beta},
\]

(\(0 \leq \tau \leq \beta\)) for the anti-periodic boundary condition sector (\(\eta = \pi\)), the large \(N\) analysis shows that no modes \(\varphi_n\) can develop expectation values. The expectation value \(\langle \alpha \rangle_\beta\) is a monotonically decreasing function of the temperature \(1/\beta\) and becomes negative for \(\beta < \beta_c(\pi)\). Note that a negative \(\langle \alpha \rangle_\beta\) is not a trouble in this case since the effective (mass gap)\(^2\) is given by \(\langle \alpha \rangle_\beta + (\pi/\beta)^2\), which is seen to be always positive. The discrepancy between real-time and imaginary-time formalisms in the case of twisted boundary conditions may seem strange, but it is not a problem since the twisted sector is not an ordinary statistical mechanics system. This discrepancy, however, is disappointing for an attempt to analyze the original chiral model by the Monte Carlo simulation since it is possible only in the imaginary-time formalism.

4 Summary and discussion

In this paper, as a first step toward the understanding of the deconfining transition in Yang-Mills theory in the sense of the breakdown of the color confinement condition of Kugo and Ojima, we have studied the model obtained by taking the limit of vanishing gauge coupling constant. This model at zero temperature is known to satisfy the KO confinement condition. Adopting a special gauge-fixing function, the system in 3+1 dimensions in the real-time formalism is reduced to a “sum” of chiral models in 1+1 dimensions with various boundary conditions concerning the time contour. As a qualitative approximation to the \(SU(2)\) chiral
model we analyzed the equivalent $O(4)$ non-linear $\sigma$-model using the large-$N$ like analysis. We found that the $O(4)$ model with a twisted boundary condition undergoes a transition to Nambu-Goldstone phase because the infrared singularity is softened by the boundary condition. This implies the breakdown of the KO confinement condition in the 3+1 dimensional model.

The pure-gauge model we considered in this paper, namely the vanishing $g_{\text{YM}}$ limit of the Yang-Mills theory, is physically trivial, and the transition we have found is not the singularity of the free energy for the 3+1 dimensional pure-gauge model. However, since the confinement mechanism of KO has an intimate relationship with the gauge field disorder in the direction of local gauge transformation [14], the breakdown of the KO confinement condition we observed should suggest the deconfining transition in the real Yang-Mills theory.††

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References

[1] A. M. Polyakov, Phys. Lett. 72B (1978) 447.

[2] L. Susskind, Phys. Rev. D20 (1979) 2610.

[3] T. Kugo and I. Ojima, Prog. Theor. Phys. Suppl. 66 (1979) 1.

[4] H. Hata and I. Niigata, Nucl. Phys. B389 (1993) 133.

[5] H. Hata and T. Kugo, Phys. Rev. D21 (1980) 3333.

[6] A. J. Niemi and G. W. Semenoff, Ann. Phys. 152 (1984) 105.

[7] Y. Takahashi and H. Umezawa, Collective Phenomena 2 (1975) 55.

[8] H. Umezawa, H. Matsumoto and M. Tachiki, “Thermo Field Dynamics and Condensed States” (North-Holland, Amsterdam, 1982).

[9] I. Ojima, Ann. Phys. 137 (1981) 1.

†† See ref. [19] for an attempt to show the confinement by the KO mechanism in the real Yang-Mills theory on the basis of the pure-gauge model.
[10] H. Hata, *Phys. Lett.* B143 (1984) 171.

[11] H. Hata and T. Kugo, *Phys. Rev.* D32 (1985) 938.

[12] G. Parisi and N. Sourlas, *Phys. Rev. Lett.* 43 (1979) 744.

[13] A. Polyakov and P. B. Wiegmann, *Phys. Lett.* B131 (1983) 121; P. B. Wiegmann, *Phys. Lett.* B141 (1984) 217; B142 (1984) 173.

[14] H. Hata, *Prog. Theor. Phys.* 67 (1982) 1607; 69 (1983) 1524.

[15] R. Delbourgo and P. D. Jarvis, *J. Phys.* A15 (1982) 611

[16] J. L. Cardy, *Phys. Lett.* 125B (1983) 470.

[17] L. Dolan and R. Jackiw, *Phys. Rev.* D9 (1974) 3320.

[18] L. Jacobs, *Phys. Rev.* D10 (1974) 3956; B. Harrington and A. Yildiz, *Phys. Rev.* D11 (1975) 1499.

[19] K. I. Izawa, *Prog. Theor. Phys.* 90 (1993) 911.