On 3-crossed modules of algebras

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Abstract

In this paper we define 3-crossed modules for commutative (Lie) algebras and investigate the relation between this construction and the simplicial algebras. Also we define the projective 3-crossed resolution for investigate a higher dimensional homological information and show the existence of this resolution for an arbitrary $k$-algebra.

Keywords: crossed module, 2-crossed module, simplicial algebra, Moore complex.

1 Introduction

As an extension of crossed modules (Whitehead) and 2-crossed modules (Conduché); Arvasi, Kuzpınarı and Uslu in [4], defined 3-crossed modules as a model for homotopy 4-types. Kan, in [17], also proved that simplicial groups are algebraic models for homotopy types. It is known from [3, 14, 23] that simplicial algebras with Moore complex of length 1, 2 lead to crossed module and 2-crossed modules which are related to Koszul complex and Andre-Quillen homology constructions for use in homotopical and homological algebra.

PJ.L.Doncel, A.R. Grandjean and M.J.Vale in [10] extent the 2-crossed modules of groups to commutative algebras. As in indicated in [10], they defined a homology theory and obtain the relation with Andre-Quillen homology for $n = 0, 1, 2$. This homology theory includes the projective 2-crossed resolution and the homotopy operator given in [15]. Of course these results based on the work of T.Porter [23], which involves the relation between Koszul complex and the Andre-Quillen homology by means of free crossed modules of commutative algebra. In this vein, we hope that it would be possible to generalise these results by using commutative algebra case of higher dimensional crossed algebraic gadgets.

The present work involves the relation between 3-crossed modules and simplicial algebra without details since the most calculation are same as group case given in [4]. Furthermore the work involves the existence of projective 3-crossed resolution of a $k$-algebra to obtain an higher dimensional homological information about commutative algebras. Here the construction is a bit different from the 2-crossed resolution given in [10] because of the number of Peiffer liftings. At the end of the work we give the Lie algebra 3-crossed modules.

The main results of this work are:

1. Introduce the notion of 3-crossed modules of commutative algebras and Lie algebras;

2. Construct the passage from 3-crossed modules of algebras to simplicial algebras and the converse passage as an analogue result given in [4];

3. Define the projective 3-crossed resolution for investigate a higher dimensional homological information and show the existence of this resolution for an arbitrary $k$-algebra which was shown for two crossed modules in [10].
2 Preliminaries

In this work \( k \) will be a fixed commutative ring with identity 1 not equal to zero and all algebras will be commutative \( k \)-algebras, we accept they are not required to have the identity 1.

2.1 Simplicial Algebras

See [20], [9] for most of the basic properties of simplicial structures.

A simplicial algebra \( E \) consists of a family of algebras \( \{E_n\} \) together with face and degeneracy maps \( d^i_n : E_n \to E_{n-1}, 0 \leq i \leq n, (n \neq 0) \) and \( s^i_n : E_{n-1} \to E_n, 0 \leq i \leq n \), satisfying the usual simplicial identities given in [1], [16]. The category of simplicial algebras will be denoted by \( \text{SimpAlg} \).

Let \( \Delta \) denotes the category of finite ordinals. For each \( k \geq 0 \) we obtain a subcategory \( \Delta_{\leq k} \) determined by the objects \([i]\) of \( \Delta \) with \( i \leq k \). A \( k \)-truncated simplicial algebra is a functor from \( \Delta_{\leq k}^{op} \) to \( \text{Alg} \) (the category of algebras). We will denote the category of \( k \)-truncated simplicial algebras by \( \text{Tr}_k \text{SimpAlg} \). By a \( k \)-truncation of a simplicial algebra, we mean a \( k \)-truncated simplicial algebra \( \text{tr}_k E \) obtained by forgetting dimensions of order \( > k \) in a simplicial algebra \( E \).

Then we have the adjoints situations

\[
\text{SimpAlg} \xrightarrow{\text{tr}_k} \text{Tr}_k \text{SimpAlg} \cong \xleftarrow{\text{cost}_k} \text{SimpAlg}
\]

where \( \text{st}_k \) and \( \text{cost}_k \) are called the \( k \)-skeleton and the \( k \)-coskeleton functors respectively. For detailed definitions see [11].

2.2 The Moore Complex.

The Moore complex \( NE \) of a simplicial algebra \( E \) is defined to be the normal chain complex \((NE, \partial)\) with

\[
NE_n = \bigcap_{i=0}^{n-1} \ker d_i
\]

and with differential \( \partial_n : NE_n \to NE_{n-1} \) induced from \( d_n \) by restriction.

We say that the Moore complex \( NE \) of a simplicial algebra \( E \) is of \textit{length} \( k \) if \( NE_n = 0 \) for all \( n \geq k + 1 \). We denote the category of simplicial algebras with Moore complex of length \( k \) by \( \text{SimpAlg}_{\leq k} \).

The Moore complex, \( NE \), carries a hypercrossed complex structure (see Carrasco [5]) from which \( E \) can be rebuilt. Now we will have a look to this construction slightly. The details can be found in [5].

2.3 The Poset of Surjective Maps

The following notation and terminology is derived from [5].

For the ordered set \([n] = \{0 < 1 < \cdots < n\}\), let \( \alpha^n_i : [n+1] \to [n] \) be the increasing surjective map given by:

\[
\alpha^n_i(j) = \begin{cases} j & \text{if } j \leq i, \\ j - 1 & \text{if } j > i. \end{cases}
\]

Let \( S(n, n-r) \) be the set of all monotone increasing surjective maps from \([n]\) to \([n-r]\). This can be generated from the various \( \alpha^n_i \) by composition. The composition of these generating maps is subject to the following rule: \( \alpha_j \alpha_i = \alpha_{i-1} \alpha_j, j < i \). This implies that every element \( \alpha \in S(n, n-r) \) has a unique expression as \( \alpha = \alpha_{i_1} \circ \alpha_{i_2} \circ \cdots \circ \alpha_{i_r} \) with \( 0 \leq i_1 < i_2 < \cdots < i_r \leq n - 1 \), where the indices \( i_k \) are the elements of \([n]\) such that \( \{i_1, \ldots, i_r\} = \{i : \alpha(i) = \alpha(i + 1)\} \). We thus can identify \( S(n, n-r) \) with the set \( \{(i_r, \ldots, i_1) : 0 \leq i_1 < i_2 < \cdots < i_r \leq n - 1\} \). In particular, the
single element of $S(n, n)$, defined by the identity map on $[n]$, corresponds to the empty 0-tuple ( ) denoted by $\emptyset_n$. Similarly the only element of $S(n, 0)$ is $(n - 1, n - 2, \ldots, 0)$. For all $n \geq 0$, let

$$S(n) = \bigcup_{0 \leq r \leq n} S(n, n-r).$$

We say that $\alpha = (i_r, \ldots, i_1) < \beta = (j_r, \ldots, j_1)$ in $S(n)$ if $i_1 = j_1, \ldots, i_k = j_k$ but $i_{k+1} > j_{k+1}$, $(k \geq 0)$ or if $i_1 = j_1, \ldots, i_r = j_r$ and $r < s$. This makes $S(n)$ an ordered set. For example

- $S(2) = \{\phi_2 < (1) < (0) < (1,0)\}$
- $S(3) = \{\phi_3 < (2) < (1) < (2,1) < (0) < (2,0) < (1,0) < (2,1,0)\}$
- $S(4) = \{\phi_4 < (3) < (2) < (3,2) < (1) < (3,1) < (2,1) < (3,2,1) < (0) < (3,0) < (2,0) < (3,2,1,0)\}$

## 2.4 The Semidirect Decomposition of a Simplicial Algebra

The fundamental idea behind this can be found in Conduché [3]. A detailed investigation of this for the case of simplicial groups is given in Carrasco and Cegarra [6]. The algebra case of the structure is also given in [5].

**Proposition 1** If $E$ is a simplicial algebra, then for any $n \geq 0$

$$E_n \cong (\ldots (NE_n \rtimes s_{n-1}NE_{n-1}) \times \ldots \times s_{n-2} \ldots s_0 NE_1) \rtimes ((s_0 NE_2 \rtimes s_2 s_0 NE_1) \rtimes (s_1 s_0 NE_1 \times s_2 s_1 s_0 NE_0)).$$

**Proof.** This is by repeated use of the following lemma. ■

**Lemma 2** Let $E$ be a simplicial algebra. Then $E_n$ can be decomposed as a semidirect product:

$$E_n \cong \ker d_n^n \rtimes s_{n-1}^{n-1}(E_{n-1}).$$

The bracketing and the order of terms in this multiple semidirect product are generated by the sequence:

- $E_1 \cong NE_1 \rtimes s_0 NE_0$
- $E_2 \cong (NE_2 \rtimes s_1 NE_1) \rtimes (s_0 NE_1 \rtimes s_1 s_0 NE_0)$
- $E_3 \cong ((NE_3 \rtimes s_2 NE_2) \rtimes (s_1 NE_2 \rtimes s_2 s_1 NE_1)) \rtimes ((s_0 NE_2 \rtimes s_2 s_0 NE_1) \rtimes (s_1 s_0 NE_1 \times s_2 s_1 s_0 NE_0)).$

and

$$E_4 \cong (((NE_4 \rtimes s_3 NE_3) \rtimes (s_2 NE_3 \rtimes s_3 s_2 NE_2)) \rtimes ((s_1 NE_3 \times s_3 s_1 NE_2) \rtimes (s_2 s_1 NE_2 \rtimes s_3 s_2 s_1 NE_1))) \rtimes s_0 (\text{decomposition of } E_3).$$

Note that the term corresponding to $\alpha = (i_r, \ldots, i_1) \in S(n)$ is

$$s_\alpha(NE_{n-\#} \alpha) = s_{i_r \ldots i_1}(NE_{n-\#} \alpha) = s_{i_r \ldots s_1}(NE_{n-\#} \alpha),$$

where $\# \alpha = r$. Hence any element $x \in E_n$ can be written in the form

$$x = y + \sum_{\alpha \in S(n) \setminus \{\emptyset_n\}} s_\alpha(x) \text{ with } y \in NE_n \text{ and } x_\alpha \in NE_{n-\# \alpha}.$$
2.5 Hypercrossed Complex Pairings

In the following we recall from \cite{3} hypercrossed complex pairings for commutative algebras. The fundamental idea behind this can be found in Carrasco and Cegarra (cf. \cite{6}). The construction depends on a variety of sources, mainly Conduché \cite{3}, Z. Arvasi and T. Porter, \cite{8}. Define a set \( P(n) \) consisting of pairs of elements \((\alpha, \beta)\) from \( S(n) \) with \( \alpha \cap \beta = \emptyset \) and \( \beta < \alpha \), with respect to lexicographic ordering in \( S(n) \) where \( \alpha = (i_1, \ldots, i_n), \beta = (j_1, \ldots, j_n) \in S(n) \). The pairings that we will need,

\[
\{ C_{\alpha, \beta} : NE_{n-\sharp \alpha} \otimes NE_{n-\sharp \beta} \to NE_n : (\alpha, \beta) \in P(n), n \geq 0 \}
\]

are given as composites by the diagram

\[
\begin{array}{ccc}
NE_{n-\sharp \alpha} \otimes NE_{n-\sharp \beta} & \xrightarrow{C_{\alpha, \beta}} & NE_n \\
\downarrow{s_{\alpha} \otimes s_{\beta}} & & \downarrow{p} \\
E_n & \xrightarrow{\mu} & E_n
\end{array}
\]

where \( s_{\alpha} = s_{i_1}, \ldots, s_{i_n} : NE_{n-\sharp \alpha} \to E_n \), \( s_{\beta} = s_{j_1}, \ldots, s_{j_n} : NE_{n-\sharp \beta} \to E_n \), \( p : E_n \to NE_n \) is defined by composite projections \( p(x) = p_{n-1} \ldots p_0(x) \), where \( p_j(z) = z s_j d_j(z)^{-1} \) with \( j = 0, 1, \ldots, n-1 \). \( \mu : E_n \otimes E_n \to E_n \) is given by multiplication map and \( \sharp \alpha \) is the number of the elements in the set of \( \alpha \), similarly for \( \sharp \beta \). Thus

\[
C_{\alpha, \beta}(x_\alpha \otimes y_\beta) = p\mu(s_{\alpha} \otimes s_{\beta})(x_\alpha \otimes y_\beta) = p(s_{\alpha}(x_\alpha) \otimes s_{\beta}(y_\beta)) = (1 - s_{n-1}d_{n-1}) \ldots (1 - s_0d_0)(s_{\alpha}(x_\alpha)s_{\beta}(y_\beta))
\]

Let \( I_n \) be the ideal in \( E_n \) generated by elements of the form

\[
C_{\alpha, \beta}(x_\alpha \otimes y_\beta)
\]

where \( x_\alpha \in NE_{n-\sharp \alpha} \) and \( y_\beta \in NE_{n-\sharp \beta} \).

We illustrate this for \( n = 3 \) and \( n = 4 \) as follows:

For \( n = 3 \), the possible Peiffer pairings are the following

\[
C(1,0)(2), C(2,0)(1), C(0,2,1), C(2,0), C(2,1), C(1,0)
\]

For all \( x_1 \in NE_1, y_2 \in NE_2 \), the corresponding generators of \( I_3 \) are:

\[
\begin{align*}
C(1,0)(2)(x_1 \otimes y_2) &= (s_1s_0x_1 - s_2s_0x_1)s_2y_2, \\
C(2,0)(1)(x_1 \otimes y_2) &= (s_2s_0x_1 - s_2s_1x_1)(s_1y_2 - s_2y_2) \\
C(0,2,1)(x_2 \otimes y_1) &= s_2s_1x_2(s_0y_1 - s_1y_1 + s_2y_1) \\
C(1,0)(0)(x_2 \otimes y_2) &= [s_1x_2(s_0y_2 - s_1y_2) + s_2(x_2y_2), \\
C(2,0)(0)(x_2 \otimes y_2) &= (s_2x_2)(s_0y_2), \\
C(2,1)(x_2 \otimes y_2) &= s_2x_2(s_1y_2 - s_2y_2).
\end{align*}
\]

For \( n = 4 \), the key pairings are thus the following

\[
\begin{align*}
C(3,2,1)(0), \ C(3,2,0)(1), \ C(3,1,0)(2), \ C(2,1,0)(3), \ C(3,0)(2,1), \\
C(3,1,2)(0), \ C(3,2,1)(0), \ C(3,2)(1), \ C(3,0)(2), \ C(3,1)(0), \\
C(0,2,1)(0), \ C(3,1,2)(0), \ C(2,1,3)(0), \ C(3,0)(2), \ C(3,1)(0), \\
C(2,0,3)(3), \ C(2,0,1)(3), \ C(1,0,3)(3), \ C(1,0)(2), \ C(3,2)(0), \\
C(3,1)(3), \ C(3,0)(0), \ C(2,1)(0), \ C(2,0)(0), \ C(1,0)(0).
\end{align*}
\]
Theorem 3 (13) Let $E$ be a simplicial algebra with Moore complex $NE$ in which $E_n = D_n$, is an ideal of $E_n$ generated by the degenerate elements in dimension $n$, then

$$\partial_n(NE_n) = \sum_{I,J} [K_I, K_J]$$

for $I, J \subseteq [n-1]$ with $I \cup J = [n-1]$, $I = [n-1] - \{\alpha\}$ $J = [n-1] - \{\beta\}$ where $(\alpha, \beta) \in P(n)$ for $n = 2, 3$ and $4$.

Remark 4 Shortly in [21] they defined the normal subgroup $\partial_n(NG_n \cap D_n)$ by $F_{\alpha,\beta}$ elements which were defined first by Carrasco in [5]. Castiglioni and Ladra generalised this inclusion in [7].

Following [21] we have

Lemma 5 Let $E$ be a simplicial algebra with Moore complex $NE$ of length 3. Then for $n = 4$ the images of $C_{\alpha,\beta}$ elements under $\partial_4$ given in Table 1 are trivial.

Proof. Since $NG_4 = 1$ by the results in [3] result is trivial.
\begin{align*}
1. \quad d_4(C_{(3,2,1)}(0)(x_1 \otimes y_3)) &= s_2s_1x_1(s_0d_3y_3 - s_1d_3y_3 + s_2d_3y_3 - y_3) \\
2. \quad d_4(C_{(3,2,0)}(1)(x_1 \otimes y_3)) &= (s_1s_0x_1 - s_2s_1x_1)(s_1d_3y_3 - s_2d_3y_3 + y_3) \\
3. \quad d_4(C_{(3,1,0)}(2)(x_1 \otimes y_3)) &= (s_1s_0x_1 - s_2s_0x_1)(s_2d_3y_3 - y_3) \\
4. \quad d_4(C_{(2,1,0)}(3)(x_1 \otimes y_3)) &= (s_2s_1s_0d_1x_1 - s_1s_0x_1)y_3 \\
5. \quad d_4(C_{(3,2,1)}(1,0)(x_2 \otimes y_2)) &= (s_1s_0d_2x_2 - s_2s_0d_2x_2 - s_0x_2)s_3y_2 \\
6. \quad d_4(C_{(3,1,0)}(2,0)(x_2 \otimes y_2)) &= (s_1s_2x_1 - s_0x_2 + s_2s_0d_2x_2 - s_2s_1d_2x_2)(s_1y_2 - s_2y_2) \\
7. \quad d_4(C_{(3,0,0)}(2,1)(x_2 \otimes y_2)) &= (s_2s_1d_2x_2 - s_1x_2)(s_0y_2 - s_1y_2 + s_2y_2) \\
8. \quad d_4(C_{(3,2,1)}(1,1)(x_2 \otimes y_2)) &= s_2x_2(s_1d_3y_3 - s_2d_3y_3 - y_3) \\
9. \quad d_4(C_{(3,2,0)}(1)(x_2 \otimes y_3)) &= s_2x_2(s_2d_3y_3 - s_1d_3y_3 + s_0d_3y_3 - y_3) \\
10. \quad d_4(C_{(3,1,1)}(2)(x_2 \otimes y_3)) &= (s_1x_2 - s_2x_2)(s_2d_3y_3 - y_3) \\
11. \quad d_4(C_{(3,1,0)}(3)(x_2 \otimes y_3)) &= (s_1x_2 - s_2x_2)(s_2d_3y_3 - s_1d_3y_3 + s_0d_3y_3 - y_3) \\
12. \quad d_4(C_{(2,0,0)}(2)(x_2 \otimes y_3)) &= (s_0x_2 - s_1x_2 + s_2x_2)(s_2d_3y_3 - y_3) \\
13. \quad d_4(C_{(2,0,1)}(3)(x_2 \otimes y_3)) &= (s_0x_2 - s_1x_2 + s_2x_2)(s_1d_3y_3 - s_2d_3y_3 + y_3) \\
14. \quad d_4(C_{(2,1,0)}(3)(x_2 \otimes y_3)) &= (s_2s_1d_2x_2 - s_1x_2)y_3 \\
15. \quad d_4(C_{(0,0,0)}(2,1)(x_2 \otimes y_3)) &= (s_2s_1d_2x_2 - s_1x_2)(s_2d_3y_3 - s_1d_3y_3 + s_0d_3y_3 - y_3) \\
16. \quad d_4(C_{(2,0,0)}(3)(x_2 \otimes y_3)) &= (s_2s_0d_2x_2 - s_0x_2 + s_1x_2 - s_1s_2d_2x_2)y_3 \\
17. \quad d_4(C_{(2,0,1)}(1,0)(x_2 \otimes y_3)) &= (s_2s_0d_2x_2 - s_0x_2 + s_1x_2 - s_2s_1d_2x_2) \\
&\quad (s_1d_3y_3 - s_2d_3y_3 + y_3) \\
18. \quad d_4(C_{(0,1,0)}(3)(x_2 \otimes y_3)) &= (s_2s_0d_2x_2 - s_0x_2 - s_1s_0d_0x_2)y_3 \\
19. \quad d_4(C_{(0,1,0)}(1,0)(x_2 \otimes y_3)) &= (s_1s_0d_2x_2 - s_2s_0d_2x_2 + s_0x_2)(s_2d_3y_3 - y_3) \\
20. \quad d_4(C_{(3,0,1)}(2)(x_3 \otimes y_3)) &= x_3(s_2d_3y_3 - y_3) \\
21. \quad d_4(C_{(3,1,0)}(1)(x_3 \otimes y_3)) &= x_3(s_1d_3y_3 - s_2d_3y_3 + y_3) \\
22. \quad d_4(C_{(3,0,0)}(1)(x_3 \otimes y_3)) &= x_3(s_2d_3y_3 - s_1d_3y_3 + s_0d_3y_3 - y_3) \\
23. \quad d_4(C_{(2,1,0)}(1)(x_3 \otimes y_3)) &= (s_2d_3x_3 - x_3)(s_1d_3y_3 - s_2d_3y_3 + y_3) \\
24. \quad d_4(C_{(2,0,0)}(1)(x_3 \otimes y_3)) &= (s_2d_3x_3 - x_3)(s_2d_3y_3 - s_1d_3y_3 + s_0d_3y_3 - y_3) \\
25. \quad d_4(C_{(1,0,0)}(1)(x_3 \otimes y_3)) &= (s_1d_3x_3 - s_2d_3x_3 + x_3)(s_2d_3y_3 - s_1d_3y_3 + s_0d_3y_3 - y_3)
\end{align*}

Table 1

where $x_3, y_3 \in NG_3, x_2, y_2 \in NG_2, x_1 \in NG_1.$
2.6 Crossed Modules

Here we will recall the notion of crossed modules of commutative algebras given in [23] and [12].

Let \( R \) be a \( k \)-algebra with identity. A crossed module of commutative algebra is an \( R \)-algebra \( C \), together with a commutative action of \( R \) on \( C \) and \( R \)-algebra morphism \( \partial : C \rightarrow R \) together with an action of \( R \) on \( C \), written \( r \cdot c \) for \( r \in R \) and \( c \in C \), satisfying the conditions.

\[ \text{CM1)} \quad \text{for all } r \in R, c \in C \quad \partial(r \cdot c) = r \partial c \]

\[ \text{CM2)} \quad \text{(Peiffer Identity) for all } c, c' \in C \quad \partial c \cdot c' = cc' \]

We will denote such a crossed module by \((C,R,\partial)\).

A morphism of crossed module from \((C,R,\partial)\) to \((C',R',\partial')\) is a pair of \( k \)-algebra morphisms \( \phi : C \rightarrow C' \), \( \psi : R \rightarrow R' \) such that \( \phi(r \cdot c) = \psi(r) \cdot \phi(c) \) and \( \partial' \phi(c) = \psi \partial(c) \).

We thus get a category \( \text{XMod} \) of crossed modules.

**Examples of Crossed Modules**

(i) Any ideal, \( I \), in \( R \) gives an inclusion map \( I \rightarrow R \), which is a crossed module then we will say \((I,R,i)\) is an ideal pair. In this case, of course, \( R \) acts on \( I \) by multiplication and the inclusion homomorphism \( i \) makes \((I,R,i)\) into a crossed module, an “inclusion crossed modules”.

Conversely,

**Lemma 6** If \((C,R,\partial)\) is a crossed module, \( \partial(C) \) is an ideal of \( R \).

(ii) Any \( R \)-module \( M \) can be considered as an \( R \)-algebra with zero multiplication and hence the zero morphism \( 0 : M \rightarrow R \) sending everything in \( M \) to the zero element of \( R \) is a crossed module. Again conversely:

**Lemma 7** If \((C,R,\partial)\) is a crossed module, \( \ker \partial \) is an ideal in \( C \) and inherits a natural \( R \)-module structure from the \( R \)-action on \( C \). Moreover, \( \partial(C) \) acts trivially on \( \ker \partial \), hence \( \ker \partial \) has a natural \( R/\partial(C) \)-module structure.

As these two examples suggest, general crossed modules lie between the two extremes of ideal and modules. Both aspects are important.

(iii) In the category of algebras, the appropriate replacement for automorphism groups is the multiplication algebra defined by Mac Lane [19]. Then automorphism crossed module correspond to the multiplication crossed module \((R,M(R),\mu)\).

To see this crossed module, we need to assume \( \text{Ann}(R) = 0 \) or \( R^2 = R \) and let \( M(R) \) be the set of all multipliers \( \delta : R \rightarrow R \) such that for all \( c, c' \in C \), \( \delta(rr') = \delta(r)r' \). \( M(R) \) acts on \( R \) by

\[ M(R) \times R \rightarrow R \]

\[ (\delta, r) \mapsto \delta(r) \]

and there is a morphism \( \mu : R \rightarrow M(R) \) defined by \( \mu(r) = \delta_r \), with \( \delta_r(r') = rr' \) for all \( r, r' \in R \).

2.7 2-Crossed Modules

Now we recall the commutative algebra case of 2-crossed modules due to A.R.Grandjean and Vale, [14].

A 2-crossed module of \( k \)-algebras is a complex

\[ C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \]
of $C_2$-algebras with $\partial_2, \partial_1$ morphisms of $C_0$-algebras, where $C_0$ acts on $C_0$ by multiplication, with a bilinear function
\[
\{ \otimes \} : C_1 \otimes C_0 C_1 \to C_2
\]
called as Peiffer lifting which satisfies the following axioms:

1. \[\partial_2 \{ y_0 \otimes y_1 \} = y_0 y_1 - \partial_1(y_1) y_0\]
2. \[\{ \partial_2(x_1) \otimes \partial_2(x_2) \} = x_1 x_2\]
3. \[\{ y_0 \otimes y_1 y_2 \} = \{ y_0 y_1 \otimes y_2 \} + \partial_1 y_2 \{ y_0 \otimes y_1 \}\]
4. \[\{ \partial_2(x) \otimes y \} = y \cdot x - \partial_1(y) x\]
5. \[\{ y \otimes \partial_2(x) \} = y \cdot x\]
6. \[\{ z y_0 \otimes y_1 \} = \{ z y_0 \otimes y_1 \} = \{ y_0 \otimes z y_1 \}\]

for all $x, x_1, x_2 \in C_2$, $y, y_0, y_1, y_2 \in C_1$ and $z \in C_0$. A morphism of 2-crossed modules can be defined in an obvious way. We thus define the category of 2-crossed modules denoting it by $X_2\text{Mod}$.

The proof of the following theorem can be found in [3].

**Theorem 8** The category of 2-crossed modules is equivalent to the category of simplicial algebras with Moore complex of length 2.

Now we will give some remarks on 2-crossed modules where the group case can be found in [22].

1) Let $C_1 \xrightarrow{\partial_1} C_0$ be a crossed module. If we take $C_2$ trivial then
\[C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0\]
is a 2-crossed module with the Peiffer lifting defined by $\{ x \otimes y \} = 0$ for $x, y \in C_1$.

2) If
\[C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0\]
is a 2-crossed module then
\[C_1 \xrightarrow{\partial_1} C_0\]
is a crossed module.

3) Let
\[C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0\]
be a 2-crossed module with trivial Peiffer lifting then $C_1 \xrightarrow{\partial_1} C_0$ will be a crossed module. Also in this situation we have the trivial action of $C_0$ on $C_2$.

## 3 Three Crossed Modules

As a consequence of [4], here we will define 3-crossed modules of commutative algebras. The way is similar but some of the conditions are different.

Let $E$ be a simplicial algebra with Moore complex of length 3 and $NE_0 = C_0$, $NE_1 = C_1$, $NE_2 = C_2$, $NE_3 = C_3$. Thus we have a $k$-algebra complex
\[C_3 \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0\]
Let the actions of $C_0$ on $C_3$, $C_2$, $C_1$, $C_1$ on $C_2$, $C_3$ and $C_2$ on $C_3$ be as follows:
\[
\begin{align*}
[x_0 x_1] & = s_0 x_0 x_1 \\
[x_0 x_2] & = s_1 s_0 x_2 \\
[x_0 x_3] & = s_2 s_1 s_0 x_2 \\
[x_1 x_2] & = s_1 x_1 x_2 \\
[x_1 x_3] & = s_2 s_1 x_2 \\
x_2 \cdot x_3 & = s_2 x_2 x_3
\end{align*}
\]
Then, since

\[
\begin{align*}
(s_2s_1s_0 \partial_1 x_1 - s_1s_0 x_1) y_3 &= 0 \\
(s_2s_1 \partial_2 x_2 - s_1 x_2) y_3 &= 0 \\
x_3(s_2 \partial_3 y_3 - y_3) &= 0
\end{align*}
\]

we get

\[
\begin{align*}
\partial_1 x_1 y_3 &= s_1 s_0 x_1 y_3 \\
\partial_2 x_2 y_3 &= s_1 x_2 y_3 \\
\partial_3 x_3 y_3 &= x_3 y_3
\end{align*}
\]

and using the simplicial identities we get,

\[
\partial_3(x_2 \cdot x_3) = \partial_3(s_2 x_2 x_3) = \partial_3(s_2 x_2) \partial_3(x_3) = x_2 \partial_3(x_3)
\]

Thus \( \partial_3 : C_3 \to C_2 \) is a crossed module.

**Definition 9** Let \( C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \) be a complex of \( k \)-algebras defined above. We define Peiffer liftings as follows:

\[
\begin{align*}
\{ \otimes \} & : C_1 \otimes C_1 & \to & C_2 \\
\{ x_1 \otimes y_1 \} & = s_1 x_1 (s_0 y_1 - s_1 y_1) \\
\{ \otimes \}_{(1,0)(2)} & : C_1 \otimes C_2 & \to & C_3 \\
\{ x_1 \otimes y_2 \}_{(1,0)(2)} & = (s_1 s_0 x_1 - s_2 s_0 x_1) s_2 y_2 \\
\{ \otimes \}_{(2,0)(1)} & : C_1 \otimes C_2 & \to & C_3 \\
\{ x_1 \otimes y_2 \}_{(2,0)(1)} & = (s_2 s_0 x_1 - s_2 s_1 x_1)(s_1 y_2 - s_2 y_2) \\
\{ \otimes \}_{(0)(2,1)} & : C_1 \otimes C_2 & \to & C_3 \\
\{ x_1 \otimes y_2 \}_{(0)(2,1)} & = s_2 s_1 x_1 (s_0 y_2 - s_1 y_2 + s_2 y_2) \\
\{ \otimes \}_{(1)(0)} & : C_2 \otimes C_2 & \to & C_3 \\
\{ x_2 \otimes y_2 \}_{(1)(0)} & = (s_0 x_2 - s_1 x_2)s_1 y_2 + s_2(x_2 y_2) \\
\{ \otimes \}_{(2)(0)} & : C_2 \otimes C_2 & \to & C_3 \\
\{ x_2 \otimes y_2 \}_{(2)(0)} & = s_2 x_2 s_0 y_2 \\
\{ \otimes \}_{(2)(1)} & : C_2 \otimes C_2 & \to & C_3 \\
\{ x_2 \otimes y_2 \}_{(2)(1)} & = s_2 x_2 (s_1 y_2 - s_2 y_2)
\end{align*}
\]

where \( x_1, y_1 \in C_1, x_2, y_2 \in C_2 \).

Then using Table 1 we get the following identities.
A 3-crossed module consist of a complex

\[
\begin{align*}
x_2 \otimes \partial_2 y_2 & \quad x_2 \otimes y_2 + y_2 \otimes x_2, \\
x_1 \otimes \partial_1 y_1 & \quad x_1 \otimes y_1 + y_1 \otimes x_1, \\
\end{align*}
\]

\[
\begin{align*}
\partial_3 x_3 & = x_3, \\
\partial_2 x_2 & = \partial_2 y_1 - \partial_2 y_2, \\
\partial_3 x_3 + \partial_2 y_1 - \partial_2 y_2 & = 0, \\
\partial_3 x_3 - \partial_2 x_2 & = \partial_3 y_1 - \partial_3 y_2, \\
\partial_3 x_3 & = \partial_3 y_1 + \partial_3 y_2, \\
\end{align*}
\]

Table 2

Table 3

Table 4

where \(x_0 \in C_0, x_1, y_1 \in C_1, x_2, y_2 \in C_2, x_3, y_3 \in C_3\). From these results all liftings given in definition 1 are \(C_0, C_1\)-bilinear maps.

**Definition 10** A 3-crossed module consist of a complex

\[
\begin{align*}
C_3 & \rightarrow C_2 & \partial_3 & \rightarrow C_1 & \partial_2 & \rightarrow C_0 \\
\end{align*}
\]

together with \(\partial_3, \partial_2, \partial_1\) which are \(C_0, C_1\)-algebra morphisms, an action of \(C_0\) on \(C_3, C_2, C_1\), an
action of $C_1$ on $C_2, C_3$ and an action of $C_2$ on $C_3$, further $C_0, C_1$-bilinear maps

$$\{ \otimes \}(0): C_2 \otimes C_2 \rightarrow C_3, \quad \{ \otimes \}(0)(2): C_2 \otimes C_2 \rightarrow C_3, \quad \{ \otimes \}(2)(1): C_2 \otimes C_2 \rightarrow C_3,$$

$$\{ \otimes \}(1,0)(2): C_1 \otimes C_2 \rightarrow C_3, \quad \{ \otimes \}(2,0)(1): C_1 \otimes C_2 \rightarrow C_3,$$

$$\{ \otimes \}(0)(2,1): C_2 \otimes C_1 \rightarrow C_3, \quad \{ \otimes \}: C_1 \otimes C_1 \rightarrow C_2$$

called Peiffer liftings which satisfy the following axioms for all $x_1 \in C_1, x_2, y_2 \in C_2$, and $x_3, y_3 \in C_3$:

3CM1) $C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1$ is a 2-crossed module with the Peiffer lifting $\{ \otimes \}(2,1)$

3CM2) $\partial_2 \{ x_1 \otimes y_1 \} = \partial_1 x_1 - x_1 y_1$

3CM3) $\{ x_2 \otimes \partial_2 y_2 \}(0)(2,1) = \{ x_2 \otimes y_2 \}(2)(1) - \{ x_2 \otimes y_2 \}(1)(0)$

3CM4) $\partial_3 \{ x_2 \otimes y_2 \}(1)(0) = \{ \partial_2 x_2 \otimes \partial_2 y_2 \} + x_2 y_2$

3CM5) $\{ x_1 \otimes \partial_3 y_3 \}(2,0)(1) = \{ x_1 \otimes \partial_3 y_3 \}(1)(0)(2) + \{ x_1 \otimes \partial_3 y_3 \}(1,0)(2) - \partial_1 x_1 y_3$

3CM6) $\{ \partial_2 x_2 \otimes \partial_3 y_3 \}(2,0)(1) = -\{ x_2 \otimes y_2 \}(0)(2) + \{ x_2 y_2 \}(2)(1) + \{ x_2 \otimes y_2 \}(1)(0)$

3CM7) $\{ \partial_3 x_3 \otimes \partial_3 y_3 \}(1)(0) = y_3 x_3$

3CM8) $\{ \partial_3 y_3 \otimes \partial_3 y_3 \}(0)(2,1) = -\partial_2 x_2 y_3$

3CM9) $\{ \partial_2 x_2 \otimes \partial_3 y_3 \}(1,0)(2) = -\{ x_2 \otimes \partial_3 y_3 \}(0)(2)$

3CM10) $\{ \partial_2 x_2 \otimes \partial_3 y_3 \}(2,0)(1) = \partial_2 x_2 y_3 - \{ x_2 \otimes \partial_3 y_3 \}(0)(2)$

3CM11) $\{ \partial_3 y_3 \otimes x_1 \}(0)(2,1) = -x_1 y_3$

3CM12) $\{ y_2 \otimes \partial_3 x_3 \}(1)(0) = -y_2 \cdot x_3$

3CM13) $\{ \partial_3 x_3 \otimes y_2 \}(1)(0) = y_2 \cdot x_3$

3CM14) $\{ \partial_3 x_3 \otimes y_2 \}(2)(0) = 0$

3CM15) $\partial_3 \{ x_1 \otimes y_2 \}(2,0)(1) = \partial_3 \{ x_1 \otimes y_2 \}(1,0)(2) + \{ x_1 \otimes \partial_2 y_2 \} - \partial_1 x_1 y_2 + x_1 y_2$

3CM16) $\partial_3 \{ x_1 \otimes y_2 \}(0)(2,1) = \{ x_1 \otimes \partial_2 y_2 \} - x_1 y_2$

We denote such a 3-crossed module by $(C_3, C_2, C_1, C_0, \partial_3, \partial_2, \partial_1)$.

A morphism of 3-crossed modules of groups may be pictured by the diagram

```
C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0
```

```
  \downarrow f_2 \quad \downarrow f_1 \quad \downarrow f_0
```

```
C'_3 \xrightarrow{\partial'_3} C'_2 \xrightarrow{\partial'_2} C'_1 \xrightarrow{\partial'_1} C'_0
```

where

$$f_1(c_0 c_1) = (f_0(c_0)) f_1(c_1) = f_2(c_0 c_2) = (f_0(c_0)) f_2(c_2), f_3(c_0 c_3) = (f_0(c_0)) f_3(c_3)$$

for $\{ \otimes \}(0)(2), \{ \otimes \}(2)(1), \{ \otimes \}(1)(0)$

$$\{ \otimes \} f_2 \otimes f_2 = f_3 \{ \otimes \}$$

for $\{ \otimes \}(1,0)(2), \{ \otimes \}(2,0)(1), \{ \otimes \}(0)(2,1)$

$$\{ \otimes \} f_1 \otimes f_2 = f_3 \{ \otimes \}$$

for $\{ \otimes \}$

$$\{ \otimes \} f_1 \otimes f_1 = f_2 \{ \otimes \}$$

for all $c_3 \in C_3, c_2 \in C_3, c_1 \in C_3, c_0 \in C_3$. These compose in an obvious way. So we can define the category of 3-crossed modules of commutative algebras, which we will be denoted by $X_3 ModAlg$. 
4 Applications

4.1 Simplicial Algebras

As an application we consider the relation between simplicial algebras and 3-crossed modules which were given in [4] for group case. So proofs in this section are omitted, since can be checked easily by using the proofs given in [4].

Proposition 11 Let $\mathbf{E}$ be a simplicial algebra with Moore complex $\mathbf{NE}$. Then the complex

$$NE_3/\partial_4(NE_4 \cap D_4) \overset{\partial_3}{\longrightarrow} NE_2 \overset{\partial_2}{\longrightarrow} NE_1 \overset{\partial_1}{\longrightarrow} NE_0$$

is a 3-crossed module with the Peiffer liftings defined below:

- $\{ \otimes \}$ : $NE_1 \otimes NE_1 \rightarrow NE_2$
  $\{ x_1 \otimes y_1 \}_{(1)(0)} \mapsto s_1 x_1(s_1 y_1 - s_0 y_1)$

- $\{ \otimes \}_{(1,0)(2)}$ : $NE_1 \otimes NE_2 \rightarrow NE_3/\partial_4(NE_4 \cap D_4)$
  $\{ x_1 \otimes y_2 \}_{(1)(0)(2)} \mapsto (s_2 s_0 x_1 - s_1 s_0 x_1) s_2 y_2$

- $\{ \otimes \}_{(2,0)(1)}$ : $NE_1 \otimes NE_2 \rightarrow NE_3/\partial_4(NE_4 \cap D_4)$
  $\{ x_1 \otimes y_2 \}_{(2,0)(1)} \mapsto (s_2 s_1 x_1 - s_2 s_0 x_1)(s_1 y_2 - s_2 y_2)$

- $\{ \otimes \}_{(0,2,1)}$ : $NE_1 \otimes NE_2 \rightarrow NE_3/\partial_4(NE_4 \cap D_4)$
  $\{ x_1 \otimes y_2 \}_{(0)(2,1)} \mapsto s_2 s_1 x_1(s_1 y_2 - s_0 y_2 - s_2 y_2)$

- $\{ \otimes \}_{(1)(0)}$ : $NE_2 \otimes NE_2 \rightarrow NE_3/\partial_4(NE_4 \cap D_4)$
  $\{ x_2 \otimes y_2 \}_{(1)(0)} \mapsto (s_1 x_2 - s_2 x_2) s_1 y_2 + s_2(s_2 x_2 y_2)$

- $\{ \otimes \}_{(2)(0)}$ : $NE_2 \otimes NE_2 \rightarrow NE_3/\partial_4(NE_4 \cap D_4)$
  $\{ x_2 \otimes y_2 \}_{(2)(0)} \mapsto -s_2 x_2 s_0 y_2$

- $\{ \otimes \}_{(2)(1)}$ : $NE_2 \otimes NE_2 \rightarrow NE_3/\partial_4(NE_4 \cap D_4)$
  $\{ x_2 \otimes y_2 \}_{(2)(1)} \mapsto s_2 x_2(s_2 y_2 - s_1 y_2)$

(The elements denoted by $\{ \otimes \}$ are cosets in $NE_3/\partial_4(NE_4 \cap D_4)$ and given by the elements in $NE_3$.)

Proof. Here we will check some of the conditions. The others can be checked easily.

3CM9) Since

$$\partial_4(C(3,2)(1) \langle x_2 \otimes y_3 \rangle) = (s_2 s_0 d_2 x_2 - s_1 s_0 d_2 x_2 - s_0 x_2) s_2 d_3 y_3$$

we find

$$\{ \partial_2 x_2 \otimes \partial_3 y_3 \}_{(1,0)(2)}^3 = (s_2 s_0 d_2 x_2 - s_1 s_0 d_2 x_2 - s_0 x_2) s_2 d_3 y_3$$

$$s_0 x_2 s_2 d_3 y_3 \in \text{mod}\partial_4(NE_4 \cap D_4)$$

$$= \{ x_2 \otimes \partial_3 y_3 \}_{(0)(2)}^3$$

3CM13) Since

$$\partial_4(C(3,2)(1) \langle x_2 \otimes y_3 \rangle) = s_2 x_2(s_2 y_3 - s_1 y_3 - y_3)$$

we find

$$\{ x_2 \otimes \partial_3 y_3 \}_{(2)(1)}^3 = s_2 x_2(s_1 d_3 y_3 - s_2 d_3 y_3)$$

$$\equiv s_2 x_2 y_3 \in \text{mod}\partial_4(NE_4 \cap D_4)$$

$$= x_2 \cdot y_3 \quad (3.13)$$

Theorem 12 The category of 3-crossed modules is equivalent to the category of simplicial algebras with Moore complex of length 3.
4.2 Projective 3-crossed Resolution

Here as an application we will define projective 3-crossed resolution of commutative algebras.

This construction was defined by P.J.L. Doncel, A.R. Grandjean and M.J. Vale in [10] for 2-crossed modules.

**Definition 13** A projective 3-crossed resolution of an \( \mathbb{k} \)-algebra \( E \) is an exact sequence

\[
\ldots \rightarrow C_{k+1} \xrightarrow{\partial_{k+1}} C_k \rightarrow \ldots \rightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \rightarrow E \rightarrow 0
\]

of \( \mathbb{k} \)-modules such that

1) \( C_0 \) is projective in the category of \( \mathbb{k} \)-algebras
2) \( C_i \) is a \( C_{i-1} \)-algebras and projective in the category of \( C_{i-1} \) algebras for \( i = 1,2 \)
3) For any epimorphism \( F = (f, id, id, id) : (C'_3, C_2, C_1, C_0, \partial'_3, \partial_2, \partial_1) \rightarrow (C''_3, C_2, C_1, C_0, \partial''_3, \partial_2, \partial_1) \)
and morphism \( H = (h, id, id, id) : (C_3, C_2, C_1, C_0, \partial_3, \partial_2, \partial_1) \rightarrow (C'_3, C_2, C_1, C_0, \partial'_3, \partial_2, \partial_1) \) there exist a morphism \( Q = (q, id, id, id) : (C_3, C_2, C_1, C_0, \partial_3, \partial_2, \partial_1) \rightarrow (C'_3, C_2, C_1, C_0, \partial'_3, \partial_2, \partial_1) \) such that \( FQ = H \)
4) for \( k \geq 4, C_k \) is a projective \( \mathbb{k} \)-module
5) \( \partial_1 \) is a homomorphism of \( C_0 \)-module where the action of \( C_0 \) on \( C_4 \) is defined by \( \partial_0 \)
6) For \( k \geq 5, \partial_k \) is a homomorphism of \( \mathbb{k} \)-modules

**Proposition 14** Any commutative \( \mathbb{k} \)-algebra with a unit has a projective 3-crossed resolution.

**Proof.** Let \( E \) be a \( \mathbb{k} \)-algebra and \( C_0 = \mathbb{k}[X_i] \) a polynomial ring such that there exist an epimorphism \( \partial_0 : C_0 \rightarrow B \).

Now let define \( C_1 \) as \( C_0 = [ker \partial_0] \) the positively graded part of polynomial ring on \( ker \partial_0 \) and define \( \partial_1 : C_1 \rightarrow C_0 \) by inducing from the inclusion \( i : ker \partial_0 \rightarrow C_0 \).

Now define \( K_2 = C_0(C_1 \times C_1 \bigcup ker \partial_1) \), the free \( C_0 \)-module on the disjoint union \( (C_1 \times C_1) \bigcup ker \partial_1 \) and define

\[
\partial'_2 : K_2 \rightarrow ker \partial_1
\]

as

\[
\partial'_2(x_1y_1) = x_1y_1 - \partial_2(y_1)x_1, \quad x_1y_1 \in C_1
\]

\[
\partial'_2(x) = x, \quad x \in ker \partial_1
\]

Let \( R' \) be the \( C_0 \)-module generated by the relations

\[
(\alpha x_1 + \beta y_1, z_1) = (\alpha(x_1, z_1) - \beta(y_1, z_1)
\]

\[
(x_1, \alpha y_1 + \beta z_1) = (\alpha(x_1, y_1) - \beta(x_1, z_1
\]

when \( \alpha, \beta \in \mathbb{C} \) and \( x_1, y_1, z_1 \in C_1 \). Now define \( C_2 = K_2/R' \). Now define \( \partial_2 : C_2 \rightarrow ker \partial_1 \) with \( \partial_2 \pi = \partial'_2 \) where \( \pi : K_2 \rightarrow (C_2 = K_2/R') \) is projection.

Now we will define \( C_3 \). Let \( K_3 \) is the \( C_0 \)-module defined on the disjoint union \( C_0(A_{(1,0)} \cup A_{(0,2)} \cup A_{(2,1)} \cup A_{(1,2)} \cup A_{(1,0)(2)} \cup A_{(2,0)(1)} \cup A_{(0,2)1} \cup ker \partial_2) \) where

\[
A_{(1,0)} = A_{(0,2)} = A_{(2,1)} = C_2 \times C_2,
\]

\[
A_{(1,0)(2)} = A_{(2,0)(1)} = C_1 \times C_2
\]

and \( A_{(0,2)1} = C_2 \times C_1 \).

We have \( \partial'_3 : K_3 \rightarrow (C_1 \times C_1 \cup ker \partial_1) \)

\[
\partial'_3(x, y) = xy - \partial_3yx, \quad (x, y) \in A_{(2)(1)}
\]

\[
\partial'_3(x, y) = (\partial_2x, \partial_2y) + xy, \quad (x, y) \in A_{(1)(0)}
\]

\[
\partial'_3(x, y) = (\partial_2x, y) + xy, \quad (x, y) \in A_{(0)(2,1)}
\]

\[
\partial'_3(x, y) = (x, \partial_2y) - \partial_1xy + xy, \quad (x, y) \in A_{(2)(0)(1)}
\]

\[
\partial'_3(x, y) = 0, \quad (x, y) \in A_{(0)(2)}
\]

\[
\partial'_3(x, y) = 0, \quad (x, y) \in A_{(1)(0)(2)}
\]

\[
\partial'_3(x, y) = x, \quad x \in ker \partial_2
\]
Now define a $C_0$-module $R$, generated by the relations, where $\alpha, \beta \in C_0$. Now define $C_3 = K_3/R$ we have $\partial_3 : C_3 \rightarrow \ker \partial_2$ with $\partial_3 \pi : \partial_3 \pi$ where $\pi : K_3 \rightarrow (C_3 = K_3/R)$ is projection. With these constructions

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

is a projective 3-crossed module. If we define $C_4$ as the projection resolution of the $E$-module $\ker \partial_3$ then we have the projective crossed resolution

$$\cdots C_k \xrightarrow{q} C_{k-1} \cdots C_4 \xrightarrow{q} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} E \rightarrow 0$$

where $q$ is the projection. ■

### 4.3 Lie Algebra Case

Lie algebraic version of crossed modules were introduced by Kassel and Loday in, [13], and that of 2-crossed modules were introduced by G. J. Ellis in, [13]. Also higher dimensional Peiffer elements in simplicial Lie algebras introduced in [2]. As a consequence of the commutative algebra version of 3-crossed modules in this section we will define the 3-crossed modules of Lie algebras by using the results given in [2] with a similar way used for defining the commutative algebra case. The relations for commutative algebra are given in the previous section can be applied to the Lie algebra case with some slight differences in the proofs up to the definition.

**Definition 15** A 3-crossed module over Lie algebras consists of a complex of Lie algebras

$$L_3 \xrightarrow{\partial_3} L_2 \xrightarrow{\partial_2} L_1 \xrightarrow{\partial_1} L_0$$

together with an action of $L_0$ on $L_3, L_2, L_1$ and an action of $L_1$ on $L_3, L_2$ and an action of $L_2$ on $L_3$ so that $\partial_3, \partial_2, \partial_1$ are morphisms of $L_0, L_1$-groups and the $L_1, L_0$-equivariant liftings

$$\{ , \}_{(1)(0)} : L_2 \times L_2 \rightarrow L_3, \quad \{ , \}_{(0)(2)} : L_2 \times L_2 \rightarrow L_3, \quad \{ , \}_{(2)(1)} : L_2 \times L_2 \rightarrow L_3,$$

$$\{ , \}_{(1,0)(2)} : L_1 \times L_2 \rightarrow L_3, \quad \{ , \}_{(2,0)(1)} : L_1 \times L_2 \rightarrow L_3,$$

$$\{ , \}_{(0,2,1)} : L_2 \times L_1 \rightarrow L_3, \quad \{ , \} : L_1 \times L_1 \rightarrow L_2$$

called 3-dimensional Peiffer liftings. This data must satisfy the following axioms:

**3CM1)** $C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1$ is a 2-crossed module with the Peiffer lifting $\{ \times \}_{(2),(1)}$

**3CM2)** $\partial_2 \{ l_1 \otimes m_1 \} = \partial_2 m_1 l_1 - [l_1, m_1]$

**3CM3)** $\{ l_2 \otimes \partial_2 m_2 \}_{(0),(2,1)} = \{ l_2 \otimes m_2 \}_{(2),(1)} - \{ l_2 \otimes m_2 \}_{(1),(0)}$

**3CM4)** $\partial_2 \{ l_2 \otimes m_2 \}_{(1),(0)} = \{ \partial_2 l_2 \otimes \partial_2 m_2 \} + [l_2, m_2]$

**3CM5)** $\{ l_1 \otimes \partial_3 l_3 \}_{(2),(0)} = \{ l_1 \otimes \partial_3 l_3 \}_{(0),(2,1)} + \{ l_1 \otimes \partial_3 l_3 \}_{(1),(0)} - \partial_1 l_1 l_3$

**3CM6)** $\partial_2 \{ l_2 \otimes m_2 \}_{(2),(0),(1)} = - \{ l_2 \otimes m_2 \}_{(2),(0)} + [l_2, m_2] \cdot \{ l_2 \otimes m_2 \}_{(2),(1)} + \{ l_2 \otimes m_2 \}_{(1),(0)}$

**3CM7)** $\partial_3 l_3 \otimes m_3 \}_{(1),(0)} = [m_3, l_3]$

**3CM8)** $\partial_3 l_3 \otimes \partial_3 l_3 \}_{(0),(2,1)} = -\partial_3 l_3 l_3$

**3CM9)** $\partial_2 \partial_2 \otimes \partial_3 l_3 \}_{(1),(0),(2)} = - \{ l_2 \otimes \partial_3 l_3 \}_{(0),(2)}$

**3CM10)** $\partial_2 \partial_2 \otimes \partial_3 l_3 \}_{(2),(0),(1)} = \{ \partial_2 l_2 \otimes \partial_3 l_3 \}_{(0),(2)} - \partial_2 l_2 l_3 - \{ l_2 \otimes \partial_3 l_3 \}_{(2),(0)}$

**3CM11)** $\partial_3 \partial_2 l_3 \otimes l_1 \}_{(0),(2,1)} = -l_1 l_3$

**3CM12)** $\{ l_2 \otimes \partial_3 l_3 \}_{(1),(0)} = -l_2 \otimes l_3$

**3CM13)** $\partial_3 l_3 \otimes l_2 \}_{(1),(0)} = l_2 \cdot l_3$

**3CM14)** $\{ l_3 l_3 \otimes l_2 \}_{(2),(0)} = 0$

**3CM15)** $\partial_3 \{ l_1 \otimes l_2 \}_{(0),(2,1)} = \partial_3 \{ l_1 \otimes l_2 \}_{(1),(0),(2)} + \{ l_1 \otimes \partial_2 l_2 \} - \partial_2 l_2 + l_1 l_2$

**3CM16)** $\partial_3 \{ l_1 \otimes l_2 \}_{(0),(2,1)} = \{ l_1 \otimes \partial_2 l_2 \} - l_1 l_2$
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