Beyond the triangle and uniqueness relations: non-zeta counterterms at large $N$ from positive knots*)

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Abstract Counterterms that are not reducible to $\zeta_n$ are generated by $3F_2$ hypergeometric series arising from diagrams for which triangle and uniqueness relations furnish insufficient data. Irreducible double sums, corresponding to the torus knots $(4,3) = 8_{19}$ and $(5,3) = 10_{124}$, are found in anomalous dimensions at $O(1/N^3)$ in the large-$N$ limit, which we compute analytically up to terms of level 11, corresponding to 11 loops for 4-dimensional field theories and 12 loops for 2-dimensional theories. High-precision numerical results are obtained up to 24 loops and used in Padé resummations of $\varepsilon$-expansions, which are compared with analytical results in 3 dimensions. The $O(1/N^3)$ results entail knots generated by three dressed propagators in the master two-loop two-point diagram. At higher orders in $1/N$ one encounters the uniquely positive hyperbolic 11-crossing knot, associated with an irreducible triple sum. At 12 crossings, a pair of 3-braid knots is generated, corresponding to a pair of irreducible double sums with alternating signs. The hyperbolic positive knots $10_{139}$ and $10_{152}$ are not generated by such self-energy insertions.

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1 Introduction

Triangle \([1]\) and uniqueness \([2]\) relations make the \textit{analytical} work of renormalizing a field theory: \textit{elementary} to 3 loops \([3]\); subject to a comprehensive \textit{algorithm} \([4]\) at 4 loops \([5]\); and \textit{achievable}, with ingenuity \([6]\), in the case of \(\phi^4\)-theory at 5 loops \([7]\). Then an obstacle \([8]\) occurs. Fig. 1 shows a 6-loop \(\phi^4\) diagram whose counterterm cannot be obtained by mere differentiation of Euler’s formula, \(\ln \Gamma(1-z) = \gamma z + \sum_{n>1} \zeta(n) z^n/n\), from which all counterterms to 5 loops ultimately derive their transcendentals.

The precise location of this obstacle has recently been confirmed by knot theory \([8,9,10]\). According to \([8,9]\) counterterms are rational if the link diagrams, encoding the intertwining of momenta of the relevant Feynman diagrams, give no non-trivial knots, when skeined. Conversely, knots obtained by skeining link diagrams are in correspondence \([10]\) with distinct transcendental counterterms. The simplest example is the \((2n-3,2)\) torus knot, with \(2n-3\) crossings, which is responsible for the appearance of \(\zeta_{2n-3}\) in counterterms at \(n\) loops \([9]\).

Only special circumstances, such as a gauge symmetry, or supersymmetry, lead one to expect lesser transcendental complexity. For example, knot theory relates \([11]\) the cancellation of transcendentals in quenched QED to the cancellation of subdivergences entailed by the Ward identity. In \(\phi^4\)-theory \([10]\), where no such privileges apply, one expects a new transcendental at the \(n\)-loop level for \textit{each} positive knot with \(2n-3\) or \(2n-4\) crossings, together with products of lower-level transcendentals, corresponding to factor knots \([11]\). This is confirmed by the discovery \([10]\) of the torus knot \((4,3) = 8_{19}\) in a positive 3-braid obtained from the 6-loop diagram of Fig 1. High-precision evaluation of this counterterm reveals \([10]\) it to be expressible in terms of a double sum, previously encountered \([12,13]\) in the \(\epsilon\)-expansion of two-loop diagrams in \(4-2\epsilon\) dimensions. At 7 loops one expects 4 new transcendentals, corresponding to the 10-crossing knots \(10_{124}, 10_{139}, 10_{152}\), and a uniquely positive 11-crossing hyperbolic knot, with braid word \(\sigma_1^2\sigma_2\sigma_1\sigma_3^2\sigma_3^2\sigma_3^2\), which was denoted by \(11_{353}\) in \([10]\), on account of the triple sum that it entails. The double sum associated with \((5,3) = 10_{124}\) has also been identified \([10]\). Transcendentals associated with knots with more crossings are given in \([14,15,16]\).

The presence of further transcendentals, associated with the hyperbolic knots \(10_{139}\) and \(10_{152}\), has been detected in only three highly complex non-planar 7-loop diagrams \([10]\). The precise multiple sums that are involved have not yet been identified, since we have at present only 10-digit accuracy for the relevant counterterms, which is far less than that achieved for the conclusive identification of the new transcendentals arising from \(8_{19}, 10_{124}\) and \(11_{353}\).

In this paper we address the following questions:

Q1 What is the simplest type of analytical structure that produces non-zeta transcendentals in counterterms?

Q2 Which anomalous dimensions entail such a structure?

Q3 Are the resulting new transcendentals at least as sparse as knot theory requires?

Q4 How can one calculate the rational coefficients that multiply the transcendentals in the anomalous dimensions?
In brief, our answers are as follows:

A1 Double sums, associated with the knots $8_{19}$ and $10_{124}$, and the triple sum associated with $11_{353}$, occur in the $\varepsilon$-expansions of Saalschützian $3F_2$ series generated by diagrams reducible to two-loop form.

A2 Double sums contribute to anomalous dimensions, in the large-$N$ limit [17], at $O(1/N^3)$; triple sums do not appear until one goes to higher order in $1/N$.

A3 Up to level 11, the non-zeta transcendental that are generated by $3F_2$ series are sparser than required by the enumeration of positive knots with up to 11 crossings. In particular, more complex analytical structures are needed to generate the transcendental knot-numbers corresponding to $10_{139}$ and $10_{152}$.

A4 Combining knot, field, number and group theory, we obtain the exact rational coefficients of all transcendental that occur in anomalous dimensions at $O(1/N^3)$, up to level 11, corresponding to 11 loops in 4-dimensional theories and 12 loops in 2-dimensional theories. We exemplify the result in the simplest case of a supersymmetric theory [18], which does not entail level-mixing. Other cases lead to lengthier expressions, with no further analytical complexity.

The remainder of the paper demonstrates these answers, as follows.

In Section 2 we derive and solve a pair of recurrence relations for the two-loop two-point diagram with 3 dressed lines, corresponding to arbitrary exponents $\alpha_n$ of dressed propagators $1/(p_n^2)^\alpha_n$ carrying momenta $p_n$. The solution, in terms of a pair of $3F_2$ series, is sufficient to determine anomalous dimensions to all orders at $O(1/N^3)$. From it, we extract a new class of zeta-reducible two-point integrals.

Section 3 reviews previous work on $1/N$ expansions of critical exponents and establishes the connection between $O(1/N^3)$ results and a specific case of the hypergeometric solution of Section 2.

Section 4 uses the wreath product group, $S_3 \wr Z_2$, of transformations of Saalschützian $3F_2$ series [19], which enables us to enumerate the minimal set of Taylor coefficients in the expansion of the two-loop diagram that are not reducible to zetas. This enumeration is compared with expectations based on the relation between counterterms and positive knots [8, 9, 10]. Detailed analysis confirms these expectations and provides further information about the relations between knots and numbers, realized by field theory. We obtain a convenient all-order reduction, to non-alternating double sums, of the integral that is the source of all non-zeta terms in $O(1/N^3)$ anomalous dimensions.

Section 5 exploits these results to obtain $\varepsilon$-expansions of critical exponents, and the related coupling-constant expansions of anomalous dimensions. In the case of the 2-dimensional supersymmetric $\sigma$-model, for which analytical expansions are the most compact, we give the explicit reduction of a 12-loop result to $\{\zeta_n \mid 3 \leq n \leq 11\}$, together with the double sums $\zeta_{5,3} = \sum_{m,n>0} 1/m^5 n^3$ and $\zeta_{7,3} = \sum_{m,n>0} 1/m^7 n^3$, at levels 8 and 10. Extending the analysis to 24 loops, we investigate the utility of Padé resummation in the $\sigma$ model, its supersymmetric extension, the Gross-Neveu model, and $\phi^4$ theory.

Section 6 gives our conclusions.
2 Beyond the triangle and uniqueness relations

We begin by considering the two-loop integral

\[ I_6(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = \frac{d^{2(\mu-\alpha_6)}}{\pi^{2\mu}} \int \int \frac{d^2k \, d^2\mu}{(k-p)^{2\alpha_1}(l-p)^{2\alpha_2}(k-l)^{2\alpha_3}l^{2\alpha_4}k^{2\alpha_5}}, \quad (1) \]

with \( \alpha_6 \equiv 3\mu - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 \) determining the dependence on the number, \( d = 2\mu \), of (euclidean) dimensions. \( I_6 \) is clearly independent of the norm of the external momentum \( p \). Remarkably, it has a 1,440-element symmetry group \([12]\), corresponding to all permutations of 6 linear combinations \([13]\) of its arguments, combined with the total reflection \( \alpha_n \to \bar{\alpha}_n \equiv \mu - \alpha_n \). A function that is invariant under these \( S_6 \times Z_2 \) transformations is

\[ T_6(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \equiv \frac{(2\mu - 3) [G(1)]^2}{\prod_{n=1}^{10} [G(\alpha_n)]^{1/2}} I_6(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6), \quad (2) \]

where \( G(\alpha_n) \equiv \Gamma(\bar{\alpha}_n)/\Gamma(\alpha_n) \), and the auxiliary variables, \( \alpha_7 \equiv \alpha_1 + \alpha_5 - \bar{\alpha}_6 \), \( \alpha_8 \equiv \alpha_2 + \alpha_4 - \bar{\alpha}_6 \), \( \alpha_9 \equiv \alpha_4 + \alpha_5 - \bar{\alpha}_3 \), \( \alpha_{10} \equiv \alpha_1 + \alpha_2 - \bar{\alpha}_3 \), are associated with the vertices of the tetrahedral vacuum diagram \([12]\) of Fig. 2b, formed by joining the external lines of Fig. 2a.

Practically everything that is known about Laurent expansions of multi-loop diagrams can be derived from expanding \([1]\), in \( \varepsilon = 2 - \mu \), with arguments differing from integers by multiples of \( \varepsilon \) resulting from integrations by parts that collapse more complex diagrams to combinations of diagrams of two-loop form, with self-energy insertions. For example, there is only one \([1]\) independent 3-loop diagram that cannot be reduced to combinations of \([1]\), and it enters 4-loop counterterms only at \( O(1/\varepsilon) \), where its value, \( 20\zeta_5 \), is the same as for other 3-loop diagrams, which can be so reduced. Moreover the expansion of \([1]\) is easily achieved to the level of \( \zeta_5 \), which is the highest-level transcendental that enters 4-loop counterterms, corresponding to the torus knot \((5, 2)\).

Group theory \([12, 13]\) greatly reduces the burden of expanding \( I_6 \). Only at the level of \( \zeta_{13} \) does one encounter an expansion coefficient of the six-argument function \([1]\) that cannot be determined by expanding the four-argument function

\[ I_4(\alpha, \beta, \gamma, \delta) \equiv I_6(\alpha, \beta, \gamma, 1, 1, 2\mu - \delta - 2); \quad \mu = \alpha + \beta + \gamma - \delta, \quad (3) \]

whose expansion is therefore sufficient to identify transcendentals in \( I_6 \) corresponding to knots with up to 12 crossings. Triangle relations \([1]\), or equivalently uniqueness relations \([1]\), reduce \( I_4 \) to \( \Gamma \) functions when any element of \( \{\gamma, \delta, 2\mu - \gamma - 2, 2\mu - \delta - 2\} \) is equal to unity, but such zeta-reducible cases are insufficient to expand \( I_6 \) beyond the level of \( \zeta_7 \). We shall expand it to level 11 and study the non-zeta transcendentals, corresponding to positive knots with up to 11 crossings, sufficient for 11 loops at \( O(1/N^3) \) in 4-dimensional field theory, and 12 loops in 2-dimensional theories.

We proceed as follows. First we solve a master recurrence relation, obtaining \( I_4 \) as a pair of \( _3F_2 \) series. This provides a new, two-argument, zeta-reducible result. Reduction of the large-\( N \) results of Section 3 to hypergeometric series enables us to exploit the wreath-product group \( S_3 \wr Z_2 \) \([14]\) to study the knot theory of Section 4. In Section 5 we compute anomalous dimensions to 12 loops, analytically, and to 24 loops, numerically.
2.1 Solution of the master recurrence relation

Operating on the integrand of (1) with $k_\mu(\partial/\partial k_\mu)$ we obtain, from integration by parts,

$$\alpha I_4(\alpha + 1, \beta, \gamma, \delta) + (\alpha + \gamma + 2 - 2\mu)I_4(\alpha, \beta, \gamma, \delta) = \gamma G(1, \delta + 1) \left( \frac{\alpha G(\alpha + 1, \gamma + 1)}{\delta - \beta + 1} - G(\beta, \gamma + 1) \right),$$

(4)

where $G(\alpha_1, \alpha_2) \equiv G(\alpha_1)G(\alpha_2)G(\bar{\alpha}_1 + \bar{\alpha}_2)$ is the general one-loop integral, obtained by multiplication of propagators in $x$-space. The recurrence relation in $\beta$ is obtained from (4) by $\alpha \leftrightarrow \beta$ symmetry.

We now operate on integral (1) with $(\partial/\partial p_\mu)^2$ and use (4) to restore $\alpha$ and $\beta$ to their original values, to obtain a recurrence relation in $\gamma$:

$$\gamma I_4(\alpha, \beta, \gamma, \delta) + \left( \alpha + \gamma + 1 - 2\mu \right) I_4(\alpha, \beta, \gamma - 1, \delta - 1) = \gamma (\delta + \gamma + 1 - 2\mu) G(1, \delta) \left( \frac{G(\alpha, \gamma + 1)}{\delta - \beta} + \frac{G(\beta, \gamma + 1)}{\delta - \alpha} \right).$$

(5)

From (4,5) all other identities can be obtained systematically. One can now forget about triangle or uniqueness relations; they contain no further information about $I_4$. The recurrence relations are solved by finding a function, $S$, with the properties:

$$S(a, b, c, d) = S(b, a, c, d),$$

(6)

$$S(c, d, a, b) + S(a, b, c, d) = 0,$$

(7)

$$a(a + b + c + d) S(a, b, c, d) = a + b + c + d + (a + c)(a + d) S(a - 1, b, c, d).$$

(8)

Given such a function, it is straightforward to verify that (4,5) are satisfied by the Ansatz

$$\frac{I_4(\alpha, \beta, \gamma, \delta)}{\gamma \delta G(1, \delta + 1)} = \frac{G(\alpha, \gamma + 1)}{2\mu - 3} S(\mu - \alpha - 1, \beta - 1, \mu + \alpha - \delta - 2, \delta - \beta) + (\alpha \leftrightarrow \beta),$$

(9)

where $2\mu = 2(\alpha + \beta + \gamma - \delta)$ is the number of dimensions.

Now the problem is reduced to solving the master recurrence relation (8), subject to symmetry properties (6,7), which generate all other recurrence relations. The solution involves a $3F_2$ hypergeometric series$^1$, with one restriction on the 5 parameters, which permits transformations discovered by Saalschütz [21, 22]. Scouring standard texts [23, 24] and formularies [25, 26, 27] for relevant properties of Saalschützian $3F_2$ series, we find that they are all contained in Hardy’s elegant study [22] of chapter XII of Ramanujan’s notebook. Defining the series [19]

$$F(a, b, c, d) \equiv \sum_{n=1}^{\infty} \frac{(-a)_n(-b)_n}{(1 + c)_n(1 + d)_n} = 3F_2 \left[ \begin{array}{c} -a, -b, 1 \\ 1 + c, 1 + d \end{array} ; 1 \right] - 1,$$

(10)

we encapsulate its properties as follows:

$$F(a, b, c, d) = F(b, a, c, d),$$

(11)

$^1$Reduction of a non-trivial two-loop diagram to $3F_2$ series was first achieved in [20].
\[(a + c)(b + c)F(a, b, c, d) = abF(-b - c, -a - c, a + b + c + d), \quad (12)\]
\[(a + c)(a + d)F(a, b, c, d) = ab + a(a + b + c + d)F(a - 1, b, c, d), \quad (13)\]
\[F(c, d, a, b) + F(a, b, c, d) = H(a, b, c, d) - 1, \quad (14)\]
\[F(a - b, -2b, a + b, 2b) = 2b[\psi(1 + a) - \psi(1 + 2a) - \psi(1 + b) + \psi(1 + 2b)], \quad (15)\]

where \(\psi(z) \equiv \Gamma'(z)/\Gamma(z)\), and
\[H(a, b, c, d) \equiv \frac{\Gamma(1 + a)\Gamma(1 + b)\Gamma(1 + c)\Gamma(1 + d)\Gamma(1 + a + b + c + d)}{\Gamma(1 + a + c)\Gamma(1 + a + d)\Gamma(1 + b + c)\Gamma(1 + b + d)} \quad (16)\]
generates all known special cases, save that of (15).

A solution to (16), and hence to (17), is given by
\[S(a, b, c, d) = \frac{\pi \cot \pi c}{H(a, b, c, d)} - \frac{1}{c} - \frac{b + c}{bc}F(a + c, -b, -c, b + d), \quad (17)\]

where (12) gives (6), (13) gives (8), and (14) gives (7), via the elementary identity
\[\cot \pi b + \cot \pi c = \left[\frac{1}{\pi b} + \frac{1}{\pi c}\right]H(a, b, c, d)H(a + c, -b, -c, b + d). \quad (18)\]

### 2.2 A new case of zeta-reducibility

Properties (12,14) ensure that \(F(a, b, c, d)\) is reducible to \(\Gamma\) functions, or zero, when any element of \(\{a, b, c, d, a + b + c + d\}\) vanishes. Hence (17) makes \(S(a, b, c, d)\) reducible when any element of \(\{a + c, a + d, b + c, b + d\}\) vanishes. Hence (14) makes \(I_4(\alpha, \beta, \gamma, \delta)\) reducible when any element of \(\{\gamma, \delta, 2\mu - \gamma - 2, 2\mu - \delta - 2\}\) is equal to unity. Evaluating these special instances, we obtain agreement with known results from the triangle and uniqueness relations, thereby checking that (12,17) give the correct solution to (12,14), i.e. that no homogeneous solution to the recurrence relations has been omitted. We have also checked (12,17) in instances that are not reducible, by comparing results from single sums of the form (10) with the far less convenient double \(\sum\) sums that result from Gegenbauer polynomial \(4\) methods. Likewise we find numerical agreement with a more cumbersome reduction to \(4F_3\) series given in (13).

There is one more case in which \(I_4\) has an expansion that is reducible to zetas. It results from (13), obtained from identity 8.4 of [22], which gives the 4-dimensional result
\[T_6(1 + u, 1 + u, 1 - u - v, 1, 1, 1 - u + v) = 2\sum_{s=1}^{\infty} \zeta_{2s+1}(1 - 4^{-s})(u + v)^{2s} - (u - v)^{2s}uv. \quad (19)\]

Single-argument special cases of this proved useful in [1,12,13,20]. The two-argument result is new and enables us to derive, by simple algebra, the \(S_0 \times Z_2\)-invariant expansion of (2) that was given to level 9 in [13], where only one non-zeta transcendental was encountered. This is shown in [1,10] to be associated with the 8-crossing knot \((4,3) = 8_{19}\). At level 9 we achieve reduction to zetas without the need of the numerical searching that was used in [13]. What happens at higher levels is the subject of Section 4.

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2A transformation from double sums to single sums was found in [28], following communication that we had obtained \(3F_2\) series by solving recurrence relations.
3 Critical exponents at large $N$

Before searching for level-10 knot-transcendentals in the expansion of (9), we review the manner in which such a two-loop integral enters the computation of critical exponents in the large-$N$ limit.

Results from multi-loop perturbation theory are important for high-precision computation of critical exponents in statistical physics, allowing comparison with accurate experimental values. A widely studied example is the Heisenberg ferromagnet. It undergoes a phase transition, from a disordered to an ordered phase, with properties of the transition characterized solely by critical exponents. The field theories underlying the transition are the O(3) nonlinear $\sigma$-model, or $(\phi^2)^2$-theory with an O(3) symmetry, in $d = 2\mu = 3$ dimensions. The $\beta$-function of each theory has a non-trivial zero, which is identified as the phase transition of the Heisenberg model. At their fixed points, the two theories are equivalent, or said to be in the same universality class. In other words, an exponent derived from one theory has the same value as that derived from the other. Consequently, analytic calculations may be performed in either model to predict values for the exponents.

In computing the exponents from $(\phi^2)^2$-theory, a model with a more general O($N$) symmetry may be considered, either directly in three dimensions [31], or near four dimensions [6, 32, 33]. We describe the procedure in the latter case, as it entails the expansion in $\varepsilon$ of diagrams calculated in $d = 2\mu = 4 - 2\varepsilon$ dimensions, where non-zeta terms will occur, via the expansion of (9). First, the theory is renormalized, to some order in perturbation theory, at $d = 4$ dimensions, using dimensional regularization and minimal subtraction of powers of $1/\varepsilon = 2/(4 - d)$. From the $\beta$-function, one then deduces the location of the non-trivial $d$-dimensional fixed point, $g_c(\varepsilon)$, as a power series in $\varepsilon$. Knowledge of the wavefunction and mass anomalous dimensions, $\gamma_2(g)$ and $\gamma_m(g)$, to corresponding orders in the coupling constant $g$, then yields the critical exponents $\eta = 2\gamma_2(g_c), 1/\nu \equiv 2\lambda = 2[1 - \gamma_m(g_c)]$, and $\omega = 2\beta'(g_c)$ as power series in $\varepsilon$. Values for three dimensions are obtained by improving the convergence of these series, using methods of accelerated convergence, after which one sets $\varepsilon = 1/2$. This procedure was applied initially at the three-loop level, yielding values in impressive agreement with experiment [34]. Later, four- and five-loop results were used to improve the accuracy [33, 35]. In addition to the $N = 3$ case of the Heisenberg ferromagnet, O($N$)-symmetric $(\phi^2)^2$-theory underlies the critical behaviour of polymers, the Ising model, and superfluid Bose liquids, at $N = 0, 1$ and 2, respectively. Moreover, the $N = 1$ case of the three-dimensional Ising model is in the same universality class as the deconfinement transition of pure SU(2) Yang-Mills theory in $d = 4$ spacetime dimensions.

As an alternative to the $\varepsilon$-expansion at fixed $N$, one can develop a large-$N$ expansion at fixed $\varepsilon$. In this approach, the critical exponents are expanded in powers of $1/N$, with the coefficients in $\eta = \sum k \eta_k/N^k$ determined for a particular value of $\varepsilon$, or for an arbitrary value. With a reasonable number of terms, one hopes to improve the convergence of the series, in a manner similar to that for the $\varepsilon$-expansion, before setting $N = 3$, or some other low value. Initially, this approach was developed for the O($N$) $\sigma$-model in strictly three dimensions, with exponents determined to O($1/N^2$) [37, 38]. Later these results were extended in an impressive series of papers [33, 39, 40] by Vasil’ev and co-workers.
who obtained \( \eta \) at \( O(1/N^3) \) and \( 1/\nu \) at \( O(1/N^2) \), in an arbitrary spacetime dimension. The method is based on ideas from [13], and on the application of the conformal bootstrap programme of [13, 14, 15]. It is possible to compute in \( d = 2\mu \) dimensions by exploiting the conformal symmetry that exists at the fixed point, making use of so-called uniqueness relations for conformal integration, to determine the values of three- and four-loop diagrams that occur in the corrections to the Dyson-Schwinger equations and hence to the critical exponents. An obvious check on these calculations is the reproduction of previous results on restriction to three dimensions.

There are further checks on the \( d \)-dimensional results at large-\( N \), based on the equivalence of the two underlying field theories at their fixed points. For concreteness, we recall that the lagrangians are

\[
\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)^2 + \frac{1}{2}\sigma(\phi_i^2 - 1/g),
\]

for the \( \sigma \)-model, which is perturbatively renormalizable only in two dimensions, and

\[
\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)^2 + \frac{1}{2}\sigma(\phi_i^2 - \sigma/h),
\]

for \( (\phi^2)^2 \)-theory. The flavour index \( i \) is summed from 1 to \( N \), and \( \sigma \) acts as a Lagrange-multiplier field in (20), but as an auxiliary field in (21), with \( g \) and \( h \) as the respective coupling constants. The \( (\phi^2)^2 \) formulation (21) was used in the initial study of the relation of knots to counterterms [8, 9, 10]. To obtain exponents from the \( \sigma \)-model formulation (20), one follows the procedure described above for \( (\phi^2)^2 \)-theory in \( d \)-dimensions, but sets \( d = 2 - 2\varepsilon \) to obtain \( \varepsilon \)-expansions that yield three-dimensional values at \( \varepsilon = -\frac{1}{2} \). In the case of the \( \sigma \)-model, it is the slope of the \( \beta \)-function that now determines the critical exponent \( 2\lambda = 1/\nu \).

Highly non-trivial checks result from comparing large-\( N \) expansions of critical exponents, at arbitrary \( \varepsilon \), with \( \varepsilon \)-expansions at arbitrary \( N \), obtained from either field theory. Agreement was demonstrated in [38, 40, 41] up to orders then known in perturbation theory. Now that corrected [9] five-loop results have recently been given for \( \phi^4 \)-theory, we have checked all available large-\( N \) results for the exponents, namely \( \eta \) to \( O(1/N^3) \) [11], \( 2\lambda \equiv 1/\nu \) to \( O(1/N^2) \) [11], and \( \omega \) to \( O(1/N^2) \), where we calculated \( \omega \) by extending the techniques of [11] to include corrections to the asymptotic scaling forms of the propagators due to insertion of an operator with dimension \((\mu - 2)\), obtaining a result that is reported in Section 5.1, where full agreement with the 5-loop results of [11] is found.

Given this impressive agreement between all-order results at large \( N \) and 5-loop perturbation theory at fixed \( N \), we may confidently take the \( \varepsilon \)-expansion of the \( O(1/N^3) \) result for \( \eta \) [11] as containing transcendentals that will be encountered in the loop expansions of four-dimensional \( \phi^4 \)-theory, and the two-dimensional \( \sigma \)-model, beyond the levels thus far computed in fixed-\( N \) minimally-subtracted perturbation theory. Hence large-\( N \) results provide a window on transcendentals from knots with many crossings.

It is at \( O(1/N^3) \) in the critical exponent \( \eta \) that one first encounters an integral, in \( d = 2\mu \) dimensions, which has resisted all attempts at reduction to \( \Gamma \) functions and their logarithmic derivatives, the polygamma functions \( \psi^{(n)}(z) \equiv (d/dz)^{n+1} \ln \Gamma(z) \). From the point of view of the authors of [11], this irreducibility was understandably annoying. From our point of view, it is a great bonus, since it leads us into the domain of knots more complex than the \((2n-3, 2)\) torus knots that produce \( \zeta_{2n-3} \) at \( n \) loops in \( \phi^4 \)-theory [3, 11]. The irreducible term encountered at \( O(1/N^3) \) was denoted by \( I(\mu) \) in [11] and is the
logarithmic derivative of a two-loop self-energy diagram, \( \Pi(\mu, \Delta) \), with respect to the exponent, \( \Delta \), of the completely internal line. In terms of the two-loop momentum-space integral (22), it is defined by

\[
I(\mu) \equiv \frac{d \ln \Pi(\mu, \Delta)}{d \Delta} \bigg|_{\Delta=0}; \quad \Pi(\mu, \Delta) \equiv I_4(\mu - 1, \mu - 1, \mu - 1 + \Delta, 2\mu - 3 + \Delta),
\]

and results from non-planar diagrams in the skeleton Dyson-Schwinger equation for the \( \sigma \phi^2 \) vertex \([41]\). The same integral occurs, with different \( \mu \)-dependent coefficients, in \( \eta_3 \) for the Gross-Neveu model \([46, 47]\) and for the supersymmetric \( \sigma \)-model \([18]\).

There is a simple rule (which will be derived later) relating the number of loops in perturbative expansions to \( \varepsilon \)-expansions of \( I(\mu) \): for 2-dimensional theories, the term of \( \mathcal{O}(\varepsilon^n) \) in the expansion of \( \varepsilon I(1 - \varepsilon) \) is of level \( n \) and first appears at \( n + 1 \) loops; for 4-dimensional theories, this level-\( n \) term first appears at \( n \) loops.

## 4 Non-zeta terms and positive knots

Now we have shown how large-\( N \) anomalous dimensions involve the expansion of (9), at \( \mathcal{O}(1/N^3) \), we proceed to enumerate the possible non-zeta terms in the expansion of (10) and compare the tally with the numbers allowed by knot theory in the \( \varepsilon \)-expansion of (22).

### 4.1 The wreath-product group \( S_3 \wr \mathbb{Z}_2 \)

It is clear that \([1, 2, 14]\) relate a large number of \( 3F_2 \) series, with transformed arguments. In fact the group of transformations has 72 elements, as can be seen by considering its operation on the matrix

\[
M \equiv \begin{bmatrix}
-(b + d) & -(b + c) & a \\
-(a + d) & -(a + c) & b \\
\phantom{-(a + d)} & c & d \\
(a + b + c + d)
\end{bmatrix}.
\]

(23)

The symmetry \([11]\) corresponds to exchange of rows 1 and 2; Saalschütz’s transformation \([12]\) to exchange of columns 2 and 3. Combining these with the transformation entailed in relation \([14]\), we may transpose the matrix, permute its rows, and permute its columns. The group of such transformations is the wreath product \( S_3 \wr \mathbb{Z}_2 \) \([19]\). To generate its polynomial invariants, we define

\[
\begin{align*}
\lambda_1 &= \frac{1}{3}(+a - 2b - c - d); \quad \lambda_2 = \frac{1}{3}(-2a + b - c - d); \quad \lambda_3 = -\lambda_1 - \lambda_2; \\
\mu_1 &= \frac{1}{3}(-a - b + c - 2d); \quad \mu_2 = \frac{1}{3}(-a - b - 2c + d); \quad \mu_3 = -\mu_1 - \mu_2; \\
N_n^\pm &= \sum_{i=1}^3 \frac{\lambda_i^n}{2}; \quad N(p_2, m_2, p_3, m_3) = [N_2^+]^{p_2} [N_2^-]^{m_2} [N_3^+]^{p_3} [N_3^-]^{m_3}. \quad (24)
\end{align*}
\]

By construction, the matrix elements of (23) are \( M_{i,j} = \lambda_i + \mu_j \), and \( N_i^\pm \) vanish. With arguments of order \( \varepsilon \), a complete set of linearly independent wreath-product invariants, at \( \mathcal{O}(\varepsilon^n) \), is given by

\[
\{ N(p_2, m_2, p_3, m_3) \mid n = 2(p_2 + m_2) + 3(p_3 + m_3), (-1)^{m_2} = (-1)^{m_3} \}.
\]

(25)
It remains to relate the $3F_2$ series \([10]\) to an invariant function, whose expansion in terms of \((23)\) contains all the non-zeta terms. To achieve this, we define

$$F(a, b, c, d) = \frac{(ab - cd)F(a, b, c, d) + ab}{abcd(a + b + c + d)} - \frac{\Gamma(c)\Gamma(d)\Gamma(a + b + c + d)}{\Gamma(1 + a + c + d)\Gamma(1 + b + c + d)}, \quad (26)$$

\[
\left\{ \begin{array}{l}
\{ a, b \\
\{ c, d \\
\end{array} \right\} = F(c, d, a, b) - F(a, b, c, d), \quad (27)
\]

$$W(a, b, c, d) = F(a, b, c, d) + \frac{1}{6} \left\{ \begin{array}{l}
\{ -a - c, -a - d \\
\{ a, a + b + c + d \\
\end{array} \right\} + \frac{1}{6} \left\{ \begin{array}{l}
\{ -b - c, -b - d \\
\{ b, a + b + c + d \\
\end{array} \right\} + \frac{1}{2} \left\{ \begin{array}{l}
\{ a, b \\
\{ c, d \\
\end{array} \right\} + \frac{1}{6} \left\{ \begin{array}{l}
\{ -a - c, -b - c \\
\{ c, a + b + c + d \\
\end{array} \right\} + \frac{1}{6} \left\{ \begin{array}{l}
\{ -a - d, -b - d \\
\{ d, a + b + c + d \\
\end{array} \right\}. \quad (28)
\]$$

By construction, \((26)\) is regular when any element of \(\{a, b, c, d, a + b + c + d\}\) vanishes, \((14)\) reduces \((27)\) to $\Gamma$ functions, and \((28)\) is a wreath-product invariant, which is reducible to $\Gamma$ functions if $N_2^0 = 0$ or $N_3^+ N_3^- = 0$. The vanishing of $N_2^0 = ab - cd$ clearly removes $F$ from \((28)\), while the vanishing of both $N_3^+$ and $N_3^-$ corresponds to the two-argument zeta-reducible case in \((15)\). To remove from $W$ the trivial case $ab = cd$, we define

$$\mathcal{W}(a, b, c, d) = W(a, b, c, d) - \frac{3}{x} \sum_{n=1}^{\infty} \frac{T(M_{1, n}, M_{2, n}, M_{3, n}) + T(M_{n, 1}, M_{n, 2}, M_{n, 3})}{\Gamma(1 + x)\Gamma(1 + y)\Gamma(1 + z)}, \quad (29)$$

$$T(x, y, z) = \frac{1}{xyz} - \frac{\Gamma(x)\Gamma(y)\Gamma(z)}{\Gamma_+ (x, y, z)\Gamma_- (x, y, z)}, \quad (30)$$

$$\Gamma_\pm (x, y, z) = \Gamma \left( 1 + \frac{1}{2} (x + y + z) \pm \frac{1}{2} \left[ x^2 + y^2 + z^2 - 2xy - 2yz - 2zx \right]^{1/2} \right), \quad (31)$$

with a subtraction in \((29)\) that is also a wreath-product invariant, thanks to the symmetry of \((30)\). When $N_2^0 = 0$, all instances of the square root in \((31)\) give rational differences of matrix elements, making $\mathcal{W}$ vanish. In any case, expanding \((30)\) removes the square root, producing the invariants \((23)\) at $O(\varepsilon^n)$.

The expansion of $\mathcal{W}$, in terms of the invariants \((23)\), contains only terms with $m_2 > 0$ and even values of $m_2 + m_3$. Moreover, those with $p_3 = m_3 = 0$ are zeta-reducible. The leading term occurs at level 7 and is determined by Hardy’s result \((15)\), which gives

$$\mathcal{W}(a, b, c, d) = \left[ -\frac{75}{16} \zeta_7 + 3\zeta_5 \zeta_2 - \frac{3}{2} \zeta_4 \zeta_3 \right] (ab - cd)^2 + O(\varepsilon^5), \quad (32)$$

with transcendentals of level $n + 3$ entering at $O(\varepsilon^n)$.

It is now straightforward to count the expansion coefficients, at any given level, and to determine how many are zeta-reducible. Up to level 20, the tally is

| level | coefficients | zeta-reducible | non-zeta |
|-------|--------------|----------------|----------|
| 7     | 1            | 1              | 0 |
| 8     | 1            | 1              | 0 |
| 9     | 1            | 1              | 0 |
| 10    | 3            | 3              | 0 |
| 11    | 5            | 5              | 0 |
| 12    | 6            | 6              | 0 |
| 13    | 7            | 7              | 0 |
| 14    | 9            | 9              | 0 |
| 15    | 11           | 11             | 0 |
| 16    | 12           | 12             | 0 |
| 17    | 16           | 16             | 0 |
| 18    | 18           | 18             | 0 |
| 19    | 19           | 19             | 0 |
| 20    | 20           | 20             | 0 |

(33)

whose non-zeta terms we expect to involve transcendentals associated with knots more complex than the $(2n - 3, 2)$ knot that generates $\zeta_{2n-3}$ in sub-divergence-free counterterms of 4-dimensional theories at $n$ loops.
4.2 Expectations from knot theory

To see what knot theory has to tell us in this context, we consider dressings of the three-loop tetrahedron vacuum diagram by chains of one-loop self-energy insertions, since counterterms with this structure may be obtained from the $\varepsilon$-expansion of the master two-loop diagram, by the method of infrared rearrangement [18].

Unadorned, the tetrahedron delivers $\zeta_3$, which corresponds to the trefoil knot [8, 9]. For example, the 3-loop coefficient of the $\beta$-function of $\phi^4$ theory, in 4 dimensions, receives a scheme-independent contribution $6\zeta_3$ from the counterterm that cancels the overall divergence of the subdivergence-free four-point function obtained by attaching external lines to the vertices of the tetrahedron of Fig. 2b. Its evaluation is straightforward, since it is merely the finite value of the diagram of Fig. 2a, with massless propagators, $1/p_n^2$, and unit external momentum, which is easily obtained by the triangle rule [1], or the method of uniqueness [2].

Now we dress this three-loop vacuum diagram with chains of one-loop self-energy insertions, thus extending the domain of investigation to counterterm diagrams with subdivergences. Since we are interested, at this stage, in the momentum flow, we do not specify the particle content of a specific field theory, though we have in mind our experience [10] of $(\phi^2)^2$ theory, with an $O(N)$ symmetry group, whose self-energy insertions would involve shrinking some of the lines in Figs. 3–8, without affecting the flow patterns.

Such self-energy insertions modify the powers of propagators in the 3-loop vacuum diagram, so that we are now dealing with transcendentals generated by closing a two-loop two-point diagram that contains non-integer lines, which [22] confirms as the arena explored by large-$N$ analyses.

Experience with diagrams with subdivergences is reported in [8], where it was observed that the dressing of ladder topologies turns rational [19] counterterms into counterterms containing both odd and even zetas. The occurrence of odd zetas in subdivergence-free diagrams was studied in [8, 9, 10], where $\zeta_{2n+1}$ was associated with the torus knot $(2n+1, 2)$. The even zetas, $\{\zeta_{2n} | n \geq 2\}$, are restricted to diagrams with subdivergences, leading to link diagrams with non-zero writhe numbers [8, 9]. In general, a dressed ladder diagram with $n \geq 4$ loops may contribute $\{\zeta_k | 2 < k < n\}$ to the overall counterterm, after subtraction of subdivergences from dressed ladders in 4 dimensions. For the dressed tetrahedron, we find that non-zeta transcendentals of level $n \geq 8$ occur at $n$ loops for 4-dimensional field theories, and $n+1$ loops for 2-dimensional theories.

The question now at issue is this: how does the insertion of self-energy dressings in the 3-loop tetrahedron turn a zeta-reducible counterterm into one that is not reducible to the torus knots $(2n+1, 2)$? We shall exhibit, in Figs. 3–8, self-energy insertions that generate the knots $8_{19}, 10_{124}$, associated with irreducible [20] double Euler sums [21], the knot $11_{353}$, associated with an irreducible triple Euler sum [32], and a pair of 12-crossing knots, whose appearance is associated with alternating double sums.

We start our investigation with Fig. 3a, where three propagators are dressed. With such a dressing we find that all momentum routings yield link diagrams whose skeining produces the double-sum knot $8_{19}$, familiar from the 6-loop subdivergence-free contributions to the $\beta$-function of $\phi^4$ theory [10]. Note also that this pattern of insertions leaves
no vertex, nor any triangle, unmodified, and hence reduction to zetas by the method of
uniqueness is no longer possible.

Next, in Fig. 3b, we assign a 8-component link diagram to the 8-loop Feynman dia-
gram of Fig. 3a, following the methodology of [8, 9]. The momentum flows of the Feynman
diagram become components of the link diagram, and each vertex in the diagram corre-
sponds to a crossing in the link diagram (though the converse is not always the case). In
Fig. 3b we see three rings generated by the three dressings in the upper half of Fig. 3a.
Their presence forces non-trivial entanglement elsewhere in the link diagram. Having
served that purpose they play no further role, since the skeining process, associated with
removal of subdivergences, removes all trace of them from the final knots that are in
correspondence with the transcendentals in the overall counterterm. The entanglement
generated by the remaining pair of insertions, in the bottom right of Fig. 3a, is of the
essence in this example. These insertions sit on a propagator that necessarily carries the
flow of two of the three loop momenta of the tetrahedral skeleton, when each of the upper
propagators carries only a single loop momentum of the tetrahedron. To generate $S_{19}$, we
have chosen one of the dressings to have a non-local momentum flow, and one to have a
local flow. The local momentum flow runs through only two propagators (i.e. it remains
in the smallest forest of the associated subdivergence, in the language of renormalization
theory). If we had chosen a local flow for both self-energy insertions, we would have gen-
erated a 2-braid knot, giving merely the zeta content of the overall counterterm, whereas
its non-zeta content corresponds to the 3-braid that is obtained by skeining Fig. 3b.

Next, we use skeining to reduce the 8-component link diagram of Fig. 3b to the 3-
component link diagram of Fig. 3c. Three components are trivially removed by skeining
the upper rings of Fig. 3b to give curls that can be untwisted; a further skeining and
untwisting removes all trace of the central propagator; a final skeining, at the bottom right,
generates a non-trivial entanglement. In Fig. 3c, a central dot is marked, as an origin for
reading the positive braid word for the 10-crossing 3-component link, $\sigma_1^3 \sigma_2^2 \sigma_1 \sigma_2^4$, starting
at bottom of the figure and reading anti-clockwise, with a crossing of the outermost line
over the next-to-outermost denote by $\sigma_1$, and a crossing of the next-to-outermost over the
innermost by $\sigma_2$.

Finally, we perform two more skeinings to generate the knot $S_{19}$ of Fig. 3d, with braid
word $\sigma_1^2 \sigma_2^2 \sigma_1 \sigma_2^3$. Using the relation $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$, between braid group generators, one
may prove that $\sigma_1^{2k} \sigma_2^2 \sigma_1 \sigma_2^{2l+1} = \sigma_1 \sigma_2^{2k+1} \sigma_1 \sigma_2^{2l+1}$ and hence obtain the more familiar braid
words $(\sigma_1 \sigma_2^3)^2 = (\sigma_1 \sigma_2^4)^4$, which show that $S_{19}$ is indeed the $(4, 3)$ torus knot [53].

To generate the next member of the series of non-zeta knots, we proceed as illustrated
in Fig. 4, repeating the methodology of Fig. 3. Note that we add two more self-energy
insertions to Fig. 3a to obtain Fig. 4a: one spectates; the other participates in the entan-
glement. Each is involved, since the spectator forces a pair of tetrahedron loop momenta
to flow through a chain of three self-energy insertions. The 10-loop Feynman diagram of
Fig. 4a then gives the 10-component link of Fig. 4b, which in turn skeins to the 12-crossing
3-component link of Fig. 4c, with braid word $\sigma_1^5 \sigma_2^2 \sigma_1 \sigma_2^4$. Two final skeinings (omitted from
Fig. 4) then deliver the knot $\sigma_1^4 \sigma_2^2 \sigma_1 \sigma_2^3 = \sigma_1 \sigma_2^3 \sigma_1 \sigma_2^3 = 10_{124}$, which is, coincidentally, the
$(5, 3)$ torus knot. Note that we again chose one dressing to have non-local momentum
flow. In the case of Fig. 4b it is the rightmost self-energy insertion. We have verified
that each of the other choices leads to the same knot. Moreover, choosing more dressings
to have non-local flows does not produce any more complicated knot. In particular the
positive hyperbolic knots $10_{139}$ and $10_{152}$ cannot be obtained by such a process.

At 12 loops a new feature emerges: a pair of 12-crossing knots is generated by self-
energy insertions, as illustrated by Figs. 5 and 6, which refer to the same Feynman dia-
gram. The momentum flow encoded by the link diagram of Fig. 5b gives the 3-component
link of Fig. 5c, which delivers the knot $\sigma_2^3 \sigma_3^2 \sigma_1^3 = \sigma_1 \sigma_2^3 \sigma_1 \sigma_3^2$. On the other hand, the
momentum flow of Fig. 6b delivers a different knot: $\sigma_2^3 \sigma_3^2 \sigma_1^3 = \sigma_2^3 \sigma_1 \sigma_3^2$. Other choices
of routing and skeining deliver no further non-zeta knots.

From explicit calculation of HOMFLY polynomials [53] of positive braid wor-
ds of length 12 we find that there are precisely 7 positive non-torus knots with crossing number
12. All are 3-braids and are listed in Table 1. However, only the first two are obtained
by dressing three propagators of the tetrahedron.

| crossings | knots | numbers |
|-----------|-------|---------|
| $2a + 1$  | $\sigma_1^{2a+1}$ | $\zeta_{2a+1}$ |
| 8         | $\sigma_1 \sigma_2^3 \sigma_1 \sigma_3^2 = 8_{19}$ | $N_{5,3}$ |
| 9         | none                                               | none |
| 10        | $\sigma_1^3 \sigma_2^3 \sigma_1 \sigma_3^2 = 10_{124}$ | $N_{7,3}$ |
|           | $\sigma_1^2 \sigma_2^3 \sigma_1 \sigma_3^2 = 10_{139}$ | ? |
|           | $\sigma_2^2 \sigma_3^3 \sigma_1^2 = 10_{152}$ | ? |
| 11        | $\sigma_1^2 \sigma_2^3 \sigma_1 \sigma_3 \sigma_2^3 \sigma_3^2 = N_{3,5,3}$ | |
| 12        | $\sigma_1 \sigma_2^3 \sigma_1 \sigma_3^2 = N_{9,3}$ |
|           | $\sigma_1 \sigma_2^3 \sigma_1 \sigma_3^2 = N_{7,5} - \frac{\pi^{12}}{2^{10}10!}$ |
|           | $\sigma_1 \sigma_2^3 \sigma_1 \sigma_3^2 = ?$ |
|           | $\sigma_1 \sigma_2^3 \sigma_1 \sigma_3^2 = ?$ |
|           | $\sigma_1 \sigma_2^3 \sigma_1 \sigma_3^2 = ?$ |
|           | $\sigma_1 \sigma_2^3 \sigma_1 \sigma_3^2 = ?$ |
|           | $\sigma_1 \sigma_2^3 \sigma_1 \sigma_3^2 = ?$ |
|           | $\sigma_1 \sigma_2^3 \sigma_1 \sigma_3^2 = ?$ |

Hence we expect that 12-loop counterterms, generated in the manner of Figs. 5 and 6,
will contain up to two double-sum level-12 transcendentals that are not reducible to zetas.
More generally, we expect $[m/2]$ irreducibles of level $2m + 4$, associated with the knots

$$\{ \sigma_1 \sigma_2^{2k+1} \sigma_1 \sigma_2^{2l+1} \mid l + k = m, \ k \geq l \geq 1 \}. \quad (34)$$

At levels 8 and 10, we find that subdivergence-free diagrams deliver the knot numbers
$N_{5,3}$ and $N_{7,3}$, where [14]

$$N_{a,b} \equiv \sum_{m>n>0} \left( \frac{(-1)^m}{m^a n^b} - \frac{(-1)^n}{n^a m^b} \right). \quad (35)$$

In [14] it was found that the knots $8_{19}$ and $10_{124}$ are associated with the combinations
$29 \zeta_{5,3} - 12 \zeta_{5,3}$ and $94 \zeta_{10} - 793 \zeta_{7,3}$, in subdivergence-free counterterms, where

$$\zeta_{a,b} \equiv \sum_{m>n>0} \frac{1}{m^a n^b} \quad (36)$$
are non-alternating double sums. Subsequently it was realized that the $\pi^2$ terms in these combinations are precisely those generated by the alternating sums $N_{5,3}$ and $N_{7,3}$. In higher-loop subdivergence-free diagrams we encounter $N_{9,3}$ and $N_{7,5} - \frac{\pi^4}{2100}$.

In general, we conjecture that the knots (34) are associated, via subdivergence-free counterterms, with the numbers

$$\{N_{2k+3,2l+1} \mid l + k = m, \; k \geq l \geq 1\}$$

modulo a rational multiple of $\pi^{2k+2l+4}$ that vanishes when $k = l$.

Analytical and numerical investigations [14] strongly support the irreducibility of (37). Moreover, it was the methodology of Figs. 3-6 that led to the discovery that double-sum irreducibles increase by the ‘rule of two’ entailed by the knots (34), with $[m/2]$ knot-numbers at level $2m + 4$. This is in marked contrast to the ‘rule of three’ that governs the non-alternating sums (36), of which $[(m+1)/3]$ are irreducible [50] at level $2m + 4$. From this we conclude that one should look beyond non-alternating sums, when studying the transcendental content of counterterms. Interestingly, it appears that non-alternating sums do not inhabit a cosy world of their own: in the course of this work one of us discovered [14] a four-fold non-alternating sum, $\zeta_{4,4,2,2} = \sum_{k>l>m>n>0} 1/k^4l^4m^2n^2$, which is reducible to alternating double sums, but not to non-alternating double sums. This led to an enumeration [14] of irreducible multiple sums that is much simpler than might have been suspected when attention was restricted to non-alternating sums.

Having associated the knots (34) with the numbers (37), by consideration of the tetrahedron with three dressed propagators, we now proceed to consider cases in which more than three propagators are dressed, having in mind that we now probe higher terms in the large-$N$ expansion than are entailed by the $O(1/N^3)$ analysis that produces (22). We expect to find the 4-braid knot $\sigma_1^2\sigma_2^2\sigma_1\sigma_3\sigma_2^3\sigma_3^2 \equiv 11_{353}$, associated [10] with a single [52] irreducible level-11 triple sum, which may be taken as $\zeta_{3,5,3} = \sum_{l>m>n>0} 1/l^3m^5n^3$. It is also interesting to see whether the 3-braids $10_{139}$ and $10_{152}$ emerge from complicated dressings of the tetrahedron.

Detailed investigation suggests that four dressed propagators are insufficient to generate $11_{353}$. Investigating the five dressings of Fig. 7a, we obtain the link diagram of Fig. 7b, in which three self-energy insertions spectate, whilst the other two necessarily lead to entanglements. The resulting 4-component 12-crossing link diagram of Fig. 7c then delivers the 4-braid 9-crossing factor knot $\sigma_1^4\sigma_2^4\sigma_1\sigma_3\sigma_2^3\sigma_3^2$, corresponding to the level-9 factorized transcendental $\zeta_3^3$. Fig. 8 shows how the 4-braid 11-crossing prime knot $\sigma_1^4\sigma_2^4\sigma_1\sigma_3\sigma_2^3\sigma_3^2 \equiv 11_{353}$ can be generated when there are dressings on all 6 lines of the tetrahedron, forcing even more entanglements. In Fig. 8b, only two of the 6 self-energy insertions spectate, while the remaining 4 yield the 4-component 14-crossing link diagram of Fig. 8c. Three final skeinings deliver the knot $11_{353}$. From this we conclude that the associated knot number [14]

$$N_{3,5,3} = \zeta_{3,5,3} - \zeta_3\zeta_{5,3} - 7\zeta_5\zeta_3^2$$

will appear in the expansion of the master two-loop integral (1), though not in the $O(1/N^3)$ anomalous dimensions.

By adding further self-energy insertions to Fig. 8, we can generate a pair of 13-crossing 4-braids. Their braid words and knot numbers are given in [14]. Here we are content to
stop at 12 crossings, with results summarized by Table 1. Comparing these findings with
the numbers of possible non-zeta expansion coefficients in (33), we arrive at the following
expectations:

E8 The $\varepsilon$-expansion of $\varepsilon I(1 - \varepsilon)$ will contain the double sum $\zeta_{5,3}$, associated with $8_{19}$, at
$O(\varepsilon^8)$. From (33) we see that is the sole level-8 irreducible in the generic expansion
of $3F_2$ series.

E9 It will contain no new irreducible at $O(\varepsilon^9)$, since the only positive prime knot with
crossing number 9 is the $(9, 2)$ torus knot, which delivers $\zeta_9$.

E10 It will contain the double sum $\zeta_{7,3}$, associated with $10_{124}$, at $O(\varepsilon^{10})$. The second level-10 non-zeta in the generic expansion of $3F_2$ series, indicated by (33), presumably
involves the product $\zeta_2\zeta_{5,3}$. Such a product cannot occur in MS counterterms, since
$\zeta_2$ is avoided in the $G$-scheme [1], which is equivalent to the MS scheme for the
calculation of anomalous dimensions.

E11 It will not contain the triple sum $\zeta_{3,5,3}$, at $O(\varepsilon^{11})$. This term corresponds to the
level-11 non-zeta term in (32), but will not show up at $O(1/N^3)$, since it requires
more than 3 dressed propagators to generate the entanglement of momentum flows
that skeins to produce the positive hyperbolic$^3$ knot $\sigma_1^2\sigma_2^2\sigma_1\sigma_3\sigma_2^3\sigma_3^2$, dubbed 11$_{353}$
in [10].

E12 Two level-12 irreducible Euler sums are expected in the expansion of the master two-
loop diagram. From the point of view of knot theory, it appears most natural to take
these as the alternating sums $N_{9,3}$ and $N_{7,5}$. If one wishes to stay within the realm
of non-alternating sums, the irreducibles must be taken from different depths, with
$\zeta_{9,3}$ appearing at depth 2, and $\zeta_{4,4,2,2}$ at depth 4, which appears somewhat unnatural
for a pair of 3-braids with very similar structure. In any case, no more than two
new transcendentals should appear at 12 loops in the anomalous dimensions of $(\phi^2)^2$
theory to order $1/N^3$ in the large-$N$ limit.

To check these expectations, we must tackle the generic problem of expanding $3F_2$ series.

4.3 Expansion to level 11 of any Saalschützian $3F_2$ series

To extend expansion (32) to level 11, we evaluated two instances of (15), which gave all
the zeta-reducible coefficients at levels 7, 9, and 11. Choosing a case that involved only
double and triple sums, we further reduced the remaining coefficients, save that of the
invariant $N(0, 2, 1, 0)$, to zetas and the canonical knot-theoretic set of non-zetas up to
level 11, namely [10]

\[
\zeta_{5,3} = \sum_{m>n>0} \frac{1}{m^3n^3} \approx 0.037707672984847544011304782293659915, \quad (39)
\]

\[
\zeta_{7,3} = \sum_{m>n>0} \frac{1}{m^7n^3} \approx 0.008419668503096332423968579714670651, \quad (40)
\]

$^3$In [10], 10$_{139}$, 10$_{152}$ and 11$_{353}$ were wrongly called satellite knots. All three are, in fact, hyperbolic.
\[ \zeta_{3,5,3} = \sum_{l>m>n>0} \frac{1}{l^3m^5n^3} \approx 0.002630072587647467345248476381643627, \quad (41) \]

associated with the knots 8_{19}, 10_{124}, and 11_{353}. Finally, we reduced the remaining level-10 coefficient to this set and a single further non-zeta term:

\[ \sum_{m>n>0} \frac{1}{m^5n^5} = \frac{301509}{5600} \zeta_{10} - \frac{119}{4} \zeta_{7} \zeta_{3} - \frac{1413}{56} \zeta_{5}^2 - 8 \zeta_{5} \zeta_{3} \zeta_{2} + \frac{33}{2} \zeta_{4} \zeta_{3}^2 - \frac{4}{5} \zeta_{2} \zeta_{5,3} - \frac{191}{56} \zeta_{7,3}, \quad (42) \]

whose value was obtained by high-precision evaluation and integer-relation searching, enabled by Bailey’s mpfun routines. Excluding double and triple sums, there are 42 sums of the form (42), with inverse powers of \( m \) and \( n_i < m \) whose exponents sum to 10. Thanks to mpfun [54], we discovered that all are reducible to the basis set that appears in (2), which was the only case needed for our present field-theoretic purposes. It is noteworthy that such sums do not suggest candidates for new knot-transcendentals to associate with 10_{139} or 10_{152}.

For the development of the invariant expansion (32), we obtain

\[
W(a, b, c, d) = \sum C(p_2, m_2, p_3, m_3) N(p_2, m_2, p_3, m_3) + O(\varepsilon^9), \quad (43)
\]

\[
C(0, 2, 0, 0) = -\frac{25}{48} \zeta_7 + 3 \zeta_5 \zeta_2 - \frac{9}{5} \zeta_4 \zeta_3,
\]

\[
C(0, 1, 0, 1) = \frac{27}{20} \zeta_8 - \frac{9}{5} \zeta_{5,3},
\]

\[
C(1, 2, 0, 0) = -\frac{662}{48} \zeta_9 + 10 \zeta_7 \zeta_2 - \frac{7}{5} \zeta_6 \zeta_3 - \frac{19}{4} \zeta_5 \zeta_4 - \frac{1}{2} \zeta_3^2,
\]

\[
C(0, 2, 1, 0) = -\frac{12023}{1120} \zeta_{10} + \frac{91}{8} \zeta_7 \zeta_3 + \frac{981}{125} \zeta_5^2 - 6 \zeta_5 \zeta_3 \zeta_2 + \frac{5}{2} \zeta_4 \zeta_3^2 + \frac{75}{112} \zeta_7 \zeta_3,
\]

\[
C(1, 1, 0, 1) = \frac{234}{35} \zeta_{10} + \frac{153}{28} \zeta_8^2 - \frac{27}{11} \zeta_7 \zeta_3,
\]

\[
C(2, 2, 0, 0) = -\frac{2003}{64} \zeta_11 + \frac{105}{4} \zeta_8 \zeta_2 - \frac{1}{4} \zeta_6 \zeta_3 - \frac{43}{8} \zeta_7 \zeta_4 - \frac{77}{8} \zeta_6 \zeta_5 - \frac{13}{4} \zeta_5 \zeta_2^2,
\]

\[
C(0, 4, 0, 0) = -\frac{177}{256} \zeta_11 + \frac{35}{36} \zeta_8 \zeta_2^3 + \frac{325}{288} \zeta_8 \zeta_3 + \frac{11}{8} \zeta_7 \zeta_4 - \frac{17}{16} \zeta_6 \zeta_5 - \frac{7}{8} \zeta_8 \zeta_2^2 + \frac{4}{5} \zeta_3 \zeta_2,
\]

\[
C(0, 1, 1, 1) = \frac{568}{10} \zeta_11 - \frac{71}{5} \zeta_8 \zeta_3 - 16 \zeta_7 \zeta_4 - 38 \zeta_6 \zeta_5 + 4 \zeta_5 \zeta_2^2 + \frac{9}{5} \zeta_3 \zeta_5, + \frac{9}{5} \zeta_5 \zeta_3,.
\]

which enables us to expand, to level 11, any Saalschützian \(3F_2\) series (10) whose parameters differ from integers by multiples of \( \varepsilon \), and hence to expand the master two-loop diagram (3) to this order, thanks to the reduction of the case (3) to a pair of series in (9), each of which is given by a \(3F_2\) series in (17).

To expand (3) in \(2\mu = 4 - 2\varepsilon = \frac{2}{3} \sum \alpha_n\) dimensions, one constructs \(\sum_{k=1}^6 \Delta^6_k\), where \(\Delta_1 \equiv \frac{1}{6} (\alpha_1 + 3\alpha_2 - \alpha_3 + \alpha_4 - 3\alpha_5 - \alpha_6)\), with \(\Delta_2 \to \Delta_6\) obtained by cyclic permutation of subscripts. In 4 dimensions, we obtain

\[
\begin{align*}
T_6 \big|_{\varepsilon = 0} &= 6 \zeta_3 + \frac{15}{2} N_2 \zeta_5 + \frac{43}{22} \left[ 5 N_2^2 - 4 N_4 \right] \zeta_7 + \frac{3}{4} N_3^2 - \frac{9}{7} N_4 N_2 + 6 N_6 - N_3^2 \zeta_3 \\
&+ \left[ \frac{85}{36} N_3^2 - 28 N_4 N_2 + \frac{64}{9} N_6 \right] - \frac{83}{9} N_2^2 \zeta_9 \\
&+ \left[ \frac{14}{9} \left( N_4^2 - 6 N_4 N_2^2 + 8 N_6 N_2 \right) - \frac{2}{7} N_3^2 N_2 - \frac{18}{7} N_5 N_3 \right] \zeta_5 \zeta_2^2 \\
&+ \left[ \frac{1023}{1024} \left( \frac{151}{6} N_2^4 - 92 N_4 N_2^2 + \frac{256}{3} N_6 N_2 + 8 N_4^2 - \frac{224}{9} N_5^2 N_2 \right) - \frac{32}{8} N_5 N_3 \right] \zeta_{11} \\
&+ \frac{2}{5} N_5 N_3 - \frac{4}{3} N_3^2 N_2 \left\{ \frac{9}{5} (\zeta_{3,5,3} - \zeta_3 \zeta_5) + \frac{64}{63} \zeta_{11} - \zeta_5 \zeta_3^2 \right\} + O(\Delta_1^{10}),
\end{align*}
\]

where the final brace is the constant called \(K_{353}\) in (11), and was found in all those subdivergence-free 7-loop \(\phi^4\) counterterms whose skeinings produce the knot 11_{353}. The
even-level transcendentals \( \zeta_{5,3} \) and \( \zeta_{7,3} \) occur at levels 8 and 10 in the \( \varepsilon \)-expansion. Note, however, that only odd levels are encountered in (44), at \( \varepsilon = 0 \). It follows that a counterterm producing the knot \( 8_{19} \) or \( 10_{124} \) cannot be reduced to a two-loop two-point form merely by analytic regularization [55] in 4 dimensions, which is sufficient for the 6-loop zig-zag [7, 10] counterterm, 168\( \zeta_9 \), of \( \phi^4 \)-theory. This no-go theorem may help to prevent fruitless searches for analytically regularized evaluations of counterterms that skein to prime 3-braids.

4.4 Expansion to level 11 of the large-\( N \) integral \( I(\mu) \)

The particular case of (3) required for \( \eta \) at \( O(1/N^3) \) is given in (22). Near 4 and 2 dimensions, we obtain

\[
\Pi(2 - \varepsilon, \Delta) = \frac{2}{1 - 2\varepsilon} \frac{\Gamma(1 + \Delta - \varepsilon)\Gamma(1 - \Delta + \varepsilon)}{\Gamma(1 + \Delta - 2\varepsilon)\Gamma(1 - \Delta)} S(\Delta - \varepsilon, -\Delta),
\]

(45)

\[
\Pi(1 - \varepsilon, \Delta) = \frac{2(\Delta - 3\varepsilon)}{\Delta + \varepsilon} \frac{\Gamma(\Delta - \varepsilon)\Gamma(-\Delta + \varepsilon)}{\Gamma(1 + \Delta - 2\varepsilon)\Gamma(1 - \Delta)} + \frac{2\varepsilon^2(1 - 2\varepsilon)}{\Delta + \varepsilon} \Pi(2 - \varepsilon, \Delta),
\]

(46)

respectively, where the symmetric function

\[
S(a, b, a/a, 0) = \frac{3\zeta_3 + O(a, b)}{ab}
\]

is reducible if any element of \( \{a, b, a+b, a+2b, 2a+b\} \) vanishes, since

\[
2aS(a, 0) = 3\psi'(1) - 3\psi'(1 + a),
\]

(48)

\[
4aS(2a, -2a) = 2\psi'(1 - 2a) + \psi'(1 + a) - 2\psi'(1 + 2a) - \psi'(1 - a),
\]

(49)

\[
\frac{1 + 2a^2S(a, -2a)}{\cos \pi a} = \frac{\Gamma(1 + a)\Gamma(1 - 3a)}{\Gamma(1 - 2a)}.
\]

(50)

Note that the coefficient of the last term in (46) is \( O(\varepsilon) \), when \( \Delta \sim \varepsilon \). Thus level-\( n \) terms, occurring at \( n \) loops in 4-dimensional theories, occur at \( n + 1 \) loops in 2-dimensional theories.

The special cases (48, 49, 50) determine the expansion of (17) to level 7, giving

\[
\varepsilon I(1 - \varepsilon) = \varepsilon^2 + \frac{2}{3} \zeta_3 \varepsilon^3 + \zeta_4 \varepsilon^4 + \frac{12}{3} \zeta_5 \varepsilon^5 + \left[ \frac{25}{6} \zeta_6 + \frac{11}{3} \zeta_3^2 \right] \varepsilon^6 + \left[ \frac{25}{2} \zeta_7 + 11 \zeta_4 \zeta_3 \right] \varepsilon^7 + \sum_{n \geq 8} X_n \varepsilon^n.
\]

(51)

The first term was obtained in [11], the second in [54], the third in [18, 57], and the fourth in [18]. The zeta terms at levels 6 and 7 are new. At level 8 we expect to encounter the first non-zeta term (39), corresponding to the knot \( (4, 3) = 8_{19} \).

Using the general expansion (43), we obtain the coefficients at levels 8 to 11:

\[
X_8 = \frac{797}{15} \zeta_8 + \frac{74}{3} \zeta_5 \zeta_3 + \frac{18}{5} \zeta_{5,3},
\]

\[
X_9 = \frac{227}{2} \zeta_9 + \frac{130}{3} \zeta_6 \zeta_3 + 10 \zeta_5 \zeta_4 - \frac{22}{3} \zeta_3^2,
\]

\[
X_{10} = \frac{5553}{28} \zeta_{10} + \frac{165}{2} \zeta_7 \zeta_3 + \frac{336}{21} \zeta_6^2 + 33 \zeta_4 \zeta_3^2 + \frac{54}{7} \zeta_{7,3},
\]

\[
X_{11} = \frac{5875}{12} \zeta_{11} + \frac{2827}{30} \zeta_8 \zeta_3 - \frac{153}{4} \zeta_7 \zeta_4 + \frac{200}{3} \zeta_6 \zeta_5 - 64 \zeta_5 \zeta_3^2 - \frac{3}{5} \zeta_3 \zeta_{5,3},
\]

(52)
with $\zeta_{3,5,3}$ conspicuous by its absence at level 11, as anticipated.

Referring back to the expectations E8 to E11 of Section 4.2, one sees that knot theory is indeed a good guide, up to level 11. To progress to level 12, and beyond, much more analysis was needed. In fact, we eventually succeeded in reducing the $3F_2$ series that occurs at $O(1/N^3)$ to an elementary double sum, for all values of $d = 2 - 2\varepsilon$.

## 4.5 Reduction of $I(\mu)$ to a double sum, for all orders

For all $|\varepsilon| < 1$, we define

$$S_\pm(\varepsilon) \equiv \sum_{m>n>0} \frac{\varepsilon^3}{(m + \varepsilon)^2(n - \varepsilon)} \pm (\varepsilon \rightarrow -\varepsilon) = \sum_{r,s>0} (r - 1)\zeta_{r,s} [(-1)^r \pm (-1)^s] \varepsilon^{r+s}, \quad (53)$$

where $\zeta_{r,s} = \sum_{m>n>0} m^{-r}n^{-s}$, and $S_-(\varepsilon)$, involving odd (and hence zeta-reducible) double sums, may be reduced to products of polygammas, by solving its recurrence relation. We find that

$$S_-(\varepsilon) = \frac{1}{2}(\psi_2 + \psi_2 - \psi_1 - \psi_1)(\psi_1' + \psi_1' + 1) + \frac{3}{4}(\psi_1' - \psi_1') + (\psi_2' - \psi_2')(\psi_1 - \psi_1 - 1), \quad (54)$$

where we use the shorthand $\psi_p^{(n)} \equiv [\partial/\partial p]^{n+1} \ln \Gamma(1 + p \varepsilon) = \varepsilon^{n+1} \psi_p(n)(1 + p \varepsilon)$ for the polygamma functions. On the other hand, $S_+(\varepsilon)$ is not reducible; its expansion in $\varepsilon$ involves even-level double sums, $\zeta_{r,s}$ with $(-1)^r = (-1)^s$, which cannot all be expressed as linear combinations of products of zetas. In fact [1] the number of independent irreducible non-alternating double sums, at level $r + s = 2n$, is the integer part of $(n - 1)/3$. In particular, one cannot further reduce the expansion

$$S_+(\varepsilon) = \frac{1}{2} \zeta_4 \varepsilon^4 + \left[6 \zeta_3^2 - \frac{47}{6} \zeta_6\right] \varepsilon^6 + \left[-\frac{36}{7} \zeta_5 \zeta_3 + 24 \zeta_5 \zeta_3 - \frac{863}{30} \zeta_8\right] \varepsilon^8$$

$$+ \left[-\frac{108}{7} \zeta_7 \zeta_3 + \frac{96}{7} \zeta_5 \zeta_3 + 36 \zeta_7 \zeta_3 - \frac{4019}{70} \zeta_{10}\right] \varepsilon^{10} + O(\varepsilon^{12}), \quad (55)$$

which will contain one more non-zeta term at level 12; two at each of levels 14, 16, and 18; three at levels 20, 22 and 24; and so on. Since no four-fold sum occurs in expansion (53), no alternating double sum occurs, which is why only one of the two knot-numbers occurs at level 12. In the generic case of expanding the master two-loop diagram, both are expected to occur.

We now show that (53) is the sole source of non-zetas in the expansion of $I(1 - \varepsilon)$. Consider first the $3F_2$ series $F(2a + b, -b, -a - b, -b)$, which gives (17), via (17). Its expansion to $O(a^3)$ is required, to obtain $I(\mu)$. This entails two distinct series in the expansion of the wreath-product invariant (29), of the form

$$\prod(2a + b, -b, -a - b, -b) = \sum_{n=0}^{\infty} c_n N(n, 2, 0, 0) + \sum_{n=0}^{\infty} d_n N(n, 1, 0, 1) + O(a^3). \quad (56)$$

The coefficients $c_n$ of the first series are those giving the $O(a^2)$ terms in the zeta-reducible case (13); those of the second series give the $O(hk)$ terms in $F(h + \varepsilon, -k - \varepsilon, -\varepsilon, k + \varepsilon)$, which can in turn be related to the expansion of $S_+(\varepsilon)$. Thus one may use wreath-product invariance to relate $I(1 - \varepsilon)$ to $S_+(\varepsilon)$, using Hardy’s result (13). The algebra
is rather demanding: for each of the three series one must determine the zeta-reducible terms of \( \{24, 25, 29\} \), which involves taking third derivatives of a very large number of \( \Gamma \) functions, generating \( \{\psi^{(n)}_p \mid n = 0, 1, 2; p = 0, \pm 1, \pm 2\} \) and their many products. Some of these products are related by trigonometric identities, involving the even-zeta parts of

\[
\psi^{(n)}_p = \frac{1}{2} \left( \frac{\partial}{\partial \rho} \right)^{n+1} \left[ \ln \frac{\Gamma(1 + p \varepsilon)}{\Gamma(1 - p \varepsilon)} + \ln \frac{\rho \pi \varepsilon}{\sin \rho \pi \varepsilon} \right].
\]

(57)

After considerable use of REDUCE \([58]\), we obtained

\[
(\varepsilon I(1 - \varepsilon) + \psi_1 + \psi_2 - \psi_1 - \psi_0) \left( 1 + \psi_1' - \psi_0 \right) - \frac{2}{3} \equiv
\]

\[
E(\varepsilon) = -\frac{1}{2} S_+ (\varepsilon) + \frac{1}{16} \left( 2 \psi_1 - 2 \psi_0 + 6 \psi_1' + 10 \psi_0' - 12 \psi_0' + \psi_1'' + \frac{7}{3} (3 \psi_1' - 2 \psi_0') \right)
\]

\[
+ \frac{1}{4} (\psi_2 - \psi_1) (1 + \psi_1' + \psi_0' - 6 \psi_0') + \frac{1}{4} (\psi_2 - \psi_1) (1 + \psi_1' + \psi_0' + 2 \psi_0'),
\]

(58)

and verified that the expansion coefficients of \( \{51, 52, 55\} \) satisfy this remarkable identity to \( O(\varepsilon^{11}) \). It is hard to imagine how it might have been obtained without systematizing the 72 wreath-product transformations of matrix \( \{23\} \). We conclude that the group-theory of Saalschützian \( _3F_2 \) series is useful for obtaining all-order results; not just for perturbative expansions. Comprehensive checks are provided by the recurrence relation

\[
\frac{E(\varepsilon) + \frac{2}{3}}{\varepsilon^3} = \frac{E(\varepsilon + 1)}{(\varepsilon + 1)^3} + \frac{\psi(-\varepsilon) - \psi(1)}{2 \varepsilon (1 + 2 \varepsilon)},
\]

(59)

which follows from \( \{13, 19\} \), and is satisfied by \( \{53, 58\} \), and by evaluating \( \{58\} \) at \( \varepsilon = -\frac{1}{2} \), where the zeta-reducibility of \( \{53\} \) yields a finite 3-dimensional result:

\[
S_+ \left( \pm \frac{1}{2} \right) = -\frac{7}{8} \zeta_3 + \frac{9}{8} \zeta_2 - \frac{3}{4} \quad \Rightarrow \quad I \left( \frac{3}{2} \right) = 2 \ln 2 - \frac{7 \zeta_3}{2 \zeta_2},
\]

(60)

in agreement with \( \{11\} \).

Thus we conclude the comparison with knot theory with the finding that only one new non-zeta level-12 transcendental occurs in the critical exponent \( \eta \) at \( O(1/N^3) \). It may be taken as the double sum \( \zeta_{9,3} \), or indeed any member of \( \{\zeta_{12-s,s} \mid s = 2 \ldots 5\} \), since this set contains only one \( \{50\} \) independent non-zeta term. We incline to associate it with \( \sigma_1 \sigma_2 \sigma_3 \sigma_2 \sigma_1 \sigma_4^3 \), though there is at present no strong reason for preferring this knot to \( \sigma_1 \sigma_2 \sigma_3 \sigma_2 \sigma_5 \), or some linear combination of their knot numbers. The braid words of the remaining five 12-crossings knots in Table 1 make them unlikely candidates to be associated exclusively with double sums.

### 5 Computation of anomalous dimensions

We now use the all-order result \( \{58\} \) to compute critical exponents and anomalous dimensions in four field theories: the bosonic \( \phi^4 \) model in 2-dimensions; its 4-dimensional cousin, which is \( \phi^4 \) theory; its fermionic cousin, which is the Gross-Neveu model; and its supersymmetric extension, which has the attractive feature that at \( L \) loops all terms have transcendentality level \( L - 1 \). In this last case we give analytic results to 12 loop;
space does not permit us to write the explicit perturbation expansions in the other cases, though they may be obtained from the results that we give. Using (53), it is possible to develop the expansions numerically, to any desired accuracy and number of loops. Hence we are able to investigate the behaviour of Padé resummations, up to 24 loops, and apply them outside the domain of convergence of the original perturbation series.

5.1 Bosonic σ-model and φ⁴-theory

The critical exponent \( \eta = \sum_k \eta_k/N^k \) of the bosonic σ-model, in \( 2\mu = 2 - 2\varepsilon \) dimensions, is given to \( O(1/N^3) \) by [11]

\[
\eta_1 \equiv -2\varepsilon A \frac{R_0}{1-\varepsilon}; \quad A \equiv \frac{\Gamma(1-2\varepsilon)}{\Gamma(1+\varepsilon)[\Gamma(1-\varepsilon)]^3}, \tag{61}
\]

\[
\eta_2 \equiv -4\varepsilon A^2 \frac{R_1 + R_2B}{(1-\varepsilon)^2}; \quad B^{(n)} \equiv \psi_1^{(n)} + (-1)^n\psi_2^{(n)} - (-1)^n\psi_1^{(n)} - \psi_0^{(n)}, \tag{62}
\]

\[
\eta_3 \equiv -8\varepsilon A^3 \frac{R_3 + R_4B + R_5B^2 + R_6C + R_7B' + R_8BC + R_9D + R_{10}E}{(1-\varepsilon)^3}, \tag{63}
\]

where \{B, B', B'', C, D\} entail the polygammas (57), \( E \) has been reduced to \( S_+ (\varepsilon) \) in (58), and \( R_n \) are rational functions of \( \varepsilon = 1 - \mu \), which we obtain from [10], [11] as:

\[
R_0 = 1 + \varepsilon; \quad R_1 = (1 - \varepsilon^2 + 2\varepsilon^3 - 4\varepsilon^4)/(1-\varepsilon); \quad R_2 = -(1 - \varepsilon + 2\varepsilon^2)(1+\varepsilon);
\]

\[
R_3 = (1 - \varepsilon - \varepsilon^2 + 5\varepsilon^3 - 9\varepsilon^4 + \varepsilon^5 - 8\varepsilon^6 + 25\varepsilon^7 - 11\varepsilon^8 + 2\varepsilon^9)/(1-\varepsilon^2);
\]

\[
R_4 = -4(1 - \varepsilon^2 + 13\varepsilon^3 - 11\varepsilon^4 - 7\varepsilon^5 + 2\varepsilon^6 + 2\varepsilon^7)/(1-\varepsilon);
\]

\[
R_5 = -2\varepsilon(3 + 4\varepsilon - 7\varepsilon^2 + 2\varepsilon^3); \quad R_6 = -\frac{11}{4}(3 + \varepsilon - 25\varepsilon^2 + 49\varepsilon^3 - 18\varepsilon^4 - 2\varepsilon^5);
\]

\[
R_7 = 1 + 5\varepsilon - 5\varepsilon^2 - 13\varepsilon^3 + 4\varepsilon^4 + 3\varepsilon^5; \quad R_8 = -(1 + 4\varepsilon + 7\varepsilon^2)(1-\varepsilon^2);
\]

\[
R_9 = -\frac{2}{3}(1 + 2\varepsilon)^2(1-\varepsilon)^2; \quad R_{10} = \frac{3}{2}(1 + 4\varepsilon)(1-\varepsilon^2)(1+\varepsilon). \tag{65}
\]

From [10], we obtain

\[
\lambda_1 = \frac{2\varepsilon^2(1 - 2\varepsilon)A}{(1 - \varepsilon)}; \quad \lambda_2 = \frac{2\varepsilon(S_1 + S_2B + S_3(B^2 - B') + S_4C)A^2}{(1 + \varepsilon)(1-\varepsilon)^3}, \tag{66}
\]

for \( \lambda = 1/2\nu = \sum_k \lambda_k/N^k \), to \( O(1/N^2) \), with polynomial coefficients

\[
S_1 = 2\varepsilon(1 - 2\varepsilon)(1 - \varepsilon + 4\varepsilon^2 - 7\varepsilon^3 - 2\varepsilon^4 + 3\varepsilon^5);
\]

\[
S_2 = -2(1 - \varepsilon)(2 + 5\varepsilon - 6\varepsilon^2 - 9\varepsilon^3 + 4\varepsilon^5);
\]

\[
S_3 = 2(1-\varepsilon)^3(1 + 2\varepsilon)^2; \quad S_4 = -3(1 - \varepsilon)^3(1 + 5\varepsilon + 8\varepsilon^2). \tag{67}
\]

Extending the techniques of [10] to include corrections to the asymptotic scaling forms of the propagators due to insertion of an operator with dimension \( (\mu - 2) \), we obtained

\[
\omega_1 = \frac{4\varepsilon(1 - 2\varepsilon)^2(1+\varepsilon)A}{(1 - \varepsilon)}; \quad \omega_2 = -\frac{4\varepsilon T_0A + 4\varepsilon(T_1 + T_2B + T_3(B^2 - B') + T_4C)A^2}{(2 + \varepsilon)^2(1 + \varepsilon)^2(1-\varepsilon)^3}, \tag{68}
\]
with polynomial coefficients

\[ T_0 = -16(1 + 2\varepsilon)^2(1 - \varepsilon)^3 ; \quad T_1 = 8 + 30\varepsilon - 94\varepsilon^2 - 49\varepsilon^3 + 372\varepsilon^4 - 3\varepsilon^5 - 612\varepsilon^6 - 271\varepsilon^7 + 556\varepsilon^8 + 609\varepsilon^9 - 70\varepsilon^{10} - 260\varepsilon^{11} - 72\varepsilon^{12} ; \]
\[ T_2 = -2(1 - \varepsilon)(20 + 118\varepsilon + 177\varepsilon^2 - 36\varepsilon^3 - 298\varepsilon^4 - 278\varepsilon^5 - 131\varepsilon^6 + 52\varepsilon^7 + 136\varepsilon^8 + 80\varepsilon^9 + 16\varepsilon^{10}) ; \quad T_3 = 8(2 + \varepsilon)(1 + \varepsilon)(1 + 2\varepsilon)(1 - \varepsilon)^2(1 + 3\varepsilon + \varepsilon^2) ; \]
\[ T_4 = -3(2 + \varepsilon)(1 + \varepsilon)(1 - \varepsilon)^3(2 + 19\varepsilon + 74\varepsilon^2 + 89\varepsilon^3 + 28\varepsilon^4 + 4\varepsilon^5) , \]

also in \( d = 2 - 2\varepsilon \) dimensions.

Working to 5 loops at \( O(1/N^3) \) we obtain the \( \varepsilon \)-expansion

\[ \eta_3 = -8\varepsilon - 32\varepsilon^2 - 72\varepsilon^3 - 4(40 + 27\zeta_3)\varepsilon^4 - 18(16 + 4\zeta_3 + 9\zeta_4)\varepsilon^5 + O(\varepsilon^6) , \]

for the \( \sigma \)-model in \( 2 - 2\varepsilon \) dimensions. Using (53) to shift the dimensionality by two units, we obtain the \( \varepsilon \)-expansions

\[ \eta_3 = 320\varepsilon^2 - 1984\varepsilon^3 + 4(683 - 240\zeta_3)\varepsilon^4 + 2(343 + 4720\zeta_3 + 1280\zeta_4)\varepsilon^5 + O(\varepsilon^6) , \]
\[ \lambda_2 = -48\varepsilon + 242\varepsilon^2 + (-283 + 240\zeta_3)\varepsilon^3 + (59 - 1328\zeta_3 + 360\zeta_4 - 640\zeta_5)\varepsilon^4 + 8(-22 + 279\zeta_3 - 32\zeta_4^2 - 249\zeta_4 + 620\zeta_5 - 200\zeta_6)\varepsilon^5 + O(\varepsilon^6) , \]
\[ \omega_2 = 408\varepsilon^2 + 4(-259 + 240\zeta_3)\varepsilon^3 + 96(-18 - 63\zeta_3 + 15\zeta_4 - 40\zeta_5)\varepsilon^4 + 2(1591 + 6432\zeta_3 - 1152\zeta_4^2 - 4536\zeta_4 + 19520\zeta_5 - 4800\zeta_6)\varepsilon^5 + O(\varepsilon^6) , \]

for \( (\phi^2)^2 \) theory, in \( 4 - 2\varepsilon \) dimensions. Results (71,72,73) agree with [1].

### 5.2 Gross-Neveu model

For the Gross-Neveu model, with a four-fermion interaction, one has merely to replace, in (61,62,63), the rational factors \( R_n \) by [47]

\[ \tilde{R}_0 = -\varepsilon ; \quad \tilde{R}_1 = -\frac{1}{2}\varepsilon(3 - 4\varepsilon)(1 - 2\varepsilon)/(1 - \varepsilon) ; \quad \tilde{R}_2 = \varepsilon(1 - 2\varepsilon) ; \]
\[ \tilde{R}_3 = -\frac{1}{2}\varepsilon(4 - 18\varepsilon + 24\varepsilon^2 - 2\varepsilon^3 - 14\varepsilon^4 + 6\varepsilon^5 - \varepsilon^6)(1 - 2\varepsilon)/(1 - \varepsilon)^2 ; \]
\[ \tilde{R}_4 = \frac{1}{2}\varepsilon(8 - 26\varepsilon + 19\varepsilon^2 + 3\varepsilon^3 - \varepsilon^4)(1 - 2\varepsilon)/(1 - \varepsilon) ; \quad \tilde{R}_5 = -3\tilde{R}_7 = -\frac{3}{2}\varepsilon(1 - 2\varepsilon)^2 ; \]
\[ \tilde{R}_6 = -\frac{1}{4}\varepsilon(15 - 32\varepsilon + 17\varepsilon^2 - \varepsilon^3) ; \quad \tilde{R}_8 = \tilde{R}_{10} = \frac{3}{2}\varepsilon(1 - \varepsilon)^2 ; \quad \tilde{R}_9 = 0 . \]

### 5.3 Supersymmetric \( \sigma \)-model

Beyond 5 loops, the bosonic and fermionic expansions become lengthy to write, because of the mixing of transcendentals of different levels. To exemplify the generic transcendentality structure we consider, instead, the supersymmetric \( N \)-component \( \sigma \)-model of [18, 54, 60], in \( 2\mu = 2 - 2\varepsilon \) dimensions, which is free of such level mixing. We expand the anomalous dimensions, \( \beta(g) \) and \( \gamma(g) \), of the coupling and field, in powers of \( 1/N \), as

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The critical coupling is determined by $\beta$ in terms of the polygammas of (62,64). The results of [18,59] give

$$\gamma_1(\varepsilon) = A - 1 = 2\zeta_3\varepsilon^3 + 3\zeta_4\varepsilon^4 + 6\zeta_5\varepsilon^5 + O(\varepsilon^6).$$

The critical coupling is determined by $\beta(g_c) = 0$, which gives

$$-\frac{N}{2}g_c = \varepsilon \left[ 1 + \frac{2}{N} + \frac{4 + \beta_2(\varepsilon)}{N^2} + O(1/N^3) \right],$$

where the $O(1/N^2)$ term in the critical exponent $\beta'(g_c) \equiv -2\lambda$ gives

$$\varepsilon\beta'_2(\varepsilon) = -2A^2(4B - 2B^2 + 3C + 2B') = 36\zeta_3\varepsilon^3 + 54\zeta_4\varepsilon^4 + 232\zeta_5\varepsilon^5 + O(\varepsilon^6),$$

in terms of the polygammas of (62,64). The results of [18,59] give

$$\frac{(N-1)\gamma(g_c)}{(N-2)\beta(g_c)} = 1 - \frac{\eta}{2\varepsilon} = 1 + \frac{A}{N} + 2(1 - B)A^2/N^2$$

$$+ 4(1 - 4B - \frac{3}{4}C + B' - BC - \frac{3}{2}D + \frac{3}{2}E)A^3/N^3 + O(1/N^4),$$

where the coefficients of the polygammas are obtained by the simple device [18] of setting $\varepsilon = 0$ in the rational functions of [65]. From (80) we obtain the $O(1/N^2)$ terms in $\gamma$:

$$\gamma_2(\varepsilon) = \beta_2(\varepsilon) - (2[\gamma_1(\varepsilon) - \varepsilon d/\varepsilon] + 5)\gamma_1(\varepsilon) + 2A^2B = 6\zeta_3\varepsilon^3 + \frac{21}{2}\zeta_4\varepsilon^4 + \frac{222}{5}\zeta_5\varepsilon^5 + O(\varepsilon^6),$$

so that $\gamma_2(\varepsilon) = 3\gamma_1(\varepsilon) + O(\varepsilon^4)$, which means that the 4-loop term in $\gamma$ vanishes for $N = 3$.

The non-zeta transcendentals $\zeta_{5,3}$ and $\zeta_{7,3}$ are found at 9 and 11 loops, respectively, in the expansion of $\beta_3 - \gamma_3$. Using [25,28] in (80), we obtain

$$\beta_3(\varepsilon) - \gamma_3(\varepsilon) = \frac{45}{2}\zeta_4\varepsilon^4 + \frac{1328}{5}\zeta_5\varepsilon^5 + \frac{5}{3}\left[212\zeta_2^2 + 389\zeta_6\right]\varepsilon^6 + \frac{1}{114}\left[17232\zeta_4\zeta_3 + 30629\zeta_7\right]\varepsilon^7$$

$$+ \frac{2}{7}\left[-54\zeta_{5,3} + 12613\zeta_5\zeta_3 + 16633\zeta_8\right]\varepsilon^8$$

$$+ \frac{1}{9}\left[16532\zeta_3^3 + 73494\zeta_5\zeta_4 + 97730\zeta_6\zeta_3 + 129669\zeta_9\right]\varepsilon^9$$

$$+ \frac{1}{35}\left[-1620\zeta_{7,3} + 309797\zeta_4\zeta_3^2 + 449566\zeta_5^2 + 1033435\zeta_7\zeta_3 + 1896379\zeta_{10}\right]\varepsilon^{10}$$

$$+ \frac{1}{70}\left[-99792\zeta_3\zeta_{5,3} + 22188152\zeta_5\zeta_3^2 + 40049660\zeta_6\zeta_5$$

$$+ 35783955\zeta_7\zeta_4 + 64668044\zeta_8\zeta_3 + 65848405\zeta_{11}\right]\varepsilon^{11} + O(\varepsilon^{12}),$$

to 12 loops in the perturbation expansion, with a coupling $g = -2N\varepsilon$. Note that we cannot separate the non-zeta contributions of $\beta_3$ and $\gamma_3$ to $\eta_3$, without also knowing $\lambda_3$. If we parametrize the unknown terms of the 6-loop anomalous dimension by $\delta_{1,2,3}$ in

$$\gamma(g) = (N-1)g - \frac{1}{2}\zeta_2(N-1)(N-2)(N-3)g^4$$

$$+ \frac{3}{80}\zeta_4(N-1)(N-2)[(N-3)(2N-1) + \delta_1]g^5$$

$$+ \frac{3}{80}\zeta_5(N-1)(N-2)[(N-3)(5N^2 - 22(N-1) + \delta_2) + \delta_3]g^6 + O(g^7),$$

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then \( \beta(g)/g = 2\mu - 2 - (N - 2)g - \frac{3}{4}\zeta_3(N - 2)(N - 3)g^4 \)
\[ + \frac{27}{32}\zeta_4(N - 2)(N - 3)
\left( N - (2 + \frac{1}{5}\delta_1) \right) g^5 \]
\[ - \frac{29}{320}\zeta_5(N - 2)(N - 3)
\left( N^2 - (8 + \frac{3}{14}\delta_2)N + \delta_4 \right) g^6 + O(g^7), \]

where \( \delta_4 \) involves \( \lambda_4 \). The vanishing of the 4-loop term in \( \gamma \) at \( N = 3 \), like all terms beyond one loop in \( \beta \), suggests \( \delta_1 = 0 \), giving simple factors of \( N - 3 \) and \( N - 2 \) in the 5-loop terms of \( (83,84) \), respectively. Similarly, \( \delta_3 = 0 \) gives a factor of \( N - 3 \) at 6 loops in \( \gamma \), and \( \delta_2 = 0 \) simplifies \( \beta \). In such a case, the 6-loop term in \( \beta_3 - \gamma_3 \) comes from that in \( 8\beta_2 - 4(\gamma_2 - 3\gamma_1) \), with \( (77,79,81) \) giving \( 8(232/5) - 4(222/5 - 18) = 1328/5 \) in \( (82) \).

### 5.4 Padé approximation of \( \varepsilon \)-expansions

Suppose that one knows the expansions of anomalous dimensions to some order in the perturbation theory of the 2-dimensional field theory. How accurately can one estimate 3-dimensional exponents?

From \( L \)-loop perturbation theory one may clearly construct \( L \) terms of the \( \varepsilon \)-expansion of the \( O(1/N^3) \) term \( \eta_3 \). Setting \( \varepsilon = -\frac{1}{2} \) will not give a reasonable estimate of the 3-dimensional value, since there are singularities at \( 2\mu = 1 \) that limit the convergence of the series to \( |\varepsilon| < \frac{1}{2} \). These are severe: in the supersymmetric case, the coefficient of \( \varepsilon^n \) increases like \( n^52^n \), on account of a \( (1 - 2\varepsilon)^{-6} \) singularity in the \( A^3D \) term of \( (80) \). It is for this reason that the coefficients of \( (82) \) are so large.

In such a situation, one may proceed by calculating Padé approximants of the form

\[ \eta_3 \approx \frac{\sum_{n=1}^{L-M} a_n\varepsilon^n}{1 + \sum_{m=1}^{M} b_m\varepsilon^m} \equiv [L - M\backslash M], \]

with coefficients chosen to fit the \( L \)-loop \( \varepsilon \)-expansion. Extending \( (55) \), we generate

\[
\begin{array}{cccccc}
L & [L\backslash 0] & [L-2\backslash 2] & [L-4\backslash 4] & [L-6\backslash 6] & [L-8\backslash 8] \\
12 & 50.6 & 1.618343 & 1.816265 & 1.739471 & 1.501598 \\
16 & 190.6 & 1.737426 & 1.737543 & 1.722441 & 1.734508 \\
20 & 520.2 & 1.733445 & 1.728054 & 1.728646 & 1.728264 \\
24 & 1178.6 & 1.733935 & 1.728344 & 1.728335 & 1.728337 \\
\end{array}
\]

as approximants to the supersymmetric 3-dimensional result \[ [18] \]

\[ \eta_1 = \frac{8}{\pi^2}; \quad \eta_2 = \eta_1^2; \quad \eta_3 = \left[ 2 - (3\ln 2 + 1)\zeta_2 + \frac{21}{4}\zeta_3 \right] \eta_1^3 \approx 1.728337. \]

Direct summation, corresponding to \([L\backslash 0]\), is clearly not an option. Note that at \( L = 12 \) loops the second significant figure is in doubt; it is sobering to realize how deep into the \( \varepsilon \)-expansion one must go to get close to the non-perturbative value.

\footnote{JAG regrets errors in previous work. To correct these: in Eq (14) of \[ [18] \] replace \( \frac{3}{5} \) by \( \frac{1}{5} \); hence in Eq (15) replace \((2N - 3)(N - 2)\) by \((2N - 1)(N - 3)\); in Eq (5.11) of \[ [60] \] delete the \( \zeta_3 \) term at 6 loops and change the sign of the \( \zeta_3^2 \) term at 7 loops; hence in Eq (5.15) delete the \( \zeta_3 \) term at 6 loops.}
For the Gross-Neveu model, the results \[17\] of \[74\] give the 3-dimensional values:

$$\eta_1 = \frac{8}{3\pi^2}; \quad \eta_2 = \frac{28}{3}\zeta_1; \quad \eta_3 = -\left[\frac{653}{18} - (27\ln 2 + \frac{47}{4})\zeta_2 + \frac{189}{4}\zeta_3\right] \eta_1^3 \approx -0.847408, \quad (88)$$

to be compared with the Padé table

| L  | [L\0] | [L-2\2] | [L-4\4] | [L-6\6] | [L-8\8] |
|----|--------|---------|---------|---------|---------|
| 12 | 12.86  | -1.768424 | -0.816904 | -0.7050237 | 0.715507 |
| 16 | 26.58  | -0.844409 | -0.850763 | -0.839889 | -0.841839 |
| 20 | 46.98  | -0.847379 | -0.846989 | -0.847284 | -0.847540 |
| 24 | 75.42  | -0.847306 | -0.847406 | -0.847409 | -0.847405 |

which is similarly slow to settle down.

Continuing the bosonic $\varepsilon$-expansion of \[70\], we obtain the Padé table

| L  | [L\0] | [L-2\2] | [L-4\4] | [L-6\6] | [L-8\8] |
|----|--------|---------|---------|---------|---------|
| 12 | 233.3  | -1.514539 | -1.725373 | -2.423265 | -0.770594 |
| 16 | 991.4  | -1.839365 | -2.008766 | -1.971604 | -1.870298 |
| 20 | 2888.6 | -1.847849 | -1.890675 | -1.881383 | -1.881535 |
| 24 | 6847.2 | -1.842803 | -1.881245 | -1.881240 | -1.881215 |

5to be compared with the exact 3-dimensional result \[41\]

$$\eta_1 = \frac{8}{3\pi^2}; \quad \eta_2 = -\frac{8}{3}\zeta_1; \quad \eta_3 = -\left[\frac{707}{18} - (27\ln 2 - \frac{61}{4})\zeta_2 + \frac{189}{4}\zeta_3\right] \eta_1^3 \approx -1.881235. \quad (91)$$

The convergent $\varepsilon$-expansion of \[71\], for $\phi^4$-theory in $4 - 2\varepsilon$ dimensions, gives

| L  | [L\0] | [L-2\2] | [L-4\4] | [L-6\6] | [L-8\8] |
|----|--------|---------|---------|---------|---------|
| 12 | -3.205048 | -2.899329 | -1.449140 | -2.033950 | -2.675782 |
| 16 | -2.067376 | -1.861968 | -1.881627 | -1.885853 | -1.876600 |
| 20 | -1.880789 | -1.881162 | -1.881278 | -1.881218 | -1.881232 |
| 24 | -1.881231 | -1.881235 | -1.881234 | -1.881234 | -1.881235 |

for the same exponent. The convergence is almost as slow as for the $\sigma$-model, in \[90\].

6 Conclusions

Our findings have consequences for field theory, number theory, and knot theory.

From the point of view of field theory, we have reduced the two-loop diagram of Fig. 2a, with up to three dressed lines, and two adjacent lines free of dressings, to a pair of $F_2$ series. A particular case yields $O(1/N^3)$ critical exponents in terms of $\Gamma$ functions, their derivatives, and a single source of non-zetas, which we have reduced, via \[58\], to an elementary double sum in \[53\], whose $\varepsilon$-expansion generates non-alternating Euler double sums. The 12-loop analytical result \[52\] ensues in the supersymmetric $\sigma$-model. In other

5In Eq (4.7) of \[61\], 244 should be replaced by 224. In Eq (5.1) of \[46\], the final term should be divided by 2; hence 167 should be replaced by 653 in Eq (5.2).
field theories, perturbative expansions contain the same transcendentals but are lengthier, due to level mixing. Numerical results are now obtainable to any desired order and level of precision. Padé resummation of $\varepsilon$-expansions, in four distinct field theories, to 24 loops, reproduces analytical 3-dimensional results, albeit with painfully slow convergence.

From the point of view of number theory, we have given a systematic method to exploit the group theory of Saalschützian $3\mathrm{F}_2$ series, so as to obtain the expansion of (10) from (26–31), in terms of $\Gamma$ functions and the wreath-product invariant expansion (43), whose non-zeta Taylor coefficients are enumerated in (33), whilst the zeta-reducible ones are obtained from (11–15). The non-zeta terms are irreducible Euler sums: at level 8, and again at level 10, one encounters an irreducible double sum; at level 11, an irreducible triple sum; and at level 12, two irreducibles, which may be taken as a pair of alternating double sums, or as a non-alternating double sum and a non-alternating quadruple sum.

From the point of view of knot theory, all five irreducibles with levels up to 12 are associated with positive knots more complex than the $(2n-3,2)$ torus knot that produces $\zeta_{2n-3}$ in $n$-loop counterterms. Table 1 shows the associations of the 3-braids $8_{19}$ and $10_{124}$ with alternating double sums of the form (33), and of the uniquely positive hyperbolic 11-crossing 4-braid with the knot-number (38), which is first irreducible member of a class of triple sums whose general form is given in [14]. At 12-crossings, we associate the first two of the seven 3-braid knots of Table 1 with linear combinations of the knot numbers $N_{9,3}$ and $N_{7,5} - \frac{\pi^{12}}{2^{10}11!}$, though we have no method, at present, to determine these linear combinations, since diagrams that entail double sums give link diagrams whose skeinings generate both knots. If only one knot is associated with the non-alternating double sum of the $O(1/N^3)$ results, then it is probably $\sigma_1\sigma_2^4\sigma_1\sigma_2^3$, whose Jones polynomial is simpler than that of the other candidate, $\sigma_1\sigma_2^5\sigma_1\sigma_2^5$. However, there is no compelling reason to suppose that the absence of alternating double sums at $O(1/N^3)$ is a signal that only one member of the pair is entailed. At higher order in $1/N$, we confidently expect that alternating double sums will occur. More generally, at crossing number $2m + 4$, we associate the $|m/2|$ knots, in (34), with linear combinations of the $|m/2|$ numbers, in (33), modulo a multiple of $\pi^{2k+2l+4}$ that is required to make $N_{2k+3,2l+1}$ a knot-number, for $k > l$. Further arguments in favour of this association will be presented in [15], where knots with up to 14 crossings are studied. Conspicuous among our findings is the impossibility of generating the knots $10_{139}$ and $10_{152}$ from counterterms obtained by arbitrary dressings of the tetrahedron whose skeleton delivers $\zeta_3$. This strengthens our belief that their knot-numbers entail transcendentals more complex than Euler sums, as is expected from the fact that they have been generated, to date, only by the most complex of 7-loop counterterms, with multiple sums weighted by the squares of 6–j symbols [10].

In conclusion: expansions of critical exponents and anomalous dimensions at $O(1/N^3)$ entail non-zeta terms because the link diagrams that encode the intertwining of momenta in the associated Feynman diagrams yield the knots (34). The coefficients of the associated transcendentals are now obtainable, via (58), to arbitrarily high order, thanks to application of the wreath-product transformations of [19]. The knot theory of [8, 9, 10], applied to this problem, has led to discoveries in number theory [14].

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Figure 1: A 6-loop graph for the coupling of $\phi^4$ theory, giving a non-zeta counterterm associated with the 8-crossing positive 3-braid knot, $8_{19}$. 
Figure 2: The two-loop two-point function (a) is obtained by cutting the log-divergent tetrahedral vacuum diagram (b) on the line with index $\alpha_6 = 3d/2 - \sum_{n=1}^{5} \alpha_n$. The index $\alpha_{10} = \alpha_1 + \alpha_2 + \alpha_3 - d/2$ is associated with the vertex where lines 1,2,3 meet.
Figure 3: Generation of $S_{19} = \sigma_1 \sigma_2^3 \sigma_1 \sigma_2^3$ by self-energy insertions.
Figure 4: Generation of $10_{124} = \sigma_1 \sigma_2^5 \sigma_1 \sigma_2^3$ by self-energy insertions.
Figure 5: Generation of the 12-crossing 3-braid $\sigma_1\sigma_2^7\sigma_1\sigma_2^3$ by self-energy insertions.
Figure 6: Generation of the 12-crossing 3-braid $\sigma_1 \sigma_2^5 \sigma_1 \sigma_2^5$ by self-energy insertions.
Figure 7: Generation of the 9-crossing 4-braid positive factor knot $\sigma_1^3 \sigma_2^3 \sigma_3^3$ by self-energy insertions.
Figure 8: Generation of the 11-crossing 4-braid positive prime knot $\sigma_1^2\sigma_2^2\sigma_1\sigma_3^3\sigma_3^3$ by self-energy insertions.