Collapsing Shells and the Isoperimetric Inequality for Black Holes

G.W. GIBBONS
D.A.M.T.P.
University of Cambridge
Silver Street
Cambridge CB3 9EW
U.K.

ABSTRACT

Recent results of Trudinger on Isoperimetric Inequalities for non-convex bodies are applied to the gravitational collapse of a lightlike shell of matter to form a black hole. Using some integral identities for co-dimension two surfaces in Minkowski spacetime, the area $A$ of the apparent horizon is shown to be bounded above in terms of the mass $M$ by the $16\pi G^2 M^2$, which is consistent with the Cosmic Censorship Hypothesis. The results hold in four spacetime dimensions and above.

0 Introduction

Some years ago Penrose suggested a means of producing a counter-example to the Cosmic Censorship Hypothesis by considering the collapse at the speed of light of a thin shell, $\mathcal{N}$, of matter with total mass $M$ [1]. During the collapse a closed marginally outer trapped outer 2-surface $T$ is formed with area $A(T)$. Penrose pointed out that consistency with our conventional ideas demands that

$$GM \geq \sqrt{\frac{A}{16\pi}}, \quad (0.1)$$

with equality only in the spherically symmetric case. This inequality, which is sometimes called the Penrose inequality, is expected to hold for the ADM mass of a general asymptotically flat data set containing an apparent horizon. It has thus come to be of interest in its own right, quite independently of its connection with
Cosmic Censorship. Its general validity would constitute a significant and geometrically elegant strengthening of the Positive Mass Theorem. It has an appealing interpretation in terms of the variational characterization of the static black hole equilibrium states. Finally, and more speculatively, one may attempt to extend the notion of the Bekenstein-Hawking entropy $S$ for stationary event horizons to the time dependent case

$$S(t) = \frac{1}{4G} A(t)$$

where now $A(t)$ is the area of a cross section of the event horizon, i.e of a black hole $B(t)$, at time $t$. One might then attempt to bound $A(t)$ below by the area $A(t)$ of an apparent horizon lying inside the black hole $B(t)$.

For these reasons, and because it actually coincides with it in some particular cases, I have suggested [2] calling the inequality (1) the ‘isoperimetric inequality for black holes’. This would not only indicate its intrinsic importance but puts it in the more general context of the basic geometric inequalities, such as the eponymous example, which play such a significant role in geometry and physics. Moreover emphasising this connection might lead to the importation into black hole theory of further useful ideas and techniques from global analysis.

The initial evidence for the isoperimetric inequality was limited to instructive particular examples some of them reported in [1] and [3,4]. Since that time more evidence has accumulated but we still lack a general proof. It has also become clear that the conjecture need not be restricted to four space-time dimensions. With the obvious appropriate adjustment of the factor $16\pi$ to take into account conventions about the definition of the ADM mass and the areas of unit spheres in higher dimensions the inequality should continue to hold. In five spacetime dimensions this would lead to some interesting properties of the euclidean gravitational action. It is also possible to generalize the inequality to incorporate a cosmological term.

Penrose took the interior $M^-$ of the shell to be isometric to the interior of a null hypersurface $N$ in flat Minkowski spacetime $\mathbb{E}^{3,1}$. This allows one to calculate both the mass $M$ and the area $A$ using Minkowski geometry. In [3] I pointed out that if the marginally outer-trapped surface $T$ lay in a flat spacelike hyperplane, then the inequality is equivalent to a well known inequality of Minkowski [5] relating the mean curvature integral of a convex body $T$ in euclidean 3-space $\mathbb{E}^3$ to its area. In fact Minkowski’s inequality is, in the case of a convex body, intimately
connected with the standard isoperimetric inequality for the volume enclosed by a closed surface of area $A(T)$. I also showed, by means of some calculations that the more general case in which $T$ did not lie in a flat spacelike hyperplane would also hold if Minkowski’s inequality continued to hold for a body in euclidean 3-space $E^3$ whose mean curvature

$$J = \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$$  \hspace{1cm} (0.3)$$

is non-negative. Here $R_1$ and $R_2$ are the principal radii of curvature, both of which must be positive if the body is to be (strictly) convex. In fact if $T$ lies in a flat hyperplane the conditions of the problem demand that $T$ be convex but if $T$ does not lie in a flat hyperplane then my procedure was to project $T$ orthogonally onto an arbitrarily chosen spacelike hyperplane in its past to give a projected 2-surface $\hat{T}$ and then to work with $\hat{T}$.

My calculations showed that $\hat{T}$ must have non-positive mean curvature and a few numerical calculations, especially with a catenoid capped with two flat discs, encouraged me to believe that indeed Minkowsksi’s inequality continued to hold but all the proofs in the literature that I could find at that time [6] made essential use of convexity and did not seem to generalize. For that reason I did not publish my calculations.

Subsequently Tod [7,8,9,10,11] made further progress. Following work by Penrose he considered the special case when $\mathcal{N}$ is the past light cone of a point in Minkowski spacetime $E^{3,1}$. Remarkably he was able to show the equivalence in that case with another example of the iso-perimetric inequality.

Recently Robert Bartnik has drawn my attention to some new work of Trudinger [12] in which he establishes a large number of strengthened iso-perimetric inequalities. Among them is the case I needed. Thus we now have the general proof for the example considered originally by Penrose. Moreover Trudinger’s work is valid in any dimension and so encourages one to generalize the the original setup to any spacetime dimension and check the inequality in that case. That is the purpose of this present paper which thus includes my original unpublished calculations as a special case.

The paper begins in section (1) with a description of the basic set up in $n + 1$ spacetime dimensions and then in section (2) develops a convenient formalism for describing the integral differential geometry of a spacelike $n - 1$ surface $T$ embedded
in \( n + 1 \) dimensional Minkowski-spacetime \( \mathbb{E}^{n,1} \). In the original case of \( n = 3 \) this is a fairly straightforward adaptation of a discussion in [13] for 2-surfaces embedded in 3-dimensional euclidean space \( \mathbb{E}^3 \). It is hoped that this formalism may prove useful for other purposes. This machinery is used in section (3) to establish some useful integral identities. In sections (4) and (5) we establish our basic result and section (6) is a conclusion.

1 Collapsing Shells.

The situation envisaged by Penrose [1] which has been elaborated upon in detail by Barrabés and Israel [14] is that of a thin shell of matter which collapses at the speed of light in an asymptotically flat spacetime \( \mathcal{M} \). The shell is an ingoing null hypersurface \( \mathcal{N} \) and the spacetime \( \mathcal{M} \) is the union of the interior \( \mathcal{M}^- \) and the exterior \( \mathcal{M}^+ \) of \( \mathcal{N} \). The interior \( \mathcal{M}^- \) is taken to be isometric to a null hypersurface, which we also call \( \mathcal{N} \), in flat Minkowski spacetime \( \mathbb{E}^{n,1} \). The exterior region \( \mathcal{M}^+ \) is not flat and contains (unless the collapse is spherically symmetric) gravitational radiation. The shell \( \mathcal{N} \) carries energy and momentum of the form

\[
T_{\mu\nu} = \epsilon n_{\nu} n_{\mu}
\]

with \( n_{\mu} n^{\mu} = 0 \). In the limit that the shell becomes indefinitely thin we have distributional source and the spacetime has a discontinuity across \( \mathcal{N} \).

The world lines of the null matter are then taken to coincide with the null geodesic generators \( n^{\alpha} \) of the null hypersurface \( \mathcal{N} \) and it is convenient to choose them to be affinely parameterized so that

\[
n^{\alpha} ;_{\beta} n^{\beta} = 0.
\]

Because \( \mathcal{N} \) is ingoing the expansion should be negative, i.e.

\[
n^{\alpha} ;_{;\alpha} < 0.
\]

Note that the divergence coincides with the expansion because we are using an affine parameterization.

One now considers the outgoing null hypersurfaces \( \mathcal{L} \) starting out from an \((n - 1)\)-dimensional surface \( S \) lying just outside the shell \( \mathcal{N} \). Let \( l^{\alpha} \) be the geodesic generators of \( \mathcal{L} \) which we can also take to be affinely parameterized. These outgoing
null geodesics may be expected to escape to infinity if $S$ is well to the past but eventually one would expect, as one moves it to the future along $\mathcal{N}$, that $S$ would become outer trapped, i.e. everywhere on $T$ one has, because we are using an affine parameterization,

$$l^\alpha;_\alpha < 0.$$ (1.4)

Since the outgoing null geodesics started out inside $\mathcal{N}$ which is isometric to a portion of Minkowski spacetime they must have started out diverging i.e. with

$$l^\alpha;_\alpha > 0.$$ (1.5)

By applying the Raychaudhuri equation to a bundle of null geodesic rays parallel to $l^\alpha$ one deduces that the change in the expansion is proportional to the energy carried by the shell. Now imagine that there is a marginally trapped surface $T$ lying on $\mathcal{N}$, i.e. one for which everywhere on it

$$l^\alpha;_\alpha = 0.$$ (1.6)

One deduces that the energy carried by the shell is proportional to expansion of the outward null normals of $T$, regarded as embedded in Minkowski spacetime. Moreover because the surface is marginally trapped it cannot lie outside the event horizon and thus its area $A(T)$, again regarded as embedded in Minkowski spacetime, should provide a lower bound for the final area of the event horizon. Note that it is only consistent to assume that the metric inside the shell is flat if no points in $\mathcal{M}^-$ are in the timelike future of the collapsing shell. This means that the null geodesic generators of $\mathcal{N}$ can have no caustics or focal points to the past of the marginally outer trapped surface $T$. If it happens that $T$ lies in a flat spacelike hyperplane then $T$ must therefore be convex.

In this way one reduces the general relativity problem to one about the geometry of surfaces in Minkowski spacetime. Specifically one needs to know the ratio of the $\frac{1}{n-1}$ root of the area $A(T)$ to $\frac{1}{n-2}$ root of the the integral

$$\int_T l^\alpha;_\alpha dA.$$ (1.7)

One does not really need the factor of proportionality since this is determined by the spherically symmetric, i.e. $SO(n)$-invariant, case. Therefore we shall not work it out explicitly here.
2 Co-dimension two surfaces in Minkowski spacetime.

We are now going to discuss the differential geometry of an $n - 1$ dimensional sub-manifold $T$ of $n + 1$ dimensional Minkowski spacetime. We are of course interested in the case when the surface $T$ is spacelike but most of the general formalism goes through if the surface is timelike and would thus be applicable to the motion of an $n - 2$-brane. In fact it would not be difficult to generalize much of the formalism to the case of arbitrary $p$-branes. The general description of embeddings is of course not new but what is important for us and perhaps not as widely known is the discussion of integral formulae.

An immersion, or in our case an embedding, $X : T \to \mathbb{E}^{n,1}$ of a co-dimension surface $T$ into Minkowski spacetime $\mathbb{E}^{n,1}$ in local coordinates is specified by the embedding functions $X^\alpha(q^a)$ where $(q^a)$, $a = 1, \ldots, n - 1$ are coordinates on $T$ and $X^\alpha$, $\alpha = 0, 1, \ldots, n$ are inertial coordinates on $\mathbb{E}^{n,1}$. In what follows we shall make repeated and unacknowledged use of the fact that vectors in Minkowski spacetime may be parallelly propagated unambiguously. The induced metric $g_{ab}$ on $T$ is given by

$$g_{ab} = \frac{\partial X^\alpha}{\partial q^a} \eta_{\alpha\beta} \frac{\partial X^\beta}{\partial q^b},$$

where $\eta_{\alpha\beta} = \text{diag}(-1, 1, \ldots, 1)$ is the flat metric, and is assumed to be spacelike. Therefore $T$ has two future directed lightlike normals $l^\alpha$ and $n^\beta$ with $\eta_{\alpha\beta}l^\alpha l^\beta = 0 = \eta_{\alpha\beta}n^\alpha n^\beta$ which are assumed outward and inward respectively. For convenience we have partially fixed the normalization by the condition

$$l^\alpha n_\beta = -1$$

where indices have been lowered using the metric $\eta_{\alpha\beta}$. In what follows we shall not, for brevity, always distinguish verbally between covariant or contravariant Minkowski tensors, although the correct placing of greek indices in formulae will be strictly adhered to.

Later we will fix the remaining freedom in the scaling of the of the null normals by introducing an arbitrary constant future directed timelike vector $t^\alpha$ such that $t^\alpha t_\alpha = -1$ and then fix the scale of the inward null normal by $n^\alpha t_\alpha = -1$. If we define $\gamma = -t^\alpha l_\alpha$ one has:

$$t^\alpha = s^\alpha + \gamma n^\alpha + l^\alpha$$

(2.3)
where \( s^\alpha \) is a spacelike vector tangent to \( T \) and hence orthogonal to the null normals \( l^\alpha \) and \( n^\alpha \). Since

\[
\gamma = \frac{1}{2} \left( 1 + s^\alpha s_\alpha \right),
\]

one has \( \gamma \geq \frac{1}{2} \).

Now, regardless of whether we introduce \( t^\alpha \) or not, we may invert the pulled back metric \( g_{ab} \) to give \( g^{ab} \) and then push it forward to give a rank \( (n-1) \) projection operator

\[
g^{\alpha\beta} = g^{ab} \frac{\partial X^\alpha}{\partial q^a} \frac{\partial X^\beta}{\partial q^b} = \eta^{\alpha\beta} + l^\alpha n^\beta + n^\alpha l^\beta.
\]

The projection operator is idem-potent in the sense that

\[
g^{\alpha\beta} \eta_{\beta\gamma} g^{\gamma\delta} = g^{\alpha\delta},
\]

and its kernel is spanned by the two null normals:

\[
g^{\alpha\beta} l_\beta = 0 = g^{\alpha\beta} n_\beta,
\]

and thus using it any Minkowski vector \( F^\alpha \) may be projected onto \( T \)

\[
F^\alpha = \overline{F}^\alpha + n^\alpha (F^\beta n_\beta) + l^\alpha (F^\beta l_\beta).
\]

The essence of our formalism is to regard extrinsic quantities such as \( l^\alpha \) as functions, or in this case \((n+1)\)-tuples of functions, on \( T \). Therefore we introduce a derivative operator \( \mathcal{D}^\alpha \) acting on scalars and differentiating them along \( T \) by

\[
\mathcal{D}^\alpha = \frac{\partial X^\alpha}{\partial q^a} g^{ab} \frac{\partial}{\partial q^b} = g^{\alpha\beta} \partial_\beta.
\]

It is not difficult to give \( \mathcal{D} \) an invariant characterization in terms of the pull-back under the embedding \( X \) but since this does not greatly expedite the calculations I shall not do so. Since \( l^\alpha \) and \( n^\alpha \) are normal to \( T \) one has

\[
l_\alpha \mathcal{D}^\alpha = 0 = n_\alpha \mathcal{D}^\alpha.
\]

Since \( T \) has co-dimension 2 it has two second fundamental forms. Using the two lightlike normals we may define

\[
L_{ab} = l_\alpha \frac{\partial^2 X^\alpha}{\partial q^a \partial q^b} = -\frac{\partial X^\alpha}{\partial q^a} \frac{\partial l_\alpha}{\partial q^b} = -\frac{\partial X^\alpha}{\partial q^a} \frac{\partial X^\beta}{\partial q^b} \partial_\alpha l_\beta
\]
and
\[ N_{ab} = n_\alpha \frac{\partial^2 X^\alpha}{\partial q^a \partial q^b} = -\frac{\partial X^\alpha}{\partial q^a} \frac{\partial n_\alpha}{\partial q^b} = -\frac{\partial X^\alpha}{\partial q^a} \frac{\partial X^\beta}{\partial q^b} \partial_\alpha n_\beta. \] (2.12)

The two second fundamental forms \( L_{ab} \) and \( N_{ab} \) are symmetric by virtue of the fact that
\[ \frac{\partial X^\alpha}{\partial q^a} l_\alpha = 0 = \frac{\partial X^\alpha}{\partial q^a} n_\alpha. \] (2.13)

We also define their traces
\[ \mu = -\frac{1}{2} D^\alpha n_\alpha = -\frac{1}{2} g^{ab} N_{ab}, \] (2.14)
and
\[ \rho = \frac{1}{2} D^\alpha l_\alpha = -\frac{1}{2} g^{ab} L_{ab}. \] (2.5)

As we have defined them the second fundamental forms depend only on the behaviour of the null normals on the surface \( T \). However in the problem we are considering \( l_\alpha \) and \( n_\alpha \) are defined off the surface \( T \) as the tangents to the null geodesic generators of the ingoing \( N \) and outgoing \( L \) null hypersurfaces through \( T \).

In particular if we chose them both to be affinely parameterized then
\[ l^\beta \partial_\beta l^\alpha = 0 = n^\beta \partial_\beta n^\alpha. \] (2.16)

It then follows that
\[ 2 \rho = \partial_\alpha l^\alpha, \] (2.17)
and
\[ -2 \mu = \partial_\alpha n^\alpha. \] (2.18)

The quantities \( \rho \) and \( -\mu \) may thus be regarded as the expansions of the outward and inward normals \( l^\alpha \) and \( n^\alpha \) respectively. In the usual case both \( \mu \) and \( \rho \) will be positive. If \( T \) is marginally outer trapped then \( \rho \) will vanish. If both vanish then the spacelike surface \( T \) will be extremal with respect to the area functional.
3 Integral Formulae

We have the following identity for any Minkowski vector $F^\alpha$

$$D_\alpha F^\alpha = D^\alpha \overline{F}_\alpha - 2\rho(F_\alpha n^\alpha) + 2\mu(F_\alpha t^\alpha)$$  \hspace{1cm} (3.1)

Now the first term on the righthand side is just the covariant divergence of the projected vector $\overline{F}^\alpha$. To check this explicitly one writes

$$\overline{F}^\alpha = \frac{\partial X^\alpha}{\partial q^a} F^a$$  \hspace{1cm} (3.2)

and recalls that the Christoffel symbols of the metric $g_{ab}$ are given by

$$[ab, c] = \frac{\partial X^\alpha}{\partial q^c} \eta_{\alpha\beta} \frac{\partial^2 X^\beta}{\partial q^a \partial q^b}. \hspace{1cm} \text{ (3.3)}$$

Thus if $T$ is closed (i.e. compact without boundary) then Stokes’s theorem on $T$ yields:

$$\int_T (D^\alpha F_\alpha) dA = -2 \int_T \rho(F_\alpha n^\alpha) dA + 2 \int_T \mu(F_\alpha t^\alpha) dA,$$  \hspace{1cm} (3.4)

where $dA$ is the area element on $T$. This formula leads to some interesting integral identities.

If we take $F^\alpha = X^\alpha$ we obtain

$$A = -2 \int_T \rho(X_\alpha n^\alpha) dA + 2 \int_T \mu(X_\alpha t^\alpha) dA.$$  \hspace{1cm} (3.5)

If we choose for $F^\alpha$ the arbitrary constant future directed timelike vector $t^\alpha$ introduced above we obtain

$$\int_T \rho dA = \int_T \mu \gamma dA.$$  \hspace{1cm} (3.6)

As an illustration we note that these formulae yield a simple proof that Minkowski spacetime admits no closed trapped or marginally trapped spacelike surfaces. By definition this requires $\mu$ positive and $\rho$ non-positive. This can only happen if both $\rho$ and $\mu$ vanish which can only occur if the area of the spacelike surface $T$ also vanishes which is a contradiction. If we choose for $F^\alpha$ an arbitrary constant Minkowski vector we obtain:

$$\int_T \mu l^\alpha dA = \int_T \rho n^\alpha dA.$$  \hspace{1cm} (3.7)
By taking $F^\alpha = G^{\mu_1 \cdots \mu_p} C^\alpha_{\mu_1 \cdots \mu_p}$ where $C_{\alpha \mu_1 \cdots \mu_p}$ is an arbitrary constant Minkowski tensor we obtain

$$\int_T \mathcal{D}^\alpha G^{\mu_1 \cdots \mu_p} dA = -2 \int_T \rho G^{\mu_1 \cdots \mu_p} n^\alpha dA + 2 \int_T G^{\mu_1 \cdots \mu_p} l^\alpha dA. \quad (3.8)$$

Taking $G^{\beta} = X^{\beta}$ and recalling that $\mathcal{D}^\alpha X^{\beta} = \frac{\partial X^{\alpha}}{\partial q^a} g^{ab} \frac{\partial X^{\beta}}{\partial q^b}$ is symmetric in $\alpha$ and $\beta$ we obtain

$$\int_T \mu X^{[\mu \nu]} dA = \int_T \rho X^{[\mu \nu]} dA. \quad (3.9)$$

A physical interpretation of these formulae will be given shortly.

4 The case when $T$ lies in a hyperplane.

Clearly in this case we are on the familiar ground of a hypersurface in $n$-dimensional euclidean space $\mathbb{E}^n$. If $n = 3$ we recover the formalism developed in chapter XII of Weatherburn [13]. The following description will use the notation of three spatial dimensions but the discussion generalizes immediately to higher dimensions. Thus I have followed Weatherburn in using the symbol $J$ for the divergence of the normal.

If $t^\alpha$ is the timelike normal to the hyperplane and one has $s^\alpha = 0$ whence $\gamma = \frac{1}{2}$ and

$$t^\alpha = \frac{1}{2} n^\alpha + l^\alpha. \quad (4.1)$$

The surface $T$ has spacelike unit normal $\nu^\alpha$ which is orthogonal to $t^\alpha$

$$\nu^\alpha = l^\alpha - \frac{1}{2} n^\alpha. \quad (4.2)$$

Since $t^\alpha$ is constant we thus have $2 \rho = \mu$, moreover the operator $\mathcal{D}^\alpha$ is orthogonal to $t^\alpha$, i.e. $t^\alpha \mathcal{D}_\alpha = 0$ and thus

$$\rho = \frac{1}{4} \mathcal{D}^\alpha \nu_\alpha. \quad (4.3)$$

where the greek index may now be taken to range from 1 to $n$. The right hand side equals one quarter the trace of the second fundamental form of $T$ regarded as embedded in $\mathbb{E}^n$. If $n = 3$ we have

$$\rho = \frac{1}{4} J = \frac{1}{4} \left( \frac{1}{R_1} + \frac{1}{R_2} \right). \quad (4.4)$$
where \( R_1 \) and \( R_2 \) are the principal radii of curvature, i.e. the inverses of the eigenvalues of the second fundamental form.

The integral identity:

\[
\int_T (\mu l^\alpha - \rho n^\alpha) dA = 0 \tag{4.5}
\]

reduces to the known vector identity:

\[
\int_T dAJ \nu = 0 \tag{4.6}
\]

and the identity:

\[
\int_T X^{[\alpha}(\mu l^{\alpha]} - \rho n^{\alpha]} dA = 0 \tag{4.7}
\]

to the known vector identity:

\[
\int_T dAJ x \times \nu. \tag{4.8}
\]

These have the following physical interpretation. If the surface \( T \) were a perfectly elastic shell, such as a soap film, the net force per unit area it exerts is

\[
\left( \frac{1}{R_1} + \frac{1}{R_2} \right) \nu. \tag{4.9}
\]

The identities (4.6) and (4.8) are an expression of Newton’s third law: the total force and the total moment of this force must vanish. The full relativistic formulae have a similar interpretation. One may regard the quantity

\[
P^\alpha = \frac{1}{4\pi} \int_T \rho n^\alpha dA
\]

as the total ingoing four momentum associated with the surface \( T \). Note that it is independent of the scaling freedom of \( l^\alpha \) and \( n^\alpha \). Moreover from (3.7) \( P^\alpha \) is the same as the outgoing normal momentum, i.e.

\[
P^\alpha = \frac{1}{4\pi} \int_T \mu l^\alpha dA.
\]
Similarly
\[ M^{\mu\nu} = \frac{1}{2\pi} \int_T \rho X^{\mu} n^{\nu} dA \]
may be interpreted as the total ingoing relativistic angular momentum associated to \( T \). By (3.9) this equals the total outgoing relativistic angular momentum.

As noted above, by the conditions of the physical problem the surface \( T \) can have no focal points in its past and therefore in the present case \( T \) must be a convex surface. The isoperimetric inequality now reduces to the original Minkowski inequality [5] for convex bodies:
\[ \frac{1}{8\pi} \int_T J dA \geq \sqrt{\frac{A}{4\pi}}. \] (4.10)

5 The case when \( T \) does not lie in a hyperplane.

In this case one may arbitrarily choose a spacelike hyperplane with unit normal \( t^\alpha \) and project \( T \) orthogonally onto it by means of the projection operator:
\[ h^\beta_\alpha = \delta^\beta_\alpha + t_\alpha t^\beta. \] (5.1)
The projected surface we call \( \hat{T} \). Thus if \( T \) is given by \( q^a \rightarrow (t(q^a), X(q^a)) \), then \( \hat{T} \) is given by \( q^a \rightarrow (0, X(q^a)) \). The unit normal to \( \hat{T} \) orthogonal to \( t^\alpha \) must satisfy \( t^\alpha \hat{\nu}_\alpha = 0 \) and \( \hat{\nu}_\alpha h^{\alpha\beta} a_\beta = 0 \) for all vectors \( a_\beta \) tangent to \( T \) and is therefore given by
\[ \hat{\nu}^\alpha = \frac{1}{\sqrt{2\gamma}} (l^\alpha - \gamma n^\alpha). \] (5.2)

The area element \( d\hat{A} \) of the projected surface \( \hat{T} \) is less than the area element \( dA \) of the surface \( T \). The induced metric \( \hat{g}_{ab} \) on \( \hat{T} \) is related to that on \( T \), \( g_{ab} \), by
\[ \hat{g}_{ab} = g_{ab} + \frac{\partial t}{\partial q^a} \frac{\partial t}{\partial q^b}. \] (5.3)
Thus
\[ \det \hat{g}_{ab} = \det \hat{g}_{ab} (1 + g^{ab} \frac{\partial t}{\partial q^a} \frac{\partial t}{\partial q^b}). \] (5.4)
Now \( \frac{\partial t}{\partial q^a} \) is the spatial projection \( s^\alpha \) of the unit normal \( t^\alpha \) onto \( T \). One has
\[ 1 + s_\alpha s^\alpha = 2\gamma \] (5.5)
and therefore

\[ dA = \sqrt{\frac{1}{2\gamma}} d\hat{A}. \quad (5.6) \]

We now evaluate the mean curvature \( \hat{J} \) of the projected surface \( \hat{T} \). If \( \hat{\partial}_\alpha = h_\alpha^\beta \partial_\beta \) is the purely spatial derivative orthogonal to \( t^\alpha \) one has the standard result that

\[ \hat{J} = \hat{\partial}_\alpha \hat{\nu}^\alpha. \quad (5.7) \]

But

\[ \hat{\partial}_\alpha = \partial_\alpha + t_\alpha t_\beta \partial_\beta. \quad (5.8) \]

Using the fact that \( \hat{\nu}^\alpha t_\alpha = 0 \),

\[ \partial_\alpha \hat{\nu}^\alpha + t_\alpha (t_\beta \partial_\beta \hat{\nu}^\alpha) = \partial_\alpha \hat{\nu}^\alpha - \nu_\alpha (t_\beta \partial_\beta t^\alpha). \quad (5.9) \]

But \( t^\alpha \) is a constant vector and therefore

\[ \hat{J} = \partial_\alpha \hat{\nu}^\alpha = \partial_\alpha (\frac{1}{\sqrt{2\gamma}} (l^\alpha - \gamma n^\alpha)). \quad (5.10) \]

Therefore finally

\[ \frac{1}{2} \hat{J} = \frac{1}{\sqrt{2\gamma}} \rho + \sqrt{\frac{\gamma}{2}} \mu. \quad (5.11) \]

Note that \( \hat{J} \) is necessarily non-negative. We may now use our integral identities and the relation (5.6) between \( dA \) and \( d\hat{A} \) to show that

\[ \int_T \rho dA = \int_T \gamma \mu dA = \frac{1}{4} \int_T \hat{J} d\hat{A}. \quad (5.12) \]

We are now done because the ratio

\[ (4 \int_T \rho dA)^{\frac{1}{n-2}} / A^{\frac{1}{n-1}} \quad (5.13) \]

is clearly never less than the ratio

\[ (\int_T \hat{J} d\hat{A})^{\frac{1}{n-2}} / \hat{A}^{\frac{1}{n-1}}, \quad (5.14) \]
and this latter ratio is, by Trudinger’s [10] strengthened form of the Minkowski inequality, never less than the value it takes for the standard round embedding of the \((n - 1)\)-sphere.

I am grateful to Paul Tod for suggesting the following formulation of the result. The energy \(P^0\) of the shell in the frame determined by \(t^\alpha\) is given by

\[
P^0 = \frac{1}{4\pi} \int T \rho dA.
\] (5.15)

We have shown that in all frames

\[
P^0 \geq \sqrt{\frac{A}{16}}.
\] (5.16)

Thus \(P^\alpha\) is future directed timelike and

\[
-P^\alpha P_\alpha \geq \frac{A}{16\pi G^2}.
\] (5.17)

6 Conclusion

After some delay it is now clear that the answer to Penrose’s original question: ‘can one set up a contradiction to Cosmic Censorship using collapsing shells?’ is definitely no. The reason is the existence of an inequality which, links the isoperimetric properties of black hole with the second law of thermodynamics.

An intriguing question for further study is whether there exist further geometric inequalities which constrain gravitational collapse. For charged bodies there is of course a Bogomol’nyi bound [2]. Perhaps this has a geometrical interpretation for collapsing shells. We also know that in four spacetime dimensions black holes cannot have greater angular momentum than

\[
G^2 M^2.
\] (6.1)

However this cannot be a general upper bound for all possible data sets because two particles scattering against one another with very little energy may carry a large amount of angular momentum if their impact parameter is sufficiently great. Nevertheless (6.1) suggests that it might be worthwhile investigating the angular momentum using the methods of this paper. In fact there are some similarities with
a classical Regge inequality for the angular momentum of a string [15]. However while the Regge inequality holds in all dimensions there is no upper bound like (6.1) for black holes in higher dimensions [16]. Presumably a Regge inequality holds for $p$-branes as well. Finally, in connection with higher dimensions it is worth pointing out that Trudinger establishes inequalities for integrals of other elementary symmetric functions of the principal radii of curvature. It would be interesting to know what, if any, is their physical significance.

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