ATOMIC BASES IN CLUSTER ALGEBRAS OF TYPES $A$ AND $\tilde{A}$

GRÉGOIRE DUPONT AND HUGH THOMAS

Abstract. We give explicit atomic bases of arbitrary coefficient-free cluster algebras of types $A$ and $\tilde{A}$.

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1. INTRODUCTION AND MAIN RESULTS

1.1. Cluster algebras. Cluster algebras were introduced by Fomin and Zelevinsky in the early 2000’s in order to provide a combinatorial framework for studying total positivity and dual canonical bases in semisimple groups [FZ02]. Since then, cluster algebras have shown interactions with various areas of mathematics like combinatorics, Lie theory, Poisson geometry, Teichmüller theory, mathematical physics and representation theory.

A (skew-symmetric) cluster algebra is defined from a seed, that is a pair $(Q, \mathbf{x})$ where $Q$ is a finite connected quiver with $n$ vertices and without oriented cycles of length $l \leq 2$ and where $\mathbf{x} = (x_1, \ldots, x_n)$ is a $n$-tuple of variables, called the cluster of the seed. A combinatorial process, called mutation, allows one to define recursively a (possibly infinite) family of seeds. The (coefficient-free) cluster algebra $\mathcal{A}_Q$ is the $\mathbb{Z}$-subalgebra of the ambient field $\mathbb{Q}(x_1, \ldots, x_n)$ generated by the union of all the clusters of the seeds arising from this mutation procedure. It is therefore naturally equipped with a $\mathbb{Z}$-module structure. A free generating set for this $\mathbb{Z}$-module structure is called a $\mathbb{Z}$-linear basis of $\mathcal{A}_Q$.

The cluster structure of $\mathcal{A}_Q$ naturally endows it with a distinguished set of elements, the cluster monomials, which are the monomials in cluster variables all belonging to a single cluster. The set of cluster monomials in $\mathcal{A}_Q$ is denoted by $\mathcal{M}_Q$. As was proved for instance by Lampe for (quantum) cluster algebras of type $A$, see [Lam11], this set plays a prominent role in the construction of $\mathbb{Z}$-linear bases of $\mathcal{A}_Q$ which are of interest with respect to the study of dual canonical bases.

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A remarkable fact about cluster algebras is the so-called *Laurent phenomenon*, proved in [FZ02], which asserts that for any cluster \( c = (c_1, \ldots, c_n) \) in \( \mathcal{A}_Q \), the cluster algebra \( \mathcal{A}_Q \) is a subring of \( \mathbb{Z}[c_1^{\pm 1}, \ldots, c_n^{\pm 1}] \). An element in \( \mathcal{A}_Q \) is called *positive* if it belongs to the semiring \( \mathbb{Z}_{\geq 0}[c_1^{\pm 1}, \ldots, c_n^{\pm 1}] \) for any cluster \( c = (c_1, \ldots, c_n) \) in \( \mathcal{A}_Q \). We denote by \( \mathcal{A}_Q^+ \) the cone of positive elements in \( \mathcal{A}_Q \). The *positivity conjecture* asserts that every cluster monomial in \( \mathcal{A}_Q \) is a positive element of \( \mathcal{A}_Q \), see [FZ02]. This conjecture was in particular established for cluster algebras with a bipartite seed [Nak10] and for cluster algebras arising from surfaces [ST09, MSW09] but remains open in general. Note that this latter class of cluster algebras contains the class of cluster algebras of types \( A \) and \( \tilde{A} \) which are considered in the present article.

A \( \mathbb{Z} \)-basis \( \mathcal{B} \) of \( \mathcal{A}_Q \) is called an *atomic basis* (or a *canonically positive basis*) of \( \mathcal{A}_Q \) if

\[
\mathcal{A}_Q^+ = \bigoplus_{b \in \mathcal{B}} \mathbb{Z}_{\geq 0}b.
\]

The definition of atomic bases, which first appeared in [SZ04], was motivated by the positivity of the structure constants for multiplication of dual canonical bases elements. Note that it follows from the definition that if an atomic basis exists, then it is unique. Therefore, under the existence hypothesis, we can speak of *the* atomic basis of \( \mathcal{A}_Q \). However, the problem of showing the existence of this atomic basis of \( \mathcal{A}_Q \) remains wide open in general.

If \( \mathcal{A}_Q \) is of finite type in the sense of [FZ03], Cerulli recently proved that the atomic basis coincides with the set of cluster monomials of \( \mathcal{A}_Q \), see [Cer11b]. If \( \mathcal{A}_Q \) is not of finite type, it was observed in [SZ04] that the set of cluster monomials does not necessarily generate the cluster algebra as a \( \mathbb{Z} \)-module, and therefore is not the atomic basis of \( \mathcal{A}_Q \). In the particular cases where \( Q \) is an affine quiver of type \( \tilde{A}_1 \) or \( \tilde{A}_2 \), the atomic bases were made explicit in [SZ04] and [Cer11a] respectively. In this article, we generalise this construction to arbitrary quivers of affine type \( \tilde{A} \) and we provide a new, short and elementary proof of Cerulli’s result for cluster algebras of type \( \tilde{A} \).

The article is organised as follows. In the remainder of this section we recall the necessary background on cluster algebras from surfaces and their connection to representation theory in order to state our main results. In Section 2, we prove that cluster monomials form the atomic basis in a cluster algebra of type \( A \). In Section 3 we give a combinatorial interpretation to a conjectural formula provided in [Dup10a] for the atomic basis in a cluster algebra of type \( \tilde{A} \). Finally, we prove in Section 4 that this conjectural atomic basis is indeed the atomic basis in type \( \tilde{A} \).

1.2. Marked surfaces. Following [FST08], we define an (unpunctured) *marked surface* as a pair \((S, M)\) where \( S \) is a connected oriented 2-dimensional Riemann surface with non-empty boundary \( \partial S \) and \( M \) is a finite set of marked points on \( \partial S \) such that each connected component of \( \partial S \) contains at least one marked point. Moreover, we assume that \((S, M)\) is not homeomorphic to a disc with less than three marked points.

For any \( n \geq 1 \), we denote by \( \Pi_n \) the marked surface consisting of a disc with \( n+3 \) marked points on the boundary, which we sometimes refer to as the \( n+3 \)-gon. For \( p, q \geq 1 \), we denote by \( C_{p,q} \) the marked surface consisting of an annulus with \( p \) marked points \( \iota_1, \ldots, \iota_p \) on a boundary component \( \iota \), called the *inside* and \( q \) marked
points $o_1, \ldots, o_q$ on the other boundary component, called the outside, see Figure 1 below.

Let $(S, M)$ be a marked surface. When we consider curves “up to isotopy” in $(S, M)$, we always mean “up to isotopy with respect to the set $M$ of marked points”. Given two isotopy classes $\gamma$ and $\gamma'$ of curves in $(S, M)$, we define the number of intersections $|\gamma \cap \gamma'|$ as the minimal number of intersections in the interior of $S$ of representatives of the two isotopy classes. Note that this number is reached if, once a hyperbolic structure is fixed on $(S, M)$, we consider geodesic representatives of $\gamma$ and $\gamma'$. Therefore, if one does not want to work up to isotopy, one can fix a hyperbolic structure and work with geodesic representatives. We say that two isotopy classes of curves are compatible if they are the same or if their number of intersections is zero. An isotopy class $\gamma$ of curves in $(S, M)$ is called without self-intersection if $|\gamma \cap \gamma'| = 0$.

A boundary segment is a connected component of $\partial S \setminus M$. We denote by $C(S, M)$ the set of isotopy classes of curves joining two marked points, not isotopic to a boundary segment. An arc in $(S, M)$ is an isotopy class of curves in $(S, M)$ joining two marked points, which is without self-intersection, and which is not isotopic to a boundary segment. We denote by $A(S, M)$ the set of all arcs in $(S, M)$ and by $T(S, M)$ the set of finite (possibly empty) families of pairwise compatible elements in $A(S, M)$, considered with multiplicity. A curve (or its isotopy class) is called peripheral if both its endpoints lie on a same boundary component and it is called bridging otherwise.

A triangulation $T$ of $(S, M)$ is a maximal set of pairwise distinct compatible arcs in $(S, M)$. Given a triangulation $T$ of $(S, M)$, one can associate to it a certain quiver $Q_T$ without loops and 2-cycles and thus a (coefficient-free) cluster algebra $\mathcal{A}_{Q_T}$, see [FST08]. The cluster algebra constructed in this way is independent of the choice of the triangulation $T$. It is called the cluster algebra associated to the marked surface $(S, M)$ and is denoted by $\mathcal{A}(S, M)$.

It is well-known that there is a bijection between the set $A(S, M)$ of arcs in $(S, M)$ and the set of cluster variables in $\mathcal{A}(S, M)$. Moreover, this bijection induces a bijection between the set of triangulations of $(S, M)$ and the set of clusters of $\mathcal{A}(S, M)$: inducing a bijection between $T(S, M)$ and the set of cluster monomials in $\mathcal{A}(S, M)$. Moreover, in this context, mutations of clusters corresponds to flips of triangulations, see [FST08]. Using these bijections, we will usually abuse notations and identify arcs in $(S, M)$ with cluster variables in $\mathcal{A}(S, M)$, triangulations in

![Figure 1. The marked surfaces $\Pi_5$ and $C_{2,1}$.](image_url)
With these notations, the cluster algebra $\mathcal{A}_{\Pi n}$ is of Dynkin type $A_n$ for any $n \geq 1$ and the cluster algebra $\mathcal{A}_{C_{p,q}}$ is of affine type $\tilde{A}_{p,q}$ for any $p, q \geq 1$.

Following [ST09], for any cluster $T$ in $\mathcal{A}(S,M)$, we can define an explicit map $x^T : \{ A(S,M) \rightarrow A(S,M) \gamma \mapsto x^T_\gamma \}$ which sends an arc $\gamma$ in $A(S,M)$ to the $T$-expansion of the corresponding cluster variables in $\mathcal{A}(S,M)$. This map does not depend on the choice of the triangulation so that we simply write $x^T_\gamma$ for $x^T_\gamma$. Given a family $\Gamma \in T(S,M)$, we denote by $x^\Gamma$ the corresponding cluster monomial in $\mathcal{A}(S,M)$; in other words

$$x^\Gamma = \prod_{\gamma \in \Gamma} x^\gamma$$.

1.3. **Atomic bases in type $A$.** Our first main result is a short elementary combinatorial proof of the fact that cluster monomials form the atomic basis of a cluster algebra of type $A$.

**Theorem 1.1.** Let $n \geq 1$, then

$$\{ x^\Gamma \mid \Gamma \in T(\Pi_n) \}$$

is the atomic basis of the cluster algebra $\mathcal{A}_{\Pi_n}$.

Equivalently, the atomic basis of a (coefficient-free) cluster algebra $\mathcal{A}$ of type $A$ is the set of cluster monomials in $\mathcal{A}$.

Note that this result was first obtained by Cerulli [Cer11b] using representations of quivers with potential in the sense of [DWZ08].

1.4. **Atomic bases in type $\tilde{A}$.** In the context of cluster algebras of type $\tilde{A}_{p,q}$, we will consider an additional family of isotopy classes of curves in $C_{p,q}$, which we call loops. A loop in $C_{p,q}$ is the isotopy class of a non-contractible closed curve which lies in the interior of $C_{p,q}$. Note that for any $m \geq 1$, there is a unique loop $z_m$ in $C_{p,q}$ with $m - 1$ self-intersections. We set $z = z_1$ and we denote by

$$\hat{A}(C_{p,q}) = A(C_{p,q}) \cup \{ z_m \mid m \geq 1 \}$$

the set of all arcs and loops in $C_{p,q}$ and by $\hat{T}(C_{p,q})$ the set of finite (possibly empty) families of pairwise compatible arcs or loops in $\hat{A}(C_{p,q})$, considered with multiplicity, containing at most one loop.

In Section 4.3 for any triangulation $T$ of $C_{p,q}$, we will extend the domain of definition of the maps $x^T_\gamma$ to $\hat{A}(C_{p,q})$ and prove that these new maps still do not depend on the choice of the triangulation $T$, thus defining a map $x^\Gamma$ on $\hat{A}(C_{p,q})$. As before, in order to simplify notations, if $\Gamma$ is any collection in $\hat{T}(C_{p,q})$, we adopt the following notations:

$$x^\Gamma = \prod_{\gamma \in \Gamma} x^\gamma$$

with the convention that $x^\emptyset = 1$.

Our main result is an explicit realisation of the atomic basis in any (coefficient-free) cluster algebra of type $\tilde{A}$:
Theorem 1.2. Let $p, q \geq 1$, then

$$\{ x_T \mid \Gamma \in \hat{T}(C_{p,q}) \}$$

is the atomic basis of the cluster algebra $\mathcal{A}_{p,q}$.

1.5. A representation-theoretic interpretation. An explicit expression for the atomic basis of a cluster algebra associated to an affine quiver was conjectured in [Dup10a, Conjecture 7.10] in terms of representation theory of algebras. Our third main result is the proof of this conjecture for cluster algebras of type $\tilde{A}$. Before stating it precisely, we need to recall some background concerning the representation-theoretic approach to cluster algebras of type $\tilde{A}$.

Let $p, q \geq 1$ and let $Q$ be a quiver of type $\tilde{A}_{p,q}$, that is, an orientation of the cyclic diagram with $n = p + q$ vertices with $p$ arrows going clockwise and $q$ arrows going counterclockwise. We fix an algebraically closed field $k$.

Since $Q$ is an acyclic quiver of type $\tilde{A}$, its path algebra $kQ$ is a tame hereditary algebra so that the category $\text{mod-}kQ$ of finitely generated right $kQ$-modules is well-understood. We refer the reader to [Rin84] or [SS07] for classical results on the representation theory of such algebras.

A $kQ$-module $M$ is called rigid if $\text{Ext}^1_{kQ}(M, M) = 0$. A connected component of the Auslander-Reiten quiver of $\text{mod-}kQ$ is called regular if it contains neither a projective nor an injective $kQ$-module. A $kQ$-module is called regular if all its indecomposable direct summands belong to regular components of Auslander-Reiten quiver of $\text{mod-}kQ$. We denote by $\text{reg-}kQ$ the set of regular $kQ$-modules and by $\text{reg}^0-kQ$ the set of regular rigid $kQ$-modules. The regular components of the Auslander-Reiten quiver of $\text{mod-}kQ$ form a $P^1(k)$-family of tubes. At most two tubes have a rank strictly larger than one and these have respective ranks $p$ and $q$. A tube with a rank equal to 1 is called homogeneous; otherwise, it is called exceptional. It is known that an indecomposable module in a tube is rigid if and only if its quasi-length is strictly smaller than the rank of the tube in which it is contained. In particular, homogeneous tubes do not contain any rigid modules.

Let $D^b(\text{mod-}kQ)$ denote the bounded derived category of $\text{mod-}kQ$. It is a triangulated category with suspension functor $[1]$ and Auslander-Reiten translation $\tau$. The cluster category $\mathcal{C}_Q$ is the orbit category of the functor $F = \tau^{-1}[1]$ in $D^b(\text{mod-}kQ)$. It is a triangulated 2-Calabi-Yau category [Kel05] [BMR+06] and, up to isomorphisms, the set of indecomposable objects in $\mathcal{C}_Q$ can be identified with the disjoint union of the set of indecomposable $kQ$-modules and the set of shifts of indecomposable projective modules. Therefore, we view $kQ$-modules as objects in $\mathcal{C}_Q$.

An object $T$ in $\mathcal{C}_Q$ is called cluster-tilting if for any object $X$ in $\mathcal{C}_Q$, the equality $\text{Ext}^1_{\mathcal{C}_Q}(T, X) = 0$ holds if and only if $X$ belongs to the additive category $\text{add}(T)$. It is well-known that there is a bijection between the set of cluster-tilting objects in $\mathcal{C}_Q$ and the set of clusters in $\mathcal{A}_Q$ [BMR+06] [CK06]. Thus, we will usually identify cluster-tilting objects in $\mathcal{C}_Q$ with clusters in $\mathcal{A}_Q$ (and with triangulations of $C_{p,q}$). This bijection induces a bijection between the set of isomorphism classes of rigid objects in $\mathcal{C}_Q$ (that is, objects $M$ such that $\text{Ext}^1_{\mathcal{C}_Q}(M, M) = 0$) and the set of cluster monomials in $\mathcal{A}_Q$.

This bijection can be made explicit by using the so-called cluster characters, first introduced in [CC06] and whose definition was generalised in [CK06] [Pal08]. For
any cluster-tilting object $T$ in $\mathcal{C}_Q$ we denote by $X_T^T$ the cluster character on $\mathcal{C}_Q$ associated to $T$ with values in the ring of Laurent polynomials in the cluster $T$. If $(c_1, \ldots, c_n)$ is the cluster in $\mathcal{A}_Q$ corresponding to the cluster-tilting object $T$, the cluster character is a map

$$X_T^T : \mathrm{Ob}(\mathcal{C}_Q) \rightarrow \mathbb{Z}[c_1^{\pm 1}, \ldots, c_n^{\pm 1}]$$

which endows the cluster algebra $\mathcal{A}_Q$ with a structure of a Hall algebra on the cluster category $\mathcal{C}_Q$ in the sense that for any objects $M, N$ in $\mathcal{C}_Q$, the product $X_M^T X_N^T$ is a linear combination of $X_Y^T$ where $Y$ runs over the middle terms of triangles involving $M$ and $N$ [CK08, Pal11]. In particular, if $\text{Ext}^1_{\mathcal{C}_Q}(M, N) \simeq \mathbb{k}$, then $X_M^T X_N^T = X_B^T + X_{B'}^T$, where $B$ and $B'$ are the unique objects in $\mathcal{C}_Q$ such that there exist triangles $M \rightarrow B \rightarrow N \rightarrow M[1]$ and $N \rightarrow B' \rightarrow M \rightarrow N[1]$, see [CK06, Pal08]. We refer the reader to [Pal08] for the precise definition.

Since $Q$ is an affine quiver, the cluster character $X_T^T$ takes its values in the cluster algebra $\mathcal{A}_Q$ and for any object $M$ in $\mathcal{C}_Q$, if $T$ and $T'$ are two distinct cluster-tilting objects in $\mathcal{C}_Q$, then $X_M^T = X_{M}^{T'}$, see [Dup11]. We will thus omit the reference to the cluster-tilting object $T$ and simply denote by $X_M$ the corresponding element in the cluster algebra $\mathcal{A}_Q$.

We denote by $X_\delta$ the so-called generic variable of dimension $\delta$ in $\mathcal{A}_Q$, which is given by the image of any quasi-simple module in a homogeneous tube of the Auslander-Reiten quiver of mod-$\mathbb{k}Q$, see [Dup11].

For any $m \geq 1$, we denote by $F_m$ the $m$th normalised Chebyshev polynomial of the first kind defined by

$$F_0(z) = 2, \quad F_1(z) = z \quad \text{and} \quad F_{m+1}(z) = zF_m(z) - F_{m-1}(z), \quad \text{for any} \quad m \geq 1.$$ 

They are characterised by

$$F_m(t + t^{-1}) = t^m + t^{-m}, \quad \text{for any} \quad m \geq 0.$$ 

In [Dup10a, Conjecture 7.10], it is conjectured that the set

$$\mathcal{B}_Q = \mathcal{A}_Q \sqcup \{X_R F_m(X_\delta) \mid m \geq 1, R \in \text{reg}^0 - \mathbb{k}Q\}$$

is the atomic basis of $\mathcal{A}_Q$.

Out third main result is the following theorem:

**Theorem 1.3.** Let $Q$ be a quiver of type $\tilde{A}_{p,q}$, then

$$\mathcal{B}_Q = \{x_\Gamma \mid \Gamma \in \hat{T}(G_{p,q})\}.$$ 

Thus, combined with Theorem 1.2, this proves [Dup10a, Conjecture 7.10].

2. **Proof of Theorem 1.1**

In this section, $Q$ is a quiver of type $A_n$ with $n \geq 1$. Proving Theorem 1.1 amounts to showing the following three points:

(A1) The cluster monomials form a $\mathbb{Z}$-linear basis of $\mathcal{A}_Q$.

(A2) All cluster monomials are positive elements of $\mathcal{A}_Q$.

(A3) Every positive element of $\mathcal{A}_Q$ can be written as a $\mathbb{Z}_{\geq 0}$-linear combination of cluster monomials.
In this setting, (A1) and (A2) are already well-known. (A1) follows from [CK08]. (A2) follows from the explicit positive combinatorial formulas of [Sch08]; positivity can also be shown directly by using the Ptolemy relations. The remaining step is therefore to prove (A3).

2.1. A combinatorial formula for $A_n$ cluster variables. $\Pi_n$ is the disc with $n + 3$ marked points on the boundary. Recall that the cluster variables of an $A_n$ cluster algebra are in bijection with $C(\Pi_n)$. We begin by recalling a formula for expanding the cluster variable corresponding to a given curve $\gamma$ in terms of the cluster variables corresponding to a triangulation $T$ of $\Pi_n$.

We write $W_T^{\gamma}$ for the set of walks joining the two endpoints of $\gamma$ satisfying the following:

1. Each edge of the walk is either an arc of $T$ or a boundary segment.
2. No edge of the walk is immediately followed by the same edge in the reverse direction.
3. The walk is of odd length; we number the edges $\alpha_1, \ldots, \alpha_{2m+1}$.
4. Each even-numbered edge crosses $\gamma$.
5. The arc $\gamma$ crosses the even-numbered edges of the path in the same order that they appear on the walk.

Such a walk is called a coloured $\gamma$-walk on $T$.

For each $w \in W_T^{\gamma}$, define a Laurent monomial

$$p(w) = \frac{x_{\alpha_1}x_{\alpha_3} \cdots x_{\alpha_{2m+1}}}{x_{\alpha_2}x_{\alpha_4} \cdots x_{\alpha_{2m}}}.$$ 

Then:

$$(2.1) \quad x_\gamma = \sum_{w \in W_T^{\gamma}} p(w).$$

This result first appeared in print in [Sch08], but it had also been noticed by others previously.

2.2. Technical lemmas. The following lemmas are at the heart of our argument.

Lemma 2.1. Fix a triangulation $T$ of the disc $\Pi_n$, and let $\gamma \in A(\Pi_n)$ which is not in $T$. Then any term in the $T$-expansion of $x_\gamma$ has negative degree with respect to arcs of $T$ which cross $\gamma$.

Proof. Consider $w$ a coloured $\gamma$-walk on $T$, and suppose that its length is $2m + 1$. The corresponding term in the expansion of $x_\gamma$ has $m + 1$ factors in the numerator, and $m$ factors in the denominator. All the factors in the denominator correspond to arcs in $\Pi_n$ which cross $\gamma$. Neither the first nor the last edge of $w$ contributes to the degree, proving the lemma. \qed

Lemma 2.2. Fix a triangulation $T$ of the disc $\Pi_n$, and let $\gamma \in A(\Pi_n)$, with $\gamma \notin T$. Suppose that $\beta$ is an arc of $\Pi_n$ which is compatible with $\gamma$. Then each term in the $T$-expansion of $x_\beta$ has non-positive degree with respect to arcs of $T$ which cross $\gamma$.

Proof. If $\beta$ is in $T$, then it cannot cross $\gamma$, so its degree is zero. If $\beta = \gamma$, we are done by the previous lemma. Otherwise, choose $w$ a coloured $\beta$-walk on $T$. Let $P$ denote the union of the triangles through which $\gamma$ passes.

Suppose first that $\beta$ and $\gamma$ do not share an endpoint. Consider the even-position edges of $w$ which lie in the interior of $P$. Since edges which cross $\beta$ must be...
encountered on $w$ in the same order as on $\beta$, these even-position edges must be a consecutive string of even positions, say $w_{2i}, w_{2i+2}, \ldots, w_{2j}$. Since $w_{2j-2}$ crosses $\beta$ but doesn’t cross $\gamma$, it lies outside $P$, and so do all previous edges; similarly for $w_{2j+2}$ and all subsequent edges. It follows that the total degree of $p(w)$ with respect to the edges crossed by $\gamma$ would be positive only if all the odd-numbered edges $w_{2i-1}, \ldots, w_{2j+1}$ also cross $\gamma$. But it would follow that $w$ crosses $\gamma$ an odd number of times, and thus that so does $\beta$, contradicting the fact that $\beta$ and $\gamma$ are compatible.

Suppose next that $\beta$ and $\gamma$ share an endpoint, which we assume is the starting point. Suppose that the even-position edges of $w$ which lie in $P$ are $w_{2i}, \ldots, w_{2j}$. As in the previous case, the corresponding term in the expansion of $x_\beta$ would be positive only if $w_{1}, \ldots, w_{2j+1}$ all also cross $\gamma$. But $w_1$ is incident to the endpoint of $\gamma$, so it does not cross $\gamma$. □

2.3. Proof of (A3). Let $y$ be a positive element in $A_Q$. Write:

$$y = \sum_{\Gamma \in T(\Pi_n)} \lambda_\Gamma(y)x_\Gamma.$$

Choose a particular $\Gamma$ appearing in the sum. We wish to show that $\lambda_\Gamma(y)$ is positive. Let $T$ be a triangulation of $\Pi_n$ which is compatible with $\Gamma$.

Consider some collection $\Sigma \in T(\Pi_n)$, with $\Sigma \neq \Gamma$. We wish to show that the $T$-expansion of $x_\Sigma$ does not include any term $x_\Gamma$.

If $\Sigma$ consists of arcs from $T$, then $x_\Sigma$ is its own $T$-expansion, and we are done. Suppose otherwise. Let $\sigma$ be an arc in $\Sigma$ which is not an arc of $T$. Lemma 2.1 tells us that each term in the $T$-expansion of $x_\sigma$ is of negative degree with respect to the edges which cross $\sigma$. At the same time, Lemma 2.2 tells us that each term in the $T$-expansion of the other factors of $x_\Sigma$ are of non-positive degree with respect to the same grading. It follows that each term in the $T$-expansion of $x_\Sigma$ is of negative degree with respect to this grading, which implies in particular that it contains no term $x_\Gamma$.

Therefore, $\lambda_\Gamma(y)$ equals the coefficient of $x_\Gamma$ in the $T$-expansion of $y$, which is therefore non-negative, as desired. □

3. Proof of Theorem 1.3

3.1. A combinatorial formula for curves. We begin by recalling the extension of the combinatorial formula which we stated in Section 2.1, to the case of a general marked surface $(S, M)$, following [ST09].

There, a formulation is given of the rules (1)–(5) for coloured $\gamma$-walks on a triangulation $T$. However, in practice, it is easier to use the following reformulation, which is immediate from Lemma 4.7 of [ST09].

Take $\gamma$, and lift it to an arc $\tilde{\gamma}$ in the universal cover. Lift the triangulation $T$ to a triangulation $\tilde{T}$. Then define the coloured $\gamma$-walks on $T$ to be the images on $S$ of the coloured $\tilde{\gamma}$-walks on $\tilde{T}$. The main theorem of [ST09] is that (2.1) still holds in this case, i.e., $x_\gamma$ is the sum of the terms $p(w)$, as $w$ runs through the coloured $\gamma$-walks on $T$.

We also want to assign an element of the cluster algebra to a curve which runs between two marked points and which has self-intersections. For such a curve $\gamma$, we take the same definition of $\mathcal{A}_\gamma^T$ as above, and define $x_\gamma^T$ to be the sum of $p(w)$
over all coloured $\gamma$-walks $w$. Note that, a priori, this definition is not independent of the choice of $T$.

**Lemma 3.1.** Let $T$ and $T'$ be two triangulations of $(S, M)$. Then for any curve $\gamma \in \mathcal{C}(S, M)$, we have $x^T_\gamma = x^{T'}_\gamma$.

**Proof.** Fix $\hat{\gamma}$, a lift of $\gamma$ to the universal cover of $(S, M)$. Let $\hat{T}$ and $\hat{T}'$ be lifts of the triangulations $T$ and $T'$ to the universal cover of $(S, M)$. Let $P$ be the union of the triangles of $\hat{T}$ which intersect $\hat{\gamma}$. Consider the expansion of $x_{\hat{\gamma}}$ in terms of cluster variables corresponding to edges of $\hat{T}$. Only edges from $P$ are involved in the expansion, and the expansion is independent of the triangulation outside $P$.

Now, if we take $D$ to be the union of the triangles of $T'$ which intersect $P$, we have a disc with marked points, which can be triangulated either by the restriction of $T'$, or by extending the triangulation of $P$ by $T$. In the type $A$ cluster algebra associated to $D$, we know that the expansions of $x_{\hat{\gamma}}$ with respect to these two triangulations agree.

When we pass down to $(S, M)$, these two expansions yield the expansions of $x_\gamma$ with respect to $T$ and $T'$, so these also coincide. $\square$

From now on, we will in general omit the reference to the triangulation and for any curve $\gamma$ in $(S, M)$, the notation $x_\gamma$ will designate the element in the ambient field corresponding to $x^T_\gamma$ for any choice of triangulation $T$. The notation $x^T_\gamma$ will be kept only in order to specify the the explicit Laurent expansion in the cluster $T$.

### 3.2. Curves in $C_{p,q}$ and objects in $\mathcal{C}_Q$.

From now on, we assume that $(S, M) = C_{p,q}$ for some $p, q \geq 1$ and that $Q$ is an affine quiver of type $\tilde{A}_{p,q}$. It is known that there is a bijection $\gamma \mapsto M_\gamma$ from the set $\mathcal{C}(C_{p,q})$ to the set of isomorphism classes of indecomposable objects in $\mathcal{C}_Q$ which are not contained in an homogeneous tube $[BZ11]$. Let us make this bijection explicit for objects in exceptional tubes.

If $p = q = 1$, there are no exceptional tubes in $\Gamma(\mathcal{C}_Q)$. Thus, without loss of generality, we assume that $p > 1$. We consider the set $\mathcal{C}'(C_{p,q})$ of curves in $\mathcal{C}(C_{p,q})$ both of whose endpoints are the inside boundary $\iota$ of $C_{p,q}$ containing $p$ points. We denote by $m_i$ with $i \in \mathbb{Z}/p\mathbb{Z}$ the marked points on $\iota$. The orientation of $C_{p,q}$ induces an orientation of $\iota$ and following this orientation we can assume that the successor of $m_i$ is $m_{i+1}$ for any $i \in \mathbb{Z}/p\mathbb{Z}$. We denote by $\gamma_m^{(0)}$ the oriented arc with starting point $m_i$ and endpoint $m_{i+1}$ for any $i \in \mathbb{Z}/p\mathbb{Z}$. Finally, for any $l \geq 1$, we set $\gamma_m^{(l)} = \gamma_m^{(0)} \circ \gamma_m^{(l-1)}$. Figure $2$ depicts the situation.
In $\Gamma(\mathcal{C}_Q)$ there is an exceptional tube $\mathcal{T}_p$ of rank $p$. We denote by $R_i$, with $i \in \mathbb{Z}/p\mathbb{Z}$ the quasi-simple objects in $\mathcal{T}_p$. For any $l \geq 1$, we denote by $R_i^{(l)}$ the unique indecomposable object in $\mathcal{T}_p$ with quasi-socle $R_i$ and quasi-length $l$. Then the above bijection is given by $M_{\gamma^{(l)}}^{(i)} = R_i^{(l)}$ for any $i \in \mathbb{Z}/p\mathbb{Z}$ and any $l \geq 1$. We adopt the convention that $R_i^{(0)} = 0$ for any $i \in \mathbb{Z}/p\mathbb{Z}$. The situation is depicted in Figure 3.

If $q \geq 1$, one can write down a similar bijection between the set $\mathcal{C}_o(C_{p,q})$ of arcs whose both endpoints lie on the outside component $o$ containing $q$ points and the corresponding tube $\mathcal{T}_q$ of rank $q$ in $\Gamma(\mathcal{C}_Q)$.

We now compare the formula $x_? \gamma$ with the cluster character on the category $\mathcal{C}_Q$.

**Lemma 3.2.**

1. Assume that $p > 1$. Then $x_? \gamma = X_{M_i}$ for any $\gamma \in \mathcal{C}_i^{(l)}(C_{p,q})$.
2. Assume that $q > 1$. Then $x_? \gamma = X_{M_o}$ for any $\gamma \in \mathcal{C}_o(C_{p,q})$.

**Proof.** By symmetry it is enough to prove the first point. Fix a curve in $\mathcal{C}_i^{(l)}(C_{p,q})$. It is of the form $\gamma^{(l)}_{m_i}$ for some $i \in \mathbb{Z}/p\mathbb{Z}$ and some $l \geq 1$. We prove the result by
induction on $l$. If $l = 1$ then $\gamma_{m_i}^{(1)}$ has no self-intersection so that $\gamma_{m_i}^{(1)} \in \mathcal{A}(C_{p,q})$. In particular, $x_{\gamma_{m_i}^{(1)}}$ and $X_{R_i^{(1)}}$ are cluster variables. In order to prove that they are the same, it is enough to consider their expressions as Laurent polynomials with respect to a fixed cluster/triangulation. We fix the following triangulation $T$ in $C_{p,q}$:

![Diagram](attachment:image.png)

Then it is known that the denominator vector of $x_{\gamma_{m_i}^{(1)}}^T$ is given by the number of intersections of $\gamma_{m_i}^{(1)}$ with the arcs of $T$ [FST08, Theorem 8.6] and that the denominator vector of $X_{R_i^{(1)}}^T$ is the dimension vector of the $kQ$-module $R_i$ [CK06, Theorem 3]. Then, up to a cyclic permutation of the indices, the denominator vectors of $x_{\gamma_{m_i}^{(1)}}^T$ and $X_{R_i^{(1)}}^T$ coincide. Thus, the two cluster variables are the same.

Now assume that the result holds for $l \geq 1$. Using a standard covering argument and Ptolemy relations in the covering, it is easily verified that

$$x_{\gamma_{m_i}^{(1)}}^T x_{\gamma_{m_i+1}^{(1)}}^T = x_{\gamma_{m_i}^{(1)}}^{T_{l+1}} x_{\gamma_{m_i+1}^{(1)}}^{T_{l}} + 1$$

for any $i \in \mathbb{Z}/p\mathbb{Z}$. But on the other hand, it follows from [Pal11, Theorem 1] that

$$X_{R_i^{(1)}}^T X_{R_{i+1}^{(1)}}^T = X_{R_i^{(1)}}^{T_{l+1}} X_{R_{i+1}^{(1)}}^{T_{l}} + 1$$

for any $i \in \mathbb{Z}/p\mathbb{Z}$. Thus,

$$x_{\gamma_{m_i+1}^{(1)}}^T = X_{R_{i+1}^{(1)}}^T,$$

which proves the induction step. 

### 3.3. A formula for loops.

Let $m \geq 1$ be an integer and let $T$ be a triangulation of $C_{p,q}$. Consider the annulus $C_{mp,mq}$ which is the $m$-fold cover of $C_{p,q}$. The triangulation $T$ induces naturally a triangulation of $C_{mp,mq}$, which is denoted by $\tilde{T}$. Let $[0,1] \rightarrow C_{mp,mq}$ denote a parametrisation of the meridian $\tilde{z}$ in $C_{mp,mq}$. The order in which $\tilde{z}$ intersects the bridging arcs of the triangulation $T$ with respect to this parametrisation induces an order on the bridging arcs of $\tilde{T}$.

**Definition 3.3.** A **coloured $m$-walk** on $T$ is a walk of even length along the edges of the triangulation $\tilde{T}$ and along the boundary components of $C_{mp,mq}$, whose edges are decorated with alternating $+$ and $-$ signs and such that:

(P1) The walk is homotopic to $\tilde{z}$.
(P2) Every edge decorated with a $-$ is a bridging arc of $T$.
(P3) The walk goes forward in the sense that if two bridging arcs $\alpha$ and $\beta$ appear in the walk in this order, then $\alpha$ strictly precedes $\beta$ in the order induced by the parametrisation of $\tilde{z}$.

We denote by $\mathcal{W}_{m}^T$ the set of coloured $m$-walks on the triangulation $T$ in $C_{p,q}$. 


Definition 3.4. For any triangulation $T$ of $C_{p,q}$ and any $m \geq 1$, the Laurent polynomial $x^T_{zm}$ in the cluster $T$ is the sum over all coloured $m$-walks $p$ on $T$ satisfying (P1)–(P3), of the product $x(w)$ of the cluster variables in $T$ corresponding to $+$ edges in $w$, divided by the product of the cluster variables in $T$ corresponding to $-$ edges in $w$, with the convention that boundary arcs contribute as 1. In other words,

$$x^T_{zm} = \sum_{w \in \mathcal{W}^T_{zm}} x(w).$$

Example 3.5. We consider the triangulation $T$ of $C_{1,1}$ depicted in Figure 4. The coloured 1-walks on $T$ are depicted in Figure 4 where edges with a negative colour appear in blue and edges with a positive colour appear in red.

Thus, we have

$$x^T_z = x_{\beta} + x_{\alpha} + \frac{1}{x_{\alpha}x_{\beta}} = \frac{x^2_{\alpha} + x^2_{\beta}}{x_{\alpha}x_{\beta}}.$$ 

For $m = 2$, the two-fold covering of $C_{1,1}$ is

Thus, the set of coloured 2-walks on $T$ consists of the following elements:

- $\alpha^+\beta^-\alpha^-\beta^+\alpha^+\beta^-\alpha^-\beta^+$
- $\alpha^-\beta^+\alpha^-\beta^+$
- $\alpha^-\beta^+\alpha^-\beta^+$
- $\beta^-\alpha^+\beta^-\alpha^+\beta^-\alpha^+\beta^-\alpha^+$
- $\iota^+\beta^-\alpha^+\iota^+\beta^-\alpha^-$

Thus, we have

$$x^T_z = \frac{2}{x_{\alpha}^2} + \frac{2}{x_{\beta}^2} + \frac{x^2_{\beta}}{x^2_{\alpha}} + \frac{x^2_{\alpha}}{x^2_{\beta}} + \frac{1}{x^2_{\alpha}x^2_{\beta}} = (x^T_z)^2 - 2 = F_2(x^T_z).$$
3.4. Invariance under cluster change.

**Lemma 3.6.** Let $T$ and $T'$ be two triangulations of $C_{p,q}$. Then for any $\gamma$ in $\hat{A}(C_{p,q})$, we have $x^T_\gamma = x^{T'}_\gamma$.

**Proof.** For $\gamma$ an arc (not a loop), as we have already discussed, $x^T_\gamma$ and $x^{T'}_\gamma$ are two different expansions of the same cluster variable, so they are equal. Thus, we only need to prove that $x^T_{z_m} = x^{T'}_{z_m}$ for any $m \geq 1$. Since any two triangulations are related by a sequence of mutations, it is enough to prove it in the case when $T$ and $T'$ are related by a single mutation.

Suppose first that the mutation replaces a peripheral arc by another peripheral arc. This means that all the arcs of the quadrilateral in which the mutation takes place are peripheral. It follows that neither of the edges being mutated can appear in the formulas for $x^T_{z_m}$ and $x^{T'}_{z_m}$, so in this case we are done.

Next suppose that the mutation replaces a peripheral arc $\sigma$ by a bridging arc $\tau$, and suppose the other four arcs are labelled $\alpha, \beta, \gamma, \delta$ as shown in Figure 5 below.

![Figure 5. Replacing a peripheral arc by a bridging arc](image.png)

Suppose also that the coloured $m$-walk passes locally from left to right in the diagram. Since $\sigma$ is peripheral, it is decorated with a $\sigma^+$. Each coloured $m$-walk on $T$ which includes $\sigma$ induces two coloured $m$-walks on $T'$ where $\sigma$ is replaced respectively by $\delta\tau\beta$ and by $\alpha\tau\gamma$. (If these replacements result in some edge being used twice consecutively in opposite directions, we cancel them out.) We claim that any coloured $m$-walk on $T'$ is obtained from a coloured $m$-walk on $T$ in this way. Suppose we start with a coloured $m$-walk on $T'$ containing $\tau$. It can only occur with a $-$ decoration, since on one end it is not adjacent to any bridging edges. Each coloured $m$-walk $w$ on $T'$ which includes $\tau$ from $i$ to $o$ must be of the form $\cdots \alpha^+ \tau^+ \cdots$. Replacing $\alpha^+ \tau^-$ by $\sigma^+ \gamma^-$ in this expression, we obtain the desired coloured $m$-walk on $T$ inducing $w$. If $w$ uses $\tau$ from $o$ to $i$, then it must be of the form $\cdots \tau^- \beta^+ \cdots$. Replacing $\tau^- \beta^+$ by $\delta^- \sigma^+$ provides the desired coloured $m$-walk on $T$. This proves the claim.

Finally, suppose that the mutation replaces one bridging $\sigma$ in $T$ arc by another bridging arc $\tau$ in $T'$, and suppose the other four arcs are labelled $\alpha, \beta, \gamma, \delta$ as shown in Figure 6 below.
Assume first that $w$ is a coloured $m$-walk on $T$ in which $\sigma$ arises with a $+$ decoration. Replacing $\sigma^+$ by $\alpha^+\tau^-\gamma^+$ and $\delta^+\tau^-\beta^+$ provides two coloured $m$-walks on $T'$. We claim that every coloured $m$-walk on $T'$ including $\tau$ with a $-$ sign is obtained in this way. Fix thus a coloured $m$-walk on $T'$ with $\tau$ appearing with a $-$ sign. If $\tau$ is crossing from $o$ to $\iota$, then it must be of the form $\alpha^+\tau^-\gamma^+$. If $\tau$ is crossing from $\iota$ to $o$ then consider the coloured $m$-walk on $T$ obtained by replacing $\tau^-$ by $\delta^-\sigma^+\beta^-$. Then the coloured $m$-walk on $T'$ is obtained by replacing $\sigma^+$ by $\delta^+\tau^-\beta^+$, which proves the claim. The same argument works for walks with $\tau$ having a $+$ decoration (and with $\sigma$ decorated with a $-$).

Thus, for any $\gamma \in \hat{A}(C_{p,q})$ we simply write $x_\gamma$ for the element in $x_T^\gamma$ and for any collection $\Gamma$ of elements in $\hat{A}(C_{p,q})$, we set $x_\Gamma = \prod_{\gamma \in \Gamma} x_\gamma$.

### 3.5. From loops to elements in $\mathcal{B}_Q$.

**Lemma 3.7.** For any $m \geq 1$, we have $x_{zm} = F_m(X_\delta)$.

**Proof.** We first assume that $p + q \geq 3$ so that $Q$ is not the Kronecker quiver and by symmetry, we can assume that $p > 1$.

It follows from Lemma 3.6 that it is enough to prove the lemma for a particular choice of triangulation $T$. We fix an integer $m \geq 1$.

We consider the following triangulation $T$ of $C_{p,q}$ (viewed in the universal cover):

We denote by $M_m$ the object in $\mathcal{C}_Q$ corresponding to $\gamma_{t_1}^{(mp)}$, which, with the previous notations amounts to saying that $M_m = R_1^{(mp)}$ in the tube $\mathcal{F}_p$. We set $N_m = R_2^{(mp-1)}$. It follows from the so-called higher difference properties proved in [Dup10b, Proposition 3.3] that

$$F_m(X_\delta) = X_T^T M_m - X_T^T N_m.$$
We now prove the analogous identity for the formula $x^T$. Identifying objects in $F_p$ with arcs in $C'(C_{p,q})$, we are in the situation depicted in Figure 7.

As before, we denote by $\mathcal{W}_N^T$ the set of coloured walks on $T$ which are considered in the formula $x_N^T$, and by $\mathcal{W}_M^T$ the set of coloured walks on $T$ which are considered in the formula $x_M^T$. As in Figure 7, we denote by $\alpha$ the edge of $T$ joining $o_1$ to $t_1$ and by $\beta$ the edge of $T$ joining $t_1$ to $o_2$. Since in the formula for $x^T$ boundary components contribute as 1, in order to simplify notations we will always denote by $o$ a boundary segment on $o$ and by $t$ a boundary segment on $t$.

For any coloured walk $w$ in $\mathcal{W}_N^T$, the coloured walk $\alpha^+ \beta^- w \alpha^- \beta^+$ is in $\mathcal{W}_M^T$ and the respective contributions in $x_N^T$ and $x_M^T$ are the same.

We denote by $\mathcal{W}_m^T$ the set of $m$-coloured walks on $T$ (i.e., those which are considered in the formula $x_m^T$). We denote by $\mathcal{W}_{m,\alpha}^T$ the set of coloured walks in $\mathcal{W}_m^T$ going through the first lift of $t_1$ in the $m$-fold cover $C_{mp,mq}$ and by $\mathcal{W}_{m,\alpha}^T$ its complement in $\mathcal{W}_m^T$ (which consists of walks passing through the first lift of $o_1$ but not all such walks).

If $w \in \mathcal{W}_{m,\alpha}^T$ then $w^o+ \in \mathcal{W}_{m,\alpha}^T$ and the respective contributions in $x^T_m$ and $x^T_m$ are the same. If $w \in \mathcal{W}_{m,\alpha}^T$, then $\alpha^+ \beta^- w \beta^+ \in \mathcal{W}_M^T$ and the respective contributions in $x^T_m$ and $x^T_m$ are the same. Moreover, we have

$$\mathcal{W}_M^T = \alpha^+ \beta^- (\mathcal{W}_{m,\alpha}^T) \beta^+ \cup (\mathcal{W}_{m,\alpha}^T) \alpha^+ \cup \alpha^+ \beta^- (\mathcal{W}_{N,m}^T) \alpha^- \beta^+.$$  

Thus,

$$x^T_{M,m} = \sum_{w \in \mathcal{W}_M^T} x(w)$$

$$= \sum_{w \in \mathcal{W}_{m,\alpha}^T} x(o^+ \beta^- w \beta^+) + \sum_{w \in \mathcal{W}_{m,\alpha}^T} x(w^o+) + \sum_{w \in \mathcal{W}_{N,m}^T} x(\alpha^+ \beta^- w \alpha^- \beta^+)$$

$$= \sum_{w \in \mathcal{W}_{m,\alpha}^T} x(w) + \sum_{w \in \mathcal{W}_{m,\alpha}^T} x(w) + \sum_{w \in \mathcal{W}_{N,m}^T} x(w)$$

$$= \left( \sum_{w \in \mathcal{W}_{m,\alpha}^T} x(w) + \sum_{w \in \mathcal{W}_{m,\alpha}^T} x(w) \right) + \sum_{w \in \mathcal{W}_{N,m}^T} x(w)$$

$$= x^T_m + x^T_{N,m}.$$
Now, since $M_m$ and $N_m$ are curves in $C^1(C_{p,q})$, we can apply Lemma 3.2 and we get

$$x^T \zeta_m = x^T M_m - x^T N_m = X^T M_m - X^T N_m = F_m(X_\delta)$$

and the lemma is proved for $p + q \geq 3$.

Assume now that $p = q = 1$ and consider the same triangulation $T$ as in Figure 4. Consider the annulus $C_{2,1}$ equipped with the triangulation $T'$ containing two bridging edges $\alpha$ and $\beta$ and one peripheral edge $\tau$, as in Figure 8 below.

![Figure 8. The triangulation $T'$ of $C_{2,1}$.](image)

Consider the ring homomorphism $\pi$ from the ring of Laurent polynomials in the cluster $T'$ of $\mathcal{A}_{C_{2,1}}$ to the ring of Laurent polynomials in the cluster $T$ of $\mathcal{A}_{C_{1,3}}$ sending $x_\alpha$ to $x_\alpha$, $x_\beta$ to $x_\beta$ and $x_\tau$ to 1. We denote by $\tilde{z}_m$, with $m \geq 1$ the loops in $C_{2,1}$. It follows that $\pi(x^T \tilde{z}_m) = x^T m$ for any $m \geq 1$. In particular, it follows from the above discussion that

$$x^T \zeta_m = \pi(x^T \tilde{z}_m) = \pi(F_m(x^T \zeta')) = F_m(\pi(x^T \zeta')) = F_m(x^T \zeta)$$

which finishes the proof.

□

Example 3.8. We illustrate the combinatorial interpretation of these higher difference properties for $m = 1$ in the following example of type $\tilde{A}_{1,3}$. Consider the following triangulation $T$ of $C_{1,3}$:

![Diagram](image)

The quiver $Q$ of the triangulation $T$ is the following quiver of type $\tilde{A}_{3,1}$:

$$Q: \begin{array}{ccc}
1 & \rightarrow & 4 \\
\downarrow & & \downarrow \\
2 & \rightarrow & 3 \\
\end{array}$$

Now, consider the following arcs $M_1$ and $N_1$:

![Diagram](image)
The corresponding representations of $Q$ (viewed here as objects in the cluster category $\mathcal{C}_Q$), are

$$M_1: \quad \begin{array}{c}
\downarrow 1_k \\
\downarrow 1_k \\
\downarrow \end{array} \quad k \quad \begin{array}{c}
\rightarrow 1_k \\
\rightarrow \\
\rightarrow \end{array} \quad k, \quad N_1: \quad \begin{array}{c}
\downarrow 1_k \\
\downarrow \\
\downarrow \\
\downarrow \end{array} \quad 0 \quad \begin{array}{c}
\rightarrow \\
\rightarrow 0. \\
\rightarrow \end{array} \quad k
$$

We now list the coloured walks corresponding to $M_1$, the monomials that they give rise to in the formula $x_{M_1}$ and the dimension vector $e$ giving the same Laurent monomial in the formula $X_{M_1}$. We also make explicit the factorisations in terms of $\mathcal{W}_{N_1}^T$ and $\mathcal{W}_z^T$:

| $w \in \mathcal{W}_{M_1}^T$ | $x(w)$ | $e$ |
|-----------------------------|--------|-----|
| $1^+ o^+$ | $x_1/x_4$ | (0000) |
| $(1+3-o^+4-o^+1^-)o^+$ | $1/x_3x_4$ | (0001) |
| $(1+2-o^+3-o^+1^-)o^+$ | $1/x_2x_3$ | (0011) |
| $(o^+2-o^+1^-)o^+$ | $1/x_1x_2$ | (0111) |
| $o^+2^- (4+1^-)2^+$ | $x_4/x_1$ | (1111) |
| $1^+2^- (2+3^-o^+1^-2^-)1^+2^+$ | $x_2/x_3$ | (1001) |
| $1^+2^- (o^+3^-4^+)1^-2^+$ | $x_4/x_3$ | (1011) |

Similarly for $N_1$ and $z$, we have

| $w \in \mathcal{W}_{N_1}^T$ | $x(w)$ | $e$ |
|-----------------------------|--------|-----|
| $2^+3^-o^+$ | $x_2/x_3$ | (0000) |
| $o^+3^-4^+$ | $x_4/x_3$ | (0010) |

and for $z$,

| $w \in \mathcal{W}_z^T$ | $x(w)$ | $e$ |
|-----------------------------|--------|-----|
| $1^+4^-$ | $x_1/x_4$ | (0000) |
| $1^+3^-o^+4^-o^+1^-1^-$ | $1/x_3x_4$ | (0001) |
| $1^+2^-o^+3^-o^+1^-1^-$ | $1/x_2x_3$ | (0011) |
| $o^+2^-o^+1^-1^-$ | $1/x_1x_2$ | (0111) |
| $4^+1^-$ | $x_4/x_1$ | (1111) |

where the first four lines in the last table correspond to $\mathcal{W}_z^{T:o}$ and the last one to $\mathcal{W}_z^{T:o}$.

3.6. **Proof of Theorem 1.3.** We now finish the proof of Theorem 1.3. Recall from the definition that

$$\mathcal{B}_Q = \mathcal{M}_Q \sqcup \{x_R F_m(X_b) \mid m \geq 1, R \in \text{reg}^0-kQ\}.$$  

As we already mentioned, we know that

$$\mathcal{M}_Q = \{x_\Gamma \mid \Gamma \in T(C_{p,q})\}$$

and that for any $M = \bigoplus_{i \in I} M_i \in \text{reg}-kQ$, we have $X_M = x_\Gamma$ where $\Gamma$ is the family of arcs in $C(C_{p,q}) \sqcup C^0(C_{p,q})$ corresponding to $\{M_i\}_{i \in I}$. Moreover, it is known that $M$ is rigid if and only if the corresponding curves in $C_{p,q}$ do not intersect, see for instance [BZII]. Therefore, to finish the proof of Theorem 1.3, it is enough to observe that Lemma 3.7 implies that $F_m(X_b) = x_{z_m}$ for any $m \geq 1$. \qed
4. Proof of Theorem 1.2

In this section $Q$ still denotes a quiver of type $\tilde{A}_{p,q}$ with $p, q \geq 1$.

Proving Theorem 1.2 is equivalent to proving the following three points:

(B1) $B_Q$ is a $\mathbb{Z}$-linear basis of $A_Q$;

(B2) $B_Q$ is contained in the positive cone of $A_Q$;

(B3) Every element in the positive cone of $A_Q$ can be written as a $\mathbb{Z}_{\geq 0}$-linear combination of elements of $B_Q$.

4.1. Proof of (B1). It follows from [Dup08, Theorem 4.21] (see also [GLS10]) that the set

$$B_Q = \mathcal{M}_Q \cup \{ X_R X_R^m | m \geq 1, R \in \text{reg}^0 - kQ \}$$

is a $\mathbb{Z}$-linear basis of $A_Q$. Since for any $m \geq 1$, $F_m$ is a monic polynomial of degree $m$, it easily follows that $B_Q$ is a $\mathbb{Z}$-linear basis of $A_Q$, see [Dup11, §6] for a precise description of the base change. This proves (B1).

4.2. Proof of (B2). We need to prove that for any cluster $T$ in $A_Q$, the Laurent expansion of any element of $B_Q$ in the cluster $T$ has positive coefficients. An element $b \in B_Q$ is a certain $x_T$ where $\Gamma$ is a collection of elements in $\tilde{A}(C_{p,q})$ and its $T$-expansion is given explicitly by the formula $x_T^\gamma$. Now it follows from the definition of the map $x_T^\gamma$ that $x_T^\gamma$ is a subtraction-free Laurent polynomial in the cluster $T$ for any $\gamma \in \tilde{A}(C_{p,q})$. Since subtraction-free Laurent polynomials in $T$ form a semiring, it follows that $x_T^\gamma$ is also a subtraction-free Laurent polynomial in $T$. This proves (B2).

4.3. Proof of (B3). This is the long part of the proof and it will be divided in several intermediate results.

4.3.1. Beginning the proof. Let $y$ be a positive element in the cluster algebra $A_Q$. According to (B1), we can write

$$y = \sum_{\Gamma \in \mathcal{T}(C_{p,q})} \lambda_\Gamma(y) x_\Gamma$$

with $\lambda_\Gamma(y) \in \mathbb{Z}$. We need to prove that $\lambda_\Gamma(y) \geq 0$ for any $\Gamma \in \mathcal{T}(C_{p,q})$.

When $\Gamma \in \mathcal{T}(C_{p,q})$ the situation is slightly easier, so we explain that situation first. We find some cluster $T$ which is compatible with $\Gamma$, and such that $x_T$ does not appear in the $T$-expansion of any other element $x_\Sigma$ of $B_Q$. Then, $\lambda_\Gamma(y)$ coincides with the coefficient of $x_T$ in the $T$-expansion of $y$, which is non-negative by assumption. One point to bear in mind about this strategy is that the same $T$ must work for all choices of $\Sigma$.

When $\Gamma \in \mathcal{T}(C_{p,q}) \setminus \mathcal{T}(C_{p,q})$, the situation is more complicated for two reasons. First of all, there are infinitely many clusters compatible with $\Gamma$, and it is not enough to pick one. Rather, we define an infinite family of clusters $T_r$ for $r \in \mathbb{Z}$, any of which is compatible with $\Gamma_r$.

Secondly, the expansion of $x_T$ in any cluster will include multiple terms. We will therefore pick one, and call it $t_{\Gamma,r}$. For any $r$, $t_{\Gamma,r}$ will appear in $T_r$-expansion of $x_T$ with coefficient one.

We will then show that, for any $\Sigma \in \mathcal{T}(C_{p,q})$, if $r$ is sufficiently large, then $t_{\Gamma,r}$ does not appear in the $T_r$ expansion of $x_\Sigma$. Since, in our sum for $y$, only finitely many terms appear, it follows that we can choose $r$ large enough for all possible $\Sigma$.
simultaneously. Then the coefficient of $x_T$ in the cluster expansion of $y$ agrees with the coefficient of $t_{T,r}$ in the $T_r$-expansion of $y$, which is positive, so we are done.

The remainder of the proof fills in the details of the discussion above. First we treat the case that $\Gamma \in T(C_{p,q})$, and then we treat the case $\Gamma \in T(C_{p,q}) \setminus T(C_{p,q})$. Within each of these cases, there are two subcases, depending on whether $\Sigma$ belongs to $T(C_{p,q}) \setminus T(C_{p,q})$.

4.3.2. Technical lemmas. Before we can complete the proof, we need to collect some technical lemmas.

**Lemma 4.1.** Fix $T$ a triangulation of $C_{p,q}$, and let $\gamma$ be a peripheral arc in $C_{p,q}$, with $\gamma \not\in T$. Then any term in the $T$-expansion of $x_\gamma$ has negative degree with respect to the cluster variables corresponding to arcs of $T$ which cross $\gamma$.

**Proof.** The proof is essentially the same as that of Lemma 2.1 for any coloured $\gamma$-walk $w$ on $T$, the first and last segments of $w$ run along arcs which do not cross $\gamma$, and therefore do not contribute to the degree of the corresponding term $p(w)$. □

We now prove an analogue of Lemma 2.2.

**Lemma 4.2.** Fix a triangulation $T$ of $C_{p,q}$. Let $\gamma$ be a peripheral arc in $C_{p,q}$. Suppose that $\beta$ is an arc compatible with $\gamma$. Then each term in the $T$-expansion of $\beta$ has non-positive degree with respect to the set of edges which cross $\gamma$.

**Proof.** It is possible that the beginning and ending segments of $\gamma$ lie in the same triangle of $T$: this only happens if $\gamma$ is homotopic to the entire inner boundary or the entire outer boundary. In this case, we call $\gamma$ a near-loop.

Suppose first that $\gamma$ is not a near-loop. For each lifting of $\gamma$ to an arc $\tilde{\gamma}$ of the universal cover of $C_{p,q}$, define the polygon $\tilde{P}_\gamma$ as in the proof of Lemma 2.2. These polygons do not overlap (except possibly at a vertex).

Let $w$ be a coloured $\beta$-walk on $T$. The argument from Lemma 2.2 goes through — in order for the degree of the term corresponding to $w$ to be positive with respect to the edges crossed by $\gamma$, there would have to be some $\tilde{\gamma}$ such that $w$ crosses $\tilde{\gamma}$ an odd number of times, which is impossible.

Suppose next that $\gamma$ is a near-loop. For convenience, fix that $\gamma$ is attached to the inside boundary component. There are three possibilities for $\beta$: it might be peripheral on the inside, peripheral on the outside, or bridging.

Suppose first that $\beta$ is peripheral on the inside. The proof of Lemma 2.2 goes through without any changes.

Suppose next that $\beta$ is peripheral on the outside. Let $w$ be a $T$-walk for $\beta$. Define $P$ to consist of the union of those triangles through which $\gamma$ passes. Note that this includes all triangles of $T$ which have vertices lying on both boundary components. As in Lemma 2.2 the even-numbered edges of $w$ which cross $\beta$ form a consecutive string, say $w_{2i}, w_{2i+2}, \ldots, w_{2j}$. In order for the degree of the corresponding term $p(w)$ to be positive, it must be the case that all the odd-numbered edges from $w_{2i-1}$ and $w_{2i+1}$ also cross $\gamma$, resulting in a subsequence $\overline{w} = w_{2i-1}, \ldots, w_{2j+1}$ of $w$ where all the steps of $\overline{w}$ cross $\gamma$, while the step before and the one after do not. Note also that $\overline{w}$ begins on the outside component (since $w_{2i-2}$ crosses $\beta$ but not $\gamma$). The one extra observation to make in this case is that there are edges of $T$ which run from the inside component to the inside component, and cross $\gamma$ twice. (This type of phenomenon does not arise in a disc.) Such an edge would foil the argument of Lemma 2.2 which depends on the parity of the total number of crossings. However,
such an edge cannot appear in even position in $\overline{w}$, because it does not cross $\beta$. We also claim that it cannot appear in odd position in $\overline{w}$. Suppose it appears in position $2k+1$. Up until that point, we assume that each edge $w_{2i-1}, \ldots, w_{2k-1}$ crosses $\gamma$ once — but this means that $w_{2k+1}$ begins on the outside edge, so it cannot be one of these pathological edges.

Suppose next that $\beta$ is bridging. Since $\gamma$ is a near-loop, and $\beta$ is compatible with $\gamma$, $\beta$ shares an endpoint with $\gamma$. The argument from the proof of Lemma 2.2 for the case that $\beta$ and $\gamma$ share an endpoint goes through without changes. □

We prove a similar lemma, involving a peripheral arc $\gamma$ and a loop.

**Lemma 4.3.** Fix a triangulation $T$ of $C_{p,q}$. Let $\gamma$ be an arc in $A(C_{p,q})$ which is peripheral. For any positive integer $m$, each term in the $T$-expansion of $xz^m$ has non-positive degree with respect to the set of edges which cross $\gamma$.

**Proof.** Consider a term in the $T$-expansion of $xz^m$, and let $w$ be the corresponding walk. If all the even-numbered edges of $w$ cross $\gamma$, then the result is true, so suppose otherwise.

Consider a maximal-length subsequences of even-numbered edges of $w$ which cross $\gamma$, say $w_{2i}, \ldots, w_{2j}$. It will suffice to show that it is impossible to have both $w_{2i-1}$ and $w_{2j+1}$ crossing $\gamma$. Suppose both of them do cross $\gamma$. It would follow the endpoints of the walk $w_{2i-1} \ldots w_{2j+1}$ are on opposite sides of $\gamma$. Therefore, one of the endpoints is in the region cut out by $\gamma$ and the boundary component to which $\gamma$ is connected. But then the next even-numbered edge, which crosses from one boundary component to the other, necessarily also crosses $\gamma$, contrary to our assumption. □

Before we go on, we also recall a representation-theoretic lemma from [Cer11b], and for this, we need to recall some additional notions. A fuller account of them can be found in [Cer11b] and in [DWZ08 DWZ10] from which it draws.

Let $T$ be a triangulation of $C_{p,q}$, whose arcs are numbered 1 to $n$. Associated to it is a quiver $Q$, whose vertices are associated to the arcs of $T$, a potential $S$, and a collection of (complex) decorated quiver representations $M_\alpha$, where $\alpha$ ranges over the isotopy classes of curves in $C_{p,q}$. For us, the decorations will not play a role, so we will not distinguish the decorated quiver representation from the representation itself.

The cluster variable associated to $\alpha$ can be recovered in the following way. First, for any $e$ in $\mathbb{Z}^n$, thought of as a dimension vector, we can consider the $Gr_e(M)$, the Grassmannian of subrepresentations of $M$ whose dimension at vertex $i$ is $e_i$. Then the $T$-expansion of $x_\alpha$ can be expressed as follows:

$$x_\alpha = \sum e \chi(Gr_e(M_\alpha))x^{g_\alpha + Be}$$

Here $\chi$ denotes the Euler-Poincaré characteristic, $B$ is the $B$-matrix corresponding to $T$, and $g_\alpha$ is the $g$-vector associated to $\alpha$. For $v \in \mathbb{Z}^n$, we write $x^v$ for $x_1^{v_1} \ldots x_n^{v_n}$, where $x_1, \ldots, x_n$ are the cluster variables associated to the arcs of $T$. Inversely, given a monomial $x^v$, we let $\exp(x^v) = v$.

A formula of a similar form can be given for any cluster monomial. Let $\Sigma \in T(C_{p,q})$. Define $M_\Sigma$ to be the sum of the corresponding $M_\alpha$'s, with multiplicity,
and define $g_{\Sigma}$ to be the sum of the corresponding $g_{\alpha}$'s. Then

$$x_\Sigma = \sum_e \chi(\text{Gr}_e(M_\Sigma)) \prod_{i=1}^n x_i^{g_{\Sigma} + Be}$$

We write $x_\Sigma(e)$ for the term in the above sum corresponding to $e$.

We endow $\mathbb{Z}^n$ with the standard inner product given by

$$\mathbf{e} \cdot \mathbf{f} = \sum_{i=1}^n e_i f_i$$

for any $\mathbf{e} = (e_i)_{1 \leq i \leq n}$ and $\mathbf{f} = (f_i)_{1 \leq i \leq n}$ in $\mathbb{Z}^n$.

The following lemma was proved in [Cer11b, Lemma 5.1] for finite type cluster algebras. Its proof remains valid in the context of cluster algebras of type $\tilde{A}$: condition (12) of [Cer11b] is satisfied by [Lab09, Theorem 36] or (in a more special case sufficient for our purposes) [ABCP10, Lemma 2.4].

**Lemma 4.4 (Cer11b).** Let $T$ be a triangulation of $C_{p,q}$, and let $\Sigma \in \mathcal{T}(C_{p,q})$, such that none of the arcs in $\Sigma$ is in $T$. Then for any $\mathbf{e} \neq 0$ such that $\text{Gr}_e(M_\Sigma) \neq \emptyset$, we have:

$$\mathbf{e} \cdot \exp(x_\Sigma(\mathbf{e})) < 0.$$

We also need the following proposition from [DWZ10]. Let $\Sigma \in \mathcal{T}(C_{p,q})$ such that it shares no arcs with $T$. Then define $E(M_\Sigma) = \dim(M_\Sigma) \cdot g_{\Sigma} + \dim \text{Hom}(M_\Sigma, M_\Sigma)$.

**Proposition 4.5 (DWZ10, Corollary 7.2).** For any $\Sigma \in \mathcal{T}(C_{p,q})$ containing no arcs from $T$, we have that $E(M_\Sigma) = 0$.

We can now complete our proof.

4.3.3. The case when $\Gamma \in \mathcal{T}(C_{p,q})$. In this case, $x_\Gamma$ is a cluster monomial. Depending on $\Gamma$, there may or may not be any choice for a triangulation $T$ compatible with $\Gamma$, but in any case, we choose it arbitrarily.

If $\Gamma$ and $\Sigma$ have arcs in common, we can remove corresponding arcs from each, and $x_\Sigma$ will appear in the $T$-expansion of $x_\Sigma$ iff this is true after the cancellation. So we may assume that $\Gamma$ and $\Sigma$ have no arcs in common.

The case where $\Gamma \in \mathcal{T}(C_{p,q})$ and $\Sigma \in \mathcal{T}(C_{p,q})$. In this case, both $\Gamma$ and $\Sigma$ are cluster monomials and the argument is the same as in [Cer11b], but we give it for completeness.

Suppose $\Sigma$ contains an arc $\sigma$ of $T$. By assumption, this arc does not appear in $\Gamma$. Snip the annulus open along $\sigma$. This results in either a disc or a disc together with an annulus with fewer marked points than before. Specializing the cluster variable $x_\sigma$ to one and applying induction to the annulus if necessary, we deduce that $\Sigma$ and $\Gamma$ coincide in the interior of the new surface(s). This implies that $\Sigma$ and $\Gamma$ coincide except that $\Sigma$ contains an extra arc not appearing in $\Gamma$, which is obviously impossible.

We may therefore assume that $\Sigma$ contains no arcs of $T$. Thus, it follows from Lemma 4.4 that $X_M^T$ is the sum of $x_\Gamma^T(0)$ and of proper Laurent monomials in the cluster $T$, since $\mathbf{e} \cdot \exp(x_\Sigma(\mathbf{e})) < 0$ implies that some term of $\exp(x_\Sigma(\mathbf{e})) < 0$.

It thus remains to prove that $x_\Sigma(0)$ is a proper Laurent monomial. As in [Cer11b], this is equivalent to showing that $g_{\Sigma}$ has at least one negative entry. By Proposition 4.5 we know $0 = E(M_\Sigma) = \dim M \cdot g_{\Sigma} + \dim \text{Hom}(M_\Sigma, M_\Sigma)$. Since
the second term is strictly positive (except in the degenerate case that \( M_\Sigma = 0 \)), the first is negative, implying that \( g_\Sigma \) has at least one negative term, as desired.

Therefore, \( x_\Sigma \) is a sum of proper Laurent monomials in the cluster \( T \) and \( x_\Gamma \) is not a summand of the \( T \)-expansion of \( x_\Sigma \).

The case where \( \Gamma \in \mathcal{T}(C_{p,q}) \) and \( \Sigma \in \mathcal{T}(C_{p,q}) \setminus \mathcal{T}(C_{p,q}) \). Since \( \Sigma \in \mathcal{T}(C_{p,q}) \setminus \mathcal{T}(C_{p,q}) \), it cannot contain any bridging edges.

If all the arcs of \( \Sigma \) were in \( T \), then, since all the terms in the expansion of the loop have bridging edges of \( T \) in the denominator, the same will be true for all the terms of \( x_\Sigma \), since all the arcs of \( \Sigma \) are necessarily peripheral.

Suppose now that \( \Sigma \) has a peripheral edge \( \sigma \) which is not in \( T \). By Lemmas 4.1, 4.2 and 4.3 the total degree of any term in the expansion of \( \Sigma \) with respect to the edges of \( T \) which cross \( \sigma \), is negative. It follows that \( x_\Gamma \) does not appear in the \( T \)-expansion of \( x_\Sigma \).

This completes the proof that \( \lambda_\Gamma(y) \), the coefficient of \( x_\Gamma \), in the expansion of \( y \), is non-negative, in the case where \( x_\Gamma \) is a cluster monomial.

4.3.4. The case where \( \Gamma \in \mathcal{T}(C_{p,q}) \setminus \mathcal{T}(C_{p,q}) \). We write \( \Gamma = \{ z_m \} \cup \mathcal{T} \) for some \( m \geq 1 \) and some \( \mathcal{T} \in \mathcal{T}(C_{p,q}) \).

Since \( \mathcal{T} \) contains no bridging arcs, there is some freedom to choose a triangulation of \( C_{p,q} \). First of all, add enough peripheral edges to the edges of \( C_{p,q} \) so that the region which is not triangulated is a subannulus which has one vertex on each boundary component. Denote these vertices by \( O \) and \( I \).

Now, we are going to define a \( \mathbb{Z} \)-indexed family of triangulations. For \( r \in \mathbb{Z} \), define \( T^{(r)} \) to be the triangulation obtained by adding an edge \( \alpha \) which starts at \( I \), wraps \( r \) times around the annulus, then goes to \( O \), and let \( \beta \) be the edge which starts at \( O \) and wraps \( -r+1 \) times around the annulus before returning to \( I \). These two edges are compatible and define a triangulation.

The \( T^{(r)} \)-expansion of \( x_\Gamma \) is \( x_\mathcal{T} \) times the \( T^{(r)} \)-expansion of \( x_{z_m} \). We note that the \( T^{(r)} \)-expansion of \( x_{z_m} \) contains a term \( x_\beta^{m}/x_\alpha^{m} \). We define \( t_{r,\beta} = x_\mathcal{T} x_\beta^{m}/x_\alpha^{m} \).

The case where \( \Gamma \in \mathcal{T}(C_{p,q}) \setminus \mathcal{T}(C_{p,q}) \) and \( \Sigma \in \mathcal{T}(C_{p,q}) \). If \( \Sigma \) has any arcs in common with \( T \), then we can proceed as in the case that \( \Gamma \) is a cluster monomial, so we may assume that \( \Sigma \) has no such arcs. Therefore, for any \( e \neq 0 \) in \( \mathbb{Z}^n \) such that \( \text{Gr}_e(M_\Sigma) \neq 0 \), we know from Lemma 4.4 that \( e \exp(x_\Sigma(e)) < 0 \).

We claim that for any such \( e \), we have \( e \exp(t_{r,\beta}) \geq 0 \). Clearly \( e \exp(t_{r,\beta}) \geq 0 \), since all the entries in both vectors are non-negative. It is therefore enough to show that \( e \exp(x_\beta^{m}/x_\alpha^{m}) \geq 0 \). Write \( Q_{T^{(r)}} \) for the quiver associated to \( T^{(r)} \), and write \( \tilde{Q}_{T^{(r)}} \) for its full subquiver on the vertices \( v_\alpha, v_\beta \) corresponding to \( \alpha \) and \( \beta \). This is a Kronecker quiver with two arrows from \( v_\alpha \) to \( v_\beta \). For \( M \) a representation of \( Q_{T^{(r)}} \), write \( \tilde{M} \) for the representation restricted to the subquiver.

For \( \sigma \in \Sigma \), consider \( \tilde{M}_\sigma \). By [BZ11], it is indecomposable, and its dimensions at \( v_\alpha, v_\beta \) count the number of intersections of \( \sigma \) with \( \alpha, \beta \). By choosing \( r \) large enough, we may guarantee that \( \sigma \) crosses \( \beta \) at least as many times as \( \alpha \). This implies that \( \tilde{M}_\sigma \) is preprojective or regular, which implies that the same is true of any of its indecomposable subobjects.

Consider the Grothendieck group for representations of \( \tilde{Q} \), which we think of as \( \mathbb{Z}^2 \), by fixing the basis \( [S_\alpha], [S_\beta] \). We also think of this as the multiplicative group of monomials in \( x_\alpha, x_\beta \). Then \( \exp(x_\beta^{m}/x_\alpha^{m}) = (-m, m) \), which is \(-m \) times the class of the null root for the Kronecker quiver. Therefore, if \( d \) is a dimension
vector of an indecomposable preprojective $\tilde{Q}_{T(r)}$-representation, $d.(-m,m) = m$, while if $d$ is the dimension vector of a regular indecomposable $\tilde{Q}_{T(r)}$-representation, $d.\exp(x^m_\alpha / x^m_\beta) = 0$. It follows that, for any $e$ such that $Gr_e(M_\Sigma) \neq \emptyset$, we have $e.\exp(t_{\Gamma,r}) \geq 0$ for $r$ sufficiently large.

This shows that the only term in the $T$-expansion of $x_\Sigma$ which could coincide with $t_{\Gamma,r}$ is $x_\Sigma(0)$. To treat the case $e = 0$, we also follow Cerulli. Specifically, we show that $\dim M_\Sigma.\exp(t_{\Gamma,r}) \geq 0$, while $\dim M_\Sigma.\exp(x^m_\alpha / x^m_\beta(0)) < 0$.

As in the $e \neq 0$ case, the first of these statements reduces to showing that $\dim M.\exp(x^m_\beta / x^m_\alpha) \geq 0$, and this will hold for $r$ sufficiently large, because we can arrange $\tilde{M}$ to be preinjective or regular. For the second statement, we see that $\dim M_\Sigma.\exp(x_\Sigma(0)) = \dim M_\Sigma.g_\Sigma$. Proposition 4.5 tells us that $0 = E(M) = \dim M_\Sigma.g_\Sigma + \dim \text{Hom}(M,M)$. Thus $\dim M.\exp(x_\Sigma(0)) < 0$ as desired.

The case where $\Gamma \in T(C_{p,q}) \setminus T(C_{p,q})$ and $\Sigma \in T(C_{p,q}) \setminus T(C_{p,q})$. Note that in this case $\Sigma$ does not contain any bridging arc. As before, we may assume that $\Gamma$ and $\Sigma$ do not contain any arcs in common.

Suppose that $\Sigma$ contains some arc which does not appear in $T$. We can therefore apply Lemma 4.3 to conclude that any term in the expansion of $x_\Sigma$ has negative degree with respect to arcs that cross $\gamma$. Therefore $t_{\Gamma,r}$ cannot appear as such a term, since $x^m_\beta / x^m_\alpha$ has zero degree with respect to edges of $T$ which cross $\gamma$ (either both or neither of $\alpha$ and $\beta$ cross $\gamma$), and $x_\Gamma$ has non-negative degree with respect to arcs that cross $\gamma$.

On the other hand, if $\Sigma$ contains only a loop and edges from $T^{(r)}$ (disjoint from those of $\Gamma$), the claim is clear. Thus this case is established.

This completes the proof that $\lambda_\Gamma(y)$, the coefficient of $x_\Gamma$ in the expansion of $y$, is non-negative, in the case that $\Gamma$ includes a loop. We have therefore completed the proof of (B3), and thus the proof of Theorem 1.2.

5. AN EXAMPLE IN TYPE $\tilde{A}_{2,2}$

In this short section, we give an explicit description of the atomic basis in a cluster algebra $\mathcal{A}$ of type $\tilde{A}_{2,2}$. Such a cluster algebra is associated to the annulus $C_{2,2}$ with two marked points on each boundary component. We denote by $\mathcal{M}$ the set of cluster monomials in $\mathcal{A}$. Moreover, we distinguish four particular elements in the cluster algebra $\mathcal{A}$ which correspond to the following curves:

In terms of representation theory, these four curves correspond to the four indecomposable rigid objects which belong to tubes in the Auslander-Reiten quiver of a cluster category of type $\tilde{A}_{2,2}$. As usual, for any $m \geq 1$, we denote by $z_m$ the unique loop in $C_{2,2}$ going $m$ times around the annulus. Then it follows from Theorem 1.2 that the atomic basis of $\mathcal{A}$ is:

$$\mathcal{B} = \mathcal{M} \cup \left\{ x_{zm} x^a_\alpha x^b_\beta \mid m \geq 1, a, b \geq 0, 1 \leq i,j \leq 2 \right\}.$$
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References

[ABCP10] Ibrahim Assem, Thomas Brüstle, Gabrielle Charbonneau-Jodoin, and Pierre-Guy Plamondon. Gentle algebras arising from surface triangulations. Algebra and Number Theory, 4(2):201–229, 2010.

[BMR+06] Aslak Buan, Robert Marsh, Markus Reineke, Iain Reiten, and Gordana Todorov. Tilting theory and cluster combinatorics. Adv. Math., 204(2):572–618, 2006.

[BrZ11] Thomas Brüstle and Jie Zhang. On the cluster category of a marked surface. Algebra and Number Theory, to appear, 2011.

[CC06] Philippe Caldero and Frédéric Chapoton. Cluster algebras as Hall algebras of quiver representations. Commentarii Mathematici Helvetici, 81:596–616, 2006.

[Cer11a] Giovanni Cerulli Irelli. Cluster algebras of type $A_2^{(1)}$. Algebras and Representation Theory, to appear, 2011.

[Cer11b] Giovanni Cerulli Irelli. Positivity in skew-symmetric cluster algebras of finite type. arXiv:1102.3050v2 [math.RA], 2011.

[CK06] Philippe Caldero and Bernhard Keller. From triangulated categories to cluster algebras II. Annales Scientifiques de l’École Normale Supérieure, 39(4):83–100, 2006.

[CK08] Philippe Caldero and Bernhard Keller. From triangulated categories to cluster algebras. Inventiones Mathematicae, 172:169–211, 2008.

[Dup08] Grégoire Dupont. Generic variables in acyclic cluster algebras and bases in affine cluster algebras. arXiv:0811.2909v2 [math.RT], 2008.

[Dup10a] Grégoire Dupont. Quantized Chebyshev polynomials and cluster characters with coefficients. Journal of Algebraic Combinatorics, 31(4):501–532, June 2010.

[Dup10b] Grégoire Dupont. Transverse quiver Grassmannians and bases in affine cluster algebras. Algebra and Number Theory, 4(5):599–624, July 2010.

[Dup11] Grégoire Dupont. Generic variables in acyclic cluster algebras. Journal of Pure and Applied Algebra, 215(4):628–641, April 2011.

[DWZ08] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky. Quivers with potentials and their representations. I. Mutations. Selecta Math. (N.S.), 14(1):59–119, 2008.

[DWZ10] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky. Quivers with potentials and their representations II: Applications to cluster algebras. J. Amer. Math. Soc., 23(3):749–790, 2010.

[FST08] Sergey Fomin, Michael Shapiro, and Dylan Thurston. Cluster algebras and triangulated surfaces. I. Cluster complexes. Acta Math., 201(1):83–146, 2008.

[FZ02] Sergey Fomin and Andrei Zelevinsky. Cluster algebras I: Foundations. J. Amer. Math. Soc., 15:497–529, 2002.

[FZ03] Sergey Fomin and Andrei Zelevinsky. Cluster algebras II: Finite type classification. Inventiones Mathematicae, 154:63–121, 2003.

[GLS10] Christof Geiss, Bernard Leclerc, and J. Schröer. Generic bases for cluster algebras and the Chamber Ansatz. arXiv:1004.2781v2 [math.RT], 2010.

[Kel05] Bernhard Keller. On triangulated orbit categories. Documenta Mathematica, 10:551–581, 2005.

[Lab09] Daniel Labardini-Fragoso. Quivers with potentials associated to triangulated surfaces. Proc. London Math. Soc., 98(3):797–839, 2009.

[Lam11] Philipp Lampe. Quantum cluster algebras of type A and the dual canonical basis. arXiv:1101.0580v1 [math.RT], 2011.
[MSW09] Gregg Musiker, Ralf Schiffler, and Lauren Williams. Positivity for cluster algebras from surfaces. arXiv:0906.0748v1 [math.CO], 2009.

[Nak10] Hiraku Nakajima. Quiver varieties and cluster algebras. arXiv:0905.0002v5 [math.QA], 2010.

[Pal08] Yann Palu. Cluster characters for 2-Calabi-Yau triangulated categories. Ann. Inst. Fourier (Grenoble), 58(6):2221–2248, 2008.

[Pal11] Yann Palu. Cluster characters II: A multiplication formula. Proc. LMS, to appear, 2011.

[Rin84] Claus Michael Ringel. Tame algebras and integral quadratic forms. Lecture Notes in Mathematics, 1099:1–376, 1984.

[Sch08] Ralf Schiffler. A cluster expansion formula (A_n case). Electron. J. Combin., 15(1):Research paper 64, 9, 2008.

[SS07] Daniel Simson and Andrzej Skowroński. Elements of the Representation Theory of Associative Algebras, Volume 2: Tubes and Concealed Algebras of Euclidean type, volume 71 of London Mathematical Society Student Texts. Cambridge University Press, 2007.

[ST09] Ralf Schiffler and Hugh Thomas. On cluster algebras arising from unpunctured surfaces. Int. Math. Res. Not., (17):3160–3189, 2009.

[SZ04] Paul Sherman and Andrei Zelevinsky. Positivity and canonical bases in rank 2 cluster algebras of finite and affine types. Mosc. Math. J., 4:947–974, 2004.

Université de Sherbrooke, Sherbrooke QC, Canada
E-mail address: gregoire.dupont@usherbrooke.ca

University of New Brunswick, Fredericton NB, Canada
E-mail address: hthomas@unb.ca