EXISTENCE THEOREM FOR NON-ABELIAN VORTICES
IN THE AHARONY–BERGMAN–JAFFERIS–MALDACENA
THEORY

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Abstract. In this paper, we discuss the existence theorem for multiple vortex solutions in the non-Abelian Chern–Simons–Higgs field theory developed by Aharony, Bergman, Jafferis, and Maldacena, on a doubly periodic domain. The governing equations are of the BPS type and derived by Auzzi and Kumar in the mass-deformed framework labeled by a continuous parameter. Our method is based on fixed point method.

1. Introduction

Vortices in non-Abelian gauge field theory play important roles in confinement mechanism and are governed by systems of nonlinear elliptic equations of complicated structures [2, 4, 7, 8, 10, 11, 12, 13, 14, 16, 22, 28, 30]. In this paper, we will focus on the vortex equations in the non-Abelian Chern–Simons–Higgs field theory developed by Aharony, Bergman, Jafferis, and Maldacena [1], known as the ABJM model, on a doubly periodic domain. The governing equations are of the BPS type and derived by Auzzi and Kumar [5] in the mass-deformed framework labeled by a continuous parameter. Developing and extending the methods of [6, 15, 17, 18, 19, 20, 21, 24, 27], we obtain the existence of a multiple vortex solution.

Recall that the ABJM model [1] is a Chern–Simons–Higgs theory within which the matter fields are four complex scalars,

\[ C^I = (Q^1, Q^2, R^1, R^2), \quad I = 1, 2, 3, 4, \]

in the bifundamental matter field \( (\mathbf{N}, \overline{\mathbf{N}}) \) representation of the gauge group \( U(N) \times U(N) \), which hosts two gauge fields, \( A_\mu \) and \( B_\mu \). The Chern–Simons action associated to the two gauge group \( A_\mu \) and \( B_\mu \) of levels \(+k\) and \(-k\) is...
given by the Lagrangian density

\[ L_{\text{CS}} = \frac{k}{4\pi} \epsilon^{\mu\nu\gamma} \text{Tr} \left( A_\mu \partial_\nu A_\gamma - \frac{2i}{3} A_\mu A_\nu A_\gamma - B_\mu \partial_\nu B_\gamma - \frac{2i}{3} B_\mu B_\nu B_\gamma \right), \]

where the gauge-covariant derivatives on the bifundamental field \( s \) are defined as

\[ D_\mu C^I = \partial_\mu C^I + i A_\mu C^I - i C^I B_\mu, \quad I = 1, 2, 3, 4. \]

The scalar potential of the mass deformed theory can be written in a compact way as [9]

\[ V = \text{Tr}(M^\alpha M^\alpha + N^\alpha N^\alpha), \]

where

\[ M^\alpha = \rho Q^\alpha + \frac{2\pi}{k} (2Q^\alpha Q^\beta Q^\beta + R^\alpha R^\beta R^\beta - Q^\alpha R^\beta R^\beta) \]

\[ N^\alpha = -\rho R^\alpha + \frac{2\pi}{k} (2R^\alpha R^\beta R^\beta + Q^\alpha Q^\beta R^\beta - R^\alpha Q^\beta Q^\beta) \]

where the Kronecker symbol \( \epsilon^{\alpha\beta} \) (\( \alpha, \beta = 1, 2 \)) is used to lower or raise indices, and \( \rho > 0 \) a massive parameter. Thus, when the spacetime metric is of the signature \((+ - -)\), the total (bosonic) Lagrangian density of ABJM model can be written as

\[ L = -L_{\text{CS}} + \text{Tr}([D_\mu C^I]'^* [D^\mu C^I]) - V, \]

which is of a pure Chern–Simons type for the gauge field sector. The equations of motion of the Lagrangian (1.7) are rather complicated. As in [5] and [6], we concentrate on a reduced situation where (say) \( R^\alpha = 0, N = 3 \). In the static limit, Auzzi and Kumar [5] showed that these equations may be reduced into the first-order BPS vortex equations without assuming radial symmetry

\[ (\partial_1 + i\partial_2)\kappa = i(a_1 + ia_2)\kappa, \]

\[ (\partial_1 + i\partial_2)\phi = -i([a_1 + ia_2] - [b_1 + ib_2])\phi, \]

\[ a_{12} = -\frac{\lambda}{2}(2|\kappa|^2 - |\phi|^2 - 1), \]

\[ b_{12} = -\lambda(|\phi|^2 - 1), \]

where \( \kappa \) is a real-valued scalar field, \( \phi \) a complex-valued scalar field, and \( a_j \) and \( b_j \) are two real-valued gauge potential vector fields, \( a_{jk} = \partial_j a_k - \partial_k a_j \) and \( \lambda = 4\rho^2 \).

We shall look for solutions of these equations so that \( \kappa \) never vanishes but \( \phi \) vanishes exactly at the finite set of points

\[ Z = \{p_1, p_2, \ldots, p_n\}. \]
Set $u = \ln \kappa^2$ and $w = \ln |\phi|^2$ and note that $|\phi|$ behaves like $|x - p_s|$ for $x$ near $p_s$ ($s = 1, \ldots, n$). We see that $u$ and $w$ satisfy the equations [6]

$\Delta u = \lambda (2e^u - e^w - 1)$, \hspace{1cm} (1.13)

$\Delta u + \Delta w = 2\lambda (e^w - 1) + 4\pi n \sum_{s=1}^{n} \delta_{p_s}(x)$, \hspace{1cm} (1.14)

where we have included our consideration of the zero set $Z$ of $\phi$ as given in (1.12).

Chen, Zhang and Zhu [6] studied vortex equations in a supersymmetric Chern–Simons–Higgs theory in the ABJM model. They obtained a series of existence and uniqueness theorems for multiple vortex solutions of the ABJM model, over $\mathbb{R}^2$ and on a doubly periodic domain using the methods of calculus of variations.

In the present paper, we are going to discuss the non-Abelian BPS vortex equations of the ABJM model on a doubly periodic domain. We shall show how to approach the existence problem by a fixed point method via the Leray–Schauder theorem. Our approach is of independent interest because the a priori estimates obtained in the process may provide additional information on the governing equations. It’s interesting that, our method is completely applicable to the self-dual equations governing multiple vortices in a product Abelian Higgs model may be regarded as a generalized Ginzburg–Landau theory [25, 26, 29].

2. Fixed point method

In this section, we approach the existence problem of the multiple vortex solutions in a doubly periodic domain $\Omega$ by a fixed point method where we apply the maximum principle and the Poincaré inequality to derive suitable a priori estimates. We introduce a background function $w_0$ satisfying

$\Delta w_0 = -\frac{4\pi n}{|\Omega|} + 4\pi \sum_{s=1}^{n} \delta_{p_s}(x)$, \hspace{1cm} (2.1)

where $\delta_p$ is the Dirac distribution concentrated at the point $p$. Using the new variable $v$ so that $w = w_0 + v$, we can modify (1.13) and (1.14) into

$\Delta u = \lambda (2e^u - e^{w_0+v} - 1)$, \hspace{1cm} (2.2)

$\Delta v = \lambda (3e^{w_0+v} - 2e^u - 1) + \frac{4\pi n}{|\Omega|}$, \hspace{1cm} (2.3)

which are now in a regular (singularity-free) form. Note that, since the singularity of $w_0$ at $p_s$ is of the type $\ln |x - p_s|^2$, the weight function $e^{w_0}$ is everywhere smooth.
Let \((u, v)\) be a solution of (2.2) and (2.3). Then \((u, w)\) solves (1.13) and (1.14). We first derive a necessary condition for the solvability of (2.2) and (2.3). Integrating (2.2) and (2.3), we have

\[
\int_{\Omega} e^{u_{0} + v} \, dx = |\Omega| - \frac{2\pi n}{\lambda} \equiv C_{1} > 0,
\]
\[
\int_{\Omega} e^{u} \, dx = \frac{1}{2} \int_{\Omega} e^{u_{0} + v} \, dx + \frac{1}{2} |\Omega| = \frac{1}{2} (C_{1} + |\Omega|) \equiv C_{2} > 0.
\]

Of course, the conditions (2.4) and (2.5) imply that the existence of an \(n\)-vortex solution requires that \(C_{1} > 0\) and \(C_{2} > 0\), which is simply

\[
|\Omega| - \frac{2\pi n}{\lambda} \equiv C_{1} > 0,
\]

since \(C_{1} > 0\) contains \(C_{2} > 0\).

We now proceed to prove that (2.4) and (2.5) are also sufficient for the existence of a solution to the equations (2.2) and (2.3).

We use \(W^{1,2}(\Omega)\) to denote the usual Sobolev space of scalar-valued or vector-valued \(\Omega\)-periodic \(L^{2}\)-functions whose derivatives are also in \(L^{2}(\Omega)\). For this purpose, we rewrite each \(f \in W^{1,2}(\Omega)\) as follows

\[
f = f + f',
\]

where \(f\) denotes the integral mean of \(f\), \(f = \frac{1}{|\Omega|} \int_{\Omega} f \, dx\) and \(\int_{\Omega} f' \, dx = 0\). We can derive from (2.4) and (2.5) the expressions

\[
u = \ln C_{1} - \ln \left( \int_{\Omega} e^{u_{0} + v'} \, dx \right),
\]
\[
u = \ln C_{2} - \ln \left( \int_{\Omega} e^{u} \, dx \right).
\]

For \(X = \{ f' \in W^{1,2}(\Omega) \mid \int_{\Omega} f' \, dx = 0 \}\) and \(Y = X \times X\) define a operator \(T : Y \rightarrow Y\) by setting

\[
(U', V') = T(u', v'), \quad (u', v') \in Y,
\]

where \((U', V') \in Y\) is the unique solution of the system of the equations

\[
\Delta U' = \lambda \left( \frac{2C_{2}e^{u'}}{\int_{\Omega} e^{u} \, dx} - \frac{C_{1}e^{u_{0} + v'}}{\int_{\Omega} e^{u_{0} + v} \, dx} - 1 \right),
\]
\[
\Delta V' = \lambda \left( \frac{3C_{1}e^{u_{0} + v'}}{\int_{\Omega} e^{u} \, dx} - \frac{2C_{2}e^{u'}}{\int_{\Omega} e^{u} \, dx} - 1 \right) + \frac{4\pi n}{|\Omega|}.
\]

The existence and uniqueness of a solution of the system of equations (2.10) and (2.11) may easily be seen since the right-hand sides of (2.10) and (2.11) have zero average value on \(\Omega\) as a consequence of the definitions of (2.7) and (2.8). By the Poincaré inequality [23], we may define the norm of \(Y\) as follow

\[
\|(u', v')\|_{Y} = \|\nabla u'\|_{L^{2}(\Omega)} + \|\nabla v'\|_{L^{2}(\Omega)}.
\]
Theorem 2.1. The system of equation (1.13) and (1.14) has a solution if and only if the conditions (2.4) and (2.5) are valid.

We will prove Theorem 2.1 in terms of two lemmas as follows.

Lemma 2.1. The operator $T : Y \mapsto Y$ is completely continuous.

Proof. Let $(u'_n, v'_n) \rightarrow (u'_0, v'_0)$ weakly in $Y$ as $n \rightarrow \infty$. Then $(u'_n, v'_n) \rightarrow (u'_0, v'_0)$ strongly in $L^p(\Omega) \times L^p(\Omega)$ ($p \geq 1$). The Egorov theorem imply that for any $\epsilon > 0$ there is a sufficiently large number $K_\epsilon > 0$ and a subset $\Omega_\epsilon \subset \Omega$ such that $|u'_n|, |v'_n| \leq K_\epsilon$, $x \in \Omega - \Omega_\epsilon$, $|\Omega_\epsilon| < \epsilon$.

Set $(U'_n, V'_n) = T(u'_n, v'_n)$ and $(U'_0, V'_0) = T(u'_0, v'_0)$. Then

\begin{align}
\Delta(U'_n - U'_0) &= \lambda \left( \frac{2C_2 e^{u'_n}}{f^u_{\Omega e}} - \frac{C_1 e^{u'_0 v'_n + v'_n}}{f^u_{\Omega e}} \right) \Delta e^{u'_n} dx + \frac{C_1 e^{u'_0 v'_n + v'_n}}{f^u_{\Omega e}} \Delta e^{u'_n} dx, \\
\Delta(V'_n - V'_0) &= \lambda \left( \frac{-2C_2 e^{v'_n}}{f^v_{\Omega e}} + \frac{3C_1 e^{u'_0 v'_n + v'_n}}{f^v_{\Omega e}} \right) \Delta e^{v'_n} dx + \frac{2C_2 e^{u'_0 v'_n + v'_n}}{f^v_{\Omega e}} \Delta e^{v'_n} dx.
\end{align}

Multiplying (2.13) and (2.14) by $U'_n - U'_0$ and $V'_n - V'_0$, and integrating by parts, respectively, we obtain

\begin{align}
\int_{\Omega} |\nabla(U'_n - U'_0)|^2 dx &= \int_{\Omega} \lambda \left( \frac{2C_2 e^{u'_n}}{f^u_{\Omega e}} - \frac{2C_2 e^{v'_n}}{f^v_{\Omega e}} \right) (U'_n - U'_0) dx, \\
\int_{\Omega} |\nabla(V'_n - V'_0)|^2 dx &= \int_{\Omega} \lambda \left( \frac{2C_2 e^{u'_n}}{f^u_{\Omega e}} - \frac{2C_2 e^{v'_n}}{f^v_{\Omega e}} \right) (V'_n - V'_0) dx.
\end{align}

Note that the boundedness of $\{(u'_n, v'_n)\}$ in $Y$ and the Trudinger-Moser inequality [3] imply that

\begin{align}
\sup_n \int_{\Omega} e^{u'_n} dx &\leq C < \infty, \\
\sup_n \int_{\Omega} e^{v'_n} dx &\leq C < \infty.
\end{align}

For any $\epsilon > 0$, let $\Omega_\epsilon$ be a neighborhood of the points $p_1, p_2, \ldots, p_n$ so that $p_n \in \Omega_\epsilon(\forall \epsilon)$ and $|\Omega_\epsilon| < \epsilon$. On the other hand, since there is a constant $\epsilon_0 > 0$ such that $e^{u_0(x)} \geq \epsilon_0$ for all $x \in \Omega - \Omega_\epsilon$.

Therefore, from (2.14), we obtain

\begin{align}
\int_{\Omega} |\nabla(U'_n - U'_0)|^2 dx &\leq \lambda \left( \frac{4C_2}{f^u_{\Omega e}} \int_{\Omega} e^{u'_n} dx \right) (U'_n - U'_0)^2 dx.
\end{align}
Inserting (2.20) and (2.21) into (2.19), and letting
$$\varepsilon > 0,$$
where \( \tilde{u}_n \) used the inequalities
\[
\int_{\Omega} e^{u_n} dx \geq |\Omega| \exp\left(\frac{1}{|\Omega|} \int_{\Omega} u_n^4 dx\right) = |\Omega|,
\]
and
\[
\int_{\Omega} e^{u_n + v_n^0} dx \geq \int_{\Omega - \Omega_\varepsilon} e^{u_n + v_n^0} dx \geq \varepsilon_0 |\Omega - \Omega_\varepsilon| \exp(-K_\varepsilon) = K_{\Omega_\varepsilon}.\]

Applying the Cauchy inequality and Hölder inequality, and (2.17), we have
\[
\int \int_{\Omega} \int e^{2u_n}|u_n^4 - u_0^4| dx \leq \frac{1}{2\varepsilon} \int \int_{\Omega} e^{2u_n}|u_n^4 - u_0^4|^2 dx + \varepsilon \int \int_{\Omega} (U_n' - U_0')^2 dx
\]
\[
\leq \frac{1}{2\varepsilon} \left( \int \int_{\Omega} e^{4u_n} dx \right)^\frac{1}{2} \left( \int \int_{\Omega} |u_n^4 - u_0^4|^4 x \right)^\frac{1}{4}
\]
\[
+ \frac{C_3 \varepsilon}{2} \| \nabla(U_n' - U_0') \|^2_{L^2(\Omega)}
\]
(2.20)
\[
\leq C_4 \| u_n^4 - u_0^4 \|^2_{L^4(\Omega)} + \frac{C_3 \varepsilon}{2} \| \nabla(U_n' - U_0') \|^2_{L^2(\Omega)}.
\]

Similarly,
\[
\int \int_{\Omega} \int e^{u_n + v_n^0}|u_n - v_0^0||U_n' - U_0'| dx \leq C_5 \| u_n^4 - v_0^4 \|^2_{L^4(\Omega)} + \frac{C_4 \varepsilon}{2} \| \nabla(U_n' - U_0') \|^2_{L^2(\Omega)}.
\]
Inserting (2.20) and (2.21) into (2.19), and letting \( \varepsilon > 0 \) be small enough, we have
\[
\| \nabla(U_n' - U_0') \|^2_{L^2(\Omega)} \leq C \left( \| u_n^4 - u_0^4 \|^2_{L^4(\Omega)} + \| v_n^4 - v_0^4 \|^2_{L^4(\Omega)} \right),
\]
where \( C > 0 \) is a constant.

For (2.16), we have
\[
\| \nabla(V_n' - V_0') \|^2_{L^2(\Omega)} \leq C \left( \| u_n^4 - u_0^4 \|^2_{L^4(\Omega)} + \| v_n^4 - v_0^4 \|^2_{L^4(\Omega)} \right).
\]

From (2.22) and (2.23), we arrive at
\[
\| (U_n' - U_0', V_n' - V_0') \|^2_\mathcal{Y} \leq C \left( \| u_n^4 - u_0^4 \|^2_{L^4(\Omega)} + \| v_n^4 - v_0^4 \|^2_{L^4(\Omega)} \right),
\]
(2.24)
where $C > 0$ is a constant. This proves that $(U'_n, V'_n) \to (U'_0, V'_0)$ strongly in $Y$ and the lemma follows. □

We now study the fixed point equation labeled by a parameter $t$,

$$\begin{align*}
(u'_t, v'_t) &= tT(u'_t, v'_t), \quad 0 \leq t \leq 1.
\end{align*}$$

**Lemma 2.2.** There is a constant $C > 0$ independent of $t \in [0, 1]$ so that

$$\begin{align*}
\|(u'_t, v'_t)\|_Y &\leq C, \quad 0 < t \leq 1.
\end{align*}$$

Consequently, $T$ has a fixed point in $Y$.

**Proof.** When $t > 0$, it is straightforward to check that $(u'_t, v'_t)$ satisfies the equations

$$\begin{align*}
\Delta u'_t &= \lambda t \left( \frac{2C_2 e^{u'_t}}{\int_\Omega e^{u'_t} dx} - \frac{C_1 e^{w_0 + v'_t}}{\int_\Omega e^{w_0 + v'_t} dx} - 1 \right), \\
\Delta v'_t &= \lambda t \left( -\frac{2C_2 e^{v'_t}}{\int_\Omega e^{v'_t} dx} + \frac{3C_1 e^{w_0 + v'_t}}{\int_\Omega e^{w_0 + v'_t} dx} - 1 \right) + \frac{4\pi n}{|\Omega|} t.
\end{align*}$$

Set $w'_t = w_0 + v'_t$. Then the equations (2.27) and (2.28) are modified into

$$\begin{align*}
\Delta u'_t &= \lambda t \left( \frac{2C_2 e^{u'_t}}{\int_\Omega e^{u'_t} dx} - \frac{C_1 e^{w'_t}}{\int_\Omega e^{w'_t} dx} - 1 \right), \\
\Delta w'_t &= \lambda t \left( -\frac{2C_2 e^{w'_t}}{\int_\Omega e^{w'_t} dx} + \frac{3C_1 e^{w'_t}}{\int_\Omega e^{w'_t} dx} - 1 \right) + \frac{4\pi n}{|\Omega|} (t - 1) + 4\pi \sum_{s=1}^n \delta_{p_s}(x),
\end{align*}$$

where $\Delta w_0 = -\frac{4\pi n}{|\Omega|} + 4\pi \sum_{s=1}^n \delta_{p_s}(x)$. In the doubly periodic domain $\Omega$, we let $p, q \in \Omega$ so that

$$u'_t(p) = \max\{u'_t(x)|x \in \Omega\}, \quad w'_t(q) = \max\{w'_t(x)|x \in \Omega\}.$$  

To facilitate our computation, we adopt the notation

$$\begin{align*}
h'_t(x) &= \frac{C_2 e^{u'_t}}{\int_\Omega e^{u'_t} dx}, \\
g'_t(x) &= \frac{C_1 e^{w'_t}}{\int_\Omega e^{w'_t} dx}.
\end{align*}$$

Then from (2.29), we have

$$0 \geq (\Delta u'_t)(p) = \lambda t (2h'_t(p) - g'_t(p) - 1).$$

Therefore

$$2h'_t(p) \leq g'_t(p) + 1 \leq \frac{C_1 e^{w'_t(q)}}{\int_\Omega e^{w'_t} dx} + 1 = g'_t(q) + 1.$$ 

Hence, for any $x \in \Omega$, we have

$$2h'_t(x) \leq g'_t(q) + 1, \quad \forall x \in \Omega.$$  

From (2.30), using (2.32), we obtain

$$g'_t(q) \leq 1 + \frac{2\pi n}{\lambda |\Omega|} \frac{1 - t}{t}, \quad 0 < t \leq 1.$$
In view of (2.32) and (2.33), for any \( x \in \Omega \), we have
\[
(2.34) \quad g_t'(x) \leq 1, \quad h_t'(x) \leq 1 + \frac{\pi n}{\lambda|\Omega|} \frac{1-t}{t}, \quad x \in \Omega.
\]

Multiplying (2.27) and (2.28) by \( u'_t, v'_t \) and integrating by parts, respectively, and using (2.34), we have
\[
\left\| \left( \nabla u'_t, \nabla v'_t \right) \right\|_{L^2(\Omega) \times L^2(\Omega)}^2 \leq \int_\Omega \left\{ (1 + 1 + 2)\lambda|u'_t| + \left[ (1 + 3 + 2)\lambda + \frac{4\pi n}{|\Omega|} \right]|v'_t| \right\} dx \\
\leq \tilde{C}_1 \int_\Omega |u'_t| dx + \tilde{C}_2 \int_\Omega |v'_t| dx \\
(2.35) \quad \leq C \varepsilon + \tilde{C}_\varepsilon \left\| \left( \nabla u'_t, \nabla v'_t \right) \right\|_{L^2(\Omega) \times L^2(\Omega)}^2.
\]

Let \( \varepsilon > 0 \) be small enough, we have
\[
(2.36) \quad ||(u'_t, v'_t)||_Y = ||(\nabla u'_t, \nabla v'_t)||_{L^2(\Omega) \times L^2(\Omega)} \leq C,
\]
where \( C > 0 \) is a constant. The existence of a fixed point is a consequence of Lemma 2.2, the apriori estimate (2.26) and the Leray–Schauder theory. In particular, the existence of a fixed point of \( T \), say \((u', v')\), follows. \( \square \)

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