Zero modes and low-energy resolvent expansion for three dimensional Schrödinger operators with point interactions

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Abstract We investigate the low-energy behavior of the resolvent of Schrödinger operators with finitely many point interactions in three dimensions. We also discuss the occurrence and the multiplicity of zero energy obstructions.

1 Introduction and main results

A central topic in quantum mechanics is the study of quantum systems subject to very short-range interactions, supported around a submanifold of the ambient space. A relevant situation occurs when the singular interaction is supported on a set of points in the Euclidean space \( \mathbb{R}^d \). This leads to consider, formally, operators of the form

\[
-\Delta + \sum_{y \in Y} \mu_y \delta_y(z),
\]

where \( Y \) is a discrete subset of \( \mathbb{R}^d \), and \( \mu_y, y \in Y \), are real coupling constants.

Heuristically, \( \text{(1)} \) can be interpreted as the Hamiltonian for a non-relativistic quantum particle interacting with “point obstacles” of strengths \( \mu_y \), located at \( y \in Y \).

From a mathematical point of view, Schrödinger operators with point (delta-like) interactions have been intensively studied, since the first rigorous realization by Berezin and Faddeev \( [5] \), and subsequent characterizations by many other authors \( [2, 3, 16, 17, 8, 23] \) (see the surveys \( [11, 3] \), the monograph of Albeverio, Gesztesy, Høegh-Krohn, and Holden \( [4] \), and references therein for a thorough discussion).

In this work we focus on the case of finitely many point interactions in three dimensions. Our aim is to provide a detailed spectral analysis at the bottom of the continuous spectrum, i.e. at zero energy. A similar analysis has been done in \( [6] \)
for the two dimensional case, with application to the $L^p$-boundedness of the wave operators.

We start by recalling some well-known facts on the construction and the main properties of Schrödinger operators with point interactions.

We fix a natural number $N \geq 1$ and the set $Y = \{y_1, \ldots, y_N\} \subseteq \mathbb{R}^3$ of distinct centers of the singular interactions. Consider

$$T_Y := (-\Delta) \left| C_0^\infty(\mathbb{R}^3 \setminus \{Y\}) \right.$$ (2)

as an operator closure with respect to the Hilbert space $L^2(\mathbb{R}^3)$. It is a closed, densely defined, non-negative, symmetric operator on $L^2(\mathbb{R}^3)$, with deficiency index $N$. Hence, it admits an $N^2$-real parameter family of self-adjoint extensions. Among these, there is an $N$-parameter family of local extension, denoted by

$$\{ -\Delta_{\alpha,Y} \mid \alpha = (\alpha_1, \ldots, \alpha_N) \in (\mathbb{R} \cup \{\infty\})^N \},$$ (3)

whose domain of self-adjointness is qualified by certain local boundary conditions at the singularity centers.

The self-adjoint operators $-\Delta_{\alpha,Y}$ provide rigorous realizations of the formal Hamiltonian [1], the coupling parameters $\alpha_j, j = 1, \ldots, N$, being now proportional to the inverse scattering length of the interaction at the center $y_j$. In particular, if for some $j \in \{1, \ldots, N\}$ one has $\alpha_j = \infty$, then no actual interaction is present at the point $y_j$, and in practice things are as if one discards the point $y_j$. When all $\alpha_j = \infty$, one recovers the Friedrichs extension of $T_Y$, namely the self-adjoint realization of $-\Delta$ on $L^2(\mathbb{R}^3)$. Owing to the discussion above, we may henceforth assume, without loss of generality, that $\alpha$ runs over $\mathbb{R}^N$.

We review the basic properties of $-\Delta_{\alpha,Y}$, from [4, Section II.1.1] and [26] (see also [11, 9, 18, 12]). We introduce first some notations.

For $z \in \mathbb{C}$ and $x, y, y' \in \mathbb{R}^3$, set

$$\mathcal{G}_z^x := \frac{e^{iz|x-y|}}{4\pi|x-y|}, \quad \mathcal{G}_z^{y'y} := \begin{cases} 
\begin{array}{ll}
\frac{e^{i|z|y'-y'}}{4\pi|y-y'|} & \text{if } y' \neq y \\
0 & \text{if } y' = y,
\end{array}
\end{cases} \quad (4)$$

and

$$\Gamma_{\alpha,Y}(z) := \left( \alpha_j - \frac{iz}{4\pi} \right) \delta_{j,k} - \mathcal{G}_z^{y'y_k} \right)_{j,k=1,\ldots,N}. \quad (5)$$

The function $z \mapsto \Gamma_{\alpha,Y}(z)$ has values in the space of $N \times N$ symmetric, complex valued matrices and is clearly entire, whence $z \mapsto \Gamma_{\alpha,Y}(z)^{-1}$ is meromorphic in $\mathbb{C}$. It is known that $\Gamma_{\alpha,Y}(z)^{-1}$ has at most $N$ poles in the open upper half-plane $\mathbb{C}^+$, which are all located along the positive imaginary semi-axis. We denote by $\mathfrak{e}^+$ the set of such poles. Moreover, we denote by $\mathfrak{e}^0$ the set of poles of $\Gamma_{\alpha,Y}(z)^{-1}$ on the real line. Observe that $\mathfrak{e}^0$ is finite and symmetric with respect to $z = 0$. Actually, either $\mathfrak{e}^0 = \emptyset$ or $\mathfrak{e}^0 = \{0\}$. This follows by a generalization of the Rellich Uniqueness Theorem [29, Theorem 2.4], valid for a large class of compactly supported perturbations of
the Laplacian, introduced by Sjöstrand and Zworski in [30]. For an introduction to the classical theory of the Rellich Uniqueness Theorem, we refer to the monograph of Lax and Phillips [22]. More recently, the absence of non-zero real poles for $\Gamma^{-1}_{\alpha,Y}$ has been proved through different techniques by Galtbayar-Yajima [14], and by the author in collaboration with Michelangeli [24].

The following facts are known.

**Proposition 1**

(i) The domain of $-\Delta_{\alpha,Y}$ has the following representation, for any $z \in \mathbb{C}^+ \setminus \mathbb{D}^+$:

$$\mathcal{D}(-\Delta_{\alpha,Y}) = \left\{ g = F_z + \sum_{j,k=1}^{N} (\Gamma_{\alpha,Y}(z)^{-1})_{jk} F_z(\mathcal{G}^y_j), F_z \in H^2(\mathbb{R}^3) \right\}. \quad (6)$$

Equivalently, for any $z \in \mathbb{C}^+ \setminus \mathbb{E}^+$,

$$\mathcal{D}(-\Delta_{\alpha,Y}) = \left\{ g = F_z + \sum_{j=1}^{N} c_j \mathcal{G}^y_j \left| \begin{array}{c} F_z \in H^2(\mathbb{R}^3) \\ (c_1, \ldots, c_N) \in \mathbb{C}^N \\ \left( \begin{array}{c} F_z(y_1) \\ \vdots \\ F_z(y_N) \end{array} \right) = \Gamma_{\alpha,Y}(z) \left( \begin{array}{c} c_1 \\ \vdots \\ c_N \end{array} \right) \end{array} \right. \right\}. \quad (7)$$

At fixed $z$, the decompositions above are unique.

(ii) With respect to the decompositions (6)-(7), one has

$$(-\Delta_{\alpha,Y} - z^2) g = (-\Delta - z^2) F_z. \quad (8)$$

(iii) For $z \in \mathbb{C}^+ \setminus \mathbb{D}^+$, we have the resolvent identity

$$(-\Delta_{\alpha,Y} - z^2)^{-1} - (-\Delta - z^2)^{-1} = \sum_{j,k=1}^{N} (\Gamma_{\alpha,Y}(z)^{-1})_{jk} |\mathcal{G}^y_j \rangle \langle \mathcal{G}^y_k| \cdot (9)$$

(iv) The spectrum $\sigma(-\Delta_{\alpha,Y})$ of $-\Delta_{\alpha,Y}$ consists of at most $N$ non-positive eigenvalues and the absolutely continuous part $\sigma_{ac}(-\Delta_{\alpha,Y}) = [0, \infty)$, the singular continuous spectrum is absent.

Parts (i) and (ii) of Proposition 1 above originate from [17] and are discussed in [4] Theorem II.1.1.3, in particular (7) is highlighted in [11]. Part (iii) was first proved in [16, 17] (see also [4] equation (II.1.1.33)). Part (iv) is discussed in [4] Theorem II.1.1.4, where it is stated that $\sigma_p(-\Delta_{\alpha,Y}) \subset (-\infty, 0)$. An errata at the end of the monograph (see also [13, 15]) specifies that a zero eigenvalue embedded in the continuous spectrum can actually occur: in fact for every $N \geq 2$ one can find a configuration $Y$ of the $N$ centers, and coupling parameters $\alpha_1, \ldots, \alpha_N$ such that $0 \in \sigma_p(-\Delta_{\alpha,Y})$ – see the discussion in Section [4].

Next, let us discuss in detail the spectral properties of $-\Delta_{\alpha,Y}$, whose resolvent is characterized by (9) as an explicit rank-$N$ perturbation of the free resolvent. For negative eigenvalues, the situation is well-understood [4] Theorem II.1.1.4.
Proposition 2 There is a one to one correspondence between the poles $i\lambda \in \mathcal{E}^+$ of $\Gamma_{\alpha,Y}(z)^{-1}$ and the negative eigenvalues $-\lambda^2$ of $-\Delta_{\alpha,Y}$, counting the multiplicity. The eigenfunctions associated to the eigenvalue $-\lambda^2 < 0$ have the form
\[ \psi = \sum_{j=1}^{N} c_j \varphi_{i\lambda}^j, \]
where $(c_1, \ldots, c_N) \in \text{Ker} \Gamma_{\alpha,Y}(i\lambda) \setminus \{0\}$.

Our main purpose is to analyze the spectral behavior of $-\Delta_{\alpha,Y}$ at $z = 0$, and more generally when $z$ approaches the real line. The starting point is a classical version of the Limiting Absorption Principle for the free Laplacian. Given $\sigma > 0$, we consider the Banach space
\[ B_\sigma := \mathcal{B}(L^2(\mathbb{R}^3, \langle x \rangle^{2+\sigma} dx); L^2(\mathbb{R}^3, \langle x \rangle^{-2-\sigma} dx)), \]
where $\langle x \rangle := \sqrt{1+|x|^2}$, and $\mathcal{B}(X;Y)$ denotes the space of linear bounded operators from $X$ to $Y$. We have the following result [1, 21, 19].

Proposition 3 (Limiting Absorption Principle for $-\Delta$) Let $\sigma > 0$. For any $z \in \mathbb{C}^+$, we have $(-\Delta - z^2)^{-1} \in B_\sigma$. Moreover, the map $\mathbb{C}^+ \ni z \mapsto (-\Delta - z^2)^{-1} \in B_\sigma$ can be continuously extended to the real line.

Owing to the resolvent formula (9), and observing that for any $z \in \mathbb{C}^+ \cup \mathbb{R}$ the projectors $|\varphi_{i\lambda}^j \rangle \langle \varphi_{i\lambda}^k|$ belong to $B_\sigma$, it is easy to deduce that also $-\Delta_{\alpha,Y}$ satisfies a Limiting Absorption Principle.

Proposition 4 (Limiting Absorption Principle for $-\Delta_{\alpha,Y}$) Let $\sigma > 0$. For every $z \in \mathbb{C}^+ \setminus \mathcal{E}^+$, we have $(-\Delta_{\alpha,Y} - z^2)^{-1} \in B_\sigma$. Moreover, the map
\[ \mathbb{C}^+ \setminus \mathcal{E}^+ \ni z \mapsto (-\Delta_{\alpha,Y} - z^2)^{-1} \in B_\sigma \]
can be continuously extended to $\mathbb{R} \setminus \mathcal{E}^0$.

Our main result is a resolvent expansion in a neighborhood of $z = 0$, which in view of the previous discussion is the only possible singular point on the real line for the map $z \mapsto (-\Delta_{\alpha,Y} - z^2)^{-1} \in B_\sigma$.

Theorem 1 In a (real) neighborhood of $z = 0$, we have the expansion
\[ (-\Delta_{\alpha,Y} - z^2)^{-1} = \frac{R_{-2}}{z^2} + \frac{R_{-1}}{z} + R_0(z), \]
where $R_{-2}, R_{-1} \in B_\sigma$ and $z \mapsto R_0(z)$ is a continuous $B_\sigma$-valued map. Moreover, $R_{-2} \neq 0$ if and only if zero is an eigenvalue for $-\Delta_{\alpha,Y}$.

Remark 1 For Schrödinger operators of the form $-\Delta + V$, the Limiting Absorption Principle and the analogous of Theorem 1 can be proved under suitable short-range
assumptions on the scalar potential $V$ (see e.g. the classical papers \cite{[1],19}). In this case, moreover, it is well-known that $R_{-1} \neq 0$ if and only if there exists a generalized eigenfunction at $z = 0$ (a zero-energy resonance for $-\Delta + V$), namely a function $\psi \in L^2(\mathbb{R}^3, (x)^{-1-\sigma} \, dx) \setminus L^2(\mathbb{R}^3)$, for any $\sigma > 0$, which satisfies $(-\Delta + V)\psi = 0$ as a distributional identity on $\mathbb{R}^3$. As it will be clear from the proof of Theorem 1, a similar characterization holds true also for $-\Delta_{\alpha,Y}$ (see Remark 2).

2 Asymptotic for $\Gamma_{\alpha,Y}(z)^{-1}$ as $z \to 0$

We fix $N \geq 1$, $\alpha \in \mathbb{R}^N$ and $Y \subseteq \mathbb{R}^3$, and we set $\Gamma(z) := \Gamma_{\alpha,Y}(z)$.

We shall use the notation $O(z^k)$, $k \in \mathbb{Z}$, to denote a meromorphic $M_N(\mathbb{C})$-valued function whose Laurent expansion in a neighborhood of $z = 0$ contains only terms of degree $\geq k$. In particular, $O(1)$ denotes an analytic map in a neighborhood of $z = 0$. We also write $\Theta(z^k)$ to denote a function of the form $A z^k$, with $A \in M_N(\mathbb{C}) \setminus \{0\}$.

In a neighborhood of $z = 0$, we can expand

$$\Gamma(z) = \Gamma_0 + z \Gamma_1 + z^2 \Gamma_2 + O(z^3).$$

Explicitly, we have

$$(\Gamma_0)_{jk} = a_j \delta_{jk} - \Theta_0^{Y_j}, \quad (\Gamma_1)_{jk} = (4\pi i)^{-1}, \quad (\Gamma_2)_{jk} = (8\pi)^{-1} |y_j - y_k|.$$ 

In particular, $\Gamma_0$, $\Gamma_2$ are real symmetric matrices, while $\Gamma_1$ is skew-Hermitian, i.e. $\Gamma_1^* = -\Gamma_1$. Our aim is to characterize the small $z$ behavior of $\Gamma(z)^{-1}$. Preliminary, we recall the following useful result due to Jensen and Nenciu \cite{[20]}.

Lemma 1 \textbf{(Jensen-Nenciu)} Let $A$ be a closed operator in a Hilbert space $\mathcal{H}$ and $P$ a projection, such that $A + P$ has a bounded inverse. Then $A$ has a bounded inverse if and only if

$$B = P - P(A + P)^{-1}$$

has a bounded inverse in $P\mathcal{H}$, and in this case

$$A^{-1} = (A + P)^{-1} + (A + P)^{-1} P(B \mid P\mathcal{H})^{-1} P(A + P)^{-1}.$$ 

We can state now the main result of this Section.

Proposition 5 \textbf{In a neighborhood of} $z = 0$ \textbf{we have the Laurent expansion}

$$\Gamma(z)^{-1} = A_{-2} \frac{1}{z^2} + A_{-1} \frac{1}{z} + O(1),$$

where $A_{-2}, A_{-1} \in M_N(\mathbb{C})$. Moreover,

(i) $A_{-2} \neq 0$ if and only if $\text{Ker} \Gamma_0 \cap \text{Ker} \Gamma_1 \neq \{0\}$,

(ii) $A_{-1} \neq 0$ if and only if $\text{Ker} \Gamma_0 \not\subseteq \text{Ker} \Gamma_1$. 

Proof If \( I_0 = \Gamma(0) \) is non-singular, then \( \Gamma(z)^{-1} \) is analytic in a sufficiently small neighborhood of \( z = 0 \). Assume now that \( I_0 \) is singular. Let us distinguish two cases:

**Case 1:** \( \text{Ker} I_0 \cap \text{Ker} I_1 = \{0\} \). Let us set \( \Gamma_{\leq 1}(z) := I_0 + zI_1 \), and observe that for \( z \) small enough, \( z \neq 0 \), the matrix \( \Gamma_{\leq 1}(z) \) is non-singular. Suppose indeed that \( \Gamma_{\leq 1}(z)v = 0 \) for some \( v \in \mathbb{C}^N \). If \( I_0v \neq 0 \), then for small \( z \) we also have \( \Gamma_{\leq 1}(z)v \neq 0 \), a contradiction. Hence \( I_0v = 0 \), which for \( z \neq 0 \) implies \( I_1v = 0 \), and using the hypothesis \( \text{Ker} I_0 \cap \text{Ker} I_1 = \{0\} \) we deduce that \( v = 0 \). Observe also that for \( z \) small enough, \( z \neq 0 \), the matrix \( \Gamma(z) \) is non-singular, with \( \Gamma(z)^{-1} = \Gamma_{\leq 1}(z)^{-1} + O(1) \).

In order to invert \( \Gamma_{\leq 1}(z) \), we use the Jensen-Nenciu Lemma. Let \( P : \mathbb{C}^N \to \mathbb{C}^N \) be the orthogonal projection onto \( \text{Ker} I_0 \). Since \( I_0^* = I_0 \), we have that \( I_0 + P \) is non-singular, whence the same is \( \Gamma_{\leq 1}(z) + P \) for small \( z \), with \( (\Gamma_{\leq 1}(z) + P)^{-1} = O(1) \). More precisely,

\[
(\Gamma_{\leq 1}(z) + P)^{-1} = [I + z(I_0 + P)^{-1}I_1]^{-1}[I_0 + P]^{-1}
= [I - z(I_0 + P)^{-1}I_1][I_0 + P]^{-1} + O(z^2). \tag{13}
\]

By Lemma 1 we get

\[
\Gamma_{\leq 1}(z)^{-1} = (\Gamma_{\leq 1}(z) + P)^{-1} + (\Gamma_{\leq 1}(z) + P)^{-1}P \left( (P - P(\Gamma_{\leq 1}(z) + P)^{-1}P) \mid P \mathbb{C}^N \right)^{-1} P(\Gamma_{\leq 1}(z) + P)^{-1}. \tag{14}
\]

Observe that \( (I_0 + P)^{-1}P = P \), and since \( I_0^* = I_0 \) we also have \( P(I_0 + P)^{-1} = P \). Using these relations and \( \text{(13)} \), we compute

\[
P - P(\Gamma_{\leq 1}(z) + P)^{-1}P = zP\Gamma_1P + O(z^2).
\]

Substituting into \( \text{(14)} \) we obtain

\[
\Gamma_{\leq 1}^{-1}(z) = (\Gamma_{\leq 1}(z) + P)^{-1}
+ (\Gamma_{\leq 1}(z) + P)^{-1}P \left( (zP\Gamma_1P \mid P \mathbb{C}^N)^{-1} + O(1) \right) P(\Gamma_{\leq 1}(z) + P)^{-1}
= z^{-1}P(\Gamma_1P \mid P \mathbb{C}^N)^{-1}P + O(1) = \Theta(z^{-1}) + O(1). \tag{15}
\]

**Case 2:** \( \text{Ker} I_0 \cap \text{Ker} I_1 \neq \{0\} \). We start by proving that \( \text{Ker} I_1 \cap \text{Ker} I_2 = \{0\} \). Since \( I_2 \) is real symmetric, and \( I_1 \) is purely imaginary and skew-symmetric, it is sufficient to show that the quadratic form associated to \( I_2 \) is strictly negative on

\[
(\text{Ker} I_1 \cap \mathbb{R}^N) \setminus \{0\} = \{v \in \mathbb{R}^N \setminus \{0\} \mid v_1 + \ldots + v_N = 0 \}.
\]

To this aim, we prove preliminary that for any \( v \in \mathbb{R}^N \), with \( v_1 + \ldots + v_N = 0 \),

\[
\langle I_2v, v \rangle := (8\pi)^{-1} \sum_{1 \leq j, k \leq N} |y_j - y_k| |v_j v_k| \leq 0. \tag{16}
\]
The key point is to use the so called averaging trick. By rotational and scaling invariance, we can see that there exists a positive constant $c$ such that, for any $y \in \mathbb{R}^3$,
\[
\int_{S^2} |\langle w, y \rangle| dw = c|y|.
\]
It follows that
\[
(8\pi)^{-1} \sum_{1 \leq j,k \leq N} |y_j - y_k| v_j v_k = (8\pi c)^{-1} \int_{S^2} \sum_{1 \leq j,k \leq N} |\langle w, y_j - y_k \rangle| v_j v_k dw, \quad (17)
\]
and then it is sufficient to prove that, for a fixed $w \in S^2$,
\[
\sum_{1 \leq j,k \leq N} |\tilde{y}_j - \tilde{y}_k| v_j v_k \leq 0,
\]
where we set $\tilde{y}_j := \langle w, y_j \rangle$ for $j = 1, \ldots, N$. We have
\[
\sum_{1 \leq j,k \leq N} |\tilde{y}_j - \tilde{y}_k| v_j v_k = 2 \sum_{1 \leq j,k \leq N} \max\{\tilde{y}_j - \tilde{y}_k, 0\} v_j v_k
\]
\[= 2 \int_{t \in \mathbb{R}} \sum_{1 \leq j,k \leq N} [\tilde{y}_k < t < \tilde{y}_j] v_j v_k, \quad (18)
\]
where we use the Iverson bracket notation $[P]$, which equals 1 if the statement $P$ is true and 0 if it is false. So it is enough to prove that, for almost every $t \in \mathbb{R}$,
\[
\sum_{\tilde{y}_k < t < \tilde{y}_j} v_j v_k \leq 0.
\]
For every $t \in \mathbb{R} \setminus \{\tilde{y}_1, \ldots, \tilde{y}_N\}$, define $J_t := \{j \mid \tilde{y}_j > t\}$, $K_t := \{k \mid \tilde{y}_k < t\}$. We have
\[
\sum_{\tilde{y}_k < t < \tilde{y}_j} v_j v_k = \sum_{j \in J_t, k \in K_t} v_j v_k = \left( \sum_{j \in J_t} v_j \right) \left( \sum_{k \in K_t} v_k \right) = -\left( \sum_{j \in J_t} v_j \right)^2 \leq 0, \quad (19)
\]
where we use, in the last equality, the hypothesis $v_1 + \ldots + v_N = 0$.

Assume now that we have the equality in (16), for a suitable vector $v \in \mathbb{R}^N$ with $v_1 + \ldots + v_N = 0$. It follows from the identity (17) that for almost every $w \in S^2$
\[
\sum_{1 \leq j,k \leq N} |\langle w, y_j - y_k \rangle| v_j v_k = 0. \quad (20)
\]
In particular, we can choose $w \in S^2$ satisfying (20), and such that the quantities $\tilde{y}_j = \langle w, y_j \rangle$ are pairwise distinct, say $\tilde{y}_1 > \tilde{y}_2 > \ldots > \tilde{y}_N$. Owing to (18)-(19), we deduce that
\[
\sum_{\tilde{y}_k < t < \tilde{y}_j} v_j v_k = 0, \quad (21)
\]
for every \( t \) in a full-measure set \( \mathcal{T} \subset \mathbb{R} \). In particular, choosing \( t_1, \ldots, t_{n-1} \in \mathcal{T} \), with \( t_n \in (\tilde{y}_{n+1}, \tilde{y}_n) \) for \( n = 1, \ldots, N - 1 \), we obtain from (21) and (19) that

\[
\sum_{j=1}^{n} v_j = 0 \quad \forall n \in \{1, \ldots, N\}.
\]

This implies \( v = 0 \), concluding the proof that \( \text{Ker} \Gamma_1 \cap \text{Ker} \Gamma_2 = \{0\} \).

Now, let us set \( \Gamma_{\leq 2}(z) := \Gamma_{\leq 1}(z) + z^2 \Gamma_2 \). Arguing as in Case 1, and using the property \( \text{Ker} \Gamma_1 \cap \text{Ker} \Gamma_2 = \{0\} \), we deduce that for \( z \) small enough, \( z \neq 0 \), the matrix \( \Gamma_{\leq 2}(z) \) is non-singular. In particular, for \( z \neq 0 \) small enough, also \( \Gamma(z) \) is non-singular, with \( \Gamma(z)^{-1} = \Gamma_{\leq 2}(z)^{-1} + O(1) \).

In order to invert \( \Gamma_{\leq 2}(z) \), we use the Jensen-Nenciu Lemma. Let \( P : \mathbb{C}^N \to \mathbb{C}^N \) be the orthogonal projection onto \( \text{Ker} \Gamma_0 \cap \text{Ker} \Gamma_1 \). Owing to the relations \( \Gamma_0^* = \Gamma_0 \), \( \Gamma_1^* = -\Gamma_1 \), we deduce that for \( z \) small enough \( \Gamma_{\leq 1}(z) + P \) is non-singular, with

\[
(\Gamma_{\leq 1}(z) + P)^{-1} = \begin{cases} 
\Theta(z^{-1}) + O(1) & \text{Ker} \Gamma_0 \not\subseteq \text{Ker} \Gamma_1 \\
O(1) & \text{Ker} \Gamma_0 \subseteq \text{Ker} \Gamma_1
\end{cases}.
\]

(22)

For small \( z \), also \( \Gamma_{\leq 2}(z) + P \) is non-singular, with

\[
(\Gamma_{\leq 2}(z) + P)^{-1} = (\Gamma_{\leq 1}(z) + P)^{-1} + O(1).
\]

With similar computations as in Case 1, we get

\[
\Gamma_{\leq 2}(z)^{-1} = (\Gamma_{\leq 2}(z) + P)^{-1} + z^{-2}P(\Gamma_2 P \upharpoonright \mathbb{C}^N)^{-1}P + O(1)
\]

\[
= \begin{cases} 
\Theta(z^{-2}) + \Theta(z^{-1}) + O(1) & \text{Ker} \Gamma_0 \not\subseteq \text{Ker} \Gamma_1 \\
\Theta(z^{-2}) + O(1) & \text{Ker} \Gamma_0 \subseteq \text{Ker} \Gamma_1
\end{cases}.
\]

(23)

Expansion (12) is thus proved in any case. Moreover, statements (i) and (ii) easily follow from the discussion above. \( \square \)

3 Proof of the main Theorem

This Section is devoted to the proof of Theorem 1. Let us fix \( N \geq 1 \), \( \alpha \in \mathbb{R}^N \) and \( Y \subseteq \mathbb{R}^3 \), and set \( \Gamma(z) := \Gamma_{\alpha,Y}(z) \). Preliminary, observe that the low-energy expansion (11) follows by combining the resolvent formula (9) with the small \( z \) expansion (12) for \( \Gamma(z)^{-1} \). We prove now that \( R_{-2} \neq 0 \) if and only if \( 0 \in \sigma(-\Delta_{\alpha,Y}) \), which in view of Proposition 5 part (i), is equivalent to prove that \( \text{Ker} \Gamma_0 \cap \text{Ker} \Gamma_1 \neq \{0\} \) if and only if \( 0 \in \sigma(-\Delta_{\alpha,Y}) \).

Suppose first that there exists \( c = (c_1, \ldots, c_N) \neq 0 \in \text{Ker} \Gamma_0 \cap \text{Ker} \Gamma_1 \). We are going to show that the non-zero function
belongs to $\text{Ker}(-\Delta_{\alpha,Y})$. First of all, observe that the condition $\Gamma_1 c = 0$ is equivalent to $c_1 + \ldots + c_N = 0$, which implies $\psi \in L^2(\mathbb{R}^3)$.

Let us fix $z \in \mathbb{C}^+ \setminus \mathbb{R}^+$, and write

$$
\psi = F_z + N \sum_{j=1}^{N} c_j \xi_j,
$$

where

$$
F_z := \sum_{j=1}^{N} c_j (\xi_j - \xi_z).
$$

Observe that $F_z \in H^2(\mathbb{R}^3)$. Moreover, for every $k \in \{1, \ldots, N\}$,

$$
F_z(y_k) = \sum_{j=1}^{N} c_j (\xi(y_k)_j - \xi_z(y_k)) = \sum_{k=1}^{N} \Gamma(z)_{kj} c_j,
$$

where in the second equality we use that $\Gamma_0 c = \Gamma_1 c = 0$. By virtue of representation (7), we conclude that $\psi \in \mathcal{D}(-\Delta_{\alpha,Y})$. Moreover, formula (8) yields

$$
-\Delta_{\alpha,Y} \psi = (-\Delta - z^2) F_z + \lambda^2 2 \sum_{j=1}^{N} c_j \xi_j = \sum_{j=1}^{N} c_j \left[ (-\Delta - z^2) \xi_j - \Delta \xi_j \right] = 0,
$$

which shows that $\psi \in \text{Ker}(-\Delta_{\alpha,Y})$.

Let us discuss now the opposite implication. To this aim, consider a function $\psi \in \text{Ker}(-\Delta_{\alpha,Y}) \setminus \{0\}$. For a fixed $z = i\lambda \in \mathbb{C}^+ \setminus \mathbb{R}^+$, we can write

$$
\psi = F_{i\lambda} + N \sum_{j=1}^{N} c_j \xi_{i\lambda}^j,
$$

for some non-zero $F_{i\lambda} \in H^2(\mathbb{R}^3)$, where

$$
c_j = \sum_{k=1}^{N} \Gamma(z)_{jk}^{-1} F_z(y_k).
$$

Observe that the $c_j$’s are necessarily independent of $z$, since $\xi_{i\lambda}^j \notin H^2(\mathbb{R}^3)$ for any $j$. Moreover, the condition $\psi \in L^2(\mathbb{R}^3)$ implies $c_1 + \ldots + c_n = 0$, namely $\Gamma_1 c = 0$. Owing to (8) and the representation (25), the relation $-\Delta_{\alpha,Y} \psi = 0$ is equivalent to

$$
-\Delta F_{i\lambda} = \lambda^2 \sum_{j=1}^{N} c_j \xi_{i\lambda}^j.
$$

(26)
We show now that \( \| F_i \|_{H^2} \to 0 \) as \( \lambda \downarrow 0 \), whence also \( F_{i\lambda} \to 0 \) as \( \lambda \downarrow 0 \), uniformly on compact subsets of \( \mathbb{R}^3 \). This implies
\[
\Gamma_0 c = \lim_{\lambda \downarrow 0} \Gamma(i\lambda)c = 0,
\]
and the identity
\[
\psi = \sum_{j=1}^N c_j \phi_j^\gamma,
\]
which conclude the proof.

In order to show that \( \| F_i \|_{H^2} \to 0 \) as \( \lambda \downarrow 0 \), we start with the estimate
\[
\| \Delta F_{i\lambda} \|_{L^2} = \| \lambda^2 \Delta (-\Delta + \lambda^2)^{-1} \psi \|_{L^2} \leq \lambda^2 \| \psi \|_{L^2}.
\]
(27)

Observe moreover that \( \hat{F}_{i\lambda}(p) = \lambda^2 (p^2 + \lambda^2)^{-1} \hat{\psi}(p) \). By dominate convergence we get \( \| F_{i\lambda} \|_{L^2} = o(1) \), which combined with (27) yields \( \| F_{i\lambda} \|_{H^2} = o(1) \), as desired.

**Remark 2** By Proposition 5(ii), there is a \( \Theta(z^{-1}) \) term in the expansion of \( \Gamma(z)^{-1} \) at \( z = 0 \) if and only if there exists \( c \in \mathbb{R}^n \) such that \( \Gamma_0 c = 0, \Gamma_1 c \neq 0 \). In this case, the function defined by (24) belongs to \( L^2(\mathbb{R}^3, (x^{-1} - \sigma \, dx) \setminus L^2(\mathbb{R}^3) \), for any \( \sigma > 0 \), and formally satisfies \( -\Delta_{\alpha,Y} \psi = 0 \), whence \( \psi \) can be interpreted as a zero energy resonance for \( -\Delta_{\alpha,Y} \). Hence, as anticipated in Remark 1, we have that \( \mathcal{R} \neq 0 \) in expansion (11) if and only if there exists a zero energy resonance, analogously to the case of classical Schrödinger operators.

### 4 Occurrence and multiplicity of zero energy obstructions

In this Section we discuss the occurrence and the multiplicity of obstructions at zero energy for the resolvent of \( -\Delta_{\alpha,Y} \), depending on the choice of the set \( Y \) of centers of interactions and the coupling parameters \( \alpha_1, \ldots, \alpha_N \).

In the single center case, it is easy to check that the only possible obstruction at \( z = 0 \) is a resonance, attained if and only if \( \alpha = 0 \). In general, for any \( N \) and for any given configuration of the centers, there exists a measure zero set of choices of the parameters \( \alpha_1, \ldots, \alpha_N \) which leads to a zero-energy resonance. By means of the discussion in Section 2 and Section 3, we can define the multiplicity of the zero-energy resonance as
\[
\rho_{\alpha,Y} \eqdef \dim (\ker \Gamma_0) - \dim (\ker \Gamma_0 \cap \ker \Gamma_1).
\]

We conjecture that, as \( N \) increases, one can find \( Y \) and \( \alpha \) such that \( \rho_{\alpha,Y} \) becomes arbitrarily large.

As anticipated in Section 1 when \( N = 2 \) we can find a simple zero eigenvalue by choosing \( \alpha_1 = \alpha_2 = -(4\pi d)^{-1} \), where \( d \) is the distance between the two centers. For \( N \geq 3 \), a zero eigenvalue occurs for specific geometric configurations of the
centers of interactions and for a measure zero set of choices of $\alpha_1, \ldots, \alpha_N$. Owing to the discussion in Section 2 and Section 3, the multiplicity of the zero eigenvalue is given by

$$e_{\alpha, Y} := \dim \text{Ker}(-\Delta_{\alpha, Y}) = \dim \left(\text{Ker} \Gamma_0 \cap \text{Ker} \Gamma_1\right).$$

Let us discuss now the maximal possible value for $e_{\alpha, Y}$ as the number of centers of interactions increases.

- $N = 3$. We can take $Y$ as the vertices of an equilateral triangle of side-length one, and $\alpha_1 = \alpha_2 = \alpha_3 = -(4\pi)^{-1}$. With this choice we get $e_{\alpha, Y} = 2$.
- $N = 4$. We can take $Y$ as the vertices of a regular tetrahedron of side-length one, and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -(4\pi)^{-1}$. With this choice we get $e_{\alpha, Y} = 3$.
- $N = 5$. Observe that we can not find five points in $\mathbb{R}^3$ with constant pairwise distances. It easily follows that the maximal value for $e_{\alpha, Y}$ is still three.

One could conjecture that for $N \geq 4$ the maximal value of $e_{\alpha, Y}$ is three. Nevertheless, it is also conceivable that for large $N$ there exist complicated geometrical configurations which lead to a higher multiplicity. Such kind of mechanism is well-known in similar contexts. Consider, for example, the problem in combinatorics to determine the chromatic number of the unit distance graph on $\mathbb{R}^3$, that is the graph with vertices set $V = \mathbb{R}^3$ and edges set $E = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x - y| = 1\}$. Owing to the compactness principle by De Bruijn and Erdős [10] this is equivalent, under the axiom of choice, to determine the highest chromatic number of a finite graph embedded in $\mathbb{R}^3$ in such a way all its edges have length one. For a graph with $N$ vertices, we have the following situation:

- $N = 3$. We can consider an equilateral triangle of side-length one, which has chromatic number three.
- $N = 4$. We can consider a regular tetrahedron of side-length one, which has chromatic number four.
- $N = 5$. The highest possible chromatic number is still four.
- $N = 14$. There is a configuration of 14 points in $\mathbb{R}^3$, the Moser-Raiskii spindle, with chromatic number five [28, 31].
- For large $N$, the highest possible chromatic number is known to be between 6 and 12 [25, 27, 7].

It is evident that there are similarities between the two problems, and it would be interesting to understand if they are actually related. In particular, one may take $Y$ as the vertices of the Moser-Raiskii spindle and wondering whether there exists $\alpha = (\alpha_1, \ldots, \alpha_{14})$ such that $e_{\alpha, Y} = 4$.

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