A REMARK ON CLASSICAL PLÜCKER’S FORMULAE

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Abstract. For any reduced curve $C \subset \mathbb{P}^2$, we define the notions of the number of its virtual cusps $c_v$ and the number of its virtual nodes $n_v$, which are non-negative, coincide respectively with the numbers of ordinary cusps and nodes in the case of cuspidal curves, and if $\hat{C}$ is the dual curve of an irreducible curve $C$ and $\hat{n}_v$ and $\hat{c}_v$ are the numbers of its virtual nodes and virtual cusps, then the integers $c_v, n_v, \hat{c}_v, \hat{n}_v$ satisfy Classical Plücker’s formulae.

INTRODUCTION.

Let $C \subset \mathbb{P}^2$ be a reduced curve defined over the field of complex numbers $\mathbb{C}$. A curve $C$ is called cuspidal if the singular points of $C$ are only the ordinary cusps and nodes.

In modern textbooks on algebraic geometry, classical Plücker’s formulae are stated as follows (see, for example, [1], [2]).

Classical Plücker’s formulae. Let $C \subset \mathbb{P}^2$ be an irreducible cuspidal curve of genus $g$, degree $d \geq 2$, having $c$ ordinary cusps and $n$ nodes. Assume that the dual curve $\hat{C}$ of $C$ is also a cuspidal curve. Then

$$\hat{d} = d(d - 1) - 3c - 2n;$$ (1)

$$g = \frac{(d - 1)(d - 2)}{2} - c - n;$$ (2)

$$d = \hat{d}(\hat{d} - 1) - 3\hat{c} - 2\hat{n};$$ (3)

$$g = \frac{(\hat{d} - 1)(\hat{d} - 2)}{2} - \hat{c} - \hat{n},$$ (4)

where $\hat{c}$ and $\hat{n}$ are the numbers of ordinary cusps and nodes of $\hat{C}$ and $\hat{d} = \deg \hat{C}$.

Denote by $V(d, c, n) \subset \mathbb{P}^{\frac{d(d+1)}{2}}$ the variety parametrizing the irreducible cuspidal curves of degree $d$ with $c$ ordinary cusps and $n$ nodes. Very often, if for given $d$, $c$, and $n$ one of the invariants $\hat{c}$ or $\hat{n}$, obtained as the solution of (1) – (4), is negative, then it is claimed that this is sufficient for the "proof" of the emptiness of $V(d, c, n)$. However, the correctness of the following statement is unknown: "the dual curve $\hat{C}$ of a curve $C$ corresponding to a generic point of

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V(d, c, n) is cuspidal”. Therefore, in general case, it is impossible to conclude the non-existence of cuspidal curve C if \( \hat{c} \) or \( \hat{n} \) is negative. Of course, to avoid this problem, it is possible to use generalized Plücker’s formulae including the numbers of all possible types of singular points of \( \hat{C} \). But, we again have a difficulty, namely, in this case we must take into account too many unknown invariables.

To obviate the arising difficulty, in section 1 for any reduced plane curve C we define the notions of the number of its virtual cusps \( c_v \) and the number of its virtual nodes \( n_v \) which are non-negative, coincide respectively with the numbers of ordinary cusps and nodes in the case of cuspidal curves, and if the dual curve \( \hat{C} \) of an irreducible curve C has \( \hat{n}_v \) virtual nodes and \( \hat{c}_v \) virtual cusps, then the integers \( c_v, n_v, \hat{c}_v, \) and \( \hat{n}_v \) satisfy Classical Plücker’s formulae.

In section 2, we investigate the behaviour of the Hessian curve \( H_C \) of a cuspidal curve C at cusps and nodes of C, and in section 3, we give a proof of some inequalities for the numbers of cusps and nodes of plane cuspidal curves of degree d which was obtained early in [3] under additional assumption that the dual curve of a generic cuspidal curve is also cuspidal.

1. THE NUMBERS OF VIRTUAL CUSPS AND NODES

Let \((C, p) \subset (\mathbb{P}^2, p)\) be a germ of a reduced plane singularity. It splits into several irreducible germs: \((C, p) = (C_1, p) \cup \cdots \cup (C_k, p)\). Denote by \(m_j\) the multiplicity of the singularity \((C_j, p)\) at the point p and let \(\delta_p\) be the \(\delta\)-invariant of the singularity \((C, p)\). By definition, the integers

\[
c_{v,p} := \sum_{i=1}^k (m_i - 1)
\]

and

\[
n_{v,p} := \delta_p - \sum_{i=1}^k (m_i - 1)
\]

are called respectively the numbers of virtual cusps and virtual nodes of the singularity \((C, p)\). We have \(\delta_p = c_{v,p} + n_{v,p}\).

**Lemma 1.** Let \((C, p) \subset (\mathbb{P}^2, p)\) be a germ of a reduced plane singularity, \(c_{v,p}\) be the number of its virtual cusps and \(n_{v,p}\) be the number of its virtual nodes. Then

(i) \(c_{v,p} \geq 0, n_{v,p} \geq 0\);

(ii) if \((C, p)\) is an ordinary cusp, then \(c_{v,p} = 1\) and \(n_{v,p} = 0\);

(iii) if \((C, p)\) is an ordinary node, then \(c_{v,p} = 0\) and \(n_{v,p} = 1\).
Proof. We prove only the inequality $n_v \geq 0$, since all the other claims of Lemma 1 are obvious. Let $(C, p) = (C_1, p) \cup \cdots \cup (C_k, p)$ and $m_i$ be the multiplicity of its irreducible branch $(C_i, p)$. Then the multiplicity of $(C, p)$ at $p$ is equal to $m_p = \sum_{i=1}^k m_i$ and we have

$$n_{v, p} = \delta_p - \sum_{i=1}^k (m_i - 1) \geq \delta_p - \sum_{i=1}^k m_i + 1 = \delta_p - (m_p - 1) \geq \delta_p - \frac{m_p(m_p - 1)}{2} \geq 0,$$

since $m_p \geq 2$ for singular points and $\delta_p \geq \frac{m_p(m_p - 1)}{2}$. Therefore, we have

$$n_v = \sum_{p \in \text{Sing} C} n_{v, p} \geq 0. \quad \Box$$

Let $C \subset \mathbb{P}^2$ be a reduced curve. Denote by $\text{Sing} C$ the set of its singular points. By definition, we put

$$c_v := \sum_{p \in \text{Sing} C} c_{v, p},$$

$$n_v := \sum_{p \in \text{Sing} C} n_{v, p}$$

and call these integers respectively the number of virtual cusps and the number virtual nodes of the curve $C$. If $C$ is an irreducible curve of degree $d$ and geometric genus $g$, then we have $g = \frac{(d-1)(d-2)}{2} - \delta_C$, where $\delta_C = \sum_{p \in \text{Sing} C} \delta_p$ is the $\delta$-invariant of $C$. Therefore, we have

$$g = \frac{(d-1)(d-2)}{2} - c_v - n_v. \quad (5)$$

The following proposition is a corollary of Lemma 1.

**Proposition 1.** Let $c_v$ be the number of virtual cusps and $n_v$ be the number of virtual nodes of a reduced curve $C \subset \mathbb{P}^2$. We have

(i) $c_v \geq 0$ and $n_v \geq 0$,

(ii) if $C$ is a cuspidal curve, then $c_v$ and $n_v$ are equal respectively to the number $c$ of cusps and the number $n$ of nodes of $C$. 


Theorem 1. (Plücker’s formulae). Let $C$ and $\hat{C}$ be irreducible dual curves of genus $g$, $\deg C = d \geq 2$, $\deg \hat{C} = \hat{d}$, and $c_v$, $n_v$, $\hat{c}_v$, $\hat{n}_v$ are the numbers of their virtual cusps and nodes, respectively. Then we have the following equalities:

\begin{align*}
\hat{d} &= d(d - 1) - 3c_v - 2n_v; \\
2g &= (d - 1)(d - 2) - 2c_v - 2n_v; \\
d &= \hat{d}(\hat{d} - 1) - 3\hat{c}_v - 2\hat{n}_v; \\
2g &= (\hat{d} - 1)(\hat{d} - 2) - 2\hat{c}_v - 2\hat{n}_v.
\end{align*}

Proof. To prove Plücker’s formulae, we need the following

Lemma 2. For an irreducible plane curve $C$ we have

\begin{align*}
\hat{d} &= 2d + 2(g - 1) - c_v, \\
\hat{c}_v &= 3d + 6(g - 1) - 2c_v, \\
d &= 2\hat{d} + 2(g - 1) - \hat{c}_v, \\
c_v &= 3\hat{d} + 6(g - 1) - 2\hat{c}_v.
\end{align*}

Proof. Denote by $\nu : \overline{C} \to C$ and $\hat{\nu} : \overline{C} \to \hat{C}$ the normalization morphisms, consider generic (with respect to $C$ and $\hat{C}$) linear projections $pr : \mathbb{P}^2 \to \mathbb{P}^1$ and $\hat{pr} : \hat{\mathbb{P}}^2 \to \hat{\mathbb{P}}^1$, and put $\pi = pr \circ \nu$ and $\hat{\pi} = \hat{pr} \circ \hat{\nu}$. We have $\deg \pi = d$ and $\deg \hat{\pi} = \hat{d}$.

Let $\nu^{-1}(x_i) = \{y_{i,1}, \ldots, y_{i,m_i}\}$ for $x_i \in \text{Sing } C$. For each point $y_{i,j}$ denote by $r_{i,j}$ the ramification index of $\pi$ at $y_{i,j}$. It is easy to see that $r_{i,j}$ coincides with the multiplicity $m_{i,j}$ at $x_i$ of the irreducible germ $(C_{i,j}, x_i) \subset (C, x_i)$ corresponding to the point $y_{i,j}$. Therefore, we have

$$c_v = \sum_{i,j} (r_{i,j} - 1).$$

Applying Hurwitz formula to $\pi$ and $\hat{\pi}$, we obtain

$$2(g - 1) = -2d + c_v + \hat{d}$$

and

$$2(g - 1) = -2\hat{d} + \hat{c}_v + d$$

which give formulae (10) and (12).

To prove (11), note that $\hat{c}_v = 2\hat{d} + 2(g - 1) - d$ by (12). Therefore

$$\hat{c}_v = 2(2d + 2(g - 1) - c_v) + 2(g - 1) - d$$

by (10), that is, $\hat{c}_v = 3d + 6(g - 1) - 2c_v$. Formula (13) is obtained similarly. □

It follows from (5) that

$$2(g - 1) + 2c_v + 2n_v = d(d - 3),$$

$$2(g - 1) + 2\hat{c}_v + 2\hat{n}_v = \hat{d}(\hat{d} - 3)$$
which are equivalent to (7) and (9). To complete the proof of Plücker’s formulae, notice that formulae (6) and (8) easily follow from equations (10) – (13) and (16).

2. On the Hessian curve of a cuspidal curve

Let $C \subset \mathbb{P}^2$ be an irreducible cuspidal curve of degree $d$ with $c$ cusps and $n$ nodes. It follows from (7) and (11) that

$$8c + 6n + \hat{c}_v = 3d(d - 2).$$

Equality (17) has a natural geometric meaning. To explain it, let the curve $C$ is given by equation $F(x_0, x_1, x_2) = 0$, where $x_0, x_1, x_2$ are homogeneous coordinates in $\mathbb{P}^2$. Consider the Hessian curve $H_C \subset \mathbb{P}^2$ of the curve $C$. It is given by equation $\det(\frac{\partial^2 F}{\partial x_i \partial x_j}) = 0$. We have $\deg H_C = 3(d - 2)$. Therefore the intersection number $(C, H_C)_{P^2}$ is equal to $3d(d - 2)$. On the other hand, it is well-known (see, for example, [1]) that the curves $C$ and $H_C$ meet at the singular points and at the inflection points of the curve $C$. Therefore we have

$$\sum'(C, H_C)_p + \sum''(C, H_C)_p + \sum'''(C, H_C)_p = (C, H_C)_{P^2} = 3d(d - 2),$$

where $(C, H_C)_p$ is the intersection number of the curves $C$ and $H_C$ at a point $p \in C$ and the sum $\sum'$ is taken over all cusps of $C$, the sum $\sum''$ is taken over all nodes of $C$, and the sum $\sum'''$ is taken over all inflection points of $C$.

Let us show that the coefficients involving in equation (17) have the following geometric meaning: equality (17) is the same as equality (18), that is, the coefficient $8$ in (17) is the intersection number $(C, H_C)_p$ at a cusp $p \in C$, the coefficient $6$ is the intersection number $(C, H_C)_p$ at a node $p \in C$, and $\hat{c}_v = \sum'''(C, H_C)_p$. Indeed, let $p$ be a cusp of $C$. Without loss of generality, we can assume that $p = (0, 0, 1)$ and

$$F(x_0, x_1, x_2) = x_0^2U(x_0, x_1, x_2) + x_0x_1^2V(x_0, x_1, x_2) + x_1^3W(x_0, x_1, x_2),$$

where $U$ is a homogeneous polynomial of degree $d - 2$ such that $U(0, 0, 1) = 1$ and $V$ and $W$ are homogeneous polynomials of degree $d - 3$ such that $W(0, 0, 1) = 1$. Put $a = V(0, 0, 1)$, then in non-homogeneous coordinates $x = \frac{x_0}{x_2}, y = \frac{x_1}{x_2}$ we have $p = (0, 0, 1)$, the curve $C$ is given by equation of the form

$$x^2 + y^3 + axy^2 + bx^2y + cx^3 + \text{terms of higher degree} = 0,$$

and the curve $H_C$ is given by equation of the form

$$x^2(6y + 2ax) + \text{terms of higher degree} = 0.$$ 

Easy computation (applying $\sigma$-process with center at $p$) gives the following inequality:

$$(C, H_C)_p \geq 8$$

if $p$ is a cusp of $C$. 

Let \( p \) be a node of \( C \). Again, without loss of generality, we can assume that \( p = (0, 0, 1) \) and
\[
F(x_0, x_1, x_2) = x_0 x_1 U(x_0, x_1, x_2) + V(x_0, x_1) W(x_0, x_1, x_2),
\]
where \( U \) is a homogeneous polynomial of degree \( d - 2 \) such that \( U(0, 0, 1) = 1 \), \( V \) is a homogeneous polynomial of degree 3, and \( W \) is a homogeneous polynomial of degree \( d - 3 \). In non-homogeneous coordinates \( x = \frac{x_0}{x_2}, y = \frac{x_1}{x_2} \) we have \( p = (0, 0) \), the curve \( C \) is given by equation of the form
\[
xy + \text{terms of higher degree} = 0,
\]
and the curve \( H_C \) is given by equation of the same form
\[
xy + \text{terms of higher degree} = 0.
\]

Easy computation (applying \( \sigma \)-process with center at \( p \)) gives the following inequality:
\[
(C, H_C)_p \geq 6 \tag{20}
\]
if \( p \) is a node of \( C \).

If \( p \) is an \( r \)-tuple inflection point of \( C \) (that is, \( (C, L_p)_p = r + 2 \), where the line \( L_p \) is tangent to \( C \) at \( p \)), then by Theorem 1 on page 289 in [1], we have \( (C, H_C)_p = r \). On the other hand, the branch \((\hat{C}, \hat{p})\) of the dual curve \( \hat{C} \), corresponding to an irreducible branch \((C, p) \subset (C, p)\) at a point \( p \) of a cuspidal curve \( C \), is singular if and only if \( p \) is an inflection point of \( C \); and the branch \((\hat{C}, \hat{p})\), corresponding to the branch \((C, p)\) at \( r \)-tuple inflection point \( p \in C \), has a singularity of type \( u^{r+1} - v^{r+2} = 0 \). The multiplicity \( m_{\hat{p}} \) of this singularity is equal to \( r + 1 \). Therefore, we have
\[
\Sigma''(C, H_C)_p = \sum_{(\hat{C}, \hat{p})} (m_{\hat{p}} - 1) = \hat{c}_v. \tag{21}
\]

Finally, it follows from (17) – (21) that inequalities (19) and (20) are the equalities in the case of cuspidal curves.

3. Lefschetz’s Inequalities

As above, let \( C \subset \mathbb{P}^2 \) be an irreducible cuspidal curve of degree \( d \) and genus \( g \) having \( c \) cusps and \( n \) nodes.

In [3], assuming that for a generic cuspidal curve with given numerical invariants the dual curve is also cuspidal, Lefschetz proved the following inequalities
\[
c \leq \frac{3}{2} d + 3(g-1) \tag{22}
\]
if \( d \) is even and
\[
c \leq \frac{3d-1}{2} + 3(g-1) \tag{23}
\]
if $d$ is odd. It follows from (11) that these inequalities occur for any plane cuspidal curve, since $\hat{c}_v$ is a non-negative integer.

Note also that equality (17) gives rise to the following statement: for a plane cuspidal curve of degree $d \geq 2$ the following inequality holds:

$$8c + 6n \leq 3d(d - 2) - \frac{1 - (-1)^d}{2}. \quad (24)$$

**Remark 1.** One can show that for any $d = 2k$, $k \geq 3$, and for any $g \geq 0$ such that $2 \leq 3g \leq k - 4$ or $g \leq 1$, there exist a cuspidal curve of degree $d$ having $c = 3(k + g - 1)$ cusps and $n = 2(k - 1)(k - 2) - 4g$ nodes for which inequality (24) becomes the equality. If $d = 2k + 1$, $k \geq 3$, then for any $g$ such that $2 \leq 3g \leq k - 4$ or $g \leq 1$, there exist a cuspidal curve of degree $d$ having $c = 3(k + g) - 2$ cusps, $n = 2(k - 1)^2 - 4g$ nodes, and for which inequality (24) becomes the equality. The proof of these statements follows from the fact that the genus of such curves $C$ is equal to $g$ and for these curves the dual curves $\hat{C}$ have degree $\hat{d} = 2(g - 1) + 7 + \frac{1 - (-1)^d}{2}$ and the number of virtual cusps $\hat{c}_v = \frac{1 - (-1)^d}{2}$. Therefore such curves can be obtained as the image of a generic linear projection to $\mathbb{P}^2$ of a smooth curve $\overline{C} \subset \mathbb{P}^{\hat{d} - g}$ of degree $\hat{d}$ birationally isomorphic to $C$. Standard computations (which we leave to the reader) of codimension of the locus of "bad" projections shows that in this case there is a linear projection $\text{pr} : \mathbb{P}^{\hat{d} - g} \to \mathbb{P}^2$ such that $\text{pr}(\overline{C}) = \hat{C}$ is a cuspidal curve with $\hat{c} = \frac{1 - (-1)^d}{2}$ and its dual curve $\hat{C}$ is also cuspidal.

For completeness, let me remind also the following well-known inequalities which we have for plane cuspidal curves:

$$3c + 2n < d(d - 1) - \sqrt{d}, \quad (25)$$

$$2c + 2n \leq (d - 1)(d - 2), \quad (26)$$

$$d(d - 2)(d^2 - 9) + (3c + 2n)^2 + 27c + 20n \geq 2d(d - 1)(3c + 2n) \quad (27)$$

which are consequences of equalities (6) – (9) and the inequalities $\hat{d} \geq \sqrt{d}$, $g \geq 0$, $\hat{n}_v \geq 0$;

$$16c + 9n \leq d(5d - 6) \quad (\text{Hirzebruch – Ivinskis inequality (4)})$$

which is true in the case of even $d$.

Note also that inequalities (24) – (27) hold for any irreducible plane curve if we substitute $c_v$ and $n_v$ instead of $c$ and $n$. 
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