ON SOME LOW DIMENSIONAL QUANTUM GROUPS

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ABSTRACT. This paper is an adaptation of a chapter from an upcoming monograph on noncommutative geometry and quantum groups. We present examples of non compact quantum groups which are deformations of low dimensional Lie groups. The paper is of expository nature and provides both particular examples and some general procedures for constructing them.

INTRODUCTION

This article is devoted to the description of topological quantum deformations of a large class of low dimensional Lie groups. It mainly concerns the groups ‘$az + b$’ and ‘$ax + b$’ of affine (orientation preserving) transformations of the complex and real line respectively. They form a family of interesting locally compact quantum groups and the methods of construction as well as analysis of these quantum groups bear a lot of similarities. It has to be emphasized, however, that those similarities are often superficial and do not allow easy transition from one example to another.

One reason that we focus on the quantum ‘$az + b$’ and ‘$ax + b$’ groups is that they do provide insight into many interesting phenomena of the theory of locally compact quantum groups, but are still relatively easy to construct and study. They will illustrate, in particular, some of the technical difficulties encountered in the process of constructing new examples of quantum groups on the $C^*$-algebra level. These problems are related to realization various commutation relations by unbounded operators acting on some Hilbert space. In particular in the case of ‘$az + b$’ groups we will encounter the spectral conditions restricting spectrum of some operators to special subsets of $\mathbb{C}$, while the ‘$ax + b$’ groups will touch on the problems of extending symmetric operators to selfadjoint ones.

Another reason is that these “affine” quantum groups can be used as building blocks for further constructions. In Section 2 one example of such a construction is presented. It is the so called quantum double group construction. Applying this construction to one of the quantum ‘$az + b$’ groups we get a quantum deformation of $GL(2, \mathbb{C})$. Moreover the double group construction will also be used in Section 3 to describe examples of two different quantum deformations of the Lorentz group.

1. QUANTUM ‘$az + b$’ GROUPS

1.1. Classical ‘$az + b$’ group. The classical ‘$az + b$’-group is the group $G$ of all transformations

$\mathbb{C} \ni z \mapsto az + b \in \mathbb{C}$

where $a$ and $b$ are complex numbers with $a \neq 0$. It is convenient to realize $\mathbb{C}$ as a subset of $\mathbb{C}^2$ via

$\mathbb{C} \ni z \mapsto \begin{pmatrix} z \\ 1 \end{pmatrix} \in \mathbb{C}^2$.

Then the $G$ group becomes the group of all matrices of the form

$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$

with $a \neq 0$. 

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The passage to a quantum deformation of this locally compact group means that we replace the algebra of continuous function vanishing at infinity on the group by some non commutative algebra. Let us take a close look at the undeformed algebra first. Consider the two functions

\[
\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto a, \quad \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto b.
\]

With a slight abuse of notation we will call them \(a\) and \(b\) respectively. They are continuous, but certainly not vanishing at infinity. However, it is easy to see that the set of functions

\[
\{ f(a)g(b) : f, g \in C_0(\mathbb{C} \setminus \{0\}) \}
\]

is linearly dense in \(C_0(G)\). In fact, \(a\) and \(b\) are affiliated with this \(C^*\)-algebra and \(C_0(G)\) is “generated” by the three functions. The notion of “generation” we use here is quite involved. We refer to [18] for details of this concept.

1.2. Quantum deformations. Quantum deformations of the \('az + b'\) group on the purely algebraic level are labeled by a complex parameter and are introduced by considering an associative \(*\)-algebra \(A\) generated by three normal elements \(a\), \(a^{-1}\) and \(b\) subject to the relations

\[
ab = q^2 ba \quad \text{and} \quad ab^* = b^* a,
\]

where \(q\) is a fixed nonzero complex number. The algebra \(A\) can be endowed with a Hopf \(*\)-algebra structure by defining the comultiplication

\[
\Delta(a) = a \otimes a,
\]

\[
\Delta(b) = a \otimes b + b \otimes I.
\]

In order to proceed with the construction on \(C^*\)-algebra level one must give a precise operator meaning to the relations (1.2) and construct the \(C^*\)-algebra \(A\) “generated” by elements satisfying these relations. Moreover, this \(C^*\)-algebra must then be endowed with a comultiplication \(\Delta \in \text{Mor}(A, A \otimes A)\) acting on generators in the way prescribed by (1.3). In particular this means that the operators \(\Delta(a)\) and \(\Delta(b)\) have to satisfy the relations of the form (1.2) as well. It should be stressed that giving precise meaning to relations (1.2) and finding all operator solutions satisfying the relations will not be sufficient to have comultiplication on the algebra generated by \(a\) and \(b\).

The known approaches to solve these problems strongly depend on the value of the deformation parameter. Nevertheless they may be seen as realizations of a more general scheme which works for different special values of deformation parameter \(q\). Any such value defines a self dual multiplicative subgroup \(\Gamma_q\) of \(\mathbb{C} \setminus \{0\}\). Namely \(\Gamma_q\) is the subgroup generated by \(q\) and \(\{q^t : t \in \mathbb{R}\}\) (the choice of \(q\) involves the choice of logarithm of \(q\)). The resulting quantum groups naturally form three families reflecting the three types of the “shape” of the corresponding subgroup. We will refer to them as cases (I), (II) and (III) and the corresponding sets \(\Sigma\) of admissible deformation parameters will be denoted by \(\Sigma_1\), \(\Sigma_{11}\) and \(\Sigma_{111}\) respectively, and we let \(\Sigma = \Sigma_1 \cup \Sigma_{11} \cup \Sigma_{111}\). The three cases are described in Table 1.

The quantum deformations \('az + b'\) group of type I and II were introduced in [21] and type III in [9].

| Case | Set \(\Sigma\) of admissible values of \(q\) | Group \(\Gamma_q\) |
|------|---------------------------------|-----------------|
| (I)  | \(\{e^{\frac{2\pi n i}{N}} : N = 6, 8, \ldots\}\) | bunch of \(N\) half lines |
| (II) | \([0, 1]\) | set of concentric circles with radii \(q^n\) \((n \in \mathbb{Z})\) |
| (III)| \(\{e^{\frac{2\pi i \rho}{N}} : \Re \rho < 0, \exists \rho = \frac{\pi}{N}, N = \pm 2, \pm 4, \ldots\}\) | \(|N|\) logarithmic spirals |

Table 1. Types of deformations corresponding to values of \(q\)

Figure 5 shows part of the set of admissible values of \(q\) in case (III). Examples of \(\Gamma_q\) for different values of \(q\) are given in Figures 1, 2, 3 and 4.
Figure 1. $\Gamma_q$ in case (I), $q = e^{2\pi i/12}$

Figure 2. $\Gamma_q$ in case (II), $q = 0.8$
Figure 3. $\Gamma_q$ in case (III), $\rho = -\frac{3}{2} - i\frac{6}{2\pi}$

Figure 4. $\Gamma_q$ in case (III), $\rho = -2 + i\frac{8}{2\pi}$
It is reasonable to conjecture existence of appropriate limiting procedures which connect the three families. However, so far, the three cases must be treated separately despite striking similarities one encounters in all three constructions.

1.3. Weyl relations and Schrödinger pairs. For a fixed admissible \( q \in \Sigma \) we can give precise operator meaning to the commutation relations (1.2). In order to do that we will first describe a canonical pair of operators which satisfy such type of relations. Remembering that \( \Gamma_q \) is an abelian locally compact group we set \( H = L^2(\Gamma_q, \mu) \) where \( \mu \) denotes the Haar measure on \( \Gamma_q \).

Let \((S.R)\) be a pair of operators on \( H \) defined by

\[
\begin{align*}
(R\psi)(\gamma) &= \gamma \psi(\gamma), \\
(\text{Phase}(S)\psi)(\gamma) &= \psi(q\gamma), \\
(\text{Phase}(S)\psi)|_{it} &= \psi(q^it\gamma)
\end{align*}
\]

for all \( \psi \in H \) and \( t \in \mathbb{R} \). Then \( R = \text{Phase}(R) |R| \) and \( S = \text{Phase}(S) |S| \) are unbounded normal operators with trivial kernels. Therefore \( \text{Phase}(R) \) and \( \text{Phase}(S) \) are unitaries. Moreover \( \text{Sp}R = \overline{\Gamma_q} = \text{Sp}S \) where \( \overline{\Gamma_q} = \Gamma_q \cup \{0\} \) is the closure of \( \Gamma_q \). Now one can verify that the commutation relations described in Table 2 are satisfied.
Commutation relations

| Case   | Commutation relations                                                                 |
|--------|---------------------------------------------------------------------------------------|
| Case (I) | \(\text{Phase}(S)|R| = |R|\text{Phase}(S), \text{Phase}(S)|R| \neq \text{Phase}(R)|S|, \text{Phase}(S)\text{Phase}(R) = q\text{Phase}(R)\text{Phase}(R)\) |
|        | \(|S|^it|R|^iu = q^{it}\text{Phase}(R)|S|^it\)                                    |
| Case (II) | \(\text{Phase}(S)|R| = q|R|\text{Phase}(S), \text{Phase}(S)|R| = \text{Phase}(R)|S|, \text{Phase}(S)\text{Phase}(R) = \text{Phase}(R)\text{Phase}(S)\) |
|        | \(|S|^it|R|^iu = \text{Phase}(q)^{-it\prime}|R|^iu\text{Phase}(S)^{it}\)         |
| Case (III) | \(\text{Phase}(S)|R| = \text{Phase}(S)\text{Phase}(R), \text{Phase}(S)|R| = \text{Phase}(R)|S|, \text{Phase}(S)\text{Phase}(R) = \text{Phase}(q)\text{Phase}(R)\text{Phase}(S)\) |
|        | \(|S|^it|R|^iu = \text{Phase}(q)^{i\prime}|R|^it\text{Phase}(S)^{it}\)         |

Table 2. Precise meaning of commutation relations between \(S\) and \(R\)

One can prove that in each case the products \(S\circ R\), \(R\circ S\), \(S\circ R^*\) and \(R^*\circ S\) are well defined, closable operators and their closures \(SR\), \(RS\), \(SR^*\) and \(R^*S\) satisfy relations (cf. (1.2)):

\[SR = q^2RS, \quad SR^* = R^*S.\]

The pair \((S, R)\) constructed in this way is called the Schrödinger pair. This construction follows the construction of well known pair of position and momentum operators of quantum mechanics. In this case the precise meaning of the Heisenberg canonical commutation relations is achieved by formulating them in the Weyl form using the selfduality of the \(\mathbb{R}^N\) group. Since \(\Gamma_q\) is a selfdual group a similar formulation is also possible in this case.

The isomorphism of \(\Gamma_q\) with its dual \(\widehat{\Gamma}_q\) can be described by a non degenerate bicharacter on \(\Gamma_q\). In fact it can be shown that there exists a continuous function \(\chi : \Gamma_q \times \Gamma_q \rightarrow \mathbb{T}\) such that

\[
\chi(\gamma, \gamma') = \chi(\gamma', \gamma),
\]

\[
\chi(\gamma, \gamma'\gamma'') = \chi(\gamma, \gamma')\chi(\gamma, \gamma''),
\]

and

\[
\chi(q, \gamma) = \text{Phase}\gamma,
\]

\[
\chi(q^{it}, \gamma) = |\gamma|^{it}.
\]

Then \(\chi\) is a non degenerate bicharcter on \(\Gamma_q\), i.e. it is a character with respect to each variable (with the other variable fixed) and the condition that \(\chi(\gamma, \gamma') = 1\) for all \(\gamma\) implies that \(\gamma' = 1\).

Remembering that the spectra of operators \(S\) and \(R\) from the Schrödinger pair are contained in \(\bar{\Gamma}_q\) and the point 0 is of spectral measure 0 for both \(R\) and \(S\) then by functional calculus of normal operators we obtain strongly continuous one parameter groups of unitary operators \(\chi(R, \gamma')\) and \(\chi(S, \gamma)\). Now the relations from Table 2 can be written as

\[
\chi(S, \gamma)\chi(R, \gamma') = \chi(\gamma, \gamma')\chi(R, \gamma')\chi(S, \gamma).
\]

(1.4)

for any \(\gamma, \gamma' \in \Gamma_q\).

We refer to (1.4) as the Weyl form of the commutation relations between \(S\) and \(R\). This leads to the notion of a non degenerate \(q^2\)-pair.

A pair \((S, R)\) of normal invertible operators acting on a Hilbert space \(H\) is called a non degenerate \(q^2\)-pair if \(S\) and \(R\) satisfy spectral condition: \(\text{Sp } R \subset \overline{\Gamma}_q\), \(\text{Sp } S \subset \overline{\Gamma}_q\) and the relation (1.4) holds for all \(\gamma, \gamma' \in \Gamma_q\).

Let us note that relations described in Table 2 are more general than the notion of a \(q^2\)-pair. In fact they can be satisfied also by operators not fulfilling the spectral condition. On the other hand due to the spectral condition any non degenerate \(q^2\)-pair is unique up to a multiplicity by Mackey-Stone-Von Neumann theorem, i.e. any non degenerate \(q^2\)-pair is unitarily equivalent to a direct sum of some number of copies of the Schrödinger pair.
Theorem 1.1. Let \( q \in \Sigma \) be an admissible parameter (cf. Table 1). There exists a unique \( C^* \)-algebra \( A \) with three affiliated elements \( a, a^{-1}, b \) such that

1. \( a \) and \( b \) are normal,
2. \( S_p a \) and \( S_p b \) are contained in \( \Gamma_q \),
3. for any representation \( \pi \) of \( A \) the pair \( (\pi(a), \pi(b)) \) is an semi-non degenerate \( q^2 \)-pair,
4. for any semi-non degenerate \( q^2 \)-pair \( (a_0, b_0) \) acting on a Hilbert space \( H \) there is a representation \( \pi \) of \( A \) on \( H \) such that \( \pi(a) = a_0 \) and \( \pi(b) = b_0 \).

The reasoning leading to the proof of Theorem 1.1 uses heavily properties of certain special functions, some of which we will describe later. The conclusion, however, can be stated in very plain words. The \( C^* \)-algebra \( A \) of Theorem 1.1 is simply the crossed product \( C_0(\Gamma_q) \rtimes \sigma \Gamma_q \), where \( \sigma \) is the natural action coming from multiplication of complex numbers. The elements \( b \) and \( a \) are the natural “generators” of \( C_0(\Gamma_q) \) and \( C^*(\Gamma_q) \) respectively.

1.5. Quantum group structure. We have constructed the \( C^* \)-algebra \( A \) describing the “quantum space” of our quantum \( 'az + b' \) group. In order to give this quantum space a group structure we need to define a morphism \( \Delta \in \text{Mor} (A, A \otimes A) \) corresponding to (130). It turns out to be quite a hard problem to solve.

The biggest difficulty lies in forming the sum \( a \otimes b + b \otimes I \). Namely, this element should be affiliated to \( A \otimes A \) and the pair \( (a \otimes a, a \otimes b + b \otimes I) \) should be a semi-non degenerate \( q^2 \)-pair. Unfortunately the sum \( a \otimes b + b \otimes I \) is not even a closed operator (in a Hilbert space representation).

In order to deal with this problem one must return to Hilbert space considerations. We will define a special function \( F_q \) on \( \Gamma_q \) such that \( F_q \) will be continuous and its values will be complex numbers of modulus one. In Table 3 we give formulas defining \( F_q \) in cases (I)–(III). The auxiliary function \( f_0 \) used in case (I) is defined as

\[
    f_0(z) = \exp\left( \frac{1}{\pi i} \int_0^\infty \log(1 + i^{-t}) \frac{dt}{t + z^{-1}} \right),
\]

where \( N \) is the even natural number determining \( q \) for this case.

Let us stress here that in order to unify the treatment of cases (I), (III), and (III) we must introduce in case (I) a different special function from the corresponding one used in [21]. Namely we have in case (I)

\[
    F_q(\gamma) = F_N(q^{-2} \gamma)
\]

for all \( \gamma \in \Gamma_q \) (where \( F_N \) is the special function considered in [21], cf. also Section 2).
Theorem 1.2. Let $(S, R)$ be a non degenerate $q^2$-pair. Then

1. the sum $S + R$ is closable and its closure $S + R$ satisfies

$$S + R = Φ_q(RS^{-1})^* SΦ_q(RS^{-1}) = Φ_q(R^{-1}S) RΦ_q(R^{-1}S)^*.$$  

In particular $S + R$ is a normal operator with spectrum contained in $T_q$.

2. We have the exponential property:

$$Φ_q(S + R) = Φ_q(R) Φ_q(S).$$

3. We have the multiplicative property:

$$Φ_q(RS) = Φ_q(R)^* Φ_q(S) Φ_q(R) Φ_q(S)^*.$$  

The function $Φ_q$ is referred to as the quantum exponential function. This name is justified by point 2 of Theorem 1.2 (cf. [20]).

Using Theorem 1.2 one can prove that for any $C^*$-algebra $A$ and any two affiliated elements $a, b$ such that $(a, b)$ is semi-non degenerate $q^2$-pair (in the sense that they form such a pair in any representation of $A$) the elements $\hat{a} + \hat{b}$ and $\hat{a} \hat{b}$ are affiliated with $A$. In particular we have

Proposition 1.3. Let $A$ be the $C^*$-algebra described in Theorem 1.1 and let $a, b \in A$ be the elements described in that theorem. Then the element $a \otimes b + b \otimes I$ is affiliated with $A \otimes A$.

Now we can describe the main element of the quantum group structure of our quantum ‘$aq + b’$ groups. We have

Theorem 1.4. Let $A$, $a$ and $b$ be as in Proposition 1.3. Then there exists a unique $Δ \in $ Mor $(A, A \otimes A)$ such that

$$Δ(a) = a \otimes a,$$

$$Δ(b) = a \otimes b + b \otimes a.$$  

Theorem 1.4 can be proven directly. It is a consequence of the universal property of $A$. However one can also put in some more work and obtain the following:

Theorem 1.5. Let $H$ be a Hilbert space and let $(a, b)$ be a non degenerate $q^2$-pair of acting on $H$. Then the unitary operator

$$W = Φ_q(b^{-1} a \otimes b) \chi(b^{-1} \otimes I , I \otimes a)$$  

is a modular multiplicative unitary. Moreover

$$W(a \otimes I) W^* = a \otimes a,$$

$$W(b \otimes I) W^* = a \otimes b + b \otimes a.$$  

| Case     | Special function $Φ_q(γ)$                      |
|----------|-----------------------------------------------|
| Case (I) | $Φ_q(γ) = \left\{ \begin{array}{ll}
\prod_{s=1}^{k} \left( \frac{1 + q^{2s+1} r}{1 + q^{2s+1} r} \right) f_0(r) & \text{for } k \text{-even}, \\
\prod_{s=0}^{k} \left( \frac{1 + q^{2s+1} r}{1 + q^{2s+1} r} \right) f_0(r) & \text{for } k \text{-odd},
\end{array} \right. $ |
|           | for $q^{-2γ} = q^k r$                        |
| Case (II) | $Φ_q(γ) = \prod_{k=0}^{∞} \frac{1 + q^{2k+1} r}{1 + q^{2k+1} r}$ |
| Case (III)| $Φ_q(γ) = \prod_{k=0}^{∞} \frac{1 + q^{2k+1} r}{1 + q^{2k+1} r}$ |

Table 3. Special functions $Φ_q$
and the $C^*$-algebra
\[
\{(\omega \otimes \text{id})W : \omega \in B(H)_+\text{ norm closure} \}
\]
is isomorphic to the $C^*$-algebra $A$ described in Theorem 1.1. The operators $a$, $a^{-1}$ and $b$ are affiliated to (1.6).

Modular multiplicative unitaries provide a very convenient framework for constructing new examples of quantum groups. Given $W$ we can construct our quantum group. This procedure is described in [19, 10]. Thus having chosen $q$ from one of the sets in Table 1 we can define a multiplicative unitary $W$ by formula (1.5). The quantum group obtained from $W$ is then called the quantum ‘$aq + b$’ group for the deformation parameter $q$.

One needs to do some extra work in order to arrive at the level of locally compact quantum groups as defined in [2]. More precisely we need to find Haar weights for our quantum ‘$aq + b$’ group. Let us recall that it is not known whether every quantum group arising from a modular multiplicative unitary is a locally compact quantum group (the converse, however, is true). In the case of quantum ‘$aq + b$’ groups the problem of existence of Haar weights has been solved successfully by A. Van Daele and later by S.L. Woronowicz ([12, 22]).

1.6. Locally compact quantum group structure. According to the general theory ([19, 10]) a modular multiplicative unitary gives rise to an object $(\Lambda, \Delta)$ which has many (in fact most) features of a locally compact quantum group. In particular, for our quantum ‘$aq + b$’ groups we have the scaling group
\[
\tau_t(a) = a, \quad \tau_t(b) = q^{2it}b,
\]
the coinverse and unitary coinverse
\[
\kappa(a) = a^{-1}, \quad a^R = a^{-1},
\]
\[
\kappa(b) = -a^{-1}b, \quad b^R = -qa^{-1}b.
\]

We can examine the reduced dual quantum group to find that it is isomorphic to the opposite quantum group (the same quantum group with opposite comultiplication). Finally one can show that the reduced dual is also the universal dual (cf. [21, 5, 4]).

It turns out that the framework of modular multiplicative unitaries can be very well suited to study the question whether $(\Lambda, \Delta)$ is a locally compact quantum group as defined in [2]. Recall from the definition of modularity (Definition [10, Definition 2.1]) that there is a positive self-adjoint operator $\hat{Q}$ on $H$ such that
\[
W^*(\hat{Q} \otimes Q)W = \hat{Q} \otimes Q,
\]
where $Q$ satisfies the other conditions of definition of modularity of $W$. From the results of [22] we know that the formula
\[
h : A_+ \ni c \mapsto \text{Tr}(\hat{Q}^* c\hat{Q}) \in [0, \infty]
\]
defines a weight on $A$ which is right invariant. It is the right Haar weight if it is locally finite (finite on a norm dense subset of $A_+$).

It turns out that in the case of the modular multiplicative unitary $W$ defined by (1.5) the operator $\hat{Q}$ is simply equal to $|b|$. Moreover since (1.6) is the crossed product $C_0(\Gamma_q) \rtimes \Gamma_q$, it contains the linearly dense subset
\[
\{ f(a)g(b) : f \in C_0(\Gamma_q), \ g \in C_0(\Gamma_q) \}
\]
(cf. (1.1)). We can compute $h(c^*c)$ for $c$ of the form $c = f(a)g(b)$.

Using the Haar measure $\mu$ on $\Gamma_q$ we have
\[
h(c^*c) = \int_{\Gamma_q} |f(\gamma)|^2 d\mu(\gamma) \int_{\Gamma_q} |g(\gamma)|^2 d\mu(\gamma) \quad (1.7)
\]
the point $0 \in \Gamma_q$ is of measure 0).

It is now quite obvious that the set $\{ c \in A : h(c^*c) < \infty \}$ is norm dense in $A$ and so $h$ defined by (1.7) is the right Haar measure of the quantum ‘$az + b$’ group. The left Haar measure is $h^L = h \circ R$.

Finally let us note that the quantum ‘$az + b$’ groups provide illustration for the phenomenon forseen by the theory of locally compact groups ([2]) of existence of the so called scaling constant (cf. [2]).

As we know from [2, Proposition 6.8.3] there exists a positive number $\nu$ such that $h \tau_t = \nu^{-t} h$. This number is equal to 1 in most examples. It was noticed first by A. Van Daele ([12]) that for quantum ‘$az + b$’ groups we may have $\nu \neq 1$. More precisely

$$\nu = |q^{4t}|$$

so for $q$ not real the scaling constant is different from 1.

2. Quantum $GL(2, \mathbb{C})$ Group

When attempting to construct a quantum group on the $C^*$-algebra level one starts very often with generators and relations. Then detailed inspection of an operator meaning of the relations allows to describe an universal $C^*$-algebra corresponding to given set of generators and relations or additional constraints such as spectral conditions imposed on generators. The proper operator meaning of the relations ensures also the existence of a group structure (comultiplication, counit and coinverse (antipode)).

On the other hand one can look for general constructions allowing to construct new more complicated examples starting from known simpler ones. The quantum double group construction is of this type. It may be applied to the quantum ‘$az + b$’ groups described in the previous section. To be more concrete we shall consider the first family of quantum ‘$az + b$’ groups i.e. admissible deformation parameter is a special root of unity, $q \in \Sigma_I$. The purpose of this section is to show that the new quantum group obtained as a result of the construction we obtain a quantum $GL(2, \mathbb{C})$ group at roots of unity. In fact we shall see that both approaches to constructing quantum $GL(2, \mathbb{C})$ give the same quantum group. The exposition is based on [7]. To simplify presentation we shall focus only on constructing underlying $C^*$-algebras and comultiplications.

2.1. The first construction. The classical $GL(2, \mathbb{C})$ group is a collection of all invertible matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$  

Let $\mathcal{A}$ be the $*$-algebra of commutative polynomials on $GL(2, \mathbb{C})$ generated by four normal elements $\alpha, \beta, \gamma$ and $\delta$ subject to the relation

$$\det := \alpha \delta - \gamma \beta \text{ is invertible.}$$

The group structure of $GL(2, \mathbb{C})$ leads to the unique comultiplication

$$\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$$

such that

$$\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma, \quad \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \delta,$$
$$\Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma, \quad \Delta(\delta) = \gamma \otimes \beta + \delta \otimes \delta.$$  

It turns out that $(\mathcal{A}, \Delta)$ is a Hopf $*$-algebra with counit $e$ and coinverse $\kappa$ are given by

$$e(\alpha) = 1 = e(\delta), \quad e(\beta) = 0 = e(\gamma),$$
$$\kappa(\alpha) = \det^{-1} \delta, \quad \kappa(\beta) = -\det^{-1} \beta,$$
$$\kappa(\gamma) = -\det^{-1} \gamma, \quad \kappa(\delta) = \det^{-1} \alpha.$$  

On the Hopf algebra level $GL(2, \mathbb{C})$ admits a large two-parameter family of standard quantum deformations known for a long time ([14]). The considered quantum $GL(2, \mathbb{C})$ at roots of unity
corresponds to one-parameter subfamily of the standard deformations which can be described as follows: Let as before

\[ q = e^{\frac{2\pi i}{N}}, \quad N = 6, 8, 10, \ldots \]

and \( \alpha, \beta, \gamma, \delta \) be elements subject to the relations:

\[
\begin{align*}
\alpha \beta &= q^2 \beta \alpha, & \alpha \gamma &= \gamma \alpha, \\
\gamma \delta &= q^2 \delta \gamma, & \beta \delta &= \delta \beta, \\
\gamma \beta &= q^2 \beta \gamma, & \alpha \delta - \delta \alpha &= (q^2 - 1) \beta \gamma.
\end{align*}
\]

Define

\[ \det := \alpha \delta - \gamma \beta, \]

which will be the quantum determinant. Then also

\[ \det = \delta \alpha - \beta \gamma \]

and

\[ \alpha \det = \det \alpha, \quad \delta \det = \det \delta, \quad \beta \det = q^{-2} \det \beta, \quad \gamma \det = q^2 \det \gamma. \]

Assume that \( \det \) is invertible, i.e.

\[ \det^{-1} (\alpha \delta - \gamma \beta) = (\alpha \delta - \gamma \beta) \det^{-1} = 1. \]

Let \( A_{\text{hol}} \) be the algebra generated by \( \det^{-1} \) and four elements \( \alpha, \beta, \gamma, \delta \) satisfying above relations. One can check that:

- \( \alpha^{\frac{2\pi i}{N}}, \beta^{\frac{2\pi i}{N}}, \gamma^{\frac{2\pi i}{N}}, \delta^{\frac{2\pi i}{N}} \) are central elements of \( A_{\text{hol}} \).
- \( \det^{\frac{2\pi i}{N}} = \begin{cases} (\alpha \delta - \gamma \beta)^{\frac{2\pi i}{N}} = \alpha^{\frac{2\pi i}{N}} \delta^{\frac{2\pi i}{N}} - \gamma^{\frac{2\pi i}{N}} \beta^{\frac{2\pi i}{N}}, \\ (\delta \alpha - \beta \gamma)^{\frac{2\pi i}{N}} = \delta^{\frac{2\pi i}{N}} \alpha^{\frac{2\pi i}{N}} - \beta^{\frac{2\pi i}{N}} \gamma^{\frac{2\pi i}{N}}. \end{cases} \)
- \( \det \) is not in the center of \( A_{\text{hol}} \) but \( \det^{\frac{2\pi i}{N}} \) is a central element of \( A_{\text{hol}} \).

Now \( \Delta \) defined in a standard way (as in the classical case) respects above relations and \( (A_{\text{hol}}, \Delta) \) is a Hopf algebra with counit and coinverse described by the same expressions as given in the classical case. It corresponds to deformation \( GL_{q^2,1}(2, \mathbb{C}) \) in notation of \([14]\). Moreover \( \det \) is a character, \( \Delta(\det) = \det \otimes \det \) and

\[
\begin{align*}
\Delta(\alpha^{\frac{2\pi i}{N}}) &= \alpha^{\frac{2\pi i}{N}} \otimes \alpha^{\frac{2\pi i}{N}} + \beta^{\frac{2\pi i}{N}} \otimes \gamma^{\frac{2\pi i}{N}}, \\
\Delta(\beta^{\frac{2\pi i}{N}}) &= \alpha^{\frac{2\pi i}{N}} \otimes \beta^{\frac{2\pi i}{N}} + \beta^{\frac{2\pi i}{N}} \otimes \delta^{\frac{2\pi i}{N}}, \\
\Delta(\gamma^{\frac{2\pi i}{N}}) &= \gamma^{\frac{2\pi i}{N}} \otimes \alpha^{\frac{2\pi i}{N}} + \delta^{\frac{2\pi i}{N}} \otimes \gamma^{\frac{2\pi i}{N}}, \\
\Delta(\delta^{\frac{2\pi i}{N}}) &= \gamma^{\frac{2\pi i}{N}} \otimes \beta^{\frac{2\pi i}{N}} + \delta^{\frac{2\pi i}{N}} \otimes \delta^{\frac{2\pi i}{N}}.
\end{align*}
\]

A \( \ast \)-structure is obtained by “complexifying” \( A_{\text{hol}} \). More precisely let \( A_o \) be the \( \ast \)-algebra generated by \( \alpha, \beta, \gamma, \delta \) and \( \det^{-1} \) satisfying described relations and

\[ cc' = c'c \quad \text{for any } c \in \{ \alpha, \beta, \gamma, \delta \} \text{ and } c' \in \{ \alpha^*, \beta^*, \gamma^*, \delta^* \}. \]

Then

- \( A_{\text{hol}} \) is a subalgebra of \( A_o \).
- \( A_o \) is identified with \( A_{\text{hol}} \otimes A_{\text{hol}}^* \) by the multiplication map

\[ A_{\text{hol}} \otimes A_{\text{hol}}^* \ni a \otimes b^* \mapsto ab^* \in A_o. \]

- \( \alpha, \beta, \gamma, \delta \) and \( \det \) are normal elements of \( A_o \).
- \( \alpha^{\frac{2\pi i}{N}}, \beta^{\frac{2\pi i}{N}}, \gamma^{\frac{2\pi i}{N}}, \delta^{\frac{2\pi i}{N}} \) and \( \det^{\frac{2\pi i}{N}} \) are central elements of \( A_o \).
- The formula \( \Delta(ab^*) = \Delta(a)\Delta(b)^* \) extends \( \Delta \) to \( A_o \).

This way \( (A_o, \Delta) \) becomes a Hopf \( \ast \)-algebra.

It turns out that \( A_o \) can not be used as the starting point for the description of quantum \( GL(2, \mathbb{C}) \) on the \( C^* \)-algebra level because the existence of the comultiplication is not guaranteed. To finish the construction of the proper Hopf \( \ast \)-algebra we impose the \textit{hermiticity conditions}
following the construction of quantum 'az + b' group: elements $\alpha \frac{d}{2}, \beta \frac{d}{2}, \gamma \frac{d}{2}$ and $\delta \frac{d}{2}$ are hermitian, i.e.

$$(\alpha \frac{d}{2})^* = \alpha \frac{d}{2}, \quad (\beta \frac{d}{2})^* = \beta \frac{d}{2},$$

$$(\gamma \frac{d}{2})^* = \gamma \frac{d}{2}, \quad (\delta \frac{d}{2})^* = \delta \frac{d}{2}.$$

Let us note that

- $\alpha \frac{d}{2}$ and $(\alpha \frac{d}{2})^*$ have the same commutation relations with all generators of $\mathcal{A}_o$ (both are in the center of $\mathcal{A}_o$) and relation $(\alpha \frac{d}{2})^* = \alpha \frac{d}{2}$ is compatible with the algebraic structure of $\mathcal{A}_o$. The same holds for the remaining relations.

- $(\det \frac{d}{2})^* = \det \frac{d}{2}$.

- Hermiticity conditions of $(\alpha \frac{d}{2}), (\beta \frac{d}{2}), (\gamma \frac{d}{2})$ and $\Delta(\delta \frac{d}{2})^*$ follow.

Therefore the hermiticity conditions are compatible with algebra and coalgebra structures. Now let $\mathcal{A}$ be the $^*$-algebra generated by five normal elements $\alpha, \beta, \gamma, \delta$ and $\det^{-1}$ satisfying relations and hermiticity conditions. Then $(\mathcal{A}, \Delta)$ is a Hopf $^*$-algebra.

In particular

$$u = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is a two dimensional corepresentation of $(\mathcal{A}, \Delta)$.

The Hopf $^*$-algebra $(\mathcal{A}, \Delta)$ is called the algebra of polynomials on the quantum $GL(2, \mathbb{C})$. Now we shall assign precise operator meaning to the commutation relations. At first we observe that

- $\alpha, \beta, \gamma, \delta$ and $\det$ are normal operators.

- By hermiticity conditions $\alpha \frac{d}{2}, \beta \frac{d}{2}, \gamma \frac{d}{2}, \delta \frac{d}{2}$ and $\det \frac{d}{2}$ are selfadjoint operators. This imposes spectral conditions localizing spectra of $\alpha, \beta, \gamma, \delta$ and $\det$:

$$\operatorname{Sp} \alpha, \operatorname{Sp} \beta, \operatorname{Sp} \gamma, \operatorname{Sp} \delta, \operatorname{Sp} \det \subset \Gamma_q,$$

where $\Gamma_q$ is the subset of $\mathbb{C}$ considered in the construction of quantum 'az + b' group (see Subsection 1.2).

Next we should give the precise meaning to the relations of the form

$$XY = q^2 YX \quad \text{and} \quad XY^* = Y^* X,$$

where $X$ and $Y$ are normal operators on a Hilbert space $H$, and $\operatorname{Sp} X, \operatorname{Sp} Y \subset \Gamma_q$.

The reader should notice that under additional assumption of invertibility of $X$ and $Y$ this reduces to problem of $(\mathbb{S}, R)$ pairs considered in Subsection 1.3 and solved by introducing the Weyl form of the above commutation relations. We can follow this idea also in more general case (without assumption of invertibility) and such pair $(X,Y)$ will be called a $q^2$-pair on $H$.

Extending the bicharacter $\chi$ from $\Gamma_q$ to $\Gamma_q$ by a formula

$$\tilde{\chi}(z, z') = \begin{cases} 
\chi(z, z') & \text{for } z, z' \in \Gamma_q, \\
0 & \text{otherwise}.
\end{cases}$$

we define a measurable symmetric function $\tilde{\chi} : \Gamma_q \times \Gamma_q \to \mathbb{T}^1 \cup \{0\}$, such that

$$\tilde{\chi}(z z', z'') = \chi(z, z'') \chi(z', z'')$$

for any $z, z', z'' \in \Gamma_q$. Then for $z' \in \Gamma_q$ and a normal operator $X$ on $H$ with $\operatorname{Sp} X \subset \Gamma_q$ we obtain

$$\tilde{\chi}(X, z') = (\text{unitary operator}) \oplus 0.$$

Moreover

$$\tilde{\chi}(z X, z') = \tilde{\chi}(z, z') \chi(X, z').$$

Now by definition a pair $(X,Y)$ of normal operators acting on a Hilbert space $H$ is a $q^2$-pair on $H$ if

1. $\operatorname{Sp} X \subset \Gamma_q, \operatorname{Sp} Y \subset \Gamma_q$;
2. $\tilde{\chi}(X, z) \tilde{\chi}(Y, z') = \chi(z, z') \tilde{\chi}(Y, z') \tilde{\chi}(X, z)$ for any $z, z' \in \Gamma_q$. 
We let $\mathcal{D}_H$ denote the set of all $q^2$-pairs on $H$.

One can easily show that any $q^2$-pair $(X, Y)$ on the Hilbert space $H$ is a direct sum of at most four components of the form

$$X = S \oplus X_o \oplus 0 \oplus 0,$$

$$Y = R \oplus 0 \oplus Y_o \oplus 0,$$

where $R, S, X_o$ and $Y_o$ are normal invertible operators with spectra localized in $\bar{\Gamma}$. Remembering that irreducible $(S, R)$ pair is unique (the Schrödinger pair) this gives the complete description of a general $q^2$-pair.

The components of a $q^2$-pairs despite being in general unbounded, behave in a very regular way with respect to the multiplication and addition operations. In particular let us mention the following results for $(X, Y) \in \mathcal{D}_H$:

- The compositions $X \circ Y, Y \circ X, X \circ Y^*, Y^* \circ X$ and $Y^* \circ X^*$ are densely defined closeable operators and denoting by $XY, YX, XY^*, Y^*X$ and $Y^*X^*$ their closures we have

$$XY = q^2 YX, \quad XY^* = Y^*X, \quad (XY)^* = Y^*X^*.$$  

Moreover

- $XY$ is a normal operator, $\text{Sp} XY \subset \bar{\Gamma}_q$ and $XY$ is invertible if and only if $(X, Y)$ is non-degenerate.

- $(X^*, Y^*)$, $(XY, Y)$ and $(X, YX)$ are $q^2$-pairs on $H$.

- If $X$ is invertible then $(Y, X^{-1}) \in \mathcal{D}_H$.

- If $Y$ is invertible then $(Y^{-1}, X) \in \mathcal{D}_H$.

- $X + Y$ is a densely defined closeable operator, its closure $X + Y$ is a normal operator with $\text{Sp}(X + Y) \subset \bar{\Gamma}_q$. Moreover

$$X + Y = \begin{cases} F_N(X^{-1}Y)^* X F_N(X^{-1}Y) & \text{if } X \text{ is invertible}, \\ F_N(XY^{-1}) Y F_N(XY^{-1})^* & \text{if } Y \text{ is invertible}. \end{cases}$$

In particular $X + Y$ is an invertible operator if $X$ or $Y$ is invertible.

- $F_N(Y + X) = F_N(Y) F_N(X)$

In the above equations $F_N$ is the quantum exponential function related to the exponential function $\mathbb{F}_q$, introduced in Table 3 for the case (I) of quantum ‘$a z + b$’ group by the formula

$$\mathbb{F}_q(\gamma) = F_N(q^{kr})$$

where $q^{-2} \gamma = q^{kr}$.

Products and sums of operators forming $q^2$-pairs play an essential role in the further discussion. First we consider implications of the invertibility of the quantum determinant. In contrast to the classical case invertibility of $\delta$ and $\alpha$ follows as a result of simple observations:

- $\ker \delta = \ker \delta^*$ is invariant under the action of $\gamma \beta, (\beta \gamma)^*$, det and det$^*$ therefore $\ker \delta$ is an invariant subspace for det $+ \gamma \beta$ and det$^* + (\beta \gamma)^*$.

- formulae $\alpha \delta = \text{det} + \gamma \beta$ and $\alpha^* \delta^* = \text{det}^* + (\beta \gamma)^*$ indicate that

$$\text{det} + \gamma \beta = 0, \quad \text{and} \quad \text{det}^* + (\beta \gamma)^* = 0$$

on $\ker \delta \subset (\ker \alpha \delta) \cap (\ker \alpha^* \delta^*)$.

- $\text{det} + \gamma \beta = \text{det} + \beta \gamma$ on $\ker \delta$.

- $\beta \gamma = 0$ since $\gamma \beta = q^2 \beta \gamma$.

- $\text{det} = 0$ on $\ker \delta$.

- $\ker \delta = \{0\}$ due to invertibility of det, i.e. $\delta$ is an invertible operator.

- In the same manner $\alpha$ is an invertible operator.

Now we are ready to give a precise operator meaning to all commutation relations. These are encoded in the notion of $G$-matrix.

In the classical case the determinant is expressed in terms of matrix elements, but in our approach it turns out to be more convenient to include the quantum determinant det into the set of parameters and then determine one of the matrix elements. In the definition below $\alpha$ is such a distinguished element (equivalently one can use $\delta$).
Consider a matrix
\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\]
(2.1)
where \(\alpha, \beta, \gamma\) and \(\delta\) are normal operators acting on a Hilbert space \(H\).

**Definition 2.1.** We say that (2.1) is a \(G\)-matrix whenever there exists a normal operator \(\text{det}\) such that

1. \(\text{Sp}\beta, \text{Sp}\gamma, \text{Sp}\delta, \text{Sp}\text{det} \subset \overline{\Gamma_q}\);
2. \(\delta\) strongly commutes with \(\beta\) and \(\text{det}\);
3. \((\gamma, \beta), (\gamma, \delta), (\text{det}, \beta), (\gamma, \text{det})\) are \(q^2\)-pairs;
4. \(\text{det}\) and \(\delta\) are invertible operators;
5. If \(x \in D(\delta) \cap D(\gamma\beta) \cap D(\text{det})\) then \(\delta(x) \in D(\alpha)\) and
   \[\alpha\delta(x) = \text{det}(x) + \gamma\beta(x)\]

One can show that
- If \(\text{det}\) exists it is unique.
- The fifth condition implies a stronger (therefore equivalent) form
  \[\alpha = \text{det}\delta^{-1} + \gamma\beta\delta^{-1}\]

Consequently \(\alpha\) is determined and satisfies the relations:
- \(\text{Sp}\alpha \subset \overline{\Gamma_q}\);
- \(\alpha\) is an invertible operator;
- \(\alpha\) strongly commutes with \(\gamma\) and \(\text{det}\);
- \((\alpha, \beta)\) is a \(q^2\)-pair;
- The set \(D_\alpha := \{x \in D(\delta) \cap D(\gamma\beta) : \delta(x) \in D(\alpha)\}\) is a core for \(\text{det}\) and
  \[\text{det}(x) = \alpha\delta(x) - \gamma\beta(x)\]

for any \(x \in D_\alpha\).

The basic fact concerning the \(G\)-matrices states that tensor product of two \(G\)-matrices is again a \(G\)-matrix. More precisely, let
\[
u_1 = \begin{pmatrix}
\alpha_1 & \beta_1 \\
\gamma_1 & \delta_1
\end{pmatrix}, \quad \nu_2 = \begin{pmatrix}
\alpha_2 & \beta_2 \\
\gamma_2 & \delta_2
\end{pmatrix}
\]
be \(G\)-matrices on Hilbert spaces \(H_1\) and \(H_2\) respectively and
\[
\tilde{\nu} = \begin{pmatrix}
\tilde{\alpha} & \tilde{\beta} \\
\tilde{\gamma} & \tilde{\delta}
\end{pmatrix} = \begin{pmatrix}
\alpha_1 \otimes \alpha_2 + \beta_1 \otimes \gamma_2 & \alpha_1 \otimes \beta_2 + \beta_1 \otimes \delta_2 \\
\gamma_1 \otimes \alpha_2 + \delta_1 \otimes \gamma_2 & \gamma_1 \otimes \beta_2 + \delta_1 \otimes \delta_2
\end{pmatrix}
\]
Then \(\tilde{\nu}\) is a \(G\)-matrix on \(H_1 \otimes H_2\). Moreover
\[
\tilde{\text{det}} = \text{det}_1 \otimes \text{det}_2
\]
where \(\text{det}, \text{det}_1\) and \(\text{det}_2\) are the corresponding quantum determinants for \(\tilde{\nu}, \nu_1\) and \(\nu_2\) respectively.

To reveal the consequence of this fact assume that the universal \(C^*\)-algebra \(B\) generated by \(G\)-matrices exists. Then due to the universality of \(B\) (since the matrix elements and quantum determinant of the right hand side matrix are affiliated with \(B \otimes B\)), there exists a unique \(\Delta \in \text{Mor}(B, B \otimes B)\) such that
\[
\Delta(\alpha) = \alpha \otimes \alpha \dagger \beta \otimes \gamma, \quad \Delta(\beta) = \alpha \otimes \beta \dagger \beta \otimes \delta,
\]
\[
\Delta(\gamma) = \gamma \otimes \alpha \dagger \delta \otimes \gamma, \quad \Delta(\delta) = \gamma \otimes \beta \dagger \delta \otimes \delta.
\]
Then one shows that \(\Delta\) is coassociative. Therefore \(\Delta\) is a comultiplication.

Universal \(C^*\)-algebra \(B\) related to \(G\)-matrices is generated by \(\alpha, \beta, \gamma, \delta\) and \(\text{det}^{-1}\) and corresponds to “the algebra of continuous functions vanishing at infinity” on the quantum group \(GL_{q^2}(2, \mathbb{C})\). The existence of such an algebra follows from the fact that if (2.1) is a \(G\)-matrix on a Hilbert space \(H\) and \(C\) is a non degenerate \(C^*\)-subalgebra of \(B(H)\) then
\[
\begin{pmatrix}
\alpha, \beta, \gamma, \delta, \text{det}^{-1} \eta C
\end{pmatrix} \Longleftrightarrow \begin{pmatrix}
\beta\delta^{-1}, \gamma, \delta^{-1}, \text{det}\delta^{-1}, \text{det}^{-1} \delta \eta C
\end{pmatrix}
\]
This means that
\* \(\beta\delta^{-1}, \gamma, \delta, \delta^{-1}, \det\delta^{-1}\) and \(\det\delta^{-1}\) also parameterize \(G\)-matrices;
\* \((\delta^{-1}, \gamma), (\det\delta^{-1}, \beta\delta^{-1})\) are \(q^2\)-pairs and \(\delta^{-1}, \det\delta^{-1}\) are invertible;
\* \(\gamma\) and \(\delta^{-1}\) strongly commute with \(\beta\delta^{-1}\) and \(\det\delta^{-1}\).

Therefore \(B\) should be a tensor product of two copies of the \(C^*\)-algebra \(A\) generated by a \(q^2\)-pair \((X, Y)\) where \(X\) is invertible. The reader should notice that such an algebra was described in Subsection 1.4 and corresponds to the “algebra of continuous functions vanishing at infinity” on quantum ‘\(aq + b\)’ group. It coincides with the crossed product algebra \(C_\infty(\Gamma_q) \rtimes_\sigma \Gamma_q\). Therefore

\[ B := A \otimes A \]

and \((B, \Delta)\) is the quantum \(GL(2, \mathbb{C})\) group at roots of unity on the \(C^*\)-algebra level.

2.2. **Quantum \(GL(2, \mathbb{C})\) as a double group.** Let us now turn to the second description of the quantum \(GL(2, \mathbb{C})\) group at roots of unity. It is based on the double group construction (cf. \([5]\)).

The quantum double construction was introduced by Drinfeld \([1]\) in the framework of deformed enveloping algebras. Podleś and Woronowicz proposed \([4]\) its dual version, the double group construction, which is more useful in \(C^*\)-algebra approach. It involved a compact quantum group and its dual. In fact, this type of construction can be applied in more general situation. In particular such a construction may be described in terms of modular multiplicative unitaries.

To gain a better understanding of the double group construction we shall describe underlying classical situation first.

Assume that \(G\) is a topological group and \(K, \hat{K}\) its closed subgroups such that the maps

\[
\begin{align*}
\phi &: K \times \hat{K} \ni (x, \hat{x}) \mapsto x \cdot \hat{x} \in G \\
\psi &: \hat{K} \times K \ni (\hat{x}, x) \mapsto \hat{x} \cdot x \in G
\end{align*}
\]

are homeomorphisms. Then

\[
\sigma_* := \phi^{-1} \circ \psi : \hat{K} \times K \to K \times \hat{K}
\]

is a homeomorphism compatible with multiplication rules in \(K\) and \(\hat{K}\) respectively. Moreover \(\sigma_*\) encodes group structure of \(G\) in terms of that for \(K\) and \(\hat{K}\):

\[
(x_1, \hat{x}_1) \cdot (x_2, \hat{x}_2) = (x_1 x'_2, \hat{x}_1 \hat{x}_2),
\]

where

\[
(x'_2, \hat{x}_1') = \sigma_*(\hat{x}_1, x_2).
\]

Conversely starting with topological groups \(K, \hat{K}\) and a homeomorphism \(\sigma_*\) compatible with group structures of \(K\) and \(\hat{K}\) one can put a group structure on \(K \times \hat{K}\) using above formulas. This way a new group \(G\), the twisted product of \(K\) and \(\hat{K}\), is constructed.

The main steps of the quantum double construction are as follows:

\[
K = (A, \Delta) \quad \quad \hat{K} = (\hat{A}, \hat{\Delta})
\]

(\(\Delta\) is the group coadjoint action and \(\hat{\Delta}\) is the Pontryagin dual of \(\Delta\)).

Let

\[
B := A \otimes \hat{A}
\]

and let \(W\) be a bicharacter on \(K \times K\), i.e.

\[
\text{\(W\) is a unitary element in } M(\hat{A} \otimes A)
\]

and

\[
(id \otimes \Delta)W = W_{12}W_{13},
\]

\[
(\hat{\Delta} \otimes id)W = W_{23}W_{13}.
\]

Define \(\sigma \in \text{Mor} \left( A \otimes \hat{A}, \hat{A} \otimes A \right) \) by

\[
\sigma(x \otimes \hat{x}) := W(\hat{x} \otimes x)W^*
\]  

(2.2)
for any $x \in A$ and $\hat{x} \in \hat{A}$ and

$$\Delta := (\text{id} \otimes \sigma \otimes \text{id})(\Delta \otimes \hat{\Delta}). \quad (2.3)$$

Then $\Delta \in \text{Mor}(B, B \otimes B)$ and it is coassociative. The double group build over $K$ is by definition $G = (B, \Delta)$.

The double group construction can be nicely described in the framework of locally compact quantum groups. The formula for the multiplicative unitary defining the double group in terms of the Kac-Takesaki operator of the original quantum group can be found in [3]. Let us note also that similar ideas are basis of introducing of matched pairs considered e.g. in [11].

Now we apply above construction to build the double group over ‘az + b’ quantum group.

| Quantum ‘az + b’ group | Dual of quantum ‘az + b’ group |
|------------------------|--------------------------------|
| $(a, b) - q^2$-pair, $a$ - invertible | $(\hat{b}, \hat{a}) - q^2$-pair, $\hat{a}$ - invertible |
| $\Delta(a) = a \otimes a$ | $\hat{\Delta}(\hat{a}) = \hat{a} \otimes \hat{a}$ |
| $\Delta(b) = a \otimes b + b \otimes I$ | $\hat{\Delta}(\hat{b}) = \hat{a} \otimes \hat{b} + \hat{b} \otimes \hat{I}$ |
| $v = \begin{pmatrix} a & b \\ 0 & I \end{pmatrix}$ | $\hat{v} = \begin{pmatrix} \hat{I} & 0 \\ \hat{b} & \hat{a} \end{pmatrix}$ |
| $K = (A, \Delta)$ | $\hat{K} = (\hat{A}, \hat{\Delta})$ |

Table 4. Ingredients for the double group construction

It is known that $v$ and $\hat{v}$ are two dimensional fundamental representations of $K$ and $\hat{K}$ respectively. Moreover the crossed product algebra $A$ is isomorphic to $\hat{A}$, $\hat{A} = A$ (cf. [21] and Section [11]). Therefore $B = A \otimes \hat{A} = A \otimes A$.

Let

$$W = F_N(\hat{b}\hat{a}^{-1} \otimes b)\chi(\hat{a}^{-1} \otimes I, \hat{I} \otimes a).$$

Then $W$ is a bicharacter on $\hat{K} \times K$. This way we get a quantum group $G = (B, \Delta)$.

Now we shall show that this new group coincides with that obtained in the first approach. Clearly $C^*$-algebras are the same in both cases. It remains to prove that the comultiplications coincide. To this end let us define a matrix

$$u = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} := \begin{pmatrix} a & b \\ 0 & I \end{pmatrix} \begin{pmatrix} \hat{I} & 0 \\ \hat{b} & \hat{a} \end{pmatrix}$$

i.e. $u = v \bigoplus \hat{v}$. At first let us discuss the problem of whether $u$ is a $G$-matrix. The right hand side of the above expression can be regarded as a Gauss decomposition for $u$. This decomposition leads to considering two important types of $G$-matrices:

1. $K$-matrices:

$$v = \begin{pmatrix} a & b \\ 0 & I \end{pmatrix},$$

where $a$ and $b$ are normal operators acting on a Hilbert space $H$ such that $(b, a)$ is a $q^2$-pair and $a$ is an invertible operator. Then the matrix $v$ is a $G$-matrix. In this case $\det = a$.

More precisely a $G$-matrix $u$ is called $K$-matrix if and only if $\gamma = 0$ and $\delta = I$.

2. $\hat{K}$-matrices:

$$\hat{v} = \begin{pmatrix} \hat{I} & 0 \\ \hat{b} & \hat{a} \end{pmatrix},$$

where $a$ and $b$ are normal operators acting on a Hilbert space $H$ such that $(b, a)$ is a $q^2$-pair and $a$ is an invertible operator. Then the matrix $v$ is a $G$-matrix. In this case $\det = a$.
where $\hat{a}$ and $\hat{b}$ are normal operators acting on a Hilbert space $H$ such that $(\hat{a}, \hat{b})$ is a $q^2$-pair and $\hat{a}$ is an invertible operator. Then the matrix $\hat{v}$ is a $G$-matrix. In this case $\det = \hat{a}$. In other words a $G$-matrix $u$ is a $\hat{K}$-matrix if and only if $\alpha = I$ and $\beta = 0$.

It turns out that $u$ is $G$-matrix if and only if it is of the form $u = v\hat{v}$ where $v$ and $\hat{v}$ are $K$- and $\hat{K}$-matrices respectively and matrix elements of $v$ commute with those of $\hat{v}$. Moreover the decomposition is unique.

Now it is interesting that if $u$ is representation of $G$, i.e. the new comultiplication $\Delta$ (cf. (2.3)) acts on matrix elements of $u$ in the standard way:

$$
\begin{align*}
\Delta(\alpha) &= \alpha \otimes \alpha + \beta \otimes \gamma, \\
\Delta(\beta) &= \beta \otimes \beta + \beta \otimes \gamma, \\
\Delta(\gamma) &= \gamma \otimes \alpha + \delta \otimes \gamma, \\
\Delta(\delta) &= \gamma \otimes \beta + \delta \otimes \delta.
\end{align*}
$$

Remembering that $\nu$ and $\hat{v}$ are (co)-representations one has to check only that the compatibility condition

$$(\text{id} \otimes \sigma)(v \otimes \hat{v}) = \hat{v} \oplus v$$

holds. In expanded form

$$(\text{id} \otimes \sigma) \left[ \begin{pmatrix} a & b \\ 0 & I \end{pmatrix} \oplus \begin{pmatrix} \hat{a} & 0 \\ b & \hat{a} \end{pmatrix} \right] = \begin{pmatrix} \hat{a} & 0 \\ b & \hat{a} \end{pmatrix} \oplus \begin{pmatrix} a & b \\ 0 & I \end{pmatrix}$$

i.e.

$$
\begin{align*}
\sigma(a \otimes \hat{a} + b \otimes \hat{b}) &= \hat{a} \otimes a, \\
\sigma(b \otimes \hat{a}) &= \hat{b} \otimes a, \\
\sigma(I \otimes \hat{a}) &= \hat{a} \otimes I + \hat{b} \otimes b.
\end{align*}
$$

The result follows from the properties of $W$.

This shows that both comultiplications coincide on generators. Therefore on the $C^*$-algebra level the quantum $GL(2, \mathbb{C})$ group at roots unity coincides with the double group build over quantum ‘$az + b$’ group.

### 3. Quantum Lorentz groups

Let us now turn to other examples of quantum groups. Historically compact quantum groups were the first objects of investigation, but since the theory of compact quantum groups is by now relatively well developed we shall focus on non compact examples.

#### 3.1. Quantum Lorentz group with Iwasawa decomposition.

This quantum group was constructed and studied in [4]. It is closely related to the quantum $SU(2)$. More precisely quantum Lorentz group is the double group built over $SU_q(2)$ for a deformation parameter $q \in [-1, 1] \setminus \{0\}$ (cf. [15]). The double group construction was described in Subsection 2.2.

The ingredients for the construction are the quantum $SU(2)$ group $(A, \Delta) = (A_c, \Delta_c)$ (script “c” stands for “compact”) introduced in [15], its dual $(\hat{A}, \hat{\Delta}) = (A_d, \Delta_d)$ (“d” for “discrete”) and a bicharacter $W \in M(A_d \otimes A_c)$ given by

$$W = \sum_{u^s} \oplus u^s,$$

where the sum is taken over all classes of non isomorphic, finite dimensional representations $u^s$ of $SU_q(2)$ (cf. [15] and [4]). The algebra $A_d$ is generated by affiliated elements $a_d$ and $n_d$ satisfying relations:

$$
\begin{align*}
a_d^* a_d &= a_d a_d^*, \\
ap_d n_d &= q n_d a_d, \\
q n_d a_d^* &= q^{-1} a_d^* n_d, \\
n_d n_d^* &= n_d^* n_d + (1 - q^2)((a_d^* a_d)^{-1} - a_d^* a_d).
\end{align*}
$$

The analog of Table 4 is the following:
Quantum SU(2) group | Dual of quantum SU(2) group
---|---
\( (\alpha_c, \gamma_c) \) – generators of \( A_c \) | \( (a_d, n_d) \) – generators of \( A_d \)
\( \Delta(\alpha_c) = \alpha_c \otimes \alpha_c - q_{\gamma_c}^* \otimes \gamma_c \) | \( \widehat{\Delta}(a_d) = a_d \otimes a_d \)
\( \Delta(\gamma_c) = \gamma_c \otimes \alpha_c + \alpha_c^* \otimes \gamma_c \) | \( \widehat{\Delta}(n_n) = a_d \otimes n_d + n_d \otimes a_d^{-1} \)
\( w_c = \begin{pmatrix} \alpha_c & -q_{\gamma_c}^* \\ \gamma_c & \alpha_c^* \end{pmatrix} \) | \( w_d = \begin{pmatrix} a_d & n_d \\ 0 & a_d^{-1} \end{pmatrix} \)

Table 5. Ingredients for the double group construction

The morphism \( \sigma \) is introduced by formula (2.2) (cf. Subsection 2.2).

The fundamental representation of the quantum Lorentz group is the matrix
\[
w = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},
\]
where \( \alpha, \beta, \gamma \) and \( \delta \) are unbounded elements affiliated with the \( C^* \)-algebra \( A \otimes \widehat{A} \) satisfying a long list of relations (equations (1.9)–(1.25) of [4]). The Iwasawa decomposition property of the quantum Lorentz group means that
\[
w = w_c w_d,
\]
where \( w_c \) and \( w_d \) are introduced in Table 5.

The commutation relations between \( \alpha, \beta, \gamma \) and \( \delta \) can be derived from those for matrix elements of \( w_c \) and \( w_d \). In particular, matrix elements of \( w_c \) commute with matrix elements of \( w_d \) and their adjoints and all these matrix elements are affiliated with the algebra generated by \( \alpha, \beta, \gamma \) and \( \delta \) (for details cf. [4]).

3.2. Quantum \( E(2) \) group. Quantum \( E(2) \) group was the second example of a non compact quantum group. It is a deformation of the (two fold covering) of the group of motions of the Euclidean plane. Its algebra was generated by two elements \( v \) and \( n \) with \( v \) unitary and \( n \) normal. The defining relation
\[
vnv^* = qn,
\]
(where \( 0 < q < 1 \) is the deformation parameter) turned out not to be sufficient to have a comultiplication
\[
\Delta(v) = v \otimes v,
\]
\[
\Delta(n) = v \otimes n + n \otimes v^*.
\]
It was necessary to introduce a spectral condition. More precisely the comultiplication exists on the \( C^* \)-algebra level if and only if \( \text{Sp}
p_n \subset \mathbb{C}', \) where
\[
\mathbb{C}' = \{ z \in \mathbb{C} : |z| \in q^\mathbb{Z} \} \cup \{0\}.
\]
This was the first example of a phenomenon which appeared in the theory of \( C^* \)-algebraic quantum groups and was not present in the Hopf algebra picture.

This quantum group was defined and investigated in [10]. Its Pontriagin dual was found and identified with a deformation of the group of transformations of the plane generated by translations and dilations.

The dual quantum group \( (\widehat{A}, \widehat{\Delta}) \) is a little more complicated to describe. The algebra \( \widehat{A} \) is generated by two affiliated elements \( N \) and \( b \), where \( N \) is selfadjoint and \( b \) is normal with polar decomposition \( b = ub \) such that \( N \) and \( |b| \) strongly commute and \( uNu^* = N - 2I \). Moreover the joint spectrum of \( (N, |b|) \) is contained in the set \( \Sigma_q \),
\[
\Sigma_q = \{ (s, q^r) : \ s \in \mathbb{Z}, \ r \in \mathbb{Z} + \frac{s}{2} \}.
\]
The comultiplication on $\hat{A}$ is given by
\[
\hat{\Delta}(N) = N \otimes I + I \otimes N,
\]
\[
\hat{\Delta}(b) = b \otimes q^{\frac{N}{2}} + q^{-\frac{N}{2}} \otimes b.
\]

The paper [13] is devoted to a direct proof that the dual of this last quantum group is the quantum $E(2)$, i.e. the Pontriagin duality holds.

3.3. Quantum Lorentz group with Gauss decomposition. The double group construction applied to the quantum $E(2)$ group gave a quantum Lorentz group which was different from the one described above. Its characteristic feature was the so called Gauss decomposition. Again this means that the fundamental representation decomposes
\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} =
\begin{pmatrix}
v & n \\
0 & v^*
\end{pmatrix}
\begin{pmatrix}
a & 0 \\
b & a
\end{pmatrix},
\tag{3.1}
\]
where $v$ and $n$ are the generators of the quantum $E(2)$, $N$ and $b$ are the generators of the dual quantum group (described in Subsection 3.2) and $q \in [0,1]$ is the deformation parameter.

Moreover $a = q^N$, where $N$ is a self adjoint element affiliated with the algebra generated by $a$ and $b$ and $\text{Sp} N \subset \mathbb{Z}$. The matrices appearing on the right hand side of (3.1) are fundamental representations of the quantum $E(2)$ and its dual respectively. The last ingredient of the double group construction – the bicharacter $W \in M(\hat{A} \otimes A)$ is given by
\[
W = F_q(q^{\frac{N}{2}} b \otimes v n)(I \otimes v)^{N \otimes I},
\]
where $F_q$ is the quantum exponential function studied in Subsection 1.5 (case (II)).

Again we can group ingredients for the double group construction in the following table:

| Quantum $E(2)$ group | Dual of quantum $E(2)$ group |
|----------------------|-----------------------------|
| $(v, n)$ – generators of $A$ | $(N, b)$ – generators of $\hat{A}$ |
| $\Delta(v) = v \otimes v$ | $\hat{\Delta}(N) = N \otimes I + I \otimes N$ |
| $\Delta(n) = v \otimes n + n \otimes v^*$ | $\hat{\Delta}(b) = b \otimes q^N + q^{-N} \otimes b$ |
| $v = \begin{pmatrix} v & n \\ 0 & v^* \end{pmatrix}$ | $\hat{v} = \begin{pmatrix} q^N & 0 \\ b & q^{-N} \end{pmatrix}$ |

Table 6. Ingredients for the double group construction

As described in Subsection 2.2 we also need a morphism $\sigma \in \text{Mor} \big( A \otimes \hat{A}, \hat{A} \otimes A \big)$. It is given, as before, by formula (2.2). The quantum group resulting from applying the double group construction is the quantum Lorentz group with Gauss decomposition. It was first defined and studied on $C^*$-algebra level by S.L. Woronowicz and S. Zakrzewski in [23].

4. Quantum ‘$ax + b$’ group

The section is devoted to a description of locally compact quantum groups that are related to classical group $G$ of affine transformations ‘$ax + b$’ of real line. This is one parameter family of deformations where the deformation parameter $q^2$ runs over an interval in the unit circle.

Such a quantum ‘$ax + b$’ groups for some special values of a deformation parameter was presented first in [24]. In our presentation we shall follow [8] where the more general family was constructed. The reader should be warned that despite the similarity of notation used in the further considerations to that used in construction of quantum ‘$az + b$’ groups the investigated objects have quite different properties.

Let us recall that on the classical level $G$ consists of all maps of the form $\mathbb{R} \ni x \mapsto ax + b \in \mathbb{R}$, where $a$ and $b$ are real parameters. Moreover we shall assume that $a > 0$. In what follows
by the same letters we shall denote also two unbounded real continuous functions on $G$ defined by assigning to any element of the group the corresponding values of the parameters. Then the functions $a$ and $b$ may be considered as elements affiliated with the $C^*$-algebra $C_0(G)$ of all continuous vanishing at infinity functions on $G$ and one can check that $C_0(G)$ is generated by $\log a$ and $b$:

$$C_0(G) = \{ f(\log a)g(b) : f, g \in C_0(\mathbb{R}) \} \text{ norm closed linear envelope}.$$ 

Now the group composition rule leads to the formulae describing the comultiplication:

$$\begin{align*}
\Delta(a) &= a \otimes a, \\
\Delta(b) &= a \otimes b + b \otimes I. \tag{4.1}
\end{align*}$$

For further generalizations it is important to note that $G$ equivalently can be realized as the group of unitary operators acting on a Hilbert space. More precisely, any affine transformation we identify with the unitary operator $V(a, b) \in B(L^2(\mathbb{R}))$:

$$[V(a, b)f](x) = a^{-1/2}f(a^{-1}(x - b))$$

where $f \in L^2(\mathbb{R})$. Impose the strong operator topology on the set of all such operators. Then this identification preserves the group structure and topology. Therefore $G$ may be identified with the set of unitary operators:

$$G = \{ V(a, b) : a, b \in \mathbb{R}; a > 0 \}. \tag{4.2}$$

Clearly

$$V(a_1, b_1)V(a_2, b_2) = V(a_1a_2, a_1b_2 + b_1). \tag{4.3}$$

Let $\mathcal{K}(H)$ denote the $C^*$-algebra of all compact operators acting on a Hilbert space $H$. It is known \[18\] that the strongly continuous family of unitaries \[12\] is described by a single unitary $V \in M(\mathcal{K}(L^2(\mathbb{R})) \otimes C_0(G))$. Moreover $C_0(G)$ is generated by $V$ and formula \[4.3\] means that

$$V(a \otimes I, b \otimes I)V(I \otimes a, I \otimes b) = V(a \otimes a, [a \otimes b + b \otimes I]). \tag{4.4}$$

Now using leg numbering notation and formula \[4.1\] we get

$$(\text{id} \otimes \Delta)V = V_{12}V_{13}.$$ 

For any $C^*$-algebra $A$ a unitary element $V \in M(\mathcal{K}(K) \otimes A)$ may be considered as a “strongly continuous” quantum family (labelled by the quantum space related to $A$) of unitary operators acting on the Hilbert space $K$. Now the above considerations lead to the notion of quantum group of unitary operators.

**Definition 4.1.** Let $A$ be a $C^*$-algebra, $K$ be a Hilbert space and let $V$ be a unitary element of $M(\mathcal{K}(K) \otimes A)$. Assume that

1. $A$ is generated by $V$.
2. $V$ is closed with respect to operator multiplication, i.e. there exists a morphism $\Delta \in \text{Mor} (A, A \otimes A)$ such that

$$(\text{id} \otimes \Delta)V = V_{12}V_{13}. \tag{4.5}$$

Then we say that $(A, V)$ is a quantum group of unitary operators whenever the pair $(A, \Delta)$ is a quantum group, i.e. $(A, \Delta)$ is related to some modular multiplicative unitary operator.

Let us note that by generating property any morphism $\Delta \in \text{Mor} (A, C)$ is determined by the value of $(\text{id} \otimes \Delta)$ on $V$. Therefore there is at most one $\Delta \in \text{Mor} (A, A \otimes A)$ satisfying \[4.5\]. On the other hand if $\Delta$ exists then it is co-associative. Indeed, $\Delta_1 = (\text{id} \otimes \Delta)\Delta$ and $\Delta_2 = (\Delta \otimes \text{id})\Delta$ are both elements of $\text{Mor} (A, A \otimes A \otimes A)$ and

$$(\text{id} \otimes \Delta_1)V = V_{12}V_{13}V_{14} = (\text{id} \otimes \Delta_2)V.$$ 

Therefore they coincide on $V$ and $\Delta_1 = \Delta_2$. This means that $(A, \Delta)$ is a $C^*$-bialgebra and $V$ is a co-representation. Now it remains to study whether $G = (A, \Delta)$ is a quantum group.
Using the concept of a quantum group of unitary operator we shall describe a quantum deformations of ‘$ax + b$’ group. To this end, at first functions $a$ and $b$ are replaced by a pair of non commuting selfadjoint operators $a = a^* > 0$ and $b = b^*$ such that

$$ab = q^2 ba,$$  \hspace{1cm} (4.6)

where $q^2$ is the deformation parameter. We assume that $q^2$ is a complex number of modulus 1. Clearly $a$ and $b$ are unbounded operators and the above formula is rather formal due to domain problems. The precise meaning of (4.6) is clarified by the so called Zakrzewski relation (cf. [20]).

**Definition 4.2.** Let $R$ and $S$ be selfadjoint operators acting on a Hilbert space $H$ and assume that $\ker R = \{0\}$. Let $R = (\text{sgn} R) |R|$ be the polar decomposition of $R$ and let $\hbar \in \mathbb{R}$. We say that $R$ and $S$ are in Zakrzewski relation, $R \simeq S$ whenever

1. $\text{sgn} R$ commutes with $S$,
2. $|R|^{i \lambda} S |R|^{-i \lambda} = e^{i \lambda} S$ for any $\lambda \in \mathbb{R}$.

Note that $\text{sgn} R$ in the above definition is unitary selfadjoint operator. It is known that if both operators have trivial kernels and $R \simeq S$ then [20] Example 3.1] the operators $e^{i \hbar/2 S^{-1}} R$ and $e^{i \hbar/2 SR^{-1}}$ are selfadjoint and

$$\text{sgn} \left( e^{i \hbar/2 S^{-1}} R \right) = \text{sgn} \left( e^{i \hbar/2 SR^{-1}} \right) = (\text{sgn} R)(\text{sgn} S).$$

Now the precise meaning of (4.6) is that $a \leadsto b$, i.e. for any $\lambda \in \mathbb{R}$

$$a^{i \lambda} ba^{-i \lambda} = e^{i \lambda} b,$$

where $\hbar$ is a real constant such that

$$q^2 = e^{-i \hbar}.$$  \hspace{1cm} (4.7)

Note that setting $\lambda = -i$ we get (4.6). In what follows we shall assume (mainly for technical reasons) that $0 < \hbar < \frac{\pi}{2}$.

Next we expect that formula for comultiplication $\Delta$ will be of the same form as (4.1). In particular $\Delta(a)$ and $\Delta(b)$ should be selfadjoint operators and $\Delta(a) \leadsto \Delta(b)$. But, in general, $a \otimes b + b \otimes I$ is a symmetric and not selfadjoint operator. Nevertheless, if it admits selfadjoint extensions, then a properly chosen one should be used in formula for $\Delta(b)$. One can look for such an extension in a class associated with reflection operators.

Let us recall that if $Q$ is a symmetric operator acting on a Hilbert space $H$ then any unitary selfadjoint operator $\rho$, i.e. $\rho^* = \rho$ and $\rho^2 = I$ is a reflection operator for $Q$ if $\rho$ and $Q$ anticommute. Let

$$[Q]_\rho = Q^* \left| \{ x \in \mathcal{D}(Q^*): (\rho - I)x \in \mathcal{D}(Q) \} \right..$$

Then it is known (cf. [20] Proposition 5.1) that $[Q]_\rho$ is a selfadjoint extension of $Q$.

Now the right hand side of the formula (4.1) for $\Delta(b)$ should be replaced by some selfadjoint extension of the form

$$\Delta(b) = [a \otimes b + b \otimes I]_\rho.$$

It turns out that to define the reflection operator $\rho$ one have to use an additional operator $\beta$ which is independent of $a$ and $b$: $\beta$ is a selfadjoint unitary commuting with $a$ and anticommuting with $b$. In particular this means that the algebra $A$ has to be enlarged, i.e. it is no longer generated by log $a$ and $b$.

Such an approach to the quantum ‘$ax + b$’ group using additional operator $\beta$ was presented in [23]. Its main result states that within this scheme the quantum group exists only for a very special values of the deformation parameter: $\hbar = \frac{\pi}{2k+1}$, $k = 0, 1, \ldots$. On the other hand considerations of [24] seemed to indicate that to remove the quantization of the deformation parameter a further enlargement of the $C^*$-algebra $A$ is required. In fact the $C^*$-algebra constructed in [24] admits $S^1$ as group of automorphisms and these automorphisms play a crucial role in a construction of the more general quantum ‘$ax + b$’ group. This was presented in [8]. It was shown that now the quantum ‘$ax + b$’ groups do exist for $\hbar$ running over an interval in $\mathbb{R}$.

In further considerations $s$ denotes a fixed element of $S^1$. This is a new deformation parameter and later on we describe its relation to $\hbar$. At the first step in the construction we describe a
$C^*$-algebra $A$. It depends on four operators, beside $a$ and $b$ it involves a reflection operator $\beta$ and a unitary operator $w$.

Let us consider the Hilbert space $L^2(\mathbb{R} \times S^1)$ and let

$$
\begin{align*}
(a^{i\tau}x)(t, z) &= e^{i\hbar \tau/2}x(e^{i\hbar \tau}t, z), \\
(bx)(t, z) &= tx(t, z), \\
(\beta x)(t, z) &= x(-t, z), \\
(wx)(t, z) &= s^{1(t<0)}x(t, z).
\end{align*}
$$

(4.8)

for any $\tau \in \mathbb{R}$ and any $x \in L^2(\mathbb{R} \times S^1)$. Therefore $a$ is the analytic generator of the one-parameter group of unitary operators corresponding to homothetic transformations of $\mathbb{R}$ and the multiplication operator $b$ is selfadjoint on its natural domain consisting of all $x$ such that $|tx(t, z)|$ is square integrable over $\mathbb{R} \times S^1$. Operators $\beta$ and $w$ are unitary and $\beta^* = \beta$. One can verify that

$$
\begin{align*}
a > 0 & \quad \text{and} \quad a \to b, \\
a\beta = \beta a & \quad \text{and} \quad b\beta = -\beta b, \\
waw = a, \quad w^*bw = b, \quad w^*\beta w = s^{\text{sgn} \beta}.
\end{align*}
$$

(4.9)

Note that $\text{Ad}_w$ is an automorphism of an algebra related to operators $a$, $b$ and $\beta$ due to the last equations of (4.9). Now one can prove (cf. [8, Theorem 4.1])

**Theorem 4.3.** Let

$$
A = \left\{ [f_1(b) + \beta f_2(b)]g(\log a) w^k : f_1, f_2, g \in C_0(\mathbb{R}) \atop f_2(0) = 0, \quad k \in \mathbb{Z} \right\}_{\text{norm closed linear envelope}}.
$$

(4.10)

Then

1. $A$ is a non degenerate $C^*$-algebra of operators acting on $L^2(\mathbb{R} \times S^1)$,
2. $\log a, b, ib\beta$ and $w$ are affiliated with $A$: $\log a, b, ib\beta, w\eta A$,
3. $\log a, b, ib\beta$ and $w$ generate $A$ (in the sense of [18, 17]).

At the second step of the construction, quantum ‘$ax + b$’ group is presented as a quantum group of unitary operators. To this end for some Hilbert space $K$, according to the Definition 4.1 one has to describe a unitary element $V \in M(K(K) \otimes A)$ satisfying corresponding conditions. Clearly $V$ should depend on operators $a$, $b$, $\beta$ and $w$ and possibly a special structure of $K$ is required.

At first we recall some special function. This is a modified version of the quantum exponential function introduced in [20]. For $h \in \mathbb{R}$ such that $0 < h < \frac{\pi}{2}$ let $G_h$ be the function defined for any $(r, \varrho) \in \mathbb{R} \times \{-1, 1\}$ by the formula

$$
G_h(r, \varrho) = \begin{cases} 
V_\varrho(\log r) & \text{for } r > 0 \\
[1 + i\varrho |r|^{1/2}] V_\varrho(\log |r| - \pi i) & \text{for } r < 0, \\
1 & \text{for } r = 0,
\end{cases}
$$

(4.11)

where $\theta = \frac{2\pi}{h}$ and $V_\varrho$ is the meromorphic function on $\mathbb{C}$ such that

$$
V_\varrho(z) = \exp \left\{ \frac{1}{2\pi i} \int_0^\infty \log(1 + t^{-\theta}) \frac{dt}{t + e^{-z}} \right\}
$$

for all $z \in \mathbb{C}$ such that $|3z| < \pi$.

It is known that $G_h$ is a continuous function which takes the values in the unit circle of the complex plane, $G_h \in C(\mathbb{R} \times \{-1, 1\}, S^1)$. Therefore for any pair of commuting selfadjoint operators $(T, \tau)$ and $\tau$ unitary acting on a Hilbert space $H$ the operator $G_h(T, \tau)$ makes sense by the functional calculus and is unitary. The function $G_h$ plays a key role in a theory involving operators $R$ and $S$ satisfying Zakrzewski relation. It has many interesting properties. We shall recall two of them.

Let $R$ and $S$ be selfadjoint operators acting on a Hilbert space $H$ such that $\ker R = \{0\} = \ker S$ and $R \sim R$. Let

$$
T = e^{i\hbar/2}S^{-1}R.
$$

Then $T$ is a selfadjoint operator with trivial kernel, $\text{sgn} T = (\text{sgn} R)(\text{sgn} S)$, $T \sim R$ and $T \sim S$. 


• Let $\tau \in B(H)$ be unitary and selfadjoint operator such that $R \tau = -\tau R$ and $S \tau = -\tau S$. Then
  
  (1) $T$ commutes with $\tau$.
  (2) $R + S$ is a closed symmetric operator, $\tau$ is a reflection operator for $R + S$ and the
  corresponding selfadjoint extension is unitary equivalent to $R$ and $S$:
  
  \[
  [R + S]_\tau = G_h(T, \tau)^*SG_h(T, \tau) = G_h(T^{-1}, \tau)RG_h(T^{-1}, \tau)^*.
  \]  
  (4.12)

• Let $\rho$ and $\sigma$ be unitary and selfadjoint operators on $H$ such that
  
  $\rho R = R\rho, \quad \rho S = -S\rho$ and $\sigma R = -R\sigma, \quad \sigma S = S\sigma$.

For $\alpha = i e^{\frac{2\pi}{\sqrt{\hbar}}}$ we set:
  
  $\tau = \alpha \rho \sigma \chi(S < 0) + \bar{\alpha} \sigma \rho \chi(S > 0)$.

Then
  
  (1) $\tau$ is unitary and selfadjoint operator, $\tau$ commutes with $T$ and $R \tau = -\tau R, \quad S \tau = -\tau S$.
  (2) $\tilde{\tau} := G_h(T, \tau)^*\sigma G_h(T, \tau)$ is unitary selfadjoint operator commuting with the selfadjoint extension $[R + S]_\tau$ corresponding to the reflection operator $\tau$.
  (3) $G_h$ satisfies an exponential type equality:
  
  \[
  G_h(R, \rho)G_h(S, \sigma) = G_h([R + S]_\tau, \tilde{\tau}) = G_h(T, \tau)^*G_h(S, \sigma)G_h(T, \tau).
  \]  
  (4.13)

Now we describe the relevant structure of the Hilbert space $K$. It is determined by a quadruple of selfadjoint operators $(\hat{a}, \hat{b}, \hat{\beta}, \hat{\tilde{L}})$ acting on $K$ and such that
  
  i) $\hat{a} > 0, \quad \ker \hat{a} = \{0\} = \ker \hat{\tilde{b}}$ and $\hat{a} \sim \hat{b}$,
  ii) $\hat{\beta}$ is a unitary and selfadjoint, $\hat{\beta} \hat{a} = \hat{a} \hat{\beta}$ and $\hat{\beta} \hat{b} = -\hat{b} \hat{\beta}$,
  iii) $\Sp \hat{L} \subset \mathbb{Z}$ and $\hat{\tilde{L}}$ strongly commutes with $\hat{a}$ and $\hat{b}$,
  iv) $\hat{\beta} \hat{\tilde{L}} \hat{\beta} = \hat{L} - \sgn \hat{b}$.

Using the above structure we have (cf. [8] Theorem 4.2 and 4.3])

**Theorem 4.4.** Let
  
  \[
  V = G_h(\hat{b} \otimes b, \hat{\beta} \otimes \beta)^* \exp \left( \frac{i}{\hbar} \log \hat{a} \otimes \log a \right) (I \otimes w)^{\hat{L} \otimes I}.
  \]  
  (4.15)

Then
  
  (1) $V$ is a unitary operator and $V \in M(\mathcal{K}(K) \otimes A)$.
  (2) $A$ is generated by $V \in M(\mathcal{K}(K) \otimes A)$.
  (3) There exists $\Delta \in \text{Mor} (A, A \otimes A)$ such that $(\id \otimes \Delta)V = V_{12}V_{13}$ if and only if
  
  $\hbar = \frac{\pi}{p}$, where $p \in \mathbb{R}, \quad p > 2$ and $e^{\pi p} = -s$.

We focus only on the last statement and sketch the main points of the proof.
  
  The basic idea is to find a unitary operator $W'$ acting on $K \otimes L^2(\mathbb{R} \times S^1) \otimes L^2(\mathbb{R} \times S^1)$ such that
  
  \[
  V_{12}V_{13} = W'V_{12}W'^*.
  \]  
  (4.16)

Let $L$ be an operator on $L^2(\mathbb{R} \times S^1)$ introduced by the formula
  
  \[
  (Lx)(t, z) = \frac{\partial}{\partial z} \chi(t, z).
  \]  
  (4.17)

Then $L$ is a selfadjoint operator such that $\Sp L \subset \mathbb{Z}$. Moreover it commutes with $a, b$ and $\beta$ and $w^*Lw = L + I$. In particular the last relation implies that
  
  \[
  (I \otimes w)^{L \otimes I}(w \otimes I)(I \otimes w)^{-L \otimes I} = w \otimes w.
  \]  
  (4.18)
For
\[ \alpha = i \exp \left( \frac{i \pi^2}{2\hbar} \right) \] (4.19)
we set
\[ T = I \otimes e^{i\theta/2}b^{-1}a \otimes b, \]
\[ \tau = (I \otimes \beta w^{-\text{sgn} b} \otimes \beta) \left[ \alpha s^{-1} \chi(b \otimes b \otimes I < 0) + \overline{\alpha} \chi(b \otimes b \otimes I > 0) \right]. \] (4.20)
Then \( T \) and \( \tau \) are selfadjoint operators, \( \tau \) is unitary and \( T \tau = \tau T \). Let
\[ W' = G_h(T, \tau)^* \exp \left( -\frac{i}{\hbar} I \otimes \log |b| \otimes \log a \right) (I \otimes I \otimes w)^{L \otimes I}. \] (4.21)
Then \( W' \) is a unitary operator and using properties of \( G_h \), relations (4.9) and (4.14) one can show that equation (4.16) is satisfied. Now assume that there exists \( \Delta \in \text{Mor} (A, A \otimes A) \) such that \( (id \otimes \Delta)V = V_{12}V_{13} \). We have to analyze a formula (cf. (4.16))
\[ (id \otimes \Delta)V = W'V_{12}W'^*. \] (4.22)
\[ C = K(K) \otimes A \text{ and let} \]
\[ \Delta_1(c) = (id \otimes \Delta)(c), \quad \Delta_2(c) = W'(c \otimes I)W'^*. \]
for any \( c \in C \). Then \( \Delta_1 \) and \( \Delta_2 \) are representations of \( C \) acting on the same Hilbert space \( K \otimes L^2(\mathbb{R} \times S^1) \otimes L^2(\mathbb{R} \times S^1) \) and \( \Delta_1(V) = \Delta_2(V) \). Using the generating property of \( V \) and definition of \( W' \) one can show that this implies
\[ \hat{b} \otimes \Delta(b) = W'(\hat{b} \otimes b \otimes I)W'^* = G_h(T, \tau)(\hat{b} \otimes b \otimes I)G_h(T, \tau)^*. \]
On the other hand using properties of \( G_h \) (cf. formula (4.12) with \( R = \hat{b} \otimes b \otimes I \) and \( S = \hat{a} \otimes I \otimes I \)) we get:
\[ \hat{b} \otimes \Delta(b) = \left[ \hat{b} \otimes a \otimes b + \hat{b} \otimes b \otimes I \right] \tau. \] (4.23)
Taking into account the sign of \( \hat{b} \) this implies in particular
\[ \Delta(b) = [a \otimes b + b \otimes I]_{\tau_+} = [a \otimes b + b \otimes I]_{\tau_-} \] (4.24)
where
\[ \tau_+ = (\beta w^{-\text{sgn} b} \otimes \beta) \left[ \alpha s^{-1} \chi(b \otimes I < 0) + \overline{\alpha} \chi(b \otimes I > 0) \right], \]
\[ \tau_- = (\beta w^{-\text{sgn} b} \otimes \beta) \left[ \alpha s^{-1} \chi(b \otimes I > 0) + \overline{\alpha} \chi(b \otimes I < 0) \right]. \]
Now one proves that in this case the reflection operator is determined by the selfadjoint extension, \( \tau_+ = \tau_- \). Therefore \( s = \alpha^2 \) and remembering that \( 0 < \hbar < \frac{\pi}{2} \), we obtain that \( h \) is of the form
\[ h = \frac{\pi}{p}, \] where \( p \in [2, \infty) \) and \( e^{i\pi p} = -s \).
Conversely, assuming that \( h \) is of such a form we check that \( \alpha s^{-1} = \overline{s}, \tau = I \otimes \overline{\beta} w^{-\text{sgn} b} \otimes \beta = I \otimes \alpha w^{\text{sgn} b} \otimes \beta \) and (cf. (4.21)) \( W' = W_{23} = I \otimes W \), where
\[ W = G_h \left( e^{i\theta/2}b^{-1}a \otimes b, \alpha w^{\text{sgn} b} \otimes \beta \right)^* \exp \left( -\frac{i}{\hbar} \log |b| \otimes \log a \right) (I \otimes w)^{L \otimes I}. \] (4.26)
Therefore (cf. (4.16))
\[ V_{12}V_{13} = W_{23}V_{12}W_{23}^*. \] (4.27)
Now let
\[ \Delta(c) = W(c \otimes I)W'^*. \] (4.28)
for any \( c \in A \). Clearly \( \Delta \) is a representation of \( A \) acting on \( L^2(\mathbb{R} \times S^1) \otimes L^2(\mathbb{R} \times S^1) \) Remembering that \( V \in M(K(K) \otimes A) \) we have
\[ (id \otimes \Delta)V = V_{12}V_{13}. \]
Since the right hand side of the above formula belongs to \( M(K(K) \otimes A \otimes A) \), the operator \( (id \otimes \Delta)V \in M(K(K) \otimes A \otimes A) \) and \( \Delta \in \text{Mor} (A, A \otimes A) \) due to the fact that \( A \) is generated by \( V \).
Let us note that formula (4.28) applies to any element affiliated with $A$ as well. Then for generators of $A$ one obtains

$$\Delta(a) = a \otimes a,$$
$$\Delta(b) = [a \otimes b + b \otimes I]_{\alpha w^{\text{sgn} b} \beta},$$
$$\Delta(\beta |b|^p) = [(w \otimes I)^{-I} \otimes \text{sgn} b (a^p \otimes b |b|^p) + \beta |b|^p \otimes I]_{-\text{sgn}(b \otimes b)},$$
$$\Delta(w) = w \otimes w.$$  

(4.29)

This way we have constructed the $C^*$-bialgebra $(A, \Delta)$. To prove that this is a quantum group one has to look for the multiplicative unitary. Let us observe that the possible choice for $(\hat{a}, \hat{b}, \hat{\beta}, \hat{L})$ is $K = L^2(\mathbb{R} \times S^1)$ and for $\alpha \in S^1$ such that $\alpha^2 = s$:

$$(\hat{a}, \hat{b}, \hat{\beta}, \hat{L}) = (|b|^{-1}, e^{i\theta/2}b^{-1}a, \alpha w^{\text{sgn} b} \beta, L).$$  

(4.30)

Then all properties (1.13) are satisfied and in this case operators $V$ and $W$ coincide, $V = W$ (cf. formulae (4.15) and (4.26)) and by (4.27) operator $W$ satisfies the pentagon equation:

$$W_{12}W_{13} = W_{23}W_{12}^*.$$  

In fact [8] Theorem 5.2):

**Theorem 4.5.** $W$ is a modular multiplicative unitary operator acting on $L^2(\mathbb{R} \times S^1) \otimes L^2(\mathbb{R} \times S^1)$.

We conclude the section with a comment on the “size” of the group $'ax+b'$. This notion reflects the fact that the construction of the $C^*$-algebra $A$ for quantum $'ax+b'$ group besides operators $a$ and $b$ involves additional operators such as $\beta$ and $w$. Let us consider any representation $\pi$ of $A$ such that

1. $\ker \pi(b) = \{0\}$,
2. $\pi$ is faithful.

Then due to Zakrzewski relation operators $\log \pi(a)$ and $\log \pi(|b|)$ satisfy canonical commutation relations. Therefore by Stone-von Neumann theorem $\pi$ is a multiple $m_\pi$ of the unique irreducible representation of such relations. By definition, the size of $'ax+b'$ is the smallest possible $m_\pi$. As a result the quantum $'ax+b'$ groups constructed above are of infinite size. Nevertheless for the special cases of deformation parameter $q^2 = e^{-i\theta}$ being a root of unity one may pass to the groups with a smaller size. If this is the case then the parameter $s' = \alpha = -e^{2\pi i}q$ is the root of unity as well. Assume that $N$ is the smallest number such that $s^N = 1$. Then using (4.19) one can verify that $w^N$ is in the center of $A$. Let $C_N$ denote the closed ideal in $A$ generated by the relation $w^N - 1 = 0$ and $A_N = A/C_N$ be the quotient $C^*$-algebra. Then the canonical map $\pi$ is a morphism, $\pi \in \text{Mor} (A, A_N)$ and there exists $\Delta_N \in \text{Mor} (A_N, A_N \otimes A_N)$ such that

$$\Delta_N(\pi(c)) = (\pi \otimes \pi)\Delta(c)$$

for any $c \in A$. Now quantum $'ax+b'$ group at roots of unity may be described as $(A_N, \Delta_N)$. One can prove that its size is $2N$.

The minimal value of the size is 2. Let us note that the old quantum $'ax+b'$ groups described in [24] are of size 2 and it is known that they are the only ones with size 2.

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