Efficient detection of inhomogeneous magnetic fields from the single spin with Dicke states

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The efficient detection of a single spin is a significant goal of improving the sensitivity of quantum magnetic fields sensors. Recent results show that a specific type of entanglement such as Greenberger-Horne-Zeilinger (GHZ) states can be used as a resource to improve the performance of the single spin detection. However, scalable generation of the GHZ states is experimentally difficult to realize. It is desirable to use a practical entangled state that can be easily generated. In this paper, we propose the efficient detection of a single spin with Dicke states. We show a way to prepare and measure the Dicke states via a global control. Moreover, we investigate how dephasing due to the unwanted coupling with environment affects the performance of our proposal, and show that single spin detection with Dicke states with dephasing has a significant advantage over the classical strategy with separable states. Our results are important toward realizing entanglement enhanced single-spin detection.

I. INTRODUCTION

A great deal of effort has long been devoted to improving the accuracy of the measurement for weak magnetic fields and many types of magnetic sensors have been developed so far [1]. The precise measurement is not only fundamentally interesting (as it is related to the exploring the ultimate precision allowed by quantum mechanics) but also is important for practical applications in various fields of study such as condensed matter physics, material science, and life sciences [2]. Particularly, the efficient detection of the single (electron or nuclear) spin [3–18] is an extremely important task and also one of the ultimate goals in quantum metrology. However, the magnetic fields from the single spin is weak, and a long repetition time is required to detect in the current technology. Therefore, it is essential to improve the sensitivity of the magnetic fields for more rapid detection of the single spin.

It is known that entanglement can be a resource to achieve the sensitivity about homogeneous magnetic fields beyond the standard quantum limit (SQL) [19–31]. Especially, Greenberger-Horne-Zeilinger (GHZ) state (which is also called a cat state) achieves the highest sensitivity without any noise, which is determined by the Heisenberg uncertainty relation, so-called Heisenberg limit. Recently, it was shown that the GHZ states can overcome the SQL even under the effect of the realistic decoherence [24, 25]. In addition to the case of the homogeneous magnetic fields, the GHZ states are also useful for the detection of the inhomogeneous magnetic fields from single spin [32]. Due to the dipole-dipole interaction between target single spin and probe spins, the magnetic fields are inversely proportional to the cubic of the distance from the target single spin, and therefore the magnetic fields affected by probe spins are quite different from the homogeneous magnetic fields. Despite this great difference, the GHZ states can also detect single spin efficiently.

However, it is known that the accurate control and measurement of the GHZ state is experimentally difficult to be realized, because it typically requires accurate individual control of the qubits [33–35]. To achieve a high sensitivity much better than the SQL, a large entangled state is necessary, which might be difficult to obtain as long as we use the GHZ states. Toward realizing practical entanglement-based quantum sensors, it is essential to use an entangled state that can be measured just by a global control.

Here, we propose the single spin detection by using Dicke states [36–63] that can be created and measured by a global and deterministic control. The Dicke states are related to the well-known cooperative phenomenon, superradiance that has been discussed for a long time [38–42]. The Dicke states are also highly entangled states, known to be a resource to measure spatially homogeneous magnetic fields with an accuracy beyond the SQL without decoherence [44–49]. In this paper, we show that Dicke states are also useful resources to detect the inhomogeneous magnetic fields from the single spin even under the effect of dephasing. Although the Dicke states are created in various methods of previous experiments [51–55] or theories [56–62] such as by continuous measurement [56] and quantum algorithm implemented as a quantum circuit [61–63], we propose a scheme to create and measure the Dicke states by a global and deterministic control. To implement our protocol for the spin detection, the necessary number of the operational steps are constant against the size of the Dicke states, and so our scheme is efficient in terms of scalability.

The remainder of this paper is organized as follows. In section II, we describe the set-up of our scheme for the single spin detection with Dicke states. In section III, we show our results about the sensitivity of the single spin detection with Dicke states under the effect of dephasing. In the section IV, we describe our proposal to create and measure Dicke states via a global control. Finally, we summarize and conclude our paper in section V.

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II. SINGLE SPIN DETECTION WITH DICKE STATES

In this section, we explain the details of our proposal for single-spin detection with Dicke states. Especially, we describe the Hamiltonian, decoherence model, and our measurement basis. We consider a single target spin and an ensemble of probe $L$ spins. For simplicity, we assume $L$ is an even number through the paper. As shown in Fig. 1 (a) and (b), the target spin is located in the origin of the coordinate, and the probe spins are uniformly distributed inside a columnar substrate with the spin density of $\rho$. Each probe spin is located at $r_j = (x_j, y_j, z_j)$. Since there is the dipole-dipole interaction between the target spin and the probe spins, the Hamiltonian of the total system is given by

$$\hat{H} = \hat{H}_T + \hat{H}_P + \hat{H}_D,$$

$$\hat{H}_T = \frac{\omega^{(T)}}{2} \hat{\sigma}_z^{(T)}, \quad \hat{H}_P = \sum_{j=1}^{L} \frac{\omega^{(P)}}{2} \hat{\sigma}_z^{(P)} r_{j},$$

$$\hat{H}_D = G \sum_{j=1}^{L} \left( \hat{\sigma}_z^{(T)} \cdot \hat{\sigma}_z^{(P)} + 3 \left( \frac{\hat{\sigma}_z^{(T)} \cdot \hat{r}_{j}}{|\hat{r}_{j}|^3} \right) \left( \frac{\hat{\sigma}_z^{(P)} \cdot \hat{r}_{j}}{|\hat{r}_{j}|^3} \right) \right),$$

where $\omega^{(T)}$ ($\omega^{(P)}$) is a Zeeman energy of the target (probe) spin, $G = \frac{\omega^{(T)} \omega^{(P)}}{4s}$ (here, we choose $\hbar = 1$) is a constant determined by the magnetic moments of the target spins $\gamma^{(T)}$ and the probe spins $\gamma^{(P)}$, and $\hat{\sigma}_z^{(P)} = (\hat{\sigma}_{x,j}, \hat{\sigma}_{y,j}, \hat{\sigma}_{z,j})$ is a set of the Pauli matrices of the probe spins at $\vec{r}_j = (x_j, y_j, z_j)$, and expresses the inner product. Here, we assume a large detuning between the target spin and the probe spins $\omega^{(P)} \gg \omega^{(T)}$. In the rotating frame and under the rotating wave approximation, we can remove the terms oscillating with $\omega^{(T)}$ and $\omega^{(P)}$, and therefore the Hamiltonian in Eq. (3) effectively has only the Ising type interaction

$$\hat{H}^{(\text{eff})} = G \sum_{j=1}^{L} \frac{x_j^2 + y_j^2 - 2z_j^2}{(x_j^2 + y_j^2 + z_j^2)^{3/2}} \hat{\sigma}_z^{(T)} \hat{\sigma}_z^{(P)}. \quad (4)$$

In this paper, we consider a case either the target spin is up or down, and so we replace $\hat{\sigma}_z^{(T)}$ in Eq. (4) with a classical value $s = 1$ or $-1$;

$$\hat{H}_s^{(\text{eff})} = \sum_{j=1}^{L} \frac{\omega_s(r_j, z_j)}{2} \hat{\sigma}_z^{(P)},$$

$$\omega_s(r_j, z_j) = 2Gs \frac{r_j^2 - 2z_j^2}{(r_j^2 + z_j^2)^{5/2}}, \quad (6)$$

where we use the cylindrical coordinates $r = \sqrt{x^2 + y^2}$ because of the rotational symmetry along $z$ axis. $\omega_s(r_j, z_j)$ denotes the inhomogeneous magnetic fields from the target spin, and we show the $r$ and $z$ dependence of $\omega_s(r_j, z_j)/(2Gs)$ in the Fig. 1 (c). This graph shows that the $\omega_s(r, z)/(2Gs)$ decreases as the distance from the origin increases. We will estimate the parameter $s$ through the results of the readout using the probe spins with the Dicke state along $x$ axis:

$$|D^{L/2}_L\rangle_x = \left( \frac{L}{L/2} \right)^{-1/2} \sum_{\text{all permutations}} \left( |++\cdots+--\cdots\rangle \right), \quad (7)$$

where the $\sum_{\text{all permutations}}$ represents all permutations of the spins and $|\pm\rangle$ are the eigenstates of $\sigma_x$. For example, when $L = 4$, we have $|D^{L/2}_L\rangle_x = \frac{1}{\sqrt{6}}(|+++\rangle + |---\rangle + |++-\rangle + |--+\rangle + |+-+\rangle + |-+-\rangle + |---\rangle + |++-\rangle)$. The probe spins with this state are exposed to the inhomogeneous magnetic fields described by the Hamiltonian Eq. (5). Here, we assume the non-Markov dephasing model, which is one of the most typical decoherence in the solid state systems [64-69]. The dynamics of the probe state under the effect of such a dephasing is given by the following master equation:

$$\frac{d}{dt} \hat{\rho}(t) = i [\hat{\rho}(t), \hat{H}_s^{(\text{eff})}] - \frac{t}{T_2^*} \sum_{j=1}^{L} \left( \hat{\rho}(t) - \hat{\sigma}_z^{(T)} \hat{\rho}(t) \hat{\sigma}_z^{(P)} \right), \quad (8)$$

where $\hat{\rho}(t)$ is the density operator at time $t$ and $T_2^*$ denotes the time of free induction decay. The first term of the right-hand side in Eq. (8) describes the interaction with the inhomogeneous magnetic fields from the target spin and the second term describes the decoherence.

We describe the measurement sequence. Firstly, prepare an initial state of the probe spins Eq. (7). Secondly, let the quantum state evolve according to the master equation Eq. (8) for a time $t$. Thirdly, measure the quantum state by a specific readout basis

$$|SQ\text{read}\rangle = \frac{1}{\sqrt{2}} \left[ |D^{L/2}_{L/2}\rangle_x + i |D^{L/2}_{L/2+1}\rangle_x \right], \quad (9)$$
where \( |D^L_{L/2+1}\rangle_x \) are also the Dicke states defined as
\[
|D^L_{L/2+1}\rangle_x = (L/2+1)^{-1/2} \sum_{\text{all permutations}} (|+\cdots + +\cdots +\rangle)_{L/2+1} (|+\cdots + -\cdots -\rangle)_{L/2-1}
\]
(\( \sum_{\text{all permutations}} \) represents all permutations of the spins). \( |\text{SQread}\rangle \) represents the superposition of two Dicke states and is a kind of spin squeezed states \([70]\) (a similar state analyzed in \([50]\)). Finally, repeat 1-3 steps \( N \) times. We assume that the preparation time of the initial state and the readout time is negligibly small, and we can approximately obtain \( N \approx T/t \) where \( T \) is a given total measurement time.

III. CALCULATION OF THE SENSITIVITY

In this section, we show our results about the sensitivity to detect single spin with the Dicke states. We will explain the outline of the calculation of our results in the text and show the details of that derivations in Appendices A and B.

According to the prescription described above, we prepare an initial state \( \hat{\rho}(0) = |D^L_{L/2+1}\rangle_x \langle D^L_{L/2+1}| \) let this state evolve, and measure the state by the basis of \(|\text{SQread}\rangle\), which provides us with a probability
\[
p = \langle \text{SQread} | \hat{\rho}(t) | \text{SQread} \rangle.
\]
The exact form of the \( p \) is described in Appendix A. In order to estimate the uncertainty of the estimation of \( s \) from the \( N \) measurement values, we calculate the following
\[
\delta s^{(\text{Dicke})} = \frac{\sqrt{p(1-p)}}{\sqrt{N}},
\]
where \( \sqrt{p(1-p)} \) is the standard deviation of \( p \). In order to distinguish whether the target spin is up or down, \( \delta s^{(\text{Dicke})} \) should be smaller than 1. We will minimize \( \delta s^{(\text{Dicke})} \) by optimizing \( t \) and the form of probe spins. To rescale the time \( t \), we set \( t = uT_* / \sqrt{L} \) where \( u \) denotes a dimensionless parameter. Throughout this paper, we only consider the limit of the large \( L \) and small \( G \). In this assumption, we obtain
\[
\delta s^{(\text{Dicke})} = \frac{F(u) \times L^{1/4}}{\sqrt{TT_*}} \sum_j \frac{\partial \omega_s(r_j, z_j)}{\partial s},
\]
Here, the explicit form of \( F(u) \) is shown in Eq. (A4) and the derivation of \( F(u) \) is shown in Appendix A. Moreover, we calculate the \( \sum_j \frac{\partial \omega_s(r_j, z_j)}{\partial s} \) to take a continuous limit about the sum of the probe spins:
\[
2Gp \int \int dx dy dz \frac{r^2 z^2}{(p+z^2)^{3/2}} = 4\pi Gp \frac{\sqrt{r_{\text{max}}}}{\sqrt{\sqrt{r_{\text{max}}^2 + z_{\text{max}}^2}}} - \frac{\sqrt{r_{\text{min}}}}{\sqrt{\sqrt{r_{\text{min}}^2 + z_{\text{min}}^2}}}.
\]
This approximation is justified as \( r_{\text{max}} \approx \sqrt{z_{\text{min}}} \gg \rho^{-1/3} \), where \( \rho^{-1/3} \) is the average distance among each probe spin. Finally, we optimize the form of the columnar substrate (that is, the number of the probe spins). Using \( L = \rho \pi r_{\text{max}}^2 \langle z_{\text{max}} - z_{\text{min}} \rangle \), we can obtain
\[
\delta s^{(\text{Dicke})} = \frac{F(u_{\text{min}}) \times f_{\text{min}}(r_{\text{max}}, z_{\text{max}})}{4G^3/4 \times \sqrt{TT_*}} \frac{3^{3/4}}{\rho^{3/4}},
\]
where
\[
f(r_{\text{max}}, z_{\text{max}}) = \left[ \frac{1}{2} \left( \frac{r_{\text{max}}^2 z_{\text{max}} - 1}{r_{\text{max}}^2} \right) \right]^{1/4} \times \left( \frac{1}{\sqrt{r_{\text{max}}^2 + z_{\text{max}}^2}} - \frac{1}{\sqrt{r_{\text{min}}^2 + z_{\text{min}}^2}} \right)^{-1},
\]
and \( r_{\text{max}}, z_{\text{max}} \) are the normalized parameters. \( r_{\text{max}} = r_{\text{max}}/z_{\text{max}}, z_{\text{max}} = z_{\text{max}}/z_{\text{min}} \). As a comparison, the explicit form of the single spin detection with separable states is given as follows \([32, 71]\):
\[
\delta s_{\text{min}} = \frac{\sqrt{2}e^{1/4} \times g_{\text{min}}(r_{\text{max}}, z_{\text{max}})^{3/2}}{4G^3/4 \times \sqrt{TT_*}} \frac{\rho}{\sqrt{\rho^2 + 1}},
\]
where
\[
g(r_{\text{max}}, z_{\text{max}}) = \left[ \frac{1}{2} \left( \frac{r_{\text{max}}^2 z_{\text{max}} - 1}{r_{\text{max}}^2} \right) \right]^{1/2} \times \left( \frac{1}{\sqrt{r_{\text{max}}^2 + z_{\text{max}}^2}} - \frac{1}{\sqrt{r_{\text{min}}^2 + z_{\text{min}}^2}} \right)^{-1}.
\]
According to \([32, 71]\), this was numerically minimized \( g_{\text{min}}(r_{\text{max}}, z_{\text{max}}) = 5.32 \) as \( r_{\text{max}} = 0.928, z_{\text{max}} = 1.89 \). We can see that the scaling with \( \rho \) and \( z_{\text{min}} \) for the entanglement scheme is different from that of the separable scheme.

In our expression of the uncertainty of the estimation of \( s \), we need to minimize the functions of \( f(r_{\text{max}}, z_{\text{max}}) \) and \( F(u) \). Importantly, the form of \( f(r_{\text{max}}, z_{\text{max}}) \) has been determined by the choice of the interaction time \( t = uT_* / \sqrt{L} \) and the shape of the columnar substrate. In the previous results about the single spin detection with the GHZ states \([32]\), there was the same form as \( f(r_{\text{max}}, z_{\text{max}}) \) in the sensitivity, and this was numerically minimized \( f_{\text{min}}(r_{\text{max}}, z_{\text{max}}) = 4.14 \) as \( r_{\text{max}} = 1.87, z_{\text{max}} = 4.30 \). We adopt the same minimization for our spin detection with the Dicke states, and we obtain the number of the probe spins \( L = \rho \pi r_{\text{max}}^2 (z_{\text{max}} - 1) \rho^{3/2} = 35.9 \times \rho^{3/2} \).

On the other hand, we have derived \( F(u) \) after we fix the initial state, the decoherence model, and the readout basis. Since \( F(u) \) only depends on \( u \), we can easily minimize it by a numerical method, and we obtain \( F(u_{\text{min}}) = 3.35 \) when \( u_{\text{min}} = 0.357 \). It is worth mentioning that, if we replace \( F(u) \) with \( \sqrt{2}e^{1/4} = 1.82 \) in the expression \( \delta s \), we obtain the uncertainty of \( s \) when we use the GHZ state for the probe \([32]\). This means that, even if we use the Dicke states that is experimentally feasible to realize, we can obtain the sensitivity comparable with the GHZ states that are typically considered as the best resource for quantum metrology.

To evaluate the performance of the single spin detection with the Dicke states, we will show the numerical result using realistic parameters. We consider the nitrogen-vacancy centers in diamond \([3–16, 72–78]\) as probe spins. According to the previous experiment \([78]\), \( T_2 \) has a linear relation with \( \rho^{-1} \), and the experimental value is \( \rho = (1.98 \times 10^{12} \text{cm}^{-3} \text{s})/T_2 \) \((10^2 \text{cm}^{-3} \leq \rho \leq 10^8 \text{cm}^{-3})\). By taking into account of the relation between \( T_2 \) and \( \rho \), we investigate how the sensitivity of the single spin detection changes by varying \( \rho \). It is worth mentioning that, as the total measurement time \( T \) increases, \( \delta s \) decreases. To quantify the performance of the single spin detection, we define the necessary measurement time \( T_0 \) such that \( \delta s = 1 \) should be satisfied. If \( T_0 \) is smaller, we can detect
IV. CREATION AND MEASUREMENT OF THE DICKE STATES BY A GLOBAL AND DETERMINISTIC CONTROL

In this subsection, we explain how to create and measure the Dicke states by a global and deterministic control. Firstly, we consider the Hamiltonian of an ancillary qubit collectively coupled with many probe spins

\[ \hat{H}_{SS} = \hat{H}_p + \hat{H}_A + \hat{H}_{TR} \]

(15)

\[ \hat{H}_p = \omega^{(P)} \hat{J}_z, \quad \hat{H}_A = \frac{\omega^{(A)}}{2} \hat{\sigma}_z^{(A)}, \]

(16)

\[ \hat{H}_{TR} = \lambda \sum_j^{L} \left( \hat{\sigma}_z^{(A)} \hat{\sigma}_x^{(P)} + \hat{\sigma}_z^{(A)} \hat{\sigma}_x^{(P)} \right), \]

(17)

where \( \omega^{(A)} \) and \( \hat{\sigma}_z^{(A)} \) are the resonant frequency and the Pauli \( Z \) operator of the ancillary qubit, \( \lambda \) denotes the transverse coupling strength between the ancillary qubit and the probe spins, and \( \hat{J}_z^{(P)} = \sum_{\ell=1}^{L} \hat{\sigma}_z^{(P)} \). \( \hat{H}_{SS} \) is so-called the spin star model \([79, 80]\). Moreover, we add the driving terms so as to perform the pulse operation

\[ \hat{H}_d = \lambda_d \hat{\sigma}_x^{(A)} \cos \omega^{(d)} t \]

(18)

\[ \hat{H}_d^{(P)} = \lambda_d^{(P)} \hat{J}_z^{(P)} \cos \omega^{(d')}(t) \]

(19)

where \( \omega^{(d)} \) (\( \omega^{(d')} \)) denotes the frequency of driving fields for the ancillary qubit (probe spins), \( \hat{J}_z^{(P)} = \sum_{\ell=1}^{L} \hat{\sigma}_z^{(P)} / 2 \) denotes the summation of the Pauli operators, and \( \lambda_d \) (\( \lambda_d^{(P)} \)) denotes the Rabi frequency for the ancillary qubit (probe spins). We assume \( \omega^{(A)} \gg \lambda^{(d)} \) and \( \omega^{(P)} \gg \lambda^{(d')} \). Also, we assume that we can turn on and off these Rabi frequencies. In our scheme, when we drive the ancillary spins (probe spin) by setting a finite value of \( \lambda_d (\lambda_d^{(P)}) \), we turn of the driving of the probe spin (ancillary qubit) by setting \( \lambda_d^{(P)} = 0 \) (\( \lambda_d = 0 \)). We define that, if \( \lambda_d \) or \( \lambda_d^{(P)} \) is much larger (smaller) than \( \lambda \), we call it a hard (soft) pulse. Intuitively, when we perform the hard pulses, the effect of the coupling between the ancillary qubit and probe spins is negligible during the pulse operations. It is known that this type of Hamiltonian was experimentally realized by a hybrid system composed of a superconducting qubit coupled with an electron spin ensemble in diamond \([81–84]\). By using this Hamiltonian, we will show how to prepare the initial state of \( |D_{L/2}^L \rangle_x \) and to readout the state with the basis of \( |SQ_{\text{read}} \rangle \).

1. Preparation of the initial state \( |D_{L/2}^L \rangle_x \)

We show how to prepare the state \( |D_{L/2}^L \rangle_x \). The basic idea of our protocol is to repeat an energy transfer from the ancillary qubit to the probe spins with the flip-flop interaction. Using the Dicke states, \( \hat{H}_{SS} \) can be easily diagonalized. Particularly when the detuning is zero \( (\omega^{(A)} = \omega^{(P)}) \), the energy eigenvalues

FIG. 2. (color online) This graph shows the necessary measurement time \( T_s \) (s) against \( \rho \) (cm\(^{-3}\)) and \( z_{\text{min}} \) for the case of the Dicke states of the probe spins. We assume that the target spin is an electron spin.

FIG. 3. (color online) This graph shows the necessary measurement time \( T_s \) (s) against \( \rho \) (cm\(^{-3}\)) and \( z_{\text{min}} \) for the case of the separable states of the probe spins.

the target single spin for shorter measurement time, which is considered as a more efficient single-spin detection scheme. Fig. 2 and 3 show the detection time \( T_s \) against \( \rho \) for the case of the Dicke states and the separable states \([32, 71]\). From this graph, \( T_s \) with the Dicke states becomes smaller as \( \rho \) increases with \( z_{\text{min}} \) fixed, because \( T_s \propto \rho^{-1/2} z_{\text{min}}^{-3/2} \) from Eq. (13). On the other hand, \( T_s \) with the separable states does not change because \( \delta_{\text{sep}} \) in Eq. (14) depends only on \( \rho T_s^2 \) and therefore \( T_s \propto \rho^{1/3} z_{\text{min}}^{-1} \).
1. Preparation of the initial state $|D_{L/2}⟩_x$

\[
\begin{array}{c}
\text{Ancilla} \quad \omega^{(A)} = \omega^{(P)} \\
\text{Probe} \quad \omega^{(A)} = \omega^{(P)} = \omega^{(d)}
\end{array}
\]

\[
\begin{array}{cccc}
\pi & \pi & \pi & \pi \\
\text{t}_L/2+1 & \text{t}_L/2+2 & \cdots & \text{t}_0
\end{array}
\]

\[
\omega^{(A)} = \omega^{(P)} = \omega^{(d)}
\]

FIG. 4. (color online) A schematic of the pulse sequence of the preparation of $|D_{L/2}⟩_x$. First, we perform a hard $\pi$ pulse to the ancillary qubit. Second, let the system evolve by the Hamiltonian $H_{SS}$. Third, we repeat the first and second process $\times$ times. Finally, we perform hard $\pi/2$ pulse along $y$ axis into the probe spin ensemble.

and eigenstates are given by

\[
E_{n,x} = \left(n - \frac{1}{2}\right) \frac{\omega^{(P)}}{2} + \frac{1}{2} \mu_n \quad (-L/2 < n < L/2),
\]

\[
\mu_n = 2\sqrt{\frac{L}{2}}(L/2 + 1 - n)(n - 1),
\]

\[
|E_{n,x}\rangle = \frac{1}{\sqrt{2^n}} \left( |1⟩ \left| D_{L-1}^L⟩_z + |0⟩ \left| D_{L+1}^L⟩_z \right) \right.
\]

\[
E_{-L/2} = -\frac{L}{2} + \frac{1}{2} \omega^{(P)}, \quad E_{L/2} = L + \frac{1}{2} \omega^{(P)},
\]

\[
|E_{-L/2}\rangle = |0⟩ \langle D_{L}^L|, \quad |E_{L/2+1}\rangle = |1⟩ \langle D_{L}^L|,
\]

where $|D_{n}^L⟩_z$ is the Dicke state along $z$ axis $|D_{n}^L⟩_z = \frac{1}{\sqrt{2^n}} \sum_\text{all permutations} \left( |1⟩ \cdots |1⟩ \right)$. Here, $n$ is the eigenvalue of $\hat{J}_z^{(P)}$ for $-L/2 < n \leq L/2$.

Fig. 4 shows the pulse sequence of the preparation of $|D_{L/2}⟩_x$. Firstly, we prepare an initial state

\[
|E_{-L/2}\rangle = |0⟩ \langle D_{L}^L|.
\]

Secondly, we excite the ancillary qubit by a hard $\pi$ pulse

\[
|1⟩ \langle D_{0}^L| = e^{-\frac{i\pi}{2}n} |0⟩ \langle D_{0}^L|,
\]

which can be realized by turning on $\lambda^{(d)}$ and choosing $\omega^{(d)} = \omega^{(A)}$. Thirdly, let the system evolve by the Hamiltonian $H_{SS}$ for a certain time until the excitation of the ancillary qubit is completely transferred to the spin ensemble, and we obtain

\[
|0⟩ \langle D_{1}^L| = \exp[-i\hat{H}'t_{m}] |0⟩ \langle D_{0}^L|.
\]

(The interaction time is $t_{m} = \pi/(E_{-L/2+1} + E_{L/2+1} - E_{-L/2+1} - E_{L/2+1})$. In this case.) Fourthly, repeat the step 2 and 3 by changing the evolution time $t_{m}$ for the energy excitation transfer,

\[
|0⟩ \langle D_{L/2}^L| = \prod_{m=0}^{L/2-1} \left[ \exp[-i\hat{H}'t_{m}] e^{-i\pi \sigma_z^{(A)/2}} \right] |0⟩ \langle D_{0}^L|.
\]

where $t_{m} = \pi/(E_{-m+1} + E_{-m+1}) = \pi/\mu_{-m+1}$. We repeat these $L/2$ times. Finally, by turning on $\lambda_p$ in order to perform hard $\pi/2$ pulse along $y$ axis into the probe spin ensemble, we obtain

\[
|0⟩ \langle D_{L/2}^L| = e^{-i\pi \sigma_z^{(P)/2}} |0⟩ \langle D_{0}^L|.
\]

2. Readout by $|SQ\rangle$

We show how to readout the state with the basis of $|SQ\rangle$. If we can construct a unitary operator $U_{SQ}(|\bar{0}\rangle) = U_{SQ}(|\bar{0}\rangle) |D_{L}^L⟩_z$, the expectation value $p$ (already defined by Eq. (10)) can be rewritten by

\[
p = \langle D_{L}^L| \hat{P}(t) U_{SQ} |D_{0}^L⟩_z.
\]

This means that the combination of the inverse operation $U_{SQ}^\dagger$ and the global projection measurement to all-down state $|D_{L}^L⟩_z = \frac{1}{\sqrt{2^n}} \sum_\text{all permutations} \left( |1⟩ \cdots |1⟩ \right)$ provides us with a way to obtain the value of $p$, which is the probability to measure the state with the basis of $|SQ\rangle$. So we consider how to construct $U_{SQ}^\dagger$.

The construction of $U_{SQ}^\dagger$ is as follows. The basic idea is to use $|0⟩ \langle D_{L}^L|_z$ and $<0| \langle D_{L}^L|_z$ as an effective qubit due to the frequency selectivity. If we choose the large detuning $\omega^{(A)} \gg \omega^{(P)}$ and $\lambda$, we can obtain the effective Hamiltonian as follows

\[
\hat{H}_{SS}^{(eff)} = \hat{H}_{P} + \hat{H}_{A} - \frac{\lambda^2}{\omega^{(A)} - \omega^{(P)}} \sigma_z^{(A)} \left( J_z^{(P)} \right)^2.
\]

Here, in the large detuning case $\omega^{(A)} \gg \omega^{(P)}$ and $\lambda$, the energy eigenstates of $\hat{H}_{SS}^{(eff)}$ are expressed by the separable states of $|0⟩ \langle 1|$ and the Dicke states, such as $|0⟩ \langle D_{L}^L|_z$ or $|0⟩ \langle D_{L}^L|_z$. The difference of the eigenvalues between $|0⟩ \langle D_{L}^L|_z$ and $|0⟩ \langle D_{L}^L|_z$ is $\omega^{(P)} + \lambda^2 \frac{\omega_{\text{A} \omega_{\text{P}}} - \omega_{\text{P} \omega_{\text{A}}}}{\omega_{\text{P} \omega_{\text{A}}}}$, which is detuned from other energy eigenstates. For example, the difference of the eigenvalues between $|0⟩ \langle D_{L}^L|_z$ and $|0⟩ \langle D_{L}^L|_z$ is $\omega^{(P)} - \lambda^2 \frac{\omega_{\text{A} \omega_{\text{P}}} - \omega_{\text{P} \omega_{\text{A}}}}{\omega_{\text{P} \omega_{\text{A}}}}$.

Fig. 5 shows that the pulse sequence of the construction of $|SQ\rangle$. Firstly, we prepare the state in Eq. (28). Secondly, we globally perform the soft $\pi/2$ pulse to the spin ensemble by turning on $\lambda_p$ with $\omega^{(d)} = \omega^{(P)} + \frac{\lambda^2}{\omega_{\text{A} \omega_{\text{P}}} - \omega_{\text{P} \omega_{\text{A}}}}$

\[
\frac{1}{\sqrt{2}} \left( |0⟩ \langle D_{L}^L| + i |D_{L}^L|_z \right) = U_{\text{pulse}} |0⟩ \langle D_{L}^L|_z.
\]

Finally, by turning off $\lambda$ and choosing $\omega^{(d)} = \omega^{(P)}$ in order to perform the hard $\pi/2$ pulse along $y$ axis into each of the probe
2. Construction of |SQread⟩

![Diagram of pulse sequence]

FIG. 5. (color online) A schematic of the pulse sequence of the construction of |SQread⟩. Firstly, we perform a hard π pulse to the ancillary qubit. Secondly, let the system evolve by the Hamiltonian $\hat{H}_{SS}$. Thirdly, repeat the second and third process $\frac{L}{2}$ times. Thirdly, we increase the frequency of the ancillary qubit to induce the detuning and globally perform the soft π/2 pulse to the spin ensemble with $\omega^{(d')} = \omega^{(P)} + \lambda_2^2 \omega^{(A)} - \omega^{(P)}$. Finally, we perform a hard π/2 pulse along the y axis into the probe spin ensemble.

Spins, we obtain

$$|0⟩|\text{SQread}⟩ = \frac{e^{-i\pi J_{P}^{P}/2}}{\sqrt{2}} |0⟩ \left( |D_{L/2}^{P})^{d}_{z}⟩ + i |D_{L/2+1}^{P})^{d}_{z}⟩ \right) .$$

(33)

V. SUMMARY AND CONCLUSION

To summarizes, we propose a single spin detection by using Dicke states as a probe, and evaluate its performance. Particularly, we investigate the necessary time $T_s$ to readout the target spin with a probe of Dicke states and compare it with that of the classical strategy where only separable states are used as the probe. Assuming a relationship of $\rho \propto (T_s)^{-1}$ (which has been experimentally observed in some systems), we show that $T_s$ becomes smaller as $\rho$ increases for the case of Dicke states, while $T_s$ does not depend on $\rho$ for the classical strategy. Therefore, we conclude that by using the dense probe spins, the Dicke states provides higher sensitivity than the separable states when we aim to detect a single spin. Moreover, we propose how to create and measure the Dicke states by a global and deterministic control. Our results pave the way for a rapid spin detection that is useful for many areas such as condensed matter physics, material science, and life sciences.

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Appendix A: The explicit form of $p$ and $F(u)$

Using explicit form of $p$ (see the derivation in Appendix B), we can derivate $F(u)$ in Eq. (12). The explicit form of $p$ is given by

$$p = \frac{e^{-2u^2}}{2} I_0(u^2/4) \left[ I_0(u^2/4) + I_1(u^2/4) \right] + \frac{T_2}{2 \sqrt{L}} ve^{-u^2} \frac{1}{2} \left( I_0(u^2/4) - I_1(u^2/4) \right) \left[ \sum_j \frac{d\omega_j(r_j, z_j)}{d\theta} \right] ,$$

(A1)

where $I_n(x)$ is the modified Bessel function. From this, we obtain

$$p(1-p) = \frac{e^{-2u^2}}{2} I_0(u^2/4) \left[ I_0(u^2/4) + I_1(u^2/4) \right] \left( 1 - \frac{e^{-2u^2}}{2} I_0(u^2/4) \left[ I_0(u^2/4) + I_1(u^2/4) \right] \right) \left[ 1 - \frac{e^{-2u^2}}{2} I_0(u^2/4) \left[ I_0(u^2/4) + I_1(u^2/4) \right] \right) ,$$

(A2)

$$\left| \frac{\partial p}{\partial s} \right| = \frac{T_2}{2 \sqrt{L}} ve^{-u^2} \frac{1}{2} \left( I_0(u^2/4) - I_1(u^2/4) \right) \left[ \sum_j \frac{d\omega_j(r_j, z_j)}{d\theta} \right] ,$$

(A3)
and $\sqrt{N} = \sqrt{T/(T_{2u}/\sqrt{L})}$, we can derive Eq. (12) by using

$$F(u) = \frac{2\sqrt{\frac{x}{2}}}{\sqrt{x}e^{-\frac{x^2}{2}} \left( l_0 \left( \frac{u}{T} \right) - I_1 \left( \frac{u}{T} \right) \right)}.$$

(A4)

**Appendix B: Derivation of the expectation value $\rho$ in Eq. (A1)**

From Eq. (10), $p$ can be rewritten by

$$p = \frac{1}{2} \langle D_{L/2}^L \rangle_x \rho(t) | D_{L/2}^L \rangle_x + \frac{1}{2} \langle D_{L/2+1}^L \rangle_x \hat{\rho}(t) | D_{L/2+1}^L \rangle_x - \text{Im} \left[ \langle D_{L/2}^L \rangle_x \hat{\rho}(t) | D_{L/2+1}^L \rangle_x \right],$$

(B1)

where $\text{Im}[\cdot]$ denotes the imaginary part. We will calculate these three terms.

Here, we rewrite $|D_{L/2}^L \rangle_x$ with the basis of $|0\rangle$ or $|1\rangle$:

$$|D_{L/2}^L \rangle_x = \frac{1}{2^L} \sum_{m} \zeta(m) |m\rangle,$$

(B2)

$$\zeta(m) = \frac{1}{\sqrt{\binom{L}{L/2}}} \sum_{t} (-1)^{(j_m)},$$

(B3)

$$\langle j, m \rangle = j_1 m_1 + j_2 m_2 + \cdots + j_L m_L,$$

(B4)

$$j = j_1 j_2 \cdots j_L = 00 \cdots 11 \cdots 1,$$

(B5)

where $m = m_1 m_2 \cdots m_L$ is a $L$ bit sequence and $m_i = 0$ or 1 ($i = 1, \cdots, L$) means the eigenvalues of the $\sigma_z$, and $j = j_1 j_2 \cdots j_L$ ($j_i = 0$ or 1, $i = 1, \cdots, L$). We assume that half of components of $j$ is 1 and the other half of components of $j$ is 0. $\sum_m$ means $\sum_{m_1, m_2, \cdots, m_L}$ (the sum of the $2^L$ terms), and $\sum_j$ means all the permutations of $j$ corresponding to Eq. (7) (the sum of the binomial coefficient terms), which means all permutation such that half of components of $j$ is 1 and the other half of $j$ is 0. For example, when $L = 4$, all the permutation of $j$ are $j = 0011, 0101, 0110, 1001, 1010, 1100$, and when $j = 0011, j_1 = j_2 = 0$, and $j_3 = j_4 = 1$. In this case, $|D_{L/2}^L \rangle_x = \frac{1}{2^L} (3(|0000\rangle + |1111\rangle) - (|0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle))$. Moreover, we rewrite $|D_{L/2+1}^L \rangle_x$ with the basis of $|0\rangle$ or $|1\rangle$:

$$|D_{L/2+1}^L \rangle_x = \frac{1}{2^L} \sum_{m} \zeta(m) |m\rangle,$$

(B6)

$$\zeta(m) = \frac{1}{\sqrt{\binom{L}{L/2+1}}} \sum_{t} (-1)^{(j_m)},$$

(B7)

$$l = 00 \cdots 011 \cdots 1,$$

(B8)

where $l = l_1 l_2 \cdots l_L$ and $\sum_l$ means all the permutations of $l$ (the sum of the binomial coefficient terms). For example, when $L = 4$, all the permutation of $l$ are $l = 0111, 1011, 1101, 1110$, and when $l = 0111, l_1 = 0$, and $l_2 = l_3 = l_4 = 1$. In this case, $|D_{L/2+1}^L \rangle_x = \frac{1}{2^L} (2(|0000\rangle - |1111\rangle) + (|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle + |0110\rangle + |1010\rangle + |1100\rangle + |1110\rangle)).$

The solution of Eq. (8) is given by

$$\hat{\rho}(t) = \frac{1}{2^L} \sum_{m, m'} \zeta(m) \zeta(m') |m\rangle \langle m'| \times \exp \left[ i \sum_n \frac{\omega_n \xi_n}{2} t (-1)^{m_n} - (-1)^{m_n} \right] \times \prod_n \left( \delta_{m_n, m_n'} + (1 - \delta_{m_n, m_n'}) e^{-\left( i \xi_n \right)^2} \right),$$

(B9)

where $\exp[\cdots]$ term expresses the unitary time evolution and $\prod_n (\cdots)$ term expresses the decoherence corresponding to the first and second term in Eq. (8), respectively.
1. First term calculation: \( \langle D^{L}_{L/2} | \hat{\rho}(t) | D^{L}_{L/2} \rangle \)

The first term \( \langle D^{L}_{L/2} | \hat{\rho}(t) | D^{L}_{L/2} \rangle \) gives

\[
\frac{1}{2L} \sum_{m,m'} \xi(m)^2 \xi(m')^2 \times \exp \left[ i \sum_n \frac{\omega_n (r_n, z_n)}{2} t \left( (-1)^{m_n} - (-1)^{m'_n} \right) \right] \times \prod_n \left( \delta_{m_n,m'_n} + (1 - \delta_{m_n,m'_n}) e^{-i \frac{\pi}{2}} \right)
\]

(B10)

\[
= \frac{1}{2L} (L/L)^2 \sum_{m,m',j_{(1)} j_{(2)} j_{(3)} j_{(4)}} \sum_{m,n} (-1)^{j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)}} e \left[ i \sum_n \omega_n (r_n, z_n) \left( (-1)^{m_n} - (-1)^{m'_n} \right) \right] \prod_n \left( \delta_{m_n,m'_n} + (1 - \delta_{m_n,m'_n}) e^{-i \frac{\pi}{2}} \right)
\]

(B11)

\[
= \frac{1}{2L} (L/L)^2 \sum_{j_{(1)} j_{(2)} j_{(3)} j_{(4)}} \prod_n \left( \sum_{m,n,m'_n=0} (-1)^{j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)}} e \left[ i \sum_n \omega_n (r_n, z_n) \left( (-1)^{m_n} - (-1)^{m'_n} \right) \right] \prod_n \left( \delta_{m_n,m'_n} + (1 - \delta_{m_n,m'_n}) e^{-i \frac{\pi}{2}} \right) \right)
\]

(B12)

\[
= \frac{1}{2L} (L/L)^2 \sum_{j_{(1)} j_{(2)} j_{(3)} j_{(4)}} \prod_n \left( \delta_{j_{(1)} - j_{(2)} + j_{(3)} + j_{(4)} = 0} + \delta_{j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)} = 1} \right) e^{i \omega_n (r_n, z_n) t} e^{-i \frac{\pi}{2}} + \delta_{j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)} = 0} \right) e^{-i \omega_n (r_n, z_n) t} e^{-i \frac{\pi}{2}}
\]

(B13)

\[
= \frac{1}{2L} (L/L)^2 \sum_{j_{(1)} j_{(2)} j_{(3)} j_{(4)}} \prod_n \left( \delta_{j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)} = 0} - \delta_{j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)} = 1} \right) e^{-i \omega_n (r_n, z_n) t} e^{-i \frac{\pi}{2}} + \delta_{j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)} = 0} \right) e^{-i \omega_n (r_n, z_n) t} e^{-i \frac{\pi}{2}}
\]

(B14)

\[
= \frac{1}{2L} (L/L)^2 \sum_{j_{(1)} j_{(2)} j_{(3)} j_{(4)}} \prod_n \left( \delta_{j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)} = 0} + \delta_{j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)} = 1} \right) e^{-i \omega_n (r_n, z_n) t} e^{-i \frac{\pi}{2}} + \delta_{j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)} = 1} \right) e^{-i \omega_n (r_n, z_n) t} e^{-i \frac{\pi}{2}}
\]

(B15)

Here, \( \delta_{j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)} = 0} = 1 \) (or 0) if \( j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)} \equiv 0 \) (or 1) \( \pmod{2} \). We assume that \( O((\omega_n (r_n, z_n) t)^2) \) is negligibly small. Table I shows four cases of the contents of \( \prod_n [ \cdots ] \) in Eq. (B15). From the Eq. (B15), we have a term of \( \delta_{j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)} = 0} \) \( \omega_n (r_n, z_n) t \)

| Four cases | values |
|------------|--------|
| \( j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)} = 0 \) | \( 1 + e^{-i \frac{\pi}{2}} \) |
| \( j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)} = 1 \) | \( i \omega_n (r_n, z_n) t e^{-i \frac{\pi}{2}} \) |
| \( j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)} = 0 \) | \( (i \omega_n (r_n, z_n) t e^{-i \frac{\pi}{2}} \) |
| \( j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)} = 1 \) | \( -i \omega_n (r_n, z_n) t e^{-i \frac{\pi}{2}} \) |
| \( j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)} = 0 \) | \( 1 - e^{-i \frac{\pi}{2}} \) |

and also have a term of \( \delta_{j_{(1)} + j_{(2)} + j_{(3)} + j_{(4)} = 0} \omega_n (r_n, z_n) t \). After the summation of \( j_{(1)}, j_{(2)}, j_{(3)}, j_{(4)} \), these terms cancel out each other so that we should not have a term of \( O((\omega_n (r_n, z_n) t)^2) \). Therefore, in the Table I, we can just consider the first line and fourth line. This means that we can consider only following condition

\[
\sum_n \omega_n (r_n, z_n) t \equiv 0 \pmod{2}
\]

(for all \( n \)).

(B16)

We need to count how many set of \( j_{(1)}, j_{(2)}, j_{(3)}, j_{(4)} \) exists to satisfy the condition of Eq. (B16).

Firstly, we fix the sequence \( j_{(1)} \) to

\[
\begin{bmatrix} j_{(1)} \end{bmatrix} = \frac{000 \cdots 0111 \cdots 1}{L/2 \ L/2}
\]

(B17)

Secondly, we consider the sequences \( j_{(2)} \) which satisfy the condition that the sequences of \( j_{(1)} + j_{(2)} \) contain \( L - 2n \) number of 0
and $2n$ number of 1. For example, when

$$j^{(2)} = \underbrace{000\cdots011\cdots1}_{L/2-n} \underbrace{100\cdots011\cdots1}_{L/2-n},$$

we obtain

$$j^{(1)} + j^{(2)} \equiv \underbrace{000\cdots011\cdots1}_{L/2-n} \underbrace{111\cdots100\cdots0}_{L/2-n},$$

and this sequence surely contains $L - 2n$ number of 0 and $2n$ number of 1. Since $j^{(1)}$ is fixed, let’s consider how many number of the configuration of $j^{(2)}$ is possible. Of course, the total number of the configuration of $j^{(2)}$ is $\binom{L/2}{n}$. However, we consider a condition such that the number of 1 should be $n$ in the left side, as seen in the Eq. (B18). In this condition, the number of possible configuration of $j^{(2)}$ is $\binom{L/2}{n}^2$.

$$j^{(2)} = \underbrace{000\cdots011\cdots1}_{L/2-n} \underbrace{100\cdots011\cdots1}_{L/2-n},$$

\[ \binom{L/2}{n} \text{ combinations} \]
\[ \binom{L/2}{n} \text{ combinations} \]

It is worth mentioning that, of course, we satisfy a condition of $\binom{L}{L/2} = \sum_{n=0}^{L/2} \binom{L/2}{n}^2$. Thirdly, we change the sequence $j^{(1)}$ and for each sequence $j^{(1)}$, the number of the sequences $j^{(2)}$ is $\binom{L/2}{n}^2$. Hence, the number of the sequences is $\binom{L}{L/2} \times \binom{L/2}{n}^2$. Finally, let’s count the number of a set of $j^{(1)}$ and $j^{(2)}$ such that Eq. (B19) should be satisfied. This is calculated as follows

$$\frac{\binom{L}{L/2} \times \binom{L/2}{n}^2}{\binom{L}{2n}},$$

and this is summarized in Table II.

| $n$ | sequence | combination | degree of duplication |
|-----|----------|-------------|----------------------|
| 0   | $L$      | $\binom{L}{0}$ | $\frac{\binom{L}{L/2} \times \binom{L/2}{0}^2}{\binom{L}{2n}}$ |
| 1   | $L/2$   | $\binom{L/2}{1}$ | $\frac{\binom{L}{L/2} \times \binom{L/2}{1}^2}{\binom{L}{2n}}$ |
| 2   | $L/4$   | $\binom{L/4}{2}$ | $\frac{\binom{L}{L/2} \times \binom{L/2}{2}^2}{\binom{L}{2n}}$ |
| :  | :       | :           | :                    |
| $L/2$ | $L$     | $\binom{L}{L/2}$ | $\frac{\binom{L}{L/2} \times \binom{L/2}{L/2}^2}{\binom{L}{2n}}$ |

If we fix a sequence $j^{(1)} + j^{(2)} = \underbrace{000\cdots011\cdots1}_{L/2-n} \underbrace{100\cdots011\cdots1}_{L/2-n}$, then the sequence $j^{(3)} + j^{(4)} = \underbrace{000\cdots011\cdots1}_{L/2-n} \underbrace{100\cdots011\cdots1}_{L/2-n}$ is uniquely determined.
such that \( j^{(1)} + j^{(2)} + j^{(3)} + j^{(4)} \equiv 00 \cdots 0 \). From this, we obtain

\[
\langle D_{L/2}^L \hat{\rho}(t) | D_{L/2}^L \rangle_x = \frac{1}{2^{L/2}} \sum_{n=0}^{L/2} \frac{1}{2^n} \left( 1 + e^{-i \frac{L}{2} \omega} \right)^{L-2n} \left( 1 - e^{-i \frac{L}{2} \omega} \right)^{2n} \frac{\left( L/2 \right) \times \left( L/2 \right)}{\left( \frac{L}{2} \right)} \times \frac{\left( \frac{L}{2n} \right)}{\left( \frac{L}{2n} \right)} + O((\omega_x(r_n, z_n)t)^2)
\]

Table I

duplication Eq. (B21)

\[
= \frac{1}{2^L} \sum_{n=0}^{L/2} \left( \frac{L/2}{2n} \right) \left( 1 + e^{-i \frac{L}{2} \omega} \right)^{L-2n} \left( 1 - e^{-i \frac{L}{2} \omega} \right)^{2n} + O((\omega_x(r_n, z_n)t)^2)
\]

(B22)

\[
e^{-\frac{L}{L/2} \left( \frac{L}{2} \right)} \sum_{n=0}^{L/2} \left( \frac{L/2}{2n} \right) \left( \frac{L/2}{2n} \right) \left( 1 + e^{-i \frac{L}{2} \omega} \right)^{L-2n} \left( 1 - e^{-i \frac{L}{2} \omega} \right)^{2n} + O((\omega_x(r_n, z_n)t)^2) + O(L^{-1})
\]

(B23)

2. Second term calculation: \( \langle D_{L/2+1}^L \hat{\rho}(t) | D_{L/2+1}^L \rangle_x \)

The second term \( \langle D_{L/2+1}^L \hat{\rho}(t) | D_{L/2+1}^L \rangle_x \) gives

\[
\frac{1}{2^{L+1}} \sum_{m,m'} \xi(m) \xi(m') \xi(m) \xi(m') \times \exp \left[ \sum_n \frac{\omega_x(r_n, z_n)}{2} \left( (-1)^{m_n} + (-1)^{m'_n} \right) \right] \times \prod_n \left( \delta_{m_n, m'_n} + (1 - \delta_{m_n, m'_n}) e^{-i \frac{L}{2} \omega} \right)
\]

(B25)

\[
e^{-\frac{L}{L/2+1} \left( \frac{L}{2} \right)} \sum_{n=0}^{L/2+1} \prod_n \left( \delta_{j_n^{(1)} + j_n^{(2)} + j_n^{(3)} + j_n^{(4)} = 0} + \delta_{j_n^{(1)}, j_n^{(2)} = 0} (1 + i \omega_x(r_n, z_n)t) e^{-i \frac{L}{2} \omega} + \delta_{j_n^{(1)} = 0} (1 - i \omega_x(r_n, z_n)t) e^{-i \frac{L}{2} \omega} \right) + O((\omega_x(r_n, z_n)t)^2).
\]

(B26)

Note that Eq. (B26) is equal to Eq. (B15) except the range of the sum \( \sum_{j^{(1)}, j^{(2)}, j^{(3)}, j^{(4)}} \). Therefore, we investigate the sequence \( j^{(1)} + j^{(1)} \) such that \( j^{(1)} + j^{(1)} + j^{(2)} + j^{(2)} \equiv 00 \cdots 0 \). From the same discussion as Eq. (B21), the degree of duplication for each sequence is given as

\[
\frac{\left( L/2 \right) \times \left( L/2 \right) \times \left( L/2 \right)}{\left( 2n-1 \right)}
\]

(B27)

and this is summarized in Table III. From this, we obtain

| \( n \) | sequence | combination | degree of duplication |
|-------|----------|-------------|----------------------|
| 1     | 000 \cdots 0 0  | \( \frac{L}{L/2} \times \frac{L/2}{2} \times \frac{L/2}{2} \) | \( (\frac{L}{L})_{L/2} \) |
| 2     | 000 \cdots 0 111 | \( \frac{L}{L/2} \times \frac{L/2}{2} \times \frac{L/2}{2} \) | \( (\frac{L}{L})_{L/2} \) |
| 3     | 000 \cdots 0 11111 | \( \frac{L}{L/2} \times \frac{L/2}{2} \times \frac{L/2}{2} \) | \( (\frac{L}{L})_{L/2} \) |
| \vdots | \vdots | \vdots | \vdots |
| \( L/2 \) | 0 \cdots 0 111 \cdots 0 | \( \frac{L}{L/2} \times \frac{L/2}{2} \times \frac{L/2}{2} \) | \( (\frac{L}{L})_{L/2} \) |
\[\langle D_{L/2}^L \mid \dot{\rho}(t) \mid D_{L/2+1}^L \rangle_x = \frac{1}{2L(L/2) \prod_{n=1}^{L/2} \frac{1 + e^{-\left(\pi_T \right)^2}}{1 - e^{-\left(\pi_T \right)^2}}} \frac{\left(1 - e^{-\left(\pi_T \right)^2}\right)^{2n} \left(L/L_{n+1/2}\right)^{2n} \left(L/L_{n-1/2}\right)^{2n} \times \left(L/L_n\right)^{2n}}{\left(L/L_{n+1}\right)^{2n+1} \times O((\omega_x r_m z_n r)^2)} \]  \hspace{1cm} (B28)
4. Derivation of the explicit form \( p \) in Eq. (A1)

As described in the main text, we set \( t = \frac{T_2}{\sqrt{L}} u \) in Eqs. (B24), (B30), (B32),

\[
p = \frac{e^{-\frac{u^2}{2}}}{2} \sum_{n=0}^{L/2} \left( \frac{L/2}{2n} \right)^4 \left( \frac{u^2}{2L} \right)^{2n} + \frac{e^{-\frac{u^2}{2}}}{2} \sum_{n=1}^{L/2} \left( \frac{L/2}{2n-1} \right)^2 \left( \frac{L/2}{2n} \right)^2 \left( \frac{u^2}{2L} \right)^{2n-1} + \frac{T_2}{2L\sqrt{L}} e^{-\frac{u^2}{2}} \left( \sum_j \omega_s(r_j, z_j) \right) \sum_{n=1}^{L/2} \left( \frac{L/2}{2n-1} \right)^3 \left( \frac{L/2}{2n} \right)(2n) \left( \frac{u^2}{2L} \right)^{2n-1}
\]

\[
- \frac{T_2}{2L\sqrt{L}} e^{-\frac{u^2}{2}} \left( \sum_j \omega_s(r_j, z_j) \right) \sum_{n=1}^{L/2} \left( \frac{L/2}{2n} \right)^2 \left( \frac{L/2}{2n-1} \right)(2n-1) \left( \frac{u^2}{2L} \right)^{2n-2} + O(\omega_s(r_n, z_n) t^2) + O(L^{-1}).
\]

(B33)

Here, we rewrite the first two terms

\[
e^{-\frac{u^2}{2}} \sum_{n=0}^{L/2} \left( \frac{L/2}{2n} \right)^4 \left( \frac{u^2}{2L} \right)^{2n} + e^{-\frac{u^2}{2}} \sum_{n=1}^{L/2} \left( \frac{L/2}{2n-1} \right)^2 \left( \frac{L/2}{2n} \right)^2 \left( \frac{u^2}{2L} \right)^{2n-1}
\]

\[
= e^{-\frac{u^2}{2}} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \left( \frac{u^2}{8} \right)^{2n} + e^{-\frac{u^2}{2}} \sum_{n=1}^{\infty} \frac{(2n-1)!}{(n!)^2 ((n-1)!)^2} \left( \frac{u^2}{8} \right)^{2n-2} + O(L^{-1})
\]

(B34)

\[
= e^{-\frac{u^2}{2}} I_0(u^2/4) \left[ I_0(u^2/4) + I_1(u^2/4) \right],
\]

(B35)

where \( I_0(x) \) is the modified Bessel function \( I_0(x) = \sum_{m=0}^{\infty} \frac{1}{m! (m+\alpha+1)} \left( \frac{x}{2} \right)^{2m+\alpha} \), and \( \Gamma(x) \) is the Gamma function. Also, we also rewrite

\[
\sum_{n=1}^{L/2} \left( \frac{L/2}{2n} \right)^2 \left( \frac{L/2}{2n-1} \right)(2n-1) \left( \frac{u^2}{2L} \right)^{2n-2}
\]

\[
= \frac{L}{2} \left( I_0(u^2/4) - I_1(u^2/4) \right)^2 + O(1),
\]

(B36)

and consequently we obtain \( p \) in Eq. (A1).

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