Variable projection methods for approximate (greatest) common divisor computations

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Abstract

We consider the problem of finding for a given \(N\)-tuple of polynomials the closest \(N\)-tuple that has a common divisor of degree at least \(d\). Extended weighted Euclidean seminorm of the coefficients is used as a measure of closeness. Two equivalent representations of the problem are considered: (i) direct optimization over the common divisors and cofactors (image representation), and (ii) Sylvester low-rank approximation (kernel representation). We use the duality between least-squares and least-norm problems to show that (i) and (ii) are closely related to mosaic Hankel low-rank approximation. This allows us to apply to the approximate common divisor problem recent results on complexity and accuracy of computations for mosaic Hankel low-rank approximation. We develop optimization methods based on the variable projection principle both for image and kernel representation. These methods have linear complexity in the degrees of the polynomials if either \(d\) is small or \(d\) is of the same order as the degrees of the polynomials. We provide a software implementation of the developed methods, which is based on a software package for structured low-rank approximation.

Key words: approximate GCD, structured low-rank approximation, variable projection, mosaic Hankel matrices, least-squares problem, least-norm problem, weighted 2-norm, computational complexity

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1. Introduction

In many applied problems there is a need to compute a (greatest) common divisor of a set of polynomials with inexact coefficients. An approximation may be needed because the input polynomial coefficients are subject to measurement noise or they have a representation with limited precision. Examples include problems in signal processing and system identification (Agrawal et al., 2004; Gaubitch et al., 2005), computer-aided geometric design (Emiris et al., 2013), blind image deblurring (Li et al., 2010), control of linear systems (Khare et al., 2010) and approximate factorization of polynomials (Zeng, 2008).

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In this paper, we consider the problem of finding the closest polynomials to given ones such that they have a common divisor of degree at least \( d \), where \( d \) is a fixed parameter. This problem is an important part of the approximate greatest common divisor problem. We do not consider the problem of selecting the degree of the common divisor. (See the discussion in Section 1.1.) The measure of closeness between given and approximating \( N \)-tuples of polynomials is chosen to be the weighted Euclidean semi-norm. (For a review of other possible closeness measures, see [Pan, 2001].)

1.1. Problem statement and related formulations

Let \( \mathbb{P}_n \) be the set of univariate complex (or real) polynomials of degree at most \( n \geq 0 \), i.e.
\[
\mathbb{P}_n := \{ p_0 + p_1 z + \cdots + p_n z^n \mid p_j \in \mathbb{F} \} \subset \mathbb{F}[z],
\]
where \( \mathbb{F} \) is \( \mathbb{C} \) or \( \mathbb{R} \). As discussed in Section 2.2, \( \mathbb{P}_n \) is isomorphic to the space of coefficients \( \mathbb{F}^{n+1} \).

Let \( n = [n_1 \cdots n_N]^{\top} \in \mathbb{N}^N \) be fixed degrees, \( n_{\text{min}} := \min n_k \), and denote \( \mathbb{P}_n := \mathbb{P}_{n_1} \times \cdots \times \mathbb{P}_{n_N} \) be the set of \( N \)-tuples of polynomials with these degrees. We assume that \( \mathbb{P}_n \) is equipped with some distance \( \text{dist}(\cdot, \cdot) \), which is continuous in the Euclidean topology.

In this paper, we address the problem of finding the distance from a given \( N \)-tuple to the set of \( N \)-tuples that have a greatest common divisor (GCD) of degree at least \( d \geq 0 \):
\[
\mathcal{G}_d := \left\{ (\hat{p}^{(1)}, \ldots, \hat{p}^{(N)}) : \deg \gcd(\hat{p}^{(1)}, \ldots, \hat{p}^{(N)}) \geq d \right\}.
\]

**Problem 1** (Approximate GCD with bounded degree). Given \( p = (p^{(1)}, \ldots, p^{(N)}) \in \mathbb{P}_n \) and \( d : 0 \leq d \leq n_{\text{min}} \), find the distance
\[
\text{dist} (p, \mathcal{G}_d) := \min_{\hat{p} \in \mathcal{G}_d} \text{dist} (p, \hat{p}).
\]

The set \( \mathcal{G}_d \) is closed (see Lemma 15 in A) in the Euclidean topology, and hence Problem 1 is well-posed. Closely related to Problem 1 is the problem of finding the maximal GCD degree in the \( \varepsilon \)-ball around the given \( N \)-tuple.

**Problem 2** (\( \varepsilon \)-GCD degree). Given \( p = (p^{(1)}, \ldots, p^{(N)}) \in \mathbb{P}_n \) and \( \varepsilon > 0 \), find
\[
d^* (\varepsilon) = \max_{\hat{p} = (\hat{p}^{(1)}, \ldots, \hat{p}^{(N)})} \deg \gcd (\hat{p}^{(1)}, \ldots, \hat{p}^{(N)}) \text{ subject to } \text{dist} (p, \hat{p}) \leq \varepsilon.
\]

As noted by Emiris et al. (1996); Rupprecht (1999), \( \mathcal{G}_d \) form a descending chain of sets:
\[
\mathbb{P}_n = \mathcal{G}_0 \supset \mathcal{G}_1 \supset \cdots \supset \mathcal{G}_{n_{\text{min}}} \supset \mathcal{G}_{n_{\text{min}}+1} = \emptyset.
\]

Therefore, the solution of Problem 2 is the only integer \( d^* = d^* (\varepsilon) \) that satisfies
\[
\text{dist} (p, \mathcal{G}_{d^*}) \leq \varepsilon \quad \text{and} \quad \text{dist} (p, \mathcal{G}_{d^*+1}) > \varepsilon.
\]

Hence, being able to solve Problem 1 allows us to solve Problem 2, and vice versa (see also Rupprecht (1999) and Markovsky (2012, §2.3)). Indeed, we can find \( d^* \) satisfying (6) by solving Problem 1 for different \( d \) (for example, using bisection). Vice versa, by solving Problem 2 for varying \( \varepsilon \), we can detect a jump of \( d^* (\varepsilon) \) from \( d \) to \( d + 1 \), which will correspond to the solution of Problem 1.

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1 Emiris et al. (1996) provide an exposition of algebraic-geometric properties of the sets \( \mathcal{G}_d \) in the case \( N = 2 \).
Many authors were focused on finding maximal GCD degree within a given ball (see Problem 2), however it was recognized that besides the maximal degree of interest is also the closest $N$-tuple of polynomials with GCD of this degree. More precisely, the following problem formulation was adopted by Rupprecht (1999); Zeng and Dayton (2004); Bini and Boito (2010).

**Problem 3** (Approximate GCD with tolerance $\varepsilon$). Given $p = (p^{(1)}, \ldots, p^{(N)}) \in \mathbb{P}_n$ and $\varepsilon > 0$
1. Find the degree $d^*(\varepsilon)$ of the $\varepsilon$-GCD (solve Problem 2).
2. Find the closest $N$-tuple $\hat{p}^*$ with the computed degree (solve Problem 1 with $d = d^*$).

Usually, the candidate degree $d^*$ on the first step of Problem 3 is found by some heuristic. The heuristic can be based on finding the gap between the singular values of the Sylvester matrix [Corless et al., 1995], tracking the last singular value of the sequence of Sylvester sub-resultants [Rupprecht, 1999; Zeng and Dayton, 2004] or, more recently, looking at the principal angles between the columns of sub-resultant matrices [Winkler et al., 2012].

The distance measures used are typically induced by a norm $\| \cdot \|$ in $\mathbb{P}_n$, where in most cases $\| \cdot \|$ is a weighted Euclidean norm. This choice can be justified from a statistical point of view [Agrawal et al., 2004], or because it makes the problem more computationally tractable. In addition, it can be shown that the Euclidean distance is equivalent to minimizing a weighted sum of squared sines of angles between the polynomials, therefore this approximation criterion depends only on the roots of the approximating polynomials (see the discussion in Section 4.4). Other norms were also considered in the literature, for example Hitz et al. (1999) use $\ell_\infty$ norm for the nearest singular polynomial problem (computing the approximate GCD of a polynomial and its derivatives).

In this paper, we consider the distance defined by the weighted extended semi-norm on $\mathbb{P}_n$:

$$\text{dist}(p, q) := \|p - q\|^2_w = \sum_{k=1}^{N} \|p^{(k)} - q^{(k)}\|^2_{w^{(k)}},$$

where $w^{(k)} \in [0, +\infty][^{n+1}, w = (w^{(1)}, \ldots, w^{(N)})$ and

$$\|p\|^2_w := \sum_{j=0}^{n} w_j p_j p^*_j.$$  \hspace{1cm} (7)

Note that if $w_j \in (0, \infty)$, then $\| \cdot \|^2_w$ is a norm.

If an infinite weight $w_j^{(k)} = \infty$ is present, we formally assume $0 \cdot \infty = 0$ in (7) and require the distance to be finite in (3). Then, $w_j^{(k)} = \infty$ is equivalent to the equality constraint $p_j^{(k)} = p_j^{(k)}$ imposed on a separate coefficient (for example, a monicity constraint). A zero weight $w_j^{(k)} = 0$ makes the coefficient $p_j^{(k)}$ irrelevant, and the resulting solution $\hat{p}$ does not depend on $p_j^{(k)}$. Therefore, we may assume that the coefficient is undefined (which corresponds to the case of missing observations in data modeling problems [Markovsky and Usevich, 2013]).

1.2. Previous work on approximate GCD with bounded degree

Problem 1 is non-convex and has no analytic solution. Therefore, except for recent papers Cheze et al. 2011; Li et al., 2008; Kaltofen et al., 2008, the problem is typically solved with local optimization methods, starting from an initial approximation. (In the methods for solving Problem 3, the initial approximation is computed on the first step of the $\varepsilon$-GCD degree selection.)
Most of the optimization methods for Problem 1 are based on two different equivalent representations of the set $\mathcal{G}_d$ (mentioned in Emiris et al. [1996], Rupprecht [1999]): (i) the set of products of common divisors and cofactors and (ii) the set of vectors with rank-deficient Sylvester sub-resultant matrices. We will use the term *image representation* for (i) since $\mathcal{G}_d$ is represented as the image of a multiplication mapping, and *kernel representation* for (ii) since $\mathcal{G}_d$ is represented as the kernel of the product of maximal minors of a structured matrix.

In the kernel representation, the following set is used instead of $\mathcal{G}_d$:

$$\{ \mathbf{p} \in \mathbb{P}_n : \mathcal{S}^{(1)}_d(\mathbf{p}) \text{ is rank deficient} \}, \quad (8)$$

where $\mathcal{S}^{(1)}_d(\mathbf{p})$ is a Sylvester sub-resultant matrix (for $N = 2$) or a generalized Sylvester subresultant matrix proposed by Rupprecht (1999); Kaltofen et al. (2006) (for $N > 2$). Note that for $N > 2$ the set defined in (8) includes extra polynomials which do not have a common divisor (along with the polynomials from $\mathcal{G}_d$). An alternative matrix structure, proposed by Agrawal et al. (2004), overcomes this deficiency and handles properly the non-generic cases. (For more details, see the discussion in Section 5.2.)

In optimization methods, the rank deficiency constraint is usually parametrized by a vector $\hat{\mathbf{u}}$ in the right kernel of $\mathcal{S}^{(1)}_d(\mathbf{p})$, and the following problem is considered:

$$\min_{\hat{\mathbf{p}}, \hat{\mathbf{u}} \neq 0} \text{dist}(\hat{\mathbf{p}}, \mathbf{p}) \quad \text{subject to} \quad \mathcal{S}^{(1)}_d(\hat{\mathbf{p}})\hat{\mathbf{u}} = 0. \quad (9)$$

For local optimization of (9), in a number of papers the constraint is replaced by a penalty, and a Gauss-Newton iteration is applied (Kaltofen et al., 2006; Li et al., 2007). This is known as structured total-least norm approach (STLN), which has complexity $O(n^2)$. Note that in the STLN approach the last coordinate of $\mathbf{u}$ is fixed to 1 (which corresponds to a monic polynomial), and some solutions of (9) correspond to ill-posed cases in STLN. Another approach to (9) is used in the recently proposed GPGCD method with complexity $O(n^3)$ (Terui, 2013).

In the image representation, the set $\mathcal{G}_n$ is replaced by the following set:

$$\mathcal{F}_d := \left\{ (g^{(1)} h, \ldots, g^{(N)} h) : g^{(1)} \in \mathbb{P}_{n_1-d}, \ldots, g^{(N)} \in \mathbb{P}_{n_N-d}, h \in \mathbb{P}_d \setminus \{0\} \right\}. \quad (10)$$

Note that for complex polynomials $\mathcal{G}_d = \mathcal{F}_d$, but for real polynomials $\mathcal{G}_d = \mathcal{F}_d$ only if $d$ is even, and $\mathcal{G}_d = \mathcal{F}_d \cup \mathcal{F}_{d+1}$ if $d$ is odd (see Lemma 14 in A).

Problem 1 thus can be reduced to finding the distance from an $N$-tuple $\mathbf{p}$ to the set $\mathcal{F}_d$, which is the problem of minimization of the function

$$f(g^{(1)}, \ldots, g^{(N)}, h) := \text{dist}(\mathbf{p}, (g^{(1)} h, \ldots, g^{(N)} h)). \quad (11)$$

This is a nonlinear least-squares problem. A preliminary analysis of problem (11) and different optimization strategies was performed by Chin et al. [1998]. In the method UVGCD (Zeng and Dayton, 2004) implemented in the package ApaTools (Zeng, 2008), the cost function is minimized using the Gauss-Newton method. In the Fastgcd method (Bini and Boito, 2010), the authors use a combination of the Gauss-Newton method and a line search method.

A large class of methods is based on the variable projection principle (going back to the work of Golub and Pereyra [1973]). The variable projection principle is based on the fact that finding the minimum of $f$ for a given $h$

$$\tilde{f}(h) = \min_{g^{(1)}, \ldots, g^{(N)}} f(g^{(1)}, \ldots, g^{(N)}, h) \quad (12)$$
is a linear least squares problem and has a closed-form solution. Thus the variables $g^{(1)}, \ldots, g^{(N)}$ are eliminated and the search space is reduced from $O(n)$ to $d + 1$ optimization variables, which is beneficial if $d$ is small compared to $n$. Variable projection can also be applied if $d$ is large and the cofactors have small dimensions (in this case, the common divisor $h$ is eliminated (Stoica and Söderström, 1997; Agrawal et al., 2004)).

Corless et al. (1995) showed that the function $f(h)$ can be evaluated in $O(dn)$ flops for uniform weights (unweighted Euclidean norm), and a preliminary analysis of the accuracy of the computational procedure was presented (Section 3.3 contains more extensive analysis). Pan (2001) showed that for general weights, the complexity is $O(dn \log n)$. Independently, Markovsky and Van Huffel (2006) showed that not only the cost function, but also its gradient can be computed in $O(dn)$ flops for uniform weights. Unfortunately, there is no publicly available package for $N$ polynomials and non-uniform weights.

The variable projection principle is also used in the classical paper of Karmarkar and Lakshman (1998) (and also in Hitz and Kaltofen, 1998), where the authors show that for $d = 1$ ($h(x) = x - \alpha$) the function $f(h)$ has a simple expression through $\alpha$ and the coefficients of the given polynomials. This approach is extended by Zhi et al. (2004); Li and Zhi (2013) for symbolic computation of nearest singular polynomial (approximate GCD of a polynomial and its derivatives), where $h$ has the form $(x - \alpha)^d$.

The variable projection was also used in recent methods of global optimization for Problem 1. (Note that all the methods reviewed above are local optimization methods.) The global optimization methods include subdivision methods of Chèze et al. (2011) (for $d = 1$) and semi-definite programming relaxations of Li et al. (2008); Kaltofen et al. (2008).

1.3. Contribution of this paper

In this paper, we show how the nonlinear least squares problem (minimization of (11)) is related to the weighted mosaic Hankel low-rank approximation. Using recent results on the complexity of mosaic Hankel low-rank approximation (Usevich and Markovsky, 2013), we show that the cost function (12), its first derivatives and an approximation of the Hessian can be evaluated in $O(d^2 n)$ flops, which is an improvement of the results of Corless et al. (1995); Pan (2001) for $d \ll n$. We also provide more detailed than in Corless et al. (1995) coverage on the accuracy of the computational process. We also show that the same results apply to the case of variable projection with respect to elimination of the common factor (Agrawal et al., 2004). In this case, the computational complexity per iteration is $O((\sum_{k=1}^{N} (n_k - d))^2 n)$, which is linear for $d$ with $(n_k - d) \ll n$.

Next, we show that the variable projection can be also applied in the kernel representation (for Sylvester low-rank approximation), by eliminating $\hat{p}$ from (9). For $N = 2$ polynomials, the complexity of the evaluation of the cost function, its first- and second-order derivatives is also linear in $n$, if $(n_k - d) \ll n$. However, for $N > 2$ polynomials, the complexity is not linear in $n$ even if $(n_k - d) \ll n$, and there are some intrinsic computational issues.

This paper is composed as follows. Section 2 contains necessary background, including commonly used notation, details on the space $\mathbb{P}_n$ and common divisors, and an overview of least-squares and least-norm problems. In Section 3, we consider a class of least-squares and least-norm problems with matrix-polynomial multiplication matrices. We show their correspondence to mosaic-Hankel low-rank approximation, and provide a summary of results on complexity and accuracy of the evaluation of the cost function, gradient, and a Gauss-Newton approximation of the Hessian. In Sections 4 and 5, we show that the variable projection principle applied both
to optimization over common divisors/cofactors and Sylvester low-rank approximation leads to least-squares and least-norm problems with matrix-polynomial multiplication matrix.

In Section 4, we consider optimization in the image representation (11) and apply the variable projection principle either to common divisor, or to the cofactors. We also show that the same technique can be applied in the complex case. In Section 4.4, we show how minimization of the norm (7) is connected to minimization of the angles. In Section 5, we consider optimization in the kernel representation. In Section 5.1, we compare generalized Sylvester subresultant matrices (Rupprecht, 1999; Kaltofen et al., 2006) and alternative structured matrices of Agrawal et al. (2004). In Section 5.2, we show how low-rank approximation of generalized Sylvester subresultant matrices can be reduced to mosaic Hankel low-rank approximation. In Section 5.4, we show how an initial approximation is obtained for local optimization methods. In Section 6, we provide numerical experiments that include comparison with the methods Fastgcd (Bini and Boito, 2010) and UVGCD (Zeng and Dayton, 2004) implemented in MATLAB.

The methods developed in this paper are implemented in MATLAB and are based on the SLRA package (SLR, 2013) described in (Markovsky and Usevich, 2012). The source code of the methods and experiments is publicly available and is embedded in the extended version of the paper using literate programming. The methods are currently implemented only for real-valued polynomials. (For complex polynomials only theoretical results are presented.)

2. Preliminaries

2.1. Matrix and vector notation

In the paper, we adopt the following notation for matrices and vectors.

- \( \mathbb{F}^{m \times n} \) — the space of \( m \times n \) matrices with elements in a field \( \mathbb{F} \) (where \( \mathbb{F} \) is \( \mathbb{R} \) or \( \mathbb{C} \ ));
- \( 0_{m \times n}, 0_n \) — \( m \times n \) matrix of zeros, vector of \( n \) zeros;
- \( 1_{m \times n}, 1_n \) — \( m \times n \) matrix of ones, vector of \( n \) ones;
- \( I_n \) — \( n \times n \) identity matrix;
- \( J_n \) — \( n \times n \) matrix with \( 1_n \) on the main anti-diagonal and all other elements equal to zero;
- \( \text{diag}(v) \) — the diagonal matrix constructed with the elements of \( v \) on the main diagonal;
- \( \text{blkdiag}(A_1, \ldots, A_N) \) — the block-diagonal matrix composed of matrices \( A_1, \ldots, A_N \);
- \( \text{col}(p_1, \ldots, p_N) \) — concatenation of vectors \( [p_1 \cdots p_N]^\top \).

For a vector \( v = [v_1 \cdots v_N]^\top \in \mathbb{K}^N \) and a scalar \( b \in \mathbb{K} \) we define the following operations:

- \( v + b := [v_1 + b \cdots v_N + b]^\top \);
- \( \Sigma(v) \) — sum of the elements in the vector \( v \);
- \( v^{-1} = [v_1^{-1} \cdots v_N^{-1}]^\top \) — sum of the elements in the vector \( v \);
- \( v \leq b \iff v_k \leq b \) for all \( k \in \{1, \ldots, N\} \).

We also summarize some commonly used symbols:

- \( n = [n_1 \cdots n_N]^\top \) — polynomial degrees;
- \( p = (p^{(1)}, \ldots, p^{(N)}) \in \mathbb{P}_n \) — given \( N \)-tuple of polynomials;
- \( d \) — common divisor degree;
- \( \hat{h} \) — tentative common divisor;
- \( \hat{p} = (p^{(1)}, \ldots, p^{(N)}) \in \mathbb{P}_n \) — approximating \( N \)-tuple;
- \( \hat{g} = (g^{(1)}, \ldots, g^{(N)}) \in \mathbb{P}_{n-d} \) — tentative cofactors.
2.2. The set of polynomials with bounded degree

The space $P_n$ defined in (1) is isomorphic to $F^{n+1}$ through the correspondence

$$p(z) = p_0 + p_1 z + \cdots + p_n z^n \in P_n \leftrightarrow p = [p_0 \cdots p_n]^T \in F^{n+1}.$$ 

Note that we use the term “degree” for $n$ (it may be different from the polynomial degree, since the leading coefficients are allowed to be zero).

**Remark 4.** $P_n \setminus \{0\}$ can be viewed as the space of univariate polynomials of degree $d$ with the roots on the Riemannian sphere $\mathbb{C} \cup \{\infty\}$ (where the number of zero leading coefficients is the multiplicity of the root $\infty$), or more formally the homogeneous bivariate polynomials $p(y, z)$ of degree $n$.

The standard multiplication of polynomials $(p^{(1)} \cdot p^{(2)})(z) = p^{(1)}(z)p^{(2)}(z)$ (acting as $P_{n_1} \times P_{n_2} \to P_{n_1+n_2}$) has the following matrix representation:

$$p^{(1)} \cdot p^{(2)} = M_{n_2}(p^{(1)})p^{(2)} = M_{n_1}(p^{(2)})p^{(1)},$$

where $M_m(h)$ is the multiplication matrix

$$M_m(h) := \begin{bmatrix} h_0 & \vdots & \vdots & \vdots & \vdots \\ \vdots & h_0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & h_0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & h_d \\ \end{bmatrix} \in F^{(m+d+1)\times(m+1)}.$$ 

"multmat.m" 7≡

```matlab
function M = multmat(h, m)
M = toeplitz([h(:);zeros(m,1)], [h(1); zeros(m,1)]);
end
```

For $0 \leq d \leq n$, we say that a polynomial $h \in P_d \setminus \{0\}$ divides a polynomial $p \in P_n$ (or $h$ is a divisor of $p$), denoted by $h \mid p$, if there exists a polynomial $q \in P_{n-d}$ such that $p = q \cdot h$. In particular, this definition includes the following special cases:

- All polynomials $h \in P_d \setminus \{0\}$, $0 \leq d \leq n$, are the divisors of a zero polynomial $0 \in P_n$.
- A nonzero polynomial of zero degree $h \in P_0 \setminus \{0\}$ is a divisor of any polynomial.

2.3. N-tuples and greatest common divisors

We also introduce the operation of multiplication of an $N$-tuple $g = (g^{(1)}, \ldots, g^{(N)}) \in P_{n-d}$ by a polynomial $h \in P_d$ as follows:

$$g \cdot h := (g^{(1)} \cdot h, \ldots, g^{(N)} \cdot h).$$

The polynomials $p = (p^{(1)}, \ldots, p^{(N)}) \in P_n := P_{n_1} \times \cdots \times P_{n_N}$ have a common divisor of degree $d$, if there exists $h \in P_d \setminus \{0\}$ that divides all the polynomials $p^{(k)}$, or equivalently $p = g \cdot h$ for some $g \in P_{n-d}$. 

7
We also say that \( h \in \mathbb{P}_d \setminus \{0\} \) is a greatest common divisor (GCD) if there is no common divisors in \( \mathbb{P}_{d'} \) for any \( d' > d \). Since \( 1 \in \mathbb{P}_d \) is a divisor of any polynomial, a GCD always exists (but it is not unique). We denote by \( \deg \gcd p \) the degree of GCDs, and denote by \( \gcd p \) the set of all GCD (which is a subset of \( \mathbb{P}_{\deg \gcd p} \)). In particular, if all \( p^{(1)}, \ldots, p^{(N)} \) are zero, then

\[ \gcd p = \mathbb{P}_{n,m} \setminus \{0\}. \]

Otherwise, \( \gcd p \) is a punctured one-dimensional linear subspace of \( \mathbb{P}_{\deg \gcd p} \)

\[ \gcd p = \{ \alpha h : \alpha \in \mathbb{F} \setminus \{0\}, \quad h \text{ is a GCD of } p^{(1)}, \ldots, p^{(N)} \}. \]

2.4. Least-squares and least-norm problems

In this paper, we use the duality between least-squares and least-norm problems. Next, we give an overview of these problems.

**Problem 5** (Weighted least-squares problem). Let \( A \in \mathbb{R}^{m \times n}, m \geq n, b \in \mathbb{R}^m, \) and \( v \in [0, \infty)^m \), such that rank \( \text{diag}(\sqrt{v})A = n \)

\[
\text{minimize } \|Ax - b\|_v^2.
\]

The problem in (13) is the orthogonal projection of \( b \) on the image of \( A \) in the seminorm \( \| \cdot \|_v \). The solution can be easily seen if we rewrite the cost function in (13) as \( \|y^{(LS)}\|_2^2 \), where \( y^{(LS)} = \text{diag}(\sqrt{v})(Ax - b) \). Therefore, the solution of (13) is the following

\[
x_* = (A^\top \text{diag}(v)A)^{-1}A^\top \text{diag}(v)b,
\]

\[
y^{(LS)}_* = \text{diag}(\sqrt{v}) (A(A^\top \text{diag}(v)A)^{-1}A^\top \text{diag}(v) - I) b,
\]

\[
M_*^{(LS)} = \|y^{(LS)}_*\|_2^2 = \|b\|_v^2 - b^\top \text{diag}(v)A(A^\top \text{diag}(v)A)^{-1}A^\top \text{diag}(v)b.
\]

The least-squares problem (13) is closely connected to the following dual problem:

**Problem 6** (Weighted least-norm problem). Let \( A \in \mathbb{R}^{m \times n}, m \geq n, c \in \mathbb{R}^m, \) and \( w \in (0, \infty)^m \), such that rank \( \text{diag}(\sqrt{w^{-1}})A = n \)

\[
\text{minimize } \|c - z\|_w^2 \quad \text{subject to } \quad A^\top z = 0.
\]

The problem (14) is to find the orthogonal projection of the vector \( c \) on the kernel of the matrix \( A \) in the norm \( \| \cdot \|_w \). Changing variables as \( c - z = \text{diag}(\sqrt{w^{-1}})y^{(LN)} \) The cost function can be rewritten as \( \|y^{(LN)}\|_2^2 \), and the constraint as \( A^\top \text{diag}(\sqrt{w^{-1}})y^{(LN)} = A^\top c \). Therefore, the solution of (14) is the following

\[
z_* = c - \text{diag}(w^{-1})A(A^\top \text{diag}(w^{-1})A)^{-1}A^\top c,
\]

\[
y^{(LN)}_* = \text{diag}(\sqrt{w^{-1}})A(A^\top \text{diag}(w^{-1})A)^{-1}A^\top c,
\]

\[
M_*^{(LN)} = \|y^{(LN)}_*\|_2^2 = c^\top A(A^\top \text{diag}(w^{-1})A)^{-1}A^\top c.
\]

One can see that the expressions for the solutions of (13) and (14) have a similar form. (Note the inversion of the weights.) In particular, if we have \( c = \text{diag}(w^{-1})b \) and \( v = w^{-1} \), then

\[
M^{(LS)}_* = \|b\|_v^2 - M^{(LN)}_*,
\]

\[
y^{(LS)}_* = \text{diag}(\sqrt{v})b - y^{(LN)}_*.
\]
3. Least-squares and least-norm problems with parameters and multiplication matrices

In this section, we consider the least-squares and least-norm problems for matrices depending on a parameter $A = A(P)$. As we will show in Sections 4 and 5, application of the variable projection principle both in optimization over common divisors/cofactors (11) and Sylvester low-rank approximation (9) leads to minimization over $P$ of the least-squares solution $M^s(LS)(P)$ or the least-norm solution $M^s(LN)(P)$.

We show how the least-norm problem is connected to mosaic Hankel low-rank approximation, which allows to use the efficient algorithms for evaluating $M^s(LN)(P)$, its gradient and a Gauss-Newton approximation of the Hessian developed in [Usevich and Markovsky, 2013]. The software implementation is described in [Markovsky and Usevich, 2012]. We also provide a summary of known results on the accuracy of computations.

3.1. Mosaic Hankel matrices and least-squares problems with multiplication matrices

Given a matrix $P \in \mathbb{R}^{k \times l}$, integer vector $k = [k_1 \cdots k_K] \in \mathbb{N}^K$, such that $\Sigma(k) = k$, and a number $l \geq 1$, we define the matrix-polynomial multiplication matrix:

$$A_{k,l}(P) := \begin{bmatrix} M_{l-1}(P^{(1,1)}) & \cdots & M_{l-1}(P^{(1,t)}) \\ \vdots & \ddots & \vdots \\ M_{l-1}(P^{(K,1)}) & \cdots & M_{l-1}(P^{(K,t)}) \end{bmatrix},$$

where $P^{(i,j)} \in \mathbb{R}^{k_i \times 1}$ are the following sub-matrices of the matrix $P$:

$$P = \begin{bmatrix} P^{(1,1)} & \cdots & P^{(1,t)} \\ \vdots & \ddots & \vdots \\ P^{(q,1)} & \cdots & P^{(q,t)} \end{bmatrix}.$$

For two integer vectors $k$ and $l = [l_1 \cdots l_L]^\top$, $l = \Sigma(p)$, we define $A_{k,l}(P)$ as

$$A_{m,n}(P) := \text{blkdiag}(A_{k,l_1}(P), \ldots, A_{k,l_L}(P)).$$

Let $\mathcal{H}_{k,l}(c) \in \mathbb{R}^{k \times l}$ denote a Hankel matrix, generated from the vector $c \in \mathbb{R}^{k+l-1}$, i.e.

$$\mathcal{H}_{k,l}(c) = \begin{bmatrix} c_1 & c_2 & \cdots & c_l \\ c_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{l-1} \\ c_k & \cdots & \cdots & c_{k+l-1} \end{bmatrix}.$$

For two vectors $k$ and $l$ and a vector $c = \text{col}(c^{(1,1)}, \ldots, c^{(q,1)}, \ldots, c^{(1,N)}, \ldots, c^{(q,N)})$ with $c^{(k,l)} \in \mathbb{R}^{m_k+n_l-1}$ we define the mosaic Hankel matrix [Usevich and Markovsky, 2013] $\mathcal{H}_{k,l}(P)$ as

$$\mathcal{H}_{k,l}(P) := \begin{bmatrix} \mathcal{H}_{k_1,l_1}(c^{(1,1)}) & \cdots & \mathcal{H}_{k_1,l_L}(c^{(1,L)}) \\ \vdots & \ddots & \vdots \\ \mathcal{H}_{k_K,l_1}(c^{(K,1)}) & \cdots & \mathcal{H}_{k_K,l_L}(c^{(K,L)}) \end{bmatrix}.$$
Then the following lemma holds true (see also [Usevich and Markovsky, 2013, Sec.3]).

**Lemma 7.**

\[ A_{k,1}^T(P)c = 0 \iff P^T \mathcal{H}_{k,1}(c) = 0. \]  

(18)

By Lemma 7 and the results of Section 2.4, we have that the following problem

\[ \min_c \|c - \hat{c}\|_w^2 \text{ subject to } P^T \mathcal{H}_{k,1}(\hat{c}) = 0 \]  

(19)

is a least-norm problem (14) with \( A = A_{k,1}(P) \), and its solution (15) is given by

\[ M_*^{(LN)}(P) = \frac{\|y_*^{(LN)}(P)\|_2^2}{s(P)\Gamma(P)s(P)}, \]

(20)

\[ y_*^{(LN)}(P) = \text{diag}(\sqrt{w-1})A_{k,1}(P)\Gamma(P)s(P)\]

(21)

where

\[ \Gamma(P) := A_{k,1}^T(P)\text{diag}(w^{-1})A_{k,1}(P), \]

\[ s(P) := A_{k,1}^T(P)c. \]

(22)

The solution of the following problem

\[ M_*^{(LS)}(P) = \min_x \|A_{k,1}(P)x - b\|_v^2, \]

(23)

can be expressed using the correspondence (16), with \( c = \sqrt{vb} \) and \( w = v^{-1} \).

### 3.2. Computations and their complexity

In [Usevich and Markovsky, 2013; Markovsky and Usevich, 2012], the following low-rank approximation problem is considered: given \( c, w, k \) and \( l \)

\[ \min_c \|c - \hat{c}\|_w^2 \text{ subject to } \text{rank} \mathcal{H}_{k,1}(\hat{c}) \leq k - t. \]  

(24)

By using the kernel representation of the rank constraint,

\[ \text{rank} \mathcal{H}_{k,1}(\hat{c}) \leq k - t \iff P^T \mathcal{H}_{k,1}(\hat{c}) = 0, \text{ for some } P \in \mathbb{R}^{k \times t} \text{ with rank } P = t, \]

the problem (24) can be rewritten in an equivalent form (using Lemma 7)

\[ \min_{P \in \mathcal{R}_{k,t}} M_*^{(LN)}(P), \]  

(25)

where \( \mathcal{R}_{k,t} \subset \mathbb{R}^{k \times t} \) is the set representing all \( t \)-dimensional subspaces of \( \mathbb{R}^k \) (the Grassmann manifold). See [Markovsky and Usevich, 2012] for the parametrizations used in the software implementation.

The efficient algorithms of [Usevich and Markovsky, 2013] for local optimization of (25) are based on the fact that for \( k_i \ll l_j \) the matrix \( \Gamma(P) \) is \( 2\mu d \)-banded, where \( \mu = \max_i k_i \). Therefore, the Cholesky factor of \( \Gamma(P) \) is \( \mu d \)-banded, and the term \( \Gamma^{-1}(P)s(P) \) in the cost function (20) can be computed efficiently using the Cholesky factorization of \( \Gamma(P) \). In [Usevich and Markovsky, 2013] has been shown that the cost function of (19), its gradient and a Gauss-Newton approximation of the Hessian \((2J^TJ, \text{ where } J \text{ is the Jacobian of (21)}) can all be evaluated in \( O(t^3kl) \) flops. Also the complexity of the cost function and gradient evaluation is \( O(t^3kl) \) when the weights are uniform across the subvectors \( c^{(i,j)} \) of the vector \( c \).

As we have shown in Section 3.1, the cost function in (23) has the same form as the cost function in (19). Note that in the special case of problem (23), when \( t = 1 \) and \( K = 1 \) and...
the norm is Euclidean, our complexity results for the cost function evaluation coincide with the ones of Corless et al. (1995). For the case of non-trivial weights (and also \( t = 1 \) and \( K = 1 \)), the complexity results for the cost function were proposed to be \( O(l^2 \log(l)) \) by Pan (2001). Therefore, our results improve over the results of Pan (2001). For computation of the gradient, the complexity results for \( t = 1 \) and \( K = 1 \) and the Euclidean norm was presented by Markovsky and Van Huffel (2006). The algorithms and their complexity for computation of an approximation of the Hessian was first considered by Usevich and Markovsky (2013). However, an approximation of the Hessian allows to use more robust and efficient methods (for example, in (Markovsky and Usevich, 2012) the Levenberg-Marquardt method is used for minimization of \( y^*_k(\mathbb{L}_N)(P) \)).

Note that the cost function and its derivatives in the problem

\[
\text{minimize } M^{(LS)}_k(P) = \| y^{(LS)}_k(P) \|_2^2, \quad (26)
\]

coincide (up to a linear transformation) with the cost function and its derivatives for the problem (25). The software (Markovsky and Usevich, 2012) allows to perform local optimization for both problems (25) and (26).

### 3.3. Accuracy of the computations

The key computational procedure in evaluation of the cost function and its derivatives is the solution of the system of equations \( \Gamma u = v \) using the Cholesky factorization. In what follows we consider the case of the Euclidean norm (which can also be generalized to the case of block-wise weights (Usevich and Markovsky, 2013)). In this case,

\[
\Gamma(P) = \text{blkdiag} \left( \Gamma^{(1)}(P), \ldots, \Gamma^{(K)}(P) \right),
\]

where \( \Gamma^{(i)}(P) := A_{k,i}^T(P) A_{k,i}(P) \). Therefore, we can solve the system \( \Gamma u = v \) separately, block by block.

The accuracy of solving the system of equations \( \Gamma^{(i)}(P) u = v \) with Cholesky factorization mainly depends on the condition number of \( \Gamma^{(i)} \) (Golub and Van Loan, 1996). In Corless et al. (1995), only a preliminary analysis of the conditioning of \( \Gamma^{(i)} \in \mathbb{R}^{lxlt} \) was performed (for the case \( t = 1 \) and \( K = 1 \)).

Let us introduce the matrix polynomial

\[
P(z) := \begin{bmatrix}
P^{(1,1)}(z) & \ldots & P^{(1,t)}(z) \\
\vdots & \ddots & \vdots \\
P^{(K,1)}(z) & \ldots & P^{(K,t)}(z)
\end{bmatrix},
\]

where \( P^{(i,j)}(z) \) are the generating functions of the vectors \( P^{(i,j)} \). Then \( \Gamma^{(i)} \) is block-Toeplitz with the symbol (Miranda and Tilli, 2000)

\[
F(z) = P^T(z^{-1}) P(z).
\]

Since \( F(z) \) is Hermitian for all \( z \) and \( F \) is continuous on the unit circle \( \mathbb{T} \), we can define

\[
a_F := \min_{z \in \mathbb{T}} \lambda_{\min}(F(z)) \geq 0, \quad b_F := \max_{z \in \mathbb{T}} \lambda_{\max}(F(z)) < \infty.
\]

Where \( \lambda_{\min}(B) \) and \( \lambda_{\max}(B) \) are the minimal and maximal eigenvalues of a matrix \( B \). The results of (Miranda and Tilli, 2000) imply that \( \lambda_{\min}(\Gamma^{(i)}) \prec a_F \) and \( \lambda_{\max}(\Gamma^{(i)}) \succ b_F \).
i.e. the eigenvalues are in the interval \([a_F; b_F]\) and converge to the endpoints as \(l \to \infty\). Therefore, the condition number \(\kappa_2(\Gamma(l)) := \frac{\lambda_{\text{max}}(\Gamma(l))}{\lambda_{\text{min}}(\Gamma(l))}\) behaves as

\[
\kappa_2(\Gamma(l)) \sim \frac{b_F}{a_F}.
\]

If \(F(z)\) is positive definite on \(T\), then \(\kappa_2(\Gamma(l)) \leq \frac{b_F}{a_F} < \infty\). Otherwise, \(\kappa_2(\Gamma(l)) \to \infty\), and the rate of the convergence depends on the order of the zeros of \((F(z))\) on \(T\) (Serra, 1998).

4. Optimization in the image representation

In this section, we consider the problem of finding the distance to the set \(\mathcal{F}_d\), defined in (10). As noted in Section 1.2, this corresponds to solving Problem 1 using the image representation.

**Problem 8** (Approximate common divisor). Given \(p\), find

\[
\min_{\hat{g} \in \mathbb{P}_{n-d}, \hat{h} \in \mathbb{P}_d} \| \hat{g} \cdot \hat{h} - p \|_w^2. \tag{28}
\]

Note that we can include the zero polynomial \(\hat{h}\) in the search space of (28) (compared to the definition in (10)), since the zero polynomial is necessarily in \(\mathcal{F}_d\).

We see that the problem is a nonlinear least squares problem in both \(\hat{h}\) and \(\hat{g}\). However, if we fix one of the variables, then the problem becomes linear least squares. Next, we show that in both cases problem (28) can be reduced to minimization of \(M^{(LS)}_v(P)\) for suitably chosen \(k, l\) and \(t\).

4.1. Variable projection with respect to \(\hat{h}\) for real polynomials

In the case of variable projection with respect to \(\hat{h}\) for real polynomials, the problem (8) is rewritten as the following double minimization problem

\[
\min_{h \in \mathbb{P}_d \setminus \{0\}} f_1(h), \quad \text{where}
\]

\[
f_1(h) := \text{minimum of (28) for fixed } \hat{h}. \tag{30}
\]

Hence,

\[
f_1(\hat{h}) = M_{v}^{(LS)}(\hat{h}) \quad \text{for } k = \left[ d + 1 \right], l = n - d \quad \text{and } v = w.
\]

Note that \(f\) is invariant to multiplication by a scalar, i.e. \(f_1(\alpha \hat{h}) = f_1(\hat{h})\) for any \(\alpha \neq 0\), since the columns of \(A_{k,1}(\hat{h})\) and \(A_{k,1}(\alpha \hat{h})\) span the same subspace. Therefore, (29) is a minimization on the projective space (which is a special case of the Grassmann manifold). Thus, the problem (29) is equivalent to the problem (26) and can be solved by the software package [Markovsky and Usevich, 2012].

Thus we have reduced the number of optimization variables from \(\Sigma(n - d + 1) + d\) to \(d\), which is beneficial if \(d \ll n\). Indeed, as shown in Section 3.2, the cost function and its first- and second-order derivatives can be evaluated in linear time in \(n\).
\textit{GCD common parameters 13a} \equiv
\begin{verbatim}
if ~exist('opt', 'var')
opt = struct();
end
p = reshape(p,length(p), 1);
bfp = cell2mat(p);
bfn = cellfun(@(length, p) - 1;
bfell = bfn - d);
end
\end{verbatim}

Fragment referenced in 14a, 15, 19b, 20, 21a.

\textit{Calculate opt.hini if it is empty 13b} \equiv
\begin{verbatim}
if (~isfield(opt, 'hini') || isempty(opt.hini) )
\textit{Calculate opt.gini if it is empty 14b}
opt.hini = lsdivmult(p, d, opt.gini);
end
\end{verbatim}

Fragment referenced in 14a.

\textit{Split ph into separate polynomials 13c} \equiv
\begin{verbatim}
ph = mat2cell(ph, bfn+1, [1]);
\end{verbatim}

Fragment referenced in 14a, 15, 19b.

\textit{Optimize NLS ph with SLRA solver 13d} \equiv
\begin{verbatim}
s.gcd = 1;
if isempty(w)
    [ph, info] = slra(bfp, s, sum(s.m)-1, opt);
else
    bw = cell2mat(w);
s.w = 1./bw;
    [ph, info] = slra(bfw.*bfp, s, sum(s.m)-1, opt);
end
\end{verbatim}

Fragment referenced in 14a, 15.
function [ph, info] = gcd_nls(p, w, d, opt)
⟨GCD common parameters 13a⟩
⟨Calculate opt.hini if it is empty 13b⟩
opt.Rini = opt.hini';
s = struct('m', d+1, 'n', bfn-d+1);
⟨Optimize NLS ph with SLRA solver 13d⟩
⟨Split ph into separate polynomials 13c⟩
end

From the results of Section 3.3, we have that the accuracy of the computations are determined by the properties of the function $F(z)$, which in this case has the form $F(z) = |\hat{h}(z)|^2$. Therefore, the computations become ill-conditioned if the tentative common divisor has roots on or close to the complex unit circle $\mathbb{T}$. Note that this is similar to the results on conditioning of the Sylvester matrix (Zeng and Dayton, 2004, Prop. 6).

4.2. Variable projection with respect to $\hat{g}^{(k)}$ for real polynomials

In the case of variable projection with respect to $\hat{g}^{(k)}$ for real polynomials, the problem (8) is rewritten as the following double minimization problem

$$\minimize_{\hat{g} \in \mathbb{P}_{n-d}} f_2(\hat{g}), \text{ where}$$

$$f_2(\hat{g}) := \minimize_{\hat{g}} \text{ of (28) for fixed } \hat{g}. \quad (32)$$

Then the cost function has the form

$$f_2(\hat{g}) = M_{k,l}^{LS}(\hat{g}) \quad \text{for} \quad k = n - d, \ l = [d + 1] \quad \text{and} \quad v = w.$$  

This type of variable projection was used in (Stoica and Söderström, 1997; Agrawal et al., 2004) (under the name of Common Factor Estimation).

We also have that $f_2(\alpha \hat{g}) = f_2(\hat{g})$ for $\alpha \neq 0$, since the columns of $A_{k,1}(\hat{g})$ and $A_{k,1}(\alpha \hat{g})$ span the same subspace. Therefore, problem (31) is also an instance of problem (26).

⟨Calculate opt.gini if it is empty 14b⟩

```matlab
if (~isfield(opt, 'gini') || isempty(opt.gini) )
opt.gini = g_ini(p, d);
end
```

Fragment referenced in 13b, 15.
function [ph, info] = gcd_cofe(p, w, d, opt)
\[\text{GCD common parameters 13a}\]
\[\text{Calculate opt.gini if it is empty 14b}\]
opt.Rini = opt.gini';
s = struct('m', bfn-d+1, 'n', d+1);
\[\text{Optimize NLS ph with SLRA solver 13d}\]
\[\text{Split ph into separate polynomials 13c}\]
end

The number of optimization variables is reduced from \(\sum(n - d + 1) + d\) to \(\sum(n - d + 1)\), which is beneficial if \((n_k - d) \ll n\) (the degree of the common divisor is high). This is the case, for example, in deconvolution problems (Stoica and Söderström, 1997). In this case, the computational complexity per iteration is also linear in \(n\).

The symbol (27) in this case equals to \(F(z) = \sum_{k=1}^{Nk} |\hat{g}(k)(z)|^2\), therefore the computations are ill-conditioned if the polynomials \(\hat{g}(k)(z)\) have common roots on the unit circle \(T\). Note that this is less likely to occur in practice than \(\hat{h}\) having roots on the unit circle.

4.3. Variable projection with respect to \(\hat{h}\) for complex polynomials

Now we assume that the polynomials in (28) are complex. Then they can be represented by the vectors
\[
p^{(k)} = p^{(R,k)} + i \cdot p^{(I,k)}, \quad \hat{h} = \hat{h}^R + i \cdot \hat{h}^I, \quad g^{(k)} = g^{(R,k)} + i \cdot g^{(I,k)},
\]
where \(i = \sqrt{-1}\). If we set
\[
\begin{align*}
p^{(R)} &= \text{col} \left( p^{(R,1)}, p^{(I,1)}, \ldots, p^{(R,N)}, p^{(I,N)} \right), \\
g^{(R)} &= \text{col} \left( g^{(R,1)}, g^{(I,1)}, \ldots, g^{(R,N)}, g^{(I,N)} \right), \\
w^{(R)} &= \text{col} \left( w^{(1)}, w^{(1)}, \ldots, w^{(N)}, w^{(N)} \right), \\
P^{(\hat{h})} &= \begin{bmatrix} \hat{h}^R & -\hat{h}^I \\
\hat{h}^I & \hat{h}^R \end{bmatrix} \in \mathbb{R}^{(d+1) \times 2}.
\end{align*}
\]

Then we have that the problem (28) is equivalent to
\[
\min_{\hat{h}, \hat{g}} \|A_{k,1}(P^{(\hat{h})})p^{(R)} - g^{(R)}\|_{w^{(R)}}, \quad (33)
\]
for \(k = \begin{bmatrix} d+1 & d+1 \end{bmatrix}^T\), \(1 = n - d\) and \(v = w\). The problem (33) can be equivalently rewritten as
\[
\text{minimize } f_3(P^{(\hat{h})}), \quad \text{where } \hat{h} \in \mathbb{P}_{\mathbb{C}} \backslash \{0\} \quad (34)
\]
\[
f_3(P^{(\hat{h})}) := \text{minimum of (33) for fixed } P^{(\hat{h})} = M^{(LS)}(P^{(\hat{h})}). \quad (35)
\]
It can be seen that \(f_3(P^{(\hat{h})}) = f_3(P^{(\alpha \hat{h})})\) for any \(\alpha \in \mathbb{C} \backslash \{0\}\). Therefore, \(f_3\) can be minimized on the complex projective plane. The cost function can be evaluated in the software package.
\(\text{(Markovsky and Usevich, 2012), but minimization over the projective complex plane is not yet implemented.}

However, something can be said about conditioning of \(\Gamma\). We have that for \(P = P(\hat{h})\)
\[
F(z) = \begin{bmatrix}
\hat{h}^\Re(z^{-1}) & \hat{h}^\Im(z^{-1}) \\
-\hat{h}^\Im(z^{-1}) & \hat{h}^\Re(z^{-1})
\end{bmatrix}
\begin{bmatrix}
\hat{h}^\Re(z) - \hat{h}^\Im(z) \\
\hat{h}^\Im(z) \hat{h}^\Re(z)
\end{bmatrix},
\]
and
\[
det F(z) = |\hat{h}^\Re(z)|^4 + |\hat{h}^\Im(z)|^4 + 2\Re\left(\hat{h}^\Re(z^{-1})\hat{h}^\Im(z)\right)^2 \text{ on } \mathbb{T}.
\]
Therefore, the \(\Gamma\) is ill-conditioned if the polynomials \(\hat{h}^\Re(z)\) and \(\hat{h}^\Im(z)\) have common roots on the unit circle \(\mathbb{T}\).

### 4.4. Angles between polynomials as an approximation criterion

In this section, we consider the problem for finding the distance to the set \(\mathcal{F}_d\) (10), in the following distance based on angles between vectors
\[
\text{dist}_{\text{sin}}(p, q) = \sum_{k=1}^{N} \sin^2(\angle(p^{(k)}, q^{(k)})). \tag{36}
\]
(For simplicity we consider only the case of real polynomials.) The distance (36) depends only on the roots of the polynomials. By applying the variable projection principle with respect to \(\hat{h}\), the problem becomes minimization of \(f_4(\hat{h})\), where
\[
f_4(\hat{h}) := \min_\mathcal{G} \sum_{k=1}^{N} \sin^2(\angle(p^{(k)}, \hat{h} \cdot g^{(k)})) = \sum_{k=1}^{N} \min_\mathcal{G} \sin^2(\angle(p^{(k)}, M_{n_k-d-1}(\hat{h})g^{(k)})�).
\]
It is well known (Kendall and Stuart, 1977, §17.26), that the least-squares solution minimizes the angle between the approximating vector and the given vector. Let \(\hat{g}^{(k)}\) be the solution of a least-squares problem with matrix \(A_k = M_{n_k-d-1}(\hat{h})\) and right-hand side \(b = p^{(k)}\). Then \((p^{(k)} - A_k \hat{g}^{(k)}) \perp A_k \hat{g}^{(k)}\) and we have that
\[
f_4(\hat{h}) := \sum_{k=1}^{N} \frac{||A_k \hat{g}^{(k)} - p^{(k)}||^2}{||p^{(k)}||^2} = \min_\mathcal{G} \sum_{k=1}^{N} \frac{||p^{(k)} - \hat{h} \cdot \hat{g}^{(k)}||^2}{||p^{(k)}||^2},
\]
which is an instance of the weighted 2-norm (7). Therefore, minimizing the relative distance (which is done by many authors, see for example (Bini and Boito, 2010)) is equivalent to minimizing the distance (36) based on angles.

### 5. Optimization in the kernel representation

#### 5.1. Kernel representation: an overview

First we will present a reformulation of Problem 1 using Sylvester identity and its generalizations. The following lemma is well-known in the computer algebra community (Rupprecht, 1999, Kaltofen et al., 2006). We present it in adjusted form (suitable for our definition of \(\mathbb{P}_n\), which allows zero polynomials).
Lemma 9 ([Kaltofen et al., 2006, Lemma 2.1, adjusted]). Let \( p \in \mathbb{P}_n \), \( u^{(1)} \in \mathbb{P}_{n_1-d} \setminus \{0\} \) and \( u^{(2)} \in \mathbb{P}_{n_2-d}, \ldots, u^{(N)} \in \mathbb{P}_{n_N-d} \) such that

\[
  u^{(k)} p^{(1)} - u^{(1)} p^{(k)} = 0, \quad \text{where} \quad u^{(k)} \in \mathbb{P}_{n_k}, \quad \forall k = 2, \ldots, N. \quad (u_k)
\]

Then the polynomials \( p^{(1)}, \ldots, p^{(N)} \) have a common divisor of degree at least \( d \).

**Proof.** The proof is given in A. \( \square \)

Note that the condition \((u_k)\) is equivalent to

\[
  S_d^{(1)}(p) u = 0,
\]

where \( S_d^{(1)}(p) \) is the following (generalized Sylvester subresultant) matrix:

\[
  S_d^{(1)}(p) := \begin{bmatrix}
    M_{n_1-d}(p^{(2)}) & -M_{n_2-d}(p^{(1)}) & 0 & 0 \\
    \vdots & 0 & \ddots & 0 \\
    M_{n_1-d}(p^{(N)}) & 0 & 0 & -M_{n_N-d}(p^{(1)})
  \end{bmatrix} \in \mathbb{R}^{K \times L}. \tag{37}
\]

In the literature it is assumed in ([Rupprecht, 1999; Kaltofen et al., 2006]) that we can replace Problem 1 with the structured low-rank approximation of \( S_d^{(1)} \). However, the rank deficiency of \( S_d^{(1)} \) does not imply in general that \( \deg \gcd p \geq d \), because \( u^{(1)} \) may be zero in the right kernel of \( S_d(p) \) (which is required in Lemma 9). A better alternative was proposed by [Agrawal et al., 2004]

\[
  S_d(p) := \begin{cases}
    0 & S_d^{(1)}(p^{(2)}, \ldots, p^{(N)}) \\
    S_d^{(1)}(p), & \text{if} \quad N = 2.
  \end{cases}
\]

Here we show that the low-rank approximation of \( S_d(p) \) is indeed equivalent to Problem 1.

**Proposition 10.** Homogeneous polynomials \( p = (p^{(1)}, \ldots, p^{(N)}) \in \mathbb{P}_n \) have \( \deg \gcd p \geq d \) if and only if \( S_d(p) \) is rank-deficient

**Proof.** The proof is given in A. \( \square \)

Therefore, Problem 1 is equivalent to the following structured low-rank approximation problem.

**Problem 11** (Sylvester low-rank approximation problem).

\[
  \min_{\tilde{p}} \| \tilde{p} - p \|^2_w \quad \text{subject to} \quad S_d(p) \text{ is rank deficient}. \tag{38}
\]

However, in what follows we consider the Sylvester resultant matrices \( S_d^{(1)}(p) \).

5.2. Sylvester low-rank approximation as a mosaic Hankel low-rank approximation

In this subsection, we reduce the low-rank approximation of \( S_d^{(1)}(p) \) to a form that can be solved by the methods of ([Markovsky and Usevich, 2012]).
\( L_b := \ell_2 + 1 + \sum_{k=3}^{N} (\ell_k + 1 + n_1), \)

\( K_b := \sum_{k=2}^{N} (n_k + \ell_1 + 1), \)

\( q^{(1)} := \text{col}(0_{K_b-n_1-1}, -p^{(1)}, 0_{K_b-n_1-1}), \)

\( q^{(2)} := \text{col}(0_{\ell_1}, p^{(2)}, 0_{\ell_1}, p^{(3)}, \ldots, 0_{\ell_1}, p^{(N)}, 0_{\ell_1}). \)

\( \langle \text{Sylvester GCD parameters 18a} \rangle \equiv \)

\[
\begin{align*}
L_b &= \text{bfell}(2) + 1 + \text{sum}(\text{bfell}(3:end) + 1 + \text{bfn}(1)); \\
K_b &= \text{sum}(\text{bfn}(2:end) + (\text{bfell}(1) + 1)); \\
q_1 &= \text{zeros}(L_b-1,1); -p{1}; \text{zeros}(L_b-1,1)]; \\
q_2 &= \text{zeros}(\text{bfell}(1),1)]; \\
\text{for } k=2:length(p) \\
q_2 &= [q_2; p{k}; \text{zeros}(\text{bfell}(1),1)]; \\
\text{end}
\end{align*}
\]

\( \diamond \)

\( \langle \text{Create Sylvester structure 18b} \rangle \equiv \)

\[ (\mathcal{S}_d^{(1)}(p))^\top = \Phi \mathcal{H}_{m,K_b} \begin{bmatrix} q^{(1)} \\ q^{(2)} \end{bmatrix}, \quad \text{where} \quad m := \begin{bmatrix} L_b & \ell_1 & 1 \end{bmatrix}^\top \quad \text{and} \]

\( \Phi := J \text{blkdiag} \left( \begin{bmatrix} I_{\ell_1+1} & 0_{(\ell_1+1)\times d} \\ 0_{(\ell_3+1)\times d} & \ldots & \ldots & \ldots \\ I_{\ell_2+\ell_1+2} \end{bmatrix} \right) \] (39)

**Proof.** The proof is given in A. \( \square \)
Hence, the problem (38) can be solved as a weighted mosaic low-rank approximation of the matrix in (39), if we fix the zero elements. This can be accomplished by putting infinite weights in the approximation criterion.

\[ w = \text{col}(\infty 1_{n_b-n_1}, w^{(1)}, \infty 1_{n_b-n_1}, \infty 1_{\ell_1}, w^{(2)}, \infty 1_{\ell_1}, w^{(3)}, \ldots, \infty 1_{\ell_1}, w^{(N)}, \infty 1_{\ell_1}). \]

Create weights for Sylvester structure 19a

\[
\langle \text{Create weights for Sylvester structure 19a} \rangle \equiv \\
\text{if (\text{exist}(w, 'var') || isempty(w))} \\
\quad w = \text{cellfun}(@(x){\text{ones(size(x))}}, p); \\
\text{end} \\
w_1 = [\text{Inf} \ast \text{ones(Kb-bfn(1)-1,1)}; w(1); \text{Inf} \ast \text{ones(Kb-bfn(1)-1,1)}]; \\
w_2 = [\text{Inf} \ast \text{ones(bfell(1),1)}]; \\
\text{for } k=2:\text{length(bfn)} \\
\quad w_2 = [w_2; w(k); \text{Inf} \ast \text{ones(bfell(1),1)}]; \\
\text{end} \\
s.w = [w_1; w_2]; \\
\]

Fragment referenced in 19b.

Therefore, the problem (38) is equivalent to the following problem

\[
\min_{u} f_{LN}(\Phi^T u),
\]

where \( f_{LN} \) is defined in (19). The minimization of \( f_{LN}(\Phi^T u) \) can be performed by the SLRA package (SLR, 2013; Markovsky and Usevich, 2012).

Create Sylvester structure

\[
\langle \text{Create Sylvester structure 18b} \rangle
\]

Create weights for Sylvester structure 19a

\[
\langle \text{Create weights for Sylvester structure 19a} \rangle
\]

\[
\text{opt.reggamma} = 10e4; \\
[ph, info] = \text{slra}([q_1;q_2], s, \text{size}(s.phi,1)-1, \text{opt}); \\
ph = ph(s.w=\text{Inf}); \\
\langle \text{Split ph into separate polynomials 13c} \rangle \\
ph(1) = -ph(1); \\
\]

5.3. Computational complexity

The complexity of the computations using the variable projection principle is \( O(\ell_0^2 n_b) \). In terms of degrees of the polynomials, it is \( O((n-d)^2 n) \) if \( N = 2 \). Therefore for 2 polynomials this approach is beneficial if \( (n_k - d) \ll n \). In the general case \( N > 2 \), the complexity is \( O(n^3) \), even if \( (n_k - d) \ll n \).
5.4. Initial approximation for the developed methods

The formulation (38) also has advantages that it yields and initial approximation which can be obtained from the following problem:

\[
\begin{array}{ll}
\text{minimize} & \frac{u^T S_d^T(p) S_d(p) u}{u^T u}.
\end{array}
\] (40)

The solution of (40) can be obtained from the unstructured low-rank approximation of the matrix \( S_d(p) \) (for example, taking the last vector in the SVD, or using the QR decomposition). The vector \( u_{LRA} \) obtained from the unstructured low-rank approximation can serve as an initial approximation for \( u \) in (38).

On the other hand, \( u_{LRA} \) can be used as an initial approximation for the problem (31), as shown by the following lemma.

**Lemma 13.** If \( \deg \gcd(\hat{p}) = d \) and \( S_d(\hat{p}) = 0 \), then the polynomials \( u^{(k)} \) are the factors of \( p^{(k)} \) from (\( u_k \)), i.e. \( p^{(k)} = u^{(k)} h \), where \( h \in \gcd(\hat{p}) \).

**Proof.** The proof is given in A. □

Since point with \( \deg \gcd(\hat{p}) = d \) represent generic points in \( \mathcal{G}_n \), we may assume that an approximation \( u_{LRA} \) of the “true” \( u \) is an approximation of the cofactors. In order to obtain an initial approximation for the problem (29), we can solve a least-squares problem, which yields and initial guess for \( \hat{h} \). This can be summarized in the following two-step algorithm: The following algorithm is used to obtain initial approximation:

- Compute \( u_{LRA} \) — last right singular vector of \( S_d(p) \) (or \( S_d^{(1)}(p) \));
- put \( \hat{h}_0 = \arg\min_N \sum_{k=1}^N \| M_d(\hat{u}^{(k)}) \hat{h}_0 - p^{(k)} \|_w^2 \).

"g_ini.m" 20

```matlab
function [g] = g_ini(p, d)
\langle GCD common parameters 13a \rangle

\langle Create Sylvester structure 18b \rangle
opt = struct(’disp’, ’off’, ’maxiter’, 0);
[bfph, info] = slra([q1;q2], s, size(s.phi,1)-1, opt);
g = info.Rh(:);
end
```

20
6. Numerical examples

6.1. Example of ill-conditioned polynomials (Bini and Boito, 2010, Terui, 2013)

First we consider Example 4.2 from Bini and Boito (2010) (Test 5 from Terui, 2013). In this test the following two polynomials are considered:

\[ u(x) = 10 \prod_{1}^{10} (x - x_j), \quad v(x) = 10 \prod_{1}^{10} (x - x_j), \quad x_j = (-1)^j (j/2). \]

We define \( p^{(1)} = u(x)/\|u\|_2 \) and \( p^{(2)} = v(x)/\|u\|_2 \) (note that this differs from normalization in Bini and Boito, 2010), and compare several methods according to Euclidean distance \( \| (p^{(1)}, p^{(2)}) - (\hat{p}^{(1)}, \hat{p}^{(2)}) \|_2 \).

Define test 1 polynomials

```matlab
function [h] = lsdivmult(p, d, g)

% GCD common parameters
    st2 = cumsum([0;bfell]+1);
    mm = [];
    for k=1:length(bfn)
        mm = [mm; multmat(g(st2(k):st2(k+1)-1), d)];
    end
    h = mm \ bfp;
end
```

All the methods are started from the same initial approximation computed in Section 5.4. In Table 1 we present the results of comparison for \( d = \{1, \ldots, 10\} \), and in Table 2. “LRA” stands for using initial approximation in Section 5.4 without optimization (refinement). “STLN” denoted the figures for the STLN method of Kaltofen et al. (2006) reported in Terui (2013) (scaled by the factor \( \|u\|_2 \)). “UVGCD” denoted the figures for the UVGCD method of Zeng.
Table 1. Optimal approximations of the methods

| \(d\) | LRA       | STLN      | UVGCD     | IM\([\hat{g}]\)/IM\([g]\) | KER        | GN+LS     |
|---|-----------|-----------|-----------|-----------------------------|------------|-----------|
| 1 | \(2.37 \cdot 10^{-3}\) | \(4.46 \cdot 10^{-5}\) | \(3.54 \cdot 10^{-16}\) | \(3.63 \cdot 10^{-16}\) | \(1.36 \cdot 10^{-16}\) | \(8.85 \cdot 10^{-16}\) |
| 2 | \(1.1 \cdot 10^{-4}\)   | \(5.99 \cdot 10^{-8}\) | \(2.28 \cdot 10^{-14}\) | \(2.28 \cdot 10^{-14}\) | \(2.61 \cdot 10^{-17}\) | \(6.32 \cdot 10^{-14}\) |
| 3 | \(1.09 \cdot 10^{-4}\) | \(1.7 \cdot 10^{-9}\) | \(1.55 \cdot 10^{-12}\) | \(1.55 \cdot 10^{-12}\) | \(1.91 \cdot 10^{-16}\) | \(3.12 \cdot 10^{-12}\) |
| 4 | \(1.79 \cdot 10^{-5}\) | \(2.49 \cdot 10^{-10}\) | \(8.53 \cdot 10^{-11}\) | \(3.64 \cdot 10^{-6}\) | \(1.05 \cdot 10^{-14}\) | \(1.79 \cdot 10^{-5}\) |
| 5 | \(4.65 \cdot 10^{-5}\) | \(4.55 \cdot 10^{-9}\) | \(4.55 \cdot 10^{-9}\) | \(4.55 \cdot 10^{-9}\) | \(1.75 \cdot 10^{-5}\) | \(4.65 \cdot 10^{-5}\) |
| 6 | \(3.3 \cdot 10^{-5}\)  | \(1.85 \cdot 10^{-7}\) | \(1.86 \cdot 10^{-7}\) | \(1.86 \cdot 10^{-7}\) | \(2.75 \cdot 10^{-5}\) | \(3.3 \cdot 10^{-5}\) |
| 7 | \(4.96 \cdot 10^{-4}\) | \(7.19 \cdot 10^{-6}\) | \(7.19 \cdot 10^{-6}\) | \(7.19 \cdot 10^{-6}\) | \(3.98 \cdot 10^{-4}\) | \(1.13 \cdot 10^{-5}\) |
| 8 | \(2.89 \cdot 10^{-3}\) | \(1.76 \cdot 10^{-4}\) | \(1.76 \cdot 10^{-4}\) | \(1.76 \cdot 10^{-4}\) | \(1.76 \cdot 10^{-4}\) | \(3.27 \cdot 10^{-4}\) |
| 9 | \(2.2 \cdot 10^{-2}\)  | \(4.05 \cdot 10^{-3}\) | \(4.05 \cdot 10^{-3}\) | \(4.05 \cdot 10^{-3}\) | \(4.05 \cdot 10^{-3}\) | \(4.29 \cdot 10^{-3}\) |
| 10| \(6.67 \cdot 10^{-2}\) | \(6.67 \cdot 10^{-2}\) | \(6.67 \cdot 10^{-2}\) | \(6.67 \cdot 10^{-2}\) | \(6.67 \cdot 10^{-2}\) | \(6.67 \cdot 10^{-2}\) |

Table 2. Number of iterations of the methods

| \(d\) | IM\([\hat{g}]\)/IM\([g]\) | KER | GN+LS |
|---|----------------|-----|-------|
| 1 | 4              | 5   | 5     |
| 2 | 4              | 2   | 13    |
| 3 | 4              | 2   | 7     |
| 4 | 100            | 3   | 1     |
| 5 | 5              | 6   | 1     |
| 6 | 0              | 16  | 1     |
| 7 | 4              | 100 | 20    |
| 8 | 3              | 5   | 6     |
| 9 | 3              | 4   | 7     |
| 10| 1              | 1   | 1     |

“IM\([\hat{g}]\)” denotes the variable projection method w.r.t. \(\hat{h}\) in the image representation (Section 4.1). “IM\([g]\)” denotes the variable projection method w.r.t. \(\hat{g}\) in the image representation (Section 4.2). “KER” stands for the variable projection method in the kernel representation (Section 5.2). “GN+LS” denotes the combination of the Gauss-Newton method and line search, used for the optimization step in (Bini and Boito, 2010). (We modified the code of Bini and Boito (2010) in order to obtain the correct stopping criterion.)

The constraints in the current version of the software in (SLR, 2013) require \(k_i \leq l_j\) in (17). (These constraints are reasonable for mosaic Hankel low-rank approximation (Usevich and Markovsky, 2013).) These constraints exclude the cases \(d \in \{6, \ldots, 10\}\) in the method of Section 4.1 and \(d \in \{1, \ldots, 6\}\) the method of Section 4.2. Hence, the results of the image representation methods (IM\([\hat{g}]\) and IM\([g]\)) are combined in one column in Table 1 and Table 2.

The results in Table 1 show that for all \(d\) except 4 (and also 6, which was not computed) the methods based on the image representation (Sections 4.1 and 4.2), match the results of...
the UVGCD method (which are the overall best). Note that for all \( d \) (including \( 4 \)) the methods based on image representation improve over the Gauss-Newton method from (Bini and Boito, 2010). (However, it should be noted that for \( d = 4 \) the image representation method runs into maximum number of iterations.)

The method based on the kernel representation (Section 5.2) fails to produce good results for \( d < 8 \). For \( d = 1, \ldots, 3 \), it has ill-conditioned \( \Gamma \) matrix, and the result is computed with regularization of \( \Gamma \) in the package (SLR, 2013). This means that the computed approximating polynomials may not have a common divisor.

\[
\langle \text{SLRA include directories 23a} \rangle \equiv \\
\quad \text{addpath } 
\ldots 
\text{addpath } 
\ldots 
\text{addpath } 
\ldots
\langle \text{Prepare arrays 23b} \rangle \equiv \\
\quad \text{res} = \text{zeros}(10,7); \\
\quad \text{iters} = \text{zeros}(10,4); \\
\quad \text{res}(:,1) = (1:10)'; \\
\quad \text{iters}(:,1) = (1:10)'; \\
\quad \text{res}(:,3) = \left[ 5.17e-1; 6.95e-4; 1.97e-5; 2.89e-6; 5.28e-5; \ldots \\
\quad \quad 2.15e-3; 8.34e-2; 2.04e0; 4.70e1; 7.73e2 \right] / \text{sccoef};
\]
\[
\quad \text{opt.disp} = 'iter'; \\
\quad \text{opt.epsgrad} = 1e-20;
\]

Fragment referenced in 25.
\begin{verbatim}
for d=1:5
    [ph, info] = gcd_nls({p,q}, [], d, opt);
    res(d,5) = norm([p;q] - cell2mat(ph),2);
    iters(d,2) = info.iter;
end
for d=7:10
    [ph, info] = gcd_cofe({p,q}, [], d, opt);
    res(d,5) = norm([p;q] - cell2mat(ph),2);
    iters(d,2) = info.iter;
end
for d=1:10
    [ph, info] = gcd_syl({p,q}, [], d, opt);
    res(d,6) = norm([p;q] - cell2mat(ph),2);
    iters(d,3) = info.iter;
end
\end{verbatim}

Fragment referenced in 25.
6.2. **Comparison of the speed of the computations**

In this section we consider the following family of polynomials (which is similar to the one used in [Bini and Boito, 2010, § 4.6]):

\[
g^{(1,k)}(z) := h(z)g^{(1,k)}(z), \quad g^{(2,k)}(z) := h(z)g^{(2,k)}(z),
\]
\[
h := z^4 + 10z^2 + z - 1,
\]
\[
g^{(1,k)}(z) := (z^{k\ell_1} - 1)(z^{k\ell_2} - 2)(z^{k\ell_3} - 3),
\]
\[
g^{(2,k)}(z) := (z^{k\ell_1} + 1)(z^{k\ell_2} + 5)(z^{k\ell_3} + 2),
\]

where \(\ell_1 = 25, \ell_2 = 15, \ell_3 = 10\). The test polynomials are defined to be

\[
p^{(1,k)} := q^{(1,k)} + \varepsilon_{1,k}, \quad p^{(2,k)} := q^{(2,k)} + \varepsilon_{2,k},
\]

where each \(\varepsilon_{j,k}\) is a single realization of the Gaussian zero-mean random vector with the covariance matrix \(10^{-4}I\).
Define test 2 polynomials 26a

\[
\begin{align*}
n_1 &= 25k; n_2 = 15k; n_3 = 10k; 
\end{align*}
\]

\[
\begin{align*}
g1 &= \text{conv}(\text{conv}([-1; \text{zeros}(n_1 - 1, 1); 1], ... \\
   & \quad [-2; \text{zeros}(n_2 - 1, 1); 1]), [-3; \text{zeros}(n_3 - 1, 1); 1]); \\
g2 &= \text{conv}(\text{conv}([1; \text{zeros}(n_1 - 1, 1); 1], ... \\
   & \quad [5; \text{zeros}(n_2 - 1, 1); 1]), [2; \text{zeros}(n_3 - 1, 1); 1]); \\
p &= \text{conv}(g1, h0); q = \text{conv}(g2, h0); \\
sigma &= 1e-2; \\
p &= p + \sigma \text{randn(size(p))}; \\
q &= q + \sigma \text{randn(size(q))}; \\
\end{align*}
\]

Fragment referenced in 27.

We consider the test polynomials for \( k = \{1, \ldots, 8\} \), thus the degrees of the polynomials range between 50 and 400. We compare two methods: “IM\( ^h \)” (Section 4.1) and “GN” (the Gauss-Newton method from [Bini and Boito, 2010], without line search). We limit the number of iterations to 1 in order to measure the time needed per iteration. All the methods are started from the same initial approximation. In Fig. 1, the time needed to perform 1 iteration of the local optimization procedure is plotted versus \( k \).

![Graph](image.png)

**Fig. 1.** Comparison of the computational complexity of the methods

It can be seen from Fig. 1, the time growth is sublinear in \( k \) for the variable projection method, which confirms the results of Section 4. The time growth for the Gauss-Newton-based method is similar to quadratic. Note that the time needed for one iteration of the local optimization is of the same order as the total time reported in [Bini and Boito, 2010] (including initial approximation), and therefore cannot be neglected.

```
function t = meas_func(func, pars)
    tic; eltm = toc; n = 0;
    while (eltm < 1)
        func(pars{:});
        n = n+ 1;
        eltm = toc;
    end
    t = eltm / n;
end
```

\( \text{meas\_func.m} \) 26b

26
"test_speed.m" 27≡

⟨SLRA include directories 23a⟩
addpath 'fastgcd';

ntest = 8;
times = zeros(ntest,3);
iters = zeros(ntest,3);
res = zeros(ntest,3);

h0 = [-1; 1; 0; 10; 1];
d = length(h0) - 1;
times(:,1) = 1:ntest;
iters(:,1) = 1:ntest;
res(:,1) = 1:ntest;
for k=1:ntest
⟨Define test 2 polynomials 26a⟩

gini = g_ini({p,q}, d);
h = lsdivmult({p,q}, d, gini)'
g1 = gini(1:length(p)-d)'
g2 = gini(length(p)-d + (1:length(q)-d))'

hh = h; glh = g1; g2h = g2;

opt.hini = hh';
opt.epsgrad = 1e-20;
opt.maxiter = 1;

[ph, info] = gcd_nls({p,q}, [], d, opt);
res(k,2) = norm([p;q] -cell2mat(ph),2);
iters(k,2) = info.iter;

[hh, glh, g2h, resn, iter] = c_f_newton_iter_mod(p',q', ...
    hh,glh,g2h, 0, opt.maxiter+1, 0);
iters(k,3) = iter-1;
res(k,3) = norm([p;q] -[conv(hh,glh)';conv(hh,g2h)'],2);

hh = h; glh = g1; g2h = g2;
times(k,2) = meas_func(@gcd_nls, {{p,q}, [], d, opt});
times(k,3) = meas_func(@c_f_newton_iter_mod, {p',q', ...
    hh,glh,g2h, 0, opt.maxiter+1, 0});
end

save('res_speed.txt', 'res', '-ascii');
save('iter_speed.txt', 'iters', '-ascii');
save('times_speed.txt', 'times', '-ascii');
6.3. Singularity of $\Gamma$ matrix for $N > 2$ in kernel representation

In this subsection we show that on a particular example of $N = 3$ polynomials that in the kernel representation approach the corresponding $\Gamma$ matrix in (20) is essentially singular. Consider three polynomials $p^{(1)}, p^{(2)}, p^{(3)} \in \mathbb{P}_2$, and $d = 1$. Then the corresponding generalized Sylvester subresultant matrix is

$$S_1^{(1)}(p) = \begin{bmatrix} M_1(p^{(2)}) & -M_1(p^{(1)}) \\ M_1(p^{(3)}) & -M_1(p^{(1)}) \end{bmatrix}.$$ 

By Proposition 12 and the results of (Usevich and Markovsky, 2013), we have that the corresponding $\Gamma$ matrix in (20) has the form $\Gamma(\hat{u}) = G(u)G^\top(\hat{u})$, where $\hat{u} \in \mathbb{P}_1$ is the annihilating vector from (9), where

$$G(u) = \begin{bmatrix} -M_1(\hat{u}^{(2)}) & M_1(\hat{u}^{(1)}) \\ -M_1(\hat{u}^{(3)}) & M_1(\hat{u}^{(1)}) \end{bmatrix}.$$ 

It can be easily checked that the polynomial matrix $\Gamma(\hat{u})$ has (symbolic) determinant 0. This is also confirmed by running the kernel representation optimization method.

```
"test_det.m" 28

```
Acknowledgements

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A. Proofs

Lemma 14.

\[ G_d = \begin{cases} \mathcal{F}_d, & \text{if } \mathbb{F} = \mathbb{C} \text{ or } (\mathbb{F} = \mathbb{R} \text{ and } d \text{ is even}), \\ \mathcal{F}_d \cup \mathcal{F}_{d+1}, & \text{if } \mathbb{F} = \mathbb{R} \text{ and } d \text{ is odd}. \end{cases} \]

Proof. [Proof of Lemma 14] Evidently, \( \mathcal{F}_d \subset G_d, \mathcal{F}_{d+1} \subset G_d \) and \( 0 \in \mathcal{F}_d \). Let \( p = (p^{(1)}, \ldots, p^{(N)}) \in G_d \setminus \{0\} \) and \( h \in \mathbb{F}_d^* \) be a GCD of the \( p^{(k)} \), where \( d' > d \). Then, \( h \) can be factorized as

\[ h(z) = \sum_{k=1}^{d'} (z - \alpha_k), \]

where \( \alpha_k \in \mathbb{C} \cup \{\infty\} \). If \( \mathbb{F} = \mathbb{C} \) then the \( p^{(k)} \) have a common divisor of degree \( d \) and \( p \in \mathcal{F}_d \). If \( \mathbb{F} = \mathbb{R} \), then every \( \alpha_k \) has its conjugate counterpart. Hence, \( p \) have common divisors of degrees \( 2l \) for any \( l : 0 \leq 2l \leq d' \). Therefore, \( p \in \mathcal{F}_d \) if \( d \) is even and \( p \in \mathcal{F}_{d+1} \) if \( d \) is odd. \( \square \)

Lemma 15. For any \( 0 \leq d \leq n_{\min} \), the sets \( \mathcal{F}_d \) and \( G_d \) are closed subsets of \( \mathbb{P}_n \).

Proof. [Proof of Lemma 15] Denote by \( S_d \subset \mathbb{P}_d \) the polynomials with 2-norm equal to 1. Then \( \mathcal{F}_d \) is the image of the infinitely smooth map \( \mathbb{P}_{n-d} \times S_d \to \mathbb{P}_n \) defined as \( (\hat{g}, \hat{h}) \to \hat{g} \cdot \hat{h} \). Since the domain of the definition is closed and the map is continuous, \( \mathcal{F}_d \) is closed.

By Lemma 14, \( G_d \) can be expressed as a union of at most two sets \( \mathcal{F}_d \) and \( \mathcal{F}_{d+1} \), and therefore it is also closed. \( \square \)

Proof. [Proof of Lemma 9] If \( p^{(1)} = 0 \), then all polynomials \( p^{(k)} \) are zero by \( (u_k) \).

It is left to consider \( p^{(1)} \neq 0 \). The case when all other polynomials are zero is trivial. Without loss of generality assume that there exists \( 2 \leq K \leq N \) such that \( p^{(k)} \neq 0 \) for all \( 2 \leq k \leq N \). Then we need to prove that \( \deg \gcd(p^{(1)}, \ldots, p^{(K)}) \geq d \).

Since \( u^{(1)} p^{(k)} \neq 0 \), we have that \( u^{(k)} \neq 0 \) for \( k \geq 2 \). Let us rewrite \( (u_k) \) as

\[ \frac{p^{(1)}}{u^{(1)}} = \frac{p^{(2)}}{u^{(2)}} = \cdots = \frac{p^{(K)}}{u^{(K)}} = \frac{a}{b}, \]

where \( a/b \) is an irreducible fraction. Then we have that

\[ p^{(k)} = \frac{u^{(k)} a}{b}, \]

and \( u^{(k)} \) should be divisible by \( b \). Therefore, \( p^{(k)} \) have a common divisor \( a \), with \( \deg a \geq d \), which can be established by counting dimensions in (A.1). \( \square \)
Proof. [Proof of Proposition 10] The “only if” part is trivial. Indeed, one can construct \( \mathbf{u} \), as in Lemma 9.

In order to prove the “if”, let us look at the shifted Sylvester matrices

\[
\mathcal{J}_d(k)(p^{(1)}, \ldots, p^{(N)}) := \mathcal{J}_d(1)(p(k), p^{(1)}, p^{(k-1)}, p^{(k+1)}, \ldots, p^{(N)}),
\]

where \( l_k := n_k - d + 1 \), \( m_1 := \sum ((l_2, \ldots, l_{k-1})) \) and \( m_2 := \sum ((l_{k+1}, \ldots, l_{N})) \). Note that \( \mathcal{J}_d(k) \) coincides with the definition in (37). For example, for three polynomials we have that

\[
\begin{align*}
\mathcal{J}_d^{(2)}(p^{(1)}, p^{(2)}, p^{(3)}) &:= \begin{bmatrix}
-M_{n_1-d}p^{(2)} & M_{n_2-d}p^{(3)} \\
0 & M_{n_3-d}p^{(3)}
\end{bmatrix}, \\
\mathcal{J}_d^{(3)}(p^{(1)}, p^{(2)}, p^{(3)}) &:= \begin{bmatrix}
-M_{n_1-d}p^{(2)} & M_{n_3-d}p^{(3)} \\
-M_{n_2-d}p^{(2)} & M_{n_3-d}p^{(3)}
\end{bmatrix}.
\end{align*}
\]

Note that any matrix \( \mathcal{J}_d(k)(p) \) can be extracted from the matrix \( \mathcal{J}_d(p) \) by selecting corresponding block rows. Therefore, if \( \mathbf{u} \in \mathbb{F}_{n-d} \) is a right annihilating vector \( \mathbf{u} \in \mathbb{F}_{n-d} \), it is also annihilating vector of \( \mathcal{J}_d(k)(p) \). Let us select \( k \) such that \( u^{(k)} \neq 0 \). Then we have that

\[
\mathcal{J}_d(1)(p(k), p^{(1)}, p^{(k-1)}, p^{(k+1)}, \ldots, p^{(N)}) \mathbf{u}^{(k)}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k-1)}, \mathbf{u}^{(k+1)}, \ldots, \mathbf{u}^{(N)} = 0,
\]

and by Lemma 9, the polynomials have deg gcd \( \geq d \). \( \Box \)

Proof. [Proof of Proposition 12] For \( p \in \mathbb{F}_n \) and \( \ell \) we have

\[
\mathbf{M}_m(p)^T = \mathcal{H}_{\ell+1, \ell+n+1}(\mathbf{col}(\mathbf{0}_\ell, p, \mathbf{0}_\ell)).
\]

If we denote \( \ell_k := n_k - d \), then

\[
J \mathcal{J}_d(p)^T = \begin{bmatrix}
0 & 0 & \mathcal{H}_{\ell_N+1}(\mathbf{col}(\mathbf{0}_{\ell_N}, p^{(1)}, \mathbf{0}_{\ell_N})) \\
0 & \ddots & 0 \\
\mathcal{H}_{\ell_2+1}(\mathbf{col}(\mathbf{0}_{\ell_2}, p^{(1)}, \mathbf{0}_{\ell_2})) & 0 & 0 \\
\mathcal{H}_{\ell_1+1}(\mathbf{col}(\mathbf{0}_{\ell_1}, p^{(2)}, \mathbf{0}_{\ell_1})) & \cdots & \mathcal{H}_{\ell_1+1}(\mathbf{col}(\mathbf{0}_{\ell_1}, p^{(N)}, \mathbf{0}_{\ell_1}))
\end{bmatrix}
\]

\[
= \begin{bmatrix}
I_{\ell_N+1} \mathbf{0}_{(\ell_N+1) \times d} \\
\vdots \\
I_{\ell_3+1} \mathbf{0}_{(\ell_3+1) \times d} \\
I_{\ell_2+1} \\
I_{\ell_1+1}
\end{bmatrix}
\begin{bmatrix}
\mathcal{H}_{\ell_k}(q^{(1)}) \\
\mathcal{H}_{\ell_{k+1}}(q^{(2)})
\end{bmatrix}.
\]

\( \Box \)
Proof. [Proof of Lemma 13] If $\deg \gcd(p) = d$, then $b$ from (A.1) should be necessarily in $P_0$ (otherwise, by (A.2), the polynomials would have a common divisor of degree higher than $d$). Therefore, $u^{(k)}$ are factors of $p^{(k)}$ by (A.2). \[\square\]

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