Reproducing kernel Hilbert spaces and Mercer theorem

C. Carmeli,* E. De Vito† A. Toigo‡
29th March 2022

Abstract

We characterize the reproducing kernel Hilbert spaces whose elements are $p$-integrable functions in terms of the boundedness of the integral operator whose kernel is the reproducing kernel. Moreover, for $p = 2$ we show that the spectral decomposition of this integral operator gives a complete description of the reproducing kernel.

1 Introduction

In recent years there is a new interest for the theory of reproducing kernel Hilbert spaces in different frameworks, like statistical learning theory (Cucker and Smale, 2002), signal analysis (Daubechies, 1992) and quantum mechanics (Ali et al., 2000). In particular, for these applications there is often the need of reproducing kernel Hilbert spaces having some additional regularity property, as continuity or square-integrability.

This paper is both a research article and a self-contained survey about the characterization of the reproducing kernel Hilbert spaces whose elements are continuous, measurable or $p$-integrable functions ($1 \leq p \leq \infty$). As briefly reviewed in Section 2, this problem is equivalent to study the weak regularity properties of maps taking values in an arbitrary Hilbert space (Saitoh, 1988).

---

*C. Carmeli, Dipartimento di Fisica, Università di Genova, and I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy. e-mail: carmeli@ge.infn.it
†E. De Vito, Dipartimento di Matematica, Università di Modena e Reggio Emilia, Via Campi 213/B, 41100 Modena, Italy, and I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy. e-mail: devito@unimo.it
‡A. Toigo, Dipartimento di Fisica, Università di Genova, and I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy. e-mail: toigo@ge.infn.it
In the following sections, we take this last point of view and, given a set $X$ and a Hilbert space $\mathcal{H}$, we characterize the weak regularity properties of a map $\gamma : X \to \mathcal{H}$ in terms of the corresponding properties of the associated kernel
\[ X \times X \ni (x, y) \mapsto \langle \gamma_y, \gamma_x \rangle_{\mathcal{H}} \in \mathbb{C}. \tag{1} \]

More precisely, in Section 3 we prove that, if $X$ is a measurable set and $\mathcal{H}$ is a separable Hilbert space, $\gamma$ is weakly measurable if and only if the associated kernel (1) is separately measurable. Our proof is an easy consequence of the equivalence between weak and strong measurability. Moreover, if $\mathcal{H}$ is a space of square-integrable functions, we recall a result of Pettis (1938) clarifying the relation between vector valued maps and the theory of integral operators (Halmos and Sunder, 1978).

In Section 4 we show that, if $X$ is a measurable set endowed with a $\sigma$-finite measure $\mu$ and $\mathcal{H}$ is a separable Hilbert space, $\gamma$ is weakly $p$-integrable if and only if the integral operator with kernel given by Eq. (1) is bounded from $L^{p-1}(X, \mu)$ into $L^p(X, \mu)$ where $1 \leq p \leq \infty$. Up to our knowledge this result is new, though is deeply based on the theory of Pettis integral (Hille and Phillips, 1974). Moreover, for $p = 1$ we prove that the operator
\[ \mathcal{H} \ni v \mapsto \langle v, \gamma(\cdot) \rangle_{\mathcal{H}} \in L^1(X, \mu) \]
is always compact. For finite measures this result is due to Pettis (1938), but it appears original for non finite measures. A brief discussion of the compactness for $1 < p < \infty$ is also given to show that the strong $p$-integrability of the map $\gamma$ is a sufficient condition, but is not necessary.

In Section 5, if $X$ is a locally compact space, it is proved that $\gamma$ is weakly continuous if and only if the kernel (1) is locally bounded and separately continuous, whereas the continuity of the kernel on $X \times X$ is equivalent to the compactness of the operator
\[ \mathcal{H} \ni v \mapsto \langle v, \gamma(\cdot) \rangle_{\mathcal{H}} \in \mathcal{C}(X). \]

The results we present are due to Schwartz (1964) in the framework of reproducing kernel Hilbert spaces and we review them giving an elementary proof based on standard functional analysis tools.

Finally, in Section 6 if $X$ is a locally compact second countable Hausdorff space endowed with a positive Radon measure $\mu$ and $\mathcal{H}$ is a reproducing kernel Hilbert space such that $\mathcal{H} \subset L^2(X, \mu) \cap \mathcal{C}(X)$, we characterize the space $\mathcal{H}$ and the reproducing kernel $\Gamma$ in terms of the spectral decomposition of the integral operator of kernel $\Gamma$. When $X$ is compact, this kind of result is known as Mercer theorem (Hochstadt, 1989). Extensions of Mercer theorem
can be found in Novitskii and Romanov (1999) and references therein. Our proof is very simple and general since it is based on the polar decomposition of the canonical inclusion of $\mathcal{H}$ into $L^2(X, \mu)$.

2 Notations

In this section we fix the notation, give the main definitions and review the connection between reproducing kernel Hilbert spaces and vector valued maps.

If $E$ is a (complex) Banach space, $\|\cdot\|_E$ denotes the norm of $E$, and $E^*$ is the Banach space of continuous antilinear functionals of $E$. We let $\langle \cdot, \cdot \rangle : E^* \times E \to \mathbb{C}$ be the canonical pairing. If $F$ is another Banach space and $A : E \to F$ is a bounded linear operator, then $A^* : F^* \to E^*$ is the adjoint of $A$. If $\mathcal{H}$ is a Hilbert space, $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ denotes the scalar product, linear in the first argument and, by means of the scalar product, $\mathcal{H}^*$ is canonically identified with $\mathcal{H}$.

If $X$ is a set, $\mathbb{C}^X$ is the vector space of all the complex functions on $X$. If $X$ is a measurable space endowed with a $\sigma$-finite positive measure $\mu$, $L^0(X, \mu)$ denotes the topological vector space of all measurable complex functions$^1$ on $X$ endowed with the topology of the convergence in measure on subsets of finite measure [Schwartz, 1993]. Given $1 \leq p < \infty$, $L^p(X, \mu)$ is the Banach space of functions $f \in L^0(X, \mu)$ such that $\|f\|^p$ is $\mu$-integrable, and $L^\infty(X, \mu)$ is the Banach space of elements $f \in L^0(X, \mu)$ that are bounded $\mu$-almost everywhere.

If $X$ is a locally compact Hausdorff space, $\mathcal{C}(X)$ denotes the space of continuous functions on $X$ endowed with the open-compact topology [Kelley, 1955]. If $X$ is second countable locally compact Hausdorff space, a positive Radon measure on $X$ is a positive measure $\mu$ defined on the Borel $\sigma$-algebra of $X$ and finite on compact subsets, and supp $\mu$ denotes the support of $\mu$.

Given a map $\gamma$ from a set $X$ into a Hilbert space $\mathcal{H}$, we denote by $A_\gamma : \mathcal{H} \to \mathbb{C}^X$ the linear operator

$$ (A_\gamma v)(x) = \langle v, \gamma_x \rangle_{\mathcal{H}} \quad \forall x \in X \ v \in \mathcal{H}, \tag{2} $$

by $\Gamma : X \times X \to \mathbb{C}$ the kernel

$$ \Gamma(x, t) = \langle \gamma_t, \gamma_x \rangle_{\mathcal{H}} = A_\gamma(\gamma_t)(x) \quad \forall x, t \in X \tag{3} $$

and we let

$$ \mathcal{H}_\gamma = \overline{\text{span}} \{ \gamma_x \in \mathcal{H} \mid x \in X \}. \tag{4} $$

$^1$As usual, a function is identified with its equivalence class $\mu$-almost everywhere.
where \( \overline{\text{span}} \) is the closure of the linear span.

We now recall some basic definitions about weak regularity properties of vector valued maps.

**Definition 1** Let \( X \) be a set, \( H \) a Hilbert space and \( \gamma : X \to H \).

1. Assume \( X \) be a measurable space. The map \( \gamma \) is weakly measurable if the function \( A_\gamma v \) is measurable for all \( v \in H \).

2. Assume \( X \) be a measurable space endowed with a \( \sigma \)-finite measure \( \mu \) and \( 1 \leq p \leq \infty \). The map \( \gamma \) is weakly \( p \)-integrable if the function \( A_\gamma v \) is \( p \)-integrable for all \( v \in H \).

3. Assume that \( X \) is a locally compact Hausdorff space. The map \( \gamma \) is weakly continuous if the function \( A_\gamma v \) is continuous for all \( v \in H \).

The assumptions that \( \mu \) is a \( \sigma \)-finite measure and \( X \) is locally compact will avoid technical problems. Moreover, the above properties clearly depend only on \( H_\gamma \). If \( X \) is a measurable space with a \( \sigma \)-finite measure \( \mu \) and \( \gamma \) is measurable, we let

\[
S = \{ v \in H | \mu(\gamma^{-1}(B(v, \epsilon))) > 0 \forall \epsilon > 0 \}, \quad H_\mu = \overline{\text{span}} S, \tag{5}
\]

where \( B(v, \epsilon) = \{ w \in H | \|w - v\|_H < \epsilon \} \). The closed subset \( S \) is the essential range of \( \gamma \) and \( S \subset H_\gamma \), so that \( H_\mu \subset H_\gamma \).

We need also the definition of kernel of positive type.

**Definition 2** Given a set \( X \), a complex kernel \( \Gamma : X \times X \to \mathbb{C} \) is called of positive type if

\[
\sum_{i,j}^{\ell} c_i \overline{c_j} \Gamma(x_i, x_j) \geq 0 \tag{6}
\]

for any \( \ell \in \mathbb{N}, x_1, \ldots, x_\ell \in X \) and \( c_1, \ldots, c_\ell \in \mathbb{C} \).

**Remark 1** In the complex case, the positivity condition \((6)\) ensures that

\[
\Gamma(x, t) = \overline{\Gamma(t, x)} \quad \forall x, t \in X. \tag{7}
\]

This is no longer true in the real case. In this case, a kernel \( \Gamma : X \times X \to \mathbb{R} \) is called of positive type if

\[
\sum_{i,j}^{\ell} c_i c_j \Gamma(x_i, x_j) \geq 0 \quad \forall \ell \in \mathbb{N}, \ x_1, \ldots, x_\ell \in X, \ c_1, \ldots, c_\ell \in \mathbb{R}.
\]
In the framework of harmonic analysis, kernel of positive type are also called positive definite, and, in the context of reproducing kernel Hilbert spaces, Aronszajn kernel.

We now recall the definition of bounded kernel from the theory of integral operators.

**Definition 3** Let $X$ and $Y$ two measurable spaces endowed with $\sigma$-finite measures $\mu$ and $\nu$, respectively. Fixed $1 \leq p, r \leq \infty$, a measurable kernel $K : X \times Y \rightarrow \mathbb{C}$ is called $(r, p)$-bounded if, for all $\phi \in L^r(Y, \nu)$,

1. there is a $\mu$-null set $X_\phi \subset X$ such that the function $K(x, \cdot)\phi(\cdot)$ is in $L^1(Y, \nu)$ for all $x \notin X_\phi$;
2. the map $x \mapsto \int_Y K(x, y)\phi(y) \, d\nu(y)$ is in $L^p(X, \mu)$.

Moreover, if the following condition holds

1') for $\mu$-almost all $x \in X$, $K(x, \cdot) \in L(Y, \nu)^{\frac{r}{r-1}}$ (if $r = 1$, $\frac{r}{r-1} = \infty$ and, if $r = \infty$, $\frac{r}{r-1} = 1$),

$K$ is called a Carleman $(r, p)$-bounded kernel.

Since $\nu$ is $\sigma$-finite, Condition 1') is equivalent to Condition 1) and the fact that the $\mu$-null set $X_\phi$ can be chosen in such a way to be independent of $\phi$. In Halmos and Sunder (1978) there is an example of $(2, 2)$-bounded kernel, which is not a Carleman kernel. In the above definition, boundedness refers to the fact that the integral operator of kernel $K$ is bounded from $L^r(Y, \nu)$ to $L^p(X, \mu)$, as shown in Proposition 4.

Finally, we recall that a reproducing kernel Hilbert space $\mathcal{H}$ on $X$ is a subspace of $\mathbb{C}^X$ such that $\mathcal{H}$ is a Hilbert space and, for all $x \in X$, there is a function $\gamma_x \in \mathcal{H}$ satisfying

$$f(x) = \langle f, \gamma_x \rangle_{\mathcal{H}} \quad \forall f \in \mathcal{H}. \quad (8)$$

The corresponding reproducing kernel of $\mathcal{H}$ is defined by

$$\Gamma(x, t) = \langle \gamma_t, \gamma_x \rangle_{\mathcal{H}} = \gamma_t(x) \quad x, t \in X. \quad (9)$$

We now review the connection between reproducing kernel Hilbert spaces, Hilbert space valued maps and kernels of positive type. The following result has been obtained by many authors, see Aronszajn (1950); Godement (1948); Kolmogorov (1922); Krein (1949, 1950); Schoenberg (1938) and, for a complete list of references, Bekka and de la Harpe (2003); Hille (1972); Saitoh (1988, 1997); Schwartz (1964).
Proposition 1 Let $X$ be a set.

1. Given a kernel $\Gamma : X \times X \to \mathbb{C}$ of positive type, there is a unique reproducing kernel Hilbert space $\mathcal{H}$ on $X$ with reproducing kernel $\Gamma$.

The inclusion of $\mathcal{H}$ into $\mathbb{C}^X$ is the operator $A_{\gamma}$ associated with the map $\gamma : X \to \mathcal{H}$ defined by

$$
\gamma_x = \Gamma(\cdot, x) \quad x \in X. 
$$

The kernel (5) associated with $\gamma$ is precisely the reproducing kernel of $\mathcal{H}$ and

$$
\mathcal{H} = \mathcal{H}_{\gamma}. 
$$

2. Given a Hilbert space $\mathcal{H}$ and a map $\gamma : X \to \mathcal{H}$, the associated kernel $\Gamma$ is of positive type and $\ker A_{\gamma} = \mathcal{H}_{\gamma}^\perp$. In particular, $A_{\gamma}$ is a unitary operator from $\mathcal{H}_{\gamma}$ onto the reproducing kernel Hilbert space with reproducing kernel $\Gamma$.

Proof. We report the proof given in Bekka and de la Harpe (2003).

1. Let $\gamma_x = \Gamma(\cdot, x) \in \mathbb{C}^X$ with $x \in X$ and

$$
\mathcal{H}_0 = \text{span}\{ \gamma_x \mid x \in X \} \subset \mathbb{C}^X.
$$

If $f = \sum_i c_i \gamma_{x_i} \in \mathcal{H}_0$ and $g = \sum_j d_j \gamma_{t_j} \in \mathcal{H}_0$, the definition of $\gamma_x$ implies that

$$
\sum_{ij} c_i \overline{d_j} \Gamma(t_j, x_i) = \sum_j \overline{d_j} f(t_j) = \sum_i c_i g(x_i),
$$

so the following sequilinear form on $\mathcal{H}_0 \times \mathcal{H}_0$

$$
(f, g) \mapsto \langle f, g \rangle := \sum_{ij} c_i \overline{d_j} \Gamma(t_j, x_i)
$$

is well defined. Equation (7) ensures that $\langle \cdot, \cdot \rangle$ is hermitian and Eq. (6) that $\langle f, f \rangle \geq 0$ for all $f \in \mathcal{H}_0$. Let now $x \in X$, the choice $g = \gamma_x$ in the above equation implies that

$$
f(x) = \langle f, \gamma_x \rangle \quad \forall x \in X, 
$$

for all $f \in \mathcal{H}_0$. Assume now that $\langle f, f \rangle = 0$, the Cauchy-Schwarz inequality gives

$$
|f(x)| \leq \sqrt{\langle f, f \rangle} \sqrt{\langle \gamma_x, \gamma_x \rangle} = 0,
$$

6
and, hence, \( f = 0 \). This shows that \( \langle \cdot, \cdot \rangle \) is a scalar product on \( \mathcal{H}_0 \).

Finally, if \( (f_n)_{n \in \mathbb{N}} \) is a Cauchy sequence in \( \mathcal{H}_0 \), Eq. (12) implies that, for all \( x \in X \), \( (f_n(x))_{n \in \mathbb{N}} \) converges to \( f(x) \in \mathbb{C} \), so that the completion of \( \mathcal{H}_0 \) is a subspace of \( \mathbb{C}^X \) and Eq. (12) holds for all \( f \in \mathcal{H} \), showing that \( \mathcal{H} \) is a reproducing kernel Hilbert space with kernel \( \Gamma \). The uniqueness is evident. Finally, let \( \gamma \) as in Eq. (10), Eq. (8) implies that the corresponding operator \( A_\gamma \) is the inclusion of \( \mathcal{H} \) into \( \mathbb{C}^X \). Equation (11) follows by density of \( \mathcal{H}_0 \).

2. A simple check shows that the kernel \( \Gamma \) associated with \( \gamma \) is of positive type and that \( \ker A_\gamma = \mathcal{H}_\gamma \perp \). In particular, \( A_\gamma \) is a bijective map from \( \mathcal{H}_\gamma \) onto \( \text{Im} \ A_\gamma \). Let now \( \mathcal{H}_\Gamma \) be the reproducing kernel Hilbert space with kernel \( \Gamma \) given by point 1 of this proposition. Both \( \text{Im} \ A_\gamma \) and \( \mathcal{H}_\Gamma \) are subspaces of \( \mathbb{C}^X \). In particular, for all \( x \in X \), the function \( \Gamma(\cdot, x) = A_\gamma \gamma_x \) is both in \( \text{Im} \ A_\Gamma \) and in \( \mathcal{H}_\Gamma \), see Eq. (2) and Eq. (9), respectively. Let now \( x, t \in X \), the cited equations give

\[
\langle A_\gamma \gamma_t, A_\gamma \gamma_x \rangle_{\mathcal{H}_\Gamma} = \langle \Gamma(\cdot, t), \Gamma(\cdot, x) \rangle_{\mathcal{H}_\Gamma} = \Gamma(x, t) = \langle \gamma_t, \gamma_x \rangle_{\mathcal{H}}
\]

showing that \( A_\gamma \) is an isometry from \( \text{span}\{ \gamma_x \mid x \in X \} \) onto \( \text{span}\{ \Gamma(\cdot, x) \mid x \in X \} \). The claim follows by definition of \( \mathcal{H}_\gamma \) and Eq. (11) applied to \( \mathcal{H}_\Gamma \).

The construction of the Hilbert space \( \mathcal{H} \) given the kernel \( \Gamma \) is related to the GNS construction and it is also known as Kolmogorov theorem [Kolmogorov, 1992]. In the above proof, \( \mathcal{H} \) is directly defined as a subspace of \( \mathbb{C}^X \). In doing so, one has to prove that the scalar product is well defined, but, due to Eq. (8), it is strictly positive, so that there is no need to quotient with respect to the vectors of null norm. Another possibility is to define \( \mathcal{H} \) as a subspace of the formal linear combinations of elements \( \Gamma(\cdot, x_i) \), see, for example, Dutkay (2004).

Proposition II shows that there is a one-to-one correspondence between Hilbert space valued maps, complex kernels of positive type and reproducing kernel Hilbert spaces. In particular, given a map \( \gamma \), the Hilbert space where the map takes value can be identified by means of the operator \( A_\gamma \) with a unique reproducing kernel Hilbert space, which is a subspace of \( \mathbb{C}^X \). Conversely, any reproducing kernel Hilbert space defines uniquely a Hilbert space valued map \( \gamma \) such that the operator \( A_\gamma \) is the inclusion into \( \mathbb{C}^X \). In both cases, the weak regularity properties of \( \gamma \) stated in Definition I are in correspondence with the regularity properties of the complex functions in the reproducing kernel Hilbert space. In the following, we will show that these properties can be completely characterized in terms of the kernel \( \Gamma \).
the above equivalence, we will state the results for an arbitrary Hilbert space valued map. Here, for the convenience of the reader, we summarize them in the case of a separable reproducing kernel Hilbert space \( \mathcal{H} \) and for \( X \) being a locally compact second countable Hausdorff space endowed with a positive Radon measure \( \mu \).

1. Proposition 2 \( \mathcal{H} \subset L^0(X, \mu) \) if and only if \( \Gamma \) is separately measurable;

2. Proposition 5 \( \mathcal{H} \subset L^p(X, \mu) \) if and only if \( \Gamma \) is a \( \left( \frac{p}{p-1}, p \right) \)-bounded kernel, for all \( 1 \leq p \leq \infty \);

3. Proposition 8 \( \mathcal{H} \subset C(X) \) if and only if \( \Gamma \) is separately continuous and locally bounded.

The assumption that \( \mathcal{H} \) is separable is essential (see Example 1) and, in general, the inclusion of \( \mathcal{H} \) into \( L^0(X, \mu) \) is not injective. However, if \( \gamma \) is weakly continuous and \( \text{supp } \mu = X \), \( \mathcal{H} \) is separable and the inclusion is injective (see Eqs. (11), (13), (22)).

### 3 Measurability

In this section we characterize the weak measurable maps and we discuss the relation with the theory of integral operators. The following proposition is based on the well-known equivalence between weak and strong measurability for maps with separable range (see, for example, Hille and Phillips (1974); Pettis (1938)).

**Proposition 2** Let \( X \) be a measurable space, \( \mathcal{H} \) a Hilbert space and \( \gamma : X \to \mathcal{H} \). Assume that \( \mathcal{H}_\gamma \) is separable, then the following conditions are equivalent:

1. the map \( \gamma \) is weakly measurable;

2. the map \( \gamma \) is measurable from \( X \) to \( \mathcal{H} \);

3. the function \( \Gamma \) is measurable from \( X \times X \) into \( C \);

4. for all \( x \in X \), the function \( \Gamma(x, \cdot) \) is measurable from \( X \) into \( C \).

If \( X \) is endowed with a \( \sigma \)-finite measure \( \mu \) and one of the above conditions holds, then \( \text{Im } A_\gamma \subset L^0(X, \mu) \), the operator \( A_\gamma \) is a continuous from \( \mathcal{H} \) into \( L^0(X, \mu) \) and

\[
\ker A_\gamma = \mathcal{H}_\mu^\perp.
\] (13)

**Proof.**
1) \( \Rightarrow \) 2) Since \( \mathcal{H}_\gamma \) is separable, there is a denumerable Hilbert basis \((e_n)_{n \in I}\) of \( \mathcal{H}_\gamma \) and
\[
\gamma_x = \sum_{n \in I} (A_\gamma e_n)(x) e_n \quad \forall x \in X.
\]
By definition \( A_\gamma e_n = \langle e_n, \gamma(\cdot) \rangle \) and is measurable by assumption and \( \gamma \) is measurable since \( I \) is denumerable.

2) \( \Rightarrow \) 3) Since scalar product is continuous and \( \mathcal{H}_\gamma \) is separable, the function \( \Gamma(x, t) = \langle \gamma_t, \gamma_x \rangle_\mathcal{H} \) is measurable.

3) \( \Rightarrow \) 4) Given \( x \in X \), the map \( t \mapsto (x, t) \) is clearly measurable, so Condition (4) follows.

4) \( \Rightarrow \) 1) Let \( v = \sum_{i=1}^N c_i \gamma_{x_i} \) with \( x_i \in X \). By assumption \( \Gamma(\cdot, x_i) \) is measurable for all \( i \), so is \( A_\gamma v = \sum_i c_i \Gamma(\cdot, x_i) \). By density and Eq. (4), \( A_\gamma v \) is measurable for all \( v \in \mathcal{H}_\gamma \). If \( v \in \mathcal{H}_\gamma^\perp \), \( A_\gamma v = 0 \) and, hence, is measurable.

Assume now one of the four equivalent conditions. Since pointwise convergence implies convergence in measure on subsets of finite measure [Schwartz 1993], Eq. (2) gives that \( A_\gamma \) is continuous from \( \mathcal{H} \) to \( L^0(X, \mu) \).

We now prove \( \ker A_\gamma = \mathcal{H}_\mu^\perp \). Since \( \mathcal{H}_\gamma^\perp \subset \ker A_\gamma \), without loss of generality, we can assume that \( \mathcal{H}_\gamma = \mathcal{H} \) with \( \mathcal{H} \) separable and, by definition of \( \mathcal{H}_\mu \), it is enough to show that \( \ker A_\gamma = S^\perp \), where \( S \) is the essential range of \( \gamma \).

The set \( X_0 = \{ x \in X \mid \gamma_x \not\in S \} \) is of null measure. Indeed, since \( S \) is closed and \( \gamma \) is (strongly) measurable, \( X_0 \) is measurable. Moreover, by definition, if \( x \in X_0 \), there is \( \epsilon_x > 0 \) such that \( \mu(\gamma^{-1}(B(\gamma_x, \epsilon_x))) = 0 \). Since \( \mathcal{H}_\gamma \) is separable, each member of the family \( \{B(\gamma_x, \epsilon_x)\}_{x \in X_0} \) is a union of members of a fixed countable family of open sets \( \{\Omega_i\}_{i \in \mathbb{N}} \), with each \( \Omega_i \) contained in \( B(\gamma_x, \epsilon_x) \) for some \( x \in X_0 \). Clearly \( \mu(\gamma^{-1}(\Omega_i)) = 0 \) and \( X_0 \subset \bigcup_{i \in \mathbb{N}} \gamma^{-1}(\Omega_i) \), so that \( \mu(X_0) = 0 \).

Let now \( v \in S^\perp \), then \( (A_\gamma v)(x) = 0 \) for all \( x \not\in X_0 \) since \( \gamma_x \in S \), but \( \mu(X_0) = 0 \), so that \( v \in \ker A_\gamma \). Conversely, let \( v \in \ker A_\gamma \). By contradiction, assume there is \( w_0 \in S \) such that \( \langle v, w_0 \rangle \neq 0 \). Since the scalar product is continuous,
\[
\langle v, w \rangle \neq 0 \quad \forall w \in B(w_0, \epsilon),
\]
for some \( \epsilon > 0 \). In particular, \( \langle v, \gamma_x \rangle \neq 0 \) for all \( x \in \gamma^{-1}(B(w_0, \epsilon)) \). However, \( \mu(\gamma^{-1}(B(w_0, \epsilon))) > 0 \), since \( w_0 \in S \), so that \( A_\gamma v \neq 0 \). Since this last fact contradicts the assumption, it follows that \( \langle v, w_0 \rangle = 0 \) for all \( w_0 \in S \), that is, \( v \in S^\perp \). \( \blacksquare \)
The following example (see Bourbaki (2004)) shows that the separability of $\mathcal{H}_\gamma$ is essential in the above proposition.

**Example 1** Let $X = [0, 1]$ with the Lebesgue measure and $\mathcal{H}$ be the Hilbert space of functions $v : [0, 1] \rightarrow \mathbb{C}$ such that

$$\sum_{x \in X} |v(x)|^2 < +\infty,$$

with scalar product

$$\langle v, w \rangle_{\mathcal{H}} = \sum_{x \in X} v(x)\overline{w(x)}.$$

Let $A$ be a non measurable subset of $[0, 1]$ and $\gamma$

$$\gamma_x = \begin{cases} 
\delta_x & x \in A \\
0 & x \notin A 
\end{cases}$$

where $\delta_x(y) = 0$ if $y \neq x$ and $\delta_x(x) = 1$. Since $(A_\gamma v)(x) = 0$ for all but denumerable $x$, $A_\gamma v$ is measurable. However, the function

$$\Gamma(x, x) = \begin{cases} 
1 & x \in A \\
0 & x \notin A 
\end{cases}$$

is not measurable and, hence, Condition (2) and Condition (3) of Proposition 2 can not be true.

We now clarify the relation with the theory of integral operators (for a complete account see Halmos and Sunder (1978), where weakly integrable maps are called Carleman functions). The following proposition, due to Pettis (1938), characterizes the measurable maps taking value in $L^2(Y, \nu)$ (see also Theorem 11.5 of Halmos and Sunder (1978)). For a converse result, see Proposition 7 below.

**Proposition 3** Let $X$ and $Y$ two measurable sets endowed with two $\sigma$-finite measures $\mu$ and $\nu$, respectively. Suppose $L^2(Y, \nu)$ is separable. Given a weakly measurable map $\gamma : X \rightarrow L^2(Y, \nu)$, there exists a measurable function $K_\gamma : X \times Y \rightarrow \mathbb{C}$ and a $\mu$-null set $N \subset X$ such that, for $x \in X \setminus N$, one has $\gamma_x(y) = K_\gamma(x, y)$ for $\nu$-almost all $y \in Y$.

The operator $A_\gamma : L^2(Y, \nu) \rightarrow L^0(X, \mu)$ associated with $\gamma$ is the integral operator with kernel $K_\gamma$ and the kernel associated with $\gamma$ is

$$\Gamma(x, x') = \int_Y K_\gamma(x', y)K_\gamma(x, y) \, d\nu(y),$$

for $\mu \otimes \mu$-almost all $(x, x') \in X \times X$. 

Proof. Let \((C_n)_{n \geq 1}\) be an increasing sequence of measurable subsets of \(X\) such that \(\mu(C_n) < +\infty\) and \(X = \bigcup_n C_n\). For all \(n \in \mathbb{N}\), define by induction
\[
A_0 = \emptyset \\
A_n = \left\{ x \in X \mid x \in C_n, x \notin A_{n-1} \text{ and } \|\gamma_x\|_{L^2(Y)} \leq n \right\}
\]
Clearly, each \(A_n\) is a subset of finite measure, \(A_n \cap A_m = \emptyset\) if \(n \neq m\), and \(\bigcup_n A_n = X\). Define
\[
\gamma^*_n = \left\{ \begin{array}{ll} \gamma_x & \text{if } x \in A_n \\ 0 & \text{elsewhere} \end{array} \right.
\]
If \(f \in L^2(X \times Y, \mu \otimes \nu)\), by Fubini theorem \(f(x, \cdot) \in L^2(Y, \nu)\) for \(\mu\)-almost all \(x \in X\). Moreover, by separability of \(L^2(Y, \nu)\) the map \(x \mapsto \langle f(x, \cdot), \gamma^*_x \rangle_{L^2(Y)}\) is measurable. Let \(\lambda_n\) be the linear form on \(L^2(X \times Y, \mu \otimes \nu)\) given by
\[
\lambda_n(f) = \int_X \langle f(x, \cdot), \gamma^*_x \rangle_{L^2(Y)} \, d\mu(x).
\]
The linear form \(\lambda_n\) is bounded since
\[
|\lambda_n(f)| \leq \int_X \left| \langle f(x, \cdot), \gamma^*_x \rangle_{L^2(Y)} \right| \, d\mu(x) \\
\leq \int_X \|f(x, \cdot)\|_{L^2(Y)} \|\gamma^*_x\|_{L^2(Y)} \, d\mu(x) \\
\leq \left[ \int_X \|f(x, \cdot)\|^2_{L^2(Y)} \, d\mu(x) \right]^{1/2} \left[ \int_X \|\gamma^*_x\|^2_{L^2(Y)} \, d\mu(x) \right]^{1/2} \\
\leq n \sqrt{\mu(A_n)} \|f\|_{L^2(X \times Y)}.
\]
By Riesz lemma, there exists unique \(\Lambda_n \in L^2(X \times Y, \mu \otimes \nu)\) such that
\[
\lambda_n(f) = \int_{X \times Y} f(x, y) \Lambda_n(x, y) \, d\mu(x) \, d\nu(y).
\]
It is not restrictive to assume that \(\Lambda_n(x, \cdot) \in L^2(Y, \nu)\) for all \(x\). Let \(u \in L^2(Y, \nu)\). For every measurable subset \(C\) of \(X\) with \(\mu(C) < +\infty\), choosing \(f(x, y) = \chi_C(x) u(y)\), the above relations and Fubini theorem give
\[
\int_C \langle u, \gamma^*_x \rangle_{L^2(Y)} \, d\mu(x) = \int_C d\mu(x) \int_Y u(y) \Lambda_n(x, y) \, d\nu(y) = \int_C \langle u, \Lambda_n(x, \cdot) \rangle_{L^2(Y)} \, d\mu(x),
\]
11
which implies \( \langle u, \gamma^n \rangle_{L^2(Y)} = \left\langle u, \Lambda_n(x, \cdot) \right\rangle_{L^2(Y)} \) for \( \mu \)-almost all \( x \). Letting \( u \) vary over a denumerable dense subset of \( L^2(Y, \nu) \), we find that there exists a \( \mu \)-null set \( N_n \subset X \) such that, for \( x \in X \setminus N_n \),

\[
\gamma^n_x(y) = \Lambda_n(x, y)
\]

for \( \nu \)-almost all \( y \in Y \). Hence the map

\[
K_\gamma(x, y) = \sum_n \chi_{A_n}(x) \Lambda_n(x, y)
\]

is finite, measurable and, for all \( x \in A_n \setminus N_n \),

\[
K_\gamma(x, y) = \gamma^n_x(y) = \gamma_x(y)
\]

for \( \nu \)-almost all \( y \in Y \). Since \( X = \bigcup_n A_n \) and \( N = \bigcup_n N_n \) is a \( \mu \)-null set, the first part of the proposition follows. The definition of \( A_\gamma \) implies that \( A_\gamma \) is the integral operator of kernel \( K_\gamma \) and the last equation easily follows.

## 4 Integrability

In this section we characterize weakly \( p \)-integrable maps. First of all, we recall that the integral operator with a \((r, p)\)-bounded kernel is continuous.

**Proposition 4** With the assumptions of Definition 3, let \( K \) be a \((r, p)\)-bounded kernel, then the operator \( L_K : L^r(Y, \nu) \to L^p(X, \mu) \)

\[
(L_K \phi)(x) = \int_Y K(x, y) \phi(y) \, d\nu(y) \quad \text{for } \mu-\text{a.a. } x \in X
\]

is bounded.

**Proof.** We report the proof of Halmos and Sunder (1978). Since \( L_K \) is defined on all the space \( L^r(Y, \nu) \), it is enough to prove that it is closed. Let \( (\phi_n)_{n \in \mathbb{N}} \) be a sequence such that it converges to \( \phi \) in \( L^r(Y, \nu) \) and \( \psi_n = L_K \phi_n \) converges to \( \psi \) in \( L^p(X, \mu) \). Possibly passing to a double subsequence, we can assume that both \( (\phi_n)_{n \in \mathbb{N}} \) and \( (\psi_n)_{n \in \mathbb{N}} \) converges almost everywhere and that there is \( \omega \in L^r(Y, \nu) \) for which \( |\phi_n(y)| \leq \omega(y) \) for \( \nu \)-almost all \( y \in Y \) (if \( r < \infty \) it is a consequence of Fischer-Riesz theorem, see Lemma 3.9 of Halmos and Sunder (1978), for \( r = +\infty \), it is trivial). Condition 1 of
Definition 3 and the fact that denumerable union of null sets is a null set imply that, for \( \mu \)-almost all \( x \in X \), the sequence \( (K(x, \cdot)\phi_n(\cdot))_{n \in \mathbb{N}} \) converges to \( K(x, \cdot)\phi(\cdot) \) for \( \nu \)-almost all \( y \in Y \) and it is bounded almost everywhere by \( |K(x, \cdot)\omega(\cdot)| \in L^1(Y, \nu) \). The dominated convergence theorem gives that

\[
\lim_{n \to +\infty} \psi_n(x) = \lim_{n \to +\infty} \int_Y K(x,y)\phi_n(y) \, d\nu(y) = \int_Y K(x,y)\phi(y) \, d\nu(y)
\]

for \( \mu \)-almost all \( x \in X \), so that \( \psi(x) = \int_Y K(x,y)\phi(y) \, d\nu(y) \) in \( L^p(X, \mu) \). \( \blacksquare \)

We are now in position to state the main result of the paper.

**Proposition 5** Let \( X \) be a measurable space, \( \mu \) a \( \sigma \)-finite measure on \( X \) and \( \mathcal{H}_\gamma \) a separable Hilbert space. Let \( 1 \leq p \leq \infty \), then the following conditions are equivalent

1. the map \( \gamma \) is weakly \( p \)-integrable;
2. the kernel \( \Gamma \) is \( (q, p) \)-bounded with \( q = \frac{p}{p-1} \).

If one of the above conditions holds, then \( \text{Im} \ A_\gamma \subset L^p(X, \mu) \) and

(i) \( A_\gamma \) is a bounded linear operator from \( \mathcal{H} \) into \( L^p(X, \mu) \);

(ii) its adjoint \( A_\gamma^* : L^q(X, \mu) \to \mathcal{H} \) is given by

\[
A_\gamma^* \phi = \int_X \phi(x)\gamma_x \, d\mu(x),
\]

where \( \phi \in L^q(X, \mu) \) and the integral has to be understood in the weak sense (if \( p = \infty \) and \( q = 1 \), in Eq. (16) \( A_\gamma^* \) is the restriction of the adjoint to \( L^1(X, \mu) \subset L^\infty(X, \mu)^* \));

(iii) \( \Gamma \) is a Carleman kernel and \( A_\gamma A_\gamma^* = L_\Gamma \), where \( L_\Gamma \) is the integral operator of kernel \( \Gamma \) given by Eq. (15).

**Proof.**

1) \( \Rightarrow \) 2) To show that \( A_\gamma \) is a bounded operator from \( \mathcal{H} \) into \( L^p(X, \mu) \) we follow the proof of [Hille and Phillips (1974)]. Since \( \gamma \) is weakly \( p \)-integrable, \( A_\gamma \) is a linear operator from \( \mathcal{H} \) to \( L^p(X, \mu) \). We claim that it is closed. Indeed, let \( (v_n)_{n \in \mathbb{N}} \) be a sequence that converges to \( v \in \mathcal{H} \) and the
sequence \((A_{\gamma} v_n)_{n \in \mathbb{N}}\) converges to \(\phi \in L^p(X, \mu)\). By construction, for \(\mu\)-almost all \(x \in X\),

\[
\lim_{n \to +\infty} (A_{\gamma} v_n)(x) = \lim_{n \to +\infty} \langle v_n, \gamma_x \rangle_{\mathcal{H}} = \langle v, \gamma_x \rangle_{\mathcal{H}} = (A_{\gamma} v)(x).
\]

By uniqueness of the limit, \(A_{\gamma} v = \phi\), so that \(A_{\gamma}\) is closed. The closed graph theorem implies that \(A_{\gamma}\) is bounded.

We show Eq. \([16]\). Given \(\phi \in L^q(X, \mu)\) and \(v \in \mathcal{H}\), by assumption \(A_{\gamma} v \in L^p(X, \mu)\), so that the function \(\phi A_{\gamma} v = \phi(\cdot) \langle \gamma(\cdot), v \rangle_{\mathcal{H}}\) is in \(L^1(X, \mu)\). Since \(v\) is arbitrary, it follows that the function \(\phi(\cdot) \gamma(\cdot)\) is weakly integrable and

\[
\int_X \phi(x) \langle \gamma_x, v \rangle_{\mathcal{H}} \, d\mu(x) = \langle \phi, A_{\gamma} v \rangle = \langle A_{\gamma}^\ast \phi, v \rangle_{\mathcal{H}},
\]

so that Eq. \([16]\) holds.

The fact that \(\Gamma\) is a Carleman kernel follows observing that, for all \(x \in X\), \(\Gamma(\cdot, x) = A_{\gamma} (\gamma_x)(\cdot)\), which is in \(L^p(X, \mu)\) by assumption. Moreover, given \(\phi \in L^q(X, \mu)\), Eqs. \([3]\) and \([16]\) imply that

\[
\int_X \Gamma(x, t) \phi(t) \, d\mu(t) = \langle A_{\gamma}^\ast \phi, \gamma_x \rangle_{\mathcal{H}} = (A_{\gamma} A_{\gamma}^\ast \phi)(x) \quad \text{for } \mu-\text{a.a. } x \in X.
\]

Hence, \(\int_X \Gamma(\cdot, t) \phi(t) \, d\mu(t)\) is in \(L^p(X, \mu)\), that is, \(\Gamma\) is a Carleman \((q, p)\)-bounded kernel, and \(A_{\gamma} A_{\gamma}^\ast = L_{\Gamma}\).

2) \(\Rightarrow 1)\) Since \(\Gamma\) is measurable and \(\mathcal{H}\) is separable, Proposition \([2]\) ensures the function \(\langle \gamma(\cdot), v \rangle_{\mathcal{H}}\) be measurable for all \(v \in \mathcal{H}\). So it is enough to prove that \(|\langle \gamma(\cdot), v \rangle_{\mathcal{H}}| \in L^p(X, \mu)\)

To this aim, let \((A_{n})_{n \in \mathbb{N}}\) be the sequence of sets defined in the proof of Proposition \([3]\). For all \(n \in \mathbb{N}\) let \(D_n = \bigcup_{k=1}^n A_k\), which is of finite measure and \(\gamma\) is bounded by \(n\) on \(D_n\).

Let now \(\phi \in L^q(X, \mu)\), then the map \(x \mapsto \chi_{D_n}(x) \phi(x) \gamma_x\) is clearly measurable and bounded by \(\chi_{D_n} \, |\phi| \in L^1(X, \mu)\) since \(D_n\) has finite measure.

Hence the linear operator \(B_n : L^q(X, \mu) \to \mathcal{H}\)

\[
B_n \phi = \int_X \chi_{D_n}(x) \phi(x) \gamma_x \, d\mu(x) \quad \phi \in L^q(X, \mu),
\]

where the integral is in the strong sense, is well defined and bounded by

\[
\|B_n \phi\|_{\mathcal{H}} \leq n \int_{D_n} |\phi(x)| \, d\mu(x) \leq n \mu(D_n)^{1/p} \|\phi\|_q.
\]

14
Since $\Gamma$ is a bounded kernel, $L_\Gamma$ is a bounded operator and the following inequality holds

$$
\|B_n\phi\|_H^2 = \int_X \chi_{D_n}(y) \overline{\phi(y)} \left( \int_X \chi_{D_n}(x) \phi(x) \langle \gamma_x, \gamma_y \rangle_H \ d\mu(x) \right) \ d\mu(y) = \langle L_\Gamma(\chi_{D_n}\phi), (\chi_{D_n}\phi) \rangle \leq \|L_\Gamma\| \|\phi\|_q^2,
$$

for all $\phi \in L^q(X,\mu)$, so that

$$
\sup_{n \in \mathbb{N}} \|B_n\| \leq M,
$$

with $M = \sqrt{\|L_K\|}$ and, hence, $\sup_n \|B_n^*\| \leq M$.

Let now $v \in \mathcal{H}$ and $\phi \in L^q(X,\mu)$, then

$$
\langle \phi, B_n^*v \rangle = \langle B_n\phi, v \rangle_{\mathcal{H}} = \int_X \phi(x) \langle \chi_{D_n}(x)\gamma_x, v \rangle_{\mathcal{H}} \ d\mu(x) \quad \forall \phi \in L^q(X,\mu),
$$

so $B_n^*v(x) = \chi_{D_n}(x)\langle v, \gamma_x \rangle_{\mathcal{H}}$ for $\mu$-almost all $x \in X$ (in particular, $B_n^*v \in L^1(X,\mu)$ if $q = \infty$), and

$$
\|B_n^*v\|_p \leq \|B_n^*\| \|v\|_{\mathcal{H}} \leq M \|v\|_{\mathcal{H}}. \quad (17)
$$

Since $\bigcup_{n \in \mathbb{N}} D_n = X$, $\lim_{n \to \infty} B_n^*v(x) = \langle v, \gamma_x \rangle_{\mathcal{H}} \mu$-almost everywhere, and Eq. (17) immediately ensures that $\langle v, \gamma(\cdot) \rangle_{\mathcal{H}} \in L^\infty(X,\mu)$ for $p = \infty$. If $1 \leq p < \infty$, the monotone convergence theorem gives that the map $x \mapsto |\langle v, \gamma_x \rangle_{\mathcal{H}}|^p$ is in $L^1(X,\mu)$.

If $\gamma$ is a weakly measurable function such that

$$
\int_{X \times X} |\Gamma(x,t)|^p \ d(\mu \otimes \mu)(x,t) < +\infty,
$$

Fubini theorem and Holder inequality imply that $\Gamma$ is a Carleman $(q,p)$-bounded kernel and, hence, $\gamma$ is a weakly $p$-integrable function. The following proposition discusses the problem of compactness, compare with Corollary 2. For finite measures the first statement is due to Pettis (1938). The second statement is well known. See, for example, Halmos and Sunder (1978) for a complete discussion about the compactness of integral operators in $L^2(X,\mu)$.

**Proposition 6** With the notation of Proposition 5, if $\mathcal{H}_\gamma$ is separable and $\gamma$ is weakly integrable, then $A_\gamma$ is a compact operator in $L^1(X,\mu)$. If $p < \infty$ and $\gamma$ is strongly $p$-integrable, then $A_\gamma$ is a compact operator in $L^p(X,\mu)$.
Proof. Assuming that $\mathcal{H}_\gamma$ is separable and $\gamma$ is weakly integrable, we prove that $A_\gamma$ is compact. First, suppose that $\gamma$ takes only a countable number of values, that is,

$$\gamma = \sum_{n \in I} \chi_{E_n} v_n,$$

where $I$ is denumerable, the sequence $(E_n)_{n \in I}$ is disjoint, each $E_n$ is of finite measure and $v_n \in \mathcal{H}$. The condition that $\gamma$ is weakly integrable implies that

$$\|A_\gamma v\|_{L^1(X)} = \sum_n \mu(E_n) |\langle v, v_n \rangle| = \|Tv\|_{\ell^1},$$

where $T : \mathcal{H} \to \ell^1$, $(Tv)_n = \mu(E_n) \langle v, v_n \rangle$. Since $T$ is a bounded operator from the Hilbert space $\mathcal{H}$ to $\ell^1$, it is known that $T$ is compact (see Conway (1990)). Hence $A_\gamma$ maps weakly convergent sequences into strongly convergent ones, so that it is compact.

Assume now that $\gamma$ is arbitrary, we claim there exists $\gamma_1$ and $\gamma_2$ such that $\gamma_1$ takes only a countable number of values, $\gamma_2$ is strongly integrable and $\gamma = \gamma_1 + \gamma_2$. To this aim, let $(A_n)_{n \in \mathbb{N}}$ be the sequence of sets defined in the proof of Proposition 3. Given $n \in \mathbb{N}$, since $A_n$ has finite measure and $\gamma$ is bounded by $n$ on it, $\chi_{A_n} \gamma$ is strongly integrable, so there is a map $\eta_n : A_n \to \mathcal{H}$, which takes only a finite number of values and

$$\int_{A_n} \|\gamma(x) - \eta_n(x)\|_\mathcal{H} \, d\mu(x) \leq \frac{1}{2^n}.$$

Since the sequence $(A_n)_{n \in \mathbb{N}}$ is disjoint and measurable, the map

$$\gamma_1 = \sum_{n \in \mathbb{N}} \chi_{A_n} \eta_n$$

is well defined, measurable and takes only a countable number of values. Let $\gamma_2 = \gamma - \gamma_1$, then

$$\int_X \|\gamma_2(x)\|_\mathcal{H} \, d\mu(x) = \sum_n \int_{A_n} \|\gamma(x) - \eta_n(x)\|_\mathcal{H} \, d\mu(x) \leq \sum_n \frac{1}{2^n} = 2,$$

so that $\gamma_2$ is strongly integrable and $\gamma = \gamma_1 + \gamma_2$.

Finally, since $\gamma$ and $\gamma_2$ are weakly integrable, so is $\gamma_1$ and, by the previous result, $A_{\gamma_1}$ is compact. Since $\gamma_2$ is strongly integrable, it is easy to check that $A_{\gamma_2}$ is compact (see the second part of this proposition). The thesis follows since $A_\gamma = A_{\gamma_1} + A_{\gamma_2}$.

The second statement of the proposition is standard. Indeed, if $\gamma$ is a strongly $p$-integrable map, that is, $\gamma$ is measurable and

$$\int_X \|\gamma(x)\|_{\mathcal{H}}^p \, d\mu(x) = \int_X \Gamma(x, x)^{p/2} \, d\mu(x) < +\infty,$$

16
then $A_\gamma : H \to L^p(X, \mu)$ is a compact operator. Indeed, if $(v_n)_{n\in \mathbb{N}}$ is a sequence in $H$ which converges weakly to 0, then $\langle v_n, \gamma_x \rangle_H \to 0$ for all $x \in X$. Since $|\langle v_n, \gamma_x \rangle_H| \leq \|v_n\|_H \|\gamma_x\|_H$ and $\sup_n \|v_n\|_H < +\infty$, it follows by dominated convergence theorem that $A_\gamma v_n \to 0$ in $L^p(X, \mu)$. This shows that $A_\gamma$ maps weakly convergent sequences into norm convergent sequences, so $A_\gamma$ is compact.

Example 2 below shows that, if $p > 1$ and $\gamma$ is only weakly $p$-integrable, $A_\gamma$ can be a non-compact operator in $L^p(X, \mu)$. However, if $\mu(X) < +\infty$, since $L^p(X, \mu) \subset L^1(X, \mu)$, $A_\gamma$ is a compact operator from $H$ into $L^1(X, \mu)$ (Halmos and Sunder, 1978).

If $p = \infty$ and $\gamma$ is essentially bounded, $A_\gamma$ is not necessarily compact.

For $p = 2$ we can compute $A_\gamma^* A_\gamma$, which is known as frame operator in the context of frame theory (see, for example, Young (2001)). The following result can be found in Halmos and Sunder (1978).

**Corollary 1** With the notation of Proposition 5, assume that $\gamma$ is weakly square-integrable, then

$$A_\gamma^* A_\gamma = \int_X \langle \cdot, \gamma_x \rangle_H \gamma_x \, d\mu(x),$$

(18)

where the integral converges in the weak operator topology.

In particular, $A_\gamma$ is a Hilbert-Schmidt operator if and only if $\gamma$ is strongly square-integrable, that is,

$$\int_X \Gamma(x, x) \, d\mu(x) < +\infty.$$ (19)

If this last condition holds, the integral in Equation 16 converges in norm and the integral in Equation 18 converges in trace norm.

Finally, $\gamma$ is a (strongly) square-integrable function if and only if $L_\Gamma$ is a trace class operator from $L^2(X, \mu)$ in $L^2(X, \mu)$.

**Proof.** Eq. (18) follows from Eq. (16) and the definition of $A_\gamma$. To prove Condition (19), let $(e_n)_{n \in \mathbb{N}}$ be a Hilbert basis of $H_\gamma$. Since $A_\gamma^* A_\gamma$ is a positive operator and $|\langle \gamma(\cdot), e_n \rangle_H|^2$ is a positive function, by monotone convergence
theorem, one has that
\[
\text{Tr}(A^*_\gamma A_\gamma) = \sum_n \int_X |\langle e_n, \gamma_x \rangle_H|^2 d\mu(x)
\]
\[
= \int_X \sum_n |\langle e_n, \gamma_x \rangle_H|^2 d\mu(x)
\]
\[
= \int_X \langle \gamma_x, \gamma_x \rangle_H d\mu(x)
\]
\[
= \int_X \Gamma(x, x) d\mu(x)
\]
and the thesis follows. Finally, we prove the statements about Eqs. (16) and (18). Indeed, since \(H_\gamma\) is separable, by Proposition 2, the map \(\gamma\) is measurable, so that the maps \(x \mapsto \phi(x)\gamma_x\) and \(x \mapsto \langle \cdot, \gamma_x \rangle_H \gamma_x\) are measurable as maps taking values in \(H_\gamma\) and in the separable Banach space of trace class operators on \(H_\gamma\), respectively. Moreover,
\[
\|\phi(x)\gamma_x\|_H = |\phi(x)|\sqrt{\Gamma(x, x)} \in L^1(X, \mu)
\]
\[
\|\langle \cdot, \gamma_x \rangle_H \gamma_x\|_{\text{tr}} = \Gamma(x, x) \in L^1(X, \mu),
\]
where \(\|\cdot\|_{\text{tr}}\) is the trace norm. So the integrals are well defined and the claims concerning convergence follow.

The above result can be restated in the following way. The function \(\gamma\) is (strongly) square-integrable, that is, Eq. (19) holds, if and only if \(A^*_\gamma A_\gamma\) is a trace class operator on \(H\) and, by polar decomposition, this latter condition is equivalent to the fact that \(A_\gamma A^*_\gamma\) is of trace class on \(L^2(X, \mu)\). This proves the last statement of the corollary.

The following example shows that the weakly square-integrability is not sufficient for compactness of \(A_\gamma\) and that strong square-integrability is not necessary.

**Example 2** Let \(H\) be an infinite dimensional separable Hilbert space and \(\mu\) a measure on \(X\) such that there exists a disjoint sequence \((X_n)_{n \in \mathbb{N}}\) of measurable sets satisfying
\[
0 < \mu(X_n) = a_n < +\infty \quad n \in \mathbb{N}.
\]
Let \((e_n)_{n \in \mathbb{N}}\) be a basis of \(H\) and \((\sigma_n)_{n \in \mathbb{N}}\) be a positive sequence of \(\mathbb{R}\).
Define the map \(\gamma\) from \(X\) to \(H\) as
\[
\gamma_x = \frac{1}{\sqrt{a_n}} \sigma_n e_n \quad \text{if } x \in X_n.
\]
Clearly $\gamma$ is measurable and, for all $v \in \mathcal{H}$,
\[
\langle v, \gamma_x \rangle_{\mathcal{H}} = \frac{1}{\sqrt{a_n}} \sigma_n \langle v, e_n \rangle_{\mathcal{H}} \quad \text{if } x \in X_n.
\]
It follows that
\[
\int_X |\langle v, \gamma_x \rangle_{\mathcal{H}}|^2 d\mu(x) = \sum_n \sigma_n^2 |\langle v, e_n \rangle_{\mathcal{H}}|^2.
\]
Since the sequence $(|\langle v, e_n \rangle_{\mathcal{H}}|^2)_{n\in\mathbb{N}}$ is in $\ell_1$, $\gamma$ is a weakly square-integrable function if and only if the sequence $(\sigma_n)_{n\in\mathbb{N}}$ is bounded.
Assume from now on that $(\sigma_n)_{n\in\mathbb{N}}$ is bounded. It follows that, given $v \in \mathcal{H}$,
\[
A_{\gamma}v = \sum_n \sigma_n \langle v, e_n \rangle_{\mathcal{H}} \phi_n
\]
where
\[
\phi_n(x) = \begin{cases} 
\frac{1}{\sqrt{a_n}} & x \in X_n \\
0 & x \notin X_n
\end{cases}
\]
and the series converges in $L^2(X, \mu)$.
Since $(\phi_n)_{n\in\mathbb{N}}$ is an orthonormal sequence of $L^2(X, \mu)$, the above equation shows that $(e_n, \sigma_n^2)$ is the spectral decomposition of $A_{\gamma}^* A_{\gamma}$. It follows that $A_{\gamma}$ is a compact operator if and only if
\[
\lim_{n \to +\infty} \sigma_n = 0.
\]
Finally, $A_{\gamma}$ is a Hilbert-Schmidt operator if and only if
\[
\sum_{n=1}^{+\infty} \sigma_n^2 < +\infty.
\]
We end the section with a simple proposition showing the connection between integral operators and weakly integrable functions (see Halmos and Sunder (1978) for a discussion).

**Proposition 7** With the notation of Proposition 3, let $K : X \times Y \to \mathbb{C}$ be a Carleman $(p, 2)$-bounded kernel with $1 \leq p \leq \infty$ and let $\gamma : X \to L^2(Y, \nu)$
\[
\gamma_x = K(x, \cdot)
\]
for $\mu$-almost all $x \in X$, then $\gamma$ is a weakly $p$-integrable function.
Proof. By separability of $L^2(Y, \nu)$, the map $x \mapsto \gamma_x$ is measurable. By definition of Carleman $(p, 2)$-bounded kernel, for $\mu$-almost all $x \in X$ $K(x, \cdot)$ is square-integrable with respect to $\nu$ and
\[
    \langle v, \gamma_x \rangle_{L^2(Y)} = \int_Y K(x, y)v(y) \, d\nu(y) = (L_Kv)(x) \quad \forall v \in L^2(Y, \nu).
\]
Since $K$ is a $(p, 2)$-bounded kernel, the range of $L_K$ is in $L^p(X, \mu)$, so that $\gamma$ is a weakly $p$-integrable function. 

5 Continuity

In this section we study the weak continuity of $\gamma$. The following result is due to Schwartz (1964, Proposition 24), but our proof is based on elementary tools.

Proposition 8 Let $X$ be a locally compact space and $\mathcal{H}$ a Hilbert space. The following facts are equivalent:

1. the map $\gamma$ is weakly continuous;
2. the function $\Gamma$ is locally bounded and separately continuous.

If one of the above conditions holds, $\text{Im} \, A_\gamma \subset C(X)$, the operator $A_\gamma$ is continuous from $\mathcal{H}$ into $C(X)$, and
\[
    \ker A_\gamma = \overline{\text{span}} \{ \gamma_x \in \mathcal{H} | x \in X \} = \mathcal{H}_\gamma^\perp. \tag{20}
\]

If $X$ is separable, then $\mathcal{H}_\gamma$ is separable.

Proof.

1) $\Rightarrow$ 2) By assumption, $A_\gamma v$ is a continuous function for all $v \in \mathcal{H}$. Let $x \in X$, since $\Gamma(\cdot, x) = A_\gamma(\gamma_x)(\cdot)$, clearly $\Gamma$ is separately continuous. Fix now a compact set $C$. Since $A_\gamma v = \langle v, \gamma(\cdot) \rangle_{\mathcal{H}}$ is continuous, it is bounded on $C$ for all $v \in \mathcal{H}$, so the Banach-Steinhaus theorem ensures
\[
    \|\gamma_x\|_{\mathcal{H}} \leq M \quad \forall x \in C,
\]
for some constant $M > 0$. Moreover, by Cauchy-Schwartz inequality,
\[
    |\Gamma(x, t)| = |\langle \gamma_t, \gamma_x \rangle_{\mathcal{H}}| \leq \|\gamma_x\|_{\mathcal{H}} \|\gamma_t\|_{\mathcal{H}} \leq M^2,
\]
for all $t, x \in C$, so that $\Gamma$ is locally bounded.
2) ⇒ 1) Let \( w = \sum_{i=1}^{n} a_i \gamma_{x_i} \). The fact that \( \Gamma \) is separately continuous and Eq. (11) imply that the function \( A_{\gamma}w = \sum_{i=1}^{n} a_i \Gamma(\cdot, x_i) \) is continuous.

Let now \( v \in H_\gamma \) and \( x_0 \in X \), we prove that \( A_{\gamma}v \) is continuous at \( x_0 \).

Let \( C \) be a compact neighborhood \( C \) of \( x_0 \) and

\[
M = \sup_{x \in C} \left\| \gamma_{x} \right\|_{\mathcal{H}} = \sup_{x \in C} \sqrt{\Gamma(x, x)},
\]

where \( M \) is finite due to locally boundedness of \( \Gamma \). Fixed \( \epsilon > 0 \), there is a finite linear combination \( w = \sum a_i \gamma_{x_i} \) such that \( \left\| v - w \right\|_{\mathcal{H}} \leq \epsilon \).

By the above observation, \( A_{\gamma}w \) is continuous, so, possibly replacing \( C \) with a smaller neighborhood,

\[
\left| (A_{\gamma}v)(x) - (A_{\gamma}v)(x_0) \right| \leq \left| (A_{\gamma}w)(x) - (A_{\gamma}w)(x_0) \right| + \left\| \gamma_{x} - \gamma_{x_0} \right\|_{\mathcal{H}} \left\| w - v \right\|_{\mathcal{H}} \leq \epsilon (1 + 2M) \quad \forall x \in C.
\]

Finally, if \( v \in H_\gamma^\perp \), \( A_{\gamma}v = 0 \), so that \( \gamma \) is weakly continuous.

If one of the above equivalent conditions holds, \( A_{\gamma} \) is a bounded operator from \( \mathcal{H} \) into \( C(X) \). Indeed, let \( C \) be a compact set, by locally boundedness of \( \Gamma \) and \( \left\| \gamma_{x} \right\|^2 = \Gamma(x, x) \), there is a constant \( M > 0 \) such that \( \left\| \gamma_{x} \right\|_{\mathcal{H}} \leq M \) for all \( x \in C \). Finally,

\[
\sup_{x \in C} \left| (A_{\gamma}v)(x) \right| = \sup_{x \in C} |\langle v, \gamma_{x} \rangle_{\mathcal{H}}| \leq \left\| v \right\|_{\mathcal{H}} \sup_{x \in C} \left\| \gamma_{x} \right\|_{\mathcal{H}} \leq M \left\| v \right\|_{\mathcal{H}},
\]

so that \( A_{\gamma} \) is bounded. The fact that \( \ker A_{\gamma} = H_\gamma^\perp \) is clear. Finally, assume that \( X \) is separable, we prove that \( H_\gamma \) is separable. Indeed, let \( X_0 \) be a dense denumerable subset of \( X \) and define

\[
H_0 = \overline{\text{span}} \{ \gamma_{x} \mid x \in X_0 \} \subset H_\gamma.
\]

Clearly, \( H_0 \) is separable. We claim that \( H_0 = H_\gamma \). Choose \( v \in H_0^\perp \) so that \( (A_{\gamma}v)(x) = \langle v, \gamma_{x} \rangle_{\mathcal{H}} = 0 \) for all \( x \in X_0 \). Since \( A_{\gamma}v \) is continuous and \( X_0 \) is dense, \( \langle v, \gamma_{x} \rangle_{\mathcal{H}} = 0 \) for all \( x \in X \) so that \( v \in H_\gamma^\perp \). It follows that \( H_0^\perp \subset H_\gamma^\perp \) so that \( H_\gamma \subset H_0 \) and the claim follows. \( \blacksquare \)

The following corollary characterizes the strong continuity of \( \gamma \) (see Proposition 24 of Schwartz [1964]).

**Corollary 2** With the assumptions of Proposition VIII, the following facts are equivalent:

1. the function \( \Gamma \) is continuous;
2. the function $\Gamma$ is continuous on the diagonal of $X \times X$;

3. the map $\gamma$ is continuous;

4. the map $\gamma$ is weakly continuous and the restriction of $\Gamma$ on the diagonal $x \mapsto \Gamma(x, x)$ is continuous;

5. the operator $A_\gamma : \mathcal{H} \to \mathcal{C}(X)$ is compact.

Proof.

1) $\Rightarrow$ 2) Trivial.

2) $\Rightarrow$ 3) It follows observing that Eq. (3) gives
\[
\|\gamma_x - \gamma_t\|^2 = \Gamma(x, x) + \Gamma(t, t) - \Gamma(x, t) - \Gamma(t, x) \quad x, t \in X.
\]

3) $\Rightarrow$ 1) Use the fact that the scalar product is continuous and Eq. (3).

3) $\iff$ 4) It is a restatement of the equivalence between strong convergence and weak convergence plus convergence of norms (so called $\mathcal{H}$-property).

3) $\iff$ 5) Let $B_1$ be the closed unit ball in $\mathcal{H}$. The operator $A_\gamma$ is compact if and only if $A_\gamma(B_1)$ is compact in $\mathcal{C}(X)$. Since a locally compact Hausdorff space is regular, by Ascoli theorem (Kelley, 1955) the set of functions $A_\gamma(B_1)$ is compact if and only if $A_\gamma(B_1)$ is closed, the set $(A_\gamma(B_1))(x)$ is bounded for all $x \in X$, and $A_\gamma(B_1)$ is equicontinuous. First two conditions are always satisfied since $A_\gamma$ is continuous from the Hilbert space $\mathcal{H}$ to $\mathcal{C}(X)$. Moreover, given $x, t \in X$, we have that
\[
\|\gamma_x - \gamma_t\|_{\mathcal{H}} = \sup_{v \in B_1} |(v, \gamma_x - \gamma_t)_{\mathcal{H}}| = \sup_{v \in B_1} |(A_\gamma v)(x) - (A_\gamma v)(t)| = \sup_{f \in A_\gamma(B_1)} |f(x) - f(t)|.
\]

Eq. (21) shows that $A_\gamma(B_1)$ is equicontinuous if and only if map $\gamma$ is continuous.

Let now $X$ be a locally compact second countable Hausdorff space endowed with a positive Radon measure $\mu$. Assume that $\gamma$ is a weakly continuous map from $X$ into a Hilbert space $\mathcal{H}$ and regard $A_\gamma$ as an operator from $\mathcal{H}$ into $L^0(X, \mu)$. Since $\mathcal{H}_\gamma$ is separable, Proposition 2 holds, so that $\ker A_\gamma = \mathcal{H}_\mu^\perp$. 

22
On the other hand, since $f \in C(X)$ is equal to 0 in $L^0(X, \mu)$ if and only if $f(x) = 0$ for all $x \in \text{supp } \mu$,

$$\ker A_\gamma = \{ v \in \mathcal{H} | \langle v, \gamma_x \rangle_\mathcal{H} = 0 \forall x \in \text{supp } \mu \} = \overline{\text{span} \{ \gamma_x | x \in \text{supp } \mu \}},$$

hence

$$\mathcal{H}_\mu = \overline{\text{span} \{ \gamma_x | x \in \text{supp } \mu \}}.$$ (22)

Finally, assume $X$ compact, so that $\mu$ is finite. Since $\gamma$ is weakly continuous, it is bounded and, hence, strongly $p$-integrable, so the operator $A_\gamma$ is always compact as a map in $L^p(X, \mu)$. However, in order $A_\gamma$ be compact as a map in $C(X)$, it is necessary (and sufficient) that $\gamma$ is strongly continuous.

### 6 Mercer theorem

In this section we characterize those reproducing kernel Hilbert spaces that are subspaces of $L^2(X, \mu)$ in terms of the spectral decomposition of the integral operator $L_\Gamma$. We recall the definition of complex Radon measure [Dieudonné, 1968]. Let $R > 0$ and $C_c((0, R])$ the space of compactly supported functions on the interval $(0, R]$, a complex Radon measure on $(0, R]$ is a complex linear form on $C_c((0, R])$ such that its restriction to $C_c([a, R])$ is bounded for all $0 < a < R$. If $\rho$ is a complex Radon measure, there is a unique positive Radon measure $|\rho|$ and a complex measurable function $h$ on $(0, R]$, such that

$$\rho(\phi) = \int_{(0,R]} \phi(\lambda) h(\lambda) d|\rho| (\lambda) \quad \phi \in C_c((0, R]).$$

and $|h(\lambda)| = 1$ for all $\lambda \in (0, R]$. If $\phi \in L^1((0, R], |\rho|)$, the integral with respect to $\rho$ is defined by

$$\int_{(0,R]} \phi(\lambda) d\rho(\lambda) = \int_{(0,R]} \phi(\lambda) h(\lambda) d|\rho| (\lambda).$$

The measure $|\rho|$ is called the absolute value of $\rho$.

**Proposition 9** Let $X$ be a locally compact second countable Hausdorff space endowed with a positive Radon measure $\mu$ such that $\text{supp } \mu = X$. Let $\mathcal{H}$ be a reproducing kernel Hilbert space with kernel $\Gamma$. The following conditions are equivalent.

1. $\mathcal{H}$ is a subspace of $L^2(X, \mu) \cap C(X)$;

2. the kernel $\Gamma$ is locally bounded, separately continuous, and $(2, 2)$-bounded.
If one of the above assumptions holds, the integral operator \( L_\Gamma \) of kernel \( \Gamma \) is a positive operator with spectral decomposition

\[
L_\Gamma = \int_{[0,R]} \lambda dP(\lambda) \quad R = \|L_\Gamma\|
\]

and

\[
\mathcal{H} = \{ v \in L^2(X, \mu) \cap C(X) \mid \int_{[0,R]} \frac{1}{\lambda} \langle dP(\lambda)v, v \rangle_{L^2(X)} < +\infty \}\]

\[
\|v\|_{\mathcal{H}}^2 = \int_{[0,R]} \frac{1}{\lambda} \langle dP(\lambda)v, v \rangle_{L^2(X)} \quad \forall v \in \mathcal{H}.
\] (23)

If \( x, y \in X \) there is a complex Radon measure \( \rho_{x,y} \) on \((0, R]\) such that

\[
\Gamma(x, y) = \int_{(0,R]} \lambda d\rho_{x,y}(\lambda)
\] (25)

and, for all \( E \in \mathcal{B}((0,R]) \) such that \( \overline{E} \subset (0, R]\), then

\[
\int_E d\rho_{x,y}(\lambda) = \sum_{n \in I} \phi_n(x) \overline{\phi_n(y)}
\] (26)

where \( \phi_n \in \mathcal{H} \) and \( (\phi_n)_{n \in I} \) is an orthonormal basis in \( L^2(X, \mu) \) for the range of \( P(E) \).

If \( x \in X \), \( \rho_{x,x} \) is a positive Radon measure on \((0, R]\) such that

\[
\rho_{x,x}((0, R]) = \sum_{n \in I} |\phi_n(x)|^2,
\] (27)

where \( \phi_n \in \mathcal{H} \) and \( (\phi_n)_{n \in I} \) a basis in \( L^2(X, \mu) \) for \( \ker L_\Gamma^\perp \). In particular \( \rho_{x,x} \) is finite if and only if \( \Gamma(\cdot, x) \in \text{Im } L_\Gamma \).

**Proof.** Propositions ensure that the inclusion of \( \mathcal{H} \) into \( C(X) \) is the operator \( A_\gamma \) associated with the map \( \gamma : X \to \mathcal{H} \)

\[
\gamma_x = \Gamma(\cdot, x) \quad x \in X.
\]

Propositions imply that \( \mathcal{H} \subset C(X) \) if and only if \( \Gamma \) is separately continuous and locally bounded. In particular, since \( X \) is separable, both the above conditions and Eq. (11) ensure that \( \mathcal{H} \) is separable. So Proposition with \( p = 2 \) gives that \( \mathcal{H} \subset L^2(X, \mu) \) if and only if \( \Gamma \) is a \((2,2)\)-bounded kernel. Hence the equivalence of Condition 1 and Condition 2 is now clear. Assume one of them and regard the inclusion \( A_\gamma \) as an operator from \( \mathcal{H} \) into \( L^2(X, \mu) \).
By Eqs. (22), (13), \( A_\gamma \) is injective and the polar decomposition of the adjoint \( A_\gamma^* \) gives

\[
A_\gamma^* = W(A_\gamma A_\gamma^*)^{\frac{1}{2}}
\]

(28)

where \( W \) is a partial isometry from \( L^2(X, \mu) \) to \( \mathcal{H} \) with kernel being equal to the kernel of \( A_\gamma^* \) and with range being equal to the closure of the range of \( A_\gamma^* \).

Proposition 5 states that \( A_\gamma A_\gamma^* = L_\Gamma \), so that \( L_\Gamma \) is a positive operator, the kernel of \( W \) is the kernel of \( L_\Gamma \) and, since \( A_\gamma \) is injective, \( W \) is surjective, that is, \( WW^* \) is the identity. Equation (28) gives

\[
A_\gamma = L_\Gamma^{\frac{1}{2}} W^*.
\]

(29)

Since \( A_\gamma \) is the inclusion of \( \mathcal{H} \) into \( L^2(X, \mu) \cap C(X) \) and is injective, Eq. (29) implies that we can identify \( \mathcal{H} \) with \( \text{Im} \ L_\Gamma^{\frac{1}{2}} \) and, by means of spectral theorem, Eq. (28) follows. In particular, for all \( v \in \mathcal{H} \), \( v \in (\text{ker} \ L_\Gamma)^\perp = (\text{Im} \ P(\{0\}))^\perp \).

Let now \( v \in \mathcal{H} \), then \( v = L_\Gamma^{\frac{1}{2}} W^* v \) and

\[
\int_{[0,R]} \frac{1}{\lambda} \langle dP(\lambda) v, v \rangle_{L^2(X)} = \int_{[0,R]} \frac{1}{\lambda} \left\langle dP(\lambda)L_\Gamma^{\frac{1}{2}} W^* v, L_\Gamma^{\frac{1}{2}} W^* v \right\rangle_{L^2(X)} = \int_{[0,R]} \langle dP(\lambda) W^* v, W^* v \rangle_{L^2(X)} = \langle W^* v, W^* v \rangle_{L^2(X)} = \langle v, v \rangle_{\mathcal{H}},
\]

where we used that \( P(\{0\}) W^* = 0 \). So Eq. (21) follows.

Given \( x, y \in X \), let \( \rho_{x,y} \) be the linear form on \( C_c((0, R)] \) given by

\[
\rho_{x,y}(\phi) = \int_{[0,R]} \frac{1}{\lambda} \phi(\lambda) \langle dP(\lambda) W^* \gamma_y, W^* \gamma_x \rangle_{L^2(X)}.
\]

Since \( \langle dP(\lambda) W^* \gamma_y, W^* \gamma_x \rangle \) is a bounded complex measure and \( \phi \) has compact support, \( \rho_{x,y} \) is well defined and is a complex Radon measure. By definition, \( |\rho_{x,y}| \) has density \( \frac{1}{\lambda} \) with respect to the positive bounded measure \( |\langle dP(\lambda) W^* \gamma_y, W^* \gamma_x \rangle| \). Since the function 1 is integrable with respect to \( |\langle dP(\lambda) W^* \gamma_y, W^* \gamma_x \rangle| \), the function \( \lambda \) is integrable with respect to \( |\rho_{x,y}| \) and, hence, with respect to \( \rho_{x,y} \). In particular,

\[
\int_{[0,R]} \lambda d\rho_{x,y} = \int_{[0,R]} \langle dP(\lambda) W^* \gamma_y, W^* \gamma_x \rangle_{L^2(X)} = \int_{[0,R]} \langle dP(\lambda) W^* \gamma_y, W^* \gamma_x \rangle_{L^2(X)} = \langle W^* \gamma_y, W^* \gamma_x \rangle_{L^2(X)} = \langle \gamma_y, \gamma_x \rangle_{\mathcal{H}},
\]

25
so that Eq. (26) follows by definition of $\Gamma$.

Let now $E \in \mathcal{B}((0, R])$ be such that $E \subset (0, R]$. This last fact and the spectral theorem imply that $\text{Im} \ P(E) \subset \text{Im} \ L_\Gamma^\frac{1}{2} = \mathcal{H}$. Hence, there is a sequence $(\phi_n)_{n \in I}$ of $\mathcal{H}$ such that $(\phi_n)_{n \in I}$ is an orthonormal basis for $\text{Im} \ P(E)$. Since $E \subset (0, R]$, $\chi_E$ is integrable with respect to $\rho_{x,y}$ and

$$
\int_{(0,R]} \chi_E(\lambda) d\rho_{x,y} = \int_{[0,R]} \frac{\chi_E(\lambda)}{\lambda} \langle dP(\lambda)W^*\gamma_y, W^*\gamma_x \rangle_{L^2(X)} \\
= \int_{[0,R]} \frac{1}{\lambda} \langle dP(\lambda)P(E)W^*\gamma_y, P(E)W^*\gamma_x \rangle_{L^2(X)} \\
= \left\langle L_\Gamma^{-\frac{1}{2}} P(E)W^*\gamma_y, L_\Gamma^{-\frac{1}{2}} P(E)W^*\gamma_x \right\rangle_{L^2(X)}, \tag{30}
$$

where $P(E)W^*\gamma_x$ and $P(E)W^*\gamma_y$ are in $\text{Im} \ L_\Gamma^\frac{1}{2}$.

Let now $J$ a finite subset of $I$, taking into account the properties of the sequence $(\phi_n)_{n \in I}$,

$$
\sum_{n \in J} \left\langle L_\Gamma^{-\frac{1}{2}} P(E)W^*\gamma_y, \phi_n \right\rangle_{L^2(X)} \left\langle \phi_n, L_\Gamma^{-\frac{1}{2}} P(E)W^*\gamma_x \right\rangle_{L^2(X)} \\
= \sum_{n \in J} \left\langle W^*\gamma_y, L_\Gamma^{-\frac{1}{2}} \phi_n \right\rangle_{L^2(X)} \left\langle L_\Gamma^{-\frac{1}{2}} \phi_n, W^*\gamma_x \right\rangle_{L^2(X)} \\
(\phi_n = L_\Gamma^\frac{1}{2} W^* \phi_n) \\
= \sum_{n \in J} \left\langle W^*\gamma_y, W^* \phi_n \right\rangle_{L^2(X)} \left\langle W^* \phi_n, W^*\gamma_x \right\rangle_{L^2(X)} \\
= \sum_{n \in J} \langle \gamma_y, \phi_n \rangle_{\mathcal{H}} \langle \phi_n, \gamma_x \rangle_{\mathcal{H}} \\
(\text{Eq. (26)}) \\
= \sum_{n \in J} \phi_n(x) \phi_n(y).
$$

Eq. (26) follows observing that the series

$$
\sum_{n \in I} \left\langle L_\Gamma^{-\frac{1}{2}} P(E)W^*\gamma_y, \phi_n \right\rangle_{L^2(X)} \left\langle \phi_n, L_\Gamma^{-\frac{1}{2}} P(E)W^*\gamma_x \right\rangle_{L^2(X)}
$$

is summable with sum $\left\langle L_\Gamma^{-\frac{1}{2}} P(E)W^*\gamma_y, L_\Gamma^{-\frac{1}{2}} P(E)W^*\gamma_x \right\rangle_{L^2(X)} = \rho_{x,y}(E)$, by means of Eq. (30).

Finally, if $x \in X$, clearly $\rho_{x,x}$ is a positive Radon measure on $(0, R]$ having density $\frac{1}{\lambda}$ with respect to the positive bounded measure $\langle dP(\lambda)W^*\gamma_x, W^*\gamma_x \rangle$. 

26
Hence $\rho_{x,x}$ is bounded if and only if $W^*\gamma_x \in \text{Im} \ L^\frac{1}{2}_\Gamma$. Equation (29) implies that this condition is equivalent to $\gamma_x \in \text{Im} \ L^\frac{1}{2}_\Gamma$ and, if it is satisfied,

$$\rho_{x,x}(0, R) = \left\| L^{-\frac{1}{2}}_\Gamma W^*\gamma_x \right\|_{L^2(X)}^2.$$  \hspace{1cm} (31)

Let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{H}$ such that $(\phi_n)_{n \in \mathbb{N}}$ is a basis in $L^2(X, \mu)$ of ker $L^\perp_\Gamma$. Given $N \in \mathbb{N}$, reasoning as above,

$$\sum_{n=1}^{N} |\phi_n(x)|^2 = \sum_{n=1}^{N} \left| \left\langle L^{-\frac{1}{2}}_\Gamma \phi_n, W^*\gamma_x \right\rangle_{L^2(X)} \right|^2.$$  

The series in the right side converges if and only if $W^*\gamma_x \in \text{Im} \ L^\frac{1}{2}_\Gamma$ and, if it is so, its sum is $\left\| L^{-\frac{1}{2}}_\Gamma W^*\gamma_x \right\|_{L^2(X)}^2$. Eq. (31) implies Eq. (27). \hspace{1cm} \blacksquare

Equation (23) allows us to identify the elements of $\mathcal{H}$ with the only continuous functions on $X$ whose equivalence class belongs to the range of $L^\frac{1}{2}_\Gamma$, extending a result of Cucker and Smale (2002). The assumptions that supp $\mu = X$ and $\mathcal{H} \subset C(X)$ ensure that the identification between functions in $\mathcal{H}$ and equivalence classes in $L^2(X, \mu)$ is well defined. With this identification, Eq. (24) implies that $L^\frac{1}{2}_\Gamma$ is a unitary operator from ker $L^\perp_\Gamma$ onto $\mathcal{H}$.

If supp $\mu \neq X$, the statements of the above proposition hold replacing X with supp $\mu$ and $\mathcal{H}$ with span $\{\gamma_x | x \in \text{supp } \mu\}$. We use the assumption that $\gamma$ is weakly continuous only in two steps: to show the separability of $\mathcal{H}$ and to identify elements of $\mathcal{H}$, which are functions on $X$, with elements in $L^2(X, \mu)$, which are equivalence classes.

If the integral operator $L_\Gamma$ has a pure point spectrum

$$L_\Gamma = \sum_{n \in I} \lambda_n \langle \cdot, \phi_n \rangle \phi_n,$$

where $\lambda_n \geq 0$ and $(\phi_n)_{n \in \mathbb{N}}$ is a basis of $L^2(X, \mu)$, it is possible to choose $\phi_n \in \mathcal{H}$ for all $\lambda_n > 0$ and, with this choice, Proposition 9 gives that

$$\mathcal{H} = \{ v \in L^2(X, \mu) \cap C(X) | \sum_n \frac{1}{\lambda_n} | \langle v, \phi_n \rangle_{L^2(X)} |^2 < +\infty \},$$

$$\|v\|^2_{\mathcal{H}} = \sum_n \frac{1}{\lambda_n} | \langle v, \phi_n \rangle_{L^2(X)} |^2,$$

$$\Gamma(x, y) = \sum_n \lambda_n \phi_n(x)\phi_n(y).$$

The last series converges absolutely and, by Dini theorem and Condition 4 of Corollary 2 of Corollary 2 converges uniformly on compact subsets if and only if $\gamma$ is strongly continuous.
References

Syed Twareque Ali, Jean-Pierre Antoine, and Jean-Pierre Gazeau. *Coherent states, wavelets and their generalizations*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 2000. ISBN 0-387-98908-0.

N. Aronszajn. Theory of reproducing kernels. *Trans. Amer. Math. Soc.*, 68:337–404, 1950.

M. Bachir Bekka and Pierre de la Harpe. Irreducibility of unitary group representations and reproducing kernels Hilbert spaces. *Expo. Math.*, 21(2):115–149, 2003. ISSN 0723-0869. Appendix by the authors in collaboration with Rostislav Grigorchuk.

Nicolas Bourbaki. *Integration. I. Chapters 1–6*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2004. ISBN 3-540-41129-1. Translated from the 1959, 1965 and 1967 French originals by Sterling K. Berberian.

John B. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990. ISBN 0-387-97245-5.

Felipe Cucker and Steve Smale. On the mathematical foundations of learning. *Bull. Amer. Math. Soc. (N.S.)*, 39(1):1–49 (electronic), 2002. ISSN 0273-0979.

Ingrid Daubechies. *Ten lectures on wavelets*, volume 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992. ISBN 0-89871-274-2.

J. Dieudonné. *Éléments d’analyse. Tome II: Chapitres XII à XV*. Cahiers Scientifiques, Fasc. XXXI. Gauthier-Villars, Éditeur, Paris, 1968.

Dorin Ervin Dutkay. Positive definite maps, representations and frames. *Rev. Math. Phys.*, 16(4):451–477, 2004. ISSN 0129-055X.

Roger Godement. Les fonctions de type positif et la théorie des groupes. *Trans. Amer. Math. Soc.*, 63:1–84, 1948.

Paul Richard Halmos and Viakalathur Shankar Sunder. *Bounded integral operators on L² spaces*, volume 96 of *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*. Springer-Verlag, Berlin, 1978. ISBN 3-540-08894-6.
Einar Hille. Introduction to general theory of reproducing kernels. Rocky Mountain J. Math., 2(3):321–368, 1972.

Einar Hille and Ralph S. Phillips. Functional analysis and semi-groups. American Mathematical Society, Providence, R. I., 1974. Third printing of the revised edition of 1957, American Mathematical Society Colloquium Publications, Vol. XXXI.

Harry Hochstadt. Integral equations. Wiley Classics Library. John Wiley & Sons Inc., New York, 1989. ISBN 0-471-50404-1. Reprint of the 1973 original, A Wiley-Interscience Publication.

John L. Kelley. General topology. D. Van Nostrand Company, Inc., Toronto-New York-London, 1955.

A.N. Kolmogorov. Stationary sequences in Hilbert space. In Selected works. Probability theory and mathematical statistics, volume II, pages 228–271. Kluwer, 1992.

M. G. Krein. Hermitian positive kernels on homogeneous spaces. I. Ukrain. Mat. Žurnal, 1(4):64–98, 1949.

M. G. Krein. Hermitian-positive kernels in homogeneous spaces. II. Ukrain. Nat. Žurnal, 2:10–59, 1950.

I.M. Novitskii and M.A. Romanov. An extension of Mercer’s theorem to unbounded operator. Far Eastern Mathematical Reports, 7:123–132, 1999. (in Russian, English translation: arXiv:math.FA/0303159v3).

B. J. Pettis. On integration in vector spaces. Trans. Amer. Math. Soc., 44(2):277–304, 1938. ISSN 0002-9947.

S. Saitoh. Integral transforms, reproducing kernels and their applications, volume 369 of Pitman Research Notes in Mathematics Series. Longman, Harlow, 1997. ISBN 0-582-31758-4.

Saburou Saitoh. Theory of reproducing kernels and its applications, volume 189 of Pitman Research Notes in Mathematics Series. Longman Scientific & Technical, Harlow, 1988. ISBN 0-582-03564-3.

I. J. Schoenberg. Metric spaces and positive definite functions. Trans. Amer. Math. Soc., 44(3):522–536, 1938. ISSN 0002-9947.
Laurent Schwartz. Sous-espaces hilbertiens d’espaces vectoriels topologiques et noyaux associés (noyaux reproduisants). *J. Analyse Math.*, 13:115–256, 1964.

Laurent Schwartz. *Analyse. III*, volume 44 of *Collection Enseignement des Sciences* [*Collection: The Teaching of Science*]. Hermann, Paris, 1993. ISBN 2-7056-6163-2. Calcul intégral.

Robert M. Young. *An introduction to nonharmonic Fourier series*. Academic Press Inc., San Diego, CA, first edition, 2001. ISBN 0-12-772955-0.