The automorphism group of the Andrásfai graph

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Abstract

Let \( k \geq 1 \) be an integer and \( n = 3^k - 1 \). Let \( \mathbb{Z}_n \) denote the additive group of integers modulo \( n \) and let \( C \) be the subset of \( \mathbb{Z}_n \) consisting of the elements congruent to 1 modulo 3. The Cayley graph \( \text{Cay}(\mathbb{Z}_n; C) \) is known as the Andrásfai graph \( \text{And}(k) \). In this note, we determine the automorphism group of this graph. We will show that \( \text{Aut}(\text{And}(k)) \) is isomorphic with the dihedral group \( D_{2n} \).

1 Introduction

In this paper, a graph \( \Gamma = (V, E) \) is considered as an undirected simple graph where \( V = V(\Gamma) \) is the vertex-set and \( E = E(\Gamma) \) is the edge-set. For all the terminology and notation not defined here, we follow [4,15].

Let \( m > 0 \) be an integer. Let \( \mathbb{Z}_m \) denote the additive group of integers modulo \( m \). Let \( k > 1 \) be an integer and \( n = 3^k - 1 \). Let \( C = \{3t + 1 \mid 0 \leq t \leq k - 1\} \) be the subset of \( \mathbb{Z}_n \) consisting of the elements congruent to 1 modulo 3. It is easy to see that \( C \) is a symmetric set, that is, \( C \) is an inverse closed subset of the group \( \mathbb{Z}_n \). The Cayley graph \( \text{Cay}(\mathbb{Z}_n; C) \) is known as the Andrásfai graph \( \text{And}(k) \). It is easy to check that the graph \( \text{And}(2) \) is isomorphic to the 5-cycle and the graph \( \text{And}(3) \) is the Möbius ladder of order 8. The graph \( \text{And}(4) \) is depicted in Figure 1.
Figure 1. And(4)

The Cayley graphs And(k) were first used by Andrásfai in [1], and also appeared in his book [2]. It is not hard to show that the graph And(k) has diameter 2 and girth 4. The Andrásfai graph And(k) has some interesting properties and is a classic example in the subject of graph homomorphism [4]. For a given graph, one of the problems concerning it is determination its automorphism group. To the best of our knowledge, the automorphism group of the graph And(k) is still unknown. The main aim of the present paper is to determine the automorphism group of And(k). We will show that Aut(And(k)) ∼= D_{2n}, n = 3k − 1, where D_{2n} denotes the dihedral group of order 2n.

2 Preliminaries

The graphs Γ_1 = (V_1, E_1) and Γ_2 = (V_2, E_2) are called isomorphic, if there is a bijection ω : V_1 → V_2 such that {a, b} ∈ E_1 if and only if {ω(a), ω(b)} ∈ E_2 for all a, b ∈ V_1. In such a case, the bijection ω is called an isomorphism. An automorphism of a graph Γ is an isomorphism of Γ with itself. The set of automorphisms of Γ with the operation of composition of functions is a group called the automorphism group of Γ and denoted by Aut(Γ).

The group of all permutations of a set V is denoted by Sym(V) or just Sym(n) when |V| = n. A permutation group G on V is a subgroup of Sym(V). In this case we say that G acts on V. If G acts on V we say that G is transitive on V (or G acts transitively on V), when there is just one orbit. This means that given any two elements u and v of V, there is an element β of G such that β(u) = v. If Γ is a graph with vertex-set V then we can view each automorphism of Γ as a permutation on V and so Aut(Γ) = G is a permutation group on V.

A graph Γ is called vertex-transitive if Aut(Γ) acts transitively on V(Γ). For v ∈ V(Γ) and G = Aut(Γ) the stabilizer subgroup G_v is the subgroup of G containing of all automorphisms fixing v. In the vertex-transitive case all stabilizer subgroups G_v are conjugate in G, and consequently isomorphic. In this case, the index of G_v
in \( G \) is given by the equation, \(|G : G_v| = \frac{|G|}{|G_v|} = |V(\Gamma)|\). This fact is known as the Orbit-Stabilizer theorem which is a useful tool in finding the automorphism group of vertex-transitive graphs.

Let \( G \) be any abstract finite group with identity 1, and suppose \( \Omega \) is a set of \( G \), with the properties:

(i) \( x \in \Omega \implies x^{-1} \in \Omega \); (ii) \( 1 \notin \Omega \).

The Cayley graph \( \Gamma = \Gamma(G; \Omega) \) is the (simple) graph whose vertex-set and edge-set are defined as follows:

\[
V(\Gamma) = G, \quad E(\Gamma) = \{\{g, h\} \mid g^{-1}h \in \Omega\}.
\]

Although in most situations it is difficult to determine the automorphism group of a graph \( G \), and how it acts on its vertex-set or edge-set, there are various papers in the literature and some of the recent works include \([3,5,6,7,8,9,10,11,12,13,14,16,17]\).

The group \( G \) is called a semidirect product of \( N \) by \( Q \), denoted by \( G = N \rtimes Q \), if \( G \) contains subgroups \( N \) and \( Q \) such that: (i) \( N \unlhd G \) (\( N \) is a normal subgroup of \( G \)); (ii) \( NQ = G \); and (iii) \( N \cap Q = 1 \).

### 3 Main Results

**Definition 3.1.** Let \( k > 1 \) be an integer and \( n = 3k - 1 \). Let \( C = \{3t + 1 \mid 0 \leq t \leq k - 1\} \) be the subset of \( \mathbb{Z}_n \) consisting of the elements congruent to 1 modulo 3. It is easy to see that \( C \) is a symmetric set, that is, \( C \) is an inverse closed subset of the group \( \mathbb{Z}_n \). The Cayley graph \( \text{Cay}(\mathbb{Z}_n; C) \) is known as the Andrásfai graph \( \text{And}(k) \).

It follows from Definition 3.1, that the graph \( \text{And}(k) \) is a regular graph of valency \( k \) and the vertex-set of \( \text{And}(k) \) is the set \( V = V_0 \cup V_1 \cup V_2 \), where \( V_0 = \{3t \mid 0 \leq t \leq k - 1\} \), \( V_1 = \{3t + 1 \mid 0 \leq t \leq k - 1\} \) and \( V_2 = \{3t + 2 \mid 0 \leq t \leq k - 2\} \). Thus, we have \( |V_0| = |V_1| = k \) and \( |V_2| = k - 1 \). If \( v \in V_0 \), then \( v = 3j \), for some \( j \), \( 0 \leq j \leq k - 1 \). Now it is easy to see that

\[
N(v) = \{3i + 1 \mid j \leq i \leq k - 1\} \cup \{3l + 2 \mid 0 \leq l \leq j - 1\}. \quad (*)
\]

Also, if \( w \in V_2 \), then \( w = 3j + 2 \), \( 0 \leq j \leq k - 2 \), and thus we have

\[
N(w) = \{3i + 1 \mid 0 \leq i \leq j\} \cup \{3l \mid j + 1 \leq l \leq k - 1\}. \quad (**)\]

Now, from (*) and (**), it follows that the graph induced by the set \( V_0 \cup V_2 \) in \( \text{And}(k) \) is a bipartite graph such that the vertex \( 3j = v \in V_0 \) has \( j \) neighbors in \( V_2 \) and the vertex \( 3j + 2 = w \in V_2 \) has \( k - j - 1 \) neighbors in \( V_0 \). Note that all the neighbors of the vertex \( v = 0 \) are in \( V_1 \). Let \( H = (V_0 - \{0\}) \cup V_2 \) be the subgraph induced by the set \( (V_0 - \{0\}) \cup V_2 \) in \( \text{And}(k) \). Thus, \( H \) is a connected bipartite graph such that if \( v, w \) are distinct vertices in \( H \), then we have \( N(v) \neq N(w) \) (note that the vertex \( v = 3(k - 1) \) is adjacent to every vertex in \( V_2 \) and the vertex \( w = 2 \) is adjacent to any vertex in \( V_0 \)).

In the sequel, we need the following fact.
Lemma 3.2. Let $\Gamma = (U \cup W, E)$, $U \cap W = \emptyset$ be a connected bipartite graph. If $f$ is an automorphism of the graph $\Gamma$, then $f(U) = U$ and $f(W) = W$, or $f(U) = W$ and $f(W) = U$.

Proof. Automorphisms of $\Gamma$ preserve distance between vertices and since two vertices are in the same part if and only if they are at even distance from each other, the result follows.

Theorem 3.3. Let $k > 1$ be an integer and $n = 3k - 1$. Then for the automorphism group of the graph $\operatorname{And}(k)$ we have, $\operatorname{Aut}(\operatorname{And}(k)) \cong \mathbb{D}_{2n}$, where $\mathbb{D}_{2n}$ denotes the dihedral group of order $2n$.

Proof. Let $\Gamma = (V, E) = \operatorname{And}(k)$ and $A = \operatorname{Aut}(\Gamma)$ be the automorphism group of $\Gamma$. Consider the vertex $v = 0$ and let $A_0$ be its stabilizer subgroup, that is, $A_0 = \{a \in A \mid a(0) = 0\}$. We know that $\Gamma$ is a Cayley graph, hence it is a vertex-transitive graph. From the well known Orbit-Stabilizer theorem, we know that $|V| = |A|/|A_0|$ and hence $|A| = |V||A_v|$, where $v$ is a vertex in $\Gamma$. In the first step of our work we determine $|A_0|$. Let $f \in A_0$. Let $V_0, V_1$ and $V_2$ be the subsets of $V$ which are defined preceding (*) and $W_0 = V_0 - \{0\}$. Thus for the restriction of $f$ to $N(0) = V_1$ we have $f(V_1) = V_1$ and hence $f(W_0 \cup V_2) = W_0 \cup V_2$. Let $H$ be the subgraph induced by the set $W_0 \cup V_2$ in the graph $\Gamma = \operatorname{And}(k)$ and $g = f|_{W_0 \cup V_2}$. Hence $g$ is an automorphism of $H$. It is clear that $H$ is a connected bipartite graph with parts $W_0$ and $V_2$ such that $|W_0| = |V_2| = k - 1$. In each part of the graph $H$ there is exactly one vertex $x_j$ of degree $j$, $1 \leq j \leq k - 1$. In other words, the vertex $v_j = 3j$ is the unique vertex in $W_0$ of degree $j$, also the vertex $w_j = 3k - 1 - 3j = 3(k - j) - 1 = 3(k - j - 1) + 2$ is the unique vertex in $V_2$ of degree $j$. Note that $w_j = 3k - 1 - 3j$ is the inverse of $v_j = 3j$ in the cyclic group $\mathbb{Z}_{3k - 1}$, hence we denote it by $-v_j$. We know that the mapping $g$ is an automorphism of the connected bipartite graph $H$. Thus from Lemma 3.2, it follows that

(i) $g(W_0) = W_0$ or (ii) $g(W_0) = V_2$.

(i) If $g(W_0) = W_0$, then since the vertex $v = 3j$ is the unique vertex of $W_0$ of degree $j$, hence for every $w \in W_0$ we have $g(w) = w$. Similarly, for every $v \in V_2$ we have $g(v) = v$. In other words, the restriction of the automorphism $f$ to the set $W_0 \cup V_2$ is the identity mapping. We show that if $x \in V_1$, then $f(x) = x$. Note that we have $f(V_1) = V_1$. If $v = 3j + 1$ is a vertex in $V_1$, then the set of neighbors of $v$ in $W_0$ is $N_1 = \{3t \mid 1 \leq t \leq j\}$. Hence $v$ has exactly $j$ neighbors in $W_0$. Since the number of neighbors of $v$ and $f(v)$ in $W_0$ are equal, hence we must have $v = f(v)$. From our discussion it follows that if $g(W_0) = W_0$, then the automorphism $f$ is the identity automorphism of the graph $\operatorname{And}(k)$, that is, $f = 1$.

(ii) Now, suppose that $g(W_0) = V_2$. Let $v \in W_0$. We saw that the vertex $-v$ is the unique vertex in $V_2$ such that its degree in the graph $H$ is equal to the degree of $v$, that is, $\deg_H(-v) = \deg_H(v)$. It follows that for every vertex $x$ of $H$ we have $g(x) = -x$. Since $\Gamma = \operatorname{And}(k)$ is an Abelian Cayley graph, then the mapping $a : V(\Gamma) \to V(\Gamma)$ defined by the rule $a(v) = -v$ is an automorphism of the graph $\Gamma$. Let $b = af$. Thus $b$ is an automorphism of $\Gamma$ such that its restriction to $W_0$ is
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the identity automorphism. Now, by what is proved in (i), it follows that $b = 1$. Since $a$ has order 2 in $Aut(\Gamma)$, then $f = a$.

We now conclude that if $A = Aut(\Gamma)$ and $A_0$ is the stabilizer subgroup of the vertex $v = 0$, then $A_0 = \{1, a\}$, and hence we have $|A_0| = 2$. Now, from Orbit-Stabilizer theorem it follows that $|A| = |A_0||V(\Gamma)| = 2(3k - 1)$.

On the other hand, we know that $Aut(\Gamma)$ has a subgroup isomorphic to the cyclic group $\mathbb{Z}_{3k - 1}$, that is, $S = \{f_v \mid v \in \mathbb{Z}_{3k - 1}\}$, where $f_v : V(\Gamma) \rightarrow V(\Gamma)$, $f_v(x) = x + v$ for every $x \in \mathbb{Z}_{3k - 1}$. It is easy to check that $A_0 \cap S = \{1\}$, hence $|SA_0| = \frac{|S||A_0|}{|S \cap A_0|} = 2(3k - 1) = |A|$. Thus, we have $A = SA_0$. Since the index of $S$ in $A$ is 2, so $S$ is a normal subgroup of $A$. Now we have $A \cong S \times A_0 \cong \mathbb{Z}_{3k - 1} \rtimes \mathbb{Z}_2 \cong \mathbb{D}_{2n}$, where $n = 3k - 1$.

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References

[1] B. ANDRÁSFAI, Gmphentheoretische Extremalprobleme, Acta Math. Acad. Sci. Hungar, 15 (1964), 413-438.

[2] B. ANDRÁSFAI, Introductory Graph Theory, Pergamon Press Inc., Elmsford, N.Y. 1977.

[3] A. Ganesan, Automorphism group of the complete transposition graph, J. Algebraic Combin. 42 (2015) 793-801.

[4] C. Godsil, G. Royle, Algebraic Graph Theory, Springer, Berlin (2001).

[5] L. Lu, Q. Huang, Automorphisms and Isomorphisms of Enhanced Hypercubes, Filomat. 34:8, (2020) 2805-2812.

[6] S. M. Mirafzal, Some other algebraic properties of folded hypercubes, Ars Comb. 124, (2016), 153-159.

[7] S. M. Mirafzal, A. Zafari, Some algebraic properties of bipartite Kneser graphs, Ars Combin. 153 (2020), 3-13 (Available from: arXiv: 1804.04570 [math.GR] (12 Apr 2018)).

[8] S. M. Mirafzal, A note on the automorphism groups of Johnson graphs, Ars Comb. 154 (2021), 245-255 (Available from: arXiv: 1702.02568v4 (2017)).

[9] S. M. Mirafzal, The automorphism group of the bipartite Kneser graph, Proc. Math. Sci. (2019), doi.org/10.1007/s12044-019-0477-9.

[10] S. M. Mirafzal, On the automorphism groups of us-Cayley graphs, arXiv: 1910.12563.v4 1702.02568v4 [math.GR] (2019).

[11] S. M. Mirafzal, On the automorphism groups of connected bipartite irreducible graphs, Proc. Math. Sci. (2020). https://doi.org/10.1007/s12044-020-0589-1
[12] S. M. Mirafzal, Cayley properties of the line graphs induced by consecutive layers of the hypercube, Bull. Malays. Math. Sci. Soc. DOI: 10.1007/s40840-020-01009-3, (2020).

[13] S. M. Mirafzal, On the distance-transitivity of the square graph of the hypercube, arXiv: 2101.01615v4.

[14] S. M. Mirafzal, M. Ziaee, A note on the automorphism group of the Hamming graph, Trans. Comb. Vol. 10 No. 2 (2021), pp. 129-136.

[15] J. Rotman, An Introduction to the Theory of Groups, 4th ed., Springer-Verlag, New York, 1995.

[16] J. X. Zhou, Y. Q. Feng, The automorphisms of bi-Cayley graphs, J. Combin. Theory Ser. B 116 (2016) 504-532.

[17] J.X. Zhou, J.H. Kwak, Y.Q. Feng, Z.L. Wu, Automorphism group of the balanced hypercube, Ars Math. Contemp. 12 (2017) 145-154.