ON CARMICHAËL’S CONJECTURE

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Introduction.
Carmichaël’s conjecture is the following: “the equation \( \varphi(x) = n \) cannot have a unique solution, \((\forall)n \in \mathbb{N}\), where \( \varphi \) is the Euler’s function”. R. K. Guy presented in [1] some results on this conjecture; Carmichaël himself proved that, if \( n_0 \) does not verify his conjecture, then \( n_0 > 10^{37} \); V. L. Klee [2] improved to \( n_0 > 10^{400} \), and P. Masai & A. Valette increased to \( n_0 > 10^{1000} \). C. Pomerance [4] wrote on this subject too.

In this article we prove that the equation \( \varphi(x) = n \) admits a finite number of solutions, we find the general form of these solutions, also we prove that, if \( x_0 \) is the unique solution of this equation (for a \( n \in \mathbb{N} \)), then \( x_0 \) is a multiple of \( 2^2 \cdot 3^2 \cdot 7^2 \cdot 43^2 \) (and \( x_0 > 10^{10000} \) from [3]).

In the last paragraph we extend the result to: \( x_0 \) is a multiple of a product of a very large number of primes.

§1. Let \( x_0 \) be a solution of the equation \( \varphi(x) = n \). We consider \( n \) fixed. We’ll try to construct another solution \( y_0 \neq x_0 \).

The first method:
We decompose \( x_0 = a \cdot b \) with \( a, b \) integers such that \( (a, b) = 1 \).
we look for an \( a' \neq a \) such that \( \varphi(a') = \varphi(a) \) and \( (a', b) = 1 \); it results that \( y_0 = a' \cdot b \).

The second method:
Let’s consider \( x_0 = q_1^{\beta_1} \cdots q_r^{\beta_r} \), where all \( \beta_i \in \mathbb{N}^+ \), and \( q_1, \ldots, q_r \) are distinct primes two by two; we look for an integer \( q \) such that \( (q, x_0) = 1 \) and \( \varphi(q) \) divides \( x_0 / (q_1, \ldots, q_r) \); then \( y_0 = x_0q / \varphi(q) \).

We immediately see that we can consider \( q \) as prime.

The author conjectures that for any integer \( x_0 \geq 2 \) it is possible to find, by means of one of these methods, a \( y_0 \neq x_0 \) such that \( \varphi(y_0) = \varphi(x_0) \).

Lemma 1. The equation \( \varphi(x) = n \) admits a finite number of solutions, \((\forall)n \in \mathbb{N}\).
Proof. The cases \( n = 0,1 \) are trivial.
Let’s consider \( n \) to be fixed, \( n \geq 2 \). Let \( p_1 < p_2 < \ldots < p_s \leq n + 1 \) be the sequence of prime numbers. If \( x_0 \) is a solution of our equation (1) then \( x_0 \) has the form 
\[ x_0 = p_1^{\alpha_1} \ldots p_s^{\alpha_s}, \]
with all \( \alpha_i \in \mathbb{N} \). Each \( \alpha_i \) is limited, because:
\[ (\forall i \in \{1,2,\ldots,s\}, \ (\exists) a_i \in \mathbb{N} : p_i^{\alpha_i} \geq n. \]
Whence \( 0 \leq \alpha_i \leq a_i + 1 \), for all \( i \). Thus, we find a wide limitation for the number of solutions:
\[ \prod_{i=1}^{s} (a_i + 2) \]

**Lemma 2.** Any solution of this equation has the form (1) and (2):
\[ x_0 = n \cdot \left( \frac{p_1}{p_1-1} \right)^{\epsilon_1} \ldots \left( \frac{p_s}{p_s-1} \right)^{\epsilon_s} \in \mathbb{Z}, \]
where, for \( 1 \leq i \leq s \), we have \( \epsilon_i = 0 \) if \( \alpha_i = 0 \), or \( \epsilon_i = 1 \) if \( \alpha_i \neq 0 \).

Of course, \( n = \varphi(x_0) = x_0 \left( \frac{p_1}{p_1-1} \right)^{\epsilon_1} \ldots \left( \frac{p_s}{p_s-1} \right)^{\epsilon_s} \),
whence it results the second form of \( x_0 \).

From (2) we find another limitation for the number of the solutions: \( 2^s - 1 \) because each \( \epsilon_i \) has only two values, and at least one is not equal to zero.

§2. We suppose that \( x_0 \) is the unique solution of this equation.

**Lemma 3.** \( x_0 \) is a multiple of \( 2^2 \cdot 3^2 \cdot 7^2 \cdot 43^2 \).

*Proof.* We apply our second method.

Because \( \varphi(0) = \varphi(3) \) and \( \varphi(1) = \varphi(2) \), we take \( x_0 \geq 4 \).

If \( 2 \mid x_0 \) then there is \( y_0 = 2x_0 \neq x_0 \) such that \( \varphi(y_0) = \varphi(x_0) \), hence \( 2 \mid x_0 \); if \( 4 \mid x_0 \), then we can take \( y_0 = x_0 / 2 \).

If \( 3 \mid x_0 \) then \( y_0 = 3x_0 / 2 \), hence \( 3 \mid x_0 \); if \( 9 \mid x_0 \) then \( y_0 = 2x_0 / 3 \), hence \( 9 \mid x_0 \); whence \( 4 \cdot 9 \mid x_0 \).

If \( 7 \mid x_0 \) then \( y_0 = 7x_0 / 6 \), hence \( 7 \mid x_0 \); if \( 49 \mid x_0 \) then \( y_0 = 6x_0 / 7 \) hence \( 49 \mid x_0 \); whence \( 4 \cdot 9 \cdot 49 \mid x_0 \).

If \( 43 \mid x_0 \) then \( y_0 = 43x_0 / 42 \), hence \( 43 \mid x_0 \); if \( 43^2 \mid x_0 \) then \( y_0 = 42x_0 / 43 \), hence \( 43^2 \mid x_0 \); whence \( 2^2 \cdot 3^2 \cdot 7^2 \cdot 43^2 \mid x_0 \).

Thus \( x_0 = 2^\gamma_1 \cdot 3^\gamma_2 \cdot 7^\gamma_3 \cdot 43^\gamma_4 \cdot t \), with all \( \gamma_j \geq 2 \) and \( (t, 2 \cdot 3 \cdot 7 \cdot 43) = 1 \) and \( x_0 > 10^{10000} \) because \( n_0 > 10^{10000} \).

§3. Let’s consider \( \gamma_j \geq 3 \). If \( 5 \mid x_0 \) then \( 5x_0 / 4 = y_0 \), hence \( 5 \mid x_0 \); if \( 25 \mid x_0 \) then \( y_0 = 4x_0 / 5 \), whence \( 25 \mid x_0 \).

We construct the recurrent set \( M \) of prime numbers:

a) the elements \( 2,3,5 \in M \);

b) if the distinct odd elements \( e_1,\ldots,e_n \in M \) and \( b_m = 1 + 2^m \cdot e_1,\ldots,e_n \) is prime, with \( m = 1 \) or \( m = 2 \), then \( b_m \in M \);
c) any element belonging to \( M \) is obtained by the utilization (a finite number of times) of the rules a) or b) only.

The author conjectures that \( M \) is infinite, which solves this case, because it results that there is an infinite number of primes which divide \( x_0 \). This is absurd.

For example 2, 3, 5, 7, 11, 13, 23, 29, 31, 43, 47, 53, 61, \( \ldots \) belong to \( M \).

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The method from §3 could be continued as a tree (for \( \gamma_2 \geq 3 \) afterwards \( \gamma_3 \geq 3 \), etc.) but its ramifications are very complicated…

§4. A Property for a Counter-Example to Carmichael Conjecture.

Carmichael has conjectured that:

\[
\forall n \in \mathbb{N}, \quad \exists m \in \mathbb{N}, \quad \text{with } m \neq n, \quad \text{for which } \varphi(n) = \varphi(m), \quad \text{where } \varphi \text{ is Euler’s totient function.}
\]

There are many papers on this subject, but the author cites the papers which have influenced him, especially Klee’s papers.

Let \( n \) be a counterexample to Carmichael’s conjecture.

Grosswald has proved that \( n_0 \) is a multiple of 32, Donnelly has pushed the result to a multiple of \( 2^{14} \), and Klee to a multiple of \( 2^{42} \cdot 3^{47} \), Smarandache has shown that \( n \) is a multiple of \( 2^2 \cdot 3^3 \cdot 7^2 \cdot 43^2 \). Masai & Valette have bounded \( n > 10^{10000} \).

In this paragraph we will extend these results to: \( n \) is a multiple of a product of a very large number of primes.

We construct a recurrent set \( M \) such that:

a) the elements \( 2, 3 \in M \);

b) if the distinct elements \( 2, 3, q_1, \ldots, q_r \in M \), and \( p = 1 + 2^a \cdot 3^b \cdot q_1 \cdot \ldots \cdot q_r \) is a prime, where \( \{0, 1, 2, \ldots, 41\} \) and \( \{0, 1, 2, \ldots, 46\} \), then \( p \in M \) ; \( r \geq 0 \);

c) any element belonging to \( M \) is obtained only by the utilization (a finite number of times) of the rules a) or b).

Of course, all elements from \( M \) are primes.

Let \( n \) be a multiple of \( 2^{42} \cdot 3^{47} \);

- if \( 5 \mid n \) then there exists \( m = 5n/4 \neq n \) such that \( \varphi(n) = \varphi(m) \); hence \( 5 \mid n \); whence \( 5 \in M \);
- if \( 5 \mid n \) then there exists \( m = 4n/5 \neq n \) with our property; hence \( 5^2 \mid n \);

analogously, if \( 7 \mid n \) we can take \( m = 7n/6 \neq n \), hence \( 7 \mid n \); if \( 7 \mid n \) we can take \( m = 6n/7 \neq n \); whence \( 7 \in M \) and \( 7^2 \mid n \); etc.

The method continues until it isn’t possible to add any other prime to \( M \), by its construction.

For example, from the 168 primes smaller than 1000, only 17 of them do not belong to \( M \) (namely: 101, 151, 197, 251, 401, 491, 503, 601, 607, 677, 701, 727, 751, 809, 883, 907, 983); all other 151 primes belong to \( M \).
Note $M = \{2, 3, p_1, p_2, \ldots, p_s, \ldots\}$, then $n$ is a multiple of $2^{42} \cdot 3^{17} \cdot p_1^2 \cdot p_2^2 \cdots p_s^2 \cdots$

From our example, it results that $M$ contains at least 151 elements, hence $s \geq 149$.

If $M$ is infinite then there is no counterexample $n$, whence Carmichaël’s conjecture is solved.

(The author conjectures $M$ is infinite.)

Using a computer it is possible to find a very large number of primes, which divide $n$, using the construction method of $M$, and trying to find a new prime $p$ if $p - 1$ is a product of primes only from $M$.

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[First part published in “Gamma”, XXV, Year VIII, No. 3, June 1986, pp. 4-5; and second part in “Gamma”, XXIV, Year VIII, No. 2, February 1986, pp. 13-14.]