LOCALLY SYMMETRIC FAMILIES OF CURVES AND JACOBIANS

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1. Introduction

In this paper we study locally symmetric families of curves and jacobians. For the purposes of this paper, a jacobian is an abelian variety that is \( \text{Pic}^0 \) of a semi-stable curve. Denote the moduli space of principally polarized abelian varieties of dimension \( g \) by \( \mathcal{A}_g \). It is a locally symmetric variety. By a **locally symmetric family of jacobians**, we mean a family of jacobians parameterized by a locally symmetric variety \( X \) where the period map \( X \to \mathcal{A}_g \) is a map of locally symmetric varieties — cf. (8.2). A **locally symmetric family of curves** is a family of semi-stable curves over a locally symmetric variety where \( \text{Pic}^0 \) of each curve in the family is an abelian variety and where the corresponding family of jacobians is a locally symmetric family — cf. (7.1). Not every locally symmetric family of jacobians can be lifted to a locally symmetric family of curves, even if one passes to arbitrary finite unramified covers of the base, as can be seen by looking at the universal abelian variety of dimension 3 — see (8.3).

**Definition 1.1.** A locally symmetric variety \( X \) is **bad** if it has a locally symmetric divisor. Otherwise, we shall say that \( X \) is **good**.

Each locally symmetric variety \( X \) arises from a semisimple \( \mathbb{Q} \)-group \( G \) whose associated symmetric space is hermitian. We may suppose that \( G \) is simply connected as a linear algebraic group. In this case we can write \( G \) as a product \( G_1 \times \cdots \times G_m \) of almost simple \( \mathbb{Q} \)-groups. There is a corresponding splitting \( X' = X_1 \times \cdots \times X_m \) of a finite covering \( X' \) of \( X \), and \( X \) is good if and only if each \( X_i \) is good. Since each \( G_i \) is almost simple, each \( X_i \) is also “simple” in the sense that it has no finite cover which splits as a product of locally symmetric varieties.

**Remark 1.2.**

(i) Suppose that \( X \) is a locally symmetric variety whose associated \( \mathbb{Q} \)-group \( G \) is almost simple. Nolan Wallach [2] has communicated to me a proof that if the \( \mathbb{Q} \)-rank of \( G \) is \( \geq 3 \), then \( X \) is good.

(ii) It follows from the classification of hermitian symmetric spaces [12, p. 518] that if the non-compact factor of \( G(\mathbb{R}) \) is simple and not \( \mathbb{R} \)-isogenous to \( \text{SO}(n,2) \) or \( \text{SU}(n,1) \) for any \( n \), then \( X \) is also good — see (5.2).

Recall that the symmetric space \( \text{SU}(n,1)/\text{U}(n) \) is the complex \( n \)-ball.

**Theorem 1.** Suppose that \( p : C \to X \) is a non-constant locally symmetric family of curves over the locally symmetric variety \( X \). Suppose that the corresponding \( \mathbb{Q} \)-group is almost simple. If every fiber of \( p \) is smooth, or if \( X \) is good and the generic fiber of \( p \) is smooth, then \( X \) is a quotient of the complex \( n \)-ball.

*Date:* March 1997.

This work was supported in part by a grant from the National Science Foundation.
Note that the simplicity of the group $G$ implies that the period map $X \to A_g$ is finite.

After studying the geometry of the locus of jacobians, we are able to prove the following result about a locally symmetric family $B \to X$ of jacobians. In this result, $X^{\text{dec}}$ denotes the set of points of $X$ where the fiber is the jacobian is that of a singular curve and $X^*$ denotes its complement. The locus of hyperelliptic jacobians in $X$ will be denoted by $X_H$, and $X^* \cap X_H$ will be denoted by $X^*_H$.

**Theorem 2.** Suppose that $B \to X$ is a non-constant locally symmetric family of jacobians and that $X^*$ is non-empty. Suppose that the corresponding $\mathbb{Q}$-group is almost simple. If $X$ is good, then either:

(i) $X$ is a quotient of the complex $n$-ball, or

(ii) $g \geq 3$, each component of $X^{\text{dec}}$ is of complex codimension $\geq 2$, $X^*_H$ is a non-empty divisor in $X^*$ which is smooth if $g > 3$, and the family does not lift to a locally symmetric family of curves.

Of course, we can also apply this theorem to all locally symmetric subvarieties of $X$. Note that the second case in the theorem does occur; take $g = 3$ and $X$ to be $A_3$ (cf. [8.3]). I do not know any examples where the second case of the theorem holds and $g > 3$, but suspect there are none.

These results depend upon the following kind of rigidity result for mapping class groups. We shall denote the mapping class group associated to a surface of genus $g$ with $n$ marked points and $r$ non-zero boundary components by $\Gamma_{g,n,r}$.

**Theorem 3.** Suppose that $G$ is a simply connected, almost simple $\mathbb{Q}$-group, that the symmetric space associated to $G(\mathbb{R})$ is hermitian, and that $\Gamma$ is an arithmetic subgroup of $G$. If $G(\mathbb{R})$ has real rank $> 1$ (that is, it is not $\mathbb{R}$-isogenous to the product of $SU(n,1)$ with a compact group) and if $g \geq 3$, then the image of every homomorphism $\Gamma \to \Gamma_{g,n,r}$ is finite.

This is a special case of a much more general result which was conjectured by Ivanov and proved by Farb and Masur [4]. (They do not assume hermitian symmetric nor that $\Gamma$ is arithmetic.) Their methods are very different from ours, so we have included a proof of this theorem in the case when $r + n > 0$, and a weaker statement that is sufficient for our applications in the case $r + n = 0$.

Our approach is via group cohomology. The basic idea is that if $C \to X$ is a locally symmetric family of curves, then there is a homomorphism from $\pi_1(X,*)$ to the mapping class group $\Gamma_g$, which is the orbifold fundamental group of the moduli space of curves. The rigidity theorem above shows that there are very few homomorphisms from arithmetic groups to mapping class groups. To get similar results for locally symmetric families of jacobians, we exploit the fact that the fundamental group of a smooth variety does not change when a subvariety of codimension $\geq 2$ is removed. It is for this reason that we are interested in good locally symmetric varieties.

This work arose out of a question of Frans Oort, who asked if there are any positive dimensional Shimura subvarieties of the jacobian locus not contained in the locus of reducible jacobians. He suspects that once the genus is sufficiently large, there may be none. His interest stems from a conjecture of Coleman [1] Conj. 6 which asserts that the set of points in $M_g$ whose jacobians have complex multiplication is finite provided that $g \geq 4$. This is false as stated as was shown by de Jong and Noot [12] who exhibited infinitely many smooth curves of genus 4.
and genus 6 whose Jacobians have complex multiplication. (Note that when \( g \leq 3 \) there are infinitely many curves whose Jacobian has complex multiplication as \( \mathcal{M}_g \) is dense in \( \mathcal{A}_g \) in these cases.) However, Oort tentatively believes Coleman’s conjecture may be true when \( g \) is sufficiently large. Partial results on Oort’s question, which are complementary to results in this paper, have been obtained by Ciliberto, van der Geer and Teixidor i Bigas in [3] and [4].

I am very grateful to Frans Oort for asking me this question, and also to my colleagues Les Saper, Mark Stern (especially) and Jun Yang for helpful discussions about symmetric spaces, arithmetic groups and their cohomology. I would also like to thank Nolan Wallach for communicating his result to me, and Nikolai Ivanov for telling me about his belief that rigidity for mapping class groups held in this form. Finally, thanks to Ben Moonen, Carel Faber and Gerard van der Geer for their very helpful comments on this manuscript, which helped improve the exposition and saved me from many a careless slip.

2. Background and Definitions

Here we gather together some definitions to help the reader. A locally symmetric variety is a locally symmetric space of the form

\[
\Gamma \backslash G(\mathbb{R})/K
\]

where \( G \) is a semi-simple \( \mathbb{Q} \)-group, \( \Gamma \) is an arithmetic subgroup of \( G \), \( K \) is a maximal compact subgroup of \( G(\mathbb{R}) \), and the symmetric space \( G(\mathbb{R})/K \) is hermitian. The moduli space of abelian varieties with a level \( l \) structure \( \mathcal{A}_g[l] = \text{Sp}_g(\mathbb{Z})[l]/\text{Sp}_g(\mathbb{R})/U(g) \) is a primary example. Here \( \text{Sp}_g(\mathbb{Z})[l] \) denotes the level \( l \) subgroup of \( \text{Sp}_g(\mathbb{Z}) \). This moduli space is well known to be a quasi-projective variety. A fundamental theorem of Baily and Borel [1] asserts that every locally symmetric variety is a quasi-projective algebraic variety. The imbedding into projective space is given by automorphic forms.

Suppose that

\[
X_1 = \Gamma_1 \backslash G_1(\mathbb{R})/K_1 \quad \text{and} \quad X_2 = \Gamma_2 \backslash G_2(\mathbb{R})/K_2
\]

are two locally symmetric varieties. A map of locally symmetric varieties \( X_1 \to X_2 \) is a map induced by a homomorphism of \( \mathbb{Q} \)-algebraic groups \( G_1 \to G_2 \).

Recall that a semi-simple \( k \)-group \( G \) is almost simple if its adjoint form is simple. An algebraic \( k \)-group is absolutely (almost) simple if the corresponding group over \( \overline{k} \) is (almost) simple.

Every simply connected \( \mathbb{Q} \)-group \( G \) is the product of simply connected, almost simple \( \mathbb{Q} \)-groups:

\[
G = \prod_{i=0}^{n} G_i
\]

This implies that if \( \Gamma \) is an arithmetic subgroup of \( G \), then

\[
\prod_{i=0}^{n} \Gamma \cap G_i
\]
has finite index in $\Gamma$. It follows that if $X$ is the locally symmetric space associated to $\Gamma$ and $X_i$ the locally symmetric space associated to $\Gamma_i$, then the natural mapping
\[
\prod_{i=0}^{n} X_i \to X
\]
is finite. If $X$ is a locally symmetric variety (i.e., $G(\mathbb{R})/K$ is hermitian), then each $X_i$ will also be a locally symmetric variety.

We shall call a locally symmetric variety associated to an arithmetic subgroup of an almost simple $\mathbb{Q}$-group a *simple* locally symmetric variety. They have no finite covers which are products of locally symmetric varieties. Also, since the associated $\mathbb{Q}$-group is almost simple, every map from a simple locally symmetric variety to a locally symmetric variety will be either constant or finite.

We shall make critical use of a vanishing theorem proved by Raghunathan [21]. Suppose that $\Gamma$ is an arithmetic subgroup of an almost simple $\mathbb{Q}$-group $G$. If $G(\mathbb{R})$ has real rank $\geq 2$, then $H^1(\Gamma, V) = 0$ for all finite dimensional representations of $G(\mathbb{R})$.

The classification of hermitian symmetric spaces [12, p. 518] implies that if $X = \Gamma \backslash G(\mathbb{R})/K$ is a simple locally symmetric variety where $G(\mathbb{R})$ has real rank 1, then the associated symmetric space $G(\mathbb{R})/K$ is the complex $n$-ball $SU(n, 1)/U(n)$. This rank condition, and the corresponding failure of Raghunathan’s Vanishing Theorem, is the principal reason we cannot handle ball quotients.

Margulis’s Rigidity Theorem [17] has the following important consequence for mappings between locally symmetric varieties. Suppose that $X_1$ and $X_2$ are locally symmetric varieties as above and that $X_1$ is simple. If the real rank of $G_1(\mathbb{R})$ is 2 or more, then for every homomorphism $\phi : \Gamma_1 \to \Gamma_2$, there is a finite covering
\[
\Gamma'_1 \backslash G_1(\mathbb{R})/K_1 \longrightarrow \Gamma_1 \backslash G_1(\mathbb{R})/K_1
\]
\[
X'_1 \longrightarrow X_1
\]
of locally symmetric varieties corresponding to a finite index subgroup $\Gamma'_1$ of $\Gamma_1$, and a map of locally symmetric varieties $X'_1 \to X_2$ which induces the restriction
\[
\phi|_{\Gamma'_1} : \Gamma'_1 \to \Gamma_2
\]
of $\phi$ to $\Gamma'_1$.

Every arithmetic group $\Gamma$ has a torsion free subgroup of finite index. When $\Gamma$ is torsion free, the locally symmetric variety $\Gamma \backslash G(\mathbb{R})/K$ is a $K(\Gamma, 1)$. That is, it has fundamental group $\Gamma$ and all of its higher homotopy groups are trivial.

The level $l$ subgroup $\Gamma_{g,r}[l]$ of the mapping class group $\Gamma_{g,r}^n$ is the kernel of the natural homomorphism $\Gamma_{g,r}^n \to Sp_g(\mathbb{Z}/l\mathbb{Z})$. It is torsion free when $l \geq 3$. In this case it is isomorphic to the fundamental group of $\mathcal{M}_{g,r}[l]$, the moduli space of smooth projective curves of genus $g$ over $\mathbb{C}$ with $n$ marked points, $r$ non-zero tangent vectors and a level $l$ structure. (As is customary, shall omit $r$ and $n$ when they are zero.) This isomorphism is unique up to an inner automorphism. We shall often fix a level $l \geq 3$ to guarantee that $A_g[l]$ and $\mathcal{M}_{g,r}[l]$ are smooth.
3. Homomorphisms to Mapping Class Groups

Suppose that $\Gamma$ is a discrete group and that $\phi: \Gamma \to \Gamma_{g,r}$ is a homomorphism. Denote the composite of $\phi$ with the natural homomorphism $\Gamma_{g,r} \to Sp_g(\mathbb{Z})$ by $\psi$. Denote the $k$th fundamental representation of $Sp_g(\mathbb{R})$ by $V_k$ ($1 \leq k \leq g$). Each of these can be viewed as a $\Gamma$ module via $\psi$.

**Theorem 3.1.** If $H^1(\Gamma, V_3) = 0$ and $g \geq 3$, then $\psi^*: H^2(Sp_g(\mathbb{Z}), \mathbb{Q}) \to H^2(\Gamma, \mathbb{Q})$ vanishes.

**Proof.** It suffices to prove the result when $r = n = 0$. I'll give a brief sketch. We can consider the group $H^1(\Gamma g, V_3)$. One has the cup product map $H^1(\Gamma g, V_3) \otimes H^1(\Gamma g, V_3) \to H^2(\Gamma g, \Lambda^2 V_3)$. Since $V_3$ has an $Sp_g$ invariant, skew symmetric bilinear form, we have a map $H^2(\Gamma g, \Lambda^2 V_3) \to H^2(\Gamma g, \mathbb{R})$. We therefore have a map

$$c: H^1(\Gamma g, V_3) \otimes H^1(\Gamma g, V_3) \to H^2(\Gamma g, \mathbb{R}).$$

It follows from Dennis Johnson’s results [14] (cf. [9, (5.2)]) that the left hand group has dimension 1 when $g \geq 3$, and from [8, §§7–8] that this map is injective. It is also known that the natural map $H^2(Sp_g(\mathbb{Z}), \mathbb{R}) \to H^2(\Gamma g, \mathbb{R})$ is an isomorphism for all $g \geq 3$ and that both groups are isomorphic to $\mathbb{R}$ (cf. [11], [18], [19]). The result now follows from the commutativity of the diagram.

**Corollary 3.2.** For all $g \geq 3$ and all $l > 0$, the homomorphism $\Gamma g[l] \to Sp_g(\mathbb{Z})[l]$ does not split.

**Proof.** By a result of Raghunathan [21], if $g \geq 2$, $\Gamma$ is a finite index subgroup of $Sp_g(\mathbb{Z})$ and $V$ is a rational representation of $Sp_g(\mathbb{R})$, then $H^1(\Gamma, V)$ vanishes. In particular,

$$H^1(Sp_g(\mathbb{Z})[l], V_3) = 0$$

when $g \geq 3$. If there were a splitting of the canonical homomorphism $\Gamma g \to Sp_g(\mathbb{Z})[l]$, then, by Theorem 3.1, the homomorphism

$$H^2(Sp_g(\mathbb{Z}), \mathbb{Q}) \to H^2(Sp_g(\mathbb{Z})[l], \mathbb{Q})$$

induced by the inclusion would be trivial. But when $g \geq 3$, Borel’s stability theorem [2] (see also [11, 3.1]) implies that this mapping is an isomorphism. It follows that no such splitting can exist.

**Remark 3.3.** There are cases when the cohomological obstruction above vanishes but there is still no lift. An example is given in the appendix.
Suppose that $X$ is a topological space with fundamental group $\pi$ and that $V$ is the local system corresponding to the $\pi$ module $V$. It is a standard fact that there is a natural map

$$H^k(\pi, V) \to H^k(X, V)$$

which is an isomorphism when $k \leq 1$ and injective when $k = 2$. We thus have the following corollary.

**Corollary 3.4.** Suppose that $g \geq 3$, $l \geq 3$ and that $\phi : X \to M_g[l]$ is a continuous map from a topological space $X$ to $M_g[l]$. If $H^1(X, V_3) = 0$, then the map

$$H^2(A_g, \mathbb{Q}) \to H^2(X, \mathbb{Q})$$

induced by the composition $X \to A_g$ of $\phi$ with the period map vanishes. \(\square\)

4. Maps of Lattices to Mapping Class Groups

In this section we apply the results of the previous section to the case where $\Gamma$ is the arithmetic group associated to a locally symmetric variety. We begin by recalling a result of Yang \[25\].

Suppose that $G$ is a semisimple group over a number field $k$. Yang \[25, (3.3)\] defined an invariant $\ell(G/k)$ of $G$ as follows. Fix a minimal parabolic $P$ of $G$. Let $T$ be the corresponding maximal $k$-split torus. Denote the relative root system of $G$ by $\Phi$, the base of $\Phi$ corresponding to $P$ by $\Delta$, and the set of positive relative roots by $\Phi^+$. Set

$$\rho_P = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$ 

Then we can define

$$\ell(G/k) = \max \{ q \in \mathbb{N} : 2\rho_P - \sum_{\alpha \in J} \alpha \text{ is strictly dominant for all subsets } J \text{ of } \Phi^+ \text{ of cardinality } q \}.$$ 

Denote the Lie algebra of $G$ by $\mathfrak{g}$ and its maximal compact subgroup by $K$. The importance of this invariant lies in the following result of Yang \[25\].

**Theorem 4.1** (Yang). If $G$ is a semisimple $\mathbb{Q}$-group and $\Gamma$ an arithmetic subgroup of $G$, then the natural map $H^m(\mathfrak{g}, K) \to H^m(\Gamma, \mathbb{R})$ is injective whenever $m \leq \ell(G/\mathbb{Q})$. \(\square\)

We will need the following consequence:

**Lemma 4.2.** Suppose that $\Gamma$ is an arithmetic subgroup of an almost simple $\mathbb{Q}$-group $G$. If $G \neq SL_2(\mathbb{Q})$, then $H^2(\mathfrak{g}, K) \to H^2(\Gamma, \mathbb{R})$ is injective.

**Proof.** When the $\mathbb{Q}$-rank of $G$ is zero, the corresponding locally symmetric space is compact and the relative Lie algebra cohomology injects. So we suppose that $G$ has positive $\mathbb{Q}$-rank. Note that $\ell(G/\mathbb{Q}) \geq 1$ for all such $G$.

Every almost simple $\mathbb{Q}$-group is of the form $R_{k/\mathbb{Q}}G$, where $G$ is an absolutely almost simple $k$-group (\[23\] p. 46]). It is not difficult to see that if $G$ is a $k$-group, then

$$\ell((R_{k/\mathbb{Q}}G)/\mathbb{Q}) \geq [k : \mathbb{Q}] \ell(G/k).$$

So, by Yang’s Theorem, it suffices to prove that for every absolutely almost simple group $G$, $\ell(G/k) \geq 2$. This follows from Tits’ classification of the decorated Dynkin
This is the statement we set out to prove when this with (4.2), we see that the map $p$ is injective or trivial. If $Y$ is connected and has $\geq H$, the map $\text{completion} \ldots$ [10, (14.9)] when $r$. But this group is residually torsion free nilpotent (i.e., injects into its unipotent completion). Proof of Theorem 3. Since every simply connected almost simple $\mathbb{Q}$-group $G$ is of the form $R_{k/\mathbb{Q}}G'$ where $k$ is a number field and $G'$ is a simply connected, absolutely almost simple $k$-group, the classification of Hermitian symmetric spaces [12, p. 518] implies that either $G(\mathbb{R})$ is isogenous to the product of $SU(n,1)$ and a compact group, or has real rank $\geq 2$.

By the superrigidity theorem of Margulis [17], we know that, after replacing $\Gamma$ by a finite index subgroup if necessary, there is a $\mathbb{Q}$-group homomorphism $G \to Sp_{g}(\mathbb{Q})$ that induces the homomorphism $\Gamma \to \Gamma_{g,r} \to Sp_{g}(\mathbb{Z})$. It follows that the representation $V_{3}$ of $\Gamma$ is the restriction of a representation of $G$. Since the real rank of $G$ is at least 2, it follows from Raghunathan’s result [21] that $H^{1}(\Gamma,V_{3})$ also vanishes. So, by (3.1), $H^{2}(Sp_{g}(\mathbb{Z}),\mathbb{R}) \to H^{2}(\Gamma,\mathbb{R})$ vanishes. But we have the commutative diagram

$$
\begin{align*}
H^{2}(Sp_{g}(\mathbb{Z}),\mathbb{C}) & \longrightarrow H^{2}(\Gamma,\mathbb{C}) \\
\| & \| \\
H^{2}(sp_{g}(\mathbb{R}),U(g);\mathbb{C}) & \longrightarrow H^{2}(g(\mathbb{R}),K;\mathbb{C}) \\
\| & \| \\
H^{2}(X,\mathbb{C}) & \longrightarrow H^{2}(Y,\mathbb{C})
\end{align*}
$$

where $X$ is the compact dual of Siegel space $Sp_{g}(\mathbb{R})/U(g)$, $Y$ is the compact dual of $G(\mathbb{R})/K$, and the vertical maps are the standard ones. Consider the homomorphism $G \to Sp_{g}(\mathbb{R})$. If it is trivial, then the image of $\Gamma$ in $Sp_{g}(\mathbb{Z})$ is finite. (Remember that we passed to a finite index subgroup earlier in the argument.) We will show that it must be trivial.

Since $G(\mathbb{R})$ is semisimple, it has a finite cover which is a product of simple groups $G_{i}$. Since the symmetric space associated to $G(\mathbb{R})$ is hermitian, the symmetric space associated to each $G_{i}$ is also hermitian. Denote the compact dual of the symmetric space associated to $G_{i}$ by $Y_{i}$. Then $Y$ is the product of the $Y_{i}$. Since each $G_{i}$ is simple, the induced map $G_{i} \to Sp_{g}(\mathbb{R})$ is either trivial or has finite kernel. It follows that the corresponding maps $Y_{i} \to X$ of compact duals are either injective or trivial. If $G \to Sp_{g}(\mathbb{R})$ is non-trivial, there is an $i$ such that $G_{i} \to Sp_{g}(\mathbb{R})$ is injective. The corresponding map $Y_{i} \to X$ of compact duals is injective. Since $X$ and $Y_{i}$ are Kähler, and since $H^{2}(Sp_{g}(\mathbb{Z}),\mathbb{C})$ is one dimensional, the map $H^{2}(X,\mathbb{C}) \to H^{2}(Y_{i},\mathbb{C})$ must be injective as the Kähler form cannot vanish on $Y_{i}$. It follows that the bottom map in the diagram is also injective. Combining this with (4.2), we see that the map $p$ is injective, which in turn implies that the top map is injective. But this contradicts (3.1). So $G \to Sp_{g}(\mathbb{R})$ must be trivial. This is the statement we set out to prove when $r + n = 0$.

Since $G \to Sp_{g}(\mathbb{R})$ is trivial, we conclude that $\Gamma$ maps to the Torelli group $T_{g,r}$. But this group is residually torsion free nilpotent (i.e., injects into its unipotent completion — see [10, (14.9)]) when $r + n > 0$. Since $H_{1}(\Gamma,\mathbb{Q}) = 0$, the image of

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1In the case $r + n = 0$ we shall prove that the image of $\Gamma \to \Gamma_{g} \to Sp_{g}(\mathbb{Q})$ is finite. If $\Gamma \to \Gamma_{g}$ is the monodromy representation of a family of jacobians, the finiteness of the image of this homomorphism implies the isotriviality of the family. This is all we shall need in the sequel.
Γ in every unipotent group over \(\mathbb{Q}\) is trivial. It follows that the image of \(\Gamma\) in the unipotent completion of \(T_{g,r}^n\), and therefore in \(T_{g,r}^n\), is trivial.  

**Corollary 4.3.** If we have a non-constant family of non-singular curves of genus \(\geq 3\) over a locally symmetric variety whose corresponding \(\mathbb{Q}\)-group \(G\) is almost simple, then the base is a quotient of the complex \(n\)-ball.

**Proof.** The base \(X\) of the family is \(X = \Gamma \backslash G(\mathbb{R})/K\) where \(G\) is the associated \(\mathbb{Q}\)-group, \(\Gamma\) is an arithmetic subgroup of \(G\), and \(K\) is a maximal compact subgroup of \(G(\mathbb{R})\). By passing to a finite index subgroup if necessary, we may assume that \(\Gamma\) is torsion free. Since the family is non trivial, the homomorphism \(\Gamma \to Sp_g(\mathbb{Z})\) induced by the period map of the family is not finite. It follows from Theorem 3.1 (see also the footnote to its proof) that \(G(\mathbb{R})\) must have real rank 1. This implies that \(G(\mathbb{R})/K\) is the complex \(n\)-ball.

5. Locally Symmetric Hypersurfaces in Locally Symmetric Varieties

In order to extend (4.3) to locally symmetric families of stable curves, we will need to know when a locally symmetric variety has a locally symmetric hypersurface. Recall that every almost simple \(\mathbb{Q}\)-group is of the form \(R_k/\mathbb{Q}G\) where \(k\) is a number field and \(G\) is an absolutely almost simple group over \(k\).

**Proposition 5.1.** If \(G' = R_{k/\mathbb{Q}}G\), where \(G\) is a \(k\)-group, then \(G(\mathbb{R})\) is hermitian symmetric space if and only if \(k\) is a totally real field and the symmetric space associated to each real imbedding of \(k\) into \(\mathbb{R}\) is hermitian.

**Proof.** We have
\[
G'(\mathbb{R}) = \prod_{\nu:k \to \mathbb{C}} G_{\nu}.
\]
where \(\nu\) ranges over complex conjugate pairs of imbeddings of \(k\) into \(\mathbb{C}\). Note that \(G_{\nu}\) is an absolutely simple real group if \(\nu\) is a real imbedding and \(G_{\nu}\) is not an absolutely simple real group if \(\nu\) is not real. The symmetric space associated to \(G'(\mathbb{R})\) is the product of the symmetric spaces of the \(G_{\nu}\). It is hermitian symmetric if and only if each of its factors is. But if \(\nu\) is not real, then \(G_{\nu}\) cannot be compact, and its symmetric space is not hermitian (see the list in Helgason [12, p. 518]). So \(k\) is totally real.  

**Proposition 5.2.** Suppose that \(G\) is a simple real Lie group whose associated symmetric space is hermitian. If \(G\) is not isogenous to \(SO(n,2)\) or \(SU(n,1)\) for any \(n\), then the symmetric space associated to \(G\) has no hermitian symmetric hypersurfaces.

**Proof.** This follows from the classification of irreducible hermitian symmetric spaces [12, p. 518] using [22, Theorem 10.5].

\(^2\)It would be interesting to know if \(T_g\) is residually torsion free nilpotent when \(g\) is sufficiently large.
6. GEOMETRY OF THE JACOBIAN LOCUS

The geometry of the locus of jacobians will play a significant role in the sequel. The jacobian locus $J_g[l]$ in $A_g[l]$ is the closure of the image of the period map $M_g[l] \to A_g[l]$. Denote the locus of jacobians of singular curves by $J^{dec}_g[l]$, and the locus $J_g[l] - J^{dec}_g[l]$ of jacobians of smooth curves by $J^*_g[l]$. In this section we are concerned with the geometry of $J^*_g[l]$ along the hyperelliptic locus of $J^*_g[l]$ and along $J^{dec}_g[l]$.

**Proposition 6.1.** If $g \geq 3$ and $l \geq 3$, then the locus of hyperelliptic jacobians in $J^*_g[l]$ is smooth, and its projective normal cone in $A_g[l]$ at the point $[C]$ is $\mathbb{P}(S^2(V_C))$, where $V_C$ is a vector space of dimension $g - 2$. Further, its projective normal cone in $J^*_g[l]$ at the point $[C]$ is $\mathbb{P}(V_C)$, which is imbedded in $\mathbb{P}(S^2V_C)$ via the “Plucker imbedding.”

**Proof.** This follows directly from [20].

The following fact must be well known.

**Lemma 6.2.** If $V$ is a complex vector space, then there is no line $L$ in $\mathbb{P}(S^2V)$ that is contained in the image of the Veronese imbedding $\nu : \mathbb{P}(V) \hookrightarrow \mathbb{P}(S^2V)$.

**Proof.** Denote the hyperplane class of $\mathbb{P}(V)$ by $H_1$ and that of $\mathbb{P}(S^2V)$ by $H_2$. Since a hyperplane section of $\mathbb{P}(V)$ in $\mathbb{P}(S^2V)$ is a quadric, $\nu^*H_2 = 2H_1$. If $L$ is a line in $\mathbb{P}(S^2V)$ that is contained in $\mathbb{P}(V)$, then

$$H_2|_L = 2H_1|_L \in H^2(L, \mathbb{Z}).$$

This is impossible as $H_2|_L$ generates $H^2(L, \mathbb{Z})$. So no such line can exist.

These two results combine to give us geometric information about how smooth subvarieties of $A_g[l]$ that are contained in $J^*_g[l]$ intersect the hyperelliptic locus.

**Proposition 6.3.** Suppose that $g \geq 4$ and $l \geq 3$ and that $Z$ is a connected complex submanifold of $A_g[l]$. If $Z$ is contained in $J^*_g[l]$, then one of the following holds:

(i) $Z$ is contained in the hyperelliptic locus,
(ii) $Z$ does not intersect the hyperelliptic locus,
(iii) the intersection of $Z$ with the hyperelliptic locus is a smooth subvariety of $Z$ of pure codimension 1.

**Proof.** Suppose that $Z$ is neither contained in the locus of hyperelliptic jacobians nor disjoint from it. Suppose that $P$ is in the intersection of $Z$ and the hyperelliptic locus. Denote by $V$ the $g - 2$ dimensional vector space corresponding to the point $P$ as in (6.1). Denote the quotient of the tangent space $T_PZ$ by the Zariski tangent space of the intersection of $Z$ with the hyperelliptic jacobians by $N_P$. Since $Z$ is contained in $J^*_g[l]$, it follows from (6.3) that

$$\mathbb{P}(N_P) \hookrightarrow \mathbb{P}(V) \hookrightarrow \mathbb{P}(S^2V).$$

Since $Z$ is a smooth subvariety of $A_g[l]$, $\mathbb{P}(N_P)$ is a linear subspace of $\mathbb{P}(S^2V)$. It follows from (1.2) that $N_P$ has dimension 0 or 1. If the dimension of $N_P$ is 0 for generic $P$, then $Z$ is contained in the hyperelliptic locus. Since $Z$ is not contained in the hyperelliptic locus, it must be that $N_P$ is one dimensional for generic $P$ in the intersection. But since the dimension of $N_P$ is bounded by 1, it is one for all $P$.
in the intersection of $Z$ with the locus of hyperelliptic jacobians. This implies that the intersection is a smooth divisor. 

Denote the point in $\mathcal{A}_g[l]$ corresponding to the principally polarized abelian variety $A$ with level $l$ structure by $[A]$. If $g' + g'' = g$, there is a generically finite-to-one morphism $\mu : \mathcal{A}_{g'}[l] \times \mathcal{A}_{g''}[l] \to \mathcal{A}_g[l]$. It takes $([A'], [A''])$ to $[A' \times A'']$. Suppose that $[A' \times A'']$ is a simple point of the image of $\mu$. Let 

$$V' = H^{0,1}(A')$$

and 

$$V'' = H^{0,1}(A'')$$

Lemma 6.4. The projective normal cone to the image of $\text{im} \mu$ at $[A \times B]$ is naturally isomorphic to $\mathbb{P}(V' \otimes V'')$. 

Proof. This follows directly by looking at period matrices. 

Now suppose that $C$ is a genus $g$ stable curve with one node and whose normalization is $C' \cup C''$ where $g(C') = g'$ and $g(C'') = g''$. Provided the level $l$ is sufficiently large, $\text{Jac}(C')$ will be a simple point of $\text{im} \mu$. We now continue the discussion above with $A' = \text{Jac}(C')$ and $A'' = \text{Jac}(C'')$. Note that we have canonical imbeddings

$$C' \to \mathbb{P}(V')$$

and 

$$C'' \to \mathbb{P}(V'').$$

Composing the product of these with the Segre imbedding

$$\mathbb{P}(V') \times \mathbb{P}(V'') \to \mathbb{P}(V' \otimes V'')$$

we obtain a morphism

$$(1) \quad \delta : C' \times C'' \to \mathbb{P}(V' \otimes V'')$$

which is a finite onto its image when both $g'$ and $g''$ are 2 or more, and is a projection onto the second factor when $g' = 1$.

We can project the tangent cone of $\mathcal{J}_g[l]$ into the normal bundle of the image of $\mathcal{A}_{g'}[l] \times \mathcal{A}_{g''}[l]$ in $\mathcal{A}_g[l]$. We shall call this image the relative normal cone of $\mathcal{J}_g[l]$ in $\mathcal{A}_g[l]$. We can then projectivize to get a map from the projectivized relative normal cone of $\mathcal{J}_g[l]$ to the projective normal cone of $\mathcal{A}_{g'}[l] \times \mathcal{A}_{g''}[l]$ in $\mathcal{A}_g[l]$. 

Proposition 6.5. The image of the projectivized relative normal cone of $\mathcal{J}_g[l]$ in $\mathcal{A}_g[l]$ at $\text{Jac}(C' \times C'')$ in the projective normal cone of $\mathcal{A}_{g'}[l] \times \mathcal{A}_{g''}[l]$ in $\mathcal{A}_g[l]$ at $\text{Jac}(C' \times C'')$ is the image of the map $\delta$.

Proof. This follows directly from [3, Cor. 3.2]. 

Suppose that $Z$ is a complex submanifold of $\mathcal{A}_g[l]$ which is contained in the jacobian locus. Set $Z_{\text{dec}}^{\text{dec}} = Z \cap J_{\text{dec}}^{\text{dec}}[l]$. Then

$$Z_{\text{dec}}^{\text{dec}} = \bigcup_{j=1}^{[g/2]} Z_j^{\text{dec}}$$

where $Z_j^{\text{dec}}$ is the intersection of $Z$ with the image of $\mathcal{A}_j[l] \times \mathcal{A}_{g-j}[l]$ in $\mathcal{A}_g[l]$. Set

$$Z_j^{\text{dec}} = Z_j^{\text{dec}} - \bigcup_{i \neq j} Z_i^{\text{dec}}.$$ 

This map is generically one-to-one if $g' \neq g''$ and generically two-to-one if $g' = g''$. 

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3This map is generically one-to-one if $g' \neq g''$ and generically two-to-one if $g' = g''$. 

Corollary 6.6. Suppose that \( g \geq 4 \) and \( 1 \leq j \leq g/2 \) and that \( Z \) is a complex submanifold of \( \mathbb{A}_g[l] \) that is contained in \( \mathcal{J}_g[l] \). If \( Z \) is not contained in \( \mathcal{J}_2^{\text{dec}} \) and \( j \neq 2 \), then \( \mathcal{Z}_j^{\text{dec}} \) is empty or is smooth and has pure codimension 1 in \( Z \). If \( Z_2^{\text{dec}} \) is non-empty, it has codimension at most 2 in \( Z \).

Proof. As in the proof above, it follows from the fact that the image of \( \delta \) contains no lines in \( \mathbb{P}(V' \otimes V'') \) except when \( j = 2 \). This follows from the fact that canonical curves contain no lines except in genus 2.

Remark 6.7. One can find further restrictions on the largest strata of \( Z_2^{\text{dec}} \), but so far I have not been able to find a use for them. Partial results in this direction have been obtained in [3] and [4].

7. Locally Symmetric Families of Curves

Definition 7.1. A \textit{locally symmetric family of curves} is a family of stable curves \( p : C \rightarrow X \) where

(i) \( X \) is a locally symmetric variety;
(ii) the Picard group \( \text{Pic}^0 \) of each curve in the family is an abelian variety (i.e., the dual graph of each fiber is a tree);
(iii) the period map \( X \rightarrow \mathbb{A}_g \) is a map of locally symmetric varieties (i.e., it is induced by a \( \mathbb{Q} \)-algebraic group homomorphism).

We lose no generality by assuming that the base \( X \) is a simple locally symmetric variety.

By (4.3), every locally symmetric family of curves with all fibers smooth has a ball quotient as base. It becomes more difficult to understand locally symmetric families when the generic fiber is smooth, but some fibers are singular. Let \( X_j^{\text{dec}} \) be the closure of the locus in \( X \) where the fiber has two irreducible components, one of genus \( j \), the other of genus \( g - j \). We will assume that \( j \leq g/2 \). Set \( \circ X_j^{\text{dec}} = X_j^{\text{dec}} - \bigcup_{i \neq j} X_i^{\text{dec}} \).

Proposition 7.2. Suppose that \( X \) is simple and that \( C \rightarrow X \) is a locally symmetric family of curves of genus \( g \) over whose generic fiber is smooth. If \( g \geq 4 \), then each \( X_j^{\text{dec}} \) is a locally symmetric subvariety of \( X \). Moreover, \( X_j^{\text{dec}} \) is a locally symmetric hypersurface when \( j \neq 2 \), and \( \circ X_j^{\text{dec}} \) has codimension at most 2.

Proof. Since \( X \) is simple, the period map \( X \rightarrow \mathbb{A}_g \) is constant or finite onto its image. If it is constant, the result is immediate, so we assume that it is finite. Note that \( X_j^{\text{dec}} \) is just the preimage of \( \mathbb{A}_g^{\text{dec}} \), the locus of reducible principally polarized abelian varieties. Since the period map \( X \rightarrow \mathbb{A}_g \) is a map of locally symmetric varieties, each \( X_j^{\text{dec}} \) is a locally symmetric subvariety of \( X \). The final statement follows from (6.6).

Proof of Theorem [4]. The theorem is easily seen to be true when \( g \leq 2 \) as all simple, locally symmetric varieties of complex dimension \( \leq 3 \) and real rank \( \leq 2 \) are ball quotients. Thus we can assume that \( g \geq 3 \). Fix a level \( l \geq 3 \) so that \( \mathbb{A}_g[l] \) is smooth. By replacing \( X \) by a finite covering, we may assume that \( X \) is smooth and that the family \( C \rightarrow X \) is a family of curves with a level \( l \) structure, so that we have a period map \( X \rightarrow \mathbb{A}_g[l] \). The fundamental group \( \Gamma \) of \( X \) is a torsion free
arithmetic group in a simple $\mathbb{Q}$-group $G$. Since the period map is not constant, the homomorphism $\Gamma \to Sp_g(\mathbb{Z})$ induced by it has infinite image. (If the image were finite, the family would be isotrivial.)

If all fibers of $p : C \to X$ are smooth, then we have a lift of the homomorphism $\Gamma \to Sp_g(\mathbb{Z})$ to a homomorphism $\Gamma \to \Gamma_g$. If $X$ is good, $X^{\text{dec}}$ has codimension $\geq 2$, so that the inclusion of $X - X^{\text{dec}}$ into $X$ induces an isomorphism $\pi_1(X - X^{\text{dec}}, *) \cong \Gamma$. Since we have a map $X - X^{\text{dec}} \to M_g[l]$, we have a homomorphism $\Gamma \to \Gamma_g$ which lifts the homomorphism induced by the period map. Thus in both cases, we have a lift $\Gamma \to \Gamma_g$ of the homomorphism induced by the period map.

Since the image of $\Gamma$ in $Sp_g(\mathbb{Z})$ is not finite, Theorem 3 implies that the corresponding real group $G(\mathbb{R})$ is isogenous to the product of $SU(n, 1)$ with a compact group. It follows that the symmetric space $G(\mathbb{R})/K$ of $X$ is the complex $n$-ball.

8. Locally Symmetric Families of Jacobians

Definition 8.1. A locally symmetric family of abelian varieties is a family $B \to X$ of principally polarized abelian varieties where the corresponding map $X \to A_g$ is a family of locally symmetric varieties. We will call such a family essential when the period map is generically finite (and therefore finite) onto its image.

The universal family of abelian varieties over $A_g[l]$ is essential. If $X$ is simple, then every locally symmetric family of abelian varieties over $X$ is either essential or constant.

Definition 8.2. A locally symmetric family of jacobians is a locally symmetric family $B \to X$ of abelian varieties where the image of $B \to A_g$ lies in the jacobian locus $J_g$. It is essential when its family of jacobians is essential.

The family of jacobians of a locally symmetric family of curves is also a locally symmetric family of jacobians. It is natural to ask whether every locally symmetric family of jacobians can be lifted to a (necessarily locally symmetric) family of curves. The answer is no. The problem is that the period map $M_g[l] \to A_g[l]$ is 2:1 and ramified along the hyperelliptic locus when $g \geq 3$.

Proposition 8.3. Suppose that $l \geq 3$. There is no family of curves over $A_3[l]$ whose family of jacobians is the universal family of abelian varieties over $A_3[l]$.

Proof. Let $U = A_3[l] - A_3^{\text{dec}}[l]$. If the universal family of abelian varieties over $M_3[l]$ could be lifted to a family of curves, then one would have a section of the period map $M_3[l] \to U$. This is impossible. One can explain this in two different ways:

(i) When $l \geq 3$, the period map $M_3[l] \to U$ is 2:1 and ramified along the hyperelliptic locus. So it can have no section.

(ii) If there were such a section, one would have a splitting of the group homomorphism $\Gamma_3[l] \to \pi_1(U)$. But since each component of $A_3^{\text{dec}}$ has codimension $\geq 2$ in $A_3[l]$, it follows that $\pi_1(U)$ is isomorphic to $\pi_1(A_3[l])$, which is isomorphic to $Sp_g(\mathbb{Z})[l]$. The homomorphism $\Gamma_3[l] \to \pi_1(U)$ does not split by (3.2).

Suppose that $C \to X$ is an essential locally symmetric family of jacobians (or curves). Set $X^* = X \cap J^*_f[l]$. Let $X_H$ be the locus of points in $X$ where the fiber is hyperelliptic, and let $X^*_H = X^* \cap X_H$. 

Proof of Theorem 3. The theorem is easily seen to be true when \( g \leq 2 \). So we suppose that \( g \geq 3 \). By passing to a finite index subgroup if necessary, we may suppose that the arithmetic group \( \Gamma \) associated with \( X \) is torsion free and that the period map induces a homomorphism \( \Gamma \to Sp_g(\mathbb{Z})[l] \) where \( l \geq 3 \). With these assumptions, we have a period map \( X \to \mathcal{A}_g[l] \) and both \( X \) and \( \mathcal{A}_g[l] \) are smooth. Since \( X \) is smooth and good, the inclusion \( X^* \to X \) induces an isomorphism on fundamental groups. Since \( \Gamma \) is torsion free, \( \pi_1(X,*) \cong \Gamma \).

Now, if \( X^* \) is contained in the locus of hyperelliptic jacobians, or disjoint from it, then the period map \( X \to \mathcal{A}_g[l] \) has a lift to a map \( X \to \mathcal{M}_g[l] \). That is, we can lift the family of jacobians over \( X^* \) to a family of curves. This implies that the homomorphism \( \Gamma \to Sp_g(\mathbb{Z}) \) induced by the period map, lifts to a homomorphism \( \Gamma \to \Gamma_g \). Since the period map is not constant, the image of \( \Gamma \) in \( Sp_g(\mathbb{Z}) \) is not finite (otherwise the family of abelian varieties would be isotrivial). Applying Theorem 3, we see that \( X \) is a quotient of the complex \( n \)-ball.

The remaining case is where \( X^* \) is neither disjoint from the locus of hyperelliptic jacobians nor contained in it. When \( g = 3 \), the locus of hyperelliptic curves is a divisor in \( \mathcal{M}_g[l] \) from which it follows that \( X^*_\mathcal{H} \) is of pure codimension 1 in \( X^* \). When \( g > 3 \), it follows from \( \Box \) that \( X^*_\mathcal{H} \) is smooth of pure codimension 1 in \( X^* \).

Let \( Y \) be the fibered product

\[
Y = X \times_{\mathcal{A}_g[l]} \mathcal{M}_g[l]
\]

The map \( Y \to X \) is 2:1 and ramified above the divisor \( X^*_\mathcal{H} \) in \( X^* \). This implies that \( Y \to X^* \) cannot be the restriction of a covering of locally symmetric varieties as such coverings of \( X \) are unramified as \( \Gamma \) is torsion free.

\[\square\]

Remark 8.4. To try to eliminate the case where \( X^*_\mathcal{H} \) is non-empty, one should study the second fundamental form (with respect to the canonical metric of Siegel space) of the hyperelliptic locus in \( \mathcal{A}_g[l] \) and also that of its normal cone. If one is lucky, this will give an upper bound on \( \dim X^*_\mathcal{H} \), and therefore of \( \dim X \) as well.

Appendix A. An example

Suppose that

\[
1 \to \mathbb{Z} \to G \to \Gamma \to 1
\]

is a non-trivial central extension of a finite index subgroup \( \Gamma \) of \( Sp_g(\mathbb{Z}) \) by \( \mathbb{Z} \). Suppose that the class of this extension is non-trivial in \( H^2(\Gamma, \mathbb{Q}) \). One can ask whether the natural homomorphism \( G \to Sp_g(\mathbb{Z}) \) lifts to a homomorphism \( G \to \Gamma_g \).

We first show that the obstructions of Theorem 3.1 vanish when \( g \geq 3 \).

Since \( g \geq 3 \), \( H^2(\Gamma, \mathbb{Q}) = \mathbb{Q} \). Since the class of the extension is non-trivial, an easy spectral sequence argument shows that \( H^2(G, \mathbb{Q}) \) vanishes. Raghunathan’s work \( \Box \) implies that that \( H^1(\Gamma, V_3) \) vanishes where \( V_3 \) is the third fundamental representation of \( Sp_g(\mathbb{R}) \). Another easy spectral sequence argument implies that \( H^1(G, V_3) \) vanishes. The obstruction to a lift \( G \to \Gamma_g \) given in Theorem 3.1 therefore vanishes.

Theorem A.1. There is an integer \( g_0 \leq 7 \) such that if \( g \geq g_0 \), there is no lift of the natural homomorphism \( p : G \to Sp_g(\mathbb{Z}) \) to a homomorphism \( G \to \Gamma_g \).

Proof. Suppose that the homomorphism \( G \to \Gamma \) lifts to a homomorphism \( \phi : G \to \Gamma_g \). The latter homomorphism induces maps on cohomology

\[
H^\bullet(\Gamma_g, V) \to H^\bullet(G, V)
\]
for all irreducible representations $V$ of $Sp_g(\mathbb{R})$. Recall that the cohomology group $H^1(\Gamma_g, V_3)$ is isomorphic to $\mathbb{R}$. For each $Sp_g(\mathbb{R})$ equivariant map $p : V_3^{\otimes 6} \to \mathbb{R}$, there is a natural map

$$p_* : H^1(\Gamma_g, V_3)^{\otimes 6} \to H^6(\Gamma_g, \mathbb{R}).$$

As is traditional, we denote the $i$th Chern class of the Hodge bundle over $A_g$ by $\lambda_i$. It follows from the work of Kawazumi and Morita [14], that for a suitable choice of $p$, the image of $p_*$ is spanned by $\lambda_3$.

Since the diagram

$$
\begin{array}{ccc}
H^1(\Gamma_g, V_3)^{\otimes 6} & \longrightarrow & H^6(\Gamma_g, \mathbb{R}) \\
\downarrow & & \downarrow \phi^* \\
H^1(G, V_3)^{\otimes 6} & \longrightarrow & H^6(G, \mathbb{R})
\end{array}
$$

commutes, and since $H^1(G, V_3)$ vanishes when $g \geq 3$, it follows that $\phi^* \lambda_3$ is trivial in $H^6(G, \mathbb{R})$. We show that this leads to a contradiction.

It follows from the work of Borel [2] that the ring homomorphism

$$\mathbb{R}[\lambda_1, \lambda_3, \lambda_5, \ldots] \to H^\bullet(\Gamma, \mathbb{R})$$

is an isomorphism in degrees $< g$. Since the characteristic class of the extension $G$ of $\Gamma$ by $\mathbb{Z}$ is a non-zero multiple of $\lambda_1$, it follows that the ring homomorphism

$$\mathbb{R}[\lambda_3, \lambda_5, \ldots] \to H^\bullet(G, \mathbb{R})$$

is an isomorphism in degrees $< g$. In particular, $\lambda_3$ is not zero in $H^6(G, \mathbb{R})$ when $g \geq 7$. It follows that there is no lifting $\phi : G \to \Gamma_g$ when $g \geq 7$.

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