Collapsing Inhomogeneous Dust Fluid in the Background of Dark Energy

Tanwi Bandyopadhyay* and Subenoy Chakraborty†
Department of Mathematics, Jadavpur University, Calcutta-32, India.

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In the present work, gravitational collapse of an inhomogeneous spherical star model, consisting of inhomogeneous dust fluid (dark matter) in the background of dark energy is considered. The collapsing process is examined first separately for both dark matter and dark energy and then under the combined effect of dark matter and dark energy with or without interaction. The dark energy is considered in the form of perfect fluid and both marginally and non-marginally bound cases are considered for the collapsing model. Finally dark energy in the form of anisotropic fluid is investigated and it is found to be similar to ref. [12].

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I. INTRODUCTION

In recent past, there are remarkable observational evidences which contradict the present day prediction of Standard Cosmology. The observed data from the high red shift of type I-a supernova [1] suggests that the universe at present is accelerating instead of deceleration (a prediction of Standard Cosmology). This was confirmed by the observed data from measurements of the fluctuations in the power spectrum of the cosmic microwave background radiation [2] and large scale structure [3].

To incorporate such accelerating phase within the frame-work of Einstein’s general relativity one requires source of repulsive gravity, termed as dark energy. This component of the matter distribution of the universe should have a large negative pressure and hence violates the strong energy condition and dominates over matter at present. Also astronomical observations predict that at present the universe contains approximately $\frac{2}{3}$ dark energy and $\frac{1}{3}$ dark matter. Although, the nature of both dark matter and dark energy are still unknown, yet based on their characteristics, different models have been suggested namely tiny positive cosmological constant, quintessence, phantoms, chaplygin gas, dark energy in brane-worlds and many others (see the references [4]).

The dark energy, by virtue of its repulsive gravitational nature, is interesting to study gravitational collapse and formation of black hole. As all massive stars do not form black holes (may be neutron stars or white dwarfs), so it is generally speculated [5] that the dark energy may play an important role in the collapsing stars.

In the present study, a collapsing spherical star is considered having finite thickness. The star is made of dust cloud in the background of dark energy. Let $\Sigma$ be the boundary of the star and $V^+$ and $V^-$ indicate the exterior and interior region of the star. The spherically symmetric inhomogeneous space-time (in $V^-$) can be described by the Lemaitre-Tolman-Bondi (LTB) metric element as

$$V^- : \quad ds^2 = -dt^2 + \frac{R^2}{1 - f(r)} dr^2 + R^2 d\Omega^2$$

with $R = R(r, t)$.

Due to the choice of the co-moving coordinates, the surface $\Sigma$ can be identified as

$$\Sigma : \quad r = r_\Sigma, \quad \text{a constant.}$$

* b._tanwi@yahoo.com
† subenoyc@yahoo.co.in
and the metric on it is in the form

\[ ds^2_{\Sigma} = -d\tau^2 + R^2_{\Sigma}(\tau) \, d\Omega^2 \]  \hspace{1cm} (3)

Thus we have

\[ t = \tau \quad \text{and} \quad R_{\Sigma}(\tau) = R(r_\Sigma, \tau) \]  \hspace{1cm} (4)

In \( V^+ \), the metric of the space-time in general can be taken as

\[ V^+ : \quad ds^2_+ = -A^2(T, z)dt^2 + B^2(T, z)(dz^2 + z^2d\Omega^2) \]  \hspace{1cm} (5)

So the bounding surface \( \Sigma \) for the coordinates of the exterior space can be expressed as

\[ z = z_0(T) \]  \hspace{1cm} (6)

Then the junction conditions namely \( ds^2_{\Sigma} = ds^2_+ \) give

\[ i) \quad \frac{dT}{dt} = \frac{1}{\sqrt{A^2 - (\frac{dz_0}{dT})^2 B^2}} \]  \hspace{1cm} (7)

\[ ii) \quad R(r_\Sigma, t) = Z_0(T) \ B(T, z_0(T)) \]  \hspace{1cm} (8)

The other junction conditions (relating to extrinsic curvature) due to Israel [6] depend on the choice of the space-time outside the star. If the bounding surface \( \Sigma \) is an energy layer i.e., an infinitely thin matter shell appears on \( \Sigma \), then the extrinsic curvature is not continuous across \( \Sigma \) but the jump discontinuity depends on the matter of the thin shell over \( \Sigma \). On the other hand, the continuity of the extrinsic curvature components demand that no such shell exists on \( \Sigma \). In fact, whenever the space-time \( V_- \) is fixed, then the space-time \( V_+ \) will determine whether this shell appears or not. In the present paper, it is assumed that no such thin shell exists on \( \Sigma \) and extrinsic curvature is continuous across \( \Sigma \). The paper is organized as follows: In Section II, basic equations for collapsing inhomogeneous spherical star model are presented. Sections III and IV deal with marginally bound and non-marginally bound cases with dark energy in the form of perfect fluid while anisotropic fluid form is used for dark energy in Section V. Finally, the paper ends with conclusion in Section VI.

\section*{II. INHOMOGENEOUS SPHERICALLY SYMMETRIC STAR MODEL WITH DARK ENERGY IN THE FORM OF PERFECT FLUID}

The energy-momentum tensor for the matter field consisting of an inhomogeneous dust (dark matter) \( \rho_{DM}(r, t) \) and homogeneous dark energy in the form of perfect fluid is given by

\[ T_{\mu\nu} = (\rho_{DM} + \rho + p)u_\mu u_\nu + pg_{\mu\nu} \]  \hspace{1cm} (9)

where in the comoving representation, four-velocity \( u_\mu = \delta^t_\mu \).

The explicit form of the energy conservation equations \( T^\mu_{\nu;\mu} = 0 \) for the metric (1) are

\[ \rho_{DM} + \left( 2 \frac{\dot{R}}{R} + \frac{\ddot{R}}{R} \right) \rho_{DM} = Q(r, t) \]  \hspace{1cm} (10)

and
\[ \dot{\rho} + \left( 2 \frac{\dot{R}}{R} + \frac{\dot{R}'}{R'} \right) (\rho + p) = -Q(r, t) \]  

(11)

where (') represents differentiation with respect to time while ('') represents the same with respect to \( r \) and \( Q(r, t) \) stands for the interaction between dark matter and dark energy.

The non-vanishing components of the Einstein field equations for the metric (1) having matter field in the form (9) are given by (with \( \kappa = \frac{8\pi G}{c^4} = 1 \))

\[ \frac{f}{R^2} + \frac{f'}{R'R} + \frac{\dot{R}^2}{R^2} + 2 \frac{\dot{R} \dot{R}'}{R R'} = \rho_{DM} + \rho \]  

(12)

\[ \frac{f}{R^2} + \frac{\dot{R}^2}{R^2} + 2 \frac{\ddot{R}}{R} = -p \]  

(13)

\[ \frac{f'}{2R'R^2} + \frac{\dot{R} \dot{R}'}{R R'} + \frac{\dot{R}}{R} + \frac{\dot{R}'}{R'} = -p \]  

(14)

In general the interaction term \( Q(r, t) \) is non-zero. But if \( Q(r, t) \) is zero then from the conservation equation (11), \( R \) should be in the product form.

Since in the present work, one mainly considers gravitational collapse, so one assumes \( \dot{R} < 0 \).

The apparent horizon is characterized by

\[ R_{\alpha} R^\alpha = 0 \]  

(15)

If \( t = t_{ah}(r) \) be the time of formation of apparent horizon, then from the above equation (15)

\[ \dot{R}^2(r, t_{ah}(r)) = 1 - f(r) \]  

(16)

The mass function at comoving coordinate 'r' is given by [7]

\[ m(r, t) = \frac{1}{2} R \left( 1 - R_{\alpha} R^{\alpha} \right) = \frac{1}{2} R \left( \dot{R}^2 + f(r) \right) \]  

(17)

So the total mass of the collapsing star at any time \( \tau \) is [8],

\[ M(\tau) = \frac{1}{2} R_{\Sigma}(\tau) \left( \dot{R}_{\Sigma}^2 + f(r) \right) \]  

(18)

Since the collapsing star is assumed to be not trapped initially (at \( \tau = \tau_i \)), so on the initial hypersurface \( \tau = \tau_i \) one should have

\[ \dot{R}^2(r, \tau_i) + f(r) - 1 < 0 \]  

(19)
III. MARGINALLY BOUND CASE: \( f = 0 \)

In this case the hypersurfaces \( t = \text{constant} \) have zero curvature. Now equating the field equations (13) and (14), we have a differential equation in \( R \) which has a first integral

\[
2 \frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} = \mu(t) \tag{20}
\]

Integrating once more, the evolution equation for \( R \) is

\[
\dot{R}^2 = \frac{\mu(t)}{3} R^2 + \frac{g(r)}{R} \tag{21}
\]

where \( \mu(t) \) and \( g(r) \) are arbitrary functions of \( t \) and \( r \) respectively.

Now from the field equation (13) using (20), one finds

\[
p(t) = -\mu(t) \tag{22}
\]

As for dark energy \( p(t) \) is negative, so the arbitrary function \( \mu(t) \) is positive. Then from the field equation (12) one gets

\[
\rho_{DM} + \rho = \mu(t) + \frac{g'(r)}{R^2 R'} \tag{23}
\]

It is to be noted that if a barotropic equation of state of the form \( p = \epsilon \rho \) is taken for dark energy, then one arrives at a contradiction to satisfy the conservation equations (10) and (11).

In particular, if \( \mu \) is taken to be constant and the dark energy is in the form of cosmological constant i.e, \( \rho = -p = \mu \), then from (23)

\[
\rho_{DM} = \frac{g'(r)}{R^2 R'}
\]

and from conservation equations \( Q = 0 \).

Thus when the dark energy is in the form of cosmological constant then the model corresponds to inhomogeneous dust collapse with a cosmological constant.

Now, apparent horizon will form if the cubic equation

\[
\mu R^3 - 3R + 3g = 0
\]

has at least one positive root. Various cases are as follows:

i) For \( \mu > 0 \) and \( 3g = \frac{2}{\sqrt{\mu}} \), a unique apparent horizon will form, given by

\[
R_{ah}(r) = \frac{1}{\sqrt{\mu}}.
\]

ii) If \( \mu > 0 \) and \( 3g < \frac{2}{\sqrt{\mu}} \), then the cubic equation has two positive real roots, which correspond to cosmological and black hole horizons given by

\[
R_c(r) = \frac{2}{\sqrt{\mu}} \cos \left[ \frac{1}{3} \cos^{-1} \left( -\frac{3}{2} g \sqrt{\mu} \right) \right]
\]

and

\[
R_b(r) = \frac{2}{\sqrt{\mu}} \cos \left[ \frac{4\Pi}{3} + \frac{1}{3} \cos^{-1} \left( -\frac{3}{2} g \sqrt{\mu} \right) \right].
\]
iii) For $\mu > 0$ and $3g > \frac{2}{\sqrt{\mu}}$ there are no positive roots and hence there are no apparent horizons.

The details of this collapsing process has been studied in Ref. [13].

The effect on the collapsing star will be studied separately for a dust cloud and dark energy to examine the different roles that may play during the evolution. Also their joint effects (with or without interaction) will be investigated subsequently.

**Case I:** $(\rho_{DM} \neq 0, \rho = 0 = p)$

This case corresponds to inhomogeneous dust collapse in LTB model. This has been studied exhaustively by Joshi and collaborators [9] (for higher dimensional study, see Ref. [10]).

**Case II:** $(\rho_{DM} = 0, \ p = \epsilon \rho \neq 0)$

As $Q(r, t) = 0$ [from equation (10)], so area radius $R$ must be in the separable form

$$R(r, t) = \sqrt[3]{g(r)} R_t(t)$$

for consistency of the conservation equation (11) and field equations (12)-(14).

Then from the conservation equation $\rho(t) = \rho_0/R_t^{3(1+\epsilon)}$

and

$$\mu(t) = -\frac{\epsilon \rho_0}{R_t^{3(1+\epsilon)}}$$

Thus the differential equation in $R_t$ is

$$R_t^2 = -\frac{\epsilon \rho_0}{3} R_t^{-(3\epsilon + 1)} + \frac{1}{R_t}$$

In the following, this differential equation in $R_t$ is solved for $\epsilon = -1, 0, 1$ and $-\frac{1}{3}$.

**a) Dark energy in the form of cosmological constant:** ($\epsilon = -1$)

Equation (26) can be solved easily as

$$R_t^2 = \sqrt{\frac{3}{\rho_0}} Sinh \left[ \frac{\sqrt{3\rho_0}}{2} (t - t_0) \right] \quad \text{or} \quad \sqrt{\frac{3}{\rho_0}} Sinh \left[ \frac{\sqrt{3\rho_0}}{2} (t_0 - t) \right]$$

according as $\hat{R} > 0$ or $< 0$.

The mass function has the expression

$$m(r, t) = \frac{1}{2} g(r) \left( 1 + \frac{\rho_0 R_t^2}{3} \right).$$

Hence it represents either an ever expanding model of the universe starting from the big-bang at $t = t_0$ or a collapsing model of the universe having infinite volume at $t = -\infty$ and collapse to big crunch at $t = t_0$ with a finite mass.

**b) $\epsilon = 0$:**

One can see easily that it corresponds to a pure homogeneous dust (not dark energy) with $R_t \sim t^\frac{2}{3}$ as expected in standard FRW model.
c) **Dark energy in the form of perfect fluid: \( \epsilon = 1 \)**

Here the scale factor grows with time as

\[
R_t^3 = \frac{\rho_0}{3} + \frac{9}{4} (t - t_0)^2
\]

with \( \rho(t) = \rho_0/R_t^6 \).

d) \( \epsilon = -\frac{1}{3} \): In this case we have

\[
\sqrt{R_t^2 \rho_0^2 + R_t \rho_0 - \sinh^{-1} \left( \sqrt{R_t \rho_0} \right)} = \frac{\rho_0^2}{3} (t_0 - t)
\]

The mass function has the expression

\[
m(r, t) = \frac{1}{2} g(r) \left( 1 + \frac{\rho_0 R_t}{3} \right).
\]

**Case III**: \( (\rho \neq 0, \rho_{DM} \neq 0, Q = 0) \)

As \( Q = 0 \), so similar to Case II, \( R \) must be in the product separable form (i.e, eq. (24)) for consistency of the conservation equation (10). The expressions for the matter densities become

\[
\rho_{DM} = h(r) R_t^{-3} \quad \text{and} \quad \rho = \rho_0 R_t^{-3(1+\epsilon)}
\]

with \( \rho_0 \), an arbitrary constant and \( h(r) \), an arbitrary function of \( r \) alone. Also the evolution equation for \( R_t \) takes the form of equation (26).

The consistency of the field equation (12) demands \( \rho_{DM} \) to be also homogeneous with \( h(r) = 3 \). Thus it is very similar to case II and physical consequences are identical. Note that here the role of dark matter is insignificant.

**Case IV**: \( (\rho \neq 0, \rho_{DM} \neq 0, Q \neq 0) \)

For the interacting dark matter and dark energy, the evolution equation for \( R \), the expressions for \( p(t) \) and \( (\rho(t) + \rho_{DM}) \) are given by equations (21)-(23) respectively.

Now for explicit solution of \( R \) one assumes (see Ref. [11])

\[
\mu(t) = \mu_0 t^{-s}, \quad (\mu_0 \text{ and } s \text{ are positive constants})
\]

and the general solution for \( R \) has the form [12]

\[
R_t^2 = \begin{cases}
\sqrt{T} \left\{ C_1 J_\xi \left[ \frac{2\sqrt{\lambda}}{|s-2|} t^{-\frac{\lambda}{s-2}} \right] + C_2 Y_\xi \left[ \frac{2\sqrt{\lambda}}{|s-2|} t^{-\frac{\lambda}{s-2}} \right] \right\} \\
\sqrt{T} \left\{ C_1 J_{-\xi} \left[ \frac{2\sqrt{\lambda}}{|s-2|} t^{-\frac{\lambda}{s-2}} \right] + C_2 J_{-\xi} \left[ \frac{2\sqrt{\lambda}}{|s-2|} t^{-\frac{\lambda}{s-2}} \right] \right\} \\
C_1 t^{q_1} + C_2 t^{1-q_1}
\end{cases}
\]

according as \( \xi \) is an integer, non-integer and \( s = 2 \). Hence \( C_1 \) and \( C_2 \) are arbitrary functions of \( r \) and

\[
\xi = \frac{1}{s-2}, \quad \lambda = \frac{3\mu_0}{4}, \quad q_1 = \frac{1}{2} \left( 1 + \sqrt{1 - 3\mu_0} \right).
\]

It can be shown (for details see Ref. [12]) that the model approaches isotropy along the fluid world line as \( t \to \infty \).
IV. NON-MARGINALLY BOUND CASE: \( f \neq 0 \)

In this case, the first integral of equation (13) gives the dynamical equation for the area radius as

\[
\dot{R}^2 = -\frac{p(t)R^2}{3} + \frac{g(r)}{R} - f(r)
\]  

(33)

with \( g(r) \) as arbitrary function of \( r \). The explicit form of total energy density is (obtained from equation (12))

\[
\rho + \rho_{DM} = \frac{g'(r)}{R^2 R'} - p(t)
\]  

(34)

In particular, if one chooses \( \rho(t) = -p(t) \), then

\[
\rho_{DM} = \frac{g'(r)}{R^2 R'}
\]  

(35)

and consequently from the conservation equations (10) and (11) one obtains respectively

\[
Q = 0 \quad \text{and} \quad \rho = \text{constant } \lambda \text{ (say).}
\]

Hence as in marginally bound case, this can also corresponds to a collapsing dust (inhomogeneous) with a cosmological constant. Note that, due to the difference in the evolution equations [see eqns. (33) and (21)], the collapsing processes will not be identical.

One may note that if the separable (product) form of \( R \) is assumed, i.e, \( R = \sqrt{f(r)} \xi(t) \), then \( \xi(t) \) satisfies

\[
\dot{\xi}^2 + 1 = -\frac{p(t)\xi^2}{3} + \frac{\eta(t)}{\xi}
\]  

(36)

where \( \eta(t) \) is an arbitrary function of \( t \) alone. The form of energy density becomes

\[
\rho + \rho_{DM} = \frac{3\eta(t)}{\xi^3} - p(t)
\]  

(37)

In particular if \( \rho(t) = -p(t) = \lambda(t) \), then

\[
\rho_{DM} = \frac{3\eta(t)}{\xi^3}
\]  

(38)

and the arbitrary function \( \eta(t) \) is related to \( \lambda(t) \) by the relation

\[
\eta(t) = -\frac{\lambda(t)\xi^3}{3}
\]  

(39)

Thus dark energy behaves as time dependent cosmological constant and the model describes homogeneous dust collapse with time dependent cosmological term.

V. DARK ENERGY IN THE FORM OF ANISOTROPIC FLUID

Here if the dark energy is in the form of anisotropic fluid with energy momentum tensor
\[ T^\nu_{(DE)}_{\mu} = \text{diag}(-\rho, p_r, p_T, p_T) \]  
\[ (40) \]

then the total energy momentum tensor is

\[ T^\nu_{(T)}_{\mu} = \rho_{DM} u^\mu u^\nu + T^\nu_{(DE)}_{\mu} \]  
\[ (41) \]

So the conservation equation gives

\[
\begin{align*}
\dot{\rho} + \left( \frac{2R'}{R} + \frac{R''}{R} \right) \rho_R &= Q(r,t) \\
\dot{\rho} + \left( \frac{R'}{R} \rho_r + \frac{2R}{R} \rho_T \right) &= -Q(r,t)
\end{align*}
\]  
\[ (42) \]

and

\[ p'_r + 2(p_r - p_T) \frac{R'}{R} = 0 \]  
\[ (43) \]

One may note that the dark energy in the form of anisotropic fluid must be inhomogeneous, otherwise from the conservation equation (43), \( p'_r = 0 \) implies \( p_r = p_T \) i.e., fluid is isotropic (discussed in the previous sections). Conversely, if the dark energy fluid is inhomogeneous (i.e., \( p'_r \neq 0 \)), then again from equation (43), \( p_r \neq p_T \) i.e., the fluid must be anisotropic.

Now from the field equation (13) (replacing \( p \) by \( p_r \)), the evolution equation for \( R \) becomes (integrating once),

\[
\dot{R}^2 = g(r) \frac{R}{R} - \frac{1}{R} \int p_r(r,t) R^2 dR - f(r)
\]  
\[ (44) \]

Now assuming the regularity of the initial radial pressure at the centre and blowing up at the singularity, the form of \( p_r \) can be taken as (for details see Ref. [12])

\[ p_r = \frac{\alpha(r)}{R^n} \]  
\[ (45) \]

where \( \alpha(r) \) is an arbitrary function of \( r \) such that \( \alpha(r) \sim r^n \) near \( r = 0 \). and \( n(> 0) \) is any constant.

Thus the radial velocity of the collapsing shells at a distance \( r \) from the centre is given by

\[
\dot{R}^2 = g(r) \frac{R}{R} - \frac{\alpha(r)}{3-n} R^{2-n} - f(r)
\]  
\[ (46) \]

Also from the conservation equations, the expressions for the energy densities become

\[ \rho_{DM} = \frac{\rho_0(r)}{R^2 R'} + \frac{1}{R^2 R'} \int Q R^2 R' dt \]  
\[ (47) \]

and

\[ \rho = \frac{H'(R, t)}{R^2 R'} - \frac{1}{R^2 R'} \int Q R^2 R' dt \]  
\[ (48) \]

where \( H(R, t) = \rho_1(r) - \frac{\alpha(r)}{3-n} R^{3-n} \) and the arbitrary functions \( \rho_0 \) and \( \rho_1 \) are restricted by the relation [consistency of equation (12)]:

[...]
\[ \rho_1 + \rho_0 = g(r) \]  

The expression for tangential stress becomes

\[ p_T = \frac{\alpha(r)}{R^n} \left[ 1 - \frac{nR'}{2(R' + R)} \right] + \frac{\alpha'(r)}{2R^{n-1}(R' + R'} \right], \quad (n \neq 3) \]

A similar collapsing process has been studied extensively in Ref. [12] for marginally bound case only \((f = 0)\) and it is calculated that in general, pressure tries to resist the formation of naked singularity. Thus formation of a black hole from the collapsing star due to the presence of dark energy in the form of anisotropic fluid is more favorable than formation of a naked singularity.

VI. CONCLUSION

The paper deals with a detail study of an inhomogeneous spherically symmetric star model having dark matter in the background of dark energy as the matter content. The dark matter is taken in the form of inhomogeneous dust while for dark energy, both homogeneous perfect fluid and anisotropic fluid models are considered separately. One may note that the present study can easily be extended to quasi-spherical collapsing star models and the results will be identical.

For dark energy in the form of perfect fluid, both marginally bound \((f = 0)\) and non-marginally bound \((f \neq 0)\) cases are considered in two different sections. When \(f = 0\), dark matter and dark energy are considered both separately and in combined form. It is observed that due to the presence of dark energy, trapped surfaces do not form at all and collapsing models lead to big crunch singularity. In a particular situation (when \(\epsilon = -1\)), the non-marginally bound case represents inhomogeneous dust collapse with a cosmological constant while assuming separable form of area radius, the model corresponds to homogeneous dust collapse with time dependent cosmological term. As for inhomogeneous dust with cosmological constant, trapped surfaces may be possible while for homogeneous dust with time dependent cosmological term, formation of trapped surfaces is not possible. Hence naked singularity is more favorable in second case than in first case.

Lastly, when anisotropic fluid represents dark energy, it is found that black hole formation is more favorable than naked singularity. Therefore, one may conclude that inhomogeneity, both in dark matter and dark energy favors formation of trapped surfaces, while homogeneous cases support the singularity to be naked.

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