Variable Modified Newtonian Mechanics from an interpolating metric II Galactic rotational curve

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We use an interpolating metric from a previous work [1] with a variable modified Newtonian potential (VMOND) to explore the issue of flatten rotational curve of a galaxy. We examine a model of a large protogalactic cloud with growing density perturbation, in which a central sphere collapses while the outer protogalaxy continues to grow until density perturbation reaches $O(1)$. This cloud then follows a Secondary Infall to collapse into a disk which will move towards the stellar disk. The resulting mass profile matches the mass to light ratio of the baryonic scaling model observations and the predicted rotational curve matches the Milky Way (MW) data.

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INTRODUCTION

A common response to non-Newtonian rotational curve of a galaxy is that its dynamics requires dark matter (DM) particles. Challenges remain for the ΛCDM model in galaxy formation in which a DM halo potential has developed before baryons arrive. During infalling the baryons receive heat shock to virial temperature, with their angular momentum mixing with DM halo and accrete into the galactic centre after cooling, the resulting disk size does not match observations. This is known as the "Angular Momentum Catastrophe", see [2]. The DM mass function provides an additional tunable potential for the flatten rotational curve but also acts as a source of unwanted shock heating for baryon’s early gravitational collapse. Alternatively one expects that cold streaming gas can escape heat shock from DM halo and directly accrete onto the galactic centre and produce the correct disk size [4]. (Recent observation of coplanar dwarf galaxies around Centaurus A [3] adds a new challenge to the ΛCDM model.) MOND [5]-[8] with its constant acceleration where Newtonian acceleration is below a certain scale and Conformal Gravity [9]-[10] with a tunable parameter can provide the required non-Newtonian gravitational potential without heat shock. Recent observations suggest something more complicated is at work. Genzel et.al. [11] and Lang [12] show that early type galaxies have non-flatten rotational curves outside its scale length $r_0$, which in fact fall off faster than the (Newtonian) exponential disk from $2r_0 \sim 5r_0$, where $5r_0$ is the disk size. By modelling DM density profile based on the NSW model, the authors of [12] obtain the following equation for the rotational speed $v_{rot}$, speed due to disk mass $v_{disk}$ and speed due to dark matter $v_{DM}$

$$v^2_{rot}(r) = v^2_{disk}(r) + v^2_{DM}(r) - 2\sigma^2\left(\frac{r}{r_0}\right),$$

where $\sigma$ is the velocity dispersion usually taken to be $v^2_{rot} = 3\sigma^2$ and the last term in the Eq. (1) represents pressure support inside the disk. Using simulations, the authors in [12] reproduce the observations by using maximal baryonic density (no DM allowed) inside the disk and including pressure support term. One conclusion is that "at $z \sim 2$ (around 10 Gyr look back time) there is no need for dark matter inside the galactic disk upto 5$r_0". Work by Simons et.al. [13] demonstrates that from $z \sim 2$ to $z \sim 0.2$ we have
i) galactic rotational velocity growth correlates with galactic mass growth over the said period,
ii) velocity dispersion $\sigma$ decreases sufficiently fast over cosmological time with an universal profile for all mass range. This is also confirmed by Kassin et. al. [14].

Still more interestingly, for a class of galaxies called pure disk galaxies (no bulge) [15] where observations show that central surface brightness (proportional to the central mass density) remains unchanged over $1 > z > 0.4$ and fits well with the Freeman exponential function, but over $0.4 > z > 0.02$ the stellar disk mass increases upto 40% while the central brightness and scale length remain unchanged. This interesting observation suggests that early arriving accretion of gas materials only impacts the peripheral (outer distance $r > 2r_0$) of the stellar disk. It is the late arriving accretions that eventually managed to filter through the scale length boundary. Comparing the rotational velocity and dispensor of late type with early type spirals, one notices that the rotational curves are evolving from an early Newtonian profile to the late profile of a flat-curve over a period of around 10 Gyr (from $z \sim 2$.
These observations provide a serious challenge to the constant acceleration proposals from MOND and Conformal gravity, as well as to the CDM paradigm in which a typical CDM halo is already settled in the central region before the baryons fall towards the centre. Continuous cold gas accretion is known to be a crucial ingredient in the evolution of spiral galaxies, see for example [16], although accretion modelling is difficult without knowing where the cold gas comes from. Observational support for continuous accretion is also found for MW [17] in which $1.1 \sim 2.2 \times 10^{11} M_\odot$ accreted material is estimated over a period of 8 Gyrs.

In a previous work [1] we find an interpolating metric between Schwarzschild metric and Friedmann-Robertson-Walker (FRW) metric in which a variable MOND (called VMOND) term arises in its potential similar to Conformal Gravity, except that this VMOND has no tunable parameter and depends strictly on the central baryonic density and the cosmological background density. This metric only modifies the Newtonian potential and does not introduce new scalar particles such as from a Scalar-Tensor theory approach. We show in [1] that VMOND can play a similar role of DM in density perturbation growth and lift the higher CMB acoustic peaks. We show also [18] that VMOND produces a consistent strong lensing effect for galaxy and clusters. At late time, we find that the VMOND is negligible comparing to a large central Newtonian potential and does not automatically lead to flatten rotational curve. However, it is not necessarily problematic since the early time rotational curve is consistent with a pure central Newtonian potential. The challenge is whether the subsequent central galactic mass growth will lead to the right rotational curve. The origin of a constant specific angular momentum in galaxies are thought to be due to tidal torque of galaxies from their neighbouring protogalaxies [19]-[22]. There are suggestions [23] that the origin comes from galaxy attached to the surface of large bubbles. The general view is that once the galaxy comes free from its source, the overall angular momentum becomes conserved. There are also observed accretion that cold streaming gas from filaments and sheets having high specific angular momentum [24], but it is not clear whether these effects are critical in galactic mass growth. Before the advent of DM, one of the most favoured theories for galaxy formation is the Mestel theory for an initial spherical uniform density (or constant mass) gas with constant angular momentum to collapse into a disk. Typically, there is a central core which collapses earlier to form an exponential disk with a constant angular speed while a relatively small amount of late arrival gas which due to a constant mass cloud collapses to a disk with constant rotational speed. The model is found to work well by Crampin and Hoyle [25] and Freeman [26], but the challenge remains in the high M/L (missing mass) required in the flat disk. In this work, we propose that the m'estel theory works better in conjunction with a Secondary Infall scenario, first proposed by Gunn and Gott [27], with a VMOND potential. We find satisfactory match with both observed M/L and MW rotational curve data.

In section 2, we discuss the main features of the interpolating metric and its equation of motion in the slow speed regime. In section 3, we discuss the mass modelling of spiral galaxies. We examine a model with a large protogalaxy with growing density perturbation towards $O(1)$ while a central region collapses following the Mestel theory. In section 4, we discuss the Secondary Infall of the un-collapsed shells and the VMOND effect on such model. We compare the resulting rotational curve prediction with MW data. The last section is a short summary and conclusion.

THE MODEL

In a previous work, we find a metric that interpolates between the Schwarzschild metric and the Friedmann-Robertson-Walker (FRW) metric. In a cosmological constant background this metric takes a familiar form

$$ds^2 = Zc^2dt^2 - \frac{dr^2}{Z} - r^2d\phi^2,$$

where

$$Z = 1 - \left(\frac{2GM}{rc^2} - \frac{H^2r}{c^2}\right)^2, H^2 = \frac{\Lambda}{3},$$

and $\Lambda$ is the cosmological constant.

For matter and radiation dominant epochs, the radial acceleration can be obtained in Tolman-Lemaître metric and can be written in terms of densities as follows

$$\ddot{r} = \frac{h^2}{r^3} - \left(\frac{GM}{r^3} + H\sqrt{\frac{GM}{2r}} - \frac{\dot{a}}{a}\right)r = \frac{h^2}{r^3} - \left(\rho_b + \rho_b\sqrt{\frac{\rho_H}{\rho_b} + q\rho_H}\right)\frac{4\pi Gr}{3},$$

where $\rho_b$ and $\rho_H$ are the baryonic and dark matter densities, and $q$ is the equation of state parameter.
where \( a(t) \) is the scale factor, \( \rho_b \) is the baryon density. We use \( q \) as a book keeping parameter so that \( q\rho_H = \rho \) is background density in matter dominant epoch, and \( q\rho_H = -\rho_\Lambda \) is background density the in the dark energy dominant epoch. The VMOND radial acceleration is given as

\[
\dot{r}_{VM} = -Hr\sqrt{\frac{GM}{2r}} = -\frac{4\pi Gr}{3}\sqrt{\rho_b\rho_\Lambda}.
\]

(5)

For a central mass system such as MW with \( M = 1.5 \times 10^{12}M_\odot \), at distance \( r_0 = 10\text{kpc} \), this acceleration is \( \dot{r} = 1.15 \times 10^{-12}\text{m/s}^2 \) and is \( \dot{r} = 1.15 \times 10^{-13}\text{m/s}^2 \) at 1Mpc. Within the scales of interest this VMOND acceleration is significantly less than the Milgrom MOND acceleration of \( \dot{r} = 2 \times 10^{-10}\text{m/s}^2 \). Therefore VMOND does not directly lead to Milgrom MOND hypothesis. In Conformal Gravity program \[9\], one obtains the metric \( \gamma \) of the densities due to the central mass and cosmological background. One could tune this to a desired value in cases where Newtonian acceleration is not enough to provide the gravitational attraction needed to fit the flatten rotational curves \[10\] and \[28\]. However at this tuned value \( \gamma \) is not tunable.

When comparing with the Schwarzschild’s de-Sitter(SdS) metric, here \( Z(r) \) takes on an additional term of the form \(-\gamma r\). Deriving radial acceleration from Eq.(6) and write in terms of densities, one has

\[
\dot{r} = \frac{\hbar^2}{r^3} - \frac{4\pi Gr}{3}(\rho_b - \rho_\Lambda) - \gamma.
\]

(7)

We note that the \( \gamma r \) term in the conformal Gravity metric plays the role of a constant acceleration independent of the densities due to the central mass and cosmological background. One could tune this \( \gamma \) to a desired value in cases where Newtonian acceleration is not enough to provide the gravitational attraction needed to fit the flatten rotational curves \[10\] and \[28\]. However at this tuned value \( \gamma \), the corresponding strong lensing deflection angles for galaxies and clusters are negative which are not compatible with observations \[29\].

By contrast, in Eq.(4) when comparing with Newtonian acceleration, the VMOND acceleration

\[
\dot{r}_{VM} = -\frac{4\pi Gr}{3}\sqrt{\frac{\rho_H}{\rho_b}},
\]

(8)

which depends only on the ratio \( \rho_b/\rho_H \) and is not tunable.

Crucial features of this model:

Comparing to the Newtonian term, VMOND is negligible when \( \rho_b/\rho_H \gg 1 \) such as in a large central point mass potential. However, effects of VMOND is completely different for \( \rho_b/\rho_H \leq 1 \). At the baryon overdensity region with \( \rho_b/\rho = \delta (\delta \sim 10^{-5}) \) at very early time, we see that the VMOND density dominates over Newtonian gravitational attraction. For a protogalactic cloud with baryonic density \( \rho_b/\rho = \delta \sim O(1) \) where VMOND density plays the role of adding a term \( \rho_b/\sqrt{\delta} \) to the baryonic density. When an outer gas shell of the protogalaxy comes close to the central mass after a turnaround point, the VMOND term will diminish as \( \rho_b/\rho_\Lambda \) grows larger. For \( \rho_b/\rho_\Lambda \gg 1 \) the VMOND density contribution becomes negligible and Newtonian gravity dominates. Thus we can see that the evolution of VMOND term from early time is critical to galactic formation theory.

In \[11\], we show that for \( \rho_b = \delta \rho \), in the linear perturbation regime where the overdensity \( \delta \ll 1 \) the term \( \sqrt{\frac{\rho_b}{\rho}} = \delta^{-1/2} \gg \delta \), VMOND dominates over Newton. We find that overdensity grows as \( \delta \propto a(t)^2 \) and can account for the effects attributed to DM to improve the \( \delta \propto a(t) \) growth. The increasing VMOND term in radiation dominant epoch also account for the matter loading effects for high acoustic peaks in CMB. For galactic rotational curve near a high mass density central potential, we have shown above that the VMOND acceleration in Eq.(5) is much smaller than the Newtonian acceleration whilst MOND or Conformal Gravity acceleration is larger and constant at outside stellar disk scales, so that VMOND acceleration does not lead automatically to a flattened rotational curve. While this VMOND acceleration commensurates with the near stellar disk observations \[11\]-\[12\] at early times (\( z \sim 2 \)), the task is to explain how the rotational curve flattens gradually over time. By incorporating the VMOND effect, we build on the Mestel models \[30\] of a spherical protogalaxy with uniform density for which the inner region will fragment and collapses to an exponential disk while the outer gas shells follow a Secondary Infall (Gunn and Gott \[27\]) to reach an asymptotic mass profile similar to Mestel’s flat disk.
GALACTIC MASS MODEL AND FORMATION

Mass modelling for late time spiral galaxies have been studied extensively. A review on late time disk galaxy mass modelling can be found in [31]. For later use, here we recapulate some key features of the Freeman disk mass modelling of Spiral galaxy up to \( \sim 5r_0 \) (hereafter we use \( 5r_0 \) as the galactic disk size) in the following: The central mass of a thin disk is measured by its surface density which is written as

\[
\Sigma_0 = \frac{M}{L} I_0
\]  

(9)

where the surface density is given by \( \Sigma = \int_{h_0}^{\infty} \rho dz \), \( 2h_0 \) is the disk thickness and \( I_0 \) is the central surface brightness. \( M/L \) is the mass to light ratio which is assumed roughly constant with radius for different galaxy type. The surface brightness (and thus its surface density) is found to fit by an exponential disk profile which is

\[
I(r) = I_0 e^{-r/r_0}; \quad (\Sigma(r) = \Sigma_0 e^{-r/r_0}).
\]  

(10)

First assuming spherical symmetry and the Newtonian relation for rotational speed \( v^2 = \frac{GM(r)}{r} \) for a centralised mass density \( M(r) \), and we have

\[
\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) = \frac{v^2}{G}
\]  

(11)

To match an observed constant density central region \( \rho \sim \rho_0 \) upto \( r = r_0 \) we have velocity \( v_\phi \) rises as \( r \) increases similar to a solid body, with constant angular velocity

\[
\Omega = \frac{v}{r} = \sqrt{4\pi G \rho_0},
\]  

(12)

which is consistent with observations.

Outside the constant density region, we expect Newtonian gravity resumes with the mass density behaves as \( \rho(r) \propto 1/r^3 \) and \( v^2 \propto \frac{1}{r^2} \).

To obtain a flat rotational curve, \( v = constant \), outside the central disk at \( 2r_0 \), one needs \( \rho(r) \sim 1/r^2 \) \( [M(r) \sim M_0 r] \), where \( M_0 \) is constant. One can reproduce the constant rotational speed profile from various potentials [32] and the Finite Mestel disk [33], or from models with protogalactic clouds such as the constant-mass Mestel sphere [30] and the Secondary Infall model [34]-[36].

As the stellar disk exhibits a solid body behaviour, one wants to estimate its the angular momentum. Freeman [26] estimates the angular momentum of the disk, by considering stars in circular orbits at the scale length \( r_0 \) with a centralised mass \( M \),

\[
v^2 \approx \frac{GM}{r_0}
\]  

(13)

The square of angular momentum per unit mass (specific angular momentum) is given by

\[
h^2 = v^2 r_0^2 = \left( \frac{GM}{r_0} \right) r_0^2 = GMr_0
\]  

(14)

For a rotating exponential disk, the following relation is derived by Freeman [26]

\[
h = 1.109 \sqrt{GMr_0}
\]  

(15)

which says that the specific angular momentum of the disk is closely connected to the disk scale length.

The Freeman disk describes that the stellar disk is a region of constant \( \rho \) where \( r < r_0 \) \( v \approx r \) which is rising as \( r \to r_0 \), and beyond \( r > 2r_0 \) one expects a newtonian fall-off \( v^2 = \frac{GM}{r} \).

Virialised galactic disk:
At late times the scale of the specific angular momentum of the virialised Freeman disk (which is the whole galactic disk) is estimated in the form of Eq. \[15\]. From \[23\], \[37\] and \[38\] we use \(v^2_{\text{vir}} = GM_{\text{vir}}/R_{\text{vir}}\) and \(v_c = 1.2v_{\text{vir}}\) is the peak circular velocity. For a virial radius \(R_{\text{vir}} = 10r_0 \sim 13r_0\) and \(M_{\text{vir}}/M_{\text{disk}} = 2.79\) for MW, one obtains the specific angular momentum for the virial disk
\[
h^2 = GM_{\text{disk}}l; \quad l = 1.1r_0 \sim 1.6r_0.
\]

We see from the specific angular momentum of the virial mass of MW, which is the Freeman disk plus HI gas disk, varies from the Freeman value \(1.109GMr_0\) to a slightly higher value \(1.6GM_{\text{vir}}r_0\) at a larger virial distance. This suggests that materials at distances further away from the scale length have a higher specific angular momentum. This virialised exponential Freeman disk model is largely successful upto \(r \sim 2r_0\). The main challenge is to explain the flatten rotation curve outside \(r_0 \sim 2r_0\) upto \(5r_0\).

The simplest mass model for galactic flatten rotational curves outside \(r_0\) can come from the "maximal disk hypothesis” \[39\]-\[41\], which proposes that by taking the maximal M/L ratio, one could explain the flat rotational curves by circulating baryonic matter alone. However, although there is a strong correlation between baryonic mass and rotational curve, works by many (see Freeman \[31\], \[42\] ) etc. suggest that most spirals are submaximal and an extra halo mass of \(\sim 1\) to a slightly higher value \(1\) can come from the “maximal disk + HI gas disk model (baryonic scaling model)” can be found in \[40\] where they use the formula
\[
v_{\text{rot}} = \sqrt{\gamma v(r)^2_{\text{disk}} + \eta v(r)^2_{\text{HI}} + \epsilon v(r)^2_{\text{bulge}}},
\]
where \(\gamma, \eta\) and \(\epsilon\) are the mass/light ratio of the disk, the HI gas and the bulge respectively. For bulgeless galaxies, a 2-parameter fit is sufficient to model the rotational curves of 43 spirals and dwarfs of both early and late type, with \(\gamma\) falling within a stable maximal range and \(\eta\) in R band mostly falls in \(5 \sim 14\) (with an average value of \(8\) range. Combined with more detailed work from the same group \[37\] the authors find that for early spirals the HI gas mass peaks at a distance outside \(5r_0\), but for late spirals the mass profile of the HI gas is gaussian and peaked around \(r \sim 2r_0\) with a gas disk radius at \(R_{\text{HI}} \sim 5r_0\). The authors conclude that there is no compelling evidence for dark particles within \(2r_0\) and the HI gas disk size \(R_{\text{HI}}\) increase is seen to improve rotational curve flatness. A major challenge is to explain the HI mass profiles with their peaks shown up at different distance scales at different red-shifts. One possibility is that the HI mass peak also moves towards the stellar disk over time. The DiskMass Survey authors also find that using a "Freeman disk plus HI gas plus MOND model” on disk galaxies requires a large M/L ratio which leads to a disk thickness that is far too large than observed in the Disk mass Survey \[43\].

The early works of Hoyle \[19\], Peebles \[20\], Eggen et. al. \[49\] established that a galaxy such as the MW can originate from the collapse of a large uniform density sphere protogalaxy. Mestel’s idea \[30\] is to derive equilibrium distributions of mass and angular momentum of the disk given the conditions in the primeval sphere. In spherical coordinates \((r, \theta, \phi)\), with \(\theta\) the angle extended from the mid-plane. By considering the ratio, based on small eccentricity and energy dissipation
\[
\frac{h^2 \sin \theta / r^3}{GM(r) \sin \theta r^2} = \frac{h^2}{GMr},
\]
one notices that radial collapse will stop at some equilibrium distance \(r\), while the along the direction of the angular momentum \(z\)-axis, the collapse will stop at a small distance. Using the poisson equation and equation of hydrostatic support normal to the mid-plane, one obtains
\[
\Sigma = 2\ell \bar{\rho}, \quad \bar{\rho} = \frac{\pi GS^2}{2c^2},
\]
where \(c\) is the sound speed, \(2\ell\) is thickness of the disk, \(\bar{\rho}\) is local density independent of \(z\) and the local surface density \(\Sigma \sim O(M_{\text{tot}}/\pi r_0^2)\), where \(r_0\) is the scale length and \(M_{\text{tot}}\) is the total mass. One assumes that \(z \ll r_0\) as long as \(GM/r_0 \gg c^2\). Then Mestel proceeds to calculate the consequences of this equilibrium state. The equilibrium angular speed is
\[
\Omega^2 = \frac{v^2}{r^2} = -\frac{1}{r} \frac{d\phi}{dr},
\]
where the equilibrium potential at $r$ of a thin disk with size $R_0$ and surface density $\Sigma(r)$ is given by

$$
\phi(r) = G \int_0^{2\pi} \int_0^{r_0} \frac{\Sigma(q)q}{(r^2 + q^2 - 2rq \cos \theta)} d\theta dq.
$$

(21)

For uniform density sphere, one obtains

$$
\Sigma(r) = \Sigma_0 \left[ 1 - \left( \frac{r}{r_0} \right) ^{2\gamma/2} \right].
$$

(22)

This is a limit of an uniform density spheroid of semi-axes $r_0$, $r_0$, and $(1 - e^2)^{1/2}r_0$, which has equilibrium constant angular velocity $\Omega$ in the $e \to 0$, $\rho \to \infty$ limit which describes solid body rotation where $r_0$ is the size of the disk with $(r < r_0)$.

For constant mass spheroid, where density decreases slightly as radius increases, Mestel [30] obtains

$$
\Sigma(r) = \frac{\Sigma_0 r_0}{r},
$$

(23)

which describes a flat rotational curve. Mestel also obtains a finite disk potential, [30] Eq.(56)

$$
\Sigma(r) = \frac{v^2}{2\pi Gr} \left( 1 - \frac{2}{\pi} \sin^{-1} \frac{r}{R_0} \right),
$$

(24)

which reproduces Eq. [23] for $R_0 \to \infty$. These Mestel’s results received good observational supports from work of Hoyle and Crampin [29] and Freeman [26]. Apart from an uniform isothermal sphere, there is suggestion that one could start with a finite Mestel disk (FMD) instead and for this FMD Schulz [33] obtains an analytic solution which describes a flat disk with a sharp edge. Given that the central disk is well described by a solid body with constant angular velocity $\Omega$, we believe that an uniform density sphere is a more appropriate starting object to collapse into an exponential stellar disk.

The Mestel theory is recently revisited by Davies using numerical calculations [50] in which the initial MW central disk mass is taken to be $M = 3 \times 10^{11} M_\odot$ at $z \sim 14$. The density of the cloud at $z \sim 14$ is taken as $\rho_0 \sim 7 \times 10^{-23} kg/m^3$ and the critical density is $\rho_c \sim 2 \times 10^{-26} kg/m^3$ at $z \sim 0.5$. The radius of this initial galactic region is calculated as $R_0 = 40 kpc$ (we have $R_0 = 30 kpc$, if one takes $M = 4\pi \rho R_0^3 = 2\Sigma_0 R_0^2$ where surface density $\Sigma_0 = 2\rho R_0$). The collapse time $t_{col}$ can be calculated using

$$
t_{col} = \sqrt{\frac{3\pi}{32 G \rho}} = 0.5427 \frac{1}{\sqrt{G \rho}},
$$

(25)

which gives $t_{col} \sim 0.3 Gyr$. For a MW virial mass taken from [51] at $1.2 \times 10^{12} M_\odot$, the initial outer radius $R_0 \sim 63 kpc$. (We note that our analysis using VMOND should require much less MW mass than the usual virial mass quoted as there is no DM present, however using a larger viral mass does not affect the analysis.)

Fragmentation:

The observations that star forming clouds are usually not undergoing rapid collapse supports the view that gravitational collapse can occur in the densest part of a cloud even when the cloud as a whole is not collapsing. see for example [51]. The Jean Mass $M_J$ estimate after recombination is $M_J < 10^6 M_\odot$ so that higher density objects like globular clusters with $M_J = 10^5 \sim 10^6 M_\odot$ in principle can form at $z \sim 100$ when the cosmic matter density is much higher than the critical density. Fragmentation can be further sustained by the Jean Mass relation to its density with adiabatic index $\gamma_0$

$$
M_J \sim \rho^{\frac{2}{\gamma_0 - 1}}
$$

(26)

where the adiabatic index for astrophysical objects is lower than 1, so that a localised density increase will lead to a smaller Jean Mass.

The break-up due to fragmentation:

When a central region where density perturbation and radius stop growing while the overall cloud continues to expand,
fragmentation could occur. In this case, we envisage that the overall cloud will continue to expand following the density perturbation growth equation until \( \delta \sim O(1) \), while a central region with \( \delta < 1 \) will break up with the overall cloud and collapses to an exponential disk. An observational support for this scenario is the 2015 Nature report of an observed protogalaxy \([32]\) in which the central disk is in place while the external cloud spans up to a distance of 130 kpc.

Incorporating VMOND, we know that \( \delta \propto a(t)^2 \). At recombination, we take \( \delta = 3 \frac{\Delta T}{T} = 3 \times 10^{-5} a^2 \), where \( a(z = 1000) = 1 \), \( \Delta T \) is the temperature fluctuation and \( a(z) \) is the scale factor relative to the scale factor \( a(z = 1000) \) at recombination. It is generally believed that galaxies can begin to form at \( z \sim 100, a(z) = 1000/(1 + z) \sim 10 \), here \( \delta = 3 \times 10^{-5} \times 10^2 \), where \( \rho = 2 \times 10^{-20} \text{kg/m}^3 \), \( \delta_0 = 6 \times 10^{-23} \text{kg/m}^3 \), \( \delta^{1/2} \rho = 1.09 \times 10^{-21} \text{kg/m}^3 \). Under VMOND with \( \delta^{1/2} \rho \), the Jean Mass is much smaller and thus more favourable for gravitational collapse.

To collapse to a large stellar disk the size of the MW, we need to start with a central sphere that is significantly larger than the disk. We look at the case, \( z = 9 \), here \( a(z) \) has grown by a factor of 100 since recombination, so that \( \delta_0 = 3 \times 10^{-5} \times 10^4 = 0.3 \) with \( \delta_0^{1/2} = 0.54 \), and we have an effective \( \delta = 0.84 \) to drive acceleration. We note that from the latest observational estimate for MW masses \( M_* \) (within distance \( r \)) from \([33]\) and \([34]\) are given as \( M_{r0} = 6.08 \times 10^{10} M_\odot, M_{50kpc} = 4.48 \times 10^{11} M_\odot; M_{vir} = 1.20 \times 10^{12} M_\odot \).

To be compatible with the observed stellar disk mass, we take the inner sphere mass \( M = 1 \times 10^{11} M_\odot \) which has its initial radius at \( R_I = 100 \text{kpc} \) (Had we chosen \( M = 6 \times 10^{10} M_\odot \), we have \( R_I = 84.3 \text{kpc} \) which may also be a good choice for our MW). For a MW mass of \( 1.2 \times 10^{12} M_\odot \), the initial outer radius is \( R_E = 228 \text{kpc} \). At this point the cosmic background density is \( \rho_0 = 5.92 \times 10^{-24} \text{kg/m}^3 \). Since the effect of \( \delta^{1/2} \) is to speed up the collapse time and that \( \delta^{1/2} \propto r^{-2/3} \), as \( r \) decreases, based on Newtonian gravity alone we can obtain an upper bound for collapse time for the inner sphere by

\[
t_{col} = 0.5427 \frac{1}{\sqrt{G \rho_0 \delta}},
\]

i.e. \( t_{col} \) takes maximal value at \( \delta = 0.3 \) at \( t_{col} = 1 \text{Gyr} \) from Eq. \([27]\) by neglecting \( \delta^{1/2} \). (Note: Had we chosen the collapse to start at \( z = 10 \), we will have a shorter upper bound collapse time \( t_{col} \leq 0.47 \text{Gyr} \). Without VMOND, we will have \( \delta \sim 0.3 \times 10^{-2} \) with a central mass at \( O(10^8) M_\odot \) which may be less favourable for breaking up with the overall cloud.) We expect that this central sphere will follow the Mestel sphere analysis and collapses into an exponential stellar disk.

**DENSITY PERTURBATION GROWTH AND TURNAROUND OF PROTO-GALAXY SHELL**

Here we take the simpler approach of taking \( \delta = 3 \left( \frac{\Delta T}{T} \right) \sim 1 \), and follow the protogalaxy to reach its turnaround via the energy equation. At \( z \sim 9 \) when the inner spherical collapses, we have a small \( \delta \sim 0.3 \) and \( a(z) = 10^2 \). It is reasonable to expect the density perturbation of the overall protogalaxy will continue to grow until it reaches \( O(1) \). When \( a(z) \) grows to the point \( \delta = 3 \) we have \( z = 2.16 \). Using the equation for radius growth in a central mass

\[
r \approx \frac{1}{2} \left( GM(1 + \delta^{1/2}) \right)^{1/3} (6t)^{2/3} \propto a(t).
\]

we obtain \( a(z = 2.16)/a(z = 9) = 3.16 \). Thus up to this point the radius of the outer shells should grow by a factor of 3.16. Here we have the inner shell radius at \( R_I \sim 316 \text{kpc} \) and outer shell at \( R_E \sim 229 \text{kpc} \). After this time we have a constant mass within these spherical shells which provides a density decrease for larger radius shell and reaching their turnaround radius \( r_t \). The cosmic background density is estimated at \( \rho_0 = 1.86 \times 10^{-25} \text{kg/m}^3 \) and \( \rho_b = 3\rho_0 \). If the sphere collapses at this point, the upper bound for estimated collapse time \( t_{col} = 2.24 \text{Gyr} \).

Following the idea that this sphere can be considered as independent mass shells, each of which will continue to expand until it is stopped by gravity. The initial velocity \( \dot{r} \) is usually taken as the cosmic background speed,

\[
\frac{1}{2} \dot{r}^2 = \frac{1}{2} H^2 r^2.
\]
$R$ is used to denote the turnaround radius, $H_R$ the Hubble constant at turnaround and the energy equation

$$\frac{1}{2} \dot{r}^2 + \frac{\dot{h}^2}{2r^2} - \frac{GM}{r} - \frac{1}{2} H \sqrt{2GMr} - \frac{1}{2} H^2 r^2 = \frac{1}{2} \dot{r}^2 + \frac{\dot{h}^2}{2r^2} - \frac{GM}{R} - \frac{1}{2} H_R \sqrt{2GM\rho} - \frac{1}{2} H^2 R^2$$  \hspace{1cm} (30)

For $h^2 = 1.1GM_0$, at $r > R_E$ the angular speed term is much smaller comparing to other terms. At $z = 2.16$ we have the baryonic density $\delta \rho(z = 2.16)$ and $H = \sqrt{\frac{2GM}{\delta \rho}}$. The value of $\delta$ affects significantly the time the mass shell approaches its turnaround. If for example for large $\delta$, we can neglect the Hubble flow and consider the mass shell in a central gravitational potential,

$$r^2 = H^2 r^2 = \frac{2GM}{r} - \frac{2GM}{R} + H \sqrt{2GMr} - H_R \sqrt{2GM\rho}.$$  \hspace{1cm} (31)

So that $H^2 = \frac{2GM}{\delta \rho}$ where $H_R$ is the Hubble constant at turnaround and as an estimate we assume that $H_R = \frac{2GM}{4R}$, so that

$$R = \frac{1 + 1/\sqrt{3}}{1 + 1/\sqrt{3} - 1/3}r = 1.26 r$$  \hspace{1cm} (32)

On the other hand if $\delta$ small, the mass shell instead will follow the Hubble flow until the baryonic density matches the background density, we have from Eq. (30)

$$H^2 R^2 = \frac{2GM}{r} - \frac{2GM}{R} + H \sqrt{2GMr} - H_R \sqrt{2GM\rho}.$$  \hspace{1cm} (33)

For $H^2 = \frac{2GM}{R^2}$, $H^2 = \frac{2GM}{\delta \rho}$

$$R = \frac{3}{1 + 1/\sqrt{\delta}}r = 1.90 r, \text{ for } \delta = 3.$$  \hspace{1cm} (34)

I) Firstly, we assume that the material that breaks away from Hubble flow, so that the expansion stops at $\rho_b(R) = \frac{1}{6} \rho_b$ and the shell radius will expand by another factor of 1.26 to reach their turnaround points. Here the shell radii have grown to $R_I \sim 398 \, kpc$ and $R_E \sim 909 \, kpc$. The red-shift estimate at turnaround is $z \sim 0.96$ by using Newtonian calculation only,

$$\cos \eta = \sqrt{\frac{0.26}{1.26}}, \quad \tau_c = \frac{t}{c} \left( \eta + \frac{1}{4} \sin 2\eta \right) = 0.956 t_{\text{col}} = 3.02 \text{Gyr}.$$  \hspace{1cm} (35)

The collapse time from the turnaround point at $z \sim 0.96$ is given by $t_{\text{col}} = 3.16 \text{Gyr}$ and the collapse will complete at $z \sim 0.42$.

In summary, at $z = 9$, the central sphere will fragment and the outer sphere will grow until $z = 2.16$, then it will reach its turnaround point at $z \sim 0.96$ and the sphere will collapse onto the stellar disk at $z \sim 0.42$.

II) If the sphere follows the Hubble flow and turnaround at $R = 1.9 \, r$ given above, with $R_I = 600 \, kpc$ and $R_E = 1.37 \, Mpc$, where the background density $\rho_H = \rho_b(R) = \frac{3}{8.86} \rho(z = 2.16)$, $a(z)$ has expanded by a factor of 1.32, so that the redshift is $z \sim 1.4$.

We use $\tau = \xi$, and the energy equation to calculate collapse time $t_{\text{col}}$

$$\left( \frac{dr}{dt} \right)^2 = \frac{2GM}{r} - \frac{2GM}{R} + H \sqrt{2GMr} - H_R \sqrt{2GM\rho} + H^2 r^2 - H_R^2 R^2 = \frac{2GM}{R} \left( \frac{1}{c^2} - 3 \right)$$  \hspace{1cm} (36)

c = 1/(1 + \delta^{-1/2} + \delta^{-1})

$$\left( \frac{d\xi}{dt} \right)^2 = \frac{2GM}{R^3} \left( \frac{1}{c^2} - 3 \right) = \frac{8\pi G \rho_b}{3} \left( \frac{1}{c^2} - 3 \right)$$  \hspace{1cm} (37)
For $\delta = 1$, $3c = 1$

$$t_{col} = \left(\frac{3}{8\pi G \rho_0}\right)^{1/2} \int_1^\xi \left(1 - 3c\xi\right) \frac{1}{2} d\xi = \left(\frac{3}{8\pi G \rho_0}\right)^{1/2} \int_1^\xi \left(1 - \xi\right) \frac{1}{2} d\xi$$

(38)

$$t_{col} = 0.5427 \sqrt{\frac{6.68}{G \rho_b}} = 5.78 \, \text{Gyr},$$

(39)

so that the end of the collapse occurs at $z \sim 0.3$.

The alternative scenario is the sphere will reach its turnaround point at $z = 1.4$ and collapse onto the stellar disk at $z = 0.3$. The Infalling gas will reach the stellar disk after dark energy dominant epoch starts at $z \sim 0.3$ and this is consistent with the pure disk galaxies observations [15].

Asymptotic behaviour of Infalling gas:
The behaviour of the protogalactic shell in a secondary infall are considered by [27], [34]-[36]. In this model, a gas cloud of constant mass is represented by spherical shells moving under the effect of cosmic expansion. A mass shell in a central potential continues to move towards its turnaround radius after which point the shell starts to follow shrinking oscillatory cycles and moves towards its radius of closest approach. In our case, high eccentricity shells will make only one oscillation to approach its radius of closest approach which is smaller than the stellar disk size and thus they will simply collapse onto the stellar disk. The gas density in the shells at turnaround point is around 4 times the critical density. Following the work of Nusser [36], we consider a particle mass shell at distance $r_i$ and time $t_i$, with enclosed mass $M_i$ and overdense mass $\delta M_i$. An idealised choice is that these initial mass perturbations to be scale free except at the turnaround point. One assumes that the perturbation $\delta_i$ take a power-law form,

$$\delta_i = \left(\frac{M_i}{M_l}\right)^{-\epsilon}$$

(40)

where $M_l$ is some reference mass scale. The equation of motion in Newtonian potential is

$$\frac{d^2r}{dt^2} = -H^2 r - \frac{GM(r,t)}{r^2} + h^2 \frac{r^3}{r^3}$$

(41)

where $M(r,t) = M(<r,t)$ is the mass inside radius $r$ at time $t$. where the initial velocity of a shell is assumed to be the cosmological background expansion rate

$$\frac{dr_i}{dt} = H r = \frac{2}{3t_i} r_i$$

(42)

Nusser makes the assumption that

$$h^2 = 2\alpha G\delta_i M r_i = 2\alpha G M_i r_i$$

(43)

$$\epsilon^2 = 1 + 4\alpha(\alpha - 1)\delta_i^2.$$  

(44)

(note: in Newtonian gravity for a circular orbit, $\alpha = 1/2$, larger $\alpha$ value denotes higher eccentricity orbits). To obtain asymptotic behaviour at large $t$, one further assumes that the mass distribution takes the a power-law function of both $r$ and $t$

$$M = k(t)r^\gamma, \quad k(t) = k_0 t^{-s}.$$  

(45)

where $\gamma$, $k_0$, $s$ are constant and $k(t)$ is a slow varying function of $t$. For an unit mass particle in a central potential the Lagrangian ”action” variable $J_r$, Nusser FG shows that for invariant $h$ and no energy loss

$$J_r = \int_{r_h}^{r_a} dr \left(\frac{dr}{dt}\right) = \int dr \left[2 \left(\frac{E - k(t)}{\gamma - 1} r^{\gamma - 1} - \frac{h^2}{r^3}\right)^{1/2}\right]$$

(46)

$$J_r = 2\pi h \int_1^\xi du \left[(\xi^2 - 1)\left(\frac{1 - u^{\gamma - 1}}{1 - \xi^{\gamma - 1}}\right) + (1 - u^{-2})\right]^{1/2}$$

(47)
is adiabatically invariant (slowly varying over time) where \( u = 1/r \). This also implies that \( \xi \) the ratio of distance of closest approach \( r_{\text{b}} \) to distance of furthest approach \( r_{\text{a}} \) is constant,

\[
\frac{r_{\text{a}}}{r_{\text{b}}} = \left( \frac{t}{t_{\text{a}}} \right)^q, \quad \frac{r_{\text{b}}}{r_{\text{a}}} = \xi = \text{constant}, \quad q = \frac{s}{\gamma + 1} > 0,
\]

These mass shells in principle would oscillate between \( r_{\text{a}} \) and \( r_{\text{b}} \) while individually they will shrink in scale over time.

\[
s = \frac{2}{3\epsilon} \left[ \frac{\gamma - n}{n} + (\gamma - n + 3)\epsilon \right], \quad q = \frac{2}{3\epsilon(\gamma - n + 4)} \left[ \frac{\gamma - n}{n} + (\gamma - n + 3)\epsilon \right]
\]

By defining \( P(r, r_{\text{a}}, t) \) to be the fraction of time a particle with further distance \( r_{\text{a}} \) spends inside \( r \):

\[
P(v) = \frac{I(v)}{I(1)} \quad (v \leq 1), \quad P(v) = 1 \quad (v > 1); \quad I(v) = \int_0^v \frac{du}{(1 - u^{\gamma-n+2})^{1/2}} \left( \frac{r}{r_t} \right)^{\gamma-p} = \frac{1}{p} \int_{r/r_t}^{\infty} du u^{-(1+p)} P(u, \xi)
\]

In the asymptotic large \( t \) regime, one finds the exponents \([35]\) when \( n = 1, 2, 3 \) denotes planar, cylindrical and spherical geometry,

\( n = 1 \):

\[
\gamma = p = \frac{3}{3 + \epsilon} \leq 1, \quad q = \frac{4}{9};
\]

\( n = 2 \):

\[
\gamma = 1, \quad p = \frac{3}{2 + \epsilon} \geq 1, \quad q = \frac{4\epsilon - 1}{9\epsilon};
\]

\( n = 3 \):

\[
\gamma = p = \frac{3}{1 + 3\epsilon}, \quad s = q = 0, \quad \text{for} \ \epsilon \geq \frac{2}{3}
\]

\[
\gamma = 1, \quad p = \frac{6}{4 + 3\epsilon} \geq 1, \quad s = 2q, \quad q = \frac{3\epsilon - 2}{9\epsilon}, \quad \text{for} \ \epsilon < \frac{2}{3}
\]

We take \( \gamma = 1 \) for the time being.

A key requirement above is the adiabatically invariant of \( J_r \), in which \( k(t) \) is slow varying in \( t \). In our case we have initially \( \delta \sim 1, \rho_H/\rho = 1 \) and the effective mass inside a mass shell at \( R_t \) becomes

\[
M = (1 + \delta^{-1/2})k(t)r^\gamma = \left[ 1 + \sqrt{\frac{\rho_H}{\delta \rho}} \left( \frac{r}{R_t} \right)^{2/3} \right] k(t)r^\gamma = \left[ 1 + \left( \frac{r}{R_t} \right)^{2/3} \right] k(t)r^\gamma.
\]

A typical shell would shrink from 500 kpc to 50 kpc while the multiplicative factor \( (1 + (r/R_t)^{2/3}) \) changes from 2 to 1.215 in 5 Gyr. So that with VMOND the effective \( k(t) \) term in Eq. \([46]\) is also slow varying function in \( t \). The arguments for the evolution of secondary infall shells above should remain valid. We expect that the scaling behaviour and the exponents remain the same, especially for the shells that arrive at the stellar disk in the matter dominant epoch.

In the above analysis, the asymptotic state of the Infall mass function has the form \( M(r) \propto r^{\gamma} \). In practice, we expect that a disk will be formed. In general it is not easy to deal with asymmetric potential \([32]\) analytically and one usually uses a thin disk approximation which is numerically shown to provide a good approximation. In section 6, one can scale-transform a nearly uniform sphere into a Mestel disk with potential Eq. \([24]\). Specifically it is shown that the sphere will start flattening from the outer regions and leaving a central region going through a quasi-spherical collapse. The asymptotic states of this mestel disk potential yields a mass function with the same exponent \( \gamma = 1 \). Based on this scenario, we consider a thin disk approximation in which the mass shells moving in a thin disk following the Nusser’s analysis, resulting in the asymptotic states of the shells having the mass function \( M(r) = M_0 r \). Here the asymptotic shells in radial direction does not follow an equilibrium circular path as in an
For high eccentricity shells to reach the inner region, if there is no low eccentricity orbits oscillating outside 5$r_0$ but remains in motion moving towards its distance of closest approach. The surface density of asymptotic disk with thickness $2\ell$ with density $M(r)/(\pi r^2 \ell)$ is thus

$$\Sigma = \int_{-\ell}^{\ell} \rho dz = 2\ell \frac{M_0 r}{\pi r^2 \ell} = 2\ell \rho = \left(\frac{2\ell M_0}{\pi}\right) \frac{1}{r}$$

which is simply the Mestel constant velocity disk at Eq. \([15]\) and Eq. \([23]\). The central region of the disk will have lower density due to earlier stellar disk collapse, and also with little radial velocity components. There is an outer region of mass annuli with higher mass densities and radial speeds. It is helpful to point out that from \([30]\) (section 6) that the constant velocity Mestel disk solution is also not at equilibrium apart from at the centre. The outer annuli at asymptotic state will continue to follow their equation of motion to approach the stellar disk until some form of equilibrium can be attained. Qualitatively this is consistent with observations in \([46]-[47]\), where it is observed that an early time HI mass profile grows with increasing distance and peaks outside 5$r_0$ (we take $r_0 = 8.5 kpc$ for MW to be more precise) and the rotational curve flattens where the HI gas surface density is flat. At late times, the mass profile inside 5$r_0$ strongly peaks around 2$r_0$ and we do not have a mass distribution growing as distance grows within 5$r_0$. Within 5$r_0$, this is consistent with the consideration that there is nothing to stop the arriving mass shells to continue to follow the equation of motion Eq. \([41]\) and collapse onto the stellar disk. With small central pressure, the collapse can take the form of a shock \([55]\) or mass shell accumulation \([56]\). An equilibrium can also be achieved by the fountain mechanism proposed by Pezzulli and Fraternali in \([16]\). We will come back to this mechanism below.

We note in \([1]\) that in Dark energy dominant epoch, MW has a cut-off radius of $\geq 1.3 Mpc$ so that all the outer shell turnaround radii are well inside the MW cut-off radius and no material will be lost due to the repulsive background acceleration in the dark energy dominant epoch.

Effective angular momentum and M/L ratio:
For a particle in mass shells starting at $r_c \sim 30 r_0$ and coming to $r_0$ scales, we have the energy equation of the metric,

$$\frac{1}{2} v^2 + \frac{h^2}{2r^2} = \left(\frac{c^2 - 1}{2}\right)\frac{GM^2}{2h^2} + \frac{GM}{r} + \frac{1}{2} H^2 \sqrt{2GMr} + \frac{1}{2} H^2 r^2.$$ \hspace{1cm} (57)

For $r_c \gg r_{cl}$ where $r_{cl}$ is the distance of closest approach, eccentricity $e$ is given by $E = -GM/(r_c + r_{cl}) \approx -GM/r_c$,

$$e^2 = 1 + \frac{2EH^2}{GM^2} = 1 - \frac{2r_0}{r_c}$$ \hspace{1cm} (58)

As the turnaround radii assume the values $R_E > R_\ell \geq 398 kpc$, eccentricity $e \sim 1$, $\dot{r} = 0$, and assuming negligible energy and angular momentum losses we obtain for distance outside $r > r_0$

$$v_\phi^2 \approx \frac{2GM}{r}.$$ \hspace{1cm} (59)

Eq.\([59]\) means that these collapsing mass shells close to the stellar disk have an effectively higher rotational speed. For a single particle in a point mass central potential, this speed would be temporary before the particle moves back to large distances after passing its point of closest approach. However, in this case the collapsing mass shells are still trying to reach its radius of closest approach inside the stellar disk. The general gas density is now given by

$$\rho(r) = \frac{3}{4\pi} \frac{M(r)}{r^3} \propto r^{\gamma - 3}.$$ \hspace{1cm} (60)

For high eccentricity shells to reach the inner region, if there is no low eccentricity orbits oscillating outside 5$r_0$, it is more appropriate to take the cylindrical or spherical coordinates and we have $\gamma = 1$.

As the radial velocity of an ellipse will slow down as a mass shell comes near its distance of closest approach, we assume that the shock on the stellar disk due to the mass shell arrival is weak and the eccentricity loss is minimal. After the arrival of some mass shells to the stellar disk, the modelling of central disk mass $M_\ast$ and stellar disk density $\rho_\ast$ with HI gas with density $\rho_g$ using Eq.\([60]\) gives

$$\dot{r} = \frac{4\pi G}{3} \left(\rho_\ast + \rho_g\right) r = \frac{G}{r} \left(\frac{M_\ast}{r} + M_0\right)$$ \hspace{1cm} (61)
where $M_0$ only exists within the gas disk. From Eq. (61) the $M_0$ term leads to a flatten rotational curve at sufficiently large $r$.

However, taking into account Eq. (59) Eq. (61) becomes

$$v_0^2 = \frac{2GM_*}{r} + 2GM_0$$  \hspace{1cm} (62)$$

where there is also a factor of 2 for $M_0$ since the high eccentricity of the collapsing outer shells also applies to the flat disk, where the gas mass profile simply has a different form $M(r) = M_0r$. In this case, for the baryonic scaling model observations [46], the M/L requirement is reduced by a half to 2.5 $\sim$ 7 (or an average of 4) which is a desirable value.

To compare the analysis above with rotational curve observations, For comparison purposes, we take a reasonable mass estimate $M_{16kpc} = 1.9 \times 10^{11} M_\odot$ to match rotational curve data in [57] based on Eq. (57), where we have

$$v_{16kpc} = 321.0 km/s, \quad v_{obs} = 321.4(\pm 25.27) km/s.$$ \hspace{1cm} (63)

where $v_r$ is calculated rotational speed at $r$ and $v_{obs}$ is rotational speed from observation. At distances greater than $2r_0$ (ignoring the effect of additional mass density at larger scales) we obtain a good match as follows,

$$v_{28kpc} = 245 km/s, \quad v_{obs} = 238.0(\pm 1.54) km/s,$$ \hspace{1cm} (64)

$$v_{50kpc} = 183.6 km/s, \quad v_{obs} = 178.5(\pm 17.6) km/s.$$ \hspace{1cm} (65)

The above results are depicted in Fig. 1. in which we observe that Eq. (62) provides a good match to the rotational curves for the range $16kpc \sim 50kpc$ ($2r_0 \sim 6r_0$) where gas surface density is high. Although we know that most disks are not maximal, a question remains as to "why assuming a maximum M/L ratio for disk works so well?", that is if one can raise the mass of the central disk, a Newtonian near disk curve suffices to match observations. The above match between Eq. (62) and observation provides an answer to this question. (Outside $50kpc$ the surface density is much lower. The velocities have large uncertainty at $9r_0 \sim 10r_0$, but these velocities could indicate the existence of a mass profile of the HI from Eq. (62). In Fig. 1, similar plot including the VMOND term shows that the VMOND’s direct gravitational effect is insignificant at these scales.)

Accretion and radial flow:

The basic scenario is that there is a layer $2r_0 \sim 5r_0$ of high angular speed gas circling the stellar disk and outside this layer there is another (low metallicity) HI accretion gas layer (called hot Corona) at scales $5r_0 \sim 10r_0$ with lower angular speed with a higher temperature.

Following Pitts and Taylor [58]-[59] one defines the "effective accretion rate" surface density as

$$\dot{\Sigma}_{eff}(t,r) = \frac{\partial}{\partial t} \left( \dot{\Sigma}_s(t,r) + \dot{\Sigma}_g(t,r) \right)$$ \hspace{1cm} (66)

which is the amount of gas that is needed at a given time $t$ and a given radius $r$ in order to sustain a given structural evolution at late times, where $\dot{\Sigma}_s$ and $\dot{\Sigma}_g$ are surface density of the stellar disk and outer HI gas disk respectively. This fact is conveniently formalised by the equation of conservation of mass. If, in the evolution of the gas component, a sink term $\dot{\Sigma}_s$ and a source term $\dot{\Sigma}_{acc}$ are accounted for, to describe star formation and accretion, respectively, then the continuity equation can be written:

$$\dot{\Sigma}_{eff} = \dot{\Sigma}_{acc} - \frac{1}{2\pi r} \frac{\partial \mu}{\partial r}$$  \hspace{1cm} (67)$$

where

$$\dot{M} = \mu = 2\pi r \dot{\Sigma}_g u_r$$ \hspace{1cm} (68)$$

is the radial gaseous mass flux, with $u_r$ being the net radial velocity of the gas. The basic equations describing the dynamical consequences of accretion onto a rotating disk has been introduced in the context of galaxy evolution, by [60] and improved by Lacey and Fall [61] can be rewritten as

$$\mu = -2\pi \alpha r^2 \dot{\Sigma}_{acc}; \quad u_r = -r \frac{\dot{\Sigma}_{acc}}{\alpha \dot{\Sigma}_g}$$ \hspace{1cm} (69)$$
where
\[ \alpha = \frac{h_{\text{disk}} - h_{\text{acc}}}{r \frac{\partial h_{\text{disk}}}{\partial r}} \] (70)

There is a major simplification for a flat rotational curve between the disk and the accreting gas layer, one has
\[ h_{\text{disk}} = v_{\text{disk}} r; \quad h_{\text{acc}} = v_{\text{acc}} r, \quad \alpha = \frac{v_{\text{disk}} - v_{\text{acc}}}{v_{\text{disk}}} = 1 - \frac{v_{\text{acc}}}{v_{\text{disk}}} \] (71)

In [16] in order to produce the right chemical abundance profile for MW up to Corona scale, it is found that \( \alpha \) lies in the small range \( \alpha = 0.26 \sim 0.3 \).

The analysis in the last section provides a natural scenario of high angular speed gas outside stellar disk at scales \( r_0 \sim 5r_0 \). However the rotational curve beyond \( 5r_0 \) appears to support a potential of Eq. (60) with possible angular momentum taken from high rotational velocity region. Taking data from [57] for \( v_{\text{acc}} \) at \( r_{\text{acc}} = 8.5 \text{kpc} \) and \( v_{\text{d}} \) at \( r_{\text{d}} = 16 \text{kpc} \),
\[ \alpha = \frac{h_{\text{disk}} - h_{\text{acc}}}{h_{\text{disk}}} = \frac{h_{\text{disk}}^2 - h_{\text{acc}}^2}{h_{\text{disk}}(h_{\text{disk}} + h_{\text{acc}})} = \frac{2 h_{\text{acc}}^2 (1 - \frac{v_{\text{acc}}^2}{v_{\text{d}}^2})}{h_{\text{acc}}^2 (1 + \frac{v_{\text{acc}}^2}{v_{\text{d}}^2})} = 0.318 \pm 0.12. \] (72)

Although the observed velocity uncertainty is too large to confirm that \( \alpha \) does fall into the required range of \( 0.26 \sim 0.3 \), the above calculation demonstrates qualitatively how an out-flow can occur from \( r_0 \) toward \( 2r_0 \). At a constant rotational region, one expects the outflow radial velocity can be maintained. Based on observed rotational velocities differences, similar mechanism can exist around \( 6r_0 \sim 9r_0 \).

**SUMMARY AND CONCLUSION**

Recent observations suggest that the early time galactic rotational curve is consistent with Newtonian gravity and it gradually flattens as the galactic disk grows in mass over cosmological time. Using the metric in a previous work we explore its implication for galactic rotational curve. In a high density centralised mass the VMOND term is negligible comparing to the Newtonian potential so it is consistent with early time rotational curve. By following the idea that a large protogalaxy fragments into an early collapsing Mestel sphere with its outer cloud following a secondary infall, we find that VMOND potential enhances the gravitational acceleration when density perturbation \( \delta \) is much less than one at high redshift and that the initial sphere can have a much larger radius. The outer cloud which separates from the central sphere will continue to grow in density perturbation until it reaches \( O(1) \). The Infall starts when the outer mass shells reach their turnaround radii. In this case the outer gas density is diluted and the collapse time increases significantly. The large time secondary infall mass profile is similar to that of a flat Mestel disk, but the mass shells in this disk will collapse onto the stellar disk before equilibrium is attained. Due to the high eccentricity of the secondary infall mass shells, there is a factor of 2 added to the mass terms in the acceleration equation leading to a higher rotational speed. The late time mass profile of HI gas with a peak near \( 2r_0 \) and the M/L ratio improvement is consistent with the baryonic scaling model observations. The resulting rotational curve profile matches MW data.

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Fig. 1: Plot of the rotational speeds vs the distances from Milky Way centre, where the solid line is the results from Eq.(59) for mass at $0.76 \times 10^{12}$ kg, the dashed line is the Newtonian speeds and the dots with error bars are the observational results from Ref.57 Table 2. The dotted line is a MOND estimate at $v = 220$ km/s.