MODELING GRAVITATIONAL CLUSTERING WITHOUT COMPUTING GRAVITATIONAL FORCE

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Received 1996 February 16; accepted 1996 May 8

ABSTRACT

The large-scale structure in the universe is believed to have arisen out of small, random density perturbations generated in the very early universe that are amplified by gravity. Large and usually intricate N-body simulations are typically employed to model the complex nonlinear dynamics in a self-gravitating medium. We suggest a very simple model that predicts, on large scales, the correct density and velocity distributions. The model does not involve an explicit computation of the gravitational force. It is based on a simple transformation of the variables and local conservation laws of mass and generalized linear momentum. The model demonstrates that the overall appearance of large-scale structure in the universe is explicitly determined by the initial velocity field, and it reveals the most significant large-scale effects of gravity on the formation of structure.

Subject headings: cosmology: theory — large-scale structure of universe

1. INTRODUCTION

Understanding the formation of the large-scale structure (LSS) in the universe is one of the most important problems in modern cosmology (see, e.g., Zeldovich & Novikov 1983; Peebles 1993). On scales of galaxies, complex nonlinear physical processes like turbulence, cooling and heating of the gas, star formation, and supernova explosions determine the shape and other properties of objects. However, on the scale of superclusters, these processes become considerably less important, and it is reasonable to assume that gravity primarily shapes the structures. In addition, a variety of observational evidence supports the hypothesis that most of the mass in the universe is in the form of weakly interacting particles that feel only gravity. Therefore, a simple physical model assuming that the universe is filled with a cold dustlike gas interacting only gravitationally is most often used to study the LSS formation. The baryonic component plays no crucial role except on the scale of galaxies.

When the amplitude of the density fluctuations is small, their growth is adequately described by linear theory. However, as soon as the amplitude of the perturbations reaches the order of unity ($\sigma_s = \langle (\rho - \bar{\rho})^2 \rangle^{1/2} \approx 1$, where $\rho$ is the density field and $\bar{\rho}$ is its mean), the linear theory breaks down (see, e.g., Zeldovich & Novikov 1983; Shandarin & Zeldovich 1989). The most widespread method to deal with the complex dynamics at the nonlinear stage is to run N-body simulations, generating the initial conditions as a realization of a Gaussian random process (Klypin & Shandarin 1983). In N-body simulations of this type, the gravitational forces generated by the density distribution are calculated at each time step. The trajectory of every particle is integrated in a self-consistently varying gravitational field. Cosmological N-body simulations have played the most significant role in testing (and in most cases rejecting) the models for dark matter. Here we are interested in a different aspect of the problem of the LSS formation. We are trying to formulate simple, physical, macroscopic principles controlling the nonlinear stage of gravitational clustering.

Long ago, Zeldovich (1970) suggested a very elegant, analytic approximation to describe the beginning of the nonlinear stage in cosmological scenarios, assuming smooth initial conditions. Quantitatively, it can be expressed as a requirement that the initial power spectrum of density fluctuations have a steep cutoff on small scales [steeper than $P(k) \propto k^{-3}$]. Mathematically, the Zeldovich approximation (ZA) is a one-step mapping from Lagrangian space into Eulerian space at a time $t$, given by

$$r(q, t) = a(t)|q - D(t)\nabla\Phi(q)|,$$

where $r$ and $q$ are the Eulerian and Lagrangian coordinates, respectively, $a(t)$ is the scale factor describing the homogeneous expansion of the universe, $D(t)$ is a known function of time describing the growth of perturbations, and $\Phi(q)$ is the potential of the initial velocity field: $v_0 \propto -\nabla\Phi(q)$. With the aid of the above mapping, one can calculate the density at the final time $t$ by using the conservation of mass. However, cosmological observations almost certainly exclude scenarios having no perturbations on small scales. Small-scale power is required to explain the existence of quasars and galaxies at very high redshifts.

Two modifications—the truncated Zeldovich approximation (TZA) (Kofman et al. 1992; Coles, Melott, & Shandarin 1993) and the adhesion approximation (AA) (Gurbatov, Saichev, & Shandarin 1989)—have been suggested in an attempt to extend the scope of the ZA and to make it more useful for general cosmological scenarios (for a brief review see, e.g., Shandarin 1994, and for more exhaustive discussion of various approximations, see Sathyaprakash et al. 1995 and Sahni & Coles 1995). A comparison with gravitational N-body simulations shows that these two approximations fairly accurately describe nonlinear gravitational clustering (Melott, Shandarin, & Weinberg 1994; Sathyaprakash et al. 1995). Here we describe another approximation to the nonlinear gravitational
evolution of the density and velocity perturbations that is numerically almost as simple as the ZA but completely universal and much superior to any approximation method suggested so far. It consists of two elements:

1. A transformation of variables;
2. An assumption that in the dense regions the local gravitational interaction can be approximated by the diffusion of a generalized momentum (see below for a definition of the generalized momentum).

The latter assumption is similar to the AA, but in this model the generalized momentum is locally conserved.

2. CONSERVING MOMENTUM APPROXIMATION

It is well known that one can exclude many effects of the homogeneous and isotropic expansion of the universe by introducing the comoving coordinate $x = r/a(t)$ and peculiar velocity $v_p = u - H(t)r$, where $u$ is the physical velocity and $H(t) = \dot{a}/a$ is the Hubble parameter characterizing the rate of expansion of the universe. In addition, it is convenient to parameterize time by $D(t)$, which describes the growth of perturbations, and rescale the density and peculiar velocity (Gurbatov et al. 1989) by

$$\rho(x, t) = a^7 \eta[x, D(t)], \quad v_p(x, t) = (aD)\mathbf{v}[x, D(t)]. \quad (2)$$

In terms of the new variables, the dynamics of gravitational clustering is described by (1) the conservation of mass, (2) a dynamical equation analogous to the Euler equation

$$\frac{\partial v_i}{\partial D} + v_j \frac{\partial v_i}{\partial x_k} = \frac{3}{2} \frac{\Omega_0}{D(d \ln D/d \ln a)^2} \frac{\partial \varphi}{\partial x_i} + v_i, \quad (3)$$

where $\Omega_0 = 8\pi G \rho_c / 3H^2$ is the dimensionless mean density of the universe, and (3) an equation for the gravitational field $\varphi$ analogous to the Poisson equation (for derivations see, e.g., Shandarin & Zeldovich 1989).

As in the ZA, the conserving momentum approximation (COMA) assumes that the gravitational force is approximated by the velocity,

$$\frac{\partial \varphi}{\partial x_i} \approx u_i(x, D), \quad (4)$$

which sets the right-hand side of equation (3) to zero. Thus, on large scales, the model mimics the dynamics of a self-gravitating medium by the effectively kinematic model of a noninteracting medium.

However, on small scales the approximation assumes that particles collide inelastically, resulting in the exchange and diffusion of the generalized momentum defined as

$$p = \eta \mathbf{v} = \frac{a^7}{D} \rho \mathbf{v}_p, \quad (5)$$

which mimics the effects of gravity in dense regions (Shandarin & Zeldovich 1989).

In the AA, the right-hand side of equation (3) is supplemented by the viscosity term, resulting in Burgers’s equation,

$$\frac{\partial v_i}{\partial D} + v_j \frac{\partial v_i}{\partial x_k} = \nu \mathbf{v}^2 v_i. \quad (6)$$

In the AA, $\nu$ is assumed to be constant, which results in the violation of momentum conservation (Kofman & Shandarin 1990), and in this respect the model is not physical. The COMA assumes physical viscosity is described by an equation similar to the Navier-Stokes equation (here we ignore the second viscosity):

$$\frac{\partial v_i}{\partial D} + v_j \frac{\partial v_i}{\partial x_k} = \frac{1}{\eta} \frac{\partial}{\partial x_k} \left[ \mu \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \frac{\delta}{\delta x_k} \frac{\partial}{\partial x_i} \right) \right]. \quad (7)$$

where $\mu$ is the dynamical viscosity. The Navier-Stokes equation conserves the linear momentum automatically but does not have an analytic solution in the general case; it also may generate vorticity.

According to the ZA, in the single-stream flow regime each particle moves along a straight line with constant velocity. However, as soon as a particle enters a multistream flow region, its trajectory becomes very complex, resembling a random walk. Collectively this type of motion can be labeled as a sort of violent relaxation (Lynden-Bell 1967) and can be roughly approximated by the diffusion of the momentum. It is convenient to split the force exerted on the particle into two parts: the first component represents its interaction with the other particles of the clump of which it is a member, and the second represents its interaction with particles that belong to other clumps or voids. One can reasonably assume that the irregular component of the motion is mostly determined by the gravitational force induced by the particles from the same clump. If so, the momentum of the clump must be approximately conserved in interactions of this type. The change of the momentum of the clump is due to interaction with the surrounding matter, which can be roughly described by equation (4). In practice, the diffusion of momentum is important in all regions of high density (filaments and pancakes) and aids in preventing the formation of multistream flows, as in the AA. The major physical difference between the COMA and AA is that the COMA conserves momentum locally and the AA, based on Burgers’s equation, does not (instead, the latter conserves the velocity). We believe that this is an advantage of the COMA.

Another advantage of the COMA is mainly practical. The numerical code that realizes the model is extremely simple and efficient. It operates with the density and velocity distributions (no particles) on a cubic grid in the Eulerian space: $\eta = \eta(x, D)$ and $v = v(x, D)$. At each time step, it calculates the flows of mass and momentum from a mesh cell to its neighbors and computes the new density and velocity fields on the grid by using a cloud-in-cell algorithm (see, e.g., Hockney & Eastwood 1988), thereby explicitly conserving the mass and the generalized momentum (simulation of perfectly inelastic collisions). Thus one circumvents the problem of having to compute the gravitational force after each time step. A full description of the code, as well as the code itself, will be published elsewhere; here we list its major features. The algorithm that implements the new approximation is

- **Universal**, in terms of the initial and boundary conditions as well as the shape of the computational box, provided that the initial velocity field is somehow generated;
- **Extremely simple**, both conceptually and practically;
- **Very economical** in terms of memory (just four arrays, each as big as the size of the box, are sufficient to implement the algorithm);
- **Very fast**;
- **Local**, therefore 100% parallelizable.

Generalization of the algorithm to deal with particles rather than densities on the grid, as well as introducing partially
Inelastic collisions, is straightforward but would require additional memory and slightly slow down the code.

The only disadvantage of the COMA (compared to the AA) is that it does not have an analytic solution.

3. COMPARISON WITH N-BODY SIMULATIONS

In order to test the strength of our approximate model, we compared it with gravitational N-body simulations. The N-body simulations were performed using 128³ particles on a 128³ mesh with periodic boundary conditions. The power spectrum of primordial density fluctuations is assumed to be a simple power law $|\delta_k|^2 \propto k^n$, where $\delta_k$ is the Fourier transform of the density contrast $\delta$ covering the range $n = +1, 0, -1, -2$. The initial fluctuations are evolved gravitationally, and comparisons are made when different scales, determined by the parameter $k_{nl}$ [defined by the equation $\sigma_8^2 = a^{-2} \int_0^{\infty} P(k) d^3k = 1$], go nonlinear: $k_{nl} = 64, 32, 16, 8,$ and $4$ (in units of the fundamental frequency $k_f = 2\pi/L$, where $L$ is the size of the box). For a detailed discussion of the N-body simulations, see Melott & Shandarin (1993). At each epoch, the density field is computed on a reduced grid of size 64³ by a cloud-in-cell method.

We ran the model code with identical initial conditions and compared its results with those of N-body simulations at different stages. Here we illustrate this comparison for three different values of the power-law index of the spectrum of fluctuations, namely, $n = 0, -1,$ and $-2$, and at two stages, $k_{nl} = 8$ and $4$. Although it is unlikely that such power-law models can explain the real universe, they serve as toy models to understand generic features of gravitational instability. The following normalizations provide a rough idea of the scales involved and how such toy models may relate to the real universe. One can view the density fields corresponding to different stages in the evolution of such power-law density fluctuations as equivalent to the density distribution at the present epoch, but obtained after smoothing with a top-hat filter of progressively smaller radii $R_{TH}$, with smaller smoothing lengths corresponding to later epochs. For instance, the epochs $k_{nl} = 8$ and $4$ correspond to smoothing radii $R_{TH} \approx 2$ and $1\ h^{-1}\ Mpc$, respectively, within volumes $(200\ h^{-1}\ Mpc)^3$ and $(100\ h^{-1}\ Mpc)^3$.

In Figure 1, we compare the N-body density distributions (open circles) with those obtained using the COMA (crosses) in a typical two-dimensional slice, one mesh size thick, in the $n = -1$ case, when the scale of nonlinearity corresponds to $k_{nl} = 8$. The panels show sites of density $\rho$ larger than a certain threshold ($\rho_{true}$). Filled circles are sites where both the N-body and model densities are larger than the threshold. One can see that there is generally a good overall agreement between the two distributions at nonlinear but not extremely high density thresholds. However, the COMA shows a rather smoother distribution, and the density peaks that it produces are somewhat lower, especially at higher densities. This behavior is not unexpected: local nonlinear gravitational effects, ignored by the COMA, make the density distribution more clumpy, and the clumps are more compact. A detailed comparison shows that the vast majority of density peaks in the N-body simulations have their counterparts in the COMA,
We conclude that the proposed approximation mimics the large-scale gravitational evolution at late nonlinear stages quite well (positions of clumps, filaments, pancakes)—better than any other known approximation. This implies that two simple assumptions,

1. The generalized gravitational force on large scales \( (l \approx k_{nl}^{-1}) \) equals the velocity, \( \varphi / \Delta x_i \approx -v_i(x, D) \); 
2. The conservation of mass and the local \( (l \lesssim k_{nl}^{-1}) \) conservation of the generalized momentum, \( p = \eta \varphi \); explain fairly well the nonlinear gravitational clustering on large scales.

In a very general sense, the COMA falls in the class of the sticky-particle methods used in numerical hydrodynamics and also resembles the lattice gas models used in modeling turbulence and similar phenomena. An interesting question is whether the COMA guarantees complete hierarchical clustering or not. Because of numerical errors, some small clumps may miss merging with the larger ones, passing by without collision. We have not seen this phenomena in our simulations and believe that it should not be a serious problem for the method.

We believe that, after a thorough testing, the approximation can be a practical tool for cosmological studies of large-scale processes that do not require a resolution better than a few megaparsecs, such as large-scale streaming velocities, spatial distribution of rich clusters of galaxies, and statistics of voids. In such low-resolution calculations, the COMA can be more efficient than \( N \)-body simulations if very large volumes and large statistical ensembles are required.

We are grateful to Lev Kofman, Dmitri Pogosyan, Varun Sahni, and Capp Yess for discussions. Acknowledgments are due to the Smithsonian Institution, Washington, for international travel assistance under the ongoing Indo-US exchange program at IUCAA, Pune, India. S. S. acknowledges financial support from NASA grant NAGW-3832, NSF grant AST 90-21414, and University of Kansas grant GRF 96.

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