COHOMOLOGY AND OBSTRUCTIONS II:
CURVES ON $K$-TRIVIAL THREEFOLDS

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Abstract. On a threefold with trivial canonical bundle, Kuranishi theory
gives an algebro-geometry construction of the (local analytic) Hilbert scheme
curves at a smooth holomorphic curve as a gradient scheme, that is, the
zero-scheme of the exterior derivative of a holomorphic function on a (finite-
dimensional) polydisk. (The corresponding fact in an infinite dimensional set-
ting was long ago discovered by physicists.) An analogous algebro-geometric
construction for the holomorphic Chern-Simons functional is presented giving
the local analytic moduli scheme of a vector bundle. An analogous gradient
scheme construction for Brill-Noether loci on ample divisors is also given. Fi-
nally, using a structure theorem of Donagi-Markman, we present a new for-
mulation of the Abel-Jacobi mapping into the intermediate Jacobian of a threefold
with trivial canonical bundle.

1. Introduction

1.1. The problem. This paper computes the local analytic deformation theory of
closest related objects on a $K$-trivial threefold $X_0$. The three are:
1) Deformations of pairs $(X_0, Y_0)$ where $Y_0$ is a smooth curve in $X_0$.
2) Deformations of pairs $(X_0, E_0)$ where $E_0$ is a holomorphic vector bundle on
$X_0$.
3) Deformations of triples $(X_0, S_0, L_0)$ where $S$ is a smooth very ample divisor
on $X_0$ and $L$ is a line bundle on $S_0$ whose Chern class is an algebraic 1-cycle on $S_0$
which goes to zero in $H_2(X_0; \mathbb{Z})$. In this case we make the additional assumption
that
$$h^1(\mathcal{O}_{X_0}) = h^2(\mathcal{O}_{X_0}) = 0.$$ 

The special properties of all these local analytic deformation schemes in the
case of $K$-trivial threefolds derive from the (Serre) duality between the first-order
deformations for fixed $K$-trivial threefold $X_0$, given by
$$\text{Ext}^1(A, A),$$
and the obstruction space to extension to higher orders, given by
$$\text{Ext}^2(A, A).$$

This duality is given by the natural pairing
$$\text{Ext}^1(A, A) \times \text{Ext}^2(A, A) \to \text{Ext}^3(A, A)$$
coupled with a trace map
$$\text{Ext}^3(A, A) \to H^3(\mathcal{O}_{X_0}) = \mathbb{C}.$$
This point of view is that of, for example, [KS]. In case 1), for the inclusion map
\[ i : Y_0 \to X_0 \]
apply
\[ R\text{hom}_{\mathcal{O}_{X_0}} (\cdot, i_*\mathcal{O}_{Y_0}) \]
to the exact sequence
\[ 0 \to \mathcal{I}_{Y_0} \to \mathcal{O}_{X_0} \to i_*\mathcal{O}_{Y_0} \to 0 \]
to obtain
\[ \text{Ext}^1 (i_*\mathcal{O}_{Y_0}, i_*\mathcal{O}_{Y_0}) = \mathcal{N}_{Y_0 \setminus X_0} \] .

Apply
\[ R\text{hom}_{\mathcal{O}_{X_0}} (i_*\mathcal{O}_{Y_0}, \cdot) \]
to the exact sequence
\[ 0 \to \mathcal{I}_{Y_0} \to \mathcal{O}_{X_0} \to i_*\mathcal{O}_{Y_0} \to 0 \]
to obtain a surjection
\[ \omega_{Y_0} \to \text{Ext}^2 (i_*\mathcal{O}_{Y_0}, i_*\mathcal{O}_{Y_0}) . \]

Since a Koszul resolution shows that \( \text{Ext}^2 (i_*\mathcal{O}_{Y_0}, i_*\mathcal{O}_{Y_0}) \) is locally of rank 1 we conclude
\[ \text{Ext}^2 (i_*\mathcal{O}_{Y_0}, i_*\mathcal{O}_{Y_0}) = \omega_{Y_0} . \]

So
\[ A = i_*\mathcal{O}_{Y_0} . \]

In case 2),
\[ A = E_0 \]
and the trace map
\[ H^3 (\text{End} (E_0)) \to H^3 (\mathcal{O}_{X_0}) \]
is the obvious one.

Finally, in case 3), let
\[ j : S_0 \to X_0 \]
denote the inclusion and put
\[ A = j_*L_0 . \]

Now
\[ R\text{hom}_{\mathcal{O}_{X_0}} (j_*L_0, j_*L_0) = R\text{hom}_{\mathcal{O}_{X_0}} (j_*\mathcal{O}_S, j_*\mathcal{O}_S) \]
and, as above,
\[ \text{Ext}^1 (j_*\mathcal{O}_S, j_*\mathcal{O}_S) = \mathcal{N}_{S \setminus X_0} . \]

Also
\[ \text{hom} (j_*L, j_*L) = j_*\mathcal{O}_S . \]

The behavior of \( L_0 \) is reflected at the \( E_2 \)-term of the local-to-global spectral sequence, namely
\[ E^{0,1}_{2} = H^0 (\text{Ext}^1 (j_*L_0, j_*L_0)) = H^0 (\mathcal{N}_{S \setminus X_0}) \]
\[ E^{2,0}_{2} = H^2 (\text{hom} (j_*L_0, j_*L_0)) = H^2 (\mathcal{O}_S) , \]
and, letting \( \nabla \) denote the Gauss-Manin connection,
\[ d_2 : H^0 (\mathcal{N}_{S \setminus X_0}) \to H^2 (\mathcal{O}_S) \]
\[ \zeta \mapsto \nabla_{\zeta} (c_1 (L_0)) . \]

For more details of this computation (applied to a special case), see the Appendix of [KS].
1.2. **The setting for curves on $K$-trivial threefolds.** Let

$$X_0$$

be a smooth $K$-trivial Kähler threefold. The deformations of $X_0$ are unobstructed (see, for example, [3]). We let

$$(1) \quad s : X \to X'$$

be a versal deformation of $X_0$ over an analytic polydisk $X'$, that is, the natural map

$$T_{X_0} \to H^1(T_{X_0})$$

is an isomorphism. Let

$$F^3 H^3 = H^{3,0}(X/X') \oplus \ldots \oplus H^{3-j}(X/X') \subseteq H^3(X/X'; \mathbb{C})$$

denote the Hodge filtration and let

$$n' := \dim X' + 1 = h^0(\Omega^3_{X_0}) + h^1(\Omega^2_{X_0}).$$

Let $Y_0 \subseteq X_0$ be a smooth irreducible curve which is *homologically ample*. By this last we mean that there is a family $L/X'$ of smooth curves in $X/X'$ such that $L_0$ is disjoint from $Y_0$ and

$$\{Y_0\} \equiv \{rL_0\} \in H_2(X_0; \mathbb{Z})$$

for some $r \in \mathbb{Q}$. We let

$$Y'$$

denote an analytic neighborhood of $\{Y_0\}$ in the relative Hilbert scheme of (proper) curves in $X/X'$. Let

$$p : Y \to Y'$$

denote the universal curve and

$$\pi : Y' \to X'$$

the induced map. Then, since the tangent space to the deformation space of the pair $(X_0, Y_0)$ is

$$H^1(T_{X_0} \to N_{Y_0/X_0}),$$

which sits in the exact sequence

$$(2) \quad 0 \to H^0(N_{Y_0/X_0}) \to H^1(T_{X_0} \to N_{Y_0/X_0}) \to H^1(T_{X_0}),$$

$Y'$ can be realized as a closed analytic subscheme of an analytic polydisk $U'$ of dimension equal to

$$n' + h^0(N_{Y_0/X_0})$$

for which there is a smooth (surjective) morphism

$$U' \to X'$$

extending $\pi$ above. Abusing notation we denote this extension again as $\pi$. Thus we have an exact sequence

$$0 \to T_{Y'} \to T_{U'} \to \pi^* T_{X'} \to 0.$$
By (2) we have a natural inclusion of sequences
\[ 0 \to H^0(N_{Y_0}\setminus X_0) \to \mathbb{H}^1(T_{X_0} \to N_{Y_0}\setminus X_0) \to H^1(T_{X_0}) \]
\[ 0 \to T_{\pi|_{\{(Y_0),\{X_0\})}} \to \pi^*T_{U'}|_{\{(Y_0),\{X_0\})} \to \pi^*T_{X'}|_{\{(Y_0),\{X_0\})} \to 0 \]
such that \( \lambda \) and \( \mu \) are both isomorphisms.

1.3. **Choice of a 3-form.** Finally we let
\[ \tilde{X}' \]
denote the analytic manifold obtained by removing the zero-section from the total space on the analytic line bundle
\[ s_*\Omega^3_{X/X'} \]
on \( X' \) and use ‘tilde’ to denote base extension by \( \tilde{X}'/X' \), that is,
\[ \tilde{X} = X \times_{X'} \tilde{X}' \]
\[ \tilde{Y}' = Y' \times_{X'} \tilde{X}' \]
\[ \tilde{Y} = Y \times_{X'} \tilde{X}' \]
\[ \tilde{\pi} : \tilde{U}' \to \tilde{X}' \]

1.4. **The “potential function” \( \Phi \).** Our first main goal in this paper is to construct (shrinking \( X' \) and \( U' \) as necessary) a holomorphic function
\[ \Phi \]
on \( \tilde{U}' \) such that, with respect to the exact sequence,
\[ 0 \to \tilde{\pi}^*\Omega^1_{\tilde{X}'} \to \Omega^1_{\tilde{U}'} \to \Omega^1_{\tilde{U}'/\tilde{X}'} \to 0, \]
we have:

**Property 1:** The relative Hilbert scheme \( \tilde{Y}' \), considered as an analytic sub-scheme of \( \tilde{U}' \) is the zero-scheme of the section
\[ d\tilde{U}'/\tilde{X}', \Phi \]
of
\[ \Omega^1_{\tilde{U}'/\tilde{X}'}. \]

**Property 2:** Under a natural isomorphism
\[ F^2H^3\left( \tilde{X}/\tilde{X}' \right) \cong T_{\tilde{X}'}, \]
\[ \Omega^1_{\tilde{X}'} \cong \left( F^2H^3\left( \tilde{X}/\tilde{X}' \right) \right)^\vee \]
given by Donagi-Markman (see §4.1 below or §1 of [V]), the section
\[ d\Phi|_{\tilde{Y}'}, \]
of
\[ \tilde{\pi}^*\Omega^1_{\tilde{X}'}, \]
is the normal function
\[ \int^{Y'/Y'}_{r(L \times_{X'} Y')/Y'} : \tilde{Y}' \to \left( F^2H^3\left( \tilde{X}/\tilde{X}' \right) \right)^\vee. \]
An immediate corollary of Property 2 is that the image of \( \partial \Phi \mid \tilde{Y} \), under the natural map

\[
\left( F^2 H^3 \left\{ \tilde{U} \times_{X'} X' \right\} \right)^\vee \to \left( F^2 H^3 \left( \tilde{X} / \tilde{X}' \right) \right)^\vee
\]

is Lagrangian with respect to the symplectic structure of Donagi-Markman (again see below or §1 of [V]).

1.5. Holomorphic Chern-Simons and Brill-Noether theory. Reacting to a preliminary version of the above result for deformations of \((Y_0, X_0)\), both Richard Thomas and Claire Voisin saw wider settings in which analogous results were true. As Thomas pointed out, an analogous theorem, proved below, must hold for deformations of \((E_0, X_0)\) where the “holomorphic Chern-Simons functional” plays the role of the potential function. Additionally, at the author’s invitation, Richard Thomas authored an additional chapter for this paper explaining the direct link between the deformation problem for \((Y_0, X_0)\) and that for \((E_0, X_0)\). This is accomplished through an analogue of Abel’s theorem, as anticipated in [T]. Finally Claire Voisin authored an final chapter establishing the analogous results for deformations of \((X_0, S, L)\), the Brill-Noether loci for hypersurface sections of a Calabi-Yau threefold.

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2. The trivialization \( F \)

Proposition 2.1. Let \((0, 0) \in U'\) represent the basepoint \((\{Y_0\}, \{X_0\})\). Shrinking \( U' \) as necessary, there is a \( C^\infty \)-isomorphism

\[
F : (U' \times_{X'} X') / U' (\sigma, \text{ident}_U) \to X_0 \times U'
\]

such that

i) \( F \mid_{\pi^{-1}(0,0)} = \text{identity}_{X_0} \),

ii) \( \sigma^{-1}(x_0) \)

is an analytic submanifold for all \( x_0 \in X_0 \),

iii) \( \sigma^{-1}(Y_0) \supset Y \)

iv) \( \sigma^{-1}(L_0) = L \times_{X'} U' \).
Proof: The proof is just as in Theorem 13.1 of [C], using a patching argument on local data to construct a convergent $C^\infty$-isomorphism
\[(U' \times X', X)/U' \to X_0 \times U',\]
satisfying i)-iv).

As in Part One of [C], the deformation of $\mathcal{O}$ determined by the trivialization $F$ is given by an operator
\[\mathcal{O} - L\xi\]
on $X_0 \times U'$ where
\[\xi \in A^{0,1}_{X_0} (T_{X_0}) \otimes \mathbb{C}[[U']] .\]
In fact, by Appendix A of [C], the trivialization has the additional property that $\xi$ is convergent, that is, $\xi$ is given by a well-defined holomorphic mapping
\[\xi : U' \to A^{0,1}_{X_0} \left( T^{1,0}_{X_0} \right) \]
comprised of pointwise holomorphic mappings
\[\xi (x) : U' \to \left( T^{0,1}_{X_0} \right)_{|x} \otimes \left( T^{1,0}_{X_0} \right)_{|x} \]
for each $x \in X_0$. By construction,
\[\xi|_{Y_0 \times U'} \in A^{0,1}_{Y_0} (T_{Y_0}) \otimes \mathcal{O}_{Y'}, \]
\[\xi|_{L_0 \times U'} \in A^{0,1}_{L_0} (T_{L_0}) \otimes \mathcal{O}_{Y'} .\]
An important point to notice is that, if $I_{Y'}$ denotes the ideal of $Y'$ in $U'$ and $m$ is the (maximal) ideal of $(0, 0) \in U'$, then
\[\xi|_{Y_0 \times U'} \in A^{0,1}_{Y_0} (T_{Y_0}) \otimes \mathcal{O}_{Y'},\]
gives the element in
\[H^1 \left( N_{Y_0 \setminus X_0} \right) \otimes \frac{I_{Y'}}{m \cdot I_{Y'}},\]
which is the obstruction to extending the family $Y/Y'$. (See [C], §11.) Furthermore $F$ determines a $C^\infty$-mapping
\[f : Y_0 \times U' \to X \times X', U'\]
by the rule
\[f (u') = F^{-1}|_{Y_0 \times \{u'\}} .\]
We denote the family of $C^\infty$-deformations of $Y_0$ as
\[U/U',\]
with fibers $U_{u'}$ which are no longer algebraic curves but simply smooth 2-real-dimensional manifolds when $u' \notin Y'$. 
3. Hodge theory in terms of the trivialization $F$

3.1. The holomorphic 3-form on $X/X'$. Let $\eta$ be an everywhere non-zero global section of

$$F^3H^3(X/X').$$

Abusing notation we shall continue to denote by $\eta$ the pullback of this form to

$$F^3H^3((U' \times_{X'} X)/U').$$

In Appendix A of [C] the trivialization $F : (U' \times_{X'} X)/U' \to X_0 \times U'$

in Proposition 2.1 is constructed the property that

$$(F^{-1})^* (\eta)$$

is a holomorphic family of 3-forms on $X_0$, that is, is given by a holomorphic mapping from $U'$ into the (infinite-dimensional) complex vector space

$$A^3_{X_0}$$

of global $C^\infty$-three-forms on $X_0$. Thus, if $\Gamma$ is a fixed 3-chain in $X_0$, then by differentiating under the integral sign,

$$\int_{\Gamma} (F^{-1})^* (\eta)$$

is a holomorphic function on $U'$.

3.2. Abel-Jacobi map. As in [C], we define for the trivialization given by Proposition 2.1 the “Hodge spaces”

$$K^{p,q} = H^q \left( A^{p,*}_{X_0}, \mathcal{J}_{X_0} - L^1_{\xi} \right).$$

Then in [C], Lemma 8.2, it is shown that the $C^\infty$-isomorphism $F$ induces the formal correspondence

$$H^q \left( \Omega^p_{(X \times X')/U'} \right) \leftrightarrow e^\langle \xi \rangle K^{p,q}.$$

Here it is important to note that this correspondence is to be understood modulo $u'$ where $u'$ is a system of holomorphic coordinates on $U'$ centered at our given basepoint $(\{Y_0\}, \{X_0\})$—indeed, for $q > 0$, the Hodge spaces $H^q \left( \Omega^p_{(X \times X')/U'} \right)$ do not vary holomorphically in $u'$. So writing

$$\eta_0 = \eta|_{X_0}$$

there is an element

$$\beta \in A^3_{x_0} \otimes \mathfrak{m} \cdot \mathbb{C}[[U']]$$

such that

$$(\ref{eq:6}) \quad \eta_0 + \beta \in K^{3,0}$$

and, modulo $u'$,

$$\eta = F^* \left( e^\langle \xi \rangle (\eta_0 + \beta) \right).$$

Also we can rewrite the pull-back of the (relative) intermediate Jacobian

$$\mathcal{J}(X/X') = \frac{(F^2H^3(X/X'))^\vee}{H_3(X/X'; \mathbb{Z})}$$
to a complex torus bundle over $U'$ as

$$J((X \times X', U')/U') = \frac{\text{Hom}_{O_{\tilde{X}'}}(e^{\langle \xi \rangle} (K^{3,0} \oplus K^{2,1}), O_{\tilde{X}'})}{H_3(X_0; \mathbb{Z})}.$$ 

In this last formulation the normal function associated to the cycle $Y'/Y' - L \times \tilde{Y}'$ is given by the restriction to $Y' \subseteq (U' \times X')$ of the (formal) function

$$\int_{rL_0}^{Y_0} : e^{\langle \xi \rangle} (K^{3,0} \oplus K^{2,1}) \to \mathbb{C}. \quad (7)$$

Let $\Gamma$ be a rational 3-chain such that $\partial \Gamma = Y_0 - rL_0$.

So for our choice of $\eta$ above the holomorphic function $\int_{\Gamma} (F^{-1})^* \eta$ can be rewritten as

$$\int_{\Gamma} \left( e^{\langle \xi \rangle} (\eta_0 + \beta) \right). \quad (8)$$

4. The ‘potential function’ $\Phi$ for curves on $K$-trivial threefolds

4.1. The tautological section. Following [DM] (see §1 of [V]), $F^3H^3(\tilde{X}/\tilde{X}')$ has a tautologial section

$$\tau : (x', \omega_{x'}) \mapsto \omega_{x'}$$

and, following Donagi-Markman [DM], the Gauss-Manin connection $\nabla$ induces a holomorphic bundle isomorphism

$$\nabla : T_{\tilde{X}'} \to F^2H^3(\tilde{X}/\tilde{X}')$$

$$\zeta \mapsto \nabla_{\zeta} \tau \quad (9)$$

which restricts to an isomorphism

$$T_{\tilde{X}'/X} \to F^3H^3(\tilde{X}/\tilde{X}').$$

Since $\nabla$ is a real operator, we extend (9) to a real isomorphism

$$T_{\tilde{X}'/X} \oplus T_{\tilde{X}'} \to F^2H^3(\tilde{X}/\tilde{X}') \oplus F^2H^3(\tilde{X}/\tilde{X}') = H^3(\tilde{X}/\tilde{X}'; \mathbb{C}) \quad (10)$$

which restricts to

$$T_{\tilde{X}'}(\mathbb{R}) \to H^3(\tilde{X}/\tilde{X}'; \mathbb{R}).$$

4.2. Distinguished coordinates for $\tilde{X}'$. The isomorphism (10) implies that, for our non-zero holomorphic section $\eta_0$ of $F^3H^3(X_0)$, the maps

$$\tilde{x}_i = \left( \int_{\gamma_i} \tau - \int_{\gamma_i} \eta_0 \right) \quad (11)$$

give distinguished local coordinates for $\tilde{X}'$ centered at $(0, \eta_0)$. If

$$\left\{ {\omega}_i = p^{F^2H^3(\tilde{X}/\tilde{X}')} (\gamma^i) \right\}_{i=1,...,n'}.$$
denotes the framing of $F^2H^3\left(\tilde{X}/\tilde{X}'\right)$ dual to $\{\gamma_i\}_{i=1,...,n'}$, then we have
\[
\omega_i = \frac{\nabla \tau}{\partial \tilde{x}_i}.
\]
since
\[
\int_{\gamma_i} \frac{\nabla \tau}{\partial \tilde{x}_i'} = \frac{\partial \tilde{x}_i}{\partial \tilde{x}_i'} = \delta_{ii}'.
\]
Thus under the isomorphism
\[
(F^2H^3\left(\tilde{X}/\tilde{X}'\right))^\vee \cong \Omega^1_{\tilde{X}'}.
\]
induced by (9) we have
\[
\int_{\gamma_i} \leftrightarrow d\tilde{x}_i.
\]

4.3. Definition of $\Phi$. In what follows we will wish to study the function
\[
\Phi : \tilde{U}' \to \mathbb{C}
\]
\[
(\tilde{u}') \mapsto \left(\int_{U'} F^{-1}(L_0) \cap X_{\tilde{u}'} \tau\right).
\]
For $\eta$ as above, denote
\[
q = \frac{\tau}{\eta}.
\]
Then (14) can be rewritten as
\[
\int_{\Gamma} (F^{-1})^* \tau,
\]
which, by (8), yields
\[
\Phi = q \int_{\Gamma} e^{\xi} \cdot (\eta_0 + \beta)
\]
The function (14) at $(0, 0, \eta_0) \in \tilde{U}'$ can be thought of first as a formal power series in $\tilde{u}'$ and $\overline{\tilde{u}}'$, where as above $\tilde{u}'$ is a set of holomorphic coordinates on $\tilde{U}'$, then as an equivalence class modulo the ideal generated by the anti-holomorphic coordinates $\overline{\tilde{u}}'$ (see [C]). Notice that $\tilde{X}'$ is a $\mathbb{C}^*$-bundle over $X'$. If we denote the (Euler) vector field which is the derivative of the $\mathbb{C}^*$-action as $\chi$, then by definition, $\Phi$ is a function on the pull-back this bundle to $U'$ satisfying $\chi(\Phi) = \Phi$.

That is, $\Phi$ is a section of the dual bundle of $\tilde{U}'/U'$. Let $\Psi$ denote the composite function
\[
F^2H^3\left(\tilde{X}/\tilde{X}'\right)^\vee \to F^3H^3\left(\tilde{X}/\tilde{X}'\right)^\vee \to \mathbb{C}
\]
induced by the inclusion $F^3H^3\left(\tilde{X}/\tilde{X}'\right) \subseteq F^2H^3\left(\tilde{X}/\tilde{X}'\right)$. If we let

$$\varphi = \int_{\sigma^{-1}(\mathcal{L}_0) \cap X_w}^{\mathcal{U}_w} \in F^2H^3(X/X')^\vee$$

then

$$\Phi = \Psi \circ \varphi.$$  

Notice that $\varphi$ is an extension to $U'$ of the Abel-Jacobi map

$$\int_{Y/Y'}^{Y'/Y'}$$

on $Y'$.

Another way to write the function $\Phi$ is as follows. Use the trivialization $F$ to rewrite the operator

$$\varphi = \sum_{i=1}^{n'} \left( \int_{\sigma^{-1}(\mathcal{L}) \cap U'}^{\mathcal{U}_w} \frac{\nabla \tau}{\partial \tilde{x}_i} \right) \int_{\gamma_i} = \sum_{i=1}^{n'} \left( \int_{\Gamma} \left( (F^{-1})^* \omega_i \right) \right) \int_{\gamma_i}.$$  

Furthermore, referring to (11) and (12), we define the coordinates

$$(\tilde{x}_i), (w_i) = \left( \sum_{i=1}^{n'} w_i d\tilde{x}_i \right)$$

for $\left( F^2H^3\left(\tilde{X}/\tilde{X}'\right)\right)^\vee$. In terms of these coordinates, the extension

$$\varphi = \int_{\sigma^{-1}(\mathcal{L}) \cap U'}^{\mathcal{U}_w} = \sum_{i=1}^{n'} \left( \int_{\Gamma} \left( (F^{-1})^* \omega_i \right) \right) \int_{\gamma_i}$$

of the Abel-Jacobi map on $\tilde{Y}'$ is given by

$$w_i = \int_{\Gamma} \frac{\partial \left( qe^{(x)} (\eta_0 + \beta) \right)}{\partial \tilde{x}_i}.$$  

5. **LAGRANGIAN SUBMANIFOLD**

Choose local coordinates $u' = (\{z_i\}, x')$ for $U'$ where $x'$ is the pullbacks to $U'$ of a coordinate system on $X'$. If we denote by $\tilde{x}'_i$ the pullbacks to $\tilde{U}'$ of the distinguished coordinate system $\{\tilde{x}_i\}$ for $\tilde{X}'$ defined above, then

$$\tilde{u}' = (\{z_i\}, \{\tilde{x}_i\})$$

gives a coordinate system on $\tilde{U}'$. Suppose the coordinate system $\tilde{u}'$.is centered at $(0, 0, \eta_0)$. We consider the 1-form

$$\Theta \in H^0\left( \Omega^1_0(F^2H^3(\tilde{X}/\tilde{X}'))^\vee \right)$$

induced by (12) and the natural inclusion

$$\tilde{\pi}_* \Omega^1_{\tilde{X}'} \subseteq \Omega^1_{T_{\tilde{X}'}^{\vee}}$$

where

$$\tilde{\pi} : T_{\tilde{X}'}^{\vee} \to \tilde{X}'$$
is the natural projection map on the geometric cotangent bundle. Said otherwise, under (12) we have

\[ d\bar{x}_i = \int_{\gamma_i} \nabla \tau \to \left( \int_{\gamma_i} F^2 H^3 \left( \bar{X}/\bar{X}' \right) \to \mathbb{C} \right) \in H^0 \left( \left( F^2 H^3 \left( \bar{X}/\bar{X}' \right) \right)^\vee \right). \]

So the one-form on \( T_{\bar{X}'} \), whose value at the point \( (\bar{x}_i, \left( \sum_{i=1}^{n'} w_i d\bar{x}_i \right) \) is \( \sum_{i=1}^{n'} w_i (\bar{\pi}^* d\bar{x}_i) \) corresponds under (12) to the one-form \( \Theta \) whose value at the point \( (\bar{x}_i, \left( \sum_{i=1}^{n'} w_i \int_{\gamma_i} \right) \) is \( \sum_{i=1}^{n'} w_i (\bar{\rho}^* d\bar{x}_i) \) where

\( \bar{\rho} : \left( F^2 H^3 \left( \bar{X}/\bar{X}' \right) \right)^\vee \to \bar{X}'. \)

Then

\[ d\Theta = \sum_{i=1}^{n'} dw_i (\bar{\rho}^* d\bar{x}_i) \]

descends to give a canonical symplectic structure on the bundle \( J \left( \bar{X}/\bar{X}' \right) \) of intermediate Jacobians. In what follows, we will use the notation \( \Theta \) and \( d\Psi \) (see (17) for both the one-forms on \( \left( F^2 H^3 \left( \bar{X}/\bar{X}' \right) \right)^\vee \) and their pull-backs to the product \( U' \times \bar{X}' \)).

**Theorem 5.1.** i) The normal function associated to the family of curves \( (Y - r (L \times \bar{X}', Y')) / Y' \) is the restriction to \( \bar{Y}' \) of the holomorphic mapping \( \varphi : \bar{U}' \to \left( F^2 H^3 \left( \bar{X}/\bar{X}' \right) \right)^\vee \) (18). Under the Donagi-Markman isomorphism

\[ \left( F^2 H^3 \left( \bar{X}/\bar{X}' \right) \right)^\vee \cong \Omega_{\bar{X}'}^1, \]

we have the identification

\[ \varphi \leftrightarrow \varphi^* \Theta. \]

Furthermore, \( \varphi^* \Theta = d\bar{\varphi} \Phi. \)

ii) The image of \( \bar{U}' \) under the “Abel-Jacobi” map

\( \bar{U}' \xrightarrow{\varphi} \left( F^2 H^3 \left( \bar{X}/\bar{X}' \right) \right)^\vee \)

is “quasi-Lagrangian,” that is,

\[ \varphi^* d\Theta = d\varphi^* (d\varphi \Phi). \]
Proof. i) The assertion follows directly from (21) and (20). Said otherwise,
\[ \varphi = \sum_{i=1}^{n'} \left( \int_{\gamma_i} \partial \left( (F^{-1})^* \tau \right) \right) \int_{\gamma_i}, \]
by (19) and, under the isomorphism
\[ \left( F^2 H^3 \left( \bar{X}/\bar{X}' \right) \right)^\vee \cong \Omega_{\bar{X}'}^1, \]
we have, by (13) that
\[ \int_{\gamma_i} \leftrightarrow d\bar{x}_i. \]

\[ ii) \quad \varphi^* d\Theta = d\varphi^* \Theta = d(d_{\bar{X}' \Phi}) = d_{U'} (d_{\bar{X}' \Phi}). \]

In our distinguished local coordinates we have by (20) that
\[ \varphi^* \Theta = \sum_{i=1}^{n'} \left( \int_{\Gamma} \partial \left( qe^{\xi_1} (\eta_0 + \beta) \right) \right) d\bar{x}_i. \]

6. Hilbert scheme

**Theorem 6.1.** i) The relative Hilbert scheme \( \bar{Y}' \) in \( \bar{U}' \) is given by the gradient ideal of \( \Phi \), that is, the (formal) germ of \( \bar{Y}' \) at \((0,0,\eta_0)\) is the zero-scheme of the section
\[ d_{U'/\bar{X}'} \Phi \]
of \( \Omega_{U'/\bar{X}'}^1 \).

ii) The image of \( \bar{Y}'_{\text{red}} \) under the map
\[ \varphi : \bar{U}' \to \left( F^2 H^3 \left( \bar{X}/\bar{X}' \right) \right)^\vee \]
is Lagrangian in the sense that the section \( \varphi^*(d\Theta) \) of \( \Omega_U^2 \), restricts to zero on \( \bar{Y}'_{\text{red}} \).
(Compare with [3], Theorem 1.12.)

iii) For the inclusion
\[ \iota : \bar{Y}'_{\text{red.sm.}} \to \bar{U}' \]
of the smooth points of \( \bar{Y}'_{\text{red}} \), we have
\[ (\varphi \circ \iota)^* (\Theta) = d(\iota^* (\Phi)). \]

**Proof.** i) Let \( \mathfrak{m} \) denote the ideal of \{0,0\} in \( U' \). Now
\[ \xi |_{Y_0 \times X'} \in A^{0,1}_{Y_0} (T_{Y_0}) \otimes \mathbb{C} [[u']] + I_{Y'} : A^{0,1}_{Y_0} (T_{X_0}) \]
and the obstruction to extending \( Y/Y' \) to a larger family of curves is the injective map
\[ \{ \xi \} : \frac{I_{Y'}}{\mathfrak{m} : I_{Y'}} \to H^1 (N_{Y_0 \setminus X_0}) \]
induced by $\xi$ and the natural map

$$T_{0}|_{Y_0} \rightarrow N_{Y_0\setminus X_0}.$$ 

Under the isomorphism

$$T_{X'/X'} \rightarrow \Omega^2_{X'/X'},$$

$$\zeta \mapsto \langle \zeta|\tau \rangle$$

we can reinterpret (23) as induced by the map

(24) \[ \frac{I_{Y'}}{m \cdot I_{Y'}} \rightarrow A^{0,1} \left( \frac{\Omega^2_{X_0}}{\Lambda^2 N_{Y_0\setminus X_0}} \right) \]

given by

$$\langle \zeta|\eta_0 \rangle|_{Y_0} \in \frac{I_{Y'}}{m \cdot I_{Y'}} \cdot A^{0,1} \left( \frac{\Omega^1_{Y_0}}{Y_0 \otimes N_{Y_0\setminus X_0}} \right)$$

or, equivalently, by

$$\langle \xi|\eta_0 \rangle|_{Y_0} \in \frac{I_{Y'}}{m \cdot I_{Y'}} \cdot A^{0,1} \left( \frac{\Omega^1_{Y_0}}{Y_0 \otimes N_{Y_0\setminus X_0}} \right)$$

since

$$\xi, \beta$$

both vanish along $u' = x' = 0$.

On the other hand, for the function

$$\Phi (u') = \Phi (z', \tilde{x}') = \int_{\sigma^{-1}(L_0) \cap X_{x'}} \tau,$$

when we change $z'$ and leave $\tilde{x}'$ constant, there is no change in the complex structure or holomorphic 3-form $\tau_{X'}$ on $X_{x'}$ nor in the position of $\sigma^{-1}(L_0) \cap X_{x'}$. Only the position

$$Y_{z',x'} := \sigma^{-1}(Y_0) \cap X_{x'}$$

moves in $X_{x'}$. Thus, for a real parameter $t$ we have

$$\frac{\partial \Phi}{\partial z'_j} (z', \tilde{x}') = q (\tilde{x}') \lim_{t \to 0} \int_{Y_{z'+t\epsilon_j,x'}} \eta_{z'}$$

$$= q (\tilde{x}') \int_{Y_{z',x'}} \langle \gamma_j (z', x') | \eta_{z'} \rangle.$$

Here, as in §3 of [C],

$$\gamma_j (z', x') + \gamma_j (z', x')$$

is the real vector field which is the derivative of the family of diffeomorphisms

$$F_{-1}^{-1}(z', x') \circ F_{z'+t\epsilon_j, x'} : X_{x'} \rightarrow X_{x'}$$

at $t = 0$. Recall now that

$$\left( F^{-1} \right)^* (\eta) = \xi (z', x') \langle \eta_0 + \beta (z', x') \rangle$$

so that

$$\frac{\partial \Phi}{\partial z'_j} (z', \tilde{x}') = q (\tilde{x}') \int_{Y_0} \langle \lambda_k (z', x') | \xi (z', x') \rangle \langle \eta_0 + \beta (z', x') \rangle$$

where

$$\lambda_j (z', x') = \left( F_{(z', x')} \right)_* (\gamma_j (z', x'))$$
is again of type $(1,0)$ by construction. Thus by considerations of type, we obtain

$$\frac{\partial \Phi}{\partial z_j'} (z', \tilde{x}') = q (\tilde{x}') \int_{Y_0} \langle \lambda_j (z', \tilde{x}') | \langle \xi (z', \tilde{x}') | (\eta_0 + \beta (z', \tilde{x}')) \rangle \rangle .$$

Finally we can consider

$$\left\langle \xi | e^{(\Vert \cdot \Vert)} (\eta_0 + \beta) \right\rangle$$

as a section of $\Omega^1_{U_\tilde{x}'./X'}$ by sending $\frac{\partial}{\partial z_k'}$ to

$$\int_{Y_0} \langle \lambda_j (0,0) | \langle \xi | (\eta_0 + \beta) \rangle \rangle .$$

Considering the functions (25) modulo $m \cdot I_{\tilde{Y}'}$, they can be rewritten as

$$\int_{Y_0} \langle \lambda_j (0,0) | \langle \xi | (\eta_0 + \beta) \rangle \rangle .$$

By (3) the normal vector fields $\lambda_j (0,0)$ give a basis of $H^0 (N_{Y_0 \setminus X_0})$. So we obtain functions

$$\left\langle \frac{\partial}{\partial z_j'} \Bigg| \Phi (y', \tilde{x}') \right\rangle$$

which, modulo $m \cdot I_p$, are equivalent to

$$z \int_{Y_0} \langle \lambda_j (0,0) | \langle \xi | (\eta_0 + \beta) \rangle \rangle .$$

The injectivity of the map (25) (rewritten as (24)) then implies that the functions

$$\int_{Y_0} \langle \lambda_j (0,0) | \langle \xi | (\eta_0 + \beta) \rangle \rangle$$

generate

$$\frac{I_{\tilde{Y}'}}{m \cdot I_{\tilde{Y}'}}$$

and so, by Nakayama’s lemma, generate

$$I_{\tilde{Y}'}.$$

ii) By Theorem 5.1),

$$\varphi^* d\Theta = d_{z'} (d_{z'} \Phi) = -d (d_{z'} \Phi) .$$

But using i) we can conclude that

$$\left. \left( \text{image of } d_{z'} \Phi \text{ in } \Omega^1_{U_\tilde{x}'} \right) \right|_{\tilde{Y}'} = 0 .$$

So the image of $\varphi^* d\Theta$ in $\Omega^2_{Y_\text{red.sm.}}$ is

$$d \left( \left. \text{image of } d_{z'} \Phi \text{ in } \Omega^1_{Y_\text{red.sm.}} \right|_{\tilde{Y}'} \right) = 0 .$$

iii) By Theorem 5.1

$$\varphi^* \Theta = d_{z'} \Phi$$
and by the proof of ii) just above the image of $\iota$ lies in the zero-locus of $d_x \Phi$. So
\[ (\varphi \circ \iota)^* \Theta = d_x \iota^* \Phi = d_x \iota^* \Phi + d_x \iota^* \Phi = \iota^* d\Phi. \]

\[ \Box \]

7. The holomorphic Chern-Simons invariant

The potential function $\Phi$ above is closely related to a variant of the functional (4) of [DT]. (See also §7 of [T].) To explain this connection, suggested to the author by R. Thomas, we proceed as follows. Let $E_0$ be a holomorphic vector bundle on $X_0$ (for the moment $X_0$ can be any compact complex manifold) and the dual bundle. For example, $X_0$ could be a $K$-trivial threefold and $E_0$ could be such that, computing its Chern classes as algebraic cycles modulo rational equivalence,
\[ c_1(E_0) = 0, \]
\[ c_2(E_0) \equiv \{Y_0\} - r \{L_0\}. \]

Let
\[ \rho : U' \rightarrow X' \]
be a smooth morphism of polydisks as above, but this time let the fiber dimension of $\rho$ be $h^1(\text{End}(E_0))$, the embedding dimension at $\{E_0\}$ of the local analytic scheme parametrizing holomorphic vector bundles on $X_0$. We can compute deformations of the pair $(X_0, E_0)$ as follows. Let $\text{End}^0(E_0)$ denote the sheaf of trace-0 endomorphisms so that
\[ \text{End}(E_0) = \mathcal{O}_{X_0} \oplus \text{End}^0(E_0). \]

We call the symbol map
\[ \mathcal{D}_1(E_0) \rightarrow \text{End}(E_0) \otimes T_{X_0} \]
followed by the projection
\[ \text{End}(E_0) \otimes T_{X_0} \rightarrow \text{End}^0(E_0) \otimes T_{X_0} \]
the \textit{reduced symbol map}. Call the kernel of the composition
\[ \mathcal{D}_1^1(E_0). \]

We have the exact sequence of left $\mathcal{O}_{X_0}$-modules
\[ 0 \rightarrow \mathfrak{gl}(E_0^\vee) \rightarrow \mathcal{D}_1^1(E_0) \rightarrow T_{X_0} \rightarrow 0. \]

We consider sections of $E_0$ as functions $f$ on $E_0^\vee$ which are complex linear on fibers. For any locally defined $C^\infty$-vector field $\beta$ of type $(1, 0)$ on $X_0$ we can lift to a vector field $\beta^\dagger$ on $E_0^\vee$ such that
\[ \pi_*(\beta^\dagger) \in A^0\left(\mathcal{D}_1^1(E_0)\right). \]

The tangent space to the deformations of $(X_0, E_0)$ is
\[ H^1\left(\mathcal{D}_1^1(E_0)\right). \]
induced by the reduced symbol map. It sits in an exact sequence

\[(29) \quad H^1(\text{End}(E_0)) \to H^1\left(\mathcal{D}_1^+(E_0)\right) \to H^1(T_{X_0}).\]

Let 

\[
\bar{\partial}_0
\]

be the \(\bar{\partial}\)-operator for the complex structure on the total space of \(E_0^\vee\). The action of \(\bar{\partial}_0\) on functions and forms pulled back from \(X_0\) gives the \(\partial\)-operator for the complex structure on \(X_0\). We fix a \(C^\infty\)-trivialization 

\[
F : X \to X' \times X_0
\]

with corresponding Kuranishi data \(\xi_\cdot\). We embed the first-order deformations of \((X_0, E_0)\), which are parametrized by \(H\), into an artinian subscheme of \(U'\) in a way compatible with the exact sequence \(29\) and the differential of \(\rho : U' \to X'\) at \((0, 0)\). Next take a maximal extension \(E_{Y'/Y' \times X'}\) of this first-order deformation of \((X_0, E_0)\) over an analytic subscheme \(Y'\) of \(U'\). Analogous to the deformation of the pair \((Y_0, X_0)\) considered earlier, Kuranishi data

\[
(30) \quad \bar{\partial}_0 - L_{\xi^\dagger}
\]

associated with this maximal formation of \((X_0, E_0)\) corresponds to a \(C^\infty\)-isomorphism of vector bundles

\[
(31) \quad E := F^*(\mathcal{O}_{U'} \boxtimes E_0) \xrightarrow{\rho^*F} \mathcal{O}_{U'} \boxtimes E_0 \\
U' \times X' \times X_0 \xrightarrow{\rho^*F} U' \times X_0
\]

such that

\[
E_{Y'} := E|_{Y' \times X'}
\]

has a complex structure induced by the \(\bar{\partial}\)-operator \(\{31\}\). Since locally in \(X_0\), 

\[
\xi^\dagger = \bar{\partial}_0 \left(\pi_\# \beta^\dagger\right)
\]

for some vector field \(\beta^\dagger\) on \(E_0^\vee\) with \(\pi_\# \beta^\dagger \in A^0\left(\mathcal{D}_1^+(E_0)\right)\), \(\xi^\dagger\) is a holomorphic mapping

\[
\xi^\dagger : U' \to A^{0,1}(\mathcal{D}_1^+(E_0)) \subseteq \pi^*A^{0,1}_{X_0} \otimes T_{E_0^\vee}
\]

lying over (the pull-back to \(U'\) of) Kuranishi data \(\xi\) for the trivalization \(F\) of \(X/X'\). The scheme \(Y'\) is then defined as the maximal solution scheme of the equation 

\[
(\bar{\partial}_0 - L_{\xi^\dagger})^2 = 0.
\]

Next we fix a hermitian metric on \(E_0\) and, pulling back the hermitian metric on \(E_0\) via \(\{31\}\), we induce a hermitian structure on \(E\). Let 

\[
\nabla_0 : A^0(E_0) \to A^0(E_0 \otimes T_{E_0}^\vee(X_0))
\]

be the metric \((1, 0)\)-connection for the Kähler manifold \(X_0\) and the given metric. So, in particular,

\[
\nabla_0^{0,1} = \bar{\partial}_0
\]

and, with respect to a local holomorphic frame \(e_0\) of \(E\), the hermitian structure is given by a hermitian-matrix-valued function \(H\) and

\[
\nabla_0^{1,0}(e_0) = (\partial H \cdot H^{-1}) \cdot e_0 = \nabla_0(e_0).
\]
Abusing notation, let $\nabla_0$ also denote the pull-back of the connection $\nabla_0$ on $E_0$ via the product structure on $(\mathcal{O}_{U'} \boxtimes E_0)/ (U' \times X_0)$. By \cite{C} we have

$$
\nabla_0^{1,0} : A^0_{X_0}(E_0) \otimes \mathbb{C}[[U']] \to \mathcal{E}^{(\xi^\dagger)} (A^1_{X_0}(E_0) \otimes \mathbb{C}[[U']])
$$

$$
\nabla_0^{0,1} = \overline{\partial}_0 : A^0_{X_0}(E_0) \otimes \mathbb{C}[[U']] \to A^{0,1}_{X_0} (E_0) \otimes \mathbb{C}[[U']].
$$

For a local holomorphic section $s(u')$ over $Y'$ we have

$$(\nabla_0 - L_{\xi^d}) (s(u')) = \nabla_0^{1,0} (s(u'))$$

so that

$$\nabla := \nabla_0 - L_{\xi^d} =: d + L_0$$

is a $(1,0)$-connection. By this we mean

$$\nabla_0^{0,1} = \overline{\partial}_0 - L_{\xi^d}.$$

As in Lemma 8.2ii) of \cite{C}, the integrability conditions for the almost complex structure given by $(\overline{\partial}_0 - L_{\xi^d})$ on $E'_0$ are given by the vanishing of the tensor

$$\left(\overline{\partial}_0 - L_{\xi^d}\right)^2.$$

The critical remark is therefore that the $(0,2)$ component of the curvature form $R$ defined by

$$L_R = \nabla^2 = L_{d\alpha + \frac{1}{2}[\alpha,\alpha]}$$

is simply the obstruction tensor

$$\left(\overline{\partial}_0 - L_{\xi^d}\right)^2 = -L_{\overline{\partial}_0 \xi^d - \frac{1}{2}[\xi^d,\xi^d]}$$

whose vanishing defines $Y'$.

Also

$$d_{U'/X'} (\overline{\partial}_0 - L_{\xi^d}) = d\alpha^{0,1} \in A^{1,0}_{U'/X'} \otimes A^{0,1}_{X_0} (\text{End} (E_0))$$

induces the isomorphism

(32)

$$T_{U'/X'} |_{(X_0), (E_0)} \to H^1 (\text{End} (E_0))$$

determined by the first-order deformations of $E_0$ with $X_0$ fixed. If as above $\{z'_j, x'_i \circ \rho\}$ are a system of holomorphic coordinates on $U'$ we have

(33)

$$\sum_j \frac{\partial \xi^d_j}{\partial z'_j} |_{u'=0} dz'_j$$

where $\left\{\frac{\partial \xi^d_j}{\partial z'_j} |_{u'=0}\right\}$ is a basis of $H^1 (\text{End} (E_0))$.

As before, the maximality of $Y'$ implies that the map

(34)

$$\{R^{0,2} \} : \left(\mathcal{I}_{Y'} \overline{m} \mathcal{I}_{Y'} \right)^{\vee} \to H^2 (\text{End} (E_0))$$

is injective. For a minimal generating set $\{g_k\}$ of $\mathcal{I}_{Y'}$, Nakayama’s lemma says that the $g_k$ are linearly independent in $\mathcal{I}_{Y'}$, and so we can write

(35)

$$\{(G^{-1})^* R^{0,2}\} = \sum_k \mu_k g_k$$

for a partial basis $\{\mu_k\}$ of $H^2 (\text{End} (E_0))$. 
Now we return to the situation in which $X_0$ is a $K$-trivial threefold. Following [DT] define the holomorphic Chern-Simons functional as follows. For each fixed $u' = (z', x') \in U'$ embed $[0, 1]$ in $U'$ via
\begin{equation}
[0, 1] \rightarrow U' \\
t \mapsto (tz', x')
\end{equation}
and form the connection
\begin{align*}
\tilde{\nabla}_{u'} & : = \nabla_{(tz', x')} + dt \\
& = \nabla_0 - L_{\xi^i (tz', x')} + dt
\end{align*}
on
$$([0, 1] \cdot z', x') \times X'. $$

**Definition 7.1.** The holomorphic Chern-Simons functional $CS_{\nabla_{u'}} (u')$ on $U'$ is given by the formula
\[CS_{\nabla_{u'}} (u') = \int_{X/X'} \tau \wedge \left( \int_0^1 \left( \tilde{R}_{u'} \wedge \tilde{R}_{u'} \right) dt \right)\]
where
\[\tilde{R}_{u'} = \tilde{\nabla}_{u'}^2.\]

We can decompose $\tilde{R}_{u'}$ by type to obtain that
\[\left( (\nabla_0^1 - L_{\xi^j} + dt)^2 \right)^{(0,2),0+(0,1),dt}\]
is given by the expression
\[R_0 - \left( \partial_0 \xi^j - \frac{1}{2} [\xi^j, \xi^j] \right) - \sum_j z_j \frac{\partial \xi^j}{\partial z_j} dt.\]
So by type
\begin{equation}
(37) \left( \int_0^1 (\tilde{R} \wedge \tilde{R}) dt \right)^{0,3} = 2 \int_0^1 \left( \sum_j z_j \frac{\partial \xi^j}{\partial z_j} \right) \wedge \left( \partial_0 \xi^j - \frac{1}{2} [\xi^j, \xi^j] \right) dt \\
= 2 \sum_j z_j \int_0^1 \left( \frac{\partial \xi^j}{\partial z_j} \wedge R^{0,2} \right) dt
\end{equation}
where the $\xi^j$ and $R^{0,2}$ under the integral sign are functions of $t$ via the map (36).

**Definition 7.2.** We define
\[\Phi_{DT} (\tilde{u}') := CS_{D_0} (\nabla, \tau)\]
where $D_0$ is a “base” $(1,0)$-connection on $E_0$ and the right-hand expression is the holomorphic Chern-Simons functional associated to the variation $G^* (\nabla)$ of the connection $G^* (\nabla_0)$ on the bundle
\[G^* (E_0 \times U')\]
over
\[X \times X', U'/U'.\]
By (37) it is computed as $CS_{\nabla_0} (\nabla, \tau) + CS_{\nabla_0} (\nabla, \tau)$ where

$$
CS_{\nabla_0} (\nabla, \tau) = \int_{X_0 \times ([0,1] \cdot u')} \text{tr} \left( \tilde{R} \wedge \tilde{R} \right) \wedge (G^{-1})^*(\tau)
$$

$$
= 2 \int_{X_0} \sum_j z_j' \int_0^1 \text{tr} \left( \frac{\partial \xi^i}{\partial z_j} \wedge R^{0,2} (u') \right) dt \wedge q \epsilon (\eta_0 + \beta)
$$

$$
= 2 \int_{X_0} \sum_j z_j' \int_0^1 \text{tr} \left( \frac{\partial \xi^i}{\partial z_j} \wedge R^{0,2} (u') \right) dt \wedge (\eta_0 + \beta).
$$

and, as above, $\tau$ is the tautological $\langle 3, 0 \rangle$-form on $\tilde{X}/\tilde{X}'$ and $\beta$ and $q$ are as in (22).

As before, $R^{0,2} (u') \in m \cdot I_{Y'} \cdot A^{0,2} (X_0)$

where we consider

$$
\{ R^{0,2} \}
$$

as in (34). And so, modulo $m^2 \cdot I_{Y'}$, we have

$$
\Phi_{DT} \equiv CS_{D_0} (\nabla_0, \tau) + q \sum_j \int_{X_0} \text{tr} \left( \frac{\partial \xi^i}{\partial z_j} (u') \wedge R^{0,2} (u') \right) \wedge \eta_0.
$$

Thus since Serre duality of $H^1 (\text{End} (E_0))$ and $H^2 (\text{End} (E_0))$ is expressed by the pairing

$$
\int_{X_0} \text{tr} \langle \cdot, \cdot \rangle \wedge \eta_0
$$

we have by (38) and (39) we have modulo $m^2 \cdot I_{Y'}$ that

$$
\Phi_{DT} \equiv CS_{D_0} (\nabla_0, \tau) + q \sum_k \sum_j \left( \sum \left( \frac{\partial \xi^i}{\partial z_j} (0) \right) \mu_k \right) g_k.
$$

So, modulo $m \cdot I_{Y'}$, we have

$$
\frac{\partial \Phi_{DT}}{\partial z_j} = q \sum_{j,k} \left( \frac{\partial \xi^i}{\partial z_j} (0) \right) \mu_k g_k
$$

where, by the injectivity of (34), the matrix

$$
\begin{bmatrix}
\frac{\partial \xi^i}{\partial z_j} (0) \\
\mu_k
\end{bmatrix}
$$

is of maximal rank.

Nakayama’s lemma then gives an alternate proof of the result, established independently by Witten and Donaldson-Thomas [DT], that $Y'$ is the zero-scheme of the section

$$
d_{U'/\tilde{X}} \Phi_{DT}
$$

of

$$
\Omega^1_{U'/\tilde{X}}.
$$

Tjurin’s construction in §7 of [1] of an Abel-Jacobi map

$$
Y' \cap \rho^{-1} (0) \rightarrow (F^2 H^3 (X_0))^\vee
$$
is given via (9) by 
\[ d\tilde{\Upsilon}' \cap \tilde{\rho}^{-1}(0) \to (F^2H^3(X_0))^\vee. \]
Since we are in a relative setting, we can use the isomorphism (9) to give a more general definition of an Abel-Jacobi or normal function map. Namely 
\[ d\tilde{\Upsilon}' \cap \tilde{\rho}^{-1}(0) : \tilde{\Upsilon}' \to (F^2H^3(\tilde{X}/\tilde{X}'))^\vee. \]
and so, via (9),
\[ d\tilde{\Upsilon}' \cap \tilde{\rho}^{-1}(0) : \tilde{\Upsilon}' \to (F^2H^3(\tilde{X}/\tilde{X}'))^\vee. \]

8. Relation between \(\Phi\) and \(\Phi_{DT}\), an analogue of Abel’s theorem (by Richard Thomas)

There is at least one setting in which \(\Phi\) and \(\Phi_{DT}\) are exactly the same. Let \(E_0\) be a \(C^\infty\) rank-2 complex vector bundle with a fixed trivialisation of its determinant; thus its determinant has a natural holomorphic structure
\[ \det E_0 = \mathcal{O}_{X_0}. \]
For most of this section we may assume that \(X_0\) is any compact complex 3-fold. Choosing a hermitian structure on \(E_0\) reduces the structure group of \(E_0\) to \(U(2)\); the choice of a distinguished
\[ 1 \in H^0(\mathcal{O}_{X_0}) \]
and a hermitian metric compatible with this (i.e. such that 1 has constant unit norm in the induced metric on \(\Lambda^2E_0\)) further reduces the structure group to \(SU(2)\). Thus \(E_0\) has the structure of a (left) quaternionic line bundle with the multiplicative group
\[ \mathbb{H}^* = (0, \infty) \times S^3 \]
of the quaternions
\[ \mathbb{H} = \left\{ \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} : a, b \in \mathbb{C} \right\} \]
acting on the left on \(E_0\). This \(\mathbb{H}\)-structure is determined by left multiplication by \(j\) which is given by the equations
\[ (j \cdot s) \perp s \]
\[ (j \cdot s) \wedge s = \|s\|^2. \]
(Notice that multiplication by \(j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) is not \(\mathbb{C}\)-linear.) The proof of the equality of \(\Phi\) and \(\Phi_{DT}\) is exactly parallel of the classical proof of Abel’s theorem in which \(\mathbb{C}\) is replaced by \(\mathbb{H}\). Every \(SU(2)\) connection
\[ D : A^0_{X_0}(E_0) \to A^1_{X_0}(E_0) \]
is quaternionic, that is, commutes with the left \(\mathbb{H}^*\)-action. Thus the connection is completely determined by its value on a single non-vanishing section, just as is the case for \(\mathbb{C}\)-line bundles. For any locally defined non-zero \(C^\infty\)-section \(s_0\) of \(E_0\) there is a unique \(SU(2)\)-connection \(D_0\) on \(E_0\) such that
\[ D_0(s_0) = D^{1,0}_0(s_0) \oplus D^{0,1}_0(s_0) = 0. \]
Then we also have
\[ D_0(j \cdot s_0) = D^{1,0}_0(j \cdot s_0) \oplus D^{0,1}_0(j \cdot s_0) = 0. \]
If $E_0$ has a holomorphic structure given by the del-bar operator $\overline{\partial}_0$ (on sections of $E_0$), and $s_0$ is a holomorphic section of $E_0$ with zero scheme $Y_0$ which is smooth of codimension 2, then there is a unique $\mathbb{H}$-connection $D_0$ over $X_0 - Y_0$ for which $s_0$ is flat. $D_0$ is not necessarily a $(1,0)$-connection. We compute

$$s_0 \wedge (\overline{\partial}_0 - D_0^{0,1}) (j \cdot s_0) = \overline{\partial}_0 (s_0 \wedge j \cdot s_0)$$

$$= -\overline{\partial}_0 (\|s_0\|^2)$$

so that, in terms of the basis $(s_0, j \cdot s_0)$ we have

$$\overline{\partial}_0 - D_0^{0,1} = \begin{bmatrix} 0 & * \\ * & \overline{\partial}_0 \log \|s_0\|^2 \end{bmatrix}.$$

So for any $(1,0)$-connection $\nabla_0$, in terms of the basis $(s_0, j \cdot s_0)$ we have

$$\nabla_0^{0,1} - D_0^{0,1} = \begin{bmatrix} 0 & 0 \\ * & \overline{\partial}_0 \log \|s_0\|^2 \end{bmatrix}.$$

Then for any $(3,0)$-form $\tau$, the Chern-Simons functional $CS_{\nabla_0} (D_0, \tau)$ is given by the expression

$$- \int_{X_0 - Y_0} tr \left( A^{0,1} \wedge \overline{\partial}_0 A^{0,1} + \frac{2}{3} (A^{0,1})^3 \right) \wedge \tau$$

since

$$(\nabla_0^{2})^{0,2} = \overline{\partial}_0^2 = 0.$$  

By (38), this reduces to

$$- \frac{2}{3} \int_{X_0 - Y_0} \left( \overline{\partial}_0 \log \|s_0\|^2 \right)^3 \wedge \tau = 0,$$

and so by the basic additivity $CS_A(C) = CS_A(B) + CS_B(C)$ of Chern-Simons functionals,

$$CS_{\nabla_0} (D_{u'}, \tau) = CS_{\overline{\partial}_0} (D_{u'}, \tau)$$

for any connection $D_{u'}$.

Pick a family of sections $\{s_{u'}\}_{u' \in U'}$ of $E_0$ (not necessarily holomorphic with respect to any del-bar operator now), with zero set $Y_{u'}$. If there happens to be a finite del-bar operator $\overline{\partial}_{u'}$ with respect to which $s_{u'}$ is holomorphic, then these will also equal the holomorphic Chern-Simons invariant of $\overline{\partial}_{u'}$, by the above computation. Now $D_{u'} - D_0$ can be computed as follows.

$$f = s_{u'} \cdot s_0^{-1}$$

is a $C^\infty$-function on $X_0 - (Y_0 \cup Y_{u'})$ with values in the multiplicative group $\mathbb{H}^*$. Then

$$0 = D_{u'} (f \cdot s_0) = df \cdot s_0 + f \cdot D_{u'} s_0$$

$$D_{u'} s_0 = -f^{-1} \cdot df \cdot s_0$$
and since $H$-connections are determined by their values on a single section
\[ D_{u'} - D_0 = -f^{-1} \cdot df = -f^* (g^{-1} \cdot dg) \]
for the invariant one-form
\[ g = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}^{-1} \cdot d \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \]
on $H^*$. Then
\[ 0 = d (g^{-1} \cdot g) = (dg^{-1}) \cdot g + g^{-1} \cdot dg \]
so that
\[ d (g^{-1} \cdot dg) = -g^{-1} \cdot dg \cdot g^{-1} \cdot dg \]
and, since $D_0$ is integrable ($(D_0)^2 = 0$), we have the formula
\[ (41) \quad CS_{D_0} (D_{u'}, \tau) = \int_{X_0} tr \left( f^{-1} \cdot df \wedge D_0 (f^{-1} \cdot df) - \frac{2}{3} (f^{-1} \cdot df)^\wedge 3 \right) \wedge \tau 
= -\frac{5}{3} \int_{X_0} f^* tr \left( (g^{-1} \cdot dg)^\wedge 3 \right) \wedge \tau. \]
But
\[ tr \left( (g^{-1} \cdot dg)^\wedge 3 \right) \]
is the pull-back to $H^*$ of the invariant 3-form on $S^3$ which generates $H^3 (S^3)$ and becomes exact on
\[ S^3 - \{1\} \]
so that, on $H^*$ we have, for some non-zero constant $c'$ that, as distributions,
\[ tr \left( (g^{-1} \cdot dg)^\wedge 3 \right) \sim c' \int_{(0, \infty)} \tau. \]
Pulling back via $f$ we therefore have by (41) that
\[ (42) \quad CS_{D_0} (D_{u'}, \tau) = c \int_{L_0} Y_{u'} \tau. \]
Here we integrate over the 3-chain $f^{-1} (0, \infty)$, which indeed bounds $Y_{u'} - L_0$. This is of course the complete analogue, for rank-2 vector bundles on threefolds with trivial determinant, of the classical Abel’s theorem for line bundles on curves.

Suppose now that $X_0$ is a Calabi-Yau threefold, and $Y_0 \subseteq X_0$ is a disjoint unions of smooth elliptic curves, and $Y/Y'$ is as above the maximal deformation of $Y_0$. Since $H^1 (O_{X_0}) = 0 = H^2 (O_{X_0})$, Serre’s construction gives a unique holomorphic rank-2 vector bundle $E \times_{X'} Y'$ over $X \times_{X'} Y'$ via the extension
\[ 0 \to O_{X \times X'} Y' \to E \times_{X'} Y' \to I_{Y' \times X \times X'} \to 0. \]
Here we assume that
\[ E \times_{X'} Y' \]
is the restriction to $X \times_{X'} Y'$ of some $C^\infty$-vector bundle $E$ on $X \times_{X'} U'$. Since $H^1 (E_0) = 0$, all sections extend and so this family is a maximal holomorphic deformation of its restriction $E_0$ over $X_0$. Then
\[ \det (E \times_{X'} Y') = O_{X \times X'} Y', \]
and $E \times_{X'} Y'$ has a section $s$ whose vanishing scheme is $Y$. 
Construct a $C^\infty$-trivialization
\[ F : X \times X', U' \to X_0 \times U' \]
such that the maximal deformation $Y/Y'$ of $Y_0$ satisfies
\[ Y \subseteq F^{-1} (Y_0 \times U') \]
and then construct a compatible trivialization
\[ G : E \to E_0 \times U'. \]
We can assume that we have chosen this trivialization so that $G \circ s$ is given over $Y'$ by
\[ s_{u'} = (s_0, u'). \]
Then, for the hermitian structure pulled back from $E_0$, we have the hermitian connection $D_{u'}$ defined above (such that $D_{u'} s_{u'} = 0 = D_{u'} (j \cdot s_{u'})$) inducing the possibly singular del-bar operator $\partial_{\bar{\partial}}^{D_{u'}}$. By (40) we have $CS_{\nabla_0} (D_{u'}) = CS_{D_0} (D_{u'})$.

We therefore have by (42) that, for $\tilde{u}' \in U'$,
\[ \Phi_{DT} (\tilde{u}') = \Phi (\tilde{u}') - \Phi (0, 0). \]
Differentiating we have
\[ d\Phi_{DT} |_{\tilde{\gamma}'} = d\Phi |_{\tilde{\gamma}'}, \]
that is, that the two Abél-Jacobi maps coincide.

9. A relative gradient scheme structure on the Noether-Lefschetz locus (by Claire Voisin)

Let $X_0$ be a Ca\-li-Yau threefold. Let $L_0$ be a very ample line bundle on $X_0$. We are assume that $H^1 (X_0; \mathcal{O}_{X_0}) = H^2 (X_0; \mathcal{O}_{X_0}) = 0$ so that there is no obstruction to deforming $L_0$ with $X_0$. We are interested in the deformation theory of the triple $(S_0, X_0, \lambda)$ where
\[ S_0 \to X_0 \]
is the inclusion of a smooth member of $|L_0|$, and
\[ \lambda \in H^2 (S_0; \mathbb{Z})_{\text{van}} \cap H^{1,1} (S_0) \]
is an integral cohomology class on $S_0$ which is both vanishing (annihilated by $j_*$) and of type $(1,1)$. Notice that we have an orthogonal decomposition
\[ H^2 (S_0; \mathbb{Q}) = \text{image} (j^*) \oplus H^2 (S_0; \mathbb{Q})_{\text{van}}, \]
and that, by the assumption that $H^2 (X_0; \mathcal{O}_{X_0}) = 0$, the first term, namely image $(j^*)$, is made of classes which stay of type $(1,1)$ under any deformation of the pair $(S_0, X_0)$. Hence the restriction to vanishing classes is not really restrictive.

The locus of the pairs $(S_0, X_0)$ such that there exists a non zero $\lambda$ as above is called the Noether-Lefschetz locus. It splits locally as a countable union over all $\lambda$’s of the locus where the locally constant class $\lambda$ remains of $(1,1)$-type. This last locus is called the component of the Noether-Lefschetz locus determined by $\lambda$.

We make it now more precise. The deformation space of the pair $(S_0, X_0)$ is smooth: indeed the deformation theory of $X_0$ is unobstructed, $L_0$ deforms with $X_0$ (uniquely since $H^1 (X_0; \mathcal{O}_{X_0}) = 0$). And, since by Kodaira vanishing we have $H^1 (X_0; L_0) = 0$, $i > 0$, the sections of $L_0$ deform with $X_0$. Hence we get a vector bundle over any local deformation space $X'$ of $X_0$, with fiber $H^0 (X_{x'}, L_{x'})$ over
$x' \in X'$, and the deformations of the pair $(S, X_0)$ are parametrized by a Zariski open set of the projective bundle

$$\mathbb{P}H^0(X/X', L) \to X'$$

with fiber $\mathbb{P}H^0(X_{x'}, L_{x'})$.

Let now $U' \subseteq \mathbb{P}H^0(X/X', L)$ be an open ball giving the local deformation space of $(S_0, X_0)$. As above we have a smooth morphism

$$U' \to X'$$

$$u' \mapsto x' = x'(u').$$

Over $U'$, the local systems

$$H^2_{S, Z, \text{van}}, H^3_{X, S, Z}$$

with fibers $H^2(S_{u'}, Z)_{\text{van}}$ and $H^3(X_{x'}, S_{u'}, Z)$ respectively, are trivial. Choose $\lambda \in H^2(S_0, Z)_{\text{van}} \cap H^1(S_0)$. The class $\lambda$ provides a locally constant section $(\lambda_{u'})_{u' \in U'}$ of $H^2_{S, Z, \text{van}}$ and inside $U'$, the Noether-Lefschetz component determined by $\lambda$ is the scheme

$$U'_{\lambda} = \{u' \in U', \lambda_{u'} \in F^1H^2(S_{u'})\},$$

where $F^\circ$ denotes the Hodge filtration. This is also equivalent to the condition

$$\lambda_{u'} \perp H^2_{2, 0}(S_{u'}),$$

where the orthogonality is with respect to the intersection pairing.

Next identifying via Poincaré duality $H^2(S_{u'}, Z)_{\text{van}}$ with

$$\ker(j_\ast): H^2(S_{u'}; Z) \to H^2(X_{x'}; Z),$$

the exact sequence of relative homology provides a surjective map of local systems

$$\partial: H^3_{X, S, Z} \to H^2_{S, Z, \text{van}}.$$

Hence the section $(\lambda_{u'})_{u' \in U'}$ lifts to a section

$$(\gamma_{u'})_{u' \in U'} \in H^3_{X, S, Z}$$

Note that the homology class which is Poincaré dual to $\lambda_{u'}$ is represented by a closed differentiable 2-chain $z_{u'}$ in $S_{u'}$ which is homologous to 0 in $X_{x'}$, and that such a lifting $\gamma_{u'}$ is provided by a differentiable 3-chain $\Gamma_{u'}$ in $X_{u'}$ (varying continuously with $u'$) satisfying the condition

$$\partial\Gamma_{u'} = z_{u'}.$$
the Hodge bundle $H^3_{X/S}$ associated with the local system with fiber $H^3(X_{u'}, S_{u'}, \mathbb{Z})$. There is a natural bundle map

$$f : H^3_{X,S} \to H^3_X$$

on $U'$. Now we observe that since each $\tau_{\tilde{u}} \in H^3(X_{u'})$ vanishes on $S_{u'}$, it lifts naturally to a class in $H^3(X_{u'}, S_{u'}, \mathbb{C})$. Hence we get a natural lifting

$$\tau_{rel} \in H^3_{X,S}$$

of $\tau$. This section $\tau_{rel}$ is easily seen to be in $F^3H^3_{X,S}$ where $F^3$ here is the Deligne-Hodge filtration on relative cohomology. Denoting by $\langle, \rangle$ the pairing between $H^3(X_{u'}, S_{u'})$ and $H^3(X_{u'}, S_{u'})$, we have now a function $\Phi_{BN}$ on $\tilde{U}'$ which is defined by

$$\Phi_{BN}(\tilde{u}') = \int_{\Gamma_{u'}} \tau,$$

where $\Gamma_{u'}$ is a differentiable 3-chain defined above.

We can now state the following analogue of the results in the previous sections:

**Proposition 9.1.**

i) The function $\Phi_{BN}$ is holomorphic on $\tilde{U}'$.

ii) The inverse image $\tilde{U}'_{\lambda}$ of $U'_{\lambda} \subseteq U'$ is equal to the relative gradient scheme associated to the function $\Phi_{BN}$ on $\tilde{U}' \to \tilde{X}'$, that is

$$\tilde{U}'_{\lambda} = V(d\tilde{U}'/\tilde{X}'\Phi_{BN})$$

where “$V$” denotes the vanishing scheme.

**Proof.**
i) This follows immediately from the formula

$$\Phi_{BN} = \langle \tau_{rel}, \tilde{\gamma} \rangle,$$

where $\tilde{\gamma}$ is a flat section of the local system $H_{3,X,S,Z}$ pulled back to $\tilde{U}'$, while $\tau_{rel}$ is a holomorphic section of the bundle $H^3_{X,S}$.

ii) Differentiating the equation (43), we get

$$d\tilde{U}'/\tilde{X}'\Phi_{BN} = \langle \nabla \tilde{U}'/\tilde{X}'\tau_{rel}, \tilde{\gamma} \rangle,$$

where $\nabla$ is the Gauss-Manin connection on the Hodge bundle $H^3_{X,S}$ on $\tilde{U}'$. So it suffices to understand

$$\nabla \tilde{U}'/\tilde{X}'\tau_{rel} \in \Omega^{1}_{\tilde{U}'/\tilde{X}'} \otimes H^3_{X,S}.$$
is surjective.

Admitting this lemma, we conclude the proof of the proposition as follows: We have
\[ \nabla_{\tilde{U}'/\tilde{X}', \tau_{\text{rel}}} = i(\alpha), \]
where
\[ \alpha \in \Omega^1_{\tilde{U}'/\tilde{X}'} \otimes i(F^2H^2_S) \]
is surjective as an element of $\text{Hom}(T_{\tilde{U}'/\tilde{X}'}, F^2H^2_S)$. Since $i$ is dual to $\partial$ it follows that
\[ \langle d_{\tilde{U}'/\tilde{X}', \tau_{\text{rel}}}, \tilde{\gamma} \rangle = \langle \alpha, \partial \tilde{\gamma} \rangle = \langle \alpha, \tilde{\lambda} \rangle, \]
where the pairing in the second and third terms are the intersection pairing on $H^2_S$, and $\tilde{\lambda}$ is the pull-back to $\tilde{U}'$ of the section $(\lambda_{u'})_{u' \in U'}$ of $H^2_S, \text{van}$. It follows that the vanishing of $d_{\tilde{U}'/\tilde{X}', \Phi_BN}$ at a point $\tilde{u}'$ over $u' \in U'$ is equivalent to the vanishing of
\[ \langle \alpha(\zeta), \lambda_{u'} \rangle, \]
for any $\zeta \in T_{\tilde{U}'/\tilde{X}', \tilde{u}'}$. But since $\alpha$ is surjective, this is equivalent to the vanishings
\[ \langle \mu, \lambda_{u'} \rangle = 0, \forall \mu \in H^{2,0}(S_{u'}), \]
which provide exactly the equations defining the locus $\tilde{U}'_{\lambda}$. \hfill \square

We next prove Lemma 9.2.

Proof. By Griffiths transversality, we have
\[ \nabla_{\tilde{U}'/\tilde{X}', \tau_{\text{rel}}} \in \Omega^1_{\tilde{U}'/\tilde{X}'} \otimes F^2H^3_{X,S}. \]
On the other hand, since $f(\tau_{\text{rel}}) = \tau$ is pulled-back from $\tilde{X}'$, the differential $\nabla_{\tilde{U}'/\tilde{X}', \tau_{\text{rel}}}$ vanishes. Hence we have
\[ f(\nabla_{\tilde{U}'/\tilde{X}', \tau_{\text{rel}}}) = \nabla_{\tilde{U}'/\tilde{X}', \tau} = 0, \]
so that
\[ \nabla_{\tilde{U}'/\tilde{X}', \tau_{\text{rel}}} \in \Omega^1_{\tilde{U}'/\tilde{X}'} \otimes i(H^2_S). \]
On the other hand, we have
\[ F^2H^3_{X,S} \cap \text{image}(i) = i(F^2H^2_S). \]
Hence
\[ \nabla_{\tilde{U}'/\tilde{X}', \tau_{\text{rel}}} \in \Omega^1_{\tilde{U}'/\tilde{X}'} \otimes i(F^2H^2_S), \]
which proves the first statement.

It remains to see that
\[ \nabla_{\tilde{U}'/\tilde{X}', \tau_{\text{rel}}} : T_{\tilde{U}'/\tilde{X}'} \rightarrow i(F^2H^2_S) \cong F^2H^2_S \]
is surjective. We claim that at a point $\tilde{u}' = (u', \tau_{\tilde{u}'}) \in \tilde{U}'$, this maps identifies up to sign to the composed isomorphism
\[ H^0(S_{u'}, N_{S_{u'}/X_{u'}}) \overset{\theta'}{\rightarrow} H^0(S_{u'}, K_{X_{u'}} \otimes N_{S_{u'}/X_{u'}}) \cong H^0(S_{u'}, K_{S_{u'}}). \]
To see this, let $F := \tilde{U}'_{\tilde{u}'}$ be the fiber of the map $\tilde{U}' \rightarrow \tilde{X}'$ passing through $\tilde{u}'$. Then $F$ identifies to the fiber of $U' \rightarrow X'$ passing through $u'$, that is, to an open set
of \( \mathbb{P}(H^0(X_{x'}, L_{x'})) \), and the map \( \eta_u \) is given along \( F \) by contraction with a fixed \( \eta \in H^{3,0}(X_{x'}) \). Let us consider the universal family 

\[ S \subset F \times X_{x'} . \]

The form \( pr_1^* \eta \in H^0(\Omega^3_{F \times X}) \) vanishes under restriction on each fiber of this family, hence provides at \( u' \) an element of \( \Omega^1_F \otimes H^0(S_{u'}, \Omega^2_{S_{u'}}) \cong \text{Hom}(T_F, H^0(S_{u'}, \Omega^2_{S_{u'}})) \) given by

\[ \zeta' \mapsto \left \langle \zeta' \mid \left( pr_1^* \eta \right)_{| S_{u'} } \right \rangle , \]

where \( \zeta \in T_{F,u'} \) and \( \zeta' \) is any \( C^\infty \)-lifting of \( \zeta \) in \( T_{S_{| S_{u'} }} \). It is a standard fact that up to sign, this element is equal to \( \nabla_F(\eta) \), where \( \nabla_F \) here is the Gauss-Manin connection on the bundle of relative cohomology \( \mathcal{H}^3_{X,S} \) restricted to \( F \).

So we have found that \( \nabla_F(\eta) \in \Omega^1_F \otimes F^2\mathcal{H}^3_{X} \) is computed at \( u' \) as the map (44). Notice that here \( \zeta' \) is any lifting of \( \zeta \) and, in particular, the right hand side can be computed using local holomorphic liftings of \( \zeta \) in \( T_S \). But if \( \sigma \) is a local defining equation for \( S \), we can take for \( \zeta' \) the vector field

\[ \zeta' = (\zeta, 0) - (0, \nu) , \]

where \( \nu \) is a vertical vector field in \( F \times X \), which is normal to \( S \) and satisfies \( d\sigma(\nu) = d\sigma(\zeta, 0) \). For this choice we have that

\[ \left \langle \zeta' \mid \left( pr_1^* \eta \right)_{| S_{u'} } \right \rangle = - \left \langle \nu \mid \left( \eta \right)_{| S_{u'} } \right \rangle . \]

Now we note that the map which to \( \zeta \) associates \( \nu \in N_{S_{u'}/X_{x'}} \) such that

\[ d\sigma(\zeta, 0) = d\sigma(\nu) \]

is exactly the identification of \( T_{F,u'} \) with \( H^0(S_{u'}/N_{S_{u'}/X_{x'}}) \), and that the map which to \( \nu \) associates \( - \left \langle \nu \mid \left( \eta \right)_{| S_{u'} } \right \rangle \in K_{S_{u'}} \) is exactly the adjunction isomorphism. Hence our claim is proved. This concludes the proof of the lemma.

To complete the parallel with the results of the previous sections, we now relate \( d\Phi_{BN} \) along the Noether-Lefschetz locus with the Abel-Jacobi map. Note first of all that along the Noether-Lefschetz component \( \tilde{U}' \), the class \( \lambda \) is the cohomology class of a divisor \( D_{\lambda} \) on the surface \( S_{u'} \), which is well-defined up to rational equivalence, since under our assumptions, the surfaces \( S_{u'} \) are regular. This divisor, or 1-cycle, becomes homologous to 0 in \( X_{x'} \), hence provides a cycle

\[ Z_{\lambda} = (j_{u'})_{*}(D_{\lambda}) \in CH_1(X_{x'})_{\text{hom}}, \]

which has an Abel-Jacobi invariant

\[ \varphi_{x'}(Z_{\lambda}) = \int_{\Gamma_{u'}} F^2H^3(X_{x'})^{\nu} / H^3(X_{x'}; \mathbb{Z}) = J(X_{x'}). \]

Next, as in (7) there is a natural identification

\[ \nabla_{\tau}: T_{\tilde{X}} \cong F^2H^3(\tilde{X}/\tilde{X}') \]

and dually

\[ \Omega^1_{\tilde{X}} \cong F^2H^3(\tilde{X}/\tilde{X}')^{\nu} , \]

which we can pull-back to \( \tilde{U}' \).

We have now
Proposition 9.3. At the point \( \tilde{u}' \in \tilde{U}_\lambda \), with images \( \tilde{x}' \in \tilde{X}' \) and \( u' \in U' \), the differential

\[
d\Phi_{BN}|_{\tilde{u}'} \in \Omega^1_{\tilde{X}',\tilde{x}'}
\]

identifies via the above isomorphism to a lifting in \( F^2H^3(X_{x'})^\vee \) of the Abel-Jacobi invariant \( \varphi_{x'}(Z_{\lambda}) \in J(X_{x'}) \).

Proof. We have

\[
d\Phi_{BN} = <\nabla_{\tau_{\text{rel}}}, \tilde{\gamma}>.
\]

But the right hand side is equal to

\[
\int_{\Gamma_{u'}} (\nabla_{\tau_{\text{rel}}}).
\]

We now use the fact that \( f(\nabla_{\tau_{\text{rel}}}) = \nabla_{\tau} \) and the fact that the integrals

\[
\int_{\Gamma_{\mu}} \mu, \mu \in \mathcal{H}^3_{X,S,u'}
\]

vanish on \( i(F^2\mathcal{H}^2_{S,u'}) = \ker f \) to rewrite this as

\[
d\Phi_{BN}(\tilde{u}') = \int_{\Gamma_{u'}} \nabla_{\tau}(\tilde{u}').
\]

But this means exactly that, via the isomorphism \( [13] \) given by \( \nabla_{\tau} \), the differential \( d\Phi_{BN}(\tilde{u}') \in \Omega^1_{\tilde{X}',\tilde{x}'} \) identifies with \( \int_{\Gamma_{u'}} \in F^2H^3(X_{x'})^\vee \). Since the boundary of the chain \( \Gamma_{u'} \) is equal to any differentiable chain associated to \( D_{\lambda} \) by triangulation, the last term is by definition the Abel-Jacobi invariant of \( Z_{\lambda} \in CH_1(X_{x'})_{\text{hom}} \). \( \square \)

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