Opinion formation and spread: Does randomness of behaviour and information flow matter?

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We propose an opinion dynamics model based on Latané’s social impact theory. Agents in this model are heterogeneous and, in addition to opinions, are characterised by their varying levels of persuasion and support. The model is tested for two and three initial opinions randomly distributed among agents. We examine how the randomness of behaviour and the flow of information between agents affect the formation and spread of opinions. Our main research involves the process of opinion evolution, opinion cluster formation and studying the probability of sustaining opinion. The results show that opinion formation and spread are influenced by both flow of information between agents (interactions outside the closest neighbours) and randomness in adopting opinions.

Keywords: Complex systems; Opinion dynamics; Social modelling; Nowak–Szamrej–Latané model; Long range interactions; Agent based simulations

I. INTRODUCTION

Understanding how opinions are formed and spread in society is very important in studying consumer behaviour, organisational behaviour, predicting election results, and many others. As pointed out by Acemoglu and Ozdaglar [1], we acquire our opinions and beliefs in the process of social learning, during which people get information and update their opinions as a result of their own experience, as well as observation of other people’s activities and from their experience. This process takes place in a social network consisting of friends, co-workers, family members and a certain group of leaders that we listen to and respect [1, 2]. Units update and create their views by communicating with other people who belong to their social network. It is communication that connects people and creates relationships [3].

It should be noted that people often copy the choices of others [4, 5]. This applies, for example, to the choice of names for children [6, 7], a popular book, dishes ordered in a restaurant (instead of studying the menu, we look at what the others have ordered), and even ideological beliefs [5]. This copying of opinions and behaviours often takes place in a network of informal contacts and it is based on social relations between people [8] and plays an important role in forming opinions. In addition, we are often dealing with unpredictability or indifference in opinion-forming or decision-making (despite the positive attitude towards the proposed actions). This applies, among others, to electricity tariffs, eco-innovations or pro-environmental attitudes [9–11], as well as voting behaviour [12], in which human rationality is bounded.

One of the most active discussions in psychology of the opinion dynamics is also about the irrational processing of information [13], therefore, this aspect should be taken into account in studying opinion formation. Furthermore, individuals belong to many groups, or have many interactions outside the main group (a group of closest neighbours). Such a connection with people from other groups (neighbourhoods) increases the information advantage [14], and can be interesting in disseminating information.

We therefore propose a model of forming an opinion based on the social impact theory formulated by Latané [15], in which we take into account the randomness of the actors’ behaviour by introducing a social temperature, as well as interactions with agents; not only close neighbours, but in the whole network by α parameter (scale the distance function). Our agents are heterogeneous through a different level of persuasion intensity and support intensity, as well as the possibility of having different opinions.

Recently, the multi-choice opinion dynamics model [16, 17] based on Latané theory [15, 18–21] was proposed. In this model, it is possible to test the diffusion of opinions in case there are more than two opinions available in the system. The earlier attempts to modelling multiple-choice of opinions include among others multi-state and discrete-state opinions models [22–32] or discrete vector-like variables [33–36]. The rest of huge literature (see papers by Sirbu et al. [37], Castellano et al. [38], Stauffer [39], Anderson et al. [40], and Galam [41] for reviews) is devoted to the systems with binary opinions (see for example Refs. 42–45) or the continuous space of opinions (see for example Refs. 13, 46–62).

In this paper we study how opinions are formed and how they spread in the community. We take into account the flow of information in the community (interactions outside the closest neighbours) and randomness of
human behaviour. Agent based model with lattice fully populated by actors has been adopted, where each of the network nodes refers to one person.

II. MODEL

To study the diffusion of opinions, the theory of social influence introduced by Latané [15] in the dynamic manner proposed by Nowak et al. [21]—as implemented by Bańcerowski and Malarz [16]—has been used.

Social influence is a process that results in a change in the behaviour, opinion or feelings of a human being as a result of what other people do, think or feel. The essence of social influence is of course not only exerting social influence, but also succumbing to it, which will be taken into account in the used model by means of appropriate parameters (intensity of persuasion and intensity of support). The Latané [15] theory rely on three experimentally proven [18–20] assumptions:

social force principle: it says that social impact \( I \) (details are given in description of Eq. (2)) on \( i \)-th actors is a function of the product of strength \( S \), immediacy \( J \), and the number of sources \( N \); The strength of influence is the intensity, power or importance of the source of influence. This concept may reflect socio-economical status of the one that affects our opinion, his/her age, prestige or position in the society. The immediacy determines the relationship between the source and the goal of influence. This may mean closeness in the social relationship, lack of communication barriers and ease of communication among actors;

psycho-social law: it states that each next actor \( j \) sharing the same opinion as actor \( i \) exerts the lower impact on the \( i \)-th actor;

division of impact theory: it is based on the bystander effect and is observed as the errors of reacting to crisis events, along with an increase in the number of witnesses to this event.

Based on these assumptions Nowak et al. [21] proposed computerised model of opinion dynamics based on Latané [18–20] social impact theory (see Ref. 63 for review).

Every agent \( i \) is characterised by the following parameters:

opinion \( \xi_i \): the current opinion supported by agent \( i \),

intensity of support \( s_i \): the strength of the agent \( i \) influence on other agents, which determines the ability of this agent to convince other agents not to change his opinion if this opinion is identical to his/her opinion \( (0 \leq s_i \leq 1) \),

intensity of persuasion \( p_i \): the strength of agent \( i \) influence on agents, which determines the ability of this agent to convince other agents to accept his/her opinions \( (0 \leq p_i \leq 1) \).

Each agent is influenced by all other agents on the network. The strength of this influence decreases as the distance between agents increases. In the presented model a cellular automaton was used, which consists of a square grid of \( L^2 \) cells, where exactly one agent is assigned to each cell. The distance \( d_{ij} \) between agents \( i \) and \( j \) is calculated as the Euclidean distance between cells.

To take into account the varied flow of information in the community, we use the \( \alpha \) parameter, which was adapted to scale the distance function. Parameter \( \alpha \) talks about the influence of close and distant neighbours in the community. Small \( \alpha \) values mean good communication between agents and good access to information, because it allows for an exchange of information with a large number of agents in the network. The larger values of \( \alpha \), weaker the communication among the groups of agents, weaker effective exchange of information and weaker access to information, because the exchange of information takes place only in the closest neighbourhood of actors, although we still keep long-range interactions among actors.

Here we are on a position to recapitulate the formal model composition as proposed in Ref. 16.

A. Formal model description

Actors occupy the nodes of the square lattice with linear size \( L \). Every actor \( 1 \leq i \leq L^2 \) is characterised by his/her discrete opinion \( \xi_i \in \{\Xi_1, \Xi_2, \ldots, \Xi_K\} \), where \( K \) is the number of opinions available in the system. Additionally, we assign random real value \( p_i \in [0, 1] \) and \( s_i \in [0, 1] \) describing actor’s persuasiveness and his/her supportiveness, respectively.

The system evolution depends on the social temperature \( T \). If \( T = 0 \), then a lack of randomness is assumed, and the agent \( i \) adopts an opinion \( \Xi_k \) that has the most impact on it:

\[
\xi_i(t + 1) = \Xi_k \iff \quad I_{i,k}(t) = \max(I_{i,1}(t), I_{i,2}(t), \ldots, I_{i,K}(t)),
\]

where \( k \) is the label of this opinion which believers exert the largest social impact on \( i \)-th actor and \( I_{i,k} \) are the social influence on actor \( i \) exerted by actors sharing opinion \( \Xi_k \).

The social impact on actor \( i \) from actors \( j \) sharing opinion of actor \( i \) (\( \xi_j = \xi_i \)) is calculated as
\[ I_{i,k}(t) = 4\mathcal{J}_s \left( \sum_{j=1}^{N} \frac{q(s_j)}{g(d_{i,j})} \delta(\Xi_k, \xi_j(t)) \delta(\xi_j(t), \xi_i(t)) \right) \]  

(2a)

while \( K - 1 \) social impacts on actor \( i \) from all other actors having \( K - 1 \) different opinions (\( \xi_j \neq \xi_i \)) is given as

\[ I_{i,k}(t) = 4\mathcal{J}_p \left( \sum_{j=1}^{N} \frac{q(p_j)}{g(d_{i,j})} \delta(\Xi_k, \xi_j(t))[1 - \delta(\xi_j(t), \xi_i(t))] \right), \]

(2b)

where \( 1 \leq k \leq K \) enumerates the opinions and Kronecker’s delta \( \delta(x, y) = 1 \) if \( x = y \) and zero otherwise [16].

As in Ref. 16 we assume identity function for scaling functions \( \mathcal{J}_S(x) \equiv x, \mathcal{J}_P(x) \equiv x, q(x) \equiv x \). The distance scaling function should be an increasing function of its argument. Here, we assume the distance scaling function

\( g(x) = 1 + x^\alpha \),

what ensures non-zero values \( g(0) = 1 \) of denominator for self-supportiveness in Eq. (2a).

The exponent \( \alpha \) is an arbitrary quantity which characterise the long-range interaction among actors. For small values of \( \alpha \) (for instance for \( \alpha = 2 \)) we assume good communication among actors, good access to information in the society and effective exchange of information. In contrary, for larger values of \( \alpha \) (for instance for \( \alpha = 6 \)) discussion and information exchange takes place only in the actors’ nearest neighbourhood.

For \( T > 0 \), the larger the social temperature \( T \), the more often the opinions, that do not have the greatest impact are selected. As it was shown in Ref. 16 in the modelled system the phase transition occurs: below critical temperature \( T \ll T_c \), the ordered phase is observed with domination of one of the available opinion, while for \( T \gg T_c \) all opinions become equally supported by agents. Critical temperatures \( T_c \) (but for homogeneous society with \( \forall i : s_i = p_i = 0.5 \)) are \( T_c = 6.1 \) and \( T_c = 4.7 \), for two and for three opinions, respectively [16]. In this article, simulations were carried out for \( T \leq 5 \) to take into account different levels of randomness in adopting opinions by agents, reaching the critical level at which agents more often take random opinions than guided by the opinion of their neighbours.

For finite values of social temperature \( T > 0 \) we apply the Boltzmann choice

\[ p_{i,k}(t) = \exp \left( \frac{I_{i,k}(t)}{T} \right), \]

(4)

which yields probabilities

\[ P_{i,k}(t) = \frac{p_{i,k}(t)}{\sum_{j=1}^{K} p_{i,j}(t)} \]

(5)

of choosing by \( i \)-th actor in the next time step \( k \)-th opinion:

\[ \xi_i(t+1) = \Xi_k, \text{ with probability } P_{i,k}(t). \]

The form of dependence (4) in statistics and economy is called logit function [11, 40].

Both, for \( T = 0 \) and \( T > 0 \) the calculated social impacts \( I_{i,k}(t) \) influence the \( i \)-th actor opinion \( \xi_i(t+1) \) at the subsequent time step. Newly evaluated opinions are applied synchronously to all actors. The simulations takes one hundred time steps which ensures reaching a plateau in time evolution of several observables including average probability of sustaining opinion (described in Sec. IIIB) and the number of clusters (discussed in Sec. IIID).

The simulations are carried out on square lattice of linear size \( L = 41 \) with open boundary conditions. We assume random values of supportiveness \( s_i \) and persuasiveness \( p_i \) for all actors. The studies for homogeneous society, i.e. with \( \forall i : p_i = s_i = 0.5 \) were carried out in Ref. 16.

The example of social impact calculations for a small system (with nine actors and three opinions) is given in Appendix A. The model implementation in Fortran95 [64] is attached as Listing 1 in Appendix C.

III. RESULTS

A. Spatial distribution of opinions

We start presentation of our results by showing the spatial distribution of opinions for \( K = 2, 3 \) (various numbers of opinions available in the system), for \( \alpha = 2, 3, 6 \) (various flow of information), for \( T = 0, 1, 2, 3, 5 \) (various level of randomness in adopting the opinion).

1. \( K = 2 \)

In Fig. 1 the simulation results for \( K = 2, \alpha = 3,6 \) and \( T = 0,1,2,3,5 \) are presented. All results are for the same initial random distribution of agents.

Both, \( \alpha \) (information flow) and \( T \) (randomness of actors behaviours) influence opinion formation and polarisation in the groups (self-organisation of opinion clusters). For \( \alpha = 2 \) the consensus takes place (all agents accept one of two opinions, except of few actors for large \( T \) values). The greater the \( \alpha \), the more agents interact
FIG. 1: (Colour online) Spatial distribution of opinions $\xi$ for $K = 2$ opinions for various $\alpha$ and various social temperatures ($T$) after $t = 100$ time steps of the system evolution. Various colours correspond to various agents' opinions.
more effectively with only their closest neighbours, and this leads to the formation of more opinion clusters (the polarization of opinions is therefore smaller). In addition, small clusters are able to survive, although they are surrounded by large clusters of agents with different opinion. An analogous situation occurs when the \(T\) parameter is increased. With larger \(T\), there is more heterogeneity in the areas where actors with different opinions coexist and the division into clusters is less pronounced (less polarization of opinions in the groups is observed). An increase of the social temperature \(T\) often results in the emergence of small clusters of agents with a minority of opinion. As a result, a minority opinion can survive (see Figs. 1k and 1n for \(\alpha = 3\)). In general, the increase of \(T\) and \(\alpha\) causes that more clusters are formed, and the polarization of opinions in groups is weaker.

An interesting phenomenon occurs for \(T = 1\) and \(T = 2\). Low values of the social temperature \(T\) cause a clearer division into supporters opposing the opinion (better polarization) than for \(T = 0\). This is explained in more details in Section III D.

In general, we can observe four types of structures in the formation of opinions for \(K = 2\), after hundred time steps:

- formation of single cluster, when all agents adopt one opinion and consensus takes place \((\alpha = 1\) and \(T = 1, 2, 3, 5)\);
- formation of several large clusters of agents with different opinions—polarisation of the group opinion \((\alpha = 3, T = 1, 2)\);
- formation of plenty small clusters with all opinions \((\alpha = 6, T = 0, 1, 2, 5)\);
- the majority of agents with the same opinion and single agents with opposing opinions scattered across the lattice (other cases).

2. \(K = 3\)

The simulation results for three opinions among agents (where \(K = 3, \alpha = 3, 6\) and \(T = 0, 1, 2, 3, 5\)) are presented in Fig. 2. For \(\alpha = 2\) \((T = 0, 1, 2, 5)\), two separate clusters are formed, in which there are agents with two out of three opinions. Similarly to \(K = 2\), the formation of opinions (the formation of clusters of opinion) depends on the level of randomness of agents’ behaviour and the influence of close and distant neighbours and the influence of close and distant neighbours. For \(K = 3\) (different \(\alpha\) and \(T\)) the consensus among agents with three different opinions is possible only for \(\alpha = 1\). For \(\alpha = 2\) two separate clusters are formed, in which there are agents with two out of three opinions. In the case of larger \(\alpha\) values, smaller clusters are formed but include representatives of all opinions.

In general, we can observe five types of structures in the formation of opinions for \(K = 3\), after hundred time steps:

- formation of single cluster, when all agents adopt one opinion and consensus takes place \((\alpha = 1, T = 1, 2, 3, 5)\);
- formation of two clusters with two opinions—polarisation of the group opinion \((\alpha = 2, T = 1, 2, 3, 5)\);
- formation of few large clusters with all possible opinions—polarisation of the group opinion \((\alpha = 3, T = 1, 2)\);
- formation of plenty small clusters with all opinions \((\alpha = 6, T = 1, 2, 3)\);
- formation of two large clusters with two opinions and single agents with three possible opinions scattered across the lattice \((\alpha = 3 \text{ and } T = 3; \alpha = 2 \text{ and } T = 2)\).

In other results obtained, it is difficult to talk about clusters of opinion—opinions are random and resemble the initial state of simulation.

B. Probability of sustaining opinion

For Figs. 1 and 2 discussed in the previous section, corresponding heat-maps presenting probability of sustaining opinion have been created (see Figs. 3 and 4). Each agent is assigned to one point in the network with a certain colour. This colour depends on the probability with which the agent will sustain his opinion. The colours range from yellow (high probability of sustaining opinion) to black (low probability of sustaining opinion). We do not show heat-maps for \(T = 0\), because the probability of sustaining the opinion has then only two states: 0—the agent will change opinions and 1—the agent will sustain the opinion.

As can be seen in Fig. 3 in the case of two opinions, the probability of sustaining the opinion is affected by both, \(T\) and \(\alpha\). The higher \(T\), the less probability of sustaining the opinion (less yellow), which is especially visible for \(\alpha = 6\) in the Figs. 3i and 3l. The chances of sustaining the current opinion of agents also decrease with the increase of \(\alpha\) (there is more and more darker colour). To sum up, the larger the \(T\) and \(\alpha\), the less yellow colour in the heat-maps is observed (a lower probability of sustaining opinion).

Similarly to two opinions, for \(K = 3\) (see Fig. 4) the higher the social temperature \(T\) and the \(\alpha\) exponent, the less yellow colour in the heat-maps (a lower probability of sustaining opinion).
FIG. 2: (Colour online) Spatial distribution of opinions $\xi$ for $K = 3$ opinions for various $\alpha$ and various social temperatures ($T$) after $t = 100$ time steps of the system evolution. Various colours correspond to various agents’ opinions.
FIG. 3: (Colour online) Spatial distribution of probabilities of sustaining opinion $\mathcal{P}$ for $K = 3$ opinions, social temperature $T$ and for various values of $\alpha$ after $t = 100$ time steps of the system evolution.
FIG. 4: (Colour online) Spatial distribution of probabilities of sustaining opinion $P$ for $K = 3$ opinions, social temperature $T$ and for various values of $\alpha$ after $t = 100$ time steps of the system evolution.

which is particularly evident in Figs. 4h–4i ($T = 3$) and Figs. 4k–4l ($T = 5$).

The probability of sustaining opinion is also visible in Fig. 5. This is the average probability $\langle P \rangle$, that is averaged over all $L^2$ actors. $\langle P \rangle$ depends on both $T$ and $\alpha$. Sustaining opinions by agents on the lattice is definitely higher for low values $\alpha$. The higher $\alpha$, the less probability of sustaining opinion is both for $K = 2$ (Fig. 5a) and $K = 3$ (Fig. 5b). The formation of opinions under the influence of more agents (when a better flow of information in the community is available) results in forming more stable opinions. The social temperature $T$ also affects the average probability of sustaining opinion: the higher the $T$, the lower the probability of sustaining opinion.

To sum up the impact of $T$ and $\alpha$, it should be noted that for high values of $T$ and for low values $\alpha$ the proba-
Generally, the average probability of sustaining opinion after hundred steps of simulation is higher than for low values of $T$ and high values of $\alpha$ (see Fig. 5a and 5b for $T = 5$, $\alpha = 2$ and $T = 1$, $\alpha = 6$ or $T = 5$, $\alpha = 3$).

Now let us look at the average probability of sustaining opinion over time (with hundred simulation steps averaged over hundred initial conditions) in the Fig. 5. In simulations for $K = 2$ and $K = 3$, two main phases are visible. In the first phase a rapid increase of $\langle P \rangle$ is observed, in the second phase the stabilisation of $\langle P \rangle$ takes place. For $K = 2$, the first phase takes place for $t < 20$ iterations (for most cases), and the second phase after $t > 20$ iteration. For $K = 3$, the first phase takes place for $t < 15$ iterations, and the second phase after $t > 15$. Generally, the average probability $\langle P \rangle$ of sustaining opinion over time for $K = 2$ and $K = 3$ generates similar plots, although it is higher for two opinions. This is quite understandable, as for high temperature limit we expect $\lim_{T \to \infty} \langle P \rangle = 1/K$. As considered social temperatures $T \ll T_c$, the results of simulations of $\langle P \rangle$ are still above $1/K$.

C. Spatial distribution of social impact

Figs. 6, 7 and 8 present the spatial distribution of social impact $I_n$ for $n = 1, 9, 25, 49$ and $K = 2, 3, \alpha = 2, 3, 6$ and $T = 0$ after hundred time steps of the system evolution. The social impact $I_n$ on actor $i$ from opinion $\Xi_k$ believers is calculated not based on all actors sharing opinion $\Xi_k$ but based only on those of them who are in $n$-elements large neighbourhood presented in Fig. 9. It means, that for $n = 1$ only impact of self-supportiveness is presented, while for $n = 9$ all actors from the Moore’s neighbourhood influence the central actor (at central, red site) opinion evolution, etc.

1. $K = 2$

For $\alpha = 2$, the opinion $\Xi_1$ ultimately vanishes (not shown), and only social impact from actors with $\xi = \Xi_2$ has non-zero value (as in this case the consensus take place, see Fig. 1a). With increasing number of sites $n$ in the neighbourhood an increase of both $I_n$ and $I_n/I$ is observed. This observation is valid also for larger values of $\alpha$ and also for more opinions $K$ available in the community.

Now let us merge information from the second, fourth and sixth columns of Figs. 6 also from (not shown) influence from supporters of opinion $\Xi_1$. In Fig. 7 (for $K = 2$, $\alpha = 2, 3, 6$) the opinion independent relative impacts $I_n/I$ are presented. The total impact $I$ is calculated without any spatial restrictions—i.e. according to Eq. (2)—from all actors independently on their distance to the central (red in Fig. 9) site. For the averaging procedure we sum $I_n$ and $I$ for site $i$ from all $K$ opinions available in the community. Namely, we calculate the

\[
\langle P \rangle = \frac{1}{K} \sum_{k=1}^{K} I_n
\]
Table I: Average fraction \( \langle I_n/I \rangle \) of total (i.e. summed over all \( K \) opinions) social impact \( I \) as dependent on the number \( n \) of the nearest-neighbours and the distance function exponent \( \alpha \). The average \( \langle \cdots \rangle \) symbol stands for the spatial average over all agents.

| \( n \)   | \( K = 2 \) | \( K = 3 \) | \( K = 6 \) |
|---------|------------|------------|------------|
|         | \( \alpha = 2 \) |            |            |
| 1       | 0.061      | 0.148      | 0.280      |
| 9       | 0.255      | 0.587      | 0.956      |
| 25      | 0.399      | 0.764      | 0.993      |
| 49      | 0.501      | 0.846      | 0.998      |
|         | \( \alpha = 3 \) |            |            |
| 1       | 0.061      | 0.149      | 0.283      |
| 9       | 0.256      | 0.588      | 0.956      |
| 25      | 0.401      | 0.765      | 0.993      |
| 49      | 0.503      | 0.847      | 0.998      |
|         | \( \alpha = 6 \) |            |            |

We repeat the calculations of \( I_n \) also for \( K = 3 \) opinions available in the community.

For \( K = 3 \) we present only the influence \( I_n \) from believers of opinion \( \Xi_3 \) (see Fig. 8) omitting presenting \( I_n/I \).

The impacts from all opinions \( \Xi_1, \Xi_2, \Xi_3 \) available in the community are again summed-up over all actors and presented as \( \langle I_n/I \rangle \) at the bottom of Tab. I. As we can see, an increase in number of available opinions \( K \) does not influence fraction \( I_n/I \), when \( \alpha \) and \( n \) are fixed.

For \( K = 3 \), we can clearly see the influence of \( \alpha \) on the results. For \( \alpha = 6 \) over 95% of social impact is gathered from nine actors in the Moore’s neighbourhood (presented in Fig. 9b), but for \( \alpha = 2 \) even \( n = 49 \) of the closest neighbours exerts only half of the total social impact.

D. Clustering of opinions

Analysing the results in the previous section, the clustering of opinions is influenced by both the level of randomness in agents’ decisions \( T \) and the influence coming from neighbours \( \alpha \).

We apply the Hoshen–Kopelman algorithm [65] for clusters detection. In Hoshen–Kopelman algorithm each actor is labelled in such way, that actors with the same opinions and in the same cluster have identical labels. The algorithm allows for cluster detection in multidimensional space and for complex neighbourhoods [66–70], here however, we assume the simplest case, i.e. square lattice with von Neumann neighbourhood (see Fig. 10).

In Table II, the relative size \( S_{max}/L^2 \) of the largest cluster after \( t = 100 \) simulations time steps for different values of \( T \), and \( \alpha \) for \( K = 2, 3 \) has been shown. These results coincide with Figs. 1 and 2. In all cases, for \( \alpha = 1 \) (\( K = 2 \) and \( K = 3 \) opinions), the largest cluster fills the entire lattice in 100% (i.e. there is a consensus in opinion). In these cases, the histogram \( H(S) \) of cluster sizes \( S \) is given by Kronecker’s delta

\[
H(S) = R \cdot \delta(S, L^2),
\]

where \( R \) is the number of independent runings (here \( R = 10^3 \)).

It should be noted that this is a situation in which most agents in the lattice influence the opinion of the selected agent (the flow of information between agents is very good). For \( \alpha = 2 \), the influence of other agents in the lattice is still greater than the influence of the closest neighbourhood (see Table I, \( n = 9 \) and \( n = 25 \)). In the case of simulations for \( K = 2 \) (two opinions), the results are analogous to \( \alpha = 1 \), while for \( K = 3 \) (three opinions) there is no consensus, but the largest cluster consists of more than 50% of all agents in the lattice (except of when \( T = 5 \) — where we have 48.9%). In general, the size of the
\[ \alpha = 2 \]

(a) \( n = 1 \)

(b) \( I_1 \)

(c) \( \frac{I_1}{T} \)

(d) \( I_1 \)

(e) \( \frac{I_1}{T} \)

(f) \( I_1 \)

(g) \( n = 9 \)

(h) \( I_9 \)

(i) \( \frac{I_9}{T} \)

(j) \( \frac{I_9}{T} \)

(k) \( I_9 \)

(l) \( \frac{I_9}{T} \)

(m) \( n = 25 \)

(n) \( I_{25} \)

(o) \( \frac{I_{25}}{T} \)

(p) \( \frac{I_{25}}{T} \)

(q) \( I_{25} \)

(r) \( \frac{I_{25}}{T} \)

(s) \( n = 49 \)

(t) \( I_{49} \)

(u) \( \frac{I_{49}}{T} \)

(v) \( \frac{I_{49}}{T} \)

(w) \( I_{49} \)

(x) \( \frac{I_{49}}{T} \)

FIG. 6: (Colour online) Spatial distribution of influence \( I_n \) and \( I_n/I \) for opinion \( \Xi_2 \), \( n = 1, 9, 25 \) and 49 for \( K = 2 \) opinions, \( \alpha = 2, 3, 6 \), social temperatures \( T = 0 \) after \( t = 100 \) time steps of the system evolution. \( L = 41 \)

TABLE II: The relative size of the largest cluster \( S_{\text{max}}/L^2 \) for \( L = 41 \).

| \( \alpha \) | 1 | 2 | 3 | 6 |
|---|---|---|---|---|
| \( T = 0 \) | 100% | 100% | 42.5% | 34.0% |
| 1 | 100% | 100% | 43.6% | 43.5% |
| 2 | 100% | 100% | 53.8% | 30.9% |
| 3 | 100% | 100% | 97.7% | 16.9% |
| 5 | 99.8% | 85.4% | 10.1% | |
| \( T = 0 \) | \( K = 3 \) | | | |
| 1 | 66.7% | 20.6% | 9.1% | |
| 2 | 59.9% | 37.9% | 34.1% | |
| 3 | 72.3% | 74.1% | 27.0% | |
| 3 | 75.6% | 73.4% | 2.0% | |
| 5 | 48.9% | 8.1% | 1.7% | |

The largest cluster of opinions decreases with the increase of \( \alpha \).

Let us look at the impact of social temperature \( T \)—which is responsible for randomness in taking opinions—on the results of the simulation (Figs. 1–2). Interesting phenomenon for \( K = 3 \) opinions is visible. For \( \alpha = 2, 3, 6 \), the size of the largest cluster increases with \( T \) to a certain point, and then decreases. This point of inflection takes place around \( T = 2, T = 3 \), which can also be seen in Figs. 2. It means, that slight randomness in taking opinions helps in the clustering of opinions but too high randomness in taking opinions destroys this clustering. For \( \alpha \leq 3 \) (independently on considered social temperature \( T \)) the left side of histograms \( H(S) \) of cluster sizes for \( S < 10 - 100 \) may be approximated by power laws

\[ H(S) \propto S^{-\gamma}, \tag{9} \]

with various \( \gamma \) exponents—for example \( \gamma = 0.987 \) (for \( K = 2, \alpha = 3 \) and \( T = 0 \)) and \( \gamma = 1.69 \) (for \( K = 2, \alpha = 6 \) and \( T = 5 \)).

In order to get a better look at the clustering of opinions, we also studied the number of small and large clusters after hundred steps of simulation (see Fig. 11 and 12). The example of cluster counting and their size measurement is provided in Appendix B.

The randomness in accepting opinions \( (T) \) often results in the formation of small clusters of opposing opinions (as can be seen in the figures in the previous section). We assumed that small clusters contain no more than five agents with the same opinion (\( S \leq 5 \)). Figs. 11 and 12 show the simulation results for \( K = 2 \), and for \( K = 3 \), respectively. The number of clusters in both cases increases with an increase of \( \alpha \), i.e. the smaller the influence of all agents in the network on the selected agent, the more difficult it is for clustering opinions. In general, the number of clusters increases with \( T \), but for \( \alpha = 3 \) and \( \alpha = 6 \) and for \( T = 1 \), the number of clusters is lower than for \( T = 0 \). First of all, the probability of a new cluster form-
ing (with different opinion) inside another cluster is still very small. Secondly, the probability of changing opinions is greater at the border between two clusters than inside them. If the adjacent clusters have different sizes, then the probability of changing the agent’s opinion in a smaller cluster is greater than in the larger one. This situation leads to the disappearance of small clusters, or in general to a reduction in number of clusters. We can see similar phenomena in terms of only the number of small clusters (Figs. 11 and 12). The number of small clusters increases with $T$ (of course, apart from low $T$ values and for $\alpha > 1$).

Fig. 13 shows the dynamics of changing the number of clusters $n_c$, averaged over hundred runnings with different initial conditions for different values of $T$ and $\alpha$, and simulations for $K = 2$ (left panel) and $K = 3$ (right
(a) $\alpha = 2, n = 1$

(b) $\alpha = 3, n = 1$

(c) $\alpha = 6, n = 1$

(d) $\alpha = 2, n = 9$

(e) $\alpha = 3, n = 9$

(f) $\alpha = 6, n = 9$

(g) $\alpha = 2, n = 25$

(h) $\alpha = 3, n = 25$

(i) $\alpha = 6, n = 25$

(j) $\alpha = 2, n = 49$

(k) $\alpha = 3, n = 49$

(l) $\alpha = 6, n = 49$

FIG. 8: (Colour online) Spatial distribution of influence $I_n$, for opinion $\Xi_3$, $n = 1, 9, 25$ and 49 for $K = 3$ opinions, $\alpha = 2, 3, 6$ social temperatures $T = 0$ after $t = 100$ time steps of the system evolution. $L = 41$

On the basis of the plots in Fig. 13, two main phases can be distinguished in the dynamics of the number $n_c$ of opinion clusters:

- The first phase is the formation of larger clusters of opinion (the smallest clusters disappear quickly). The number of clusters is rapidly dropping. This process takes between 0 – 20 iterations for $K = 2$ and between 0 – 15 iterations for $K = 3$;

- The second phase is the polarity of the system. The rate of changes in the number of clusters stabilises. This process takes place after 15 – 20 iterations.

In addition, as can be seen in Fig. 13, the number of
clusters \( n_c \) depends on \( T \) and \( \alpha \). The bigger \( T \) and \( \alpha \), the more clusters of opinion are formed for both two and three possible initial opinions.

**IV. SUMMARY AND CONCLUSIONS**

In this paper, we are interested in how opinions are formed and how they spread in the community. We were investigating how flow of information in the community and randomness of human behaviour influence formation of opinions, its spreading and its polarisation. The community was presented as a square lattice of linear size \( L \) with open boundary conditions, which is fully filled by agents.

The flow of information was control by the parameter \( \alpha \). This parameter reflects the effective impact of the neighbourhood on the opinion of the agents. In case of low values of this parameter, agents shape their opinion basing on a large number of agents (including distant neighbours). In our research, we also take into account the randomness in adopting opinions, which is expressed in the parameter \( T \). The larger \( T \), the more often agents adopt opinions which have no greatest impact on them.

Each agent in our model is characterised, in addition to the opinion, by two parameters. They are the intensity of persuasion \( (p_i) \) and the intensity of support \( (s_i) \). The higher the value of persuasiveness \( p_i \), the agent more easier convincing other agents to accept his/her opinion. With bigger \( (s_i) \), the agent convinces more strongly other agents. These parameters therefore determine the effectiveness of which an individual may interact with or influence other individuals by changing or confirming their opinions. In all performed simulations, we adopted random values of \( (p_i) \) and \( (s_i) \) parameters, which brings us closer to the social reality, in which we do not usually have data on the strength with which the unit affects other units. Simulations have been carried out when agents have a choice of two or three opinions on a given topic. First, the spatial distribution of opinions after hundred steps of simulation was analysed. The simulations showed how clusters of opinion are formed depending on the flow of information in the agents’ network, as well as after considering the randomness in forming the opinion. Consensus (one large cluster) was possible for \( K = 2 \) (two opinions) for low values \( \alpha \) \((\alpha = 1, 2)\), when agents formed their opinions also contacting more distant agents (that is, when the flow of information was good in the whole agents community). For three opinions \((K = 3)\), the consensus was possible only when \( \alpha = 1 \).

Generally, the greater \( \alpha \) and \( T \), the more clusters, i.e. groups of agents with the same opinions are observed (less polarisation of opinions in the groups). An interesting phenomenon occurs for \( T = 1 \) and \( \alpha > 2 \) (both for two and three opinions). In this case, the polarization of opinions is more clearly visible than for \( T = 0 \). This phenomenon is also confirmed by the study on the number of clusters of opinion (Fig. 13). If we assume a low level of randomness in adopting opinions \((T = 1)\), there is a little chance that agents with opposing opinion will appear in polarised opinion groups. Therefore, there is a low probability that new clusters (with opposite opinions) will be created among existing ones.

With the formation of opinion clusters, the probability of sustaining opinion is closely related. This probability is greater within the clusters than at their borders and it is larger in the larger clusters (Figs. 4 and 5). This leads to the disappearance of small clusters, and thus to reduction of the number of clusters.

This phenomenon can also be observed by analysing the average probability of sustaining opinion over time (see Fig. 5). Both for \( K = 2 \) and \( K = 3 \), two main phases are visible. The first phase is a rapid increase in \( \langle P \rangle \), the second phase is stabilisation \( \langle P \rangle \). In the first phase (except of when \( T \) and \( \alpha \) is large), \( \langle P \rangle \) is growing rapidly, because larger clusters of opinion are formed, and small clusters disappear, and as mentioned earlier, the probability of sustaining opinion is higher in larger clusters.
Now let us look at the Figs. 6 (for $K = 2$) and 8 (for $K = 3$), on which the spatial distribution of social impact is presented. With the exception of cases for $K = 2$ and $\alpha = 2$, (when the consensus takes place), decisions of individual agents are influenced by more agents inside the clusters of opinions than on their outskirts. The relative impact $I_n/I$ is also greater in clusters than on their borders. In addition, this effect increases with $\alpha$ (more and more bright colour).

The formulation of opinions and its spread is also described by the number and size of clusters of opinion, as well as the change in the number of clusters in time. The clustering of opinions is influenced by both the level of randomness in agents’ decisions and the influence coming from neighbours. Generally, the size of the largest cluster of opinions decreases with the increase of $\alpha$. This, of course, corresponds with the spatial distribution of opinions (see Figs. 1 and 2). Furthermore, the number of clusters for both $K = 2$ and $K = 3$ increases with $\alpha$, i.e. the smaller the influence of all agents in the network on the selected agent, the more difficult it is for forming clusters of opinions. An interesting phenomenon occurs in the case of analysis of the impact of $T$. The number of clusters increases with $T$, but for $\alpha = 3$ and $\alpha = 6$ and for low values of $T$, the number of clusters is lower than for $T = 0$. The probability of a new cluster forming (with different opinion) inside another cluster is then small and the probability of changing opinions is greater at the border between two clusters than inside them. Additionally, the probability of changing the agent’s opinion in a smaller cluster is greater than in the larger one, which leads to the disappearance of small clusters, or in general to a reduction in the number of clusters.

Analysing the change in the number of clusters over
time, as in the case of the average probability of sustaining opinion, the two phases take place. In the first phase, the number of clusters is rapidly dropping, and in the second phase, the number of clusters stabilises, although there are fluctuations, especially for the smaller $\alpha$ and $T$ (similar to the average probability of sustaining opinion). Comparing Fig. 13 and Fig. 5, in the first phase the number of clusters decreases rapidly and the average probability of sustaining opinion increases, because larger opinion clusters are formed in which it is easier to sustaining opinion.

In summary, the simulations showed that opinion formation and spread is influenced by both flow of information between agents (interactions outside the closest neighbours) and randomness in adopting opinions (what is shown in Table III). Better information flow, i.e. contacts with more agents in the network representing the community, facilitates the spread of opinion and its formation. In the case of small values of $\alpha$ (when information flow is very good), the result of the simulation is a consensus, as in most socio-physical models of social dynamics [38], for both the two and three initial opinions. For large values of $\alpha$ (when opinions are consulted only in the close neighbourhood), the polarisation of opinions is weak and there are many small groups of agents with the same opinion. In addition, the presence of many small clusters causes a lower probability of sustaining opinion for individual agents, i.e. they change their opinions more often.

The lack of consensus in models is mainly caused by the introduction of ‘noise’ [71] or anti-conformism [72]. In the presented model there is no global agreement also for $T = 0$ (when there is no noise). Clusters of both opinions (or three for $K = 3$) appear for large values of
α, i.e. when opinion consultations take place only with close neighbours.

As it was mentioned earlier, many studies indicate irrationality and unpredictability in the process of forming opinions [9–13]. As our simulations have shown, this randomness in adopting opinions plays a big role. The higher the level of randomness, the more opinion clusters and the lack of large clusters. However, low level of randomness (low values of $T$) cause that less clusters of opinions are created than in the absence of randomness ($T = 0$). This is the case for $\alpha \geq 2$. So randomness in a sense favours the polarisation of opinions in groups

FIG. 13: (Colour online) Time evolution of the number of clusters $n_c$ for various values of the number $K$ of available opinions, temperature $T$ and exponent $\alpha$. $L = 41$. The results (lines) are averaged over hundred various initial conditions while symbols show only single lattice realisation.
TABLE III: The impact of both $T$ and $\alpha$ on the formation and spread of opinions.

| Impact on                                    | $T$                                   | $\alpha$                      |
|----------------------------------------------|---------------------------------------|-------------------------------|
| Spatial distribution of opinions             | The greater the $T$, there are more and more clusters | The greater the $\alpha$, there are more and more clusters |
| Probability of sustaining opinion            | The greater $T$, the less probability of sustaining the opinion | The greater $\alpha$, the chances of sustaining the current opinion of agents decrease |
| Spatial distribution of social impact        | —                                     | The relative impact $I_{\alpha}/I$ increases with $\alpha$ |
| Clustering of opinions                       | The number of clusters increases with $T$ | The number of clusters increases with $\alpha$. The size of the largest cluster decreases with the increase of $\alpha$ |
| Summary                                      | Generally randomness hinders polarization of opinions in groups | The better the flow of information in the community, the easier it is to form and spread opinions, which can also lead to consensus |

but only when the influence from distant neighbours is smaller.

In future research, we intend to take into account the impact of strong leaders on the opinion dynamics. Also the influence of external sources of information (for instance the impact of mass media) is worth of investigation.

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exerted on actor $i = 5$ for three opinions available in the system. As $\xi_i(t) = \Xi_2$ (‘blue’) we use Eq. (2a) to calculate impact

$$I_{5,\text{blue}}(t) = 4J_p \left( \frac{q(s_5)}{g(d_{5,1})} + \frac{q(s_6)}{g(d_{5,3})} + \frac{q(s_9)}{g(d_{5,9})} \right),$$

from all actors with ‘blue’ opinions (i.e. for $i = 6,9$), including actor $i = 5$ himself/herself. The impacts from actors with ‘red’ and ‘green’ opinions are calculated basing on Eq. (2b):

$$I_{5,\text{red}}(t) = 4J_p \left( \frac{q(p_1)}{g(d_{5,1})} + \frac{q(p_3)}{g(d_{5,3})} + \frac{q(p_4)}{g(d_{5,4})} + \frac{q(p_7)}{g(d_{5,7})} \right),$$

$$I_{5,\text{green}}(t) = 4J_p \left( \frac{q(p_2)}{g(d_{5,2})} + \frac{q(p_8)}{g(d_{5,8})} \right),$$

We assume identity function for scaling functions $J_S(x) \equiv x$, $J_P(x) \equiv x$, $q(x) \equiv x$ and the distance scaling function $g(x) = 1 + x^\alpha$, with $\alpha = 2$. These assumptions yield

$$I_{5,\text{blue}}(t) = 4 \left( \frac{s_5}{1 + d_{5,1}^2} + \frac{s_6}{1 + d_{5,3}^2} + \frac{s_9}{1 + d_{5,9}^2} \right) = 4 \left( \frac{0.5}{1 + 0^2} + \frac{0.6}{1 + 1^2} + \frac{0.9}{1 + \sqrt{2}^2} \right) = 4.4,$$

$$I_{5,\text{red}}(t) = 4 \left( \frac{p_1}{1 + d_{5,1}^2} + \frac{p_3}{1 + d_{5,3}^2} + \frac{p_4}{1 + d_{5,4}^2} + \frac{p_7}{1 + d_{5,7}^2} \right) = 4 \left( \frac{0.9}{1 + \sqrt{2}^2} + \frac{0.7}{1 + \sqrt{2}^2} + \frac{0.6}{1 + 1^2} + \frac{0.3}{1 + \sqrt{2}^2} \right) = 7.3,$$

$$I_{5,\text{green}}(t) = 4 \left( \frac{p_2}{1 + d_{5,2}^2} + \frac{p_8}{1 + d_{5,8}^2} \right) = 4 \left( \frac{0.8}{1 + 1^2} + \frac{0.2}{1 + 1^2} \right) = 2.$$
For $T = 0$ the largest impact on actor $i = 5$ is exerted by ‘red’ actors and thus—according to Eq. (1)—actor $i = 5$ in the next time step will change his/her opinion from ‘blue’ ($\xi_5(t) = \xi_2$) to ‘red’ ($\xi_5(t+1) = \xi_1$).

For $T > 0$ we calculate probabilities $P_{5,\text{blue}}$, $P_{5,\text{red}}$ and $P_{5,\text{green}}$ of choosing opinion by actor $i = 5$ (see Eqs. (4)–(5)). For example, for $T = 1$ these probabilities are

$$P_{5,\text{blue}} = \frac{\exp(I_{5,\text{blue}}/1)}{P_1},$$

$$P_{5,\text{red}} = \frac{\exp(I_{5,\text{red}}/1)}{P_1},$$

$$P_{5,\text{green}} = \frac{\exp(I_{5,\text{green}}/1)}{P_1},$$

while for $T = 10$ we have

$$P_{5,\text{blue}} = \frac{\exp(I_{5,\text{blue}}/10)}{P_{10}},$$

$$P_{5,\text{red}} = \frac{\exp(I_{5,\text{red}}/10)}{P_{10}},$$

$$P_{5,\text{green}} = \frac{\exp(I_{5,\text{green}}/10)}{P_{10}},$$

where normalisation constants are

$$P_1 = \exp(I_{5,\text{blue}}/1) + \exp(I_{5,\text{red}}/1) + \exp(I_{5,\text{green}}/1)$$

and

$$P_{10} = \exp(I_{5,\text{blue}}/10) + \exp(I_{5,\text{red}}/10) + \exp(I_{5,\text{green}}/10).$$

The calculated probabilities for $T = 1$ are

$$P_{5,\text{blue}} = \frac{\exp(4.4/1)}{e^{4.4} + e^{7.3} + e^2} \approx 0.050,$$

$$P_{5,\text{red}} = \frac{\exp(7.3/1)}{e^{4.4} + e^{7.3} + e^2} \approx 0.945,$$

$$P_{5,\text{green}} = \frac{\exp(2/1)}{e^{4.4} + e^{7.3} + e^2} \approx 0.005,$$

while for $T = 10$ we have

$$P_{5,\text{blue}} = \frac{\exp(44/10)}{e^{0.44} + e^{0.73} + e^{0.2}} \approx 0.320,$$

$$P_{5,\text{red}} = \frac{\exp(73/10)}{e^{0.44} + e^{0.73} + e^{0.2}} \approx 0.429,$$

$$P_{5,\text{green}} = \frac{\exp(2/10)}{e^{0.44} + e^{0.73} + e^{0.2}} \approx 0.251.$$

For non-deterministic version of algorithm (i.e. for $T > 0$) still the most probably state $\xi_5(t+1)$ is $\xi_1$ (‘red’). But probability of such evolution for actor $i = 5$ decreases from 100% for $T = 0$ to 94.5% for $T = 1$ and to 42.9% for $T = 10$ to become 33.3% = $1/K$ for $T \to \infty$.

Let us repeat these calculation for actor $i = 9$:

$$I_{9,\text{blue}}(t) = 4J \left( \frac{q(s_9)}{g(d_{9,5})} + \frac{q(s_6)}{g(d_{9,6})} + \frac{q(s_8)}{g(d_{9,9})} \right),$$

$$I_{9,\text{red}}(t) = 4J \left( \frac{q(p_1)}{g(d_{9,1})} + \frac{q(p_3)}{g(d_{9,3})} + \frac{q(p_4)}{g(d_{9,4})} + \frac{q(p_7)}{g(d_{9,7})} \right),$$

$$I_{9,\text{green}}(t) = 4J \left( \frac{q(p_2)}{g(d_{9,2})} + \frac{q(p_8)}{g(d_{9,8})} \right),$$

For $T = 0$ the largest impact on actor $i = 9$ is exerted by ‘blue’ actors and thus—according to Eq. (1)—actor $i = 9$ in the next time step will sustain his/her ‘blue’ opinion ($\xi_9(t+1) = \xi_9(t) = \xi_2$). Two factors influence the difference in actors $i = 5$ and $i = 9$ opinion in time ($t+1$). Namely, the difference in supportiveness of these two actors and their distance to ‘red’ actors: agent $i = 5$ has moderate supportiveness ($s_5 = 0.5$) and his/her distance to ‘red’ actors is no longer than $\sqrt{2}$. In contrast, actor $i = 9$ has very high supportiveness ($s_9 = 0.9$) and distance to ‘red’ actors no shorter than 2. Please note however, that ultimate fate of the system is the state with the unanimity of opinions. As we have shown above, in the next time step at least the actor in the middle of the system ($i = 5$) will convert his opinion to the ‘red’ one. The same presumably will occur for actor $i = 2$ who has low supportiveness ($s_2 = 0.2$) and who has only a single supporter. Thus in time ($t+3$) all actors will convert to the supporters of the ‘red’ opinion.

For $T > 0$ we calculate probabilities $P_{9,\text{blue}}$, $P_{9,\text{red}}$ and $P_{9,\text{green}}$ of choosing opinion by actor $i = 9$ (see Eqs. (4)–(5)). For example, for $T = 1$ these probabilities are

$$P_{9,\text{blue}} = \frac{\exp(I_{9,\text{blue}}/1)}{P_1},$$

$$P_{9,\text{red}} = \frac{\exp(I_{9,\text{red}}/1)}{P_1},$$

$$P_{9,\text{green}} = \frac{\exp(I_{9,\text{green}}/1)}{P_1},$$

$$I_{9,\text{red}}(t) = 4J \left( \frac{q(p_1)}{g(d_{9,1})} + \frac{q(p_3)}{g(d_{9,3})} + \frac{q(p_4)}{g(d_{9,4})} + \frac{q(p_7)}{g(d_{9,7})} \right),$$

$$I_{9,\text{green}}(t) = 4J \left( \frac{q(p_2)}{g(d_{9,2})} + \frac{q(p_8)}{g(d_{9,8})} \right),$$

$$I_{9,\text{blue}}(t) = 4J \left( \frac{q(s_9)}{g(d_{9,5})} + \frac{q(s_6)}{g(d_{9,6})} + \frac{q(s_8)}{g(d_{9,9})} \right).$$
while for $T = 10$ we have

$$P_{9,\text{blue}} = \frac{\exp(I_{9,\text{blue}}/10)}{P_{10}},$$

$$P_{9,\text{red}} = \frac{\exp(I_{9,\text{red}}/10)}{P_{10}},$$

$$P_{9,\text{green}} = \frac{\exp(I_{9,\text{green}}/10)}{P_{10}},$$

where normalisation constants are

$$P_1 = \exp(I_{9,\text{blue}}/1) + \exp(I_{9,\text{red}}/1) + \exp(I_{9,\text{green}}/1)$$

and

$$P_{10} = \exp(I_{9,\text{blue}}/10) + \exp(I_{9,\text{red}}/10) + \exp(I_{9,\text{green}}/10).$$

The calculated probabilities for $T = 1$ are

$$P_{9,\text{blue}} = \frac{\exp(5(4)/6)}{e^{5.4(6)}/1 + 1.6 + e^{0.9(3)}} \approx 0.969,$$

$$P_{9,\text{red}} = \frac{\exp(1.6/1)}{e^{5.4(6)}/1 + 1.6 + e^{0.9(3)}} \approx 0.020,$$

$$P_{9,\text{green}} = \frac{\exp(0.9(3)/1)}{e^{5.4(6)}/1 + 1.6 + e^{0.9(3)}} \approx 0.011.$$

while for $T = 10$ we have

$$P_{9,\text{blue}} = \frac{\exp(5(4)/6)}{e^{5.4(6)}/10 + 1.6 + e^{0.9(3)}} \approx 0.432,$$

$$P_{9,\text{red}} = \frac{\exp(1.6/10)}{e^{5.4(6)}/10 + 1.6 + e^{0.9(3)}} \approx 0.293,$$

$$P_{9,\text{green}} = \frac{\exp(0.9(3)/10)}{e^{5.4(6)}/10 + 1.6 + e^{0.9(3)}} \approx 0.275.$$

Similarly to the actor $i = 5$, the increase of the social temperature reduces chance of keeping initial opinion for actor $i = 9$. For $T = 10$ these probabilities do not differ from $1/K$ for more than 0.1.

### Appendix B: Small example of clustering ($L = 10, K = 3$)

Two sites are in the same cluster if they are adjacent (in von Neumann neighbourhood) to each other and simultaneously actors at these sites share the same opinion. The Hoshen–Kopelman algorithm allows for sites labelling in such way that sites in the same cluster have the same labels and sites in different cluster have different labels. Example of sites labelling for $L = 10$ and $K = 3$ is presented in Fig. 15, where $n_c = 11$ clusters have been identified. The time evolution of this number $n_c$ is presented in Fig. 13.

The number of sites in each cluster defines its size $S$. In given example this histogram is presented in Table IV.

### Appendix C: Source code

In Listing 1 the Fortran 95 code allowing for reproductions of data for Figs. 1, 2, 3, 4, 5, 11, 12, 13 (for non-deterministic version of simulations, i.e. for $T > 0$) is presented.

The module settings provides model parameters including lattice size $L$, number of opinions $K$, social temperature $T$, number of time steps $t_{\text{max}}$ and exponent in distance scaling function $\alpha$.

In module utils the scaling functions $g(x)$ and $q(x)$ as well as the Euclidean distance $d(x,y)$ are defined. Also the reclassify function for Hoshen–Kopelman algorithm is defined there.

The main program starts in line 52. The actors supportiveness ($s_i$) and persuasiveness ($p_i$) are initialised randomly in lines 85–90, while initial actors opinions ($\xi_i$) are given in lines 96–100. Loop 88 provides time evolution of the system. Loop 77 realises Hoshen–Kopelman algorithm of sites (actors) labelling. In lines 181–193 Eqs. (2a) and (2b) are implemented. In lines 216–222 heatmaps of social impact are printed. In lines 232–235 heatmaps of probability of sustaining opinions are printed while in lines 407–410 the average probability of sustaining opinions ($\langle P \rangle$) are calculated and printed. In lines 412–415 histograms of clusters size are printed and number of cluster is calculated. In loop 99 the system characteristics after the system time evolution is completed are calculated. Loop 777 realises averaging procedure over independent runnings for various initial conditions.
Listing 1: Fortran95 code implementing Eq. (2) i.e. for $T > 0$

```fortran
module settings

implicit none

integer, parameter :: Xmax=41, Ymax=41, Tmax=100, Kmax=3, x_L=1+Xmax/2, y_L=1+Ymax/2, L2=(Xmax+1)*(Ymax+1), Run=1000
real*8, parameter :: alpha=1.0d0
real*8 :: T
end module settings

module utils

use settings

implicit none
contains

real*8 function g(x)
    real*8 :: x
    g=1.0d0+x**alpha
end function

real*8 function q(x)
    real*8 :: x
    q=x
end function

real*8 function d(x1,y1,x2,y2)
    integer :: x1,y1,x2,y2
    d=dsqrt((1.d0*x1-1.d0*x2)**2 + (1.d0*y1-1.d0*y2)**2)
end function

integer function reclassify (ix)
    integer :: ix
    integer, dimension (0:Xmax,0:Ymax) :: label
    integer, dimension (L2) :: iclass
    common/block/ label , iclass
    reclassify=iclass(ix)
do
    if(iclass(reclassify).eq.reclassify) return
    reclassify=iclass(reclassify)
goto 90
end do

end function

end module utils

program Latane_Hoshen_Kopelmann

use settings
use utils

implicit none
integer :: x,y, it , xx,yy,k,kk, strongest_k , irun , maxlabel , nc
real :: r
real*8, dimension (0:Tmax) :: ave_Probsustain

integer, dimension (0:Xmax,0:Ymax) :: label
integer, dimension (L2) :: iclass
integer, dimension (0:Xmax*Ymax) :: isize , histogram , ave_histogram
integer, dimension (0:Xmax*Ymax,Kmax) :: histogramK
integer, dimension (0:Tmax) :: ave_nc

end program Latane_Hoshen_Kopelmann
```

real*8, dimension (Xmax, Ymax) :: p, s
real*8, dimension (Xmax, Ymax, Kmax) :: I, prob
common/block/ label, iclass
ave_nc=0
ave_Probsustain=0.d0
ave_histogram=0
s_L=0.50d0
p_L=0.50d0
read *,T
print '(A3,1A11,2I11) ', '###', 'leader/uni2423@', x_L, y_L
print '(A3,7A11) ', '###', 'Xmax', 'Ymax', 'K', 'alpha', 'T', 's_L', 'p_L'
print '(A3,3I11,4F11.3) ', '###', Xmax, Ymax, Kmax, alpha, T, s_L, p_L

do 777 irun=1,Run
do x=1,Xmax !! initial state
do y=1,Ymax
  s(x,y)=rand()
  p(x,y)=rand()
endo
endo
s(x_L,y_L)=s_L
p(x_L,y_L)=p_L
it=0
xi=0
do x=1,Xmax
do y=1,Ymax
  xi(x,y)=1+Kmax*rand()
endo
endo
xi(x_L,y_L)=Kmax

!! print *, '# it=', it, 'xi: '
!! do x=1,Xmax
!! print '(41I5)', (xi(x,y), y=1,Ymax)
!! enddo
histogram=0
histogramK=0

do 66 k=1,Kmax
  isize=0
  if(k.eq.xi(x_L,y_L)) print *, "###.leader*
  label=L2
  do kk=1,L2
    iclass(kk)=kk
  enddo
  maxlabel=0
  do x=1,Xmax
do y=1,Ymax
    if(xi(x,y).eq.k) then !! labeling clusters
      if(xi(x-1,y).eq.k .or. xi(x,y-1).eq.k) then
        !! reclassifying neighbouring sites
        if(xi(x-1,y).eq.k) label(x-1,y)=reclassify(label(x-1,y))
        if(xi(x,y-1).eq.k) label(x,y-1)=reclassify(label(x,y-1))
        label(x,y)=min(label(x-1,y),label(x,y-1))
        iclass(label(x-1,y))=label(x,y)
        iclass(label(x,y-1))=label(x,y)
      else
        maxlabel=maxlabel+1
        label(x,y)=maxlabel
      endif
    endif
  endif
endo
endo
!! reclassifying all occupied sites
do x=1,Xmax
  do y=1,Ymax
    if((xi(x,y) .eq. k) .and. (label(x,y) .gt. iclass(label(x,y)))) label(x,y)=reclassify(label(x,y))
  enddo
enddo

!!!  do x=1,Xmax
!!!  print '(41 I5 ) ',label(x,y),y=1,Ymax
!!!  enddo

!!!  do x=1,Xmax
!!!  isize(label(x,y))=isize(label(x,y))+1
!!!  enddo
enddo

! do kk=1,Xmax*Ymax
!  if(isize(kk)>0) print *,#_histogram ,_k=",k
! enddo

! do kk=1,Xmax*Ymax
!  histogram(isize(kk))=histogram(isize(kk))+1
!  histogramK(isize(kk),k)=histogramK(isize(kk),k)+1
! enddo
print *,"#_histogram , _k=",k
do kk=1,Xmax*Ymax
  if(histogramK(kk,k)>0) print *,kk,histogramK(kk,k)
enddo

nc=0
print *,"#_histogram_before:",nc=0
do k=1,Xmax*Ymax
  if(histogram(k)>0) print *,k,histogram(k)
  nc=nc+histogram(k)
enddo
print *,it ,nc,"#_it(nc"
ave_nc(it)=ave_nc(it)+nc

do 88 it=1,Tmax  ! ! time evolution
  I=0.0d0
  do x=1,Xmax
    do y=1,Ymax
      do xx=1,Xmax
        do yy=1,Ymax
          if((xi(x,y) .eq. xi(xx,yy)) then
            I(x,y,xx,yy)=I(x,y,xx,yy)+q(s(xx,yy))/g(d(x,y,xx,yy))
          else
            I(x,y,xx,yy)=I(x,y,xx,yy)+q(p(xx,yy))/g(d(x,y,xx,yy))
          endif
        enddo
      enddo
    enddo
  enddo
  I(x,y,k)=4.0d0*I(x,y,k)
enddo
enddo
enddo

do x=1,Xmax
  do y=1,Ymax
    do k=1,Kmax
      prob(x,y,k)=dexp(I(x,y,k)/T)
sump=sump+prob(x,y,k)
    enddo
  enddo
enddo
enddo
  do k=1,Kmax
    prob(x,y,k)=prob(x,y,k)/sump
  enddo
enddo
enddo

enddo

! ! ! print *,‘## it =’,it,’I:’
! ! ! do k=1,Kmax
! ! ! print *,‘# k=',k
! ! ! do x=1,Xmax
! ! ! print ’(41F11.2),(I(x,y,k),y=1,Ymax)
! ! ! enddo
! ! ! enddo

! ! ! print *,‘## it =’,it,’p:’
! ! ! do k=1,Kmax
! ! ! print *,‘# k=',k
! ! ! do x=1,Xmax
! ! ! print ’(41F6.3),(prob(x,y,k),y=1,Ymax)
! ! ! enddo
! ! ! enddo

! ! ! print *,‘# it =’,it,’prob of sustaining the opinion:’
! ! ! do x=1,Xmax
! ! ! print ’(41F6.3),(prob(x,y,xi(x,y)),y=1,Ymax)
! ! ! enddo

Probsustain=0.0d0
do x=1,Xmax
do y=1,Ymax
  Probsustain=Probsustain+prob(x,y,xi(x,y))
enddo
enddo

print *,it,Probsustain,’average_prob_of_sustaining_opinion’
ave_Probsustain(it)=ave_Probsustain(it)+Probsustain

do x=1,Xmax
do y=1,Ymax
  r=rand()
sump=0.0d0
  do k=1,Kmax
    sump=sump+prob(x,y,k)
    if (r.lt.sump) goto 666
  enddo
666 xi(x,y)=k
enddo
enddo

histogram=0
histogramK=0

do 77 k=1,Kmax
! ! Hoshen–Kopelman algorithm
  isize=0
  print *,’#_k=’’,k
  if(k.eq.xi(x_L,y_L)) print *,’##_leader’
  label=L2
  label=L2
  do kk=1,L2
    iclass(kk)=kk
  enddo
maxlabel=0

  do x=1,Xmax
  do y=1,Ymax
    if(xi(x,y).eq.k) then
      if(xi(x-1,y).eq.k) or (xi(x,y-1).eq.k) then
        !! reclassifying neighbouring sites
        if(xi(x-1,y).eq.k) label(x-1,y)=reclassify(label(x-1,y))
        if(xi(x,y-1).eq.k) label(x,y-1)=reclassify(label(x,y-1))
      enddo
    enddo
  enddo
enddo
label(x,y) = min(label(x-1,y), label(x,y-1))

iclass (label(x-1,y)) = label(x,y)
iclass (label(x,y-1)) = label(x,y)
else
    maxlabel = maxlabel + 1
    label(x,y) = maxlabel
endif
endif
dodo
!

reclassifying all occupied sites

do x=1,Xmax
do y=1,Ymax
    if ((xi(x,y).eq.k) .and. (label(x,y).gt.iclass(label(x,y))))
        label(x,y) = reclassify(label(x,y))
    endif
enddo
dodo

! reclassifying all occupied sites

do x=1,Xmax
do y=1,Ymax
    isize(label(x,y)) = isize(label(x,y)) + 1
enddo
dodo

!
do kk=1,Xmax*Ymax
    histogram(isize(kk)) = histogram(isize(kk)) + 1
    histogramK(isize(kk),kk) = histogramK(isize(kk),kk) + 1
enddo
print *,"#/uni2423histogram ,/uni2423k=",kk,"/uni2423at/uni2423" ,it
!
do kk=1,Xmax*Ymax
    if (histogramK(kk,kk).gt.0) print *,kk,histogramK(kk,kk)
enddo

nc=0
print *,"#/uni2423histogram/at_",it
!
do kk=1,Xmax*Ymax
    if (histogram(kk).gt.0) print *,kk,histogram(kk)
enddo
!
nc=nc+histogram(kk)
!
ave_nc(it) = ave_nc(it) + nc

88 enddo ! ! ! time evolution

!!! print *,"# it=",it,"/uni2423xi:"
!!! do x=1,Xmax
!!! print '(4115)',(xi(x,y),y=1,Ymax)
!!! enddo

histogram=0
histogramK=0
!
do 99 k=1,Kmax
    isize=0
    print *,"#/uni2423k=" ,k
    if (k.eq.xi(x_L,y_L)) print *,"###/uni2423leader*
        label = L2
    do kk=1,L2
        iclass(kk) = kk
    enddo
    maxlabel = 0
    do x=1,Xmax
        do y=1,Ymax
            if (xi(x,y).eq.k) then ! labeling clusters
if \((x_i(x-1,y).eq.k \text{ or } x_i(x,y-1).eq.k)\) then

\textit{reclassifying neighbouring sites}

if \((x_i(x-1,y).eq.k)\) label\((x-1,y)=\text{reclassify}(\text{label}(x-1,y))\)

if \((x_i(x,y-1).eq.k)\) label\((x,y-1)=\text{reclassify}(\text{label}(x,y-1))\)

label\((x,y)=\text{\(\min\)}(\text{label}(x-1,y),\text{label}(x,y-1))\)

iclass\((\text{label}(x-1,y))=\text{label}(x,y)\)

iclass\((\text{label}(x,y-1))=\text{label}(x,y)\)

else

maxlabel=maxlabel+1

label\((x,y)=\text{maxlabel}\)

endif
endif
enddo
enddo

\textit{reclassifying all occupied sites}

do \(x=1,X_{\text{max}}\)

do \(y=1,Y_{\text{max}}\)

if \(((x_i(x,y).eq.k) \text{ and } (\text{label}(x,y).gt.\text{iclass}(\text{label}(x,y))))\) label\((x,y)=\text{reclassify}(\text{label}(x,y))\)

endif
enddo
enddo

!!! do \(x=1,X_{\text{max}}\)

!!! print '!\(41\)I5', '(\text{label}(x,y),y=1,Y_{\text{max}})'

!!! enddo

do \(x=1,X_{\text{max}}\)

do \(y=1,Y_{\text{max}}\)

if \((x_i(x,y).eq.k)\) isize\((\text{label}(x,y))=\text{isize}(\text{label}(x,y))+1\)

endif
enddo
enddo

! do \(kk=1,X_{\text{max}}Y_{\text{max}}\)
!
! if (\text{isize}(kk).gt.0) print *, '#', kk, isize(kk)
!
enddo

do \(kk=1,X_{\text{max}}Y_{\text{max}}\)

histogram\((\text{isize}(kk))=\text{histogram(\text{isize}(kk))}+1\)

histogramK\((\text{isize}(kk),k)=\text{histogramK(\text{isize}(kk),k)}+1\)

enddo

print *, '#histogram, _k=#, k'

do kk=1,X_{\text{max}}Y_{\text{max}}

if (histogramK\((kk,k).gt.0)\) print *, kk, histogramK\((kk,k)\)

endif

enddo

nc=0

print *, '#histogram\_after:*'

do k=1,X_{\text{max}}Y_{\text{max}}

if (histogram\((k).gt.0)\) print *, k, histogram\((k)\)

nc=nc+histogram\((k)\)

ave\_histogram\((k)=\text{ave\_histogram\((k)+histogram\((k)\)}\)

enddo

print *, it, nc, '#_it\_nc'

ave\_nc\((\text{it})=\text{ave\_nc(\text{it})}+\text{nc}\)

enddo

print *, 't\_nc\_Probsustain'

do it=0,T_{\text{max}}

print *, it, ave\_nc\((\text{it}),\text{ave\_Probsustain}(\text{it})\)

enddo

print *, 's\_H(s)\_histogram\_of\_cluster\_sizes'

do k=1,X_{\text{max}}Y_{\text{max}}

if (ave\_histogram\((k).gt.0)\) print *, k, ave\_histogram\((k)\)

enddo

end program Latane_Hoshen_Kopelmann