ON SUM OF TWO SUBNORMAL KERNELS

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Abstract. We show, by means of a class of examples, that if $K_1$ and $K_2$ are two positive definite kernels on the unit disc such that the multiplication by the coordinate function on the corresponding reproducing kernel Hilbert space is subnormal, then the multiplication operator on the Hilbert space determined by their sum $K_1 + K_2$ need not be subnormal. This settles a recent conjecture of Gregory T. Adams, Nathan S. Feldman and Paul J. McGuire in the negative. We also discuss some cases for which the answer is affirmative.

1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space. Let $B(\mathcal{H})$ denote the Banach algebra of bounded linear operators on $\mathcal{H}$. Recall that an operator $T$ in $B(\mathcal{H})$ is said to be subnormal if there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a normal operator $N$ in $B(\mathcal{K})$ such that $N(\mathcal{H}) \subset \mathcal{H}$ and $N|_\mathcal{H} = T$. For the basic theory of subnormal operators, we refer to [12].

Completely hyperexpansive operators were introduced in [6]. An operator $T \in B(\mathcal{H})$ is said to be completely hyperexpansive if

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} T^{*j} T^j \leq 0 \quad (n \geq 1).$$

The theory of subnormal and completely hyperexpansive operators are closely related with the theory completely monotone and completely alternating sequences (cf. [5], [6]).

Let $\mathbb{Z}_+$ denote the set of non-negative integers. A sequence $\{a_k\}_{k \in \mathbb{Z}_+}$ of positive real numbers is said to be a completely monotone if

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} a_{m+j} \geq 0 \quad (m, n \geq 0).$$

It is well-known that a sequence $\{a_k\}_{k \in \mathbb{Z}_+}$ is completely monotone if and only if the sequence $\{a_k\}_{k \in \mathbb{Z}_+}$ is a Hausdorff moment sequence, that is, there exists a positive measure $\nu$ supported in $[0,1]$ such that $a_k = \int_{[0,1]} x^k d\nu(x)$ for all $k \in \mathbb{Z}_+$ (cf. [9]).

Similarly, a sequence $\{a_k\}_{k \in \mathbb{Z}_+}$ of positive real numbers is said to be completely alternating if

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} a_{m+j} \leq 0 \quad (m \geq 0, n \geq 1).$$

2010 Mathematics Subject Classification: Primary 46E20, 46E22; Secondary 47B20, 47B37

Key words and phrases: Completely alternating, Completely hyperexpansive, Completely monotone, Positive definite kernel, Spherically balanced spaces, Subnormal operators

Work of the first author was supported by CSIR SPM Fellowship and work of the second author was supported by Inspire Faculty Fellowship.
Note that \( \{a_k\}_{k \in \mathbb{Z}^+} \) is completely alternating if and only if the sequence \( \{\Delta a_k\}_{k \in \mathbb{Z}^+} \) is completely monotone, where \( \Delta a_k := a_{k+1} - a_k \).

Let \( \mathcal{H}(K) \) be a reproducing kernel Hilbert space consisting of holomorphic functions on the open unit disc \( \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\} \) with reproducing kernel \( K(z, w) = \sum_{k \in \mathbb{Z}^+} a_k (z \bar{w})^k \).

Thus \( K \) is holomorphic in the first variable and anti-holomorphic in the second. We make the assumption that a kernel function is always holomorphic in the first variable and anti-holomorphic in the second throughout this note. Consider the operator \( M_z \) of multiplication by the coordinate function \( z \) on \( \mathcal{H}(K) \). As is well-known, such a multiplication operator is unitarily equivalent to a weighted shift operator \( W \) on the sequence space

\[
\ell^2(\mathbf{a}) := \{\mathbf{x} := (x_0, x_1, \ldots) : \|\mathbf{x}\|^2 = \sum_{k=0}^{\infty} \left| x_k \right|^2 a_k < \infty \}
\]

with \( W e_n = \sqrt{\frac{a_n}{a_{n+1}}} e_{n+1} \), where \( e_n \) is the standard unit vector. Assume that \( M_z \) is bounded. Then \( M_z \) is a contractive subnormal if and only if the sequence \( \{\frac{1}{\sqrt{a_k}}\}_{k \in \mathbb{Z}^+} \) is a Hausdorff moment sequence. On the other hand, \( M_z \) on \( \mathcal{H}(K) \) is completely hyperexpansive if and only if the sequence \( \{\frac{1}{\sqrt{a_k}}\}_{k \in \mathbb{Z}^+} \) is completely alternating (cf. [9, Proposition 3]).

For any two positive definite kernels \( K_1 \) and \( K_2 \), their sum \( K_1 + K_2 \) is again a positive definite kernel and therefore determines a Hilbert space \( \mathcal{H}(K_1 + K_2) \) of functions. It was shown in [4] that

\[
\mathcal{H}(K_1 + K_2) = \{f = f_1 + f_2 : f_1 \in \mathcal{H}(K_1), f_2 \in \mathcal{H}(K_2)\},
\]

is a Hilbert space with the norm given by

\[
\|f\|_{\mathcal{H}(K_1 + K_2)}^2 := \inf \left\{ \|f_1\|_{\mathcal{H}(K_1)}^2 + \|f_2\|_{\mathcal{H}(K_2)}^2 : f = f_1 + f_2, f_1 \in \mathcal{H}(K_1), f_2 \in \mathcal{H}(K_2) \right\}.
\]

Sum of two kernel functions is also discussed by Salinas in [19]. He proved that if \( K_1 \) and \( K_2 \) are generalized Bergman kernels (for definition, refer to [16]), then so is \( K_1 + K_2 \). Although not explicitly stated in [4], it is not hard to verify that the multiplication operator \( M_z \) on \( \mathcal{H}(K_1 + K_2) \) is unitarily equivalent to the operator \( P_{M_z} (M_{z,1} + M_{z,2}) |_{M_z} \), where \( M_{z,i} \) is the operator of multiplication by the coordinate function \( z \) on \( \mathcal{H}(K_i) \) and

\[
M = \{(g, -g) : g \in \mathcal{H}(K_1) \ominus \mathcal{H}(K_2) : g \in \mathcal{H}(K_1) \cap \mathcal{H}(K_2) \} \subseteq \mathcal{H}(K_1) \oplus \mathcal{H}(K_2).
\]

Evidently, if \( M_{z,1} \) and \( M_{z,2} \) are subnormal, then so is \( M_{z,1} + M_{z,2} \). Here, we discuss the subnormality of the compression \( P_{M_z} (M_{z,1} + M_{z,2}) |_{M_z} \) for a class of kernels. In particular, we show that the subnormality of \( M_{z,1} \) and \( M_{z,2} \) need not imply \( P_{M_z} (M_{z,1} + M_{z,2}) |_{M_z} \) is subnormal.

A similar question on subnormality involving the point-wise product of two positive definite kernels was raised in [19]. Recall that the product \( K_1 K_2 \) of two positive definite kernels defined on, say the unit disc \( \mathbb{D} \), is also a positive definite kernel on \( \mathbb{D} \). Indeed, if \( \mathcal{H}(K_1) \otimes \mathcal{H}(K_2) \subseteq \text{Hol}(\mathbb{D} \times \mathbb{D}) \) be the usual tensor product of the two Hilbert spaces \( \mathcal{H}(K_1) \) and \( \mathcal{H}(K_2) \), and \( N \) is the subspace \( \{h \in \mathcal{H}(K_1) \otimes \mathcal{H}(K_2) : h(z, z) = 0, z \in \mathbb{D}\} \), then the operator \( M_{z,1} \otimes I \) acting on the Hilbert space \( \mathcal{H}(K_1) \otimes \mathcal{H}(K_2) \) compressed to \( N \) is unitarily equivalent to the multiplication operator \( M_z \) on the Hilbert space \( \mathcal{H}(K_1 K_2) \). If \( M_{z,1} \) is subnormal on \( \mathcal{H}(K_1) \), then so is the operator \( M_{z,1} \otimes I \).
The answer to the question of subnormality, both in the case of the sum as well as the product, is affirmative in several examples. For over thirty years, the question of whether the compression to $\mathcal{N}_0$ is subnormal had remained open. Recently, a counter-example has been found, see [3, Theorem 1.5]. The conjecture below is similar except that it involves the sum of two kernels.

**Conjecture 1.1.** ([1, pp. 22]). Let $K_1(z, w) = \sum_{k \in \mathbb{Z}^+} a_k(z\overline{w})^k$ and $K_2(z, w) = \sum_{k \in \mathbb{Z}^+} b_k(z\overline{w})^k$ be any two reproducing kernels satisfying:

(a) $\lim_{k \to \infty} \frac{a_k}{a_{k+1}} = \lim_{k \to \infty} \frac{b_k}{b_{k+1}} = 1$

(b) $\lim_{k \to \infty} a_k = \lim_{k \to \infty} b_k = \infty$

(c) $\frac{1}{a_k} = \int_{[0,1]} t^k d\nu_1(t)$ and $\frac{1}{b_k} = \int_{[0,1]} t^k d\nu_2(t)$ for all $k \in \mathbb{Z}^+$.

Then the multiplication operator $M_z$ on $\mathcal{H}(K_1 + K_2)$ is a subnormal operator.

An equivalent formulation, in terms of the moment sequence criterion, of the conjecture is the following. If $\{\frac{1}{a_k}\}_{k \in \mathbb{Z}^+}$ and $\{\frac{1}{b_k}\}_{k \in \mathbb{Z}^+}$ are Hausdorff moment sequences, does it necessarily follow that $\{\frac{1}{a_k+b_k}\}_{k \in \mathbb{Z}^+}$ is also a Hausdorff moment sequence?

Like the case of the product of two kernels, here we give a class of counter examples to the conjecture stated above. Paul McGuire, in a private communication, has informed the authors of an example that they had found. In fact, he says that their example is one of the examples discussed in this note.

These two cases suggest that it may be fruitful to ask when the compression of a subnormal operator to an invariant subspace is again subnormal.

The paper is organized as follows. In section 2, we provide a class of counter-examples which settles the Conjecture 1.1. We also discuss some cases for which answer to this conjecture is affirmative. In the last section, we try to answer analogously in a certain class of weighted multi-shifts.

## 2. Sum of Two Subnormal Reproducing Kernels Need Not Be Subnormal

For the construction of counter-examples to the conjecture, we make use of the following result, borrowed from [2] Proposition 4.3].

**Proposition 2.1.** For distinct positive real numbers $a_0, \ldots, a_n$ and non-zero real numbers $b_0, \ldots, b_n$, consider the polynomial $p(x) = \prod_{k=0}^n (x + a_k + ib_k)(x + a_k - ib_k)$. Then the sequence $\{\frac{1}{p(l)}\}_{l \in \mathbb{Z}^+}$ is never a Hausdorff moment sequence.

For $r > 0$, let $K_r$ be a positive definite kernel given by

$$K_r(z, w) := \sum_{k \in \mathbb{Z}^+} \frac{k+r}{r} (z\overline{w})^k \quad (z, w \in \mathbb{D}).$$

The case $r = 1$ corresponds to the Bergman kernel. It is easy to see that the multiplication operator $M_z$ on $\mathcal{H}(K_r)$ is a contractive and subnormal and the representing measure is $rx^{r-1}dx$. 

The multiplication operator

Theorem 2.2.

Proof. Notice that

The case \( s = 1 \) and \( t = 2 \), corresponds to the kernel \((1 - zw)^{-3}\). Note that \( M_z \) is a contractive subnormal with the representing measure \( \nu \) is given by

One easily verifies that \( K_r \) and \( K_{s,t} \) both satisfy all the conditions (a), (b) and (c) of the Conjecture \[4\]. But the multiplication operator on their sum need not be subnormal for all possible choices of \( s, t > 0 \). This follows from the following theorem.

Theorem 2.2. The multiplication operator \( M_z \) on \( \mathcal{H}(K_r + K_{s,t}) \) is subnormal if and only if

\[
(rs + st + tr)^2 \geq 8r^2 st. 
\]

Proof. Notice that

\[
(K_r + K_{s,t})(z, w) = \sum_{k \in \mathbb{Z}_+} \left( \frac{k^2 + (s + t + \frac{st}{r})k + 2st}{st} \right)(z\bar{w})^k \quad (z, w \in \mathbb{D}).
\]

The roots of the polynomial \( x^2 + (s + t + \frac{4t}{r})x + 2st \) are

\[
x_1 := \frac{-(s + t + \frac{4t}{r}) + \sqrt{(s + t + \frac{4t}{r})^2 - 8st}}{2} \quad \text{and} \quad x_2 := \frac{-(s + t + \frac{4t}{r}) - \sqrt{(s + t + \frac{4t}{r})^2 - 8st}}{2}.
\]

Suppose that \( (rs + st + tr)^2 \geq 8r^2 st. \) Then the kernel \( K_r + K_{s,t} \) will be of the form \( 2K_{s',t'} \) where \( s' = -x_1 \) and \( t' = -x_2 \). Hence, \( M_z \) on \( \mathcal{H}(K_r + K_{s,t}) \) is a subnormal operator.

Conversely, assume that \( (rs + st + tr)^2 < 8r^2 st. \) By Proposition \[4\], it follows that \( M_z \) on \( \mathcal{H}(K_r + K_{s,t}) \) can not be subnormal.

Remark 2.3: If we choose \( s = 1, \ t = 2 \) and \( r > 2 \), then the inequality \[4\] is not valid.

We also point out that if \( K_1 \) and \( K_2 \) are any two reproducing kernels such that the multiplication operators on \( \mathcal{H}(K_1) \) and \( \mathcal{H}(K_2) \) are hyponormal, then the multiplication operator on \( \mathcal{H}(K_1 + K_2) \) need not be hyponormal. An example illustrating this is given below.

Example 2.4. For any \( s, t > 0 \), consider the reproducing kernel \( K^{s,t} \) given by

\[
K^{s,t}(z, w) := 1 + sz\bar{w} + s^2(z\bar{w})^2 + t(z\bar{w})^3 \left( \frac{1}{1 - z\bar{w}} \right).
\]

Note that \( K^{s,t} \) defines a reproducing kernel on the unit disc \( \mathbb{D} \) and the multiplication operator \( M_z \) on \( \mathcal{H}(K^{s,t}) \) can be seen as a weighted shift operator with weights \( \left( \sqrt{\frac{1}{s}}, \sqrt{\frac{1}{s}}, \sqrt{\frac{1}{t}}, 1, 1, \cdot \cdot \cdot \right) \). Thus, it follows that \( M_z \) on \( \mathcal{H}(K^{s,t}) \) is hyponormal if and only if \( s^2 \leq t \leq s^3 \).
Theorem 2.5. Let \( K(z, w) = \sum_{k \in \mathbb{Z}_+} a_k(z \bar{w})^k \) be a positive definite kernel on \( \mathbb{D} \) and \( M_z \) be the multiplication operator on \( \mathcal{H}(K) \). Assume that \( M_z \) is left invertible. Then the followings are equivalent:

(i) \( \{a_k\}_{k \in \mathbb{Z}_+} \) is a completely alternating sequence.

(ii) The Cauchy dual \( M_z^* \) of \( M_z \) is completely hyponormal.

(iii) For all \( t > 0, \{\frac{1}{t(a_k - 1)}\}_{k \in \mathbb{Z}_+} \) is a completely monotone sequence.

(iv) For all \( t > 0, \) the multiplication operator \( M_z \) on \( \mathcal{H} (tK + (1 - t)S) \) is contractive subnormal, where \( S \) is the Szegő kernel on \( \mathbb{D} \).

Remark 2.6: If \( \{a_k\}_{k \in \mathbb{Z}_+} \) is a completely alternating sequence then by putting \( t = 1 \) in part (iii) of Theorem 2.5, it follows that \( \left\{ \frac{1}{a_k} \right\}_{k \in \mathbb{Z}_+} \) is a completely monotone sequence.

Corollary 2.7. Let \( K_1(z, w) = \sum_{k \in \mathbb{Z}_+} a_k(z \bar{w})^k \) and \( K_2(z, w) = \sum_{k \in \mathbb{Z}_+} b_k(z \bar{w})^k \) be any two reproducing kernels such that \( \{a_k\}_{k \in \mathbb{Z}_+} \) and \( \{b_k\}_{k \in \mathbb{Z}_+} \) are completely alternating sequences, then the multiplication operator \( M_z \) on \( \mathcal{H}(K_1 + K_2) \) is subnormal.

Proof. It is easy to verify that the sum of two completely alternating sequences is completely alternating. The desired conclusion follows immediately from Remark 2.6.

Remark 2.8: Note that \( \{\frac{k+r}{r}\}_{k \in \mathbb{Z}_+} \) is a completely alternating sequence but the sequence \( \{\frac{(k+s)(k+t)}{st}\}_{k \in \mathbb{Z}_+} \) is not completely alternating. So, the reproducing kernels \( K_r \) and \( K_{s,t} \) discussed in Theorem 2.7 does not satisfy the hypothesis of Corollary 2.7.

Proposition 2.9. Let \( K(z, w) = \sum_{k \in \mathbb{Z}_+} a_k(z \bar{w})^k \) be any positive definite kernel such that the multiplication operator \( M_z \) on \( \mathcal{H}(S + K) \) is subnormal. Then the multiplication operator on \( \mathcal{H}(K) \) is subnormal.

Proof. From subnormality of \( M_z \) on \( \mathcal{H}(S + K) \), it follows that \( \left\{\frac{1}{1 + a_k}\right\}_{k \in \mathbb{Z}_+} \) is a completely monotone sequence. Thus, \( \left\{1 - \frac{1}{1 + a_k}\right\}_{k \in \mathbb{Z}_+} \) is completely alternating. Note that

\[
(a_k)^{-1} = (1 + a_k)^{-1}(1 - \frac{1}{1 + a_k})^{-1} = \sum_{j=1}^{\infty} \left(\frac{1}{1 + a_k}\right)^j.
\]
Observe that $\{\frac{1}{(1+a_k)^j}\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence for all $j \geq 1$. Now, being the limit of completely monotone sequences, $\{a_k^{-1}\}_{k\in\mathbb{Z}_+}$ is completely monotone.

\[ \square \]

**Remark 2.10:** We have the following remarks:

(i) The converse of the Proposition 2.9 is not true (see the example discussed in part (ii) of the Remark 2.13).

(ii) If we replace Szegő kernel $S$ by Bergman kernel then the conclusion of the Proposition 2.9 need not be true. For example, by using Proposition 2.11 one may choose $\alpha > 0$ such that the sequence $\{\frac{1}{k^2+\alpha k+1}\}_{k\in\mathbb{Z}_+}$ is not completely monotone but the sequence $\{\frac{1}{k^{2+\alpha k+1}}\}_{k\in\mathbb{Z}_+}$ is completely monotone.

We use the convenient Pochhammer symbol given by $(x)_n := \frac{\Gamma(x+n)}{\Gamma(x)}$, where $\Gamma$ denotes the gamma function.

For $\lambda, \mu > 0$, consider the positive definite kernel

$$K_{\lambda,\mu}(z, w) = \sum_{k\in\mathbb{Z}_+} \frac{(\lambda)_k}{(\mu)_k} (z\bar{w})^k \quad (z, w \in \mathbb{D}).$$

It is easy to see that the case $\mu = 1$ corresponds to the kernel $(1 - z\bar{w})^{-\lambda}$. Note that the multiplication operator $M_z$ on $\mathcal{H}(K_{\lambda,\mu})$ may be realized as a weighted shift operator with weight sequence $\{\sqrt{\frac{k+\mu}{k+\lambda}}\}_{k\in\mathbb{Z}_+}$.

First part of the following theorem is proved in [15] and the representing measure is given in [13, Lemma 2.2]. Here, we provide a proof for the second part only.

**Theorem 2.11.** The multiplication operator $M_z$ on $\mathcal{H}(K_{\lambda,\mu})$ is

(i) subnormal if and only if $\lambda \geq \mu$. In this case, the representing measure $\nu$ of $M_z$ is given by

$$d\nu(x) = \begin{cases} \frac{\Gamma(\lambda)}{\Gamma(\mu)\Gamma(\lambda-\mu)} x^{\mu-1}(1-x)^{\lambda-\mu-1} dx & \text{if } \lambda > \mu \\ \delta_1(x) dx & \text{if } \lambda = \mu, \end{cases}$$

where $\delta_1$ is the Dirac delta function.

(ii) completely hyperexpansive if and only if $\lambda \leq \mu \leq \lambda + 1$.

**Proof.** The multiplication operator $M_z$ on $\mathcal{H}(K_{\lambda,\mu})$ is completely hyperexpansive if and only if the sequence $\{(\mu)_k / (\lambda)_k\}_{k\in\mathbb{Z}_+}$ is completely alternating. Here

$$\Delta \left( \frac{(\mu)_k}{(\lambda)_k} \right) = (\mu)_{k+1} - \frac{(\mu)_k}{(\lambda)_{k+1}} = \frac{(\mu)_{k+1} - (\mu)_k(\lambda + k - 1)}{(\lambda)_{k+1}} = \frac{\mu - \lambda}{\lambda} \frac{\mu_k}{(\lambda + 1)_k}.$$  

By first part of this theorem, $\{\mu - \lambda \frac{(\mu)_k}{(\lambda+1)_k}\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence if and only if $\lambda \leq \mu \leq \lambda + 1$. This completes the proof.  

\[ \square \]
The following proposition gives a sufficient condition for the subnormality of multiplication operator on Hilbert space determined by sum of two kernels belonging to the class $K_{\lambda,\mu}$.

**Proposition 2.12.** Let $0 < \mu \leq \lambda' \leq \lambda \leq \lambda' + 1$. Then the multiplication operator $M_z$ on $\mathcal{H}(K_{\lambda,\mu} + K_{\lambda',\mu})$ is contractive subnormal.

**Proof.** Observe that

$$(K_{\lambda,\mu} + K_{\lambda',\mu})(z, w) = \sum_{k \in \mathbb{Z}_+} \frac{(\lambda)_k + (\lambda')_k}{(\mu)_k} (zw)^k \quad (z, w \in \mathbb{D})$$

and

$$\frac{(\mu)_k}{(\lambda)_k + (\lambda')_k} = \frac{(\mu)_k}{(\lambda')_k + 1 + \frac{(\lambda')_k}{(\lambda')_k}}.$$  

Since $\mu \leq \lambda'$, it follows from part (i) of Theorem 2.11 that $\left\{ \frac{(\mu)_k}{(\lambda')_k} \right\}_{k \in \mathbb{Z}_+}$ is completely monotone.

If $\lambda' \leq \lambda \leq \lambda' + 1$, then by part (ii) of Theorem 2.11 the sequence $\left\{ \frac{(\lambda')_k}{(\lambda')_k} \right\}_{k \in \mathbb{Z}_+}$ is a completely alternating sequence. So is the sequence $\left\{ 1 + (\lambda')_k \right\}_{k \in \mathbb{Z}_+}$. Hence, by Remark 2.6,

$$\left\{ 1 + \frac{(\lambda')_k}{(\lambda')_k} \right\}_{k \in \mathbb{Z}_+}$$

is a completely monotone sequence. Thus, being a product of two completely monotone sequences, $\left\{ \frac{(\mu)_k}{(\lambda)_k + (\lambda')_k} \right\}_{k \in \mathbb{Z}_+}$ is a completely monotone sequence. This completes the proof. \qed

**Remark 2.13:** Here are some remarks:

(i) The case when $\mu < \lambda'$ and $\lambda = \lambda' + 1$. The representing measure for the sequence $\left\{ \frac{1}{1 + \frac{(\lambda')_k}{(\lambda')_k}} \right\}_{k \in \mathbb{Z}_+}$ is $\lambda' x^{2\lambda' - 1} dx$. The representing measure for the sequence $\left\{ \frac{(\mu)_k}{(\lambda')_k} \right\}_{k \in \mathbb{Z}_+}$ is given in part (i) of Theorem 2.11. Thus, using Remark 2.4 of [2], one may obtain the representing measure for $M_z$ on $\mathcal{H}(K_{\lambda,\mu} + K_{\lambda',\mu})$ to be given by

$$d\nu(x) = \frac{\lambda' \Gamma(\lambda')}{\Gamma(\mu)\Gamma(\lambda' - \mu)} x^{2\lambda' - 1} \left( \int_0^{x^{-1}} t^{\lambda' - 1} (1 - t)^{\mu - 2\lambda' - 1} dt \right) dx.$$  

But in general, when $\lambda < \lambda' + 1$, we do not know the representing measure for the sequence $\left\{ \frac{1}{1 + \frac{(\lambda')_k}{(\lambda')_k}} \right\}_{k \in \mathbb{Z}_+}$ as well as for the sequence $\left\{ \frac{(\mu)_k}{(\lambda)_k + (\lambda')_k} \right\}_{k \in \mathbb{Z}_+}$.

(ii) Consider the kernel $(K_{1,1} + K_{3,1})(z, w) = \sum_{k \in \mathbb{Z}_+} \frac{k^2 + 3k + 4}{2} (zw)^k$ for all $z, w \in \mathbb{D}$. It follows from Proposition 2.11 that the sequence $\left\{ \frac{2}{k^2 + 3k + 4} \right\}_{k \in \mathbb{Z}_+}$ is not completely monotone. Consequently, the multiplication operator $M_z$ on $\mathcal{H}(K_{1,1} + K_{3,1})$ is not subnormal.

For $\lambda > 1$, consider the kernel $K_{\lambda,1} + K_{3,1}$. We claim that there exists a $\lambda_0 > 1$ such that $\left\{ \frac{(1)_k}{(\lambda_0)_k + (3)_k} \right\}_{k \in \mathbb{Z}_+}$ is not completely monotone. If not, assume that it is a completely monotone sequence for all $\lambda > 1$. As $\lambda$ goes to 1, one may get that $\left\{ \frac{2}{k^2 + 3k + 4} \right\}_{k \in \mathbb{Z}_+}$ is completely monotone, which is a contradiction. Therefore, we conclude that there exists a $\lambda_0 > 1$ such that the multiplication operator $M_z$ on $\mathcal{H}(K_{\lambda_0,1} + K_{3,1})$ is not subnormal. By using properties of gamma function, one may verify that $K_{\lambda_0,1}$ and $K_{3,1}$ both satisfy
Note that

\[ \sum \]

Proposition 2.14. Let \( 0 < p \leq q \leq p + 1 \). Suppose \( K_1(z, w) = \sum_{k \in \mathbb{Z}^+} a_k^p(z \bar{w})^k \) and \( K_2(z, w) = \sum_{k \in \mathbb{Z}^+} a_k^q(z \bar{w})^k \) are any two reproducing kernels such that \( \{a_k\}_{k \in \mathbb{Z}^+} \) is a completely alternating sequence. Then the multiplication operator \( M_z \) on \( \mathcal{H}(K_1 + K_2) \) is subnormal.

Proof. Note that

\[
\frac{1}{a_k^p + a_k^q} = \frac{1}{a_k^p(1 + a_k^{q-p})}
\]

Since \( 0 \leq q - p \leq 1 \) and \( \{a_k\}_{k \in \mathbb{Z}^+} \) is completely alternating, it follows from [7, Corollary 1] that \( \{a_k^{-p}\}_{k \in \mathbb{Z}^+} \) is also completely alternating. Thus so is \( \{1 + a_k^{q-p}\}_{k \in \mathbb{Z}^+} \). Hence, by Remark 2.6, \( \{(1 + a_k^{q-p})^{-1}\}_{k \in \mathbb{Z}^+} \) is completely monotone. By [8, Corollary 4.1], \( \{a_k^{-p}\}_{k \in \mathbb{Z}^+} \) is completely monotone. Now the proof follows as the product of two completely monotone sequences is also completely monotone. \( \square \)

Example 2.15. For any \( p > 0 \), let \( K_p(z, w) \) be the positive definite kernel given by

\[
K_p(z, w) := \sum_{k \in \mathbb{Z}^+} (k + 1)^p(z \bar{w})^k \quad (z, w \in \mathbb{D}).
\]

Then it is known that the multiplication operator \( M_z \) on \( \mathcal{H}(K_p) \) is subnormal with the representing measure \( d\nu(x) = \frac{(-\log x)^{p-1}}{\Gamma(p)} dx \) (cf. [14, Theorem 4.3]). By Proposition 2.14, it follows that \( M_z \) on \( \mathcal{H}(K_p + K_q) \) is subnormal if \( p \leq q \leq p + 1 \).

The next result also provides a class of counter-examples to the Conjecture 1.1.

Theorem 2.16. Consider the positive definite kernel \( K_p \) given in Example 2.15. Then the multiplication operator \( M_z \) on \( \mathcal{H}(K_p + K_{p+2}) \) is subnormal if and only if \( p \geq 1 \).

Proof. For \( x \in (0, 1] \), let \( g(x) := \frac{1}{\Gamma(p)} \int_{-\log x}^{\infty} (-\log x - y)^{p-1} \sin y dy \) and \( d\nu(x) = g(x) dx \). Then

\[
\int_0^1 x^k d\nu(x) = \frac{1}{\Gamma(p)} \int_{-\log x}^{\infty} \int_{y=0}^{\infty} x^k (-\log x - y)^{p-1} dx \sin y dy = \frac{1}{\Gamma(p)} \int_{y=0}^{\infty} \int_{x=0}^{\infty} x^{-(k+1)y} \sin y dy = \frac{1}{\Gamma(p)} \int_{y=0}^{\infty} \int_{u=0}^{\infty} e^{-(k+1)u} \sin y du dy = \frac{1}{\Gamma(p)} \frac{1}{(k+1)^p} \sin y dy = \frac{1}{(k+1)^p} \frac{1}{(k+1)^2 + 1}.
\]

Thus the sequence \( \{\frac{1}{(k+1)^p} \frac{1}{(k+1)^2 + 1}\}_{k \in \mathbb{Z}^+} \) is completely monotone if and only if the function \( g(x) \) is non-negative a.e. Note that the function \( g(x) \) is non-negative on \( (0, 1] \) a.e. if and only if the function \( h(x) := g(e^{-x}) \) is non-negative a.e. on \( (0, \infty) \). Now

\[
h(x) = \frac{1}{\Gamma(p)} \int_0^x (x - y)^{p-1} \sin y dy = \frac{xp}{\Gamma(p)} \int_0^1 (1 - y)^{p-1} \sin(xy) dy.
\]

By [17, Chapter 3, pp 439], we have \( h(x) = \sqrt{x \frac{\pi}{\Gamma(p)}} s_{p-1}^{\frac{2}{p^2}} (x) \), where \( s_{p-1}^{\frac{2}{p^2}} (x) \) is the Lommel’s function of first kind. Thus, the sequence \( \{\frac{1}{(k+1)^p} \frac{1}{(k+1)^2 + 1}\}_{k \in \mathbb{Z}^+} \) being completely monotone is
equivalent to the non-negativity of the function \( s \frac{p-1}{2} (x) \). If \( p \geq 1 \) then by [20, Theorem A], we get that \( s \frac{p-1}{2} (x) \geq 0 \) for all \( x > 0 \). The converse follows from [20, Theorem 2], which completes the proof.

\[ \square \]

3. Multi-variable case

Let \( Z_+^d \) denote the cartesian product \( Z_+ \times \cdots \times Z_+ \) (\( d \) times). Let \( \alpha = (\alpha_1, \cdots, \alpha_d) \in Z_+^d \), we write \( |\alpha| := \alpha_1 + \cdots + \alpha_d \) and \( \alpha! = \alpha_1! \cdots \alpha_d! \).

If \( T = (T_1, \cdots, T_d) \) is a \( d \)-tuple of commuting bounded linear operators \( T_j \) (\( 1 \leq j \leq d \)) on \( \mathcal{H} \) then we set \( T^* \) to be \((T_1^*, \cdots, T_d^*)\) and \( T^\alpha \) to be \( T_1^{\alpha_1} \cdots T_d^{\alpha_d} \).

Given a commuting \( d \)-tuple of bounded linear operators \( T_1, \cdots, T_d \) on \( \mathcal{H} \), set
\[
Q_T(X) := \sum_{i=1}^d T_i^* X T_i \quad (X \in B(\mathcal{H})).
\]

For \( X \in B(\mathcal{H}) \) and \( k \geq 1 \), one may define \( Q_T^k(X) := Q_T(Q_T^{k-1}(X)) \), where \( Q_T^0(X) = X \).

Recall that \( T \) is said to be

(i) spherical contraction if \( Q_T(I) \leq I \).

(ii) jointly left invertible if there exists a positive number \( c \) such that \( Q_T(I) \geq cI \).

For a jointly left invertible \( T \), the spherical Cauchy dual \( T^s \) of \( T \) is the \( d \)-tuple \((T_1^s, T_2^s, \cdots, T_d^s)\), where \( T_i^s := T_i(Q_T(I))^{-1} \) (\( i = 1, 2, \cdots, d \)). We say that \( T \) is a joint complete hyperexpansion if
\[
B_n(T) := \sum_{k=0}^n (-1)^k \binom{n}{k} Q_T^k(I) \leq 0 \quad (n \geq 1).
\]

Throughout this section \( \mathbb{B} \) denotes the open unit ball \( \{ z \in \mathbb{C}^d : |z_1|^2 + \cdots + |z_d|^2 < 1 \} \) and \( \partial \mathbb{B} \) denotes the unit sphere \( \{ z \in \mathbb{C}^d : |z_1|^2 + \cdots + |z_d|^2 = 1 \} \) in \( \mathbb{C}^d \).

Let \( \{ \alpha \} \in Z_+^d \) be a multi-sequence of positive numbers. Consider the Hilbert space \( H^2(\beta) \) of formal power series \( f(z) = \sum_{\alpha \in Z_+^d} \hat{f}(\alpha) z^\alpha \) such that
\[
\| f \|^2_{H^2(\beta)} = \sum_{\alpha \in Z_+^d} |\hat{f}(\alpha)|^2 \beta_\alpha^2 < \infty.
\]

The Hilbert space \( H^2(\beta) \) is said to be spherically balanced if the norm on \( H^2(\beta) \) admits the slice representation \([\nu, H^2(\gamma)]\), that is, there exist a Reinhardt measure \( \nu \) and a Hilbert space \( H^2(\gamma) \) of formal power series in one variable such that
\[
\| f \|^2_{H^2(\beta)} = \int_{\partial \mathbb{B}} \| f \|^2_{H^2(\gamma)} d\nu(z) \quad (f \in H^2(\beta)),
\]
where \( \gamma = \{ \gamma_k \}_{k \in Z_+} \) is given by the relation \( \beta_\alpha = \gamma_\alpha \| z^\alpha \|_{L^2(\partial \mathbb{B}, \nu)} \) for all \( \alpha \in Z_+^d \). Here, by the Reinhardt measure, we mean a \( \mathbb{T}^d \)-invariant finite positive Borel measure supported in \( \partial \mathbb{B} \), where \( \mathbb{T}^d \) denotes the the unit \( d \)-torus \( \{ z \in \mathbb{C}^d : |z_1| = 1, \cdots, |z_d| = 1 \} \). For more details on spherically balanced Hilbert spaces, we refer to [11].
The following lemma has been already recorded in \cite{11} Lemma 4.3. We include a statement for ready reference.

**Lemma 3.1.** Let $H^2(\beta)$ be a spherically balanced Hilbert space and let $[\nu, H^2(\gamma)]$ be the slice representation for the norm on $H^2(\beta)$. Consider the $d$-tuple $M_z = (M_{z_1}, \ldots, M_{z_d})$ of multiplication by the co-ordinate functions $z_1, \ldots, z_d$ on $H^2(\beta)$. Then for every $n \in \mathbb{Z}_+$ and $\alpha \in \mathbb{Z}_+^d$,

$$
\langle B_n(M_z)z^\alpha, z^\alpha \rangle = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \langle Q^k(M_z)z^\alpha, z^\alpha \rangle = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \gamma_{k+|\alpha|}^2 \|z^\alpha\|^2_{L^2(\partial B, \nu)}.
$$

If the interior of the point spectrum $\sigma_p(M_z^*)$ of $M_z^*$ is non-empty then $H^2(\beta)$ may be realized as a reproducing kernel Hilbert space $\mathcal{H}(K)$ \cite{18} Propositions 19 and 20, where the reproducing kernel $K$ is given by

$$
K(z, w) = \sum_{\alpha \in \mathbb{Z}_+^d} \frac{z^\alpha \bar{w}^\alpha}{\beta_\alpha} (z, w \in \sigma_p(M_z^*)).
$$

This has lead to the following definition.

**Definition 3.2:** Let $\mathcal{H}(K)$ be a reproducing kernel Hilbert space defined on the open unit ball $B$ with reproducing kernel $K(z, w) = \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha z^\alpha \bar{w}^\alpha$ for all $z, w \in B$. We say that $K$ is a balanced kernel if $\mathcal{H}(K)$ is a spherically balanced Hilbert space. Further, the multiplication $d$-tuple $M_z$ on $\mathcal{H}(K)$ may be called as balanced multiplication tuple.

**Remark 3.3:** The spherical Cauchy dual $M_z^*$ of a jointly left invertible balanced multiplication tuple $M_z$ can be seen as a multiplication $d$-tuple $M_z^* = (M_{z_1}^*, \ldots, M_{z_d}^*)$ of multiplication by the co-ordinate functions $z_1, \ldots, z_d$ on $H^2(\beta^*)$, where

$$
\beta_\alpha = \frac{1}{\gamma_\alpha} \|z^\alpha\|_{L^2(\partial B, \nu)} (\alpha \in \mathbb{Z}_+^d).
$$

In other words, the norm on $H^2(\beta^*)$ admits the slice representation $[\nu, H^2(\gamma)]$, where $\gamma_k = 1/\gamma_k$ for all $k \in \mathbb{Z}_+$.

**Proposition 3.4.** If $K_1(z, w) = \sum_{\alpha \in \mathbb{Z}_+^d} a_{\alpha} z^\alpha \bar{w}^\alpha$ and $K_2(z, w) = \sum_{\alpha \in \mathbb{Z}_+^d} b_{\alpha} z^\alpha \bar{w}^\alpha$ are any two balanced kernels with the slice representations $[\nu, H^2(\gamma_1)]$ and $[\nu, H^2(\gamma_2)]$ respectively. Then $K_1 + K_2$ is a balanced kernel with the slice representation $[\nu/2, H^2(\gamma)]$, where $\gamma = \{\gamma_k\}$ is given by the relation

$$
\gamma_k = \frac{\sqrt{2} \gamma_{k,1} \gamma_{k,2}}{\left(\gamma_{k,1} + \gamma_{k,2}\right)^{1/2}} (k \in \mathbb{Z}_+).
$$

**Proof.** For every $\alpha \in \mathbb{Z}_+^d$, we have

$$
a_{\alpha} + b_{\alpha} = \frac{1}{\gamma_{\alpha,1}^2 \|z^\alpha\|^2_{L^2(\partial B, \nu)}} + \frac{1}{\gamma_{\alpha,2}^2 \|z^\alpha\|^2_{L^2(\partial B, \nu)}} = \frac{\gamma_{\alpha,1}^2 + \gamma_{\alpha,2}^2}{\gamma_{\alpha,1}^2 \gamma_{\alpha,2}^2 \|z^\alpha\|^2_{L^2(\partial B, \nu)}}.
$$

Therefore

$$
\|z^\alpha\|^2_{\mathcal{H}(K_1+K_2)} = \frac{2 \gamma_{\alpha,1}^2 \gamma_{\alpha,2}^2}{\gamma_{\alpha,1}^2 + \gamma_{\alpha,2}^2} \|z^\alpha\|^2_{L^2(\partial B, \nu/2)} = \gamma_{\alpha}^2 \|z^\alpha\|^2_{L^2(\partial B, \nu/2)}
$$
for all $\alpha \in \mathbb{Z}_+^d$. Since $\{z^\alpha\}_{\alpha \in \mathbb{Z}_+^d}$ forms an orthogonal subset of $L^2(\partial B, \nu/2)$, the conclusion follows immediately. \hfill \Box

**Remark 3.5:** The conclusion of the Proposition 3.4 still holds even if we chose two different Reinhardt measures $\nu_1$ and $\nu_2$ in the slice representations of $K_1$ and $K_2$, such that for some sequence of positive real numbers $\{h_k\}_{k \in \mathbb{Z}_+}$, $\|z^\alpha\|_{L^2(\partial B, \nu_1)} = h_{|\alpha|}\|z^\alpha\|_{L^2(\partial B, \nu_2)}$ for all $\alpha \in \mathbb{Z}_+^d$. For every $j = 1, 2$, it is easy to verify that

$$\sum_{i=1}^d \frac{\|z^{\alpha+\epsilon_i}\|_{L^2(\partial B, \nu_j)}}{\|z^\alpha\|_{L^2(\partial B, \nu_j)}} = 1.$$  

This implies that $\{h_k\}_{k \in \mathbb{Z}_+}$ is a constant sequence, say $c$. Now, a routine argument, using the Stone-Weierstrass theorem, we conclude that $\mu_1 = c^2\mu_2$.

Let $\mathcal{K}_\nu$ denote the class of all balanced kernels with the following properties:

(i) For all $K \in \mathcal{K}_\nu$, the norm on $\mathcal{H}(K)$ admits the slice representations with fixed Reinhardt measure $\nu$.

(ii) For every member $K$ of $\mathcal{K}_\nu$, the multiplication operator $M_z$ defined on $\mathcal{H}(K)$ is jointly left invertible.

(iii) The Cauchy dual tuple $M_z^\nu$ of is a joint complete hyperexpansion.

**Lemma 3.6.** For every member $K$ of $\mathcal{K}_\nu$, the multiplication operator tuple $M_z$ defined on $\mathcal{H}(K)$ is a subnormal spherical contraction.

**Proof.** Let $K \in \mathcal{K}_\nu$ and $[\nu, H^2(\gamma)]$ be the slice representation for the norm on $\mathcal{H}(K)$. Note that the Cauchy dual $M_z^\nu$ of $M_z$ is a balanced multiplication tuple with slice representation $[\nu, H^2(1/\gamma)]$ (see Remark 3.3). Since $M_z^\nu$ is a joint complete hyperexpansion. It follows from Lemma 3.1 that $\{1/\gamma_k^2\}_{k \in \mathbb{Z}_+}$ is a completely alternating sequence. Therefore, by Remark 2.6 $\{\gamma_k^2\}_{k \in \mathbb{Z}_+}$ is completely monotone sequence. Now again by applying Lemma 3.1 we conclude that the multiplication operator $M_z$ is a subnormal spherical contraction. \hfill \Box

**Theorem 3.7.** If $K_1$ and $K_2$ are any two members of $\mathcal{K}_\nu$, then the multiplication operator $M_z$ on $\mathcal{H}(K_1 + K_2)$ is a subnormal spherical contraction.

**Proof.** Note that the norm on $\mathcal{H}(K_1 + K_2)$ admits the slice representation $[\nu/2, H^2(\gamma)]$, where $\gamma_k^2 = 2\gamma_{k,1}^2\gamma_{k,2}^2/\gamma_{k,1}^2 + \gamma_{k,2}^2$ for all $k \in \mathbb{Z}_+$ (see Proposition 3.4). It follows from the proof of Lemma 3.6 that $\{1/\gamma_{k,1}^2\}_{k \in \mathbb{Z}_+}$ and $\{1/\gamma_{k,2}^2\}_{k \in \mathbb{Z}_+}$ are completely alternating. So their sum, that is, $\{1/\gamma_k^2\}_{k \in \mathbb{Z}_+}$ is a completely alternating sequence. Now the conclusion follows by imitating the argument given in Lemma 3.6. \hfill \Box

For $\lambda > 0$, consider the positive definite kernel $K_\lambda$ given by

$$K_\lambda(z, w) = \frac{1}{(1 - (z, w))^\lambda} \quad (z, w \in \mathbb{B}).$$

The norm on $H(K_\lambda)$ admits the slice representation $[\sigma, H^2(\gamma)]$, where $\sigma$ denotes the normalized surface area measure on $\partial \mathbb{B}$ and $\gamma_k^2 = (d_k/\lambda_k)$ for all $k \in \mathbb{Z}_+$. It is well known that the multiplication
operator $M_{z, \lambda}$ on $H(K_{\lambda})$ is a subnormal contraction if and only if $\lambda \geq d$. The same can also be verified by using Lemma 3.1 and part (i) of Theorem 2.11. Similarly, by using Lemma 3.1 and part (ii) of Theorem 2.11 one may conclude that the Cauchy dual tuple $M_{z, \lambda}^2$ is a joint completely hyperexpansion if and only if $d \leq \lambda \leq d + 1$. Thus, if we choose $\lambda$ and $\lambda'$ are such that $d \leq \lambda, \lambda' \leq d + 1$. Then $K_{\lambda}$ and $K_{\lambda'}$ are in $\mathcal{K}_\sigma$. It now follows from Theorem 3.7 that the multiplication operator $M_z$ on $\mathcal{H}(K_{\lambda} + K_{\lambda'})$ is subnormal. This is also included in the following example.

**Example 3.8.** Let $0 < d \leq \lambda' \leq \lambda \leq \lambda' + 1$. Note that the norm on $\mathcal{H}(K_{\lambda} + K_{\lambda'})$ admits the slice representation $[\sigma/2, H^2(\gamma)]$, where $\gamma_k^2 = \frac{2(d)k}{(\lambda)k + (\lambda')k}$ for all $k \in \mathbb{Z}_+$. From the proof of Proposition 2.12 it is clear that $\{\gamma_k^2\}_{k \in \mathbb{Z}_+}$ is completely monotone. Hence, the multiplication operator $M_z$ on $\mathcal{H}(K_{\lambda} + K_{\lambda'})$ is subnormal.

A $d$-tuple $S = (S_1, \cdots, S_d)$ of commuting bounded linear operators $S_1, \cdots, S_d$ in $\mathcal{B}(\mathcal{H})$ is a spherical isometry if $S_1^*S_1 + \cdots + S_d^*S_d = I$. In other words, $Q_{S}(I) = I$. The most interesting example of a spherical isometry is the Szegö $d$-shift; that is, the $d$-tuple $M_z$ of multiplication operators $M_{z_1}, \cdots, M_{z_d}$ on the Hardy space $H^2(\partial \mathbb{D})$ of the unit ball.

Let $\nu$ be a Reinhardt measure. Consider the multiplication $d$-tuple $M_z$ on a reproducing kernel Hilbert space $\mathcal{H}(K^\nu)$ determined by the reproducing kernel

$$K^\nu(z, w) = \sum_{\alpha \in \mathbb{Z}_+^d} \frac{z^\alpha \bar{w}^\alpha}{\|z^\alpha\|^2_{L^2(\partial \mathbb{D}, \nu)}} \quad (z, w \in \mathbb{D}).$$

Note that $M_z$ is a spherical isometry. In this case, the norm on $\mathcal{H}(K^\nu)$ admits the slice representation $[\nu, H^2(\mathbb{D})]$, where $H^2(\mathbb{D})$ is the Hardy space of the unit disc.

**Theorem 3.9.** Let $K^\nu$ be the reproducing kernel given as in equation (3.2) and $\tilde{K}$ be any balanced kernel with the slice representation $[\nu, H^2(\gamma)]$. Assume that the multiplication operator $M_z$ on $\mathcal{H}(K^\nu + \tilde{K})$ is subnormal. Then the multiplication operator on $\mathcal{H}(\tilde{K})$ is subnormal.

**Proof.** Observe that the norm on $\mathcal{H}(K^\nu + \tilde{K})$ admits the slice representation $[\nu/2, H^2(\gamma)]$, where $\gamma_k^2 = 2(1 + 1/\gamma_k^2)^{-1}$ for all $k \in \mathbb{Z}_+$. Since $M_z$ on $\mathcal{H}(K^\nu + \tilde{K})$ is subnormal, it follows from Lemma 3.1 that $\{\gamma_k^2\}_{k \in \mathbb{Z}_+}$ is a completely monotone sequence. Hence, $\{(1 + 1/\gamma_k^2)^{-1}\}_{k \in \mathbb{Z}_+}$ is a completely monotone sequence. If we replace $\alpha_k$ by $1/\gamma_k^2$ in the proof of the Proposition 2.9 we get that $\{\tilde{\gamma}_k^2\}_{k \in \mathbb{Z}_+}$ is completely monotone. Now, by applying Lemma 3.1 we conclude that the multiplication operator on $\mathcal{H}(\tilde{K})$ is subnormal.

We conclude the paper with the following questions:

**Question 3.10.** In view of Proposition 2.12 and Theorem 2.12, it is natural to ask that

(i) what is the necessary and sufficient condition for the multiplication operator $M_z$ on $\mathcal{H}(K_{\lambda, \mu} + K_{\lambda', \mu})$ to be subnormal?

(ii) what is the necessary and sufficient condition for the multiplication operator $M_z$ on $\mathcal{H}(K_p + K_q)$ to be subnormal?
Question 3.11. Let $K^\nu$ be the reproducing kernel given as in equation (3.2) and $\tilde{K}$ be any positive definite kernel given by
\[
\tilde{K}(z, w) := \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha z^\alpha \bar{w}^\alpha \quad (z, w \in \mathbb{B}).
\]
Assume that the $d$-tuple $M_z = (M_{z_1}, \cdots, M_{z_d})$ of multiplication by the co-ordinate functions $z_1, \cdots, z_d$ on $\mathcal{H}(K^\nu + \tilde{K})$ is subnormal. Is it necessary that the multiplication operator on $\mathcal{H}(\tilde{K})$ subnormal?

Acknowledgments. We express our sincere thanks to Prof. G. Misra for many fruitful conversations and suggestions in the preparation of this paper. We would also like to thank Prof. S. Chavan for his many useful comments and careful reading of the manuscript.

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