FIRST $\ell^2$-BETTI NUMBERS AND PROPER PROXIMALITY

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Abstract. We show that for a countable exact group, having positive first $\ell^2$-Betti number implies proper proximality in this sense of [BIP21]. This is achieved by showing a cocycle superstiffness result for Bernoulli shifts of non-properly proximal groups. We also obtain that Bernoulli shifts of countable, nonamenable, i.c.c., exact, non-properly proximal groups are $O\!E$-superrigid.

1. Introduction

The group measure space construction associates to every probability measure preserving (p.m.p.) action $\Gamma \acts (X, \mu)$ of a countable group $\Gamma$, a finite von Neumann algebra $M = L^\infty(X) \rtimes \Gamma$ [MVN36]. If the action is free and ergodic, then $M$ is a II$_1$ factor and $L^\infty(X)$ is a Cartan subalgebra. During the last two decades, Popa’s deformation/rigidity theory has led to spectacular progress in the classification and structural results of II$_1$ factors (see surveys [Pop07b, Vae10, Ioa18]). In particular, several large families of group measure space II$_1$ factors $L^\infty(X) \rtimes \Gamma$ have been shown to have a unique Cartan subalgebra, up to unitary conjugacy [OP10a, OP10b, CS13, CSU13, PV14a, PV14b, BIP21]. Such unique Cartan subalgebra results play a crucial role in the classification of group measure space II$_1$ factors, as they allow one to reduce the classification of the factors $L^\infty(X) \rtimes \Gamma$, up to isomorphism, to the classification of the corresponding actions $\Gamma \acts X$, up to orbit equivalence [Sin55, FM77].

Partially motivated by this question, Boutonnet, Ioana and Peterson introduce the notion of properly proximal groups in [BIP21], and they show that, among other results, $L^\infty(X) \rtimes \Gamma$ has a unique weakly compact Cartan subalgebra, up to unitary conjugacy, provided $\Gamma$ is properly proximal and $\Gamma \acts X$ is a free ergodic p.m.p action. Properly proximal groups form a robust family, which includes lattices in noncompact semisimple Lie groups, nonamenable biexact groups, nonelementary convergence groups [BIP21], CAT(0) cubical groups, nonelementary mapping class groups [HHL20], wreath products $\Lambda \ltimes \Gamma$ with $\Lambda$ nontrivial and $\Gamma$ nonamenable [DKE22], and is stable under measure equivalence and $W^*$-equivalence [IPR19], while as shown in [IPR19], inner amenability is not the only obstruction to proper proximality. Notably, [BIP21] demonstrates the first $W^*$-strong rigidity result for $\text{SL}_n(\mathbb{Z})$ with $n \geq 3$.

Another class of groups whose associated II$_1$ factors have been extensively studied is the class of groups with positive first $\ell^2$-Betti numbers [Pet09, PS12, Ioa13a, Ioa12b, CP13, CS13, Vae13]. For a nonamenable countable group $\Gamma$, having positive first $\ell^2$-Betti number, $\beta_1^{(2)}(\Gamma) > 0$, is equivalent to the existence of unbounded cocycle into its left regular representation. Popa and Vaes conjecture that $L^\infty(X) \rtimes \Gamma$ has a unique Cartan subalgebra, up to unitary conjugacy, for any free ergodic p.m.p. action $\Gamma \acts X$ given $\beta_1^{(2)}(\Gamma) > 0$ (see also [Ioa18, Problem 1]). In their breakthrough work [PV14a], Popa and Vaes verify this conjecture if $\Gamma$ is, in addition, weakly amenable.

In this paper, we establish the connection between first $\ell^2$-Betti numbers and proper proximality, under a mild technical assumption.

**Theorem 1.1.** Let $\Gamma$ be a countable exact group. If $\beta_1^{(2)}(\Gamma) > 0$, then $\Gamma$ is properly proximal.
One concrete class of groups that satisfy the assumption of Theorem 1.1 is the class of one relator groups with at least 3 generators \cite{Gue02, DL07}, which was not known to be properly proximal before.

Since weak amenability implies exactness (see e.g. \cite{Kir95} Proposition 2 and \cite{Oza07}), Theorem 1.1 together with \cite{BIP21} Theorem 1.5 implies that for a weakly amenable group $\Gamma$ with $\beta^2(\Gamma) > 0$, $L\Gamma$ has no Cartan subalgebras and $L^\infty(X) \rtimes \Gamma$ has a unique Cartan subalgebra, up unitary conjugacy, for any action free ergodic p.m.p. $\Gamma \rtimes (X, \mu)$, which recovers the result in \cite{PV14} concerning groups with positive first $\ell^2$-Betti numbers. Although it should be noted that \cite{BIP21} Theorem 1.5 follows the same general strategy as laid out in \cite{PV14}.

Our approach to Theorem 1.1 is rather indirect. In fact, we first obtain the following cocycle superrigidity result for non-properly proximal groups, from which Theorem 1.1 follows in combination with \cite[Corollary 1.2]{PS12}.

**Theorem 1.2.** Let $\Gamma$ be a countable group, $(X_0, \mu_0)$ be a diffuse standard probability space and $\Gamma \rtimes (X_0^1, \mu_0^1) := (X, \mu)$ be the Bernoulli action. If $\Gamma$ is exact and contains a nonamenable non-properly proximal $wq$-normal subgroup, then $\Gamma \rtimes (X, \mu)$ is $\{\mathbb{T}\}$-cocycle superrigid, i.e., any $1$-cocycle $w: \Gamma \times X \to \mathbb{T}$ is cohomologous to a homomorphism.

Another theme that we explore is the rigidity of Bernoulli shifts of non-properly proximal group. Much of the work is heavily inspired by Popa’s pioneering work on Bernoulli shifts of rigid groups \cite{Pop06a, Pop06b, Pop07a}.

**Theorem 1.3.** Let $\Gamma$ be a countable group with infinite conjugacy classes (i.c.c.), $(X_0, \mu_0)$ be a diffuse standard probability space and $\Gamma \rtimes (X_0^1, \mu_0^1) := (X, \mu)$ be the Bernoulli action. Let $\Lambda$ be a countable group and $\Lambda \rtimes (Y, \nu)$ a free ergodic p.m.p. action such that $L^\infty(Y) \rtimes \Lambda \cong (L^\infty(X) \rtimes \Gamma)^t$ for some $0 < t \leq 1$. If $\Lambda$ is exact and contains a nonamenable, non-properly proximal normal subgroup, then $t = 1$ and $\Gamma \rtimes X$ and $\Lambda \rtimes Y$ are conjugate.

In particular, it follows that the fundamental group of $L^\infty(X) \rtimes \Gamma$ is trivial if $\Gamma$ is a countable, nonamenable, i.c.c., exact and non-properly proximal group and $\Gamma \rtimes X$ is the Bernoulli action. Furthermore, since proper proximality and exactness are stable under measure equivalence \cite{IPR19, Oza07}, Theorem 1.3 also implies the following OE-superrigidity result.

**Theorem 1.4.** Let $\Gamma$ be a countable nonamenable i.c.c. group $(X_0, \mu_0)$ be a diffuse standard probability space and $\Gamma \rtimes (X_0^1, \mu_0^1) := (X, \mu)$ be the Bernoulli action. If $\Gamma$ is exact and non-properly proximal, then $\Gamma \rtimes X$ is OE-superrigid, i.e., if a free ergodic p.m.p. action $\Lambda \rtimes (Y, \nu)$ is orbit equivalent to $\Gamma \rtimes (X, \mu)$, then these two actions are conjugate.

As a consequence, every countable nonamenable i.c.c. exact group has at least one desirable rigidity property, depending on whether or not it is properly proximal: either every group measure space II$_1$ factor has at most one weakly compact Cartan subalgebra, or else Bernoulli shifts are OE-superrigid.

All the above theorems are derived from the following von Neumann algebraic statement.

**Theorem 1.5.** Let $\Gamma$ be a countable group, $(X_0, \mu_0)$ be a diffuse standard probability space and $\Gamma \rtimes (X_0^1, \mu_0^1) := (X, \mu)$ be the Bernoulli action. Suppose $\Gamma$ is exact and $N \subset M := L^\infty(X) \rtimes \Gamma$ is a von Neumann subalgebra that has no amenable and no properly proximal direct summand. Then there exists a $s$-malleable deformation $\{\alpha_t\}_{t \in \mathbb{R}}$ on $M$ such that $\alpha_t \to \text{id}_M$ uniformly on the unit ball of $N$, as $t \to 0$. 
Here, the s-malleable deformation is in the sense of Popa [Po p06c, Po p06a] and this specific deformation is the one associated with Gaussian actions [Fur07] (see Section 4 for details). Proper proximality is for von Neumann algebras, in the sense of [DKE P22]. We note that Theorem 1.2 follows from Theorem 1.5 together with Popa’s seminal work on cocycle superrigidity [Pop07a, Pop08], and Theorem 1.3 is a combination of Theorem 1.5 and with Popa’s conjugacy criterion for Bernoulli actions [Pop06b]. Exactness of groups is crucial to our proof as we exploit the fact that \( \mathbb{Z} \rtimes \Gamma \) is biexact relative to \( \Gamma \), provided \( \Gamma \) is exact. Let us finish with some comparisons between our results and some existing results on inner-amenable groups. The family of exact, non-properly proximal groups is strictly larger than the family of exact, inner-amenable groups, due to [IPR19], [DT DW20] and [GHW05], as well as the permanence properties of exactness of groups (see e.g. [BO08, Section 5.1]). Moreover, exactness and proper proximality are both stable under measure equivalence and \( \mathcal{W}^* \)-equivalence [Oza07, IPR19], while inner-amenability is not preserved under measure equivalence [DTDW20] and is not known to be stable under \( \mathcal{W}^* \)-equivalence. Therefore, Theorem 1.2 can be seen as a generalization of [TD20, Theorem 11] in the case of \( T \)-valued cocycles associated with Bernoulli shifts of exact groups. And under the mild assumption on exactness, Theorem 1.1 generalizes [Dri22, Corollary F] and Theorem 1.5 extends [Dri22, Theorem E] in the case of wreath products.

Comments on the proofs. Let us outline the proof of Theorem 1.5, which uses the recently developed notion of proper proximality in [BIP21, IPR19, DKEP22] and Popa’s deformation/rigidity theory. The proof is divided into three steps. First we observe in Proposition 2.3 that for any von Neumann subalgebra \( N \) in \( L(\mathbb{Z} \rtimes \Gamma) \), with \( \Gamma \) exact, if \( N \) has no amenable direct summand, then it must be properly proximal relative to \( L(\mathbb{Z} \rtimes \Gamma) \) in the sense of [DKEP22]. This is a direct adaptation of [DKEP22, Theorem 7.1], since \( \mathbb{Z} \rtimes \Gamma \) is biexact relative to \( \Gamma \) [BO08, Proposition 15.3.6]. Next in Section 3, we use techniques from [DKE22], which extends the idea in [DKE21, Lemma 3.3] to the von Neumann algebra setting. Continuing in the above setting with \( N \) proper proximal relative to \( L(\mathbb{Z} \rtimes \Gamma) \), we show that \( N \) either has a properly proximal direct summand or is amenable relative to \( L(\mathbb{Z} \rtimes \Gamma) \) inside \( L(\mathbb{Z} \rtimes \Gamma) \). In this step, the notion of normal bidual developed in [DKEP22, Section 2] is extensively used. Lastly, using a technique from [Ioa15], we conclude in Section 4 that if \( N \subset L(\mathbb{Z} \rtimes \Gamma) \) is amenable relative to \( L(\mathbb{Z} \rtimes \Gamma) \), then \( N \) must be rigid with respect to the s-malleable deformation \( \{\alpha_t\} \) associated with \( L(\mathbb{Z} \rtimes \Gamma) \). Altogether, we obtain that if \( N \subset L(\mathbb{Z} \rtimes \Gamma) \) has no amenable or properly proximal direct summand, then \( N \) must be \( \alpha_t \)-rigid.

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2. Preliminaries

2.1. Popa’s intertwining-by-bimodules.

**Theorem 2.1** ([Pop06a]). Let \((M,\tau)\) be a tracial von Neumann algebra and \( P \subset pMp, Q \subset M \) be von Neumann subalgebras. Then the following are equivalent:

1. There exist projections \( p_0 \in P, q_0 \in Q \), a \(*\)-homomorphism \( \theta : p_0 P p_0 \to q_0 Q q_0 \) and a non-zero partial isometry \( v \in q_0 M p_0 \) such that \( \theta(x)v = vx \), for all \( x \in p_0 P p_0 \).
2. There is no sequence \( u_n \in U(P) \) satisfying \( \|E_Q(x^*u_n y)\|_2 \to 0 \), for all \( x, y \in pM \).

If one of these equivalent conditions holds, we write \( P \prec_M Q \).
2.2. Relative amenability. Let $P \subset M$ and $Q \subset M$ be a von Neumann subalgebras. Following [OP10a], we say that $P$ is amenable relative to $Q$ inside $M$ if there exists a sequence $\xi_n \in L^2(\mathcal{M}, e_Q)$ such that $\langle x \xi_n, \xi_n \rangle \to \tau(x)$, for every $x \in M$, and $\|y \xi_n - \xi_n y\|_2 \to 0$, for every $y \in P$, or equivalently if there exists a $P$-central state on $\mathcal{M}$ that is normal when restricted to $M$ and faithful on $\mathcal{Z}(P' \cap M)$.

2.3. Mixing subalgebras of finite von Neumann algebras. Let $M$ be a finite von Neumann algebra and $N \subset M$ a von Neumann subalgebra. Recall the inclusion $N \subset M$ is mixing if $L^2(M \otimes N)$ is mixing as an $N$-$N$ bimodule, i.e., for any sequence $u_n \in \mathcal{U}(N)$ converging to 0 weakly, one has $\|E_N(x u_n y)\|_2 \to 0$ for any $x, y \in M \otimes N$. When $M$ and $N$ are both diffuse, we may replace sequence of unitaries with any uniformly bounded sequence in $N$ converging to 0 weakly by the proof of (4) $\Rightarrow$ (1) in [DKEP22, Theorem 5.9].

Examples of mixing subalgebras include $L \Lambda \subset L \Gamma$, where $\Lambda < \Gamma$ is almost malnormal i.e., $|t \Lambda t^{-1} \cap \Lambda| < \infty$ for any $t \in \Gamma \setminus \Lambda$ (see e.g. [Bon14, Appendix A]).

2.4. Proper proximality. We recall the notion of properly proximal von Neumann algebras from [DKEP22].

2.4.1. Boundary pieces. Given $M$ a von Neumann algebra, an $M$-boundary piece $X$ is a hereditary $C^*$-subalgebra of $B(L^2M)$ such that $M(X) \cap M \subset M$ and $M(X) \cap JM \subset JM$ are weakly dense, where $M(X)$ is the multiplier of $X$. To avoid pathological examples, we will always assume that $X \neq \{0\}$ and it follows that $K(L^2M) \subset X$ for any $M$-boundary piece $X$.

Let $M$ be a finite von Neumann algebra and $X$ an $M$-boundary piece. Denote by $K^L_X(M) \subset B(L^2M)$ the $\|\cdot\|_{\infty,2}$-closure of the norm closed left ideal $B(L^2M)X$, where $\|T\|_{\infty,2} = \sup_{a \in (M)} \|Ta\|_2$ for any $T \in B(L^2M)$, and set $K_X(M) = (K^L_X(M))^* \cap K^L_X(M)$ to be the hereditary $C^*$-subalgebra generated by $K^L_X(M)$. The multiplier algebra of $K_X(M)$ contains both $M$ and $JM$ and we denote by $K_{\infty,1}^X(M)$ the $\|\cdot\|_{\infty,1}$-closure of $K_X(M)$, where $\|T\|_{\infty,1} = \sup_{a,b \in (M)} \langle Ta, b \rangle$ for $T \in B(L^2M)$, and $K_{\infty,1}^X(M)$ coincides with $\overline{X} \|\cdot\|_{\infty,1}$. And denote by $S_X(M)$ the following operator system that contains $M$, $S_X(M) = \{T \in B(L^2M) \mid \langle T, x \rangle \in K^L_{\infty,1}(M), \text{ for any } x \in JM\}$. When $X = K(L^2M)$, we omit $X$ in the above notations for simplicity.

Let $N \subset M$ be a von Neumann subalgebra. We say $N \subset M$ is properly proximal relative to $X$ if there does not exist any $N$-central state $\phi$ on $S_X(M)$ such that $\phi|_M$ is normal. And we say $M$ is properly proximal if $M \subset M$ properly proximal relative to $K(L^2M)$. By [DKEP22, Theorem 6.2], a group $\Gamma$ is properly proximal in the sense of [BIP21] if and only if $L \Gamma$ is properly proximal.

One particular type of boundary pieces arise from subalgebras. Let $N \subset M$ be a von Neumann subalgebra and we may associate with $N$ an $M$-boundary piece $X_N$, which is the hereditary $C^*$-subalgebra of $B(L^2M)$ generated by $xJyJ$, for $x, y \in M$, where $e_N \in B(L^2M)$ is the orthogonal projection from $L^2M$ onto $L^2N$.

Remark 2.2. Let $\Gamma$ be a group that is not properly proximal, then $L \Gamma$ has no proper proximal direct summand. Indeed, suppose $z \in \mathcal{Z}(L \Gamma)$ in a nonzero central projection such that $zL \Gamma$ is not properly proximal, i.e., there exists a $zL \Gamma$-central state $\phi : S(zL \Gamma) \to \mathbb{C}$ that is normal on $zL \Gamma$. We may consider the $\Gamma$-equivariant embedding $i : S(\Gamma) \to S(L \Gamma)$ and $E := \text{Ad}(z) : S(L \Gamma) \to S(zL \Gamma)$ [DKEP22, Section 6]. Then $\phi \circ E \circ i : S(\Gamma) \to \mathbb{C}$ is then a $\Gamma$-invariant state, showing $\Gamma$ is not properly proximal. A similar argument shows that if $\Gamma$ is nonamenable, then $L \Gamma$ has no amenable direct summand.
Recall that a group $\Gamma$ is biexact relative to a subgroup $\Lambda < \Gamma$ if the left action of $\Gamma$ on $S_\Lambda(\Gamma) = \{ f \in L^\infty(\Gamma) | f - R_t f \in c_0(\Gamma, \{ A \}) \}$ is topologically amenable, where $c_0(\Gamma, \{ A \})$ is functions on $\Gamma$ that converge to 0 when $t \in \Gamma$ escapes subsets of $\Gamma$ that are small relative to $\Lambda$ (See [BO08, Chapter 15] for the precise definition). We remark that this is equivalent to $\Gamma \rtimes S_I(\Gamma)$ is amenable. Indeed, since we may embed $L^\infty(\Gamma) \rightarrow L^\infty(\Gamma)$ in a $\Gamma$-equivariant way, we have $\Gamma \rtimes I^\infty(\Gamma) \oplus (S_I(\Gamma)/I)^\infty = S_I(\Gamma)^\infty$ is amenable, and it follows that $\Gamma \rtimes S_I(\Gamma)$ is an amenable action [BEW19, Proposition 2.7].

The following is an easy adaptation of [DKEP22, Theorem 7.1]. For completeness, we include the proof.

**Proposition 2.3.** Suppose a group $\Gamma$ is biexact relative to a subgroup $\Lambda < \Gamma$. Then for every von Neumann subalgebra $N \subset \ell^2(\Gamma)$, either the inclusion $N \subset \ell^2(\Gamma)$ is properly proximal relative to $X_{LA}$ or else $N$ has an amenable direct summand.

**Proof.** Suppose the inclusion $N \subset \ell^2(\Gamma)$ is not properly proximal relative to $X_{LA}$, and let $\phi : S_{S_{LA}}(\ell^2(\Gamma)) \rightarrow (p(\ell^2(\Gamma)) p, e_{NP})$ be a $p(\ell^2(\Gamma)) p$-bimodular u.c.p. map, where $p \in Z(N)$ is a non-zero central projection.

If we consider the $\Gamma$-equivariant diagonal embedding $L^\infty(\Gamma) \subset \ell(\ell^2(\Gamma))$, we see that $c_0(\Gamma, \{ A \})$ is mapped to $X_{LA}$. Restricting to $S_{\Lambda}(\Gamma)$ then gives a $\Gamma$-equivariant embedding into $S_{S_{LA}}(\ell^2(\Gamma))$. We therefore obtain a $*$-homomorphism $S_{\Lambda}(\Gamma) \rtimes f \Gamma \rightarrow \ell(\ell^2(\Gamma))$ whose image is contained in $S_{S_{LA}}(\ell^2(\Gamma))$. Composing this $*$-homomorphism with the u.c.p. map $\phi$ then gives a u.c.p. map $\bar{\phi} : S_{\Lambda}(\Gamma) \rtimes f \Gamma \rightarrow (p(\ell^2(\Gamma)) p, e_{NP})$ such that $\bar{\phi}(t) = pu_t p$ for all $t \in \Gamma$.

Since $\Gamma$ is biexact relative to $\Lambda$, the action $\Gamma \rtimes S_{\Lambda}(\Gamma)$ is topologically amenable. Hence, $S_{\Lambda}(\Gamma) \rtimes f \Gamma = S_{\Lambda}(\Gamma) \rtimes_\alpha \Gamma$ is a nuclear $\mathcal{C}^*$-algebra. We set $\phi(\cdot) := \frac{1}{\tau(p)} (\phi(\cdot) \bar{\phi}, \bar{\phi})$ and note that for $x \in C^*_\Lambda \Gamma$ we have $\phi(x) = \frac{1}{\tau(p)} \tau(x p)$. Since $C^*_\Lambda \Gamma$ is weakly dense in $\ell^2(\Gamma)$ an argument similar to Proposition 3.1 in [BCL9] then gives a representation $\pi_\phi : S_{\Lambda}(\Gamma) \rtimes_\alpha \Gamma \rightarrow \ell(\ell^2(\Gamma))$, a state $\pi_\phi \in \ell(\ell^2(\Gamma))$, with $\phi = \pi_\phi \circ \pi_\phi$, and a projection $q \in \pi_\phi(S_{\Lambda}(\Gamma) \rtimes_\alpha \Gamma)'''$ with $\phi(q) = 1$ such that there is a normal unital $*$-homomorphism $i : \ell^2(\Gamma) \rightarrow q \pi_\phi(S_{\Lambda}(\Gamma) \rtimes_\alpha \Gamma)'' q$. Since $\pi_\phi(S_{\Lambda}(\Gamma) \rtimes_\alpha \Gamma)''$ is nuclear, we have that $\pi_\phi(S_{\Lambda}(\Gamma) \rtimes_\alpha \Gamma)'''$ is injective, and so there is a u.c.p. map $i : \ell(\ell^2(\Gamma)) \rightarrow \pi_\phi(S_{\Lambda}(\Gamma) \rtimes_\alpha \Gamma)''' q$ that extends $i$. Notice that $\psi := \bar{\phi} \circ i$ is then an $NP$-central state on $\ell(\ell^2(\Gamma))$ and $\psi(x) = \frac{1}{\tau(p)} \tau(x p)$ for $x \in \ell^2(\Gamma)$. Therefore, $NP$ is amenable.\[\square\]

### 2.4.2. A bidual characterization.

Next we collect some basics of the normal bidual from [DKEP22, Section 2].

Given a finite von Neumann algebra $M$ and a $\mathcal{C}^*$-subalgebra $A \subset \ell^2(\mathbb{M})$ such that $M$ and $JM J$ are contained in the multiplier algebra $\mathbb{M}(A)$, we recall that $A^{JM \dagger}$ (resp. $A^{JM J, JM J}$) denotes the space of $\varphi \in A^*$ such that for each $T \in A$ the map $M \times M \ni (a, b) \mapsto \varphi(a T b)$ (resp. $JM J \times JM J \ni (a, b) \mapsto \varphi(a T b)$) is separately normal in each variable. When there is no confusion about the von Neumann algebra that we are referring to, we will denote $A^{JM \dagger}$ by $A^*$ and $A^{JM \dagger} \cap A^{JM J, JM J}$ by $A^J_J$.

We may view $(A^J_J)^* \dagger$ as a von Neumann algebra as follows. Denote by $p_{nor} \in \mathbb{M}(A)^*$ the supremum of support projections of states in $\mathbb{M}(A)^*$ that restrict to normal states on $M$ and $JM J$, so that $M$ and $JM J$ may be viewed as unital von Neumann subalgebras of $p_{nor}\mathbb{M}(A)^*$, which is canonically identified with $(\mathbb{M}(A)^J_J)^*$. Let $q_A \in \mathcal{P}(\mathbb{M}(A)^*)$ be the central projection such that $q_A(\mathbb{M}(A)^*) = A^*$ and we may then identify $(A^J_J)^* \dagger$ with $p_{nor} q_A \mathbb{M}(A)^* p_{nor} = p_{nor} A^* \dagger p_{nor}$. Furthermore, if $B \subset A$ is another $\mathcal{C}^*$-subalgebra with $M$, $JM J \subset \mathbb{M}(B)$, we may identify $(B^J_J)^* \dagger$ with $q_B p_{nor} A^* \dagger p_{nor} q_B$, which is a non-unital subalgebra of $(A^J_J)^* \dagger$. 
If we denote by $\iota : \mathbb{M}(A) \to \mathbb{M}(A)^{\ast \ast}$ the canonical embedding, we may then view $\mathbb{M}(A)$ as an operator subsystem of $(\mathbb{M}(A)^{\ast \ast})_J^\sharp$ through the isometric u.c.p. map $\iota_{\text{nor}} : \mathbb{M}(A) \ni T \to p_{\text{nor}} \iota(T)p_{\text{nor}} \in (\mathbb{M}(A)^{\ast \ast})_J^\sharp$, and its restriction to $M$ gives a natural embedding of $A \subset (A^J_N)^{\ast \ast}$ as an operator system.

It is worth noting that $\iota_{\text{nor}}$ and $\iota$ are different in a few ways. On one hand, $\iota : \mathbb{M}(A) \to \mathbb{M}(A)^{\ast \ast}$ is a $*$-homomorphism while $\iota_{\text{nor}} : \mathbb{M}(A) \to (\mathbb{M}(A)^{\ast \ast})_J^\sharp$ is a u.c.p. map; on the other hand, $\iota_{\text{nor}}$ gives rise to normal faithful representations when restricted to $M$ and $M_{J\mathcal{M}}$, but $\iota|_M$ and $\iota|_{M_{J\mathcal{M}}}$ are not normal in general.

The following is a bidual characterization of properly proximal.

**Lemma 2.4.** [DKEP22, Lemma 8.5] Let $M$ be a separable tracial von Neumann algebra with an $M$-boundary piece $\mathbb{X}$. Then $M$ is properly proximal relative to $\mathbb{X}$ if and only if there is no $M$-central state $\varphi$ on \(\mathbb{S}_\mathbb{X}(M) := \{T \in \mathbb{B}(L^2M)^{\ast \ast} \mid [T,a] \in (\mathbb{K}_\mathbb{X}(M)^{\ast \ast})_J^\sharp \text{ for all } a \in M_{J\mathcal{M}}\}\) such that $\varphi|_M$ is normal.

When $\mathbb{X} = \mathbb{K}(L^2M)$, we will abbreviate $\mathbb{X}$ for simplicity. It is worth noting that $\mathbb{S}_\mathbb{X}(M)$ is a von Neumann algebra which contains $M$ as a von Neumann subalgebra, while $\mathbb{S}_\mathbb{X}(M)$ is only an operator system. Following the above discussion, we note that $\mathbb{S}_\mathbb{X}(M)$ may be identified with $\mathbb{S}_\mathbb{X}(M) = \{T \in p_{\text{nor}} \mathbb{B}(L^2M)^{\ast \ast} p_{\text{nor}} \mid [T,a] \in q_X(\mathbb{M}(\mathbb{K}_\mathbb{X}(M)))^{\ast \ast} q_X, \text{ for any } a \in M_{J\mathcal{M}}\}$, where $q_X$ is the identity of $(\mathbb{K}_\mathbb{X}(M)^{\ast \ast})_J^\sharp \subset (\mathbb{M}(\mathbb{K}_\mathbb{X}(M)))^{\ast \ast}$. If we set $q_X = q_{\mathbb{K}(L^2M)}$ to be the identity of $(\mathbb{K}(L^2M)^{\ast \ast})_J^\sharp \subset (\mathbb{B}(L^2M)^{\ast \ast})_J^\sharp$, then using the above description of $\mathbb{S}_\mathbb{X}(M)$, we have $q_X^\dagger \mathbb{S}_\mathbb{X}(M) q_X^\dagger \subset q_X^\dagger \mathbb{S}(M)$, as $q_X$ commutes with $M$ and $M_{J\mathcal{M}}$.

**Lemma 2.5.** Let $M$ be a separable tracial von Neumann algebra. Suppose $M$ has no properly proximal direct summand, then there exists an $M$-central state $\varphi$ on $\mathbb{S}(M)$ such that $\varphi|_M$ is faithful and normal.

**Proof.** First we show that there exists an $M$-central state $\varphi$ on $\mathbb{S}(M)$ such that $\varphi|_Z(M)$ is faithful. Consider a pair $(\varphi,p)$, where $\varphi \in \mathbb{S}(M)^{\ast \ast}$ is an $M$-central state such that $\varphi|_M$ is normal and $p \in Z(M)$ is support projection of $\varphi$. And we may order such pairs by the order on $Z(M)$, i.e., $(\varphi_1,p_1) \leq (\varphi_2,p_2)$ if $p_1 \leq p_2$. If $\{(\varphi_i,p_i)\}_{i \in I}$ is a chain, then we may find a subsequence $p_i(n)$ such that $\lim_n p_i(n) = \vee_{i \in I} p_i$, and $\varphi_0(\cdot) = \sum_{n \geq 1} 2^{-n} \varphi_i(n)(p_{i(n)} \cdot p_{i(n)})$ then is an $M$-central state on $\mathbb{S}(M)$ such that $\varphi_{0|M}$ is normal and $\vee_{i \in I} p_i$ is the support of $\varphi_0$. Suppose $(\varphi,p)$ is a maximal element and $q = p^\perp > 0$. Denote by $E_q : \text{Ad}(q) : \mathbb{B}(L^2M) \to \mathbb{B}(L^2(qM))$ and one checks that $(E_q)^{\ast \ast}$ maps $\mathbb{B}(L^2(qM))^J_N$ to $\mathbb{B}(L^2M)^J_N$. Therefore dualizing $E_q$ yields a u.c.p. map $E_q : (\mathbb{B}(L^2M)^J_N)^{\ast \ast} \to (\mathbb{B}(L^2(qM))^J_N)^{\ast \ast}$, and $(E_q)|_{\mathbb{S}(M)} : \mathbb{S}(M) \to \mathbb{S}(qM)$. Since $qM$ is not properly proximal, there exists a state $\psi \in \mathbb{S}(qM)^{\ast \ast}$ that is $qM$-central and $\psi|qM$ is normal. Set $\varphi'(T) = \varphi(pTP) + \psi(E_q(qTq))$, which is an $M$-central state on $\mathbb{S}(M)$ that is normal on $M$ with support strictly larger than $p$, which is a contradiction.

Now suppose $\varphi$ is such a state with $\varphi|_Z(M)$ faithful, and $\varphi(p) = 0$ for some $p \in \mathcal{P}(M)$, then we may write the central support $z(p) = \sum_{i=1}^\infty v_i v_i^*$, where $v_i \in M$ are partial isometries such that $v_i^* v_i \leq p$. Since $\varphi$ is normal and tracial on $M$, we have $\varphi(z(p)) = \sum_{i=1}^\infty \varphi(v_i v_i^*) \leq \sum_{i=1}^\infty \varphi(p) = 0$, which shows that $p \leq z(p) = 0$. \(\square\)
3. From non-proper proximality to relative amenability

In this section, we connect non-proper proximality with relative amenability using the following result.

**Proposition 3.1.** Let $\Gamma$ be a nonamenable countable group and $\Lambda < \Gamma$ an infinite almost malnormal subgroup. Let $M = L\Gamma$, $\mathbb{X} = \mathbb{X}_{\Lambda}$, the $M$-boundary piece associated with $L\Lambda$, and $N \subset M$ a von Neumann subalgebra. Suppose $N \subset M$ is properly proximal relative to $\mathbb{X}$. If $N$ does not have any properly proximal direct summand, then $N$ is amenable relative to $\Lambda$ inside $M$.

The above proposition, which grew out of discussions with Srivatsav Kunnapawalkam Elayavalli. A more general version appears in [DKEP22] and we only present the form that is sufficient for our purpose. Before proceeding to the proof, we collect a few auxiliary lemmas.

### 3.1. Boundary pieces in the bidual

Let $M$ be a finite von Neumann algebra, $\mathbb{X}$ an $M$-boundary piece, $N \subset M$ a von Neumann subalgebra and $E := \text{Ad}(e_N) : \mathbb{B}(L^2M) \to \mathbb{B}(L^2N)$. Notice that $(E^*)_{|B(L^2M)^{\sharp}_j} : \mathbb{B}(L^2M)^{\sharp}_j \to \mathbb{B}(L^2N)^{\sharp}_j$ since $E$ is a normal conditional expectation when restricted to $M$ and $JMJ$, and a state $\varphi \in \mathbb{B}(L^2M)^{\ast}$ lies in $\mathbb{B}(L^2M)^{\sharp}_j$ if and only if $\varphi|_M$ and $\varphi|_{JMJ}$ are normal. Thus we may consider the u.c.p. map

$$\tilde{E} := (E^*)_{|B(L^2M)^{\sharp}_j}^* : \mathbb{B}(L^2M)^{\ast}_j \to \mathbb{B}(L^2N)^{\ast}_j.$$  

**Lemma 3.2.** Using the above notations, we have $\tilde{E}|_{\mathbb{B}(M)} : \tilde{\mathbb{S}}(M) \to \tilde{\mathbb{S}}(N)$.

**Proof.** First observe that $\tilde{E}$ is weak* continuous and $E : \mathbb{B}(L^2M) \to \mathbb{B}(L^2N)$. It follows that $\tilde{E}$ maps $(\mathbb{K}(L^2M)^{\ast}_j)$ to $(\mathbb{K}(L^2N)^{\ast}_j)^*$. Furthermore, since $\tilde{E}|_{JMJ} = E|_{JMJ} = \text{id}_{JMJ}$, we have $\tilde{E}([T,x]) = [\tilde{E}(T),x]$ for any $x \in JMJ$ and $T \in (\mathbb{B}(L^2M)^{\ast}_j)^*$. The statement follows from the definition of $\tilde{\mathbb{S}}(M)$. \hfill $\square$

Recall from Section 2.1.2 that $\iota : \mathbb{K}_X(M) \to \mathbb{K}_X(M)^{\ast\ast}$ is the canonical embedding, $p_{\text{nor}} \in \mathbb{B}(L^2M)^{\ast\ast}$ is the projection such that $p_{\text{nor}} \mathbb{K}_X(M)^{\ast\ast} p_{\text{nor}} = (\mathbb{K}_X(M)^{\ast}_j)^*$ and the embedding $\iota_{\text{nor}} : \mathbb{K}_X(M) \to (\mathbb{K}_X(M)^{\ast}_j)^*$ is given by $\iota_{\text{nor}} = \text{Ad}(p_{\text{nor}}) \circ \iota$.

**Lemma 3.3.** Let $M$ be a finite von Neumann algebra and $\mathbb{X}$ an $M$-boundary piece. Let $\mathbb{X}_0 \subset \mathbb{K}_X(M)$ be a C$^*$-subalgebra and $\{e_n\}_{n \in I}$ an approximate unit of $\mathbb{X}_0$. If $\mathbb{X}_0 \subset \mathbb{K}_X^{\infty}(M)$ is dense in $\|\cdot\|_{\infty,1}$ and $\iota(e_n)$ commutes with $p_{\text{nor}}$ for each $n \in I$, then $\lim_n \iota_{\text{nor}}(e_n) \in (\mathbb{K}_X(M)^{\ast}_j)^*$ is the identity, where the limit is in the weak* topology.

**Proof.** Since $\iota_{\text{nor}}(\mathbb{K}_X(M)) \subset (\mathbb{K}_X(M)^{\ast}_j)^*$ is weak* dense and functionals in $\mathbb{K}_X(M)^{\ast}_j$ are continuous in $\|\cdot\|_{\infty,1}$ topology by [DKEP22] Proposition 3.1], we have $\iota_{\text{nor}}(\mathbb{X}_0) \subset (\mathbb{K}_X(M)^{\ast}_j)^*$ is also weak* dense. Let $e = \lim_n \iota_{\text{nor}}(e_n) \in (\mathbb{K}_X(M)^{\ast}_j)^*$ be a weak* limit point and for any $T \in \mathbb{X}_0$, we have

$$e_{\text{nor}}(T) = \lim_n p_{\text{nor}} \iota(e_n) \iota(T) p_{\text{nor}} = \lim_n p_{\text{nor}} \iota(e_n T) p_{\text{nor}} = e_{\text{nor}}(T),$$

and similarly $e_{\text{nor}}(T)e = e_{\text{nor}}(T)$. By density of $\iota_{\text{nor}}(\mathbb{X}_0) \subset (\mathbb{K}_X(M)^{\ast}_j)^*$, we conclude that $e$ is the identity in $(\mathbb{K}_X(M)^{\ast}_j)^*$. \hfill $\square$

**Lemma 3.4.** Let $M$ be a finite von Neumann algebra and $B \subset M$ a von Neumann subalgebra. Let $e_B \in \mathbb{B}(L^2M)$ be the orthogonal projection onto $L^2B$. Then $\iota(e_B) \in \mathbb{B}(L^2M)^{\ast\ast}$ commutes with $p_{\text{nor}}$. 

Proof. Suppose \( \mathbb{B}(L^2M)^{**} \subset \mathbb{B}(H) \) and notice that \( \xi \in H \) is in the range of \( p_{\text{nor}} \) if and only if \( M \ni x \rightarrow \langle \langle x \rangle \xi, \xi \rangle \) and \( JMJ \ni x \rightarrow \langle \langle x \rangle \xi, \xi \rangle \) are normal. For \( \xi \in p_{\text{nor}}H \), we have \( \varphi(x) := \langle \langle x \rangle \xi, \xi \rangle \) is also normal for \( x \in M \) and \( JMJ \), which implies that \( \varphi(e_B)p_{\text{nor}} = p_{\text{nor}}\varphi(e_B)p_{\text{nor}} \). It follows that \( \varphi(e_B) \) and \( p_{\text{nor}} \) commutes.

Lemma 3.5. Let \( \Gamma \) be a representative and \( \Lambda < \Gamma \) a subgroup. Let \( M = L\Gamma', B = L\Lambda \) and \( X = X_B \). Denote by \( \{ t_k \}_{k \in K} \) a representative of \( \Gamma/\Lambda \), i.e., \( \Gamma = \bigcup_{k \in K} t_k\Lambda \) and \( u_k := \lambda t_k \in \Gamma \) the canonical unitaries. For each finite subset \( F \subset K \), let \( e_F = \bigvee_{k \in F} u_k J u_k J e' B J u' J_k \). Then \( \lim_F t_{\text{nor}}(e_F) \in (\mathbb{K}_{X}(M)_j^\times)^* \) is the identity.

Proof. Denote by \( X_0 \subset \mathbb{B}(L^2M) \) the hereditary \( C^* \)-subalgebra generated by \( x J y J e_N \) for \( x, y \in C^*_\rho(\Gamma) \). It is clear that \( X_0 \) is an \( M \)-boundary piece and by hereditariness we have \( e_F \in X_0 \) for each \( F \).

First we show that \( \mathbb{K}_{X_0}(M)^{\times} = \mathbb{K}_{X}(M)^{\times} \), where \( \mathbb{K}_{X_0}(M)^{\times} \) is obtained from \( X_0 \) in the way described in Section 2.4.1. Notice that \( \mathbb{B}(L^2M)X_0 \subset \mathbb{K}_{X}(M)^{\times} \) is dense in \( \| \cdot \|_{\infty,2} \). Indeed, for any contractions \( T \in \mathbb{B}(L^2M) \) and \( x, y \in L\Gamma \), we may find a net of contractions \( T_s \in \mathbb{B}(L^2M)X_0 \) such that \( T_s \rightarrow T \) in \( \mathbb{B}(L^2M)X_0 \) in \( \| \cdot \|_{\infty,2} \), as it follows directly from [DKP22, Proposition 3.1], the non-commutative Egorov theorem and the Kaplansky density theorem. It then follows that \( \mathbb{K}_{X_0}(M) \subset \mathbb{K}_{X_0}(M)^{\times} \) is dense in \( \| \cdot \|_{\infty,1} \) and hence \( \mathbb{K}_{X_0}(M)^{\times} = \mathbb{K}_{X}(M)^{\times} = \mathbb{K}_{X}(M)^{\times} \) by [DKP22, Proposition 3.6].

Next we show that \( \{ e_F \} \subset X_0 \) forms an approximate unit of \( X_0 \). Indeed, every element in \( X_0 \) can be written as a norm limit of linear spans consisting of elements of the from \( x_1 J y_1 J T J y_2 J x_2 \), where \( x_i, y_i \in C^*_\rho(\Gamma) \) and \( T \in \mathbb{B}(L^2B) \). Write each \( x_i, y_i \) as summations of \( u_k \lambda \), \( t \in \Lambda \), it suffices to check \( e_F(u_k J u_s J e_B) \) and \( (e_B J u_u J u_k)e_F \) agree with \( u_k J u_s J e_B \) and \( e_B J u_u J u_k \) when \( F \) is large enough, respectively, which follows easily from the construction of \( e_F \).

By Lemma 3.3, it is easy to check that \( \varphi(e_F) \) commutes with \( p_{\text{nor}} \) for every \( F \). And it follows from Lemma 3.3 that \( \lim_F t_{\text{nor}}(e_F) \in (\mathbb{K}_{X}(M)_j^\times)^* \) is the identity. \( \Box \)

Lemma 3.6. Let \( \Gamma \) be a group and \( \Lambda < \Gamma \) a subgroup. Denote by \( q_{\Lambda} \in (\mathbb{K}(L^2M)^{\times})^* \) the identity, \( M = L\Gamma' \) and \( B = L\Lambda \). Then \( p_{t,s} = q_{\Lambda} \) is a projection for \( t, s \in \Gamma \).

Proof. Since \( p_{\text{nor}} \) commutes with \( \iota(M) \) and \( \iota(JMJ) \) and \( q_{\Lambda} \in (\mathbb{B}(L^2M)^{\times})^* \) is a central projection, together with Lemma 3.6, we see that \( p_{t,s} \) is a projection.

Not that \( p_{t,s}p_{t',s'} = q_{\Lambda}^{\frac{1}{2}} p_{\text{nor}} \iota(\text{Proj}_{\Lambda_\Lambda' \cap t's'}) \) by Lemma 3.2. Thus \( \varphi = \mu \circ \tilde{E} : \tilde{S}(M) \rightarrow \tilde{S}(N) \) defines a \( N \)-central state that is faithful and normal on \( M \). Let \( \tilde{S} \) be the corresponding identities in these von Neumann algebras. Note that \( q_{\Lambda} \leq \tilde{S} \) and \( q_{\Lambda} \) commutes with \( M \) and \( JMJ \).
First we analyze the support of $\varphi$. Observe that $\varphi(q_K^+)=1$. Indeed, if $\varphi(q_K^+) > 0$, i.e., $\varphi$ does not vanish on $(\mathbb{K}(L^2M)^{†})_J^*$, then we may restrict $\varphi$ to $\mathbb{B}(L^2M)$, which embeds into $(\mathbb{K}(L^2M)^{†})_J^*$ as a normal operator $M$-system \cite[Section 8]{DKEP22}, and this shows that $N$ would have an amenable direct summand. We also have $\varphi(q_K^+) = 1$, since if $\varphi(q_K^+) > 0$, we would then have an $N$-central state

$$\frac{1}{\varphi(q_K^+)} \varphi \circ \text{Ad}(q_K^+) : \tilde{S}_X(M) \to \mathbb{C},$$

whose restriction to $M$ is normal. This contradicts the assumption that $N \subset M$ is properly proximal relative to $X$, since $\tilde{S}_X(M)$ embeds unitaly into $\tilde{S}_X(M)$ through $\iota_{\text{nor}}$ in Section 3.1 Therefore we conclude that $\varphi(q_K^+) = 1$.

Let $B := LA \subset M$ and $e_B : L^2M \to L^2B$ the orthogonal projection.

**Claim.** There exists a u.c.p. map $\phi : (M,e_B) \to q_K^+q_K^+\tilde{S}(M)q_K^+$ such that $\phi(x) = q_K^+q_K^+x$ for any $x \in M$.

This claim clearly implies that $N$ is amenable relative to $B$ inside $M$, as $\nu = \varphi \circ \phi \in (M,e_B)^*$ is an $N$-central state, which is a normal faithful state when restricted to $M$.

**Proof of claim.** Recall from Section 2.4.2 that we may embed $\mathbb{B}(L^2M)$ into $(\mathbb{B}(L^2M)^{†})_J^*$ through the u.c.p. map $\iota_{\text{nor}}$, which is given by $\iota_{\text{nor}} = \text{Ad}(p_{\text{nor}}) \circ \iota$, where $\iota : \mathbb{B}(L^2M) \to \mathbb{B}(L^2M)^{**}$ is the canonical $*$-homomorphism into the universal envelope, and $p_{\text{nor}}$ is the projection in $\mathbb{B}(L^2M)^{**}$ such that $p_{\text{nor}} \mathbb{B}(L^2M)^{**} p_{\text{nor}} = (\mathbb{B}(L^2M)^{†})_J^*$. We have that $(\iota_{\text{nor}})_M$ and $(\iota_{\text{nor}})_{JM,J}$ are faithful normal representations of $M$ and $JM,J$, respectively, and to eliminate possible confusion, we will denote by $\iota_{\text{nor}}(M)$ and $\iota_{\text{nor}}(JM,J)$ the copies of $M$ and $JM,J$ in $(\mathbb{B}(L^2M)^{†})_J^*$. Restricting $\iota_{\text{nor}}$ to $C^*$-subalgebra $A \subset \mathbb{B}(L^2M)$ satisfying $M, JM,J \subset M(A)$ give rise to the embedding of $A$ into $(\mathbb{A}_J^e)^*$. Furthermore, although $\iota_{\text{nor}}$ is not a $*$-homomorphism, by Lemma 3.4 sp$Me_B M$ is in the multiplicative domain of $\iota_{\text{nor}}$.

Denote by $\{t_k\}_{k \geq 0} \subset \Gamma$ a representative of the cosets $\Gamma/\Lambda$ with $t_0$ being the identity of $\Gamma$, i.e., $\Gamma = \bigsqcup_{k \geq 0} t_k \Lambda$, and $u_k := \lambda_{t_k} \in U(L\Gamma)$. We will construct the map $\psi$ in the following steps.

**Step 1.** For each $n \geq 0$, consider the u.c.p. map $\psi_n : (M,e_B) \to (M,e_B)$ given by $\psi_n(x) = \sum_{a \in B,k \geq 0} u_{k \ell} e_B u_k^* x \sum_{a \in B,k \geq 0} u_{k \ell} e_B u_k^*$, and notice that $\psi_n$ maps $(M,e_B)$ into the $*$-subalgebra $A_0 := \text{sp}\{u_k e_B u_k^* | a \in B,k, \ell \geq 0\}$.

**Step 2.** By Lemma 3.6 we have $\{\iota_{\text{nor}}(Ju_k J e_B J u_k^* J)\}_{k \geq 0} \subset (\mathbb{B}(L^2M)^{†})_J^*$ are pairwise orthogonal projections. Set $e = \sum_{k \geq 0} \iota_{\text{nor}}(Ju_k J e_B J u_k^* J) \in (\mathbb{B}(L^2M)^{†})_J^*$ and notice that $e$ is independent of the choice of the representative $\Gamma/\Lambda$. Put $\phi_0 : A_0 \to q_K^+(\mathbb{B}(L^2M)^{†})_J^*$ to be $\phi_0(u_k e_B u_k^*) = q_K^+ \iota_{\text{nor}}(u_k e_B u_k^*)$.

It is easy to see that $\phi_0$ is well-defined. We then check that $\phi_0$ is a $*$-homomorphism. For any $x \in M$, we claim that

$$q_K^+ e_{\iota_{\text{nor}}}(x) e = q_K^+ e_{\iota_{\text{nor}}}(E_B(x)) e.$$

Indeed,

$$q_K^+ e_{\iota_{\text{nor}}}(x) e = q_K^+ \sum_{k, \ell \geq 0} \iota_{\text{nor}}((Ju_k J e_B J u_k^* J)x(Ju_k J e_B J u_k^* J)))$$

$$= q_K^+ \iota_{\text{nor}}(E_B(x)) \sum_{k \geq 0} \iota_{\text{nor}}(Ju_k J e_B J u_k^* J) + \sum_{k \neq \ell} \iota_{\text{nor}}((Ju_k J e_B J u_k^* J)x(Ju_k J e_B J u_k^* J)).$$
Since $\Lambda < \Gamma$ is almost malnormal which implies that $L^2(M \otimes B)$ is a mixing $B$-bimodule, one may check that $(Ju_kJe_BJu_k^*)^*(x - E_B(x))(Ju_kJe_BJu_k^*) \in \mathcal{B}(L^2M)$ is a compact operator from $M$ to $L^2M$ if $\ell \neq k$. We also have $(Ju_kJe_BJu_k^*)E_B(x)(Ju_kJe_BJu_k^*) = 0$ if $\ell \neq k$, and it follows that $\sum_{k \neq \ell} q_{k,\ell}^*t_{\text{nor}}(Ju_kJe_BJu_k^*)^*Ju_kJe_BJu_k^*) = 0$.

It then follows from (1) that $\phi_0$ is a $*$-homomorphism.

We also show $\phi_0$ is norm continuous. Set $\sum_{i=1}^d u_{k_i}a_i e_Bu_{k_i}^* \in A_0$, and note that we may assume $k_i \neq k_j$ and $\ell_i \neq \ell_j$ for $i \neq j$. Consider $P_k = q_{k,\ell}^*\sum_{i=1}^d t_{\text{nor}}(\text{Proj}_{k_iA_i^{-1}})$ and $Q_k = q_{k,\ell}^*\sum_{i=1}^d t_{\text{nor}}(\text{Proj}_{k_iA_i^{-1}})$, where $\text{Proj}_{k_iA_i^{-1}} \in \mathcal{B}(\ell^2\Gamma)$ is the orthogonal projection onto the subspace $\text{sp}\{\delta_\ell | \ell \in k_iA_i^{-1}\}$, i.e., $\text{Proj}_{k_iA_i^{-1}} = Ju_kJu_k^*$. By Lemma 5.6, we have $P_k$ and $Q_k$ are projections and $P_kP_r = Q_kQ_r = 0$ if $k \neq r$. Moreover, note that for each $i$, $t_{\text{nor}}(e_Bu_{k_i}^*Ju_k^*)P_k = q_{k,\ell}^*t_{\text{nor}}(e_Bu_{k_i}^*Ju_k^*)Q_k = q_{k,\ell}^*t_{\text{nor}}(e_Bu_{k_i}^*Ju_k^*)$. Let $\mathcal{H}$ be the Hilbert space where $(\mathcal{B}(L^2M))^*$ is represented on. For $\xi, \eta \in (\mathcal{H}_1)$, we compute

$$\langle \langle \phi_0(\sum_{i=1}^d u_{k_i}a_i e_Bu_{k_i}^*)\xi, \eta \rangle \rangle \leq \sum_{k \geq 0} \sum_{i=1}^d \langle \langle q_{k,\ell}^*t_{\text{nor}}(e_Bu_{k_i}^*Ju_k^*)\xi, t_{\text{nor}}(Ju_kJu_k^*e_Ba_i)^*\eta \rangle \rangle$$

$$= \sum_{k \geq 0} \sum_{i=1}^d \langle \langle t_{\text{nor}}(e_Bu_{k_i}^*Ju_k^*)P_k\xi, t_{\text{nor}}(Ju_kJu_k^*e_Ba_i)^*Q_k\eta \rangle \rangle$$

$$\leq \sum_{k \geq 0} \|t_{\text{nor}}(Ju_kJ(\sum_{i=1}^d u_{k_i}a_i e_Bu_{k_i}^*)Ju_k^*)\| \|P_k\xi\| \|Q_k\eta\|$$

$$\leq \sum_{k \geq 0} \sum_{i=1}^d \|u_{k_i}a_i e_Bu_{k_i}^*\| \|P_k\xi\|^2 \|Q_k\eta\|^2$$

where the last inequality follows from the orthogonality of $\{P_k\}$ and $\{Q_k\}$.

Lastly, notice that $\phi_0$ maps $A_0$ into $q_{k,\ell}^*\tilde{S}(M)$. In fact, for any $s \in \Gamma$, we have

$$t_{\text{nor}}(\rho_s)e_{\text{nor}}(\rho_s) = \sum_{k \geq 0} t_{\text{nor}}(J(\lambda_s u_k)Je_BJ(\lambda_s u_k)^*J) = e,$$

as $\bigsqcup_{k \geq 0} st_k\Lambda_j = \Gamma$, and it follows that $\phi_0(A_0)$ commutes with $t_{\text{nor}}(JM^*J)$. Therefore, we conclude that $\phi_0$ is a norm continuous $*$-homomorphism from $A_0$ to $q_{k,\ell}^*\tilde{S}(M)$ and hence extends to the C*-algebra $A := \mathfrak{T}_0^{\|\| \|$.

**Step 3.** For each $n \geq 0$, set $\phi_n := \phi_0 \circ \psi_n : \langle M, e_B \rangle \rightarrow q_{k,\ell}^*\tilde{S}(M)$, which is c.p. and subunital by construction. We may then pick $\phi \in CB(\langle M, e_B \rangle, q_{k,\ell}^*\tilde{S}(M))$ a weak* limit point of $\{\phi_n\}_n$, which exists as $q_{k,\ell}^*\tilde{S}(M)$ is a von Neumann algebra.

We claim that

$$\text{Ad}(q_{k,\ell}^*) \circ \phi : \langle M, e_B \rangle \rightarrow q_{k,\ell}^*q_{k,\ell}^*\tilde{S}(M)q_{k,\ell}^*$$

is an $M$-bimodular u.c.p. map, which amounts to showing $\phi(x) = q_{k,\ell}^*q_{k,\ell}^*\phi(x)$ for any $x \in M$. 
In fact, for any $x \in M$, we have
\[
\phi(x) = \lim_{n \to \infty} \phi_0 \left( \sum_{0 \leq k, \ell \leq n} \langle u_k E_B(u^*_k x u^*_t) e_B u^*_t \rangle \right)
\]
\[
= q^\perp K_\infty \lim_{n \to \infty} \sum_{0 \leq k, \ell \leq n} t_{nor}(u_k E_B(u^*_k x u^*_t)) e_{t_{nor}}(u^*_t)
\]
\[
= q^\perp K_\infty \lim_{n \to \infty} \sum_{0 \leq k, \ell \leq n} \left( t_{nor}(u_k) e_{t_{nor}}(u^*_k) \right) t_{nor}(x) \left( t_{nor}(u^*_k) e_{t_{nor}}(u^*_t) \right),
\]
where the last equation follows from (1). Finally, note that by Lemma 3.6 \{p_k\}_{k \geq 0} is a family of pairwise orthogonal projections, where
\[
p_k := q^\perp K_\infty t_{nor}(u_k) e_{t_{nor}}(u^*_k) = q^\perp K_\infty \sum_{r \geq 0} t_{nor}(J u_r J u_k e_B u^*_k J u^*_r J),
\]
and $\sum_{k \geq 0} p_k = \sum_{k, r \geq 0} q^\perp K_\infty t_{nor}(J u_r J u_k e_B u^*_k J u^*_r J) = q^\perp K_\infty$ by Lemma 3.5. Therefore, we conclude that $\phi(x) = q^\perp K_\infty e_{t_{nor}}(x)$, as desired. \hfill \Box

4. From relative amenability to rigidity

In this section, we show that for von Neumann algebras arising from Gaussian actions, the associated s-malleable deformations converge uniformly on subalgebras that are amenable relative to the acting group, provided that the orthogonal representations are weakly contained in the left regular.

First we recall the construction of Gaussian actions and the associated s-malleable deformations. See e.g., [KL10] for details on Gaussian actions.

Let $H$ be a real Hilbert space, the Gaussian process gives a tracial abelian von Neumann algebra $A_H$, together with an isometry $S : H \to L^2(A_H)$ so that orthogonal vectors are sent to independent Gaussian random variables, and so that the spectral projections of vectors in the range of $S$ generate $A_H$ as a von Neumann algebra.

In this case, the complexification of the isometry $S$ extends to a unitary operator from the symmetric Fock space $\mathcal{G}(H) = \mathcal{C} \Omega + \bigoplus_{n=1}^{\infty} (H \otimes \mathcal{C})^{\otimes n}$ into $L^2(A_H)$. If $H = H_1 \oplus H_2$, then conjugation by the unitary implementing the canonical isomorphism $\mathcal{G}(H_1 \oplus H_2) \cong \mathcal{G}(H_1) \otimes \mathcal{G}(H_2)$ implements a canonical isomorphism $A_{H_1 \oplus H_2} \cong A_{H_1} \overline{\otimes} A_{H_2}$.

If $V : K \to H$ is an isometry, then we obtain an isometry $V^\mathcal{G} : \mathcal{G}(K) \to \mathcal{G}(H)$ on the level of the symmetric Fock spaces, and conjugation by this isometry gives an embedding of von Neumann algebras $\text{Ad}(V^\mathcal{G}) : A_K \to A_H$. If $V$ were a co-isometry the conjugation by $V^\mathcal{G}$ implements instead a conditional expectation from $A_H$ to $A_K$. In particular, if $U \in \mathcal{O}(H)$ is an orthogonal operator, then we obtain a trace-preserving *-isomorphism $\sigma_U = \text{Ad}(U^\mathcal{G}) \in \text{Aut}(A_H)$. If $\pi : \Gamma \to \mathcal{O}(H)$ is an orthogonal representation, then the Gaussian action associated to $\pi$, denoted by $\sigma_\pi$, is given by $\Gamma \ni t \mapsto \sigma_{\pi(t)} \in \text{Aut}(A_H)$. When $\pi$ is the left regular representation, the Gaussian action coincides with the Bernoulli action with diffuse base.

Now let $\pi : \Gamma \to \mathcal{O}(H)$ be a fixed orthogonal representation of a countable group $\Gamma$ and $A_H \simeq A_H$ the associated Gaussian action. We recall the construction of the s-malleable deformation from [PS12].

Consider orthogonal matrices
\[
V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{O}(H \oplus H) \quad \text{and} \quad U_t = \begin{pmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{pmatrix} \in \mathcal{O}(H \oplus H),
\]
for $t \in \mathbb{R}$. Let $\alpha_t = \sigma U_t$ and $\beta = \sigma V$ be the associated automorphisms of $A_{\Pi} \otimes A_{\Pi} \cong A_{\Pi} \otimes A_{\Pi}$, and both extend to $\text{Aut}((A_{\Pi} \otimes A_{\Pi}) \rtimes_{\sigma_\tau \otimes \sigma_\tau} \Gamma)$, still denoted by $\alpha_t$ and $\beta$, as $V$ and $U_t$ commute with $(\pi \otimes \pi)(\Gamma)$ and $\sigma_{\pi \otimes \pi} = \sigma_\pi \otimes \sigma_\pi$. And $\alpha_t$ and $\beta$ form a s-malleable deformation in the sense of Popa [Pop06a] for $M := A_{\Pi} \rtimes_{\sigma_\tau} \Gamma$ inside $\tilde{M} := (A_{\Pi} \otimes A_{\Pi}) \rtimes_{\sigma_\tau \otimes \sigma_\tau} \Gamma$.

The following is an abstraction of [Io15] Corollary 2.12. Nevertheless we include the proof for completeness.

**Lemma 4.1.** Let $(M, \tau)$ be a tracial von Neumann algebra, $B \subset M$ and $N \subset pMp$ von Neumann subalgebras with $p \in \mathcal{P}(M)$. Suppose there exist another tracial von Neumann algebra $(\tilde{M}, \tilde{\tau})$ such that $M \subset \tilde{M}$ and $\tilde{\tau} | M = \tau$, and a net of trace preserving automorphisms $\{\theta_t\} \subset \text{Aut}(M)$ such that $\theta_t | B \subset \text{Aut}(B)$, and such that $\theta_t | M \rightarrow \text{id}_M$ in the point-$\| \cdot \|$ topology, as $t \rightarrow 0$. If $N$ is amenable relative to $B$ inside $M$, the for any $0 < \delta \leq 1$, one of the following is true.

1. There exists $t_\delta > 0$ such that $\inf_{u \in H(N)} \| E_M(\theta_{t_\delta}(u)) \|_2 > (1-\delta) \| p \|_2$.
2. There exists a net $\{\eta_k\} \subset K^1$, where $K$ is the closure of $M \otimes M$ inside $L^2((M, e_M))$, such that $\| x\eta_k - \eta_k x \|_2 \rightarrow 0$ for all $x \in N$, $\limsup_k \| \eta_k \|_2 \leq 2 \| y \|_2$ for all $y \in pMp$ and $\limsup_k \| p\eta_k \|_2 > 0$.

**Proof.** Since $N$ is amenable relative to $B$, there exists a net $\{\xi_n\} \subset L^2(p(M, e_B))$ such that $\| x\xi_n - \xi_n x \|_2 \rightarrow 0$ for all $x \in N$ and $(y\xi_n, \xi_n) \rightarrow \tau(y)$ for all $y \in pMp$ by [OP10a]. We may extend $\alpha_t$ to an automorphism on $(M, e_B)$ as $\alpha_t$ leaves $B$ globally fixed. Denote by $e$ the orthogonal projection from $L^2((\tilde{M}, e_{\tilde{M}}))$ to $K$.

**Claim.** For any $x \in N$, $y \in \tilde{M}$, $t \in \mathbb{R}$, we have

1. $\lim_n \| y\alpha_t(\xi_n) \|_2^2 = \tau(y^* y \alpha_t(\xi_n)) \leq \| y \|_2^2$ and $\lim_n \| \alpha_t(\xi_n)y \|_2^2 = \tau(y^* \alpha_t(p)) \leq \| y \|_2^2$.
2. $\limsup_n \| y(e\alpha_t(\xi_n)) \|_2 \leq \| y \|_2$.
3. $\limsup_n \| x\alpha_t(\xi_n) - \alpha_t(\xi_n)x \|_2 \leq 2 \| \alpha_t(x) - x \|_2$.

**Proof of the claim.** (1) Note that since $\xi_n \in pK$, we have

$\| y\alpha_t(\xi_n) \|_2^2 = \langle \alpha_t^{-1}(y^* y)\xi_n, \xi_n \rangle = \langle E_M(\alpha_t^{-1}(y^* y))p\xi_n, p\xi_n \rangle = \tau(pE_M(\alpha_t^{-1}(y^* y))p) = \tau(y^* y \alpha_t(p))$, and the second one follows similarly.

(2) Observe that $(\tilde{M} \ominus M)K \perp K$, and hence

$\| y(e\alpha_t(\xi_n)) \|_2^2 = \langle y^* y e\alpha_t(\xi_n), e\alpha_t(\xi_n) \rangle = \langle E_M(y^* y)e\alpha_t(\xi_n), e\alpha_t(\xi_n) \rangle$,

and $\| E_M(y^* y)^{1/2}\alpha_t(\xi_n) \|_2 \leq \| E_M(y^* y)^{1/2}\alpha_t(\xi_n) \|_2 = \| y \|_2$ by (1).

(3) Compute

$\| x\alpha_t(\xi_n) - \alpha_t(\xi_n)x \|_2 \leq \| (x - \alpha_t(x))\alpha_t(\xi_n) \|_2 + \| \alpha_t(\xi_n)(x - \alpha_t(x)) \|_2 + \| x\xi_n - \xi_n x \|_2$.

For each pair of $(t, n)$ with $t > 0$ and $n \in \mathbb{N}$, let $\eta_{t,n} = \alpha_t(\xi_n) - e\alpha_t(\xi_n)$. Fix a $0 < \delta \leq 1$ and consider the following two cases.

**Case 1.** There exists $t > 0$ such that $\limsup_n \| \eta_{t,n} \|_2 \leq \delta \| p \|_2/2$.

**Case 2.** For all $t > 0$, $\limsup_n \| \eta_{t,n} \|_2 \geq \delta \| p \|_2/2$. 

In Case 1, fix \( x \in \mathcal{U}(N) \) and compute
\[
\|E_M(\alpha_t(x))\alpha_t(\xi_n)\|_2 \geq \|eE_M(\alpha_t(x))\alpha_t(\xi_n)\|_2 = \|e\alpha_t(x)e\alpha_t(\xi_n)\|_2 \geq \|e\alpha_t(x)e\alpha_t(\xi_n)\|_2 - \|\eta_{t,n}\|_2 \\
\geq \|e(\alpha_t(\xi_n)\alpha_t(x))\|_2 - \|x\xi_n - \xi_n x\|_2 - \|\eta_{t,n}\|_2,
\]
and
\[
\|e(\alpha_t(\xi_n)\alpha_t(x))\|_2 = \|e(\alpha_t(\xi_n)\alpha_t(x))\|_2 \geq \|\alpha_t(\xi_n)\alpha_t(x)\|_2 - \|\eta_{t,n}\|_2 = \|\xi_n x\|_2 - \|\eta_{t,n}\|_2.
\]
Altogether, we conclude that for any \( x \in \mathcal{U}(N) \),
\[
\|E_M(\alpha_t(x))\|_2 \geq \lim_n \|E_M(\alpha_t(x))\alpha_t(\xi_n)\|_2 \geq \lim_n \|\xi_n x\|_2 - \|\eta_{t,n}\|_2 = \|\xi_n x\|_2 - \|\eta_{t,n}\|_2.
\]

In Case 2, let \( k = (X, Y, \varepsilon) \) be a triple such that \( X \subset N \), \( Y \subset pMp \) are finite subsets and \( \varepsilon > 0 \). Then we may find \( 0 < t_k \leq 1 \) such that \( \|x - \alpha_{t_k}(x)\|_2 < \varepsilon/2 \) for all \( x \in X \) and \( \|\alpha_{t_k}(p) - p\|_2 < (1 - \sqrt{1 - \delta^2/4})\|p\|_2/2 \). Observe that for any \( x \in X \)
\[
\|x\eta_{k,n} - \eta_{k,n} x\|_2 = \|(1 - \varepsilon)(x\alpha_{t_k}(\xi_n) - \alpha_{t_k}(\xi_n)x)\|_2 \leq \|x\alpha_{t_k}(\xi_n) - \alpha_{t_k}(\xi_n)x\|_2,
\]
and by (3)
\[
\limsup_n \|x\alpha_{t_k}(\xi_n) - \alpha_{t_k}(\xi_n)x\|_2 \leq \|x - \alpha_{t_k}(x)\|_2 < \varepsilon.
\]
For \( y \in Y \), by (1) and (2) we have
\[
\|y\eta_{k,n}\|_2 \leq \|y\alpha_{t_k}(\xi_n)\|_2 + \|ye\alpha_{t_k}(\xi_n)\|_2 \leq 2\|y\|_2.
\]
Furthermore, from (1) we also have
\[
\limsup_n \|p\eta_{k,n}\| \geq \limsup_n (\|p\alpha_{t_k}(\xi_n)\|_2 - \|e\alpha_{t_k}(\xi_n)\|_2) = \|p\alpha_{t_k}(p)\|_2 - \liminf_n \|e\alpha_{t_k}(\xi_n)\|_2,
\]
and
\[
\liminf_n \|e\alpha_{t_k}(\xi_n)\|_2^2 = \liminf_n (\|\alpha_{t_k}(\xi_n)\|_2^2 - \|\eta_{t,n}\|_2^2) = \lim \|\xi_n\|_2^2 - \limsup \|\eta_{t,n}\|_2^2 \leq (1 - \delta^2/4)\|p\|_2^2.
\]
It follows that
\[
\limsup_n \|p\eta_{k,n}\| \geq \|p\alpha_{t_k}(p)\|_2 - \sqrt{1 - \delta^2/4}\|p\|_2 \\
\geq \|p\|_2 - \|p - \alpha_{t_k}(p)\|_2 - \sqrt{1 - \delta^2/4}\|p\|_2 \\
> (1 - \sqrt{1 - \delta^2/4})\|p\|_2/2.
\]
Altogether, we may find some \( n \in I \) such that by putting \( \eta_k = \eta_{t_k,n} \) we have
\[
(1) \quad \|x\eta_k - \eta_k x\|_2 \leq \varepsilon \text{ for all } x \in X,
\]
\[
(2) \quad \|y\eta_k\|_2 \leq 2\|y\|_2 + \varepsilon \text{ for all } y \in Y,
\]
\[
(3) \quad \|p\eta_k\| \geq (1 - \sqrt{1 - \delta^2/4})\|p\|_2/2.
\]
\]

\[ \square \]

**Proposition 4.2.** Let \( \Gamma \) be a nonamenable group and \( \pi : \Gamma \to \mathcal{O}(\mathcal{H}) \) be an orthogonal representation such that \( \pi \prec \lambda \). Denote by \( \Gamma \curvearrowright \sigma_p A_N \) the associated Gaussian action and \( M = A_N \rtimes_{\sigma_p} \Gamma \). Suppose \( N \subset pMp \) is a von Neumann subalgebra, for some \( p \in \mathcal{P}(M) \), with no amenable direct summand, such that \( N \) is amenable relative to \( L\Gamma \) inside \( M \). Then we have \( \alpha_t \to \text{id}_N \) uniformly on the unit ball of \( N \) as \( t \to 0 \), where \( \alpha_t \) is the \( s \)-malleable deformation described above.
Proof. Let $\tilde{M} = (A_{\mathcal{H}} \otimes A_H) \rtimes_{\sigma_\oplus \sigma_\otimes} \Gamma$ and $\alpha_\beta \in \text{Aut}(\tilde{M})$ be as above. Suppose there exists some $0 < \delta \leq 1$ such that case (1) of Lemma 4.1 does not hold. Then we have that there exists $\{\eta_k\} \in \mathcal{K}$ as in the second case of Lemma 4.1. Note that the $M$-$M$ bimodule $L^2((\tilde{M}, e_{LR}) \otimes \mathcal{K}$ is isomorphic to $L^2(\tilde{M} \otimes M) \otimes_{LR} L^2 \tilde{M}$. It is shown in [Bon12, Lemma 3.3] that $L^2(\tilde{M} \otimes M)$ is weakly contained in the coarse $M$-$M$ bimodule as $\pi \prec \lambda$, and hence we have

$$L^2((\tilde{M}, e_{LR}) \otimes \mathcal{K} \prec L^2 M \otimes (L^2 M \otimes_{LR} L^2 \tilde{M}) \prec L^2 M \otimes L^2 M,$$

as $M$-$M$ bimodules. It follows that there exists a u.c.p. map

$$\phi : \mathbb{B}(L^2 M) \to \mathbb{B}(L^2((\tilde{M}, e_{LR}) \otimes \mathcal{K}) \cap (M^{op})',$$

such that $\phi|_M = \text{id}_M$. Therefore, we obtain a state $\varphi$ on $\mathbb{B}(L^2 M)$ given by

$$\varphi(\cdot) = \lim_k \|p_{\eta_k}\|_2^2 \langle \phi(\cdot)p_{\eta_k}, p_{\eta_k}\rangle,$$

which is $\mathcal{N}$-central and restricts to a normal state on $pMp$. This contradicts the assumption that $\mathcal{N}$ has no amenable direct summands.

Therefore, we have that $\lim_{t \to 0} \inf_{u \in \mathcal{U}(\mathcal{N})} \|E_M(\alpha_t(u))\|_2 = \|p\|_2$. It follows that $\sup_{u \in \mathcal{U}(\mathcal{N})} \|\alpha_t(u) - E_M(\alpha_t(u))\|_2 \to 0$ as $t \to 0$ and hence $\alpha_t \to \text{id}$ uniformly on $(\mathcal{N})_1$ by Popa’s transversality inequality [Pop08, Lemma 2.1].

**Corollary 4.3.** Let $M$, $p \in \mathcal{P}(M)$ and $N \subset pMp$ be as in Proposition 4.2. Denote $Q = \mathcal{N}pMp(N)'$. If $\pi$ is mixing, then $Q \prec_M \Lambda$. Moreover, if $\Gamma$ is an i.c.c. group, then there exists $u \in \mathcal{U}(M)$ such that $u^\ast Q u \subset \Lambda$. 

**Proof.** Since $A_{\mathcal{H}}$ is abelian, $N$ is diffuse and $Q$ is type II$_1$, the assertion $Q \prec_M \Lambda$ follows directly from [Bon12 Theorem 3.4] and Proposition 4.2. The proof for the moreover part is contained in [Bon13 Proposition 2.3].

5. **Proofs of main theorems**

Now we are ready to prove Theorem 1.5 and its corollaries.

**Proof.** First we may realize $M$ as $L(\mathbb{Z} \wr \Gamma)$ and note that $\mathbb{Z} \wr \Gamma$ is biexact relative to $\Gamma$ [BO08 Corollary 15.3.9]. By Proposition 2.3 we have that $N \subset M$ is properly proximal relative to $\mathbb{Z} \wr \Gamma$ as $\mathcal{N}$ has no amenable direct summand, where $\mathbb{Z} \wr \Gamma$ is the $M$-boundary piece associated with $\Lambda$. Moreover, since $\Gamma < \mathbb{Z} \wr \Gamma$ is almost malnormal and $\mathcal{N}$ has no proper proximal direct summand, we have $\mathcal{N}$ is amenable relative to $\Lambda$ inside $M$ by Proposition 3.1. The rest follows from Proposition 4.2 by setting $\pi = \lambda$. 

**Proof of Theorem 4.3.** Let $\sigma : \Gamma \curvearrowright X$ be the Bernoulli action, $M = L^\infty(X) \rtimes_\sigma \Gamma$. Set $\tilde{M} = (L^\infty(X) \otimes L^\infty(X)) \rtimes_\sigma \Gamma$, where $\tilde{\sigma} = \sigma \otimes \sigma$. If we denote by $\sigma_t \in \mathcal{U}(L^2(X))$ for each $t \in \Gamma$ the unitary that implements the action $\sigma$, then we have $M \subset \tilde{M}$ is generated by canonical unitaries $\{u_t = \delta_t \otimes \lambda_t \mid t \in \Gamma\}$ and $L^\infty(X) \otimes \mathbb{C}$, where $\delta_t = \sigma_t \otimes \sigma_t$.

Let $\Gamma_0 < \Gamma$ be a nonamenable wq-normal subgroup that is not properly proximal. If $\omega : \Gamma \times X \to T$ is a 1-cocycle associated with $\sigma$, then for each $t \in \Gamma$, we may consider $\omega_t \in \mathcal{U}(L^\infty(X))$ given by $\omega_t(x) = \omega(t, t^{-1}x)$ and $\tilde{L}(\Gamma_0) := \{u_t := \omega_t u_t \mid t \in \Gamma_0\}' \subset M$, which is a von Neumann subalgebra isomorphic to $L(\Gamma_0)$.

Since $\tilde{L}(\Gamma_0) \cong L(\Gamma_0)$ has no amenable and no properly proximal direct summand by Remark 2.2, it follows from Theorem 4.3 that $\alpha_t$ converges to identity uniformly on the unit ball of $L(\Gamma_0)$. The result follows from [Pop07a].
Proof of Theorem 1.4. This is an immediate result of Theorem 1.2 and [PST12, Theorem 1.1]. □

Proof of Theorem 1.3. Since Λ is exact, we have $L^\infty(Y) \rtimes \Gamma$ is an exact C*-algebra (e.g. [BO08, Theorem 10.2.9]) and it follows that $(L^\infty(X) \rtimes \Gamma)'' \cong L^\infty(Y) \rtimes \Lambda$ is a weakly exact von Neumann algebra [Kir95]. Since weak exactness is stable under amplifications and passes to von Neumann subalgebras (with normal conditional expectations) [BO08, Corollary 14.1.5], we have $L\Gamma$ is weakly exact, which implies $\Gamma$ is exact [Oza07].

Let $M = L^\infty(X) \rtimes \Gamma$, $N = L^\infty(Y) \rtimes \Lambda$ and $\Lambda_0 \subset \Lambda$ be the nonamenable normal subgroup that is not properly proximal. Since $N \cong M^1$, we may denote by $\theta : N^{1/t} \to M$ a $*$-isomorphism, and identify $N^{1/t}$ with $pM_n(N)p$, where $n = [1/t]$, $p = \text{diag}(1, \ldots, 1, p_0) \in M_n(N)$ and $p_0 \in L(\Lambda_0)$ with $\tau_N(p_0) = 1/t - [1/t]$.

Note that by Remark 2.2.2 $\theta(pM_n(L(\Lambda_0))) \subset M$ satisfies the assumption of Theorem 1.3 and hence by Corollary 1.3 we may find some $u \in U(M)$ such that $\alpha(pM_n(LA)) \subset L\Gamma$, where $\alpha := \text{Ad}(u) \circ \theta$. Set $e = \text{diag}(1, 0, \ldots, 0) \in M_n(LA)$ and we have $\alpha(LA) = \alpha(eM_n(LA)e) \subset qL\Gamma q$, where $q = \alpha(e) \in L\Gamma$ and $\tau_M(q) = \tau_{N^{1/t}}(e) = t$.

It then follows from Popa’s conjugacy criterion for Bernoulli actions [Pop06] Theorem 0.7 [see also [Io11, Theorem 6.3]] that $t = 1$ and there exist a unitary $v \in M$, a character $\eta \in \Lambda$ and a group isomorphism $\delta : \Lambda \to \Gamma$ such that $\alpha(L^\infty(Y)) = vL^\infty(X)\eta^*$ and $\alpha(\lambda_t) = \eta(t)v\lambda_{\delta(t)}v^*$ for any $t \in \Lambda$. □

Proof of Theorem 1.4. A direct consequence of Theorem 1.3 [IPR19] and [Oza07]. □
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