Isometry groups and geodesic foliations of Lorentz manifolds.
Part I: Foundations of Lorentz dynamics

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Abstract

This is the first part of a series on non-compact groups acting isometrically on compact Lorentz manifolds. This subject was recently investigated by many authors. In the present part we investigate the dynamics of affine, and especially Lorentz transformations. In particular we show how this is related to geodesic foliations. The existence of geodesic foliations was (very succinctly) mentioned for the first time by D’Ambra and Gromov, who suggested that this may help in the classification of compact Lorentz manifolds with non-compact isometry groups. In the Part II of the series, a partial classification of compact Lorentz manifolds with non-compact isometry group will be achieved with the aid of geometrical tools along with the dynamical ones presented here.

1 Lorentz dynamics

We are interested here in the following question: when is the isometry group of a compact Lorentz manifold non-compact? Many progress has been made towards answering it, see for instance the works by, Zimmer [22], Gromov ([6], [7]), D’Ambra [8], Kowalsky [10], Adams-Stuck ([1], [2]), and the author ([16], [17]). At this stage, from the different investigations due to these authors, we know the list of non-compact connected Lie groups acting isometrically on compact Lorentz manifolds. For some groups in the list, we also understand completely the geometric structure of the underlying Lorentz manifold. However, we do not know enough things about this structure, in the case of the remaining groups, such as, for example, the case of abelian groups. In the case of non-connected groups, no significant thing is known on neither their algebraic structure, nor the geometric structure of the Lorentz manifold on which they act.

Our approach here, is to study the dynamics of lorentz transformations, i.e. diffeomorphisms preserving Lorentz metrics. By this approach, we intend to understand, at the same time, the structure of Lorentz manifolds with large isometry groups, and the dynamics of the individual isometries themselves.

It is natural to think about Lorentz metrics as the nearest to Riemannian metrics, among all the geometric structures. This is why, among geometric dynamics, one might claim that the Lorentz dynamics is the simplest one after the Riemannian dynamics. It is worth saying that this latter dynamics is very poor. More precisely, let \( f \) be an isometry of a compact Riemannian manifold \( M \). Then the closure of an orbit of \( f \) is diffeomorphic to a torus, on which \( f \) induces a linear translation. One can then say that there is only one pattern of Riemannian dynamics: a linear translation on a torus. On the other hand, one can (roughly) characterize Riemannian isometries, by topological properties, such as equicontinuity.

We know many patterns of Lorentz dynamics. For instance, on the unit tangent bundle of a hyperbolic compact surface, co-exist, the circle, the horocycle and the geodesic flows. All of
them preserve the Killing Lorentz metric, although they present antagonistic dynamics. Now the question is: find (fundamental) patterns with the help of which, one can build up any Lorentz isometry. Also, what are the topological properties characterizing diffeomorphisms that preserve Lorentz metrics?

Let’s firstly observe that the Liapunov theory is not efficient for our purpose. Indeed, this theory is only sensitive to hyperbolicity, since it deals with the asymptotic exponential behavior of vectors under the derivative of a dynamical system. For instance, this theory does not distinguish between the circle and horocycle flows!

Here we will introduce another growth notion, which appears to be efficient in studying some dynamical systems with geometric properties (at least permitting to detect a difference between the horocycle and circle flows!).

Essentially, our fundamental growth notion is the following: Consider a diffeomorphism $f$ of a compact manifold $M$. We say that a vector $v \in TM$ is **$f$-approximately stable**, if $v$ is a limit of a sequence $(v_n)$ (in $TM$) such that the image sequence $(Df^n v_n)$ is bounded (i.e. lies in a compact subset of $TM$). We denote the set of such vectors by $AS(f) \subset TM$ (observe that $AS(x, f) = AS(f) \cap T_x M$ is not a priori a vector subspace of $T_x M$).

The weakness of this notion with regard to the Liapunov theory is clear, the set $AS(f)$ may behave very badly. Its efficiency appears in Lorentz dynamics. For a circle flow, the situation is trivial: $AS = TM$. For the horocycle flow, $AS$ is the codimension 1 weakly stable or unstable bundle of the geodesic flow, that contains the given horocycle flow (so $AS$ has dimension 2 and not 1). For the geodesic flow, $AS$ is exactly its weak stable bundle.

The following theorem is one of our main results. It claims that the codimension 1 property of $AS$ is a generic situation in Lorentz dynamics. It seems also, that this same property, gives a (rough) topological characterization of Lorentz isometries (however, we won’t develop this point of view here).

**Theorem 1.1** Let $M$ be a compact Lorentz manifold (i.e. $M$ is endowed with a non degenerate symmetric 2-tensor of signature $- + . . +$). Let $f$ be an isometry of $M$ generating a non-equicontinuous subgroup of isometries of $M$ (that is $\{f^n/n \in \mathbb{Z}\}$ is not pre-compact in $Isom(M)$). Then the following statements hold:

**Existence of approximately stable foliations.** The approximately stable set $AS(f)$ is a codimension one bundle tangent to a codimension 1 Lipschitz foliation $AS(f)$, called the approximately stable foliation of $f$. The leaves of this foliation are geodesic and lightlike (that is the restriction of the Lorentz metric to them is degenerate). The isometry $f$ preserves both its approximately stable foliation, and that of $f^{-1}$, which will be called the approximately unstable foliation of $f$.

**Dynamics of $f$.** In the subset of points where $AS(f)$ and $AS(f^{-1})$ are transverse, their (one dimensional) normal spaces $AS(f)^{\perp}$ and $AS(f^{-1})^{\perp}$ are respectively the negative and positive Liapunov spaces of $f$.

The dynamics of $Df$ on the projective tangent bundle $P(TM)$ has the following “north-south” description: Let $U$ and $V$ be respectively neighborhoods of the associated projective bundles $P(AS(f))$ and $P(AS(f^{-1}))$, then $Df^n(P(TM) - U) \subset V$, for $n$ sufficiently large.

**Ergodic properties.** Let $O$ be an open $f$-invariant subset and $\sigma$ a continuous $f$-invariant function defined in $O$. Then $\sigma$ is constant along the leaves of the restriction to $O$ of the two 1 dimensional foliations tangent to $AS(f)^{\perp}$ and $AS(f^{-1})^{\perp}$.

Finally, the topological entropy of $f$ vanishes exactly when the approximately stable and unstable foliations of $f$ are identical.

It is a nuisance that we were not able to exclude, a priori, the approximately stable and unstable foliations of an isometry, may be different but coincide somewhere!
Generalized dynamical systems. The following terminology will be useful. Consider a sequence $(f_n)$ of diffeomorphisms of a compact manifold $M$. To emphasize the fact that we are treating such a sequence in a dynamical viewpoint, we shall call it a **generalized dynamical system**. For instance we say that a vector $v \in TM$ is $(f_n)$-**approximately stable**, if $v$ is a limit of a sequence $(v_n)$ in $TM$ such that the image sequence $(Df_nv_n)$ is bounded. We denote the set of such vectors by $AS((f_n)) \subset TM$.

So, classical dynamical systems correspond to the case: $f_n = f^n$, where $f$ is a diffeomorphism. But, even the less classical case $f_n = f^{k_n}$, for a sequence $(k_n)$ of integers (for example the return times to some subset) is very interesting. Our philosophy here is that, sometimes, one doesn’t need the rich structure of classical dynamical systems, especially the various associated cocycles. Conversely, sometimes (in fact in many situations in dynamical systems), one can’t avoid use (implicitly) of generalized dynamical systems. For instance, in our situation, it might happen that the theorem above is dramatically empty, because every isometry $f$ of $M$ generates a pre-compact subgroup, although the group $\text{Isom}(M)$ itself is non-compact (for example, $\text{Isom}(M)$ might be an infinite discrete torsion group). In fact the proof of the theorem above, will come via the following generalized version.

**Theorem 1.2** Let $M$ be a compact Lorentz manifold and $(f_n)$ a non-equicontinuous sequence of isometries of $M$ (that is $(f_n)$ is not contained in a compact subset of $\text{Isom}(M)$).

Then there is a subsequence $(\phi_n)$ such that the approximately stable set $AS((\phi_n))$ is a codimension one bundle tangent to a codimension 1 Lipschitz foliation $AS((\phi_n))$, called the approximately stable foliation of $(\phi_n)$. The leaves of this foliation are geodesic and lightlike (that is the restriction of the Lorentz metric to the leaves is degenerate).

After passing to a subsequence, we can assume that the same is true for $(\phi_n^{-1})$. In this case, if $v \in TM - AS((\phi_n))$, $D\phi_nv$ tends to $\infty$, and converges projectively (i.e. after normalization) to $AS(\phi_n^{-1})$. The convergence is uniform in compact subsets of $TM - AS((\phi_n))$. That is, if $U$ and $V$ are neighborhoods of the associated projective bundles $P(AS((\phi_n)))$ and $P(AS((\phi_n^{-1})))$, respectively, then $D\phi_n(P(TM) - U) \subset V$, for $n$ sufficiently large.

In addition, we have the following property. Let $\sigma : O \to \mathbb{R}$ be a continuous function defined in an open subset $O$. Suppose that $\sigma$ and $O$ are invariant by each $f_n$. Then $\sigma$ is constant along the leaves of the restriction to $O$ of the two 1 dimensional foliations tangent to $AS((\phi_n))$ and $AS((\phi_n^{-1}))$.

More details and a non compact variant of this theorem are in §8.

2 Results on isometry groups

2.1 Compactification

The space of codimension 1 geodesic foliations of a compact affine manifold $M$ (i.e. $M$ is endowed with a torsion free connection) is naturally compact (this is because such a foliation is uniformly Lipschitz, see §3). The same is true for the space of codimension 1 geodesic lightlike foliations if $M$ is Lorentzian. We denote this space by $FG$. We construct a topology on $\text{Isom}M \cup FG$, essentially by the following rule: a sequence $(\phi_n)$ in $\text{Isom}M$ converges to $F \in FG$ if $F$ is the approximately stable foliation of $(\phi_n)$ (that is if $AS((\phi_n)) = TF$). One then shows that $\text{Isom}M \cup FG$ is a (metrizable) compact space, and that the action of $\text{Isom}M$ on it is continuous. Therefore, by taking its closure we get a natural compactification of $\text{Isom}M$. The same construction yields a compactification and a boundary $\partial_{\infty}\Gamma$ for any closed subgroup of $\text{Isom}M$. 
As example, in the case of the unit tangent bundle \( PSL(2, \mathbb{R})/\Gamma \) of a hyperbolic surface \( \mathbb{H}^2/\Gamma \), the isometry group is \( PSL(2, \mathbb{R}) \). Its boundary is the circle, endowed with the usual projective action, see §15 for further examples.

### 2.2 Elementary groups

The “north-south” property described in the two preceding theorems tends to suggest that the action of Isom\( M \) on its boundary is of convergence type, as in the case of Fuschian groups \([15]\). This analogy between Lorentz isometry groups and Fuschian groups may arise in dynamical as well as in geometrical contexts. In §9 we will see how a Lorentz geometry gives rise to a fiberwise hyperbolic geometry, i.e. a family of hyperbolic spaces... This suggests that one could translate and update notions on Fuschian groups to Lorentz isometry groups. Let’s formulate the simplest one.

**Definition 2.1** A compact Lorentz manifold \( M \) is called polarized if the boundary of its isometry group is one point.

As example, an important geometrical class of polarized Lorentz manifolds (that we won’t consider here) consists of manifolds for which the holonomy group is reducible, but non-decomposable (this can’t happen in Riemannian geometry). In this case the polarizing foliation is parallel, i.e. its tangent bundle is invariant by parallel transport. (in this case the term “polarization” may be justified by physical, symplectical or optical considerations).

The Fuschian notion which corresponds to polarization is that of elementary parabolic groups. Now, elementary hyperbolic Fuschian groups are those with limit sets of cardinality 2. In the Lorentz case, it is not obvious (at least at this stage) how to check that parabolic and hyperbolic behaviors don’t co-exist. So, we were able to define elementary groups only via the following somewhat technical notion.

**Definition 2.2** A codimension 1 lightlike geodesic bi-foliation is a map \( x \in M \rightarrow L(x) = \{T_xF_1, T_xF_2\} \). We will denote such a bi-foliation by \( F^1 \cup F^2 \).

For instance, a pair of two (perhaps identical) foliations gives a bi-foliation. Notice, however, that in our definition, the foliations \( F^1 \) and \( F^2 \) are not part of the data. Any foliation \( F \) such that \( T_xF \in L(x) \), for any \( x \in M \), is called tangent to the given bi-foliation. The nuisance here is that the coincidence locus \( C \) where the bi-foliation is one-valued, i.e. where the two defining foliations are tangent, may be non-empty, or even worse, \( M - C \) may have infinitely many connected components (because one hasn’t assumed things are analytic). In this case the space of tangent foliations (to the given bi-foliation) is an infinite compact space! Here, by analogy, not with Fuschian groups, but with higher rank groups, one may call this space, the apartment generated by the bi-foliation, and denote it by \( Apa(F^1, F^2) \) (in despite of the reference in notation to the foliations \( F^1 \) and \( F^2 \), the apartment is clearly related to the bi-foliation, only).

**Definition 2.3** We say that a closed non compact subgroup of Isom\( M \) is elementary if it preserves a codimension 1 lightlike geodesic bi-foliation. We say that \( M \) is bi-polarized if its isometry group is elementary.

The following is a characterization of elementary groups.
**Theorem 2.4** Let \( \Gamma \) be a closed non-compact subgroup of \( \text{Isom} M \). Then \( \Gamma \) is elementary if and only if its boundary has cardinality 1 or 2.

In the case \( \partial_\infty \Gamma = 1 \) point, there is no other 1-codimension geodesic lightlike foliation, preserved by \( \Gamma \). All the elements of \( \Gamma \) have vanishing topological entropy.

In the case \( \partial_\infty \Gamma = 2 \) points, up to a finite index subgroup, \( \Gamma \) is a direct product of a compact group by \( \mathbb{Z} \) or \( \mathbb{R} \). Any element of the \( \mathbb{Z} \) or the \( \mathbb{R} \) factor has positive entropy.

**Remark 2.5** In the case where the boundary of \( \Gamma \) is is one point, we obtain nothing but the Alexandroff compactification. In fact the group \( \Gamma \) may be very large in this case. For instance, there are many compact homogeneous polarized Lorentz manifolds. Their isometry groups contain solvable groups obtained as semi-direct products of the circle \( S^1 \) with Heisenberg groups (see [1] and [16]).

### 2.3 Amenable groups. Dynamical structure of one parameter groups

We also have the following convergence aspect of the action of \( \text{Isom} M \) on its boundary.

**Theorem 2.6** A closed non-compact amenable subgroup of \( \text{Isom} M \) is elementary.

Theorem 1.1 concerning groups isomorphic to \( \mathbb{Z} \) will be a consequence of the two above theorems. Similarly, the following result describes the nice dynamical structure of non-equicontinuous isometric flows.

**Theorem 2.7** Let \( X \) be a Killing field on a compact Lorentz manifold \( M \), generating a non equicontinuous flow \( f^t \).

Then \( X \) is everywhere isotropic or spacelike, that is \( <X(x),X(x)> \geq 0 \). Furthermore, \( X \) is contained in (and hence \( f_t \) preserves) two codimension 1 geodesic lightlike foliations, called the approximately stable and approximately unstable foliations of \( f_t \).

These two foliations coincide on the (invariant) set of points where \( X \) is isotropic (but the coincidence may occur in a larger set). (In particular, if \( X \) is everywhere isotropic, then \( X^\perp \) is integrable. This is false for equicontinuous isotropic Killing fields).

**Remark 2.8** It may happen that the approximately stable and unstable foliations are identical, as in the case of the horocycle flow. They might also, a priori, coincide in some proper subset of \( M \).

### 2.4 Ergodic properties of non bi-polarized manifolds

Let \( M \) be a compact non bi-polarized manifold, that is by definition, it has a non compact and non elementary isometry group. Let \( S \) be the set of isotropic vector fields \( X \), which are tangent to foliations \( \mathcal{F} \) belonging to the boundary of \( \text{Isom}(M) \) (note that because \( X \) is isotropic, then \( X \) tangent to \( \mathcal{F} \) is equivalent to \( X \) normal to \( \mathcal{F} \)). We denote \( S_x = \{X(x)/X \in S\} \) the evaluation of \( S \) at \( x \), and \( TAS_x \subset T_x M \) the linear space that it generates. The “bundle” \( TAS \), obtained in this way will be called the total approximately stable bundle. In a straightforward way, one sees that somewhere the dimension of \( TAS \) is \( \geq 3 \), because \( \text{Isom}(M) \) is not elementary. In the open subset where its dimension is locally constant, \( TAS \) is a Lipschitz bundle. The same is true for its orthogonal \( TAS^\perp \). Observe that this last bundle is integrable. Indeed the geodesic foliation obtained as the intersection of all the foliations belonging to \( \partial_\infty \text{Isom}(M) \) is a natural candidate to be tangent to \( TAS^\perp \). The starting point of Part II of this work, will be the claim that \( TAS \) is
also integrable, in fact with umbilical leaves, which yields a local warped product structure for \( M \). Here we won’t deal with these geometric details but instead with the following ergodic property behind their proofs.

**Theorem 2.9** Let \( E \) be a measurable linear subbundle of \( TM \), containing \( TAS \) and invariant by \( \text{Isom}(M) \). Let \( t : E \times E \to TM \) be a measurable invariant bilinear bundle map. Let \( \mathcal{U} \) be the open set \( \{ x \in M/ \dim TAS_x \geq 3 \} \). Then there is a vector field \( n \) defined almost everywhere in \( \mathcal{U} \), such that \( t(X,Y) = \langle X,Y \rangle n \), when \( X \in TAS \) and \( Y \in E \). In addition, we have the following ergodic property. Let \( \sigma : O \to \mathbb{R} \) be a \( C^1 \) function defined in an open set \( O \), and invariant by \( \text{Isom}(M) \). Then \( \sigma \) is constant along \( TAS \), that is \( TAS \) is contained in \( \ker (d\sigma) \).

### 2.5 Approximate isometries

Questions on isometry groups of (fixed) Lorentz metrics on a manifold \( M \), may be translated in a more stable way to questions on the dynamics of the action of the group of diffeomorphisms \( \text{Diff}(M) \) on the space of Lorentz metrics \( \text{Lor}(M) \). So, the compactness of \( \text{Isom}(M,g) \) for any Lorentz metric \( g \) on \( M \), means that the stabilizers of the \( \text{Diff}(M) \)-action are compact. A well known result of D’Ambra [5] states this is true, for compact simply connected manifolds, but, only in the analytic category. In fact one could ask for the more stable:

**Question 2.10** Let \( M \) be a compact simply connected manifold. Is the action of \( \text{Diff}^2(M) \) on \( \text{Lor}^2(M) \) proper? (the exponent 2 stands for the \( C^2 \) differentiability and the \( C^2 \) topology).

Also here the language of generalized dynamical systems is pertinent.

**Definition 2.11** Let \( g \) be an element of \( \text{Lor}^2(M) \) and \( (f_i) \subset \text{Diff}^2(M) \) a generalized dynamical system. We say that \( (f_i) \) is approximately isometric for \( (M,g) \) if \( f_i^*g \to g \) in \( \text{Lor}^2(M) \).

Such dynamical systems occur exactly when the action of \( \text{Diff}^2(M) \) on \( \text{Lor}^2(M) \) is not proper. In this article, we will not treat approximately isometric dynamical systems (to avoid interference with the approximate stability notion, which is more central for us). However, most of the results, in particular Theorem [1.2] generalizes, with the same proofs, to this context. As a corollary we have:

**Theorem 2.12** Let \( M \) be a compact manifold such that \( \text{Diff}^2(M) \) acts non properly on \( \text{Lor}^2(M) \). Then, \( M \) admits a codimension 1 lightlike geodesic foliation, in the sense of some metric \( g \in \text{Lor}^2(M) \).

From a result of [18], a 3-manifold admitting such a foliation is covered by \( \mathbb{R}^3 \). So, we have the following partial positive answer.

**Theorem 2.13** Let \( M \) be a compact 3-manifold not covered by \( \mathbb{R}^3 \). Then \( \text{Diff}^2(M) \) acts properly on \( \text{Lor}^2(M) \).

### 2.6 Organization of the article

In the sequel, in order to simplify notations, we shall restrict our presentation to compact manifolds. At certain points, we shall remark on the differences in definitions and statements, for the non compact case. Our investigation of approximate stability starts with the linear case and is contained in \( \S 4 \). The general affine (i.e. connection preserving) appears in \( \S 5, 6 \) and \( 7 \) and is followed by
the Lorentz case in §8. Next, we introduce a bundle compactification (§9), a first step towards the (foliation) compactification of isometry groups of Lorentz manifolds (§10). Theorem 1.2 is proved in §8 and §9. In §§11 and 12 we study elementary groups. This part is somewhat technical due to the fact of the possible co-existence of parabolic and hyperbolic behavior for elementary groups. The last sections are devoted to proofs of theorems and exposition of illustrative examples.

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3 Approximate stability

Here we give precise definitions of approximately stable objects associated to a generalized dynamical system \((M, (f_n))\) (that is \((f_n)\) is just a sequence of diffeomorphisms on \(M\)).

**Definition 3.1** Let \((f_n)\) be a generalized dynamical system.

1. **Stability.** A sequence of vectors \((v_n)\) in \(TM\) is stable if \((f_n(v_n))\) is bounded. Similarly, a sequence \(P_n \subset T_{x_n}M\) of tangent linear subspaces (of some fixed dimension) is stable, if \(D_{x_n}f_n|P_n\) is bounded.

2. **Approximate stability.** A vector \(v \in T_xM\) is approximately stable if \(v\) is a limit of a stable sequence of vectors, that is there exists a convergent sequence \(v_n \to v\), such that \(Df_n v_n\) is bounded. We denote by \(\text{AS}((f_n))\) the set of approximately stable vectors of \((f_n)\) and by \(\text{AS}(x, (f_n)) \subset T_xM\) its intersection with \(T_xM\).

We say that \(v\) is approximately strongly stable if in addition \(Df_n v_n \to 0\) (in \(TM\)).

3. **Punctual approximate stability.** A vector \(v \in T_xM\) is punctually approximately stable, if it is a limit of stable sequence in the same tangent space \(T_xM\): \(v_n \in T_xM\), \(v_n \to v\) and \(Df_n v_n\) bounded. As above we introduce the notations \(\text{PAS}((f_n))\) and \(\text{PAS}(x, (f_n))\). We define analogously a notion of punctually approximately strongly stable.

Similarly, we define the same notions for tangent subspaces in \(TM\).

**Remark 3.2 (Punctual and non punctual)** Observe that \(\text{PAS}(x, (f_n))\) is a linear space but not \(\text{AS}(x, (f_n))\) is. It is also important to observe that if \(P \subset T_xM\) is such that each \(v \in P\) is approximately stable (i.e. \(P \subset \text{AS}(x, (f_n))\)), then one cannot infer that \(P\) is an approximately stable subspace. The vectors of \(P\) may be approximated by stable sequences of vectors with different base-points, and there is no way to sum (or even approximately sum) them (since we have no control on the distance between base-points). In the punctual case, it is clear that \(P \subset \text{PAS}(x, (f_n))\) is equivalent to \(P\) being an approximately stable subspace.

**Remark 3.3 (the non-compact case)** If \(M\) is not compact, then in the definition of stable sequence of vectors, we require that the sequence \((f_n(x_n))\) lies in a compact subset of \(M\).

3.1 Sizes and modulus of stability

Although, when considering approximate stability, we are far from uniform estimates, we will sometimes need to specify the sizes of objects. Since we are working on a compact manifold, we choose a fixed continuous norm (on \(TM\)) with respect to which, we compare sizes. In this case, we define the size of a subset of \(M\) to simply be its diameter. Now, let \(P_n \subset T_{x_n}M\) be a stable sequence with respect to a generalized dynamical system \((M, (f_n))\), converging to an approximately stable
subspace $P \subset T_x M$. By the **modulus of stability** of $P$ (or of $(P_n)$), we mean the inverse of the supremum of the norms of $D_{x_n} f_n | P_n$.

However, to avoid explicit use of a particular norm, which would make the statements rather technical, and which has no natural relation with data, we prefer to think of sizes in a “rough” sense, that is up to multiplicative constants, given by the geometric context.

For example, given a stable sequence $(P_n)$ (or even a family of such sequences with uniform modulus of stability), one can associate a continuous family $(V_x)_{x \in M}$ where $V_x$ is a neighborhood of 0 in $T_x M$, with size proportional to that of the stable sequence. So, if $P_n$ is a subspace of $T_{x_n} M$, then $Df_n (P_n \cap V_{x_n})$ is contained in a neighborhood of 0 in $T_{f_n x_n} M$, with uniformly bounded size. For instance, in the affine context of §2, 3 and 8, we insist that the image by the exponential map of the later neighborhood, is contained in a convex neighborhood of $f_n x_n$ in $M$.

### 4 Examples. First calculations

**The fundamental example.** We consider here (classical) linear dynamical systems on $\mathbb{R}^d$, where $A \in GL(d, \mathbb{R})$. The fundamental example is the following. Let $d = 3$, endow $\mathbb{R}^3$ with its canonical basis $\{e_1, e_2, e_3\}$, and take: $f_n = A^n = \begin{pmatrix} 1 & n & n^2/2 \\ 0 & 1 & -n \\ 0 & 0 & 1 \end{pmatrix} = \exp n \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$

Here approximate stability and pointed approximate stability coincide. At $x = 0$, we have $AS(0) = PAS(0) = \mathbb{R} e_1 \oplus \mathbb{R} e_2$. For instance to see that $e_2$ is approximately stable, observe that the vector sequence $v_n = (0, 1, -2/n)$ is stable. Indeed, $A^n(v_n) = (0, 3, -2/n)$ is bounded.

Observe that $e_1$ which is obviously stable, is in fact strongly approximately stable. To check this consider $v_n = (1, 1/n^2, -2/n^2)$, and note that $A^n(v_n) = (0, 1/n^2 + 2/n, -2/n^2)$.

Note, however, that in the case of $\mathbb{R}^2$ and $A_n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \exp n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the approximately stable space is reduced to $\mathbb{R} e_1$.

In general, consider on $\mathbb{R}^d$, $A_n = \exp n B$, where $B$ is a Jordan block, $B = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 1 & \ldots \end{pmatrix}$.

Then $AS(0) = PAS(0) = \mathbb{R} e_1 \oplus \mathbb{R} e_2$, if $d \geq 3$.

Finally, for $f_n = A^n$, with $A$ semi-simple, approximate stability coincides with stability.

#### 4.1 Generalized linear systems of $\mathbb{R}^d$

In this section, we consider a general sequence $(A_n) \in GL(d, \mathbb{R})$ of linear transformations of $\mathbb{R}^d$.

Let $A^+ \subset GL(d, \mathbb{R})$ denote the semi-group of diagonal matrices $\text{diag}\{\lambda_1, \ldots, \lambda_d\}$ where $0 < \lambda_1 \leq \ldots \leq \lambda_d$. Then we can write $A_n = L_n D_n R_n$ where $D_n \in A^+$ and $L_n$ and $R_n$ belong to $SO(d)$. Therefore, a sequence of vectors $(v_n)$ is stable in the sense of $(A_n)$ if and only if $(R^{-1}v_n)$ is stable in the sense of $(D_n)$.

Now, it is straightforward to calculate the approximately stable space for a diagonal sequence. Indeed, as in the classical case above, approximate stability coincides with stability. We point out the following special case.

**Fact 4.1** Let $A_n = L_n D_n R_n$ be such that $D_n$ has only one eigenvalue (with multiplicity) $> 1$, that is there exists $i$ such that $\lambda_i \leq 1 < \lambda_{i+1} = \ldots = \lambda_d$. Assume that $A_n$ is divergent (i.e. has no convergent subsequence), then a subspace $P$ is approximately stable if and only there is a subsequence $Q_n$ of subspaces of $\mathbb{R}^i \times \{0\}$, such that $P = \lim_{n \to \infty} R_n^{-1} Q_n$. In particular, if $\lim_{n \to \infty} R_n^{-1} (\mathbb{R}^i \times \{0\})$
Hence, it is one dimensional (because the group of the Lorentz quadratic form on $\mathbb{R}^d$: $q = -x_1^2 + x_2^2 + \ldots + x_d^2$. We note firstly that the above observation holds.

**Fact 4.2** An element $A \in SO(1,d-1)$ can be written as $A = LDR$, with $L, R \in SO(d)$ and $D = \text{diag}\{\lambda, 1, \ldots, 1, \lambda^{-1}\}$, with $\lambda \leq 1$.

**Proof.** The $KA^+K$-decomposition for semi-simple Lie groups yields in our case, $K = SO(1, d-1) \cap SO(d) = SO(d-1)$, and $A^+$ any one parameter group of symmetric matrices that belongs to $SO(1, d-1)$. So, after conjugation by a rotation $r \in SO(d)$, $rA^+r^{-1}$ becomes diagonal. The eigenspace associated to an eigenvalue $< 1$ (resp. eigenvalues $> 1$) is isotropic (for the form $q$). Hence, it is one dimensional (because $q$ is Lorentz). This proves the Fact.

**Corollary 4.3** Let $(A_n)$ be a divergent sequence in $SO(1,d-1)$. Then, there is a subsequence $(B_n)$ such that the approximately stable space $AS(0,(B_n))$ is a lightlike hyperplane. The strongly approximately stable space of $(B_n)$ is the orthogonal of $AS(0,(B_n))$. It is an isotropic one-dimensional space. The approximately stable space of $(A_n)$ is the intersection of the hyperplanes obtained from all the subsequences $(B_n)$. In particular, if all the approximately stable hyperplanes involved coincide, then, this equals the approximately stable space of $(A_n)$ itself.

Finally, all the moduli of stability are uniform.

**Discompactness.** In what follows we give an equivalent definition of the approximately stable space, reminiscent to the Carrière’s notion of discompactness. The codimension 1 fact in the Lorentz case translates to that $SO(1, d-1)$ has discompactness 1. This fact is as crucial for our work, as it was for Carrière’s.

Consider a sequence $(A_n)$ in $GL(d, \mathbb{R})$, and let $U$ be the unit ball of $\mathbb{R}^d$. Then $E_n = U \cap A_nU$ is a $d$-dimensional ellipse. A limit (in the sense of Hausdorff) of $(E_n)$ is an ellipse of dimension $\leq d$. Let $U'$ be the intersection of all the limits of all convergent subsequences of $(E_n)$. It is an ellipse of certain dimension. Observe that $AS(0,(A_n))$ is the linear space generated by $U'$.

**Graphs.** The approach of consists of taking graphs. Keeping the notation above, consider the Graphs $Gr(A_n) = \{(x, A_n(x)/x \in \mathbb{R}^d) \subset \mathbb{R}^d \times \mathbb{R}^d$. Then $AS(0,(A_n))$ is the intersection of all the projections of all the limits of subsequences of $(Gr(A_n))$. In the Lorentz case, we endow $\mathbb{R}^d \times \mathbb{R}^d$ with the product $\lt, > \oplus <, >$. Thus, an element $A$ of $GL(d, \mathbb{R})$ belongs to $SO(1,d-1)$ iff $Gr(A)$ is isotropic. Observe that if for some sequence $(A_n)$, $(Gr(A_n))$ converges to a (d–plane) $E \subset \mathbb{R}^d \times \mathbb{R}^d$, then the intersection of $E$ with each of the factors $\mathbb{R}^d \times \{0\}$ and $\{0\} \times \mathbb{R}^d$ are isotropic and hence of dimensions $\leq 1$. This is the content of discompactness 1.

**Rank one groups.** The analogue of the fact above is valid for all the simple groups of non-compact type and of rank one, but now we allow diagonal matrices of the form: $\text{diag}\{\lambda, \ldots, \lambda, 1, \ldots, 1, \lambda^{-1}, \ldots, \lambda^{-1}\}$. The multiplicity of $\lambda$ (or $\lambda^{-1}$) may then be 1, 2, 4 or 8 (thanks to the classification of simple Lie groups).
**Chaos.** Consider the two Lorentz linear systems: $B_n = \begin{pmatrix} 1 & b_n & b_n^2/2 \\ 0 & 1 & -b_n \\ 0 & 0 & 1 \end{pmatrix}$ and $C_n = \begin{pmatrix} c_n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c_n^{-1} \end{pmatrix}$. They are orthogonal for the form: $q = x_1x_3 + x_2^2$. We suppose that both $(b_n)$ and $(c_n)$ go to $\infty$ when $n \to \infty$. So we have as above, $AS(0, (B_n)) = \mathbb{R}e_1 \oplus \mathbb{R}e_2$, and $AS(0, (C_n)) = \mathbb{R}e_2 \oplus \mathbb{R}e_3$.

Let now $A_n = C_nB_n$ and observe that as above, we have $AS(0, (A_n)) = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ and that $e_1$ is strongly approximately stable, that is, there is a sequence $u_n \to e_1$ such that $A_n(u_n) \to 0$. Of course, $A_n(e_1) = c_ne_1 \to \infty$!

**4.2 The derivative cocycle.**

In the (Liapunov) measurable theory one uses measurable trivializations of $TM$, with respect to which the derivative of a diffeomorphism $f$, is written as a map $C_f : M \to GL(d)$, where $d = \dim M$. In consideration of approximate stability, one needs some control of the “continuity” of the trivialization. So, to treat the punctual approximate stability, it suffices to consider a bounded trivialization. That is a frame-field with image in a compact set of the frame bundle. To handle approximate stability at some point, we further assume that the frame-field is continuous at this point. So we will always suppose that the trivialization satisfy the needed requisitions.

Let now $(M, (f_n))$ be a generalized dynamical system on a compact manifold $M$. We denote $C_n = C_{f_n}$. In the classical case, i.e. $f_n = f^n$, we obtain a cocycle $C : M \times \mathbb{Z} \to GL(d)$. In the generalized case, we just obtain a collection of linear systems $(C_n(x))$, for $x$ running over $M$. So one can relate the punctually approximately stable space of $(f_n)$ at a point $x$ with the analogous one of $(C_n(x))$ at $0$.

**Fact 4.4** We have: $\text{PAS}(x, (f_n)) = \text{PAS}(0, (C_n(x))) = AS(0, (C_n(x)))$.

**The Lorentz case.** From the above facts we deduce:

**Proposition 4.5** Let $(f_n)$ be a non-equicontinuous sequence of isometries of a compact Lorentz manifold $M$. Let $M'$ be a countable subset of $M$. Then there is a subsequence $(\phi_n)$ of $(f_n)$ such that, for $x \in M'$, $PAS(x, (\phi_n))$ is a lightlike hyperplane, and $PAS(x, (\phi_n))^{-1} = SPAS(x, (\phi_n))$ (= the strongly punctually approximately stable space of $(\phi_n)$ at $x$). All these hyperplanes have a uniform modulus of stability.

*Proof.* Keeping the notations above and using a Lorentz trivialization, we have at each $x$, a derivative sequence $(C_n(x))$. We shall see in §2, since $(f_n)$ is not equicontinuous, that for any $x$, $(C_n(x))$ is not equicontinuous. The proof follows by using diagonal procedure and Corollary 4.3.

**5 Affine dynamics: uniformity.**

Henceforth, we will only consider affine generalized dynamical systems, that is $M$ is endowed with a linear torsion free connection $\nabla$ and $(f_n)$ is a sequence of connection preserving transformations.

**Equicontinuity. Divergent sequences.** The first fundamental property of affine dynamics is the following:

**Proposition 5.1 (uniformity)** Let $(f_n)$ be an affine generalized dynamical system on a compact manifold $M$. Suppose that there is a sequence $(x_n)$ such that $(D_{x_n}f_n)$ is equicontinuous (that is $(D_{x_n}f_n)$ and $(D_{x_n}f_n^{-1}) = D_{f_n(x_n)}f_n^{-1}$ are bounded). Then $(f_n)$ is equicontinuous, that is, $(f_n)$ lies in a compact subset of the affine group Affin(M).
The proof follows easily from the fact that Affin$(M)$ acts properly on the frame bundle of $M$. 

Corollary 5.2 (the unimodular case) Let $(f_n)$ be an affine unimodular generalized dynamical system on a compact manifold $M$, that is all the $f_n$ preserve a volume form. (This is for example the case if there is a parallel volume form, e.g. the connection derives from a pseudo-Riemannian metric). Suppose that there is a sequence $(x_n)$ such that $(D_{x_n} f_n)$ is bounded. Then $(f_n)$ is equicontinuous.

Proof. The unimodularity and the boundedness of $(D_{x_n} f_n)$, imply that $(D_{x_n} f_n^{-1})$ is also bounded, and hence $(D_{x_n} f_n)$ is equicontinuous.

In the sequel we will be only interested in the opposite situation of equicontinuity. Specifically, we say that a sequence $(f_n)$ is divergent if $\{f_n/n \in \mathbb{N}\}$ is a closed discrete subset of the group of homeomorphisms of $M$. So $(f_n)$ is not divergent if it contains a convergent subsequence (in the group of homeomorphisms of $M$).

The codimension 0 case. We use the notations of the Proposition above. By definition $D_{x_n} f_n$ is bounded if and only if $(T_x M)$ is a $(f_n)$-stable sequence of subspaces (of codimension 0). In particular if $x_n \to x$, then $T_x M$ is an approximately stable subspace. So the corollary above translates to the fact that an unimodular affine generalized dynamical system is equicontinuous whenever $T_x M$ is an approximately stable subspace, for some $x \in M$. So, in particular a divergent sequence $(f_n)$ can only have approximately stable subspaces of codimension $> 0$.

Modulus of stability. One may then ask whether or not the unimodularity condition is necessary. The answer is yes, as we shall see below in the case of Hopf manifolds. These examples also explain why the phenomena of “propagation of stability” (expressed in [5.1]), which generalizes the uniformity fact [5.1], is only local (more precisely, proportional to modulus of stability). In fact, the pathology of Hopf manifolds is due to their non-completeness. In affine flat dynamics, non-unimodularity and non-completeness are generally thought of as being equivalent phenomena.

An example: Hopf manifolds. Recall that an affine (flat) Hopf manifold is the quotient of $\mathbb{R}^d - \{0\}$ by a linear contraction. The simplest case is when this contraction is given by a multiplication map: $x \to \alpha x$, $0 < \alpha < 1$. The quotient $H_\alpha$ is thus endowed with an affine action of $GL(d, \mathbb{R})$. This action does not preserve any (non trivial) measure. One can see this in the following way. For the sake of simplicity, we only consider the the case $d = 2$, so that the Hopf manifold is topologically a torus. Let $A'$ be a non compact one parameter group of $SL(2, \mathbb{R})$. Then, its orbits determine a Reeb foliation, and hence has only finitely many recurrent leaves.

Let $f$ be the diffeomorphism of $H_\alpha$ corresponding to a matrix \[
\begin{pmatrix}
1 & 0 \\
0 & \lambda
\end{pmatrix}
\] with $\lambda > 1$. Then $f^n$ corresponds to any of the matrices \[
\begin{pmatrix}
\alpha^{-m} & 0 \\
0 & \lambda^n \alpha^{-m}
\end{pmatrix}
\] for any integer $m$. Let $x \in H_\alpha$, be the projection of a point $(a, b) \in \mathbb{R}^2 - \{0\}$. For calculation, one chooses a fundamental domain containing $(a, b)$ and for each $n$, one chooses $m$ such that \[
\begin{pmatrix}
\alpha^{-m} & 0 \\
0 & \lambda^n \alpha^{-m}
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
\] belongs to the same domain. If $b = 0$, $x$ is a fixed point of $f$ and so $D_x f^n$ is identified with \[
\begin{pmatrix}
1 & 0 \\
0 & \lambda^n
\end{pmatrix}
\]. Therefore, the approximately stable space is the $x$-axis. For $a \neq 0$, in order to return to the fundamental domain, we will need an integer $m$ increasing with $n$ and such that $\lambda^n \alpha^{-m} b$ is proportional to $b$. Therefore,
$D_x f^n$ equals $\begin{pmatrix} \alpha^{-m} & 0 \\ 0 & \lambda^n \alpha^{-m} \end{pmatrix}$ and is thus equivalent to $\begin{pmatrix} \alpha^{-m} & 0 \\ 0 & 1/b \end{pmatrix}$ modulo multiplication by $\alpha$. In particular, $(D_x f^n)$ is bounded (but not equicontinuous). Despite the fact the modulus of stability is not uniform.

6 Approximate stability in affine dynamics: partial uniformity

In the present section, we prove some integrability and geodesibility properties for the approximately stable objects, due to a propagation of stability phenomenon, which holds in affine dynamics, and generalizing the uniformity Proposition [5.1].

For $x \in M$, the exponential map $\exp_x$ is defined in an open subset of $\mathcal{D}ef_x \subset T_x M$. We recall that a submanifold $V$ of $M$ is geodesic if whenever a geodesic $c : [a, b] \to M$ is somewhere tangent to $M$: $c'(t_0) \in T_{c(t_0)} V$ then $c(t) \in M$, when $t$ belongs to some neighborhood of $t_0$. Notice that in general, if $P$ is a linear subspace of $T_x M$, then $\exp_x P \cap \mathcal{D}ef_x$ is not geodesic except when $\dim P = 1$ or for specific nice manifolds (see Part II).

**Notations.** In the sequel, for any $y \in M$, we choose $V_y \subset T_y M$ a neighborhood of 0, with size proportional to the modulus of stability of a given $(f_n)$-stable sequence of linear subspaces $(P_n)$.

6.1 Propagation of stability

**Proposition 6.1 (Propagation of stability)** Let $(P_n)$ be a $(f_n)$-stable sequence of linear spaces, with $P_n \subset T_{x_n} M$. Let $\mathcal{P}_n = \exp_{x_n}(P_n \cap V_{x_n})$ and $y_n \in \mathcal{P}_n$. Consider the restrictions $h_n = f_n|\mathcal{P}_n$. Then, the derivatives $D_{y_n} h_n$ are uniformly bounded by the size of $(P_n)$.

In particular, let $v_n \in P_n$ be a convergent sequence of vectors, $y_n = \exp_{x_n} v_n$ and $P'_{n} = T_{y_n}(\exp_{x_n} P_{n} \cap V_{y_n})$. Then $(P'_n)$ is a $(f_n)$-stable sequence, with size controlled by means of that of $P_n$.

**Proof.** Firstly, observe that the claim is obvious if $f_n$ are linear transformations of an Euclidean space $\mathbb{R}^d$, and $x_n = 0$ (here, without size restriction).

The proof of the general case follows by linearization. Assume to begin with that the sequence $(x_n)$ is stationary: $x_n = x_0$, and furthermore, $x_0$ is fixed by all the $f_n$, $f_n(x_0) = x_0$, so that the problem becomes linear after conjugation by $\exp_{x_0}$. More precisely, let $g_n = \exp_{x_0}^{-1} f_n \exp_{x_0}$ be defined on some neighborhood $U$ of 0 in $T_{x_0} M$. It follows from the stability of $(P_n)$, we may choose $U$ so that $g_n(P_n \cap U)$ is contained in some fixed small neighborhood $U'$ of 0. Hence, the derivatives of $f_n = \exp_{x_0} g_n \exp_{x_0}^{-1}$ along points of $\exp_{x_0}(P_n \cap U)$ are comparable to the corresponding ones for $g_n$, because of the fact that all things stay in a compact set, where the derivatives of $\exp_{x_0}$ and $\exp_{x_0}^{-1}$ are controlled.

When $(x_n)$ is not stationary, we consider the family of derivatives $g_n = D_{x_n} f_n : T_{x_n} M \to T_{f_n(x_n)} M$. Inasmuch these spaces are equipped with norms induced from a metric on $TM$, with respect to which we define stability, these norms are defined up to a bounded distortion. Now, since by definition $x_n$ and $f_n(x_n)$ stay in a compact set, we can find identifications of bounded distortion of all our linear tangent spaces with a fixed Euclidean space. Therefore the stability notions are preserved and the proof goes as in the previous case.

**Proposition 6.2 (compatibility with parallel transport)** Keeping the notations of the proposition above, let $(c_n)$ be a sequence of curves, such that the image of $c_n : [0,1] \to M$ is contained in
\( \exp_{x_n} P_n \cap V_{x_n} \) with \( c_n(0) = x_n \). Consider \( P''_n \) the parallel transport of \( P_n \) along \( c_n \). Suppose that \( (c_n) \) is bounded in the \( C^1 \) topology. Then \( P''_n \) is a stable sequence.

**Proof.** From the proposition above the image curves \( d_n = f_n(c_n) \) are bounded in the \( C^1 \) topology. Let \( \tau_n = T_{c_n(0)} M \rightarrow T_{c_n(1)} M \) be the parallel transport along \( c_n \) and let \( \tau'_n \) be the analogous parallel transport along \( d_n \). Then, \( \tau_n \) and \( \tau'_n \) are uniformly bounded (since \( c_n \) and \( d_n \) are \( C^1 \) bounded). Now because \( f_n \) are affine, they commute with parallel transport, in particular: \( D_{c_n(1)} f_n = (\tau'_n)^D_{c_n(0)} \tau_n^{-1} \). Therefore, \( (\tau_n(P_n)) \) is a stable sequence.

### 6.2 Geodesibility properties

**Proposition 6.3** Suppose that the stable sequence \( (P_n) \) converges to an approximately stable linear subspace \( P \subset T_x M \), which is maximal among such subspaces (that is not strictly contained in another approximately stable subspace). Then \( \exp_x P \cap V_x \) is geodesic in \( M \). The same result is true in the punctual approximately stable case, that is, for any \( x \in M \), \( \exp_x (PAS(x, (f_n)) \cap V_x) \) is geodesic.

**Proof.** Let \( c_n : [0,1] \rightarrow \exp_{x_n} P_n \cap V_{x_n} \) be a sequence of curves converging in the \( C^1 \) topology to a curve \( c : [0,1] \rightarrow \exp_x P \cap V_x \), and such that \( c_n(0) = x_n \). Let \( y_n = c_n(1) \), \( P'_n \subset T_{y_n} M \) be the tangent space of \( \exp_{x_n} P_n \cap V_{x_n} \) at \( y_n \), and \( P''_n \subset T_{y_n} M \) the parallel transport of \( P_n \) along \( c_n \). By the above propositions, both \( (P'_n) \) and \( (P''_n) \) are stable sequences of linear subspaces.

Denote the analogous objects at \( y = c(1) \) by \( P' \) and \( P'' \), which are obviously the limits of \( P'_n \) and \( P''_n \), respectively.

To prove that \( \exp_x P \cap V_x \) is geodesic, it suffices to check the equality: \( P' = P'' \). Indeed, if this is true for arbitrary \( c \), then the tangent space of \( \exp_x P \cap V_x \) is parallel (along itself), which is equivalent to the geodesibility (for torsion free connections).

But, if \( P' \neq P'' \), then \( P' \oplus P'' \) is an approximately stable linear subspace. Indeed, let \( \{e_i\} \) be a basis of this later subspace, and choose \( \{e'_n\} \) vectors of \( P'_n \oplus P''_n \), such that \( e_i \rightarrow e'_n \). Denote by \( E_n \) the vector subspace generated by the \( \{e'_n\} \). Then \( (E_n) \) is a stable sequence converging to \( P' \oplus P'' \). This contradicts the fact that \( P \) is maximal. Therefore \( P' = P'' \).

### 6.3 The codimension 1 case: lamination properties

**Fact 6.4** Assume that \( (f_n) \) has no codimension 0 approximately stable subspace (that is for no \( x \in M, T_x M \) is an approximately stable space). Let \( P \subset T_x M \) be an approximately stable hyperplane. Then \( AS(x, (f_n)) = P \). In particular, in this case \( P \) is maximal and hence from the Proposition above, \( \exp_x P \cap V_x \) is a geodesic hypersurface.

**Proof.** Assume the contrary, that is there exists an approximately stable vector \( v \in T_x M \) which is transverse to \( P \). Thus, \( P \) and \( v \) are respectively limits of stable sequences \( P_n \subset T_{x_n} M \), and \( v_n \in T_{y_n} M \). By transversality (of \( P \) and \( v \)), \( \exp_{x_n} P_n \cap V_{x_n} \) and \( \exp_{y_n} R_{v_n} \cap V_{y_n} \) intersect, in an uniform transverse meaner at a point \( z_n \) (near \( x \), for \( n \) large). Moreover by Proposition 6.3, \( D_{z_n} f_n \) is uniformly bounded along the tangent spaces at \( z_n \), of each of the submanifolds \( \exp_{x_n} P_n \cap V_{x_n} \) and \( \exp_{y_n} R_{v_n} \cap V_{y_n} \). By definite transversality, \( D_{z_n} f_n \) is bounded in \( T_{z_n} M \), that is \( (T_{z_n} M) \) is a \( (f_n) \)-stable sequence, a fact that is excluded by hypothesis.

■
Corollary 6.5 With the same hypothesis, if \( y \in \exp_x P \cap V_x \), then \( AS(y) = T_y(\exp_x P \cap V_x) \).

Proof. It follows from Proposition [B.1] that \( T_y(\exp_x P \cap V_x) \subset AS(y) \), and we infer from the fact above, the equality: \( AS(y) = T_y(\exp_x P) \), as desired.

By the same argument we get:

Corollary 6.6 If \( P_x \subset T_x M \) and \( P_y \subset T_x M \) are approximately stable hyperplanes, then the two geodesic hypersurfaces \( \exp_x P_x \cap V_x \) and \( \exp_y P_y \cap V_y \) are either disjoint or tangent, in which case their intersection is open in each of them.

The method of Graphs. As in the linear case \( (\S 4) \), there is a graph approach leading to some geometric proofs of the above properties of the approximately stable submanifolds \( \exp_x P \cap V_x \). To see this, note that if we endowed \( M \times M \) with the product connection. Then \( (\Graph(f_n)) \) is a sequence of geodesic submanifolds in \( M \times M \). Any approximately stable submanifold \( \exp_x P \cap V_x \) is obtained as a projection of a limit of a sequence of connected components of \( U \cap \Graph(f_n) \), where \( U \) is an open subset of \( M \times M \).

The geodesic character of \( \exp_x P \cap V_x \) is thus obvious. Nevertheless, neither the lamination properties \( (6.4) \), nor the control of sizes by modulus of stability, seems to be easy to treat, via this approach.

7 The approximately stable foliation theorem

Here follows a fundamental existence and regularity result:

Theorem 7.1 Suppose that there is a dense subset \( M' \subset M \), in which \(\PAS(x,(f_n)) \) is a hyperplane, with uniform modulus of stability. Then \(\AS((f_n)) \) is a Lipschitz codimension 1 subbundle of \( T M \), tangent to a geodesic foliation \( \AS((f_n)) \), called the approximately stable foliation of \( (f_n) \).

For the proof we need:

Digression: Codimension 1 geodesic laminations. Geodesic laminations (and in particular foliations) of codimension 1, enjoy some remarkable regularity properties. These properties are well known for laminations of hyperbolic surfaces \([3]\), but are in fact valid in the general context of codimension 1 geodesic laminations, in the sense of a connection (and also some some codimension 1 foliations with other geometric origins). For proofs and related questions, see \([13]\) and \([12]\). Behind all of the regularity results for geodesic laminations, is the following fundamental Lipschitz regularity fact.

Lemma 7.2 Let \( M \) be a compact manifold endowed with a torsion free connection and an auxiliary norm \( |.| \) on \( T M \). Let \( M' \) be a subset of \( M \) and suppose given a real \( r \) and for \( x \in M' \) a hyperplane \( P_x \subset T_x M \) and let \( L_{x,r} = \exp_x P_x \cap B_x(r) \) where \( B_x(r) \) is the ball of \( T_x M \) centered at \( 0 \) and with radius \( r \). Also, suppose that \( L_{x,r} \) is geodesic and that if two plaques \( L_{x,r} \) and \( L_{y,r} \) intersect at some point, then they are tangent at that point (and hence by geodesibility, the intersection \( L_{x,r} \cap L_{y,r} \) is open in both \( L_{x,r} \) and \( L_{y,r} \)). Then, along \( M' \), the map \( x \rightarrow P_x \) is Lipschitz, with Lipschitz constant depending only on the geometry of the connection, the auxiliary norm, and \( r \).

We infer from the above lemma the following corollary:
Corollary 7.3 (i) A codimension 1 geodesic lamination on $M$ is Lipschitz, with Lipschitz constant depending only on the geometry of the connection and the auxiliary norm. It then follows that the space of codimension 1 geodesic foliations of $M$, endowed with the $C^0$ topology (or equivalently the Lipschitz topology) on hyperplane fields, is compact.

(ii) With the same hypothesis as in the lemma above, suppose that the set $M'$ is dense. Then, the geodesic plaques $L_{x,r}$ extend to a geodesic foliation of $M$.

Proof of Theorem 7.1 Since the modulus of stability is uniform, the geodesic plaques $\exp_x PAS(x, (f_n)) \cap V_x$ given by 6.3, satisfy the conditions of 7.2. Therefore, we have a geodesic foliation $F$ of $M$ such that $T_x F = PAS(x, (f_n))$ for $x \in M'$. Again by uniformity of the modulus of stability, and the assumption that $M'$ is dense, for any $x \in M$, $T_x F$ is an approximately stable hyperplane at $x$. Therefore, it follows from Fact 6.4 that for any $x \in M$, $AS(x, (f_n)) = T_x F$. □

8 Lorentz dynamics. Proof of the existence part in Theorem 1.2

Henceforth, we will only deal with transformations preserving a Lorentz structure. So $M$ is now a compact Lorentz manifold and $f_n \in \text{Isom} M$. In the present section, we shall prove the existence part of Theorem 1.2 that is the existence of approximately stable foliations for subsequences (cf. Fact 8.1). The remaining part of Theorem 1.2 will be proved in §9.3. To begin with, we recall some facts about lightlike geodesic foliations.

Lightlike geodesic foliations. The compact space $FG$. A (codimension 1) foliation $F$ is lightlike if the restriction of the metric to $TF$ is degenerate. This metric is thus positive non-definite and its Kernel is a 1-dimensional sub-foliation of $F$, denoted by $\text{Nor} F$ and called the normal foliation of $F$. This 1-dimensional foliation is isotropic (i.e. the metric vanishes along it) and completely determines $F$, since $TF$ is just the orthogonal of $T(\text{Nor} F)$.

Here, we consider lightlike geodesic foliations [18]. One of their basic properties is that their (1-dimensional) normal foliations are also geodesic.

We denote by $FG$ the space of all lightlike geodesic foliations on $M$. It is a closed subset in the space of all the geodesic foliations, and so it is compact whenever $M$ is.

Existence of approximately stable lightlike geodesic foliations. The existence part in Theorem 1.2 is a consequence of:

Fact 8.1 Let $M$ be a compact Lorentz manifold and $(f_n)$ a non-equicontinuous sequence of isometries. Then, there is a subsequence $\phi_n$ admitting an approximately stable lightlike (codimension 1) geodesic foliation $AS((\phi_n))$, that is the approximately stable set $AS((\phi_n))$ is a codimension one bundle tangent to a codimension 1 geodesic lightlike foliation.

In fact, $AS((\phi_n))$ coincide with $PAS((\phi_n))$, the punctually approximately stable bundle. Furthermore, the 1-dimensional normal foliation of $AS((\phi_n))$ equals the (punctually or not) approximately strongly stable bundle of $(\phi_n)$.

Proof. Choose a countable dense subset $M'$ of $M$. Then, by 4.3, there is a subsequence $(\phi_n)$ such that $PAS(x, (\phi_n))$ is a lightlike hyperplane, with uniform modulus of stability, when $x \in M'$. We may infer from Theorem 7.1, the existence on $M$ of a codimension 1 lightlike geodesic foliation $AS((\phi_n))$ tangent to $AS((\phi_n))$. 
On $M'$, we have the equality: $\text{AS}(x, (\phi_n)) = \text{PAS}(x, (\phi_n))$. To prove the equality for an arbitrary $x \in M$, we apply Lemma 4.3 for $(\phi_n)$ (instead of $(f_n)$) and for $M' = \{x\}$. We obtain a subsequence $(\psi_n)$ for which $\text{PAS}(x, (\psi_n))$ is a hyperplane and hence equals $\text{AS}(x, (\phi_n))$ (since obviously $\text{AS}(x, (\phi_n)) \subset \text{AS}(x, (\psi_n))$ and $\text{AS}(x, (\psi_n)) = \text{PAS}(x, (\psi_n))$ from Lemma 4.4. In particular $\text{PAS}(x, (\psi_n))$ does not depend on the subsequence $(\psi_n)$ (provided it is a hyperplane). By Lemma 4.3, this implies that $\text{PAS}(x, (\phi_n))$ is itself a hyperplane, and thus equals $\text{AS}(x, (\phi_n))$.

The fact that the normal direction of $\text{AS}(\phi_n)$ is strongly approximately stable follows from the analogous statement in the linear case, Lemma 4.3. In the same fashion, one can prove the coincidence between strong approximate stability and strong punctual approximate stability.

The non compact case. If $M$ is not compact, it may happen that for some $x$, $\text{AS}(x, (\phi_n)) = \emptyset$ for any subsequence $(\phi_n)$ of $(f_n)$. Indeed, from our definition 3.1, this happens when $x$ satisfies the uniform escaping property: for any sequence $x_n \to x$ and a subsequence $(\phi_n)$ of $(f_n)$, $(\phi_n(x_n))$ tends to $\infty$ (i.e. leaves all compact subsets of $M$).

“Recurrence” for generalized dynamical systems. There is no natural way to define recurrence notions for generalized dynamical systems, so that they satisfy a kind of Poincaré recurrence Lemma. For our purpose, the following notion seems interesting:

Definition 8.2 Let $K$ be a compact subset of $M$. Define its non-escaping subset $\text{NE}(K, (f_n))$ as the (compact) subset of points $x \in K$ such that there is $x_n \in K$, $x_n \to x$ and $f_n x_n \in K$. In other words $\text{NE}(K, (f_n))$ is the Hausdorff limit of the sequence of compact sets $K \cap f_n^{-1} K$. In particular $\text{Vol}(\text{NE}(K, (f_n))) \geq \text{Vol}(K \cap f_n^{-1} K) \geq 2 \text{Vol}(K) - \text{Vol}(M)$ (because the $f_n$ are volume preserving). (In particular, letting $K$ having a large relative volume, we see that almost every $x \in M$ is non-escaping for some compact $K$).

We lose uniformity when $M$ is not compact, and so to estimate sizes, we will choose a norm $|.|$ on $TM$. As sizes may depend upon the choice of such a norm, the previous uniformity of modulus of stability (Lemma 4.3) fails. A straightforward localization of the previous arguments yields:

Proposition 8.3 Let $M$ be a (not necessarily compact) Lorentz manifold, $(f_n)$ a sequence in $\text{Isom}(M)$ and $K$ a compact subset of $M$. Then there is a subsequence $(\phi_n)$ such that, on $\text{NE}(K, (f_n))$, $\text{AS}((\phi_n))$ is a Lipschitz codimension 1 bundle, and there is $r = r(K)$ (not depending on $(f_n)$), such that the family of plaques $L_{x,r} = \exp_x AS(\phi_n) \cap B_x(r)$ determine a codimension 1 lightlike geodesic lamination of a neighborhood of $\text{NE}(K, (f_n))$, tangent to $\text{AS}((\phi_n))$ in $\text{NE}(K, (f_n))$.

This applies at least in the finite volume case, because for instance if $\text{Vol}(K) > \text{Vol}(M) - \epsilon$, then, $\text{Vol}(\text{NE}(K, (f_n))) > \text{Vol}(M) - 2\epsilon$. Therefore, approximately stable bundles of some subsequences of $(f_n)$, give rise to laminations, along big volume subsets of $M$. However, in order to obtain a foliation, on the whole of $M$, we must find volume exhausting sequence of compact sets $(K_j)$ with non collapsing radii $r(K_j))$. This is generally impossible because of non compatibility of the auxiliary metric $|.|$ with the natural data.

Therefore, to avoid use of such a norm, we introduce for $x \in M$, $\text{Def}_x \subset T_x M$ to be the domain of definition of the exponential map $\exp_x$. It is open (by definition) and star shaped in $T_x M$, that is, if $u \in \text{Def}_x$, then $tu \in \text{Def}_x$, for $t \in [0,1]$). Consider also the regular domain of definition $\text{Def}_x^r$ defined as the (open) set of vectors $u \in \text{Def}_x$, such that $tu$ is a regular point of $\exp_x$ for any $t \in [0,1]$. Thus, if $E$ is a linear subspace of $T_x M$, then $\exp_x(E \cap \text{Def}_x^r)$ is an immersed submanifold of $M$. This defines open subsets $\text{Def}$ and $\text{Def}^r$ of $TM$, which are invariant by $\text{Isom}(M)$.
The propagation of stability is valid a priori only in a domain having a size proportional to the modulus of stability. The obstruction lies in fact essentially in non-completeness, as this was shown in the examples of Hopf manifolds. Therefore, one may hope for a propagation of stability in Def. Indeed, this was shown in [20] in the context of (strict) stability (instead of approximate stability here), but only generically. The point is that we just need to ensure continuity (by using the Lusin theorem) of the map \( x \rightarrow \text{Def}^*_x \), and also the continuity of its variants obtained by intersection with stable (here approximately stable) bundles. The same proof yields the following fact:

**Proposition 8.4** Let \( M \) be a (not necessarily compact) Lorentz manifold, \((f_n)\) a sequence of \( \text{Isom}(M) \). Let \( \mu \) be a Borel finite measure (not related to data, but serves in applying the Lusin almost everywhere continuity theorem). Given a positive real \( \epsilon \), there is a compact \( K \) in \( M \), with \( \mu(K) > \mu(M) - \epsilon \) and there is a subsequence \((\phi_n)\) such that, the family of plaques \( \mathcal{L}_x = \exp_x \text{AS}((\phi_n)) \cap \text{Def}^*_x \), for \( x \in \text{NE}(K,(f_n)) \), determines a codimension 1 lightlike geodesic lamination of a neighborhood of \( \text{NE}(K,(f_n)) \), tangent to \( \text{AS}((\phi_n)) \) in \( \text{NE}(K,(f_n)) \) (that is if \( y \in \text{NE}(K,(f_n)) \cap \mathcal{L}_x \), then \( \text{AS}(y,(\phi_n)) = T_y \mathcal{L}_x \)).

In the finite volume case, we use this proposition for the Lorentz measure. As we remarked earlier, we get \( \text{Vol}(\text{NE}(K,(f_n)) > \text{Vol}(M) - 2\epsilon \). By letting \( K \) get larger and larger, and applying a diagonal process, we obtain a foliation:

**Theorem 8.5 (The finite volume case)** The Theorem 1.2, extends to the case of \( M \) of finite volume: a divergent sequence \((f_n)\) of \( \text{Isom}(M) \) possesses a subsequence \((\phi_n)\) admitting an approximately stable foliation \( \text{AS}((\phi_n)) \). Furthermore, almost everywhere: \( \text{AS}(x,(\phi_n)) = \text{PAS}(x,(\phi_n)) \) (this equality is everywhere true in the compact case).

**A relative version.** Here we give a relative version:

**Theorem 8.6** Let \( M \) be a finite volume Lorentz manifold, \((f_n)\) a divergent sequence of isometries that leaves invariant a finite volume Lorentz submanifold \( N \). Let \((\phi_n)\) be a subsequence of \((f_n)\) having an approximately stable foliation \( \text{AS}((\phi)) \). Then, the restriction \((\phi_n|N)\) admits an approximately stable foliation, which is just the trace of \( \text{AS}((\phi_n)) \) on \( N \). In particular, the intersection of a leaf of \( \text{AS}((\phi_n)) \) with \( N \) is geodesic in \( N \).

This result follows from the fact that: if \( x \in N \), then \( \text{PAS}(x,(\phi_n|N)) = \text{PAS}(x,(\phi_n)) \cap T_x N \)

**A foliated version.** We also have the following foliated version:

**Theorem 8.7** Let \( M \) be a finite volume Lorentz manifold and \((f_n)\) a divergent sequence of isometries. Let \( U \) be an open subset of \( M \) and \( L \subset TM|U \) a \( C^0 \) subbundle over \( U \), invariant by each \( Df_n \) (in particular \( U \) itself is invariant by \( f_n \)). Let \( M_0 \) be the subset of \( U \) where \( L \) is of lorentzian type (i.e. the restriction of the metric has signature \(- + \ldots + \)). Let \((\phi_n)\) be a subsequence of \((f_n)\) having an approximately stable foliation \( \text{AS}((\phi)) \). Then in \( U - M_0 \), \( L \subset \text{AS}((\phi_n)) \) and in \( M_0 \), \( L^\perp \subset \text{AS}((\phi_n)) \) (where \( L^\perp \) is the orthogonal of \( L \)).

Suppose that \( L \) is integrable. Then, in \( M_0 \), the leaves of \( \text{AS}((\phi_n)) \cap L \) are geodesic inside the leaves of \( L \).


9 Bundle compactification

Let $M$ be a manifold. One may naturally construct a fiber metric on its (principal) frame bundle $P_M \to M$, so that any $C^1$ diffeomorphism acts isometrically on the fibers. In fact, instead of $P_M$, it is more convenient to consider the associated bundle $S_M \to M$ whose typical fiber the universal symmetric Riemannian space $S_k = SL(k)/SO(k)$, where $k = \dim M$. So, $S_M$ may be interpreted as the bundle of conformal structures on (the fibers of) $TM$.

One may then naturally compactify the fibers to get a bundle $\overline{S_M} \to M$ with a compact fiber $\overline{S_k}$, the Hadamard compactification of $S_k$ which is topologically a closed ball.

It is also interesting to interpret recurrence properties of the action of $f$, or more generally, of a generalized dynamical system $(f_n)$ on $\overline{S_M}$. The philosophy is that recurrence conditions may be related to classical notions as, Oseledec’s decomposition, invariant metrics (of some regularity)...In the sequel we will rather use this construction to interpret the approximate stability.

Notice that these constructions generalize to fiber dynamical systems on a linear bundle $E \to M$. It is also possible to consider the case of bundles with a $G$-structure (i.e a reduction of the structural group of the principal bundle $P_M \to M$ to $G$) where $G$ is a semi-simple subgroup of $SL(k,\mathbb{R})$ of non compact type. In which case, one can build the same constructions with the symmetric space associated to $G$. In fact, instead of developing the general situation, we shall henceforth restrict ourselves to the case where $G$ is $SO(1,k-1)$, and $E$ is the tangent bundle of $M$. This exactly means that $M$ has a Lorentz structure. We mention at this stage that some but not all of the next results are valid in the general case.

Fiberwise hyperbolic geometry. Let $(M^{1+d},<,>)$ be a Lorentz manifold. Note $T^r M = \{v \in TM/ <v,v> = r\}$. After passing if necessary to a double cover of $M$, we can choose a sheet of $T^{-1} M$ that we note $\overline{HM}$. It is a bundle over $M$ with type fiber the hyperbolic space $\mathbb{H}^d$ (recall that the Lorentz metric has signature $- + \ldots +$). It is compactified by adding the the projectivization $\overline{S^\infty M}$ of the isotropic cone $T^0 M$. We denote it by $\overline{HM}$ the bundle over $M$ with type fiber a topological closed ball of dimension $d$.

The group $\text{Isom}(M)$ acts on $\overline{HM}$ and on $\overline{HM}$, by preserving the hyperbolic metric and the conformal structure on the fibers, respectively. This is reminiscent, as the following analogies will confirm, to a Kleinian group acting on the Riemann sphere.

9.1 Limit sets

Definition 9.1 Consider $s : M \to \overline{HM}$ a continuous section (this always exists since $\overline{H^d}$ is contractible). Let $A$ be a closed non compact subset of $\text{Isom}(M)$. The limit set $L_A$ of $A$ is the set of the limits in $\overline{HM}$ of the sequences $(\gamma_n(s(x_n)))$, for $(x_n)$ a sequence in $M$ and $(\gamma_n)$ a divergent sequence of $A$ (i.e. $\{\gamma_n/n \in \mathbb{N}\}$ is a closed discrete subset of $\text{Isom}(M)$).

Fact 9.2 The definition above does not depend on the choice of the section $s$. The limit set $L_A$ is a non empty closed subset of $\overline{S^\infty M}$.

Proof. $L_A$ is non-empty since $\overline{HM}$ is compact. One easily see that if for a sequence $(\gamma_n)$, there is $s_0 \in \overline{HM}$, such that the elements $\{\gamma_n(s_0)/n \in \mathbb{N}\}$ remain in a compact subset of $\overline{HM}$, then $(\gamma_n)$ is equicontinuous. This shows that $L_A$ is contained in $\overline{S^\infty}$. To see that $L_A$ does not depend on the choice of the section $s$, consider a distance $d$ on $\overline{HM}$ inducing the hyperbolic metric on the fibers. If $s'$ is another section, then $d(f_n s(x), f_n s'(x)) = d(s(x), s'(x))$. As usual, this implies that if $(\gamma_n(s(x)))$ converges in $\overline{HM}$, then $(\gamma_n(s'(x)))$ also tends to the same limit. ■
Proposition 9.3 Let $\Gamma$ be a closed non compact subgroup of $\text{Isom}(M)$. Then the limit set $L_\Gamma$ is the smallest $\Gamma$-invariant closed subset of $S^\infty M$ which projects surjectively onto $M$.

Proof. The proof is the same as for Kleinian groups. Let $F$ be a $\Gamma$-invariant subset of $S^\infty M$ projecting onto $M$. One naturally construct $H(F) \subset \overline{\Pi M}$, the fiberwise convex hull of $F$. We can choose the section $s$ (in the definition of limit sets) to have image in $H(F) \cap \Pi M$ (since the fibers of $H(F)$ are still contractible and since $F$ projects onto $M$). Therefore $L_\Gamma \subset H(F)$ since $H(F)$ is closed and invariant. In fact $L_\Gamma \subset H(F) \cap S^\infty M = F$.

In the definition of limit sets, we used continuous sections. In fact, less regular sections may be equally useful. We will need later the particular following statement

Fact 9.4 (measurable sections) Suppose that a closed subgroup $\Gamma$ of $\text{Isom}(M)$ preserves a measurable section $N \to \Pi M$, defined over a measurable $\Gamma$-invariant subset $N$ with a positive volume. Then, $\Gamma$ is compact.

Proof. Let $s : M \to \Pi M$ be a measurable $\Gamma$-invariant section. Let $K$ be a compact subset of $M$, along which $s$ is continuous and such that $\text{Vol}(K) > 1/2 \text{Vol}(M)$. Let $(f_n)$ be sequence in $\Gamma$ and $x_n \in f_n^{-1} K \cap K$, this exist because of the volume condition. By continuity of $s|K$, the set $(f_n(s(x_n)), n \in \mathbb{N}) \subset \Pi M$ is precompact. This implies that $(f_n)$ is equicontinuous. Therefore, $\Gamma$ is compact.

9.2 Limit sets and approximately stable foliations

Proposition 9.5 Let $(f_n)$ be a sequence of $\text{Isom}(M)$ such that $(f_n^{-1})$ admits an approximately stable foliation $\text{AS}((f_n^{-1}))$. Then the limit set $L(f_n)$ is the image of a section $M \to S^\infty M$, i.e. a field of isotropic lines, which is in fact the normal direction of $\text{AS}((f_n^{-1}))$.

Assume furthermore that $(f_n)$ admits an approximately stable foliation. Then if $v \in TM - \text{AS}((f_n))$, $Df_n v$ tends to $0$, and converges projectively (i.e. after normalization) to $L(f_n)$. The convergence is uniform in compact subsets of $TM - \text{AS}((f_n))$.

This convergence is in particular valid in $\Pi M$ (since it is missed by lightlike hyperplanes and hence $\Pi M \subset TM - \text{AS}((f_n))$. More precisely, let $U$ and $V$ be neighborhoods of respectively $\text{AS}((f_n^{-1}))^\perp$ and $\text{AS}((f_n))^\perp$ in $S^\infty$. Then there is $N$, such for, $f_n(S^\infty M - V) \subset U$, for $n > N$.

Proof. We argue by contradiction to prove that the direction $L(f_n)$ is the normal direction of $\text{AS}((f_n))$. Let $s : M \to \Pi M$ be a section, that is a vector field $< s(x), s(x) >= -1$ for $x \in M$. We have to prove that $Df_n s$ converges projectively (i.e. in direction) to $\text{AS}((f_n^{-1}))^\perp$. If this were false, it would exist a sequence $x_n$ such that $Dx_n f_n(s(x_n))$ converges projectively to a vector $u \notin \text{AS}((f_n^{-1}))^\perp$. Hence, $Dx_n f_n (s(x_n)) = \alpha_n u_n$, with $u_n \to u$ and $\alpha_n \to \infty$ (because if not $(f_n)$ would be equicontinuous). Thus $Df_n^{-1} u_n = \alpha_n^{-1} s(x_n)$. Thus, by definition $u$ is strongly approximately stable for $(f_n^{-1})$, and hence by $[\text{ES}]$ belongs to $\text{AS}((f_n^{-1}))^\perp$, which yields a contradiction. The last statement may be checked in a similar fashion.

From the above theorem, one can conclude the two following “continuity” corollaries:

Corollary 9.6 Let $(f_n)$ be a sequence of $\text{Isom}(M)$ such that $(f_n)$ and $(f_n^{-1})$ admit approximately stable foliations $\text{AS}((f_n))$ and $\text{AS}((f_n^{-1}))$. Let $X$ (resp. $Y$) be a continuous vector field tangent to $\text{AS}((f_n))$ (resp. $\text{AS}((f_n))$. Then there is sequence $(X_n)$ (resp. $(Y_n)$) of continuous vector fields converging in the $C^0$ topology to $X$ (resp. $Y$), and such that $(Df_n X_n)$ converges in the $C^0$ topology to 0 and $(Df_n Y_n)$ is bounded.
Proof. We will only prove the claim for $X$, as the same argument works for $Y$. To simplify notation, we will do the proof for $g_n = f_n^{-1}$ instead of $(f_n)$. By the Proposition above, after passing to subsequences, we may find sequences of neighborhoods $(U_n)$ and $(V_n)$, collapsing to $AS((f_n))$ and $AS((f_n)^{-1})$ respectively, and such that $Df_n(TM - U_n) \subset V_n$. Now, let $(X'_n)$ be a sequence of vector fields such that $X'_n \notin U_n$, but $(X'_n)$ converges to some non singular vector field tangent to $AS((f_n))$. Thus, from the Proposition above, we infer that $X''_n = Df_nX'_n$ is a vector field with a big norm and converging in direction in $AS((f_n^{-1}))$. Therefore, for suitable distortion functions $\lambda_n$, going uniformly to $\infty$, we may get $X_n = \lambda_nX''_n \to X$. But $Dg_nX_n = (1/\lambda_n)X'_n \to 0$. 

\textbf{Corollary 9.7} Let $A$ be a closed non compact subset of $\text{Isom}M$ and $u \in L_A \cap S^\infty_X M$. Then, there is a continuous (in fact Lipschitz) section $\sigma : M \to L_A$ with $\sigma(x) = u$.

\textbf{Proof.} $u$ is a (projective) limit of a sequence $(f_n(s(x_n)))$ where $(f_n)$ is a sequence in $A$ and $x_n \to x$ (and $s$ is fixed section of $HM$). After passing to a subsequence we may assume that $(f_n^{-1})$ admits an approximately stable foliation. We set $\sigma(x) = AS((f_n^{-1}))$. From the Proposition above, we conclude that $\sigma(x) = u$.

\section{End of the proof of Theorem 1.2}

The statement in Theorem 1.2, about the dynamics of $(D\phi_n)$ on $TM$, follows from Proposition 9.3.

To finish the proof of Theorem 1.2, it remains to check its ergodicity statement. For this end, to simplify notation, suppose that $\phi_n = f_n$. Let $\sigma : O \to R$ be a continuous function, invariant by all the $f_n$.

Consider a vector field $X_n$ as in 9.6. Let $\alpha$ be a real such that $\exp_x \alpha X_n(x)$ exist for any $x \in M$. Then, $\sigma(\exp_x \alpha X_n(x)) - \sigma(x) = \sigma(\exp_{f_nx} \alpha Df_nX_n(x)) - \sigma(f_nx)$. Denote this difference by $\Delta_n(x)$.

Let $(x_n)$ be sequence such that $f_nx_n$ belong to the same compact $K$. Then $\Delta_n(x_n) \to 0$, as $Df_nX_n \to 0$, in the $C^0$ topology and because $\sigma$ is uniformly continuous on $K$.

Suppose that $x_n \to x$ and $X_n(x_n) \to u \in AS(f)$, $\perp$. Then $\sigma(\exp_x \alpha u) - \sigma(x) = \lim \sigma(\exp_x \alpha u) - \sigma(x_n)) = \lim \Delta_n(x_n) = 0$, for any $\alpha$ such that $\exp_x \alpha u \in O$. That is $\sigma$ is locally constant along the leaf of $AS(f)$ passing through $x$. Remember finally that, from 8.2, almost every $x$ is a limit of a sequence as $(x_n)$, for some compact $K$.

\section{Foliation compactification}

Here we define a topology on $\text{Isom}(M) \cup FG$ and then an ideal boundary $\partial_\infty \Gamma$ for $\Gamma$ a closed non compact subgroup of $\text{Isom}(M)$.

We fix a section $s : M \to HM$ and we choose a distance $d$ on $HM$ (in fact the choice of $s$ permit to construct a natural Euclidean fiberwise distance, we then tensorize by a distance on $M$ and thus we get a semi-canonical distance $d$). Now we embed $IsomM \cup FG$ in the Sect$\overline{HM}$, the space of sections of $\overline{HM}$. To $f \in IsomM$, we associate the section $fs$, and to a foliation $F$ we associate its normal direction field $F^\perp$. We then endow $IsomM \cup FG$ with the $C^0$ topology defined by the distance $p(\sigma_1, \sigma_2) = \sup_{x \in M} d(\sigma_1(x), \sigma_2(x))$.

From 9.3, we deduce:

\textbf{Fact 10.1} If a sequence $(f_n)$ of $IsomM$ converges to $F \in FG$, then $F = AS((f_n^{-1}))$.

Since $FG$ is compact, we have:
Corollary 10.2. \( \Isom M \cup \mathcal{F}G \) is a (metrizable) compact space endowed with a natural continuous action of \( \Isom M \).

Definition 10.3. Let \( \Gamma \) be a closed non compact subgroup of \( \Isom M \). Its boundary \( \partial_\infty \Gamma \) is the intersection with \( \mathcal{F}G \) of the closure of \( \Gamma \) in \( \Isom M \cup \mathcal{F}G \).

11 Preliminaries on elementary groups. Proof of Theorem 2.6

Let \( \mathcal{F}^1 \cup \mathcal{F}^2 \) be a bi-foliation (Definition 2.2). Its set of tangent foliations \( \Apa(\mathcal{F}^1, \mathcal{F}^2) \) has the following description. Let \( C \) be the coincidence set \( \{ x \in M : T_x \mathcal{F}^1 = T_x \mathcal{F}^2 \} \). Let \( \pi_0(M - C) \) be the set of connected components of \( M - C \), and consider a map: \( c : \pi_0(M - C) \to \{1, 2\} \). This allows to construct a foliation, \( \mathcal{F}_c \), by the rule: \( T_x \mathcal{F}_c = T_x \mathcal{F}^c(U) \), in the connected component \( U \) of \( M - C \), and \( T_x \mathcal{F}_c = T_x \mathcal{F}^1 = T_x \mathcal{F}^2 \) in \( C \). It is straightforward to verify that \( \mathcal{F}_c \in \Apa(\mathcal{F}^1, \mathcal{F}^2) \) (in particular there is no loss of continuity or Lipschitz character in this construction). In fact we have a topological identification of \( \Apa(\mathcal{F}^1, \mathcal{F}^2) \) with \( \{1, 2\}^{\pi_0(M - C)} \).

Observe that we have a natural distance in \( \Apa(\mathcal{F}^1, \mathcal{F}^2) \) defined by \( d(\mathcal{F}_c, \mathcal{F}_c') = \sum_{U \in \pi_0(M - C)} (c(U) - c'(U)) \text{Vol}(U) \). Any group preserving the volume and the bi-foliation, preserves this distance.

Fact 11.1. Let \( \Gamma \subset \Isom(M) \) be a closed non compact subgroup with the property that for any \( x \in M \), \( L_\Gamma(x) = L_\Gamma \cap \overline{HM} = S^\infty M \), has exactly 1 or 2 points. Then, \( \Gamma \) is elementary.

Proof. Let \( u \in L_\Gamma(x) \). From the corollary above there is a foliation \( \mathcal{F}(u) \), such that the normal direction \( \mathcal{F}^\perp \) is contained in \( L_\Gamma \) and equals \( u \) at \( x \). Let \( C \) be the coincidence set of all these foliations. Obviously, it equals the set of points \( x \) where \( L_\Gamma(x) \) has cardinality 1. Let \( U \) be a component of \( M - C \), then the restrictions to \( U \) of two foliations \( \mathcal{F}(u) \) and \( \mathcal{F}(u') \) are identical or everywhere transverse. Therefore, as above, one may construct two foliations \( \mathcal{F}^1 \) and \( \mathcal{F}^2 \), which are transverse in each component \( U \), and each of them equals a foliation of the form \( \mathcal{F}(u) \) (in the component \( U \)). Therefore, \( L_\Gamma \) can be defined by a bi-foliation.

Proof of Theorem 2.6. Let \( \Gamma \subset \Isom(M) \) be a closed non compact amenable subgroup. To prove that it is elementary, we will check that it satisfies the condition of the fact above. By amenability, there is a \( \Gamma \)-invariant probability measure \( \mu \) onto \( S^\infty M \), projecting onto the Lorentz measure of \( M \). Let \( \mu_x \) the conditionals of \( \mu \) on the fibers \( S^\infty_x M \).

Let us show that for almost all \( x \in M \) (in the sense of the Lorentz measure), the support of \( \mu_x \) has exactly 1 or 2 points. Indeed, if not a fiberwise barycenter construction (see [1] for the classical one) yields a measurable \( \Gamma \)-invariant section \( \sigma : N \subset M \to HM \) over a \( \Gamma \) invariant subset \( N \). This is impossible by Fact 11.1.

The bi-foliation of an elementary group. Observe that an elementary group may preserve many bi-foliations. However we have the following fact allowing to define the bi-foliation associated to an elementary group:

Fact 11.2. Let \( \Gamma \) be an elementary group. Then \( \Gamma \) preserves a unique bi-foliation \( \mathcal{F}^1 \cup \mathcal{F}^2 \) determining a maximal apartment among that determined by all the bi-foliations preserved by \( \Gamma \). It is characterized by: \( L_\Gamma(x) = \{(T_x \mathcal{F}^1)^\perp, (T_x \mathcal{F}^2)^\perp\} \) for any \( x \in M \). We call it the bi-foliation of \( \Gamma \).
Proof. For a bi-foliation $\mathcal{G}^1 \cup \mathcal{G}^2$, denote by $T(\mathcal{G}^1, \mathcal{G}^2) = \{x \in M \mid T_x \mathcal{G}^1 \neq T_x \mathcal{G}^2\}$ its transversality (open) set. Now, we will consider only bi-foliations preserved by $\Gamma$. Let $\mathcal{G}^1 \cup \mathcal{G}^2$ and $\mathcal{L}^1 \cup \mathcal{L}^2$ be two of them. Observe then that along $T(\mathcal{G}^1, \mathcal{G}^2)$, $\mathcal{L}^1$ and $\mathcal{L}^2$ are tangent to $\mathcal{G}^1 \cup \mathcal{G}^2$. Indeed, the opposite situation, would give a $\Gamma$-invariant 3- or 4-valued section of $\mathbf{S}^\infty \mathcal{M}$ along some open subset of $T(\mathcal{G}^1, \mathcal{G}^2)$. As above, the barycenter construction (the classical one in this case), yields a $\Gamma$-invariant section of $\mathbf{H} \mathcal{M}$, over some open subset of $\mathcal{M}$. This implies $\Gamma$ is compact, by Fact 11.4.

Observe now that by the above gluing process, one may construct a bi-foliation $\mathcal{M}^1 \cup \mathcal{M}^2$ such that $T(\mathcal{M}^1, \mathcal{M}^2) = T(\mathcal{G}^1, \mathcal{G}^2) \cup T(\mathcal{L}^1, \mathcal{L}^2)$. We construct the wished maximal foliation as one having a maximal transversality set. This exists by compactness of the space of bi-foliations.

Remark 11.3 We close this section by making some remarks on the general set-up of the “bundle compactification” (§11), when we let a generalized dynamical system of $C^1$ diffeomorphisms acting on the bundle $\overline{S_k} \mathcal{M} \to \mathcal{M}$. One considers a given measure $\nu$ preserved by the dynamical system, and choose an invariant measure $\mu$ on $\overline{S_k} \mathcal{M}$ of maximal support and projecting onto $\nu$. It may happen (in fact generically) that $\nu$ has a full measure in $\overline{S_k} \mathcal{M}$, or even worse, $\nu$ can be ergodic. If not, i.e. $\nu$ has support inside $S_k \mathcal{M}$ (that is the support of $\nu$ does not interest $\partial \mathcal{M}$), then the barycenter construction (in general Hadamard spaces) yields an invariant measurable metric. However, if $k > 2$, the universal symmetric space $S_k = SL(k, \mathbf{R})/SO(k)$ is not of negative curvature (i.e. it has a higher rank), and there is no way to construct barycenters for measures supported in the Hadamard boundary. In fact, there are alternative boundaries which may be efficient in this matter. Depending on the interpretation of such a boundary, that is by modeling it as a kind of flag spaces, the construction of limit sets yields, roughly speaking, flag-fields on $\mathcal{M}$. For example, the oselldec’s decomposition for a diffeomorphism $f$, may be handled by looking to points in the limit set (of the group $\{f^n, n \in \mathbf{Z}\}$), with special “approach”. For instance, the conical and horospherical limit sets may be interesting in this regard (see [13]).

12 Partial hyperbolicity

Definition 12.1 An elementary group $\Gamma$ is called partially hyperbolic if its associated bi-foliation $\mathcal{F}^1 \cup \mathcal{F}^2$ is non trivial, that is the (open) transversality locus $T = \{x \in \mathcal{M} \mid T_x \mathcal{F}^1 \neq T_x \mathcal{F}^2\}$ is non-empty, or equivalently, the (closed) coincidence locus $C = \{x \in \mathcal{M} \mid T_x \mathcal{F}^1 = T_x \mathcal{F}^2\}$ is a proper subset of $\mathcal{M}$.

Now, we will justify the word “partially hyperbolic” in a dynamical viewpoint (of course, partially hyperbolic, is also reminiscent to the term “elementary hyperbolic” in the theory of Fuschian groups). We keep the notations above. Denote by $\mathcal{N}^1$ and $\mathcal{N}^2$ the two (one dimensional) normal directions of $\mathcal{F}^1$ and $\mathcal{F}^2$, respectively. We may assume that they are orientable, after passing to a finite covering. Let $X^1$ and $X^2$ be two Lipschitz non singular vector-fields orienting $\mathcal{N}^1$ and $\mathcal{N}^2$, respectively.

The group $\Gamma$ respects the set of (open) connected components of the transversality locus $T$. Since these components have positive Lorentz volume and since $\Gamma$ preserves the volume, then each component $U$ is preserved by a finite index subgroup of $\Gamma$, say $\Gamma$ itself. It then follows that $\Gamma$ preserves the closure $\overline{U}$ and the directions of $X^1$ and $X^2$, along it. Therefore we get two derivative cocycles: $\lambda^1$ and $\lambda^2 : \overline{U} \times \Gamma \to \mathbf{R}$, defined by $D_x f(X^i(x)) = \lambda^i(f, x)X^i(f(x))$, for $i \in \{1, 2\}$. This section is devoted to the proof of:

Theorem 12.2 Let $\Gamma$ be a partially hyperbolic elementary group. Then, up to a subgroup of finite index, $\Gamma$ is a direct product of a compact group by $\mathbf{Z}$ or $\mathbf{R}$. Furthermore, we can find two foliations
\( \mathcal{L}^1 \) and \( \mathcal{L}^2 \) generating the same bi-foliation \( \mathcal{F}^1 \cup \mathcal{F}^2 \), and defining two cocycles \( c^1 \) and \( c^2 \) satisfying the following condition. Let \( g \) be a non trivial element of the \( Z \) or the \( R \) part of \( \Gamma \) and \( K \) a compact subset in the transversality set of the bi-foliation. Then, there is an integer \( p = p(K) > 0 \), such that for \( f = g^p \), we have:

\[
\begin{align*}
\lambda^1(f^n, x) < 1/2, & \quad \text{if } n > 0, x \in K \text{ and } f^n x \in K, \text{ and} \\
\lambda^2(f^n, x) > 2, & \quad \text{if } n > 0, x \in K \text{ and } f^n x \in K.
\end{align*}
\]

Finally, in the transversality set \( T \), the normal foliation of \( \mathcal{L}^1 \) and \( \mathcal{L}^2 \) are respectively the negative and positive Liapunov spaces of \( g \). In particular, \( \mathcal{L}^1 \) and \( \mathcal{L}^2 \) are preserved by \( g \).

**Beginning of the proof.** Consider an auxiliary complete distance \( \rho \) on \( \Gamma \). For example embed \( \Gamma \) is the frame bundle \( P_M \), and take \( \rho \) to be the restriction of a distance on \( P_M \) induced by a complete Riemannian metric. We denote open balls around the identity by \( B^\rho(1, r) \).

**Fact 12.3** With the notations above let \( K \) be a compact of \( U \). There are:

(i) a function: \( r \in \mathbb{R}^+ \to c^K_r \in \mathbb{R}^+ \) such that \( \lim_{r \to \infty} c^K_r = \infty \), and

(ii) a map \( s : K \times \Gamma \to \{1, 2\} \), defined for \( (f, x) \) such that \( f(x) \in K \). We denote \( u(f, x) = s(f, x) + 1 \mod 2 \).

These maps satisfy the condition that whenever, \( f \notin B^\rho(1, r) \), then:

\[
\lambda^u(f, x)(f, x) < (c^K_r)^{-1} \text{ and } \lambda^u(f, x)(f, x) > c^K_r
\]

**Proof.** In \( U \), \( \Gamma \) preserves the directions \( \mathcal{N}^1 \) and \( \mathcal{N}^2 \), and hence also the orthogonal \( (\mathcal{N}^1 \oplus \mathcal{N}^2)^\perp \). This last space is spacelike, i.e. the Lorentz metric restricted to it is positive definite. Therefore, in the compact \( K \) of \( U \), we have an uniform bound of the restriction \( D_x f |(\mathcal{N}^1 \oplus \mathcal{N}^2)^\perp \) and \( (D_x f)^{-1}|(\mathcal{N}^1 \oplus \mathcal{N}^2)^\perp \), for \( x \in K \) and \( f \in \Gamma \), such that \( f(x) \in K \).

It then follows by the volume preservation property that the product \( \lambda^1(f, x)\lambda^2(f, x) \) belongs to some fixed compact interval (around 1) in \( [0, \infty[ \), for \( x \in K \), and \( f \in \Gamma \), with \( f(x) \in K \). That is the cocycles \( \lambda^1 \) and \( \lambda^2 \) are almost one the inverse of the other, provided we restrict ourself to \( K \).

To prove the estimates contained in the statement of the Fact, we argue by contradiction. Suppose that for a divergent sequence \( (f_n) \) in \( \Gamma \), there is a sequence \( (x_n) \) of points of \( K \), such that \( f_n(x_n) \in K \) and such both \( \lambda^1(f_n, x_n) \) and \( \lambda^2(f_n, x_n) \) remain bounded. Hence, \( (D_{x_n} f_n) \) is equicontinuous and thus it follows from \( \lambda^2 \) that \( (f_n) \) is not divergent ! Therefore, there is a function \( c^K_r \), satisfying the property (i) of the Fact, and such that for any \( x \) and \( f \) as in the Fact, there is some \( s = s(f, x) \in \{1, 2\} \), such that the following inequalities hold: \( \lambda^u(f, x) < (c^K_r)^{-1} \) and \( \lambda^u(f, x) > c^K_r \) with \( u = s + 1 \mod 2 \).

**Fact 12.4** There is a partition \( f = A_1 \cup A_2 \) satisfying the following conditions. Let \( U_1 \) (resp. \( U_2 \)) be a neighborhood in (the projective isotropic cone) \( S^\infty M \) of \( \mathcal{N}^1 \) (resp. \( \mathcal{N}^2 \)) along the closure \( U \).

There is \( r \) such for \( f \in \Gamma - B^\rho(1, r) \), we have:

\[
\text{if } f \in A_1, \text{ then: } Df(S^\infty M - U_1) \subset U_2, \text{ and,}
\]

\[
\text{if } f \in A_2, \text{ then: } Df(S^\infty M - U_2) \subset U_1.
\]

**Proof.** It is easily seen that one can localize the compactification of isometry groups of \( M \) to that of \( U \). So, here the boundary of \( \Gamma \) (acting on \( U \)) consists of the two foliations \( \mathcal{F}^1|\bar{U} \) and \( \mathcal{F}^2|\bar{U} \). Let \( B_1 \) and \( B_2 \) be two disjoint neighborhoods of these last foliations in the compact \( \text{Isom}(|\bar{U}|) \). So, for \( r \) big enough, every element \( f \), with \( \rho(1, f) > r \), belongs to exactly one of the neighborhoods \( B_1 \) or \( B_2 \). Suppose that we can’t find \( r \) (big enough) satisfying the claim. Then there is a divergent
sequence $f_n$ contained in $B_1$ or $B_2$ (say $B_1$) and not satisfying the desired inclusions. Thus the approximately stable foliation of $(f_n)$ is $\mathcal{F}^1|\hat{U}$. Now we apply the expanding property of the approximately stable foliation $\mathcal{F}$. This would lead to a contradiction if we check that for some subsequence $(\phi_i)$ of $(f_n)$, the approximately stable foliation $\mathcal{F}(\phi_i^{-1})$ is $\mathcal{F}^2$. For this last fact, we just remark that the opposite situation, i.e. $\mathcal{F}(f_n^{-1}) = \mathcal{F}^1$ is impossible, indeed this would imply that the complementary of a small neighborhood of $\mathcal{N}^1$ is mapped by $Df_n$ to a small neighborhood of $\mathcal{N}^2$. This contradicts the fact that $Df_n$ preserves $\mathcal{N}^2$.

Finally, extend arbitrarily the partial partition $B_1 \cup B_2$ to a partition $\Gamma = A_1 \cup A_2$. ■

**Fact 12.5** $s(f,x)$ and $u(f,x)$ are independent of $x$.

**Proof.** It is clear that by choosing $U_1$ and $U_2$ small enough that for $f \in A_2$ far from the identity map, $s(f,x)$ cannot be 2 for any $x \in K$. Therefore, $s(f) = 1$, and thus does not depend on $x$. ■

**Liapunov exponents.** Henceforth, we will assume that the following hypothesis holds:

**Hypothesis.** $\text{Vol}(K) > (1/2)\text{Vol}(U)$ and choose $r_0$ such that $c^K r_0 > 2$.

**Lemma 12.6** Let $f$ and $g$ be two elements of $\Gamma$ outside the ball $B^0(1,r_0)$ such that $s(f) = s(g)$. Then there is $x \in K$ such that $f(x)$ and $gf(x)$ belong to $K$ and $\lambda^{\mathcal{F}(f)}(gf,x) < 1/4$. In particular, if $gf \notin B^0(1,r_0)$, then $s(gf) = s(f) = s(g)$. Furthermore, if $f_n \notin B^0(1,r_0)$ for all $n > 0$, then $s(f^n) = s(f)$ for all $n > 0$.

**Proof.** Observe that if three subsets $A$, $B$, and $C$ of $U$ have volume $(1/2)\text{Vol}(U)$, then $A \cap B \cap C \neq \emptyset$. Apply this to $K$, $f^{-1}K$ and $gf^{-1}K$, we get a point $x \in K$ such that $f(x) \in K$, and $gf(x) \in K$. Thus $\lambda_1(gf,x) = \lambda_1(g,f(x))\lambda_1(f,x)$, and hence $\lambda(gf,x) < 1/4$, if $f$ and $g$ satisfy the conditions of the Lemma. Of course if $gf \notin B^0(1,r_0)$, then $s(gf) \neq 2$, and thus $s(gf) = 1$. Using this, the last part of the lemma is proved by induction. ■

Consider $l^1(f,x) = \lim_{n \rightarrow +\infty}(\log \lambda(f^n,x))/n$. Define analogously $l^2(f,x)$. They are Liapunov exponents and thus exist almost everywhere.

**Fact 12.7** Let $f$ be such that $f_n \notin B^0(1,r_0)$ for all $n > 0$. Suppose that $s(f) = 1$. Then, for almost every $x \in U$, $l^1(f,x) < 0$, that is $\mathcal{N}^1$ is the negative Liapunov space of $f|U$. Furthermore, $\int_U l^1(f,x)dx < (-(\log 2)\text{Vol}(K))$.

**Proof.** To $x \in K$, associate its sequence of positive return times $(n_i(x))_{i \in \mathbb{N}}$. To simplify notation, fix $x$ and denote the sequence by $(n_i)$. Thus, $f^{n_i}x \in K$. From the Lemma 12.6, $s(f^n) = 1$ for all $n > 0$, and hence $\lambda(f^{n_i+1-n_i},f^{n_i}x) < 1/2$. Thus, by the cocycle property of $\lambda$, $\lambda(f^{n_i},x) < (1/2)^{n_i}$, and hence, $\log \lambda(f^{n_i},x)/n_i < i/n_i(\log 2)$. Let $\chi_K$ denote the characteristic function of $K$. Observe that $i/n_i$ equals the partial Birkhoff sum $(\sum_{i=1}^{n_i}(\chi_K + \chi(fx) + \cdots + \chi(f^{n_i}x))/n_i$. So, we have proved $l^1(f,x) < (-(\log 2)\chi^*_K(x)$, where $\chi^*_K$ stands for the Birkhoff sum of $\chi_K$. In particular $l^1(f,x) < 0$ for almost every $x \in K$. Let $K^*$ be the saturation of $K$, by $f$, $l^1(f,x) < (-(\log 2)\chi_K^*(x)$, for $x \in K^*$, because both of the two functions $l^1(f,x)$ and $\chi_K^*$ are $f$-invariant.

In particular: $\int_{K^*} l^1(f,x)dx < -(\log 2)\int_{K} \chi^*_K = \text{Vol}(K)$.

It remains to prove that $l^1(f,x) < 0$, almost everywhere in $U$. To this end, observe that the sequence generated by $f$ is divergent because $f$ has non vanishing exponents. Hence if we replace $K$ by a bigger compact $K'$, then for some $g = f^p$, $p > 0$, the powers $\{g^n, n > 0\}$ lie outside the ball analogous to $B^0(1,r_0)$ associated to $K'$. Obviously, the index $s^{K'}(g)$ is the same as $s(f)$ and thus equals 1. Therefore, almost everywhere in $K'$, $l^1(f^p,x) < 0$. But $l^1(f,x) = l^1(f^p,x)/p$. ■

Consider now the map $\Lambda^1: \Gamma \rightarrow \mathbb{R}$, $\Lambda^1(f) = \int_U \log \lambda(f,x)dx$. 

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Fact 12.8 \( \Lambda^1 \) is a homomorphism and for \( f \) such that \( f^n \notin B^0(1,r_0) \) for all \( n > 0 \), we have \( |\Lambda^1(f)| > \log 2 \, \text{Vol}(K) \). Furthermore, \( \Lambda^1(f) \neq 0 \) if and only if \( f \) generates a non precompact group (that is \( \{f^n/n \in \mathbb{Z}\} \) is non compact).

Proof. We have \( \log \lambda^1(fg,x) = \log \lambda^1(f,gx) + \log \lambda^1(f,x) \). Thus \( \Lambda^1 \) is a homomorphism because \( g \) is volume preserving and hence: \( f_U \log \lambda^1(f,gx) = f_U \log \lambda^1(f,x) \).

The remaining parts of the fact are obvious or follow from the preceding Fact.  

Corollary 12.9 Suppose that \( \text{Ker} \Lambda^1 \) is compact. Then, up to a subgroup of finite index, \( \Gamma \) is a direct product of a compact group by \( \mathbb{Z} \) or \( \mathbb{R} \).

Proof. Suppose to start with that \( \text{Ker} \Lambda^1 \) is trivial, that is every element \( f \in \Gamma \) generates a non precompact group. Therefore, from Fact 12.8 \( \Lambda^1 \) is injective, and thus \( \Gamma \) is abelian and torsion free. Let \( G \) be a closed subgroup of \( \Gamma \) isomorphic to \( \mathbb{Z}^2 \). It is obvious that for a big enough radius \( r, f \in G - B^0(1,r) \implies f^n \notin B^0(1,r_0) \). But, from Lemma 12.6, \( |\Lambda| \) is bounded from below for elements satisfying this condition. This means that \( \Lambda^1 : G \to \mathbb{R} \) is proper, which is impossible. Thus, because it can’t contain a closed copy of \( \mathbb{Z}^2 \), \( \Gamma \) must be isomorphic to \( \mathbb{Z} \) or \( \mathbb{R} \) (remember that it is torsion free).

Now when \( \text{Ker} \Lambda^1 \) is merely compact, we may argue with the quotient \( \Gamma/\text{Ker} \Lambda^1 \) which enjoys the same properties as \( \Gamma \). So, we obtain that \( \Gamma \) is a semi-direct product of \( \mathbb{Z} \) or \( \mathbb{R} \) by a (normal) compact group. We may find in a standard way a subgroup of finite index which is a direct product of \( \mathbb{Z} \) or \( \mathbb{R} \) by a compact group.  

Torsion. We now check that \( \text{Ker} \Lambda^1 \) is compact. Equivalently, we suppose that \( \Lambda^1 = 0 \) and thus show that \( \Gamma \) is compact. This is standard for the identity component of \( \Gamma \). Indeed, a Lie group for which every element generates a precompact subgroup, is compact (see for instance [3]). Hence, without loss of generality, we may restrict ourselves to the case where \( \Gamma \) is discrete. It is thus a torsion group, and we have to check it is finite.

Let \( f \) be an element of order \( k \). Consider \( I_1(f) = \{i \in \{1, \ldots, k-1\}/f^i \notin B^0(1,r_0), s(f) = 1\} \). Define analogously \( I_2(f) \).

From Lemma 12.6, we have the following “semi-group” property:

Fact 12.10 Let \( \alpha, \beta \in I_1(f) \), then \( \alpha + \beta \in I_1(f) \), unless, \( f^{\alpha+\beta} \in B^0(1,r_0) \). The same statement holds for \( I_2(f) \).

Consider \( P = \{f^n/f \in B^0(1,r_0), n \in \mathbb{Z}\} \). Since \( B^0(1,r_0) \) is finite and any element has finite order, it follows that \( P \) is finite.

We deduce from the Fact above that every (finite cyclic) group intersects non trivially the ball \( B^0(1,r_0) \). Indeed, if not, we would obtain a partition \( \{1, \ldots, k-1\} = I_1(f) \cup I_2(f) \), where \( f \) is a generator of order \( k \) of the given cyclic subgroup. The Fact above implies that only one part, say \( I_1(f) \) is non empty. So, we apply, Lemma 12.6 to \( f \) and \( f^{k-1} \) and we obtain that \( \lambda^1(\text{identity}, x) < 1/4 \) which is obviously impossible.

If \( f \) is of prime order, then, all its non trivial powers generate the same group, and hence \( f \) is a power of some element of \( B^0(1,r_0) \), that is \( f \in P \).

To treat the general case, represent the congruence group \( \mathbb{Z}/k\mathbb{Z} \) as \( \{0, 1, \ldots, (k-1)\} \). Let \( \mathcal{G} = \{\alpha/\langle \alpha, k \rangle = 1\} \) be the set of generators of \( \mathbb{Z}/k\mathbb{Z} \).

Let \( f \) be an element of order \( k \) that does not belong to \( P \), then \( f^\alpha \notin B^0(1,r_0) \), for \( \alpha \in \mathcal{G} \). Hence, we have a partition \( \mathcal{G} = (\mathcal{G} \cap I_1(f)) \cup (\mathcal{G} \cap I_2(f)) \), as above.
Suppose that \( k = p^m \) with \( p \) prime and \( \neq 2 \). Then \( \hat{\alpha} \in G \) if and only \( \alpha \) is not a multiple of \( p \). Therefore, we can’t have \( \hat{\alpha} \) and \( \alpha + 1 \) in \( G \), noting that \( p \neq 2 \). Also, suppose that \( \hat{1} \in I_1(f) \), then \( \hat{2} \in I_2(f) \). Let \( \alpha \) be the smallest number such that \( \hat{\alpha} + 1 \notin I_1(f) \). Then necessarily, \( \hat{\alpha} \in I_1(f) \) and \( \hat{\alpha} + 1 \notin G \), and hence \( \hat{\alpha} + 2 \in I_1(f) \). By recurrence, we get \( I_2(f) \cap G = \emptyset \). In particular \( f \) and \( f^{k-1} \) belong to \( I_1(f) \). We get a contradiction as in the above case where \( k \) was prime.

To treat the case \( p = 2 \), the previous combinatorial approach fails. However, this may be adapted, if we suppose a stronger “semi-group” property of \( I_1(f) \) and \( I_2(f) \), involving three instead of two elements. That is given three elements \( \hat{\alpha} \), \( \hat{\beta} \), and \( \hat{\gamma} \) of \( I_1(f) \), then each of the elements \( \hat{\alpha} + \hat{\beta} \), and \( \hat{\alpha} + \hat{\beta} + \hat{\gamma} \) belongs to \( I_1(f) \), unless it belongs to \( B^0(1, r_0) \). To have this, we choose \( K \) in the Lemma [12.6] with relative big volume (that is \( \text{Vol}(K) > 3/4\text{Vol}(U) \)), and we obtain a statement (of the Lemma) involving three elements of \( \Gamma \).

We have thus proved that \( P \) contains all the elements having order of the form \( p^m \) with \( p \) prime. One may push forward the combinatorial argument to prove that \( P = \Gamma \). Instead, we prefer to argue as follows. Since every cyclic group is generated by groups with order of the form \( p^m \), we deduce that every subgroup \( \Gamma' \) of \( \Gamma \) is generated by its intersection with \( P \). In particular, \( \Gamma' \) is a finitely generated torsion group. Consider the adjoint action of \( \Gamma \) on itself. It preserves the finite set \( P \). Its Kernel \( \Gamma' \) is of finite index in \( \Gamma \), and centralizes \( \Gamma \) because it centralizes the generating set \( P \). Therefore, \( \Gamma' \) is finite as it is a finitely generated abelian torsion group. It then follows that \( \Gamma \) is finite.

End of the proof of Theorem [12.2]. Let \( g \in \Gamma \) be as in Theorem [12.2]. From the previous development, for each component \( U \) of the transversality set \( T \), we can associate \( s(U) \in \{1, 2\} \), such that the normal direction of \( F^{s(U)}|U \) is the negative Liapunov space of the restriction \( g|U \). As in [11], this allows us to construct two foliations \( \mathcal{L}^1 \) and \( \mathcal{L}^2 \), elements of \( \text{Ap}(\mathcal{F}^1, \mathcal{F}^2) \) whose normal directions are the negative and positive spaces of \( g|T \), respectively. Now let \( K \) be a compact subset of \( T \), then it meets only finitely many components \( U \), and hence for a sufficiently big positive integer \( p \), we have the estimates stated in the theorem for the power \( f = g^p \). This ends the proof of the theorem.

13  Proofs of Theorems 2.4, 1.1 and 2.7

The case where the bi-foliation of \( \Gamma \) is trivial. This means there is a foliation \( \mathcal{F}^1 \) such that everywhere \( L(x) = T_x \mathcal{F}^1 \). It then follows from Proposition [13] that the approximately stable foliation of any sequence of \( \text{of } \Gamma \) is \( \mathcal{F}^1 \) and therefore the boundary of \( \Gamma \) consists exactly of \( \mathcal{F}^1 \).

To show the vanishing of the entropy of the elements of \( \Gamma \), we argue by contradiction. Suppose that some \( f \in \Gamma \) has positive entropy with respect to some invariant measure. Then \( f \) must have somewhere non trivial negative and positive Liapunov spaces \( E^1 \) and \( E^2 \). Observe then that these spaces must be isotropic (in the sense of the Lorentz metric) and hence are 1-dimensional. At most only one of these directions, say \( E^1 \), is contained in \( \mathcal{F} \). However, by the uniform attraction of \( \mathcal{F} \) (Proposition [15]), the direction \( E^2 \) is mapped by powers of \( f \), near \( \mathcal{F} \), which contradicts the fact that it is preserved by \( f \).

The case of partially hyperbolic groups. In order to estimate its boundary, it is easy to see that we may replace \( \Gamma \) by a subgroup \( \{ g^n/n \in \mathbb{Z} \} \) generated by an element \( g \) as in Theorem [12.2]. We will thus prove: \( \partial \infty \Gamma = \{ \mathcal{L}^1, \mathcal{L}^2 \} \) (following the notations of Theorem [12.2]). More precisely, we will prove that: \( \lim_{n \to +\infty} g^n = \mathcal{L}^2 \) and \( \lim_{n \to -\infty} g^n = \mathcal{L}^1 \).
Let \((f_n)\) be a sequence of \(\Gamma\) of the form \(f_n = g^{k_n}\), for some sequence of integers \((k_n)\). Suppose that \((f_n)\) has an approximately stable foliation, that is, \((f_n^{-1})\) converges in \(\Gamma \cup \partial_\infty \Gamma\) to some foliation \(F\).

Suppose that \(k_n \to +\infty\), when \(n \to +\infty\). Then, by localization to larger and larger compact subsets, and using Theorem 12.2, one deduces that the limit of \((g^{k_n})\) cannot be different from \(L^2\). Therefore, compactness, \(g^n \to L^1\), when \(n \to +\infty\) (because \(L^2\) is the unique limit of its convergent subsequences). By the same argument \(\lim_{n \to -\infty} g^n = L^1\).

It then follows that any sequence \((f_n = g^{k_n})\) for \((k_n)\) oscillating between \(-\infty\) and \(+\infty\) is not convergent. Hence, the boundary of \(\Gamma\) is \(\{L^1, L^2\}\).

**Proof of Theorems 1.1 and 2.7.** Let \(f\) be an isometry of \(M\) generating a non-equicontinuous subgroup \(\Gamma\). Firstly, observe that \(\Gamma\) is closed. Indeed, the closure of \(\Gamma\) is an abelian Lie group and hence, up to finite index, it can be written as a product \(\Gamma = T^i \times \mathbb{R}^j \times \mathbb{Z}^k\), where \(T\) is the torus part. But, because \(\Gamma\) is non compact and has \(\mathbb{Z}\) as a dense subgroup, then we must have \(\Gamma = \mathbb{Z}\). Therefore, \(\Gamma\) is closed and amenable, and hence elementary.

We apply Theorem 2.4 and we get, with the help of the previous notations, \(L^1\) (resp. \(L^2\)) as an approximately stable (resp. unstable) foliation for \(f\).

The weak partial ergodicity part of Theorem 1.1 follows from the analogous statement in Theorem 1.2. This ends the proof of Theorem 1.1.

The same argument yields approximately stable and unstable foliations for isometric flows, as stated in Theorem 2.7. The statement concerning the causal character of non-equicontinuous flows, that is, their infinitesimal generators must be non-timelike, was noticed in [19].

### 14 Non bi-polarized manifolds. Proof of Theorem 2.9

Observe that, over \(U\), TAS is Lorentzian, that is the restriction of the metric along TAS is of Lorentzian type. Indeed, for \(x \in U\), TAS\(x\) contains at least two different isotropic directions.

Let \(K\) be a compact subset of \(U\) over which \(E\) is continuous. We will consider as in 8.2, the non escaping set subset \(NE(K, (f_n))\), where \((f_n)\) is sequence in Isom\((M)\). We firstly present the following relative version of Corollary 9.6.

**Fact 14.1** Let \(K\) be a compact subset of \(U\) over which \(E\) is continuous. Moreover, let \(F = \lim f_n \in \partial_{\infty} \text{Isom}(M)\), and \(X\) (resp. \(Y\)) a vector field tangent to \(F^\perp\) (resp. \(F\)). Then, there is a sequence of continuous sections \((X_n)\) (resp. \((Y_n)\)) of \(E\) over \(K\), such that \(X_n \to X\) (resp. \(Y_n \to Y\)) in the \(C^0\) topology, and satisfying the following. Choose an auxiliary norm \(|.|\) on \(TM\). Then, there is a real sequence \((a_n)\) converging to 0 (resp. a bounded real sequence \((b_n)\)) such that \(|Df_n X_n(x_n)| < a_n\) (resp. \(|Df_n Y_n(x_n)| < b_n\) whenever \(x_n \in K \cap f_n^{-1}K\).

**Fact 14.2** Let \(F \in \partial_{\infty} \text{Isom}(M)\). Then: \(t(u, v) = 0\) for \(u \in T_x F^\perp, v \in T_x F \cap E_x\), and almost every \(x \in U\),

**Proof.** Let \(K\) be a compact subset over which \(E\) and \(t\) are continuous. Let \(x \in NE(K, (f_n))\) and \(u, v\) as in the Fact. Extend \(u\) and \(v\) to local vector fields \(X\) and \(Y\) tangent to \(F^\perp\) and \(F\), respectively. Approximate \(X\) and \(Y\) as in Corollary 9.6, and consider a sequence \(x_n \to x\) in \(K\), such that \(f_n x_n \to x\). Thus, by continuity, \(t(u, v) = \lim_{n \to \infty} t(X(x_n), Y(x_n))\) which equals 0, from the properties of \(X_n\) and \(Y_n\) in Corollary 9.6.
Finally, recall that (see 8.2) \( \text{Vol}NE(K, (f_n)) > \text{Vol}(U) - 2\epsilon \), whenever \( \text{Vol}(K) > \text{Vol}(U) \). Therefore, the property holds is almost everywhere because we can choose the volume of \( K \) arbitrarily approaching that of \( U \).

It then follows that \( t(u, v) = 0 \) if \( u \in S_x \) and \( v \in u^\perp \cap E_x \), for almost every \( x \in U \).

Let \( \{ e_i \} \) be a basis of \( T_x M \), and write \( t(X, Y) = \Sigma b_i(X, Y)e_i \) where the \( b_i \) are bilinear scalar forms. To prove Theorem 2.9, it suffices to show that for each \( b_i \) there is \( \alpha_i \) such that \( b_i(u, v) = \alpha_i < u, v > \), whenever \( u \in TAS_x \).

So let \( b \) be one of these forms and write it \( b(u, v) = < Au, v > \) for some linear endomorphism of \( E_x \). From the above, one sees that every \( u \in S_x \), is an eigenvector for \( A \), \( Au = \lambda_u u \). Now, if \( \lambda_u \) does not depend on \( u \), \( A \) is a homothety on \( TAS_x \) and we are done. If not, \( A \) induces an eigenspace decomposition on \( E_x \).

By considering all the endomorphisms corresponding to the \( b_i \), and by letting \( x \) vary over \( U \), we get an invariant measurable decomposition \( E = E^1 \oplus \ldots \oplus E^k \) of \( TAS \) such that \( S_x \subset E^1_x \cup \ldots \cup E^k_x \). To show that this is impossible (and hence finishing the proof of the theorem) we use the following irreducibility fact:

**Fact 14.3** There are no \( \text{Isom}(M) \)-invariant measurable subbundles \( E^1, \ldots, E^k \) of \( TAS \), with \( k \geq 2 \), such that \( S_x \subset E^1_x \cup \ldots \cup E^k_x \) in some subset of \( U \) with positive volume (it is here where we use that \( \text{Isom}(M) \) is non elementary).

**Proof.** We may suppose \( k = 2 \) and that \( E^1 \) is Lorentzian in some subset of positive volume. Since we are dealing with measurable bundles, we may suppose this is everywhere true, just restricting domains of definition. So, we have an orthogonal decomposition \( TM = E^1 \oplus E^1 \perp \).

Let \( F = \lim f_n \in \partial_\infty \text{Isom}(M) \) and apply Fact 14.1 to a compact \( K \) over which \( E^1 \) and \( E^2 \) are continuous. Observe that if \( u_n \in E^1_{x_n} \rightarrow u \neq 0 \), and \( f_n x_n \in K \), then \( (Df_n u_n) \) can’t tend to 0 because the metric on \( E^1 \perp \) is Riemannian (i.e. positive definite). This implies as in the proof of the Fact above, that, over \( NE(K, (f_n)) \), \( F \perp \) is contained in \( E^1 \). Choosing \( K \) with larger volume as necessary, we may conclude that this is always true.

To finish the proof of Theorem 2.9, it remains to check its last partial ergodicity statement. Let \( \sigma \) be a function as in the theorem. From the ergodicity property in Theorem 1.2, \( \sigma \) is constant along any (1-dimensional) foliation defined by a vector field \( X \in S \) (see the notations of §2.4). Therefore, by definition \( d\sigma|TAS = 0 \).

### 15 Examples

Here we will give some examples of boundaries of isometry groups of compact Lorentz manifolds. As we will see in the part II of this work, the non-trivial (i.e. non bi-polarized) cases, involve constant curvature manifolds. In what follows, we therefore investigate the structure of isometry groups of manifolds which are locally isometric to a product of a Riemannian manifold by a constant curvature Lorentz manifold. The following theorem summarizes the non trivial cases.

**Theorem 15.1** Let \( M \) be a compact Lorentz manifold whose universal cover is a product of a simply connected Riemannian manifold \( \tilde{N} \) and a complete simply connected space \( X_c \) of constant curvature \( c \). Let \( \pi \subset \text{Isom}\tilde{N} \times \text{Isom}(X_c) \). Then,

(i) Such a manifold does not exist if \( c > 0 \).

(ii) If \( c > 0 \) and \( M \) is not bi-polarized, then \( \dim M = 3 \), and \( M \) is a quotient of \( \tilde{N} \times SL(2, \mathbb{R}) \) by a subgroup of \( \text{Isom}\tilde{N} \times SL(2, \mathbb{R}) \). The isometry group of \( M \) is a product of a compact group
by a finite cover of $\text{PSL}(2, \mathbb{R})$. Its boundary is the circle $S^1$ endowed with the usual action of this latter group.

(iii) If $c = 0$ and $M$ is not bi-polarized, then there is a metric decomposition $\tilde{M} = \tilde{N} \times \mathbb{R}^d$, where $\tilde{N}$ is a Riemannian manifold, and $\mathbb{R}^d$ is a Minkowski space, and such that $\pi = \pi_1(M) \subset \text{Isom}\tilde{N} \times \mathbb{R}^d$. The action of $\text{Isom}(M)$ on its boundary factors through the action of a (arithmetic) lattice of a $(d - 1)$-dimensional hyperbolic space on its sphere at infinity.

Remark 15.2 (Erratum) The case (iii), i.e. the flat non-bi-polarized case, was forgotten in [21]. So, the statement of the principal result in this reference has to be modified to take into account this case, see the part II of the present article.

Observe that that we don’t address completeness questions, which will be treated in detail in the part II of this article. The theorem will be proved through the present section, which contains further details. Let’s start with the following general fact about de Rham decomposition of Lorentz manifolds, which follows from the foliated version 8.7.

**Proposition 15.3** Let $M$ be a compact Lorentz manifold whose universal cover admits a de Rham decomposition $\tilde{M} = \tilde{N} \times X$, where $\tilde{N}$ is Riemannian and $X$ is Lorentzian. Then any foliation $\mathcal{F}$, element of the boundary of $\text{Isom}(M)$ lifts to a foliation $\mathcal{F}'$ containing the factor $\tilde{N}$, that is its leaves have the form $\tilde{N} \times \mathcal{L}$, where $\mathcal{L}$ is a codimension one lightlike geodesic foliation of $X$, invariant under the action of the projection of $\pi_1(M)$ in $\text{Isom}(X)$.

### 15.1 De Sitter space

Let $\mathbb{R}^{p,q}$ be the space $\mathbb{R}^{p+q}$ endowed with a non degenerate quadratic form of signature $(-p, q)$ (for example $\mathbb{R}^{1, q}$ is endowed with a form of signature $- + \ldots +$). For a real $r$, we denote by $S^{p,q}(r)$ the level $r$ in $\mathbb{R}^{p,q}$. The (universal) de Sitter space of dimension $q$ is $S^{1,q}(+1)$. We only consider the case $q > 2$, and hence, the de Sitter space is simply connected. It inherits from $\mathbb{R}^{1,q}$ a Lorentz metric of positive constant curvature, and has isometry group $O(1, q)$.

The well known Calabi-Markus phenomena states that a Lorentz manifold covered by $S^{1,q}(+1)$ has a finite fundamental group. Let $\tilde{N}$ be a complete simply connected manifold and consider the product $\tilde{M} = \tilde{N} \times S^{1,q}(+1)$. This may have quotients with large fundamental group, but no compact ones. The proof of this claim resembles that of the Calabi-Markus phenomena and goes as follows. Suppose that $\pi \subset \text{Isom}\tilde{N} \times O(1, q)$ is the fundamental group of a compact manifold $M$. Then, the projection of $\pi$ on $\text{Isom}\tilde{N}$ is not discrete, otherwise, $M$ would fiber over a quotient of $\tilde{N}$, with fiber a compact quotient of $S^{1,q}(+1)$, which does not exist. Hence, there is a divergent sequence $\gamma_n = (g_n, h_n) \in \pi$ such that $g_n \to 1$ in $\text{Isom}\tilde{N}$. The basic fact behind the Calabi-Markus phenomena is that if $K = S^{1,q}(+1) \cap \mathbb{R}^{0,q}$, then $h(K) \cap K \neq \emptyset$ for any $h \in O(1, q)$ (because both $K$ and $h(K)$ are traces in $S^{1,q}(+1)$ of linear hyperplanes in $\mathbb{R}^{1,q}$). Therefore, for any open set $U \subset \tilde{N}$, $\gamma_n(U \times K) \cap (U \times K) \neq \emptyset$, because $g_n \to 1$. This means that $\pi$ does not act properly on $\tilde{N} \times S^{1,q}(+1)$.

### 15.2 Anti de Sitter manifolds

The anti de Sitter space of dimension $1 + q$ corresponds to the level $-1$ in $\mathbb{R}^{2,q}$. More precisely, the (universal) anti de Sitter space $\mathbb{H}^{1,q}$ is the universal cover of $S^{2,q}(-1)$. However, it is more convenient to work in the “linear model” $S^{2,q}(-1)$ and then, translate into $\mathbb{H}^{1,q}$. In fact one would hope that, one needs to pass to $\mathbb{H}^{1,q}$ only in pathological situations. Indeed, as we will recall below
compact quotients of $H^{1,q}$ are in fact quotients of $S^{2,q}(-1)$. So let’s work in this latter space. Let $Q = -x_1^2 - x_2^2 + x_3^2 + ... x_{q+2}^2$ be a quadratic form defining $R^{2,q}$. Then $S^{2,q}(-1)$ inherits a Lorentz metric of negative constant curvature and its isometry group is the orthogonal group of $Q$, that is $O(2,q)$. The totally geodesic subspaces of $S^{2,q}(-1)$ are exactly the traces on it of linear subspaces of $R^{2,q+1}$. Furthermore, the lightlike geodesic hypersurfaces of $S^{2,q}(-1)$ have the form: $H_u = Ru^1 \cap S^{2,q}(-1)$ for $u \in R^{2,q}$ an isotropic vector.

Two hypersurfaces $H_u$ and $H_v$ (for $u$ and $v$ isotropic) are disjoint if and only if $u$ and $v$ are orthogonal but not collinear. It follows that a codimension one lightlike geodesic foliation $F$ of $S^{2,q}(-1)$ is determined by hypersurfaces $H_u$, for $u$ running over an isotropic 2-plane $P$. The group $O(2,q)$ acts transitively on the space of isotropic 2-planes. Its action on the space of pairs $(P_1, P_2)$ of isotropic 2-planes has exactly 3 orbits, one for $P_1 = P_2$, one for $\text{dim}(P_1 \cap P_2) = 1$, and the last for $P_1 \cap P_2 = 0$.

We can write $R^{2,q} = R^{2,2} \oplus R^{0,q}$. Because the factor $R^{0,2}$ is Riemannian, most of the dynamics happens on $R^{2,2}$. Note that the decomposition above is not canonical, but we will see this does not matter. So we now decorticate the case $q = 2$, i.e. the 3-dimensional anti de Sitter space. Instead of the standard form $Q$, one consider the form $Q'$ on $R^4 = R^2 \times R^2$ defined by $Q'(u,v) = \omega(u,v)$, where $\omega$ is the volume form on $R^2$. An element $A \in SL(2,R)$ acts diagonally: $A(u,v) = (Au, Av)$, by preserving $Q'$. The group $SL(2,R)$ acts freely on the non-vanishing levels of $Q'$. Hence, the $SL(2,R)$ orbits coincide (for dimension reasons) with the components of non vanishing levels. It turns out that metrics defined in that way on $SL(2,R)$ are multiples of its Killing form. Indeed, one verifies that the identity component of $O(2,2)$ contains another copy of $SL(2,R)$ commuting with the given $SL(2,R)$-action. More precisely, this identity component is isomorphic to $SL(2,R) \times SL(2,R)$, which turns out to be the identity component of the isometry group of the Killing form of $SL(2,R)$, acting by: $(g,h)x = gxh^{-1}$, where $g$, $h$, and $x$ belong to $SL(2,R)$.

A lightlike geodesic foliation is determined by the orbits of a subgroup of the form $A \times \{1\}$ or $\{1\} \times A$ where $A$ is conjugate to the affine group $AG \subset SL(2,R)$. The case of two different subgroups lying in the same factor $\{1\} \times SL(2,R)$ or $SL(2,R) \times \{1\}$ corresponds to two isotropic 2-planes such that $P_1 \cap P_2 = 0$. The case of different factors corresponds to two intersecting 2-planes. It then follows that the symmetry group of a pair or isotropic 2-planes $(P_1, P_2)$ is conjugate up to swith of factors to either $SL(2,R) \times \{1\}$ or $AG \times AG$.

For example, the group $SL(2,R)$ acting as above on $(R^4, Q')$ preserves each isotropic 2-plane of the form $\{\{(u, au), u \in R^2\}$ for $a$ a real number, together with the plane $\{0\} \times R^2$. The set of the so defined foliations is a circle, on which the other factor $SL(2,R)$ acts as usually.

From this we deduce that the symmetry group of 3 distinguish isotropic 2-planes is contained up to swith of factors in $SL(2,R) \times \{1\}$. In particular, this symmetry group is centralized by the other factor $\{1\} \times SL(2,R)$. Therefore, if a compact quotient $M$ of a product $\tilde{N} \times S^{2,q}(-1)$, is not bi-polarized and has non-compact isometry group, then Isom$(M)$ contains $SL(2,R)$. From (see also [1] and [10]), Isom$(M)$ is a product of a compact group by a finite cover of $PSL(2,R)$.

In higher dimensions the symmetry group of 3 isotropic 2-planes is contained in a product of $SL(2,R)$ by a compact group. This does not act co-compactly on $S^{2,q}(-1)$. By translating the argumentation to the universal cover $H^{1,q}$, one checks that a co-compact subgroup cannot preserve 3 distinct codimensions a lightlike geodesic foliations. This proves the claim that if a compact Lorentz manifold has a non-compact isometry group and is not bi-polarized, then the anti de Sitter factor of its universal cover has dimension 3.

**Finiteness of levels.** As mentionned above, a compact Lorentz manifold $M$, which is a quotient of the universal anti de Sitter space, is in fact up to finite covers, a quotient of the more concrete one $S^{2,q}(-1)$. This property was called in [1] the finitness of levels of compact anti de Sitter manifolds.
It is related to isometry groups as follows. The fundamental group of $S^{2,q}(-1)$ is cyclic, generated by an element $\sigma$. The statement is that some power of $\sigma$ belongs to $\pi_1(M)$. Because $\sigma$ is central in Isom$(H^{1,q})$, it defines an isometry $f$ of $M$. The finiteness of the level of $M$ is equivalent to $f$ having finite order. Thus the opposite situation implies Isom$(M)$ is infinite. If the identity component of Isom$(M)$ is not trivial, we get a connected subgroup of Isom$(H^{1,q})$ centralizing $\pi_1(M)$. One can thus completely understand this latter group, and in particular get a contradiction to the hypothesis that $f$ is of infinite order. If Isom$(M)$ is discrete, then it is non-compact, and therefore, it preserves a foliation, and thus $\pi_1(M)$ is contained in the symmetry group of a foliation as described above. Again here, by working algebraically, one gets a contradiction to the fact that $f$ has an infinite order.

15.3 The flat case

Let $g$ be a Lorentz quadratic form on $\mathbb{R}^k$. It is called rational, if some multiple $\alpha g$ has rational coefficients when expressed in the canonical basis. The isometry group of the Lorentz space $(\mathbb{R}^k, g)$ is the semi-direct product $\mathbb{R}^k \rtimes O(g)$ where $O(g) \subset GL(n, \mathbb{R})$ is the orthogonal group of $g$. Of course all these spaces are isometric to the Minkowski space. However, this representation may help to understand the modulus of flat Lorentz structures. For instance, consider a topological torus $T^k = \mathbb{R}^k / \mathbb{Z}^k$, then the family $(T^k, g)$, for $g$ as above, exhausts the space of Lorentz flat structures on $T^k$.

A lightlike geodesic foliation in $\mathbb{R}^k$ consists obviously of parallel lightlike affine hyperplanes, and hence it is determined by specifying an isotropic direction of $g$. Let $C(g)$ be the space of such directions. It is a sphere of dimension $k - 2$. The action of $\mathbb{R}^k \rtimes O(g)$ on it, factors via the usual conformal action of $O(g)$ (i.e. it is isomorphic to the conformal action of $O(1, k - 1)$ on $S^{k-2}$).

Let $M$ be a flat Lorentz manifold obtained as a quotient of $(\mathbb{R}^k, g)$ by a subgroup $\pi \subset \mathbb{R}^k \rtimes O(g)$. The space $\mathcal{F}G(M)$ of codimension 1 lightlike geodesic foliations of $M$, is identified with the fixed points of the action of $\pi$ (via its linear part) on $C(g)$. It follows in particular, that in the torus case, this space is identified with $C(g)$, for any $g$. Conversely, if $\operatorname{card}(\mathcal{F}G(M)) \geq 3$, then up to a subgroup of finite index, $\pi$ is a lattice in the translation part $\mathbb{R}^k$ of $\mathbb{R}^k \rtimes O(g)$. To see this, let $E$ be the linear space generated by the isotropic directions determined by the elements of $\mathcal{F}G(M)$. Because of the dimension, the quadratic form $g$ is of Lorentz type on $E$ and is positive on $E^\perp$.

Observe now that if an element $A \in O(g)$ fixes 3 isotropic directions $X_1$, $X_2$ and $X_3$, then $A = \pm$ Identity on on the linear space $F$ that they generate. Indeed, write $AX_i = \lambda_i X_i$, then $\lambda_i \lambda_j = 1$ for $i \neq j$, because $<X_i, X_j> \neq 0$, and hence $\lambda_i = \pm 1$.

It then follows that $\Gamma$ preserves a positive scalar product. Therefore, by the Bieberbach Theorem, $\pi$ is a lattice in $\mathbb{R}^k$. Equivalently $M$ is covered by a torus.

As a corollary, we get that a compact flat manifold is bi-polarized, unless it is, up to a finite cover, a torus. Now for a torus $(T^k, g)$, its isometry group is $(\mathbb{Z}^k) \rtimes \mathbb{R}^k$, where $O(g, \mathbb{Z}) = O(g) \cap GL(k, \mathbb{Z})$.

The action of Isom$(T^k, g)$ on its boundary can be identified with that of $O(g, \mathbb{Z})$ on its limit set in $C(g)$. So, we are not leaving Kleinian groups, but what kind of group could be $O(g, \mathbb{Z})$ and what it might have as a limit set?

A classical theorem of Harish-Chandra and Borel states that $O(g, \mathbb{Z})$ is a lattice in $O(g)$ if $g$ is rational. Hence its limit set in this case is the whole of $C(g)$. Let’s now treat the general case (that is when $g$ may be partially rational).

**Fact 15.4** Let $G$ be a non-compact connected Lie subgroup of $O(g)$. Then either $G$ fixes a point of $C(g)$ (that is $G$ is elementary) or $G$ is reductive. In this last case, there is a Lorentzian plane $E$ in $\mathbb{R}^k$ (that is $g$ restricted to $E$ is of Lorentzian type) which is preserved by $G$ and such that
G = O(g|E) × K, where K is a compact subgroup of O(g|E^⊥). (Here for a non degenerate plane F, we denote O(g|F) = \{A ∈ O(g)/A(F) = F and A|F^⊥ = Identity\}).

**Proof.** The proof is standard and briefly the idea is as follows. The radical R of G fixes a point of C(g). If it is non compact, R contains a parabolic or hyperbolic one parameter group. It then follows that R fixes at most two points, and in particular, G also fixes these points, and is thus elementary. Therefore, if G is not elementary, then its radical is compact. Hence, it is by definition reductive. The remaining part of the Fact is standard.

Let G be the Zariski closure of O(g, Z). It is defined over Q (see for instance [8]). Therefore, if non elementary, O(g, Z) is a lattice in G, by the Harish-Chandra-Borel Theorem, for general reductive Q-groups. Thus O(g, Z) may be thought as a lattice of O(g|E), and so its limit set is the projective isotopic cone of g|E.

**Partial rationality.** In fact, it seems that we have more precise partial rationality when O(g, Z) is non elementary, that is E and E^⊥ seem to be rational. We were able to check it, assuming O(g, Z) non-cocompact in G. Indeed in this case, O(g, Z) possesses parabolic elements, that is there exist an element of O(g, Z) of the form A = (B, C) ∈ O(g|E) × K such that B is unipotent, i.e. B − 1 is nilpotent. In fact unipotent elements of orthogonal groups of Lorentz forms, have degree of nilpotency 3. Let A be the set of A ∈ O(g, Z) such that B is unipotent.

Then E is contained in the vector subspace F = ∩_{A ∈ A} ker(A − 1)^3, which is rational.

Observe that if A ∈ A, then its projection C in K, acts trivially on F' = F ∩ E^⊥, because there it detemines an unipotent element in a compact group. In other words, F^⊥ ⊂ ∩_{A ∈ A}(ker(A − 1). In fact, we have equality. Otherwise, the intresection E ∩ (∩_{A ∈ A} ker(A − 1) would give a proper subspace of E invariant by O(g, Z); which is easily seen to be impossible. It then follows that F' is rational. By similar arguments, one sees that E = ∩_{A ∈ A}(A − 1)(F), and is hence rational. Again, consider F'' = ∩_{A ∈ A} Image(A − 1)^3. This is a rational space, which is (by similar argument) a supplementary of F' in E^⊥. Hence E^⊥ is rational.

In conclusion, at least when O(g, Z) is non elementary and non co-compact, then up to a finite cover, M is a metric product of a Lorentz rational torus with a Riemannian torus.

**The product case.** Suppose now that M is a quotient of a product N × R^k, where N is Riemannian and R^k is a Minkowski space. As above, assuming M is not bi-polarized, we construct another invariant metric decomposition ˜M = N' × R^{d'}, where the projection of π_1(M) on the isometry group of the Minkowski space R^d consists of translations, only. Here R^d corresponds to the space E defined above and N' = N × E^⊥. Now we can write Isom(M) = Nor(π_1(M))/π_1(M) where Nor(π_1(M)) is the normalizer of π_1(M) in Isom(N') × Isom(R^{d'}). Nevertheless, it is not obvious how to exploit further such a formula in an algebraic way, and so we argue as follows. Observe that the translations along R^d centralize π_1(M) and thus determine an isometric action of R^d on M. This action is contained in a compact group because of its obvious equicontinuity. Therefore, we get a torus T^k in Isom(M). This torus inherits a metric defined on its Lie algebra T^k, by means of the formula: <X, Y> = f <X(x), Y(x)>, where X and Y are Killing fields generating flows in T^k. Parallel fields tangent to R^d, allow us to embed isometrically the Minkowski space R^d in (T^k, <,>). In fact, as M itself, T^k admits an orthogonal decomposition A ⊕ R^d, such that the scalar product <,> on A is positive definite, and hence the metric on T^k is Lorentzian. This construction is natural in all its steps, and therefore we have succeeded to essentially incorporate questions about the isometry group of M into ones relative to T^k. In particular one sees that the action of Isom(M) on its boundary factors through the action of an arithmetic lattice of the hyperbolic space H^{d−1} on its boundary S^{d−2}.
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