Core and core-EP inverses of tensors

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Received: 21 June 2019 / Revised: 16 August 2019 / Accepted: 5 October 2019 / Published online: 29 October 2019
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Abstract
Specific definitions of the core and core-EP inverses of complex tensors are introduced. Some characterizations, representations and properties of the core and core-EP inverses are investigated. The results are verified using specific algebraic approach, based on proposed definitions and previously verified properties. The approach used here is new even in the matrix case.

Keywords Generalized inverse \cdot Core-EP inverse \cdot Core inverse \cdot Tensor \cdot Einstein product

Mathematics Subject Classification 15A69 \cdot 15A09

Communicated by Jinyun Yuan.

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1 Introduction, background and motivation

The core inverse for matrices was introduced by Baksalary and Trenkler (2010, 2014). However, Ben-Israel and Greville (2003) and Cline (1968) proposed the first right weak generalized inverse, which was renamed as the core inverse. Several properties of the core inverse and interconnections with other generalized inverses were further studied by many researchers (Baksalary and Trenkler 2010; Kurata 2018; Rakić et al. 2014a; Wang and Liu 2015; Xu et al. 2017). Perturbation bounds of the core inverse were considered in Ma (2018). An extension of the core inverse in the finite dimensional space to the set of bounded Hilbert space operators was proposed in Rakić et al. (2014b). Rakić et al. (2014a) extended the notion of the core inverse from the complex matrix space to an arbitrary $\ast$-ring.

Prasad and Raj (2018) introduced a bordering method for computing the core-EP inverse by relating this generalized inverse with a suitable bordered matrix. Some further properties of the core-EP inverse were investigated in Ferreyra et al. (2018). Prasad et al. (2018) introduced an iterative method to approximate the core-EP inverse. Three limit representations of the core-EP inverse were proposed in Zhou et al. (2018). The notion of the core inverse was generalized to the core-EP inverse. Many characterizations of the core-EP inverse with other inverses are discussed in Gao and Chen (2018) and Prasad and Mohana (2014). The core-EP was extended to rectangular matrices in Ferreyra et al. (2018) and Gao et al. (2018a). Some new characterizations, representations and the perturbation bounds of the core-EP and the weighted core-EP were investigated in Ma (2019) and Ma and Stanimirović (2019).

The tensor inversion and generalized inversion based on different tensor products have been a frequent topic for investigation in the available literature. The representations and properties of the tensor inverse were considered in Brazell et al. (2013). Further, Liang et al. (2019) investigated necessary and sufficient conditions for the invertibility of a given tensor. Representations and properties of the Moore–Penrose inverse of tensors were derived in Behera and Mishra (2017) and Sun et al. (2016). Representations for the weighted Moore–Penrose inverse of tensors were considered in Ji and Wei (2017). Panigrahy and Mishra (2018) investigated the Moore–Penrose inverse of the Einstein product of two tensors. Ji and Wei (2018) investigated the Drazin inverse of even-order square tensors under the Einstein product. More details can be found in two recent monographs (Qi and Luo 2017; Wei and Ding 2016).

So, it is observable that the core and core-EP inverse are not investigated in the tensor case, so far. The targets of our research, in the present article, are core and core-EP inverses of tensors. Corresponding definitions are introduced and main properties are investigated.

Main contributions of this manuscript can be summarized as follows.

(1) Definitions of the core and core-EP inverses of tensors are introduced.
(2) Some characterizations and properties of the core and core-EP inverses are investigated.
(3) The results are verified using one specific algebraic approach, which is new even in the matrix case.

The rest of the paper is organized as follows. Some necessary notations, useful known results and definitions are presented in Sect. 2. Definition, basic properties and representations of the core inverse are considered in Sect. 3. The core inverse of the sum of two tensors is investigated in Sect. 3.1. Section 4 is aimed to the core-EP inverse of a square tensor. Some properties, representations and characterizations of the core-EP inverse are given. Illustrative numerical examples are given in Sect. 5. Concluding remarks are stated in Sect. 6.
2 Preliminaries

For convenience, we first briefly explain some of the terminologies which will be used here onwards. We refer to $\mathbb{C}^{I_1 \times \cdots \times I_N}$ (resp. $\mathbb{R}^{I_1 \times \cdots \times I_N}$) as the set of order $N$ complex (resp. real) tensors. Particularly, a matrix is a second-order tensor and a vector is a first-order tensor. Each entry of a tensor $A \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ is denoted by $A_{i_1 \ldots i_N}$. Note that throughout the paper, tensors are represented in calligraphic letters like $\mathcal{A}$ and the notation $\mathcal{A}_{i_1 \ldots i_N}$ represents the scalars. The Einstein product (Einstein 1916) $\mathcal{A} \ast_N \mathcal{B}$ of tensors $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_N}$ and $\mathcal{B} \in \mathbb{C}^{K_1 \times \cdots \times K_N \times J_1 \times \cdots \times J_M}$ is defined by the operation $\ast_N$ via

$$\quad (\mathcal{A} \ast_N \mathcal{B})_{i_1 \ldots i_N j_1 \ldots j_M} = \sum_{k_1 \ldots k_N} A_{i_1 \ldots i_N k_1 \ldots k_N} B_{k_1 \ldots k_N j_1 \ldots j_M} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_M}. \quad (2.1)$$

Specifically, if $\mathcal{B} \in \mathbb{C}^{K_1 \times \cdots \times K_N}$, then $\mathcal{A} \ast_N \mathcal{B} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ and

$$\quad (\mathcal{A} \ast_N \mathcal{B})_{i_1 \ldots i_N} = \sum_{k_1 \ldots k_N} A_{i_1 \ldots i_N k_1 \ldots k_N} B_{k_1 \ldots k_N}.$$

The Einstein product is discussed in the area of continuum mechanics (Einstein 1916) and the theory of relativity (Lai et al. 2009). Further, the sum of two tensors $\mathcal{A}$, $\mathcal{B} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_N}$ is

$$\quad (\mathcal{A} + \mathcal{B})_{i_1 \ldots i_N k_1 \ldots k_N} = A_{i_1 \ldots i_N k_1 \ldots k_N} + A_{i_1 \ldots i_N k_1 \ldots k_N}. \quad (2.2)$$

For a tensor $\mathcal{A} = (A_{i_1 \ldots i_M j_1 \ldots j_N}) \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$, the tensor $\mathcal{A}^T = (A_{j_1 \ldots j_N i_1 \ldots i_M}) \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ is the transpose of $\mathcal{A}$. The conjugate transpose of a tensor $\mathcal{A} = (A_{i_1 \ldots i_M j_1 \ldots j_N}) \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ is denoted by $\mathcal{A}^*$ and elementwise defined as $(\mathcal{A}^*)_j = (\overline{A})_{i_1 \ldots i_M j} = (\mathcal{A})_{i_1 \ldots i_M j} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ where the over-line means the conjugate operator. Further, a tensor $\mathcal{O}$ denotes the zero tensor if all the entries are zero. A tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ is symmetric if $\mathcal{A} = \mathcal{A}^T$, skew-symmetric if $\mathcal{A} = -\mathcal{A}^T$, and orthogonal if $\mathcal{A}^* \mathcal{A} = \mathcal{A}^T \mathcal{A} = \mathbb{I}$. A tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ is idempotent if $\mathcal{A} \ast \mathcal{A} = \mathcal{A}$ and tripotent if $\mathcal{A}^3 = \mathcal{A}$. Further, a tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ is called Hermitian idempotent if $\mathcal{A}^2 = \mathcal{A} = \mathcal{A}^\dagger$ or $\mathcal{A}^2 = \mathcal{A} = \mathcal{A}^\ast$.

The additional notation $I(N) = I_1 \times \cdots \times I_N$ will be useful in increasing the efficiency of the presentation. Accordingly, the tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ will be denoted in a simpler form as $\mathcal{A} \in \mathbb{C}^{I(M) \times J(N)}$. Tensors of the form $\mathbb{C}^{I(N) \times I(N)}$ are known as even-order tensors. Following the terminology from Ji and Wei (2018), even-order tensors of the shape $\mathbb{C}^{I(N) \times I(N)}$ will be termed as even-order square tensors, or simply square tensors.

**Definition 2.1** Let $\mathcal{A} \in \mathbb{C}^{I(N) \times I(N)}$. Then, the tensor $\mathcal{A}$ is called EP if $\mathcal{A} \ast_N \mathcal{A}^\dagger = \mathcal{A}^\ast \ast_N \mathcal{A}$.

**Definition 2.2** A tensor $\mathcal{A} \in \mathbb{C}^{I(M) \times J(N)}$ is called partial isometry if $\mathcal{A} \ast_N \mathcal{A}^\ast \ast_M \mathcal{A} = \mathcal{A}$ or $\mathcal{A}^\dagger = \mathcal{A}^\ast$.

The definition of a diagonal tensor follows from Sun et al. (2016).

**Definition 2.3** (Sun et al. 2016) A tensor with entries $\mathcal{D}_{i_1 \ldots i_N j_1 \ldots j_N}$ is called a diagonal tensor if $\mathcal{D}_{i_1 \ldots i_N j_1 \ldots j_N} = 0$ for $(i_1, \ldots, i_N) \neq (j_1, \ldots, j_N)$.

We recall the definition of an identity tensor below.
Definition 2.4 (Brazell et al. 2013, Definition 3.13) A tensor with entries $I_{i_1i_2\cdots i_Nj_1j_2\cdots j_N} = \prod_{k=1}^{N} \delta_{i_kj_k}$, where
\[
\delta_{i_kj_k} = \begin{cases} 1, & i_k = j_k, \\ 0, & i_k \neq j_k, \end{cases}
\]
is called a unit tensor or identity tensor.

Definition 2.5 [Stanimirovic et al. (2018), Definition 2.1] The range and null space of a tensor $A \in \mathbb{R}^{I(M) \times J(N)}$ are defined as per the following:
\[
\mathcal{R}(A) = \left\{ A*_{N} \chi : \chi \in \mathbb{R}^{J_1 \times \cdots \times J_N} \right\} \quad \text{and} \quad \mathcal{N}(A) = \left\{ \chi : A*_{N} \chi = 0 \in \mathbb{R}^{I_1 \times \cdots \times I_M} \right\},
\]
where $O$ is an appropriate zero tensor.

The index of a tensor $A \in \mathbb{R}^{I(N) \times J(N)}$ is defined as the smallest positive integer $k$, such that $\mathcal{R}(A^k) = \mathcal{R}(A^{k+1})$. If $k = 1$, then the tensor is called index one or group, or core tensor. The index of a square tensor $A$ is denoted as $\text{ind}(A)$.

A tensor $A \in \mathbb{C}^{I \times \cdots \times I \times J_1 \times \cdots \times J_N}$ is invertible if there exists a tensor $\chi$ such that $A*_{N} \chi = \chi*_{N} A = I$. In this case, $\chi$ is called the inverse of $A$ and denoted by $A^{-1}$.

Lemma 2.1 [Stanimirovic et al. (2018), Lemma 2.2.] Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$, $B \in \mathbb{R}^{I_1 \times \cdots \times I_M \times K_1 \times \cdots \times K_L}$. Then, $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ if and only if there exists $U \in \mathbb{R}^{I_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}$ such that $B = A*_{N} U$.

Let $A \in \mathbb{C}^{I(M) \times J(N)}$, $\chi \in \mathbb{C}^{J(N) \times I(M)}$ and consider the following conditions:
\[
(1^T) \quad A*_{N} \chi*_{N} A = A; \quad (2^T) \quad \chi*_{N} A*_{N} \chi = \chi; \\
(3^T) \quad (A*_{N} \chi)* = A*_{N} \chi; \quad (4^T) \quad (\chi*_{N} A)^* = \chi*_{N} A.
\]

Definition 2.6 (Sun et al. 2016, Definition 2.2) The tensor $\chi$ satisfying the tensor equations:
\[
(1^T) \quad \text{is called inner or generalized inverse of } A \text{ and denoted by } A^{(1^T)}.
\]
\[
(2^T) \quad \text{is called outer inverse of } A \text{ and denoted by } A^{(2^T)}.
\]
\[
(1^T) \quad \text{and } (2^T) \quad \text{is called reflexive generalized inverse of } A \text{ and denoted by } A^{(1^T, 2^T)}.
\]
\[
(1^T) \quad \text{and } (2^T) \quad \text{is called Moore–Penrose inverse of } A \text{ and denoted by } A^*.
\]

We use the notation $A^{(1^T)}$, $A^{(1^T, 2^T)}$ and $A^{(1^T, 2^T, 3^T)}$, respectively, for the set of inner inverses, reflexive generalized inverses of $A$, and {1, 3} inverses of $A$. Similarly, the sets $A^{(1^T, 4^T)}$ and $A^{(1^T, 2^T, 3^T)}$ are defined. The group and Drazin inverse are defined as follows for an even-order square tensor $A \in \mathbb{C}^{I(N) \times J(N)}$.

Definition 2.7 Let $A \in \mathbb{C}^{I(N) \times J(N)}$ be a core tensor. A tensor $\chi \in \mathbb{C}^{I(N) \times J(N)}$ satisfying
\[
(1^T) \quad A*_{N} \chi*_{N} A = A, \quad (2^T) \quad \chi*_{N} A*_{N} \chi = \chi, \quad (3^T) \quad A*_{N} \chi = \chi*_{N} A,
\]
is called the group inverse of $A$ and it is denoted by $A^\#$.

Definition 2.8 (Ji and Wei 2018) Let $A \in \mathbb{C}^{I(N) \times J(N)}$ with $\text{ind}(A) = k$. A tensor $\chi \in \mathbb{C}^{I(N) \times J(N)}$ satisfying
\[
((k^T^T) \quad A^{k+1} A*_{N} A = A^k, \quad (2^T) \quad \chi*_{N} A*_{N} \chi = \chi, \quad (3^T) \quad \chi*_{N} A = A*_{N} \chi
\]
is called the Drazin inverse of $A$ and it is denoted by $A^D$.  

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It is important to mention that the matrix equations corresponding to \((1)^T, (2)^T, (3)^T, (4)^T, (5)^T, (1^k)^T\) are denoted by \((1), (2), (3), (4), (5), (1^k)\), respectively.

The orthogonal projection onto a subspace \(\mathcal{R}(A)\) is denoted by \(P_{\mathcal{R}(A)}\) and defined as 

\[ P_{\mathcal{R}(A)} = A *_{\mathcal{N}} A^T. \]

One specific and useful algorithm for a proper matricization of an arbitrary tensor was proposed in Stanimirovic et al. (2018).

**Definition 2.9** Stanimirovic et al. (2018) Let \(I_1, \ldots, I_M, K_1, \ldots, K_N\) be given integers and \(\mathcal{I}, \mathcal{R}\) are the integers defined as 

\[ \mathcal{I} = I_1I_2\cdots I_M, \quad \mathcal{R} = K_1K_2\cdots K_N. \]

The reshaping operation 

\[ \text{rsh} : \mathbb{C}^{I(M) \times K(N)} \mapsto \mathbb{C}^{\mathcal{I} \times \mathcal{R}} \]

transforms a tensor \(A \in \mathbb{C}^{I(M) \times K(N)}\) into the matrix \(A \in \mathbb{C}^{\mathcal{I} \times \mathcal{R}}\) using the Matlab function \text{reshape} as follows:

\[ \text{rsh}(A) = A = \text{reshape}(A, \mathcal{I}, \mathcal{R}), \quad A \in \mathbb{C}^{I(M) \times K(N)}, \quad A \in \mathbb{C}^{\mathcal{I} \times \mathcal{R}}. \]

The inverse reshaping of \(A \in \mathbb{C}^{\mathcal{I} \times \mathcal{R}}\) is the tensor \(A \in \mathbb{C}^{I(M) \times K(N)}\) defined by 

\[ \text{rsh}^{-1}(A) = A = \text{reshape}(A, I_1, \ldots, I_M, K_1, \ldots, K_N). \]

Also, an appropriate definition of the tensor rank, arising from the reshaping operation, was proposed in Stanimirovic et al. (2018).

**Definition 2.10** Stanimirovic et al. (2018) Let \(A \in \mathbb{C}^{I(N) \times K(N)}\) and \(A = \text{reshape}(A, \mathcal{I}, \mathcal{R}) = \text{rsh}(A) \in \mathbb{C}^{\mathcal{I} \times \mathcal{R}}\). Then, the tensor rank of \(A\), denoted by \(\text{rshrank}(A)\), is defined by \(\text{rshrank}(A) = \text{rank}(A)\).

### 3 Tensor core inverse

Let \(A\) be an \(n \times n\) complex matrix of index 1. The core inverse \(A^\# \in \mathbb{C}^{n \times n}\) of \(A \in \mathbb{C}^{n \times n}\) was introduced in (Baksalary and Trenkler 2010, Definition 1) as the matrix satisfying 

\[ AA^\# = P_A, \quad \mathcal{R}(A^\#) \subseteq \mathcal{R}(A), \]

where \(P_A = AA^\dagger\) denotes an orthogonal projection onto the range \(\mathcal{R}(A)\). Various characterizations of the core inverse, including its correlations with another generalized inverses, were originated in Baksalary and Trenkler (2010). The results obtained in Baksalary and Trenkler (2010) were developed on the basis of the Hartwig and Spindelböck decomposition. In Rakić et al. (2014b), the authors introduced an equivalent definition of the core inverse, by proving in Theorem 3.1 that (3.1) is equivalent to 

\[ AA^\# A = A, \quad \mathcal{R}(A^\#) = \mathcal{R}(A), \quad \mathcal{N}(A^\#) = \mathcal{N}(A^\ast). \]

Analogous definition of the core inverse in a ring \(\mathcal{R}\) with involution was introduced in Rakić et al. (2014a):

\[ aa^\# a = a, \quad a^\# R = aR, \quad Ra^\# = Ra^\ast. \]
Finally, the core inverse in the set $C_n^{CM} = \{ A \in \mathbb{C}^{n \times n} | \text{rank}(A^2) = \text{rank}(A) \}$ was also defined (Rakić et al. 2014a):

$$AA^\odot A = A, \ A^\odot C_n^{CM} = A C_n^{CM}, \ C_n^{CM} A^\odot = C_n^{CM} A^*.$$  \hfill (3.4)

In Xu et al. (2017, Theorem 3.1), the authors introduced the following representation of the core inverse in a ring $R$ with involution:

$$(aa^\odot)^* = aa^\odot, \ a^\odot a^2 = a, \ a (a^\odot)^2 = a^\odot.$$  \hfill (3.5)

In this section, we will introduce the core inverse of tensors and investigate its properties and representations. Also, several characterizations of the tensor core inverse as well as some relationships with other generalized inverses will be discussed.

A tensor $A$ is said to be a **core invertible** or a **core tensor** if the core inverse of $A$ exists. For this purpose, we will say that a tensor $A$ is a **core tensor** if its reshaping index is equal to 1, i.e.,

$$\text{rshrank}(A) = \text{rshrank}(A^2) \iff \text{rank}(\text{rsh}(A)) = \text{rank}(\text{rsh}(A)^2).$$

In the present paper, the tensor core inverse is introduced in Definition 3.1, using the approach analogous to (3.5).

**Definition 3.1** Let $A \in C(I(N) \times I(N))$ be a given core tensor. A tensor $X \in C(I(N) \times I(N))$ satisfying

\begin{align*}
(C1) \ X*_{N} A^2 &= A \\
(C2) \ A*_{N} X^2 &= X \\
(3T) \ (A*_{N} X)^* &= A*_{N} X
\end{align*}

is called the core inverse of $A$ and denoted by $A^\odot$.

It is known that the core inverse $A^\odot$ of a square matrix $A$ is the unique $[1, 2]$-inverse which satisfies $R(A^\odot) = R(A), \ N(A^\odot) = N(A^*)$. The goal of Lemma 3.1 is to verify that the tensor core inverse, introduced by the tensor equations (C1), (C2), (3T) in Definition 3.1, satisfies (1$^T$) and (2$^T$).

**Lemma 3.1** Let $A \in C(I(N) \times I(N))$ be given. Then, $A^\odot$ satisfies (1$^T$) and (2$^T$).

**Proof** The proof follows after the following verification:

\begin{align*}
A*_{N} A^\odot*_{N} A &= A*_{N} A^\odot*_{N} A^\odot*_{N} A^2 = A*_{N} (A^\odot)^*_{N} A^2 = A^\odot*_{N} A^2 = A; \\
A^\odot*_{N} A*_{N} A^\odot &= A^\odot*_{N} A*_{N} A^\odot*_{N} (A^\odot)^2 = A^\odot*_{N} A^2*_{N} (A^\odot)^2 = A*_{N} (A^\odot)^2 = A^\odot. \\
\end{align*}

(3.6) \hspace{1cm} (3.7)

The uniqueness of the core inverse is proved in Theorem 3.1.

**Theorem 3.1** An arbitrary core invertible tensor $A \in C(I(N) \times I(N))$ has one and only one core inverse.

**Proof** Let $X_1$ and $X_2$ be two tensors satisfying (C1), (C2), (3$^T$). Using these properties in conjunction with (3.6) and (3.7), it follows that

\begin{align*}
A*_{N} X_1 &= A*_{N} X_2*_{N} A*_{N} X_1 = (A*_{N} X_2)^*_{N} (A*_{N} X_1)^* = X_2*_{N} A^*_{N} X_1^*_{N} A^* \\
&= X_2*_{N} (A*_{N} X_1*_{N} A^*) = X_2^*_{N} A^* = (A*_{N} X_2)^* = A*_{N} X_2.
\end{align*}
The last one follows from (a) and the definition of the Moore–Penrose inverse.

\[ \chi_1 = \chi_1 A_N^* N \chi_1 = \chi_1 A_N^* N \chi_2 = \chi_1 A_N^* N N^2 = A_N^* N \chi_2 = \chi_2. \]

Hence, the statement is proved.

Corollary 3.1 can be obtained by combining (3.6), (3.7) and (3^T).

**Corollary 3.1** Let \( A \in \mathbb{C}^{I(N) \times I(N)} \) be a core tensor, then \( A^\circ \in \mathcal{A} \{1^T, 2^T, 3^T\}. \)

Theorem 3.2 gives some characterizations of the core inverse in terms of other generalized inverses and generalizes Theorem 1 from Baksalary and Trenkler (2010). It is important to mention that the results of Theorem 3.2 are verified using one specific algebraic approach, which is new even in the matrix case.

**Theorem 3.2** Let \( A \in \mathbb{C}^{I(N) \times I(N)} \) be a core tensor. Then, the following holds:

(a) \( A^\circ = A N^2 A_N^* A_N^* N A^\circ = A N^2 P_{R(A)} = A N^2 A N^2 A_N^* \);
(b) \( A^\circ N A^\circ = A^\circ N A^\circ \);
(c) \( (A^\circ)^\circ = A^2 N A^\circ \);
(d) \( A^\circ \) is EP;
(e) \( (A^\circ)^\circ = A^2 N A^\circ \);
(f) \( (A^\circ)^2 = A^\circ \);
(g) \( A^\circ N A = A^\circ N A^\circ \).

**Proof** (a) Since \( A \) is a core tensor, the existence of \( A^\circ \) is ensured and let \( X := A N^2 A_N^* A_N^* \). It is necessary to verify that \( X \) satisfies (C1), (C2), (3^T). Indeed

\[ X N A^2 = A N^2 A_N^* A_N^* N A^2 = A N^2 A^2 = A, \]
\[ A N X^2 = A N^2 A_N^* A_N^* N A^2 = A N^2 A_N^* A^2 = A. \]

(b) The proof follows from the representations given in (a) and simple transformations:

\[ A^\circ N N A^\circ = A^\circ N N A_N^* A_N^* N A^\circ = A^\circ N N A^\circ. \]

(c) Let \( Z := A^2 N A^\circ \). It is easy to show \( A^\circ N Z = Z N A^\circ = A N^2 A^\circ \). Since

\[ A^\circ N Z N A^\circ = A N^2 A_N^* N A^2 N A_N^* A N^2 = A N^2 A_N^* N A N^2 = A^\circ, \]
\[ Z N A^\circ N Z = A N^2 A_N^* N A^2 N A_N^* A = Z, \]
\[ (A^\circ N Z)^\circ = (Z N A^\circ)^\circ = (A N^2 A^\circ)^\circ = (A^\circ N A^\circ), \]

the tensor \( Z \) is the Moore–Penrose inverse of \( A^\circ \).

(d) This part can be demonstrated after easy verification of the equalities

\[ A^\circ N N (A^\circ)^\circ = (A^\circ)^\circ N A^\circ = A^\circ N A^\circ \].

(e) It is enough to show \( A^\circ N (A^2 N A^\circ)^2 = A^2 N A^\circ \) and \( (A^2 N A^\circ)^2 = A^\circ \). Indeed

\[ A^\circ N (A^2 N A^\circ)^2 = A N^2 A^\circ N A^2 N A^\circ = A^2 N A^\circ, \]
\[ A^2 N A^\circ N (A^\circ)^2 = A N^2 A^\circ N A N^2 A^\circ = A^2 N A^\circ N A^\circ = A^\circ. \]

(f) This statement follows from \( (A^\circ)^2 = A N^2 A_N^* A_N^* N A^\circ = A^\circ \).

(g) The last one follows from (a) and the definition of the Moore–Penrose inverse. \( \square \)
Remark 3.1 It is easy to verify that $A^\circ = A^{\#}A^{\ast N}A^\ast N = A^{\#}P_{R(A)} = A^{\ast N}A^{\#}A^\dagger$ satisfies (3.1).

Corollary 3.2 If $A$ is a core tensor with full rank factorization $A = P\mathcal{Q}$, then
\[ A^\circ = A^{(2)}_{R(P), N(P^\ast)}. \]

Proof Using full rank factorizations of the group inverse and the Moore–Penrose inverse, it follows that
\[ A^\circ = A^{\ast N}A^{\#}A^\ast N = P^{\ast N}Q^{\ast N}P^{\ast N}(Q^{\ast N}P)^{-2}NQ^{\ast N}Q^{\ast N} \]
\[ = P^{\ast N}(P^{\ast N}A^{\ast N}P)^{-1}N^\ast P^\ast, \]
which clearly coincides with $A^{(2)}_{R(P), N(P^\ast)}$.

The following representation of $A^\circ$ follows from Theorem 3.2(c).

Corollary 3.3 The core inverse can be obtained by $A^\circ = (A^{\ast N}A^\ast N)^\dagger$.

We can also generalize Theorem 3.2(a) using $\{1, 3\}$ inverses of $A$ instead of the Moore–Penrose inverse. The result is proved in Theorem 3.3.

Theorem 3.3 Let $A \in \mathbb{C}^{I(N)\times I(N)}$ be a core tensor. If there exists a tensor $X := A^{(1^T, 3^T)} \in \mathbb{C}^{I(N)\times I(N)}$ such that $X \in A[1^T, 3^T]$, then $A^\circ = A^{\#}A^{\ast N}X$.

Proof Let us assume that there exists a tensor $X := A^{(1^T, 3^T)}$ satisfying $A^{\ast N}X^{\ast N}A$ and $(A^{\ast N}X)^\ast = A^{\ast N}X$. Consider $Y := A^{\#}A^{\ast N}A^{(1^T, 3^T)}$. We can notice the following expressions:
\[ A^{\ast N}Y^2 = A^{\ast N}A^{\#}A^{\ast N}A^{(1^T, 3^T)}A^{\#}A^{\ast N}A^{(1^T, 3^T)} = A^{\ast N}A^{\#}A^{\ast N}A^{(1^T, 3^T)} = Y, \]
\[ Y^{\ast N}A^2 = A^{\#}A^{\ast N}A^{(1^T, 3^T)}A^{\#}A^{\ast N}A^{(1^T, 3^T)} = A^{\#}A^{\ast N}A^{(1^T, 3^T)} = Y, \]
\[ (A^{\ast N}Y)^\ast = (A^{\#}A^{(1^T, 3^T)})^\ast = A^{\#}A^{\ast N}A^{(1^T, 3^T)} = A^{\#}A^{\ast N}X. \]

Therefore, $Y$ satisfies $(C1)$, $(C2)$ and $(3^T)$ with respect to $A$. By the uniqueness of the core inverse, it follows that $A^\circ = A^{\#}A^{\ast N}X$.

Corollary 3.4 Let $A \in \mathbb{C}^{I(N)\times I(N)}$ be a core tensor and $X \in \mathbb{C}^{I(N)\times I(N)}$. If $X$ satisfies one of the following two systems (i) or (ii)
\[ (i) \quad X^{\ast N}A^2 = A, \quad \text{and} \quad (3^T) \quad (A^{\ast N}X)^\ast = A^{\ast N}X; \]
\[ (ii) \quad X^{\ast N}A = A^{\#}A, \quad \text{and} \quad (3^T) \quad (A^{\ast N}X)^\ast = A^{\ast N}X; \]
then $A^\circ = A^{\#}A^{\ast N}X$.

Proof Let the system (i) be satisfied. If $X^{\ast N}A^2 = A$ is true, then
\[ X^{\ast N}A = X^{\ast N}A^{\#}A^{\ast N}A^{\#} = X^{\ast N}A^{\#} = A^{\#}A. \]

Therefore, $A^{\ast N}X^{\ast N}A = A$ and $X \in A[1, 3]$. 

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Further if the conditions (ii) are satisfied \( X_N A = A_N A^\dagger \), then
\[
A_N X_N A = A_N A^\dagger \; \ast_N A = A,
\]
which again implies \( X \in A\{1, 3\} \).

Now, the proof in both cases follows from Theorem 3.3. \( \square \)

Corollary 3.5 follows from [Stanimirovic et al. (2018), Theorem 4.3] and the properties of the core tensor inverse.

**Corollary 3.5** Let \( A \in \mathbb{C}^{I(N) \times I(N)} \) is a given core tensor.

(a) The following statements are equivalent:

(i) there exists an outer inverse \( X \in \mathbb{C}^{I(N) \times I(N)} \) : \( A^\odot = A^{(2)}_{\mathcal{R}(A) \mathcal{N}(A^\ast)} \);

(ii) there exists a tensor \( U \in \mathbb{C}^{I(N) \times I(N)} \) that fulfils the tensor equations
\[
A_N U_N A^\ast N A^2 = A, \quad A^\ast N A^2 \ast N U_N A^\ast = A^\ast;
\]

(iii) there exist tensors \( U, V \in \mathbb{C}^{I(N) \times I(N)} \) such that
\[
A_N U_N A^\ast N A^2 = A, \quad A^\ast N A^2 \ast N V_N A^\ast = A^\ast;
\]

(iv) there exist \( U \in \mathbb{C}^{I(N) \times I(N)} \) and \( V \in \mathbb{C}^{I(N) \times I(N)} \) such that
\[
A_N U_N A^2 = A, \quad A^\ast N A^2 \ast N V_N A^\ast = A^\ast, \quad A_N U = V_N A^\ast;
\]

(v) there exist \( U \in \mathbb{C}^{I(N) \times I(N)} \) and \( V \in \mathbb{C}^{I(N) \times I(N)} \) satisfying
\[
A^\ast N A^2 \ast N U = A^\ast, \quad V \ast N A^\ast N A^2 = A;
\]

(i) \( \mathcal{N}(A^\ast N A^2) = \mathcal{N}(A) \);

(ii) \( \mathcal{R}(A^\ast N A^2) = \mathcal{R}(A^\ast) \);

(iii) \( A^\ast N (A^\ast \ast N A^2)^{(1)} \ast N A^\ast N A^2 = A \) and \( A^\ast \ast N A^2 \ast N (A^\ast N A^2)^{(1)} \ast N A^\ast = A^\ast \);

(iii) \( rshrank(A^\ast \ast N A^2) = rshrank(A) \);

(ix) \( A \) is core invertible;

(b) If the statements in (a) are true, then
\[
A^\odot = A^\ast N (A^\ast \ast N A^2)^{(1)} \ast N A^\ast = A_N U_N A^\ast,
\]
for arbitrary \( U \in \mathbb{C}^{K(k) \times L(l)} \) satisfying (3.8).

The next result gives a correlation between the core inverse of a tensor power and the power of its core inverse.

**Theorem 3.4** Let \( A \in \mathbb{C}^{I(N) \times I(N)} \) be a core tensor. Then, the following holds for an arbitrary integer \( m \geq 1 \):

(a) \( A^m \ast_N (A^\circ)^m = A\ast_N A^\dagger \),

(b) \( (A^\circ)^m = (A^m)^\circ \).

**Proof** (a) Using the statement of Theorem 3.2(a), we obtain
\[
A^m \ast_N (A^\circ)^m = A^m \ast_N (A^\dagger \ast_N A \ast N A^\dagger)^m
\]
\[
= A^m \ast_N A^\dagger \ast_N (A^\dagger \ast N A \ast N A^\dagger)^{m-1} = A^m \ast_N A^\dagger \ast N (A^\dagger \ast N A \ast N A^\dagger)^{m-2}
\]
\[
= A^{m-1} \ast_N A^\dagger \ast_N (A^\dagger \ast N A \ast N A^\dagger)^{m-2} = A^{m-2} \ast_N A^\dagger \ast N (A^\dagger \ast N A \ast N A^\dagger)^{m-3}.
\]
Continuing in the same way, one can obtain
\[ A^m_{*N}(A^{\#})^m = A^2_{*N}A^T_{*N}(A^{\#}_{*N}A_{*N}A^\dagger) = A_{*N}A^\dagger. \]

(b) Let \( X = (A^{\#})^m \). Now, using the part (a) of this theorem, we get
\[
X_{*N}(A^m)^2 = (A^{\#}_{*N}A_{*N}A^\dagger)^m_{*N}A^{2m} = (A^{\#}_{*N}A_{*N}A^\dagger)^{m-1}_{*N}A^{2m-1} = \ldots = A^m.
\]

\[ A^m_{*N}X^2 = A_{*N}A^\dagger_{*N}X = A_{*N}A^\dagger_{*N} (A^{\#}_{*N}A_{*N}A^\dagger)^m = (A^{\#}_{*N}A_{*N}A^\dagger)^m = X. \]

Therefore, \( X \) is the core inverse of \( A^m \), which completes the proof.

**Example 3.1** Consider a tensor \( A = (A_{ijkl}) \in \mathbb{R}^{(2 \times 3) \times (2 \times 3)} \) with entries
\[
A_{i1j1} = \begin{bmatrix} 1 & 6 & 2 \\ -1 & 3 & 0 \end{bmatrix}, \quad A_{i1j2} = A_{i1j3} = A_{i2j1} = A_{i2j2} = a_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Then, \( A^{\#} = A \in \mathbb{R}^{(2 \times 3) \times (2 \times 3)} \). However, \( A^\dagger = (X_{ijkl}) \in \mathbb{R}^{(2 \times 3) \times (2 \times 3)} \), where
\[
X_{i1j1} = \begin{bmatrix} 1/51 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_{i1j2} = \begin{bmatrix} 2/17 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_{i1j3} = \begin{bmatrix} 2/51 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]
\[
X_{i2j1} = \begin{bmatrix} -1/51 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_{i2j2} = \begin{bmatrix} 1/17 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_{i2j3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

and \( A^{\#} = (Y_{ijkl}) \in \mathbb{R}^{(2 \times 3) \times (2 \times 3)} \), where
\[
Y_{i1j1} = \begin{bmatrix} 1/51 & 2/17 & 2/51 \\ -1/51 & 1/17 & 0 \end{bmatrix}, \quad Y_{i1j2} = \begin{bmatrix} 2/17 & 12/17 & 4/17 \\ -2/17 & 6/17 & 0 \end{bmatrix}.
\]
\[
Y_{i1j3} = \begin{bmatrix} 2/51 & 4/17 & 4/51 \\ -2/51 & 2/17 & 0 \end{bmatrix}, \quad Y_{i2j1} = \begin{bmatrix} -1/51 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]
\[
Y_{i2j2} = \begin{bmatrix} 1/17 & 6/17 & 2/17 \\ -1/17 & 3/17 & 0 \end{bmatrix}, \quad Y_{i2j3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Hence, it is clear that \( A^{\#} \), \( A^\dagger \) and \( A^\# \) are different.

Now, a natural question is: are these inverses the same under some assumptions? The next theorem will provide an affirmative answer to this along with some characterizations for EP tensors. The results of Theorem 3.5 generalize the results of Theorem 2 from Baksalary and Trenkler (2010).

**Theorem 3.5** Let \( A \in \mathbb{R}^{I(N) \times N} \) be a core tensor. Then, the following statements hold:

(a) \( A^{\#} = 0 \) if and only if \( A = 0 \),
(b) \( A^{\#} = A_{*N}A^\dagger \) if and only if \( A \) idempotent,
(c) \( A^{\#} = A^\dagger \) if and only if \( A \) is EP,
(d) \( A^{\#} = A^\# \) if and only if \( A \) is EP,
(e) \( A^{\#} = A \) if and only if \( A \) is EP and tripotent,
(f) \( A^{\#} = A^* \) if and only if \( A \) is EP and partial isometry.
Remark 3.2 Let $A = 0$, then $A^\circ = 0$ follows trivially from Theorem 3.2(a). To show the converse, let $A^\circ = 0$. This assumption leads to $A^\#_{*N} A_{*N} A^\dagger = 0$. Post-multiplying and pre-multiplying by $A$, we get $A = 0$. Thus, the statement in (a) is completed.

(b) Now let $A^2 = A$. Then, $A^\circ = A^\#_{*N} A_{*N} A^\dagger = A^\#_{*N} A_{*N} A^\dagger$. The opposite statement is true since

$$A = A_{*N} A^\#_{*N} A = A_{*N} A A^\dagger_{*N} A = A_{*N} A A^\dagger_{*N} A = A^2.$$  

(c) Let us assume $A^\circ = A^\dagger$. So, by Theorem 3.2(d), it follows that $A^\circ$ is EP. This implies that $A^\#_{*N} (A^\circ)^\dagger = (A^\circ)^\dagger_{*N} A^\circ$. This yields $A^\dagger_{*N} A = A_{*N} A^\circ$ since $A^\circ = A^\dagger$. Therefore, $A$ is EP.

Now the converse follows by the following expression:

$$A^\circ = A^\#_{*N} A_{*N} A^\dagger = A^\#_{*N} A_{*N} A^\dagger_{*N} A = A_{*N} A^\dagger_{*N} A = A^\circ_{*N} A^\dagger = A^\circ.$$  

(d) Again, assume that $A$ is EP. Then, the core inverse of $A$ is equal to

$$A^\circ = A^\#_{*N} A_{*N} A^\dagger = A^\#_{*N} A_{*N} A^\dagger_{*N} A = (A^\#_{*N} A^\dagger)^2_{*N} A^\dagger_{*N} A = (A^\#_{*N} A)^2_{*N} A = A^\#.$$  

Consider $A^\circ = A^\dagger$. To claim that $A$ is EP, it is enough to show $A^\# = A^\dagger$. Since $A^\circ$ is $\{1^T, 2^T, 3^T\}$ inverse of $A$, we only need to claim that $(A^\#_{*N} A)^*_{*N} A = A^\#_{*N} A$. Since $A^\#_{*N} A = A_{*N} A^\circ$, it implies $A_{*N} A^\circ = A^\#_{*N} A$. Now

$$(A^\#_{*N} A)^*_{*N} A = (A_{*N} A^\circ)^*_{*N} A = A_{*N} A^\circ = A_{*N} A^\circ = A_{*N} A^\# = A^\#_{*N} A.$$  

This completes the proof for (d).

(e) Next assume that $A^\circ = A$. Since $A^\circ$ is EP which yields $A$ is EP. From Definition 3.1, it follows that $A_{*N} (A^\circ)^2 = A^\circ$. This turns to $A^3 = A$. So $A$ is EP and tripotent. The converse statement holds since

$$A^\circ = A^\#_{*N} A_{*N} A^\dagger = A^\#_{*N} A^3_{*N} A^\dagger = A^\#_{*N} A^2_{*N} A^\dagger A = A^\#_{*N} A^2 = A.$$  

(f) To show the last part, first consider $A^\circ = A^\dagger$. Since $A_{*N} A^\circ_{*N} A = A$, it gives $A_{*N} A^\circ_{*N} A = A$. This gives $A^\circ = A^\dagger = A^\dagger$. Then, by part (c), $A$ is an isometry and EP. The converse statement follows from part (c) and the definition of isometry.

Combining the results of Theorem 3.5 (c) and (d), we state the following remark.

**Remark 3.2** Let $A \in C^{I(N) \times I(N)}$ be a core tensor. Then, $A$ is EP if and only if $A^\circ = A^\dagger = A^\#$.

Further characterizations of EP tensors are provided in the next theorem. The results of Theorem 3.6 generalize the results of Theorem 2 from Baksalary and Trenkler (2010).
Theorem 3.6 Let $A \in \mathbb{C}^{I(N) \times I(N)}$ be a core tensor. Then, the following are equivalent:

(i) $A$ is EP,
(ii) $(A^\#)^\# = A$,
(iii) $A^\# * N A = A * N A^\#$,
(iv) $(A^\dagger)^\dagger = A$,
(v) $(A^\#)^\dagger = A$.

Proof Let $A$ is EP. By Theorem 3.2(d), $(A^\#)^\# = A^2 * N A^\dagger = A * N A^\# * N A = A$. Thus, $(i) \Rightarrow (ii)$.

Assume $(ii)$ is true, i.e., $A^2 * N A^\dagger = A$. Now

$A^\# * N A = A^# * N A = A^# * N A^2 * N A^\dagger = A * N A^\# * N A * N A^\dagger = A * N A^\#$.

This completes $(ii) \Rightarrow (iii)$.

Next, we will claim (iii) $\Rightarrow$ (i). Consider $A^\# * N A = A * N A^\#$. This turns out $A * N A^\# = A * N A^\dagger$. Now $A^\# = A^\# * N A^\# * N A^\dagger = A^\# * N A * N A^\dagger = A^\#$. So by Theorem 3.5(d), the tensor $A$ is EP.

Next, we will claim the equivalence between $(i)$ and $(iv)$. Let us assume $A$ is EP. Then, $A^\# = A^\dagger$ by Theorem 3.5(c). This yields $(A^\dagger)^\dagger = (A^\#)^\# = A$. To show the converse, it is enough to show that $A^\dagger$ is the core inverse of $A$. Let $(A^\#)^\# = A$. which implies $A^\# * N A^\dagger = A$ and $A^\# * N (A^\dagger)^2 = A^\dagger$. As $(A^\# * N A^\dagger)^2 = A^\# * N A^\dagger$, it follows that $A^\dagger$ satisfies all the conditions of the core inverse. Therefore, $(i) \Leftrightarrow (iv)$.

Finally, we will show $(i) \Leftrightarrow (v)$. The tensor $A$ is EP if and only if $A^\# = A^\dagger$. This turns out $A$ is EP if and only if $(A^\#)^\dagger = (A^\dagger)^\dagger = A$. □

In view of the above Theorem 3.6, we state the following remark.

Remark 3.3 Let $A \in \mathbb{C}^{I(N) \times I(N)}$ be a core tensor. If $A$ is EP, then $(A^\#)^\dagger = (A^\dagger)^\#$.

We will discuss some equivalent characterization of core inverse in the remaining part of this section. The condition $A^\# * N A^2 = X'$ can be replaced by an alternative condition as stated in Theorem 3.7.

Theorem 3.7 Let $A \in \mathbb{C}^{I(N) \times I(N)}$ be a core tensor. If there exists a tensor $X'$ satisfying $(2^T) X^* N A * N X' = X'$, $(3^T)$ $(A * N X)^* = A * N X$ and $(C1)$ $X^* N A^2 = A$, then $X'$ is the core inverse of $A$.

Proof Post-multiplying $A = X^* N A^2$ by $A^\#$, we obtain $A * N A^\# = X^* N A^2 * N A^\# = X^* N A$. Now $A * N X^* N A = A * N A * N A^\# = A$. Therefore, $X' \in A^T, X^T$. So, by Theorem 3.3, the core inverse of $A$ is $A^\# = A^\# * N A * N X = A * N A^\# * N X$. Replacing $A * N A^\#$ by $X * N A$, one can obtain $A^\# = X^* N A * N X = X'$. Hence, the proof is complete. □

Theorem 3.7 leads to the following corollary.

Corollary 3.6 Let $A \in \mathbb{C}^{I(N) \times I(N)}$ be a core tensor. If there exists a tensor $X'$ satisfying $(2^T) X^* N A * N X' = X'$, $(3^T)$ $(A * N X)^* = A * N X$ and $(C1)$ $X^* N A^2 = A$, then $A^\# * N A^2 = X'$.}

Remark 3.4 Corollary 3.6 can be proved separately and the proof is the following. Since $X = X^* N A * N X = X^* N A^2 * N X$. Pre-multiplying by $X^\#$, we obtain

$X^\# * N X = X^\# * N A^2 * N X = X^\# * N A^2 * N X = X^\# * N X = X'$. The assumptions of Corollary 3.6 can also be used as an equivalent definition of core inverse of $A$. □
Next, we discuss another equivalent definition of the core inverse, which was proved in Wang and Liu (2015) for the matrix case using the singular decomposition.

**Proposition 3.1** Let \( A \in \mathbb{C}^{I(N) \times I(N)} \) be a core tensor. If there exists a tensor \( \mathcal{X} \) satisfying
\[
(1^T) \ A^* N \mathcal{X} * N A = A, \quad (3^T) \ (A^* N \mathcal{X})^* = A^* N \mathcal{X}, \quad \text{and} \quad (C2) \ A^* N \mathcal{X}^2 = \mathcal{X}, \text{then } \mathcal{X} \text{ is core inverse of } A.
\]

**Proof** It is sufficient to show that \( A^* N \mathcal{X}^2 = A \). Combining the result \( A^* N \mathcal{X} N A = A \) and \( A^* N \mathcal{X}^2 = \mathcal{X} \), we get \( A = A^2 N \mathcal{X}^2 * N A \). Pre-multiplying \( A^* \) both sides, we obtain \( A^* N \mathcal{X}^2 = A^* N \mathcal{X}^2 * N A = A^* N \mathcal{X} \mathcal{X}^2 * N A = \mathcal{X}^* N A \mathcal{X}^2 = \mathcal{X} N A \mathcal{X}^2 = \mathcal{X} N A \mathcal{X} \mathcal{X} \mathcal{X} = \mathcal{X} \mathcal{X} \mathcal{X} = \mathcal{X} \mathcal{X} = \mathcal{X} \). Hence, \( \mathcal{X} \) is the core inverse of \( A \).

By combining Theorem 3.7 and Proposition 3.1, we state the following corollary which can be used as an equivalent characterization of the core inverse.

**Corollary 3.7** Let \( A \in \mathbb{C}^{I(N) \times I(N)} \) be a core tensor and \( \mathcal{X} \in \mathbb{C}^{I(N) \times I(N)} \). If \( \mathcal{X} \) satisfies any one of the following triads of equations:

- (i) \( (C1) \ A^* N \mathcal{X}^2 = A, \,(C2) \ A^* N \mathcal{X}^2 = \mathcal{X}, \text{ and } (3^T) \ (A^* N \mathcal{X})^* = A^* N \mathcal{X}, \)
- (ii) \( (2^T) \ A^* N \mathcal{X}^2 = \mathcal{X}, \,(C1) \ A^* N \mathcal{X}^2 = A, \text{ and } (3^T) \ (A^* N \mathcal{X})^* = A^* N \mathcal{X}, \)
- (iii) \( (1^T) \ A^* N \mathcal{X}^2 = A, \,(2^T) \ A^* N \mathcal{X}^2 = \mathcal{X}, \text{ and } (3^T) \ (A^* N \mathcal{X})^* = A^* N \mathcal{X}, \)

then \( \mathcal{X} \) is the core inverse of \( A \).

If we replace the condition \( A^* N \mathcal{X}^2 = \mathcal{X} \) in Corollary 3.7 (iii) by \( A^* N \mathcal{X}^2 = A \), then one can easily verify that \( \mathcal{X}^* N A^* N \mathcal{X} \) is the core inverse of \( A \). In conclusion, we pose the following remark.

**Remark 3.5** Let \( A \in \mathbb{C}^{I(N) \times I(N)} \) be a core tensor. If there exists a tensor \( \mathcal{X} \) satisfying \( A^* N \mathcal{X}^* N A = A, \,(A^* N \mathcal{X})^* = A^* N \mathcal{X}, \text{ and } A^* N \mathcal{X}^2 = A \), then \( \mathcal{X}^* N A^* N \mathcal{X} \) is the core inverse of \( A \).

The core inverse can be further characterized using the range condition as in Theorem 3.8.

**Theorem 3.8** Let \( A \in \mathbb{C}^{I(N) \times I(N)} \) be a core tensor and \( \mathcal{X} \in \mathbb{C}^{I(N) \times I(N)} \). If \( \mathcal{X} \) satisfies \( (A^* N \mathcal{X})^* = A^* N \mathcal{X}, \mathcal{X}^* N A^2 = A, \text{ and } \mathcal{X}^* N A \subseteq \mathcal{X}^* N A \), then \( \mathcal{X} \) is the core inverse of \( A \).

**Proof** To show \( \mathcal{X} \) is the core inverse of \( A \), it is enough to show only \( \mathcal{X}^* N A^* N \mathcal{X} = \mathcal{X} \). Since \( \mathcal{X}^* N A \subseteq \mathcal{X}^* N A \), by Lemma 2.1, there exists a tensor \( \mathcal{Z} \in \mathbb{C}^{I_1 \times \cdots \times I_N} \) such that \( \mathcal{X} = A^* N \mathcal{Z} \). From the condition \( \mathcal{X}^* N A^2 = A \), we obtain
\[
\mathcal{X}^* N A^* N \mathcal{X} = \mathcal{X}^* N A^* N A^2 = \mathcal{X}^* N A^2 N A^2 = A^2 N A^2 = A^2 N A = A^2 N A^* N A^* N \mathcal{Z} = A^* N \mathcal{Z} = \mathcal{X}.
\]

Therefore, by Theorem 3.7, \( \mathcal{X} \) is the core inverse of \( A \).

Note that the converse of Theorem 3.8 is also true and can be verified from the definition of the core inverse. In the next subsection, we discuss the computation of core inverse for sum of two tensors.

### 3.1 Tensor core inverse of the sum of two tensors

Group inverse of sum of two elements over a ring was first introduced by Drazin (1958). Some generalization also discussed in Chen et al. (2009). Then, the same idea was extended to core inverse on a ring in Xu et al. (2017). The result of Xu et al. (2017) was further extended to tensors in [4] and stated below.
Theorem 3.9 Let $A, B \in \mathbb{C}^{l_1 \times \cdots \times l_N}$ be core tensors with $A \ast_N B = O$. Then
\[(A + B)^\# = (I - B \ast_N B^\#) \ast_N A^\# + B^\# \ast_N (I - A \ast_N A^\#).\]

In the next theorem, we generalize the same result with one more additional condition.

Theorem 3.10 Let $A, B \in \mathbb{C}^{l_1 \times \cdots \times l_N}$ be core tensors with $A \ast_N B = O$ and $A^\# \ast_N B = O$. Then, $(A + B)^\# = (I - B \ast_N B^\#) \ast_N A^\# + B^\#$.

Proof By Theorem 3.9, it follows that
\[(A + B)^\# = (I - B \ast_N B^\#) \ast_N A^\# + B^\# \ast_N (I - A \ast_N A^\#).
\]
Using the fact of the Theorem 3.2(f) (i.e., $A^\# = (A^\#)^2 \ast_N A$) and the definition of core inverse, we obtain
\[(A + B)^\# = (I - (B^\#)^2 \ast_N B^2) \ast_N (A^\#)^2 \ast_N A + (B^\#) \ast_N (I - (A^\#)^2 \ast_N A^2).
\]
Since $A \ast_N B = O$ and $A^\# \ast_N B = O$, the transformations are as follows:
\[A \ast_N B^\# = A \ast_N B \ast_N (B^\#)^2 = O,
\]
\[B^\# \ast_N A = B^\# \ast_N B \ast_N B^\# \ast_N A = B^\# \ast_N (B \ast_N B^\#)^\# \ast_N A
\]
\[= B^\# \ast_N (B^\#)^\# \ast_N B^\# \ast_N A = B^\# \ast_N (B^\#)^\# \ast_N (A \ast_N B)^\# = O,
\]
\[A^\# \ast_N B = A^\# \ast_N A \ast_N A^\# \ast_N B = A^\# \ast_N (A \ast_N A^\#)^\# \ast_N A^\# \ast_N B = O.
\]
Let $\chi = (I - B \ast_N B^\#) \ast_N A^\# + B^\#$. Now, we have
\[(A + B)^\# \ast_N \chi = (A + B)^\# (I - B \ast_N B^\#) \ast_N A^\# + B^\# = A \ast_N (I - B \ast_N B^\#) \ast_N A^\# + A \ast_N B^\# + B \ast_N B^\#
\]
\[= A \ast_N (I - B \ast_N B^\#) \ast_N A^\# + B \ast_N B^\# = A \ast_N A^\# + B \ast_N B^\#.
\]
Thus, $(A + B)^\# \ast_N \chi = (A \ast_N A^\#)^\# + (B \ast_N B^\#)^\# = (A + B)^\# \ast_N \chi$. Also, we have
\[(A + B)^\# \ast_N \chi = (A \ast_N A^\# + B \ast_N B^\#)^\# (A + B) = A \ast_N A^\# + B \ast_N B^\#
\]
\[= A \ast_N A^\# + B \ast_N B^\# = (I - B \ast_N B^\#) \ast_N A \ast_N B = A + B.
\]
Since $\chi = \{1^T, 3^T\}$ inverse of $(A + B)$, by Theorem 3.3,
\[(A + B)^\# = (A + B)^\# \ast_N (A + B)^\# \ast_N \chi
\]
\[= [(I - B^\# \ast_N B) \ast_N (A^\#)^2 \ast_N A + (B^\#)^2 \ast_N B \ast_N (I - A \ast_N A)] \ast_N (A + B)^\# \ast_N
\]
\[= [(I - B^\# \ast_N B) \ast_N (A^\#)^2 \ast_N A^2 + (B^\#)^2 \ast_N B^2] \ast_N [(I - B \ast_N B^\#) \ast_N A^\# + B^\#]
\]
\[= [(I - B^\# \ast_N B) \ast_N A \ast_N A^\# + B^\# \ast_N B^\#] \ast_N [(I - B \ast_N B^\#) \ast_N A^\# + B^\#]
\]
\[= (A \ast_N A - B \ast_N B \ast_N A \ast_N A + B \ast_N B \ast_N A \ast_N A) \ast_N (A \ast_N B \ast_N A \ast_N A + B \ast_N B \ast_N A \ast_N A)
\]
\[= A \ast_N B \ast_N A \ast_N A + B \ast_N B \ast_N A \ast_N A = (I - B \ast_N B) \ast_N A \ast_N A = A + B.
\]
\[\square\]
The necessity of the condition $A\ast NB = O$ in Theorem 3.10 can be verified in the following example:

**Example 3.2** Consider the tensor $A = (a_{ijkl}) \in \mathbb{R}^{(2 \times 3) \times (2 \times 3)}$ defined in Example 3.1 and a tensor $B = (b_{ijkl}) \in \mathbb{R}^{2 \times 3 \times 2 \times 3}$ with entries

$$b_{ij11} = \begin{bmatrix} -1 & -1 & -3 \\ -1 & -2 & 0 \end{bmatrix}, \quad b_{ij12} = b_{ij13} = b_{ij21} = b_{ij22} = b_{ij23} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$ 

It is easy to verify $A \ast_2 B \neq O$, $A + B \neq O$, and $(A + B)^2 = O$. Since $\mathcal{R}(A + B) \neq \mathcal{R}(A + B)^2$, so the core inverse of $A + B$ does not exist, whereas $A$ and $B$ are core invertible.

Notice that, if $B \ast_N A = O$, then $B \ast_N A^\oplus = B \ast_N A \ast_N (A^\oplus)^2 = O$. So, with this additional assumption, we have the following corollary.

**Corollary 3.8** Let $A, B \in \mathbb{C}^{I(N) \times I(N)}$ be core tensors with $A \ast_N B = O = B \ast_N A$ and $A \ast_N B = O$. Then, $(A + B)^\oplus = A^\oplus + B^\oplus$.

### 4 Tensor core-EP inverse

The core-EP inverse $X$ of a square matrix $A$ with $\text{ind}(A) = k$ is defined in Prasad and Mohana (2014) as the outer inverse satisfying $\mathcal{R}(X) = \mathcal{R}(A^k), \mathcal{N}(X) = \mathcal{N}((A^k)^*)$. Also, the core-EP inverse $X$ is the unique $(1^k), (2), (3)$ inverse satisfying $\mathcal{R}(X) \subset \mathcal{R}(A^k)$ (Prasad and Mohana 2014).

In this section, we will extend the notion of the core inverse to the core-EP inverse for square tensors. The core-EP inverse, as the core inverse, is unique. Some characterizations of the tensor core-EP inverse with respect to the tensor Drazin inverse, group inverse and the tensor Moore–Penrose inverse will be presented in this section.

The definition of the core-EP inverse of square tensors is introduced in Definition 4.1.

**Definition 4.1** Let $A \in \mathbb{C}^{I(N) \times I(N)}$ and $\text{ind}(A) = k$. A tensor $X \in \mathbb{C}^{I(N) \times I(N)}$ satisfying

1. **(EP)** $X \ast_N A^{k+1} = A^k$
2. **(C2)** $A \ast_N X^2 = X$
3. **(T)** $(A \ast_N X)^* = A \ast_N X$

is called core-EP inverse of $A$ and it is denoted as $A^{\oplus}$.

Clearly, if $k = 1$, then $A^{\oplus} = A^\oplus$.

Before proving the uniqueness of the core-EP inverse, we first prove some preliminary results.

**Lemma 4.1** Let $A \in \mathbb{C}^{I(N) \times I(N)}$ satisfy $\text{ind}(A) = k$. If a tensor $X \in \mathbb{C}^{I(N) \times I(N)}$ satisfies (EP) and (C2), then the following holds:

(a) $A \ast_N X = A^m \ast_N X^m, \quad m \in \mathbb{N}$.
(b) $X \ast_N A \ast_N X = X$.
(c) $A^m \ast_N X^m \ast_N A^m = A^m$ for $m \geq k$.
(d) $X^{m+1} \ast_N A^m = A^D$ when $m \geq k = \text{ind}(A)$.

**Proof** (a) Using (C2) repetitively, we obtain

$A \ast_N X = A \ast_N A \ast_N X = A^2 \ast_N X = A^2 \ast_N X = A^3 \ast_N X^3$
Theorem 4.1

Let \( \mathcal{A} \in \mathbb{C}^{I(N) \times I(N)} \) and \( \text{ind}(\mathcal{A}) = k \). Then, \( \mathcal{A} \) has a unique core-EP inverse in \( \mathbb{C}^{I(N) \times I(N)} \).

Proof Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two distinct core-EP inverses of \( \mathcal{A} \). Then

\[
\mathcal{X} \ast_{N} \mathcal{A}^{k+1} = \mathcal{A}^{k}, \quad \mathcal{A} \ast_{N} \mathcal{X}^{2} = \mathcal{X}, \quad (\mathcal{A} \ast_{N} \mathcal{X})^{*} = \mathcal{A} \ast_{N} \mathcal{X}
\]

and

\[
\mathcal{Y} \ast_{N} \mathcal{A}^{k+1} = \mathcal{A}^{k}, \quad \mathcal{A} \ast_{N} \mathcal{Y}^{2} = \mathcal{Y}, \quad (\mathcal{A} \ast_{N} \mathcal{Y})^{*} = \mathcal{A} \ast_{N} \mathcal{Y}.
\]
By Lemma 4.1(d), \( X^{m+1} \circ A^m = \mathcal{A}^D \) and \( Y^{m+1} \circ A^m = \mathcal{A}^D \) for any \( m \geq k \). Again by Lemma 4.1(c), we have \( A^k \circ A^{k+1} \circ A^m = \mathcal{A}^k \). Now
\[
(A^{k+1} \circ X)^* = (A \circ X)^* = A \circ X = A \circ A^{k} X^k,
\]
\[
X^{k+1} \circ (A^k)^2 = X^k \circ A^k \circ X^k
\]
\[
= X^k \circ A^k \circ A^k = X^k \circ A^{k+1} X^k = X^k \circ A^{k+1} X^k = X^k \circ A^k X^k = X^k.
\]
This yield \( X^k \) is the core inverse of \( A^k \). Similarly, we can show that \( Y^k \) is also the core inverse of \( A^k \). From the uniqueness of the core inverse, it follows \( X^k = Y^k \). Now
\[
X^k = X^k \circ A^k \circ X^k = X^k \circ A^k \circ X^k
\]
\[
= X^k \circ A^k \circ X^k = X^2 \circ A^k \circ X^k = X^2 \circ A^k \circ X^k.
\]
Continuing in the same way, we obtain
\[
X^{k+1} = X^k \circ A^k \circ X^k = X^{k+1} \circ A^k \circ X^k
\]
\[
= A^D \circ A^k \circ X^k = A^D \circ A^k \circ X^k = A^D \circ A^D \circ X^k
\]
\[
= Y^{k+1} \circ A^k \circ X^k = Y^{k+1} \circ A^k \circ X^k = Y^k.
\]
Hence, the core-EP inverse is unique. \( \square \)

The existence of the core-EP inverse and its computation can be discussed through other generalized inverses. Theorem 4.2 gives some characterizations of the core inverse in that direction.

**Theorem 4.2** Let \( A \in C(I^{(N)} \times I^{(N)}) \) and \( \text{ind}(A) = k \). Then, \( A^{\circ} = A^D \circ A^{l} \circ (A^l)^\dagger \), where \( l \) is an integer satisfying \( l \geq k \). Furthermore, \( A \circ A^{\circ} = A^{l} \circ (A^l)^\dagger \).

**Proof** It is necessary to verify that \( X' := A^D \circ A^{l} \circ (A^l)^\dagger \) satisfies \((EP)\), \((C2)\) and \((3T)\).

The equation \((EP)\) can be verified using
\[
X' = A^D \circ A^{l} \circ (A^l)^\dagger.
\]

Further, \((C2)\) follows from
\[
A \circ X^2 = A \circ A^D \circ A^{l} \circ (A^l)^\dagger \circ A \circ A^D \circ A^{l} \circ (A^l)^\dagger
\]
\[
= A^D \circ A^{l} \circ A^D \circ A^{l} \circ (A^l)^\dagger \circ A \circ A^D \circ A^{l} \circ (A^l)^\dagger
\]
\[
= A^D \circ A^{l} \circ A^D \circ A^{l} \circ (A^l)^\dagger
\]
\[
= A^D \circ A^{l} \circ (A^l)^\dagger = X.
\]
The equation (3') is verified as
\[
(A \ast N X)^* = (A \ast N A D \ast N A^l \ast N (A^l)^\dagger)^* = (A D \ast N A \ast N A^l \ast N (A^l)^\dagger)^* \\
= (A^l \ast N (A^l)^\dagger)^* = A^l \ast N (A^l)^\dagger = A \ast N X.
\]
So, \( \mathcal{X} \) is the core-EP inverse of \( A \).

The representations obtained in Corollary 4.1 generalize Prasad and Mohana (2014, Corollary 3.3).

**Corollary 4.1** Let \( A \in \mathbb{C}^{I(N) \times I(N)} \) and \( \text{ind}(A) = k \). Then
\[
A^\circ = A \ast N (A^{l+1})^\ast = A \ast N (A^{l+1})^\dagger, \tag{4.1}
\]
where \( l \) is an integer satisfying \( l \geq k \).

**Proof** Using known representation of the core inverse from Theorem 3.2 in conjunction with known properties of the Drazin and the group inverse, one can verify
\[
A \ast N (A^{l+1})^\ast = A \ast N (A^{l+1})^\# \ast N A^{l+1} \ast N (A^{l+1})^\dagger \\
= A \ast N (A^D)^{l+1} \ast N A^{l+1} \ast N (A^{l+1})^\dagger \\
= A D \ast N A^{l+1} \ast N (A^{l+1})^\dagger.
\]

Now, according to the representation of the core-EP inverse from Theorem 4.2, it follows that \( A \ast N (A^{l+1})^\circ = A^\circ \). Finally, in view of \( A^D \ast N A^{l+1} = A^l \), it follows that \( A^\circ = A \ast N (A^{l+1})^\dagger \).

One can observe that the proof of Theorem 4.2 requires only \{1, 3\} inverse of \( A^l \). Hence, using similar lines, we generalize the theorem as follows:

**Theorem 4.3** Let \( A \in \mathbb{C}^{I(N) \times I(N)} \) and \( \text{ind}(A) = k \). Then, \( A^\circ = A D \ast N A^l \ast N (A^l)^{(1,3)} \), where \( l \) is an integer with \( l \geq k \). Furthermore, \( A \ast N A^\circ = A \ast N (A^l)^{(1,3)} \).

The core-EP inverse can be computed through the core inverse. The next result is in this direction.

**Theorem 4.4** Let \( A \in \mathbb{C}^{I(N) \times I(N)} \) and \( \text{ind}(A) = m \). Then \( A \) is core-EP invertible if and only if \( A^m \) is core invertible. Furthermore, \( (A^m)^\circ = (A^\circ)^m \) and \( A^\circ = A^{m-1} \ast N (A^m)^\circ \)

**Proof** Let \( (A^\circ)^\circ = \mathcal{X} \) and \( \mathcal{Y} = \mathcal{X}^m \). Now using Lemma 4.1, we obtain
\[
\mathcal{Y} \ast N (A^m)^2 = \mathcal{X}^m \ast N (A^m)^2 = \mathcal{X}^{m-1} \ast N (\mathcal{X}^m \ast N A^2)A^{m-2} \ast N A^m = \cdots = A^m,
\]
\[
A^{m} \ast N \mathcal{Y}^2 = A^{m} \ast N (\mathcal{X}^{m})^2 = (A^{m} \ast N \mathcal{X}^{m}) \ast N A^{m} = A \ast N A \ast N \mathcal{X}^{m} = \mathcal{X}^{m} = \mathcal{Y}, \text{ and}
\]
\[
(A^{m} \ast N \mathcal{Y})^* = (A^{m} \ast N \mathcal{X}^{m})^* = (A^{m} \ast N \mathcal{X})^* \ast N A^{m} = A \ast N \mathcal{X} = A^{m} \ast N \mathcal{X}^{m} = A^{m} \ast N \mathcal{Y}.
\]
Therefore, \( A^m \) is core invertible and \( (A^m)^\circ = \mathcal{Y} = (A^\circ)^m \). To show the converse part, let \( A^m \) be core invertible. So by Definition 3.1, we have the following expressions:
\[
(A^{m})^\circ \ast N (A^m)^2 = A^{m}, \ A^{m} \ast N ((A^m)^\circ)^2 = (A^m)^\circ, \text{ and } (A^{m} \ast N (A^m)^\circ)^* = A^{m} \ast N (A^m)^\circ.
\]
Now, consider $\mathcal{X} := A^{m-1}*_{N}(A^{m})^\circ$. Then, we have

\[
\mathcal{X}*_{N}A^{m+1} = A^{m-1}*_{N}(A^{m})^\circ*_{N}A^{m+1} = A^{m-1}*_{N}(A^{m})*_{N}A^{m+1} = A^m,
\]

\[
A*_{N}\mathcal{X}^2 = A*_{N}(A^{m-1}*_{N}(A^{m})^\circ)^2 = A*_{N}(A^{m})(*_{N}A^{m-1}*_{N}(A^{m})^\circ) = A^{m-1}*_{N}(A^{m})^\circ = \mathcal{X},
\]

\[
(A*_{N}\mathcal{X})^* = (A*_{N}A^{m-1}*_{N}(A^{m})^\circ)^* = (A*_{N}(A^{m}))^* = A*_{N}(A^{m})^\circ = A*_{N}\mathcal{X}.
\]

Thus, $\mathcal{X}$ is the core-EP inverse of $A$ and hence $A^{\odot} = \mathcal{X} = A^{m-1}*_{N}(A^{m})^\circ$. $\square$

The power of the core-EP inverse of a tensor and the core-EP of a tensor power is discussed in the next theorem.

**Theorem 4.5** Let $A \in \mathbb{C}^{I(N) \times I(N)}$ and $k$ be any positive integer. Then, $A$ is core-EP invertible if and only if $A^k$ is core-EP invertible. In particular, $(A^k)^{\odot} = (A^{\odot})^k$ and $A^{\odot} = A^{k-1}*_{N}(A^{k})^{\odot}$.

**Proof** Let $\text{ind}(A) = m$ and $A^{\odot} = \mathcal{X}$. Then by definition of core-EP inverse, we have $\mathcal{X}*_{N}A^{m+1} = A^m$, $A*_{N}\mathcal{X}^2 = \mathcal{X}$, $(A*_{N}\mathcal{X})^* = A*_{N}\mathcal{X}$. Choose an integer $n$ such that $0 \leq kn - m < k$. Now, we have the following:

\[
(A^k)^n = A^{kn} = A^{m}*_{N}A^{kn-m} = \mathcal{X}*_{N}A^{m+1}*_{N}A^{kn-m} = \mathcal{X}*_{N}A^{kn+1} = \mathcal{X}*_{N}(A^k)^n*_{N}A
\]

\[
= \mathcal{X}^2*_{N}(A^k)^n*_{N}A^2 = \cdots = \mathcal{X}^k*_{N}(A^k)^n*_{N}A^k = \mathcal{X}^k*_{N}(A^k)^{n+1};
\]

\[
A^k*_{N}(A^k)^2*_{N}A^k = A^k*_{N}\mathcal{X}*_{N}A^k = (A*_{N}\mathcal{X}^2)*_{N}A^k-1 = \mathcal{X}*_{N}A^k-1 = \mathcal{X}^k;
\]

\[
(A^k*_{N}A^k)^* = (A*_{N}\mathcal{X})^* = A*_{N}\mathcal{X} = A^k*_{N}A^k.
\]

Therefore, $\mathcal{X}^k$ is the core-EP inverse of $A^k$ and hence $(A^k)^{\odot} = \mathcal{X}^k = (A^{\odot})^k$ with $\text{ind}(A^k) \leq n$.

Conversely, suppose $\text{ind}(A^k) = n$ and $\mathcal{Y} := (A^k)^{\odot}$. So by the definition, we have $\mathcal{Y}*_{N}(A^k)^{n+1} = (A^k)^n$, $A^k*_{N}\mathcal{Y}^2 = \mathcal{Y}$, and $(A^k*_{N}\mathcal{Y})^* = A^k*_{N}\mathcal{Y}$. To claim the converse part, it is enough to show that $\mathcal{X} := A^{k-1}*_{N}\mathcal{Y}$ is the core-EP inverse of $A$. The equation $(EP)$ is satisfied because of

\[
\mathcal{X}*_{N}A^{kn+1} = A^{k-1}*_{N}\mathcal{Y}*_{N}A^{kn+1} = A^{k-1}*_{N}A^{k}*_{N}\mathcal{Y}^2*_{N}A^{kn+1}
\]

\[
= A^{k-1}*_{N}A^{k}*_{N}\mathcal{Y}*_{N}A^{kn+1} = A^{k-1}*_{N}(A^{2k}*_{N}\mathcal{Y}*_{N}\mathcal{Y}^2)*_{N}A^{kn+1}
\]

\[
= \cdots = A^{k-1}*_{N}(A^{nk}*_{N}\mathcal{Y}^n)*_{N}A^{kn+1}
\]

\[
= A^{kn+k-1}*_{N}\mathcal{Y}*_{N}A^{kn+1} = A^{kn+k-1}*_{N}\mathcal{Y}^n*_{N}(A^k)^n*_{N}A
\]

\[
= A^{kn+k-1}*_{N}(A^k)^D*_{N}A^k = (A^k)^D*_{N}A^{kn+k} = (A^k)^D*_{N}(A^k)^2*_{N}A^{kn-k}
\]

\[
= A^k*_{N}A^{kn-k} = A^{kn},
\]
Also, (C2) is satisfied
\[
\mathcal{A}\mathcal{N}^2 = \mathcal{A}^{*N}A^{k-1*N}\mathcal{V} = \mathcal{A}^{k*N}\mathcal{V} = \mathcal{A}^{k*N}\mathcal{V} + \mathcal{A}^{*N}A^{k-1*N}\mathcal{V}
\]
\[
= \mathcal{A}^{k*N}\mathcal{V} + \mathcal{A}^{*N}(A^{k*N}\mathcal{V})^2
\]
\[
= \mathcal{A}^{k*N}\mathcal{V} + \mathcal{A}^{*N}(A^{k*N}\mathcal{V})^{n+2}
\]
\[
= \mathcal{A}^{k*N}\mathcal{V} + (A^{k*N}\mathcal{V})^{n+2}
\]
\[
= \mathcal{A}^{k*N}\mathcal{V} + \mathcal{A}^{k*N}\mathcal{V} + \mathcal{A}^{k*N}\mathcal{V} = \mathcal{A}^{k*N}\mathcal{V} + \mathcal{A}^{k*N}\mathcal{V}
\]
Finally, (T) is verified by
\[
(A^{*N}\mathcal{V})^* = (A^{*N}A^{k-1*N}\mathcal{V})^* = (A^{k*N}\mathcal{V})^* = A^{k*N}\mathcal{V} = A^{*N}\mathcal{V}
\]
Thus, \(\mathcal{A}\) is the core-EP inverse of \(\mathcal{A}\) with \(\text{ind}(\mathcal{A}) \leq kn\). Therefore, \(A^{\oplus} = \mathcal{A}^{k-1*N}\mathcal{V} = A^{k-1*N}(A^{k})^\oplus\).

**Theorem 4.6** Let \(\mathcal{A} \in C(I(N) \times I(N))\) with \(\text{ind}(\mathcal{A}) = m\). Then, \((A^{\oplus})^\oplus = A^{2*N}A^{\oplus}\)

**Proof** For \(m = 0\), the result is trivial, i.e.,
\[
(A^{-1})^{-1} = A = A^{*N}A^{*N}A^{-1} = A^{2*N}A^{-1}
\]
Further, for \(m = 1\), the result follows from Theorem 3.2, i.e.,
\[
(A^{\oplus})^\oplus = A^{2*N}A^\oplus = A^{2*N}A^{\oplus}.
\]
Let \(m \geq 2\) and \(\mathcal{V} := \mathcal{A}^\#.\) So by definition, we have \(\mathcal{V}A^{m+1} = A^m\), \(A^{*N}\mathcal{V}^2 = \mathcal{V}\), and \((A^{*N}\mathcal{V})^* = A^{*N}\mathcal{V}\). Let us assume \(\mathcal{V} = A^{2*N}\mathcal{V}\). To claim the result, we need to show \(\mathcal{V}\) is core-EP inverse of \(\mathcal{V}\). As
\[
\mathcal{V}A^m = A^{2*N}\mathcal{V}A^m = A^{*N}A^{*N}A^{2*N}\mathcal{V} = A^{*N}\mathcal{V}A^m = \mathcal{V}A^m
\]
\[
\mathcal{V}A^m = A^{2*N}\mathcal{V}A^m = A^{*N}A^{*N}A^{2*N}\mathcal{V} = A^{*N}\mathcal{V}A^m = \mathcal{V}A^m
\]
\[
\mathcal{V}A^m = A^{2*N}\mathcal{V}A^m = A^{*N}A^{*N}A^{2*N}\mathcal{V} = A^{*N}\mathcal{V}A^m = \mathcal{V}A^m
\]
\[
\mathcal{V}A^m = A^{2*N}\mathcal{V}A^m = A^{*N}A^{*N}A^{2*N}\mathcal{V} = A^{*N}\mathcal{V}A^m = \mathcal{V}A^m
\]
\[
\mathcal{V}A^m = A^{2*N}\mathcal{V}A^m = A^{*N}A^{*N}A^{2*N}\mathcal{V} = A^{*N}\mathcal{V}A^m = \mathcal{V}A^m
\]
Therefore, \((A^{\oplus})^\oplus = \mathcal{V} = A^{2*N}\mathcal{V} = A^{2*N}A^{\oplus}\).

Since \(A^{\oplus}\) is of index 1 and \(\mathcal{V}A^m = A^{2*N}\mathcal{V} = A^{*N}A^{*N}A^{2*N}\mathcal{V} = A^{*N}\mathcal{V}\), we conclude as a corollary.

**Corollary 4.2** Let \(\mathcal{A} \in C(I(N) \times I(N))\) with \(\text{ind}(\mathcal{A}) = m\). Then, \((A^{\oplus})^\oplus = A^{2*N}A^{\oplus} = (A^{\oplus})^\oplus\).
Corollary 4.3 Let $A \in \mathbb{C}^{I(N) \times I(N)}$ with $\text{ind}(A) = m$. Then, $((A^\oplus)^\ominus)^\ominus = A^\ominus$.

**Proof** Let $B = A^\ominus$. By Theorem 4.6, we have

\[
((A^\ominus)^\ominus)^\ominus = (B^\ominus)^\ominus = B^\ominus \circ N B^\ominus = (A^\ominus)^2 \circ N (A^\ominus)^\ominus = (A^\ominus)^2 \circ N (A^\ominus)^\ominus = A^{\ominus \circ N} A^{\ominus \circ N} A^{m+1} \circ N (A^\ominus)^m = A^{\ominus \circ N} A^{\ominus \circ N} (A^\ominus)^m = A^{\ominus \circ N} A^{\ominus \circ N} A^\ominus = A^\ominus \quad \text{(by Lemma 4.1)}.
\]

$\square$

In the case of the Drazin inverse, the following result was proved in [4].

**Theorem 4.7** If $A \circ N B = O = B \circ N A$, then $(A + B)^D = A^D + B^D$.

Last part of this section discusses additive property of the core-EP inverse.

**Theorem 4.8** Let $A \circ N B = O = B \circ N A$, and $A^* \circ N B = O$. Then, $(A + B)^\ominus = A^\ominus + B^\ominus$.

**Proof** Let us assume $A \circ N B = O = B \circ N A$, and $A^* \circ N B = O = (A^* \circ N B)^* = B^* \circ N A$. Using these assumptions, we obtain

\[
A \circ N B^\ominus = A \circ N B^\ominus (B^\ominus)^2 = O,
B^\ominus \circ N A^\ominus = B^\ominus \circ N A^\ominus (A^\ominus)^2 = O,
B^\ominus \circ N A = B^\ominus \circ N B^\ominus B^\ominus \circ N A = B^\ominus \circ N (B^\ominus)^* \circ N B^\ominus A = O,
A^\ominus \circ N B = A^\ominus \circ N A^\ominus A^\ominus \circ N B = A^\ominus \circ N (A^\ominus)^* \circ N A^\ominus B = O,
A^\ominus \circ N B^\ominus = A^\ominus \circ N (A^\ominus)^* \circ N A^\ominus B^\ominus = O^\ominus \circ N B^\ominus A^\ominus (A^\ominus)^2 = O.
\]

Let $m = \max\{\text{ind}(A, B)\}$. Then by Lemma 4.1, $A^m \circ N (A^\ominus)^m \circ N A^m = A^m$ and $B^m \circ N (B^\ominus)^m \circ N B^m = B^m$. Now we see that

\[
(A + B)^m \circ N \left( (A^\ominus)^m + (B^\ominus)^m \right) \circ N (A + B)^m
= (A^m + B^m) \circ N (A^\ominus)^m + (B^\ominus)^m \circ N (A^m + B^m)
= (A^m \circ N (A^\ominus)^m + B^m \circ N (B^\ominus)^m) \circ N (A^m + B^m)
= (A^\ominus \circ N A^m + B^\ominus \circ N B^m) \circ N (A^m + B^m)
= A^\ominus \circ N A^m + B^\ominus \circ N B^m = A^m \circ N (A^\ominus)^m \circ N A^m + B^m \circ N (B^\ominus)^m \circ N B^m
= A^m + B^m.
\]

and

\[
((A + B)^m \circ N (A^\ominus)^m + (B^\ominus)^m))^* = (A^\ominus \circ N A^\ominus + B^\ominus \circ N B^\ominus)^*
= (A^\ominus \circ N A^\ominus)^* + (B^\ominus \circ N B^\ominus)^* = A^\ominus \circ N A^\ominus + B^\ominus \circ N B^\ominus
= (A + B)^m \circ N (A^\ominus)^m + (B^\ominus)^m).
\]
Thus, \((A^\oplus)^m + (B^\oplus)^m\) is \(\{1^T, 3^T\}\) inverse of \((A + B)^m\). So, by Theorem 4.3, we obtain
\[
(A + B)^\oplus = (A + B) D_N (A + B)_N (A^\oplus)^m + (B^\oplus)^m \\
= (A D + B D) D_N (A^m + B^m)_N (A^\oplus)^m + (B^\oplus)^m \\
= (A D_N A^m + B D_N B^m)_N (A^\oplus)^m + (B^\oplus)^m \\
= A D_N A^m \oplus + B D_N B^m \oplus (B^\oplus)^m \\
= A D_N A^m \oplus (A^m)^{\oplus} + B D_N B^m \oplus (B^m)^{\oplus} \\
= A D_N A^m \oplus (A^m)^{(1,3)} + B D_N B^m \oplus (B^m)^{(1,3)} = A^\oplus + B^\oplus.
\]
This completes the proof. \(\square\)

5 Numerical examples

In this section, we present several numerical examples to verify selected properties discussed in the previous sections. All examples were carefully implemented in MATLAB.

Example 5.1 This example is aimed to the verification of parts (a), (d) and (g) of Theorem 3.2. Let \(A \in \mathbb{R}^{(2 \times 2) \times (2 \times 2)}\) with
\[
A(:, :, 1, 1) = 10^2 \cdot \begin{bmatrix}
0.985940927109977 & 1.682512984915278 \\
1.42072484319284 & 1.962489222569553
\end{bmatrix}, \quad A(:, :, 2, 1) = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix},
\]
\[
A(:, :, 1, 2) = 10^2 \cdot \begin{bmatrix}
8.929224052859770 & 5.557379427193866 \\
7.032232245652910 & 1.844336677576532
\end{bmatrix}, \quad A(:, :, 2, 2) = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}.
\]

Verification of part (a) of Theorem 3.2:

We shall compute the core inverse \(A^\ominus\) with the formula \(A^\ominus = A^\#_N A^\#_N A^\dagger\), which turns out to be \(A^\ominus = A^\#_N A^3 \dagger N A^3 \dagger\), since it holds \(A^\# = A^3 \dagger A\). We have that
\[
A^\dagger(:, :, 1, 1) = \begin{bmatrix}
-0.002139621590719 & 0.000961949275511 \\
0 & 0
\end{bmatrix},
\]
\[
A^\dagger(:, :, 2, 1) = 10^{-3} \cdot \begin{bmatrix}
0.154116174683626 & 0.400242571625946 \\
0 & 0
\end{bmatrix},
\]
\[
A^\dagger(:, :, 1, 2) = \begin{bmatrix}
0.001721913469812 & 0.000005405961529 \\
0 & 0
\end{bmatrix},
\]
\[
A^\dagger(:, :, 2, 2) = \begin{bmatrix}
0.004582706318709 & -0.000777570778470 \\
0 & 0
\end{bmatrix},
\]
and
\[
(A^3)^\dagger(:, :, 1, 1) = 10^{-6} \cdot \begin{bmatrix}
-0.169528528162183 & 0.042750108332711 \\
0 & 0
\end{bmatrix},
\]
\[
(A^3)^\dagger(:, :, 2, 1) = 10^{-7} \cdot \begin{bmatrix}
-0.179043837563866 & 0.051654956705111 \\
0 & 0
\end{bmatrix},
\]
\[
(A^3)^\dagger(:, :, 1, 2) = 10^{-7} \cdot \begin{bmatrix}
0.864318723074744 & -0.207154961229323 \\
0 & 0
\end{bmatrix},
\]
\[
(a^3)^\dagger(:, :, 2, 2) = 10^{-6} \cdot \begin{bmatrix}
0.280825799119855 & -0.069038738080892 \\
0 & 0
\end{bmatrix}.
\]
Then, the core inverse of \( A \) is the following:

\[
A^\#(\cdot ; \cdot, 1, 1) = \begin{bmatrix}
-0.002139621590719 & 0.000961949275511 \\
-0.000303613542775 & 0.00332187328937 \\
\end{bmatrix},
\]

\[
A^\#(\cdot ; \cdot, 2, 1) = 10^{-3} \begin{bmatrix}
0.154116174683627 & 0.400242571625946 \\
0.304669984681755 & 0.532596423124327 \\
\end{bmatrix},
\]

\[
A^\#(\cdot ; \cdot, 1, 2) = \begin{bmatrix}
0.001721913469812 & 0.000005405961529 \\
0.000713871950958 & -0.001398898794183 \\
\end{bmatrix},
\]

\[
A^\#(\cdot ; \cdot, 2, 2) = \begin{bmatrix}
0.004582706318709 & -0.00077570778470 \\
0.001422901360057 & -0.005026199525584 \\
\end{bmatrix}.
\]

Now, we shall verify the identities \((C1), (C2), (3^T)\) for the resulted core inverse \(A^\#\).

We have that

\[
A^2(\cdot ; \cdot, 1, 1) = 10^5 \begin{bmatrix}
1.599561492590487 & 1.100922146057665 \\
1.323212683603806 & 0.503801885212158 \\
\end{bmatrix}, \quad A^2(\cdot ; \cdot, 1, 2) = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix},
\]

\[
A^2(\cdot ; \cdot, 2, 1) = 10^5 \begin{bmatrix}
5.842677349321679 & 4.590800151195201 \\
5.176271403733929 & 2.777318467849203 \\
\end{bmatrix}, \quad A^2(\cdot ; \cdot, 2, 2) = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix}.
\]

Then, we have that \(A^\#(\cdot, N) A^2 = A\); hence, \((C1)\) is true.

Also, it holds that

\[
(A^\#)^2(\cdot ; \cdot, 1, 1) = 10^{-4} \begin{bmatrix}
0.2148508181358990 & -0.047655377387934 \\
0.059851986462700 & -0.253823166679777 \\
\end{bmatrix},
\]

\[
(A^\#)^2(\cdot ; \cdot, 2, 1) = 10^{-5} \begin{bmatrix}
0.284712034659642 & -0.014177387864857 \\
0.108958616205944 & -0.256102470356689 \\
\end{bmatrix},
\]

\[
(A^\#)^2(\cdot ; \cdot, 1, 2) = 10^{-4} \begin{bmatrix}
-0.099756585933128 & 0.030298875490167 \\
-0.022919369812014 & 0.131415268596208 \\
\end{bmatrix},
\]

\[
(A^\#)^2(\cdot ; \cdot, 2, 2) = 10^{-4} \begin{bmatrix}
0.339584711910359 & 0.08881859082190 \\
-0.086647284748517 & 0.423786927376011 \\
\end{bmatrix}.
\]

hence, we have that \(A^\#(\cdot, N)(A^\#)^2 = A^\#\) and \((C2)\) is true.

Finally, it holds that

\[
A^\#(\cdot, N)A^\#(\cdot ; \cdot, 1, 1) = \begin{bmatrix}
0.647992011370661 & 0.174597600453917 \\
0.372580504169106 & -0.242482598137053 \\
\end{bmatrix},
\]

\[
A^\#(\cdot, N)A^\#(\cdot ; \cdot, 2, 1) = \begin{bmatrix}
0.372580504169107 & 0.248360229853187 \\
0.303348568052670 & 0.10406338661755 \\
\end{bmatrix},
\]

\[
A^\#(\cdot, N)A^\#(\cdot ; \cdot, 1, 2) = \begin{bmatrix}
0.174597600453919 & 0.292718475124185 \\
0.248360229853188 & 0.338920703982751 \\
\end{bmatrix},
\]

\[
A^\#(\cdot, N)A^\#(\cdot ; \cdot, 2, 2) = \begin{bmatrix}
-0.242482598137050 & 0.338920703982753 \\
0.10406338661757 & 0.755940945452490 \\
\end{bmatrix}.
\]

Therefore, equation \((3^T)\) is true.

**Verification of part (d) of Theorem 3.2:**
According to our calculations for part (a), it is easy to see that
\[
\mathcal{A}^\otimes N \mathcal{A}^2 N \mathcal{A}^\dagger (:, 1, 1) = \mathcal{A}^2 N \mathcal{A}^\dagger N \mathcal{A}^\otimes (:, 1, 1)
\]
\[
= \begin{bmatrix}
0.647992011370665 & 0.174597600453918 \\
0.372580504169107 & -0.242482598137055
\end{bmatrix},
\]
\[
\mathcal{A}^\otimes N \mathcal{A}^2 N \mathcal{A}^\dagger (:, 2, 1) = \mathcal{A}^2 N \mathcal{A}^\dagger N \mathcal{A}^\otimes (:, 2, 1)
\]
\[
= \begin{bmatrix}
0.372580504169109 & 0.248360229853187 \\
0.303348568052670 & 0.104063338661752
\end{bmatrix},
\]
\[
\mathcal{A}^\otimes N \mathcal{A}^2 N \mathcal{A}^\dagger (:, 1, 2) = \mathcal{A}^2 N \mathcal{A}^\dagger N \mathcal{A}^\otimes (:, 1, 2)
\]
\[
= \begin{bmatrix}
0.174597600453919 & 0.292718475124185 \\
0.248360229853188 & 0.338920703982751
\end{bmatrix},
\]
\[
\mathcal{A}^\otimes N \mathcal{A}^2 N \mathcal{A}^\dagger (:, 2, 2) = \mathcal{A}^2 N \mathcal{A}^\dagger N \mathcal{A}^\otimes (:, 2, 2)
\]
\[
= \begin{bmatrix}
-0.242482598137050 & 0.338920703982753 \\
0.104063338661757 & 0.755940945452490
\end{bmatrix}.
\]

Hence, from part (c) we have \( \mathcal{A}^\otimes N (\mathcal{A}^\dagger) = (\mathcal{A}^\dagger) N \mathcal{A}^\otimes \) and \( \mathcal{A}^\otimes \) is EP.

**Verification of part (g) of Theorem 3.2:**

Since, it holds \( \mathcal{A}^\# = \mathcal{A}(\mathcal{A}^3)^\dagger \mathcal{A} \), we have that
\[
\mathcal{A}^\# (:, 1, 1) = \begin{bmatrix}
-0.005822728386101 & 0.001762848163527 \\
-0.001341208843236 & 0.007661282558445
\end{bmatrix},
\]
\[
\mathcal{A}^\# (:, 2, 1) = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix},
\]
\[
\mathcal{A}^\# (:, 1, 2) = \begin{bmatrix}
0.009355568940286 & -0.00103301678392 \\
0.003238756270141 & -0.009348711331342
\end{bmatrix},
\]
\[
\mathcal{A}^\# (:, 2, 2) = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

and
\[
\mathcal{A}^\otimes N \mathcal{A} (:, 1, 1) = \mathcal{A}^\# N \mathcal{A} (:, 1, 1) = \begin{bmatrix}
1.000000000000000 & -0.000000000000000 \\
0.412689678913960 & -0.817575617868222
\end{bmatrix},
\]
\[
\mathcal{A}^\otimes N \mathcal{A} (:, 2, 1) = \mathcal{A}^\# N \mathcal{A} (:, 2, 1) = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix},
\]
\[
\mathcal{A}^\otimes N \mathcal{A} (:, 1, 2) = \mathcal{A}^\# N \mathcal{A} (:, 1, 2) = \begin{bmatrix}
-0.000000000000000 & 1.000000000000001 \\
0.602304320244615 & 1.645497247305027
\end{bmatrix},
\]
\[
\mathcal{A}^\otimes N \mathcal{A} (:, 2, 2) = \mathcal{A}^\# N \mathcal{A} (:, 2, 2) = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}.
\]

Hence, part (g) of Theorem 3.2 is verified.

**Example 5.2** In this example, we will verify the representation \( \mathcal{A}^\odot = \mathcal{A}^D * \mathcal{A}^l * \mathcal{A}^\dagger (\mathcal{A}^l)^* \), \( l \geq \text{ind}(\mathcal{A}) \), from Theorem 4.2. We consider the tensor \( \mathcal{A} \in \mathbb{R}^{(2\times 2) \times (2\times 2)} \) with
\[
\mathcal{A} (:, 1, 1) = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}, \quad \mathcal{A} (:, 2, 1) = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix},
\]
\[
\mathcal{A} (:, 1, 2) = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}, \quad \mathcal{A} (:, 2, 2) = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}.
\]

We observe that \( \text{rshrank}(\mathcal{A}) = 2 \), while \( \text{rshrank}(\mathcal{A}^2) = \text{rshrank}(\mathcal{A}^3) = 1 \). Hence, \( \text{ind}(\mathcal{A}) = 2 \). The Drazin inverse of a tensor is defined by the formula \( \mathcal{A}^D = \mathcal{A}^l (\mathcal{A}^{2l+1})^\dagger \mathcal{A}^l \). So, we
need to verify the identities \((EP), (C2)\) and \((3^T)\) for the tensor defined by the formula \(A^\oplus = A^2_{\ast N} (A^5)_{\ast N} A^4_{\ast N} (A^2)_{\ast N}\). It holds \((A^2)^\dagger = (A^5)^\dagger\), where

\[
(A^2)^\dagger(:, 1, 1) = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \end{bmatrix}, \quad (A^2)^\dagger(:, 2, 1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

\[
(A^2)^\dagger(:, 1, 2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad (A^2)^\dagger(:, 2, 2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

Therefore, according to the identity \(A^\oplus = A^2_{\ast N} (A^5)_{\ast N} A^4_{\ast N} (A^2)_{\ast N}\), the core-EP inverse of \(A\) is

\[
A^\oplus(:, 1, 1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^\oplus(:, 2, 1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
A^\oplus(:, 1, 2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^\oplus(:, 2, 2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

Because of the previous calculations, the validity of \((EP), (C2)\) and \((3^T)\) is clear.

**Example 5.3** The aim of this example is to verify the representation \(A^\oplus = A^l_{\ast N} (A^l+1)_{\ast N}\), presented in Corollary 4.1. The input tensor \(A \in \mathbb{R}^{(2 \times 2) \times (2 \times 2)}\) with the entries

\[
A(:, 1, 1) = \begin{bmatrix} 2 & 8 \\ 4 & 16 \end{bmatrix}, \quad A(:, 2, 1) = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix},
\]

\[
A(:, 1, 2) = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}, \quad A(:, 2, 2) = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}.
\]

It holds that \(\text{rshrank}(A) = 3\) and \(\text{rshrank}(A^2) = \text{rshrank}(A^3) = 2\). Hence, \(\text{ind}(A) = 2\). For \(l = 2\), we will compute the core-EP inverse of \(A\) with the formula \(A^\oplus = A^l_{\ast N} (A^{l+1})_{\ast N}\). So, we need to verify the identities \((EP), (C2)\) and \((3^T)\) for the tensor defined by the formula \(A^\oplus = A^2_{\ast N} (A^3)_{\ast N}\).

We have that

\[
A^2(:, 1, 1) = \begin{bmatrix} 8 & 64 \\ 24 & 128 \end{bmatrix}, \quad A^2(:, 2, 1) = \begin{bmatrix} 4 & 32 \\ 12 & 64 \end{bmatrix},
\]

\[
A^2(:, 1, 2) = \begin{bmatrix} 1 & 12 \\ 4 & 24 \end{bmatrix}, \quad A^2(:, 2, 2) = \begin{bmatrix} 0 & 4 \\ 1 & 8 \end{bmatrix},
\]

\[
A^3(:, 1, 1) = \begin{bmatrix} 40 & 416 \\ 144 & 832 \end{bmatrix}, \quad A^3(:, 2, 1) = \begin{bmatrix} 20 & 208 \\ 72 & 416 \end{bmatrix},
\]

\[
A^3(:, 1, 2) = \begin{bmatrix} 6 & 72 \\ 24 & 144 \end{bmatrix}, \quad A^3(:, 2, 2) = \begin{bmatrix} 1 & 20 \\ 6 & 40 \end{bmatrix}
\]
and

\[(A^3)^\dagger(\cdot; 1, 1) = \begin{bmatrix} 0.048094852307653 & -0.270803595540296 \\ 0.024047426153826 & -0.276815452078752 \end{bmatrix},\]

\[(A^3)^\dagger(\cdot; 2, 1) = \begin{bmatrix} 0.047143242930443 & -0.264910690173993 \\ 0.023571621465222 & -0.270803595540299 \end{bmatrix},\]

\[(A^3)^\dagger(\cdot; 1, 2) = \begin{bmatrix} -0.003806437508841 & 0.023571621465222 \\ -0.001903218754421 & 0.024047426153827 \end{bmatrix},\]

\[(A^3)^\dagger(\cdot; 2, 2) = \begin{bmatrix} -0.007612875017682 & 0.047143242930443 \\ -0.003806437508841 & 0.048094852307653 \end{bmatrix}.\]

Therefore, the core-EP inverse \(A^{\odot} = A^*_N (A^3)^\dagger\) of \(A\) is equal to

\[A^{\odot}(\cdot; 1, 1) = \begin{bmatrix} 0.210144927536232 & -0.509316770186336 \\ 0.082815734989648 & -1.018633540372671 \end{bmatrix},\]

\[A^{\odot}(\cdot; 2, 1) = \begin{bmatrix} 0.206521739130437 & -0.490683229813671 \\ 0.083850931677020 & -0.981366459627341 \end{bmatrix},\]

\[A^{\odot}(\cdot; 1, 2) = \begin{bmatrix} -0.014492753623189 & 0.074534161490684 \\ 0.004140786749482 & 0.149068322981367 \end{bmatrix},\]

\[A^{\odot}(\cdot; 2, 2) = \begin{bmatrix} -0.028955072463777 & 0.149068322981367 \\ 0.008281573498965 & 0.298136645962735 \end{bmatrix}.\]

Now, we shall verify the identities \((EP)\), \((C2)\) and \((3^T)\) for the resulted core-EP inverse \(A^{\odot}\).

We have that

\[A^{\odot}A^3(\cdot; 1, 1) = \begin{bmatrix} 8 \\ 24 \end{bmatrix}, \quad A^{\odot}A^3(\cdot; 2, 1) = \begin{bmatrix} 4 \\ 12 \\ 64 \end{bmatrix},\]

\[A^{\odot}A^3(\cdot; 1, 2) = \begin{bmatrix} 1 \\ 4 \\ 24 \end{bmatrix}, \quad A^{\odot}A^3(\cdot; 2, 2) = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}.\]

Then, we have that \(A^{\odot}_N A^3 = A^2\); hence, \((EP)\) is true.

Also, it holds that

\[(A^{\odot})^2(\cdot; 1, 1) = \begin{bmatrix} 0.098171152518979 & -0.337474120082818 \\ 0.013802622498275 & -0.674948240165636 \end{bmatrix},\]

\[(A^{\odot})^2(\cdot; 2, 1) = \begin{bmatrix} 0.096273291925468 & -0.329192546583857 \\ 0.013975155279503 & -0.658385093167713 \end{bmatrix},\]

\[(A^{\odot})^2(\cdot; 1, 2) = \begin{bmatrix} -0.007591442374051 & 0.033126293995860 \\ 0.000690131124914 & 0.066252587991719 \end{bmatrix},\]

\[(A^{\odot})^2(\cdot; 2, 2) = \begin{bmatrix} -0.015182884748102 & 0.066252587991719 \\ 0.001380262249827 & 0.132505175983439 \end{bmatrix}.\]

Hence, we have that \(A^*_N(A^{\odot})^2 = A^{\odot}\) and \((C2)\) is satisfied.
Finally, it holds that
\[
\begin{align*}
A^*_N A^\odot(\cdot; 1, 1) &= \begin{bmatrix} 0.503105590062112 & -0.024844720496894 \\ 0.496894409937889 & -0.049689440993788 \end{bmatrix}, \\
A^*_N A^\odot(\cdot; 2, 1) &= \begin{bmatrix} 0.503105590062117 & 0.496894409937894 \\ 0.024844720496893 & 0.049689440993786 \end{bmatrix}, \\
A^*_N A^\odot(\cdot; 1, 2) &= \begin{bmatrix} -0.024844720496895 & 0.198757763975156 \\ 0.024844720496894 & 0.397515527950311 \end{bmatrix}, \\
A^*_N A^\odot(\cdot; 2, 2) &= \begin{bmatrix} -0.049689440993790 & 0.397515527950311 \\ 0.496894409937889 & 0.795031055900622 \end{bmatrix}.
\end{align*}
\]

Therefore, the equation \((3^T)\) is true.

6 Conclusion

So far, the core and core-EP inverses in the matrix case are introduced in different ways. Our research introduces the core and the core-EP inverses of complex square tensors by means of specific definitions. Additionally, some of their characterizations, representations and properties are investigated. The results are verified using specific algebraic approach, based on the proposed definitions and previously verified properties. The approach used here is new even in the matrix case. Illustrative numerical examples are presented.

To finalize our conclusion, it will be useful to mention some possibilities for further research.

1. According to Theorem 3.2(e), the tensor core inverse \(A^\#\) is EP. Therefore, it is reasonable to expect that \(A^\#\) inherits all properties of EP matrices stated in Tian and Wang (2011). These properties would be an interesting topic for further research.

2. It will be interesting to continue the results from Pappas et al. (2018) and investigate properties of the \(\lambda\)-Aluthge transform of the core inverse in both the matrix and the tensor case.

3. Also, the reverse order law of multiple tensor products would be an interesting area for further research.

Acknowledgements Rikitanka Behera acknowledges the support provided by Science and Engineering Research Board (SERB), Department of Science and Technology, India, under the Grant No. EEQ/2017/000747.

Predrag Stanimirović gratefully acknowledges support from the Ministry of Education, Science and Technological Development, Republic of Serbia, Grant No. 174013.

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