Quantum $j$-invariant in positive characteristic II:
formulas and values at the quadratics

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Abstract. In this sequel to Demangos and Gendron (Arch Math 107:23–35, 2016), the multi-valued quantum $j$-invariant in positive characteristic is studied at quadratic elements. For every quadratic $f$, an explicit expression for each of the values of $j_{\text{qt}}(f)$ is given as a limit of rational functions of $f$. It is proved that the number of values of $j_{\text{qt}}(f)$ is finite.

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1. Introduction. Let $\mathbb{F}_q$ be the field with $q$ elements, $q = p^r$ a prime power, $k = \mathbb{F}_q(T)$ the function field over $\mathbb{F}_q$, $k_{\infty}$ the completion of $k$ with respect to the valuation $v(f) = -\deg_T(f)$. In [2], the quantum $j$-invariant was introduced as a multi-valued modular function

$$j_{\text{qt}} : \text{GL}_2(A) \backslash k_{\infty} \rightarrow k_{\infty} \cup \{\infty\}$$

obtained as the limit of a sequence of approximating functions

$$j_{\varepsilon} : k_{\infty} \rightarrow k_{\infty} \cup \{\infty\}$$ (1)

as $\varepsilon \rightarrow 0$. Explicit formulas for the approximants (1) in terms of the sequence $\{q_i\} \subset A$ of best approximations of $f$ were obtained, and using them, it was shown that $f \in k$ if and only if eventually $j_{\varepsilon}(f) = \infty$.

In this paper we consider $j_{\text{qt}}(f)$ in the special case of $f \in k_{\infty}$ quadratic. Here we are able to derive explicit formulas not just for the approximants but for all the values of $j_{\text{qt}}(f)$, each of which expressed as a limit of certain rational functions of $f$. Using these formulas, we are able to prove that for $f$ quadratic, $\# j_{\text{qt}}(f) < \infty$.

In what follows, we fix notation used in [2].
2. Values at quadratic units. Fix \( a \in A - \mathbb{F}_q \), \( b \in \mathbb{F}_q^\times \) with
\[
d := \deg a > 0,
\]
and write \( D = a^2 + 4b \). The solutions \( f \) and \( f' \) of \( X^2 - aX - b = 0 \) are quadratic units, and every quadratic unit in \( k_\infty - k \) occurs in this way. Since \( ff' = -b \), we have \( |ff'| = 1 \). Moreover, since \( f + f' = a \), we have as well that \( |f|, |f'| \neq 1 \). Therefore, if we assume that \( |f| > 1 \), then
\[
|f| = |a| = q^d > 1 \quad \text{and} \quad |f'| = |b/a| = q^{-d} < 1.
\]

Consider the recursive sequence
\[
Q_0 = 1, Q_1 = a, \ldots, Q_{n+1} = aQ_n + bQ_{n-1}.
\]
Note that when \( b = 1 \), \( Q_n = q_n \) is the sequence of best approximations for \( f = [a, a, \ldots] \); see [2, §3]. It is clear that for all \( n \)
\[
|Q_n| = |a|^n = q^{dn}.
\]
Recall Binet’s formula
\[
Q_n = \frac{f^{n+1} - (f')^{n+1}}{\sqrt{D}}, \quad n = 0, 1, \ldots,
\]
which, while usually stated over \( \mathbb{Q} \), is equally true in this setting, by a very simple induction on \( n \). The root \( \sqrt{D} \) is chosen so that in odd characteristic, \( f = (a + \sqrt{D})/2 \); in even characteristic, we simply take \( \sqrt{D} = a \).

Recall that \( \|x\| \) is distance of \( x \) to the nearest element of \( A \); see [2, §2] for this and other relevant notation. Binet’s formula immediately gives
\[
\|Q_n f\| = q^{-(n+1)d}. \quad (2)
\]
Indeed,
\[
|Q_n f - Q_{n+1}| = \frac{|f'|^{n+1}|f - f'|}{\sqrt{D}} = |f'|^{n+1} = q^{-(n+1)d} < 1. \quad (3)
\]
In particular, \( Q_n^\perp = Q_{n+1} \), where \( \lambda^\perp \) denotes the element of \( A \) closest to \( \lambda f \).

The remainder of this section is devoted to deriving an explicit formula for each value of \( j^{\text{un}}(f) \). In view of our need to work with monic polynomials, we make the following modification to the above sequence. For each \( n \), we write \( Q_n \) for the unique monic polynomial obtained as \( c_n Q_n \) for some \( c_n \in \mathbb{F}_q^\times \). Then \( |Q_n| = |Q_n| \) and \( \|Q_n f\| = \|Q_n f\| \) since \( Q_n^\perp = c_n Q_n^\perp \). (N.B. \( Q_n^\perp \) is not necessarily monic.) Note that if we choose \( c \in \mathbb{F}_q^\times \) so that \( ca \) is monic, then \( c_n = c^n \). It follows from Binet’s formula that
\[
Q_n = c^n \frac{f^{n+1} - (f')^{n+1}}{\sqrt{D}} = \frac{\tilde{f}^{n+1} - (\tilde{f'})^{n+1}}{c\sqrt{D}}, \quad \tilde{f} := cf, \quad \tilde{f'} := cf'. \quad (4)
\]
The set
\[
\mathcal{B} = \{ T^{d-1}Q_0, \ldots, TQ_0, Q_0; T^{d-1}Q_1, \ldots, TQ_1, Q_1; \ldots \} \quad (5)
\]
is a basis for $A$ as an $\mathbb{F}_q$-vector space, as the degree map $\deg : B \to \mathbb{N} = \{0, 1, \ldots\}$ is a bijection. The order in which we have presented the elements of $B$ corresponds to decreasing errors: for $0 \leq l \leq d - 1$ and $n \geq 0$,

$$\| T^l \mathcal{Q}_n f \| = | T^l \mathcal{Q}_n f - T^l \mathcal{Q}_{n+1} | = q^{l-(n+1)d} < 1. \quad (6)$$

In particular, $(T^l \mathcal{Q}_n)^\perp = T^l \mathcal{Q}_n^\perp$, and the map

$$B \rightarrow q^{-\mathbb{N}}, \quad T^l \mathcal{Q}_n \mapsto \| T^l \mathcal{Q}_n f \| \quad (7)$$

defines a bijection between $B$ and the set of possible errors.

Write

$$B(i) = \{ T^{d-1} \mathcal{Q}_i, \ldots, \mathcal{Q}_i \}$$

for the $i$th block of $B$. Furthermore, for $0 \leq \tilde{d} \leq d - 1$, denote

$$B(i)_{\tilde{d}} = \{ T^d \mathcal{Q}_i, \ldots, \mathcal{Q}_i \}.$$ 

**Lemma 1.** Let $l \in \{0, \ldots, d - 1\}$ and write

$$d_l = d - 1 - l.$$ 

Then

$$\Lambda_{q^{-Nd-l}}(f) = \text{span}_{\mathbb{F}_q}(B(N)_{d_l}, B(N + 1), \ldots).$$

**Proof.** First observe that

$$\text{span}_{\mathbb{F}_q}(B(N)_{d_l}, B(N + 1), \ldots) \subset \Lambda_{q^{-Nd-l}}(f).$$

Indeed, for $T^d \mathcal{Q}_{N+r}$ with $\tilde{d} \leq d - 1$ and $\tilde{d} \leq d_l$ if $r = 0$, we have

$$\| T^d \mathcal{Q}_{N+r} f \| \leq q^{d_l}, q^{-(N+1)d} = q^{-Nd-l-1} < q^{-Nd-l}.$$ 

Moreover, by (6), $\Lambda_{q^{-Nd-l}}(f)$ contains no other elements of $B$. In view of the bijection (7), no linear combination of the excluded basis elements could appear in $\Lambda_{q^{-Nd-l}}(f)$. This proves the equality in the statement of the lemma. \(\square\)

We recall the main definitions established in [2]. For each $\varepsilon$, $\Lambda^{\text{mon}}_\varepsilon(f)$ is the subset of monic polynomials in $\Lambda_\varepsilon(f)$. We defined

$$\zeta_{f, \varepsilon}(n) := \sum_{\lambda \in \Lambda^{\text{mon}}_\varepsilon(f) \setminus \{0\}} \lambda^{-n}, \quad n \in \mathbb{N},$$

and

$$\tilde{j}_\varepsilon(f) = \frac{1}{Tq - T} - j_\varepsilon(f);$$

where

$$j_\varepsilon(f) = \frac{Tq^2 - T}{(Tq - T)^{q+1}} \cdot \tilde{j}_\varepsilon(f), \quad \tilde{j}_\varepsilon(f) := \frac{\zeta_{f, \varepsilon}(q^2 - 1)}{\zeta_{f, \varepsilon}(q - 1)^{q+1}}.$$ 

Then $j^{\text{at}}(f)$ is the set of limits of $j_\varepsilon(f)$ for $\varepsilon \to 0$. It follows from Lemma 1 that to calculate the values of $j^{\text{at}}(f)$, it suffices to consider the possible limits of $j_\varepsilon(f)$ formed from the spans of the initial truncations of $B$.

Let $\varepsilon = q^{-Nd-l}$. As in [2], we decompose

$$\Lambda^{\text{mon}}_\varepsilon(f) = \bigcup \Lambda^{\text{bas}}_\varepsilon(f) \bigcup \Lambda^{\text{non-bas}}_\varepsilon(f).$$
where
\[ \Lambda^\text{bas}_\varepsilon(f) = \{B(N)_d, B(N + 1), \ldots\} \]
and
\[ \Lambda^\text{non-bas}_\varepsilon(f) = \Lambda^\text{mon}_\varepsilon(f) - (\Lambda^\text{bas}_\varepsilon(f) \cup \{0\}) . \]
Any element \( \lambda \in \Lambda^\text{non-bas}_\varepsilon(f) \) may therefore be written in the form
\[ \lambda = c_0(T)Q_N + c_1(T)Q_{N + 1} + \cdots + c_m(T)Q_{N + m}, \]
where the \( c_i(T) \in A \) are polynomials that are not all zero and satisfy the following conditions:

I. \( c_m(T) \) is monic and if \( \lambda = c_m(T)\bar{Q}_{N + m} \), \( c_m(T) \neq T^j \) for all \( j \leq d - 1 \).

II. \( \deg(c_i(T)) \leq d - 1, \ i = 0, \ldots, m \).

III. \( \deg(c_0(T)) \leq d_l \).

In view of this characterization, sums over \( \lambda \in \Lambda^\text{non-bas}_\varepsilon(f) \) may be understood as sums indexed by tuples \( c_0(T), \ldots, c_m(T) \) subject to conditions I, II, and III, and so we will abbreviate
\[ \sum_{\text{Conditions I, II, III}} := \sum_{\Lambda^\text{non-bas}_\varepsilon(f)} \cdot \]

Using this notation, we may now express
\[ \tilde{J}_\varepsilon(f) = \frac{\sum_{\lambda \in \Lambda^\text{bas}_\varepsilon(f)} \lambda^{1-q^2} + \sum_{\text{Conditions I, II, III}} \left( \sum_{i=0}^{m} c_i(T)Q_{N + i} \right)^{1-q^2}}{\left( \sum_{\lambda \in \Lambda^\text{bas}_\varepsilon(f)} \lambda^{1-q} + \sum_{\text{Conditions I, II, III}} \left( \sum_{i=0}^{m} c_i(T)Q_{N + i} \right)^{1-q} \right)^{q+1}}. \]  

Write
\[ \zeta_{T,l}(n) := 1 + T^{-n} + \cdots + T^{-nd_l}, \quad \zeta_T(n) := \zeta_{T,0}(n), \]
then define
\[ H_l(n) := \zeta_{T,l}(n) + \sum_{\text{Conditions I, II, III}} \left( \sum_{c_0(T) \neq 0}^{m} c_i(T)\bar{f}^i \right)^{-n} \]
and
\[ H(n) := \frac{\zeta_T(n)}{f^n - 1} + \sum_{\text{Conditions I, II}} \left( \sum_{i=1}^{m} c_i(T)\bar{f}^i \right)^{-n} . \]

Now let
\[ \tilde{J}(f)_l := H_l(q^2 - 1) + H(q^2 - 1) \]
\[ \frac{H_l(q - 1) + H(q - 1))^{1+q}}{H_l(q - 1) + H(q - 1))^{1+q}} . \]

Note that \( \tilde{J}(f)_l \) converges. Indeed, both numerator and denominator have absolute value 1, and \( |\tilde{J}(f)_l| = 1 \). Finally, we write
\[ J(f)_l := \frac{Tq^2 - T}{(Tq - T)^{q+1}} \cdot \tilde{J}(f)_l . \]
**Theorem 1.** Let $f$ be a quadratic unit which is a solution of $X^2 - aX - b$ with $\deg a = d > 0$ and $\deg b = 0$. Then $j^{qt}(f)$ has precisely $d = \deg a = \log_q |\sqrt{D}|$ values, and they are

$$j^{qt}(f) = \left\{ j(f)_l := \frac{1}{T^{q^l} - J(f)_l} \right\}_{l=0}^{d-1}.$$

**Proof.** The conjugate solution satisfies $f' = -bf^{-1}$ with $b \in \mathbb{F}_q^\times$, so $f'$ is $GL_2(A)$-equivalent to $f$, hence by modularity, $j^{qt}(f') = j^{qt}(f)$. Thus, in what follows, we will suppose $f$ satisfies $|f| > 1$. As above, $\varepsilon = q^{-Nd-l}$. We begin by noting that

$$\sum_{\lambda \in \Lambda^b_{bas}(f)} \lambda^{-n} = \zeta_{T,l}(n) Q_N^{-n} + \zeta_T(n) \sum_{i=1}^{\infty} Q_{N+i}^{-n}.$$

Then if we define

$$H_{N,l}(n) := \zeta_{T,l}(n) Q_N^{-n} + \sum_{\text{Conditions I, II, III}} \left( \sum_{i=0}^{m} c_i(T) Q_{N+i} \right)^{-n},$$

and

$$H_N(n) := \zeta_T(n) \sum_{i=1}^{\infty} Q_{N+i}^{-n} + \sum_{\text{Conditions I, II}} \left( \sum_{i=1}^{m} c_i(T) Q_{N+i} \right)^{-n},$$

we may re-write (8) as

$$\tilde{J}_\varepsilon(f) = \frac{H_{N,l}(q^2 - 1) + H_{N}(q^2 - 1)}{(H_{N,l}(q-1) + H_{N}(q-1))^{q+1}}.$$

Replace in (9) each $\bar{Q}_{N+i}$ by $(f^{N+i+1} - f^{i(N+i+1)})/c\sqrt{D}$ using (4): equivalently (since the numerator and denominator of (9) have homogeneous degree $1 - q^2$), we may omit the constant $1/c\sqrt{D}$ and simply replace $\bar{Q}_{N+i}$ by $f^{N+i+1} - f^{i(N+i+1)}$. Then dividing out the numerator and denominator by $f^{(1-q^2)(N+1)}$ yields

$$\tilde{J}_\varepsilon(f) = \frac{\hat{H}_{N,l}(q^2 - 1) + \hat{H}_{N}(q^2 - 1)}{(\hat{H}_{N,l}(q-1) + \hat{H}_{N}(q-1))^{q+1}},$$

where

$$\hat{H}_{N,l}(n) := \zeta_{T,l}(n) (1 - (\bar{f}/f)^{N+1})^{-n}$$

$$+ \sum_{\text{Conditions I, II, III}} \left( \sum_{i=0}^{m} c_i(T) \bar{f}_i (1 - (\bar{f}/f)^{N+i+1}) \right)^{-n}.$$
and

\[
\hat{H}_N(n) := \left\{ \zeta_T(n) \sum_{i=1}^{\infty} \tilde{f}^{-ni} \left(1 - \left( \tilde{f}'/\tilde{f} \right)^{N+i+1} \right)^{-n} + \sum_{\text{Conditions } 1, II} \left( \sum_{i=1}^{m} c_i(T) \tilde{f}^i \left(1 - \left( \tilde{f}'/\tilde{f} \right)^{N+i+1} \right) \right)^{-n} \right\}.
\]

Since \( |\tilde{f}'/\tilde{f}| < 1 \),

\[
1 - \left( \tilde{f}'/\tilde{f} \right)^{N+i+1} \longrightarrow 1
\]
as \( N \to \infty \), uniformly in \( i \), and it follows that

\[
\lim_{N \to \infty} \hat{H}_{N,l}(n) = H_l(n) \quad \text{and} \quad \lim_{N \to \infty} \hat{H}_N(n) = H(n),
\]

and therefore

\[
\lim_{N \to \infty} \tilde{J}^{\epsilon-N-d-l}(f) = \tilde{J}(f). \quad \square
\]

\textbf{Note 1.} In the number field case, PARI GP experiments \([3]\) performed on fundamental quadratic units \( \theta \) indicate that \( j^{\epsilon}(\theta) \) appears to have \( D \) values, where \( D \) is the corresponding fundamental discriminant.

\textbf{3. Arbitrary real quadratics.} Let \( h \in k_{\infty} - k \) be an arbitrary quadratic. By the Dirichlet unit theorem \([1]\), the group of units in the quadratic extension \( k(h) \) is isomorphic to \( \mathbb{F}_q^* \times \mathbb{Z} \). Thus there exists a unit \( f \) such that \( k(h) = k(f) \), i.e. \( h \) can be written in the form

\[
h = \frac{x + yf}{z}, \quad x, y, z \in A,
\]

where \( f \) satisfies \( X^2 - aX - b = 0 \), \( d = \deg(a) > 0 \), and \( \deg(b) = 0 \).

Note that for \( h \) and \( f \) as in (11) and \( \varepsilon \) satisfying \( |z| \varepsilon < 1 \), we have the following inclusions of \( \mathbb{F}_q \)-vector spaces

\[
z\Lambda_{|y|^{-1}\varepsilon}(f) \subset \Lambda_{\varepsilon}(h) \subset y^{-1}\Lambda_{|z|\varepsilon}(f).
\]

The first inclusion follows upon noting that if \( z\lambda \in z\Lambda_{|y|^{-1}\varepsilon}(f) \), we have

\[
\|(z\lambda)h\| = |(z\lambda)h - (y\lambda^\perp + \lambda x)| = |\lambda(x + yf) - (y\lambda^\perp + \lambda x)| = |y||\lambda f - \lambda^\perp| < \varepsilon.
\]

The second inclusion follows from noting that if \( \lambda \in \Lambda_{\varepsilon}(h) \), \( y\lambda \in \Lambda_{|z|\varepsilon}(f) \), as

\[
\|(y\lambda)f\| = |(y\lambda)f - (\lambda^\perp z - \lambda x)| = |\lambda(zh - x) - (\lambda^\perp z - \lambda x)| = |z||\lambda h - \lambda^\perp| < |z|\varepsilon.
\]

\textbf{Lemma 2.} The inclusion of \( \mathbb{F}_q \)-vector spaces

\[
z\Lambda_{|y|^{-1}\varepsilon}(f) \subset \Lambda_{\varepsilon}(h)
\]

has index bounded by a constant which depends only on \( y \) and \( z \).
Proof. By (12), it suffices to show that the induced inclusion 
\[ z\Lambda_{|y|^{-1}\varepsilon}(f) \subset y^{-1}\Lambda_{|z|\varepsilon}(f) \]
is of index bounded by a constant which only depends on \( y, z \). For any \( \delta < 1 \), denote 
\[ \Lambda_\delta^2(f) := \{ (\lambda, \lambda^\perp) \in A^2 | \lambda \in \Lambda_\delta(f) \}. \]
Note that the map \( \lambda \mapsto (\lambda, \lambda^\perp) \) induces an isomorphism \( \Lambda_\delta(f) \cong \Lambda_\delta^2(f) \) of \( \mathbb{F}_q \)-
vector spaces: this, in fact, follows from [2, Proposition 1]. In turn, we obtain an induced isomorphism of \( y^{-1} \)-rescalings 
\[ y^{-1}\Lambda_\delta(f) \cong y^{-1}\Lambda_\delta^2(f). \]
Taking \( \delta = |z|\varepsilon \), the corresponding isomorphism takes \( z\Lambda_{|y|^{-1}\varepsilon}(f) \) to \( z\Lambda_{|y|^{-1}\varepsilon}(f) \). Thus 
\[ y^{-1}\Lambda_{|z|\varepsilon}(f)/z\Lambda_{|y|^{-1}\varepsilon}(f) \cong y^{-1}\Lambda_{|z|\varepsilon}(f)/z\Lambda_{|y|^{-1}\varepsilon}(f). \]
On the other hand, the natural map 
\[ y^{-1}\Lambda_{|z|\varepsilon}(f)/z\Lambda_{|y|^{-1}\varepsilon}(f) \hookrightarrow (y^{-1}A)^2/(zA)^2 \]
is injective: for if \( (\lambda_1, \lambda_1^\perp) \) and \( (\lambda_2, \lambda_2^\perp) \in \Lambda_{|z|\varepsilon}(f) \) satisfy 
\[ y^{-1}(\lambda_1, \lambda_1^\perp) - y^{-1}(\lambda_2, \lambda_2^\perp) = z(\beta_1, \beta_2) \in (zA)^2, \]
then since \( z(\beta_1, \beta_2) \in y^{-1}\Lambda_{|z|\varepsilon}(f) \) (being a difference of elements of the latter), 
\[ |(yz\beta_1) - yz\beta_2| < |z|\varepsilon \]
which implies \( (\beta_1, \beta_2) \in \Lambda_{|y|^{-1}\varepsilon}(f) \). Therefore the index of \( z\Lambda_{|y|^{-1}\varepsilon}(f) \) in 
\( y^{-1}\Lambda_{|z|\varepsilon}(f) \) is bounded by \( \dim_{\mathbb{F}_q}((y^{-1}A)^2/(zA)^2) \). This proves the lemma. \( \square \)

Theorem 2. Let \( h \in k_\infty - k \) be quadratic. Then \#\( j^{qt}(h) \) < \( \infty \).

Proof. By Lemma 2, for each \( \varepsilon \), we may find \( \lambda_{1,\varepsilon} = 0, \ldots, \lambda_{r,\varepsilon} \in \Lambda_{\varepsilon}(h), \ r < M = M(y, z), \) such that 
\[ \Lambda_{\varepsilon}(h) = [z\Lambda_{|y|^{-1}\varepsilon}(f) + \lambda_{1,\varepsilon}] \bigcup \cdots \bigcup [z\Lambda_{|y|^{-1}\varepsilon}(f) + \lambda_{r,\varepsilon}]. \quad (13) \]
Thus we have 
\[ \tilde{J}_{\varepsilon}(h) = \frac{\zeta_{h,\varepsilon}(q^2 - 1)}{\zeta_{h,\varepsilon}(q - 1)q^{1+1}} = -\frac{\sum_{\lambda \in \Lambda_{\varepsilon}(h)} - \lambda^{1-q^2}}{(\sum_{\lambda \in \Lambda_{\varepsilon}(h)} - \lambda^{1-q^2})^{q+1}} \]
\[ = -\frac{\sum_{i=1}^{r} \sum_{\lambda \in \Lambda_{|y|^{-1}\varepsilon}(f)} (\lambda + \lambda_{i,\varepsilon}/z)^{1-q^2}}{(\sum_{i=1}^{r} \sum_{\lambda \in \Lambda_{|y|^{-1}\varepsilon}(f)} (\lambda + \lambda_{i,\varepsilon}/z)^{1-q})^{1+q}}. \]
By (12), for \( i \neq 1 \), 
\[ \lambda_{i,\varepsilon} \in y^{-1}\Lambda_{|z|\varepsilon}(f) - z\Lambda_{|y|^{-1}\varepsilon}(f), \quad (14) \]
so we may write 
\[ \lambda_{i,\varepsilon} = y^{-1} \sum_{j=m}^{m'} c_{ij,\varepsilon}(T)\tilde{q}_i, \quad (15) \]
where \( \deg(c_{ij}(T)) \leq d - 1 \) and \( m \) is the smallest index so that \( \mathcal{Q}_m \in \Lambda_{\{z|\varepsilon(f)\}} \). Note that by (14) and Lemma 2, the difference \( m' - m \) has a uniform bound which is independent of \( \varepsilon \). In particular, the set of all possible coefficients \( c_{ij,\varepsilon}(T) \) is finite in number.

Consider a sequence \( \{q^{-N(d-l)}\}_{N>1} \) of values for \( \varepsilon \) giving a limiting value for \( j^{q^t}(f) \) as described in Theorem 1: recall that the limit is calculated by using Binet’s formula to replace occurrences of \( \mathcal{Q}_{N+i} \) by \( \mathcal{f}^{N+i+1} \) in \( J_{\varepsilon}(f) \) and dividing out by \( \mathcal{f}^{1-\mathcal{q}^2}(N+1) \). We apply exactly the same process to \( J_{\varepsilon}(h) \): and in view of (15), we obtain approximations

\[
-z^{-1} \frac{\lambda_i,\varepsilon}{\mathcal{f}^{N+1}} \sim (yz)^{-1} \sum_{j=m}^{m'} c_{ij,\varepsilon}(T) \mathcal{f}^{j-N}, \quad m, m' \geq 0,
\]

where \( \deg(c_{ij,\varepsilon}(T)) \leq d - 1 \). Therefore, the set of possible limits of the \( z^{-1} \frac{\lambda_i,\varepsilon}{\mathcal{f}^{N+1}} \) as \( N \to \infty \) is contained in the finite set

\[
\left\{(yz)^{-1} \sum_{j=-n'}^{n'} c_{j}(T) \mathcal{f}^{j} \left| \deg(c_{j}(T)) \leq d - 1 \right.\right\}
\]

where \( n' + n = m' - m \). Thus within the family \( \{\varepsilon = q^{-N(d-l)}\}_{N>1} \), there are only finitely many possible limits of sub-sequences of \( \{J_{\varepsilon}(h)\} \) giving rise to elements of \( j^{q^t}(h) \). This proves the Theorem. \( \square \)

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