Point-Contact Conductance in Asymmetric Chalker-Coddington Network Model

Koji Kobayashi, Tomi Ohtsuki, and Keith Slevin

Department of Physics, Sophia University, Tokyo 102-8554, Japan
1Department of Physics, Osaka University, Osaka 560-0043, Japan

We study the transport properties of disordered two-dimensional electron systems with a perfectly conducting channel. We introduce an asymmetric Chalker-Coddington network model and numerically investigate the point-contact conductance. We find that the behavior of the conductance in this model is completely different from that in the symmetric model. Even in the limit of a large distance between the contacts, we find a broad distribution of conductance at saddle points of the random potential. Assuming the amplitudes of incoming and outgoing currents for a node to be $c_1, c_4$ and $c_2, c_3$, respectively (see Fig. 2), scattering at a node is described by a $2 \times 2$ unitary scattering matrix $s$,

$$
\begin{pmatrix}
  c_2 \\
  c_4
\end{pmatrix} = s 
\begin{pmatrix}
  c_1 \\
  c_3
\end{pmatrix},
$$

(1)

$$
\begin{pmatrix}
  e^{i\phi_2} & 0 \\
  0 & e^{i\phi_3}
\end{pmatrix} \begin{pmatrix}
  \sqrt{1-p} & \sqrt{p} \\
  -\sqrt{1-p} & -\sqrt{p}
\end{pmatrix} \begin{pmatrix}
  e^{i\phi_1} & 0 \\
  0 & e^{i\phi_4}
\end{pmatrix}.
$$

(2)

The effect of disorder is included in the phases $\phi_i$, which are independently and uniformly distributed between $0$ and $2\pi$. The scattering probability $p$ controls whether the system is in the insulating regime ($0 \leq p < 0.5$), at criticality ($p = 0.5$), or in the quantum Hall insulating regime ($0.5 < p \leq 1$).

1. Introduction

The critical behavior of the transport properties of two-dimensional electron systems under quantum Hall conditions has been investigated in various models. The Chalker-Coddington (CC) network model\(^3\) (Fig. 1) is especially suited to the calculation of transport properties\(^3\) because current amplitudes are calculated directly. This is in contrast to the tight binding model where the wave functions must be calculated first.

The CC model consists of links corresponding to equipotential lines and nodes describing the scattering at saddle points of the random potential. Assuming the numbers of right-going and left-going channels are different. Under certain conditions, however, some of the channels decouple. One example is a graphene sheet with zigzag edges,\(^4\)

\(^{*}\)E-mail address: k-koji@sophia.ac.jp

1.1 Asymmetric Chalker-Coddington network model

In tight binding models, the numbers of the right-going and left-going channels are always the same. Under certain conditions, however, some of the channels decouple. One example is a graphene sheet with zigzag edges,\(^4\)

where there are $l$, say, left-going and $l + 1$ right-going channels near $ka = 2\pi/3$, and $l + 1$ left-going and $l$ right-going ones near $ka = -2\pi/3$, where $k$ is the wave number and $a$ the lattice constant. For long ranged scatterers, states near $2\pi/3$ and $-2\pi/3$ do not mix, and hence the numbers of right-going and left-going channels become, in effect, asymmetric. This asymmetric situation has been studied numerically for quantum railroads\(^5\) and analytically\(^6,7\) on the basis of the DMPK equation.\(^8,9\) Here we realize such an asymmetric situation in the CC model\(^10\) (Fig. 3).
For asymmetric systems with two-terminal geometry, terminals at the ends of the system are also asymmetric in the numbers of incoming and outgoing channels and the two-terminal conductances measured with current flowing left to right $G_{L\rightarrow R}$, and right to left $G_{R\rightarrow L}$ are related by

$$G_{L\rightarrow R} = G_{R\rightarrow L} + (n_{L}^{in} - n_{L}^{out}),$$

(3)

where $G$ is measured in units of $e^2/h$. Here, $L$ and $R$ refer to the left and right terminals, and $n_{L}^{in}$ and $n_{L}^{out}$ are the number of incoming and outgoing channels, respectively, in the left terminal. It follows from current conservation that

$$n_{L}^{in} + n_{R}^{in} = n_{L}^{out} + n_{R}^{out}.$$  

(4)

Using this equation, we can rewrite eq. (3) as

$$G_{R\rightarrow L} = G_{L\rightarrow R} + (n_{R}^{in} - n_{R}^{out}).$$  

(5)

If we suppose that $n_{L}^{in} > n_{L}^{out}$, it follows that

$$G_{L\rightarrow R} \geq n_{L}^{in} - n_{L}^{out}.$$  

(6)

Thus we expect $G_{L\rightarrow R}$ to be finite even in the limit of infinite length (see Table I). The analysis of the transmission eigenvalues shows that the system has $n_{L}^{in} - n_{L}^{out}$ perfectly conducting channels. However, the formula (6) makes it appear that this property is a consequence of the asymmetry of the terminals rather than the sample.

In this paper, we calculate the point-contact conductance $G_{pc}$ of an asymmetric CC network model. The point-contact conductance is the conductance measured between two interior probes. Just like the probes of a scanning tunneling microscope, the probes make contact with the sample at a point. The probes work as symmetric terminals ($n_{L}^{in} = n_{L}^{out} = n_{R}^{in} = n_{R}^{out} = 1$) and $G_{pc}$ varies between 0 and 1 in units of $e^2/h$. In the next section, we explain how to calculate the point-contact conductance. In § 3, we show that the asymmetry of the network is reflected in a broad distribution of $G_{pc}$ with finite averaged values in the long distance limit. In the final section, we summarize and conclude.

2. Method

We denote the numbers of links in the $x$ and $y$ directions by $L_x$ and $L_y$, respectively. We impose periodic boundary condition (PBC) in the $x$ direction and fixed boundary conditions in the $y$ direction. This corresponds to a ring geometry. For PBC in the $x$ direction, $L_x$ must be even. We regard the links at $x = L_x + 1$ as the ones at $x = 1$. In the standard CC model, $L_y$ is even and the system is symmetric. Here we set $L_y$ odd so that the system is asymmetric. The state of the network is specified by the complex current amplitudes $c_i$ on the $L_x \times L_y = N$ links.

2.1 Point-contact conductance

To introduce point-contacts into the network, we cut link $L$ at $(x_1, y_1)$ and link $R$ at $(x_2, y_2)$. We then define incoming current amplitudes $c_{L}^{in}, c_{R}^{in}$ and outgoing current amplitudes $c_{L}^{out}, c_{R}^{out}$ on the corresponding links (Fig. 4). The current amplitudes satisfy the equation

$$
\begin{pmatrix}
 c_1 \\
 c_2 \\
 \vdots \\
 c_N
\end{pmatrix}
 = S
\begin{pmatrix}
 c_1^{in} \\
 c_2^{in} \\
 \vdots \\
 c_N^{in}
\end{pmatrix},
$$

(7)

where $S$ is the $N \times N$ scattering matrix consisting of $2 \times 2$ scattering matrices $s$ at each node. For given $(c_{L}^{in}, c_{R}^{in})$, the remaining current amplitudes $(c_1, c_2, \ldots, c_{L}^{out}, \ldots, c_{R}^{out}, \ldots, c_N)$ are uniquely determined by the following set of $N$ simultaneous linear equa-
For convenience, we consider only two values of $y$ conductance $G$ taking averaged conductance disorder, translational symmetry is recovered, and the point-contact conductance depends on the position $(x_1, y_1)$ and $(x_2, y_2)$ of contacts in addition to the parameters of the network $L_x, L_y,$ and $p,$

$$G_{pc} = G_{pc}(x_1, y_1, x_2, y_2, L_x, L_y, p).$$

This is a sample dependent quantity. If we average over disorder, translational symmetry is recovered, and the averaged conductance $\langle G_{pc} \rangle$ depends only on the distance $|x_1 - x_2| \equiv x_d$ for fixed $y_1$ and $y_2$ (see Fig. 5). Taking $y_1 = y_2 \equiv y_p,$ $\langle G_{pc} \rangle$ is a function of $x_d, y_p, L_x, L_y,$ and $p$

$$\langle G_{pc} \rangle = F(x_d, y_p, L_x, L_y, p).$$

For convenience, we consider only two values of $y_p,$ corresponding to edge conductance $G^e_{pc};$ $y_p = 1$ and bulk conductance $G^b_{pc};$ $y_p = \frac{L_y + 1}{2}$.

As a consequence of the structure of these equations, there is a linear relationship between the incoming and outgoing current amplitudes

$$c^\text{in} \quad \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} = S \begin{pmatrix} c^\text{out} \\ c^\text{out} \\ \vdots \\ c^\text{out} \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}.$$ 

In the insulating limits $p \to 0$ and $p \to 1,$ only the edge channels (in Fig. 1(c), Figs. 3(a) and 3(c)) carry current and the point-contact conductance is bi-modal (see Table II).

In Fig. 8, we show the squared flux amplitudes $|c_i|^2 (i = 1, ..., L^\text{out}, ..., R^\text{out}, ..., N).$ Note that in the asymmetric case, the current is distributed all across the sample even in the limit $x_d \gg L_y$ (Fig. 8(a)). This is in sharp contrast to the symmetric case where the current quickly decays (Fig. 8(b)).

Fig. 5. The geometry of our model. Coordinate $x$ is measured along the ring and $y$ across the ring. The circumference of the ring is $L_x$ and the width is $L_y.$ The distance between the contacts is $x_d (< L_x/2).$

| $p$ | $\delta_{y_p}$ | $\delta_{y_p} + \delta_{y_p} L_y$ |
|-----|----------------|------------------------|
| $p \to 0$ | $\delta_{y_p}$ | $\delta_{y_p}$ |
| $p \to 1$ | $\delta_{y_p} L_y$ | $\delta_{y_p} L_y$ |

Table II. The point-contact conductances in the insulating limits (see Figs. 1 and 3). $G_{pc}$ is unity only when the contacts are directly attached to the edge states. Otherwise, $G_{pc} = 0.$

Fig. 6. The distribution of the edge conductance for $L_y = 9,$ (a) $x_d = 3$ for $L_x = 12,$ (b) $x_d = 3$ for $L_x = 12,$ (c) $x_d = 9$ for $L_x = 24,$ (d) $x_d = 9$ for $L_x = 24.$ Ensemble averages over 1,000,000 systems have been taken. Irrespective of the value of $x_d,$ the dependence on $L_x$ disappears for $L_x \gg L_y.$
averaged conductance converges exponentially, of the averaged conductance. We have found that the convergence to its limiting form, we study the dependence of averaged edge conductance. An example is shown in Fig. 9. Note that the values of $G_{\text{pc}}^b$ correspond to lower squared amplitudes. Probes, indicated by wedges, are attached at the middle of the system. Current flows from the left probe to the right probe ($c_{i}^L$, $c_{i}^R$ = 0, see eq. (10)).

3.2 Dependence of $\langle G_{\text{pc}} \rangle$ on $x_d$

To quantify how the conductance distribution converges to its limiting form, we study the $x_d$ dependence of the averaged conductance. We have found that the averaged conductance converges exponentially,

$$\langle G_{\text{pc}} \rangle = \langle G_{\infty} \rangle \left[ 1 + b \exp \left( - \frac{x_d}{\lambda} \right) \right]. \quad (15)$$

An example is shown in Fig. 9. Note that the values of $\langle G_{\infty} \rangle$, $b$, and $\lambda$ depend, in principle, on $y_p$ and $L_y$. We emphasize that $G_{\infty} = 0$ in the symmetric CC model.

3.3 Dependence of $\langle G_{\text{pc}} \rangle$ on $L_y$

We now analyze the $x_d$-dependence of averaged edge conductance ($G_{\text{pc}}^b$) and similarly for the bulk conductance ($G_{\text{pc}}^b$) for various $L_y$.

Fig. 9. The average of the bulk conductance as a function of $x_d$ for $L_y = 9$. The data are an average over an ensemble of 2,000,000 systems. The solid line is the fit to eq. (15) with $\langle G_{\infty} \rangle = 0.3208 \pm 0.0001$, $b = 0.670 \pm 0.017$, and $\lambda = 4.67 \pm 0.05$ (goodness of fit $Q = 0.81$).

Scaling form describing the dependence of $\langle G_{\text{pc}} \rangle$ on $x_d$ and $L_y$ can be derived by assuming following factorization,

$$\langle G_{\text{pc}} \rangle = h(y_p, L_y) f(x_d, L_y). \quad (16)$$

To eliminate the ambiguity in this factorization, we set $f(x_d \rightarrow \infty, L_y) = 1$. Taking the limit $x_d \rightarrow \infty$,

$$\langle G_{\infty} \rangle = h(y_p, L_y). \quad (17)$$

Comparing with eq. (15), we can write

$$f(x_d, L_y) = 1 + b \exp (-X_d/\Lambda), \quad (18)$$

$$X_d = x_d/L_y, \quad \Lambda = \lambda/L_y. \quad (19)$$

We have found that data for different $x_d$ and different $L_y$ collapse onto a single curve (Fig. 10) with the following values,

$$b = 0.675 \pm 0.019, \quad \Lambda = 0.518 \pm 0.006. \quad (20)$$

Fig. 10. The ratio $f = \langle G_{\text{pc}} \rangle/\langle G_{\infty} \rangle$ as a function of $X_d = x_d/L_y$ for $L_y = 13(\times)$, $25(\circ)$, $45(\times)$ (edge conductance) and $L_y = 13(\triangle)$, $25(\Box)$, $45(\diamond)$ (bulk conductance). The solid line is a fit to eq. (18) (goodness of fit probability $Q = 0.59$).
criticality.

\begin{equation}
\langle G_{\infty} \rangle = C_0 L_y^{\alpha} + \frac{C_1}{L_y}
\end{equation}

fits our data. The best-fit values of parameters are listed in Table III. Here \( i \) denotes whether \( G_{pc} \) (edge) or \( G_{b} \) (bulk). The first term is a non-trivial power law decay that reflects the multi-fractal nature of the conducting states. The second term is a correction for the discreteness of the model and the effect of the boundary.

The difference between edge and bulk conductance may originate from the difference between the surface and bulk multi-fractality.

3.4 Dependence of \( \langle G_{\infty} \rangle \) on \( L_y \)

The dependence of \( \langle G_{\infty} \rangle \) on \( L_y \) for edge and bulk is shown in Fig. 11. We have found that the following form

\begin{equation}
\langle G_{\infty} \rangle = C_0 L_y^{\alpha} + \frac{C_1}{L_y}
\end{equation}

So far we have focused on criticality \( p = 0.5 \). When \( p \) deviates from 0.5, the states are localized in the transverse direction. When system width \( L_y \) exceeds the transverse localization length, the perfectly conducting state is localized along one of the edges, \( y = 1 \) for \( p > 0.5 \) and \( y = L_y \) for \( p < 0.5 \) (Figs. 3(a) and 3(c) are extreme examples). In this case, \( \langle G_{pc} \rangle \) decays quickly with the distance from the conducting edge. Broad distributions \( P(G_{pc}) \) are observed only when we attach contacts near the conducting edge. Note that even in the ordinary quantum Hall effect, such fluctuations in point-contact conductances are expected near the edges.

It is known that the conductance distribution is sensitive to the symmetry class (unitary, orthogonal, or symplectic) classified according to the presence or absence of time-reversal and spin-rotation symmetries. Since the time-reversal symmetry is broken in the scattering matrix eq. (7), the asymmetric Chalker-Coddington model belongs to the unitary class.\(^{19}\) A perfectly conducting channel also arises in the symplectic class, which is realized in carbon nanotubes.\(^{20–23}\) The distribution of the point-contact conductance in the symplectic class, especially in the metal phase, may also be worth investigating.

Acknowledgment

This work was supported by Grant-in-Aid No. 18540382. We would like to thank Dr. H. Obuse and Mr. K. Hirose for useful discussions and fruitful comments.

1) K. v. Klitzing, G. Dorda, and M. Pepper: Phys. Rev. Lett. 45 (1980) 494.
2) J. T. Chalker and P. D. Coddington: J. Phys. C 21 (1988) 2665.
3) B. Kramer, T. Ohtsuki, and S. Kettemann: Physics Reports 417 (2005) 211.
4) K. Wakabayashi, Y. Takane, and M. Sigrist: Phys. Rev. Lett. 99 (2007) 036601.
5) C. Barnes, B. L. Johnson, and G. Kirzenow: Phys. Rev. Lett. 70 (1993) 1159.
6) T. Imamura and M. Wadati: J. Phys. Soc. Jpn. 71 (2002) 1511.
7) Y. Takane and K. Wakabayashi: J. Phys. Soc. Jpn. 76 (2007) 053701.
8) P. A. Mello, P. Pereyra, and N. Kumar: Ann. Phys. (N.Y.) 181 (1988) 290.
9) O. N. Dorokhov: JETP. Lett. 36 (1982) 318.
10) K. Hirose, T. Ohtsuki, and K. Slevin: Physica E 40 (2008) 1677.
11) M. Jansen, M. Metzler, and M. R. Zirnbauer: Phys. Rev. B 59 (1999) 15836.
12) R. Kliesel and M. R. Zirnbauer: Phys. Rev. Lett. 86 (2001) 2094.
13) H. Aoki: J. Phys. C 16 (1983) L205.
14) F. Evers, A. Mildenberger, and A. D. Mirlin: Phys. Rev. Lett. 101 (2008) 116803.
15) F. Evers and A. D. Mirlin: Rev. Mod. Phys. 80 (2008) 1355.
16) H. Obuse, A. R. Subramaniam, A. Purusasi, I. A. Gruzberg, and A. W. W. Ludwig: Phys. Rev. Lett. 98 (2007) 156802.
17) H. Obuse, A. R. Subramaniam, A. Purusasi, I. A. Gruzberg, and A. W. W. Ludwig: Phys. Rev. Lett. 101 (2008) 116802.
18) K. Slevin, T. Ohtsuki, and T. Kawarabayashi: Phys. Rev. Lett. 84 (2000) 3915.
19) Y. Takane and K. Wakabayashi: J. Phys. Soc. Jpn. 76 (2007) 083710.
20) T. Ando and T. Nakanishi: J. Phys. Soc. Jpn. 67 (1998) 1704.
21) T. Ando and H. Suzuura: J. Phys. Soc. Jpn. 71 (2002) 2753.
22) H. Suzuura and T. Ando: Phys. Rev. Lett. 89 (2002) 266603.
23) Y. Takane: J. Phys. Soc. Jpn. 73 (2004) 2366.