NONCOMMUTATIVE SYMMETRIC FUNCTIONS AND AN
AMAZING MATRIX

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Abstract. We present a simple way to derive the results of Diaconis and Fulman
[arXiv:1102.5159] in terms of noncommutative symmetric functions.

1. Introduction

In [3, 4], Diaconis and Fulman investigate a remarkable family of matrices intro-
duced by Holte [9] in his analysis of the process of “carries” in the addition of random
integers in base \( b \).

The aim of this note is to show that the results of [4] can be derived in a simple
and natural way within the formalism of noncommutative symmetric functions [8].

This is possible thanks to the following equivalent characterization of the “amazing
matrix” \( P \) (Theorem 2.1 of [3]):

The number of descents in successive \( b \)-shuffles of \( n \) cards form a
Markov chain on \( \{0, 1, \ldots, n-1\} \) with transition matrix \( P(i, j) \).

Such random processes involving descents of permutations can usually be inter-
preted in the descent algebra of the symmetric group. Here, it is only the number of
descents which is involved, so that one can in fact compute in the Eulerian subalgebra.

We assume that the reader is familiar with the notations of [8].

2. The Eulerian algebra

This is a commutative subalgebra of dimension \( n \) of the group algebra of the
symmetric group \( \mathfrak{S}_n \), and in fact of its descent algebra \( \Sigma_n \). It was apparently first
introduced in [1] under the name algebra of permutors [1]. It is spanned by the Eulerian
idempotents, or, as well, by the sums of permutations having the same number of
descents.

It is easier to work with all symmetric groups at the same time, with the help of
generating functions. Recall that the algebra of noncommutative symmetric functions
\( \text{Sym} \) is endowed with an internal product \( * \), for which each homogeneous component
\( \text{Sym}_n \) is anti-isomorphic to \( \Sigma_n \).

Recall also the following definitions from [8]. We denote by \( \sigma_t \) or \( \sigma_t(A) \) the gener-
ating series of the complete symmetric functions \( S_n \):

\[
\sigma_t(A) = \sum_{n \geq 0} t^n S_n(A).
\]

\[1\] A self-contained and elementary presentation of the main results of [1] can be found in [8].
The Eulerian idempotents $E_n^{[k]}$ are the homogenous components of degree $n$ in the series $E^{[k]}$ defined by

$$\sigma_t(A)^x = \sum_{k \geq 0} x^k E^{[k]}(A).$$

We have

$$E_n^{[k]} \ast E_n^{[l]} = \delta_{kl} E_n^{[k]}, \quad \text{and} \quad \sum_{k=1}^{n} E_n^{[k]} = S_n,$$

so that the $E_n^{[k]}$ span a commutative $n$-dimensional $*$-subalgebra of $\text{Sym}_n$, denoted by $\mathcal{E}_n$ and called the Eulerian subalgebra.

The noncommutative Eulerian polynomials are defined by

$$A_n(t) = \sum_{k=1}^{n} t^k \left( \sum_{|I|=n} R_I \right) = \sum_{k=1}^{n} A(n, k) t^k,$$

where $R_I$ is the ribbon basis. The following facts can be found (up to a few misprints) in [8]. The generating series of the $A_n(t)$ is

$$A(t) := \sum_{n \geq 0} A_n(t) = (1 - t) (1 - t \sigma_1 - t)^{-1}.$$

Let $A^*_n(t) = (1 - t)^{-n} A_n(t)$. Then,

$$A^*(t) := \sum_{n \geq 0} A^*_n(t) = \sum_{I} \left( \frac{t}{1-t} \right)^{\ell(I)} S^I.$$

This last formula can also be written in the form

$$A^*(t) = \sum_{k \geq 0} \left( \frac{t}{1-t} \right)^k (S_1 + S_2 + S_3 + \cdots)^k$$

or

$$\frac{1}{1 - t \sigma_1(A)} = \sum_{n \geq 0} A_n(t) (1-t)^{n+1}.$$

Let $S^{[k]} = \sigma_1(A)^k$ be the coefficient of $t^k$ in this series. In degree $n$,

$$S_n^{[k]} = \sum_{I: n, \ell(I) \leq k} \left( \frac{k}{\ell(I)} \right) S^I = \sum_{i=1}^{n} k^i E_n^{[i]}.$$

This is another basis of $\mathcal{E}_n$. Expanding the factors $(1 - t)^{-(n+1)}$ in the right-hand side of [8] by the binomial theorem, and taking the coefficient of $t^k$ in the term of weight $n$ in both sides, we get

$$S_n^{[k]} = \sum_{i=0}^{k} \binom{n+i}{i} A(n, k-i).$$
Conversely,

\[ \mathcal{A}_n(t) = \frac{1}{(1-t)^{n+1}} = \sum_{k \geq 0} t^k S_n^{[k]}, \]

so that

\[ \mathbf{A}(n, p) = \sum_{i=0}^{p} (-1)^i \binom{n+1}{i} S_n^{[p-i]}. \]

The expansion of the \( E_n^{[k]} \) on the basis \( \mathbf{A}(n, i) \), which is a noncommutative analog of Worpitzky’s identity (see [7] or [10]) is

\[ \sum_{k=1}^{n} x^k E_n^{[k]} = \sum_{i=1}^{n} \binom{x + n - i}{n} \mathbf{A}(n, i). \]

Indeed, when \( x \) is a positive integer \( N \),

\[ \sum_{k=1}^{n} N^k E_n^{[k]} = S_n(N,A) = \sum_{f \vdash n} F_I(N) R_f(A) \]

where \( F_I \) are the fundamental quasi-symmetric functions, and for a composition \( I = (i_1, \ldots, i_r) \) of \( n \),

\[ F_I(N) = \binom{N + n - r}{n}. \]

3. The \( b \)-shuffle process

For a positive integer \( b \), the \( b \)-shuffle permutations in \( \mathfrak{S}_n \) are the inverses of the permutations with at most \( b - 1 \) descents. Thus, the \( b \)-shuffle operator can be identified with \( S_n^{[b]} \) (i.e., with \( \ast \)-multiplication by \( S_n^{[b]} \)). It belongs to the Eulerian algebra, so that it preserves it, and it makes sense to compute its matrix in the basis \( \mathbf{A}(n, k) \). Note that since \( \mathcal{E}_n \) is commutative, it does not matter whether we multiply on the right or on the left.

Note that the \( b \)-shuffle process is an example of what Stanley has called the QS-distribution [11]. It is the probability distribution on permutations derived by assigning probability \( b^{-1} \) to the first \( b \) positive integers, see [3] for a simplified version.

Summarizing, we want to compute the coefficients \( P_{ij}(b) \) defined by

\[ S_n^{[b]} \ast \mathbf{A}(n, j) = \sum_{i=1}^{n} P_{ij}(b) \mathbf{A}(n, i). \]

From (15), it is clear that

\[ S_n^{[p]} \ast S_n^{[q]} = S_n^{[pq]} \]
so that, using (12), we obtain

\[
S_n^{[b]} * A(n, j) = \sum_{r=0}^{j} (-1)^r \binom{n+1}{r} S_n [b(j - r)]
\]

(18)

\[
= \sum_{r=0}^{j} (-1)^r \binom{n+1}{r} \sum_{k=0}^{b(j-r)} \binom{n+k}{k} A(n, b(j-r) - k).
\]

The coefficient of \(A(n, i)\) in this expression is therefore

\[
P_{ij}(b) = \sum_{r=0}^{j} (-1)^r \binom{n+1}{r} \binom{n+b(j-r)-i}{n}
\]

(19)

These are the coefficients of the amazing matrix (up to a shift of 1 on the indices \(i, j\), and a global normalization factor \(b^n\) so as the probabilities sum up to 1).

Since the \(E_n^{[k]}\) form a basis of orthogonal idempotents in \(E_n\), it is reasonable to introduce a scalar product such that

\[
\langle E_n^{[i]} | E_n^{[j]} \rangle = \delta_{ij}.
\]

(20)

Then, the \(b\)-shuffle operator is self-adjoint. Its orthonormal basis of eigenvectors is clearly \(E_n^{[k]}\) (with eigenvalues \(b^k\)).

In terms of coordinates, since we are working in the non-orthogonal basis \(A(n, i)\), its right eigenvector of eigenvalue \(b^j\) is the column vector whose \(i\)th component is the coefficient of \(E_n^{[j]}\) on \(A(n, i)\), that is, the coefficient of \(x^j\) in \((x^{n-i})\)^\(n\), thanks to (13).

By duality, its left eigenvector associated with the eigenvalue \(b^i\) is the row vector whose \(j\)th component is

\[
\langle A(n, j) | E_n^{[i]} \rangle = \sum_{r=0}^{j} (-1)^r \binom{n+1}{r} (j - r)^i.
\]

(21)

This is precisely the Foulkes character table (up to indexation, the Frobenius characteristic of \(\chi_{n,k}\) is the commutative image\(^2\) of \(A(n, n - k)\)).

4. OTHER EXAMPLES

4.1. Determinant of the Foulkes character table. This is the determinant of the matrix \(F\)

\[
F(i, j) = \langle A(n, i), E_n^{[j]} \rangle \quad i, j = 1, \ldots, n.
\]

(22)

Because of the triangularity property

\[
A(n, i) = S_n^{[i]} + \sum_{r=1}^{i} (-1)^r \binom{n+1}{r} S_n^{[i-r]},
\]

(23)

we have as well

\[
\det F = \det G \quad \text{where } G(i, j) = \langle S_n^{[i]}, E_n^{[j]} \rangle = i^j.
\]

\(^2\) These commutative symmetric functions have been studied in [2].
a Vandermonde determinant which evaluates to \(n!(n-1)!\cdots2!1\).

4.2. Descents of \(b^r\)-riffle shuffles. Recall from [5] that \(\text{FQSym}\) is an algebra based on all permutations and that it has two bases

\[
G_\sigma = \sum_{\text{std}(w)=\sigma} w = F_{\sigma^{-1}}
\]

which are mutually adjoint for its natural scalar product

\[
\langle F_\sigma, G_\tau \rangle = \delta_{\sigma,\tau}.
\]

Under the embedding of \(\text{Sym}\) into \(\text{FQSym}\), the \(b^r\)-shuffle operator is

\[
(S_n^{[b]^r})^r = S_n^{[b^r]} = \sum_{\sigma} F_\sigma.
\]

The generating function of \(b^r\)-shuffle by number of descents is therefore its scalar product in \(\text{FQSym}\) with the noncommutative Eulerian polynomial

\[
A_n(t) = \sum_{k=1}^{n} t^k A(n, k) = \sum_{\tau \in S_n} t^{d(\tau)+1} G_\tau.
\]

Recall that

\[
A_n(t) = (1-t)^{n+1} \sum_{k=1}^{n} t^k S_n^{[k]}]
\]

so that

\[
\langle S_n^{[b^r]}, A_n(t) \rangle = (1-t)^{n+1} \sum_{k=1}^{n} t^k \langle S_n^{[b^r]}, S_n^{[k]} \rangle
\]

Now, when one factor \(P\) of a scalar product \(\langle P, Q \rangle\) in \(\text{FQSym}\) is in \(\text{Sym}\), one has \(\langle P, Q \rangle = \langle p, Q \rangle\) where \(p = \underline{P}\) is the commutative image of \(P\) in \(\text{QSym}\), and the bracket is now the duality between \(\text{Sym}\) and \(\text{QSym}\). Furthermore, when \(p\) in in \(\text{Sym}\), then, the scalar product reduces to \(\langle p, q \rangle\), where \(q = \underline{Q}\) is the commutative image of \(Q\) in \(\text{Sym}\), and the bracket is now the ordinary scalar product of symmetric functions (see [5]). Thus,

\[
\langle S_n^{[b^r]}, S_n^{[k]} \rangle = \langle h_n(b^r X), h_n(kX) \rangle = h_n(b^r k) = \binom{b^r k + n - 1}{n}
\]

(\(\lambda\)-ring notation) and we are done:

\[
\langle S_n^{[b^r]}, A_n(t) \rangle = (1-t)^{n+1} \sum_{k=1}^{n} t^k \binom{b^r k + n - 1}{n}.
\]
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