GEOMETRIC STRUCTURES ON NILPOTENT LIE GROUPS: ON THEIR CLASSIFICATION AND A DISTINGUISHED COMPATIBLE METRIC

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Abstract. Let \((N, \gamma)\) be a nilpotent Lie group endowed with an invariant geometric structure (cf. symplectic, complex, hypercomplex or any of their ‘almost’ versions). We define a left invariant Riemannian metric on \(N\) compatible with \(\gamma\) to be minimal, if it minimizes the norm of the invariant part of the Ricci tensor among all compatible metrics with the same scalar curvature. We prove that minimal metrics (if any) are unique up to isometry and scaling, they develop soliton solutions for the ‘invariant Ricci’ flow and are characterized as the critical points of a natural variational problem. The uniqueness allows us to distinguish two geometric structures with Riemannian data, giving rise to a great deal of invariants.

Our approach proposes to vary Lie brackets rather than inner products; our tool is the moment map for the action of a reductive Lie group on the algebraic variety of all Lie algebras, which we show to coincide in this setting with the Ricci operator. This gives us the possibility to use strong results from geometric invariant theory. We describe the moduli space of all isomorphism classes of geometric structures on nilpotent Lie groups of a given class and dimension admitting a minimal compatible metric, as the disjoint union of semi-algebraic varieties which are homeomorphic to categorical quotients of suitable linear actions of reductive Lie groups. Such special geometric structures can therefore be distinguished by using invariant polynomials.

1. Introduction

Invariant structures on nilpotent Lie groups, as well as on their compact versions, nilmanifolds, play an important role in symplectic and complex geometry. The aims of this paper are the search for the ‘best’ compatible metric and the classification of such structures up to isomorphism, which are, in the end, two intimately related problems. Symplectic, complex and hypercomplex cases, and their respective ‘almost’ versions, will be treated in some detail, and although we will always have these particular cases in mind, the main results will be proved for a geometric structure in general. Contact and complex symplectic structures will be studied in a forthcoming paper.

1.1. Geometric structures and compatible metrics. Let \(N\) be a real \(n\)-dimensional nilpotent Lie group with Lie algebra \(n\), whose Lie bracket will be denoted by...
A natural question arises:

Given a class-$\gamma$ nilpotent Lie group $(N, \gamma)$, are there distinguished left invariant Riemannian metrics on $N$ compatible with $\gamma$?,

where the meaning of ‘distinguished’ is of course part of the problem. The Ricci tensor has always been a very useful tool to deal with this kind of questions, and since the answer should depend on the metric and on the structure under consideration, we consider the invariant Ricci operator $\text{Ric}^\gamma_{\langle \cdot, \cdot \rangle}$ (and the invariant Ricci tensor $\text{ric}^\gamma_{\langle \cdot, \cdot \rangle} = \langle \text{Ric}^\gamma_{\langle \cdot, \cdot \rangle} \cdot \cdot \rangle$), that is, the orthogonal projection of the Ricci operator $\text{Ric}_{\langle \cdot, \cdot \rangle}$ onto the subspace of those symmetric maps of $n$ leaving $\gamma$ invariant.

D. Blair, S. Ianus and A. Ledger [BI, BL] have proved in the compact case that metrics satisfying

$$\text{ric}^\gamma_{\langle \cdot, \cdot \rangle} = 0$$

are very special in symplectic (so called metrics with hermitian Ricci tensor) and contact geometry, as they are precisely the critical points of two very natural curvature functionals on $C$: the total scalar curvature functional $S$ and a functional $K$ measuring how far are the metrics of being Kähler or Sasakian, respectively (see also [B] and Section 5.1).

We will show that for a non-abelian nilpotent Lie group, condition (3) cannot hold for the classes of structures we have in mind, and hence it is natural to try to get as close as possible to this unattainable goal. In this light, a metric $\langle \cdot, \cdot \rangle \in C(N, \gamma)$ is called minimal if it minimizes the functional $\| \text{ric}^\gamma_{\langle \cdot, \cdot \rangle} \|^2 = \text{tr}(\text{Ric}^\gamma_{\langle \cdot, \cdot \rangle})^2$ on the set of all compatible metrics with the same scalar curvature. It turns out that minimal metrics are the elements in $C$ closest to satisfy the ‘Einstein-like’ condition $\text{ric}^\gamma_{\langle \cdot, \cdot \rangle} = c\langle \cdot, \cdot \rangle$, $c \in \mathbb{R}$. We may also try to improve the metric via the evolution flow

$$\frac{d}{dt} \langle \cdot, \cdot \rangle_t = \pm \text{ric}^\gamma_{\langle \cdot, \cdot \rangle}_t,$$

whose fixed points are precisely metrics satisfying (3). In the symplectic case, this flow is called the anticomplexified Ricci flow and has been recently studied by H-V Le and G. Wang [LW]. Of particular significance are then those metrics for which the solution to the normalized flow (under which the scalar curvature is constant in time) remains isometric to the initial metric. Such special metrics will be called invariant Ricci solitons. The main result in this paper can be now stated.
Theorem 1.1. Let \((N, \gamma)\) be a nilpotent Lie group endowed with an invariant geometric structure \(\gamma\) (non-necessarily integrable). Then the following conditions on a left invariant Riemannian metric \(\langle \cdot, \cdot \rangle\) which is compatible with \((N, \gamma)\) are equivalent:

(i) \(\langle \cdot, \cdot \rangle\) is minimal.
(ii) \(\langle \cdot, \cdot \rangle\) is an invariant Ricci soliton.
(iii) \(\text{Ric}\gamma\langle \cdot, \cdot \rangle = cI + D\) for some \(c \in \mathbb{R}\), \(D \in \text{Der}(n)\).

Moreover, there is at most one compatible left invariant metric on \((N, \gamma)\) up to isometry (and scaling) satisfying any of the above conditions.

A major obstacle to classify geometric structures is the terrible lack of invariants. The uniqueness result in the above theorem gives rise to a very useful tool to distinguish two geometric structures; indeed, if they are isomorphic then their respective minimal compatible metrics (if any) have to be isometric. One therefore can eventually distinguish geometric structures with Riemannian data, which suddenly provides us with a great deal of invariants. This will be used to find explicit continuous families depending on 1, 2 and 3 parameters of pairwise non-isomorphic geometric structures in low dimensions, mainly by using only one Riemannian invariant: the eigenvalues of the Ricci operator (see Sections 1.3-1.5 for a summary in each case).

A remarkable weakness of this approach is however the existence problem; the theorem does not even suggest when such a distinguished metric does exist. How special are the symplectic or (almost-) complex structures admitting a minimal metric? So far, we know how to deal with this ‘existence question’ only by giving several explicit examples, for which the neat ‘algebraic’ characterization (iii) will be very useful. It turns out that in low dimensions the structures in general tend to admit a minimal compatible metric, and the only obstruction we know at this moment is when \(G_\gamma \subset SL(n)\) and the nilpotent Lie algebra is characteristically nilpotent, that is, \(\text{Der}(n)\) is nilpotent. Anyway, we could not find a non-existence example yet.

1.2. Variety of compatible metrics and the moment map. A class-\(\gamma\) metric structure on a nilpotent Lie group is determined by a triple \((\mu, \gamma, \langle \cdot, \cdot \rangle)\) of tensors on \(n\). The proof of Theorem 1.1 is based on an approach which proposes to vary the Lie bracket \(\mu\) rather than the inner product \(\langle \cdot, \cdot \rangle\).

Let us consider as a parameter space for the set of all real nilpotent Lie algebras of a given dimension \(n\), the set

\[
\mathcal{N} = \{\mu \in V : \mu \text{ satisfies Jacobi and is nilpotent}\},
\]

where \(n\) is a fixed \(n\)-dimensional real vector space and \(V = \Lambda^2 n^* \otimes n\) is the vector space of all skew-symmetric bilinear maps from \(n \times n\) to \(n\). Since the Jacobi identity and the nilpotency condition are both determined by zeroes of polynomials, \(\mathcal{N}\) is an algebraic subset of \(V\). We fix a tensor \(\gamma\) on \(n\) (or a set of tensors), and denote by \(G_\gamma\) the subgroup of \(\text{GL}(n)\) preserving \(\gamma\). The reductive Lie group \(G_\gamma\) acts naturally on \(V\) leaving \(\mathcal{N}\) invariant and also the algebraic subset \(\mathcal{N}_\gamma \subset \mathcal{N}\) given by

\[
\mathcal{N}_\gamma = \{\mu \in \mathcal{N} : \text{IC}(\gamma, \mu) = 0\},
\]

that is, those nilpotent Lie brackets for which \(\gamma\) is integrable (see (3)).

For each \(\mu \in \mathcal{N}\), let \(N_\mu\) denote the simply connected nilpotent Lie group with Lie algebra \((n, \mu)\). Fix an inner product \(\langle \cdot, \cdot \rangle\) on \(n\) compatible with \(\gamma\), that is, such that
(2) holds. We identify each $\mu \in \mathcal{N}_\gamma$ with a class-$\gamma$ metric structure on a nilpotent Lie group

$$\mu \longleftrightarrow (N_\mu, \gamma, \langle \cdot, \cdot \rangle),$$

where all the structures are defined by left invariant translation. The orbit $G_\gamma, \mu$ parameterizes then all the left invariant metrics which are compatible with $(N_\mu, \gamma)$ and hence we may view $\mathcal{N}_\gamma$ as the space of all class-$\gamma$ metric structures on nilpotent Lie groups of dimension $n$. Two metrics $\mu, \lambda \in \mathcal{N}_\gamma$ are isometric if and only if they live in the same $K_\gamma$-orbit, where $K_\gamma = G_\gamma \cap O(n, \langle \cdot, \cdot \rangle)$.

We now go back to our search for the best compatible metric. It is natural to consider the functional $F : \mathcal{N}_\gamma \mapsto \mathbb{R}$ given by $F(\mu) = \text{tr}(\text{Ric}_\gamma \mu)^2$, which in some sense measures how far the metric $\mu$ is from satisfying (3). The critical points of $F/||\mu||^4$ on the projective algebraic variety $\mathbb{P}\mathcal{N}_\gamma \subset \mathbb{P}V$ (which is equivalent to normalize by the scalar curvature since $\text{sc}(\mu) = -\frac{1}{4}||\mu||^2$), may therefore be considered compatible metrics of particular significance.

A crucial fact of this approach is that the moment map $m_\gamma : V \mapsto p_\gamma$ for the action of $G_\gamma$ on $V$, where $p_\gamma$ is the space of symmetric maps of $(n, \langle \cdot, \cdot \rangle)$ leaving $\gamma$ invariant (i.e. $g_\gamma = \mathfrak{k}_\gamma \oplus p_\gamma$ is a Cartan decomposition), satisfies

$$m_\gamma(\mu) = 8 \text{Ric}_\gamma \mu, \quad \forall \mu \in \mathcal{N}_\gamma,$$

where $\text{Ric}_\gamma \mu$ is the invariant Ricci operator of $\mu$. This allows us to use strong and well-known results on the moment map due to F. Kirwan [K1] and L. Ness [N], and proved by A. Marian [M] in the real case (see Section 3.2 for an overview). Indeed, since $F$ becomes a scalar multiple of the square norm of the moment map, we obtain the following

**Theorem 1.2.** [M] Let $F : \mathbb{P}\mathcal{N}_\gamma \mapsto \mathbb{R}$ be defined by $F([\mu]) = \text{tr}(\text{Ric}_\gamma \mu)^2/||\mu||^4$. Then for $\mu \in \mathcal{N}_\gamma$ the following conditions are equivalent:

(i) $[\mu]$ is a critical point of $F$.
(ii) $F_{[G_\gamma, [\mu]]}$ attains its minimum value at $[\mu]$.
(iii) $\text{Ric}_\gamma \mu = cI + D$ for some $c \in \mathbb{R}, D \in \text{Der}(\mu)$.

Moreover, all the other critical points of $F$ in the orbit $G_\gamma, [\mu]$ lie in $K_\gamma, [\mu]$.

Theorem 1.1 follows then almost directly from this result, except for the equivalence between (ii) and (iii), which will be proved separately. We note that Theorem 1.2 also gives a variational method to find minimal compatible metrics, by characterizing them as the critical points of a natural curvature functional (see Example 4.9 for an explicit application).

Most of the results obtained in this paper are still valid for general Lie groups, although some considerations have to be carefully taken into account (see Remark 4.6).

1.3. Symplectic structures. We first prove that a symplectic non-abelian nilpotent Lie group $(N, \omega)$ can never admit a compatible left invariant metric with hermitian Ricci tensor. We also find a minimal compatible metric for the two 4-dimensional symplectic nilpotent Lie groups and exhibit curves of pairwise non-isomorphic symplectic structures on the 6-dimensional nilpotent Lie groups denoted by $(0, 0, 0, 12, 13, 23)$ and $(0, 0, 12, 13, 14 + 23, 24 + 15)$ in [S].
1.4. **Complex structures.** We find two different curves of pairwise non-isomorphic abelian complex structures on the Iwasawa manifold; for only one of them their minimal compatible metrics are modified H-type. A third curve of pairwise non-isomorphic non-abelian complex structures on the Iwasawa manifold is also given. The initial point is the bi-invariant complex structure, which is the only point for which the minimal metric is modified H-type. We also exhibit a curve of pairwise non-isomorphic abelian complex structures on \( h_3 \oplus h_3 \), where \( h_3 \) is the 3-dimensional Heisenberg Lie algebra, and a curve of non-abelian ones on the group denoted by \((0,0,0,12,14+23)\) in \([S]\).

1.5. **Hypercomplex structures.** We prove that any hypercomplex 8-dimensional nilpotent Lie group admits a minimal compatible metric (being actually the only compatible metric up to isometry and scaling), which is modified H-type if and only if the structure is in addition abelian. Let \( g_1, g_2 \) and \( g_3 \) denote the 8-dimensional Lie algebras obtained as the direct sum of an abelian factor and the 5-dimensional Heisenberg Lie algebra, the 6-dimensional complex Heisenberg Lie algebra and the 7-dimensional quaternionic Heisenberg Lie algebra, respectively. A curve and a surface of pairwise non-isomorphic abelian hypercomplex structures on \( g_2 \) and \( g_3 \) respectively, are given. We also find curves of pairwise non-isomorphic non-abelian hypercomplex structures on \( g_3 \) and \( n = u(2) \oplus \mathbb{C}^2 \). By using results due to I. Dotti and A. Fino \([DF1, DF3]\), we actually prove that the moduli space of all hypercomplex 8-dimensional nilpotent Lie groups, up to isomorphism, is parameterized by the 9-dimensional quotient
\[
P \left( (\mathfrak{su}(2) \otimes \mathbb{R}^4) \oplus \mathbb{R}^4 \right) / (SU(2) \times SU(2)),
\]
where \( \mathfrak{su}(2) \) is the adjoint representation and \( \mathbb{R}^4 \) is the standard representation of \( SU(2) \) on \( \mathbb{C}^2 \) viewed as real. The abelian ones are parameterized by the quotient
\[
P \left( \mathfrak{su}(2) \otimes \mathbb{R}^4 \right) / (SU(2) \times SU(2)),
\]
which has dimension 5. Explicit continuous families depending on 5 parameters on \( g_2 \) and \( g_3 \) are given.

1.6. **Einstein solvmanifolds.** If one considers no structure (i.e. \( \gamma = 0 \)), then we show that the ‘moment map’ approach proposed in this paper can be also applied to the study of Einstein solvmanifolds. Each \( \mu \in \mathcal{N} \) is identified via \((4)\) with the Riemannian manifold \((N_\mu, \langle \cdot, \cdot \rangle)\), but we also have in this case another identification with a solvmanifold: for each \( \mu \in \mathcal{N} \), there exists a unique rank-one metric solvable extension \( S_\mu = (S_\mu, \langle \cdot, \cdot \rangle) \) of \((N_\mu, \langle \cdot, \cdot \rangle)\) standing a chance of being Einstein, and every \((n+1)\)-dimensional rank-one Einstein solvmanifold can be modelled as \( S_\mu \) for a suitable \( \mu \in \mathcal{N} \). The functional \( F \) measures how far is the metric \( \mu \) from being Einstein. We obtain, as a consequence of the above theorems, many of the uniqueness and structure results proved by J. Heber in \([Hb]\), as well as the variational result in \([L4]\) and the relationship between Ricci soliton metrics on nilpotent Lie groups and Einstein solvmanifolds proved in \([L3]\).

1.7. **On the classification of geometric structures.** Everything seems to indicate that the moduli space of isomorphism classes of \( n \)-dimensional, class-\( \gamma \), nilpotent Lie groups is a very complicated space for most of the classes of geometric structures, even in low dimensions. Anyway, what can be said about such a moduli
space?. Can we show that it is really unmanageable? Can we at least find subspaces which are manifolds or algebraic varieties and obtain lower bounds for its ‘dimension’? This kind of questions belong to invariant theory. For a fixed nilpotent Lie group $N$, the isomorphism between geometric structures is determined by the action of $\text{Aut}(n)$, which is a group in general unknown and ‘very ugly’ from an invariant-theoretic point of view since it is far from being semisimple or reductive. We then propose to consider the class-$\gamma$ nilpotent Lie groups of a given dimension all together, by using the variety of nilpotent Lie algebras, as in the study of compatible metrics (see Section 1.2). The advantage of this unified approach is that the group giving the isomorphism is the reductive Lie group $G_{\gamma}$; the price to pay is that the space $N_{\gamma}$ where $G_{\gamma}$ is acting on, is really wild. Fortunately, $N_{\gamma}$ is at least a real algebraic variety, and so the classification problem for such structures may be approached by using tools from invariant theory (see Section 3.3 for an overview).

We may view $N_{\gamma}$ as the variety of all class-$\gamma$ $n$-dimensional nilpotent Lie groups by identifying each element $\mu \in N_{\gamma}$ with a class-$\gamma$ nilpotent Lie group,

$$(5) \quad \mu \leftrightarrow (N_{\mu}, \gamma).$$

Two class-$\gamma$ structures $\mu, \lambda$ are isomorphic if and only if they live in the same $G_{\gamma}$-orbit, and hence the quotient $N_{\gamma}/G_{\gamma}$ parameterizes the moduli space of all class-$\gamma$ nilpotent Lie groups up to isomorphism. Recall that a nilpotent Lie group $N_{\mu}$ admits a class-$\gamma$ structure if and only if the orbit $G_{\gamma,\mu}$ meets the variety $N_{\gamma}$. The data set $(k_1 < \ldots < k_r; d_1, \ldots, d_r)$ is called the type of the critical point $[\mu]$. The set of types of critical points is in bijection with the finite set of strata determined by the negative gradient flow of $F = ||\text{ric}^\gamma||^2$ stays in the $G_{\gamma}$-orbit of the starting point, every class-$\gamma$ nilpotent Lie group degenerates via such a flow into one of these special structures.
and the reductive Lie group given by

\[ \tilde{G}_\alpha := \left\{ g \in G_\gamma \cap (GL(d_1) \times \ldots \times GL(d_r)) : \prod_{i=1}^{r} (\det g_i)^{k_i} = 1 = \det g \right\}. \]

The quotient \( \mathcal{N}_\gamma // G_\gamma \) decomposes as a disjoint union of semi-algebraic varieties

\[ \mathcal{N}_\gamma // G_\gamma = X_1 \cup \ldots \cup X_s, \]

where each \( X_i \) is homeomorphic to the categorical quotient \((V_\alpha \cap N_\gamma) // \tilde{G}_\alpha_i\). This allows us to approach the classification of invariant geometric structures on nilpotent Lie groups by using invariant-theoretic methods. By considering each \( X_i \) separately, we have for instance that geometric structures of type \( \alpha_i \) are precisely the closed \( \tilde{G}_\alpha_i \)-orbits, and two different orbits give rise to non-isomorphic structures. We therefore have that two non-isomorphic structures can always be separated by a \( \tilde{G}_\alpha \)-invariant polynomial on \( V_\alpha \). Moreover, \( X_i \) can be described by using a set of generators and relations of \( \mathbb{R}[V_\alpha \cap N_\gamma]^{\tilde{G}_\alpha_i} \), the ring of all invariant polynomials. It is shown that some of the simplest types already lead to wide open problems in invariant theory.

We do not know how far is \( \mathcal{N}_\gamma // G_\gamma \) from the whole quotient \( \mathcal{N}_\gamma / G_\gamma \). The crucial question is how strong is, for a class-\( \gamma \) structure, the property of admitting a minimal compatible metric (see end of Section 1.2).

2. Geometric structures and compatible metrics

Let \( N \) be a real \( n \)-dimensional nilpotent Lie group with Lie algebra \( \mathfrak{n} \), whose Lie bracket is denoted by \( \mu : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{n} \). An invariant geometric structure on \( N \) is defined by left translation of a tensor \( \gamma \) on \( \mathfrak{n} \) (or a set of tensors), usually non-degenerate in some way, which satisfies a suitable integrability condition

\[ \text{IC}(\gamma, \mu) = 0, \]

involving only \( \mu \) and \( \gamma \). In this paper, we will focus on the following classes of geometric structures: symplectic, complex and hypercomplex, as well as on their respective ‘almost’ versions, that is, when condition (6) is not required. In this way, \( \text{IC}(\gamma, \mu) \) can be for instance the differential of a 2-form or the Nijenhuis tensor associated to some \((1,1)\)-tensor. The contact case is somewhat different because the condition is ‘open’, but it becomes an equation of the form (6) when one considers fixed the underlying almost-contact structure. We shall deal with contact and complex symplectic structures in a forthcoming paper.

The pair \((N, \gamma)\) will often be called a class-\( \gamma \) nilpotent Lie group, and \( N \) will be assumed to be simply connected for simplicity. The group \( GL(n) := GL(n, \mathbb{R}) = GL(\mathfrak{n}) \) of invertible maps of \( \mathfrak{n} \) acts on the vector space of tensors on \( \mathfrak{n} \) of a given class, preserving the non-degeneracy, and if \( \gamma \) is integrable then \( \varphi.\gamma \) is so for any \( \varphi \in \text{Aut}(\mathfrak{n}) \), the group of automorphisms of \( \mathfrak{n} \). In view of this fact, two class-\( \gamma \) nilpotent Lie groups \((N, \gamma)\) and \((N', \gamma')\) are said to be isomorphic if there exists a Lie algebra isomorphism \( \varphi : \mathfrak{n} \to \mathfrak{n}' \) such that \( \gamma' = \varphi.\gamma \). Also, given two geometric structures \( \gamma, \gamma' \) of the same class on \( N \), we say that \( \gamma \) degenerates to \( \gamma' \) if \( \gamma' \in \overline{\text{Aut}(\mathfrak{n}).\gamma} \), the closure of the orbit \( \text{Aut}(\mathfrak{n}).\gamma \) relative to the natural topology.

A left invariant Riemannian metric on \( N \) is said to be compatible with \((N, \gamma)\) if the corresponding inner product \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{n} \) satisfies an orthogonality condition

\[ \text{OC}(\gamma, \langle \cdot, \cdot \rangle) = 0, \]
in which only $\langle \cdot, \cdot \rangle$ and $\gamma$ are involved. We denote by $\mathcal{C} = \mathcal{C}(N, \gamma)$ the set of all left invariant metrics on $N$ which are compatible with $(N, \gamma)$. The pair $(\gamma, \langle \cdot, \cdot \rangle)$ with $\langle \cdot, \cdot \rangle \in \mathcal{C}$ will often be referred to as a class-$\gamma$ metric structure. It is clear from (8) that for an invariant geometric structure there always exist a compatible metric, since the condition is independent from $\mu$. Moreover, the space $\mathcal{C}$ is usually huge; recall for instance that the group $G_\gamma = \{ \phi \in GL(n) : \phi.\gamma = \gamma \}$ acts on $\mathcal{C}$, and it is easy to see that actually for any $\langle \cdot, \cdot \rangle \in \mathcal{C}$ we have that

$$C = G_\gamma.\langle \cdot, \cdot \rangle = \{ \langle \phi^{-1} \cdot, \phi^{-1} \cdot \rangle : \phi \in G_\gamma \}.$$  

A natural question takes place:

Given a class-$\gamma$ nilpotent Lie group $(N, \gamma)$, are there distinguished left invariant Riemannian metrics on $N$ compatible with $\gamma$?

As usual, the meaning of the word ‘distinguished’ is part of the question. This problem may be (and it is) stated for differentiable manifolds in general, and does not only present some interest in Riemannian geometry; indeed, the existence of a certain nice compatible metric could eventually help to distinguish two geometric structures as well as to find privileged geometric structures on a given manifold.

The aim of this section is to propose two properties which make a compatible metric very distinguished, one is obtained by minimizing a curvature functional and the other as a soliton solution for a natural evolution flow. The Ricci tensor will be used in both approaches. In Appendix 10, we have reviewed some well known properties of left invariant metrics on nilpotent Lie groups, which will be used constantly from now on.

Fix a class-$\gamma$ nilpotent Lie group $(N, \gamma)$. Let $g_\gamma$ be the Lie algebra of $G_\gamma$,

$$g_\gamma = \{ A \in \mathfrak{gl}(n) : A.\gamma = 0 \}.$$  

Definition 2.1. For each compatible metric, we consider the orthogonal projection $\text{Ric}^\gamma(\cdot, \cdot)$ of the Ricci operator $\text{Ric}(\cdot, \cdot)$ on $g_\gamma$, called the invariant Ricci operator, and the corresponding invariant Ricci tensor given by $\text{ric}^\gamma = \langle \text{Ric}^\gamma \cdot, \cdot \rangle$.

The role of the Ricci tensor has always been crucial in defining privileged (compatible) metrics; we have for example Einstein metrics, extremal Kähler metrics in complex geometry, and more recently metrics with hermitian Ricci tensor and $\phi$-invariant Ricci tensor in symplectic and contact geometry, respectively. These two last notions are equivalent to $\text{ric}^\gamma = 0$ and have been characterized in the compact case by D. Blair, S. Ianus and A. Ledger \cite{BI, BL} as the critical points of two very natural curvature functionals on $\mathcal{C}$: the total scalar curvature functional $S$ and a functional $K$ for which the global minima are precisely Kähler or Sasakian metrics, respectively (see also \cite{B} and Section 5.1).

In this light, condition

$$\text{ric}^\gamma(\cdot, \cdot) = 0,$$

involves both the geometric structure and the metric, and seems to be very natural to require to a compatible metric. Nevertheless, if $RI \subset g_\gamma$, then $\text{tr Ric}^\gamma(\cdot, \cdot) = \text{sc}(\cdot, \cdot)$, and so it is forbidden for instance for non-abelian nilpotent Lie groups (where always $\text{sc}(\cdot, \cdot) < 0$) in the complex and hypercomplex cases. We shall prove that this condition is forbidden in the symplectic case as well. We therefore have to consider (6) as an unreachable goal and try to get as close as possible, for
instance, by minimizing $||\text{ric}^\gamma\langle\cdot,\cdot\rangle||^2 = \text{tr}(\text{Ric}^\gamma\langle\cdot,\cdot\rangle)^2$. In order to avoid homothetical changes, we must normalize the metrics some way. In the noncompact homogeneous case, the scalar curvature always appears as a very natural choice. We then propose the following

**Definition 2.2.** A left invariant metric $\langle\cdot,\cdot\rangle$ compatible with a class-$\gamma$ nilpotent Lie group $(N,\gamma)$ is called *minimal* if

$$\text{tr}(\text{Ric}^\gamma\langle\cdot,\cdot\rangle)^2 = \min\{\text{tr}(\text{Ric}^\gamma\langle\cdot,\cdot\rangle')^2 : \langle\cdot,\cdot\rangle' \in \mathcal{C}(N,\gamma), \ \text{sc}(\langle\cdot,\cdot\rangle') = \text{sc}(\langle\cdot,\cdot\rangle)\}.$$  

Recall that the existence and uniqueness (up to isometry and scaling) of minimal metrics is far from being clear from the definition. The uniqueness shall be proved in Section 4, but the ‘existence question’ is still nebulous. Minimal metrics are the compatible metrics closest to satisfy the ‘Einstein-like’ condition $\text{ric}^\gamma\langle\cdot,\cdot\rangle = c\langle\cdot,\cdot\rangle$, for some $c \in \mathbb{R}$. Indeed,

$$||\text{Ric}^\gamma\langle\cdot,\cdot\rangle - \frac{\text{tr} \text{Ric}^\gamma\langle\cdot,\cdot\rangle}{n} I||^2 = \text{tr}(\text{Ric}^\gamma\langle\cdot,\cdot\rangle)^2 - \frac{(\text{tr} \text{Ric}^\gamma\langle\cdot,\cdot\rangle)^2}{n}$$

and $\text{tr} \text{Ric}^\gamma\langle\cdot,\cdot\rangle$ equals either 0 or $\text{sc}(\langle\cdot,\cdot\rangle)$, depending on $g_\gamma$ contains or not $\mathbb{R}I$.

We now consider an evolution approach. Motivated by the famous Ricci flow introduced by R. Hamilton [H1], we consider the *invariant Ricci flow* for our left invariant metrics on $N$, given by the following evolution equation

$$\frac{d}{dt}\langle\cdot,\cdot\rangle_t = \pm \text{ric}^\gamma\langle\cdot,\cdot\rangle_t,$$

which coincides for example with the anticomplexified Ricci flow studied in [LW] in the symplectic case (see Section 5.1). The choice of the best sign might depend on the class of structure. This is just an ordinary differential equation and hence the existence for all $t$ and uniqueness of the solution is guaranteed. It follows from (8) that

$$T_{\langle\cdot,\cdot\rangle} C = \{\alpha \in \text{sym}(n) : A_{\alpha,\gamma} = 0\},$$

and therefore, if $\langle\cdot,\cdot\rangle_0 \in C$ then the solution $\langle\cdot,\cdot\rangle_t \in C$ for all $t$ since $\text{ric}^\gamma\langle\cdot,\cdot\rangle_t \in T_{\langle\cdot,\cdot\rangle} C$ (see Appendix 10).

**Remark 2.3.** If we had however the uniqueness of the solution for the flow (10) in the non-compact general case, then we would not need to restrict ourselves to left invariant metrics. Indeed, if $f$ is an isometry of the initial metric $\langle\cdot,\cdot\rangle_0$ which also leaves $\gamma$ invariant, then since $f^*\langle\cdot,\cdot\rangle_t$ is also a solution and $f^*\langle\cdot,\cdot\rangle_0 = \langle\cdot,\cdot\rangle_0$ we would get by uniqueness of the solution that $f$ is an isometry of all the metrics $\langle\cdot,\cdot\rangle_t$ as well. Left invariance of the starting metric would be therefore preserved along the flow.

When $M$ is compact, a normalized Ricci flow is often considered, under which the volume of the solution metric is constant in time. Actually, the normalized equation differs from the Ricci flow only by a change of scale in space and a change of parametrization in time (see [H2, CC]). In our case, where the manifold is noncompact but the metrics are homogeneous, it seems natural to do the same thing but normalizing by the scalar curvature, which is just a single number associated to the metric. We recall that a left invariant metric $\langle\cdot,\cdot\rangle$ on a nilpotent Lie group $N$ has always $\text{sc}(\langle\cdot,\cdot\rangle) < 0$, unless $N$ is abelian (see [H8]).
Proposition 2.4. The solution to the normalized invariant Ricci flow

\[
\frac{d}{dt} \langle \cdot, \cdot \rangle_t = \pm \text{ric} \gamma \langle \cdot, \cdot \rangle_t \mp \frac{\text{tr}(\text{Ric} \gamma \langle \cdot, \cdot \rangle_t)^2}{\text{sc}(\langle \cdot, \cdot \rangle_t)} \langle \cdot, \cdot \rangle_t
\]

satisfies \( \text{sc}(\langle \cdot, \cdot \rangle_t) = \text{sc}(\langle \cdot, \cdot \rangle_0) \) for all \( t \). Moreover, this flow differs from the invariant Ricci flow \( (10) \) only by a change of scale in space and a change of parametrization in time.

Proof. It follows from the formula for the gradient of the scalar curvature functional \( \text{sc} : \mathcal{P} \to \mathbb{R} \) given in (39) that if \( \langle \cdot, \cdot \rangle_t \) is a solution of (12), then the function \( f(t) = \text{sc}(\langle \cdot, \cdot \rangle_t) \) satisfies

\[
\frac{d}{dt} f(t) = g_{\langle \cdot, \cdot \rangle_t} \left( \frac{d}{dt} \langle \cdot, \cdot \rangle_t - \text{ric} \langle \cdot, \cdot \rangle_t \right)
\]

\[
= \mp g_{\langle \cdot, \cdot \rangle_t} (\text{ric} \gamma \langle \cdot, \cdot \rangle_t, \text{ric} \langle \cdot, \cdot \rangle_t) + \frac{\text{tr}(\text{Ric} \gamma \langle \cdot, \cdot \rangle_t)^2}{\text{sc}(\langle \cdot, \cdot \rangle_t)} g_{\langle \cdot, \cdot \rangle_t}(\langle \cdot, \cdot \rangle_t, \text{ric} \langle \cdot, \cdot \rangle_t)
\]

\[
= \mp \text{tr}(\text{Ric} \gamma \langle \cdot, \cdot \rangle_t \text{ric} \langle \cdot, \cdot \rangle_t) + \frac{\text{tr}(\text{Ric} \gamma \langle \cdot, \cdot \rangle_t)^2}{\text{sc}(\langle \cdot, \cdot \rangle_t)} \text{tr}(\text{Ric} \langle \cdot, \cdot \rangle_t)
\]

\[
= \mp \text{tr}(\text{Ric} \gamma \langle \cdot, \cdot \rangle_t)^2 (1 - \frac{f(t)}{\text{sc}(\langle \cdot, \cdot \rangle_t)}) = 0, \quad \forall t,
\]

and thus \( f(t) \equiv f(0) = \text{sc}(\langle \cdot, \cdot \rangle_0) \). The last assertion follows as in [H2] in a completely analogous way. \( \square \)

The fixed points of this normalized flow (12) are those metrics satisfying \( \text{Ric} \gamma \langle \cdot, \cdot \rangle \in \mathbb{R}I \), and so in particular, if \( g_t \subset \mathfrak{s}(n) \), then this is equivalent to \( \text{Ric} \gamma \langle \cdot, \cdot \rangle = 0 \). Indeed, if \( \text{Ric} \gamma \langle \cdot, \cdot \rangle = \pm \frac{\text{tr}(\text{Ric} \gamma \langle \cdot, \cdot \rangle)}{\text{sc}(\langle \cdot, \cdot \rangle_t)} I \) then \( \text{Ric} \gamma \langle \cdot, \cdot \rangle = 0 \) since \( \text{tr} \text{Ric} \gamma \langle \cdot, \cdot \rangle = 0 \). We should also note that for the flow (12), \( \frac{d}{dt} \langle \cdot, \cdot \rangle_t \in T_{\langle \cdot, \cdot \rangle_t} C + \mathbb{R}I \) for all \( t \), which implies that the solution \( \langle \cdot, \cdot \rangle_t \) stays in the set of all scalar multiples of compatible metrics. Recall that if \( \mathbb{R}I \subset \mathfrak{g}_3 \), then the solution stays anyway in \( C \).

In these evolution approaches always appear naturally the soliton metrics, which are not fixed points of the flow but are close to, and they play an important role in the study of singularities (see the surveys [H2] [C] for further information). The idea is that if one is trying to improve a metric via an evolution equation, then those metrics for which the solution remains isometric to the initial point may be certainly considered as very distinguished.

Definition 2.5. A metric \( \langle \cdot, \cdot \rangle \) compatible with \( (N, \gamma) \) is called an invariant Ricci soliton if the solution \( \langle \cdot, \cdot \rangle_t \) to the normalized invariant Ricci flow (12) with initial metric \( \langle \cdot, \cdot \rangle_0 = \langle \cdot, \cdot \rangle \) is given by \( \varphi_t^* \langle \cdot, \cdot \rangle \), the pullback of \( \langle \cdot, \cdot \rangle \) by a one parameter group of diffeomorphisms \( \{ \varphi_t \} \) of \( N \).

We now give a neat characterization of invariant Ricci soliton metrics, which will be very useful in Section 2 to prove the equivalence with the property of being minimal (see Definition 2.2), and to find explicit examples in the subsequent sections.

Proposition 2.6. Let \( (N, \gamma) \) be a class-\( \gamma \) nilpotent Lie group. A compatible metric \( \langle \cdot, \cdot \rangle \) is an invariant Ricci soliton if and only if \( \text{Ric} \gamma \langle \cdot, \cdot \rangle = cI + D \) for some \( c \in \mathbb{R} \) and \( D \in \text{Der}(n) \). In such a case, \( c = \frac{\text{tr}(\text{Ric} \gamma \langle \cdot, \cdot \rangle)}{\text{sc}(\langle \cdot, \cdot \rangle)} \).
Proof. We first note that the assertion on the value of the number $c$ follows from (40); in fact,
\[
\text{tr}(\text{Ric}^\gamma(\cdot,\cdot))^2 = \text{tr}(\text{Ric}_t(\cdot,\cdot) \text{Ric}^\gamma(\cdot,\cdot)) = c \text{tr} \text{Ric}_t(\cdot,\cdot) + \text{tr}(\text{Ric}_t(\cdot,\cdot) D) = c \text{sc}(\cdot,\cdot).
\]
Assume that there exists a one-parameter group of diffeomorphisms $\varphi_t$ on $N$ such that $\langle \cdot, \cdot \rangle_t = \varphi_t^* \langle \cdot, \cdot \rangle$ is a solution to the flow (12). By the uniqueness of the solution we have that $\varphi_t^* g$ is also $N$-invariant for all $t$ (see Remark 2.3). Thus $\varphi_t$ normalizes $N$ and so it follows from ([W], Thm 2.4) that $\varphi_t \in \text{Aut}(N,N)$. This implies that there exists a one-parameter group $\psi_t$ of automorphisms of $N$ such that $\varphi_t^* \langle \cdot, \cdot \rangle = \psi_t^* \langle \cdot, \cdot \rangle$ for all $t$. Now, if $\psi_t = e^{-\frac{t}{2}D}$ with $D \in \text{Der}(N)$ then $\frac{d}{dt} \psi_t^* \langle \cdot, \cdot \rangle = \langle D \cdot, \cdot \rangle$, and using that $\psi_t^* \langle \cdot, \cdot \rangle$ is a solution of (12) in $t = 0$ we obtain that $\text{ric}^\gamma(\cdot,\cdot) = c \langle \cdot, \cdot \rangle + \langle D \cdot, \cdot \rangle$ for some $c \in \mathbb{R}$, or equivalently, $\text{Ric}^\gamma = c I + D$.

Conversely, if $\text{Ric}^\gamma = c I + D$ then we will show that the curve $\langle \cdot, \cdot \rangle_t = e^{-\frac{t}{2}D} \langle \cdot, \cdot \rangle$ is a solution to the flow (12). For any $t$, it follows from $\frac{d}{dt} D = \frac{1}{2} \text{Ric}^\gamma(\cdot,\cdot) - \frac{1}{2} c I \in g_\gamma + \mathbb{R} I$ that
\[
\gamma = b(t)e^{-\frac{t}{2}D} \gamma,
\]
for some $b(t) \in \mathbb{R}$. This implies that
\[
\text{Ric}^\gamma(\cdot,\cdot) = e^{-\frac{t}{2}D} \text{Ric}^\gamma(\cdot,\cdot) e^{\frac{t}{2}D} = e^{-\frac{t}{2}D}(c I + D)e^{\frac{t}{2}D} = c I + D
\]
for all $t$. Therefore
\[
\frac{d}{dt} \langle \cdot, \cdot \rangle_t = \langle D \cdot, \cdot \rangle_t = \langle (\text{Ric}^\gamma(\cdot,\cdot) - c I) \cdot, \cdot \rangle_t
\]
\[
= \text{ric}^\gamma(\cdot,\cdot) - c \langle \cdot, \cdot \rangle_t = \text{ric}^\gamma(\cdot,\cdot) + \frac{\text{tr}(\text{Ric}^\gamma(\cdot,\cdot))^2}{\text{sc}(\cdot,\cdot)} \langle \cdot, \cdot \rangle_t,
\]
as was to be shown. \qed

Recall that the condition in the above proposition can be replaced by
\[
\text{Ric}^\gamma(\cdot,\cdot) - \frac{\text{tr}(\text{Ric}^\gamma(\cdot,\cdot))^2}{\text{sc}(\cdot,\cdot)} I \in \text{Der}(N),
\]
which gives a computable method to check whether a metric is an invariant Ricci soliton or not.

3. Real geometric invariant theory and the moment map

In this section, we overview some results from (geometric) invariant theory over the real numbers. We refer to [RS] for a detailed exposition. These will be our tools to study metrics compatible with geometric structures on nilpotent Lie groups, as well as to approach the classification problem for such structures.

3.1. Closed orbits and minimal vectors. Let $G$ be a real reductive Lie group acting on a real vector space $V$ (see [RS] for a precise definition of the situation). Let $\mathfrak{g}$ denote the Lie algebra of $G$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup $K$ of $G$. Endow $V$ with a fixed from now on inner product $\langle \cdot, \cdot \rangle$ such that $\mathfrak{k}$ and $\mathfrak{p}$ act by skew-symmetric and symmetric transformations, respectively. Let $\mathcal{M} = \mathcal{M}(G,V)$ denote the set of minimal vectors, that is
\[
\mathcal{M} = \{ v \in V : ||v|| \leq ||g.v|| \quad \forall g \in G \}.
\]
For each $v \in V$ define
\[
\rho_v : G \mapsto \mathbb{R}, \quad \rho_v(g) = ||g.v||^2 = \langle g.v, g.v \rangle.
\]
In [RS], R. Richardson and P. Slodowy showed that the nice interplay between closed orbits and minimal vectors found by G. Kempf and L. Ness for actions of complex reductive algebraic groups, is still valid in the real situation.

**Theorem 3.1.** [RS] Let $V$ be a real representation of a real reductive Lie group $G$, and let $v \in V$.

(i) $v \in \mathcal{M}$ if and only if $\rho_v$ has a critical point at $e \in G$.
(ii) If $v \in \mathcal{M}$ then $G.v \cap \mathcal{M} = K.v$.
(iii) The orbit $G.v$ is closed if and only if $G.v$ meets $\mathcal{M}$.
(iv) The closure of any orbit $G.v$ always meets $\mathcal{M}$. Indeed, there exists $A \in \mathfrak{p}$ such that $\lim_{t \to -\infty} \exp(tA).v = v_0$ exists and $G.v_0$ is closed.
(v) $G.v \cap \mathcal{M}$ is a single $K$-orbit, or in other words, $G.v$ contains a unique closed $G$-orbit.

As usual in the real case, classical topology of $V$ is always considered rather than Zariski topology, unless explicitly indicated otherwise.

### 3.2. Critical points of the moment map.

We keep the notation of the above subsection. Let $(d\rho_v)_e : \mathfrak{g} \to \mathbb{R}$ denote the differential of $\rho_v$ at the identity $e$ of $G$. It follows from the $K$-invariance of $\langle \cdot, \cdot \rangle$ that $(d\rho_v)_e$ vanishes on $\mathfrak{k}$, and so we can assume that $(d\rho_v)_e \in \mathfrak{p}^*$, the vector space of real-valued functionals on $\mathfrak{p}$. We therefore may define a function called the *moment map* for the action of $G$ on $V$ by

$$m : V \to \mathfrak{p}, \quad \langle m(v), A \rangle_p = (d\rho_v)_e(A),$$

where $\langle \cdot, \cdot \rangle_p$ is an $\text{Ad}(K)$-invariant inner product on $\mathfrak{p}$. Since $m(tv) = t^2m(v)$ for all $t \in \mathbb{R}$, we also may consider the moment map

$$m : \mathbb{P}V \to \mathfrak{p}, \quad m(x) = \frac{m(v)}{|v|^2}, \quad 0 \neq v \in V, \; x = [v],$$

where $\mathbb{P}V$ is the projective space of lines in $V$. If $\pi : V \setminus \{0\} \to \mathbb{P}V$ denotes the usual projection map, then $\pi(v) = x$. In the complex case, under the natural identifications $\mathfrak{p} = \mathfrak{p}^* = (i\mathfrak{k})^* = \mathbb{R}^*$, the function $m$ is precisely the moment map from symplectic geometry, corresponding to the Hamiltonian action of $K$ on the symplectic manifold $\mathbb{P}V$ (see for instance the survey [K2] or [MFK, Chapter 8] for further information).

Consider the functional square norm of the moment map

$$F : V \to \mathbb{R}, \quad F(v) = ||m(v)||^2 = \langle m(v), m(v) \rangle_p,$$

which is easily seen to be a 4-degree homogeneous polynomial. Recall that $\mathcal{M}$ coincides with the set of zeros of $F$. It then follows from Theorem 3.1, parts (i) and (iii), that an orbit $G.v$ is closed if and only if $F(w) = 0$ for some $w \in G.v$, and in that case, the set of zeros of $F|_{G.v}$ coincides with $K.v$. We furthermore have the following result due to A. Neeman and G. Schwarz (see [RS]).

**Theorem 3.2.** Let $X$ be a closed $G$-invariant subset of $V$ and set $\mathcal{M}_X = \mathcal{M} \cap X$. Then the negative gradient flow of $F : V \to \mathbb{R}$ defines a $K$-equivariant deformation retraction $\psi : X \times [0, 1] \to X$ from $X$ onto $\mathcal{M}_X$ along $G$-orbits, that is,

(i) $\psi_0 = \text{Id}$, $\psi_1(X) = \mathcal{M}_X$, $\psi(v, t) = v$ for any $v \in \mathcal{M}_X$.
(ii) $\psi(v, t) \in G.v$ for all $t < 1$ and so $\psi(v, 1) \in \mathcal{M}_X$ for any $v \in X$.

$\psi$ also determines a deformation retraction of $X/G$ onto $\mathcal{M}_X/K$. 

A natural question arises: what is the role played by the remaining critical points of $F: \mathbb{P}V \to \mathbb{R}$ (i.e., those for which $F(x) > 0$) in the study of the $G$-orbit space of the action of $G$ on $V$, as well as on other real $G$-varieties contained in $V$? This was studied independently by F. Kirwan [K1] and L. Ness [N], and it is shown in the complex case that the non-minimal critical points share some of the nice properties of minimal vectors stated in Theorem 3.1. In the real case, which is actually the one we need to apply in this paper, the analogous of some of these results have been proved by A. Marian [M].

**Theorem 3.3.** [M] Let $V$ be a real representation of a real reductive Lie group $G$, $m: \mathbb{P}V \to \mathfrak{p}$ the moment map and $F = \|m\|^2: \mathbb{P}V \to \mathbb{R}$.

(i) If $x \in \mathbb{P}V$ is a critical point of $F$ then the functional $F|_{G,x}$ attains its minimum value at $x$.

(ii) If nonempty, the critical set of $F|_{G,x}$ consists of a unique $K$-orbit.

We endow $\mathbb{P}V$ with the Fubini-Study metric defined by $\langle \cdot, \cdot \rangle$ and denote by $x \mapsto A_x$ the vector field on $\mathbb{P}V$ defined by $A \in \mathfrak{g}$ via the action of $G$ on $\mathbb{P}V$, that is, $A_x = \frac{d}{dt}|_{t=0} \exp(tA).x$.

**Lemma 3.4.** [M] The gradient of the functional $F = \| m \|^2: \mathbb{P}V \to \mathbb{R}$ is given by

$$\text{grad}(F)_x = 4m(x)_x, \quad x \in \mathbb{P}V,$$

and therefore $x$ is a critical point of $F$ if and only if $m(x)_x = 0$, if and only if $\exp tm(x)$ fixes $x$.

We now develop some examples which are far to be the natural ones, but they are those ones will be considered in this paper to study left invariant structures on nilpotent Lie groups.

**Example 3.5.** Let $\mathfrak{n}$ be an $n$-dimensional real vector space and $V = \Lambda^2 \mathfrak{n}^* \otimes \mathfrak{n}$ the vector space of all skew-symmetric bilinear maps from $\mathfrak{n} \times \mathfrak{n}$ to $\mathfrak{n}$. There is a natural action of $GL(n) := GL(n, \mathbb{R})$ on $V$ given by

$$g.\mu(X,Y) = g\mu(g^{-1}X, g^{-1}Y), \quad X,Y \in \mathfrak{n}, \quad g \in GL(n), \quad \mu \in V.$$

Any inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ defines an $O(n)$-invariant inner product on $V$, denoted also by $\langle \cdot, \cdot \rangle$, as follows:

$$\langle \mu, \lambda \rangle = \sum_{ijk} \langle \mu(X_i, X_j), X_k \rangle \langle \lambda(X_i, X_j), X_k \rangle,$$

where $\{X_1, \ldots, X_n\}$ is any orthonormal basis of $\mathfrak{n}$. A Cartan decomposition for the Lie algebra of $GL(n)$ is given by $\mathfrak{gl}(n) = \mathfrak{so}(n) \oplus \mathfrak{sym}(n)$, that is, in skew-symmetric and symmetric transformations respectively, and we consider the following $\text{Ad}(O(n))$-invariant inner product on $\mathfrak{p} := \mathfrak{sym}(n)$,

$$\langle A, B \rangle_{\mathfrak{p}} = \text{tr} AB, \quad A, B \in \mathfrak{p}.$$

The action of $\mathfrak{gl}(n)$ on $V$ obtained by differentiation of (16) is given by

$$A.\mu = -\delta_\mu(A) := A\mu(\cdot, \cdot) - \mu(A, \cdot) - \mu(\cdot, A), \quad A \in \mathfrak{gl}(n), \quad \mu \in V.$$

If $\mu \in V$ satisfies the Jacobi condition, then $\delta_\mu: \mathfrak{gl}(n) \to V$ coincides with the cohomology coboundary operator of the Lie algebra $(\mathfrak{n}, \mu)$ from level 1 to 2, relative to cohomology with values in the adjoint representation. Recall that $\text{Ker} \delta_\mu = \text{Der}(\mu)$, the Lie algebra of derivations of the algebra $\mu$. Let $A^t$ denote the transpose.
relative to $\langle \cdot, \cdot \rangle$ of a linear transformation $A : n \mapsto n$ and consider the adjoint map $\text{ad}_\mu X : n \mapsto n$ (or left multiplication) defined by $\text{ad}_\mu X(Y) = \mu(X, Y)$.

**Proposition 3.6.** The moment map $m : V \mapsto p$ for the action (10) of $\text{GL}(n)$ on $V = \Lambda^2 n^* \otimes n$ is given by

$$m(\mu) = -4 \sum_i (\text{ad}_\mu X_i)^t \text{ad}_\mu X_i + 2 \sum_i \text{ad}_\mu X_i (\text{ad}_\mu X_i)^t,$$

where $\{X_1, \ldots, X_n\}$ is any orthonormal basis of $n$, and it is a simple calculation to see that

$$\langle m(\mu)X, Y \rangle = -4 \sum_{ij} \langle \mu(X, X_i), X_j \rangle \langle \mu(Y, X_i), X_j \rangle$$

$$+ 2 \sum_{ij} \langle \mu(X_i, X_j), X \rangle \langle \mu(X_i, X_j), Y \rangle, \quad \forall X, Y \in n.$$

**Proof.** For any $A \in p$ we have that

$$(d\rho_\mu)_I(A) = \frac{d}{dh}\big|_0 (e^{tA} \mu, e^{tA} \mu) = -2 \langle \mu, \delta_\mu(A) \rangle$$

$$= -2 \sum_{prij} \langle \mu(X_p, X_i), X_j \rangle \langle \delta_\mu(A)(X_p, X_i), X_j \rangle$$

$$= -2 \left( \sum_{prij} \langle \mu(X_p, X_i), X_j \rangle \langle \mu(AX_p, X_i), X_j \rangle \right.$$  

$$+ \langle \mu(X_p, X_i), X_j \rangle \langle \mu(X_p, AX_i), X_j \rangle$$

$$- \langle \mu(X_p, X_i), X_j \rangle \langle A\mu(X_p, X_i), X_j \rangle \bigg)$$

$$= -2 \left( \sum_{prij} \langle \mu(X_p, X_i), X_j \rangle \langle X_r, X_i \rangle \langle X_r, X_j \rangle \langle AX_p, X_r \rangle 

+ \langle \mu(X_p, X_i), X_j \rangle \langle X_r, X_i \rangle \langle X_r, X_j \rangle \langle AX_i, X_r \rangle 

- \langle \mu(X_p, X_i), X_j \rangle \langle X_r, X_i \rangle \langle X_r, X_j \rangle \langle AX_j, X_r \rangle \right).$$

By interchanging indexes $p$ and $i$ in the second line, and $p$ and $j$ in the third one, we get

$$(d\rho_\mu)_I(A) = -4 \sum_{prij} \langle \mu(X_p, X_i), X_j \rangle \langle X_r, X_i \rangle \langle X_r, X_j \rangle \langle AX_p, X_r \rangle$$

$$+ 2 \sum_{prij} \langle \mu(X_i, X_j), X_p \rangle \langle \mu(X_i, X_j), X_r \rangle \langle AX_p, X_r \rangle.$$

If we call $M$ the right hand side of (20), then we obtain from (21) that

$$(d\rho_\mu)_I(A) = \sum_{pr} \langle MX_p, X_r \rangle \langle AX_p, X_r \rangle = tr MA = \langle M, A \rangle_p,$$

which implies that $m(\mu) = M$ (see (10)).

**Example 3.7.** We keep the notation as in Example 3.4. Let $\gamma$ be a tensor on $n$ and let $G_\gamma \subset \text{GL}(n)$ denote the subgroup leaving $\gamma$ invariant, with Lie algebra $\mathfrak{g}_\gamma$. The group $G_\gamma$ is reductive and $K_\gamma = G_\gamma \cap O(n)$ is a maximal compact subgroup of $G_\gamma$, whose Lie algebra will be denoted by $\mathfrak{k}_\gamma$. A Cartan decomposition is given by

$$\mathfrak{g}_\gamma = \mathfrak{k}_\gamma \oplus \mathfrak{p}_\gamma, \quad \mathfrak{p}_\gamma = \mathfrak{p} \cap \mathfrak{g}_\gamma.$$
If \( p : p \mapsto p_\gamma \) is the orthogonal projection relative to \( \langle \cdot , \cdot \rangle_p \), then it is easy to see that
\[
m_\gamma : V \mapsto p_\gamma, \quad m_\gamma = p \circ m_x.
\]
is precisely the moment map for the action of \( G_\gamma \) on \( V \).

In the cases considered in detail in this paper we will have \( (G_\gamma, K_\gamma) \) equal to \((\text{Sp}(\mathbb{H}, \mathbb{R}), U(\mathbb{H}))\) (symplectic), \((\text{GL}(\mathbb{H}, \mathbb{C}), U(\mathbb{H}))\) (complex), \((\text{GL}(\mathbb{H}, \mathbb{H}), \text{Sp}(\mathbb{H}))\) (hypercomplex) and \((\text{GL}(n, \mathbb{R}), \text{O}(n))\) \((\gamma = 0)\).

3.3. Quotients of \( G \)-varieties. Let \( G \) be a real reductive Lie group acting on a finite dimensional real vector space \( V \), and let \( X \subset V \) be a \( G \)-variety, that is, a real algebraic variety which is \( G \)-invariant. The main problem of geometric invariant theory is to understand the orbit space of the action of \( G \) on \( X \), parameterized by the quotient \( X/G \) (see [PV] for further information). In the real situation, the main references are [RS], on which this brief overview is based, and papers due to D. Luna and G. Schwarz cited in [RS].

The standard quotient topology of \( X/G \) can be very ugly, for instance, if \( y \in \overline{G.x} \) and \( G.x \neq G.y \) then they can not be separated by \( G \)-invariant open neighborhoods and so \( X/G \) is not usually a \( T_1 \)-space. In order to avoid this problem one may consider a smaller quotient \( X/G \) parametrizing only closed orbits. For any \( x \in X \) there exists a unique \( G \)-orbit in \( \overline{G.x} \), and so a natural map \( q : X \mapsto X/G \) can be defined. Consider on \( X/G \) the quotient topology for the map \( q \). \( X/G \) is called the categorical quotient for the action of \( G \) on \( X \) since it satisfies the following universal property in the category of all \( T_1 \)-spaces: for any continuous map \( \alpha : X \mapsto Y \) that is constant on \( G \)-orbits, there exists a unique continuous map \( \beta : X/G \mapsto Y \) such that \( \alpha = \beta \circ q \). The uniqueness of an object with such a property is clear from the definition. Recall that the usual quotient \( X/G \) would be the categorical quotient in the category of all topological spaces.

The space \( X/G \) is actually Hausdorff. Moreover, the map \( \mathcal{M}_X/K \mapsto X/G \) determined by the inclusion \( \mathcal{M}_X \mapsto X \) is a homeomorphism, and hence it follows from properties of actions of compact Lie groups that \( X/G \) is a real semi-algebraic variety (i.e. defined by finitely many polynomial equations and inequalities).

It is clear that \( X/G \) is the ‘correct’ quotient to consider from many points of view, but the price to pay for a \( T_1 \) quotient is that in some cases \( X/G \) classifies only a very few orbits. In that case, it is then natural to try to understand the bigger set \( X/G \) parametrizing those orbits which contain a (non-necessarily zero) critical point of \( F = ||m||^2 : PV \mapsto \mathbb{R} \). Let \( C \subset PV \) denote the set of all such critical points and set \( C_X := C \cap PX, \) where \( PX = \pi(X) \) is the projection of \( X \) on \( PV \). Since there is at most one \( K \)-orbit of critical points on each \( G[v], \) we have that \( C_X/K = GC_X/G \) and thus we may define
\[
X/G := \{ v \in X : [v] \in C_X \}/K = \pi^{-1}(C_X/K), \quad PX/G := C_X/K.
\]
Recall that \( X/G \) is invariant by scalar multiplication and so the fact that two vectors in a line \( \mathbb{R}^n v \) were in the same \( G \)-orbit or not is disregarded.

This is not the place for deep considerations on the topology of this wider quotient \( X/G \) (nor are we qualified to do it). We do not know for instance if \( X/G \) will be the categorical quotient for a suitable category of non-necessarily \( T_1 \) topological spaces; or if the quotient topology for the map \( q : PX \mapsto PX/G \) defined by the negative gradient flow of \( F \) coincides with the standard topology of \( C_X/K \).
Anyway, although the space $X/\!/G$ is rather nebulous, the following result due to L. Ness helps us to describe it, by giving a decomposition of $X/\!/G$ in finitely many disjoint subsets each one being a categorical quotient for a suitable action and so homeomorphic to a semi-algebraic variety. These results are proved in the complex case, but it is not hard to see that they remain valid over $\mathbb{R}$.

**Theorem 3.8.** The negative gradient flow of $F = ||m||^2 : \mathbb{P}V \to \mathbb{R}$ determines a stratification of the projection $\mathbb{P}X$ of $X$ on $\mathbb{P}V$ given by

$$\mathbb{P}X = S_{(A_1)} \cup \ldots \cup S_{(A_s)}, \quad A_1, \ldots, A_s \in \mathfrak{p},$$

where each stratum $S_{(A)}$, $A \in \mathfrak{p}$, is the set of all the points $x \in \mathbb{P}X$ which flow into $C(A)$, the set of critical points $y \in \mathbb{P}X$ of $F$ such that $m(y) \in \text{Ad}(K).A$. Moreover, for each stratum $S_{(A)}$ there exists a subspace $V_A \subset V$ and a reductive subgroup $\tilde{G}_A \subset G$ such that $C(A)/K = \pi((V_A \cap X)/\tilde{G}_A)$, the projection of the categorical quotient for the action of $\tilde{G}_A$ on $V_A \cap X$.

It then follows that $X/\!/G$ can be decomposed as a disjoint union

$$X/\!/G = \pi^{-1}(C(A_1)/K) \cup \ldots \cup \pi^{-1}(C(A_s)/K), \quad A_1, \ldots, A_s \in \mathfrak{p},$$

where each subset $\pi^{-1}(C(A)/K)$, $i = 1, \ldots, s$, is homeomorphic to the categorical quotient $(V_A \cap X)/\tilde{G}_A$, and hence is a semi-algebraic variety.

In what follows, we explain how to construct the subspace $V_A$ and the subgroup $\tilde{G}_A$. For each $A \in \mathfrak{p}$ consider

$$V_A = \{v \in V : A.v = \frac{||A||^2}{2}v\}, \quad C_A = \{[v] \in C_X : m([v]) = A\},$$

and

$$G_A = \{g \in G : \text{Ad}(g)A = A\}, \quad \mathfrak{g}_A = \text{Lie}(G_A) = \{B \in \mathfrak{g} : [B, A] = 0\}.$$  

Let $\tilde{G}_A$ be the subgroup of $G_A$ with Lie algebra $\tilde{\mathfrak{g}}_A = \mathfrak{g}_A/\mathbb{R}A$. The corresponding Cartan decompositions will be denoted by $\mathfrak{g}_A = \mathfrak{t}_A \oplus \mathfrak{p}_A$ and $\tilde{\mathfrak{g}}_A = \mathfrak{t}_A \oplus \tilde{\mathfrak{p}}_A$, where $\mathfrak{p}_A = \mathfrak{t}_A \oplus \mathbb{R}A$ is an orthogonal decomposition and $\mathfrak{t}_A$ is the Lie algebra of $K_A = \{g \in K : \text{Ad}(g)A = A\}$, the maximal compact subgroup of both $G_A$ and $\tilde{G}_A$.

A crucial point here proved in 

is that $m(V_A) \subset \mathfrak{p}_A$ and hence the moment map $m_A : V_A \mapsto \tilde{\mathfrak{p}}_A$ for the action of $G_A$ on $V_A$ is just given by

$$m_A = p \circ m|_{V_A},$$

where $p : \mathfrak{p}_A \mapsto \tilde{\mathfrak{p}}_A$ is the orthogonal projection and $m : V \mapsto \mathfrak{p}$ is the moment map for the action of $G$ on $V$. This implies that

$$C_A = \{[v] \in \mathbb{P}(V_A \cap X) : m_A([v]) = 0\} = \pi(\{v \in V_A \cap X : m_A(v) = 0\}),$$

and hence

$$C(A)/K = C_A/K_A = \pi((V_A \cap X)/\tilde{G}_A),$$

the projection of the corresponding categorical quotient.
4. Variety of compatible metrics

Let us consider as a parameter space for the set of all real nilpotent Lie algebras of a given dimension $n$, the set $\mathcal{N}$ of all nilpotent Lie brackets on a fixed $n$-dimensional real vector space $n$. If

$$V = \Lambda^2 n^* \otimes n = \{\mu : n \times n \mapsto n : \mu \text{ skew-symmetric bilinear map}\},$$

then

$$\mathcal{N} = \{\mu \in V : \mu \text{ satisfies Jacobi and is nilpotent}\}$$

is an algebraic subset of $V$. Indeed, the Jacobi identity and the nilpotency condition are both determined by zeroes of polynomials.

We fix a tensor $\gamma$ on $n$ (or a set of tensors), and let $G_\gamma$ denote the subgroup of $GL(n)$ preserving $\gamma$. These groups act naturally on $V$ by (10) and leave $\mathcal{N}$ invariant. Consider the subset $\mathcal{N}_\gamma \subset \mathcal{N}$ given by

$$\mathcal{N}_\gamma = \{\mu \in \mathcal{N} : IC(\gamma, \mu) = 0\},$$

that is, those nilpotent Lie brackets for which $\gamma$ is integrable (see (3)). $\mathcal{N}_\gamma$ is also an algebraic variety since $IC(\gamma, \mu)$ is always linear on $\mu$. Recall that

$$W_\gamma = \{\mu \in V : IC(\gamma, \mu) = 0\}$$

is a $G_\gamma$-invariant linear subspace of $V$, and $\mathcal{N}_\gamma = \mathcal{N} \cap W_\gamma$.

For each $\mu \in \mathcal{N}_\gamma$, let $N_\mu$ denote the simply connected nilpotent Lie group with Lie algebra $(n, \mu)$. We now consider an identification of each point of $\mathcal{N}_\gamma$ with a compatible metric. Fix an inner product $\langle \cdot, \cdot \rangle$ on $n$ compatible with $\gamma$, that is, such that (3) holds. We identify each $\mu \in \mathcal{N}_\gamma$ with a class-$\gamma$ metric structure

$$\mu \leftrightarrow (N_\mu, \gamma, \langle \cdot, \cdot \rangle),$$

where all the structures are defined by left invariant translation. Therefore, each $\mu \in \mathcal{N}_\gamma$ can be viewed in this way as a metric compatible with the class-$\gamma$ nilpotent Lie group $(N_\mu, \gamma)$, and two metrics $\mu, \lambda$ are compatible with the same geometric structure if and only if they live in the same $G_\gamma$-orbit. Indeed, the action of $G_\gamma$ on $\mathcal{N}_\gamma$ has the following interpretation: each $\varphi \in G_\gamma$ determines a Riemannian isometry preserving the geometric structure

$$(N_\varphi \mu, \gamma, \langle \cdot, \cdot \rangle) \mapsto (N_\mu, \gamma, \langle \varphi \cdot, \varphi \cdot \rangle)$$

by exponentiating the Lie algebra isomorphism $\varphi^{-1} : (n, \varphi, \mu) \mapsto (n, \mu)$. We then have the identification $G_\gamma \mu = C(N_\mu, \gamma)$, and more in general the following

**Proposition 4.1.** Every class-$\gamma$ metric structure $(N', \gamma', \langle \cdot, \cdot \rangle')$ on a nilpotent Lie group $N'$ of dimension $n$ is isometric-isomorphic to a $\mu \in \mathcal{N}_\gamma$.

**Proof.** We can assume that the Lie algebra of $N'$ is $(n, \lambda)$ for some $\lambda \in \mathcal{N}$. There exist $\varphi \in GL(n)$ and $\psi \in O(n, \langle \cdot, \cdot \rangle)$ such that $\varphi \cdot = \langle \cdot, \cdot \rangle = (\cdot, \cdot)$ and $\psi(\varphi, \gamma') = \gamma$. Thus the Lie algebra isomorphism $\psi \varphi : (n, \lambda) \mapsto (n, \mu)$, where $\mu = \psi \varphi \lambda$, satisfies $\psi \varphi (\cdot, \cdot') = (\cdot, \cdot)$ and $\psi \varphi, \gamma' = \gamma$ and so it defines an isometry

$$(N', \gamma', \langle \cdot, \cdot \rangle') \mapsto (N_\mu, \gamma, \langle \cdot, \cdot \rangle)$$

which is also an isomorphism between the class-$\gamma$ nilpotent Lie groups $(N', \gamma')$ and $(N, \gamma)$, concluding the proof. \qed
According to the above proposition and identification (23), the orbit \( G_\gamma.\mu \) parameterizes all the left invariant metrics which are compatible with \((N_\mu,\gamma)\) and hence we may view \( N_\gamma \) as the space of all class-\( \gamma \) metric structures on nilpotent Lie groups of dimension \( n \). Since two metrics \( \mu, \lambda \in N_\gamma \) are isometric if and only if they live in the same \( K_\gamma \)-orbit, where \( K_\gamma = G_\gamma \cap O(n,\langle \cdot, \cdot \rangle) \) (see Appendix 10), we have that \( N_\gamma/K_\gamma \) parameterizes class-\( \gamma \) metric nilpotent Lie groups of dimension \( n \) up to isometry and \( G_\gamma.\mu/K_\gamma \) do the same for all the compatible metrics on \((N_\mu,\gamma)\).

We now recall a crucial fact which is the link between the study of left invariant compatible metrics for geometric structures on nilpotent Lie groups and the results from real geometric invariant theory exposed in Section 3. The interplay is based on the identification given in (23), and it will be used in the proofs of the remaining results of this section. The proof of the following proposition follows just from a simple comparison between formulas (38) and (21).

**Proposition 4.2.** Let \( m : V \to p \) and \( m_\gamma : V \to p_\gamma \) be the moment maps for the actions of \( GL(n) \) and \( G_\gamma \) on \( V = \Lambda^2 n^* \otimes n \), respectively (see Examples 3.5, 3.7), where \( p \) is the space of symmetric maps of \((n,\langle \cdot, \cdot \rangle)\) and \( p_\gamma \) the subspace of those leaving \( \gamma \) invariant.

(i) For each \( \mu \in N \subset V \),

\[
m(\mu) = 8 \operatorname{Ric}_\mu,
\]

where \( \operatorname{Ric}_\mu \) is the Ricci operator of the Riemannian manifold \((N_\mu,\langle \cdot, \cdot \rangle)\).

(ii) For each \( \mu \in N_\gamma \subset V \),

\[
m_\gamma(\mu) = 8 \operatorname{Ric}_\gamma^\mu,
\]

where \( \operatorname{Ric}_\gamma^\mu \) is the invariant Ricci operator of \((N_\mu,\gamma,\langle \cdot, \cdot \rangle)\), that is, the orthogonal projection of the Ricci operator \( \operatorname{Ric}_\mu \) on \( p_\gamma \).

Part (i) will be applied in Section 8 to consider the case \( \gamma = 0 \), which is strongly related with the study of Einstein solvmanifolds.

Let us now go back to our search for the best compatible metric. The identification (23) allows us to view each point of the variety \( N_\gamma \) as a class-\( \gamma \) metric structure on a nilpotent Lie group of dimension \( n \). In this light, it is natural to consider the functional \( F : N_\gamma \to \mathbb{R} \) given by \( F(\mu) = \operatorname{tr}(\operatorname{Ric}_\gamma^\mu)^2 \), which in some sense measures how far is the metric \( \mu \) from having \( \operatorname{Ric}_\gamma^\mu = 0 \), which is the goal proposed in Section 2 (see (9)). The critical points of \( F \) may be therefore considered compatible metrics of particular significance. However, we should consider some normalization since \( F(t\mu) = t^4 F(\mu) \) for all \( t \in \mathbb{R} \).

For any \( \mu \in N_\gamma \) we have that \( \operatorname{sc}(\mu) = -\frac{1}{4}||\mu||^2 \) (see (38)), which says that normalizing by scalar curvature and by the spheres of \( V \) is equivalent:

\[
\{\mu \in N_\gamma : \operatorname{sc}(\mu) = s\} = \{\mu \in N_\gamma : ||\mu||^2 = -4s\}, \quad \forall s < 0.
\]

The critical points of \( F : PV \to \mathbb{R} \), \( F(||\mu||) = \operatorname{tr}(\operatorname{Ric}_\gamma^\mu)^2/||\mu||^4 \), which lie in \( \mathbb{P}N_\gamma = \pi(N_\gamma) \) appears then as very natural candidates, since it is like we are restricting \( F \) to the subset of all class-\( \gamma \) metric structures having a given scalar curvature. It follows from Proposition 4.2, (ii), that

\[
F(||\mu||) = \frac{1}{64}||m_\gamma(||\mu||)||^2,
\]
where \( m_\gamma : \mathbb{P}V \mapsto \mathfrak{p}_\gamma \) is the moment map for the action of \( G_\gamma \) on \( \mathbb{P}V \). We then obtain from Lemma 3.4 and (19) that

\[
\text{grad}(F)(\mu) = -\frac{1}{16} \pi^* \delta_\mu (\text{Ric}^\gamma_\mu), \quad ||\mu|| = 1,
\]

where \( \pi^* : V \mapsto T[\mu] \mathbb{P}V \) denotes the derivative of the projection map \( \pi : V \mapsto \mathbb{P}V \). This shows that \( [\mu] \in \mathbb{P}V \) is a critical point of \( F \) if and only if \( \text{Ric}^\gamma_\mu = cI + D \) for some \( c \in \mathbb{R} \) and \( D \in \text{Der}(\mu) (= \text{Ker}\delta_\mu) \). By applying Theorem 3.3 to our situation we obtain the main result of this section.

**Theorem 4.3.** Let \( F : \mathbb{P}V \mapsto \mathbb{R} \) be defined by \( F([\mu]) = \text{tr}(\text{Ric}^\gamma_\mu)^2/||\mu||^4 \). Then for \( \mu \in V \) the following conditions are equivalent:

(i) \( [\mu] \) is a critical point of \( F \).
(ii) \( F|_{G_\gamma[\mu]} \) attains its minimum value at \( [\mu] \).
(iii) \( \text{Ric}^\gamma_\mu = cI + D \) for some \( c \in \mathbb{R} \), \( D \in \text{Der}(\mu) \).

Moreover, all the other critical points of \( F \) in the orbit \( G_\gamma[\mu] \) lie in \( K_\gamma[\mu] \).

We now rewrite the above result in geometric terms, by using the identification (23), Proposition 2.6 and Definition 2.2.

**Theorem 4.4.** Let \( (N, \gamma) \) be a nilpotent Lie group endowed with an invariant geometric structure \( \gamma \). Then the following conditions on a left invariant Riemannian metric \( \langle \cdot, \cdot \rangle \) which is compatible with \( (N, \gamma) \) are equivalent:

(i) \( \langle \cdot, \cdot \rangle \) is minimal.
(ii) \( \langle \cdot, \cdot \rangle \) is an invariant Ricci soliton.
(iii) \( \text{Ric}^\gamma_{\langle \cdot, \cdot \rangle} = cI + D \) for some \( c \in \mathbb{R} \), \( D \in \text{Der}(\mathfrak{n}) \).

Moreover, there is at most one compatible left invariant metric on \( (N, \gamma) \) up to isometry (and scaling) satisfying any of the above conditions.

Recall that the proof of this theorem does not use the integrability of \( \gamma \), and so it is valid for the ‘almost’ versions as well.

We also note that part (i) of Theorem 4.3 makes possibly the study of minimal compatible metrics by a variational method. Indeed, the projective algebraic variety \( \mathbb{P}N_\gamma \) may be viewed as the space of all class-\( \gamma \) metric structures on \( n \)-dimensional nilpotent Lie groups with a given scalar curvature, and those which are minimal are precisely the critical points of \( F : \mathbb{P}N_\gamma \mapsto \mathbb{R} \). This variational approach will be used quite often in the search for explicit examples in the following sections.

The above theorems propose then as privileged these compatible metrics called minimal, which have a neat characterization (see (iii)), are critical points of a natural curvature functional (square norm of Ricci), minimize such a functional when restricted to the compatible metrics for a given geometric structure, and are solitons for a natural evolution flow. Moreover, the uniqueness up to isometry of such special metrics holds. But a remarkable weakness of this approach is the existence problem; the theorems do not even suggest when there do exist such a distinguished metric. How special are the symplectic or (almost-) complex structures admitting a minimal metric?. So far, we know how to deal with this ‘existence question’ only by giving several examples, which is the goal of Sections 5-8. The only obstruction we have found is in the case \( \mathbb{R} I \not\subset \mathfrak{g}_\gamma \), namely when \( \text{Der}(\mathfrak{n}) \) is nilpotent. These Lie algebras are called characteristically nilpotent and have been extensively studied in the last years, but we could not find any example of a characteristically nilpotent
Lie algebra admitting a symplectic structure. Thus we have no any non-existence example yet.

Corollary 4.5. Let $\gamma, \gamma'$ be two geometric structures on a nilpotent Lie group $N$, and assume that they admit minimal compatible metrics $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$, respectively. Then $\gamma$ is isomorphic to $\gamma'$ if and only if there exists $\varphi \in \text{Aut}(n)$ and $c > 0$ such that $\gamma' = \varphi \gamma$ and

$$\langle \varphi X, \varphi Y \rangle' = c \langle X, Y \rangle \quad \forall X, Y \in n.$$  

In particular, if $\gamma$ and $\gamma'$ are isomorphic then their respective minimal compatible metrics are necessarily isometric up to scaling (recall that $c = 1$ when $\text{sc}(\langle \cdot, \cdot \rangle) = \text{sc}(\langle \cdot, \cdot \rangle')$).

We have here a very useful tool to distinguish two geometric structures. Indeed, the corollary allows us to do it by looking at their respective minimal compatible metrics, that is, with Riemannian data. This is a remarkable advantage since we suddenly have a great deal of invariants. This method will be used in the subsequent sections to find explicit continuous families depending on 1, 2 and 3 parameters of pairwise non-isomorphic geometric structures in low dimensions, mainly by using only one Riemannian invariant: the eigenvalues of the Ricci operator.

Remark 4.6. The Ricci curvature operator of a left invariant metric $\langle \cdot, \cdot \rangle$ on a Lie group $G$ is given by

$$\text{Ric}_{\langle \cdot, \cdot \rangle} = R_{\langle \cdot, \cdot \rangle} - \frac{1}{2} B_{\langle \cdot, \cdot \rangle} - D_{\langle \cdot, \cdot \rangle},$$

where $R_{\langle \cdot, \cdot \rangle}$ is defined by (35), $B_{\langle \cdot, \cdot \rangle}$ is the Killing form of the Lie algebra $g$ of $G$ in terms of $\langle \cdot, \cdot \rangle$, $D_{\langle \cdot, \cdot \rangle}$ is the symmetric part of $\text{ad} Z_{\langle \cdot, \cdot \rangle}$ and $Z_{\langle \cdot, \cdot \rangle} \in g$ is defined by $\langle Z_{\langle \cdot, \cdot \rangle}, X \rangle = \text{tr}(\text{ad} X)$ for any $X \in g$. Recall that $Z_{\langle \cdot, \cdot \rangle} = 0$ if and only if $g$ is unimodular, and that $\text{Ric}_{\langle \cdot, \cdot \rangle} = R_{\langle \cdot, \cdot \rangle}$ in the nilpotent case. If we consider the tensor $R$ instead of the Ricci tensor, and the variety of all Lie algebras $L$ rather than just the nilpotent ones, to define and state all the notions, flows, identifications and results in Sections 2 and 4, then everything is still valid for Lie groups in general, with the only exception of the first part of Corollary 4.5. We only have to consider the corresponding invariant part $R^\gamma$ and replace $\text{sc}(\langle \cdot, \cdot \rangle)$ with $\text{tr} R_{\langle \cdot, \cdot \rangle}$ each time it appears. The only detail to be careful with is that if two $\mu, \lambda \in L_\gamma$ lie in the same $K_\gamma$-orbit then they are isometric, but the converse might not be true. Recall that the uniqueness result in Theorem 4.4 is nevertheless valid.

The reason why we decided to work only in the nilpotent case is that, at least at first sight, the use of this ‘unnamed’ tensor $R$ make minimal and soliton metrics, as well as the functionals and evolution flows, into concepts lacking in geometric sense. For instance, we have found ourselves with the unpleasant fact that some Kähler metrics on solvable Lie groups would not be minimal viewed as compatible metrics for the corresponding symplectic structures, in spite of $\text{Ric}_{\langle \cdot, \cdot \rangle}^\text{ac} = 0$.

For a compact simple Lie group, $-B$ is minimal for the case $\gamma = 0$, and if $g = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition for a non-compact semi-simple Lie algebra $g$, then it is easy to see that the metric $\langle \cdot, \cdot \rangle$ given by $\langle \mathfrak{k}, \mathfrak{p} \rangle = 0$, $\langle \cdot, \cdot \rangle|_{\mathfrak{k} \times \mathfrak{k}} = -B$ and $\langle \cdot, \cdot \rangle|_{\mathfrak{p} \times \mathfrak{p}} = B$ is minimal as well.

5. Symplectic structures

5.1. Metrics with hermitian Ricci tensor and the anti-complexified Ricci flow. Let $(M, \omega)$ be a symplectic manifold, that is, a differentiable manifold $M$
endowed with a global 2-form $\omega$ which is closed ($d\omega = 0$) and non-degenerate ($\omega^n \neq 0$). A Riemannian metric $g$ on $M$ is said to be compatible with $\omega$ if there exists an almost-complex structure $J_g$ (i.e. a $(1,1)$-tensor field with $J_g^2 = -I$) such that

$$\omega = g(\cdot, J_g \cdot).$$

In that case $J_g$ is uniquely determined by $g$, and one may also define that an almost-complex structure $J$ is compatible with $\omega$ if

$$g_J = \omega(\cdot, J \cdot)$$

determines a Riemannian metric, which is again uniquely determined by $J$. In such a way we are really talking about compatible pairs $(g, J_g) = (g, J_g)$, and the triple $(\omega, g, J_g)$ is called an almost-Kähler structure on $M$.

It is well known that for any symplectic manifold there always exist a compatible metric. Moreover, the space $C = C(M, \omega)$ of all compatible metrics is usually huge; recall for instance that the group of all symplectomorphisms (i.e. diffeomorphisms $\varphi$ of $M$ such that $\varphi^* \omega = \omega$) acts on $C$.

We fix from now on a symplectic manifold $(M, \omega)$. Let $\text{Ric}_g$ and $\nabla_g$ denote the Ricci operator and the Levi-Civita connexion of a compatible metric $g$, respectively. The most famous conditions to ask $g$ to satisfy are Einstein (i.e. $\text{Ric}_g = cI$) and Kähler (i.e. $\nabla_g J_g = 0$), which are both very strong and share the following property.

**Definition 5.1.** We say that $g$ has hermitian (or $J$-invariant) Ricci tensor or that $J_g$ is harmonic, if $\text{Ric}_g J_g = J_g \text{Ric}_g$.

Examples of compatible metrics with hermitian Ricci tensor which are neither Einstein nor Kähler are known in any dimension $2n \geq 6$ (see [AG, DM, LW]). We will show in Section 5.2 that a symplectic nilpotent Lie group can never admit a compatible metric with hermitian Ricci tensor unless it is abelian.

A classical approach to searching for distinguished metrics is the variational one, that is, to consider critical points of natural functionals of the curvature on the space of all metrics of a given class. For instance, if $M$ is compact, D. Hilbert [Hil] proved that Einstein metrics on $M$ are precisely the critical points of the total scalar curvature functional

$$S : \mathcal{M}_1 \mapsto \mathbb{R}, \quad S(g) = \int_M \text{sc}(g) \, d\nu_g,$$

where $\mathcal{M}_1$ is the space of all Riemannian metrics on $M$ with volume equal to 1. Since the set of compatible metrics $C$ is smaller, one should expect a weaker critical point condition for $S : C \mapsto \mathbb{R}$. Another natural functional in our setup would be

$$K : C \mapsto \mathbb{R}, \quad K(g) = \int_M ||\nabla_g J_g||^2 \, d\nu_g,$$

for which Kähler metrics are precisely the global minima. D. Blair and S. Ianus proved that, curiously enough, both functionals $S$ and $K$ have the same critical points on $C$.

**Theorem 5.2.** [BI] Let $(M, \omega)$ be a symplectic manifold and $C$ the set of all compatible metrics. Then $g \in C$ is a critical point of $S : C \mapsto \mathbb{R}$ or $K : C \mapsto \mathbb{R}$ if and only if $g$ has hermitian Ricci tensor.
This result and the above considerations do suggest that the compatible metrics with hermitian Ricci tensor (if any) are really ‘good friends’ of the symplectic structure.

In [LW], H-V Le and G. Wang approach the problem of the existence of such metrics by considering an evolution flow inspired in the Ricci flow introduced by R. Hamilton [H1]. If \( \text{ric}_g \) is the Ricci tensor of a compatible metric \( g \), then consider the orthogonal decomposition

\[
\text{ric}_g = \text{ric}^{ac}_g + \text{ric}^c_g,
\]

where \( \text{ric}^{ac}_g = \frac{1}{2}(\text{ric}_g - \text{ric}_g(J_g \cdot, J_g \cdot)) \) and \( \text{ric}^c_g = \frac{1}{2}(\text{ric}_g + \text{ric}_g(J_g \cdot, J_g \cdot)) \) are the anti-complexified and complexified parts of \( \text{ric}_g \), respectively. In this way, \( g \) has hermitian Ricci tensor if and only if \( \text{ric}^{ac}_g = 0 \), and since the gradient of the functional \( K \) equals \( -\text{ric}^{ac}_g \), it is natural to consider the negative gradient flow equation

\[
\frac{d}{dt} g(t) = \text{ric}^{ac}_g g(t),
\]

for a curve \( g(t) \) of metrics, which is called in [LW] the anti-complexified Ricci flow. Recall that the fixed points of (25) are precisely the metrics with hermitian Ricci tensor. The main result in [LW] is the short time existence and uniqueness of the solution to (25) when \( M \) is compact.

5.2. Symplectic nilpotent Lie groups. Let \( N \) be a real \( 2n \)-dimensional nilpotent Lie group with Lie algebra \( n \), whose Lie bracket is denoted by \( \mu : n \times n \rightarrow n \). An invariant symplectic structure on \( N \) is defined by a 2-form \( \omega \) on \( n \) satisfying

\[
\omega(X,Y) = 0 \quad \text{if and only if} \quad X = 0 \quad \text{(non-degenerate),}
\]

and for all \( X,Y,Z \in n \)

\[
\omega(\mu(X,Y),Z) + \omega(\mu(Y,Z),X) + \omega(\mu(Z,X),Y) = 0 \quad \text{(closed, d} \omega = 0).\]

Fix a symplectic nilpotent Lie group \( (N,\omega) \). A left invariant Riemannian metric which is compatible with \( (N,\omega) \) is determined by an inner product \( \langle \cdot , \cdot \rangle \) on \( n \) such that if

\[
\omega(X,Y) = \langle X, J(\cdot,Y) \rangle \quad \forall X,Y \in n \quad \text{then} \quad J^2 = -I.
\]

For the geometric structure \( \gamma = \omega \) we have that

\[
G_\gamma = \text{Sp}(n,\mathbb{R}) = \{ g \in \text{GL}(2n) : g^t J g = J \}, \quad K_\gamma = \text{U}(n),
\]

and the Cartan decomposition of \( g_\gamma = \text{sp}(n,\mathbb{R}) = \{ A \in \text{gl}(2n) : A^t J + JA = 0 \} \) is given by

\[
\text{sp}(n,\mathbb{R}) = \mathfrak{u}(n) \oplus \mathfrak{p}_\gamma, \quad \mathfrak{p}_\gamma = \{ A \in \mathfrak{p} : AJ = -JA \}.
\]

Thus the invariant Ricci tensor \( \text{ric}^\gamma \) coincides with the anti-complexified Ricci tensor (see [24]) and for any \( \langle \cdot , \cdot \rangle \in \mathcal{C} \),

\[
\text{Ric}^\gamma = \text{Ric}^{ac} = \frac{1}{2}(\text{Ric} + J(\cdot,Y) \text{Ric}(\cdot,Y) J(\cdot,Y)).
\]

This implies that our ‘goal’ condition \( \text{Ric}^\gamma = 0 \) (see [9]) is equivalent to have hermitian Ricci tensor. Also, the evolution flow considered in Section 2 is not other than the anti-complexified Ricci flow.

Concerning the search for the best compatible left invariant metric for a symplectic nilpotent Lie group, and in view of the facts exposed in Section 5.1, our first result is negative.
Proposition 5.3. Let \((N, \omega)\) be a symplectic nilpotent Lie group. Then \((N, \omega)\) does not admit any compatible left invariant metric with hermitian Ricci tensor, unless \(N\) is abelian.

Proof. We first note that since \(\mu\) is nilpotent the center \(z\) of \((n, \mu)\) meets non-trivially the derived Lie subalgebra \(\mu(n, n)\), unless \(\mu = 0\) (i.e. \(n\) abelian). Assume that \(\langle \cdot, \cdot \rangle \in C(N, \omega)\) has hermitian Ricci tensor and consider the orthogonal decomposition \(n = v \oplus \mu(n, n)\). If \(Z \in z\) then \(JZ \in v\). In fact, it follows from (24) that
\[
\langle \mu(X, Y), JZ \rangle = \omega(\mu(X, Y), Z) = 0 \quad \forall X, Y \in n.
\]
Now, the above equation, the fact that \(\text{Ric}_{\langle \cdot, \cdot \rangle} = J \text{Ric}_{\langle \cdot, \cdot \rangle}\) and the definition of \(\text{Ric}_{\langle \cdot, \cdot \rangle}\) (see (28)) imply that
\[
0 \leq \langle \text{Ric}_{\langle \cdot, \cdot \rangle} Z, Z \rangle = \langle \text{Ric}_{\langle \cdot, \cdot \rangle} JZ, JZ \rangle \leq 0,
\]
and hence
\[
\frac{1}{4} \sum_{ij} \langle \mu(X_i, X_j), Z \rangle^2 = \langle \text{Ric}_{\langle \cdot, \cdot \rangle} Z, Z \rangle = 0 \quad \forall Z \in z.
\]
Thus \(\mu(n, n) \perp z\) and so \(n\) must be abelian by the observation made in the beginning of the proof. \(\square\)

We now review the variational approach developed in Section 4 and obtain some applications. Fix a non-degenerate 2-form \(\omega\) on \(n\), and let \(\text{Sp}(n, \mathbb{R})\) denote the subgroup of \(\text{GL}(2n)\) preserving \(\omega\), that is,
\[
\text{Sp}(n, \mathbb{R}) = \{\varphi \in \text{GL}(2n) : \omega(\varphi X, \varphi Y) = \omega(X, Y) \quad \forall X, Y \in n\}.
\]
Consider the algebraic subvariety \(N_{\ast} := N_{\gamma} \subset N\) given by
\[
N_{\ast} = \{\mu \in N : d_{\mu} \omega = 0\},
\]
that is, those nilpotent Lie brackets for which \(\omega\) is closed (see (26)). By fixing an inner product \(\langle \cdot, \cdot \rangle\) on \(n\) satisfying that
\[
\omega = \langle \cdot, J\cdot \rangle \quad \text{with} \quad J^2 = -I,
\]
(23) identify each \(\mu \in N_{\ast}\) with the almost-Kähler manifold \((N_{\mu}, \omega, \langle \cdot, \cdot \rangle, J)\). The action of \(\text{Sp}(n, \mathbb{R})\) on \(N_{\ast}\) has the following interpretation: each \(\varphi \in \text{Sp}(n, \mathbb{R})\) determines a Riemannian isometry which is also a symplectomorphism
\[
(N_{\varphi, \mu}, \omega, \langle \cdot, \cdot \rangle, J) \mapsto (N_{\mu}, \omega, \langle \varphi \cdot, \varphi \cdot \rangle, \varphi^{-1} J \varphi)
\]
by exponentiating the Lie algebra isomorphism \(\varphi^{-1} : (n, \varphi, \mu) \mapsto (n, \mu)\).

We have seen in Proposition 5.3 that the hermitian Ricci condition \(\text{Ric}_{\text{ac}} = 0\) is a very nice but forbidden condition for a symplectic non-abelian nilpotent Lie group. However, we can get as close as we want.

Proposition 5.4. Let \((N, \omega)\) be a symplectic nilpotent Lie group. Then for any \(\epsilon > 0\) there exists a compatible left invariant metric \(\langle \cdot, \cdot \rangle\) such that \(\text{tr}(\text{Ric}_{\text{ac}})^2 < \epsilon\).

Proof. In view of the identification (23), this is equivalent to prove that for any \(\mu \in N_{\ast}\) there exists \(\lambda \in \text{Sp}(n, \mathbb{R}), \mu\) such that \(\text{tr}(\text{Ric}_{\text{ac}})^2 < \epsilon\). We have that an orbit \(\text{Sp}(n, \mathbb{R}), \mu\) is closed if and only if \(\text{Ric}_{\text{ac}} = 0\) for some \(\lambda \in \text{Sp}(n, \mathbb{R}), \mu\) (see Theorem 3.1, (iii) and Proposition 4.2, (ii)). It then follows from Proposition 5.3 that for \(\mu \in N_{\ast}\) the orbit \(\text{Sp}(n, \mathbb{R}), \mu\) can never be closed, unless \(\mu = 0\). Thus \(0 \in \text{Sp}(n, \mathbb{R}), \mu\) for any \(\mu \in N_{\ast}\) (see Theorem 3.1, (v)), and the continuity of the function \(\lambda \mapsto \text{tr}(\text{Ric}_{\text{ac}})^2\) concludes the proof. \(\square\)
Proposition 5.5. Let \((N, \omega)\) be a symplectic nilpotent Lie group. For any real number \(s < 0\) there exists a compatible metric \(\langle \cdot, \cdot \rangle\) such that \(sc(\langle \cdot, \cdot \rangle) = s\).

Proof. By considering the identification (23) and taking in account that \(sc(\mu) = -\frac{1}{4}\|\mu\|^2\) (see [RS]), we get that this proposition is equivalent to the following fact: for each \(\mu \in \mathcal{N}\), the orbit \(Sp(n, \mathbb{R}).\mu\) meets all the spheres of \(V\). In the proof of Proposition 4.4 we have seen that \(0 \in Sp(n, \mathbb{R}).\mu\) for any \(\mu \in \mathcal{N}\), and so by Theorem 3.1 (iv), there exists \(A \in \mathfrak{p}\) such that \(\lim_{t \to -\infty} \exp tA.\mu = 0\).

If \(f(t) = \| \exp tA.\mu \|^2\) then \(f''(t) > 0\) for all \(t \in \mathbb{R}\) (see [RS, Lemma 3.1]) and so \(\lim_{t \to -\infty} f(t) = +\infty\), concluding the proof.

5.3. Examples. Let \(\mathfrak{n}\) be a 2\(n\)-dimensional vector space with basis \(\{X_1, \ldots, X_{2n}\}\) over \(\mathbb{R}\), and consider the non-degenerate 2-form

\[
\omega = \alpha_1 \wedge \alpha_{2n} + \ldots + \alpha_n \wedge \alpha_{n+1},
\]

where \(\{\alpha_1, \ldots, \alpha_{2n}\}\) is the dual basis of \(\{X_i\}\). For the compatible inner product \(\langle X_i, X_j \rangle = \delta_{ij}\) we have that \(\omega = \langle \cdot, J \cdot \rangle\) for

\[
J = \begin{bmatrix}
0 & 1 & & \\
-1 & 0 & & \\
& & \ddots & \\
& & & 0
\end{bmatrix}.
\]

In all the examples the symplectic structure will be \(\omega\), the almost-complex structure \(J\) and the compatible metric \(\langle \cdot, \cdot \rangle\). We will vary Lie brackets and use constantly identification (23).

Example 5.6. Let \(\mu_n\) the 2\(n\)-dimensional Lie algebra whose only non-zero bracket is

\[
\mu_n(X_1, X_2) = X_3,
\]

that is, \(\mu_n\) is isomorphic to \(\mathfrak{h}_3 \oplus \mathbb{R}^n\), where \(\mathfrak{h}_3\) is the 3-dimensional Heisenberg Lie algebra. Recall that \((N_{\mu_2}, \omega)\) is precisely the simply connected cover of the famous Kodaira-Thurston manifold. It is easy to prove that \(Sp(n, \mathbb{R}).\mu_n = Sp(n, \mathbb{R}).\mu_n \cup \{0\}\), and so the orbit \(Sp(n, \mathbb{R}).\mu_n\) is closed in \(P.V\). This implies that the functional \(F\) from Theorem 4.3 must attain its minimum value on \(Sp(n, \mathbb{R}).\mu_n\) and hence there exists a metric compatible with \((N_{\mu_n}, \omega)\) which is minimal. In fact, the inner product \(\langle X_i, X_j \rangle = \delta_{ij}\) satisfies

\[
\text{Ric}^{ac}\langle \cdot, \cdot \rangle = -\frac{1}{4} I - \frac{3}{4} I + \left[ \begin{array}{cccc}
2 & 4 & \cdots & 4 \\
4 & 2 & \cdots & 4 \\
\vdots & \vdots & \ddots & \vdots \\
4 & 4 & \cdots & 2
\end{array} \right],
\]

\(2n \geq 8,\)

\[
\text{Ric}^{ac}\langle \cdot, \cdot \rangle = -\frac{1}{4} I - \frac{3}{4} I + \left[ \begin{array}{cccc}
2 & 4 & \cdots & 4 \\
4 & 2 & \cdots & 4 \\
\vdots & \vdots & \ddots & \vdots \\
4 & 4 & \cdots & 2
\end{array} \right],
\]

\(2n = 6,\)

and hence \(\text{Ric}^{ac}\langle \cdot, \cdot \rangle \in \mathbb{R}I + \text{Der}(\mu_n)\) in all the cases. Moreover, it follows from the closeness of \(Sp(n, \mathbb{R}).\mu_n\) that \(F\) must also attain its maximum value, and therefore \(Sp(n, \mathbb{R}).\mu_n = U(n).\mu_n\) by uniqueness in Theorem 4.3. This implies that there is only one left invariant metric compatible with \((N_{\mu_n}, \omega)\) up to isometry, often called the Abbena metric in the case \(n = 2\).
Example 5.7. Consider the 4-dimensional Lie algebra given by
\[ \lambda(X_1, X_2) = X_3, \quad \lambda(X_1, X_3) = X_4. \]
The compatible metric \( \langle X_i, X_j \rangle = \delta_{ij} \) is minimal for \( (N_\lambda, \omega) \) since
\[ \text{Ric}^{ac}_{\omega} (\eta, \eta) = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 & -3 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -3 & -1 & 1 & 3 \end{bmatrix} = -\frac{5}{4} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \in \mathbb{R}I + \text{Der}(\lambda). \]

It is well-known that \( (N_{\mu_2}, \omega) \) and \( (N_\lambda, \omega) \) are the only symplectic nilpotent Lie groups in dimension 4, and then the existence of minimal compatible metrics in the case \( 2n = 4 \) follows.

Example 5.8. Let \( \mu = \mu(a, b, c) \) be the 6-dimensional 2-step nilpotent Lie algebra defined by
\[ \mu(X_1, X_2) = aX_4, \quad \mu(X_1, X_3) = bX_5, \quad \mu(X_2, X_3) = cX_6. \]
It is easy to check that \( \mu \in \mathcal{N}_s \) if and only if \( a - b + c = 0 \). We can also get from a simple calculation that
\[ \text{Ric}^{ac}_{\mu} = -\frac{1}{4} (a^2 + b^2 + c^2) \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \in \mathbb{R}I + \text{Der}(\mu), \]
and so the whole family \( \{\mu(a, b, c) : a - b + c = 0\} \subset \mathbb{P}\mathcal{N}_s \) consists of critical points of \( F \). We assume that \( a^2 + b^2 + c^2 = 2 \) in order to avoid homothetical changes, which is equivalent to \( \text{sc}(\mu) = -1 \). The Ricci operator on the center \( \mathfrak{z} = \langle X_4, X_5, X_6 \rangle_\mathbb{R} \) is given by
\[ \text{Ric}_{\mu} = \frac{1}{2} \begin{bmatrix} a^2 \\ b^2 \\ c^2 \end{bmatrix}, \]
and thus the curve \( \{\mu_{st} = \mu(s, s+t, t) : s^2 + st + t^2 = 1, \quad 0 \leq t \leq \frac{1}{\sqrt{2}}\} \) is pairwise non-isometric. It then follows from Corollary 4.3 that \( (N_{\mu_{st}}, \omega) \) is a curve of pairwise non-isomorphic symplectic nilpotent Lie groups. In terms of the notation in [KGM, Table A.1], we have that \( \mu_{01} \simeq (0, 0, 0, 0, 12, 13) \) and \( \mu_{st} \simeq (0, 0, 0, 0, 12, 13, 23) \) for any \( 0 < t \leq \frac{1}{\sqrt{2}} \). We note that this curve coincides with the curve of pairwise non-isomorphic symplectic structures denoted by \( \omega_1(t) \) in [KGM, Theorem 3.1, 18], and then this example shows that any symplectic structure in such a curve admits a compatible metric which is minimal.

Example 5.9. We now take advantage of the variational nature of Theorem 4.3 to find explicit examples of minimal compatible metrics. Consider for each 6-upla \( \{a, \ldots, f\} \) of real numbers the skew-symmetric bilinear form \( \mu = \mu(a, b, c, d, e, f) \in V = \Lambda^2(\mathbb{R}^6)^* \otimes \mathbb{R}^6 \) defined by
\[ \mu(X_1, X_2) = aX_3, \quad \mu(X_1, X_3) = bX_4, \quad \mu(X_1, X_4) = cX_5, \]
\[ \mu(X_1, X_5) = dX_6, \quad \mu(X_2, X_3) = eX_5, \quad \mu(X_2, X_4) = fX_6. \]
Our plan is to find first the critical points of \( F \) restricted to the set \( \{\mu(a, \ldots, f) : a, \ldots, f \in \mathbb{R}\} \) and after that to show by using the characterization given in part (iii)
of the theorem that they are really critical points of $F : \mathbb{R}^V \mapsto \mathbb{R}$. We can see by a simple computation that $\text{Ric}^{ac}$, given by the diagonal matrix with entries

$$\text{Ric}^{ac} = -\frac{1}{4} \begin{bmatrix} a^2 + b^2 + c^2 + d^2 + e^2 \\ a^2 + b^2 - c^2 - d^2 + e^2 \\ a^2 + c^2 + e^2 + f^2 \\ a^2 + 2c^2 - e^2 + f^2 \\ a^2 - 2b^2 + c^2 - e^2 + f^2 \\ a^2 - 2b^2 + c^2 - d^2 - f^2 \\ a^2 - b^2 - c^2 - d^2 - f^2 \\ a^2 - b^2 - c^2 - d^2 - f^2 \end{bmatrix},$$

and hence we are interested in the critical points of

$$F(\mu) = \text{tr}(\text{Ric}^{ac})^2 = F(a, ..., f)$$

$$= \frac{1}{8} \left((a^2 + b^2 + c^2 + 2d^2 + f^2)^2 + (a^2 + c^2 - d^2 + 2e^2 + f^2)^2ight)$$

$$+ (-a^2 + 2b^2 - c^2 + e^2 - f^2)^2$$

restricted to any leaf of the form $a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \equiv \text{const.}$, which are easily seen to depend on three parameters. We still have to impose the Jacobi and closeness conditions on these critical points (or equivalently to find the intersection with $N_s$), after which we obtain the following ellipse of symplectic structures:

$$\{ \mu_{xy} = \mu(x, 1, x + y, 1, 1, y) : x^2 + y^2 + xy = 1 \}.$$ 

It follows from the formula for $\text{Ric}^{ac}$ given above that

$$\text{Ric}^{ac}_{\mu_{xy}} = -\frac{1}{4} \begin{bmatrix} 5 & 1 \\ 1 & -3 \\ -1 & 3 \\ -1 & -5 \end{bmatrix} = \frac{7}{4} I + \frac{1}{2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}I + \text{Der}(\mu_{xy}),$$

showing definitely that this is a curve of minimal compatible metrics. We furthermore have that the Ricci tensor of the metrics $\mu_{xy}$ is given by

$$\text{Ric}_{\mu_{xy}} = -\frac{1}{4} \begin{bmatrix} 4 - y^2 & 2 - xy & 2 - xy \\ 2 - xy & 1 - x^2 & 1 - xy \\ 2 - xy & 1 - xy & -1 - y^2 \end{bmatrix},$$

which clearly shows that they are pairwise non-isometric for $x, y \geq 0$. It then follows from Corollary 4.5 that

$$\{ (N_{\mu_{xy}}, \omega) : x^2 + y^2 + xy = 1, \ x, y \geq 0 \}$$

is a curve of pairwise non-isomorphic symplectic nilpotent Lie groups. There are three 6-dimensional nilpotent Lie groups involved, $N_{\mu_{00}}$, $N_{\mu_{01}}$ and $N_{\mu_{xy}}$, $x, y > 0$, denoted in Table A.1 by $(0, 0, 12, 13, 14, 23 + 15)$, $(0, 0, 0, 12, 14 - 23, 15 + 34)$ and $(0, 0, 12, 13, 14 + 23, 24 + 15)$, respectively.

6. Complex structures

Let $N$ be a real $2n$-dimensional nilpotent Lie group with Lie algebra $\mathfrak{n}$, whose Lie bracket is denoted by $\mu : \mathfrak{n} \times \mathfrak{n} \mapsto \mathfrak{n}$. An invariant almost-complex structure on $N$ is defined by a map $J : \mathfrak{n} \mapsto \mathfrak{n}$ satisfying $J^2 = -I$. If in addition $J$ satisfies the integrability condition

$$(28) \quad \mu(JX, JY) = \mu(X, Y) + J\mu(JX, Y) + J\mu(X, JY), \quad \forall X, Y \in \mathfrak{n},$$

then $J$ is said to be a complex structure.
Fix an almost-complex nilpotent Lie group \((N, J)\). A left invariant Riemannian metric which is compatible with \((N, J)\), also called an almost-hermitian metric, is given by an inner product \(\langle \cdot, \cdot \rangle\) on \(\mathfrak{n}\) such that
\[
\langle JX, JY \rangle = \langle X, Y \rangle \quad \forall X, Y \in \mathfrak{n}.
\]
We have for this particular geometric structure \(\gamma = J\) that
\[
G_\gamma = GL(n, \mathbb{C}) = \{ \varphi \in GL(2n) : \varphi J = J \varphi \}, \quad K_\gamma = \text{U}(n),
\]
and the Cartan decomposition of \(\mathfrak{g}_\gamma = \mathfrak{gl}(n, \mathbb{C}) = \{ A \in \mathfrak{gl}(2n) : AJ = JA \}\) is given by
\[
\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{p}_\gamma, \quad \mathfrak{p}_\gamma = \{ A \in \mathfrak{p} : AJ = JA \}.
\]
The invariant Ricci operator is then given by the complexified Ricci operator
\[
\text{Ric}^\gamma(\cdot, \cdot) = \text{Ric}^c(\cdot, \cdot) = \frac{1}{2}(\text{Ric}(\cdot, \cdot) - J \text{Ric}(\cdot, \cdot) J)
\]
(see (24)). In this way, condition \(\text{Ric}^\gamma(\cdot, \cdot) = 0\) is equivalent to the Ricci operator anti-commute with \(J\). We do not know if this property has any special significance in complex geometry, but for instance it holds for a Kähler metric if and only if the metric is Ricci flat. Anyway, as in the symplectic case, the condition \(\text{Ric}^c(\cdot, \cdot) = 0\) is also forbidden for non-abelian \(N\), since \(\text{tr} \text{Ric}^c(\cdot, \cdot) = \text{sc}(\langle \cdot, \cdot \rangle) < 0\).

We now fix a map \(J : \mathfrak{n} \to \mathfrak{n}\) integrable and consider the algebraic subvariety \(\mathcal{N}_c := \mathcal{N}_\gamma \subset \mathcal{N}\) given by
\[
\mathcal{N}_c = \{ \mu \in \mathcal{N} : (28) \text{ holds} \},
\]
that is, those nilpotent Lie brackets for which \(J\) is integrable and so define a complex structure on \(\mathcal{N}_c\), the simply connected nilpotent Lie group with Lie algebra \((\mathfrak{n}, \mu)\).

Fix also an inner product \(\langle \cdot, \cdot \rangle\) on \(\mathfrak{n}\) compatible with \(J\), then (23) identifies each \(\mu \in \mathcal{N}_c\) (or \(\mathcal{N}\)) with the hermitian (or almost-hermitian) manifold \((\mathcal{N}_\mu, J, \langle \cdot, \cdot \rangle)\). If we use the same triple \((\omega, J, \langle \cdot, \cdot \rangle)\) to define and identify \(\mathcal{N}_c\) (see Section 5.2) and \(\mathcal{N}_c\), then the intersection of these varieties is \(\mathcal{N}_c \cap \mathcal{N}_c = \{0\}\) since no non-abelian nilpotent Lie group can admit a Kähler metric.

We now give some examples.

**Example 6.1.** Let \(\mu_n\) be the 2\(n\)-dimensional Lie algebra considered in Example 6.6. It is easy to check that \(\langle \cdot, \cdot \rangle\) is also minimal as a compatible metric for the almost-complex nilpotent Lie group \((N_{\mu_n}, J)\). For the 4-dimensional Lie algebra in Example 6.7, we have that \(\text{Ric}^c(\cdot, \cdot) = -\frac{1}{4}I\) and hence this metric is minimal for the almost-complex nilpotent Lie group \((N_\Lambda, J)\) as well.

For \(\mathfrak{n}_1 = \mathbb{R}^4\) and \(\mathfrak{n}_2 = \mathbb{R}^2\), consider the vector space \(W = \Lambda^2 \mathfrak{n}_1^* \otimes \mathfrak{n}_2\) of all skew-symmetric bilinear maps \(\mu : \mathfrak{n}_1 \times \mathfrak{n}_1 \to \mathfrak{n}_2\). Any 6-dimensional 2-step nilpotent Lie algebra with \(\dim \mu(\mathfrak{n}, \mathfrak{n}) \leq 2\) can be modelled in this way. Fix basis \(\{X_1, \ldots, X_4\}\) and \(\{Z_1, Z_2\}\) of \(\mathfrak{n}_1\) and \(\mathfrak{n}_2\), respectively. Each element in \(W\) will be described as \(\mu = \mu(a_1, a_2, \ldots, f_1, f_2)\), where
\[
\begin{align*}
\mu(X_1, X_2) &= a_1 Z_1 + a_2 Z_2, \\
\mu(X_1, X_3) &= c_1 Z_1 + c_2 Z_2, \\
\mu(X_1, X_4) &= e_1 Z_1 + e_2 Z_2, \\
\mu(X_1, X_3) &= b_1 Z_1 + b_2 Z_2, \\
\mu(X_2, X_3) &= d_1 Z_1 + d_2 Z_2.
\end{align*}
\]

The complex structure and the compatible metric will be always defined by
\[
J = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}, \quad \langle X_i, X_j \rangle = \langle Z_i, Z_j \rangle = \delta_{ij}.
\]
If \( A = (a_1, a_2), \ldots, F = (f_1, f_2) \) and \( JA = (-a_2, a_1), \ldots, JF = (-f_2, f_1) \), then it is easy to check that \( J \) is integrable on \( N_\mu \) (or \( (N_\mu, J) \) is a complex nilpotent Lie group) if and only if

\[(29) \quad E = B + JD + JC,\]

\( J \) is bi-invariant (i.e. \( \mu(JX,Y) = J\mu(X,Y) \)) if and only if

\[(30) \quad A = F = 0, \quad C = D = JB, \quad E = -B,\]

and \( J \) is abelian (i.e. \( \mu(JX,JY) = \mu(X,Y) \)) if and only if

\[(31) \quad E = B, \quad D = -C.\]

We note that the above conditions determine \( GL(2,\mathbb{C}) \times GL(1,\mathbb{C}) \)-invariant linear subspaces of \( W \) of dimensions 10, 2 and 8, respectively. For each \( \mu \in W \), it follows from \((38)\) that the Ricci operator of the almost-hermitian manifold \( (N_\mu, J, \langle \cdot, \cdot \rangle) \) (see identification \((23)\)) restricted to \( n_1 \), \( \text{Ric}_\mu |_{n_1} \), is given by

\[(32) \quad \text{Ric}_\mu |_{n_2} = \frac{1}{2}\begin{bmatrix}
||A||^2 + ||B||^2 + ||C||^2 & \langle B,D \rangle + \langle C,E \rangle & -\langle A,D \rangle + \langle C,F \rangle & -\langle A,E \rangle - \langle B,F \rangle \\
\langle B,D \rangle + \langle C,E \rangle & ||A||^2 + ||B||^2 + ||E||^2 & \langle A,D \rangle + \langle E,F \rangle & -\langle A,C \rangle - \langle D,F \rangle \\
-\langle A,D \rangle + \langle C,F \rangle & \langle A,D \rangle + \langle E,F \rangle & ||B||^2 + ||D||^2 + ||F||^2 & \langle B,C \rangle + \langle D,E \rangle \\
-\langle A,E \rangle - \langle B,F \rangle & -\langle A,C \rangle - \langle D,F \rangle & \langle B,C \rangle + \langle D,E \rangle & ||C||^2 + ||E||^2 + ||F||^2
\end{bmatrix}\]

and

\[\text{Ric}_\mu |_{n_2} = \frac{1}{2} \begin{bmatrix}
||v_1||^2 & \langle v_1, v_2 \rangle & ||v_2||^2 \\
\langle v_1, v_2 \rangle & ||v_1||^2 & ||v_2||^2 \\
||v_2||^2 & ||v_2||^2 & ||v_2||^2
\end{bmatrix}, \quad v_i = (a_i, b_i, c_i, d_i, e_i, f_i), \quad i = 1, 2.\]

Recall that if the complexified Ricci operator satisfied \( \text{Ric}_\mu |_{n_1} = pI, \ p \in \mathbb{R} \), then \( \mu \) is minimal. Indeed, since always \( \text{Ric}_\mu |_{n_2} = qI \) for some \( q \in \mathbb{R} \), we would have that

\[(33) \quad \text{Ric}_\mu = \begin{bmatrix}
pI & \langle q-p \rangle I \\
\langle q-p \rangle I & (2p-q)I + \frac{(q-p)^2}{2(q-p)}I
\end{bmatrix} \in \mathbb{R}I + \text{Der}(\mu).\]

In particular, any bi-invariant complex nilpotent Lie group \( (N_\mu, J) \) (see \((31)\)) admits a compatible metric which is minimal.

We will now focus on the abelian complex case (see \((31)\)). It is not hard to see that these conditions imply that \( \text{Ric}_\mu |_{n_1} = \text{Ric}_\mu \), and so in this case, to get \( \text{Ric}_\mu |_{n_1} \in \mathbb{R}I \) is necessary and sufficient that

\[\langle A + F, B \rangle = 0, \quad \langle A + F, C \rangle = 0, \quad ||A||^2 = ||F||^2.\]

In order to avoid homothetical changes we will always ask for \( ||v_1||^2 + ||v_2||^2 = 2 \), which is equivalent to \( sc(\mu) = -1 \).

**Example 6.2.** If we put \( A = (s, t), \ F = (-s, t), \ B = C = D = E = 0, \ s^2 + t^2 = 1 \), then the corresponding curve \( \mu_{st} \) of minimal compatible metrics satisfies

\[\text{Ric}_{\mu_{st}} |_{n_2} = \begin{bmatrix}
s^2 & 0 \\
0 & t^2
\end{bmatrix},\]

proving that \( \{\mu_{st} : s^2 + t^2 = 1, \ 0 \leq s \leq \frac{1}{\sqrt{2}} \} \) is a curve of pairwise non-isometric metrics. It then follows from Corollary \(1.3\) that \((N_{\mu_{st}}, J)\) is a curve of pairwise non-isomorphic abelian complex nilpotent Lie groups. Recall that \( \mu_{st} \simeq \mathfrak{h}_3 \oplus \mathfrak{h}_3 \) for all \( 0 < s \) and \( \mu_{01} \simeq \mathfrak{h}_3 \oplus \mathbb{R}^3 \).

**Example 6.3.** For \( A = (s, t), \ F = (-s, t), \ B = \frac{1}{\sqrt{2}}, \ 0 = C = -D = E, \ s^2 + t^2 = \frac{1}{2} \), the curve \( \mu_{st} \) of minimal compatible metrics satisfies

\[\text{Ric}_{\mu_{st}} |_{n_2} = \begin{bmatrix}
s^2 + \frac{1}{2} & 0 \\
0 & t^2
\end{bmatrix},\]
which implies that the family \( \{ \mu_{st} : s^2 + t^2 = \frac{1}{2} \} \) is pairwise non-isometric. It is easy to see that for \( t \neq 0 \), \( \mu_{st} \) is isomorphic to the complex Heisenberg Lie algebra, and hence \((N_{\mu_{st}}, J)\) defines a curve of pairwise non-isomorphic abelian complex structures on the Iwasawa manifold. Since \( j_{\mu_{st}}(Z_2)^2 \notin \mathbb{R}I \), we have that the hermitian manifolds \((N_{\mu_{st}}, J, \langle \cdot, \cdot \rangle)\) are not modified H-type (see Appendix 10).

**Example 6.4.** Consider the abelian complex structures defined by \( A = -F, E = B \) and \( D = -C \). In this case, the Hermitian manifolds \((N_\mu, J, \langle \cdot, \cdot \rangle)\) are modified H-type and \( \mu \) is always isomorphic to the complex Heisenberg Lie algebra when \( v_1, v_2 \neq 0 \). In fact, by assuming for simplicity that \( \langle v_1, v_2 \rangle = 0 \), then

\[
j_{\mu}(Z)^2 = -\frac{1}{2}((Z, Z_1)^2||v_1||^2 + (Z, Z_2)^2||v_2||^2)I, \quad \forall Z \in n_2.
\]

For \( A = (s, 0) = -F, B = (0, t) = E, s^2 + t^2 = 1, D = C = 0 \), the corresponding curve \( \mu_{st} \) of minimal compatible metrics satisfies

\[
\text{Ric}_{\mu_{st}} |_{n_2} = \begin{bmatrix} s^2 & 0 \\ 0 & t^2 \end{bmatrix},
\]

and so the family \( \{ \mu_{st} : s^2 + t^2 = 1, 0 \leq s \leq \frac{1}{\sqrt{2}} \} \) is pairwise non-isometric and the abelian complex structures \((N_{\mu_{st}}, J)\) are pairwise non-isomorphic. Each modified H-type metric is compatible with two spheres of abelian complex structures of this type which can be described by

\[
\{ \pm v_1 \times v_2 : v_i \in \mathbb{R}^3, ||v_1||^2 = 2s^2, ||v_2||^2 = 2t^2, \langle v_1, v_2 \rangle = 0 \},
\]

where \( v_1 \times v_2 \) denotes the vectorial product, but one can see that these structures are all isomorphic to \((N_{\mu_{st}}, J)\) (compare with [KS]). We finally recall that \( \mu_{01} \cong h_3 \oplus \mathbb{R} \), and so \( \langle \cdot, \cdot \rangle \) is a minimal compatible metric for the abelian complex nilpotent Lie group \((N_{\mu_{01}}, J)\).

We finally give a curve of non-abelian complex structures on the Iwasawa manifold.

**Example 6.5.** Let \( \mu_t \) be the curve defined by

\[
\begin{align*}
\mu_t(X_1, X_3) &= -tsZ_2, \\
\mu_t(X_2, X_3) &= sZ_1, \\
\mu_t(X_1, X_4) &= sZ_1, \\
\mu_t(X_2, X_4) &= s(2-t)Z_2,
\end{align*}
\]

\[
s = \sqrt{2 + t^2 + (2-t)^2}, \quad t \in \mathbb{R}.
\]

We then have that \( A = F = 0, C = D = (s, 0), B = -tJC, E = (2-t)JC \), and so \((N_{\mu_t}, J)\) is a non-abelian complex nilpotent Lie group for all \( t \in \mathbb{R} \) (see (23)). Moreover, \( \text{Ric}_{\mu_t} |_{n_1} \) is diagonal and hence \( \text{Ric}^\sigma_{\mu_t} |_{n_1} \) is a multiple of the identity, which implies that \( \langle \cdot, \cdot \rangle \) is a minimal compatible metric for all \((N_{\mu_t}, J)\) (see (23)). It follows from

\[
\text{Ric}_{\mu_t} |_{n_2} = \frac{1}{2} \begin{bmatrix} 2s^2 & 0 \\ 0 & s^2(t^2 + (2-t)^2) \end{bmatrix},
\]

that the hermitian manifolds \( \{(N_{\mu_t}, J, \langle \cdot, \cdot \rangle) : 1 \leq t < \infty \} \) are pairwise non-isometric since

\[
s^2(t^2 + (2-t)^2) - 2s^2 = (t^2 + (2-t)^2)^2 - 4
\]

is a strictly increasing non-negative function for \( 1 \leq t \), which vanishes if and only if \( t = 1 \). We therefore obtain a curve \( \{(N_{\mu_t}, J) : 1 \leq t < \infty \} \) of pairwise non-isomorphic non-abelian complex nilpotent Lie groups. A natural question is which
are the nilpotent Lie groups involved. We have for all \( t \) that
\[
j_{\mu_t}(Z_1) = \begin{bmatrix} 0 & -s & 0 \\ s & 0 & -t \\ 0 & t & 0 \end{bmatrix}, \quad j_{\mu_t}(Z_2) = \begin{bmatrix} 0 & t_s & 0 \\ -t_s & 0 & -(2-t)s \\ 0 & (2-t)s & 0 \end{bmatrix},
\]
and hence \( j_{\mu_t}(Z) \) is non-singular if and only if
\[
-t(2-t)(Z, Z_1)^2 - (Z, Z_2)^2 \neq 0.
\]
This implies that \( \mu_t \) is isomorphic to the complex Heisenberg Lie algebra (i.e. when \( j_{\mu_t}(Z) \) is non-singular for any non-zero \( Z \in \mathfrak{n}_2 \)) if and only if \( 1 \leq t < 2 \), providing a curve on the Iwasawa manifold. Furthermore, \((N_{\mu_t}, J)\) is the bi-invariant complex structure and it can be showed by computing \( j_{\mu_t}(Z)^2 \) that \((N_{\mu_t}, \langle \cdot, \cdot \rangle)\) is not modified H-type for any \( 1 < t \). We finally note that \( \mu_2 \) is isomorphic to the group denoted by \((0, 0, 0, 12, 14 + 23)\) in \( \mathfrak{S} \), and one can easily see by discarding any other possibility that actually \( \mu_1 \simeq \mu_2 \) for all \( 2 \leq t < \infty \), which gives rise a curve of pairwise non-isomorphic structures on such a group.

Although it has not been mentioned, most of the curves given in this section have been obtained via the variational method provided by Theorem 4.3, by using an approach very similar to that in Example 5.3.

7. Hypercomplex structures

Let \( N \) be a real \( 4n \)-dimensional nilpotent Lie group with Lie algebra \( \mathfrak{n} \), whose Lie bracket is denoted by \( \mu : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{n} \). An invariant hypercomplex structure on \( N \) is defined by a triple \( \{J_1, J_2, J_3\} \) of complex structures on \( \mathfrak{n} \) (see Section 5) satisfying the quaternion identities
\[
J^2_i = -I, \quad i = 1, 2, 3, \quad J_1J_2 = J_3 = -J_2J_1.
\]
An inner product \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{n} \) is said to be compatible with \( \{J_1, J_2, J_3\} \), also called an hyper-hermitian metric, if
\[
\langle J_iX, J_iY \rangle = \langle X, Y \rangle \quad \forall X, Y \in \mathfrak{n}, \ i = 1, 2, 3.
\]
Two hypercomplex nilpotent Lie groups \((N, \{J_1, J_2, J_3\})\) and \((N', \{J'_1, J'_2, J'_3\})\) are said to be isomorphic if there exists an automorphism \( \varphi : \mathfrak{n}' \to \mathfrak{n} \) such that
\[
\varphi J'_i \varphi^{-1} = J_i, \quad i = 1, 2, 3.
\]
For \( \gamma = \{J_1, J_2, J_3\} \) we therefore have that
\[
G_\gamma = GL(\mathfrak{n}, \mathbb{H}) = \{\varphi \in GL(4n) : \varphi J_i = J_i \varphi, \ i = 1, 2, 3\}, \quad K_\gamma = Sp(n),
\]
and the Cartan decomposition of
\[
\mathfrak{g}_\gamma = \mathfrak{gl}(\mathfrak{n}, \mathbb{H}) = \{A \in \mathfrak{gl}(4n) : AJ_i = J_iA, \ i = 1, 2, 3\}
\]
is given by
\[
\mathfrak{gl}(\mathfrak{n}, \mathbb{H}) = \mathfrak{sp}(n) \oplus \mathfrak{p}_\gamma, \quad \mathfrak{p}_\gamma = \{A \in \mathfrak{p} : AJ_i = J_iA, \ i = 1, 2, 3\}.
\]
The invariant Ricci operator for a compatible metric \( \langle \cdot, \cdot \rangle \in \mathcal{C} \) is then given by
\[
\text{Ric}^\gamma(\cdot, \cdot) = \frac{1}{4}(\text{Ric}(\cdot, \cdot) - J_1 \text{Ric}(\cdot, \cdot) J_1 - J_2 \text{Ric}(\cdot, \cdot) J_2 - J_3 \text{Ric}(\cdot, \cdot) J_3),
\]
and hence condition \( \text{Ric}^\gamma(\cdot, \cdot) = 0 \) can never holds since \( \text{tr} \text{Ric}^\gamma(\cdot, \cdot) = \text{sc}(\cdot, \cdot) < 0 \) for a non-abelian nilpotent Lie group.
We now fix \( \{J_1, J_2, J_3, \langle \cdot, \cdot \rangle \} \) satisfying (34) and (35) and consider the algebraic subvariety \( N_h := N_3 \cap N \) given by

\[
N_h = \{ \mu \in N : J_i \text{ is integrable for } i = 1, 2, 3 \},
\]

that is, those nilpotent Lie brackets for which \( \{J_1, J_2, J_3\} \) is an hypercomplex structure on the corresponding nilpotent Lie group \( N_\mu \). Thus (23) identifies each \( \mu \in N_h \) with the hyper-hermitian manifold \( (N_\mu, \{J_1, J_2, J_3\}, \langle \cdot, \cdot \rangle) \).

7.1. Hypercomplex 8-dimensional nilpotent Lie groups. There are no non-abelian nilpotent Lie groups of dimension 4 admitting an hypercomplex structure. In dimension 8, hypercomplex nilpotent Lie groups have been determined by I. Dotti and A. Fino in [DF1] and [DF3]. They proved the following strong restrictions on an 8-dimensional nilpotent Lie algebra \( \mathfrak{n} \) which admits an hypercomplex structure: \( \mathfrak{n} \) has to be 2-step nilpotent, \( \dim \mu(n, n) \leq 4 \), there exists a decomposition \( \mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \) such that \( \dim \mathfrak{n}_1 = 4 \), \( \mathfrak{n}_1 \) is \( \{J_1, J_2, J_3\} \)-invariant and \( \mu(n, n) \subset \mathfrak{n}_2 \subset \mathfrak{g} \), where \( \mathfrak{g} \) is the center of \( \mathfrak{n} \). Thus the Lie bracket of \( \mathfrak{n} \) is just given by a skew-symmetric bilinear form \( \mu : \mathfrak{n}_1 \times \mathfrak{n}_1 \rightarrow \mathfrak{n}_2 \), and those for which a fixed \( \{J_1, J_2, J_3\} \) is integrable are also completely described in [DF3] as a 16-dimensional subspace. Such a description has been successfully used in [DF2] to prove that the associated Obata connections are always flat.

What shall be studied here are the isomorphism classes of such structures and the existence of minimal compatible metrics. As we have seen in the previous sections, these two problems are intimately related, and the hypercomplex case will not be an exception.

For \( \mathfrak{n}_1 = \mathbb{R}^4 \) and \( \mathfrak{n}_2 = \mathbb{R}^4 \) consider the vector space \( W = \Lambda^2 \mathfrak{n}_1^\ast \otimes \mathfrak{n}_2 \) of all skew-symmetric bilinear maps \( \mu : \mathfrak{n}_1 \times \mathfrak{n}_1 \rightarrow \mathfrak{n}_2 \). Any 8-dimensional 2-step nilpotent Lie algebra with \( \dim \mu(n, n) \leq 4 \) can be modelled in this way. Fix basis \( \{X_1, X_2, X_3, X_4\} \) and \( \{Z_1, Z_2, Z_3, Z_4\} \) of \( \mathfrak{n}_1 \) and \( \mathfrak{n}_2 \), respectively. Each element in \( W \) will be denoted as \( \mu = (a_1, ..., a_4, f_1, ..., f_4) \), where

\[
\begin{align*}
\mu(X_1, X_2) &= a_1 Z_1 + a_2 Z_2 + a_3 Z_3 + a_4 Z_4, & \mu(X_2, X_3) &= d_1 Z_1 + d_2 Z_2 + d_3 Z_3 + d_4 Z_4, \\
\mu(X_1, X_3) &= b_1 Z_1 + b_2 Z_2 + b_3 Z_3 + b_4 Z_4, & \mu(X_2, X_4) &= c_1 Z_1 + c_2 Z_2 + c_3 Z_3 + c_4 Z_4, \\
\mu(X_1, X_4) &= c_1 Z_1 + c_2 Z_2 + c_3 Z_3 + c_4 Z_4, & \mu(X_3, X_4) &= f_1 Z_1 + f_2 Z_2 + f_3 Z_3 + f_4 Z_4.
\end{align*}
\]

The compatible metric will be \( \langle X_i, X_j \rangle = \langle Z_i, Z_j \rangle = \delta_{ij} \) and the hypercomplex structure will always act on \( \mathfrak{n}_i \) by

\[
J_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.
\]

If \( A = (a_1, ..., a_4) \), ..., \( F = (f_1, ..., f_4) \), then it is easy to prove that \( J_i \) is integrable for all \( i = 1, 2, 3 \) on \( N_\mu \) (or \( (N_\mu, \{J_1, J_2, J_3\}) \) is a hypercomplex nilpotent Lie group) if and only if

\[
E = B + J_1 D + J_1 C, \quad D = -C - J_2 A - J_2 F, \quad F = -A - J_2 B + J_3 E.
\]

If we define \( T := D + C \), then the above conditions are equivalent to

\[
(36) \quad D = -C + T, \quad E = B + J_1 T, \quad F = -A + J_2 T.
\]

In order to use a notation as similar as possible to [DF2], [DF3], we should put \( T = (t_3, t_2, -t_1, t_4) \). It is easy to check that \( (N_\mu, \{J_1, J_2, J_3\}) \) is abelian (i.e. \( \mu(J_i, J_i) = \mu, i = 1, 2, 3 \)) if and only if \( T = 0 \). We note that \( \dim \mathfrak{m} = 24 \), and so condition (36) determine a \( GL(1, \mathbb{H}) \times GL(1, \mathbb{H}) \)-invariant linear subspace \( W_h \) of \( W \) of dimension 16, and a 12-dimensional subspace \( W_{ab} \) if we ask in addition abelian.
For each \( \mu \in W \), the Ricci operator of \((N, \{J_1, J_2, J_3\}, \langle \cdot, \cdot \rangle)\) (see identification (23)) restricted to \( n_1 \), \( \text{Ric}_\mu |_{n_1} \), is given by formula (32), and
\[
\text{Ric}_\mu |_{n_2} = \frac{1}{2}[(v_i, v_j)], \quad 1 \leq i, j \leq 4, \quad v_i := (a_i, b_i, c_i, d_i, e_i, f_i).
\]

Since the only symmetric transformations of \( n_1 = \mathbb{R}^4 \) commuting with all the \( J_i \)'s are the multiplies of the identity, we obtain that the invariant Ricci operator satisfies \( \text{Ric}_\gamma |_{n_1} \in \mathbb{R}I \) for any \( \mu \in W \). By arguing as in (33), one obtains that any \( \mu \in W \) is minimal. We summarize in the following proposition the consequences of this fact.

**Proposition 7.1.**

(i) Every hypercomplex 8-dimensional nilpotent Lie group admits a minimal compatible metric, which is actually its unique compatible left invariant metric up to isometry and scaling.

(ii) Two hypercomplex 8-dimensional nilpotent Lie groups \((N, \{J_1, J_2, J_3\})\) and \((N', \{J_1, J_2, J_3\})\) are isomorphic if and only if \( \mu \) and \( c\lambda \) lie in the same \( \text{Sp}(1) \times \text{Sp}(1) \)-orbit for some non-zero \( c \in \mathbb{R} \).

(iii) The moduli space of all 8-dimensional hypercomplex nilpotent Lie groups up to isomorphism is parameterized by
\[
\mathbb{P} W_h / \text{Sp}(1) \times \text{Sp}(1).
\]

The representation \( W_h \) is equivalent to \((\mathfrak{su}(2) \otimes \mathbb{R}^4) \oplus \mathbb{R}^4 \), where \( \mathfrak{su}(2) \) is the adjoint representation and \( \mathbb{R}^4 \) is the standard representation of \( \text{SU}(2) = \text{Sp}(1) \) viewed as real. Since the isotropy of an element in general position is finite, the dimension of this quotient is \( 15 - 6 = 9 \).

(iv) The moduli space of all 8-dimensional abelian hypercomplex nilpotent Lie groups up to isomorphism is parameterized by
\[
\mathbb{P} W_{ah} / \text{Sp}(1) \times \text{Sp}(1).
\]

The representation \( W_{ah} \) is equivalent to \( \mathfrak{su}(2) \otimes \mathbb{R}^4 \), and since the isotropy of an element in general position is again finite, the dimension of this quotient is \( 11 - 6 = 5 \) (see Example 7.3 for explicit 5-dimensional families).

The proofs of the above results follow from Corollary 4.3. It is easy to see that \( j_\mu(Z)^2 \in \mathbb{R}I \) for all \( Z \in n_2 \) if and only if \( T = 0 \), and so only abelian hypercomplex structures can admit (and does) modified H-type compatible metrics. Recall that we are considering here a weaker modified H-type condition by allowing \( c(Z) = 0 \) (see Appendix 10).

We will give now explicit continuous families of hypercomplex structures on some particular nilpotent Lie groups. Let \( g_1, g_2 \) and \( g_3 \) denote the 8-dimensional Lie algebras obtained as the direct sum of an abelian factor and the following H-type Lie algebras: the 5-dimensional Heisenberg Lie algebra, the 6-dimensional complex Heisenberg Lie algebra and the 7-dimensional quaternionic Heisenberg Lie algebra. In order to avoid homothetical changes we will always ask for \( ||v_1||^2 + \ldots + ||v_4||^2 = 2 \), which is equivalent to \( sc(\mu) = -1 \).

**Example 7.2.** If we put \( T = 0, A = (0, r, 0, 0), B = (0, 0, s, 0), C = (0, 0, 0, t) \), we have for each \( \mu_{rst} \) that
\[
\text{Ric}_{\mu_{rst}} |_{n_2} = \frac{1}{2} \begin{bmatrix}
0 & r^2 \\
0 & s^2 \\
0 & t^2
\end{bmatrix},
\]
and thus the family
\[
\{(N_{\mu_{rest}}, \{J_1, J_2, J_3\}, \langle \cdot, \cdot \rangle) : 0 \leq r \leq s \leq t, \quad r^2 + s^2 + t^2 = 2\}
\]
of minimal compatible metrics is pairwise non-isometric. This gives rise then a
surface of pairwise non-isomorphic abelian hypercomplex nilpotent Lie groups (see
Corollary 4.5). If 0 < r then \(\mu_{rest} \simeq \mathfrak{g}_3\), for \(r = 0 < s\) we get a curve on \(\mathfrak{g}_2\) and for
\(r = s = 0, t = 1\), a single structure on \(\mathfrak{g}_1\).

Example 7.3. We now set \(T = 0\) and choose \(A, B, C\) such that
\[
v_1 = \left(\frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}\right), \quad v_2 = (0, \sqrt[3]{\frac{3}{8}}, 0, 0, \sqrt[3]{\frac{3}{8}}), \quad ||v_3||^2 + ||v_4||^2 = \frac{1}{4}, \quad ||v_3||^2 > ||v_4||^2.
\]
Assume that two of such elements \(\lambda = \mu(v_3, v_4)\) and \(\lambda' = \mu'(v_3', v_4')\) are in the same
\(\text{Sp}(1) \times \text{Sp}(1)-\text{orbit}, \) say \(\lambda = \varphi.\lambda\) for some \(\varphi = (\varphi_1, \varphi_2) \in \text{Sp}(1) \times \text{Sp}(1)\). Recall that
\[
j_{\lambda}(Z) = \varphi_1 \lambda Z \varphi_1^{-1}, \quad \forall Z \in \mathfrak{n}_2,
\]
(see Appendix 10) and \(\langle v_i, v_j \rangle = -\frac{1}{2} tr \mu(Z_i) \mu(Z_j)\), \(1 \leq i, j \leq 4\). It then follows from
\[
||v_1|| > ||v_2|| > ||v_3|| > ||v_4||, \quad ||v'_1|| > ||v'_2|| > ||v'_3|| > ||v'_4||
\]
that \(j_{\varphi.\lambda}(Z_i) = \pm j_{\lambda}(Z_i)\) for all \(i = 1, \ldots, 4\), and hence \(v'_1 = \pm v_3\) and \(v'_4 = \pm v_4\). Thus
we have a family of pairwise non-isomorphic abelian hypercomplex structures on \(\mathfrak{g}_3\)
depending on 5 parameters (see Corollary 4.3). Analogously, we get a 5-dimensional family on \(\mathfrak{g}_2\) by putting \(v_1 = v_2 = 0\).

We are now concerned with explicit continuous families of hypercomplex structures which are non-abelian.

Example 7.4. Consider the family defined by
\[
\mu_{rest}(X_1, X_2) = rz_2, \quad \mu_{rest}(X_2, X_3) = (1 - t)z_4,
\]
\[
\mu_{rest}(X_1, X_3) = sz_3, \quad \mu_{rest}(X_2, X_4) = -(1 - s)z_3,
\]
\[
\mu_{rest}(X_1, X_4) = tz_4, \quad \mu_{rest}(X_3, X_4) = (1 - r)z_2,
\]
which is easily seen to satisfy (30) but it is not abelian since \(t_1 = t_2 = t_3 = 0\) but
\(t_4 = 1\). The Ricci operator on the center is given by
\[
\text{Ric}_{\mu_{rest}}|_{\mathfrak{n}_2} = \frac{1}{2} \begin{bmatrix}
0 & r^2 + (1 - r)^2 \\
-(1 - s)^2 & s^2 + (1 - s)^2
\end{bmatrix}.
\]
and hence the family
\[
\{(N_{\mu_{rest}}, \{J_1, J_2, J_3\}, \langle \cdot, \cdot \rangle) : 0 \leq r \leq s \leq t, \quad r^2 + s^2 + t^2 - r - s - t = -\frac{1}{2}\}
\]
is pairwise non-isometric. This gives rise a surface of pairwise non-isomorphic non-abelian hypercomplex structures on \(\mathfrak{g}_3\) (see Corollary 4.3), since \(j_{\mu_{rest}}(Z)\) is invertible for any non-zero \(Z \in \mathfrak{n}_2\).

Example 7.5. Let \(\mu_t\) be the curve defined for \(0 \leq t \leq \frac{1}{\sqrt{3}}\) by
\[
\mu_t(X_1, X_2) = \sqrt{1 - 3t^2}z_1 + tz_2, \quad \mu_t(X_2, X_3) = t\bar{z}_4,
\]
\[
\mu_t(X_1, X_3) = tz_3, \quad \mu_t(X_2, X_4) = -t\bar{z}_3,
\]
\[
\mu_t(X_1, X_4) = t\bar{z}_4, \quad \mu_t(X_3, X_4) = -\sqrt{1 - 3t^2}z_1 + tz_2.
\]
It is easy to check that \((N_{\mu_1}, \{J_1, J_2, J_3\})\) is always non-abelian hypercomplex (recall that \(t_1 = t_2 = t_3 = 0, t_4 = 2t\)) and the curve is pairwise non-isomorphic since it follows from

\[
\text{Ric}_{\mu_1} |_{\mathfrak{a}_2} = \begin{bmatrix}
1-3t^2 & t^2 \\
t^2 & t^2
\end{bmatrix}
\]

that the curve \((N_{\mu_1}, \langle \cdot, \cdot \rangle)\) is pairwise non-isometric (see Corollary \(4.18\)). The starting and ending points are \(\mu_0 \simeq \mathfrak{g}_1\) and \(\mu_{\frac{3}{\sqrt{3}}} \simeq \mathfrak{g}_3\), respectively, and \(\mu_t \simeq \mathfrak{u}(2) \oplus \mathbb{C}^2\) for any \(0 < t < \frac{1}{\sqrt{3}}\) (see \(\text{L}3\) for further information on these 2-step nilpotent Lie algebras constructed via representations of compact Lie groups).

8. Einstein solvmanifolds

Our goal in this section is to show that the ‘moment map’ approach proposed in this paper can be also applied to the study of Einstein solvmanifolds. After a brief overview of such spaces, we will follow the same path used to study compatible Einstein solvmanifolds into a problem on nilpotent Lie algebras. More specifically, a chance of being an Einstein space (see \(\text{L}3\)). This fact turns the study of rank-one solitons coincide with the metrics studied in \(\text{L}3\).

In other words, none geometric structure is considered (or \(\gamma = 0\)).

A solvmanifold is a solvable Lie group \(S\) endowed with a left invariant Riemannian metric, and \(S\) is called standard if \(\mathfrak{a} := \mathfrak{n}^\perp\) is abelian, where \(\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]\) and \(\mathfrak{s}\) is the Lie algebra of \(S\). Curiously enough, all known examples of non-compact homogeneous Einstein manifolds are isometric to standard Einstein solvmanifolds. These spaces have been extensively studied by J. Heber in \(\text{H}4\), obtaining remarkable structure and uniqueness results.

Let \(N\) be a nilpotent Lie group with Lie algebra \(\mathfrak{n}\) of dimension \(n\), whose Lie bracket is denoted by \(\mu : \mathfrak{n} \times \mathfrak{n} \mapsto \mathfrak{n}\). We have in this case \(\gamma = 0\), thus any inner product is ‘compatible’, \(G_\gamma = \text{GL}(n), K_\gamma = \text{O}(n), \mathfrak{p}_\gamma = \mathfrak{p}, N_\gamma = N, \text{Ric}^\gamma = \text{Ric}\) and then condition \(\text{Ric}^\gamma = 0\) is clearly forbidden for non-abelian \(N\). Moreover, the evolution equation is precisely the Ricci flow and the corresponding invariant Ricci solitons coincide with the metrics studied in \(\text{L}3\).

Given a metric nilpotent Lie algebra \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\), a metric solvable Lie algebra \((\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, \langle \cdot, \cdot \rangle')\) is called a metric solvable extension of \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) if the restrictions of the Lie bracket of \(\mathfrak{s}\) and the inner product \(\langle \cdot, \cdot \rangle\) to \(\mathfrak{n}\) coincide with the Lie bracket of \(\mathfrak{n}\) and \(\langle \cdot, \cdot \rangle\), respectively. It turns out that for each \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) there exists a unique rank-one (i.e. \(\dim \mathfrak{a} = 1\)) metric solvable extension of \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) which stand a chance of being an Einstein space (see \(\text{L}4\)). This fact turns the study of rank-one Einstein solvmanifolds into a problem on nilpotent Lie algebras. More specifically, a nilpotent Lie algebra \(\mathfrak{n}\) is the nilradical of a standard Einstein solvmanifold if and only if \(\mathfrak{n}\) admits an inner product \(\langle \cdot, \cdot \rangle\) such that \(\text{Ric}_{\langle \cdot, \cdot \rangle} = cI + D\) for some \(c \in \mathbb{R}\) and \(D \in \text{Der}(\mathfrak{n})\), that is, \(\langle \cdot, \cdot \rangle\) is minimal.

Let \(\mathcal{N}\) be the variety of all nilpotent Lie algebras of dimension \(n\). Fix an inner product \(\langle \cdot, \cdot \rangle\) on \(\mathfrak{n}\). Each \(\mu \in \mathcal{N}\) is then identified via \(\text{L}3\) with the Riemannian manifold \((N_{\mu}, \langle \cdot, \cdot \rangle)\), but we also have in this case another identification with a solvmanifold: for each \(\mu \in \mathcal{N}\), there exists a unique rank-one metric solvable extension \(S_\mu = (S_\mu, \langle \cdot, \cdot \rangle)\) of \((N_{\mu}, \langle \cdot, \cdot \rangle)\) standing a chance of being Einstein, and every \((n+1)\)-dimensional rank-one Einstein solvmanifold can be modelled as \(S_\mu\) for a suitable \(\mu \in \mathcal{N}\). We recall that the study of standard solvmanifolds reduces to the rank-one case (see \(\text{L}4\) \(4.18\)).
The functional $F: \mathcal{P}N \mapsto \mathbb{R}$ given by $F(\mu) = \text{tr} \text{Ric}^2/||\mu||^4$ measures how far is the metric $\mu$ from being Einstein (see [L3]).

We obtain in this way in Theorem 4.3 the uniqueness up to isometry of Einstein metrics on standard solvable Lie groups proved in [Hb], as well as the variational result given in [L4] characterizing Einstein solvmanifolds as critical points of a natural curvature functional.

Theorem 4.4 gives the relationship between Ricci soliton metrics on nilpotent Lie groups and Einstein solvmanifolds proved in [L3]. Part (i) is a new characterization of these privileged metrics, claiming that they minimize the square norm of the Ricci tensor along all left invariant metrics on $\mathcal{N}$ of a given scalar curvature.

Finally, we note that the quotient $\mathcal{N}/\mathbb{R}GL(n)$ defined in Section 3.3 parameterizes rank-one Einstein solvmanifolds of dimension $(n+1)$ up to isometry, as they are precisely the critical points of $F : ||m||^2 : \mathcal{P}N \mapsto \mathbb{R}$ (see Theorem 4.3). The decomposition of $\mathcal{N}/\mathbb{R}GL(n)$ in categorical quotients described in (22) correspond then to the eigenvalue-types introduced in [Hb] (see Section 9 for further information).

For explicit examples of Einstein solvmanifolds obtained by this method we refer to [W], where it is proved that any nilpotent Lie group of dimension 6 admits a minimal metric (see also [L6] for dimension $\leq 5$ and a 7-dimensional curve).

9. ON THE CLASSIFICATION OF INVARIANT GEOMETRIC STRUCTURES ON NILPOTENT LIE GROUPS

There are only two 4-dimensional symplectic nilpotent Lie groups. In [KGM], Y. Khakimdjanov, M. Goze and A. Medina classified the 6-dimensional symplectic nilpotent Lie groups up to isomorphism, and have appeared several continuous families depending on one and two parameters (see also [14] for a classification in the $\mathbb{N}$-graded filiform case). The complex case is not different, S. Salamon determined which 6-dimensional nilpotent Lie algebras admit a complex structure, but the moduli space of all complex structures up to isomorphism on each one of them is still nebulous. For the Iwasawa manifold for example, such a moduli space seem to be particularly rich (see Section 9). The set of isomorphism classes of $2n$-dimensional complex nilpotent Lie groups contains anyway the isomorphism classes of $n$-dimensional nilpotent Lie algebras over $\mathbb{C}$, which is far to be achieved for $n \geq 8$. All the hypercomplex nilpotent Lie groups have been recently described in dimension 8 by I. Dotti and A. Fino as a family depending on 16 parameters. It is proved in Section 7.1 that we still need 9 parameters to describe the moduli space of 8-dimensional hypercomplex nilpotent Lie groups up to isomorphism, and 5 just for the abelian ones.

All this suggests that the moduli space of isomorphism classes of $n$-dimensional class-$\gamma$ nilpotent Lie groups is probably a very complicated space for most of the classes of geometric structures, even in low dimensions. So that, without any hope of an explicit description of such a moduli, what may be said about it?. Can we show that it is really unmanageable?. Can we at least find subspaces which are manifolds or algebraic varieties and obtain lower bounds for its ‘dimension’?.

This kind of questions belong to invariant theory. Given a nilpotent Lie group $\mathcal{N}$, the set of all class-$\gamma$ geometric structures on $\mathcal{N}$ is parameterized by a relatively nice space: a closed subset of the symmetric space $\mathbb{G}L(n)/G_{\gamma}$. The isomorphism is however determined by the natural action of $\text{Aut}(\mathcal{N})$, which is a group in general unknown and ‘very ugly’ from an invariant-theoretic point of view since it is far from
being semisimple or reductive. We then propose to consider the class-$\gamma$ nilpotent Lie groups of a given dimension all together, by using the variety of nilpotent Lie algebras as we did in the study of compatible metrics in Section 4. The advantage of this unified approach is that the group giving the isomorphism is the reductive Lie group $G_\gamma$; the price to pay is that the space

$$N_\gamma = \{ \mu \in \mathcal{N} : IC(\gamma, \mu) = 0 \}$$

where $G_\gamma$ is acting on is really wild. Fortunately, $N_\gamma$ is at least a real algebraic variety, and so the classification problem for such structures may be approached by using the tools from invariant theory given in Section 3.

Let $n, \mathcal{N}, \gamma, \mathcal{N}_\gamma, K_\gamma$ and $N_\mu$ be as in Section 4. We may view $N_\gamma$ as the variety of all class-$\gamma$ $n$-dimensional nilpotent Lie groups by identifying each element $\mu \in N_\gamma$ with a class-$\gamma$ nilpotent Lie group,

$$(37) \quad \mu \longleftrightarrow (N_\mu, \gamma),$$

where the geometric structure is defined by left invariant translation. The action of $G_\gamma$ on $N_\gamma$ has the following interpretation: each $\varphi \in G_\gamma$ determines an isomorphism of class-$\gamma$ geometric structures

$$(N_{\varphi \cdot \mu}, \gamma) \mapsto (N_\mu, \gamma)$$

by exponentiating the Lie algebra isomorphism $\varphi^{-1} : (\mathfrak{n}, \varphi \cdot \mu) \mapsto (\mathfrak{n}, \mu)$. In this way, two class-$\gamma$ structures $\mu, \lambda$ are isomorphic if and only if they live in the same $G_\gamma$-orbit, and hence the quotient

$$N_\gamma / G_\gamma$$

parameterizes the moduli space of all $n$-dimensional class-$\gamma$ nilpotent Lie groups up to isomorphism. Recall that a nilpotent Lie group $N_\mu$ admits a class-$\gamma$ structure if and only if the orbit $GL(n).\mu$ meets the variety $N_\gamma$, and $(GL(n).\mu \cap N_\gamma)/G_\gamma$ classifies left invariant class-$\gamma$ geometric structures on $N_\mu$ up to isomorphism.

According to the overview given in Section 3.3, our first natural question would be which are the closed orbits, or in other words, what kind of class-$\gamma$ nilpotent Lie groups does the categorical quotient $N_\gamma / G_\gamma$ classify?

Theorem 3.1, (iii) asserts that an orbit $G_\gamma.\mu$ is closed if and only if $m_\gamma(\lambda) = 0$ for some $\lambda \in G_\gamma.\mu$, where $m_\gamma$ is the moment map for the action of $G_\gamma$ on $N_\gamma$. In view of identification (37) and the formula $m_\gamma = 8 \text{Ric}^\gamma$ (see Proposition 4.2, (ii)), what we are saying is that the categorical quotient $N_\gamma / G_\gamma$ parameterizes precisely those class-$\gamma$ nilpotent Lie groups $(N_\mu, \gamma)$ which admit a compatible metric $\langle \cdot, \cdot \rangle_{\gamma} = 0$. In the case when $RI \subset \mathfrak{g}_\gamma$, we then have that

$$N_\gamma / G_\gamma = \{ (\mathbb{R}^n, \gamma) \},$$

and in the symplectic case, it follows from Proposition 5.3 that for $\mu \in N_s$ the orbit $Sp(n, \mathbb{R}).\mu$ can never be closed, unless $\mu = 0$, and hence also

$$N_s / Sp(n, \mathbb{R}) = \{ (\mathbb{R}^{2n}, \omega) \}.$$ 

Moreover, $0 \in Sp(n, \mathbb{R}).\mu$ for any $\mu \in N_s$ (see Theorem 3.1, (v)), that is, any $2n$-dimensional symplectic nilpotent Lie group degenerates to $(\mathbb{R}^{2n}, \omega)$. We also obtain from Theorem 5.2 the following topological result.

**Proposition 9.1.** The moduli space $N_s / Sp(n, \mathbb{R})$ is contractible.
Recall that all these properties are obviously satisfied when \( R I \subset g_\gamma \).

Due to the absence of closed orbits, it is natural to consider the wider quotient \( N_\gamma \sslash G_\gamma \) parameterizing orbits which contain a critical point of the functional square norm of the moment map (see Section 3.3). Recall that the moment map \( m_\gamma : N_\gamma \to p_\gamma \) for the action of \( G_\gamma \) on \( N_\gamma \) is given by \( m_\gamma(\mu) = 8 \text{Ric}_\gamma \mu \), and therefore the following nice ‘geometric’ characterization of this quotient follows from Theorems 4.3, 4.4.

**Proposition 9.2.** \( N_\gamma \sslash G_\gamma \) classifies precisely those class-\( \gamma \) nilpotent Lie groups admitting a minimal compatible metric.

Moreover, every class-\( \gamma \) nilpotent Lie group degenerates via the negative gradient flow of \( F = \| \text{ric}_\gamma \|^2 \) to one of these special structures admitting a minimal metric (see Theorem 3.8).

We now describe the decomposition of \( N_\gamma \sslash G_\gamma \) in categorical quotients given in (22). Notation of Section 3.3 will be constantly used from now on. The proof of the following rationality result is given in [N, Section 4] for the general case over \( \mathbb{C} \), but it is seen to be valid over \( \mathbb{R} \) (see also [L7, Theorem 3.5] for an alternative proof in this particular case which is clearly valid over \( \mathbb{R} \)).

**Theorem 9.3.** \([N]\) Let \( [\mu] \in \mathbb{P}N_\gamma \) be a critical point of \( F \), with \( \text{Ric}_\gamma \mu = c_\mu I + D_\mu \) for some \( c_\mu \in \mathbb{R} \) and \( D_\mu \in \text{Der}(\mu) \). Then there exists \( c > 0 \) such that the eigenvalues of \( cD_\mu \) are all integers prime to each other, say \( k_1 < ... < k_r \in \mathbb{Z} \) with multiplicities \( d_1, ..., d_r \in \mathbb{N} \).

**Definition 9.4.** The data set \( (k_1 < ... < k_r; d_1, ..., d_r) \) in the above theorem is called the type of the critical point \([\mu]\).

It follows from Theorem 3.8 that the set of types of critical points is in bijection with the set of strata \( S_{A_1}, ..., S_{A_s} \) given in Theorem 3.8, and it will be denoted by \( \{\alpha_1, ..., \alpha_s\} \).

Fix a type \( \alpha = \alpha_i = (k_1 < ... < k_r; d_1, ..., d_r) \). Since \( C_{(A_{\alpha})} \neq \emptyset \), there exists \( \mu \in N_\gamma, \|\mu\| = 1 \), such that \( m_\gamma(\mu) = 8 \text{Ric}_\gamma \mu = A_\alpha \), where

\[
A_\alpha = c_\alpha I + D_\alpha, \quad D_\alpha = \begin{bmatrix}
k_1 I_{d_1} & & \\
& \ddots & \\
& & k_r I_{d_r}
\end{bmatrix}, \quad c_\alpha = \begin{cases}
\frac{\text{tr } D_\alpha}{n} & \text{if } R I \subset g_\gamma, \\
\frac{\text{tr } D_\alpha + 2}{n} & \text{if } R I \not\subset g_\gamma,
\end{cases}
\]

and \( I_d \) denotes the \( d \times d \) identity matrix. The formula for \( c_\alpha \) follows from taking trace and using that \( \text{tr } \text{Ric}_\gamma \mu = 0 \) in the first case and \( \text{tr } \text{Ric}_\gamma \mu = 4 \text{Ric}_\gamma \mu = -\frac{1}{2} \|\mu\|^2 \) in the second one. Thus the set \( C_{(A_{\alpha})} \) given in Theorem 3.8 is precisely the set of critical points \([\mu]\) of \( F : \mathbb{P}V \to \mathbb{R} \) such that \( 8 \text{Ric}_\gamma \mu \) is conjugate to \( A_\alpha \), and

\[
G_{A_\alpha} := G_{A_{\alpha}} = G_\gamma \cap (\text{GL}(d_1) \times ... \times \text{GL}(d_r)).
\]

**Proposition 9.5.** For each type \( \alpha = \alpha_i = (k_1 < ... < k_r; d_1, ..., d_r) \) we have that

\[
V_\alpha := V_{A_\alpha} = \{ \mu \in V : D_\alpha \in \text{Der}(\mu) \}
\]

and

\[
\tilde{G}_{A_\alpha} = \tilde{G}_{A_{\alpha}} = \left\{ g = (g_1, ..., g_r) \in G_\alpha : \prod_{i=1}^{r}(\det g_i)^{k_i} = \det g^{-c_\alpha} \right\}.
\]
we have obtained in the symplectic case see Table 1.

| n | k |
|---|---|
| 2 | 0 |
| 3 | 1 |
| 4 | 2 |

In this paper a complete description has been achieved only for symplectic and complex types. It would be interesting to do this at least in low dimensions. Recall that in each dimension

\[ \text{Example 9.6.} \]

Consider the type \( \alpha = (1 < 2; n, n) \) in the symplectic case. If we put \( n = n_1 \oplus n_2 \) with \( \dim n_1 = n \), then

\[ V_\alpha = A^2 n_1^* \otimes n_2 = A(\mathbb{R}^n)^* \otimes \mathbb{R}^n, \quad D_\alpha = \begin{bmatrix} I_n & c_\alpha \end{bmatrix}, \quad c_\alpha = -\frac{3}{2}. \]

Let \( \{X_1, ..., X_n\} \) and \( \{Z_1, ..., Z_n\} \) be basis of \( n_1 \) and \( n_2 \) respectively, with dual basis \( \{\alpha_1, ..., \alpha_n\} \) and \( \{\beta_1, ..., \beta_n\} \). With respect to the inner product \( \langle \cdot, \cdot \rangle \) which makes
Example | type
---|---
5.6 (2n \geq 8) | (2 < 3 < 4; 3, 2n - 6, 3)
5.6 (2n = 6) | (1 < 2; 3, 3)
5.6 (2n = 4) | (3 < 4 < 6; 1, 1, 1)
5.7 | (1 < 2 < 3 < 4; 1, 1, 1, 1)
5.8 | (1 < 2; 3, 3)
5.9 | (1 < 2 < 3 < 4 < 5 < 6; 1, 1, 1, 1, 1)

| Table 1. Types of symplectic structures |
| --- |
| the basis \{X_i, Z_i\} orthonormal the 2-form |
| \[ \omega = \alpha_1 \wedge \beta_1 + ... + \alpha_n \wedge \beta_n \] |
| satisfies \( \omega = \langle \cdot, J \cdot \rangle \), where |
| \[ J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \] |
| This implies that |
| \( G_\alpha = Sp(n, \mathbb{R}) \cap (GL(n) \times GL(n)) = \{ (g, (g^t)^{-1}) : g \in GL(n) \} = GL(n) \) |
| and |
| \( \tilde{G}_\alpha = \{ (g, (g^t)^{-1}) : \det g \det g^{-2} = 1 \} = \{ (g, (g^t)^{-1}) : g \in SL(n) \} = SL(n) \). |
| A very special feature of this example is that any \( \mu \in V_\alpha \) satisfies the Jacobi condition, it is indeed a two-step nilpotent Lie algebra of dimension \( 2n \) and \( \dim \mu(n, n) \leq n \). Moreover, any Lie algebra of this kind can be represented by an element in \( V_\alpha \). Thus \( \mathcal{N}_s \cap V_\alpha \) (i.e. those Lie brackets for which \( \omega \) is closed) is a \( GL(n) \)-invariant subspace of \( V_\alpha \), which is easily seen to be of dimension \( \dim V_\alpha - \binom{n}{3} \). It follows from the decomposition |
| \( V_\alpha = \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n = W_{e_1+2e_2} + W_{e_3}, \quad W_{e_3} = \Lambda^3 \mathbb{R}^n, \) |
| in irreducible \( GL(n) \)-modules (\( \epsilon_i \): fundamental weights) that \( \mathcal{N}_s \cap V_\alpha = W_{e_1+2e_2} \). |
| We then obtain that the categorical quotient corresponding to the type \( (1 < 2; n, n) \) parameterizing symplectic nilpotent Lie groups up to isomorphism which admits a minimal compatible metric is given by |
| \( W_{e_1+2e_2} / SL(n). \) |
| The description of this quotient is intimately related with the study of the ring \( \mathbb{R}[W_{e_1+2e_2}]^{SL(n)} \) of invariant polynomials. This is a wide open problem for arbitrary \( n \) in invariant theory, even over the complex numbers, and shows that an explicit
classification of symplectic structures on nilpotent Lie groups would not be feasible. It can be showed however that
\[
\dim W_{r_1 + 2r_2} / SL(n) \geq \frac{1}{3}(n^3 - 3n^2 - n), \quad \forall n \geq 4.
\]

**Example 9.7.** We now consider the type \( \alpha = (1 < 2; 2k, 2m) \) in the complex case, \( n = k + m \). If we put \( n = n_1 \oplus n_2 \) with \( \dim n_1 = 2k \), \( \dim n_2 = 2m \), then
\[
V_\alpha = \Lambda^2 n_1^* \otimes n_2 = \Lambda(\mathbb{R}^{2k})^* \otimes \mathbb{R}^{2m}, \quad D_\alpha = \begin{bmatrix} 1_{2k} & 2_{2m} \end{bmatrix}, \quad c_\alpha = \frac{k + 2m + 2}{k + m}.
\]
Let \( \{X_1, ..., X_{2k}\} \) and \( \{Z_1, ..., Z_{2m}\} \) be basis of \( n_1 \) and \( n_2 \) respectively, and consider \( J \) acting on each \( n_i \) by
\[
\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ \vdots & \vdots \\ 0 & -1 \\ 1 & 0 \end{bmatrix},
\]
and the compatible inner product \( \langle \cdot, \cdot \rangle \) which makes the basis \( \{X_i, Z_i\} \) orthonormal. This implies that
\[
G_\alpha = GL(n, \mathbb{C}) \cap (GL(2k) \times GL(2m)) = GL(k, \mathbb{C}) \times GL(m, \mathbb{C})
\]
and
\[
\tilde{G}_\alpha = \{ (g_1, g_2) : \det g_1 \det g_2^2 = (\det g_1 \det g_2)^{-c_\alpha} \},
\]
but we can assume that \( \tilde{G}_\alpha = SL(k, \mathbb{C}) \times SL(m, \mathbb{C}) \) by discarding the elements which act trivially. Any \( \mu \in V_\alpha \) is again a two-step nilpotent Lie algebra of dimension \( 2n \) and \( \dim \mu(n, n) \leq 2m \). Thus \( N_c \cap V_\alpha \) (i.e. those Lie brackets for which \( J \) is integrable) is a \( GL(k, \mathbb{C}) \times GL(m, \mathbb{C}) \)-invariant subspace of \( V_\alpha \), as well as the subspaces \( N_{ac} \) and \( N_{bc} \) of abelian and bi-invariant complex structures, respectively. One can easily see that the dimensions of these subspaces are given by
\[
\dim N_c = k(3k - 1)m, \quad \dim N_{ac} = 2k^2m, \quad \dim N_{bc} = k(k - 1)m,
\]
and moreover, \( N_c = N_{ac} \oplus N_{bc} \). The decomposition of \( V_\alpha \) in irreducible \( SL(k, \mathbb{C}) \times SL(m, \mathbb{C}) \)-modules is
\[
V_\alpha = \Lambda^2(\mathbb{R}^{2k})^* \otimes \mathbb{R}^{2m} = (u(k) \otimes \mathbb{R}^{2m}) \oplus (W \otimes \mathbb{R}^{2m}) \oplus (W \otimes \mathbb{R}^{2m}),
\]
where the action of \( SL(k, \mathbb{C}) \) on the skew-hermitian matrices \( u(k) \) is given by \( A \mapsto g^*Ag \), \( \Lambda^2 \mathbb{C}^k = W \oplus W \) as a real representation of \( SL(k, \mathbb{C}) \) and \( \mathbb{R}^{2m} \) is the standard representation of \( SL(m, \mathbb{C}) \) on \( \mathbb{C}^m \) viewed as real. It follows from a simple dimension argument that \( N_{ac} = u(k) \otimes \mathbb{R}^{2m} \) and \( N_{bc} = W \otimes \mathbb{R}^{2m} \).

We then obtain for the type \( (1 < 2; 2k, 2m) \), that the categorical quotients parameterizing abelian complex and bi-invariant complex nilpotent Lie groups up to isomorphism which admits a minimal compatible metric are respectively given by
\[
u(k) \otimes \mathbb{R}^{2m} / SL(k, \mathbb{C}) \times SL(m, \mathbb{C}), \quad W \otimes \mathbb{R}^{2m} / SL(k, \mathbb{C}) \times SL(m, \mathbb{C}),
\]
and for complex structures
\[
(u(k) \otimes \mathbb{R}^{2m}) \oplus (W \otimes \mathbb{R}^{2m}) / SL(k, \mathbb{C}) \times SL(m, \mathbb{C}).
\]
The description of these quotients also lead to open problems in invariant theory, except for \( m = 1 \) and maybe for other small values of \( k \) and \( m \).
Example 9.8. We can argue analogously to the above example for the type given by \((1 < 2, 4k, 4m)\) in the abelian hypercomplex case, and obtain that the corresponding categorical quotient is given by
\[
\mathfrak{sp}(k) \otimes \mathbb{R}^{1/m} / \mathbb{SL}(k, \mathbb{H}) \times \mathbb{SL}(m, \mathbb{H}).
\]
Recall that the case \(k = m = 1\) has been studied in Section 7.3, and \(\mathbb{SL}(1, \mathbb{H}) = \mathbb{Sp}(1)\). For a type of the form \(\alpha = (k_1 < \ldots < k_r; 4, \ldots, 4)\) in the hypercomplex case, we will always have that \(\tilde{G}_\alpha\) is compact as it is a closed subgroup of \(\mathbb{Sp}(1) \times \ldots \times \mathbb{Sp}(1)\) \((r \text{ times})\). This implies that any \(\mu \in V_\alpha \cap \mathcal{N}_\gamma\) admits a minimal compatible metric, which is actually its unique compatible metric up to isometry and scaling. Also, the categorical quotient \(V_\alpha \cap \mathcal{N}_\gamma / \tilde{G}_\alpha\) coincides with the whole quotient \(V_\alpha \cap \mathcal{N}_\gamma / \tilde{G}_\alpha\) since any orbit is closed. All this makes intriguing enough the study of these particular types.

10. Appendix

We briefly recall in this appendix some features of Riemannian geometry of left invariant metrics on nilpotent Lie groups.

Consider the vector space \(\text{sym}(\mathfrak{n})\) of symmetric real valued bilinear forms on \(\mathfrak{n}\), and \(\mathcal{P} \subset \text{sym}(\mathfrak{n})\) the open convex cone of the positive definite ones (inner products), which is naturally identified with the space of all left invariant Riemannian metrics on \(N\). Every \(\langle \cdot, \cdot \rangle \in \mathcal{P}\) induces a natural inner product \(g_{\langle \cdot, \cdot \rangle}\) on \(\text{sym}(\mathfrak{n})\) given by \(g_{\langle \cdot, \cdot \rangle}(\alpha, \beta) = \text{tr} A_\alpha A_\beta\) for all \(\alpha, \beta \in \text{sym}(\mathfrak{n})\), where \(\alpha(X, Y) = g(A_\alpha X, Y)\).

We endow \(\mathcal{P}\) with the Riemannian metric \(g\) given by \(g_{\langle \cdot, \cdot \rangle}\) on the tangent space \(T_{\langle \cdot, \cdot \rangle} \mathcal{P} = \text{sym}(\mathfrak{n})\) for any \(\langle \cdot, \cdot \rangle \in \mathcal{P}\). Thus \((\mathcal{P}, g)\) is isometric to the symmetric space \(\text{GL}(n) / \text{O}(n)\). E. Wilson proved that \((N, \langle \cdot, \cdot \rangle)\) and \((N, \langle \cdot, \cdot \rangle')\) are isometric if and only if \(\langle \cdot, \cdot \rangle' = \varphi. \langle \cdot, \cdot \rangle := \langle \varphi^{-1} \cdot, \varphi^{-1} \cdot \rangle\) for some \(\varphi \in \text{Aut}(\mathfrak{n})\) (see the proof of [W]). Therefore, although the Lie bracket \(\mu\) does not play any role in the definition of a compatible metric, it is crucial in the study of the moduli space of compatible metrics on \((N, \gamma)\) up to isometry.

The Ricci curvature tensor \(\text{ric}_{\langle \cdot, \cdot \rangle}\) and the Ricci operator \(\text{Ric}_{\langle \cdot, \cdot \rangle}\) of \((N, \langle \cdot, \cdot \rangle)\) are given by (see [B, 7.39]),
\[
\text{ric}_{\langle \cdot, \cdot \rangle}(X, Y) = \langle \text{Ric}_{\langle \cdot, \cdot \rangle} X, Y \rangle = -\frac{1}{2} \sum_{i,j} \langle \mu(X, X_i), X_j \rangle \langle \mu(Y, X_i), X_j \rangle \\
+ \frac{1}{4} \sum_{i,j} \langle \mu(X_i, X_j), Y \rangle \langle \mu(X_i, X_j), Y \rangle,
\]
(38)
for all \(X, Y \in \mathfrak{n}\), where \(\{X_1, ..., X_n\}\) is any orthonormal basis of \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\). Notice that always \(\text{sc}(N, \langle \cdot, \cdot \rangle) < 0\), unless \(N\) is abelian. It is proved in [H] that the gradient of the scalar curvature functional \(\text{sc} : \mathcal{P} \mapsto \mathbb{R}\) is given by
\[
\text{grad}(\text{sc})_{\langle \cdot, \cdot \rangle} = -\text{ric}_{\langle \cdot, \cdot \rangle},
\]
(39)
and hence it follows from the properties of \(\mathcal{P}\) described above that
\[
\text{tr} \text{Ric}_{\langle \cdot, \cdot \rangle} D = 0, \quad \forall \text{ symmetric } D \in \text{Der}(\mathfrak{n}),
\]
(40)
where \(\text{Der}(\mathfrak{n})\) is the Lie algebra of derivations of \(\mathfrak{n}\) (see for instance [L, (2)] for a proof of this fact).

Assume now that \(\mathfrak{n}\) is 2-step nilpotent, and let \(\langle \cdot, \cdot \rangle\) an inner product on \(\mathfrak{n}\). Consider the orthogonal decomposition \(\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2\), where \(\mathfrak{n}_2\) is the center of
\( n \). Thus the Lie bracket of \( n \) can be viewed as a skew-symmetric bilinear map 
\( \mu : n_1 \times n_1 \mapsto n_2 \). For each \( Z \in n_2 \) we define \( j_\mu(Z) : n_1 \mapsto n_1 \) by
\[
\langle j_\mu(Z)X, Y \rangle = \langle \mu(X, Y), Z \rangle, \quad X, Y \in n_1.
\]
\((N, \langle \cdot, \cdot \rangle)\) is said to be a modified H-type Lie group if for any non-zero \( Z \in n_2 \)
\[
\langle j_\mu(Z)^2 = c(Z)I \quad \text{for some } c(Z) < 0,
\]
and it is called H-type when \( c(Z) = -\langle Z, Z \rangle \) for all \( Z \in n_2 \). These metrics, introduced by A. Kaplan, play a remarkable role in the study of Riemannian geometry on nilpotent and solvable Lie groups (see for instance [BTV] for further information and [L1] for the ‘modified’ case).

If \( \mu' = \varphi \mu \) for some \( \varphi = (\varphi_1, \varphi_2) \in GL(n_1) \times GL(n_2) \), then it is easy to see that
\[
j_{\mu'}(Z) = \varphi_1 j_\mu(\varphi_2^t Z) \varphi_1^t, \quad \forall Z \in n_2.
\]

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