In this work, a general method is described for obtaining degenerate solutions of the Dirac equation, corresponding to an infinite number of electromagnetic 4-potentials and fields, which are explicitly calculated. More specifically, using four arbitrary real functions, one can automatically construct a spinor that satisfies the Dirac equation for an infinite number of electromagnetic 4-potentials, defined by those functions. An interesting characteristic of these solutions is that, in the case of Dirac particles with nonzero mass, the degenerate spinors should be localized, both in space and time. The method is also extended to the cases of massless Dirac and Weyl particles, where the localization of the spinors is no longer required. Finally, two experimental methods are proposed for detecting the presence of degenerate states.

1. Introduction

We consider the Dirac equation in the form

\[ i\gamma^\mu \partial_\mu \Psi + a_\mu \gamma^\mu \Psi - m\Psi = 0 \]  \hspace{1cm} (1)

where \( \gamma^\mu \), \( \mu = 0, 1, 2, 3 \), are the standard Dirac matrices, \( m \) is the mass of the particle, and \( a_\mu \) is the electromagnetic 4-potential multiplied by the electric charge \( q \) of the particle. It should also be noted that Equation (1) is written in natural units, where the speed of light in vacuum \( c \) and the reduced Planck constant \( \hbar \) are both set equal to one.

In a recent article\cite{1} we have shown that all solutions to the Dirac equation satisfying the conditions \( \Psi^T \gamma^2 \Psi \neq 0 \), where \( \gamma = \gamma^0 + iy^1 \gamma^2 \gamma^3 \), are degenerate, corresponding to an infinite number of electromagnetic 4-potentials, which are explicitly calculated through Theorem 5.4. We have also shown that all solutions to the Weyl equations are degenerate. In the latter case, the corresponding electromagnetic 4-potentials are calculated using Theorem 3.1\cite{1}.

In recent articles\cite{2,3,4,5} we have extended these results providing several classes of degenerate solutions to the Dirac and Weyl equations for massive\cite{2,5} and massless\cite{1,4} particles, and describing their physical properties and potential applications. Furthermore, we discuss some very interesting properties of Weyl particles, mainly regarding their localization\cite{4}. Here, it should be noted that in the present work, as well as in all the aforementioned articles\cite{1–5}, we use the term degenerate not in its conventional sense, but in a novel way, first introduced by Kechriniotis et al.,\cite{1} describing quantum states which are invariant under a wide variety of electromagnetic fields.

In this work, we provide a general method for obtaining degenerate solutions to the Dirac equation for real 4-potentials, which are explicitly calculated. The method is described in detail in Section 2, whereas in Section 3 it is extended to massless Dirac and Weyl particles. In Section 4 we discuss two experimental techniques for detecting the presence of degenerate states and the transition between these states and the non-degenerate ones. Our conclusions are presented in Section 5. We have also added two appendices, to provide the necessary mathematical background.

2. Formulation of the Method and Description of the Degenerate Spinors and the Corresponding Electromagnetic 4-Potentials in the Case of Massive Dirac Particles

Let \( T, R \) be arbitrary complex functions of the spatial coordinates and time, and \( \phi \neq n\pi + \pi/2, n \in \mathbb{Z} \) be an arbitrary real constant.

It is easy to verify that any spinor of the form

\[ \Psi = T \begin{pmatrix} \cos \phi \\ 1 - \sin \phi \\ \cos \phi \\ 1 - \sin \phi \end{pmatrix} + R \begin{pmatrix} -\cos \phi \\ 1 + \sin \phi \\ \cos \phi \\ -1 - \sin \phi \end{pmatrix} \]  \hspace{1cm} (2)

is degenerate.
Substituting the spinor given by Equation (2) into the Dirac equation, we obtain the following system of equations

$$(\cos \varphi \partial_0 + \sin \varphi \partial_3 + \partial_0) R = i (a_1 \cos \varphi + a_3 \sin \varphi + a_0) R$$

$$(\cos \varphi \partial_0 + \sin \varphi \partial_3 + \partial_0) T = i (a_1 \cos \varphi + a_3 \sin \varphi + a_0) T$$

$$(\cos \varphi \partial_0 - \partial_3 - \sin \varphi \partial_0) R = (a_2 \cos \varphi - i a_3 - i a_0 \sin \varphi) R + im (1 - \sin \varphi) T$$

$$(\cos \varphi \partial_0 - \partial_3 + \sin \varphi \partial_0) T = (a_2 \cos \varphi + i a_3 + i a_0 \sin \varphi) T + im (1 + \sin \varphi) R$$

Defining the matrix

$$\Pi = \begin{pmatrix} \cos \varphi & 0 & -i \cos \varphi & 0 \\ 0 & 1 & - \sin \varphi & 0 \\ -i \cos \varphi & 1 & \sin \varphi & 0 \\ \sin \varphi & 0 & -\sin \varphi & 1 \end{pmatrix}$$

and setting

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_0 \end{pmatrix} = \Pi^T \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \\ \partial_0 \end{pmatrix}$$

where $\Pi^T$ is the transpose of $\Pi$, the system of Equations (3)–(6) can be written as

$$D_1 R = A_1 R$$

$$D_2 T = A_1 T$$

$$D_3 R = A_2 R + im (1 - \sin \varphi) T$$

$$D_3 T = A_3 T + im (1 + \sin \varphi) R$$

where

$$A_1 = i (a_1 \cos \varphi + a_3 \sin \varphi + a_0)$$

$$A_2 = a_2 \cos \varphi - i a_3 - i a_0 \sin \varphi$$

$$A_3 = a_2 \cos \varphi + i a_3 + i a_0 \sin \varphi$$

As shown in Appendix A, using the following transformation of the coordinates $x_0, x_1, x_2, x_3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_0 \end{pmatrix} = \Pi \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_0 \end{pmatrix}$$

the linear differential operators $D_i, i = 1, 2, 3$ can be written as $\partial_i \partial_0 = \tilde{\partial}_i$. Consequently, the system of Equations (9)–(12) takes the form

$$\tilde{\partial}_i \tilde{T} = \tilde{A}_i \tilde{T}$$

$$\tilde{\partial}_i \tilde{R} = \tilde{A}_i \tilde{R} + im (1 - \sin \varphi) \tilde{T}$$

$$\tilde{\partial}_i \tilde{T} = \tilde{A}_i \tilde{T} + im (1 + \sin \varphi) \tilde{R}$$

where $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{R}, \tilde{T}$ are the functions $A_1, A_2, A_3, R, T$ expressed in the coordinates $s_0, s_1, s_2, s_3$.

Multiplying Equations (17) and (18) with $\exp(-\int \tilde{A}_i ds_i)$ we obtain

$$\tilde{\partial}_i \left( \mathbb{R} \exp \left( - \int \tilde{A}_i ds_i \right) \right) = 0$$

$$\tilde{\partial}_i \left( \tilde{T} \exp \left( - \int \tilde{A}_i ds_i \right) \right) = 0$$

Consequently, the functions $\mathbb{R}, \tilde{T}$ can be written as

$$\mathbb{R} = \exp \left( \int \tilde{A}_1 ds_1 \right) \tilde{g}_R$$

$$\tilde{T} = \exp \left( \int \tilde{A}_1 ds_1 \right) \tilde{g}_T$$

where $\tilde{g}_R, \tilde{g}_T$ are arbitrary complex functions of the coordinates $s_0, s_1, s_2, s_3$. Substituting Equations (23) and (24) into Equations (19) and (20) and supposing that $\tilde{A}_1$ depends only on $s_0, s_1$, we obtain the following system of equations for the functions $\tilde{g}_R, \tilde{g}_T$

$$\tilde{\partial}_2 - \tilde{A}_2 \tilde{g}_R = im (1 - \sin \varphi) \tilde{g}_T$$

$$\tilde{\partial}_3 - \tilde{A}_3 \tilde{g}_R = im (1 + \sin \varphi) \tilde{g}_T$$

Multiplying Equation (25) by $\mathbb{R}(1 + \sin \varphi)$ and Equation (26) by $\mathbb{R}(1 - \sin \varphi)$, yields that

$$\tilde{\partial}_2 - \tilde{A}_2 \tilde{g}_R = \mathbb{R}m^2 \cos^2 \varphi \tilde{g}_T$$

$$\tilde{\partial}_3 - \tilde{A}_3 \tilde{g}_R = \mathbb{R}m^2 \cos^2 \varphi \tilde{g}_T$$

which, according to Equations (25) and (26), can be written as

$$\tilde{\partial}_2 - \tilde{A}_2 \tilde{g}_R = \mathbb{R}m^2 \cos^2 \varphi \tilde{g}_T$$

$$\tilde{\partial}_3 - \tilde{A}_3 \tilde{g}_R = \mathbb{R}m^2 \cos^2 \varphi \tilde{g}_T$$

Multiplying Equations (29) and (30) by $\exp(-\int \tilde{A}_2 ds_2 - \int \tilde{A}_3 ds_3)$ and assuming that $\tilde{\partial}_2 \tilde{A}_2 = 0$ and $\tilde{\partial}_3 \tilde{A}_3 = 0$, the above system of equations takes the following form

$$\tilde{\partial}_2 \tilde{\partial}_3 \left( \exp \left( - \int \tilde{A}_2 ds_2 - \int \tilde{A}_3 ds_3 \right) \tilde{g}_T \right) = -m^2 \cos^2 \varphi \exp \left( - \int \tilde{A}_2 ds_2 - \int \tilde{A}_3 ds_3 \right) \tilde{g}_T$$

$$\tilde{\partial}_2 \tilde{\partial}_3 \left( \exp \left( - \int \tilde{A}_2 ds_2 - \int \tilde{A}_3 ds_3 \right) \tilde{g}_R \right) = -m^2 \cos^2 \varphi \exp \left( - \int \tilde{A}_2 ds_2 - \int \tilde{A}_3 ds_3 \right) \tilde{g}_R$$
Consequently, the functions \( \tilde{g}_R, \tilde{g}_T \) can be written as

\[
\tilde{g}_R = \exp \left( \int \tilde{A}_1 ds_2 + \int \tilde{A}_3 ds_3 \right) \tilde{W}_R
\]

\[
\tilde{g}_T = \exp \left( \int \tilde{A}_1 ds_2 + \int \tilde{A}_3 ds_3 \right) \tilde{W}_T
\]

where \( \tilde{W}_R(s_0, s_1, s_2) \), \( \tilde{W}_T(s_0, s_1, s_2) \) are solutions to the differential equation

\[
\tilde{\partial}_1 \tilde{\partial}_1 \tilde{W} = -m^2 \cos^2 \varphi \tilde{W}
\]

Here, we have also assumed that \( \tilde{\partial}_1 \tilde{A}_2 = 0 \) and \( \tilde{\partial}_1 \tilde{A}_3 = 0 \), because the functions \( \tilde{g}_R, \tilde{g}_T \) depend only on \( s_0, s_2, s_3 \).

Thus, assuming that \( \tilde{\partial}_2 \tilde{A}_1 = 0, \tilde{\partial}_2 \tilde{A}_2 = 0, \tilde{\partial}_2 \tilde{A}_3 = 0, \tilde{\partial}_2 \tilde{A}_4 = 0 \), the functions \( \tilde{R}, \tilde{T} \) can be written as

\[
\tilde{R} = \exp \left( \int \tilde{A}_1 ds_2 + \int \tilde{A}_2 ds_3 + \int \tilde{A}_3 ds_4 \right) \tilde{W}_R \tag{37}
\]

\[
\tilde{T} = \exp \left( \int \tilde{A}_1 ds_2 + \int \tilde{A}_2 ds_3 + \int \tilde{A}_3 ds_4 \right) \tilde{W}_T \tag{38}
\]

Finally, substituting the above expressions into Equation (20), yields that the functions \( \tilde{W}_R, \tilde{W}_T \) should be related through the following formula

\[
\tilde{\partial}_1 \tilde{W}_T = im (1 + \sin \varphi) \tilde{W}_R \tag{39}
\]

Thus, any spinor of the form

\[
\Psi = \exp \left( \int \tilde{A}_1 ds_2 + \int \tilde{A}_3 ds_4 \right) \tilde{W} \left( \begin{array}{c}
\cos \varphi \\
1 - \sin \varphi \\
\cos \varphi \\
1 + \sin \varphi
\end{array} \right) + \tilde{W} \left( \begin{array}{c}
-\cos \varphi \\
-1 + \sin \varphi \\
-\cos \varphi \\
-1 - \sin \varphi
\end{array} \right)
\]

\[
\times \left( \begin{array}{c}
im (1 + \sin \varphi) \int \tilde{W} ds_3
\end{array} \right)
\]

where \( \tilde{A}_1, \tilde{A}_2, \tilde{A}_3 \) satisfy the conditions given by Equation (36), and \( \tilde{W}(s_0, s_2, s_3) \) is an arbitrary solution to the differential equation (35), is degenerate solution to the Dirac equation.

An interesting remark is that, according to Equations (13)–(15), assuming that the 4-potentials \( (a_0, a_1, a_2, a_3) \) are real, the function \( A_1 \) becomes imaginary and the function \( A_2 \) becomes the complex conjugate of \( A_2 \). Thus, the 4-potentials \( (a_0, a_1, a_2, a_3) \) are given by the formulas

\[
a_0 = h
\]

\[
a_1 = -h \cos \varphi + \text{Im} (A_1) \sec \varphi + \text{Im} (A_2) \tan \varphi
\]

\[
a_2 = \text{Re} (A_2) \sec \varphi
\]

\[
a_3 = -h \sin \varphi - \text{Im} (A_2)
\]

where \( h \) is an arbitrary real function of the spatial coordinates and time, \( \text{Im}(A_1), \text{Im}(A_2) \) are the imaginary parts of \( A_1, A_2 \), respectively, and \( \text{Re}(A_3) \) is the real part of \( A_2 \). Here, it should be mentioned that the functions \( A_1, A_2, A_3 \), and consequently the 4-potentials \( (a_0, a_1, a_2, a_3) \) could also depend on the mass of the particles.

Another interesting remark is that the coordinates \( s_0, s_2 \) are real functions of the coordinates \( x_0, x_2, x_2, x_3 \), as it can be easily verified from the matrix \( \Pi \). Consequently, assuming that \( \tilde{A}_2, \tilde{A}_4 \) depend only on \( s_0 \) and defining the real functions \( \tilde{f}_1(s_0, s_1), \tilde{f}_2(s_0) \), \( \tilde{f}_3(s_0) \) through the formulas

\[
\tilde{f}_1(s_0, s_1) = -i \tilde{A}_1
\]

\[
\tilde{f}_2(s_0) = (\tilde{A}_2 + \tilde{A}_4)/2
\]

\[
\tilde{f}_3(s_0) = -i (\tilde{A}_2 - \tilde{A}_4)/2
\]

it is easy to verify that the spinor

\[
\Psi = \exp \left( i \int \tilde{f}_1(s_0, s_1) ds_1 + \int \tilde{f}_2(s_0) (s_2 + s_3 + 1) + \int \tilde{f}_3(s_0) (s_2 - s_3) \right)
\]

\[
\times \left( \begin{array}{c}
im (1 + \sin \varphi) \int \tilde{W} ds_3 \\
\cos \varphi \\
1 - \sin \varphi \\
\cos \varphi \\
1 + \sin \varphi \\
-\cos \varphi \\
-1 + \sin \varphi \\
-\cos \varphi \\
-1 - \sin \varphi
\end{array} \right)
\]

is a degenerate solution to the Dirac equation, for the real 4-potentials given by the following expressions

\[
a_0 = h
\]

\[
a_1 = -h \cos \varphi + \tilde{f}_1(s_0, s_1) \sec \varphi + \tilde{f}_2(s_0) \tan \varphi
\]

\[
a_2 = \tilde{f}_2(s_0) \sec \varphi
\]

\[
a_3 = -h \sin \varphi - \tilde{f}_3(s_0)
\]

Here, \( \tilde{f}_1, \tilde{f}_2, \tilde{f}_3 \) are real functions of the spatial coordinates \( x_0, x_2, x_2, x_3 \), and time, connected to the functions \( \tilde{f}_1(s_0, s_1), \tilde{f}_2(s_0), \tilde{f}_3(s_0) \) through the following transformation of the coordinates

\[
\begin{pmatrix}
s_1 \\
s_2 \\
s_3 \\
s_0
\end{pmatrix} = \Pi^{-1}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_0
\end{pmatrix}
\]

where \( \Pi^{-1} \) is the inverse matrix of \( \Pi \), given by the formula

\[
\Pi^{-1} = \begin{pmatrix}
\sec \varphi & 0 & 0 & 0 \\
0 & \frac{1}{2} \tan \varphi & \frac{1}{2} \sec \varphi & \frac{1}{2} \\
0 & \frac{1}{2} \tan \varphi & \frac{1}{2} \sec \varphi & 0 \\
-\cos \varphi & 0 & -\sin \varphi & 1
\end{pmatrix}
\]

Consequently, for any combination of the arbitrary real functions \( \tilde{f}_1(s_0, s_1), \tilde{f}_2(s_0), \tilde{f}_3(s_0) \), one can automatically construct the spinor given by Equation (48), which is degenerate solution to the Dirac equation for the infinite number of real 4-potentials given by the formulas (49)–(52). Finally, using the coordinate transformation described by Equation (16), the degenerate spinor (48) can be expressed in terms of the spatial and temporal coordinates \( x_0, x_1, x_2, x_3 \).
In addition, according to Theorem 5.4, the spinor given by Equation (48) will also be solution to the Dirac equation for the 4-potentials
\[ b_\mu = a_\mu + \kappa_\mu, \mu = 0, 1, 2, 3 \]  
(55)
where
\[
(\kappa_0, \kappa_1, \kappa_2, \kappa_3) = \left( 1, -\frac{\psi^{\gamma^0 \gamma^1 \gamma^2 \sigma_3}}{\psi^{\gamma^0}}, -\frac{\psi^{\gamma^0 \gamma^1 \gamma^2 \sigma_3}}{\psi^{\gamma^0}}, \frac{\psi^{\gamma^0 \gamma^1 \gamma^2 \sigma_3}}{\psi^{\gamma^0}} \right) \ 
(56)
\]
and \( s \) is an arbitrary real function of the spatial coordinates and time. It is evident that the 4-potentials \( b_\mu \) coincides with the 4-potentials \( a_\mu \), given by Equations (49)–(52).

The electromagnetic fields, in Gaussian units, corresponding to the 4-potentials \( a_\mu \), or \( b_\mu \), can be easily calculated through the formulas\(^6.7\)
\[
E = -\nabla U - \frac{\partial A}{\partial t}, \quad B = \nabla \times A \ 
(57)
\]
where \( U = a_\mu / q \) is the electric potential and \( A = -(1/q)(a_1 + a_2 + a_3) \) is the magnetic vector potential. Here, it should be reminded that we are working in the natural system of units, and consequently the speed of light has been set equal to one in the above formulas. It should also be noted that the 4-potentials (49)–(52) are not connected through a gauge transformation and consequently they generate a whole family of electromagnetic fields, infinite in number, depending on the choice of the arbitrary function \( \kappa \). Thus, particles described by the degenerate spinor (48) have the remarkable property to be able to exist in the same quantum state under the whole variety of the electromagnetic 4-potentials given by Equations (49)–(52) and the corresponding electromagnetic fields, calculated through Equation (57).

Furthermore, as shown in Appendix B, the differential equation (35) has solutions of the form
\[
\tilde{W}(s_0, s_2, s_3) = g(s_0) \exp \left( -\frac{m^2 \cos^2 \varphi}{k} s_2 + k(s_0) s_3 \right) \ 
(58)
\]
where \( g(s_0), k(s_0) \neq 0 \) are arbitrary complex functions of \( s_0 \). Using the above expression for \( \tilde{W}(s_0, s_2, s_3) \) and assuming that \( k(s_0) \) is constant, the spinor given by Equation (48) becomes
\[
\Psi = \exp \left( i \int_{t_1}^{t_2} \left( s_0, s_1 \right) ds_1 + \int_{s_2}^{s_3} (s_0) (s_2 + s_3) + \int_{s_2}^{s_3} (s_0) (s_2 - s_1) \right) 
\times g(s_0) \exp \left( -\frac{m^2 \cos^2 \varphi}{k} s_2 \right) \exp (k s_3) 
\times \left( \begin{array}{c} \cos \varphi \\ 1 - \sin \varphi \\ \cos \varphi \\ 1 + \sin \varphi \end{array} \right) 
\times \left( \begin{array}{c} \cos \varphi \\ 1 - \sin \varphi \\ \cos \varphi \\ 1 + \sin \varphi \end{array} \right) \ 
(59)
\]
Another interesting remark is that, according to the transformation given by Equation (53), the coordinates \( s_1, s_2, s_3, s_0 \) can be written as
\[
s_1 = x \sec \varphi \ 
(60)
s_2 = \frac{1}{2} x \tan \varphi + i \frac{1}{2} y \cos \varphi - \frac{1}{2} z \ 
(61)
s_3 = -\frac{1}{2} x \tan \varphi + i \frac{1}{2} y \cos \varphi + \frac{1}{2} z \ 
(62)
s_0 = t - x \cos \varphi - z \sin \varphi \ 
(63)
\]
where we have also made the substitution \( x_1 \rightarrow x, x_2 \rightarrow y, x_3 \rightarrow z, x_0 \rightarrow t \). Consequently,
\[
s_2 + s_3 = iy \sec \varphi \ 
(64)
\]
and
\[
s_2 - s_3 = x \tan \varphi + z \ 
(65)
\]
Additionally, it is evident that the coordinates \( s_1, s_0 \) are real function of \( x, y, z, t \). Therefore, the factor
\[
\exp \left( i \int_{t_1}^{t_2} \left( s_0, s_1 \right) ds_1 + \int_{s_2}^{s_3} (s_0) (s_2 + s_3) + \int_{s_2}^{s_3} (s_0) (s_2 - s_1) \right) 
\times \exp \left( -\frac{m^2 \cos^2 \varphi}{k} s_2 + k(s_0) s_3 \right) 
\times \exp \left( -\frac{m^2 \cos^2 \varphi + k^2}{2k} (x \sec \varphi + z) \right) \ 
(66)
\]
in the spinors given by Equations (48) and (59) is a phase factor depending on the coordinates and the 4-potentials. Thus, all the information regarding the 4-potentials is incorporated into the phase of the spinor.

Furthermore, the factor
\[
g(s_0) \exp \left( -\frac{m^2 \cos^2 \varphi}{k} s_2 \right) \exp (k s_3) \ 
(67)
\]
in Equation (59), can be written in terms of the coordinates \( x, y, z, t \)
\[
g(t - x \cos \varphi - z \sin \varphi) \exp \left( -\frac{m^2 \cos^2 \varphi + k^2}{2k} (x \tan \varphi + z) \right) \ 
\times \exp \left( -\frac{m^2 \cos^2 \varphi - k^2}{2k} y \sec \varphi \right) \ 
(68)
\]
Consequently, the spinor given by Equation (59) tends to infinity, as \( x, y, z \) tend to infinity. To overcome this problem, we must use an appropriate form for the function \( g(t - x \cos \varphi - z \sin \varphi) \), e.g.,
\[
g(t - x \cos \varphi - z \sin \varphi) = c_1 \exp \left( -k_0 (t - x \cos \varphi - z \sin \varphi)^2 \right) \times \exp (-ik t (t - x \cos \varphi - z \sin \varphi)) \ 
(69)
\]
where \( c_1 \) is an arbitrary complex constant, \( k_0 \) is an arbitrary real and positive constant, and \( k_1 \) is an arbitrary real constant.

In the above expression we have also introduced the factor \( \exp (-ik t (t - x \cos \varphi - z \sin \varphi)) \) to ensure the wave-nature of the spinor.
As an example, we consider the special case that \( f_1(t_0, s_i) = k_1 t_0, f_2(t_0, s_i) = k_1 t_0, \) where \( k_1, k_2, k_3 \) are arbitrary real constants. Then, the 4-potentials given by Equations (49)–(52) take the following form

\[
\begin{align*}
  a_0 &= h \\
  a_1 &= -h \cos \varphi - (k_1 + k_2 \sin \varphi) (x - t \sec \varphi + z \tan \varphi) \\
  a_2 &= -k_1 (x + z \tan \varphi - t \sec \varphi) \\
  a_3 &= -h \sin \varphi + k_2 (x \cos \varphi + z \sin \varphi - t)
\end{align*}
\]

According to Equation (57), the electromagnetic fields corresponding to these 4-potentials are given by the formulas

\[
\begin{align*}
  E &= \frac{1}{q} \left( k_1 \sec \varphi + k_2 \tan \varphi - \cos \varphi \frac{\partial h}{\partial t} - \frac{\partial h}{\partial x} \right) i \\
  &\quad + \frac{1}{q} \left( k_1 \sec \varphi + \frac{\partial h}{\partial y} \right) j - \frac{1}{q} \left( k_2 + \sin \varphi \frac{\partial h}{\partial t} + \frac{\partial h}{\partial z} \right) k \\
  B &= \frac{1}{q} \left( k_2 \sec \varphi + k_1 \tan \varphi + \cos \varphi \frac{\partial h}{\partial x} - \sin \varphi \frac{\partial h}{\partial y} \right) j \\
  &\quad + \frac{1}{q} \left( -k_1 \tan \varphi + \sin \varphi \frac{\partial h}{\partial y} \right) i + \frac{1}{q} \left( k_1 - \cos \varphi \frac{\partial h}{\partial t} + \frac{\partial h}{\partial z} \right) k
\end{align*}
\]

Furthermore, according to Equation (59), the spinor that is solution to the Dirac equation for the above 4-potentials, takes the following form

\[
\Psi = c \exp \left( i \varphi \left( (k_1 + k_2 \sin \varphi) x + k_2 y \right) \right) \exp \left( -i k_1 \left( t - x \cos \varphi - y \sin \varphi \right) \right)
\]

\[
\times \exp \left( -m_i^2 \cos^2 \varphi + \frac{k_i^2}{2} \left( x \tan \varphi + z \right) \right) \exp \left( -i m_i^2 \cos^2 \varphi - \frac{k_i^2}{2} \right) y \sec \varphi
\]

\[
\times \left( \frac{\cos \varphi}{1 - \sin \varphi} \left( \begin{array}{cc}
  \cos \varphi & -\cos \varphi \\
  1 - \sin \varphi & 1 + \sin \varphi
\end{array} \right) + \frac{-\cos \varphi}{1 - \sin \varphi} \left( \begin{array}{cc}
  \cos \varphi & \cos \varphi \\
  1 - \sin \varphi & -1 + \sin \varphi
\end{array} \right) \right)
\]

where we have also considered the coordinate transformation given by Equation (53).

Thus, particles described by the degenerate spinor given by Equation (76) can exist in the same quantum state in the wide variety of electromagnetic fields described by Equations (74) and (75).

An important remark is that particles described by the above spinors are localized, both in space and time, since \( \lim_{t_0 \to \pm \infty} \Psi = 0 \). Equivalently, the family of degenerate solutions for massive Dirac particles presented in this article, describes particles in localized states. As it will be shown in the following section, this is no longer required in the cases of massless Dirac and Weyl particles.

Another characteristic of these solutions is that the expected values of the projections of the spin of the particles along the \( x, y, \) and \( z \) axes, as calculated by the following formulas [8, 9]

\[
S_x = \frac{i}{2} \Psi \gamma^x \Psi = \frac{1}{2} \cos \varphi \left( \frac{m^2 (1 + \cos 2\varphi) - 2k^2}{m^2 (1 + \cos 2\varphi) + 2k^2} \right) \left| \Psi \right|^2
\]

\[
S_y = \frac{i}{2} \Psi \gamma^y \Psi = 0
\]

\[
S_z = \frac{i}{2} \Psi \gamma^z \Psi = \frac{1}{2} \sin \varphi \left( \frac{m^2 (1 + \cos 2\varphi) - 2k^2}{m^2 (1 + \cos 2\varphi) + 2k^2} \right) \left| \Psi \right|^2
\]

are functions of the mass of the particles and the modulus of the spinor \( \left| \Psi \right|^2 = \Psi \Psi^{\dagger} \), which is also a function of the mass of the particles and the spatial and temporal coordinates. However, setting \( k = m \cos \varphi \), causes the expected values of the projections of the spin of the particles along the \( x, y, \) and \( z \) axes to become all equal to zero.

### 3. Extension of the Method to Massless Dirac and Weyl Particles

In the special case that the mass of the particles becomes zero, it can be easily verified through Equations (19), (20), (23), (24), (33), and (34) that a degenerate solution to the massless Dirac equation for the real 4-potentials given by Equations (49)–(51) is as follows

\[
\Psi = \exp \left( i \int \mathbf{f}_1 (s_0, s_i) ds_1 + \mathbf{f}_2 (s_0) (s_2 + s_3) + \mathbf{f}_3 (s_0) (s_2 - s_3) \right)
\]

\[
\times \left( \begin{array}{cc}
  W_\gamma (s_0, s_i) & \cos \varphi \\
  \cos \varphi & 1 + \sin \varphi \\
  \cos \varphi & -1 - \sin \varphi \\
\end{array} \right)
\]

\[
\begin{aligned}
  W_\gamma (s_0, s_i) &= \begin{cases}
    1 & \text{if } s_0 \text{ is even} \\
    0 & \text{if } s_0 \text{ is odd}
  \end{cases} \\
  W_\gamma (s_0, s_i) &= \begin{cases}
    1 & \text{if } s_0 \text{ is even} \\
    0 & \text{if } s_0 \text{ is odd}
  \end{cases}
\end{aligned}
\]

where \( W_\gamma (s_0, s_i) \) and \( W_\gamma (s_0, s_i) \) are arbitrary complex functions of the coordinates \( s_0, s_1 \) and \( s_0, s_1 \), respectively. Obviously, special care must be taken to ensure that the spinor given by Equa-
tation (80) is bound for all values of the spatial and temporal coordinates. The simplest choice satisfying this condition is setting the functions $\mathcal{W}_T(s_y, s_z)$ and $\mathcal{W}_R(s_y, s_z)$ as follows

$$
\mathcal{W}_T (s_y, s_z) = c_T \exp \left(-i k \left(t - x \cos \varphi - z \sin \varphi \right) \right) 
$$

$$
\mathcal{W}_R (s_y, s_z) = c_R \exp \left(-i k \left(t - x \cos \varphi - z \sin \varphi \right) \right)
$$

where we have used the term $\exp(-ik \left(t - x \cos \varphi - z \sin \varphi \right))$ to ensure the wave-nature of the spinor. Here, $c_T, c_R$ are arbitrary complex constants. As an example, we consider the following spinor

$$
\Psi = \exp \left(i \sec \varphi \left( (k_1 + k_2 \sin \varphi) x + k_2 y - k_2 z \cos \varphi \right) \left( t - x \cos \varphi - z \sin \varphi \right) \right)
$$

$$
\times \exp \left(-i k \left(t - x \cos \varphi - z \sin \varphi \right) \right) \begin{pmatrix}
\cos \varphi & - \cos \varphi \\
\sin \varphi & 1 + \sin \varphi
\end{pmatrix}
$$

$$
\Psi = \exp \left(i \sec \varphi \left( (k_1 + k_2 \sin \varphi) x + k_2 y - k_2 z \cos \varphi \right) \left( t - x \cos \varphi - z \sin \varphi \right) \right)
$$

$$
\times \begin{pmatrix}
\cos \varphi & - \cos \varphi \\
\sin \varphi & 1 + \sin \varphi
\end{pmatrix}
$$

which is degenerate solution to the massless Dirac equation for the real 4-potentials (70)–(73), corresponding to the electromagnetic fields (74) and (75). Thus, a massless Dirac particle described by the above spinor will exist in the same quantum state in the wide variety of electromagnetic fields given by Equations (74) and (75).

An important remark is that, contrary to the case of massive particles, massless particles are free to move in all space and time, there is no need to impose any spatial or temporal restrictions. Additionally, in the case of massless particles, the expected values of the projections of the spin along the $x$, $y$, and $z$ axes become all constants, taking the following values

$$
S_x = \frac{i}{2} \Psi^\dagger \gamma^y \gamma^z \Psi = -2 \cos \varphi \left( |c_{y1}|^2 + |c_{y2}|^2 \right) \sin \varphi
$$

$$
S_y = \frac{i}{2} \Psi^\dagger \gamma^y \gamma^z \Psi = 0
$$

$$
S_z = \frac{i}{2} \Psi^\dagger \gamma^y \gamma^z \Psi = -2 \sin \varphi \left( |c_{z1}|^2 + |c_{z2}|^2 \right) \sin \varphi
$$

Furthermore, it is important to mention that, in the case of $\mathcal{W}_R(s_y, s_z) = 0$ or $\mathcal{W}_T(s_y, s_z) = 0$, the degenerate spinors given by Equation (80) take the form $\Psi = (\psi_T, \psi_R)^T$ or $\Psi = (\psi_R, -\psi_T)^T$, respectively, where

$$
\psi_T = \exp \left(i \int f_{11} \left(s_y, s_z \right) ds_1 + f_{2R} \left(s_y \right) \left(s_2 + s_1 \right) + f_{21} \left(s_y \right) \left(s_2 - s_1 \right) \right)
$$

$$
\times \mathcal{W}_T \left(s_y, s_z \right) \begin{pmatrix}
\cos \varphi & - \cos \varphi \\
\sin \varphi & 1 + \sin \varphi
\end{pmatrix}
$$

and

$$
\psi_R = \exp \left(i \int f_{11} \left(s_y, s_z \right) ds_1 + f_{2R} \left(s_y \right) \left(s_2 + s_1 \right) + f_{21} \left(s_y \right) \left(s_2 - s_1 \right) \right)
$$

$$
\times \mathcal{W}_R \left(s_y, s_z \right) \begin{pmatrix}
- \cos \varphi & \cos \varphi \\
1 + \sin \varphi & \sin \varphi
\end{pmatrix}
$$

According to Theorem 3.1 given by Kechriniotis et al.[11] the spinors $\psi_T$ are solutions to the Weyl equation in the form

$$
i \sigma^\mu \partial_\mu \Psi + a_\mu \sigma^\mu \Psi = 0
$$

$$
\sigma^\mu \partial_\mu \Psi - 2i \sigma^\mu \partial_\mu \Psi + a_\mu \sigma^\mu \Psi - 2a_\mu \sigma^\mu \Psi = 0
$$

corresponding to particles with positive helicity, whereas the spinors $\psi_R$ are solutions to the Weyl equation in the form

$$
\sigma^\mu \partial_\mu \Psi - 2i \sigma^\mu \partial_\mu \Psi + a_\mu \sigma^\mu \Psi - 2a_\mu \sigma^\mu \Psi = 0
$$

are solutions to the Weyl equation for particles with positive and negative helicity, respectively. In both cases, the 4-potentials corresponding to these solutions are given by Equations (70)–(73).

An interesting remark is that the phase factor

$$
\exp \left(i \int f_{11} \left(s_y, s_z \right) ds_1 + f_{2R} \left(s_y \right) \left(s_2 + s_1 \right) + f_{21} \left(s_y \right) \left(s_2 - s_1 \right) \right)
$$

containing the information regarding the electromagnetic 4-potentials is the same in all cases, namely, for massive Dirac particles, massless Dirac particles, and Weyl particles.
Furthermore, according to Theorem 3.1,[31] all Weyl spinors are degenerate, corresponding to the 4-potentials
\[ b_{\mu} = a_{\mu} + sk_{\mu} \]
where
\[ (\kappa_{01}, \kappa_{11}, \kappa_{21}, \kappa_{31}) = \left( 1, -\frac{\psi^-_{\alpha} \sigma^1 \psi^+}{\psi^+_{\alpha} \psi^-}, -\frac{\psi^+_{\alpha} \sigma^1 \psi^-}{\psi^-_{\alpha} \psi^+}, -\frac{\psi^-_{\alpha} \sigma^1 \psi^+}{\psi^+_{\alpha} \psi^-} \right) \]
\[ = (1, -\cos \varphi, 0, -\sin \varphi) \]
in the case of particles with positive helicity, and
\[ b_{\mu} = a_{\mu} + sk_{\mu} \]
where
\[ (\kappa_{02}, \kappa_{12}, \kappa_{22}, \kappa_{32}) = \left( 1, \frac{\psi_{\alpha} \sigma^1 \psi_R}{\psi_R}, \frac{\psi_R \sigma^1 \psi_{\alpha}}{\psi_R}, -\frac{\psi_{\alpha} \sigma^1 \psi_R}{\psi_R} \right) \]
\[ = (1, -\cos \varphi, 0, -\sin \varphi) \]
in the case of particles with negative helicity. It is evident that, in both cases, the 4-potentials, and consequently the electromagnetic fields, are the same to those corresponding to Dirac particles.

Thus, for any combination of the arbitrary real functions \( f_1(s_0), f_2(s_1), f_3(s_2) \), one can automatically construct spinors which are degenerate solutions to the massive Dirac, massless Dirac and Weyl equations, for the infinite number of 4-potentials given by Equations (49)–(52).

4. On the Experimental Detection of Degenerate States

In the special case that the function \( h \) depends only on time and \( k_i = k_i = k_i = 0 \), the electromagnetic fields of Equations (74) and (75) are simplified, taking the following form
\[ E = -\frac{d}{dt} (\cos \varphi \mathbf{i} + \sin \varphi \mathbf{k}) \cdot \mathbf{B} = 0 \]

Additionally, in the above case, the spinors defined by Equations (83), (91), and (92) describe massless Dirac or Weyl particles moving parallel to the vector \( \cos \varphi \mathbf{i} + \sin \varphi \mathbf{k} \). Consequently, the state of the particles will not be affected by the presence of an electric field of arbitrary time dependence, applied along their direction of motion. This practically means that the electric current transferred by charged massless Dirac or Weyl particles in degenerate states, which always move at a critical velocity \( c \), equal to one in natural units, will not change if a voltage of arbitrary magnitude and time dependence, is applied along the direction of motion of the particles. Obviously, this is not the case for “ordinary” charged particles, where according to Ohm’s law, the electric current is proportional to the applied voltage. Consequently, the dependence of the electric current on the applied voltage is completely different for particles in degenerate states and “ordinary” charged particles, enabling the experimental detection of degenerate states in materials supporting massless Dirac or Weyl particles, such as graphene sheets and Weyl semimetals.[10–15]

Furthermore, the state of Weyl and massless Dirac particles described by degenerate spinors, will not be affected by the presence of a plane electromagnetic wave, e.g., a laser beam of arbitrary polarization, propagating along the direction of motion of the particles.[3] Thus, particles in degenerate states, as well as electromagnetic waves can propagate along the same direction without interacting with each other, which obviously is not the case for charged particles in nondegenerate states.

This result can be used to detect the presence of degenerate states through an interferometric method. In more detail, if we place a material where charged particles are in degenerate states in one arm of a Michelson interferometer, the electromagnetic wave propagating through this arm will behave as if it was propagating in vacuum. On the other hand, if the particles are not in degenerate states, they will interact with the electromagnetic wave affecting its velocity and consequently its phase. Therefore, the transition from nondegenerate to degenerate states and vice versa could be easily detected through the changes in the interference pattern produced by the interferometer, as shown in Figure 1.

It should be noted that not all particles are expected to move parallel to the electromagnetic wave and consequently some of them will interact with the wave, even in the degenerate states. However, interferometric experiments are so sensitive, that even if a small fraction of the particles stops interacting with the electromagnetic wave, it should be sufficient to induce a measurable change in the diffraction pattern.

In more detail, the shift in the interference fringes is equal to
\[ \text{Shift} = \frac{\Delta d}{\lambda} = \frac{\Delta n}{\alpha} \]

where \( \Delta d \) and \( \Delta n \) are the changes in the optical path and the refractive index, respectively, due to the transition from nondegenerate to generate states and vice versa, \( L \) is the physical length of the material containing particles in degenerate states and \( \lambda \) is the wavelength of the laser radiation in vacuum. Assuming that the experimental setup is able to resolve fringe shifts equal to 0.01, and using typical values for \( L, \lambda \), e.g., \( L = 1 \text{cm} \) and \( \lambda = 500 \text{nm} \), the above equation implies that the Michelson interferometer can detect changes in the refractive index of the order of \( 5 \times 10^{-7} \). Consequently, even if a tiny fraction of the particles stops interacting with the electromagnetic wave, it should be sufficient to induce a detectable change in the diffraction pattern.

Thus, the techniques described above can be used to detect the presence of degenerate states, as well as the transition between degenerate and nondegenerate states. In addition, these methods can be used for determining if quantum states corresponding to massless Dirac or Weyl particles are excited in a material. More details on these techniques, as well as the potential applications of our theory in various fields of physics involving the interaction of charged particles with electromagnetic fields, will be provided in future works.

5. Conclusions

We have provided a general method for obtaining degenerate solutions to the Dirac equation, corresponding to an infinite num-
A proposed method for experimentally detecting the transition between degenerate and nondegenerate states using a Michelson interferometer.

The number of electromagnetic 4-potentials and fields, which are also explicitly calculated. The electromagnetic 4-potentials are constructed using four arbitrary real functions, three of which appear in the phase of the degenerate spinors, corresponding to these 4-potentials. Furthermore, the method is extended to the cases of massless Dirac and Weyl particles, where the information regarding the 4-potentials is also encoded in the phase of the spinors. An additional, important remark is that, in the case of massive Dirac particles, the degenerate spinors describe localized states, which is no longer required for massless Dirac and Weyl particles. Finally, we describe two experimental methods that could be used for detecting the presence of degenerate states, as well as the transition between degenerate and nondegenerate states.

Appendix A

We suppose that $A = (a_{ij}) \in \text{Mat}_n(C)$ is an invertible $n \times n$ matrix. We also define the linear differential operators $D_i, i = 1, \ldots, n$ as

$$
\begin{pmatrix}
D_1 \\
\vdots \\
D_n
\end{pmatrix}
= A^T
\begin{pmatrix}
\partial_1 \\
\vdots \\
\partial_n
\end{pmatrix}
$$

where $\partial_i = \frac{\partial}{\partial x_i}$. Then, using the following linear transformation of the coordinates

$$
\begin{pmatrix}
k_1 \\
\vdots \\
k_n
\end{pmatrix}
= A
\begin{pmatrix}
s_1 \\
\vdots \\
s_n
\end{pmatrix}
$$

we can verify that

$$D_i = \partial_i, i = 1, \ldots, n$$

where $\partial_i = \frac{\partial}{\partial n_i}$. Indeed, defining the matrix

$$B = \left( b_{ij} \right) = A^{-1}$$

and using Equation (A2) we obtain that

$$s_i = \sum_{k=1}^n b_{ik} x_k, i = 1, \ldots, n$$

Then, using Equations (A1), (A4), and (A5), it is easy to verify that, for any function $U(x_1, \ldots, x_n) = \tilde{U}(s_1, \ldots, s_n)$, the following is true

$$
\begin{pmatrix}
D_1 \\
\vdots \\
D_n
\end{pmatrix}
U = A^T
\begin{pmatrix}
\partial_1 U \\
\vdots \\
\partial_n U
\end{pmatrix}
= A^T
\begin{pmatrix}
\sum_{i=1}^n \partial_i \tilde{U} \partial_i s_i \\
\vdots \\
\sum_{i=1}^n \partial_i \tilde{U} \partial_i s_i
\end{pmatrix}

= A^T B^T
\begin{pmatrix}
\partial_1 U \\
\vdots \\
\partial_n U
\end{pmatrix}
= \tilde{U}
$$

Appendix B

We consider the differential equation

$$\tilde{\partial}_2 \tilde{\partial}_3 \tilde{W} = -m^2 \cos^2 \varphi \tilde{W}$$

$$\tilde{\partial}_2 \tilde{\partial}_3$$
where \( \tilde{W} \) is an arbitrary complex function of \( s_0, s_2, s_3 \). Assuming that \( \tilde{W} \) can be written in the form

\[
\tilde{W}(s_0, s_2, s_3) = \tilde{W}_2(s_0, s_2) \tilde{W}_3(s_0, s_3)
\]  \hspace{1cm} (B2)

the differential equation (B1) takes the form

\[
\frac{\partial_3 \tilde{W}_2 \partial_2 \tilde{W}_3}{\tilde{W}_3} = -m^2 \cos^2 \varphi \tilde{W}_2 \tilde{W}_3
\] \hspace{1cm} (B3)

Under the condition that

\[
\frac{\partial_3 \tilde{W}_3}{\tilde{W}_3} = k(s_0)
\] \hspace{1cm} (B4)

the solution of Equation (B3) for \( \tilde{W}_2 \) is

\[
\tilde{W}_2(s_0, s_2) = g_2(s_0) \exp \left( -\frac{m^2 \cos^2 \varphi}{k(s_0)} s_2 \right)
\] \hspace{1cm} (B5)

where \( g_2(s_0) \), \( k(s_0) \not= 0 \) are arbitrary complex functions of \( s_0 \). Additionally, the solution of Equation (B4) for \( \tilde{W}_3 \) is

\[
\tilde{W}_3(s_0, s_3) = g_3(s_0) \exp \left( k(s_0) s_3 \right)
\] \hspace{1cm} (B6)

where \( g_3(s_0) \) is an arbitrary complex function of \( s_0 \).

Thus, the solution of Equation (B1) for \( \tilde{W} \) can be expressed as

\[
\tilde{W}(s_0, s_2, s_3) = g(s_0) \exp \left( -\frac{m^2 \cos^2 \varphi}{k(s_0)} s_2 + k(s_0) s_3 \right)
\] \hspace{1cm} (B7)

where \( g(s_0) = g_2(s_0)g_3(s_0) \) is an arbitrary complex function of \( s_0 \).

**Conflict of Interest**

The authors declare no conflict of interest.

**Data Availability Statement**

The data that support the findings of this study are available from the corresponding author upon reasonable request.