QUADRATIC TRANSFORMATIONS
AND GUILLERA’S FORMULAE FOR $1/\pi^2$

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Abstract. We prove two new series of Ramanujan type for $1/\pi^2$.

In a series of works [3]–[5], J. Guillera discovered several Ramanujan-type formulae for $1/\pi^2$, some of which he proved. For instance, the following two identities are proven in [3], [5]:

\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})^n}{n!^5} (20n^2 + 8n + 1) \left( \frac{1}{2^2} \right)^n = \frac{8}{\pi^2};
\]

\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})^n}{n!^5} (820n^2 + 180n + 13) \left( -\frac{1}{2^{10}} \right)^n = \frac{128}{\pi^2},
\]

where $(a)_n = \Gamma(a+n)/\Gamma(a) = \prod_{k=0}^{n-1}(a+k)$ denotes Pochhammer’s symbol. It happens that, in all Guillera’s formulae expressions, the left-hand sides are given by certain hypergeometric $5F4$-series, with the sole exception [6]

\[
\sum_{n=0}^{\infty} A_n \frac{36n^2 + 12n + 1}{2^{10n}} = \frac{32}{\pi^2}, \quad \text{where } A_n = \left( \frac{2n}{n} \right)^2 \sum_{k=0}^{n} \frac{2k}{k} \left( \frac{2n-2k}{n-k} \right)^2.
\]

(this formula remains unproven). Recall that a generalized hypergeometric series [7] is defined for $z \in \mathbb{C}, |z| < 1$, by

\[
_{q+1}F_q \left( a_0, a_1, \ldots, a_q \left| b_1, \ldots, b_q \right| z \right) = \sum_{n=0}^{\infty} \frac{(a_0)_n(a_1)_n \cdots (a_q)_n}{n!(b_1)_n \cdots (b_q)_n} z^n.
\]

Formula (3) looks a natural extension of Sato’s formula for $1/\pi$ involving Apéry’s numbers (see [2] and [2]) as well as of several other examples like

\[
\sum_{n=0}^{\infty} B_n \frac{4n + 1}{36^n} = \frac{18}{\pi \sqrt{15}}, \quad \text{where } B_n = \sum_{k=0}^{n} \frac{n^4}{k^4},
\]

(proven by Y. Yang [10]).

The aim of this note is to derive two new identities of type (3) from (1), (2) using hypergeometric technique.

1. Quadratic Transformations

Recall quadratic transformations for $2F_1$-series,

\[
_{2}F_1 \left( a, b \left| 1 + a - b \right| z \right) = (1 - z)^{-a} \cdot _{2}F_1 \left( \frac{1}{2}a, \frac{1}{2} + \frac{1}{2}a - b \left| 1 + a - b \right| -\frac{4z}{(1 - z)^2} \right)
\]

due to Gauss, and for $3F_2$-series,

\[
_{3}F_2 \left( a, b, c \left| 1 + a - b, 1 + a - c \right| z \right) = (1 - z)^{-a} \cdot _{3}F_2 \left( \frac{1}{2}a, \frac{1}{2} + \frac{1}{2}a, 1 + a - b - c \left| 1 + a - b, 1 + a - c \right| -\frac{4z}{(1 - z)^2} \right)
\]

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due to Whipple. Their higher-order analogues necessarily involve multiple hypergeometric series in the right-hand side (cf., e.g., [1], Proposition 6, and [4], Theorem 5). Here we indicate the following result.

**Theorem 1.** The following quadratic transformation is valid:

\[
5F_4\left(\begin{array}{c}
a, b, c, d, e \\
1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e
\end{array} \mid z \right) = (1 - z)^{-a} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}a)_n(\frac{1}{2} + \frac{1}{2}a)_n}{(1 + a - b)_n(1 + a - c)_n} \frac{(-4z)^n}{(1 - z)^2} \times \sum_{\nu=0}^{n} \frac{(b)_\nu(c)_\nu(1 + a - d - e)_\nu}{\nu!(1 + a - d)_\nu(1 + a - e)_\nu} (1 + a - b - c)_{n-\nu},
\]

whenever both series converge.

**Remark.** Theorem [4] may be stated in the form of Orr-type theorem (cf. [7], Section 2.5): If

\[
(1 - z)^{b+c-a-1} \cdot 3F_2\left(\begin{array}{c}
b, c, 1 + a - d - e \\
1 + a - d, 1 + a - e
\end{array} \mid z \right) = \sum_{n=0}^{\infty} f_n z^n,
\]

then

\[
5F_4\left(\begin{array}{c}
a, b, c, d, e \\
1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e
\end{array} \mid z \right) = (1 - z)^{-a} \sum_{n=0}^{\infty} f_n \frac{(\frac{1}{2}a)_n(\frac{1}{2} + \frac{1}{2}a)_n}{(1 + a - b)_n(1 + a - c)_n} \frac{(-4z)^n}{(1 - z)^2} \times \sum_{\nu=0}^{n} \frac{(b)_\nu(c)_\nu(1 + a - d - e)_\nu}{\nu!(1 + a - d)_\nu(1 + a - e)_\nu} (1 + a - b - c)_{n-\nu},
\]

It follows from [4] that \(|f_n|^{1/n} \rightarrow 1\) as \(n \rightarrow \infty\), hence the condition \(|4z/(1 - z)^2| < 1\) ensures convergence of the double series in [4].

**Proof.** It is sufficient to prove the identity in a neighbourhood of the origin. Using the Pfaff–Saalschitz theorem [7], formula (2.3.1.3),

\[
3F_2\left(\begin{array}{c}
-n, a + n, 1 + a - d - e \\
1 + a - d, 1 + a - e
\end{array} \mid 1 \right) = \frac{(-d - n + 1)n(e)_n}{(1 + a - d)_n(e - a - n)_n} = \frac{(d)_n(e)_n}{(1 + a - d)_n(1 + a - e)_n},
\]

write

\[
5F_4\left(\begin{array}{c}
a, b, c, d, e \\
1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e
\end{array} \mid z \right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{n!(1 + a - b)_n(1 + a - c)_n} z^n \cdot 3F_2\left(\begin{array}{c}
-n, a + n, 1 + a - d - e \\
1 + a - d, 1 + a - e
\end{array} \mid 1 \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{n!(1 + a - b)_n(1 + a - c)_n} z^n \sum_{\nu=0}^{n} \frac{(-n)_\nu(a + n)_\nu(1 + a - d - e)_\nu}{\nu!(1 + a - d)_\nu(1 + a - e)_\nu} \times \sum_{m=0}^{\infty} \frac{(a)_{n+\nu}(b)_m(c)_m}{m!(1 + a - b)_m(1 + a - c)_m} z^m
\]

\[
= \sum_{\nu=0}^{\infty} \frac{(1 + a - d - e)_\nu(-1)^\nu}{\nu!(1 + a - d)_\nu(1 + a - e)_\nu} \sum_{m=0}^{\infty} \frac{(a)_{m+\nu}(b)_m(c)_m}{m!(1 + a - b)_m(1 + a - c)_m} z^{m+\nu}
\]
Applying the quadratic transformation \[4\] to the latter
which is the required formula \[5\].

\[\text{Theorem 2.}\]

We summarize saying above in satisfied by both expressions in \[7\]. (Proof of \[8\] uses the algorithm of creative telescoping \[8\],
followed from \[7\], formula \(2.5.18\). Another way to prove \[7\] is based on the recurrence relation

\[
n \quad \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} n!^5 z^n = \frac{1}{(1-z)^{1/2}} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} n!^2 \left(\frac{-4z}{(1-z)^2}\right)^n \sum_{\nu=0}^{n} \binom{\frac{1}{2}}{\nu} \binom{\frac{1}{2}}{n-\nu} \frac{n!^3 (\frac{1}{2})^{n-\nu}}{(n-\nu)!},
\]

which is the required formula \[5\].

Plugging in \(a = b = c = d = e = \frac{1}{2}\) we obtain

\[
\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} n!^5 z^n = \frac{1}{(1-z)^{1/2}} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} z^n \left(\frac{-4z}{(1-z)^2}\right)^n \sum_{\nu=0}^{n} \binom{\frac{1}{2}}{\nu} \binom{\frac{1}{2}}{n-\nu} \frac{n!^3 (\frac{1}{2})^{n-\nu}}{(n-\nu)!},
\]

Note the equality

\[
u_n = \sum_{\nu=0}^{n} \binom{\frac{1}{2}}{\nu} \binom{\frac{1}{2}}{n-\nu} \frac{n!^3 (\frac{1}{2})^{n-\nu}}{(n-\nu)!} = \sum_{\nu=0}^{n} \binom{\frac{1}{2}}{\nu} \binom{\frac{1}{2}}{n-\nu} \frac{n!^3 (\frac{1}{2})^{n-\nu}}{(n-\nu)!},
\]

followed from \[7\], formula \(2.5.18\). Another way to prove \[7\] is based on the recurrence relation

\[
8(n+1)^3 u_{n+1} - (2n+1)(8n^2 + 8n + 5)u_n + 8n^3 u_{n-1} = 0
\]
satisfied by both expressions in \[7\]. (Proof of \[8\] uses the algorithm of creative telescoping \[8\],
Chapter 6.) We summarize saying above in

\[\textbf{Theorem 2.} \text{ Suppose } |z| < 1 \text{ and } |4z/(1-z)^2| < 1. \text{ The following identity holds:}
\]

\[
\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} n!^5 z^n = \frac{1}{(1-z)^{1/2}} \sum_{n=0}^{\infty} u_n \binom{\frac{1}{2}}{n} n!^2 \left(\frac{-4z}{(1-z)^2}\right)^n,
\]

where \(u_n\) are given in \[7\].
Remark. It is worth mentioning the following curious hypergeometric identity:

\[
\sum_{n=0}^{\infty} u_n \frac{(\frac{1}{3})_n (\frac{2}{3})_n}{n!^2} z^n = 3F_2 \left( \begin{array}{c} \frac{1}{6}, \frac{1}{2}, \frac{5}{6} \\ 1, 1, 1 \end{array} \right) z^2 = 2F_1 \left( \begin{array}{c} \frac{1}{12}, \frac{5}{12} \\ 1 \end{array} \right) z^4,
\]

(10)

since the double hypergeometric series in the left-hand side is very close to the one used in (9). Formula (10) was born in our correspondence with G. Almkvist and Y. Yang.

2. NEW FORMULAE FOR 1/\pi^2

For the differential operator \( \theta = \frac{d}{dz} \), we have

\[
\theta(1-z)^{-1/2} = \frac{z}{2(1-z)} \cdot (1-z)^{-1/2} \quad \text{and} \quad \theta \left( \frac{-4z}{(1-z)^2} \right) = \frac{1+z}{1-z} \cdot \left( \frac{-4z}{(1-z)^2} \right).
\]

Therefore,

\[
\theta \left( (1-z)^{-1/2} \sum_{n=0}^{\infty} C_n \left( \frac{-4z}{(1-z)^2} \right)^n \right) = (1-z)^{-1/2} \sum_{n=0}^{\infty} C_n \cdot \left( \frac{1+z}{1-z} + \frac{z}{2(1-z)} \right) \cdot \left( \frac{-4z}{(1-z)^2} \right)^n,
\]

\[
\theta^2 \left( (1-z)^{-1/2} \sum_{n=0}^{\infty} C_n \left( \frac{-4z}{(1-z)^2} \right)^n \right) = (1-z)^{-1/2} \sum_{n=0}^{\infty} C_n \cdot \left( n^2 \frac{(1+z)^2}{(1-z)^2} + n \frac{z(3+z)}{1-z} + \frac{z(2+z)}{4(1-z)^2} \right) \cdot \left( \frac{-4z}{(1-z)^2} \right)^n.
\]

We now apply the results to the functions

\[
200\theta^2 f(z) + 80f(z) + f(z) \quad \text{and} \quad 820\theta^2 f(z) + 180\theta f(z) + 13f(z),
\]

where \( f(z) \) is given in (4), and substitute \( z = -1/2^2 \) and \( z = -1/2^{10} \), respectively. Using then evaluations (11, 2) and formulae

\[
\frac{(\frac{1}{3})_n (\frac{2}{3})_n}{n!^2} = 2^{-6n} \frac{(4n)!}{n!^2(2n)!}, \quad u_n = \sum_{k=0}^{n} \frac{(\frac{1}{3})_k (\frac{2}{3})_n-k}{k!^3 (n-k)!} = 2^{-6n} \sum_{k=0}^{n} \binom{2k}{k}^3 \binom{2n-2k}{n-k} 2^{4(n-k)},
\]

we arrive at

Theorem 3. The following identities are valid:

\[
\sum_{n=0}^{\infty} U_n \frac{(4n)!}{n!^2(2n)!} = \frac{18n^2 - 10n - 3}{(2^8 5^2)^n} = \frac{10\sqrt{5}}{\pi^2},
\]

\[
\sum_{n=0}^{\infty} U_n \frac{(4n)!}{n!^2(2n)!} = \frac{1046529n^2 + 227104n + 16032}{(5^4 41^2)^n} = \frac{5^4 41\sqrt{41}}{\pi^2},
\]

where the sequence of integers

\[
U_n = \sum_{k=0}^{n} \binom{2k}{k}^3 \binom{2n-2k}{n-k} 2^{4(n-k)}, \quad n = 0, 1, 2, \ldots,
\]

satisfies the recurrence relation

\[
(n+1)^3 U_{n+1} - 8(2n+1)(8n^2 + 8n + 5)U_n + 4096n^3 U_{n-1} = 0, \quad n = 1, 2, \ldots.
\]
It seems quite likely that the hypergeometric machinery could lead to several other formulae for $1/\pi^2$.

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