2-CAPABILITY AND 2-NILPOTENT MULTIPLIER OF FINITE DIMENSIONAL NILPOTENT LIE ALGEBRAS

PEYMAN NIROOMAND AND MOHSEN PARVIZI

ABSTRACT. In the present context, we investigate to obtain some more results about 2-nilpotent multiplier \( M^{(2)}(L) \) of a finite dimensional nilpotent Lie algebra \( L \). For instance, we characterize the structure of \( M^{(2)}(H) \) when \( H \) is a Heisenberg Lie algebra. Moreover, we give some inequalities on \( \dim M^{(2)}(L) \) to reduce a well known upper bound on 2-nilpotent multiplier as much as possible. Finally, we show that \( H(m) \) is 2-capable if and only if \( m=1 \).

1. Introduction

For a finite group \( G \), let \( G \) be the quotient of a free group \( F \) by a normal subgroup \( R \), then the \( c \)-nilpotent multiplier \( M^{(c)}(G) \) is defined as

\[
R \cap \gamma_{c+1}(F)/\gamma_{c+1}[R, F],
\]

in which \( \gamma_{c+1}[R, F] = [\gamma_c[R, F], F] \) for \( c \geq 1 \). It is an especial case of the Baer invariant \( [2] \) with respect to the variety of nilpotent groups of class at most \( c \). When \( c = 1 \), the abelian group \( M(G) = M^{(1)}(G) \) is more known as the Schur multiplier of \( G \) and it is much more studied, for instance in [11, 14, 18].

Since determining the \( c \)-nilpotent multiplier of groups can be used for the classification of group into isoclinism classes(see [2]), there are multiple papers concerning this subject.

Recently, several authors investigated to develop some results on the group theory case to Lie algebra. In [22], analogues to the \( c \)-nilpotent multiplier of groups, for a given Lie algebra \( L \), the \( c \)-nilpotent multiplier of \( L \) is defined as

\[
M^{(c)}(L) = R \cap F^{c+1}/[R, F]^{c+1},
\]

in which \( L \) presented as the quotient of a free Lie algebra \( F \) by an ideal \( R \), \( F^{c+1} = \gamma_{c+1}(F) \) and \( [R, F]^{c+1} = \gamma_{c+1}[R, F] \). Similarly, for the case \( c = 1 \), the abelian Lie algebra \( M(L) = M^{(1)}(L) \) is more studied by the first author and the others (see for instance [4, 5, 6, 8, 9, 10, 15, 16, 17, 24]).

The \( c \)-nilpotent multiplier of a finite dimensional nilpotent Lie algebra \( L \) is a new field of interest in literature. The present context is involving the 2-nilpotent multiplier of a finite dimensional nilpotent Lie algebra \( L \). The aim of the current paper is divided into several steps. In [22, Corollary 2.8], by a parallel result to the group theory result, showed for every finite nilpotent Lie algebra \( L \), we have

\[
(1.1) \quad \dim(M^{(2)}(L)) + \dim(L^3) \leq \frac{1}{3}n(n-1)(n+1).
\]

Date: March 26, 2021.

Key words and phrases. 2-nilpotent multiplier; Schur multiplier; Heisenberg algebras; derived subalgebra; 2-capable Lie algebra.

Mathematics Subject Classification 2010. Primary 17B30 Secondary 17B60, 17B99.
Here we prove that abelian Lie algebras just attain the bound 1. It shows that always $\ker \theta = 0$ in [22, Corollary 2.8 (ii)a].

Since Heisenberg algebras $H(m)$ (a Lie algebra of dimension $2m + 1$ with $L^2 = Z(L)$ and $\dim (L^2) = 1$) have interest in several areas of Lie algebra, similar to the result of [21, Example 3] and [13, Theorem 24], by a quite different way, we give explicit structure of 2-nilpotent multiplier of these algebras. Among the other results since the Lie algebra which attained the upper bound 1 completely described in Lemma 2.6 (they are just abelian Lie algebras), by obtaining some new inequalities on dimension $\mathcal{M}(2)(L)$, we reduce bound 1 for non abelian Lie algebras as much as possible.

Finally, among the class of Heisenberg algebras, we show that which of them is 2-capable. It means which of them is isomorphic to $H/Z_2(H)$ for a Lie algebra $H$. For more information about the capability of Lie algebras see [18, 21]. These generalized the recently results for the group theory case in [19].

2. Further investigation on 2-nilpotent multiplier of finite dimensional nilpotent Lie algebra

The present section illustrates to obtain further results on 2-nilpotent multiplier of finite dimensional nilpotent Lie algebra. At first we give basic definitions and known results for the seek of convenience the reader.

Let $F$ be a free Lie algebra on an arbitrary totaly ordered set $X$. Recall from [25], the basic commutator on the set $X$, which is defined as follows, is a basis of $F$.

The elements of $X$ are basic commutators of length one and ordered relative to the total order previously chosen. Suppose all the basic commutators $a_i$ of length less than $k \geq 1$ have been defined and ordered. Then the basic commutators of length $k$ to be defined as all commutators of the form $[a_i, a_j]$ such that the sum of lengths of $a_i$ and $a_j$ is $k$, $a_i > a_j$, and if $a_i = [a_s, a_t]$, then $a_j \geq a_t$. Also the number of basic commutators on $X$ of length $n$, namely $l_d(n)$, is

$$\frac{1}{n} \sum_{m|n} \mu(m)d^\frac{n}{m},$$

where $\mu$ is the Möbius function.

From [8], let $F$ be a fixed field, $L, K$ be two Lie algebras and $[\ , \ ]$ denote the Lie bracket. By an action of $L$ on $K$ we mean an $F$-bilinear map

$$(l, k) \in L \times K \mapsto ^{l}k \in K$$
satisfying

$$\mathcal{[l, l']}k = ^{l'}([l'k]) - ^{l'}(1k) \text{ and } ^{l}[k, k'] = [^{l}k, ^{l}k'] + [k, ^{l}k'],$$

for all $c \in F, l, l' \in L, k, k' \in K$.

When $L$ is a subalgebra of a Lie algebra $P$ and $K$ is an ideal in $P$, then $L$ acts on $K$ by Lie multiplications $^{l}k = [l, k]$. A crossed module is a Lie homomorphism $\sigma : K \rightarrow L$ together with an action of $L$ on $K$ such that

$$\sigma([l, k]) = [l, \sigma(k)] \text{ and } \sigma([k, k']) = [k, k'] \text{ for all } k, k' \in K \text{ and } l \in L.$$

Let $\sigma : L \rightarrow M$ and $\eta : K \rightarrow M$ be two crossed modules, $L$ and $K$ act on each other and on themselves by Lie. Then these actions are called compatible provided that

$$^{k^l}k' = ^{k'}(^{l}k) \text{ and } ^{k^l}l' = ^{l'}(^{k}l).$$
The non-abelian tensor product $L \otimes K$ of $L$ and $K$ is the Lie algebra generated by the symbols $l \otimes k$ with defining relations
\[
e(l \otimes k) = cl \otimes k = l \otimes ck; \quad (l + l') \otimes k = l \otimes k + l' \otimes k,
\]
\[
l \otimes (k + k') = l \otimes k + l \otimes k', \quad l' \otimes k = l \otimes l' \cap k - l' \otimes k, \quad l \otimes k = k' l \otimes k - k' l \otimes k',
\]
\[
[l \otimes k, l' \otimes k'] = -k l \otimes l' k', \quad \text{for all } c \in F, l, l' \in L, k, k' \in K.
\]

The non-abelian tensor square of $L$ is a special case of tensor product $L \otimes K$ when $K = L$. Note that we denote the usual abelian tensor product $L \otimes_Z K$, when $L$ and $K$ are abelian and the actions are trivial.

Let $L \square K$ be the submodule of $L \otimes K$ generated by the elements $l \otimes k$ such that $\sigma(l) = \eta(k)$. The factor Lie algebra $L \wedge K \cong L \otimes K / L \square K$ is called the exterior product of $L$ and $K$, and the image of $l \otimes k$ is denoted by $l \wedge k$ for all $l \in L, k \in K$.

Throughout the paper $\Gamma$ is denoted the universal quadratic functor (see [8]).

Recall from [13], the exterior centre of a Lie algebra $Z^\wedge(L) = \{ l \in L \mid l \wedge l' = 1_{L/L}, \forall l' \in L \}$ of $L$. It is shown that in [13] the exterior centre $L$ is a central ideal of $L$ which allows us to decide when Lie algebra $L$ is capable, that is, whether $L \cong H / Z(H)$ for a Lie algebra $H$.

The following Lemma is a consequence of [13] Lemma 3.1.

**Lemma 2.1.** Let $L$ be a finite dimensional Lie algebra, $L$ is capable if and only if $Z^\wedge(L) = 0$.

The next two lemmas are special cases of [22 Proposition 2.1 (i)] when $c = 2$ and that is useful for proving the next theorem.

**Lemma 2.2.** Let $I$ be an ideal in a Lie algebra $L$. Then the following sequences are exact.

(i) $\text{Ker}(\mu^2_I) \to \mathcal{M}^{(2)}(L) \to \mathcal{M}^{(2)}(L/I) \to \frac{\text{dim } I \cap L^3}{[[I, L], L]} = 0$.

(ii) $(I \wedge L/L^3) \wedge L/L^3 \to \mathcal{M}^{(2)}(L) \to \mathcal{M}^{(2)}(L/I) \to I \cap L^3 \to 0$, when $[[I, L], L] = 0$.

**Lemma 2.3.** Let $I$ be an ideal of $L$, and put $K = L/I$. Then

(i) $\dim \mathcal{M}^{(2)}(K) \leq \dim \mathcal{M}^{(2)}(L) + \frac{\text{dim } I \cap L^3}{[[I, L], L]}$.

(ii) Moreover, if $I$ is a 2-central subalgebra. Then

(a) $(I \wedge L) \wedge L \to \mathcal{M}^{(2)}(L) \to \mathcal{M}^{(2)}(K) \to \text{dim } I \cap L^3 \to 0$.

(b) $\dim \mathcal{M}^{(2)}(L) + \text{dim } I \cap L^3 \leq \dim \mathcal{M}^{(2)}(K) + \text{dim } (I \otimes L/L^3) \otimes L/L^3$.

**Proof.** (i). Using Lemma 2.2 (i).

(ii)(a). Since $[I, L] \subseteq Z(L)$, $\text{Ker } \mu^2_I = (I \wedge L) \wedge L$ and $[[I, L], L] = 0$ by Lemma 2.2. It follows the result.

(ii)(b). Since there is a natural epimorphism $(I \otimes L/L^3) \otimes L/L^3 \to (I \wedge L/L^3) \wedge L/L^3$, the result deduces from Lemma 2.2 (ii).

The following theorem gives the explicit structure of the Schur multiplier of all Heisenberg algebra.

**Theorem 2.4.** [3 Example 3] and [13 Theorem 24] Let $H(m)$ be Heisenberg algebra of dimension $2m + 1$. Then

(i) $\mathcal{M}(H(1)) \cong A(2)$.

(ii) $\mathcal{M}(H(m)) = A(2m^2 - m - 1)$ for all $m \geq 2$. 
The following result comes from [20] Theorem 2.8 and shows the behavior of 2-nilpotent multiplier respect to the direct sum of two Lie algebras.

**Theorem 2.5.** Let $A$ and $B$ be finite dimensional Lie algebras. Then

$$\mathcal{M}^{(2)}(A \oplus B) \cong \mathcal{M}^{(2)}(A) \oplus \mathcal{M}^{(2)}(B) \oplus (A/A^2 \otimes Z A/A^2) \otimes B/B^2 \oplus (B/B^2 \otimes Z B/B^2) \otimes Z A/A^2).$$

The following theorem is proved in [22] and will be used in the next contribution. At this point, we may give a short proof with a quite different way of [22, Proposition 1.2] as follows.

**Theorem 2.6.** Let $L = A(n)$ be an abelian Lie algebra of dimension $L$. Then

$$\mathcal{M}^{(2)}(L) \cong A(\frac{1}{3}n(n-1)(n+1)).$$

**Proof.** We perform induction on $n$. Assume $n = 2$. Then Theorem 2.5 allows us to conclude that

$$\mathcal{M}^{(2)}(L) \cong \mathcal{M}^{(2)}(A(1)) \oplus (A(1) \otimes Z A(1)) \oplus A(1) \oplus A(1) \cong A(2).$$

Now assume that $L \cong A(n) \cong A(n-1) \oplus A(1)$. By using induction hypothesis and Theorem 2.5, we have

$$\mathcal{M}^{(2)}(A(n-1) \oplus A(1)) \cong \mathcal{M}^{(2)}(A(n-1)) \oplus (A(n-1) \otimes Z A(n-1)) \oplus A(1) \oplus A(1) \cong A(2).$$

The main strategy, in the next contribution, is to give the similar argument of Theorem 2.4 for the 2-nilpotent multiplier. In the first theorem, we obtain the structure of $\mathcal{M}^{(2)}(L)$ when $L$ is non-capable Heisenberg algebra.

**Theorem 2.7.** Let $L = H(m)$ be a non-capable Heisenberg algebra. Then

$$\mathcal{M}^{(2)}(H(m)) \cong A(\frac{8m^3 - 2m}{3}).$$

**Proof.** Since $L$ is non-capable, Lemma 2.1 implies $Z^\wedge(L) = L^2 = Z(L)$. Invoking Lemma 2.3 by putting $I = Z^\wedge(L)$, we have $\mathcal{M}^{(2)}(H(m)) \cong \mathcal{M}^{(2)}(H(m)/H(m)^2)$. Now result follows from Theorem 2.6. □

The following theorem from [18] Theorem 3.4] shows in the class of all Heisenberg algebras which one is capable.

**Theorem 2.8.** $H(m)$ is capable if and only if $m = 1$.

**Corollary 2.9.** $H(m)$ is not 2-capable for all $m \geq 2$.

**Proof.** Since every 2-capable Lie algebra is capable, the result follows from Theorem 2.8. □
Since \( H(m) \) for all \( m \geq 2 \) is not 2-capable, we just need to discus about the 2-capability of \( H(1) \). Here, we obtain 2-nilpotent multiplier of \( H(1) \) and in the next section we show \( H(1) \) is 2-capable.

**Theorem 2.10.** Let \( L = H(1) \). Then
\[
\mathcal{M}^{(2)}(H(1)) \cong A(5).
\]

**Proof.** We know that \( H(1) \) is in fact the free nilpotent Lie algebra of rank 2 and class 2. That is \( H(1) \cong F/F^3 \) in which \( F \) is the free Lie algebra on 2 letters \( x, y \). The second nilpotent multiplier of \( H(1) \) is \( F^4 \cap F^3/[F^3, F, F] \) which is isomorphic to \( F^3/F^5 \) ant the latter is the abelian Lie algebra on the set of all basic commutators of weights 3 and 4 which is the set \( \{[y, x, x], [y, x, y], [y, x, x, x], [y, x, y, y], [y, x, y, y]\} \). So the result holds. \( \square \)

We summarize our result as below

**Theorem 2.11.** Let \( H(m) \) be Heisenberg algebra of dimension \( 2m + 1 \). Then

1. \( \mathcal{M}^{(2)}(H(1)) \cong A(5) \).
2. \( \mathcal{M}^{(2)}(H(m)) = A(\frac{8m^3 - 2m}{3}) \) for all \( m \geq 2 \).

The following Lemma lets us to obtain the structure of the 2-nilpotent multiplier of all nilpotent Lie algebras with \( \text{dim} \ L^2 = 1 \).

**Lemma 2.12.** [16] Lemma 3.3] Let \( L \) be an \( n \)-dimensional Lie algebra and \( \text{dim} \ L^2 = 1 \). Then
\[
L \cong H(m) \oplus A(n - 2m - 1).
\]

**Theorem 2.13.** Let \( L \) be an \( n \)-dimensional Lie algebra with \( \text{dim} \ L^2 = 1 \). Then
\[
\mathcal{M}^{(2)}(L) \cong \begin{cases} 
A\left(\frac{1}{2}n(n-1)(n-2)\right) & \text{if } m > 1, \\
A\left(\frac{1}{2}n(n-1)(n-2) + 3\right) & \text{if } m = 1.
\end{cases}
\]

**Proof.** By using Lemma 2.12 we have \( L \cong H(m) \oplus A(n - 2m - 1) \). Using the behavior of 2-nilpotent multiplier respect to direct sum
\[
\mathcal{M}^{(2)}(L) \cong \mathcal{M}^{(2)}(H(m)) \oplus \mathcal{M}^{(2)}(A(n - 2m - 1))
\]
\[
\oplus \ (H(m)/H(m^2) \otimes Z H(m)/H(m^2)) \otimes Z A(n - 2m - 1))
\]
\[
\oplus \ ((A(n - 2m - 1) \otimes Z A(n - 2m - 1)) \otimes Z H(m)/H(m^2))
\]
First assume that \( m = 1 \), then by virtue of Theorems 2.10 and 2.11
\[
\mathcal{M}^{(2)}(H(1)) \cong A(5) \text{ and } \mathcal{M}^{(2)}(A(n - 3)) \cong A(\frac{1}{3}n(n - 2)(n - 3)(n - 4)).
\]
Thus
\[
\mathcal{M}^{(2)}(L) \cong A(5) \oplus A\left(\frac{1}{2}(n-2)(n-3)(n-4)\right)
\]
\[
\oplus \ (A(2) \otimes Z A(2)) \otimes Z A(n - 3))
\]
\[
\oplus \ ((A(n - 3) \otimes Z A(n - 3)) \otimes Z A(2))
\]
\[
\cong A(\frac{1}{3}n(n - 1)(n - 2) + 3).
\]
The case \( m \geq 1 \) is obtained by a similar fashion. \( \square \)
Theorem 2.14. Let $L$ be a $n$-dimensional nilpotent Lie algebra such that $\dim L^2 = m(m \geq 1)$. Then

$$\dim M^{(2)}(L) \leq \frac{1}{3}(n-m)((n+2m-2)(m-1) + 3(m-1)) + 3.$$ 

In particular, $\dim M^{(2)}(L) \leq \frac{1}{3}n(n-1)(n-2) + 3$. The equality holds in last inequality if and only if $L \cong H(1) \oplus A(n-3)$.

Proof. We do induction on $m$. For $m = 1$, the result follows from Theorem 2.14. Let $m \geq 2$, and taking $I$ a 1-dimensional central ideal of $L$. Since $I$ and $L/I^3$ act to each other trivially we have $(I \otimes L/I^3) \otimes L/L^3 \cong (I \otimes_Z \frac{L/L^3}{(L/L^3)^2}) \otimes_Z \frac{L/L^3}{(L/L^3)^2}$. Thus by Lemma 2.3 (ii)(b)

$$\dim M^{(2)}(L) + \dim I \cap L^3 \leq \dim M^{(2)}(L/I) + \dim (I \otimes_Z \frac{L/L^3}{(L/L^3)^2}) \otimes_Z \frac{L/L^3}{(L/L^3)^2}.$$ 

Since

$$\dim M^{(2)}(L/I) \leq \frac{1}{3}(n-m)((n+2m-5)(n-m-1) + 3(m-2)),$$ 

we have

$$\dim M^{(2)}(L) \leq \frac{1}{3}(n-m)((n+2m-5)(n-m-1) + 3(m-2)) + 3 + (n-m)^2$$

$$= \frac{1}{3}(n-m)((n+2m-2)(n-m-1) + 3(m-1)) + 3,$$ 

as required. \qed

The following corollary shows that the converse of [22] Proposition 1.2 for $c = 2$ is also true. In fact it proves always $\ker \theta = 0$ in [22] Corollary 2.8 (ii)a.

Corollary 2.15. Let $L$ be a $n$-dimensional nilpotent Lie algebra. If $\dim M^{(2)}(L) = \frac{1}{3}n(n-1)(n+1)$, then $L \cong A(n)$.

3. 2-CAPABILITY OF LIE ALGEBRAS

Following the terminology of [7] for groups, a Lie algebra $L$ is said to be 2-capable provided that $L \cong H/Z_2(H)$ for a Lie algebra $H$. The concept $Z^*_2(L)$ was defined in [23] and it was proved that if $\pi : F/[F, F] \to F/R$ be a natural Lie epimorphism then

$$Z^*_2(L) = \pi(Z_2(F/[[[F, F], F]], \pi).$$ 

The following proposition gives the close relation between 2-capability and $Z_2^*(L)$.

Proposition 3.1. A Lie algebra $L$ is 2-capable if and only if $Z_2^*(L) = 0$.

Proof. Let $L$ has a free presentation $F/R$, and $Z_2^*(L) = 0$. Consider the natural epimorphism $\pi : F/[[F, F], F] \to F/R$. Obviously

$$\ker \pi = R/[[[F, F], F] = Z_2(F/[[[F, F], F]],$$ 

and hence $L \cong F/[[[F, F], F]]Z_2(F/[[[F, F], F]])$, which is a 2-capable.

Conversely, let $L$ is 2-capable and so $H/Z_2(H) \cong L$ for a Lie algebra $H$. Put $F/R \cong H$ and $Z_2(H) \cong S/R$. There is natural epimorphism $\eta : F/[[[S, F], F] \to F/S \cong L$. Since $Z_2(F/[[[F, F], F]]) \subseteq \ker \eta$, $Z_2^*(L) = 0$, as required. \qed

The following Theorem gives an instrumental tools to present the main.
Theorem 3.2. Let $I$ be an ideal subalgebra of $L$ such that $I \subseteq Z_2^*(L)$. Then the natural Lie homomorphism $\mathcal{M}(2)(L) \to \mathcal{M}(2)(L/I)$ is a monomorphism.

Proof. Let $S/R$ and $F/R$ be two free presentations of $L$ and $I$, respectively. Looking the natural homomorphism $\phi: \mathcal{M}(2)(L) \cong R \cap F^2/[[R, F], F] \to \mathcal{M}(2)(L/I) \cong R \cap S^2/[[S, F], F]$ and the fact that $S/R \subseteq Z_2(F/R)$ show $\phi$ has trivial kernel. The result follows. □

Theorem 3.3. A Heisenberg Lie algebra $H(m)$ is 2-capable if and only if $m = 1$.

Proof. Let $m \geq 2$, by Corollary 2.9 $H(m)$ is not capable so it is not 2-capable as well. Hence we may assume that $L \cong H(1)$. Let $I$ be an ideal of $L$ of dimension 1. Then $L/I$ is abelian of dimension 2, and hence $\dim \mathcal{M}(2)(L) = 2$. On the other hands, Theorem 2.1 implies $\dim \mathcal{M}(2)(L) = 5$, and Theorem 3.2 deduces $\mathcal{M}(2)(L) \to \mathcal{M}(2)(L/I)$ can not be a monomorphism, as required. □

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School of Mathematics and Computer Science, Damghan University, Damghan, Iran
E-mail address: nirooand@du.ac.ir, pniroomand@yahoo.com

Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran.
E-mail address: parvizi@math.um.ac.ir