ON THE AUSLANDER-REITEN QUIVER OF THE REPRESENTATIONS OF AN INFINITE QUIVER

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ABSTRACT. Let $Q$ be a strongly locally finite quiver and denote by $\text{rep}(Q)$ the category of locally finite dimensional representations of $Q$ over some fixed field $k$. The main purpose of this paper is to get a better understanding of $\text{rep}(Q)$ by means of its Auslander-Reiten quiver. To achieve this goal, we define a category $\text{rep}_+(Q)$ which is a full, abelian and Hom-finite subcategory of $\text{rep}(Q)$ containing all the almost split sequences of $\text{rep}(Q)$. We give a complete description of the Auslander-Reiten quiver of $\text{rep}_+(Q)$ by describing its connected components. Finally, we prove that these connected components are also connected components of the Auslander-Reiten quiver of $\text{rep}(Q)$. We end the paper by giving a conjecture describing the Auslander-Reiten components of $\text{rep}(Q)$ that cannot be obtained as Auslander-Reiten components of $\text{rep}_+(Q)$.

INTRODUCTION

It follows from a result of Gabriel that any basic and finite dimensional algebra over an algebraically closed field $k$ is given by a quiver with relations, that is, it is a quotient of a path algebra $kQ$ by an admissible ideal $I$ of $kQ$, where $Q$ is a finite quiver; see, for example, [2, Section 3, Theorem 1.9]. Therefore, the algebras of the form $kQ$ where $Q$ is a quiver and $k$ is any field are of particular interest. If $Q$ is finite and contains no oriented cycle, $kQ$ is a finite dimensional hereditary algebra and the Auslander-Reiten theory of $kQ$ is well established; see [4, 12]. In [6], the Auslander-Reiten theory of $kQ$ where $Q$ is infinite but strongly locally finite is studied. Indeed, the category $\text{rep}^+(Q)$ consisting of the finitely presented representations is studied by means of its Auslander-Reiten theory. A complete description of the Auslander-Reiten quiver of $\text{rep}^+(Q)$ is given. There is a unique preprojective component, some preinjective components and four types of regular components. Dually, the category $\text{rep}^-(Q)$ of the finitely co-presented representations has a well-understood Auslander-Reiten theory and its Auslander-Reiten quiver is completely described. In [10], it is shown that an almost split sequence in $\text{rep}(Q)$ necessarily starts with a finitely co-presented representation and ends with a finitely presented one. Therefore, it seems that the categories $\text{rep}^+(Q)$ and $\text{rep}^-(Q)$ somehow control the Auslander-Reiten theory of the whole category $\text{rep}(Q)$. The main goal of this paper is to show that this is indeed the case. One can construct a full subcategory $\text{rep}_+(Q)$ of $\text{rep}(Q)$ which is abelian and Hom-finite and contains the Auslander-Reiten theory of $\text{rep}(Q)$ (and hence the Auslander-Reiten theory of both $\text{rep}^+(Q)$ and $\text{rep}^-(Q)$). We give a complete description of the Auslander-Reiten quiver of $\text{rep}_+(Q)$ and show how the knowledge of this helps one to get a partial description of the irreducible morphisms in the whole category $\text{rep}(Q)$. In particular, we provide all the Auslander-Reiten components of $\text{rep}(Q)$ for which the translation acts non-trivially.
In the first section, we provide some background on representations of quivers and recall some key facts concerning the categories \( \rep^+(Q) \), \( \rep^-(Q) \) and \( \rep(Q) \) and the existence of almost split sequences in \( \rep(Q) \). In Section 2, we define and study finite extensions, and provide some properties of these extensions. In Section 3, we define a full subcategory \( \overline{\rep}(Q) \) of \( \rep(Q) \) which consists of the finite extensions of objects in \( \rep^-(Q) \) by those in \( \rep^+(Q) \). The additive \( k \)-category \( \overline{\rep}(Q) \) is shown to be abelian and Hom-finite. We also provide a result saying that each representation in \( \overline{\rep}(Q) \) is built from a projective representation in \( \rep^+(Q) \), an injective representation in \( \rep^-(Q) \) and a finite dimensional representation. In Section 4, we show how to control the domain and co-domain of an irreducible morphism in \( \overline{\rep}(Q) \). In particular, if \( M \rightarrow N \) is irreducible in \( \overline{\rep}(Q) \) with both \( M, N \) indecomposable, then either \( M \in \rep^-(Q) \) or \( N \in \rep^+(Q) \). In Section 5, we give a complete description of the Auslander-Reiten quiver of \( \overline{\rep}(Q) \). If \( Q \) is connected, there is a unique preprojective component, a unique preinjective component and four possible types of regular components. Finally, in Section 6, we describe partially the Auslander-Reiten quiver of \( \overline{\rep}(Q) \), by showing that the connected components of the Auslander-Reiten quiver of \( \overline{\rep}(Q) \) are connected components of the Auslander-Reiten quiver of \( \rep(Q) \). We propose a conjecture for the shapes of the other connected components of the Auslander-Reiten quiver of \( \rep(Q) \).

1. Representations of quivers

Let \( Q = (Q_0, Q_1) \) be a strongly locally finite quiver, that is, a locally finite quiver for which the number of paths between any given pair \( x, y \) of vertices of \( Q \) is finite. This last property will be referred to as \( Q \) being interval-finite. Throughout the paper, we fix \( k \) to be any field. Recall that a representation \( M \) of \( Q \) (over \( k \)) is defined by the following data. For each \( x \in Q_0 \), \( M(x) \) is a \( k \)-vector space and for each arrow \( \alpha : x \rightarrow y \in Q_1 \), \( M(\alpha) : M(x) \rightarrow M(y) \) is a \( k \)-linear map. If \( M, N \) are representations in \( \rep(Q) \), then a map \( f : M \rightarrow N \) is a family \( \{f_x : M(x) \rightarrow N(x) \mid x \in Q_0 \} \) of \( k \)-linear maps such that for any arrow \( \alpha : x \rightarrow y \in Q \), we have \( f_y M(\alpha) = N(\alpha)f_x \). The category of all representations of \( Q \) over \( k \) is denoted by \( \Rep(Q) \). Indeed, \( \Rep(Q) \) is the category of all \( k \)-linear covariant functors from the path category \( kQ \) to the category of all \( k \)-vector spaces. The importance of the category \( \Rep(Q) \) relies on the fact that it is equivalent to the category \( \Mod A \) of all unitary left \( A \)-module, where \( A \) is the path algebra \( kQ \) (and has no identity if \( Q \) is infinite). Here, a left \( A \)-module \( M \) is unitary if \( AM = M \). A representation \( M \in \Rep(Q) \) is said to be locally finite dimensional if \( M(x) \) is finite dimensional for all \( x \in Q_0 \); and finite dimensional if \( \sum_{x \in Q_0} M(x) \) is finite dimensional. The full subcategory of \( \Rep(Q) \) of all locally finite dimensional representations of \( Q \) is denoted by \( \rep(Q) \). In some sense, the objects in \( \rep(Q) \) are close from being finite dimensional and they have nice properties. For example, every indecomposable object in \( \rep(Q) \) has a local endomorphism algebra; see \[7\], and there exists a (pointwise) duality \( D_Q : \rep(Q) \rightarrow \rep(Q^{op}) \) where \( Q^{op} \) is the opposite quiver of \( Q \); see \[8\]. It is also shown in \[7\] Section 3.6 that \( \Rep(Q) \) (and \( \rep(Q) \)) is abelian and hereditary, that is, \( \text{Ext}^2_{\Rep(Q)}(\cdot, \cdot) \) vanishes. Note however that, in general, \( \rep(Q) \) is not Hom-finite.

Let us now introduce two important families of representations in \( \rep(Q) \). First, given \( x, y \in Q_0 \), let us denote by \( Q(x, y) \) the (finite) set of paths from \( x \) to \( y \) in \( Q \).
For \( a \in Q_0 \), let \( P_a \) denote the representation defined as follows. For \( x \in Q_0 \), \( P_a(x) = k(Q(a,x)) \) and for an arrow \( \alpha : x \to y \), \( P_a(\alpha) \) is the right multiplication by \( \alpha \). Since \( Q \) is interval-finite, \( P_a \in \text{rep}(Q) \). It is easy to show that \( P_a \) is indecomposable projective in \( \text{rep}(Q) \); see, for example, [6, Proposition 1.3]. Dually, for \( a \in Q_0 \), denote by \( I_a \) the following representation. For \( x \in Q_0 \), \( I_a(x) = k(Q(x,a)) \) and for an arrow \( \alpha : x \to y \), \( I_a(\alpha) \) is the transpose of the map \( kQ(y,a) \to kQ(x,a) \) which is the left multiplication by \( \alpha \). Since \( Q \) is interval-finite, \( I_a \in \text{rep}(Q) \). Moreover, using the duality \( DQ \), we see that \( I_a \) is indecomposable injective in \( \text{rep}(Q) \); see also [6].

The full subcategory of \( \text{Rep}(Q) \) whose objects are the finitely presented (finitely co-presented, respectively) representations is denoted by \( \text{rep}^+(Q) \) (\( \text{rep}^-(Q) \), respectively). Since \( Q \) is strongly locally finite, \( \text{rep}^+(Q) \) and \( \text{rep}^-(Q) \) are indeed full subcategories of \( \text{rep}(Q) \). Since \( \text{rep}(Q) \) is hereditary, \( M \in \text{rep}^+(Q) \) if and only if there exists a short exact sequence
\[
0 \to \bigoplus_{i=1}^r P_{x_i} \to \bigoplus_{i=1}^s P_{x_i} \to M \to 0
\]
in \( \text{rep}(Q) \) where the \( x_i \) are vertices in \( Q \). Similarly, \( M \in \text{rep}^-(Q) \) if and only if there exists a short exact sequence
\[
0 \to M \to \bigoplus_{i=1}^{r'} I_{y_i} \to \bigoplus_{i=1}^{s'} I_{y_i} \to 0
\]
in \( \text{rep}(Q) \) where the \( y_i \) are vertices in \( Q \). The indecomposable projective representations in \( \text{rep}^+(Q) \), up to isomorphisms, are the \( P_x \) for \( x \in Q_0 \). Similarly, the indecomposable injective representations in \( \text{rep}^-(Q) \), up to isomorphisms, are the \( I_x \) for \( x \in Q_0 \). However, \( \text{rep}(Q) \) may have indecomposable projective representations which are not isomorphic to the \( P_x \), and dually, may have indecomposable injective representations which are not isomorphic to the \( I_x \).

Let us now introduce more definitions. For this purpose, fix \( M \in \text{rep}(Q) \). The support of \( M \), written as \( \text{supp}(M) \), is the full subquiver of \( Q \) generated by the vertices \( x \in Q_0 \) for which \( M(x) \neq 0 \). If \( M \in \text{rep}^+(Q) \), then it follows from the definition of \( \text{rep}^+(Q) \) and the fact that \( Q \) is strongly locally finite that \( \text{supp}(M) \) is top-finite, that is, contains finitely many source vertices and every vertex \( x \) in \( \text{supp}(M) \) is a successor of one such source vertex. Dually, if \( M \in \text{rep}^-(Q) \), then \( \text{supp}(M) \) is socle-finite, that is, contains finitely many sink vertices and every vertex \( x \) in \( \text{supp}(M) \) is a predecessor of one such sink vertex. Observe that a socle-finite quiver may contain a left-infinite path, that is, a path of the form
\[
\cdots \to \circ \to \circ \to \circ \to \circ
\]
but does not contain a right-infinite path, that is, a path of the form
\[
\circ \to \circ \to \circ \to \circ \to \cdots .
\]
Dually, a top-finite quiver may contain a right-infinite path but does not contain left-infinite paths. It is easy to see that a full subquiver of \( Q \) which is top-finite and socle-finite needs to be finite. In particular,
\[
\text{rep}^h(Q) := \text{rep}^+(Q) \cap \text{rep}^-(Q)
\]
consists of all the finite dimensional representations of \( Q \). If \( x \) is a vertex in \( Q \), then one says that \( M \) is supported by \( x \) if \( M(x) \neq 0 \), or equivalently, if \( x \in \text{supp}(M) \). Finally, if \( \alpha \in Q_1 \), we say that \( M \) is supported by \( \alpha \) if \( M(\alpha) \neq 0 \). Note that if \( \alpha \) is an arrow in \( \text{supp}(M) \), then it does not necessarily mean that \( M \) is supported by \( \alpha \).
In this paper, we shall use many basic results concerning $\text{rep}^+(Q)$ and $\text{rep}^-(Q)$ appearing in [6]. This is our main reference. We shall, however, recall the main definitions and results we need at the appropriate place in the sequel. Here are some of them, that we will use freely. Both $\text{rep}^+(Q)$ and $\text{rep}^-(Q)$ are Hom-finite hereditary abelian $k$-categories. For any indecomposable non-projective representation $X$ in $\text{rep}^+(Q)$, there exists an almost split sequence

$$0 \rightarrow X' \rightarrow E \rightarrow X \rightarrow 0$$

in $\text{rep}(Q)$ where $X' \in \text{rep}^-(Q)$. This sequence is almost split in $\text{rep}^-(Q)$ if and only if $X$ is finite dimensional. Dually, for any indecomposable non-injective representation $Y$ in $\text{rep}^-(Q)$, there exists an almost split sequence

$$0 \rightarrow Y \rightarrow E' \rightarrow Y' \rightarrow 0$$

in $\text{rep}(Q)$ where $Y' \in \text{rep}^+(Q)$. This sequence is almost split in $\text{rep}^+(Q)$ if and only if $Y$ is finite dimensional.

2. Finite extensions and their properties

In this section, we define the main category (a subcategory of $\text{rep}(Q)$) of interest of the paper. As we will see, this category contains all the almost split sequences of $\text{rep}(Q)$ and has a nice Auslander-Reiten theory. Let $L, M, N \in \text{rep}(Q)$. An extension of $N$ by $L$ is a short exact sequence

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

in $\text{rep}(Q)$, where $M$ is called an extension-representation of $N$ by $L$. Two extensions $\eta, \gamma$ of $N$ by $L$ are equivalent if there exists a commutative diagram

$$\eta : \begin{array}{ccc}
0 & \rightarrow & L \\
\downarrow & & \downarrow \\
M & \rightarrow & N \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\gamma : & & \end{array}$$

in $\text{rep}(Q)$. If $M$ is an extension-representation of $N$ by $L$ such that all but a finite number of arrows supporting $M$ are arrows supporting $L \oplus N$, then $M$ is said to be a finite extension-representation of $N$ by $L$, while the corresponding short exact sequence is a finite extension of $N$ by $L$. Recall that $\text{Ext}(N, L)$ is the set of all extensions of $N$ by $L$ modulo the equivalence relation given above. It is an abelian group under the Baer sum; see [9]. Observe that if $\eta, \gamma$ are equivalent extensions of $N$ by $L$, then $\eta$ is finite if and only if $\gamma$ is finite. Moreover, one can easily check that if $\gamma, \eta \in \text{Ext}(N, L)$ are finite extensions, then their Baer sum $\gamma + \eta$ is also a finite extension. Hence, the subset $\mathcal{E}(N, L)$ of $\text{Ext}(N, L)$ of all equivalence classes of finite extensions of $N$ by $L$ is a subgroup of $\text{Ext}(N, L)$. The following lemma tells us how to recognize such a finite extension.

**Lemma 2.1.** Let $M$ be an extension-representation of $N$ by $L$ and $S$ be the arrows $x \rightarrow y$ in $Q$ with $x \in \text{supp}(N)$ and $y \in \text{supp}(L)$. Then $M$ is a finite extension-representation if and only if $M(\alpha) = 0$ for all but a finite number of $\alpha \in S$.

**Proof.** Let $E$ be the set of all arrows in $Q$ supporting $M$ but not supporting $L \oplus N$, and let $\alpha : x \rightarrow y$ with $\alpha \in E$. If $x \in \text{supp}(L) \setminus \text{supp}(N)$, then $M(x) = L(x)$. Moreover, $L$ being a sub-representation of $M$ implies that $L(\alpha) \neq 0$, a contradiction. Hence, $x \notin \text{supp}(L) \setminus \text{supp}(N)$. Similarly, $y \notin \text{supp}(N) \setminus \text{supp}(L)$. In particular, since $\text{supp}(M) = \text{supp}(L) \cup \text{supp}(N)$, $x \in \text{supp}(N)$ and $y \in \text{supp}(L)$. \qed
Let $\text{rep}(Q)$ be the full subcategory of $\text{rep}(Q)$ with objects $M$ such that $M$ is a finite extension-representation of $N$ by $L$ for some $L \in \text{rep}^+(Q)$ and $N \in \text{rep}^-(Q)$. We will use this category to get a better understanding of the Auslander-Reiten theory of $\text{rep}(Q)$. Observe that Auslander has defined a similar category in [1], for a similar purpose but in a different setting. Given a noetherian algebra $\Lambda$, Auslander defined $\text{armo}(\Lambda)$ as the full subcategory of the module category whose objects are the middle terms of the short exact sequences of the form $0 \to L \to M \to N \to 0$ with $L$ an artinian $\Lambda$-module and $N$ a noetherian $\Lambda$-module. Note that in general, a finitely presented representation is not artinian and, more importantly, is not noetherian; see [11]. Hence our definition is slightly different. When $\text{rep}^+(Q)$ is noetherian and $\text{rep}^-(Q)$ is artinian, then Lemma 2.3 below implies that our category $\text{rep}(Q)$ contains all representations occurring as a middle term of a short exact sequence of the form $0 \to L \to M \to N \to 0$ with $L$ an artinian representation and $N$ a noetherian representation.

**Lemma 2.2.** Let $M$ be a representation in $\text{rep}(Q)$. Then any short exact sequence $0 \to L \to M \to N \to 0$ with $L \in \text{rep}^+(Q)$ and $N \in \text{rep}^-(Q)$ is a finite extension.

**Proof.** By assumption, there exists a finite extension

$$0 \to L' \to M \to N' \to 0$$

with $L' \in \text{rep}^+(Q)$ and $N' \in \text{rep}^-(Q)$. Let

$$0 \to L \to M \to N \to 0$$

be an extension with $L \in \text{rep}^+(Q)$ and $N \in \text{rep}^-(Q)$. Observe that

$$\text{supp}(L') \setminus \text{supp}(L) \subseteq \text{supp}(N).$$

We claim that $\text{supp}(L') \setminus \text{supp}(L)$ is finite. Suppose the contrary. Since $\text{supp}(N)$ is socle-finite, there exists a left-infinite path $p$ in $\text{supp}(N)$ ending at $y \in Q_0$ such that infinitely many vertices of $p$, say $\{x_i\}_{i \geq 1}$, lie in $\text{supp}(L') \setminus \text{supp}(L)$. Since $\text{supp}(L')$ is top-finite, infinitely many of the $x_i$ are successor of a fixed vertex $x$ in $\text{supp}(L')$. Therefore, there are infinitely many vertices $\{y_i\}_{i \geq 1}$ such that, for $i \geq 1$, there is a path from $x$ to $y$ passing through $y_i$. This contradicts the fact that $Q$ is interval-finite. Hence, $\text{supp}(L') \setminus \text{supp}(L)$ is finite. Similarly, $\text{supp}(L) \setminus \text{supp}(L')$ is finite. In a dual way, we can show that $\text{supp}(N) \setminus \text{supp}(N')$ and $\text{supp}(N') \setminus \text{supp}(N)$ are finite. Therefore, all but a finite number of arrows starting at a vertex in $\text{supp}(N)$ and ending at a vertex in $\text{supp}(L)$ are arrows starting at a vertex in $\text{supp}(N')$ and ending at a vertex in $\text{supp}(L')$. The result follows immediately from Lemma 2.1. □

**Lemma 2.3.** Let $X \in \text{rep}^+(Q)$ and $Y \in \text{rep}^-(Q)$. Then $\text{Ext}(X,Y) = \mathcal{E}(X,Y)$.

**Proof.** From Lemma 2.1 it is sufficient to show that there are finitely many arrows starting from a vertex in $\text{supp}(X)$ and ending to a vertex in $\text{supp}(Y)$. Assume the contrary. Let $\{\alpha_i : x_i \to y_i\}_{i \geq 1}$ be an infinite family of arrows with $x_i \in \text{supp}(X)$ and $y_i \in \text{supp}(Y)$. Since $\text{supp}(X)$ is top-finite and $\text{supp}(Y)$ is socle-finite, there exist a source vertex $x$ in $\text{supp}(X)$ and a sink vertex $y$ in $\text{supp}(Y)$ such that, for infinitely many $i \geq 1$, $x_i$ is a successor of $x$ in $\text{supp}(X)$ and $y_i$ is a predecessor of $y$ in $\text{supp}(Y)$. Hence, there are infinitely many paths starting from $x$ and ending at $y$. This contradicts the fact that $Q$ is interval-finite. □
Recall that since \( Q \) is interval-finite, \( Q(x, y) \) is finite for \( x, y \in Q_0 \). However, \( Q(x, y) \) can be arbitrarily large when \( x, y \) run through \( Q_0 \times Q_0 \). If there exists a global bound on \( |Q(x, y)| \), then every extension of a finitely co-presented representation by a finitely presented one is finite.

**Proposition 2.4.** Suppose that \( Q \) is such that there exists a positive integer \( r \) such that \( |Q(x, y)| \leq r \) for any \( x, y \in Q_0 \). Then \( E(X, Y) = \text{Ext}(X, Y) \) for any \( X \in \text{rep}^{-}(Q) \) and \( Y \in \text{rep}^{+}(Q) \).

**Proof.** Let

\[
0 \to L \to M \to N \to 0
\]

be an extension with \( L \in \text{rep}^{+}(Q) \) and \( N \in \text{rep}^{-}(Q) \). Let \( A \) be the set of all arrows in \( Q \) starting at a vertex in \( \text{supp}(N) \) and ending at a vertex in \( \text{supp}(L) \). It is sufficient to show that \( A \) is finite. If one of \( \text{supp}(N) \), \( \text{supp}(L) \) is infinite, then the claim follows since \( Q \) is locally finite. Hence, we may assume that both \( \text{supp}(N) \), \( \text{supp}(L) \) are infinite. Suppose to the contrary that \( A \) is infinite. This implies that \( S = \{ s(\alpha) \mid \alpha \in A \} \) and \( E = \{ e(\alpha) \mid \alpha \in A \} \) are both infinite subsets of \( Q_0 \). Since \( \text{supp}(N) \) is socle-finite and \( S \subseteq \text{supp}(N) \) is infinite, there exists a left infinite path

\[
p : \cdots \to a_3 \to a_2 \to a_1
\]

in \( \text{supp}(N) \) with an infinite number of \( a_j \) being in \( S \). Then \( J := p \cap S \) is infinite. Since \( \text{supp}(L) \) is top-finite and

\[
J' = \{ e(\alpha) \mid \alpha \in A, s(\alpha) \in J \} \subseteq E \subseteq \text{supp}(L)
\]

is infinite, there exists a right-infinite path

\[
b_1 \to b_2 \to b_3 \to \cdots
\]

in \( \text{supp}(L) \) with an infinite number of \( b_j \) being in \( J' \). It is now easy to see that the sets \( Q(a_j, b_j) \), \( j \geq 1 \), have arbitrary large cardinality. This is a contradiction.

Let us introduce some definitions. Let \( \Sigma \) be a quiver and \( \Omega \) a full subquiver of \( \Sigma \). One says that \( \Omega \) is predecessor-closed (resp. successor-closed) in \( \Sigma \) if for any path \( p : x \rightsquigarrow y \) in \( \Sigma \) with \( y \in \Omega \) (resp. \( x \in \Omega \)), we have \( x \in \Omega \) (resp. \( y \in \Omega \)). We say that \( \Omega \) is co-finite in \( \Sigma \) if \( \Sigma_0 \setminus \Omega_0 \) is finite. If \( \Omega \) is a full subquiver of \( \Sigma \), and \( M \) is a representation of \( \Sigma \), we denote by \( M_\Omega \) the restriction of \( M \) to \( \Omega \). It is a representation of \( \Omega \) but can be seen as a representation of \( \Sigma \) by setting \( M_\Omega(x) = 0 \) if \( x \not\in \Omega \).

**Lemma 2.5.** Suppose that \( M \in \text{rep}(Q) \). Then there exists a top-finite full subquiver \( \Omega \) of \( Q \) such that \( M_\Omega \in \text{rep}^{+}(Q) \) is a sub-representation of \( M \) and \( M/M_\Omega \in \text{rep}^{-}(Q) \). In particular,

\[
0 \to M_\Omega \to M \to M/M_\Omega \to 0
\]

is a finite extension.

**Proof.** One has a finite extension

\[
0 \to L \to M \to N \to 0
\]

with \( L \in \text{rep}^{+}(Q) \) and \( N \in \text{rep}^{-}(Q) \). Let \( \Omega \) be a top-finite and successor-closed subquiver of \( Q \) supporting \( L \). Then \( L \) is a sub-representation of \( M_\Omega \). Consider the
diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & L & \rightarrow & M & \rightarrow & N & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow g & \\
0 & \rightarrow & M_{\Omega} & \rightarrow & M & \rightarrow & M/M_{\Omega} & \rightarrow & 0
\end{array}
\]

where \( g \) is the induced morphism. Observe that \( g \) is an epimorphism. By the snake lemma, \( \text{Ker} \, g \cong M_{\Omega}/L \) and hence,

\[
\text{supp}(\text{Ker} \, g) \subseteq \text{supp}(N) \cap \text{supp}(M_{\Omega}) \subseteq \text{supp}(N) \cap \Omega,
\]

which is finite. In particular, \( \text{Ker} \, g \) is finite dimensional and hence lies in \( \text{rep}^{-}(Q) \). Since \( \text{rep}^{-}(Q) \) has cokernels, \( M/M_{\Omega} \in \text{rep}^{-}(Q) \). The fact that \( M_{\Omega} \in \text{rep}^{+}(Q) \) follows from the fact that \( \text{rep}^{+}(Q) \) is closed under extensions. Hence, the bottom row of the above diagram yields that \( M \) is an extension-representation of \( M/M_{\Omega} \in \text{rep}^{-}(Q) \) by \( M_{\Omega} \in \text{rep}^{+}(Q) \) which is finite by Lemma 2.2.

\[\square\]

3. Properties of \( \text{rep}(Q) \)

The purpose of this section is to collect some properties of \( \text{rep}(Q) \) and its objects. Using the fact that both \( \text{rep}^{+}(Q) \) and \( \text{rep}^{-}(Q) \) are abelian and Hom-finite, we can prove that \( \text{rep}(Q) \) is abelian and Hom-finite and consequently is Krull-Schmidt, that is, the Krull-Remak-Schmidt theorem holds: every object decomposes uniquely (up to isomorphism and permutation) as a finite direct sum of objects having local endomorphism algebras. Let us first prove that it is abelian.

**Proposition 3.1.** The category \( \text{rep}(Q) \) is abelian.

**Proof.** Let us first prove that \( \text{rep}(Q) \) has kernels. Let \( f : M_{1} \rightarrow M_{2} \) be a morphism with \( M_{1}, M_{2} \in \text{rep}(Q) \). It is easy to see that \( M_{1} \oplus M_{2} \in \text{rep}(Q) \). By Lemma 2.3, there exists a full subquiver \( \Omega \) of \( Q \) such that \( (M_{1} \oplus M_{2})_{\Omega} \in \text{rep}^{+}(Q) \) and \( (M_{1} \oplus M_{2})/(M_{1} \oplus M_{2})_{\Omega} \in \text{rep}^{-}(Q) \). Now, \( (M_{i})_{\Omega} \in \text{rep}^{+}(Q) \), for \( i = 1, 2 \), since \( \text{rep}^{+}(Q) \) is closed under direct summand. Similarly, \( M_{i}/(M_{i})_{\Omega} \in \text{rep}^{-}(Q) \), for \( i = 1, 2 \). By Lemma 2.2 we have finite extensions

\[
0 \rightarrow (M_{i})_{\Omega} \rightarrow M_{i} \rightarrow M_{i}/(M_{i})_{\Omega} \rightarrow 0
\]

for \( i = 1, 2 \). Thus, one has a diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & (M_{1})_{\Omega} & \rightarrow & M_{1} & \rightarrow & M_{1}/(M_{1})_{\Omega} & \rightarrow & 0 \\
\uparrow & & \uparrow u & & \uparrow f & & \uparrow h & & \\
0 & \rightarrow & (M_{2})_{\Omega} & \rightarrow & M_{2} & \rightarrow & M_{2}/(M_{2})_{\Omega} & \rightarrow & 0
\end{array}
\]

where \( u \) is the restriction of \( f \) to \( \Omega \) and \( h \) is the induced morphism. This yields an exact sequence

\[
0 \rightarrow \text{Ker} \, u \rightarrow \text{Ker} \, f \xrightarrow{\psi} \text{Ker} \, h \xrightarrow{\psi^{-1}} \text{Coker} \, u
\]

in \( \text{rep}(Q) \). Now, \( \text{Ker} \, u, \text{Coker} \, u \in \text{rep}^{+}(Q) \) while \( \text{Ker} \, h \in \text{rep}^{-}(Q) \). Since

\[\text{supp}(\text{Im} \, v) \subseteq \text{supp}(\text{Coker} \, u) \cap \text{supp}(\text{Ker} \, h),\]

\( \text{Im} \, v \) is finite dimensional. Hence, \( \text{Ker} \, v \in \text{rep}^{-}(Q) \) since \( \text{Ker} \, h/\text{Ker} \, v \) is finite dimensional. Therefore, \( \text{Im} \, v' \in \text{rep}^{-}(Q) \), meaning that \( \text{Ker} \, f \) is an extension-representation of \( \text{Im} \, v' \in \text{rep}^{-}(Q) \) by \( \text{Ker} \, u \in \text{rep}^{+}(Q) \). Now, if \( \text{Ker} \, f \) is not a finite extension-representation of \( \text{Im} \, v' \) by \( \text{Ker} \, u \), it means that there exists an infinite
family of arrows \( \{ \alpha_i \}_{i \geq 1} \) such that for \( i \geq 1 \), \( \alpha_i \) starts at a vertex in \( \text{supp}(\text{Im} \; v') \), ends at a vertex in \( \text{supp}(\text{Ker} \; u) \) and \( \langle \text{Ker} \; f \rangle(\alpha_i) \) is non-zero. Observe that

\[
\text{supp}(\text{Im} \; v') \subseteq \text{supp}(M_1/(M_1)_\Omega) \quad \text{and} \quad \text{supp}(\text{Ker} \; u) \subseteq \text{supp}((M_1)_\Omega).
\]

Moreover, \( \langle \text{Ker} \; f \rangle(\alpha_i) \neq 0 \) implies that \( M_1(\alpha_i) \neq 0 \). Hence, for \( i \geq 1 \), \( \alpha_i \) is an arrow starting at a vertex in \( \text{supp}(M_1/(M_1)_\Omega) \) and ending at a vertex in \( \text{supp}((M_1)_\Omega) \) such that \( M_1(\alpha_i) \neq 0 \). This contradicts the fact that the given extension of \( M_1/(M_1)_\Omega \) by \( (M_1)_\Omega \) is finite. This shows that \( \text{Ker} \; f \in \text{rep}(Q) \).

Similarly, one can prove that \( \text{rep}(Q) \) has cokernels. Hence, being a full subcategory of \( \text{rep}(Q) \), \( \text{rep}(Q) \) is abelian. \( \square \)

**Proposition 3.2.** The category \( \text{rep}(Q) \) is Hom-finite and Krull-Schmidt.

**Proof.** Recall that both \( \text{rep}^+(Q) \) and \( \text{rep}^-(Q) \) are Hom-finite; see [6]. Moreover, if \( M \in \text{rep}^+(Q) \) and \( N \in \text{rep}^-(Q) \), then \( \text{Hom}(M, N) \) and \( \text{Hom}(N, M) \) are finite dimensional since \( \text{supp}(M) \cap \text{supp}(N) \) is finite. Let \( M_1, M_2 \in \text{rep}(Q) \) such that \( M_i \) is a finite extension-representation of \( N_i \) by \( L_i \), with \( L_i \in \text{rep}^+(Q) \) and \( N_i \in \text{rep}^-(Q) \), \( i = 1, 2 \). Then one has

\[
0 \to \text{Hom}(N_1, L_2) \to \text{Hom}(M_1, L_2) \to \text{Hom}(L_1, L_2)
\]

showing that \( \text{Hom}(M_1, L_2) \) is finite dimensional. Similarly, \( \text{Hom}(M_1, N_2) \) is finite dimensional. The exact sequence

\[
0 \to \text{Hom}(M_1, L_2) \to \text{Hom}(M_1, M_2) \to \text{Hom}(M_1, N_2)
\]

yields that \( \text{Hom}(M_1, M_2) \) is finite dimensional. Now, it is well known that a Hom-finite abelian category is Krull-Schmidt. \( \square \)

**Proposition 3.3.** The category \( \text{rep}(Q) \) is closed under finite extensions and direct summands.

**Proof.** Let

\[
(*) \; : \; 0 \to M_1 \to M_2 \to M_3 \to 0
\]

be a finite extension in \( \text{rep}(Q) \) with \( M_1, M_2 \in \text{rep}(Q) \). By Lemma 2.5 there exist top-finite successor-closed subquivers \( \Omega', \Omega'' \) of \( Q \) such that \( (M_1)_{\Omega'}, (M_2)_{\Omega''} \in \text{rep}^+(Q) \) and \( M_1/(M_1)_{\Omega'}, M_2/(M_2)_{\Omega''} \in \text{rep}^-(Q) \). Let \( \Omega \) be the union of \( \Omega' \) and \( \Omega'' \), which is top-finite and successor-closed in \( Q \). Consider the following commutative diagram with exact rows:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & (M_1)_{\Omega'} & \longrightarrow & M_1 & \longrightarrow & M_1/(M_1)_{\Omega'} & \longrightarrow & 0 \\
0 & \longrightarrow & (M_1)_{\Omega} & \longrightarrow & M_1 & \longrightarrow & M_1/(M_1)_{\Omega} & \longrightarrow & 0.
\end{array}
\]

Since the inclusion \( (M_1)_{\Omega'} \to (M_1)_{\Omega} \) has a cokernel which is supported by the intersection of \( \text{supp}(M_1/(M_1)_{\Omega'}) \) with \( \Omega'' \), which is a finite quiver, we see that \( (M_1)_{\Omega} \in \text{rep}^+(Q) \). Also, the kernel of the projection \( M_1/(M_1)_{\Omega'} \to M_1/(M_1)_{\Omega} \) is finite dimensional showing that \( M_1/(M_1)_{\Omega} \) is finitely co-presented. Similarly, \( (M_2)_{\Omega} \in \text{rep}^+(Q) \) and \( (M_2)_{\Omega'} \in \text{rep}^-(Q) \). By Lemma 2.2 the extensions

\[
0 \to (M_1)_{\Omega} \to M_i \to M_i/(M_i)_{\Omega} \to 0
\]

is a finite extension in \( \text{rep}(Q) \).
are finite for \( i = 1, 2 \). By restricting the extension \((*)\) to \( \Omega \) and \( Q \setminus \Omega \), one gets a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & (M_1)_{\Omega} & \longrightarrow & (M_3)_{\Omega} & \longrightarrow & (M_2)_{\Omega} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_3 & \longrightarrow & M_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1/(M_1)_{\Omega} & \longrightarrow & M_3/(M_3)_{\Omega} & \longrightarrow & M_2/(M_2)_{\Omega} & \longrightarrow & 0
\end{array}
\]

where all rows and all columns are exact. Since \( \text{rep}^+(Q) \) and \( \text{rep}^-(Q) \) are closed under extensions, \( (M_3)_{\Omega} \in \text{rep}^+(Q) \) and \( M_3/(M_3)_{\Omega} \in \text{rep}^-(Q) \). If \( M_3 \) is not a finite extension-representation of \( M_3/(M_3)_{\Omega} \), then there exists an infinite family of arrows \( \{\alpha_i\}_{i \geq 1} \) all starting at a vertex in \( \text{supp}(M_3/(M_3)_{\Omega}) \subseteq Q \setminus \Omega \) and ending at a vertex in \( \text{supp}(M_3/(M_3)_{\Omega}) \subseteq \Omega \) such that \( M_3(\alpha_i) \neq 0 \). Since the middle row is a finite extension, \( (M_3 \oplus M_2)(\alpha_i) \neq 0 \) for all a finite number of \( i \geq 1 \), which yields that \( M_1(\alpha_i) \neq 0 \) for all but a finite number of \( i \geq 1 \) or \( M_2(\alpha_i) \neq 0 \) for all but a finite number of \( i \geq 1 \). In the first case, we get that the first column is not a finite extension, and in the second case, that the third column is not a finite extension. This is a contradiction. This shows that \( M_3 \in \overline{\text{rep}}(Q) \). The last part of the proposition follows from the fact that \( \overline{\text{rep}}(Q) \) is abelian. 

\[\square\]

In order to have a better understanding of the objects in \( \overline{\text{rep}}(Q) \), we first state the following result, which can be derived easily from [6, Theorem 1.12].

**Proposition 3.4** (Bautista-Liu-Paquette). Let \( 0 \neq M \in \text{rep}^+(Q) \) have support \( \Sigma \). Then there exists a co-finite and successor-closed subquiver \( \Omega \) of \( \Sigma \) such that

1. \( M_{\Omega} \) is projective,
2. \( M_{\Sigma \setminus \Omega} \) is non-zero, finite dimensional and is indecomposable when \( M \) is.

Moreover, any co-finite and successor-closed subquiver \( \Omega' \) of \( \Omega \) also satisfies properties (1) and (2).

Of course, the dual result for \( \text{rep}^-(Q) \) holds true. We will show that a similar statement in \( \overline{\text{rep}}(Q) \) can be obtained. Before going further, we need a lemma.

We say that a representation \( M \in \text{rep}(Q) \) is *indecomposable up to projectives* if \( M = M_1 \oplus M_2 \) implies that \( M_1 \) or \( M_2 \) is projective in \( \text{rep}^+(Q) \). We also have the dual notion of *indecomposable up to injectives*. Finally, \( M \in \text{rep}(Q) \) is *indecomposable up to projectives and injectives* if \( M = M_1 \oplus M_2 \) implies that one of \( M_1, M_2 \) is projective in \( \text{rep}^+(Q) \) or injective in \( \text{rep}^-(Q) \).

**Lemma 3.5.** Let \( M \in \overline{\text{rep}}(Q) \) have support \( \Sigma \). Then there exists a top-finite and successor-closed subquiver \( \Omega \) of \( \Sigma \) such that

1. \( M_{\Omega} \in \text{rep}^+(Q) \) is projective,
2. \( M/M_{\Omega} \in \text{rep}^-(Q) \),
3. If \( M/M_{\Omega} \) is indecomposable, then \( M \) is indecomposable up to projectives,
4. If \( M \) is indecomposable, then \( M/M_{\Omega} \) is indecomposable.

Moreover, any successor-closed and co-finite subquiver of \( \Omega \) also satisfies properties (1) to (4).

**Proof.** By Lemma [2.3], there exists a top-finite full subquiver \( \Omega' \) of \( \Sigma \) such that \( M_{\Omega'} \in \text{rep}^+(Q) \) and \( M/M_{\Omega'} \in \text{rep}^-(Q) \). We may choose \( \Omega' \) to be successor-closed in \( \Sigma \). If \( \Omega' \) is empty, then we are done. Suppose that \( \Omega' \) is non-empty. By
Proposition 3.4, let $\Sigma_P$ be a co-finite and successor-closed subquiver of $\Omega$ such that $(M_{\Omega'})_{\Sigma_P} = M_{\Sigma_P}$ is projective in rep$^+(Q)$ and $M_{\Omega'}/M_{\Sigma_P}$ is finite dimensional, non-zero and indecomposable whenever $M_{\Omega'}$ is. Clearly, $M_{\Sigma_P} \in \text{rep}^+(Q)$ and $M/M_{\Sigma_P} \in \text{rep}^-(Q)$. Suppose that $\Omega$ is a co-finite and successor-closed subquiver of $\Sigma_P$ such that if $\alpha : x \to y$ is an arrow with $y \in \Omega$ and $M(\alpha) \neq 0$, then $x \in \Sigma_P$. Such a co-finite subquiver of $\Sigma_P$ exists since $M$ is a finite extension-representation of $M/M_{\Sigma_P}$ by $M_{\Sigma_P}$ by Lemma 3.3. Moreover, one can chose $\Omega$ so that $\Sigma_P \setminus \Omega$ contains the support of the top of the projective representation $M_{\Sigma_P}$. Being a sub-representation of $M_{\Sigma_P}$, $M_{\Omega}$ is projective. Moreover, it lies in rep$^+(Q)$ since $\Omega$ is co-finite in $\Sigma_P$. Similarly, $M/M_{\Omega} \in \text{rep}^-(Q)$.

Now, suppose that $M$ decomposes non-trivially as $M = M_1 \oplus M_2$ with $M_1, M_2$ non-projective. If the support of $M_1$ is included in $\Omega$, then $M_2$ is a sub-representation of $M_{\Omega}$ and hence is projective, a contradiction. Thus, the support of $M_1$ has an intersection with $\Sigma \setminus \Omega$, meaning that the restriction $M_1/(M_1)_{\Omega}$ of $M_1$ to $\Sigma \setminus \Omega$ yields a non-zero direct summand of $M/M_{\Omega}$. If $M_1/(M_1)_{\Omega} = M/M_{\Omega}$, then $M_2$ has a support included in $\Omega$ and hence is projective, a contradiction. This shows (3).

Conversely, let $N = M/M_{\Omega} = M_{\Sigma_P \setminus \Omega}$ with a non-trivial decomposition $N = N_1 \oplus N_2$. Let $M'$ be a the sub-representation of $M$ generated by the elements $N_1(x) \subseteq N(x) = M(x)$ for $x$ in the support of $N$. Let $i_x : M'(x) \to M(x)$, for $x \in Q_0$, be the inclusions defining the inclusion morphism $M' \to M$. Observe now that $(M')_{\Sigma_P \setminus \Omega} = N_1$. For $x \in Q \setminus \Omega$, there are maps $t_x : M(x) = N(x) \to M'(x) = N_1(x)$ such that $t_x i_x = 1_{M'(x)}$ and for any arrow $\alpha : x \to y$ with $x, y \in Q \setminus \Omega$, $M'(\alpha) t_x = t_y M(\alpha)$. Observe that $(M')_{\Sigma_P}, M_{\Sigma_P}$ are projective representations whose tops are supported by $\Sigma_P \setminus \Omega$. Hence, the maps $t_x, x \in \Sigma_P \setminus \Omega$, provide an epimorphism $M_{\Sigma_P \setminus \Omega} \to (M')_{\Sigma_P \setminus \Omega}$ which could be extended to an epimorphism $t : M_{\Sigma_P} \to (M')_{\Sigma_P}$ between projective representations. Therefore, $t$ is a retraction. Thus, for $x \in Q$, we have maps $t_x : M(x) \to M'(x)$ which are compatible with the arrows in $Q \setminus \Omega$ and in $\Omega$. Since any other arrow of $Q_1$ is not supporting $M$, the $t_x$ define an epimorphism $M \to M'$ and is such that $t_x i_x = 1_{M'(x)}$ for $x \in Q \setminus \Omega$. It only remains to show that $t_x i_x = 1_{M'(x)}$ for $x \in \Omega$. Recall that $\Omega$ is top-finite and hence, for $x \in \Omega$, there is a non-negative integer $n_x$ such that every path $y \to x$ with $y \in \Omega$ has length bounded by $n_x$. We proceed by induction on $n_x$. If $n_x = 0$ and every arrow $\alpha : y \to x$ in $Q$ is not supporting $M$, then the elements in $M(x)$ are top elements of $M_{\Sigma_P}$, contradicting the fact the $\Omega$ does not contain the vertices supporting the top of $M_{\Sigma_P}$. Let $\{\alpha_i : y_i \to x\}_{1 \leq i \leq r}$ be the non-empty set of arrows ending in $x$ and supporting $M$. Since $(M')_{\Sigma_P}$ is a projective representation whose top is supported by $\Sigma_P \setminus \Omega$ and $y_i \in \Sigma_P$ for $1 \leq i \leq r$, the map

$$(M'(y_i)) : \oplus_{i=1}^r M'(y_i) \to M'(x)$$

is bijective. Moreover, $t_y i_y = 1_{M'(y_i)}$ for $1 \leq i \leq r$. Now,

$$t_x i_x (M'(y_i)) = t_x (M(y_i)) (i_y) = (M'(y_i)) (i_y) = (M'(y_i))$$

showing that $t_x i_x = 1_{M'(x)}$. Now, if $n_x > 0$, then every arrow $\alpha : y \to x$ supporting $M$ is such that $y \in \Sigma_P$ or $y \in \Omega$ with $n_y < n_x$. Hence, by induction, $t_y i_y = 1_{M'(y)}$. The proof then uses the same argument as above to show that $t_x i_x = 1_{M'(x)}$. This shows that $M'$ is a direct summand of $M$, which is non-trivial and proper since $N_1$
is a non-trivial proper direct summand of \( N \). Hence, \( M \) is decomposable, showing (4). The last part of the lemma is easy to see. \( \square \)

The following result says how the representations in \( \text{rep}(Q) \) are constructed.

**Proposition 3.6.** Let \( M \in \text{rep}(Q) \) with support \( \Sigma \). There exist full subquivers \( \Sigma_P \) and \( \Sigma_I \) of \( \Sigma \) such that

1. \( \Sigma_P \) is top-finite and successor-closed in \( \Sigma \) such that \( M_{\Sigma_P} \) is projective,
2. \( \Sigma_I \) is socle-finite and predecessor-closed in \( \Sigma \setminus \Sigma_P \) such that \( M_{\Sigma_I} \) is injective,
3. \( \Omega := \Sigma \setminus (\Sigma_P \cup \Sigma_I) \) is finite and non-empty,
4. \( M_{\Omega} \) is indecomposable whenever \( M \) is indecomposable; and \( M \) is indecomposable up to projectives and injectives whenever \( M_{\Omega} \) is indecomposable.

Moreover, if \( \Sigma'_P \) is co-finite and successor-closed in \( \Sigma_P \) and \( \Sigma'_I \) is co-finite and predecessor-closed in \( \Sigma_I \), then \( \Sigma'_P \) and \( \Sigma'_I \) also satisfies properties (1) to (4).

**Proof.** By Lemma 3.5, there exists a successor-closed and top-finite subquiver \( \Sigma_P \) of \( \Sigma \) such that \( M_{\Sigma_P} \) is projective in \( \text{rep}^+(Q) \) and \( M/M_{\Sigma_P} \in \text{rep}^-(Q) \). Moreover, if \( M/M_{\Sigma_P} \) is indecomposable, then \( M \) is indecomposable up to projectives; and if \( M \) is indecomposable, then \( M/M_{\Sigma_P} \) is. Since \( M/M_{\Sigma_P} \in \text{rep}^-(Q) \), by the dual of Proposition 3.4, there exists a predecessor-closed subquiver \( \Sigma_I \) of \( \Sigma \setminus \Sigma_P \) such that \( \Sigma_I \) is co-finite in \( \Sigma \setminus \Sigma_P \) and \( (M/M_{\Sigma_P})_{\Sigma_I} = M_{\Sigma_I} \) is injective in \( \text{rep}^-(Q) \). It is easy to see that \( \Sigma \setminus (\Sigma_P \cup \Sigma_I) \) is finite. By Proposition 3.4, \( \Sigma_P \) and \( \Sigma_I \) can be chosen so that \( \Omega := \Sigma \setminus (\Sigma_P \cup \Sigma_I) \) is non-empty. Moreover, since \( M \in \text{rep}(Q) \), we can assume that \( \Omega \) is large enough so that any arrow attached to \( \Sigma_P \) and \( \Sigma_I \) does not support \( M \). Suppose now that \( M \) is indecomposable. Then \( M/M_{\Sigma_P} \) is indecomposable by Lemma 3.5. Moreover, by the dual of Proposition 3.4, one can choose \( \Sigma_I \) in such a way that

\[
(M/M_{\Sigma_P})_{\Sigma \setminus \Sigma_I} = M_{\Omega}
\]

is indecomposable. Conversely, suppose that \( M_{\Omega} \) is indecomposable. Then any non-trivial decomposition of \( M \) yields an indecomposable direct summand \( Z \) of \( M \) supported by \( \Sigma_P \cup \Sigma_I \). Since no arrow supporting \( Z \) joins \( \Sigma_P \) to \( \Sigma_I \), we see that \( \text{supp}(Z) \subseteq \Sigma_P \) or \( \text{supp}(Z) \subseteq \Sigma_I \). In the first case, \( Z \) is a direct summand of \( M_{\Sigma_P} \) and hence is projective in \( \text{rep}^+(Q) \). In the second case, \( Z \) is injective in \( \text{rep}^-(Q) \). The last part of the statement follows similarly. \( \square \)

### 4. Irreducible Morphisms in \( \text{rep}(Q) \)

Let \( C \) be any additive \( k \)-category. A morphism \( f: X \to Y \) is said to be *irreducible* if it is neither a section nor a retraction, and any factorization \( f = gh \) imply that \( h \) is a section or \( g \) is a retraction. In this section, we prove that the irreducible morphisms in \( \text{rep}(Q) \) are all contained in the Auslander-Reiten sequences of \( \text{rep}(Q) \).

For simplicity, an indecomposable representation \( M \in \text{rep}(Q) \) which is neither finitely presented nor finitely co-presented will be called *doubly-infinite*, since its support contains a left-infinite path and a right-infinite path.

Now, we need some notations for the lemmas presented in this section. Fix \( M, N \) two doubly-infinite indecomposable representations in \( \text{rep}(Q) \) with a non-isomorphism \( f: M \to N \). Let \( \Sigma \) be the support of \( M \oplus N \). We can deduce from Proposition 3.4 that there exist a successor-closed subquiver \( \Sigma_P \) of \( \Sigma \) and a predecessor-closed subquiver \( \Sigma_I \) of \( \Sigma \setminus \Sigma_P \) such that \( M_{\Sigma_P}, N_{\Sigma_P} \) are projective in
Figure 1. The subdivisions of the quiver $\Sigma$

rep$^+(Q)$, $M_{\Sigma_I}$, $N_{\Sigma_I}$ are injective in rep$^-(Q)$ and $M_{\Theta}$, $N_{\Theta}$, where $\Theta = \Sigma \setminus (\Sigma_P \cup \Sigma_I)$, are finite dimensional and indecomposable.

Since $\Sigma_I$ is clearly infinite, there exists a finite successor-closed subquiver $\Theta'$ of $\Sigma_I$ such that:

1. $\Theta'$ supports the socle of $(M \oplus N)_{\Sigma_I}$,
2. every arrow $x \to y$ supporting $M \oplus N$ with $x \in \Sigma_I \setminus \Theta'$ is such that $y \in \Sigma_I$.

Such a finite quiver $\Theta'$ exists since $M \oplus N$ is a finite extension-representation of $(M \oplus N)_{\Sigma_I}$ by $(M \oplus N)_{\Sigma_P}$, and $\Theta = \Sigma \setminus (\Sigma_P \cup \Sigma_I)$.

First, let us show that $f : \Sigma \setminus \Sigma_I$ can be assumed not to be an isomorphism. Otherwise, there exists a co-finite and predecessor-closed subquiver $\Sigma'_I$ of $\Sigma_I$ such that $f : \Sigma'_I$ is not an isomorphism. By the second part of Proposition 3.6, $\Sigma_P$ and $\Sigma'_I$ satisfy the same properties stated in the introduction of this section. We then set $\Sigma_I := \Sigma'_I$.

The subquivers $\Theta$ and $\Lambda$ need also to be changed according to the new definition of $\Sigma_I$. We start with the following lemma.

**Lemma 4.1.** Let $L \in \text{rep}(Q)$ such that $\text{supp}(L) \subseteq \Sigma$ and any arrow $\alpha : x \to y$ supporting $L$ with $x \in \Sigma_I \setminus \Lambda$ is such that $y \in \Sigma_I$. Then the restriction map:

$$\psi_L : \text{Hom}(L, X) \to \text{Hom}(L, X)$$

is an isomorphism of $k$-vector spaces, where $X = M$ or $X = N$.

**Proof.** We only consider the case where $X = N$. It is clear that $\psi_L$ is $k$-linear. If $h : L \to N$ is such that $h_\Lambda = 0$, then $h_\Delta = 0$. Now, consider the morphism $h_{\Sigma_I} : L_{\Sigma_I} \to N_{\Sigma_I}$ where $N_{\Sigma_I}$ is injective in rep$(Q)$. By the construction of $\Lambda$, $N_\Lambda$ is an essential sub-representation of $N_{\Sigma_I}$. Therefore, if $\text{Im}(h_{\Sigma_I})$ is non-zero, then it has a non-zero intersection with $N_\Lambda$, contradicting the fact that $h_\Delta = 0$. Hence, $h_{\Sigma_I} = 0$ yielding $h = 0$. This shows that $\psi_L$ is injective. Conversely, let $g : L_\Delta \to N_\Delta$ be any morphism. Consider the canonical inclusions $i_L : L_\Lambda \to L_{\Sigma_I}$ and $i_N : N_\Lambda \to N_{\Sigma_I}$. Since $N_{\Sigma_I}$ is injective, there exists a morphism $v : L_{\Sigma_I} \to N_{\Sigma_I}$ such that $vi_L = i_N g_\Lambda$. Now, the morphisms $g : L_\Delta \to N_\Delta$ and $v : L_{\Sigma_I} \to N_{\Sigma_I}$...
Hence, \( f_{L} \) extremity in \( \Sigma \) \( I \) such that \((L)\) the morphisms \((\alpha)\) and the other in \( \Sigma \) \( \setminus I \). Since there is no arrow from \( \Sigma \) \( I \) to \( \Sigma \) \( P \) supporting \( M \), we see that \( Z \) is a direct summand of \( M \), a contradiction. Thus, \( M_{\Delta} \) is indecomposable and similarly, \( N_{\Delta} \) is indecomposable. \( \square \)

**Lemma 4.3.** If \( f \) is an irreducible monomorphism, then \( f_{\Delta} \) is irreducible in \( \text{rep}^{\ast}(\Delta) \).

**Proof.** Suppose that \( f : M \rightarrow N \) is an irreducible monomorphism, which will be assumed to be an inclusion. By Lemma 4.1, \( f_{\Delta} \) is a monomorphism which is neither a section nor a retraction. Suppose that \( f_{\Delta} = vu \) where \( u : M_{\Delta} \rightarrow L, v : L \rightarrow N_{\Delta} \) and \( L \in \text{rep}^{\ast}(\Delta) \). Since \( \Lambda \) is successor-closed in \( \Sigma \), it is a convex full subquiver of \( \Sigma \). Hence, \( M_{\Delta} \) and \( N_{\Delta} \) are injective in \( \text{rep}(\Delta) \) since \( M_{\Sigma_{j}} \) and \( N_{\Sigma_{j}} \) are injective in \( \text{rep}(Q) \). Therefore, \( u_{\Lambda} \) and \( f_{\Lambda} \) are section maps. Hence, \( L_{\Lambda} = M_{\Lambda} \oplus Z \) where \( Z \in \text{rep}(\Lambda) \). Thus, one may choose to write \( u_{\Lambda} = (\mathbf{1}, 0)^{T} \) and \( v_{\Lambda} = (s, g) \) where \( s \) is a section. Since \( N_{\Sigma_{j}} \) is injective, there exists \( s' : M_{\Sigma_{j}} \rightarrow N_{\Sigma_{j}} \) such that \((s')_{\Lambda} = s \). Let \( i : N_{\Lambda} \rightarrow N_{\Sigma_{j}} \) be the canonical inclusion. Consider the representation \( L' = M_{\Sigma_{j}} \oplus Z \) of \( \Sigma \) with \( u' : M_{\Sigma_{j}} \rightarrow L' \) and \( v' : L' \rightarrow N_{\Sigma_{j}} \) where \( u' = (\mathbf{1}, 0)^{T} \) and \( v' = (s', ig) \). It is clear that \((v'u')_{\Lambda} = vu \). Now, the representation \( L' \) of \( \Sigma \) together with the representation \( L \) of \( \Delta \) are such that \((L')_{\Lambda} = L_{\Lambda} \) and thus yield a representation \( L'' \) of \( \Sigma \) such that if \( \alpha : x \rightarrow y \) is an arrow with one extremity in \( \Sigma \setminus \Lambda \) and the other in \( \Sigma \setminus \Sigma_{j} \), then \( L''(\alpha) = 0 \). It is easy to see that \( L'' \in \text{rep}(Q) \). Observe also that if \( \alpha : x \rightarrow y \) is an arrow with one extremity in \( \Sigma \setminus \Lambda \) and the other in \( \Sigma \setminus \Sigma_{j} \), then \( M(\alpha) = 0 \). Using this and the fact that \( \Delta \cap \Sigma_{j} = \Lambda \), the morphisms \( u : M_{\Delta} \rightarrow L \) and \( u' : M_{\Sigma_{j}} \rightarrow L' \) give a morphism \( u'' : M \rightarrow L'' \) such that \((u'')_{\Delta} = u \) and \((u'')_{\Sigma_{j}} = u' \). Similarly, the morphisms \( v : L \rightarrow N_{\Delta} \) and \( v' : L' \rightarrow N_{\Sigma_{j}} \) give a morphism \( v'' : L'' \rightarrow N \) such that \((v'')_{\Delta} = v \) and \((v'')_{\Sigma_{j}} = v' \). Hence, \( f = v''u'' \) by Lemma 4.1. Since \( f \) is irreducible in \( \text{rep}(Q) \), either \( u'' \) is a section, or \( v'' \) is a retraction. Thus, \( u \) is a section or \( v \) is a retraction. This shows that \( f_{\Delta} \) is irreducible. \( \square \)

We are now ready to prove the main result of this section.

**Proposition 4.4.** Let \( f : M \rightarrow N \) be an irreducible morphism in \( \text{rep}(Q) \) with \( M, N \) indecomposable. Then we have four possible cases.

1. \( M, N \in \text{rep}^{\ast}(Q) \),
2. \( M, N \in \text{rep}^{-}(Q) \),
3. \( M \in \text{rep}^{-}(Q) \) is infinite dimensional and \( N \) is doubly-infinite,
4. \( N \in \text{rep}^{\ast}(Q) \) is infinite dimensional and \( M \) is doubly-infinite.

**Proof.** Suppose that \( N \in \text{rep}^{\ast}(Q) \). If \( N \) is projective, then the inclusion \( \text{rad}(N) \rightarrow N \) is minimal right almost split in \( \text{rep}(Q) \) and hence also minimal right almost split in \( \text{rep}(Q) \). Therefore, \( M \) is a direct summand of the radical of \( N \) and hence is
projective in \( \text{rep}^+(Q) \). If \( N \) is not projective, then one has an almost split sequence
\[
0 \to N' \to E \to N \to 0
\]
in \( \text{rep}(Q) \) with \( N' \in \text{rep}^-(Q) \), which is also an almost split sequence in \( \text{rep}(Q) \) by Lemma 2.3. Then \( M \) is a direct summand of \( E \). If \( N' \) is finite dimensional, then \( M \in \text{rep}^+(Q) \). If \( N \) is finite dimensional, then \( M \in \text{rep}^-(Q) \). Otherwise, by [6, Corollary 3.3], either \( M \) is finite dimensional or doubly-infinite. Hence, if \( N \in \text{rep}^+(Q) \), then (1), (2) or (4) hold. Dually, if \( M \in \text{rep}^-(Q) \), (1), (2) or (3) hold.

We need to show that \( M, N \) cannot be both doubly-infinite. Suppose it is the case. We only consider the case where \( f \) is a monomorphism. The case where \( f \) is an epimorphism is treated in a similar way.

We may assume that \( f \) is an inclusion, that is, \( M \) is a sub-representation of \( N \). We set \( \Sigma \) to be the support of \( M \oplus N \) and use the notation introduced at the beginning of this section for the subquivers \( \Sigma_P, \Sigma_I, \Delta, \Delta', \Theta \) and \( \Lambda \) of \( \Sigma \). By Lemma 4.2, \( M_\Delta, N_\Delta \) are indecomposable. By Lemma 4.3, \( f_\Delta \) is an irreducible monomorphism. Let us first assume that \( N_\Delta \) is projective in \( \text{rep}^+(\Delta) \). Then \( M_\Delta \) is also projective in \( \text{rep}^+(\Delta) \) and \( M_\Delta \) is a direct summand of the radical of \( N_\Delta \).

Since \( a \) is a source vertex in \( \Delta \), \( N_\Delta \cong P_a \) and hence \( a \) is not in the support of \( M_\Delta \). Now, there exists an arrow \( a \to c \) in \( \Delta \) with \( c \in \text{supp}(M_\Delta) \). By the construction of \( \Lambda, c \in \Lambda \). Since \( M_\Sigma \) is injective in \( \text{rep}(Q) \), \( M_\Delta \) is injective in \( \text{rep}(\Lambda) \). Therefore, \( f_\Lambda \) is a section. Since \( N_\Lambda \) has a simple top, it is indecomposable. Hence \( f_\Lambda \) is an isomorphism. But this is impossible since \( M_\Lambda(a) = 0 \). This contradiction shows that \( N_\Delta \) is not projective. Therefore, we have an almost split sequence
\[
\eta: 0 \to W' \to E' \to N_\Delta \to 0
\]
in \( \text{rep}(\Delta) \).

Observe that \( M_\Delta \) is a direct summand of \( E' \) and \( W' \) is finite dimensional since \( W' \in \text{rep}^-(\Delta) = \text{rep}^+(\Delta) \). Thus, we get an irreducible monomorphism \( W' \to M_\Delta \) whose image \( W \) is a proper sub-representation of \( M_\Delta \). We claim that \( W_\Lambda \neq M_\Lambda \).

Suppose the contrary. In particular, \( W_\Lambda \) is injective in \( \text{rep}(\Lambda) \), and also in \( \text{rep}^+(\Delta) \) since \( \Lambda \) is predecessor-closed in \( \Delta \). By the dual of [3, Lemma 2.5], \( N_\Delta \) is constructed from \( W \cong W' \) in the following way. Take a minimal injective co-resolution
\[
(*) : 0 \to W \to \bigoplus_{i=1}^r I'_{x_i} \to \bigoplus_{j=1}^s I'_{y_j} \to 0
\]
of \( W \) where the \( x_i, y_j \) are vertices in \( \Delta \) and \( I'_{x_i} \), for \( x \in \Delta_0 \), denotes the injective representation at \( x \) in \( \text{rep}^-(\Delta) \). Then \( N_\Delta \) is (isomorphic to) the cokernel of the corresponding map
\[
\bigoplus_{i=1}^r P'_{x_i} \to \bigoplus_{j=1}^s P'_{y_j}
\]
where \( P'_{x_i} \), for \( x \in \Delta_0 \), denotes the projective representation at \( x \) in \( \text{rep}^+(\Delta) \). In particular, the support of \( \text{top}(N_\Delta) \) consists of the \( y_j \). Recall that \( \Lambda \) is finite, contains a source vertex \( a \), an arrow \( a \to b \) and has the property that every arrow in \( Q \) starting in \( a \) and supporting \( M \oplus N \) has an ending point in \( \Lambda \). Moreover, \( a \) does not lie in \( \text{supp}(\text{soc}(M \oplus N)_\Lambda) = \text{supp}(\text{soc}(N_{\Sigma_\Lambda})) \). Since \( a \) is a source vertex in \( \Delta \) supporting \( N_\Delta, a \in \text{supp}(\text{top}N_\Delta) \), which means that \( a = y_t \) for some \( t \). For simplicity, set \( I_0 = \bigoplus_{i=1}^r I'_{x_i} \), and \( I_1 = \bigoplus_{j=1}^s I'_{y_j} \). Let \( \alpha_1, \ldots, \alpha_q \) be the arrows in \( \Delta \) starting in \( a \), where \( \alpha_i : a \to b_i, i = 1, 2, \ldots, q \). Since \( W \) is indecomposable

non-injective in $\text{rep}(\Delta)$ and $a$ is a source vertex in $\Delta$, $a \not\in \text{supp}(\text{soc}W)$. This means that no $x_i$ is equal to $a$. Thus, by the injectivity of $I_0$, we get that

$$\sum_{i=1}^{q} I_0(\alpha_i) : I_0(a) \to \bigoplus_{i=1}^{q} I_0(b_i)$$

is bijective. In particular,

$$\dim_k(I_0(a)) = \sum_{i=1}^{q} \dim_k(I_0(b_i)).$$

Now, since $W_\Lambda = M_\Lambda$,

$$\text{supp}(\text{soc}(W_\Lambda)) \subseteq \text{supp}(\text{soc}(N_\Lambda)) = \text{supp}(\text{soc}(N_{\Sigma_j})).$$

By the definition of the vertex $a$, $a \not\in \text{supp}(\text{soc}W_\Lambda)$. Also, since $W_\Lambda$ is injective and every non-zero $W(\alpha_i)$ is equal to $W_\Lambda(\alpha_i)$, one also has

$$\dim_k(W(a)) = \sum_{i=1}^{q} \dim_k(W(b_i)).$$

Therefore, using $(*)$,

$$\dim_k(I_1(a)) = \sum_{i=1}^{q} \dim_k(I_1(b_i)).$$

The last equality is true if and only if $a$ does not support the socle of $I_1$, that is, if and only if $I'_a$ is not a direct summand of $I_1$. This means that $a \neq y_j$ for all $j$, a contradiction to $a = y_t$. This proves the claim, that is, $W_\Lambda \neq M_\Lambda$.

Suppose now that the support of $W$ is contained in $\Lambda$. By restricting $\eta$ to $\Sigma \setminus \Sigma_I$, one gets $(E')_{\Sigma \setminus \Sigma_I} \cong N_{\Sigma \setminus \Sigma_I}$. This yields $M_{\Sigma \setminus \Sigma_I} = N_{\Sigma \setminus \Sigma_I}$, since $N_{\Sigma \setminus \Sigma_I}$ is indecomposable. But as observed above, $f_{\Sigma \setminus \Sigma_I}$ is not an isomorphism. This contradiction shows that $\text{supp}(W)$ is not contained in $\Lambda$. Let $L$ be the sub-representation of $M_\Delta$ generated by $M_{\Sigma \setminus \Sigma_I}$ and $W$. Since $W$ is finite dimensional, we have $L \in \text{rep}^+(\Delta)$.

Since $M_{\Sigma \setminus \Sigma_I}$ is indecomposable, any proper decomposition of $L$ would yield a proper section $s_L : L' \to L$ with $L'$ supported by $\Lambda$. Hence $s_L$ factors through $W$, which means that $L'$ is a direct summand of $W$. Since $W$ is indecomposable, $L' = W$, contradicting $\text{supp}(W) \not\subseteq \Lambda$. This shows that $L$ is indecomposable. Hence, we have a proper inclusion $W \to L$ between indecomposable representations. Since $W_\Lambda \neq M_\Lambda$, we have another proper inclusion $L \to M_\Delta$ between indecomposable representations of $\text{rep}^+(\Delta)$. This contradicts the fact that the inclusion $W \to M_\Delta$ is irreducible. $\square$

5. The Auslander-Reiten Quiver of $\overline{\text{rep}}(Q)$

In this section, we assume that $Q$ is a connected strongly locally finite quiver and $\overline{\text{rep}}(Q)$ is the full abelian subcategory of $\text{rep}(Q)$ of those objects being finite extension-representations of objects in $\text{rep}^-(Q)$ by objects in $\text{rep}^+(Q)$. We give a complete description of the Auslander-Reiten quiver of $\overline{\text{rep}}(Q)$ by giving the possible shapes of its connected components.

Let us recall some definitions. Let $\mathcal{C}$ be any skeletally small abelian $k$-category such that every indecomposable object has a local endomorphism algebra. We do not assume that $\mathcal{C}$ is Hom-finite. Let us denote by $\text{rad}_\mathcal{C}$ (or simply $\text{rad}$ when no risk of confusion) the ideal of $\mathcal{C}$ defined as follows. A morphism $f : X \to Y$ lies in
rad(C)(X, Y) if and only if, for every morphism \( g : Y \to X \), \( 1_X - gf \) is an isomorphism. Now, if \( 1_X - gf \) is an isomorphism of inverse \( h \), then a straightforward argument yields that \( 1_Y - fg \) is an isomorphism of inverse \( 1_Y +_fhg \). Hence, \( f \in \text{rad}(X, Y) \) if and only if, for every morphism \( g : Y \to X \), \( 1_Y - fg \) is an isomorphism. The ideal \( \text{rad}(\cdot) \) is known as the radical of \( C \) and a morphism \( f \in \text{rad}(X, Y) \) is said to be a radical morphism. When \( C \) is \( \text{Hom}\)-finite, the description of the radical of \( C \) is given in [7].

It is well known that when \( C \) is \( \text{Hom}\)-finite with \( X, Y \in C \), then \( f : X \to Y \) is non-zero in \( \text{rad}(C)(X, Y)/\text{rad}^2(C)(X, Y) \) if and only if \( f \) is irreducible. This is also true in our setting when \( X, Y \) are indecomposable.

**Lemma 5.1.** A morphism \( f : X \to Y \) in \( C \) with \( X, Y \) indecomposable is irreducible if and only if it is a radical morphism whose class in \( \text{rad}(C)(X, Y)/\text{rad}^2(C)(X, Y) \) is non-zero.

**Proof.** Let \( f : X \to Y \) with \( X, Y \) indecomposable. Assume first that \( f \) is irreducible. Then \( f \) is not an isomorphism. For \( g : Y \to X \), \( gf \in \text{End}(X) \) is not an isomorphism, and hence \( 1_X - gf \) is an isomorphism since \( \text{End}(X) \) is a local algebra. This shows that \( f \in \text{rad}(X, Y) \). Suppose that \( f \in \text{rad}^2(X, Y) \). Hence, \( f = h_1g_1 + \cdots + h_rg_r \) where \( h_i : L_i \to Y \) and \( g_i : X \to L_i \) are radical morphisms. Let \( L = L_1 \oplus \cdots \oplus L_r \), \( h = (h_1, \ldots, h_r) \) and \( g = (g_1, \ldots, g_r)^T \). Then \( f = hg \) and \( g \) is a section or \( h \) is a retraction. Assume that \( g \) is a section. Then, for \( 1 \leq i \leq r \), there exist \( g'_i : L_i \to X \) such that \( g'_ig_1 + \cdots + g'_ig_r = 1_X \). Since \( \text{End}(X) \) is local, this means that \( g'_ig_j \) is invertible (of inverse \( g_j \)) for some \( j \), and hence that \( 1_X - g'_ig_j \) is not invertible, contradicting the fact that \( g_j \) is a radical morphism. Hence, \( g \) is not a section. Similarly, \( h \) is not a retraction. This shows that \( f \) is non-zero in \( \text{rad}(X, Y)/\text{rad}^2(X, Y) \). Conversely, assume that \( f \) is non-zero in \( \text{rad}(X, Y)/\text{rad}^2(X, Y) \). It is clear that \( f \) is not an isomorphism since \( f \in \text{rad}(X, Y) \). Assume that \( f = hg \) with \( g : X \to L \) and \( h : L \to Y \). By assumption, one of \( g, h \) is not a radical morphism. Assume that \( g \) is not a radical morphism. Hence, there exists \( g' : L \to X \) such that \( 1_X - g'g \) is not an isomorphism, meaning that \( g'g \) is an isomorphism, since \( \text{End}(X) \) is local. But then \( g \) is a section. Similarly, if \( h \) is not a radical morphism, then \( h \) is a retraction. \( \square \)

Hence, for \( X, Y \) indecomposable in \( C \), it makes sense to define

\[ \text{irr}(X, Y) := \text{rad}(X, Y)/\text{rad}^2(X, Y) \]

and call it the set of irreducible maps from \( X \) to \( Y \) in \( C \). Let us now turn our attention to the main object of study of the rest of the paper. The Auslander-Reiten quiver of \( C \), denoted \( \Gamma_C \), is a partially valued translation quiver defined as follows; compare [8] (2.1)). The vertex set is a complete set of representatives of the isomorphism classes of the indecomposable objects in \( C \). If \( Z \) is a vertex of \( \Gamma_C \), we denote by \( k_Z \) the division \( k \)-algebra \( \text{End}(Z)/\text{rad}(Z, Z) \). Let \( X, Y \) be two vertices in \( \Gamma_C \). By definition, there exists a unique arrow \( X \to Y \) in \( \Gamma_C \) if and only if \( \text{irr}(X, Y) \) is non-zero. In this case, if \( \text{irr}(X, Y) \) is of finite length over \( k_X \) and \( k_Y \), we attach to the arrow \( X \to Y \) a valuation \( (d_{XY}, d'_{XY}) \) where \( d_{XY} \) and \( d'_{XY} \) are the dimension of \( \text{irr}(X, Y) \) over \( k_X \) and \( k_Y \), respectively. In this case, \( d_{XY} \) and \( d'_{XY} \) are the maximal integers such that \( C \) admits an irreducible morphism \( X^{d_{XY}} \to Y \) and an irreducible morphism \( X \to Y^{d'_{XY}} \), respectively; see [5] (3.4). A valuation \( (d_{XY}, d'_{XY}) \) is called symmetric if \( d_{XY} = d'_{XY} \), and trivial if \( d_{XY} = d'_{XY} = 1 \). For
technical reasons, we replace each arrow \( X \to Y \) having a symmetric valuation \((d_{XY}, d_{YX}) \) by \( d_{XY} \) unvalued arrows from \( X \) to \( Y \). The translation \( \tau \) is defined in such a way that \( \tau Z = X \) if and only if \( C \) has an almost split sequence
\[
0 \to X \to Y \to Z \to 0.
\]

Hence, \( \Gamma_C \) is actually a partially valued translation quiver with multiple arrows in which all possible valuations are non-symmetric. If \( C \) is Hom-finite, then each arrow of \( \Gamma_C \) has a valuation attached to it (which is then replaced by multiple arrows if it is symmetric). A connected component of \( \Gamma_C \) is called an \textit{Auslander-Reiten component} of \( C \).

In this section, we study the Auslander-Reiten quiver of \( \text{rep}(Q) \), which is a Hom-finite abelian \( k \)-category. We will show that all arrows of \( \Gamma_{\text{rep}(Q)} \) have symmetric valuation, and hence that \( \Gamma_{\text{rep}(Q)} \) is a quiver with no valuation. We first need the following lemmas.

**Lemma 5.2.** An irreducible map between indecomposable objects of \( \text{rep}(Q) \) is irreducible in \( \text{rep}(Q) \).

**Proof.** Suppose that \( f : M \to N \) is irreducible in \( \text{rep}(Q) \) with \( M, N \) indecomposable. From Lemma 4.4 either \( M \in \text{rep}^-(Q) \) or \( N \in \text{rep}^+(Q) \). We only consider the first case, that is, \( M \in \text{rep}^-(Q) \). If \( M \) is injective in \( \text{rep}^-(Q) \), then \( h : M \to M/\text{soc}M \) is a minimal left almost split map in \( \text{rep}(Q) \) and hence also a minimal left almost split map in \( \text{rep}(Q) \). Hence, there exists a retraction \( r : M/\text{soc}M \to N \) such that \( f = rh \) which shows that \( f \) is irreducible in \( \text{rep}(Q) \).

Otherwise, we have an almost split sequence
\[
\zeta : 0 \to M \xrightarrow{h} E \to L \to 0
\]
in \( \text{rep}(Q) \) with \( L \in \text{rep}^+(Q) \). In particular, \( \zeta \) is almost split in \( \text{rep}(Q) \). Hence, there exists a retraction \( r : E \to N \) such that \( f = rh \) which shows that \( f \) is irreducible in \( \text{rep}(Q) \).

Recall from [6] that an indecomposable representation \( M \in \text{rep}^+(Q) \) is \textit{regular} in \( \Gamma_{\text{rep}^+(Q)} \) if the connected component of \( \Gamma_{\text{rep}^+(Q)} \) containing \( M \) does not contain a representation of the form \( P_x \) or \( I_x \), \( x \in Q_0 \).

**Lemma 5.3.** Let \( M \) be an infinite dimensional regular representation in \( \Gamma_{\text{rep}^+(Q)} \). Then \( \text{rep}(Q) \) has a minimal right almost split morphism \( h : N_1 \oplus N_2 \to M \) with \( N_1 \) indecomposable doubly-infinite and \( N_2 \) finite dimensional.

**Proof.** There exists an almost split sequence
\[
\eta : 0 \to M' \to E \to M \to 0
\]
in \( \text{rep}(Q) \) where \( M' \) is finitely co-presented. Since \( M \) is infinite dimensional, \( E \) is infinite dimensional. Let \( L \) be an infinite dimensional direct summand of \( E \). There are irreducible morphisms \( f : M' \to L \) and \( g : L \to M \) in \( \text{rep}(Q) \). Let us first assume that \( M' \) is infinite dimensional. We claim that in this case, there exists a left infinite path \( p \) in \( \text{supp}(M') \) such that \( p \cap \text{supp}(L) \) is infinite. Assume first that \( f \) is an epimorphism. Since \( \text{supp}(L) \) is infinite and \( \text{supp}(L) \subseteq \text{supp}(M') \) with \( \text{supp}(M') \) socle-finite, there exists a left infinite path
\[
p : \cdots \to x_3 \to x_2 \to x_1
\]
in \( \text{supp}(M') \) such that infinitely many \( x_i \) lie in the support of \( L \), showing the claim in this case. Suppose now that \( f \) is a monomorphism. Since \( M' \) is infinite dimensional, \( \text{supp}(M') \) contains a left infinite path \( p \), and \( \text{supp}(M') \subseteq \text{supp}(L) \) yields the claim in this case.

Now, we show that \( g \) is an epimorphism. Suppose first that \( M' \) is finite dimensional. Then \( \eta \) is an almost split sequence in \( \text{rep}^+ (Q) \), and hence \( L \) is finitely presented. Now, since \( \text{rep}^+ (Q) \) is Krull-Schmidt, \( L \) has an indecomposable infinite dimensional direct summand \( L_0 \) and from [6 Lemma 4.13(2)], the restriction of \( g \) to \( L_0 \) is an epimorphism. In particular, \( g \) is an epimorphism. Suppose that \( M' \) is infinite dimensional while \( g \) is a monomorphism. By the above claim, there exists a left infinite path \( p \) in \( \text{supp}(M') \) such that infinitely many vertices of \( p \) lie in \( \text{supp}(L) \subseteq \text{supp}(M) \), contradicting the fact that \( \text{supp}(M) \) is top-finite and \( Q \) is interval-finite. Hence, \( g \) is an epimorphism.

Suppose now that \( L_1, L_2 \) are two infinite dimensional representations such that \( L_1 \oplus L_2 \) is a direct summand of \( E \). Then we have epimorphisms \( g_1 : L_1 \to M \) and \( g_2 : L_2 \to M \). Since \( M \) is infinite dimensional, there exists a right infinite path \( y_1 \to y_2 \to y_3 \to \cdots \in \text{supp}(M) \). Since \( \text{supp}(M') \) is socle-finite, there exists some \( y_j \) with \( y_j \notin \text{supp}(M') \). Then,

\[
\dim M(y_j) = \dim (M')(y_j) + \dim M(y_j)
\]

\[
= \dim E(y_j)
\]

\[
\geq \dim L_1(y_j) + \dim L_2(y_j)
\]

\[
\geq \dim M(y_j) + \dim M(y_j),
\]

which is a contradiction. This shows that if \( E = E_1 \oplus E_2 \), then one of \( E_1, E_2 \) is finite dimensional. This proves the lemma. \( \square \)

The following proposition says that the category \( \text{rep}(Q) \) contains most of the Auslander-Reiten theory of \( \text{rep}(Q) \).

**Proposition 5.4.** Let \( \eta : 0 \to X \twoheadrightarrow Y \twoheadrightarrow Z \to 0 \) be a short exact sequence in \( \text{rep}(Q) \). Then \( \eta \) is almost split in \( \text{rep}(Q) \) if and only if it is almost split in \( \text{rep}^+(Q) \). In this case, \( X \in \text{rep}^-(Q) \), \( Z \in \text{rep}^+(Q) \) and either

1. \( Y = Y_1 \oplus Y_2 \), with \( Y_1 \) indecomposable doubly-infinite and \( Y_2 \) zero or indecomposable finite dimensional,
2. The sequence lies in \( \text{rep}^-(Q) \),
3. The sequence lies in \( \text{rep}^+(Q) \).

**Proof.** Suppose that \( \eta \) is almost split in \( \text{rep}(Q) \). By [10 Theorem 3.5], \( X \in \text{rep}^-(Q) \) and \( Z \in \text{rep}^+(Q) \). By Lemma 2.3, \( \eta \) lies in \( \text{rep}(Q) \) and hence is almost split in \( \text{rep}(Q) \). Suppose now that \( \eta \) is almost split in \( \text{rep}(Q) \). Then \( X, Z \) are (strongly) indecomposable. If \( Z \in \text{rep}^+(Q) \), there is an almost split sequence

\[
\xi : 0 \to X' \to Y' \to Z \to 0
\]

in \( \text{rep}(Q) \) which lies in \( \text{rep}(Q) \) by what we have shown. By the unicity of almost split sequences, \( \xi = \eta \) and we are done. We can treat similarly the case where \( X \in \text{rep}^-(Q) \). Assume now that \( X \notin \text{rep}^-(Q) \) and \( Z \notin \text{rep}^+(Q) \). By Proposition 4.4, we must have that \( Y \in \text{rep}^+(Q) \cap \text{rep}^-(Q) \), which is impossible. This shows that \( \eta \) is almost split in \( \text{rep}(Q) \).
Assume now that \( \eta \) is almost split in \( \text{rep}(Q) \) (and hence in \( \text{rep}(Q) \)). If \( X \) is finite dimensional, then (3) holds and if \( Z \) is finite dimensional, then (2) holds. Otherwise, since both \( X, Z \) are infinite dimensional, \( Z \) is regular in \( \Gamma_{\text{rep}^+(Q)} \) and from Lemma 5.3 \( Y = Y_1 \oplus Y_2 \) with \( Y_1 \) indecomposable infinite dimensional and \( Y_2 \) finite dimensional. If \( Y_1 \in \text{rep}^+(Q) \), then \( Y \in \text{rep}^+(Q) \) and hence \( X \in \text{rep}^+(Q) \). Being in \( \text{rep}^-(Q) \), we get \( X \in \text{rep}^-(Q) \), a contradiction. Hence, \( Y_1 \not\in \text{rep}^+(Q) \). Similarly, \( Y_1 \not\in \text{rep}^-(Q) \). Therefore, \( Y_1 \) is doubly-infinite. Now, if \( Y_2 \) is non-zero, then there is an irreducible map \( Y_2 \to Z \) in \( \text{rep}^+(Q) \). From [6 Theorem 4.14] (see also Theorem 5.9), \( Y_2 \) is indecomposable. This proves that one of (1), (2) or (3) hold.

Unfortunately, there may be irreducible maps \( M \to N \) in \( \text{rep}(Q) \) with \( M, N \) indecomposable but not in \( \text{rep}(Q) \). Hence, the Auslander-Reiten quiver of \( \text{rep}(Q) \) misses some irreducible maps of \( \text{rep}(Q) \). However, in the next section, we shall see that these irreducible morphisms are isolated from the irreducible morphisms in \( \text{rep}(Q) \).

**Example 5.5.** Let \( Q \) be the following quiver

![Diagram of the quiver](image)

of type \( A_\infty \) with zigzag orientation. Consider the indecomposable sincere representation \( M \) such that \( M(i) = k \) for all \( i \in \mathbb{N} \). Let \( N \) be the quotient of \( M \) by the simple representation at the vertex 0. Then the morphism \( M \to N \) is irreducible in \( \text{rep}(Q) \) with \( M, N \not\in \text{rep}(Q) \). Observe, however, that there is no almost split sequence in \( \text{rep}(Q) \) starting or ending in \( M \) or \( N \), by Proposition 5.4.

**Lemma 5.6.** Let \( f : M \to N \) be a morphism in \( \text{rep}^+(Q) \). Then \( f \) is irreducible in \( \text{rep}^+(Q) \) if and only if \( f \) is irreducible in \( \text{rep}^+(Q) \).

*Proof.* We only need to prove the necessity. Suppose that \( f \) is irreducible in \( \text{rep}^+(Q) \). Let \( L \in \text{rep}^+(Q) \) with two morphisms \( u : M \to L, v : L \to N \) such that \( f = vu \). Let \( S \) be the set of vertices \( x \) in \( Q \) such that there exists an arrow \( \alpha : x \to y \) with \( x \not\in \text{supp}(M \oplus N) \), \( y \in \text{supp}(M \oplus N) \) and \( L(\alpha) \neq 0 \). Since \( L \in \text{rep}^+(Q) \), there exists a top-finite successor-closed subquiver \( \Omega \) of \( \text{supp}(L) \) such that \( L_\Omega \in \text{rep}^+(Q) \) and there is a finite number of arrows \( \beta : a \to b \) with \( a \in \text{supp}(L) \setminus \Omega \), \( b \in \Omega \) and \( L(\beta) \neq 0 \). The successor-closed subquiver \( \Omega' \) of \( Q \) generated by \( \Omega \) and \( \text{supp}(M \oplus N) \) is top-finite and is such that \( L_{\Omega'} \in \text{rep}^+(Q) \). By Lemma 2.2 since \( L \in \text{rep}^+(Q) \), \( L \) is a finite extension-representation of \( L/L_{\Omega'} \) by \( L_{\Omega'} \). In particular, \( S \cap \text{supp}(L) \setminus \Omega' \) is finite. Therefore, the successor-closed subquiver \( \Sigma \) of \( Q \) generated by \( S \) and \( \Omega' \) is top-finite with \( L_\Sigma \in \text{rep}^+(Q) \). Thus, we have a factorization \( f = v_\Sigma u_\Sigma \in \text{rep}^+(Q) \). Therefore, \( u_\Sigma \) is a section or \( v_\Sigma \) is a retraction. Since \( \Sigma \) is successor-closed in \( Q \) and contains the vertices in \( S \), one easily checks that \( u_\Sigma \) is a section if and only if \( u \) is a section; and \( v_\Sigma \) is a retraction if and only if \( v \) is a retraction. This shows that \( f \) is irreducible in \( \text{rep}^+(Q) \).

Let \( \Gamma \) be a connected component of \( \Gamma_{\text{rep}^+(Q)} \). Then \( \Gamma \) is said to be preprojective if it contains a projective object in \( \text{rep}^+(Q) \) and preinjective if it contains an injective object in \( \text{rep}^-(Q) \). Otherwise, it is called regular. A full convex and connected subquiver \( \Delta \) of \( \Gamma \) is a section if it contains no oriented cycle and meets every \( \tau \)-orbit of \( \Gamma \) exactly once. It is right-most if \( \tau X \) is not defined for every \( X \in \Delta \); and left-most if \( \tau^{-1} X \) is not defined for every \( X \in \Delta \).
**Theorem 5.7.** Let $Q$ be connected infinite and strongly locally finite. Then $\Gamma_{\text{rep}}(Q)$ contains a unique preprojective component $\mathcal{P}_Q$ having a left-most section $P_Q$ consisting of all the indecomposable projective objects in $\text{rep}^+(Q)$. Moreover,

1. If $Q$ has no right infinite path, then $\mathcal{P}_Q$ is of shape to $NQ^{op}$.
2. Otherwise, it is a predecessor-closed subquiver of $\mathcal{P}_Q$ having a right-most section consisting of the infinite dimensional representations of $\mathcal{P}_Q$.

**Proof.** The statement has been proven for the category $\text{rep}^+(Q)$ in [6]. Let $\Gamma$ be the unique preprojective component of $\Gamma_{\text{rep}}^+(Q)$ and $X \in \Gamma$. If $X$ is not projective in $\text{rep}^+(Q)$, then one has an almost split sequence

$$\eta: 0 \to X' \to E \to X \to 0$$

in $\text{rep}(Q)$. Since $X$ is preprojective in $\text{rep}^+(Q)$, $X'$ is finite dimensional and the sequence is almost split in $\text{rep}^+(Q)$ and also in $\text{rep}(Q)$. In particular, we have a minimal right almost split morphism $E \to X$ in $\text{rep}^+(Q)$ which is also minimal right almost split in $\text{rep}(Q)$. This will also be the case if $X$ is projective in $\text{rep}^+(Q)$.

Suppose first that we have an arrow $\alpha: Y \to X$ in $\Gamma_{\text{rep}}(Q)$. Using what we just proved, $Y \in \text{rep}^+(Q)$ and we get an arrow $\alpha': Y \to X$ in $\Gamma$. The valuations of $\alpha$ and $\alpha'$ need to coincide by Lemma 5.6. Hence, $\Gamma$ is a predecessor-closed subquiver of $\Gamma_{\text{rep}}(Q)$.

Suppose now that we have an arrow $\beta: X \to Y$ in $\Gamma_{\text{rep}}(Q)$. We have an irreducible map $f: X \to Y$ in $\text{rep}(Q)$, which needs to be irreducible in $\text{rep}^+(Q)$ by Proposition 4.4. Therefore, we have an arrow $\beta': X \to Y$ in $\Gamma$, and the valuations of $\beta$ and $\beta'$ coincide by Lemma 5.4. This shows that $\Gamma$ is a successor-closed subquiver of $\Gamma_{\text{rep}}(Q)$. Therefore, $\Gamma$ is a connected component of $\Gamma_{\text{rep}}(Q)$, and consequently, since it contains all the $P_x$, $x \in Q_0$, is the unique preprojective component of $\Gamma_{\text{rep}}(Q)$.

A dual argument yields the following dual result for the preinjective component.

**Theorem 5.8.** Let $Q$ be connected infinite and strongly locally finite. Then $\Gamma_{\text{rep}}(Q)$ contains a unique preinjective component $\mathcal{I}_Q$ having a right-most section $I_Q$ consisting of all the indecomposable injective objects in $\text{rep}^-(Q)$. Moreover,

1. If $Q$ has no left infinite path, then $\mathcal{I}_Q$ is isomorphic to $N^{-1}Q^{op}$.
2. Otherwise, it is a successor-closed subquiver of $N^{-1}Q^{op}$ having a left-most section consisting of the infinite dimensional representations of $\mathcal{I}_Q$.

Recall from [6] or [12] that a valued translation quiver is said to be of (finite) **wing type** if it is isomorphic to the following translation quiver with trivial valuations:

![Diagram](https://via.placeholder.com/150)

The following theorem was proven in [6]. An indecomposable representation $M$ in $\text{rep}^+(Q)$ is **pseudo-projective** if it is not projective and the almost split sequence

$$0 \to M' \to E \to M \to 0$$

in $\text{rep}(Q)$ is such that $M'$ is infinite dimensional.
**Theorem 5.9** (Bautista, Liu and Paquette). *Let $Q$ be an infinite, connected and strongly locally finite quiver. Let $\Gamma$ be a regular component of $\Gamma_{\text{rep}^+(Q)}$.  

(1) If $\Gamma$ has no infinite dimensional or pseudo-projective representation, then it is of shape $\mathbb{Z}A_\infty$.

(2) If $\Gamma$ has infinite dimensional but no pseudo-projective representations, then it is of shape $\mathbb{N}^-A_\infty$ and its right-most section is a left infinite path.

(3) If $\Gamma$ has pseudo-projective but no infinite dimensional representations, then it is of shape $\mathbb{N}A_\infty$ and its left-most section is a right infinite path.

(4) If $\Gamma$ has both pseudo-projective and infinite dimensional representations, then $\Gamma$ is finite of wing type.*

*We have a similar theorem for the category $\text{rep}(Q)$.*

**Theorem 5.10.** *Let $Q$ be a connected infinite and strongly locally finite quiver. Let $\Gamma$ be a regular component of $\Gamma_{\text{rep}(Q)}$.  

(1) If $\Gamma$ has no infinite dimensional representation, then it is of shape $\mathbb{Z}A_\infty$.

(2) If $\Gamma$ has infinite dimensional representations all lying in $\text{rep}^+(Q)$, then it is of shape $\mathbb{N}^-A_\infty$ and its right-most section is a left infinite path.

(3) If $\Gamma$ has infinite dimensional representations all lying in $\text{rep}^-(Q)$, then it is of shape $\mathbb{N}A_\infty$ and its left-most section is a right infinite path.

(4) Otherwise, $\Gamma$ is finite of wing type and contains exactly one doubly-infinite representation.*

*Proof. If $\Gamma$ contains only finite dimensional representations, then for any $X \in \Gamma$, one has almost split sequences

$$0 \to X' \to E \to X \to 0$$

and

$$0 \to X \to E' \to X'' \to 0$$

in $\text{rep}(Q)$ which are almost split in $\text{rep}^+(Q)$. Therefore, $\Gamma$ is a component of the Auslander-Reiten quiver of $\text{rep}^+(Q)$ by Lemma 5.6. Hence, $\Gamma$ is of shape $\mathbb{Z}A_\infty$ by Theorem 5.9.

Suppose now that $\Gamma$ has infinite dimensional representations all lying in $\text{rep}^+(Q)$. Then for any $X \in \Gamma$, the almost split sequence

$$0 \to X' \to E \to X \to 0$$

in $\text{rep}(Q)$ is an almost split sequence in $\text{rep}^+(Q)$. Moreover, by Proposition 4.4, every irreducible morphism $X \to Y$ in $\text{rep}(Q)$ with $Y$ indecomposable is such that $Y \in \text{rep}^+(Q)$. This shows that $\Gamma$ is a (predecessor-closed) connected component of the Auslander-Reiten quiver of $\text{rep}^+(Q)$ by Lemma 5.6. Hence $\Gamma$ is of shape $\mathbb{N}^-A_\infty$ and its right-most section is a left infinite path by Theorem 5.9.

Dually, if $\Gamma$ has infinite dimensional representations all lying in $\text{rep}^-(Q)$, then $\Gamma$ is of shape $\mathbb{N}A_\infty$ and its left-most section is a right infinite path.

Consider now the case where $\Gamma$ contains an infinite dimensional representation in $\text{rep}^+(Q)$ and an infinite dimensional representation in $\text{rep}^-(Q)$. From Proposition 4.4, we get that the full subquiver $\Gamma'$ of $\Gamma$ consisting of the representations in $\text{rep}^+(Q)$ is successor-closed in $\Gamma$. Similarly, the full subquiver $\Gamma''$ of $\Gamma$ consisting of the representations in $\text{rep}^-(Q)$ is predecessor-closed in $\Gamma$. Now, by Lemma 5.6, $\Gamma'$ is a connected component of the Auslander-Reiten quiver of $\text{rep}^+(Q)$. If $\Gamma''$ is left stable as a translation quiver, then we see that $\Gamma = \Gamma'$, a contradiction. Hence,
\( \Gamma' \) is not left stable and contains infinite dimensional representations in \( \text{rep}^+(Q) \). Since \( \Gamma' \) does not contain any of the \( P_i \) and \( I_x, x \in Q_0 \), we see that \( \Gamma' \) is a regular component of the Auslander-Reiten quiver of \( \text{rep}^+(Q) \). Hence, \( \Gamma' \) is of wing type with trivial valuations by Theorem 5.9. We get similarly that \( \Gamma'' \) is of wing type. Hence, \( \Gamma' \cup \Gamma'' \) is a full subquiver of \( \Gamma \) which has trivial valuations and which is of the form

\[
\begin{array}{cccccc}
X_{n,1} & X_{n,2} & \cdots & X_{n-1,2} & X_{n-1,1} & X_{n-1,3} \\
\vdots & & \ddots & & \vdots & \vdots \\
X_{2,1} & & \cdots & & \cdots & \cdots \\
X_{1,1} & & \cdots & & \cdots & X_{1,n} \\
\end{array}
\]

where the \( X_{1,1}, \ldots, X_{n,1} \) are the infinite dimensional representations in \( \Gamma'' \) and \( X_{n,2}, X_{n-1,3}, \ldots, X_{1,n+1} \) are the infinite dimensional representations in \( \Gamma' \). By Proposition 5.4, the vertices in \( \Gamma \setminus (\Gamma' \cup \Gamma'') \) are all doubly-infinite representations. Since \( \Gamma' \) is successor-closed in \( \Gamma \) and \( \Gamma'' \) is predecessor-closed in \( \Gamma \), the arrows in \( \Gamma \) attached to a vertex in \( \Gamma' \cup \Gamma'' \) and which are not in \( \Gamma' \cup \Gamma'' \) start in \( X_{n,1} \) or end in \( X_{n,2} \). By Proposition 5.4 since \( X_{n,1}, X_{n,2} \) are infinite dimensional, there is an almost split sequence

\[ \eta: 0 \to X_{n,1} \to E \to X_{n,2} \to 0 \]

in \( \text{rep}(Q) \) where \( E \cong X_{n-1,2} \oplus E' \) and \( E' \) is doubly infinite. By Proposition 5.4 it is clear that the only arrow ending in \( E' \) is \( X_{n,1} \to E' \) and the only arrow starting in \( E' \) is \( E' \to X_{n,2} \). Hence, \( \Gamma \) is of shape

\[
\begin{array}{cccccc}
E' & \cdots & X_{n,2} & X_{n,1} & \cdots & X_{n-1,2} & X_{n-1,1} & X_{n-1,3} \\
\vdots & \ddots & \cdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
X_{2,1} & & \cdots & & \cdots & X_{2,n} & \cdots & X_{2,2} \\
X_{1,1} & & \cdots & & \cdots & X_{1,n} & X_{1,n+1} \\
\end{array}
\]

and it remains to show that the arrows \( X_{n,1} \to E' \) and \( E' \to X_{n,2} \) are trivially valued. Let \( (e,d) \) be the valuation of \( E' \to X_{n,2} \). It is clear that \( d = 1 \). Suppose that there is an irreducible map \( f: E' \to X_{n,2}^r, r \geq 2, \) in \( \text{rep}(Q) \). Since \( \text{supp}(E') \) contains \( \text{supp}(X_{n,2}) \) properly, \( f \) is an epimorphism. Let

\[ p: x_0 \to x_1 \to \cdots \]

be a right infinite path in \( \text{supp}(X_{n,2}) \). Since \( \text{supp}(X_{n,1}) \) is socle-finite and \( X_{n-1,2} \) is finite dimensional, there exists an integer \( j \) with \( x_j \not\in \text{supp}(X_{n,1} \oplus X_{n-1,2}) \). The almost split sequence \( \eta \) then yields \( \dim(E'(x_j)) = \dim(X_{n,2}(x_j)) \neq 0 \). However, since \( f \) is an epimorphism,

\[ \dim(X_{n,2}(x_j)) = \dim(E'(x_j)) \geq 2 \dim(X_{n,2}(x_j)), \]
a contradiction. Hence, we get that the valuation of the arrow $E' \rightarrow X_{n,2}$ in $\Gamma$ is $(1,1)$. Similarly, the valuation of the arrow $X_{n,1} \rightarrow E'$ in $\Gamma$ is $(1,1)$. This shows that $\Gamma$ is of wing type.

If $\Gamma$ does not contain representations in $\text{rep}^+(Q) \cup \text{rep}^-(Q)$, then it contains only doubly-infinite representations. By Proposition 4.4, $\Gamma$ is a trivial component, and hence is necessarily of wing type.

**Remark 5.11.** (1) The last theorem says in particular that there exists a bijection between the isomorphism classes of doubly-infinite representations and the regular components of wing type in $\Gamma_{\text{rep}(Q)}$.

(2) If $Q$ is connected strongly locally finite, then there exists a doubly-infinite representation in $\text{rep}(Q)$ if and only if $Q$ has a left-infinite path and a right-infinite path. Hence, if $\Gamma_{\text{rep}(Q)}$ has both a regular component of shape $\mathbb{NA}_\infty$ and of shape $\mathbb{NA}_\infty^-$, then it needs to have a regular component of wing type.

(3) The Auslander-Reiten quiver of $\text{rep}^+(Q)$ is a successor-closed subquiver of $\Gamma_{\text{rep}(Q)}$. It is obtained by removing the infinite dimensional non-finitely presented representations in $\Gamma_{\text{rep}(Q)}$. Similarly, the Auslander-Reiten quiver of $\text{rep}^-(Q)$ is a predecessor-closed subquiver of $\Gamma_{\text{rep}(Q)}$. It is obtained by removing the infinite dimensional non-finitely co-presented representations in $\Gamma_{\text{rep}(Q)}$.

(4) The Auslander-Reiten quiver of $\text{rep}(Q)$ has a symmetric valuation by Theorems 5.7, 5.8 and 5.10.

(5) If $M$ is doubly-infinite and has no simple submodule and no simple quotient, then $\{M\}$ is a trivial component of $\Gamma_{\text{rep}(Q)}$.

An additive Krull-Schmidt $k$-category $C$ is said to be a left Auslander-Reiten category if every indecomposable object in $C$ is the domain of a minimal left almost split epimorphism or is the starting term of an almost split sequence; a right Auslander-Reiten category if every indecomposable object in $C$ is the co-domain of a minimal right almost split monomorphism or the ending term of an almost split sequence; and an Auslander-Reiten category if it is a left and a right Auslander-Reiten category; compare [8, (2.6)].

The following proposition follows easily from our previous results.

**Proposition 5.12.** Let $Q$ be a strongly locally finite quiver.

(1) The category $\text{rep}^+(Q)$ is a left Auslander-Reiten category if and only if $Q$ has no right-infinite path.

(2) The category $\text{rep}^-(Q)$ is a right Auslander-Reiten category if and only if $Q$ has no left-infinite path.

(3) The category $\text{rep}(Q)$ is an Auslander-Reiten category if and only if $Q$ has no infinite path.

6. The Auslander-Reiten Quiver of $\text{rep}(Q)$

In this section, again, $Q$ stands for a connected strongly locally finite quiver. Although $\text{rep}(Q)$ is, in general, not Hom-finite, it is true that every indecomposable object in $\text{rep}(Q)$ has a local endomorphism algebra; see [7]. Thus, one can construct the Auslander-Reiten quiver $\Gamma_{\text{rep}(Q)}$ of $\text{rep}(Q)$ as defined in Section 5. The objective of this section is to show that the Auslander-Reiten components of $\Gamma_{\text{rep}(Q)}$ are connected components of $\Gamma_{\text{rep}(Q)}$. 

We start with the following lemma, where the proof is inspired from the proof of [10, Proposition 2.1].

**Lemma 6.1.** Let $N \in \text{rep}(Q)$ with a sub-representation $M$ and suppose we have a chain

$$0 = L_0 \subset L_1 \subset L_2 \subset \cdots$$

of finitely generated proper sub-representations of $N$ with every inclusion $M \to M + L_i$ being a section. Suppose moreover that the union of the $M + L_i$ is $N$. Then the inclusion $M \to N$ is a section.

**Proof.** Set $M_i = M + L_i$ for $i \geq 0$. For $0 \leq i < j$, denote the inclusion $M \to M_i$ by $q_i$ and the inclusion $M_i \to M_j$ by $q_{i,j}$. For $i \geq 0$, we have a short exact sequence

$$0 \to M \cap L_i \to M \oplus L_i \to M_i \to 0$$

giving a monomorphism

$$\varphi_i : \text{Hom}(M_i, M) \to \text{Hom}(M, M) \oplus \text{Hom}(L_i, M),$$

sending a morphism $g$ to $(g_1, g_2)$ where $g_1$ is the restriction of $g$ to $M$ and $g_2$ is the restriction of $g$ to $L_i$. Let $V_i$ denote the subspace of $\text{Hom}(M_i, M)$ of the morphisms $g$ for which $gq_i$ is a scalar multiple of $1_M$. Since $V_i$ is the pre-image of $k\{1_M\} \oplus \text{Hom}(L_i, M)$, which is finite dimensional, we see that $V_i$ is finite dimensional. A morphism $g \in V_i$ for which $gq_i = 1_M$ is called normalized. By assumption, each $V_i$ contains a normalized morphism and hence is non-zero. Now, one has a non-zero map

$$g_i : V_{i+1} \to V_i$$

which is induced by $g_{i,i+1}$ and sends a normalized map to a normalized one. By assumption, we have a normalized map $v_i : M_i \to M$ in $V_i$ such that

$$v_i q_{i,j} = v_i q_{i-1,j} q_{i-2,j-1} \cdots q_{j,j+1} = g_j \cdots g_i - 1(v_i)$$

is normalized in $V_j$ for $0 \leq j < i$. Let $0 \neq M_{ij} = \text{Im}(g_j \cdots g_i - 1)$ for $0 \leq j < i$ with $M_{ii} = V_i$. The chain

$$M_{jj} \geq M_{j+1,j} \geq M_{j+2,j} \cdots$$

of finite dimensional $k$-vector spaces yields an integer $r_j \geq j$ for which $0 \neq M_{r_j,j} = M_{k,j}$ whenever $k \geq r_j$. Moreover, each such $M_{r_j,j}$ contains a normalized map. Then the maps $g_i$ clearly induce non-zero maps

$$\overline{g}_i : M_{r_{i+1},i+1} \to M_{r_i,i}.$$

We claim that these maps are surjective. Let $u \in M_{r_i,i}$. For every positive integer $r > i + 1$, $u \in \text{Im}(g_i g_{i+1} \cdots g_{r-1})$ and hence, there exists an element $u_r \in \text{Im}(g_i g_{i+1} \cdots g_{r-1})$ such that $g_i(u_r) = u$. But then $u_{r+1} \in M_{r_{i+1},i+1}$ is such that $\overline{g}_i(u_{r+1}) = u$, showing the claim. Now, set $u_0 \in M_{r_0,0}$ be a normalized map. Then there exists $u_1 \in M_{r_1,1}$ such that $\overline{g}_0(u_1) = u_0$. Observe that if $u_1$ is not normalized, then there exists $\alpha \in k \setminus \{0\}$ such that $\alpha u_1$ is normalized and hence that $\overline{g}_0(\alpha u_1) = \alpha u_0$ is normalized, showing that $\alpha = 1$. Hence, $u_1$ is normalized. Choose such $u_i \in M_{r_i,i}$ for all positive integers $i$. Hence, for $i \geq 0$, we have that $u_i q_i = 1_M$ and $u_{i+1} q_{i+1} = u_i$. Since $N$ is the union of the $M_i$, it is also the direct limit of the $M_i$. Therefore, the family of morphisms $u_i : M_i \to M$ yields a unique morphism $h : N \to M$ such that $hr_i = u_i$ for $i \geq 0$, where $r_i : M_i \to N$ is the inclusion. This shows that the inclusion $M \to N$ is a section. \[\square\]
As an immediate consequence of the preceding lemma, we get the following.

**Corollary 6.2.** Let $f : M \to N$ be an irreducible monomorphism with $M, N \in \text{rep}(Q)$. Then $N = \text{Im}(f) + L$ where $L$ is finitely generated. In particular, $\text{Coker}(f)$ is finitely generated indecomposable.

**Proof.** Since $Q$ is connected and strongly locally finite, it has a countable number of vertices and we can find, for $i \geq 0$, a chain

$$0 = L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$$

of finitely generated sub-representations of $N$ such that the union of the $M + L_i$ is equal to $N$. If the above chain is not stationary, then using Lemma 6.1 we get that $f$ is a section, which is impossible. This shows the first part of the statement. The fact that $\text{Coker}(f)$ is indecomposable follows from the well known result that the cokernel of an irreducible monomorphism in an abelian category is indecomposable; see [3]. □

**Lemma 6.3.** Let $f : M \to N$ be an irreducible monomorphism with $M \in \text{rep}(Q)$ and $N \in \text{rep}(Q)$. Then $N \in \text{rep}(Q)$.

**Proof.** We may assume that $f$ is the inclusion. By Corollary 6.2, $N = M + L$ where $L$ is finitely generated. In particular, $\text{supp}(L)$ is top-finite.

Let $\Sigma$ be the support of $N$. Since $M \in \text{rep}(Q)$ and $N = M + L$ it exists a top-finite successor-closed subquiver $\Omega$ of $\Sigma$ such that $M$ is a finite extension-representation of $M/\text{supp}(L)$ and $\text{supp}(L)$ contains the support of $L$. Moreover, by Proposition 3.6 there exists a co-finite successor-closed subquiver $\Sigma^\prime$ of $\Theta$ such that for every arrow $x \to y$ supporting $N$ with $y \in \Sigma^\prime$ it is such that $x \in \Theta$. We want to show that $N_\Theta$ is finitely presented and equivalently, that $N_\Theta$ is finitely presented. Suppose it is not the case. First, since $L$ is finitely generated, we see that $L$ is finitely generated since the inclusion $L_\Theta \to L$ has a finite dimensional cokernel. Therefore, $N_\Theta$ is finitely generated since $N_\Theta = M_\Theta + L_\Theta$. Hence, we have a projective resolution of the form

$$0 \to \bigoplus_{i=1}^\infty P_{y_i} \xrightarrow{h} \bigoplus_{j=1}^\infty P_{x_j} \to N_\Theta \to 0$$

where $h$ is a radical morphism and all $y_i$ lie in $\Theta$ since $\Theta$ is successor-closed. There exists infinitely many $i$ with $y_i \in \Sigma^\prime$. Set $I$ to be the set of all such $i$. For each $i \in I$, the pushout of the the projection $\bigoplus_{j=1}^\infty P_{y_j} \to S_{y_i}$ with $h$ then yields an exact sequence

$$0 \to S_{y_i} \to E_i \xrightarrow{\delta_i} N_\Theta \to 0$$

which is non-split since $h$ is a radical morphism. Let $E_i^\prime$ be the following representation. For $x \in Q_0$, we set

$$(E_i^\prime)(x) = \begin{cases} E_i(x), & \text{if } x \in \Theta_0; \\ N(x), & \text{otherwise.} \end{cases}$$
and for \( \alpha \in Q_1 \),
\[
(E'_i)(\alpha) = \begin{cases} 
E_i(\alpha), & \text{if } \alpha \in \Theta_1; \\
N(\alpha), & \text{if } \alpha : x \to y \text{ with } x, y \not\in \Theta_1; \\
(v_i)^{-1}N(\alpha), & \text{if } \alpha : x \to y \text{ with } x \not\in \Theta_1 \text{ and } y \in \Theta_1, y \neq y_i; \\
0, & \text{otherwise.}
\end{cases}
\]

Since every arrow attached to \( y_i \) and supporting \( N \) is clearly in \( \Theta \), we can extend the last extension to a non-split short exact sequence
\[ 0 \to S_{y_i} \to E'_i \xrightarrow{v'_i} N \to 0 \]
where \((v'_i)_e = (v_i)_e \) if \( e \in \Theta_0 \) and \((v'_i)_e = 1_{N_i(e)} \), otherwise. Hence, we have a pullback diagram
\[
\begin{array}{cccccc}
0 & \to & S_{y_i} & \to & E'_i & \xrightarrow{v'_i} & N & \to 0 \\
& & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & S_{y_i} & \to & E'_i & \xrightarrow{v'_i} & N & \to 0
\end{array}
\]
where the restriction of the top row to \( \Theta \) splits since \( M_\Theta \) is projective. Since \( y_i \in \Sigma' \) and every arrow attached to \( y_i \) and supporting \( E'_i \) (hence \( F_i \)) is entirely contained in \( \Theta \), we see that the top row splits. Let \( w'_i : M \to F_i \) such that \( w_iu'_i = 1_M \). We have \( f = v'_iu'u'_i \) where \( v'_i \) is not a retraction. Hence, \( u_iw'_i \) is a section since \( f \) is irreducible in \( \text{rep}(Q) \). For \( i \in I \), let \( \Sigma(i) = \Sigma \setminus \{y_i\} \). Since \((w'_i)_{\Sigma(i)}\) and \((v'_i)_{\Sigma(i)}\) are isomorphisms, we see that \( f \Sigma(i) \cong (u_i)_{\Sigma(i)} \) is a section for all \( i \in I \). In particular, for \( i \in I \), \( y_i \in \Sigma \) since \( f \) is not a section. Hence, there exists a sequence
\[ \Sigma \setminus \Theta = \Delta(0) \subset \Delta(1) \subset \Delta(2) \subset \ldots \]
of predecessor-closed subquivers of \( \Sigma \) such that \( \Delta(0) \) is co-finite in \( \Delta(j) \) for all \( j \geq 1 \) and the union of the \( \Delta(j) \) is \( \Sigma \). Moreover, for \( j \geq 0 \), there exists \( i_j \in I \) such that \( y_{i_j} \) is not in \( \Delta(j) \). By what we have shown, \( f_{\Delta(j)} \) is a section for all \( j \).

For \( j \geq 0 \), set \( M_j = M_{\Delta(j)}, N_j = N_{\Delta(j)} \) and \( f_j = f_{\Delta(j)} \). Observe that \( M_j, N_j \in \text{rep}^{-}(Q) \) for all \( j \). Let \( V_j \) denote the subspace of \( \text{Hom}(N_j, M_j) \) of the morphisms \( g \) for which \( gf_j \) is a multiple of \( 1_{M_j} \). Observe that \( V_j \) is finite dimensional. A morphism \( g \in V_j \) for which \( gf_j = 1_{M_j} \) is called normalized. By assumption, each \( V_j \) contains a normalized morphism and hence is non-zero. Then one has a non-zero map
\[ g_j : V_{j+1} \to V_j \]
which is the restriction to \( \Delta(j) \) and sends a normalized map to a normalized one.

By assumption, each \( V_j \) contains a normalized map \( v_j : N_j \to M_j \) such that
\[ (v_j)_{\Delta(i)} = g_l \cdots g_{j-1}(v_j) \]
is normalized in \( V_l \) for \( 0 \leq l < j \). Using a similar argument as in the proof of Lemma
\[ 0.1 \]
we can choose \( u_j \in V_j \), for \( j \geq 0 \), such that \( u_jf_j = 1_{M_j} \) and \((u_{j+1})_{\Delta(j)} = u_j \).
Since \( \Sigma \) is the union of the \( \Delta(j) \), we see that \( f \) is a section, a contradiction. This shows that \( N_{\Theta} \) is finitely presented, and so is \( N_{\Omega} \). Since \( N_{\Sigma \setminus \Omega} = M_{\Sigma \setminus \Omega}, N_{\Sigma \setminus \Omega} \) is finitely co-presented. Since \( M \) is a sub-representation of \( N = M + L \), we see that
\[ 0 \to N_{\Omega} \to N \to N_{\Sigma \setminus \Omega} \to 0 \]
is finite, showing that \( N \in \text{rep}(Q) \). \( \square \)

**Lemma 6.4.** Let \( f : M \to N \) be an irreducible monomorphism in \( \text{rep}(Q) \) with \( M \) indecomposable and \( N \in \text{rep}(Q) \). Then \( M \in \text{rep}(Q) \).
Proof. By Corollary 6.2, we can assume that \( f \) is the inclusion and \( N = M + L \) with \( L \) finitely generated. Let \( \Sigma \) be the support of \( N \) and \( \Sigma_P \) be a successor-closed subquiver of \( \Sigma \) such that \( N_{\Sigma_P} \) is projective in \( \text{rep}^+(Q) \) and \( N/N_{\Sigma_P} \) is finitely co-presented. Since \( N \) is a finite extension-representation of \( N/N_{\Sigma_P} \) by \( N_{\Sigma_P} \), there exists a co-finite successor-closed subquiver \( \Omega \) of \( \Sigma \) such that every arrow supporting \( N \) with a vertex in \( \Omega \) lies entirely in \( \Sigma_{\Omega} \). Being a sub-representation of \( N_{\Sigma_P} \), \( M_{\Sigma_P} \) is projective. If \( M_{\Sigma_P} \) is not finitely generated, then there exists a vertex \( x \) in \( \Omega \) such that \( P_x \) is a direct summand of \( M_{\Sigma_P} \). Since every arrow supporting \( M \) and attached to \( \text{supp}(P_x) \subseteq \Omega \), we see that \( P_x \) is a direct summand of \( M \), giving \( M = P_x \), a contradiction. Hence, \( M_{\Sigma_P} \) is finitely generated and being projective, is finitely presented. Since \( N = M + L \) where \( L \) is finitely generated, there exists a co-finite predecessor-closed subquiver \( \Sigma' \) of \( \Sigma \) such that \( M_{\Sigma'} = N_{\Sigma'} \) is finitely co-presented. Thus, \( M/M_{\Sigma_P} \) is also finitely co-presented. The extension
\[
0 \to M_{\Sigma_P} \to M \to M/M_{\Sigma_P} \to 0
\]
is finite since
\[
0 \to N_{\Sigma_P} \to N \to N/N_{\Sigma_P} \to 0
\]
is finite. \( \square \)

The preceding lemmas with their dual versions and Proposition 4.4 then give the following interesting result.

**Proposition 6.5.** Every irreducible morphism \( M \to N \) between indecomposable objects in \( \text{rep}(Q) \) with one of \( M, N \) in \( \text{rep}(Q) \) lies entirely in \( \Gamma_{\text{rep}}(Q) \). In particular, either \( M \in \text{rep}^-(Q) \) or \( N \in \text{rep}^+(Q) \).

Therefore, we have proven the promised main result of this section.

**Theorem 6.6.** Any Auslander-Reiten component of \( \Gamma_{\text{rep}}(Q) \) is a connected component of \( \Gamma_{\text{rep}}(Q) \).

By Proposition 5.3, any other connected component of \( \Gamma_{\text{rep}}(Q) \) is such that the translation \( \tau \) is nowhere defined.

**Example 6.7.** Let \( Q \) be the quiver
\[
\begin{array}{c}
5 \\
\downarrow \\
4 \quad 3 \\
\downarrow \quad \downarrow \\
2 \quad 1 \\
\downarrow \quad \downarrow \\
0
\end{array}
\]
of the last example. For \( i \geq 0 \), let \( M_i \) be the indecomposable representation of \( \text{rep}(Q) \) such that \( M(j) = k \) for all \( j \geq i \) and \( M(j) = 0 \), otherwise. Since \( Q \) has no infinite path, \( \text{rep}(Q) = \text{rep}^b(Q) \) are the finite dimensional representations. The only indecomposable infinite dimensional representations of \( \text{rep}(Q) \), up to isomorphisms, are the \( M_i \). The only connected component of \( \Gamma_{\text{rep}(Q)} \) which is not a connected component of \( \Gamma_{\text{rep}}(Q) \) is the following component with trivial valuations:
\[
\cdots \to M_4 \to M_2 \to M_0 \to M_1 \to M_3 \to M_5 \to \cdots
\]

We end this paper with the following conjecture.

**Conjecture 6.8.** Let \( Q \) be a strongly locally finite quiver. The connected components of \( \Gamma_{\text{rep}(Q)} \) containing representations not in \( \text{rep}(Q) \) are connected subquivers of the quiver
\[
\cdots \to \circ \to \circ \to \circ \to \cdots
\]
and all have trivial valuations.
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