Linearized stability for a multi-dimensional free boundary problem modelling two-phase tumour growth

Shangbin Cui

Department of Mathematics, Sun Yat-Sen University, Guangzhou, Guangdong 510275, People's Republic of China

E-mail: cuishb@mail.sysu.edu.cn

Received 14 October 2013, revised 7 February 2014
Accepted for publication 19 March 2014
Published 25 April 2014

Recommended by J Lowengrub

Abstract

This paper is concerned with a multi-dimensional free boundary problem modelling the growth of a tumour with two species of cells: proliferating cells and quiescent cells. This free boundary problem has a unique radial stationary solution. By using the Fourier expansion of functions on unit sphere via spherical harmonics, we establish some decay estimates for the solution of the linearized system of this tumour model at the radial stationary solution, so proving that the radial stationary solution is linearly asymptotically stable when neglecting translations.

Keywords: free boundary problem, tumour model, two phases, linearized stability
Mathematics Subject Classification: 34B15, 35C10, 35Q80

1. Introduction

Since Greenspan first used free boundary problems of partial differential equations to model the growth of solid tumours in 1972 [33, 34], many different tumour models in terms of free boundary problems of partial differential equations have been proposed by different groups of researchers, see the reviewing articles [2, 19, 23–25, 36] and references cited therein. Rigorous mathematical analysis of such models has made great progress during the past 20 or more years, and many interesting results have been obtained, see [5–13, 15, 26–32, 45, 46] and references cited therein. From these mentioned references it can be seen that dynamics of such tumour models are usually very rich, and some recently developed mathematical tools have played
important roles in their rigorous analysis. A major concern in this topic is the asymptotic behavior of solutions as time goes to infinity.

Models describing the growth of tumours possessing homogeneous structures or consisting of only one species of cells can usually be reduced into differential equations in Banach spaces possessing parabolic structures. It turns out that asymptotic behavior of their solutions can be well treated by using the abstract theory for parabolic differential equations in Banach spaces, see [8, 11, 12, 45, 46] and references therein (see also [26, 27, 30–32] and references therein for analysis by some classical methods). In contrast, for tumours with inhomogeneous structures, their growth models are more complex and the corresponding rigorous analysis is much more difficult. Here we particularly mention the models reviewed in [23–25], which describe the growth of tumours consisting of more than one species of cells. For those models, some interesting results have been obtained in the spherically symmetric case ([5–7, 9, 13, 15]). For the general non-symmetric case, however, only local well-posedness is established ([4]), and large-time behavior of the solution is totally unclear at the present time.

In this paper we study a tumour model describing the growth of spherically non-symmetric tumours consisting of two species of cells: proliferating cells and quiescent cells. Mathematical formulation of this model is the following multi-dimensional free boundary problem:

\[
\begin{align*}
\Delta c &= F(c) \quad \text{for} \quad x \in \Omega(t), \ t > 0, \\
c &= 1 \quad \text{for} \quad x \in \partial\Omega(t), \ t > 0, \\
\frac{\partial p}{\partial t} + \nabla \cdot (\vec{v}p) &= [K_B(c) - K_Q(c)]p + K_P(c)q \quad \text{for} \quad x \in \Omega(t), \ t > 0, \\
\frac{\partial q}{\partial t} + \nabla \cdot (\vec{v}q) &= K_Q(c)p - [K_P(c) + K_D(c)]q \quad \text{for} \quad x \in \Omega(t), \ t > 0, \\
p + q &= 1 \quad \text{for} \quad x \in \Omega(t), \ t > 0, \\
\vec{v} &= -\nabla \sigma \quad \text{for} \quad x \in \Omega(t), \ t > 0, \\
\sigma &= \gamma \kappa \quad \text{for} \quad x \in \partial\Omega(t), \ t > 0, \\
V_n &= \vec{v} \cdot \vec{n} \quad \text{for} \quad x \in \partial\Omega(t), \ t > 0.
\end{align*}
\]

Here \(\Omega(t)\) is the domain occupied by the tumour at time \(t\), \(c = c(x, t), \ p = p(x, t)\) and \(q = q(x, t)\) are the concentration of nutrient, the density of proliferating cells and the density of quiescent cells respectively, \(\vec{v} = \vec{v}(x, t)\) is the velocity of tumour cell movement, \(\sigma = \sigma(x, t)\) is the pressure distribution in the tumour, \(\kappa\) is the mean curvature of the tumour surface whose sign is designated by the convention that \(\kappa \geq 0\) at points where \(\partial\Omega(t)\) is convex, \(\vec{n}\) is the unit outward normal vector of \(\partial\Omega(t)\), and \(V_n\) is the normal velocity of the tumour surface. Besides, \(F(c)\) is the consumption rate of nutrient by tumour cells, \(K_B(c)\) is the birth rate of tumour cells, \(K_P(c)\) and \(K_Q(c)\) are transferring rates of tumour cells from quiescent state to proliferating state and from proliferating state to quiescent state respectively, and \(K_D(c)\) is the death rate of quiescent cells. Typically we have ([39])

\[
\begin{align*}
F(c) &= \lambda c, \\
K_B(c) &= k_B c, \quad K_D(c) = k_D(1 - c), \quad K_P(c) = k_P c, \quad K_Q(c) = k_Q(1 - c).
\end{align*}
\]

where \(\lambda, k_B, k_D, k_P\) and \(k_Q\) are positive constants. Finally, \(\gamma\) is a positive constant and is referred to as surface tension coefficient. For an illustration of the biological implications of each equation in the above model, we refer the reader to [9, 23–25, 39, 42] and references therein.

By summing up (1.3), (1.4) and using (1.5), we get

\[
\nabla \cdot \vec{v} = K_B(c) p - K_D(c) q \quad \text{for} \quad x \in \Omega(t), \ t > 0.
\]
Besides, we note that due to (1.5), the unknown variables $p$ and $q$ are not independent. In what follows we shall keep $p$ only. It follows that the system (1.1)–(1.8) reduces into the following one:

$$\Delta c = F(c) \quad \text{for } x \in \Omega(t), \quad t > 0,$$

$$c = 1 \quad \text{for } x \in \partial \Omega(t), \quad t > 0,$$

$$\frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p = f(c, p) \quad \text{for } x \in \Omega(t), \quad t > 0,$$

$$\nabla \cdot \vec{v} = g(c, p) \quad \text{for } x \in \Omega(t), \quad t > 0,$$

$$\vec{v} = -\nabla \sigma \quad \text{for } x \in \Omega(t), \quad t > 0,$$

$$\sigma = \gamma \kappa \quad \text{for } x \in \partial \Omega(t), \quad t > 0,$$

$$V_n = -\partial_n \sigma \quad \text{for } x \in \partial \Omega(t), \quad t > 0,$$

where

$$f(c, p) = K_P(c) + \left[ K_M(c) - K_N(c) \right] p - K_M(c) p^2,$$

$$g(c, p) = K_M(c) p - K_D(c),$$

and

$$K_M(c) = K_B(c) + K_D(c), \quad K_N(c) = K_P(c) + K_Q(c).$$

We note that the equations (1.15)–(1.17) can be regarded as an elliptic boundary value problem for the unknown $\sigma$. Thus, since $\kappa$ is a (quasi-linear) second-order elliptic operator for the unknown function $\rho$ describing the free boundary $\partial \Omega(t)$, by solving (1.15)–(1.17) to get $\sigma$ as a functional of $c, p, \rho$ and next substituting it into (1.18), we see that the equation (1.18) can be reduced into a (quasi-linear) third-order parabolic pseudo-differential equation for $\rho$ (containing other unknown functions), [17, 18]. On the other hand, the equation (1.14) is clearly a quasi-linear hyperbolic equation for the unknown function $p$ (containing other unknown functions). This determines that the above model can neither be treated as a purely parabolic type equation as in [11, 12, 45, 46], nor can it be dealt with as a purely hyperbolic equation, which is the main point where the difficulty of the above problem lies.

As we have mentioned before, local existence and uniqueness of a classical solution of the initial value problem for the above system has been well established by Chen and Friedman in [4] in a more general setting. In [13] and [5] it was proved that the above system has a unique radial stationary solution under the following general conditions on the given functions $F, K_B, K_D, K_P,$ and $K_Q$:

$$F, \quad K_B, \quad K_D, \quad K_P \quad \text{and} \quad K_Q \quad \text{are} \ C^\infty \text{-functions};$$

$$F(0) = 0 \quad \text{and} \quad F'(c) > 0 \quad \text{for} \quad 0 \leq c \leq 1;$$

$$\begin{cases} K_B(c) > 0 \quad \text{and} \quad K_D(c) < 0 \quad \text{for} \quad 0 \leq c \leq 1, \quad K_B(0) = 0 \quad \text{and} \quad K_D(1) = 0; \quad K_P \quad \text{and} \quad K_Q \quad \text{satisfy the same conditions as} \quad K_B \quad \text{and} \quad K_D \quad \text{respectively}; \end{cases}$$

Moreover, in [7] it was proved that this unique radial stationary solution is asymptotically stable under radial perturbations (see [9] for an extension of this result). Naturally, we want to know if this unique radial stationary solution is also asymptotically stable under non-radial perturbations. To date we have been unable to give a satisfactory answer to such a difficult question. The purpose of this paper is to prove the following weaker result:
Theorem 1.1. Let $K_B$, $K_D$, $K_P$ and $K_Q$ be given by (1.10) with coefficients satisfying the following conditions:

$$k_B > k_D \geq 2k_Q > 0, \quad k_B > k_P, \quad k_Bk_Q \leq k_Dk_P.$$  \hfill (1.25)

There exists a constant $\gamma^* > 0$ such that for $\gamma > \gamma^*$, the unique radial stationary solution of the system (1.12)–(1.18) is linearly asymptotically stable modulo translations, i.e., the trivial solution of the linearized system of (1.12)–(1.18) at its unique radial stationary solution is asymptotically stable modulo translations in suitable function spaces.

Here the phrase ‘modulo translations’ is used to refer to the following property of the system (1.12)–(1.18): since this system is invariant under translations in the coordinate space, its stationary solutions are not isolated, and by translating a given stationary solution we get a $n$-parameter family of stationary solutions. Thus stationary solutions of the above system form a $n$-dimensional manifold—the so-called centre manifold. It follows that the trivial solution of the linearized system is also not an isolated stationary solution, but instead, all stationary solutions of the linearized system make up a $n$-dimensional linear space. Hence, to study asymptotic stability of the trivial solution for the linearized system, we must make an analysis in certain quotient spaces. See theorem 8.1 for more explicit statement of the above result.

The idea of the proof of theorem 1.1 is as follows. First we reduce the linearized equations of the system (1.12)–(1.18) (see (2.11)–(2.17) in section 2) into a 2-system of integral partial differential equations (see (2.19)). Next we use spherical harmonic expansions of functions on unit sphere to convert this 2-system into a sequence of 2-systems of integral differential equations. The main part of this paper is to establish decay estimates for solutions of these systems of integral differential equations. Note that the idea of using Fourier expansion of functions on unit sphere via spherical harmonics to reduce tumour models of PDE form into systems of ordinary differential equations is not new; it has already been used more than once in previous literatures such as \cite{8,11,12,45,46}. However, the reduced equations in the present work are more difficult to treat than the corresponding ones in previous works. One of the major new difficulties we encounter is that the integral differential equations are not only non-local but also singular. To overcome this difficulty we appeal to the techniques for solving singular differential equations developed in \cite{5,7,13}; see sections 4–7 for details.

We remark that in (1.25), the condition $k_B > k_D$ is essential and it cannot be removed. Indeed, if this condition is removed then the system (1.12)–(1.18) does not have a stationary solution and the tumour will finally disappear, \cite{14}. Unlike this, the other conditions in (1.25) are imposed just for technical reasons; see lemmas 3.2 and 6.2–6.4. We conjecture that the rest conditions can be removed without affecting the validity of the above result.

Throughout this paper we shall make mention of the general $n$-dimension version of the system (1.12)–(1.18) with $n \geq 2$. This will enable us to use some abstract theory of differential equations and spherical harmonic functions and avoid using concrete expressions of Bessel functions and three-dimensional spherical harmonics. We note that all discussions made in the literature \cite{4,5,7,13} can be easily extended to the general $n$-dimension case, so that, in particular, the conditions (1.22)–(1.24) ensure that the system (1.12)–(1.18) also has a unique radial stationary solution in the general $n$-dimension case.

The structure of the paper is as follows: in the following section we compute the linearization of the system (1.12)–(1.18) around the radial stationary solution and use the spherical harmonic expansion to make reduction to the linearized system. In section 3 we collect a few preliminary lemmas. In sections 4–7 we step-by-step establish decay estimates for each mono-mode system obtained from the spherical harmonic expansion of the linearized system. In section 8 we combine all the mono-mode estimates to reach the desired result. Some concluding remarks are given in the last section.
2. Linearization and reduction

We denote by \((c_s(r), p_s(r), v_s(r), \sigma_s(r), \Omega_1s)\) \((\Omega_1s = \{x \in \mathbb{R}^n : r = |x| < R_s\})\) the unique radial stationary solution of the system (1.12)–(1.18), namely, the solution of the following system of equations:

\[
\begin{align*}
  c_s''(r) + \frac{n-1}{r} c_s'(r) &= F(c_s(r)), & 0 < r < R_s, \\
  c_s'(0) &= 0, & c_s(R_s) = 1, \\
  v_s(r) p_s'(r) &= f(c_s(r), p_s(r)), & 0 < r < R_s, \\
  v_s'(r) + \frac{n-1}{r} v_s(r) &= g(c_s(r), p_s(r)), & 0 < r < R_s, \\
  v_s(r) &= -\sigma_s'(r), & 0 < r < R_s. \\
\end{align*}
\]

Later on we shall use the following notations:

\[
\begin{align*}
  f^*(r) &= f(c_s(r), p_s(r)), & g^*(r) &= g(c_s(r), p_s(r)), \\
  f^*_c(r) &= f_c(c_s(r), p_s(r)), & f^*_p(r) &= f_p(c_s(r), p_s(r)), \\
  g^*_c(r) &= g_c(c_s(r), p_s(r)), & g^*_p(r) &= g_p(c_s(r), p_s(r)).
\end{align*}
\]

As we mentioned before, the existence and uniqueness of the above system has been proved in [5,13] in the three-dimension case. For the general \(n\)-dimension case \((n \geq 2)\), the argument is quite similar so we omit it here. Moreover, this solution satisfies the following properties (see [13]):

\[
\begin{align*}
  0 < c_s(r) < 1 & \text{ for } 0 \leq r < R_s, & c_s'(r) > 0 & \text{ for } 0 < r < R_s; \\
  0 < p_s(r) < 1 & \text{ for } 0 \leq r < R_s, & p_s'(r) > 0 & \text{ for } 0 < r < R_s; \\
  p_s(r) > \alpha(c_i(r)) & \text{ for } 0 < r < R_s, & p_s(0) = \alpha(c_i(0)) & \text{ and } p_s(R_s) = 1, \\
\end{align*}
\]

where

\[
\alpha(\lambda) = \frac{1}{2K_M(\lambda)} [K_M(\lambda) - K_N(\lambda) + \sqrt{(K_M(\lambda) - K_N(\lambda))^2 + 4K_M(\lambda)K_P(\lambda)}] \tag{2.10}
\]

(\(0 \leq \lambda \leq 1\)), and there exist positive constants \(c_1, c_2\) such that

\[-c_1r(R_s - r) \leq v_s(r) \leq -c_2r(R_s - r) \quad \text{for } 0 \leq r \leq R_s.\]

It is easy to verify that the linearization of the system (1.12)–(1.18) around the stationary solution \((c_s(r), p_s(r), v_s(r), P_s, \Omega_1s)\) is as follows:

\[
\begin{align*}
  \Delta \sigma &= F'(c_s(r))\sigma, & x \in \Omega_1s, & t > 0, \\
  \sigma|_{t=R_\omega}^\omega &= -c_s'(R_s)\eta(\omega, t), & \omega \in \mathbb{S}^{n-1}, & t > 0, \\
  \psi_t + v_s(r)\psi_r &= p_s'(r)\psi_r + f^*_c(r)\sigma + f^*_p(r)\psi, & x \in \Omega_1s, & t > 0, \\
  \tilde{w}_t &= -\nabla \psi, & x \in \Omega_1s, & t > 0, \\
  \Delta \psi &= -g^*_c(r)\sigma - g^*_p(r)\psi, & x \in \Omega_1s, & t > 0, \\
  \psi|_{t=R_\omega}^\omega &= -\frac{\gamma}{R^2_s}[\eta(\omega, t) + \frac{1}{n-1}\Delta_\omega \eta(\omega, t)], & \omega \in \mathbb{S}^{n-1}, & t > 0, \\
  \eta_t(\omega, t) &= -\psi_r(\Omega_1s, \omega, t) + g(1, 1)\eta(\omega, t), & \omega \in \mathbb{S}^{n-1}, & t > 0, \\
\end{align*}
\]

(2.11)–(2.17)
where \( \sigma = \sigma(x,t), \varphi = \varphi(x,t), \bar{\varphi} = \bar{\varphi}(x,t), \psi = \psi(x,t) \) and \( \eta = \eta(\omega,t) (x \in \Omega_s, \omega \in S^{n-1}, t \geq 0) \) are unknown variables, the subscript \( r \) denotes derivative in radial direction (e.g., \( \varphi_r = \frac{\partial \varphi}{\partial r} \)), \( \Delta_w \) denotes the Laplace–Beltrami operator on the unit sphere \( S^{n-1} \). To obtain the above equations, we let

\[
\begin{aligned}
c(x,t) &= c_i(r) + \varepsilon \sigma(x,t), \\
p(x,t) &= p_i(r) + \varepsilon \sigma(x,t), \\
\bar{v}(x,t) &= v_i(r) \omega + \varepsilon \bar{w}(x,t), \\
v(x,t) &= \sigma(x) + \varepsilon \psi(x,t),
\end{aligned}
\]

where \( r = |x|, \omega = x/|x|, \) and \( \varepsilon \) is a small real parameter. Substituting these expressions into (1.12)–(1.18) and using some similar arguments as in [11], we obtain (2.11)–(2.17). As an example we give only the deduction of the equation (2.13). Substituting the first three relations in (2.18) into the equation (1.14), we get

\[
\varepsilon \varphi_i + [v_i(r) \omega + \varepsilon \bar{w}] \cdot [\nabla p_s(r) + \varepsilon \nabla \varphi] = f(c_i(r) + \varepsilon \sigma, p_i(r) + \varepsilon \psi),
\]

or

\[
v_i(r) \omega \cdot \nabla p_s(r) + \varepsilon [\varphi_i + v_i(r) \omega \cdot \nabla \varphi + \bar{w} \cdot \nabla p_s(r)] + o(\varepsilon) = f(c_i(r), p_i(r)) + \varepsilon [f^*_{\sigma}(r) \sigma + f^*_{p}(r) \varphi] + o(\varepsilon).
\]

Since \( v_i(r) \omega \cdot \nabla p_s(r) = v_i(r) p'_s(r) = f(c_i(r), p_i(r)), \) by first removing these terms in the above equation, next dividing both sides with \( \varepsilon \), and finally letting \( \varepsilon \to 0 \), we get

\[
\varphi_i + v_i(r) \omega \cdot \nabla \varphi + \bar{w} \cdot \nabla p_s(r) = f^*_{\sigma}(r) \sigma + f^*_{p}(r) \varphi.
\]

Since \( \omega \cdot \nabla \varphi = \varphi, \) and \( \bar{w} \cdot \nabla p_s(r) = -\nabla \psi \cdot p'_s(r) \omega = -p'_s(r) \psi, \) (by (2.14)), we see that (2.13) follows.

The system (2.11)–(2.17) can be reduced into a 2-system of linear evolution equations containing only the unknowns \( \varphi \) and \( \eta \). To see this we denote by \( \mathcal{K} \), \( \mathcal{K}_0 \) and \( \mathcal{G} \) respectively the following operators: for \( \eta \in C^2(S^{n-1}), \) let \( u = \mathcal{K} (\eta) \in C^2(\Omega_s) \) and \( v = \mathcal{K}_0(\eta) \in C^2(\Omega_s), \) where \( C^2(\Omega_s) \) represents the second-order Zygmund space in \( \Omega_s, \) be respectively solutions of the following elliptic boundary value problems:

\[
\begin{aligned}
\Delta u &= u'(c_i(r)) u, \quad x \in \Omega_s, \\
\left. u \right|_{r=R_s} &= \eta(\omega), \quad \omega \in S^{n-1}; \\
\Delta v &= 0, \quad x \in \Omega_s, \\
\left. v \right|_{r=R_s} &= \eta(\omega), \quad \omega \in S^{n-1}.
\end{aligned}
\]

For \( h \in C(\Omega_s), \) let \( w = \mathcal{G}(\eta) \in C^2(\Omega_s) \) be the solution of the following elliptic boundary value problem:

\[
\begin{aligned}
\Delta w &= h, \quad x \in \Omega_s, \\
w &= 0, \quad x \in \partial \Omega_s.
\end{aligned}
\]

Then from (2.11) and (2.12) we have

\[
\sigma = -c'_s(R_s) \mathcal{K}(\eta),
\]

and from (2.15) and (2.16) we have

\[
\psi = \Phi + \Upsilon + \Psi,
\]

where

\[
\begin{aligned}
\Phi &= -\mathcal{G}_s^* \mathcal{G}_s^*(r) \psi, \\
\Upsilon &= -\mathcal{G}_s^* \mathcal{G}_s^*(r) \sigma = c'_s(R_s) \mathcal{G}_s^* \mathcal{K}(\eta), \\
\Psi &= -\frac{\gamma}{R_s^2} \mathcal{K}_0[\eta(\omega,t) + \frac{1}{n-1} \Delta_w \eta(\omega,t)].
\end{aligned}
\]
Substituting these expressions into (2.13) and (2.17), we see that the system (2.11)–(2.17) reduces into the following 2-system:

\[
\begin{align*}
\frac{\partial \varphi}{\partial t} &= \mathcal{A}(\varphi, \eta), \\
\frac{\partial \eta}{\partial t} &= \mathcal{B}(\varphi, \eta),
\end{align*}
\]

(2.19)

where

\[
\mathcal{A}(\varphi, \eta) = -v_s(r)\frac{\partial \varphi}{\partial t} + f_s^*(r)\varphi + p'_s(r)\frac{\partial \varphi}{\partial r} + p'_s(r)\frac{\partial \varphi}{\partial \eta} + p'_s(r)\frac{\partial \varphi}{\partial \xi} + \frac{1}{\Delta_0} Y_k(\omega)\frac{\partial \varphi}{\partial \xi} + c'_s(R_{s})p'_s(r)\frac{\partial \varphi}{\partial \eta} - \gamma \frac{\partial \varphi}{\partial \xi},
\]

\[
\mathcal{B}(\varphi, \eta) = -\frac{\partial \Phi}{\partial r}\bigg|_{r=R_s} - \frac{\partial \gamma \varphi}{\partial r}\bigg|_{r=R_s} - \frac{\partial \gamma \varphi}{\partial r}\bigg|_{r=R_s} + g(1, 1)\eta,
\]

\[
= \frac{\partial}{\partial r}\left[ g_s^*(r)\varphi\right]_{r=R_s} - c'_s(R_{s})\frac{\partial \varphi}{\partial \eta} - \frac{\partial \varphi}{\partial \xi} - \frac{\partial \varphi}{\partial \xi} - \frac{\partial \varphi}{\partial \xi} + \gamma \frac{\partial \varphi}{\partial \xi} + g(1, 1)\eta.
\]

Let \( Y_k(\omega) \), where \( \omega \) represents a variable in the sphere \( S^{n-1} \), be a spherical harmonics of degree \( k \) (see [41]), i.e., \( Y_k(\omega) \) is a nontrivial solution of the following equation:

\[
\Delta_0 Y_k(\omega) = -\lambda_k Y_k(\omega), \quad \text{where } \lambda_k = (n + k - 2)k
\]

\((k = 0, 1, 2, \ldots)\). Consider a solution of (2.19) of the form

\[
\varphi(x, t) = \varphi_k(r, t)Y_k(\omega), \quad \eta(\omega, t) = \eta_k(t)Y_k(\omega),
\]

where \( r = |x| \) and \( \omega = \frac{x}{|x|} \). Using the identity

\[
\Delta \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{n-1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \Delta_0 \varphi
\]

we easily see that

\[
\mathcal{X}_c(Y_k) = R^{-k-k}Y_k(\omega), \quad \mathcal{X}(Y_k) = R^{-k-k}Y_k(\omega),
\]

where \( u_k \) is the solution of the following problem:

\[
\begin{cases}
\frac{u_k}{\mu}(r) + \frac{n+2k-1}{r} u_k(r) = F(c_s(r))u_k(r), & 0 < r < R_s \\
u_k(0) = 0, & u_k(R_s) = 1.
\end{cases}
\]

(2.20)

Moreover, for any \( f \in C[0, R_s]\) we have

\[
\mathcal{I}(f)(Y_k) = -\int_0^R \int_0^{\xi} (\xi^{n+2k-1} f(\rho) \frac{d\rho}{d\xi}) Y_k(\omega)
\]

\[
= -\frac{1}{n+2(k-1)} \left[ \left( \frac{1}{\mu_{n+2(k-1)}} - \frac{1}{\mu_{n+2(k-1)}} \right) \int_0^R \rho^{n+2k-1} f(\rho) d\rho 
\]

\[
+ \frac{1}{\mu_{n+2k-1}} \int_0^R \rho^{n+2k-1} f(\rho) d\rho - \frac{n-k+1}{n-k+1} \int_0^R \rho^{n-k+1} f(\rho) d\rho \right] Y_k(\omega).
\]

(2.21)

Hence

\[
\sigma = -c'_s(R_{s})R^{-k-k}u_k(r)Y_k(\omega)\eta_k(t),
\]

\[
\Phi = \left( \int_0^R \int_0^{\xi} \rho^{n+2k-1} \frac{d\rho}{d\xi} \right) Y_k(\omega),
\]

\[
\Upsilon = -c'_s(R_s)R^{-k-k} \left( \int_0^R \int_0^{\xi} \rho^{n+2k-1} \frac{d\rho}{d\xi} \right) u_k(\rho) \eta_k(t),
\]

\[
\Psi = -\left( 1 - \frac{\lambda_k}{n-k} \right) R^{-k-2k} Y_k(\omega) \eta_k(t).
\]
It follows that
\[
\mathcal{A}(\psi, \eta) = -v_s(r) \delta \psi + f_s^*(r) \phi(r) + p_s^*(r) \phi(r) \frac{d}{dr} \left( r^k \int_0^r \frac{\rho^{n+2k-1}}{\xi^{n+2k-1}} g^*_p(\rho) \phi(\rho) \, d\rho \, d\xi \right)
\]
\[
-\frac{c'_i(R_s) \rho^{n+2k-1}}{\xi^{n+2k-1}} g^*_c(\rho) u_k(\rho) \, d\rho \, d\xi \right)\eta_k(t)
\]
\[
- \left(1 - \frac{\lambda_k}{n-1}\right) \gamma k R_s^{-k-1} p'_i(\rho) \eta_k(t) - c'_i(R_s) R_s^{-k} f^*_c(\rho) r^k u_k(r) \eta_k(t) \right) Y_k(\omega),
\]
\[
\mathcal{B}(\psi, \eta) = -\frac{\partial}{\partial r} \Phi|_{r=R_s} - \partial \gamma|_{r=R_s} - \partial \Psi|_{r=R_s} + g(1, 1) \eta
\]
\[
= \left\{ -\frac{\partial}{\partial r} \left( r^k \int_0^r \frac{\rho^{n+2k-1}}{\xi^{n+2k-1}} g^*_c(\rho) \phi(\rho) \, d\rho \, d\xi \right) \right\}_{r=R_s}
\]
\[
+ \frac{c'_i(R_s) \rho^{n+2k-1}}{\xi^{n+2k-1}} g^*_c(\rho) u_k(\rho) \, d\rho \, d\xi \right)\eta_k(t)
\]
\[
+ \left(1 - \frac{\lambda_k}{n-1}\right) \gamma k R_s^{-k-1} \eta_k(t) + g(1, 1) \eta_k(t) \right) Y_k(\omega).
\]
Hence we get
\[
\frac{\partial \phi_k}{\partial t} = \mathcal{L}_k(\phi_k) + b_k(\eta, \gamma) \eta_k.
\]
\[
\frac{d \eta_k}{dr} = \alpha_k(\gamma) \eta_k + J_k(\psi_k).
\]
where
\[
\alpha_k(\gamma) = \left(1 - \frac{\lambda_k}{n-1}\right) Y R_s + g(1, 1) - \frac{c'_i(R_s)}{R_s^{n+2k-1}} \int_0^{R_s} \rho^{n+2k-1} g^*_c(\rho) u_k(\rho) \, d\rho,
\]
\[
b_k(\gamma, \gamma) = -\frac{c'_i(R_s) \rho^{n+2k-1}}{\xi^{n+2k-1}} g^*_c(\rho) u_k(\rho) \, d\rho \, d\xi \right)\eta_k(t)
\]
\[
= \left(1 - \frac{\lambda_k}{n-1}\right) \gamma k R_s^{-k-2} p'_i(\rho) - c'_i(R_s) R_s^{-k} f^*_c(\rho) r^k u_k(r)
\]
\[
= \left(1 - \frac{\lambda_k}{n-1}\right) \gamma k R_s^{-k-2} p'_i(\rho) - c'_i(R_s) R_s^{-k} f^*_c(\rho) r^k u_k(r)
\]
\[
- c'_i(R_s) R_s^{-k} p'_i(\rho) \left[ \theta_k \int_0^{R_s} \rho g^*_c(\rho) u_k(\rho) \, d\rho - \frac{1 - \theta_k}{\rho^{n+2k-1}} \right]
\]
\[
\times \int_0^{R_s} \rho^{n+2k-1} g^*_c(\rho) u_k(\rho) \, d\rho
\]
\[
- \frac{\theta_k}{\rho^{n+2k-1}} \int_0^{R_s} \rho^{n+2k-1} g^*_c(\rho) u_k(\rho) \, d\rho
\]
\[
\right],
\]
where \( \theta_k = \frac{k}{n+2(k-1)} \), and for \( \phi = \phi(r) \),
\[
\mathcal{L}_k(\phi) = -v_s(r) \phi'(r) + f_s^*(r) \phi(r) + p_s^*(r) \phi(r) \frac{d}{dr} \left( r^k \int_0^r \frac{\rho^{n+2k-1}}{\xi^{n+2k-1}} g^*_p(\rho) \phi(\rho) \, d\rho \, d\xi \right)
\]
\[
= -v_s(r) \phi'(r) + f_s^*(r) \phi(r) + r^{k-1} p'_i(\rho) \theta_k \int_0^{R_s} \rho^{-k+1} g^*_c(\rho) \phi(\rho) \, d\rho
\]
\[
- \frac{1 - \theta_k}{\rho^{n+2k-1}} \int_0^{R_s} \rho^{n+2k-1} g^*_c(\rho) \phi(\rho) \, d\rho
\]
and

\[ J_\epsilon(\phi) = \frac{1}{R_s^{2+nk-1}} \int_0^{R_s} \rho^{nk-1} g^*_\rho(\rho) \phi(\rho) \, d\rho. \]

Multiplying (2.23) with \( R_s^{-k(\ell-1)} r^{k-1} p'_r(r) \) and adding it into (2.22), we see that the system (2.22)–(2.23) reduces into the following equivalent one:

\[
\frac{\partial \tilde{\phi}_k}{\partial t} = \tilde{L}_k(\tilde{\phi}_k) + c_k(t) \eta_k,
\]

where \( \tilde{\phi}_k = \phi_k + R_s^{-k(\ell-1)} r^{k-1} p'_r(r) \eta_k(t) \),

\[
\tilde{L}_k(\phi) = L_k(\phi) + R_s^{-k(\ell-1)} r^{k-1} p'_r(r) J_k(\phi)
\]

\[
= -v_k(r) \phi'(r) + f_k^*(r) \phi(r) + \frac{r^{k-1}}{R_s^{2+nk-1}} \int_0^{R_s} \rho^{nk-1} g^*_\rho(\rho) \phi(\rho) \, d\rho
\]

\[
+ \frac{1 - \theta_k}{R_s^{2+nk-1}} \int_0^{R_s} \rho^{nk-1} g^*_\rho(\rho) \phi(\rho) \, d\rho - \frac{1 - \theta_k}{r^{k+1}} \int_{R_s}^{r} \rho^{nk-1} g^*_\rho(\rho) \phi(\rho) \, d\rho\]

\[(2.26)\]

\[ c_k(r) = b_k(r, \gamma) + \alpha_k(\gamma) R_s^{-k(\ell-1)} r^{k-1} p'_r(r) - R_s^{-k(\ell-1)} \tilde{\phi}_k[r^{k-1} p'_r(r)] \]

\[
= \frac{r^{k-1}}{R_s^{2+nk-1}} \left[ (1 - g^*(r)) p'_r(r) + \frac{n+k-2}{r} f^*(r) + f_k^*(r) \left[ c'_k(r) - c'_k(R_s) R_s^{-1} r u_k(r) \right] \right]
\]

\[
- p'_r(r) \int_0^{R_s} v_k(r) \, d\rho + \frac{1 - \theta_k}{R_s^{2+nk-1}} \int_0^{R_s} \rho^{nk-1} v_k(r) \, d\rho
\]

\[
- \frac{1 - \theta_k}{r^{k+1}} \int_{R_s}^{r} \rho^{nk-1} v_k(r) \, d\rho\]

\[(2.27)\]

where \( f^*(r) = f(c_r(r), p_r(r)), g^*(r) = g(c_r(r), p_r(r)), \)

\[ u_k(r) = g_k^*(r) p'_r(r) + c'_k(R_s) R_s^{-1} g_k^*(r) r u_k(r), \]

\[(2.28)\]

and

\[
\tilde{\alpha}_k(\gamma) = \alpha_k(\gamma) - R_s^{-k(\ell-1)} J_k(r^{k-1} p'_r(r))
\]

\[
= \left( 1 - \frac{\theta_k}{n-1} \right) \frac{k \gamma}{R_s^{2+nk-1}} + g(1, 1) - \frac{1}{R_s^{2+nk-1}} \int_0^{R_s} \rho^{nk-1} v_k(r) \, d\rho.
\]

\[(2.29)\]

A simple computation shows that \( u_1(r) = \frac{R_s c'_1(r)}{r c'_1(R_s)} \) (see the assertion (4) of lemma 3.3 below), so that \( v_1(r) = g_k^*(r) p'_r(r) + g_k^*(r) c'_k(r) = \frac{d}{dr} g^*(r) \). Using these facts, one may easily check that

\[ c_1(r) \equiv 0, \quad \tilde{\alpha}_1(\gamma) = 0. \]

This implies that in the case \( k = 1 \), (2.24)–(2.25) has the following stationary solution:

\[ \tilde{\phi}_1(t, \gamma) \equiv 0, \quad \eta_1(t) \equiv \text{const.}. \]

\[(2.30)\]

Or equivalently, in the case \( k = 1 \) the system (2.22)–(2.23) has the following stationary solutions:

\[ \varphi_1(r, t) = -c p'_1(r), \quad \eta_1(t) = c, \]

\[(2.31)\]

where \( c \) is an arbitrary real constant. This means that the system (2.19) has infinite stationary solutions, and all its stationary solutions form a \( n \)-dimensional linear space (see (8.2)).
3. Some preliminary lemmas

In this section we collect some preliminary lemmas.

Lemma 3.1. The following inequalities hold for all $0 \leq r \leq 1$:

$$f_p^+(r) < 0, \quad f_p^-(r) > 0, \quad g_p^+(r) > 0 \quad \text{and} \quad g_p^-(r) > 0.$$  \hfill (3.1)

Proof. From (1.16) and (1.17) we see that

$$f_p^+(r) = K_M(c_+(r)) - K_N(c_+(r)) - 2K_M(c_+(r))p_s(r),$$
$$f_p^-(r) = K'_M(c_+(r)) - K'_N(c_+(r))p_s(r) - K_M(c_+(r))p_p^+(r),$$
$$g_p^+(r) = K_M(c_+(r)),$$
$$g_p^-(r) = K'_M(c_+(r))p_s(r) - K_D(c_+(r)).$$

Since $K_M(c) > 0$, $K'_M(c) > 0$, $K_M(c) < 0$, $K'_M(c) < 0$ for $c > 0$ and $0 < p_s(r) \leq 1$ for $0 \leq r \leq 1$, the last three inequalities in (3.1) are immediate.

Next, since $v_1(r) < 0$ for $0 < r < 1$ and $p'_s(r) > 0$ for $0 < r < 1$, we see that $f(c_+(r), p_s(r)) = v_1(r)p'_s(r) < 0$ for $0 < r < 1$, which implies that

$$p_s(r) \geq \frac{K_M(c_+(r)) - K_N(c_+(r)) + \sqrt{[K_M(c_+(r)) - K_N(c_+(r))]^2 + 4K_M(c_+(r))K'_M(c_+(r))}}{2K_M(c_+(r))}$$

for $0 \leq r \leq 1$. Hence

$$f_p^+(r) \leq -\sqrt{[K_M(c_+(r)) - K_N(c_+(r))]^2 + 4K_M(c_+(r))K'_M(c_+(r))} < 0.$$

□

Lemma 3.2. Let the conditions in (1.25) be satisfied. There exists a constant $c_0 > 0$ such that as $r \to 0^+$,

$$p_s^+(r) = (c_0 + o(1))r.$$  \hfill (3.2)

Proof. Let $\theta = f_p^+(0)/v_1^+(0)$. Since $v_1^+(0) < 0$ and $f_p^+(0) < 0$, we have $\theta > 0$. By lemma 5.2 of [5] we know that there exists a constant $c_0 > 0$ such that as $r \to 0^+$,

$$p_s^+(r) = \begin{cases} r & \text{if } \kappa > 2, \\ r \ln r & \text{if } \kappa = 2, \\ r^{\kappa-1} & \text{if } \kappa < 2. \end{cases}$$

Hence, we need only prove that $\theta > 2$, or equivalently, $f_p^+(0) < 2v'_1(0)$. We have

$$f_p^+(0) = K_M(c_+(0)) - K_N(c_+(0)) - 2K_M(c_+(0))p_s(0),$$
$$v'_1(0) = \frac{1}{n} g^{*}(0) = \frac{1}{n} [K_M(c_+(0))p_s(0) - K_D(c_+(0))].$$

Hence $\theta > 2$ if and only if

$$n[K_M(c_+(0)) - K_N(c_+(0))] + 2K_D(c_+(0)) < 2(n + 1)K_M(c_+(0))p_s(0).$$  \hfill (3.3)

Since (see (2.9))

$$p_s(0) = \frac{K_M(c_+(0)) - K_N(c_+(0)) + \sqrt{[K_M(c_+(0)) - K_N(c_+(0))]^2 + 4K_M(c_+(0))K'_M(c_+(0))}}{2K_M(c_+(0))}.$$
we see that (3.3) is equivalent to
\[ K_M(c_1(0)) - K_N(c_1(0)) + (n + 1)\sqrt{[K_M(c_1(0)) - K_N(c_1(0))]^2 + 4K_M(c_1(0))K_P(c_1(0))} \]
\[ > 2K_D(c_2(0)). \]

This is equivalent to
\[ n(n + 2)[K_M(c_1(0)) - K_N(c_1(0))]^2 + 4(n + 1)^2K_M(c_1(0))K_P(c_1(0)) \]
\[ + 4K_B(c_1(0))K_D(c_2(0)) > 4K_D(c_2(0))K_N(c_1(0)). \]

It is easy to check that the conditions in (1.25) ensure that the above inequality holds. Hence the desired assertion follows. □

**Lemma 3.3.** For the solution \( u_k \) of the problem (2.20), we have the following assertions:

1. \( u_k \in C^\infty[0, R_s] \), and \( 0 < u_k(r) \leq 1 \) for \( 0 \leq r \leq R_s \).
2. There exists a constant \( C > 0 \) independent of \( k \) such that
\[ 1 - \frac{C}{n + 2k}(R_s - r) \leq u_k(r) \leq 1 \quad \text{for} \quad 0 \leq r \leq R_s, \]
\[ (3.4) \]
\[ 0 \leq u'_k(r) \leq \frac{Cr}{n + 2k} \quad \text{for} \quad 0 \leq r \leq R_s, \]
\[ (3.5) \]
3. \( u_k(r) \) is monotone non-decreasing in \( k \), i.e., \( u_k(r) \geq u_l(r) \) for \( 0 \leq r \leq R_s \) and \( k > l \).
4. \( u_1(r) = \frac{R_sc'_s(r)}{rc'_s(R_s)} \).
5. \( u_0(r) > \frac{r c'_s(r)}{r c'_s(R_s)} \) for \( 0 \leq r < R_s \).

**Proof.** The problem (2.20) can be regarded as the spherically symmetric form of the \( n + 2k \)-dimensional elliptic boundary value problem
\[ \begin{cases} \Delta u(x) = F'(c_s(r))u(x) & \text{for} \quad |x| < R_s, \\ u(x) = 1 & \text{for} \quad |x| = R_s, \end{cases} \]

From this fact the assertion (1) immediately follows. Next, from the first equation in (2.20) we have
\[ u'_k(r) = \frac{1}{r^{n+2k-1}} \int_0^r \rho^{n+2k-1} F'(c_s(\rho))u_k(\rho) \, d\rho. \]

Let \( C_0 = \max_{0 \leq c \leq 1} F'(c) \). Then we get
\[ 0 \leq u'_k(r) \leq \frac{C_0}{r^{n+2k-1}} \int_0^r \rho^{n+2k-1} \, d\rho \leq \frac{C_0 r}{n + 2k} \quad \text{for} \quad 0 \leq r \leq R_s, \]

This proves (3.5). Since \( u_k(R_s) = 1 \), by integrating (3.5) over \((r, R_s)\) we get (3.4). This proves the assertion (2). The assertion (3) follows from the fact that \( u'_k(r) \geq 0 \) and the maximum principle for second-order elliptic equations. The assertion (4) follows from direct computation. Indeed, a direct computation shows that the function \( \tilde{u}_1(r) = \frac{R_sc'_s(r)}{rc'_s(R_s)} \) satisfies the same equation as \( u_1(r) \) in the region \( 0 < r < R_s \), and it is clear that \( \tilde{u}_1(R_s) = 1 \). Since
\[ \lim_{r \to 0^+} \left( \frac{c'_s(r)}{r} \right)' = \lim_{r \to 0^+} \frac{r c'_s(r) - c'_s(r)}{r^2} = \frac{1}{2} c''_s(0) = 0, \]
we see that \( \tilde{u}'_0(0) = 0 \). Hence, by uniqueness of the solution of the elliptic boundary value problem we get \( \tilde{u}_1(r) = u_1(r) \) for \( 0 \leq r \leq R_s \). Finally, it is easy to check that the function 
\[
\tilde{u}_0(r) = \frac{r_c(1)}{R_s c(R_s)}
\] satisfies the inequality
\[
\tilde{u}_0''(r) + \frac{n - 3}{r} \tilde{u}_0'(r) \geq F'(c_s(r))\tilde{u}_0(r) \quad \text{for } 0 < r < R_s.
\]
Using the fact that \( u_0'(r) \geq 0 \) we can also easily see that \( u_0(r) \) satisfies the inequality
\[
u_0''(r) + \frac{n - 3}{r} u_0'(r) \leq F'(c_s(r))u_0(r) \quad \text{for } 0 < r < R_s.
\]
Since \( u_0'(0) = \tilde{u}_0'(0) \) and \( u_0(R_s) = \tilde{u}_0(R_s) \), by the maximum principle we see that the desired assertion follows. This completes the proof of lemma 3.3. □

**Corollary 3.4.** For the function \( v_k \) given by (2.28), there exists a positive constant \( C \) independent of \( k \) such that \( 0 \leq v_k(r) \leq C[1 + p'(r)] \) for \( 0 \leq r \leq R_s \).

4. Decay estimates for some positive semigroups

In this preliminary section we establish decay estimates for some positive semigroups in \( C[0, R_s] \) and \( L^1([0, R_s], r^{n-1} \, dr) \). Let \( v_k = v_k(r) \) be as before and \( a = a(r) \) be a real-valued continuous function defined in \( [0, R_s] \). Let \( L_0 \) be the following differential operator in \([0, R_s]\):

For any function \( \varphi \) defined in \([0, R_s]\) such that the right-hand side of the following equality makes sense,

\[
L_0 \varphi(r) = -v_k(r)\varphi'(r) + a(r)\varphi(r) \quad \text{for } 0 \leq r \leq R_s.
\]

We shall regard \( L_0 \) both as an unbounded closed linear operator in \( C[0, R_s] \) with domain \( D(L_0) = \{ \varphi \in C[0, R_s] \cap C^1(0, R_s) : v_k(r)\varphi'(r) \in C[0, R_s] \} \) and as an unbounded closed linear operator in \( L^1([0, R_s], r^{n-1} \, dr) \) with domain \( D(L_0) = \{ \varphi \in L^1([0, R_s], r^{n-1} \, dr) \cap W^{1,1}_{loc}(0, R_s) : v_k(r)\varphi'(r) \in L^1([0, R_s], r^{n-1} \, dr) \} \). We denote

\[
\lambda_0(a) = a(R_s), \quad \lambda_1(a) = \max\{a(0), a(R_s)\}, \quad \lambda^*(a) = \max_{0 \leq r \leq R_s} a(r).
\]

It is clear that \( \lambda_0(a) \leq \lambda_1(a) \leq \lambda^*(a) \). In what follows the notation \( \lambda \) denotes a complex number, and \( h \) denotes a complex-valued function defined in \([0, R_s]\).

**Lemma 4.1.** We have the following assertions:

1. If \( \text{Re} \lambda > \lambda_0(a) \) then for any \( h \in C(0, R_s) \) the equation

   \[
   \lambda \varphi(r) - L_0 \varphi(r) = h(r) \quad \text{for } 0 < r < R_s
   \]

   has a unique solution \( \varphi = \varphi_\lambda \in C(0, R_s) \cap C^1(0, R_s) \), with boundary value

   \[
   \varphi_\lambda(R_s) = \frac{h(R_s)}{\lambda - a(R_s)}.
   \]

   Moreover, for any \( 0 < r_0 < R_s \) there exists a corresponding constant \( C_\lambda(r_0) > 0 \) such that

   \[
   \max_{r_0 \leq r \leq R_s} |\varphi_\lambda(r)| \leq C_\lambda(r_0) \max_{r_0 \leq r \leq R_s} |h(r)|.
   \]

   If furthermore \( \lambda \) is real and \( h(r) \geq 0 \) for \( 0 < r \leq R_s \) then also \( \varphi_\lambda(r) \geq 0 \) for \( 0 < r \leq R_s \).
(2) If $\text{Re} \lambda > \lambda_1(a)$ then for any $h \in C[0, R_s]$ the unique solution of (4.1) ensured by the above assertion belongs to $C[0, R_s] \cap C^1(0, R_s)$, and in addition to (4.2) we have also that

$$\varphi_\lambda(0) = \frac{h(0)}{\lambda - a(0)}. \quad (4.4)$$

Moreover, there exists a constant $C_\lambda > 0$ such that

$$\max_{0 \leq r \leq R_s} |\varphi_\lambda(r)| \leq C_\lambda \max_{0 \leq r \leq R_s} |h(r)|. \quad (4.5)$$

(3) If $\text{Re} \lambda > \lambda^*(a)$ then the estimate (4.5) can be improved as follows:

$$\max_{0 \leq r \leq R_s} |\varphi_\lambda(r)| \leq [\text{Re} \lambda - \lambda^*(a)]^{-1} \max_{0 \leq r \leq R_s} |h(r)|. \quad (4.6)$$

**Proof.** We first assume that $\text{Re} \lambda > \lambda_0(a)$. Choose a number $r_0 \in (0, R_s)$ and set

$${W_\lambda}(r) = \exp \left( \int_{r_0}^{r} \frac{\lambda - a(\rho)}{v_s(\rho)} \, d\rho \right) \quad \text{for} \quad 0 < r < R_s.$$ (4.7)

It is easy to see that $W_\lambda \in C^1(0, R_s)$, and

$${W_\lambda}(r) = C(R_s - r)^{\alpha_1} \left( 1 + o(1) \right) \quad \text{as} \quad r \to R_s^-,$$ (4.8)

where $\alpha_1 = \frac{\lambda - a(R_s)}{v_s'(R_s)}$, and $C$ is a nonzero constant (depending on the choice of $r_0$). Note that $\text{Re} \alpha_1 > 0$. Clearly, the equation (4.1) can be rewritten as follows:

$$\frac{d}{dr} \left( W_\lambda(r) \varphi(r) \right) = h(r) W_\lambda(r) v_s(r).$$

Letting $c = \varphi(r_0)$ and integrating both sides of this equation from $r_0$ to an arbitrary point $0 < r < R_s$, we see that the general solution of the equation (4.1) is given by

$$\varphi(r) = \frac{1}{W_\lambda(r)} \left[ c + \int_{r_0}^{r} \frac{h(\eta)}{v_s(\eta)} W_\lambda(\eta) \, d\eta \right] = \frac{1}{W_\lambda(r)} \left[ c - \int_{r_0}^{r} \frac{h(\eta)}{v_s(\eta)} W_\lambda(\eta) \, d\eta \right]$$ (4.9)

(for $0 < r < R_s$). Since $\lim_{r \to R_s^-} W_\lambda(r) = 0$ and

$$\lim_{r \to R_s^-} \int_{r_0}^{r} \frac{h(\eta)}{v_s(\eta)} W_\lambda(\eta) \, d\eta = \int_{r_0}^{R_s} \frac{h(\eta)}{v_s(\eta)} W_\lambda(\eta) \, d\eta$$

is a finite number (by (4.7) and (2.10)), we see that $\lim \varphi(r) = \infty$ unless

$$c = \int_{r_0}^{R_s} \frac{h(\eta)}{v_s(\eta)} W_\lambda(\eta) \, d\eta,$$ in which case

$$\varphi(r) = \frac{h(R_s)}{v_s'(R_s) \alpha_1} \left[ 1 + o(1) \right] = \frac{h(R_s) [1 + o(1)]}{\lambda - f_\lambda'(R_s)} \quad \text{as} \quad r \to R_s^-.$$ (4.10)

Hence the solution, which is bounded near $r = R_s$, is unique, and this unique bounded solution is given by

$$\varphi_\lambda(r) = \frac{1}{W_\lambda(r)} \int_{r}^{R_s} \frac{h(\eta) W_\lambda(\eta)}{|v_s(\eta)|} \, d\eta \quad \text{for} \quad 0 < r < R_s, \quad (4.9)$$

which is continuous in $(0, R_s)$, continuously differentiable in $(0, R_s)$, and satisfies (4.2) (as we have seen above). The estimate (4.3) easily follows from (4.9) because the function $r \mapsto \frac{1}{|W_\lambda(r)|} \int_{r}^{R_s} \frac{|W_\lambda(\eta)|}{|v_s(\eta)|} \, d\eta$ is continuous in $(0, R_s)$. Moreover, if $\lambda$ is real then $W_\lambda(r) > 0$ for $0 < r < R_s$. 

1057
for $0 < r < R_s$. Using this fact and the expression (4.9) we easily see that if $h \geq 0$ then also $\phi_\lambda \geq 0$. This proves the assertion (1).

Next we assume that $\text{Re} \lambda > \lambda_1(a)$. Then in addition to (4.7) we have also that

$$W_\lambda(r) = C r^{\alpha_0} (1 + o(1)) \quad \text{as } r \to 0^*,$$

(4.10)

where $\alpha_0 = a(0) - \lambda$, and $C$ is a nonzero constant (depending on the choice of $r_0$). Note that $\text{Re} \alpha_0 > 0$. Using this fact we can easily deduce that for any constant $c$ the function $\phi$ given by (4.8) satisfies

$$\phi(r) = -\frac{h(0)}{\nu_j(0) \alpha_0} [1 + o(1)] = \frac{h(0) [1 + o(1)]}{\lambda - f^*_p(0)} \quad \text{as } r \to 0^*.$$

Hence, all solutions given by (4.8) are continuous at $r = 0$ and satisfy (4.4). In particular, the solution $\phi_\lambda$ given by (4.9) belongs to $C[0, R_s] \cap C^1(0, R_s)$ and satisfy both (4.2) and (4.4). The estimate (4.5) easily follows from (4.9) because in the present situation the function

$$\int_{R_s}^{\infty} |W_\lambda(\eta)| \frac{d\eta}{|v_j(\eta)|} \leq \int_{R_s}^{\infty} |v_j(\eta)|^{-1} e^{-\frac{\text{Re} \lambda - \lambda^*_1(a)}{|v_j(\eta)|} \int_{R_s}^{\infty} d\eta} d\eta$$

$$\leq \int_{R_s}^{\infty} |v_j(\eta)|^{-1} e^{-\frac{\text{Re} \lambda - \lambda^*_1(a)}{1 - |v_j(\eta)|^{-1} d\eta} d\eta}$$

$$= \frac{1}{\text{Re} \lambda - \lambda^*_1(a)} \left[ 1 - e^{-\frac{\text{Re} \lambda - \lambda^*_1(a)}{1 - |v_j(\eta)|^{-1} d\eta} d\eta} \right].$$

From this estimate and the expression (4.9) we easily see that (4.6) follows. This completes the proof of lemma 4.1. □

**Corollary 4.2.** The operator $L_0$ generates a positive $C_0$-semigroup $e^{t L_0}$ in $C[0, R_s]$ satisfying the following estimate: For any $\mu > \lambda_1(a)$ there exists a corresponding constant $C_\mu > 0$ such that

$$\max_{0 \leq r \leq R_s} |e^{t L_0} \phi(r)| \leq C_\mu e^{\mu t} \max_{0 \leq r \leq R_s} |\phi(r)| \quad \text{for } \phi \in C[0, R_s], \ t \geq 0.$$

(4.11)

**Proof.** By the Hille–Yosida theorem, the assertion (3) of lemma 4.1 ensures that $L_0$ generates a $C_0$-semigroup $e^{t L_0}$ in $C[0, R_s]$. The assertion (1) of lemma 4.1 ensures that this semigroup is positive (see theorem 1.8 in chapter VI of [16]). The assertion (2) of lemma 4.1 implies that the spectral bound of $L_0$ (see definition 1.12 in chapter II of [16] for this concept) is not greater than $\lambda_1(a)$: $\sigma(L_0) \leq \lambda_1(a)$. It follows by proposition 1.14 in chapter VI of [16] that for any $\mu > \lambda_1(a)$ there exists a corresponding constant $C_\mu > 0$ such that

$$\max_{0 \leq r \leq R_s} |e^{t L_0} |1\rangle \leq C_\mu e^{\mu t} \quad \text{for } t \geq 0.$$  

(4.12)

The positivity of $e^{t L_0}$ implies that the comparison principle holds for it. Hence, since

$$- \max_{0 \leq r \leq R_s} |\phi(r)| \leq \phi(r) \leq \max_{0 \leq r \leq R_s} |\phi(r)|$$

for every $\phi \in C[0, R_s]$, (4.11) is an immediate consequence of (4.12). □
Lemma 4.3. The operator $L_0$ also generates a positive $C_0$-semigroup $e^{L_0 t}$ in $L^1([0, R], r^{n-1} \, dr)$ satisfying the following estimate: For any $\phi \in L^1([0, R], r^{n-1} \, dr)$,
\[
\int_0^R |e^{L_0 t} \phi(r)| r^{n-1} \, dr \leq e^{\lambda_2 t} \int_0^R |\phi(r)| r^{n-1} \, dr \quad \text{for } t \geq 0,
\]
where $\lambda_2 = \max_{0 \leq r \leq R} [g^*(r) + a(r)]$.

Proof. By the density of $C[0, R]$ in $L^1([0, R], r^{n-1} \, dr)$, we need only prove the estimate (4.13). Let $\varphi(r, t) = e^{L_0 t} \phi(r)$. Then $\varphi$ is the solution of the following initial value problem:
\[
\begin{cases}
\partial_t \varphi = -v_s(r) \partial_r \varphi + a(r) \varphi & \text{for } 0 < r < R, \ t > 0, \\
\varphi|_{t=0} = \phi & \text{for } 0 \leq r \leq R.
\end{cases}
\]

Multiplying the first equation in (4.14) with $(\text{sgn}\varphi)r^{n-1}$ and next integrating it over $[0, R]$, we get
\[
\frac{d}{dr} \int_0^R |\varphi(r, t)| r^{n-1} \, dr = - \int_0^R v_s(r)r^{n-1} \frac{\partial}{\partial r} |\varphi(r, t)| \, dr + \int_0^R a(r)|\varphi(r, t)|r^{n-1} \, dr
\]
\[
= \int_0^R \frac{\partial}{\partial r} \left[ v_s(r)r^{n-1} \right] |\varphi(r, t)| \, dr + \int_0^R a(r)|\varphi(r, t)|r^{n-1} \, dr
\]
\[
= \int_0^R [g^*(r) + a(r)]|\varphi(r, t)|r^{n-1} \, dr
\]
\[
\leq \lambda_2 \int_0^R |\varphi(r, t)| r^{n-1} \, dr.
\]
From this estimate, (4.13) immediately follows. \qed

For every non-negative integer $k$, we let $\hat{\mathcal{L}}_k^*$ be the following linear differential–integral operator in $(0, R)$: For $\varphi \in C(0, R_1) \cap C^1(0, R_1)$,
\[
\hat{\mathcal{L}}_k^*(\varphi) = -v_s(r)\varphi'(r) + a_k(r)\varphi(r) + \theta_k \int_r^R \left( \frac{r}{\rho} \right)^{n+2(k-1)} g_p^*(\rho) p'_p(\rho) \varphi(\rho) \, d\rho
\]
\[
+ (1 - \theta_k) \int_r^R g_p^*(\rho) p'_p(\rho) \varphi(\rho) \, d\rho \quad \text{for } 0 < r < R_1.
\]
where
\[
a_k(r) = \frac{k}{r} v_s(r) + g^*(r) - \frac{f_p^*(r)}{p'_p(r)}.
\]
Since $g^*(0) < 0$, $v'_s(0) = \frac{1}{n} g^*(0) < 0$ and $f_p^*(r) \frac{c'_p(r)}{p'_p(r)} > 0$, we easily see that $a_k(0) = \lim_{r \to 0^+} a_k(r) < 0$ (note that since $p'_p(0) \neq 0$ in case $p'_p(0) = 0$ (see (4.10) in [13]), this limit exists). We have also that $a_k(R_1) < 0$. Indeed, since $f_p^*(r)c'_p(r) + f_p^*(r)p'_p(r) = (v_s(r)p'_p(r) + v_s(0)p'_p(0)) + v_s(R_1) = 0$ and $v'_s(R_1) = g^*(R_1)$, we see that
\[
a_k(R_1) = g^*(R_1) - v'_s(R_1) + f_p^*(R_1) = f_p^*(R_1) < 0.
\]
We denote
\[
\mu_k = \max \{a_k(0), a_k(R_1)\}.
\]

The above argument shows that $\mu_k < 0$, $k = 0, 1, 2, \ldots$, and, in fact,
\[
\mu_k \leq \max \left\{ \left( 1 + \frac{k}{n} \right) g^*(0), f_p^*(R_1) \right\} \leq \max \{g^*(0), f_p^*(R_1)\} \equiv \mu_0^* < 0, \quad k = 0, 1, 2, \ldots.
\]
**Lemma 4.4.** Let \( \text{Re}\lambda > \mu_k \). For any \( h \in C[0, R_s] \) the equation
\[
\lambda \varphi - \hat{L}_k^*(\varphi) = h \quad \text{in} \quad (0, R_s)
\]  
has a unique solution \( \varphi = \varphi_h \in C[0, R_s] \cap C^1(0, R_s) \), and there exists a constant \( C_{k, \lambda} > 0 \) such that
\[
\max_{0 \leq r \leq R_s} |\varphi_h(r)| \leq C_{k, \lambda} \max_{0 \leq r \leq R_s} |h(r)|. 
\]  
Moreover, if \( \lambda \) is real and \( h \geq 0 \) then also \( \varphi_h \geq 0 \).

**Proof.** We fulfil the proof through three steps.

**Step 1.** We first prove that the equation (4.11) has a unique solution in the class 
\( C(0, R_s) \cap C^1(0, R_s) \). To this end, we explicitly write out the equation (4.11) as follows:
\[
v_s(r)\varphi'(r) + [\lambda - a_k(r)]\varphi(r) - \theta_k \int_r^{R_s} \left( \frac{r}{\rho} \right)^{n+2(k-1)} g^*_p(\rho)p'_s(\rho)\varphi(\rho) d\rho \\
- (1 - \theta_k) \int_r^{R_s} g^*_p(\rho)p'_s(\rho)\varphi(\rho) d\rho = h(r). 
\]  
Let \( W_s(r) \) be as before but with \( a(r) \) replaced with \( a_k(r) \). By rewriting the above equation in the form
\[
\frac{d}{dr} \left( W_s(r)\varphi(r) \right) = \frac{W_s(r)}{v_s(r)} \left[ h(r) + \theta_k \int_r^{R_s} \left( \frac{r}{\rho} \right)^{n+2(k-1)} g^*_p(\rho)p'_s(\rho)\varphi(\rho) d\rho \\
+ (1 - \theta_k) \int_r^{R_s} g^*_p(\rho)p'_s(\rho)\varphi(\rho) d\rho \right],
\]  
we easily see that, as far as solutions which are bounded near \( r = R_s \) are concerned, the differential–integral equation (4.22) is equivalent to the following integral equation:
\[
\varphi(r) = \frac{1}{W_s(r)} \int_r^{R_s} \frac{W_s(\eta)}{|v_s(\eta)|} \left[ h(\eta) + \theta_k \int_\eta^{R_s} \left( \frac{\rho}{\eta} \right)^{n+2(k-1)} g^*_p(\rho)p'_s(\rho)\varphi(\rho) d\rho \\
+ (1 - \theta_k) \int_\eta^{R_s} g^*_p(\rho)p'_s(\rho)\varphi(\rho) d\rho \right] d\eta. 
\]  
It follows from a standard contraction mapping argument (similar to that used in the proof of theorem 5.3 (1) of [5]) that there exists a sufficiently small \( \delta > 0 \) such that (4.19) has a unique bounded solution in the interval \( (R_s - \delta, R_s) \), such that \( \varphi \in C(R_s - \delta, R_s) \cap C^1(R_s - \delta, R_s) \), and
\[
\varphi(r) = \frac{h(R_s)}{v_s(R_s)} \left[ 1 + o(1) \right] = \frac{h(R_s)}{\lambda - a(R_s)} \left[ 1 + o(1) \right] \quad \text{as} \quad r \to R_s^-. 
\]  
Since \( v_s(r) \neq 0 \) for \( 0 < r < R_s \), the equation (4.19) is a regular linear differential–integral equation at any point in \( (0, R_s) \), so that by standard ODE theory we can uniquely extend the solution to the whole interval \( (0, R_s) \) such that \( \varphi \in C(0, R_s) \cap C^1(0, R_s) \). This fulfils the task of the first step.

We note that if \( \lambda \) is real, \( \lambda > \mu_k \), and \( h \geq 0 \), then also \( \varphi \geq 0 \). Indeed, since \( \lambda \) is real, we have \( W_s(r) > 0 \) for \( 0 < r < R_s \). If \( h(R_s) > 0 \) then by (4.21) we see that \( \varphi(R_s) > 0 \). Let \( r_0 \) be the smallest number such that \( \varphi(r) > 0 \) for \( r_0 < r \leq R_s \). Then by (4.20) we must have \( r_0 = 0 \). The assertion for the case \( h(R_s) = 0 \) follows from a limit argument.

**Step 2.** We next prove that the solution ensured by the above step satisfies
\[
\int_0^{R_s} g^*_p(\rho)p'_s(\rho)|\varphi(\rho)| d\rho < \infty.
\]
To prove this assertion we note that from (4.20) we have
\[ |\varphi(r)| \leq \frac{1}{|W_s(r)|} \int_0^R |W_s(\eta)| \left| \frac{1}{|v_s(\eta)|} \left[ |h(\eta)| + \alpha \int_\eta^R \left( R_s \right) p_\alpha(\rho) |\varphi(\rho)| \right] \right| d\rho \]
\[ + (1 - \alpha) \int_\eta^R |W_s(\varphi)| \left| \frac{1}{|v_s(\eta)|} \left[ |h(\eta)| + \alpha \int_\eta^R |g_\alpha(\rho) p_\alpha(\rho) |\varphi(\rho)| | \right| d\rho \right| d\eta \]
\[ \leq \frac{1}{|W_s(r)|} \int_0^R |W_s(\eta)| \left[ |h(\eta)| + \alpha \int_\eta^R |g_\alpha(\rho) p_\alpha(\rho) |\varphi(\rho)| | \right| d\rho \right| d\eta. \]

It follows that for any \( 0 < r < r' \leq R_s \) we have
\[ \int_r^{r'} g_\alpha(\xi) p_\alpha(\xi) |\varphi(\xi)| d\xi \leq \int_r^{r'} \int_\xi^R g_\alpha(\xi) |p_\alpha(\xi)| |W_s(\eta)| \left| \frac{1}{|v_s(\eta)|} \left[ |h(\eta)| + \alpha \int_\eta^R |g_\alpha(\rho) p_\alpha(\rho) |\varphi(\rho)| | \right| d\rho \right| d\eta \]
\[ + C \int_r^{r'} \int_\xi^R \int_\eta^R |W_s(\xi)| |v_s(\eta)| |g_\alpha(\rho) p_\alpha(\rho) |\varphi(\rho)| | \right| d\rho d\eta d\xi \]
\[ \leq C \left( \int_r^{r'} \int_\xi^R g_\alpha(\xi) |p_\alpha(\xi)| |W_s(\eta)| \left| \frac{1}{|v_s(\eta)|} \left[ |h(\eta)| + \alpha \int_\eta^R |g_\alpha(\rho) p_\alpha(\rho) |\varphi(\rho)| | \right| d\rho \right| d\eta \right) \]
\[ + C \left( \int_r^{r'} \int_\xi^R p_\alpha(\xi) |W_s(\eta)| \left| \frac{1}{|v_s(\eta)|} \left[ |h(\eta)| + \alpha \int_\eta^R |g_\alpha(\rho) p_\alpha(\rho) |\varphi(\rho)| | \right| d\rho \right| d\eta \right) \]
\[ < \infty. \]

Hence there exists a constant \( \delta > 0 \) independent of \( k \) such that if \( 0 < r' - r \leq \delta \) then
\[ C \int_r^{r'} \int_\xi^R g_\alpha(\xi) p_\alpha(\xi) |\varphi(\xi)| d\xi \leq \frac{1}{2}, \]
which implies that
\[ \int_r^{r'} g_\alpha(\xi) p_\alpha(\xi) |\varphi(\xi)| d\xi \leq \max_{0 \leq r \leq R_s} |h(\eta)| + \int_r^{R_s} g_\alpha(\xi) p_\alpha(\xi) |\varphi(\xi)| | \right| d\xi. \]

Hence, by dividing the interval \([0, R_s]\) into finite number (depending on \( k \) because \( W_s \) depends on \( k \) of subintervals and using an iteration argument, we see that there exists a constant \( C_{k, s} > 0 \) depending on \( k \) such that
\[ \int_0^{R_s} g_\alpha(\rho) p_\alpha(\rho) |\varphi(\rho)| | \right| d\rho \leq C_{k, s} \max_{0 \leq r \leq R_s} |h(\eta)|. \]

This fulfills the task of the second step.
Step 3. From the assertion obtained in the above step, it follows that the function

\[ h_1(r) = \theta_k \int_r^{R_s} \left( \frac{r}{\rho} \right)^{n+2(k-1)} g_\rho^*(\rho) p'_s(\rho) \psi(\rho) \, d\rho + (1 - \theta_k) \int_r^{R_s} g_\rho^*(\rho) p'_s(\rho) \psi(\rho) \, d\rho \]

belongs to \( C[0, R_s] \), and

\[ \max_{0 \leq r \leq R_s} |h_1(r)| \leq C_{k,\lambda} \max_{0 \leq r \leq R_s} |h(r)|. \]

Hence, by rewriting the equation (4.19) into the form

\[ v_s(r) \psi'(r) - \lambda \psi(r) + a_k(r) \psi(r) = h(r) + h_1(r) \]

and applying the assertion (2) of lemma 4.1, we get the desired assertion. This completes the proof of lemma 4.3. \( \square \)

Corollary 4.5. The operator \( \hat{L}_k^* \) generates a positive \( C_0 \)-semigroup \( e^{t\hat{L}_k^*} \) in \( C[0, R_s] \) satisfying the following estimate: for any \( \mu > \mu_k \) there exists a corresponding constant \( C_{k,\mu} > 0 \) such that

\[ \max_{0 \leq r \leq R_s} |e^{t\hat{L}_k^*} \psi(r)| \leq C_{k,\mu} e^{\mu t} \max_{0 \leq r \leq R_s} |\psi(r)| \quad \text{for} \quad \psi \in C[0, R_s], \ t \geq 0. \] (4.23)

Proof. We note that \( \hat{L}_k^* = \hat{L}_k^{\ast 0} + \mathcal{B}_k \), where

\[ \hat{L}_k^{\ast 0} \phi = -v_s(r) \phi'(r) + a_k(r) \phi(r) \quad \text{for} \quad \phi \in C[0, R_s], \] (4.24)

and \( \mathcal{B}_k \) is the integral part of \( \hat{L}_k^* \). By corollary 4.2 we see that \( \hat{L}_k^{\ast 0} \) generates a positive \( C_0 \)-semigroup in \( C[0, R_s] \). Since clearly \( \mathcal{B}_k \) is a positive bounded linear operator in \( C[0, R_s] \), by a standard perturbation theorem (see corollary 1.11 in chapter VI of [16]) for \( C_0 \)-semigroups we see that \( \hat{L}_k^* \) also generates a positive \( C_0 \)-semigroup in \( C[0, R_s] \). By lemma 4.4 we see that the spectral bound of \( \hat{L}_k^* \) is not greater than \( \mu_k \):

\[ s(\hat{L}_k^*) \leq \mu_k. \]

Hence by a similar argument as in the proof of corollary 4.2 we obtain the estimate (4.23). \( \square \)

Let \( J \) be the following continuous linear functional in \( L^1[0, R_s] \):

\[ J(\phi) = \int_0^{R_s} g_\rho^*(\rho) p'_s(\rho) \phi(r) \, dr \quad \text{for} \quad \phi \in L^1[0, R_s]. \]

 Lemma 4.6. Assume that the conditions in (1.25) are satisfied. For \( \psi_0 \in C[0, R_s] \) we let \( \psi = e^{t\hat{L}_k^*} \psi_0 \). There exists a constant \( \kappa_0 > 0 \) independent of \( k \) such that the following assertion holds for any \( \psi_0 \in C[0, R_s] \): If \( \psi_0 \geq 0 \) then

\[ \frac{d}{dt} J(\psi(t)) \leq -\kappa_0 J(\psi) \quad \text{for} \ t \geq 0, \] (4.25)

and for general \( \psi_0 \in C[0, R_s] \) we have

\[ \frac{d}{dt} J(|\psi(t)|) \leq -\kappa_0 J(|\psi|) \quad \text{for} \ t \geq 0. \] (4.26)

Proof. Since the semigroup \( e^{t\hat{L}_k^*} \) is positive, we see that \( \psi_0 \geq 0 \) implies \( \psi \geq 0 \). To prove (4.25) we note that \( \psi \) is a solution of the following equation:

\[ \partial_t \psi = \hat{L}_k^* \psi \quad \text{for} \ 0 < r < R_s, \ t > 0. \] (4.27)
Using this fact we compute
\[
\frac{d}{dr} J(\psi) = \int_0^R g_p^*(r) p_s'(r) \partial_r \psi(r, t) \, dr \\
= - \int_0^R v_s(r) g_p^*(r) p_s'(r) \partial_r \psi(r, t) \, dr + \int_0^R a_k(r) g_p^*(r) p_s'(r) \psi(r, t) \, dr \\
+ \theta_k \int_0^R \int_0^r \left( \frac{r}{\rho} \right)^{n+2(k-1)} g_p^*(r) p_s'(r) g_p^*(\rho) p_s'(\rho) \psi(\rho) \, d\rho \, dr \\
+ (1-\theta_k) \int_0^R \int_0^r g_p^*(r) p_s'(r) g_p^*(\rho) p_s'(\rho) \psi(\rho) \, d\rho \, dr \\
= \int_0^R \int_0^r \left( \frac{r}{\rho} \right)^{n+2(k-1)} g_p^*(r) p_s'(r) g_p^*(\rho) p_s'(\rho) \psi(\rho, t) \, d\rho \, dr \\
+ \int_0^R \int_0^r a_k(\rho) + \theta_k \int_0^r \left( \frac{r}{\rho} \right)^{n+2(k-1)} g_p^*(r) p_s'(r) \, d\rho \, dr \\
+ (1-\theta_k) \int_0^r g_p^*(r) p_s'(r) \psi(\rho, t) \, d\rho \\
\leq \int_0^R \tilde{a}_k(r) g_p^*(r) p_s'(r) \psi(r, t) \, dr, \tag{4.28}
\]
where
\[
\tilde{a}_k(r) = \frac{k}{r} v_s(r) + \frac{g_p^*(r)}{g_p^*(\rho)} v_s(r) + f_p^*(r) + g^*(r) + \int_0^r g_p^*(\rho) p_s'(\rho) \, d\rho.
\]

It is clear that the first two terms on the right-hand side of the above equality are negative for \(0 < r < R_r \). In what follows we prove that if the conditions in (1.25) are satisfied then
\[
f_p^*(r) + g^*(r) + \int_0^r g_p^*(\rho) p_s'(\rho) \, d\rho < 0 \quad \text{for } 0 \leq r \leq R_r. \tag{4.29}
\]

We first note that, since \(g_p^*(r) = K_M(c_r(r)) \) is monotone increasing in \(r \), we have
\[
\int_0^r g_p^*(\rho) p_s'(\rho) \, d\rho \leq g_p^*(r) \int_0^r p_s'(\rho) \, d\rho = g_p^*(r)[p_s(r) - p_s(0)] \\
= g^*(r) + K_D(c_r(r)) - p_s(0) K_M(c_r(r)).
\]

Hence
\[
f_p^*(r) + g^*(r) + \int_0^r g_p^*(\rho) p_s'(\rho) \, d\rho \leq f_p^*(r) + 2g^*(r) + K_D(c_r(r)) - p_s(0) K_M(c_r(r)) \\
= K_M(c_r(r)) - K_N(c_r(r)) - K_D(c_r(r)) - p_s(0) K_M(c_r(r)) \\
= K_M(c_r(r))[1 - p_s(0)] - K_N(c_r(r)) - K_D(c_r(r)).
\]

Note that the conditions in (1.25) imply that the function \( c \mapsto \frac{K_M(c)}{K_M(0)} = \frac{(k_n-k_s)c+k_2}{k_n-k_{0}} \) is monotone increasing for \( c > 0 \). Hence
\[
p_s(0) = \frac{1}{2K_M(c)(c)} \left[ K_M(c(0)) - K_N(c(0)) \right] \\
+ \sqrt{\left[ K_M(c(0)) - K_N(c(0)) \right]^2 + 4K_M(c(0)) K_P(c(0))} \\
> 1 - \frac{K_N(c(0))}{K_M(c(0))} \geq 1 - \frac{K_N(c(0))}{K_M(c(0))}.
\]

1063
From these estimates we see that (4.29) follows. Having proved (4.29), we see that
\[ \kappa_0 = - \max_{0 \leq r \leq R_i} \left( f'_p(r) + g^*(r) + \int_0^{R_i} g_p^*(\rho) p'_i(\rho) d\rho \right) > 0, \]
and
\[ \tilde{a}_k(r) \leq -\kappa_0 \quad \text{for} \quad 0 \leq r \leq R_i. \]
Using this result and (4.28), we see that (4.25) follows. To prove (4.26) we multiply the equation in (4.27) with \( \text{sgn} \psi \), which yields the following relation:
\[ \partial_t |\psi| \leq \tilde{L}_k^+ |\psi| \quad \text{for} \quad 0 < r < R_i, \quad t > 0. \]
Using this fact and a similar argument as above we obtain (4.26). This completes the proof of lemma 4.6. \( \square \)

5. Decay estimates for the equation \( \partial_t \phi = \tilde{\mathcal{L}}_k(\phi) \) for large \( k \)

In this and subsequent sections we establish decay estimates for the solution of the following initial value problem:
\[
\begin{cases}
\partial_t \phi = \tilde{\mathcal{L}}_k(\phi) & \text{for} \quad 0 < r < R_i, \quad t > 0, \\
\phi|_{t=0} = \phi_0 & \text{for} \quad 0 \leq r \leq R_i.
\end{cases}
\]
(5.1)

In what follows we consider the case that \( k \) is sufficiently large; the remaining cases will be treated in the next subsection. We denote
\[ v_0 = \max\{f'_p(0), f'_p(R_i)\}. \]
Since \( f'_p(r) < 0 \) for all \( 0 \leq r \leq R_i \), we see that \( v_0 < 0 \).

**Lemma 5.1.** For any \( \mu > v_0 \) there exist corresponding positive integer \( k_\mu \) and positive constant \( C = C_\mu \) such that for any \( k \geq k_\mu \) and \( \varphi_0 \in C[0, R_i] \), the solution of the initial value problem (5.1) satisfies the following estimate:
\[ \max_{0 \leq r \leq R_i} |\varphi(r, t)| \leq C \max_{0 \leq r \leq R_i} |\varphi_0(r)| e^{\mu t} \quad \text{for} \quad t \geq 0. \]
(5.2)

**Proof.** Let \( \mathcal{L}_0 \) be the following unbounded closed linear operator in \( C[0, R_i] \) with domain \( D(\mathcal{L}_0) = \{ \varphi \in C[0, R_i] \cap C^1([0, R_i]) : v_i(r) \varphi'(r) \in C[0, R_i]\} \): For \( \phi \in D(\mathcal{L}_0) \),
\[ \mathcal{L}_0(\phi) = -v_i(r) \phi'(r) + f'_p(r) \phi(r) \quad \text{for} \quad 0 \leq r \leq R_i. \]

By corollary 4.2 we see that \( \mathcal{L}_0 \) generates a positive \( C_0 \)-semigroup \( e^{\mathcal{T}t} \) in \( C[0, R_i] \) satisfying the following estimate: For any \( \mu > v_0 \),
\[ \| e^{\mathcal{T}t} \|_{L(C[0, R_i])} \leq C_\mu e^{\mu t} \quad \text{for} \quad t \geq 0. \]
(5.3)

Now for each integer \( k \geq 2 \) we denote by \( \mathcal{K}_k \) the following bounded linear operator in \( C[0, R_i] \):
For any \( \varphi \in C[0, R_i] \),
\[
\mathcal{K}_k(\varphi)(r) = \int_{r}^{R_i} \rho^{-k+1} g^*_p(\rho) \varphi(\rho) \, d\rho - \frac{1 - \theta_k}{\rho^{\alpha+2(k-1)}} \int_0^r \rho^{\alpha+1-k} g^*_p(\rho) \varphi(\rho) \, d\rho \\
+ \frac{1 - \theta_k}{R_i^{\alpha+2(k-1)}} \int_0^{R_i} \rho^{\alpha+1-k} g^*_p(\rho) \varphi(\rho) \, d\rho.
\]
(5.4)

It is easy to see that there exists a positive constant \( C \) independent of \( k \) such that for any \( k \geq 3 \),
\[ \max_{0 \leq r \leq R_i} |\mathcal{K}_k(\varphi)(r)| \leq C k^{-1} \max_{0 \leq r \leq R_i} |\varphi(r)| \quad \text{for any} \quad \varphi \in C[0, R_i]. \]
Since $\mathcal{L}_k = \mathcal{L}_0 + \mathcal{K}_k (k = 2, 3, \ldots)$, by using a standard perturbation theorem for $C_0$-semigroups we deduce from the above estimate and (5.3) that for any $\mu > \nu_0$,
\[ \|e^{\mathcal{L}_k^t}\|_{L^\infty([0, R_s]; r^{n-1} dr)} \leq C_\mu e^{(\alpha + C_\mu) t} \quad \text{for } t \geq 0. \]

The desired assertion immediately follows from this result. This proves lemma 5.1. \hfill \Box

For every $\alpha \geq 1$ we denote
\[ \mu^*_\alpha = \max_{0 \leq r \leq R_s} \left( f^*_\alpha(r) + \frac{1}{\alpha} g^*_\alpha(r) \right). \]

Note that
\begin{align*}
 f^*_\alpha(r) + \frac{1}{\alpha} g^*_\alpha(r) &= [K_M(c_s(r)) - K_N(c_s(r))] - 2K_M(c_s(r))p_s(r) \\
 &\quad + \frac{1}{\alpha} [K_M(c_s(r))p_s(r) - K_D(c_s(r))] \\
 &= - \left( 1 - \frac{1}{\alpha} \right) [K_M(c_s(r))p_s(r) - \frac{1}{\alpha} K_D(c_s(r))].
\end{align*}

Hence $\mu^*_\alpha < 0$ for all $\alpha \geq 1$.

**Lemma 5.2.** For any $\alpha \geq 1$ and $\mu > \mu^*_\alpha$ there exist corresponding positive integer $k_{\alpha, \mu}$ and positive constant $C = C_{\alpha, \mu}$ such that for any $k \geq k_{\alpha, \mu}$ and $\psi_0 \in L^\alpha([0, R_s]; r^{n-1} dr)$, the solution of the initial value problem (5.1) satisfies the following estimate:
\[ \left( \int_0^R |\psi(r, t)|^\alpha r^{n-1} dr \right)^{\frac{1}{\alpha}} \leq C \left( \int_0^R |\psi_0(r)|^\alpha r^{n-1} dr \right)^{\frac{1}{\alpha}} e^{\mu t} \quad \text{for } t \geq 0. \]

**Proof.** We first establish a $L^\alpha$-estimate for $e^{t \mathcal{L}_0} \psi_0$. Given $\psi_0 \in C[0, R_s]$ we let $\psi(r, t) = e^{t \mathcal{L}_0}\psi_0(r)$. Then $\psi$ is a solution of the following initial value problem:
\[ \begin{cases} 
 \partial_t \psi = \mathcal{L}_0(\psi) & \text{for } 0 < r < R_s, \ t > 0, \\
 \psi|_{t=0} = \psi_0 & \text{for } 0 \leq r \leq R_s. 
\end{cases} \]

By a standard argument we have
\begin{align*}
 \frac{1}{\alpha} \frac{d}{dt} \int_0^R r^{n-1} |\psi(r, t)|^\alpha dr &= - \frac{1}{\alpha} \int_0^R r^{n-1} v_s(r) \frac{\partial}{\partial r} |\psi(r, t)|^\alpha dr + \int_0^R r^{n-1} f^*_\alpha(r) |\psi(r, t)|^\alpha dr \\
 &= \frac{1}{\alpha} \int_0^R \frac{\partial}{\partial r} \left( r^{n-1} v_s(r) |\psi(r, t)|^\alpha \right) dr + \int_0^R r^{n-1} f^*_\alpha(r) |\psi(r, t)|^\alpha dr \\
 &= \int_0^R \left( \frac{1}{\alpha} g^*_\alpha(r) + f^*_\alpha(r) \right) r^{n-1} |\psi(r, t)|^\alpha dr \\
 &\leq \mu^*_\alpha \int_0^R r^{n-1} |\psi(r, t)|^\alpha dr.
\end{align*}

Hence
\[ \left( \int_0^R |\psi(r, t)|^\alpha r^{n-1} dr \right)^{\frac{1}{\alpha}} \leq \left( \int_0^R |\psi_0(r)|^\alpha r^{n-1} dr \right)^{\frac{1}{\alpha}} e^{\mu t} \quad \text{for } t \geq 0. \]

This means that, by using the abbreviation $L^\alpha$ for $L^\alpha([0, R_s]; r^{n-1} dr)$, we have
\[ \|e^{t \mathcal{L}_0}\psi_0\|_{L^\alpha} \leq \|\psi_0\|_{L^\alpha} e^{\mu t} \quad \text{for } t \geq 0. \]
Next, it is not hard to prove that there exists a constant $C_\alpha > 0$ independent of $k$ such that for any $k \geq 3$ and $\phi \in L^\infty([0, R_s]; r^{n-1} \, dr)$,
\begin{equation}
\| \mathcal{X}_k \phi \|_{L^\infty} \leq C_\alpha k^{-1} \| \phi \|_{L^\infty}.
\end{equation}
Since $\mathcal{L}_k = \mathcal{L}_0 + \mathcal{X}_k \ (k = 2, 3, \cdots)$, from (5.7), (5.8) and a standard perturbation theorem for $C_0$-semigroups we see that the desired assertion follows.

6. Decay estimates for the equation $\partial_t \varphi = \mathcal{L}_k(\varphi)$ for small $k$

A similar estimate for small $k$ is much more involved. In what follows we consider this case.

For every non-negative integer $k$, we denote by $\mathcal{E}_k$ the following differential–integral operator:
\begin{equation}
\mathcal{E}_k \phi(r) = - v_s(r) \phi'(r) + a_k(r) \phi(r) + \theta_k \int_{r}^{R_s} \left( \frac{r}{\rho} \right)^{n+2(k-1)} g_\rho^*(\rho) p'_p(\rho) \phi(\rho) \, d\rho \\
+ (1 - \theta_k) \left( \frac{r}{R_s} \right)^{n+2(k-1)} \int_0^{R_s} g_\rho^*(\rho) p'_p(\rho) \phi(\rho) \, d\rho \\
- (1 - \theta_k) \int_0^{r} g_\rho^*(\rho) p'_p(\rho) \phi(\rho) \, d\rho,
\end{equation}
where $a_k(r)$ is as before (see (4.16)). By using the relations
\begin{align*}
v_s(r) p'_p(r) &= f(c_s(r), p_s(r)), \\
v'_s(r) + \frac{n-1}{r} v_s(r) &= g^*(r),
\end{align*}
we can easily get the following relation:
\begin{equation}
\mathcal{L}_k r^{-(n+1)} p'_p(\varphi)(r) = r^{-(n+1)} p'_p(\varphi) \mathcal{E}_k \phi(r).
\end{equation}
It follows that if we let $\varphi(r,t) = r^{-(n+1)} p'_p(\varphi) \psi(r,t)$ and $\psi_0(r) = r^{-(n+1)} p'_p(\varphi_0(r))$, then $\psi$ is a solution of the problem (5.1) if and only if $\varphi$ is a solution of the following problem:
\begin{equation}
\begin{cases}
\partial_t \psi = \mathcal{E}_k \psi & \text{for } 0 < r < R_s, \\
\psi |_{t=0} = \psi_0.
\end{cases}
\end{equation}
Note that by denoting
\begin{equation*}
e_k(r) = (1 - \theta_k) \left[ 1 - \left( \frac{r}{R_s} \right)^{n+2(k-1)} \right],
\end{equation*}
we have
\begin{equation*}
\mathcal{E}_k \phi(r) = \mathcal{E}_k^* \phi(r) - J(\phi) e_k(r) \quad \text{for } \phi \in D(\mathcal{E}_k) = D(\mathcal{E}_k^*).
\end{equation*}

**Lemma 6.1.** Let $\psi(r,t)$ be the solution of the problem (6.3), $\kappa(r,t) = e^{\mathcal{E}_k^*} e_k(r)$, and $\tilde{\varphi}(r,t) = e^{\mathcal{E}_k^*} \varphi_0(r)$. Let $\Psi(t) = J(\psi)$, $K(t) = J(\kappa)$ and $\tilde{\Psi}(t) = J(\tilde{\varphi})$. Then the following relation holds:
\begin{equation}
\Psi(t) + \int_0^t \Psi(\tau) K(t - \tau) \, d\tau = \tilde{\Psi}(t) \quad \text{for } t \geq 0.
\end{equation}

**Proof.** The proof is similar to that of lemma 8.2 of [5], so that is omitted.
Lemma 6.2. Let $K \in C^1[0, \infty)$ and assume that

$$K(t) \geq 0, \quad \frac{d}{dt}(e^{\sigma t} K(t)) \leq 0 \quad \text{for} \quad t \geq 0$$

for some real constant $\sigma$. Then for any $\tilde{\Psi} \in C[0, \infty)$, the unique solution $\Psi$ of the Volterra integral equation (6.4) satisfies the following estimate:

$$|\Psi(t) - \tilde{\Psi}(t)| \leq K(0) \int_0^t e^{-\sigma(t-\tau)} |\tilde{\Psi}(\tau)| \, d\tau \quad \text{for} \quad t \geq 0.$$

Proof. See lemma 8.3 of [5]. □

Lemma 6.3. Assume that the conditions in (1.25) are satisfied. There exists a constant $\mu^* < 0$ independent of $k$ such that for every non-negative integer $k$ and any $\mu > \mu^*$ there exists a corresponding constant $C = C_{k, \mu} > 0$ such that for the solution of the problem (6.3) the following estimate holds:

$$\max_{0 \leq r \leq R_s} |\psi(r, t)| \leq Ce^{\mu t} \max_{0 \leq r \leq R_s} |\psi_0(r)| \quad \text{for} \quad t \geq 0. \quad (6.5)$$

Proof. Let the notation be as in lemma 6.1. Since $e_k \geq 0$, by lemmas 4.5 and 4.6 we see that $K(t) \geq 0$ and $d/dt(e^{\sigma t} K(t)) \leq 0$ for $t \geq 0$. By lemma 6.2, it follows that the following estimate holds:

$$|\Psi(t)| \leq |\tilde{\Psi}(t)| + K(0) \int_0^t e^{-\sigma(t-\tau)} |\tilde{\Psi} (\tau)| \, d\tau \quad \text{for} \quad t \geq 0. \quad (6.6)$$

Moreover, applying lemma 4.5 to $e^{\hat{\mathcal{L}}_t e_k(r)}$ and $e^{\hat{\mathcal{L}}_t \psi_0(r)}$ we see that for any non-negative integer $k$ and any $\mu > \mu_k$ there exists a corresponding constant $C = C_{k, \mu} > 0$ such that

$$\max_{0 \leq r \leq R_s} |e^{\hat{\mathcal{L}}_t e_k(r)}| \leq Ce^{\mu t} \quad \text{for} \quad t \geq 0, \quad \text{(6.7)}$$

$$\max_{0 \leq r \leq R_s} |e^{\hat{\mathcal{L}}_t \psi_0(r)}| \leq Ce^{\mu t} \max_{0 \leq r \leq R_s} |\psi_0(r)| \quad \text{for} \quad t \geq 0. \quad \text{(6.8)}$$

The latter implies that

$$|\tilde{\Psi}(t)| \leq Ce^{\mu t} \max_{0 \leq r \leq R_s} |\psi_0(r)| \quad \text{for} \quad t \geq 0. \quad \text{(6.9)}$$

Since $\mu_k \leq \mu_0^*$ for all non-negative integer $k$, the above estimate holds for any $\mu > \mu_0^*$. Substituting (6.9) into (6.6) we easily see that for any non-negative integer $k$ and any $\mu > \mu^* \equiv \max\{\mu_0^*, -\kappa_0\}$ there exists the corresponding constant $C = C_{k, \mu} > 0$ such that

$$|\Psi(t)| \leq Ce^{\mu t} \max_{0 \leq r \leq R_s} |\psi_0(r)| \quad \text{for} \quad t \geq 0. \quad \text{(6.10)}$$

Now, noticing that

$$\mathcal{L}_t \psi(r, t) = \hat{\mathcal{L}}_t e_k(r) - J(\psi(\cdot, t)) e_k(r) = \hat{\mathcal{L}}_t e_k(r) - \Psi(t) e_k(r),$$

by Duhamel’s formula we have

$$\psi(\cdot, t) = e^{\hat{\mathcal{L}}_t \psi_0} - \int_0^t \left( e^{(t-\tau)\hat{\mathcal{L}}_t} e_k \right) \Psi(\tau) \, d\tau.$$

From this relation and the estimates (6.7), (6.8) and (6.10), we immediately obtain (6.5). This proves lemma 6.3. □
Lemma 6.4. Assume that the conditions in (1.25) are satisfied and let \( \kappa_0 \) be as in lemma 4.6. For every non-negative integer \( k \) there exists a corresponding constant \( C = C_k > 0 \) such that for the solution of the problem (6.3) the following estimate holds:

\[
J(\{\psi(\cdot, t)\}) \leq CJ(\{\psi_0\})(1 + t)^2e^{-\kappa_0 t} \quad \text{for } t \geq 0.
\]  

(6.11)

Proof. Let the notation be as in lemma 6.1. Applying lemma 4.5 to \( \bar{\psi}(r, t) = e^{\int J(\psi_0)} \psi_0(r) \) we see that

\[
\frac{d}{dt}J(\{\bar{\psi}(\cdot, t)\}) \leq -\kappa_0 J(\{\bar{\psi}(\cdot, t)\}) \quad \text{for } t \geq 0.
\]

This implies that

\[
|\bar{\psi}(t)| \leq J(\{\bar{\psi}(\cdot, t)\}) \leq J(\{\psi_0\})e^{-\kappa_0 t} \quad \text{for } t \geq 0.
\]

From this estimate and lemmas 6.1, 6.2 we get

\[
|\psi(t)| \leq CJ(\{\psi_0\})(1 + t)e^{-\kappa_0 t} \quad \text{for } t \geq 0.
\]

(6.12)

Now, we rewrite the equation \( \partial_t \psi = \mathcal{L}_{\kappa} \psi \) as follows:

\[
\partial_t \psi(r, t) = \mathcal{L}_{\kappa}^+ \psi(r, t) - \Psi(t)e_k(r).
\]

Multiplying this equation with \( \text{sgn} \psi(r, t) \), we get

\[
\partial_t |\psi(r, t)| \leq \mathcal{L}_{\kappa}^+ |\psi(r, t)| + |\Psi(t)|e_k(r).
\]

Using this relation and a similar argument as in the proof of lemma 4.6 we get

\[
\frac{d}{dt}J(\{\psi(\cdot, t)\}) \leq -\kappa_0 J(\{\psi(\cdot, t)\}) + C|\Psi(t)| \quad \text{for } t \geq 0.
\]

(6.13)

From (6.12) and (6.13) we easily see that (6.11) follows. \( \square \)

We are now ready to study the problem (5.1) for small \( k \).

Lemma 6.5. Assume that the conditions in (1.25) are satisfied. There exists a constant \( \mu^* < 0 \) such that for every non-negative integer \( k \) and any \( \mu > \mu^* \) there exists a corresponding constant \( C = C_{k,\mu} > 0 \) such that for the solution of the problem (5.1) the following estimate holds:

\[
\max_{0 \leq r \leq R_i} |\varphi(r, t)| \leq Ce^{\mu t} \max_{0 \leq r \leq R_i} |\varphi_0(r)| \quad \text{for } t \geq 0.
\]

(6.14)

Proof. Let \( \varphi \) be the solution of the problem (5.1) and set \( \varphi(r, t) = r^{n+k-1} \psi(r, t)/p'_i(r) \). By (6.2), \( \psi \) is a solution of the problem (6.3) with initial data \( \psi_0(r) = r^{n+k-1} \psi_0(r)/p'_i(r) \). Hence, by lemma 6.3 we see that for any \( \mu > \max\{\mu^*, -\kappa_0\} \) there holds

\[
\max_{0 \leq r \leq R_i} r^{n+k-1} |\varphi(r, t)|/p'_i(r) \leq Ce^{\mu t} \max_{0 \leq r \leq R_i} r^{n+k-1} |\varphi_0(r)|/p'_i(r) \quad \text{for } t \geq 0.
\]

This implies that for any \( 0 < \delta < R_i \) there exists a corresponding constant \( C = C_{k,\mu,\delta} > 0 \) such that

\[
\max_{\delta \leq r \leq R_i} |\varphi(r, t)| \leq Ce^{\mu t} \max_{0 \leq r \leq R_i} |\varphi_0(r)| \quad \text{for } t \geq 0.
\]

(6.15)

Leaving \( \delta \) to be specified later, we take a non-negative cut-off function \( \chi \in C[0, R_i] \) such that \( \chi(r) \leq 1 \) for \( 0 \leq r \leq R_i \), \( \chi(r) = 1 \) for \( 0 \leq r \leq \delta \) and \( \chi(r) = 0 \) for \( 2\delta \leq r \leq R_i \).
and split the operator $\mathcal{K}_k$ introduced in (5.4) into the sum of two operators $\mathcal{K}'_k$ and $\mathcal{K}''_k$, where

$$\mathcal{K}'_k(\phi) = r^{k-1}p'_1(r)\left[\theta_k \int_r^{\max[r,\delta]} \rho^{-k+1}g^*_p(\rho)\phi(\rho)d\rho - \frac{(1-\theta_k)\chi(r)}{\rho^{n+2(k-1)}}\right] \times \int_0^{\min[r,\delta]} \rho^{n+2(k-1)}g^*_p(\rho)\phi(\rho)d\rho,$$

$$\mathcal{K}''_k(\phi) = r^{k-1}p'_1(r)\left[\theta_k \int_{\max[r,\delta]}^{R_k} \rho^{-k+1}g^*_p(\rho)\phi(\rho)d\rho + \frac{1-\theta_k}{R_k^{n+2(k-1)}} \int_0^{R_k} \rho^{n+2(k-1)}g^*_p(\rho)\phi(\rho)d\rho \right] - \frac{(1-\theta_k)[1-\chi(r)]}{\rho^{n+2(k-1)}} \int_0^{\min[r,\delta]} \rho^{n+2(k-1)}g^*_p(\rho)\phi(\rho)d\rho.$$

Next we let $f(r, t) = \mathcal{K}_k''(\phi_0(t)) \phi(r)$. Since $\tilde{\mathcal{K}}_k = \mathcal{K}_0 + \mathcal{K}_k$, from (5.1) we see that $\phi$ is the solution of the following problem:

$$\begin{cases}
\partial_t \phi = L_0(\phi) + \mathcal{K}'_k(\phi) + f(r, t) & \text{for } 0 \leq r < 1, \ t > 0, \\
\phi |_{t=0} = \phi_0 & \text{for } 0 \leq r \leq R_k.
\end{cases}$$

By using (6.11) and (6.15) we easily see that for any $\mu > \max\{\mu^*, -\kappa_0\}$ there exists a corresponding constant $C = C(k, \mu, \delta) > 0$ such that

$$\max_{0 \leq r \leq R_k} |f(r, t)| \leq Ce^{\mu t} \max_{0 \leq r \leq R_k} |\phi_0(r)| \quad \text{for } t \geq 0. \quad (6.17)$$

By corollary 4.3 we see that for any $\mu > \mu_0$ there holds

$$\max_{0 \leq r \leq R_k} |e^{t L_0 \phi(r)}| \leq C e^{\mu t} \max_{0 \leq r \leq R_k} |\phi(r)| \quad \text{for } t \geq 0. \quad (6.18)$$

Besides, it is easy to check that there exists a positive function $\varepsilon(\delta)$ of $\delta$ which converges to zero as $\delta \to 0^+$, such that

$$\max_{0 \leq r \leq R_k} |\mathcal{K}_k''(\phi)| \leq \varepsilon(\delta) \max_{0 \leq r \leq R_k} |\phi(r)| \quad \text{for } \phi \in C[0, R_k]. \quad (6.19)$$

For instance, for the cases $k \geq 2$ we have

$$\max_{0 \leq r \leq R_k} \left| r^{k-1}p'_1(r) \int_r^{\max[r,\delta]} \rho^{-k+1}g^*_p(\rho)\phi(\rho)d\rho \right| \leq C \max_{0 \leq r \leq R_k} \left( rp'_1(r) \int_0^{\delta} \rho^{-1}d\rho \right) \max_{0 \leq r \leq R_k} |\phi(r)| \leq C \varepsilon_1(\delta) \max_{0 \leq r \leq R_k} |\phi(r)|,$$

where $\varepsilon_1(\delta) = \max_{0 \leq r \leq \delta} (rp'_1(r) \ln \frac{\delta}{r}) \to 0$ as $\delta \to 0^+$. For the cases $k = 0, 1$ we can use lemma 3.2 to get a similar inequality. By a standard perturbation theorem for $C_0$-semigroups, from (6.18) and (6.19) we have

$$\max_{0 \leq r \leq R_k} |e^{t(L_0 + X_1') \phi(r)}| \leq C'_\mu e^{[\mu + C_\mu(\delta)]t} \max_{0 \leq r \leq R_k} |\phi(r)| \quad \text{for } t \geq 0. \quad (6.20)$$

Now, from (6.16) we have

$$\psi(\cdot, t) = e^{t(L_0 + X_1') } \psi_0 + \int_0^t e^{(t-\tau)(L_0 + X_1') } f(\cdot, \tau)d\tau \quad \text{for } t \geq 0. \quad (6.21)$$

From (6.17), (6.20) and (6.21) we can easily deduce that for any given $\mu > \mu^* \equiv \max\{\mu^*_i, -\kappa_0\}$, by first choosing $\mu' > \mu^*$ such that $\mu > \mu'$ and next choosing $\delta$ sufficiently small so that $\mu' + C_\mu \varepsilon(\delta) < \mu$, (6.14) follows. This completes the proof. \qed
Lemma 6.6. Assume that the conditions in (1.25) are satisfied. There exists a constant \( \mu_0^* < 0 \) such that for every non-negative integer \( k \), any \( \alpha \geq 1 \) and \( \mu > \mu_0^* \) there exists a constant \( C_{k,\alpha,\mu} \) such that for any initial data \( \varphi_0 \) such that \( \varphi_0 \in L^\infty([0, R_*]; r^{n-1} \, dr) \), the solution of the initial value problem (5.1) satisfies the following estimate:

\[
\left( \int_0^{R_*} |\varphi(r, t)|^{\alpha} r^{n-1} \, dr \right)^{\frac{1}{\alpha}} \leq C \left( \int_0^{R_*} |\varphi_0(r)|^{\alpha} r^{n-1} \, dr \right)^{\frac{1}{\alpha}} e^{\mu t} \quad \text{for } t \geq 0.
\]

(6.22)

Proof. We split the proof into three steps.

Step 1. We first prove that

\[
\int_0^{R_*} |\varphi(r, t)| r^{n+k-1} \, dr \leq Ce^{\mu t} \int_0^{R_*} |\varphi_0(r)| r^{n-k+1} \, dr \quad \text{for } t \geq 0.
\]

(6.23)

Indeed, by the relation (6.2) it follows that if \( \varphi \) is a solution of the problem (5.1) then \( \psi(r, t) = r^{n+k-1} \varphi(r, t)/p'_k(r) \) is a solution of the problem (6.3) with initial data \( \psi_0(r) = r^{n+k-1} \varphi_0(r)/p'_k(r) \). Due to this fact, the above estimate is an immediate consequence of lemma 6.4.

Step 2. We next prove that

\[
\int_0^{R_*} |\varphi(r, t)| r^{n-1} \, dr \leq Ce^{\mu t} \int_0^{R_*} |\varphi_0(r)| r^{n-1} \, dr \quad \text{for } t \geq 0.
\]

(6.24)

Let \( \mathcal{X}_k', \mathcal{X}_k'' \) and \( f \) be as in the proof of lemma 6.5. Using (6.23) we easily see that

\[
\int_0^{R_*} |f(r, t)| r^{n-1} \, dr \leq C_{k,\alpha,\mu} e^{\mu t} \int_0^{R_*} |\varphi_0(r)| r^{n-1} \, dr \quad \text{for } t \geq 0.
\]

(6.25)

By lemma 4.3 we see that with \( v^* = \max_{0 \leq r \leq R_*} \{ g^*(r) + f^*_p(r) \} < 0 \) there holds

\[
\int_0^{R_*} |e^{tX_k}(\varphi(r))| r^{n-1} \, dr \leq e^{v^*t} \int_0^{R_*} |\varphi_0(r)| r^{n-1} \, dr \quad \text{for } t \geq 0.
\]

(6.26)

Besides, we have

\[
\int_0^{R_*} |\mathcal{X}_k'(\phi(r))| r^{n-1} \, dr \leq \theta_k \int_0^{\delta} \int_{r_{\delta}}^{r} \rho^{n+k-2} \rho^{-k+1} g^*_p(\rho)|\phi(\rho)| \, d\rho \, dr + (1-\theta_k) \int_0^{\delta} \int_{r_{\delta}}^{r} r^{-k} \rho^{n+k-1} g^*_p(\rho)|\phi(\rho)| \, d\rho \, dr
\]

\[
\leq C \int_0^{\delta} \left( \int_{r_{\delta}}^{r} \rho^{n+k-2} p'_k(\rho) \, d\rho \right) \rho^{-k+1} |\phi(\rho)| \, d\rho + C \int_0^{\delta} \left( \int_{r_{\delta}}^{r} r^{-k} \rho^{n+k-1} p'_k(\rho) \, d\rho \right) |\phi(\rho)| \, d\rho
\]

\[
\leq C[p_\delta(\delta) - p_\delta(0)] \int_{r_{\delta}}^{\delta} \rho^{n-1} |\phi(\rho)| \, d\rho + C[p_\delta(2\delta) - p_\delta(0)]
\]

\[
\times \int_0^{\delta} \rho^{n-1} |\phi(\rho)| \, d\rho
\]

\[
\leq C[p_\delta(2\delta) - p_\delta(0)].
\]

(6.27)

where \( \varepsilon(\delta) = 2C[p_\delta(2\delta) - p_\delta(0)] \). By using (6.21), (6.25), (6.26), (6.27) and a similar argument as in the proof of lemma 6.5, we obtain (6.24).

Step 3. Interpolating the inequalities (6.21) and (6.24), we see that (6.22) follows. \( \square \)
7. Decay estimates for the system (2.22)–(2.23)

Lemma 7.1. Assume that the conditions in (1.25) are satisfied. There exist constants $\gamma^* > 0$ and $\lambda > 0$ such that for any $\gamma > \gamma^*$, any integer $k \geq 2$ and any $\alpha \geq 1$, the solution of the system of equations (2.22)–(2.23) satisfies the following estimates:

$$\max_{0 \leq r \leq R_i} |\tilde{\psi}_k(r, t)| + |\eta_k(t)| \leq C e^{-\eta^* \lambda t} \left[ \max_{0 \leq r \leq R_i} |\tilde{\psi}_{k0}(r)| + |\eta_{k0}| \right] \quad \text{for } t \geq 0,$$

(7.1)

$$\left( \int_0^R |\tilde{\psi}_k(r, t)|^\alpha r^\alpha - 1 \, dr \right)^\frac{1}{\alpha} + |\eta_k(t)| \leq C e^{-\eta^* \lambda t} \left[ \left( \int_0^R |\tilde{\psi}_{k0}(r)|^\alpha r^\alpha - 1 \, dr \right)^\frac{1}{\alpha} + |\eta_{k0}| \right]$$

for $t \geq 0$,

(7.2)

where $\tilde{\psi}_{k0}$ and $\eta_{k0}$ are the initial data of $\tilde{\psi}_k$ and $\eta_k$ respectively, and $C, C_\alpha$ represent positive constants independent of $k$.

Proof. We need only prove that the solution of the system of equations (2.24)–(2.25) satisfies the following estimates:

$$\max_{0 \leq r \leq R_i} |\tilde{\psi}_k(r, t)| + |\eta_k(t)| \leq C e^{-\eta^* \lambda t} \left[ \max_{0 \leq r \leq R_i} |\tilde{\psi}_{k0}(r)| + |\eta_{k0}| \right] \quad \text{for } t \geq 0,$$

(7.3)

$$\left( \int_0^R |\tilde{\psi}_k(r, t)|^\alpha r^\alpha - 1 \, dr \right)^\frac{1}{\alpha} + |\eta_k(t)| \leq C e^{-\eta^* \lambda t} \left[ \left( \int_0^R |\tilde{\psi}_{k0}(r)|^\alpha r^\alpha - 1 \, dr \right)^\frac{1}{\alpha} + |\eta_{k0}| \right]$$

for $t \geq 0$,

(7.4)

where $\tilde{\psi}_{k0}$ and $\eta_{k0}$ are the initial data of $\tilde{\psi}_k$ and $\eta_k$ respectively. Indeed, since the solutions of the systems (2.22)–(2.23) and (2.24)–(2.25) satisfy the relations

$$\tilde{\psi}_k(r, t) = \psi_k(r, t) + \left( \frac{r}{R_i} \right)^{k-1} p_k'(r) \eta_k(t), \quad k = 0, 1, 2, \ldots$$

(7.5)

we see that (7.1) and (7.2) are immediate consequences of (7.3) and (7.4).

Fix two constants $\lambda$ and $\mu$ such that $\lambda > 0$ and $\mu^* < \mu < -\lambda$, where $\mu^*$ is as in lemma 6.5. We re-denote the constant appearing on the right-hand side of (6.14) as $C_0$. Let $\tilde{\psi}_{k0} \in C[0, R_i]$ and $\eta_{k0} \in \mathbb{R}$ be given. Given $\psi \in C([0, R_i] \times [0, \infty))$ satisfying the condition

$$\max_{0 \leq r \leq R_i} |\psi(r, t)| \leq 2 C_0 e^{-\lambda t} \left[ \max_{0 \leq r \leq R_i} |\tilde{\psi}_{k0}(r)| + |\eta_{k0}| \right] \quad \text{for } t \geq 0,$$

(7.6)

we consider the following initial value problems:

$$\frac{d\eta}{dt} = \tilde{a}_k(\gamma) \eta + J_k(\psi) \quad \text{for } t \geq 0, \quad \text{and} \quad \eta|_{t=0} = \eta_{k0},$$

(7.7)

$$\frac{d\tilde{\psi}}{dt} = \tilde{Z}_k(\tilde{\psi}) + c_k(r) \eta \quad \text{for } 0 < r < R_i, \quad t \geq 0, \quad \text{and} \quad \tilde{\psi}|_{t=0} = \tilde{\psi}_{k0}.$$  

(7.8)

The solution of (7.7) is given by

$$\eta(t) = e^{\tilde{a}_k(\gamma) t} \eta_{k0} + \int_0^t e^{\tilde{a}_k(\gamma)(t-\tau)} J_k(\psi(\cdot, \tau)) \, d\tau \quad \text{for } t \geq 0.$$ 

(7.9)

It is clear that there exists a constant $c > 0$ independent of $k$ and $\gamma$ such that for $\gamma$ sufficiently large,

$$\tilde{a}_k(\gamma) \leq -ck^2 \gamma \quad \text{for } k \geq 2.$$  

(7.9)

Using this fact and (7.6) we easily see that for $\gamma$ sufficiently large and $k \geq 2$,

$$|\eta(t)| \leq e^{-ck^2 \gamma t} |\eta_{k0}| + C C_0 e^{-\lambda t} \left[ \max_{0 \leq r \leq R_i} |\tilde{\psi}_{k0}(r)| + |\eta_{k0}| \right] \quad \text{for } t \geq 0.$$  

(7.10)
Having solved the initial value problem (7.7), we substitute its solution into (7.8) and next solve that initial value problem. The solution is given by
\[ \tilde{\phi}(\cdot, t) = e^{2\tilde{Z}t} \tilde{\phi}_{t=0} + \int_0^t [e^{t-\tau} \tilde{Z}_1] \eta(\tau) d\tau \quad \text{for } t \geq 0. \] (7.11)

It is clear that there exists constant \( C > 0 \) independent of \( k \) such that for any \( k \geq 2 \),
\[ \max_{0 \leq r \leq R_k} |c_k(r)| \leq Ck. \] (7.12)

By using lemmas 5.1, 6.5 and the estimates (7.10), (7.12) we easily deduce from (7.11) to get
\[ \max_{0 \leq r \leq R_k} |\tilde{\phi}(r, t)| \leq C_0 e^{\lambda t} \max_{0 \leq r \leq R_k} |\tilde{\phi}_{t=0}(r)| + \frac{C C_0 ke^{-\lambda t}}{ck^3} [\max_{0 \leq r \leq R_k} |\tilde{\phi}_{t=0}(r)| + \eta_{t=0}]. \]

From this estimate we easily see that by choosing \( \gamma^* > 0 \) sufficiently large, we have that for any \( \gamma > \gamma^* \) and any \( k \geq 2 \),
\[ \max_{0 \leq r \leq R_k} |\tilde{\phi}(r, t)| \leq 2 C_0 e^{-\lambda t} [\max_{0 \leq r \leq R_k} |\tilde{\phi}_{t=0}(r)| + \eta_{t=0}] \quad \text{for } t \geq 0, \]
i.e., \( \tilde{\phi} \) satisfies the condition (7.6). Furthermore, it can also be easily seen that if we choose \( \gamma^* > 0 \) so large that for any \( \gamma > \gamma^* \) and any \( k \geq 2 \),
\[ \frac{Ck}{|\lambda + \mu|(ck^3) - \lambda)} < 1, \]
then the mapping \( \varphi \mapsto \tilde{\phi} \) is a contraction. Hence, by using the standard contraction mapping argument we see that the system of equations (2.24)–(2.25) subject to the initial conditions in (7.7) and (7.8) has a unique solution satisfying (7.6) (with \( \varphi \) replaced by \( \tilde{\phi} \)) and (7.10). This proves the estimate (7.3). To prove the estimate (7.4), we replace the condition (7.6) with the following one:
\[ \left( \int_0^R |\tilde{\phi}(r, t)|^{\alpha r^{n-1}} dr \right)^{\frac{1}{\alpha}} \leq 2 C_0 e^{-\lambda t} \left[ \left( \int_0^R |\tilde{\phi}_{t=0}(r)|^{\alpha r^{n-1}} dr \right)^{\frac{1}{\alpha}} + |\eta_{t=0}| \right] \quad \text{for } t \geq 0, \]
and use a similar argument as above but instead of using lemmas 5.1 and 6.5 we now use lemmas 5.2 and 6.6. We omit the details. This completes the proof of lemma 7.1.

The above results do not work for the cases \( k = 0, 1 \). We first note that in these cases \( \tilde{\alpha}_k \) are independent of \( \gamma \). For these special cases, we have the following results:

**Lemma 7.2.** In the case \( k = 1 \), there exist constants \( \lambda > 0 \) and \( C > 0 \) such that the solution of the system of equations (2.24)–(2.25) satisfies the following estimates:
\[ \max_{0 \leq r \leq R_k} |\varphi_1(r, t) - \varphi_\infty(r)| \leq C e^{-\lambda t} \left( \max_{0 \leq r \leq R_k} |\varphi_{t=0}(r)| + |\eta_{t=0}| \right) \quad \text{for } t \geq 0, \] (7.13)
\[ \left( \int_0^R |\varphi_1(r, t) - \varphi_\infty(r)|^{\alpha r^{n-1}} dr \right)^{\frac{1}{\alpha}} \leq C_0 e^{-\lambda t} \left( \int_0^R |\varphi_{t=0}(r)|^{\alpha r^{n-1}} dr + |\eta_{t=0}| \right)^{\frac{1}{\alpha}} \quad \text{for } \alpha \geq 1, \ t \geq 0, \] (7.14)
\[ |\eta_1(t) - \eta_\infty| \leq C e^{-\lambda t} \left( \max_{0 \leq r \leq R_k} |\varphi_{t=0}(r)| + |\eta_{t=0}| \right) \quad \text{for } t \geq 0, \] (7.15)
where \( \varphi_\infty(r) = -p'_1(r) \eta_\infty \), and \( \eta_\infty \) is a real constant uniquely determined by the initial data \( \varphi_{t=0} \) and \( \eta_{t=0} \).
Proof. Because of the relations in (7.5) and lemma 3.2, we see that the above estimates follow if we prove that the solution of the system (2.24)–(2.25) satisfies the following estimates:

$$\max_{0 \leq r \leq R_s} |\tilde{\psi}_1(t, r)| \leq Ce^{-\lambda t} \max_{0 \leq r \leq R_s} |\tilde{\psi}_{10}(r)| \quad \text{for } t \geq 0,$$

(7.16)

$$\left( \int_0^R |\tilde{\psi}_1(t, r)|^{\alpha r^{\alpha-1}} \, dr \right)^{\frac{1}{\alpha}} \leq C_\alpha e^{-\lambda t} \left( \int_0^R |\tilde{\psi}_{10}(r)|^{\alpha r^{\alpha-1}} \, dr \right)^{\frac{1}{\alpha}} \quad \text{for } \alpha \geq 1, \ t \geq 0,$$

(7.17)

$$|\eta_1(t) - \eta_{10} - \int_0^\infty J_1(\tilde{\psi}_1(t, r)) \, dt| \leq Ce^{-\lambda t} \max_{0 \leq r \leq R_s} |\tilde{\psi}_{10}(r)| \quad \text{for } t \geq 0.$$  

(7.18)

These estimates are immediate consequences of the relations in (2.29) and lemmas 6.4–6.6 applied to the case \( k = 1 \).

Lemma 7.3. In the case \( k = 0 \), there exist constants \( \lambda > 0 \) and \( C > 0 \) such that the solution of the system of equations (2.22)–(2.23) satisfies the following estimate:

$$\max_{0 \leq r \leq R_s} |\psi_0(t, r)| + |\eta_0(t)| \leq Ce^{-\lambda t} \max_{0 \leq r \leq R_s} |\psi_{00}(r)| + |\eta_{00}| \quad \text{for } t \geq 0,$$

(7.19)

$$\left( \int_0^R |\psi_0(t, r)|^{\alpha r^{\alpha-1}} \, dr \right)^{\frac{1}{\alpha}} + |\eta_0(t)| \leq C_\alpha e^{-\lambda t} \left[ \left( \int_0^R |\psi_{00}(r)|^{\alpha r^{\alpha-1}} \, dr \right)^{\frac{1}{\alpha}} + |\eta_{00}| \right]$$

for \( t \geq 0, \)  

(7.20)

where \( \psi_{00} \) and \( \eta_{00} \) are the initial data of \( \psi_0 \) and \( \eta_0 \) respectively.

Proof. In the case \( n = 3 \), the estimate (7.19) follows from lemma 6.2 of [7], which is an improvement of theorem 5.1 of [5]. This is because the system (2.22)–(2.23) is the linearization of the radial version of the system (1.12)–(1.18), so that it is equivalent to the system (5.1)–(5.2) in [5]. The estimate (7.20) follows from this fact and a similar argument as in the proof of lemma 6.6 and (7.2) in lemma 7.1. In the general dimension case, the arguments are similar.

In what follows we give a sketch of them. Since \( b_0(r, \gamma) \) and \( \alpha_0(\gamma) \) are actually independent of \( \gamma \), in the sequel we briefly write them as \( b_0(r) \) and \( \alpha_0 \) respectively.

Firstly, by using some similar arguments as in the proofs of lemmas 7.1–7.3 of [5] (see also the proof of lemma 4.4 of this work) we can prove that the equation

$$\mathcal{L}_0 \phi^*(r) - [\alpha_0 + J_0(\phi^*)] \phi^*(r) + b_0(r) = 0 \quad \text{for } 0 < r < R_s,$$

(7.21)

has a unique solution \( \phi^*(r) = \psi^*(r) - \left( \frac{R_s}{r} \right)^{n-1} p'_s(r) \) with \( \alpha_0 + J_0(\phi^*) < 0 \), where \( \psi^* \in C^1(0, R_s), \psi^*(r) > 0 \) for \( 0 < r < R_s \), and \( J_0(\psi^*) < \infty \). More precisely, the above equation can be rewritten into the following equivalent system of equations:

$$[\lambda - \mathcal{L}^*_0] \psi^*(r) = h_0(r) \quad \text{for } 0 < r < R_s$$

(7.22)

$$\lambda = \alpha_0 + J_0(\psi^*) - J_0 \left[ \left( \frac{R_s}{r} \right)^{n-1} p'_s(r) \right]$$

(with \( \psi^* \) and \( \lambda \) being the unknowns), where

$$\mathcal{L}^*_0 \psi(r) = -v_s(r) \psi'(r) + f_p^*(r) \psi(r) + \frac{p'_s(r)}{r^{n-1}} \int_r^R \rho^{n-1} f_p^*(\rho) \psi(\rho) \, d\rho,$$

$$h_0(r) = b_0(r) - \mathcal{L}^*_0 \left[ \left( \frac{R_s}{r} \right)^{n-1} p'_s(r) \right] + \alpha_0 \left( \frac{R_s}{r} \right)^{n-1} p'_s(r).$$
By using the assertions (4) and (5) of lemma 3.3 and the equations (2.3), (2.4) and (2.5) as well as their equivalent integral forms, we can prove that \( h_0(r) > 0 \) for \( 0 < r < R_s \). Hence, by using some similar arguments as in the proofs of lemmas 7.1–7.3 of [5], we see that the above system has a unique solution \((\psi^*, \lambda)\) with the properties that

\[
nv'(0) + f^*_p(0) < \lambda < 0, \quad \lambda \geq f^*_p(R_s),
\]

and

\[
c_1 r^{-(n-1)}(R_s - r)p'(r) \leq \psi^*(r) \leq c_2 r^{-\theta} \quad \text{for } 0 < r < R_s,
\]

where \( c_1, c_2 \) and \( \theta \) are positive constants, \( 0 < \theta < n \), so that \( J_0(\psi^*) < \infty \). This proves the above statement. We now make a transformation of unknown variables \((\varphi_0, \eta_0) \mapsto (\psi_0, \eta_0)\) such that

\[
\psi_0(r, t) = \varphi_0(r, t) - \phi^*(r) \eta(t).
\]

Then the system (2.22)–(2.23) with \( k = 0 \) is transformed into the following equivalent system:

\[
\frac{\partial \psi_0}{\partial t} = \mathcal{L}_0(\psi_0) - \phi^*(r) J_0(\psi_0),
\]

\[
\frac{d\eta_0}{dt} = \lambda \eta_0 + J_0(\psi_0).
\]

Note that

\[
\mathcal{L}_0(\psi_0) - \phi^*(r) J_0(\psi_0) = \mathcal{L}_0^0(\psi_0) - \psi^*(r) J_0(\psi_0).
\]

Using this fact and some similar arguments as in the proofs of lemmas 6.5 and 6.6, we obtain from (7.23) the following estimates:

\[
\max_{0 \leq r \leq R_s} |\psi_0(r, t)| \leq C e^{-\lambda t} \max_{0 \leq r \leq R_s} |\psi_0(r, 0)| \quad \text{for } t \geq 0,
\]

\[
\left( \int_0^{R_s} |\psi_0(r, t)| n r^{n-1} dr \right)^{\frac{1}{n}} \leq C a e^{-\lambda t} \left( \int_0^{R_s} |\psi_0(r, 0)| n r^{n-1} dr \right)^{\frac{1}{n}} \quad \text{for } t \geq 0.
\]

By (7.24), this further implies that

\[
|\eta_0(t)| \leq C e^{-\lambda t} \left[ \int_0^{R_s} |\psi_0(r, 0)| n r^{n-1} dr + |\eta_{00}| \right] \quad \text{for } t \geq 0.
\]

The estimates (7.19) and (7.20) now follow from (7.22) and (7.26)–(7.28). \( \square \)

8. Decay estimates for the system (2.19)

For every non-negative integer \( k \), we let \( Y_{lI}(\omega) \) \((l = 1, 2, \cdots, d_k)\) be a normalized orthogonal basis of the linear space of spherical harmonics of degree \( k \) (! [41]), i.e.

\[
\Delta_{\omega} Y_{lI}(\omega) = -\lambda_k Y_{lI}(\omega), \quad \lambda_k = (n + k - 2)k,
\]

\[
\int_{S^{n-1}} Y_{lI}(\omega) Y_{l'I}(\omega) d\omega = 0 \quad (l \neq l'), \quad \int_{S^{n-1}} Y_{lI}^2(\omega) d\omega = 1.
\]

Here \( d_k \) is the dimension of the linear space of spherical harmonics of degree \( k \), i.e.

\[
d_0 = 1, \quad d_1 = n \quad \text{and} \quad d_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2} \quad \text{for } k \geq 2,
\]

\[
1074
\]
and $d\omega$ is the induced element on the unit sphere $S^{n-1}$ of the Lebesque measure $dx$ in $\mathbb{R}^n$.

Note that in particular,

$$Y_{0l}(\omega) = \frac{1}{\sqrt{\sigma_n}} \quad \text{and} \quad Y_{1l}(\omega) = \frac{\sqrt{n} \omega_l}{\sqrt{\sigma_n}}, \quad l = 1, 2, \ldots, n,$$

where $\sigma_n$ denotes the surface area of the unit sphere $S^{n-1}$, i.e. $\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, and $\omega_l$ denotes the $l$th component of $\omega$.

For any $1 \leq \alpha < \infty$ and $1 \leq \beta < \infty$, we denote by $X_{\alpha\beta}$ the space of all measurable functions $u(x)$ in the ball $B(0, R_s) \subseteq \mathbb{R}^n$ satisfying the following conditions:

$$u(x) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} u_{kl}(r) Y_{kl}(\omega) \quad \text{in} \quad S'(\mathbb{B}(0, R)), \quad r = |x|, \quad \omega = \frac{x}{|x|}, \quad (8.2)$$

$$\|u\|_{X_{\alpha\beta}} = \left[ \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} \left( \int_0^{R_s} |u_{kl}(r)|^\alpha r^{n-1} dr \right)^{\frac{\beta}{\alpha}} \right]^{\frac{1}{\beta}} < \infty.$$ 

We also let $X_{\infty\beta}, X_{\alpha\infty}$ and $X_{\infty\infty}$ be, respectively, the following spaces:

$$\|u\|_{X_{\infty\beta}} = \left[ \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} \left( \sup_{0 \leq r \leq R_s} |u_{kl}(r)| \right)^\beta \right]^{\frac{1}{\beta}} < \infty,$$

$$\|u\|_{X_{\alpha\infty}} = \sup_{k,l} \left( \int_0^{R_s} |u_{kl}(r)|^\alpha r^{n-1} dr \right)^{\frac{1}{\alpha}} < \infty,$$

$$\|u\|_{X_{\infty\infty}} = \sup_{k,l} \sup_{0 \leq r \leq R_s} |u_{kl}(r)| < \infty.$$ 

It is clear that for any $1 \leq \alpha \leq \infty$ and $1 \leq \beta \leq \infty$, $X_{\alpha\beta}$ is a Banach space. For $1 \leq \beta \leq \infty$, we denote by $X_{\infty\beta}$ the closure of $C(\mathbb{B}(0, R_s))$ in $X_{\infty\beta}$. If $u \in X_{\infty\beta}$ then from the relation

$$u_{kl}(r) = \int_{S^{n-1}} u(r\omega) Y_{kl}(\omega) d\omega, \quad k = 0, 1, 2, \ldots, \quad l = 1, 2, \ldots, d_k,$$

we see that in the expansion (8.2), all coefficients $u_{kl}(r)$ are continuous functions in $[0, R_s]$.

Note that $X_{22} = L^2(\mathbb{B}(0, R_s))$.

Next, for any $1 \leq \beta < \infty$, we denote by $Y_\beta$ the space of all measurable functions $\varphi(\omega)$ on the sphere $S^{n-1}$ satisfying the following conditions:

$$\varphi(\omega) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} a_{kl} Y_{kl}(\omega) \quad \text{in} \quad D'(S^{n-1}), \quad (8.3)$$

$$\|\varphi\|_{Y_\beta} = \left( \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} |a_{kl}|^\beta \right)^{\frac{1}{\beta}} < \infty.$$ 

We also denote by $Y_\infty$ the following space:

$$\|\varphi\|_{Y_\infty} = \sup_{k,l} |a_{kl}| < \infty.$$ 

It is clear that for any $1 \leq \beta \leq \infty$, $Y_\beta$ is a Banach space. Note that $Y_2 = L^2(S^{n-1})$.

We can now give the precise statement of the main result of this paper:
Theorem 8.1. Assume that the conditions in (1.25) are satisfied. There exist constants \( \gamma^* > 0 \) and \( \lambda > 0 \) such that for any \( \gamma > \gamma^* \), \( 1 \leq \alpha < \infty \) and \( 1 \leq \beta \leq \infty \), the solution \((\varphi, \eta)\) of the system (2.19) satisfies the following estimates:

\[
\|\varphi(\cdot, t) - \varphi_\infty\|_{X_{\alpha, \beta}} + \|\eta(\cdot, t) - \eta_\infty\|_{Y_{\gamma}} \leq C_{\text{ip}} e^{-\gamma t} [\|\varphi_0\|_{X_{\alpha, \beta}} + \|\eta_0\|_{Y_{\gamma}}] \quad \text{for} \ t \geq 0.
\]

where \( \varphi_0 = \varphi(\cdot, 0) \) and \( \eta_0 = \eta(0) \) are the initial data of \( \varphi \) and \( \eta \) respectively, \( \varphi_\infty \) and \( \eta_\infty \) are functions in \( \mathbb{H}(0, R_s) \) and \( \mathbb{S}^{n-1} \) respectively, having the following expressions:

\[
\varphi_\infty(x) = -p_s'(r) \sum_{l=1}^{n} c_l Y_{1l}(\omega), \quad \eta_\infty(\omega) = \sum_{l=1}^{n} c_l Y_{1l}(\omega),
\]

where \( c_1, c_2, \ldots, c_n \) are real constants uniquely determined by the initial data \( \varphi_0 \) and \( \eta_0 \), and \( C_{\text{ip}} \) is a positive constant. Moreover, in case \( \alpha = \infty \) we have also the following estimate:

\[
\|\varphi(\cdot, t) - \varphi_\infty\|_{X_{\alpha, \beta}} + \|\eta(\cdot, t) - \eta_\infty\|_{Y_{\gamma}} \leq C_{\text{ip}} e^{-\lambda t} [\|\varphi_0\|_{X_{\alpha, \beta}} + \|\eta_0\|_{Y_{\gamma}}] \quad \text{for} \ t \geq 0.
\]

Proof. This is an immediate consequence of lemmas 7.1–7.3. □

Hence, we have finished proving theorem 1.1. □

9. Conclusions and further discussion

We have studied asymptotic stability of the trivial solution of the linearized equations (2.11)-(2.17) of the tumour model (1.12)-(1.18) at its unique spherically symmetric stationary solution \((c(r), p(r), v_s(r), \sigma_s(r), \Omega_1)\) of the tumour model (1.12)-(1.18) at its unique spherically symmetric stationary solution, which is nonlinearly asymptotically stable. Actually, trying to get an answer to this question is not encountered in previous similar works on tumour model analysis such as [8,11,12,45,46]. This is an immediate consequence of lemmas 7.1–7.3.

The problem then reduces into studying asymptotic stability of the trivial solution of the system of equations (2.22)-(2.23). A main feature of these equations is that they are not only non-local but also singular. This causes some new difficulties which are not encountered in previous similar works on tumour model analysis such as [8,11,12,45,46].

These difficulties are overcome with the aid of application of techniques for solving singular differential equations recently developed in [4,7,13]. It is proved that if the conditions in (1.25) are satisfied then there exists a constant \( \gamma^* > 0 \) such that if the surface tension coefficient \( \gamma > \gamma^* \), then for any \( 1 \leq \alpha < \infty \) and \( 1 \leq \beta \leq \infty \), the solution of the system (2.19) satisfies the decay estimate in (8.4), so proving that the unique radial stationary solution of the system (1.12)-(1.18) is linearly asymptotically stable modula translations in the function spaces introduced in section 8.

We point out once again that in the group of conditions (1.25), the condition \( k_B > k_D \) is essential to ensure the validity of theorem 1.1, whereas the other conditions are imposed just for technical reasons: the main role of the remaining conditions other than \( k_B > k_D \) in (1.25) is to ensure that the stationary solution \( p_s(r) \) is differentiable at \( r = 0 \); see lemma 3.2. Linearity of the functions \( k_B(c), k_D(c), k_P(c) \) and \( k_Q(c) \) are assumed also mainly for this purpose. It might be true that other conditions in (1.25) than \( k_B > k_D \) and the linearity assumption of the functions \( k_B(c), k_D(c), k_P(c) \) can be removed without affecting the validity of theorem 1.1.

A natural question is whether the radial stationary solution of the system (1.12)-(1.18) is nonlinearly asymptotically stable. Actually, trying to get an answer to this question is the author’s original motivation of writing this paper. However, presently we are unable to
give a satisfactory answer to this question due to some difficulties which we are temporarily unable to overcome. The main difficulty lies at the point that when the system of equations (1.12)–(1.18) is reduced into the simplest form, which is a 2-system of nonlinear integral partial and pseudo-differential equations, one of the two reduced equations is a quasilinear hyperbolic equation. Unlike quasilinear parabolic equations to which the Lyapunov theorem for differential equations in $\mathbb{R}^n$ (concerning asymptotic stability of stationary solutions) has already been successfully extended ([1, 37]), there is not a similar extension to quasilinear hyperbolic equations. This can be roughly seen as follows: since all uniformly elliptic operators of same order in the same domain have same strength or can be mutually estimated, two different parabolic operators

$$L_1u = \partial_t u - P_1(D_x)u \quad \text{and} \quad L_2u = \partial_t u - P_2(D_x)u$$

(where $P_1(D_x)$, $P_2(D_x)$ are two different pseudo-differential operators of the negative elliptic type with same order) have also same strength so that they can also be mutually estimated. For two different hyperbolic operators

$$\mathcal{L}_1u = \partial_t u - \sum_{j=1}^n a_{1j}(x)\partial_j u \quad \text{and} \quad \mathcal{L}_2u = \partial_t u - \sum_{j=1}^n a_{2j}(x)\partial_j u \quad (9.1)$$

(where $\sum_{j=1}^n a_{ij}(x)\partial_j$ ($i = 1, 2$) are two different vector fields), however, it is well-known that neither of them can be estimated by the other. This is actually a common difficulty encountered in every quasilinear hyperbolic problem. As far as the tumour model (1.1)–(1.8) is concerned, recall that in the spherically symmetric case, this difficulty was overcome by using the similarity transformation technique ([7]). Such a transformation is used to transform one hyperbolic operator into another so that the above-mentioned difficulty is bypassed. It is possible to extend this concept to a multi-dimension case. However, when we try to follow this approach for the present spherically asymmetric case, a new difficulty is encountered: it is difficult to establish estimates in the function spaces used in theorem 8.1 for partial differential operators of the general irradial form in (9.1). Currently, we are still working on this problem.

Another interesting question is can the result of this paper be extended to other related tumour models? One should recall that a more realistic multi-phase tumour model than (1.1)–(1.8) contains three species of cells: proliferating cells, quiescent cells and dead cells; [23–25], for instance. The model (1.1)–(1.8) is the simplest simplification of such multi-phase tumour models which assumes that no dead cells exist in the tumour. This assumption reduces the difficulty of rigorous mathematical analysis to a great extent. Indeed, another apparently similar simplification assumes that the tumour contains other two species of cells: living cells and dead cells, and all living cells are in the proliferating state. Such a simplified multi-phase tumour model is called the modified Ward–King model, because it is a modification of the tumour model proposed by Ward and King in [43, 44]. Even such a greatly simplified multi-phase tumour model is much more difficult than the apparently similar model (1.1)–(1.8) for mathematical treatment. In [6] we proved that it has a radial stationary solution but failed to prove the uniqueness of such a solution. Asymptotic stability of this stationary solution under radial perturbations is still an open problem, i.e., similar results as those in [5, 7, 9] for the model (1.1)–(1.8) have not been well-established for the modified Ward–King model, though it has been conjectured to be true. Due to this reason, the problem of how to extend the result of the present paper to the modified Ward–King model is far from overcome. For more complex 3-phase models, even the early-stage problem of existence of a radial stationary solution is completely open to investigation.
In the model (1.1)–(1.8), Darcy’s law is used as the constitutive relation of tumour’s configuration. Surely, using other constitutive relations such as the stress–strain relation from elasticity theory as in [3, 35, 40] or the Stokes equations as in [21, 22] to replace Darcy’s law is also reasonable, see the literature [23–25, 36] for reviews and comments on these approaches of tumour modelling, and [28, 29, 45, 46] and references therein for rigorous mathematical analysis of some models of this kind. If we make such modifications to the model (1.1)–(1.8), we then obtain some new tumour models whose rigorous analysis is completely open. Whether the results of [4, 6, 7, 9, 13] and the present work can be extended to such new tumour models is a very interesting topic, and one for which the author does not have an idea at hand. Interested readers are encouraged to contribute their intelligence and wisdom on this valuable topic.

Acknowledgments

The author is glad to acknowledge his gratitude to the anonymous referee and editors for their valuable suggestions on improving the writing of this paper. This work is supported by the China National Natural Science Foundation under grant number 11171357.

References

[1] Amann H 1993 Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems Function Spaces, Differential Operators and Nonlinear Analysis ed H J Schmeisser and H Triebel (Leipzig: BSB Teubner) pp 9–126
[2] Araujo R P and McElwain D L 2004 A history of the study of solid tumor growth: the contribution of mathematical modeling Bull. Math. Biol. 66 1039–91
[3] Chaplain M A J and Sleeman B D 1993 Modelling the growth of solid tumours and incorporating a method for their classification using nonlinear elasticity theory J. Math. Biol. 31 431–79
[4] Chen X and Friedman A 2003 A free boundary problem for an elliptic–hyperbolic system: an application to tumor growth SIAM J. Math. Anal. 35 974–86
[5] Chen X, Cui S and Friedman A 2005 A hyperbolic free boundary problem modeling tumor growth: asymptotic behavior Trans. Am. Math. Soc. 357 4771–804
[6] Cui S 2006 Existence of a stationary solution for the modified Ward–King tumor growth model Adv. Appl. Math. 36 421–45
[7] Cui S 2008 Asymptotic stability of the stationary solution for a hyperbolic free boundary problem modeling tumor growth SIAM J. Math. Anal. 40 1692–724
[8] Cui S 2009 Lie group action and stability analysis of stationary solutions for a free boundary problem modelling tumor growth J. Diff. Eqns 246 1845–82
[9] Cui S 2013 Asymptotic stability of the stationary solution for a parabolic–hyperbolic free boundary problem modeling tumor growth SIAM J. Math. Anal. 45 2870–93
[10] Cui S and Escher J 2007 Bifurcation analysis of an elliptic free boundary problem modelling the growth of avascular tumors SIAM J. Math. Anal. 39 210–35
[11] Cui S and Escher J 2008 Asymptotic behavior of solutions of multidimensional moving boundary problem modeling tumor growth Commun. Part. Diff. Eqns 33 636–55
[12] Cui S and Escher J 2009 Well-posedness and stability of a multidimensional moving boundary problem modeling the growth of tumors Arch. Rat. Mech. Anal. 191 173–93
[13] Cui S and Friedman A 2002 A free boundary problem for a singular system of differential equations: an application to a model of tumor growth Trans. Am. Math. Soc. 355 3537–90
[14] Cui S and Friedman A 2003 A hyperbolic free boundary problem modeling tumor growth Interfaces Free Bound. 5 159–81
[15] Cui S and Wei X 2005 Global existence for a parabolic–hyperbolic free boundary problem modeling tumor growth Acta Math. Appl. Sin. Engl. Ser. 21 597–614
[16] Engel K J and Nagel R 2000 One-Parameter Semigroups for Linear Evolution Equations (New York: Springer)
[17] Escher J 2004 Classical solutions to a moving boundary problem for an elliptic–parabolic system Interfaces Free Bound. 6 175–93
[18] Escher J and Simonett G 1997 Classical solutions for Hele-Shaw models with surface tension Adv. Diff. Eqns 2 619–42
[19] Fasano A, Bertuzzi A and Gandolfi A 2006 Mathematical modelling of tumour growth and treatment Lect. Notes Math. 1872 71–106
[20] Fontelos M A and Friedman A 2003 Symmetry-breaking bifurcations of free boundary problems in three dimensions Asympt. Anal. 35 187–206
[21] Franks S J H et al 2003 Modelling the early growth of ductal carcinoma in situ of the breast J. Math. Biol. 47 424–52
[22] Franks S J H et al 2005 Biological inferences from a mathematical model of comedo ductal carcinoma in situ of the breast J. Theor. Biol. 232 523–43
[23] Friedman A 2004 A hierarchy of cancer models and their mathematical challenges Discuss Cont. Dyna. Syst. B 4 147–59
[24] Friedman A 2006 Cancer models and their mathematical analysis Lect. Notes Math. 1872 223–46
[25] Friedman A 2007 Mathematical analysis and challenges arising from models of tumor growth Math. Modelling Methods Appl. Sci. 17 (suppl.) 1751–72
[26] Friedman A and Hu B 2006 Bifurcation from stability to instability for a free boundary problem arising in tumor model Archive Ration Mech. Anal. 180 293–330
[27] Friedman A and Hu B 2006 Asymptotic stability for a free boundary problem arising in a tumor model J. Diff. Eqns 227 598–639
[28] Friedman A and Hu B 2007 Bifurcation for a free boundary problem modeling tumor growth by Stokes equation SIAM J. Math. Anal. 39 174–94
[29] Friedman A and Hu B 2007 Bifurcation from stability to instability for a free boundary problem modeling tumor growth by Stokes equation J. Math. Anal. Appl. 327 643–64
[30] Friedman A and Hu B 2008 Stability and instability of Lyapounov–Schmidt and Hopf bifurcation for a free boundary problem arising in a tumor model Trans. Am. Math. Soc. 360 5291–342
[31] Friedman A and Rietich F 1999 Analysis of a mathematical model for growth of tumors J. Math. Biol. 38 262–84
[32] Friedman A and Rietich F 2000 Symmetry-breaking bifurcation of analytic solutions to free boundary problems: An application to a model of tumor growth Trans. Am. Math. Soc. 353 1587–634
[33] Greenspan H P 1972 Models for the growth of solid tumor by diffusion Stud. Appl. Math. 51 317–40
[34] Greenspan H P 1976 On the growth and stability of cell cultures and solid tumors J. Theor. Biol. 56 229–42
[35] Jones A F, Byrne H M, Gibson J S and Dold J W 2000 Mathematical model for the stress induced during avascular tumor growth J. Math. Biol. 40 473–99
[36] Lowengrub J S et al 2010 Nonlinear modelling of cancer: bridging the gap between cells and tumours Nonlinearity 23 R1–91
[37] Lunardi A 1995 Analytic Semigroups and Optimal Regularity in Parabolic Problems (Basel: Birkhäuser)
[38] Pazy A 1983 Semigroups of Linear Operators and Applications to Partial Differential Equations (New York: Springer)
[39] Pettet G, Please C and McElwain M 2001 The migration of cells in multicell tumour spheroids Bull. Math. Biol. 63 231–57
[40] Shannon M A and Rubinsky B 1992 The effect of tumor growth on the stress distribution in tissue Adv. Biol. Heat Mass Transfer 231 35–8
[41] Stein E M and Weiss G 1971 Introduction to Fourier Analysis on Euclidean Spaces (Princeton; NJ: Princeton University Press) Chapter IV
[42] Tindall M J and Please C P 2007 Modelling the cell cycle and cell movement in multicellular tumour spheroids Bull. Math. Biol. 69 1147–65
[43] Ward J P and King J R 1997 Mathematical modelling of avascular tumor growthIMA J. Math. Appl. Med. Biol. 14 39–70
[44] Ward J P and King J R 1998 Mathematical modelling of avascular-tumor growth II: Modelling growth saturation IMA J. Math. Appl. Med. Biol. 15 1–42
[45] Wu J and Cui S 2009 Asymptotic stability of stationary solutions of a free boundary problem modelling the growth of tumors with fluid tissues SIAM J. Math. Anal. 41 391–414
[46] Wu J and Zhou F 2013 Asymptotic behavior of solutions of a free boundary problem modelling the growth of tumors with fluid-like tissue under the action of inhibitors Trans. Am. Math. Soc. 365 4181–207