A DYNAMICAL ARGUMENT FOR A RAMSEY PROPERTY

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Abstract. We show by a dynamical argument that there is a positive integer valued function $q$ defined on positive integer set $\mathbb{N}$ such that $q([\log n]+1)$ is a super-polynomial with respect to positive $n$ and

$$\limsup_{n \to \infty} r((2n+1)^2, q(n)) < \infty,$$

where $r(\cdot, \cdot)$ is the opposite-Ramsey number function.

1. Introduction and Preliminaries

For positive integers $p$ and $q$, we define the opposite-Ramsey number $r(p, q)$ to be the maximal number $k$ for which every edge-coloring of the complete graph $K_q$ with $p$ colors yields a monochromatic complete subgraph of order $k$ (the order of a graph means the number of its vertices).

The following is implied by the well-known Ramsey’s theorem.

Theorem 1.1. Let $p$ be a fixed positive integer. Then

$$\liminf_{q \to \infty} r(p, q) = \infty.$$

One may expect that if $p = p(n)$ and $q = q(n)$ are positive integer valued functions defined on $\mathbb{N}$ and the speed of $q(n)$ tending to infinity is much faster than that of $p(n)$ as $n$ tends to infinity, then we still have

$$\liminf_{n \to \infty} r(p(n), q(n)) = \infty.$$

The purpose of the paper is to show by a dynamical argument that this is not true in general even if $p(n)$ is a polynomial and $q([\log n]+1)$ is a super-polynomial. By a super-polynomial, we mean a function $f : \mathbb{N} \to \mathbb{R}$ such that for any polynomial $g(n)$,

$$\liminf_{n \to \infty} \frac{|f(n)|}{|g(n)|} = \infty.$$

Let $(X, d)$ be a compact metric space. For any $\varepsilon > 0$, let $N(\varepsilon)$ denote the minimal number of subsets of diameter at most $\varepsilon$ needed to cover $X$. The lower box dimension of $X$ is defined to be

$$\dim_B(X, \varepsilon) = \liminf_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log 1/\varepsilon}.\tag{1.1}$$

For a subset $E$ of $X$ and $\varepsilon > 0$, we say $E$ is $\varepsilon$-separated if for any distinct $x, y \in E$, $d(x, y) \geq \varepsilon$. Let $S(\varepsilon)$ denote the cardinality of a maximal $\varepsilon$-separated subset of $X$. It is easy to verify $N(\varepsilon) \leq S(\varepsilon) \leq N(\varepsilon/2)$. Thus
There is a function \( q: \mathbb{N} \to \mathbb{R} \) such that \( q(\lfloor \log n \rfloor + 1) \) is a super-polynomial and
\[
\limsup_{n \to \infty} \left( (2n + 1)^2, q(n) \right) < \infty.
\]

**Proof.** Let \( T: \mathbb{Z}^2 \times X \to X \) be an expansive continuous action on a compact metric space \((X, d)\) of infinite dimension (see [5] where an expansive \( \mathbb{Z}^2 \)-action on \( \mathbb{T}^\infty \) was constructed). By Lemma 1.2, there exist \( \alpha > 1 \) and a compatible metric \( D \) on \( X \) such that for any positive integer \( n \) and any two distinct points \( x, y \in X \) with \( D(x, y) \geq \alpha^{-n} \),
\[
\max_{v \in \mathbb{Z}^2 : |v| \leq n} D(T^v x, T^v y) \geq \frac{1}{4\alpha}.
\]
$C_n = \{ v \in \mathbb{Z}^2 : |v| \leq n \}$ to color the edges of $G_n$. Since $V_n$ is $\alpha^{-n}$-separated, for any two distinct points $x, y \in V_n$, $D(x, y) \geq \alpha^{-n}$. By Lemma 1.2, there exists $v \in C_n$ such that $D(T^v x, T^v y) \geq \frac{1}{4\alpha}$. Then we color the edge $\{x, y\}$ by $v$. By the definition of opposite-Ramsey number, there is a monochromatic complete subgraph $H_n$ of order $r(\alpha^{-n})$. 

By Lemma 1.3, $S(1/n)$ is a super-polynomial. Let $q(n) = S(\alpha^{-n})$. Thus $q([\log n] + 1)$ is a super-polynomial with respect to positive $n$. Assuming that the conclusion of the Theorem is false, we have

$$\lim sup_{n \to \infty} r(\alpha^{-n}) = \infty.$$ 

Therefore, there is an increasing subsequence $(n_i)$ of positive integers such that the sequence of orders of $H_{n_i}$ is unbounded. Since $H_{n_i}$ is monochromatic, there exists $v_{n_i} \in C_{n_i}$ such that the image of vertex set of $H_{n_i}$ under $T^{v_{n_i}}$ is $\frac{1}{4\alpha}$-separated. These imply that there are arbitrarily large $\frac{1}{4\alpha}$-separated subsets of $X$, which contradicts the compactness of $X$. Thus we complete the proof. 

3. Comparison with classical Ramsey number

For any positive integers $k$ and $g$, the Ramsey number $R_g(k)$ is defined to be the minimal number $n$ for which every edge-coloring of the complete graph $K_n$ with $g$ colors yields a monochromatic complete subgraph of order $k$.

By Corollary 3 of Greenwood and Gleason in [1], $R_g(k)$ has an upper bound $g^k$. In [2] Lefmann and Rödl obtained a lower bound $2^\Omega(gk)$ for $R_g(k)$. Thus

$$2^\Omega((2n+1)^2k) \leq R((2n+1)^2(k) \leq (2n+1)^2(k).$$

Suppose $r(\alpha^{-n}) = r < \infty$. Then it implies that

(1) every edge-coloring of complete graph $K_{q(n)}$ with $(2n+1)^2$ colors yields a monochromatic complete subgraph of order $r$, hence

$$q(n) \geq R((2n+1)^2(r));$$

(2) there exists an edge-coloring of $K_{q(n)}$ with $(2n+1)^2$ colors such that there is no monochromatic complete subgraph of order $r + 1$, hence

$$q(n) \leq R((2n+1)^2(r + 1)).$$

Thus $q(n)$ gives a lower bound of $R((2n+1)^2(r + 1))$. By Theorem 2.1, every expansive $\mathbb{Z}^2$-action on a compact metric space of infinite dimension gives rise to such a $q(n)$. In addition, there is a positive integer $r$ and an increasing subsequence $(n_i)$ of positive integers such that for any $i \in \mathbb{N}$, $r((2n_i+1)^2, q(n_i)) = r$. Therefore, we obtain a lower bound of $R((2n_i+1)^2(r + 1))$ and an upper bound of $R((2n_i+1)^2(r)$ for each $i \in \mathbb{N}$.

If $q(\log n)$ is a super-polynomial, then we claim that for any $A \geq 0$,

$$\lim inf_{n \to \infty} \frac{q(n)}{A^n} = \infty.$$ 

In fact, take a positive integer $m$ such that $e^m \geq A$. Then

$$\lim inf_{n \to \infty} \frac{q(n)}{A^n} \geq \lim inf_{n \to \infty} \frac{q(n)}{e^{m n}} = \lim inf_{n \to \infty} \frac{q(n)}{(e^m)^n} = \infty.$$
The lower bound of $R_{2^{n+1}}(r+1)$ obtained by (3.1) is $2^{\Omega\left(\frac{1}{2}n(r+1)\right)}$ which is also faster than any exponential growth. If there is an expansive $\mathbb{Z}$-action on a compact metric space of infinite dimension, then we can get a lower bound for $R_{2^{n+1}}(r+1)$ which is faster than the classical bound $2^{\Omega\left(\frac{1}{2}n(r+1)\right)}$. Unfortunately, in [3] Mañé showed that such action does not exist. If we can construct an expansive $\mathbb{Z}^2$-action on a compact metric space of infinite dimension such that the condition $D(x,y) \geq \alpha^{-n}$ in Lemma 1.2 can be replaced by $D(x,y) \geq \alpha^{-n^2}$, then we can show that $q(n)$ obtained in Theorem 2.1 satisfies that $q\left(\sqrt{\log n}+1\right)$ is a super-polynomial. Then it satisfies $\liminf_{n \to \infty} q(n) = \infty$. Hence $q(n)$ is faster than the classical lower bound $2^{\Omega\left(\frac{1}{2}n(r+1)\right)}$. Therefore, we leave the following question.

**Question 3.1.** Is there an expansive $\mathbb{Z}^2$-action on a compact metric space $(X,d)$ of infinite dimension and $\alpha > 1$ such that for any positive integer $n$ and any two distinct points $x,y \in X$ satisfying $d(x,y) \geq \alpha^{-n^2}$, we have

$$\max_{v \in \mathbb{Z}^2, |v| \leq n} d(T^v x, T^v y) \geq \frac{1}{4\alpha}.$$ 

A positive answer to Question 3.1 can give a better estimate of the lower bound of $R_{2^{n+1}}(r+1)$, where $(n_i)$ and $r$ come from the system. By (3.2), a negative answer also gives a better estimate of the upper bound of $R_{2^{n+1}}(r+1)$. Finally, we remark that the above comparison between $q(n)$ and the bound of Ramsey is only for a subsequence of positive integers. However, dealing with a concrete system we may obtain more information and can get a special edge-coloring of $K_{q(n)}$ as the proof of Theorem 2.1. Our method may give a new direction to estimate the bounds of Ramsey numbers and construct edge-colorings of big graphs.

**Acknowledgements.** We would like to thank Professor Bingbing Liang and Professor Xin Wang for their helpful comments.

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