Complexes of graphs with bounded independence number

Minki Kim* and Alan Lew†

Department of Mathematics, Technion, Haifa 32000, Israel

Abstract

Let $G = (V,E)$ be a graph and $n$ a positive integer. Let $I_n(G)$ be the abstract simplicial complex whose simplices are the subsets of $V$ that do not contain an independent set of size $n$ in $G$. We study the collapsibility numbers of the complexes $I_n(G)$ for various classes of graphs, focusing on the class of graphs with maximum degree bounded by $\Delta$. As an application, we obtain the following result:

Let $G$ be a claw-free graph with maximum degree at most $\Delta$. Then, every collection of $\lfloor \left(\frac{\Delta}{2} + 1\right)(n-1) \rfloor + 1$ independent sets in $G$ has a rainbow independent set of size $n$.

1 Introduction

Let $X$ be a simplicial complex and $d$ a non-negative integer. A face $\sigma$ that is contained in a unique maximal face $\tau$ of $X$ is called a free face of $X$. If $\sigma$ is a free face of $X$ with $|\sigma| \leq d$, we say that the complex

$$X' = X \setminus \{\eta \in X : \sigma \subset \eta \subset \tau\}$$

is obtained from $X$ by an elementary $d$-collapse, and we write $X \sigma \rightarrow X'$. A complex $X$ is called $d$-collapsible if there exists a sequence of elementary $d$-collapses

$$X = X_1 \xrightarrow{\sigma_1} X_2 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_{k-1}} X_k = \emptyset$$

reducing $X$ to the void complex $\emptyset$.

We define the collapsibility number $C(X)$ as the minimum integer $d$ such that $X$ is $d$-collapsible.

Let $G = (V,E)$ be a (simple, undirected) graph. A vertex subset $I \subset V$ is called an independent set in $G$ if no two vertices in $I$ are adjacent in $G$.

*kmiminski@campus.technion.ac.il M.K. was supported by ISF grant no. 2023464 and BSF grant no. 2006099.
†alan@campus.technion.ac.il A.L. was supported by ISF grant no. 326/16.
The independence number of $G$, denoted by $\alpha(G)$, is the maximal size of an independent set in $G$. For $U \subset V$, we denote by $G[U]$ the subgraph of $G$ induced by $U$. For every integer $n \geq 1$, we define the simplicial complex

$$I_n(G) = \{ U \subset V : \alpha(G[U]) < n \}.$$ 

For example, $I_2(G)$ is the clique complex of $G$, i.e. $U \in I_2(G)$ if and only if $G[U]$ is a complete graph. For any graph $G$, the complex $I_1(G)$ is just the empty complex $\{\emptyset\}$.

In this paper we study the collapsibility numbers of the complexes $I_n(G)$, for several classes of graphs. Our main motivation is the following problem, presented by Aharoni, Briggs, Kim and Kim in [4]:

Let $F = \{A_1, \ldots, A_m\}$ be a family of (not necessarily distinct) non-empty subsets of some finite set $V$. For a positive integer $n \leq m$, a rainbow set of size $n$ for $F$ is a set of $n$ distinct elements in $V$ of the form \{a_{i_1}, \ldots, a_{i_n}\},$ where $1 \leq i_1 < i_2 < \cdots < i_n \leq m$ and $a_{i_j} \in A_{i_j}$ for each $j \leq n$.

Let $G$ be a graph, and let $F$ be a finite family of independent sets in $G$. A rainbow independent set in $G$ with respect to $F$ is a rainbow set for $F$ that forms an independent set in $G$. For a positive integer $n$, let $f_G(n)$ be the minimum integer $t$ such that every collection of $t$ independent sets of size $n$ in $G$ has a rainbow independent set of size $n$. For a graph class $\mathcal{G}$ and a positive integer $n$, let

$$f_{\mathcal{G}}(n) = \sup_{G \in \mathcal{G}} f_G(n).$$

The connection between the complexes $I_n(G)$ and the parameters $f_{\mathcal{G}}(n)$ is given by the following version of Kalai and Meshulam’s “topological colorful Helly theorem”:

**Theorem 1.1** (Kalai and Meshulam [11]). Let $X$ be a $d$-collapsible simplicial complex on vertex set $V$, and let $X^c = \{ \sigma \subset V : \sigma \notin X \}$. Then, every collection of $d + 1$ sets in $X^c$ has a rainbow set belonging to $X^c$.

Theorem [11] is a special case of Theorem 2.1 in [11]. A detailed derivation of Theorem [11] from the general case can be found in [5]. An immediate application of Theorem [11] gives us:

**Proposition 1.2.** Let $G$ be a graph and $n \geq 1$ an integer. Then,

$$f_G(n) \leq C(I_n(G)) + 1.$$

The study of rainbow independent sets originated as a generalization of the “rainbow matching problem” in graphs (note that a matching in a graph is an independent set in its line graph); see e.g. [2, 3, 6]. The application
of collapsibility numbers in the study of rainbow matchings was initiated in [5], and further developed in [9].

In [4], Aharoni et al. prove some results about \( f_G(n) \) for different classes of graphs. One of the main conjectures in [4] is the following.

**Conjecture 1.3** (Aharoni, Briggs, Kim, Kim [4]). Let \( D(\Delta) \) be the class of graphs with maximum degree at most \( \Delta \), and let \( n \) be a positive integer. Then,

\[
f_{D(\Delta)}(n) = \left\lceil \frac{\Delta + 1}{2} \right\rceil (n - 1) + 1.
\]

It is shown in [4] that Conjecture 1.3 is true for \( \Delta \leq 2 \) and for \( n \leq 3 \). In the general case, the best bounds observed by Aharoni et al. are given by

\[
\left\lceil \frac{\Delta + 1}{2} \right\rceil (n - 1) + 1 \leq f_{D(\Delta)}(n) \leq \Delta(n - 1) + 1.
\]

It is natural to ask whether the following extension of Conjecture 1.3 holds:

**Conjecture 1.4.** Let \( G \) be a graph with maximum degree at most \( \Delta \), and let \( n \) be a positive integer. Then,

\[
C(I_n(G)) \leq \left\lceil \frac{\Delta + 1}{2} \right\rceil (n - 1).
\]

Our main results are the following:

**Theorem 1.5.** Let \( G = (V, E) \) be a chordal graph and \( n \geq 1 \) an integer. Then,

\[
C(I_n(G)) \leq n - 1.
\]

**Theorem 1.6.** Let \( G = (V, E) \) be a graph with maximum degree at most \( \Delta \) and \( n \geq 1 \) an integer. Then,

\[
C(I_n(G)) \leq \Delta(n - 1).
\]

The bound in Theorem 1.6 is tight only for \( \Delta \leq 2 \). In the case \( n \leq 3 \) we can prove the following tight bounds, for general \( \Delta \):

**Theorem 1.7.** Let \( G = (V, E) \) be a graph with maximum degree at most \( \Delta \). Then,

\[
C(I_2(G)) \leq \left\lceil \frac{\Delta + 1}{2} \right\rceil.
\]

**Theorem 1.8.** Let \( G = (V, E) \) be a graph with maximum degree at most \( \Delta \). Then,

\[
C(I_3(G)) \leq \begin{cases} 
\Delta + 2 & \text{if } \Delta \text{ is even,} \\
\Delta + 1 & \text{if } \Delta \text{ is odd.}
\end{cases}
\]
Theorems 1.6, 1.7, and 1.8 confirm Conjecture 1.4 in the special cases where $\Delta \leq 2$ or $n \leq 3$. Unfortunately, Conjecture 1.4 does not hold in general: In Section 7 we present a family of counterexamples to the case $\Delta = 3$.

Combining these results with Proposition 1.2, we obtain corresponding upper bounds for $f_G(n)$, thus recovering several results first proved in [4].

The following bound, however, is new:

**Theorem 1.9.** Let $G$ be a claw-free graph with maximum degree at most $\Delta$, and let $n \geq 1$ be an integer. Then,

$$f_G(n) \leq \left\lfloor \left( \frac{\Delta}{2} + 1 \right) (n - 1) \right\rfloor + 1.$$ 

Theorem 1.9 shows that Conjecture 1.3 holds for the subclass of claw-free graphs with maximum degree at most $\Delta$, in the case where $\Delta$ is even.

The proof of Theorem 1.9 relies on bounding the collapsibility numbers of certain subcomplexes of the complex $I_n(G)$.

The paper is organized as follows. In Section 2 we introduce some basic definitions about graphs and simplicial complexes that we will use throughout the paper. In Section 3 we present several tools for bounding the collapsibility numbers of a general simplicial complex. Section 4 contains the proof of Theorem 1.5 dealing with the case of chordal graphs. Section 5 focuses on the class of graphs with bounded maximum degree. It contains the proofs of Theorems 1.6, 1.7, and 1.8. In Section 6 we prove our main application, Theorem 1.9. Section 7 deals with the Leray number, a homological variant of the collapsibility number, of the complex $I_n(G)$. In particular, it presents extremal examples determining the tightness of our main results (Theorems 1.7, 1.8, and 1.9), and examples of 3-regular graphs for which the complexes $I_n(G)$ do not satisfy the bound in Conjecture 1.4 (for various values of $n$). In Section 8 we discuss some open problems arising from our work and possible directions for further research.

## 2 Preliminaries

### 2.1 Simplicial complexes

A (finite) abstract simplicial complex is a family $X$ of subsets of some finite set that is closed downward. That is, if $\tau \in X$ and $\sigma \subseteq \tau$, then $\sigma \in X$.

The vertex set of $X$ is the set $V = \bigcup_{\sigma \in X} \sigma$. A set $\sigma \in X$ is called a simplex or a face of $X$. The dimension of a simplex $\sigma \in X$ is $\dim(\sigma) = |\sigma| - 1$. The dimension of the complex $X$ is the maximal dimension of a simplex in $X$.

A missing face of a complex $X$ is a set $\tau \subset V$ such that $\tau \notin X$ but $\sigma \in X$ for any $\sigma \subsetneq \tau$. If all the missing faces of $X$ are of size 2, then $X$ is called a flag complex.
Let \( U \subset V \). The subcomplex of \( X \) induced by \( U \) is the complex 
\[
X[U] = \{ \sigma \in X : \sigma \subset U \}.
\]
For any vertex \( v \in V \), we define the deletion of \( v \) in \( X \) to be the subcomplex 
\[
X \setminus v = \{ \sigma \in X : v \notin \sigma \} = X[V \setminus \{v\}].
\]
Let \( \tau \subset V \). We define the link of \( \tau \) in \( X \) to be the subcomplex 
\[
\text{lk}(X, \tau) = \{ \sigma \in X : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in X \}.
\]
Note that \( \text{lk}(X, \tau) = \emptyset \) unless \( \tau \in X \). If \( \tau = \{v\} \), we write \( \text{lk}(X, v) = \text{lk}(X, \{v\}) \).

Let \( X \) and \( Y \) be two simplicial complexes on disjoint vertex sets. We define the join of \( X \) and \( Y \) to be the simplicial complex 
\[
X * Y = \{ \sigma \cup \tau : \sigma \in X, \tau \in Y \}.
\]
Let \( v \in V \). If \( v \in \tau \) for every maximal face \( \tau \in X \) we say that \( X \) is a cone over \( v \).

For \( U \subset V \), we denote by \( 2^U = \{ \sigma : \sigma \subset U \} \) the complete complex on vertex set \( U \).

### 2.2 Homology and Leray numbers

For \( i \geq -1 \), let \( \tilde{H}_i(X) \) be the \( i \)-th reduced homology group of \( X \) with real coefficients. We say that \( X \) is \( d \)-Leray if every induced subcomplex \( Y \) of \( X \) has trivial homology in dimensions \( d \) and above, namely \( \tilde{H}_i(Y) = 0 \) for \( i \geq d \). The Leray number of \( X \), denoted by \( L(X) \), is the minimum integer \( d \) such that \( X \) is \( d \)-Leray.

The notions of \( d \)-collapsibility and \( d \)-Lerayness of simplicial complexes were introduced by Wegner in [16]. He observed the following simple fact:

**Lemma 2.1** (Wegner [16]). Let \( X \) be a simplicial complex. Then,
\[
C(X) \geq L(X).
\]

In Section 7 we will use some well known facts about the homology of simplicial complexes (see e.g. [7]):

**Theorem 2.2.** Let \( X = X_1 \ast X_2 \ast \cdots \ast X_m \). Then,
\[
\tilde{H}_i(X) \cong \bigoplus_{i_1 + \cdots + i_m = i - m + 1, -1 \leq i_j \leq \dim(X_j) \forall j \in [m]} \tilde{H}_{i_1}(X_1) \otimes \cdots \otimes \tilde{H}_{i_m}(X_m).
\]
Let $\mathcal{F} = \{A_1, \ldots, A_m\}$ be a family of sets. The nerve of $\mathcal{F}$ is the simplicial complex

$$N(\mathcal{F}) = \{I \subset [m] : \cap_{i \in I} A_i \neq \emptyset\}.$$ 

The following is a simple version of the Nerve Theorem:

**Theorem 2.3.** Let $X$ be a simplicial complex with maximal faces $\sigma_1, \ldots, \sigma_m$. Then,

$$\tilde{H}_i(X) \cong \tilde{H}_i(N(\{\sigma_1, \ldots, \sigma_m\}))$$

for all $i \geq -1$.

Let $X$ be a simplicial complex on vertex set $V$. The combinatorial Alexander dual of $X$ is the complex

$$D(X) = \{\sigma \subset V : V \setminus \sigma \notin X\}.$$ 

It is easy to check that the maximal faces of $D(X)$ are the complements of the missing faces of $X$. Similarly, the missing faces of $D(X)$ are the complements of the maximal faces of $X$. Alexander duality relates the homology groups of $X$ with those of $D(X)$ (see e.g. [8]):

**Theorem 2.4** (Alexander duality). If $V \notin X$ then for all $-1 \leq i \leq |V| - 2$

$$\tilde{H}_i(D(X)) \cong \tilde{H}_{|V|-i-3}(X).$$

### 2.3 Graphs

Throughout this paper, we assume every graph is simple and undirected. Let $G = (V, E)$ be a graph.

For a vertex subset $U \subset V$, the subgraph of $G$ induced by $U$ is the graph $G[U] = (U, \{e \in E : e \subset U\})$. The subset $U \subset V$ is called a clique in $G$ if the induced subgraph $G[U]$ is the complete graph on vertex set $U$.

For any vertex $v \in V$, we define the deletion of $v$ in $G$ to be the induced subgraph $G \setminus v = G[V \setminus \{v\}]$.

For each $v \in V$, we define the open neighborhood of $v$ in $G$ as the vertex subset

$$N_G(v) = \{u \in V : u \text{ is adjacent to } v\},$$

and we define the closed neighborhood of $v$ in $G$ as

$$N_G[v] = \{v\} \cup N_G(v).$$

For a set $A \subset V$, let

$$N_G(A) = \bigcup_{u \in A} N_G(u).$$

A vertex $v \in V$ is called a simplicial vertex if $N_G[v]$ is a clique. The degree of $v$ in $G$ is the number $\deg_G(v) = |N_G(v)|$. 

6
We say $G$ is $k$-colorable (or $k$-partite) if we can partition the vertex set $V$ into $k$ parts so that each part is independent in $G$. The following is a classical result in graph theory that states a relation between the maximum degree and the $k$-colorability of $G$.

**Theorem 2.5** (Brooks’ Theorem [10]). Let $G$ be a connected graph with maximum degree $k$. Then $G$ is $k$-colorable unless $G$ is the complete graph $K_{k+1}$ or an odd cycle.

The complete bipartite graph $K_{1,3}$ is called a claw. A graph is said to be claw-free if it does not have a claw as an induced subgraph.

We say a graph is chordal if it does not contain a cycle of length at least 4 as an induced subgraph. Chordal graphs satisfy the following special property:

**Theorem 2.6** (Lekkerkerker, Boland [13]). Every chordal graph contains a simplicial vertex.

## 3 Upper bounds for collapsibility numbers

In this section we present our main technical tools for proving $d$-collapsibility of a simplicial complex. Most of the bounds presented in this section rely on the inductive application of the following two basic results, due to Tancer:

**Lemma 3.1** (Tancer [14, Prop.1.2]). Let $X$ be a simplicial complex on vertex set $V$, and let $v \in V$. Then,

$$C(X) \leq \max\{C(X \setminus v), C(\text{lk}(X, v)) + 1\}.$$  

**Lemma 3.2** (Tancer [14, Prop. 3.1]). Let $X$ be a simplicial complex on vertex set $V$, and let $v \in V$ such that $X$ is a cone over $v$. Then,

$$C(X) = C(X \setminus v).$$

It will be helpful to state the following straightforward generalization of Lemma 3.2.

**Lemma 3.3.** Let $U$ and $V$ be two disjoint sets, and let $X$ be a simplicial complex on vertex set $V$. Then,

$$C(X \ast 2^U) = C(X).$$

**Proof.** We argue by induction on $|U|$. If $|U| = 0$ then $X \ast 2^U = X$, so the claim holds. Now, assume $|U| > 1$. Let $u \in U$ and $U' = U \setminus \{u\}$. Note that $X \ast 2^U$ is a cone over $u$; therefore, by Lemma 3.2

$$C(X \ast 2^U) = C((X \ast 2^U) \setminus u) = C(X \ast 2^{U'}).$$

By the induction hypothesis, $C(X \ast 2^{U'}) = C(X)$. Hence, $C(X \ast 2^U) = C(X)$, as wanted. 

\[\Box\]
Lemma 3.4. Let $X$ be a simplicial complex, and let $\sigma = \{ v_1, \ldots, v_k \} \in X$. For every $0 \leq i \leq k$, define $\sigma_i = \{ v_j : 1 \leq j \leq i \}$. Let $d \geq k$. If for all $0 \leq i \leq k-1$,  
\[ C(\text{lk}(X \ \setminus \ v_{i+1}, \sigma_i)) \leq d - i, \]
and  
\[ C(\text{lk}(X, \sigma_k)) \leq d - k, \]
then $C(X) \leq d$.

Proof. We will show that, for any $i \in \{0, \ldots, k\}$,  
\[ C(\text{lk}(X, \sigma_i)) \leq d - i. \]

We argue by backwards induction on $i$. For $i = k$, $C(\text{lk}(X, \sigma_k)) \leq d - k$ by assumption. Let $i < k$. By Lemma 3.1, we have  
\[ C(\text{lk}(X, \sigma_i)) \leq \max\{ C(\text{lk}(X \ \setminus \ v_{i+1}, \sigma_i)), C(\text{lk}(X, \sigma_{i+1})) + 1 \}. \]
But $C(\text{lk}(X \ \setminus \ v_{i+1}, \sigma_i)) \leq d - i$ by assumption, and $C(\text{lk}(X, \sigma_{i+1})) \leq d - i - 1$ by the induction hypothesis. Therefore,  
\[ C(\text{lk}(X, \sigma_i)) \leq d - i. \]
Setting $i = 0$, we obtain (since $\sigma_0 = \emptyset$),  
\[ C(X) = C(\text{lk}(X, \sigma_0)) \leq d - 0 = d. \]

As a consequence of Lemma 3.4, we obtain:

Proposition 3.5. Let $X$ be a simplicial complex on vertex set $V$. If all the missing faces of $X$ are of dimension at most $d$, then  
\[ C(X) \leq \left\lfloor \frac{|V|}{d+1} \right\rfloor. \]
Moreover, equality $C(X) = \frac{|V|}{d+1}$ is obtained if and only if $X$ is the join of $r = \frac{|V|}{d+1}$ disjoint copies of the boundary of a $d$-dimensional simplex (or equivalently, if the set of missing faces of $X$ consists of $r$ disjoint sets of size $d+1$).

Proof. We argue by induction on $|V|$. If $|V| = 0$, then $X$ is 0-collapsible, and the inequality holds. Assume $|V| > 0$. If $X$ is a complete complex, then it is 0-collapsible, and the inequality holds. Otherwise, let $\sigma = \{ v_1, \ldots, v_{k+1} \} \subset V$ be a missing face of $X$. Since all the missing faces of $X$ are of dimension
at most \(d\), we have \(k \leq d\). For each \(0 \leq i \leq k\), let \(\sigma_i = \{v_j : 1 \leq j \leq i\} \in X\). Let \(V_i\) be the vertex set of \(\text{lk}(X \setminus v_{i+1}, \sigma_i)\). Note that for every \(0 \leq i \leq k\),

\[
V_i \subset V \setminus \sigma_{i+1}.
\]

Therefore, by the induction hypothesis,

\[
C(\text{lk}(X \setminus v_{i+1}, \sigma_i)) \leq \frac{d}{d+1}|V_i| \leq \frac{d}{d+1}(|V| - i - 1).
\]

Since \(i \leq k \leq d\), we obtain

\[
C(\text{lk}(X \setminus v_{i+1}, \sigma_i)) \leq \frac{d}{d+1}|V| - \frac{i}{i+1}(i+1) = \frac{d}{d+1}|V| - i.
\]

Also, since \(\sigma\) is a missing face, we have

\[
\text{lk}(X, \sigma_k) = \text{lk}(X \setminus v_{k+1}, \sigma_k),
\]

and in particular \(C(\text{lk}(X, \sigma_k)) \leq \frac{d}{d+1}|V| - k\). Therefore, by Lemma 3.4, we obtain

\[
C(X) \leq \frac{d}{d+1}|V|.
\]

Since \(C(X)\) is an integer, we obtain \(C(X) \leq \left\lfloor \frac{d|V|}{d+1} \right\rfloor\).

Now, assume \(C(X) = \frac{d}{d+1}|V|\). Note that, since \(C(X)\) is an integer and \(\gcd(d, d+1) = 1\), then \(d+1\) must divide \(|V|\).

Then, there exists some \(0 \leq i \leq k\) such that

\[
C(\text{lk}(X \setminus v_{i+1}, \sigma_i)) = \frac{d}{d+1}(|V| - i - 1)
\]

(otherwise, by the same argument as above, we could prove that \(C(X) < \frac{d}{d+1}|V|\)). Since \(d + 1\) divides \(|V|\), it must also divide \(i + 1\). Hence, we must have \(i = k = d\). By the induction hypothesis, the missing faces of

\[
\text{lk}(X, \sigma_d) = \text{lk}(X \setminus v_{d+1}, \sigma_d)
\]

form a set of \(r - 1\) disjoint sets of size \(d+1\). Therefore, the set of missing faces of \(X\) consists of \(r\) disjoint sets of size \(d+1\) plus, possibly, some additional faces of the form \(\tau \cup \{v'_{d+1}\}\), where \(\tau \in V \setminus \sigma\). But the choice of the order \(v_1, \ldots, v_{d+1}\) on the vertices of \(\sigma\) was arbitrary. Thus, repeating the same argument with a different order (e.g. \(v'_i = v_i\) for \(i \leq d - 1\), \(v'_d = v_{d+1}\), \(v'_{d+1} = v_d\)), we obtain that the set of missing faces of \(X\) consists exactly of \(r\) disjoint sets of size \(d + 1\).

\[\square\]

**Remark.** An analogous bound in terms of Leray numbers was proved in [1, Prop. 5.4].
Lemma 3.6. Let $X$ be a complex on vertex set $V$, and let $B \subset V$. Let $<$ be a linear order on the vertices of $B$. Let $\mathcal{P} = \mathcal{P}(X, B)$ be the family of partitions $(B_1, B_2)$ of $B$ satisfying:

- $B_2 \in X$.
- For any $v \in B_2$, the complex
  \[
  \text{lk}(X[V \setminus \{u \in B_1 : u < v\}], \{u \in B_2 : u < v\})
  \]
  is not a cone over $v$.

If for every $(B_1, B_2) \in \mathcal{P}$,
\[
C(\text{lk}(X[V \setminus B_1], B_2)) \leq d - |B_2|,
\]
then $C(X) \leq d$.

Proof. We argue by induction on $|B|$. If $|B| = 0$ there is nothing to prove. So, assume $|B| > 0$, and let $v$ be the minimal vertex in $B$ (with respect to the order <). Let $X' = X \setminus v$, and let $V' = V \setminus \{v\}$ be its vertex set. Let $B' = B \setminus \{v\}$, and let $(B'_1, B'_2) \in \mathcal{P}(X', B')$. Define $B_1 = B'_1 \cup \{v\}$ and $B_2 = B'_2$. Then, $B_2 \in X \setminus v \subset X$, and for any $u \in B_2$, the complex
\[
\text{lk}(X[V \setminus \{w \in B_1 : w < u\}], \{w \in B_2 : w < u\})
= \text{lk}(X'[V' \setminus \{w \in B'_1 : w < u\}], \{w \in B'_2 : w < u\})
\]
is not a cone over $u$ (since $(B'_1, B'_2) \in \mathcal{P}(X', B')$). Therefore $(B_1, B_2) \in \mathcal{P}(X, B)$. So,
\[
C(\text{lk}(X'[V' \setminus B'_1], B'_2)) = C(\text{lk}(X[V \setminus B_1], B_2)) \leq d - |B_2| = d - |B'_2|.
\]
Hence, by the induction hypothesis, $C(X \setminus v) = C(X') \leq d$.

If $X$ is a cone over $v$ then, by Lemma 3.2, $C(X) = C(X \setminus v) \leq d$, as wanted. Otherwise, let $X'' = \text{lk}(X, v)$, and let $V'' \subset V \setminus \{v\}$ be its vertex set. Let $B'' = B \cap V''$, and let $(B''_1, B''_2) \in \mathcal{P}(X'', B'')$. Let $B_2 = B''_2 \cup \{v\}$ and $B_1 = B \setminus B_2$.

Since $B''_2 \in X'' = \text{lk}(X, v)$, we have $B_2 = B''_2 \cup \{v\} \in X$. Let $u \in B_2$. If $u = v$, then
\[
\text{lk}(X[V \setminus \{w \in B_1 : w < u\}], \{w \in B_2 : w < u\}) = X
\]
is not a cone over $u = v$. If $u > v$, then
\[
\text{lk}(X[V \setminus \{w \in B_1 : w < u\}], \{w \in B_2 : w < u\})
= \text{lk}(X''[V'' \setminus \{w \in B''_1 : w < u\}], \{w \in B''_2 : w < u\})
\]
is not a cone over $u$ (since $(B'_{1}, B'_{2}) \in \mathcal{P}(X', B'_{2}))$. Therefore, $(B_{1}, B_{2}) \in \mathcal{P}(X, B)$. So,

$$C(\text{lk}(X'[V \setminus B'_{1}], B'_{2})) = C(\text{lk}(X[V \setminus B_{1}], B_{2})) \leq d - |B_{2}| = (d - 1) - |B'_{2}|.$$  

Thus, by the induction hypothesis, $C(\text{lk}(X, v)) = C(X') \leq d - 1$. Hence, by Lemma 3.1, $C(X) \leq d$.

The following bound is proved in [12]. For completeness, we include here the proof.

**Proposition 3.7** (Khmelnitsky [12]). Let $X$ be a simplicial complex on vertex set $V$, and let $\sigma \in X$. Then,

$$C(\text{lk}(X, \sigma)) \leq C(X).$$

**Proof.** Let $d \geq 0$, and assume that $X$ can be reduced to the void complex by a sequence of $k$ elementary $d$-collapses. We will show that $\text{lk}(X, \sigma)$ is $d$-collapsible. We argue by induction on $k$. If $k = 1$, we must have $d = 0$ and $X = 2^{V}$; so, $C(\text{lk}(X, \sigma)) = C(2^{V \setminus \sigma}) = 0 = d$, and the claim holds. Assume $k > 1$. Then, there exists a free face $\eta \in X$ with $|\eta| \leq d$, such that the complex

$$X' = X \setminus \{\xi \in X : \eta \subset \xi\}$$

can be reduced to the void complex by a sequence of $k - 1$ elementary $d$-collapses.

Let $\tau$ be the unique maximal face of $X$ containing $\eta$.

Assume that $\eta \subset \sigma$. Let $\xi \in \text{lk}(X, \sigma)$. Then $\eta \subset \sigma \cup \xi \in X$. Therefore, since $\eta$ is contained in the unique maximal face $\tau \in X$, we have $\sigma \cup \xi \subset \tau$. So, $\xi \subset \tau \setminus \sigma$. Since $\tau \setminus \sigma \in \text{lk}(X, \sigma)$, we obtain $\text{lk}(X, \sigma) = 2^{\tau \setminus \sigma}$. In particular, $C(\text{lk}(X, \sigma)) = 0 \leq d$.

Otherwise, assume $\eta \not\subset \sigma$. We divide into two cases:

1. If $\eta \notin \text{lk}(X, \sigma)$, then

$$\text{lk}(X', \sigma) = \{\xi \in \text{lk}(X, \sigma) : \eta \not\subset \xi\} = \text{lk}(X, \sigma).$$

Thus, by the induction hypothesis, $\text{lk}(X, \sigma)$ is $d$-collapsible.

2. If $\eta \in \text{lk}(X, \sigma)$, let $\tau' = \tau \setminus \sigma$. Let $\eta \subset \xi \in \text{lk}(X, \sigma)$. Then $\eta \subset \sigma \cup \xi \in X$; therefore, since $\eta$ is contained in the unique maximal face $\tau$ of $X$, we obtain $\sigma \cup \xi \subset \tau$. That is, $\xi \subset \tau'$. Hence, since $\tau' \in \text{lk}(X, \sigma)$, $\eta$ is a free face in $\text{lk}(X, \sigma)$. So, we can perform the elementary $d$-collapse

$$\text{lk}(X, \sigma) \xrightarrow{\eta} \text{lk}(X, \sigma) \setminus \{\xi \in \text{lk}(X, \sigma) : \eta \subset \xi\} = \text{lk}(X', \sigma).$$

By the induction hypothesis, $\text{lk}(X', \sigma)$ is $d$-collapsible. Thus, $\text{lk}(X, \sigma)$ is $d$-collapsible.
Lastly, we will need the following simple bound:

**Lemma 3.8.** Let $X$ be a simplicial complex. Then,

$$C(X) \leq \dim(X) + 1.$$  

**Proof.** We argue by induction on the number of faces of $X$. If $X$ contains a unique face (that is, $X = \{\emptyset\}$), then $C(X) = 0 = \dim(X) + 1$.

Now, assume that $X$ contains more than one face. Let $d = \dim(X) + 1$. Let $\tau$ be a maximal face of $X$. Then, $\tau$ is a free face in $X$ of size $|\tau| \leq d$; so, we can perform the elementary $d$-collapse

$$X \xrightarrow{\tau} X' = X \setminus \{\tau\}.$$  

By the induction hypothesis, $X'$ is $d$-collapsible. Hence, $X$ is $d$-collapsible. That is,

$$C(X) \leq d = \dim(X) + 1.$$  

\[\Box\]

### 4 Chordal graphs

In this section we prove Theorem [1.5] which bounds the collapsibility of $I_n(G)$ in the case that $G$ is a chordal graph. The proof relies on the next result.

**Lemma 4.1.** Let $G = (V, E)$ be a graph, and let $v \in V$ be a simplicial vertex in $G$. Then, for any $n \geq 2$,

$$C(I_n(G)) \leq \max\{C(I_n(G \setminus \{v\}), C(I_{n-1}(G[V \setminus N_G[v]])) + 1\}.$$  

**Proof.** Let $W \subset V \setminus \{v\}$. Then, $W$ belongs to $\text{lk}(I_n(G), v)$ if and only if $W \setminus N_G(v) \in I_{n-1}(G)$. Indeed, assume that $W \setminus N_G(v) \notin I_{n-1}(G)$; that is, $G[W \setminus N_G(v)]$ contains an independent set $A$ of size $n - 1$. Then, $A \cup \{v\}$ is an independent set of size $n$ in $G$, and therefore $W \notin \text{lk}(I_n(G), v)$. For the opposite direction, suppose $W \notin \text{lk}(I_n(G), v)$. Then, $W \cup \{v\}$ contains an independent set $A$ of size $n$ in $G$. Since $N_G[v]$ is a clique in $G$, $A$ contains at most one vertex from $N_G[v]$. Thus, $A \setminus N_G[v] \subset W \setminus N_G(v)$ is an independent set of size at least $n - 1$. So, $W \setminus N_G(v) \notin I_{n-1}(G)$.

It follows that $\text{lk}(I_n(G), v) = 2N_G(v) * I_{n-1}(G[V \setminus N_G[v]])$. By Lemma [3.3] we have

$$C(\text{lk}(I_n(G), v)) = C(I_{n-1}(G[V \setminus N_G[v]])).$$

12
So, by Lemma 3.1 we obtain

\[ C(I_n(G)) \leq \max\{C(I_n(G \setminus v)), C(lk(I_n(G), v)) + 1\} \]

\[ = \max\{C(I_n(G \setminus v)), C(I_n-1(G[V \setminus N_G[v]])) + 1\}. \]

\[ \square \]

**Proof of Theorem 1.5.** We argue by induction on \(|V|\). For \(|V| = 0\) the statement is obvious. Suppose \(|V| > 0\). For \(n = 1\), \(C(I_1(G)) = C(\emptyset) = 0\), so the claim holds. Let \(n \geq 2\). Since \(G\) is a chordal graph, there exists a simplicial vertex \(v\) in \(G\). By the induction hypothesis,

\[ C(I_n(G - v)) \leq n - 1 \]

and

\[ C(I_n-1(G[V \setminus N_G[v]])) \leq n - 2. \]

Hence, by Lemma 4.1

\[ C(I_n(G)) \leq \max\{C(I_n(G \setminus v)), C(I_n-1(G[V \setminus N_G[v]])) + 1\} \leq n - 1. \]

\[ \square \]

**Remark.** Let \(G\) be a graph with \(\alpha(G) \geq n\), and let \(A\) be an independent set of size \(n\) in \(G\). Then \(I_n(G)[A]\) is the boundary of an \((n - 1)\)-dimensional simplex, and in particular \(H_{n-2}(I_n(G)[A]) \neq 0\). Hence, \(C(I_n(G)) \geq L(I_n(G)) \geq n - 1\). So, the bound in Theorem 1.5 is tight: any chordal graph \(G\) with \(\alpha(G) \geq n\) has \(C(I_n(G)) = n - 1\).

5 Graphs with bounded maximum degree

In this section we prove our main results about graphs with bounded maximum degree, Theorems 1.6, 1.7 and 1.8. We also prove an auxiliary result about claw-free graphs (Proposition 5.5), which will be later used for the proof of Theorem 1.9.

We begin with the following related problem: Let \(\mathcal{X}(k)\) be the class of all \(k\)-colorable graphs. In [1] it was observed that \(f_{\mathcal{X}(k)}(n) = k(n - 1) + 1\). The following proposition (combined with Proposition 1.2) offers an alternative proof for this result.

**Proposition 5.1.** Let \(G\) be a \(k\)-colorable graph and \(n \geq 1\) an integer. Then,

\[ C(I_n(G)) \leq k(n - 1). \]
Proof. Take a proper vertex-coloring of $G$ with $k$ colors. Note that each color class forms an independent set in $G$. Let $\sigma \in I_n(G)$. Since $\sigma$ contains no independent set of size $n$ in $G$, it contains at most $n - 1$ vertices from each color class. It follows that $|\sigma| \leq k(n - 1)$. Hence, by Lemma 3.8

$$C(I_n(G)) \leq \dim(I_n(G)) + 1 \leq k(n - 1).$$

\[ \square \]

Next, we present the proof of Theorem 1.6. We deal with the case $\Delta = 2$ separately:

**Theorem 5.2.** Let $G = (V, E)$ be a graph with maximum degree at most 2 and $n \geq 1$ an integer. Then $I_n(G)$ is $2(n - 1)$-collapsible.

Recall that a graph with maximum degree bounded by 2 is a disjoint union of cycles and paths. In order to apply an inductive argument, we state the following more general claim:

**Proposition 5.3.** Let $G = (V, E)$ be a graph with maximum degree at most 2. Let $A$ be an independent set in $G$ of size at most $n - 1$ that is contained in the union of all the components of $G$ that are paths. Then,

$$C(\text{lk}(I_n(G), A)) \leq 2(n - 1) - |A|. $$

**Proof.** We argue by induction on the number of cycles $c$ in $G$.

If $c = 0$, then $G$ is a disjoint union of paths. In particular, it is a chordal graph, and by Theorem 1.5 $C(I_n(G)) \leq n - 1$. By Proposition 3.7, we obtain

$$C(\text{lk}(I_n(G), A)) \leq C(I_n(G)) \leq n - 1 \leq 2(n - 1) - |A|. $$

Let $c \geq 1$, and assume that the claim holds for all graphs with less than $c$ cycles. Let $C = \{v_1, \ldots, v_k\}$ be the vertex set of a cycle in $G$ (such that $\{v_i, v_{i+1}\} \in E$ for all $i \in [k]$, where the indices are taken modulo $k$). Let

$$r = \min \left\{ \left\lfloor \frac{k}{2} \right\rfloor, n - |A| - 1 \right\},$$

and let

$$U = \{v_{2i-1} : 1 \leq i \leq r\}.$$

So, $U$ is an independent set in $G$ of size $r$.

For each $0 \leq i \leq r$, let $U_i = \{v_{2j-1} : 1 \leq j \leq i\}$. Let $0 \leq i \leq r - 1$. The graph $G \setminus v_{2i+1}$ has $c - 1$ cycles, and the set $A \cup U_i$ is an independent set contained in components of $G \setminus v_{2i+1}$ that are paths. Therefore, by the induction hypothesis,

$$C(\text{lk}(I_n(G \setminus v_{2i+1}), A \cup U_i)) \leq 2(n - 1) - |A| - i.$$
Next, we divide into two cases. First, assume \( r = n - |A| - 1 < \left\lceil \frac{k}{2} \right\rceil \). Then \( 2r + 1 \leq k \), and, by the same argument as before, we obtain

\[
C(\text{lk}(I_n(G \setminus v_{2r+1}), A \cup U_r)) \leq 2(n - 1) - |A| - r.
\]

Since \( r = n - |A| - 1 \), the set \( A \cup U_r \cup \{v_{2r+1}\} \) is an independent set of size \( n \) in \( G \); therefore, \( v_{2r+1} \notin \text{lk}(I_n(G), A \cup U_r) \). Hence,

\[
\text{lk}(I_n(G), A \cup U_r) = \text{lk}(I_n(G \setminus v_{2r+1}), A \cup U_r).
\]

So,

\[
C(\text{lk}(I_n(G), A \cup U_r)) \leq 2(n - 1) - |A| - r.
\]

Now, assume \( r = \left\lceil \frac{k}{2} \right\rceil \). Then, \( U_r \) is a maximum independent set in \( G[C] \), and we have

\[
\text{lk}(I_n(G), A \cup U_r) = 2^{C(U_r)} \ast \text{lk}(I_{n-1}(G[V \setminus C]), A).
\]

Therefore, by Lemma 3.8 we obtain

\[
C(\text{lk}(I_n(G), A \cup U_r)) = C(\text{lk}(I_{n-1}(G[V \setminus C]), A)) \leq 2(n - r - 1) - |A|
\]

\[
= 2(n - 1) - |A| - 2r \leq 2(n - 1) - |A| - r,
\]

where the first inequality follows by the induction hypothesis (since the number of cycles in \( G[V \setminus C] \) is \( c - 1 \)).

In both cases we obtained

\[
C(\text{lk}(I_n(G), A \cup U_r)) \leq 2(n - 1) - |A| - r.
\]

So, by Lemma 3.4 we obtain

\[
C(\text{lk}(I_n(G), A)) \leq 2(n - 1) - |A|,
\]

as wanted.

\[\square\]

Theorem 5.2 follows from Proposition 5.3 by setting \( A = \emptyset \).

Now we can prove the general case of Theorem 1.6.

**Proof of Theorem 1.6.** We argue by induction on \( n \). For \( n = 1 \) the claim is trivial. Assume \( n \geq 2 \).

If \( \Delta = 1 \) then the edges of \( G \) are pairwise disjoint. In particular, \( G \) is a chordal graph; therefore, the claim follows from Theorem 1.5. If \( \Delta = 2 \), the claim follows from Theorem 5.2. Assume \( \Delta \geq 3 \), and let \( G \) be a graph with maximum degree at most \( \Delta \). We will show that \( C(I_n(G)) \leq \Delta(n - 1) \). Let \( c(G) \) be the number of connected components of \( G \) that are isomorphic to the complete graph \( K_{\Delta+1} \). We argue by induction on \( c(G) \).
If \( c(G) = 0 \), then by Brooks’ Theorem (Theorem 2.5) \( G \) is \( \Delta \)-colorable. Then, by Proposition 5.1, \( I_n(G) \) is \( \Delta(n - 1) \)-collapsible, as wanted.

Otherwise, assume there exists a component of \( G \) that is isomorphic to \( K_{\Delta+1} \), and let \( v \) be a vertex in that component. Note that \( v \) is a simplicial vertex in \( G \). Since \( c(G \setminus v) = c(G) - 1 \), we obtain by the induction hypothesis

\[
C(I_n(G \setminus v)) \leq \Delta(n - 1).
\]

Also, by the (first) induction hypothesis, we have

\[
C(I_{n-1}(G[V \setminus N_G[v]])) \leq \Delta(n - 2) \leq \Delta(n - 1) - 1.
\]

So, by Lemma 4.1, we obtain

\[
C(I_n(G)) \leq \max\{C(I_n(G \setminus v)), C(I_{n-1}(G[V \setminus N_G[v]])) + 1\} \leq \Delta(n - 1).
\]

5.1 The \( n \leq 3 \) case and claw-free graphs

Next, we prove Theorems 1.7 and 1.8, which give tight upper bounds on the collapsibility of \( I_n(G) \) for graphs \( G \) with bounded maximum degree, for \( n \leq 3 \). We also prove Proposition 5.5, bounding the collapsibility of certain subcomplexes of \( I_n(G) \), in the case where \( G \) is a bounded degree claw-free graph.

Proof of Theorem 1.7. We argue by induction on \( |V| \). For \( |V| = 0 \) the bound holds trivially. Assume \( |V| > 0 \), and let \( v \in V \). By Lemma 3.1, we have

\[
C(I_2(G)) \leq \max\{C(I_2(G \setminus v)), C(\text{lk}(I_2(G), v)) + 1\}. \quad (1)
\]

Note that \( \text{lk}(I_2(G), v) \) is a flag complex on vertex set \( N_G(v) \). Thus, by Proposition 3.5, we have

\[
C(\text{lk}(I_2(G), v)) \leq \left\lfloor \frac{|N_G(v)|}{2} \right\rfloor \leq \left\lfloor \frac{\Delta}{2} \right\rfloor = \left\lceil \frac{\Delta + 1}{2} \right\rceil - 1.
\]

Also, by the induction hypothesis,

\[
C(I_2(G \setminus v)) \leq \left\lceil \frac{\Delta + 1}{2} \right\rceil.
\]

Hence, by (1), we obtain

\[
C(I_2(G)) \leq \left\lceil \frac{\Delta + 1}{2} \right\rceil.
\]
Lemma 5.4. Let $G = (V, E)$ be a graph and $n \geq 2$ an integer. Let $A$ be an independent set of size $n - 1$ in $G$, such that any vertex in $V \setminus A$ is adjacent to at most two vertices in $A$. Let

$$B = \bigcup_{\{u,v\} \in \binom{A}{2}} N_G(u) \cap N_G(v).$$

Then, $\text{lk}(I_n(G), A \cup B)$ is a flag complex.

Proof. Let $X = \text{lk}(I_n(G), A \cup B)$, and let $\tau$ be a missing face of $X$. Then, there exists an independent set $I$ of $G$ of size $n$, such that $\tau \subset I \subset \tau \cup A \cup B$. We may choose $I$ such that $|A \cap I|$ is maximal. Each vertex in $A \setminus I$ is adjacent to at least two vertices in $I \setminus A$: otherwise, assume there exists $a \in A \setminus I$ that is adjacent to at most one vertex in $I \setminus A$. We divide into two cases:

- If $a$ is not adjacent to any vertex in $I \setminus A$, let $\tau' = \tau \setminus \{a\}$ for any vertex $u \in \tau$.
- If $a$ is adjacent to a single vertex $u \in I \setminus A$, observe that $u$ should be contained in $\tau$. If not, we can take an independent set $I' = I \setminus \{a\} \cup \{u\}$ of size $n$ in $G$ such that $\tau \subset I' \subset \tau \cup A \cup B$. Since $|A \cap I'| = |A \cap I| + 1$, this contradicts the maximality assumption of $|A \cap I|$. Hence, $u \in \tau$.

Now, let $\tau' = \tau \setminus \{u\}$.

In both cases, $I \setminus \{a\} \cup \{u\}$ is an independent set of size $n$ satisfying $\tau' \subset I \setminus \{a\} \cup \{u\} \subset \tau' \cup A \cup B$. It follows that $\tau' \notin X$, which is a contradiction to $\tau$ being a missing face.

Let $|\tau| = k$ and $|A \cap I| = t$. Then, $|A \setminus I| = n - t - 1$; so, there are at least $2(n - t - 1)$ edges between $A$ and $I \setminus A$.

By assumption, each vertex $v \in I \setminus (A \cup \tau)$ is adjacent to at most $2$ vertices in $A$. Therefore, since $|I \setminus (A \cup \tau)| = n - t - k$, there are at least $2(n - t - 1) - 2(n - t - k) = 2k - 2$ edges between $A$ and $\tau$. But, since $\tau \subset V \setminus B$, each vertex in $\tau$ is adjacent to at most one vertex in $A$. Therefore, we must have $2k - 2 \leq k$; that is, $|\tau| = k \leq 2$. Thus, $X$ is a flag complex. \(\square\)

Proposition 5.5. Let $G = (V, E)$ be a claw-free graph with maximum degree at most $\Delta$, and let $n \geq 1$ be an integer. Let $A$ be an independent set of size $n - 1$ in $G$. Then,

$$C(\text{lk}(I_n(G), A)) \leq \left\lfloor \frac{(n-1)\Delta}{2} \right\rfloor.$$ 

Proof. For $n = 1$ the claim holds trivially. Assume $n \geq 2$.

Let $v \in V \setminus (A \cup N_G(A))$. Then, $A \cup \{v\}$ is an independent set of size $n$ in $G$; hence, $v \notin \text{lk}(I_n(G), A)$. So, we may assume without loss of generality that $V = N_G(A) \cup A$. Let

$$B = \bigcup_{\{u,v\} \in \binom{A}{2}} N_G(u) \cap N_G(v).$$

17
and $U = N_G(A) \setminus B$. Since $G$ is claw-free, each vertex is adjacent to at most 2 vertices in $A$. Hence, we have

$$|N_G(A)| = \sum_{v \in A} |N_G(v)| - \sum_{\{u,v\} \in \binom{A}{2}} |N_G(u) \cap N_G(v)| = \sum_{v \in A} |N_G(v)| - |B|.$$ 

So, since the maximum degree in $G$ is at most $\Delta$, we obtain

$$|U| \leq (n - 1)\Delta - 2|B|.$$ 

Let $(B_1, B_2)$ be a partition of $B$ such that $B_2 \in \text{lk}(I_n(G), A)$. Let $G' = G[V \setminus B_1]$, and let

$$X = \text{lk}(I_n(G)[V \setminus B_1], A \cup B_2) = \text{lk}(I_n(G'), A \cup B_2).$$

Note that

$$B_2 = \bigcup_{\{u,v\} \in \binom{A}{2}} N_{G'}(u) \cap N_{G'}(v)$$

Also, since $G'$ is claw-free and $A$ is independent in $G'$, then every vertex in $V \setminus B_1$ is adjacent to at most 2 vertices in $A$. Therefore, by Lemma 5.4, $X$ is a flag complex.

The vertex set of $X$ is contained in $U = N_G(A) \setminus B$. Thus, by Proposition 3.5, we obtain

$$C(X) \leq \left\lfloor \frac{|U|}{2} \right\rfloor \leq \left\lfloor \frac{n - 1)\Delta - 2|B|}{2} \right\rfloor \leq \left\lfloor \frac{(n - 1)\Delta}{2} \right\rfloor - |B_2|.$$ 

Therefore, by Lemma 3.6,

$$C(\text{lk}(I_n(G), A)) \leq \left\lfloor \frac{(n - 1)\Delta}{2} \right\rfloor.$$ 

\hfill \Box

**Proposition 5.6.** Let $G = (V, E)$ be a graph with maximum degree at most $\Delta$. Let $A = \{a_1, a_2\}$ be an independent set of size 2 in $G$. Assume that there exists an independent set in $G$ of the form $\{a_1, w, w'\}$, where $w, w' \in N_G(a_2)$, or there exists an independent set of the form $\{a_2, v, v'\}$, where $v, v' \in N_G(a_1)$. Then,

$$C(\text{lk}(I_3(G), A)) \leq \begin{cases} \Delta & \text{if } \Delta \text{ is even}, \\ \Delta - 1 & \text{if } \Delta \text{ is odd}. \end{cases}$$

**Proof.** Let $v \in V \setminus (N_G(A) \cup A)$. Then $A \cup \{v\}$ is an independent set of size 3 in $G$; hence, $v \notin \text{lk}(I_3(G), A)$. So, we may assume without loss of generality that $V = N_G(A) \cup A$. 18
Let 

\[ B = N_G(a_1) \cap N_G(a_2) \]

and \( U = N_G(A) \setminus B \). Since the maximum degree of a vertex in \( G \) is at most \( \Delta \), we have

\[
|N_G(A)| = |N_G(a_1)| + |N_G(a_2)| - |N_G(a_1) \cap N_G(a_2)| \leq 2\Delta - |B|.
\]

So, \( |U| \leq 2\Delta - 2|B| \).

Write \( B = \{u_1, \ldots, u_k\} \). Let \( P = P(\text{lk}(I_3(G), A), B) \) be the family of partitions \((B_1, B_2)\) of \( B \) satisfying:

- \( B_2 \in \text{lk}(I_3(G), A) \).
- For any \( u_i \in B_2 \), the complex 
  \[
  \text{lk}(I_3(G)[V \setminus \{u_j \in B_1 : j < i\}], A \cup \{u_j \in B_2 : j < i\})
  \]
  is not a cone over \( u_i \).

Let \((B_1, B_2)\) \( \in \mathcal{P} \). Let \( G' = G[V \setminus B_1] \), and let

\[
X = \text{lk}(I_3(G)[V \setminus B_1], A \cup B_2) = \text{lk}(I_3(G'), A \cup B_2).
\]

Note that \( B_2 = N_{G'}(a_1) \cap N_{G'}(a_2) \). Also, since \( A \) is of size 2, then every vertex in \( V \setminus B_1 \) is adjacent to at most 2 vertices in \( A \). Therefore, by Lemma 5.4, \( X \) is a flag complex.

The vertex set of \( X \) is contained in \( U = N_G(A) \setminus B \). So, by Proposition 3.5, we obtain

\[
C(X) \leq \frac{|U|}{2} \leq \frac{2\Delta - 2|B|}{2} = \Delta - |B| \leq \Delta - |B_2|.
\]

Therefore, by Lemma 3.6,

\[
C(\text{lk}(I_3(G), A)) \leq \Delta.
\]

Now, assume \( \Delta \) is odd. Again, let \((B_1, B_2)\) \( \in \mathcal{P} \), and let

\[
X = \text{lk}(I_3(G)[V \setminus B_1], A \cup B_2).
\]

If \( B_2 \neq B \) then, by (2),

\[
C(X) \leq \Delta - |B| \leq \Delta - 1 - |B_2|.
\]

So, assume \( B_2 = B \). By the equality case of Proposition 3.5, we have \( C(X) \leq \Delta - 1 - |B| \) unless \( X \) contains exactly \( 2\Delta - 2|B| \) vertices, and its set of missing faces consists of \( \Delta - |B| = \Delta - k \) pairwise disjoint sets of size 2.

Assume for contradiction that this is the case. Then, \( X \) is a simplicial complex on vertex set \( U = U_1 \cup U_2 \), where \( U_1 = N_G(a_1) \setminus N_G(a_2) \) and \( U_2 = N_G(a_2) \setminus N_G(a_1) \), and \( |U_1| = |U_2| = \Delta - k \).
Proof. Since Claim 5.8. There exist distinct vertices of the following forms:

\[ J = \{a_1, v, w\}, \text{ where } v, w \in U_2, \]
\[ J = \{a_2, v, w\}, \text{ where } v, w \in U_1, \text{ or} \]
\[ J = \{u_i, v, w\} \text{ for some } i \in [k], \text{ where } v, w \in U. \]

Proof. Since \( B_2 = B \) and \( (B_1, B_2) \in \mathcal{P} \), we have \( B \in \text{lk}(I_3(G), A) \). Thus, any independent set \( J \) of size 3 in \( G \) contains at least one vertex from \( U \).

Also, since \( X \) is a flag complex, at least one vertex in \( J \) must belong to \( A \cup B \) (otherwise \( J \) is a missing face of size 3 in \( X \)).

Note that since \( U \subset N_G(A) \), each independent set of size 3 contains at most one of the vertices \( a_1 \) or \( a_2 \).

Assume that \( a_1 \in J \). Then, for all \( i \in [k] \), \( u_i \notin J \). Otherwise, the unique vertex \( v \) in \( J \setminus \{a_1, u_i\} \) does not belong to \( X \), a contradiction to the assumption that the vertex set of \( X \) is the whole set \( U \). So, the two vertices in \( J \setminus \{a_1\} \) must belong to \( U \). And, since all the vertices in \( U_1 \) are adjacent to \( a_1 \), they must in fact belong to \( U_2 \), as wanted.

Similarly, if \( a_2 \in J \), then the two vertices in \( J \setminus \{a_2\} \) must belong to \( U_1 \).

Now, assume that \( a_1, a_2 \notin J \). Then, there exists some \( i \in [k] \) such that \( u_i \in J \). For all \( j \in [k] \setminus \{i\} \), \( u_j \notin J \), otherwise the unique vertex \( v \) in \( J \setminus \{u_i, u_j\} \) does not belong to \( X \), a contradiction to the assumption that the vertex set of \( X \) is the whole set \( U \). So, the two vertices in \( J \setminus \{u_i\} \) must belong to \( U \), as wanted.

Claim 5.8. There exist distinct vertices \( v_1, \ldots, v_k \in U_1 \) and \( w_1, \ldots, w_k \in U_2 \) such that:

\[ \text{For all } i \in [k], \{u_i, v_i, w_i\} \text{ is an independent set in } G. \]
\[ \text{For all } 1 \leq j < i \leq k, \{u_j, v_i, w_i\} \text{ is not independent in } G. \]

Proof. We define the vertices \( v_1, \ldots, v_k, w_1, \ldots, w_k \) recursively, as follows. Let \( i \in [k] \), and assume that we already defined \( v_1, \ldots, v_{i-1} \) and \( w_1, \ldots, w_{i-1} \). Since \( (B_1, B_2) = (\emptyset, B) \in \mathcal{P} \), then the complex

\[ X' = \text{lk}(I_3(G), A \cup \{u_j \in B : j < i\}) \]

is not a cone over \( u_i \). Therefore, there exists a missing face \( \tau \) of \( X' \) containing \( u_i \). Since \( \tau \) is a missing face of \( X' \), there exists an independent set \( J \) of size 3 in \( G \) containing \( \tau \). By Claim 5.7, \( J \) is of the form \( J = \{u_i, v_i, w_i\} \), for some \( v_i, w_i \in U \).

Note that actually \( J = \tau \). Otherwise, assume without loss of generality that \( \tau = \{u_i, v_i\} \). Then \( w_i \notin X' \). But then \( w_i \notin X \), a contradiction to the assumption that the vertex set of \( X \) is the whole set \( U \).
If both \( v_i \) and \( w_i \) belong to \( U_1 \), or both of them belong to \( U_2 \), then \( \{v_i, w_i\} \not\in X' \), a contradiction to \( \{u_i, v_i, w_i\} \) being a missing face. So, we may assume that \( v_i \in U_1 \) and \( w_i \in U_2 \). Moreover, for all \( j < i \), \( \{u_j, v_i, w_i\} \) is not independent in \( G \), otherwise \( \{v_i, w_i\} \not\in X' \), a contradiction to \( \{u_i, v_i, w_i\} \) being a missing face.

The pairs \( \{\{v_i, w_i\}\}_{i \in [k]} \) are missing faces of the complex \( X \). Hence, they must be pairwise disjoint. Thus, the vertices \( v_1, \ldots, v_k, w_1, \ldots, w_k \) are all distinct.

**Claim 5.9.** There exist some \( i_0 \in [k] \) and vertices \( v'_{i_0} \in U_1 \setminus \{v_1, \ldots, v_k\} \), \( w'_{i_0} \in U_2 \setminus \{w_1, \ldots, w_k\} \) such that \( \{u_{i_0}, v'_{i_0}, w'_{i_0}\} \) is independent in \( G \).

**Proof.** Recall that, by assumption, the missing faces of \( X \) consist of \( \Delta - k \) pairwise disjoint sets of size 2. In particular, each vertex \( v \in U \) belongs to exactly one missing face of \( X \).

Assume for contradiction that the only missing faces of \( X \) of the form \( \{v, w\} \), where \( v \in U_1 \) and \( w \in U_2 \), are the pairs \( \{v_i, w_i\}, i \in [k] \), from Claim 5.8.

Then, the \( \Delta - 2k \) remaining missing faces must be of the form \( \{v, w\} \), where \( v, w \in U_1 \) or \( v, w \in U_2 \). In particular, the set \( U_1 \setminus \{v_1, \ldots, v_k\} \) must be of even size (otherwise, there exists a vertex \( v \in U_1 \setminus \{v_1, \ldots, v_k\} \) that does not belong to any missing face of \( X \), a contradiction). But

\[
|U_1 \setminus \{v_1, \ldots, v_k\}| = \Delta - 2k
\]

is odd, since \( \Delta \) is odd.

Therefore, there exists some additional missing face of the form \( \{v, w\} \), where \( v \in U_1, w \in U_2 \). That is, there is some \( i_0 \in [k] \) such that \( \{u_{i_0}, v, w\} \) is independent in \( G \). So, we can choose \( v'_{i_0} = v \) and \( w'_{i_0} = w \).

**Claim 5.10.** \( \Delta \geq 2k + 3 \).

**Proof.** Assume without loss of generality that there exists an independent set in \( G \) of the form \( \{a_1, w, w'\} \), where \( w, w' \in N_G(a_2) \). Then, the set \( \{w, w'\} \) is a missing face in \( X \). Since the missing faces of \( X \) are all disjoint, the vertices \( w_1, \ldots, w_k, w'_{i_0}, w, w' \in U_2 \) must be all distinct. Therefore,

\[
\Delta - k = |U_2| \geq k + 3.
\]

Hence, \( \Delta \geq 2k + 3 \). 

Let

\[
S = \{j \in [k] \setminus \{i_0\} : \{v_{i_0}, u_j\} \not\in E \text{ or } \{w_{i_0}, u_j\} \not\in E\}.
\]
Claim 5.11. There exists a set $N_1$ consisting of exactly one vertex from each pair $\{w_j, u_j\}$, for all $j \in S$, such that

$$N_G(v_{i_0}) = \{a_1\} \cup \{w'_i\} \cup (U_1 \setminus \{v_{i_0}\}) \cup \{u_j : j \in [k] \setminus (S \cup \{i_0\})\} \cup N_1.$$  

In particular, $|N_G(v_{i_0})| = \Delta$.

Similarly, there exists a set $N_2$ consisting of exactly one vertex from each pair $\{v_j, u_j\}$, for all $j \in S$, such that

$$N_G(w_{i_0}) = \{a_2\} \cup \{v'_i\} \cup (U_2 \setminus \{w_{i_0}\}) \cup \{u_j : j \in [k] \setminus (S \cup \{i_0\})\} \cup N_2.$$  

And, in particular, $|N_G(w_{i_0})| = \Delta$.

Proof. We prove the claim for $v_{i_0}$. The proof for $w_{i_0}$ is identical.

First, since $v_{i_0} \in U_1$, then $a_1$ is adjacent to $v_{i_0}$. Also, for every $v_{i_0} \neq v \in U_1$, $v$ is adjacent to $v_{i_0}$, since otherwise the set $\{u_2, v_{i_0}, v\}$ is independent in $G$, but then the set $\{v, v_{i_0}\}$ is a missing face of $X$ that intersects the missing face $\{v_{i_0}, w_{i_0}\}$, a contradiction to the assumption that the missing faces are pairwise disjoint.

The vertex $w'_i$ must also be adjacent to $v_{i_0}$, otherwise $\{u_{i_0}, v_{i_0}, w'_i\}$ is an independent set in $G$. But then, $\{v_{i_0}, w'_i\}$ is a missing face of $X$ intersecting the missing face $\{v_{i_0}, w_{i_0}\}$, again a contradiction.

By the definition of $S$, $v_{i_0}$ is adjacent to $u_j$ for all $j \in [k] \setminus (S \cup \{i_0\})$.

Finally, let $j \in S$. If $\{v_{i_0}, u_j\} \notin E$ and $\{v_{i_0}, w_j\} \notin E$, then $\{v_{i_0}, u_j, w_j\}$ is independent in $G$; therefore, $\{v_{i_0}, w_j\}$ is a missing face of $X$, a contradiction. So, $v_{i_0}$ is adjacent to either $u_j$ or $w_j$. Let $S' = \{j \in S : \{u_j, v_{i_0}\} \in E\}$. Let

$$N_1 = \{u_j : j \in S'\} \cup \{w_j : j \in S \setminus S'\}.$$  

Then, $N_1 \subset N_G(v_{i_0})$. Let

$$N = \{a_1\} \cup \{w'_i\} \cup (U_1 \setminus \{v_{i_0}\}) \cup \{u_j : j \in [k] \setminus (S \cup \{i_0\})\} \cup N_1.$$  

We showed that $N \subset N_G(v_{i_0})$. Note that

$$|N| = 1 + 1 + (\Delta - k - 1) + (k - |S| - 1) + |S| = \Delta.$$  

Since the maximal degree of a vertex in $G$ is at most $\Delta$, then we must have $N_G(v_{i_0}) = N$, as wanted.

Claim 5.12. For all $j \in [k] \setminus \{i_0\}$, $u_{i_0}$ is adjacent in $G$ to at least one of the vertices $v_j$ or $w_j$.

Proof. Let $j \neq i_0$. Assume for contradiction that $u_{i_0}$ is not adjacent to any of the two vertices $v_j$ and $w_j$. Then $\{u_{i_0}, v_j, w_j\}$ is independent in $G$. So, by Claim 5.11, we must have $i_0 > j$. Moreover, either $\{v_{i_0}, u_j\} \in E$ or $\{w_{i_0}, u_j\} \in E$ (otherwise $\{u_j, v_{i_0}, w_{i_0}\}$ is independent in $G$, a contradiction
to Claim 5.8. Assume without loss of generality that \( \{v_{i_0}, u_j\} \in E \). The vertex \( v_{i_0} \) must be also adjacent to \( w_j \), since otherwise the set \( \{u_{i_0}, v_{i_0}, w_j\} \) is independent in \( G \). But then \( \{v_{i_0}, w_j\} \) is a missing face of \( X \), a contradiction to the assumption that the missing faces are pairwise disjoint.

But, by Claim 5.11, the set of neighbors of \( v_{i_0} \) in \( G \), \( N_G(v_{i_0}) \), contains at most one of the vertices \( u_j \) or \( w_j \), a contradiction.

So, \( u_{i_0} \) must be adjacent in \( G \) to at least one of the vertices \( v_j \) or \( w_j \). □

Claim 5.13. There is some vertex

\[ w \in U \setminus \{\{v_j, w_j : j \in S\} \cup \{v_{i_0}, w_{i_0}, v'_{i_0}, w'_{i_0}\}\} \]

such that \( \{u_{i_0}, w\} \notin E \).

Proof. Let \( U' = U \setminus \{\{v_j, w_j : j \in S\} \cup \{v_{i_0}, w_{i_0}, v'_{i_0}, w'_{i_0}\}\} \). The vertex \( u_{i_0} \) is adjacent in \( G \) to both \( a_1 \) and \( a_2 \) (since \( u_{i_0} \in B = N_G(a_1) \cap N_G(a_2) \)). Also, by Claim 5.12, it is adjacent to at least \( |S| \) vertices from the set \( \{v_j, w_j : j \in S\} \).

By the definition of \( S \), for each \( j \in S \), \( u_j \) is not adjacent to one of the vertices \( v_{i_0} \) or \( w_{i_0} \). Thus \( u_{i_0} \) must be adjacent in \( G \) to \( u_j \) (otherwise, one of the sets \( \{u_j, u_{i_0}, v_{i_0}\} \) or \( \{u_j, u_{i_0}, w_{i_0}\} \) is independent in \( G \), in contradiction to Claim 5.7).

So, \( u_{i_0} \) is adjacent to at least \( 2|S| + 2 \) vertices outside of \( U' \). Since the degree of \( u_{i_0} \) is at most \( \Delta \), \( u_{i_0} \) is adjacent to at most \( \Delta - 2 - 2|S| \) vertices in \( U' \).

But \( |U'| = |U| - 2|S| - 4 = 2\Delta - 2k - 2|S| - 4 \). So, \( u_{i_0} \) is not adjacent to at least \( \Delta - 2k - 2 \) vertices in \( U' \). By Claim 5.10, \( \Delta \geq 2k + 3 \). Therefore, \( u_{i_0} \) is not adjacent to at least one vertex \( w \in U' \). □

Assume without loss of generality that the vertex \( w \) from Claim 5.13 belongs to \( U_2 \). If \( \{v_{i_0}, w\} \notin E \), then \( \{u_{i_0}, v_{i_0}, w\} \) is independent in \( G \). But then, \( \{v_{i_0}, w\} \) is a missing face of \( X \) intersecting \( \{v_{i_0}, w_{i_0}\} \), a contradiction to the assumption that all the missing faces are disjoint. So, \( w \in N_G(v_{i_0}) \). But this is a contradiction to Claim 5.11.

Therefore, \( C(X) \leq (\Delta - 1) - |B| \); so, by Lemma 3.6, \( \text{lk}(I_3(G), A) \) is \((\Delta - 1)\)-collapsible. □

Proposition 5.14. Let \( \Delta \geq 2 \). Let \( G = (V, E) \) be a graph with maximum degree at most \( \Delta \), and let \( a_1 \in V \). Then,

\[ C(\text{lk}(I_3(G), a_1)) \leq \begin{cases} \Delta + 1 & \text{if } \Delta \text{ is even,} \\ \Delta & \text{if } \Delta \text{ is odd.} \end{cases} \]

Proof. Let \( d = \Delta + 2 \) if \( \Delta \) is even, and \( d = \Delta + 1 \) if \( \Delta \) is odd. Let \( V' \) be the vertex set of \( \text{lk}(I_3(G), a_1) \); we argue by induction on \( |V'| \). If \( |V'| \leq \Delta \),
then by Proposition 3.5
\[ C(\text{lk}(I_3(G), \{a_1, a_2\})) \leq \frac{2|V'|}{3} \leq \frac{2\Delta}{3} \leq d - 1, \]
as wanted. Otherwise, let |V'| > \Delta. We divide into three different cases:

Case 1: There exists an independent set in \( G \) of the form \( \{u, v, a_2\} \), where \( u, v \in N_G(a_1) \) and \( a_2 \notin N_G(a_1) \). Then, by Proposition 5.6 we have
\[ C(\text{lk}(I_3(G), \{a_1, a_2\})) \leq d - 2. \]

Case 2: There exists a triple \( \{u, v, a_2\} \subset V' \) such that \( u, v, a_2 \notin N_G(a_1) \), \( \{u, v\} \notin E \), \( \{u, a_2\} \in E \) and \( \{v, a_2\} \in E \). Then, \( \{a_1, u, v\} \) is an independent set in \( G \), and \( u, v \in N_G(a_2) \). Thus, by Proposition 5.6
\[ C(\text{lk}(I_3(G), \{a_1, a_2\})) \leq d - 2. \]

Case 3: Assume none of the two first cases holds. Since \( |V'| > \Delta \), there exists a vertex \( a_2 \in \text{lk}(I_3(G), \{a_1, a_2\}) \) such that \( a_2 \notin N_G(a_1) \) (otherwise \( \text{deg}_G(a_1) = |N_G(a_1)| > \Delta \), a contradiction).

Let \( w \in N_G(a_2) \setminus N_G(a_1) \). Note that \( w \in \text{lk}(I_3(G), \{a_1, a_2\}) \).

Assume for contradiction that there exists a missing face \( \tau \) of the complex \( \text{lk}(I_3(G), \{a_1, a_2\}) \) that contains \( w \). First, assume that \( \tau = \{u, v, w\} \) is an independent set of size 3. Then, both \( u \) and \( v \) must belong to \( N_G(a_1) \). Otherwise, assume without loss of generality that \( v \notin N_G(a_1) \). Then \( \{w, v, a_1\} \) is an independent set in \( G \), and therefore \( \{v, w\} \notin \text{lk}(I_3(G), \{a_1, a_2\}) \), a contradiction to \( \tau \) being a missing face. But then, the existence of the independent set \( \{u, v, w\} \) is a contradiction to the assumption that Case 1 does not hold.

Now, assume \( \tau = \{v, w\} \) is of size 2. Then there exists an independent set \( J \) of size 3 such that \( \tau \subset J \subset \tau \cup \{a_1, a_2\} \). Since \( w \in N_G(a_2) \), we must have \( J = \{a_1, v, w\} \). In particular \( v \notin N_G(a_1) \). So, we must have \( v \in N_G(a_2) \). But then, the triple \( \{a_2, v, w\} \) satisfies \( a_2, v, w \notin N_G(a_1) \), \( \{v, w\} \notin E \), \( \{a_2, v\} \in E \) and \( \{a_2, w\} \in E \). This is a contradiction to the assumption that Case 2 does not hold.

Therefore, \( w \) is not contained in any missing face of \( \text{lk}(I_3(G), \{a_1, a_2\}) \).

Let \( U = N_G(a_1) \cup \{a_1, a_2\} \). Then, we have
\[ \text{lk}(I_3(G), \{a_1, a_2\}) = 2^{N_G(a_2) \setminus N_G(a_1)} \ast \text{lk}(I_3(G[U]), \{a_1, a_2\}). \]
So, by Lemma 3.3 we have
\[ C(\text{lk}(I_3(G), \{a_1, a_2\})) = C(\text{lk}(I_3(G[U]), \{a_1, a_2\})). \]
By Proposition 3.5, we obtain
\[ C(\text{lk}(I_3(G), \{a_1, a_2\})) \leq \frac{2|N_G(a_1)|}{3} \leq \frac{2\Delta}{3}. \]
Note that \( \frac{2\Delta}{3} \leq \Delta \), and \( \frac{2\Delta}{3} \leq \Delta - 1 \) for \( \Delta \geq 3 \). Hence, we obtain
\[ C(\text{lk}(I_3(G), \{a_1, a_2\})) \leq \frac{2\Delta}{3} \leq d - 2 \]
for all \( \Delta \geq 2 \).

For any of the three cases we have \( C(\text{lk}(I_3(G \setminus a_2), a_1)) \leq d - 1 \) by the induction hypothesis. Also, we showed that \( C(\text{lk}(I_3(G), \{a_1, a_2\})) \leq d - 2 \) in all three cases. So, by Lemma 3.1,
\[ C(\text{lk}(I_3(G), a_1)) \leq \max\{C(\text{lk}(I_3(G \setminus a_2), a_1)), C(\text{lk}(I_3(G), a_1)) + 1\} \leq d - 1, \]
as wanted. \( \qed \)

**Theorem 5.15.** Let \( G = (V, E) \) be a graph with maximum degree \( \Delta \). Then,
\[ C(I_3(G)) \leq \begin{cases} \Delta + 2 & \text{if } \Delta \text{ is even}, \\ \Delta + 1 & \text{if } \Delta \text{ is odd}. \end{cases} \]

**Proof.** For \( \Delta = 1 \) the claim holds by Theorem 1.6. Assume \( \Delta \geq 2 \).

Let \( d = \Delta + 2 \) if \( \Delta \) is even, and \( d = \Delta + 1 \) if \( \Delta \) is odd. We argue by induction on \( |V| \). If \( |V| = 0 \) the claim holds trivially. Otherwise, let \( a_1 \in V \).
By the induction hypothesis, \( C(I_3(G \setminus a_1)) \leq d \). Also, by Proposition 5.14 \( C(\text{lk}(I_3(G), a_1)) \leq d - 1 \). So, by Lemma 3.1
\[ C(I_3(G)) \leq \max\{C(I_3(G \setminus a_1)), C(\text{lk}(I_3(G), a_1)) + 1\} \leq d. \]
\( \qed \)

## 6 Rainbow independent sets in claw-free graphs

Now we are ready to prove Theorem 1.9.

**Proof of Theorem 1.9** We argue by induction on \( n \). The case \( n = 1 \) is trivial. Now, assume \( n > 1 \). Let \( t = \left\lceil \left( \frac{\Delta}{2} + 1 \right) (n - 1) \right\rceil + 1 \) and let \( J_1, \ldots, J_t \)
be independent sets of size \( n \) in \( G \). Since \( t \geq \left\lceil \left( \frac{\Delta}{2} + 1 \right) (n - 2) \right\rceil + 1 \), then, by the induction hypothesis, there exists a rainbow independent set \( A \) of size \( n - 1 \). Without loss of generality, we may assume that \( A = \{v_1, \ldots, v_{n-1}\} \), where \( v_i \in J_i \) for all \( i \in [n-1] \).

Let \( X = \text{lk}(I_n(G), A) \). By Proposition 5.5 \( X \) is \( \left\lceil \frac{\Delta}{2} (n - 1) \right\rceil \)-collapsible.
The family \( \{J_i\}_{n \leq i \leq t} \) consists of \( \lfloor \frac{\Delta}{2} (n-1) \rfloor + 1 \) sets not belonging to \( X \).

Thus, by Theorem 1.1, there exists a set \( R = \{v_n, \ldots, v_t\} \), where \( v_i \in J_i \) for all \( n \leq i \leq t \), such that \( R \notin X \). Therefore, the set \( A \cup R \) contains a set \( I \) of size \( n \) that is independent in \( G \). \( I \) is a rainbow independent set of size \( n \) in \( G \), as wanted.

7 Lower bounds on Leray numbers

In this section we present some examples establishing the sharpness of our different bounds on the collapsibility of \( I_n(G) \). Also, we present a family of counterexamples to Conjecture 1.4 in the case of graphs with maximum degree at most 3.

7.1 Extremal examples

Let \( n \) be an integer, and \( k \) be an even integer. Let \( G_{k,n} \) be the graph obtained from a cycle of length \( \left( \frac{k}{2} + 1 \right) n \) by adding all edges connecting any two vertices of distance at most \( \frac{k}{2} \) in the cycle. Note that \( G_{k,n} \) is a \( k \)-regular graph, i.e., every vertex has degree exactly \( k \). Moreover, \( G_{k,n} \) is claw-free.

In [4] it is shown that \( f_{G_{k,n}}(n) \geq \left( \frac{k}{2} + 1 \right) (n-1) + 1 \). In particular, this shows the tightness of Theorem 1.9, in the case that \( k \) is even. Moreover, by Proposition 1.2, we obtain

\[
C(I_n(G_{k,n})) \geq f_{G_{k,n}}(n) - 1 \geq \left( \frac{k}{2} + 1 \right) (n-1).
\]

This shows that the bound in Conjecture 1.4 whenever it holds, is tight. A different way to show this is as follows.

**Proposition 7.1.**

\[
\tilde{H}_i(I_n(G_{k,n})) = \begin{cases} \mathbb{R} & \text{if } i = \left( \frac{k}{2} + 1 \right) (n-1) - 1, \\ 0 & \text{otherwise}. \end{cases}
\]

In particular, \( L(I_n(G_{k,n})) \geq \left( \frac{k}{2} + 1 \right) (n-1) \).

**Proof.** Let \( t = \frac{k}{2} + 1 \). It is easy to check that there are precisely \( t \) independent sets of size \( n \) in \( G_{k,n} \), and they are pairwise disjoint. Therefore, \( I_n(G_{k,n}) \) can be described as the join of \( t \) disjoint copies of the boundary of an \((n-1)\)-dimensional simplex. Since the boundary of an \((n-1)\)-dimensional simplex is an \((n-2)\)-dimensional sphere, we obtain by Theorem 2.2

\[
\tilde{H}_i(I_n(G_{k,n})) = \begin{cases} \mathbb{R} & \text{if } i = t(n-1) - 1, \\ 0 & \text{otherwise}. \end{cases}
\]

26
Thus, $L(I_n(G)) \geq t(n - 1) = \left(\frac{k}{2} + 1\right)(n - 1)$.

By Lemma 2.1 we obtain

$$C(I_n(G_{k,n})) \geq L(I_n(G_{k,n})) \geq \left(\frac{k}{2} + 1\right)(n - 1).$$

On the other hand, $I_n(G_{k,n})$ is a \((\left(\frac{k}{2} + 1\right)(n - 1) - 1)\)-dimensional complex, and therefore it is \((\frac{k}{2} + 1)(n - 1)\)-collapsible. So,

$$C(I_n(G_{k,n})) = \left(\frac{k}{2} + 1\right)(n - 1).$$

Proposition 7.1 also shows that the bound in Proposition 5.1 is tight, since $G_{2k-2,n}$ is a $k$-partite graph with $C(I_n(G_{2k-2,n})) = k(n - 1)$. Another such extremal example is the complete $k$-partite graph $K_{n,\ldots,n}$. In this case, it easy to see that $I_n(K_{n,\ldots,n}) \cong I_n(G_{2k-2,n})$.

7.2 A counterexample to Conjecture 1.4

Let $G = (V, E)$ be the dodecahedral graph. It will be convenient to represent $G$ as a generalized Petersen graph (see [15]), as follows:

$$V = \{a_1, \ldots, a_{10}, b_1, \ldots, b_{10}\}$$

and

$$E = \{\{a_i, b_i\}, \{a_i, a_{i+1}\}, \{b_i, b_{i+2}\} : i = 1, 2, \ldots, 10\},$$

where the indices are taken modulo 10.

Every vertex in $G$ is adjacent to exactly 3 vertices; that is, $G$ is 3-regular. The maximal independent sets in $G$ are the sets

$$I_i = \{a_i, a_{i+2}, a_{i+5}, a_{i+7}, b_{i-2}, b_{i-1}, b_{i+3}, b_{i+4}\}$$

for $i = 1, \ldots, 5$ (also here, the indices are to be taken modulo 10). In particular, $\alpha(G) = 8$.

**Proposition 7.2.** Let $G = (V, E)$ be the dodecahedral graph. Then,

$$\check{H}_i(I_8(G)) = \begin{cases} \mathbb{R}^4 & \text{if } i = 15, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $L(I_8(G)) \geq 16$. 
Proof. Let $\mathcal{F} = \{V \setminus I_1, V \setminus I_2, \ldots, V \setminus I_5\}$. The family $\mathcal{F}$ is the set of maximal faces of $D(I_8(G))$. So, by the Nerve Theorem (Theorem 2.3),

$$\tilde{H}_i(N(\mathcal{F})) \cong \tilde{H}_i(D(I_8(G)))$$

for all $i \geq -1$. So, by Alexander duality (Theorem 2.4),

$$\tilde{H}_i(N(\mathcal{F})) \cong \tilde{H}_{|V| - i - 3}(I_8(G)) = \tilde{H}_{17 - i}(I_8(G)) \quad (3)$$

for all $-1 \leq i \leq |V| - 2 = 18$. We have

$$N(\mathcal{F}) = \left\{ A \subseteq [5] : \bigcap_{i \in A} V \setminus I_i \neq \emptyset \right\} = \left\{ A \subseteq [5] : \bigcup_{i \in A} I_i \neq V \right\}.$$  

It is easy to check that $N(\mathcal{F})$ is the complete 2-dimensional complex on 5 vertices. So,

$$\tilde{H}_i(N(\mathcal{F})) = \begin{cases} \mathbb{R}^4 & \text{if } i = 2, \\ 0 & \text{otherwise}. \end{cases}$$

Thus, by (3),

$$\tilde{H}_i(I_8(G)) = \begin{cases} \mathbb{R}^4 & \text{if } i = 15, \\ 0 & \text{otherwise}, \end{cases}$$

as wanted. □

We obtain $C(I_8(G)) \geq L(I_8(G)) \geq 16 > 2 \cdot (8 - 1) = 14$. Therefore, $I_8(G)$ does not satisfy the bound in Conjecture 1.4. However, this is not a counterexample for Conjecture 1.3. Indeed, it is not hard to check that $f_G(8) = 11$.

### 7.3 Leray number of the disjoint union of graphs

The following result will help us in constructing more counterexamples to Conjecture 1.3.

**Theorem 7.3.** Let $G$ be the disjoint union of the graphs $G_1, \ldots, G_m$. For $i \in [m]$, let $t_i = \alpha(G_i)$ and let $\ell_i = L(I_{t_i}(G_i))$. Let $t = \sum_{i=1}^m t_i = \alpha(G)$ and $\ell = L(I_{t}(G))$. Then,

$$\ell = \sum_{i=1}^m \ell_i + m - 1.$$  

The proof relies on the following result.

**Proposition 7.4.** Let $G$ be the disjoint union of the graphs $G_1, \ldots, G_m$. For $i \in [m]$, let $t_i = \alpha(G_i)$. Let $t = \sum_{i=1}^m t_i = \alpha(G)$. Then, $H_k(I_t(G)) = 0$ if and only if for every choice of integers $k_1, \ldots, k_m$ satisfying $\sum_{i=1}^m k_i = k - 2m + 2$, $H_{k_i}(I_{t_i}(G_i)) = 0$ for all $i \in [m]$.
Proof. For all \( i \in [m] \), let \( V_i \) be the vertex set of \( G_i \), and let \( V = \bigcup_{i=1}^{m} V_i \) be the vertex set of \( G \). Let \( N_i = |V_i| \) for all \( i \in [m] \), and \( N = |V| = \sum_{i=1}^{m} N_i \).

A set \( U \subset V \) contains an independent set of size \( t \) in \( G \) if and only if \( U \cap V_i \) contains an independent set of size \( t_i \) in \( G_i \) for all \( i \in [m] \). That is, \( U \notin I_t(G) \) if and only if \( U \cap V_i \notin I_{t_i}(G_i) \) for all \( i \in [m] \). Equivalently, a set \( W \subset V \) belongs to \( D(I_t(G)) \) if and only if \( W \cap V_i \in D(I_{t_i}(G_i)) \) for all \( i \in [m] \). Thus, we have

\[
D(I_t(G)) = D(I_{t_1}(G_1)) \ast \cdots \ast D(I_{t_m}(G_m)).
\]

Note that for every \( i \in [m] \), \( V_i \notin I_{t_i}(G_i) \) (since \( G_i \) contains an independent set of size \( t_i = \alpha(G_i) \)). Similarly, \( V \notin I_t(G) \). So, by Alexander duality (Theorem 2.4), we have

\[
\tilde{H}_j(D(I_{t_i}(G_i))) = \tilde{H}_{N_i-j-3}(I_{t_i}(G_i))
\]

for all \( i \in [m] \) and \(-1 \leq j \leq |V_i| - 2 \), and

\[
\tilde{H}_j(D(I_t(G))) = \tilde{H}_{N-j-3}(I_t(G))
\]

for all \(-1 \leq j \leq |V| - 2 \).

Therefore, by Theorem 2.2, we obtain

\[
\tilde{H}_{N-j-3}(I_t(G)) = \bigoplus_{j_1+\cdots+j_m=j-m+1} \tilde{H}_{j_1}(D(I_{t_1}(G_1))) \otimes \cdots \otimes \tilde{H}_{j_m}(D(I_{t_m}(G_m)))
\]

\[
= \bigoplus_{j_1+\cdots+j_m=j-m+1} \tilde{H}_{N_i-j_1-3}(I_{t_i}(G_i)) \otimes \cdots \otimes \tilde{H}_{N_m-j_m-3}(I_{t_m}(G_m)).
\]

Setting \( k = N - j - 3 \) and \( k_i = N_i - j_i - 3 \) for all \( i \in [m] \), we obtain

\[
\tilde{H}_k(I_t(G)) = \bigoplus_{k_1+\cdots+k_m=k-2m+2} \tilde{H}_{k_1}(I_{t_1}(G_1)) \otimes \cdots \otimes \tilde{H}_{k_m}(I_{t_m}(G_m)).
\]

In particular, \( \tilde{H}_k(I_t(G)) = 0 \) if and only if for every choice of \( k_1, \ldots, k_m \) satisfying \( \sum_{i=1}^{m} k_i = k - 2m + 2 \), \( \tilde{H}_{k_i}(I_{t_i}(G_i)) = 0 \) for all \( i \in [m] \).

\[ \square \]

Proof of Theorem 7.3: For all \( i \in [m] \), let \( V_i \) be the vertex set of \( G_i \), and let \( V = \bigcup_{i=1}^{m} V_i \) be the vertex set of \( G \).

Since \( L(I_t(G)) = \ell \), there exists a subset \( U \subset V \) such that

\[
\tilde{H}_{\ell-1}(I_t(G[U])) \neq 0.
\]

Let \( G' = G[U] \) and \( G'_i = G_i[U \cap V_i] \) for all \( i \in [m] \). Note that \( I_t(G') \) is not the complete complex, since it has non-trivial homology; hence, \( \alpha(G') = \ldots \)
Since $G'$ is the disjoint union of the graphs $G_1', \ldots, G_m'$, we must have $\alpha(G_i') = t_i$ for all $i \in [m]$. By Proposition 7.4 there exists $k_1, \ldots, k_m$ satisfying $\sum_{i=1}^m k_i = \ell - 2m + 1$ such that

$$\tilde{H}_{k_i} (I_t(G_i')) \neq 0.$$ 

In particular, $\ell_i = L(I_{t_i}(G_i)) \geq k_i + 1$. Summing over all $i \in [m]$, we obtain

$$\sum_{i=1}^m \ell_i \geq \sum_{i=1}^m k_i + m = \ell - m + 1.$$ 

Now, let $i \in [m]$. Since $\ell_i = L(I_{t_i}(G_i))$, there exists a subset $U_i \subset V_i$ such that

$$\tilde{H}_{\ell_i-1} (I_{t_i}(G_i[U_i])) \neq 0.$$ 

Let $G'_i = G_i[U_i]$ Note that $I_{t_i}(G_i')$ is not the complete complex, since it has non-trivial homology. Therefore, $\alpha(G_i') = t_i$. Let $U = U_1 \cup \cdots \cup U_m$, and let $G' = G[U]$. Then, $G'$ is the disjoint union of $G_1', \ldots, G_m'$. By Proposition 7.4 we have

$$\tilde{H}_{\sum_{i=1}^m (\ell_i - 1) + 2m - (2I_{t}(G'))} = \tilde{H}_{\sum_{i=1}^m \ell_i + m - 2(I_{t}(G'))} \neq 0.$$ 

Thus, $\ell = L(I_{t}(G)) \geq \sum_{i=1}^m \ell_i + m - 1$. 

**Corollary 7.5.** Let $G_k$ be the union of $k$ disjoint copies of the dodecahedral graph. Then,

$$L(I_{8k}(G_k)) \geq 17k - 1.$$ 

**Proof.** Let $H_1, \ldots, H_k$ be $k$ disjoint copies of the dodecahedral graph. Then, by Propositions 7.2 and 7.3 we obtain

$$L(I_{8k}(G_k)) = L(I_{8k}(H_1 \cup H_2 \cup \cdots \cup H_k))$$

$$= \sum_{i=1}^k L(I_{8}(H_i)) + k - 1 \geq 16k + k - 1 = 17k - 1.$$ 

Note that the graphs $G_k$ are 3-regular, and

$$\frac{L(I_{8k}(G_k))}{8k - 1} \geq \frac{17k - 1}{8k - 1} > 2 \frac{1}{8} > 2.$$ 

Thus, the complexes $I_{8k}(G_k)$ do not satisfy the bound in Conjecture 1.3.

Note that the graphs $G_k$ are not counterexamples for Conjecture 1.3. This can be shown by the following observation.
Proposition 7.6. Let $G$ be the disjoint union of two graphs $G_1$ and $G_2$ with $\alpha(G_1) = t_1$ and $\alpha(G_2) = t_2$. Then,

$$f_G(t_1 + t_2) \leq \max\{f_{G_1}(t_1), f_{G_2}(t_2) + t_1\}.$$ 

Proof. Let $V_1$ and $V_2$ denote the vertex sets of $G_1$ and $G_2$ respectively. Let $t = \max\{f_{G_1}(t_1), f_{G_2}(t_2) + t_1\}$.

Let $A = \{A_1, \ldots, A_t\}$ be a family of independent sets of size $t_1 + t_2$ in $G$. Note that any independent set of size $t_1 + t_2 = \alpha(G)$ in $G$ has $t_1$ vertices in $V_1$ and $t_2$ vertices in $V_2$.

Thus, $A_1 \cap V_1, A_2 \cap V_1, \ldots, A_t \cap V_1$ is a family of $t \geq f_{G_1}(t_1)$ independent sets of size $t_1$ in $G_1$. Hence, it contains a rainbow independent set $R_1$ of size $t_1$. Without loss of generality, we may assume that $R_1 = \{a_{t-t_1+1}, \ldots, a_t\}$, where $a_i \in A_i$ for all $i \in \{t-t_1+1, \ldots, t\}$.

The family $A_1 \cap V_2, A_2 \cap V_2, \ldots, A_{t-t_1} \cap V_2$ is a family of $t-t_1 \geq f_{G_2}(t_2)$ independent sets of size $t_2$ in $G_2$; therefore, it contains a rainbow independent set $R_2$ of size $t_2$.

Then, the set $R_1 \cup R_2$ is a rainbow independent set of size $t_1 + t_2$ in $G$ with respect to $A$, as wanted. \hfill $\Box$

Applying Proposition 7.6 repeatedly, we obtain that $f_G(8k) \leq 8k + 3 < 16k - 1$.

8 Open problems

We showed that the bound in Conjecture 1.4 holds in some special cases, but not in general. It would be interesting to decide for which values of $\Delta$ and $n$ the inequality holds. Alternatively, one could try to characterize the graphs satisfying the bound for all values of $n$.

A weaker result, which may hold for general bounded degree graphs, is the following:

Conjecture 8.1. Let $G = (V, E)$ be a graph with maximum degree at most $\Delta$, and let $n \geq 1$ be an integer. Let $A$ be an independent set of size $n-1$ in $G$. Then,

$$C(\text{lk}(I_n(G), A)) \leq \left\lfloor \frac{(n-1)\Delta}{2} \right\rfloor.$$ 

For the subclass of claw-free graphs, this is proved in Proposition 5.5. Conjecture 8.1 would imply the bound $f_G(n) \leq \left\lfloor \left(\frac{\Delta}{2} + 1\right)(n-1) \right\rfloor + 1$ (by the same argument as the one used to prove Theorem 1.9), settling Conjecture 1.3 in the case of even $\Delta$.

Another possible direction is to focus on the family of claw-free bounded degree graphs. We showed in Theorem 1.9 that Conjecture 1.3 holds for graphs in this family when $\Delta$ is even. In the case of odd $\Delta$, although we
obtain good upper bounds for $f_G(n)$, the question remains unsettled. It would also be interesting to prove the corresponding tight upper bound on the collapsibility number of $I_n(G)$, at least for the case of even $\Delta$.

We know, by Proposition 7.2, that Conjecture 1.4 does not hold for graphs with maximum degree at most 3. The following problem arises:

**Problem 8.2.** Find the smallest positive integer $g(n)$ such that the following holds: for every graph $G$ with maximum degree at most 3,

$$C(I_n(G)) \leq g(n).$$

By Theorem 1.6 and Proposition 7.1 we have $2(n-1) \leq g(n) \leq 3(n-1)$ for all $n \geq 1$, and, by Corollary 7.5, $g(8k) \geq 17k - 1$ for all $k \geq 1$. Improving either the upper or lower bounds for $g(n)$ may be of interest.

**Acknowledgment**

The authors thank Jinha Kim for her insightful comments during the early stages of this research.

**References**

[1] M. Adamaszek. Extremal problems related to betti numbers of flag complexes. *Discrete Appl. Math.*, 173:8–15, 2014.

[2] R. Aharoni and E. Berger. Rainbow matchings in r-partite r-graphs. *Electron. J. Combin.*, 16(1):R119, 2009.

[3] R. Aharoni, E. Berger, M. Chudnovsky, D. Howard, and P. Seymour. Large rainbow matchings in general graphs. *European J. Combin.*, 79:222 – 227, 2019.

[4] R. Aharoni, J. Briggs, J. Kim, and M. Kim. Rainbow independent sets in certain classes of graphs. preprint, [https://arxiv.org/abs/1909.13143](https://arxiv.org/abs/1909.13143), 2019.

[5] R. Aharoni, R. Holzman, and Z. Jiang. Rainbow fractional matchings. *Combinatorica*, 2019.

[6] J. Barát, A. Gyárfás, and G. N. Sárközy. Rainbow matchings in bipartite multigraphs. *Period. Math. Hungar.*, 74(1):108–111, 2017.

[7] A. Björner. Topological methods. In R. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of combinatorics*, chapter 34, pages 1819–1872. North-Holland, Amsterdam, 1994.
[8] A. Björner and M. Tancer. Note: Combinatorial Alexander duality– a short and elementary proof. Discrete Comput. Geom., 42(4):586, 2009.

[9] J. Briggs and M. Kim. Choice functions in the intersection of matroids. Electron. J. Combin., 26(4):P4.26, 2019.

[10] R. L. Brooks. On colouring the nodes of a network. Proc. Cambridge Philos. Soc., 37:194–197, 1941.

[11] G. Kalai and R. Meshulam. A topological colorful Helly theorem. Adv. Math., 191(2):305–311, 2005.

[12] I. Khmelnitsky. d-collapsibility and its applications. Master’s thesis, Technion, Haifa, 2018.

[13] C. G. Lekkerkerker and J. C. Boland. Representation of a finite graph by a set of intervals on the real line. Fund. Math., 51(1):45–64, 1962.

[14] M. Tancer. Strong d-collapsibility. Contrib. Discrete Math., 6(2):32–35, 2011.

[15] M. E. Watkins. A theorem on Tait colorings with an application to the generalized Petersen graphs. J. Combin. Theory, 6(2):152–164, 1969.

[16] G. Wegner. d-collapsing and nerves of families of convex sets. Arch. Math., 26(1):317–321, 1975.