CONSTRAINTS ON THE MAGNITUDE OF $\alpha$ IN DYNAMO THEORY

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ABSTRACT

We consider the back-reaction of the magnetic field on the magnetic dynamo coefficients and the role of boundary conditions in interpreting whether numerical evidence for suppression is dynamical. If a uniform field in a periodic box serves as the initial condition for modeling the back-reaction on the turbulent EMF, then the magnitude of the turbulent EMF, and thus the dynamo coefficient $\alpha$, have a stringent upper limit that depends on the magnetic Reynolds number $R_M$ to a power of order $-1$. This is not a dynamic suppression but results just because of the imposed boundary conditions. In contrast, when mean field gradients are allowed within the simulation region, or nonperiodic boundary conditions are used, the upper limit is independent of $R_M$ and takes its kinematic value. Thus only for simulations of the latter types could a measured suppression be the result of a dynamic back-reaction. This is fundamental for understanding a long-standing controversy surrounding $\alpha$ suppression. Numerical simulations that do not allow any field gradients and invoke periodic boundary conditions appear to show a strong $\alpha$ suppression (e.g., Cattaneo & Hughes). Simulations of accretion disks that allow field gradients and allow free boundary conditions (Brandenburg & Donner) suggest a dynamo $\alpha$ that is not suppressed by a power of $R_M$. Our results are consistent with both types of simulations.

Subject headings: MHD — magnetic fields — turbulence

1. INTRODUCTION

A leading candidate to explain the origin of large-scale magnetic fields in stars and galaxies is mean-field turbulent magnetic dynamo theory (Moffatt 1978; Parker 1979; Krause & Rädler 1980; Zeldovich, Ruzmaikin, & Sokoloff 1983; Ruzmaikin, Shukurov, & Soloff 1988; Beck et al. 1996). The theory appeals to a combination of helical turbulence (leading to the $\alpha$ effect), differential rotation (the $\Omega$ effect), and turbulent diffusion to exponentiate an initial seed mean magnetic field. The total magnetic field is split into a mean component and a fluctuating component, and the rate of growth of the mean field is sought. The mean field grows on a length scale much larger than the outer scale of the turbulent velocity, with a growth time much larger than the eddy turnover time at the outer scale. A combination of kinetic and current helicity provides a statistical correlation of small-scale loops favorable to exponential field growth. Turbulent diffusion is needed to redistribute the amplified mean field rapidly to ensure a net magnetic flux gain inside the system of interest. Rapid growth of the fluctuating field necessarily accompanies the mean-field dynamo. Its impact upon the growth of the mean field, and the impact of the mean field itself on its own growth, are controversial.

The controversy results because Lorentz forces from the growing magnetic field react back on and complicate the turbulent motions driving the field growth (e.g., Cowling 1957; Piddington 1981; Kulsrud & Anderson 1992). It is tricky to disentangle the back-reaction of the mean field from that of the fluctuating field. Analytic studies and numerical simulations seem to disagree as to the extent to which the dynamo coefficients are suppressed by the back-reaction of the mean field. Some numerical studies (e.g., Cattaneo & Vainshtein 1991; Vainshtein & Cattaneo 1992; Cattaneo 1994; Cattaneo & Hughes 1996) and analytic studies (e.g., Gruzinov & Diamond 1994; Bhattacharjee & Yuan 1995; Vainshtein 1998) argue that the suppression of $\alpha$ takes the form $\alpha \sim \alpha^{(0)}/(1 + R_M B^2/v_0^2)$, where $\alpha^{(0)}$ is the value of $\alpha$ in the absence of a mean-field, $R_M$ is the magnetic Reynolds number, $B$ is the mean field in velocity units, $v_0$ is the rms turbulent velocity, and $p$ is a number of order 1. Such a strong dependence on $R_M$ would prevent astrophysical dynamos from working, as $R_M$ is usually $\gtrsim 1$.

Other numerical studies (Brandenburg & Donner 1997) and analytic studies (e.g., Kraichnan 1979a, 1979b; Field, Blackman, & Chou 1999; Chou & Fish 1999) suggest that $p = 0$, so $\alpha \sim \alpha^{(0)}/(1 + B^2/v_0^2)$ in the fully dynamic regime. In particular, Field et al. (1999) considered an expansion in the mean magnetic field (see also Vainshtein & Kitchatinov 1983; Montgomery & Chen 1984; Blackman & Chou 1997), and were able to derive the effect of the nonlinear back-reaction on $\alpha$ in the case for which $\nabla B = 0$. Their result is expressed in terms only of the difference between the zeroth-order kinetic and current helicities. They find that $R_M$ does not enter strongly, except possibly by suppressing the difference between the zeroth-order helicities, an effect that cannot depend upon $\nabla B$ and is not therein constrained.

Blackman & Field (1999) have shown that some of the analytic approaches (e.g., Bhattacharjee & Yuan 1995; Gruzinov & Diamond 1994) that employ a use of Ohm’s law dotted with the fluctuating component of the magnetic field do not distinguish between turbulent quantities of the base (zeroth-order) state and quantities that are of higher order in the mean field. This distinction is important. When it is made, many arguments for suppression via such approaches fall through. Note that Blackman & Field (1999) do not prove that the dynamo survives back-reaction as a result of their considerations, only that some analytic approaches to the problem can be challenged.
Despite this challenge, the apparent result of extreme $\alpha$ suppression is seen in the simulation of Cattaneo & Hughes (1996). These authors externally force the turbulence, imposing periodic boundary conditions and a uniform mean field, and find that the suppression of $\alpha$ involves $R_M$, in the form given above. By contrast, the simulation of Brandenburg & Donner (1997) suggests that an $\alpha - \Omega$ dynamo may in fact be operative in an accretion disk whose turbulence is self-generated by a shearing instability, without $R_M$ entering the suppression. The latter simulation does not employ periodic boundary conditions and allows gradients in mean fields.

In this paper we show that the suppression of $\alpha$ depends crucially on the boundary conditions. We find that the mean quantities are defined by averaging over a periodic box $x$ has an upper limit that depends on a factor of $R^{-p}_M$, with $p \sim 1$. In the presence of mean-field gradients and non-periodic boundary conditions, however, we find that the upper limit on the dynamo coefficients is significantly larger, and $R_M$ is not involved. The small upper limit in the periodic box case does not represent a dynamical suppression but rather an apparent suppression that is just a result from the boundary conditions. The results herein may be a step toward resolving controversies surrounding numerical suppression experiments.

Central to the discussion is the equation for the time evolution of magnetic helicity. This equation was also employed by Seehafer (1994), who derived a suppression of $\alpha$ apparently consistent with that of Keinigs (1983) (and qualitatively consistent with the Cattaneo & Hughes [1996] simulation). The techniques of these additional two papers are different, and one should note that they do not separate zeroth from higher order quantities.

Section 2 reviews the basic formalism of the dynamo coefficient expansion in orders of $\mathbf{B}$. Section 3 shows that constraints on the magnitude of the EMF (and thus $\alpha$) results from the presence of a nonzero mean field (see Field et al. 1999). The expansion order parameter is $|\mathbf{B}|/|\mathbf{b}^{(0)}|$, which is indeed $\leq 1$ for the early dynamo evolution, and less than 1 in the Galaxy at present. In particular, we have

$$\langle v \times b \rangle = \alpha_{ij} \mathbf{B}_j - \beta_{ijk} \mathbf{B}_k + \gamma_{ijkl} \mathbf{O}(\mathbf{B}/R^2) + \ldots,$$

where $\alpha_{ij}$, $\beta_{ijk}$, and $\gamma_{ijkl}$ are explicit functions of correlations of turbulent quantities but can depend implicitly on $\mathbf{B}$ (Moffatt 1978) through their dependence on the induction equation for the fluctuating field. The order at which there is no implicit dependence on $\mathbf{B}$ is the zeroth-order base state (see Field et al. 1999). The expansion order parameter is $|\mathbf{B}|/|\mathbf{b}^{(0)}|$, which is indeed $\leq 1$ for the early dynamo evolution, and less than 1 in the Galaxy at present. In particular, we have $b = b^{(0)} + \sum_n b^{(n)}$, and similarly for $v$, where $\sum_n b^{(n)} < b^{(0)}$ and $n$ indicates the number of powers of $|\mathbf{B}|/|\mathbf{b}|$. The zeroth-order base state correlations are composed of products of $b^{(0)}$ and $v^{(0)}$ and have no dependence on the mean field. The zeroth-order base state is taken to be homogeneous and isotropic—the violation of isotropy comes from the contributions due to higher order fluctuating quantities, whose isotropy is broken by the mean field. Note that the zeroth-order state need not be reflection invariant, and it is important for dynamo theory that it is not.

Correlations between higher order quantities can be reduced to correlations of zeroth-order quantities times the respective products of $n$ linear functions of $\mathbf{B}$. Thus for example, $b^{(2)}$ is the anisotropic component of the fluctuating magnetic field that depends on two powers of $\mathbf{B}$, and is found by twice iterating terms like $b \cdot \mathbf{B}$ in the induction equation to obtain an approximate solution in terms of $b^{(0)}$ and $v^{(0)}$.

To zeroth order, the $\alpha$ tensor can be written

$$\alpha_{ij}^{(0)} = v^{(0)} \delta_{ij},$$

which highlights the isotropy of this zeroth-order quantity. In our previous work (Field et al. 1999) we have used the induction equation for the fluctuating components of the magnetic field and the Navier-Stokes equation for the fluctuating velocity to find the form of $\alpha$ in terms of correlations of the zeroth-order products (see also Blackman & Chou 1997). Calculating the turbulent EMF in the absence of
gradients of \( \mathbf{B} \), to first order in \( \mathbf{B} \), gives \( \langle \mathbf{v} \times \mathbf{b} \rangle^{(1)} = \mathbf{a}^{(0)} \mathbf{B} \), where \( \mathbf{a}^{(0)} \) is the sum of kinetic and current helicities associated with the zeroth-order state, namely,

\[
\alpha^{(0)} = -\frac{1}{3} \left[ \mathbf{v}^{(0)}(t) \cdot \nabla \times \mathbf{v}^{(0)}(t)dt' \right] \\
- \frac{1}{3} \mathbf{b}^{(0)}(t) \cdot \nabla \times \mathbf{b}^{(0)}(t)dt' \\
\approx -\frac{1}{3} t_c \left[ \langle \mathbf{v}^{(0)} \cdot \mathbf{v} \rangle^{(0)} \right. \\
- \langle \mathbf{b}^{(0)} \cdot \mathbf{v} \rangle^{(0)} \left. \right],
\]

where \( v \) is the turbulent velocity and \( t_c \) is defined as the correlation time of the scale of the turbulence, which dominates the averaged quantity. If we adopt a Kolmogorov energy spectrum (i.e., \( k b_0^2, k_v^2 \propto k^2 E_k \propto k^{1/3} \)), then it might appear that the dominant contributions to the terms of (2) come from large \( k \). However, Pouquet, Frisch, & Leorat (1976) showed that if the forcing is at the outer wavenumber \( k_0 \), most of the energy and helicity is concentrated there, and the turbulence for \( k > 3 k_0 \) is locked up into Alfvén waves that do not contribute to correlations. It is therefore likely reasonable to assume that any helicity in the zeroth-order state is concentrated near \( k_0 \), in which case \( t_c \approx (v_0 k_0)^{-1} \approx L/v_0 \).

The first term in equation (2) was first derived by Steenbeck, Krause, & Rädler (1966). The second, current helicity, term in equation (2) was first derived by Pouquet et al. (1976); neither paper made the necessary distinction between zeroth- and higher order quantities.

In the next sections we will not rederive the form of \( \alpha^{(0)} \) in terms of \( \mathbf{b}^{(0)} \) and \( \mathbf{e}^{(0)} \); instead we will derive an independent upper limit on \( \alpha^{(0)} \) from the use of Ohm’s law, the definition of the electric field in terms of the vector potential, and the equation for magnetic helicity evolution. We will invoke the assumption that the zeroth-order state is isotropic and homogeneous, and we will assume that all anisotropies and inhomogeneities of higher order correlations are due to mean fields.

We will need the Reynolds relations (Rädler 1980), i.e., that derivatives with respect to \( x \) or \( t \) obey \( \partial_{t,x} \langle X_i, X_j \rangle = \langle \partial_{t,x} X_i, X_j \rangle \) and \( \langle X_i, x_i \rangle = 0 \) where \( X_i = \mathbf{X}_i + x_i \) are components of vector functions of \( x \) and \( t \). For statistical ensemble means, these hold when correlation times are small compared to the times over which mean quantities vary. For spatial means, defined by \( \langle X_i(x,t) \rangle = V^{-1} \int X_i(x+s,t)ds \), the relations hold when the average is over a large enough \( V \) that \( L < V^{1/3} \leq R \leq D \), where \( D \) is the size of the system \( R \) is the scale of mean-field variation and \( L \) is the outer scale of the turbulence. Note that the scale of averaging is less than the overall system size.

3. CONSTRAINTS ON THE TURBULENT EMF FOR PERIODIC AND NONPERIODIC BOUNDARY CONDITIONS

3.1. Constraint Equations

Let the electric field \( \mathbf{E} \), like \( \mathbf{B} \), be divided into a mean component \( \bar{\mathbf{E}} \) and a fluctuating component \( \mathbf{e} \). Ohm’s law for the mean field is thus

\[
\mathbf{E} = -\mathbf{c}^{-1} \mathbf{v} \times \mathbf{b} + \eta \mathbf{J} = -\mathbf{c}^{-1} \mathbf{v} \times \mathbf{b} + \eta \mathbf{J} 
\]

for the case \( \bar{\mathbf{v}} = 0 \), where \( \mathbf{J} \) is the current density and \( \eta \) is the resistivity. We also have

\[
\langle \mathbf{E} \cdot \mathbf{B} \rangle = \bar{\mathbf{E}} \cdot \bar{\mathbf{B}} + \langle \mathbf{e} \cdot \mathbf{b} \rangle \\
= -c^{-1} \langle \mathbf{v} \times \mathbf{b} \rangle \cdot \bar{\mathbf{B}} + \eta \mathbf{J} \cdot \bar{\mathbf{B}} + \langle \mathbf{e} \cdot \mathbf{b} \rangle 
\]

where we have used equation (7).

A second expression for \( \langle \mathbf{E} \cdot \mathbf{B} \rangle \) also follows from Ohm’s law without first splitting into mean and fluctuating components, that is

\[
\langle \mathbf{E} \cdot \mathbf{B} \rangle = -c^{-1} \langle \mathbf{v} \times \mathbf{b} \rangle \cdot \bar{\mathbf{B}} + \eta \mathbf{J} \cdot \bar{\mathbf{B}} \\
= \eta \langle \mathbf{J} \cdot \mathbf{B} \rangle = \eta \bar{\mathbf{J}} \cdot \bar{\mathbf{B}} + \eta \mathbf{j} \cdot \mathbf{b} \\
+ c^{-1} \lambda \langle \mathbf{b} \cdot \nabla \times \mathbf{b} \rangle. 
\]

By substituting equation 9 into equation 8, we obtain

\[
c^{-1} \langle \mathbf{v} \times \mathbf{b} \rangle \cdot \mathbf{B} = -c^{-1} \lambda \langle \mathbf{b} \cdot \nabla \times \mathbf{b} \rangle + \langle \mathbf{e} \cdot \mathbf{b} \rangle, 
\]

an equation that will now constrain \( \langle \mathbf{v} \times \mathbf{b} \rangle \). However, we must expand equation (10) to second order in \( \mathbf{B} \) (as defined in § 2) to constrain the turbulent EMF \( \langle \mathbf{v} \times \mathbf{b} \rangle \). This is because to zeroth order, the left-hand side of equation (10) vanishes directly. To first order, the left side would be \( \langle \mathbf{v} \times \mathbf{b} \rangle^{(0)} \mathbf{B} \), but \( \langle \mathbf{v} \times \mathbf{b} \rangle^{(0)} = 0 \), since vector averages of zeroth-order quantities vanish. To second order in \( \mathbf{B} \) then, equation (10) implies that

\[
c^{-1} \langle \mathbf{v} \times \mathbf{b} \rangle^{(1)} \cdot \mathbf{B} = -c^{-1} \langle \mathbf{b} \cdot \nabla \times \mathbf{b} \rangle^{(2)} + \langle \mathbf{e} \cdot \mathbf{b} \rangle^{(2)}. 
\]

Because \( \mathbf{R} \gg 1 \), significant limits on \( \langle \mathbf{v} \times \mathbf{b} \rangle^{(1)} \) and thus on \( \alpha^{(0)} \) come from the \( \langle \mathbf{e} \cdot \mathbf{b} \rangle^{(2)} \) term above. The result of Seehafer (1994) and Keinig (1983) amount to equation (11) with the last term equal zero, but without distinguishing the order in mean fields (i.e., without the superscripts). We now focus on this last, term keeping in mind that it is second order in mean fields.

Since \( \langle \mathbf{e} \cdot \mathbf{b} \rangle^{(2)} \) is second order in \( \mathbf{B} \), its simplest general form will be expressible as a sum of terms that each involve products of two types of quantities: (1) correlations of scalar or pseudoscalar products of zeroth-order quantities, and (2) quadratic scalar or pseudoscalar products of \( \mathbf{B} \). Now \( \langle \mathbf{e} \cdot \mathbf{b} \rangle \) can be written as a sum a time derivative and spatial divergence. Consider \( e \) in terms of the vector and scalar potentials \( \mathbf{a} \) and \( \phi \):

\[
e = -\nabla \phi - (1/c) \partial_t \mathbf{a} . 
\]

Dotting with \( \mathbf{b} = \mathbf{V} \times \mathbf{a} \) and averaging we have

\[
\langle \mathbf{e} \cdot \mathbf{b} \rangle = -\langle \nabla \phi \rangle \cdot \mathbf{b} - \langle 1/c \rangle \langle \partial_t \mathbf{a} \rangle .
\]

After straightforward algebraic manipulation, application of Reynolds rules and \( \nabla \cdot \mathbf{b} = 0 \), this equation implies

\[
\langle \mathbf{e} \cdot \mathbf{b} \rangle = -1/2 \mathbf{V} \cdot \langle \phi \mathbf{b} \rangle + 1/2 \mathbf{V} \cdot \langle \mathbf{a} \times \mathbf{e} \rangle \\
- 1/2 \mathbf{c} \partial_t \langle \mathbf{a} \cdot \mathbf{e} \rangle \equiv -\bar{\mathbf{c}}_0 \mathbf{R}^0 + \bar{\mathbf{c}}_t \mathbf{R}^t = \bar{\mathbf{c}}_0 \mathbf{R}^0 .
\]

where we have defined a helicity density 4-vector for fluctuating quantities

\[
[\mathbf{h}_0, h] = [(1/2c)\mathbf{a} \cdot \mathbf{b}, (1/2) \langle \phi \mathbf{b} \rangle - \langle 1/2 \rangle (\mathbf{a} \times \mathbf{e})] ,
\]

and the overbar is used, as always, to mean the same thing as the brackets.
3.2. Constraints for Periodic Boundary Conditions

We now investigate the implications of equation (14) for simulations of type performed by Cattaneo & Hughes (1996), where the brackets are interpreted as a spatial average over a periodic box. Under these conditions, there are two important consequences. First, note that the second two terms of equation (14) vanish upon conversion to surface integrals and we have

$$\langle e \cdot b \rangle = -(1/2c)\bar{e} \langle a \cdot b \rangle,$$

which is gauge invariant. The second consequence of the periodic box is that $\partial_t \bar{B} = 0$ for incompressible flows. This follows simply from equation (2): the last three terms of equation (2) would vanish, as they are all surface integrals. Using Reynolds rules and vector identities, the second term can be written $[\nabla \times (b \times b)] = \langle \partial_t (b \cdot v) \rangle - \langle \partial_t (v \cdot b) \rangle$, which also vanishes by surface integration.

The two consequences just discussed can be used to show that equation (16) vanishes for a periodic box, and thus the only contribution to the right-hand side of equation (11) will come from the first term on the right. To second order in mean quantities, assuming $b(t = 0) = 0$ and that all times are far enough from $t = 0$ such that $b(t)$ does not correlate with any finite $a(0)$, we have

$$\langle a \cdot b \rangle^{(2)} = \int \partial_t \langle a^2 \rangle(t') \cdot b^0(t') dt' + \int \partial_t \langle a^1(t') \cdot \partial_t b^1(t') \rangle dt' dt'' + \int \langle a^0(t) \cdot \partial_t b^2(t') \rangle dt'. \quad (17)$$

To express equation (17) explicitly in terms of mean fields, we use the equations of motion for $b$ and $a$. The use of $\partial_t \bar{B}$ from equation (3) for the last two terms of (17) leads directly to contributions depending on products of the mean fields $\bar{B}$ or $\bar{V}$ and turbulent quantities $b$ and $v$. Consider now the equation for $a$, which comes from uncurling the equation for $b$, namely,

$$\partial_t a = (v \times b) - (v \times b) + (v \times b) + (V \times b) + \nabla \theta,$$

where $\theta$ is an arbitrary scalar field. When equation (18) is used in equation (17) in the first and second terms on the right-hand side of equation (17), the periodic box nullifies the contribution from $\nabla \theta$. All other contributions depend only on products of $v$, $b$, $\bar{B}$, and $\bar{V}$. Thus when $\bar{V} = 0$, the only remaining mean field is $\bar{B}$. Thus for a periodic box, $\langle a \cdot b \rangle^{(2)}$ must be second order in $\bar{B}$. Then, when plugged into equation (16), the time derivative will act on some quadratic function of $\bar{B}$ multiplied by correlations of zeroth order. Since the zeroth-order quantities are time-independent, isotropic, and homogeneous, the function of $\bar{B}$ must be a scalar, denoted $F$, and we have

$$\partial_t \langle a \cdot b \rangle^{(2)} = \partial_t (F(\bar{B})^2) \bar{Q}_1 = \bar{Q}_1 \partial_t (F(\bar{B})^2) = 0,$$

where $\bar{Q}_1$ is a scalar or pseudoscalar correlation of zeroth-order quantities. The last equality of (19) follows from stationarity of zeroth-order quantities, and our proof that $\bar{B}$ is time-independent over the timescales of interest for the periodic box. We therefore conclude that $\partial_t \langle a \cdot b \rangle = 0$ in equation (16). This result relates to the fact that for a periodic box, there is no periodic mean vector field $\bar{A}$ whose curl is everywhere equal to $\bar{B}$. The divergence of $\bar{B}$ is still equal to zero, so Maxwell’s equations are satisfied, but $\bar{B}$ is the only nontrivial mean field.

Since in the Cattaneo & Hughes (1996) simulation $\bar{B}$ = constant in both space and time, $\langle v \times b \rangle^{(2)} = \bar{b}^0 \nabla \theta$. Using this, and equations (19), (16) and (11), we obtain

$$\bar{a}^0 = \lambda \bar{b} \cdot \nabla \times \bar{b}^{(2)} / \bar{B}^2.$$ \hspace{1cm} (20)

Field et al. (1999) showed that conclusions about $\bar{a}^0$ are also conclusions about $a$ to all orders, by relating the fully nonlinear $a$ to $\bar{a}^0$ and showing that in the limit of large $\bar{B}$, $a$ is not catastrophically affected when $\bar{B}$ is large. Thus $\bar{a}^0$ is an upper limit to $a$, and so the result (eq. [20]) shows that $a$ will be small when the brackets indicate an average over a periodic box. The important point is that this is not a dynamical suppression from the back-reaction but a constraint on the magnitude of $\bar{a}^0$ that is imposed by the boundary conditions. Notice that it is a constraint on the zeroth-order quantity, and so it cannot represent the effect of back-reaction.

3.3. Constraints for Nonperiodic Boundary Conditions

If the averaging brackets are not over a periodic box, or if the scale of the averaging is $< \bar{L}$ than the overall scale size of the system, then the divergence terms in equation (14) do not vanish. In addition, the $\nabla \phi \cdot \bar{V}$ term in equation (18) will contribute to equation (16). In this case, each term on the right-hand side of equation (14) is not gauge invariant. Thus, the only constraint we can make on the magnitude of the right-hand side of (14) is on the sum of all the terms together. Writing down all possible second order terms up to one spatial derivative in $\bar{B}$, we have

$$c \langle e \cdot b \rangle^{(2)} = \bar{Q}_2 \bar{B}^2 + \bar{Q}_3 \bar{B} \cdot \bar{V} \times \bar{B} + O(\bar{B}^2 L^2/R^2) + \ldots,$$

where $\bar{Q}_2$ and $\bar{Q}_3$ are correlations of zeroth-order averages, $L$ is the outer turbulent scale and $R$ is the mean-field variation scale. The quantity $\bar{Q}_2$ must have units of velocity, and thus be of maximum order $v_0$, since it depends only on turbulent quantities. The quantity $\bar{Q}_3$ must have dimensions of viscosity, and must of maximum $\sim v_0 L$, since it too depends only on turbulent quantities. The combination of terms in equation (21) is the same form of the combination of terms entering on the left-hand side of equation (10) which would result from using equation (4). That is, since $\langle v \times b \rangle \cdot \bar{B} = \bar{a}^0 \bar{B}^2 - \bar{b}^0 \bar{B} \cdot \bar{V} \times \bar{B}$, we have $\bar{Q}_2^0 = \bar{a}^0$ and $\bar{Q}_3^0 = -\bar{b}^0$. Thus for simulations in which the mean values are not taken over a periodic box, there is certainly no a priori restriction on $\bar{a}^0$. Since now $\alpha \leq \bar{a}^0$, any simulation result indicating suppression on $a$ under these relaxed boundary conditions would indeed be a dynamical suppression. So far there are no simulations that invoke such boundary conditions that show catastrophic suppression (see Brandenburg & Donner 1997).

4. DISCUSSION

Section 3 shows that periodic boundary conditions impose an upper limit on $a$ that does not represent a dynamical suppression. Nonperiodic boundary conditions or a finite scale separation between system size and mean-field gradient scale allow for a much higher upper limit on $a$, namely the kinematic limit. The dynamical back-reaction is testable only in simulations of the latter type.
4.1. Relation to Magnetic Helicity

Here we point out a connection to magnetic helicity. Repeating equations (12), (13) and (14) for the total $E$ and $B$ gives

$$
\langle E \cdot B \rangle = E \cdot B + \langle e \cdot b \rangle = \frac{1}{2} \partial_\mu H^\mu = \frac{1}{2} \partial_\mu \tilde{H}^\mu + \frac{1}{2} \partial_\mu \tilde{H}^\mu \approx 0,
$$

(22)

where $H^\mu$ is the total magnetic helicity 4-vector (Field 1986) defined exactly as in equation (15), but with all fluctuating quantities replaced by their total values. Similarly, $\tilde{h}^\mu$ is the helicity 4-vector associated with the mean fields. The last similarity in equation (22) follows because $R_M > 1 (\lambda \sim 0)$ in the astrophysical plasmas of interest. Using $\langle e \cdot b \rangle = \partial_\mu \tilde{h}^\mu$, equation (22) then shows that any nonnegligible $\langle e \cdot b \rangle$ requires a finite but opposite $\partial_\mu \tilde{h}^\mu$.

In general, for nonvanishing turbulent EMF, the $\langle e \cdot b \rangle$ must be nonzero, and thus the 4-divergences in equation (22) cannot vanish separately. Interestingly, when equation (22) is integrated over the total volume inside and outside of the rotator, and interpreted in terms of a flow of relative magnetic helicity (Berger & Field 1984), it can be shown that a working dynamo implies an associated magnetic energy flow through the magnetic rotator of interest, which likely leads to an active corona (Field & Blackman 2000; Blackman & Field 2000).

4.2. Implications for Previous and Future Studies

The use of periodic boundary conditions in simulations appears to be unsuitable for testing the suppression of $\alpha$ in a real dynamo unless the scale of mean-field variations is much smaller than the scale of the periodicity. If periodic boundary conditions are used, one must also be careful about causality issues. The scale separation should at minimum be large enough such that the Alfvén crossing time across the box is longer than correlation time of the fluctuating quantities, and possibly even longer than the timescale for mean-field variation. Thus, the box could be periodic, but the dynamics of interest would occur in a nonperiodic subregion. The brackets that we have used to indicate averages, would thus represent averages over this subregion, not the entire volume. Alternatively, the box could be nonperiodic.

The numerical simulations of Cattaneo & Hughes (1996) do not allow for any mean-field gradients and employ periodic boundary conditions. The strong $\alpha$ reduction seen there is consistent with our suggestion that the suppression may not be dynamical, but may instead be a result of the boundary conditions. In contrast, the shearing box accretion disk simulations of Brandenburg & Donner (1997) do employ nonperiodic boundary conditions and allow mean-field gradients. Interestingly, they do find that something like a mean-field dynamo is operating therein. The limited suppression that they find does not involve $R_M$.

5. Conclusions

We have suggested that the cause for apparent $\alpha$ suppression in numerical simulations that use periodic boundary conditions may not result from dynamics but rather from a choice of boundary conditions. If the boundary conditions enforce all mean-field gradients and spatial divergences to vanish, then the upper limit on $\alpha$ is given by equation (20). For nonperiodic boundary conditions or a box with significant scale separation between the mean field and box size, the upper limit on the turbulent EMF is given by the kinematic value. This would be a consistent interpretation of the large suppression reported by Cattaneo & Hughes (1996). In contrast, Brandenburg & Donner (1997) interpret disk simulations that use nonperiodic boundary conditions and do not find such a strong suppression. In summary, our results are consistent with seemingly contradictory simulations.

Working dynamos in real astrophysical bodies (even in the kinematic approximation) require mean-field gradients and scale separations between the overall system scale, mean-field averaging scale, and fluctuating scale. In order to disentangle boundary effects from dynamical ones, future simulations of $\alpha$ suppression should include nonperiodic boundary conditions or allow the mean field to change over scales smaller than the size of the overall box. This is a challenging task.

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