FLOATING BODY, ILLUMINATION BODY, 
AND POLYTOPAL APPROXIMATION

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Abstract. Let $K$ be a convex body in $\mathbb{R}^d$ and $K_t$ its floating bodies. There is a polytope with at most $n$ vertices that satisfies

$$K_t \subset P_n \subset K$$

where

$$n \leq e^{16d \frac{\text{vol}_d(K \setminus K_t)}{t \text{vol}_d(B_{2}^d)}}$$

Let $K^t$ be the illumination bodies of $K$ and $Q_n$ a polytope that contains $K$ and has at most $n$ $d-1$-dimensional faces. Then

$$\text{vol}_d(K^t \setminus K) \leq cd^4 \text{vol}_d(Q_n \setminus K)$$

where

$$n \leq \frac{c}{dt} \text{vol}_d(K^t \setminus K)$$

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1. Introduction

We investigate the approximation of a convex body $K$ in $\mathbb{R}^d$ by a polytope. We measure the approximation by the symmetric difference metric. The symmetric difference metric between two convex bodies $K$ and $C$ is

$$d_S(C, K) = \text{vol}_d((C \setminus K) \cup (K \setminus C))$$

We study in particular two questions: How well can a convex body $K$ be approximated by a polytope $P_n$ that is contained in $K$ and has at most $n$ vertices and how well can $K$ be approximated by a polytope $Q_n$ that contains $K$ and has at most $n$ $d-1$-dimensional faces. Macbeath [Mac] showed that the Euclidean Ball $B_d^2$ is an extremal case: The approximation for any other convex body is better. We have for the Euclidean ball

$$c_1 d^{d-1} \text{vol}_d(B_d^2)n^{-\frac{d-1}{2}} \leq d_S(P_n, B_d^2) \leq c_2 d^{d-1} \text{vol}_d(B_d^2)n^{-\frac{d-1}{2}}$$

provided that $n \geq (c_3 d^{d-1})^{-2}$. The right hand inequality was first established by Bronshtein and Ivanov [BI] and Dudley [D1,D2]. Gordon, Meyer, and Reisner [GMR1,GMR2] gave a constructive proof for the same inequality. Müller [Mü] showed that random approximation gives the same estimate. Gordon, Reisner, and Schütt [GRS] established the left hand inequality. Gruber [Gr2] obtained an asymptotic formula. If a convex body $K$ in $\mathbb{R}^d$ has a $C^2$-boundary with everywhere positive curvature, then

$$\inf\{d_S(K, P_n) \mid P_n \subset K \text{ and } P_n \text{ has at most } n \text{ vertices}\}$$

is asymptotically the same as

$$\frac{1}{2}\text{del}_{d-1}\left(\int_{\partial K} \kappa(x) \frac{1}{\pi} d\mu(x)\right)^{\frac{d+1}{d}} \left(\frac{1}{n}\right)^{-\frac{d-1}{2}}$$

where $\text{del}_{d-1}$ is a constant that is connected with Delone triangulations. In this paper we are not concerned with asymptotic estimates, but with uniform.

Int($M$) denotes the interior of a set $M$. $H(x, \xi)$ denotes the hyperplane that contains $x$ and is orthogonal to $\xi$. $H^+(x, \xi)$ denotes the halfspace that contains the vector $x - \xi$, and $H^-(x, \xi)$ the halfspace containing $x + \xi$. $e_i, i = 1, \ldots, d$ denotes the unit vector basis in $\mathbb{R}^d$. $[A, B]$ is the convex hull of the sets $A$ and $B$. The convex floating body $K_t$ of a convex body $K$ is the intersection of all halfspaces whose defining hyperplanes cut off a set of volume $t$ from $K$.

The illumination body $K^t$ of a convex body $K$ is [W]

$$\{x \in \mathbb{R}^d \mid \text{vol}_d([x, K] \setminus K) \leq t\}$$

$K^t$ is a convex body. It is enough to show this for polytopes. Let $F_i$ denote the faces of a polytope $P$, $\xi_i$ the outer normal and $x_i$ an element of $F_i$. Then we have

$$\text{vol}_d([x, P] \setminus P) = \frac{1}{d} \sum_{i=1}^n \max\{0, <\xi_i, x - x_i>\} \text{vol}_{d-1}(F_i)$$

The right hand side is a convex function.
2. The Floating Body

**Theorem 2.1.** Let $K$ be a convex body in $\mathbb{R}^d$. Then we have for every $t$, $0 \leq t \leq \frac{1}{4} e^{-4} \text{vol}_d(K)$, that there are $n \in \mathbb{N}$ with

$$n \leq e^{16d} \frac{\text{vol}_d(K \setminus K_t)}{t \text{vol}_d(B^d_2)}$$

and a polytope $P_n$ that has $n$ vertices and such that

$$K_t \subset P_n \subset K$$

We want to see what kind of asymptotic estimate we get for bodies with smooth boundary from Theorem 1. We have [SW]

$$\text{vol}_d(K \setminus K_t) \sim \frac{d + 1}{2} \left( \frac{1}{\text{vol}_{d-1}(B^d_2)} \right)^{\frac{d^2}{d+1}} \int_{\partial K} \kappa(x)^{\frac{d}{d+1}} d\mu(x)$$

$$\sim t^{\frac{d}{d+1}} d \int_{\partial K} \kappa(x)^{\frac{d}{d+1}} d\mu(x)$$

Since

$$n \sim d^{\frac{d}{t}} \frac{1}{t} \text{vol}_d(K \setminus K_t)$$

we get

$$\text{vol}_d(K \setminus K_t) \sim d \left( \frac{d + 1}{n} \text{vol}_d(K \setminus K_t) \right)^{\frac{d^2}{d+1}} \int_{\partial K} \kappa(x)^{\frac{d}{d+1}} d\mu(x)$$

$$\text{vol}_d(K \setminus K_t)^{\frac{d}{d+1}} \sim d^2 n^{-\frac{d}{d+1}} \int_{\partial K} \kappa(x)^{\frac{d}{d+1}} d\mu(x)$$

Thus we get

$$\text{vol}_d(K \setminus P_n) \leq \text{vol}_d(K \setminus K_t) \sim d^2 n^{-\frac{d}{d+1}} \left( \int_{\partial K} \kappa(x)^{\frac{d}{d+1}} d\mu(x) \right)^{\frac{d^2}{d+1}}$$

In case that $K$ is the Euclidean ball we get

$$\text{vol}_d(B^d_2 \setminus P_n) \leq c d^2 n^{-\frac{d^2}{d+1}} \text{vol}_d(B^d_2)$$

where $c$ is an absolute constant. If one compares this to the optimal result (1.1) one sees that there is an additional factor $d$.

The volume difference $\text{vol}_d(P) - \text{vol}_d(P_t)$ for a polytope $P$ is of a much smaller order than for a convex body with smooth boundary. In fact, we have [S] that it is of the order $t |\ln t|^{d-1}$. In [S] this has been used to get estimates for approximation of convex bodies by polytopes.

The same result as in Theorem 2.1 holds if we fix the number of $(d-1)$-dimensional faces instead of the number of vertices. This follows from the economic cap covering for floating bodies [BL, Theorem 6]. The constants are not as good as in Theorem 2.1.

The following lemmata are not new. They have usually been formulated for symmetric, convex bodies [B,H,MP].
Lemma 2.2. Let $K$ be a convex body in $\mathbb{R}^d$ and let $H(cg(K), \xi)$ be the hyperplane passing through the center of gravity $cg(K)$ of $K$ and being orthogonal to $\xi$. Then we have for all $\xi \in \partial B^d_2$

(i)

$$(1 - \frac{1}{d+1})^d \text{vol}_d(K) \leq \text{vol}_d(K \cap H^+(cg(K), \xi)) \leq (1 - \frac{1}{d+1})^d \text{vol}_d(K)$$

(ii) for all hyperplanes $H$ in $\mathbb{R}^d$ that are parallel to $H(cg(K), \xi)$

$$(1 - \frac{1}{d+1})^{d-1} \text{vol}_{d-1}(K \cap H) \leq \text{vol}_{d-1}(K \cap H(cg(K), \xi))$$

The sequence $(1 - \frac{1}{d+1})^d$, $d = 2, 3, \ldots$ is monotonely decreasing. Indeed, by Bernoulli’s inequality we have $1 - \frac{1}{d} \leq (1 - \frac{1}{d+1})^d$, or $\frac{d}{d+1} \leq \frac{d}{d+1} \frac{d}{d+1}$. Therefore we get $(\frac{d}{d+1})^d \leq (\frac{d}{d+1})^{d-1}$, which implies $(1 - \frac{1}{d+1})^d \leq (1 - \frac{1}{d})^{d-1}$.

Therefore we get for the inequalities (i)

$$(2.1) \quad \frac{1}{e} \text{vol}_d(K) \leq \text{vol}_d(K \cap H^+(cg(K), \xi)) \leq (1 - \frac{1}{e}) \text{vol}_d(K)$$

By the above $(1 + \frac{1}{d})^d$ is a monotonely increasing sequence. Thus we get $(1 + \frac{1}{d})^d < e$. For (ii) we get

$$(2.2) \quad \text{vol}_{d-1}(K \cap H) \leq e \text{vol}_{d-1}(K \cap H(cg(K), \xi))$$

Proof. (i) We can reduce the inequality to the case that $K$ is a cone with a Euclidean ball of dimension $d - 1$ as base. To see this we perform a Schwarz symmetrization parallel to $H(cg(K), \xi)$ and denote the symmetrized body by $S(K)$. The Schwarz symmetrization replaces a section parallel to $H(cg(K), \xi)$ by a $d - 1$-dimensional Euclidean sphere of the same $d - 1$-dimensional volume. This does not change the volume of $K$ and $K \cap H^+(cg(K), \xi)$ and the center of gravity $cg(K)$ is still an element of $H(cg(K), \xi)$. Now we consider the cone

$$[z, S(K) \cap H(cg(K), \xi)]$$

such that

$$\text{vol}_d([z, S(K) \cap H(cg(K), \xi)]) = \text{vol}_d(K \cap H^-(cg(K), \xi))$$

and such that $z$ is an element of the axis of symmetry of $S(K)$ and of $H^-(cg(K), \xi)$. See figure 2.1.

$$\tilde{K} = (K \cap H^+(cg(K), \xi)) \cup [z, S(K) \cap H(cg(K), \xi)]$$

is a convex set such that $\text{vol}_d(K) = \text{vol}_d(\tilde{K})$ and such that the center of gravity $cg(\tilde{K})$ of $\tilde{K}$ is contained in $[z, S(K) \cap H(cg(K), \xi)]$. Thus

$$\text{vol}_d(\tilde{K} \cap H^+(cg(\tilde{K}), \xi)) > \text{vol}_d(\tilde{K} \cap H^+(cg(K), \xi)) = \text{vol}_d(K \cap H^+(cg(K), \xi))$$
We apply a similar argument to the set $S(K) \cap H^+(cg(K), \xi)$ and show that we may assume that $S(K)$ is a cone with $z$ as its vertex. Thus we may assume that

$$K = [(0, \ldots, 0, 1), \{(x_1, \ldots, x_{d-1}, 0) | \sum_{i=1}^{d-1} |x_i|^2 \leq 1\}] \text{ and } \xi = (0, \ldots, 0, 1)$$

Then

$$\text{vol}_d(K) = \frac{1}{d} \text{vol}_{d-1}(B_2^{d-1})$$

and

$$\frac{1}{\text{vol}_d(K)} \int_K x_d dx_d = d \int_0^1 t(1 - t)^{d-1} dt = d \int_0^1 (1 - s)^{d-1} ds = \frac{1}{d+1}$$

We obtain that

$$\text{vol}_d(K \cap H^-(cg(K), (0, \ldots, 0, 1)) = (1 - \frac{1}{d+1})^d \text{vol}_d(K)$$

(ii) Let $H$ be a hyperplane that is parallel to $H(cg(K), \xi)$ and such that $\text{vol}_{d-1}(K \cap H) > \text{vol}_{d-1}(K \cap H(cg(K), \xi))$. Otherwise there is nothing to prove. We apply a Schwarz symmetrization parallel to $H(cg(K), \xi)$ to $K$. The symmetrized body is denoted by $S(K)$. Let $z$ be the element of the axis of symmetry of $S(K)$ such that

$$[z, S(K) \cap H] \cap H(cg(K), \xi) = S(K) \cap H(cg(K), \xi)$$

Since $\text{vol}_{d-1}(K \cap H) > \text{vol}_{d-1}(K \cap H(cg(K), \xi))$ there is such a $z$. We may assume that $H^+(cg(K), \xi)$ is the half space containing $z$. Then we have

$$[z, S(K) \cap H] \cap H^-(cg(K), \xi) \subset S(K) \cap H^-(cg(K), \xi)$$

$$[z, S(K) \cap H] \cap H^+(cg(K), \xi) \supset S(K) \cap H^+(cg(K), \xi)$$

Therefore we have that

$$cg([z, S(K) \cap H]) \in H^+(cg(K), \xi)$$

Therefore, if $h_{cg}$ denotes the distance of $z$ to $H(cg(K), \xi)$ and $h$ the distance of $z$ to $H$, we get as in the proof of (i) that

$$h_{cg} \geq h(1 - \frac{1}{d+1})$$

Thus we get

$$\text{vol}_{d-1}(K \cap H(cg(K), \xi)) = \text{vol}_{d-1}(S(K) \cap H(cg(K), \xi)) \geq (1 - \frac{1}{d+1})^{d-1} \text{vol}_{d-1}(K \cap H)$$

$\square$
Lemma 2.3. Let \( K \) be a convex body in \( \mathbb{R}^d \) and let \( \Theta(\xi) \) be the infimum of all numbers \( t \), \( 0 < t \), such that

\[
\text{vol}_{d-1}(K \cap H(\text{cg}(K), \xi)) \geq e \text{vol}_{d-1}(K \cap H(\text{cg}(K) + t\xi, \xi))
\]

Then we have

\[
\frac{1}{2e^3} \text{vol}_d(K) \leq \Theta(\xi) \text{vol}_{d-1}(K \cap H(\text{cg}(K), \xi)) \leq e \text{vol}_d(K)
\]

Proof. The right hand inequality follows from Fubini’s theorem and Brunn-Minkowski’s theorem. Now we verify the left hand inequality. We consider first the case that we have for \( t \), \( t > \Theta(\xi) \),

\[
K \cap H(\text{cg}(K) + t\xi, \xi) = \emptyset
\]

Then we have by (2.1) and (2.2)

\[
\frac{1}{e^3} \text{vol}_d(K) \leq \text{vol}_d(K \cap H^+(\text{cg}(K), \xi))
\]

\[
= \int_0^\Theta(\xi) \text{vol}_{d-1}(K \cap H(\text{cg}(K) + t\xi, \xi)) dt \leq e \Theta(\xi) \text{vol}_{d-1}(H(\text{cg}(K), \xi))
\]

If for some \( t \), \( t > \Theta(\xi) \), we have \( K \cap H(\text{cg}(K) + t\xi, \xi) \neq \emptyset \) then we have

\[
\text{vol}_{d-1}(K \cap H(\text{cg}(K), \xi)) = e \text{vol}_{d-1}(K \cap H(\text{cg}(K) + \Theta(\xi)\xi, \xi))
\]

We perform a Schwarz symmetrization parallel to \( H(\text{cg}(K), \xi) \). We consider the cone

\[
[z, S(K) \cap H(\text{cg}(K), \xi)]
\]

such that \( z \) is an element of the axis of symmetry of \( S(K) \) and such that

\[
[z, S(K) \cap H(\text{cg}(K), \xi)] \cap H(\text{cg}(K) + \Theta(\xi)\xi, \xi) = S(K) \cap H(\text{cg}(K) + \Theta(\xi)\xi, \xi)
\]

Let \( H^+(\text{cg}(K), \xi) \) and \( H^+(\text{cg}(K) + \Theta(\xi)\xi, \xi) \) be the half spaces that contain \( z \). Then we get by convexity

\[
[z, S(K) \cap H(\text{cg}(K), \xi)] \cap H^+(\text{cg}(K) + \Theta(\xi)\xi, \xi) \supset S(K) \cap H^+(\text{cg}(K) + \Theta(\xi)\xi, \xi)
\]

(2.3)

We get by (2.1)

\[
\frac{1}{e^3} \text{vol}_d(K) \leq \text{vol}_d(K \cap H^+(\text{cg}(K), \xi)) = 
\]
By the hypothesis of the lemma we have for all \( s \) with \( 0 \leq s \leq \Theta(\xi) \)

\[
vol_{d-1}(K \cap H(cg(K), \xi)) \leq e \ vol_{d-1}(K \cap H(cg(K) + s\xi, \xi))
\]

Using this and (2.2) we estimate the first summand. The second summand is estimated by using (2.3). Thus the above expression is not greater than

\[
e^2 \ vol_d([z, S(K) \cap H(cg(K), \xi)] \cap H^-(cg(K) + \Theta(\xi)\xi, \xi)) + \\
vol_d([z, S(K) \cap H(cg(K), \xi)] \cap H^+(cg(K) + \Theta(\xi)\xi, \xi))
\]

By an elementary computation for the volume of a cone we get that the latter expression is smaller than

\[
2e^2 vol_d([z, S(K) \cap H(cg(K), \xi)] \cap H^-(cg(K) + \Theta(\xi)\xi, \xi))
\]

We use (2.2) again and get that the above expression is smaller than

\[
2e^3 \Theta(\xi) vol_{d-1}(K \cap H(cg(K), \xi))
\]

\( \square \)

**Lemma 2.4.** Let \( K \) be a convex body in \( \mathbb{R}^d \). Then there is a linear transform \( T \) with \( \det(T) = 1 \) so that we have for all \( \xi \in \partial B^d_2 \)

\[
\int_{T(K)} | < x, \xi > |^2 dx = \frac{1}{d} \int_{T(K)} \sum_{i=1}^d | < x, e_i > |^2 dx
\]

We say that a convex body is in an isotropic position if the linear transform \( T \) in Lemma 2.4 can be chosen to be the identity. See [B, H].

**Proof.** We claim that there is a orthogonal transform \( U \) such that we have for all \( i, j = 1, \ldots, d \) with \( i \neq j \),

\[
\int_{U(K)} < x, e_i > < x, e_j > dx = 0
\]

Clearly, the matrix

\[
(\int_K < x, e_i > < x, e_j > dx)_{i,j=1}^d
\]

is symmetric. Therefore there is an orthogonal \( d \times d \)-matrix \( U \) so that

\[
U(\int_K < x, e_i > < x, e_j > dx)_{i,j=1}^d U^t
\]
is a diagonal matrix. We have

\[ U \left( \int_K \langle x, e_i \rangle \langle x, e_j \rangle \, dx \right)_{i,j=1}^d U^t = \left( \int_K \sum_{i,j=1}^d u_{i,i} \langle x, e_i \rangle \langle x, e_j \rangle \, dx \right)_{i,k=1}^d \]

\[ = \left( \int_K \langle x, U^t(\langle x, U^t(\langle x, u_{k,k} \rangle \rangle d)_{i,k=1}^d = \left( \int_{U(K)} \langle y, e_i \rangle \langle y, e_j \rangle \, dy \right)_{i,i=1}^d \]

So the latter matrix is a diagonal matrix. All the diagonal elements are strictly positive. This argument is repeated with a diagonal matrix so that the diagonal elements turn out to be equal. Therefore there is a matrix \( T \) with \( \det T = 1 \) such that

\[ \int_{T(K)} \langle x, e_i \rangle \langle x, e_j \rangle \, dx = \left\{ \begin{array}{ll} 0 & \text{if } i \neq j \\ \frac{1}{d} \int_{T(K)} \sum_{j=1}^d |\langle x, e_j \rangle|^2 \, dx & \text{if } i = j \end{array} \right. \]

From this the lemma follows.

\[ \square \]

**Lemma 2.5.** Let \( K \) be a convex body in \( \mathbb{R}^d \) that is in an isotropic position and whose center of gravity is at the origin. Then we have for all \( \xi \in \partial B_2^d \)

\[ \frac{1}{24 e^{10}} \ vol_d(K)^3 \leq \ vol_{d-1}(K \cap H(\text{cg}(K), \xi)) \leq \frac{1}{6} e^3 \ vol_d(K)^3 \]

**Proof.** By Lemma 2.4 we have for all \( \xi \in \partial B_2^d \)

\[ \frac{1}{d} \int_K \sum_{i=1}^d |\langle x, e_i \rangle|^2 \, dx = \int_K |\langle x, \xi \rangle|^2 \, dx \]

By Fubini’s theorem we get that this equals

\[ \int_{-\infty}^{\infty} t^2 \ vol_{d-1}(K \cap H(t\xi, \xi)) \, dt \geq \int_0^{\Theta(\xi)} t^2 \ vol_{d-1}(K \cap H(t\xi, \xi)) \, dt \]

where \( \Theta(\xi) \) is as defined in Lemma 2.3. By the definition of \( \Theta(\xi) \) the above expression is greater than

\[ \frac{1}{e} \ vol_{d-1}(K \cap H(\text{cg}(K), \xi)) \int_0^{\Theta(\xi)} t^2 \, dt \geq \frac{1}{3e} \Theta(\xi)^3 \ vol_{d-1}(K \cap H(\text{cg}(K), \xi)) \]

By Lemma 2.3 this is greater than

\[ \frac{1}{24 e^{10}} \ vol_d(K)^3 \]
Now we show the right hand inequality. By Lemma 2.4 we have

\[ \frac{1}{d} \int_{K} \sum_{i=1}^{d} |< x, e_i >|^2 \, dx = \int_{K} |< x, \xi >|^2 \, dx = \int_{-\infty}^{\infty} t^2 \, vol_{d-1}(K \cap H(t\xi, \xi)) \, dt = \]

\[ \int_{\Theta(\xi)}^{d} t^2 \, vol_{d-1}(K \cap H(t\xi, \xi)) \, dt + \int_{\Theta(-\xi)}^{0} t^2 \, vol_{d-1}(K \cap H(t\xi, \xi)) \, dt + \]

\[ \int_{\Theta(-\xi)}^{0} t^2 \, vol_{d-1}(K \cap H(t\xi, \xi)) \, dt + \int_{-\infty}^{\Theta(-\xi)} t^2 \, vol_{d-1}(K \cap H(t\xi, \xi)) \, dt \]

By (2.2) this is not greater than

\[ \frac{e}{3} \Theta(\xi)^3 vol_{d-1}(K \cap H(cg(K), \xi)) + \int_{\Theta(\xi)}^{\infty} t^2 \, vol_{d-1}(K \cap H(t\xi, \xi)) \, dt + \]

\[ \frac{e}{3} \Theta(-\xi)^3 vol_{d-1}(K \cap H(cg(K), \xi)) + \int_{-\infty}^{\Theta(-\xi)} t^2 \, vol_{d-1}(K \cap H(t\xi, \xi)) \, dt \]

The integrals can be estimated by

\[ 2 \Theta(\xi)^3 vol_{d-1}(K \cap H(cg(K), \xi)) \quad \text{and} \quad 2 \Theta(-\xi)^3 vol_{d-1}(K \cap H(cg(K), \xi)) \]

respectively. We treat here only the case \( \xi \), the case \(-\xi\) is treated in the same way. If the integral equals 0 then there is nothing to show. If the integral does not equal 0 then we have

\[ vol_{d-1}(K \cap H(cg(K), \xi)) = e \, vol_{d-1}(K \cap H(cg(K) + \Theta(\xi)\xi, \xi)) \]

We consider the Schwarz symmetrization \( S(K) \) of \( K \) with respect to the plane \( H(cg(K), \xi) \). We consider the cone \( C \) that is generated by the Euclidean spheres \( S(K) \cap H(cg(K), \xi) \) and \( S(K) \cap H(cg(K) + \Theta(\xi)\xi, \xi) \). We have that

\[ S(K) \cap H^+(cg(K) + \Theta(\xi)\xi, \xi) \subset C \]

and that the height of \( C \) is equals

\[ \frac{\Theta(\xi)}{1 - e^{-\frac{1}{d-1}}} \]

Since \((1 + \frac{1}{d-1})^{d-1} < e \) we have \( 1 - e^{-\frac{1}{d-1}} > \frac{1}{d} \). Thus the height of the cone \( C \) is less than \( d \, \Theta(\xi) \). Thus we get for all \( t \) with \( \Theta(\xi) \leq t \leq d \, \Theta(\xi) \)

\[ vol_{d-1}(K \cap H(cg(K) + t\xi, \xi)) \leq (1 - \frac{t}{\Theta(\xi)})^{d-1} vol_{d-1}(K \cap H(cg(K), \xi)) \]
Now we get
\[
\int_{\Theta(\xi)}^{\infty} t^2 \, \text{vol}_{d-1}(K \cap H(t\xi, \xi)) \, dt \leq \\
\int_{\Theta(\xi)} \frac{\Theta(\xi)}{t} \, t^2 \, (1 - \frac{t}{d\Theta(\xi)})^{d-1} \text{vol}_{d-1}(K \cap H(\text{cg}(K), \xi)) \, dt \leq \\
\text{vol}_{d-1}(K \cap H(\text{cg}(K), \xi)) (d \Theta(\xi))^3 \int_{0}^{1} s^2 (1 - s)^{d-1} \, ds = \\
\text{vol}_{d-1}(K \cap H(\text{cg}(K), \xi)) (d \Theta(\xi))^3 \frac{2}{d(d+1)(d+2)} \leq \\
2 \, \text{vol}_{d-1}(K \cap H(\text{cg}(K), \xi)) \Theta(\xi)^3
\]

Therefore we get
\[
\frac{1}{d} \int_{K} \sum_{i=1}^{d} | < x, e_i > |^2 \, dx \leq \left( \frac{e}{3} + 2 \right) (\Theta(\xi)^3 + \Theta(-\xi)^3) \text{vol}_{d-1}(K \cap H(\text{cg}(K), \xi))
\]

Now we apply Lemma 2.3 and get
\[
2 \left( \frac{e}{3} + 2 \right) e^3 \frac{\text{vol}_{d}(K)^3}{\text{vol}_{d-1}(K \cap H(\text{cg}(K), \xi))^2}
\]

□

**Lemma 2.6.** Let $K$ be a convex body in $\mathbb{R}^d$ such that the origin is an element of $K$. Then we have
\[
\frac{1}{d} \int_{K} \sum_{i=1}^{d} | < x, e_i > |^2 \, dx \geq \frac{d^2}{d+2} \text{vol}_{d-1}(\partial B_2^d)^{-\frac{d}{2}} \text{vol}_{d}(K)^{\frac{d+2}{d}}
\]

**Proof.** Let $r(\xi)$ be the distance of the origin to the boundary of $K$ in direction $\xi$. By passing to spherical coordinates we get
\[
\frac{1}{d} \int_{K} \sum_{i=1}^{d} | < x, e_i > |^2 \, dx = \frac{1}{d} \int_{\partial B_2^d} \int_{0}^{r(\xi)} \rho^{d+1} \, d\rho \, d\xi = \frac{1}{d(d+2)} \int_{\partial B_2^d} r(\xi)^{d+2} \, d\xi
\]

By Hölder’s inequality we get that the above expression is greater than
\[
\frac{\text{vol}_{d-1}(\partial B_2^d)}{d(d+2)} \left( \frac{1}{\text{vol}_{d-1}(\partial B_2^d)} \int_{\partial B_2^d} r(\xi)^d \, d\xi \right)^{\frac{d+2}{d}} = \frac{d^2}{d+2} \text{vol}_{d-1}(\partial B_2^d)^{-\frac{d}{2}} \text{vol}_{d}(K)^{\frac{d+2}{d}}
\]

□

The following lemma can be found in [MP]. It is formulated there for the case of symmetric convex bodies.
Lemma 2.7. Let $K$ be a convex body in $\mathbb{R}^d$ such that the origin coincides with the center of gravity of $K$ and such that $K$ is in an isotropic position. Then we have

$$B^d_2(cg(K), \frac{1}{24e^3\sqrt{\pi}} vol_d(K)^{\frac{1}{3}}) \subset K \frac{1}{4e} vol_d(K)^{\frac{1}{3}}$$

Proof. As in Lemma 2.3 let $\Theta(\xi)$ be the infimum of all numbers $t$ such that

$$vol_{d-1}(K \cap H(cg(K), \xi)) \geq e vol_{d-1}(K \cap H(cg(K) + t\xi, \xi))$$

By Lemma 2.3 we have

$$\Theta(\xi) \geq \frac{1}{2e^3} \frac{vol_d(K)}{vol_{d-1}(K \cap H(cg(K), \xi))}$$

By Lemma 2.5 we get

$$\Theta(\xi) \geq \frac{1}{2e^3\sqrt{6e^3}} \left( \frac{1}{vol_d(K)} \frac{1}{d} \int_K \sum_{i=1}^d |<x, e_i>|^2 dx \right)^{\frac{1}{d}}$$

We have

$$vol_d(B^d_2) = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} \leq \frac{\pi^{d-1}(2e)^{d/2}}{d^{d+1}}$$

and thus

$$vol_d(B^d_2)^{\frac{1}{d}} \leq \sqrt{\frac{2\pi e}{d}}$$

Therefore we get by Lemma 2.6

$$\Theta(\xi) \geq \frac{1}{2e^3\sqrt{6e^3}} \frac{d^\frac{d}{2}}{\sqrt{d+2}} \left( \frac{vol_d(K)}{vol_{d-1}(\partial B^d_2)} \right)^{\frac{1}{d}} \geq \frac{1}{12e^5\sqrt{\pi}} vol_d(K)^{\frac{1}{d}}$$

On the other hand, we have

$$vol_d(K \cap H^-(cg(K) + \frac{\Theta(\xi)}{2} \xi, \xi)) \geq \int_0^{\Theta(\xi)} vol_{d-1}(K \cap H(cg(K) + t\xi, \xi)) dt$$

where $H^-(cg(K) + \frac{\Theta(\xi)}{2} \xi, \xi)$ is the half space not containing the origin. By the definition of $\Theta(\xi)$ this expression is greater than

$$\frac{\Theta(\xi)}{2e} vol_{d-1}(K \cap H(cg(K), \xi))$$

By Lemma 2.3 we get that this is greater than

$$\frac{1}{4e} vol_d(K)$$
Therefore, every hyperplane that has distance
\[
\frac{1}{24e^5 \sqrt{\pi}} \text{vol}_d(K)^{\frac{1}{d}}
\]
from the center of gravity cuts off a set of volume greater than \(\frac{1}{4e} \text{vol}_d(K)\).

□

**Proof of Theorem 2.1.** We are choosing the vertices \(x_1, \ldots, x_n \in \partial K\) of the polytope \(P_n\). \(N(x_k)\) denotes the normal to \(\partial K\) at \(x_k\). \(x_1\) is chosen arbitrarily. Having chosen \(x_1, \ldots, x_{k-1}\) we choose \(x_k\) such that
\[
\{x_1, \ldots, x_{k-1}\} \cap \text{Int}(K \cap H^- (x_k - \Delta_k N(x_k), N(x_k))) = \emptyset
\]
where \(\Delta_k\) is determined by
\[
\text{vol}_d(K \cap H^- (x_k - \Delta_k N(x_k), N(x_k))) = t
\]
It could be that the hyperplane \(H(x_k - \Delta_k N(x_k), N(x_k))\) is not tangential to the floating body \(K_t\), but this does not affect the computation. We claim that this process terminates for some \(n\) with
\[
(2.4) \quad n \leq e^{16d \frac{\text{vol}_d(K \setminus K_t)}{t \text{vol}_d(B_2^d)}}
\]
This claim proves the theorem: If we cannot choose another \(x_{n+1}\), then there is no cap of volume \(t\) that does not contain an element of the polytope \(P_n = [x_1, \ldots, x_n]\). By the theorem of Hahn-Banach we get \(K_t \subset P_n\). We show now the claim. We put
\[
(2.5) \quad S_n = K \cap H^- (x_n - \Delta_n N(x_n), N(x_n))
\]
\[
S_k = K \cap \left( \bigcap_{i=k+1}^{n} H^+(x_i - \Delta_i N(x_i), N(x_i)) \right) \cap H^- (x_k - \Delta_k N(x_k), N(x_k))
\]
for \(k = 1, \ldots, n - 1\). We have for \(k \neq l\) that
\[
\text{vol}_d(S_k \cap S_l) = 0
\]
Let \(k < l < n\). Then we have
\[
S_k \cap S_l = K \cap \left( \bigcap_{i=k+1}^{n} H^+(x_i - \Delta_i N(x_i), N(x_i)) \right) \cap H^- (x_k - \Delta_k N(x_k), N(x_k))
\]
\[
\cap K \cap \left( \bigcap_{i=l+1}^{n} H^+(x_i - \Delta_i N(x_i), N(x_i)) \right) \cap H^- (x_l - \Delta_l N(x_l), N(x_l))
\]
\[
\subset H^+(x_l - \Delta_l N(x_l), N(x_l)) \cap H^- (x_l - \Delta_l N(x_l), N(x_l))
\]
\[
= H(x_l - \Delta_l N(x_l), N(x_l))
\]
Thus we have

\[(2.6) \quad \text{vol}_d(S_k \cap S_l) \leq \text{vol}_d(H(x_l - \Delta l N(x_l), N(x_l))) = 0\]

The case \(k < l = n\) is shown in the same way. We have for \(k = 1, \ldots, n - 1\)

\[
S_k = K \cap \left( \bigcap_{i=k+1}^{n} H^+(x_i - \Delta_i N(x_i), N(x_i)) \right) \cap H^-(x_k - \Delta_k N(x_k), N(x_k))
\]

\[
\supset [x_k, K_l] \cap H^-(x_k - \Delta_k N(x_k), N(x_k))
\]

\[
\supset [x_k, (K \cap H^-(x_k - \tilde{\Delta}_k N(x_k), N(x_k))_t] \cap H^-(x_k - \Delta_k N(x_k), N(x_k))
\]

where \(\tilde{\Delta}_k\) is determined by

\[
\text{vol}_d(K \cap H^-(x_k - \tilde{\Delta}_k N(x_k), N(x_k))) = 4e^4 t
\]

By Lemma 2.7 there is an ellipsoid \(\mathcal{E}\) contained in \((K \cap H^-(x_k - \tilde{\Delta}_k N(x_k), N(x_k)))_t\) whose center is \(cg(K \cap H^-(x_k - \tilde{\Delta}_k N(x_k), N(x_k)))\) and that has volume

\[
\text{vol}_d(\mathcal{E}) = \frac{4e^4}{(24e^5 \sqrt{\pi})^d} t \text{vol}_d(B_2^d)
\]

Since \((K \cap H^-(x_k - \tilde{\Delta}_k N(x_k), N(x_k)))_t\) is contained in \(K_t\), \(\mathcal{E}\) is contained in \(K_t\). Thus

\[
S_k \supset [x_k, \mathcal{E}] \cap H^-(x_k - \Delta_k N(x_k), N(x_k))
\]

We claim now that \([x_k, \mathcal{E}] \cap H^-(x_k - \Delta_k N(x_k), N(x_k))\) contains an ellipsoid \(\tilde{\mathcal{E}}\) such that

\[
\text{vol}_d(\tilde{\mathcal{E}}) = \frac{4e^4}{(24e^5 \sqrt{\pi})^d \frac{1}{(4e^5)^d}} t \text{vol}_d(B_2^d)
\]

and consequently

\[(2.7) \quad \text{vol}_d(S_k) \geq \frac{4e^4}{(24e^5 \sqrt{\pi})^d \frac{1}{(4e^5)^d}} t \text{vol}_d(B_2^d) = \frac{4e^4}{(96e^{10} \sqrt{\pi})^d} t \text{vol}_d(B_2^d)
\]

For this we have to see that \(\tilde{\Delta}_k \leq 4e^5 \Delta_k\). By the assumption \(t \leq \frac{1}{4} e^{-5} \text{vol}_d(K)\) we get that

\[
\text{vol}_d(K \cap H^-(x_k - \tilde{\Delta}_k N(x_k), N(x_k))) \leq \frac{1}{e} \text{vol}_d(K)
\]

Therefore we get by (2.1) that \(cg(K) \in H^+(x_k - \tilde{\Delta}_k N(x_k), N(x_k))\). We consider two cases. If

\[
\text{vol}_d((K \cap H(x_k - \Delta_k N(x_k), N(x_k)))) < \text{vol}_d((K \cap H(x_k - \Delta_k N(x_k), N(x_k))))
\]
then we have for all $t$, $\Delta_k \leq t \leq \tilde{\Delta}_k$, by the theorem of Brunn-Minkowski

\begin{equation}
vol_{d-1}(K \cap H(cg(K), N(x_k))) \leq vol_{d-1}(K \cap H(x_k - \tilde{\Delta}_k N(x_k), N(x))) \\
\leq vol_{d-1}(K \cap H(x_k - t N(x_k), N(x)))
\end{equation}

We get by (2.2)

\[ \Delta_k \geq \frac{t}{e \cdot vol_{d-1}(K \cap H(cg(K), N(x)))} \]

By (2.8)

\[(\Delta_k - \tilde{\Delta}_k)vol_{d-1}(K \cap H(cg(K), N(x))) \leq \]

\[vol_d(K \cap H^{-}(x_k - \tilde{\Delta}_k N(x_k), N(x))) - vol_d(K \cap H^{-}(x_k - \Delta_k N(x_k), N(x))) \]

This implies

\[\tilde{\Delta}_k - \Delta_k \leq \frac{(4e^4 - 1)t}{vol_{d-1}(K \cap H(cg(K), N(x)))} \]

Therefore we get

\[\tilde{\Delta}_k \leq \frac{(4e^4 - 1)t}{vol_{d-1}(K \cap H(cg(K), N(x)))} + \Delta_k \leq 4e^5 \Delta_k \]

If

\[vol_{d-1}(K \cap H(x_k - \Delta_k N(x_k), N(x))) \leq vol_{d-1}(K \cap H(x_k - \tilde{\Delta}_k N(x_k), N(x))) \]

then by the theorem of Brunn-Minkowski we have for all $t$, $0 \leq t \leq \Delta_k$, and all $s$, $\Delta_k \leq s \leq \tilde{\Delta}_k$,

\[vol_{d-1}(K \cap H(x_k - t N(x_k), N(x))) \leq vol_{d-1}(K \cap H(x_k - \Delta_k N(x_k), N(x))) \]

\[\leq vol_{d-1}(K \cap H(x_k - s N(x_k), N(x))) \]

We get

\[\Delta_k \geq \frac{t}{vol_{d-1}(K \cap H(x_k - \Delta_k N(x_k), N(x)))} \]

and

\[\tilde{\Delta}_k - \Delta_k \leq \frac{(4e^4 - 1)t}{vol_{d-1}(K \cap H(x_k - \Delta_k N(x_k), N(x)))} \]

Therefore we get

\[\tilde{\Delta}_k \leq \frac{(4e^4 - 1)t}{vol_{d-1}(K \cap H(x_k - \Delta_k N(x_k), N(x)))} + \Delta_k \leq 4e^4 \Delta_k \]

We have verified (2.7). From (2.6) and (2.7) we get

\[vol_d(K \setminus K_t) \geq vol_d(\bigcup_{k=1}^{n} S_k) = \sum_{k=1}^{n} vol_d(S_k) \geq n \frac{4e^4}{(96e^{10} \sqrt{\pi})^d} t \cdot vol_d(B_2^d) \]

Thus we get (2.4)

\[vol_d(K \setminus K_t) \geq e^{-16d} n t \cdot vol_d(B_2^d) \]
3. The Illumination Body

**Theorem 3.1.** Let $K$ be a convex body in $\mathbb{R}^d$ such that

$$\frac{1}{c_1}B^d_2 \subset K \subset c_2B^d_2$$

Let $0 \leq t \leq (5c_1c_2)^{-d-1}vol_d(K)$ and let $n \in \mathbb{N}$ with

$$\left(\frac{128}{7}\pi\right)^{\frac{d}{d+1}} \leq n \leq \frac{1}{32ed}vol_d(K^t \setminus K)$$

Then we have for every polytope $P_n$ that contains $K$ and has at most $n \ d-1$ dimensional faces

$$vol_d(K^t \setminus K) \leq 10^7 \ d^2(c_1c_2)^{2+\frac{1}{d+1}}vol_d(P_n \setminus K)$$

We want to see what this result means for bodies with a smooth boundary. We have the asymptotic formula [W]

$$\lim_{t \to 0} \frac{vol_d(K^t) - vol_d(K)}{t^{\frac{2}{d+1}}} = \frac{1}{2} \left( \frac{d(d+1)}{vol_{d-1}(B^d_2)} \right)^{\frac{2}{d+1}} \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} d\mu(x)$$

Thus we get

$$vol_d(K^t) - vol_d(K) \sim t^{\frac{2}{d+1}} \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} d\mu(x)$$

And by the theorem we have

$$n \sim \frac{1}{dt} vol_d(K^t \setminus K)$$

Thus we get

$$vol_d(K^t) - vol_d(K) \sim d\left(\frac{1}{dn} vol_d(K^t \setminus K)\right)^{\frac{2}{d+1}} \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} d\mu(x)$$

Or

$$vol_d(K^t \setminus K)^{\frac{d}{d+1}} \sim d\left(\frac{1}{dn} \right)^{\frac{2}{d+1}} \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} d\mu(x)$$

$$vol_d(K^t \setminus K) \sim d\left(\frac{1}{n} \right)^{\frac{2}{d+1}} \left( \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} d\mu(x) \right)^{\frac{d+1}{d}}$$

By Theorem 3.1 we get now

$$vol_d(P_n \setminus K) \geq \frac{1}{d} \left(\frac{1}{c_1c_2}\right)^{1+\frac{d}{d+1}} \left(\frac{1}{n} \right)^{\frac{2}{d+1}} \left( \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} d\mu(x) \right)^{\frac{d+1}{d}}$$

By a theorem of F. John [J] we have $c_1c_2 \leq d$.

The following lemma is due to Bronshtein and Ivanov [BI] and Dudley [D1, D2]. It can also be found in [GRS].
Lemma 3.2. For all dimensions \(d, d \geq 2\), and all natural numbers \(n, n \geq 2d\), there is a polytope \(Q_n\) that has \(n\) vertices and is contained in the Euclidean ball \(B_2^d\) such that
\[
d_H(Q_n, B_2^d) \leq \frac{16}{7} \left( \frac{\text{vol}_{d-1}(\partial B_2^d)}{\text{vol}_{d-1}(B_2^{d-1})} \right)^{\frac{2}{d-1}} n^{-\frac{2}{d-1}}
\]

We have that
\[
\text{vol}_{d-1}(\partial B_2^d) = d \text{vol}_d(B_2^d) = d \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2} + 1)}
\]
\[
= d \sqrt{\pi} \frac{\Gamma(\frac{d-1}{2} + 1)}{\Gamma(\frac{d}{2} + 1)} \text{vol}_{d-1}(B_2^{d-1}) \leq d \sqrt{\pi} \text{vol}_{d-1}(B_2^{d-1})
\]
(3.1)

Since \(d^{\frac{2}{d-1}} \leq 4\) and \((1 - t)^d \geq 1 - dt\) we get from (3.1)
\[
d_H(B_2^d, Q_n) \leq \frac{16}{7} \left( \frac{d \sqrt{\pi}}{n} \right)^{\frac{2}{d-1}} \leq \frac{64}{7} \pi n^{-\frac{2}{d-1}}
\]
(3.2)

Proof of Theorem 3.1. We denote the \(d - 1\)-dimensional faces of \(P_n\) by \(F_i, i = 1, \ldots, n\), and the cones generated by the origin and a face \(F_i\) by \(C_i, i = 1, \ldots, n\). Let \(x_i \in F_i\) and \(\xi_i, \|\xi_i\|_2 = 1\), orthogonal to \(F_i\) and pointing to the outside of \(P_n\). Then \(H(x_i, \xi_i)\) is the hyperplane containing \(F_i\) and \(H^+(x_i, \xi_i)\) the halfspace containing \(P_n\). See figure 3.1. We may assume that the hyperplanes \(H(x_i, \xi_i), i = 1, \ldots, n\), are supporting hyperplanes of \(K\). Otherwise we can choose a polytope of lesser volume. Let \(\Delta_i\) be the height of the set
\[
K^t \cap H^-(x_i, \xi_i) \cap C_i
\]
i.e. the smallest number \(s\) such that
\[
K^t \cap H^-(x_i, \xi_i) \cap C_i \subset H^+(x_i + s\xi_i, \xi_i)
\]

Let \(z_i\) be a point in \(\partial K^t \cap C_i\) where the height \(\Delta_i\) is attained. We may assume that
\[
B_2^d \subset K \subset cB_2^d\]
where \(c = c_1 c_2\). Also we may assume that
(3.3)
\[
P_n \subset 2cB_2^d
\]
if we allow twice as many faces. This follows from (3.2): There is a polytope \(Q_k\) such that \(\frac{1}{2}B_2^d \subset Q_k \subset B_2^d\) and the number of vertices \(k\) is smaller than \(\left(\frac{128}{7} \pi\right)^{\frac{d-1}{2}}\). Thus \(Q_k^*\) satisfies \(B_2^d \subset Q_k^* \subset 2B_2^d\) and has at most \(\left(\frac{128}{7} \pi\right)^{\frac{d-1}{2}}\) \(d - 1\)-dimensional faces. As the new polytope \(P_n\) we choose the intersection of \(cQ_k^*\) with the original polytope \(P_n\). Since we have by assumption that \(n\) is greater than \(\left(\frac{128}{7} \pi\right)^{\frac{d-1}{2}}\) the new polytope has at most
\[
\frac{1}{\text{vol}_{d}(K^t \setminus K)}
\]
(3.4)
$d - 1$-dimensional faces.

We show first that for $t$ with $0 \leq t \leq (5cd)^{-d-1}vol_d(K)$ and all $i, i = 1, \ldots, n$ we have

\begin{equation}
\Delta_i \leq \frac{1}{d}
\end{equation}

Assume that there is a face $F_i$ with $\Delta_i > \frac{1}{d}$. Consider the smallest infinite cone $D_i$ having $z_i$ as vertex and containing $K$. Since $H(x_i, \xi_i)$ is a supporting hyperplane to $K$ and $K \subset cB_2^d$ we have

\[ K \subset D_i \cap H^+(x_i, \xi_i) \cap H^-(x_i - 4c\xi_i, \xi_i) \]

and

\[ D_i \cap H^-(x_i, \xi) = [z_i, K] \cap H^-(x_i, \xi) \]

We have

\[ t = vol_d([z_i, K] \setminus K) \geq vol_d([z_i, K] \cap H^-(x_i, \xi_i)) = vol_d(D_i \cap H^-(x_i, \xi_i)) = \frac{1}{d} \Delta_i vol_{d-1}(D_i \cap H(x_i, \xi_i)) \geq \frac{1}{d^2} vol_{d-1}(D_i \cap H(x_i, \xi_i)) \]

Thus

\begin{equation}
vol_{d-1}(D_i \cap H(x_i, \xi_i)) \leq d^2 t
\end{equation}

Since (3.5) does not hold we have

\[ vol_{d-1}(D_i \cap H(x_i - 4c\xi_i, \xi_i)) = (\frac{4c + \Delta_i}{\Delta_i})^{d-1} vol_{d-1}(D_i \cap H(x_i, \xi_i)) \leq (4cd + 1)^{d-1} vol_{d-1}((D_i \cap H(x_i, \xi_i)) \]

By (3.6) we get

\[ vol_{d-1}(D_i \cap H(x_i - 4c\xi_i, \xi_i)) \leq (4cd + 1)^{d-1} d^2 t \leq (5cd)^{d-1} d^2 t \]

Thus we get

\[ vol_d(K) \leq vol_d(D_i \cap H^+(x_i, \xi_i) \cap H^-(x_i - 4c\xi_i, \xi_i)) \leq 2c(5cd)^{d-1} d^2 t \leq (5cd)^{d+1} t \]

Thus

\[ t \geq (5cd)^{-d-1} vol_d(K) \]
This is a contradiction to the assumption on \( t \) in the hypothesis of the theorem. Thus we have shown (3.5). We consider now two cases: All those heights \( \Delta_i \) that are smaller than \( \frac{2dt}{\text{vol}_{d-1}(F_i)} \) and those that are greater. We may assume that \( \Delta_i, i = 1, \ldots, k \) are smaller than \( \frac{2dt}{\text{vol}_{d-1}(F_i)} \) and \( \Delta_i, i = k+1, \ldots, n \) are strictly greater. We have

\[
\text{vol}_d((K^t \setminus P_n) \cap C_i) = \int_0^{\Delta_i} \text{vol}_{d-1}((K^t \setminus P_n) \cap C_i \cap H(x_i + s\xi_i, \xi_i))ds
\]

Since \( B_2^d \subset K \subset P_n \) we get

\[
\text{vol}_d((K^t \setminus P_n) \cap C_i) \leq \int_0^{\Delta_i} \text{vol}_{d-1}(F_i)(1 + s)^{d-1}ds \leq \Delta_i(1 + \Delta_i)^{d-1}\text{vol}_{d-1}(F_i)
\]

By (3.5) we get

\[
\text{vol}_d((K^t \setminus P_n) \cap C_i) \leq \Delta_i(1 + \frac{1}{d})^{d-1}\text{vol}_{d-1}(F_i)
\]

For \( i = 1, \ldots, k \) we get

\[
\text{vol}_d((K^t \setminus P_n) \cap C_i) \leq \frac{2dt}{\text{vol}_{d-1}(F_i)}(1 + \frac{1}{d})^{d-1}\text{vol}_{d-1}(F_i) \leq 2edt
\]

Thus we get

\[
\text{vol}_d((K^t \setminus P_n) \cap (\bigcup_{i=1}^k C_i)) \leq 2kedt \leq 2nedt
\]

By (3.4) we get

\[
(3.7) \quad \text{vol}_d((K^t \setminus P_n) \cap (\bigcup_{i=1}^k C_i)) \leq \frac{1}{8}\text{vol}_d(K^t \setminus K)
\]

Now we consider the other faces. We have for \( i = k+1, \ldots, n \)

\[
(3.8) \quad \Delta_i \geq \frac{2dt}{\text{vol}_{d-1}(F_i)}
\]

We show that we have for \( i = k+1, \ldots, n \)

\[
(3.9) \quad \Delta_i \leq 5c \left( \frac{5c\text{vol}_{d-1}(F_i)}{2d\text{vol}_d(K)} \right)^{\frac{1}{d-1}}
\]

Suppose that there is a face \( F_i \) so that (3.9) does not hold. Then we have

\[
t = \text{vol}_d([z_i, K] \setminus K) \geq \text{vol}_d([z_i, K] \cap H^-(x_i, \xi_i)) = \frac{\Delta_i}{d}\text{vol}_{d-1}([z_i, K] \cap H(x_i, \xi_i))
\]

Therefore we get by (3.8)

\[
(3.10) \quad \text{vol}_{d-1}([z_i, K] \cap H(x_i, \xi_i)) \leq \frac{dt}{\text{vol}_{d-1}(F_i)} \leq \frac{1}{8}\text{vol}_d(K)
\]
By (3.3) we have that

\[ K \subset D_i \cap H^+(x_i, \xi_i) \cap H^-(x_i - 4c\xi_i, \xi_i) \]

Thus

\[ \text{vol}_d(K) \leq \text{vol}_d(D_i \cap H^-(x_i - 4c\xi_i, \xi_i)) \]

The cone \( D_i \cap H^-(x_i - 4c\xi_i, \xi_i) \) has a height equal to \( 4c + \Delta_i \). Therefore we get

\[ \text{vol}_d(K) \leq \frac{1}{d}(4c + \Delta_i)(\frac{4c + \Delta_i}{\Delta_i})^{d-1}\text{vol}_{d-1}(D_i \cap H(x_i, \xi_i)) \]

By (3.5) we have \( \Delta_i \leq 1 \). Therefore we get

\[ \text{vol}_d(K) \leq \frac{5c}{d}(\frac{5c}{\Delta_i})^{d-1}\text{vol}_{d-1}(D_i \cap H(x_i, \xi_i)) \]

By (3.10) we get

\[ \text{vol}_d(K) \leq \frac{5c}{2d}(\frac{5c}{\Delta_i})^{d-1}\text{vol}_{d-1}(F_i) \]

This inequality implies (3.9).

Let \( y_i \) be the unique point

\[ y_i = [0, z_i] \cap H(x_i, \xi_i) \]

We want to make sure that \( y_i \in F_i \cap [z_i, K] \). This holds since \( z_i \in C_i \cap H^-(x_i, \xi_i) \) and \( \Delta_i > 0 \). Since \( y_i \in F_i \) we have

\[ \text{vol}_{d-1}(F_i) = \frac{\text{vol}_{d-1}(B_2^{d-1})}{\text{vol}_{d-2}(\partial B_2^{d-1})} \int_{\partial B_2^{d-1}} r_i(\eta)^{d-1}d\mu(\eta) \]

where \( r_i(\eta) \) is the distance of \( y_i \) to the boundary \( \partial F_i \) in direction \( \eta, \eta \in \partial B_2^{d-1} \), and, since \( y_i \in F_i \cap [z_i, K] \), we have

\[ \text{vol}_{d-1}(F_i \cap [z_i, K]) = \frac{\text{vol}_{d-1}(B_2^{d-1})}{\text{vol}_{d-2}(\partial B_2^{d-1})} \int_{\partial B_2^{d-1}} \rho_i(\eta)^{d-1}d\mu(\eta) \]

where \( \rho_i(\eta) \) is the distance of \( y_i \) to the boundary \( \partial(F_i \cap [z_i, K]) \). Consider the set

\[ A_i = \{ \eta \mid (1 - \frac{1}{4d})r_i(\eta) \leq \rho_i(\eta) \} \]

We show that

\[ \frac{1}{d}\text{vol}_{d-1}(F_i) \leq \frac{\text{vol}_{d-1}(B_2^{d-1})}{\text{vol}_{d-2}(\partial B_2^{d-1})} \int_{\partial B_2^{d-1}} r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1}d\mu(\eta) \]
We have

\[
\frac{\text{vol}_{d-1}(B_2^{d-1})}{\text{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i} r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1} \, d\mu(\eta) \\
\leq \frac{\text{vol}_{d-1}(B_2^{d-1})}{\text{vol}_{d-2}(\partial B_2^{d-1})} \int_{\partial B_2^{d-1}} r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1} \, d\mu(\eta)
\]
\[
\leq \frac{1}{4} \frac{\text{vol}_{d-1}(B_2^{d-1})}{\text{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i} r_i(\eta)^{d-1} \, d\mu(\eta) \leq \frac{1}{4} \text{vol}_{d-1}(F_i)
\]

Therefore we get

\[
\frac{\text{vol}_{d-1}(B_2^{d-1})}{\text{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i^c} r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1} \, d\mu(\eta) \geq \\
\frac{1}{4} \frac{\text{vol}_{d-1}(B_2^{d-1})}{\text{vol}_{d-2}(\partial B_2^{d-1})} \int_{\partial B_2^{d-1}} r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1} \, d\mu(\eta) - \\
\frac{1}{4} \frac{\text{vol}_{d-1}(B_2^{d-1})}{\text{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i} r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1} \, d\mu(\eta) \geq \\
\text{vol}_{d-1}(F_i) - \text{vol}_{d-1}(F_i \cap [z_i, K]) - \frac{1}{4} \text{vol}_{d-1}(F_i)
\]

By (3.10) we get that this is greater than \(\frac{1}{4} \text{vol}_{d-1}(F_i)\). This implies

\[
\frac{1}{4} \text{vol}_{d-1}(F_i) \leq \frac{\text{vol}_{d-1}(B_2^{d-1})}{\text{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i^c} r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1} \, d\mu(\eta)
\]

Thus we have established (3.11).

We shall show that

\[
(3.12) \quad \text{vol}_d((K^t \setminus P_n) \cap C_i) \leq 20480 \, ed^2c^{2 + \frac{1}{2}} \pi \text{vol}_d((P_n \setminus K) \cap C_i)
\]

We have

\[
\text{vol}_d(D_i^c \cap H^+(x_i, \xi_i) \cap C_i) \leq \text{vol}_d((P_n \setminus K) \cap C_i)
\]

Compare figure 3.2. Therefore, if we want to verify (3.12) it is enough to show

\[
\text{vol}_d((K^t \setminus P_n) \cap C_i) \leq 20480 \, ed^2c^{2 + \frac{1}{2}} \pi \text{vol}_d(D_i^c \cap H^+(x_i, \xi_i) \cap C_i)
\]

We may assume that \(y_i\) and \(z_i\) are orthogonal to \(H(x_i, \xi_i)\). This is accomplished by a linear, volume preserving map: Any vector orthogonal to \(\xi_i\) is mapped onto itself and \(y_i\) is mapped to \(\xi_i\). See figure 3.3.
Let \( w_i(\eta) \in D_i^c \cap H^+(x_i, \xi_i) \cap C_i \) such that \( w_i(\eta) \) is an element of the 2-dimensional subspace containing 0, \( y_i \), and \( y_i + \eta \). Let \( \delta_i(\eta) \) be the distance of \( w_i(\eta) \) to the plane \( H(x_i, \xi_i) \). Then we have

\[
\frac{1}{d} \frac{\text{vol}_{d-1}(B_2^{d-1})}{\text{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i^c} (r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1}) \delta_i(\eta) \, d\mu(\eta) \leq \text{vol}_d(D_i^c \cap H^+(x_i, \xi_i) \cap C_i)
\]

Thus, in order to verify (3.12), it suffices to show

\[
\text{vol}_d((K^t \setminus P_n) \cap C_i) \leq
\]

(3.13)

\[
20480 \, ed^2c^2+\frac{1}{2} \frac{1}{d} \frac{\text{vol}_{d-1}(B_2^{d-1})}{\text{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i^c} (r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1}) \delta_i(\eta) \, d\mu(\eta)
\]

In order to do this we shall show that for all \( i = k + 1, \ldots, n \) and all \( \eta \in A_i^c \) there is \( w_i(\eta) \) such that the distance \( \delta_i(\eta) \) of \( w_i \) from \( H(x_i, \xi_i) \) satisfies

\[
\frac{\Delta_i}{\delta_i} \leq \begin{cases} 
32dc & \text{if } 0 \leq \alpha_i \leq \frac{\pi}{4} \\
160 \, dc^2 \left( \frac{5c \, \text{vol}_{d-1}(F_i)}{2d \, \text{vol}_d(K)} \right) \cdot \frac{1}{\rho_i} & \text{if } \frac{\pi}{4} \leq \alpha_i \leq \frac{\pi}{2}
\end{cases}
\]

(3.14)

The angles \( \alpha_i(\eta) \) and \( \beta_i(\eta) \) are given in figure 3.3. We have for all \( \eta \in A_i^c \)

\[
\delta_i = (r_i - \rho_i) \frac{\sin(\alpha_i) \sin(\beta_i)}{\sin(\pi - \alpha_i - \beta_i)} \quad 0 \leq \alpha_i, \beta_i \leq \frac{\pi}{2}
\]

(3.15)

Thus we get

\[
\frac{\Delta_i}{\delta_i} \leq \frac{\rho_i \sin(\pi - \alpha_i - \beta_i)}{r_i - \rho_i \cos(\alpha_i) \sin(\beta_i)} \leq \frac{\rho_i}{(r_i - \rho_i) \cos(\alpha_i) \sin(\beta_i)}
\]

By (3.11) we have \( \rho_i \leq (1 - \frac{1}{4d})r_i \). Therefore we get

\[
\frac{\Delta_i}{\delta_i} \leq \frac{4d}{\cos(\alpha_i) \sin(\beta_i)}
\]

Since \( B_2^d \subset K \subset P_n \subset 2c \, B_2^d \) we get that \( \tan \beta_i \geq \frac{1}{4c} \). Here we have to take into account that we applied a transform to \( K \) mapping \( y_i \) to < \( \xi_i, y_i > \xi_i \). That leaves the distance of \( F_i \) to the origin unchanged and \( r_i(\eta) \) is less than \( 4c \). If \( \beta_i \geq \frac{\pi}{4} \) we have \( \sin \beta_i \geq \frac{1}{\sqrt{2}} \). If \( \beta_i \leq \frac{\pi}{4} \) then \( \frac{1}{4c} \leq \tan \beta_i = \frac{\sin \beta_i}{\cos \beta_i} \leq \sqrt{2} \sin \beta_i \). Therefore we get

\[
\frac{\Delta_i}{\delta_i} \leq \frac{16\sqrt{2} \, dc}{\sin \beta_i}
\]
Therefore we get for all $0 \leq \alpha_i \leq \frac{\pi}{4}$

$$\frac{\Delta_i}{\delta_i} \leq 32 dc$$

By (3.9) and (3.15) we get

$$\frac{\Delta_i}{\delta_i} \leq \frac{1}{r_i - \rho_i} \sin(\pi - \alpha_i - \beta_i) \delta_i \left( \frac{5c \ vol_{d-1}(F_i)}{2d \ vol_d(K)} \right)^{\frac{1}{d-1}}$$

We proceed as in the estimate above and obtain

$$\frac{\Delta_i}{\delta_i} \leq \frac{16\sqrt{2} \ dc}{r_i} \frac{5c}{\sin(\alpha_i)} \left( \frac{5c \ vol_{d-1}(F_i)}{2d \ vol_d(K)} \right)^{\frac{1}{d-1}}$$

Thus we get for $\frac{\pi}{4} \leq \alpha_i \leq \frac{\pi}{2}$

$$\frac{\Delta_i}{\delta_i} \leq \frac{32 \ dc}{r_i} \frac{5c}{\sin(\alpha_i)} \left( \frac{5c \ vol_{d-1}(F_i)}{2d \ vol_d(K)} \right)^{\frac{1}{d-1}}$$

We verify now (3.13). By the definition of $A_i$ we get

$$\frac{\ vol_{d-1}(B_2^{d-1})}{\ vol_{d-2}(\partial B_2^{d-1})} \int_{A_i^c} (r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1}) \delta_i(\eta) \ d\mu(\eta) \geq$$

$$(1 - e^{-\frac{1}{8}}) \frac{\ vol_{d-1}(B_2^{d-1})}{\ vol_{d-2}(\partial B_2^{d-1})} \int_{A_i} r_i(\eta)^{d-1} \delta_i(\eta) \ d\mu(\eta)$$

We get by (3.15)

$$\frac{1}{320 dc} \frac{\ vol_{d-1}(B_2^{d-1})}{\ vol_{d-2}(\partial B_2^{d-1})} \left\{ \int_{\ A_i^c \atop \alpha_i \leq \frac{\pi}{4}} r_i^{d-1} d\mu + \frac{1}{5c} \left( \frac{2d \ vol_d(K)}{5c \ vol_{d-1}(F_i)} \right)^{\frac{1}{d-1}} \int_{\ A_i^c \atop \alpha_i > \frac{\pi}{4}} r_i^{d-1} d\mu \right\}$$

By (3.11) we get that either

$$\frac{\ vol_{d-1}(B_2^{d-1})}{\ vol_{d-2}(\partial B_2^{d-1})} \int_{A_i^c \atop \alpha_i \leq \frac{\pi}{4}} r_i^{d-1} d\mu \geq \frac{1}{8} \ vol_{d-1}(F_i)$$

or

$$\frac{\ vol_{d-1}(B_2^{d-1})}{\ vol_{d-2}(\partial B_2^{d-1})} \int_{A_i^c \atop \alpha_i > \frac{\pi}{4}} r_i^{d-1} d\mu \geq \frac{1}{8} \ vol_{d-1}(F_i)$$

In the first case we get for the above estimate

$$\frac{\ vol_{d-1}(B_2^{d-1})}{\ vol_{d-2}(\partial B_2^{d-1})} \int_{A_i^c} (r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1}) \delta_i(\eta) \ d\mu(\eta) \geq$$

$$\frac{\Delta_i}{\ vol_{d-1}(F_i)} \geq \frac{1}{\ vol_d((K^t \setminus P_n) \cap C_i)}$$
The last inequality is obtained by using (3.5): Since $B_2^d \subset K$ we have for all hyperplanes $H$ that are parallel to $F_i$,$\ vol_{d-1}(K^t \cap H \cap C_i) \leq (1 + \Delta_i)^{d-1} vol_{d-1}(F_i)$. By (3.5) we get $vol_{d-1}(K^t \cap H \cap C_i) \leq \epsilon \ vol_{d-1}(F_i)$. In the second case we have

$$\frac{vol_{d-1}(B_2^{d-1})}{vol_{d-2}(\partial B_2^{d-1})} \int_{A_i^c} (r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1}) \delta_i(\eta) \ d\mu(\eta) \geq$$

$$\frac{1}{5c} \left( \frac{2d \ vol_d(K)}{5c \ vol_{d-1}(F_i)} \right) \frac{\Delta_i}{320dc} \ vol_{d-1}(B_2^{d-1}) \ int_{A_i^c \ \alpha_i > \frac{\pi}{4}} r_i^{d-1} d\mu \geq$$

$$\frac{1}{5c} \left( \frac{2d \ vol_d(K)}{5c \ vol_{d-1}(F_i)} \right) \frac{\Delta_i}{320dc} \ vol_{d-1}(B_2^{d-1}) \ \frac{\Delta_i}{2560dc} \ vol_{d-1}(F_i) \geq$$

$$\frac{1}{5c} \left( \frac{d \ vol_d(K)}{20c \ vol_{d-1}(B_2^{d-1})} \right) \ \frac{\Delta_i}{2560dc} \ vol_{d-1}(F_i) \geq$$

Since $B_2^d \subset K$ we get

$$\frac{vol_{d-1}(B_2^{d-1})}{vol_{d-2}(\partial B_2^{d-1})} \ int_{A_i^c} (r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1}) \delta_i(\eta) \ d\mu(\eta) \geq$$

$$\frac{1}{5c} \left( \frac{d \ vol_d(B_2^d)}{20c \ vol_{d-1}(B_2^{d-1})} \right) \ \frac{1}{2560edc} \ vol_d((K^t \ P_n) \cap C_i) \geq$$

$$(20480 \ ed^2 c^{2+\frac{1}{d-1}})^{-1} \ vol_d((K^t \ P_n) \cap C_i)$$

The second case gives a weaker estimate. Therefore we get for both cases

$$vol_d((K^t \ P_n) \cap C_i) \leq$$

$$20480 \ ed^2 c^{2+\frac{1}{d-1}} \ vol_{d-1}(B_2^{d-1}) \ int_{A_i^c} (r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1}) \delta_i(\eta) \ d\mu(\eta)$$

Thus we have verified (3.13) and by this also (3.12). By (3.12) we get

$$vol_d((K^t \ P_n) \cap \bigcup_{i=k+1}^{n} C_i) \leq 20480 \ ed^2 c^{2+\frac{1}{d-1}} \ vol_d((K^t \ P_n) \cap (P_n \ K))$$

(3.16)

$$\leq 20480 \ ed^2 c^{2+\frac{1}{d-1}} \ vol_d((P_n \ K))$$

If the assertion of the theorem does not hold we have

$$vol_d(P_n \ K) \ \\leq \ \frac{1}{\delta^d} \ vol_d(K \ \cap \ K)$$

(3.17)
Thus we get
\[ \text{vol}_d((K^t \setminus P_n) \cap \left( \bigcup_{i=k+1}^n C_i \right)) \leq \frac{1}{8} \text{vol}_d(K^t \setminus K) \]
Together with (3.7) we obtain
\[ (3.18) \quad \text{vol}_d(K^t \setminus P_n) \leq \frac{1}{4} \text{vol}_d(K^t \setminus K) \leq \frac{1}{4} \{ \text{vol}_d(K^t \setminus P_n) + \text{vol}_d(P_n \setminus K) \} \]
By (3.17) we have
\[ \text{vol}_d(P_n \setminus K) \leq \frac{1}{8} \text{vol}_d(K^t \setminus K) \leq \frac{1}{2} \text{vol}_d(K^t \setminus P_n) + \frac{1}{2} \text{vol}_d(P_n \setminus K) \]
This implies
\[ \text{vol}_d(P_n \setminus K) \leq \text{vol}_d(K^t \setminus P_n) \]
Together with (3.18) we get now the contradiction
\[ \text{vol}_d(K^t \setminus P_n) \leq \frac{1}{2} \text{vol}_d(K^t \setminus P_n) \]
\[ \square \]

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Figure 2.1
Figure 2.2

\[ S(K) \]

\[ H(\text{cg}(K), \xi) \]

\[ H(\text{cg}(K) + \Theta(\xi)\xi, \xi) \]
Figure 2.3
Figure 3.1

\[ \Delta_i \]

\[ P_n \rightarrow K \rightarrow H(x_i, \xi_i) \]

\[ C_i \]

\[ 0 \]

\[ Z_i \]
Figure 3.2
Figure 3.3

\[ y_i + r_i(\eta)\eta \]