Zipf’s law and phase transition

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Abstract

We describe the link between the Zipf law and statistical distributions for the Fortuin-Kasteleyn clusters in Ising as well as Potts models. From these results it is seen that Zipf’s law can be a criterion of a phase transition, but it does not determine its order. We present the corresponding histograms for fixed domain configurations.

1. Introduction

Originally the Zipf law was used for analyzing the hierarchy of word’s occurrence in a language, i.e. the relative population of words ranking from these used most frequently to the ones used less frequently [1]. Existence of very similar linear hierarchy distributions was found in other domains of science, for example the Zipf law has appeared in the description of distributions of city populations [2], distributions of turnovers of Europe’s largest companies [3], in linguistic features of noncoding DNA sequences [4], molecular biology networks [5] and in physics in multifragmentation of atomic nuclei in nuclear reactions [6], [7] or liquid crystal [8]. The statistics of domains was numerically investigated for percolation [9], [10], for Ising and Potts models [11], [12], [13], [14]. Various formulation of cluster theories one can find in papers dealing with Fisher droplets [15] or percolation [9]. The Zipf law states that the mass \( m \) of the largest, second largest, ..., \( k \)–largest clusters decreases according to their rank, \( k = 1, 2, 3, .., n \), as

\[
x(k) \sim \frac{1}{k^\lambda}, \quad \text{where } \lambda \sim 1.
\]
We wish to underline that Zipf’s law is a special case of the more general Zipf-Mandelbrot law \[16\]

\[
x(k) = C/(k + \alpha)^{-\lambda},
\]
(2)

where the offset \(\alpha\) is an additional constant parameter. The value of \(\lambda\) is asymptotically approximated by the function of the critical exponent \(\tau\) of power law cluster size distribution\[16\] and is given by formula \[16\]

\[
\lambda = \frac{1}{\tau - 1}.
\]
(3)

Zipf’s law is a consequence of power law cluster size distribution with the exponent \(\tau = 2\). There is a connection of the critical exponent \(\tau\) with critical exponents of scaling theory, similarly as in percolation \[9\].

We study the size distributions of the Fortuin-Kasteleyn (FK) clusters obtained by the Monte Carlo simulations in a pair of models - two-dimensional Ising and Potts models- considered as well in our recent publications \[13\], \[14\]. In our case the measure of size is the number of spins in the cluster, defining the mass of the cluster. For the FK spin clusters, near the critical point, the probability density distribution of the number of clusters with mass \(x\) has the asymptotic form

\[
\rho(x) \sim x^\tau \exp[-\theta x].
\]
(4)

The first factor characterized by exponent \(\tau = \frac{d}{D} + 1\) is the entropy factor in where \(d\) is the dimension of the system and \(D\) describes its fractal dimension of the system. The second factor is the Boltzman weight, which suppress large clusters when parameter \(\theta\) is finite. When parameter \(\theta\) tends to zero we can use the approximation \(\theta \sim |T - T_c|^\frac{1}{\sigma}\) which defines a critical exponent \(\sigma\).

We investigate the statistics of the domain masses for Ising and Potts models in the critical region as well beyond it. Connection of the exponent \(\tau\) of the cluster size distribution with critical exponents of scaling theory causes that Zipf’s law can be treated as providing some criterion of a phase transition.

We notice that the power law distributions are not the only forms of broad distribution. Zipf’s law suggests also modeling by distributions of hyperbolic type, as in linguistics \[17\] and economy \[18\]. Generalized inverse Gaussian distributions play an important role in the theory of generalized hyperbolic distributions. Their right hand tail behaviour spans a range of formulas from exponential decrease to a Pareto tail. They are much slower than in the case of normal distribution and therefore suitable to describe model phenomena, where numerically large values are more probable.

2. Application of Zipf’s law to Ising and Potts models

The Potts model \[19\] is a generalization of the Ising model \[20\] to spins with more than two components; for a detailed review of the Potts model see \[21\].

The Hamiltonian for the \(q\)-state Potts model \[21\] is the following

\[
H = -\sum_{ij} J_{ij} \delta_{\sigma_i, \sigma_j},
\]
(5)
where \( \sigma_i \in \{1, 2, ..., q\} \), \( \delta_{x,y} \) is the Kronecker delta

\[
\delta_{x,y} = \begin{cases} 
1 & \text{if } x = y, \\
0 & \text{otherwise.}
\end{cases}
\]

and

\[
J_{ij} = \begin{cases} 
J & \text{if } i, j \text{ describe neighbour pairs of spins,} \\
0 & \text{in opposite case.}
\end{cases}
\]

The case \( q = 2 \) describes the Ising model.

A number of exact results for two-dimensional Potts models are known in the infinite volume limit. For example, the phase transition appears at the critical temperature \( T = T_c \) (\( T_c = 2J/k_B \ln(1 + \sqrt{q}) \)). It means that for Ising model \( T_c = 2.2692 \) while for Potts model with \( q = 3 \), \( T_c = 1.9899 \) and with \( q = 6 \), \( T_c = 1.6152 \) It is the second order phase transition for \( q \leq 4 \) and the first order one for \( q \geq 5 \), (see [21]). We use the Fortuin-Kasteleyn random cluster model representation [22]. Clusters were generating by Monte Carlo techniques, based on the Swendsen-Wang cluster algorithm [23] applied to the two-dimensional models with periodic boundary conditions. We examine the changes in the cluster size distribution as the system approaches the critical point of the phase transition indicated by the critical temperature.

The power law is presented by the Zipf- Mandelbrot law (2) where \( C \) and \( \alpha \) are some constants, \( \mu = \lambda^{-1} \) and \( x \) denotes the value (mass) of the object, \( k \) is the rank order of the object. The object with the largest variable value is ranked as the first, the next with smaller value is ranked as the second and so on. In this way we determine the rank order \( k \) of the object. For \( \alpha = 0 \) and \( \mu = 1 \) the Zipf-Mandelbrot law takes the form of the Zipf law:

\[
x \sim C/k
\]

This equation describes a straight line in a double logarithmic plot with the slope equal to \(-1\).

We can consider the Zipf law in the Bouchaud notation [3]. The distribution of the cluster size appears to be \([\mu]\)-distribution and the indeks \( \mu \) is a critical exponent. Suppose that a set of \( N \) \([\mu]\)-variables \( x_k \) is ordered as decreasing sequence, then

\[
x_k \sim x_0(N/k)^{1/\mu}
\]

The formula (8) means that the largest variable is of the order \( x_0N^{1/\mu} \), while the smallest is of order \( x_0 \).

In our case \( x_k \) is the number of Ising or Potts spins in the domain with rank \( k \), where the greatest cluster, the one with greatest mass, has rank equal to 1. The Pareto distribution implies the Zipf law [24], [13], but inverse conclusion is not valid - for instance the Zipf law is satisfied as well in the case of the hyperbolic distributions [17], [18].

The probability density of the Pareto distributions is defined as follows:

\[
\rho(x) = (\mu/x_0)(x_0/x)^{\mu+1}
\]
for \( x > x_0 \), where \( x_0 \) denotes a typical scale, \( x \) is the mass and \( \mu \sim 1 \).

One can define \( \mu \)-variable as in [3]. The main property of \( [\mu] \)-variable is that all its momenta \( m_q = \langle X^q \rangle \) with \( q \geq \mu \) are infinite.

3. Numerical considerations

In Ising and Potts models the clusters form sets of nearest neighbour sites occupied by spins with the same orientation.

The \( q \)-state Potts model has the grand state degeneracy \( q \) and after quench from high temperature phase small domains start to grow. There is well established connection between thermal and geometrical phase transitions when the big clusters appear.

![Graphs showing log-log distribution of domain masses (a) Ising, (b) 3-Potts, (c) 6-Potts](image)

Figure 1: The log-log distribution of the domain masses \( x_k \) versus the rank order index \( k \), for (a) the two-dimensional Ising model for \( T = 2.2692 \) (the critical temperature) (△), \( T = 2.5000 \) (○), \( T = 2.8571 \) (□) and \( T = 4.0000 \) (○) for \( L = 1000 \), (b) for the 3-state Potts model for \( T = 2.5000 \) (○), 2.2222 (●), 2.0408 (□) and critical \( T \) (○) and (c) for the 6-state Potts model for \( T = 2.0000 \) (○), 1.8181 (●), 1.6949 (□) and critical \( T \) (○), for one configuration (see [14], [13]).

In both models \( x_k \) denotes the domain mass of rank \( k \). Figure 1 describes the plot of domain masses versus its rank for the Ising (a) for 3 state Potts (b) and 6-state Potts (c) models. The values of the parameter \( \mu \) which is equal to

\[ 4 \]
–(\tan \alpha)^{-1}$, where $\alpha$ is the angle of the slope of regression line, are significantly smaller than 1 for temperatures different from critical temperature, while for the critical temperature the value of parameter $\mu$ is approximately equal to 1. We start from the paramagnetic phase and we decrease the temperature of the system. For Ising model the calculations were done for four different temperatures $T = 4.0000$, $T = 2.8571$, $T = 2.5000$ and $T_c = 2.2692$. The first three absolute values of slopes are significantly smaller than 1 and equal respectively to 0.149, 0.213 and 0.261, but for the phase transition temperature $T_c = 0.2692$ the slope is near to $-1$.

We have similar results for 3–state Potts and 6–state Potts models presented on Figure 1b and Figure 1c. For 3-statePotts model the calculations were done for four different temperatures $T = 2.5000$, $T = 2.2222$, $T = 2.0408$ and $T_c = 1.9900$ respectively with slopes 0.249, 0.310, 0.591 and 1.095. For systems in their critical temperatures the slopes are equal to $-1$ what means that only for this temperatures the Zipf law is satisfied. We observe that Zipf’s law can be used as additional criterion to determine the location of phase transition. Further we can notice that for temperatures higher than critical the domains are ordered along strait lines.

It is interesting, that Zipf law helps to determine location of phase transition without determining its order. As was noticed by Ma from lattice gas simulations [6], [7], it appears that some questions concerning the order phase transition still remain open.

![Figure 2](image_url)

Figure 2: The dependence of the number of domains with mass $x$ normalized by the number of configurations ($N_x/N$) on the mass($x$) in the log-log scale, 2(a) for the two dimensional Ising model, 2(b) for the three state Potts model and 2(c) for the 3-state Potts model, near its critical temperatures $T_c$ and $L = 600$. (for 2b and 2c see [13])
Figure 2 presents the dependence between the number of domain with mass $x$ normalized by number of configurations and the mass $x$ in log-log scale for Ising model 2(a), for 3-state Potts model 2(b) and 6-state Potts models 2(c) in critical temperatures. Similar plots were presented in [12]. As it is depicted on Figures 2 the power law distribution is complicated by a large fluctuations of large but rare events which occur in the tail and it appears as noisy curve on the plot. The straight line on Figure 2 represents the Pareto distribution.

Figure 3: The dependence between the number of domain with mass $x$ normalized by number of configurations ($N_x/N$) and mass $x$ in log-log scale 3(a) for Ising model for temperature $T = 3.3333$, 3(b) for the 3-state Potts model for the temperature $T = 2.5000$ and 3(c) for the 6-state Potts model, for the temperature $T = 2.0000$. (for 2b and 2c see [13])

Figure 3 presents the dependence between the number of domain with mass $x$ normalized by number of configurations and mass $x$ in log-log scale 3(a) for Ising model for temperature $T = 3.3333$ and $L = 500$, 3(b) for 3-state Potts model for $T = 2.5$ and 3(c) for 6-state Potts models for $T = 2$ and $L = 600$. As it is in Figure 2 the noisy curve appears on the plot. For both models, distributions of domain have a smaller tails than the one for critical temperature, but still bulk of distributions occurs for fairly small sizes. Similar as in the percolation theory [9].
beyond a critical region the probability of the event with spin in a fixed position belonging to a domain with mass \( x \) is asymptotically equal \( \rho(x) \sim x^\tau \exp[-\theta x] \), where \( \theta \sim (T - T_c)^{\frac{1}{2}} \) when \( T \to T_c \).

**Figure 4:** Histogram of the number of cluster configurations for fixed mass \( m = 5 \) (a), \( m = 10 \) (b), \( m = 50 \) (c) and \( m = 70 \) (d), for the critical temperature \( T = 2.2692 \) for the system with \( L = 600 \) and for 10000 configurations.

**Figure 5:** Histogram of the number of cluster configurations for fixed mass \( m = 5 \) (a), \( m = 10 \) (b), \( m = 50 \) (c) and \( m = 70 \) (d), for the temperature \( T = 4.000 \), for the system with \( L = 600 \), for 10000 configurations.

Figures 4 and 5 present histograms of the numbers of cluster configurations with fixed mass for the Ising model. Histograms are presented for masses \( x = 5 \), \( x = 10 \), \( x = 50 \) and \( x = 70 \) for total number of configurations 10000 of the the system with \( L = 600 \) near critical point and beyond it, respectivelly. For example from histogram 4c it can be seen that five clusters of mass \( m = 50 \) appear in the 1250 configurations out of 10000.

The test for the gaussian distribution was performed using the Kolmogorov-Smirnov and the chi-square tests. Near the critical point the gaussian distributions for the number of cluster configurations of a fixed masses are present only for the sizes smaller than 3. Beyond the critical point the gaussian distributions are present
only for masses smaller than 5. For example for the temperature \( T = 4.000 \) and the cluster of size 5 the p-value of the chi-squared test was equal to 0.0771, while for critical temperature and the cluster 5 the p-value of the chi-squared test was equal to 0.0117.

For the clusters of larger size both for the critical point and outside of it, the histograms of distributions of numbers of domain configurations provide right-skewed distributions. For very small masses approximate by the gaussian distributions are present.

Experimentally Zipf and Pareto laws in the context of phase transitions distributions were investigated in physics on various occasions, for example: in multifragmentation of atomic nuclear reactions [6], [7], experimental test for Zipf’s law in liquid crystal [8], the discontinuous metal films on dielectric substraces [25].

4. Conclusions

In our paper we investigate numerically the statistical distributions of domains in Ising and Potts models for critical region as well as beyond it. The main conclusions are following:

(i) In the critical region as well beyond it the log-log plot (Figure 1) of distributions for domain masses presents a straight lines, what means that domains forms a hierarchical order. When \( T < T_c \) slopes of the lines are bigger than for critical temperature slope which is equal to \(-1\), when Zipf’s law is valid. Zipf’s law can indicate the presence of phase transition.

(ii) The distributions of domain masses near the critical point (Figure 2) is well approximated by Pareto distribution, presented in log-log by straight lines. In such case the phase transition occurs.

Beyond the critical region (Figure 3) we do not have straight lines and the tails of the distributions are smaller than the ones described by Pareto tail.

(iii) Histograms of the numbers of domain configurations with the same number of domains with fixed mass present in the critical point as well as beyond it the right-skewed distributions

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