A NOTE ON FREE TIME EVOLUTION OF THE QUANTUM WAVE FUNCTION AND OPTIMAL TRANSPORTATION

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Abstract. It is shown that, in the absence of nodes and under regularity assumptions, a solution in a finite interval of time of the free Schroedinger equation solves a minimization problem which is a stochastic generalization of the classical optimal transportation problem with quadratic cost.
1. Introduction

Consider the free Schroedinger equation on $\mathbb{R}^d$

$$i\partial_t \psi + \frac{1}{2}\nabla^2 \psi = 0, \quad \psi(x, 0) = \psi_o$$ (1.1)

Putting $\rho := |\psi|^2$ and denoting by $S$ the principal argument of $\psi$ we can write

$$\psi = \rho^{\frac{1}{2}} \exp iS$$

so that, for all $(x, t)$ such that $\rho(x, t)$ is different from zero, (1.1) is equivalent to

$$\partial_t \rho(x, t) + \nabla (\rho(x, t) \nabla S(x, t)) = 0$$ (1.2)

$$\partial_t S(x, t) + \frac{1}{2}(\nabla S(x, t))^2 - \frac{1}{2} \frac{\nabla^2 \sqrt{\rho(x, t)}}{\sqrt{\rho(x, t)}} = 0$$ (1.3)

which are the (free) Madelung-fluid equations.

Then $(\rho, \nabla S)$ acts as a "fluid-dynamical couple" and the first Madelung equation represents its continuity equation.

We address the question of whether $(\rho, \nabla S)$ is optimal in some sense among all time dependent fluid-dynamical couples $(\rho', v')$ which belong to a non trivial set such that $\rho'_o = \rho_o$ and $\rho'_1 = \rho_1$, with $\rho_o := |\psi_o|^2$ and $\rho_1 := |\psi_1|^2$.

We do this by exploiting some features of Nelson’s Stochastic Mechanics (see [4] for a review).

Introducing the drift-field

$$b[\rho, \nabla S] := \nabla S + \frac{1}{2} \nabla \log \rho$$

we know, thanks to a general result due to Carlen [3], that, if the quantum energy is finite at $t = 0$, then there exists a Markov diffusion process $q^b$ with drift field $b = b[\rho, \nabla S]$, diffusion matrix equal to the identity matrix and time dependent probability density $\rho$. We call this the "Nelson diffusion associated to $\psi \equiv \rho^{\frac{1}{2}} \exp iS"$.

To be more precise, let $\Omega$ be the set of continuous functions from $[0, 1]$ to $\mathbb{R}^d$. Let $\mathcal{F}$ denote the associated Borel $\sigma$-algebra and the filtration $(\mathcal{F}_t)_t$ be defined in the natural way. Let also $X_o$ be equal to $\omega(0)$. Then we know that there exists a probability measure $\mathbb{P}$ and a standard Brownian Motion $W$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t)$, such that the configuration process $X$ satisfies the equality

$$X_t = X_0 + \int_0^t b[\rho, \nabla S](X_s, s) ds + W_t, \quad \mathbb{P}\text{-a.s.}$$ (1.4)
and \( X_t \) has a probability density equal to \( \rho(\cdot, t) \) for all \( t \in [0, 1] \). Then Nelson’s diffusion is defined by identifying \( q^{[\rho, \nabla S]} \) with \( X \).

In this paper we consider only the very regular case when \( \rho_o \) is smooth and strictly positive and the stochastic differential equation with coefficients \( (b[\rho, \nabla S], I) \) has a strong solution. The more general case, when in particular \( \psi \) has nodes, is currently the subject of further work.

To formulate the optimization problem, we introduce a suitable set \( \Xi_o(\rho_o, \rho_1) \) of smooth time dependent fluid-dynamical couples which connect \( \rho_o \) to \( \rho_1 \) (see Definition 1).

Then, for a given choice of \( \mathbb{F} \) and \( W \) on \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t) \), such that the law of \( X_o \) has probability density \( \rho_o \) and \( W \) is independent of \( X_o \), we define, for any \((\rho, v)\) in \( \Xi_o \), the Nelson diffusion \( q^b, b \equiv b[\rho, v] \), as the solution of the S.D.E.

\[
q^b_t = X_0 + \int_0^t b(q^b_s, s)ds + W_t
\] (1.5)

Introducing the equipartition \( \{t_i\}_{i=0}^n \) of \([0, 1]\) and putting, with \( b \equiv b[\rho, v] \),

\[
\Delta q^b_i := \int_{t_i}^{t_{i+1}} b(q^b_s, s)ds + (W_{t_{i+1}} - W_{t_i})
\]

we consider for every \( \omega \in \Omega \) the classical action in discrete time

\[
\frac{1}{n} \sum_{i=0}^{n-1} (\Delta q^b_i(\omega))^2
\] (1.6)

We take the average with respect to the initial configurations and all possible Brownian paths, and leave \( n \) going to infinity. Exploiting Nelson’s renormalization formula [12] to get rid of the divergent term, one gets

\[
\lim_{n \to \infty} n\mathbb{E} \sum_{i=0}^{n-1} (\Delta q^b_i)^2 - nd = \int_0^1 \int_{\mathbb{R}^d} (v^2 - (\frac{1}{2} \nabla \ln \rho)^2) \rho dx dt
\] (1.7)

Then we consider on \( \Xi_o(\rho_o, \rho_1) \) the functional

\[
A^Q(\rho, v) := \int_{\mathbb{R}^d} \int_0^1 (v^2 - (\frac{1}{2} \nabla \ln \rho)^2) \rho dt dx
\] (1.8)

The functional \( A^Q \) and equivalent expressions of it, usually generalized by adding terms related to scalar and vector potentials and possibly extended to the case when the configuration space is a Riemannian manifold, were considered in the literature on S.M., mainly during the ’80s and ’90s. In fact, within a stochastic control approach, their critical points were shown to be related to the solutions of Schroedinger’s
equation [7][13]. A fluid-mechanical reformulation of part of the results
given in [7] was proposed by Loffredo [9] and was frequently adopted
in the literature on S.M.. An approach based on stochastic differential
games was proposed in [14] and, recently, a relationship with Fisher
information was also suggested [15].

The problem of establishing whether the critical points of $A^Q$, and
of its equivalent expressions, correspond to minimizers or not has re-

mained unsolved. The main difficulty comes from the non convexity of
the functional.

It is worth mentioning that a variational method which allows deriv-
ing the Schroedinger equation in the framework of Nelson's Stochastic
Mechanics starting from a convex functional, was proposed by Yasue
in [16]. Unfortunately in his approach the ”variations of a Nelson dif-
fusion” are assumed to be smooth functions of the diffusion itself. This
allows to exploit a nice integration by parts formula. But, as a con-
sequence, the ”varied motions” are Markov diffusions with a new, non
constant, diffusion coefficient. Thus the ”varied motions”, at variance
with what happens within the stochastic control approach adopted in
[7] starting from $A^Q$, they are no longer Nelson diffusions and this
characteristic makes difficult formulating a minimization problem.

In this work we construct on a proper space a convex functional such
that a suitable restriction of it is equivalent, in a proper sense, to $A^Q$
(see (2.13), (3.3) and (3.4)). We prove that, if $\psi \equiv \rho^\frac{1}{2} \exp iS$, with
satisfies (1.1) with $|\psi_o|^2 = \rho_o$, $|\psi_1|^2 = \rho_1$ and $(\rho, \nabla S) \in \Xi_o(\rho_o, \rho_1)$, then

$$A^Q(\rho, \nabla S) \leq A^Q(\rho', \nu'), \quad \forall (\rho', \nu') \in \Xi_o(\rho_o, \rho_1) \quad (1.9)$$.
2. A CONVEX ASYMPTOTIC FUNCTIONAL

We consider the probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_t)\), where \(\Omega\) is the set of continuous functions from \([0, 1]\) to \(\mathbb{R}^d\), \(\mathcal{F}\) denotes the associated Borel \(\sigma\)-algebra and the filtration \((\mathcal{F}_t)_t\) is defined in the natural way.

Let also \(X_0 := \omega(0)\). Let \(\mathbb{P}\) be a probability measure such that the law of \(X_0\) has the probability density \(\rho_0\) and let \(W\) be a standard Brownian Motion independent of \(X_0\).

**Remark 1.** Let \(\rho_0\) be smooth and strictly positive and \(b : \mathbb{R}^d \times [0, 1] \rightarrow \mathbb{R}^d\) a smooth time dependent drift field with sublinear growth at infinity. Then the Stochastic Differential Equation with coefficients \((b, I)\) has a unique strong solution, so that there exists a unique continuous square integrable Markov process \(q^b\) s.t.

\[
q^b_t = X_0 + \int_0^t b(q^b_s, s)ds + W_t, \quad \mathbb{P}\text{-a.s.} \quad (2.1)
\]

The process admits a time continuous probability density \(\rho\) which is smooth and strictly positive. Defining the ”current velocity field” \(v\) by \(v := b - \frac{1}{2} \nabla \ln \rho\), the Fokker-Planck equation, describing the time evolution of the probability density \(\rho\), takes the form the continuity equation for the pair \((\rho, v)\), i.e.

\[
\partial_t \rho + \nabla (\rho v) = 0 \quad (2.2)
\]

(see [11]).

**Definition 1.** We denote by \(\Xi_\omega\) the set of pairs \((\rho, v)\) where \(\rho\) is a strictly positive smooth time dependent probability density on \(\mathbb{R}^d\), \(v\) is a smooth time dependent velocity field on \(\mathbb{R}^d\) such that

\[
\partial_t \rho + \nabla (\rho v) = 0 \quad \text{(continuity equation)} \quad (2.3)
\]

and

\[
\int_0^1 \int_{\mathbb{R}^d} (v^2 + \frac{1}{2} \nabla \ln \rho)^2 \rho dx dt < \infty, \quad \text{(finite action condition)} \quad (2.4)
\]

Moreover \(v\) and \(\nabla \ln \rho\) are assumed to have a sublinear growth at infinity.

The set \(\Xi_\omega\) is a small subset of the ”set of proper infinitesimal characteristics” introduced by Carlen in [5].

We now introduce the following space of processes

\[
L^2_{[0, 1]}(\mathbb{P}) := \{\beta : \Omega \times [0, 1] \rightarrow \mathbb{R}^d \text{ s.t. } \int_\Omega \int_0^1 \beta^2(\omega, s)ds d\mathbb{P}(d\omega) < \infty\} \quad (2.5)
\]
Let define

\[ q^\beta_t := X_0 + \int_0^t \beta_s ds + W_t, \quad X_0(\omega) = \omega(0) \]

and

\[ \Delta q^\beta_t := q^\beta_{t+1} - q^\beta_t \]

Considering the equipartition \( \{ t_i \}_{i=0}^n \) of \([0, 1]\) and denoting by \( E \) the integration with respect to \( \mathbb{P} \), we introduce the convex functional \( F_n : L^2_{[0,1]}(\mathbb{P}) \to \mathbb{R} \)

\[ F_n(\beta) := nE \sum_{i=0}^{n-1} (\Delta q^\beta_i)^2 \] (2.6)

Let now \( \beta \) be a "Markovian drift", i.e. such that \( \beta_t = \beta^b_t \) where

\[ \beta^b_t := b(q^b_t, t), \] (2.7)

\( q^b \) being the solution of the stochastic differential equation (2.1).

Introducing the notation

\[ b[\rho, v] := v + \frac{1}{2} \nabla \log \rho, \] (2.8)

if \( b \) is equal to \( b[\rho, v] \) with \( (\rho, v) \in \Xi_o \), then, by the finite action condition, \( \beta^b := (\beta^b_t)_{t \in [0,1]} \) belongs to \( L^2_{[0,1]}(\mathbb{P}) \).

Moreover, exploiting Nelson’s renormalization formula (12) we can take the limit for \( n \) going to infinity, getting, with \( b \equiv b[\rho, v] \),

\[ \lim_{n \to \infty} nE \sum_{i=0}^{n-1} (\Delta q^b_i)^2 - nd = E \int_0^1 (b^2(q^b_t, t) + \nabla b(q^b_t, t)) dt \] (2.9)

and, integrating by parts,

\[ E \int_0^1 (b^2(q^b_t, t) + \nabla b(q^b_t, t)) dt = \int_0^1 \int_{\mathbb{R}^d} (v^2 - (\frac{1}{2} \nabla \ln \rho)^2) \rho dx dt < \infty \] (2.10)

Then, for any "Markovian element" \( \beta^b \), \( b \equiv b[\rho, v] \), \( (\rho, v) \) in \( \Xi_o \), we have

\[ \lim_{n \to \infty} (F_n(\beta^b) - nd) < \infty \] (2.11)

and, for any \( \lambda \in [0, 1] \) and \( (\beta^{b_1}, \beta^{b_2}) \in L^2_{[0,1]}(\mathbb{P}) \), \( b_1 := b[\rho_1, v_1] \) and \( b_2 := b[\rho_2, v_2] \) with \( (\rho_1, v_1) \) and \( (\rho_2, v_2) \) in \( \Xi_o \),
\[
\lim_{n \to \infty} (F_n(\lambda\beta^{b_1} + (1 - \lambda)\beta^{b_2}) - nd) \leq \\
\leq \lim_{n \to \infty} \{ \lambda(F_n(\beta^{b_1}) - nd) + (1 - \lambda)(F_n(\beta^{b_2}) - nd) \} < \infty \quad (2.12)
\]

Since the convex combination of three elements is equal to the convex combination of proper two elements, one can see by induction that for any finite convex combination \( \sum_{i=1}^{m} \alpha_i \beta^{b_i} \), \((\beta^{b_i})_{i=1}^{m}\) being Markovian elements in \( L^2_{[0,1]}(\mathbb{P}) \), we have

\[
\lim_{n \to \infty} [F_n(\sum_{i=1}^{m} \alpha_i \beta^{b_i}) - nd] < \infty
\]

Denoting by \( \Sigma \) the convex set given by all finite convex combinations of Markovian elements in \( L^2_{[0,1]}(\mathbb{P}) \), we can define the convex functional

\[
\hat{F}_\infty : L^2_{[0,1]}(\mathbb{P}) \to \bar{\mathbb{R}}
\]

\[
\hat{F}_\infty(\beta) := \begin{cases} 
\lim_{n \to \infty} (F_n(\beta) - nd) & \forall \beta \in \Sigma \\
+\infty & \text{otherwise}
\end{cases} \quad (2.13)
\]

The elements of \( \Sigma \) are not markovian in general and they are somehow reminiscent of the quantum mixtures, but in fact describing quantum mixtures would require an enlarged probability space (see for example [3]).
3. Critical points and minima

Definition 2. Let $(Ω, F, (F_t)_{t}), (P, W)$ and $X_o$ be defined as in the beginning of Section 2. Assume also $b \equiv b[\rho, v]$, with $(\rho, v) \in Ξ_o$ and let $q^b$ be defined by (2.1).

We define the functionals

$$I : \{ b = b[\rho, v] : (\rho, v) \in Ξ_o \} \to ℜ$$

$$b \mapsto ℰ \int_0^1 (b^2(q^b_t, t) + \nabla b(q^b_t, t)) dt \quad (3.1)$$

and,

$$\Lambda^Q : \{(\rho, v) : \text{finite action condition holds}\} \to ℜ$$

$$(\rho, v) \mapsto \int_0^1 \int_{R^d} (v^2 - \frac{1}{2} \nabla (\ln \rho)^2) \rho dx dt \quad (3.2)$$

By (2.10) we have, $\forall (\rho, v) \in Ξ_o$

$$\Lambda^Q(\rho, v) = I(b[\rho, v]) \quad (3.3)$$

and, defining $\beta^{b[\rho, v]}$ by (2.7) and (2.8),

$$I(b[\rho, v]) = \hat{F}_\infty(\beta^{b[\rho, v]}) \quad (3.4)$$

Let $Ξ_o(\rho_0, \rho_1)$ be constituted by all elements $(\rho, v)$ in $Ξ_o$ such that $\rho(., 0)$ is equal to $\rho_0$ and $\rho(., 1)$ is equal to $\rho_1$. We face the problem of looking for possible minima of $\Lambda^Q$ on $Ξ_o(\rho_0, \rho_1)$.

As a first step we revisit the fluid dynamical version of the variational principle given in [7], that was proposed in [9].

For any sufficiently regular function $f$ of a pair $(\rho, v)$ in $Ξ_o$ and for any sufficiently regular pair $(\delta \rho : (R^d \times [0, 1] \to R), \delta v : (R^d \times [0, 1] \to R^d))$ we will use the short-hand notation

$$\tilde{D}f(\rho, v)(\delta \rho, \delta v) := \lim_{\epsilon \to 0} \frac{1}{\epsilon} [f(\rho + \epsilon \delta \rho, v + \epsilon \delta v) - f(\rho, v)] \quad (3.5)$$

Definition 3. We will say that $(\rho, v) \in Ξ_o(\rho_0, \rho_1)$ is a critical point of $\Lambda^Q$ if

$$\tilde{D} \Lambda^Q(\rho, v)(\delta \rho, \delta v) = 0$$

for all $(\delta \rho, \delta v)$ satisfying the conditions

a) $\delta \rho \in C^\infty_c(R^d \times [0, 1] \to R)$, $\delta v \in C^\infty_c(R^d \times [0, 1] \to R^d)$ with $\delta \rho_o = 0$ and $\delta \rho_1 = 0$

b) $\frac{\partial}{\partial y} \{ \partial_t (\rho + y \delta \rho) + \nabla [(\rho + y \delta \rho)(v + y \delta v)] \} |_{y=0} = 0$. 
Lemma 1. Let \((\delta \rho, \delta v)\) satisfy conditions a) and b).

Then a sufficient condition in order that an element \((\rho, v)\) of \(\Xi_o(\rho_0, \rho_1)\) satisfies the equality
\[
\tilde{A}^Q(\rho, v)(\delta \rho, \delta v) = 0, \quad \forall (\delta \rho, \delta v) \text{ s.t. a) and b) hold}
\] (3.6)
is that, for all \((x, t) \in \mathbb{R}^d \times [0, 1] \)

i) There exists a smooth \(S: \mathbb{R}^d \times [0, 1] \to \mathbb{R}\) such that
\[
v(x, t) = \nabla S(x, t)
\]

ii) \((\rho, \nabla S)\) satisfies Madelung’s equations, so that
\[
\partial_t \rho(x, t) + \nabla(\rho(x, t)\nabla S(x, t)) = 0 \quad (3.7)
\]
\[
\partial_t S(x, t) + \frac{1}{2}(\nabla S(x, t))^2 - \frac{1}{2} \frac{\nabla^2 \sqrt{\rho(x, t)}}{\sqrt{\rho(x, t)}} = 0 \quad (3.8)
\]

Proof. Let \(\lambda : \mathbb{R}^d \times [0, 1] \to \mathbb{R}\) be smooth. Define
\[
F(\rho, v, \lambda) := A^Q(\rho, v) + \int_0^1 \int_{\mathbb{R}^d} \lambda(\partial_t \rho + \nabla(\rho v))dxdt
\]
In the given assumptions one has, for every \((\rho, v)\) in \(\Xi_o\)
\[
F(\rho, v, \lambda) := A^Q(\rho, v)
\]
and, for all \((\delta \rho, \delta v)\) which satisfy a) and b),
\[
\tilde{D}F(\rho, v, \lambda)(\delta \rho, \delta v, 0) = \tilde{D}A^Q(\rho, v)(\delta \rho, \delta v)
\]

Expliciting \(\tilde{D}\) and integrating by parts, one gets
\[
\tilde{D}F(\rho, v, \lambda)(\delta \rho, 0, 0) = \int_0^1 \int_{\mathbb{R}^d} \left[(-\partial_t \lambda - \nabla \lambda)v + v^2 + \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \right] \delta \rho dxdt + [\lambda \delta \rho]_0^1 + [\lambda \nu \delta \rho]_{-\infty}^{+\infty} \quad (3.9)
\]
and
\[
\tilde{D}F(\rho, v, \lambda)(0, \delta v, 0) = \int_0^1 \int_{\mathbb{R}^d} (2v - \nabla \lambda) \rho \delta v dxdt + [\lambda \rho \delta v]_{-\infty}^{+\infty} \quad (3.10)
\]
where all boundary terms are equal to zero. Then putting \(\lambda = 2S\) and \(v = \nabla S\), we get, by (3.8),
\[
\tilde{D}A^Q(\rho, \nabla S)(\delta \rho, \delta v) = \tilde{D}F(\rho, \nabla S, 2S)(\delta \rho, \delta v, 0) = 0
\]
\(\square\)
Corollary 1. If the Schrödinger equation

\[ i\partial_t\psi + \frac{1}{2}\nabla^2\psi = 0, \quad |\psi_0|^2 = \rho_0, \quad |\psi_1|^2 = \rho_1 \]  

(3.11)
is satisfied by \( \psi := \rho^2 \exp(iS) \) with \( (\rho, \nabla S) \in \Xi_o \) then (3.6) holds.

As a second step we show that Lemma 1 and the convexity of \( \hat{F}_\infty \) allows to solve a minimization problem for \( A^Q \).

Theorem 1. Let \( (\rho_o, \rho_1) \) be probability densities on \( \mathbb{R}^d \) with finite variance.

Assume that \( \psi := \rho^2 \exp^{iS} \) satisfies the free Schrödinger equation

\[ i\partial_t\psi + \frac{1}{2}\nabla^2\psi = 0, \quad |\psi_0|^2 = \rho_0, |\psi_1|^2 = \rho_1, \]  

(3.12)

and that \( (\rho, \nabla S) \) belongs to \( \Xi_o(\rho_o, \rho_1) \).

Then

\[ A^Q(\rho, \nabla S) \leq A^Q(\rho', v') \]  

(3.13)

\( \forall (\rho', v') \in \Xi_o(\rho_o, \rho_1) \)

Proof. Let \( (\rho', v') \) belong to \( \Xi_o(\rho_o, \rho_1) \). Put \( g := \rho' - \rho \). Being \( \rho' \) and \( \rho \) normalized to 1 one has that \( \int_{\mathbb{R}^d} g dx \) is equal to 0 and that \( \rho + yg \) is also normalized to 1 and strictly positive for all \( y \in [-1, 1] \). We consider firstly the case when \( g \) is of class \( C^\infty \).

Introducing a time dependent vector field \( X^g_y \), we consider the family \( (\rho'_y, v'_y)_{y \in [-1, 1]} \) defined by

\[
\begin{cases}
\rho'_y := \rho + yg \\
v'_y := \nabla S + X^g_y
\end{cases}
\]  

(3.14)

Requiring \( (\rho'_y, v'_y) \) to satisfy, for all \( y \in [-1, 1] \), the continuity equation

\[ \partial_t \rho'_y + \nabla_1 (\rho'_y v'_y) = 0 \]  

(3.15)

and putting

\[ u := (\rho + yg)(X^g_y) \]

one gets

\[ \sum_{i=1}^d \partial_{x_i} u_i = -y[\partial_t g + \sum_{i=1}^d \partial_{x_i} (g \partial_{x_i} S)] \]

This equation admits solutions \( u \) in \( C^\infty_K(\mathbb{R}^d \times [0, 1] \to \mathbb{R}^d) \) so that \( X^g_y \) can be chosen in \( C^\infty_K(\mathbb{R}^d \times [0, 1] \to \mathbb{R}^d) \). Then \( (\rho'_y, v'_y) \) belongs to \( \Xi_o(\rho_o, \rho_1) \) for all \( y \in [-1, 1] \).
Moreover one can check that $X^g_y$ depends smoothly on $y$ and that $\frac{d}{dy} X^g_y|_{y=0}$ also belongs to $C^\infty(\mathbb{R}^d \times [0,1] \to \mathbb{R}^d)$. Defining

$$\delta \rho := g$$

and

$$\delta v := \frac{d}{dy} X^g_y|_{y=0}$$

one can see that conditions a) and b) in Lemma 1 are satisfied. Then

$$\lim_{y \to 0} \frac{1}{y} (A^Q(\rho'_y, v'_y) - A^Q(\rho, \nabla S)) = \tilde{D} A^Q(\rho, \nabla S)(\delta \rho, \delta v) = 0 \quad (3.16)$$

We can exploit (3.3) to get

$$\lim_{y \to 0} \frac{1}{y} (I(b[\rho'_y, v'_y]) - I(b[\rho, \nabla S])) = 0$$

which, by (3.4), is equivalent to

$$\lim_{y \to 0} \frac{1}{y} (\hat{F}_\infty(\beta^{bl}[\rho'_y, v'_y]) - \hat{F}_\infty(\beta^{bl}[\rho, \nabla S])) = 0$$

Since by the chain of equalities (3.3) and (3.4), for all $y \in [0,1]$

$$\hat{F}_\infty(\beta^{bl}[\rho'_y, v'_y]) = A^Q(\rho'_y, v'_y) \quad (3.17)$$

with $(\rho'_y, v'_y)$ defined by (3.14), the dependence of $\hat{F}_\infty(\beta^{bl}[\rho'_y, v'_y])$ on $y \in [-1,1]$ is smooth by construction.

Finally, by the convexity of $\hat{F}_\infty$,

$$\hat{F}_\infty(\beta^{bl}[\rho, \nabla S]) \leq \hat{F}_\infty(\beta^{bl}[\rho'_y, v'_y]), \quad \forall y \in [-1,1] \quad (3.18)$$

Then, again by the chain of equalities (3.3) and (3.4) and putting $y = 1$ we find

$$A^Q(\rho, \nabla S) \leq A^Q(\rho', v')$$

for all $(\rho', v')$ in $\Xi(\rho_0, \rho_1)$ such that $\rho' := \rho + g$, $g \in C^\infty_K$.

For a generic element $(\rho', v')$ of $\Xi(\rho_0, \rho_1)$ put again $g := \rho' - \rho$. Then $g$ is smooth and such that $\int_{\mathbb{R}^d} g dx = 0$ but its support is non necessarily compact.

Let $(D_j)_j$ be a sequence of open bounded subsets of $\mathbb{R}^d \times [0,1]$ such that $D_j \uparrow \mathbb{R}^d \times [0,1]$.

Let also $(h_j)_j \subset C^\infty_K(\mathbb{R} \times [0,1] \to \mathbb{R})$ be such that $0 \leq h_j \leq 1$ and

$$h_j(x, t) = \begin{cases} 1 & (x, t) \in D_j \\ 0 & (x, t) \in U^\infty_{m=j+2}(D_m \setminus D_{m-1}) \end{cases} \quad (3.19)$$
Then, putting $g_j := gh_j$, $\rho'_j := \rho + g_j$ and $v'_j := v + X^g_j$, where $X^g_j$, $y \in [0,1]$, is constructed as before, one has, for all $j \in \mathbb{N}$,

$$A^Q(\rho, \nabla S) \leq A^Q(\rho'_j, v'_j) =$$

$$= \int_{D_j} (v'^2 - (\frac{1}{2} \nabla \log \rho'_j)^2) \rho'_j dx \, dt + \int_{D_j} (v'^2 - (\frac{1}{2} \nabla \log \rho'_j)^2) \rho'_j dx \, dt$$

Then

$$A^Q(\rho, \nabla S) \leq \lim_{j \to \infty} A^Q(\rho'_j, v'_j) = A^Q(\rho', v')$$

for all $(\rho', v')$ in $\Xi_\rho(\rho_0, \rho_1)$. □

4. COMPARISON WITH THE CLASSICAL OPTIMAL TRANSPORTATION PROBLEM

The classical Monge-Kantorovich Optimal Transportation Problem [10] [8] with quadratic cost, in the Monge formulation, is

$$\inf_{T: T\sharp \rho_0 = \rho_1} \int_{\mathbb{R}^d} |T x - x|^2 \rho_0(x) dx$$

(4.1)

where the infimum is taken on the set of maps $T: \mathbb{R}^d \to \mathbb{R}^d$ that "transport $\rho_0$ onto $\rho_1"."

In [1] Brénière proved that the solution $T_0$ of problem (4.1) is the unique map $T_0: T_0\sharp \rho_0 = \rho_1$, which is a gradient of a convex function $\phi$.

Moreover, by introducing the "time variable" $t \in [0,1]$, in [2] a computational fluid-mechanical solution of the Monge problem was constructed.

The new problem can be formulated as follows (see [17] chap 8):

$$\inf_{(\rho, v) \in V(\rho_0, \rho_1)} \int_{\mathbb{R}^d} \int_0^1 v^2 \rho dt dx$$

where $\mathbb{V}(\rho_0, \rho_1)$ is a very general set of fluid-dynamical couples $(\rho, v)$ which in particular satisfy the continuity equation in distributional sense. The "set of proper infinitesimal characteristics" connecting $\rho_0$ to $\rho_1$, and in particular $\Xi_\rho(\rho_0, \rho_1)$, are subsets of $\mathbb{V}(\rho_0, \rho_1)$.

Assuming that $\rho_0$ and $\rho_1$ have finite variance and denoting by $\tau_2(\rho_0, \rho_1)$ the infimum in (4.1), we have the Bénamou-Brénière formula

$$\tau_2(\rho_0, \rho_1) = \inf_{(\rho,v) \in \mathbb{V}(\rho_0, \rho_1)} A(\rho, v)$$

where
\[ A(\rho, v) := \int_{\mathbb{R}^d} \int_0^1 v^2 \rho dt dx \]

which, when restricted to \( \Xi_\omega(\rho_0, \rho_1) \), is the classical limit of \( A^Q \). By a simple change of variables the functional \( A \) becomes convex.

Moreover if \((\rho, v)\) is the solution to the Bénamou-Brézis problem, it turns to be a gradient-flow and, in the smooth case, the classical limit of the free Madelung fluid equations holds, which are equivalent to

\[
\begin{align*}
\partial_t \rho + \nabla (\rho v) &= 0 \\
\partial_t v + v \nabla v &= 0, \quad v_o(x) = \nabla \phi(x) - x
\end{align*}
\]

where \( \nabla \phi \) is the solution of the Monge problem.
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