GROUPS OF ORDER $p^3$ ARE MIXED TATE

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Abstract. Let $G$ be a finite group. A natural place to study the Chow ring of the classifying space $BG$ is Voevodsky’s triangulated category of motives, inside which Morel–Voevodsky and Totaro have defined motives $M(BG)$ and $M_c(BG)$, respectively. We show that, for any group $G$ of order $p^3$ over a field of characteristic not $p$ which contains a primitive $p^3$-th root of unity, the motive $M(BG)$ is a mixed Tate motive. We also show that, for a finite group $G$ over a field of characteristic zero, $M(BG)$ is a mixed Tate motive if and only $M_c(BG)$ is a mixed Tate motive.

1. Introduction

1.1. Mixed Tate groups. The group cohomology of a group $G$ can be computed as the cohomology (with twisted coefficients) of the classifying space $BG$. One would like to understand what part of the group cohomology of $G$ comes from algebraic geometry. Morel–Voevodsky [17] and Totaro [21] defined the motive of a classifying space $M(BG)$ and the motive of a classifying space with compact supports $M_c(BG)$, respectively, as objects in $DM(k; R)$, Voevodsky’s “big” triangulated category of motives over the field $k$ with coefficients in a commutative ring $R$ [23]. One can recover the motivic (co)homology groups of $BG$ as defined by Edidin–Graham [7] by computing the motivic (co)homology groups of these motives.

Inside $DM(k; R)$, one can define the subcategory of mixed Tate motives $DMT(k; R)$ as the smallest triangulated and closed under arbitrary direct sums subcategory which contains all the objects $R(j)$ with $j \in \mathbb{Z}$. We prove in Theorem 5.1 that the motive $M(BG)$ is mixed Tate if and only if $M_c(BG)$ is mixed Tate for a finite group $G$. We will simply say that a finite group $G$ is mixed Tate if $M_c(BG)$ is in the category $DMT(k; R)$. From now on, we will restrict the discussion in the Introduction to finite groups. Our main result is:

Theorem 1.1. Let $G$ be a group of order $p^3$ and let $k$ be a field of characteristic not $p$ which contains a primitive $p^3$-root of unity. Then $M_c(BG)$ is mixed Tate.

One is interested in understanding $p$-groups because one recovers important information about a given finite group by studying its Sylow groups. The precise form of this philosophy which is applicable in our case is [21, Lemma 9.3] which says that $BG$ is mixed Tate with $\mathbb{Z}/p$ or $\mathbb{Z}(p)$ coefficients if $BH$ is, where $H$ is a $p$-Sylow subgroup of $G$.

1.2. Other properties of finite groups. A group $G$ is called stably rational if it has a faithful representation $V$ such that $V \oplus G$ is stably rational over $\mathbb{C}$. A group $G$ has the weak Chow–K"unneth property if $CH^*(BG) \to CH^*(BG_E)$ is surjective for every extension of fields $E/k$. If $G$ is mixed Tate, then $BG$ is stably rational, satisfies the weak Chow–Kunneth property, and has trivial unramified cohomology.
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see [21] Section 9] for definitions and references. We do not know whether any of these properties of a finite group $G$ are equivalent.

1.3. Related results. In all the following examples, we assume that $k$ is a field in which $p$ is invertible and which contains $|G|$-roots of unity, where $G$ is the group studied.

The starting point for studying these properties of a group $G$ are Bogomolov’s [2] and Saltman’s [19] examples of groups of order $p^5$ and $p^9$, respectively, which are not stably rational. Chu–Kang, and Chu et. al. [4], [3] showed that for every $p$-group $G$ of order $\leq p^4$ or 2-group of order $\leq 2^5$ and for every $G$-representation $V$, the quotient $V \sslash G$ is rational. This property is stronger than saying that $BG$ is stably rational.

Bogomolov [2] showed (with a further correction in [10]) that every $p$-group of order $\leq p^3$, for $p$ odd prime, or $\leq 2^5$ for $p$ equal to 2, has trivial unramified cohomology, and that these are the best possible bounds.

Totaro [21, Section 10] showed that all 2-groups of order $\leq 2^5$ and all $p$-groups of order $\leq p^4$ have the weak Chow-Künneth property. He also showed [21, Corollary 9.10] that all abelian $p$-groups are mixed Tate. There are groups of order $p^3$ for $p$ odd which do not have the weak Chow-Künneth property [21, Discussion after Corollary 3.1] and thus which are not mixed Tate.

In view of these examples, it is worth investigating whether all $p$-groups of order $\leq p^4$ and all 2-groups of order $\leq 2^5$ are actually mixed Tate. Our methods only apply to $p$-groups of order $\leq p^3$ and to some groups of order $p^4$ as explained in Section 4.

1.4. Structure of the paper. In Section 2 we recall the definitions of linear schemes and of the motives $M(X)$ and $M^c(X)$ for a quotient stack $X$ in $\text{DM}(k; R)$. In Section 3 we reduce the proof of Theorem 1.1 to Theorem 3.3 and we prove three technical preliminary Propositions. Section 4 contains the proof of Theorem 3.3 which says that for a group $G$ of order $p^3$ and $V$ an irreducible $G$-representation of dimension $p$, the scheme $V \sslash G$ is a linear scheme. The proof is inspired by a result of Chu–Kang [4] that says that $V \sslash G$ is rational for $G$ of order $p^3$ and $V$ a $G$-representation. In Section 5 we show that $M(BG)$ is mixed Tate if and only if $M^c(BG)$ is mixed Tate.

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2. Definitions and notations

2.1. Fix $p$ a prime number. Unless otherwise stated, we will denote by $k$ a field of characteristic not $p$ which contains a primitive $p^2$-root of unity. In Section 4 we assume that the characteristic of $k$ is zero.

All the schemes considered will be separated schemes of finite type over $k$. One can define the Chow groups $CH_i(X)$ as the group of dimension $i$ algebraic cycles modulo rational equivalence [8]. One can further define the higher Chow groups [1], or the motivic (co)homology groups of such a scheme [23], see [21, Section 5] for a brief overview of these topics.
Let $A$ be an affine $k$-scheme with a linear action of a reductive group $G$. We denote by $A \sslash G := \text{Spec} \left( \mathcal{O}_A^G \right)$ the quotient scheme and by $A/G$ the corresponding quotient stack.

For a finite group $G$, we denote by $|G|$ the order of $G$. We denote by $[n]$ the set $\{1, \ldots, n\}$.

2.2. We will work in the category $\text{DM}(k; R)$, the “big” triangulated category of motives over the field $k$ with coefficients in the commutative ring $R$ \cite[Section 5]{21}, see also the general references \cite{16, 23}.

The exponential characteristic of $k$ is 1 if $k$ has characteristic zero and $p$ if $k$ has characteristic $p > 0$. We will assume throughout the paper that the exponential characteristic of $k$ is invertible in $R$. Voevodsky defined two natural functors from the category of schemes to $\text{DM}(k; R)$, which we will write as $M$ and $M^\heartsuit$ \cite[Section 5]{21}, see also \cite[Section 5]{21}.

We can associate a motive to any quotient stack $X = Y/G$, with $Y$ a quasi-projective scheme over $k$ and $G$ an affine group scheme of finite type over $k$ such that there is a $G$-equivariant ample line bundle on $Y$, as follows \cite[Section 8]{21}. Choose $G$-representations $V_1 \hookrightarrow V_2 \hookrightarrow \ldots$ of $G$ such that $\text{codim} (S_i \subset V_i)$ increases to infinity, where $S_i$ is the locus of $V_i$ where $G$ does not act freely. Denote by $M_i(X) := M \left( \left( \left( (V_i - S_i) \times Y \right) /G \right) \right)$ and define

$$M(X) = \text{hocolim} \left( \cdots \leftarrow M_2(X) \leftarrow M_1(X) \right) \right),$$

where the maps are induced by the inclusions $V_i \hookrightarrow V_{i+1}$. To define $M^\heartsuit(X)$, choose $G$-representations $\cdots \to V^2 \to V^1$ with loci $S^i$ with the same property as above. Let $M^\heartsuit_i(X) := M \left( \left( \left( V^i - S^i \right) \times Y \right) /G \right)$. Let $n_i$ be the rank of the bundle $V^i$. Define

$$M^\heartsuit(X) = \text{holim} \left( \cdots \to M_2^\heartsuit(X)(-n_2)[-2n_2] \to M_1^\heartsuit(X)(-n_1)[-2n_1] \right),$$

where the maps are induced by the projections $V^{i+1} \to V^i$. The definitions of $M^\heartsuit(X)$ and $M(X)$ are independent of the choices of $V_i$ and $V^i$; see \cite[Theorem 8.4 and the discussion in Section 8]{21}.

2.3. A linear scheme over $k$ is defined inductively as follows \cite[Section 5, page 17-18]{21}: all the affine spaces are linear; if $Z \subset X$ is closed, and $X$ and $Z$ are linear, then $X \setminus Z$ is linear; further, if $X \setminus Z$ and $Z$ are linear, then $X$ is linear \cite[page 17]{21}. There are examples of schemes with mixed Tate motive, but which are not linear schemes \cite{9}.

Let $X$ be a linear scheme over $k$ and let $R$ be a ring whose exponential characteristic is invertible in $R$. Then $M^\heartsuit(X)$ is a mixed Tate motive.

Let $I$ be a finite set, let $X_i \subset X$ be locally closed irreducible subschemes of $X$, and let $d = \dim(X)$. For $e \leq d$, let $Y_e$ be the union of $X_i$ for $i \in I$ such that $\dim(X_i) = e$. We say that $X$ has a stratification $(X_i)_{i \in I}$ if there is a partition of underlying topological spaces

$$X = \bigcup_{i \in I} X_i$$

and $Y_e$ is open in $X \setminus \bigcup_{f > e} Y_f = \bigcup_{g \leq e} Y_g$ for every $e \leq d$. 

3. The plan of the proof and preliminaries

3.1. Theorem \[11\] is known for abelian groups \[21\] Corollary 9.10. The two non-abelian groups of order 8 are the dihedral and the quaternion group. Theorem \[11\] holds for them by \[21\] Corollary 9.7. It thus suffices to show the following:

**Theorem 3.1.** Let \( p \) be an odd prime, let \( k \) be a field of characteristic not \( p \) which contains a primitive \( p^2 \)-root of unity, and let \( G \) be a non-abelian group of order \( p^3 \). Then \( M_i'(BG) \) is mixed Tate.

There are sufficient conditions on \( G \) which imply that \( G \) is mixed Tate. For example, by \[21\] Theorem 9.6] it is enough to show that every proper subgroup \( H \subset G \) is mixed Tate and that there exists a faithful representation \( V \) of \( G \) such that the variety \( (V - S) \parallel G \) is mixed Tate, where \( S \) is the closed subset of \( V \) where \( G \) does not act freely.

For \( K \subset G \) a subgroup, let \( N_K := \{ g \in G | gKg^{-1} = K \} \) be the normalizer of \( K \) and let \( N'_K := N K / K \).

**Proposition 3.2.** Let \( G \) be a finite group such that \( N'_K \) is abelian for every subgroup \( 1 < K \subset G \). Let \( V \) be a representation of \( G \) and let \( S \subset V \) be the locus of points with non-trivial stabilizer. Then \( (V - S) \parallel G \) is a linear scheme if and only if \( V \parallel G \) is a linear scheme.

**Proof.** It suffices to check that \( S \parallel G \) is a linear scheme. We use induction on \( |G| \). The statement is clear if \( |G| \) is a prime number, because then \( G \) is a cyclic group and \( S \) is a subspace of \( V \), and so \( S \parallel G \cong S \) is an affine space.

For \( K \subset G \) a subgroup, let \( V^K \subset V \) be the subspace of points fixed by \( K \) and let

\[
V_K := V^K - \bigcup_{K < L \subset G} V^L.
\]

If \( K' \) is a subgroup of \( G \) conjugate to \( K \), the images of \( V_K \parallel N'_K \) and \( V_{K'} \parallel N'_{K'} \) in \( V \parallel G \) are the same. Let \( I \) be a set of subgroups of \( G \) such that any subgroup \( K \) of \( G \) is conjugate to a unique group in \( I \). We have that \( S = \bigsqcup_{i < K \in G} V_K \) and there is a stratification

\[
S \parallel G = \bigsqcup_i V_K \parallel N'_K
\]

It suffices to check that \( V_K \parallel N'_K \) is a linear scheme for any \( 1 < K \subset G \). The group \( N'_K \) is abelian, so it satisfies the hypothesis of the Proposition. We have that \( |N'_K| < |G| \), so by the induction hypothesis we know that \( V_K \parallel N'_K \) is a linear scheme if and only if \( V^K \parallel N'_K \) is a linear scheme. By Proposition 3.3, the quotient \( V^K \parallel N'_K \) is a linear scheme, thus \( V_K \parallel N'_K \) is a linear scheme. \( \square \)

Any non-abelian group of order \( p^3 \) has a faithful irreducible representation. Indeed, a \( p \)-group has a faithful irreducible representation if and only if its center is cyclic \[11 \ p.29\], and \( Z(G) \) has order \( p \) for any non-abelian group of order \( p^3 \). Moreover, all irreducible representations of a group \( G \) of order \( \leq p^4 \) have dimension 1 or \( p \). Any group of order \( p^3 \) satisfies the hypothesis of Proposition 3.2 because for every subgroup \( 1 < K \subset G \), the quotient \( N_K / K \) has order \( 1, p, \) or \( p^2 \), and thus it is abelian. It is thus sufficient to prove the following:

**Theorem 3.3.** Let \( k \) be a field of characteristic not \( p \) which contains a primitive \( p^2 \)-root of unity. Let \( G \) be a non-abelian group of order \( p^3 \) and let \( V \) be an irreducible representation of degree \( p \). Then \( V \parallel G \) is a linear scheme.
3.2. There are two non-abelian groups of order $p^3$. For a classification of $p$-groups of order $\leq p^4$ and their representations, see [4].

3.2.1. The first group is $G \cong (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/p$, which can be also written as

$$G = \langle \sigma, \pi, \tau | \sigma^p = \pi^p = \tau^p = 1, \sigma \pi = \pi \sigma, \sigma \tau = \tau \sigma, \tau \pi \tau^{-1} = \sigma \pi \rangle.$$  

It has a faithful irreducible representation $(\rho, V)$ which can be written explicitly on a basis $(e_i)_{i=1}^p$ of $V$ as follows:

$$\rho(\sigma) = \text{diag}(\zeta, \ldots, \zeta),$$

$$\rho(\pi) = \text{diag}(1, \zeta, \ldots, \zeta^{p-1}),$$

$$\rho(\tau) = P,$$

where $P$ is the matrix which permutes the basis $e_1 \mapsto e_2 \mapsto \cdots \mapsto e_p \mapsto e_1$, and $\zeta$ is a primitive $p$-th root of unity.

3.2.2. The second group is $G \cong \mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$, which can be also written as

$$G = \langle \sigma, \tau | \sigma^{p^2} = \tau^p = 1, \tau \sigma \tau^{-1} = \sigma^{1+p} \rangle.$$  

It has a faithful irreducible representation $(\rho, V)$ given by

$$\rho(\sigma) = \text{diag}(\omega, \omega^{1+p}, \ldots, \omega^{1+p(p-1)}),$$

$$\rho(\tau) = P,$$

where $\omega$ is a primitive $p^2$-root of unity and $P$ is the permutation matrix defined above.

3.3. The proof of Theorem 3.3 will be given in Section 4. In the rest of this Section, we include proofs for two Propositions used in its proof. The first one gives a proof of the already known fact that $BG$ is mixed Tate for $G$ abelian group [21, Corollary 9.10]. Recall that the exponent of a group is defined as the least common multiple of the orders of all elements of the group.

**Proposition 3.4.** Let $N$ be an abelian $p$-group, and let $V$ be a $N$-representation over a field $k$ of characteristic not $p$ which contains the $p^e$ roots of unity, where $p^e$ is the exponent of $N$. Then $\text{Spec } k[V]^N$ is a linear scheme.

**Proof.** As $\text{char } k \neq p$, the representation $V$ decomposes as a sum of one dimensional representations, and thus we can choose a basis $x_1, \ldots, x_d$ of $V$ on which $N$ acts diagonally. We prove the statement by induction on $|N|$. The base case, when $N$ is the trivial group, is clear. In general, choose $\sigma \in N$ such that $N = \langle \sigma \rangle \oplus M$, where $\langle \sigma \rangle$ denotes the subgroup of $N$ generated by $\sigma$. Assume that $\sigma$ has order $p^e$. We will use the following stratification:

$$\text{Spec } k[x_1, \ldots, x_d] = \bigsqcup_{J \subset [d]} \text{Spec } k \left[ x_j^\pm 1 \big| j \in J \right],$$

where the disjoint union is taken after all sets $J \subset [d]$. This stratification is the partition of the affine space $\mathbb{A}^d_k$ into $2^d$ schemes $P_J$ with $x_j \neq 0$ for $j \in J$ and $x_j = 0$ for $j \notin J$. We obtain a stratification

$$(3.1) \quad \text{Spec } k[x_1, \ldots, x_d]^{(\sigma)} = \bigsqcup_{J \subset [d]} \text{Spec } k \left[ x_j^\pm 1 \big| j \in J \right]^{(\sigma)}.$$
It is enough to show that
\[(3.2) \quad \text{Spec } k \left[ x_1^{\pm 1}, \ldots, x_d^{\pm 1} \right]^{(\sigma)} \cong \text{Spec } k \left[ y_j^{\pm 1} \right], \]
where the $y_j$ are monomials in $x_i$. The analogous statement holds for any stratum on the right hand side of (3.1). Once we show (3.2), we can reduce the problem from $N$ to $M$ for various representations of $M$.

To find such a decomposition, let $\sigma \cdot x_i = \zeta^{a_i} x_i$, where $\zeta$ is a primitive $p^s$-root of unity chosen such that $a_1 = 1$. Then
\[
k \left[ x_1^{\pm 1}, \ldots, x_d^{\pm 1} \right]^{\sigma} = k \left[ x_1^{p^s}, x_2 x_1^{-a_2}, \ldots, x_d x_1^{-a_d}, \frac{1}{x_1^p x_2 \ldots x_d} \right],
\]
where $Q := p^s - a_2 - \cdots - a_d$. The right hand side is included in the left hand side, and $k \left[ x_1^{\pm 1}, \ldots, x_d^{\pm 1} \right]$ is a free $k \left[ x_1^{p^s}, x_2 x_1^{-a_2}, \ldots, x_d x_1^{-a_d}, \frac{1}{x_1^p x_2 \ldots x_d} \right]$-module of rank $p^s$, so the two sides are indeed equal. \(\square\)

Consider the torus $(\mathbb{G}_m)^p$ with coordinates $w_1, \ldots, w_p$ and let $W \subset (\mathbb{G}_m)^p$ be the subtorus with $w_1 \cdots w_p = 1$. The action of the cyclic group $\mathbb{Z}/p$ of order $p$ which permutes the factors of $(\mathbb{G}_m)^p$ by $w_i \mapsto w_{i+1}$ for $1 \leq i \leq p$, where $w_{p+1} := w_1$, extends to an action of $\mathbb{Z}/p$ on $W$.

**Proposition 3.5.** The schemes $S := W \sslash \mathbb{Z}/p$ and $T := ((\mathbb{G}_m)^p - W) \sslash \mathbb{Z}/p$ are linear schemes.

**Proof.** Let $\tau$ be a generator of the cyclic group $\mathbb{Z}/p$. Define
\[W_d = 1 + \zeta^d w_1 + \cdots + \zeta^{d(p-1)} w_1 \cdots w_{p-1}\]
for $d = 0, \ldots, p-1$. The stratification we are going to use is
\[S = \bigsqcup_{d=0}^{p-1} S_d,\]
where the schemes $S_d$ are defined as
\[S_d := \text{Spec } \left( k \left[ w_1^{\pm 1}, \ldots, w_{p-1}^{\pm 1}, \frac{1}{W_d} \right] \right)^\tau / (W_0, \ldots, W_{d-1}).\]
We will show that every such piece is a linear scheme.

**Step 1.** We first explain the argument for $S_0 = \text{Spec } k \left[ w_1^{\pm 1}, \ldots, w_{p-1}^{\pm 1}, \frac{1}{W_0} \right]^\tau$. Define
\[s_i := \frac{\prod_{j \leq i} w_j}{W_0},\]
for $i \in \{0, \ldots, p-1\}$, $w_0 := 1$. Observe that $s_0 + \cdots + s_{p-1} = 1$ and that $k \left[ w_1^{\pm 1}, \ldots, w_{p-1}^{\pm 1}, \frac{1}{W_0} \right] \cong k \left[ s_0^{\pm 1}, \ldots, s_{p-1}^{\pm 1} \right] / (s_0 + \cdots + s_{p-1} - 1)$. Further, $\tau$ acts via $\tau : s_0 \mapsto s_1 \mapsto \cdots \mapsto s_{p-1} \mapsto s_0$. To show that
\[
\text{Spec } \left( k \left[ s_0^{\pm 1}, \ldots, s_{p-1}^{\pm 1} \right] / (s_0 + \cdots + s_{p-1} - 1) \right)^\tau
\]
is a linear scheme, we linearize the action by introducing the variables
\[v_i = s_0 + \zeta^i s_1 + \cdots + \zeta^{i(p-1)} s_{p-1},\]
\( v_0 = 1 \). Then \( \tau v_i = \zeta^{-i} v_i \) and

\[
  s_i = \frac{v_0 + \zeta^{-i} v_1 + \cdots + \zeta^{-i(p-1)} v_{p-1}}{p}.
\]

In this basis, \( S_0 \) becomes

\[
  \text{Spec} \left( k \left[ v_0, \ldots, v_{p-1}, \frac{1}{\prod_{i=0}^{p-1} \tau^i(l)} \right] / (v_0 - 1) \right) \cong \text{Spec} \left[ v_1, \ldots, v_{p-1}, \frac{1}{\prod_{i=0}^{p-1} \tau^i(l)} \right],
\]

where \( l = 1 + v_1 + \cdots + v_{p-1} \) is the equation of a hyperplane. Now, we can realize \( S_0 \) as the complement of a linear scheme inside an affine space. Indeed,

\[
  \text{Spec} \left[ v_1, \ldots, v_{p-1} \right] = \text{Spec} k \left[ v_1, \ldots, v_{p-1}, \frac{1}{\prod_{i=0}^{p-1} \tau^i(l)} \right] \bigcup \text{Spec} \left( k \left[ v_1, \ldots, v_{p-1} \right] / \prod_{i=0}^{p-1} \tau^i(l) \right)
\]

and \( \tau \) acts on both terms on the bottom line.

Observe that Spec \( k \left[ v_1, \ldots, v_{p-1} \right] / \prod_{i=0}^{p-1} \tau^i(l) \) is the union of the hyperplanes \( l, \tau(l), \ldots, \tau^{p-1}(l) \), which are cyclically permuted by \( \tau \). Both Spec \( k \left[ v_1, \ldots, v_{p-1} \right] \) and Spec \( k \left[ v_1, \ldots, v_{p-1} \right] / \prod_{i=0}^{p-1} \tau^i(l) \) are linear schemes, so \( S_0 \) is indeed a linear scheme.

**Step 2.** Fix \( 0 \leq d \leq p - 1 \). The proof that \( S_d \) is a linear scheme is similar to the one in Step 1. Define

\[
  s_i = \frac{\prod_{j \leq i} w_j}{W_d},
\]

for \( i = 0, \ldots, p - 1, \ w_0 := 1 \). Observe that \( s_0 + \cdots + \zeta^{d(p-1)} s_{p-1} = 1 \) and that

\[
  k \left[ w_1^{\pm 1}, \ldots, w_{p-1}^{\pm 1}, \frac{1}{W_d} \right] = k \left[ s_0^{\pm 1}, \ldots, s_{p-1}^{\pm 1} \right] / (s_0 + \cdots + \zeta^{d(p-1)} s_{p-1} - 1).
\]

Furthermore, we have that

\[
  W_e = \frac{s_0 + \cdots + \zeta^{e(p-1)} s_{p-1}}{s_0}
\]

for \( e \leq d \), so computations similar to those for \( S_0 \) show that

\[
  S_d \cong \text{Spec} \left( k \left[ s_0^{\pm 1}, \ldots, s_{p-1}^{\pm 1} \right] / I \right),
\]

where \( I \) is the ideal generated by \( s_0 + \zeta^e s_1 + \cdots + \zeta^{e(p-1)} s_{p-1} \) for all \( 0 \leq e \leq d - 1 \), and by \( s_0 + \zeta^d s_1 + \cdots + \zeta^{d(p-1)} s_{p-1} - 1 \). Changing the basis to \( v_j \) defined as in Step 1, we find out that

\[
  S_d \cong \text{Spec} \left( k \left[ v_{d+1}, \ldots, v_{p-1}, \frac{1}{\prod_{i=0}^{p-1} \tau^i(l)} \right] \right).
\]

The end of the argument in Step 1 shows that \( S_d \) is a linear scheme.

**Step 3.** The proof that \( T \) is a linear scheme is already contained in the above argument. Indeed, introduce the basis

\[
  v_j = s_0 + \zeta^j s_1 + \cdots + \zeta^{j(p-1)} s_{p-1},
\]
for \( j = 0, \ldots, p - 1 \). Then we need to show that

\[
\text{Spec} \left( k \left[ v_0, \ldots, v_{p-1}, \frac{1}{\prod_{i=0}^{p-1} \tau^i(l)} \right] \right)
\]

is a linear scheme, where \( \tau \) acts on the \( v_i \) by \( \tau(v_i) = \zeta^{-i} v_i \) and \( l = v_0 + \cdots + v_{p-1} \) is a hyperplane. The same argument as in Step 1 shows this is a linear scheme. \( \square \)

4. Proof of Theorem 3.3

4.1. In the beginning, we will work in a little more general framework which also covers some groups of order \( p^4 \). Thus, assume for the moment that \( G \) has order \( \leq p^4 \) and has an irreducible representation of dimension \( p \). We may assume that \( V \) is faithful, and let \( \rho : G \to GL(V) \). As \( \rho \) is irreducible, it is induced from a one dimensional representation of a subgroup \( N \subset G \), that is, \( \rho = \text{Ind}^G_N \psi \) with \( \psi : N \to GL(W) \) and with \( W \) one dimensional \([14]\). As \( V \) has dimension \( p \), the subgroup \( N \) has index \( p \) in \( G \), and so \( N \leq G \).

Choose representatives \( \{1, t, \ldots, t^{p-1}\} \) for the cosets of \( G/N \). The explicit form of \( \rho \) is

\[
\rho(g) = (\psi(t^{-i}gt^j))_{0 \leq i, j \leq p-1},
\]

where \( \psi(g) = 0 \) if \( g \notin N \).

If \( Z(G) \not\subset N \), we can choose \( t \in Z(G) \). Then \( \rho(g) = (\psi(gt^{-i}j)) \), so \( \rho(g) = \psi(g)I \), for every \( g \in N \). As \( \rho \) is faithful, this implies that \( N \subset Z(G) \), and further that \( G \) is abelian, contradicting that \( G \) has an irreducible representation of dimension \( p \).

We thus have that \( Z(G) \subset N \). In order for \( \rho \) to be faithful, \( \psi|_{Z(G)} \) needs to be faithful, too, so \( Z(G) \) is cyclic.

Using the explicit description of \( \rho \), we have that \( \rho(G) \subset T \cdot W \), where \( T \) is the group of diagonal matrices and \( W \) is the group of permutation matrices. By identifying \( G \) with its image \( \rho(G) \), \( G \) can be written as a semi-direct product \( N \rtimes M \), with \( M \cong \mathbb{Z}/p \), and \( N \) an abelian \( p \)-group with \( |N| \leq p^3 \).

4.2. The plan is to construct a decomposition of \( V \sslash G \) into smaller linear schemes. We isolate one open subset of \( V \sslash G \) and decompose its complement in linear schemes. After that, we show that the open subset is itself a linear scheme.

Choose a basis \( x_1, \ldots, x_p \) of \( V \) on which \( N \) acts diagonally and which is cyclically permuted by \( \tau \), the generator of \( M \). Observe that

\[
V \sslash G = \text{Spec} \ k[x_1, \ldots, x_p]^G = \text{Spec} \ (k[x_1, \ldots, x_p]^N)\tau.
\]

As we have already discussed in the proof of Proposition 3.4, there is a stratification

\[
\text{Spec} \ k[x_1, \ldots, x_p] = \bigcup_{J \subset [p]} \text{Spec} k\left[ x_j^{\pm 1} \middle| j \in J \right],
\]

where the disjoint union is taken after all sets \( J \subset [p] \). This stratification is the partition of the affine space \( \mathbb{A}^p_k \) in the \( 2^p \) schemes \( P_J \) with \( x_j = 0 \) for \( j \notin J \) and \( x_j \neq 0 \) for \( j \notin J \). As \( N \) acts linearly on the functions \( x_i \) for \( 1 \leq i \leq p \), we have that

\[
\text{Spec} k[x_1, \ldots, x_p]^N = \text{Spec} k\left[ x_1^{\pm 1}, \ldots, x_p^{\pm 1} \right]^N \sqcup \bigcup_{J \subset [p]} \text{Spec} k\left[ x_j^{\pm 1} \middle| j \in J \right]^N.
\]

By Proposition 3.4 each \( \text{Spec} k\left[ x_j^{\pm 1} \middle| j \in J \right]^N \) for \( J \subset [p] \) is a linear scheme.
Let $t: [p] \to [p]$ be the function $t(x) = x + 1$ for $x \leq p - 1$ and $t(p) = 1$. For $J \subset [p]$, let $t(J) := \{t(x) | x \in J\} \subset [p]$. Observe that $\tau$ permutes the schemes $S_J = \text{Spec} \ k \left[ x_j^{\pm 1} | j \in J \right]^N$ by sending $S_J$ to $S_{t(J)}$. Consequently, there is a stratification

$$\text{Spec} \ k[x_1, \ldots, x_p]^G = \text{Spec} \ k \left[ x_1^{\pm 1}, \ldots, x_p^{\pm 1} \right]^G \sqcup \bigcup_A \text{Spec} \ S_J,$$

where $A$ is a set of representatives of the equivalence classes of the action of $t$ on the set of proper subsets of $[p]$. This means that, in order to show that $V \parallel G$ is a linear scheme, we have to prove that $\text{Spec} \ k \left[ x_1^{\pm 1}, \ldots, x_p^{\pm 1} \right]^G$ is a linear scheme. We do this in the next Subsection.

4.3. The study of this open piece is inspired by [4]. We begin by analyzing the $Z(G)$-invariants. If we can conveniently reduce the dimension of the scheme $\text{Spec} \ k \left[ x_1^{\pm 1}, \ldots, x_p^{\pm 1} \right]$ on which $G$ acts from $p$ to $p - 1$, for example by finding a $G$-invariant element among the $Z(G)$-invariants, then the resulting ring will give a natural $\mathbb{Z}[\tau]$-representation on $\mathbb{Z}^{p-1}$. This representation was shown in [4, page 687] to be generated by one element. By a theorem of Reiner [18], this representation is the canonical representation of $\mathbb{Z}[\tau]$ on $\mathbb{Z}[\zeta]$. This reduction can be done for the group $G \cong (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/p$.

If all the elements of $N$ act by the same character of the $Z(G)$-invariants, then we can make a change of variables to reduce to the case of $\text{Spec} \ k \left[ w_1^{\pm 1}, \ldots, w_p^{\pm 1} \right]^\tau$, where $\tau$ cyclically permutes the $w_i$. For example, this is the case for $G \cong (\mathbb{Z}/p^2) \rtimes \mathbb{Z}/p$. In both situations, the final ingredient will be Proposition 3.3.

4.3.1. Assume that $G$ has order $p^3$. Then $Z(G)$ acts on $V$ via multiples of the identity, so

$$k \left[ x_1^{\pm 1}, \ldots, x_p^{\pm 1} \right]^{Z(G)} = k \left[ x_1^p, x_1^p, x_2, \ldots, x_1 \right] \cong k \left[ y_2^{\pm 1}, \ldots, y_p^{\pm 1} \right] [y_1^{\pm 1}],$$

for $y_1 = x_1^p$, $y_i = x_i^{x_1^p}$, $i = 2, \ldots, p$. Assume that we can replace $y_1$ with a $G$-invariant monomial $z_1$ such that

$$k \left[ y_2^{\pm 1}, \ldots, y_p^{\pm 1} \right] [y_1^{\pm 1}] = k \left[ y_2^{\pm 1}, \ldots, y_p^{\pm 1} \right] [z_1^{\pm 1}].$$

This can be done when $G \cong (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/p$. Recall the notations from Subsection 3.2.1. Indeed, in this case $Z(G) = \langle \sigma \rangle$. For the representation $(\rho, \pi)$ described in Subsection 3.2.1, $\pi$ acts on any $y_i$, $i = 2, \ldots, p$, by multiplication with $\zeta$ and it fixes $y_1$, while

$$\tau : y_2 \mapsto \cdots \mapsto y_p \mapsto \frac{1}{y_2 \cdots y_p}$$

and $\tau(y_1) = y_1 y_2^p$. If we replace $y_1$ by $z_1 = y_1 y_2^{p-1} \cdots y_{p-1}^2 y_p$, then $z_1$ is indeed $G$-invariant and

$$k \left[ y_2^{\pm 1}, \ldots, y_p^{\pm 1} \right] [y_1^{\pm 1}] = k \left[ y_2^{\pm 1}, \ldots, y_p^{\pm 1} \right] [z_1^{\pm 1}].$$

Even more, the same argument works for a $p$-group of cardinal $p^4$ with $Z(G) \cong \mathbb{Z}/p^2$ and $N$ different from $\mathbb{Z}/p^3$. Indeed, in this case, $N \cong Z(G) \oplus \langle \pi \rangle$, and the $Z(G)$-invariants of $k \left[ x_1^{\pm 1}, \ldots, x_p^{\pm 1} \right]$ are

$$k \left[ x_1^{p^2}, x_1^{p-2}, \frac{x_2}{x_1}, \ldots, \frac{x_1}{x_p} \right] = k \left[ y_2^{\pm 1}, \ldots, y_p^{\pm 1} \right] [y_1^{\pm 1}],$$
for $y_1 = x_1^{p^2}$, $y_i = x_{i+1}^i$ for $i = 2, \ldots, p$. Observe that $\pi$ acts trivially on $y_1$ and by a $p$-root of unity on the others $y_i$, and that

$$\tau : y_2 \mapsto \cdots \mapsto y_p \mapsto \frac{1}{y_2 \cdots y_p}$$

and $\tau(y_1) = y_1y_2^{p^2}$. In particular, this implies that $y_1y_2^p \cdots y_p^{p(p-1)}$ is $G$-invariant, so the above argument works.

4.3.2. Assume $G \cong \mathbb{Z}/p^2 \times \mathbb{Z}/p$. Recall the notations from Subsection 4.2.2. The center is generated by $\sigma^p$. The element $\sigma$ acts on any $y_i$, $i = 1, \ldots, p$, by multiplication with $\zeta$, while

$$\tau : y_2 \mapsto \cdots \mapsto y_p \mapsto \frac{1}{y_2 \cdots y_p}$$

and $\tau(y_1) = y_1y_2^p$. Replace $y_1$ with $y_1y_2^{p-1} \cdots y_{p-1}^{p-1}$. Then $\sigma(y_1) = \zeta y_1$, and $\tau(y_1) = y_1$. Taking $\sigma$-invariants,

$$k \left[ y_1^{\pm 1}, \ldots, y_p^{\pm 1} \right]^{\sigma} = k \left[ y_1^{p}, \frac{y_2}{y_1}, \ldots, \frac{y_p}{y_1}, \text{their inverses} \right],$$

which can be further written as $k \left[ w_1^{\pm 1}, \ldots, w_p^{\pm 1} \right]$ for $w_1 = y_1^p$, $w_i = \frac{y_i}{y_1}$, for $i = 2, \ldots, p$. Observe that

$$\tau : w_2 \mapsto w_3 \mapsto \cdots \mapsto w_p \mapsto \frac{1}{w_1 \cdots w_p},$$

and thus, by replacing $w_1$ with $\frac{1}{w_1 \cdots w_p}$, we need to show that Spec $k \left[ w_1^{\pm 1}, \ldots, w_p^{\pm 1} \right]^\tau$, where $\tau$ acts by $\tau : w_1 \mapsto \cdots \mapsto w_p \mapsto w_1$, is a linear scheme. This follows from Proposition 3.5. The same argument shows that any group of the form $\mathbb{Z}/p^a \times \mathbb{Z}/p$ is mixed Tate. In particular, this means that any group $G$ of order $p^4$ and center of order $p^3$ is mixed Tate.

4.4. Assume from now on that we are in the situation from Subsection 4.3.1 in which the dimension of the scheme we want to prove is linear was reduced from $p$ to $p - 1$. We will explain how to obtain a $\mathbb{Z}[\tau]$-representation on $\mathbb{Z}^{p-1}$. The argument works for any $p$-group and $V$ a $p$-dimensional representation, just that in this case we will get a representation of $\mathbb{Z}[\tau]$ on $\mathbb{Z}^p$. In order to compute the $\tau$-invariants of $k \left[ y_2^{\pm 1}, \ldots, y_p^{\pm 1} \right]^N$, write $N = N_1 \oplus N_2$ with $N_1$ cyclic. As in the proof of the Proposition 3.4, we have that

$$k \left[ y_2^{\pm 1}, \ldots, y_p^{\pm 1} \right]^{N_1} = k \left[ y_2^{a_2}, y_2^{a_3}y_3, \ldots, y_2^{a_p}y_p, \text{their inverses} \right].$$

If we repeat the computation for $N_2$ instead of $N_1$, we find that

$$k \left[ y_2^{\pm 1}, \ldots, y_p^{\pm 1} \right]^{N_2} = k \left[ y_2^{b_2}, y_2^{b_3}y_3, \ldots, y_2^{b_p}y_p, \text{their inverses} \right].$$

Let $z_i := y_2^{b_i}y_i$ for $2 \leq i \leq p$. Observe that $\tau$ acts on $z_i$ in the following way:

$$\tau(z_2) = y_2^{a_2a_3},$$

$$\tau(z_3) = y_2^{b_2b_3},$$

for some explicit integer exponents. For any $N$-invariant $z$, the element $\tau(z)$ is also $N$-invariant because

$$n\tau z = \tau n_0 z = \tau z.$$
for some \( n_0 \in N \). In particular, \( \tau(z_2) \) is \( N \)-invariant, so \( y_{2,2}^{b_{2,2}} \) is an integer power of \( z_2 \). This implies that \( \tau(z_3) \) is a monomial in \( z_2, z_3, \) and \( z_4 \), and a similar computation shows that this is true for any \( 2 \leq k \leq p \), namely that there are integer exponents such that
\[
\tau(z_k) = z_2^{a_{2,k}} \cdots z_k^{a_k+1,k}.
\]
Now, we can construct a \( \mathbb{Z}[\mathbb{Z}/p] \cong \mathbb{Z}[\tau] \) representation
\[
W := \mathbb{Z}^{p-1} = \mathbb{Z} \log(z_2) + \cdots + \mathbb{Z} \log(z_2)
\]
by defining
\[
\tau(\log(z_k)) = a_{2,k} \log(z_2) + \cdots + a_{k+1,k} \log(z_{k+1}).
\]
By a theorem of Reiner [18], the representation \( W \) is isomorphic to an ideal of \( \mathbb{Z}[\zeta] \), where \( \zeta \) is a primitive \( p \)-root of unity. Chu–Kang have shown in [4, page 687] that all such representations coming from groups of order \( \leq p^3 \) are generated by one element, so \( W \cong \mathbb{Z}[\zeta] \). Then we can choose monomials \( w_i \) in the \( z_i \) on which \( \tau \) acts via
\[
\tau : w_1 \mapsto w_2 \mapsto \cdots \mapsto w_{p-1} \mapsto \frac{1}{w_1 \cdots w_{p-1}}
\]
and such that
\[
k \begin{pmatrix} z_2^{\pm 1} & \cdots & z_p^{\pm 1} \end{pmatrix} = k \begin{pmatrix} w_1^{\pm 1} & \cdots & w_{p-1}^{\pm 1} \end{pmatrix}.
\]
We know that \( \text{Spec } k \left[ w_1^{\pm 1}, \ldots, w_{p-1}^{\pm 1} \right] \tau \) is a linear scheme by Proposition 3.5, so \( V / G \) is indeed a linear scheme in our case.

### 5. More on mixed Tate motives of a classifying space

In this Section, we assume that the base field \( k \) has characteristic zero.

#### 5.1. Define the triangulated category of geometrical motives

\[
\text{DM}_{gm}(k; R) \subset \text{DM}(k; R)
\]
as the smallest thick subcategory which contains all the motives \( M(X)(a) \) for \( X \) a separated scheme of finite type over \( k \) and \( a \) an integer [23, 21 Section 5]. In general, the motive of a quotient stack is not a geometric motive. For example, for a finite non-trivial group \( G \), the Chow groups (with \( \mathbb{Z} \) coefficients) \( CH^i(BG) \) are non-trivial for infinitely many values of \( i \) [24 Theorem 3.1], and thus the motive \( M(BG) \in \text{DM}(k, \mathbb{Z}) \) is not geometric. For an explicit computation of the motive of a quotient stack, let \( k(1) \) be the one-dimensional representation on which \( G_m \) acts with weight one. Observe that \( (k(1) \oplus (n+1) - 0) / G_m \cong \mathbb{P}^n \) “approximate” the motives associated to \( G_m \). We thus have that
\[
M(BG_m) = \bigoplus_{j \geq 0} R(j)[2j],
\]
\[
M^c(BG_m) = \prod_{j \leq -1} R(j)[2j].
\]
None of these motives are geometric.

Even if the motives associated to a quotient stack are not geometric motives, they exhibit some properties which resemble geometric motives. Indeed, recall that for \( X \) a proper scheme, \( M^c(X) \cong M(X) \), and for \( X \) a smooth scheme of pure dimension \( n \) over \( k \), \( M^c(X) \cong M(X)^*(n)[2n] \) [21 Section 5].
Let \( X = Y / G \) be a smooth quotient stack for which we can define motives \( M(X) \) and \( M^c(X) \), see Subsection 2.2. There is an isomorphism
\[
M(X)^* \cong M^c(X)(- \dim(X))[-2 \dim(X)].
\]
The isomorphism in (5.1) follows from the fact that the dual of a direct sum in \( \text{DM}(k, R) \) is a product, so the dual of a homotopy colimit is a homotopy limit.

Furthermore, the dual of a mixed Tate motive in \( \text{DM}(k; R) \) is not necessarily mixed Tate. For example, if \( k \) is algebraically closed, \( M := \bigoplus_{i \in \mathbb{N}} \mathbb{Z} \) is an element of \( \text{DM}(k; \mathbb{Z}) \), but its dual in \( \text{DM}(k; \mathbb{Z}) \) is \( M^* = \prod_{i \in \mathbb{N}} \mathbb{Z} \), which is not an element of \( \text{DM}(k; \mathbb{Z}) \) [22, Corollary 4.2].

However, \( \text{DMT}_{gm}(k; R) := \text{DMT}(k; R) \cap \text{DMT}_{gm}(k; R) \) is closed under taking duals [15, Section 5.1]. The main result of this Section is:

**Theorem 5.1.** Let \( G \) be a finite group, let \( k \) be a field of characteristic 0, and let \( R \) be an arbitrary ring. Then \( M^c(BG) \in \text{DMT}(k; R) \) is mixed Tate if and only if \( M(BG) \in \text{DMT}(k; R) \) is mixed Tate.

In light of the above counterexample of a mixed Tate motive whose dual is not mixed Tate, we see that mixed Tate motives of finite groups exhibit finiteness properties. A related result [23, Theorem 3.1] says that any scheme \( X \) of finite type over a field \( k \) with \( M^c(X) \) mixed Tate has finitely generated Chow groups \( CH^*(X; R) \) as \( R \)-modules. This implies that \( CH^*(BG; R) \) are finitely generated over \( R \), when \( G \) is a finite group with \( BG \) mixed Tate.

5.2. We reduce the proof of Theorem 5.1 to:

**Theorem 5.2.** Let \( X \) be a smooth quotient stack and let \( E \) be a \( \mathbb{G}_m \)-bundle over \( X \). Then \( M(X) \) is mixed Tate if and only if \( M(E) \) is mixed Tate.

Totaro has shown in [21, Corollary 8.13] that for a finite group \( G \), \( M^c(BG) \) is mixed Tate if and only if \( M^c(GL(n)/G) \) is mixed Tate for a faithful representation \( G \to GL(n) \). One knows that the category of geometric Tate motives \( \text{DMT}_{gm}(k; R) \) is closed under taking duals, as mentioned above. Recall that for any geometric motive \( X \in \text{DMT}_{gm}(k; R) \), the map \( X \to X^{**} \) is an isomorphism [21, Lemma 5.5]. As \( GL(n)/G \) is a smooth scheme, and for any smooth scheme \( S \) one has
\[
M(S)^* \cong M^c(S)(- \dim(S))[-2 \dim(S)],
\]
we see that it is enough to prove that \( M(BG) \) is mixed Tate if and only if \( M(GL(n)/G) \) is mixed Tate for a faithful representation \( G \to GL(n) \). The strategy is to show the more general result, that for \( X \) a quotient stack and \( E \) a principal \( GL(n) \)-bundle over \( X \), \( M(X) \) is mixed Tate if and only if \( M(E) \) is mixed Tate. The next Proposition, inspired by [21, Lemma 7.13], shows that Theorem 5.1 follows from Theorem 5.2.

**Proposition 5.3.** Assume that for any smooth quotient stack \( X \) and any principal \( \mathbb{G}_m \)-bundle \( F \) over \( X \), \( M(X) \in \text{DMT}(k; R) \) if and only if \( M(F) \in \text{DMT}(k; R) \). Then, for any smooth quotient stack \( X \) and any principal \( GL(n) \)-bundle \( E \) over \( X \), \( M(X) \in \text{DMT}(k; R) \) if and only if \( M(E) \in \text{DMT}(k; R) \).

**Proof.** Denote by \( B \) the subgroup of upper triangular matrices in \( GL(n) \). Then \( E/B \) is an iterated projective bundle over \( X \). Recall that \( GL(n) \)-bundles are Zariski locally trivial. We obtain the following Leray-Hirsch decomposition for motives
\[
M(E/B) \cong \bigoplus M(X_a) [a_j],
\]
where $a_j$ are the dimensions of the $n!$ Bruhat cells of the flag manifold $GL(n)/B$, see also [21, Proof of Lemma 7.13].

Now, because $\text{DMT}(k; R)$ is closed under arbitrary direct sums, $M(X) \in \text{DMT}(k; R)$ implies $M(E/B) \in \text{DMT}(k; R)$. Conversely, $\text{DMT}(k; R)$ is thick [21, Discussion after Lemma 5.4], so $M(E/B) \in \text{DMT}(k; R)$ implies $M(X) \in \text{DMT}(k; R)$.

Next, let $U$ be the subgroup of strictly upper triangular matrices in $\text{GL}(n)$. Since $B/U \cong \mathbb{G}_m^n$, $E/U$ is a principal $\mathbb{G}_m^n$-bundle over $E/B$. Using the assumption about $\mathbb{G}_m^n$-bundles, we deduce that $M(E/U) \in \text{DMT}(k; R)$ if and only if $M(X) \in \text{DMT}(k; R)$. Finally, $U$ is an extension of copies of the additive group $\mathbb{G}_a$, so $M(E) \cong M(E/U)$, which means that $M(E) \in \text{DMT}(k; R)$ if and only if $M(X) \in \text{DMT}(k; R)$. \hfill $\Box$

5.3. We will also need the following vanishing result:

**Proposition 5.4.** If $Y$ is a smooth quasi-projective scheme, then

$$\text{Hom} \ (R(i)[j], M(Y)) = 0,$$

for $j \leq i - 2$.

**Proof.** Choose a smooth compactification $Z$ of $Y$ such that the complement $W := Z \setminus Y$ is a divisor with simple normal crossings, which can be done since $k$ has characteristic 0 [13, Theorem 3.35]. Then, the Gysin distinguished triangle [23, pg. 10] gives, for $c = \text{codim} W$:

$$M(W) \to M(Z) \to M(Y)(c)[2c] \to M(W)[1].$$

Taking the dual of this triangle we obtain, for $n = \dim Y$:

$$M^c(W)^*(n)[2n - 1] \to M(Y) \to M(Z) \to M^c(W)^*(n)[2n].$$

Both $\text{Hom} \ (R(i)[j], M(Z)[-1])$ and $\text{Hom} \ (R(i)[j], M(Z))$ are zero because $Z$ is projective. Indeed, in our case $M(Z) \cong M^c(Z)$ and $j \leq i - 2$, and it is known that $\text{Hom} \ (R(i)[j], M^c(Z)) = 0$ for any scheme $Z$ and any integers $i$ and $j$ with $j \leq i - 1$ [21, pg. 16]. Thus, the Hom long exact sequence obtained from this distinguished triangle gives that

$$\text{Hom} \ (R(i)[j], M^c(W)^*(n)[2n - 1]) \cong \text{Hom} \ (R(i)[j], M(Y)).$$

Observe that $W$ is proper, so $M(W) \cong M^c(W)$. Further,

$$\text{Hom} \ (R(i)[j], M^c(W)^*(n)[2n - 1]) \cong \text{Hom} \ (M^c(W), R(n - i)[2n - 1 - j]).$$

Thus, it is enough to prove

$$\text{Hom} \ (M^c(W), R(a)[b]) = 0,$$

for $b - a \geq n + 1$. Further, $\dim W < n$ and $W$ is a divisor with simple normal crossings, so there are at most $n$ divisor through any point of $W$. To show this, we will use induction on $n$, the maximal number of divisors which pass through a given point, and then on the number of connected components of $W$. If $n = 1$ or if $W$ has only one component, then $W$ is smooth; in this case, $M(W) \cong M^c(W)$ and $M(W)^* \cong M(W)(-\dim(W))[-2\dim(W)]$. We need to show that

$$\text{Hom} \ (R(i + \dim W - n)[j + 1 + 2(\dim W - n)], M^c(W)) = 0,$$

for $j \leq i - 2$, where $i = n - a$ and $j = 2n - 1 - b$. This follows from the vanishing property of motivic homology

$$\text{Hom} \ (R(i)[j], M^c(Z)) = 0$$
for any scheme $Z$ and any integers $i$ and $j$ with $j \leq i - 1$ [21 page 16]. In our case, $b - a \geq n + 1$ is equivalent to $j \leq i - 2$, and we know that $\dim W < n$, thus $i + \dim W - n \geq j + 1 + 2(\dim W - n) + 1$. 

For the general case, let $U$ a smooth connected component of $W$ and let $V$ the closure of $W \setminus U$ inside $W$. Then $V$ will be also be a divisor with simple normal crossings such that there are at most $n$ divisors passing through a given point, but will have less components than $W$. Further, $T := U \cap V$ will be a divisor with simple normal crossings, with at most $n - 1$ divisors passing through any point. By the induction hypothesis, $\text{Hom}(M(T)[1], R(a)[b]) = 0$ for $b - a \geq n$, and $\text{Hom}(M(V)[1], R(a)[b]) = 0$ for $b - a \geq n + 1$. Recall that we want to show $\text{Hom}(M(W)[1], R(a)[b]) = 0$ for $b - a \geq n + 1$. For this, use the following two distinguished triangles

\[
M^c(U) \to M^c(W) \to M^c(W - U) \to M^c(U)[1],
\]
\[
M^c(T) \to M^c(V) \to M^c(W - U) \to M^c(T)[1].
\]

From the second triangle, we get

\[
\text{Hom}(M^c(T)[1], R(a)[b]) \to \text{Hom}(M^c(W - U), R(a)[b]) \to \text{Hom}(M^c(V), R(a)[b]) \to \text{Hom}(M^c(T), R(a)[b]).
\]

We deduce that $\text{Hom}(M^c(W - U), R(a)[b]) = 0$ for $b - a \geq n + 1$, Similarly, we can use the first triangle to deduce that $\text{Hom}(M^c(W), R(a)[b]) = 0$ for $b - a \geq n + 1$. □

5.4. In this Subsection, we prove Theorem 5.2. We split its proof in a sequence of steps.

5.4.1. Let $T$ be the total space of a line bundle over $X$ such that $T - X \cong E$, where $X \hookrightarrow T$ is embedded as the zero section. We claim that there is a Gysin distinguished triangle

(5.2) \[ M(T - X) \to M(T) \to M(X)[1] \cong (T - X)[1]. \]

Indeed, let $X = \mathbb{Y}/G$ and $T = W/G$ with $Y$ smooth and $W$ an $\mathbb{A}^1$-bundle over $Y$. Consider the (smooth) approximations

\[
X_i = ((V_i - S_i) \times Y)/G,
\]
\[
T_i = ((V_i - S_i) \times W)/G.
\]

Then we have the Gysin distinguished triangles [23 Theorem 3.5.4]:

(5.3) \[ M(T_i - X_i) \to M(T_i) \to M(X_i)[1] \cong (T_i - X_i)[1]. \]

The category $\text{DM}(k; R)$ is a model category with arbitrary direct sums and products [21 Subsection 5], so it has an underlying triangulated derivator [5 Theorem 6.11], [12 Appendix 2, page 25]. Thus the homotopy colimit of distinguished triangles is a distinguished triangle [12 Corollary 11.4], and we thus obtain the Gysin triangle (5.2). Using $M(X) \cong M(T)$, the distinguished triangle (5.2) becomes:

(5.4) \[ M(E) \to M(X) \to M(X)[1] \cong (E)[1]. \]
5.4.2. The inclusion
\[ \text{DMT}(k; R) \hookrightarrow \text{DM}(k; R) \]
has a right adjoint
\[ C : \text{DM}(k; R) \rightarrow \text{DMT}(k; R). \]
We will sometimes write \( C(Z) \) instead of \( C(M(Z)) \) for \( Z \) a quotient stack. Let \( U \) be the cone of \( C(E) \rightarrow M(E) \) and let \( W \) be the cone of \( C(X) \rightarrow M(X) \). There is a distinguished triangle
\[ U \rightarrow W \rightarrow W(1)[2] \rightarrow U[1]. \]
Indeed, this triangle is induced from the triangle (5.4), the diagram
\[
\begin{array}{cccccc}
C(E) & \rightarrow & C(X) & \rightarrow & C(E)(1)[2] & \rightarrow & C(E)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M(E) & \rightarrow & M(X) & \rightarrow & M(E)(1)[2] & \rightarrow & M(E)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
U & \rightarrow & W & \rightarrow & W(1)[2] & \rightarrow & U[1]
\end{array}
\]
and the 3 \times 3 lemma.

5.4.3. Observe that \( C(W) = 0 \). Indeed,
\[ M(X) \rightarrow C(X) \rightarrow W \rightarrow M(X)[1] \]
and, for any \( i \) and \( j \) integers,
\[ \text{Hom}(R(i)[j], M(X)) \rightarrow \text{Hom}(R(i)[j], C(X)). \]
This implies that \( W \) has trivial motivic homology groups.

Then the Tate motive \( C(W) \) has trivial homology groups and so \( C(W) = 0 \). Indeed, because \( \text{Hom}(R(a)[b], C(X)) = 0 \) and \( R(a)[b] \) generate the category \( \text{DMT}(k; R) \), we get that \( \text{Hom}(M, C(X)) = 0 \) for any mixed Tate motive \( M \), and, in particular, that \( \text{Hom}(C(X), C(X)) = 0 \), so \( C(X) = 0 \).

5.4.4. We need to show \( U = 0 \) if and only if \( W = 0 \). If \( W = 0 \), then it is immediate that \( U = 0 \). Conversely, suppose \( U = 0 \). In this case,
\[ W \cong W(1)[2]. \]
In [6, Proposition 7.10], Dugger and Isaksen have shown that one can compute, via a spectral sequence, the motivic homology of \( X \otimes M \) from the motivic homology of \( M \) and \( X \), for any motive \( X \) and any mixed Tate motive \( M \). A related result [21, Theorem 7.2] says that if
\[ C(W) \otimes C(M(Z)) \cong C(W \otimes M(Z)), \]
for any \( Z \) a smooth, projective scheme, then \( W \) is mixed Tate. We will use both these results in our argument below.

The plan is the following: it is enough to show that
\[ C(W) \otimes C(M(Z)) \cong C(W \otimes M(Z)), \]
for \( Z \) a smooth, projective scheme. Taking into account that \( C(W) \cong 0 \), we will need to show that the motivic homology groups of any product \( W \otimes M(Z) \) are trivial.
We show that the motive $W$ has a vanishing property similar to the one of $M^c$ of a geometrical motive, namely that $\text{Hom}(R(i)[j], W) = 0$ for $j \leq i - 2$. Even more, we will be able to show that $\text{Hom}(R(i)[j], W \otimes M(Z)) = 0$ for $j \leq i - 2$ for $Z$ a smooth projective scheme. This will imply that all the motivic homology groups of $W \otimes M(Z)$ are trivial, because $W \cong W(1)[2]$. Consequently, we only need to show (5.6) $\text{Hom}(R(i)[j], W \otimes M(Z)) = 0$ for $j \leq i - 2$, where $Z$ is a smooth projective scheme.

5.4.5. First, by Proposition 5.4, we have that $\text{Hom}(R(i)[j], M(Y)) = 0$ for $j \leq i - 2$ for a quasi-projective scheme $Y$. There is a distinguished triangle:

$$M(X \times Z) \to C(M(X)) \otimes M(Z) \to W \otimes M(Z) \to M(X \times Z)[1].$$

It is enough to show

$$\text{Hom}(R(i)[j], M(X \times Z)) = 0,$$

$$\text{Hom}(R(i)[j], C(M(X)) \otimes M(Z)) = 0$$

for $j \leq i - 2$. To show that $\text{Hom}(R(i)[j], M(X \times Z)) = 0$ for $j \leq i - 2$, write $M(X \times Z)$ as the cone of a morphism

$$\bigoplus_{l \in I} M(S_l) \to \bigoplus_{l \in I} M(S_l) \to M(X \times Z) \to \left(\bigoplus_{l \in I} M(S_l)\right)[1],$$

where $S_l$ are quasi-projective schemes for $l$ in a set $I$. Because $R(i)[j]$ is a compact object inside $\text{DM}(k; R)$, we have that

$$\text{Hom}\left(R(i)[j], \bigoplus_{l \in I} M(S_l)\right) = \bigoplus_{l \in I} \text{Hom}(R(i)[j], M(S_l)) = 0$$

for $j \leq i - 2$. Finally,

$$\text{Hom}\left(R(i)[j], \bigoplus_{l \in I} M(S_l)\right) \to \text{Hom}(R(i)[j], M(X \times Z)) \to \text{Hom}\left(R(i)[j], \left(\bigoplus_{l \in I} M(S_l)\right)[1]\right),$$

which immediately implies $\text{Hom}(R(i)[j], M(X \times Z)) = 0$ for $j \leq i - 2$.

5.4.6. To show $\text{Hom}(R(i)[j], C(M(X)) \otimes M(Z)) = 0$ for $i \leq j - 2$, use the motivic Künneth spectral sequence [21] Theorem 6.1:

$$E_2^{pq} = \text{Tor}^H_{-p,-q,i}(H_{-}(C(X), R(-)), H_{-}(Z, R(-))) \Rightarrow \text{H}_{-p-q}(C(X) \otimes Z, R(i)),$$

where $\text{Tor}^H_{-p,-q,i}$ denotes $(-q, i)$ bigraded piece of $\text{Tor}_{-p}$. The vanishing properties for the motivic homology of $C(M(X))$ and $M(Z)$ imply the desired result. Indeed, assume $i < 0$. On the sheet $E_2^{pq}$, all nontrivial $H_{-}(k, R(-))$ modules are concentrated in the lower left corner $j \leq i - 2$, $p \leq 0$. Every page $E_n^{pq}$ will be concentrated in the same lower left square, which implies the vanishing of motivic homology groups for $C(M(X)) \otimes M(Z)$ for $j \leq i - 2$. In particular, $\text{Hom}(R(i)[j], C(M(X)) \otimes M(Z)) = 0$ for $j \leq i - 2$. Using the triangle (5.7) and the discussion in Subsection 5.4.5, we see that (5.6) holds.
5.4.7. Finally, let $i$ and $j$ be arbitrary integers, and choose $a \leq i - j - 2$. By (5.3) and (5.6), we have that

$$\text{Hom}(R(i)[j], W \otimes M(Z)) \cong \text{Hom}(R(i + a)[j + 2a], W \otimes M(Z)) \cong 0.$$ 

Thus the motivic homology of $W \otimes M(Z)$ is trivial for every smooth projective scheme $Y$. As discussed in Subsection 5.4.4, this implies that $W \cong 0$, and thus Theorem 5.2 follows.

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