Solutions of the Yang–Baxter equation for \((n + 1)(2n + 1)\)-vertex models using a differential approach

R S Vieira and A Lima-Santos

Universidade Federal de São Carlos, Departamento de Física, Caixa Postal 676, CEP 13569-905, São Carlos, Brazil
E-mail: dals@df.ufscar.br

Received 15 December 2020
Accepted for publication 23 March 2021
Published 17 May 2021

Abstract. The formal derivatives of the Yang–Baxter equation with respect to its spectral parameters, evaluated at some fixed point of these parameters, provide us with two systems of differential equations. The derivatives of the \(R\) matrix elements, however, can be regarded as independent variables and eliminated from the systems, after which, two systems of polynomial equations are obtained in their place. In general, these polynomial systems have a non-zero Hilbert dimension, which means that not all elements of the \(R\) matrix can be fixed through them. Nevertheless, the remaining unknowns can be found by solving a few simple differential equations that arise as consistency conditions of the method. The branches of the solutions can also be easily analyzed by this method, which ensures the uniqueness and generality of the solutions. In this work, we consider the Yang–Baxter equation for \((n + 1)(2n + 1)\)-vertex models with a generalization based on the \(A_{n-1}\) symmetry. This differential approach allows us to solve the Yang–Baxter equation in a systematic way.

Keywords: algebraic structures of integrable models, exact results, integrable spin chains and vertex models
1. Introduction ............................ 2

A short life story:
- ‘I was privileged to work with Ricardo Soares Vieira or Ricardinho, as he was called by colleagues and friends.

Ricardinho died prematurely on October 21, 2020, due to a post-operative complication arising from a procedure to remove stomach cancer.

I followed his graduate studies in physics at the Federal University of São Carlos and was his advisor for his master’s degree [1] and doctorate [2].

During this period, it was possible to observe his interests in various areas of knowledge.

I was often impressed by his ability to solve complex problems quickly’.

I’m very sorry for his death!

A few months before his death, Vieira came to me with a proposal for me to work on the Yang–Baxter equation (YBE). The objective was to consider the possible vertex models, taking into account the structure of the $R$ matrices associated with the symmetries of the non-exceptional affine Lie algebras. The new $R$ matrix solution was recalculated by the Yang–Baxter equation using a differential approach. He had just

https://doi.org/10.1088/1742-5468/abf7be
solved this problem for two-state models [3]. We then started looking at the $A_{n-1}^{(1)}$ models, due to the current interest in them [4, 5] and in the certainty that they are the simplest. We have organized this paper as follows. In sections 2 and 3, we make the usual presentation of Yang–Baxter’s equation and its corresponding differential equations [3]. In section 4, we present the calculations for the 15-vertex model. In section 5, we consider the case $n=3$, or the 28-vertex model. In section 6, we present the general case. In section 7, we present some properties of the regular solutions and discuss their free parameters. We conclude in section 8.

2. The Yang–Baxter equation

The YBE is one of the most important equations of contemporary mathematical physics. It originally emerged in two different contexts of theoretical physics: in quantum field theory, the YBE appeared as a sufficient condition for the many-body scattering amplitudes to factor into the product of pairwise scattering amplitudes [6–8]; in statistical mechanics it represented a sufficient condition for the transfer matrix of a given statistical model to commute for different values of the spectral parameters [9, 10]. Since the pioneering works on quantum integrable systems—see [11–13] for a historical background—the YBE has become a cornerstone in several fields of physics and mathematics: it is most known for its fundamental role in the quantum inverse scattering method and in the algebraic Bethe ansatz [14–16], although it has also been revealed to be important in the formulation of Hopf algebras and quantum groups [17–21], in knot theory [22], in quantum computation [23], in AdS-CFT correspondence [24, 25] and, more recently, in gauge theory [26–28]. The YBE can be seen as a matrix relation defined in $\text{End}(V \otimes V)$, where $V$ is an $n$-dimensional complex vector space.

In the most general case, it reads:

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v).$$

(1)

Using this non-differential form, [29] recently generalized the work of [3], where the boost operator method was discussed.

However, in this paper we are interested in the cases where $R_{12}(u,v) = R_{12}(u-v)$, in order to get the particular form

$$\text{YB} = R_{12}(u)R_{13}(u+v)R_{23}(v) - R_{23}(v)R_{13}(u+v)R_{12}(u),$$

(2)

where the arguments $u$ and $v$, called spectral parameters, have values in $C$. The solution of the YBE is an $R$ matrix defined in $\text{End}(V \otimes V)$. The indexed matrices $R_{ij}$ that appear in (1) are defined in $\text{End}(V \otimes V)$ through the formulas

$$R_{12} = R \otimes I, \quad R_{23} = I \otimes R, \quad R_{13} = P_{23}R_{12}P_{23}$$

(3)

where $I \in \text{End}(V)$ is the identity matrix, $P \in \text{End}(V \otimes V)$ is the permutation matrix (defined by the relation $P(A \otimes B)P = B \otimes A$ for $A, B \in \text{End}(V)$) and $P_{12} = P \otimes I, \quad P_{23} = I \otimes P$.

https://doi.org/10.1088/1742-5468/abf7be
Each solution of the YBE can be associated with a given integrable system. In fact, in statistical mechanics, the $R$ matrix represents the Boltzmann weights of a given statistical model, while in quantum field theory, the $R$ matrix is associated with factorizable scattering amplitudes between relativistic particles. From the YBE, we can prove that systems described by an $R$ matrix possess infinitely many conserved quantities in involution—the Hamiltonian being one of them—which is the reason why they are called integrable [30]. We say that a given solution $R(u)$ of the YBE (1) is regular if $R(0) = P$. Regular solutions of the YBE have several important properties [13].

3. The differential Yang–Baxter equations

The YBE corresponds to a system of nonlinear functional equations. Several particular solutions of the YBE are known [11–13]. The first solutions were found by a direct inspection of the functional equations, which are, in fact, very simple because the $R$ matrix is assumed to have many symmetries. Nevertheless, there are other more advanced methods for solving the YBE: we can cite, for instance, the Baxterization of braid relations [31], the use of Lie algebras and super algebras [32–34], the construction of Hopf algebras and quantum groups [18–20], and also techniques that rely on algebraic geometry [35–37]; see also [38]. The methods mentioned above usually require that the $R$ matrix presents one or more symmetries from the very start. From a mathematical point of view, it would be desirable to develop a method that requires, in principle, as few symmetries as possible and which, at the same time, is powerful enough to find and classify the solutions of the YBE. This paper is concerned with the development and extensive use of such a method, which is based on a differential approach.

From the quantum group invariant representation for non-exceptional affine Lie algebra $A^{(1)}_{n-1}$ [32], we consider the following generalization for an $R$-matrix solution of the YBE

$$R(u) = \sum_{i=1}^{n+1} a_{ii}(u) e_{ii} \otimes e_{ii} + \sum_{i \neq j}^{n+1} b_{ij}(u) e_{ii} \otimes e_{jj} + \sum_{i \neq j}^{n+1} c_{ij}(u) e_{ii} \otimes e_{jj}$$

(4)

where $e_{ij}$ are the Weyl matrices $(e_{ij})_{ab} = \delta_{ia} \delta_{jb}$, acting for an $n + 1$ dimensional vector space $V$ at the site $n$. The $R$-matrix elements $a_{ii}(u)$, $b_{ij}(u)$ and $c_{ij}(u)$ are fixed by the YBE.

In particular, when $a_{ii}(u) = a(u)$, $b_{ij}(u) = b(u)$, $c_{ij}(u) = c(I < j)$ and $c_{ij}(u) = cc_{2u}(i > j)$, we have the quantum group invariant representation for the non-exceptional affine Lie algebra $A^{(1)}_{n-1}$.

To be more precise, this method mainly consists of the following: if we take the formal derivatives of (2) with respect to the spectral parameters $u$ and $v$ and then evaluate the derivatives at some fixed point of those variables (say, at zero), then we get two systems of ordinary non-linear differential equations for the elements of the $R$ matrix. The derivatives of the $R$ matrix elements, however, can be regarded as independent variables, so that after they are eliminated, two systems of polynomial equations are obtained in their
Solutions of the Yang–Baxter equation for \((n + 1)(2n + 1)\)-vertex models using a differential approach

Thus, these polynomial systems can be analyzed—for instance, using computational algebraic geometry techniques [39]—and eventually completely solved. It happens, however, that these polynomial systems usually have a positive Hilbert dimension, which means that the systems are satisfied even when some of the \(R\) matrix elements are still arbitrary. The remaining unknowns, nonetheless, can be found by solving a small number of differential equations that arise from the expressions for the derivatives we had eliminated before. These auxiliary differential equations, therefore, can be thought of as consistency conditions of the method. For example, if we take the formal derivatives of (2) with respect to \(v\) and then evaluate the result at the point \(v = 0\), then we get the equation,

\[
YB_v = R_{12}(u)D_{13}(u)P_{23} + R_{12}(u)R_{13}(u)H_{23} - H_{23}R_{13}(u)R_{12}(u) - P_{23}D_{13}(u)R_{12}(u)
\]

and from its derivative with respect \(u\) at the point \(u = 0\), we get

\[
YB_u = H_{12}R_{13}(v)R_{23}(v) + P_{12}D_{13}(v)R_{23}(v) - R_{23}(v)D_{13}(v)P_{12} - R_{23}(v)R_{13}(v)H_{12}
\]

where

\[
D(u) = \left. \frac{dR(u + v)}{dv} \right|_{v=0}, \quad D(v) = \left. \frac{dR(u + v)}{du} \right|_{u=0}, \quad P = R(0), \quad H = D(0).
\]

Now, using the definitions

\[
\left. \frac{da_{ii}(u)}{du} \right|_{u=0} = \alpha_{ii}, \quad \left. \frac{db_{ij}(u)}{du} \right|_{u=0} = \beta_{ii} \quad \text{and} \quad \left. \frac{dc_{ij}(u)}{du} \right|_{u=0} = \mu_{ij}
\]

we can write the matrices \(D(u)\) and \(H\) as:

\[
D(u) = \sum_{i=1}^{n+1} a'_{ii}(u)e_{ii} \otimes e_{ii} + \sum_{i \neq j}^{n+1} b'_{ij}(u)e_{ii} \otimes e_{jj} + \sum_{i \neq j}^{n+1} c'_{ij}(u)e_{ij} \otimes e_{ji}
\]

and

\[
H = \sum_{i=1}^{n+1} \alpha_{ii} e_{ii} \otimes e_{ii} + \sum_{i \neq j}^{n+1} \beta_{ij} e_{ii} \otimes e_{jj} + \sum_{i \neq j}^{n+1} \mu_{ij} e_{ij} \otimes e_{ji}
\]

where \(a', b', \text{ and } c'\) are the derivatives of \(a, b, \text{ and } c\).

Here, we note that (5) and (6) are the derivatives of (2). They are different and both should be used equally in the method used to solve the functional equation.

We also highlight that \(\mathcal{H} = PH\), where \(\mathcal{H}\) is nothing but the local Hamiltonian associated with the model—see, for instance, [11, 13].

The idea of transforming a functional equation into a differential one goes back to the works of Niels Henrik Abel, who solved several functional equations in this way. Abel’s method presents many advantages, compared to other methods of solving functional equations. For instance, it consists of a general method that can be applied to a huge
Solutions of the Yang–Baxter equation for \((n + 1)(2n + 1)\)-vertex models using a differential approach class of functional equations; it establishes the generality and uniqueness of the solutions (which would be difficult, if not impossible, to establish in other ways) by reducing the problem to the theory of differential equations, and so on—see [38] for more. Notice, moreover, that although Abel’s method requires the solutions to be differential (there can be non-differential solutions of some functional equations), this restriction is not a problem when dealing with the YBE, as its solutions are always assumed to be differentiable because of the connection between the \(R\) matrix and the corresponding local Hamiltonian. Concerning the theory of integrable systems, the differential method is perhaps most known in connection with the boundary YBE [40–42].

Now we can look for the matrices (4) that are solutions of (2). First, we will explain the calculations for the deformed \(A_{1}^{(1)}\), or 15-vertex model.

4. The 15-vertex model

The corresponding matrices are:

\[
R(u) = \begin{pmatrix}
a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b_{12} & 0 & c_{12} & 0 & 0 & 0 & 0 \\
0 & 0 & b_{13} & 0 & 0 & c_{13} & 0 & 0 \\
0 & c_{21} & 0 & b_{21} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{22} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_{23} & 0 & c_{23} & 0 \\
0 & 0 & c_{31} & 0 & 0 & b_{31} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{32} & b_{32} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{33} & a_{33}
\end{pmatrix}
\] (11)

\[
D(u) = \begin{pmatrix}
a'_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b'_{12} & 0 & c'_{12} & 0 & 0 & 0 & 0 \\
0 & 0 & b'_{13} & 0 & 0 & c'_{13} & 0 & 0 \\
0 & c'_{21} & 0 & b'_{21} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a'_{22} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b'_{23} & 0 & c'_{23} & 0 \\
0 & 0 & c'_{31} & 0 & 0 & b'_{31} & 0 & 0 \\
0 & 0 & 0 & 0 & c'_{32} & b'_{32} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c'_{33} & a'_{33}
\end{pmatrix}
\] (12)

\[
H = \begin{pmatrix}
\alpha_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \beta_{12} & 0 & \mu_{12} & 0 & 0 & 0 & 0 \\
0 & 0 & \beta_{13} & 0 & 0 & \mu_{13} & 0 & 0 \\
0 & \mu_{21} & 0 & \beta_{21} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_{22} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \beta_{23} & 0 & \mu_{23} \\
0 & 0 & \mu_{31} & 0 & 0 & 0 & \beta_{31} & 0 \\
0 & 0 & 0 & 0 & \mu_{32} & 0 & \beta_{32} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{33}
\end{pmatrix}
\] (13)
Solutions of the Yang–Baxter equation for \((n+1)\cdot(2n+1)\)-vertex models using a differential approach

Note that the entries of \(R(u)\) and \(D(u)\) are functions of \(u\) and the conditions \(a_{ii}(0) = 1, b_{ij}(0) = 0,\) and \(c_{ij}(0) = 1\) define the matrix \(P.\)

For this model, we have three 27 by 27 matrix equations \(Y_B = 0, Y_B u = 0\) and \(Y_B v = 0.\) Looking at their diagonals \(Y_B[i, i] = 0,\) we can find several equations that only contain the \(c_{ij}(u)\) amplitudes. For their derivatives, \(Y_B[i, i] = 0,\) we find

\[
c_{ij}(u)c_{ji}(u)(\mu_{ij} - \mu_{ji}) + c_{ij}(u)c'_{ji}(u) - c'_{ij}(u)c_{ji}(u) = 0, \quad i \neq j = \{1, 2, 3\}.
\]

(14)

With the regularity condition, the solutions are

\[
c_{ij}(u) = \exp(\mu_{ij} u) \quad \text{or} \quad c_{ij}(u) = 1 + \mu_{ij} u.
\]

(15)

In this paper, we will only use the exponential functions, because the rational solution drastically decreases the number of free parameters.

After replacing the \(c_{ij}(u),\) we look at the conjugate equations \(Y_B[i, 28 - i] = 0,\) \(Y_B u[i, 28 - i] = 0,\) and \(Y_B v[i, 28 - i] = 0,\) which are satisfied by the following relations

\[
b_{23}(u) = \frac{\beta_{23}}{\beta_{21}}b_{21}(u), \quad b_{32}(u) = \frac{\beta_{32}}{\beta_{12}}b_{12}(u)
\]

(16)

with the constraint \(\beta_{23}\beta_{32} = \beta_{12}\beta_{21}.\) From other equations, we find \(b_{13}(u), b_{23}(u)\) and its derivatives

\[
b_{13}(u) = \frac{\beta_{13}}{\beta_{12}}b_{12}(u), \quad b_{31}(u) = \frac{\beta_{31}}{\beta_{21}}b_{21}(u),
\]

(17)

with a second constraint \(\beta_{13}\beta_{31} = \beta_{12}\beta_{21}.\)

We notice that these relations can be written in a more compact notation

\[
b_{ij}(u) = \beta_{ij}K(u), \quad \beta_{ji} = \frac{\beta_{12}\beta_{21}}{\beta_{ij}}, \quad i \neq j = \{1, 2, 3\}
\]

(18)

where \(K(u)\) is an arbitrary function, to be determined.

Using (18), all the remaining equations only contain \(a_{ii}(u)\) and \(K(u),\) as well as their derivatives. In particular, the differential equation \(Y_B[u, 11, 25] = 0\) allows us to find the function \(K(u):\)

\[
-\frac{d}{du}K(u) + (\mu_{13} + \mu_{21} - \mu_{23})K(u) + \exp((\mu_{12} + \mu_{23} - \mu_{13})u) = 0.
\]

(19)

For regular solutions, we have

\[
K(u) = \frac{\exp((\mu_{12} - \mu_{13} + \mu_{23})u) - \exp((\mu_{21} + \mu_{13} - \mu_{23})u)}{(\mu_{12} - 2\mu_{13} - \mu_{21} + 2\mu_{23})}.
\]

(20)

Substituting in the YBEs, we have to fix two \(\mu_{ij}\)

\[
\mu_{31} = \mu_{21} + \mu_{12} - \mu_{13} \quad \text{and} \quad \mu_{32} = \mu_{21} + \mu_{12} - \mu_{23}.
\]

(21)
Solutions of the Yang–Baxter equation for \((n+1)(2n+1)\)-vertex models using a differential approach

Before the computation of \(a_{ii}(u), i = 1, 2, 3\), we can still simplify the notation by defining two parameters

\[
\kappa_1 = \mu_{13} + \mu_{21} - \mu_{23}, \quad \kappa_2 = \mu_{12} - \mu_{13} + \mu_{23}.
\]

(22)

It follows that

\[
K(u) = \frac{e^{\kappa_1 u} - e^{\kappa_2 u}}{\kappa_1 - \kappa_2}, \quad \text{and} \quad \mu_{3j} = \kappa_1 + \kappa_2 - \mu_{3j}, \quad j = 1, 2
\]

(23)

and from \(YB_u[7,15] = 0\), we get the relation between the parameters \(\alpha_{11}, \beta_{ij}\) and \(\mu_{ij}\):

\[
\beta_{21}\beta_{12} = (\kappa_1 - \alpha_{11})(\kappa_2 - \alpha_{11}).
\]

(24)

We can now easily calculate \(a_{ii}(u)\), as they satisfy a recurrence relationship

\[
a_{ii}(u) = a_{11}(u) + (\alpha_{ii} - \alpha_{11})K(u), \quad i = 2, 3
\]

(25)

with

\[
a_{11}(u) = \frac{(\kappa_1 - \alpha_{11})e^{\kappa_2 u} - (\kappa_2 - \alpha_{11})e^{\kappa_1 u}}{\kappa_1 - \kappa_2}.
\]

(26)

Finally, we have a system of two equations whose solutions will determine the \(R\) matrices of the model.

\[
(\alpha_{11} - \alpha_{kk})(\kappa_1 + \kappa_2 - \alpha_{11} - \alpha_{kk}) = 0, \quad k = \{2, 3\}.
\]

(27)

In this case, we get four solutions:

4.1. Solution 1: \(\alpha_{22} = \alpha_{11}, \alpha_{33} = \alpha_{11}\)

For this solution we have (11) with the following entries

\[
c_{ij}(u) = \exp(\mu_{ij}(u)), \quad b_{ij}(u) = \beta_{ij}K(u), \quad i \neq j = \{1, 2, 3\}
\]

(28)

and

\[
a_{33}(u) = a_{22}(u) = a_{11}(u) = \frac{\kappa_1 - \alpha_{11}}{\kappa_1 - \kappa_2} \exp(\kappa_2 u) - \frac{\kappa_2 - \alpha_{11}}{\kappa_1 - \kappa_2} \exp(\kappa_1 u)
\]

(29)

where

\[
\kappa_1 = \mu_{13} + \mu_{21} - \mu_{23}, \quad \kappa_2 = -\mu_{13} + \mu_{12} + \mu_{23}, \quad K(u) = \frac{\exp(\kappa_1 u) - \exp(\kappa_2 u)}{\kappa_1 - \kappa_2}
\]

(30)

and the fixed parameters

\[
\mu_{31} = \mu_{12} + \mu_{21} - \mu_{13}, \quad \mu_{32} = \mu_{12} + \mu_{21} - \mu_{32}
\]

(31)

\[
\beta_{ji} = \frac{\beta_{12}\beta_{21}}{\beta_{ij}}, \quad i < j = \{1, 2, 3\}, \quad \beta_{12}\beta_{21} = (\kappa_1 - \alpha_{11})(\kappa_2 - \alpha_{11}).
\]

(32)
Solutions of the Yang–Baxter equation for \((n + 1)(2n + 1)\)-vertex models using a differential approach

For a particular choice of parameters
\[
\mu_{ij} = \eta, \quad \mu_{ji} = 0, \quad (i < j), \quad \beta_{ij} = \xi, \quad i \neq j = \{1, 2, 3\}
\] (33)

with
\[
\eta = \frac{\xi^2 - \alpha_{11}^2}{\alpha_{11}}
\] (34)

we get the quantum group invariant solution of [32].

4.2. Solution 2: \(\alpha_{22} = \alpha_{11}, \alpha_{33} = \kappa_1 + \kappa_2 - \alpha_{11}\)

In this case, we get
\[
a_{22}(u) = a_{11}(u) = \frac{(\kappa_1 - \alpha_{11}) \exp(\kappa_2 u) - (\kappa_2 - \alpha_{11}) \exp(\kappa_1 u)}{\kappa_1 - \kappa_2}
\] (35)

and
\[
a_{33}(u) = \frac{(\kappa_1 - \alpha_{11}) \exp(\kappa_1 u) - (\kappa_2 - \alpha_{11}) \exp(\kappa_2 u)}{\kappa_1 - \kappa_2}
\] (36)

4.3. Solution 3: \(\alpha_{22} = \kappa_1 + \kappa_2 - \alpha_{11}, \alpha_{33} = \alpha_{11}\)

Here, we get
\[
a_{33}(u) = a_{11}(u) = \frac{(\kappa_1 - \alpha_{11}) \exp(\kappa_2 u) - (\kappa_2 - \alpha_{11}) \exp(\kappa_1 u)}{\kappa_1 - \kappa_2}
\] (37)

and
\[
a_{22}(u) = \frac{(\kappa_1 - \alpha_{11}) \exp(\kappa_1 u) + (\kappa_2 - \alpha_{11}) \exp(\kappa_2 u)}{\kappa_2 - \kappa_1}
\] (38)

4.4. Solution 4: \(\alpha_{22} = \kappa_1 + \kappa_2 - \alpha_{11}, \alpha_{33} = \kappa_1 + \kappa_2 - \alpha_{11}\)

In this case, we have
\[
a_{11}(u) = \frac{(\kappa_1 - \alpha_{11}) \exp(\kappa_2 u) - (\kappa_2 - \alpha_{11}) \exp(\kappa_1 u)}{\kappa_1 - \kappa_2}
\] (39)

and
\[
a_{33}(u) = a_{22}(u) = \frac{(\kappa_1 - \alpha_{11}) \exp(\kappa_2 u) - (\kappa_2 - \alpha_{11}) \exp(\kappa_1 u)}{\kappa_1 - \kappa_2}
\] (40)

Here, we remember that \(\kappa_1 + \kappa_2 = \mu_{12} + \mu_{21}\) and from (27) that \(a_{ii}(u)\) have only two values, namely
\[
a_{ii}(u) = A(u) = \frac{(\kappa_1 - \alpha_{11}) \exp(\kappa_2 u) - (\kappa_2 - \alpha_{11}) \exp(\kappa_1 u)}{\kappa_1 - \kappa_2}
\] (41)
Solutions of the Yang–Baxter equation for \((n+1)(2n+1)\)-vertex models using a differential approach

when \(\alpha_{ii} = \alpha_{11}\), and

\[
a_{jj}(u) = B(u) = \frac{(\kappa_1 - \alpha_{11}) \exp(\kappa_1 u) - (\kappa_2 - \alpha_{11}) \exp(\kappa_2 u)}{\kappa_1 - \kappa_2}
\]  
(42)

when \(\alpha_{jj} = \kappa_1 + \kappa_2 - \alpha_{11}\).

Therefore, we have

|       | \(a_{11}(u)\) | \(a_{22}(u)\) | \(a_{33}(u)\) |
|-------|----------------|----------------|----------------|
| Solution 1 | \(A(u)\)      | \(A(u)\)      | \(A(u)\)      |
| Solution 2 | \(A(u)\)      | \(A(u)\)      | \(B(u)\)      |
| Solution 3 | \(A(u)\)      | \(B(u)\)      | \(A(u)\)      |
| Solution 4 | \(A(u)\)      | \(B(u)\)      | \(B(u)\)      |

Solutions 2 and 3 are equivalent, which means that there are three different solutions.

For these \(R\)-matrices, we started with 15 parameters, the derivatives of the matrix elements at the point \(u = 0\). We fixed \(\{\alpha_{22}, \alpha_{33}, \beta_{21}, \beta_{31}, \beta_{32}, \mu_{31}, \mu_{32}\}\). Therefore, these \(R\) matrices have eight free parameters.

Several particular solutions of the YBE associated with the 15-vertex models are known. We can cite, for example, the fifteen-vertex \(R\) matrices of Cherednik [43], Babelon [44], Chudnovsky and Chudnovsky [45] and Perk and Schultz [46–48] (these solutions hold for higher vertex models as well). These \(R\) matrices contain fewer parameters than the solutions we found, so they can be thought of as reductions of a more general solution.

5. The 28 vertex-model

In the \(A_2^{(1)}\) model, we have a 28-vertex model and its Yang–Baxter solutions are obtained by following the procedures used in the 15-vertex model. The 16 by 16 matrices \(R, D, H\) are given by (4), (9) and (10), with \(n = 3\), respectively.

The diagonal matrix entry equations

\[
\text{YB}_u[i, i] = 0, \quad \text{and} \quad \text{YB}_v[i, i] = 0
\]  
(43)

are verified by the amplitudes

\[
c_{ij}(u) = \exp(\mu_{ij} u), \quad i \neq j = \{1, 2, 3, 4\}
\]  
(44)

where \(\mu_{ij}\) are arbitrary parameters to be fixed.

The \(b_{ij}(u)\) vertices are computed in the same way as the case of the 15-vertex model. Their form is

\[
b_{ij}(u) = \beta_{ij} K(u), \quad i \neq j = \{1, 2, 3, 4\}.
\]  
(45)
The parameters $\beta_{ij}$ satisfy the relation
\[ \beta_{ji} = \frac{\beta_{12}\beta_{21}}{\beta_{ij}} = \frac{(\kappa_1 - \alpha_{11})(\kappa_2 - \alpha_{11})}{\beta_{ij}}, \quad i < j = \{1, 2, 3, 4\}. \tag{46} \]

We can now get the relations between the parameters $\mu_{ij}$ from the derived equations:
\[ \mu_{3i} = \mu_{12} + \mu_{21} - \mu_{23}, \quad i < 3 \quad \text{and} \quad \mu_{4i} = \mu_{12} + \mu_{21} - \mu_{43} \quad i < 4. \tag{47} \]

Replacing the expressions in (4) and its derivatives (5) and (6), we find, for instance, from $\text{YB}_v[46, 55] = 0$ the same function $K(u)$
\[ K(u) = \frac{e^{\kappa_1 u} - e^{\kappa_2 u}}{\kappa_1 - \kappa_2}, \tag{48} \]
where $\kappa_1 = \mu_{13} + \mu_{21} - \mu_{23}, \kappa_2 = \mu_{12} - \mu_{13} + \mu_{23}$. The $a_{ii}(u)$ functions still satisfy the recurrence
\[ a_{kk}(u) = a_{11}(u) + (\alpha_{kk} - \alpha_{11})K(u), \quad k = 2, 3, 4 \tag{49} \]
with
\[ a_{11}(u) = \frac{(\kappa_1 - \alpha_{11})e^{\kappa_2 u} - (\kappa_2 - \alpha_{11})e^{\kappa_1 u}}{\kappa_1 - \kappa_2}. \tag{50} \]

As we can see, the results are the same as those of the 15-vertex model, but include the index $n + 1 = 4$.

Unlike the previous case, the parameters $\mu_{ij}, i < j$ are insufficient to fix the solutions. Looking at the equations $\text{YB}_v[I, j] = 0$, we see that many of them are of the type $(\alpha_{kk} - \alpha_{11})(\alpha_{11} + \kappa_1 + \kappa_2 - \alpha_{kk})G_{ij}(u) = 0$ and the remaining ones are used to find $\mu_{ij}$, but now with $i < j$. These calculations are very annoying, but the results are simple. We find two possibilities:
\[ \mu_{ij} = \kappa_1 - \mu_{1i} + \mu_{1j} \quad \text{and} \quad \mu_{ij} = \kappa_2 - \mu_{1i} + \mu_{1j}. \tag{51} \]

For this model, we have to fix two parameters $\mu_{24}$ and $\mu_{34}$. After this, we have three equations
\[ (\alpha_{kk} - \alpha_{11})(\alpha_{11} + \kappa_1 + \kappa_2 - \alpha_{kk}) = 0 \tag{52} \]
and eight solutions for each set of fixed parameters.

Taking into account the equivalences for the solutions with the same number of $A(u)$ and $B(u)$, we have four different solutions.

Remember that
\[ A(u) = \frac{(\kappa_1 - \alpha_{11})\exp(\kappa_2 u) - (\kappa_2 - \alpha_{11})\exp(\kappa_1 u)}{\kappa_1 - \kappa_2} \tag{53} \]
for $\alpha_{kk} = \alpha_{11}$, and
\[ B(u) = \frac{(\kappa_1 - \alpha_{11})\exp(\kappa_1 u) - (\kappa_2 - \alpha_{11})\exp(\kappa_2 u)}{\kappa_1 - \kappa_2} \tag{54} \]
for $\alpha_{kk} = \alpha_{11} + \kappa_1 + \kappa_2$.

Note that we have fixed $7\mu_{ij}$, $6\beta_{ij}$ and $3\alpha_{ii}$. Therefore, our $R$-matrix solutions have 12 free parameters.

Now we know how to generalize the results:

6. The $(n+1)(2n+1)$-vertex models

For each value of $n > 1$, the $R$ matrix has $n+1$ diagonal entries $a_{ii}(u)$, that are determined by recurrence relative to $a_{11}(u)$

$$a_{kk}(u) = a_{11}(u) + (\alpha_{kk} - \alpha_{11})K(u), \quad k = 2, \ldots, n+1$$

where

$$\alpha_{jj} = \frac{d}{du}a_{jj}(u) \bigg|_{u=0} \quad \text{and} \quad K(u) = \frac{e^{\kappa_1u} - e^{\kappa_2u}}{-\kappa_2 + \kappa_1}$$

where $\kappa_1 = \mu_{13} + \mu_{21} - \mu_{23}$ and $\kappa_2 = -\mu_{13} + \mu_{12} + \mu_{23}$.

The remaining $n(n+1)$ diagonal entries

$$b_{ij}(u) = \beta_{ij}K(u), \quad i \neq j = \{1, \ldots, n+1\}$$

where

$$\beta_{ij} = \frac{d}{du}b_{ij}(u) \bigg|_{u=0}$$

with the constraints

$$\beta_{ji}\beta_{ij} = \beta_{21}\beta_{12} = (\kappa_1 - \alpha_{11})(\kappa_2 - \alpha_{11}).$$

The number of fixed parameters $\beta_{ij}$ is $(n-1)(n+2)/2 + 1$.

The $n(n+1)$ off-diagonal matrix elements

$$c_{ij}(u) = e^{\mu_{ij}u}.$$
All $\mu_{ij}$ are fixed by the relation

$$\mu_{ji} = \kappa_1 + \kappa_2 - \mu_{ij}, \quad j > i. \quad (61)$$

The number is $(n - 1)(n + 2)/2$ and some $\mu_{ij}$ are fixed by two relations

$$\mu_{ij} = \kappa_1 - \mu_{1i} + \mu_{1j} \quad \text{and} \quad \mu_{ij} = \kappa_2 - \mu_{1i} + \mu_{1j}, \quad j < i. \quad (62)$$

The number is $(n + 1)(n - 2)/2$. This means that we have two sets of solutions. With these relations, the YBE and its derivatives are solved by two sets of $2^n$ solutions of the following $n$ equations

$$(\alpha_{kk} - \alpha_{11})(\alpha_{kk} + \alpha_{11} - \kappa_1 - \kappa_2) = 0. \quad (63)$$

Therefore, we have two different values for $a_{kk}(u)$, and $n$ parameters $\alpha_{ii}$ are fixed. This means that our $R$-matrix solutions have $n(n + 3)/2 + 3$ free parameters

$$a_{kk}(u) = A(u) = \frac{(\kappa_1 - \alpha_{11})\exp(\kappa_2 u) - (\kappa_2 - \alpha_{11})\exp(\kappa_1 u)}{\kappa_1 - \kappa_2} \quad (64)$$

when $\alpha_{ii} = \alpha_{11}$ and

$$a_{kk}(u) = B(u) = \frac{(\kappa_1 - \alpha_{11})\exp(\kappa_1 u) - (\kappa_2 - \alpha_{11})\exp(\kappa_2 u)}{\kappa_1 - \kappa_2} \quad (65)$$

when $\alpha_{kk} = \kappa_1 + \kappa_2 - \alpha_{11}$.

Using the identity

$$2^n = \sum_{k=0}^{n+1} \binom{n}{k} = \sum_{k=0}^{n+1} \frac{n!}{k!(n-k)!} \quad (66)$$

we can identify $n + 1$ different solutions.

### 7. Symmetries and free parameters

The derivative equations (5) and (6) are sufficiently general as to include the Sutherland equations [49], but they are different.

Our approach to the problem does not choose a particular model (symmetry), but rather the structure of the $R$ matrix. Therefore, we can expect many free parameters that will be fixed by the links imposed by the desired symmetries. For example, the symmetry $R_{12}(u) = P_{12}R_{12}(u)P_{12}$ is not valid, but we have sufficient free parameters that can be fixed in order to include that symmetry in our $n + 1$ solutions.

In an $R$-matrix, the position of a particular entry is determined by the symmetries and defines the Boltzmann weight associated with a vertex. In the vector representation, we can say that an $R$ with $n$ non-null entries defines an $n$-vertex model.

For $n \geq 2$, the $n + 1$ solutions do not enjoy $P$ and $T$ symmetry, but $PT$ invariance

$$P_{12}R_{12}(u)P_{12} = R_{21}(u) = R_{12}(u)^{1!12} \quad (67)$$
Solutions of the Yang–Baxter equation for \((n + 1)(2n + 1)\)-vertex models using a differential approach and unitarity

\[
R_{12}(u)R_{21}(-u) = \xi(u) = a_{11}(u)a_{11}(-u).
\]  

(68)

They are not crossing invariant, either, but obey the weaker property \([50]\)

\[
\{\{R_{12}(u)\}^{12}\}^{-1} = \frac{\xi(u + \varrho)}{\xi(u + 2\varrho)} M_2 R_{12}(u + 2\varrho) M_2^{-1}
\]

(69)

where \(M\) is a symmetry of the \(R\)-matrix

\[
[R(u), M \otimes M] = 0, \quad M_{ij} = \delta_{ij} \chi^{(n+1-2i)/2}
\]

(70)

where \(\chi = \kappa_1 + \kappa_2 + 1\), and with the choice \(\chi = e^{\eta \kappa}\) we have the weaker crossing parameter \(\varrho = n\eta\).

We now turn to the parameters that were not fixed by the YB equation and its two derivative equations. We now notice that two of these parameters were used in the definition of the crossing parameter through the relation \(\kappa_1 + \kappa_2\).

From this information, the \(\frac{n(n+3)}{2} + 3\) free parameters are simply values of the derivatives of the Boltzmann weights in a particular value of the spectral parameter \(u\). However, this allows an important result of this work to be understood, because with these free parameters, we find a Hamiltonian integrable with several constant couplings. They will be fixed in the same way as was done for solution 1 of the 15-vertex model, which leaves us with the same number of free parameters as the model with the corresponding symmetry. In other words, they are available for possible reductions.

8. Conclusion

We experienced the power of the derivative method to solve functional equations. It is not over-pretentious to say that all models with an \(R\) matrix with its \((n + 1)(2n + 1)\) matrix elements positioned as in the \(A^{(1)}_{n-1}\) symmetry matrix are obtained after setting some of the free parameters in the solutions. We believe that there are enough parameters to do this reduction.

It may also be interesting to consider a generalization of this work using \((1)\) instead of \((2)\), in order to compare it with the ‘boost operator’ method presented in \([29, 51]\). Perhaps the calculation of the reflection matrices for these \(R\) matrices would also be interesting, as well as their Bethe ansatz solutions. The reflection matrix for our fifteen-vertex model has already been calculated by the first author \([52]\).

We also believe that the rational case is interesting.

In the literature, we find articles with these reductions, for example, in \([53]\), ten reductions are presented for the nineteen-vertex model, but the vertex number was not preserved.
Solutions of the Yang–Baxter equation for \((n + 1)(2n + 1)\)-vertex models using a differential approach

Acknowledgments

Posthumous thanks are extended to Ricardinho. We thank R A Pimenta for correcting the text. This work was supported in part by Conselho Nacional de Desenvolvimento-CNPq-Brasil, Grant #305856/2017-0.

References

[1] Vieira R S and Lima-Santos A 2017 Reflection matrices with \(U_q[osp(2|2m)]\) symmetry J. Phys. A: Math. Theor. 50 375204
[2] Vieira R S and Lima-Santos A 2015 Where are the roots of the Bethe ansatz equations? Phys. Lett. A 379 2150–3
[3] Vieira R S 2018 Solving and classifying the solutions of the Yang–Baxter equation through a differential approach. Two-state systems J. High Energy Phys. JHEP10(2018)110
[4] Bittleston R and Skinner D 2019 Gauge theory and boundary integrability J. High Energy Phys. JHEP05(2019)195
[5] Bittleston R and Skinner D 2020 Gauge theory and boundary integrability. Part II. Elliptic and trigonometric cases J. High Energy Phys. JHEP06(2020)080
[6] Yang C N 1967 Some exact results for the many-body problem in one dimension with repulsive delta-function interaction Phys. Rev. Lett. 19 1312
[7] Yang C N 1968 S matrix for the one-dimensional \(N\)-body problem with repulsive or attractive \(\delta\)-function interaction Phys. Rev. 168 1920
[8] Zamolodchikov A B and Zamolodchikov A B 1979 Factorized \(S\)-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models Ann. Phys., NY 120 253–91
[9] Baxter R J 1972 Partition function of the eight-vertex lattice model Ann. Phys., NY 70 193–228
[10] Baxter R J 1978 Solvable eight-vertex model on an arbitrary planar lattice Phil. Trans. R. Soc. A 289 315–46
[11] Kulish P P and Sklyanin E K 1982 Solutions of the Yang–Baxter equation J. Math. Sci. 19 1596–620
[12] Jimbo M 1990 Yang–Baxter Equation in Integrable Systems vol 10 (Singapore: World Scientific)
[13] Kulish P P 1996 Yang–Baxter equation and reflection equations in integrable models Low-dimensional Models in Statistical Physics and Quantum Field Theory (Berlin: Springer) pp 125–44
[14] Sklyanin E K, Takhtadzhyan L A and Faddeev L D 1979 Quantum inverse problem method I Theor. Math. Phys. 40 88
[15] Takhtadzhyan L A and Faddeev L D 1979 Quantum method of the inverse problem and the Heisenberg XYZ model Russ. Math. Surv. 34 11–68
[16] Sklyanin E K 1982 Quantum versus the method of inverse scattering problem J. Math. Sci. 19 1546–96
[17] Sklyanin E K 1982 Some algebraic structures connected with the Yang–Baxter equation Funct. Anal. Appl. 16 263–70
[18] Jimbo M 1985 A \(q\)-difference analogue of \(U(g)\) and the Yang–Baxter equation Lett. Math. Phys. 10 63–9
[19] Drinifel’d V G 1985 Hopf algebra and Yang–Baxter equation Sov. Math. Dokl. 32 254–8
[20] Drinifel’d V G 1988 Quantum groups J. Sov. Math. 41 898–915
[21] Faddeev L D, Reshetikhin N Y and Takhtjsjan L 1988 Quantization of lie groups and lie algebras Algebraic Analysis vol 1 (Amsterdam: Elsevier) pp 129–39
[22] Turaev V G 1988 The Yang–Baxter equation and invariants of links Invent Math. 92 527–53
[23] Kauffman L H and Lomonaco S J Jr 2004 Braiding operators are universal quantum gates New J. Phys. 6 134
[24] Minahan J A and Zarembo K 2003 The Bethe-ansatz for script \(N\) = 4 super Yang–Mills J. High Energy Phys. JHEP03(2003)013
[25] Beisert N et al 2012 Review of AdS/CFT integrability: an overview Lett. Math. Phys. 99 3–32
[26] Witten E 1989 Gauge theories and integrable lattice models Nucl. Phys. B 322 629–97
[27] Costello K, Witten E and Yamazaki M Gauge theory and integrability, I. (arXiv:1709.09993)
[28] Costello K, Witten E and Yamazaki M Gauge theory and integrability, II. (arXiv:1802.01570)
[29] de Leeuw M, Pribytok A and Ryan P 2019 Classifying integrable spin-1/2 chains with nearest neighbour interactions J. Phys. A: Math. Theor. 52 505201
[30] Korepin V E, Bogoliubov N M and Izergin A G 1997 Quantum Inverse Scattering Method and Correlation Functions vol 3 (Cambridge: Cambridge University Press)
Solutions of the Yang–Baxter equation for \((n + 1)(2n + 1)\)-vertex models using a differential approach

[31] Jones V F R 1990 Baxterization *Int. J. Mod. Phys.* B **04** 701–13

[32] Jimbo M 1986 Quantum \(R\) matrix for the generalized Toda system *Commun. Math. Phys.* **102** 537–47

[33] Bazhanov V V 1987 Integrable quantum systems and classical Lie algebras *Commun. Math. Phys.* **113** 471–503

[34] Bazhanov V V and Shadrikov A G 1987 Trigonometric solutions of triangle equations. Simple Lie superalgebras *Theor. Math. Phys.* **73** 1302–12

[35] Krichever I M 1981 Baxter’s equations and algebraic geometry *Funct. Anal. Appl.* **15** 92–103

[36] Cox D A, Little J B and O’Shea D 2015 *Ideals, Varieties, and Algorithms (An Introduction to Computational Algebraic Geometry and Commutative Algebra)* 4th edn (Berlin: Springer)

[37] Aczél J 1966 *Lectures on Functional Equations and Their Applications* vol 19 (New York: Academic)

[38] Pimenta R A and Martins M J 2011 The Yang–Baxter equation for \(\mathcal{PT}\) invariant 19-vertex models *J. Phys. A: Math. Theor.* **44** 085205

[39] Abel N H 1823 Méthode générale pour trouver des fonctions d’une seule quantité variable, lorsqu’une propriété de ces fonctions est exprimée par une équation entre deux variables *Magazin for Naturvidenskaberne* **1** 1–10

[40] Sklyanin E K 1988 Boundary conditions for integrable quantum systems *J. Phys. A: Math. Gen.* **21** 2375

[41] Mezincescu L and Nepomechie R I 1991 Integrable open spin chains with nonsymmetric \(R\)-matrices *J. Math. Phys.* **24** L17

[42] Vieira R S and Lima-Santos A 2013 On the multiparametric \(U_q(D_{n+1}^{(2)})\) vertex model *J. Stat. Mech.* P02011

[43] Cherednik I V 1980 On a method of constructing factorized \(S\) matrices in elementary functions *Theor. Math. Phys.* **43** 356–8

[44] Babelon O, de Vega H J and Viallet C M 1981 Solutions of the factorization equations from Toda field theory *Nucl. Phys.* B **190** 542–52

[45] Chudnovsky D V and Chudnovsky G V 1980 Characterization of completely \(X\)-symmetric factorized \(S\)-matrices for a special type of interaction applications to multicomponent field theories *Phys. Lett.* A **79** 36–8

[46] Perk J H H and Schultz C L 1981 New families of commuting transfer matrices in \(q\)-state vertex models *Phys. Lett.* A **84** 407–10

[47] Perk J H and Schultz C L 1990 Families of commuting transfer matrices in \(q\)-state vertex models *Yang-Baxter Equation in Integrable Systems* (Singapore: World Scientific) pp 326–43

[48] Perk J H H and Au-Yang H 2006 Yang–Baxter equations *Encycl. Math. Phys.* **5** 465–73

[49] Schutherland B 1970 *J. Math. Phys.* **11** 3183–6

[50] Reshetikhin N Y and Semenov-Tian-Shansky M A 1990 *Lett. Math. Phys.* **19** 133

[51] de Leeuw M, Paletta C, Prilutko A, Retore A L and Ryan P 2020 Yang–Baxter and the Boost: splitting the difference (arXiv:2010.11231)

[52] Vieira R S 2019 Fifteen-vertex models with non-symmetric \(R\) matrices (arXiv:1908.06932)

[53] Idzumi M, Tokihiro T and Arai M 1994 Solvable nineteen-vertex models and quantum spin chains of spin one *J. Phys.* **4** 1151–9