Fault-Tolerant Quantum Computation

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I give a brief overview of fault-tolerant quantum computation, with an emphasis on recent work and open questions.

The world is a dangerous place, particularly if you are a qubit. Organized vibrations (or just thermal phonons) are constantly trying to shake you up for all you’re worth. Gangs of unemployed photons fly about, flipping over any qubit foolish enough to get in their way. A moment’s relaxation can send you spontaneously plummeting to your ground state. And as for having a friend keep an eye on you, forget about it — a qubit being watched is no better than a conventional classical bit.

Indeed, it is unlikely that a single qubit can survive for long on its own. But by teaming up as a quantum error-correcting code (QECC), groups of qubits can work together to fight off the dangers of their environment. Indeed, by acting in concert, the qubits of a QECC can not only survive, but flourish, performing together complex quantum computations without losing their integrity. In order to do so, however, all their interactions need to be carefully structured according to the dictates of a fault-tolerant protocol. If the qubits break this code of behavior, they run the risk of, well, breaking the code, exposing them once again to the chaotic environment of the world outside the quantum computer.

The basic techniques of quantum error correction and fault-tolerant quantum computation were first developed in 1995–6, in response to the surge of interest in quantum computation that followed Shor’s 1994 factoring algorithm. Those theoretical results showed that there was unlikely to be any barrier in principle to building large quantum computers, although of course serious practical difficulties remain even today. After the breakthroughs of the mid-90s, the field experienced a period of relative inactivity, but in the past few years there has been a burst of new research aimed mostly at bringing the theoretical results to experimental reality.

Fault tolerance is not the only approach that has been discovered to deal with errors, but it is the most general one. A properly designed fault-tolerant protocol can correct arbitrary small errors (the precise meaning of a “small error” will be discussed in section 3). Other sorts of error control techniques, such as the self-correcting pulses now employed in a variety of different quantum computer implementations, tend to be good for certain kinds of errors, but will leave other residual errors behind. Given the extremely high accuracy required for a large quantum computation, it is likely that fault-tolerant quantum computation will be needed to correct these remaining errors in order for the computation to succeed. Thus, it seems likely that future quantum computers will use a variety of specific control techniques to eliminate the most prevalent types of errors and will, on top of that, employ fault-tolerant protocols to eliminate the remaining sources of noise.

I. QUANTUM ERROR-CORRECTING CODES

The first element in a fault-tolerant protocol is the quantum error-correcting code. Space does not permit a full exposition here of the principles of quantum error correction. Instead, I will just present one specific code, the 7-qubit code, for which fault-tolerant operations are particularly straightforward.

The 7-qubit code encodes a single qubit as follows:

\[ |0\rangle = |0000000\rangle + |1111000\rangle + |1100110\rangle + |1010101\rangle \]
\[ |1\rangle = |0011111\rangle + |0101101\rangle + |0110011\rangle + |1001101\rangle \]

The encoded zero state (or logical zero) \( |0\rangle \) is the superposition of the even weight codewords from the classical 7-bit Hamming code, whereas the logical one \( |1\rangle \) is the superposition of the odd weight codewords from the Hamming code. The weight of a codeword is the number of 1s in the codeword. The classical Hamming codes have the useful property that to get from any codeword in the code to any other codeword, you must flip at least 3 bits. Thus, if we are given a codeword with only one flipped bit (or no flipped bits) then we can identify unambiguously which bit was flipped and what the original codeword was, and the Hamming code can correct one error on an arbitrary bit.

It is a bit less straightforward to see how the 7-qubit code can correct a single error on an arbitrary qubit. First of all, note that an arbitrary encoded state \( \alpha |0\rangle + \beta |1\rangle \) is a superposition of codewords from the classical Hamming code. Therefore, if the code experiences a bit flip or X error (\( |0\rangle \rightarrow |1\rangle, |1\rangle \rightarrow |0\rangle \)) on a specific qubit, the state is a superposition of Hamming codewords which have been flipped at the corresponding bit location. By the error-correcting properties of the Hamming code, it is therefore possible to make a measurement to determine the location of the error. However, it is critical that the measurement we make does not tell us what the original classical codeword was, as that would destroy the quantum superposition that gives us an encoded qubit.

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rather than an encoded classical bit. One way to do this measurement will be discussed in section II.

As it happens, the code can be used to correct phase errors in nearly the same way. Because of the particular superpositions of classical codewords chosen in the encoding, if we perform a Hadamard transform \((|0\rangle \rightarrow |0\rangle + |1\rangle, |1\rangle \rightarrow |0\rangle - |1\rangle)\) on each of the physical qubits comprising the code, we again get a superposition of codewords from the classical Hamming code. The Hadamard transform will convert a phase flip or \(Z\) error \((|0\rangle \rightarrow |0\rangle, |1\rangle \rightarrow -|1\rangle)\) into an \(X\) error, so by checking for errors in the Hadamard-transformed basis, we can identify the location of any single \(Z\) error.

The phase correction procedure can be done independently of measuring the location of an \(X\) error, so the code can therefore correct not just \(X\) and \(Z\) errors, but also \(Y = iXZ\) errors, or indeed one \(X\) error and one \(Z\) error on different qubits. (The global phase factor \(i\) has no physical significance, but it makes some of the mathematics nicer.) In addition, in the case that there is no error (the identity \(I\)), the state is kept safe as well, and is not destroyed by our error correction procedure. The four types of errors \(I, X, Y, \text{ and } Z\), acting on the \(n\) qubits of the code, generate a group of possible errors, frequently known as the Pauli group. We can define the weight of an operation from the Pauli group analogously to the weight of a classical bit string as the number of frequently known as the Pauli group. We can define the qubits of the code, generate a group of possible errors, \(n\) thus correct an arbitrary weight 0 or 1 Pauli operator.

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However, \(X, Y, \text{ and } Z\) are not the only possible things that can go wrong on a single qubit. The most general possible error would be described by some unitary interaction between the faulty qubit and some environment. However, \(I, X, Y, \text{ and } Z\) form a basis for the set of \(2 \times 2\) matrices, so any unitary \(U\) can be written as

\[
U = \alpha I \otimes A_I + \beta X \otimes A_X + \gamma Y \otimes A_Y + \delta Z \otimes A_Z,
\]

with \(A_I, A_X, A_Y, \text{ and } A_Z\) operators acting on the environment. Suppose we were to forget about the possibility of this general error, and instead just perform error correction as if only Pauli errors could occur. Quantum mechanics is linear, and the error correction procedure accurately records any Pauli errors, so the complete state of the system will be entangled between the encoded state, the environment, and the extra register (the ancilla) which we are using to record information we have measured about the error, and will be a superposition of cases where the incorrect qubit has each possible Pauli error or no error at all, and where the ancilla’s state correctly records which error is present. But then when we measure the ancilla, we collapse not just the ancilla itself, but also the data and the environment, into the case where one particular Pauli error \(I, X, Y, \text{ or } Z\) has occurred. Our naive belief that only a Pauli error was possible has become a self-fulfilling prophecy.

Of course, this 7-qubit code only encodes one qubit, but if we wish to encode many qubits, we can simply use this code repeatedly, expanding each logical qubit in our system into 7 physical qubits. Each set of 7 qubits is called a block of the code, and this system is capable of correcting one error in each block. Because of the linearity of quantum mechanics, entanglement between separate blocks is protected from errors as well. Of course, if we happen to have two errors in a given block, that block may fail, and if we attempt to decode the failed block, we will get the wrong quantum state out. However, when errors are somewhat rare, two errors close together are even rarer, so a QECC can convert a small physical error rate into an even smaller logical error rate.

II. FAULT-TOLENTANT PROTOCOLS

A quantum error-correcting code by itself is only useful when we can consider the quantum gates we perform to be effectively perfect. That is unlikely to be a good approximation for all but the smallest quantum computations, so we must supplement the QECC with a fault-tolerant protocol. A fault-tolerant protocol must contain components to perform fault-tolerant error correction, fault-tolerant state preparation, fault-tolerant measurement, and a universal set of fault-tolerant gates. Typically, completing this set involves a rather large collection of different tricks, but most of them are devoted to conquering the same problem: error propagation. Usually, we assume that a single faulty gate can cause errors only in the qubits involved in the gate, so a bad single-qubit gate would cause errors in at most one qubit and a two-qubit gate in at most two qubits. However, even a perfect two-qubit gate can propagate a pre-existing error from one of the qubits involved in the gate to the other one, so even a single erroneous gate can thus indirectly cause errors in many qubits.

A QECC can only correct a limited number of errors per block (one in the case of the 7-qubit code described above), so we must be particularly careful that errors do not spread within a block. The usual solution to this is to use transversal gates — gates which interact only the \(i\)th qubit of one block with the \(i\)th qubit of another block. With a system composed completely of transversal gates, a single bad gate anywhere in the system can only spread to produce one error per block of the code, which avoids overwhelming the error tolerance of any single block. Of course, two bad gates could then cause a large number of blocks to fail, so we must also perform error correction periodically to get rid of the errors before they build up.

The 7-qubit code described in section II is particularly convenient for fault-tolerance because a number of gates can be performed transversally. In particular, the logical Hadamard transform can be performed by just performing the Hadamard transform on each of the 7 physical qubits in the code, and the logical CNOT gate \((|a\rangle|b\rangle \rightarrow |a\rangle|a \oplus b\rangle, \text{ with } a \text{ and } b \text{ bits})\) can be performed transversally with CNOT gates from each of the 7 qubits.
of the control block to the corresponding qubits in the target block. The logical $\pi/4$ phase rotation, $|0\rangle \rightarrow |0\rangle$, $|1\rangle \rightarrow i|1\rangle$, can be done by performing the $-\pi/4$ phase rotation on each physical qubit. Products of these gates are equally easy; for instance, the $Z$ gate is the square of the $\pi/4$ gate and can be performed simply by a $Z$ gate on each qubit, and similarly for $X$ and $Y$. The theory of fault tolerance makes frequent reference to these gates and to the group (frequently called the Clifford group) formed by all possible products of the CNOT, the Hadamard, and the $\pi/4$ gate.

Unfortunately, the Clifford group is not universal for quantum computation, and only the gates from the Clifford group can be performed transversally on the 7-qubit code. Other transversal combinations of gates would take a correct codeword to one with errors in it. Adding even a single additional gate to the mix is sufficient, however, to allow us to approximate any unitary operation to arbitrary accuracy. A frequently used additional gate is the $\pi/4$ phase rotation, and the $\pi/4$ phase rotation, which in some sense encapsulates the Clifford group, can be done transversally. The ancilla preparation requires some more complicated tricks, however, which I will not describe here due to lack of space.

Fault-tolerant measurement is straightforward for the 7-qubit code. Since the logical codewords are superpositions of classical Hamming codewords, measuring each qubit gives us (in the absence of errors) a random classical codeword, with an even-weight codeword corresponding to a logical zero and an odd-weight codeword corresponding to a logical one. If there is an error, we may instead get a classical codeword with a bit flip in it, but that can be corrected easily using classical error correction. This is a destructive measurement procedure, but if we wish to perform a non-demolition measurement, we can prepare an encoded ancilla in the state $|0\rangle$, perform a logical CNOT from the data block to be measured to the ancilla, and then destructively measure the ancilla.

Fault-tolerant state preparation is a bit more complicated, and a variety of ways have been suggested to do it. Fault-tolerantly encoding an unknown quantum state is not possible, since a single error at the very start of our encoding procedure will ruin the state regardless of how cleverly we design the rest of the procedure. Instead, we typically perform fault-tolerant quantum computations by first creating a number of encoded $|0\rangle$ states and using classical control and fault-tolerant gates to produce the correct initial state for our algorithm. We thus need only consider how to fault-tolerantly produce $|0\rangle$ states.

We can start with a non-fault-tolerant encoding circuit, for instance, but even a single error in the procedure could produce another state, such as the $|1\rangle$. One way to avoid this is to produce two $|0\rangle$ states and check them against each other using the non-demolition measurement procedure described above. If one of the states is an encoded $|1\rangle$ while the other is a $|0\rangle$, the measurement will identify that there is a problem (although it does not tell us which one was wrong), and we discard both and try again. Of course, if both states have errors in the preparation procedure, we may be fooled into accepting a bad state, but that requires two separate errors, which could ruin the state anyway.

Finally, we can perform fault-tolerant error correction by combining the fault-tolerant preparation, measurement, and Clifford group constructions from above. Using fault-tolerant preparation and Clifford group gates we can make reliable $|0\rangle$ and $|0\rangle + |1\rangle$ ancilla states. Note that a logical CNOT with the data block as a control block and a $|0\rangle + |1\rangle$ ancilla as the target would not have any effect at all in the absence of any errors. However, if there are $X$ errors in the data, this operation will cause them to propagate to the corresponding locations in the ancilla block. Then a transversal measurement of the ancilla will give us a random classical Hamming codeword with bit flip errors in the locations corresponding to the $X$ errors in the data, and if there is only one such error, we can then identify its location using classical error correction. Note that because we used an ancilla in the state $|0\rangle + |1\rangle$, the measurement gives us no information about the encoded state of the data, only the errors on the physical qubits making up the data block.

Similarly, performing a logical CNOT from a $|0\rangle$ ancilla to a data block copies the phase errors without disturbing the logical data qubit. Then measuring the ancilla in the Hadamard basis again gives a random Hamming codeword with errors in the places corresponding to the locations of $Z$ errors in the data block. Of course, while errors are propagating from the data block to the ancilla block, they are also propagating the other way, from the ancilla blocks into the data block. This procedure can never make the data more reliable than the ancillas, but the ancillas we use are always built from scratch, so their error rates are never too high, and fault-tolerant error correction will prevent the frequency of errors in the data blocks from building up to a level where they are likely to cause our computation to fail.

We can understand fault-tolerant error correction from a thermodynamic point of view: Errors introduce entropy, heating up the state of our computer. We introduce cool ancilla states, and error correction pumps heat from the data into the ancillas, acting like a refrigerator. We can never cool the data to below the temperature of the ancillas, but we can prevent it from heating up to arbitrarily high temperatures.

III. THE THRESHOLD FOR FAULT TOLERANCE

Of course, the 7-qubit code has a definite limit to its usefulness. If two errors occur in a block (either directly
or by propagating from another block) before we have the opportunity to do error correction, the block will fail, potentially introducing an error in the state we were trying to protect. If the probability for a single error in a single physical gate (or a time step without a gate) is \( p \), then the probability of having two errors in two particular physical gates is \( p^2 \). However, by encoding the state and by using a fault-tolerant protocol, we have introduced many extra places something could go wrong, so the total probability of having two errors accumulate in a block despite our attempts at error correction becomes something like \( C p^2 \) per logical gate, where \( C \) represents roughly the number of pairs of locations (gates or time steps) where two physical errors can cause a logical error in the course of performing the gate. \( C \) will depend on the code and the fault-tolerant circuitry we use, but if we perform error correction at regular intervals, \( C \) will not depend on the length of the overall computation.

Now, replacing a physical error rate of \( p \) per gate with a logical error rate of \( C p^2 \) per gate is an improvement when \( p < 1/C \) (although it is actually worse when \( p > 1/C \)); in that case, the extra qubits and gates we introduce cause extra errors faster than we can correct them). Before, we could do a computation of length about \( 1/p \) before we would expect to see an error, and now we can last for a time about \( 1/(C p^2) \). However, we would like to do better still.

One way to achieve this is to use concatenated codes. We can encode each logical qubit in 7 physical qubits, as before, but then we can encode each of those 7 qubits again using the same code (or a different one, if we so desire). Now the logical qubit is encoded by 49 physical qubits, but the error rate has dropped again, to \( C(C p^2)^2 \), or \( C^3 p^4 \). If necessary, we can encode a third or fourth time, giving a logical error rate \( O(p^8) \) or \( O(p^{16}) \). After \( k \) levels of concatenation, the effective logical error rate is

\[
p_k = p_T (p/p_T)^{2^k},
\]

where \( p_T = 1/C \) is called the threshold. When \( p < p_T \), the logical error rate can be made arbitrarily small. To achieve an error rate \( \epsilon \), we need \( \log \log 1/\epsilon \) levels of concatenation; of course, the number of qubits in a concatenated block is exponential in the number of levels, but that still means that we need only \( \log(\log 1/\epsilon) \) extra qubits to achieve error rate \( \epsilon \).

This result is known as the threshold theorem. A complete rigorous proof is more difficult than the above argument, as it must define, for instance, the meaning of error rate in a code block which is itself part of a larger concatenated code block, but the conclusion is the same: There is a threshold error rate such that, if the physical error rate per gate and per time step is below the threshold value, then arbitrarily long universal quantum computations are possible with only polylogarithmic overhead. A word of caution is necessary, however, regarding the overhead: Estimates for plausible error rates and computation lengths frequently require overhead of 1000:1 or more using concatenation. The advantage of the threshold result is in the scaling — much longer computations require only slightly more overhead.

IV. THE VALUE OF THE THRESHOLD

As you may imagine, a considerable amount of research has been devoted to determining the numerical value of the threshold, as that sets a target value of accuracy that experimentalists will try to achieve. However, citing a single number as the threshold value is a bit deceptive, as there are a large number of variables involved. Also, it is important to bear in mind that thresholds are derived for specific QECCs and fault-tolerant protocols. Future developments in these areas might increase our estimates of the threshold, allowing fault-tolerant quantum computation with noisier devices.

Threshold calculations frequently make a number of assumptions about the properties of the quantum computer; while any given implementation may satisfy some of these, I don’t know of any that satisfies all of them. Below, I list some common assumptions worth discussing further. These are not the only assumptions necessary to prove the threshold theorem. Indeed, they are interesting to discuss precisely because none of them is completely necessary. Because we may not be able to satisfy all of these assumptions at once, it is important to study the tradeoffs between them.

1. Multiple-qubit gates can interact any pair of qubits in the computer. This is not unreasonable for optical quantum computers, since photons move around so easily, but is a poor assumption in many other models, where qubits are constrained to interact only with other qubits that are nearby in a 1-dimensional, 2-D, or conceivably 3-D arrangement.

2. Classical computation and measurement are fast and reliable. In some systems, gate times are so fast that classical computation cannot keep up; in others, measurement takes a long time compared to the decoherence time, or is not possible on individual qubits.

3. There is an ample supply of extra qubits. Perhaps someday this will be true, but for the time being, extra qubits are at a premium.

4. Errors occur independently on separate gates and at separate times, and \( X \), \( Y \), and \( Z \) errors are equally likely, each with probability \( p/3 \). This error model is known as the depolarizing channel, and is a very common simple one to consider. There are some variations on how it is extended to treat errors in two-qubit gates, but generally there is a total probability \( p \) of error, which will likely affect both qubits involved in the gate. A somewhat more realistic error model has errors occur with probability \( p \),
independently on each gate, but does not specify the probability of various different kinds of errors. Even this model will rarely hold to very high precision, as correlated errors are likely between qubits which are located near each other.

Much of the recent work on the threshold has focused on considering different subsets of these assumptions or others like them, and studying the effect on the threshold. However, even if we make all of the above assumptions, the threshold is still not a single number, since different types of gates could have different error rates. Other work is devoted to studying fault-tolerance in specific types of systems, such as ion traps or linear optical quantum computation.

Another divide is between different methods of studying the threshold. Many of the results are derived using simulations to estimate the threshold error rate. If done correctly, simulations can be very informative, but it is easy to make mistakes, for instance to consider insufficiently large systems or insufficiently general gate networks. Since the simulations are usually performed without a rigorous proof of correctness, there is always the possibility that something critical has been omitted. The other main approach is to mathematically prove a lower bound on the threshold for a particular type of circuitry. This has the advantage that we can be sure that a quantum computer that satisfies the appropriate conditions will have a threshold at least that great. Threshold proofs can also frequently deal better with more general error models. However, proving a threshold generally requires making some conservative simplifying assumptions that bring the resulting threshold value lower than is achieved in a corresponding simulation. A factor of 10 difference is common. Most likely the true threshold for a particular fault-tolerant scheme is somewhere in between the simulated and proven threshold values. There are also a variety of analytical techniques which make some guesses about which effects are most important and prove a threshold based on those. These techniques tend to produce a number somewhere between the proof and the simulation.

The most optimistic estimates use ancilla factories to effectively trade extra qubits for error tolerance. Most of the work is done on ancilla states, which are carefully checked and discarded if found to be flawed. Simulations suggest, using all of the above assumptions, that the threshold can be pushed to at least the range of $1\%-5\%$ using an extreme version of these methods, depending on the relative error rates of different kinds of gates. These schemes, however, cause a serious blow-up in the overhead, to millions or billions of physical qubits for each logical qubit, so should be regarded primarily as a theoretical existence result for high thresholds. Recent rigorous proofs of the threshold using the extreme ancilla factory circuitry give a threshold value of around $10^{-3}$, again an order of magnitude less than the value suggested by simulations.

A fair amount of work has also now been done on systems with gates constrained to act on nearest neighbor qubits in a 1-D or 2-D lattice. One-dimensional systems are difficult to deal with, but there is still a fault-tolerant threshold. In an almost 1-D system (i.e., two parallel lines of qubits), the best current proof shows a threshold of around $10^{-6}$ (in this case, I know of no published superior simulation). In a 2-D system, the best simulation to date gives a threshold of around $7 \times 10^{-3}$, while the best proof gives around $2 \times 10^{-5}$, although using somewhat old fault-tolerant techniques. In general, then, working in a two-dimensional system seems not to hurt the threshold too much, perhaps by a factor of 2 or 3, but a one-dimensional system is substantially inferior.

There has also been some study of the case where fast measurement or reliable classical computation is unavailable. One solution is to replace the classical computation by an equivalent quantum computation. Since the quantum computer is unreliable, the computation must be performed redundantly. Luckily, it need only be redundant in the sense of classical fault-tolerance, which can tolerate a much higher error rate, so it appears that this approach does not lower the threshold by very much.

We can also consider more general error models. The depolarizing channel is overly simplistic, and is unlikely to occur in any real system. A slightly more general model known as the adversarial probabilistic channel is frequently considered in threshold proofs. It assumes the locations (time and place) of errors are chosen randomly and independently with probability $p$ per gate or time step (with an erroneous two-qubit gate having errors on both qubits), but that the actual type of error, $X$, $Y$, $Z$, or some superposition, is chosen adversarially in such a way as to cause maximum harm to the computation. The adversary may even choose the type of errors to be strongly correlated between error locations (but the locations themselves are chosen independently). Working with the adversarial error model frequently results in a factor of 2–3 drop of the threshold compared to the depolarizing channel, and is likely responsible for part of the difference between cited numbers from proofs and from simulations. (The adversarial error model is not particularly amenable to simulation, whereas most proofs have difficulty taking full advantage of the depolarizing channel.)

However, even the adversarial error model does not capture the full scope of realistic errors. For instance, a slight over- or under-rotation of a qubit leads to a coherent error, which cannot be represented as a probability $p$ of some error (even an adversarial one) and probability $1-p$ of no error. In the most general case, we should consider an environment interacting through a weak Hamiltonian coupling to the system. The environment may have an indefinitely long memory and strong internal interactions. Even in such non-Markovian systems, there is threshold for fault-tolerance, provided the system-bath Hamiltonian is bounded. (It remains unclear precisely what happens for unbounded systems once they are regularized to avoid infinite energies.)
The provable threshold value for non-Markovian systems is much worse than for probabilistic error models. To compare properly, we must convert the bound on the system-bath coupling which comes from the proof to an error probability: namely, the probability that an ideal state which undergoes the noisy gate will be projected by an ideal measurement onto the correct output state. Performing this comparison, we find that the threshold for a non-Markovian system is generally around the square of the threshold for the probabilistic error model, with perhaps an additional factor of order 1.

However, the threshold is unlikely to be that bad in reality. Coherent errors, including non-Markovian ones, are dangerous in principle because the amplitudes of errors can add up coherently, causing the error probabilities to accumulate as the square of the number of noisy gates rather than linearly. In a real system, the errors are unlikely to be able to arrange their phases so perfectly, both because the environment is not truly an adversary, and because the fault-tolerant protocol is constantly mixing things up. If the phases of consecutive errors are effectively randomized, the random walk in phase causes probabilities to again accumulate linearly, bringing the threshold back to roughly the level of probabilistic errors. It is my belief that this will be the case in practice, although there is as yet no rigorous analysis to back up that belief.

Other tradeoffs in properties relating to fault-tolerance still need more study. While ancilla factories use extra ancillas to allow higher gate error rates, we also have some indication that it is possible to go the other direction, and reduce overhead at a modest cost in threshold. To what degree is this possible? There has been almost no study of the effects of correlated errors on the threshold error rate, but in many realistic systems, some correlations are likely, at least between neighboring qubits.

And then there is the question of trading off multiple factors simultaneously. For instance, to what degree can we create ancilla factories that operate in two spatial dimensions? Does fast classical communication alter this tradeoff? We are still a long way from having a full understanding of this sort of question.

V. FOR FURTHER INFORMATION AND ACKNOWLEDGEMENTS

This has only been a very brief introduction to the accomplishments and challenges of fault-tolerant quantum computation, but there are a number of references which go into much more detail about quantum error correction and fault tolerance. For a brief introductions to quantum error correction and fault tolerance, see \[1\] and \[2\], respectively. Chapter 10 of Nielsen and Chuang \[3\] gives a more detailed overview of quantum error correction and fault-tolerant circuitry. \[4\] provides a recent detailed proof of the threshold theorem, along with a discussion of non-Markovian errors. For the modern extreme version of ancilla factories, see \[5\] or the more detailed proofs \[6\] and \[7\]. For fault tolerance in 2 dimensions, see \[8\] or \[9\], and for the 1-dimensional case, see \[10\]. Finally, for tradeoffs between overhead and error rate, see \[11\]. I have not attempted here to give full historical credit for development of the ideas, but instead list, in most cases, the most recent, most general, or otherwise best expression of each idea. Again, consult the references for more information about the historical development of quantum error-correcting codes and fault-tolerant quantum computation.

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