Error analysis between two different fuzzy multiplication operations on Triangular Fuzzy Number

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Abstract. One of the basic contents of fuzzy mathematics is the mathematical operations of fuzzy numbers. But still some of the fuzzy mathematical operations are not determined. There are some operations between fuzzy numbers are exact and there are approximated operations. In this paper, a comparison has been presented between two different fuzzy multiplication operations. One of these two fuzzy multiplication operations is exact with using interval arithmetic cut method and the other one is an approximation method which uses a standard approximation. In the two different methods, the fuzzy numbers will be defined as triangular fuzzy numbers because the triangular fuzzy number is the most popular one between the other shapes of fuzzy numbers. A numerical example has been presented with graphical result to show the difference between them and to analyze the error.

1. Introduction
The Fuzzy sets have been presented by Lotfi.A.Zadeh(1965)[1]. Since its inception 55 years ago, the of fuzzy set theory has evolved in various ways and in many fields. Computer science, decision theory, artificial intelligence, operations research, control engineering, expert systems, robotics, logic and management science are some of the applications of the fuzzy set theory. Theoretical progress has been made in numerous fields. Fuzzy numbers have a very significant role in the applications of fuzzy set theory. There was also an evolution in the mathematical operations of fuzzy numbers. The main rule on which these operations depend is the extension principle [3] or interval arithmetic [4].

During the mathematical operations of fuzzy numbers, there will be a strong dependence of the result of these operations on the shape of Membership Functions (MFs) of these fuzzy numbers. The MFs more consistent of fuzzy numbers, the easier the calculations of the mathematical operations of these numbers are. This means that fuzzy numbers that have simple shape MFs often have more natural explanation.

Considering Triangular Fuzzy Numbers (TFNs), the multiplication operation of two TFNs is not TFN. The resulting shape MF will have a different shape than these TFNs. In many conditions, the result of this problem is approximated to be TFN.

In this paper, two methods have been presented. In the first method, interval arithmetic operation has been stated as \(\alpha\)-cut method. In the other method, approximation multiplication has been mentioned as standard approximation. Finally, a comparison has been made between them TFNs.

The paper presents 6 sections as follows: in section 2 and 3, some basic definitions have been discussed, in section 4, \(\alpha\)-cut method and standard approximation have been presented. In section 5, numerical example and graphical result are shown for different cases and in section 6, error analysis between \(\alpha\)-cut method and standard approximation has been discussed.
2. Some basic definitions of fuzzy arithmetic

Now some definitions must be introduced to describe fuzzy arithmetic [5].

2.1. Fuzzy set

A fuzzy set \( A \) is a set with different degrees of membership in which each element is mapped to \([0, 1]\) by \( MF \) [6].

\[
\text{for } x \in X \quad \mu_A(x): X \rightarrow [0, 1]
\]

Usually, if elements are discrete

\[
A = \{(x, \mu_A(x))\} \quad \text{or} \quad A = \sum_{i=1}^{n} \mu_A(x_i)/x_i
\]

When the elements are not discrete

\[
A = \int \mu_A(x_i)/x_i
\]

2.2. Support (A)

Let \( A \) be a fuzzy subset of \( X \). Then support (A) will be a crisp subset of \( A \). All elements of this crisp subset have membership values greater than zero [7].

Support (A) = \( \{x \in X \mid \mu_A(x) > 0\} \)

\( \alpha \)-Cut set

Let \( A \) be a fuzzy subset of \( X \). Then the \( \alpha \)-Cut set is a crisp set consists of the elements whose membership values are not less than \( \alpha \) and \( \alpha \) is arbitrary.

\[
A_{\alpha}(x) = \{x \in X \mid \mu_A(x) \geq \alpha\}
\]

2.3. Fuzzy number

It is a fuzzy set which has the following conditions [8]:

- Convex fuzzy set.
- Normalized fuzzy set (the maximum membership value is 1).
- MF of the fuzzy number is piecewise continuous.
- Its definition is in the real number.

Triangular fuzzy number (TFN)

TFN is the most popular fuzzy number [9].

It is a fuzzy number with triangular MF and it is expressed by three points as follows:

\[ A = (a_1, a_2, a_3) \text{ shown in Fig 1.} \]

This representation is interpreted as MF \( \mu_A(x) \).

\[
\mu_A(x) = \begin{cases} 
0, & x < a_1, x > a_3 \\
\frac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\
\frac{a_3 - x}{a_3 - a_2}, & a_2 \leq x \leq a_3
\end{cases}
\]
It is a crisp interval. This crisp interval can be obtained as in equation (1)

\[ A_\alpha = [a_1^\alpha, a_2^\alpha], \quad \forall \alpha \in [0, 1] \quad a_1^\alpha = \alpha(a_2 - a_1) + a_1 \quad \text{and} \quad a_2^\alpha = a_3 - \alpha(a_3 - a_2) \]  

(1)

**Positive triangular fuzzy number (PTFN)**
A PTFN A can be defined as \( A = (a_1, a_2, a_3) \) where all \( a_i \)'s > 0.

**Negative triangular fuzzy number (NTFN)**
A NTFN A can be defined as \( A = (a_1, a_2, a_3) \) where all \( a_i \)'s < 0.

**Partial Negative triangular fuzzy number (PNTFN)**
A PNTFN A can be defined as \( A = (a_1, a_2, a_3) \), where at least one or two \( a_i \)'s are negative and not all.

3. **Fuzzy operations based on interval description of fuzzy number**
Assuming two intervals \( A = [a_1, a_2] \) and \( B = [b_1, b_2] \), where \( a_1, a_2, b_1 \) and \( b_2 \) are real numbers, then the operations can be defined as following [10].

- **Addition**
  \[ A(+)B = [a_1 + b_1, a_2 + b_2] \]

- **Subtraction**
  \[ A(-)B = [a_1 - b_2, a_2 - b_1] \]

- **Multiplication**
  \[ A(\cdot)B = [\min(a_1 \cdot b_1, a_1 \cdot b_2, a_2 \cdot b_1, a_2 \cdot b_2), \max(a_1 \cdot b_1, a_1 \cdot b_2, a_2 \cdot b_1, a_2 \cdot b_2)] \]

- **Division**
  \[ A(/)B = [\min(a_1/b_1, a_1/b_2, a_2/b_1, a_2/b_2), \max(a_1/b_1, a_1/b_2, a_2/b_1, a_2/b_2)] \]

Excluding the case \( b_1 = 0 \) or \( b_2 = 0 \).

4. **Operations of Triangular Fuzzy Number**
The result of addition and subtraction operations is a triangular MF and the definitions of these operations can be defined as in equations (2) and (3) respectively.

In case of multiplication and division operations, the actual result will be a polynomial MF. The result of these operations is approximated to be a triangular MF.

Fig 1. Triangular Membership Function of TFN \( A = (a_1, a_2, a_3) \)
The actual operation Method (α-cut method)

The actual result can be found by α-cut method. MF is rewritten to define a set of closed intervals as in equation (1). Then the interval arithmetic is used to operate on these closed intervals [11].

Assuming two PTFN A = (a₁, a₂, a₃) and B = (b₁, b₂, b₃). Then,

\[ A_\alpha = [\alpha(a₂ - a₁) + a₁, a₃ - \alpha(a₃ - a₂)] = [x₁(\alpha), x₂(\alpha)], \quad \forall \alpha \in [0, 1] \]
\[ B_\alpha = [\alpha(b₂ - b₁) + b₁, b₃ - \alpha(b₃ - b₂)] = [y₁(\alpha), y₂(\alpha)], \quad \forall \alpha \in [0, 1] \]

The operation of multiplication can be calculated as in equation (4),

\[ A(\cdot)B = \{ \min (x₁y₁, x₁y₂, x₂y₁, x₂y₂), \max (x₁y₁, x₁y₂, x₂y₁, x₂y₂) \}, \quad \forall \alpha \in [0, 1] \]  \hspace{1cm} (4)

But, when one of the two fuzzy numbers A and B is PNTFN and the other is PTFN the multiplication operation A(\cdot)B can’t be calculated according to the equation (4). Because of the minimum of the multiplication A(\cdot)B will not be the same all over the interval of α from 0 to 1.

So, The interval of \( \alpha \in [0, 1] \) will be divided into two parts, according to the intersection point of the two minimum expressions in the α-cut of A(\cdot)B. Assuming this intersection point is \( \alpha_s \). Then,

\[ A(\cdot)B = \{ \begin{array}{l}
\{ \min (x₁y₁, x₁y₂, x₂y₁, x₂y₂), \max (x₁y₁, x₁y₂, x₂y₁, x₂y₂) \}, \quad \forall \alpha \in [0, \alpha_s] \\
\{ \min (x₁y₁, x₁y₂, x₂y₁, x₂y₂), \max (x₁y₁, x₁y₂, x₂y₁, x₂y₂) \}, \quad \forall \alpha \in [\alpha_s, 1] 
\end{array} \]  \hspace{1cm} (5)

Standard approximate operations

The definitions of the standard approximate operations for multiplication will be as follows:

\[ A(\cdot)B = \{ \min (a₁\cdot b₁, a₁\cdot b₂, a₂\cdot b₁, a₂\cdot b₂), a₂\cdot b₂, \max (a₁\cdot b₁, a₁\cdot b₂, a₂\cdot b₁, a₂\cdot b₂) \} \]  \hspace{1cm} (6)

5. Numerical study

- Case study

Two PTFNs A and B with triangular MFs \( \mu_A(x) \) and \( \mu_B(x) \) respectively has been defined as follows:

\[ \mu_A(x) = \begin{cases}
0, & x < 2, x > 5 \\
\frac{x - 2}{2} + \frac{5}{2}, & 2 \leq x \leq 3 \\
-x & 3 \leq x \leq 5
\end{cases} \]

\[ \mu_B(y) = \begin{cases}
0, & y < 3, y > 6 \\
\frac{y - 3}{2}, & 3 \leq y \leq 5 \\
-y + 6, & 5 \leq y \leq 6
\end{cases} \]

Then, \( A_\alpha = [2 + \alpha, 5 - 2\alpha] \) and \( B_\alpha = [2\alpha + 3, -\alpha + 6] \)

\[ (A(\cdot)B)_\alpha = [2\alpha² + 7\alpha + 6, 2\alpha² - 17\alpha + 30] \]

MFs of \( A(\cdot)B \) in α-cut method will be as follows:
\[
\mu_{A()B}(z) = \begin{cases} 
0, & z < 6, z > 30 \\
7 + \sqrt{1 + 8z} & 6 \leq z \leq 15 \\
17 - \sqrt{49 + 8z} & 15 \leq z \leq 30 
\end{cases}
\]

In standard approximation method \(\alpha^2\) is approximated to equal \(\alpha\), so MF of \(A()B\) will be as follows:

\[
\mu_{A()B}(z) = \begin{cases} 
0, & z < 6, z > 30 \\
\frac{z - 2}{3}, & 6 \leq z \leq 15 \\
\frac{-z}{15} + 2, & 15 \leq z \leq 30 
\end{cases}
\]

Table 1. Comparison of \(\alpha\)-cut Method and Standard Approximation when both are PTFNs

| Value of \(\alpha\) | Actual Product | Standard Approximation | Error |
|---------------------|----------------|------------------------|-------|
|                     | Left | Right | Left | Right | Left | Right | Left | Right |
| 1                   | 15   | 15    | 15   | 15    | 0    | 0     | 0    | 0     |
| 0.9                 | 13.92| 16.32 | 14.1 | 16.5  | 0.18 | 0.18  | 0.18 | 0.18  |
| 0.8                 | 12.88| 17.68 | 13.2 | 18    | 0.32 | 0.32  | 0.32 | 0.32  |
| 0.7                 | 11.88| 19.08 | 12.3 | 19.5  | 0.42 | 0.42  | 0.42 | 0.42  |
| 0.6                 | 10.92| 20.52 | 11.4 | 21    | 0.48 | 0.48  | 0.48 | 0.48  |
| 0.5                 | 10   | 22    | 10.5 | 22.5  | 0.5  | 0.5   | 0.5  | 0.5   |
| 0.4                 | 9.12 | 23.52 | 9.6  | 24    | 0.48 | 0.48  | 0.48 | 0.48  |
| 0.3                 | 8.28 | 25.08 | 8.7  | 25.5  | 0.42 | 0.42  | 0.42 | 0.42  |
| 0.2                 | 7.48 | 26.68 | 7.8  | 27    | 0.32 | 0.32  | 0.32 | 0.32  |
| 0.1                 | 6.72 | 28.32 | 6.9  | 28.58 | 0.18 | 0.18  | 0.18 | 0.18  |
| 0                   | 6    | 30    | 6    | 30    | 0    | 0     | 0    | 0     |

Fig 2. Comparison when A and B are PTFNs

- Case study 2
  When A is PNTFN and B is PTFNs with triangular MFs \(\mu_A(x)\) and \(\mu_B(x)\) respectively as follows:
\[ \mu_A(x) = \begin{cases} 0, & x < -1, x > 3 \\ \frac{x + 1}{2} + rac{1}{2}, & -1 \leq x \leq 1 \\ \frac{-x}{2} + rac{3}{2}, & 1 \leq x \leq 3 \\ \end{cases} \]

\[ \mu_B(y) = \begin{cases} 0, & y < 1, y > 5 \\ \frac{y - 1}{2}, & 1 \leq y \leq 3 \\ \frac{5 - y}{2}, & 3 \leq y \leq 5 \\ \end{cases} \]

Then, \( A_\alpha = [2\alpha - 1, 3 - 2\alpha] \) and \( B_\alpha = [2\alpha + 1, -2\alpha + 5] \)

\( (A()B)_\alpha = \left[ [\frac{-4\alpha^2 + 12\alpha - 5, 4\alpha^2 - 16\alpha + 15}{4\alpha^2 - 1, 4\alpha^2 - 16\alpha + 15}], \forall \alpha \in [0, 0.5] \right] \)

MFs of \( A()B \) in \( \alpha \)-cut method will be as follows:

\[ \mu_{A()B}(z) = \begin{cases} 0, & z < -5, z > 15 \\ \frac{3 - \sqrt{4 - z}}{2}, & -5 \leq z \leq 0 \\ \frac{\sqrt{1 + z}}{2}, & 0 \leq z \leq 3 \\ \frac{4 - \sqrt{1 + z}}{2}, & 3 \leq z \leq 15 \\ \end{cases} \]

In standard approximation method \( \alpha^2 \) is approximated to equal \( \alpha \), so MF of \( A()B \) will be as follows:

\[ \mu_{A()B}(z) = \begin{cases} 0, & z < -5, z > 15 \\ \frac{z}{8} + \frac{5}{8}, & -5 \leq z \leq 3 \\ \frac{-z}{12} + \frac{5}{4}, & 3 \leq z \leq 15 \\ \end{cases} \]

**Table 2.** Comparison of \( \alpha \)-cut Method and Standard Approximation when A is PNTFN and B is PTFN

| Value of \( \alpha \) | Actual Product | Standard Approximation | Error |
|----------------------|----------------|------------------------|-------|
|                      | Left | Right | Left | Right | Left | Right |
| 1                    | 3    | 3     | 3    | 3     | 0    | 0     |
| 0.9                  | 2.24 | 3.84  | 2.2  | 4.2   | -0.04| 0.36  |
| 0.8                  | 1.56 | 4.76  | 1.4  | 5.4   | -0.16| 0.64  |
| 0.7                  | 0.96 | 5.76  | 0.6  | 6.6   | -0.36| 0.84  |
| 0.6                  | 0.44 | 6.84  | -0.2 | 7.8   | -0.46| 0.96  |
| 0.5                  | 0    | 8     | -1   | 9     | -1   | 1     |
| 0.4                  | -0.84| 9.24  | -1.8 | 10.2  | -0.96| 0.96  |
| 0.3                  | -1.76| 10.56 | -2.6 | 11.4  | -0.84| 0.84  |
| 0.2                  | -2.76| 11.96 | -3.4 | 12.6  | -0.64| 0.64  |
| 0.1                  | -3.84| 13.44 | -4.2 | 13.8  | -0.36| 0.36  |
| 0                    | -5   | 15    | -5   | 15    | 0    | 0     |
Case study 3

When A is NTFN and B is PTFN with triangular MFs $\mu_A(x)$ and $\mu_B(x)$ respectively as follows:

$$
\mu_A(x) = \begin{cases} 
0, & x < -5, x > -1 \\
\frac{x + 5}{2}, & -5 \leq x \leq -3 \\
\frac{-x - 1}{2}, & -3 \leq x \leq -1
\end{cases}
$$

$$
\mu_B(y) = \begin{cases} 
0, & y < 1, y > 3 \\
y - 1, & 1 \leq y \leq 2 \\
-y + 3, & 2 \leq y \leq 3
\end{cases}
$$

Then, $A_\alpha = [2\alpha - 5, -2\alpha - 1]$ and $B_\alpha = [\alpha + 1, -\alpha + 3]$

$$(A(\cdot)B)_\alpha = [-2\alpha^2 + 11\alpha - 15, -2\alpha^2 - 3\alpha - 1]$$

MFs of $A(\cdot)B$ in $\alpha$-cut method will be as follows:

$$
\mu_{A(\cdot)B}(z) = \begin{cases} 
0, & z < -15, z > -1 \\
\frac{11 - \sqrt{1 - 8z}}{4}, & -15 \leq z \leq -6 \\
\frac{-3 + \sqrt{1 - 8z}}{4}, & -6 \leq z \leq -1
\end{cases}
$$

In standard approximation method $\alpha^2$ is approximated to equal $\alpha$, so MF of $A(\cdot)B$ will be as follows:

$$
\mu_{A(\cdot)B}(z) = \begin{cases} 
0, & z < -15, z > -1 \\
\frac{z + 5}{3}, & -15 \leq z \leq -6 \\
\frac{-z - 1}{5}, & -6 \leq z \leq -1
\end{cases}
$$
Table 3. Comparison of α-cut Method and Standard Approximation when A is NTFN and B is PTFN

| Value of α | Actual Product | Standard Approximation | Error |
|-----------|----------------|------------------------|-------|
|           | Left | Right | Left | Right | Left | Right |        |
| 1         | -6   | -6    | -6   | -6    | 0    | 0     |        |
| 0.9       | -6.72| -5.32 | -6.9 | -5.5  | -0.18| -0.18 |        |
| 0.8       | -7.48| -4.68 | -7.8 | -5    | -0.32| -0.32 |        |
| 0.7       | -8.28| -4.08 | -8.7 | -4.5  | -0.42| -0.42 |        |
| 0.6       | -9.12| -3.52 | -9.6 | -4    | -0.48| -0.48 |        |
| 0.5       | -10  | -3    | -10.5| -3.5  | -0.5 | -0.5  |        |
| 0.4       | -10.92| -2.52 | -11.4| -3    | -0.48| -0.48 |        |
| 0.3       | -11.88| -2.08 | -12.3| -2.5  | -0.42| -0.42 |        |
| 0.2       | -12.88| -1.68 | -13.2| -2    | -0.32| -0.32 |        |
| 0.1       | -13.92| -1.32 | -14.1| -1.5  | -0.18| -0.18 |        |
| 0         | -15  | -1    | -15  | -1    | 0    | 0     |        |

Fig 4. Comparison when A is NTFN and B is positive PTFN

- Case study 4

When A and B are NTFNs with triangular MFs \( \mu_A(x) \) and \( \mu_B(x) \) respectively as follows:

\[
\mu_A(x) = \begin{cases} 
0, & x < -6, x > -2 \\
\frac{x}{2} + 3, & -6 \leq x \leq -4 \\
\frac{-x}{2} - 1, & -4 \leq x \leq -2
\end{cases}
\]

\[
\mu_B(y) = \begin{cases} 
0, & y < -5, y > -1 \\
\frac{y}{2} + \frac{5}{2}, & -5 \leq y \leq -3 \\
\frac{-y}{2} - \frac{1}{2}, & -3 \leq y \leq -1
\end{cases}
\]

Then, \( A_\alpha = [2\alpha - 6, -2\alpha - 2] \) and \( B_\alpha = [2\alpha - 5, -2\alpha - 1] \)
\[(A\cdot B)_\alpha = [4\alpha^2 + 6\alpha + 2, 4\alpha^2 - 22\alpha + 30]\]

MFs of \(A\cdot B\) in \(\alpha\)-cut method will be as follows:

\[
\mu_{A\cdot B}(z) = \begin{cases} 
0, & z < 2, z > 30 \\
-3 + \sqrt{1 + 4z} \over 4, & 2 \leq z \leq 12 \\
11 - \sqrt{1 + 4z} \over 4, & 12 \leq z \leq 30 
\end{cases}
\]

In standard approximation method \(\alpha^2\) is approximated to equal \(\alpha\), so MF of \(A\cdot B\) will be as follows:

\[
\mu_{A\cdot B}(z) = \begin{cases} 
0, & z < 2, z > 30 \\
-3 + \sqrt{1 + 4z} \over 4, & 2 \leq z \leq 12 \\
11 - \sqrt{1 + 4z} \over 4, & 12 \leq z \leq 30 
\end{cases}
\]

**Table 4.** Comparison of \(\alpha\)-cut Method and Standard Approximation when both are NTFN

| Value of \(\alpha\) | Actual Product | Standard Approximation | Error |
|---------------------|----------------|------------------------|-------|
|                     | Left | Right | Left | Right | Left | Right |
| 1                   | 12   | 12    | 12   | 12    | 0    | 0     |
| 0.9                 | 10.64| 13.44 | 11   | 13.8  | 0.36 | 0.36  |
| 0.8                 | 9.36 | 14.96 | 10   | 15.6  | 0.64 | 0.64  |
| 0.7                 | 8.16 | 16.56 | 9    | 17.4  | 0.84 | 0.84  |
| 0.6                 | 7.04 | 18.24 | 8    | 19.2  | 0.96 | 0.96  |
| 0.5                 | 6    | 20    | 7    | 21    | 1    | 1     |
| 0.4                 | 5.04 | 21.84 | 6    | 22.8  | 0.96 | 0.96  |
| 0.3                 | 4.16 | 23.76 | 5    | 24.6  | 0.84 | 0.84  |
| 0.2                 | 3.36 | 25.76 | 4    | 26.4  | 0.64 | 0.64  |
| 0.1                 | 2.64 | 27.84 | 3    | 28.2  | 0.36 | 0.36  |
| 0                   | 2    | 30    | 2    | 30    | 0    | 0     |

![Fig 5. Comparison when A and B are NTFN](image_url)
6. Error analysis for α-cut Method

The error is the difference, at a given α-level, between the approximated MF and exact results in the expressions (4), (5) and (6). Each TFN can be divided into left (L) and right (R) segments according to the LR parameter representation. The actual product of (4) and (5) will have the value x at a given α defined as $T_L$ for the left segment, and $T_R$ for the right segment. The standard approximation (6) will have a value $x$, at a given α defined as left segment $P_L$ and right segment $P_R$.

This follows us to separately analyze the left and right portions of the membership curve. Then, the left and right segment error will be as follows:

$$E_L = P_L - T_L$$
$$E_R = P_R - T_R$$

Graphically this is the horizontal distance between the two curves as shown in Figures (2, 3, 4, and 5) and numerical error is shown in tables (1, 2, 3 and 4).

7. Conclusion

A comparison between actual method and approximation method of the fuzzy multiplication operation on triangular fuzzy number has been presented and the error has been analyzed. Since fuzzy arithmetic operations have an important role in different fields of sciences and engineering, so any one can get idea about the result between actual and approximation methods. The graphs of these results have been presented.

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