Stability of steady solutions to reaction-hyperbolic systems for axonal transport

Hao Yan and Wen-an Yong

Abstract. This paper is concerned with the stability of steady solutions to initial-boundary-value problems of reaction-hyperbolic systems for axonal transport. Under proper structural assumptions, we clarify the relaxation structure of the reaction-hyperbolic systems and show the time-asymptotic stability of steady solutions or relaxation boundary-layers.

Keywords: axonal transport; reaction-hyperbolic systems; relaxation structure; boundary-layers; time-asymptotic stability.

1. Introduction

This work is concerned with the following reaction-hyperbolic system

\[ \partial_t u_i + \lambda_i \partial_x u_i = \sum_{j=1}^{r} k_{ij} u_j, \quad i = 1, 2, \cdots, r \]

on the quarter-plane \( x, t \geq 0 \). Here \( u_i = u_i(x, t)(i = 1, 2, \cdots, r) \) are unknown functions, \( \lambda_i(i = 1, 2, \cdots, r) \) and \( k_{ij}(i, j = 1, 2, \cdots, r) \) are given constants. It is well believed that such models describe the axonal transport in neuroscience.

The axonal transport is important for the maintenance and functions of nerve cells or neurons. A neuron consists of three parts mainly: cell body, dendrites and a single axon. The axon is a long and thin pipe whose length can exceed 10,000 times its diameter. It is this axon that distinguishes neurons from cells in other organs or tissues. The axon is responsible for signal transmission in the nervous system. Its cytoplasm does not contain rough endoplasmic reticulum and therefore its proteins can only be transported from the cell body, where all proteins are synthesized. In addition, the axonal transport is also used by the neuroscientists to trace the connections in the brain.

The transport proceeds as follows. Proteins are stored in vesicles as cargos. The vesicles are attached to kinesin (anterograde motors) or dynein (retrograde motors) proteins. These motor proteins drive the vesicles to walk along the cytoskeletal microtubules as track. Here the kinesin proteins move the vesicles from the cell body to synapse (anterograde transport), while the dynein proteins move the vesicles in the opposite direction (retrograde transport). During the transport, many biochemical processes are possible. For example, the cargos can leave its track, can switch its motor proteins from kinesin to dynein or vice verse, and can move back onto the track. Thus, we can divide the cargos into a number of subpopulations, such as free vesicles, vesicle-kinesin compounds off track, moving vesicle-dynein compounds on track, etc.

As the axon is long and thin, it is reasonable to assume the transport only along the longitudinal direction of the axon. Denote by \( x > 0 \) the distance down the axon from the...
cell body which is located at $x = 0$. Let $u_i = u_i(x, t)$ be the concentration at space-time $(x, t)$ of the $i$-th subpopulations. According to Reed and Blum [10], the mathematical model for axonal transport is partial differential equations of the form

\begin{equation}
\partial_t u_i + \lambda_i \partial_x u_i = F_i(u_1, u_2, \cdots, u_r), \quad i = 1, 2, \cdots, r
\end{equation}

defined on the quarter-plane $x,t \geq 0$. Here the term $\lambda_i \partial_x u_i$ accounts for the transport of the $i$-th subpopulation with constant velocity $\lambda_i$, and $F_i(u_1, u_2, \cdots, u_r)$ describes the biochemical processes of the constituents. It is well recognized that the biochemical processes are much faster than the transport in biosystems. Thus, it is more proper to rewrite the general equation (1.2) as

\begin{equation}
\partial_t u_i + \lambda_i \partial_x u_i = \frac{1}{\epsilon} F_i(u_1, u_2, \cdots, u_r), \quad i = 1, 2, \cdots, r.
\end{equation}

Here the small parameter $\epsilon > 0$ characterizes the fact that the biochemical processes are much faster than the transport.

On the other hand, in a typical experiment for investigating axonal transport, a large quantity of radiolabeled amino acids are injected into a nerve ganglion. Thus, the proteins are synthesized with the injected amino acids in the cell body and transported along the axon, and finally the radioactivity appears in the axon in a few hours. The wave of radiolabeled proteins travels at approximately constant velocity. The shapes and speeds of the profiles depend on nerves, animals, and proteins being studied. Usually the amino acids are continuously available in the experiment, and the sharp approximately parallel the wave fronts which suggest traveling waves. There are at least two such systems which carry membrane-bounded organelles and cytoplasm separately. For more details about the experiments for axonal transport, the reader can refer to [1, 7, 9].

In order to explain the approximate traveling waves observed in experiments, the authors of [3, 5, 6, 11] studied the linear case (1.1), where $F_i(u_1, u_2, \cdots, u_r)$ is linear with respect to the $u_j$’s. A concrete example can be found in [3] on neurofilament transport. Because parabolic equations typically admit traveling wave solutions, these authors all related the system (1.3) to a parabolic equation— the diffusive limit of (1.3). Especially, in [6] Friedman and Hu used parabolic-type estimates to analyse the diffusive limit of the linear systems. However, it is known [18, 4] that, unlike the conservation laws, hyperbolic systems with relaxation also allow non-trivial traveling wave solutions.

In this project, we intend to explain the experimental observation by directly studying steady solutions to the initial-boundary-value problems of the above hyperbolic-reaction systems. Steady solutions very much look like the traveling waves but they should called (relaxation) boundary-layers due to the presence of the boundary $x = 0$. Because they can be observed experimentally, we believe that the steady solutions are stable. Therefore, the goal of this paper is to investigate the time-asymptotic stability of the steady solutions under proper structural assumptions on the system.

The standard structural assumptions on the axonal transport models (1.1) read as

\begin{enumerate}
\item[(H1).] $k_{ij} \geq 0$ if $i \neq j$;
\item[(H2).] (conservation of mass) $\sum_{i=1}^r k_{ij} = 0$ for $j = 1, 2, \cdots, r$;
\item[(H3).] (irreducibility) for any $i_0 \neq i_1$, there is a sequence of indices $j_1, j_2, \cdots, j_l$
\end{enumerate}
such that $i_0 = j_1, i_1 = j_l$ and $k_{jn}j_{n+1} > 0$ for $1 \leq m \leq l - 1$;

(H4). there exist $i$ and $j$ such that $\lambda_i \neq \lambda_j$.

These assumptions are taken from [6]. See also [2, 3, 5, 10, 11]. Remark that (H4) ensures the hyperbolic-reaction system (1.1) won’t degenerate into a system of ordinary differential equations.

As a first step of this project, we consider the linear system (1.1) and additionally make the following technical assumption in this paper that

(H5). $\lambda_i > 0, \quad i = 1, 2, \ldots , r$.

In the future, we will remove or relax this (H5).

With the above assumptions, we formulate the problem as follows. Write

$$ U = (u_1, u_2, \ldots , u_r)^T, \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots , \lambda_r), \quad K = [k_{ij}]_{n \times n}. $$

Then (1.1) can be written as

$$ U_t + \Lambda U_x = KU. $$

The initial and boundary data are

$$ U|_{t=0} = U_0(x), \quad U|_{x=0} = U_0(0). $$

Here the second equality implies the consistency condition which is necessary for the above initial-boundary-value problems to have a continuous solution. Moreover, we assume that

$$ \Lambda U_{0x}(0) = KU_0(0), $$

which is necessary for the above problems to have a continuously differentiable solution. Our main results are the time-asymptotic stability of steady solutions to (1.4) together with (1.5).

For references on nonlinear systems for axonal transport, we mention that Carr in [2] discussed the existence of global classical solutions to a class of nonlinear models. Recently, in [17] we clarified the relaxation structure of nonlinear models in [2] and verified the relaxation limit of BV-solutions to the Cauchy problems.

The rest of this paper is organized as follows. In Section 2 we discuss the relaxation structure of the hyperbolic-reaction system (1.1). Section 3 is devoted to the existence and stability of the steady solutions for (1.4).

### 2. Relaxation Structure

In this section, we show that the hyperbolic-reaction system (1.1) possesses the relaxation structure formulated in [15], provided that the assumptions (H1)–(H3) hold. We start with the following elementary fact as (a) of Lemma 1 proved in [11].

**Lemma 2.1.** Under the assumptions (H1)–(H3), the kernel of the matrix $K$ is one-dimensional and is spanned by a vector with strictly positive entries.

On the basis of this fact, we can show

**Lemma 2.2.** 0 is a single eigenvalue of $K$. 

Proof. Set 
\[ L_1 = \begin{pmatrix} 1 & e_1 \\ 0 & I_{r-1} \end{pmatrix}, \]
where \( e_1 = (1, 1, \ldots, 1) \) is a vector in \( \mathbb{R}^{r-1} \), \( I_{r-1} \) is the unit matrix of dimension \((r-1)\), and the superscript \( T \) denotes the transpose of vectors or matrices. Then the inverse of \( L_1 \) is 
\[ L_1^{-1} = \begin{pmatrix} 1 & -e_1 \\ 0 & I_{r-1} \end{pmatrix}. \]
Using the partition of \( L_1 \), we rewrite \( K \) as 
\[ K = \begin{pmatrix} k_{11} & \alpha \\ \beta & K_1 \end{pmatrix}, \]
where \( \alpha = (k_{12}, k_{13}, \ldots, k_{1r}) \), \( \beta = (k_{21}, k_{31}, \ldots, k_{r1})^T \), and \( K_1 \) is the \((r-1) \times (r-1)\)-matrix. By a direct calculation using the assumption (H2), we obtain 
\[ L_1 KL_1^{-1} = \begin{pmatrix} 1 & e_1 \\ 0 & I_{r-1} \end{pmatrix} \begin{pmatrix} k_{11} & \alpha \\ \beta & K_1 \end{pmatrix} \begin{pmatrix} 1 & -e_1 \\ 0 & I_{r-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \beta & K_2 \end{pmatrix} \]
with \( K_2 = K_1 - \beta e_1 \).
Thus, it reduces to show that the sub-matrix \( K_2 \) is invertible. Otherwise, there is a vector \( \eta = (\eta_1, \eta_2, \ldots, \eta_{r-1})^T \) such that \( K_2 \eta = 0 \). Take 
\[ \xi = L_1^{-1} \begin{pmatrix} 0 \\ \eta \end{pmatrix}. \]
Then it holds that 
\[ K \xi = K L_1^{-1} \begin{pmatrix} 0 \\ \eta \end{pmatrix} = L_1^{-1} \begin{pmatrix} 0 & 0 \\ \beta & K_2 \end{pmatrix} \begin{pmatrix} 0 \\ \eta \end{pmatrix} = L_1^{-1} \begin{pmatrix} 0 \\ K_2 \eta \end{pmatrix} = 0. \]
This shows that \( \xi \) is a vector in the kernel of \( K \). On the other hand, we have 
\[ \xi = L_1^{-1} \begin{pmatrix} 0 \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & -e_1 \\ 0 & I_{r-1} \end{pmatrix} \begin{pmatrix} 0 \\ \eta_1 \\ \eta_2 \\ \ldots \\ \eta_{r-1} \end{pmatrix} = \begin{pmatrix} -\eta_1 - \eta_2 - \cdots - \eta_{r-1} \\ \eta_1 \\ \eta_2 \\ \ldots \\ \eta_{r-1} \end{pmatrix}. \]
Obviously, such a \( \xi \) can not be in the kernel of \( K \) spanned by a vector with positive entries. This contradicts Lemma 2.1. Therefore, \( K_2 \) is invertible and 0 is a single eigenvalue of \( K \). This completes the proof. \( \square \)

Remark 2.1. Following the above proof, we have 
\[ \left( \begin{pmatrix} 1 & 0 \\ K_2^{-1} \beta & I_{r-1} \end{pmatrix} \right) L_1 K = \left( \begin{pmatrix} 0 & 0 \\ 0 & K_2 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ K_2^{-1} \beta & I_{r-1} \end{pmatrix} \right) L_1. \]
On the other hand, it is not difficult to deduce from the Gershgorin circle theorem that non-zero eigenvalues of \( K \) have negative real parts. Therefore, \( K_2 \) is stable and (i) of the first stability condition in [15] is verified.

Furthermore, we have
Lemma 2.3. Under the assumptions (H1)–(H3), there exist a positive definite diagonal matrix $A_0$, an orthogonal matrix $P$, and a positive definite diagonal matrix $S$ such that

\[
A_0 \Lambda = \Lambda A_0,
\]

\[
A_0 K + K^T A_0 = -P^T \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} P.
\]

Furthermore, the first column of $P^T$ is in the kernel of $K$.

Proof. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_r)^T$ is an eigenvector of the matrix $K$, associated with the eigenvalue 0. By Lemma 2.1, we may assume that $\xi_i > 0$ for each $i$. Define

\[
D = \begin{pmatrix} \xi_1 & \xi_2 & \cdots & \xi_r \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.
\]

It is obvious that the matrix $KD$ satisfies the assumptions (H1)–(H3) as well and

\[
KD \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = K \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_r \end{pmatrix} = 0.
\]

Namely, the sum of each row of $KD$ is also 0. Moreover, it is easy to see that the symmetric matrix $KD + DK^T$ fulfils the (H1)–(H3), too.

Take $A_0 = D^{-1}$ and it is clear that $A_0$ is positive definite diagonal and $A_0 \Lambda = \Lambda A_0$. Moreover, the symmetric matrix

\[
A_0 K + K^T A_0 = D^{-1}(KD + DK^T)D^{-1}
\]

also fulfils the assumptions (H1)–(H3). According to Lemmas 2.1 and 2.2, 0 is a single eigenvalue of $A_0 K + K^T A_0$. Moreover, from the Gershgorin circle theorem it is not difficult to deduce that non-zero eigenvalues of $A_0 K + K^T A_0$ are negative. Thus, there exist an orthogonal matrix $P$ and a positive definite diagonal matrix $S$ such that

\[
A_0 K + K^T A_0 = -P^T \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} P.
\]

Furthermore, since the $r$-vector $(\xi_1, \xi_2, \ldots, \xi_r)^T$ is in the kernel of $A_0 K + K^T A_0$, it is easy to see that the last $(r-1)$ components of the column vector $P(\xi_1, \xi_2, \ldots, \xi_r)^T$ are zeros. Namely, the vector $(\xi_1, \xi_2, \ldots, \xi_r)$ is orthogonal to the last $(r-1)$ rows of the orthogonal matrix $P$ and thereby parallels to the first row of $P$. Hence, the first column of $P^T$ is in the kernel of $K$ and the proof is complete. \qed

Remark 2.2. Lemma 2.3 together with Remark 2.1 shows that the reaction-hyperbolic systems satisfying assumptions (H1)–(H3) fulfil the first stability condition in [14, 15]. However, they do not satisfy the second stability condition in [14, 15] in general, unless further assumptions are posed. An important case is that $KD$ is symmetric, which implies the second stability condition due to Theorem 5.3 in [15]. It is clear that $KD$ is symmetric, provided that the principle of detailed balance holds (see, e.g., [13]). For the neurofilament
model in [3], $K$ is tri-diagonal and one can easily see that $KD$ is symmetric. However, the assumptions (H1)–(H3) do not imply the symmetry of $KD$. In fact, the $4 \times 4$-matrix

$$K = \begin{pmatrix} -4 & 1 & 1 & 0 \\ 2 & -3 & 0 & 1 \\ 2 & 1 & -2 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix},$$

satisfies (H1)–(H3) and $(\frac{1}{2}, 1, 1, 2)^T$ is an eigenvector associated with 0. Set $D = \text{diag}(\frac{1}{2}, 1, 1, 2)$. By a direct calculation, we have

$$KD = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -3 & 0 & 2 \\ 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & -2 \end{pmatrix},$$

which is not symmetric.

But we have

**Proposition 2.4.** For $r \leq 3$, the matrix $KD$ is symmetric.

*Proof.* For $r = 2$, let $K$ be

$$K = \begin{pmatrix} a & -b \\ -a & b \end{pmatrix}$$

with $a, b > 0$. We take $\xi = (b, a)^T$ and $D = \text{diag}(b, a)$. Then

$$KD = \begin{pmatrix} ab & -ab \\ -ab & ab \end{pmatrix}$$

is symmetric.

For $r = 3$, set

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}$$

and $D = \text{diag}(\xi_1, \xi_2, \xi_3)$. Moreover, set

$$a = \xi_2k_{12} - \xi_1k_{21}, b = \xi_3k_{13} - \xi_1k_{31}, c = \xi_3k_{23} - \xi_2k_{32}.$$

Since

$$K \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1k_{11} + \xi_2k_{12} + \xi_3k_{13} \\ \xi_1k_{21} + \xi_2k_{22} + \xi_3k_{23} \\ \xi_1k_{31} + \xi_2k_{32} + \xi_3k_{33} \end{pmatrix} = 0,$$

we see immediately from the assumption (H2) that

$$\xi_1 + \xi_2 = \xi_2 + \xi_3 = \xi_3 + \xi_1 = 0,$$

and therefore

$$a = b = c = 0.$$
Hence the matrix
\[
KD = \begin{pmatrix}
\xi_1 k_{11} & \xi_2 k_{12} & \xi_3 k_{13} \\
\xi_1 k_{21} & \xi_2 k_{22} & \xi_3 k_{23} \\
\xi_1 k_{31} & \xi_2 k_{32} & \xi_3 k_{33}
\end{pmatrix}
\]
is symmetric and the proof is completed. □

Finally, we conclude this section with the following Lemma.

**Lemma 2.5.** Under the assumptions (H1)–(H4), there is a skew symmetric matrix \( H \) and a positive constant \( c \) such that
\[
H \Lambda - \Lambda H \geq cI - P^T \begin{pmatrix} 0 & 0 \\ 0 & I_{r-1} \end{pmatrix} P.
\]

**Proof.** Let \( \xi = (\xi_1, \xi_2, \cdots, \xi_r)^T \) be in the kernel of the matrix \( K \) and \( A_0 \) be the positive definite diagonal matrix in Lemma 2.2. Consider the system of equations
\[
A_0 W_t + A_0 AW_x - (A_0 K + K^T A_0) W = 0.
\]
From the proof of Lemma 2.3, we know that the kernel of the symmetric matrix \( A_0 K + K^T A_0 \) is equal to that of \( K \). By Lemma 2.1, we may assume that \( \xi_i > 0 \) for each \( i \).

Thanks to the assumption (H4), one cannot find any number \( \lambda \) such that
\[
\lambda_i \xi_i = \lambda \xi_i
\]
for all \( i \). Namely, \( \xi = (\xi_1, \xi_2, \cdots, \xi_r)^T \) is not an eigenvector of the coefficient matrix \( \Lambda \).

According to Shizuta and Kawashima (see Theorem 1.1 in [12] and also Theorem 2.3 in [16]), there exists a positive constant \( c \) and a skew symmetric matrix \( H \) such that
\[
H \Lambda - \Lambda H \geq cI + A_0 K + K^T A_0.
\]
Hence the lemma is proved by combining this with Lemma 2.3. □

### 3. Existence and Stability of steady solutions

In this section, we discuss the existence and stability of steady solutions of (1.4). The equations for steady solutions \( B = B(x) \) are
\[
\Lambda B_x = KB.
\]
Since the matrix \( \Lambda \) is invertible, \( W = \Lambda B \) satisfies
\[
W_x = (K \Lambda^{-1}) W.
\]

From the assumption (H5) and the definition of \( \Lambda \), we see that the matrix \( K \Lambda^{-1} \) satisfies the assumptions (H1)–(H3). Thus, we can use Lemma 2.2 and the Gershgorin circle theorem to show that \( K \Lambda^{-1} \) has \( (r - 1) \) stable eigenvalues and a zero-eigenvalue.

Given any boundary data \( U_0(0) \), the solution for (3.2) is
\[
W(x) = \exp(K \Lambda^{-1} x) \Lambda U_0(0),
\]
and therefore,
\[
B(x) = \Lambda^{-1} \exp(K \Lambda^{-1} x) \Lambda U_0(0).
\]
Since $K\Lambda^{-1}$ has $(r-1)$ stable eigenvalues and a zero-eigenvalue, the matrix $\exp(K\Lambda^{-1}x)$ is bounded with respect to $x \geq 0$. In conclusion, given any boundary data $B(0)$, the equations (3.1) for steady solutions have a unique bounded solution $B = B(x)$.

Now we turn to discuss the stability. Set $\Phi(x,t) = U(x,t) - B(x)$. From (1.4) and (3.1), we see that $\Phi$ satisfies

$$\Phi_t + \Lambda \Phi_x = K\Phi.$$  

(3.3)

The initial and boundary data are

$$\begin{align*}
\Phi(x,0) &= U_0(x) - B(x), \\
\Phi(0,t) &= 0.
\end{align*}$$

(3.4)

From (1.6) and (3.1), it follows that

$$\Lambda(U_0x(0) - B_x(0)) = K(U_0(0) - B(0)).$$

Since $U_0(0) = B(0)$ and $\Lambda$ is invertible, the consistency of the initial and boundary data becomes

$$\Phi_x(0,0) = \Phi(0,0) = 0.$$  

(3.5)

Thus, our task is reduced to analyzing time-asymptotic behaviors of the solution $\Phi$ to the IBVP (3.3) together with (3.4).

We start with the following local existence result. This result can be showed by slightly modifying the proof given in Section 5 of [8] and we omit it here.

**Lemma 3.1.** Suppose $U_0(x) - B(x) \in H^2$ and (3.5) holds. Then there exist a positive constant $T_*$ such that (3.3) together with (3.4) has a unique solution $\Phi(x,t) \in C(0, T_*; H^2)$. Moreover, the solution satisfies the following estimate

$$\sup_{0 \leq t \leq T_*} ||\Phi(\cdot, t)||_{H^2} \leq 2||U_0(x) - B(x)||_{H^2}.$$  

Here $T_*$ depends only on the range of $B(x)$ and any upper bound of $||U_0(x) - B(x)||_{H^2}$.

**Theorem 3.2.** Under the assumptions (H1)-(H5), if $U_0(x) - B(x) \in H^2$ and (1.6) holds, then (1.4) together with (1.5) has a unique global solution $U \in C(0, \infty; H^2)$ satisfying

$$\lim_{t \to \infty} \sup_{x \in \mathbb{R}_+} |U(x,t) - B(x)| = 0.$$  

**Proof.** Let $A_0$ be the matrix given in Lemma 2.3. We multiply (3.3) with $\Phi^T A_0$ to get

$$\Phi^T A_0 \Phi_t + \Phi^T A_0 \Lambda \Phi_x = \Phi^T A_0 K \Phi.$$  

Namely,

$$\Phi^T A_0 \Phi_t + \Phi^T A_0 \Lambda \Phi_x = \frac{1}{2} \Phi^T (A_0 K + K^T A_0) \Phi \leq -\frac{c}{2} \Phi^T P^T \begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix} P \Phi,$$

where we have used Lemma 2.3. Setting

$$V = P \Phi = (V_1, V_2, \cdots, V_r)^T$$
and integrating the above inequality with respect to \((x, t) \in [0, \infty) \times [0, t]\), we get

\[
\int_0^{+\infty} \frac{1}{2} \Phi^T A_0 \Phi(x, t) dx - \int_0^{+\infty} \frac{1}{2} \Phi^T A_0 \Phi(x, 0) dx \\
- \int_0^t \frac{1}{2} \Phi^T A_0 \Lambda \Phi(0, \tau) d\tau + \frac{c}{2} \int_0^t \int_0^{+\infty} \sum_{i=2}^r V_i^2(x, \tau) dx d\tau \leq 0.
\]

Since \(A_0\) is positive definite and \(\Phi(0, t) = 0\), there is a generic constant \(C\) such that

\[
(3.6) \quad \|\Phi(\cdot, t)\|_{L^2}^2 + \sum_{i=2}^r \int_0^t \|V_i(\cdot, \tau)\|_{L^2}^2 d\tau \leq C\|\Phi(\cdot, 0)\|_{L^2}^2.
\]

Next we estimate the derivatives. Differentiating the equation (3.3) with respect to \(x\) gives

\[
\Phi_{xt} + \Lambda \Phi_{xx} = K \Phi_x,
\]

\[
(3.7) \quad \Phi_{xxx} + \Lambda \Phi_{xxxx} = K \Phi_{xx}
\]

From the equations (3.3) and (3.7) with the data in (3.4), the boundary data for the derivative are

\[
\Phi_x(0, t) = \Lambda^{-1}(K \Phi(0, t) - \Phi_t(0, t)) = 0,
\]

\[
\Phi_{xx}(0, t) = \Lambda^{-1}(K \Phi_x(0, t) - \Phi_{xt}(0, t)) = 0.
\]

Thus, we use the same technique shown above to estimate \(||V_x(\cdot, t)||\) and \(||V_{xx}(\cdot, t)||\),

\[
\|\Phi_x(\cdot, t)\|_{L^2}^2 + \sum_{i=2}^r \int_0^t \|V_{ix}(\cdot, \tau)\|_{L^2}^2 d\tau \leq C\|\Phi_x(\cdot, 0)\|_{L^2}^2,
\]

\[
\|\Phi_{xx}(\cdot, t)\|_{L^2}^2 + \sum_{i=2}^r \int_0^t \|V_{ixx}(\cdot, \tau)\|_{L^2}^2 d\tau \leq C\|\Phi_{xx}(\cdot, 0)\|_{L^2}^2.
\]

Summing these and the inequality in (3.6), we get

\[
(3.8) \quad \|\Phi(\cdot, t)\|_{H^2}^2 + \sum_{i=2}^r \int_0^t \|V_i(\cdot, \tau)\|_{H^2}^2 d\tau \leq C\|\Phi(\cdot, 0)\|_{H^2}^2.
\]

On the other hand, we multiply the equation (3.3) with \(\Phi^T_x H\) to get

\[
(3.9) \quad \Phi^T_x H \Phi_t + \Phi^T_x H \Lambda \Phi_x = \Phi^T_x H K \Phi.
\]

Here \(H\) is the skew symmetric matrix in Lemma 2.5. Since

\[
\Phi^T_x H \Phi_t = \frac{1}{2}(\Phi^T_x H H \Phi)_t - \frac{1}{2}(\Phi^T_t H H \Phi)_x,
\]

we integrate (3.9) with respect to \((x, t)\) and use Lemma 2.5 to obtain

\[
c \int_0^t \int_0^{+\infty} \Phi^2_{xx}(x, \tau) dx d\tau \leq \int_0^t \int_0^{+\infty} \sum_{i=2}^r V_i^2(x, \tau) dx d\tau \\
+ 2 \int_0^t \int_0^{+\infty} \Phi^T_x H K \Phi(x, \tau) dx d\tau - \int_0^{+\infty} \Phi^T_x H \Phi(x, 0) dx.
\]
Recall that $\Phi = P^T V$ thanks to the orthogonality of $P$ and the first column of $P^T$ is in the kernel of $K$. Thus, $K\Phi = K P^T V$ is independent of the first component of $V$. Therefore the last inequality becomes

$$c \int_0^t \int_0^{+\infty} \Phi^2_x(x, \tau) dx d\tau \leq \int_0^t \int_0^{+\infty} \sum_{i=2}^r V^2_i(x, \tau) dx d\tau$$

$$+ 2 \int_0^t \int_0^{+\infty} \Phi^T_x K P^T V(x, \tau) dx d\tau$$

$$- \int_0^{+\infty} \Phi^T_x H \Phi(x, t) dx + \int_0^{+\infty} \Phi^T_x H \Phi(x, 0) dx$$

$$\leq \frac{c}{2} \int_0^t \|\Phi_x(\cdot, \tau)\|^2_2 d\tau + C \int_0^t \sum_{i=2}^r \|V_i(\cdot, \tau)\|^2_{H^2} d\tau$$

$$+ C\|\Phi(\cdot, t)\|^2_{H^2} + C\|\Phi(\cdot, 0)\|^2_{H^2}.$$  

Namely,

$$\int_0^t \int_0^{+\infty} \Phi^2_x(x, \tau) dx d\tau \leq C \int_0^t \sum_{i=2}^r \|V_i(\cdot, \tau)\|^2_{H^2} d\tau + C\|\Phi(\cdot, t)\|^2_{H^2} + C\|\Phi(\cdot, 0)\|^2_{H^2}.$$  

Combining this with (3.8) gives

$$\begin{equation}
(3.10) \|\Phi(\cdot, t)\|^2_{H^2} + \int_0^t \sum_{i=2}^r \|V_i(\cdot, \tau)\|^2_{H^2} d\tau + \int_0^t \|\Phi_x(\cdot, \tau)\|^2_{L^2} d\tau \leq C\|\Phi(\cdot, 0)\|^2_{H^2} 
\end{equation}$$

for all $t \geq 0$.

Furthermore, for any $t_1, t_2 > 0$, we have

$$\|\Phi_x(\cdot, t_2)\|_{L^2} - \|\Phi_x(\cdot, t_1)\|_{L^2} \leq \|\Phi_x(\cdot, t_2) - \Phi_x(\cdot, t_1)\|_{L^2}$$

$$= \| \int_{t_1}^{t_2} \Lambda^T \Phi_x(\cdot, \tau) d\tau - \int_{t_1}^{t_2} K^T \Phi_x(\cdot, \tau) d\tau \|_{L^2}$$

$$\leq C|t_2 - t_1| \max_{\tau} \|\Phi(\cdot, \tau)\|_{H^2} \leq C|t_2 - t_1|,$$

where we have used (3.7) in the second step. Thus, we can deduce that $\|\Phi_x(\cdot, t)\|_{L^2} \to 0$ as $t \to +\infty$. Moreover, from (3.10), we see that $\|\Phi\|_{L^2}$ is bounded. Following from the celebrated Gagliardo-Nirenberg inequality

$$|\Phi|_\infty \leq \sqrt{2} \|\Phi\|_{L^2}^{\frac{1}{2}} \|\Phi_x\|_{L^2}^{\frac{1}{2}},$$

we derive $|\Phi(\cdot, t)|_\infty \to 0$ as $t \to +\infty$. Recall that $\Phi(x, t) = U(x, t) - B(x)$, and we complete the proof. \hfill \Box

**References**

[1] P. Cancalon, *Influence of temperature on slow flow in populations of regenerating axons with different elongation velocities*, Developmental Brain Research, 9 (1983), pp. 279-289.

[2] D. D. Carr, *Global existence of solutions to reaction-hyperbolic systems in one space dimension*, SIAM. J. Math. Anal. 26(2) (1995), 399–414.

[3] G. Craciun & A. Brown & A. Friedman, *A dynamical system model of neurofilament transport in axons*, J. Theoretical Biology 237 (2005), 316–322.
Stability of steady solutions for Reaction-Hyperbolic Systems for Axonal Transport

[4] A. Dressel & W.-A. Yong, Existence of Traveling-Wave Solutions for Hyperbolic Systems of Balance Laws, Arch. Rational Mech. Anal 182, 49–75.

[5] A. Friedman & G. Craciun, Approximate travelling waves in linear reaction-hyperbolic equations, SIAM. J. Math. Anal. 38(3), (2006), 741–758.

[6] A. Friedman & B. Hu, Uniform convergence for approximate travelling waves in linear reaction-hyperbolic systems, Indiana Univ. Math. J. 56 (5) (2007), 2133–2158.

[7] G.W. Gross & L.M. Beidler, A quantitative analysis of isotope concentration profiles and rapid transport velocities in the C-fibers of the garfish olfactory nerve, J. Neurobiol., 6 (1975), pp. 213-232.

[8] Hailiang Liu & W.-A. Yong, Time-Asymptotic Stability of Boundary-Layers for a Hyperbolic Relaxation System, Comm. PDE. 26 (7&8) (2001), 1323-1343.

[9] S. Ochs, Rate of fast axoplasmic transport in mammalian nerve fibers, J. Physiol., 227 (1972), pp. 627-645.

[10] M. C. Reed & J. J. Blum, Mathematical Questions in Axonal Transport, In: Lectures on Mathematics in the Life Sciences, Vol. 24, 1994.

[11] M. C. Reed & S. Venakides & J. J. Blum, Approximate travelling waves in linear reaction-hyperbolic equations, SIAM. J. Appl. Math. 50(1) (1990), 167–180.

[12] Yasushi Shizuta & Shuichi Kawashima, Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation, Hokkaido Mathematical Journal. Vol.14, 1985, 249-275.

[13] G.S. Yablonsky, A.N. Gorban, D. Constales, V. Galvita, G.B. Marin Reciprocal Relations Between Kinetic Curves, EPL, 93 (2011).

[14] W.-A. Yong, Singular perturbations of first-order hyperbolic systems, Ph.D. Thesis, Universität Heidelberg, 1992.

[15] W.-A. Yong, Basic aspects of hyperbolic relaxation systems, in Advances in the Theory of Shock Waves, H. Freistühler and A. Szeppessy, eds., Progress in Nonlinear Differential Equations and Their Applications, Vol. 47, Birkhäuser, Boston, 2001, 259–305.

[16] W.-A. Yong, Entropy and Global Existence for Hyperbolic Banlance Laws, Arch.Rational Mech.Anal. 172 (2004), 247-266.

[17] Hao Yan & W.-A. Yong, Weak Entropy Solutions of Nonlinear Reaction-Hyperbolic Systems for Axonal Transport, Mathematical Models and Methods in Appl. Sci. (accepted).

[18] W.-A. Yong & K. Zumbrun Existence of relaxation shock profiles for hyperbolic conservation laws, Siam. J. Appl. Math. Vol.60, 2000, no.5, 1565-1575.

Zhou Pei-Yuan Center for Appl. Math., Tsinghua University, Beijing 100084, China
E-mail address: yanhao06@mails.tsinghua.edu.cn

Zhou Pei-Yuan Center for Appl. Math., Tsinghua University, Beijing 100084, China
E-mail address: wayong@tsinghua.edu.cn