Two-fluid model for a rotating trapped Fermi gas in the BCS phase

Michael Urban
Institut de Physique Nucléaire, F-91406 Orsay Cédex, France

We investigate the dynamical properties of a superfluid gas of trapped fermionic atoms in the BCS phase. As a simple example we consider the reaction of the gas to a slow rotation of the trap. It is shown that the currents generated by the rotation can be understood within a two-fluid model similar to the one used in the theory of superconductors, but with a position dependent ratio of normal and superfluid densities. The rather general result of this paper is that already at very low temperatures, far below the critical one, an important normal-fluid component appears in the outer regions of the gas. This renders the experimental observation of superfluidity effects more difficult and indicates that reliable theoretical predictions concerning other dynamical properties, like the frequencies of collective modes, can only be made by taking into account temperature effects.

PACS numbers: 03.75.Kk,03.75.Ss,67.40.Bz

In the last few months, experiments with trapped fermionic $^6\text{Li}$ atoms made great progress. The fact that by using the Feshbach resonance the Fermi gas can be transformed into a Bose-Einstein condensate (BEC) of molecules, which can be cooled by evaporative cooling and afterwards transformed back into a Fermi gas, allows to reach extremely low temperatures of the order of 0.03$T_F$, where $T_F = k_B \epsilon_F$ is the Fermi temperature. This allows, among other things, a detailed study of the BEC-BCS crossover. In particular, a temperature of 0.03$T_F$ should even be low enough to realize the BCS phase which is characterized by the condition $\Delta \ll \epsilon_F$, where $\Delta$ denotes the pairing gap.

However, at present it is not very clear how the transition to the BCS phase could be detected. While several observables related to collective oscillations (e.g. breathing modes) of the system have been investigated [1] [2], the most unambiguous signatures of the superfluid BCS phase seem to be those which concern the rotational properties of the system [3]. For instance, the moment of inertia of a slowly rotating Fermi gas was proposed to be a suitable observable for the detection of the BCS transition [4]. It should be mentioned that at present most of the theoretical predictions concerning possible experimental signatures of the BCS phase (e.g., [5] [6] [7] [8]) neglect temperature effects as well as possible deviations from hydrodynamic behaviour due to the discrete level spectrum in the trap.

In a previous article [9] we calculated the moment of inertia of a superfluid atomic Fermi gas in a slowly rotating trap at finite temperature. There it turned out that the rotational flow, which is characteristic for superfluidity, is realized only in the limit when the gap $\Delta$ is very large compared with the temperature $T$ and the level spacing $h\omega$ of the trap. In all other cases, the velocity field has both, rotational and irrotational components. For example, if the level spacing $h\omega$ is comparable with $\Delta$, the current has a strong rotational component even at zero temperature. On the other hand, at nonvanishing temperature $T$, a certain fraction of the Cooper pairs is broken by thermal excitations. This leads to the well-known effect that the system behaves like a mixture of normal and superfluid components [10] [11] [12]. Under rotation, the former behaves like a rigid body, while the latter can only have an irrotational velocity field.

However, in the calculation of Ref. [9] the gap $\Delta(r)$ has been replaced by a constant $\Delta$ corresponding to the average diagonal matrix element of $\Delta(r)$ at the Fermi surface. While this averaging procedure seems to be justified in cases where only one oscillator shell participates in the pairing (intrashell pairing, $\Delta < h\omega$), it is not well suited for the strong pairing regime ($\Delta > h\omega$), where the properties of the system can be described locally and depend on $r$ via the spatial dependence of $\Delta(r)$ [13]. In particular, the normal and superfluid fractions of the density, $\rho_n/\rho$ and $\rho_s/\rho$, should depend on $r$. To our knowledge this fact has not been taken into account in the existing literature.

In this article, we will concentrate on the $\hbar \to 0$ limit, i.e., we will neglect the quantum effect which is responsible for the rotational component of the velocity field at zero temperature. Anyway, if the system is sufficiently large and if the temperature is not extremely low, this quantum effect becomes much smaller than the effect resulting from the thermally created normal component of the system. The important point is that we will now take into account the $r$ dependence of the gap. In addition, we will not rely on the simplification made in our previous work that the full potential (trap + mean field) is approximately harmonic.

Let us briefly summarize the most important formulas (for more explanations and details see Ref. [13]). We assume that equal numbers of atoms in two spin states are trapped in a harmonic potential,

$$V_{\text{trap}}(r) = \sum_{i=x,y,z} \frac{m \omega_i^2 r_i^2}{2}.$$  (1)

The cigar-shaped form of the traps used in current experiments corresponds to $\omega_z \ll \omega_x = \omega_y$. However, in order to force the system to rotate around the long axis, one has to break the axial symmetry, e.g., by using a rotating laser beam as “spoon”. We will model this by taking $\omega_x \neq \omega_y$. The mean-field single-particle hamilt-
stays real, which in the case of a slow rotation implies
and

\[ p \]

\[ r \]

Hence, the price to pay for the real gap is an additional
term, the order parameter \( \Delta(\mathbf{r}) \) receiving a phase
which we will treat as a small perturbation. Because of this
parameter the order \( \Delta(\mathbf{r}) \) receives a phase
\[ \exp[-2i\phi(\mathbf{r})]. \]

The explicit form of \( \phi(\mathbf{r}) \) is unknown for the moment and will be determined below. It is
convenient to eliminate this phase by a gauge transformation,
multiplying all single-particle wave functions by \( \exp[i\phi(\mathbf{r})] \). In this way the gauge transformed gap \( \Delta(\mathbf{r}) \)
stays real, which in the case of a slow rotation implies that \( \Delta \) does not change at all [21], i.e., \( \Delta = \Delta_0(\mathbf{r}) \). On the other hand, this gauge transformation changes the momentum operator according to
\[ \hat{p} = \hat{p} - \hbar \nabla \phi(\mathbf{r}). \]

Hence, the price to pay for the real gap is an additional term in the hamiltonian. To linear order in the rotation frequency the new perturbation hamiltonian reads
\[ \hat{h}_1 = -\Omega \hat{L}_z - \frac{\hbar}{2m}(\hat{p} \cdot [\nabla \phi(\mathbf{r})] + [\nabla \phi(\mathbf{r})] \cdot \hat{p}). \]

In order to describe the system semiclassically, we make use of the Wigner-Kirkwood expansion. To that end we denote the Wigner transforms of \( \hat{h}_0, \hat{h}_1, \) etc., by \( h_0(\mathbf{r}, \mathbf{p}), \hat{h}_1(\mathbf{r}, \mathbf{p}), \) etc. We need also the Wigner transforms of the normal and abnormal density matrices in equilibrium, \( \rho_0(\mathbf{r}, \mathbf{p}) \) and \( \kappa_0(\mathbf{r}, \mathbf{p}) \), as well as their deviations from equilibrium, \( \tilde{\rho}_1(\mathbf{r}, \mathbf{p}) \) and \( \tilde{\kappa}_1(\mathbf{r}, \mathbf{p}). \) (For the sake of brevity, we will occasionally omit the arguments \( \mathbf{r} \) and \( \mathbf{p} \) if there is no risk of confusion.) Furthermore we introduce the Poisson bracket of two phase-space functions
\[ \{f, g\} = \sum_{i=x,y,z} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right). \]

Using the notations defined above, the terms linear in \( \Omega \)
of the Hartree-Fock-Bogoliubov (HFB) or Bogoliubov-de Gennes equations up to linear order in \( \hbar \) can be written as
\[ ih\{h_0, \tilde{\rho}_1\} + 2\Delta_0 \tilde{\kappa}_1 = -ih\\{\hat{h}_1, \rho_0\} \]
\[ ih\{\Delta_0, \tilde{\rho}_1\} - 2\hbar\tilde{\kappa}_1 = ih\\{\hat{h}_1, \kappa_0\}. \]

These are exactly the Eqs. (84) and (85) of Ref. [9]. The main point of the present article concerns the solution of this system of equations in the case of an \( \mathbf{r} \)-dependent gap \( \Delta_0(\mathbf{r}) \).

First we eliminate \( \tilde{\kappa}_1 \) by multiplying Eq. (7) by \( \hbar_0 \) and Eq. (8) by \( \Delta_0 \) and adding up the two resulting equations. Using the chain and product rules of differentiation, we then obtain
\[ \frac{1}{2}\{E^2, \tilde{\rho}_1\} = -h_0\\{\hat{h}_1, \rho_0\} + \Delta_0\\{\hat{h}_1, \kappa_0\}, \]
where \( E^2(\mathbf{r}, \mathbf{p}) = \hbar_0^2(\mathbf{r}, \mathbf{p}) + \Delta_0^2(\mathbf{r}) \).

To proceed further, we express \( \rho_0 \) and \( \kappa_0 \) in terms of \( h_0 \) and \( \Delta_0 \). Within the Thomas-Fermi (TF) or local-density approximation, these relations read
\[ \rho_0(\mathbf{r}, \mathbf{p}) = \frac{1}{2} - \frac{h_0(\mathbf{r}, \mathbf{p})}{2E(\mathbf{r}, \mathbf{p})}(1 - 2f[E(\mathbf{r}, \mathbf{p})]), \]
\[ \kappa_0(\mathbf{r}, \mathbf{p}) = \frac{\Delta_0(\mathbf{r})}{2E(\mathbf{r}, \mathbf{p})}(1 - 2f[E(\mathbf{r}, \mathbf{p})]), \]
where \( f(E) = 1/\exp(E/k_B T) + 1 \) denotes the Fermi function. Although Eqs. (10) and (11) are the solutions of the \( h \rightarrow 0 \) limit of the HFB equations, they are valid up to linear order in \( h \) [14] and therefore consistent with Eqs. (7) and (8). Inserting Eqs. (10) and (11) into Eq. (9), we obtain, again after repeated use of chain and product rules of differentiation, the following simple equation:
\[ \frac{1}{2}\{E^2, \tilde{\rho}_1\} = \frac{1}{2}\{E^2, \hat{h}_1 \frac{df}{dE}\}. \]

It is evident that this equation is solved by
\[ \tilde{\rho}_1 = \hat{h}_1 \frac{df}{dE}. \]

However, before this solution can be used, the gauge transformation which has been introduced in order to make the gap real must be inverted:
\[ \rho(\mathbf{r}, \mathbf{p}) = \hat{p}[\mathbf{r}, \mathbf{p} + \hbar \nabla \phi(\mathbf{r})]. \]
To linear order in \( \Omega \), this can also be written as
\[ \rho(\mathbf{r}, \mathbf{p}) = \rho_0[\mathbf{r}, \mathbf{p} + \hbar \nabla \phi(\mathbf{r})] + \tilde{\rho}_1(\mathbf{r}, \mathbf{p}) \]
\[ = \rho_0[\mathbf{r}, \mathbf{p} + \hbar \nabla \phi(\mathbf{r})] - \left( \Omega \times \mathbf{r} + \frac{\hbar}{m} \nabla \phi(\mathbf{r}) \right) \cdot \mathbf{p} \frac{df}{dE}\bigg|_{E(\mathbf{r}, \mathbf{p})}. \]

The last line has been obtained with the help of Eqs. (3) and (13).

The next step consists in calculating the corresponding current density per spin state,
\[ j(\mathbf{r}) = \int \frac{d^3p}{(2\pi \hbar)^3} \frac{\mathbf{p}}{m} \rho(\mathbf{r}, \mathbf{p}). \]
Using the explicit expression for $\rho(\mathbf{r}, p)$ given above, one easily obtains
\begin{equation}
\mathbf{j}(\mathbf{r}) = -\rho_0(\mathbf{r}) \frac{\hbar}{m} \nabla \phi(\mathbf{r}) \tag{18}
- \left( \mathbf{\Omega} \times \mathbf{r} + \frac{\hbar}{m} \nabla \phi(\mathbf{r}) \right) \int_0^\infty \frac{dp}{6\pi^2 \hbar^2 p^4} \frac{df}{dE}(E(\mathbf{r}, p)),
\end{equation}
with
\begin{equation}
\rho_0(\mathbf{r}) = \int \frac{d^3p}{(2\pi\hbar)^3} \rho_0(\mathbf{r}, p). \tag{20}
\end{equation}

The current density can therefore be written in a more suggestive way in terms of normal and superfluid densities $\rho_n(\mathbf{r})$ and $\rho_s(\mathbf{r})$,
\begin{equation}
\mathbf{j}(\mathbf{r}) = \rho_n(\mathbf{r}) \mathbf{\Omega} \times \mathbf{r} - \rho_s(\mathbf{r}) \frac{\hbar}{m} \nabla \phi(\mathbf{r}), \tag{21}
\end{equation}
if the normal and superfluid densities are defined according to the textbook result (Ref. [15], p. 459) as
\begin{equation}
\rho_n(\mathbf{r}) = \rho_0(\mathbf{r}) - \rho_s(\mathbf{r}) = -\int_0^\infty \frac{dp}{6\pi^2 \hbar^2 p^4} \frac{df}{dE}(E(\mathbf{r}, p)). \tag{22}
\end{equation}

In the BCS limit, i.e., if $\Delta_0(\mathbf{r}) \ll \epsilon_F(\mathbf{r})$, where $\epsilon_F(\mathbf{r}) = \mu - V_{\text{trap}}(\mathbf{r}) - g\rho_0(\mathbf{r})$ denotes the local Fermi energy, the ratio $\rho_n(\mathbf{r})/\rho_0(\mathbf{r})$ becomes a function of only one dimensionless argument, $T/T_c(\mathbf{r})$, where $T_c(\mathbf{r}) = 0.57\Delta_0(\mathbf{r}; T = 0)$ denotes the local critical temperature (the existence of a local critical temperature is an artifact of the TF approximation). This function, as well as the temperature dependence of the ratio $\Delta_0(\mathbf{r})/\Delta_0(\mathbf{r}; T = 0)$, is shown in Fig. 1.

Up to now, the phase $\phi(\mathbf{r})$, which determines the velocity of the superfluid component, is completely unknown. In Ref. [3], this phase was determined by calculating $\Delta_1$ from $\xi_1$ and imposing the condition $\Delta_1 = 0$. Then it was shown that with this choice the continuity equation for the current was satisfied. Here we will adopt another method which is commonly used in the literature [11] and which consists in using the continuity equation. In the rotating frame the latter reads
\begin{equation}
\nabla \cdot \mathbf{j}(\mathbf{r}) + \dot{\rho}(\mathbf{r}) - (\mathbf{\Omega} \times \mathbf{r}) \cdot \nabla \rho(\mathbf{r}) = 0, \tag{23}
\end{equation}
where $\dot{\rho}(\mathbf{r}) = 0$ in our case of a stationary rotation and $\rho(\mathbf{r}) = \rho_0(\mathbf{r})$ up to linear order in $\mathbf{\Omega}$. Taking the divergence of Eq. (21), one can see that the normal fluid component drops out and it remains a continuity equation for the superfluid component:
\begin{equation}
-\frac{\hbar}{m} \nabla \cdot [\rho_s(\mathbf{r}) \nabla \phi(\mathbf{r})] - (\mathbf{\Omega} \times \mathbf{r}) \cdot \nabla \rho_s(\mathbf{r}) = 0. \tag{24}
\end{equation}

In the case of a deformed harmonic trapping potential, this equation can be solved analytically. To see this, remember that within the TF approximation the density $\rho_0(\mathbf{r})$ and the gap $\Delta_0(\mathbf{r})$, and consequently also the superfluid density $\rho_s$, depend on $\mathbf{r}$ only via the local chemical potential $\mu_{\text{loc}}(\mathbf{r}) = \mu - V_{\text{trap}}(\mathbf{r})$, i.e., $\rho_s(\mathbf{r}) = \rho_s[\mu - V_{\text{trap}}(\mathbf{r})]$. Hence Eq. (24) can be rewritten as
\begin{equation}
\frac{d\rho_s}{d\mu_{\text{loc}}} \big|_{\mu - V_{\text{trap}}(\mathbf{r})} \left[ \nabla V_{\text{trap}}(\mathbf{r}) \right] \cdot \frac{\hbar}{m} \nabla \phi(\mathbf{r}) + \mathbf{\Omega} \times \mathbf{r}
- \frac{\hbar}{m} \rho_s(\mathbf{r}) \nabla^2 \phi(\mathbf{r}) = 0. \tag{25}
\end{equation}

In the special case of the harmonic potential [11], it can readily be shown that this equation has the same solution as in the simple case of constant $\Delta_0$ studied in Ref. [3],
\begin{equation}
\phi(\mathbf{r}) = \frac{m}{\hbar} \frac{\omega_x^2 - \omega_y^2}{\omega_x^2 + \omega_y^2} \Omega r_x r_y, \tag{26}
\end{equation}
since this solution is independent of the form of $\rho_s(\mu)$. The current density per spin state, $\mathbf{j}(\mathbf{r})$, is therefore given by
\begin{equation}
\mathbf{j}(\mathbf{r}) = \rho_n(\mathbf{r}) \mathbf{\Omega} \times \mathbf{r} - \rho_s(\mathbf{r}) \frac{\omega_x^2 - \omega_y^2}{\omega_x^2 + \omega_y^2} \Omega \nabla(\mathbf{r}_x \mathbf{r}_y). \tag{27}
\end{equation}

Due to fact that $\rho_0(\mathbf{r})$ and $\Delta_0(\mathbf{r})$ [and consequently $\rho_s(\mathbf{r})$ and $\rho_n(\mathbf{r})$] depend on $\mathbf{r}$ only via the local chemical potential, it is sufficient to perform the TF calculation for a spherical trap with the geometrically averaged trapping frequency $\bar{\omega} = (\omega_x \omega_y \omega_z)^{1/3}$. In this spherical trap, of course, the density $\rho_0$, gap $\Delta_0$, etc., depend only on the distance from the center, i.e., $\rho_0(\mathbf{r}) = \rho_0(r)$, etc. (quantities related to the spherical trap will be marked by a bar). The corresponding quantities in the deformed trap can then be obtained from
\begin{equation}
\rho_0(\mathbf{r}) = \rho_0 \left( \frac{1}{\bar{\omega}} \sqrt{\omega_x r_x^2 + \omega_y r_y^2 + \omega_z r_z^2} \right) \tag{28}
\end{equation}
and analogously for $\Delta_0(r)$, $\rho_s(r)$, etc. Note, however, that this is true only within the TF approximation.

In Fig. 2 we show the normal and superfluid densities per spin state ($\bar{\rho}_n$ and $\bar{\rho}_s$) and the gap $\Delta_0$ in the spherical trap for different temperatures. The dashed lines correspond to the TF ($\hbar \to 0$) results, while for the solid lines the gap $\Delta_0$ has been obtained by solving numerically the HFB equations [14]. At zero temperature, the trapped Fermi gas is completely superfluid, i.e., $\bar{\rho}_s = \bar{\rho}_0$ and $\bar{\rho}_n = 0$. At low but non-vanishing temperature ($T = 2\hbar \bar{\omega} \approx 0.36 T_c$, $T_c \approx 5.5\hbar \bar{\omega}$ being defined as the critical temperature within HFB), a normal fluid component appears near the surface, since there the gap is smallest and consequently the Cooper pairs are most easily broken by thermal excitations. If the temperature increases further ($T = 4\hbar \bar{\omega} \approx 0.72 T_c$), the normal-fluid component starts to extend over the whole volume, and finally, slightly above the critical temperature ($T = 6\hbar \bar{\omega}$) the superfluid component vanishes completely (solid lines). The small superfluid region near the center of the trap which survives at this temperature within the TF approximation (dashed line) is a consequence of the fact that the critical temperature predicted by the TF approximation is higher than the HFB one [17]. However, apart from this point, one can say that in general for $\bar{\rho}_n$ and $\bar{\rho}_s$ the agreement between TF and HFB is reasonable and better than for the gap $\Delta_0$ itself. The reason why the agreement between TF and HFB is better for $\bar{\rho}_n$ and $\bar{\rho}_s$ than for $\Delta_0$ is that near the critical temperature the temperature dependence of $\rho_s/\rho$ is much weaker than that of $\Delta/\Delta(T = 0)$, cf. Fig. 1.

In order to make the comparison between the gaps $\Delta_0$ calculated within the TF approximation and by solving the full HFB equations, we had to choose a rather moderate number of particles, $N = 36000$, for which the HFB calculation is feasible. If we had not been interested in the comparison between the results obtained with the HFB and TF gaps, we could of course have shown the TF results for arbitrarily large numbers of particles. We emphasize that even for much larger numbers of particles the qualitative behaviour of the TF results remains unchanged, provided that the coupling constant $g$ is tuned such that the condition $\Delta \ll \epsilon_F$ (BCS condition) remains satisfied [22] and the temperatures are scaled with respect to the critical temperature.

Using the spherical density profiles and Eqs. (25) and (27), we can immediately calculate the current distribution $\mathbf{j}(\mathbf{r})$. In Fig. 3 we show the current in the $xy$ plane ($r_z = 0$) for a deviation from axial symmetry of $\omega_x/\omega_y = 0.7$ at several temperatures. (For the cases...
shown, the current densities obtained from the HFB and TF density profiles are indistinguishable within the resolution of the plot.) At zero temperature, the current is irrotational and rather weak (it vanishes in the limit of axial symmetry, \( \omega_x = \omega_y \)). In the surface region the current reaches its ordinary (rigid-body) form already at \( T = 1.5 \hbar \omega \approx 0.27 T_c \). At \( T = 3 \hbar \omega \approx 0.54 T_c \) the current shows almost everywhere the rigid-body behavior, only near the center it is still a little bit weaker than in the normal phase, \( T = 6 \hbar \omega \).

Let us now look at the temperature dependence of the moment of inertia, \( \Theta \), which is defined as

\[
\Theta = \frac{\langle \vec{L}^2 \rangle}{\Omega} = \frac{2m}{\Omega} \int d^3r [r_x j_y(r) - r_y j_x(r)].
\]

The factor of two is a consequence of our convention that \( j \) denotes the current density per spin state. Using again Eqs. (25) and (27), we can express the moment of inertia in terms of the density profile in the corresponding spherical trap as a simple radial integral:

\[
\Theta = \frac{8 \pi m}{3} \left( \frac{\omega_x^2 + \omega_y^2}{\omega_z} \right) \int_0^\infty dr \, r^4 \left[ \tilde{\rho}_n(r) + \left( \frac{\omega_x^2 - \omega_y^2}{\omega_x^2 + \omega_y^2} \right)^2 \tilde{\rho}_s(r) \right].
\]

In Fig. 4 we show the moment of inertia of the same set of parameters that were already used in Figs. 2 and 3 as a function of temperature. The solid line has been calculated by using the HFB density profiles, while the dashed line was obtained from the density profile within TF approximation. One can see that the moment of inertia decreases strongly as the temperature goes to zero. The limiting value at zero temperature is determined by the deformation of the trap in the \( xy \) plane,

\[
\Theta(T = 0) = \left( \frac{\omega_x^2}{\omega_x^2 + \omega_y^2} \right)^2 \Theta_{\text{rigid}},
\]

where \( \Theta_{\text{rigid}} \) denotes the corresponding rigid-body moment of inertia [which can be obtained from Eq. (30) by putting \( \tilde{\rho}_n = \tilde{\rho}_0 = 0 \) and \( \tilde{\rho}_s = 0 \)]. In our case of \( \omega_x/\omega_y = 0.7 \), we have \( \Theta(T = 0) \approx 0.12 \Theta_{\text{rigid}} \). An important point to notice is that, coming from high temperatures, one does not observe an appreciable change of the moment of inertia until one reaches temperatures far below the critical one. The reason for this effect is that the main contribution to the moment of inertia comes from the outer regions of the trapped gas, where the order parameter becomes small and where consequently the normal-fluid fraction is large even far below \( T_c \). The discrepancy between the HFB (solid line) and TF results (dashed line) below \( \approx 2 \hbar \omega \) can be traced back to the effect that within the TF approximation the gap near the surface vanishes already at very low temperature, such that the normal-fluid fraction near the surface is overestimated within TF.

To conclude, we have applied the two-fluid model known from the theory of superconductivity \([10,11,12]\) to the case of ultracold trapped fermionic atoms in the BCS phase. In contrast to the usual situation, the ratio of the normal and superfluid densities is explicitly position dependent due to the inhomogeneous trapping potential. Specializing to the case of a slowly rotating system, we have shown that the linear order in \( \hbar \) of the linear response equations gives a current which can be decomposed in a natural way into normal and superfluid components. The normal component appears as a consequence of Cooper pairs which are broken by thermal excitations already below the critical temperature \( T_c \). We have shown that especially the outer region of the trapped gas behaves essentially as if it was normal-fluid, even far below \( T_c \). Only the central region of the gas keeps its superfluid character up to \( T_c \).

As a consequence, the moment of inertia decreases more slowly as it was previously expected \([9]\) if the temperature is lowered below \( T_c \), i.e., the effects of superfluidity become visible only far below \( T_c \). This important but in a certain sense negative result will apply analogously to other observables which are mainly sensitive to the surface of the system, like, e.g., collective modes. For example, the theory presented here was used in Ref. \([18]\) in order to explain the temperature dependence of the strength of the response function for the so-called “twist mode.” There the effect was even more dramatic, since the relevant integral contained an \( r^6 \) weight factor instead of \( r^4 \) in Eq. (30).

We are therefore convinced that it is not justified to compare the experimentally measured frequencies of collective modes directly with theoretical predictions obtained for zero temperature, as it is done in the current literature \([11,12]\). We rather expect that the temperature dependence is important and can be predicted by generalizing the two-fluid model presented here to the dynamic case, i.e., by performing the Wigner-Kirkwood expansion of the time-dependent HFB (TDHFB) equations up to linear order in \( \hbar \). \([11,12]\). This leads to a generalization...
of the Vlasov equation for the normal phase, which results from the Wigner-Kirkwood expansion of the time-dependent Hartree-Fock (TDHF) equation up to linear order in $\hbar$.

However, one should keep in mind that the Thomas-Fermi approximation for the ground state as well as the Wigner-Kirkwood expansion of the dynamical equations (i.e., the generalized Vlasov equation and the superfluid hydrodynamics to which it reduces in the zero-temperature limit) depend on the assumption $\hbar \omega_i \ll \Delta$ for $i = x, y, z$. Concerning the validity of the Thomas-Fermi approximation for the ground state, the condition $\hbar \omega_i \ll \Delta$ has been inferred from the requirement that the coherence length $\xi = \hbar v_F / \pi \Delta$ ($v_F$ being the Fermi velocity) must be much smaller than the typical length scale of the system, which is approximately given by the Thomas-Fermi radius $R_{TF} = \sqrt{2 \mu / m \omega_F^2}$. However, it is less evident where the assumption $\hbar \omega_i \ll \Delta$ enters into the description of the dynamics of the system within the generalized Vlasov equation.

To give a specific example, in Ref. [9], quantum corrections to the moment of inertia of higher orders in $\hbar (\omega_1 \pm \omega_2) / \Delta$ were discussed. Also in the case of the strength of the twist mode mentioned above, the fully quantum-mechanical (“microscopic”) calculation showed deviations from the two-fluid model, especially at very low temperatures. In both cases, the corrections act as if the normal-fluid component of the system was larger than predicted by Eq. (22) and in particular non-vanishing even at zero temperature. In a certain sense the accelerations acting on the Cooper pairs during their motion through the inhomogeneous potential seem to have a similar pair-breaking effect as the thermal excitations which are responsible for the normal-fluid component given by Eq. (21). From a completely different point of view, Eq. (21) is usually derived by looking at the long-wavelength limit ($q \xi \ll 1$) of the current-current correlation function in a homogeneous system [12, 13]. In the trapped system, however, the wave vectors must be of the order $q \approx 1 / R_{TF}$, and we recover the condition $\xi \ll R_{TF}$.

Deviations from superfluid hydrodynamics ($T = 0$) or from the two-fluid model ($T > 0$), respectively, may therefore be especially important in the case of the strongly elongated traps used in current experiments, which have rather high radial trapping frequencies $\omega_x$ and $\omega_y$. Therefore this kind of quantum effects should be studied in more detail. In the case of collective modes, this could be done, e.g., by comparing systematically the results obtained in quantum mechanical quasiparticle RPA (QRPA) calculations [20] with those of hydrodynamics.

### Acknowledgments

I thank P. Schuck for fruitful discussions and critical reading of the manuscript. I also acknowledge useful discussions with S. Sinha.

[1] M. Bartenstein et al., Phys. Rev. Lett. 92, 203201 (2004).
[2] J. Kinast et al., Phys. Rev. Lett. 92, 150402 (2004).
[3] M. Cozzini and S. Stringari, Phys. Rev. Lett. 91, 070401 (2003).
[4] M. Farine, P. Schuck, and X. Viñas, Phys. Rev. A 62, 013608 (2000).
[5] A. Minguzzi and M.P. Tosi, Phys. Rev. A 63, 023609 (2001).
[6] F. Zambelli and S. Stringari, Phys. Rev. A 63, 033602 (2001).
[7] C. Menotti, P. Pedri, and S. Stringari, Phys. Rev. Lett. 89, 250402 (2002).
[8] S. Stringari, Europhys. Lett. 65, 749 (2004).
[9] M. Urban and P. Schuck, Phys. Rev. A 67, 033611 (2003).
[10] A. J. Leggett, Phys. Rev. 140, A1869 (1965); Phys. Rev. 147, 119 (1966).
[11] O. Betheder-Matibet and P. Nozières, Ann. Phys. (N.Y.) 51, 392 (1969).
[12] J.R. Schrieffer, Theory of superconductivity (Benjamin, New York, 1964).
[13] G.M. Bruun and H. Heiselberg, Phys. Rev. A 65, 053407 (2002); H. Heiselberg, Phys. Rev. A 68, 053616 (2003).
[14] K. Taruishi and P. Schuck, Z. Phys. A 342, 397 (1992).
[15] A.L. Fetter and J.D. Walecka, Quantum Theory of Many-Particle Systems (McGraw-Hill, New York, 1971).
[16] M. Grasso and M. Urban, Phys. Rev. A 68, 033610 (2003).
[17] M. A. Baranov and D. S. Petrov, Phys. Rev. A 58, R801 (1998).
[18] M. Grasso, M. Urban, and X. Viñas. cond-mat/0407475 (2004).
[19] D. Vollhardt and P. Wölfle, The Superfluid Phases of Helium 3 (Taylor & Francis, London, 1990)
[20] G.M. Bruun and B.R. Mottelson, Phys. Rev. Lett. 87, 270403 (2001).
[21] Since the magnitude of $\Delta$ cannot depend on the sign of $\Omega$, its change must be at least of the order $\Omega^2$.
[22] At zero temperature and at the center of the trap we have with our choice of parameters $\Delta \approx 0.2 v_F$, i.e., we are already close to the BCS-BEC crossover region.