Free vibration of orthotropic Levy-type solution plates by using SEM

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Abstract. Orthotropic Levy-type solution plates considering shear deformation were solved by using the Spectral Element Method (SEM). In SEM, the exact solution of structural dynamic related problems can be solved by using the frequency-dependent shape functions. The solutions of SEM are accurate in decreasing the amount of Degree Of Freedoms (DOF) to discard the cost and computational disadvantages in the Finite Element Method (FEM). This study investigates the free vibration problems of orthotropic Levy-type solution plates solutions based on the two variable Refined Plate Theory (RPT). SEM for orthotropic Levy-type solution plates in the frequency domain was formulated to solve free vibration problems. The differential governing equations of the orthotropic plate element in SEM are formulated into the form of transcendental stiffness matrices. The boundary conditions of the spectral element model consist of four edge-type DOFs at each side of the orthotropic Levy-type solution plates. The natural frequencies of the plate are calculated by means of the Wittrick-Williams procedure. The numerical investigations are presented to show the accuracy, efficiency, and effectiveness of SEM without any discretization scheme.

1. Introduction
The dynamic behaviors of orthotropic plates are pronounced importance in all mechanics related subjects in science and engineering. The vibration characteristic of an orthotropic plate varies depending on its vibrational frequency. In the finite element models, the shape functions of plates are formulated by using frequency independent polynomial. Hence, solutions obtained by FEM become inaccurate, particularly in high frequencies range, where the wavelengths are very short. In particular case, FEM cannot provide the substantial high-frequency modes of vibration without refining the mesh. Therefore, the refinement will result in a substantial number of DOF, and from the computational cost perspective, to solve complex plate models by using FEM becomes unaffordable.
The alternative scheme to obtain accurate solutions of the vibrational plates is by using the frequency dependent shape functions in spectral forms. They are well-known as dynamic shape functions in the literature. Because the dynamic shape functions contain all important high-frequency wave modes, highly accurate solutions can be obtained, and the need to refine the meshes is no longer required. This idea is known as the Dynamic Stiffness Method (DSM) [1, 2]. Because DSM is formulated by using frequency dependent shape functions, it contains the mass distribution of the plate implicitly. Therefore, only a single plate element model is sufficient to obtain the exact free vibration solutions. Besides, the dynamic stiffness matrices are formulated similar to the finite element matrices, and they can be assembled similarly to that used in FEM.

In Spectral Analysis Method (SAM) [3], the superposition of an infinite number of frequencies dependent wave modes is considered as the solutions to the governing differential equations. The fundamental features of DSM and SAM are combined in [4] to introduce the concept of Spectral Element Method (SEM).

In this study, the free vibration problem of the orthotropic Levy-type solution based on the two variable Refined Plate Theory (RPT) [5] is formulated in the context of SEM. In RPT, the shear deformation is derived by the variational method consistently, hence it does not require shear correction factor and leads to parabolic transverse shear stress variation across the thickness. Formulations of the two variables RPT for bending and shear governing equations are suited for SEM. To verify the soundness of SEM, the natural frequencies of orthotropic plates computed by SEM are compared with the exact solutions which are based on the Classical Plate Theory (CPT) [6-8] by using trigonometric series. The natural frequencies of the orthotropic Levy-type solution plates are then, studied and discussed on the size aspect ratio influences.

2. Levy-type solution plate

The Levy-type solution plate is a rectangular plate which has two parallel simply supported sides and arbitrary boundary conditions at the perpendicular edges. The coupled partial differential equations of the governing equations can be decoupled by introducing supplementary functions.

2.1. Dynamic equilibrium equation of an orthotropic Levy-type solution plate

Figure 1 shows two simply supported sides of a rectangular plate which are parallel to the x-axis and the other two arbitrary boundary conditions on the perpendicular sides which are parallel to the y-axis.

The Levy-type solution plate has the dimension of \( L_x \) and \( L_y \) in the x- and y- directions, respectively. In the two variable refined plate theory [5], the transverse displacement \( w \) has two components: bending component \( w_b \) and shear component \( w_s \). Both components are the functions of coordinates \( x, y \) and time \( t \) only. The transverse displacement \( w \) is written as,

\[
w(x, y, t) = w_b(x, y, t) + w_s(x, y, t)
\] (1)

The orthotropic partial differential equations of small transverse vertical vibration \( w(x, y, t) \) of the plate considering the shear deformation are represented by,
\[
D_{11} \frac{\partial^4 w_h}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w_h}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w_h}{\partial y^4} - \frac{\rho h^3}{12} \left( \frac{\partial^2 w_h}{\partial x^2} + \frac{\partial^2 w_h}{\partial y^2} \right) + \frac{\rho h}{1008} \left( \frac{\partial^2 \ddot{w}_h}{\partial x^2} + \frac{\partial^2 \ddot{w}_s}{\partial y^2} \right) + \rho h (\ddot{w}_h + \ddot{w}_s) = 0
\]

where, \( h \) is the thickness and \( \rho \) is the mass density. The bending and shearing rigidities are given by,

\[
D_{11} = \frac{Q_{11} h^3}{12}; \quad D_{22} = \frac{Q_{22} h^3}{12}; \quad D_{12} = \frac{Q_{12} h^3}{12}; \quad D_{66} = \frac{Q_{66} h^3}{12}; \quad A_{44} = \frac{Q_{44} h}{6}; \quad A_{55} = \frac{Q_{55} h}{6}.
\]

\[
Q_{11} = \frac{E_1}{1 - \nu_{12} \nu_{21}}; \quad Q_{22} = \frac{E_2}{1 - \nu_{12} \nu_{21}}; \quad Q_{12} = \frac{E_1 \nu_{12} E_2}{1 - \nu_{12} \nu_{21}}; \quad Q_{44} = \frac{E_2}{2(1 - \nu_{23})}; \quad Q_{55} = \frac{E_1}{2(1 - \nu_{23})}
\]

where, \( E_1 \) and \( E_2 \) are elastic moduli, \( \nu_{12}, \nu_{21}, \nu_{13} \) and \( \nu_{23} \) are Poisson’s ratios.

### 2.2. Spectral element solution

Supposing the vertical displacements of the plate in \( n \)-domain spectral as

\[
w_i(x,y,t) = \frac{1}{N} \sum_{n=1}^{N} W_{sn}(x,y; \omega_n) e^{i\omega_n t}; \quad w_h(x,y,t) = \frac{1}{N} \sum_{n=1}^{N} W_{sh}(x,y; \omega_n) e^{i\omega_n t}
\]

and,

\[
W_{sn}(x,y; \omega_n) = X_{sn}(x) Y_{m}(y) \quad \text{and} \quad W_{sh}(x,y; \omega_n) = X_{hn}(x) Y_{m}(y)
\]

where \( W_{sn}(x,y; \omega_n) \) and \( W_{sh}(x,y; \omega_n) \) are the spectral components of the \( w(x,y,t) \), \( N \) is the sampling number, and \( \omega_n \) is the \( n \)-th natural frequency of the plate. The spectral components \( W_{sn}(x,y; \omega_n) \) and \( W_{sh}(x,y; \omega_n) \) can be obtained from the multiplication between \( X_{sn}(x) \), \( X_{hn}(x) \) and \( Y_{m}(y) \), \( Y_{m}(y) \) which are the displacement functions in \( x \)- and \( y \)- directions, for shear and bending transverse displacements respectively.

By assuming the \( Y_{sn}(y) \) and \( Y_{hn}(y) \) solutions in \( m \)-domain spectral form as,

\[
Y_{sn}(y) = \frac{1}{M} \sum_{m=1}^{M} Y_{snm} e^{ik_{ym} y}; \quad Y_{hn}(y) = \frac{1}{M} \sum_{m=1}^{M} Y_{hnm} e^{ik_{ym} y}
\]

where, \( M \) is the sampling number, and the wavenumber is given as,

\[
k_{ym} = \frac{m \pi}{L_y} \quad (m = 1, 2, 3 \ldots)
\]

by substituting equations (5) into the spectral components in equations (4), results in...
\[ W_{sn}(x, y; \omega_n) = X_{sn}(x) Y_{sn}(y) = \frac{1}{M} \sum_{m=1}^{M} X_{sn}(x; k_{y\text{nm}}, \omega_{nm}) Y_{snm} e^{ik_{x\text{nm}}y} \]

\[ W_{bn}(x, y; \omega_n) = X_{bn}(x) Y_{bn}(y) = \frac{1}{M} \sum_{m=1}^{M} X_{bn}(x; k_{y\text{nm}}, \omega_{nm}) Y_{bmn} e^{ik_{x\text{nm}}y} \]

(7)

where,

\[ W_{snm}(x; k_{y\text{nm}}, \omega_{nm}) = X_{sn}(x; k_{y\text{nm}}, \omega_{nm}) Y_{snm} ; \quad W_{bmn}(x; k_{y\text{nm}}, \omega_{nm}) = X_{bn}(x; k_{y\text{nm}}, \omega_{nm}) Y_{bmn} \]

(8)

by substituting equations (8) into equations (7) and further into equations (3) results in,

\[
\begin{align*}
W_i(x, y, t) &= \frac{1}{N} \sum_{n=1}^{N} \left( \frac{1}{M} \sum_{m=1}^{M} W_{snm}(x, y, t) e^{ik_{x\text{nm}}y} \right) e^{i\omega_nt} \\
W_j(x, y, t) &= \frac{1}{N} \sum_{n=1}^{N} \left( \frac{1}{M} \sum_{m=1}^{M} W_{bmn}(x, y, t) e^{ik_{x\text{nm}}y} \right) e^{i\omega_nt}
\end{align*}
\]

(9)

The free vibration differential partial equation of equation (2) can then be written as,

\[
D_{11}W_{bmn}'' - 2k_{y\text{nm}}^2 \left( D_{12} + 2D_{66} \right)W_{bmn}'' + D_{22}k_{y\text{nm}}^4 W_{bmn} + \frac{\rho h^3}{12} \omega_{nm}^2 \left( W_{bmn}'' - k_{y\text{nm}}^2 W_{bmn} \right) - \ldots = 0
\]

(10)

for the bending component and,

\[
-\left[ A_{35}W_{snm}'' - A_{44}k_{y\text{nm}}^2 W_{snm} \right] + \frac{1}{84} \left[ D_{11}W_{snm}'' - 2k_{y\text{nm}}^2 \left( D_{12} + 2D_{66} \right)W_{snm}'' + D_{22}k_{y\text{nm}}^4 W_{snm} \right] + \ldots = 0
\]

(11)

for the shear component.

By assuming the spectral solutions in the form,

\[ W_{snm}(x; k_{y\text{nm}}, \omega_{nm}) = c_{snm} e^{-ik_{x\text{nm}}x} ; \quad W_{bmn}(x; k_{y\text{nm}}, \omega_{nm}) = c_{bmn} e^{-ik_{x\text{nm}}x} \]

(12)

together with its derivations with respect to the coordinates \( x, y \) and to the time \( t \), substitution into equations (10) and (11) will give the following equations,

\[
\begin{align*}
\left[ D_{11}k_{y\text{nm}}^4 + 2k_{y\text{nm}}^2 k_{x\text{nm}}^2 \left( D_{12} + 2D_{66} \right) + D_{22}k_{y\text{nm}}^4 \right] + \frac{\rho h^3}{12} \omega_{nm}^2 \left( k_{y\text{nm}}^2 + k_{x\text{nm}}^2 \right) - \omega_{nm}^2 \rho h \right] W_{bmn} = 0 \\
\omega_{nm}^2 \rho h W_{snm} = 0
\end{align*}
\]

(13)

for the bending component and,

\[
\begin{align*}
-\left[ A_{35}W_{snm}'' - A_{44}k_{y\text{nm}}^2 W_{snm} \right] + \frac{1}{84} \left[ D_{11}W_{snm}'' - 2k_{y\text{nm}}^2 \left( D_{12} + 2D_{66} \right)W_{snm}'' + D_{22}k_{y\text{nm}}^4 W_{snm} \right] + \ldots = 0
\end{align*}
\]

(14)

for the shear component.
Representing the quadratic terms of $x_{nm}^2$ and $y_{nm}^2$ with $x_{nmK}$ and $y_{nmK}$, the solution of the fourth order equations (13) and (14) can be obtained by solving the second-degree order of polynomials. By summation, the bending and shearing components of both equations, the following relationship can be expressed in the matrix form as,

$$W_{nm}(x; k_{y_{nm}}, \omega_{nm}) = W_{nm}(x; k_{x_{nm}}, \omega_{nm}) + W_{nm}(x; k_{x_{nm}}, \omega_{nm}) = E_{nm}(x; k_{y_{nm}}, \omega_{nm}) c_{nm}$$

(15)

with,

$$E_{nm}(x; k_{y_{nm}}, \omega_{nm}) = \begin{bmatrix} e^{-ik_{x_{nm}}x} & e^{-ik_{x_{nm}}x} & e^{-ik_{x_{nm}}x} & e^{-ik_{x_{nm}}x} & e^{-ik_{x_{nm}}x} & e^{-ik_{x_{nm}}x} & e^{-ik_{x_{nm}}x} & e^{-ik_{x_{nm}}x} \end{bmatrix}$$

$$c_{nm} = [c_{nm1} \ c_{nm2} \ c_{nm3} \ c_{nm4} \ c_{nm5} \ c_{nm6} \ c_{nm7} \ c_{nm8}]^T$$

2.3. Matrix formulation of a simply supported Levy-type solution plate
In case of a simply supported condition at all sides, the spectral nodal DOF of the element can be obtained by substituting equation (15) with the boundary conditions at $x = 0$ and $x = L_x$ in the frequency $\omega$-domain results in,

$$d_{n}(k_{y_{nm}}, \omega_{n}) = \Phi_{nm}(k_{y_{nm}}, \omega_{nm}) c_{nm}$$

(16)

The resultant of bending moments and shearing forces at the boundary conditions in spectral form can be expressed by,

$$f_{n}(k_{y_{nm}}, \omega_{n}) = G_{nm}(k_{y_{nm}}, \omega_{nm}) c_{nm}$$

(17)

The constant vector $c_{nm}$ can be eliminated from both equations (16) and (17) into the form,

$$f_{nm}(k_{y_{nm}}, \omega_{n}) = S(k_{y_{nm}}, \omega_{nm}) d_{nm}(k_{y_{nm}}, \omega_{nm})$$

(18)

with

$$S(k_{y_{nm}}, \omega_{nm}) = G_{nm}(k_{y_{nm}}, \omega_{nm}) \Phi_{nm}^{-1}(k_{y_{nm}}, \omega_{nm})$$

The size of matrix $S(k_{y_{nm}}, \omega_{nm})$ is $8 \times 8$, and it consists of natural frequency $\omega_{nm}$ and wavenumber $k_{y_{nm}}$.

3. Numerical examples for free vibrations of Levy-type solution plates
Well studied examples available in the literature are selected to demonstrate the effectiveness of the present study. The examples have been shown by [8-10].

Consider the geometry of a simply supported square plate as shown in figure 2. The non-dimensional properties of the plate used in the calculation are:

Example 1: aspect ratio $h / L_x = 0.1$; size ratio $L_y / L_x = 1$; material properties $E_2 / E_1 = 1$; $G_{12} / E_1 = E / 2(1 + \nu)$; $G_{13} / E_1 = E / 2(1 + \nu)$; $G_{23} / E_1 = E / 2(1 + \nu)$; $\nu_{12} = 0.3$; $\nu_{21} = 0.3$. The non-dimensional natural frequency is computed by $\omega_{nm} = \omega_{nm} h \sqrt{2\rho(1 + \nu) / E}$.

Example 2: aspect ratio $h / L_x = 0.1$; size ratio $L_y / L_x = 1$; material properties $E_2 / E_1 = 0.52500$; $G_{12} / E_1 = 0.26293$; $G_{13} / E_1 = 0.15991$; $G_{23} / E_1 = 0.26681$; $\nu_{12} = 0.44046$; $\nu_{21} = 0.23124$. The non-dimensional natural frequency is computed by $\omega_{nm} = \omega_{nm} h \sqrt{2\rho / Q_{11}}$. 

5
The natural frequencies $\omega_{mn}$ ($m = 1, 2, 3, \ldots; n = 1, 2, 3, \ldots$) are solved from the condition of zero determinant of matrix $S(k_{ymn}, \omega_{mn})$ in equation (18) as,

$$\left|S(k_{ymn}, \omega_{mn})\right| = 0$$

The results of non-dimensional natural frequencies of the plates are compared with the other studies [5, 8-10] as shown in figure 3 and tables 1-2. One element is used in the computation using SEM. The Wittrick-Williams algorithm [11] is used to calculate the natural frequencies of the plates.

### Table 1. Non-dimensional natural frequencies of an isotropic plate.

| m | n | Exact[8] | Reddy[8] | Reissner[8] | CPT[8] | RPT[5] | Present(SEM) |
|---|---|----------|----------|------------|-------|-------|--------------|
| 1 | 1 | 0.0932   | 0.0931   | 0.0930     | 0.0955 | 0.0930 | 0.0930       |
| 1 | 2 | 0.2226   | 0.2222   | 0.2219     | 0.2360 | 0.2220 | 0.2220       |
| 2 | 2 | 0.3421   | 0.3411   | 0.3406     | 0.3406 | 0.3406 | 0.3406       |
| 1 | 3 | 0.4171   | 0.4158   | 0.4149     | 0.4629 | 0.4151 | 0.4151       |
| 2 | 3 | 0.5239   | 0.5221   | 0.5206     | 0.5951 | 0.5208 | 0.5208       |
| 1 | 4 | ---      | 0.6545   | 0.6520     | 0.7668 | 0.6525 | 0.6525       |
| 3 | 3 | 0.6889   | 0.6862   | 0.6834     | 0.8090 | 0.6840 | 0.6840       |
| 2 | 4 | 0.7511   | 0.7481   | 0.7446     | 0.8926 | 0.7454 | 0.7454       |
| 3 | 4 | ---      | 0.8949   | 0.8896     | 1.0965 | 0.8908 | 0.8908       |
| 4 | 4 | 1.0889   | 1.0847   | 1.0764     | 1.3716 | 1.0785 | 1.0785       |

### 4. Results and discussion

Figure 3 shows the results of consecutive natural frequencies of $m$ and its corresponding $n$ modes of both isotropic and orthotropic plates taking into account the shear deformation. In general, the vibration of orthotropic plates lying within the lower frequencies range compared with the isotropic material.
Example 1: isotropic plate

Example 2: orthotropic plate

$m = 1$

$m = 2$

$m = 3$

$m = 4$

Figure 3. The non-dimensional natural frequencies of a square plate examples.
Table 2. Non-dimensional natural frequencies of an orthotropic plate.

| m | n | Exact[10] | Reddy[9] | Reissner[10] | CPT[9] | RPT[5] | Present(SEM) |
|---|---|-----------|----------|-------------|--------|--------|-------------|
| 1 | 1 | 0.0474    | 0.0474   | 0.0474      | 0.0497 | 0.0477 | 0.0467      |
| 1 | 2 | 0.1033    | 0.1033   | 0.1032      | 0.1120 | 0.1040 | 0.1021      |
| 2 | 1 | 0.1188    | 0.1189   | 0.1187      | 0.1354 | 0.1198 | 0.1176      |
| 2 | 2 | 0.1694    | 0.1695   | 0.1692      | 0.1987 | 0.1722 | 0.1678      |
| 1 | 3 | 0.1888    | 0.1888   | 0.1884      | 0.2154 | 0.1898 | 0.1868      |
| 3 | 1 | 0.2180    | 0.2184   | 0.2178      | 0.2779 | 0.2197 | 0.2149      |
| 2 | 3 | 0.2475    | 0.2477   | 0.2469      | 0.3029 | 0.2520 | 0.2454      |
| 3 | 2 | 0.2624    | 0.2629   | 0.2619      | 0.3418 | 0.2675 | 0.2603      |
| 1 | 4 | 0.2969    | 0.2969   | 0.2959      | 0.3599 | 0.2980 | 0.2930      |
| 4 | 1 | 0.3319    | 0.3330   | 0.3311      | 0.4773 | 0.3340 | 0.3252      |
| 3 | 3 | 0.3320    | 0.3326   | 0.3310      | 0.4470 | 0.3407 | 0.3304      |
| 2 | 4 | 0.3476    | 0.3479   | 0.3463      | 0.4480 | 0.3534 | 0.3443      |
| 4 | 2 | 0.3707    | 0.3720   | 0.3696      | 0.5415 | 0.3774 | 0.3663      |

Comparison results of the non-dimensional natural frequencies among the theories and methods are tabulated in tables 1-2. The results obtained by the present study using SEM based on the RPT theory are seen to rise excellent values compared with the other methods reported. Through the numerical accuracies, it can be seen that the Wittrick-Williams algorithm used for solving the SEM formulation is suitable and effective, in contrast to the FEM.

SEM can be established for other fundamental theory plate elements to yield high accuracy solution with the least number of element.

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