On a question by Corson about point-finite coverings

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Abstract

We answer in the affirmative the following question raised by H. H. Corson in 1961: "Is it possible to cover every Banach space $X$ by bounded convex sets with nonempty interior in such a way that no point of $X$ belongs to infinitely many of them?"

Actually we show the way to produce in every Banach space $X$ a bounded convex tiling of order 2, i.e. a covering of $X$ by bounded convex closed sets with nonempty interior (tiles) such that the interiors are pairwise disjoint and no point of $X$ belongs to more than two tiles.

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1 Introduction, notation, main statement

Throughout the paper, by covering of a Banach space $X$ we mean a family $\{A_\lambda\}_{\lambda \in \Lambda}$ of proper subsets of $X$ ($\Lambda$ any set of indices) such that $X = \cup_\lambda A_\lambda$. By body in $X$, we mean a nonempty proper subset of $X$ that is contained in the closure of its connected interior. A covering of $X$ is called a tiling of $X$ whenever its members (tiles) are closed bodies with pairwise disjoint interiors. A covering $\tau$ of $X$ is said to be point-finite if no point of $X$ belongs to infinitely many members of $\tau$; the (possibly infinite) order of $\tau$ is the supremum of those $n$ in $\mathbb{N}$ such that there exist $n$ members of $\tau$ with a common point. A covering $\tau$ of $X$ is said to be locally finite if every point in $X$ has a neighborhood that meets only finitely many members of $\tau$ (equivalently, if every compact subset of $X$ meets only finitely many members of $\tau$). It is easy to produce examples of point-finite coverings (even by convex bodies, even tilings) that are not locally finite: for an exhaustive discussion of this topic in the general setting of spaces of any dimension see the nice paper [K1] by V. Klee; see also [N] and [Z].

A great contribution to the study of coverings of infinite-dimensional Banach spaces was given in 1961 by H. H. Corson in his classical paper [C], motivated by topological reasons. The main result of that paper states that, if a Banach space $X$ contains some infinite-dimensional reflexive subspace, then $X$ admits no locally finite covering by bounded convex sets. Such a result has been recently improved in several directions by V. P. Fonf and the second author. In fact in [FZ1] it is proved that, if every compact subset of $X$ meets only finitely many members of some covering of $X$ by bounded $w$-closed subsets, then $X$ is $c_0$-saturated. Moreover, in order that there exist a (algebraically) finite-dimensional compact set that meets infinitely many members of any covering $\tau$ of $X$ by closed convex bounded (in short CCB) sets, it is enough for $X$ to contain an infinite-dimensional separable dual space (see [FZ2]). In particular, in this case even a segment exists that meets infinitely many members of $\tau$ whenever the members of $\tau$ are CCB rotund or smooth bodies (see [FZ3]), or simply CCB bodies if $\tau$ is a tiling and $X$ itself is (infinite-dimensional) reflexive (see [N]).

Despite these results, an interesting question asked by Corson in [C] still remains unanswered. After proving his theorem, he essentially asks whether some infinite-dimensional reflexive space exists that admits a point-finite covering by CCB bodies. Note that, without any assumption on the infinite-dimensional Banach space $X$, even
locally finite tilings by CCB bodies can be exhibited, like the “lattice” tiling of $c_0$
by suitable translates of the unit ball (see [F] for a significant characterization of the
separable Banach spaces admitting such tilings as those being isomorphically poly-
hedral). Moreover, in special non-separable spaces even (not locally finite) tilings of
order 1 can be found by CCB bodies: see the surprising construction given by V.
Klee in [K2] of a tiling of $l_1(\Gamma)$ for suitable $\Gamma$ by pairwise disjoint translates of the
closed unit ball.

The aim of this paper is to answer Corson’s question in the affirmative. In fact
we prove the following

**Theorem**  Every Banach space $X$ admits a tiling of order 2 by closed convex
bounded bodies.

Our proof is obtained by combining in a suitable way two main ideas. The first
one allowed V. P. Fonf, A. Pezzotta and the second author to prove in [FPZ] that
any Banach space can be tiled by CCB bodies. Via the second one, A. H. Stone
constructed in [S] a tiling of order 2 of $\mathbb{R}^n$, $n$ any natural number, by CCB bodies.

Throughout the paper we use standard Geometry of Banach Spaces notation as
in [JLH]. All the Banach spaces under consideration are assumed to be real.

## 2 Proof of the Theorem

For finite-dimensional spaces a construction can be found in [S] (the bodies produced
there are also uniformly bounded), so we can work only in the infinite-dimensional
setting.

We recall that, for a normed space $X$ with norm $\| \cdot \|$ and $0 < \alpha \leq 1$, a set
$M \subset S_{X^*}$ is called $\alpha$–norming if $\sup\{|f(x)| : f \in M\} \geq \alpha |x| \quad \text{for every } x \in X$.
A *norming set* for $X$ is a subset of $S_{X^*}$ which is $\alpha$–norming for some $\alpha$. Moreover,
$\text{norm}(X)$ is the smallest cardinal number $c$ such that there exists a norming set $M$
for $X$ with $|M| = c$.

Finally, $\text{dens}(X)$ is the smallest cardinal number $c$ such that there exists a set
$W \subset X$, with $|W| = c$, that is (strongly) dense in $X$. 
It easy to see that $\text{norm}(X) \leq \text{dens}(X)$ for every Banach space $X$; the equality holds whenever $X$ is weakly compactly generated (see [L], Prop. 2.2).

We split our proof into four steps.

**Step 1** Let us begin producing a special covering $\tilde{\sigma}$ by pairwise disjoint convex bounded bodies of $l_\infty(\Gamma)$, $\Gamma$ any infinite set. Given any family $\{[a_\nu, b_\nu]\}_{\nu \in \Gamma}$ of bounded closed non trivial real intervals indexed on $\Gamma$ with $\inf\{b_\nu - a_\nu : \nu \in \Gamma\} > 0$, we say that the CCB body

$$S = \{t \in l_\infty(\Gamma) : t(\nu) \in [a_\nu, b_\nu], \; \nu \in \Gamma\} \quad (2.1)$$

is a “box” in $l_\infty(\Gamma)$.

Following Stone (see [S]), if in (2.1) exactly for one value of $\nu \in \Gamma$ the closed interval $[a_\nu, b_\nu]$ is replaced by the left-open interval $(a_\nu, b_\nu]$, we say that the corresponding set $S$ is a “lidless box”.

We can assume that the set $\Gamma$ is well-ordered, i.e.

$$\Gamma = \{\nu : 1 \leq \nu < \gamma\}, \; \gamma \text{ a limit ordinal.}$$

Our construction of $\tilde{\sigma}$ is totally inspired by the work that was already done in proving Proposition 1.6 in [FPZ]. Practically, we repeat that construction just replacing boxes by lidless boxes. It is worthwhile to present it again, since it will be used in Step 2 and some more work on it will be done in Step 3. A picture giving an idea of how the construction works can be found in that paper.

Put $A_0 = B_{l_\infty(\Gamma)}$ and, for $\nu \in \Gamma$ and $n = 0, 1, 2, \ldots$

$$A_\nu^{(n)} = \{t \in l_\infty(\Gamma) : |t(\mu)| \leq 2^n \text{ if } \mu < \nu, \; 2^n < t(\nu) \leq 2^{n+1}, \; |t(\mu)| \leq 2^{n+1} \text{ if } \mu > \nu\} \quad (2.2)$$

The collection

$$\tilde{\sigma} = \{A_0, \; \pm A_\nu^{(n)} : \nu \in \Gamma, \; n = 0, 1, 2, \ldots\}$$

covers $l_\infty(\Gamma)$ and its members are pairwise disjoint. In fact, trivially any point in $B_{l_\infty(\Gamma)}$ belongs to no member of $\tilde{\sigma}$ different from $A_0$. Moreover, let $t \in l_\infty(\Gamma)$ with $\|t\| > 1$ and
- in case \( \log_2|t| \) is not an integer, let \( \delta \) the first index in \( \Gamma \) such that \( |t(\delta)| > 2^{\log_2|t|} \);

- in case \( \log_2|t| \) is an integer, let \( \delta \) the first index in \( \Gamma \) such that \( |t(\delta)| > |t|/2 \).

Clearly \( t \) belongs to (\( \text{sgn} t(\delta) \)) \( A^{|\log_2|t(\delta)|}\) (resp. to (\( \text{sgn} t(\delta) \)) \( A^{|\log_2|t(\delta)|-1}\) if \( \log_2|t(\delta)| \) is not an integer (resp. \( \log_2|t(\delta)| \) is an integer) and to no other member of \( \tilde{\sigma} \).

**Step 2** Let us show how the covering \( \tilde{\sigma} \) that we have built in Step 1 plays a crucial role in providing a covering \( \sigma \) by pairwise disjoint convex bounded bodies for any Banach space. We need to recall here under the chief headings what has been done in [FPZ], Sect. 2, where detailed proofs are available.

Let \( X \) be a normed space. For a suitable \( \Gamma \) with \( |\Gamma| = \text{norm}(X) \), we want to construct an isomorphic embedding

\[
T : X \to l_{\infty}(\Gamma)
\]

such that the family

\[
\{T^{-1}(A) : A \in \tilde{\sigma}\}
\]

provides the desired covering \( \sigma \) for \( X \).

Let \( M \) be a norming set for \( X \) with \( |M| = \text{norm}(X) \); passing to the equivalent norm \( ||x|| = \sup\{|f(x)| : f \in M\} \), we may assume that \( M \) is 1–norming. With the aid of Zorn’s lemma, for some ordinal \( \gamma \) with \( |\gamma| \leq \text{dens}(X) \) we construct a totally ordered set of pairs \( \{(x_{\nu}, f_{\nu})\}_{1 \leq \nu < \gamma} \) (which, in some sense, apes a biorthogonal system) such that

1. \( x_{\nu} \in S(X) \) and \( f_{\nu} \in M, 1 \leq \nu < \gamma \);
2. \( |f_{\mu}(x_{\nu})| \leq 1/2, 1 \leq \mu < \nu < \gamma \);
3. \( f_{\nu}(x_{\nu}) \geq 3/4, 1 \leq \nu < \gamma \);
4. the set \( \{f_{\nu}\}_{1 \leq \nu < \gamma} \) is \((1/2)\)-norming for \( X \).

Set \( \Gamma = \{f_{\nu} : 1 \leq \nu < \gamma\} \). Since \( \Gamma \) is a norming set for \( X \), it follows that \( |\Gamma| = |M| = \text{norm}(X) \). The map \( T : X \to l_{\infty}(\Gamma) \) defined as follows

\[
(T(x))(f_{\nu}) = f_{\nu}(x), \ 1 \leq \nu < \gamma, \ x \in X
\]

actually is an isomorphic embedding of \( X \) into \( l_{\infty}(\Gamma) \), since we have \( (1/2)||x|| \leq ||T(x)|| \leq ||x|| \) for every \( x \in X \).
Now, let us consider the covering $\tilde{\sigma}$ of $l_\infty(\Gamma)$ that has been constructed in Step 1. It is obvious that $A_0 \cap T(X)$ has nonempty interior relative to $T(X)$. Moreover, it can be easily seen (see [FPZ]) that, for fixed $1 \leq \nu < \gamma$ and $n = 0, 1, 2, \ldots$, the point

$$z_\nu^{(n)} = (2^{n+1} - 0.4)T(x_\nu)$$

is an interior point of $A_\nu^{(n)} \in \tilde{\sigma}$, so it is an interior point of $A_\nu^{(n)} \cap T(X)$ relative to $T(X)$ too.

Hence, for every $A \in \tilde{\sigma}$, the set $T^{-1}(A)$ is a convex bounded body in $X$, so the family

$$\sigma = \{T^{-1}(A) : A \in \tilde{\sigma}\}$$

provides a covering of $X$ by pairwise disjoint convex bounded bodies.

**Step 3** Following A. H. Stone (see [S]), we now produce a refinement $\tilde{\tau}$ of $\tilde{\sigma}$ of order 2, that turns out to be a tiling of $l_\infty(\Gamma)$ by CCB bodies. To do that, it is enough to express each lidless box $A_\nu^{(n)} \in \tilde{\sigma}$ as a countable union of boxes, in such a way that any point of $A_\nu^{(n)}$ belongs to at most two of them.

For $n, \nu$ fixed, let $\epsilon_\nu^{(n)}$ be a positive number such that

$$\epsilon_\nu^{(n)} < 1, \quad z_\nu^{(n)} + 2\epsilon_\nu^{(n)} B_{T(X)} \subset A_\nu^{(n)}.$$

(2.3)

Let $\{a_\nu^{(n,j)}\}_{j=0}^{\infty}$ a strictly decreasing null sequence of positive numbers such that

$$a_\nu^{(n,0)} = 2^n$$

and

$$z_\nu^{(n)} + \epsilon_\nu^{(n)} B_{T(X)} \subset \{t \in A_\nu^{(n)} : 2^n + a_\nu^{(n,1)} < t(\nu) \leq 2^{n+1}\}.$$ 

For $j = 0, 1, 2, \ldots$ let us set

$$A_\nu^{(n,j)} = \{t \in A_\nu^{(n)} : 2^n + a_\nu^{(n,j+1)} \leq t(\nu) \leq 2^n + a_\nu^{(n,j)}\}.$$ 

(2.4)

Reasoning as in step 1, it is easy to see that the collection of CCB bodies

$$\tilde{\tau} = \{A_0, \pm A_\nu^{(n,j)} : \nu \in \Gamma, \ n, j \in \mathbb{N} \cup \{0\}\}$$

gives the desired tiling of $l_\infty(\Gamma)$ of order 2.
For $\Gamma = \{1, 2\}$, the figure gives an idea of how the construction works.

**Step 4** Finally, we have only to follow the same procedure used in Step 2, just replacing $\tilde{\tau}$ by $\tilde{\tau}$ when defining the isomorphic embedding $T$. It remains only to prove that, for any value of $\nu, n, j$, also the CCB body $A_{\nu}^{(n,j)}$ has nonempty interior relative to $T(X)$. This is clear for $j = 0$. For $j = 1, 2, \ldots$, it is easy to show that the segment $S$ of $T(X)$, having the origin and $z_{\nu}^{(n)}$ as its endpoints, meets the interior of $A_{\nu}^{(n,j)} \cap T(X)$ relative to $T(X)$. In fact set

$$r_{\nu}^{(n,j)} = \frac{1}{2} (a_{\nu}^{(n,j)} - a_{\nu}^{(n,j+1)})$$

and let $z_{\nu}^{(n,j)}$ be the point in $S$ such that

$$z_{\nu}^{(n,j)}(\nu) = 2^n + a_{\nu}^{(n,j+1)} + r_{\nu}^{(n,j)}.$$
From (2.2), (2.3) and (2.4) it easily follows that

$$z^{(n,j)}_\nu + \min\{1, r^{(n,j)}_\nu\} \epsilon^{(n)} \mathcal{B}_T(X) \subset A^{(n,j)}_\nu.$$ 

Hence, for every $A \in \overline{\tau}$, the set $T^{-1}(A)$ is a CCB body in $X$, so the family

$$\tau = \{T^{-1}(A) : A \in \overline{\tau}\}$$

is a tiling of $X$ of order 2.  \(\Box\)

**Remark** Clearly our construction cannot lead anyway to a tiling $\tau$ uniformly bounded from above or from below, i.e. we cannot get that the members of $\tau$ are uniformly bounded or that there exists some positive $r$ such that all of them contain some ball of radius $r$. So, while the choice of coefficients $2^n$ in [FPZ] was suggested by the possibility to get all members of $\tau$ uniformly bounded from below, in our Step 1 it has been made just in order to refer quickly to that paper for those proofs that have been omitted here. It is worthwhile to notice that in [FL] (Prop. 2.8) it is suggested the way to obtain, in any Banach space with the Radon-Nykodym property, a tiling uniformly bounded from above just applying a cutting procedure to each member of a given tiling in a straightforward transfinite way. This can obviously be done on the members of our tiling $\tau$. After that, to each slice that we have obtained, we can apply a new cutting procedure as described in Step 3, just referred to the linear continuous functional we used to produce the slice.

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