The Bianchi Variety

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June 9, 2010

Abstract

The totality $\text{Lie}(V)$ of all Lie algebra structures on a vector space $V$ over a field $\mathbb{F}$ is an algebraic variety over $\mathbb{F}$ on which the group $\text{GL}(V)$ acts naturally. We give an explicit description of $\text{Lie}(V)$ for $\text{dim} V = 3$ which is based on the notion of compatibility of Lie algebra structures.

Keywords: Lie Algebra, Poisson Geometry, Commutative Algebra, Cohomology of Lie Algebras, Moduli Space, Deformation.

MSC Classification: 13D10, 14D99, 17B99, 53D99.

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Introduction

Historical remarks

The problem of classifying 3–dimensional Lie algebras over $\mathbb{R}$ was firstly solved by L. Bianchi at the end of the eighteenth century. Recently, various works concerning classifications of low–dimensional Lie algebras appeared (see, for instance, [6] for a list of 4–dimensional Lie algebras and [7] for a special list of real Lie algebras of dimension $\leq 8$). Now Bianchi classification can be obtained in a more elegant coordinate–free manner. For instance, in [4] this is done on the basis of the invariants of Lie structures, in [2] the co–differential graded calculus is used, in [2] the outer derivations, etc. A shortcoming of the original Bianchi method, as well as of the above–cited works, is that they do not allow a satisfactory description of deformations of 3–dimensional Lie algebras (see [2, 3, 8] and references therein).

It should be especially stressed the recently emerged important role of Poisson geometry in various questions related with Lie algebras and, first of all, classification, representation, deformations, etc. (see [6], [13] and [9]). We shall exploit it throughout the paper.

Aim of the paper

Let $V$ be a vector space over field $\mathbb{F}$ of characteristic different from 2. All Lie algebra structures on $V$ form an algebraic variety denoted by $\text{Lie}(V)$. We call it “the Bianchi variety” if $\dim V = 3$. The aim of this paper is to describe the Bianchi variety in a geometrically transparent manner.

Our approach is based on the notion of compatibility of Lie structures (see, for instance, [13]) and differential calculus over the “manifold” $V^*$ in the spirit of [1]. First we show that all three–dimensional unimodular Lie algebra structures form an algebraic variety $\text{Lie}_0(V)$ which is naturally identified with the space of symmetric bilinear forms on $V^*$. Recall that a Lie algebra is unimodular if operators of its adjoint representation are traceless. Then we show that a generic Lie structure can be obtained by adding a non–unimodular “charge”
to a unimodular structure. This “charge” (see pag. 6) is a particular non-unimodular structure, which reduces the problem to a description of how such a “charge” can be attached to unimodular structures.

The obtained description of Lie (V) allows, besides others, to see directly peculiarities of deformations of 3–dimensionale Lie structures. Also from this point of view the Bianchi classification can be seen as moduli space $\text{Lie}(V)/\text{GL}(V)$.

**Notations and preliminaries**

We shall use the Einstein summation convention, assuming that the index “$i$” in $\frac{\partial}{\partial x_i}$ is treated as an upper one.

By a Lie structure on a vector space $V$ we mean a skew–symmetric $F$–bracket $[\cdot, \cdot]$ on $V$, which fulfills the Jacobi Identity

$$[v, [w, z]] = [[v, w], z] + [w, [v, z]] \quad \forall v, w, z \in V.$$ Fix a basis $\{x_1, x_2, \ldots, x_n\}$ of $V$. This induces a basis $\{\xi_1, \xi_2, \ldots, \xi_n\}$ of $V^*$, a volume $n$–covector $\xi \overset{\text{def}}{=} \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n$, and its dual $\nu$.

An element $c$ of $V \otimes F \wedge^2 (V^*)$ looks as $c = c^k_{ij} x_k \otimes \xi^i \wedge \xi^j$ and defines a Lie algebra structure iff

$$c^k_{aj} c^j_{bc} + c^k_{cj} c^j_{ab} + c^k_{bj} c^j_{ca} = 0, \quad a, b, c, k = 1, 2, \ldots, n. \quad (1)$$ This way Lie (V) is identified with the affine algebraic variety in $V \otimes F \wedge^2 (V^*)$ determined by equations (1), and a Lie structure $c$ identifies with the family of its structure constants $\{c^k_{ij}\}$. Obviously, a natural action of $\text{GL}(V)$ on $V \otimes F \wedge^2 (V^*)$ leaves Lie (V) invariant, and defines an action of $\text{GL}(V)$ on Lie (V).

If dim $V = 3$, we introduce the basis $\{\xi^h\}_{h=1, 2, 3}$ of $\wedge^2 (V^*)$, $\xi^h \overset{\text{def}}{=} c^h_{ij} \xi^i \wedge \xi^j$, where $c^h_{ij}$ is purely skew–symmetric symbol. Then an element $c$ of $V \otimes F \wedge^2 (V^*)$ looks as $c = c^h_{ij} x_h \otimes \xi^h$, where $c^h_{ij} = c^h_{ij} c^h_{ij}$, and (1) becomes

$$\sum_i c^h_{mi} c^m_{ij} c^k_h = 0, \quad k = 1, 2, 3. \quad (2)$$

1 **Differential Calculus over algebra $S(V)$**

In this section elements of differential calculus over $V^*$ are sketched, in the the spirit of differential calculus over commutative algebras (see [1]). Below $V$ stands for an $F$–vector space, $n = \text{dim } V$, and $S(V) = \bigoplus S_i(V)$, where $S_i(V)$ is the $i$–th symmetric power of $V$. The algebra $S(V)$ is naturally interpreted as the algebra of polynomials on $V^*$, whose $F$–spectrum identifies with $V^*$. Consequently, the necessary elements of differential calculus on the “manifold” $V^*$ are interpreted as those over commutative algebra $S(V)$. 

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Denote by $D(V^*)$ the $S(V)$–module of *derivations* of the algebra $S(V)$, which we interpret as vector fields on $V^*$. Then, obviously, the map

$$D(V^*) \longrightarrow S(V) \otimes_F V^*$$

$$X \mapsto X|_{V^*}$$

$$X_\theta = a_{i_1,\ldots,i_n,i} x_1^{i_1} \cdots x_n^{i_n} \frac{\partial}{\partial x_i} \leftrightarrow a_{i_1,\ldots,i_n,i} x_1^{i_1} \cdots x_n^{i_n} \otimes \xi^i = \theta$$

where $\frac{\partial}{\partial x_i}(v_1 v_2 \cdots v_m) \overset{\text{def}}{=} \sum_{i=1}^{n} \xi^i(v) v_1 \cdots v_{i-1} v_i \cdots v_m$, is a $S(V)$–module isomorphism. Put

$$D_*(V^*) \overset{\text{def}}{=} \bigoplus_i D_i(V^*),$$

where $D_i(V^*)$ is the $S(V)$–module of *skew–symmetric multi–derivations* of the algebra $S(V)$, which we interpret as $i$–vector fields on $V^*$. Then a similar isomorphism between $D_*(V^*)$ and $S(V) \otimes_F \bigwedge V^*$ holds. In particular, $c = c_{i,j}^k x_k \otimes \xi^i \wedge \xi^j$ corresponds to the bi–vector field

$$P^c \overset{\text{def}}{=} c_{i,j}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$  

If $n = 3$ and $c = c_{i,j}^k x_k \otimes \xi^h$, (5) reads

$$P^c \overset{\text{def}}{=} c_{i,j}^k c_{h,j}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$  

The algebra $S(V) \otimes_F \bigwedge V^*$ is $\mathbb{Z}_2$–graded. For example, linear vector fields are exactly elements of bidegree $(1, 1)$. We emphasize that accordingly to (3) linear vector fields correspond to endomorphisms of vector space $V$,

$$X_\phi \overset{\text{def}}{=} \varphi^j x_j \frac{\partial}{\partial x_i},$$

where, by definition, $X_\phi(v) = \varphi(v)$, $v \in V$.

The *Liouville vector field* on $V^*$

$$X_{id} = x^i \frac{\partial}{\partial x^i}$$

plays a special role, and is denoted by $\Delta$. A bi–vector is called *linear* when its bidegree is $(1, 2)$, *quadratic* if it is $(2, 2)$, etc. These definitions extend straightforwardly to all tensor fields over $V^*$.

Similarly,

$$\bigwedge^*(V^*) \overset{\text{def}}{=} \bigoplus_i \bigwedge^i(V^*),$$

is the $S(V)$–module of *polynomial differential forms* on $V^*$. Here the $S(V)$–module $\bigwedge^i(V^*)$ of $i$–th order differential forms on $V^*$ is identified with the $i$–th skew–symmetric power $\bigwedge^i(S(V) \otimes_F V)$ of the $S(V)$–module $S(V) \otimes_F V$, which
Hamiltonian vector field corresponding to the Hamiltonian function

\[ \text{D} \text{is a differential in} \]

is referred to as the symplectic foliation \( P \) will be denoted by \( \text{Poisson bi–vector} \).

Elements of \( S(V) \otimes F \wedge^*(V^*) \) will be called linear. A linear 1–form \( \omega \) is closed iff the matrix \( q = \|q^{ij}\| \) is symmetric.

A bivector \( P \in S(V) \otimes F \wedge^2(V^*) \) is called Poisson if \([P, P] = 0\). The following fundamental correspondence, for the first time established by S. Lie, is the starting point of the paper.

**Proposition 1.** There is a one–to–one correspondence between Lie algebra structures on \( V \) and linear Poisson bivectors on \( V^* \). Namely,

\[ c \equiv \{c^k_{ij}\} \leftrightarrow P_c = c^k_{ij} x^i \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k}. \] (10)

\( P_c \) given by \( [5] \) is called the Poisson bi–vector associated with \( c \), and the corresponding to it bracket is referred to as the Lie–Poission bracket on \( S(V) \) (see \( [3] \)).

Recall (see \( [2] \)) that the map

\[ d_P \overset{\text{def}}{=} [P, \cdot] : D_*(V^*) \longrightarrow D_*(V^*), \quad P \in D_2(V^*) \] (11)

is a differential in \( D(V^*) \), i.e., \( d_P^2 = 0 \), iff \( P \) is a Poisson bivector. Moreover we have (see \( [2] \))

**Proposition 2.** There exists an unique homomorphism \( \Gamma_P : D_*(V^*) \longrightarrow \Lambda^*(V^*) \) of \( S(V) \)–algebras which is a cochain map from \( (D_*(V^*), d_P) \) to \( (\Lambda^*(V^*), d) \).

1–cocycles (resp., 1–coboundaries) of \( d_P \) are called canonical (resp., Hamiltonian) vector fields on \( V^* \) (with respect to the Poisson structure \( P \) on \( V^* \)). The Hamiltonian vector field corresponding to the Hamiltonian function \( f \in S(V) \) will be denoted by \( P_f \), i.e., \( P_f = d_P(f) \). It is easy to see that \( P_f = -i_{df}(P) \) (the contraction of \( df \) and \( P \)).

When \( P = P_c \), the corresponding to the Hamiltonian vector fields foliation is referred to as the symplectic foliation determined by \( c \).
From now on we shall assume that \( \dim V = 3 \). The volume form \( v = dx^1 \wedge dx^2 \wedge dx^3 \) determines a standard duality between \( i \)-vector fields and \( (3-i) \)-differential forms. The linear bi–vector \( P^c \) defined by (6) is dual to the linear 1–form

\[
\alpha_c = \sum_h c_h^k x_k dx_h,
\]
i.e., \( P^c(f, g)v = df \wedge dg \wedge \alpha_c, f, g \in S(V) \).

We have (see\[13\])

**Lemma 1.** \( P \in D_2(V^*) \) is Poisson iff \( \alpha \wedge d\alpha = 0 \) for the dual to \( P \) 1–form \( \alpha \).

Denote by \( q_c \) the bilinear form on \( V^* \) corresponding to \( \alpha_c \) in (9).

**Corollary 1.** If \( q_c \) is either symmetric, or skew–symmetric, then \( c \) is a a Lie structure on \( V \).

**Proof.** Directly from Lemma \[1\] and Proposition \[1\] \( \square \)

Hence \( \text{Lie} (V) \) can be identified with a subset in the space of linear differential 1–forms. As such, it contains the subspace \( \text{Lie}_0(V) \) of differential forms which correspond to symmetric bilinear forms on \( V \) in (9), and the subspace \( N \) of those which correspond to skew–symmetric differential forms. Recall that a structure \( c \) is unimodular if and only if \( \alpha_c \) is symmetric (see\[13\]). Accordingly, elements of \( \text{Lie}_0(V) \) (resp., \( N \)) are called unimodular (resp., purely non–unimodular).

Since a bilinear form splits into the sum of a symmetric and a skew–symmetric part, a Lie structure \( c \) on \( V \) can be “disassembled” into the sum of an unimodular component with a purely non–unimodular one,

\[
\alpha_c = dF + \alpha, \quad dF \in \text{Lie}_0(V), \alpha \in N.
\]

In terms of Lie structures, (12) reads \( c = cf + c_\alpha \), where \( cf \) (resp., \( c_\alpha \)) is the Lie structure corresponding to \( dF \) (resp., \( \alpha \)), and in terms of brackets,

\[
[v, w] = [v, w]_0 + [v, w]_1, \quad v, w \in V
\]

where \([\cdot, \cdot]\) (resp., \([\cdot, \cdot]_0, [\cdot, \cdot]_1\)) is the Lie bracket on \( V \) corresponding to \( c \) (resp., \( cf, c_\alpha \)).

Recall the following

**Definition 1.** Elements \( c_1, c_2 \in \text{Lie} (V) \) are said to be compatible if \( c_1 + c_2 \in \text{Lie} (V) \).

So, the unimodular part \( cf \) of \( c \) and its purely non–unimodular part \( c_\alpha \) are compatible.

Disassembling (12) can also be read as \( \alpha_c = \pi_0(\alpha_c) + \alpha \), where

\[
V \otimes F \bigwedge^2(V^*) \xrightarrow{\pi_0} \text{Lie}_0(V)
\]

is the canonical projection of bilinear forms onto symmetric ones.
Remark 1. The possibility to identify Lie structures as a bilinear forms is a peculiarity of the three–dimensional case only.

The compatibility condition of two Lie structures are, obviously, expressed in terms of their unimodular and purely non–unimodular part as follows.

**Lemma 2.** Lie structures \( c_dF + \alpha \) and \( c_dG + \beta \) are compatible if and only if

\[
[c_F, c_\beta] + [c_G, c_\alpha] = 0, \tag{14}
\]

or, equivalently,

\[
dF \wedge \beta + dG \wedge \alpha = 0. \tag{15}
\]

The following fact is obvious as well.

**Lemma 3.** Let \( P \) and \( Q \) be commuting bi–vectors, and \( v, w \in V \), Then \( [vP, wQ] = vP(w) \wedge Q - wQ(v) \wedge P \).

**2 Finite and infinitesimal GL (V)–actions**

Fix an automorphism \( \psi \in \text{GL} (V) \). The adjoint to \( \psi \) map is a diffeomorphism of \( V^* \), which we still denote by \( \psi \). Indeed, \( \psi \), the diffeomorphism, corresponds (in the sense of [1]) to the algebra automorphism of \( S(V) \) whose restriction to \( V \) coincides with \( \psi \), the automorphism.

Then the action of \( \psi \) is naturally prolonged to differential forms and multi–vector fields on \( V^* \), and, in view of isomorphisms (3) and (9), to the algebras \( S(V) \otimes_F \Lambda^*(V^*) \) and \( S(V) \otimes_F \Lambda^*(V) \), respectively. We keep the same symbol \( \psi \) for the prolonged automorphism, except for differential forms, when the pull–back \( \psi^* \) is used.

An easy consequence of Lemma 1 is that the action of \( \text{GL} (V) \) on linear differential 1–forms restricts to \( \text{Lie} (V) \). In terms of Lie brackets this action reads

\[
[v, w]' \overset{\text{def}}{=} \psi^{-1}([\psi(v), \psi(w)]), \quad v, w \in V,
\]

where \([ \cdot, \cdot \] \) (resp., \([ \cdot, \cdot, \cdot \]' \)) corresponds to \( \alpha_c \), (resp., \( \psi^* (\alpha_c) \)). It is straightforward to verify that \( P^{\psi(c)} = \psi(P^c) \).

**Remark 2.** The identification \( \alpha_c \leftrightarrow P^c \) of linear 1–forms with linear bi–vector does not commute with actions of \( \text{GL} (V) \) on them. Namely, we have

\[
\psi^* (\alpha_c) \cdot \det \psi = \alpha_{\psi(c)}, \quad \psi \in \text{GL} (V).
\]

Denote by \( \text{Stab} (c) \overset{\text{def}}{=} \{ \psi \in \text{GL} (V) \mid \psi(c) = c \} \subseteq \text{GL} (V) \) the stabilizer of \( c \).

An endomorphism \( \varphi \in \text{End} (V) \), i.e., a linear vector field on \( V^* \) (see [7]), can be interpreted as an infinitesimal automorphism and, as such, it acts on tensor fields on \( V^* \) by Lie derivation. On the other hand, the differential \( d_{P^c} \) (see (11)) acts on \( \varphi \) and produces \( d_{P^c} (\varphi) \). It is easy to verify that \( L_{X_c} (\alpha_c) = \alpha_{d_{P^c} (\varphi)} \).

The infinitesimal counterpart of the stabilizer is the symmetry Lie sub–algebra

\[
\text{sym} (c) \overset{\text{def}}{=} \{ \varphi \in \text{End} (V) \mid L_{X_c} (\varphi) = 0 \} \subseteq \text{End} (V).
\]
3 The Canonical Disassembling of a 3–Dimensional Lie Structure

Firstly observe that GL (V) preserves the fibers of the projection $\pi_0$ of $V \otimes \mathbb{F} V$ over $\text{Lie}_0(V)$.

Put $Z^2_N(dF) \overset{\text{def}}{=} Z^2(c_F) \cap N$. The following assertion is a direct consequence of the above definitions.

Proposition 3. In the above notation the following conditions are equivalent:

- $dF + \alpha$ corresponds to a Lie structure,
- $c_F$ and $c_\alpha$ are compatible,
- $[c_F, c_\alpha] = 0$,
- $dF \wedge \alpha = 0$,
- $\alpha \in Z^2_N(dF)$.

An easy consequence of Proposition is the following

Lemma 4. $Z^2_N(dF) = \zeta^{-1}(c_F)$, with $\zeta \overset{\text{def}}{=} (\pi_0)|\text{Lie}_0(V)$.

Note that the map $\zeta$ is not of constant–rank. Namely, the dimension of $\zeta^{-1}(c_F)$ depends on the rank of the polynomial $F$. It should be stressed that $\zeta^{-1}(c_F)$ is naturally interpreted as a variety of purely non–unimodular structures compatible with $dF$. We shall show that its dimension equals $3 – \text{rank} (dF)$. 

Remark 3. Notice that $d_{P^c}(\varphi)$ is a linear bi–vector field on $V^*$, but not necessarily a Poisson one.
To this end, we compute the Schouten brackets between the basis elements \( \{ x_i dx_j \}_{i,j=1,2,3} \) of \( \text{Lie}_0(V) \) and the purely non–unimodular Lie structures \( \alpha_i \overset{\text{def}}{=} \varepsilon_i \varepsilon^i z_i x_i dx_{i_2}, \) i.e., the basis elements of \( N. \)

We usually write \( \frac{1}{2} d(x_i^2) \) instead of \( x_i dx_i, \) \( i = 1, 2, 3. \)

**Proposition 4.**

\[
[c_1 \frac{1}{2} d(x_i^2), c_{\alpha_j}] = \begin{cases} 0 & \text{if } j \neq i, \\ 2 x_j \xi & \text{otherwise;} \\ 2 x_i \xi & \text{if } j = i_2, \\ 2 x_i \xi & \text{if } j = i_1, \\ \end{cases}
\]

\[
[c_\alpha dx_i, c_\alpha] = \begin{cases} 0 & \text{if } j \neq i_1, i_2, \\ 2 x_i \xi & \text{if } j = i_2, \\ 2 x_i \xi & \text{if } j = i_1. \\ \end{cases}
\]

**Proof.** From \( d\alpha_j = \varepsilon_j \varepsilon^{i_2} z_i dx_i, dx_{i_3} \) it follows that \( dx_i^2 \wedge \alpha_j = 0 \) when \( j \neq i \) and \( dx_i, dx_{i_3} \wedge \alpha_j = 0 \) when \( j \neq i_1, i_2. \) Then, in view of Corollary \( \[3\] \) this gives the result for \( i \neq j \) and for \( i \neq i_1, i_2. \)

Next, by using Lemma \( \[3\] \) we have

\[
[c_{x_1 dx_1}, c_{x_2 dx_3 - x_3 dx_2}] = \left[ x_1 \xi^2 \wedge \xi^3, x_2 \xi^1 \wedge \xi^2 - x_3 \xi^3 \wedge \xi^1 \right] = x_1 \xi^3 \wedge \xi^1 \wedge \xi^2 + x_1 \xi^2 \wedge \xi^3 \wedge \xi^1 = 2 x_1 \xi.
\]

Similarly one computes the remaining commutators. \( \Box \)

**Lemma 5.** \( \text{codim} Z^2_N(dF) = \text{rank} (dF). \)

**Proof.** Let \( F = \frac{1}{2} (\lambda x_1^2 + \mu x_2^2 + \nu x_3^2) \) and \( \alpha = a \alpha_1 + b \alpha_2 + c \alpha_3. \) Then

\[
[c_F, c_\alpha] = (2 \lambda a x_1 + 2 \mu b x_2 + 2 \nu c x_3) \xi
\]

is zero if and only if the \( F \)-valued vector \((\lambda a, \mu b, \nu c)\) vanishes. \( \Box \)

Figure \( \[1\] \) visualizes Lemma \( \[5\] \) The four “vertical” linear spaces, crossing the “horizontal” plane \( \text{Lie}_0(V) \), represent the \( \zeta \)-fibers attached to the rank–0 Lie structure (blue point), to a rank–1 structure (green point), to a rank–2 structure (purple point), and to a non–degenerate structure (red point).

The disassembling property of Lie structures leads to a natural factorization of the action of \( \text{GL}(V) \) on \( \text{Lie}(V) \). Namely, \( \text{GL}(V) \) preserves \( \zeta \). In view of that, the study of the moduli space \( \text{Lie}_0(V)/\text{GL}(V) \) naturally splits into two steps. The first of them is to describe the moduli space of the symmetric bilinear forms (which is well–known for some fields \( F \)), while the second is to describe the moduli space \( Z^2_N(dF)/\text{Stab}(dF). \)

To this end consider the subvariety \( \Sigma \overset{\text{def}}{=} \{(dF, \psi) \mid \psi \in \text{Stab}(dF)\} \subseteq \text{Lie}_0(V) \times \text{GL}(V) \) and its natural projection \( \sigma : \Sigma \rightarrow \text{Lie}_0(V), (dF, \psi) \mapsto dF. \)

Now fix an orbit \( \Omega \overset{\text{def}}{=} \text{GL}(V) \cdot dF \) of the \( \text{GL}(V) \)-action on \( \text{Lie}_0(V) \) (see Remark \( \[2\] \). Lemma \( \[5\] \) tells precisely that \( \zeta|_{\Omega} \) is a \( (3–\text{rank } dF) \)-dimensional vector bundle over \( \Omega \).

Observe that \( \sigma|_{\Omega} \) is a principal group bundle over \( \Omega, \) acting on \( \zeta|_{\Omega}. \)

**Lemma 6.** The quotient bundle \( \frac{\Sigma|_{\Omega}}{\sigma|_{\Omega}} \) is endowed with an absolute parallelism and, therefore, it is trivial.
Proof. Take $dF, dG \in \Omega$, and choose $\varphi \in \text{GL}(V)$ such that $dG = \varphi^*(dF)$. Define parallel displacement $t : \left(\frac{\text{Lie}(V)}{\text{GL}(V)}\right)^{-1} (dF) \rightarrow \left(\frac{\text{Lie}(V)}{\text{GL}(V)}\right)^{-1} (dG)$,

$$t(\text{Stab}(dF) \cdot (dF + \alpha)) \overset{\text{def}}{=} \text{Stab}(dG) \cdot (dG + \varphi^*(\alpha)),$$

and prove that (16) does not depend on the choice of $\alpha$ and $\varphi$.

If $\alpha'$ is another choice of the non–unimodular charge of the orbit of $dF + \alpha$, then $\alpha' = \phi^*(\alpha)$, with $\phi \in \text{Stab}(dF)$. So, $\varphi^{-1} \phi \varphi \in \text{Stab}(dG)$ implies that $\text{Stab}(dG) \cdot (dG + \varphi^*(\alpha)) = \text{Stab}(dG) \cdot (dG + (\varphi^{-1} \phi \varphi)^*(\varphi^*(\alpha))) = \text{Stab}(dG) \cdot (dG + \varphi^*(\varphi^*(\alpha)))$.

If $\varphi$ is another transformation such that $dG = \varphi^*(dF)$, then $\varphi^{-1} \varphi \in \text{Stab}(dG)$. Hence, $\text{Stab}(dG) \cdot (dG + \varphi^*(\alpha)) = \text{Stab}(dG) \cdot (dG + (\varphi^{-1} \varphi)^*(\varphi^*(\alpha))) = \text{Stab}(dG) \cdot (dG + \varphi^*(\alpha))$.

Let $c = c_F + c_\alpha$ be a Lie structure. The orbit $\text{GL}(V) \cdot \alpha_c$ of $\alpha_c$ is precisely the only parallel section of $\frac{\text{Lie}(V)}{\text{GL}(V)}$ which takes the value $\text{Stab}(dF) \cdot \alpha$ at the point $dF$. In other words, we have proved the main

**Theorem 1.** The orbit space $\frac{\text{Lie}(V)}{\text{GL}(V)}$ is fibered over the orbit space $\frac{S^2(V)}{\text{GL}(V)}$, the fiber at $\Omega$ being given by the set of parallel sections of $\frac{\text{Lie}(V)}{\text{GL}(V)}$. So, we have the following algorithm for describing orbits of Lie structures:

1. find the orbits of the action of $\text{GL}(V)$ on $\text{Lie}_0(V)$;

2. find the parallel sections of $\frac{\text{Lie}(V)}{\text{GL}(V)}$, for any orbit $\Omega$ coming from the first step.
The evident advantage of this procedure is that the fibers of $\zeta$ and $\sigma$ are much smaller than Lie ($V$) and GL ($V$), respectively. Moreover, as we shall see, the second step does not depend on the field $F$.

**Remark 4.** Even in the case when the orbit space $S^2(V)_{GL(V)}$ is not known, elements of Lie$_0(V)$ are distinguished by their ranks (see [5]). Degenerate forms fill up a cubic hypersurface (purple curve in Fig. 1), which in its turn contains a closed subset of rank-one forms (green points in Fig. 1).

Let $c = c_F + c_\alpha$ be a Lie structure, and $\Omega$ the orbit of $dF$ in Lie$_0(V)$.

**Lemma 7.** $\zeta|_{GL(V)\cdot \alpha_c}$ is a bundle over $\Omega$ with the fiber $\text{Stab}(dF)\cdot \alpha$.

**Proof.** Since GL ($V$) acts as a bundle automorphism on $\zeta|_{GL(V)\cdot \alpha_c}$, it suffices to compute the fiber $\zeta|^{-1}_{GL(V)\cdot \alpha_c}(dF)$. An element $c' = c_F + c_\alpha'$ is in such a fiber if and only if $dF + \alpha' \in \text{GL}(V)\cdot \alpha_c$, i.e., $\alpha' = \psi^*(\alpha)$, with $\psi \in \text{Stab}(dF)$.

**Corollary 2.** $\dim\text{GL}(V)\cdot \alpha_c = \dim\text{GL}(V)\cdot dF + \dim\text{Stab}(dF)\cdot \alpha$.

This corollary suggests a formula for computing $\dim B^2(c)$,

$$
\dim B^2(c) = \dim B^2(c_F) + \dim\left(\frac{\text{Stab}(dF)}{\text{Stab}(dF) \cap \text{Stab}(\alpha)}\right),
$$

whose “infinitesimal version” is

$$
\dim\left(\frac{\text{End}(V)}{\text{sym}(dF + \alpha)}\right) = \dim\left(\frac{\text{End}(V)}{\text{sym}(dF)}\right) + \dim\left(\frac{\text{sym}(dF)}{\text{sym}(dF) \cap \text{sym}(\alpha)}\right). \quad (17)
$$

**Remark 5.** Notice that $\text{sym}(dF + \alpha) = \text{sym}(dF) \cap \text{sym}(\alpha)$.

## 4 Computations

### 4.1 Unimodular structures

In the case $\alpha = 0$ Lemma 7 says that the orbit of $\alpha_c$ coincides with $\Omega$. In view of (17), in order to find its dimension, it is sufficient to compute $\dim[\text{sym}(dF)]$ (Proposition 5).

**Proposition 5.**

$$
\dim B^2(dF) = \begin{cases} 
6 & \text{if rank } dF = 3 \\
5 & \text{if rank } dF = 2 \\
3 & \text{if rank } dF = 1 
\end{cases}
$$

**Proof.** We shall show that

$$
\dim[\text{sym}(dF)] = \begin{cases} 
3 & \text{if rank } dF = 3 \\
4 & \text{if rank } dF = 2 \\
6 & \text{if rank } dF = 1
\end{cases}
$$
To this end, prove that \( \varphi \in \text{sym} \left( \frac{1}{2} dx_1^2 + \lambda x_2^2 + \mu x_3^2 \right) \) if and only if
\[
X_{\varphi} = (-\lambda ax_2 - \mu bx_3) \frac{\partial}{\partial x_1} + (ax_1 + cx_2 + ex_3) \frac{\partial}{\partial x_2} + (bx_1 + fx_2 + dx_3) \frac{\partial}{\partial x_3},
\]
with the coefficients \( a, \ldots, f \) satisfying conditions
\[
\begin{array}{l}
\lambda c = 0 \\
\mu d = 0 \\
\lambda e + \mu f = 0.
\end{array}
\]
Indeed, since \( L_{X_\varphi} \left( \frac{1}{2} dx_1^2 + \lambda x_2^2 + \mu x_3^2 \right) = \frac{1}{2} d((\varphi^1_i x_j \frac{\partial}{\partial x_j})(x_k^1)) = d(\varphi^1_i x_j \delta^k_i x_k) = \varphi^1_i dx_i x_k \), the Lie derivative
\[
L_{X_\varphi} \left( \frac{1}{2} dx_1^2 + \lambda x_2^2 + \mu x_3^2 \right) = \varphi^1_1 dx_1 x_1 + \lambda \varphi^2_2 dx_2 x_2 + \mu \varphi^3_3 dx_3 x_3
\]
vanishes if and only if \( X_{\varphi} \) can be put in the form (18), with coefficients satisfying (19).

In the left side of Figure 1, the spaces \( B^2(dF) \), whose dimension was computed in Proposition 5, are drawn as tangent spaces to \( \text{Lie}_0(V) \).

**Proposition 6.** \( \dim Z^2(cF) = 9 - \text{rank } dF \).

**Proof.** Observe that \( Z^2(cF) = \text{Lie}_0(V) \oplus Z^2_N(dF) \) and apply Lemma 5.

Figure 1 makes evident Proposition 6. Indeed, \( Z^2(cF) \) is precisely the space spanned by the “horizontal” subspace \( \text{Lie}_0(V) \) and the “vertical” subspaces \( Z^2_N(dF) \).

The above results concerning the orbits of unimodular structures are summarized in the next table for \( F = \mathbb{R} \).

| Type | Bianchi type(s) | Lie structure(s) | rank \( dF \) | \( \dim \text{GL}(V) \cdot dF \) | \( \dim Z^2_N(dF) \) | \( \dim Z^2(cF) \) | \( \dim H^2(cF) \) |
|------|----------------|-----------------|-------------|-----------------|----------------|----------------|----------------|
| \( A_0 \) | AI | Abelian | 0 | 0 | 3 | 9 | 9 |
| \( A_1 \) | All | Heisenberg | 1 | 3 | 2 | 8 | 5 |
| \( A_2^-, A_2^+ \) | AVIa, AVIIa | \( \epsilon(1, 1), \epsilon(2) \) | 2 | 5 | 1 | 7 | 2 |
| \( A_3^-_1, A_3^+ \) | AVIII, AIX | \( \sigma(2, 1), \sigma(3) \) | 3 | 6 | 0 | 6 | 0 |

**Remark 6.** In this table we introduce a new notation for isomorphism classes of three–dimensional Lie algebras, hoping it will be more informative. The original Bianchi notation can be found in [10].

### 4.2 Non–unimodular structures

#### 4.2.1 rank \( dF = 0 \).

Then \( dF = 0, \Omega = \text{GL}(V) \cdot dF = \{0\}, Z^2_N(0) = N \) and \( \text{Stab}(cF) = \text{GL}(V) \). In other words, \( \frac{\tilde{\Omega}_{cF}}{\sigma_{cF}} \) consists of just one fiber, which identifies with
\[
\frac{N}{\text{GL}(V)}.
\]

\[ (20) \]
Independently on the field $F$, it can be easily proved (see [13]) the following

**Proposition 7.** The moduli space (20) consists of two orbits, one of which is $0$.

**Proposition 8.** $\varphi \in \text{sym}(\alpha_3)$ if and only if

$$X_\varphi = (ax_1 + bx_2) \frac{\partial}{\partial x_1} + (cx_1 - ax_2) \frac{\partial}{\partial x_2} + (dx_1 + ex_2 + fx_3) \frac{\partial}{\partial x_3}. \quad (21)$$

**Proof.** It directly follows from

$$L_{X_\varphi}(\alpha_3) = (\varphi_j^i x_j \frac{\partial}{\partial x_i}) (x_1 dx_2 - x_2 dx_1) =$$

$$= \varphi_j^1 x_j \delta_1^1 dx_2 + x_1 d(\varphi_j^1 x_j \delta_2^1) - \varphi_j^2 x_j \delta_2^1 dx_1 - x_2 d(\varphi_j^2 x_j \delta_1^1) =$$

$$= \varphi_1^1 (x_j dx_2 - x_2 dx_j) + \varphi_2^1 (x_1 dx_j - x_j dx_1) = -\varphi_1^3 \alpha_1 - \varphi_2^3 \alpha_2 + (\varphi_1^1 + \varphi_2^2) \alpha_3. \quad \square$$

The “vertical” blue subspace in Figure 1 is $N$. The 3–dimensional space $B^2(\alpha_3)$ is shown inside $N$.

Notice that when $F = \mathbb{R}$ or $\mathbb{C}$, Proposition 8 is sufficient to prove that the orbit of $\alpha_3$ is 3–dimensional and, therefore, it coincides with $N \setminus \{0\}$.

### 4.2.2 rank $dF = 1$.

Independently on the field $F$, all rank–1 elements of $\text{Lie}_0(V)$ belong to the same orbit $\Omega = \text{GL}(V) \cdot \frac{1}{2} d(x_1^2)$. To compute the fiber of $\text{Lie}_0(V) \cdot \frac{1}{2} d(x_1^2)$ over $\Omega$, it suffices to compute the moduli space $Z_N^2(\frac{1}{2} d(x_1^2)) \quad (22)$ (see Theorem 1).

Observe that $Z_N^2(\frac{1}{2} d(x_1^2))$ is the 2–dimensional vector space spanned by $\alpha_2$ and $\alpha_3$ (see the proof of Lemma 5). Fix a non–zero element $a \alpha_2 + b \alpha_3$. Then it is possible to choose an automorphism $\psi \in \text{GL}(V)$ which preserves $x_1$ and sends $bx_2 - ax_3$ to $x_2$. In other words, $\psi \in \text{Stab}(\frac{1}{2} d(x_1^2))$ and $\psi^*(a \alpha_2 + b \alpha_3) = \alpha_3$, thus proving the following

**Proposition 9.** The moduli space (22) consists of two orbits, one of which is $0$.

### 4.2.3 rank $dF = 2$.

The orbits of rank–2 structures in $\text{Lie}_0(V)$ are $\Omega = \text{GL}(V) \cdot dF$, $F = \frac{1}{2}(x_1^2 + cx_2^2)$, with $c \in F$ (see [5]). Recall that $Z_N^1(\epsilon dF)$ is the 1–dimensional subspace spanned by $\alpha_3$ (see the proof of Lemma 5).

We shall show that the fiber over $\Omega$ is $\mathbb{F}$.
Proposition 10. Let $\mathbb{F}$ be $\mathbb{R}$ (resp., $\mathbb{C}$). Then the moduli space

$$\frac{Z^2_N(dF)}{(\text{Stab } dF)}, \quad F = \frac{1}{2}(x_1^2 + cx_2^2), \epsilon = \pm 1 \text{ (resp. 1)}$$

coincides with $\langle \alpha_3 \rangle$.

Proof. Notice that the stabilizer in $\text{Stab } (dF)$ of an element $\lambda \alpha_3 \in Z^2_N(dF)$ coincides with $\text{Stab } (\lambda \alpha_3) \cap \text{Stab } (dF)$. To prove the result, it suffices to show that $\text{Stab } (dF)$ is contained in $\text{Stab } (\lambda \alpha_3)$.

This is obvious for $\lambda = 0$. For $\lambda \neq 0$ we, first, observe that $\text{Stab } (\lambda \alpha_3) = \text{Stab } (\alpha_3)$. Then, it follows from Propositions 5 and 8 that a symmetry of $dF$ is also a symmetry of $\alpha_3$. \hfill $\square$

The proof of the above proposition is simplified by infinitesimal arguments, which does not work if $\mathbb{F}$ is different from $\mathbb{R}$ or $\mathbb{C}$. For a generic $\mathbb{F}$ see [13].

4.2.4 Cocycles of non-unimodular Lie structures

Lemma 8. If $c$ is a non-unimodular Lie structure, then $\dim Z^2(c) = 6$.

Proof. Any non-unimodular Lie structure is equivalent to $c_1^\frac{1}{2}(\lambda x_1^2 + \mu x_2^2) + c_\alpha$. Let $dF = d\left[\frac{1}{2}(ax_1^2 + bx_2^2 + cx_3^2) + ex_2x_3 + fx_1x_3 + gx_1x_2\right]$ (resp., $\alpha = ka_1 + la_2 + ma_3$) be an arbitrary element of $\text{Lie}_0(V)$ (resp., $N$). Then, independently on $\lambda$ and $\mu$, the commutator

$$\left[c_1^\frac{1}{2}(\lambda x_1^2 + \mu x_2^2) + c_\alpha, cF + c_\alpha\right] = \left[c_1^\frac{1}{2}(\lambda x_1^2 + \mu x_2^2), c_\alpha\right]$$

vanishes if and only if the three equations $f + k\lambda = 0$, $e + l\mu = 0$ and $c = 0$ are satisfied. \hfill $\square$

The obtained results are summarized in the following table, where $c = c_F + c_\alpha$.

| Type    | Bianchi type(s) | rank $dF$ | $\dim B^2(c)$ | $\dim Z^2(c)$ | $\dim H^2(c)$ |
|---------|-----------------|-----------|---------------|---------------|---------------|
| $B_0$   | $V$             | 0         | 3             | 6             | 3             |
| $B_1$   | IV              | 1         | 5             | 6             | 1             |
| $B^1_{2,h}$ | III, VI, VII     | 2         | 5             | 6             | 1             |

5 Compatibility varieties

Let $c \in \text{Lie } (V)$. 

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Figure 2: Only when $c$ is an unimodular structure of type $A_3^\pm$, $\text{Lie}(V, c)$ is a linear space.

**Definition 2.** The affine algebraic variety $\text{Lie}(V, c) \overset{\text{def}}{=} \text{Lie}(V) \cap \mathbb{Z}^2(c) \subseteq \mathbb{Z}^2(c)$ is called the compatibility variety of $c$.

Obviously, $\text{Lie}(V, c)$ can be understood as the set of Lie structures which are compatible with $c$, or as the union of all linear subspace of $\text{Lie}(V)$ passing through $c$. So, $\text{Lie}(V, c)$ is a conic variety.

The canonical disassembling of $\text{Lie}(V)$ and other results of Section 3 are reproduced as well for the compatibility variety $\text{Lie}(V, c)$, with unimodular $c$. In particular, $\text{Lie}_0(V) \subseteq \text{Lie}(V, c_F)$ for any $F$. Consider the map $\zeta^F \overset{\text{def}}{=} \pi_0|_{\text{Lie}(V, c_F)}$. Then we have

$$(\zeta^F)^{-1}(dG) = Z_N^2(dG) \cap Z_N^2(dF).$$

### 5.1 Computations

In this subsection we shall describe the varieties $\text{Lie}(V, c)$, for all types of structures $c$. Obviously, $\text{Lie}(V, 0) = \text{Lie}(V)$, so we assume $c \neq 0$.

We introduce the notation

$$s^2 \overset{\text{def}}{=} \text{span}\{\frac{1}{2}d(x_1^2), \frac{1}{2}d(x_2^2), \frac{1}{2}d(x_1x_2)\}.$$ 

Notice that $s^2$ identifies with the space of symmetric bilinear forms on $\text{span}\{\xi_1, \xi_2\}$.

#### 5.1.1 Compatibility variety of $A_3^\pm$ structures

Let $c = c_F$. If $\text{rank}(F) = 3$, then $Z_N^2(dF) = 0$ and $\text{Lie}(V, c) = \text{Lie}_0(V)$ is a 6–dimensional vector subspace (see Fig. 2).

#### 5.1.2 Compatibility variety of $A_2^\pm$ structures

Let now $F = \frac{1}{2}(x_1^2 + \epsilon x_2^2)$, $\epsilon \in \mathbb{F} \setminus \{0\}$.

**Lemma 9.** $Z^2(\alpha_3) \cap \text{Lie}_0(V) = s^2$. 
15
\[(\zeta^F)^{-1}(dG) = Z^2_N(dG) \cap Z^3_N(dF)\] 
\[(\zeta^F)^{-1}(0) = \text{Span}\{\alpha_3\} = Z^2_N(0)\]

Proof. Immediately from Proposition 4. \(\square\)

**Proposition 11.** \(\text{Lie}(V, c_F)\) is the union

\[
\text{Lie}(V, c_F) = \text{Lie}_0(V) \cup \text{span}\{s^2, \alpha_3\}
\]

of a 6-dimensional and a 4-dimensional subspace, intersecting along the 3-dimensional subspace \(s^2\).

Proof. Obviously, the right-hand side of (24) is contained in the left one. Let \(c' = c_G + \alpha \in \text{Lie}(V, c_F)\) with \(\alpha \neq 0\).

Since \(c'\) is compatible with \(c_F\), \(dF \wedge d\alpha_{c'} = 0\). But \(dF \wedge d\alpha_{c'} = dF \wedge d\alpha\), so \(dF \wedge d\alpha = 0\), i.e. \(\alpha \in Z^2_N(dF)\). In view of Lemma 4, \(Z^2_N(dF)\) is the one-dimensional subspace generated by \(\alpha_3\). Hence \(\alpha = \lambda \alpha_3\), \(\lambda \neq 0\).

This shows that \(c_G\), being compatible with \(\alpha\), is compatible with \(\alpha_3\) and, by Lemma 9, is a linear combination of \(\frac{1}{2}d(x_2), \frac{1}{2}d(x_2^2), d(x_1x_2)\). \(\square\)

Figure 3 shows that the structure of \(\text{Lie}(V, c_F)\) is quite simple. The 3-dimensional subspace \(s^2\) is precisely the locus where the fibers of \(\zeta^F\) are non-trivial. The restriction of \(\zeta^F\) to it is a trivial bundle with fiber span \(\{\alpha_3\}\).

**5.1.3 Compatibility variety of \(A_1\) structures**

This case is more complicated (see Fig. 4). Let \(F = \frac{1}{2}x_1^2\).

**Proposition 12.** If \((0, 0) \neq (a, b) \in \mathbb{F}^2\), then \(\zeta^F\) is a rank–2 trivial bundle over the line span \(\{\frac{1}{2}d(x_1^2)\}\) with the fiber span \(\{\alpha_2, \alpha_3\}\), and over span \(\{\frac{1}{2}d(x_1^2), \frac{1}{2}d((bx_2-ax_3)^2)\}\), \(\zeta^F\) is a rank–1 trivial bundle with the fiber span \(\{a\alpha_2 + b\alpha_3\}\). Fibers of \(\zeta^F\) are trivial over the rest of \(\text{Lie}_0(V)\).
Figure 4: The compatibility variety of a structure $c_F$ of type $A_1$.

**Proof.** As it follows from Lemma 5, $Z^2_N(dF) = \text{span} \{\alpha_2, \alpha_3\}$. Therefore, the intersection $Z^2_N(dF) \cap Z^2_N(dG)$ is 2–dimensional if and only if $Z^2_N(dF) = Z^2_N(dG)$, i.e., if $dG$ belongs to the line $\text{span} \{\frac{1}{2}d(x^2_1)\}$.

The intersection $Z^2_N(dF) \cap Z^2_N(dG)$ can be of dimension 1 in the following two cases. First, $Z^2_N(dG)$ is a 2–dimensional subspace intersecting $\text{span} \{\alpha_2, \alpha_3\}$ along a line, and, second, $Z^2_N(dG)$ is a 1–dimensional subspace contained in $\text{span} \{\alpha_2, \alpha_3\}$.

In the first case, a line in $\text{span} \{\alpha_2, \alpha_3\}$ can be written as $\text{span} \{a\alpha_2 + b\alpha_3\}$, with $(0, 0) \neq (a, b)$. Then $dG = \frac{1}{2}d((bx_2 - ax_3)^2)$ is the only rank–1 structure such that $Z^2_N(dG)$ intersects $\text{span} \{\alpha_2, \alpha_3\}$ along $\text{span} \{a\alpha_2 + b\alpha_3\}$.

In the second case, $dG$ must be a rank–2 structure such that $Z^2_N(dG)$ is precisely $\text{span} \{a\alpha_2 + b\alpha_3\}$. Up to proportionality, this is $G = \frac{1}{2}(x^2_1 + (bx_2 - ax_3)^2)$.

5.1.4 Compatibility varieties of $B_0$ structures

If $\alpha_i$ (resp., $d(x_i, x_j)$) is a base vector of $N$ (resp., $\text{Lie}_0(V)$), then the dual to it covector will denoted by $\alpha_i^\circ$ (resp., $d(x_i, x_j)^\circ$).

As it follows from Lemma 8, the space of 2–cocycles of the structure $c_F + c_{\alpha_3}$, with $F = \frac{1}{2}(\lambda x^2_1 + \mu x^2_3)$, is the 6–dimensional space

$$\text{span} \{s^2, \alpha_1 - \lambda d(x_1x_3), \alpha_2 - \mu d(x_2x_3), \alpha_3\}. \quad (25)$$

If $c$ is a structure of type $B_0$, i.e., $\lambda = \mu = 0$, then

$$\text{Lie} (V, c) = \zeta^{-1}(s^2). \quad (26)$$

$\zeta|_{\text{Lie} (V, c)}$ is a stratified vector bundle over $s^2$. Indeed (see Lemma 5), $\zeta$ is of rank 3 over $\{0\}$, it is of rank 2 over the quadric $d(x^2_1)^\circ d(x^2_3)^\circ - (d(x_1x_2)^\circ)^2 = 0$, and it is of rank 1 over the rest of $s^2$ (see Fig. 5).
5.1.5 Compatibility varieties of $B_1$ structures

If $c$ is a structure of type $B_1$, then $\lambda = 1$ and $\mu = 0$. Directly from (25) it follows that $\text{Lie}(V, c)$ is the intersection of $\zeta^{-1}(\text{span}\{s^2, d(x_1x_3)\})$ with the affine hyperplane $(\alpha_1)^o = -(d(x_1x_3))^o$. Moreover, if $c_G + ad(x_1x_3) + \alpha \in \text{Lie}(V, c)$, with $c_G \in s^2$, it is easy to prove that $a = 0$. In other words,

$$\text{Lie}(V, c) = \zeta^{-1}(s^2) \cap \{(\alpha_1)^o = 0\},$$

i.e., $\zeta|_{\text{Lie}(V, c)}$ is a stratified vector bundle over $s^2$, whose fibers are subspaces of the corresponding fibers of $\zeta$.

Describe now the corresponding strata. Let $c_G + \alpha \in \text{Lie}(V, c)$. If $c_G \in \text{span}\{\frac{1}{2}d(x_1^2)\}$ then $\zeta|_{\text{Lie}(V, c)}(c_G) = \text{span}\{\alpha_2, \alpha_3\}$. If $c_G$ is a point of the quadric $d(x_1^2)^o d(x_2^2)^o - (d(x_1x_2))^o = 0$, not belonging to the line $\text{span}\{\frac{1}{2}d(x_1^2)\}$, then $\zeta|_{\text{Lie}(V, c)}(c_G)$ is the 1-dimensional subspace $(\alpha_1)^o = 0$ of $\zeta^{-1}(c_G)$. If $c_G$ is not in the quadric above, then $\zeta|_{\text{Lie}(V, c)}^{-1}(c_G)$ coincides with $\zeta^{-1}(c_G)$, i.e., $\text{span}\{\alpha_3\}$ (see Fig. 6).

5.1.6 Compatibility varieties of $B_{2, \nu}$ structures

Finally, if $\lambda = \pm \mu = \nu^{-1}$, then $c$ is a structure of type $B_{2, \nu}$. In this case $\text{Lie}(V, c)$ is the intersection of $\zeta^{-1}(\text{span}\{s^2, d(x_1x_3), d(x_2x_3)\})$ with the affine subspace

$$\begin{cases} (\alpha_1)^o = -\nu(d(x_1x_3))^o \\ (\alpha_2)^o = \mp \nu(d(x_2x_3))^o. \end{cases}$$

Moreover, if $c' = c_G + ed(x_1x_3) + fd(x_2x_3) + \alpha \in \text{Lie}(V, c)$, with $c_G \in s^2$, it is easy to prove that $e^2 = \pm f^2$. 

Figure 5: The compatibility variety of a non-unimodular Lie structure of type $B_0$. 

$$s^2 = \text{Span}\{\frac{1}{2}d(x_1^2), \frac{1}{2}d(x_2^2), \frac{1}{2}d(x_1x_2)\}$$
If \( e = f = 0 \), i.e., the unimodular component of \( c' \) belongs to \( s^2 \), then

\[
\text{Lie}(V, c) \cap \zeta^{-1}(s^2) = \zeta^{-1}(s^2) \cap \{(\alpha_1)^o = (\alpha_2)^o = 0\},
\]

i.e., the restriction of \( \zeta|_{\text{Lie}(V, c)} \) over \( s^2 \) is a trivial vector bundle with the fiber \( \text{span}\{\alpha_3\} \).

If \( ef \neq 0 \), then it is easy to prove that \( c' = ac + e(d(x_1 x_3) - \nu \alpha_1) + f(d(x_2 x_3)) + \nu \alpha_2 \). In other words, the restriction of \( \zeta|_{\text{Lie}(V, c)} \) over the degenerate quadric \( \{(d(x_1 x_3)^o)^2 = (d(x_2 x_3)^o)^2 = 0\} \subseteq \text{span}\{\frac{1}{2}d(x_1^2), \frac{1}{2}d(x_2^2), d(x_1 x_3), d(x_2 x_3)\} \) is the graph of the map

\[
a(\frac{1}{2}d(x_1^2) \pm \frac{1}{2}d(x_2^2)) + ed(x_1 x_3) + fd(x_2 x_3) \rightarrow \nu(\alpha_3 - e \alpha_1 \mp f \alpha_2).
\]

Comparing (26), (27), (28) and (29), one observes that when the rank of the unimodular component of \( c \) increases, the dimension of the fibers of \( \zeta|_{\text{Lie}(V, c)} \) over \( s^2 \) decreases. Observe that in all cases, \( \text{Lie}(V, c) \cap \text{Lie}_e(V) = s^2 \). It is worth also stressing that elements \( c' \in \text{Lie}(V, c) \) such that \( \zeta(c') \notin s^2 \) exists only for structures \( c \) of the type \( B_{2,\nu}^+ \) (see Fig. 7).

### 5.2 Deformations of Lie structures

Recall that a (algebraic, smooth, continuous) deformation of a Lie structure \( c \) is a (algebraic, smooth, continuous) curve in \( \text{Lie}(V) \), i.e., a map \( \gamma : \mathbb{F} \rightarrow \text{Lie}(V) \), passing through \( c \).

Denote by \( \mathcal{F} \) the algebra of algebraic functions on \( \text{Lie}(V) \), i.e., the quotient of \( S(V \otimes \mathbb{F} V) \) by the ideal generated by \( \{2\} \). If \( \mathbb{F} = \mathbb{R} \), define also \( C^\infty(\text{Lie}(V)) \) as the quotient of the algebra \( C^\infty(V \otimes \mathbb{R} V) \) by the ideal generated by \( \{2\} \). A map from \( \mathbb{F} \) to \( \text{Lie}(V) \) is called algebraic (resp., smooth) if it corresponds to an algebra
\[ s^2 = \text{Span} \{ \frac{1}{2} d(x_1^2), \frac{1}{2} d(x_2^2), \frac{1}{2} d(x_1 x_2) \} \]

\[ \zeta|_{\text{Lie}(V,c)}^{-1} = \text{span} \{ \alpha_3 \} \]

\[ \text{not in } S^2 \]

\[ \text{degenerate quadric } (d(x_1 x_3)^o)^2 \pm (d(x_2 x_3)^o)^2 = 0 \]

Figure 7: The compatibility variety of a non–unimodular Lie structure of type \( B_{2,\nu}^\pm \).
homomorphism $F \hookrightarrow \mathbb{F}[x]$ (resp., $C^\infty(\text{Lie}(V)) \hookrightarrow C^\infty(\mathbb{R})$) in the sense of [1]. In particular, a linear map from $F$ to $\text{Lie}(V)$, i.e., an $F$–homomorphism from $F$ to $V \otimes_F V$ whose image is contained in $\text{Lie}(V)$, is algebraic (and smooth, if $F = \mathbb{R}$).

A deformation is called linear if $\gamma$ is a straight line. Observe that the linear deformation

$$\gamma_d(t) \overset{\text{def}}{=} (1 - t)c + td$$

of $c$ is naturally associated with the element $d \in \text{Lie}(V,c)$, $d \neq 0$. Obviously, any linear deformation of $c \in \text{Lie}(V)$ is of the form $\gamma_d$.

We define an infinitesimal deformation to be tangent vector at $c$ of a deformation $\gamma$. In particular, the infinitesimal deformation associated with $\gamma_d$ is $\gamma'_d(0)$, connecting $c$ and $d$. Infinitesimal deformations must be understood as elements of the tangent space to $\text{Lie}(V)$. Two infinitesimal deformations are called equivalent if one is obtained from another by action of $d_c \psi$, with $\psi \in \text{Stab}(c)$. The tangent space to $\text{Lie}(V)$ is naturally identified with $Z^2(c)$, and the above described action of $\text{Stab}(c)$ coincides with a natural action of $\text{Stab}(c)$ on $Z^2(c)$. Moreover, the subset of $Z^2(c)$ that corresponds to the linear deformations coincides with $\text{Lie}(V,c)$, and the action of $\text{Stab}(c)$ restricts to it.

5.3 Some examples of deformations

Now we shall exploit the above description of $\text{Lie}(V,c)$ in order to describe deformations of a 3–dimensional Lie structure $c$ and their equivalence classes as well. By abusing the language we shall call the quotient $\text{Lie}(V,c)/\text{Stab}(c)$ “orbit space”.

To this end, it will be necessary to consider some special subgroups of $\text{GL}(V)$.

Remark 7. If $F = \frac{1}{2}(x_1^2 + x_2^2 \pm x_3^2)$, then $\text{Stab}(c_F)$ is $O(3)$ (resp., $O(2,1)$) (see also Proposition [5]). Similarly, for $F = \frac{1}{2}(x_1^2 \pm x_2^2)$, the group $\text{Stab}(c_F)$ will be denoted $O(2,0)$ or $O(1,1,0)$, respectively. Finally, notice that $\text{Stab}(c_F)$, for $F = \frac{1}{2}x_1^2$, coincides with the stabilizer of $x_1$. We do not describe the orbits of the action of $\text{Stab}(c_F)$ on $\text{Lie}_0(V)$, since this concerns the theory of symmetric bilinear forms (see [5]).

Denote by $p : \text{Lie}(V) \hookrightarrow \frac{\text{Lie}(V)}{\text{GL}(V)}$ a natural projection of sets. Recall that a (algebraic, smooth) deformation $\gamma$ of $c = \gamma(0)$ is called a contraction of $c$ if $p \circ \gamma$ takes two different values for $t = 0$ and $t \neq 0$.

5.3.1 Deformations of $A^+_3$ structures

Let $F = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$ and $c = c_F$. Then $\text{Lie}(V,c) = \text{Lie}_0(V)$ (see Subsection 5.1.1), and $\text{Stab}(c) = O(3)$ (see Remark [7]). Hence the orbit space identifies with $\frac{S^2(V)}{O(3)}$, i.e., with the space of diagonal 3 by 3 matrices over $\mathbb{F}$.

Observe that no deformation of $c$ is a contraction. The reader should not confuse between deformations of Lie algebras and deformations of Lie algebra structures.
5.3.2 Deformations of $A_2^+$ structures

Let $F = \frac{1}{2}(x_1^2 + x_2^2)$ and $c = c_F$.

Observe that in this case $s^2$ is $O(2, 0)$–invariant and the orbits of the restricted action of $O(2, 0)$ are the same as the orbits of the natural action of $O(2)$ on $s^2$. It is easy to prove that the set of parallel sections of $\zeta^F|\Omega$, for such an $\Omega$, is identified with $\mathbb{F}$.

Remark 8. If intersection of two subspaces of a vector space is non–trivial, then there are smooth curves passing from one subspace to the other, in contrast with the algebraic ones. In particular there are smooth curves connecting any point of $\text{Lie}_0(V)$ with any point of span $\{s^2, \alpha_3\}$ (See Figure 3). This is obviously not the case for algebraic curves. So, this example illustrates the difference between algebraic and smooth deformations.

5.3.3 Deformations of $A_1$ structures

Let $F = \frac{1}{2}(x_1^2)$ and $c = c_F$.

In this case, the line span $\{\frac{1}{2}(x_1^2)\}$ is $\text{Stab}(c)$–invariant and the restricted action is trivial, i.e., $\Omega$ is a point. Similarly to Proposition 9 one proves that there is only one nonzero parallel section of $\zeta^F|\Omega$.

Under the action of $\text{Stab}(c)$, the plane span $\{\frac{1}{2}(x_1^2), \frac{1}{2}((bx_2 - ax_3)^2)\}$ (see Fig. 4) rotates around the axis span $\{\frac{1}{2}(x_1^2)\}$. If $\Omega$ is an orbit of $\text{Stab}(c)$ not contained in this axis, then the set of parallel sections of $\zeta^F|\Omega$ is identified with $\mathbb{F}$.

5.4 Effect of deformations on symplectic foliation in the case $\mathbb{F} = \mathbb{R}$

A deformation of a Lie structure $c$ induces a deformation of the symplectic foliation of $P^c$. Note that only the solvable 3–dimensional Lie structures admit non–trivial deformations. In such a case, $P^c$ can be brought to the form

$$P^c = X_\phi \wedge \frac{\partial}{\partial x^3},$$

with $\phi \in \text{End}(\mathbb{R}^2)$. Indeed, solvable Lie structures $B_0, B_1, B_{2,\lambda}^\pm$, and $A_{2}^\pm$ are of this form, with $\phi$ being

$$\left(\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array}\right), \left(\begin{array}{cc} -1 & 1 \\ 0 & -1 \end{array}\right), \left(\begin{array}{cc} -\lambda & 1 \\ \mp 1 & -\lambda \end{array}\right), \text{ and } \left(\begin{array}{cc} 0 & 1 \\ \pm 1 & 0 \end{array}\right),$$

respectively. The nil-potent Lie structure $A_1$ corresponds to $\phi = \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right)$.

If $X_\phi = \phi^b a x_b \frac{\partial}{\partial x_a}$, $a, b = 1, 2$, then $P^c = \phi^b a x_b \frac{\partial}{\partial x_a} \wedge \frac{\partial}{\partial x_3}$, and

$$\alpha_c = \frac{1}{2} (\phi_2^b x_1^2 - \phi_1^b x_2^2) + \phi_2^a x_3 x_1 - \phi_1^a x_1 x_2,$$

(32)
and, therefore,

\[ P^c_{x_1} = -\phi(x_1) \frac{\partial}{\partial x^3}, \]
\[ P^c_{x_2} = -\phi(x_2) \frac{\partial}{\partial x^3}, \]
\[ P^c_{x_3} = X_\phi. \]

Notice that in each point \( p = (x_1, x_2, x_3) \) where \( \phi(x_1) \) and \( \phi(x_2) \) are not simultaneously zero, \( \text{span} \{ P^c_{x_1}, P^c_{x_2} \} \) is the line generated by \( \frac{\partial}{\partial x^3} \big|_p \). So, it holds the following lemma.

**Lemma 10.** *Symplectic leaves of a solvable Lie structure corresponding to Poisson bi–vector (31) are either pull–backs of trajectories of \( X_\phi \) in \( \mathbb{R}^2 \setminus \ker \phi \) via the projection \( \mathbb{R}^3 \rightarrow \mathbb{R}^2 \), or single points of the subspace \( \ker \phi \oplus \langle x_3 \rangle \).*

5.4.1 Deformation of \( B^\pm_{2,\lambda} \) to \( A^\pm_2 \)

In this case, elliptic (resp. hyperbolic) spirals converge to circles (resp. hyperbola), as \( \lambda \to 0 \). Since the \( B^\pm_2 \)'s are mutually non–isomorphic for different values of \( \lambda \), such deformation is not a contraction.

5.4.2 Deformation of \( B^\pm_{2,1} \) to \( A_1 \)

Consider the family of structures \( \{ c^\pm_{\mu} \}_{\mu \in \mathbb{R}^+} \) of the form (31), with

\[ \varphi^\pm_{\mu} = \begin{pmatrix} -1 & \mp \mu & 1 \\ 1 & 1 & -1 \end{pmatrix} \]

and

\[ \alpha_{c^\pm_{\mu}} = \frac{1}{2}(dx_1^2 \pm \mu dx_2^2) + \alpha_3. \]

Then the trajectory of \( X_{\varphi^\pm_{\mu}} \) issuing from \((x_1^0, x_2^0), x_1^0 \neq 0\), is given by

\[ x_1(t) = e^{-t} \sqrt{(x_1^0)^2 + \mu (x_2^0)^2} \cos \left( \arctan \left( \frac{\sqrt{\mu} x_2^0}{x_1^0} \right) + \sqrt{\mu} t \right) \]
\[ x_2(t) = e^{-t} \sqrt{\frac{(x_1^0)^2}{\mu} + (x_2^0)^2} \sin \left( \arctan \left( \frac{\sqrt{\mu} x_2^0}{x_1^0} \right) + \sqrt{\mu} t \right) \]

and

\[ x_1(t) = \frac{x_1^0 + \sqrt{\mu} x_2^0 e^{(\sqrt{\mu}-1)t}}{2} + \frac{x_1^0 - \sqrt{\mu} x_2^0 e^{-(\sqrt{\mu}+1)t}}{2} \]
\[ x_2(t) = \frac{x_1^0 + \sqrt{\mu} x_2^0 e^{(\sqrt{\mu}-1)t}}{2 \sqrt{\mu}} - \frac{x_1^0 - \sqrt{\mu} x_2^0 e^{-(\sqrt{\mu}+1)t}}{2 \sqrt{\mu}} \]

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Figure 8: Projection on the \((x_1, x_2)\)-plane of the symplectic leaves of the structures \(B_{2,1}^+\) (red) and \(B_{2,1}^-\) (blue), as they undergo a simultaneous deformation to \(A_1\) (red and blue overlapped).

Trajectories of \(X_{\varphi^+}\) (red) and of \(X_{\varphi^-}\) (blue), issuing from vertices of a regular hexagon centered at the origin, are represented in Figure 8 for \(\mu\) running from almost zero (first picture) to 1 (last picture). We see that both elliptic (determined by \(c_{\mu}^+\)) and hyperbolic (determined by \(c_{\mu}^-\)) spirals converge to the same foliation as \(\mu \to 0\), and the constructed deformation is a contraction.

5.4.3 Deformation of \(B_1\) to \(A_1\)

The deformation \(\{c_\lambda\}_{\lambda \in \mathbb{R}}\) of the form (31), with

\[
\varphi_\lambda = \begin{pmatrix} -\lambda & 1 \\ \mp 0 & -\lambda \end{pmatrix}
\]

and

\[
\alpha_{c_\lambda} = \frac{1}{2} dx_1^2 + \lambda \alpha_3,
\]
is a contraction. The trajectory of $X_c\lambda$ issuing from $(x_1^0, x_2^0)$, which is given by

$$x_1(t) = x_1^0 e^{-\lambda t}$$
$$x_2(t) = (x_1^0 t + x_2^0) e^{-\lambda t}$$

and converges to the vertical straight line passing through $(x_1^0, x_2^0)$, as $\lambda \to 0$.

Acknowledgements

The author is indebted to prof. Vinogradov, who carefully supervised the works on the manuscript since its conception, and to prof. Marmo, for inspiring advices.

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