Power series of the operators $U_n^\varrho$

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Abstract

We study power series of members of a class of positive linear operators reproducing linear function constituting a link between genuine Bernstein-Durrmeyer and classical Bernstein operators. Using the eigenstructure of the operators we give a non-quantitative convergence result towards the inverse Voronovskaya operators. We include a quantitative statement via a smoothing approach.

Keywords: Power series geometric series positive linear operator Bernstein-type operator genuine Bernstein-Durrmeyer operator degree of approximation eigenstructure moduli of continuity.

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1 Introduction

The present note is essentially motivated by two key papers of Păltănea which both appeared in two hardly known local Romanian journals. In the first article mentioned [9] Păltănea defined power series of Bernstein operators (with $n$ fixed) and studied their approximation behaviour for functions defined on the space $C_0[0,1] := \{f \mid f(x) = x(1-x)h(x), h \in C[0,1] \}$ to some extent. This article motivated a number of authors to study similar problems or give different proofs of Păltănea’s main result. See [1], [2], [3], [11]. In one more and most significant article Păltănea [10] introduced a very interesting link between the classical Bernstein operators $B_n$ and the so-called ”genuine Bernstein-Durrmeyer operators” $U_n$, thus also bridging the gap between $U_n$ and piecewise linear interpolation in a most elegant way for the cases $0 < \varrho \leq 1$. The operators $U_n^\varrho$ also attracted several authors to study them further. See, for example, [6], [7]. In the present note we combine both approaches of Păltănea and study power (geometric) series of the operators $U_n^\varrho$, thus bridging the gap between power series of Bernstein operators and such of the genuine operators $U_n$ mentioned above.

Our main results will concern the convergence of the series as $n$ (the degree of the polynomials inside the series) tends to infinity. The first non-quantitative theorem will essentially use the eigenstructure of the $U_n^\varrho$ which was recently studied in [8].
The second result describes the degree of convergence to the "inverse Voronovskaya operators" $-A_{\varphi}^{-1}$ using a smoothing (K-functional) approach and makes use of exact representations of the moments as presented in [6].

The quantitative statement also holds in the limiting case of Bernstein operators, thus supplementing the original work of Păltănea.

2 The operators $U_{n}^{\varphi}$ and their eigenstructure

Denote by $C[0,1]$ the space of continuous, real-valued functions on $[0,1]$ and by $\Pi_{n}$ the space of polynomials of degree at most $n \in \mathbb{N}_{0} := \{0,1,2,\ldots\}$.

**Definition 2.1.** Let $\varphi > 0$ and $n \in \mathbb{N}_{0}, n \geq 1$. Define the operator $U_{n}^{\varphi} : C[0,1] \to \Pi_{n}$ by

$$U_{n}^{\varphi}(f,x) := \sum_{k=0}^{n} F_{n,k}(f)p_{n,k}(x)$$

$$:= \sum_{k=1}^{n-1} \left( \int_{0}^{1} \frac{t^{\varphi-1}(1-t)^{(n-k)\varphi-1}}{B(k\varphi,(n-k)\varphi)} f(t)dt \right) p_{n,k}(x) + f(0)(1-x)^{n} + f(1)x^{n},$$

$f \in C[0,1], x \in [0,1]$ and $B(\cdot,\cdot)$ is Euler’s Beta function. The fundamental functions $p_{n,k}$ are defined by

$$p_{n,k}(x) = \binom{n}{k} x^{k}(1-x)^{n-k}, \quad 0 \leq k \leq n, \quad x \in [0,1].$$

For $\varphi = 1$ and $f \in C[0,1]$, we obtain

$$U_{n}^{1}(f,x) = U_{n}(f,x) = (n-1) \sum_{k=1}^{n-1} \left( \int_{0}^{1} f(t)p_{n-2,k-1}(t)dt \right) p_{n,k}(x) + (1-x)^{n} f(0) + x^{n} f(1),$$

where $U_{n}$ are the “genuine” Bernstein-Durrmeyer operators (see [9, Th. 2.3]), while for $\varphi \to \infty$, for each $f \in C[0,1]$ the sequence $U_{n}^{\varphi}(f,x)$ converges uniformly to the Bernstein polynomial

$$B_{n}(f,x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) p_{n,k}(x).$$

Moreover, for $n$ fixed and $\varphi \to 0$ one has uniform convergence of $U_{n}^{\varphi}f$ towards the first Bernstein polynomial $B_{1}f$, i.e., linear interpolation at 0 and 1 (see [7, Th. 3.2]). The eigenstructure of $U_{n}^{\varphi}$ is described in [8]. The numbers

$$\lambda_{\varphi,j}^{(n)} := \frac{\varphi^{j}n!}{(n\varphi)^{j} (n-j)!}, \quad j = 0,1,\ldots,n,$$  

(2.1)
are eigenvalues of $U_\varrho_n$. To each of them there corresponds a monic eigenpolynomial $p^{(n)}_{\varrho,j}$ such that $\deg p^{(n)}_{\varrho,j} = j$, $j = 0, 1, \ldots, n$. In particular,

$$p^{(n)}_{\varrho,0}(x) = 1, p^{(n)}_{\varrho,1}(x) = x - \frac{1}{2}, x \in [0, 1]. \quad (2.2)$$

A complete description of $p^{(n)}_{\varrho,j}(x)$, $j = 2, \ldots, n$, can be found in [8]. From (8, (3.14)) we get

$$p^{(n)}_{\varrho,j}(0) = p^{(n)}_{\varrho,j}(1) = 0, j = 2, \ldots, n. \quad (2.3)$$

Obviously $U_{\varrho}^nf$ can be decomposed with respect to the basis \{p_{\varrho,0}, p_{\varrho,1}, \ldots, p_{\varrho,n}\} of $\Pi_n$; this allows us to introduce the dual functionals $\mu^{(n)}_{\varrho,j} : C[0, 1] \rightarrow \mathbb{R}, j = 0, 1, \ldots, n$, by means of the formula

$$U_{\varrho}^nf = \sum_{j=0}^{n} \lambda^{(n)}_{\varrho,j} p^{(n)}_{\varrho,j}(f)p^{(n)}_{\varrho,j}, f \in C[0, 1]. \quad (2.4)$$

In particular, since $U_{\varrho}^n$ restricted to $\Pi_n$ is bijective, we have

$$p = \sum_{j=0}^{n} \mu^{(n)}_{\varrho,j}(p)p^{(n)}_{\varrho,j}, p \in \Pi_n. \quad (2.5)$$

Now consider the numbers

$$\lambda_{\varrho,j} := -\frac{\varrho + 1}{2\varrho}(j - 1)j, j = 0, 1, \ldots \quad (2.6)$$

and the monic polynomials

$$p_0^*(x) = 1, p_1^*(x) = x - \frac{1}{2}, p_j^*(x) = x(x - 1)P_{j-2}^{(1,1)}(2x - 1), j \geq 2, \quad (2.7)$$

where $P_i^{(1,1)}(x)$ are Jacobi polynomials, orthogonal with respect to the weight $(1 - x)(1 + x)$ on $[-1, 1]$, $i \geq 0$. Moreover, consider the linear functionals $\mu^*_j : C[0, 1] \rightarrow \mathbb{R}$, defined as

$$\mu^*_0(f) = \frac{f(0) + f(1)}{2}, \mu^*_1(f) = f(1) - f(0), \quad (2.8)$$

$$\mu^*_j(f) = \frac{1}{2} \binom{2j}{j} (-1)^j f(0) + f(1) - j \int_0^1 f(x)P_{j-2}^{(1,1)}(2x - 1)dx, j \geq 2. \quad (2.9)$$

It is easy to verify that

$$\lim_{n \to \infty} n(\lambda^{(n)}_{\varrho,j} - 1) = \lambda_{\varrho,j}, j \geq 0. \quad (2.10)$$

The following result can be found in [8].

**Theorem 2.1.** ([8]) For each $j \geq 0$ we have

$$\lim_{n \to \infty} p^{(n)}_{\varrho,j} = p^*_j, \text{ uniformly on } [0, 1], \quad (2.11)$$

$$\lim_{n \to \infty} \mu^{(n)}_{\varrho,j}(p) = \mu^*_j(p), p \in \Pi. \quad (2.12)$$
3 The power series $A^g_{n_\varrho}$

Consider the space
\[ C_0[0,1] := \{ f|f(x) = x(1-x)h(x), h \in C[0,1] \}. \quad (3.1) \]

For $f \in C_0[0,1], f(x) = x(1-x)h(x)$, define the norm
\[ ||f||_0 := ||h||_{\infty}. \quad (3.2) \]

Endowed with the norm $|| \cdot ||_0$, $C_0[0,1]$ is a Banach space. Obviously,
\[ ||f||_{\infty} \leq \frac{1}{4} ||f||_0, f \in C_0[0,1]. \quad (3.3) \]

**Lemma 3.1.** As a linear operator on $(C_0[0,1], || \cdot ||_0), U^g_{n_\varrho}$ has the norm
\[ ||U^g_{n_\varrho}||_0 = \frac{(n-1)\varrho}{n\varrho + 1} < 1. \quad (3.4) \]

**Proof.** Let $f \in C_0[0,1], f(x) = x(1-x)h(x), h \in C[0,1]$. By straightforward computation we get $U^g_{n_\varrho}f(x) = x(1-x)u(x)$, where
\[ u(x) = n(n-1) \sum_{k=1}^{n-1} \frac{1}{k(n-k)} B(k, (n-k)\varrho) \rho_{n-2,k-1}(x). \]

It follows immediately that $U^g_{n_\varrho}f \in C_0[0,1]$ and
\[ ||U^g_{n_\varrho}f||_0 = ||u||_{\infty} \leq \frac{(n-1)\varrho}{n\varrho + 1} ||h||_{\infty} = \frac{(n-1)\varrho}{n\varrho + 1} ||f||_0. \]

Thus
\[ ||U^g_{n_\varrho}||_0 \leq \frac{(n-1)\varrho}{n\varrho + 1}. \quad (3.5) \]

On the other hand, let $g(x) = x[1-x], x \in [0,1]$. Then
\[ ||g||_0 = 1 \quad \text{and} \quad U^g_{n_\varrho}g(x) = x(1-x)\frac{(n-1)\varrho}{n\varrho + 1}, \]
which entails $||U^g_{n_\varrho}g||_0 = \frac{(n-1)\varrho}{n\varrho + 1}$ and so
\[ ||U^g_{n_\varrho}||_0 \geq \frac{(n-1)\varrho}{n\varrho + 1}. \quad (3.6) \]

Now (3.4) is a consequence of (3.5) and (3.6). \hfill \Box

According to Lemma 3.1, it is possible to consider the operator $A^g_{n_\varrho} : C_0[0,1] \to C_0[0,1],$
\[ A^g_{n_\varrho} := \frac{\varrho}{n\varrho + 1} \sum_{k=0}^{\infty} (U^g_{n_\varrho})^k, n \geq 1. \quad (3.7) \]
For later purposes we also introduce the notation 

\[ A_n^\infty := \frac{1}{n} \sum_{k=0}^{\infty} (B_n)^k, \quad n \geq 1, \]

in order to have Păltănea’s power series included.

By using (3.4) we get \[ ||A^\infty_n||_0 \leq \frac{\varrho}{\varrho + 1}, \] and with the same function \( g(x) = x(1-x) \) we find

\[ ||A^\infty_n||_0 = \frac{\varrho}{\varrho + 1}, \quad n \geq 1. \]  

(3.8)

Let \( p \in \Pi_m \cap C_0[0,1] \), i.e., \( p(0) = p(1) = 0 \). Then \( m \geq 2 \). Let \( n \geq m \). From (2.2), (2.3) and (2.5) we derive

\[ p = \sum_{j=2}^{m} \mu_{e,j}^{(n)}(p) p_{e,j}^{(n)} \]  

(3.9)

and, moreover,

\[ (U^\varrho_n)^k p = \sum_{j=2}^{m} (\lambda_{e,j}^{(n)})^k \mu_{e,j}^{(n)}(p) p_{e,j}^{(n)}, \quad k \geq 0, \text{ for all } n \geq m. \]  

(3.10)

According to (3.7), for all \( p \in \Pi_m \cap C_0[0,1] \) and \( n \geq m \),

\[ A^\varrho_n p = \frac{\varrho}{n\varrho + 1} \sum_{j=2}^{m} \frac{1}{1-\lambda_{e,j}^{(n)}} \mu_{e,j}^{(n)}(p) p_{e,j}^{(n)}. \]  

(3.11)

By using (2.10), (2.11) and (2.12) we get

\[ \lim_{n \to \infty} A^\varrho_n p = \frac{\varrho}{\varrho + 1} \sum_{j=2}^{m} \frac{2}{j(j-1)} \mu_{e,j}^*(p) p_{e,j}^*, \]  

(3.12)

uniformly on \([0,1]\), for all \( p \in \Pi_m \cap C_0[0,1] \).

4 The Voronovskaya operator \( A^\varrho \)

It was proved in [7, p. 918] that

\[ \lim_{n \to \infty} n(U^\varrho_n g(x) - g(x)) = \frac{\varrho + 1}{2\varrho} x(1-x) g''(x), \quad g \in C^2[0,1], \]

uniformly on \([0,1]\). We need the following result.

**Theorem 4.1.** The operator \( \{ y \in C^2[0,1] | y(0) = y(1) = 0 \} \to C_0[0,1] \) defined by

\[ A^\varrho y(x) := \frac{\varrho + 1}{2\varrho} x(1-x) y''(x), \quad x \in [0,1], \]  

(4.1)
is bijective, and
\[ ||A^{-1}_q f||_\infty \leq \frac{\theta}{4(\theta + 1)} ||f||_0, \quad f \in C_0[0,1]. \]  \quad (4.2)

**Proof.** Obviously \( A_q \) is injective. To prove the surjectivity, let \( f \in C_0[0,1], f(x) = x(1-x)h(x), h \in C[0,1] \). It is a matter of calculus to verify that the function

\[-\frac{2\theta}{\theta + 1} F_\infty(h;x) = y(x) := -\frac{2\theta}{\theta + 1} \left( (1-x) \int_0^x t h(t) dt + x \int_0^1 (1-t) h(t) dt \right),\]

for \( x \in [0,1] \) is in \( C^2[0,1] \), \( y(0) = y(1) = 0 \), and \( A_q y = f \). Therefore \( A_q \) is bijective. Moreover, for \( x \in [0,1] \), \( y = -A^{-1}_q(f) \), i.e., \( -y(x) = -A^{-1}_q(f;x) = +\frac{2\theta}{\theta + 1} F_\infty(h;x) \). Consequently,

\[ |A^{-1}_q f(x)| \leq \frac{2\theta}{\theta + 1} \left( (1-x) \int_0^x t |h| dt + x \int_0^1 (1-t) |h| dt \right) ||h||_\infty \]

\[ = \frac{\theta}{\theta + 1} x(1-x)||h||_\infty \leq \frac{\theta}{4(\theta + 1)} ||f||_0, \]  \quad (4.3) \quad (4.4)

and this leads to (4.2).

**Remark 4.1.** Further below we will use the notation \( \Psi(x) := x(1-x) \), and

\[-A^{-1}_\infty(\Psi h) := 2 \cdot F_\infty(h), \quad h \in C[0,1],\]

in order to also cover the Bernstein case.

Another useful result reads as follows.

**Lemma 4.1.** For all \( p \in \Pi \cap C_0[0,1] \) we have

\[ \lim_{n \to \infty} A^n_q p = -A^{-1}_q p, \]  \quad (4.5)

uniformly on \([0,1] \).

**Proof.** The polynomials \( p_j^* \) from (2.7) satisfy

\[ x(1-x)(p_j^*)'(x) = -j(j-1)p_j^*(x), \quad x \in [0,1], j \geq 0 \]  \quad (4.6)

(see, e.g., [4], p.155). This yields \( A_q p_j^* = -\frac{\theta + 1}{2\theta} j(j-1)p_j^*, j \geq 0 \), and, moreover,

\[ A_q \left( \sum_{j=2}^m \frac{2}{j(j-1)} \mu_j^*(p)p_j^* \right) = -\frac{\theta + 1}{\theta} \sum_{j=2}^m \mu_j^*(p)p_j^* \]

(4.7)
for all \( p \in \Pi_m \cap C_0[0,1] \). According to (4, (4.18)),
\[
\frac{q}{q+1} \sum_{j=2}^{m} \frac{2}{j(j-1)} \mu_j^*(p)p_j^* = -A_{q}^{-1}p,
\]
(4.8)
for all \( p \in \Pi_m \cap C_0[0,1] \). Now (4.5) is a consequence of (3.12) and (4.8).
\( \square \)

5 The convergence of \( A_{n}^e \) on \( C_0[0,1] \)

One main result of the paper is contained in

**Theorem 5.1.** For all \( f \in C_0[0,1] \),
\[
\lim_{n \to \infty} A_{n}^e f = -A_{q}^{-1}f,
\]
uniformly on \([0,1]\).

**Proof.** Let \( f \in C_0[0,1], f(x) = x(1-x)h(x), h \in C[0,1] \). Consider the polynomials
\( p_i(x) := x(1-x)B_i h(x) \), where \( B_i \) are the classical Bernstein operators, \( i \geq 1 \). Then \( p_i \in C_0[0,1], i \geq 1 \), and \( \lim_{i \to \infty} ||p_i - f||_0 = \lim_{i \to \infty} ||B_i h - h||_\infty = 0 \).

Let \( \varepsilon > 0 \) and fix \( i \geq 1 \) such that
\[
||p_i - f||_0 \leq \frac{2q+2}{3q+2} \varepsilon.
\]
(5.2)

Then, according to Lemma 4.1 there exists \( n_\varepsilon \) such that
\[
||A_{n}^e p_i + A_{q}^{-1}p_i||_\infty \leq \frac{2q+2}{3q+2} \varepsilon, n \geq n_\varepsilon.
\]
(5.3)

Now using (3.3) and (3.8) we infer
\[
||A_{n}^e f - A_{n}^e p_i||_\infty \leq \frac{1}{4} ||A_{n}^e f - A_{n}^e p_i||_0 \leq \frac{1}{4} ||A_{n}^e||_0 ||f - p_i||_0 \leq \frac{q}{4(q+1)} \frac{2q+2}{3q+2} \varepsilon,
\]
so that
\[
||A_{n}^e f - A_{n}^e p_i||_\infty \leq \frac{q}{2(3q+2)} \varepsilon.
\]
(5.4)

On the other hand, (4.12) and (3.12) yield
\[
||A_{q}^{-1}f - A_{q}^{-1}p_i||_\infty \leq \frac{q}{4(q+1)} ||f - p_i||_0 \leq \frac{q}{2(3q+2)} \varepsilon.
\]
(5.5)

Finally, using (5.3), (5.4) and (5.5) we obtain, for all \( n \geq n_\varepsilon \),
\[
||A_{n}^e f + A_{q}^{-1}f||_\infty \leq ||A_{n}^e f - A_{n}^e p_i||_\infty + ||A_{n}^e p_i + A_{q}^{-1}p_i||_\infty + ||A_{q}^{-1}f - A_{q}^{-1}p_i||_\infty \leq \varepsilon,
\]
and this concludes the proof. \( \square \)
On \((C[0,1], ||\cdot||_{\infty})\) consider the linear operator \(H_n^{\infty} := A_n^{\infty} - (-A_n^{-1})\) given by

\[
C[0,1] \ni h \mapsto A_n^{\infty}(\Psi h; x) = \frac{1}{nq+1} \sum_{k=0}^{\infty} (U_n^{\infty})^k(\Psi h; x) \in C_0[0,1]
\]

\[
C[0,1] \ni h \mapsto -A_n^{-1}(\Psi h; x) = \frac{2q}{q+1} \left[ (1-x) \int_0^x th(t)dt + x \int_x^1 (1-t)h(t)dt \right]
\]

\[
H_n^{\infty}(h; x) = \frac{2q}{q+1} F_{\infty}(h; x) \in C_0[0,1]
\]

**Theorem 5.2.** Let \(h \in C[0,1], q > 0, n \geq \frac{4q+6}{q}, \epsilon = \sqrt{\frac{q+2}{nq+2}} \leq \frac{1}{2}\) and \(\Psi(x) = x(1-x)\). Then

\[
|H_n^{\infty}(h; x)| \leq \Psi(x) \left[ \frac{2q}{3(q+1)} \sqrt{\frac{q+2}{nq+2}} \omega_1(h; \epsilon) + \frac{3}{4} \left( \frac{2q}{q+1} + \frac{2q}{3(q+1)} \sqrt{\frac{q+2}{nq+2}} \omega_2(h; \epsilon) \right) \right]. \tag{5.6}
\]

**Proof.** Let \(h \in C[0,1]\) be fixed, and \(g \in C^2[0,1]\) be arbitrary.

Then \(|H_n^{\infty}(h; x)| \leq |H_n^{\infty}(h-g; x)| + |H_n^{\infty}(g; x)| = |E_1| + |E_2|\). Here

\[
|E_1| = |A_n^{\infty}(\Psi(h-g); x) - (-A_n^{-1}(\Psi(h-g); x))|
\]

\[
= |A_n^{\infty}(\Psi(h-g); x) - \frac{2q}{q+1} F_{\infty}(h-g; x)|
\]

\[
\leq ||h-g||_{\infty} A_n^{\infty}(\Psi; x) + \frac{2q}{q+1} |F_{\infty}(h-g; x)|
\]

\[
= ||h-g||_{\infty} \frac{q}{q+1} \Psi(x) + \frac{2q}{q+1} ||h-g||_{\infty} \frac{1}{2} \Psi(x)
\]

\[
= \frac{2q}{q+1} \Psi(x)||h-g||_{\infty}
\]

and

\[
|E_2| = |A_n^{\infty}(\Psi g; x) - (-A_n^{-1}(\Psi g; x))|
\]

For \(g \in C^2[0,1]\) one has \(F_{\infty} := F_{\infty}(g) \in C^4[0,1], F_{\infty}'' = -g, F_{\infty}''' = -g', F_{\infty}^{(4)} = -g''\). Moreover, by Taylor’s formula we obtain for any points \(y, t \in [0,1]\):

\[
F_{\infty}(t) = F_{\infty}(y) + F_{\infty}'(y)(t-y) + \frac{1}{2} F_{\infty}''(y)(t-y)^2 + \frac{1}{6} F_{\infty}'''(y)(t-y)^3 + \Theta_y(t) \tag{5.7}
\]

where

\[
\Theta_y(t) := \frac{1}{6} \int_y^t (t-u)^3 F_{\infty}^{(4)}(u)du.
\]
Fix \( y \) and consider (5.7) as an equality between two functions in the variable \( t \). Applying to this equality the operator \( U_n^g(\cdot, y) \) one arrives at

\[
U_n^g(F_\infty, y) = F_\infty(y) + \frac{1}{2} F''(y) U_n^g((t - y)^2; y) + \frac{1}{6} F''(y) U_n^g((t - y)^3; y) + U_n^g(\Theta_y; y)
\]

This implies

\[
\frac{1}{2} g(y) U_n^g((e_1 - y)^2; y) - F_\infty(y) + U_n^g(F_\infty, y) = -\frac{1}{6} g'(y) U_n^g((e_1 - y)^3; y) + U_n^g(\Theta_y; y).
\]

In the above equality we rewrite the left hand side as \( \frac{1}{2} g(y) U_n^g((e_1 - y)^2; y) - (I - U_n^g)(F_\infty, y) \). Thus we have

\[
g(y) U_n^g((e_1 - y)^2; y) - 2(I - U_n^g)(F_\infty, y) = -\frac{1}{3} g'(y) U_n^g((e_1 - y)^3; y) + 2U_n^g(\Theta_y; y).
\]

Application of \( A_n^g \) yields

\[
A_n^g(g(\cdot) U_n^g((e_1 - \cdot)^2; \cdot); x) - 2A_n^g(I - U_n^g)(F_\infty, x) = \frac{1}{3} A_n^g(g'(\cdot) U_n^g((e_1 - \cdot)^3; \cdot); x) + 2A_n^g(Q; x)
\]

where \( Q(y) := U_n^g(\Theta_y; y) \). The first five moments are given by (see [7, Cor. 2.1])

\[
\begin{align*}
U_n^g(e_0; y) &= 1, \\
U_n^g(e_1 - y; y) &= 0, \\
U_n^g((e_1 - y)^2; y) &= \frac{(\rho + 1) \Psi(y)}{n\rho + 1}, \\
U_n^g((e_1 - y)^3; y) &= \frac{(\rho + 1)(\rho + 2) \Psi(y) \Psi'(y)}{(n\rho + 1)(n\rho + 2)}, \\
U_n^g((e_1 - y)^4; y) &= \frac{3\rho(\rho + 1)^2 \Psi^2(y) n}{(n\rho + 1)(n\rho + 2)(n\rho + 3)} \\
&\quad + \frac{-6(\rho + 1)(\rho^2 + 3\rho + 3) \Psi^2(y) + (\rho + 1)(\rho + 2)(\rho + 3) \Psi(y)}{(n\rho + 1)(n\rho + 2)(n\rho + 3)}.
\end{align*}
\]

In the above expression we have \( 2A_n^g(I - U_n^g)(F_\infty, x) = \frac{2\rho}{n\rho + 1} F_\infty(x) = \frac{2\rho}{n\rho + 1} F_\infty(g; x) \).

Also \( A_n^g(g(\cdot) U_n^g((e_1 - \cdot)^2; \cdot); x) = A_n^g(g(\cdot) \frac{\rho + 1}{n\rho + 1} \Psi(\cdot); x) = \frac{\rho + 1}{n\rho + 1} A_n^g(\Psi g; x) \).
Hence (5.8) can be written as

\[
\left| \frac{\varrho + 1}{nq + 1} A_n^\varrho (\Psi g; x) - 2 \frac{\varrho}{nq + 1} F_\infty (g; x) \right|
\]

\[
= \left| -\frac{1}{3} g'(\cdot) U_n^\varrho (x) - 2 A_n^\varrho (Q; x) \right|
\]

\[
\leq \frac{1}{3} A_n^\varrho \left( \frac{(\varrho + 1)(\varrho + 2)}{(nq + 1)(nq + 2)} \Psi'(\cdot) \Psi(\cdot); x \right) + |2 A_n^\varrho (Q; x)|
\]

\[
\leq \frac{1}{3} (nq + 1)(nq + 2) ||g'||_\infty \frac{\varrho}{\varrho + 1} \Psi(x) + |2 A_n^\varrho (Q; x)|.
\]

Multiplying the outermost sides of the latter inequality by \(\frac{nq + 1}{\varrho + 1}\) gives

\[
|E_2| = \left| A_n^\varrho (\Psi g; x) - 2 \frac{\varrho}{\varrho + 1} F_\infty (g; x) \right|
\]

\[
\leq \frac{\varrho (\varrho + 2)}{3(nq + 2)(\varrho + 1)} \Psi(x) ||g'||_\infty + 2 \frac{nq + 1}{\varrho + 1} |A_n^\varrho (Q; x)|.
\]

In the last summand we have \(Q(y) = U_n^\varrho (\Theta y; y)\) thus

\[
|U_n^\varrho (\Theta y; y)| \leq \frac{1}{6} U_n^\varrho (x) ||g'''||_\infty
\]

\[
\leq \frac{1}{6} \cdot \frac{7}{4} \cdot \frac{(\varrho + 1)(\varrho + 2)(\varrho + 3)}{\varrho (nq + 1)(nq + 2)} \Psi(y) ||g'''||_\infty.
\]

Hence

\[
\frac{2(nq + 1)}{\varrho + 1} |A_n^\varrho (Q; x)| \leq \frac{2(nq + 1)}{\varrho + 1} \cdot \frac{7}{24} \cdot \frac{(\varrho + 1)(\varrho + 2)(\varrho + 3)}{\varrho (nq + 1)(nq + 2)} A_n^\varrho (\Psi; x) ||g'''||_\infty
\]

\[
= \frac{7}{12} \cdot \frac{(\varrho + 2)(\varrho + 3)}{(\varrho + 1)(nq + 2)} \Psi(x) ||g'''||_\infty.
\]

This leads to

\[
|E_2| \leq \frac{\varrho (\varrho + 2)}{3(nq + 2)(\varrho + 1)} \Psi(x) ||g'||_\infty + \frac{7}{12} \cdot \frac{(\varrho + 2)(\varrho + 3)}{(\varrho + 1)(nq + 2)} \Psi(x) ||g'''||_\infty.
\]

\[
= \frac{(\varrho + 2)}{3(nq + 2)(\varrho + 1)} \Psi(x) \left\{ \varrho ||g'||_\infty + \frac{7}{4} (\varrho + 3) ||g'''||_\infty \right\}.
\]

Hence for \(h \in C[0, 1]\) fixed, \(g \in C^2[0, 1]\) arbitrary we have

\[
|H_n^\varrho (h; x)| = |E_1| + |E_2|
\]

\[
\leq \frac{2 \varrho}{\varrho + 1} \Psi(x) ||h - g||_\infty + \frac{(\varrho + 2)}{3(nq + 2)(\varrho + 1)} \Psi(x) \left\{ \varrho ||g'||_\infty + \frac{7}{4} (\varrho + 3) ||g'''||_\infty \right\}.
\]
Next we choose \( g = h , 0 < \varepsilon = \sqrt{\frac{\varrho+2}{n \varrho+2}} \leq \frac{1}{2} \) and by applying Lemmas 2.1 and 2.4 in [5] we obtain

\[
\| h - g \|_\infty \leq \frac{3}{4} \omega_2(h; \varepsilon) \\
\| g' \| \leq \frac{1}{\varepsilon}[2 \omega_1(h; \varepsilon) + \frac{3}{2} \omega_2(h; \varepsilon)] \\
\| g'' \| \leq \frac{3}{2\varepsilon^2} \omega_2(h; \varepsilon).
\]

Thus

\[
| H_n^\varrho(h; x) | \leq \Psi(x) \left[ \frac{2 \varrho}{3(\varrho + 1)} \sqrt{\frac{\varrho + 2}{n \varrho + 2}} \omega_1(h; \varepsilon) + \right.
\frac{3}{4} \left( \frac{2 \varrho}{\varrho + 1} + \frac{2 \varrho}{3(\varrho + 1)} \sqrt{\frac{\varrho + 2}{n \varrho + 2}} + \frac{7(\varrho + 3)}{6(\varrho + 1)} \right) \omega_2(h; \varepsilon) \right].
\]

\[\square\]

**Remark 5.1.** If we let \( 1 \leq \varrho \to \infty \), then for all \( n \geq 10 \)

\[
\lim_{\varrho \to \infty} | H_n^\varrho(h; x) | = \lim_{\varepsilon \to \infty} | A_n^\varrho(\Psi h; x) - (-A_n^{-1})(\Psi h; x) |
\]

\[
= | A_n^\infty(\Psi h; x) - (-A_n^{-1})(\Psi h; x) |
\]

\[
\leq 3 \Psi(x) \left[ \frac{1}{\sqrt{n}} \omega_1(h; \frac{1}{\sqrt{n}}) + \omega_2(h; \frac{1}{\sqrt{n}}) \right].
\]

This is a quantitative form of Păltănea's convergence result in [3, Th. 3.2].

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