On the simultaneous identification of two space dependent coefficients in a quasilinear wave equation

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Abstract

This paper considers the Westervelt equation, one of the most widely used models in nonlinear acoustics, and seeks to recover two spatially-dependent parameters of physical importance from time-trace boundary measurements. Specifically, these are the nonlinearity parameter $\kappa(x)$ often referred to as $B/A$ in the acoustics literature and the wave speed $c_0(x)$. The determination of the spatial change in these quantities can be used as a means of imaging. We consider identifiability from one or two boundary measurements as relevant in these applications. More precisely, we provide results on local uniqueness of $\kappa(x)$ from a single observation and on simultaneous identifiability of $\kappa(x)$ and $c_0(x)$ from two measurements. For a reformulation of the problem in terms of the squared slowness $s = 1/c_0^2$ and the combined coefficient $\eta = \frac{\kappa}{c_0^2}$ we devise a frozen Newton method and prove its convergence. The effectiveness (and limitations) of this iterative scheme are demonstrated by numerical examples.

Keywords: nonlinearity parameter tomography, damped nonlinear wave equation, ultrasound.

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1 Introduction

Imaging with ultrasound has a long and successful history based on a vast range of applications. However, as is often the case, the use of lower frequencies naturally leads to lower resolution and at higher frequencies sound propagation is affected by scattering and stronger attenuation. Recently, ultrasound-based techniques such as nonlinearity parameter imaging [5, 6, 8, 16, 29, 31, 35, 36], harmonic imaging [3, 30, 31], and vibro-acoustography [11, 12, 21, 26, 27] have been developed to overcome these drawbacks and improve imaging quality. They make use of nonlinear effects that arise at higher intensities or when waves interact and are characterised by a multiplicative coefficient that is usually

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called parameter of nonlinearity and denoted by $B/A$. We will here use the mathematically convenient abbreviations $\kappa$ for a quantity containing $B/A$. This coefficient depends on tissue properties and therefore varies in the spatial direction, $\kappa = \kappa(x)$.

While ultrasound imaging relies on the propagation of sound waves and is therefore physically and mathematically correctly described by some wave-type partial differential equation (PDE), algorithms implemented in modern ultrasound scanners make use of model simplifications that allow one to apply methods from signal processing (beamforming, filtering) to generate an image based on the principles of transmission and reflection, based on differences in the acoustic impedance $Z$. These simplifications are not able to capture nonlinearity so that one has to return to the PDE model and consider $\kappa = \kappa(x)$ (and often also the speed of sound $c = c(x)$) as a spatially variable coefficient. This is actually already being done in ultrasound tomography [2, 13, 14, 17, 28] for $c$ in a linear wave equation, but with $\kappa = 0$. We mention in passing that in principle the mass density $\rho$ also varies in the spatial direction. However, in ultrasound imaging, this coefficient does not play a significant role and is therefore usually neglected.

A novel approach of ultrasound tomography [2, 13, 14, 17, 28] aims at recovering the speed of sound $c_0$ as a space dependent function in view of the fact that the impedance $Z$ responsible for transmission and reflection is tied to $c_0$ via the identity $Z = \rho c_0$, where $\rho$ is the mass density. Likewise, nonlinearity parameter imaging needs to be viewed as the identification of a space-dependent coefficient $\kappa = \kappa(x)$ in a PDE and as such it will often arise alongside together with the recovery of the spatially varying sound speed $c_0 = c_0(x)$.

In the following subsections we provide more background on the mathematical models. In particular we will show at which position in the PDE these coefficients appear which of course is a factor crucial for their recovery. We then describe the inverse problem and the basic method of its solution.

We consider, as one of the most established classical model of nonlinear acoustics, the Westervelt equation in pressure formulation

\[
    u_{tt} - c_0^2 \Delta u - b \Delta u_t = \kappa(u^2)_{tt} + g \text{ in } (0, T) \times \Omega, \tag{1}
\]

where $u$ is the acoustic pressure, $c_0$ the speed of sound, $b$ the diffusivity of sound, $\rho$ the mass density, and $\kappa = \frac{\beta}{\rho c_0^2} = \frac{1}{\rho c_0^2} (\frac{B}{2A} + 1)$ contains the nonlinearity parameter $\beta$ or $B/A$. We assume (1) to hold in a domain $\Omega \subseteq \mathbb{R}^3$ and equip it with initial conditions $u(t = 0) = u_0$, $u_t(t = 0) = u_1$, as well as absorbing or impedance boundary conditions on the rest of the boundary to enable restriction to a bounded computational domain $\Omega$, which without loss of generality we can assume to be smooth. The space- and time-dependent interior source term $g$ in (1) models excitation by a piezoelectric transducer array.

\[1\] More precisely, the PDE is $\frac{1}{\lambda(x)}u_{tt} - \nabla \cdot (\frac{\rho(x)}{\lambda(x)} \nabla u) - bD u = \kappa(u^2)_{tt} + g$ with $u$ being the pressure, $\lambda$ the bulk modulus, $\rho$ the mass density, and $c_0 = \frac{\lambda}{\rho}$ the sound speed, (cf., e.g., [3] for the linear case). As mentioned above, spatial variability of $\rho$ is not relevant in our context; rather, dependence of $c_0$ on $x$ is due to variability of $\lambda$. 

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The pressure data $h$ taken at the receiver array is expressed as a Dirichlet trace on some manifold $\Sigma$ immersed in the computational domain $\Omega$ or attached to its boundary $\Sigma \in \partial \Omega$

$$h(t, x) = u(t, x), \quad (t, x) \in (0, T) \times \Sigma.$$  \hspace{1cm} (2)

Note that $\Sigma$ may as well just be a subset of discrete points on a manifold.

The inverse problem of nonlinearity parameter tomography consists of reconstructing $\kappa = \kappa(x)$ from measurements (2). Often, the speed of sound varies in space as well $c = c(x)$ and needs to be recovered alongside with $\kappa$.

We refer to [1] [22] [23] [33] for results related to the identification of the nonlinearity coefficient $\kappa$ alone. In [1] its uniqueness from the whole Neumann-Dirichlet map (instead of the single measurement (2)) is shown; [33] provides a uniqueness and conditional stability result for the linearised problem of identifying $\kappa$ in a higher order model of nonlinear acoustics in place of the Westervelt equation. In [22] [23] we have proven injectivity of the linearised forward operator mapping $\kappa$ to $h$ in the Westervelt equation with classical strong damping and also with some fractional damping models as relevant in ultrasonics. This will serve as a basis for proving local uniqueness of $\kappa$ by means of the Inverse Function Theorem.

Besides this, the aim of this paper is to study simultaneous identification of $\kappa$ and $c_0$ as space variable functions. Indeed, we will show that $\kappa(x)$ is locally unique even for unknown $c_0(x)$ in general space dimension $d \in \{1, 2, 3\}$ in Section 3.2 and provide a result on simultaneous identifiability of $\kappa(x)$ and $c_0(x)$ in one space dimension based on inverse Sturm-Liouville theory in Section 3.3. Moreover, injectivity of the linearised forward operator for the simultaneous recovery of $\kappa(x)$ and $c_0(x)$ in dimension $d \in \{1, 2, 3\}$ from measurements with two excitations is shown in Section 3.4. This serves as a basis for applying a frozen Newton method and showing its convergence in Section 2.1. Before doing the uniqueness analysis, we provide reconstruction results in Section 2.2.

### 1.1 The inverse problem

Consider identification of the space dependent nonlinearity coefficient $\kappa(x)$ and sound speed $c_0(x)$ (contained in the operator $\mathcal{A}$, see (3) below) for the attenuated Westervelt equation in pressure form

$$\begin{align*}
(u - \kappa(x) u^2)_{tt} + c^2 \mathcal{A}u + D[u] &= r \quad \text{in } \Omega \times (0, T) \\
\partial_{\nu} u + \gamma u &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
u(0) &= 0, \quad u_t(0) = 0 \quad \text{in } \Omega.
\end{align*}$$  \hspace{1cm} (3)

where (with physical units in brackets)

- $u$ . . . pressure $[gm^{-1}s^{-2}]$
- $c$ . . . sound speed $[ms^{-1}]$
- $\kappa = \frac{B/A+2}{6c_0^2}$ . . . nonlinearity coefficient
- $\varrho$ . . . mass density $[gm^{-3}]$
\[ B/A \ldots \text{nonlinearity parameter [1]} \]
\[ b \ldots \text{diffusivity of sound [m}^2\text{s}^{-1}] \]
from observations
\[ h(x, t) = u(x, t), \quad x \in \Sigma, \quad t \in (0, T). \quad (4) \]

Here \( D \) is defined by one of the two following fractional damping models
\[
D = bA^\beta \partial_t^\alpha \quad \text{(combination of Caputo-Wismer-Kelvin and Chen-Holm – CH)}
\]
\[
D = b_1A\partial_t^{\alpha_1} + b_2\partial_t^{\alpha_2+2} \quad \text{(fractional Zener – FZ)}
\]
(for more details see, e.g., [23] and the references therein, in particular [9, 32, 10] for CH and [15, 25, 7] for FZ)

In equation (3), \( c > 0 \) is the constant mean wave speed, and
\[
\mathcal{A} = -\frac{c_0(x)^2}{c^2} \nabla^2 \quad \text{with} \quad c_0, \quad \frac{1}{c_0} \in L^\infty(\Omega) \quad (5)
\]
contains the possibly spatially varying coefficient \( c_0(x) > 0 \) and is equipped with the above impedance boundary conditions. To achieve self-adjointness of \( \mathcal{A} \), we use the weighted \( L^2 \) inner product with weight function \( c^2/c_0^2(x) \), that is, \( \|v\|_{L^2(c^2/c_0^2(\Omega))} = \int_\Omega \frac{c^2}{c_0^2(x)} v^2(x) \, dx \).

We can think of \( c_0 \) as being spatially variable and of \( \frac{c_0}{c^2} \sim 1 \) to be normalised, while the magnitude of the wave speed is given by the constant \( c \). Due to compactness of its inverse, the operator \( \mathcal{A} \) has an eigensystem \( (\lambda_j, (\varphi_j^k)_{k \in K^j})_{j \in \mathbb{N}} \) (where \( K^j \) is an enumeration of the eigenspace corresponding to \( \lambda_j \)). This allows for a diagonalisation of the operator as \( (\mathcal{A}v)(x) = \sum_{j=1}^\infty \lambda_j \sum_{k \in K^j} \langle v, \varphi_j^k \rangle \varphi_j^k(x) \).

The time fractional derivatives appearing in the damping models are defined by the Djrbashian-Caputo derivative
\[
\partial_t^\alpha u = I^{1-\alpha}u_t
\]
with the Abel integral operator
\[
(I^{1-\alpha}v)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{v(s)}{(t-s)^\alpha} \, ds
\]
and \( \alpha \in (0, 1) \). For defining fractional powers of the operator \( \mathcal{A} \) in the CH case, we use the spectral definition
\[
(\mathcal{A}^\beta v)(x) = \sum_{j=1}^\infty \lambda_j^\beta \sum_{k \in K^j} \langle v, \varphi_j^k \rangle \varphi_j^k(x).
\]

Equation (3) is the formulation we will use for the uniqueness proofs, where it is convenient to attach \( c_0^2(x) \) to \(-\nabla\) within \( \mathcal{A} \) in order to be able to apply inverse Sturm-Liouville theory. For these proofs, the inverse problem will make use of the forward operator \( F \) that takes \( \kappa \) to the residues of \( \text{tr}_\Sigma \hat{u} \) (the Laplace transform of the trace of \( u \) on the observation set \( \Sigma \)) at the poles of the Laplace transformed observation \( \hat{h} \). In order to be able to take Laplace
transforms, throughout Section 3 we will assume to have observations on the whole positive timeline, which in case of \( r \) being analytic with respect to time (for example, just vanishing) from a time instance \( T_0 \) on, follows by analytic continuation of the Fourier components of \( u \), which will be shown to satisfy linear constant coefficient ODEs from a time instance \( T_0 \) on in Section 3.

For our numerical reconstructions, we will work with the following alternative formulation. We rewrite (3) using the new variables \( s(x), \eta(x) \) as

\[
\begin{aligned}
(s(x)u - \eta(x)u^2)_{tt} - \Delta u + \tilde{D}u &= \tilde{r} & \text{in } \Omega \times (0, T) \\
\partial_\nu u + \gamma u &= 0 \text{ on } \partial \Omega \times (0, T), \quad u(0) = 0, \quad u_t(0) = 0 & \text{in } \Omega
\end{aligned}
\]

(6)

where (again with physical units in brackets)

\[
\begin{align*}
\eta &= \frac{\kappa}{c_0^2} = \frac{\beta_0}{c_0^2} = \frac{B/A+2}{c_0^4} = \frac{B/A+2}{c_0^2} s^2 \ldots \text{ (combined) nonlinearity coefficient } [g^{-1} m^{-1} s^4] \\
\tilde{D} &= -\tilde{b} \Delta D_0^c \\
\tilde{b} &= \frac{b}{c_0^2} \ldots [s].
\end{align*}
\]

Note that we neglect variability of the damping and driving terms with \( c_0(x) \) and assume \( \tilde{D} \) to come with constant and known coefficients; incorporation of this \( c_0(x) \) dependence would lead to the PDE

\[
\begin{aligned}
(s(x)u - \eta(x)u^2)_{tt} - \Delta u + \tilde{D}u &= s r & \text{in } \Omega \times (0, T).
\end{aligned}
\]

(7)

Neglecting this dependency in the damping term can be justified by smallness of the damping coefficient so that spatial variability of this term has a very minor effect. Neglecting spatial variability of \( s \) in the excitation term does not matter due to the fact that the support of \( r \) is typically remote from the region of variable (and unknown) sound speed.

The inverse problem of reconstructing \( \eta(x), s(x) \) from the observations (4) can then be written as

\[
F(\eta, s) = h,
\]

where \( F = C \circ S \) and with the parameter-to-state map \( S : (\eta, s) \mapsto u \) where \( u \) solves (4) and is subject to the observation operator \( C : u \mapsto \text{tr}_\Sigma u \).

**Notation**

Below we will make use of the spaces \( \dot{H}^s(\Omega) \) induced by the norm

\[
\|v\|_{\dot{H}^s(\Omega)} = \left( \sum_{j=1}^{\infty} \lambda_j^s \sum_{k \in K^\lambda_j} |(v, \varphi_j^k)|^2 \right)^{1/2}
\]

(8)

with the eigensystem \((\lambda_j, \varphi_j)\) of the operator \( \mathcal{A} \).
Moreover, the Bochner-Sobolev spaces $L^p(0,T; Z)$, $H^q(0,T; Z)$ with $Z$ some Lebesgue or Sobolev spaces and $T$ a finite or infinite time horizon will be used.

We denote the Laplace transform of a function $v \in L^1(0, \infty)$ by $\hat{v}(z) = \int_0^\infty e^{-zt}v(t)\,dt$ for all $z \in \mathbb{C}$ such that this integral exists.

2 Reconstruction of the nonlinearity coefficient and sound speed by a regularised Newton scheme

2.1 Well-definedeness and convergence a frozen Newton method

Since it allows us to work with a particularly simple elliptic operator - the negative Laplacian – and moreover, a formulation using the squared slowness $s = \frac{1}{c^2}$ is known to mitigate nonlinearity, in this section we use the alternative formulation (51), that is,

\[
\left(s(x)u - \eta(x)u^2\right)_{tt} - \triangle u + \bar{D}u = \bar{r} \quad \text{in } \Omega \times (0, T)
\]
\[
\partial_\nu u + \gamma u = 0 \quad \text{on } \partial\Omega \times (0, T), \quad u(0) = 0, \quad u_t(0) = 0 \quad \text{in } \Omega.
\]

Reconstruction of $\eta(x)$, $s(x)$ from observations (4) then amounts to solving the operator equation

\[
F(\eta, s) = h,
\]

where $F = C \circ S$, with the parameter-to-state map $S : (\eta, s) \mapsto u$ where $u$ solves (51) with homogeneous initial and boundary conditions, and the observation operator $C : u \mapsto \text{tr}_\Sigma u$.

By a slight extension of [18, Theorem 1.1 and Proposition 3], the parameter-to-state map $S : D(F) \to H^1(0,T; H^2(\Omega)) \cap W^{1,\infty}(0,T; H^1(\Omega)) \cap W^{2,\infty}(0,T; L^2(\Omega))$ is well-defined on

\[
D(F) := \{(\eta, s) \in L^2(\Omega)^2 : \eta \in L^\infty(\Omega), \ s, \ \frac{1}{s} \in L^\infty(0,T; L^\infty(\Omega)) \cap H^1(0,T; L^3(\Omega))\}
\]

for a smooth bounded domain $\Omega \subseteq \mathbb{R}^d$, $d \in \{1, 2, 3\}$, $\bar{r} \in L^2(0,T; L^2(\Omega)) \cup H^1(0,T; H^{-1}(\Omega))$ with $\bar{r}$ small enough in this norm. (Note that we will have to deal with potentially time-dependent $s$ below.) By Sobolev’s Lemma, this implies that evaluation of $u$ at single points or on a smooth manifold is feasible in a continuous way and thus $F : D(F) \to Y$ is well-defined for any $Y \subseteq L^\infty(0,T; C(\Sigma))$ in case $\Sigma$ is a smooth manifold or $Y \subseteq L^\infty(0,T; \ell^\infty(\Sigma))$ in case $\Sigma$ is a set of discrete points. To make use of a Hilbert space structure, we will simply set $Y = L^2(0,T; L^2(\Sigma))$ or $Y = L^2(0,T; \ell^2(\Sigma))$, respectively.

The linearisation of the forward operator $F : (s, \eta) \mapsto \text{tr}_\Sigma u$ is formally given by

\[
F'(s, \eta)(ds, d\eta) = \text{tr}_\Sigma du, \quad \text{where } du \text{ solves}
\]

\[
\left(s(x) du - 2\eta(x) u du\right)_{tt} - \triangle du + \bar{D}du = -(ds(x) u - d\eta(x) u^2)_{tt} \quad \text{in } \Omega \times (0, T).
\]
In particular for applying a frozen Newton method to (5), we linearise at \( s = 1/\tilde{c}^2 \) (for some constant \( \tilde{c} \)), \( \eta = 0 \), that is, we use \( F'(0, 1/\tilde{c}^2)(d\eta, ds) = \text{tr}_x du \), where
\[
\frac{1}{\tilde{c}^2}du_{tt} - \Delta du + \tilde{D}du = -(ds(x)u - d\eta(x)u^2)_{tt} \quad \text{in } \Omega \times (0,T)
\]

With an extension of the squared slowness to a space and time dependent function, we can verify a range invariance condition on \( F' \) which together with our linearised uniqueness Theorem 3.3 guarantees convergence of a penalised frozen Newton method, see [20]. Indeed by setting
\[
d\eta(\eta, s) := \eta - \eta_0, \quad ds(\eta, s) := \frac{1}{u_0}\left((s - s_0)u - (\eta - \eta_0)(u^2 - u_0^2) - \eta_0(u - u_0)^2\right)
\]
we can satisfy the identity
\[
F(\eta, s) - F(\eta_0, s_0) = F'(\eta_0, s_0)(d\eta(\eta, s), ds(\eta, s))
\]
with
\[
ds(\eta, s) - (s - s_0) = \frac{u - u_0}{u_0}\left((s - s_0) - (\eta - \eta_0)(u + u_0) - \eta_0(u - u_0)\right),
\]
which will allow us to establish a bound of the form
\[
\exists c \in (0, 1) \forall x \in U(\subseteq X) : \|d\eta(\eta, s), ds(\eta, s)\| \leq c\|\eta - \eta_0, s - s_0\|_X
\]
for some sufficiently small neighborhood \( U \) of the exact solution \((\eta^\dagger, s^\dagger)\) and some appropriately chosen Hilbert space \( X \). In order for the second identity in (11) to make sense, we consider \( s \) as a space and time dependent function and, in order to restore its time independence during reconstruction, use a penalization operator containing differentiation with respect to time.

In order to achieve uniqueness of \( \eta \) and \( s \), we need observations for two different excitations (see Theorem 3.3 below) and therefore define the forward operator \( \tilde{F} = (F^1, F^2) : \vec{x} := (\eta, s^1, s^2) \rightarrow (\text{tr}_x u^1, \text{tr}_x u^2) \) with \( u^i \) solving (5) with \( \tilde{r}^i, s = s^i, i \in \{1, 2\} \).

Therewith, the method
\[
x_{n+1}^\delta = x_n^\delta + (K^*K + P^*P + \alpha_nI)^{-1}\left(K^*(\tilde{h}^\delta - \tilde{F}(x_n^\delta)) - P^*\tilde{F}x_n^\delta + \alpha_n(x_0^\delta - x_n^\delta)\right)
\]
where \( \tilde{e} = (\eta, s^1, s^2) \), \( K^* \) denotes the Hilbert space adjoint of \( K = \tilde{F}'(0, 1) : X \rightarrow Y \), \( \alpha_n = \alpha_0\theta^n \) for some \( \theta \in (0, 1) \) can be shown to converge to the true solution \( \vec{x}^\dagger = (\eta^\dagger, s^{1\dagger}, s^{2\dagger}) \) in the noise free case \( \tilde{h}^\delta = \tilde{h} = \tilde{F}(\eta^\dagger, s^\dagger) \) as \( n \rightarrow \infty \). In case of noisy data with noise level \( \delta \) satisfying the bound
\[
\|\tilde{h}^\delta - \tilde{F}(\eta^\dagger, s^\dagger)\|_{L^2(0,T;L^2(\Omega))} \leq \delta
\]
we have to stop the iteration according to
\[
n_\delta(\delta) \rightarrow 0, \quad \delta \sum_{j=0}^{n_\delta(\delta)-1} c^j\alpha_n^{-1/2} \rightarrow 0 \quad \text{as } \delta \rightarrow 0
\]
with c as in (12).

As a parameter space we use the Hilbert space

\[ X = X_\eta \times X_2 \] with \( X_\eta = H^2(\Omega) \), \( X_2 = H^1(0,T;H^2(\Omega)) \) (16)

With this choice and \( Z = L^2(0,T;L^2(\Omega)) \), the operator \( P : X \to Z \), \((\eta, s^1, s^2) \mapsto (s^1_1, s^1, s^1 - s^2\)) is bounded and (12) immediately follows from the Banach algebra estimate

\[ \|a \cdot b\|_{H^1(0,T;H^2(\Omega))} \leq C \|a\|_{H^1(0,T;H^2(\Omega))} \|b\|_{H^1(0,T;H^2(\Omega))}, \quad a, b \in H^1(0,T;H^2(\Omega)). \]

From [20, Theorem 2.2] and Theorem 3.3 we thus conclude the following convergence result.

**Theorem 2.1** Let the conditions of Theorem 3.3 on the observation set \( \Sigma \) and on the excitations \( \tilde{r}^1, \tilde{r}^2 \) be satisfied. Let \( \vec{x}_0 = (\eta_0, s^1_0, s^2_0) \in U := B_\rho(\vec{x}_\dagger) \) for some \( \rho > 0 \) sufficiently small and let the stopping index \( n_\ast(\delta) \) be chosen according to (15).

Then the iterates \( (\vec{x}_n^\delta)_{n \in \{1, \ldots, n_\ast(\delta)\}} \) are well-defined by (13), remain in \( B_\rho(\vec{x}_\dagger) \) and converge in \( X \), \( \|\vec{x}_n^\delta - \vec{x}_\dagger\|_X \to 0 \) as \( \delta \to 0 \). In the noise free case \( \delta = 0, n_\ast(\delta) = \infty \) we have \( \|\vec{x}_n - \vec{x}_\dagger\|_X \to 0 \) as \( n \to \infty \).

### 2.2 Reconstructions

In this section we show reconstructions of \( \eta(x), s(x) \) in (51) with Caputo-Wismer-Kelvin damping, that is,

\[
\begin{aligned}
(s(x)u - \eta(x)u^2)_{tt} - \Delta u - b(-\Delta)\partial_t^\delta u = \tilde{r} & \quad \text{in } \Omega \times (0,T) \\
\partial_u u + \gamma u = 0 & \quad \text{on } \partial \Omega \times (0,T), \quad u(0) = 0, \quad u_t(0) = 0 \quad \text{in } \Omega.
\end{aligned}
\]

(17)

in one space dimension \( \Omega = (0,1) \) with Dirichlet-Neumann conditions \( \gamma(0) = \infty, \gamma(1) = 0 \) from measurements at two points \( \Sigma = (0,1,1) \). (Note that since we impose homogeneous Dirichlet boundary conditions at the left endpoint, measuring \( u \) there would not provide any additional information; indeed, also in practice the transducer array will be immersed into the overall computational domain \( \Omega \).)

For the numerical solution of (17), we follow the method of [23] and rewrite the equation by integrating once with respect to time

\[
\begin{aligned}
(s(x) - 2\eta(x))u_t - \tilde{b}\Delta I_1^{1-\alpha}u - \Delta I_1^u u = I_1^\delta \tilde{r} & \quad \text{in } \Omega \times (0,T) \\
\partial_u u + \gamma u = 0 & \quad \text{on } \partial \Omega \times (0,T), \quad u(0) = 0 \quad \text{in } \Omega.
\end{aligned}
\]

(18)

and apply a modified Crank-Nicolson solver taking into account the fractional integral term. Likewise for its linearisation (10).

To test the frozen Newton method from Section 2.1, we consider three scenarios A, B and C as described below. While the theory from Section 2.1 requires two excitations
and an extension of $s$ to a time dependent function, this was not needed in practical computations. The reconstructions shown here are based on a single exciation, but carrying out measurements at two points $\Sigma = \{0.1, 1\}$. Also $s$ is treated as a function of $x$ only. Both coefficients were discretized using a chapeau basis set and the starting values were $\eta_0 = 0$ and $s = 1$.

Figures 1 and 2 show a simultaneous reconstruction of both $\eta(x)$ and $s(x)$ under 1% and 0.1% noise in the time trace data. Here the value of the solution $u(x, t)$ was negative and there was therefore no cancellation effect on the $s(x)$ and the $\eta(x)$ term (test case A). In

![Figure 1](image1.png)

**Figure 1:** Reconstruction of both $\eta(x)$ and $s(x)$ under 1% data noise; test case A.

![Figure 2](image2.png)

**Figure 2:** Reconstruction of both $\eta(x)$ and $s(x)$ under 0.1% data noise; test case A.

Figure 3 we show the difference when the sign of $u(x, t)$ is reversed so that there is the potential for a cancellation effect between $\eta(x)$ and $s(x)$ (test case B). In fact this occured
resulting in a poorer reconstruction in both functions. Data noise here was again 0.1%. The final picture [4] shows a more complex function \( \eta(x) \) with two features (test case C), one at each end of the interval. For this run the function \( u(x, t) \) was zero at the endpoint \( x = 0 \) and so small in comparison at the left end as opposed to the right. Since \( \eta \) occurs in combination with \( u \) in the equation this means a relative loss of information at the left-hand endpoint. This is clearly visible from the left hand graphic. In addition this error in \( \eta(x) \) now affects the combined term \( (s - \eta u) \) and results in a similarly poor reconstruction of \( s(x) \) near \( x = 0 \). Note that a seemingly overall better match of \( \eta \) at the fourth iterate is dismissed in subsequent iterations that are much worse in approximating the left hand feature. This is due to the fact that the mismatch is weighed by the values of \( u \) which are small near the left endpoint, but penalize deviations occurring in the right half of the interval (as is the case for iteration 4) much more strongly. These reconstructions were made using 0.1% data noise.

![Figure 3: Reconstruction of both \( \eta(x) \) and \( s(x) \) under 0.1% data noise; test case B.](image)

![Figure 4: Reconstruction of both \( \eta(x) \) and \( s(x) \) under 0.1% data noise; test case C.](image)

This effect of smallness of \( u \) at the left hand endpoint is also apparent in the other figures,
particularly in the case where there is a significant feature near this endpoint. In all figures we imposed the sign constraint imposed by the physical problem that \( \eta(x) \geq 0 \).

For an \( \eta \) function with support away from \( x = 0 \) the reconstructions shown in figure 2 under 0.1% noise and in figure 1 with 1% added noise indicate a reasonable reconstruction of both \( \eta \) and \( s(x) \). Note that a poor initial guess (both these functions taken to be constant zero and one respectively) leads to a severe overshoot in the first computed approximation to \( s(x) \) although this quickly settles down. In this case both actual \( s(x) \) and \( \eta(x) \) functions have support away from the left-hand endpoint \( x = 0 \) and there is also overshoot in the first iteration of \( \eta \).

The difference between (0.1%) and 1% of added noise to the data simulated by the direct solver is quite apparent and indicates the severe ill-conditioning of the inverse problem.

Finally, we show a plot of the singular values of the Jacobian matrix frozen at \( s(x) = 1 \) and \( \eta(x) = 0 \). As Figure 5 shows there is indeed an exponential decay of the singular values and the initial steep decay of the largest values means that under even relatively small noise in the data it will be difficult to use more than about ten relevant modes as the above reconstruction figures demonstrate. However the decay rate of the singular values overall is actually more favourable for reconstructions than for classical exponentially ill-posed problems such as the backwards or sideways heat problems, the Cauchy problem for the Laplacian or inverse obstacle scattering.

All of the above reconstructions were obtained using the value \( \alpha = \frac{1}{2} \) but none were sensitive to this parameter except for the extreme ends of its range. However, it is certainly the situation if we had \( \alpha = 1 \) and thus exponential damping the usefulness of the resulting very small values of \( u \) obtained from anything beyond modest time values would be extremely limited – in particular for the reconstruction of \( \eta(x) \) as this is inherently coupled to the magnitude of \( u \).

3 Uniqueness

3.1 Well-definedness and differentiability of the forward operator \( F \)

We return to the \((\kappa, c_0^2)\) formulation and consider CH damping, that is

\[
(1 - 2\kappa(x)u)u_{tt} + c^2Au + \tilde{b}A^{\frac{3}{2}}\partial_t^2 u - 2\kappa(x)(u_t)^2 = r \quad t > 0
\]

\[
 u(0) = u_0, \quad u_t(0) = u_1. \tag{19}
\]
In order to avoid degeneracy in the leading order time derivative term due to the coefficient \( (1 - 2\kappa(x) u) \) of \( u_{tt} \) we have to assume that \( \tilde{\beta} \) is chosen sufficiently large, \( 2\tilde{\beta} - d/2 > 0 \), so that \( \hat{H}^{2\tilde{\beta}} \) embeds continuously into \( L^\infty(\Omega) \). For simplicity we will stay with the integer cases

\[
\tilde{\beta} = \begin{cases} 
1 & \text{if } d \in \{2, 3\} \\
0 & \text{if } d = 1. 
\end{cases}
\]  

(20)

As a preparation for the use of the Inverse Function Theorem for proving uniqueness of \( \kappa \), we will now study the operator \( \mathcal{F} \) that takes \( \kappa \) to the residues of \( \hat{\mathcal{F}} u \) at the poles of \( \hat{h} \). Note that this is different from the forward operator that we use in our Newton reconstruction scheme, see the reconstruction section below.

In our uniqueness proof, in order to take residues at poles \( p \) of the Laplace transformed solution to the PDE and its linearisation, we need sufficiently fast decay as time tends to infinity according to the identity

\[
\text{Res}(\hat{f}, p) = \lim_{z \to p} (z - p) \hat{f}(z) = \lim_{z \to p} \frac{\hat{f}'(z) - pf(z)}{z - p} + f(0)
\]

\[
= \lim_{T \to \infty} \lim_{z \to p} \int_0^T e^{-itz} (\hat{f}'(t) - pf(t)) \, dt + f(0)
\]

\[
= \lim_{T \to \infty} \int_0^T \frac{d}{dt} \left( e^{-ipt} f(t) \right) \, dt + f(0) = \lim_{T \to \infty} e^{-ipt} f(T),
\]

provided absolute convergence holds, which allows us to interchange the integrals. In particular, for an integrable function with finite support \([0, T]\) in time, the residue vanishes (which is clear in view of the fact that the Laplace transform of a finitely supported function has no poles). Note that the poles we consider will be single and their real parts will be negative, so that \( |e^{-pt}| \) is exponentially increasing as \( t \to \infty \).

To this end, we use a first order in time damping term \( \beta := 1 \) (which corresponds to Chen-Holm damping) and moreover modify the problem to coincide with a linear one for large times \( t > T_* \) for some properly chosen \( T_* > 0 \), which is justified by the fact that nonlinearity only takes effect for sufficiently large energy \( \mathcal{E}_0[u] \) (to be defined below, see (30)).

Before going into the details about the problem on the time interval \((0, T_*), \) in view of (21) we will study the large time behavior in the constant coefficient PDE for \( t > T_* \).

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} + c^2 \mathcal{A} u + b \mathcal{A}^\beta u_{st} &= r_* & \text{for } t > 0 \\
u_*(0) &= u_{s0}, & u_{st}(0) &= u_{s1}.
\end{align*}
\]  

(22)

Separation of variables \( u_*(x, t) = \sum_{j=1}^\infty \sum_{k \in K^{\lambda_j}} u_{kj}^j(t) \varphi_{kj}^j(x) \) with \( u_{kj}^j(t) = \langle u_*(t), \varphi_{kj}^j \rangle \) by means of an eigensystem \( (\lambda_j, (\varphi_{kj}^j)_{k \in K^{\lambda_j}})_{j \in \mathbb{N}} \) (where \( K^{\lambda_j} \) is an enumeration of the eigenspace corresponding to \( \lambda_j \)) of \( \mathcal{A} \) lead us to consider the relaxation equations

\[
\begin{align*}
\frac{\partial^2 u_{kj}^j}{\partial t^2} + c^2 \lambda_j u_{kj}^j + b \beta^\lambda_j u_{kj}^j &= r_{kj}^j & \text{for } t > 0 \\
u_{kj}^j(0) &= u_{kj0}^j, & u_{kj}^j(0) &= u_{kj1}^j.
\end{align*}
\]
whose Laplace transformed solutions are given by

\[\hat{u}_{*j}^k(z) = \frac{\hat{r}_{*j}^k(z) + u_{*j1}^k + zu_{*j0}^k}{\omega_{\lambda_j}(z)}, \text{ with } \omega_{\lambda}(z) = z^2 + b\lambda^2 z + c^2\lambda. \tag{23}\]

We recall some facts concerning the poles for actually more general damping terms \(D\).

**Lemma 3.1 (Lemma 11.4 in [24])** For CH or FZ damping, the poles of \(\frac{1}{\omega_\lambda}\) differ for different \(\lambda\).

**Lemma 3.2 (Lemma 11.5 in [24])** For CH or FZ damping, the residues of the poles of \(\frac{1}{\omega_\lambda}\) do no vanish.

Indeed, in the current setting this can be immediately seen by computing the poles and residues explicitly as

\[p_j^\pm = -\frac{b}{2}\lambda_j^\beta \pm \sqrt{\frac{b^2}{4}\lambda_j^{2\beta} - c^2\lambda_j}, \quad \text{Res}(\frac{1}{\omega_{\lambda_j}}; p_j^+) = \lim_{z \to p_j^+} \frac{z - p_j^+}{z - p_j^-}(z - p_j^-) = \frac{1}{2\sqrt{\frac{b^2}{4}\lambda_j^{2\beta} - c^2\lambda_j}}. \tag{24}\]

Since \(\lambda_j \to \infty\) as \(j \to \infty\), in space dimensions two and three, due to the condition \(2\beta - d/2 > 0\), the poles are up to finitely many – real and negative; in case \(b > 0\) their real parts tend to \(-\infty\) as \(j \to \infty\). In case \(\beta \leq \frac{1}{2}\) which is admissible in 1-d, the poles come in complex conjugate pairs and in particular for \(\beta = 0\) they have \(\lambda_j\)-independent (and negative) real part \(-b/2\).

This together with (21) allows us to compute the residues of the Laplace transformed components \(u_j^k = \langle u, \varphi_j^k \rangle\) at the poles \(p_j\) as follows. Since

\[u_j^k(t) = I_{[0,T_\ast]}(t)u_j^k(T_\ast - t) + \int_{T_\ast}^\infty e^{-z^\tau}u_j^k(t - T_\ast) d\tau = e^{-z(T_\ast - t)}u_j^k(T_\ast - t)\]

due to (21), (23), (24), we have

\[\text{Res}(\hat{u}_j^k; p_j^+) = 0 + e^{-p_j^+T_\ast}\text{Res}(\hat{u}_j^k; p_j^+) = \frac{e^{-p_j^+T_\ast}(\hat{r}_{*j}^k(p_j^+) + u_{*j1}^k + p_j^+ u_{*j0}^k)}{2\sqrt{\frac{b^2}{4}\lambda_j^{2\beta} - c^2\lambda_j}}. \tag{25}\]

We now turn to the nonlinear problem on the time interval \((0,T_\ast)\), which will provide us with the initial values \(u_{*j0}^k, u_{*j1}^k\). With a smooth cutoff function \(\chi : [0, \infty) \to [0,1]\) such that for some thresholds \(0 < m < \bar{m}\)

\[\chi([0,m]) = \{1\}, \quad \chi([\bar{m}, \infty)) = \{0\}, \quad \chi \in W^{2, \infty}(0, \infty), \tag{26}\]
we consider in place of (19)
\[(1 - 2x(x)\chi(\mathcal{E}_0[u])u)u_{tt} + c^2A u + \tilde{b}\mathcal{A}^3 u_t - 2y(x)\chi(\mathcal{E}_0[u])(u_t)^2 = r \quad t > 0 \]
\[u(0) = u_0, \quad u_t(0) = u_1, \tag{27}\]
with the time dependent function $\tilde{b}$ defined by
\[\tilde{b}(t) = b + \tilde{b}\|\kappa\|^2_{L^2(\Omega)} \chi(\mathcal{E}_0[u](t)) \text{ for some constants } \tilde{b} > 0, \ b \geq 0. \tag{28}\]

The latter means that the attenuation will also be weaker at lower energy levels, a property that we will need below to control the behavior of the pole locations and residues.

Analogously to [18, Theorem 1.3] for a smooth bounded domain $\Omega \subseteq \mathbb{R}^d$, $d \in \{1, 2, 3\}$, one obtains existence and uniqueness of a solution to (27) for all times $t \in (0, \infty)$, provided $\kappa, c_0, \frac{1}{c_0} \in L^\infty(\Omega)$, $\mathcal{E}[u](0) + \|r\|^2_{L^\infty(\Omega)}$ and $\|\kappa\|^d_{L^\infty(\Omega)}$ are sufficiently small, where
\[X_{r,t} = \begin{cases} L^1(0,t; L^2(\Omega)) \cap W^{1,1}(0,\infty; H^{-1}(\Omega)) & \text{for } d \in \{2, 3\} \\ L^1(0,t; L^2(\Omega)) & \text{for } d = 1. \end{cases} \]

and in case $d \in \{2, 3\}$ additionally $\|\nabla \kappa\|^1_{W^{1,1}(\Omega)}$ sufficiently small for $q = 3$ in case $d = 3$ and $q = 2$ in case $d = 2$. Moreover, for any function $B \in C(0, \infty)$ with
\[B(t) \geq 0, \quad B'(t) < \tilde{b}(t) \]
the following decay estimate holds.
\[\mathcal{E}_0[u](t) \leq Ce^{-B(t)}\left(\mathcal{E}_0[u](0) + \|c^2 r\|^2_{L^1(0,t; L^2(\Omega))}\right) =: C_0 e^{-B(t)} \quad t > 0 \tag{29}\]

with $C$ independent of $t$ and the low order energy $\mathcal{E}_0$
\[\mathcal{E}_0[u](t) = \frac{1}{2}\left(\|u_t(t)\|^2_{L^2(\Omega)} + c^2\|\nabla u(t)\|^2_{L^2(\Omega)}\right) =: \frac{1}{2}\langle u, u \rangle_{\mathcal{E}_0}. \tag{30}\]

In case $d \in \{2, 3\}$ we also have
\[\mathcal{E}[u](t) \leq C\left(\mathcal{E}[u](0) + \|r\|_{X_{r,\infty}}\right) \quad t > 0. \tag{31}\]

for the higher order energy functional $\mathcal{E}$ defined as
\[\mathcal{E}[u](t) := \mathcal{E}_0[u](t) + \frac{1}{2}\left(\|u_{tt}(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)} + \|Au(t)\|^2_{L^2(\infty, \Omega)}\right) =: \frac{1}{2}\langle u, u \rangle_{\mathcal{E}} \tag{32}\]

The energy functional $\mathcal{E}_0[u]$ is actually a physical one and estimate (29) can be achieved by testing the PDE with $u_t + \lambda u$ (that is, taking the $L^2(\infty, \Omega)$ inner product of the PDE with this function) for some sufficiently large factor $\lambda > 0$. To obtain an estimate on $\mathcal{E}[u]$,
we also test the PDE with $\mathcal{A}u$, and the time differentiated PDE with $u_{tt}$ and appropriately combine the resulting identities, see [19] [18] for details. By this testing strategy one obtains

$$\frac{d}{dt}e^{B(t)}\mathcal{E}_0[u](t) = e^{B(t)}(\frac{d}{dt}\mathcal{E}_0[u](t) + B'(t)\mathcal{E}_0[u](t)) \leq 0$$

(33)

if $r = 0$; in case $r \neq 0$, the time integral of (33) is bounded by $C\|e^{Bt}\|^2_{L^1(0,\varepsilon;L^2(\Omega))}$ (cf. (29)), which we assume to be small enough, in particular smaller than $\frac{m}{\lambda}$ likewise for $\mathcal{E}$. Therefore at some sufficiently large time $T_*$, the cutoff function $\chi$ switches off nonlinearity and for $t \geq T_*$, we have $u(x, t) = u_*(x, t - T_*)$, $x \in \Omega$, where $u_*$ solves the linear constant coefficient problem (22) with

$$u_{00}(x) = u(x, T_*) , \quad u_{01}(x) = u_t(x, T_*) , \quad r_*(t) = r(t + T_*) .$$

Indeed, using $B(t) := \frac{\beta\|\kappa\|^2_{L^2(\Omega)}}{2} \int_0^t \mathcal{E}_0[u](s) \, ds$ in (29) and assuming $\mathcal{E}_0[u](t) \geq m$ for all $t > 0$ via the equivalence

$$e^{-\frac{\beta\|\kappa\|^2_{L^2(\Omega)}}{2} \int_0^t \mathcal{E}_0[u](s) \, ds} \mathcal{E}_0[u](t) \leq C_0 \Leftrightarrow \ln \mathcal{E}_0[u](t) + \frac{\beta\|\kappa\|^2_{L^2(\Omega)}}{2} \int_0^t \mathcal{E}_0[u](s) \, ds \leq \ln(C_0)$$

leads to the contradiction $\ln(C_0) + \frac{\beta\|\kappa\|^2_{L^2(\Omega)}}{2} m t \leq \ln(C_0)$ for all $t > 0$. By the same reasoning, nonlinearity remains switched off after a sufficiently large time $T_*$.

We can therefore make use of the identity (25), where

$$|e^{-p_j^+T_* \hat{r}_j^k(p_j^+)}| = |e^{-p_j^+T_*} \int_0^\infty e^{-p_j^+\tau} \hat{r}_j^k(\tau + T_*) \, d\tau| = \int_{T_*}^\infty e^{-p_j^+\tau} \hat{r}_j^k(t) \, dt| \leq \int_{T_*}^\infty e^{-\Re(p_j^+)\tau} |\hat{r}_j^k(t)| \, dt = \|e^{-\Re(p_j^+)^2 \hat{r}_j^k}|_{[T_*, \infty]}\|_{L^1(0,\infty)} .$$

(34)

We still need to bound

$$|e^{-p_j^+T_*}(\langle u_t(T_*) , \varphi_j^k \rangle + p_j^+(u(T_*) , \varphi_j^k))| \leq e^{-\Re(p_j^+)^2 T_*} (\|\langle u_t(T_*) , \varphi_j^k \rangle| + \|p_j^+ \| \|\langle u(T_*) , \varphi_j^k \rangle\|) ,$$

(35)

which contains the values of $u$ and its time derivative as they have evolved nonlinearly according to (27). In case $d = 1 , \hat{\beta} = 0$, according to (24), the real parts of the poles are independent of $\lambda_j , -\Re(p_j^+) = \frac{b}{2}$, while solutions of the (weakly) damped Westervelt equation decay with time as $O(e^{-\omega t})$ for some $\omega \in (0 , \frac{b}{2})$, cf. (29). This yields a $j$ independent bound for (35). In dimensions $d \in \{2, 3\}$, we have to avoid unbounded growth of the exponent $-p_j^+ T_*$ as $j \to \infty$ by setting $b = 0$ in (28) and consequently in (24), which places the poles on the imaginary axis and yields a $j$ independent bound for (35) as well. In both cases we end up with the estimate

$$|e^{-p_j^+T_*}(\langle u_t(T_*) , \varphi_j^k \rangle + p_j^+(u(T_*) , \varphi_j^k))| \leq e^{(b/2) T_*} \sqrt{\mathcal{E}_0(\langle u , \varphi_j^k \rangle \varphi_j^k)(T_*)} ,$$

(36)
Thus with
\[2\sqrt{\frac{b^2}{4} \lambda^{2\beta} - c^2 \lambda} \geq c\sqrt{\lambda}\] for all \(\lambda \geq \frac{b^2}{3c^2}\) in (25) we have shown the following.

**Proposition 3.1** Let \(\kappa \in L^\infty(\Omega), c_0, \frac{1}{c_0} \in L^\infty(\Omega), \mathcal{E}[u](0) + \|r\|_{X_{r,t}}^2\) sufficiently small with \(b = 0\) if \(d \in \{2, 3\}\) and \(b > 0\) if \(d = 1\). Then the solution to (27) with (28), (20) exists for all \(t > 0\) and its Laplace transform \(\hat{u}\) satisfies the estimate
\[
\sum_{j=1}^\infty \lambda_j \sum_{k \in K} \text{Res}(\hat{a}_j^k; p_j^+)^2 \leq C \left( \mathcal{E}_0[U](0) + \|r\|_{X_{r,t}}^2 \right)
\]
for some \(C > 0\) independent of \(j\).

We now investigate differentiability of \(\mathbb{F}\) and for this purpose first of all study the linear problem with time dependent coefficients
\[
a_0(x, t)U_{tt} + c^2 A U + a_1(x, t) \beta U_t + a_2(x, t) U_t + a_3(x, t) U = f \quad (0, T)
\]
\[U(0) = U_0, \quad U_t(0) = U_1\] (38)

Taking the \(L^2_{c_2/c_0}(\Omega)\) inner product with \(U_t + \lambda U\) for some \(\lambda > 0\) sufficiently large, the lower order energy estimate in (29) with \(B \equiv 0\) can also be obtained for (38) provided the coefficients and right hand side satisfy
\[a_0, a_1 \in L^\infty(0, \infty, L^\infty(\Omega)) \cap W^{1, \infty}(0, \infty; L^p(\Omega)), \quad a_0 \geq a_0 > 0, \quad a_1 \geq a_1 > 0\]
and if \(d \in \{2, 3\}: a_1 \in L^\infty(0, \infty; W^{1, 2p}(\Omega))\) with \(\|\nabla a_1\|_{L^\infty(0, \infty; W^{1, 2p}(\Omega))}\) small enough,
\[f \in L^2(0, \infty; \dot{H}^{-\beta}(\Omega)), \quad a_2, a_3 \in L^\infty(0, \infty, L^p(\Omega)), \quad p \begin{cases} = \frac{3}{2} & \text{if } d = 3 \\ > 1 & \text{if } d = 2 \\ = 1 & \text{if } d = 1 \end{cases}\] (39)

**Lemma 3.3** For \(a_0, a_1, a_2, a_3, f\) satisfying (39), the solution \(u \in L^\infty(0, \infty; H^1(\Omega)) \cap W^{1, \infty}(0, \infty; L^2(\Omega)) \cap H^1(0, \infty; H^{\beta}(\Omega))\) of (38) exists, is unique, and satisfies
\[
\mathcal{E}_0[U](t) \leq C \left( \mathcal{E}_0[U](0) + \|f\|_{L^2(0, t; \dot{H}^{-\beta}(\Omega))}^2 \right) \quad t > 0,
\]
with \(C\) independent of \(t\).

Note that we do not need to show any decay of the energy here, since the cutoff time \(T_*\) is already determined (and finite) by the solution \(u\) to (27).
In order to prove Fréchet differentiability of $F$, as well as continuity of $F'$ we proceed as follows. Considering the parameter-to-state map $S: \kappa \mapsto u$ where $u$ solves (27),

$$(1 - 2\kappa(x)\chi(E_0[u])u)u_{tt} + c^2Au + \left(b + \delta\|\kappa\|_{L^2(\Omega)}^2\chi(E_0[u])\right)A^\beta u_t - 2\kappa(x)\chi(E_0[u])(u_t)^2 = r$$

we apply Lemma 3.3 to the PDEs for $du = S'(\kappa)\delta\kappa$

$$(1 - 2\kappa\chi(E_0[u])u)du_t + c^2Adu + \tilde{b}A^\beta v_t - 4\kappa\chi(E_0[u])u_t du_t - 2\kappa\chi(E_0[u])u_{tt} du_t$$

$$+ (-2\kappa(\bar{u} u_{tt} + (u_t)^2) + \tilde{b}\|\kappa\|_{L^2(\Omega)}^2A^\beta u_t)\chi'(E_0[u])\langle du, \frac{1}{2} (u + u) \rangle_{E_0}$$

$$= 2\delta\kappa\chi(E_0[u])(u_t u_{tt} + u_t^2) - \tilde{b}(\bar{\kappa} + \kappa, \kappa - \kappa)\chi(E_0[u])A^\beta u_t,$$ \hspace{1cm} $t \in (0, T)$.

We also do so for the first order Taylor remainder $w = S(\bar{\kappa}) - S(\kappa) - S'(\kappa)(\bar{\kappa} - \kappa) = v - du$

with $\delta\kappa := \bar{\kappa} - \kappa$

$$(1 - 2\kappa\chi(E_0[u])u)w_t + c^2Aw + \tilde{b}A^\beta w_t - 4\kappa\chi(E_0[u])u_t w_t - 2\kappa\chi(E_0[u])u_{tt} w$$

$$+ (-2\kappa(\bar{u} u_{tt} + (u_t)^2) + \tilde{b}\|\kappa\|_{L^2(\Omega)}^2A^\beta u_t)\chi'(E_0[u])\langle du, \frac{1}{2} (u + u) \rangle_{E_0}$$

$$= 2\delta\kappa(\bar{u} u_{tt} + (u_t)^2)\chi'(E_0[u])\langle du, \frac{1}{2} (u + u) \rangle_{E_0}$$

Now we define

$$\chi' := \int_0^1 \chi'(E_0[u] + \theta(E_0[u] - E_0[u])) \, d\theta, \hspace{1cm} \chi' := \int_0^1 \chi''(E_0[u] + \theta(E_0[u] - E_0[u])) \, d\theta$$

$$\chi'' := \int_0^1 \chi''(E_0[u] + \theta(E_0[u] - E_0[u])) \, d\theta$$
Then for the difference between two derivatives \( dv = \left[ S'(\tilde{\kappa}) - S'(\kappa) \right] \delta \kappa = \tilde{du} - du \) we have

\[
\begin{align*}
& (1 - 2\kappa \chi(E_0[u]) u) du_t + c^2 A du + \hat{b} \mathcal{A}^\delta du - 4\kappa \chi(E_0[u]) u_t dv - 2\kappa \chi(E_0[u]) u_{tt} dv \\
& \quad + (-2\kappa (u u_t + (u_t)^2) + \hat{b} \| \kappa \|^2_{L^2(\Omega)} \mathcal{A}^\beta u_t) \chi'(E_0[u]) (dv, u)_{\mathcal{E}_0} \\
& \quad = 2\delta \kappa (\tilde{u} \bar{u}_t + (\bar{u}_t)^2) \overline{\chi'(v)} \left( \frac{1}{2}(\tilde{u} + u) \right)_{\mathcal{E}_0} + 2\delta \kappa \chi(E_0[u])(v \bar{u}_t + uv_t + (\bar{u}_t + u_t)v_t) \\
& \quad - \left( 2(\kappa - \kappa)(\tilde{u} \bar{u}_t + (\bar{u}_t)^2) + \hat{b}(\kappa - \kappa, \kappa + \kappa)_{L^2(\Omega)} \mathcal{A}^\beta \bar{u}_t \right) \\
& \quad - 2\kappa \chi(E_0[u]) u_t + (u_t)^2) + \hat{b} \| \kappa \|^2_{L^2(\Omega)} \mathcal{A}^\beta v_t \right) \cdot \chi'(E_0[u]) (\tilde{du}, \bar{u})_{\mathcal{E}_0} \\
& \quad - \left( -2\kappa (u u_t + (u_t)^2) + \hat{b} \| \kappa \|^2_{L^2(\Omega)} \mathcal{A}^\beta u_t \right)
\end{align*}
\]

(43)

An application of Lemma 3.3 to (42), (43) allows to prove Fréchet differentiability of \( F \), as well as continuity of \( F' \).

Note that (40), (41), (42), (43) come with homogeneous initial conditions.

To estimate the residues from the bounds on \( du, v, w, dv \) obtained by Lemma 3.3, note that again \( du, v, w \) satisfy linear constant coefficient PDEs of the form (22) for \( t > T_\ast \).

Thus, according to (21), (23), (24), (25), (26), (27), (28) we have

\[
\text{Res}(\hat{U}^k_j; p^+_j) = 0 + e^{-p^+_j T_\ast} \text{Res}(\hat{U}^k_j; p^+_j) = e^{-p^+_j T_\ast} (f^k_j(p^+_j) + U^k_j(T_\ast) + p^+_j U^k_j(T_\ast)) \\
\leq \frac{1}{\sqrt{\lambda_j}} \left( \| e^{(b/2)T} f^k_j \|_{T^\ast, \infty} \| L^1(0, \infty) + 2e^{(b/2)T_\ast} \sqrt{\mathcal{E}_0[U^k_j \chi f^k_j(T_\ast)]} \right)
\]

for \( U \in \{ du, v, w, dv \}, U^k_j = (U, \varphi^k_j) \).

We finally apply the trace operator \( \text{tr}_\Sigma : H^1(\Omega) \to L^2(\Sigma) \) to conclude

\[
\sum_{j=1}^{\infty} \| \text{Res}(\text{tr}_\Sigma \hat{U}; p^+_j) \|_{L^2(\Sigma)}^2 \leq \| \text{tr}_\Sigma \|_{L^2(\Sigma)}^2 \sum_{j=1}^{\infty} \lambda_j \sum_{k \in K^\lambda_j} \text{Res}(\hat{U}^k_j; p^+_j)^2 \\
\leq 2 \| \text{tr}_\Sigma \|_{L^2(\Sigma)}^2 \left( \| e^{b f(T^\ast, \infty)} \|_{L^1(0, \infty; L^2(\Omega))}^2 + 4e^{b T_\ast} \mathcal{E}_0[U](T_\ast) \right)
\]

to obtain the following.
Proposition 3.2 Under the assumptions of Proposition 3.1, the forward operator $F : L^\infty(\Omega) \to \ell^2(L^2(\Sigma))$ is well-defined and continuously Fréchet differentiable.

3.2 Uniqueness of $\kappa(x)$ in $\mathbb{R}^d$, $d \in \{1, 2, 3\}$

We consider the problem of identifying $\kappa(x)$ alone (without assuming $c_0(x)$ to be known). In [23] we have shown that the linearisation of the forward operator at $\kappa = 0$ is injective. By quantifying this injectivity in the sense of showing that $F'(0)$ is in fact an isomorphism and using the inverse function theorem, we will now prove local uniqueness of the nonlinear inverse problem of reconstructing $\kappa(x)$. For this purpose, we rely on the fact that all the information required on $c_0(x)$ (actually on the whole operator $A$ as required for the reconstruction of $\kappa(x)$) is contained in conditions on the excitation $r$ and on the eigenfunctions, cf. (44) and (47) below.

We denote by $(p^*_m)_{m \in \mathbb{N}}$ the sequence of poles according to (24). Note that these are precisely the poles of the Laplace transformed data $\hat{h}$.

Moreover, as in the linearised injectivity proof of [23] we use an excitation $r$ that for $\kappa = 0$ leads to a space-time separable solution $u^0(x,t) = \phi(x)\psi(t)$ of (27), namely

$$r(x,t) := \phi(x)\psi''(t) + c^2(A\phi)(x)\psi(t) + D[\phi\psi](x,t)$$

(44)

with some given functions $\phi(x), \psi(t)$ such that $\phi \in \mathcal{D}(A)$ (in particular it satisfies the boundary conditions imposed by $A$) and vanishes nowhere in $\Omega$ and

$$\psi(0) = \psi'(0) = 0 \text{ and } (\psi^2)'(p^*_m) \neq 0 \quad \text{for all } m \in \mathbb{N}.$$  

(45)

With this, the linearisation of (27) at $\kappa = 0$ becomes

$$du_{tt} + c^2Adu + Ddu = d\kappa(x)\phi^2(x)(\psi^2)''(t) \quad \text{in } \Omega \times (0,T)$$

$$u(0) = 0, \quad u_t(0) = 0 \quad \text{in } \Omega.$$  

Moreover, using separation of variables and the relaxation functions we can write

$$\hat{d}u(x,z) = (\psi^2)''(z) \sum_{j=1}^{\infty} \frac{1}{\omega_{\lambda_j}}(z) \sum_{k \in K^{\lambda_j}} \langle d\kappa\phi^2, \varphi^k_j \rangle \varphi^k_j(x).$$

The linearised uniqueness proof relies on the fact that due to Lemma 3.1, taking the residues at the poles singles out the contributions of the individual eigenspaces to this sum, that is, after evaluation on the observation manifold

$$\text{Res}(\hat{d}u; p^*_m) = (\psi^2)''(p^*_m) \text{Res}(\frac{1}{\omega_{\lambda_m}}; p^*_m) \sum_{k \in K^{\lambda_m}} \langle d\kappa\phi^2, \varphi^k_m \rangle \varphi^k_m \text{ on } \Sigma.$$  

(46)

As a consequence of Lemma 3.2 and our assumption on $\psi$, the factor appearing here does not vanish

$$\mu_m := (\psi^2)''(p^*_m) \text{Res}(\frac{1}{\omega_{\lambda_m}}; p^*_m) \neq 0 \quad \text{for all } m \in \mathbb{N.}$$  

To resolve the contributions within each eigenspace, as in [24] we impose a linear independence assumption on the eigenspace $E^\Sigma_\lambda = \text{span}(\text{tr}_\Sigma \varphi_k)_{k \in K_\lambda}$ of each eigenvalue $\lambda$

\[
\left( \sum_{k \in K_\lambda} b_k \varphi_k(x) = 0 \text{ for all } x \in \Sigma \right) \implies (b_k = 0 \text{ for all } k \in K_\lambda).
\]

That is, for each eigenvalue $\lambda$ of $A$ with eigenfunctions $(\varphi_k)_{k \in K_\lambda}$, the restrictions of the eigenfunctions to the observation manifold must be linear independent. This implies that for each eigenvalue $\lambda$, the mapping

$$B_\lambda : \mathbb{R}^{\left| K_\lambda \right|} \rightarrow E^\Sigma_\lambda, \quad (b_k)_{k \in K_\lambda} \mapsto \sum_{k \in K_\lambda} b_k \text{tr}_\Sigma \varphi_k$$

is bijective and (as a mapping between finite dimensional spaces) its inverse is bounded. With this, the identity (46) can be rewritten as

$$\text{Res}(\text{tr}_\Sigma \hat{u}; \hat{p}_m^+) = \mu_m B_{\lambda_m} \left( (\langle d \kappa \hat{\phi}^2, \varphi_m^k \rangle)_{k \in K_\lambda} \right).$$

Now we denote by $F$ the mapping that takes $\kappa \varphi := \kappa \varphi^2$ to the residues of the Laplace transformed time trace data at the poles

$$F : X \rightarrow Y, \quad \kappa \varphi \mapsto \text{Res}(\text{tr}_\Sigma \hat{u}; \hat{p}_m^+)_{m \in \mathbb{N}}.$$ 

Here $X = \dot{H}^\sigma(\Omega)$, $Y \subseteq \ell^2(L^2(\Sigma))$. Clearly, injectivity of $F$ (which is what we will now show) implies uniqueness for the inverse problem. The topologies on $X$ and $Y$ are chosen as follows.

$$\| \kappa \varphi \|_X := \left( \sum_{j=1}^{\infty} \lambda_j^\sigma \sum_{k \in K_{\lambda_j}} \langle \kappa \varphi, \varphi_j^k \rangle^2 \right)^{1/2} = \| \kappa \varphi \|_{\dot{H}^\sigma(\Omega)}$$

$$\| (\hat{h}_m)_{m \in \mathbb{N}} \|_Y := \left( \sum_{m=1}^{\infty} \frac{\lambda_m^\sigma}{\mu_m^2} \| B_{\lambda_m}^{-1} \text{Proj}_{E^\Sigma_{\lambda_m}} \hat{h}_m \|_{E^\Sigma_{\lambda_m}}^2 \right)^{1/2}.$$ 

Due to (48), with these norms, $F'(0)$ is even an isometric isomorphism.

On the other hand, according to Proposition 3.2, $F$ is also continuously Fréchet differentiable for $\sigma > \frac{d}{2}$. The Inverse Function Theorem (see, e.g., [34, Section 4.8]) then implies that $F : X \rightarrow Y$ is locally bijective (with a continuous inverse between these spaces) and thus (due to completeness of the eigensystem and our assumption that $\phi \neq 0$ in $\Omega$ the full forward map $\kappa \rightarrow \text{tr}_\Sigma u$ is locally injective.

**Theorem 3.1** Assume that the conditions of Proposition 3.1 are satisfied, that (47) holds and $r$ takes the form (44) with $\phi \in \mathcal{D}(A)$, $\phi \neq 0$ in $\Omega$ and $\psi$ satisfying (45). Then there exists $\rho > 0$ such that for any $\kappa \varphi^2, \tilde{\kappa} \varphi^2 \in B_{\rho}^{H^\sigma}(0)$ with $\sigma > \frac{d}{2}$, equality of the observations $\text{tr}_\Sigma u(\kappa)(t) = \text{tr}_\Sigma u(\tilde{\kappa})(t)$, $t > 0$ implies $\kappa = \tilde{\kappa}$.

This does not require $c_0(x)$ nor $A$ to be known, just the conditions (44), (45), (47) to be satisfied by the (possibly unknown) operator $A$ and its eigensystem.
3.3 Uniqueness of \( \kappa(x) \) and \( c_0(x) \) in \( \mathbb{R}^1 \)

To prove identifiability of both nonlinearity coefficient and sound speed from observations with two different impedance boundary conditions, we restrict ourselves to one space dimension \( \Omega = (0, L) \) and denote by \( \Delta_i \) the Laplace operator on \( (0, L) \) equipped with impedance boundary conditions \( v'(0) - \gamma_0 v(0) = 0, v'(L) + \gamma_L v(L) = 0, i = 1, 2 \). The observations will be taken at a single point \( \Sigma = \{ x_0 \} \) for some \( x_0 \in [0, L] = \Omega \)

\[
h_i(t) = u_i(x_0, t), \quad t \in (0, T), \quad i = 1, 2.
\]

Since in one space dimension all the eigenspaces are one-dimensional, in place of the linear independence condition it suffices to assume that the eigenfunctions \( \varphi_{i,j} \) of \( A_i = -(c_0(x)^2/c^2)\Delta_i \) don’t vanish at \( x_0 \). Moreover, the excitation is supposed to be switched off after a certain time instance and \( h \) is assumed to have non-vanishing modes

\[
\forall j \in \mathbb{N}, \exists T_j > 0 : \langle r, \varphi_{i,j} \rangle|_{(T_j, \infty)} \equiv 0 \text{ and } h_{i,j} := \langle u, \varphi_{i,j} \rangle \varphi_{i,j}(x_0)|_{(T_j, \infty)} \neq 0.
\]

While the former is practically easy to satisfy by setting \( r|_{(T_j, \infty)} = 0 \) for some \( T \geq 0 \), the condition on \( h_{i,j} \) is hard to verify due to the unknown eigenfunctions \( \varphi_{i,j} \). If it is violated, the result below still provides uniqueness of the components of \( c_0(x) \) corresponding to non-vanishing \( h \) modes.

**Theorem 3.2** Assume that the conditions of Proposition 3.1 are satisfied, that (50) holds and that \( r = r_i \) takes the form (44) with \( \phi_i \in \mathcal{H}(0, L) \), \( \phi'_i(0) - \gamma_0 \phi_i(0) = 0, \phi'_i(L) + \gamma_L \phi_i(L) = 0 \), \( \phi_i \neq 0 \) in \( (0, L) \) and \( \psi_i \) satisfying (15), \( i = 1, 2 \), and that for all \( j \in \mathbb{N} \). Then the observations \( h_1, h_2 \) corresponding to these two excitations uniquely determine \( c(x) \) and \( \kappa(x) \) (the latter locally in the sense of Theorem 3.7).

**Proof:** Condition (50) guarantees that all the poles \( (p_{i,m,}^+)_{m \in \mathbb{N}} \) of the resolvent functions \( \frac{1}{\omega_{\lambda_{i,m}}} \) are visible in the observations \( h \). Indeed, assuming that \( p_{i,m,}^+ \) is a pole of \( \frac{1}{\omega_{\lambda_{i,m}}} \) but not a pole of \( \hat{h} \) implies that the (first order) residual vanishes and thus

\[
0 = \lim_{z \to p_{i,m,}^+} (z - p_{i,m,}^+) \hat{h}(z) = \lim_{z \to p_{i,m,}^+} (z - p_{i,m,}^+) \hat{h}(z)
\]

\[
= \int_\tau^\infty e^{-p_{i,m,}^+ s} r_{i,m}(s) ds + e^{-p_{i,m,}^+ \tau} (h_{i,m}(\tau) + p_{i,m,}^+ h_{i,m}(\tau))
\]

\[
= \frac{2\sqrt{\frac{b^2}{4} 2\beta_{i,m}^+ - c^2 \lambda_{i,m}^+}}{2\sqrt{\frac{b^2}{4} \lambda_{i,m}^+ - c^2 \lambda_{i,m}^+}} e^{-p_{i,m,}^+ \tau} \frac{d}{d\tau} (e^{p_{i,m,}^+ \tau} h_{i,m}(\tau))
\]

where the first equality follows from Lemma 3.1, the second one from (25), (34), which remain valid for any \( \tau \geq T_i \), and the last one from (50). Thus, the function \( \tau \mapsto e^{p_{i,m,}^+ \tau} h_{i,m}(\tau) \)
is constant on the interval \([T_{m,*}, \infty)\) with \(T_{m,*} \max\{T_m, T_s\}\), and so for any \(s \geq T_{m,*}\) we have \(|h_{i,m}(s)| = e^{(b/2)(r-s)} = |h_{i,m}(s)| \to \infty\) unless \(h_{i,m}(s) = 0\). This yields a contradiction to the boundedness result \((29)\) and \((50)\). We conclude from \((24)\) that the poles of \(\hat{h}_i\) – thus, according to what we have just shown, of \(\frac{1}{\omega_{\lambda_{i,j}}}\) – uniquely determine the eigenvalues \(\lambda_{i,j}\) of \(\mathcal{A}_i\), for \(i = 1, 2\). From this by means of classical inverse Sturm-Liouville theory (Theorem 9.15 in [24], using \((9.56)-(9.58)\) with \(p = q = 1, r = c^2/c_0^2\)) we conclude uniqueness of \(c_0(x)\).

Knowing \(c_0(x)\) and using the fact that \((47)\) trivially holds in 1-d, we can now apply Theorem 3.1 (with, e.g., the first impedance data set \(\gamma_L = \gamma_{L,1}\)) to conclude local uniqueness of \(\kappa(x)\).

\[\square\]

### 3.4 Linearised uniqueness of \(\kappa(x)\) and \(c_0(x)\) in \(\mathbb{R}^d\) from two observations

We return to higher space dimensions and more general CH or FZ damping and consider the alternative formulation \((8)\), that is,

\[
\begin{align*}
(s(x)u - \eta(x)u^2)^{tt} - \Delta u + \tilde{D}u &= \tilde{r} \quad \text{in } \Omega \times (0, T) \\
\partial_\nu u + \gamma u &= 0 \quad \text{on } \partial\Omega \times (0, T), \quad u(0) = 0, \quad u_t(0) = 0 \quad \text{in } \Omega,
\end{align*}
\]

with

\[
\begin{align*}
\frac{s}{c_0^2} = \eta = \frac{\kappa}{c_0^2}.
\end{align*}
\]

and without cutoff of the nonlinearity. We will show linearised uniqueness of \(\eta(x)\) and \(s(x)\), given two appropriately chosen excitations \(\tilde{r}_i, i \in \{1, 2\}\). On the one hand, this is essential for well-definedness and convergence of the frozen Newton method considered in the reconstruction section below. On the other hand, via \((52)\), uniqueness of \(\eta(x)\) and \(s(x)\) is equivalent to uniqueness of \(\kappa(x)\) and \(c_0(x)\).

We linearise the forward operator \(F = C \circ S\) with the parameter-to-state map \(S : (\eta, s) \mapsto u\) where \(u\) solves \((6)\) and the observation operator \(C : u \mapsto \text{tr}_\Sigma u\) at \(\eta = 0, s = 1\). This yields \(F'(0, 1)(d\eta, ds) = \text{tr}_\Sigma du\), where \(du\) solves \((10)\) with \(\eta = 0, s = 1\), that is,

\[
\begin{align*}
\frac{1}{c^2}du_{tt} - \Delta du + \tilde{D}du &= -\left(d\eta(x)u^0 - d\eta(x)(u^0)^2\right)_{tt} \quad \text{in } \Omega \times (0, T)
\end{align*}
\]

with homogeneous initial and boundary conditions; here \(u^0\) solves \((6)\) with \(\eta = 0, s = \frac{1}{c^2}\).

As in the proof of Theorem 3.1 we choose the excitations \(\tilde{r}_i, i \in \{1, 2\}\), such that they lead to space-time separable solutions \(u^0_i(x,t) = \phi_i(x)\psi_i(t)\) of \((6)\), cf. \((44)\). The corresponding components of the forward operator \(\tilde{F} = (F_1, F_2)\) are defined by \(F_i = C \circ S_i\) with \(S_i : (\eta, s) \mapsto u_i\) where \(u_i\) solves \((6)\) with \(\tilde{r} = \tilde{r}_i, i \in \{1, 2\}\).
With the relaxation functions \(1/\omega_j\), we can write the Laplace transformed solutions \(\tilde{d}_i\), \(i \in \{1, 2\}\) as

\[
\tilde{d}_i(x, z) = \sum_{j=1}^{\infty} \frac{1}{\omega_j}(z) \sum_{k \in K^\lambda_j} \left( (ds\phi_i, \varphi_j^k)\psi_i^n(z) + (d\eta\phi_i, \varphi_j^k)(\psi_i^n(z)) \right)\varphi_j^k(x).
\]

Here \((\lambda_j, \varphi_j^k)\) is an eigensystem of \(A = -\Delta\) equipped with the impedance boundary conditions of \((\mathcal{D}, \mathcal{A})\), and \(\langle \cdot, \cdot \rangle\) is the ordinary \(L^2\) inner product on \(\Omega\).

The premiss \(F_i'(0, 1/\omega_j)(d\eta, ds) = 0\) therefore reads as

\[
0 = \sum_{j=1}^{\infty} \frac{1}{\omega_j}(z) \sum_{k \in K^\lambda_j} \left( (ds\phi_i, \varphi_j^k)\psi_i^n(z) + (d\eta\phi_i, \varphi_j^k)(\psi_i^n(z)) \right)\varphi_j^k(x_0), \quad x_0 \in \Sigma. \tag{53}
\]

Again we assume this to hold for \(i \in \{1, 2\}\) and can make use of Lemmas \(\ref{lem:linear_independence} \tag{3.1} \) and \(\ref{lem:regularity} \tag{3.2} \) as well as the linear independence assumption \(\ref{ass:linear_independence} \tag{4.7} \) to single out the coefficients by taking the residues at the poles \(p_j\) of \(1/\omega_j\) in \(\ref{eq:53} \tag{53} \), which implies

\[
0 = (ds\phi_i, \varphi_j^k)\psi_i^n(p_j) + (d\eta\phi_i, \varphi_j^k)(\psi_i^n(p_j)), \quad j \in \mathbb{N}, \quad k \in K^\lambda_j, \quad i \in \{1, 2\}. \tag{54}
\]

Now we choose \(\phi_1 = \phi_2 = \phi \neq 0\) almost everywhere in \(\Omega\) so that for each \(k\) and \(j\), \(\ref{eq:54} \tag{54} \) becomes a two-by-two system of equations for the coefficients \(a_j^k := (ds\phi, \varphi_j^k)\) and \(b_j^k := (d\eta\phi^2, \varphi_j^k)\). Choosing \(\psi_1, \psi_2\) such that for all poles \(p_j\), the system matrix is regular, that is,

\[
0 \neq \det \begin{pmatrix} \psi_1^n(p_j) & (\psi_2^n(p_j)) \\ (\psi_1^n(p_j)) & (\psi_2^n(p_j)) \end{pmatrix} \quad j \in \mathbb{N}, \tag{55}
\]

we obtain \(a_j^k = 0, b_j^k = 0\) for all \(j \in \mathbb{N}, k \in K^\lambda_j\). Hence, the functions \(ds\phi\) and \(d\eta\phi^2\) vanish in \(L^2(\Omega)\) and by our choice of \(\phi \neq 0\) a.e. this implies that \(ds = 0, d\eta = 0\) almost everywhere in \(\Omega\).

Thus, with the excitations

\[
\tilde{r}_i(x, t) := \frac{1}{c^2}\phi(x)\psi_i^n(t) + (A\phi)(x)\psi_i(t) + D[\phi\psi_i](x, t), \quad i \in \{1, 2\} \tag{56}
\]

we have proven

**Theorem 3.3** Assume that \(\ref{ass:linear_independence} \tag{4.7} \) holds for the eigenvectors of \(A = -\Delta\) and the excitations \(\tilde{r}_i\) take the form \(\ref{eq:56} \tag{56} \) with \(\phi \in \mathcal{D}(A), \phi \neq 0\) a.e. in \(\Omega\) and \(\psi_1, \psi_2\) satisfying \(\ref{eq:55} \tag{55} \). Then, \(F_i'(0, 1/\omega_j)(d\eta, ds) = 0, i \in \{1, 2\}\) implies \(d\eta = 0, ds = 0\).

The same proof also works with the original \(\kappa(x)\) and \(c_0(x)\) formulation. Indeed for \(\tilde{F}_i : (\kappa, c_0^2) \mapsto \text{tr}_\Sigma u_i\) (note that we take the squared sound speed as a variable), where \(u_i\) solves \(\ref{eq:3} \tag{3} \) with

\[
r_i(x, t) := \phi(x)\psi_i^n(t) + c^2(A\phi)(x)\psi_i(t) + D[\phi\psi_i](x, t), \quad i \in \{1, 2\} \tag{57}
\]
where $\mathcal{A} = -\frac{c_0(x)^2}{c_2} \Delta$, we get $\tilde{F}_i'(0, c^2)(dk, dc_0^2) = \text{tr}_Y du_i$, where

$$
du_{i,tt} - c_2 \Delta du_i + D du_i = dk \phi^2(\psi_i'^2)'' + dc_0^2 \Delta \phi \psi_i \text{ in } \Omega \times (0, T).
$$

Thus, from $\tilde{F}_i'(0, c^2)(dk, dc_0^2) = 0$ for $i \in \{1, 2\}$, together with (47) and Lemmas 3.1, 3.2, we obtain, in place of (54), that

$$
0 = \langle dk \phi, \varphi_j^k \rangle (\psi_1'^2)''(p_j) + \langle dc_0^2 \Delta \phi_i^2, \varphi_j^k \psi_i(p_j) \rangle, \quad j \in \mathbb{N}, \quad k \in K^j, \quad i \in \{1, 2\}.
$$

Hence, under the assumption

$$
0 \neq \text{det} \left( \begin{array}{cc} \psi_1'^2''(p_j) & \psi_1(p_j) \\ \psi_2'^2''(p_j) & \psi_2(p_j) \end{array} \right) \quad j \in \mathbb{N}, \quad (58)
$$

we obtain

**Theorem 3.4** Assume that (47) holds for the eigenvectors of $\mathcal{A} = -\frac{c_0(x)^2}{c_2} \Delta$ and the excitations $r_i$ take the form (57) with $\phi \in D(\mathcal{A})$, $\phi \neq 0$, $\Delta \phi \neq 0$ a.e. in $\Omega$ and $\psi_1, \psi_2$ satisfying (58). Then, $\tilde{F}_i'(0, c^2)(dk, dc_0^2) = 0$, $i \in \{1, 2\}$ implies $dk = 0$, $dc_0^2 = 0$.

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**References**

[1] Sebastian Acosta, Gunther Uhlmann, and Jian Zhai. Nonlinear ultrasound imaging modeled by a Westervelt equation. *SIAM Journal on Applied Mathematics*, 82(2):408–426, 2022.

[2] Melody Alsaker, Diego A C Cárdenas, Sergio S Furuie, and Jennifer L Mueller. Complementary use of priors for pulmonary imaging with electrical impedance and ultrasound computed tomography. *J Comput Appl Math*, 395:113591, 2021.

[3] Arash Anvari, Flemming Forsberg, and Anthony E. Samir. A primer on the physical principles of tissue harmonic imaging. *RadioGraphics*, 35(7):1955–1964, 2015. PMID: 26562232.

[4] Alain Bamberger, Roland Glowinski, and Quang Huy Tran. A domain decomposition method for the acoustic wave equation with discontinuous coefficients and grid change. *SIAM J.NUMER.ANAL.*, 34(2):603–639, April 1997.
[5] L. Bjørnø. Characterization of biological media by means of their non-linearity. *Ultrasonics*, 24(5):254 – 259, 1986.

[6] V. Burov, I. Gurinovich, O. Rudenko, and E. Tagunov. Reconstruction of the spatial distribution of the nonlinearity parameter and sound velocity in acoustic nonlinear tomography. *Acoustical Physics*, 40:816–823, 11 1994.

[7] Wei Cai, Wen Chen, Jun Fang, and Sverre Holm. A Survey on Fractional Derivative Modeling of Power-Law Frequency-Dependent Viscous Dissipative and Scattering Attenuation in Acoustic Wave Propagation. *Applied Mechanics Reviews*, 70(3), 2018.

[8] Charles A. Cain. Ultrasonic reflection mode imaging of the nonlinear parameter b/a: I. a theoretical basis. *The Journal of the Acoustical Society of America*, 80(1):28–32, 1986.

[9] Michele Caputo. Linear models of dissipation whose q is almost frequency independent-ii. *Geophysical Journal of the Royal Astronomical Society*, 13(5):529–539, 1967.

[10] W. Chen and S. Holm. Fractional laplacian time-space models for linear and nonlinear lossy media exhibiting arbitrary frequency power-law dependency. *The Journal of the Acoustical Society of America*, 115(4):1424–1430, 2004.

[11] Mostafa Fatemi and James F Greenleaf. Ultrasound-stimulated vibro-acoustic spectrography. *Science*, 280:82–85, 1998.

[12] Mostafa Fatemi and James F. Greenleaf. Vibro-acoustography: An imaging modality based on ultrasound-stimulated acoustic emission. *Proceedings of the National Academy of Sciences*, 96(12):6603–6608, 1999.

[13] Hartmut Gemmeke, Torsten Hopp, Michael Zapf, Clemens Kaiser, and Nicole V. Ruiter. 3d ultrasound computer tomography: Hardware setup, reconstruction methods and first clinical results. *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, 873:59–65, 2017. Imaging 2016.

[14] James F. Greenleaf, S. A. Johnson, S. L. Lee, G. T. Hermant, and E. H. Woo. Algebraic reconstruction of spatial distributions of acoustic absorption within tissue from their two-dimensional acoustic projections. In Philip S. Green, editor, *Acoustical Holography: Volume 5*, pages 591–603. Springer US, Boston, MA, 1974.

[15] Sverre Holm and Sven Peter Näsholm. A causal and fractional all-frequency wave equation for lossy media. *The Journal of the Acoustical Society of America*, 130(4):2195–2202, 2011.

[16] Nobuyuki Ichida, Takuso Sato, and Melvin Linzer. Imaging the nonlinear ultrasonic parameter of a medium. *Ultrasonic Imaging*, 5(4):295–299, 1983. PMID: 6686896.
[17] Ashkan Javaherian, Felix Lucka, and Ben T. Cox. Refraction-corrected ray-based inversion for three-dimensional ultrasound tomography of the breast. Inverse Problems, 36(12):125010, 41, 2020.

[18] B. Kaltenbacher and I. Lasiecka. Global existence and exponential decay rates for the Westervelt equation. Discrete and Continuous Dynamical Systems (DCDS), 2:503–525, 2009.

[19] Barbara Kaltenbacher. Mathematics of Nonlinear Acoustics. Evolution Equations and Control Theory (EECT), 4:447–491, 2015. open access: https://www.aimsciences.org/article/doi/10.3934/eect.2015.4.447.

[20] Barbara Kaltenbacher. Convergence guarantees for coefficient reconstruction in PDEs from boundary measurements by variational and Newton type methods via range invariance. 2022. submitted and arXiv:2209.12596 [math.NA].

[21] Barbara Kaltenbacher. On the inverse problem of vibro-acoustography. Meccanica, 2022. to appear; https://doi.org/10.1007/s11012-022-01485-w; see also arXiv:2109.01907 [math.AP].

[22] Barbara Kaltenbacher and William Rundell. On the identification of the nonlinearity parameter in the Westervelt equation from boundary measurements. Inverse Problems & Imaging, 15:865–891, 2021.

[23] Barbara Kaltenbacher and William Rundell. On an inverse problem of nonlinear imaging with fractional damping. Mathematics of Computation, 91:245–276, 2022. see also arXiv:2103.08965 [math.AP].

[24] Barbara Kaltenbacher and William Rundell. Inverse Problems for Fractional Partial Differential Equations. Graduate Studies in Mathematics. AMS, 2023. to appear.

[25] F. Mainardi. Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models. Imperial College Press, 2010.

[26] Alison E. Malcolm, Fernando Reitich, Jiaqi Yang, James F. Greenleaf, and Mostafa Fatemi. Numerical modeling for assessment and design of ultrasound vibro-acoustography systems. In Biomedical Applications of Vibration and Acoustics for Imaging and Characterizations. ASME Press, New York, 2007.

[27] Alison E. Malcolm, Fernando Reitich, Jiaqi Yang, James F. Greenleaf, and Mostafa Fatemi. A combined parabolic-integral equation approach to the acoustic simulation of vibro-acoustic imaging. Ultrasonics, 48:553–558, 2008.

[28] Jennifer L Mueller, Diego A C Cárdenas, and Sergio S Furuie. A preclinical simulation study of ultrasoundsd tomography for pulmonary bedside monitoring. In Proceedings of the Second International Workshop on Medical Ultrasound Tomography (MUSTII), 2021.
[29] A Panfilova, RJG van Sloun, H Wijkstra, OA Sapozhnikov, and Mischi M. A review on b/a measurement methods with a clinical perspective. *The Journal of the Acoustical Society of America*, 149(4):2200, 2021.

[30] Uppal Talat. Tissue harmonic imaging. *Australas J Ultrasound Med*, 13(2):29–31, 2010.

[31] François Varray, Olivier Basset, Piero Tortoli, and Christian Cachard. Extensions of nonlinear b/a parameter imaging methods for echo mode. *IEEE transactions on ultrasonics, ferroelectrics, and frequency control*, 58:1232–44, 06 2011.

[32] Margaret G. Wismer. Finite element analysis of broadband acoustic pulses through inhomogenous media with power law attenuation. *The Journal of the Acoustical Society of America*, 120(6):3493–3502, 2006.

[33] Masahiro Yamamoto and Barbara Kaltenbacher. An inverse source problem related to acoustic nonlinearity parameter imaging. In Barbara Kaltenbacher, Anne Wald, and Thomas Schuster, editors, *Time-dependent Problems in Imaging and Parameter Identification*. Springer, New York, 2021.

[34] Eberhard Zeidler. *Applied Functional Analysis: Main Principles and Their Applications*, volume 109. Springer Science & Business Media, 1995.

[35] Dong Zhang, Xi Chen, and Xiu-fen Gong. Acoustic nonlinearity parameter tomography for biological tissues via parametric array from a circular piston source—theoretical analysis and computer simulations. *The Journal of the Acoustical Society of America*, 109(3):1219–1225, 2001.

[36] Dong Zhang, Xiufen Gong, and Shigong Ye. Acoustic nonlinearity parameter tomography for biological specimens via measurements of the second harmonics. *The Journal of the Acoustical Society of America*, 99(4):2397–2402, 1996.