Recent advances in description of few two-component fermions

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Abstract

Overview of the recent advances in description of the few two-component fermions is presented. The model of zero-range interaction is generally considered to discuss the principal aspects of the few-body dynamics. Particular attention is paid to detailed description of two identical fermions of mass $m$ and a distinct particle of mass $m_1$: it turns out that two $L^P = 1^-$ three-body bound states emerge if mass ratio $m/m_1$ increases up to the critical value $\mu_c \approx 13.607$, above which the Efimov effect takes place. The topics considered include rigorous treatment of the few-fermion problem in the zero-range interaction limit, low-dimensional results, the four-body energy spectrum, crossover of the energy spectra for $m/m_1$ near $\mu_c$, and properties of potential-dependent states. At last, enlisted are the problems, whose solution is in due course.

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I. INTRODUCTION

In the recent years, properties of multi-component ultra-cold quantum gases, including binary Fermi-Bose [1–3] and Fermi [4–6] mixtures and impurities embedded in a quantum gas [7–9] are under thorough experimental and theoretical investigation. In this respect, the low-energy few-body dynamics in two-species mixtures has attracted much attention. In particular, study of the energy spectrum and low-energy scattering of few two-component particles gives insight into the role of few-body processes in the many-body dynamics. One should also mention the reactions with negative atomic and molecular ions [10, 11] and the two-component model for the three-body recombination near the Feshbach resonance [12].

The aim of the paper is to present an overview of the recent advances in description of few ultra-cold two-component particles; it is assumed that identical particles are fermions interacting with a distinct particle via the $s$-wave potential, whereas interaction of identical fermions is forbidden. For the sake of generality, in some cases it is suitable to consider also a system of few non-interacting identical bosons and a distinct particle. It is worthwhile to mention that the $p$-wave interactions between fermions may be also important, in particular, an infinite number of the $1^+$ bound states was predicted [13] for three identical fermions.

To investigate the principal aspects of the few-body dynamics, it is natural to use the limit of zero-range two-body interaction, which allows one to obtain universal description of the few-body properties. In this respect, it is necessary to formulate rigorously the few-body problem in the zero-range interaction limit, which is an interesting problem in itself, especially if the identical particles are fermions.

The zero-range model is suitably defined by imposing the boundary condition at the zero inter-particle distance. Thus, the interaction depends on a single parameter, e. g., the two-body scattering length $a$, which can be chosen as a length scale. As a result, the energy scale is $1/a^2$, and the only remaining parameter is mass ratio $m/m_1$, where $m$ and $m_1$ denote masses of identical and distinct particles, respectively. The units $\hbar = |a| = 2m/(1+m/m_1) = 1$, for which the two-body binding energy $\varepsilon_2 = 1$, will be used throughout the paper.
II. UNIVERSAL PROPERTIES OF THREE TWO-COMPONENT Fermions

Investigation of two identical fermions and a distinct particle are of considerable interest for description of the multi-component ultra-cold gases. The three-body states of unit total angular momentum and negative parity \( (L^P = 1^-) \) are especially important in treatment of the low-energy processes [14, 15]. A major progress was achieved in [16], where it was shown that the zero-range model for sufficiently large mass ratio \( m/m_1 > \mu_c \) does not provide unique description of the three two-component fermions. The unambiguous description of the three-body properties for \( m/m_1 > \mu_c \) requires an additional parameter, which determines the wave function in the vicinity of the triple-collision point. As shown in [16], a number of three-body bound states is infinite and their energies differ by the scaling factor. Later on, properties of the three-body spectrum were discussed also in a number of subsequent papers [11, 17–20]. The critical mass ratio is determined by a solution of the transcendental equation for \( \sin \omega_c = 1/(1 + m_1/m) \)

\[
\frac{\pi}{2} \sin^2 \omega_c - \tan \omega_c + \omega_c = 0 ,
\]

which gives \( \omega_c \approx 1.19862376 \) and \( \mu_c = \sin \omega_c/(1 - \sin \omega_c) \approx 13.6069657 \). Furthermore, a significant achievement in this area was the construction of zero-energy solution for three two-component fermions in the interval \( 0 \leq m/m_1 \leq \mu_c \) [14], which provides the analytical expression for the low-energy recombination rate. A complete description of both the energy spectrum and the elastic and inelastic low-energy scattering is discussed below.

A. Hyperradial equations

In the zero-range limit, the two-body interaction is defined by imposing the boundary condition at the zero inter-particle distance \( r \)

\[
\lim_{r \to 0} \frac{\partial \ln(r \Psi)}{\partial r} = -\text{sign}(a) .
\]

The interaction introduced by means of the boundary condition is widely discussed in the literature [21, 24].

Both qualitative and numerical results are obtained by using the solution of hyper-radial
equations (HREs) \[25\]

\[
\left[ \frac{d^2}{d\rho^2} - \frac{\gamma_n^2(\rho) - 1/4}{\rho^2} + E \right] f_n(\rho) - \sum_{m=1}^{\infty} \left[ P_{mn}(\rho) - Q_{mn}(\rho) \frac{d}{d\rho} - \frac{d}{d\rho} Q_{mn}(\rho) \right] f_m(\rho) = 0 , \quad (3)
\]

whose terms \(\gamma_n^2(\rho), Q_{nm}(\rho), P_{nm}(\rho)\) are derived analytically \[26, 27\]. In fact, the critical mass ratio \(\mu_c\) can be determined from the condition \(\gamma_1(0) = 0\) as the first-channel effective potential at small \(\rho\) takes the form \([\gamma_1^2(0) - 1/4]/\rho^2\), which implies that a number of the bound states is finite for \(\gamma_1^2(0) > 0\) and infinite for \(\gamma_1^2(0) < 0\).

### B. Unit angular momentum

Thorough studies of three two-component fermions in the states of total angular momentum and parity \(L^P = 1^-\) were conducted in \[15\]. Only negative parity is considered, since for the positive-parity states three particles do not interact by the s-wave zero-range potential. For the problem under consideration, the functions \(\gamma_n(\rho)\) determining the effective potentials in (3) satisfy the transcendental equation

\[
\rho \frac{\text{sign}(a)}{\gamma} = \frac{1 - \gamma^2}{\gamma} \tan \frac{\pi}{2} - \frac{2}{\sin 2\omega \cos \gamma \frac{\pi}{2}} + \frac{\sin \gamma \omega}{\gamma \sin^2 \omega \cos \gamma \frac{\pi}{2}} \quad (4)
\]

In particular, taking the limit \(\gamma_1(0) \rightarrow 0\) in Eq. (4), one obtains Eq. (1) that determine \(\mu_c\).

From the solution of HREs (3) it follows that for \(a > 0\) there are no bound states in the interval \(0 < m/m_1 < \mu_1\), exactly one bound state exists in the interval \(\mu_1 \leq m/m_1 < \mu_2\), and two bound states exist in the interval \(\mu_2 \leq m/m_1 \leq \mu_c\), where \(\mu_1 \approx 8.17260\) and \(\mu_2 \approx 12.91743\). The bound-state energies decrease with increasing mass ratio on the interval \(0 < m/m_1 \leq \mu_c\), reaching the finite values \(E_1(\mu_c) \approx -5.8954, E_2(\mu_c) \approx -1.13764\) at the critical value \(m/m_1 = \mu_c\), and follow a square-root dependence \(E_i - E_i(\mu_c) \propto (\mu_c - m/m_1)^{1/2}\) near \(m/m_1 = \mu_c\). The dependence of bound-state energies on mass ratio is illustrated in Fig. 1.

For mass ratio just below \(\mu_i\) \((m/m_1 \lesssim \mu_1\) and \(m/m_1 \lesssim \mu_2\)), the relevant bound state turns to a narrow resonance, whose position \(E_i^r\) continues a linear mass-ratio dependence of the bound-state energy, \(E_i^r + 1 \propto \mu_i - m/m_1\), whereas the width \(\Gamma_i\) depends quadratically, \(\Gamma_i \propto (\mu_i - m/m_1)^2\).

Calculations at the three-body threshold reveal the two-hump structure of the mass-ratio dependencies for the elastic \((2 + 1)\)-scattering cross section and the three-body recombination rate. The dependence of the recombination rate is in accordance with the analytical
FIG. 1: Dependencies of the bound-state energies (in units of the two-body binding energy) on mass ratio \( m/m_1 \). The arrows mark mass ratios \( \mu_i \), for which the \( i \)th bound state emerges from the two-body threshold, the critical mass ratio \( \mu_c \), and the bound-state energies \( E_{ic} \) for \( m/m_1 = \mu_c \). In the inset the excited-state energy is shown on a large scale.

expression \( [14] \). The structure of both isotopic dependencies stems from the interference of the incoming and outgoing waves in the \( 2 + 1 \) channel; the effect of interference is connected with emergence of two three-body bound states due to the deepening of the effective potential with increasing mass ratio. A detailed explanation of the mass-ratio dependence can be found in \([15]\).

For negative scattering length \( a < 0 \) there are no three-body bound states in the interval \( 0 < m/m_1 \leq \mu_c \).

C. **Arbitrary angular momentum**

To obtain the comprehensive description of three two-component particles, it is necessary to study the three-body properties for arbitrary total angular momentum. In this respect, the three-body rotational-vibrational spectrum was analysed in \([28]\), where the identical particles were either fermions or bosons. It turns out that the properties of the three-body energy spectrum for arbitrary \( L \) resemble those for \( L = 1 \) \([15]\).
Method of calculations for arbitrary $L$ states in [28] was similar to that used for $L^p = 1$ states in [15], i.e., the system of HREs whose terms $\gamma_2^2(\rho)$, $Q_{nm}(\rho)$, $P_{nm}(\rho)$ are derived analytically [26, 27] was solved. By analysing the first-channel effective potential $(\gamma_1^2(\rho) - 1/4)/\rho^2$, it was shown in [28] that the bound states exist if either the identical particles are fermions and $L$ is odd or the identical particles are bosons and $L$ is even. Thus, it is suitable to treat jointly both bosonic problem for even $L$ and fermionic problem for odd $L$. The most discussed feature [16, 17, 19, 20] of the three-body problem under consideration is an appearance of the infinite number of bound states for sufficiently large $m/m_1$. For each $L$, $\gamma_1^2(0)$ decreases with increasing $m/m_1$ and crosses zero at the critical value $\mu_c(L)$. Thus, the number of bound states is infinite for mass ratio above the critical value and finite below it.

A set of critical mass ratios $\mu_c(L)$ are given in Table I for $L = 1 - 5$.

TABLE I: Upper part: Mass ratios $\mu_N(L)$ at which the $N$-th bound state arises and the critical values $\mu_c(L)$. Lower part: Bound-state energies $|E_{LN}(\mu_c(L))|$ at critical mass ratios.

| $L$ | $N = 1$ | $N = 2$ | $N = 3$ | $N = 4$ | $N = 5$ | $N = 6$ | $\mu_c(L)$ |
|-----|--------|--------|--------|--------|--------|--------|----------|
| 1   | 7.9300 | 12.789 | -      | -      | -      | -      | 13.6069657 |
| 2   | 22.342 | 31.285 | 37.657 | -      | -      | -      | 38.6301583 |
| 3   | 42.981 | 55.766 | 67.012 | 74.670 | -      | -      | 75.9944943 |
| 4   | 69.885 | 86.420 | 101.92 | 115.08 | 123.94 | -      | 125.764635 |
| 5   | 103.06 | 123.31 | 142.82 | 160.64 | 175.48 | 185.51 | 187.958355 |

| $L$ | $N = 1$ | $N = 2$ | $N = 3$ | $N = 4$ | $N = 5$ | $N = 6$ |
|-----|--------|--------|--------|--------|--------|--------|
| 1   | 5.906  | 1.147  | -      | -      | -      | -      |
| 2   | 12.68  | 1.850  | 1.076  | -      | -      | -      |
| 3   | 22.59  | 2.942  | 1.417  | 1.057  | -      | -      |
| 4   | 35.59  | 4.392  | 1.920  | 1.273  | 1.049  | -      |
| 5   | 52.16  | 6.216  | 2.566  | 1.584  | 1.206  | 1.045  |

It was shown that three particles are not bound for $a < 0$ and mass ratio in the interval $m/m_1 \leq \mu_c(L)$. There is a finite number of bound states for $a > 0$ and $m/m_1 \leq \mu_c(L)$. More precisely, three particles are unbound for sufficiently small $m/m_1$ and with increasing mass ratio the $N$th bound state arises at $m/m_1 = \mu_N(L)$. The energies of each bound state $E_{LN}(m/m_1)$ monotonically decrease with increasing mass ratio, reaching the finite values at $\mu_c(L)$ and obeying the $[\mu_c(L) - m/m_1]^{1/2}$ dependence just below $\mu_c(L)$. For $m/m_1$ just
below the critical values $\mu_N(L)$, there are resonances, whose positions depend linearly and widths depend quadratically on the mass-ratio excess $\mu_N(L) - m/m_1$.

Reasonable estimates (up to a few percent accuracy) of bound-state energies for $a > 0$ were obtained numerically by using the one-channel approximation for the total wave function. Critical values of mass ratios $\mu_N(L)$ and three-body energies $E_{LN}(m/m_1)$ at $m/m_1 = \mu_c(L)$ are given in Table I. Accuracy of the one-channel approximation can be estimated by comparing the $L = 1$ values $\mu_N(1)$ and $E_{1N}[\mu_c(1)]$ from Table I with the precise values $\mu_1$, $\mu_2$, $E_1(\mu_c)$, and $E_2(\mu_c)$ given in Section II B. Notice that for $L = 3 - 5$ the uppermost bound states are loosely bound and appear very close to the corresponding critical values $\mu_c(L)$. Thus, taking into account an estimated accuracy of the approximation, one concludes that more careful calculation is necessary to describe these loosely bound states.

As it was shown in [15] for $L = 1$, an appearance of the three-body bound states with increasing mass ratio is intrinsically connected with the oscillating behaviour of the $2 + 1$ elastic-scattering cross section and the three-body recombination rate. Analogously, the dependence of the scattering amplitudes on mass ratio for higher $L$ would exhibit the number of interference maxima which are related to the appearance of up to $N_{\text{max}}$ bound states.

A reasonably good description of the energy spectrum is obtained within the framework of the quasi-classical approximation, which allows the asymptotic expression of bound-state energies $E_{LN}(m/m_1)$ to be derived for large $L$ and $m/m_1$. As a result for $\mu_1(L) \leq m/m_1 \leq \mu_c(L)$ and $a > 0$, all the bound-state energies are described by the universal function of two scaled variables $\xi = (N - 1/2)/\sqrt{L(L + 1)}$ and $\eta = \sqrt{m/[m_1L(L + 1)]}$. This scaling dependence is confirmed by the numerical calculations for $L > 2$ and is in good agreement even for small $L = 1, 2$. The universal description implies that $\mu_c(L) \approx 6.218(L + 1/2)^2$ and $\mu_1(L) \approx 3.152(L + 1/2)^2$ for large $L$, while the number of vibrational states for given $L$ is limited as $N_{\text{max}} \leq 1.1\sqrt{L(L + 1)} + 1/2$. More details are presented in paper [28].

D. Integral equations

Description of the two-component systems was obtained also by solving the momentum-space integral equations (generalised Skorniakov-Ter-Martirosonian equations) in [29, 30]. These calculations generally confirm the results of [15, 28] and contain some additional details. One should mention that the values $\mu_N(1)$ calculated in [29] are in agreement with
those obtained in [15] for $L = 1$, while $\mu_N(L)$ from [29] for $L = 2 - 4$ slightly differ from the approximate values given in [28]. Besides, the characteristics of $(2+1)$ zero-energy scattering for $L = 0 - 3$ were obtained in [29]. These calculations provide an additional evidence for the appearance of three-body bound states, in particular, the $P$-wave scattering volume diverges exactly at $m/m_1$ tending to the critical values $\mu_1$ and $\mu_2$. The mass-ratio dependence of the elastic $(2+1)$-scattering cross sections for different energies below the three-body threshold was studied in [30]. In addition, the momentum-space integral equations were applied to solution of different few-body problems [31–35]. More details of these calculations are given in Section VI.

E. Two-dimensional problem

Study of three two-component particles with zero-range interactions confined in two dimensions was performed in [36], where the mass-ratio dependence of the three-body energies and a set of critical values $m/m_1$, at which bound states emerge, were obtained by solving the momentum-space integral equations. Similar to 3D problem [28], the bound states exist if either the identical particles are fermions and $L$ is odd or the identical particles are bosons and $L$ is even; the binding energies monotonically increase with increasing mass ratio. Calculation [36] shows that for $L = 0$ two bosons and the third particle are bound for any mass ratio, while the second and third bound state appear at $m/m_1 \approx 1.77$ and $m/m_1 \approx 8.341$. Likewise, for $L = 1$ two fermions and the third particle are bound for $m/m_1 \geq 3.34$, while the second and third bound state appear at $m/m_1 \approx 10.41$ and $m/m_1 \approx 20.85$. As in 3D problem [28], different rotational-vibrational states become quasi-degenerate for large $L$ and $m/m_1$.

More general problem for three two-component particles was considered in paper [37], where all three zero-range interactions were taken into account. In particular case of two noninteracting identical particles, a set of mass-ratio values, at which the $L = 0$ three-body bound states emerge, are consistent with those found in [36]. Furthermore, the energy spectrum of three 2D particles for different combinations of all masses and interaction strengths was considered in [38]. The $S$- and $P$-wave elastic $(2+1)$-scattering and the three-body recombination in 2D two-component mixtures were studied for mass ratio corresponding to both $^6$Li–$^{40}$K and equal-mass particles [39]. A number of 2D bound-state and scattering
properties for three and four particles were calculated in [40], where different combinations
of equal-mass bosons and fermions were considered.

The transition from two dimensions to three dimensions was studied in a quasi-two-
dimensional geometry by confining particles in a harmonic potential along one direction [41].
It was shown that $P$-wave energies of two identical fermions and one distinct particle
smoothly evolve from $3D$ to $2D$ (with increasing confining frequency) for $m/m_1 \leq \mu_c$.
Correspondingly, the mass ratio, at which the three-body bound states emerges, increases
from $2D$ value 3.33 [36] to $3D$ value 8.17260 [15]. In addition, it was estimated in [41] that
in $2D$ limit three identical fermions and one distinct particle are bound for $m/m_1 > 5$.

F. One-dimensional problem

Properties of three two-component particles confined in one dimension with contact ($\delta$-
function) interactions were considered in [42]. It is assumed that attractive interaction of
strength $\lambda < 0$ acts between each of two identical particles (either bosons or fermions) and
a distinct one, while the strength of interaction between the identical particles is arbitrary
$\lambda_1$.

The three-body energy spectrum and the scattering length $A$ for collision of a bound pair
off the third particle were calculated for two values $\lambda_1 = 0$ and $\lambda_1 \to \infty$ of the even-parity
bosonic problem. Two sets of mass-ratio values, at which the three-body bound states arise
and at which $A = 0$, were calculated both for $\lambda_1 = 0$ and $\lambda_1 \to \infty$. It is important to
recall that the bosonic problem for $\lambda_1 \to \infty$ is exactly equivalent to the fermionic problem,
for which the contact interaction between fermions is absent ($\lambda_1 = 0$). For the odd-parity
states it was shown that three particles are not bound and the $(2 + 1)$ - scattering length
for $\lambda_1 = 0$ was calculated.

In addition, few analytical results were presented, in particular, it was shown that exactly
one bound state of three equal-mass particles ($m/m_1 = 1$) exists for arbitrary $\lambda_1$. Next,
for two light fermions (in the limit of $m/m_1 \to 0$) three particles are not bound for suffi-
ciently large repulsive interaction between fermions, viz., for $\lambda_1/|\lambda|$ above the critical value
$\approx 2.66735$ [43], and exactly one bound state exists for $\lambda_1/|\lambda|$ below this value. To elucidate
the general features of three one-dimensional particles, both numerical and analytical results
were used to construct a schematic “phase” diagram, which shows the number of three-body
bound states and a sign of the \((2 + 1)\)-scattering length \(A\) in the plane of the parameters \(m/m_1\) and \(\lambda_1/|\lambda|\).

G. Dimensional analysis

Contrary to the three-body problem in 3D, the 2D and 1D solutions remain regular near the triple-collision point even in the limit of zero-range two-body interaction; therefore, there is neither Thomas nor Efimov effect. As a result, it is not necessary to introduce an additional regularisation parameter and the low-energy three-body properties in 2D and 1D are completely determined by the two-body input.

The results of calculations \([15, 28, 36, 42]\) give an opportunity to analyse the dependence of the three-body low-energy properties on the configuration-space dimension. It turns out that two identical fermions and a distinct particle are bound in 1D for \(m/m_1 \geq 1\), in 2D for \(m/m_1 \geq 3.33\) and in 3D for \(m/m_1 \geq 8.17260\). Two identical non-interacting bosons and a distinct particle are bound in 1D and 2D for any mass ratio, the first excited state appears in 1D at \(m/m_1 \approx 2.869539\) and in 2D at \(m/m_1 \approx 1.77\), while in 3D the number of bound states is infinite. If the three-body binding energy exceeds the two-body binding energy, the production of the triatomic molecules becomes energetically more favourable than diatomic ones. For two identical fermions and a distinct particle it is justified if \(m/m_1 > 49.8335\) in 1D, \(m/m_1 > 18.3\) in 2D, and \(m/m_1 > 12.69471\) in 3D.

III. POTENTIAL-DEPENDENT STATES

Analysing the few-body properties for the small interaction range \(r_0\) tending to zero, it is necessary to take into account two different length scales, \(r_0\) and \(a\) \((r_0 \ll a)\), which means that all the states should be classified as either universal ones, whose energies scale according to \(a^{-2}\) or potential-dependent ones, whose energies scale according to \(r_0^{-2}\). In the unitary limit \(a \to \infty\), the energy of universal state tends to zero, whereas the energy of potential-dependent state remains finite.

The three-body bound states for two-component fermions in the limit of the infinite two-body scattering length were considered in \([44, 45]\). The interaction between different particles was taken as the Gaussian potential, whose parameters were adjusted to provide \(a \to \infty\).
It was found that in the zero interaction-range limit the \( L^p = 1^- \) three-body bound state arises for mass ratio above \( \approx 12.314 \). This value is close to \( m/m_1 \approx 12.31310 \) determined from the condition \( \gamma_1(0) = 1/2 \), which means that above this mass ratio the first-channel effective potential in \( (\gamma_2^2 - 1/4)/\rho^2 \) becomes attractive at small \( \rho \). A similar problem was considered in [46], in which the potential-dependent three-body bound states were found to exist at least for \( m/m_1 > 13 \) and two forms of the two-body potential.

IV. CROSSOVER AT THE CRITICAL MASS RATIO

As discussed in Section II in the zero-range limit the three-body properties are essentially different for mass ratio below and above the critical value \( \mu_c \), e. g., a number of bound states abruptly increases from two to infinity [15, 28–30]. Thus, one naturally needs to describe a crossover with increasing \( m/m_1 \) beyond the critical value \( \mu_c \).

Recently, to study the crossover at \( \mu_c \), the dependence of the three-body bound state energies on mass ratio and the additional short-range three-body potential was calculated by using the momentum-space integral equations in [35]. For mass ratio above \( \mu_c \) the three-body potential is necessary to provide unambiguous description of the wave function near the triple collision point. Explicitly, the momentum cutoff in the integral equation is introduced in paper [35] by using the dimensionless parameter \( \Lambda \). Analysing the dependence of three-body energies in the plane of two parameters \( m/m_1 \) and \( \Lambda \), it was found that the universal bound states found in [15, 28–30] are located in the region \( \mu_1 \leq m/m_1 \leq \mu_c \) and \( \Lambda^{-1} \rightarrow 0 \), whereas the Efimov states are located in the region \( m/m_1 > \mu_c \) and \( \Lambda^{-1} \rightarrow 0 \). These two regions are joined by the crossover area, where the bound-state energies depend on both \( m/m_1 \) and \( \Lambda \) and are almost independent of the particular choice of the three-body potential.

V. ZERO-RANGE MODEL IN THE FEW-FERMION PROBLEM

Mathematical aspects of application of the zero-range-interaction model to few two-component fermions were investigated, e. g., in papers [18, 47–49]. In this respect, note the paper [47], in which it was shown that the zero-range model for three and four fermions can be correctly defined for sufficiently small mass ratios. Similar statement was made for an ar-
bitary number of fermions in [49]. Among discussions of properties of three two-component fermions, one should mention a paper [18], in which the original Efimov’s statement was proved, viz., the energy spectrum for the quantum numbers $L^P = 1^-$ becomes unbound from below for $m/m_1 > \mu_c$ (corresponding to $\gamma_1^2(0) < 0$). Recently, it was shown [48] that the three-fermion Hamiltonian is ambiguous in the zero-range limit if mass ratio exceeds $12.31310$ (corresponding to $\gamma_1(0) \leq 1/2$). It is important to note simple considerations [50], which indicates an existence of two square-integrable solutions in the vicinity of the triple-collision point $\rho \rightarrow 0$ for mass ratio in the interval $8.619 \leq m/m_1 \leq 13.607$ (corresponding to $1 \geq \gamma_1(0) \geq 0$). Therefore, for mass ratio that belongs to this interval, special care is needed to describe three two-component fermions.

VI. CONCLUDING REMARKS

An appearance of three-atomic molecules containing two heavy and one light particles crucially determines the equilibrium states and dynamics of both fermionic and fermionic-bosonic mixtures. One of interesting examples is the mixture of strontium and lithium isotopes. As for the $^7\text{Li} – ^{87}\text{Sr}$ mixture mass ratio ($m/m_1 \approx 12.4$) gets between $\mu_1$ and $\mu_2$ (at which the first and second bound states emerge), one expects that there is exactly one $P$-wave bound state of $^7\text{Li}^{87}\text{Sr}_2$ molecule. Calculations [15] predict energy of this molecule about $-1.793$ (recall that the binding energy of the diatomic $^7\text{Li}^{87}\text{Sr}$ molecule is taken as energy unit). For the $^6\text{Li} – ^{87}\text{Sr}$ mixture mass ratio ($m/m_1 \approx 14.5$) slightly exceeds $\mu_c$, which means that there are at least two $P$-wave bound states of the molecule $^7\text{Li}^{87}\text{Sr}_2$, whose energies should be slightly below $-5.895$ and $-1.138$ (in units of $^6\text{Li}^{87}\text{Sr}$ binding energy). Furthermore, as $m/m_1 > \mu_c$, one expects that a number of bound states and their energies depend on the details of the interactions.

Besides influence of the three-body bound states on the properties of the two-component ultra-cold gases, significance of the three-body resonances was emphasised in [31]. Furthermore, for an experimentally interesting case of $^6\text{Li} – ^{40}\text{K}$ fermionic mixture mass ratio $m/m_1 \approx 6.64$ is close to the value $\mu_1 \approx 8.17$ (at which the bound state arises), a role of the $P$-wave three-body resonance in scattering of $^6\text{Li}$ off the $^6\text{Li}^{40}\text{K}$ molecule was elucidated in [31–34]. It is worthwhile to mention also $^{173}\text{Yb}$ and $^{23}\text{Na}$ fermion-boson mixture, for which the $P$-wave resonance should be taken into account as $m/m_1 \approx 7.52$ is even closer to
\( \mu_1 \). Possible influence of the \( D \)-wave resonance can be considered for \(^{133}\)Cs and \(^6\)Li mixture, for which \( m/m_1 \approx 22.17 \) is just below \( \mu_1(2) \approx 22.34 \) (corresponding to appearance of the \( D \)-wave three-body bound state).

The effect of the \( P \)-wave three-body bound state in the problem of a light impurity atom immersed in Fermi gas was considered in [9]. By using the variational method, the ground-state properties were determined as a function of mass ratio and dimensionless scattering length and corresponding phase diagram was constructed. It was shown that for a sufficiently large mass ratio \( (m/m_1 > 7) \) the formation of a three-body molecule is energetically more preferable than a two-body molecule or polaron. As expected, on the phase diagram the boundary between regions corresponding to two- and three-body molecules in the low-density limit tends to the pure three-body result \( \mu_1 \). Furthermore, the calculated phase diagram shows that with increasing Fermi-gas density the formation of three-atomic molecules becomes more favourable than two-atomic ones.

There is only scarce information about the properties of four two-component fermions. One of the principle results on three identical fermions and distinct particle was obtained in [51], where it was shown that the four-body \( L^P = 1^+ \) spectrum is not bounded from below for \( m/m_1 \geq 13.384 \). This means that the four-body Efimov effect takes place in the interval \( 13.384 \leq m/m_1 \leq 13.607 \). Note that this peculiarity is not present for other values of total angular momentum and parity \( (L^P = 0^+, 1^-) \). Similar to the three-body problem [15, 28–30], one supposes that the universal four-body bound states could exist below the four-body critical mass ratio \( m/m_1 \approx 13.384 \). Till now, there is only the calculation [52], which indicates existence of \( L^P = 1^+ \) bound state of three identical fermions and a distinct particle for \( m/m_1 \geq 9.5 \). Besides, one should note the interesting result on the four-body scattering problem [32], where the scattering length for two colliding \(^6\)Li \(^{40}\)K molecules was calculated.

Despite a marked success in describing a few two-component fermions, there are still many problems to be solved. In particular, some questions naturally arise in consideration of the potential-dependent states, at least in the limit \( a \rightarrow \infty \). It is of interest to determine the smallest mass ratio, below which three two-component fermions are not bound by any two-body potential, and to find explicitly the corresponding potential. Likewise, it is desirable to find a maximum number of bound states, which could arise for mass ratio increasing up to \( \mu_c \), and to determine the corresponding potential.

In order to elucidate the crossover from finite to infinite number of bound states for
$m/m_1$ around $\mu_c$, it seems to be important to take into account the energy dependence on the potential range $r_0$. In a similar way, it is of interest to trace a fate of the potential-dependent states (e.g., in the unitary limit $a \to \infty$) for $m/m_1$ increasing across the critical value $\mu_c$. In view of the above discussion in Section V on application of the zero-range model in a few-fermion problem, a special care on the asymptotic dependence of the solution near the triple collision point is needed to provide a complete description in the interval $8.62 \leq m/m_1 \leq 13.607$.

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