Anomalous changing angular momentum of hairy black holes geodesics

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Abstract We study the motion of test particles and the propagation of light around neutral hairy black holes under the influence of a self-interacting real scalar field minimally coupled to gravity. It is highly relevant in the context of recent observations of the Event Horizon Telescope: The Shadow of the supermassive Black Hole and The Shadow and Mass of the Central Black Hole. The goal of the present work is to show that the time-like and null-like geodesics have an anomalous behaviour for a special range of parameters in the dense hair region, defined as \( \sqrt{\Omega(x_h)} \leq r \leq \frac{2MG_N}{c^2} \). When the geodesics are in the dense hair region, they lose their angular momenta and enter the horizon.

1 Introduction

Hairy solutions are extensively constructed in the context of different theories, in some cases with minimal and non-minimal coupling with the Einstein-Hilbert theory [1–10]. In addition, there are many exact and numerically solutions for higher curvature theories [3, 11–14]. Another relevant topic is the thermodynamic of hairy black holes [14–19] where there are an interesting window of parameters in which the asymptotically flat hairy black holes are stable [20]. In the present work, we focus on the minimal coupling with real scalar field. It was constructed in [21–23], and consists of a general hairy family of asymptotically AdS solutions. Surprisingly if we fix, at the level of theory, \( \Lambda = 0 \) we get an exact hairy black hole solution which is asymptotically flat. That hairy solutions was studied extensively in [21, 24], and it can be embedded in SUGRA theories [4, 25, 26]. The hairy solutions presented here can evade the no-hair theorem [21], and the stability of the present hairy black holes is ensured by the scalar potential and its extreme points. Clearly, if I have only a kinetic term of the (real) scalar field in the theory, would not possible to evade the no-hair theorem, but if we add a non-minimal coupling gauge field it can behave like a scalar potential and there will be a possibility to evade the no-hair theorem [28]. When we have complex scalar fields or another exotic fields, it is required another details [29, 30].

An important motivation is the construction of toy models for supermassive black holes, for example Sagittarius A* [13, 31–33]. This family of hairy black hole solutions has a horizon radius \( r_h \) which is completely different from Schwarzschild radius \( \frac{2MG_N}{c^2} \), indeed we can prove that \( r_h \leq \frac{2MG_N}{c^2} \), then exist a region \( \Omega \equiv \frac{2MG_N}{c^2} - r_h \) called the dense-hair region and It is a concrete realization described in [34]. That region has an important effect on the geodesics, they lose their angular momentum, in consequence, the trajectories become linear radial motions. In black hole horizon figures, we use gray color to show the Schwarzschild region and black, for the hairy black hole. So, the region between them is the dense hair region \( \Omega \). This simple and novel property has an interesting effect on geodesic configurations.

The qualitative nature will be studied in [35], and here we construct numerically the time-like and null-like trajectories. We use the Runge-Kutta method of fourth-order(RK4), which means that there are four parameters to obtain. The numerical plots and calculations were made in Python 3.8 [36] with the following libraries, NumPy [37], SciPy [38], Matplotlib [39]. All the plots of hairy black holes have been made using the library\textit{ hairyBH} [40] and for Schwarzschild see [41].
The present work is organized in the following form: First, in section 2 we consider a brief description of geodesics of Schwarzschild black hole, and its respective details are in the appendix Appendix A. In section 3 we describe the theory and the properties of the hairy solution like the horizon existence and the mass for each branch. In 4 and 5 we construct the orbital equation and solve it numerically, we present the plots of the effective potential and its respective trajectories for each region. In 6 we construct the near horizon geometry for hairy black holes and we solve the geodesics equations, we verify the interesting anomalous changing of the angular momenta shown in the following figures: 7 (b) and 11 (b). Finally, in 7 we present the discussion and future directions.

2 Geodesics of Schwarzschild black hole

The importance of the present section is to show the geodesic of the Schwarzschild black hole and compare it with the hairy case, in special we compare the horizon radius which for Schwarzschild (grey region) and the hairy solution are respectively, \( r_h = \frac{2GM}{c^2} \) and \( \sqrt{\Omega} \). It is interesting that the Schwarzschild horizon radius is bigger than the radius of the hairy solution, and it means that the scalar field (hair) allows the existence of more compact objects. The geodesics for hairy black holes are studied in section 2. First, we show the null (see figure 1) and time-like (see figure 2) geodesics for Schwarzschild, where we have used \( M = 10^6 M_\odot \), \( \dot{r}^2 = 5.82708 \times 10^{-13} \text{yr}^2 \) and \( r_h \approx 0.019708 \text{ AU} \), see the appendix (Appendix A.1) for details. The numerical construction of geodesics is extensively known in the literature [11–13, 42].

![Diagram](image1)

Fig. 1: (Time-like)
(a): \( E = E_2 = -0.03 \), initial conditions: \( r_0 = 0.5106936776245934 \text{ AU} \) and \( \dot{r}_0 = 0 \). Where, \( r_0 = r_a \) indicates the apoapsis, while \( r_p = 0.12158224399382238 \text{ AU} \), the periapsis. According to (A.6) the angular deviation of the apoapsis per orbital period is \( \Delta \phi = 61.1^\circ \), you can verify that in the figure (a).

(b): \( E = E_3 = 0.2 \), initial conditions: \( r_0 \to \infty \) and \( \dot{r}_0 = \sqrt{Ec^2} \). Where, \( r_2 = 0.04613484978865398 \text{ AU} \) indicates the radius of the minimum distance from the center of the black hole to the orbit.
We consider the following Einstein-Hilbert theory modified form.

\begin{align}
G_{\mu\nu} &= \kappa T_{\mu\nu}, \\
T_{\mu\nu} &= \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[ \frac{1}{2} (\partial \phi)^2 + V(\phi) \right]
\end{align}

where the Einstein tensor and the energy-momentum tensor for the scalar field are respectively

\[ G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \]
\[ T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[ \frac{1}{2} (\partial \phi)^2 + V(\phi) \right] \]

Following to [2, 16, 25, 43], we consider the exotic potential \( V(\phi) \), it presents a non-trivially self-interaction, it was first obtained and presented in [25].

\[ V(\phi) = \frac{\alpha}{x^2} \left\{ \frac{v-1}{v+2} \sinh [l_v(v+1)\phi] - \frac{v+1}{v-2} \sinh [l_v(v-1)\phi] + 4 \left( \frac{v^2-1}{v^2-4} \right) \sinh (l_v \phi) \right\} \]

where \( l_v \equiv \left( \frac{2\pi}{v-1} \right)^{1/2} \). This theory has two novels parameters, \( \alpha \), which has an important role in the existence of the horizon and \( v \), which can calibrate the scalar field \( \phi \).

Considering the following ansatz for conformal metric

\[ ds^2 = \Omega(x) \left[ -c^2 f(x) dt^2 + \frac{\eta^2 dx^2}{f(x)} + d\theta^2 + \sin^2 \theta d\phi^2 \right] \]

we can integrate the equations of motion for the metric and scalar field, such that we obtain the family of hairy solutions [21, 22, 24, 44].

\[ \phi(x) = l_v^{-1} \ln x \]

where the conformal factor \( \Omega(x) \) and the metric function \( f(x) \) are given by

\[ \Omega(x) = \frac{v^2 x^{v-1}}{\eta^2 (x^v - 1)^2} \]
\[ f(x) = \alpha \left[ \frac{1}{v^2 - 4} - \frac{x^2}{v^2} \left( 1 + \frac{x^v - v^x}{v^2 - v^2} \right) \right] + \frac{x}{\Omega(x)} \]

Actually, there are two branches of the spacetime in which the physical quantities are well defined. And the quantity \( \eta \) is a positive definite constant integration that is related to the mass of the black hole. The dimension of the parameter and constant integration are \( \dim \eta = \text{length}^{-1} \), \( \dim \alpha = \dim \eta^2 = \text{length}^{-2} \), and \( v \) is a dimensionless parameter.

1 Along the paper, we mostly use the unit system where the constants have the following values:

\[ G_N \approx 39.409 \text{AU}^3 M_\odot^{-1} \text{yr}^{-2} , \]
\[ c \approx 6.324 \times 10^4 \text{AU} \text{yr}^{-1} , \]
\[ \kappa \approx 0.192 \times 10^{-17} \text{AU}^{-1} M_\odot^{-1} \text{yr}^2 \]

where \( M_\odot \) represents the solar mass, \( \text{AU} \) is for the Astronomical Unit and \( \text{yr} \) stands for a year.
The principal characteristic of each branch are

- **Negative Branch:** The coordinate of the black hole horizon \( x_0 \) is less than 1, then the range of the coordinate \( x \) is given by \( x_0 < x < 1 \). The boundary is located at \( x = 1 \) and the singularity at \( x = 0 \). In this case the scalar field is negative definite \( \phi < 0 \).

- **Positive Branch:** The coordinate of the black hole horizon is \( x_0 \) greater than 1, then the range of the coordinate \( x \) is given by \( 1 < x < x_0 \). The boundary is located at \( x = 1 \) and the singularity at \( x = \infty \). Here the scalar field becomes positive definite \( \phi > 0 \).

The radial coordinate system is more intuitive than conformal metric (5). The equation \( r^2 = \Omega(x) \), which relates both coordinate systems, cannot be solved exactly but is easy to get the asymptotic coordinate transformation [45]

\[
x = 1 \pm \frac{1}{\eta} \frac{\sqrt{v^2 - 1}}{24\eta^3 r^3} \left[ 1 \mp \frac{2}{\eta r} \frac{9(v^2 - 9)}{80\eta^2 r^2} \right] + \mathcal{O}(r^{-6})
\]  

(9)

In [45–50] you can find a quasilocal formalism used to find the mass (energy) of the gravitational system, and for asymptotically AdS spacetime in the presence of the scalar field [47]. The scalar field is a secondary hair, therefore, there is not a constant of integration associated to it, so, we have a unique constant integration \( \eta \) which is just related with the mass. The ADM mass can be read-off from \( g_{tt} \) in canonical coordinates, in [35] we will construct a quasilocal stress tensor (asymptotically flat) in order to get \( M \), see [20, 51]

\[
\frac{1}{c^2} g_{tt} = 1 - \frac{2GM}{c^2 r} + \mathcal{O}(r^{-3})
\]

(10)

where the masses for negative branch and positive branch are respectively

\[
M = \frac{c^2}{2GN} \left( \frac{\alpha + 3\eta^2}{3\eta^3} \right), \quad \phi \leq 0
\]

(11)

\[
M = -\frac{c^2}{2GN} \left( \frac{\alpha + 3\eta^2}{3\eta^3} \right), \quad \phi \geq 0
\]

(12)

3.1 Evading the no hair theorem

\[ \frac{\kappa V(\phi)}{\alpha} \]

\[ \sqrt{\kappa \phi} \]

\[ -g_{tt}/c^2 \]

\[ \sqrt{\Omega} \]

\[ 1 < r < 2 \]

\[ \nu \to 2 \]

\[ 2 < r \]

\[ r = \sqrt{\Omega} - 1 \]

\[ 0 \]

\[ 5 \]

\[ 10 \]

\[ 15 \]

\[ 20 \]

\( c_2 = 0.04 \)

\( c_2 = 1 \)

\( c_2 = 2 \)

\( c_2 = 3 \)

\( c_2 = 4 \)

Fig. 3: Negative branch. 3a: \( \frac{\kappa V(\phi)}{\alpha} \) vs \( \sqrt{\kappa \phi} \). 3b: \( -g_{tt}/c^2 \) vs \( \sqrt{\Omega} \). We consider the hairy parameter \( \nu = 1.52 \), \( \alpha = 1\, AU^{-2} \) and the range of masses for the black holes \( 0.04\, AU \leq \frac{GM}{c^2} \leq 4.00\, AU \).

The no hair theorem can be evaded if we have a potential with a global maximum at the boundary and a minimum at the horizon. That condition is ensured by \( d^2 V/d\phi^2 \leq 0 \). Considering the following quantities for the spacetime \( 0 \leq x \leq 1 \) which is named as negative branch \( \phi \leq 0 \):

- The potential \( V(\phi) \) depends on the parameters \( \nu \) and \( \alpha \). From figure (3), we check that \( d^2 V/d\phi^2 \leq 0 \Leftrightarrow \nu \geq 1, \ \alpha > 0 \), and the scalar potential has a global minimum at the horizon \( V(\phi_h) \).
From figure (3), the horizon existence, \( -\frac{\kappa}{\Omega} = 0 \), is ensured if \( \alpha > 0 \), and there is an additional condition for the mass:

- For: \( 1 \leq \nu < 2 \) there is no restriction on the mass of the black hole.
- For: \( 2 < \nu \) in [2] they showed an interesting lower bound for the mass of hairy black holes which are asymptotically AdS, and here we present our result for asymptotically flat case. Clearly if the scalar field increase (that is dominated by hairy parameter \( \nu \)) the black hole mass has a minimum value which can hold up the horizon

\[
M > M_{cri} \equiv \frac{c^2}{2G_N} \left( \frac{\nu - 2}{\alpha} \right)^{1/2}
\]  

The existence of minimal mass giving in (13) for the negative branch \( (\phi < 0) \), with \( \nu > 2 \), can be interpreted in a similar form to Kerr-Black holes, in which the horizon existence is ensured by the inequality between the angular momentum density and mass of the black hole. In the hairy case, the horizon is ensured by the critical mass which is a function of the hairy parameters \( \nu \) and \( \alpha \). So, if the mass \( M \) is not enough, the scalar field implodes and the horizon disappears. The positive branch has the same interpretation, but in that case the minimal mass condition is given for the entire range of values of parameter \( \nu \) \( (1 \leq \nu \leq \infty) \), which describes the back reaction of the scalar field.

4 Time-like geodesics

In [35] they will show the following equations for the time-like orbits on the equatorial plane \( \theta = \pi/2 \), for hairy black hole solutions described in section (2). The first order orbital equation is given by

\[
\bar{\gamma}^2 - 1 = \left( \frac{\eta \Omega(x)}{c} \right)^2 \dot{x}^2 + U_{eff}(x),
\]

\[
U_{eff}(x) = \Omega(x)f(x) \left( 1 + \frac{\bar{\gamma}^2 c^2}{\Omega(x)} \right) - 1
\]

In order to get the second order equation we consider the following relations

\[
r = \sqrt{\Omega(x)} = r(x), \quad r(\phi) = r(x) \Rightarrow x(\phi), \quad \phi(\tau)
\]

\[\text{Here the Killing vectors are}
\]

\[
\frac{dt}{\xi} = \frac{\ddot{\phi}}{\Omega(x)f(x)}, \quad \frac{d\phi}{\xi} = \frac{\ddot{\phi} c^2}{\Omega(x)}
\]
Taking the derivative of (16) with respect to $\phi$ we get a second order orbital equation which can be easily numerically solved

\[
\frac{d^2x}{d\phi^2} + H(x, J, \eta) = 0,
\]

\[
H(x, J, \eta) = \frac{1}{2(\eta J + \eta^2)}[(\Omega f)' + (\frac{\Omega}{\eta} f)']
\]

(19)

remembering that the radial coordinate is related to $x$-coordinate in the following exact form

\[
r(x) = \sqrt{\frac{\nu^2 x^2 - 1}{\eta^2(x^2 - 1)^2}},
\]

we can plot $r(x)$ vs $\phi(x)$. The effective potential describes the following regions for negative and positive branch:

Region-I: $\bar{E}^2 - 1 > U(r_{max})$. Region-II: $U(r_{max}) > \bar{E}^2 - 1 > 0$ and Region-III: $0 > \bar{E}^2 - 1 > U_{\min}$.

### Table 1: Hairy black hole and time-like geodesic parameters

| Negative Branch | Positive Branch |
|-----------------|-----------------|
| $x < 1, \alpha = 1AU^{-2}, \nu = 1.52$ | $x > 1, \alpha = -40AU^{-2}, \nu = 1.76$ |
| $\eta \approx 12.52655373\text{AU}^{-1}$, $G_N M/c^2 = 0.04AU$ | $\eta \approx 2.52719443\text{AU}^{-1}$, $G_N M/c^2 = 0.04AU$ |
| $J = 2.6 \times 10^{-6}\text{yr}$, $\bar{J} = 2.1072 \times 10^{-6}\text{yr}$ | $J = 1 \times 10^{-7}\text{yr}$, $\bar{J} = 2.5280 \times 10^{-7}\text{yr}$ |
| $U_{\max} \approx 0.083747687$, $U_{\min} \approx -0.0668590532$ | $U_{\max} \approx 0.966450101$, $U_{\min} \approx -0.345344897$ |

Table 1: Here we consider $\bar{E} = E/mc^2$, and $\bar{J} = J/mc^2$. In figure (5) we can see the extremes of the effective potential $U_{\text{eff}}$.

\[
\frac{G_N M}{c^2} = 0.04\ [AU], \alpha = 1[AU]^{-2}, \eta = 12.527\ [AU]^{-1}
\]

\[
G_N M/c^2 = 0.04\ [AU], \alpha = -40[AU]^{-2}, \eta = 3.252719443\ [AU]^{-1}
\]

Fig. 5: We have the positive-branch (5b) and negative-branch (5a) of the effective potential $U_{\text{eff}}$. The parameters are described in (1). Apparently there is nothing new compared to the Schwarzschild solution, but we are going to show an interesting new geodesic behavior due to the scalar field in the region-I.

\[
x(\phi) = 1 - \frac{1}{\eta r(\phi)}, \quad \alpha = 3\eta^2 r_0 - 3\eta^2, \quad r(\phi) = \frac{1}{u(\phi)}
\]

(18)

\[\text{It is easy to show that with the following changes we can get the orbital equation for Schwarzschild } v = 1, \text{ see (A.7)}\]
Fig. 6: **Negative Branch (time-like)** In order to compare we plot the Schwarzschild black hole horizon \( r_h = \frac{2G\gamma M}{c^2} = 0.08 \text{ AU} \) (outer grey circle) and the hairy black hole horizon \( \sqrt{\Omega(x_h)} = 0.009 \text{ AU} \) (inner dark circle). The other constants are fixed to \( \overline{J} = 2.6 \times 10^{-6} \text{ yr} \), \( \alpha = 1 \), \( \nu = 1.52 \), \( E = \epsilon^2 - 1 \). In both cases the hairy horizon is really small, because the scalar field allows the existence of more compact objects.

Fig. 7: **Negative Branch (time-like)** In both figures we have particles that fall into the black hole. The hairy black hole is represented by the black disk of radius \( \sqrt{\Omega(x_h)} = 0.009 \text{ AU} \), while the Schwarzschild black hole, by the grey disk of radius \( r_h = \frac{2G\gamma M}{c^2} = 0.08 \text{ AU} \). We observe that the orbits enter tangentially to the hypothetical Schwarzschild black hole horizon and perpendicular to the hairy black hole horizon. The constants and parameters are fixed to \( \overline{J} = 2.6 \times 10^{-6} \text{ yr} \), \( \alpha = 1 \), \( \nu = 1.52 \), \( \eta = 12.527 \text{ AU}^{-1} \), \( E = \epsilon^2 - 1 \). In both figures, we found an amazing new phenomenon, in the region \( \sqrt{\Omega} < r < r_h \) the geodesic change suddenly losing its angular momentum and they go into a radial free fall to the black hole. Clearly, the dense hair region \( \sqrt{\Omega} < r < r_h \) is a place where the scalar field has a big back reaction, and it causes this phenomenon, that does not exist in Schwarzschild model.
Fig. 8: **Positive Branch (time-like)** In both figures the hairy black hole is represented by the black disk of radius \( \sqrt{\Omega(x_h)} = 0.006 \text{ AU} \), while the Schwarzschild black hole, by the grey disk of radius \( r_h = 2G_NM/c^2 = 0.08 \text{ AU} \). The constants and parameters are fixed to \( J = 7 \times 10^{-7} \text{ yr} \), \( \alpha = -40, \ MG_N/c^2 = 0.04, \ \nu = 1.76, \ E = \delta^2 - 1 \). In this branch exist a parabolic orbit \( E = 0 \) and the orbits are qualitative similar to the Kepler one.

Fig. 9: **Positive Branch (time-like)** In both figures we have particles that fall into the black hole. The hairy black hole is represented by the black disk of radius \( \sqrt{\Omega(x_h)} = 0.006 \text{ AU} \), while the Schwarzschild black hole, by the grey disk of radius \( r_h = 2G_NM/c^2 = 0.08 \text{ AU} \). Both figures are particles that fall to the black hole, we observe that the orbit enters quickly to the hypothetical Schwarzschild black hole horizon and almost tangentially to the hairy black hole horizon. The other constants are fixed to \( J = 7 \times 10^{-7} \text{ yr} \), \( \alpha = -40, \ MG_N/c^2 = 0.04, \ \nu = 1.76, \ E = \delta^2 - 1 \).
5 Hairy null geodesics

The parametric equation for null geodesics\(^4\) will be shown in [35], and it can be easily calculated considering \(ds^2 = 0\). We replace the proper time \(\tau\) by an affine parameter \(\lambda\) and scale it as \(\lambda \rightarrow \lambda / \dot{\lambda}\)

\[
\eta^2 \Omega^2 \left( \frac{dx}{d\lambda} \right)^2 + \mathcal{V}(x) = \frac{c^2}{\beta^2}, \quad \mathcal{V}(x) = f(x)c^2
\]

where the effective potential \(\mathcal{V}(x, \alpha, M, \nu)\) is given by

\[
\mathcal{V}(x, \alpha, M, \nu) = c^2 f(x) = \alpha x^2 \left( \frac{1}{\nu^2 - 4} - \frac{x^2}{2} + \frac{x^4}{4} \right) + \frac{x^2}{\Omega(x, M)}
\]

The null geodesics are completely determined by the impact parameter \(b^2 \equiv (c \dot{\mathcal{V}})^2 / \beta^2\). In order to integrate numerically the geodesic equation we need consider the first order orbital equation for \(x(\phi)\)

\[
\frac{1}{\eta^2} \left[ \frac{1}{\beta^2} - f(x) \right] \frac{d}{d\phi} \frac{dx}{d\phi} = 0
\]

Near to boundary we can integrate (23) and using \(x = 1 - 1/\sqrt{\nu} \tau\) we get \(\tau_0 = b\), which gives us the intuitive definition of the impact parameter. Taking the derivative with respect to \(\phi\) in (23) we can get the second order equation

\[
\frac{d^2 x}{d\phi^2} + \frac{1}{2\eta^2} \frac{df(x)}{dx} = 0
\]

The extreme points of the potential where the location is \(x_0\)

\[
\mathcal{V}(x_0) = \frac{c^2}{b_0^2}, \quad \frac{d\mathcal{V}(x_0)}{dx} = 0 \Rightarrow x_0 = \frac{\alpha + \eta^2(2 - \nu)}{\alpha + \eta^2(2 + \nu)}
\]

and the radius of the minimal unstable circular orbit is \(r_0 = \sqrt{\mathcal{V}(x_0)}\) with its respective critical impact parameter \(b_0\)

\[
r_0 = \pm \frac{1}{2\eta^2} [\alpha + \eta^2(2 - \nu)]^{\nu-1} \sqrt{[\alpha + \eta^2(2 + \nu)]^{\nu+1}}
\]

\[
x_0 < 1 (+), \quad x_0 > 1 (-)
\]

\[
\mathcal{V}(x_0) = \frac{c^2}{b_0^2} \Rightarrow b_0 = \frac{1}{\sqrt{f(x_0)}}
\]

Then

- The light can get down to black hole if \(\Rightarrow \frac{c^2}{\beta^2} = \mathcal{V}(x_0)\)
- The critical impact parameter is defined like a place in which the massless particles are trapped in an unstable circular orbit(ISCO) \(\Rightarrow \mathcal{V}(x_0) = \frac{c^2}{b_0^2}\)
- In the no-hair limit \(\nu = 1\) the ISCO radius and the critical impact parameter are \(r_0 = 3MG/\nu c^2 = 0.12\text{AU}, \quad b_0 = 3\sqrt{3}\text{MG/c}^2 = 0.207846\text{AU}\).

| Hairy black hole and null geodesic parameters |
|-----------------------------------------------|
| **Negative Branch**                            |
| \(x < 1, \alpha = 1\text{AU}^{-2}, \nu = 1.52\) |
| \(\eta = 12.52\text{MG}^{-1}\text{AU}^{-2}, \quad G_0M/c^2 = 0.04\text{AU}\) |
| \(x_0 = 0.271628, \quad r_0 = 0.1003\text{AU}\) |
| \(b_0 = 0.192949\text{AU}\) |
| \(\mathcal{V}(x_0) = 1.07423 \times 10^{11}\text{yr}^{-2}\) |
| **Positive Branch**                            |
| \(x > 1, \alpha = -40\text{AU}^{-2}, \nu = 1.76\) |
| \(\eta = 3.252719443\text{AU}^{-1}, \quad G_0M/c^2 = 0.04\text{AU}\) |
| \(x_0 = 18.5943, \quad r_0 = 0.0096397\text{AU}\) |
| \(b_0 = 0.0327298\text{AU}\) |
| \(\mathcal{V}(x_0) = 3.73333 \times 10^{25}\text{yr}^{-2}\) |

Table 2

- \(G_0 = \frac{GMC}{c^4} = 0.040\text{AU}/c^2, \alpha = 1\text{AU}^{-2}, \eta = 12.52\text{MG}^{-1}\text{AU}^{-2}\)

\[d\phi = \frac{1}{b_0 c \Omega(x)} d\lambda, \quad \frac{dx}{d\lambda} = \frac{c}{\Omega(x)} \]
Fig. 10: We have the positive-branch (10b) and negative-branch (10a) of the effective potential $V_{eff}$. The parameters are described in table (2).

Fig. 11: Negative Branch (null). We are assuming the Schwarzschild black hole horizon is $r_h = \frac{2GM}{c^2}$ (grey circle) and hairy horizon $\sqrt{\Omega(x_0)}$ (black circle). The other constants are fixed to $\alpha = 1$, $MG_N/c^2 = 0.04$, $\nu = 1.52$.

11a: According to table (1) the null geodesic is defined in the region-I, see fig(a) (10), where $c^2 \frac{\rho^2}{\Delta} = 4.10496 \cdot 10^{11}$ yr$^{-2} > \mathcal{V}(x_0)$. A null-particle fall to the black hole, we observe that the orbit is perpendicular to the hairy black hole horizon.

11b: According to table (1) the null geodesic is defined in the region-II, see fig(a) (10), where $c^2 \frac{\rho^2}{\Delta} = 0.45611 \cdot 10^{11}$ yr$^{-2} < \mathcal{V}(x_0)$. A null-particle is deflected by the hairy black hole
6 Hairy near horizon geodesics

The goal of the present section is to explain the anomalous changing angular momentum of infalling time-like or space-like geodesics shown in figures 7 and 11. In those figures the geodesics apparently go inside the black hole orthogonal to the horizon surface, here we verify that this is actually true.

Considering the solution of the hairy black hole given in section (2), the localization of the horizon \( x_h \) is such that \( f(x_h) = 0 \) and the near horizon geometry can be constructed under the following change \( x = x_h + \varepsilon \), gives us

\[
f|_{x_h+\varepsilon} \approx (x-x_h)f'(x_h),
\]

\[
\frac{f}{\Omega}|_{x_h+\varepsilon} \approx \frac{(x-x_h)f'(x_h)}{\Omega(x_h)}
\]

replacing in the hairy metric (5)

\[
ds^2 = -(x-x_h)\Omega(x_h)f'(x_h)c^2 \, dt^2 + \frac{\eta^2 dx^2}{(x-x_h)f'(x_h)} + \Omega(x_h)(d\theta^2 + \sin^2 \theta d\phi^2)
\]

(29)

taking the usual transformation to radial part

\[
d\rho^2 = \frac{\eta^2 dx^2}{(x-x_h)f'(x_h)} \Rightarrow \rho^2 = 4\eta^2 \frac{\Omega(x_h)}{f'(x_h)}(x-x_h)
\]

(30)

and for the temporal coordinate \( t_R = \int \frac{f'(x_h)}{2\eta} \, dt \), we get the Rindler geometry

\[
ds^2 = -\rho^2 \, dt^2 + d\rho^2 + \frac{\Omega(x_h)}{\rho^2}(d\theta^2 + \sin^2 \theta d\phi^2)
\]

(31)

6.1 Time-like near horizon geodesics

The near horizon geometry of (5) is described by the metric

\[
ds^2 = \Omega(x_h)\left(-F(x)c^2 \, dt^2 + \frac{\eta^2 dx^2}{F(x)} + d\theta^2 + \sin \theta d\phi^2\right)
\]

\[
F(x) = (x-x_h)f'(x_h)
\]

(32)

The Killing equations give us the following conserved quantities

\[
i = \frac{\partial}{\partial \theta}, \quad \phi = \frac{\partial}{\partial \phi}
\]

\[
\tag{33}
\]

And the parametric equation is

\[
\hat{\rho}^2 - 1 = \left(\frac{\eta \Omega(x_h)}{c}\right)^2 \left(\frac{dx}{d\tau}\right)^2 + U_{\text{eff}}(x)
\]

\[
U_{\text{eff}}(x) = \Omega(x_h)F(x) \left(1 + \frac{\hat{\rho}^2 c^2}{\Omega(x_h)}\right) - 1
\]

(34)

\[\]

Fig. 12: Positive Branch (null) We are assuming the Schwarzschild black hole horizon is \( r_h = \frac{2GM}{c^2} \) (grey disk contour) and hairy horizon \( \sqrt{\Omega(x_h)} \) (black disk contour). The other constants are fixed to \( \alpha = -40, \frac{GM}{c^2} = 0.04, \nu = 1.76 \).

12a: According to table (1) the null geodesic is defined in region-I, see fig(b) (10), where \( \frac{c^2}{2\eta} = 4.56107 \cdot 10^{12} \text{ yr}^{-2} > \sqrt{\Omega(x_0)} \). A null-particle fall to the black hole, we observe that the orbit is not perpendicular to the hairy black hole horizon.

12b: According to table (1) the null geodesic is defined in region-II, see fig(b) (10), where \( \frac{c^2}{2\eta} = 2.56560 \cdot 10^{12} \text{ yr}^{-2} < \sqrt{\Omega(x_0)} \). A null-particle is highly deflected.
Considering the following chain-rule: \( \dot{x} = \frac{d}{d \varphi} \varphi \), we can get the orbital equation with \( E \equiv \delta^2 - 1 \)

\[
E = \eta^2 \varphi^2 c^2 \left( \frac{dx}{d \varphi} \right)^2 + U_{eff}(x)
\]

(35)

taking the derivative with respect to \( \varphi \)

\[
\frac{d^2 x}{d \varphi^2} + H(x_h, \nu, \vartheta, \eta) = 0,
\]

\[
H(x_h, \nu, \vartheta, \eta) = \frac{f'(x_h)}{2(\eta \vartheta c)^2} \left[ \Omega(x_h) + (\vartheta c)^2 \right]
\]

(36)

the solution is

\[
x(\varphi) = -\frac{H(x_h, \nu, \vartheta, \eta)}{2} \varphi^2 + c_1 \varphi + c_2
\]

(37)

The initial conditions are, see the example of Schwarzschild case in the appendix Appendix B,

\[
x(0) = x_h \Rightarrow c_2 = x_h,
\]

\[
\left( \frac{dx}{d \varphi} \right)_{x_h} = \frac{\delta}{\eta \vartheta c} = c_1
\]

(38)

the near-horizon solution for time-like geodesic is

\[
x(\varphi) = -\frac{H(x_h, \nu, \vartheta, \eta)}{2} \varphi^2 + \frac{\delta}{\eta \vartheta c} \varphi + x_h
\]

(39)

6.2 Null-like near horizon geodesics

Considering the near-horizon metric (32) and the change (33) we obtain the following expressions for conserved quantities

\[
\frac{dt}{d \xi} = \frac{1}{b c \Omega(x_h) F(x)} \quad \frac{d \varphi}{d \xi} = \frac{c}{\Omega(x_h)}
\]

(40)

the first order orbital equation is

\[
\left( \frac{dx}{d \varphi} \right)^2 = \frac{1}{\eta^2} \left[ \frac{1}{b^2} - F(x) \right]
\]

(41)

the second order and its solution is

\[
\frac{d^2 x}{d \varphi^2} + \frac{f'(x_h)}{2 \eta^2} = 0 \Rightarrow x(\varphi) = -\frac{f'(x_h)}{4 \eta^2} \varphi^2 + c_1 \varphi + c_2
\]

(42)

The initial conditions are

\[
x(0) = x_h \Rightarrow c_2 = x_h,
\]

\[
\left( \frac{dx}{d \varphi} \right)_{x_h} = \frac{1}{\eta b} = c_1
\]

(43)

The near-horizon solution for time-like geodesic is, here we would like to highlight that \( x(\varphi) = x(\varphi, \nu) \) depend on the hairy parameter,

\[
x(\varphi) = -\frac{f'(x_h)}{4 \eta^2} \varphi^2 + \frac{\nu \varphi}{\eta b} + x_h, \quad r(\varphi) = \sqrt{\Omega(x(\varphi))}
\]

(44)

the above solution is very similar to the Schwarzschild case (B.16), however the constants that multiply \( \varphi^2 \) and \( \varphi \) depend on the hairy parameter \( \nu \), which clearly changes the usual Schwarzschild behaviour.

---

**Fig. 13: Hairy near horizon geodesics.** Here we plot the near horizon geodesics of the figure 7 (b) and figure 11 (b), which are null and time-like respectively. Here we consider the angular momenta per unit mass \( \vartheta = 2.6 \times 10^{-6} \) and the mass of the black hole \( G N M / c^2 = 0.04 \). For time-like geodesic we consider \( E = 0.1 \) and for null case \( b = 0.1 \), \( E = 0.1 \). The black line define the horizon of the black hole of radius \( r_h = 0.0192 \, AU \). The Schwarzschild black hole has geodesics which fall inside of the black hole with some angle with respect to the tangent to horizon, while in the hairy case we showed that geodesics cross the horizon orthogonally.
7 Discussion

In the first part of this article, we have shown the Schwarzschild geodesics in order to compare with the hairy case. Our first results are the existence of the critical mass for negative branch \(0 < x < 1\) with \(2 < \nu\) and positive branch \(1 < x < \infty\) with \(\nu > 1\)

\[
M_{\text{cri}} \equiv \frac{c^2}{2GN} \left(\frac{\nu - 2}{\alpha}\right)^{1/2},
\]

\[
M_{\text{cri}} \equiv \frac{c^2(\nu - 1)(\nu + 2)}{6GN\sqrt{-\alpha(\nu + 2)}}
\]

(45)

there is a black hole horizon if \(M > M_{\text{cri}}\), this interesting result can have a major impact on the modeling of an accretion disk in future work. The most important result shown in section 5 is the loss of angular momentum as the particles enter the dense hair region (grey disk), see 7 and 11. Both types of geodesics go inside the hairy black hole orthogonal to the horizon surface. To clarify this point, in section 6 we construct the near horizon geodesics for black holes described in figures 7 and 11, which results are shown in figure 13. There are other hairy configurations, such as 9 and 12, that do not present the loss of angular momentum, similar to the Schwarzschild black hole. This phenomenon could allow the identification, from the observational point of view, if a black hole is hairy or not.

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Appendix A: Schwarzschild solution

\[ ds^2 = -c^2N(r)dr^2 + \frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \] (A.1)

\[ N(r) = 1 - \frac{r_h}{r} \] (A.2)

Here, \( r_h \) is the horizon radius. The mass of the black hole is given by

\[ M = \frac{c^2r_h}{2G_N} \] (A.3)

Appendix A.1: Time-like geodesic

Here \( ds^2 = -c^2d\tau^2 \), where \( \tau \) is proper time. In addition, setting the geodesic at the equatorial plane (\( \theta = \pi/2 \)) due to the rotational isometry we obtain

\[ -c^2 = -N(r)c^2\dot{t}^2 + \frac{\dot{r}^2}{N(r)} + r^2\dot{\phi}^2 \] (A.4)

The conserved quantities along the isometry orbits generated by the Killing vectors \( \xi_t = \partial_t \) and \( \xi_\phi = \partial_\phi \) are given by \( \xi \) (dimensionless) and \( J \) (yr): \( dt/d\tau = \xi / N(r) \) and \( d\phi/d\tau = \sqrt{J^2/r^2} \). Here \( \xi_t = (c,0,0,0) \) and \( \xi_\phi = (0,0,0,1) \), replacing in (A.4) we can get the first order orbital equation. It describes the radial motion of a test body with energy \( E = \xi^2 - 1 \) in the effective potential, see figure (14)

\[ E = \frac{\dot{r}^2}{c^2} + U_{\text{eff}}(r), \quad U_{\text{eff}}(r) = N(r) \left( 1 + \frac{\dot{\phi}^2r^2}{c^2} \right) - 1 \] (A.5)

The polar equation can be constructed considering the chain-rule \( \frac{dr}{d\phi} = \frac{dr}{d\xi} \frac{d\xi}{d\phi} \), from that we have

\[ \left( \frac{dr}{d\phi} \right)^2 = \frac{r^4}{c^2} \left( E - U_{\text{eff}} \right) \]

\[ \Delta \phi = 2 \int_{r_a}^{r_p} \left( \frac{r^2}{c^2} \right)^{\frac{1}{2}} \sqrt{E - U_{\text{eff}}(r)} \, dr - 2\pi \] (A.6)

the left-hand equation describe the orbit in polar coordinates, and the right-hand equation is the precession of the orbits closed for each revolution, where \( r_a, r_p \) can be solved from \( \frac{dr}{d\phi} \big|_{r_a,r_p} = 0 \) or \( E = U_{\text{eff}}(r_a, r_p) \). The typical second order orbital equation can be get considering the following change of variables \( u(\phi) = 1/r(\phi) \) and \( du/d\phi = -u^2 dr/d\phi \)

\[ \frac{d^2}{d\phi^2} + u = -\frac{3r_h}{2u^2} = \frac{1}{\lambda}, \quad \lambda = \frac{2c^2J^2}{r_h} = \frac{2J^2}{m^2c^2r_h} \] (A.7)

We use the Runge-Kutta method in order to solve it

![Fig. 14: a] Region I, Above \( E = U_{\text{eff}}^{\text{max}} \). Region II: Between \( E = 0 \) and \( E = U_{\text{eff}}^{\text{min}} \). Region III: Between \( E = U_{\text{eff}}^{\text{min}} \) and \( E = 0 \). To make this plot we have considered \( J^2 = \frac{24M^2G_N^2}{c^4} \), with \( M = 10^6M_\odot \).

![Fig. 14: b] \( E_1 \) corresponds to the minimum of the effective potential. \( E_2 \) belongs to Region III. \( E_3 \) belongs to Region II. \( E_4 \) corresponds to the maximum of the effective potential. \( E_5 \) belongs to the region I.
Orbits

Here we present the plots corresponding to each energy level presented in figure 14.

Fig. 15: (Time-like)
(a) $E_1 = U_{eff}^{min} = U_{eff}(r^+_s)$, initial conditions: $r_0 = r^+_s$ and $\dot{r}_0 = 0$. $E_4 = U_{eff}^{max} = U_{eff}(r^-_s)$, initial conditions: $r_0 = r^-_s$ and $\dot{r}_0 = 0$. (b) $E = E_5 = 0.3$, initial conditions: $r_0 \to \infty$ and $\dot{r}_0 = \sqrt{E}c^2$.

Fig. 16: (Time-like)
$E = E_3 = 0.2$, initial conditions: $r_0 \to \infty$ and $\dot{r}_0 = \sqrt{E}c^2$. Where, $r_2 = 0.04613484978865398\text{AU}$ indicates the radius of the minimum distance from the center of the black hole to the orbit.
Appendix A.2: Null geodesic

Here we consider an affine parameter $\xi$ and we scale it as $\xi \rightarrow \xi / \bar{\xi}$.

$$\frac{dt}{d\xi} = \frac{\bar{\xi}}{\mathcal{J} N(r)}, \quad \frac{d\varphi}{d\xi} = \frac{c^2}{r^2} \tag{A.8}$$

replacing in (A.4) at $ds^2 = 0$ we get

$$\left(\frac{dr}{d\xi}\right)^2 + \mathcal{V}_{\text{eff}}(r) = \frac{c^2}{r^2} \tag{A.9}$$

considering the chain-rule $\frac{d}{d\xi} = \frac{d}{d\varphi} \frac{d\varphi}{d\xi}$ we have the polar equation and the deflection of a light-ray which comes from infinity, pass near to black hole $R_0$ and return to infinity

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{r^4}{b^2 c^2} - r^2 N, \quad \Delta \varphi = 2 \int_{R_0}^{\infty} \left(\frac{1}{b^2 c^2} - \frac{N(r) c^4}{r^2}\right)^{-1/2} \tag{A.10}$$

at boundary ($r \rightarrow \infty$) that polar equation can be solved $r \varphi = bc$, from that, $b$ is interpreted as an impact parameter. The right-hand side equation (A.10) is very useful in order to get the angle of deflection of light rays coming from infinity and passing close to the black hole $R_0$, where $R_0$ comes from $\frac{dr}{d\varphi} \big|_{R_0} = 0$ or $\frac{d\varphi}{d\varphi} \big|_{R_0} = 0$; in the Schwarzschild case that equation is $R_0^3 - b^2 c^2 R_0 - b^2 c^2 r_0 = 0$.

In the literature we usually have the following change of radial coordinate $r(\varphi) = 1/u(\varphi)$, in the left-hand side equation of (A.10), in order to get the second order equation. Finally, we can transform it in two differential equations in order to use the Runge-Kutta method.
and \((B.1)\) is such that \(r \to \infty\). Fig. 18: (Null-like).

18a: \(E = E_1 = 10^{21} \text{AU}^2 \text{yr}^{-4}\), initial conditions: \(r_0 \to \infty\) and \(\left(\frac{dr}{d\tau}\right)_0 = \sqrt{E}\).

18b: \(E = E_2 = \gamma_{\text{eff}}^\text{max} = \gamma_{\text{eff}}(r_h)\), initial conditions: \(r_0 = r_h\) and \(\left(\frac{dr}{d\tau}\right)_0 = 0\).

18c: \(E = E_1 = 7.5 \times 10^{21} \text{AU}^2 \text{yr}^{-4}\), initial conditions: \(r_0 \to \infty\) and \(\left(\frac{dr}{d\tau}\right)_0 = \sqrt{E}\).

Appendix B: Near-horizon geometry of Schwarzschild black hole

The present section is an interesting example which can help us to understand the hairy case. The near horizon geometry of Schwarzschild black hole (A.1) is such that \(r = r_h + \epsilon\)

\[N(r) = 1 - \frac{r_h}{r} \Rightarrow N(r_0 + \epsilon) \approx (r - r_h)N'(r_h) \quad (B.1)\]

the metric is given by

\[ds^2 = -\left(1 - \frac{r_h}{r}\right)c^2dt^2 + \frac{r_hdr^2}{(r - r_h)} + r_h^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (B.2)\]

The Lorentzian signature of the metric imposes the condition \(r \geq r_h\). Choosing a new radial coordinate \(\rho^2 = 4r_h(r - r_h)\) we get the Rindler geometry

Appendix B.1: Time-like geodesics

From (B.2) the equation for time-like geodesics near to black hole is

\[-c^2 = -N(r)c^2\hat{t}^2 + \frac{r_h^2}{N(r)} + r_h^2\hat{\theta}^2, \quad N(r) = \frac{(r - r_h)}{r_h} \quad (B.3)\]
Appendix B.2: Null geodesics

We consider an affine parameter $\xi$ and scale it as $\xi \rightarrow \xi / \mathcal{J}$

$$
\frac{dt}{d\xi} = \frac{r_h \mathcal{J}}{\mathcal{J} (r-r_h)}, \quad \frac{d\phi}{d\xi} = \frac{c^2}{r_h^2}
$$

(B.11)

in (B.2) at $d\tau^2 = 0$ we get

$$
\left( \frac{dr}{d\xi} \right)^2 + \gamma_{\text{eff}}(r) = \frac{c^2}{b^2},
$$

(B.12)

considering the chain-rule $\frac{dr}{d\xi} \frac{d\xi}{d\phi} = \frac{dt}{d\phi}$ we obtain the polar equation

$$
\left( \frac{dr}{d\phi} \right)^2 = \frac{r_h^2}{b^2 c^2} - r_h (r-r_h)
$$

Taking the derivative of the orbital equation and integrating we have

$$
\frac{d^2 r}{d\phi^2} = -\frac{r_h}{2} \Rightarrow r(\phi) = -\frac{r_h \phi^2}{4} + c_1 \phi + c_2
$$

(B.14)

The initial conditions are: from (B.13) we get $(dr/d\phi)_{r_0}$, and fixing that $r(\phi = 0) = r_h$

$$
\left( \frac{dr}{d\phi} \right)_{r_0} = \frac{r_h^2}{bc} = c_1, \quad r(\phi = 0) = r_h = c_2
$$

(B.15)

we have

$$
r(\phi) = -\frac{r_h}{2} \phi^2 + \frac{r_h^2}{bc} \phi + r_h
$$

(B.16)

Fig. 19: Near horizon geodesics. Here we plot the near horizon geodesics of the figure 2 (b) and figure 15 (b), which are null and time-like respectively. Here we consider the angular momentum per unit mas $\mathcal{J} = 7.322 \times 10^{-7}$ and the mass of the black hole $M = 10^6 M_\odot$. For time-like geodesic we consider $E = 0.3$ and for null case $b^2 = 5.47329 \times 10^{-13} \text{yr}^2$, $E = 7.5 \times 10^{13} \text{AU}^2 \text{yr}^{-4}$. The black line define the horizon of the black hole of radius $r_h = 0.0192 \text{AU}$. The Schwarzschild black hole has a geodesics which fall inside of the black hole with some angle with respect to the tangent to horizon.
