A NOTE ON WEAK SOLUTIONS OF CONSERVATION LAWS AND ENERGY/ENTROPY CONSERVATION

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Abstract. A common feature of systems of conservation laws of continuum physics is that they are endowed with natural companion laws which are in such case most often related to the second law of thermodynamics. This observation easily generalizes to any symmetrizable system of conservation laws. They are endowed with nontrivial companion conservation laws, which are immediately satisfied by classical solutions. Not surprisingly, weak solutions may fail to satisfy companion laws, which are then often relaxed from equality to inequality and overtake a role of a physical admissibility condition for weak solutions.

We want to answer the question what is a critical regularity of weak solutions to a general system of conservation laws to satisfy an associated companion law as an equality. An archetypal example of such result was derived for the incompressible Euler system by Constantin et al. \cite{8} in the context of the seminal Onsager’s conjecture.

This general result can serve as a simple criterion to numerous systems of mathematical physics to prescribe the regularity of solutions needed for an appropriate companion law to be satisfied.

Keywords: energy conservation, first order hyperbolic system, Onsager’s conjecture

1. Introduction

The passing decade has been to a significant extend directed to solving the famous conjecture of Onsager saying that solutions to incompressible Euler system conserve total kinetic energy as long as they are Hölder continuous with a Hölder exponent $\alpha > 1/3$. Otherwise they may dissipate the energy.

The ideas used to prove the celebrated Nash-Kuiper theorem appeared to have wide applicability in the context of fluid mechanics, and incompressible Euler system in particular. Interestingly, the construction of weak solutions via appropriate refinement of the method of convex integration allowed to generate solutions with a regularity as exactly prescribed by Onsager that do not conserve the energy. We shall summarize in a sequel the recent achievements in this direction, however our main interest in the current paper is aimed at an analogue and generalization of the first part of the Onsager’s statement. The positive direction of this claim was fully solved by Constantin et al. already in the early nineties, cf. \cite{8}, see also \cite{7, 10, 17}. A sufficient regularity for the energy to be conserved has been established for a variety of models, including the incompressible inhomogeneous Euler system and the compressible Euler in \cite{18}, the incompressible inhomogeneous Navier-Stokes system in \cite{24}, compressible Navier-Stokes in \cite{27} and equations of magneto-hydrodynamics in \cite{6}.

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The above list gives a flavor of how broad is the class of systems for which one can specify the regularity of weak solutions which provides the energy to be conserved. This motivates us, instead of developing tools for another dozens of systems, to look at general systems of conservation laws. Apparently one can prescribe the condition for weak solutions providing that in addition to a conservation law they will satisfy a companion conservation law. To make the statement more precise, let us consider a conservation law, not necessarily hyperbolic, in a general form

$$\text{div}_X(G(U(X))) = 0 \quad \text{for} \quad X \in \mathcal{X}$$

(1)

for an unknown (vector) function $U = U(X) : \mathcal{X} \to \mathcal{O}$ and a given matrix field $G : \mathcal{O} \to \mathbb{M}^{n \times (k+1)}$. Let us assume that $\mathcal{O}$ and $\mathcal{X}$ are open sets, $\mathcal{X} \subseteq \mathbb{R}^{k+1}$ or $\mathcal{X} \subseteq \mathbb{R} \times \mathbb{T}^k$ and $\mathcal{O} \subseteq \mathbb{R}^n$, where $\mathbb{T}^k$ denotes the flat torus of dimension $k$ (imposing the periodic boundary conditions). We denote $X = (x_0, x_1, \ldots, x_k)^T$ the standard coordinates on $\mathbb{R}^{k+1}$ or $\mathbb{R} \times \mathbb{T}^k$ and we consider on $\mathcal{O}$ the coordinates $Y = (y_1, \ldots, y_n)^T$ with respect to the canonical basis. For a matrix field $M = (M_{ij})_{i=1,\ldots,n, \ j=0,\ldots,k}$, $M_{ij} : \mathbb{R}^n \to \mathbb{R}$, we denote $M_j$ the $j$-th column vector. Moreover, we use the standard definition

$$\text{div}_X M(X) = \sum_{j=0}^k \partial_{x_j} M_j(X).$$

We denote by $D_X$ (respectively $D_Y$, $D_U$) the differential $(D_X = (\partial_{x_0}, \ldots, \partial_{x_k}))$ with respect to variables $X$ (respectively $Y$, $U$).

Following the notation in [9] we shall say that a smooth function $Q : \mathcal{O} \to \mathbb{R}^{s \times (k+1)}$ is a companion of $G$ if there exists a smooth function $B : \mathcal{O} \to \mathbb{M}^{s \times n}$ such that

$$D_U Q_j(U) = B(U) D_U G_j(U) \quad \text{for all} \quad U \in \mathcal{O}, \ j \in \{0, \ldots, k\}.$$  

(2)

Observe that for any classical solution $U$ of (1), we obtain

$$\text{div}_X (Q(U(X))) = 0 \quad \text{for} \quad X \in \mathcal{X}$$

(3)

where by a classical solution we mean a Lipschitz continuous vector field $U$ satisfying (1) for almost all $X \in \mathcal{X}$. Identity (3) is called a companion law associated to $G$ (see e.g. [9]). In many applications, which we partially recall in Section 4, some relevant companion laws are conservation of energy or conservation of entropy. Before we discuss the relations between weak solutions and companion laws, let us remark that it was observed by Godunov [19] that systems of conservation laws are symmetrizable if and only if they are endowed with nontrivial companion laws.

We consider the standard definition of weak solutions to a conservation law

**Definition.** We call the function $U : \mathcal{X} \to \mathcal{O}$ a weak solution to (1) if $G(U)$ is locally integrable in $\mathcal{X}$ and the equality

$$\int_{\mathcal{X}} G(U(X)) : D_X \psi(X) \, dX = 0$$

(4)

holds for all smooth test functions $\psi : \mathcal{X} \to \mathbb{R}^n$ with a compact support in $\mathcal{X}$.

Analogously, we can define weak solutions to (3), however weak solutions of (1) may not necessarily be weak solutions also to (3). The main question we deal with in this paper reads as follows: **What are sufficient conditions for a weak solution of (1) to satisfy also (3)?**

Let us comment in more detail results related to the question of energy conservation for weak solutions of some conservation laws. Both parts of Onsager’s conjecture for the incompressible inviscid Euler system have been resolved. Due to recent results of Isett [20] and Buckmaster et al. [5] we know there exist solutions of the incompressible Euler equations of class $C([0,T]; C^{1/3}(\mathbb{T}^3))$ which do not satisfy the energy equality. These
results were preceded with a series of papers showing firstly existence of bounded (11), later continuous (12) and Hölder continuous (13) solutions with $\alpha = 1/10$. The further results aimed to increasing the Hölder exponent, see [2, 3, 4, 21].

In the context of our studies, the second part of Onsager’s conjecture is more relevant. Constantin et al. [3] showed the conservation of the global kinetic energy if the velocity field $u$ is of the class $L^3(0, T; B_{3, \infty}^2(\mathbb{T}^3)) \cap C([0, T]; L^2(\mathbb{T}^3))$ whenever $\alpha > \frac{1}{3}$, see also [17]. Recently, similar results for the compressible Euler system were presented by Feireisl et al. in [18]. A sufficient condition for the energy conservation is that the solution belongs to $B_{3, \infty}^2((0, T) \times \mathbb{T}^3)$ with $\alpha > 1/3$. Up to our knowledge, this was the first result treating nonlinearity which is not in a multilinear form. We extend this approach to a general class of conservation laws of the form (1). Let us mention that we are not aware of any reference where the problem would be treated in such generality. We believe that this general scenario might be of interest. Moreover, at least the application on the equations of polyconvex elastodynamics (Subsection 4.3) is an original contribution of this paper.

Let us present the main results of the paper. For the notation, we refer the reader to Section 2.

**Theorem 1.1.** Let $U \in B_{3, \infty}^2(\mathcal{X}; \mathcal{O})$ be a weak solution of (1) with $\alpha > \frac{1}{3}$. Assume that $G \in C^2(\mathcal{O}; M^{n \times (k+1)})$ is endowed with a companion law with flux $Q \in C(\mathcal{O}; M^{1 \times (k+1)})$ for which there exists $B \in C^1(\mathcal{O}; M^{1 \times n})$ related through identity (2) and all the following conditions hold

$$\begin{aligned}
\mathcal{O} & \text{ is convex,} \\
B & \in W^{1, \infty}(\mathcal{O}; M^{1 \times n}), \\
|Q(V)| & \leq C(1 + |V|^3) \text{ for all } V \in \mathcal{O}, \\
\sup_{i,j \in 1, \ldots, d} \|\partial_{ij}G(U)\|_{C(\mathcal{O}; M^{n \times (k+1)})} & < +\infty.
\end{aligned}$$

Then $U$ is a weak solution of the companion law (3) with the flux $Q$.

**Remark.**

- We consider only a special case when the companion law is a scalar equation. If $Q: \mathcal{O} \to M^{s \times (k+1)}$ and $s > 1$, we can apply Theorem 1.1 to each row of (3).
- The growth condition of $Q$ can be relaxed whenever $B_{3, \infty}^2$ is embedded to an appropriate Lebesgue space.
- Under suitable assumptions, one can extend the theory on non–homogeneous fluxes $G = G(X, U)$ and equation (1) with non–zero right–hand side $h = h(X, U)$.
- Due to the definition of weak solutions, it is enough to consider the integrability and regularity of $U$ only locally in $\mathcal{X}$.

Due to the assumption on the convexity of $\mathcal{O}$, Theorem 1.1 could be straightforwardly deduced from [18], however, for the reader’s convenience, we present the proof in Section 5. It is worth noting that the convexity of $\mathcal{O}$ might not be natural for all applications (this
is e.g. the case of the polyconvex elasticity, see Section 3. To this purpose, we present a theorem dealing with the case of non-convex $\mathcal{O}$.

**Theorem 1.2.** Let the assumptions of Theorem 1.1 be satisfied, but instead of (5) we assume that

$$\text{the essential range of } U \text{ is compact in } \mathcal{O}.$$  \hspace{1cm} (6)

Then $U$ is a weak solution of the companion law (3) with the flux $Q$.

Apparently, the conclusions of the previous theorems are reasonably weaker in comparison with some known results for particular conservation laws. As an example, the result of Constantin et al. in [8] does not need the Besov-type regularity with respect to time. Having more knowledge about the nonlinear part of $G$, we may be able to relax the class of solutions in Theorem 1.1, what is discussed in Section 4.

Finally, we observe that in case we consider hyperbolic systems, the opposite direction of the Onsager’s hypothesis is almost trivial. This is of course completely different situation than the case of incompressible Euler system, which is not a hyperbolic conservation law and the construction of solutions dissipating the energy was a challenge. It is well known, cf. [9, Chapter 1] among others, that shock solutions dissipate energy. Following Dafermos again, we note that crucial properties of local behavior of shocks may be investigated, without loss of generality, within the framework of systems in one-space dimension. Thus the essence can be already seen even on a simple example of the Burger’s equation.

Let us briefly mention the outline of the rest of the paper. In Section 2, we introduce the notation. Section 3 contains proofs of the main propositions. Section 4 is devoted to some relaxation of the conditions in Theorem 1.1 and applications of the main theorems are also presented.

2. Notation and auxiliary estimates

We will briefly present some properties of the Besov spaces $B^0_{q,\infty}$. Let $\mathcal{X}$ be as above, $\alpha \in (0,1)$ and $q \in [1, \infty)$. We denote by $B^\alpha_{q,\infty}(\mathcal{X})$ the Besov space which is defined as follows

$$B^\alpha_{q,\infty}(\mathcal{X}) = \left\{ U \in L^q(\mathcal{X}) : \| U |_{B^\alpha_{q,\infty}(\mathcal{X})} < \infty \right\}$$

with

$$\| U |_{B^\alpha_{q,\infty}(\mathcal{X})} = \sup_{\xi \in \mathbb{R}^k} \frac{\| U(\cdot) - U(\cdot - \xi) \|_{L^q(\mathcal{X} \cap (\mathcal{X} + \xi))}}{|\xi|^\alpha}.$$

On $B^\alpha_{q,\infty}(\mathcal{X})$ we consider the standard norm

$$\| U \|_{B^\alpha_{q,\infty}(\mathcal{X})} = \| U \|_{L^q(\mathcal{X})} + \| U \|_{B^\alpha_{q,\infty}(\mathcal{X})}.$$
Assume that a non–negative function \( \eta \in C^\infty(\mathbb{R}^k) \) has a compact support in \( B(0, 1) \) and \( \int_{\mathbb{R}^k} \eta(X) \, dX = 1 \). For \( \varepsilon > 0 \) we denote \( \eta_\varepsilon(X) = \frac{1}{\varepsilon} \eta(\frac{X}{\varepsilon}) \) and
\[
[f]_\varepsilon(X) = f * \eta_\varepsilon(X)
\]
which is defined at least in \( \mathcal{X}_\varepsilon = \{ X \in \mathcal{X} : \text{dist}(X, \partial \mathcal{X}) > \varepsilon \} \). For vector or matrix–valued functions the convolution is defined component–wise. For \( \mathcal{X} \subseteq \mathbb{R}^k \) and \( \delta > 0 \) we also use the notation
\[
\mathcal{X}^\delta = \{ X \in \mathbb{R}^k : \text{dist}(X, \mathcal{X}) < \delta \} = \bigcup_{X \in \mathcal{X}} B(X, \delta).
\]
One easily shows that for \( f \in B^q_{\varepsilon, \infty}(\mathcal{X}) \) the following estimates hold
\[
\| D_X [f]_\varepsilon \|_{L^q(\mathcal{X}_\varepsilon)} \leq C \| f \|_{B^q_{\varepsilon, \infty}(\mathcal{X})} \varepsilon^{q-1}, \tag{7}
\]
\[
\| [f]_\varepsilon - f \|_{L^q(\mathcal{X}_\varepsilon)} \leq C \| f \|_{B^q_{\varepsilon, \infty}(\mathcal{X})} \varepsilon^q, \tag{8}
\]
\[
\| f(\cdot - y) - f(\cdot) \|_{L^q(\mathcal{X} \cap (\mathcal{X} + y))} \leq C \| f \|_{B^q_{\varepsilon, \infty}(\mathcal{X})} |y|^q \tag{9}
\]
where \( C \) depends only on \( \mathcal{X} \).

3. THE PROOF OF THE MAIN RESULTS

In what follows, we will denote by \( C \) a constant independent of \( \varepsilon \).

3.1. Commutator estimates. The essential part of the proof of Theorem 1.1 pertains the estimation of the nonlinear commutator
\[
\{G(U)\}_\varepsilon - G([U]_\varepsilon).
\]
It is based on the following observation, which appears in a special form in [18]. The rest of the proof of Theorem 1.1 is a reminiscence of the paper of [8].

**Lemma 3.1.** Let \( \mathcal{O} \) be a convex set, \( U \in L^2_{\text{loc}}(\mathcal{X}, \mathcal{O}) \), \( G \in C^2(\mathcal{O}; \mathbb{R}^n) \) and let
\[
\sup_{i,j=1, \ldots, d} \| \partial_{U_i} \partial_{U_j} G(U) \|_{L^\infty(\mathcal{O})} < +\infty. \tag{10}
\]
Then there exists \( C > 0 \) depending only on \( \eta_1 \), second derivatives of \( G \) and \( k \) (dimension of \( \mathcal{O} \)) such that

\[
\| [G(U)]_\varepsilon - G([U]_\varepsilon) \|_{L^q(\mathcal{K})} \leq C \left( \| [U]_\varepsilon - U \|_{L^{2q}(\mathcal{K})}^2 + \sup_{Y \subseteq \text{supp} \eta_\varepsilon} \| U(\cdot) - U(\cdot - Y) \|_{L^{2q}(\mathcal{K})}^2 \right)
\]
for \( q \in [1, \infty) \), where \( \mathcal{K} \subseteq \mathcal{X} \) satisfies \( K^\varepsilon \subseteq \mathcal{X} \).

**Proof.** Without loss of generality, we assume that \( G \) is a scalar function and \( U \) is finite everywhere on \( \mathcal{X} \). Then, because of (10) we get for \( X, Y \in K \)
\[
\|G(U(X)) - G([U]_\varepsilon(X)) - D_U G \circ U(X)(U(X) - [U]_\varepsilon(X))\| \leq C \|U(X) - [U]_\varepsilon(X)\|^2, \tag{11}
\]
\[
\|G(U(X)) - G(U(Y)) - D_U G \circ U(X)(U(X) - U(Y))\| \leq C \|U(X) - U(Y)\|^2. \tag{12}
\]
We convolve (12) with \( \eta_\varepsilon \) in variable \( Y \) and apply Jensen’s inequality on the left–hand side
\[
\|G(U(X)) - [G(U)]_\varepsilon(X) - D_U G \circ U(X)(U(X) - [U]_\varepsilon(X))\| \leq C \|U(X) - U(\cdot)\|_{L^2(X)}^2 \ast_Y \eta_\varepsilon. \tag{13}
\]
Finally, coupling (11) and (13) implies to
\[
\|G([U]_\varepsilon(X)) - [G(U)]_\varepsilon(X)\| \leq C \left( \|U(X) - [U]_\varepsilon(X)\|^2 + \|U(X) - U(\cdot)\|_{L^2(X)}^2 \ast_Y \eta_\varepsilon(X) \right). \tag{14}
\]
In order to complete the proof, we use Jensen’s inequality to estimate the $L^q$ norm of the second term on the right–hand side of (14)

$$\int_K \left| \int_{\text{supp } \eta_\varepsilon} |U(X) - U(X - Y)|^q \eta_\varepsilon(Y) \, dY \right|^q \, dX \leq \int_{\text{supp } \eta_\varepsilon} \int_K |U(X) - U(X - Y)|^{2q} \eta_\varepsilon(Y) \, dX \, dY \leq \sup_{Y \in \text{supp } \eta_\varepsilon} \|U(\cdot) - U(\cdot - Y)\|_{L^{2q}(K)}^{2q}.$$

3.2. Proof of Theorem 1.1 Let $\varepsilon_0 > 0$ and consider a test function $\psi \in C^\infty(\mathcal{X})$ such that $\text{supp } \psi \subseteq \mathcal{X}_{\varepsilon_0}$. Mollifying (11) by $\eta_\varepsilon$, we obtain

$$\text{div}_X [G(U)]_\varepsilon = 0 \quad \text{in } \mathcal{X}_{\varepsilon_0} \quad (15)$$

whenever $\varepsilon < \varepsilon_0$. We multiply both sides of (15) by $\psi([U]_\varepsilon)$ (where $\mathcal{B}$ comes from (2)) from the left and get

$$\int_{\mathcal{X}} \psi(X) \mathcal{B}([U]_\varepsilon(X)) \, \text{div}_X ([G(U)]_\varepsilon(X)) \, dX = 0. \quad (16)$$

We can recast the previous equality as follows

$$\int_{\mathcal{X}} \psi(X) \mathcal{B}([U]_\varepsilon(X)) \, \text{div}_X G([U]_\varepsilon(X)) \, dX = \int_{\mathcal{X}} R_\varepsilon \, dX$$

with the commutator

$$R_\varepsilon = \psi(X) \mathcal{B}([U]_\varepsilon(X)) \, \text{div}_X \left( G([U]_\varepsilon(X)) - [G(U)]_\varepsilon(X) \right). \quad (17)$$

Due to (2), equality (16) might be adjusted to the form

$$- \int_{\mathcal{X}} Q([U]_\varepsilon(X))(D_X \psi(X))^T \, dX = \int_{\mathcal{X}} R_\varepsilon \, dX. \quad (18)$$

In order to show that the right–hand side of (18) converges to zero as $\varepsilon \to 0$, we write

$$\int_{\mathcal{X}} R_\varepsilon(X) \, dX = \int_{\mathcal{X}} \left( G([U]_\varepsilon) - [G(U)]_\varepsilon \right) : \left( (D_U \mathcal{B}^T)([U]_\varepsilon) \right) D_X [U]_\varepsilon \psi \right) \, dX$$

$$+ \int_{\mathcal{X}} \left( G([U]_\varepsilon) - [G(U)]_\varepsilon \right) : \left( \mathcal{B}^T([U]_\varepsilon) \right) D_X \psi \right) \, dX \quad (19)$$

The first integral is estimated using Lemma 3.1 and (7) as follows

$$|I_2^2| \leq C\|\mathcal{B}\|_{W^{1,\infty}(\partial)} \|D_X [U]_\varepsilon\|_{L^1(\mathcal{X}_{\varepsilon_0})} \\|\psi\|_{W^{1,\infty}(\mathcal{X}_{\varepsilon_0})}$$

$$\leq C\varepsilon^{-1}e^{-2\alpha}.$$ 

Similarly, we have

$$|I_2^1| \leq C\varepsilon^\alpha,$$

hence,

$$\int_{\mathcal{X}} R_\varepsilon \, dX \to 0 \quad \text{as } \varepsilon \to 0$$

as long as $\alpha > \frac{1}{2}$.

The convergence of the left–hand side of (18) follows from the Vitali theorem. Indeed, the equi-integrability of $Q([U]_\varepsilon)$ in $\mathcal{X}_{\varepsilon_0}$ is a consequence of that of $|[U]_\varepsilon|^3$ and the growth conditions on $Q$. 

Remark. Having \( \partial \) non-convex, we face the problem that \( [U]_\varepsilon \) does not have to belong to \( \partial \). The convexity was crucial to conduct the Taylor expansion argument in Lemma 3.1. However, we will see that a suitable extension of functions \( G, B \) and \( Q \) does not alter the previous proof significantly.

3.3. Proof of Theorem 1.2 There exists \( \delta > 0 \) depending only on \( \mathcal{H} \) and \( \partial \) such that \( \mathcal{H}^{2\delta} \subseteq \partial \). Let \( \tilde{G} \in C^2(\mathbb{R}^n; M^{(k+1)\times n}) \), \( \tilde{B} \in C^1(\mathbb{R}^n; M^{1\times n}) \) and \( \tilde{Q} \in C(\mathbb{R}^n; M^{1\times(k+1)}) \) be compactly supported functions satisfying \( \tilde{G} = G, \tilde{B} = B \) and \( \tilde{Q} = Q \) on \( \mathcal{H}^{\delta} \). Such functions exist as there is a set \( \mathcal{R} \) with a smooth boundary satisfying \( \mathcal{H}^{\delta} \subseteq \mathcal{R} \subseteq \partial \).

Thus, relation (2) holds also for \( G, B \) and \( Q \) on \( \mathcal{H}^{\delta} \).

Similarly to Subsection 3.2, \( \int_{\mathcal{H}^{\delta}} R_{\varepsilon} \, dX \) vanishes as \( \varepsilon \to 0 \) due to Lemma 3.1; hence, we may turn our attention to the left-hand side of (20). We show that it converges to

\[
- \int_{\mathcal{H}^{\delta}} Q(U)(D_X \psi)^T \, dX.
\]

To this end, we put

\[
\mathcal{H}^{\delta}_\varepsilon = \{ X \in \mathcal{H} : |U(X) - [U]_\varepsilon(X)| < \delta \}
\]

and since \( D_U \tilde{Q}_j([U]_\varepsilon) = \tilde{B}([U]_\varepsilon)D_U \tilde{G}_j([U]_\varepsilon) \) on \( \mathcal{H}^{\delta}_\varepsilon \) we obtain

\[
\left| \int_{\mathcal{H}^{\delta}_\varepsilon} \psi \tilde{B}([U]_\varepsilon) \, dX \tilde{G}([U]_\varepsilon) \psi D_X \psi \, dX + \int_{\mathcal{H}^{\delta}_\varepsilon} Q(U)(D_X \psi)^T \, dX \right|
\]

\[
\leq \int_{\mathcal{H}^{\delta}_\varepsilon} \psi \tilde{B}([U]_\varepsilon) \, dX \tilde{G}([U]_\varepsilon) \psi \, dX \right| + \int_{\mathcal{H}^{\delta}_\varepsilon} Q(U)(D_X \psi)^T \, dX
\]

\[
+ \int_{\mathcal{H}^{\delta}_\varepsilon}(\tilde{Q}(U) - \tilde{Q}([U]_\varepsilon))(D_X \psi)^T \, dX = I^1_\varepsilon + I^2_\varepsilon + I^3_\varepsilon.
\]

To estimate \( I^1_\varepsilon \), recall that \( \tilde{G} \) and \( \tilde{B} \) are compactly supported, therefore

\[
I^1_\varepsilon \leq \int_{\mathcal{H}^{\delta}_\varepsilon} \left| \psi \tilde{B}([U]_\varepsilon) D_U \tilde{G}([U]_\varepsilon) \right| \, dX \leq C\|\psi\|C_1 \int_{\mathcal{H}^{\delta}_\varepsilon} \|D_X [U]_\varepsilon\| \, dX.
\]

By the means of Hölder’s and Chebyshev’s inequality, (7) and (8) we observe that

\[
I^1_\varepsilon \leq C\|\psi\|C_1 \|D_X [U]_\varepsilon\|_{L^2(\mathcal{H}^{\delta}_\varepsilon)} \|\mathcal{H}^{\delta}_\varepsilon\|_{L^2(\mathcal{H}^{\delta}_\varepsilon)} \|U - [U]_\varepsilon\|_{L^2(\mathcal{H}^{\delta}_\varepsilon)} \leq C\|\psi\|C_1 \varepsilon^{3\alpha - 1}.
\]

The integral \( I^2_\varepsilon \) vanishes, as \( \|Q(U)\|_{L^\infty(\mathcal{H})} < \infty \). Finally, we observe that

\[
I^3_\varepsilon \leq \|\psi\|C_1 \int_{\mathcal{H}^{\delta}_\varepsilon} |\tilde{Q}(U) - \tilde{Q}([U]_\varepsilon)| \, dX.
\]
Therefore, \( I^s_\varepsilon \to 0 \) due to the almost everywhere convergence of \( \tilde{Q}(U) - \tilde{Q}([U]_\varepsilon) \) to zero and boundedness of \( \tilde{Q} \).

4. Applications

Observe that we have considered so far genuinely nonlinear fluxes \( G \). The key part of the proof was to estimate

\[
\int_X \left( G([U]_\varepsilon) - [G(U)]_\varepsilon \right) : \left( (DU)B^T([U]_\varepsilon)D_X[U]_\varepsilon \psi \right) \, dX,
\]

where the integral vanishes whenever \( G \) is an affine. Using this observation we might expect to drop some conditions on \( U \) in the main theorems if some components of \( G \) are affine functions.

We present three extensions of Theorem 1.1 which follow directly from the previous observation. The first gives a sufficient condition to drop the Besov regularity with respect to some variables. It is connected with the columns of \( G \).

**Corollary 4.1.** Let \( G = (G_1, \ldots, G_s, G_{s+1}, \ldots G_k) \) where \( G_1, \ldots, G_s \) are affine vector-valued functions and \( \mathcal{X} = \mathcal{Y} \times \mathcal{Z} \) where \( \mathcal{Y} \subseteq \mathbb{R}^s \) and \( \mathcal{Z} \subseteq \mathbb{R}^{k+1-s} \). Then it is enough to assume that \( U \in L^3(\mathcal{Y}; B^s_{\alpha, \infty}(\mathcal{Z})) \) in Theorem 1.1.

Next, we specify when we can omit the Besov regularity with respect to some components of \( U \).

**Corollary 4.2.** Assume that \( U = (V_1, V_2) \) where \( V_1 = (U_1, \ldots, U_s) \) and \( V_2 = (U_{s+1}, \ldots, U_n) \). If \( B \) does not depend on \( V_1 \) and \( G = G(V_1, V_2) = G_1(V_1) + G_2(V_2) \) and \( G_1 \) is linear then it is enough to assume \( U_1, \ldots, U_s \in L^3(\mathcal{X}) \) in Theorem 1.1.

Finally, we deal with the case when some components of \( B \) are not Lipschitz on \( \mathcal{O} \), but appropriate rows of \( G \) are affine functions.

**Corollary 4.3.** Assume that a \( j \)-th row of \( G \) is an affine function. Then the statement of Theorem 1.1 holds even if we assume that \( B_j \) is only locally Lipschitz in \( \mathcal{O} \).

In the rest of this paper, we present a few examples on which the general theory applies. Some of them show how the general framework allows to recover some known results. In what follows, we consider \( \mathcal{X} = (0, T) \times \mathbb{T}^3 \), \( X = (t, x) \) and \( \alpha > \frac{3}{4} \). We also present the systems in their standard form denoting \( \nabla_x \) and \( \text{div}_x \) the correspondent operators with respect to the spatial coordinate \( x \).

### 4.1. Incompressible Euler system

Let us consider the system of equations

\[
\begin{align*}
\text{div}_x u^T &= 0 \\
\partial_t u + (u \cdot \nabla_x) u + \nabla_x p &= 0
\end{align*}
\]

for an unknown vector field \( u: (0, T) \times \mathbb{T}^3 \to \mathbb{R}^3 \) and scalar \( p: (0, T) \times \mathbb{T}^3 \to \mathbb{R} \). The system can be rewritten into the divergence form with respect to \( X = (t, x) \)

\[
\begin{align*}
\text{div}_x u^T &= 0 \\
\partial_t u + \text{div}_x (u \otimes u + pp^T) &= 0.
\end{align*}
\]

By multiplying (24) with \( B(p, u) = (p - 1/2|u|^2, uu^T) \) we obtain the conservation law for the energy

\[
\partial_t \left( \frac{1}{2} |u|^2 \right) + \text{div}_x \left( \frac{1}{2} |u|^2 + pu^T \right) = 0.
\]

Corollaries 4.1, 4.2 and 4.3 imply that any weak solution \( (p, u) \in L^3(\mathcal{X}) \times L^3(0, T; B^s_{\alpha, \infty}(\mathbb{T}^3)) \) is a weak solution to (25).

**Remark.** This result is comparable to [8].
4.2. Compressible Euler system. We consider the compressible Euler equations in the following form

\[
\begin{align*}
\partial_t \rho + \text{div}_x (\rho u^T) &= 0 \\
\partial_t u + \text{div}_x (u \otimes u) + \frac{\nabla_x p(\rho)}{\rho} &= 0
\end{align*}
\]  

(26)

for an unknown vector field \( u : \mathcal{X} \to \mathbb{R}^3 \) and scalar \( \rho : \mathcal{X} \to \mathbb{R} \). The function \( p : [0, \infty) \to \mathbb{R} \) is given. Let \( P \) be a primitive function to \( \frac{\rho}{\rho} \) such that \( P(1) = 0 \). Then the system can be rewritten into the divergence form

\[
\begin{align*}
\partial_t \rho + \text{div}_x (\rho u^T) &= 0, \\
\partial_t u + \text{div}_x (u \otimes u + P(\rho) l) &= 0.
\end{align*}
\]  

(27)

To get the conservation of the energy, we multiply (27) with \( \rho \) and 4.3. As their consequence, a weak solution \( \rho \) that is bounded Lipschitz function on \( R \) can be rewritten into the divergence form

\[
\text{B}(\rho, u) = \left( P(\rho) + \rho P'(\rho) - \frac{1}{2}|u|^2, \rho u^T \right)
\]

and obtain

\[
\begin{align*}
\partial_t \left( \frac{1}{2}\rho |u|^2 + \rho P(\rho) \right) + \text{div}_x \left[ \left( \frac{1}{2}\rho |u|^2 + \rho P(\rho) + p(\rho) \right) u^T \right] &= 0
\end{align*}
\]  

(28)

Let \( (\rho, u) \in L^3(0, T; B_{3,\infty}^3 (\mathbb{T}^3)) \times L^3(0, T; B_{3,\infty}^3 (\mathbb{T}^3; \mathbb{R}^3)) \) be a weak solution to (27) such that \( \rho \in [\underline{\rho}, \overline{\rho}] \) for some \( 0 < \underline{\rho} < \overline{\rho} < \infty \) and \( u \in B(0, R) \) for some \( R > 0 \). Moreover, if \( p \in C^2([\underline{\rho}, \overline{\rho}]) \), we use Corollary 4.1 to show that \( (\rho, u) \) is a weak solution to (28). In the contrast with the incompressible case, the continuity equation (the first equation of (26)) is not linear with respect to \( \rho \) and \( u \). Therefore, we have to assume that \( u \) is bounded to provide \( \text{B}(\rho, u) \) is Lipschitz on the range of \( (\rho, u) \).

Remark. We have considered the formulation of the compressible Euler system with the time derivative over a linear function of \( (\rho, u) \). This has lead to a slightly different sufficient condition in comparison to [18].

Remark. If \( \rho > 0 \), system (26) can be rewritten with respect to the quantities \( \rho \) and \( m = \rho u \) as follows

\[
\begin{align*}
\partial_t \rho + \text{div}_x (m) &= 0 \\
\partial_t m + \text{div}_x \left( \frac{m \otimes m}{\rho} + p(\rho) l \right) &= 0
\end{align*}
\]  

(29)

A suitable choice of \( \text{B} \) is then

\[
\text{B}(\rho, m) = \left( P(\rho) + \rho P'(\rho) - \frac{1}{2}|m|^2, \frac{m^T}{\rho} \right),
\]  

(30)

which leads to the companion law

\[
\begin{align*}
\partial_t \left( \frac{1}{2}\rho |m|^2 + \rho P(\rho) \right) + \text{div}_x \left[ \left( \frac{1}{2}\rho |m|^2 + \rho P(\rho) + p(\rho) \right) u \right] &= 0
\end{align*}
\]  

(31)

As the continuity equation is now linear with respect to \( (\rho, m) \), we can apply Corollaries 1.1 and 4.3. As their consequence, a weak solution

\[
(\rho, m) \in L^3(0, T; B_{3,\infty}^3 (\mathbb{T}^3)) \times L^3(0, T; B_{3,\infty}^3 (\mathbb{T}^3; \mathbb{R}^3))
\]

such that \( \rho \in [\underline{\rho}, \overline{\rho}] \) for some \( 0 < \underline{\rho} < \overline{\rho} < \infty \) is also a weak solution to (31).

We can extend \( p \) from \( [\underline{\rho}, \overline{\rho}] \) on \( \mathbb{R} \) such that the extended function will be of class \( C^2 \) and compactly supported in \( \mathbb{R} \). Moreover, due to the boundedness of \( |u| \) we can write \( |u|^2 = u \cdot T(u) \) in \( \mathcal{X} \) where \( T \) is a bounded Lipschitz function on \( \mathbb{R}^3 \).
4.3. Polyconvex elasticity. Let us consider the evolution equations of nonlinear elasticity, see e.g. [10] or [14],
\[
\begin{align*}
\partial_t F &= \nabla_x v \\
\partial_t v &= \text{div}_x (D_F W(F))
\end{align*}
\]  
(32)
for an unknown matrix field \( F : \mathcal{X} \to \mathbb{M}^{k \times k} \), and an unknown vector field \( v : \mathcal{X} \to \mathbb{R}^k \).

Function \( W : \mathcal{Y} \to \mathbb{R} \) is given. For many applications, \( \mathcal{Y} = \mathbb{M}^{k \times k}_+ \) where \( \mathbb{M}^{k \times k}_+ \) denotes the subset of \( \mathbb{M}^{k \times k} \) containing only matrices having positive determinant, see e.g. [1] for the discussion on the form of \( W \) and \( \mathcal{Y} \). Let us point out that \( \mathbb{M}^{k \times k}_+ \) is a non–convex connected set.

System (32) can be rewritten into the divergence form in \((t, x)\) as follows
\[
\begin{align*}
\partial_t F_{i,j} &= \partial_{x_i} u_j = \text{div}_x \left( (e^i)^T u_j \right), \quad e^i_j = \delta_{i,j}, \\
\partial_t v &= \text{div}_x (D_F W(F))^T.
\end{align*}
\]  
(33)
By considering \( F \) to have values in \( \mathbb{R}^{t^2} \) and taking \( B(F, v) = (\{D_F W(F)\}^T, v^T) \), we obtain the companion law
\[
\partial_t \left( \frac{1}{2} |v|^2 + W(F) \right) - \text{div} (D_F W(F)v) = 0.
\]  
(34)

Let \((F, v) \in B_{\alpha,\infty}^a(\mathcal{X}; \mathbb{M}^{k \times k}_+) \times B_{3,\infty}^a(\mathcal{X}; \mathbb{R}^3)\) be a weak solution to (33) such that \( F \) has a compact range in \( \mathcal{Y} \) and \( v \) in \( \mathbb{R}^k \). Directly from Theorem 1.2, \((F, v)\) is a weak solution to (34) whenever \( W \in C^3(\mathcal{Y}) \).

Note that this observation for polyconvex elasticity is up to our best knowledge an original contribution.

4.4. Magnetohydrodynamics. Let us consider the system
\[
\begin{align*}
\text{div}_x u^T &= 0 \\
\text{div}_x h^T &= 0 \\
\partial_t u + (u \cdot \nabla_x) u + \nabla_x p &= (\text{curl}_x h) \times h \\
\partial_t h + \text{curl}_x (h \times u) &= 0
\end{align*}
\]  
(35)
for unknown vector functions \( u : \mathcal{X} \to \mathbb{R}^3 \) and \( h : \mathcal{X} \to \mathbb{R}^3 \) and an unknown scalar function \( p : \mathcal{X} \to \mathbb{R} \). It describes the motion of an ideal electrically conducting fluid, see e.g. [23] Chapter VIII. Using standard vector calculus identities, (35) can be written in the divergence form as follows:
\[
\begin{align*}
\text{div}_x u^T &= 0, \\
\text{div}_x h^T &= 0, \\
\partial_t u + \text{div}_x \left( u \otimes u + p \mathbb{I} + \frac{1}{2} |h|^2 \mathbb{I} - h \otimes h \right) &= 0, \\
\partial_t h + \text{div}_x (h \otimes u - u \otimes h) &= 0.
\end{align*}
\]

With \( B(p, u, h) = (p - 1/2 |u|^2, -h \cdot u, u^T, h^T) \), the conservation of the total energy reads:
\[
\partial_t \left( \frac{1}{2} |u|^2 + \frac{1}{2} |h|^2 \right) + \text{div}_x \left[ \left( \frac{1}{2} |u|^2 + p + |h|^2 \right) u^T - (u \cdot h) h^T \right] = 0.
\]  
(36)
A combination of Corollaries 4.1, 4.2 and 4.3 implies that any weak solution
\[
(p, u, h) \in L^3(\mathcal{X}) \times \left( L^3(0, T; B_{3,\infty}^a(\mathbb{R}^3)) \right)^2
\]  
(37)
is a weak solution to (25). A similar result was obtained e.g. in [6].
4.5. **Further examples.** The list of examples is still far from being complete, however it is not our goal, and surely not an expectation of a reader, to provide an extended list. Among numerous further examples we will only mention inviscid compressible magneto-hydrodynamics. A direct combination of Subsection 4.2 and 4.4 gives a sufficient condition to satisfy the relevant energy equality. Another worth of mentioning example is heat conducting gas, see also [15].

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