Mirror Symmetry in Emergent Gravity

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ABSTRACT

Given a six-dimensional symplectic manifold \((M, B)\), a nondegenerate, co-closed four-form \(C\) introduces a dual symplectic structure \(\tilde{B} = *C\) independent of \(B\) via the Hodge duality \(*\). We show that the doubling of symplectic structures due to the Hodge duality results in two independent classes of noncommutative \(U(1)\) gauge fields by considering the Seiberg-Witten map for each symplectic structure. As a result, emergent gravity suggests a beautiful picture that the variety of six-dimensional manifolds emergent from noncommutative \(U(1)\) gauge fields is doubled. In particular, the doubling for the variety of emergent Calabi-Yau manifolds allows us to arrange a pair of Calabi-Yau manifolds such that they are mirror to each other. Therefore, we argue that the mirror symmetry of Calabi-Yau manifolds is the Hodge theory for the deformation of symplectic and dual symplectic structures.

Keywords: Emergent gravity, Mirror symmetry, Calabi-Yau manifold

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1 Holomorphic Line Bundle and Kähler Manifolds

Let $\pi : L \to M$ be a line bundle over a six-dimensional complex manifold $M$ whose connection is denoted by $A = A_a(x) dx^a$. The curvature of the line bundle $L$ is given by $\mathcal{F} = \frac{1}{2} F_{ab} dx^a \wedge dx^b = dA$. Given a complex structure of the base manifold $M$, one can canonically decompose the curvature two-form as

$$\mathcal{F} = \mathcal{F}^{(2,0)} \oplus \mathcal{F}^{(1,1)} \oplus \mathcal{F}^{(0,2)}.$$  

(1.1)

A holomorphic line bundle $L$ over $M$ is a complex vector bundle of (complex) rank one admitting holomorphic transition functions [1]. A line bundle $L$ becomes a holomorphic line bundle if

$$\mathcal{F}^{(2,0)} = \mathcal{F}^{(0,2)} = 0.$$  

(1.2)

Accordingly, the curvature of a holomorphic line bundle $L$ consists of a $(1, 1)$-form only, i.e., $\mathcal{F} = \mathcal{F}^{(1,1)}$.

For simplicity, we will assume that $M = \mathbb{C}^3$, whose complex coordinates are given by

$$z^i = y^{2i-1} + \sqrt{-1} y^{2i}, \quad \bar{z}^i = y^{2i-1} - \sqrt{-1} y^{2i}, \quad i, \bar{i} = 1, 2, 3.$$  

(1.3)

Later we will briefly discuss a generalization to a compact complex manifold, e.g., $M = T^6 \cong T_3^3$, a three-dimensional complex torus. According to the complex coordinates in Eq. (1.3), the connection of $L$, which is called $U(1)$ gauge fields, also takes the decomposition given by

$$A_i = \frac{1}{2} (A_{2i-1} - \sqrt{-1} A_{2i}), \quad A_{\bar{i}} = \frac{1}{2} (A_{2i-1} + \sqrt{-1} A_{2i}).$$  

(1.4)

Then the field strengths of $(2, 0)$ and $(1, 1)$ parts in Eq. (1.1) are, respectively, given by

$$\mathcal{F}_{ij} = \frac{1}{4} (\mathcal{F}_{2i-1,2j-1} - \mathcal{F}_{2i,2j}) - \frac{\sqrt{-1}}{4} (\mathcal{F}_{2i-1,2j} + \mathcal{F}_{2i,2j-1}),$$  

(1.5)

$$\mathcal{F}_{i\bar{j}} = \frac{1}{4} (\mathcal{F}_{2i-1,2\bar{j}-1} + \mathcal{F}_{2i,2\bar{j}}) + \frac{\sqrt{-1}}{4} (\mathcal{F}_{2i-1,2\bar{j}} - \mathcal{F}_{2i,2\bar{j}-1}).$$  

(1.6)

Therefore, the curvature of a holomorphic line bundle, i.e., $\mathcal{F}_{ij} = \mathcal{F}_{i\bar{j}} = 0$, must obey the relation

$$\mathcal{F}_{2i-1,2j-1} = \mathcal{F}_{2i,2j}, \quad \mathcal{F}_{2i-1,2j} = -\mathcal{F}_{2i,2j-1}, \quad i, j = 1, 2, 3.$$  

(1.7)

Since $\mathcal{F}_{ij} = \partial_i A_j - \partial_j A_i$ and $\mathcal{F}_{ij} = \bar{\partial}_i \bar{A}_j - \bar{\partial}_j \bar{A}_i$, the condition (1.2) for the holomorphic line bundle can be solved by

$$A_i = -\frac{\sqrt{-1}}{2} \partial_i \phi(z, \bar{z}), \quad A_{\bar{i}} = \frac{\sqrt{-1}}{2} \bar{\partial}_{\bar{i}} \phi(z, \bar{z})$$  

(1.8)

where $\phi(z, \bar{z})$ is a real smooth function on $\mathbb{C}^3$. Then the field strength of a holomorphic line bundle is given by

$$\mathcal{F}_{i\bar{j}} = \sqrt{-1} \partial_i \bar{\partial}_{\bar{j}} \phi(z, \bar{z}).$$  

(1.9)
Suppose that $\mathcal{M}$ is a six-dimensional complex manifold with the metric $ds^2 = \mathcal{G}_{\mu\nu}(x)dx^\mu dx^\nu$. On a complex manifold, the metric can also be decomposed into three types:

$$
\mathcal{G}_{\mu\nu} = \mathcal{G}_{\alpha\beta} \oplus \mathcal{G}_{\alpha\bar{\beta}} \oplus \mathcal{G}_{\bar{\alpha}\beta},
$$

(1.10)

where we have split a curved space index $\mu = 1, \cdots, 6 = (\alpha, \bar{\alpha})$ into a holomorphic index $\alpha = 1, 2, 3$ and an anti-holomorphic one $\bar{\alpha} = 1, 2, 3$, similarly a tangent space index $a = 1, \cdots, 6 = (i, \bar{i})$ into $i = 1, 2, 3$ and $\bar{i} = 1, 2, 3$. A complex manifold $\mathcal{M}$ is called a Hermitian manifold [2] if

$$
\mathcal{G}_{\alpha\beta} = \mathcal{G}_{\alpha\bar{\beta}} = 0.
$$

(1.11)

In terms of real components, the Hermitian condition (1.11) means that

$$
\mathcal{G}_{2\alpha - 1, 2\beta - 1} = \mathcal{G}_{2\alpha, 2\beta}, \quad \mathcal{G}_{2\alpha - 1, 2\beta} = -\mathcal{G}_{2\alpha, 2\beta - 1}, \quad \alpha, \beta = 1, 2, 3,
$$

(1.12)

which looks similar to Eq. (1.7) although one is antisymmetric and the other is symmetric. After all, the Hermitian metric consists of $(1, 1)$-type only, i.e.,

$$
ds^2 = \mathcal{G}_{\alpha\bar{\beta}}(z, \bar{z}) dz^\alpha d\bar{z}^\bar{\beta}.
$$

(1.13)

Given a Hermitian metric, one can introduce a fundamental two-form defined by

$$
\Omega = \sqrt{-1} \mathcal{G}_{\alpha\bar{\beta}}(z, \bar{z}) dz^\alpha \wedge d\bar{z}^\bar{\beta}.
$$

(1.14)

A Kähler manifold is defined as a Hermitian manifold with a closed fundamental two-form, i.e., $d\Omega = 0$ [2]. The so-called Kähler condition, $d\Omega = 0$, can be solved by the metric given by

$$
\mathcal{G}_{\alpha\bar{\beta}} = \partial_\alpha \bar{\partial}_\beta K(z, \bar{z}),
$$

(1.15)

where $K(z, \bar{z})$ is a real smooth function on a complex manifold $\mathcal{M}$ and is called a Kähler potential.

Now let us look at the curvature $\mathcal{F}$ of a holomorphic line bundle and the Kähler form $\Omega$ of a Kähler manifold that are, respectively, given by

$$
\mathcal{F} = \sqrt{-1} \partial_i \bar{\partial}_j \phi(z, \bar{z}) dz^i \wedge d\bar{z}^j = \sqrt{-1} \partial \bar{\partial} \phi(z, \bar{z}),
$$

(1.16)

$$
\Omega = \sqrt{-1} \partial_\alpha \bar{\partial}_\beta K(z, \bar{z}) dz^\alpha \wedge d\bar{z}^\bar{\beta} = \sqrt{-1} \partial \bar{\partial} K(z, \bar{z}).
$$

(1.17)

Since $\phi(z, \bar{z})$ and $K(z, \bar{z})$ are arbitrary smooth functions on a complex manifold in addition to a striking superficial similarity of $\mathcal{F}$ and $\Omega$, an innocent question naturally arises whether it is possible to identify them or when we can identify them. If one recalls that the Kähler form $\Omega$ is a symplectic structure, then the answer may be obvious. The curvature two-form $\mathcal{F}$ must be a symplectic structure to make sense the identification. Indeed, it was shown in [3] that one can identify $\phi(z, \bar{z})$ with $K(z, \bar{z})$ if the curvature $\mathcal{F}$ of a holomorphic line bundle is a symplectic structure, i.e., a nondegenerate, closed two-form. A nondegenerate two-form $\mathcal{F} = \frac{1}{2} \mathcal{F}_{ab}(x) dx^a \wedge dx^b$ means that $\det \mathcal{F}_{ab}(x) \neq 0$, $\forall x \in M$. 

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Of course, it is not a typical situation in the Maxwell’s electromagnetism where $F_{\alpha\beta}(x)|_{|x|\to\infty} \to 0$. To emphasize the nondegenerateness of the field strength, let us represent it by

$$F = B + F$$

(1.18)

where $B \equiv F|_{|x|\to\infty}$ is a nowhere vanishing two-form of rank 6 and $F = dA$. The identification of $\phi(z, \bar{z})$ with $K(z, \bar{z})$ means that a holomorphic line bundle with a nondegenerate curvature two-form of rank 6 is equivalent to a six-dimensional Kähler manifold. Then the real function $\phi(z, \bar{z})$ and so $K(z, \bar{z})$ will be determined by solving the equations of motion of $U(1)$ gauge fields. In other words, (generalized) Maxwell’s equations for $U(1)$ gauge fields on a holomorphic line bundle can be translated into Einstein’s equations for a Kähler manifold. For example, one may wonder what is the gauge theory object that gives rise to a Calabi-Yau manifold which is a Kähler manifold with a vanishing first Chern class. It was verified in [3, 4] that Calabi-Yau $n$-folds for $n = 2$ and 3 are emergent from the commutative limit of noncommutative (NC) $U(1)$ instantons in four and six dimensions, respectively.

Let us recapitulate why it is possible to make the identification up to holomorphic gauge transformations\footnote{Note that both $\phi(z, \bar{z})$ and $K(z, \bar{z})$ are locally defined. They may not fit together on the overlap $U_i \cap U_j$ to give a globally defined function on a complex manifold $M$ where $\bigcup_i (U_i, z^{(i)})$ is a holomorphic atlas of $M$. However, the curvature $F$ and the Kähler form $\Omega$ can be globally defined. For example, one can use a $U(1)$ gauge transformation, $A \to A + df$ where $f \in C^\infty(M)$, to glue locally defined functions $\phi^{(i)}$ on each coordinate patch $U_i$. On the overlap $U_i \cap U_j$ of two coordinate patches, the $U(1)$ gauge transformation reads as $\phi^{(i)} = \phi^{(j)} + f^{(ij)}(z) + \bar{f}^{(ij)}(\bar{z})$ where two real functions $\phi^{(i)}$ and $\phi^{(j)}$ are defined on $U_i$ and $U_j$, respectively [5]. This gluing of $U(1)$ gauge fields can be translated into that of Kähler potentials according to the identification (1.19).}

$$\phi(z, \bar{z}) = K(z, \bar{z}),$$

(1.19)

if the curvature $F$ of a holomorphic line bundle is regarded as a symplectic structure on $M$. An important fact is that a symplectic structure, for instance, the $B$-field in Eq. (1.18), provides a bundle isomorphism $B : TM \to T^*M$ by $X \mapsto A = \iota_X B$ where $X \in \Gamma(TM)$ is an arbitrary vector field, since $B$ is a nondegenerate two-form. As a result, the field strength in Eq. (1.18) can be written as

$$F = (1 + \mathcal{L}_X)B \approx e^{\mathcal{L}_X}B,$$

(1.20)

where $\mathcal{L}_X = d\iota_X + \iota_X d$ is the Lie derivative with respect to the vector field $X$. Since a vector field is an infinitesimal generator of local coordinate transformations, in other words, a Lie algebra generator of diffeomorphisms $\text{Diff}(M)$, the result (1.20) implies [6, 7] that it is possible to find a local coordinate transformation $\phi \in \text{Diff}(M)$ eliminating dynamical $U(1)$ gauge fields in $F$ such that $\phi^*(F) = B$, i.e., $\phi^* = (1 + \mathcal{L}_X)^{-1} \approx e^{-\mathcal{L}_X}$. This statement is known as the Darboux theorem or the Moser lemma in symplectic geometry [8]. It is arguably a novel form of the equivalence principle for the electromagnetic force. This fact leads to a remarkable conclusion [9] that, in the presence of $B$-fields, the “dynamical” symplectic manifold $(M, F)$ respects a (dynamical) diffeomorphism
symmetry generated by the vector field $X \in \Gamma(TM)$, so the underlying local gauge symmetry is rather enhanced. Here we mean the “dynamical” for a fluctuating field around a background like Eq. (1.18). Therefore, we fall into a situation similar to general relativity that the dynamical symplectic manifold $(M, F)$ can be locally trivialized by a coordinate transformation $\phi \in \text{Diff}(M)$.

In terms of local coordinates, the coordinate transformation $\phi \in \text{Diff}(M)$ may be represented by

$$\phi : y^a \mapsto x^a(y) = y^a + \theta^{ab}a_b(y) \quad (1.21)$$

where $\theta \equiv B^{-1}$ and the dynamical coordinates $a_b(y)$ will be called symplectic gauge fields. By using the above coordinates, the Darboux transformation obeying $\phi^*(F) = B$ is explicitly written as

$$(B_{ab} + F_{ab}(x)) \frac{\partial x^a}{\partial y^\mu} \frac{\partial x^b}{\partial y^\nu} = B_{\mu\nu}, \quad (1.22)$$

where $B$ is assumed to be constant without loss of generality. Since both sides of Eq. (1.22) are invertible, one can deduce [10, 11, 12] that

$$\Theta^{ab}(x) \equiv (F^{-1})^{ab}(x) = \{x^a(y), x^b(y)\}_\theta \quad (1.23)$$

where we have introduced the Poisson bracket defined by

$$\{\psi(y), \varphi(y)\}_\theta = \theta^{\mu\nu} \frac{\partial \psi(y)}{\partial y^\mu} \frac{\partial \varphi(y)}{\partial y^\nu} \quad (1.24)$$

for $\psi, \varphi \in C^\infty(M)$ and the field strength of symplectic gauge fields is given by

$$f_{ab}(y) = \partial_a a_b(y) - \partial_b a_a(y) + \{a_a(y), a_b(y)\}_\theta. \quad (1.25)$$

The identification (1.19) suggests a fascinating path for the quantization of Kähler manifolds. Note that the symplectic manifold $(M, F)$ is a dynamical system since it can be understood as the deformation of a symplectic manifold $(M, B)$ by the electromagnetic force $F = dA$. Thus one may quantize the dynamical system of the symplectic manifold $(M, F)$ rather than trying to quantize a Kähler manifold directly [9]. The quantization $Q$ is straightforward as the dynamical system equips with the intrinsic Poisson structure (1.24) like as quantum mechanics. An underlying math is essentially the same as quantum mechanics. It results in a NC $U(1)$ gauge theory [13] on a quantized or NC space, denoted by $\mathbb{R}^6_\theta$, whose coordinate generators satisfy the commutation relation

$$[y^a, y^b] = i\theta^{ab}. \quad (1.26)$$

The NC $\star$-algebra generated by the Moyal-Heisenberg algebra (1.26) will be denoted by $\mathcal{A}_\theta$ [14]. The NC $U(1)$ gauge theory is constructed by lifting the coordinate transformation (1.21) to a local automorphism of $\mathcal{A}_\theta$ defined by $Q : \phi \mapsto D_A$ which acts on the NC coordinates $y^a$ as [11, 15]

$$D_A(y^a) \equiv \hat{X}^a(y) = y^a + \theta^{ab}\hat{A}_b(y) \in \mathcal{A}_\theta. \quad (1.27)$$
It ascertains that NC $U(1)$ gauge fields are obtained by quantizing symplectic gauge fields, i.e., $\hat{A}_a = Q(a_a)$. Upon quantization, the Poisson bracket is similarly lifted to a NC bracket in $\mathcal{A}_\theta$. For example, the Poisson bracket relation (1.23) is now defined by the commutation relation

$$[\hat{X}^a, \hat{X}^b] = i(\theta(B - \hat{F})\theta)^{ab},$$

(1.28)

where the field strength of NC $U(1)$ gauge fields $\hat{A}_a$ is given by

$$\hat{F}_{ab} = \partial_a\hat{A}_b - \partial_b\hat{A}_a - i[\hat{A}_a, \hat{A}_b].$$

(1.29)

Here we observe [9] that NC $U(1)$ gauge fields describe a dynamical NC spacetime (1.28) which is a deformation of the vacuum NC spacetime (1.26). To sum up, a dynamical NC spacetime is defined by the quantization of a line bundle $L$ over a symplectic manifold $(M,B)$ and described by a NC $U(1)$ gauge theory.

The identification (1.19) attains its vitality in the following way. It can be shown [3] that the Darboux transformation (1.22) leads to a remarkable identity between Dirac-Born-Infeld densities (up to total derivatives):

$$\sqrt{\det(g + F)} = \sqrt{\det(G + B)}$$

(1.30)

$$= g_s \sqrt{\det(G + \hat{F} - B)},$$

(1.31)

where the flat metrics $(g, G)$ are the Kähler metric of $\mathbb{C}^3$ and $B$ its Kähler form. The identity (1.30) clearly verifies that $U(1)$ gauge fields on the left-hand side must be a connection of a holomorphic line bundle obeying $\mathcal{F}_{ij} = \mathcal{F}_{\bar{i}\bar{j}} = 0$ to give rise to a Kähler metric $\mathcal{G}$ on the right-hand side and vice versa. Thus it establishes the identification (1.19). This demonstration is also true for the identity (1.31); a connection on a NC holomorphic line bundle obeying $\hat{F}^{(2,0)} = \hat{F}^{(0,2)} = 0$ gives rise to a Kähler metric $\mathcal{G}$ [3, 4].

Although the identities in Eqs. (1.30) and (1.31) clearly illustrate how to realize the gauge/gravity duality using a NC $U(1)$ gauge theory based on an associative algebra $\mathcal{A}_\theta$, the $\star$-algebra $\mathcal{A}_\theta$ provides a more elegant approach for the gauge/gravity duality. A preliminary step to derive gravitational variables from NC $U(1)$ gauge fields [7, 9, 16, 17] is to note that the NC $\star$-algebra $\mathcal{A}_\theta$ admits a nontrivial inner automorphism $\mathfrak{A}$ defined by the map

$$\mathcal{O} \mapsto \mathcal{O}' = U \star \mathcal{O} \star U^{-1}$$

where $U \in \mathfrak{A}$ and $\mathcal{O} \in \mathcal{A}_\theta$. Its infinitesimal generators consist of an inner derivation $\mathfrak{D}$. Then the inner derivation $\mathfrak{D}$ manifests a well-known Lie algebra homomorphism defined by the map

$$\mathcal{A}_\theta \to \mathfrak{D} : \mathcal{O} \mapsto \text{ad}_\mathcal{O} = -i[\mathcal{O}, \cdot],$$

(1.32)

for any $\mathcal{O} \in \mathcal{A}_\theta$. Using the Jacobi identity of the NC $\star$-algebra $\mathcal{A}_\theta$, it is easy to verify the Lie algebra homomorphism:

$$[\text{ad}_{\mathcal{O}_1}, \text{ad}_{\mathcal{O}_2}] = -i\text{ad}_{[\mathcal{O}_1, \mathcal{O}_2]},$$

(1.33)
for any $O_1, O_2 \in A_\theta$. In particular, we define the set of NC vector fields given by

$$\hat{V}_a \equiv \text{ad}\hat{D}_a \in \mathfrak{d}|\hat{D}_a(y) = p_a + \hat{A}_a(y) \in A_\theta, \ a = 1, \cdots, 6$$

(1.34)

where $p_a = B_{ab} y^b$ and $\hat{D}_a(y) \equiv D_A(p_a) = B_{ab} \hat{X}^b(y)$. One can apply the Lie algebra homomorphism (1.33) to the commutation relation

$$-i[\hat{D}_a, \hat{D}_b] = -B_{ab} + \hat{F}_{ab}$$

(1.35)

to yield the relation [16, 17]

$$\text{ad}\hat{F}_{ab} = [\hat{V}_a, \hat{V}_b] \in \mathfrak{d}. \tag{1.36}$$

The identification (1.19) may be confirmed by using the Lie algebra of derivations in Eqs. (1.34) and (1.36). To be precise, the derivation $\mathfrak{d}$ of the associative algebra $A_\theta$ defined by a NC $U(1)$ gauge theory is associated with a (quantized) frame bundle of an emergent spacetime manifold $M$. For example, we recently verified a particular case of the identity (1.19) [3, 4] that the commutative limit of six-dimensional NC Hermitian $U(1)$ instantons obeying the Hermitian Yang-Mills equation [18, 19, 20]

$$\hat{F} = -\ast (\hat{F} \land B), \tag{1.37}$$

where $B = \frac{i}{2} I_{\mu\nu} dy^\mu \land dy^\nu$ is the Kähler form of $\mathbb{C}^3$, is equivalent to Calabi-Yau manifolds obeying the (local) Einstein equation, $\det G_{\alpha\beta} = 1$.

## 2 Doubling of Emergent Calabi-Yau Manifolds

We observed in the previous section that emergent gravity is defined by considering the deformation of a symplectic manifold $(M, B)$ by a line bundle $L \rightarrow M$. The line bundle $L$ results in a dynamical symplectic manifold $(M, F)$ by introducing a new symplectic structure $F = B + F$ where $F = dA$ is identified with the curvature of the line bundle [9]. It is important to note [8] that a symplectic manifold $(M, B)$ is necessarily an orientable manifold since the symplectic structure $B$ admits a nowhere vanishing volume form $\nu = \frac{1}{3!} B^3$. Then a globally defined volume form introduces the Hodge-dual operation $\ast : \Omega^k(M) \rightarrow \Omega^{6-k}(M)$ between vector spaces of $k$-forms and $(6 - k)$-forms. This implies that the vector space $\Lambda^2 M$ of two-forms is enlarged twice:

$$\Lambda^2 M = \Omega^2(M) \oplus \ast \Omega^4(M), \tag{2.1}$$

since there are additional two-forms from the Hodge-dual of four-forms in $\Omega^4(M)$ in addition to the original two-forms in $\Omega^2(M)$. Let $C$ be a nondegenerate four-form that is co-closed, i.e., $\delta C = 0$ where

$$\delta = -(-1)^{6k} \ast d\ast : \Omega^k(M) \rightarrow \Omega^{k-1}(M) \tag{2.2}$$
is the adjoint exterior differential operator \cite{1,2}. Define a two-form $\tilde{B} \equiv *C$. Then we have the relation
\[
\delta C = 0 \iff d\tilde{B} = 0.
\] (2.3)
Therefore $\tilde{B}$ defines another symplectic structure independent of $B$. Hence it should be possible to consider the deformation of the dual symplectic structure $\tilde{B}$ by incorporating a dual line bundle $\tilde{L} \to M$. Then an interesting question is what is a physical consequence of the doubling of symplectic structures in Eq. (2.1) due to the Hodge duality.

Let $\tilde{A}$ be a $U(1)$ connection of the dual line bundle $\tilde{L}$ and $\tilde{F} = d\tilde{A}$ its curvature. According to the vector space structure in Eq. (2.1), we identify the curvature $\tilde{F}$ with the Hodge-dual of a four-form $G$, i.e., $\tilde{F} = *G$. The Bianchi identity for $\tilde{L}$ is then equal to the co-closedness of the four-form $G$:
\[
d\tilde{F} = 0 \iff \delta G = 0.
\] (2.4)
Using the nilpotency $\delta^2 = 0$ \cite{2}, the so-called co-Bianchi identity, $\delta G = 0$, can locally be solved by $G = \delta D$ and the connection $\tilde{A}$ of the dual line bundle $\tilde{L}$ can be identified with the Hodge-dual of the five-form connection $D$, viz., $\tilde{A} = - * D$. As the usual line bundle $L$ over a symplectic manifold $(M, B)$, the dual line bundle $\tilde{L}$ will similarly deform the dual symplectic structure $\tilde{B}$ of the base manifold $M$, leading to a new symplectic structure
\[
\tilde{F} \equiv \tilde{B} + \tilde{F} = *(C + G).
\] (2.5)
Hence the dual line bundle $\tilde{L}$ also results in a dynamical symplectic manifold $(M, \tilde{F})$. Recall that the symplectic gauge fields in Eq. (1.21) have been introduced via a Darboux transformation $\phi \in \text{Diff}(M)$ such that $\phi^*(F) = B$. Similarly, one can consider a local coordinate transformation $\tilde{\phi} \in \text{Diff}(M)$ such that $\tilde{\phi}^*(\tilde{F}) = \tilde{B}$. Let us introduce Darboux coordinates $u^a (a = 1, \cdots, 6)$ so that the coordinate transformation $\tilde{\phi} \in \text{Diff}(M)$ is given by
\[
\tilde{\phi} : u^a \mapsto w^a(u) = u^a + \tilde{\theta}^{ab}c_b(u)
\] (2.6)
where $\tilde{\theta} \equiv \tilde{B}^{-1}$ and the dynamical coordinates $c_b(u)$ will be called dual symplectic gauge fields. By using the Darboux coordinates, the coordinate transformation obeying $\tilde{\phi}^*(\tilde{F}) = \tilde{B}$ is explicitly written as
\[
(B_{ab} + \tilde{F}_{ab}(w)) \frac{\partial u^a}{\partial w^\mu} \frac{\partial w^b}{\partial u^\nu} = \tilde{B}_{\mu\nu}.
\] (2.7)
It should be emphasized that the dual symplectic gauge fields in Eq. (2.6) are completely independent of the symplectic gauge fields in Eq. (1.21) to be compatible with the doubling of the vector space in Eq. (2.1).

The dual Poisson structure $\tilde{\theta} = \tilde{B}^{-1}$ defines a new Poisson bracket given by
\[
\{\psi(u), \varphi(u)\}_{\tilde{\theta}} = \tilde{\theta}^{\mu\nu} \frac{\partial \psi(u)}{\partial u^\mu} \frac{\partial \varphi(u)}{\partial u^\nu}
\] (2.8)
for $\psi, \varphi \in C^\infty(M)$. From the Darboux transformation (2.7), one can then deduce the Poisson bracket relation

$$(\tilde{F}^{-1})^{ab}(u) = \{w^a(u), w^b(u)\}_{\tilde{g}} = (\tilde{\theta}(\tilde{B} - \tilde{f})\tilde{\theta})^{ab}(u),$$

(2.9)

where the field strength of dual symplectic gauge fields is defined by

$$f_{ab}(u) = \partial_a c_b(u) - \partial_b c_a(u) + \{c_a(u), c_b(u)\}_{\tilde{g}}$$

(2.10)

with $\partial_a := \frac{\partial}{\partial u^a}$. The quantization $Q$ of the dynamical symplectic manifold $(M, \tilde{F})$ is defined by canonically quantizing a Poisson algebra $(C^\infty(M), \{-, -\}_{\tilde{g}})$ [9]. It leads to another NC $\star$-algebra $A_{\tilde{g}}$ generated by the Moyal-Heisenberg algebra satisfying the commutation relation

$$[u^a, u^b] = i\tilde{\theta}^{ab}.$$  

(2.11)

The NC $\star$-algebra $A_{\tilde{g}}$ is independent of the previous one $A_{\tilde{g}}$ by our construction. For example, a local automorphism of $A_{\tilde{g}}$ defined by $Q : \tilde{\phi} \mapsto D_{\tilde{A}}$ acts on the NC coordinates $u^a$ as [11, 15]

$$D_{\tilde{A}}(u^a) \equiv \hat{W}^a(u) = u^a + \tilde{\theta}^{ab}\hat{C}_b(u) \in A_{\tilde{g}},$$

(2.12)

where $\hat{C}_a = Q(c_a)$ are another NC $U(1)$ gauge fields obtained by quantizing dual symplectic gauge fields $c_a(u)$. The covariant dynamical coordinates $\hat{W}^a(u)$ satisfy the commutation relation

$$[\hat{W}^a, \hat{W}^b] = i(\tilde{\theta}(\tilde{B} - \hat{H})\tilde{\theta})^{ab},$$

(2.13)

where the field strength of NC $U(1)$ gauge fields $\hat{C}_a$ is given by

$$\hat{H}_{ab} = \partial_a \hat{C}_b - \partial_b \hat{C}_a - i[\hat{C}_a, \hat{C}_b].$$

(2.14)

In consequence, there exist two independent NC $\star$-algebras to define a dynamical NC spacetime. They are separately obtained by quantizing the line bundles $L$ and $\tilde{L}$ describing the deformation of symplectic structures in $\Omega^2(M)$ and $\star\Omega^4(M)$, respectively. Since the two vector spaces in Eq. (2.1) are isomorphic to each other so that they should be treated on an equal footing, the exactly same argument for the previous symplectic manifold $(M, F)$ can be equally applied to the dual symplectic manifold $(M, \tilde{F})$. It is straightforward to derive from the Darboux transformation (2.7) the following identity between Dirac-Born-Infeld densities (up to total derivatives) [3]:

$$\sqrt{\det(\tilde{g} + \tilde{F})} = \sqrt{\det(\tilde{G} + \tilde{B})} = \frac{\tilde{g}_s}{\tilde{G}_s} \sqrt{\det(\tilde{G} + \tilde{H} - \tilde{B})},$$

(2.15)  

(2.16)
where \((\tilde{g}, \tilde{G})\) are the Kähler metric of \(\mathbb{C}^3\) and \(\tilde{B}\) its Kähler form and \((\tilde{g}_s, \tilde{G}_s)\) are coupling constants in the dual gauge theories. The identity (2.15) immediately verifies that \(U(1)\) gauge fields on the left-hand side must be a connection of a holomorphic line bundle obeying \(\tilde{F}_{ij} = \tilde{F}_{ij} = 0\) to give rise to a Kähler metric \(\tilde{G}_{\alpha \beta} = \partial_{\alpha} \bar{\partial}_{\beta} \tilde{K}(z, \bar{z})\) on the right-hand side where \(\tilde{K}(z, \bar{z})\) is the Kähler potential of a Kähler manifold \(\tilde{\mathcal{M}}\). Thus the field strength of a holomorphic line bundle \(\tilde{L}\) is given by \(\tilde{F}_{ij} = \partial_i \partial_{\bar{j}} \tilde{\phi}(z, \bar{z})\) where \(\tilde{\phi}(z, \bar{z})\) is a real smooth function on \(\mathbb{C}^3\). The identity (2.15) then demands to identify the real function \(\tilde{\phi}(z, \bar{z})\) with the Kähler potential \(\tilde{K}(z, \bar{z})\) up to holomorphic gauge transformations (see the footnote 1), i.e., \(\tilde{\phi}(z, \bar{z}) = \tilde{K}(z, \bar{z})\) (2.17).

Conversely, if the metric \(\tilde{G}_{\mu \nu}\) is Kähler, \(\tilde{F}\) must be the curvature of a holomorphic line bundle. The identity (2.16) similarly requires that NC \(U(1)\) gauge fields should be a connection of a NC holomorphic line bundle satisfying \(\hat{H}_{ij} = \hat{H}_{ij} = 0\). In particular, if the NC \(U(1)\) gauge fields in Eq. (2.16) are NC Hermitian \(U(1)\) instantons obeying the Hermitian Yang-Mills equation

\[
\hat{H} = - * (\hat{H} \wedge \hat{B}),
\]

(2.18)

the Kähler metric \(\tilde{G}_{\alpha \beta}\) in Eq. (2.15) describes a Calabi-Yau manifold \(\tilde{\mathcal{M}}\) [4].

The emergent Calabi-Yau manifold \(\tilde{\mathcal{M}}\) can be demonstrated on a more concrete basis. As a counterpart of \(\hat{D}_{a}(y) = D_{A}(p_{a})\), let us introduce covariant NC momenta defined by \(\hat{K}_{a}(u) \equiv D_{A}(\tilde{p}_{a}) = \tilde{B}_{ab} \hat{W}^{b}(u)\) where \(\tilde{p}_{a} = \tilde{B}_{ab} u^{b}\). Then they satisfy the commutation relation

\[
-i [\hat{K}_{a}, \hat{K}_{b}]_{\ast} = - \tilde{B}_{ab} + \hat{H}_{ab}.
\]

(2.19)

To bear a close parallel to Eq. (1.34), let us consider the set of NC vector fields defined by

\[
\{ \hat{Z}_{a} \equiv \text{ad}_{\hat{K}_{a}} \in \mathfrak{D} | \hat{K}_{a}(u) = \tilde{p}_{a} + \hat{C}_{a}(u) \in \mathfrak{A}_{\tilde{g}}, a = 1, \cdots, 6 \}.
\]

(2.20)

One can apply the Lie algebra homomorphism (1.33) to Eq. (2.19) to yield the relation [9, 16]

\[
\text{ad}_{\hat{H}_{ab}} = [\hat{Z}_{a}, \hat{Z}_{b}] \in \mathfrak{D}.
\]

(2.21)

After all, the Hermitian Yang-Mills equation (2.18) can be transformed as [4, 20]

\[
[\hat{Z}_{a}, \hat{Z}_{b}] = - \frac{1}{2} T_{ab}^{cd} [\hat{Z}_{c}, \hat{Z}_{d}],
\]

(2.22)

where \(T_{ab}^{cd} = \frac{1}{2} e_{ab}^{cd ef} \tilde{B}_{ef}\) and \(\tilde{B} = 1_{3} \otimes \sqrt{-1} \sigma^{2}\). Following the exactly same calculation given in Ref. [4] (see Appendix A), one can show that the commutative limit of Eq. (2.22) is equivalent to geometric equations for spin connections given by

\[
\tilde{\omega}_{ab} = - \frac{1}{2} T_{ab}^{cd} \tilde{\omega}_{cd}.
\]

(2.23)
Note that the spin connections \( \tilde{\omega}^a \) are determined by solving the torsion-free conditions:

\[
\tilde{T}^a = d\tilde{E}^a + \tilde{\omega}^a \wedge \tilde{E}^b = 0
\]

for a six-dimensional manifold \( \tilde{\mathcal{M}} \) whose metric is given by

\[
ds^2 = \tilde{g}_{\mu\nu}(x)dx^\mu \otimes dx^\nu = \tilde{E}^a \otimes \tilde{E}^a.
\]

It is not difficult to show [4] that the six-dimensional manifold \( \tilde{\mathcal{M}} \) must be a Calabi-Yau manifold if its spin connections satisfy the relation (2.23).

3 Mirror Symmetry of Emergent Geometry

We showed that the doubling of symplectic structures due to the Hodge duality results in two independent classes of NC \( U(1) \) gauge fields by considering the Seiberg-Witten map [13] for each symplectic structure. It may be emphasized that this result is a direct consequence of the well-known Hodge duality stating the doubling of two-form vector spaces in Eq. (2.1). As a result, emergent gravity leads to an intriguing conclusion that the variety of six-dimensional manifolds emergent from NC \( U(1) \) gauge fields is doubled. Note that a Calabi-Yau manifold \( X \) always arises with a mirror pair \( Y \) obeying the mirror relation [21]

\[
h^{1,1}(X) = h^{2,1}(Y), \quad h^{2,1}(X) = h^{1,1}(Y)
\]

where \( h^{p,q}(M) = \dim H^{p,q}(M) \geq 0 \) is a Hodge number of a Calabi-Yau manifold \( M \). When we conceive the emergent Calabi-Yau manifolds from the mirror symmetry perspective, we cannot help investigating how the doubling for the variety of emergent geometry is related to the mirror symmetry of Calabi-Yau manifolds.

Suppose that \( M \) is a six-dimensional orientable manifold to equip a globally defined volume form. This volume form allows us to define the Hodge dual operator \( * : \Omega^k(M) \to \Omega^{6-k}(M) \) on a vector space

\[
\Lambda^*M = \bigoplus_{k=0}^{6} \Omega^k(M),
\]

where \( \Omega^k(M) \) is the space of \( k \)-forms on \( T^*M \). Consider a subspace of nondegenerate, closed two-forms and co-closed four-forms in \( \Lambda^*M \) denoted by \( S^2(M) \) and \( S^4(M) \), respectively. Let us take a direct sum

\[
S(M) \equiv S^2(M) \oplus *S^4(M).
\]

If \( \omega \in S(M) \), then \( \omega \) is a closed, \( d\omega = 0 \), and nondegenerate two-form. Therefore, \( \omega \) is a symplectic structure on \( M \). According to the Hodge decomposition theorem [2], \( \omega \in S(M) \) is decomposed as follows:

\[
\omega = \omega_H + d\alpha + \delta\beta,
\]

10
where $\omega_H$ is a harmonic two-form and $\alpha \in \Omega^1(M)$, $\beta \in \Omega^3(M)$. A harmonic $k$-form $\omega_H \in \Omega^k(M)$ is defined by $\Delta \omega_H = 0$ where the Laplace-Beltrami operator $\Delta : \Omega^k(M) \rightarrow \Omega^k(M)$ is defined by

$$\Delta = d\delta + \delta d.$$ (3.5)

A $k$-form $\omega_H$ is harmonic if and only if $d\omega_H = 0$ and $\delta \omega_H = 0$. Then $\omega_H$ is a unique harmonic representative in the $k$-th de Rham cohomology $H^k(M)$ [2]. We remark that the Hodge decomposition on the exterior algebra (3.2) is a canonical decomposition given a globally defined volume form from which an oriented inner product is defined and it is necessary to consider the direct sum (3.3) to realize the Hodge decomposition (3.4) for a symplectic structure.

Since emergent gravity is based on the symplectic geometry or more generally a Poisson geometry [9], it is necessary to exhaust, at least, all possible symplectic structures to realize a complete emergent geometry. Therefore, it is demanded to consider the direct sum (3.3) to exhaust all possible symplectic structures. For instance, $F = B + F$ in Eq. (1.18) and $\tilde{F} = \tilde{B} + \tilde{F} = \ast(C + G)$ in Eq. (2.5) belong to the vector space $S(M)$. In general, as we have shown before, the vector space (3.3) can be understood as a deformation space of primitive symplectic and dual symplectic structures $(B, \tilde{B})$ which is locally described by a line bundle $L$ over $(M, B)$ and a dual line bundle $\tilde{L}$ over $(M, \tilde{B})$. We verified how the doubling of symplectic structures in Eq. (3.3) due to the Hodge duality leads to two independent classes of NC $U(1)$ gauge fields and results in the doubling of emergent geometry. For example, we showed in Sect. 2 that NC Hermitian $U(1)$ instantons as a solution of the Hermitian Yang-Mills equation (2.18) defined by dual NC $U(1)$ gauge fields $\hat{C}_a(u)$ give rise to Calabi-Yau manifolds in the commutative limit that are independent of Calabi-Yau manifolds emergent from the line bundle $L$ over a symplectic manifold $(M, B)$. In other words, the variety of emergent Calabi-Yau manifolds is doubled thanks to the Hodge duality $\ast : S^4(M) \rightarrow S^2(M)$.

Note that the Euler characteristic of a Calabi-Yau manifold $M$ is given by [21]

$$\chi(M) = 2(h^{1,1}(M) - h^{2,1}(M)).$$ (3.6)

Since two classes of emergent Calabi-Yau manifolds are completely independent of each other, it should be possible to arrange a pair of Calabi-Yau manifolds $(X, Y)$ such that $\chi(X) = -\chi(Y)$. (A very similar doubling for the variety of Calabi-Yau manifolds was observed in [22].) Because of the fact $h^{p,q}(M) = \dim H^{p,q}(M) \geq 0$, $\chi(X) = -\chi(Y)$ implies the mirror relation (3.1). Consequently, the emergent gravity suggests a beautiful picture that the mirror symmetry of Calabi-Yau manifolds simply originates from the doubling of symplectic structures in Eq. (3.3). Furthermore, according to the Hodge decomposition theorem, a generic deformation of a symplectic structure can be written as the form (3.4). Therefore, the emergent gravity picture implies that the mirror symmetry of Calabi-Yau manifolds can be understood as the Hodge theory for the deformation of symplectic and dual symplectic structures.
4 Discussion

The identification (1.19) implies a general result [3, 4] that a holomorphic line bundle with a non-degenerate curvature two-form is equivalent to a six-dimensional Kähler manifold. Since the real function $\phi(z, \bar{z})$ will be determined by solving the equations of motion of $U(1)$ gauge fields, it means that (generalized) Maxwell’s equations for $U(1)$ gauge fields on a holomorphic line bundle can be translated into Einstein’s equations for a Kähler manifold. A particular case was verified in [3, 4] that the Einstein equations for Calabi-Yau $n$-folds for $n = 2$ and 3 are equivalent to the equations of motion for the commutative limit of NC $U(1)$ instantons in four and six dimensions, respectively. Recall that the metric for a Kähler manifold is basically determined by a single function, the so-called Kähler potential, although the gluings described in the footnote 1 must be implemented to have a globally defined metric. As a result, the Ricci tensor of a Kähler manifold is extremely simple and it is given by [2]

$$R_{\alpha\beta} = -\frac{\partial^2 \ln \det G_{\gamma\delta}}{\partial z^\alpha \partial \bar{z}^\beta}. \quad (4.1)$$

Using the identity (1.19), it must be possible to relate the Ricci tensor (4.1) to some equations of $U(1)$ gauge fields on a holomorphic line bundle. It will be interesting to find an explicit form of the equations for holomorphic $U(1)$ gauge fields.

So far we have assumed that a complex manifold $M$ is noncompact, e.g., $\mathbb{C}^3$. It is desirable to generalize the results in this paper to compact complex manifolds, e.g., $T^6$, $T^2 \times K3$, and $\mathbb{C}P^3$, which are all compact symplectic (i.e., Kähler) manifolds. We can put a holomorphic line bundle on such a compact Kähler manifold. Then, similarly to the noncompact case, the line bundle will deform an underlying symplectic (i.e., Kähler) structure of the base manifold and end in a dynamical symplectic manifold. The resulting symplectic structure can be identified with the Kähler form of a Kähler manifold emergent from the holomorphic line bundle over a compact complex manifold. Therefore, we still have a local identification (1.19) even for a compact manifold. However, an explicit construction of Poisson algebras and covariant connections on a compact Kähler manifold will be much more difficult than a noncompact case. In particular, the gluing of coordinate patches for a holomorphic atlas of a compact manifold, described in the footnote 1 will be more nontrivial compared to, e.g., $\mathbb{C}^3$. The quantization of a compact Kähler manifold will also be a more challenging issue. Thus a sophisticated mathematical tool for emergent geometry would be requested for the compact case. Nevertheless, the conclusion for the noncompact case will be true even for compact Kähler manifolds because main features such as Eqs. (3.3) and (3.4) invariably hold for any symplectic manifold.

In four dimensions, it has been possible to accomplish an explicit test of emergent gravity with known solutions in gravity and gauge theory [23, 24]. In higher dimensions, it becomes more difficult to obtain an explicit solution in gravity as well as gauge theory. Fortunately, some solutions in six dimensions are explicitly known for Ricci-flat Kähler manifolds [25] and NC Hermitian $U(1)$ instantons [19, 26]. Therefore, it will be interesting to examine an explicit test of six-dimensional emergent gravity for the known solutions in both gravity and gauge theory.
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