Distributional framework for solving fractional differential equations

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We analyse the solvability of a special form of distributed order fractional differential equations

\[
\int_0^2 \phi_1(\gamma) D^\gamma y(t) \mathrm{d}\gamma = \int_0^2 \phi_2(\gamma) D^\gamma z(t) \mathrm{d}\gamma, \quad t > 0,
\]

within $\mathcal{S}_+'$, the space of tempered distributions supported by $[0, \infty)$.

\textbf{Keywords:} distributed order fractional differential equations; tempered distribution; Laplace transform

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1. Motivation and introduction

We consider a distributed order fractional differential Equation (1) which arises in the theory of constitutive equations for viscoelastic bodies. $\phi_1, \phi_2$ are certain functions or distributions which characterize a material under consideration and, in general, are determined from experiments. $D^\gamma$, $\gamma \in \mathbb{R}$, is the left Riemann–Liouville operator of fractional differentiation or integration defined as follows.

Denote by $L_{\text{loc}}^1(\mathbb{R})$ the space of locally integrable functions $y$ on $\mathbb{R}$ such that $y(t) = 0$, $t < 0$. Then for $y \in L_{\text{loc}}^1(\mathbb{R})$ the left fractional integral of order $\gamma > 0$ is defined by

\[
I^\gamma y(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t - \tau)^{\gamma - 1} y(\tau) \mathrm{d}\tau, \quad t > 0.
\]

Here $\Gamma$ is the Euler gamma function. If $\gamma = 0$ then $I^0 y := y$. It can be shown (cf. [12]) that for $y \in L_{\text{loc}}^1(\mathbb{R})$ $\lim_{\gamma \to 0} I^\gamma y(t) = y(t)$, $t \in \mathbb{R}$ almost everywhere.

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Denote by $AC^k(\mathbb{R}_+)$ the space of functions $y$ such that $y$ has continuous derivatives on $\mathbb{R}_+ = \{x \in \mathbb{R}; x \geq 0\}$ up to the order $k - 1$ and $k$th derivative is locally integrable function. We extend such functions to $\mathbb{R}$ so that $y(t) = 0, t < 0$.

Let $y \in AC^k(\mathbb{R}_+)$. Riemann–Liouville’s fractional derivative of order $\gamma \geq 0, \gamma \leq k$ for some $k \in \mathbb{N}$, is defined by

$$D^\gamma y(t) := \frac{d^k}{dt^k} I^{k-\gamma} y(t), \quad t > 0.$$ 

It follows that $D^\gamma y \in L^1_{\text{loc}}(\mathbb{R})$. We refer to Section 2 for the definition of $D^\gamma y, y \in S'_+,$ and $\gamma \in \mathbb{R}$.

Note that $D^\gamma I^\gamma y = y$ for $y \in L^1_{\text{loc}}(\mathbb{R})$ and $I^\gamma D^\gamma y = y, \gamma > 0$ in the sense of tempered distribution. We sometimes denote $D^{-\gamma} y = I^\gamma y, \gamma > 0$.

Let $\phi$ be continuous function in $[c, d] \subset [0, k], k \in \mathbb{N}$. Distributed order fractional derivative of $y \in AC^k(\mathbb{R}_+)$ is given by

$$\int_c^d \phi(\gamma) D^\gamma y(t) \, d\gamma.$$ 

Equation (1) models various physical processes. For example, if it models a viscoelastic body then Equation (1) represents a constitutive equation of a material and connects strain $y(t)$ with corresponding stress $z(t)$ at time instant $t \geq 0$. For a standard linear viscoelastic body the constitutive equation is given as

$$y(t) + by^{(1)}(t) = z(t) + az^{(1)}(t),$$ 

where $(\cdot)^{(1)} = d/dt(\cdot)$ and $a, b$ are experimentally determined constants with the restriction $0 < a < b$ following from the second law of thermodynamics. A slight generalization (see [2] and references therein) of this equation is achieved by replacing the first derivative by a derivative of real order $\alpha > 0$

$$y(t) + bD^\alpha y(t) = z(t) + aD^\alpha z(t),$$ 

where, again $0 < a < b$. If $0 < \alpha < 1$ then Equation (2) represents viscoelastic effects while for $1 < \alpha < 2$, Equation (2) describes viscoinertial effects of a material. Standard procedure in building rheological models is to use more than one derivative on each side of constitutive equation. When this is done Equation (2) becomes

$$\sum_{n=0}^N b_n D^{\beta_n} y(t) = \sum_{m=0}^M a_m D^{\alpha_m} z(t),$$ 

where $M, N \in \mathbb{N}, a_m, b_n \in \mathbb{R}$ and $\alpha_m, \beta_n \in \mathbb{R}, 0 \leq \alpha_m, \beta_n \leq 2$.

Equation (3) is interpreted in [1] as a Riemann sum. Moreover, in [1] the constitutive equation of a linear viscoelastic body is proposed in a ‘distributed’ order form as

$$\int_0^2 \phi_1(\gamma) D^\gamma y(t) \, d\gamma = \int_0^2 \phi_2(\gamma) D^\gamma z(t) \, d\gamma, \quad t > 0.$$ 

In model (4) all derivatives of the stress $D^\gamma z$ depend on all derivatives of the strain $D^\gamma y$ for $\gamma \in [c, d]$. Since the upper bound in integrals in Equation (4) is two, both, viscoelastic and viscoinertial effects are included. The presence of integral on the left-hand side indicates, as experiments show, that dissipation properties depend on the order of the derivative. The integral on the right-hand side is a consequence of the known principle of equipresence.

In this paper we are looking for an $S'_+$ solution $z$ to Equation (4) for a given but arbitrary $y \in S'_+$. Such solution will be used in [6] for solving a differential equation of motion coupled with constitutive equation (4).
In Section 2 we extend the results obtained in [5] concerning integral in Equation (4). Afterwards we define distributed order fractional derivative in $S'_+$ and derive its main properties. In Section 3 we state without proof (which is given in [6]) a theorem on the existence and uniqueness of a solution to a linear fractional differential equation in the frame of $S'_+$. Also we derive properties of such solution. In Section 4 we connect the condition for the uniqueness with a dissipation inequality that guarantees physical admissibility of a Equation (4).

We note that the models with distributed order derivatives were analysed for example in [3,7,8,10,11].

2. Distributed order fractional derivative

We denote by $S(\mathbb{R})$ the space of rapidly decreasing functions in $\mathbb{R}$ and by $S'(\mathbb{R})$ its dual, the space of tempered distributions; $S'_+(\mathbb{R})$ denotes its subspace consisting of distributions supported by $[0, \infty)$. In the sequel we drop $\mathbb{R}$ in the notation. We consider in $S'_+$ the family

$$f_\alpha(x) = \begin{cases} H(x) \frac{x^{\alpha-1}}{\Gamma(\alpha)}, & x \in \mathbb{R}, \ \alpha > 0, \\ \frac{d^N}{dx^N} f_{\alpha + N}(x), & \alpha \leq 0, \ \alpha + N > 0, \ N \in \mathbb{N}, \end{cases} \quad (5)$$

where $H$ is Heaviside’s function. It is known that $f_\alpha * f_\beta = f_{\alpha + \beta}, \ \alpha, \beta \in \mathbb{R}$. The convolution operator $f_\alpha *$ in $S'_+$ is the operator of fractional differentiation for $\alpha < 0$ and of fractional integration for $\alpha > 0$. It coincides with the operator of derivation for $-\alpha \in \mathbb{N}$ and integration for $\alpha \in \mathbb{N}$. Let $\alpha > 0$ and $y \in L^1_{\text{loc}}(\mathbb{R})$. Then $I^\alpha y = f_\alpha * y$. Let $y \in AC^k(\mathbb{R}_+)$ and $0 < \alpha \leq k$. Then $D^\alpha y = f_{-\alpha} * y$.

Recall, if $y \in S'_+$ then its Laplace transform is defined by

$$\hat{y}(s) = \mathcal{L}y(s) = \langle y(t), \varphi(t)e^{-st} \rangle, \ \Re s > 0, \ \alpha \in \mathbb{R},$$

where $\varphi \in C^\infty, \varphi = 1$ on $(-\alpha, \infty)$ and $\varphi = 0$ in $(-\infty, -2\alpha), \alpha > 0$. Note that $\mathcal{L}y$ is an analytic function for $\Re s > 0$ and that the definition of $\mathcal{L}y$ does not depend on a chosen function $\varphi$ with given proprieties. We will often use the identity

$$\mathcal{L}(f_\alpha * y)(s) = \frac{1}{s^\alpha} \hat{y}(s), \ \Re s > 0.$$

First we analyse integral $\int_{\text{supp} \varphi} \Phi(y) D^\alpha y(\cdot) dy$. To do this we examine the mapping $\alpha \mapsto D^\alpha y : \mathbb{R} \rightarrow S'_+$, for given $y \in S'_+$ (in [5] we have considered $y \in L^1_{\text{loc}}(\mathbb{R}) \cap S'_+$).

Proposition 2.1

(a) Let $\alpha \in \mathbb{R}$ be fixed. Then the mapping $y \mapsto D^\alpha y$ is linear and continuous from $S'_+$ to $S'_+$.

(b) Let $y \in S'_+$ be fixed. Then $\alpha \mapsto D^\alpha y$ is a smooth mapping from $\mathbb{R}$ to $S'_+$.

(c) The mapping $(\alpha, y) \mapsto D^\alpha y$ is continuous from $\mathbb{R} \times S'_+$ to $S'_+$.

Proof (a) The continuity of $y \mapsto D^\alpha y = f_{-\alpha} * y$ is clear since for $g \in S'_+$, $f \mapsto f * g$ is a continuous mapping of $S'_+$ into $S'_+$.

(b) It is known that there exists a continuous function $F$, supp $F \subset [0, \infty)$ and $k \in \mathbb{N}$ such that $|F(x)| < C(1 + |x|)^k, x \in \mathbb{R}$ and $y = D^k F$. So the mapping $\alpha \mapsto D^\alpha y$ equals $\alpha \mapsto D^{\alpha + k} F$. By [5, Proposition 1] we know that for fixed $k$ and $\alpha \in \mathbb{R}, \ \alpha + k \mapsto D^{\alpha + k} F$ is smooth so the same hold for $\alpha \mapsto D^{\alpha + k} F$. 


(c) Since $\mathcal{S}$ is Fréchet space as well as locally convex, the separate continuity proved in (a) and (b) imply joint continuity (c.f. [13, Corollary to Theorem 34.1]).

By $\mathcal{E}'(\mathbb{R})$ is denoted the space of compactly supported distributions i.e. the dual space of $\mathcal{E}(\mathbb{R}) = C_0^\infty(\mathbb{R})$.

**Definition 2.2** Let $\phi \in \mathcal{E}'(\mathbb{R})$ and $y \in \mathcal{S}'_+$. Then $\int_{\text{supp}\phi} \phi(y) D^r y \, dy$ is defined as an element of $\mathcal{S}'_+$ by

$$\left( \int_{\text{supp}\phi} \phi(y) D^r y(t) \, dy, \varphi(t) \right) = \langle \phi(y), \langle D^r y(t), \varphi(t) \rangle \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}). \tag{6}$$

Such defined distribution is called distributed order fractional derivative.

By Proposition 2.1, part (b), $\gamma \mapsto D^r y : \mathbb{R} \to \mathcal{S}'_+$ is smooth as well as $\gamma \mapsto \langle D^r y(t), \varphi(t) \rangle : \mathbb{R} \to \mathbb{R}$. Since $\mathcal{S}$ is a Fréchet space it follows that in its dual space the strong and weak boundedness are the same, thus a linear functional defined by Equation (6) is continuous from $\mathcal{S}$ to $\mathbb{C}$ and therefore is a tempered distribution supported by $[0, \infty)$.

The following two examples are often used in applications.

**Example 2.3**

(a) Let $\gamma_i \in \mathbb{R}, i \in \{0, 1, \ldots, k\}$ and $\phi(\cdot) = \sum_{i=0}^k a_i \delta^{(i)}(\cdot - \gamma_i)$. Then Equation (6) gives

$$\int_{\text{supp}\phi} \phi(y) D^r y(\cdot) \, dy = \sum_{i=0}^k a_i D^r y(\cdot) \quad \text{in } \mathcal{S}'_+.$$

(b) Let $\phi$ be a continuous function in $[c, d] \subset \mathbb{R}$ for some $c < d$, then

$$\int_c^d \phi(y) D^r y(\cdot) \, dy = \lim_{N \to \infty} \sum_{i=1}^N \phi(\gamma_i) D^r y(\cdot) \Delta \gamma_i \quad \text{in } \mathcal{S}'_+,$$

where $\gamma_i$ are points of interval $[c, d]$ in usual definition of the Riemann sum for the integral.

**Proposition 2.4** Let $\phi \in \mathcal{E}'(\mathbb{R})$ and $y \in \mathcal{S}'_+$. Then:

(a) $y \mapsto \int_{\text{supp}\phi} \phi(y) D^r y \, dy$ is a linear and continuous mapping from $\mathcal{S}'_+$ to $\mathcal{S}'_+$.

(b) $\mathcal{L} \left( \int_{\text{supp}\phi} \phi(y) D^r y \, dy \right)(s) = \hat{y}(s) \langle \phi(y), s^r \rangle, \quad \text{Re } s > 0.$

(c) If $\phi$ is continuous function on $[c, d]$ and $\phi(\gamma) = 0$ for $\gamma \not\in [c, d]$ then

$\mathcal{L} \left( \int_c^d \phi(y) D^r y \, dy \right)(s) = \hat{y}(s) \int_c^d \phi(y) s^r \, dy, \quad \text{Re } s > 0.$

**Proof** (a) Clearly, this mapping is linear. Let $y_n \to 0$ in $\mathcal{S}'_+$. Then $\phi(y), D^r y_n, \varphi \to 0$, as $n \to \infty$, since by Proposition 2.1 part (a), $D^r y_n, \varphi \to 0$, as $n \to \infty$. 

By $E'(\mathbb{R})$ is denoted the space of compactly supported distributions i.e. the dual space of $E(\mathbb{R}) = C_0^\infty(\mathbb{R})$.
(b) By the definition,
\[
\mathcal{L}\left(\int_{\text{supp } \phi} \phi(\gamma) D^y y \, d\gamma\right)(s) = \left\langle \int_{\text{supp } \phi} \phi(\gamma) D^y y(t) \, d\gamma, \varphi(t)e^{-st} \right\rangle = \left\langle \phi(\gamma), \left\langle D^y y(t), \varphi(t)e^{-st} \right\rangle \right\rangle = \left\langle \phi(\gamma), s^y \hat{y}(s) \right\rangle, \quad \text{Re } s > 0.
\]

(c) In the case that \( \phi \) is continuous we have
\[
\left\langle \phi(\gamma), s^y \hat{y}(s) \right\rangle = \int_{\text{supp } \phi} \phi(\gamma)s^y \hat{y}(s) \, d\gamma
\]
and therefore the assertion follows. \( \blacksquare \)

If we assume that \( y, z \in S'_+ \) in Equation (4), put \( \phi = \phi_2 \) and \( g = \int_{\text{supp } \phi_1} \phi_1(\gamma) D^y z \) then the solvability of Equation (4) with respect to \( z \) reduces to the solvability of
\[
\int_{\text{supp } \phi} \phi(\gamma) D^y z = g, \quad g \in S'_+.
\] (7)

3. Linear fractional differential equation in \( S'_+ \)

Assuming that \( g \in S'_+ \) and that \( \phi \) is of the form as in Example 2.3, Equation (7) becomes
\[
\sum_{i=0}^{k} a_i D^{\gamma_i} z = g \quad \text{in } S'_+.
\] (8)

We suppose that \( \gamma_i \in [0, 2) \) such that \( \gamma_0 > \gamma_i > \gamma_{i+1} > \gamma_k, i \in \{1, \ldots, k-1\} \).

**Theorem 3.1** Equation (8) has a unique solution \( z \in S'_+ \) if and only if
\[
(A_0) \quad \sum_{i=0}^{k} a_i s^{\gamma_i} \neq 0, \quad s \in \mathbb{C}_+ = \{s \in \mathbb{C}; \text{Re } s > 0\}.
\]

The proof is given in [6]. The solution to Equation (8) that is obtain in Theorem 3.1 is given by \( z = l \ast g \), where
\[
l(t) = \mathcal{L}^{-1}\left(\frac{1}{\sum_{i=0}^{k} a_i s^{\gamma_i}}\right)(t), \quad t > 0,
\] (9)
is a fundamental solution to Equation (8) i.e. solution to \( \sum_{i=0}^{k} a_i D^{\gamma_i} y = \delta \).

The following lemma gives main properties of \( l \) defined by Equation (9).

**Lemma 3.2** Assume \( (A_0) \). Let \( \gamma_i \in [0, 2) \) and \( \gamma_0 > \gamma_i > \gamma_{i+1} > \gamma_k, \) for all \( i \in \{1, \ldots, k-1\} \). Let \( l \) be defined by Equation (9) and \( l(t) = 0, \) \( t < 0. \) Then:

(i) \( l \) is a locally integrable function in \( \mathbb{R} \).
(ii) Moreover, \( l \) is absolutely continuous in \( \mathbb{R} \), if \( \gamma_0 - \gamma_k > 1 \).
Proof  (i) Let \( \gamma_k = 0 \) and \( a_k \neq 0 \). Consider the integral
\[
\int_{\Gamma} \frac{e^{st}}{\sum_{i=0}^{k} a_i s^{\gamma_i}}, \quad t > 0,
\]
where \( \Gamma = \bigcup_{i=1}^{5} \Gamma_i \) and for arbitrarily chosen \( R > 0, 0 < \varepsilon < R \) and \( x_0 > 0 \), \( \Gamma_i \) are given by
\[
\Gamma_0 : \{ z; \Re z = x_0; 0 < \arg z < \phi_0 = \arcsin \frac{x_0}{R} \};
\]
\[
\Gamma_1 : z = Re^{i\phi}, -\phi_0 < \phi_0 \leq \phi < \pi; \quad \Gamma_2 : z = Re^{i\phi}, -\pi < \phi \leq -\phi_0 < 0;
\]
\[
\Gamma_3 : z = xe^{i\phi}, -\pi < \phi < \pi; \quad \Gamma_4 : z = xe^{i\phi}; \quad \Gamma_5 : z = xe^{-i\phi}, \quad x \in (\varepsilon, R).
\]
By the Cauchy residue theorem, letting \( \varepsilon \to 0 \) and \( R \to \infty \), one obtains
\[
l(t) = \sum_{s=m, m=1}^{n} \Re s \left\{ \frac{e^{st}}{\sum_{i=0}^{k} a_i s^{\gamma_i}} \right\} + l_0(t), \quad t > 0,
\]
where \( l_0(t) \) is defined by Equation (11).

Let 0 \leq a \leq b. Then
\[
\int_{a}^{b} l_0(t) \, dt = \int_{0}^{\infty} \left( e^{-sa} - e^{-sb} \right) \frac{1}{s} r(s) \, ds.
\]
Since \( \gamma_k = 0 \), this integral is finite in a neighbourhood of \( s = 0 \). In a neighbourhood of \( s = \infty \) we have \( r(s)/s \sim 1/s^{\gamma_0+1} \). Thus \( \gamma_0 + 1 > 1 \) implies that the integral in Equation (12) is finite. Therefore, \( l_0 \) is locally integrable. By Equation (10) and the fact that \( \Re s_m < 0 \) (by \( (A_0) \)), we obtain that \( l \) is locally integrable.

Let \( \gamma_k > 0 \). Then
\[
l = \mathcal{L}^{-1} \left( \frac{1}{s^{\gamma_k}} \right) \ast \mathcal{L}^{-1} \left( \frac{1}{\sum_{i=0}^{k} a_i s^{\gamma_i-\gamma_k}} \right) = f_{\gamma_k} \ast l_1,
\]
where
\[
l_1 = \mathcal{L}^{-1} \left( \frac{1}{\sum_{i=0}^{k} a_i s^{\beta_i}} \right), \quad \beta_i = \gamma_i - \gamma_k, \quad i \in \{0, 1, \ldots, k\}, \quad \Re s > 0,
\]
and \( f_{\gamma_k} \) is defined by Equation (5). Note that \( \beta_k = 0 \). By the first part of the proof, \( l_1 \) is a locally integrable function. Since \( f_{\gamma_k} \) is locally integrable, \( l \) is locally integrable as the convolution of two locally integrable functions.

(ii) Let \( \gamma_k = 0 \). Then Equation (11) is finite in a neighbourhood of \( s = 0 \). In a neighbourhood of \( s = \infty \) we have that \( r(s) \sim 1/s^{\gamma_0} \). Since \( \gamma_0 > 1 \), the integrand in Equation (11) is integrable for all
$t > 0$ and Equation (11) is finite. Let $t_0 > 0$. Since $|e^{-st}r(s)| \leq e^{-st_0}|r(s)| := g(s)$ for all $t > t_0$ and $g \in L^1((0, \infty))$, by the classical theory we obtain that Equation (11) defines a continuous function for $t > t_0$. It follows that $l_0$ and $l$ (by Equation (10)) are continuous for $t > 0$. Further on, since $|\partial_t(e^{-st}r(s))| \leq e^{-st_0}|sr(s)| := g_1(s)$, for all $t > t_0$ and $g_1 \in L^1(0, \infty)$, we obtain that $l_0$ is differentiable and

$$l'_0(t) = \int_0^\infty (-s)e^{-st}r(s), \quad t > 0.$$  

Since $-sr(s) \sim 1/s^{\gamma_0-1}$ in a neighbourhood of $s = \infty$, as in (i) we show that $l'_0$ is a locally integrable function. Therefore, the derivative $l'_0$ exists and it is a locally integrable function. It means that $l$ is absolutely continuous.

For $\gamma_k > 0$ we proceed as in (i) and obtain $l = f_{\gamma_k} * l_1$ with $l_1$ absolutely continuous. Therefore, $l$ is also absolutely continuous. $\blacksquare$

**Remark 3.3** If $\gamma_i \in [0, \infty)$ and if $\gamma_0 > p, p \in \mathbb{N}$, then $l$ is continuous in $\mathbb{R}$ as well as its derivatives up to order $p - 1$ while the $p$th derivative is a locally integrable function, i.e. $l \in AC^p$.

4. Comments from mechanics and further applications

Equation (8) represents a constitutive equation of a viscoelastic body. We will show that in the case when there exist $s_0 \in \mathbb{C}_+$ such that $\sum_{i=0}^k a_i s_0^\gamma_i = 0$ it follows that the dissipation inequality (14), (see [9]) is violated. The dissipation inequality requires that for any $T > 0$, any $y$ and $z$ the solution to $\sum_{i=0}^k a_i D^\gamma_i z(t) = y(t), \quad t > 0$, the dissipation work, $A_d$ is nonnegative, i.e.

$$A_d = \int_0^T z(t)y'(t) \, dt \geq 0. \quad (14)$$

Let $T > 0$ and $y(t) = H(t) - H(t - \tau), \quad 0 < \tau < T, \quad t > 0$. Then

$$z(t) = L^{-1}\left(\frac{1}{s} - \frac{1}{s \sum_{i=0}^k a_i s_0^\gamma_i} - \frac{e^{-ts}}{s \sum_{i=0}^k a_i s_0^\gamma_i}\right) = \int_0^t g(u) \, du, \quad t > 0,$$  

(15)

where $g(u) = l(u) - l(u - \tau)$ and $l$ is the fundamental solution to Equation (8) given by Equation (9). Assume that $w(s_0) = \sum_{i=0}^k a_i s_0^\gamma_i = 0$ for $s_0 = u + iv, \quad u > 0$. Then by Equation (10)

$$l(t) = \frac{e^{st}}{w^{(1)}(s)}\bigg|_{s=s_0} + \sum_{j=1}^k \frac{e^{st}}{w^{(1)}(s_j)}\bigg|_{s=s_j} + l_0(t), \quad t > 0.$$  

Further, note that

$$\frac{e^{st}}{w^{(1)}(s)}\bigg|_{s=s_0} = \frac{e^{ut}[\cos(\nu t) + i \sin(\nu t)]}{w^{(1)}(s_0)}, \quad t > 0,$$  

(16)

represents oscillations with increasing amplitudes. Inserting Equation (15) in Equation (14) we obtain

$$A_d = \lim_{t \to 0} \int_0^t g(u) \, du - \int_0^\tau g(u) \, du \geq 0, \quad \tau, t > 0.$$

(17)

It is obvious that due to the presence of the term (16) in $g(t)$ the inequality (17) could be violated by a suitable choice of $\tau$. 


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