Torsion of abelian varieties and Lubin-Tate extensions

Yoshiyasu Ozeki

July 8, 2019

Abstract
We show that, for an abelian variety defined over a $p$-adic field $K$ which has potential good reduction, its torsion subgroup with values in the composite field of $K$ and a certain Lubin-Tate extension over a $p$-adic field is finite.

1 Introduction
Let $p$ be a prime number and $A$ an abelian variety over a $p$-adic field $K$ (here, a $p$-adic field is a finite extension of $\mathbb{Q}_p$). For an algebraic extension $L=K$, we denote by $A(L)$ the group of $L$-rational points of $A$ and also denote by $A(L)_{\text{tor}}$ its torsion subgroup. We are interested in determining whether $A(L)_{\text{tor}}$ is finite or not. The most basic result is given by Mattuck [Ma]: $A(L)_{\text{tor}}$ is finite if $L$ is a finite extension of $K$. Thus our main interest is the case where $L$ is an infinite algebraic extension of $K$. For this, Imai’s result [Im] is well known. He showed that $A(K(\mu_{p^\infty}))_{\text{tor}}$ is finite if $A$ has potential good reduction, where $\mu_{p^\infty}$ denotes the group of $p$-power roots of unity in a fixed separable closure $\overline{K}$ of $K$. Since the field $K(\mu_{p^\infty})$ is the composite field of $K$ and the Lubin-Tate extension over $\mathbb{Q}_p$ associated with a uniformizer $\pi$ of $\mathbb{Q}_p$, we naturally have the following question.

Question. Let $A$ be an abelian variety over a $p$-adic field $K$. Let $k$ be the Lubin-Tate extension associated with a uniformizer $\pi$ of a $p$-adic field $k$. Then, is $A(K_k)_{\text{tor}}$ finite?

In the case of Imai’s theorem ($k=\mathbb{Q}_p$ and $\pi=\pi$), the answer of the question is affirmative for potential good reduction cases, that is, the case where $A$ has potential good reduction. However, the question sometimes has a negative answer. For example, if $A$ is a Tate curve over $K$, $k=\mathbb{Q}_p$ and $\pi=\pi$, then $A(K_k)[p^\infty] = A(K(\mu_{p^\infty})[p^\infty]$ is clearly infinite. We also have an example even for potential good reduction cases as given in Remark 2.10.

The aim of this paper is to give a sufficient condition on $k$ and $\pi$ so that the question has an affirmative answer for potential good reduction cases. Let $k$, $\pi$ and $\sigma$ be as above. Let $q$ be the order of the residue field of $k$. We denote by $k_G$ the Galois closure of $k/\mathbb{Q}_p$. We put $d_G = [k_G : \mathbb{Q}_p]$ and denote by $e_G$ the ramification index of the extension $k_G/k$. We fix an embedding $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}_p$.

Our main result is as follows (see Definitions 2.1 and 2.2 for some undefined notion).

Theorem 1.1. Let $A$ be an abelian variety over a $p$-adic field $K$ with potential good reduction. If $\text{Nr}_{k/\mathbb{Q}_p}(\pi)$ is not a $q$-Weil integer of weight $sd_G/t$ for any integers $1 \leq s \leq e_G$ and $1 \leq t \leq sd_G$, then $A(k_k)_{\text{tor}}$ is finite.

Applying Theorem 1.1 to the case where $k=\mathbb{Q}_p$ and $\pi=\pi$, we can recover Imai’s theorem. We should note that there is another generalization of Imai’s theorem which is given by Kubo and Taguchi [KT]. The main result of loc. cit. states that the torsion subgroup of $A(K(K^{1/p^\infty})$ is
finite, where $A$ is an abelian variety over $K$ with potential good reduction and $K(K^{1/p^\infty})$ is the extension field of $K$ by adjoining all $p$-power roots of all elements of $K$.

For the proof of the above theorem, the essential difficulty appears in the finiteness of the $p$-power torsion part $A(K_{k_\pi})[p^\infty]$ of $A(K_{k_\pi})$. For this, we proceed our arguments in more general settings. We study not only abelian varieties but also (general) proper smooth varieties.

**Theorem 1.2.** Let $X$ be a proper smooth variety over a $p$-adic field $K$ with potential good reduction. Let $V$ be a $\Gal(K/K)$-stable subquotient of $H^0_p(X, \mathbb{Q}_p(r))$ with $i \neq 2r$. Assume that $V^{\Gal(\overline{K}/L)} \neq 0$ for some finite extension $L/K_{k_\pi}$. Then $\Nr_{L/K_{k_\pi}}(\pi)$ is a $q$-Weil number of weight $-(i - 2r)/h$ for some non-zero $h \in [-i + r, r] \cap \left(\bigcup_{s \in \mathbb{Z}, 1 \leq s \leq \sigma}(1/sd_C)\mathbb{Z}\right)$. Moreover, $q^s\Nr_{L/K_{k_\pi}}(\pi)^{-h}$ is an algebraic integer.

Applying Theorem 1.2 to the case where $k = \mathbb{Q}_p$, $\pi = p$ and $i$ is odd, we obtain [CSW, Corollary 1.6]. (Note that loc. cit. studies the vanishing of not only $H^0(\Gal(\overline{K}/L), V)$ (as our result) but also $H^j(\Gal(\overline{K}/L), V)$ for all $j$.) The assumption $i \neq 2r$ in Theorem 1.2 is essential as explained in the Introduction of [KT]. The key ingredients for our proof are the theory of locally algebraic representations (cf. [Se2]) and some "weight arguments" of eigenvalues of Frobenius on various objects. For weight arguments, we use $p$-adic Hodge theory related with Lubin-Tate characters and results on weights of a Frobenius operator on crystalline cohomologies (cf. [CLS], [KM], [Na]).

We hope our results can be useful for future studies in Iwasawa theory, for example, control theorems of Selmer groups for abelian varieties over certain $p$-adic extensions of number fields. In fact, arguments of [KT, Section 6] seem to be familiar with our results.

**Notation:** In this paper, we fix algebraic closures $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}$ and $\mathbb{Q}_p$, respectively, and we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. If $F$ is a $p$-adic field, we denote by $G_F$ and $U_F$ the absolute Galois group $\Gal(\overline{\mathbb{Q}}_p/F)$ of $F$ and the unit group of the integer ring of $F$, respectively. We also denote by $F^{ur}$ and $I_F$ the maximal unramified extension of $F$ in $\overline{\mathbb{Q}}_p$ and the inertia subgroup $\Gal(\overline{\mathbb{Q}}_p/F^{ur})$ of $G_F$, respectively. We set $\Gamma_F := \Hom_{\mathbb{Q}_{p}}(F, \mathbb{Q}_p)$. If $F'/F$ is a finite extension, we denote by $f_{F'/F}$ the residual extension degree of $F'/F$, that is, the extension degree of the residue fields corresponding to $F'/F$. We put $f_F = f_{F'/\mathbb{Q}_p}$. Finally, any $p$-adic representation of $G_F$ in this paper is of finite dimension.

**Acknowledgments.** The author would like to express his sincere gratitude to Professor Yuichiro Taguchi for giving him useful advices, especially Remark 2.10. The author would like to thank the anonymous referee who gave him many valuable advices throughout this paper. This work is supported by JSPS KAKENHI Grant Number JP19K03433.

## 2 Proofs of main theorems

Our goal is to prove results in the Introduction. We often use $p$-adic Hodge theory. For the basic notion of this theory, it is helpful for the reader to refer [Fo1] and [Fo2]. In this paper, we normalize the Hodge-Tate weight so that the Hodge-Tate weight of $\mathbb{Q}_p(1)$ is one.

**Definition 2.1.** Let $q_0 > 1$ be an integer. A $q_0$-Weil number (resp. $q_0$-Weil integer) of weight $w$ is an algebraic number (resp. algebraic integer) $\alpha$ such that $|\iota(\alpha)| = q_0^{w/2}$ for all embeddings $\iota: \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$.

**Definition 2.2.** Let $F$ be a $p$-adic field with residual extension degree $f = f_F$ and $F_0/\mathbb{Q}_p$ the maximal unramified subextension of $F/\mathbb{Q}_p$. We denote by $\varphi_{F_0}: F_0 \rightarrow F_0$ the arithmetic Frobenius of $F_0$, that is, the (unique) lift of $p$-th power map on the residue field of $F_0$.

1. Let $D$ be a $\varphi$-module over $F_0$, that is, a finite dimensional $F_0$-vector space with $\varphi_{F_0}$-semilinear map $\varphi: D \rightarrow D$. Then $\varphi^j: D \rightarrow D$ is a $F_0$-linear map. We call $\det(T - \varphi^j | D)$ the characteristic polynomial of $D$. 

2.
(2) For a $\mathbb{Q}_p$-representation $U$ of $G_F$, we set $D^p_{cris}(U) := (B_{cris} \otimes_{\mathbb{Q}_p} U)^{G_r}$ and $D^p_b(U) := (B_{st} \otimes_{\mathbb{Q}_p} U)^{G_r}$, which are filtered $\varphi$-modules over $F$. Here, $B_{cris}$ and $B_{st}$ are usual $p$-adic period rings. Note that we have $D^p_{cris}(U) = D^p_b(U)$ if $U$ is crystalline.

(3) Let $S$ be a set of rational numbers. Let $U$ be a potentially semi-stable $\mathbb{Q}_p$-representation of $G_F$. Suppose that $U|_{G_{F^w}}$ is semi-stable for a finite extension $F'$ of $F$ with residue field $\mathbb{F}_q$. We say that $U$ has Weil weights in $S$ if any root of the characteristic polynomial of $D^p_{cris}(U)$ is a $q'$-Weil number of weight $w$ for some $w \in S$. (Note that this definition does not depend on the choice of $F'$.)

Let $K$ and $k$ be finite extensions of $\mathbb{Q}_p$. Let $q$ be the order of the residue field of $k$, $\pi$ a uniformizer of $k$ and $k_\pi$ the Lubin-Tate extension of $k$ associated with $\pi$. The following theorem is a key to the proof of our main results.

**Theorem 2.3.** Let $S$ be a subset of $\mathbb{Q} \setminus \{0\}$. Let $V$ be a semi-stable $\mathbb{Q}_p$-representation of $G_K$ with Hodge-Tate weights in $[h_1, h_2]$. Assume that $V$ has Weil weights in $S$ and $V^{Gal(\overline{\mathbb{Q}}/\mathbb{Q})} \neq 0$ for some finite extension $L/K_{k_\pi}$. Then

1. $\mathfrak{N}_{k/q_p}(\pi)$ is a $q'$-Weil number of weight $-w/h$ for some $w \in S$ and some non-zero $h \in [h_1, h_2] \cap \left(\bigcup_{s \in \mathbb{Z}, 1 \leq s \leq 2} (1/\mathfrak{N}_{q_s}|\mathfrak{N}_{q_s}(1/\mathfrak{N}_{q_s})\mathbb{Z})\right)$.

2. If the coefficients of the characteristic polynomial of $D^p_{cris}(V(-r))$ are algebraic integers for some integer $r$, then we can choose $h$ in (1) so that $q' \mathfrak{N}_{k/q_p}(\pi)^{-h}$ is an algebraic integer.

2.1 Proof of Theorem 2.3

In this section, we prove Theorem 2.3. We begin with some lemmas.

**Lemma 2.4.** Let $(n_\sigma)_{\sigma \in \Gamma_K}$ be a family of integers. If there exists an open subgroup $U$ of $U_K$ with the property that $\prod_{\sigma \in \Gamma_K} n_\sigma^\sigma = 1$ for any $x \in U$, then we have $n_\sigma = 0$ for any $\sigma \in \Gamma_K$.

**Proof.** Replacing $U$ by a finite index subgroup, we may assume that the $p$-adic logarithm map is defined on $U$. Then we have $\prod_{\sigma \in \Gamma_K} n_\sigma \log(x) = 0$ for any $x \in U$ by assumption. Since $\log U$ is an open ideal of the ring of integers of $K$, we obtain $\prod_{\sigma \in \Gamma_K} n_\sigma y = 0$ for any $y \in K$. Although the desired fact $n_\sigma = 0$ for any $\sigma \in \Gamma_K$ follows from Dedekind’s theorem [Bo, §6, no. 2, Corollaire 2] immediately, we also give a direct proof for this. Take any $\alpha \in K$ such that $K = \mathbb{Q}_p(\alpha)$ and let $\Gamma_K = \{\sigma = id, \sigma_1, \ldots, \sigma_n\}$ where $c := [K : \mathbb{Q}_p]$. Then we have $(n_1, n_2, \ldots, n_n)X = 0$ where $X$ is the $c \times c$ matrix with $(i, j)$-th component $\sigma_i(\alpha)^{j-1}$. Since $\det X = 1$, we obtain $n_1 = n_2 = \cdots = n_n = 0$.

We denote by $\chi_\pi : G_k \to k^\times$ the Lubin-Tate character associated with $\pi$. If we regard $\chi_\pi$ as a continuous character $k^\times \to k^\times$ by the local Artin map with arithmetic normalization, then $\chi_\pi$ is characterized by the property that $\chi_\pi(\pi) = 1$ and $\chi_\pi(u) = u^{-1}$ for any $u \in U_k$.

**Lemma 2.5.** Let $E$ be a $p$-adic field and $V$ an $E$-representation of $G_K$. Assume that $k/\mathbb{Q}_p$ is Galois, $V$ is Hodge-Tate and the $G_{K_{k_\pi}}$-action on $V$ factors through a finite quotient. Then, there exist finite extensions $K'/K$ and $E'/E$ with $K', E' \supset K$ such that any Jordan-Hölder factor of $(V \otimes_E E')|_{G_{K'}}$ is of the form $E'(\prod_{\sigma \in \Gamma_K} \sigma^{-1} \circ \chi_\pi^s)$ for some $r \in \mathbb{Z}$. Moreover, $r_\pi$ is a Hodge-Tate weight of $V$.

**Proof.** Replacing $K$ by a finite extension, we may assume that $G_{K_{k_\pi}}$ acts trivially on $V$ and $K$ is a finite Galois extension of $k$. Since the $G_K$-action on $V$ factors through the abelian group $Gal(K_{k_\pi}/K)$, it follows from Schur’s lemma that, for a finite extension $E'/E$ of sufficiently large degree, any Jordan-Hölder factor $W'$ of $V \otimes_E E'$ is of dimension 1. Our goal is to show that $W'$ is of the required form. We may assume $E' = E \supset K$.

Let $\rho : G_K \to GL_2(W) \simeq E^\times$ be the continuous homomorphism given by the $G_K$-action on $W$. Let $\overline{E}$ be the Galois closure of $E/\mathbb{Q}_p$ and take any finite extension $K'/K$ which contains $\overline{E}$. Since $W$ is Hodge-Tate, it follows from [Se2, Chapter III, A.5, Theorem 2] that there exists
Let \( E \) be a \( p \)-adic field and \( V \) an \( E \)-representation of \( G_K \). Assume that \( k/\mathbb{Q}_p \) is Galois, \( V \) is potentially semi-stable with Hodge-Tate weights in \([h_1, h_2]\) and the \( G_K \)-action on \( V \) factors through a finite quotient. Then, there exists a finite extension \( K'/Kk \) which satisfies the
following property: \( V|_{\mathcal{G}_k'} \) is semi-stable and, for any root \( \alpha \) of the characteristic polynomial of \( D^{K'}_s(V) \), we have
\[
\alpha = a^{t^{K'/k}}, \quad a = \prod_{r \in \mathcal{I}_k} \tau(\pi)^{-n_r}
\]
for some integers \( (n_r)_{r \in \mathcal{I}_k} \) such that \( dh_1 \leq \sum_{r \in \mathcal{I}_k} n_r \leq dh_2 \). Here, \( d := [k : Q_p] \).

**Proof.** By Lemma 2.5, there exist finite extensions \( K'/K \) and \( E'/E \) with \( E', K' \supset k \) which satisfy the following:

\( V|_{\mathcal{G}_k'} \) is semi-stable and any Jordan-Hölder factor \( W \) of \( (V \otimes_E E')|_{\mathcal{G}_k'} \) is of the form.

\[ E' \left( \prod_{r \in \mathcal{I}_k} a^{r} \right) \] for some \( r \in [h_1, h_2] \). In particular, \( W \) is crystalline.

Replacing \( E \) by a finite extension, we may assume \( E' = E \). Now we take a root \( \alpha \) of the characteristic polynomial of \( D^{k}_s(V) \), and choose \( W \) so that \( \alpha \) is a root of the characteristic polynomial of \( D^{k}_s(V) \).

To study \( \alpha \), we first consider the characteristic polynomial of \( D^{k'}_{\text{cris}}(E(\sigma^{-1} \circ \chi^r_{\sigma})) \) for \( \sigma \in \Gamma_k \).

We note that we have an isomorphism \( k(\sigma^{-1} \circ \chi^r_{\sigma})\text{ss} \simeq k(\chi^r_{\sigma})\text{ss} \) of \( Q_p[G_K']\)-modules (here, "ss" stands for the semi-simplification of \( Q_p[G_K']\)-modules). In fact, for any \( g \in G_K' \), we have
\[
\text{Tr}_{Q_p}(g | k(\sigma^{-1} \circ \chi^r_{\sigma})) = \text{Tr}_{k/Q_p}(\text{Tr}_k(g | k(\sigma^{-1} \circ \chi^r_{\sigma}))),
\]
\[
= \text{Tr}_{k/Q_p}(\chi^r_{\sigma}(g)) = \text{Tr}_{k/Q_p}(\text{Tr}_k(g | k(\chi^r_{\sigma})) = \text{Tr}_{Q_p}(g | k(\chi^r_{\sigma})).
\]

(Here, for a representation \( U \) of a group \( G \) over a field \( F \) and \( g \in G \), we denote by \( \text{Tr}_F(g | U) \) the trace of the \( g \)-action on the \( F \)-vector space \( U \).) Therefore, we have
\[
\det(T - \varphi^{f_{K'}} | D^{K'}_{\text{cris}}(E(\sigma^{-1} \circ \chi^r_{\sigma}))) = \det(T - \varphi^{f_{K'}} | D^{K'}_{\text{cris}}(k(\chi^r_{\sigma}))))^{[E:k]}.
\]

To study the roots of \( (2.4) \), we recall the explicit description of \( D^{k}_{\text{cris}}(k(\chi^r_{\sigma}^{-1})) \) (cf. [Con, Proposition B.4]). See also [Col, Proposition 9.10]). Let \( k_0 \) be the maximal unramified subextension of \( k/Q_p \).

By definition, we have \( f_k = [k_0 : Q_p] \) and \( q = p^{f_k} \). Then \( D^{k}_{\text{cris}}(k(\chi^r_{\sigma}^{-1})) \) is a free \( (k_0 \otimes_{Q_p} k)\)-module of rank one, and we can take a basis \( e \) of \( D^{k}_{\text{cris}}(k(\chi^r_{\sigma}^{-1})) \) such that \( \varphi^{f_k}(e) = (1 \circ \pi)e \). We claim
\[
\det(T - \varphi^{f_{K'}} | D^{k}_{\text{cris}}(k(\chi^r_{\sigma}^{-1}))) = \prod_{0 \leq i \leq f_k-1} E^{f_{K'}}(T)
\]
where \( E(T) = T^{e} + \sum_{j=0}^{e-1} a_jT^j \in k_0[T] \) is the minimal polynomial of \( \pi \) over \( k_0 \) and \( E^\pi(T) = T^{e} + \sum_{j=0}^{e-1} \varphi^j(a_j)T^j \). To show this, it suffices to show that the characteristic polynomial of the homomorphism \( 1 \otimes \pi : k_0 \otimes_{Q_p} k \to k_0 \otimes_{Q_p} k \) of \( k_0 \)-modules coincides with the right hand side of (2.5). Here, the \( k_0 \)-action on \( k_0 \otimes_{Q_p} k \) is given by \( a.(x \otimes y) := ax \otimes y \) for \( a, x \in k_0 \) and \( y \in k_0 \).

We consider a natural isomorphism
\[
k_0 \otimes_{Q_p} k_0 \cong \otimes_{j \leq f_k-1} k_0Z_{f_k}k_0, \quad a \otimes b \mapsto (a\varphi^j(b)),
\]
where \( k_{0,j} = k_0 \) for all \( j \). For \( 0 \leq s \leq f_k-1 \), let \( e_s \in k_0 \otimes_{Q_p} k_0 \) be the element which corresponds to \( (\delta_{ij})_{i} \in \otimes_{j \leq f_k-1} k_0Z_{f_k}k_0 \) where \( \delta_{ij} \) is the Kronecker delta. Then \( (e_s(1 \otimes \pi^{i}) | 0 \leq j \leq f_k-1, 0 \leq i \leq e^{-1}) \) is a \( k_0 \)-basis of \( k_0 \otimes_{Q_p} k_0 \). We see that the matrix of \( 1 \otimes \pi : k_0 \otimes_{Q_p} k \to k_0 \otimes_{Q_p} k \) associated with the ordered basis \( (e_0, \ldots, e_{f_k-1}, e_0(1 \otimes \pi), \ldots, e_{f_k-1}(1 \otimes \pi)) \) is
\[
\begin{pmatrix}
O & O & \cdots & -A_0 \\
I_{f_k} & O & \cdots & -A_1 \\
\vdots & \ddots & \ddots & \vdots \\
O & \cdots & I_{f_k} & -A_{f_k-1}
\end{pmatrix}
\]
where \( I_{f_k} \) is the \( f_k \times f_k \) identity matrix and \( A_i \) is the \( f_k \times f_k \) diagonal matrix with diagonal entries \( a_1, \varphi(a_1), \ldots, \varphi^{f_k-1}(a_1) \). Now it is an easy exercise to check that the characteristic polynomial of this matrix is \( \prod_{0 \leq i \leq f_k-1} E^{f_{K'}}(T) \) as desired.
Now we note that roots of the characteristic polynomial of $D^{K_{cris}}_{cris}(k(\chi_\tau))$ are the $f_{K'/k}$-th power of those of $D^{K_{cris}}_{cris}(k(\chi_\tau))$ since the latter describes the action of $\varphi^{f_{k'}}$ but the former describes that of $\varphi^{f_{K'/k}}$. Furthermore, we also note that all the roots of the right hand side of (2.5) is a conjugate of $\pi$ over $\bar{Q}_p$. Hence, it follows from the claim (2.5) that any root of the characteristic polynomial of $D^{K_{cris}}_{cris}(k(\chi_\tau))$ is of the form $\tau(\pi)^{-f_{K'/k}}$ for some $\tau \in \Gamma_k$. On the other hand, for crystalline characters $\psi_1, \psi_2: G_k \to k^\times$, we have a surjection $D^{K_{cris}}_{cris}(k(\psi_1)) \otimes_{k} D^{K_{cris}}_{cris}(k(\psi_2)) \to D^{K_{cris}}_{cris}(k(\psi_1 \psi_2))$ induced from the natural map $k(\psi_1) \otimes_{k} k(\psi_2) \to k(\psi_1) \otimes_k k(\psi_2) = k(\psi_1 \psi_2)$. Here, $K_0$ is the maximal unramified subextension of $K'/\bar{Q}_p$. In particular, roots of the characteristic polynomial of $D^{K_{cris}}_{cris}(k(\psi_1))$ is a product of those of $D^{K_{cris}}_{cris}(k(\psi_1))$ and $D^{K_{cris}}_{cris}(k(\psi_2))$. By this fact, we know that any root of the characteristic polynomial of $D^{K_{cris}}_{cris}(k(\chi_\sigma'))$ is of the form $\prod_{\tau \in \Gamma_k} \tau(\pi)^{-f_{K'/k}^{\sigma}}$ with $\sum_{\tau \in \Gamma_k} n_{\tau}^{\sigma} = r^{\sigma}$. By (2.4), the same holds for the roots of the characteristic polynomial of $D^{K_{cris}}_{cris}(W) = D^{K_{cris}}_{cris}(E(\sigma^{-1} \circ \chi_{\sigma}'))$. Therefore, since $\alpha$ is a root of the characteristic polynomial of $D^{K_{cris}}_{cris}(W) = D^{K_{cris}}_{cris}(E(\sigma^{-1} \circ \chi_{\sigma}'))$, we have

$$\alpha = \prod_{\tau \in \Gamma_k} \tau(\pi)^{-f_{K'/k}^{\sigma}}$$

with $\sum_{\tau \in \Gamma_k} n^{\tau} = \sum_{\sigma \in \Gamma_k} r^{\sigma}$. We note that $R$ is an integer such that $dh_1 \leq R \leq dh_2$ since we have $r^{\sigma} \in [h_1, h_2]$. This completes the proof.

We need the following two standard lemmas which describe inclusion properties of two Lubin-Tate extensions.

**Lemma 2.7.** Let $k_2/k_1$ be a finite extension of $p$-adic fields with residual extension degree $f$. For $i = 1, 2$, let $\pi_i$ be a uniformizer of $k_i$ and $k_{i, \pi_i}/k_i$ the Lubin-Tate extension associated with $\pi_i$.

1. We have $N_{k_2/k_1}(\pi_2) = \pi_1^{f_i}$ if and only if $k_{1, \pi_1} \subset k_{2, \pi_2}$.
2. $\pi_1^{-f_i}N_{k_2/k_1}(\pi_2)$ is a root of unity if and only if there exists a finite extension $M/k_{2, \pi_2}$ such that $k_{1, \pi_1} \subset M$. If this is the case, we can take $M$ to be the degree $\mu_{\infty}(k_1)$ subextension in $k_{ab}/k_{2, \pi_2}$. Here, $\mu_{\infty}(k_1)$ is the set of roots of unity in $k_1$.

**Proof.** For $i = 1, 2$, denote by $k^{ab}_{i}$ and $k_{ab}$ the maximal unramified extension of $k_i$ and the maximal abelian extension of $k_i$, respectively. We recall that the Artin map $Art_k : k_1^\times \to \text{Gal}(k_{ab}/k_1)$ associated with $k_1$ satisfies $Art_{k_1}(\pi_1)|_{k_{1, \pi_1}} = \text{id}$ and $Art_{k_1}(\pi_1)|_{k^{ur}_1} = \text{Frob}_{k_1}$, where $Frob_{k_1}$ is the geometric Frobenius of $k_1$.

1. Suppose $N_{k_2/k_1}(\pi_2) = \pi_1^{f_i}$. For any lift $\sigma \in G_{k_2}$ of $Art_{k_2}(\pi_2)$, we have

$$\sigma|_{k_{1, \pi_1}} = (Art_{k_2}(\pi_2)|_{k_{1, \pi_1}}|_{k_{1, \pi_1}} = Art_{k_1}(N_{k_2/k_1}(\pi_2)|_{k_{1, \pi_1}} = Art_{k_1}(\pi_1)|_{k_{1, \pi_1}} = \text{id}.$$ 

Since the intersection of the fixed fields (in $\bigcup_{\sigma}$) of such $\sigma$’s is $k_{2, \pi_2}$, we obtain the desired result.

Conversely, suppose $k_{1, \pi_1} \subset k_{2, \pi_2}$. Then we have

$$Art_{k_1}(N_{k_2/k_1}(\pi_2)|_{k_{1, \pi_1}} = Art_{k_2}(\pi_2)|_{k_{1, \pi_1}} = (Art_{k_2}(\pi_2)|_{k_{2, \pi_2}})|_{k_{1, \pi_1}} = \text{id}$$

and

$$Art_{k_1}(N_{k_2/k_1}(\pi_2)|_{k^{ur}_1} = Art_{k_2}(\pi_2)|_{k^{ur}_1} = (Art_{k_2}(\pi_2)|_{k^{ur}_1})|_{k^{ur}_1} = \text{Frob}_{k_2}|_{k^{ur}_1} = \text{Frob}_{k_1}.$$ 

Thus we have $Art_{k_1}(N_{k_2/k_1}(\pi_2)) = Art_{k_2}(\pi_1^{f_i})$, which shows $N_{k_2/k_1}(\pi_2) = \pi_1^{f_i}$.

2. A very similar proof to that of (1) proceeds. Suppose that $\pi_1^{-f_i}N_{k_2/k_1}(\pi_2)$ is a root of unity.

If we denote by $h$ the order of the set of roots of unity in $k_1$, then we have $N_{k_2/k_1}(\pi_2) = \pi_1^{fh}$. We see that any lift $\sigma \in G_{k_2}$ of $Art_{k_2}(\pi_2)$ fixes $k_{1, \pi_1}$. This implies that $k_{1, \pi_1}$ is contained in a degree $h$ subextension in $k_{ab}^{ur}/k_{2, \pi_2}$.

Suppose that there exists a finite extension $M/k_{2, \pi_2}$ such that $k_{1, \pi_1} \subset M$. Then $M' := k_{1, \pi_1}k_{2, \pi_2}$ is a finite subextension in $k_{ab}^{ur}/k_{2, \pi_2}$. Put $h = [M' : k_{2, \pi_2}]$. Since $Art_{k_2}(\pi_2)|_{M'}$ is the
identity map, we have $\text{Art}_{k_1}(\text{Nr}_{k_2/k_1}(\pi_1^h))|_{k_1,x_1} = \text{id}$ and $\text{Art}_{k_1}(\text{Nr}_{k_2/k_1}(\pi_2^h))|_{k_1} = \text{Frob}_{k_1}$. Thus we have $\text{Art}_{k_1}(\text{Nr}_{k_2/k_1}(\pi_2^h)) = \text{Art}_{k_1}(\pi_1^h)$, which shows $\text{Nr}_{k_2/k_1}(\pi_2^h) = \pi_1^h$.

We recall that $k_G$ is the Galois closure of $k/Q_p$ and $d_G := [k_G : Q_p]$. 

**Lemma 2.8.** There exist a finite unramified extension $k'/k_G$ and a uniformizer $\pi'$ of $k'$ which satisfy the following.

- $\text{Nr}_{k'/k}(\pi') = \pi^{f_{k'/k}}$,
- $k_\pi \subset k'_{\pi'}$, where $k'_{\pi'}$ is the Lubin-Tate extension of $k'$ associated with $\pi'$,
- the extension $k'/Q_p$ is Galois, and
- $[k' : Q_p] = sd_G$ for some integer $1 \leq s \leq e_G$.

**Proof.** Let $k_{G,0}/k$ be the maximal unramified subextension in $k_G/k$. By [Sel, Chapter V, §6, Proposition 10], there exists an unramified extension $\tilde{k}_0$ over $k_{G,0}$ of degree at most $[k_G : k_{G,0}](= e_G)$ such that $\pi = \text{Nr}_{k_{G,0}/k_0}(\pi')$ for some $\pi' \in (k')^\times$, where $k' := k_G/\tilde{k}_0$. Since $k_G/Q_p$ is Galois and $k'/k$ is unramified, we see that $k'/Q_p$ is Galois. We also see that $\pi'$ is a uniformizer of $k'$. Since $k_G \cap \tilde{k}_0 = k_{G,0}$, we have $[k' : k_G] = [\tilde{k}_0 : k_{G,0}] \leq e_G$. Thus we obtain $[k' : Q_p] = [k' : k_G][k_G : Q_p] = sd_G$ for some integer $1 \leq s \leq e_G$. Furthermore, we have $\text{Nr}_{k'/k}(\pi') = \text{Nr}_{k_{G,0}/k}(\text{Nr}_{k_G/\tilde{k}_0}(\pi')) = \text{Nr}_{k_{G,0}/k}(\pi) = \pi^{f_{k'/k}}$. By Lemma 2.7, we have $k_\pi \subset k'_{\pi'}$.

Now we are ready to prove Theorem 2.3.

**Proof of Theorem 2.3.** First we consider the case where $k/Q_p$ is Galois. Replacing $L$ by a finite extension, we may assume that $L/K$ is Galois. Then $V^{GL}$ is a $G_K$-stable submodule of $V$. By Lemma 2.6, there exists a finite extension $K'/Kk$ such that any root $\alpha$ of the characteristic polynomial of $D_{k'}^G(V^{GL})$ is of the form

$$\alpha = a^{f_{k'/k}}$$

with some integers $(n_\tau)_{\tau \in \Gamma_k}$ such that $dh_1 \leq \sum_{\tau \in \Gamma_k} n_\tau \leq dh_2$. Here, $d := [k : Q_p]$. Put $R := \sum_{\tau \in \Gamma_k} n_\tau$. Then we have

$$\prod_{\sigma \in \Gamma_k} \sigma(a) = \prod_{\tau \in \Gamma_k} \prod_{\sigma \in \Gamma_k} \sigma_\pi(n_\tau)^{-n_\tau} = \prod_{\tau \in \Gamma_k} \text{Nr}_{k/\text{Q}_p}(\pi)^{-n_\tau} = \text{Nr}_{k/\text{Q}_p}(\pi)^{-R}. \tag{2.6}$$

Since $V^{GL}$ has Weil weights in $S$, we see that $\sigma(a)$ is a $q$-Weil number of weight $w \in S$ for any $\sigma \in \Gamma_k$. Thus it follows from the condition $w \neq 0$ and the equation (2.6) that we have $R \neq 0$. Therefore, we obtain that $\text{Nr}_{k/\text{Q}_p}(\pi)$ is a $q$-Weil number of weight $-w/h$ where $h := R/d \in [h_1,h_2] \cap (1/d)\mathbb{Z}$. This shows Theorem 2.3 (1). Now Theorem 2.3 (2) follows from the fact that we have $(q^R \text{Nr}_{k/\text{Q}_p}(\pi)^{-h}) = \text{Nr}_{k/\text{Q}_p}(q^R a)$ and $(q^R a)^{f_{k'/k}} = q^{R,k}a$ is a root of the characteristic polynomial of $D_{k'}^G(V(-r))$ (here, $q^{k'}$ is the order of the residue field of $K'$). Thus we obtained a proof of Theorem 2.3 in the case where $k/Q_p$ is Galois.

Next we consider the case where $k/Q_p$ is not necessarily Galois. Take a finite extension $k'/k_G$ and a uniformizer $\pi'$ of $k'$ as in Lemma 2.8. Put $d' := [k' : Q_p]$. We have $d' = sd_G$ for some $1 \leq s \leq e_G$. Let $q'$ be the order of the residue field of $k'$. Let $L'$ be the composite field of $L$ and $k'_{\pi'}$, which is a finite extension of $Kk'_{\pi'}$. Assume that $V^{GL}$ is not zero. Since $V^{GL'}$ is also not zero and the extension $k'/Q_p$ is Galois, we know that $\text{Nr}_{k'/\text{Q}_p}(\pi')$ is a $q'$-Weil number of weight $-w/h$ for some $w \in S$ and $h \in [h_1,h_2] \cap (1/d')\mathbb{Z}$. By the equation $\text{Nr}_{k'/k}(\pi') = \pi^{f_{k'/k}}$, we have $\text{Nr}_{k'/\text{Q}_p}(\pi') = (\text{Nr}_{k/\text{Q}_p}(\pi)^{-h}) = (\text{Nr}_{k/\text{Q}_p}(\pi)^{-h})^{f_{k'/k}}$, and hence $\text{Nr}_{k/\text{Q}_p}(\pi)$ is a $q$-Weil number of weight $-w/h$. Furthermore, we have $q^d \text{Nr}_{k'/\text{Q}_p}(\pi')^{-h} = (q^d \text{Nr}_{k/\text{Q}_p}(\pi)^{-h})^{f_{k'/k}}$. This completes the proof of Theorem 2.3. \qed
2.2 Proofs of Theorems 1.1 and 1.2

We prove Theorems 1.1 and 1.2 in the Introduction. We start with a proof of Theorem 1.2.

Proof of Theorem 1.2. Let the notation be as in the theorem. Replacing $K$ by a finite extension, we may assume that $X$ has good reduction over $K$. Then we know that $V$ is crystalline with Hodge-Tate weights in $[-i + r, r]$ (cf. [Fa1], [Fa2]). We claim that $V$ has Weil weight $i - 2r$. Let $K_0$ be the maximal unramified subextension of $K/Q_p$. Put $q_K = p^{f_K}$, the order of the residue field of $K$. Let $Y$ be the special fiber of a proper smooth model of $X$ over the integer ring of $K$. By the crystalline conjecture shown by Faltings [Fa1] (cf. [Ni], [Tsu]), we have an isomorphism $Dcris(K, H^1_{dt}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)) \simeq K_0 \otimes W(F_{q_K})$. Hence we obtain the fact that the characteristic polynomial char$_{V(-r)}(T)$ of $Dcris(V(-r))$ divides char$_{X}(T)$. Thus it follows from the Weil Conjecture (cf. [De1], [De2]) that char$_{V(-r)}(T)$ has algebraic integer coefficients and its roots are $q_K$-Weil numbers of weight $i$. In particular, $V$ has Weil weight $i - 2r$ as desired. Now the result follows by Theorem 2.3.

Finally, we prove Theorem 1.1. Let $A$ be an abelian variety over a $p$-adic field $K$ and let $\ell$ be any prime number. We denote by $T_\ell(A)$ the $\ell$-adic Tate module of $A$ and set $V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell$. It is well known that we have $G_K$-equivariant isomorphisms $V_\ell(A) \simeq H^1_{\acute{e}t}(A_{\overline{\mathbb{Q}}}, \mathbb{Q})^\vee$ and $V_\ell(A)/T_\ell(A) \simeq A(\overline{\mathbb{K}})[\ell^\infty]$. Here, $A(\overline{\mathbb{K}})[\ell^\infty]$ is the $\ell$-power torsion subgroup of $A(\overline{\mathbb{K}})$.

For an algebraic extension $L/K$, the $\ell$-power torsion subgroup $A(L)[\ell^\infty]$ of $A(L)$ is finite if and only if $V_\ell(A)^{GL_\ell} = 0$. Below we denote by $L$ any finite extension of $Kk_\pi$. Assume that $A$ has potential good reduction and $N_{K/k_\pi}(\pi)$ satisfies the condition in the statement of Theorem 1.1. For the proof of Theorem 1.1, it is enough to show that both the $p$-part and the prime-to-$p$ part of $A(L)_{\text{tor}}$ are finite.

Finiteness of the $p$-part of $A(L)_{\text{tor}}$: If we put $W = V_\ell(A)^{GL_\ell}$, then it is enough to show $W = 0$. Replacing $L$ by a finite extension, we may suppose that the extension $L/K$ is Galois. Then the $G_K$-action on $V_\ell(A)$ preserves $W$, and thus the dual representation $W^\vee$ of $W$ is a quotient representation of $H^1_{\acute{e}t}(A_{\overline{\mathbb{K}}}, \mathbb{Q}_p)$. By Theorem 1.2, we have $W^\vee = (W^\vee)^{GL_\ell} = 0$, which implies $W = 0$ as desired.

Finiteness of the prime-to-$p$ part of $A(L)_{\text{tor}}$: The finiteness of the prime-to-$p$ part of $A(L)_{\text{tor}}$ follows from the following general property.

Proposition 2.9. Let $A$ be an abelian variety over $K$ with potential good reduction. Let $M$ be an algebraic extension of $K$ with finite residue field. Then the prime-to-$p$ part of $A(M)_{\text{tor}}$ is finite.

Proof. Replacing $K$ and $M$ by finite extensions, we may assume that $A$ has good reduction over $K$. It follows from the criterion of Néron-Ogg-Shafarevich [ST, Theorem 1] that the prime-to-$p$ part of $A(M)_{\text{tor}}$ has values in the maximal unramified subextension of $M/K$, which is a finite extension of $K$ by assumption on $M$. Then the result follows from the main theorem of [Ma].

Therefore, we obtained the proof of Theorem 1.1.

Remark 2.10. (This is pointed out by Yuichiro Taguchi.) We can construct an example which gives a negative answer to the question given in the Introduction for potential good reduction case. Let $E$ be an elliptic curve over $\mathbb{Q}$ with complex multiplication by the full ring of integers $\mathcal{O}_F$ of an imaginary quadratic field $F$. Let $\psi = \psi_{E/F}$ be the Grössencharacter associated with $E$. Let $p$ be a prime number such that $E$ has good ordinary reduction and $p$ a prime ideal of $\mathcal{O}_F$ above $p$. If we set $\pi := \psi(p)$, then $\pi$ is a generator of $p$ and we have $p = \pi \overline{\pi}$. Here, $\overline{\pi}$ is the complex conjugate of $\pi$. Note that $\pi$ is a $p$-Weil number of weight 1. Let $K = k$ be the completion of $F$ at $p$. By definition, we have $K = k = \mathbb{Q}_p$ and $\pi$ is a uniformizer of them. If we identify a decomposition group of $G_F$ at $p$ with $G_K$, then the action of $G_K$ on the set of $p$-power torsion points of $E(\overline{\mathbb{K}})$ is
given by the Lubin-Tate character $\chi_\pi$ associated with $\pi$. In particular, we see that $E(K_{k_\pi})[p^\infty]$ is infinite.

References

[Bo] N. Bourbaki, Algèbre. Chapitre 5, Éléments de mathématique. 23. Première partie: Les structures fondamentales de l'analyse. Livre II: Actualités Sci. Ind. no. 1261, Hermann, Paris, 1958.

[CLS] B. Chiarellotto and B. Le Stum, Sur la pureté de la cohomologie cristalline, C. R. Acad. Sci. Paris Sér. I Math. 8 (1998), 961–963.

[CSW] J. Coates, R. Sujatha and J.-P. Wintenberger, On the Euler-Poincaré characteristics of finite dimensional $p$-adic Galois representations, Publ. Math. Inst. Hautes Études Sci. 93 (2001), 107–143.

[Con] B. Conrad, Lifting global representations with local properties, preprint, 2011, available at http://math.stanford.edu/~conrad/papers/locchar.pdf

[Col] P. Colmez, Espaces de Banach de dimension finie, J. Inst. Math. Jussieu 1 (2002), 331–439.

[De1] P. Deligne, La conjecture de Weil I, Publ. Math. IHES 43 (1974), 273–308.

[De2] P. Deligne, La conjecture de Weil II, Inst. Hautes Études Sci. Publ. Math. 52 (1980), 137–252.

[Fa1] G. Faltings, $p$-adic Hodge theory, J. Amer. Math. Soc. 1 (1988), 255–299.

[Fa2] G. Faltings, Crystalline cohomology and $p$-adic Galois-representations, Algebraic analysis, geometry, and number theory (Baltimore MD, 1988), 25–80.

[Fo1] J.-M. Fontaine, Le corps des périodes $p$-adiques, With an appendix by Pierre Colmez, Périodes $p$-adiques (Bures-sur-Yvette, 1988), Astérisque 223 (1994), 59–111.

[Fo2] J.-M. Fontaine, Représentations $p$-adiques semi-stables, With an appendix by Pierre Colmez, Périodes $p$-adiques (Bures-sur-Yvette, 1988), Astérisque 223 (1994), 113–184.

[Im] H. Imai, A remark on the rational points of abelian varieties with values in cyclotomic $\mathbb{Z}_p$-extensions, Proc. Japan Acad. 51 (1975), 12–16.

[KM] N. Katz and W. Messing, Some consequences of the Riemann hypothesis for varieties over finite fields, Invent. Math. 23 (1974), 73–77.

[KT] Y. Kubo and Y. Taguchi, A generalization of a theorem of Imai and its applications to Iwasawa theory, Math. Z. 275 (2013), 1181–1195.

[Ma] A. Mattuck, Abelian varieties over $p$-adic ground fields, Ann. of Math. (2) 62 (1955), 92–119.

[Na] Y. Nakkajima, $p$-adic weight spectral sequences of log varieties, J. Math. Sci. Univ. Tokyo 12 (2005), 513–661.

[Ni] W. Niziol, Crystalline conjecture via $K$-theory, Ann. Sci. École Norm. Sup. (4) 31 (1998), 659–681.

[Se1] J.-P. Serre, Local fields, Graduate Texts in Mathematics, vol. 67, Translated from the French by Marvin Jay Greenberg, Springer-Verlag, 1979.
[Se2] J.-P. Serre, *Abelian l-adic representations and elliptic curves*, second ed., Advanced Book Classics, Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1989, With the collaboration of Willem Kuyk and John Labute.

[ST] J.-P. Serre and J. Tate, Good reduction of abelian varieties, Ann. of Math. (2) 8 (1986), 492–517.

[Tsu] T. Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. Math. 137 (1999), 233–411.