Propagation of Waves in Networks of Thin Fibers

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Abstract

The paper contains a simplified and improved version of the results obtained
by the authors earlier. Wave propagation is discussed in a network of branched
thin wave guides when the thickness vanishes and the wave guides shrink to a one
dimensional graph. It is shown that asymptotically one can describe the propagating
waves, the spectrum and the resolvent in terms of solutions of ordinary differential
equations on the limiting graph. The vertices of the graph correspond to junctions
of the wave guides. In order to determine the solutions of the ODE on the graph
uniquely, one needs to know the gluing conditions (GC) on the vertices of the graph.

Unlike other publications on this topic, we consider the situation when the spec-
tral parameter is greater than the threshold, i.e., the propagation of waves is possible
in cylindrical parts of the network. We show that the GC in this case can be ex-
pressed in terms of the scattering matrices related to individual junctions. The
results are extended to the values of the spectral parameter below the threshold
and around it.

1 Introduction

Consider the stationary wave (Helmholtz) equation

$$H_\varepsilon u = -\varepsilon^2 \Delta u = \lambda u, \quad x \in \Omega_\varepsilon, \quad Bu = 0 \quad \text{on} \ \partial \Omega_\varepsilon,$$

in a domain $\Omega_\varepsilon \subset R^d, d \geq 2$, with infinitely smooth boundary (for simplicity), which has
the following structure: $\Omega_\varepsilon$ is a union of a finite number of cylinders $C_{j,\varepsilon}$ (which will be
called channels) of lengths $l_j, 1 \leq j \leq N$, with diameters of cross-sections of order $O(\varepsilon)$
and domains $J_{1,\varepsilon}, \ldots, J_{M,\varepsilon}$ (which will be called junctions) connecting the channels into
a network. It is assumed that the junctions have diameters of the same order $O(\varepsilon)$. The
boundary condition has the form: $B = 1$ (the Dirichlet BC) or $B = \frac{\partial}{\partial n}$ (the Neumann
BC) or $B = \varepsilon \frac{\partial}{\partial n} + \alpha(x)$, where $n$ is the exterior normal and the function $\alpha \geq 0$ is real

*The authors were supported partially by the NSF grant DMS-0706928.
valued and does not depend on the longitudinal (parallel to the axis) coordinate on the boundary of the channels. One also can impose one type of BC on the lateral boundary of $\Omega_\varepsilon$ and another BC on the free ends (which are not adjacent to a junction) of the channels.

The axes of the channels form edges $\Gamma_j$, $1 \leq j \leq N$, of the limiting ($\varepsilon \to 0$) metric graph $\Gamma$. The junctions shrink to vertices of the graph $\Gamma$ when $\varepsilon \to 0$. We denote the set of vertices $v_j$ by $V$. Let $m$ channels have infinite length ($m = 0$ for bounded $\Omega_\varepsilon$). We start the numeration of $C_{j,\varepsilon}$ with the infinite channels. So, $l_j = \infty$ for $1 \leq j \leq m$.

Figure 1: An example of a domain $\Omega_\varepsilon$ with four junctions, four unbounded channels and four bounded channels.

The goal of the paper is the asymptotic analysis of the spectrum of $H_\varepsilon$, the resolvent $(H_\varepsilon - \lambda)^{-1}$, and solutions of the corresponding non-stationary problems for the heat and wave equations as $\varepsilon \to 0$. One can expect that $H_\varepsilon$ is close (in some sense) to a one dimensional operator on the limiting graph $\Gamma$ with appropriate gluing conditions (GC) at the vertices $v \in V$. The ODE on $\Gamma$ appears in a natural way from the following principle: the oscillating modes in the wave guides survive as $\varepsilon \to 0$ and the exponentially decaying and growing modes disappear. However, the justification of this fact is not always simple. In order to determine the solutions of ODE on $\Gamma$ uniquely, one needs to know the GC on the vertices of $\Gamma$. The form of the GC in the general situation was discovered quite recently in our papers [21]-[23]. It turned out that they can be expressed in terms of scattering matrices for problems of the wave propagation through individual junctions of $\Omega_\varepsilon$. These GC hold in all the cases: in the bulk of the spectrum $\lambda > \lambda_0$, and near the threshold $\lambda \approx \lambda_0$, for bounded and unbounded $\Omega_\varepsilon$.

Equation (1) degenerates when $\varepsilon = 0$. One could omit $\varepsilon^2$ in (1). However, the problem under consideration would remain singular, since the domain $\Omega_\varepsilon$ shrinks to the graph $\Gamma$ as $\varepsilon \to 0$. The presence of this coefficient in the equation is convenient, since it makes the spectrum less vulnerable to changes in $\varepsilon$. As we shall see, in some important cases (spider domains $\Omega_\varepsilon$) the spectrum of the problem (1) does not depend on $\varepsilon$, and the spectrum in
the same cases will be magnified by a factor of $\varepsilon^{-2}$ if $\varepsilon^2$ in (I) is omitted. The operator $H_\varepsilon = -\varepsilon^2 \Delta$ introduced in (I) will be considered as the operator in $L^2(\Omega_\varepsilon)$.

An important class of domains $\Omega_\varepsilon$ are the self-similar domains with only one junction and all the channels of infinite length. We shall call them spider domains. Thus, if $\Omega_\varepsilon$ is a spider domain, then there exists a point $\hat{x} = x(\varepsilon)$ and an $\varepsilon$-independent domain $\Omega$ such that

$$\Omega_\varepsilon = \{(\hat{x} + \varepsilon x) : x \in \Omega\},$$

i.e. $\Omega_\varepsilon$ is the $\varepsilon$-contraction of $\Omega = \Omega_1$.

For the sake of simplicity, we shall assume that $\Omega_\varepsilon$ is self-similar in a neighborhood of each junction. Namely, let $J_{j(v),\varepsilon}$ be the junction which corresponds to a vertex $v \in V$ of the limiting graph $\Gamma$. Consider a junction $J_{v,\varepsilon} = J_{j(v),\varepsilon}$ and all the channels adjacent to $J_{v,\varepsilon}$. If some of these channels have finite length, we extend them to infinity. We assume that, for each $v \in V$, the resulting domain $\Omega_{v,\varepsilon}$, which consists of the junction $J_{v,\varepsilon}$ and the semi-infinite channels emanating from it, is a spider domain. We also assume here that all the channels $C_{j,\varepsilon}$ have the same cross-section $\omega_\varepsilon$. This assumption is needed only to make the results more transparent (The general case is studied in [22]). From the self-similarity assumption it follows that $\omega_\varepsilon$ is an $\varepsilon$-homothety of a bounded domain $\omega \subset \mathbb{R}^{d-1}$.

Let $\lambda_0 < \lambda_1 \leq \lambda_2...$ be eigenvalues of the negative Laplacian $-\Delta_{d-1}$ in $\omega$ with the BC $B_0 u = 0$ on $\partial \omega$ where $B_0$ coincides with the boundary operator $B$ on the channels, see (I), with $\varepsilon = 1$ in the case of the third boundary condition. Let $\{\varphi_n(y)\}$, $y \in \omega \in \mathbb{R}^{d-1}$, be the set of corresponding orthonormal eigenfunctions. Then $\lambda_n$ are eigenvalues of $-\varepsilon^2 \Delta_{d-1}$ in $\omega_\varepsilon$ and $\{\varepsilon^{-d/2} \varphi_n(y/\varepsilon)\}$ are the corresponding eigenfunctions. In the presence of infinite channels, the spectrum of the operator $H_\varepsilon$ consists of an absolutely continuous component which coincides with the semi-bounded interval $[\lambda_0, \infty)$ and a discrete set of eigenvalues. The eigenvalues can be located below $\lambda_0$ and can be embedded into the absolutely continuous spectrum. We will call the point $\lambda = \lambda_0$ the threshold since it is the bottom of the absolutely continuous spectrum or (and) the first point of accumulation of the eigenvalues as $\varepsilon \to 0$. Let us consider two of the simplest examples: the Dirichlet problem in a half infinite cylinder and in a bounded cylinder of length $l$. In the first case, the spectrum of the negative Dirichlet Laplacian in $\Omega_\varepsilon$ is pure absolutely continuous and has multiplicity $n + 1$ on the interval $[\lambda_n, \infty)$. In the second case the spectrum consists of the set of eigenvalues $\lambda_{n,m} = \lambda_n + \varepsilon^2 m^2/l^2$, $n \geq 0$, $m \geq 1$.

It was shown in [21]-[22] that the wave propagation governed by the operator $H_\varepsilon$, $\varepsilon \to 0$, as well as the asymptotic behavior of the resolvent $(H_\varepsilon - \lambda)^{-1}$ and of the eigenvalues of $H_\varepsilon$ above $\lambda_0$ can be described in terms of the scattering solutions. While many particular cases of that problem with $\lambda = \lambda_0 + O(\varepsilon^2)$ or $\lambda < \lambda_0$ were considered (see [2]-[28]), the publications [21]-[23] were the first ones dealing with the case $\lambda \geq \lambda_0$, and the first ones where the significance of the scattering solutions for asymptotic analysis of $H_\varepsilon$ was established. In particular, it was shown there that in both cases $\lambda > \lambda_0$ and $\lambda \approx \lambda_0$, the GC of the operator on the limiting graph $\Gamma$ will be expressed in terms of
the scattering matrices of the auxiliary problems on the spider domains associated to individual junctions. A more profound analysis of the case \( \lambda \approx \lambda_0 \) can be found in [23].

The main goal of this paper is to overview the results of [22]-[23] and simplify the proofs. We will mostly deal with the case of \( \lambda \in (\lambda_0, \lambda_1) \) where the results and proofs are more transparent. The number of scattering solutions is the smallest in this case and the scattering matrix is of the smallest size (compared to the case \( \lambda > \lambda_1 \)). One of our main results is as follows.

**Theorem 1.** If \( \lambda_0 \leq \lambda \leq \lambda_1 \), then the resolvent \((H_0^{(1)} - \lambda)^{-1}\) can be approximated by \((H_0^{(1)} - (\lambda - \lambda_0))^{-1}\), where \(H_0^{(1)} = -\varepsilon^2 \frac{d^2}{d\varepsilon^2}\) is the operator of the second derivative defined on functions \( \varsigma \) on the limiting graph \( \Gamma \) with the GC of the form

\[
i\varepsilon[I_v + T_v(\lambda)]\frac{d}{d\varepsilon}\varsigma^v(0) - \sqrt{\lambda - \lambda_0}[I_v - T_v(\lambda)]\varsigma^v(0) = 0,
\]

Here \( T_v(\lambda) \) is the scattering matrix of the auxiliary problem on the spider domain which corresponds to the junction \( J_{v,\varepsilon} \), and \( \varsigma^v \) is the vector which consists of restrictions of the function \( \varsigma \) (defined on \( \Gamma \)) onto edges adjacent to \( v \).

To be more exact, for any compactly supported \( f \), the following relation is valid on channels outside of the support of \( f \) with exponential accuracy

\[
(H_\varepsilon - \lambda)^{-1}f \sim [(H_\varepsilon^{(1)} - (\lambda - \lambda_0))^{-1}f_0]\varphi_0(y/\varepsilon), \quad \varepsilon \to 0, \quad f_0 = = f, \varphi_0(y/\varepsilon) > .
\]

A more accurate statement of this theorem as well as some of its generalizations will be given in section 5.

Note that the eigenvalues of the problem in \( \Omega_\varepsilon \) are located not only below the threshold, but also above it. They depend on \( \varepsilon \) and move very fast on the \( \lambda \)-axis as \( \varepsilon \to 0 \). Thus one can not expect to obtain an asymptotic approximation of the resolvent \((H_\varepsilon - \lambda)^{-1}\) when \( \lambda = \lambda' > \lambda_0 \) is fixed and \( \varepsilon \to 0 \). An asymptotic approximation of the resolvent \((H_\varepsilon - \lambda)^{-1}\) as \( \varepsilon \to 0 \) can be valid only if an exponentially small (in \( \varepsilon \)), but depending on \( \varepsilon \), set on the \( \lambda \)-axis is omitted. Another option is to fix \( \lambda = \lambda' > \lambda_0 \) and pass to the limit as \( \varepsilon \to 0 \) without \( \varepsilon \) taking values in some small set which depends on \( \lambda' \).

While the condition \( \lambda > \lambda_0 \) is natural for the wave propagation, the properties of the heat and diffusion processes depend on spectrum of \( H_\varepsilon \) near \( \lambda = \lambda_0 \). As a by-product of the simpler approach to the problem introduced below, we will get a better result concerning the asymptotic behavior of the eigenvalues of \( H_\varepsilon \) in bounded domains \( \Omega_\varepsilon \) as \( \varepsilon \to 0 \), \( \lambda = \lambda_0 + O(\varepsilon^2) \). It was shown in [22], [23] that the main terms of the eigenvalues of \( H_\varepsilon \) when \( \lambda = \lambda_0 + O(\varepsilon^2), \varepsilon \to 0 \), coincide with the eigenvalues of the operator on the limiting graph with the GC defined by the scattering matrix at \( \lambda = \lambda_0 \). An explicit description of GC at \( \lambda = \lambda_0 \) for arbitrary junctions (of order \( O(\varepsilon) \)) was also given there. Significantly later (see publications in arXiv), some of our results were repeated in [14]. The new elements there are the location of the eigenvalues below the threshold and more accurate asymptotics of eigenvalues near the threshold. We will show here that the approach used
in [22] and [23] provides an approximation of the eigenvalues near the threshold with an exponential accuracy as well as the location of the eigenvalues below the threshold.

The plan of the paper is the following. The elliptic problem in \( \Omega_\varepsilon \) with a fixed \( \varepsilon = 1 \) will be studied in the next section. In particular, the scattering solutions are defined there. The asymptotic behavior of the resolvent \((H_\varepsilon - \lambda)^{-1}\), of the spectrum and of the scattering solutions as \( \varepsilon \to 0 \), \( \lambda > \lambda_0 \), are obtained in section 3 for the simplest domains with one junction (spider domains). The one dimensional problem on the limiting graph will be studied in section 4. The case of arbitrary domains \( \Omega_\varepsilon \) is considered in section 5. The last section is devoted studying the spectrum near the threshold.

2 Scattering solutions and analytic properties of the resolvent when \( \varepsilon \) is fixed.

We introduce Euclidean coordinates \((t, y)\) in channels \( C_{j,\varepsilon} \) chosen in such a way that the \( t \)-axis is parallel to the axis of the channel (so, \( t \) is not time, but space variable!), hyperplane \( R_y^{d-1} \) is orthogonal to the axis, and \( C_{j,\varepsilon} \) has the following form in the new coordinates:

\[
C_{j,\varepsilon} = \{(t, \varepsilon y) : 0 < t < l_j, \ y \in \omega\}.
\]

If a channel \( C_{j,\varepsilon} \) is bounded \((l_j < \infty)\), the direction of the \( t \) axis can be chosen arbitrarily (at least for now). If a channel is unbounded, then \( t = 0 \) corresponds to its cross-section which is attached to the junction.

Let us recall the definition of scattering solutions for the problem \((1)\) in \( \Omega_\varepsilon \) when \( \lambda \in (\lambda_0, \lambda_1) \). Consider the non-homogeneous problem

\[
( -\varepsilon^2 \Delta - \lambda )u = f, \ x \in \Omega_\varepsilon; \quad Bu = 0 \quad \text{on} \ \partial \Omega_\varepsilon.
\]

**Definition 2.** Let \( f \in L^2_{\text{comp}}(\Omega_\varepsilon) \) have a compact support, and \( \lambda < \lambda_1 \). A solution \( u \) of \((3)\) is called outgoing if it has the following asymptotic behavior at infinity in each infinite channel \( C_{j,\varepsilon} \), \( 1 \leq j \leq m \):

\[
u = a_j e^{i \sqrt{\lambda - \lambda_0} \varepsilon t} \varphi_0(y/\varepsilon) + O(e^{-\alpha t}), \quad \alpha > 0.
\]

**Remarks.** 1. Here and everywhere below we assume that

\[
\text{Im} \sqrt{\lambda - \lambda_0} \geq 0.
\]

Thus, outgoing solutions decay at infinity if \( \lambda < \lambda_0 \).

2. Obviously, if \((4)\) holds with some \( \alpha > 0 \), then it holds with any \( \alpha < \sqrt{\lambda_1 - \lambda} \).

**Definition 3.** Let \( \lambda < \lambda_1 \). A function \( \Psi = \Psi_p(\varepsilon) \), \( 1 \leq p \leq m \), is called a solution of the scattering problem in \( \Omega_\varepsilon \) if

\[
( -\varepsilon^2 \Delta - \lambda )\Psi = 0, \ x \in \Omega_\varepsilon; \quad B\Psi = 0 \quad \text{on} \ \partial \Omega_\varepsilon.
\]
and $\Psi$ has the following asymptotic behavior in infinite channels $C_{j,\varepsilon}$, $1 \leq j \leq m$:

$$
\Psi_p^{(\varepsilon)} = [\delta_{p,j} e^{-i\sqrt{\lambda - \lambda_0}/\varepsilon} + t_{p,j} e^{i\sqrt{\lambda - \lambda_0}/\varepsilon}] \varphi_0(y/\varepsilon) + O(e^{-\alpha t/\varepsilon}), \quad t \to \infty, \quad \alpha > 0.
$$

(7)

Here $\delta_{p,j}$ is the Kronecker symbol, i.e. $\delta_{p,j} = 1$ if $p = j$, $\delta_{p,j} = 0$ if $p \neq j$.

**Remark.** If $\lambda_0 < \lambda < \lambda_1$, then the term with the coefficient $\delta_{p,j}$ in (7) corresponds to the incident wave (coming through the channel $C_{p,\varepsilon}$), the terms with coefficients $t_{p,j}$ describe the transmitted waves. The transmission coefficients $t_{p,j} = t_{p,j}(\varepsilon, \lambda)$ depend on $\varepsilon$ and $\lambda$.

The matrix

$$
T = [t_{p,j}]
$$

is called the **scattering matrix**. Note that the scattering solution and scattering matrix are defined for all $\lambda < \lambda_1$. We assume that $\text{Im} \sqrt{\lambda - \lambda_0} > 0$ when $\lambda < \lambda_0$, and the incident wave is growing (exponentially) at infinity in this case.

The outgoing and scattering solutions are defined similarly when $\lambda \in (\lambda_n, \lambda_{n+1})$ (see [22]). In this case, any outgoing solution has $n + 1$ waves in each channel propagating to infinity with the frequencies $\sqrt{\lambda - \lambda_s}/\varepsilon$, $0 \leq s \leq n$. There are $m(n + 1)$ scattering solutions: the incident wave may come through one of $m$ infinite channels with one of $(n + 1)$ possible frequencies. The scattering matrix has the size $m(n + 1) \times m(n + 1)$ in this case.

Standard arguments based on the Green formula provide the following statement.

**Theorem 4.** When $\lambda_0 < \lambda < \lambda_1$, the scattering matrix $T$ is unitary and symmetric ($t_{p,j} = t_{j,p}$).

The operator $H_\varepsilon = -\varepsilon^2 \Delta$, which corresponds to the eigenvalue problem (1), is non-negative, and therefore the resolvent

$$
R_\lambda = (H_\varepsilon - \lambda)^{-1} : L^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon)
$$

(9)

is analytic in the complex $\lambda$-plane outside the positive semi-axis $\lambda \geq 0$. If $\Omega_\varepsilon$ is bounded (all the channels are finite), then operator $R_\lambda$ is meromorphic in $\lambda$ with a discrete set $\Lambda = \{\mu_{j,\varepsilon}\}$ of poles of first order at the eigenvalues $\lambda = \mu_{j,\varepsilon}$ of operator $H_\varepsilon$. If $\Omega_\varepsilon$ has at least one infinite channel, then the spectrum of $H_\varepsilon$ has more complicated structure (see Theorem 5 below). In this case, the spectrum has an absolutely continuous component $[\lambda_0, \infty)$, and the resolvent $R_\lambda$ is meromorphic in $\lambda \in C \setminus [\lambda_0, \infty)$. We are going to consider the analytic extension of the operator $R_\lambda$ to the absolutely continuous spectrum. One can extend $R_\lambda$ analytically from above ($\text{Im} \lambda > 0$) or below, if it is considered as an operator in the following spaces (with a smaller domain and a larger range):

$$
R_\lambda : L^2_{\text{com}}(\Omega_\varepsilon) \to L^2_{\text{loc}}(\Omega_\varepsilon).
$$

(10)

These extensions do not coincide on $[\lambda_0, \infty)$. To be specific, we always will consider extensions from the upper half plane ($\text{Im} \lambda > 0$). We will call (10) truncated resolvent of the operator $H_\varepsilon$, since it can be identified with the resolvent (9) multiplied from the left and right by a cut-off function.
Theorem 5. Let $\Omega_\varepsilon$ have at least one infinite channel. Then

(1) The spectrum of the operator $H_\varepsilon$ consists of the absolutely continuous component $[\lambda_0, \infty)$ and, possibly, a discrete set $\{\mu_{j,\varepsilon}\}$ of non-negative eigenvalues $\lambda = \mu_{j,\varepsilon} \geq 0$ with the only possible limiting point at infinity. The multiplicity of the a.c. spectrum changes at points $\lambda = \lambda_n$, and is equal to $m(n + 1)$ on the interval $(\lambda_n, \lambda_{n+1})$.

(2) The operator (10) admits a meromorphic extension from the upper half plane $\text{Im} \lambda > 0$ onto $[\lambda_0, \infty)$ with the branch points at $\lambda = \lambda_n$ of the second order and poles of first order at $\lambda = \mu_{j,\varepsilon}$. In particular, if $\lambda_n \in \{\mu_{j,\varepsilon}\}$, then operator (10) has the form

$$R_\lambda = \frac{A(n)}{\lambda - \lambda_n} + O\left(\frac{1}{\sqrt{|\lambda - \lambda_n|}}\right), \quad \lambda \to \lambda_n.$$

(3) If $f \in L^2_{\text{con}}(\Omega_\varepsilon), \lambda < \lambda_1$, and $\lambda$ is not a pole or the branch point ($\lambda = \lambda_0$) of the operator (10), then the problem (3), (4) is uniquely solvable and the outgoing solution $u$ can be found as the $L^2_{\text{loc}}(\Omega_\varepsilon)$ limit

$$u = R_{\lambda+i0}f.$$

(4) There exist exactly $m$ different scattering solutions for the values of $\lambda < \lambda_1$ which are not a pole or the branch point of the operator (10), and the scattering solution is defined uniquely after the incident wave is chosen.

(5) The scattering matrix $T$ is analytic in $\lambda$, when $\lambda < \lambda_1$, with a branch point of second order at $\lambda = \lambda_0$ and without real poles.

The matrix $T$ is orthogonal if $\lambda < \lambda_0$.

Remark. Let $\lambda_n \notin \{\mu_{j,\varepsilon}\}$. If the homogeneous problem (3) with $\lambda = \lambda_n$ has a nontrivial solution $u$ such that

$$u = a_j \varphi_n(y/\varepsilon) + O(e^{-\gamma t}), \quad x \in C_{j,\varepsilon}, \quad t \to \infty, \quad 1 \leq j \leq m, \quad \gamma > 0,$$

then $R_{\lambda+i0} = \frac{B(n)}{\sqrt{\lambda - \lambda_n}} + O(1),$ $\lambda \to \lambda_n$. If such a solution $u$ does not exist, then operator (10) is bounded in a neighborhood of $\lambda = \lambda_n$.

Proof of Theorem 5. All the statements above concern the problem with a fixed value of $\varepsilon$ and can be proved using standard elliptic theory arguments. A detailed proof can be found in [22], a shorter version is given below.

In order to prove the part of statement (1) concerning the absolutely continuous spectrum of the operator $H = -\Delta$, we split the domain $\Omega_\varepsilon$ into pieces by introducing cuts along the bases $t = 0$ of all infinite channels. We denote the new (not connected) domain by $\Omega'_\varepsilon$, and denote the negative Dirichlet Laplacian in $\Omega'_\varepsilon$ by $H'_{\varepsilon}$, i. e. $H'_{\varepsilon}$ is obtained from $H_{\varepsilon}$ by introducing additional Dirichlet boundary conditions on the cuts. Obviously, the operator $H'_{\varepsilon}$ has the absolutely continuous spectrum described in statement (1) of the theorem. Since the wave operators for the couple $H_{\varepsilon}, H'_{\varepsilon}$ exist and are complete (see [1]), the operator $H_{\varepsilon}$ has the same absolutely continuous spectrum as $H'_{\varepsilon}$.

The remaining part of statement (1) and statements (2) and (3) can be proved by one of the well known equivalent approaches based on a reduction of the boundary problem.
to a Fredholm equation which depends analytically on $\lambda$. For this purpose one can use a parametrix (almost inverse operator): equation (3) is solved separately in channels and junctions, and then the parametrix can be constructed from those local inverse operators using a partition of unity (allowing one to glue the local inverse operators), see [22]. A similar approach is based on gluing together these local inverse operators using Dirichlet-to-Neumann maps on the cuts of the channels which were introduced in the previous paragraph.

Statements (4) and (5) follow immediately from statement (3) and Theorem 4. Indeed, one can look for the solution $\Psi_p^{(\varepsilon)}$ of the scattering problem in the form

$$\Psi_p^{(\varepsilon)} = \chi e^{-i\sqrt{\lambda - \lambda_0} \varepsilon t} \varphi_0(y/\varepsilon) + u$$

where $\chi \in C^\infty(\Omega_\varepsilon)$, $\chi = 0$ outside $C_{p,\varepsilon}$, $\chi = 1$ in $C_{p,\varepsilon} \cap \{t > 1\}$. This reduces problem (3), (4) to problem (3), (4) for $u$ with $f$ supported on $C_{p,\varepsilon} \cap \{0 \leq t \leq 1\}$. Statement (3) of the theorem, applied to the latter problem, justifies statement (4). Function $u$, defined in (13), satisfies the homogeneous equation (3) in infinite channels $C_{j,\varepsilon}$, $j \neq p$, and in $C_{p,\varepsilon} \cap \{t > 1\}$, and it is meromorphic at the bottoms of these channels (at $t = 0$ for $j \neq p$, and $t = 1$ when $j = p$). Solving the problems in these channels by separation of variables, we obtain that the scattering matrix $T$ is meromorphic in $\lambda$, when $\lambda < \lambda_1$ with a branch point of second order at $\lambda = \lambda_0$. It also follows from here that $T$ is real valued when $\lambda < \lambda_0$. Analyticity of $T$ and Theorem 4 imply that $T$ is orthogonal when $\lambda < \lambda_0$. From the orthogonality ($\lambda < \lambda_0$) and unitarity ($\lambda_0 < \lambda < \lambda_1$) of $T$ it follows that $T$ does not have poles.

\section{Spider domains, $\varepsilon \to 0$.}

Let us recall that $\Omega_\varepsilon$ is called a spider domain if it is self-similar (see (2)) and consists of one junction and several semi-infinite channels.

\begin{theorem}
Let $\Omega_\varepsilon$ be a spider domain and $\lambda < \lambda_1$. Then

(1) the eigenvalues $\lambda = \mu_{j,\varepsilon} = \mu_j$ of operator $H_\varepsilon$ and the scattering matrix $T$ do not depend on $\varepsilon$,

(2) the truncated resolvent (10) has the following estimate: if $f$ is supported on $\varepsilon$-neighborhood of the junction, then

$$|R_\lambda f| \leq C\delta^{-1} \varepsilon^{-d/2} \|f\|_{L^2}, \quad \lambda < \lambda_1, \quad \delta = \text{dist}(\lambda, \{\mu_j\})$$

outside of $2\varepsilon$-neighborhood of the junction,

(3) the scattering solutions have the following form on the channels of the domain:

$$\Psi_p^{(\varepsilon)} = [\delta_{p,j} e^{-i\sqrt{\lambda - \lambda_0} \varepsilon t} + t_{p,j} e^{i\sqrt{\lambda - \lambda_0} \varepsilon t}] \varphi_0(y/\varepsilon) + r_{p,j}, \quad x \in C_{j,\varepsilon}, \quad 1 \leq j \leq m,$$

where $|r_{p,j}| \leq C\delta^{-1} \varepsilon^{-\alpha - \varepsilon}$ when $\varepsilon > 0$, $\frac{1}{\varepsilon} \geq 1$ and $0 \leq \lambda < \lambda_1$. Here $\alpha < \sqrt{\lambda_1 - \lambda}$ is arbitrary, $C = C(\alpha)$. \hfill $\Box$

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Remark. Formula (15) looks similar to the definition (7). In fact, the remainder in (7) decays only when \( t \to \infty \), and (7) does not allow us to single out the main term of asymptotics of scattering solutions as \( \varepsilon \to 0 \).

Proof. All the statements above follow immediately from self-similarity of the domain \( \Omega_\varepsilon \). Namely, we make the substitution

\[
x \to x - \hat{x} \varepsilon
\]

(see (2)) and reduce problem (3) in \( \Omega_\varepsilon \) to the problem in \( \Omega \) which corresponds to \( \varepsilon = 1 \). These two problems have the same eigenvalues and scattering matrices. This justifies the first statement. Let \( v_\lambda, g \) be functions \( R_\lambda f, f \) after the change of variables (16). From statement (2) of Theorem 5 it follows that

\[
||v_\lambda||_{L^2(K)} \leq C\delta^{-1}||g||_{L^2} = C\delta^{-1}e^{-d/2}||f||_{L^2},
\]

where \( K \) consists of the parts of the channels of \( \Omega \) where \( 1 < t < 3 \). Then the standard a priori estimates for the solutions of the equation \( \Delta u - \lambda u = 0 \) imply the same estimate for \( |v_\lambda| \) on the cross sections \( t = 2 \):

\[
|v_\lambda| \leq C\delta^{-1}\varepsilon^{-d/2}||f||_{L^2}, \quad t = 2.
\]

The latter implies the same estimate for \( |v_\lambda| \) when \( t > 2 \) by solving the equation \( \Delta u - \lambda u = 0 \) in the corresponding regions of the channels of \( \Omega \) with the boundary condition at \( t = 2 \). This justifies the second statement of Theorem 6. The last statement can be proved absolutely similarly. We reduce the scattering problem in \( \Omega_\varepsilon \) to the scattering problem in \( \Omega \) and use representation (13) with \( \varepsilon = 1 \). This implies (7) with \( \varepsilon = 1 \) and the remainder term \( r_{p,j} \) such that \( |r_{p,j}| \leq C\delta^{-1}e^{-\alpha t} \) for \( t > 1 \). It remains only to make the substitution inverse to (16).

In spite of its simplicity, Theorem 6 allows us to obtain two very important results: small \( \varepsilon \) asymptotics of the spectrum of \( H_\varepsilon \) and the resolvent \( (H_\varepsilon - \lambda)^{-1} \) for arbitrary networks of thin wave guides \( \Omega_\varepsilon \). For this purpose, we need to rewrite (15) in a slightly different (less explicit) form.

We denote by \( \varsigma_{p,j} \) the linear combination of exponents in the square brackets in (15). This is a function on the edge \( \Gamma_j \) of the graph. Let \( \varsigma_p \) be the vector-column with components \( \varsigma_{p,j}, 1 \leq j \leq m \). Obviously, \( \varsigma_p \) satisfies the equation

\[
(\varepsilon^2 \frac{d^2}{dt^2} + \lambda - \lambda_0)\varsigma = 0.
\]

We will use notation \( \Psi_p^{(\varepsilon)} \) for both the scattering solution and the column-vector whose components \( \Psi_{p,j}^{(\varepsilon)} \) are restrictions of the scattering solution \( \Psi_p^{(\varepsilon)} \) on the channels \( C_{j,\varepsilon} \), \( 1 \leq j \leq m \). Then (15) can be rewritten in the vector form as

\[
\Psi_p^{(\varepsilon)} = \varsigma_p \varphi_0(y/\varepsilon) + r_p^{(\varepsilon)} = [e_p e^{-i \sqrt{\lambda - \lambda_0} t} + t_p e^{i \sqrt{\lambda - \lambda_0} t}] \varphi_0(y/\varepsilon) + r_p^{(\varepsilon)},
\]
where \( x \in \bigcup_{1 \leq j \leq m} C_j, \) \( r_p^{(e)} \) is the vector with components \( r_{p,j}^{(e)} \), all components \( e_{p,j} \) of the vector \( e_p \) are zeroes except \( e_{p,p} \) which is equal to one, and \( t_p \) is the \( p \)-th column of the scattering matrix \( T \). Let us construct the \( m \times m \) matrix with columns \( \Psi_p^{(e)} \) and the matrix \( \varsigma \) with columns \( \varsigma_p \), \( 1 \leq p \leq m \). As it is easy to see, \( \varsigma(0) = (I + T) \), \( \varsigma'(0) = i\sqrt{\lambda - \lambda_0} \epsilon (I - T) \), and therefore, \( i\epsilon (I + T)\varsigma'(0) - \sqrt{\lambda - \lambda_0} (I - T)\varsigma(0) = 0 \). (19)

Of course, this equality also holds for individual columns \( \varsigma_p \) of matrix \( \varsigma \).

It is essential for extending the results to arbitrary networks of wave guides \( \Omega_\epsilon \) that the gluing condition (19) together with some condition at infinity is equivalent to the explicit form of \( \varsigma_p \) given by (18). Namely, let \( \varsigma \) satisfy (17). Then

\[
\varsigma = \alpha_p e^{-i\sqrt{\lambda - \lambda_0} \epsilon t} + \beta_p e^{i\sqrt{\lambda - \lambda_0} \epsilon t}
\]

with some constant vectors \( \alpha_p, \beta_p \). We will say that \( \varsigma = \psi_p \) is a solution of the scattering problem on the graph \( \Gamma \) with the incident wave coming through the edge \( \Gamma_p \) if \( \psi_p \) satisfies equation (17), GC (19), and \( \alpha_p = e_p \), i.e.,

\[
\psi_p = e_p e^{-i\sqrt{\lambda - \lambda_0} \epsilon t} + \beta_p e^{i\sqrt{\lambda - \lambda_0} \epsilon t} \quad (20)
\]

Thus, we specify the incident wave and impose the GC defined by the scattering problem in \( \Omega_\epsilon \), but we do not specify the scattering coefficients of the outgoing wave. The next theorem shows that the scattering problem on the graph will have the same scattering coefficients as the problem on \( \Omega_\epsilon \).

**Theorem 7.** Formulas (15), (18) and \( \Psi_p^{(e)} = \psi_p \phi_0(y/\epsilon) + r_p^{(e)} \) are equivalent.

**Proof.** It was already shown that \( \varsigma_p \) defined in (18) satisfies (19). Conversely, if we write \( \beta_p \) in (20) as \( t_p + h_p \) and put (20) into (19), we will get that \( h_p = 0 \), i.e. \( \psi_p \) coincides with \( \varsigma_p \) defined in (18). \( \square \)

### 4 One-dimensional problem on the graph.

The spectrum of the operator \( H_\epsilon \) an the asymptotic behavior of the resolvent will be expressed in terms of the solutions of a problem on the limiting graph \( \Gamma \) which is studied in this section.

Let \( \Omega_\epsilon \) be an arbitrary (bounded or unbounded) domain described in the introduction, and let \( \Gamma \) be the corresponding limiting graph. Points of \( \Gamma \) will be denoted by \( \gamma \) with \( t \) being a parameter on each edge \( \Gamma_j \) of the graph. We are going to introduce a special spectral problem

\[
h_\epsilon \varsigma := -\epsilon^2 \frac{d^2}{dt^2} \varsigma = (\lambda - \lambda_0) \varsigma \quad (21)
\]
on smooth functions \( \varsigma = \varsigma(\gamma) \) on \( \Gamma \) which satisfy the following GC at vertices. We split the set \( V \) of vertices \( v \) of the graph into two subsets \( V = V_1 \cup V_2 \), where the vertices from the set \( V_1 \) have degree 1 and correspond to the free ends of the channels, and the vertices from the set \( V_2 \) have degree at least two and correspond to the junctions \( J_{v,\epsilon} \). We keep the same BC at \( v \in V_1 \) as at the free end of the corresponding channel of \( \Omega_\epsilon \) (see \( (1) \)):

\[
B \varsigma = 0 \quad \text{at} \quad v \in V_1.
\]  

(22)

The GC at each vertex \( v \in V_2 \) will be defined in terms of an auxiliary scattering problem for a spider domain \( \Omega'_{v,\epsilon} \). This domain is formed by the individual junction \( J_{v,\epsilon} \) which corresponds to the vertex \( v \), and all channels with an end at this junction, where the channels are extended to infinity if they have a finite length. Let \( T = T_v(\lambda) \) be the scattering matrix for the problem \( (1) \) in the spider domain \( \Omega'_{v,\epsilon} \) and let \( I \) be the unit matrix of the same size as the size of \( T \).

We choose the parametrization on \( \Gamma \) in such a way that \( t = 0 \) at \( v \) for all edges adjacent to this particular vertex. Let \( d = d(v) \geq 2 \) be the order (the number of adjacent edges) of the vertex \( v \in V_2 \).

For any function \( \varsigma \) on \( \Gamma \), we form a column-vector \( \varsigma(v) = \varsigma(v)(t) \) with \( d(v) \) components which is formed by the restrictions of \( \varsigma \) on the edges of \( \Gamma \) adjacent to \( v \). We will need this vector only for small values of \( t \geq 0 \).

The components of the vector \( \varsigma(v) \) are taken in the same order as the order of channels of \( \Omega'_{v,\epsilon} \).

The GC at the vertex \( v \in V_2 \) has the form

\[
i\epsilon[I_v + T_v(\lambda)] \frac{d}{dt} \varsigma^{(v)}(t) - \sqrt{\lambda - \lambda_0} [I_v - T_v(\lambda)] \varsigma^{(v)}(t) = 0, \quad t = 0, \quad v \in V_2,
\]  

(23)

if \( \lambda \neq \lambda_0 \). Condition \( (23) \) can degenerate if \( \lambda = \lambda_0 \), and it requires some regularization in this case.

Solutions of \( (21) \) have the following form

\[
\varsigma = a_je^{i\frac{\sqrt{\lambda - \lambda_0}}{\epsilon} t} + b_je^{-i\frac{\sqrt{\lambda - \lambda_0}}{\epsilon} t}, \quad \gamma \in \Gamma_j.
\]

If \( \text{Im} \lambda > 0 \) and \( \varsigma \in L^2(\Gamma) \), then \( b_j = 0 \) for infinite edges (see \( (5) \)). Thus, if \( \varsigma \) satisfies equation \( (21) \) in a neighborhood of infinity, then

\[
\varsigma = a_je^{i\frac{\sqrt{\lambda - \lambda_0}}{\epsilon} t}, \quad \gamma \in \Gamma_j, \quad 1 \leq j \leq m, \quad t >> 1.
\]  

(24)

We will assume that condition \( (24) \) holds also when \( \lambda \) is real, i.e., we consider only those solutions of \( (21) \) with real \( \lambda = \lambda' > \lambda_0 \) which can be obtained as the limit of solutions with complex \( \lambda = \lambda' + i\epsilon \) when \( \epsilon \to 0 \).

We will call function \( g = g_\lambda(\gamma, \xi; \epsilon), \quad \gamma, \xi \in \Gamma \), the Green function of the problem \( (21)-(24) \) if it satisfies the equation (with respect to variable \( \gamma \) ):

\[
- \epsilon^2 \frac{d^2}{dt^2} g - (\lambda - \lambda_0) g = \delta_\xi(\gamma),
\]  

(25)

and conditions \( (22)-(24) \). Here \( \xi \) is a point of \( \Gamma \) which is not a vertex, and \( \delta_\xi(\gamma) \) is the delta function supported on \( \gamma = \xi \).
Lemma 8. Let $\lambda < \lambda_1$, $\lambda \neq \lambda_0$. Operator $h_\varepsilon = -\varepsilon^2 \frac{d^2}{dt^2}$ is symmetric on the space of smooth, compactly supported functions on $\Gamma$ which satisfy conditions (22) and (23).

Proof. One needs only to show that
\[
\left\langle \frac{d}{dt} \varsigma_1^{(v)}(t), \varsigma_2^{(v)}(t) \right\rangle - \left\langle \varsigma_1^{(v)}(t), \frac{d}{dt} \varsigma_2^{(v)}(t) \right\rangle = 0, \quad t = 0, \quad v \in V_2, \tag{26}
\]
for any two vector functions $\varsigma = \varsigma_1^{(v)}$, $\varsigma = \varsigma_1^{(v)}$ which satisfy GC (23) (similar relation at $v \in V_1$ obviously holds). Let $\lambda \in (\lambda_0, \lambda_1)$. Then matrix $T_v(\lambda)$ is unitary (Theorem 4). If matrix $I_v + T_v$ is non-degenerate, we rewrite (23) in the form $\frac{d}{dt} \varsigma^{(v)}(t) = A \varsigma^{(v)}(t)$, $t = 0$, where the matrix
\[
A = \frac{\sqrt{\lambda - \lambda_0}}{i \varepsilon} [I_v + T_v(\lambda)]^{-1}[I_v - T_v(\lambda)]
\]
is real. The latter immediately implies (26). Similar arguments can be used if $I_v - T_v$ is non-degenerate. If both matrices are degenerate (i.e., $T_v$ has both eigenvalues, $\pm 1$), we consider a unitary matrix $U$ such that $U T_v U^*$ is a diagonal unitary matrix. Since $\langle U \varsigma_1, U \varsigma_2 \rangle = \langle \varsigma_1, \varsigma_2 \rangle$ for any two vectors $\varsigma_1, \varsigma_2$, one can easily reduce the proof of (26) to the case when $T_v$ is a diagonal unitary matrix. Then (23) implies the following relations for coordinates $\varsigma_j(t)$ of the vector $\varsigma^{(v)}(t) : \varsigma'_j(0) = a_j \varsigma_j(0)$ or $\varsigma_j(0) = b_j \varsigma'_j(0)$, where constants $a_j, b_j$ are real. The first case occurs if the corresponding diagonal element of $T_v$ differs from $-1$, the second relation is valid if this element is $-1$. These relations for $\varsigma_j(t)$ imply (26). Similar arguments can be used to prove (26) when $\lambda < \lambda_0$, since matrix $T_v$ is orthogonal in this case (see Theorem 5).

Theorem 9. For any $\varepsilon > 0$ there is a discrete set $\Lambda(\varepsilon)$ on the interval $[-\lambda_0, \lambda_1)$ such that the Green function $g_{\lambda}(\gamma, \xi; \varepsilon)$ exists for all $\lambda < \lambda_1$, $\lambda \notin \Lambda(\varepsilon)$, and has the form
\[
g_{\lambda} = \frac{h(\gamma, \xi, \lambda, \varepsilon)}{D(\lambda, \varepsilon)}, \tag{27}
\]
where function $h$ is continuous on the set $\gamma, \xi \in \Gamma$, $\lambda < \lambda_1$, $\varepsilon > 0$ and uniformly bounded on each bounded subset, and
\[
D(\lambda, \varepsilon) = \sum_{m=1}^N c_m(\lambda) e^{i \sqrt{\lambda - \lambda_0} s_m}. \tag{28}
\]
Here $s_m$ are constants, functions $c_m(\lambda)$ are analytic in $\lambda < \lambda_1$ with a branch point of second order at $\lambda = \lambda_0$, and $D \neq 0$ if $\lambda < \lambda_0$.

Proof. We fix the parametrization on each edge $\Gamma_j$ of the graph. Then, obviously,
\[
g_{\lambda} = a_j e^{-i \sqrt{\lambda - \lambda_0} t} + b_j e^{i \sqrt{\lambda - \lambda_0} t}, \quad \gamma \in \Gamma_j, \quad \text{if} \quad \xi \notin \Gamma_j, \tag{29}
\]
\[
g_{\lambda} = a_j e^{-i \sqrt{\lambda - \lambda_0} t} + b_j e^{i \sqrt{\lambda - \lambda_0} t} + \frac{\varepsilon}{\sqrt{\lambda - \lambda_0}} \sin[\sqrt{\lambda - \lambda_0} \varepsilon (t - \tau)], \quad \text{if} \quad \xi \in \Gamma_j. \tag{30}
\]
Here \( \tau \) is the coordinate of the point \( \xi \), \( (t - \tau)_- = \min(t - \tau, 0) \), and the last term in (30) is a particular solution of (25) on \( \Gamma_j \) with a bounded support. There are \( 2N \) unknown constants in the formulas above where \( N \) is the total number of edges of the graph. Conditions (22)-(24) provide \( 2N \) linear equations for these constants. As it is easy to see, the coefficients for unknowns in all the equations have the form \( a(\lambda)e^{s}\sqrt{\lambda - \lambda_0} \), where \( a(\lambda) \) is analytic in \( \lambda < \lambda_1 \) with a branch point of second order at \( \lambda = \lambda_0 \), and \( s = 0 \) or \( \pm l_j \) (\( l_j \) are the lengths of the finite channels). The exponential factors in the coefficients appear when the formulas (29), (30) are substituted into GC at the end point of the edge \( \Gamma_j \) where \( t = l_j \).

We apply Cramer’s rule to solve this system of \( 2N \) equations. This immediately provides all the statements of the theorem with \( D(\lambda, \varepsilon) \) being the determinant of the system. One only needs to show that \( D \neq 0 \) for \( \lambda < \lambda_0 \). Note that the latter fact implies the discreteness of the set \( \Lambda(\varepsilon) = \{ \lambda : D(\lambda, \varepsilon) = 0 \} \).

Obviously, \( D = 0 \) if and only if the homogeneous problem (21)-(24) has a non-trivial solution. Let \( \lambda < \lambda_0 \). Then solutions \( \varsigma \) of the problem (21)-(24) decay at infinity, and

\[
0 = \int_{\Gamma}[-\varepsilon^2\varsigma'' - (\lambda - \lambda_0)\varsigma]d\gamma
= -\varepsilon^2\Sigma_v \left< \frac{d}{dt}\varsigma^{(v)}, \varsigma^{(v)} \right>|_v + \int_{\Gamma}[\varepsilon^2(\varsigma')^2 - (\lambda - \lambda_0)\varsigma^2]d\gamma. \tag{31}
\]

It was shown in the proof of Lemma 8 that it is enough to consider only diagonal matrices \( T \) when the terms under the sign \( \Sigma_v \) above with \( v \in V_2 \) are evaluated. Since \( T \) is orthogonal when \( \lambda < \lambda_0 \), the diagonal elements of \( T \) are equal to \( \pm 1 \). Then (23) means that each component of the vector \( \varsigma^{(v)} \) or its derivative is zero at the vertex. Hence, the terms in the sum above with \( v \in V_2 \) are equal to zero. They are zeroes also for those \( v \in V_1 \) where the boundary condition in (22) is the Dirichlet or Neumann condition. If \( v \in V_1 \) and \( B = \varepsilon\frac{d}{dt} + a \), \( a \geq 0 \), these terms are non-positive. Hence, relation (31) implies that \( \varsigma = 0 \) when \( \lambda < \lambda_0 \). \( \square \)

Theorem 9 does not contain a statement concerning the structure of the discrete set \( \Lambda(\varepsilon) \). This set becomes more and more dense when \( \varepsilon \to 0 \). In general, every point \( \lambda' \in(\lambda_0, \lambda_1) \) belongs to \( \Lambda(\varepsilon) \) for some sequence of \( \varepsilon = \varepsilon_j(\lambda') \to 0 \). However, it is not an absolutely arbitrary discrete set, but the set of zeroes of a specific analytic function (28), and this fact provides the following restriction on the set \( \Lambda(\varepsilon) \).

**Lemma 10.** For each bounded interval \( [\alpha, \lambda_1] \), each \( \sigma > 0 \) and some \( M \), there are \( c\varepsilon^{-1} \) intervals \( I_j \) of length \( \sigma \) such that

\[
|D(\lambda, \varepsilon)| > c\sigma^M \quad \text{when} \quad \varepsilon > 0, \quad \lambda \in [\alpha, \lambda_1] \cup I_j, \quad c = c(\alpha).
\]

This Lemma is a particular case of Lemma 15 from [22] (the set \( \Gamma_0 \) is empty in the case which is considered in this paper).

In order to construct the resolvent of the problem in \( \Omega_\varepsilon \), we need to represent the Green function \( g_\lambda \) of the problem (25), (22)-(24) on the graph \( \Gamma \) through the solutions of the scattering problems on the spider subgraphs of \( \Gamma \).
We will call a function \( \psi = \psi_p(\gamma) \) solution of the scattering problem on the graph \( \Gamma \) if it satisfies the equation (21), conditions (22), (23) and has the following form at unbounded edges of the graph:

\[
\psi_p(\gamma) = \delta_{p,j} e^{-i\sqrt{\lambda - \lambda_0} \varepsilon t} + a_{p,j} e^{i\sqrt{\lambda - \lambda_0} \varepsilon t}, \quad \gamma \in \Gamma_j, \quad 1 \leq p, j \leq m, \tag{32}
\]

where \( \delta_{p,j} \) is the Kronecker symbol. This scattering solution corresponds to the wave coming through the edge \( \Gamma_p \). These scattering solutions on the graphs were introduced in the previous section in the case when the graph corresponds to a spider domain. In fact, only this simple case will be needed below.

**Lemma 11.** If the graph \( \Gamma \) corresponds to a spider domain \( \Omega_\varepsilon \), then the scattering solution \( \psi_p(\gamma) \) exists and is defined uniquely for all \( \lambda < \lambda_1, \lambda \neq \lambda_0 \). Any function \( \varsigma \) on \( \Gamma \) which satisfies equation (21) and GC condition (23) is a linear combination of the scattering solutions \( \psi_p(\gamma) \).

**Remark.** For arbitrary graphs, one may have nontrivial solutions of the homogeneous problem (21)-(24) supported on the set of bounded edges of the graph. This occurs when \( \lambda \in \Lambda(\varepsilon) \). The set \( \Lambda(\varepsilon) \) is empty for spider graphs. **Proof.** If we take \( a_{p,j} = t_{p,j} \), where \( t_{p,j} \) are the scattering coefficients in the spider domain \( \Omega_\varepsilon \), then function (32) will satisfy (23) (see the derivation of (19)). Hence, the scattering solutions \( \psi_p(\gamma) \) exist for all \( \lambda < \lambda_1, \lambda \neq \lambda_0 \), since the scattering coefficients are defined for those \( \lambda \) by Theorem 5. If we put function (32) with \( a_{p,j} = t_{p,j} + h_{p,j} \) into GC (23), we immediately get that \( h_{p,j} = 0 \) (see the proof of Theorem 7). Thus, scattering solutions are defined uniquely. The space of solutions of equation (21) is \( 2m \)-dimensional. The \( m \times 2m \) dimensional matrix \((I_v + T_v(\lambda), I_v + T_v(\lambda))\) formed from coefficients in GC (23) has rank \( m \). Hence, the solution space of the problem (21), (23) is \( m \)-dimensional. Obviously, functions \( \psi_p \) are linearly independent on \( \Gamma \). Thus any solutions of (21), (23) is a linear combination of functions \( \psi_p \).

Let \( \Gamma_{j_0} \) be the edge of \( \Gamma \) which contains the point \( \xi \) (see (25)). We cut the graph \( \Gamma \) into simple graphs \( \Gamma(v) \) with one vertex \( v \) by cutting all the bounded edges at some points \( \xi_j \in \Gamma_j \). We will choose \( \xi_{j_0} = \xi \). Let us denote by \( \Gamma'(v) \) the spider graph which is obtained by extending all the edges of \( \Gamma(v) \) to infinity. Let \( \psi_{p,v}(\gamma) \) be the scattering solutions on the graph \( \Gamma'(v) \).

**Lemma 12.** There exist functions

\[
a = a_{p,v}(\lambda, \varepsilon, \xi), \quad \lambda < \lambda_1, \varepsilon > 0, \quad \xi \in \Gamma_{j_0},
\]

which are continuous, bounded on each bounded set, and such that

\[
g_\lambda = \sum_p a_{p,v}(\lambda, \varepsilon, \xi) \frac{1}{D(\lambda, \varepsilon)} \psi_{p,v}(\gamma), \quad \gamma \in \Gamma(v).
\]
Proof. It follows from the previous lemma that $g_\lambda$ can be represented as a linear combination of the scattering solutions:

$$g_\lambda = \sum_p c_{p,v} \psi_{p,v}(\gamma), \quad \gamma \in \Gamma(v).$$

In order to find the coefficients $c_{p,v}$, we note that $g_\lambda$ is equal to a combination of two exponents on the edge $\Gamma_p \subset \Gamma(v)$ with the coefficient of the incident wave equal to $c_{p,v}$:

$$g_\lambda = c_{p,v} e^{i \sqrt{\lambda - \lambda_0} t} + b_{p,v} e^{-i \sqrt{\lambda - \lambda_0} t}, \quad \gamma \in \Gamma_p \subset \Gamma(v).$$

Now $c_{p,v}$ can be found by comparing the formula above and (27) at two points of $\Gamma_p$.

5 Small $\varepsilon$ asymptotics for the problem in $\Omega_\varepsilon$.

As everywhere above, the domain $\Omega_\varepsilon$, considered below, can be bounded or unbounded. Denote by $\Lambda^0$ the union of eigenvalues of the operator (3) in all the spider domains $\Omega'_{v,\varepsilon}$ associated to $\Omega_\varepsilon$. These spider domains consist of individual junctions and all the channels adjacent to this junction. The channels are extended to infinity if they have a finite length. The set $\Lambda^0$ does not depend on $\varepsilon$ due to Theorem 6. Let us recall that $\Lambda(\varepsilon)$ is the set of eigenvalues of the one dimensional problem (21)-(24) on the limiting graph (see Theorem 9).

The eigenvalues of the operator $H_\varepsilon = -\varepsilon^2 \Delta$ of the problem (1) which are located on the interval $(-\infty, \lambda_1)$ are exponentially close to the set $\Lambda^0 \cup \Lambda(\varepsilon)$. In the process of proving this statement, we will get the asymptotic approximation of the resolvent $(H_\varepsilon - \lambda)^{-1}$ as $\varepsilon \to 0$. Namely, the following theorem will be proved.

**Theorem 13.** (1) There exists $\nu > 0$ such that the eigenvalues $\mu_{j,\varepsilon}$ of the operator $H_\varepsilon$ which belong to the interval $(-\infty, \lambda')$ with an arbitrary $\lambda' < \lambda_1$ are located in $e^{-\nu \eta \varepsilon}$-neighborhood of the set $\Lambda^0 \cup \Lambda(\varepsilon)$. Here $\alpha = \lambda_1 - \lambda'$.

(2) Let the support of $f$ belong to $\cup \Delta_j$ and let $u = R_\lambda f$ be the solution of problem (3). Here $R_\lambda$ is the truncated resolvent (4). Then for any $\eta > 0$, there exist $\nu > 0$ and $\rho = \rho(\eta) > 0$ such that $u = R_\lambda f$ has the following asymptotic behavior in all the channels outside $\eta$-neighborhood of the support of $f$

$$u = R_\lambda f = (\hat{g}_\lambda f_0) \varphi_0(\frac{y}{\varepsilon}) + O(\varepsilon^{d/2}), \quad \lambda \in (-\infty, \lambda') \setminus \Lambda^\nu, \quad \varepsilon \to 0. \quad (33)$$

Here

$$f_0 = f_0(\gamma) = \langle f, \varepsilon^{-d/2} \varphi_0(\frac{y}{\varepsilon}) \rangle, \quad \gamma \in \Gamma,$$
and $\hat{g}_\lambda$ is the integral operator on the graph $\Gamma$ whose kernel is the Green function $g_\lambda$ constructed in Theorem 9:

$$\hat{g}_\lambda f_0 = \int_\Gamma g_\lambda(\gamma, \xi; \varepsilon)f_0(\xi)d\xi.$$ 

**Remark.** Below, we also will get the asymptotics of $u = R_\lambda f$ on the support of $f$, as well as a more precise estimate of the remainder in (33).

**Proof.** Let

$$f_1 = f_1(x) = f - \varepsilon^{-d/2}f_0\varphi_0\left(\frac{y}{\varepsilon}\right), \quad x \in \Omega_\varepsilon,$$

i.e. $f_0 = f_0(\gamma)$ is the first Fourier coefficient of the expansion of $f$ with respect to the basis $\{\varepsilon^{-d/2}\varphi_j(\frac{x}{\varepsilon})\}$, and $f_1$ is the sum of all the terms of the expansion without the first one. We are going to show that $u = R_\lambda f$ has the following form on the channels of $\Omega_\varepsilon$:

$$u = R_\lambda f = (\hat{g}_\lambda f_0)\varphi_0\left(\frac{y}{\varepsilon}\right) + \chi_{R_0^\lambda f_1} + O(e^{-\rho\varepsilon}), \quad \lambda \in (-\infty, \lambda') \setminus \Lambda_\nu, \quad \varepsilon \to 0, \quad (34)$$

where $\nu, \rho > 0, \chi \in C^\infty(\Omega_\varepsilon)$ is a cut off function such that $\chi = 0$ on all the junctions, $\chi = 1$ outside of $\varepsilon$-neighborhood of junctions, and function $R_0^\lambda f_1$ is defined by solving the following simple problem in the infinite cylinder. Let $f_{1,j}$ be the restriction of $f_1$ onto the channel $C_{j,\varepsilon}$. We extend the channel $C_{j,\varepsilon}$ to infinity (in both directions) and extend $f_{1,j}$ by zero. Let $u_j$ be the outgoing solution of the equation

$$-\varepsilon^2 \Delta u - \lambda u = f_{1,j}$$

in the extended channel. Then $R_0^\lambda f_1$ is defined as $R_\lambda f_1 = u_j$ in the channel $C_{j,\varepsilon}$. Obviously, $\chi R_0^\lambda f_1$ can be considered as a function on $\Omega_\varepsilon$.

The justification of (34) and the proof of the Theorem 13 are based on an appropriate choice of the parametrix ("almost inverse operator"):

$$P_\lambda : L^2_{r,d}(\Omega_\varepsilon) \to L^2_{loc}(\Omega_\varepsilon),$$

which is defined as follows

$$P_\lambda f = (\hat{G}_\lambda f_0)\varphi_0\left(\frac{y}{\varepsilon}\right) + (\chi R_0^\lambda f_1) - \Sigma_v \chi_v R_0^\lambda \chi_v[(\varepsilon^2\Delta + \lambda)(\chi R_0^\lambda f_1) - f_1]]. \quad (35)$$

Here $L^2_{r,d}$ is a subspace of $L^2(\Omega_\varepsilon)$ which consists of functions supported on $\cup \Delta_j$. Now we are going to define and study, successively, each of the terms in the formula above. In particular, we need to show that

$$-\varepsilon^2 \Delta f - \lambda f = f_{1,j}$$

in the extended channel. Then $R_0^\lambda f_1$ is defined as $R_\lambda f_1 = u_j$ in the channel $C_{j,\varepsilon}$. Obviously, $\chi R_0^\lambda f_1$ can be considered as a function on $\Omega_\varepsilon$.

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Here $L^2_{r,d}$ is a subspace of $L^2(\Omega_\varepsilon)$ which consists of functions supported on $\cup \Delta_j$. Now we are going to define and study, successively, each of the terms in the formula above. In particular, we need to show that

$$-\varepsilon^2 \Delta f - \lambda f = f + Q_\lambda f, \quad Q_\lambda : L^2_{r,d} \to L^2_{r,d}, \quad ||Q_\lambda|| \leq Ce^{-\bar{\rho}\varepsilon}. \quad (36)$$

Operator $\hat{G}_\lambda$ is an integral operator with the kernel $G_\lambda(x, z; \varepsilon), x, z \in \Omega_\varepsilon$, which is defined as follows. We split $\Omega_\varepsilon$ onto domains $\Omega_{v,\varepsilon}$ by cutting all the finite channels $C_{j,\varepsilon}$ using the cross-sections $t = t_j$. Let $z \in \Delta_{j_0}$. Then we choose $t_{j_0}$ to be equal to the
coordinate $t = t(z)$ of the point $z$. Other cross-sections are chosen with the only condition that $\tau < t_j < l_j - \tau$, i.e., the cross-section $t = t_j$ is strictly inside of $\Delta_j$. Let $\Omega_{v,\varepsilon}$ be the spider domain which we get by extending all the finite channels of $\Omega_{v,\varepsilon}$ to infinity. Let $\Psi_{p,v}^{(e)}$ be the scattering solutions of the problem in the spider domain $\Omega_{v,\varepsilon}$. The small $\varepsilon$ asymptotics of these solutions is given by Theorems 6 and 7. We introduce the following functions $\tilde{\Psi}_{p,v}^{(e)}$ by modifying the remainder terms in these asymptotics:

$$\tilde{\Psi}_{p,v}^{(e)} = \psi_p \varphi_0(y/\varepsilon) + \chi_v r_p^{(e)},$$  \hspace{1cm} (37)$$

where $\chi_v \in C^\infty(\Omega_\varepsilon)$, $\chi_v = 1$ on $\tau$-neighborhood of the junction, $\chi_v = 0$ outside of $\Omega_{v,\varepsilon}$. Then we define $G_\lambda$ by the formula

$$G_\lambda(x, z; \varepsilon) = \sum_p a_{p,v}(\lambda, \varepsilon, \xi) \tilde{\Psi}_{p,v}^{(e)} \bigg|_{\Omega_{v,\varepsilon}}, \hspace{1cm} x \in \Omega_{v,\varepsilon},$$  \hspace{1cm} (38)$$

where $a_{p,v}$, $D$ are defined in Lemmas 12, 9. $\xi$ is the point on the graph $\Gamma$ which corresponds $z \in \Delta_j$, i.e., the point on the edge $\Gamma_j$ where $t = t_j$. Since function $\Psi_{p,v}^{(e)}$ satisfies the equation $(\varepsilon^2 \Delta + \lambda)u = 0$ on $\Omega_{v,\varepsilon}$, from Theorems 6 and 7 it follows that

$$-(\varepsilon^2 \Delta + \lambda)\tilde{\Psi}_{p,v}^{(e)} = O(\delta^{-1} e^{-2\varepsilon^2}) \hspace{1cm} \varepsilon \to 0, \hspace{1cm} -\infty < \lambda < \lambda', \hspace{1cm} x \in \Omega_{v,\varepsilon},$$

where $\alpha = \lambda_1 - \lambda'$. We choose $\nu < \frac{\tau}{4}$. Then $\delta > e^{-\frac{\alpha\tau}{2\varepsilon^2}}$ for $\lambda \in (-\infty, \lambda') \setminus \Lambda^\nu$, and

$$-(\varepsilon^2 \Delta + \lambda)\tilde{\Psi}_{p,v}^{(e)} = O(e^{-\frac{3\alpha\tau}{2\varepsilon^2}}) \hspace{1cm} \varepsilon \to 0, \hspace{1cm} \lambda \in (-\infty, \lambda') \setminus \Lambda^\nu, \hspace{1cm} x \in \Omega_{v,\varepsilon}.$$

Since coefficients $a_{p,v}$ are bounded, Lemma 10 with $\sigma = e^{-\frac{3\alpha\tau}{2\varepsilon^2}}$ implies that

$$-(\varepsilon^2 \Delta + \lambda)G_\lambda = O(e^{-\frac{3\alpha\tau}{2\varepsilon^2}}) \hspace{1cm} \varepsilon \to 0, \hspace{1cm} \lambda \in (-\infty, \lambda') \setminus \Lambda^\nu, \hspace{1cm} x \in \Omega_{v,\varepsilon}. \hspace{1cm} (39)$$

Relations (39) are valid on each domain $\Omega_{v,\varepsilon}$. Now we are going to combine them and evaluate $(\varepsilon^2 \Delta + \lambda)G_\lambda$ for all $x \in \Omega_\varepsilon$. From (37), (38) and Lemma 12 it follows that the function

$$G_\lambda - g_\lambda(\gamma, \xi; \varepsilon) \varphi_0\left(\frac{y}{\varepsilon}\right)$$

is infinitely smooth in the channels of $\Omega_\varepsilon$. Here $\gamma$ is the point on $\Gamma$ which corresponds to $x \in \Omega_\varepsilon$. Then from (39) it follows that

$$-(\varepsilon^2 \Delta + \lambda)G_\lambda = \delta_\xi(\gamma) \varphi_0\left(\frac{y}{\varepsilon}\right) + O(e^{-\frac{3\alpha\tau}{2\varepsilon^2}}), \hspace{1cm} \varepsilon \to 0, \hspace{1cm} \lambda \in (-\infty, \lambda') \setminus \Lambda^\nu, \hspace{1cm} x \in \Omega_\varepsilon. \hspace{1cm} (40)$$

As it is easy to see, the remainder in (40) is zero in the region where $\nabla \chi_v \neq 0$, i.e., support of the remainder belongs to $\cup \Delta_j$.

Now let us study the second and third terms in the left hand side of (35). Obviously,

$$-(\varepsilon^2 \Delta + \lambda)(\chi R^0_\lambda f_1) = \chi f_1 + h = f_1 + h, \hspace{1cm} h = -2\varepsilon^2 \nabla \chi \cdot \nabla R^0_\lambda f_1 - \varepsilon^2 (\Delta \chi) R^0_\lambda f_1. \hspace{1cm} (41)$$
Here we used the fact that $\chi = 1$ on the support of $f_1$. Since $f_1$ is orthogonal to $\varphi_0(\frac{y}{\varepsilon})$, function $R^{\alpha}_{\chi}f_1$ and all its derivatives decay exponentially in each channel $C_{j,\varepsilon}$ as $\frac{2}{\varepsilon} \to \infty$ where $r$ is the distance from $\Delta_j$. Hence,

$$h = O(e^{-\frac{\alpha(r-\varepsilon)}{\varepsilon}}) = O(e^{-\frac{\alpha r}{\varepsilon}}), \quad \varepsilon \to 0, \quad \lambda \in (-\infty, \lambda').$$

(42)

The remainder terms will be parts of the operator $Q_{\lambda}$, and we need the kernel of this operator to be supported on $\cup \Delta_j$. Unfortunately, $h$ is supported on $\varepsilon$-neighborhoods of the junctions. The last term in (35) is designed to correct this. Since $h$ is supported on the region where $\nabla \chi \neq 0$, function $h$ can be represented as the sum $h = \Sigma_{v}h_v$, where $h_v = \chi_vh$ has estimate (42) and is supported on the $\varepsilon$-neighborhood of the junction $J_{v,\varepsilon}$ which corresponds to the vertex $v$. Consider $\tilde{h} = \Sigma_{v}\chi_vR^0_{\lambda,v}[\chi_vh_v]$ which is defined as follows. We apply the resolvent $R^0_{\lambda,v}$ of the problem in the spider domain $\Omega_{v,\varepsilon}$ to $h_v$, multiply the result by $\chi_v$ and extend the product by zero on $\Omega_\varepsilon \setminus \Omega_{v,\varepsilon}$.

From (42) and Theorem 5 it follows that

$$|R^0_{\lambda,v}h_v| \leq C\delta^{-1}e^{-\frac{\alpha r}{\varepsilon}} \leq Ce^{-\frac{\alpha r}{\varepsilon}}, \quad \varepsilon \to 0, \quad \lambda \in (-\infty, \lambda') \setminus \Lambda^\nu,$$

(43)

if we choose $\nu < \frac{\alpha}{2}$, so that $\delta > e^{-\frac{\alpha r}{\varepsilon}}$. From standard a priori estimates for the solutions of homogeneous equation $(\varepsilon^2\Delta + \lambda)u = 0$ it follows that estimate (14) is valid also for all derivatives of $R_{\lambda}f$, since this function satisfies the homogeneous equation outside of $2\varepsilon$-neighborhood of the junction. Then (43) holds for the derivatives of $R^0_{\lambda,v}h_v$. This allows us to obtain, similarly to (41), that

$$-(\varepsilon^2\Delta + \lambda)\tilde{h} = h + h_1, \quad h_1 = O(e^{-\frac{\alpha r}{\varepsilon}}), \quad \varepsilon \to 0, \quad \lambda \in (-\infty, \lambda') \setminus \Lambda^\nu,$$

(44)

where $h_1$ is supported on the closure of the set $\nabla \chi_v \neq 0$. This set belongs to $\cup \Delta_j$. Finally, from (40), (41), (44) it follows that

$$-(\varepsilon^2\Delta + \lambda)P_{\lambda}f = f + g, \quad g = O(e^{-\frac{\alpha r}{\varepsilon}}), \quad \varepsilon \to 0, \quad \lambda \in (-\infty, \lambda') \setminus \Lambda^\nu,$$

(45)

and $g$ is supported on $\cup \Delta_j$. One can easily check that $g$ depends linearly on $f$. Besides, one can specify the dependence on the norm of $f$ in estimates of all the remainders above. This will lead to (36) instead of (15). In fact, (36) is valid when $Q_{\lambda}$ is considered as an operator in $L^2$ or as an operator in the space of continuous functions on $\cup \Delta_j$.

We are going to construct now the solution $u$ of problem (3) with $f \in \mathbb{L}^2_{\tau,\delta}$. We look for $u$ in the form $u = P_{\lambda}g$ with unknown $g \in \mathbb{L}^2_{\tau,\delta}$. Obviously, $u$ satisfies the boundary conditions and appropriate conditions at infinity. Equation (3) in $\Omega_\varepsilon$ leads to $g + Q_{\lambda}g = f$. Since the norm of operator $Q_{\lambda}$ is exponentially small, function $g$ exists, is unique and $g = f + q$, $||q|| \leq Ce^{-\frac{\alpha r}{\varepsilon}}||f||_2$, i.e.,

$$u = P_{\lambda}(f + q), \quad ||q||_{\mathbb{L}^2_{\tau,\delta}} \leq Ce^{-\frac{\alpha r}{\varepsilon}}||f||_{L^2_{\tau,\delta}}, \quad \varepsilon \to 0, \quad \lambda \in (-\infty, \lambda') \setminus \Lambda^\nu.$$

This justifies (34) and (33). The first statement of Theorem 13 follows from here. Namely, assume that an eigenvalue $\mu = \mu_{j,\varepsilon}$ of the operator $H_{\varepsilon}$ belongs to $(-\infty, \lambda') \setminus \Lambda^\nu$. Then
the truncated resolvent $R_\lambda$ (see (23)) has a pole there (see Theorem 5). The residue of this pole is the orthogonal projection on the eigenspace of $H_\varepsilon$. The pole of $R_\lambda f$ may disappear only if $f$ is orthogonal to the eigenspace which corresponds to the eigenvalue $\lambda = \mu$. Non-trivial solutions of the equation $(\Delta + \lambda)u = 0$ in $\Omega_\varepsilon$ can not be equal to zero in a subdomain of $\Omega_\varepsilon$. Thus, there is a function $f \in L^2_{\tau, d}$ which is not orthogonal to the eigenspace, and $R_\lambda f$ must have a pole at $\lambda = \mu$. This contradicts (34) and (33).

The following statement can be easily proved using Theorem 13 and reduction (13) of the scattering problem to problem (3), (4).

**Theorem 14.** For any interval $[\alpha, \lambda']$, there exist $\rho, \nu > 0$ such that scattering solutions $\Psi_{p,\varepsilon}(x)$ of the problem in $\Omega_\varepsilon$ have the following asymptotic behavior on the channels of $\Omega_\varepsilon$ as $\varepsilon \to 0$

$$\Psi(x) = \psi_{p,\varepsilon}(\gamma)\varphi_0\left(\frac{\mu}{\varepsilon}\right) + r_{p,\varepsilon}(x),$$

where $\psi_p(\gamma) = \psi_{p,\varepsilon}(\gamma)$ are the scattering solutions of the problem on the graph $\Gamma$ and

$$|r_{p,\varepsilon}(x)| \leq Ce^{-\frac{2\rho d(\gamma)}{\varepsilon}}, \quad \lambda \in [\alpha, \lambda'] \setminus \Lambda^\nu.$$

Here $\gamma = \gamma(x)$ is the point on $\Gamma$ which is defined by the cross-section of the channel through the point $x$, and $d(\gamma)$ is the distance between $\gamma$ and the closest vertex of the graph.

### 6 Eigenvalues near the threshold

In some cases, in particular when the parabolic problem is studied, the lower part of the spectrum of the operator $H_\varepsilon$ is of a particular importance. Theorem 13 provides a full description of the location of the eigenvalues. One may have a finite number of eigenvalues below $\lambda_0$. They are determined by the junctions and situated in an exponentially small neighborhood of $\Lambda^0$ (set of eigenvalues of the corresponding spider domains). The eigenvalues between $\lambda_0$ and $\lambda_1$ are situated in an exponentially small neighborhood of $\Lambda(\varepsilon)$ which is the set of eigenvalues of the one dimensional problem (21)-(24) on the limiting graph (see Theorem 9).

We will assume that $\Omega_\varepsilon$ has at least one bounded channel (for example, $\Omega_\varepsilon$ is bounded). The opposite case is studied in Theorem 6. We also assume that $\lambda = \lambda_0 + O(\varepsilon^2)$. Then the eigenvalues of the problem (21)-(24) will depend on the form of the GC (23) at $\lambda = \lambda_0$. Let put $\lambda = \lambda_0 + \mu\varepsilon^2$ in (21)-(24). Then this problem takes the form

$$-\frac{d^2}{dt^2}\zeta = \mu\zeta \quad \text{on } \Gamma, \quad (46)$$

$$B\zeta = 0 \quad \text{at } v \in V_1, \quad (47)$$

$$i[I_v + T_v(\lambda_0 + \mu\varepsilon^2)]\frac{d}{dt}\zeta^{(v)}(t) - \mu[I_v - T_v(\lambda_0 + \mu\varepsilon^2)]\zeta^{(v)}(t) = 0, \quad t = 0, \quad v \in V_2. \quad (48)$$
\[ \varsigma = a_j e^{i\omega t}, \quad \gamma \in \Gamma_j, \quad 1 \leq j \leq m, \quad t >> 1. \] (49)

The last condition is not needed if \( \Omega_\varepsilon \) is bounded (\( m = 0 \)).

Since matrix \( T_\varepsilon(\lambda_0) \) is orthogonal and its eigenvalues are \( \pm 1 \), the GC (48) with \( \varepsilon = 0 \) has the form

\[ P\varsigma^{(v)}(0) = 0, \quad P^\perp \frac{d}{dt}\varsigma^{(v)}(0) = 0, \quad v \in V_2, \] (50)

where \( P, P^\perp \) are projections onto eigenspaces of matrix \( T_\varepsilon(\lambda_0) \) with the eigenvalues \( \pm 1 \), respectively. Let \( k \) be the dimension of the operator \( P \), and \( d - k \) be the dimension of the operator \( P^\perp \), where \( d = d(v) \) is the size of the vector \( \varsigma^{(v)} \). Then (50) imposes \( k \) Dirichlet conditions and \( d - k \) Neumann conditions on the components of vector \( \varsigma^{(v)} \) written in the eigenbasis of the matrix \( T_\varepsilon(\lambda_0) \). Note that the standard Kirchhoff conditions (\( \varsigma \) is continues on \( \Gamma \), a linear combination of derivatives is zero at each vertex) has the same nature, and \( k = d - 1 \) in this case.

Problem (46)-(49) with \( \varepsilon = 0 \) has a discrete spectrum \( \{\mu_j\} \), \( j \geq 1 \), and the same problem with \( \varepsilon > 0 \) is its analytic perturbation. Thus, the following statement is valid.

**Theorem 15.** If eigenvalues \( \{\mu_j\} \) are simple, then eigenvalues \( \{\mu_j(\varepsilon)\} \) of problem (46)-(49) are analytic in \( \varepsilon \):

\[ \mu_j(\varepsilon) = \sum_{n \geq 0} \mu_{j,n} \varepsilon^n, \quad \mu_{j,0} = \mu_j. \] (51)

**Remarks 1.** This statement implies that eigenvalues \( \lambda \in \Lambda(\varepsilon) \) in \( O(\varepsilon^2) \)-neighborhood of \( \lambda_0 \) have the form

\[ \lambda = \lambda_j(\varepsilon) = \lambda_0 + \varepsilon^2 \sum_{n \geq 0} \mu_{j,n} \varepsilon^n. \] (52)

2. The assumption on simplicity of \( \mu_j \) often can be omitted. For example (51),(52) remain valid without this assumption if \( k = d \) (the limiting problem is the Dirichlet problem). In the latter case one may have multiple eigenvalues (for example, when the graph has edges of multiple lengths), but the problem with \( \varepsilon = 0 \) is split into separate problems on individual edges.

Theorem 15 makes it important to specify the value of \( k \) in the condition (50). This value depends essentially on the type of the boundary conditions at \( \partial \Omega_\varepsilon \) and on whether \( \lambda = \lambda_0 \) is a pole of the truncated resolvent (10) or not.

**Definition 16.** A ground state of the operator \( H_\varepsilon \) in a domain \( \Omega_\varepsilon \) at \( \lambda = \lambda_0 \) is the function \( \psi_0 = \psi_0(x) \), which is bounded, strictly positive inside \( \Omega_\varepsilon \), satisfies the equation \( (-\Delta - \lambda_0) \psi_0 = 0 \) in \( \Omega_\varepsilon \), and the boundary condition on \( \partial \Omega_\varepsilon \), and has the following asymptotic behavior at infinity

\[ \psi_0(x) = \varphi_0 \left( \frac{y}{\varepsilon} \right) [\rho_j + o(1)], \quad x \in C_j, \quad |x| \to +\infty, \] (53)

where \( \rho_j > 0 \) and \( \varphi_0 \) is the ground state of the operator in the cross-sections of the channels.
Obviously, if the Neumann boundary condition is imposed on \( \partial \Omega_\varepsilon \), then \( \lambda_0 = 0 \), and the ground state at \( \lambda = 0 \) exists and equal to a constant. It was shown in [22], [23] that the ground state at \( \lambda = \lambda_0 \) does not exist for generic domains \( \Omega_\varepsilon \) in the case of other boundary conditions on \( \partial \Omega_\varepsilon \). In particular, it does not exist if there are eigenvalues of \( H_\varepsilon \) below \( \lambda_0 \) or if the truncated resolvent does not have a pole at \( \lambda = \lambda_0 \). The following result was proved in [22], [23].

**Theorem 17.** (1) The ground state at \( \lambda = \lambda_0 \) implies \( k = d - 1 \). Thus the eigenvalues \( \mu_j(\varepsilon), \varepsilon \to 0 \), converge to the eigenvalues of the Kirchhoff problem in the case of Neumann condition on \( \partial \Omega_\varepsilon \) (\( \Omega_\varepsilon \) is arbitrary) and in the case of other boundary conditions on \( \partial \Omega_\varepsilon \) for special, non-generic \( \Omega_\varepsilon \).

(2) If Dirichlet or Robin condition is imposed on \( \partial \Omega_\varepsilon \) and the truncated resolvent does not have a pole at \( \lambda = \lambda_0 \) (this is a generic condition on \( \Omega_\varepsilon \)), then \( k = d \) and \( \mu_j(\varepsilon), \varepsilon \to 0 \), converge to the eigenvalues of the Dirichlet problem.

Other possible (non-generic) GC at \( \lambda = \lambda_0 \) are given by [50].

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