SIMULTANEOUS ESTIMATES FOR VECTOR-VALUED GABOR FRAMES OF HERMITE FUNCTIONS

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Abstract. We derive frame estimates for vector-valued Gabor systems with window functions belonging to Schwartz space. The main result provides frame bound estimates for windows composed of Hermite functions. The proof is based on a recently established sampling theorem for the simply connected Heisenberg group, which is translated to a family of frame estimates via a direct integral decomposition.

1. Introduction and main results

The aim of this paper is to derive frame estimates for vector-valued Gabor systems. We consider the space $L^2(\mathbb{R}; \mathbb{C}^d)$ of vector-valued signals. Elements of this space can also be understood as vectors $f = (f_1, \ldots, f_d)$, with $f_i \in L^2(\mathbb{R}; \mathbb{C})$, which amounts to identifying $L^2(\mathbb{R}; \mathbb{C}^d)$ with the $d$-fold direct sum of $L^2(\mathbb{R})$. Gabor systems in this space are obtained by picking a window function $f \in L^2(\mathbb{R}; \mathbb{C}^d)$, and applying translations and modulations.

For any $\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$, we denote the associated time-frequency shift of a function $f \in L^2(\mathbb{R}; \mathbb{C}^d)$ by

$$f_\gamma = T_{\gamma_1} M_{\gamma_2} f.$$ 

Now, given a lattice $\Gamma \subset \mathbb{R}^2$ and $f \in L^2(\mathbb{R}; \mathbb{C}^d)$, the resulting Gabor system $G(f, \Gamma)$ is given by

$$G(f, \Gamma) = (f_\gamma)_{\gamma \in \Gamma}.$$ 

Recall that a family $(\eta_i)_{i \in I}$ of vectors in a Hilbert space $\mathcal{H}$ is called a frame if it satisfies

$$A \|\varphi\|^2 \leq \sum_{i \in I} |\langle \varphi, \eta_i \rangle|^2 \leq B \|\varphi\|^2,$$

for all $\varphi \in \mathcal{H}$, with constants $0 < A \leq B$. These constants are called frame bounds; they are generally not assumed to be optimal. A Gabor system $G(f, \Gamma)$ that is also a frame is called a Gabor frame.

For the case $d = 1$, Gabor frames have been studied extensively; see e.g. [13]. For a treatment of the case $d > 1$, in somewhat different terminology, see e.g. [2].

The problem of constructing a Gabor frame in dimension $d$ contains that of constructing $d$ Gabor frames: If $G(f, \Gamma)$ is a frame with bounds $A, B$, then $G(f_i, \Gamma)$ is a frame with
frame bounds $A, B$, for each component $f_i$ of $\mathbf{f}$. More generally, $\mathcal{G}(\tilde{\mathbf{f}}, \Gamma)$ is a frame of $L^2(\mathbb{R}; \mathbb{C}^d)$, for $\tilde{\mathbf{f}} = (f_1, \ldots, f_{\tilde{d}})$, with $\tilde{d} < d$, again with frame bounds $A$ and $B$.

But the converse need not be true: Note that by definition of the scalar product in $L^2(\mathbb{R}; \mathbb{C}^d)$,

$$\langle \mathbf{g}, \mathbf{f}_\lambda \rangle = \sum_{i=1}^{d} \langle \mathbf{g}_i, f_i | \lambda \rangle ,$$

where we used $f_i | \lambda$ to denote the action of a time-frequency shift on $f_i$. Hence, cancellation may prevent the higher-dimensional system from being a frame even when all components $f_1, \ldots, f_d$ generate a frame. In fact, one can easily see that a obvious necessary requirement for $\mathbf{f}$ to generate a frame is linear independence of its entries $f_1, \ldots, f_d$.

Observe however, that at least the upper frame bound for $\mathbf{f}$ can be estimated in terms of the upper frame bounds for the $f_i$: If $B, B_1, \ldots, B_d$ are optimal upper framebounds for $\mathbf{f}, f_1, \ldots, f_d$, respectively, then the Cauchy-Schwartz inequality entails

$$B \leq d \sum_{i=1}^{d} B_i .$$

Probably the most-studied window function for the case $d = 1$ has been the Gaussian, $g(x) = \pi^{-1/4} e^{-x^2/2}$. This is partly due to historical reasons: Gabor suggested using Gaussian windows [10], and the characterization of densities for Gabor frames with Gaussian window took more than 30 years to be fully clarified [14, 16]. The choice of this window function is motivated by the way Gabor systems are employed: They are designed to measure time-frequency content in a signal. By the Heisenberg uncertainty relation a Gaussian window has optimal time-frequency concentration, and thus can be expected to yield a good time-frequency resolution. Moreover, for the Gaussian window powerful tools from complex analysis can be employed to study sampling [14, 16], which adds to its theoretical appeal.

In this paper, we intend to derive frame estimates for window functions $\mathbf{f}$ consisting of the first $d + 1$ Hermite functions. For $n \in \mathbb{N}_0$, we define the $n^{th}$ Hermite function $h_n$ by

$$h_n(x) = \frac{(-1)^n}{\sqrt{2^n n! \sqrt{\pi}}} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2} ,$$

where the normalization factor ensures $\|h_n\|_2 = 1$. The above defined Gaussian equals $h_0$, whence the problem considered here can be viewed as a generalization of Gabor’s original question.

The vector-valued windows that we are interested in are given by

$$\mathbf{h}^d = (h_0, \ldots, h_d) \in L^2(\mathbb{R}; \mathbb{C}^{d+1}) .$$

We intend to give frame conditions and estimates for $\mathcal{G}(\mathbf{h}^d, \mathcal{M}(\mathbb{Z}^2))$, with $\mathcal{M}$ an invertible matrix, in terms of a matrix norm defined by

$$\|\mathcal{M}\| = \sup \{ \|\mathcal{M}z\|_2 : \|z\|_\infty \leq 1/2 \} .$$

Here $\|z\|_p$ denotes the usual $\ell^p$-norm on $\mathbb{R}^2$. This choice of matrix norm may seem somewhat peculiar, and in fact the theorems below can be formulated with respect to
any other norm on matrix space. The use of (1) emphasizes the close connection to sampling estimates on the Heisenberg group.

Let us now state the main results of this paper. The chief purpose of Theorem 1.1 is to allow a better understanding and formulation of Theorem 1.2. Nonetheless, Theorem 1.1 is of independent interest. Even though we expect it somehow to be part of Gabor analysis folklore, we are not aware of any previous source for this result; not even for \( d = 1 \).

**Theorem 1.1.** Let \( f = (f_1, \ldots, f_d) \in L^2(\mathbb{R}, \mathbb{C}^d) \) be given, with \( f_i \in \mathcal{S}(\mathbb{R}) \), and \( \langle f_i, f_j \rangle = \delta_{i,j} \), for all \( 1 \leq i, j \leq d \). Then there exists a constant \( 0 < C_f \leq 1 \) such that for all matrices \( M \) with \( \|M\| < C_f \), the system \( \mathcal{G}(f, M(\mathbb{Z}^2)) \) is a frame with frame constants
\[
\frac{1}{|\det(A)|} \left( 1 + \frac{\|M\|^2}{C_f^2} \right)^2.
\]

Hence the tightness of the frame estimate, which is the quotient of the two frame bounds, approaches 1 as \( \|M\| \to 0 \), with speed proportional to \( \|M\|^2/C_f^2 \).

We observe that Theorem 1.1 holds for the supremum \( C^*_f \) of all possible constants, and that this choice provides the sharpest possible statement. Then the main result of this paper is the following estimate:

**Theorem 1.2.** There exists a constant \( C_H \leq 1 \) such that for all \( d \in \mathbb{N}_0 \)
\[
(5) \quad C^*_H \geq \frac{C^*_H}{\sqrt{2d + 1}}.
\]

2. Method of proof: Sampling vs. frame estimates

The proof of Theorem 1.2 relies on a combination of various techniques: It uses a sampling estimate for the Paley-Wiener space \( \mathcal{P}(\mathbb{H}) \), established in [9]. The definition of this space uses a particular differential operator on \( \mathbb{H} \), the so-called sub-Laplacian. Hermite functions enter in the spectral decomposition of this operator, and it is this connection that will allow to relate the sampling theorem to frame estimates for Hermite functions.

The connection between frames and sampling theory is not exactly new, in fact it is at the base of frame theory, which originated from nonharmonic Fourier series and their connections to irregular sampling over the reals, see the classic paper [4]. For the sake of explicitness, assume we are given a sequence \( \Lambda = (\lambda_k)_{k \in \mathbb{Z}} \) of sampling points in \( \mathbb{R} \). We are looking for criteria that allow to reconstruct Paley-Wiener functions, i.e. \( f \in L^2(\mathbb{R}) \) whose Fourier transform has support in the unit interval \([-0.5, 0.5]\), in a stable manner from its restriction to \( \Lambda \). Noting that
\[
f(\lambda_k) = \int_{-0.5}^{0.5} \hat{f}(\xi)e^{2\pi i \lambda_k \xi}d\xi = \langle \hat{f}, e_{\lambda_k} \rangle,
\]
we find the following two equivalent conditions, with identical constants \( A \) and \( B \) in both cases:

1. The sequence \( \Lambda \) fulfills the sampling estimate
\[
A\|f\|^2_2 \leq \sum_{k \in \mathbb{Z}} |f(\lambda_k)|^2 \leq B\|f\|^2_2,
\]
for all Paley-Wiener functions.

(2) The sequence \((e_{\lambda_k})_{k \in \mathbb{Z}}\) fulfills the frame estimate

\[ A \|F\|_2^2 \leq \sum_{k \in \mathbb{Z}} |\langle F, e_{\lambda_k} \rangle|^2 \leq B \|F\|_2^2 , \]

for all \(F \in L^2([-0.5, 0.5])\).

This equivalence can be used in two ways: For instance, observing that the choice \(\lambda_k = k \ (k \in \mathbb{Z})\) results in the Fourier orthonormal basis \((e_k)_{k \in \mathbb{Z}}\) of \(L^2([-0.5, 0.5])\), the implication \((2) \Rightarrow (1)\) leads to Shannon’s sampling theorem. Conversely, for irregular sampling sets, condition \((1)\) can often be checked using tools from complex analysis techniques, and then \((1) \Rightarrow (2)\) results in frame estimates for irregularly spaced exponentials.

In this paper, we use a similar approach for the Heisenberg group \(\mathbb{H}\): This time, previously established sampling estimates for Paley-Wiener functions on \(\mathbb{H}\) will allow to derive frame estimates for Hermite functions, by an analogue of the implication \((1) \Rightarrow (2)\). For this purpose we will need to work out the connections between \(PW(\mathbb{H})\) and the Hermite functions, which is the topic of Sections 4 and 5. But first let us prove Theorem 1.1.

3. Proof of Theorem 1.1

Let \(V_f : L^2(\mathbb{R}; \mathbb{C}^d) \to L^2(\mathbb{R}^2)\) denote the windowed Fourier transform,

\[ V_f(g, x, \xi) = \langle g, T_x M_\xi f \rangle . \]

The orthogonality relations for the windowed Fourier transform and the pairwise orthogonality of the components of \(f\) imply that \(V_f\) is an isometry. Hence its image \(\mathcal{H}_f\) is a closed subspace of \(L^2(\mathbb{R}^2)\). As outlined in the previous subsection, the isometry property of \(V_f\) implies that a frame estimate for \(\mathcal{G}(f, \mathcal{M}(\mathbb{Z}^2))\) is the same as a sampling estimate for \(\mathcal{H}_f\), with sampling set \(\mathcal{M}(\mathbb{Z}^2)\).

We intend to utilize the techniques from [9] for this purpose, hence we will need oscillation estimates. We define

\[ \text{osc}_r(f)(x) = \sup_{|x-y| < r} |f(x) - f(y)| . \]

Our first aim is to prove

\[ \|\text{osc}_r(F)\|_2 \leq \frac{r}{C_f} \|F\|_2 , \forall F \in \mathcal{H}_f , \]

for a suitable constant \(C_f\).

For this purpose we first observe that the projection \(P_f\) onto \(\mathcal{H}_f\) is obtained by twisted convolution [13]:

\[ (P_f G)(x, \xi) = (G \ast_f F)(x, \xi) = \int_{\mathbb{R}^2} G(x', \xi') F(x - x', \xi - \xi') e^{\pi i (x\xi' - x'\xi)} dx' d\xi' , \]

where we let \(F = V_f f\).
Hence $G = G^*_F$ for $G \in \mathcal{H}_f$, and therefore

$$
osc_r(G)(x, \xi) \leq \sup_{|(x, \xi) - (x', \xi')| < r} \left| \int_{\mathbb{R}^2} G(x', \xi') \left( F(x - x', \xi - \xi') - F(x'' - x', \xi'' - \xi') \right) e^{i(x\xi' - x'\xi')} \, dx \, d\xi' \right|
$$

where convolution is taken with reference to the abelian group structure on $\mathbb{R}^2$. Hence

$$
\|\osc_r(G)\|_2 \leq \|G\|_2 \|\osc_r(F)\|_1.
$$

Now the second factor can be estimated by

$$
\|\osc_r(F)\|_1 = \int_{\mathbb{R}^2} \sup_{(x', \xi') \in B_r(x, \xi)} |F(x, \xi) - F(x', \xi')| \, dx \, d\xi
$$

$$
\leq \int_{\mathbb{R}^2} r \sup_{(x', \xi') \in B_r(x, \xi)} \left( \left| \frac{\partial F}{\partial x}(x, \xi) \right| + \left| \frac{\partial F}{\partial \xi}(x, \xi) \right| \right) \, dx \, d\xi
$$

$$
\leq r \int_{\mathbb{R}^2} \sum_{|\alpha| \leq 1} \|D^\alpha F\|_{\infty, B_r(x, \xi)} \, dx \, d\xi
$$

$$
\leq r \int_{\mathbb{R}^2} \sum_{|\alpha| \leq 4} \|D^\alpha F\|_{1, B_r(x, \xi)} \, dx \, d\xi
$$

$$
= r \sum_{|\alpha| \leq 4} |B_r| C_{B_r} \|D^\alpha F\|_1
$$

where we used the mean value theorem for the first inequality, and the Sobolev embedding theorem for the third. Here $C_{B_r}$ denotes the norm of the embedding $W^{3,1}(B_r) \hookrightarrow C(B_r)$ [1, Theorem 5.4]. Clearly, $|B_r| = r^2|B_1|$; on the other hand, a dilation argument establishes for $r < 1$ that $C_{B_r} \leq r^{-2} C_{B_1}$. Hence

$$
\|\osc_r(F)\|_1 \leq r|B_1| C_{B_1} \sum_{|\alpha| \leq 4} \|D^\alpha F\|_1,
$$

and thus (7) implies (6), with $C_f = \frac{1}{|B_1| C_{B_1} \sum_{|\alpha| \leq 4} \|D^\alpha F\|_1}$.

Now by letting $K = \mathcal{M}([-0.5, 0.5]^2)$, we obtain $\mathbb{R}^2 = \bigcup_{\gamma \in \mathcal{M}(\mathbb{Z}^2)} \gamma + K$ as a disjoint union, and $K$ has Lebesgue measure $|\det(\mathcal{M})|$. Moreover, by definition of the norm, $K \subset B_r$, for $r = \|\mathcal{M}\|$. Now, by [9, Theorem 3.5], the oscillation estimate (6) results in
the sampling estimate
\[ \frac{1}{|\det(\mathcal{M})|} \left( 1 - \frac{r}{C_r} \right)^2 \|F\|_2^2 \leq \sum_{\gamma \in \mathcal{M}(\mathbb{Z}^2)} |F(\gamma)|^2 \leq \frac{1}{|\det(\mathcal{M})|} \left( 1 + \frac{r}{C_r} \right)^2 \|F\|_2^2, \]
which is Theorem 1.1. \( □ \)

**Remark 3.1.** As the proof shows, one can weaken the requirement \( f_i \in \mathcal{S}(\mathbb{R}) \) to the membership of \( V_f \) in a suitable Sobolev space. In addition, pairwise orthogonality of the \( f_i \) can be weakened to linear independence; in this case the frame estimate will contain a term involving the Gramian matrix of the \( f_i \). \( □ \)

### 4. Fourier transform on the Heisenberg group

The (simply connected) Heisenberg group is defined as \( H = \mathbb{R}^3 \), with group law
\[ (p, q, t)(p', q', t') = (p + p', q + q', t + t' + (pq' - p'q)/2). \]

For the following facts concerning \( H \), we refer the reader to [6]. \( H \) is a step-two nilpotent Lie group, with center \( Z(H) = \{0\} \times \{0\} \times \mathbb{R} \). \( H \) is unimodular, with two-sided invariant measure on \( H \) given by the usual Lebesgue measure of \( \mathbb{R}^3 \).

Given \( \lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\} \), the Schrödinger representation \( \rho_\lambda \) of \( H \) acts on \( L^2(\mathbb{R}) \) via
\[ \rho_\lambda(p, q, t) = e^{-2\pi i \lambda (t - pq/2)} T_{\lambda p} M_q. \]

This is an irreducible unitary representation of \( H \). The family of Schrödinger representations provides the basis for the Plancherel transform of the group, a tool that is of key importance for this paper.

Before we describe this transform in more detail, let us quickly recall the basics of Hilbert-Schmidt operators: The space of Hilbert-Schmidt operators on a Hilbert space \( \mathcal{H} \) is given by all bounded linear operators such that
\[ \|T\|_{HS}^2 = \sum_{i \in I} \|T\eta_i\|^2 \]
is finite; here \((\eta_i)_{i \in I}\) denotes an arbitrary orthonormal basis of \( \mathcal{H} \). It is well-known that the norm is independent of the choice of basis, and defines a Hilbert space structure on the Hilbert-Schmidt operators with scalar product
\[ \langle S, T \rangle = \text{trace}(T^* S) = \sum_{i \in I} \langle S\eta_i, T\eta_i \rangle. \]

Now, given \( f \in L^1 \cap L^2(\mathbb{H}) \), one defines
\[ \rho_\lambda(f) = \int_H \rho_\lambda(x)f(x)dx, \]
understood in the weak operator sense. This is just the canonical extension of the representation \( \rho_\lambda \) to the convolution algebra \( L^1(\mathbb{H}) \). The mapping \( f \mapsto (\rho_\lambda(f))_{\lambda \in \mathbb{R}^*} \) is called the group Fourier transform. This nomenclature is justified by the observation that the
euclidian Fourier transform is obtained by integration against the characters of the additive group. Moreover, it turns out that for \( f \in L^1 \cap L^2(\mathbb{H}) \), the group Fourier transform is in fact a family of Hilbert-Schmidt operators, and we have the Parseval relation

\[
\| f \|_2^2 = \int_\mathbb{R} \| \rho_\lambda(f) \|_{HS}^2 |\lambda| d\lambda .
\]

The Parseval relation allows to extend the Fourier transform to \( L^2(\mathbb{H}) \), yielding the Plancherel transform, a unitary map

\[
L^2(\mathbb{H}) \rightarrow \int_\mathbb{R} HS(L^2(\mathbb{R})) |\lambda| d\lambda ,
\]

where the right hand side denotes the direct integral of Hilbert-Schmidt spaces. We denote the Plancherel transform of \( f \in L^2(\mathbb{H}) \) as \( \hat{f} = \left( \hat{f}(\lambda) \right)_{\lambda \in \mathbb{R}^*} \). This map will play the same role as the euclidean Fourier transform in the discussion of Section 2.

Of crucial importance for this usage of the Plancherel transform are its algebraic properties, providing a decomposition of various operators and representations acting on \( L^2(\mathbb{H}) \). For instance, if we denote the left regular representation by \( L_x f(y) = f(x^{-1}y) \), for \( f \in L^2(\mathbb{H}) \) and \( x, y \in \mathbb{H} \), then

\[
\rho_\lambda(L_x f) = \rho_\lambda(x) \circ \rho_\lambda(f) ,
\]

which provides the decomposition of the left regular representation \( L \) into a direct integral,

\[
L \simeq \int_{\mathbb{R}^*} \rho_\lambda \otimes 1 |\lambda| d\lambda .
\]

Similarly, the right regular representation \( R \) decomposes by the formula \( \rho_\lambda(R_x f) = \rho_\lambda(f) \circ \rho_\lambda(x)^* \).

The decompositions extend to commuting operators: For any bounded operator \( T \) commuting with \( L \), there exists a measurable field \( (\hat{T}_\lambda)_{\lambda \in \mathbb{R}^*} \) of bounded operators on \( L^2(\mathbb{R}) \) satisfying \( \rho_\lambda(T f) = \rho_\lambda(f) \circ \hat{T}_\lambda \), or in direct integral notation

\[
T \simeq \int_{\mathbb{R}^*} \text{Id} \otimes \hat{T}_\lambda |\lambda| d\lambda .
\]

5. Paley-Wiener space on the Heisenberg group

In this section we outline the definition of Paley-Wiener space on \( \mathbb{H} \) and its relation to Hermite functions. The central role of Hermite function in the decomposition of the sub-Laplacian has been observed previously, e.g. in \( [11] \); the results presented below can be found also in \( [17] \).

We define a left-invariant differential operator \( P \) on \( \mathbb{H} \) by

\[
(Pf)(p, q, t) = \lim_{h \to 0} \frac{f((p, q, t)(h, 0, 0)) - f(p, q, t)}{h} ,
\]

corresponding to the subgroup \( \mathbb{R} \times \{0\} \times \{0\} \), and \( Q \) is a left-invariant operator associated to \( \{0\} \times \mathbb{R} \times \{0\} \) in the same manner. \( P, Q \) are viewed as elements of the Lie algebra \( \mathfrak{h} \) of
H; we have \([P, Q] = T\), the infinitesimal generator of the group center. This observation exhibits \(\mathfrak{h}\) as a stratified Lie algebra,

\[
\mathfrak{h} = V_1 + V_2
\]

with \(V_1 = \text{span}(P, Q)\), and \(V_2 = \mathbb{R} \cdot T = [V_1, V_1]\).

Of particular interest for analysis on these groups is the sub-Laplacian; as the name suggests, it can be viewed as a replacement for the Laplacian over \(\mathbb{R}^n\). For the Heisenberg group, this operator is defined by

\[
\mathcal{L} = -P^2 - \frac{Q^2}{4\pi^2}.
\]

The normalization of \(Q\) is chosen for the sake of convenience. \(\mathcal{L}\) is a left-invariant positive unbounded operator on \(L^2(\mathbb{H})\). We denote its spectral measure by \(\Pi_\mathcal{L}\). The Paley-Wiener space on \(\mathbb{H}\) is then defined as

\[
PW(\mathbb{H}) = \Pi_\mathcal{L}([0, 1])(L^2(\mathbb{H})) .
\]

Note that, up to normalization, this definition is completely analogous to the definition of bandlimited functions on \(\mathbb{R}\), since the euclidean Fourier transform can also be read as the spectral decomposition of the Laplacian. The projection \(\Pi_\mathcal{L}([0, 1])\) is left-invariant, and is therefore decomposed by the group Fourier transform into a direct integral. The following lemma provides an explicit calculation of this decomposition via Hermite functions. In the following, we use the notation \(D_a f(x) = |a|^{-1/2} f(|a|^{-1/2} x)\). As with translation and modulation operators, we use the same symbol for operators acting on scalar- and on vector-valued functions.

**Lemma 5.1.**

(a) \((h_n)_{n \in \mathbb{N}_0}\) is an orthonormal basis of \(L^2(\mathbb{R})\).

(b) The system \((h_n)_{n \in \mathbb{N}_0}\) is an eigenbasis of the Hermite operator \(H f(x) = x^2 f(x) - f''(x)\),

\[(9) \quad H h_n = (2n + 1) h_n .\]

(c) For every real \(a \neq 0\), the dilated system \((h_{n,a})_{n \in \mathbb{N}_0}\), defined by

\[
h_{n,a}(x) = (D_{|a|^{1/2}} h_n)(x) = |a|^{-1/4} h_n(|a|^{-1/2} x) ,
\]

is an eigenbasis of the scaled Hermite operator \(H_a f(x) = x^2 f(x) - a^2 f''(x)\),

\[(10) \quad H_a h_{n,a} = |a|(2n + 1) h_{n,a} .\]

(c) The sub-Laplacian decomposes into a direct integral of scaled Hermite operators:

\[
\rho_\lambda(\mathcal{L} f) = \rho_\lambda(f) \circ H_\lambda ,
\]

for all \(f \in C^\infty_c(\mathbb{H})\).

**Proof.** For part (a) confer [7, Corollary 6.2, Theorem 6.14]. Part (b) follows from this by straightforward computation. Part (c) is established by formal calculation from (8) and the analogous formula for \(Q\), using the decomposition of the right regular representation.

\(\square\)
Parts (b) and (c) contain the ingredients of the direct integral decomposition of $PW(\mathbb{H})$. For the precise formulation of this result and its proof, the tensor product notation for Hilbert-Schmidt operators will be useful. Given vectors $\eta, \varphi$ in a Hilbert space $\mathcal{H}$, we let

$$\eta \otimes \varphi : z \mapsto \langle z, \varphi \rangle \eta,$$

which is a rank-one operator on $\mathcal{H}$. Note that the notation is only conjugate linear in $\varphi$.

The Hilbert-Schmidt scalar product of two elementary tensors is

$$\langle \eta \otimes \varphi, \eta' \otimes \varphi' \rangle_{HS} = \langle \eta, \eta' \rangle_{\mathcal{H}} \langle \varphi, \varphi' \rangle_{\mathcal{H}}.$$

Moreover, for any pair $S, T$ of bounded operators, $S \circ (\eta \otimes \varphi) \circ T = (S \eta) \otimes (T^* \varphi)$.

Now, given any orthonormal basis $(\varphi_i)_{i \in I}$ of $\mathcal{H}$, every Hilbert-Schmidt operator $T$ has a unique decomposition

$$T = \sum_{i \in I} \eta_i \otimes \varphi_i.$$

Hence, if $S = \sum_{i \in I} \psi_i \otimes \varphi_i$ is another Hilbert-Schmidt operator, we obtain for the scalar product

$$\langle T, S \rangle = \sum_{i \in I} \langle \eta_i, \psi_i \rangle.$$

Observe in the formulation of the following proposition that $\widehat{P}_\lambda$ involves the first $d(\lambda) + 1$ Hermite functions. For $|\lambda| > 1$, we have $d(\lambda) = -1$, and thus $\widehat{P}_\lambda = 0$.

**Proposition 5.2.** Letting

$$d(\lambda) = \left\lfloor \frac{1}{2|\lambda|} - \frac{1}{2} \right\rfloor$$

and

$$\widehat{P}_\lambda = \sum_{n=0, \ldots, d(\lambda)} h_{n,\lambda} \otimes h_{n,\lambda},$$

the projection onto Paley-Wiener space is given by

$$\langle \Pi_L([0,1])(f) \rangle^\wedge (\lambda) = \widehat{f}(\lambda) \circ \widehat{P}_\lambda, \ \forall f \in L^2(\mathbb{H}).$$

The operator field $(\widehat{P}_\lambda)_{\lambda \in \mathbb{R}^*}$ is the Plancherel transform of a function $p \in L^2(\mathbb{H})$, whence $\Pi_L([0,1])(f) = f \ast p$.

**Proof.** We apply the above considerations to the case $\mathcal{H} = L^2(\mathbb{R})$ and its orthonormal basis $(h_{n,\lambda})_{n \in \mathbb{N}_0}$. Hence each Hilbert-Schmidt operator $T$ on $L^2(\mathbb{R})$ has a decomposition

$$T = \sum_{n \in \mathbb{N}_0} \eta_n \otimes h_{n,\lambda},$$

and we obtain from (10) that

$$T \circ H_\lambda = \sum_{n \in \mathbb{N}_0} |\lambda|(2n + 1) \eta_n \otimes h_{n,\lambda}.$$

This shows that the map $T \mapsto \eta_n \otimes h_{n,\lambda}$ can be understood as a projection onto an eigenspace of the operator $T \mapsto T \circ H_\lambda$, with associated eigenvalue $|\lambda|(2n + 1)$. By
definition of Paley-Wiener space, only eigenvalues \( \leq 1 \) are admitted, which shows that the definition of \( \hat{P}_\lambda \) indeed yields (12).

For the second statement, we compute the norm of the operator field in the direct integral space. First observe that \( \hat{P}_\lambda = 0 \) for \( |\lambda| > 1 \). Moreover, the squared Hilbert-Schmidt norm of a projection equals its rank, whence \( \|\hat{P}_\lambda\|_{HS}^2 = d(\lambda) < \frac{1}{2|\lambda|} \), and thus

\[
\int_{\mathbb{R}^*} \|\hat{P}_\lambda\|_{HS}^2 |\lambda| d\lambda < \int_{-1}^{1} \frac{1}{2|\lambda|} |\lambda| d\lambda = 1.
\]

Hence \( (\hat{P}_\lambda)_\lambda \) has a preimage \( p \) under the Plancherel transform. Finally, (12) and the convolution theorem \([8, \text{Theorem 4.18}]\) provide that \( Pf = f * p \). 

The motivation for considering \( PW(\mathbb{H}) \) is the existence of sampling estimates. The formulation of the sampling theorem requires some additional notation. We fix a quasi-norm \( | \cdot | : \mathbb{H} \to \mathbb{R}^+_0 \) by

\[
|(p,q,t)| = (p^2 + q^2 + |t|)^{1/2},
\]

and write \( B_r \) for the unit ball around 0. A discrete subset \( \Gamma \subset \mathbb{H} \) is called a quasi-lattice if there exists a relatively compact set \( K \subset \mathbb{H} \) such that \( \mathbb{H} = \bigcup_{\gamma \in \Gamma} \gamma K \), as a disjoint union. Such a set \( K \) is called complement of \( \Gamma \).

**Theorem 5.3.** \([9, \text{Theorem 5.11}]\) There exists a constant \( 0 < C_{\mathbb{H}} \leq 1 \) with the following property: For all quasi-lattices \( \Gamma \) possessing a complement \( K \) contained in a ball of radius \( r < C_{\mathbb{H}} \) and all \( f \in PW(\mathbb{H}) \)

\[
(14) \quad \frac{1}{|K|}(1 - r/C_{\mathbb{H}})^2 \|f\|_2^2 \leq \sum_{\gamma \in \Gamma} |f(\gamma)|^2 \leq \frac{1}{|K|}(1 + r/C_{\mathbb{H}})^2 \|f\|_2^2.
\]

6. **Proof of Theorem 1.2**

We will now derive Theorem 1.2 from Theorem 5.3, basically by explicit calculation. The following lemma can be seen as an analog of (1) \( \Leftrightarrow \) (2) from Section 2. A version of this result was obtained in \([8, \text{Proposition 6.11}]\).

**Lemma 6.1.** Suppose that \( \Gamma' \subset \mathbb{H} \) is of the form \( \Gamma' = \Gamma \times \alpha \mathbb{Z} \), with \( \Gamma \subset \mathbb{R}^2 \) and \( \alpha > 0 \). Consider the following statements:

(a) For all \( f \in PW(\mathbb{H}) \),

\[
(15) \quad A \|f\|_2^2 \leq \sum_{\gamma \in \Gamma'} |f(\gamma)|^2 \leq B \|f\|_2^2.
\]

(b) For all \( d \in \mathbb{N}_0 \), and for almost all \( \lambda \) with \( |\lambda| < \frac{1}{2d+1} \), the system \( \mathcal{G}(\mathbb{H}^d, |\lambda|^{1/2} \Gamma) \) is a frame of \( L^2(\mathbb{R}; \mathbb{C}^{d+1}) \) with frame bounds \( \alpha |\lambda|^{-1} A \) and \( \alpha |\lambda|^{-1} B \).

Then (a) \( \Rightarrow \) (b), and if \( \alpha < 1/2 \), (b) \( \Rightarrow \) (a).

Moreover, if \( \Gamma = \mathcal{M}(\mathbb{Z}^2) \), for a suitable invertible matrix, the frame estimates in (b) are valid for all \( \lambda < \frac{1}{2d+1} \).
Proof. For the proof of \((a) \Rightarrow (b)\), let \(f \in PW(\mathbb{H})\) be given. Then we can write

\[
\hat{f}(\lambda) = \sum_{i=0}^{d(\lambda)} \varphi_{i,\lambda} \otimes h_{i,\lambda},
\]

for suitable functions \(\varphi_{i,\lambda} \in L^2(\mathbb{R})\). In the following, we also use the notations

\[
\Phi_{\lambda} = (\varphi_{0,\lambda}, \ldots, \varphi_{d(\lambda),\lambda})
\]

and

\[
h_{\lambda} = (h_{0,\lambda}, \ldots, h_{d(\lambda),\lambda}).
\]

Let \(E \subset \mathbb{R}^\ast\) be a Borel set contained in an interval \(I\) of length \(1/\alpha\), and consider \(g\) with \(\hat{g} = \hat{f} \cdot 1_E\). Observing that \(g \in PW(\mathbb{H})\), we can compute the \(\ell^2\)-norm of its restriction to \(\Gamma'\) as follows:

\[
\sum_{\gamma \in \Gamma'} |g(\gamma)|^2 = \sum_{\gamma \in \Gamma'} |\langle g, L_{\gamma}p \rangle|^2
\]

\[
= \sum_{\gamma \in \Gamma'} |\langle \hat{g}, (L_{\gamma}p)^\ast \rangle|^2
\]

\[
= \sum_{\gamma \in \Gamma'} \left| \int_E \langle \hat{f}(\lambda), \rho_{\lambda}(\gamma)\hat{p}(\gamma) \rangle |\lambda|d\lambda \right|^2
\]

\[
= \sum_{(l,k) \in \Gamma', n \in \mathbb{Z}} \left| \int_E \langle \hat{f}(\lambda), \rho_{\lambda}(l, k, 0)\hat{p}(\gamma) \rangle e^{2\pi i \lambda n} |\lambda|d\lambda \right|^2.
\]

Applying the Parseval formula for the interval, we thus obtain

\[
\sum_{\gamma \in \Gamma'} |g(\gamma)|^2 = \alpha^{-1} \sum_{(l,k) \in \Gamma'} \int_E |\langle \hat{f}(\lambda), \rho_{\lambda}(l, k, 0)\hat{p}(\gamma) \rangle |\lambda|d\lambda
\]

\[
= \alpha^{-1} \sum_{(l,k) \in \Gamma} |\langle \Phi_{\lambda}, \ell \lambda M_k |\lambda|^1/2 h_{\lambda} \rangle e^{\pi i \lambda k}|^2 |\lambda|d\lambda
\]

\[
= \alpha^{-1} \sum_{(l,k) \in \Gamma} |\langle \Phi_{\lambda}, \ell \lambda M_k |\lambda|^1/2 h_{\lambda} \rangle |^2 |\lambda|d\lambda
\]

(17)

where the last equation used (11) to express the Hilbert-Schmidt scalar products as scalar products of vector-valued functions, as well as symmetry of \(\Gamma\) to replace \(\lambda\) by \(|\lambda|\).

On the other hand, by the Plancherel formula, we find

\[
\|g\|_2^2 = \int_E \|\hat{f}(\lambda)\|_{HS}^2 |\lambda|d\lambda
\]

\[
= \int_E \|\Phi_{\lambda}\|_{L^2(\mathbb{R}, \mathbb{C}^{d(\lambda)})}^2 |\lambda|d\lambda.
\]

Hence the lower sampling estimate yields

\[
A \int_E \|\Phi_{\lambda}\|_{L^2(\mathbb{R}, \mathbb{C}^{d(\lambda)})}^2 |\lambda|d\lambda \leq \alpha^{-1} \sum_{(l,k) \in \Gamma} |\langle \Phi_{\lambda}, \ell \lambda M_k |\lambda|^1/2 h_{\lambda} \rangle |^2 |\lambda|d\lambda
\]

(18)
Since this inequality holds true for all Borel sets \( E \) of diameter at most \( 1/\alpha \), it has to hold pointwise a.e. for the integrands, i.e. after shifting constants:

\[
(19) \quad \alpha |\lambda|^{-1}A \|\Phi_\lambda\|_{L^2(\mathbb{R}, \mathbb{C}^{d(\lambda)})}^2 \leq \sum_{(t,k) \in \Gamma} |\langle \Phi_\lambda, T_{|\lambda|^{1/2} M_k h_\lambda} \rangle|^2 \quad (\text{a.e.} \lambda).
\]

This is already quite close to the desired lower frame estimate, except that it holds on a set of \( \lambda \)'s which may depend on the choice of \( f \) (or equivalently, on the field \((\Phi_\lambda)_{\lambda \in \mathbb{R}^+}\)).

The next step is to establish (19) for all \( f \in PW(\mathbb{H}) \) and all \( \lambda \) in a set with complement of measure zero, independent of \( f \). For this purpose we pick a sequence \((f_n)_{n \in \mathbb{N}}\) with dense span in \( PW(\mathbb{H}) \), and obtain a set \( \Omega \subset [-1,1] \) with complement of measure zero such that (19) holds for all \( \lambda \in \Omega \) and all \( f \) in the \((\mathbb{Q} + i\mathbb{Q})\)-span of \((f_n)_{n \in \mathbb{N}}\). But then the \((\mathbb{Q} + i\mathbb{Q})\)-span of \( (\hat{f}(\lambda))_{n \in \mathbb{N}} \) is dense in \( HS(L^2(\mathbb{R})) \circ \hat{P}_\lambda \), for all \( \lambda \) in a Borel set \( \Omega' \subset [-1,1] \) with complement of measure zero. Hence for all \( \lambda \in \Omega \cap \Omega' \), the frame estimate holds on a dense subset of \( HS(L^2(\mathbb{R})) \circ \hat{P}_\lambda \), which is sufficient.

Thus we have finally established (19) for almost all \( \lambda \in [-1,1] \), and all \( \Phi_\lambda \in L^2(\mathbb{R}, \mathbb{C}^{d(\lambda)}) \). The same argument applies to show the upper estimate with constant \( \alpha |\lambda|^{-1}B \).

Now, using \( h_\lambda = D_{|\lambda|^{1/2}} h^{d(\lambda)} \), and the relations

\[
M_\xi D_b = D_b M_{b \xi}, \quad T_x D_b = D_b T_{b^{-1}x},
\]

we find that

\[
T_{|\lambda|^{1/2} M_k h_\lambda} = D_{|\lambda|^{1/2}} (T_{|\lambda|^{1/2} M_{|\lambda|^{1/2} k} h^{d(\lambda)})}.
\]

Since the image of a frame under a unitary map is a frame with identical constants, we finally obtain that \( (T_{|\lambda|^{1/2} M_{|\lambda|^{1/2} k} h^{d(\lambda)})_{(t,k) \in \Gamma} \) is a frame, for almost all \( |\lambda| < 1 \). Now part (b) follows from \( d \leq d(\lambda) \) for \( \lambda < \frac{1}{2d+1} \).

For the converse direction observe that by assumption on \( \alpha \), all Plancherel transforms of elements of \( PW(\mathbb{H}) \) are supported in an interval of length \( 1/\alpha \). Hence (17) holds for all \( g \in PW(\mathbb{H}) \), where this time \( E = [-1,1] \), and the field \((\Phi_\lambda)_{\lambda} \) corresponds to the Plancherel transform of \( g \). But then (b) \( \Rightarrow \) (a) is immediate.

The proof that the “almost everywhere” contained in the statement can be omitted for lattices relies on semi-continuity properties of the frame bounds.

For any unit vector \( f \in L^2(\mathbb{R}, \mathbb{C}^d) \), consider the function

\[
\Theta_f : [0, \frac{1}{2d+1}] \ni \lambda \mapsto \sum_{(l,k) \in |\lambda|^{1/2} M(\mathbb{Z}^d)} |\langle f, T_l M_k |\lambda|^{1/2} h^d \rangle|^2.
\]
We compute
\[
\Theta_f(\lambda) = |\lambda| \sum_{(l,k) \in |\lambda|^{1/2} M(\mathbb{Z}^2)} |\langle f^m, TiM_k h^d \rangle|^2
\]
\[
= |\lambda| \sum_{(l,k) \in |\lambda|^{1/2} M(\mathbb{Z}^2)} |\sum_{i=0}^d \langle f_i, TiM_k h_i \rangle|^2
\]
\[
= |\lambda| \sum_{i,j} \sum_{l,k} \langle f_i, TiM_k h_i \rangle \langle f_j, TiM_k h_j \rangle
\]
\[
= |\lambda| \sum_{i,j} \sum_{l,k} \langle f_i, f_j, TiM_k h_j \rangle TiM_k h_i
\]
\[
= |\lambda| \sum_{i,j} \langle f_i, S_{h_i,h_j;|\lambda|^{1/2} M} f_j \rangle .
\]

Here we used the linear operator \( S_{g_1,g_2;N} \) associated to functions \( g_1, g_2 \) and an invertible matrix \( N \), defined by
\[
S_{g_1,g_2;N}(f) = \sum_{(l,k) \in N(\mathbb{Z}^2)} \langle f, TiM_k g_1 \rangle TiM_k g_2 .
\]

[1] Theorem 3.6] states that \( S : M^1(\mathbb{R}) \times M^1(\mathbb{R}) \times GL(2, \mathbb{R}) \to B(L^2(\mathbb{R})) \) is continuous, where the right-hand side denotes the space of bounded operators endowed with the norm topology, and \( M^1(\mathbb{R}) \) is the Feichtinger algebra; see e.g. [13] for a definition and basic properties. Now the inclusion \( S(\mathbb{R}) \subset M^1(\mathbb{R}) \) entails that the map \( \lambda \mapsto \langle f_i, S_{h_i,h_j;|\lambda|^{1/2} M} f_j \rangle \) is continuous, for all \( 0 \leq i, j \leq d \), and then \( \Theta_f \) is continuous.

Next consider the map associating to each \( \lambda \) the optimal upper frame bound, given by
\[
B_{\text{opt}} : 0, \frac{1}{2d+1} [ \sup_{\|f\|=1} |\Theta_f(\lambda)| .
\]
The supremum is always finite: By [13] Corollary 6.2.3], the frame operator of a one-dimensional window in the Schwartz class is always bounded. Hence the upper frame bound also exists in the vector valued case, by [2].

As the supremum of a family of continuous functions, \( B_{\text{opt}} \) is lower semi-continuous, and then \( \lambda \mapsto B_{\text{opt}}(\lambda)|\lambda|^{-1/2} \) is lower semi-continuous as well. We already know that the latter map is bounded from above by \( \alpha B \) on subset of \( 0, \frac{1}{2d+1} \) with complement of measure zero. This subset is dense, hence lower semi-continuity implies \( B_{\text{opt}}(\lambda)|\lambda|^{-1/2} \leq \alpha B \) on the whole interval.

The analogous reasoning, replacing lower by upper semi-continuity, applies to the lower frame bound, and we are done. \( \square \)

**Proof of Theorem 1.2** Fix \( d \in \mathbb{N}_0 \). Suppose that \( M \) is given with \( \|M\| < C_M/\sqrt{2d+1} \). Let \( K = M([-0.5,0.5]^2) \), then \( K \) is a complement of \( M(\mathbb{Z}^2) \) in \( \mathbb{R}^2 \), contained in a ball of radius \( r_0 = \|M\| \), and with measure \( |\det(M)| \). Moreover, by choosing \( \alpha > 0 \) small enough, the set \( K' = K \times [-\alpha/2, \alpha/2] \) is contained in a ball of radius
\[ r_0 + \epsilon < C_H / \sqrt{2d + 1} \] (with respect to the quasi-norm on \( H \)). In addition, \( K' \) is a complement of \( \Gamma' = \mathcal{M}(\mathbb{Z}^2) \times \alpha\mathbb{Z} \). For any \( (p, q, t) \in H \), there exist unique \( (p_1, q_1) \in \Gamma \) and \( (p_2, q_2) \in K \) with \( (p_1 + p_2, q_1 + q_2) = (p, q) \), and finally unique \( l \in \mathbb{Z} \) and \( s \in [-\alpha/2, \alpha/2] \) with \( s + \alpha l = t - (p_1 q_2 - p_2 q_1)/2 \). But these choices imply \( (p_1, q_1, \alpha l)(p_2, q_2, s) = (p, q, t) \).

We will apply Theorem 5.3 to dilated copies of \( \Gamma' \). For this purpose, let for \( a > 0 \) and \( (p, q, t) \in H \), \( \delta_a(p, q, t) = (ap, aq, a^2 t) \). It is easy to check that this defines a group automorphism \( \delta_a \) fulfilling \( |\delta_a(p, q, t)| = a|(p, q, t)| \). Hence \( \delta_a(\Gamma') \) is a quasi-lattice, with complement \( \delta_a(K) \) contained in a ball of radius \( a(r_0 + \epsilon) \). Hence, for any \( a < C_H / (r_0 + \epsilon) \), the sampling theorem provides the estimate

\[
\frac{1}{a^4|\text{det}(\mathcal{M})|\alpha} \left( 1 - \frac{a(r_0 + \epsilon)}{C_H} \right) \|f\|_2^2 \leq \sum_{\gamma \in \delta_a(\Gamma')} |f(\gamma)|^2 \leq \frac{1}{a^4|\text{det}(\mathcal{M})|\alpha} \left( 1 + \frac{a(r_0 + \epsilon)}{C_H} \right) \|f\|_2^2 .
\]

An application of Lemma 6.1 then yields, for all \( \lambda < \frac{1}{2n+1} \), that \( \mathcal{G}(H^d, a|\lambda|^{1/2}\mathcal{M}(\mathbb{Z}^2)) \) is a frame with bounds \( \frac{1}{|\text{det}(\mathcal{M})|} \left( 1 + \frac{a(r_0 + \epsilon)}{C_H} \right) \). Letting \( a^2|\lambda| = 1 \) provides a lower frame bound for \( \mathcal{G}(H^d, \mathcal{M}(\mathbb{Z}^2)) \) given by

\[
\sup \left\{ \frac{1}{|\text{det}(\mathcal{M})|} \left( 1 - \frac{a(r_0 + \epsilon)}{C_H} \right) : \sqrt{2d + 1} \leq a < C_H / (r_0 + \epsilon) \right\} .
\]

Observe that the restriction \( a \geq \sqrt{2d + 1} \) is imposed by \( |\lambda| \leq \frac{1}{2d+1} \). By monotonicity, the supremum is \( \frac{1}{|\text{det}(\mathcal{M})|} \left( 1 - \frac{\sqrt{2d+1(r_0+\epsilon)}}{C_H} \right) \). Sending \( \epsilon \) to zero provides the lower estimate of Theorem 1.2. The upper estimate is obtained in the same fashion.

**Concluding remarks**

It is a standard observation that the construction of Gabor frames is equivalent to the discretization of the inversion formula associated to a certain discrete-series representation of the so-called **reduced Heisenberg group** \( H_r \), which is the quotient of \( H \) by a discrete central subgroup; see e.g. [13]. The proof of Theorem 1.2 shows that in working with \( H \) one needs to deal with a fair amount of additional technical details (in particular due to the occurrence of direct integrals), that one avoids by considering \( H_r \). The benefit of this approach lies in the fact that a single sampling estimate, namely (14), gives rise to a whole family of Gabor frame estimates, namely (2), valid for all \( d \geq 0 \).

The main results of this paper provide rather intuitive asymptotic estimates for Gabor frame bounds. A major drawback of these estimates is that they involve unknown constants. A “formula” for \( C_H \) is given in [9], involving operator norms for differential operators on \( PW(H) \) as well as a Sobolev constant for the unit ball \( H \); the argument is very similar to the estimate of the constant \( C_f \) in the proof of Theorem 1.2. While rough estimates for the differential operators should be obtainable from the Plancherel transform, which decomposes the differential operators as well as \( PW(H) \), we are not aware of a reasonable estimate for the Sobolev constant for \( H \). In any case, we stress
that the constant $C_H$ in the sampling theorem is the same as in Theorem 1.2; this was the chief motivation for picking the matrix norm (4).

For single Hermite functions, the results obtained here compare in an interesting way with recent results due to Gröchenig and Lyubarskii. Using complex analysis methods, they obtained the following statement [12, Theorem 3.1]:

**Theorem 6.2.** If $|\det M| < (d + 1)^{-1}$, then $G(h_d, M)$ is a frame for $L^2(\mathbb{R})$.

For the isotropic case, i.e., $M = a \cdot \text{Id}$, this result provides a criterion that is very close to our Theorem 1.2. Any $a$ below a threshold $\sim n^{-1/2}$ guarantees a frame. In the general case however, Theorem 6.2 is much more widely applicable: At the same time $|\det M|$ can be made arbitrarily small and $\|M\|$ arbitrarily large.

On the other hand, Theorem 6.2 does not provide frame bound estimates, and it only applies to the scalar-valued case.

Let us finally comment on possible generalizations. The first possible extension consists in replacing $\mathbb{R}$ by $\mathbb{R}^n$, i.e. studying vector-valued Gabor frames in $L^2(\mathbb{R}^n; \mathbb{C}^d)$. One now considers the $2n + 1$-dimensional Heisenberg group $\mathbb{H}_n$. This is a stratified Lie group, possessing a sub-Laplacian, Paley-Wiener space and, finally, a sampling theorem [9]. As for the one-dimensional case, the spectral decomposition of the sub-Laplacian involves Hermite functions, and an adaptation of the arguments for $\mathbb{H}$ should be a straightforward task, somewhat aggravated by additional bookkeeping.

A second, more interesting but also more challenging type of generalization concerns the sampling sets, which could also be irregular. There already exists an irregular sampling theorem for $\mathbb{H}$, however, in the transfer of the associated sampling estimates to Gabor frame estimates, we are crucially relying on the lattice structure of the sampling set. In this context, the key result is the continuity statement [5, Theorem 3.6], and the proof of this result makes full use of Gabor theory developed for lattices.

As a result, we can currently only prove statements of the following form: For all $d \in \mathbb{N}_0$ and all uniformly discrete and uniformly dense sets $\Gamma \subset \mathbb{R}^2$ there exists a range $(0, a_d)$ of dilation parameters such that $G(h^d, a\Gamma)$ is a frame, for almost all $a \in (0, a_d)$, including an estimate of the frame bounds. Moreover, the threshold $a_d$ is of the order $d^{-1/2}$.

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