THE WAVE MODEL OF THE STURM–LIOUVILLE OPERATOR ON AN INTERVAL

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In the paper the wave functional model of a symmetric restriction of the regular Sturm-Liouville operator on an interval is constructed. The model is based upon the notion of the wave spectrum and is constructed according to an abstract scheme, which was proposed earlier. The result of the construction is a differential operator of the second order on an interval, which differs from the original operator only by a simple transformation. Bibliography: 19 titles.

INTRODUCTION

In paper [1], the notion of the wave spectrum of a symmetric semibounded from below operator was introduced. The wave spectrum is constructed as a topological space determined by the operator. In the same work the wave spectrum was studied for the Laplace operator on a compact manifold, and it was established that in the general situation one can introduce a metric on the wave spectrum so that it becomes isometric to the original manifold. In [7], a scheme of construction of a functional model of such an operator was proposed, which is called the wave model and is based on the notion of wave spectrum. The space of functions on the wave spectrum is taken as the model space. The graph of the model operator is recovered using the method of boundary control, on which the construction of the wave spectrum also relies. This scheme was realized in [7] for the positive definite Schrödinger operator on the half-line in the limit point case. To be precise, the wave model was constructed for a symmetric restriction of such an operator with defect indices (1, 1).

In the present paper, we construct the wave model of a symmetric positive definite operator with defect indices (2, 2), namely, of the symmetric restriction of the regular Sturm–Liouville operator defined by the differential expression $-\frac{d^2}{dx^2} + q(x)$ on the interval $(0, l)$ with the boundary conditions $u(0) = u'(0) = u(l) = u'(l) = 0$. The potential $q$ is supposed to be smooth and such that the operator is positive definite. In the course of construction, we also refine and develop the abstract scheme of the wave model.

The paper consists of two parts. In the first, abstract, part we give the definition of the wave spectrum and describe the scheme of the wave model construction. Trying to keep certain level of generality, we state a number of conditions, which a symmetric operator should satisfy and under which the wave model is constructed. Conditions are given in rather abstract terms; thus one can check them only constructing the model of some particular operator. The second part of the paper is devoted to a realization of this abstract scheme for the Sturm–Liouville operator on an interval. We explicitly describe objects that were defined in the first part and directly check all the conditions.

An important feature of the wave functional model is that it turns out to be almost identical to the original operator. This happens for the example considered earlier [7] and in our case. The inverse problem data (spectral, dynamical) in some cases allows us to construct a certain “auxiliary model”, i.e., a model space and an operator acting on it, which is unitarily equivalent to the original operator. In this sense one can distinguish objects that are available to the “outer observer” (those that can be obtained knowing the inverse problem data) and available

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Translated from Zapiski Nauchnykh Seminarov POMI, Vol. 471, 2018, pp. 225–260. Original article submitted September 28, 2018.

1072-3374/19/2435-0783 ©2019 Springer Science+Business Media, LLC 783
to the “inner observer” (those that can be obtained knowing the original, the solution of the inverse problem). Knowing the auxiliary model, the “outer observer” can construct its wave model, from which it is easy to recover the original. In our examples, the wave model is a differential operator. From the coefficients of this operator one can explicitly obtain the potential of the original operator. In the case of the regular Sturm–Liouville operator, the potential can be recovered up to reflection from the middle point of the interval \((0, l)\).

The results of this work were announced in [17], where a brief description of this construction was given.

1. The abstract scheme

1.1. The operator \(L_0\). Consider a closed symmetric linear operator \(L_0\) in a separable Hilbert space \(\mathcal{H}\), and let this operator be positive definite: there exists \(\varepsilon > 0\) such that \((L_0u, u) \geq \varepsilon \|u\|^2\) for every \(u \in \text{Dom } L_0\). Denote by \(L\) the Friedrichs self-adjoint extension of the operator \(L_0\) (see [9]). For every \(u \in \text{Dom } L\) one has \((Lu, u) \geq \varepsilon \|u\|^2\), whence a bounded inverse operator \(L^{-1}\) exists.

1.2. The Green’s system. Let \(A\) be an operator in \(\mathcal{H}\), \(B\) a Hilbert space, \(\Gamma_1\) and \(\Gamma_2\) be linear operators acting from \(\mathcal{H}\) to \(B\). Let the following conditions hold:

\[
\text{Dom } A = \mathcal{H}, \quad \text{Dom } A \subseteq \text{Dom } \Gamma_1 \cap \text{Dom } \Gamma_2, \quad \text{Ran } \Gamma_1 + \text{Ran } \Gamma_2 = B.
\]

The collection \(\mathfrak{G} = \{\mathcal{H}, B; A, \Gamma_1, \Gamma_2\}\) is called the Green’s system if the relation

\[
(Au, v)_\mathcal{H} - (u, Av)_\mathcal{H} = (\Gamma_1 u, \Gamma_2 v)_B - (\Gamma_2 u, \Gamma_1 v)_B
\]

(the Green’s formula) holds for every \(u, v \in \text{Dom } A\) ([13, 10, 16]). The space \(\mathcal{H}\) is called the inner space, \(B\) is the space of boundary values, \(A\) is the basic operator, and \(\Gamma_1, \Gamma_2\) are the boundary operators.

There is a class of Green’s systems that canonically corresponds to the class of operators \(L_0\) considered. Denote

\[
\mathcal{K} := \text{Ker } L_0^*,
\]

let \(P_{\mathcal{K}}\) be the orthogonal projection to the subspace \(\mathcal{K}\) of \(\mathcal{H}\), \(\mathcal{O}\) be the zero operator in \(\mathcal{H}\), and \(\mathcal{I}\) be the identity operator. Let

\[
\Gamma_1 := L^{-1}L_0^* - \mathcal{I}, \quad \Gamma_2 := P_{\mathcal{K}} L_0^*.
\]

Then the collection \(\mathfrak{G}_{L_0} := \{\mathcal{H}, \mathcal{K}; L_0^*, \Gamma_1, \Gamma_2\}\) forms a Green’s system [5]. Such a system is related to the Vishik’s decomposition for the operator \(L_0\), which has the form

\[
\text{Dom } L_0^* = \text{Dom } L_0 + L^{-1}\mathcal{K} + \mathcal{K}\]

(\(\oplus\) denotes the direct sum of linear sets). The boundary operators can be written in terms of this decomposition as follows [19]: if \(u \in \text{Dom } L_0^*\) is represented in the form

\[
u = u_0 + L^{-1}g_u + h_u,
\]

where \(u_0 \in \text{Dom } L_0, g_u, h_u \in \mathcal{K}\), then

\[
\Gamma_1 u = -h_u, \quad \Gamma_2 u = g_u.
\]
1.3. The system with boundary control. Consider the following problem, which corresponds to the Green’s system $\mathcal{G}_{L_0}$:

\[
\begin{align*}
    u_{tt} + L_0^*u &= 0, & t > 0, \\
    u|_{t=0} = u_t|_{t=0} &= 0, \\
    \Gamma_1 u &= h, & t \geq 0,
\end{align*}
\]

where $h = h(t)$, a $\mathcal{K}$-valued function, is called the boundary control, and the $\mathcal{H}$-valued function $u(t) = u^h(t)$ is unknown. In the control theory $u^h(\cdot)$ is called the trajectory, $u^h(t)$ is the state of the system at the moment $t$; we will call $u^h$ a wave. Denote the system (1.6)–(1.8) by $\alpha_{L_0}$.

The problem (1.6)–(1.8) has a solution (see [5]) if the control $h$ belongs to the class

\[
\mathcal{M} := \{ h \in C^\infty([0, \infty); \mathcal{K}) : \supp h \subset (0, \infty) \}. 
\]

This solution can be written in the form

\[
u^h(t) = -h(t) + \int_0^t L^{-\frac{1}{2}} \sin \left[(t-s)\frac{L}{2}\right] h_{tt}(s) \, ds, \quad t \geq 0,\]

it belongs to $C^\infty([0, \infty); \mathcal{H})$ and vanishes near zero. We will call such $u^h$ classical solutions or smooth waves.

The set of states of the system $\alpha_{L_0}$

\[
\mathcal{U}_{L_0}^t := \{ u^h(t), h \in \mathcal{M} \} \subseteq \text{Dom} L_0^* 
\]

is called the reachable set at the time $t \geq 0$. It is easy to see that $\mathcal{U}_{L_0}^t$ grows with $t$. The set

\[
\mathcal{U}_{L_0} := \bigcup_{t>0} \mathcal{U}_{L_0}^t 
\]

is called the total reachable set of the system $\alpha_{L_0}$, and its orthogonal complement

\[
\mathcal{D}_{L_0} := \mathcal{H} \ominus \mathcal{U}_{L_0}
\]

is called the defect subspace of the system $\alpha_{L_0}$. The linear sets $\mathcal{U}_{L_0}^t$ and $\mathcal{U}_{L_0}$ are invariant under $L_0^*$; let $T \geq 0$ and $u = u^h(T) \in \mathcal{U}_{L_0}^T$, then

\[
\begin{align*}
    L_0^* u^h(T) &= -u_{tt}^h(T) \in \mathcal{U}_{L_0}^T, \\
    u^h(T) &= J^2[u_{tt}^h(T)] = u_{tt}^{h^2}(T) = -L_0^* u^{h^2}(T) \in L_0^* \mathcal{U}_{L_0}^T,
\end{align*}
\]

where $J : u \mapsto \int_0^t u(s) ds$. Therefore $L_0^* \mathcal{U}_{L_0} = \mathcal{U}_{L_0}$.

The system $\alpha_{L_0}$ is called controllable if $\mathcal{U}_{L_0} = \mathcal{H}$. The following fact is known [5].

**Proposition 1.** The controllability of the system $\alpha_{L_0}$ is equivalent to the fact that the operator $L_0$ is completely non-selfadjoint.

The restriction of the operator $L_0^*$ to the linear set of smooth waves $\mathcal{U}_{L_0} \subseteq \text{Dom} L_0^*$ is not necessarily a closed operator. Its closure $\overline{L_0^* | \mathcal{U}_{L_0}} \subseteq L_0^*$ is called the wave part of the operator $L_0^*$. If the operator $L_0$ is completely non-selfadjoint, the question arises as to whether the operator $L_0^*$ coincides with its wave part. This happens for the examples that we know; however, we do not have a proof of the general fact.
1.4. The wave spectrum. The functional model of the operator $L_0$ that we construct is based on the wave spectrum of the operator. For its definition we use notions of lattice theory.

A lattice is a partially ordered set every two elements $p, q$ of which have the least upper bound $\sup\{p, q\} = p \lor q$ (the least element of the set of all upper bounds) and the greatest lower bound $\inf\{p, q\} = p \land q$ (the greatest element of the set of all lower bounds). A lattice is called complete if every subset of it has the least upper and the greatest lower bounds. In a complete lattice there always exist the least and the greatest elements.

Let $\mathcal{P}$ and $\mathcal{Q}$ be partially ordered sets, $i$ be a map from $\mathcal{P}$ to $\mathcal{Q}$. The map $i$ is called isotonic if $p_1 \leq p_2$ in $\mathcal{P}$ implies $i(p_1) \leq i(p_2)$ in $\mathcal{Q}$, [8]. A family of maps $\{i^t\}_{t \geq 0}$ from $\mathcal{P}$ to $\mathcal{Q}$, is called an isotony if $p_1 \leq p_2$ and $t_1 \leq t_2$ implies $i^{t_1}(p_1) \leq i^{t_2}(p_2)$.

Let $\mathcal{P} = \mathcal{Q} = \mathcal{L}$ be a complete lattice, and let $\mathcal{O}_2$ be its least element. Then an isotony $I^t$ is called an isotony of the lattice $\mathcal{L}$ if $I^0$ is the identity map in $\mathcal{L}$ and $I^t(O_2) = O_2$ for every $t \geq 0$.

Let a partially ordered set $\mathcal{P}$ contain the least element $O_\mathcal{P}$. An element $p \neq O_\mathcal{P}$ is called an atom of $\mathcal{P}$ if there is no element $p^t \in \mathcal{P}$ such that $O_\mathcal{P} < p^t < p$.

Let $\mathcal{L}$ be a complete lattice, $\mathcal{O}_2$ be its least element, $E_\mathcal{L}$ be its greatest element. If for every $p \in \mathcal{L}$ there exists an element $p^t \in \mathcal{L}$ such that $p \lor p^t = E_\mathcal{L}$, $p \land p^t = O_\mathcal{L}$ (the complement), then $\mathcal{L}$ is called a lattice with complements.

1.4.1. The lattice of subspaces. We will work with lattices and isotonies of a special kind.

The set $\mathcal{L}(\mathcal{H})$ of all subspaces of a Hilbert space $\mathcal{H}$ with the partial order $\subseteq$ forms a complete lattice with complements: it is easy to check that $G_1 \lor G_2 = G_1 + G_2$ and $G_1 \land G_2 = G_1 \cap G_2$ for $G_1, G_2 \in \mathcal{L}(\mathcal{H})$, $\{0\}$ is the least element, $\mathcal{H}$ is the greatest element, $G^\perp$ is the complement for $G \in \mathcal{L}(\mathcal{H})$.

Let us call $\mathcal{L} \subset \mathcal{L}(\mathcal{H})$ a sublattice of the lattice $\mathcal{L}(\mathcal{H})$ with complements if $\mathcal{L}$ contains $\{0\}$, $\mathcal{H}$, $G_1 \lor G_2, G_1 \land G_2$ for every $G_1, G_2 \in \mathcal{L}$ and $G^\perp$ for every $G \in \mathcal{L}$. For every subset $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ there exists a minimal sublattice with complements $\mathcal{L}_{2\mathcal{M}}$ in $\mathcal{L}(\mathcal{H})$ that contains $\mathcal{M}$. If $I^t$ is an isotony of the lattice $\mathcal{L}(\mathcal{H})$, then there also exists a minimal lattice with complements $\mathcal{L}_{2\mathcal{M}}^I$ in $\mathcal{L}(\mathcal{H})$ that contains $\mathcal{M}$ and is invariant under $I$: for every $G \in \mathcal{L}_{2\mathcal{M}}^I$ and $t \geq 0$ one has $I^t(G) \in \mathcal{L}_{2\mathcal{M}}^I$ ([(7)]).

One can naturally define a topology on the lattice of subspaces $\mathcal{L}(\mathcal{H})$. A sequence $\{G_n\}_{n \in \mathbb{N}}$ from $\mathcal{L}(\mathcal{H})$ converges to $G \in \mathcal{L}(\mathcal{H})$ as $n \to \infty$ if the corresponding projections converge in the strong sense: $P_{G_n} \xrightarrow{\text{sdf}} P_G$. Note that the strong operator topology, restricted to orthogonal projections, satisfies the first axiom of countability and can be described in terms of converging sequences [12].

Let $\mathcal{F}(\mathcal{H})$ denote the set of functions from $[0, \infty)$ to $\mathcal{L}(\mathcal{H})$ with a pointwise partial order: $f_1 \leq f_2$ if $f_1(t) \leq f_2(t)$ in $\mathcal{L}(\mathcal{H})$ (i.e., $f_1(t) \subseteq f_2(t)$) for every $t \geq 0$. Then the lattice operations will also be pointwise:

- $(f_1 \lor f_2)(t) = f_1(t) \lor f_2(t)$,
- $(f_1 \land f_2)(t) = f_1(t) \land f_2(t)$,
- $(f^\perp)(t) = (f(t))^\perp$.

The strong operator topology generates on $\mathcal{F}(\mathcal{H})$ the product topology (the topology of pointwise convergence), which does not satisfy the first axiom of countability and can be described in terms of converging nets instead of sequences. It turns out that the objects with which we work do not require a topology on $\mathcal{F}(\mathcal{H})$ and it is possible to deal with the operation of sequential closure (topology corresponding to such an operation may be not unique). There exists a version of our construction of the wave model based on the product topology in $\mathcal{F}(\mathcal{H})$. For all
the examples known to us, both versions eventually lead to the same construction (because
the wave spectra coincide).

Let us denote by \( I \mathcal{L}(\mathcal{H}) := \{ I^t(G), G \in \mathcal{L}(\mathcal{H}) \} \) the set of isotonic \( \mathcal{L}(\mathcal{H}) \)-valued functions, obtained by applying the isotony \( I \) to the elements of the lattice \( \mathcal{L}(\mathcal{H}) \). We denote by \( [I \mathcal{L}(\mathcal{H})]_{\text{seq}} \) the sequential closure of this set in \( \mathcal{F}(\mathcal{H}) \).

**Lemma 1.** Let \( I \) be an isotony of the lattice \( \mathcal{L}(\mathcal{H}) \). Then the elements of \( [I \mathcal{L}(\mathcal{H})]_{\text{seq}} \) are isotonic functions.

**Proof.** Let \( f \in [I \mathcal{L}(\mathcal{H})]_{\text{seq}} \). Then there exists a sequence \( \{G_n\}_{n \in \mathbb{N}} \) such that for every \( t \geq 0 \) one has

\[
f(t) = \mathcal{L}(\mathcal{H}) - \lim_{n \to \infty} I^t(G_n).
\]

Let \( t_1 \leq t_2 \). Then

\[
P_{f(t_1)} = s - \lim_{n \to \infty} P_{I^t(G_n)}, \quad P_{f(t_2)} = s - \lim_{n \to \infty} P_{I^t(G_n)}.
\]

From the inclusion \( I^t(G_n) \subseteq I^t(G_{n+1}) \) and the relation \( P_{I^t(G_n)} P_{I^t(G_{n+1})} = P_{I^t(G_n)} \), for every \( x \) we obtain

\[
P_{I^t(G_n)} P_{I^t(G_{n+1})} x = P_{I^t(G_n)} (P_{I^t(G_n)} - P_{f(t_1)}) x + (P_{I^t(G_n)} - P_{f(t_2)}) P_{f(t_1)} x + P_{f(t_2)} P_{f(t_1)} x.
\]

By the convergence and boundedness of the norms, \( \|P_{I^t(G_n)}\| = 1 \), we get

\[
P_{I^t(G_n)} P_{I^t(G_{n+1})} \xrightarrow{s} P_{f(t_2)} P_{f(t_1)} \quad \text{as } n \to \infty \quad \text{and} \quad P_{f(t_2)} P_{f(t_1)} = P_{f(t_1)},
\]

which implies the inclusion \( f(t_1) \subseteq f(t_2) \). \( \square \)

Define “balls” in the set \( [I \mathcal{L}(\mathcal{H})]_{\text{seq}} \):

\[
B_r(f) = \{ g \in [I \mathcal{L}(\mathcal{H})]_{\text{seq}} : \exists t > 0 : g(t) \neq 0, g(t) \subset f(r) \}.
\]

**Lemma 2.** Let \( I \) be an isotony of the lattice \( \mathcal{L}(\mathcal{H}) \). Then the system of sets \( \{B_r(f), f \in [I \mathcal{L}(\mathcal{H})]_{\text{seq}}, r > 0 \} \) is a base of some topology on \([I \mathcal{L}(\mathcal{H})]_{\text{seq}}\).

**Proof.** Let us check the condition for a family of sets to be a base of topology: let \( f \in B_{r_1}(f_1) \cap B_{r_2}(f_2) \). Prove that there exists a radius \( r \) such that \( B_r(f) \subseteq B_{r_1}(f_1) \cap B_{r_2}(f_2) \). There exist \( t_1 \) and \( t_2 \) such that \( f(t_1), f(t_2) \neq \{0\} \), \( f(t_1) \subseteq f_1(r_1) \), \( f(t_2) \subseteq f_2(r_2) \). Since, by Lemma 1, \( f \) is an isotonic function, \( f(r) \subseteq f_1(r_1) \cap f_2(r_2) \) for \( r := \min\{t_1, t_2\} \), and \( f(r) \neq \{0\} \). Then for every \( g \in B_r(f) \) there exists \( t_g > 0 \) such that \( g(t_g) \neq \{0\} \) and \( g(t_g) \subseteq f(r) \subseteq f_1(r_1) \cap f_2(r_2) \), so that \( g(t_g) \subseteq f_1(r_1) \) and \( g(t_g) \subseteq f_2(r_2) \). This means that \( g \in B_{r_1}(f_1) \) and \( g \in B_{r_2}(f_2) \), and thus \( g \in B_{r_1}(f_1) \cap B_{r_2}(f_2) \). The lemma is proved. \( \square \)

**Remark 1.** If instead of \( [I \mathcal{L}(\mathcal{H})]_{\text{seq}} \) one considers \( \overline{I \mathcal{L}(\mathcal{H})} \), the closure of the set of functions \( I \mathcal{L}(\mathcal{H}) \) in the topology of pointwise convergence on the lattice \( \mathcal{F} \), then analogs of Lemmas 1 and 2 hold. In the proof of Lemma 1 in this case one should replace the sequence \( \{G_n\}_{n \in \mathbb{N}} \) by the net \( \{G_\alpha\} \). The ball topology on \([I \mathcal{L}(\mathcal{H})]_{\text{seq}} \subset \mathcal{F}(\mathcal{H}) \) clearly differs from the topology of pointwise convergence.

1.4.2. The wave isotony. For every positive definite self-adjoint operator \( A \), one can define an isotony of the lattice \( \mathcal{L}(\mathcal{H}) \) in the following way. Consider the system

\[
v_{tt} + Av = g, \quad t > 0, \tag{1.13}
\]

\[
v|_{t=0} = v|_{t=0} = 0, \tag{1.14}
\]

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where $g$ is an $\mathcal{H}$-valued function of time. If $g \in C^\infty([0, \infty); \mathcal{H})$, then this problem has a unique solution $v = v^g(t)$ given by the Duhamel's formula [9]:

$$v^g(t) = \int_0^t A^{-\frac{1}{2}} \sin \left[ (t - s) A^{\frac{1}{2}} \right] g(s) ds. \quad (1.15)$$

Let $\mathcal{G} \in \mathfrak{L}(\mathcal{H})$. Consider the sets

$$\mathcal{V}_L^A(\mathcal{G}) := \{ v^g(t), g \in C^\infty([0, t]; \mathcal{G}) \} \quad (1.16)$$

and define the family of maps $\{ I_A^t \}_{t \geq 0}$ as follows:

$$I_A^0 := \text{id};$$

$$I_A^t(\mathcal{G}) := \mathcal{V}_L^A(\mathcal{G}), t > 0.$$

**Proposition 2** ([1]). The family $\{ I_A^t \}_{t \geq 0}$ is an isotony of the lattice $\mathfrak{L}(\mathcal{H})$.

We call such an isotony $I_A^t$ the wave isotony of the lattice $\mathfrak{L}(\mathcal{H})$ defined by the operator $A$.

1.4.3. *The wave spectrum.* Let us return to the original problem. The family of reachable sets of the system $\alpha_{L_0}$ defines the family of subspaces $\mathfrak{M}_{L_0} = \{ U_{L_0}, t \geq 0 \} \subset \mathfrak{L}(\mathcal{H})$, and the operator $L$ defines the wave isotony $I_L^t$. As we mentioned above, there exists a minimal sublattice $\mathfrak{L}_{L_0}$ in $\mathfrak{L}(\mathcal{H})$ that contains the family $\mathfrak{M}_{L_0}$ and is invariant under $I_L^t$. Denote $I_L^t\mathfrak{L}_{L_0} = \{ I_L(\mathcal{G}), \mathcal{G} \in \mathfrak{L}_{L_0} \}$, also denote by $[I_L\mathfrak{L}_{L_0}]_{\text{seq}}$ the (sequential) closure of this set in $\mathfrak{L}(\mathcal{H})$.

The wave spectrum $\Omega_{L_0}$ of the operator $L_0$ is the set of atoms of the partially ordered set $[I_L\mathfrak{L}_{L_0}]_{\text{seq}}$.

$$\Omega_{L_0} := At[I_L\mathfrak{L}_{L_0}]_{\text{seq}}.$$  

The wave model of the operator $L_0$, which is a unitarily equivalent operator in the model space, requires for its construction some additional conditions on $L_0$. Examples that we considered earlier [1, 7] suggest that the wave model can be constructed for some class of differential operators. In the course of construction, we state these additional general conditions on $L_0$ using notions that we gradually introduce.

**Condition 1.** The wave spectrum of the operator $L_0$ is not empty: $\Omega_{L_0} \neq \emptyset$.

The ball topology on $[I_L\mathfrak{L}(\mathcal{H})]_{\text{seq}}$ induces a topology on the wave spectrum. Under additional assumptions on $\Omega_{L_0}$ one can also define a metric (in the examples mentioned above the “balls” $B_r(f)$ turn out to be open balls in this metric). Each atom $\omega \in \Omega_{L_0}$, being a function from $[0, \infty) \to \mathfrak{L}(\mathcal{H})$, defines a non-decreasing family of projections $P_{\omega(t)}$. If $P_{\omega(t)} \to I$ as $t \to +\infty$, then one can consider a self-adjoint and, generally speaking, an unbounded operator

$$\tau_\omega = \int_0^\infty t dP_{\omega(t)},$$

the eikonal. It may happen that even for unbounded $\tau$ the following holds.

**Condition 2.** $P_{\omega(t)} \to I$ as $t \to +\infty$ for every $\omega \in \Omega_{L_0}$, and $\tau_{\omega_1} - \tau_{\omega_2}$ is a bounded operator in $\mathcal{H}$ for every $\omega_1, \omega_2 \in \Omega_{L_0}$.

In such a case one can consider the function

$$\tau(\omega_1, \omega_2) = \| \tau_{\omega_1} - \tau_{\omega_2} \|.$$
as a distance in $\Omega_{L_0}$ (the properties of distance can be checked easily). For the wave spectrum one can also define the “boundary” $\partial \Omega_{L_0}$ as the set

$$\partial \Omega_{L_0} := \{ \omega \in \Omega_{L_0} : \forall t > 0 \omega(t) \subseteq \overline{U_t} \}.$$ 

In the case of the Laplace operator on a compact Riemannian manifold the “boundary” of the wave spectrum corresponds to the boundary of the manifold [1].

1.5. The wave model. Our goal is to construct the wave model so that this construction is applicable not only to the operator $L_0$, but also to its unitary copies. For this it is important to ensure that the wave model is constructed by using the objects that are available to the “outer observer.”

1.5.1. The wave representation. If Conditions 1 and 2 hold for the operator $L_0$, then its wave spectrum is a metric space with the distance $\tau$. The model space for the wave model should consist of functions on $\Omega_{L_0}$ that take values in some “natural” auxiliary spaces. The first step in constructing the model space are spaces of germs on atoms. For every $\omega \in \Omega_{L_0}$ consider the following equivalence relation on $\mathcal{H}$:

$$u_1 \sim \omega u_2 \text{ if there exists } t > 0 \text{ such that } P_{\omega(t)}u_1 = P_{\omega(t)}u_2.$$ 

The corresponding equivalence classes $\tilde{u}(\omega)$ are called germs. Germs form a linear space, which we denote by $\tilde{\mathcal{H}}: = \{ \tilde{u}(\cdot), u \in \mathcal{H} \}$.

We need the operator $W : u \mapsto \tilde{u} \in \tilde{\mathcal{H}}$ to be bijective from $\mathcal{H}$ to $\tilde{\mathcal{H}}$, and for this the following condition is imposed, which we call the completeness of the system of atoms of the wave spectrum.

**Condition 3.** For every nonzero $u \in \mathcal{H}$ there exists an atom $\omega \in \Omega_{L_0}$ such that $P_{\omega(\varepsilon)}u \neq 0$ for every $\varepsilon > 0$.

It is not convenient to work with this space, because stalks have infinite dimension. Moreover, there is no Hilbert structure there. Thus we need additional conditions. The possibility to factorize further in germs is related to the existence of gauge elements in $\mathcal{H}$. In order to define them, we need the following condition of vanishing of atoms at zero.

**Condition 4.** $\omega(t) \xrightarrow{\mathcal{H}} \{0\}$ as $t \to +0$ for every $\omega \in \Omega_{L_0}$.

By Lemma 1, this is equivalent to the condition $\bigcap_{t>0} \omega(t) = \{0\}$ for every atom. We call an element $e \in \mathcal{H}$ a gauge element of the operator $L_0$ if there exists a set of atoms $\Omega_{L_0}^e \subseteq \Omega_{L_0}$ such that its elements form a complete system in the sense of Condition 3 and that for every $u \in \mathcal{U}_{L_0}$ and $\omega \in \Omega_{L_0}^e$ the following limit exists:

$$\lim_{t \to +0} \frac{\|P_{\omega(t)}u\|_{\mathcal{H}}}{\|P_{\omega(t)}e\|_{\mathcal{H}}}.$$ 

As we see, the linear set of smooth waves starts playing an important role here.

**Condition 5.** The operator $L_0$ has a gauge element.

Let $\omega \in \Omega_{L_0}^e$. For every $u, v \in \mathcal{U}_{L_0}$ the limit

$$\langle u, v \rangle_{\omega} := \lim_{t \to +0} \frac{\langle P_{\omega(t)}u, v \rangle_{\mathcal{H}}}{\langle P_{\omega(t)}e, e \rangle_{\mathcal{H}}}$$
exists. It can be considered as a nonnegative sesquilinear form on \( \tilde{U}_{L_0,\omega} := \{ \tilde{u}(\omega), u \in U_{L_0} \} \), a linear set in the stalk above \( \omega \). After factorization of \( \tilde{U}_{L_0,\omega} \) by the neutral subspace \( \tilde{U}_{L_0,\omega}^0 \) of this form, we obtain a linear space \( \tilde{U}_{L_0,\omega}/\tilde{U}_{L_0,\omega}^0 \). Denote its elements by \([u](\omega), u \in U_{L_0} \). This space has the inner product

\[
\langle [u](\omega), [v](\omega) \rangle_{\tilde{U}_{L_0,\omega}^0} = \langle u, v \rangle_{\omega}.
\]

After completion in the corresponding norm, we obtain the space of values \( U_{L_0,\omega}^w \).

**Condition 6.** There exists a measure \( \mu \) on \( \Omega_{L_0} \) such that \( \mu(\Omega_{L_0} \setminus \Omega'_{L_0}) = 0 \) and the relation

\[
(u,v)_{\mathcal{H}} = \int_{\Omega_{L_0}} \langle [u](\omega), [v](\omega) \rangle_{U_{L_0,\omega}^w} d\mu(\omega)
\]

(1.17)

holds for every \( u,v \in U_{L_0} \).

The space

\[
\mathcal{H}^w := \bigoplus_{\Omega_{L_0}} U_{L_0,\omega}^w d\mu(\omega)
\]

is called the wave representation of the space \( \mathcal{H} \). For the operator \( W_0^w : u \mapsto [u](\cdot) \), which acts from \( U_{L_0} \) to \( \mathcal{H}^w \), one has \( \| W_0^w u \|_{\mathcal{H}} = \|[u]\|_{\mathcal{H}^w} \) owing to (1.17); therefore the operator \( W^w = \overline{W_0^w} \) is isometric.

**Condition 7.** The operator \( W^w \) of passing from \( \mathcal{H} \) to \( \mathcal{H}^w \) is unitary.

We consider the space \( \mathcal{H}^w \) as the model space. The operator \( W^w \) determines the unitary copy \( W^w L_0^s W^{w*} \) of the operator \( L_0^s \), which acts in \( \mathcal{H}^w \). Since for each \( u \in U_{L_0} \) there exists a control \( h \in \mathcal{M} \) and \( T \geq 0 \) such that \( u = u^h(T) \), we can write

\[
L_0^s u = L_0^s u^h(T) = -u_{tt}^h(T) = -u^{htt}(T).
\]

The graph of the unitary image of the wave part of the operator \( L_0^s \) can be defined via smooth waves:

\[
\text{Graph} \left( W^w L_0^s | U_{L_0} W^{w*} \right) = \{(W^w u, W^w L_0^s u), u \in U_{L_0} \}
\]

\[
= \left\{ (W^w u^h(T), -W^w u^{htt}(T)), h \in \mathcal{M}, T \geq 0 \right\}.
\]

This way of constructing the wave model is available to the “outer observer” who can apply different controls and draw graphs.

1.5.2. The coordinate representation. If the defect indices of the operator \( L_0 \) are finite, then under additional assumptions one can define coordinates in the spaces of values \( U_{L_0,\omega}^w \) and pass to the wave model, where the operator is represented as a differential operator acting in a space of square integrable functions.

**Condition 8.** The operator \( L_0 \) has defect indices \( (n,n), n < \infty \). The subspace \( \text{Ker} L_0^* \) lies in \( U_{L_0} \). There exists a basis \( e_1, e_2, ..., e_n \) in \( \text{Ker} L_0^* \) and a set \( \Omega_{L_0}^0 \subseteq \Omega_{L_0}^e \) the atoms of which form a complete system and for which \( \mu(\Omega_{L_0} \setminus \Omega_{L_0}^0) = 0 \), such that for every \( \omega \in \Omega_{L_0}^0 \) the elements \([e_1](\omega), [e_2](\omega), ..., [e_n](\omega)\) form a basis in the space of values \( U_{L_0,\omega}^w \).
For atoms $\omega \in \Omega_{L_0}$ and smooth waves $u \in \mathcal{U}_{L_0}$, the elements $[u](\omega)$ can be decomposed with respect to the basis $[e_1](\omega), [e_2](\omega), \ldots, [e_n](\omega)$. The coefficients of this decomposition can be found from the limit

$$
\tilde{u}(\omega) := \left( \frac{\langle u, e_1 \rangle_{\omega}}{\langle u, e_2 \rangle_{\omega}}, \ldots, \frac{\langle u, e_n \rangle_{\omega}}{\langle u, e_n \rangle_{\omega}} \right) = \lim_{t \to +0} \frac{1}{(P_{\omega(t)}u, e)} \left( \begin{array}{c} (P_{\omega(t)}u, e_1) \\ (P_{\omega(t)}u, e_2) \\ \vdots \\ (P_{\omega(t)}u, e_n) \end{array} \right)
$$

and the Gram matrix

$$
G(\omega) = \left( \begin{array}{cccc} \langle e_1, e_1 \rangle_{\omega} & \langle e_2, e_1 \rangle_{\omega} & \cdots & \langle e_n, e_1 \rangle_{\omega} \\ \langle e_1, e_2 \rangle_{\omega} & \langle e_2, e_2 \rangle_{\omega} & \cdots & \langle e_n, e_2 \rangle_{\omega} \\ \vdots & \vdots & \ddots & \vdots \\ \langle e_1, e_n \rangle_{\omega} & \langle e_2, e_n \rangle_{\omega} & \cdots & \langle e_n, e_n \rangle_{\omega} \end{array} \right) = \lim_{t \to +0} \frac{1}{(P_{\omega(t)}e, e)} \left( \begin{array}{cccc} (P_{\omega(t)}e_1, e_1) & (P_{\omega(t)}e_2, e_1) & \cdots & (P_{\omega(t)}e_n, e_1) \\ (P_{\omega(t)}e_1, e_2) & (P_{\omega(t)}e_2, e_2) & \cdots & (P_{\omega(t)}e_n, e_2) \\ \vdots & \vdots & \ddots & \vdots \\ (P_{\omega(t)}e_1, e_n) & (P_{\omega(t)}e_2, e_n) & \cdots & (P_{\omega(t)}e_n, e_n) \end{array} \right)
$$

this information is available to the “outer observer”. It is easier, however, to take in the coordinate representation $\tilde{u}(\omega)$ instead of these coefficients as values at $\omega$. In this way we obtain a model of the wave part of the operator $L_0^*$ in the space

$$
\mathcal{H}^c := L_2(\Omega_{L_0}, \mu, \mathbb{C}^n)
$$

of the coordinate representation, which we also call the wave model. In a perfect situation one can define on $\Omega_{L_0}$ a manifold structure or even global coordinates. This takes place for the Laplace operator on a compact Riemannian manifold [1], for the positive definite Schrödinger operator on the half-line [7], and in our case.

2. The Sturm–Liouville operator on an interval

Let us look at realization of the abstract scheme for the Sturm–Liouville operator on an interval.

2.1. The operator $L_0$. Let $0 < l < \infty$, $\mathcal{H} = L_2(0, l)$. The operator $L_0$ is defined on the domain

$$
\text{Dom } L_0 = \{ u \in H^2(0, l) : u(0) = u'(0) = u(l) = u'(l) = 0 \} \quad (2.1)
$$

by the differential expression

$$
L_0 u := -u'' + qu, \quad (2.2)
$$

where $q \in C^\infty[0, l]$ is a smooth function such that the operator $L_0$ is positive definite. Such an operator is symmetric and has the defect indices $(2, 2)$. Its adjoint operator $L_0^*$ is defined by the same differential expression on the domain

$$
\text{Dom } L_0^* = H^2(0, l).
$$

The Friedrichs extension $L$ of $L_0$ is defined on the domain

$$
\text{Dom } L = \{ u \in H^2(0, l) : u(0) = u(l) = 0 \}.
$$
2.2. The Green’s system. To describe the subspace $K = \text{Ker } L_0^*$, determine two solutions of the equation $-u'' + gu = 0$. Denote by $\phi_0$ the solution of this equation with the initial data $\phi_0(0) = 0$, $\phi'_0(0) = 1$ and by $\phi_1$ the solution with the data $\phi_1(l) = 0$, $\phi'_1(l) = 1$. Since the operator $L$ is positive definite, 0 is not its eigenvalue and these functions cannot be proportional. Therefore they form a basis in $K$.

Let us write out the Vishik’s decomposition for $u \in \text{Dom } L_0$. Let

$$\eta_0 := L^{-1}\phi_0, \quad \eta_1 := L^{-1}\phi_1.$$  

**Lemma 3.** In the decomposition of $u \in \text{Dom } L_0^*$

$$u = u_0 + L^{-1}g_u + h_u,$$

the elements $g_u, h_u \in K$ are given by the formulas

$$g_u = \frac{1}{\eta_0'(0)\eta_1(l) - \eta_1'(0)\eta_0(l)}$$

$$\times \left\{ \left[ \eta_1(l)\left( u'(l) - \frac{u(l)}{\phi_0(l)} \phi'_0(l) - \frac{u(0)}{\phi_0(l)} \phi'_0(0) \right) - \eta_0(l)\left( u'(l) - \frac{u(l)}{\phi_0(l)} \phi'_0(l) - \frac{u(0)}{\phi_0(l)} \phi'_0(0) \right) \right] \eta_0 
+ \left[ \eta_0(0)\left( u'(0) - \frac{u(l)}{\phi_0(l)} \phi'_0(l) - \frac{u(0)}{\phi_0(l)} \phi'_0(0) \right) - \eta_1(0)\left( u'(0) - \frac{u(l)}{\phi_0(l)} \phi'_0(l) - \frac{u(0)}{\phi_0(l)} \phi'_0(0) \right) \right] \eta_1 \right\}, \quad (2.3)$$

$$h_u = \frac{u(l)}{\phi_0(l)} \phi_0 + \frac{u(0)}{\phi_0(0)} \phi_1. \quad (2.4)$$

**Proof.** Since $u_0(0) = u'_0(0) = u_0(l) = u'_0(l) = 0$ and $(L^{-1}g_u)(0) = (L^{-1}g_u)(l) = 0$, we should find coefficients in the relations

$$h_u = c_0\phi_0 + c_1\phi_1, \quad g_u = d_0\phi_0 + d_1\phi_1,$$

such that

$$u(0) = h_u(0),$$

$$u(l) = h_u(l),$$

$$u'(0) = (L^{-1}g_u)'(0) + h'_u(0),$$

$$u'(l) = (L^{-1}g_u)'(l) + h'_u(l).$$

Substituting here (2.5) and $L^{-1}g_u = d_0\eta_0 + d_1\eta_1$ and taking into account the initial data for the solutions $\phi$, we find the coefficients $c_0, c_1, d_0,$ and $d_1$ and arrive at the formulas (2.3) and (2.4). \qed

From the lemma and (1.5) we get

$$\Gamma_1 u = -\frac{u(l)}{\phi_0(l)} \phi_0 - \frac{u(0)}{\phi_1(0)} \phi_1, \quad (2.6)$$

$$\Gamma_2 u = \frac{1}{\eta_0(0)\eta_1'(l) - \eta_1(0)\eta_0'(l)}$$

$$\times \left\{ \left[ u(0)\frac{\eta_0'(0) - \eta_0'(0)\phi'_0(0)}{\phi_1(0)} - \frac{u(l)}{\phi_0(l)} \phi'_0(l) - \frac{u(0)}{\phi_0(l)} \phi'_0(0) \right] \eta_0 
+ \left[ u(0)\frac{\eta_0'(0) + \eta_0'(0)\phi'_0(0)}{\phi_1(0)} - \frac{u(l)}{\phi_0(l)} \phi'_0(l) - \frac{u(0)}{\phi_0(l)} \phi'_0(0) \right] \eta_1 \right\}, \quad (2.7)$$

$$- \left[ u(0)\frac{\eta_0'(0) - \eta_0'(0)\phi'_0(0)}{\phi_1(0)} - \frac{u(l)}{\phi_0(l)} \phi'_0(l) - \frac{u(0)}{\phi_0(l)} \phi'_0(0) \right] \phi_0 
+ \left[ u(0)\frac{\eta_0'(0) + \eta_0'(0)\phi'_0(0)}{\phi_1(0)} - \frac{u(l)}{\phi_0(l)} \phi'_0(l) - \frac{u(0)}{\phi_0(l)} \phi'_0(0) \right] \phi_1 \right\}. \quad (2.7)$$
The spaces \( \mathcal{H} = L_2(0, l) \), \( \mathcal{K} = \{ c_0 \phi_0 + c_l \phi_l, c_0, c_l \in \mathbb{C} \} \) and the operators \( L_0, \Gamma_1, \Gamma_2 \) defined by (2.1), (2.2), (2.6), and (2.7) form the Green’s system \( \mathfrak{G}_{L_0} \), which canonically corresponds to the operator \( L_0 \).

2.3. The system with boundary control. Consider the system (1.6)–(1.8) in our case. The boundary control \( h(t) \in \mathcal{K} \) can be written in the form

\[
h(t) = - \frac{f_t(t)}{\phi_0(t)} \phi_0 - \frac{f_0(t)}{\phi_l(t)} \phi_l,
\]

where the functions \( f_0(t) \) and \( f_t(t) \) are taken from the class

\[
\hat{\mathcal{M}} = \{ f \in C^\infty[0, \infty) : \operatorname{supp} f \subset (0, \infty) \}.
\]

Then the system (1.6)–(1.8) takes the form of the initial boundary value problem

\[
\begin{align*}
&u_{tt} - u_{xx} + qu = 0, \quad x \in (0, l), t > 0, \\
&u|_{t=0} = u_t|_{t=0} = 0, \quad x \in [0, l], \\
&u|x=0 = f_0(t), \quad t \geq 0, \\
&u|x=l = f_l(t), \quad t \geq 0.
\end{align*}
\]

The solution of such a problem for \( t \leq l \) is given by the formula

\[
w^f(x, t) = f_0(t - x) + f_t(t - l + x) + \int_x^t w_0(x, s) f_0(t - s) ds + \int_{t-l-x}^t w_l(l - x, s) f_l(t - s) ds,
\]

where the functions \( f_0 \) and \( f_t \) are assumed to be zero on the negative half-line, the functions \( w_0(x, t) \) and \( w_l(x, t) \) are defined for \( 0 \leq x \leq t \leq l \) and are smooth.

2.3.1. Controllability of the system \( \alpha_{L_0} \). Let us find reachable sets of the system \( \alpha_{L_0} \).

Lemma 4.

\[
\mathcal{U}_{L_0}^t = \left\{ \begin{array}{c}
\{ u \in C^\infty[0, l] : \operatorname{supp} u \subset [0, t) \cup (l - t, l]\}, \quad t \leq \frac{l}{2} \\
C^\infty[0, l], \quad t > \frac{l}{2},
\end{array} \right\}
\]

Proof. Let \( t \leq \frac{l}{2} \). One can see from the expression (2.9) that for \( f_0, f_t \in \hat{\mathcal{M}} \) the solution \( u^h(\cdot, t) \) belongs to \( C^\infty[0, l] \). It also follows that its support is contained in \( [0, t) \cup (l - t, l] \). Thus

\[
\mathcal{U}_{L_0}^t \subseteq \{ u \in C^\infty[0, l] : \operatorname{supp} u \subset [0, t) \cup (l - t, l]\}.
\]

To prove the inverse inclusion, take \( u \) from the right-hand side and show that \( u(x) = u^f(x, t) \). Let us represent \( u \) in the form

\[
u = u_0 + u_l, \quad u_0, u_l \in C^\infty[0, l], \text{supp} u_0 \subseteq [0, t), \text{supp} u_l \subseteq (l - t, l] .
\]

According to (2.9), divide the equation \( u(x) = u^f(x, t) \) into two parts as follows:

\[
f_0(t - x) + \int_x^t w_0(x, s) f_0(t - s) ds = u_0(x),
\]

\[
f_l(l - x) + \int_{t-l-x}^t w_l(l - x, s) f_l(t - s) ds = u_l(x).
\]

These are Volterra equations of the second kind on the interval \((0, l)\), they have solutions from the same classes to which their right-hand sides belong (taking into account the change of
the variable; \(\text{supp } f_0, \text{supp } f_t \subseteq (0, t]\), they can be continued to \(\hat{M}\), which will not affect the relation \(u(x) = u^f(x, t)\). Thus the first assertion of the lemma is proved.

Let \(\frac{1}{2} < t \leq l\) and \(u \in C^\infty[0, l]\). There exists a function \(u_0 \in C^\infty[0, l]\) such that \(u_0|_{[0, \frac{1}{2}]} = u|_{[0, \frac{1}{2}]}\) and \(\text{supp } u_0 \subseteq [0, t]\). Take \(w = u - u_0\). Then \(w \in C^\infty[0, l]\) and \(\text{supp } w \subseteq (\frac{1}{2}, l] \subseteq \{l - t, l]\).

By the same argument as in the first part of the proof, we obtain controls \(f_0, f_t \in \hat{M}\) for which \(u(x) = u^f(x, t)\). Consequently, \(C^\infty[0, l] \subseteq \mathcal{U}_{L_0}^L\). From (2.9) it follows that \(\mathcal{U}_{L_0}^L \subseteq C^\infty[0, l]\).

For \(t > l\) the inclusion \(\mathcal{U}_{L_0}^L \subseteq C^\infty[0, l]\) holds owing to the monotonicity of reachable sets, and the inverse inclusion \(\mathcal{U}_{L_0}^L \subseteq C^\infty[0, l]\) is always true. Thus the lemma is proved. \(\Box\)

The system \(\alpha_{L_0}\) is controllable, since \(\overline{\mathcal{U}}_{L_0} = L_2(0, 1) = \mathcal{H}\), and this also follows from the fact that the operator \(L_0\) is completely non-selfadjoint. The closure of \(C^\infty[0, l]\) in the graph norm of the operator \(L_0^*\) is the Sobolev space \(H^2(0, l) = \text{Dom } L_0^*\); therefore the wave part of the operator \(L_0^*\) (which is \(I_0^*|_{L_0}\)) coincides with \(L_0^*\).

### 2.4. The wave spectrum

We turn to constructing the wave spectrum of the operator \(L_0\).

For this we have already found the family of reachable subspaces \(\overline{\mathcal{U}}_{L_0} = L_2((0, t) \cup (l - t, l))\). Now we have to find out how the wave isometry \(I_L\) acts. For a set \(E \subset [0, l]\) denote by \(E^t\) its metric neighborhood in \([0, l]\):

\[
E^t = \{x \in [0, l] : \text{dist}(x, E) < t\}, \quad t > 0,
\]

\[
\text{dist}(x, E) := \inf_{y \in E} \text{dist}(x, y).
\]

For \(t = 0\) we take \(E^t = E\).

**Lemma 5.** For \(0 \leq a < b \leq l\) and \(t \geq 0\) the following holds:

\[
I_L^t(L_2(a, b)) = L_2((a, b)^t). \tag{2.11}
\]

**Remark 2.** We identify spaces \(L_2(a, b)\) with the subspaces of \(L_2(0, l)\) that consist of functions vanishing almost everywhere outside \((a, b)\).

**Proof.** The system (1.13)–(1.14) can be written in the form of the initial boundary value problem

\[
v_{tt} - v_{xx} + qv = g, \quad x \in (0, l), t > 0, \tag{2.12}
\]

\[
v|_{t=0} = v|_{t=0} = 0, \quad x \in [0, l], \tag{2.13}
\]

\[
v|_{x=0} = v|_{x=l} = 0, \quad t \geq 0. \tag{2.14}
\]

with the right-hand side \(g(x, t)\) from the corresponding class.

An argument analogous to the proof of Lemma 2 from [7], which is based on the fact of the finiteness of the domain of influence for the hyperbolic equation (2.12), leads to the inclusion \(\mathcal{V}_L^t(L_2(a, b)) \subseteq L_2((a, b)^t)\) and thus to \(I_L^t(L_2(a, b)) \subseteq L_2((a, b)^t)\).

Consider the conjugate problem

\[
w_{tt} - w_{xx} + qw = 0, \quad x \in (0, l), t \in (0, T), \tag{2.15}
\]

\[
w|_{t=T} = 0, \quad w|_{t=T} = y, \quad x \in [0, l], \tag{2.16}
\]

\[
w|_{x=0} = w|_{x=l} = 0, \quad t \in [0, T]. \tag{2.17}
\]

For \(g \in C^\infty_0((0, l) \times (0, \infty))\) and \(y \in L_2(0, l)\), the duality relation

\[
\int_0^T \int_0^l g(x, t)w^y(x, t)dxdt = - \int_0^l v^y(x, T)y(x)dx \tag{2.18}
\]
Lemma 6. The odd continuation of the solution \( w^g \) solves the problem
\[
\begin{align*}
  &w_{tt} - w_{xx} + qw = 0, \\
  &w|_{t=T} = 0, \quad w|_{t=-T} = y, \\
  &w|_{x=0} = w|_{x=l} = 0,
\end{align*}
\]
(2.19) \( x \in (0, l), t \in (0, 2T), \)
(2.20) \( x \in [0, l], \)
(2.21) \( t \in [0, 2T]. \)

(\text{note that } w^g \text{ and } w^y \text{ retain continuity.}) \text{ If there exists } y \in L_2((a, b)^T) \supseteq V^T_L(L_2(a, b)), \text{ then an argument analogous to the proof of Lemma 2 from [7] leads to } w^g = 0, \text{ from which it follows that } y \text{ can be only zero. Therefore } V^y_L(L_2(a, b)) \text{ is dense in } L_2((a, b)^T). \text{ Thus we proved that } I^y_L(L_2(a, b)) = L_2((a, b)^T). \tag*{□}

Let us call a set \( E \subseteq [0, l] \) elementary if
\[
E = \bigcup_{k=1}^{n(E)} (a_k, b_k),
\]
where \( 0 \leq a_1 < b_1 < a_2 < b_2 < \ldots < a_{n(E)} < b_{n(E)} \leq l, \) and if the set \( E \) is symmetric with respect to the middle of the interval \((0, l)\). Let \( E[0, l] \) be the family of all elementary sets. Obviously, if \( E \in E[0, l] \), then \( E^t \in E[0, l] \) for every \( t \geq 0 \). We will also call the subspaces \( L_2(E), \ E \in E[0, l] \), elementary. The family of elementary subspaces forms the lattice \( \mathcal{L}_{E[0, l]} \subseteq \mathcal{L}(\mathcal{H}). \)

Lemma 6. For every \( E \in E[0, l] \) one has \( I^y_L(L_2(E)) = L_2(E^t). \)

Proof. By isotonicity,
\[
L_2((a_k, b_k)^t) = I^y_L(L_2(a_k, b_k)) \subseteq I^y_L(L_2(E))
\]
for every \( k \), and thus \( L_2(E^t) \subseteq I^y_L(L_2(E)) \). Using the same argument as in the proof of Lemma 5, we arrive at
\[
I^y_L(L_2(E)) = V^y_L(L_2(E)) = L_2(E^t). \tag*{□}
\]

The lattice \( \mathcal{L}_{E[0, l]} \) is invariant under the wave isotony \( I_L \) and contains all the subspaces of the form \( L_2((0, t) \cup (l - t, l)), \) i.e., all reachable subspaces. Therefore \( \mathcal{L}_{L_0} = \mathcal{L}_{E[0, l]} \).

Let \( m \) denote the Lebesgue measure, let \( \mathcal{B} \) be the Borel sigma-algebra on the segment \([0, l]\), \( \mathcal{L}_B \) be the corresponding lattice of subspaces,
\[
\mathcal{L}_B := \{ L_2(E), \ E \in \mathcal{B} \} \subseteq \mathcal{L}(\mathcal{H}),
\]
\( E \triangle F = (E \setminus F) \cup (F \setminus E) \) be the symmetric difference of sets.

Lemma 7. Let \( \{ E_n \}_{n \in \mathbb{N}} \) be a sequence of sets from \( \mathcal{B} \) and \( E \in \mathcal{B} \). Then the convergence \( L_2(E_n) \xrightarrow{n \to \infty} L_2(E) \) in the topology of \( \mathcal{L}(\mathcal{H}) \) is equivalent to \( m(E_n \triangle E) \xrightarrow{n \to \infty} 0. \)

The proof of the lemma repeats the proof of Lemma 4 from [7] almost literally.

Lemma 8. The closure of the lattice \( \mathcal{L}_{L_0} \) in the topology of \( \mathcal{L}(\mathcal{H}) \) is a subset of the lattice \( \mathcal{L}_B \):
\[
\overline{\mathcal{L}_{L_0}} \subseteq \mathcal{L}_B.
\]

Proof. Let a sequence \( L_2(E_n) \) of subspaces from \( \mathcal{L}_{L_0} = \mathcal{L}_{E[0, l]} \) be fundamental in \( \mathcal{L}(\mathcal{H}). \) Let us prove that there exists \( E \in \mathcal{B} \) such that \( L_2(E_n) \xrightarrow{\mathcal{L}(\mathcal{H})} L_2(E). \) By Lemma 7, convergence means that \( m(E_n \triangle E) \to 0. \) The symmetric difference is a pseudometric in \( \mathcal{B} \) and, after factorization with respect to the equivalence relation of the form \( E \sim F, \) if \( m(E \triangle F) = 0, \) we get \( \mathcal{B}/\sim, [14]. \) Thus there exists a measurable set \( E \subseteq [0, l] \) such that \( m(E \triangle E) \to 0, \) and by Lemma 7 this means that \( L_2(E_n) \xrightarrow{\mathcal{L}(\mathcal{H})} L_2(E). \) \tag*{□}
Remark 3. The set $E$ should be symmetric (up to a set of zero measure) with respect to the middle of the interval $(0, l)$, and therefore $\Sigma_{L_0} \neq \Sigma_B$.

Corollary 1. The functions of the family $[I, L_0]_{|E|}$ are isotonic and take values in $\Sigma_B$.

Consider the metric space $\mathcal{B}/\sim$ of equivalence classes of measurable sets with the distance $\rho(E, F) = m(E \triangle F)$ and for each $t > 0$ consider the following sets in it:

$$\mathcal{E}_{> t} := \{ E^t, E \in \mathcal{E}[0, l] \}$$

$$= \{ E \in \mathcal{E}[0, l] : b_1 > t \text{ if } a_1 = 0, b_1 - a_1 > 2t \text{ if } a_1 \neq 0,$$

$$\text{and } b_k - a_k > 2t \text{ for any } k = 2, \ldots, n - 1 \}.$$  

$$\mathcal{E}_{\geq t} := \{ E \in \mathcal{E}[0, l] : b_1 \geq t \text{ if } a_1 = 0, b_1 - a_1 \geq 2t \text{ if } a_1 \neq 0,$$

$$\text{and } b_k - a_k \geq 2t \text{ for any } k = 2, \ldots, n - 1 \}.$$  

Recall that elementary sets are symmetric with respect to the middle of the interval $(0, l)$.

Lemma 9. The closure of $(\mathcal{E}_{> t})_{\sim}$ in the metric of $\mathcal{B}/\sim$ is a subset of $(\mathcal{E}_{\geq t})_{\sim}$.

Proof. Let $\{E_n\}_{n \in \mathbb{N}}$ be a sequence from $\mathcal{E}_{> t}$ such that $(E_n)_{\sim}^{\mathcal{B}/\sim} E_\sim \in \mathcal{B}/\sim$. Each of the sets $E_n$ contains no more than $\frac{1}{t}$ component intervals. One can choose a subsequence $\{E_{n_j}\}_{j \in \mathbb{N}}$ of sets each of which contains the same number of component intervals. Denote this number by $N$.

![Fig. 1. The set $E_{n_j}$.](image1)

One can choose a subsequence $\{E_{n_{j_k}}\}_{k \in \mathbb{N}}$ of sets such that all the endpoints of the component intervals $\{a(E_{n_{j_k}})\}_{k \in \mathbb{N}}, \{b(E_{n_{j_k}})\}_{k \in \mathbb{N}}$ converge to some numbers $0 \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq \cdots \leq a_N \leq b_N \leq l$ (see Figs. 1 and 2).

![Fig. 2. The set $E_\infty$.](image2)

In this way we obtain the set

$$E_\infty := \bigcup_{k=1}^{N} (a_k, b_k).$$

It is easy to see the following estimate (see Fig. 3):

$$m(E_{n_{j_k}} \cap E_\infty) \leq \sum_{k=1}^{N} |a_k(E_{n_{j_k}}) - a_k| + |b_k(E_{n_{j_k}}) - b_k|.$$ 

Consequently, $m(E_{n_{j_k}} \cap E_\infty) \rightarrow 0$. This means that $E_\sim = (E_\infty)_{\sim}$. Since $E_\infty \in \mathcal{E}_{\geq t}$, the sequence $\{E_n\}_{n \in \mathbb{N}}$ was an arbitrary convergent sequence from $\mathcal{E}_{> t}$, the lemma is proved. \(\square\)

Lemma 10. Let $E, F \subseteq [0, l], \{E_n\}_{n \in \mathbb{N}}, \{F_n\}_{n \in \mathbb{N}}$ be sequences of subsets of the segment $[0, l]$. Let $m(E \cap F) \rightarrow 0$ and $m(F_n \cap F) \rightarrow 0$ as $n \rightarrow \infty$. If $E_n \subseteq F_n$ for every $n$, then $m(E \setminus F) = 0$. 796
Proof. We have
\[ E \setminus F \subseteq (E \cup E_n) \setminus F = (E_n \cup (E \setminus E_n)) \setminus F = (E_n \setminus F) \cup ((E \setminus E_n) \setminus F) \subseteq (F_n \setminus F) \cup (E_n \setminus E) \subseteq (F_n \triangle F) \cup (E_n \triangle E). \]
From this we immediately get the assertion of the lemma. \(\square\)

For \(x \in [0, \frac{l}{2}]\) denote
\[ \omega_x(t) := L_2((\{x\} \cup \{l-x\})^t), \]
then \(\omega_x \in [I_L \mathcal{L}_{L_0}]_{\text{seq}}\). Indeed, \(L_2((\{x\} \cup \{l-x\})^t) \in I_L \mathcal{L}_{L_0}\) for every \(n\) and \(L_2((\{x\} \cup \{l-x\})^t) \xrightarrow{n \to \infty} \omega_x\) in \(\mathcal{H}\).

**Lemma 11.** For every nonzero \(\omega \in [I_L \mathcal{L}_{L_0}]_{\text{seq}}\) there exists \(x \in [0, l]\) such that \(\omega_x \leq \omega\).

**Proof.** Let \(\omega \in [I_L \mathcal{L}_{L_0}]_{\text{seq}}\). This means that there exists a sequence \(\{E_n\}_{n \in \mathbb{N}}\) of elementary sets such that \(I_t(L_2(E_n)) = L_2(E_n^t) \to \omega(t)\) in \(\mathcal{L}(\mathcal{H})\) for every \(t \geq 0\). By Lemma 8, there exist measurable sets \(E(t) \subseteq [0, l]\) such that \(\omega(t) = L_2(E(t))\), and, by Lemma 9, \(E(t) \in \mathcal{E}_{\geq t}\).

If \(E(t) = [0, l]\) for every \(t > 0\), then the assertion of the lemma holds: every element \(\omega_x\), \(x \in [0, \frac{l}{2}]\), satisfies \(\omega_x \leq \omega\). Assume that there exists \(t_0 > 0\) such that \(E(t_0) \neq [0, l]\). Then for the right endpoint of the first interval the inequality \(b_1(E(t_0)) > t_0\) holds (two cases are possible, see Fig. 4 and 5).

The set \(E^{t_0}_n\) contains a finite number of component intervals, there exists a sequence \(\{E^{t_0}_n\}_{j \in \mathbb{N}}\) of sets all of which contain the same number of component intervals, and the endpoints of these
intervals have limits. These limits can be either endpoints of component intervals of the set $E(t_0)$ or inner points of this set.

![Fig. 5. The second case.](image)

The point $b_1(E(t_0))$ is the limit of a sequence $\{b_{k_1}(E^t_{E_{n_j}})\}_{j \in \mathbb{N}}$ of right endpoints of component intervals with some fixed number $k_1$ of the sets $E^t_{E_{n_j}}$. Denote

$$b := b_1(E(t_0)) - t_0, \quad B(E_{n_j}) := b_{k_1}(E^t_{E_{n_j}}) - t_0$$

(see Figs. 4 and 5). Then $B(E_{n_j}) \to b$ as $j \to \infty$. The sets $E^t_{E_{n_j}}$ contain $\{B(E_{n_j})\}^t$ for every $t > 0$. Since $E^t_{E_{n_j}} \to E(t)$, $m(E^t_{E_{n_j}} \Delta E(t)) \to 0$, and $m((\{B(E_{n_j})\}^t \Delta (\{b\}^t)) \to 0$ as $j \to \infty$, by Lemma 10 we derive that $\{b\}^t \subseteq E(t)$ up to a set of measure zero for every $t > 0$. Therefore $\omega_b \leq \omega$.

Now we can describe the wave spectrum of the operator $L_0$.

**Theorem 1.**

$$\Omega_{L_0} = \left\{ \omega_x, \ x \in \left[0, \frac{l}{2}\right]\right\}.$$  

![Fig. 6. $\omega_{\tilde{x}}$ and $\omega_x$.](image)

**Proof.** Let $\omega \in \Omega_{L_0}$. Since $\Omega_{L_0} \subseteq [I_{L}\mathcal{L}_{L_0}]_{\text{seq}}$, by Lemma 11 there exists $x \in [0, \frac{l}{2}]$ such that $\omega_x \leq \omega$. Since $\omega$ is an atom of the set $[I_{L}\mathcal{L}_{L_0}]_{\text{seq}}$, we have $\omega_x = \omega$. Hence

$$\Omega_{L_0} \subseteq \left\{ \omega_x, \ x \in \left[0, \frac{l}{2}\right]\right\}.$$
Let us prove the inverse inclusion. Let \( x \in [0, \frac{1}{2}] \), and let there exist an element \( \omega \in [I_L, L_0]_{seq} \) such that \( \omega < \omega_x \). By Lemma 11, there exists \( \tilde{x} \in [0, \frac{1}{2}] \) such that \( \omega_{\tilde{x}} \leq \omega \). This means that \( \omega_{\tilde{x}} < \omega_x \). But this cannot happen: for \( \tilde{x} = x \) we have \( \omega_{\tilde{x}} = \omega_x \), while for \( \tilde{x} \neq x \) and \( t < |\tilde{x} - x| \) we have \( \omega_x(t) \cap \omega_{\tilde{x}}(t) = \{0\} \) (see Fig. 6), which contradicts the inequality \( \omega_{\tilde{x}} < \omega_x \).

Therefore such \( \omega \) does not exist and \( \omega_x \) is an atom. Hence \( \{\omega_x, x \in [0, \frac{1}{2}]\} \subseteq \Omega_{L_0} \) and the theorem is proved.

Denote by \( \beta \) the bijection between \( [0, \frac{1}{2}] \) and \( \Omega_{L_0} \) established by Theorem 1, \( \beta : x \mapsto \omega_x \). Let us denote also \( x_\omega := \beta^{-1}(\omega), \omega \in \Omega_{L_0} \), \( E_\omega(t) := (\{x\} \cup \{l - x\})^t \), and \( f_\omega(x) := \text{dist}(x, (\{x_\omega\} \cup \{l - x_\omega\})^t) \). Note that

\[
E_{x_\omega}(t) = \{y \in (0, l) : f_\omega(y) < t\}
\]

(see Fig. 7).

**Fig. 7.** The set \( E_{x_\omega}(t) \) and the graph of the function \( f_\omega \).

**Lemma 12.** Let \( \omega \in \Omega_{L_0} \). Then the family of projections

\[
E_\omega(t) = \begin{cases} P_{\omega(t)}, & t \geq 0, \\ 0, & t < 0, \end{cases}
\]

is a resolution of the identity in the space \( H = L_2(0, l) \), and the corresponding eikonal

\[
\tau_\omega = \int_{\mathbb{R}} t dE_\omega(t)
\]

is the operator of multiplication by the function \( f_\omega \) in \( L_2(0, l) \).

**Proof.** As one can see from the definition of elements \( \omega_x \), for \( t > \frac{1}{2} \) one has \( \omega_x(t) = H \), and so \( E(t) \to I \) as \( t \to +\infty \). The strong left-continuity of the functions \( P_{\omega(t)} = [\chi_{E_\omega(t)}] \) also takes place. Therefore the family \( E(t) \) is indeed a resolution of the identity and defines the (Stieltjes) integral \( \int_{\mathbb{R}} t dE_\omega(t) \). If \( M_f \) is the operator of multiplication by the function \( f \), \( M_f = [f] \), in the space \( L_2(\mathbb{R}, \rho) \) with the measure \( \rho \), then the corresponding resolution of the identity is \( E(\lambda) = [\chi_{f^{-1}(\infty, \lambda)]} \). In our case, \( \rho \) is the Lebesgue measure on the segment \( [0, l] \), and for the operator \( M_{f_\omega} = [f_\omega] \) we get \( E(\lambda) = [\chi_{f_\omega^{-1}(\infty, \lambda)]} = [\chi_{E_\omega(t)}] \) by (2.22). This means that \( E(\lambda) = P_{\omega_0}(\lambda) \) for \( \lambda \geq 0 \) and \( E(\lambda) = 0 \) for \( \lambda < 0 \). Since spectral measures are the same, the operators also are, thus \( \tau_\omega = M_{f_\omega} \). \( \square \)
As one can see, for every $\omega_1, \omega_2 \in \Omega_{L_0}$ the distance 

$$
\tau(\omega_1, \omega_2) = \|\tau_{\omega_1} - \tau_{\omega_2}\| = \|[f_{\omega_1} - f_{\omega_2}]\| = |x_{\omega_1} - x_{\omega_2}|
$$

is well defined and the wave spectrum becomes a complete metric space. Thus the map $\beta$ is an isometric isomorphism between the segment $[0, \frac{l}{2}]$ and the wave spectrum $\Omega_{L_0}$. The “balls”

$$
B_r(\omega) = \{\bar{\omega} \in \Omega_{L_0} : \exists t > 0 : \bar{\omega}(t) \neq 0, \bar{\omega}(t) \subseteq \omega(r)\}
$$

obviously coincide with

$$
\left\{ \omega_{\bar{x}}, \bar{x} \in \left[ 0, \frac{l}{2} \right] : |\bar{x} - x_\omega| < r \right\} = \{\bar{\omega} \in \Omega_{L_0} : \tau(\bar{\omega}, \omega) < r\}
$$

(see Fig. 8), i.e., with the balls for the metric $\tau$, so that the “ball” topology on the wave spectrum coincides with the topology defined by this metric.

![Fig. 8. $B_r(\omega)$.](image)

From the form of reachable spaces (2.10) and the definition of the boundary of the wave spectrum $\partial \Omega_{L_0}$, it follows that in our case

$$
\partial \Omega_{L_0} = \{\omega_0\}.
$$

The atom $\omega_0$ is not a point of the boundary. Furthermore, the distance from the boundary defines the coordinate

$$
\tau(\omega) := \tau(\omega, \partial \Omega_{L_0}) = x_\omega,
$$

which parametrizes the wave spectrum for the “outer observer” (unlike the isomorphism $\beta$ available only to the “inner observer”).

### 2.5. The wave model

We begin constructing the wave model of the operator $L_0$ with the space of values. The first three Conditions from the abstract part are satisfied, which is obvious, since we explicitly know the subspaces $\omega_x(t)$. It is also clear that atoms vanish at zero. To prove the existence of a gauge element, we need the following standard lemma.

**Lemma 13.** A function $u \in \text{Ker} L_0^*$ cannot have more than one zero on the segment $[0, l]$.

**Proof.** Let $u \in \text{Ker} L_0^*$. The operator $L$ is positive definite, and thus its kernel is trivial. Therefore $u$ cannot vanish at both points 0 and $l$ simultaneously. Assume that $u$ has two zeros, $a$ and $b$, on the segment $[0, l]$, and at least one of them is an inner point. Then $u$ is in the kernel of the Strum–Liouville operator $L_{ab}$ defined on the interval $(a, b)$ by the differential expression $-\frac{d^2}{dx^2} + q(x)$ with the Dirichlet boundary conditions at the points $a$ and $b$. Such an operator is self-adjoint and semibounded from below. Let $l$ and $l_{ab}$ denote the sesquilinear
forms that correspond to the operators $L$ and $L_{ab}$. Their domains are $d[l] = \dot{H}^1(0,l)$ and $d[l_{ab}] = \dot{H}^1(a,b)$. According to the minimax principle [9],

$$\lambda_1(L_{ab}) = \min_{u \in H^1(a,b)} \frac{(l_{ab}u, u)_{L^2(a,b)}}{\|u\|_{L^2(a,b)}^2}.$$ 

If a function $u \in \dot{H}^1(a,b)$ is continued by zero to the whole segment $[0,l]$, then one gets the function $\tilde{u} \in \dot{H}^1(0,l)$ and $\|\tilde{u}\|_{L^2(0,l)} = \|u\|_{L^2(a,b)}$. Moreover,

$$(l_{ab}u, u)_{L^2(a,b)} = \|u^\prime\|_{L^2(a,b)}^2 + (qu, u)_{L^2(a,b)} = \|\tilde{u}\|_{L^2(0,l)}^2 + (\tilde{q}\tilde{u}, \tilde{u})_{L^2(0,l)} = (l\tilde{u}, \tilde{u})_{L^2(0,l)}.$$ 

Therefore

$$\min_{u \in H^1(a,b)} \frac{(l_{ab}u, u)_{L^2(a,b)}}{\|u\|_{L^2(a,b)}^2} = \min_{u \in H^1(0,l)} \frac{(l\tilde{u}, \tilde{u})_{L^2(0,l)}}{\|\tilde{u}\|_{L^2(0,l)}^2} \geq \min_{u \in H^1(0,l)} \frac{(lu, u)_{L^2(0,l)}}{\|u\|_{L^2(0,l)}^2} = \lambda_1(L).$$

We derived that $\lambda_1(L_{ab}) \geq \lambda_1(L) > 0$, which means that 0 cannot be an eigenvalue of the operator $L_{ab}$, $u \notin \text{Ker} L_{ab}$, a contradiction. Hence the function $u$ cannot have two zeros on the segment $[0,l]$, and the lemma is proved.

Let us take an element $e \in \text{Ker} L^0_0$ as a gauge element. The kernel of the operator $L^0_0$ consists of solutions of the equation $-u'' + qu = 0$, as $e$ we take a solution that does not vanish at the point $\frac{l}{2}$. The lemma just proved guarantees that $\|e(x_\omega)|^2 + |e(l-x_\omega)|^2 \neq 0$ for every $\omega \in \Omega_{L^0_0}$. The wave spectrum can be taken as the set $\Omega^e_{L_0}$. Indeed, let $u \in U_{L_0}$ and $\omega \in \Omega_{L_0}$. Then

$$\frac{\|P_{\omega(t)}u\|^2}{\|P_{\omega(t)}e\|^2} = \frac{\int_{E_{e_\omega}(t)} u(x)^2 \, dx}{\int_{E_{e_\omega}(t)} e(x)^2 \, dx} \to \frac{|u(x_\omega)|^2 + |u(l-x_\omega)|^2}{|e(x_\omega)|^2 + |e(l-x_\omega)|^2}.$$ 

Thus Condition 5 is satisfied. This allows us to define on smooth waves the sesquilinear form

$$\langle u, v \rangle_\omega := \frac{u(x_\omega)v(x_\omega) + u(l-x_\omega)v(l-x_\omega)}{|e(x_\omega)|^2 + |e(l-x_\omega)|^2}, \quad u, v \in U_{L_0}.$$ 

Factorizing with respect to the equivalence relation

$$u \sim v \iff \langle u - v, u - v \rangle_\omega = 0 \iff \left\{ \begin{array}{ll} u(x_\omega) &= v(x_\omega), \\ u(l-x_\omega) &= v(l-x_\omega), \end{array} \right.$$ 

avoiding stalks, we arrive directly at the spaces of values $U^e_{L_0,\omega} = \{ [u]_\omega, \quad u \in U_{L_0} \}$ of dimension two with the inner product

$$\langle [u]_\omega, [v]_\omega \rangle_{U^e_{L_0,\omega}} = \frac{u(x_\omega)v(x_\omega) + u(l-x_\omega)v(l-x_\omega)}{|e(x_\omega)|^2 + |e(l-x_\omega)|^2}.$$ 

This definition does not depend on the choice of the equivalence class representatives $u$ and $v$. Denoting

$$\rho(x) := (|e(x)|^2 + |e(l-x)|^2),$$

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Proof. Denote these values by \( \mu \), for every \( \mu \).

Lemma 14. The operator \( W^w \) is unitary.

Proof. Let \( y^w \in \mathcal{H}^w \) and \( u \in \mathcal{U}_{L_0} \). For every \( \omega \in \Omega_{L_0} \), the value \( y^w(\omega) \) lies in \( \mathcal{U}^w_{L_0,\omega} \), i.e., in the equivalence class of functions from \( \mathcal{U}_{L_0} \), which have certain values at the points \( x_\omega \) and \( l - x_\omega \). Denote these values by \( y(x_\omega) \) and \( y(l - x_\omega) \). Then the function \( y : [0, l] \to \mathbb{C} \) corresponds to the element \( y^w \), and

\[
[u(\omega), \omega] L_{L_0,\omega}^w \cdot \mathcal{U}^w_{L_0,\omega} \quad \text{\( \frac{u(x_\omega)v_y(x_\omega) + u(l - x_\omega)v_y(l - x_\omega)}{\rho(x)} \)}
\]

where \( v_y \) is an element from \( \mathcal{U}_{L_0} \) such that \( [v_y](\omega) = y^w(\omega) \). Therefore

\[
\int_{\Omega_{L_0}} \langle u(\omega), y^w(\omega) \rangle L_{L_0,\omega}^w d\mu(\omega) = \int_0^l \frac{u(x_\omega)v_y(x_\omega) + u(l - x_\omega)v_y(l - x_\omega)}{\rho(x)} dx = \int_0^l u(x_\omega)v_y(x_\omega) dx 
\]

for every \( u \in \mathcal{U}_{L_0} \), which implies that the integral on the right-hand side exists. This means that \( y \in L_2(0, l) = \mathcal{H} \) and \( y = W^w_0 y^w \). If \( y = 0 \), then also \( y^w = 0 \), whence \( \text{Ker} W^w_0 = \{0\} \) and \( \overline{\text{Ran} W^w_0} = \text{Ran} W^w = \mathcal{H} \). Together with the isometricity, this means that \( W^w \) is a unitary operator, and the lemma is proved.

2.6. The coordinate representation. Let \( e_1, e_2 \) be a basis in \( \text{Ker} L_0^* \). The solutions of the equation \( -u'' + qu = 0 \) are smooth functions, and thus \( e_1, e_2 \in \mathcal{U}_{L_0} \).

Lemma 15. For every \( \omega \in \Omega_{L_0} \setminus \{\omega^+\} \) the vectors \( [e_1](\omega) \) and \( [e_2](\omega) \) form a basis in \( \mathcal{U}^w_{L_0,\omega} \).

Proof. The linear dependence of \( [e_1](\omega) \) and \( [e_2](\omega) \) would mean the proportionality of the vectors \( \left( \begin{array}{c} e_1(x_\omega) \\ e_1(l - x_\omega) \end{array} \right) \) and \( \left( \begin{array}{c} e_2(x_\omega) \\ e_2(l - x_\omega) \end{array} \right) \) in \( \mathbb{C}^2 \), which would mean the existence of a solution of \( -u'' + qu = 0 \) with zeros at the points \( x_\omega \) and \( l - x_\omega \). By Lemma 13 this is impossible. \( \square \)
It follows that Condition 8 is satisfied with \( \Omega^0_{L_0} = \Omega_{L_0} \setminus \{ \omega_L \} \). Using the elements \( e_1, e_2 \), we define the coefficients
\[
\hat{u}(x_\omega) = \begin{pmatrix} \langle u, e_1 \rangle_{\omega} \\ \langle u, e_2 \rangle_{\omega} \end{pmatrix},
\]
in the spaces \( \mathcal{U}_{L_0,\omega}^w \). These coefficients are not the coordinates of the element \( [u]_{\omega} \) in the decomposition with respect to the basis \([e_1]_{\omega}, [e_2]_{\omega}\); such coordinates are given by the components of the vector \( G^{-1}(\omega)\hat{u}(x_\omega) \), where
\[
G(\omega) = \begin{pmatrix} \langle e_1, e_1 \rangle_{\omega} & \langle e_2, e_1 \rangle_{\omega} \\ \langle e_1, e_2 \rangle_{\omega} & \langle e_2, e_2 \rangle_{\omega} \end{pmatrix}
\]
is the Gram matrix. The coefficients \( \hat{u}(x) \in \mathbb{C}^2 \) are available to the “outer observer”, the linear map \( [u]_{\omega} \mapsto \hat{u}(x_\omega) \) is bijective from \( \mathcal{U}_{L_0,\omega}^w \) to \( \mathbb{C}^2 \). In the space \( \mathbb{C}^2 \) we need to determine an inner product corresponding to \( \langle [u]_{\omega}, [v]_{\omega} \rangle_{\mathcal{U}_{L_0,\omega}^w} \) in \( \mathcal{U}_{L_0,\omega}^w \).

**Lemma 16.** For every \( u, v \in \mathcal{U}_{L_0,\omega}^w \) and \( \omega \in \Omega_{L_0} \),
\[
\langle [u]_{\omega}, [v]_{\omega} \rangle_{\mathcal{U}_{L_0,\omega}^w} = (G^{-1}(\omega)\hat{u}(x_\omega), \hat{v}(x_\omega))_{\mathbb{C}^2}.
\]

**Proof.** By Lemma 15, the Gram matrix \( G(\omega_x) \) is nondegenerate for \( x \in [0, \frac{1}{2}] \). Computation gives:
\[
\langle [u]_{\omega}, [v]_{\omega} \rangle_{\mathcal{U}_{L_0,\omega}^w} = \frac{u(x_\omega)v(x_\omega) + u(l - x_\omega)v(l - x_\omega)}{\rho(x_\omega)},
\]
\[
\hat{u}(x_\omega) = \frac{1}{\rho(x_\omega)} \begin{pmatrix} u(x_\omega)e_1(x_\omega) + u(l - x_\omega)e_1(l - x_\omega) \\ u(x_\omega)e_2(x_\omega) + u(l - x_\omega)e_2(l - x_\omega) \end{pmatrix} = T(x_\omega) \begin{pmatrix} u(x_\omega) \\ u(l - x_\omega) \end{pmatrix},
\]
where
\[
T(x) := \frac{1}{\rho(x)} \begin{pmatrix} e_1(x) & e_2(x) \\ e_1(l - x) & e_2(l - x) \end{pmatrix}.
\]

Furthermore,
\[
(G^{-1}(\omega_x)\hat{u}(x), \hat{v}(x))_{\mathbb{C}^2} = \left( G^{-1}(\omega_x)T(x) \begin{pmatrix} u(x_\omega) \\ v(x_\omega) \end{pmatrix}, T(x) \begin{pmatrix} u(l - x_\omega) \\ v(l - x_\omega) \end{pmatrix} \right)_{\mathbb{C}^2} = \left( T^*(x)G^{-1}(\omega_x)T(x) \begin{pmatrix} u(x_\omega) \\ v(x_\omega) \end{pmatrix}, \begin{pmatrix} u(l - x_\omega) \\ v(l - x_\omega) \end{pmatrix} \right)_{\mathbb{C}^2}.
\]
It is easy to see that \( G(\omega_x) = \rho(x)T(x)T^*(x) \), so that \( T^*(x)G^{-1}(\omega_x)T(x) = \frac{1}{\rho(x)} \) and
\[
(G^{-1}(\omega)\hat{u}(x_\omega), \hat{v}(x_\omega))_{\mathbb{C}^2} = \frac{1}{\rho(x_\omega)} \begin{pmatrix} u(x_\omega) \\ u(l - x_\omega) \end{pmatrix}, \begin{pmatrix} v(x_\omega) \\ v(l - x_\omega) \end{pmatrix} \right)_{\mathbb{C}^2} = \langle [u]_{\omega}, [v]_{\omega} \rangle_{\mathcal{U}_{L_0,\omega}^w},
\]
and the lemma is proved. \( \square \)

Consider the space of the coordinate representation
\[
\mathcal{H}^c := L^2_2 \left( \left[ 0, \frac{1}{2} \right], G^{-1}(\omega)\rho(x_\omega)dx_\omega, \mathbb{C}^2 \right).
\]
The operator \( W^c_0 : u \mapsto \hat{u} \), from \( \mathcal{H} \) to \( \mathcal{H}^c \), defined on \( \text{Dom}W^c_0 = \mathcal{U}_{L_0} \), after closure becomes an isometric operator \( W^c = \overline{W^c_0} \) defined on the whole space \( \mathcal{H} \).

**Lemma 17.** The operator \( W^c \) is unitary and
\[
(W^c u)(x_\omega) = T(x_\omega) \begin{pmatrix} u(x_\omega) \\ u(l - x_\omega) \end{pmatrix}
\]
for every \( u \in \mathcal{H} \).

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Proof. For every \( u \in U_{L_0} \) and \( \hat{y} \in \mathcal{H}^c \), using the relation

\[
T^*(x_\omega)G^{-1}(\omega)T(x_\omega) = \frac{I}{\rho(x_\omega)}
\]

we get

\[
(W_0^c u, \hat{y})_{\mathcal{H}^c} = \int_0^1 (G^{-1}(\omega_x)\tilde{u}(x), \hat{y}(x))_C^2 \rho(x)dx
\]

\[
= \int_0^1 \left( G^{-1}(\omega_x)T(x) \left( \begin{array}{c} u(x) \\ u(l-x) \end{array} \right), \hat{y}(x) \right)_C^2 \rho(x)dx
\]

\[
= \int_0^1 \left( \frac{T^{-*}(x)}{\rho(x)} \left( \begin{array}{c} u(x) \\ u(l-x) \end{array} \right), \hat{y}(x) \right)_C^2 \rho(x)dx
\]

\[
= \int_0^1 \left( \frac{u(x)}{u(l-x)} \right)_C^2 T^{-1}(x)\hat{y}(x) \right)_C^2 dx = \int_0^1 u(x)\hat{y}(x)dx,
\]

where

\[
y(x) = \begin{cases} (T^{-1}(x)\hat{y}(x))_1, & x \in (0, \frac{1}{2}), \\ (T^{-1}(l-x)\hat{y}(l-x))_2, & x \in (\frac{1}{2}, l). \end{cases}
\]

Note that \( \hat{y} \in L_2((0, \frac{1}{2}), G^{-1}(\omega)\rho(x_\omega)dx_\omega, C^2) \) means that

\[
\int_0^1 (G^{-1}(\omega)\hat{y}(x_\omega), \hat{y}(x_\omega))_C^2 \rho(x_\omega)dx_\omega = \int_0^1 \|T^{-1}(x_\omega)\hat{y}(x_\omega)\|^2 dx_\omega = \|y\|_{L_2(0,l)}^2.
\]

Therefore \( y = W_0^c \hat{y} \). If \( y = 0 \), then \( T^{-1}\hat{y} = 0 \) and thus \( \hat{y} = 0 \), so \( \operatorname{Ker} W_0^c = \{0\} \). This means that \( \operatorname{Ran} W_0^c = \mathcal{H}^c \) and the operator \( W_0^c \) is unitary. Moreover, the operator that acts by the rule

\[
u(x) \mapsto T(x) \left( \begin{array}{c} u(x) \\ u(l-x) \end{array} \right)
\]

from \( \mathcal{H} \) to \( \mathcal{H}^c \) is isometric and coincides with \( W_0^c \) on \( U_{L_0} \). This implies that it is equal to \( W_0^c \). Therefore (2.24) holds. The lemma is proved.

Define the operator

\[
L_0^c = W^c L_0 W^{c*}
\]

in the space \( \mathcal{H}^c \). Owing to the unitarity of \( W^c \),

\[
\text{Graph } L_0^c = \text{Graph } (W^c L_0^c W^{c*})
\]

\[
= \{(W^c u^h(T), -W^c u^{h_T}(T)), h \in \mathcal{M}, T \geq 0 \}
\]

\[
= \{(u^h(T), -u^{h_T}(T)), h \in \mathcal{M}, T \geq 0 \}.
\]

The “outer observer” can construct the graph of the operator \( L_0^c \) in this form, using the boundary control method. This operator will be a differential operator of the second order, and one will be able to recover the original \( L_0^c \) from it.
Theorem 2. The operator $L_0^{\ast}$ is defined on the domain

$$\text{Dom } L_0^{\ast} = \left\{ \tilde{u}(x) = T(x) \begin{pmatrix} u(x) \\ u(l - x) \end{pmatrix}, u \in H^2(0, l) \right\},$$

where $T(x)$ is given by the formula (2.23) and acts by the rule

$$(L_0^{\ast} \tilde{u})(x) = -\tilde{u}''(x) + \tilde{P}(x)\tilde{u}'(x) + \tilde{Q}(x)\tilde{u}(x),$$

where

$$\tilde{P}(x) = -2T(x)T^{-1}'(x),$$

$$\tilde{Q}(x) = T(x)Q(x)T^{-1}(x) - T(x)T^{-1''}(x),$$

$$Q(x) = \begin{pmatrix} q(x) & 0 \\ 0 & q(l - x) \end{pmatrix}.$$

Moreover,

$$\text{Dom } L_0^{\ast} = \{ \tilde{u} \in \text{Dom } L_0^{\ast} : \tilde{u}(0) = \tilde{u}'(0) = 0 \},$$

$$\text{Dom } (W^c LW^{c\ast}) = \{ \tilde{u} \in \text{Dom } L_0^{\ast} : \tilde{u}(0) = 0 \}.$$

Proof. For $u \in \text{Dom } L_0^{\ast}$ we have

$$\tilde{u}(x) = (W^c u)(x) = T(x) \begin{pmatrix} u(x) \\ u(l - x) \end{pmatrix},$$

$$(L_0^{\ast} \tilde{u})(x) = \tilde{L_0}^{\ast} u(x) = T(x) \begin{pmatrix} -u''(x) + q(x)u(x) \\ -u''(l - x) + q(l - x)u(l - x) \end{pmatrix}$$

$$= T(x) \begin{pmatrix} -\left( \frac{u(x)}{u(l - x)} \right)'' + Q(x) \begin{pmatrix} u(x) \\ u(l - x) \end{pmatrix} \end{pmatrix}$$

$$= T(x)(-T^{-1}(x)\tilde{u}(x))'' + Q(x)T^{-1}(x)\tilde{u}(x))$$

$$= -\tilde{u}''(x) + \tilde{P}(x)\tilde{u}'(x) + \tilde{Q}(x)\tilde{u}(x).$$

The domains of the operators $L_0^{\ast}$, $L_0^c$, and $W^c LW^{c\ast}$ can be easily found from the domains of the operators $L_0^{\ast}$, $L_0$, and $L$, respectively. □

Remark 4. The domain of the operator $L_0^{\ast}$ is contained in the linear set

$$\left\{ \tilde{u} \in H^2 \left( \left[ 0, \frac{l}{2} \right], \mathbb{C}^2 \right) : \tilde{u} \left( \frac{l}{2} \right) = \tilde{u}_0 \begin{pmatrix} e_1(\frac{l}{2}) \\ e_2(\frac{l}{2}) \end{pmatrix}, \tilde{u}_0 \in \mathbb{C}, \tilde{u}' \left( \frac{l}{2} \right) = 0 \right\}.$$

Proof. Since $T \in C^\infty(0, \frac{l}{2})$,

$$T(x) \begin{pmatrix} u(x) \\ u(l - x) \end{pmatrix} \in H^2(0, \frac{l}{2}], \mathbb{C}^2)$$

holds for $u \in H^2(0, l)$. The vector-valued function $v(x) = \begin{pmatrix} u(x) \\ u(l - x) \end{pmatrix}$ belongs to $H^2(0, \frac{l}{2}], \mathbb{C}^2)$ and satisfies two other conditions: $v(\frac{l}{2}) = v_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v'(\frac{l}{2}) = v_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with some $v_0, v_1 \in \mathbb{C}$. After multiplication by the matrix $T(x)$ these conditions turn into conditions $\tilde{u}(\frac{l}{2}) = \tilde{u}_0 \begin{pmatrix} e_1(\frac{l}{2}) \\ e_2(\frac{l}{2}) \end{pmatrix}$ and $\tilde{u}'(\frac{l}{2}) = 0$ with $\tilde{u}_0 \in \mathbb{C}$. The first one follows from the substitution, for the second condition we used the symmetry of the function $\rho(x)$ with respect to the point $\frac{l}{2}$. 805
The matrix $T(x)$ degenerates at the point $\frac{l}{2}$, whence $T^{-1}(x) \notin C^\infty[0, \frac{l}{2}]$ and only inclusion, not equality, of linear sets takes place.

2.7. The inverse problem. The “outer observer” can recover the potential $q$ after construction of the wave model from the inverse data. But recovering is possible up to changing $q(x)$ to $q(l - x)$, which is natural: for these potentials the data will be the same. The wave model appears as a second order differential operator on the interval $(0, \frac{l}{2})$, which acts on vector-valued functions with two components. Thus the coefficients $\hat{P}(x)$ and $\hat{Q}(x)$ are known. Note that the Gram matrix $G(\omega_x)$ and the density of the measure $\rho(x) = \lim_{t \to +0} \frac{(P_{\omega_x}(y), e)}{2t}$ are determined in “wave” terms and thus are available to the “outer observer”.

To find the potential, it is enough to know $\hat{P}$ and $\hat{Q}$. The equation $-2TT^{-1} = \hat{P}$ is equivalent to the equation $T^{-1} = -\frac{1}{2}T^{-1}\hat{P}$ in the function $T^{-1}$. Let $M(x)$ denote its fundamental (matrix) solution:

$$M'(x) = -\frac{1}{2}M(x)\hat{P}(x),$$
$$M(0) = I.$$ 

Then $T^{-1}(x) = T_0^{-1}M(x)$ with a constant invertible matrix $T_0$ and $T(x) = M^{-1}(x)T_0$. Equation (2.26) reads

$$\hat{Q} = M^{-1}T_0QT_0^{-1}M - M^{-1}T_0(T_0M)',$$

which is equivalent to

$$M\hat{Q}M^{-1} = T_0QT_0^{-1} - M''M^{-1},$$
$$Q(x) = \begin{pmatrix} q(x) & 0 \\ 0 & q(l - x) \end{pmatrix} = T_0^{-1}(M(x)\hat{Q}(x)M^{-1}(x) + M''(x)M^{-1}(x))T_0.$$

We see that the values of the potential $q$ at the points symmetric with respect to $\frac{l}{2}$ can be found as the eigenvalues of the matrix

$$M(x)\hat{Q}(x)M^{-1}(x) + M''(x)M^{-1}(x),$$

and one can find this matrix from $\hat{P}$ and $\hat{Q}$. So we see that the potential can be recovered up to reflection from the middle of the interval.

The work was supported by the grant RFBR 18-31-00185moLa.

Translated by S. A. Simonov.

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