ON ALGEBRAIC AUTOMORPHISMS 
AND THEIR RATIONAL INVARIANTS

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Dedicated to all my family and friends

Abstract. Let $X$ be an affine irreducible variety over an algebraically closed field $k$ of characteristic zero. Given an automorphism $\Phi$, we denote by $k(X)^\Phi$ its field of invariants, i.e., the set of rational functions $f$ on $X$ such that $f \circ \Phi = f$. Let $n(\Phi)$ be the transcendence degree of $k(X)^\Phi$ over $k$. In this paper we study the class of automorphisms $\Phi$ of $X$ for which $n(\Phi) = \dim X - 1$. More precisely, we show that under some conditions on $X$, every such automorphism is of the form $\Phi = \phi g$, where $\phi$ is an algebraic action of a linear algebraic group $G$ of dimension 1 on $X$, and where $g$ belongs to $G$. As an application, we determine the conjugacy classes of automorphisms of the plane for which $n(\Phi) = 1$.

1. Introduction

Let $k$ be an algebraically closed field of characteristic zero. Let $X$ be an affine irreducible variety of dimension $n$ over $k$. We denote by $O(X)$ its ring of regular functions, and by $k(X)$ its field of rational functions. Given an algebraic automorphism $\Phi$ of $X$, denote by $\Phi^*$ the field automorphism induced by $\Phi$ on $k(X)$, i.e., $\Phi^*(f) = f \circ \Phi$ for any $f \in k(X)$. An element $f$ of $k(X)$ is invariant for $\Phi$ (or simply invariant) if $\Phi^*(f) = f$. Invariant rational functions form a field denoted $k(X)^\Phi$, and we set $n(\Phi) = \text{trdeg}_k k(X)^\Phi$.

In this paper we are going to study the class of automorphisms of $X$ for which $n(\Phi) = n - 1$. There are natural candidates for such automorphisms, such as exponentials of locally nilpotent derivations (see [M] or [Da]). More generally, one can construct such automorphisms by means of algebraic group actions as follows. Let $G$ be a linear algebraic group over $k$. An algebraic action of $G$ on $X$ is a regular map

$$\varphi : G \times X \rightarrow X$$

of affine varieties, such that $\varphi(g g', x) = \varphi(g, \varphi(g', x))$ for any $(g, g', x)$ in $G \times G \times X$. Given an element $g$ of $G$, denote by $\varphi_g$ the map $x \mapsto \varphi(g, x)$. Then $\varphi_g$ clearly defines an

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automorphism of $X$. Let $k(X)^G$ be the field of invariants of $G$, i.e., the set of rational functions $f$ on $X$ such that $f \circ \varphi_g = f$ for any $g \in G$. If $G$ is an algebraic group of dimension 1, acting faithfully on $X$, and if $g$ is an element of $G$ of infinite order, then one can prove by Rosenlicht’s theorem (see [Ro]) that

$$n(\varphi_g) = \text{trdeg}_k k(X)^G = n - 1.$$  

We are going to see that, under some mild conditions on $X$, there are no other automorphisms with $n(\Phi) = n - 1$ than those constructed above. In what follows, denote by $O(X)^\times$ the normalization of $O(X)$, and by $G(X)$ the group of invertible elements of $O(X)^\times$.

**Theorem 1.** Let $X$ be an affine irreducible variety of dimension $n$ over $k$, such that $\text{char}(k) = 0$ and $G(X)^\times = k^\times$. Let $\Phi$ be an algebraic automorphism of $X$ such that $n(\Phi) = n - 1$. Then there exist an abelian linear algebraic group $G$ of dimension 1, and an algebraic action $\varphi$ of $G$ on $X$ such that $\Phi = \varphi_g$ for some $g \in G$ of infinite order.

Note that the structure of $G$ is fairly simple. Since every connected linear algebraic group of dimension 1 is either isomorphic to $G_d(k) = (k, +)$ or $G_m(k) = (k^*, \times)$ (see [Hum, p. 131]), there exists a finite abelian group $H$ such that $G$ is either equal to $H \times G_d(k)$ or $H \times G_m(k)$. Moreover, the assumption on the group $G(X)$ is essential. Indeed, consider the automorphism $\Phi$ of $k^* \times k$ given by $\Phi(x, y) = (x, xy)$. Obviously, its field of invariants is equal to $k(x)$. However, it is easy to check that $\Phi$ cannot have the form given in the conclusion of Theorem 1.

This theorem is analogous to a result given by Van den Essen and Peretz (see [V-P]). More precisely, they establish a criterion to decide if an automorphism $\Phi$ is the exponential of a locally nilpotent derivation, based on the invariants and on the form of $\Phi$. A similar result has been developed by Daigle (see [Da]).

We apply these results to the group of automorphisms of the plane. First, we obtain a classification of the automorphisms $\Phi$ of $k^2$ for which $n(\Phi) = 1$. Second, we derive a criterion on automorphisms of $k^2$ to have no nonconstant rational invariants.

**Corollary 1.** Let $\Phi$ be an algebraic automorphism of $k^2$. If $n(\Phi) = 1$, then $\Phi$ is conjugate to one of the following forms:

- $\Phi_1(x, y) = (a^n x, a^m y)$, where $(n, m) \neq (0, 0)$, $a, b \in k$, $b$ is a root of unity but $a$ is not,
- $\Phi_2(x, y) = (ax, by + P(x))$, where $P$ belongs to $k[t] - \{0\}$, $a, b \in k$ are roots of unity.

**Corollary 2.** Let $\Phi$ be an algebraic automorphism of $k^2$. Assume that $\Phi$ has a unique fixed point $p$ and that $d\Phi_p$ is unipotent. Then $n(\Phi) = 0$.

We then apply Corollary 2 to an automorphism of $C^3$ recently discovered by Poloni and Moser-Jauslin (see [M-P]).

We may wonder whether Theorem 1 still holds if the ground field $k$ is not algebraically closed or has positive characteristic. The answer is not known for the moment. In fact, two obstructions appear in the proof of Theorem 1 when $k$ is arbitrary. First, the group $G_m(k)$ needs to be divisible (see Lemma 8), which is not always the case if $k$ is not algebraically closed. Second, the proof uses the fact that every $G_d(k)$-action on $X$
can be reconstructed from a locally nilpotent derivation on $O(X)$ (see Subsection 4.1), which is no longer true if $k$ has positive characteristic. This phenomenon is due to the existence of different forms for the affine line (see [Ru]). Note that, in case Theorem 1 holds and $k$ is not algebraically closed, the algebraic group $G$ need not be isomorphic to $H \times G_d(k)$ or $H \times G_m(k)$, where $H$ is finite. Indeed, consider the unit circle $X$ in the plane $\mathbb{R}^2$, given by the equation $x^2 + y^2 = 1$. Let $\Phi$ be a rotation in $\mathbb{R}^2$ with center at the origin and angle $\theta \not\in 2\pi \mathbb{Q}$. Then $\Phi$ defines an algebraic automorphism of $X$ with $n(\Phi) = 0$, and the subgroup spanned by $\Phi$ is dense in $SO_2(\mathbb{R})$. But $SO_2(\mathbb{R})$ is not isomorphic to either $G_d(\mathbb{R})$ or $G_m(\mathbb{R})$, even though it is a connected linear algebraic group of dimension 1.

We may also wonder what happens to the automorphisms $\Phi$ of $X$ for which $n(\Phi) = \dim X - 2$. More precisely, does there exist an action $\varphi$ of a linear algebraic group $G$ on $X$, of dimension 2, such that $\Phi = \varphi_g$ for a given $g \in G$? The answer is no. Indeed, consider the automorphism $\Phi$ of $k^2$ given by $\Phi = f \circ g$, where $f(x, y) = (x + y^2, y)$ and $g(x, y) = (x, y + x^2)$. Let $d(n)$ denote the maximum of the homogeneous degrees of the coordinate functions of the iterate $\Phi^n$. If there existed an action $\varphi$ of a linear algebraic group $G$ such that $\Phi = \varphi_g$, then the function $d$ would be bounded, which is impossible since $d(n) = 4^n$. A similar argument on the length of the iterates also yields the result.

But if we restrict to some specific varieties $X$, for instance $X = k^3$, one may ask the following question: If $n(\Phi) = 1$, is $\Phi$ birationally conjugate to an automorphism that leaves the first coordinate of $k^3$ invariant? The answer is still unknown.

2. Reduction to an affine curve $C$

Let $X$ be an affine irreducible variety of dimension $n$ over $k$. Let $\Phi$ be an algebraic automorphism of $X$ such that $n(\Phi) = n - 1$. In this section we are going to construct an irreducible affine curve on which $\Phi$ acts naturally. This will allow us to use some well-known results on automorphisms of curves. We set

$$K = \{ f \in k(X) \mid \exists m > 0, f \circ \Phi^m = f \circ \Phi \circ \cdots \circ \Phi = f \}.$$ 

It is straightforward that $K$ is a subfield of $k(X)$ containing both $k$ and $k(X)^\Phi$. We begin with some properties of this field.

**Lemma 1.** $K$ has transcendence degree $(n - 1)$ over $k$, and is algebraically closed in $k(X)$. In particular, the automorphism $\Phi$ of $X$ has infinite order.

**Proof.** First we show that $K$ has transcendence degree $(n - 1)$ over $k$. Since $K$ contains the field $k(X)^\Phi$, whose transcendence degree is $(n - 1)$, we only need to show that the extension $K/k(X)^\Phi$ is algebraic or, in other words, that every element of $K$ is algebraic over $k(X)^\Phi$. Let $f$ be any element of $K$. By definition, there exists an integer $m > 0$ such that $f \circ \Phi^m = f$. Let $P(t)$ be the polynomial of $k(X)[t]$ defined as

$$P(t) = \prod_{i=0}^{m-1} (t - f \circ \Phi^i).$$

By construction, the coefficients of this polynomial are all invariant for $\Phi$, and $P(t)$ belongs to $k(X)^\Phi[t]$. Moreover, $P(f) = 0$, $f$ is algebraic over $k(X)^\Phi$, and the first assertion follows.
Second, we show that $K$ is algebraically closed in $k(X)$. Let $f$ be an element of $k(X)$ that is algebraic over $K$. We need to prove that $f$ belongs to $K$. By the first assertion of the lemma, $f$ is algebraic over $k(X)^{\Phi}$. Let $P(t) = a_0 + a_1 t + \cdots + a_m t^m$ be a nonzero minimal polynomial of $f$ over $k(X)^{\Phi}$. Since $P(f) = 0$ and all $a_i$ are invariant, we have $P(f \circ \Phi) = P(f) \circ \Phi = 0$. In particular, all elements of the form $f \circ \Phi^i$, with $i \in \mathbb{N}$, are roots of $P$. Since $P$ has finitely many roots, there exist two distinct integers $m' < m''$ such that $f \circ \Phi^{m'} = f \circ \Phi^{m''}$. In particular, $f \circ \Phi^{m'' - m'} = f$ and $f$ belongs to $K$.

Now if $\Phi$ were an automorphism of finite order, then $K$ would be equal to $k(X)$. But this is impossible since $K$ and $k(X)$ have different transcendence degrees. \hfill \Box

Lemma 2. There exists an integer $m > 0$ such that $K = k(X)^{\Phi^m}$.

Proof. By definition, $k(X)$ is a field of finite type over $k$. Since $K$ is contained in $k(X)$, $K$ has also finite type over $k$. Let $f_1, \ldots, f_r$ be some elements of $k(X)$ such that $K = k(f_1, \ldots, f_r)$. Let $m_1, \ldots, m_r$ be some positive integers such that $f_i \circ \Phi^{m_i} = f_i$, and set $m = m_1 \ldots m_r$. By construction, all $f_i$ are invariant for $\Phi^m$. In particular, $K$ is invariant for $\Phi^m$ and $K \subseteq k(X)^{\Phi^m}$. Since $k(X)^{\Phi^m} \subseteq K$, the result follows. \hfill \Box

Let $L$ be the algebraic closure of $k(X)$, and let $\mathcal{A}$ be the $K$-subalgebra of $L$ spanned by $\mathcal{O}(X)$. By construction, $\mathcal{A}$ is an integral $K$-algebra of finite type of dimension 1. Let $m$ be an integer satisfying the conditions of Lemma 2. The automorphism $\Psi^* = (\Phi^m)^*$ of $\mathcal{O}(X)$ stabilizes $\mathcal{A}$, hence it defines a $K$-automorphism of $\mathcal{A}$, of infinite order (see Lemma 1). Let $B$ be the integral closure of $\mathcal{A}$. Then $B$ is also an integral $K$-algebra of finite type, of dimension 1, and the $K$-automorphism $\Psi^*$ extends uniquely to $B$. If $\overline{K}$ stands for the algebraic closure of $K$, we set

$$C = B \otimes_{\mathcal{K}} \overline{K}.$$  

By construction, $\mathcal{C} = \text{Spec}(C)$ is an affine curve over the algebraically closed field $\overline{K}$. Moreover, the automorphism $\Psi^*$ acts on $\mathcal{C}$ via the operation

$$\Psi^*: \mathcal{C} \longrightarrow \mathcal{C}, \quad x \otimes y \longmapsto \Psi^*(x) \otimes y.$$  

This makes sense since $\Psi^*$ fixes the field $K$. Therefore, $\Psi^*$ induces an algebraic automorphism of the curve $\mathcal{C}$. Since $K$ is algebraically closed in $k(X)$ by Lemma 1, $\mathcal{C}$ is integral (see [Z-S, Chap. VII, §11, Theorem 38]). But, by construction, $B$ and $\overline{K}$ are normal rings. Since $\mathcal{C}$ is a domain and $\text{char}(K) = 0$, $\mathcal{C}$ is also integrally closed by a result of Bourbaki (see [Bou, p. 29]). So $\mathcal{C}$ is a normal domain and $\mathcal{C}$ is a smooth irreducible curve.

Lemma 3. Let $\mathcal{C}$ be the $\overline{K}$-algebra constructed above. Then either $C = \overline{K}[t]$ or $C = \overline{K}[t, 1/t]$.

Proof. By Lemma 1, the automorphism $\Phi$ of $X$ has infinite order. Since the fraction field of $B$ is equal to $k(X)$, $\Psi^*$ has infinite order on $B$. But $B \otimes_{\mathcal{K}} C$, so $\Psi^*$ has infinite order on $\mathcal{C}$. In particular, $\Psi^*$ acts like an automorphism of infinite order on $\mathcal{C}$. Since $\mathcal{C}$ is affine, it has genus zero (see [Ro2]). Since $\overline{K}$ is algebraically closed, the curve $\mathcal{C}$ is rational (see [Che, p. 23]). Since $\mathcal{C}$ is smooth, it is isomorphic to $\mathbb{P}^1(\overline{K}) - E$, where $E$ is a finite set. Moreover, $\Psi^*$ acts like an automorphism of $\mathbb{P}^1(\overline{K})$ that stabilizes $\mathbb{P}^1(\overline{K}) - E$.  


Up to replacing \( \Psi \) by one of its iterates, we may assume that \( \Psi \) fixes every point of \( E \). But an automorphism of \( \mathbb{P}^1(\overline{K}) \) that fixes at least three points is the identity, which is impossible. Therefore, \( E \) consists of at most two points, and \( C \) is either isomorphic to \( \overline{K} \) or to \( \overline{K}^* \). In particular, either \( C = \overline{K}[t] \) or \( C = \overline{K}[t, 1/t]. \)

3. Normal forms for the automorphism \( \Psi \)

Let \( C \) and \( \Psi^* \) be the \( \overline{K} \)-algebra and the \( \overline{K} \)-automorphism constructed in the previous section. In this section we are going to give normal forms for the couple \((C, \Psi^*)\), in case the group \( G(X) \) is trivial, i.e., \( G(X) = k^* \). We begin with a few lemmas.

**Lemma 4.** Let \( X \) be an irreducible affine variety over \( k \). Let \( \Psi \) be an automorphism of \( X \). Let \( \alpha, f \) be some elements of \( k(X)^* \) such that \( (\Psi^*)^n(f) = \alpha^n f \) for any \( n \in \mathbb{Z} \). Then \( \alpha \) belongs to \( G(X) \).

**Proof.** Given an element \( h \) of \( k(X)^* \) and a prime divisor \( D \) on the normalization \( X^v \), we consider \( h \) as a rational function on \( X^v \), and denote by \( \text{ord}_D(h) \) the multiplicity of \( h \) along \( D \). This makes sense since the variety \( X^v \) is normal. Fix any prime divisor \( D \) on \( X \). Since \((\Psi^*)^n(f) = \alpha^n f \) for any \( n \in \mathbb{Z} \), we obtain

\[
\text{ord}_D((\Psi^*)^n(f)) = n \text{ord}_D(\alpha) + \text{ord}_D(f).
\]

Since \( \Psi \) is an algebraic automorphism of \( X \), it extends uniquely to an algebraic automorphism of \( X^v \), which is still denoted \( \Psi \). Moreover, this extension maps every prime divisor to another prime divisor, does not change the multiplicity, and maps distinct prime divisors into distinct ones. If \( \text{div}(f) = \sum_i n_i D_i \), where all \( D_i \) are prime, then we have

\[
\text{div}((\Psi^*)^n(f)) = \sum_i n_i (\Psi^*)^n(D_i),
\]

where all \((\Psi^*)^n(D_i)\) are prime and distinct. So the multiplicity of \((\Psi^*)^n(f)\) along \( D \) is equal to zero if \( D \) is not one of the \((\Psi^*)^n(D_i)s\), and equal to \( n_i \) if \( D = (\Psi^*)^n(D_i) \). In all cases, if \( R = \max\{|n_i|\} \), then we find that \( |\text{ord}_D((\Psi^*)^n(f))| \leq R \) and \( |\text{ord}_D(f)| \leq R \), and this implies, for any integer \( n \),

\[
|n \text{ord}_D(\alpha)| \leq 2R.
\]

In particular, we find \( \text{ord}_D(\alpha) = 0 \). Since this holds for any prime divisor \( D \), the support of \( \text{div}(\alpha) \) in \( X^v \) is empty and \( \text{div}(\alpha) = 0 \). Since \( X^v \) is normal, \( \alpha \) is an invertible element of \( \mathcal{O}(X)^* \), hence it belongs to \( G(X) \). \( \square \)

**Lemma 5.** Let \( K \) be a field of characteristic zero and \( \overline{K} \) its algebraic closure. Let \( C \) be either equal to \( \overline{K}[t] \) or to \( \overline{K}[t, 1/t] \). Let \( \Psi^* \) be a \( \overline{K} \)-automorphism of \( C \) such that \( \Psi^*(t) = at \), where \( a \) belongs to \( \overline{K} \). Let \( \sigma_1 \) be a \( K \)-automorphism of \( C \), commuting with \( \Psi^* \), such that \( \sigma_1(\overline{K}) = \overline{K} \). Then \( \sigma_1(a) \) is either equal to \( a \) or to \( 1/a \).

**Proof.** We distinguish two cases depending on the ring \( C \). First, assume that \( C = \overline{K}[t] \). Since \( \sigma_1 \) is a \( K \)-automorphism of \( C \) that maps \( \overline{K} \) to itself, we have \( \overline{K}[t] = \overline{K}[\sigma_1(t)] \). In particular, \( \sigma_1(t) = \lambda t + \mu \), where \( \lambda, \mu \) belong to \( \overline{K} \) and \( \lambda \neq 0 \). Since \( \Psi^* \) and \( \sigma_1 \) commute, we obtain

\[
\Psi^* \circ \sigma_1(t) = \lambda at + \mu = \sigma_1 \circ \Psi^*(t) = \sigma_1(a)(\lambda t + \mu).
\]
In particular, we have $\sigma_1(a) = a$ and the lemma follows in this case. Second, assume that $C = \overline{K}[t, 1/t]$. Since $\sigma_1$ is a $K$-automorphism of $C$, we find
\[ \sigma_1(t)\sigma_1(1/t) = \sigma_1(t.1/t) = \sigma_1(1) = 1. \]
Therefore, $\sigma_1(t)$ is an invertible element of $C$, and has the form $\sigma_1(t) = a_1t^{n_1}$, where $a_1 \in \overline{K}$ and $n_1$ is an integer. Since $\sigma_1$ is a $K$-automorphism of $C$ that maps $\overline{K}$ to $\overline{K}$, we have $\overline{K}[t, 1/t] = \overline{K}[\sigma_1(t), 1/\sigma_1(t)]$. In particular, $|n_1| = 1$ and either $\sigma_1(t) = a_1t$ or $\sigma_1(t) = a_1/t$. If $\sigma_1(t) = a_1t$, the relation $\Psi^* \circ \sigma_1(t) = \sigma_1 \circ \Psi^*(t)$ yields $\sigma(a) = a$. If $\sigma_1(t) = a_1/t$, then the same relation yields $\sigma(a) = 1/a$. 

**Lemma 6.** Let $X$ be an irreducible affine variety of dimension $n$ over $k$, such that $G(X) = k^*$. Let $\Phi$ be an automorphism of $X$ such that $n(\Phi) = (n - 1)$. Let $\Psi^*$ be the automorphism of $C$ constructed in the previous section. If either $C = \overline{K}[t]$ or $C = \overline{K}[t, 1/t]$, and if $\Psi^*(t) = at$, then $a$ belongs to $k^*$. 

**Proof.** We are going to prove by contradiction that $a \not\in k^*$. Let $\sigma$ be any element of $\text{Gal}(\overline{K}/K)$, and denote by $\sigma_1$ the $K$-automorphism of $C$ defined as follows:
\[ \forall (x, y) \in B \times \overline{K}, \quad \sigma_1(x \otimes y) = x \otimes \sigma_1(y). \]
Since $\Psi^* \circ \sigma_1(x \otimes y) = \Psi^*(x) \otimes \sigma_1(y) = \sigma_1 \circ \Psi^*(x \otimes y)$ for any element $x \otimes y$ of $B \otimes_K \overline{K}$, $\Psi^*$ and $\sigma_1$ commute. Moreover, if we identify $\overline{K}$ with $1 \otimes \overline{K}$, then $\sigma_1(\overline{K}) = \overline{K}$ by construction. By Lemma 5, we obtain
\[ \forall \sigma \in \text{Gal}(\overline{K}/K), \quad \sigma(a) = a \text{ or } \sigma(a) = a^{-1}. \]
In particular, the element $(a^i + a^{-r})$ is invariant under the action of $\text{Gal}(\overline{K}/K)$ for any $i$, and so it belongs to $K$ because $\text{char}(K) = 0$. Now let $f$ be an element of $B - K$. Since $f$ belongs to $C$, we can express $f$ as follows:
\[ f = \sum_{i=r}^s f_it^i. \]
Choose an $f \in B - K$ such that the difference $(s - r)$ is minimal. We claim that $(s - r) = 0$, i.e., $f = f_st^s$. Indeed, assume that $s > r$. Since $f$ is an element of $B$, the following expressions:
\[ \Psi^*(f) + (\Psi^*)^{-1}(f) - (a^s + a^{-s})f = \sum_{i=r}^{s-1} f_i(a^i + a^{-i} - a^s - a^{-s})t^i, \]
\[ \Psi^*(f) + (\Psi^*)^{-1}(f) - (a^r + a^{-r})f = \sum_{i=r+1}^s f_i(a^i + a^{-i} - a^r - a^{-r})t^i, \]
also belong to $B$. By minimality of $(s - r)$, these expressions belong to $K$. In other words, $f_i(a^i + a^{-i} - a^s - a^{-s}) = 0$ (resp., $f_i(a^i + a^{-i} - a^r - a^{-r}) = 0$) for any $i \neq 0, s$ (resp., for any $i \neq r$). Since $k$ is algebraically closed and $a \not\in k^*$ by assumption, $(a^i + a^{-i} - a^s - a^{-s})$ (resp., $(a^i + a^{-i} - a^r - a^{-r})$) is nonzero for any $i \neq s$ (resp., for any $i \neq r$). Therefore, $f_i = 0$ for any $i \neq 0$, and $f$ belongs to $K$, a contradiction. Therefore, $s = r$ and $f = f_st^s$. Since $f$ belongs to $B$, it also belongs to $k(X)$. Since $\Psi$ is an automorphism of $X$, the element $a^s = \Psi^*(f)/f$ belongs to $k(X)$. Moreover, $(\Psi^*)^n(f) = a^{ns}f$ for any $n \in \mathbb{Z}$. By Lemma 4, $a^s$ belongs to $G(X) = k^*$. Since $k$ is algebraically closed, $a$ belongs to $k^*$, hence a contradiction, and the result follows. \[ \square \]
Proposition 1. Let $X$ be an irreducible affine variety of dimension $n$ over $k$, such that $G(X) = k^*$. Let $\Phi$ be an automorphism of $X$ such that $n(\Phi) = (n - 1)$. Let $C$ and $\Psi^*$ be the $\overline{K}$-algebra and the $\overline{K}$-automorphism constructed in the previous section. Then up to conjugation, one of the following three cases occurs:

- $C = \overline{K}[t]$ and $\Psi^*(t) = t + 1$;
- $C = \overline{K}[t]$ and $\Psi^*(t) = at$, where $a \in k^*$ is not a root of unity;
- $C = \overline{K}[t, 1/t]$ and $\Psi^*(t) = at$, where $a \in k^*$ is not a root of unity.

Proof. By Lemma 3, we know that either $C = \overline{K}[t]$ or $C = \overline{K}[t, 1/t]$. We are going to study both cases.

First case: $C = \overline{K}[t]$.

The automorphism $\Psi^*$ maps $t$ to $at + b$, where $a \in \overline{K}^*$ and $b \in \overline{K}$. If $a = 1$, then $b \neq 0$ and up to replacing $t$ with $f/b$, we may assume that $\Psi^*(t) = t + 1$. If $a \neq 1$, then up to replacing $t$ with $t - c$ for a suitable $c$, we may assume that $\Psi^*(t) = at$. But then Lemma 6 implies that $a$ belongs to $k^*$. Since $\Psi^*$ has infinite order, $a$ cannot be a root of unity.

Second case: $C = \overline{K}[t, 1/t]$.

Since $\Psi^*(t) \Psi^*(1/t) = \Psi^*(1) = 1$, $\Psi^*(t)$ is an invertible element of $C$. So $\Psi^*(t) = at^n$, where $a \in \overline{K}^*$ and $n \neq 0$. Since $\Psi^*$ is an automorphism, $n$ is either equal to 1 or to $-1$. But if $n$ were equal to $-1$, then a simple computation shows that $(\Psi^*)^2$ would be the identity, which is impossible. So $\Psi^*(t) = at$, where $a \in \overline{K}^*$. By Lemma 6, $a$ belongs to $k^*$. As before, $a$ cannot be a root of unity. □

4. Proof of the main theorem

In this section we are going to establish Theorem 1. We will split its proof into two steps depending on the form of the automorphism $\Psi^*$ given in Proposition 1. But before, we begin with a few lemmas.

Lemma 7. Let $\Phi$ be an automorphism of an affine irreducible variety $X$. Let $G$ be a linear algebraic group and let $\psi$ be an algebraic $G$-action on $X$. Let $h$ be an element of $G$ such that the group $\langle h \rangle$ spanned by $h$ is Zariski dense in $G$. If $\Phi$ and $\psi_h$ commute, then $\Phi$ and $\psi_g$ commute for any $g \in G$.

Proof. It suffices to check that $\Phi^*$ and $\psi_h^n$ commute for any $g \in G$. For any $k$-algebra automorphisms $\alpha, \beta$ of $\mathcal{O}(X)$, denote by $[\alpha, \beta]$ their commutator, i.e., $[\alpha, \beta] = \alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1}$. For any $f \in \mathcal{O}(X)$, set

$$\lambda(g, f)(x) = [\Phi^*, \psi_g^n](f)(x) - f(x).$$

Since $G$ is a linear algebraic group acting algebraically on the affine variety $X$, $\lambda(g, f)(x)$ is a regular function on $G \times X$. Since $\Phi^*$ and $\psi_h^n$ commute, the automorphisms $\Phi^*$ and $\psi_h^n$ commute for any integer $n$. So the regular function $\lambda(g, f)(x)$ vanishes on $\langle h \rangle \times X$. Since $\langle h \rangle$ is dense in $G$ by assumption, $\langle h \rangle \times X$ is dense in $G \times X$ and $\lambda(g, f)(x)$ vanishes identically on $G \times X$. In particular, $[\Phi^*, \psi_g^n](f) = f$ for any $g \in G$. Since this holds for any element $f$ of $\mathcal{O}(X)$, the bracket $[\Phi^*, \psi_g^n]$ coincides with the identity on $\mathcal{O}(X)$ for any $g \in G$, and the result follows. □
Lemma 8. Let $\Phi$ be an automorphism of an affine irreducible variety $X$. Let $G$ be a linear algebraic group and let $\psi$ be an algebraic $G$-action on $X$. Let $h$ be an element of $G$ such that the group $(h)$ spanned by $h$ is Zariski dense in $G$. Assume there exists a nonzero integer $r$ such that $\Phi^r = \psi_h$, and that $G$ is divisible. Then there exists an algebraic action $\varphi$ of $G' = \mathbb{Z}/r\mathbb{Z} \times G$ such that $\Phi = \varphi_g$ for some $g'$ in $G'$.

Proof. Fix an element $b$ in $G$ such that $b^r = h$, and set $\Delta = \Phi \circ \psi_{b^{-1}}$. This is possible since $G$ is divisible. By construction, $\Delta$ is an automorphism of $X$. Since $\Phi^r = \psi_h$, $\Phi$ and $\psi_h$ commute. By Lemma 7, $\Phi$ and $\psi_g$ commute for any $g \in G$. In particular, we have

$$\Delta^i = (\Phi^i) \circ \psi_{b^{-i}} = (\Phi^i) \circ \psi_{b^{-i-1}} = \text{Id}$$

So $\Delta$ is finite, $\Phi = \Delta \circ \psi_b$, and $\Delta$ commutes with $\psi_g$ for any $g \in G$. The group $G'$ then acts on $X$ via the map $\varphi$ defined by

$$\varphi_{(i,0)}(x) = \Delta^i \circ \psi_g(x).$$

Moreover, we have $\Phi = \varphi_{g'}$ for $g' = (1, b)$.

The proof of Theorem 1 will then go as follows. In the following subsections we are going to exhibit an algebraic action $\psi$ of $G_a(k)$ (resp., $G_m(k)$) on $X$, such that $\Psi = \Phi^m = \psi_h$ for some $h$. In both cases, the group $G$ we will consider will be linear algebraic of dimension 1, and divisible. Moreover, the element $h$ will span a Zariski dense set because $h \neq 0$ (resp., $h$ is not a root of unity). With these conditions, Theorem 1 will become a direct application of Lemma 8.

4.1. The case $\Psi^*(t) = t + 1$

Assume that $C = \mathbb{K}[t]$ and $\Psi^*(t) = t + 1$. We are going to construct a nontrivial algebraic $G_a(k)$-action $\psi$ on $X$ such that $\Psi = \psi_1$. Since $\mathcal{O}(X) \subset C$, every element $f$ of $\mathcal{O}(X)$ can be written as $f = P(t)$, where $P$ belongs to $\mathbb{K}[t]$. We set $r = \deg_t P(t)$. Since $\Psi^*$ stabilizes $\mathcal{O}(X)$, the expression

$$\left(\Psi^i \right)^*(f) = P(t + i) = \sum_{j=0}^r P^{(j)}(t) \frac{t^j}{j!}$$

belongs to $\mathcal{O}(X)$ for any integer $i$. Since the matrix $M = (i^j/j!)_{0 \leq i, j \leq r}$ is invertible in $\mathcal{M}_{r+1}(\mathbb{Q})$, the polynomial $P^{(j)}(t)$ belongs to $\mathcal{O}(X)$ for any $j \leq r$. So the $\mathbb{K}$-derivation $D = \partial/\partial t$ on $C$ stabilizes the $k$-algebra $\mathcal{O}(X)$. Since $D^{r+1}(f) = 0$, the operator $D$, considered as a $k$-derivation on $\mathcal{O}(X)$, is locally nilpotent (see [Van]). Therefore the exponential map

$$\exp uD : \mathcal{O}(X) \longrightarrow \mathcal{O}(X)[u], \quad f \mapsto \sum_{j \geq 0} D^j(f) \frac{u^j}{j!},$$

is a well-defined $k$-algebra morphism. But $\exp uD$ also defines a $K$-algebra morphism from $C$ to $C[u]$. Since $\exp uD(t) = t + u$, $\exp D$ coincides with $\Psi^*$ on $C$. Since $C$ contains the ring $\mathcal{O}(X)$, we have $\exp D = \Psi^*$ on $\mathcal{O}(X)$. So the exponential map induces an algebraic $G_a(k)$-action $\psi$ on $X$ such that $\Psi = \psi_1$ (see [Van]).
4.2. The case $\Psi^*(t) = at$

Assume that $\Psi^*(t) = at$ and that $a$ is not a root of unity. We are going to construct a nontrivial algebraic $G_m(k)$-action $\psi$ on $X$ such that $\Psi = \psi_a$. First, note that either $C = \mathbb{K}[t]$ or $C = \mathbb{K}[t, 1/t]$. Let $f$ be any element of $O(X)$. Since $O(X) \subset C$, we can write $f$ as

$$f = P(t) = \sum_{i=r}^{s} f_i t^i,$$

where the $f_i t^i$ belong a priori to $C$. Since $\Psi^*$ stabilizes $O(X)$, the expression

$$(\Psi^j)^*(f) = P(a^j t) = \sum_{i=r}^{s} a^{ji} f_i t^i$$

belongs to $O(X)$ for any integer $j$. Since $a$ belongs to $k^*$ and is not a root of unity, the Vandermonde matrix $M = (a^j)_{0 \leq i, j \leq s - r}$ is invertible in $M_{s-r+1}(k)$. So the elements $f_i t^i$ all belong to $O(X)$ for any integer $i$. Consider the map

$$\psi^* : O(X) \rightarrow O(X)[v, 1/v], \quad f \mapsto \sum_{i=r}^{s} f_i t^i v^i.$$

Then $\psi^*$ is a well-defined $k$-algebra homomorphism, which induces a regular map $\psi$ from $k^* \times X$ to $X$. Moreover we have $\psi_v \circ \psi_{v'} = \psi_{vv'}$ on $X$ for any $v, v' \in k^*$. So $\psi$ defines an algebraic $G_m(k)$-action on $X$ such that $\Psi = \psi_a$.

5. Proof of Corollary 1

Let $\Phi$ be an automorphism of the affine plane $k^2$, such that $n(\Phi) = 1$. By Theorem 1, there exists an algebraic action $\varphi$ of an abelian linear algebraic group $G$ of dimension 1 such that $\Phi = \varphi_g$. We will distinguish the cases $G = \mathbb{Z}/r\mathbb{Z} \times G_m(k)$ and $G = \mathbb{Z}/r\mathbb{Z} \times G_d(k)$.

First case: $G = \mathbb{Z}/r\mathbb{Z} \times G_m(k)$.

Then $G$ is linearly reductive and $\varphi$ is conjugate to a representation in $GL_2(k)$ (see [Ka] or [Kr]). Since $G$ consists solely of semisimple elements, $\varphi$ is even diagonalizable. In particular, there exists a system $(x, y)$ of polynomial coordinates, some integers $n, m$, and some $r$-roots of unity $a, b$ such that

$$\varphi((i, a))(x, y) = (a^i u^n x, b^i u^m y).$$

Note that, since the action is faithful, the couple $(n, m)$ is distinct from $(0, 0)$. Since $k$ is algebraically closed, we can even reduce $\Phi = \varphi_g$ to the first form given in Corollary 1.

Second case: $G = \mathbb{Z}/r\mathbb{Z} \times G_d(k)$.

Let $\psi$ and $\Delta$ be, respectively, the $G_d(k)$-action and the finite automorphism constructed in Lemma 8. By Rentschler’s theorem (see [Re]), there exists a system $(x, y)$ of polynomial coordinates and an element $P$ of $k[t]$ such that

$$\psi_u(x, y) = (x, y + uP(x)).$$
For any \( f \in k[x,y] \), set \( \deg_\psi(f) = \deg_x \exp \Delta D(f) \). It is well known that this defines a degree function on \( k[x,y] \) (see [Du]). Since \( \psi \) and \( \Delta \) commute, \( \Delta^* \) preserves the space \( E_n \) of polynomials of degree \( \leq n \) with respect to \( \deg_\psi \). In particular, \( \Delta^* \) preserves \( E_0 = k[x] \). So \( \Delta^* \) induces a finite automorphism of \( k[x] \), hence \( \Delta^*(x) = ax + b \), where \( a \) is a root of unity. Since \( \Delta \) is finite, either \( a \neq 1 \) or \( a = 1 \) and \( b = 0 \). In any case, up to replacing \( x \) by \( x - \mu \) for a suitable constant \( \mu \), we may assume that \( \Delta^*(x) = ax \).

Moreover \( \Delta^* \) preserves \( E_1 = k[x] \{1,y\} \). With the same arguments as before, we obtain that \( \Delta^*(y) = cy + d(x) \), where \( c \) is a root of unity and \( d(x) \) belongs to \( k[x] \).

Composing \( \Delta \) with \( \psi_{1/m} \) then yields the second form given in Corollary 1.

6. Proof of Corollary 2

Let \( \Phi \) be an algebraic automorphism of \( k^2 \). We assume that \( \Phi \) has a unique fixed point \( p \) and that \( d\Phi_p \) is unipotent. We are going to prove that \( n(\Phi) = 0 \).

First, we check that \( n(\Phi) \) cannot be equal to 2. Assume that \( n(\Phi) = 2 \). Then \( k(x,y)^\Phi \) has transcendence degree 2, and the extension \( k(x,y)/k(x,y)^\Phi \) is algebraic, hence finite. Moreover, \( \Phi^* \) acts like an element of the Galois group of this extension. In particular, \( \Phi^* \) is finite. By a result of Kambayashi (see [Ka]), \( \Phi \) can be written as \( h \circ A \circ h^{-1} \), where \( A \) is an element of \( \text{GL}_2(k) \) of finite order and \( h \) belongs to \( \text{Aut}(k^2) \). Since \( \Phi \) has a unique fixed point \( p \), we have \( h(0,0) = p \). In particular, \( d\Phi_p \) is conjugate to \( A \) in \( \text{GL}_2(k) \). Since \( d\Phi_p \) is unipotent and \( A \) is finite, \( A \) is the identity. Therefore, \( \Phi \) is also the identity, which contradicts the fact that it has a unique fixed point.

Second we check that \( n(\Phi) \) cannot be equal to 1. Assume that \( n(\Phi) = 1 \). By the previous corollary, up to conjugacy, we may assume that \( \Phi \) has one of the following forms:

- \( \Phi_1(x,y) = (a^n x, a^m y) \), where \( (n,m) \neq (0,0) \), \( b \) is a root of unity but \( a \) is not,
- \( \Phi_2(x,y) = (ax, by + P(x)) \), where \( P \) belongs to \( k[t] - \{0\} \) and \( a, b \) are roots of unity.

Assume that \( \Phi \) is an automorphism of type \( \Phi_1 \). Then \( d\Phi_p \) is a diagonal matrix of \( \text{GL}_2(k) \), distinct from the identity. But this is impossible since \( d\Phi_p \) is unipotent. So assume that \( \Phi \) is an automorphism of type \( \Phi_2 \). Then \( d\Phi_p \) is a linear map of the form \( (u,v) \mapsto (au, bv + du) \), with \( d \in k \). Since \( d\Phi_p \) is unipotent, we have \( a = b = 1 \). So \( (\alpha, \beta) \) is a fixed point if and only if \( P(\alpha) = 0 \). In particular, the set of fixed points is either empty or a finite union of parallel lines. But this is impossible since there is only one fixed point by assumption. Therefore \( n(\Phi) = 0 \).

7. An application of Corollary 2

In this section we are going to see how Corollary 2 can be applied to the determination of invariants for automorphisms of \( \mathbb{C}^3 \). Set \( Q(x,y,z) = x^2 y - z^2 - x z^3 \) and consider the following automorphism (see [M-P]):

\[
\Phi : \mathbb{C}^3 \to \mathbb{C}^3, \quad (x,y,z) \mapsto \left( x, y(1 - xz) + \frac{Q^2}{4} + z^4, z - \frac{Q}{2} x \right).
\]

We are going to show that \( \mathbb{C}(x,y,z)^\Phi = \mathbb{C}(x) \) and \( \mathbb{C}[x,y,z]^\Phi = \mathbb{C}[x] \).
Let \( k \) be the algebraic closure of \( \mathbb{C}(x) \). Since \( \Phi^*(x) = x \), the morphism \( \Phi^* \) induces an automorphism of \( k[y, z] \), which we denote by \( \Psi^* \). The automorphism \( \Psi \) has clearly \((0, 0)\) as a fixed point, and its differential at this point is unipotent, distinct from the identity (as can be seen by an easy computation). Moreover, the set of fixed points of \( \Psi \) is reduced to the origin. Indeed, if \((\alpha, \beta)\) is a point of \( k^2 \) fixed by \( \Psi \), then \( xQ = 0 \) and \( 4\beta^4 - 8x\alpha\beta + Q^2 = 0 \). Since \( x \) belongs to \( k^* \), we have

\[
Q = x^2\alpha - \beta^2 - x\beta^3 = 0 \quad \text{and} \quad \beta^4 - x\alpha\beta = 0.
\]

If \( \beta = 0 \), then \( \alpha = 0 \) and we find the origin. If \( \beta \neq 0 \), then dividing by \( \beta \) and multiplying by \(-x\) yields the relation

\[
x^2\alpha - x\beta^3 = 0.
\]

This implies \( \beta^2 = 0 \) and \( \beta = 0 \), hence a contradiction. By Corollary 2, the field of invariants of \( \Psi \) has transcendence degree zero. So the field of invariants of \( \Phi \) has transcendence degree \( \leq 1 \) over \( \mathbb{C} \). Since this field contains \( \mathbb{C}(x) \) and that \( \mathbb{C}(x) \) is algebraically closed in \( \mathbb{C}(x, y, z) \), we obtain that \( \mathbb{C}(x, y, z)^\Phi = \mathbb{C}(x) \). As a consequence, the ring of invariants of \( \Phi \) is equal to \( \mathbb{C}[x] \).

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