ON THE ADDITIVITY OF NEWTON–OKOUNKOV BODIES

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Abstract. We study the additivity of Newton–Okounkov bodies. Our main result states that on two dimensional subcones of the ample cone the Newton–Okounkov body associated to an appropriate flag acts additively. We prove this by induction relying on the slice formula for Newton–Okounkov bodies. Moreover, we discuss a necessary condition for the additivity showing that our result is optimal in general situations. As an application, we deduce an inequality between intersection numbers of nef line bundles.

1. Introduction

For any line bundle \( \mathcal{L} \) on any irreducible projective variety \( X \) the associated Newton–Okounkov body is a convex body \( \Delta_{Y^*}(\mathcal{L}) \) in \( \mathbb{R}^{\dim X} \) containing many information about \( \mathcal{L} \), for example its volume. This notion was introduced independently by Lazarsfeld–Mustaţă [13] and by Kaveh–Khovanskii [11] based on ideas by Okounkov [16, 17]. It depends on the choice of a flag

\[ Y^*: X = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_d = \{p\} \]

of irreducible subvarieties, which are regular at \( p \), where \( d = \dim X \). Such a flag is also called an admissible flag. In general the Newton–Okounkov body satisfies the inclusion

\[ \Delta_{Y^*}(\mathcal{L}) + \Delta_{Y^*}(\mathcal{M}) \subseteq \Delta_{Y^*}(\mathcal{L} \otimes \mathcal{M}), \]

where we take the Minkowski sum on the left hand side. This inclusion makes it possible to deduce the Brunn–Minkowski inequality

\[ \text{vol}(\mathcal{L} \otimes \mathcal{M})^{1/d} \geq \text{vol}(\mathcal{L})^{1/d} + \text{vol}(\mathcal{M})^{1/d} \]

for big line bundles from its classical analogue in convex geometry. Unfortunately, inclusion (1.1) is not sufficient to translate more involved inequalities of mixed volumes in convex geometry to the algebraic geometry of line bundles. In general (1.1) is not an equality. Hence, it is a natural question in which cases it can be an equality. In this note we address this question in the special case, where the flag \( Y^* \) is obtained by divisors numerically equivalent to rational multiples of \( \mathcal{L} \). More precisely, we make the following definition.

Definition 1.1. Let \( X \) be any irreducible projective variety of dimension \( d \), \( Y^* \) an admissible flag on \( X \) and \( \mathcal{L} \) any \( \mathbb{Q} \)-line bundle on \( X \). We say that \( Y^* \) corresponds to \( \mathcal{L} \), if for all \( 0 \leq i \leq d - 2 \) the Weil divisor \( Y_{i+1} \subseteq Y_i \) is a Cartier divisor and there are rational numbers \( r_i \in \mathbb{Q} \) such that \( r_i \mathcal{O}_{Y^*}(Y_{i+1}) \equiv \mathcal{L}|_{Y_i} \), where \( \equiv \) denotes numerical equivalence.

2010 Mathematics Subject Classification. 14M25, 14C17, 52A39.

The author gratefully acknowledges support from the Swiss National Science Foundation grant “Diophantine Equations: Special Points, Integrality, and Beyond” (no 200020_184629).
We denote $N^1(X)$ for the group of numerical equivalence classes of line bundles on $X$ and $N^1(X)_\mathbb{R}$ for the real vector space induced by $N^1(X)$. We denote by $\text{Amp}(X)$, $\text{Big}(X)$ and $\overline{\text{Eff}}(X)$ the convex cones in $N^1(X)_\mathbb{R}$ of ample, big and pseudo-effective $\mathbb{R}$-line bundles on $X$. Note that $\text{Amp}(X) \subseteq \text{Big}(X) \subseteq \overline{\text{Eff}}(X)$, that $\text{Amp}(X)$ and $\text{Big}(X)$ are open and that $\overline{\text{Eff}}(X)$ is the closure of $\text{Big}(X)$.

We will always work over an algebraically closed field of any characteristic and we use additive notations for line bundles on $X$. For any two $\mathbb{R}$-line bundles $\mathcal{L}, \mathcal{M}$ we denote the convex cone

$$C_{\mathcal{L}}(\mathcal{M}) = \{ \lambda \mathcal{L} + \mu \mathcal{M} \mid \lambda \in \mathbb{R}, \mu \in \mathbb{R}_{\geq 0} \} \cap \text{Amp}(X).$$

Our main result states that the Newton–Okounkov bodies $\Delta_{Y_\bullet}(-)$ associated to an admissible flag $Y_\bullet$ corresponding to $\mathcal{L}$ act additively on $C_{\mathcal{L}}(\mathcal{M})$.

**Theorem 1.2.** Let $X$ be any irreducible projective variety, $\mathcal{L}$ and $\mathcal{M}$ any $\mathbb{Q}$-line bundles on $X$ and $Y_\bullet$ an admissible flag on $X$ corresponding to $\mathcal{L}$. Then for all $\mathcal{N}_1, \mathcal{N}_2 \in C_{\mathcal{L}}(\mathcal{M})$ we have

$$\Delta_{Y_\bullet}(\mathcal{N}_1 + \mathcal{N}_2) = \Delta_{Y_\bullet}(\mathcal{N}_1) + \Delta_{Y_\bullet}(\mathcal{N}_2).$$

To prove the theorem, we will use the description of the slices of Newton–Okounkov bodies by Lazarsfeld–Mustață [13] to construct the slices of the body on the left hand side as sum of the slices of the bodies on the right hand side, using induction on the dimension of $X$. This will give the reverse inclusion to (1.1). To show potentially applications of Theorem 1.2 let us give some examples of line bundles which admit corresponding admissible flags.

**Example 1.3.**

(i) If $X$ is any irreducible projective variety and $\mathcal{L}$ any ample line bundle on $X$, one can always construct an admissible flag $Y_\bullet$ corresponding to $\mathcal{L}$ by Bertini’s theorem. We will explain this construction in Section 3.

(ii) Let $X$ be any irreducible projective surface and $Y$ any irreducible Cartier divisor on $X$. The flag $\{ p \} \subseteq Y \subseteq X$ for any point $p$, regular in $Y$ and $X$, is an admissible flag corresponding to the line bundle $O_X(Y)$.

(iii) Let $X$ and $X'$ be two irreducible projective varieties of dimensions $d' = \dim X'$ and $d = \dim X$. We can always choose an admissible flag $Y_\bullet'$ on $X'$. Further, let $\mathcal{L}$ be a line bundle on $X$ and $Y_\bullet$ an admissible flag corresponding to $\mathcal{L}$. Then the flag

$$X \times X' \supseteq Y_1 \times X' \supseteq \cdots \supseteq Y_d \times X' \supseteq Y_d \times Y_1' \supseteq \cdots \supseteq Y_d \times Y_d'$$

is an admissible flag corresponding to the line bundle $\text{pr}_1^* \mathcal{L}$ on $X \times X'$, where $\text{pr}_1: X \times X' \to X$ denotes the projection to the first factor. The values $r_i$ for this flag are the same as for the flag $Y_\bullet$ with the additional values $r_{d-1} = \deg(\mathcal{L}|_{Y_{d-1}})$ and $r_i = 0$ for $i \geq d$.

(iv) To give a more specific example, let $X$ be any irreducible projective curve. The previous two examples show that the line bundle $O_X(d)(\Delta_{jk})$ on the self-product $X^d$ associated to the diagonal divisor

$$\Delta_{jk} = \{(x_1, \ldots, x_d) \in X^d \mid x_j = x_k\}$$

admits a corresponding admissible flag for $j \neq k$. If $j = 1$ and $k = 2$ one can give a corresponding admissible flag by

$$X^d \supseteq \Delta_{12} \times X^{d-2} \supseteq \{(p, p)\} \times X^{d-2} \supseteq \{(p, p, p)\} \times X^{d-3} \supseteq \cdots \supseteq \{(p, \ldots, p)\}$$

for some regular point $p$ of $X$. 





Previously, the additivity of Newton–Okounkov bodies has been studied only for special types of varieties. Luszcz–Świdecka and Schmitz [15] and Pokora, Schmitz and Urbinati [18] proved the existence of Minkowski bases for algebraic surfaces with rational polyhedral pseudo-effective cone respectively for smooth projective toric varieties. A Minkowski base is a finite set of pseudo-effective divisors $D_1, \ldots, D_r$ such that any pseudo-effective divisor $D$ can be written as $D = \sum_{i=1}^r a_i D_i$ for some $a_i \geq 0$ and it holds $\Delta_Y(D) = \sum_{i=1}^r a_i \Delta_Y(D_i)$, where the $\Delta_Y(D_i)$’s are indecomposable. Kiritchenko [12] studied the additivity of Newton–Okounkov bodies for Bott–Samelson varieties. The question of additivity has also been discussed for string polytopes of spherical varieties in relation to topics like toric degenerations by Alexeev and Brion [2] and the description of cohomology rings by Kaveh [9]. Note that string polytopes of spherical varieties can be realized as Newton–Okounkov bodies [10].

We would like to present Theorem 1.2 in a more systematic way. On $N^1(X)_{\mathbb{Q}}$ we have a multi-linear and symmetric intersection map

$$N^1(X)_{\mathbb{Q}}^d \to \mathbb{Q}, \quad (L_1, \ldots, L_d) \mapsto L_1 \cdot \ldots \cdot L_d$$

associating to a $d$-tuple of $\mathbb{Q}$-line bundles their intersection number. On the other side, let us write $K_d$ for the space of convex bodies in $\mathbb{R}^d$, which is a convex $\mathbb{R}_{\geq 0}$-cone. Adding formal differences we obtain the induced vector space $K_{\mathbb{Q}}^v_d$. Again, we obtain a multi-linear and symmetric map

$$(K_{\mathbb{Q}}^v_d)^d \to \mathbb{R}, \quad (K_1, \ldots, K_d) \mapsto V(K_1, \ldots, K_d),$$

which sends any $d$-tuple $(K_1, \ldots, K_d)$ of convex bodies to their mixed volume $V(K_1, \ldots, K_d)$. As a consequence of Theorem 1.2 we get linear embeddings of 2-dimensional subspaces of $N^1(X)_{\mathbb{Q}}$ into $K_{\mathbb{Q}}^v_d$ respecting the above intersection products.

**Corollary 1.4.** Let $X$ be any irreducible projective variety of dimension $d$ and $U \subseteq N^1(X)_{\mathbb{Q}}$ a linear subspace of dimension $\dim U = 2$ containing an ample class $\mathcal{L} \in U$. Then there exists an injective linear map

$$\Delta : U \to K_{\mathbb{Q}}^v_d,$$

which is compatible with the intersection products in the sense that

$$\frac{1}{d!}(\mathcal{M}_1 \cdot \ldots \cdot \mathcal{M}_d) = V(\Delta(\mathcal{M}_1), \ldots, \Delta(\mathcal{M}_d))$$

for all $\mathcal{M}_1, \ldots, \mathcal{M}_d \in U$. Moreover, for any $r \in \mathbb{N}$ and any ample $\mathbb{Q}$-line bundles $\mathcal{M}_1, \ldots, \mathcal{M}_r \in U$ the map $\Delta$ can be chosen, such that $\Delta(\mathcal{M}_j) = \Delta_Y(\mathcal{M}_j)$ is the Newton–Okounkov body associated to some fixed admissible flag $Y_*$ for any $j \leq r$.

The corollary allows us to translate inequalities of mixed volumes of general convex bodies to inequalities of intersection numbers of ample line bundles in the linear subspace $U$. As $\dim U = 2$, it is enough to consider general inequalities of mixed volumes of only two independent convex bodies. However, it has been shown by Shepard [19] that all these inequalities are induced by the Alexandrov–Fenchel inequality [1, 5], whose analogue is already known for the intersection numbers of ample line bundles, see for example [4, Theorem 6.1]. Hence, it is a natural question, in which cases the Newton–Okounkov body acts additively on a bigger set than $C_{\mathcal{L}}(\mathcal{M})$. The next proposition shows, that this only happens in special situations.
Proposition 1.5. Let $X$ be any irreducible projective variety, $\mathcal{L}$ and $\mathcal{M}$ ample $\mathbb{R}$-line bundles on $X$ and $Y_\bullet$ an admissible flag on $X$, such that $Y_1$ is a Cartier divisor on $X$. If

$$\Delta_{Y_1}(\mathcal{L} + \mathcal{M}) = \Delta_{Y_1}(\mathcal{L}) + \Delta_{Y_1}(\mathcal{M}),$$

then there exists a convex cone $C \subseteq \partial \overline{\text{Eff}}(X)$ in the boundary of the pseudo-effective cone $\partial \text{Eff}(X) = \overline{\text{Eff}}(X) \setminus \text{Big}(X)$, such that

$$\mathcal{L}, \mathcal{M} \in \{ \lambda O_X(Y_1) + \mathcal{N} \mid \lambda \in \mathbb{R}_{>0}, \mathcal{N} \in C \}.$$  

In other words, if $\Delta_{Y_1}$ acts additively on $\mathcal{L}$ and $\mathcal{M}$, then the projections of $\mathcal{L}$ and $\mathcal{M}$ by $O_X(Y_1)$ to the boundary of the pseudo-effective cone lie in a convex subcone contained in the boundary. As the pseudo-effective cone is in general not even polyhedral, one can not expect that there are such subcones in the boundary of dimension higher than 1 in general. If the convex subcone in the boundary is of dimension 1, then $\mathcal{L}$ and $\mathcal{M}$ coincide modulo $O_X(Y_1)$ up to a multiple. This is exactly the situation of Theorem 1.2. Thus, in general situations the choice of the cone $C_{\mathcal{L}}(\mathcal{M})$ in Theorem 1.2 is optimal as a subcone of $\text{Amp}(X)$.

Finally, we want to give an application of Theorem 1.2 to inequalities of intersection numbers, which we will deduce from an analogue inequality of mixed volumes of convex bodies by Lehmann–Xiao [14].

Corollary 1.6. Let $X$ be any irreducible projective variety of dimension $d$ and $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ nef $\mathbb{R}$-line bundles on $X$. Then we have the following inequality of intersection numbers

$$\mathcal{L}^d \cdot (\mathcal{M} \cdot \mathcal{N}^{d-1}) \leq d \cdot (\mathcal{M} \cdot \mathcal{L}^{d-1}) \cdot (\mathcal{L} \cdot \mathcal{N}^{d-1}).$$

We remark that the much more general inequality

$$\mathcal{L}^d \cdot (\mathcal{M}^k \cdot \mathcal{N}^{d-k}) \leq \binom{d}{k} \cdot (\mathcal{M}^k \cdot \mathcal{L}^{d-k}) \cdot (\mathcal{L}^k \cdot \mathcal{N}^{d-k})$$

for any $0 \leq k \leq d$ has recently been proved by Jiang and Li [8] by a different strategy using so-called multipoint Okounkov bodies introduced by Trusiani [20].

2. Newton–Okounkov Bodies

In this section we recall the construction of Newton–Okounkov bodies and collect some facts about them. We refer to [11, 13] for more details. Let $X$ be any irreducible projective variety of dimension $d$ defined over any algebraically closed field. We choose a flag

$$Y_\bullet : \quad X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_{d-1} \supseteq Y_d = \{ p \}$$

of irreducible subvarieties, which are regular at $p$. In the following we mean by $D$ always a Cartier divisor on $X$. We consider a valuation

$$\nu_{Y_\bullet} : H^0(X, O_X(D)) \setminus \{ 0 \} \to \mathbb{Z}^d, \quad s \to \nu_{Y_\bullet}(s) = (\nu_1(s), \ldots, \nu_d(s)),$$

which is defined as follows:

We set $\nu_1 = \nu_1(s) = \text{ord}_{Y_1}(s)$, such that the restriction of $s$ induces a non-zero section $s_1 \in H^0(Y_1, O_X(D - \nu_1 Y_1)|_{Y_1})$. Inductively, we set $\nu_i = \nu_i(s) = \text{ord}_{Y_i}(s_{i-1})$ and we write $s_i$ for the induced non-zero section

$$s_i \in H^0\left( Y_i, O_X(D)|_{Y_i} \otimes \bigotimes_{k=1}^{i} O_{Y_{k-1}}(-\nu_k Y_k)|_{Y_i} \right).$$
Now the Newton–Okounkov body of $D$ is defined by

$$\Delta_Y(D) = \text{cch} \left( \bigcup_{m \geq 1} \frac{1}{m} \nu_Y \left( H^0(X, O_X(mD)) \setminus \{0\} \right) \right) \subseteq \mathbb{R}^d,$$

where cch stands for \textit{closed convex hull}. The construction only depends on the linear equivalence class of $D$, such that we may simply write $\Delta_Y(L) = \Delta_Y(D)$ whenever $L \cong O_X(D)$. As an example, if $X$ has dimension 1, the Newton–Okounkov body of a big divisor $D$ on $X$ is just the line segment

$$\Delta_Y(D) = [0, \deg D] \subseteq \mathbb{R},$$

see [13 Example 1.14].

Let us recall some fundamental results about this construction from [13]. First, for every big divisor it holds [13 Theorem 2.3]

$$\text{vol}(\Delta_Y(D)) = \frac{1}{\dim} \text{vol}(D) := \lim_{m \to \infty} \frac{\dim H^0(X, O_X(mD))}{m^\dim}.$$

Second, the Newton–Okounkov body $\Delta_Y(D)$ only depends on the numerically equivalence class of $D$ if $D$ is ample [13 Proposition 4.1]. For any integer $p > 0$ and any big divisor $D$ the Newton–Okounkov body of $p \cdot D$

$$\Delta_Y(p \cdot D) = p \Delta_Y(D)$$

is just the scaling of the Newton–Okounkov body of $D$. The proof of [13 Proposition 4.1] shows that Equation (2.2) also holds if $D$ is any effective divisor. Hence, we may extend the definition of $\Delta_Y$ to effective $\mathbb{Q}$-line bundles. It turns out that this extension is continuous on $\text{Big}(X)$ [13 Theorem B], such that we may define $\Delta_Y$ also for big $\mathbb{R}$-divisors.

In contrast to the compatibility with scalars in (2.2), which by construction also holds for $p \in \mathbb{R}_{>0}$, the Newton–Okounkov body is in general not additive. But we obtain the weaker property that for any divisors $D_1$ and $D_2$ it always holds

$$\Delta_Y(D_1 + D_2) \supseteq \Delta_Y(D_1) + \Delta_Y(D_2),$$

where the sum on the right hand side denotes the Minkowski sum. This follows immediately from the construction, as for any sections $s_1 \in H^0(X, O_X(D_1))$ and $s_2 \in H^0(X, O_X(D_2))$ we obtain the section $s_1 \otimes s_2 \in H^0(X, O_X(D_1 + D_2))$, which induces the above inclusion.

Finally, we recall a method to study the slices of Newton–Okounkov bodies. Let $E \subseteq X$ be an irreducible Cartier divisor. Further, let $\mathcal{M}$ be a big line bundle satisfying $E \not\subseteq B_+(\mathcal{M})$, where $B_+(\mathcal{M})$ denotes the augmented base locus of $\mathcal{M}$, see [13 Section 2.4] for its definition. It always holds $B_+(\mathcal{M}) = \emptyset$ if $\mathcal{M}$ is ample. We set

$$\mu(\mathcal{M}; E) = \sup \{ s > 0 \mid \mathcal{M} - sO_X(E) \in \text{Big}(X) \}.$$ 

We choose an admissible flag $Y_\bullet$, such that $Y_1 = E$. By construction $\mu(\mathcal{M}; E)$ coincides with the endpoint $\max \{ t \geq 0 \mid \Delta_{Y_t}(\mathcal{M})_{t \neq t} \neq \emptyset \}$ of the Newton–Okounkov body $\Delta_{Y_t}(\mathcal{M})$ after projecting to the first coordinate. For any big line bundle $\mathcal{N}$
we define
\[ (2.4) \quad \Delta_{Y^*|E}(N) = \operatorname{cch} \left( \bigcup_{m \geq 1} \frac{1}{m} \nu_{Y^*} \left( \operatorname{Im} \left( H^0(X, N^\otimes m) \to H^0(E, N^\otimes m|E) \right) \setminus \{0\} \right) \right) \]
in \( \mathbb{R}^{d-1} \), where the map denotes the restriction map and \( Y^* \) denotes the admissible flag \( Y = Y_1 \supseteq \cdots \supseteq Y_d \) on \( E = Y_1 \). Noting that
\[
\Delta_{Y^*|E}(\mu N) = p \Delta_{Y^*|E}(N)
\]
for any \( p \in \mathbb{Z}_{>0} \), we can canonically define \( \Delta_{Y^*|E}(N) \) for any big \( \mathbb{Q} \)-line bundle \( N \). Then the slice of \( \Delta_{\bullet}(M) \) at \( \nu_1 = t \) for any rational \( 0 \leq t < \mu(M; E) \) is given by
\[ (2.5) \quad \Delta_{Y^*}(M)_{\nu_1 = t} = \{t\} \times \Delta_{Y^*|E}(M - tO_X(E)), \]
as proven in [13, Theorem 4.26].

3. THE ADDITIVITY OF \( \Delta_{Y^*} \)

The aim of this section is to prove Theorem 1.2. We will proceed by induction on the dimension \( d = \dim X \) of \( X \). If \( d = 1 \) we get by (2.4)
\[
\operatorname{vol}(\Delta_{Y^*}(N_1 + N_2)) = \deg(N_1 + N_2)
= \deg N_1 + \deg N_2 = \operatorname{vol}(\Delta_{Y^*}(N_1)) + \operatorname{vol}(\Delta_{Y^*}(N_2))
\]
for any ample line bundles \( N_1 \) and \( N_2 \) on \( X \). By \( \mathbb{Z} \)-linearity and by continuity this also holds for ample \( \mathbb{R} \)-line bundles. As the Newton–Okounkov bodies are line segments of the form \([0, c]\), we already obtain the equality in Theorem 1.2.

Now let \( d > 1 \) and assume that the theorem holds for projective varieties of dimension \( d-1 \). Let \( N_1, N_2 \in C_L(M) \). Since we already have the inclusion (2.3), we only need to show
\[ (3.1) \quad \Delta_{Y^*}(N_1 + N_2) \subseteq \Delta_{Y^*}(N_1) + \Delta_{Y^*}(N_2) \]
By continuity we can assume that \( N_i = \lambda_i L + \mu_i M \) for some \( \lambda_i, \mu_i \in \mathbb{Q} \) with \( \mu_i > 0 \) for any \( i = 1, 2 \). We write \( \mu = \mu(N_1 + N_2, Y_1) \). Since both sides of (3.1) are closed convex bodies, it is enough to show slice-wise
\[ (3.2) \quad \Delta_{Y^*}(N_1 + N_2)_{\nu_1 = t} \subseteq \Delta_{Y^*}(N_1) + \Delta_{Y^*}(N_2) \]
for all \( t \in (0, \mu) \cap \mathbb{Q} \), as \( (0, \mu) \cap \mathbb{Q} \) is dense in the interval \([0, \mu]\) and the slices \( \Delta_{Y^*}(N_1 + N_2)_{\nu_1 = t} \) are empty for all \( t \notin [0, \mu] \). By assumptions there is an \( r \in \mathbb{Q} \) such that \( rO_X(Y_1) \equiv L \). After interchanging \( N_1 \) and \( N_2 \) we may assume
\[
\frac{r\lambda_1}{\mu_1} \leq \frac{r\lambda_2}{\mu_2}.
\]
We set \( t_0 = r\lambda_2 - \frac{r\lambda_2}{\mu_2} r\lambda_1 \geq 0 \).

First we consider the case \( t \geq t_0 \). By (2.5) we can compute the slice at \( \nu_1 = t \) by
\[ (3.3) \quad \Delta_{Y^*}(N_1 + N_2)_{\nu_1 = t} = \Delta_{Y^*}((1 + \frac{\mu_2}{\mu_1})N_1 + t_0O_X(Y_1))_{\nu_1 = t}
= \{t\} \times \Delta_{Y^*|Y_1}((1 + \frac{\mu_2}{\mu_1})N_1 - (t - t_0)O_X(Y_1))
= t_0\nu_1 + \{t - t_0\} \times \Delta_{Y^*|Y_1}((1 + \frac{\mu_2}{\mu_1})N_1 - (t - t_0)O_X(Y_1))
= t_0\nu_1 + \Delta_{Y^*}((1 + \frac{\mu_2}{\mu_1})N_1)_{\nu_1 = t-t_0}, \]
where $e_1$ denotes the first vector of the standard basis of $\mathbb{R}^d$. Note that

$$e_1 \in \Delta_{Y^*}(\mathcal{O}_X(Y_1))$$

since the canonical section of $\mathcal{O}_X(Y_1)$ vanishes of order 1 at $Y_1$ and the line bundle $\mathcal{O}_X(Y_1) \otimes \mathcal{O}_X(-Y_1) \cong \mathcal{O}_X$ is trivial, such that

$$H^0(Y_1, \mathcal{O}_X(Y_1)|_{Y_1} \otimes \mathcal{O}_X(-Y_1)|_{Y_1}) \setminus \{0\}$$

is non-empty and consists of non-zero constants, which have order 0 restricted to any $Y_i$ for $i \geq 1$. Hence, the computation in (3.3) implies

$$\Delta_{Y^*}(N_1 + N_2)|_{Y_1} = \Delta_{Y^*}(t_0\mathcal{O}_X(Y_1)) + \Delta_{Y^*}((1 + \frac{\mu_2}{\mu_1})N_1).$$

For the right hand side we can compute

$$\Delta_{Y^*}(t_0\mathcal{O}_X(Y_1)) + \Delta_{Y^*}((1 + \frac{\mu_2}{\mu_1})N_1) = \Delta_{Y^*}(t_0\mathcal{O}_X(Y_1)) + \Delta_{Y^*}((\frac{\mu_2}{\mu_1})N_1) + \Delta_{Y^*}(N_1) \subseteq \Delta_{Y^*}(t_0\mathcal{O}_X(Y_1) + \frac{\mu_2}{\mu_1}N_1) + \Delta_{Y^*}(N_1) = \Delta_{Y^*}(N_2) + \Delta_{Y^*}(N_1)$$

Thus, we have shown inclusion (3.2) for $t \geq t_0$.

Now we consider the case $t < t_0$. In particular, we have $t_0 > 0$ and hence, $r \neq 0$. We will apply the induction hypothesis to $Y_1$. Thus, we denote

$$Y_{1,*}: Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_{d-1} \supseteq Y_d = \{p\}.$$ 

for the restriction of the flag $Y^*$ to $Y_1$. As $t < t_0$, the Q-line bundle

$$N_1 + N_2 - \frac{t}{r} \mathcal{L} = (\lambda_1 + \lambda_2 - \frac{t}{r})\mathcal{L} + (\mu_1 + \mu_2)\mathcal{M} = (1 + \frac{\mu_2}{\mu_1} \cdot \frac{t}{t_0})N_1 + \frac{t}{t_0}N_2$$

is ample by the ampleness of $N_1$ and $N_2$. It follows, that also $N_1 + N_2 - t\mathcal{O}_X(Y_1)$ is ample. Hence, we have

$$\Delta_{Y^*|Y_1}(N_1 + N_2 - t\mathcal{O}_X(Y_1)) = \Delta_{Y^*|Y_1}(N_1|Y_1 + N_2|Y_1 - t\mathcal{O}_X(Y_1)|_{Y_1}) = \Delta_{Y^*|Y_1}((1 + \frac{\mu_2}{\mu_1} \cdot \frac{t}{t_0})N_1|Y_1 + \frac{t}{t_0}N_2|Y_1),$$

where the first equality follows since the restriction map in (2.1) is surjective if $\mathcal{N}$ is ample and $m$ sufficiently large. In a similar way, we also obtain

$$\Delta_{Y^*|Y_1}(N_2 - t\mathcal{O}_X(Y_1)) = \Delta_{Y^*|Y_1}(\frac{\mu_2}{\mu_1} \cdot \frac{t}{t_0} \cdot N_1|Y_1 + \frac{t}{t_0}N_2|Y_1).$$

By the induction hypothesis we have

$$\Delta_{Y^*|Y_1}((1 + \frac{\mu_2}{\mu_1} \cdot \frac{t}{t_0})N_1|Y_1 + \frac{t}{t_0}N_2|Y_1) = \Delta_{Y^*|Y_1}(N_1|Y_1) + \Delta_{Y^*|Y_1}(\frac{\mu_2}{\mu_1} \cdot \frac{t}{t_0} \cdot N_1|Y_1 + \frac{t}{t_0}N_2|Y_1)$$

Using the above observations and the slice formula (2.5) we can finally compute

$$\Delta_{Y^*}(N_1 + N_2)|_{Y_1} = \{t\} \times \Delta_{Y^*|Y_1}(N_1 + N_2 - t\mathcal{O}_X(Y_1)) = \{t\} \times \Delta_{Y^*|Y_1}((1 + \frac{\mu_2}{\mu_1} \cdot \frac{t}{t_0})N_1|Y_1 + \frac{t}{t_0}N_2|Y_1)$$

$$= \{t\} \times \left( \Delta_{Y^*|Y_1}(N_1|Y_1) + \Delta_{Y^*|Y_1}(\frac{\mu_2}{\mu_1} \cdot \frac{t}{t_0} \cdot N_1|Y_1 + \frac{t}{t_0}N_2|Y_1) \right)$$

$$= \{0\} \times \Delta_{Y^*|Y_1}(N_1|Y_1) + \{t\} \times \Delta_{Y^*|Y_1}(\frac{\mu_2}{\mu_1} \cdot \frac{t}{t_0} \cdot N_1|Y_1 + \frac{t}{t_0}N_2|Y_1)$$

$$= \{0\} \times \Delta_{Y^*|Y_1}(N_1|Y_1) + \{t\} \times \Delta_{Y^*|Y_1}(N_2 - t\mathcal{O}_X(Y_1))$$

$$= \Delta_{Y^*|Y_1}(N_1)|_{Y_1} \cup \Delta_{Y^*|Y_1}(N_2)|_{Y_1},$$

which implies the inclusion (3.2) for $t < t_0$. Thus, the proof of Theorem 1.2 is complete.
4. A Linear Map Compatible with Intersection Products

In this section we deduce Corollary 1.4 from Theorem 1.2. Let $L \in U$ be an ample $\mathbb{Q}$-line bundle and $r_1 \in \mathbb{N}$ a positive integer, such that $r_1 L$ is very ample and hence, induces an embedding $X \to \mathbb{P}^N$. We may assume $d \geq 2$, as $\dim N^1(X)_{\mathbb{Q}} = 1$ if $d = 1$. One version of Bertini’s theorem [6, Theorem 1.1] states that any general hyperplane $H \subseteq \mathbb{P}^N$ intersects $X$ in an irreducible divisor on $X$. By another version of Bertini’s theorem [7, Theorem 17.16] for any general hyperplane $H \subseteq \mathbb{P}^N$ we have $(H \cap X)_{\text{sing}} = X_{\text{sing}} \cap (X \cap H)$. Thus, there exists a hyperplane $H_1 \subseteq \mathbb{P}^N$ such that $H_1 \cap X$ is irreducible and every regular point of $H_1 \cap X$ is also regular in $X$. We write $Y_1 = H_1 \cap X$. By construction we have $L \cong \frac{1}{r_1} O_X(Y_1)$ as $\mathbb{Q}$-line bundles. Repeating this construction inductively on $Y_i$ with $L|_{Y_i}$, we obtain a flag

$$Y_\bullet: X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_{d-1} \supseteq Y_d = \{ p \},$$

where $p$ is a point, which is regular in every $Y_i$. The flag $Y_\bullet$ corresponds to the line bundle $L$. A similar construction has been given by Anderson–Küronya–Lozovanu [4, Proposition 4].

Let $\mathcal{M} \in U$ be linearly independent to $L$. Replacing $\mathcal{M}$ by $\mathcal{M} + n L$ for some large enough $n \in \mathbb{N}$, we may assume that $\mathcal{M}$ is also ample. Since $L$ and $\mathcal{M}$ form a basis of $U$, we can write every $N \in U$ in the form $N = \lambda L + \mu \mathcal{M}$ for some $\lambda, \mu \in \mathbb{Q}$. With this notation we define the map $\Delta: U \to K_d^{\mathbb{Q}}$ by

$$\Delta(N) = \lambda \Delta_{Y_\bullet}(L) + \mu \Delta_{Y_\bullet}(\mathcal{M}).$$

This map is clearly well-defined and linear. Since $L$ and $\mathcal{M}$ are ample, Theorem 1.2 ensures that

$$\Delta(\lambda L + \mu \mathcal{M}) = \lambda \Delta_{Y_\bullet}(L) + \mu \Delta_{Y_\bullet}(\mathcal{M}) = \Delta_{Y_\bullet}(\lambda L + \mu \mathcal{M})$$

for all $\lambda, \mu \geq 0$.

Let us check the compatibility of $\Delta$ with intersection products. First, recall that the volume of an ample line bundle is its top degree intersection product, that means

$$\text{vol}((\lambda L + \mu \mathcal{M})) = (\lambda L + \mu \mathcal{M})^d$$

for all $\lambda, \mu \geq 0$. Combining this with Equations (2.1) and (4.1) we obtain

$$\frac{1}{d!}((\lambda L + \mu \mathcal{M})^d = \text{vol}(\Delta_{Y_\bullet}(\lambda L + \mu \mathcal{M})) = \text{vol}(\Delta(\lambda L + \mu \mathcal{M}))$$

for all $\lambda, \mu \geq 0$. Further, by the polarization formula we get for the intersection number of line bundles $L_1, \ldots, L_d$ as well as for the mixed volume of convex bodies $K_1, \ldots, K_d$

$$L_1 \cdot \ldots \cdot L_d = \frac{1}{d!} \sum_{J \subseteq \{1, \ldots, d\}} (-1)^{d-\# J} \left( \sum_{j \in J} L_j \right)^d,$$

$$V(K_1, \ldots, K_d) = \frac{1}{d!} \sum_{J \subseteq \{1, \ldots, d\}} (-1)^{d-\# J} \text{vol} \left( \sum_{j \in J} K_j \right),$$

where the second formula can also be taken as a definition for the mixed volume $V(K_1, \ldots, K_d)$. If we apply Equation (4.2) to every summand in the outer sum of
these formulas, we obtain
\begin{equation}
\frac{1}{d!} L^k \cdot M^{d-k} = V(\Delta(L)^k, \Delta(M)^{d-k})
\end{equation}
for any $0 \leq k \leq d$. Here, $V(K_1^k, K_2^{d-k})$ means $V(K_1, \ldots, K_1, K_2, \ldots, K_2)$ where the body $K_1$ occurs $k$-times and the body $K_2$ occurs $(d - k)$-times.

To show the compatibility in general, we regard the intersection product and the mixed volume as multi-linear maps
\[ \text{int} : S^d U \to Q, \quad V : S^d K^\infty_d \to \mathbb{R}, \]
where $S^d W$ denotes the $d$-th symmetric power of any vector space $W$. As $S^d U$ is generated by symbolic elements of the form $L^k M^{d-k}$ for all $0 \leq k \leq d$, the multi-linear map $V : S^d \Delta : S^d U \to \mathbb{R}$ is completely determined by Equation (4.3). Thus, by linearity we indeed get
\[ \frac{1}{d!} \text{int}(M_1 \cdots M_d) = \frac{1}{d!} \text{int}(M_1 \cdots M_d) = V(\Delta(M_1), \ldots, \Delta(M_d)). \]

Next we show the injectivity of the map $\Delta$. It is enough to show that $\Delta(L)$ and $\Delta(M)$ are linearly independent in $K^\infty_d$. Since $L$ and $M$ are linearly independent in $N^1(X)_{\mathbb{R}}$, it holds for their self-intersection numbers
\[ ((L + M)^d)^{1/d} \neq (L^d)^{1/d} + (M^d)^{1/d}, \]
as worked out by Cutkosky [4 Proposition 6.13] in this general situation. Using Equation (4.2) we deduce a similar inequality for the volumes of the Newton–Okounkov bodies
\begin{equation}
\text{vol}(\Delta(L) + \Delta(M))^{1/d} \neq \text{vol}(\Delta(L))^{1/d} + \text{vol}(\Delta(M))^{1/d}.
\end{equation}

Note that it holds $\text{vol}(\lambda K)^{1/d} = \lambda \text{vol}(K)^{1/d}$ for all convex bodies $K$ and all $\lambda \in \mathbb{R}_{\geq 0}$. Hence, it follows from inequality (4.4) that
\begin{equation}
s_1 \Delta(L) \neq s_2 \Delta(M)
\end{equation}
for all $(s_1, s_2) \in (\mathbb{R}_{\geq 0})^2 \setminus \{(0, 0)\}$ and hence, also for $(s_1, s_2) \in (\mathbb{R}_{\leq 0})^2 \setminus \{(0, 0)\}$. If $s_1, s_2 \in \mathbb{R}$ have different signs, we also obtain (4.4) as $s_1 \Delta(L) - s_2 \Delta(M)$ or $s_2 \Delta(M) - s_1 \Delta(L)$ is represented by a convex body of positive volume and hence non-zero. Thus, $\Delta(L)$ and $\Delta(M)$ are linearly independent.

Finally, we consider ample line bundles $M_1, \ldots, M_r \in U$ on $X$ and we want to show that the flag $Y_\bullet$ above can be chosen, such that $\Delta(M_j) = \Delta_{Y_\bullet}(M_j)$ for all $j \leq r$. For this purpose, let $C \subseteq U$ denote the convex cone in $U$ consisting of all non-negative $\mathbb{Q}$-linear combinations of the $M_i$’s. After reordering we may assume that this cone is already generated by $M_1$ and $M_2$. We set $L = M_1$ and $M = M_2$ for the construction of the flag $Y_\bullet$ and the map $\Delta$ as above. If all $M_i$’s are multiples of each other, we choose $M$ instead to be any ample class in $U$ which is linearly independent of $L$. Then every $M_i$ can be represented in the form $M_i = \lambda_i L + \mu_i M$ for some $\lambda_i, \mu_i \in \mathbb{Q}_{\geq 0}$. As $Y_\bullet$ corresponds to $L$, we get by Theorem (4.2)
\[ \Delta(M_i) = \lambda_i \Delta_{Y_\bullet}(L) + \mu_i \Delta_{Y_\bullet}(M) = \Delta_{Y_\bullet}(\lambda_i L + \mu_i M) = \Delta_{Y_\bullet}(M_i) \]
for all $i \leq r$. This completes the proof Corollary (4.3).
5. The Limits of the Additivity of \( \Delta_{Y^*} \)

In this section we prove Proposition 1.5. Let \( \mathcal{L} \) and \( \mathcal{M} \) be ample \( \mathbb{R} \)-line bundles on \( X \) and \( Y^* \) an admissible flag on \( X \), such that \( Y^* \) is Cartier divisor on \( X \). We assume

\[
\Delta_{Y^*}(\mathcal{L} + \mathcal{M}) = \Delta_{Y^*}(\mathcal{L}) + \Delta_{Y^*}(\mathcal{M}).
\]

We write \( \mu_\mathcal{L} = \mu(\mathcal{L}, Y^*), \mu_\mathcal{M} = \mu(\mathcal{M}, Y^*) \) and \( \mu_\Sigma = \mu(\mathcal{L} + \mathcal{M}, Y^*) \) for the function \( \mu \) defined in Section 2. The additivity of the Newton–Okounkov bodies assumed above implies

\[
\mu_\Sigma = \mu_\mathcal{L} + \mu_\mathcal{M}.
\]

Note that the function \( \mu \) was defined such that the \( \mathbb{R} \)-line bundles

\[
\mathcal{L}^c = \mathcal{L} - \mu_\mathcal{L} \mathcal{O}_X(Y^*), \quad \mathcal{M}^c = \mathcal{M} - \mu_\mathcal{M} \mathcal{O}_X(Y^*)
\]

lie in the boundary \( \partial \text{Eff}(X) \) of the pseudo-effective cone of \( X \). To prove Proposition 1.5, it is enough to show that the convex cone

\[
C = \{ \lambda \mathcal{L}^c + \mu \mathcal{M}^c \in \text{Eff}(X) \mid \lambda, \mu \in \mathbb{R}_{\geq 0} \}
\]

lies completely in the boundary \( \partial \text{Eff}(X) \). Assume that there are \( \lambda, \mu \in \mathbb{R}_{\geq 0} \) such that \( \lambda \mathcal{L}^c + \mu \mathcal{M}^c \notin \partial \text{Eff}(X) \). Then there exists some \( \epsilon > 0 \) such that

\[
\lambda \mathcal{L}^c + \mu \mathcal{M}^c - \epsilon \mathcal{O}_X(Y^*) \notin \text{Eff}(X).
\]

Without lost of generality we may assume \( \lambda < \mu \), in particular \( \mu > 0 \). Since \( \text{Eff}(X) \) is a convex cone, we also get that

\[
\lambda \mathcal{L}^c + \mu \mathcal{M}^c - \epsilon \mathcal{O}_X(Y^*) = \mathcal{L} + \mathcal{M} - (\mu_\Sigma + \frac{\lambda}{\mu}) \mathcal{O}_X(Y^*)
\]

lies in \( \text{Eff}(X) \) in contradiction to the definition of \( \mu_\Sigma \). Thus, we get \( C \subseteq \partial \text{Eff}(X) \) as desired.

6. An Inequality of Intersection Numbers

Finally, we give the proof of Corollary 1.6 in this section. It has been worked out by Lehmann–Xiao [14, Theorem 5.9], that for all three convex bodies \( K, L \) and \( M \) in \( \mathbb{R}^d \) we always have

\[
\text{vol}(L)V(K^k, M^{d-k}) \leq \binom{d}{k} V(K^k, L^{d-k})V(L^k, M^{d-k})
\]

for any \( 0 \leq k \leq d \). If we apply this for \( k = 1 \) to the Newton–Okounkov bodies of three ample line bundles \( \mathcal{L}, \mathcal{M} \) and \( \mathcal{N} \), we get

\[
\frac{1}{\partial} \mathcal{L}^d V(\Delta_{Y^*}(\mathcal{M}), \Delta_{Y^*}(\mathcal{N})^{d-1}) \leq dV(\Delta_{Y^*}(\mathcal{M}), \Delta_{Y^*}(\mathcal{L})^{d-1}) V(\Delta_{Y^*}(\mathcal{L}), \Delta_{Y^*}(\mathcal{N})^{d-1})
\]

for any admissible flag \( Y^* \) on \( X \). To compare the mixed volumes of Newton–Okounkov bodies with the intersection numbers of ample line bundles, we prove the following lemma.

**Lemma 6.1.** Let \( X \) be a projective variety of dimension \( d \) and \( \mathcal{L} \) and \( \mathcal{M} \) any two ample line bundles on \( X \). For any admissible flag \( Y^* \) it holds

\[
V(\Delta_{Y^*}(\mathcal{L}), \Delta_{Y^*}(\mathcal{M})^{d-1}) \leq \frac{1}{\partial} (\mathcal{L} \cdot \mathcal{M})^{d-1}.
\]

If \( Y^* \) is corresponding to \( \mathcal{L} \) or to \( \mathcal{M} \), then the above inequality is an equality.
Proof. Note, that the mixed volume is equal to the first derivative of the volume function at $t = 0$ divided by $d$

\begin{equation}
V(\Delta Y_\ast(\mathcal{L}), \Delta Y_\ast(\mathcal{M})^{d-1}) = \frac{1}{d} \cdot \frac{d\text{vol}(t\Delta Y_\ast(\mathcal{L}) + \Delta Y_\ast(\mathcal{M}))}{dt} \bigg|_{t=0}.
\end{equation}

Here and in the following we always mean the derivative for which $t$ approximates $0$ in the positive rationals. For any $t \geq 0$ the inclusion implies the inequality

\[
\text{vol}(t\Delta Y_\ast(\mathcal{L}) + \Delta Y_\ast(\mathcal{M})) \leq \text{vol}(\Delta Y_\ast(t\mathcal{L} + \mathcal{M})) = \frac{1}{d!}(t\mathcal{L} + \mathcal{M})^d,
\]

which is an equality at $t = 0$. Thus the first derivative of the left hand side has to be bounded by the first derivative of the right hand side at $t = 0$

\[
\frac{d\text{vol}(t\Delta Y_\ast(\mathcal{L}) + \Delta Y_\ast(\mathcal{M}))}{dt} \bigg|_{t=0} \leq \frac{1}{d!} \cdot \frac{d(t\mathcal{L} + \mathcal{M})^d}{dt} \bigg|_{t=0} = \frac{d}{d!}(\mathcal{L} \cdot \mathcal{M}^{d-1}).
\]

Applying this inequality to Equation (6.2), we get the inequality in the lemma. If $Y_\ast$ corresponds to $\mathcal{L}$ or $\mathcal{M}$, we get by Theorem 1.2 an equality

\[
\text{vol}(t\Delta Y_\ast(\mathcal{L}) + \Delta Y_\ast(\mathcal{M})) = \text{vol}(\Delta Y_\ast(t\mathcal{L} + \mathcal{M})) = \frac{1}{d!}(t\mathcal{L} + \mathcal{M})^d,
\]

Thus, both sides as functions in $t$ must have the same derivative at $t = 0$, such that we obtain an equality

\[
V(\Delta Y_\ast(\mathcal{L}), \Delta Y_\ast(\mathcal{M})^{d-1}) = \frac{1}{d!}(\mathcal{L} \cdot \mathcal{M}^{d-1}).
\]

This completes the proof of the lemma. \hfill \Box

We choose $Y_\ast$ in inequality to be a flag corresponding to $\mathcal{M}$. Such a flag can be constructed in the same way as described in Section 4. With this choice for $Y_\ast$ we can apply Lemma 6.1 to both sides of inequality (6.1). This yields

\[
\mathcal{L}^d \cdot (\mathcal{M} \cdot \mathcal{N}^{d-1}) \leq d \cdot (\mathcal{M} \cdot \mathcal{L}^{d-1}) \cdot (\mathcal{L} \cdot \mathcal{N}^{d-1}).
\]

By continuity of the intersection numbers, this inequality still holds true if $\mathcal{L}, \mathcal{M}$ and $\mathcal{N}$ are any nef $\mathbb{R}$-line bundles on $X$. This finishes the proof of Corollary 1.6.

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