Adiabatic groupoids and secondary invariants in K-theory

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Abstract

In this paper we define K-theoretic secondary invariants attached to a Lie groupoid \( G \). The K-theory of \( C^*_r(G_{ad}) \) (where \( G_{ad} \) is the adiabatic deformation \( G \) restricted to the interval \([0,1)\)) is the receptacle for K-theoretic secondary invariants. We give a Lie groupoid version of construction given in [27, 28] in the setting of the Coarse Geometry. Our construction directly generalises to more involved geometrical situation, such as foliations, well encoded by a Lie groupoid. Along the way we tackle the problem of producing a wrong-way functoriality between adiabatic deformation groupoid K-groups with respect to transverse maps. This extends the construction of the lower shriek map in [5]. Moreover we attach a secondary invariant to the two following operators: the signature operator on a pair of homotopically equivalent Lie groupoids; the Dirac operator on a Lie groupoid equipped with a metric that has positive scalar curvature \( s \)-fiber-wise. Furthermore we prove a Lie groupoid version of the Delocalized APS Index Theorem of Piazza and Schick. Finally we give a product formula for the secondary invariants and we state stability results about cobordism classes of Lie groupoid structures and bordism classes of Lie groupoid metric with positive scalar curvature along the \( s \)-fibers.

Introduction

Higher secondary invariants have recently been the subject of a number of papers, see for instance [4, 3, 43, 40, 12, 13, 14, 27, 28, 42, 41]. In this paper the approach to secondary invariants is given through Lie groupoids. Let us start by giving a general framework.

We begin by considering an exact sequence of C*-algebras of the following type

\[
0 \longrightarrow S \otimes B \longrightarrow E \longrightarrow A \longrightarrow 0 \tag{0.1}
\]

where \( S := C_0(0,1) \). We can investigate the following hierarchy of K-theory classes:

- a fundamental class \([D] \in K_*(A)\);
- a primary invariant \( \partial [D] \in K_{*+1}(S \otimes B) \), where \( \partial : K_*(A) \rightarrow K_{*+1}(S \otimes B) \) is the boundary map for the long exact sequence in K-theory;
- assume that the primary invariant is the zero class and that we know the “reason” why it is zero. Then we can find a canonical lift of \([D]\) in \( K_* (E) \), that we are going to call a secondary invariant and that we will denote by \( \varrho(D,w) \).

Now let us assume that the exact sequence (0.1) has a completely positive section. This implies that \( \partial \) is an element in \( KK(A,B) \) and one can prove that there exist a C*-algebra \( A' \) and two morphisms

- \( \psi : A' \rightarrow A \) that induces a KK-equivalence;
- \( \varphi : A' \rightarrow B \) that induces the boundary map for the following exact sequence

\[
0 \longrightarrow S \otimes B \longrightarrow C_\varphi(A',B) \longrightarrow A' \longrightarrow 0 \tag{0.2}
\]

whose long exact sequence in K-theory is isomorphic to the one associated to (0.1). Here \( C_\varphi(A',B) = \{ a \oplus f \in A' \oplus B | f(0) = \varphi(a) \} \) is the mapping cone C*-algebra associated to \( \varphi \). In this context a secondary invariant is a class in \( K_0(C_\varphi(A',B)) \) (the odd case is analogous) and it is explicitly given by
• the projection \( p \) over \( A' \) defining the fundamental class \([D]\).
• a path \( q_t \) of projections from \( \varphi(p) \) to a degenerate projection over \( B \), that concretely gives the reason \( \nu \) why the primary invariant is zero.

The tangent groupoid and \( \varphi \)-classes

Let us make this more concrete in a simple geometric context. Let \( X \) be a closed smooth manifold. Then consider the pair groupoid \( X \times X \rightrightarrows X \). Its smooth convolution algebra \( C^{\infty}_c(X \times X, \Omega^2(\ker dr \oplus \ker ds)) \) of the smooth compactly supported half-densities on \( X \times X \) is nothing but the \(*\)-algebra of the smoothing operators on \( L^2(X) \) and \( C^*_r(X \times X) \), its reduced \( C^*\)-algebra, is given by \( K(L^2(X)) \).

The Lie algebroid of \( X \times X \) is given by the tangent bundle \( TX \): it is a Lie groupoid and, by means of the Fourier transform, its \( C^*\)-algebra \( C^*_r(TX) \) is isomorphic to \( C_0(T^*X) \), that is the closure of 0-order symbols on \( X \). By Poincaré duality, see \[5\], we know that \( K_*(C_0(T^*X)) \) is isomorphic to \( KK_*(C(X), \mathbb{C}) \), the \( K \)-homology of \( X \).

So, following the abstract construction given in the previous subsection, we have that \( K_*(C^*_r(TX)) \) is the receptacle of the fundamental classes and the analytical index

\[
\text{Ind}: K_*(C_0(T^*X)) \to K_*(C^*_r(X \times X))
\]
gives the primary invariants. But we would like to have a realization of \( \text{Ind} \) as an element of \( KK \)-theory or, better, as the boundary map of a semi-split exact sequence as in \([0, 2]\).

Indeed the construction of A. Connes gives the solution to this problem. The tangent groupoid of the smooth manifold \( X \) is given by the following object

\[
TX := TX \times \{0\} \cup X \times X \times (0, 1] \rightrightarrows X \times (0, 1],
\]
equipped with a suitable topology. It is a deformation groupoid, whose restriction at 0 is \( TX \) and whose restriction at 1 is \( X \times X \). One can prove that

• the evaluation at 0, \( ev_0: C^*_r(TX) \to C^*_r(TX) \), induces a \( KK \)-equivalence: if \( \sigma \) is an elliptic symbol of order 0 on \( X \), then the symbol \( \sigma \times id_{[0, 1]} \) on \( T^*X \times (0, 1] \), the Lie algebroid of \( TX \), gives an elliptic pseudodifferential operator on \( TX \), in the sense of \([36]\), whose restriction at 1 is the pseudodifferential operator on \( X \) associated to \( \sigma \) and whose restriction at 0 is the Fourier transform of \( \sigma \);

• the \( KK \)-element \([ ev_0 ]^{-1} \otimes_{C^*_r(TX)} [ ev_1 ] \in KK(C^*_r(TX), C^*_r(X \times X)) \) gives the analytical index \( \text{Ind} \), where \( ev_1: C^*_r(TX) \to C^*_r(X \times X) \) is the evaluation at 1, see \([24]\).

Now let us point out that \( C^*_r(TX), C^*_r(X \times X) \), the mapping cone \( C^*\)-algebra of the evaluation at 1, is isomorphic to \( C^*_r(T^\circ X) \), where \( T^\circ X \) is the restriction of \( TX \) to the open interval \([0, 1)\).

So we have that the analytical index is the boundary morphism of the long exact sequence of \( K \)-groups associated to the exact sequence

\[
0 \longrightarrow C^*_r(X \times X) \otimes C_0(0, 1) \longrightarrow C^*_r(T^\circ X) \overset{ev_0}{\longrightarrow} C^*_r(TX) \longrightarrow 0. \tag{0.3}
\]

We have two typical geometric situations where the analytical index vanishes:

• let \( X = N \sqcup - M \) be the disjoint union of two compact simply connected smooth manifolds, that are homotopically equivalent through \( f: N \to M \), and let \( D^{sign} \) be the signature operator of \( X \); in \([17]\) the authors proved that there exists a canonical way to produce a path of operators \( D_t \) from \( D^{sign} \) to an invertible operator \( D_1 \); all that (up to passing from the language of \( K \)-theory to the language of \( KK \)-theory and from the unbounded case to the bounded one) gives a fundamental class, the symbol of the signature operator, and a path to a degenerate cycle, that is a reason why the analytical index of the signature vanishes, namely a class \( \theta(f) \) in the \( K \)-theory of \( C^*_r(T^\circ X) \);
• let \( X \) be a spin smooth compact manifolds, equipped with a Riemannian metric \( g \), such that the scalar curvature is positive everywhere; then the Dirac operator \( \mathcal{D} \) associated to the spinor bundle is invertible and this implies that its analytical index is zero; so, as for the previous case, the Dirac operator itself (no perturbations are needed) gives a class \( g(g) \) in the K-theory of \( C^*_r(T^\circ X) \).

Wrong-way functoriality

Once we have constructed such a secondary invariant, we would like to study the functoriality of these object with respect to smooth maps. In other words we would like to push forward classes from \( N \) to \( M \), through a smooth map \( f: N \to M \) at the level of the tangent groupoids. In \cite{5} the authors construct a lower shriek map \( df_! \in KK^*(C_0(T^*N), C_0(T^*M)) \), associated to any smooth map \( f: N \to M \) between compact smooth manifolds. By Poincaré duality this homomorphism corresponds to the map \( [f]: K_*(N) \to K_*(M) \) between the K-homology groups of the manifolds. Thanks to the Poincaré duality and the naturality of the index we have the following equality of morphisms \( K_*(C_0(T^*N)) \to K_*(C^*_r(M \times M)) \)

\[
df_! \otimes \text{Ind}_M = \text{Ind}_N \otimes \mu_f
\]

where \( \text{Ind}_{X \times X} \) is as in Definition and \( \mu_f \) is the Morita equivalence between \( C^*_r(N \times N) \) and \( C^*_r(M \times M) \). The problem here concerns the construction of the dotted arrow in the following diagram

\[
\cdots \rightarrow K_* (C^*_r(N \times N \times (0, 1))) \xrightarrow{\mu_f \otimes \text{id}} K_* (C^*_r(T^\circ N)) \xrightarrow{[\text{ev}]} K_* (C_0(T^*N)) \xrightarrow{df} \cdots
\]

\[
\cdots \rightarrow K_* (C^*_r(M \times M \times (0, 1))) \xrightarrow{\psi^d} K_* (C^*_r(T^\circ M)) \xrightarrow{[\text{ev}]} K_* (C_0(T^*M)) \xrightarrow{df} \cdots
\]

in such a way that all the squares commute. This dotted arrow will be implemented by a suitable deformation groupoid, as we will see in Section \ref{2}

**Cobordisms**

A second problem is the following: a natural equivalence relation among homotopy equivalences and metrics with positive scalar curvature is given by a certain cobordism equivalence. The question is if the \( g \)-classes are well defined on cobordism classes. A positive answer is given by the so-called Delocalized Atiyah-Patodi-Singer Index Theorem, firstly stated and proved by Piazza and Schick in \cite{27} \cite{28} in the setting of the coarse geometry and in this paper formalized and generalized to the context of Lie groupoids. In order to do it we need to use the Monthubert groupoid

\[
\Gamma(W, \partial W) = W \times W \sqcup \partial W \times \partial W \times \mathbb{R} \\
= W
\]

of a manifold \( W \) with boundary \( \partial W \), see Section \ref{1.4}

Let \( P \) be an elliptic pseudodifferential operator such that its restriction to the boundary \( P_0 \) is homotopic to an invertible operator, through a path \( P^t_0 \). This implies that \( P \) has a Fredholm index in the K-theory of \( C^*_r(W \times \hat{W}) \). A suitable deformation groupoid

\[
\Gamma(W, \partial W)^{P^t_0}_\text{id} \Rightarrow W \times [0, 1] \sqcup \partial W \times [0, 1]
\]

encodes both the Fredholm index of \( P \) and the \( g \)-class of the boundary in a such a way that, through convenient exact sequences, we can compare these two classes and state, roughly, that the \( g \)-class associated to the path \( P^t_0 \) is equal to the image of the Fredholm index of \( P \) into the K-theory of the tangent groupoid.

Hence if we have a spin Riemannian manifolds \( W \) that is a cobordism between \( \partial_0 W \) and \( \partial_1 W \) and if \( W \) is equipped with a metric \( G \) with positive scalar curvature that restricts to \( g_0 \) and \( g_1 \) on the boundary components, then \( \varrho(g_0) = \varrho(g_1) \) since the Dirac operator of \( W \) is invertible and its Fredholm index vanishes.
Analogously if $W$ is a smooth cobordism between two manifolds $M_0$ and $M_1$ and if there is a homoopy equivalence $F: W \to N \times [0, 1]$, such that its restrictions to the boundary components, $f_i: M_i \to N \times \{i\}$ for $i = 0, 1$, are homotopy equivalences, then $\varrho(f_0) = \varrho(f_1)$ since the index of the signature operator of $W \sqcup N \times [0, 1]$ vanishes.

Products

A last question concerns product formulas: if $g$ is a Riemannian metric with positive scalar curvature on a spin smooth compact manifold $Y$ and $h$ is any Riemannian metric on a spin smooth compact manifold $X$, we know that, up to multiply by a scalar factor $\epsilon$ the metric $h$, $g \oplus h$ is a metric with positive scalar curvature on $Y \times V$; if $f: N \to M$ is a homotopy equivalence between smooth manifolds, then so is $f \times \text{id}: N \times X \to M \times W$ for any smooth manifold $W$. What is the relation between $\varrho(g)$ and $\varrho(g \times h)$? The same question arise for $\varrho(f)$ and $\varrho(f \times \text{id})$, where $f$ is a homotopy equivalence.

Let $Z$ denote $Y$ or $N \sqcup -M$ and let $X$ denote $V$ or $W$. One can define a product

$$\boxtimes: K_i(C^r_\ast(T^\circ Z)) \times K_j(X) \to K_{i+j}(T^\circ Z \times X)$$

such that the following formulas holds:

$$\varrho(g) \boxtimes [D_h] = \varrho(g \oplus h)$$

where $[D_h]$ is the K-homology class of the Dirac operator on $(X, h)$;

$$\varrho(f) \boxtimes [D^\text{sign}_X] = \varrho(f \times \text{id})$$

where $[D^\text{sign}_X]$ is the K-homology class of the signature operator of $X$.

After proving that, one can ask when two different $\varrho$-classes on the same manifold remain distinct after making the product with a second manifold. In this specific case the answer is that if the index of the K-homology class on $X$ is non zero, then the product with this K-homology class is rationally injective.

Lie groupoids

So far we have been concerned by a very simple Lie groupoid on a smooth manifold $X$, the pair groupoid. A nice feature of the theory we explained above is that, if we take any Lie groupoid $G$ over $X$, mutatis mutandis and with some extra work, all the results hold in that generality.

Where we used the tangent groupoid, we now employ the adiabatic deformation groupoid $G_{ad}$, see Definition [1.6]. The wrong-way functoriality generalizes between the adiabatic deformation of a Lie groupoid and the adiabatic deformation of its pull-back, see the Subsection [2.1].

Cobordism relations and product formulas are established in this general context. The main examples are always given by homotopy equivalences and positive scalar curvature, in a suitable groupoid fashion. A concrete non trivial example is given by foliations: can we distinguish cobordism classes of foliations homotopically equivalent to a given one? Can we distinguish cobordism classes of foliated metrics that have longitudinally positive scalar curvature?

This paper is devoted to develop the program explained above and to tackle the technical issues one meets in generalizing it to the context of a general Lie groupoid.

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1 Groupoids

1.1 Basics

We refer the reader to [7] and bibliography inside it for the notations and a detailed overview about groupoids and index theory.

Definition 1.1. Let $G$ and $G^{(0)}$ be two sets. A groupoid structure on $G$ over $G^{(0)}$ is given by the following morphisms:

- Two maps: $r, s : G \to G^{(0)}$, which are respectively the range and source map.
- A map $u : G^{(0)} \to G$ called the unit map that is a section for both $s$ and $r$. We can identify $G^{(0)}$ with its image in $G$.
- An involution: $i : G \to G, \gamma \mapsto \gamma^{-1}$ called the inverse map. It satisfies: $s \circ i = r$.
- A map $p : G^{(2)} \to G, (\gamma_1, \gamma_2) \mapsto \gamma_1 \cdot \gamma_2$ called the product, where the set
  \[ G^{(2)} := \{(\gamma_1, \gamma_2) \in G \times G \mid s(\gamma_1) = r(\gamma_2)\} \]
  is the set of composable pair. Moreover for $(\gamma_1, \gamma_2) \in G^{(2)}$ we have $r(\gamma_1 \cdot \gamma_2) = r(\gamma_1)$ and $s(\gamma_1 \cdot \gamma_2) = s(\gamma_2)$.

The following properties must be fulfilled:

- The product is associative: for any $\gamma_1, \gamma_2, \gamma_3$ in $G$ such that $s(\gamma_1) = r(\gamma_2)$ and $s(\gamma_2) = r(\gamma_3)$ the following equality holds
  \[ (\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3). \]
- For any $\gamma$ in $G$: $r(\gamma) \cdot \gamma = \gamma \cdot s(\gamma) = \gamma$ and $\gamma \cdot \gamma^{-1} = r(\gamma)$. 

5
We denote a groupoid structure on $G$ over $G^{(0)}$ by $G \rightrightarrows G^{(0)}$, where the arrows stand for the source and target maps.

We will adopt the following notations:
$$G_A := s^{-1}(A), \quad G^B = r^{-1}(B) \quad \text{and} \quad G_A^B = G_A \cap G^B$$
in particular if $x \in G^{(0)}$, the $s$-fiber (resp. $r$-fiber) of $G$ over $x$ is $G_x = s^{-1}(x)$ (resp. $G^x = r^{-1}(x)$).

**Definition 1.2.** We call $G$ a Lie groupoid when $G$ and $G^{(0)}$ are second-countable smooth manifolds with $G^{(0)}$ Hausdorff, the structural homomorphisms are smooth.

### 1.2 Groupoid $C^*$-algebras

We can associate to a Lie groupoid $G$ the *-algebra $C^\infty_c(G, \Omega^\frac{1}{2}(\ker ds \oplus \ker dr))$ of the compactly supported sections of the half densities bundle associated to $\ker ds \oplus \ker dr$, with:

- the involution given by $f^*(\gamma) = \overline{f(\gamma^{-1})}$;
- and the product by $f \ast g(\gamma) = \int_{G_x(\gamma)} f(\gamma \eta^{-1}) g(\eta)$.

For all $x \in G^{(0)}$ the algebra $C^\infty_c(G, \Omega^\frac{1}{2}(\ker ds \oplus \ker dr))$ can be represented on $L^2(G_x, \Omega^\frac{1}{2}(G_x))$ by
$$\lambda_x(f)\xi(\gamma) = \int_{G_x} f(\gamma \eta^{-1}) \xi(\eta),$$
where $f \in C^\infty_c(G, \Omega^\frac{1}{2}(\ker ds \oplus \ker dr))$ and $\xi \in L^2(G_x, \Omega^\frac{1}{2}(G_x))$.

**Definition 1.3.** The reduced $C^*$-algebra of a Lie groupoid $G$, denoted by $C^*_r(G)$, is the completion of $C^\infty_c(G, \Omega^\frac{1}{2}(\ker ds \oplus \ker dr))$ with respect to the norm
$$||f||_r = \sup_{x \in G^{(0)}} ||\lambda_x(f)||.$$

The full $C^*$-algebra of $G$ is the completion of $C^\infty_c(G, \Omega^\frac{1}{2}(\ker ds \oplus \ker dr))$ with respect to all continuous representations.

**Remark 1.4.** From now on, if $X$ is a $G$-invariant closed subset of $G^{(0)}$ we will call $\epsilon_X : C^\infty_c(G) \to C^\infty_c(G_I\backslash X)$ the restriction map to $X$. That gives an exact sequence of full groupoid $C^*$-algebras
$$0 \longrightarrow C^*(G_{G^{(0)}\backslash X}) \longrightarrow C^*(G) \longrightarrow C^*(G_I\backslash X) \longrightarrow 0,$$
but in general this is not true for the reduced ones: for instance, the reader can find examples of this phenomenon in [10]. We want to precise that in what follows we will mainly deal with the reduced groupoid $C^*$-algebras, because there are more details in the reduced situation than in the full one that need to be carefully checked. But everything we are going to prove about the reduced $C^*$-algebras works for the full ones too.

### 1.3 Lie algebroids and the adiabatic groupoid

**Definition 1.5.** A Lie algebroid $\mathfrak{A} = (p : \mathfrak{A} \to TM, [\ , \ ]_\mathfrak{A})$ on a smooth manifold $M$ is a vector bundle $\mathfrak{A} \to M$ equipped with a bracket $[\ , \ ]_\mathfrak{A} : \Gamma(\mathfrak{A}) \times \Gamma(\mathfrak{A}) \to \Gamma(\mathfrak{A})$ on the module of sections of $\mathfrak{A}$, together with a homomorphism of fiber bundle $p : \mathfrak{A} \to TM$ from $\mathfrak{A}$ to the tangent bundle $TM$ of $M$, called the anchor map, fulfilling the following conditions:

- the bracket $[\ , \ ]_\mathfrak{A}$ is $\mathbb{R}$-bilinear, antisymmetric and satisfies the Jacobi identity,
- $[X, fY]_\mathfrak{A} = f[X, Y]_\mathfrak{A} + p(X)(f)Y$ for all $X, Y \in \Gamma(\mathfrak{A})$ and $f$ a smooth function of $M$,
Let \( G \) be a Lie groupoid. The tangent space to \( s \)-fibers, that is \( T_s G := \ker ds = \bigcup_{x \in G(0)} T_G x \) has the structure of a Lie algebroid on \( G(0) \), with the anchor map given by \( dr \). It is denoted by \( \mathfrak{A}(G) \) and we call it the Lie algebroid of \( G \). We can also think of it as the normal bundle of the inclusion \( G(0) \hookrightarrow G \).

Let \( M_0 \) be a smooth submanifold of a smooth manifold \( M \) with normal bundle \( \mathcal{N} \). As a set, the deformation to the normal cone is \( D(M_0, M) = \mathcal{N} \times \{0\} \cup M \times (0, 1] \).

In order to recall its smooth structure, we fix an exponential map, which is a diffeomorphism \( \theta \) from a neighbourhood \( V' \) of the zero section \( M_0 \) in \( N \) to a neighbourhood \( V \) of \( M_0 \) in \( M \). We may cover \( D(M_0, M) \) with two open sets \( M \times (0, 1] \) with the product structure, and \( W = \mathcal{N} \times \{0\} \cup V \times (0, 1] \), endowed with the smooth structure for which the map

\[
\Psi\left\{(m, \xi, t) \in \mathcal{N} \times [0, 1] \mid (m, t\xi) \in V'\right\} \rightarrow W
\]

(1.1)
given by \((m, \xi, t) \mapsto (\theta(m, t\xi), t)\), for \( t \neq 0 \), and by \((m, \xi, 0) \mapsto (m, 0)\), for \( t = 0 \), is a diffeomorphism. One can verify that the transition map on the overlap of these two open sets is smooth, see...

**Definition 1.6.** The adiabatic groupoid \( G_{ad} \) is given by \( D(G(0), G) \), the deformation to the normal cone of the unit map. As set it is the following

\[
\mathfrak{A}(G) \times \{0\} \cup G \times (0, 1] \subset G(0) \times [0, 1],
\]

with the smooth structure given by the construction discussed above.

**Definition 1.7.** We will use the notation \( G_{ad}^\circ \) for the restriction adiabatic groupoid to the interval open on the right side, given by

\[
\mathfrak{A}(G) \times \{0\} \cup G \times (0, 1) \subset G(0) \times [0, 1).
\]

Then we can associate to a Lie groupoid \( G \) a short exact sequence of \( C^\ast \)-algebras

\[
0 \longrightarrow C^\ast(G \times (0, 1)) \longrightarrow C^\ast(G_{ad}^\circ) \overset{ev_0}{\longrightarrow} C^\ast(\mathfrak{A}(G)) \longrightarrow 0
\]

(1.2)
that we call the (full) adiabatic extension of \( G \).

Since \( \mathfrak{A}(G) \) is an amenable groupoid, \( C^\ast(\mathfrak{A}(G)) \) is isomorphic to the reduced groupoid \( C^\ast_r \mathfrak{A}(G) \). Thanks to the fact that the map from the full \( C^\ast \)-algebra of a groupoid to the reduced one is surjective, one can deduce that the following sequence of reduced groupoid \( C^\ast \)-algebras

\[
0 \longrightarrow C^\ast_r(G \times (0, 1)) \longrightarrow C^\ast_r(G_{ad}^\circ) \overset{ev_0}{\longrightarrow} C^\ast_r(\mathfrak{A}(G)) \longrightarrow 0
\]

(1.3)
is exact.

### 1.4 Manifolds with boundary and the Monthubert groupoid

For this section we refer the reader to [23] and [29 3.1]. Let \( X \) be a manifold with boundary \( \partial X \). We can think of \( X \) as a closed subspace of an open manifold \( \tilde{X} \). Let \( \rho : \tilde{X} \rightarrow \mathbb{R} \) be a defining function of the boundary, namely a function that is zero on \( \partial X \) and only there, with nowhere vanishing differential on it.

**Definition 1.8.** The Monthubert groupoid of \( X \), denoted by \( \Gamma(X, \partial X) \), is given by

\[
\{(x, y, \alpha) \in X \times X \times \mathbb{R} \mid \rho(x) = e^{\alpha} \rho(y)\}.
\]

Notice that by [23] Proposition 3.5 the Lie groupoid \( \Gamma(X, \partial X) \) is amenable and then \( C^\ast(\Gamma(X, \partial X)) = C^\ast_r(\Gamma(X, \partial X)) \).
Definition 1.9. Let \( \hat{G} \rightrightarrows \hat{X} \) be a Lie groupoid and let \( \partial X \) be transverse with respect to \( \hat{G} \) (this means that the range map and the inclusion of \( \partial X \) in \( \hat{X} \) are transverse). Define \( G(X, \partial X) \) as the following fibered product

\[
\begin{align*}
G(X, \partial X) & \xrightarrow{\gamma} G \\
\Gamma(X, \partial X) & \xrightarrow{r \times s} X \times X
\end{align*}
\]

where \( \Gamma(X, \partial X) \) is the Monthubert groupoid of \( (X, \partial X) \), defined above. Then \( G(X, \partial X) \rightrightarrows X \) is a longitudinally smooth groupoid. As set, it is \( G_{|X} \cup G_{|\partial X} \times \mathbb{R} \). See [23] Section 3 for a detailed construction.

Remark 1.10. If we look the groupoid \( G(X, \partial X) \) near the boundary of \( X \), we can give a product structure of it as follows. Let \( \underline{n} \) be a vector field that is normal to \( \partial X \) such that \( \langle d\rho, n \rangle(x) = 1 \) for any \( x \in \partial X \). Moreover, since \( \partial X \) is transverse to the boundary, \( \underline{n}(x) \) belongs to the image of the anchor map of \( \mathfrak{X}_G \) for all \( x \in \partial X \).

Then, if \( \exp: TX_{|\partial X} \rightarrow X \) is an exponential map, one has the following diffeomorphism

\[
\phi: (x, t) \mapsto \exp_{x}(t \underline{n})
\]

from \( \partial X \times (-1,1) \) to a neighbourhood \( U \) of \( \partial X \) in \( \hat{X} \), that gives an isomorphism of groupoids \( \Phi: \partial X \times \partial X \times (-1,1) \times (-1,1) \).

There \( \{U_t\}_{t \in I} \) is a locally finite open cover of \( \partial X \) such that \( \underline{n}|_{U_t} \) lifts to a local section \( \xi_t \) of \( \mathcal{A} \hat{G}_{|\partial X} \). Then by means of a partition of the unity subordinated to \( \{U_t\} \) one can obtain a section \( \xi \) of the Lie algebroid \( \mathcal{A} \hat{G}_{|\partial X} \). Let \( \gamma_{x,t} \) be the path in \( \hat{G} \) equal to \( \exp_{x}(t \xi) \) for \( t \in (-1,1) \).

Then the map

\[
\Psi: (\gamma, t, s) \mapsto \gamma_{r(\gamma), t} \cdot \gamma_s(\gamma), s
\]

is an isomorphism of groupoids between \( \hat{G}_{|\partial X} \times (-1,1) \times (-1,1) \) and \( \hat{G}_{|U} \).

In particular, since \( \Psi \) and \( \Phi \) are compatible through the source and the target maps, we obtain the following isomorphism of Lie groupoid

\[
\Gamma([0,1], \{0\}) \times G_{|\partial X} \cong G(X, \partial X)|_{U}.
\]

In this context it is convenient to use a slight variation of the adiabatic groupoid.

Definition 1.11. Let \( G(X, \partial X) \) be as in Definition 1.9 and denote \( X \setminus \partial X \) by \( \hat{X} \):

- let \( G(X, \partial X)_{\text{ad}}^\mathcal{F} \) be the restriction of \( G(X, \partial X)_{\text{ad}} \) to the open subset \( X_\mathcal{F} := X \setminus [0,1] \setminus \partial X \setminus \{1\} \) (the superscript \( \mathcal{F} \) refers to a condition of being Fredholm that will be clear later). It is the union \( (G_{|X})_{\text{ad}} \cup (G_{|\partial X})_{\text{ad}}^0 \times \mathbb{R} \);
- let \( \mathcal{T}_{\text{ad}} G(X, \partial X) \) be the restriction of \( G(X, \partial X)_{\text{ad}} \) to \( X_\mathcal{T} := \hat{X} \setminus \{0\} \cup \partial X \setminus \{1\} \). It is the union \( (G_{|X})_{\text{ad}} \cup (G_{|\partial X})_{\text{ad}}^0 \times \mathbb{R} \);
- finally let \( G(X, \partial X)_{\text{ad}}^0 \) be the restriction of \( G(X, \partial X)^\mathcal{F}_{\text{ad}} \) to \( X \setminus \{0\} \).

Now let us give some results about the C*-algebras associated to these groupoids, that will be useful later.

Lemma 1.12. The C*-algebra \( C^*_a(\Gamma([\mathbb{R}_+, \{0\}])) \) is K-contractible.

Proof. We have that \( \Gamma([\mathbb{R}_+, \{0\}]) = \mathbb{R}_+^\times \times \mathbb{R}_+^\times \cup \{0\} \times \{0\} \times \mathbb{R} \cong \mathbb{R}_+ \). It is isomorphic to the groupoid \( \mathbb{R}_+ \times \mathbb{R}_+ \), thanks to the morphism \( \phi: \mathbb{R}_+ \times \mathbb{R}_+^\times \cup \{0\} \times \{0\} \times \mathbb{R} \rightarrow \mathbb{R}_+ \times \mathbb{R}_+^\times \) such that

\[
\begin{align*}
(y_1, y_2) & \mapsto (y_2, \frac{y_1}{y_2}) \text{ if } y_1, y_2 \neq 0; \\
(0, 0, \lambda) & \mapsto (0, e^\lambda).
\end{align*}
\]
Hence $C^*_r(\Gamma([0,1])) \cong C^*_r(\mathbb{R}_+ \times \mathbb{R}_+^*) \cong C_0(\mathbb{R}_+) \times \mathbb{R}_+^*$ and, by the Connes-Thom isomorphism, $K_* (C_0(\mathbb{R}_+) \times \mathbb{R}_+^*) \cong K_{-1} (C_0(\mathbb{R}_+^*)) = 0$.

**Remark 1.13.** By [23 Proposition 3.5], $\Gamma(\mathbb{R}_+, \{0\})$ is amenable. Then we have the following short exact sequence

$$
0 \longrightarrow C^*_r(\mathbb{R}_+^* \times \mathbb{R}_+^*) \longrightarrow C^*_r(\Gamma(\mathbb{R}_+, \{0\})) \longrightarrow C^*_r(\mathbb{R}) \longrightarrow 0,
$$

it is semi-split, the associate boundary map in KK-theory is an isomorphism by Lemma 1.12 and it given by the Bott periodicity.

As we noticed in Remark 1.4, the restriction to a saturated closed subset of $G^{(0)}$ gives an exact sequence of full $C^*$-algebras. But in general this fact is not true for the reduced groupoid $C^*$-algebras. We are going to prove that this is the case in the situations that we will encounter later. Moreover we are going to prove that in the those situations we have the completely positive lifting property. By [31 Theorem 1.1], this implies that the boundary map associated to these exact sequences is an element of KK-theory.

**Lemma 1.14.** Let $X$ be any smooth manifold and let $H \rightrightarrows X$ be a Lie groupoid. Consider the groupoid $G = H \times (-1,1) \times (-1,1) \rightrightarrows X \times (-1,1)$, the $b$-calculus groupoid associated to the restriction of $G$ to $X \times \{0\}$, denoted by $G(X \times \{0\}, X \times \{0\})$. Then we have the following semi-split exact sequence of reduced $C^*$-algebras

$$
0 \longrightarrow C^*_r(H \times (0,1) \times (0,1)) \longrightarrow C^*_r(G(X \times \{0\}, X \times \{0\})) \longrightarrow C^*_r(H \times \mathbb{R}) \longrightarrow 0
$$

and the boundary map of the long exact sequence of KK-groups associated to it is an isomorphism.

**Proof.** By Remark 1.10 the Monthubert groupoid associated to $G$ is given by $H \times \Gamma([0,1), \{0\})$, we have that $C^*_r(H \times \Gamma([0,1), \{0\})) \cong C^*_r(H) \otimes C^*_r(\Gamma([0,1), \{0\}))$. In particular the amenability of $\Gamma([0,1), \{0\})$ implies that the following sequence

$$
0 \longrightarrow C^*_r(H) \otimes C^*_r((0,1) \times (0,1)) \longrightarrow C^*_r(H) \otimes C^*_r(\Gamma([0,1), \{0\})) \longrightarrow C^*_r(H) \otimes C^*_r(\mathbb{R}) \longrightarrow 0
$$

is exact.

The Lemma 1.12 implies that $K_*(C^*_r(G(X \times \{0\}, X \times \{0\}))) = 0$, hence the result follows.

**Remark 1.15.** The boundary map of the previous Lemma is nothing but the inverse of the Bott element.

**Lemma 1.16.** The restriction to the boundary induces the following exact sequence of reduced $C^*$-algebras

$$
0 \longrightarrow C^*_r(G|_{\partial X}) \longrightarrow C^*_r(G(X, \partial X)) \overset{ev_{\partial X}}{\longrightarrow} C^*_r(G(\partial X) \times \mathbb{R}) \longrightarrow 0. \tag{1.5}
$$

Moreover this exact sequence is semisplit.

**Proof.** By Remark 1.10 there is an open neighbourhood $U$ of $\partial X$ such that the restriction of $G(X, \partial X)$ to $U$ is isomorphic to $G(\partial X) \times \Gamma([0,1), \{0\})$ and, since $C^*_r(\Gamma([0,1), \{0\}))$ is nuclear, it follows that $C^*_r(G(\partial X) \times \Gamma([0,1), \{0\})) \cong C^*_r(G(\partial X)) \otimes C^*_r(\Gamma([0,1), \{0\}))$.

We have the following commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & C^*_r(G(\partial X)) \otimes C^*_r((0,1) \times (0,1)) \longrightarrow C^*_r(G(\partial X) \otimes C^*_r(\Gamma([0,1), \{0\})) \overset{ev_{\partial X}}{\longrightarrow} C^*_r(G(\partial X) \times \mathbb{R}) \longrightarrow 0 \\
0 & \longrightarrow & C^*_r(G|_{\partial X}) \downarrow \cong & C^*_r(G(X, \partial X)) & \downarrow \cong & C^*_r(G(\partial X) \times \mathbb{R}) \longrightarrow 0
\end{array}
$$

where the top row is exact thanks to the amenability of $\Gamma([0,1), \{0\})$ and the vertical arrows are inclusions of algebras.

Let $\alpha, 1 - \alpha$ be a partition of the unity associated to the open cover $\{\partial X \times [0,1), \mathcal{X}\}$ of $X$. Let $\xi \in C^*_r(G(X, \partial X))$ be such that $ev_{\partial X}(\xi) = 0$. Observe that
• $\alpha \xi \alpha$ belongs to $C^*_r(G_{|\partial X}) \otimes C^*_r(\Gamma([0,1], \{0\}))$;

• $\xi - \alpha \xi \alpha = (1-\alpha)\xi \alpha + \xi (1-\alpha)$ belongs to the ideal $C^*_r(G_{|\hat{X}})$, because $(1-\alpha)$ is supported on the saturated submanifold $\hat{X}$.

Since the top row is exact, $\alpha \xi \alpha$ belongs to $C^*_r(G_{|\partial X}) \otimes C^*_r((0,1) \times (0,1))$. Consequently we have that $x \in C^*_r(G_{|\hat{X}})$. This proves the exactness of (1.5).

Finally let $s$ be a completely positive section for the top row in the diagram. Then $\eta \to \alpha s(\eta)\alpha$ is a completely positive section for $ev_{\partial X}$.

We need the following technical result ([6, Lemma 2.2]).

Lemma 1.17. Let $I_1$ and $I_2$ be two ideals in a separable $C^*$-algebra $A$. Let $I$ be the intersection of $I_1$ and $I_2$. If the quotient maps $q_i : A \to A/I_i$, for $i = 1, 2$, have completely positive sections, then the quotient map $\pi : A \to A/I$ has a completely positive section.

Lemma 1.18. The restriction morphism

$$ev_{X,0} : C^*_r\left(G(X, \partial X)_{ad}^F\right) \to C^*_r(\tau_{nc}G(X, \partial X))$$

induces a $KK$-equivalence.

Proof. First we have to prove that the following sequence

$$0 \longrightarrow C^*_r(G_{|\hat{X}} \times (0,1]) \longrightarrow C^*_r\left(G(X, \partial X)_{ad}^F\right) \xrightarrow{ev_{X,0}} C^*_r(\tau_{nc}G(X, \partial X)) \longrightarrow 0 .$$

is exact and semi-split.

If $\xi \in C^*_r\left(G(X, \partial X)_{ad}^F\right)$ is such that $ev_{\partial X}(\xi) = 0$, then in particular $ev_0(\xi) = 0$ in $C^*_r(G(X, \partial X))$. But we know that the sequence associated to $ev_0$ is exact. Then $\xi$ belongs to $C^*_r(G(X, \partial X) \times (0,1])$. Moreover, by hypothesis, the restriction to the boundary of $\xi$, as element of $C^*_r(G(X, \partial X) \times (0,1])$, is zero. We can use Lemma 1.16 to prove that $\xi$ belongs to $C^*_r(G_{|\hat{X}} \times (0,1])$ and then that (1.7) is exact.

Let $A$ denote the $C^*$-algebra $C^*_r\left(G(X, \partial X)_{ad}^F\right)$. To prove the fact that (1.7) is semi-split we observe that the ideal $I = \ker ev_{X,0}$ is the intersection of the two ideals $I_1 = \ker ev_{\partial X}$ and $I_2 = \ker ev_0$. By Lemma 1.17 $ev_{X,0}$ has a completely positive section.

1.5 Index as deformation

Let $G$ be a smooth deformation groupoid, namely a Lie groupoid of the following kind:

$$G = G_1 \times \{0\} \cup G_2 \times [0,1] \Rightarrow G^{(0)} = M \times [0,1].$$

One can consider the saturated open subset $M \times [0,1]$ of $G^{(0)}$. Using the isomorphisms

$$C^*(G|_{M \times [0,1]}) \simeq C^*(G_2) \otimes C_0([0,1])$$

and $C^*(G|_{M \times \{0\}}) \simeq C^*(G_1)$, we obtain the following exact sequence of $C^*$-algebras:

$$0 \longrightarrow C^*(G_2) \otimes C_0([0,1]) \xrightarrow{i} C^*(G) \xrightarrow{ev_0} C^*(G_1) \longrightarrow 0 \quad (1.8)$$

where $i$ is the inclusion map and $ev_0$ is the evaluation map at 0.

We assume now the exact sequence admits a completely positive section. Since the $C^*$-algebra $C^*(G_2) \otimes C_0([0,1])$ is contractible, the long exact sequence in KK-theory shows that the group homomorphism $KK(A, C^*_r(G)) \to KK(A, C^*_r(G_1))$, given by the Kasparov product with the element $[ev_0]$, is an isomorphism for each $C^*$-algebra $A$.

In particular with $A = C^*(G)$ we get that $[ev_0]$ is invertible in KK-theory: there is an element $[ev_0]^{-1}$ in $KK(C^*_r(G_1), C^*_r(G))$ such that $[ev_0] \otimes [ev_0]^{-1} = 1_{C^*(G)}$ and $[ev_0]^{-1} \otimes [ev_0] = 1_{C^*_r(G_1)}$.

Let $ev_1 : C^*_r(G) \to C^*_r(G_2)$ be the evaluation map at 1 and $[ev_1]$ the corresponding element of $KK(C^*_r(G), C^*_r(G_2))$. 

10
Definition 1.19. The KK-element associated to the deformation groupoid $G$ is defined by:

$$\partial_{G} = [\text{ev}_{0}]^{-1} \otimes [\text{ev}_{1}] \in KK(C^{*}(G_{1}), C^{*}(G_{2})) .$$

Remark 1.20. If the sequence (1.8) is still exact when we consider the reduced groupoid $C^{*}$-algebras instead of the full ones, all that we said is still true in the reduced setting. This happens for instance when $G_{1}$ is amenable and, as we are going to see, the adiabatic deformation is an example of this case.

Let $G \rightrightarrows X$ be a Lie groupoid and consider its adiabatic deformation $G_{ad} \rightrightarrows X \times [0, 1].$ Recall that it is of the form

$$\mathfrak{A}(G) \times \{0\} \sqcup G \times (0, 1]$$

and that $C^{*}(\mathfrak{A}(G)) \cong C_{0}(\mathfrak{A}^{*}(G))$ and then, for this case, we can consider the reduced groupoid $C^{*}$-algebras. This is a particular case of a smooth deformation groupoid. Therefore we can associate to it a KK-element as in definition 1.19.

Definition 1.21. We will denote by

$$\text{Ind}_{G} \in KK(C_{r}^{*}(\mathfrak{A}(G)), C_{r}^{*}(G))$$

the KK-element $\partial_{G_{ad}}$ and we will call the adiabatic $G$-index the homomorphism

$$K_{*}(C_{0}(\mathfrak{A}^{*}(G))) \to K_{*}(C_{r}^{*}(G)) ,$$

given by the Kasparov product with $\text{Ind}_{G}$.

Indeed this homomorphism corresponds, up to Bott periodicity, to the boundary map associated to the exact sequence (1.3).

Remark 1.22. In [24] the authors prove that the adiabatic $G$-index and the classical analytic index given by the pseudodifferential extension coincide.

2 Adiabatic groupoid and wrong-way functoriality

We will actually extend this construction to the context of a general Lie groupoid $G \rightrightarrows X$ and its pull-back along a transverse map.

2.1 The pull-back of a groupoid

Here we recall the pull-back construction for Lie groupoids. Let $G \rightrightarrows X$ be a Lie groupoid and let $\varphi: Y \to X$ be a transverse map with respect to $G$. This means that $d\varphi(T_{y}Y) + q(\mathfrak{A}(\varphi(y))(G)) = T_{\varphi(y)}X,$ where $q: \mathfrak{A}(G) \to TX$ is the anchor map of the Lie algebroid.

Definition 2.1. Let us fix the following notations:

- $G_{\varphi} = \{(\gamma, y) \in G \times Y | \varphi(y) = s(\gamma)\}$;
- $G^{r}_{\varphi} = \{(y, \gamma) \in Y \times G | \varphi(y) = r(\gamma)\}$;
- $G^{c}_{\varphi} = \{(y_{1}, \gamma, y_{2}) \in Y \times G \times Y | \varphi(y_{1}) = r(\gamma), \varphi(y_{2}) = s(\gamma)\}$.

Remark 2.2. The source and the target map for $G^{c}_{\varphi}$ are given by $s(y_{1}, \gamma, y_{2}) = y_{2}$ and $r(y_{1}, \gamma, y_{2}) = y_{1}$ respectively. Moreover $(y_{1}, \gamma, y_{2})^{-1} = (y_{2}, \gamma^{-1}, y_{1})$ and $(y_{1}, \gamma, y_{2}) \cdot (y_{2}, \gamma', y_{3}) = (y_{1}, \gamma \cdot \gamma', y_{3}).$

Since $r, s: G \to X$ are submersions, $G_{\varphi}$ and $G^{r}_{\varphi}$ are submanifolds of $G \times Y$ and $Y \times G$ respectively. We are going to prove that $G^{c}_{\varphi}$ is a smooth manifold. The space $G_{\varphi}$ in Definition 2.1 is given by the following pull-back

$$\begin{array}{ccc}
G_{\varphi} & \xrightarrow{\varphi} & G \\
\downarrow{p} & & \downarrow{s} \\
Y & \xrightarrow{\varphi} & X
\end{array}$$
and one can see that \( p \) is a surjective submersion, because \( s \) is so.

We want to prove that \( k = r \circ \varphi \) is a smooth submersion. Let \( (\gamma_0, y_0) \in G_\varphi \) such that \( \gamma_0 \) is a unit of \( G \). We define the following inclusion

- \( i: G_{s(\gamma_0)} \to G_\varphi \), \( i: \gamma \mapsto (\gamma, y_0) \) and put \( \delta = k \circ i \);
- \( j: Y \to G_\varphi \) is such that \( j: y \mapsto (id_{\varphi(y_0)}, y) \) and put \( \varepsilon = k \circ j \).

Notice that \( s(\gamma_0) = \alpha(\gamma_0) = \beta(y_0) = \psi(y_0) \) and then, by transversality, it turns out that

\[
dk_{(\gamma_0, y_0)}(di(T_{\gamma_0}G_{s(\gamma_0)}) + dj(T_{y_0}Y)) = d\delta(T_{\gamma_0}G_{s(\gamma_0)}) + d\varepsilon(T_{y_0}Y) = q(\mathfrak{A}_{\psi(y_0)} (G)) + d\varphi(T_{y_0}Y) = T_{\varphi(y_0)} X,
\]

hence that \( d_{k(\gamma_0, y_0)} k \) is onto.

Now let us consider \( (\gamma_1, y_1) \in G_\varphi \), where \( \gamma_1 \) is not necessarily a unit. Construct the following pull-back

\[
\begin{array}{ccc}
G_\varphi^{(2)} & \to & G \\
p_1 && s \\
p_2 & \downarrow & \downarrow s \\
G_\varphi & \to & X
\end{array}
\]

where \( G_\varphi^{(2)} = \{(\gamma, \gamma', y) \in G \times G_{\varphi} | s(\gamma) = r(\gamma')\} \), \( p_1: (\gamma, \gamma', y) \mapsto \gamma \) and \( p_2: (\gamma, \gamma', y) \mapsto (\gamma', y) \).

We have that \( p_2 \) is a submersion, because \( s \) is so. Moreover, at the point \( z = (\gamma_1, \varphi(y_1), y_1) \), \( d_{z} p_1 \) is onto, since \( d_z k \) is so at \( w = (\varphi(y_1), y_1) \).

Let \((m, id): G_\varphi^{(2)} \to U_\varphi\) be the map such that \((m, id): (\gamma, \gamma', y) \mapsto (\gamma \gamma', y)\). Then we have that \( r \circ p_1 = k \circ (m, id) \). But at \((\gamma_1, \varphi(y_1), y_1)\) \( r \circ p_2 \) is a submersion, hence so is \( k \) at \((m, id)(\gamma_1, \varphi(y_1), y_1) = (\gamma_1, y_1)\).

We have proven that the map \( k: (\gamma, y) \mapsto r(\gamma) \) is a submersion because of transversality. Then by the following pull-back diagram

\[
\begin{array}{ccc}
G_\varphi^{(2)} & \to & G_\varphi \\
p_1 & \downarrow & \downarrow k \\
Y & \varphi & X
\end{array}
\]

it follows that \( G_\varphi^{(2)} \) is a manifold. Moreover \( G_\varphi^{(2)} \equiv Y \) is a Lie groupoid that we will call the pull-back groupoid of \( G \) by \( \varphi \).

One can easily show that

\[
\mathfrak{A}(G_\varphi^{(2)}) \simeq \{ (\xi, \eta) \in TY \times \mathfrak{A}(G) | d\varphi(\xi) = q(\eta) \},
\]

where \( q \) is the anchor map of \( \mathfrak{A}(G) \). On the other hand the anchor map of \( \mathfrak{A}(G_\varphi^{(2)}) \) is the projection on \( TY \).

Lemma 2.3. Let \( \Phi: Y \times [0, 1] \to X \) be such that

1. \( \varphi_t := \Phi_{Y \times \{t\}}: Y \to X \) is transverse with respect to \( G \) for all \( t \in [0, 1] \);
2. for all fixed \( y_0 \in Y \) the set \( \{ \Phi(y_0, t), t \in [0, 1] \} \) is contained in an orbit of \( G \).

Then there is an isomorphism \( \alpha(\varphi_t): G_{\varphi_0}^{(2)} \to G_{\varphi_1}^{(2)} \).

Proof. By (1) \( G_{\varphi_0}^{(2)} \) is a Lie groupoid and its Lie algebroid is given by

\[
\mathfrak{A}G_{\varphi_0}^{(2)} = \{(U, V) \in T(Y \times [0, 1]) \times \mathfrak{A}G | d\Phi(U) = dr(V) \}.
\]

Let \( \partial \) bethe vector field that differentiates along the \( t \)-direction. By (2), for all fixed \( y_0 \in Y \), the integral curves of \( \partial \), namely \( \{ (y_0, t), t \in [0, 1] \} \), are contained in the orbits of \( G_{\varphi_0}^{(2)} \). Since \( Y \) is paracompact, there exists a locally finite cover \( \{ U_j \}_{j \in J} \) of \( Y \times [0, 1] \) such that for each
Lemma 2.4. Let \( \varphi : Y \rightarrow X \) and \( \psi : Y \rightarrow X \) be as in Lemma 2.3 and such that \( \varphi_1 = \psi_0 \).
Denote the concatenation of the paths \( \varphi_1 \) and \( \psi_1 \) by \( \psi_0 \). Then
\[
\alpha(\psi_0) \circ \alpha(\varphi_1) = \alpha(\psi_0 \circ \varphi_1).
\]

Proof. It is clear from the construction of \( \alpha(\psi_0) \) and \( \alpha(\varphi_1) \).

Remark 2.5. Because of this decomposition of \( G_{(G_x^\varphi)} \), we will keep the source and target notions for \( G^\varphi \) and \( G_\varphi \), though they are not groupoids.

Let \( p_Y \) be the projection given by the restriction to \( L^X \), and \( p_X \) be the projection given by the restriction to \( L^X \), they are in the multipliers algebra of \( C^*_r(L) \). Then \( E_\varphi = p_Y C^*_r(L) p_X = C^*_r(G^\varphi) \) is the \( C^*_r(G^\varphi) \)-\( C^*_r(G) \)-bimodule we were searching for.

Definition 2.6. Denote by \( \mu_\varphi \) the class of \( E_\varphi \) in \( KK \left( C^*_r(G^\varphi), C^*_r(G) \right) \).

Remark 2.7. The \( C^*_r(G^\varphi) \)-\( G^\varphi \)-valued inner product on \( E \) is given by \( \langle x, y \rangle_{C^*_r(G^\varphi)} \) = \( x^* y \) and the \( C^*_r(G) \)-valued one is given by \( \langle x, y \rangle_{C^*_r(G)} = x^* y \). It is clear that, if the image of \( \varphi \) intersects with all the orbits of \( G \), then \( E_\varphi \) is full with respect to the \( C^*_r(G) \)-valued inner product. Then \( \mu_\varphi \) is a Morita equivalence, whose inverse \( \mu^{-1}_\varphi \) is given by the bimodule \( F = p_X C^*_r(L) p_Y \).

Proposition 2.8. If \( \varphi : Y \rightarrow X \) is transverse with respect to \( G \equiv X \) and \( \psi : Z \rightarrow Y \) is transverse with respect to \( G^\varphi \equiv Y \), then
\[
E_{\varphi \circ \psi} = E_\varphi \otimes_{C^*_r(G^\varphi)} E_\psi.
\]

Proof. Let the groupoid \( H \equiv Z \sqcup Y \sqcup X \) be the pull-back of \( G \equiv X \) by the map \( \varphi \circ \psi \sqcup \varphi \sqcup id_X : Z \sqcup Y \sqcup X \rightarrow X \). We can see \( C^*_r(H) \) as \( 3 \times 3 \) matrices of the following sort
\[
\begin{pmatrix}
C^*_r(G^\varphi) & C^*_r(G^\varphi) & C^*_r(G^\varphi) \\
C^*_r(G^\varphi) & C^*_r(G^\varphi) & C^*_r(G^\varphi) \\
C^*_r(G^\varphi) & C^*_r(G^\varphi) & C^*_r(G^\varphi)
\end{pmatrix}
\]
and that \( E_{\varphi \circ \psi} = C^*_r(G^\varphi) \), \( E_\psi = C^*_r(G^\psi) \) \( \cong C^*_r(G^\varphi) \) and \( E_{\varphi} = C^*_r(G^\varphi) \) sit concretely in \( C^*_r(H) \).

Now one can see that \( E_\psi \otimes_{C^*_r(G^\varphi)} E_\varphi \) is nothing else than \( C^*_r(G^\varphi) \cdot C^*_r(G^\varphi) \), that is obviously equal to \( E_{\varphi \circ \psi} \).

Proposition 2.9. If \( \varphi : Y \rightarrow X \) is transverse with respect to \( G \equiv X \) and \( Z \) is a saturated and locally closed submanifold of \( X \) and \( \varphi' = \varphi|_{\varphi^{-1}(Z)} \), then the following equality holds
\[
\varphi^{-1}(Z) \otimes \mu_{\varphi'} = \mu_\varphi \otimes \epsilon_Z
\]
in \( KK \left( C^*_r \left( (G^\varphi) \right), C^*_r \left( G|_Z \right) \right) \).
Proof. First observe that \((G^c_\varphi)|_{\varphi^{-1}(Z)} = (G|_Z)^c_\varphi\). Then, considering the C*-algebra of \(L = G^{\phi,\text{id}_X} \to Y \sqcup X\) a 2 \times 2 matrix algebra, the equality means just that restricting \(L\) to \(W = \varphi^{-1}(Z) \sqcup Z\) and then taking the top left corner of \(C^*_r(L_W)\) is the same of taking the top left corner of \(C^*_r(L)\) an then restricting to \(W\).

**Remark 2.10.** It is worth recalling that all that we did in this section for the reduced groupoid C*-algebras is a fortiori true for the full groupoid C*-algebras.

### 2.2 Wrong-way functoriality for submersions

Let \(G \cong G^{(0)}\) be a Lie groupoid. Put \(X = G^{(0)}\) and let \(\varphi: Y \to X\) be a smooth map transverse with respect to \(G\). Consider the adiabatic groupoid of \(G\)

\[
G_{ad} = \mathfrak{X}G \times \{0\} \sqcup G \times \{0,1\} \rightrightarrows X \times \{0,1\}.
\]

Let us recall that the Lie algebroid \(\mathfrak{A}(G_{ad})\) is non canonically isomorphic to \(\mathfrak{A}(G) \times [0,1]\) and the anchor map of the adiabatic algebroid is given by \(q_{ad}: (\eta,t) \mapsto (t \cdot q(\eta),t)\), see for instance [23 Section 2.2].

We will need the pull-back along \(\varphi := \varphi \times \text{id}_{\{0,1\}}\) of the adiabatic groupoid, but the fact that \(\varphi\) is transverse with respect to \(G\) does not imply that \(\varphi\) is transverse with respect to \(G_{ad}\). Indeed at 0 the anchor map of \(\mathfrak{A}(G) \times [0,1]\) is zero. Thus \(\varphi\) is transverse if and only if \(\varphi\) is a submersion. So let \(\varphi\) be a submersion and the pull-back groupoid of \(G_{ad}\) along \(\varphi\), given by

\[
(G_{ad})^\varphi = \{((y_1,t),\gamma,(y_2,t)) \in (Y \times [0,1]) \times G_{ad} \times (Y \times [0,1]) \mid r(\gamma) = \varphi(y_1,t), s(\gamma) = \varphi(y_2,t)\},
\]

is a smooth manifold.

**Lemma 2.11.** The Lie algebroid of \((G_{ad})^\varphi\), as vector bundle, is non canonically isomorphic to \(\pi^*(\ker(d\varphi) \oplus \varphi^*\mathfrak{A}(G))\), where \(\pi: Y \times [0,1] \to Y\) is the projection.

**Proof.** We have that

\[
\mathfrak{A}((G_{ad})^\varphi) = \{((\xi,t), (\eta,t)) \in (TY \times [0,1]) \times \mathfrak{A}(G_{ad}) \mid d\varphi(\xi) = t \cdot q(\eta)\}.
\]

We deduce that, for \(t \neq 0\), the fiber of \(\mathfrak{A}((G_{ad})^\varphi)\) on \((x,t)\) is given by the following pull back

\[
\begin{array}{ccc}
T_y\mathfrak{X} & \xrightarrow{\iota_\varphi} & \mathfrak{A}(\varphi(y)) \times \{0\} \\
\downarrow & & \downarrow \iota_\varphi \\
T_y\mathfrak{X} & \xrightarrow{d\varphi} & \mathfrak{A}(\varphi(y)) \times \{1\}
\end{array}
\]

and for \(t = 0\) the fiber on \((y,0)\) is \(\ker(d\varphi)_y \oplus \mathfrak{A}(\varphi(y))\). Thanks to the following exact sequence

\[
0 \to \ker(d\varphi)_y \to T_y\mathfrak{X} \xrightarrow{d\varphi} \mathfrak{A}(\varphi(y)) \xrightarrow{\iota_\varphi} \mathfrak{A}(\varphi(y)) \to 0 ,
\]

we obtain desired isomorphism.

We are going to construct an element \(\varphi_{ad}^t \in KK(C^*_r((G_{ad})^\varphi), C^*_r(G_{ad}))\) associated to \(\varphi\) by means of a deformation groupoid, as in the Section 1.5. It is given by the following double adiabatic deformation

\[
((G_{ad})^\varphi)_{ad} \to Y \times [0,1] \times [0,1].
\]

Let us give an explicit picture of this groupoid. We fix the variables of the two deformations:

- in the horizontal direction of the square we have the parameter \(t\) of the first adiabatic deformation;
- in the vertical direction of the square we have the parameter \(u\) of the second adiabatic deformation, performed after the pull-back construction.
Then we obtain a groupoid, let us call it $H$, with objects set $Y \times [0, 1] \times [0, 1]_u$, such that

- $H$ restricted to $\{u = c\}$, for any $c \in (0, 1]$, is equal to $(G_{ad})^\ell_{\phi}$, the pull-back of the adiabatic deformation;
- $H$ restricted to $\{t = c'\}$, for any $c' \in (0, 1]$, is equal to $(G^\phi_{\phi'})_{ad}$, the adiabatic groupoid deformation of the pull-back;
- $H$ restricted to the $t$-axis, i.e. to $\{u = 0\}$, is the Lie algebroid of $(G_{ad})^\ell_{\phi}$, that we have calculated above;
- $H$ restricted to $\{t = 0\}$ is equal to the adiabatic deformation of the Lie algebroid $(\mathfrak{g}(G))^\phi_{\phi'}$.

**Definition 2.12.** Let $\mathcal{L}_\phi$ denote the reduced groupoid C*-algebra $C^*_r((G_{ad})^\phi_{\phi'})$ and let $\mathcal{L}^o$ denote the C*-algebra $\mathcal{L}^o_{\{t,u\neq(1,1)\}}$. We will drop the subscript $\phi$ when the context does not create ambiguity.

**Lemma 2.13.** The evaluation morphisms at $t = 1$, $e^1_t : \mathcal{L} \to \mathcal{L}^o_{t=1}$ and $e^1 : \mathcal{L}^o \to \mathcal{L}^o_{t=1}$ induce KK-equivalences.

**Proof.** Let us prove it for $\mathcal{L}^o$, the proof for $\mathcal{L}$ is similar. Consider the following exact sequence

$$
0 \to \mathcal{L}^o_{(t\neq 1)} \to \mathcal{L}^o \to \mathcal{L}^o_{t=1} \to 0,
$$

the evaluation at $t = 1$ has a completely positive section: since $\mathcal{L}^o_{(0,1)}$ contains as ideal $\mathcal{L}^o_{t=1} \times C_0(0,1]$, the map $\xi \mapsto t \cdot \xi$ does the job. Hence this exact sequence is semi-split and it is sufficient to prove the K-contractibility of $\mathcal{L}^o_{t=1}$.

But let us point out that the evaluation map at $u = 0$ from $\mathcal{L}^o_{\{t\neq 1\}}$ to $\mathcal{L}^o_{\{u=0,t\neq 1\}}$ is a KK-equivalence: this follows from the KK-equivalence between $C^*_r(G_{ad})$ and $C^*_r(\mathfrak{g}(G))$ in the particular case of $G = (G_{ad})^\phi_{\phi'}$.

Hence we have to prove the K-contractibility of $\mathcal{L}^o_{\{u=0,t\neq 1\}}$. But, by [2.2], $\mathcal{L}^o_{\{u=0,t\neq 1\}}$ is non canonically isomorphic to $C^*_r(\mathfrak{g}(G^\phi_{\phi'})) \otimes C[0,1)$, that is clearly K-contractible.

Let $[c_0] \in KK(\mathcal{L}^o, \mathcal{L}^o_{\{t=1\}})$ denote the KK-equivalence stated in Lemma 2.13 and let $e^1_t : \mathcal{L}^o \to \mathcal{L}^o_{\{t=1\}}$ be the evaluation map at $u = 1$.

**Definition 2.14.** Let $G \rightrightarrows X$ be a Lie groupoid and let $\varphi : Y \to X$ be a smooth submersion between smooth manifolds. Hence we can define the lower shriek map $\varphi^\ad$ as the element

$$
[e^1_{t}]^{-1} \otimes_{\mathcal{L}^o} [c_0] \otimes_{\mathcal{L}^o_{\{t=1\}}} \mu_\varphi \in KK(C^*_r((G^\phi^o_{\varphi^o})_{ad}), C^*_r((G^\phi^o_{\varphi^o})_{ad})),
$$

where $\mu_\varphi$ is as in Definition 2.9.

**Lemma 2.15.** Let $\varphi_1 : Y \to X$ be as in Lemma 2.3. If $\varphi_1 : Y \to X$ is a submersion for all $t \in [0,1]$, $(\varphi^1_0)^{\ad}$ corresponds with $(\varphi^1_1)^{\ad}$ through the isomorphism of adiabatic groupoids induced by (2.3).

**Proof.** Since $\varphi_1 : Y \to X$ is a submersion for all $t \in [0,1]$, it follows that $\varphi : Y \times [0,1] \to X \times [0,1]$ is transverse with respect to $G_{ad}$ for all $t \in [0,1]$. Then applying Lemma 2.3, we have an isomorphism $\alpha(\varphi_1)$ between $(G_{ad})^\phi_{\phi^o}$ and $(G_{ad})^\phi_{\varphi}$, consequently $\mathcal{L}^o_{\varphi^o}$ and $\mathcal{L}^o_{\varphi}$ are isomorphic. Then clearly we have that

$$(\varphi^1_0)^{\ad} = [\alpha(\varphi_1)] \otimes (\varphi^1_1)^{\ad}.$$  

**Lemma 2.16.** Let $\varphi : Y \to X$ be as in Lemma 2.3 and let $\psi : Z \to Y$ be a submersion. Then we have the following equality

$$
\psi^\ad \otimes [\alpha(\varphi^1_1)] = [\alpha(\varphi^1_1) \otimes \psi^\ad] \in KK(C^*_r((G^\phi^o_{\varphi^o})_{ad}), C^*_r((G^\phi^o_{\varphi^o})_{ad})),
$$

where $\psi^\ad$ is an element of $KK(C^*_r((G^\phi^o_{\varphi^o})_{ad}), C^*_r((G^\phi^o_{\varphi^o})_{ad}))$ on the right side and it is an element of $KK(C^*_r((G^\phi^o_{\varphi^o})_{ad}), C^*_r((G^\phi^o_{\varphi^o})_{ad}))$ on the left side.
Proof. It is enough observing that the isomorphism \( \alpha(\varphi_t) \colon (G_{\varphi_t}^g)_{ad} \to (G_{\varphi_t}^g)_{ad} \) induces an isomorphism \(((G_{\varphi_t}^g)_{ad})^\psi_{ad} \to ((G_{\varphi_t}^g)_{ad})^\psi_{ad} \). This isomorphism restricts to an isomorphism \(((G_{\varphi_t}^g)_{ad})^\psi_{ad} \to ((G_{\varphi_t}^g)_{ad})^\psi_{ad} \) that is nothing but \( \alpha(\varphi_t \circ \psi) \colon (G_{\varphi_t \circ \psi}^g)_{ad} \to (G_{\varphi_t \circ \psi}^g)_{ad} \). 

Now, by the definition of \( \psi_{ad} \), the equality \( \ref{eq:2} \) is clear.  

The following result is the Thom isomorphism in the context of the adiabatic deformation groupoid.

**Proposition 2.17.** Let \( p \colon E \to Y \) be a vector bundle and let \( G \rightrightarrows Y \) be a Lie groupoid. Then \( p^ad \in KK(C^*_r((G^g)_p), C^*_r(G^g_{ad})) \) is a KK-equivalence.

Proof. Since \( p \) is surjective \( \mu_p \) is a Morita equivalence between \( C^*_r((G^g)_p) \) and \( C^*_r(G^g_{ad}) \). Because of the definition of \( p^ad \), it is sufficient to show that the evaluation at \( u = 1 \) gives a KK-equivalence in \( KK(L, L_{(u=1)}) \). But this is equivalent to prove that the kernel of the evaluation at \( u = 1 \), \( L_{(u\neq1)} \), is KK-contractive. This turns to be equivalent to KK-contractibility of \( L_{(t=0, u\neq1)} \) because of the following exact sequence

\[
0 \longrightarrow L_{(t\neq0, u\neq1)} \longrightarrow L_{(u\neq1)} \longrightarrow L_{(t=0, u\neq1)} \longrightarrow 0,
\]

and KK-contractibility of \( L_{(t\neq0, u\neq1)} \simeq (G^p)^0 \otimes C_0(0, 1) \).

But \( L_{(t=0, u\neq1)} \) is equal to

\[
(\mathfrak{A}(G))^0_p = \{(\eta_1, \xi, \eta_2) \in E \times \mathfrak{A}(G) \times E | \, p(\eta_1) = r(\xi), p(\eta_2) = s(\xi)\}
\]

for \( u \neq 0 \) and

\[
\ker(dp) \oplus \mathfrak{A}(G)
\]

for \( u = 0 \), see \( \ref{eq:2} \). But in both case we have something that is isomorphic to \( E \oplus E \oplus \mathfrak{A}(G) \), because \( \ker(dp) \) is the vertical bundle of \( p^*E \). Then

\[
L_{(t=0, u\neq1)} \simeq C_0(E \oplus E \oplus \mathfrak{A}(G)) \otimes C_0[0, 1)
\]

is KK-contractive.

Now we want to check that the construction of the lower shriek element behaves well with respect to the composition of submersions.

**Proposition 2.18.** Let \( G \rightrightarrows Z \) be a Lie groupoid. Let \( f \colon Y \to X \) and \( g \colon X \to Z \) be two smooth submersions between smooth manifolds. Then we have that

\[
(g \circ f)^{ad} = f^{ad} \otimes g^ad \in KK(C^*_r((G^g)^{(g \circ f)}_{ad}), C^*_r(G^g_{ad})�).
\]

Proof. Consider the Lie groupoid \( K \) given by

\[
\left(([G^g_{ad}]^\psi_{ad})^f\right)_{ad} : Y \times [0, 1] \times [0, 1] \times [0, 1],
\]

where we set \( t, u, v \) as the parameters respectively of the first, the second and the third adiabatic deformation in the construction of the groupoid. For sake of clarity let us set some notations:

- \( H = Y \times ([0, 1] \times [0, 1] \times [0, 1]) \) is a cube without a point;
- \( T = \{ t = 1 \} \subset H, U = \{ u = 1 \} \subset H \) and \( V = \{ v = 1 \} \subset H \) are the right, the posterior and the top faces of the cube, respectively;

Restricting \( K \) to the previous faces and their shared edges, we recognize the following groupoids:

- \( K_T = \left(([G^g_{ad}]^\psi_{ad})^f\right)_{ad}, K_U = \left(([G^g_{ad}]^\psi_{ad})^f\right)_{ad}, K_V = \left(([G^g_{ad}]^\psi_{ad})^f\right)_{ad}; \)
Definition 2.19. Let 

\[ K_{T\cup U} = \left( (G_\phi^g)_{fg} \right)_{ad}, \quad K_{U\cap V} = \left( (G_{ad})_{f}^g \right)_{ad}, \quad K_{T\cap U} = \left( (G_\phi)_{fg} \right)_{ad}. \]

Using Lemma 2.13 we get the following KK-equivalences: 

\[ e_{T\cap U}^V, e_{T\cup U}^V, e_{T\cap U}^H \text{ and } e_{U\cap U}^H. \]

We have that \( f_{ad}^\phi \) is constructed through the groupoid \( K_T \) and it is equal to 

\[ (e_{T\cap U}^V)^{-1} \otimes e_{V\cap U}^V \otimes f^\phi. \]

Instead \( g_{ad}^\phi \) is constructed by means of the groupoid \( L = (G_{ad})^g_{ad} \) and it is equal to 

\[ (e_M^M)^{-1} \otimes e_X \otimes g^\phi, \]

where \( L_M = (G_{ad})^g_{ad}, L_N = (G_{ad})^g_{ad} \). Thus we have that 

\[ f_{ad}^\phi \otimes g_{ad}^\phi = (e_{T\cap U}^V)^{-1} \otimes e_{V\cap U}^V \otimes f^\phi \otimes (e_M^M)^{-1} \otimes e_X \otimes g^\phi. \]

Applying Proposition 2.9 we get the following equality 

\[ \mu_f \otimes (e_M^M)^{-1} \otimes e_X \otimes g^\phi = (e_{T\cap U}^V)^{-1} \otimes e_{V\cap U}^V \otimes \mu_f \otimes g^\phi \]

and, using Proposition 2.8 the term on the right side is equal to 

\[ (e_{T\cap U}^V)^{-1} \otimes e_{V\cap U}^V \otimes g^\phi f. \]

Then 

\[ f_{ad}^\phi \otimes g_{ad}^\phi = \]

\[ (e_{T\cap U}^V)^{-1} \otimes e_{V\cap U}^V \otimes (e_{T\cap U}^V)^{-1} \otimes e_{V\cap U}^V \otimes (e_{T\cap U}^V)^{-1} \otimes e_{V\cap U}^V \otimes \mu_g f = \]

\[ (e_{T\cap U}^V)^{-1} \otimes (e_{T\cap U}^V)^{-1} \otimes e_{V\cap U}^V \otimes \mu_g f = \]

\[ (e_{T\cap U}^V)^{-1} \otimes e_{V\cap U}^V \otimes \mu_g f = \]

\[ (g \circ f_{ad})^\phi. \]

\[ \square \]

2.3 Wrong-way functoriality for transverse maps

Let \( G \rightrightarrows X \) be a Lie groupoid and \( \varphi: Y \to X \) a smooth map that is transverse with respect to \( G \). Consider the following commutative square

\[
\begin{array}{ccc}
\varphi^*\mathfrak{A}G & \xrightarrow{\varphi^*} & G \\
\downarrow \scriptstyle{\varphi} & & \downarrow \scriptstyle{\varphi}
\\
Y & \xrightarrow{\varphi} & X
\end{array}
\]

where \( \varphi^* \) is defined as the composition of \( \varphi^*\mathfrak{A}G \to \mathfrak{A}G \) and the exponential map \( \exp: \mathfrak{A}G \to G \). Notice the following facts:

- \( p \) is a bundle projection, then by Lemma 2.17 \( p_{ad} \) is a KK-equivalence;
- one can prove that \( r \circ \varphi \) is a submersion as in Remark 2.2 and \( \{\psi_t: \xi \mapsto r \circ \varphi(t\xi)\}_{t \in [0,1]} \) is a path of transverse maps from \( \varphi \circ p = s \circ \varphi \) to \( r \circ \varphi \). Let \( \alpha(t) \) be the associated isomorphism between \( G_{\varphi^p}^{r\varphi} \) and \( G_{\varphi^p}^{r\varphi} \), as in 2.1.

Definition 2.19. Let \( \varphi: Y \to X \) be as above. Define the lower shriek map associated to \( \varphi \) as the element

\[ \varphi_{ad}^\phi := (p_{ad})^{-1} \otimes [\alpha(t)] \otimes (r \circ \varphi)^{ad} \in KK \left( C^*_r((G_{\varphi^p})^o), C^*_r(G_{ad}) \right) \]
Remark 2.20. If \( \varphi: Y \to X \) is a submersion, then the elements defined in Defined in 2.14 and 2.19 coincide. Indeed we have that \( \{ \psi_t : \xi \mapsto r \circ \varphi(t \xi) \}_{t \in [0,1]} \) is a path of submersions, then by Lemma 2.15 \( [\alpha(t) \varphi] \otimes (s \circ \varphi) \varphi = (s \circ \varphi) \varphi \). This element is nothing but \( (\varphi \circ p) \varphi \), that by Proposition 2.18 is equal to \( \varphi \varphi \). This proves that \( (p \varphi)^{-1} \otimes [\alpha(t) \varphi] \otimes (r \circ \varphi) \varphi = \varphi \varphi \), where the element on the left is the one defined in 2.14.

Now we can verify that our definition behaves well with respect to composition of transverse maps.

Proposition 2.21. Let \( G \rightrightarrows Z \) be a Lie groupoid. Let \( f: X \to Y \) and \( g: Y \to Z \) be two smooth maps between smooth manifolds such that \( g \) is transverse w.r.t. \( G \) and \( f \) is transverse w.r.t. \( G \). Then we have that

\[
(g \circ f) \varphi = f \varphi \otimes g \varphi \in KK \left( C^*(G^g), C^*(G^f) \right).
\]

Proof. Let \( E \) and \( F \) denote \( (g \circ f)^* \mathcal{G} \) and \( g^* \mathcal{G} \) respectively.

Consider the following diagram

\[
\begin{array}{ccc}
E \times_Y F & \xrightarrow{P} & F \\
\downarrow{k} & & \downarrow{g} \\
E & \xrightarrow{h} & Y \\
\downarrow{p} & & \downarrow{f} \\
X & & \\
\end{array}
\]

where \( P \) and \( Q \) are the obvious projections, \( h = r \circ f \), \( k = r \circ (g \circ f) \) and \( l = r \circ g \). It is commutative up to homotopy: let \( h_t \) be the homotopy between \( f \circ p \) and \( h \), let \( k_t \) be the homotopy between \( (g \circ f) \circ p \) and \( k \) and let \( l_t \) be the homotopy between \( g \circ q \) and \( l \). Moreover all the vertical arrows are vector bundle projections, then the lower shriek maps associated to them induce KK-equivalences.

Let \( f^\varphi = (p^\varphi)^{-1} \otimes [\alpha(h_t)] \otimes h^\varphi \) and \( g^\varphi = (q^\varphi)^{-1} \otimes [\alpha(l_t)] \otimes k^\varphi \) be as in Definition 2.19. Moreover let \( (g \circ f)^\varphi \) be given by \( (p^\varphi)^{-1} \otimes [\alpha(h_t)] \otimes k^\varphi \). Thanks to the following calculations

\[
\begin{align*}
(p^\varphi)^{-1} \otimes [\alpha(h_t)] & \otimes h^\varphi \otimes (q^\varphi)^{-1} \otimes [\alpha(l_t)] \otimes k^\varphi = \\
(p^\varphi)^{-1} \otimes [\alpha(h_t)] & \otimes (Q^\varphi)^{-1} \otimes P^\varphi \otimes [\alpha(l_t)] \otimes l^\varphi = \\
(p^\varphi)^{-1} \otimes (Q^\varphi)^{-1} & \otimes [\alpha(h_t \circ Q)] \otimes [\alpha(l_t \circ P)] \otimes P^\varphi \otimes l^\varphi = \\
(p^\varphi)^{-1} \otimes (Q^\varphi)^{-1} & \otimes [\alpha(h_t \circ Q)] \otimes [\alpha(l_t \circ P)] \otimes (l \circ P)^\varphi = \\
(p^\varphi)^{-1} \otimes [\alpha(k_t)] & \otimes (Q^\varphi)^{-1} \otimes (l \circ P)^\varphi = \\
(p^\varphi)^{-1} \otimes [\alpha(k_t)] & \otimes k^\varphi = \\
(g \circ f)^\varphi & \end{align*}
\]

we obtain the desired equality. We used: in the second line the fact that \( h \circ Q = q \circ P \) and that \( q \) and \( Q \) are vector bundle projections; in the third line Proposition 2.18; in the fifth line Lemma 2.14; in the fourth line Proposition 2.18; in the fifth line Lemma 2.14; one more time; finally in the seventh line Proposition 2.18.

Now we are going to state an other important property of this construction, that is its functoriality with respect to the restriction to open or closed sets. Before that let us notice the following facts: let \( \psi: Y \to X \) be transverse with respect to \( G \) and let \( X_1 \) be a closed and saturated submanifold of \( X \). Then, since \( X_1 \) is saturated, it follows that \( T_x X_1 = dr(\mathfrak{A}_x G) \) and
then that ψ and the inclusion of $X_1$ into $X$ are transverse. This implies that $Y_1 := \psi^{-1}(X_1)$ is a submanifold of $Y$; the transversality of $ψ$ with respect to $G$ means that $dψT_yY + dr\mathfrak{A}_ψ(y)G = T_ψ(y)X$. Consequently, if we consider the intersection with $T_ψ(y)X_1$ for $y \in Y_1$, we obtain the following equality $dψT_yY_1 + dr\mathfrak{A}_ψ(y)G = T_ψ(y)X_1$, namely that the restriction of $ψ$ to $Y_1$ is still transverse with respect to $G$.

**Proposition 2.22.** Let $G \rightrightarrows X$ be a Lie groupoid. And let $ψ: Y \to X$ a transverse map with respect to $G$. Let $X_1 \subset X$ be a closed and saturated submanifold and let $Y_1 = ψ^{-1}(X_1)$. Then we have the following commutative diagram

$$
\begin{array}{c}
\cdots \to K_* \left( C_r^*((G_ψ^0)_{ad})_{Y \setminus Y_1} \right) \xrightarrow{i} K_* \left( C_r^*((G_ψ^0)_{ad}) \right) \xrightarrow{j} K_* \left( C_r^*((G_ψ^0)_{ad})_{Y_1} \right) \to \cdots \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\cdots \to K_* \left( C_r^*((G_0)_{ad})_{X \setminus X_1} \right) \xrightarrow{i} K_* \left( C_r^*((G_0)_{ad}) \right) \xrightarrow{j} K_* \left( C_r^*((G_0)_{ad})_{X_1} \right) \to \cdots
\end{array}
$$

where $ψ'$ and $ψ''$ are the restriction of $ψ$ to $Y \setminus Y_1$ and $Y_1$ respectively. Actually all the arrows in the diagram are elements of $KK$-theory and the commutativity of the squares holds in the $KK$-theory framework.

**Proof.** First let us observe that it is sufficient to prove it when $ψ$ is a submersion. Let us prove the commutativity of the second square: we have to prove the equality of

$$[e_1^{-1} \otimes [e_1^*] \otimes \mu_ψ \otimes [ev_{X_1}]]$$

and

$$[ev_{Y_1}] \otimes [e_1^{-1} \otimes [e_1^*] \otimes \mu_ψ'']$$

in $KK \left( C_r^*((G_ψ^0)_{ad}), C_r^*((G_0)_{ad})_{X_1} \right)$. Here $e$ are evaluation maps in the setting of the groupoids restricted to $Y_1$, whereas we use $\tilde{e}$ for evaluation maps for groupoids over $Y$. But noticing that $[ev_{Y_1}] \otimes [e_1^{-1} \otimes [e_1^*] = [e_1^{-1} \otimes [e_1^*] \otimes [ev_{Y_1}]]$ and applying Proposition 2.9, we obtain the commutativity of the second square. For the commutativity of the first one we use a similar argument. Finally for the third one, we have to prove that

$$[e_1^{-1} \otimes [e_1^*] \otimes \mu_ψ' \otimes \partial_X$$

is equal to

$$\partial_Y \otimes [e_1^{-1} \otimes [e_1^*] \otimes \mu_ψ'$$,

where $\partial_X$ and $\partial_Y$ are the boundary map of the bottom and the top row of the diagram respectively and $\tilde{e}$ denotes the evaluation maps in the setting of the groupoids restricted to $Y \setminus Y_1$. By the naturality of the boundary map, the only thing to check is the commutativity with the Morita equivalences. The boundary maps clearly commutes with the classes $μ$. But since they are the Kasparov product of the inverse of a morphism and a morphism, using the naturality of the boundary map, we obtain the commutativity of the third square. □

Finally we have the main result of this section.

**Theorem 2.23.** If $ψ: Y \to X$ is transverse with respect to $G \rightrightarrows X$, then one has a natural transformation from the $K$-theory sequence associated to the adiabatic extension of $G_ψ^0$ to the one of $G$:

$$
\begin{array}{c}
\cdots \to K_* \left( C_r^*((G_ψ^0)) \right) \xrightarrow{k} K_* \left( C_r^*((G_ψ^0)_{ad}) \right) \xrightarrow{[evo]} K_* \left( C_r^*((\mathfrak{A}(G_ψ^0))) \right) \to \cdots \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\cdots \to K_* \left( C_r^*((G)) \right) \xrightarrow{\partial_X} K_* \left( C_r^*((G_{ad})) \right) \xrightarrow{[evo]} K_* \left( C_r^*((\mathfrak{A}(G))) \right) \to \cdots
\end{array}
$$

(2.7)
where $\mu_\psi$ is the KK-element given in Definition 2.6 and $d\psi_t \in KK\left(C^*_r(\mathfrak{A}(G^\psi_\psi)), C^*_r(\mathfrak{A}(G))\right)$ is the KK-class obtained in the obvious way, as for $\psi^{ad}$, but restricting the process to the Lie algebroids. What the more, the commutativity of the diagram still holds in the KK-theory framework.

**Remark 2.24.** Notice that one also has a wrong-way functoriality for the adiabatic deformation up to $t = 1$ included. It is given by the same construction and it enjoys the same properties. Moreover there is a commutative diagram analogous to (2.7) for the exact sequence

$$0 \longrightarrow C^*_r(G \times (0, 1]) \longrightarrow C^*_r(G_{ad}) \xrightarrow{\text{ev}_0} C^*_r(\mathfrak{A}(G)) \longrightarrow 0.$$ 

### 3 Lie groupoids and secondary invariants

#### 3.1 Pseudodifferential operators on Lie groupoids

In this section we are going to recall the definition of the pseudodifferential operators on a Lie groupoid. For more details the reader is referred to [26, 36]. Consider following data:

- a smooth embedding $\theta: U \to \mathfrak{A}^*G$, where $U$ is a tubular neighbourhood of $G^{(0)}$ in $G$, such that $\theta(G^{(0)}) = G^{(0)}$, $(d\theta)|_{G^{(0)}} = \text{Id}$ and $\theta(\gamma) \in \mathfrak{A}_{r(\gamma)}G$ for all $\gamma \in U$;
- a smooth compactly supported map $\phi: G \to \mathbb{R}_+$ such that $\phi^{-1}(1) = G^{(0)}$;
- a polyhomogeneous symbol $a$ on $\mathfrak{A}^*G$ of order $m \in \mathbb{Z}$.

Then a pseudodifferential $G$-operator $P$ is obtained by the formula:

$$Pu(\gamma) = \int_{\gamma' \in G_{r(\gamma)} \cdot \xi \in \mathfrak{A}_{s(\gamma)}(G)} e^{i\theta(\gamma'\gamma^{-1})} \xi a(r(\gamma), \xi)\phi(\gamma'\gamma^{-1})u(\gamma')d\gamma'd\xi$$

If $m > 0$, then we obtain an unbounded multiplier of $C^\infty_c(G)$; if $m = 0$, the operator $P$ is an element in the multiplier algebra of the groupoid $C^*$-algebra; finally, if $m < 0$, then $P$ lies in the groupoid $C^*$-algebra.

**Examples 3.1.** Let us recall some examples of 0-order pseudodifferential operators on Lie groupoids.

1. If $G = X \times X \rightrightarrows X$ is the pair groupoid, where $X$ is a compact smooth manifold. Then a 0-order $G$-$\Psi DO$ is simply a 0-order $\Psi DO$ on $X$.
2. Let $p: X \to Z$ a submersion, and $G = X \times_Z X = \{(x,y) \in X \times X | p(x) = p(y)\}$ the associated subgroupoid of the pair groupoid $X \times X$. Then a 0-order $G$-$\Psi DO$ is given by families $\{P_z\}_{z \in Z}$ of 0-order $\Psi DO$s on $p^{-1}(z)$.
3. Let $G$ be the fundamental groupoid of a compact smooth manifold $M$ with fundamental group $\pi_1(M) = \Gamma$. Recall that if we denote by $\tilde{M}$ a universal covering of $M$ and let $\Gamma$ act by covering transformations, then $G^{(0)} = \tilde{M}/\Gamma = M$, $G = \tilde{M} \times_{\Gamma} \tilde{M}$ and the source and the range maps are the two projections. We have a 0-order $G$-$\Psi DO$ is a properly supported $\Gamma$-invariant 0-order $\Psi DO$ on the universal covering $\tilde{M}$ of $M$.
4. Let $G = E \rightrightarrows X$ be the total space of a vector bundle $p: E \to X$ over a compact smooth manifold $X$, with $r = s = p$ and $(x,v) \cdot (x,w) = (x,v + w)$. If $P$ is a pseudodifferential $G$-operator:

$$Pf(v) = \int_{w \in E_x} k_P(v - w)f(w)$$

Thus, for all $x \in X$, $P_x$ is a translation-invariant convolution operator on the linear space $E_x$ such that the underlying distribution $k_P$ identifies with the Fourier transform of a symbol on $E$. Consequently we have that a 0-order $G$-$\Psi DO$ is given by a smooth function on $BE^*$, the unit ball bundle of $E^*$.
5. If $G$ is the holonomy groupoid of a foliation $\mathcal{F}$ on a smooth manifold $X$, then a 0-order $G$-$\Psi DO$ is just a leaf-wise 0-order $\Psi DO$ on $(X, \mathcal{F})$. 

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3.2 The $\varrho$ classes for Lie groupoids

We are going to define the $\varrho$-classes as elements of the K-theory of $C^*_r(G)\otimes(G)$, seen as a group of relative K-theory in the following sense (see [31] for more details). Let $A$ and $B$ be two separable C*-algebras and let $\varrho$ be an element in $KK(A,B)$. One can always find a C*-algebra $A'$ and two $\ast$-homomorphisms $\varphi: A'\to A$ and $\psi: A'\to B$ such that

- $\varphi$ induces a KK-equivalence between $A'$ and $A$;
- we can decompose $\varrho$ as $[\varphi]^{-1}\otimes_{A'} [\psi]$, the Kasparov product of the inverse of the KK-equivalence $[\varphi]$ and $[\psi]$.

Hence one can see $\varrho$, up to the KK-equivalence between $A$ and $A'$, as the boundary map for the long exact sequence in K-theory associated to the following short exact sequence

$$0 \longrightarrow B \otimes (0,1) \longrightarrow C_\varphi(A',B) \xrightarrow{\psi_0} A' \longrightarrow 0$$

where $C_\varphi(A',B) = \{a \otimes f \in A \otimes B[0,1] | f(0) = \psi(a)\}$ is the mapping cone C*-algebra of $\psi$ and we define the relative K-theory of $\varrho$ as elements of the K-theory of this mapping cone.

So a $\varrho$ class will be defined as a class in this K-group. More precisely, if we identify $K_r(C_\varphi(A',B))$ with the KK-group $KK^*(\mathbb{C},C_\varphi(A',B))$, such an element is given by the following data:

- a $\mathbb{C}$-$A'$ bimodule $(H,F)$;
- a $\mathbb{C}$-$B[0,1)$ bimodule $(\xi_t,G_t)$ such that $(\xi_0,G_0) = (H \otimes \psi B,F \otimes \psi 1)$ and $(\xi_1,G_1)$ is degenerate.

Remark 3.2. Of course one can equally work in the unbounded setting, following [2].

Let $G \Rightarrow X$ be a Lie groupoid. We are going to define the secondary invariants as classes in the relative K-theory of $Ind_G: KK(C^*_r(G),C^*_r(G))$, that is nothing but the K-theory of $C^*_r(G)$. Let us quickly recall the construction of the adiabatic index of a 0-order elliptic pseudodifferential $G$-operator $P$. Its principal symbol $\sigma$ defines a class in the group $KK(\mathbb{C},C_0(\mathfrak{X}(^*G)))$ that is nothing else than $KK(\mathbb{C},C^*_r(\mathfrak{X}(^*G)))$, by means of the Fourier transform.

We know that $ev_0: C^*_r(G) \to C^*_r(\mathfrak{X}(^*G))$ is a KK-equivalence. Let us give an explicit description of the inverse of the map induced in KK-theory: to the symbol $\sigma \in C_0(\mathfrak{X}(^*G))$, we can associate a symbol $\sigma_{ad}$ on $\mathfrak{X}(^*G) \times [0,1]$, the lie algebroid of the adiabatic deformation, given by $\sigma_{ad}(\xi,t) := \sigma(\xi)$; hence we obtain the unbounded regular operator $P_{ad}$ on the $C^*_r(G)$-module $C^*_r(G)$ defined by

$$\begin{align*}
(P_{ad}f)(\gamma,t) &= \int_{\gamma \in \mathfrak{X}(^*G)} \int_{\gamma' \in \mathfrak{X}(^*G)} e^{i(\exp^{-1}(\gamma')\exp^{-1}(\xi))}(\gamma\gamma'^{-1})\sigma(r(\gamma),\xi)f(\gamma',t) \frac{d\xi d\gamma'}{t} \quad (3.1)
\end{align*}$$

for $t \neq 0$ and

$$\begin{align*}
(P_{ad}f)(x,V,0) &= \int_{\gamma \in \mathfrak{X}(^*G)} \int_{V' \in \mathfrak{X}(^*G)} e^{i(\exp^{-1}(V')\xi)}\chi(\exp(V))\sigma(x,\xi)f(x,V - V',0)d\xi dV' \quad (3.2)
\end{align*}$$

for $t = 0$, with $f \in C^*_c(G,\Omega^1)$ (notice that at $t = 1$ we obtain $P$ and that $P_{ad}$ is a deformation of $P$ to the Fourier transform of its symbol).

Here we have chosen an exponential map $exp: U \to W$, from a neighbourhood of the zero section in the algebroid $\mathfrak{X}(G)$ to a tubular neighbourhood $W$ of $X$ in $G$, and a cut-off function $\chi$ with support in $W$. Furthermore we can also make this construction with coefficients in any vector bundle $E$ over $G$. The operator $P_{ad}$ defines a class in $KK(\mathbb{C},C^*_r(G))$ such that the evaluation at 0 gives the class of $\sigma$ and the evaluation at 1 gives the analytic $G$-index of $P$ while $Ind_G(P) = [\sigma(P)] \otimes [ev_0]^{-1} \otimes [ev_1]$ in the group $KK(\mathbb{C},C^*_r(G))$. 

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Notice that $C^*_r(G_{ad})$ is isomorphic to the mapping cone $C_{ev_1}(C^*_r(G_{ad}), C^*_r(G))$ of the evaluation at 1. Hence, if $P_t$ is a path of $G$-operators such that $P_0 = P$ and $P_t$ is invertible, through the above isomorphism, we obtain a class in $KK(\mathbb{C}, C^*_r(G_{ad}))$. This will be the home of the secondary invariants that we will study in the following sections. They are called secondary because they arise when the index, the primary invariant, vanishes.

**Definition 3.3.** In the situation described above, let us denote the $\varrho$-invariant associated to $\sigma$ and $P_t$ as

$$
\varrho(\sigma, P_t) \in KK(\mathbb{C}, C^*_r(G_{ad})).
$$

**Remark 3.4.** If $P$ is not bounded and it is homotopic to an invertible operator through a path $P_t$, then we can construct a $\varrho$-class in the following way. Let $\sigma$ be the symbol of $P$, consider the symbol $\sigma_{ad}$ as above and construct the unbounded $G_{ad}$-operator $P_{ad}$. This operator and the path $P_t$ fit together and we obtain an unbounded $C^*_r(G_{ad})$-operator $P_{ad}$.

Let $\psi(s) = t \cdot (1 + s^2)^{1/2}$ be a chopping function and observe that $\psi(s) := \psi(e^t \cdot s)$ is a path of continuous functions, such that $\psi_0 = \psi$ and $\psi_t(s)$ goes to sign($s$) when $t$ goes to $+\infty$.

Now, since $P_1$ is invertible and there is a gap in its spectrum near zero, the concatenation of $\psi(P_{ad})$ and $\psi(\sigma(P))$, suitably parametrized, gives a bounded Fredholm $G_{ad}$-operator and we denote its class in $KK^*(\mathbb{C}, C^*_r(G_{ad}))$ by $\varrho(\sigma, P_1)$.

### 3.3 Cobordism relations

In this section we are going to investigate the relation between the $\varrho$-invariants associated to two cobordant Lie groupoids. Let $W$ be a smooth manifold with boundary $\partial W$ and $G(W, \partial W) \equiv W$ be the Monthubert groupoid of a Lie groupoid $G$ transverse to the boundary, as in Definition 1.9. Let $P$ be an elliptic pseudodifferential $G(W, \partial W)$ operator and denote its restriction to the boundary by $P^\partial$ (this is a $G_{\partial W} \times \mathbb{R}$-operator).

Assume that there exists a homotopy $P^\partial t$ from $P^\partial_0 = P^\partial$ to an invertible operator $P^\partial_1$. Then we obtain the following classes:

- a secondary invariant $\varrho(P^\partial_1) \in K_*\left((G_{\partial W})^\partial_{ad} \times \mathbb{R}\right) \approx K_{*+1}\left((G_{\partial W})_{ad}^\partial\right)$;
- a class $[P^\partial, P^\partial]$ in $K_*\left(T_{\partial W}G(W, \partial W)\right)$ (see Definition 1.11), defined by the symbol of $P$ and the homotopy $P^\partial_t$; indeed, using the KK-equivalence in the Lemma 1.18, we can extend it to a class $[P^\partial, P^\partial]$ in $K_*\left(G(W, \partial W)^\partial_{ad}\right)$, whose restriction to the boundary is $\varrho(P^\partial_1)$;
- finally we get a class $\text{Ind}(P, P^\partial_1) \in K_*\left(G_{\partial W}\right)$. This is the generalized Fredholm index of $P$ associated to the perturbation on the boundary $P^\partial$, obtained as the Kasparov product $[P^\partial, P_{ad}] \otimes [ev_1]$.

The following elementary result is useful to prove the main formula of this section.

**Lemma 3.5.** Let

$$
\begin{array}{ccc}
1) & 0 & \longrightarrow J_B & \longrightarrow A & \longrightarrow B & \longrightarrow 0, \\
2) & 0 & \longrightarrow J_C & \longrightarrow A & \longrightarrow C & \longrightarrow 0, \\
\end{array}
$$

be exact sequences of $C^*$-algebras. Assume that $J_B + J_C = A$. We have the following exact sequences:

$$
\begin{array}{ccc}
3) & 0 & \longrightarrow J & \longrightarrow J_C & \longrightarrow B & \longrightarrow 0, \\
4) & 0 & \longrightarrow J & \longrightarrow J_B & \longrightarrow C & \longrightarrow 0, \\
5) & 0 & \longrightarrow J & \longrightarrow A & \longrightarrow B \oplus C & \longrightarrow 0, \\
\end{array}
$$

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where $J = J_B \cap J_C$. Let $\partial_B$ and $\partial_C$ be the boundary homomorphisms associated to the exact sequences 3) and 4) respectively. Then $\partial$, the boundary homomorphism associated to the exact sequences 5), is such that

$$\partial: x \otimes y \mapsto \partial_B(x) + \partial_C(y)$$

where $x \in K_n(B)$, $y \in K_n(C)$ and $\partial_B(x) + \partial_C(y) \in K_{n+1}(J)$.

Moreover if $\beta$ and $\gamma$ admit completely positive sections, we have that $\partial = p_B^*\partial_B + p_C^*\partial_C \in KK(B \oplus C, J)$, where $p_B$ and $p_C$ are the projections from $B \oplus C$ to $B$ and $C$ respectively and $\partial_B$ and $\partial_C$ are elements of $KK(B, J)$ and $KK(A, J)$ respectively.

Proof. By the following commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & J & \rightarrow & J_C & \rightarrow & B & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & J & \rightarrow & A & \rightarrow & B \oplus C & \rightarrow & 0
\end{array}
$$

we deduce that $\partial(x \oplus 0) = \partial_B(x)$. By the following one

$$
\begin{array}{cccccc}
0 & \rightarrow & J & \rightarrow & J_B & \rightarrow & C & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & J & \rightarrow & A & \rightarrow & B \oplus C & \rightarrow & 0
\end{array}
$$

we deduce that $\partial(0 \oplus y) = \partial_C(x)$. Then we obtain that $\partial: x \otimes y \mapsto \partial_B(x) + \partial_C(y)$. The second part is obvious.

We want to apply this Lemma to the following situation. We are interested in the following $C^*$-algebras:

$$A = C^r_\ast \left( G(W, \partial W)^{\ast}_{ad} \right), \ B = C^r_\ast \left( G_{\mid W}^{\ast} \right), \ C = C^r_\ast \left( (G_{\mid W})^{\ast}_{ad} \times \mathbb{R} \right) .$$

We obtain the following diagram of exact sequences:

$$
\begin{array}{cccccccccc}
0 & \rightarrow & 0 & \rightarrow & C^r_\ast \left( (G_{\mid W})^{\ast}_{ad} \right) & \rightarrow & C^r_\ast \left( G_{\mid W}^{\ast} \right) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & C^r_\ast \left( G(W, \partial W)^{\ast}_{ad} \right) & \rightarrow & C^r_\ast \left( G(W, \partial W)^{\ast}_{ad} \right) & \rightarrow & C^r_\ast \left( G_{\mid W}^{\ast} \right) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & C^r_\ast \left( (G_{\mid W})^{\ast}_{ad} \times \mathbb{R} \right) & \rightarrow & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 
\end{array}
$$

that is fit for the situation in Lemma 3.5. Using the notations in the Lemma, it follows that

- $\partial_B: K_{n+1}(C^r_\ast \left( G_{\mid W}^{\ast} \right)) \rightarrow K_{\ast}(C^r_\ast \left( (G_{\mid W})^{\ast}_{ad} \right))$ is given by $b \mapsto b \otimes \text{Bott} \otimes [i]$. Where $i: C^r_\ast \left( G_{\mid W} \times (0,1) \right) \rightarrow C^r_\ast \left( (G_{\mid W})^{\ast}_{ad} \right)$ is the obvious inclusion;

- $\partial_C: K_{n+1}(C^r_\ast \left( (G_{\mid W})^{\ast}_{ad} \times \mathbb{R} \right)) \rightarrow K_{\ast}(C^r_\ast \left( (G_{\mid W})^{\ast}_{ad} \right))$ is given by $c \mapsto c \otimes \text{Bott}^{-1} \otimes [j]$. 

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Here Bott\(^{-1}\) is given by the boundary map in Lemma \ref{lem:boundary_map} and \(j\) is the inclusion of \(C^*_r(G_{\partial W} \times (0,1) \times (0,1))\) in \(C^*_r(G_{\partial W})\). Recall that we have this inclusion because we can always assume that, near the boundary, \(W\) is of the form \(\partial W \times [0,1)\) and that \(G(W, \partial W)\) is of the form \(G_{\partial W} \times \Gamma([0,1), \{0\})\).

**Theorem 3.6** (Delocalized APS index Theorem for Lie groupoids). In the situation described above, we have the following equality

\[
\partial_C (\rho(P_r^\partial)) = -\partial_B (\text{Ind}(P, P_r^\partial)) \in K_{s+1}((G_{\partial W})_{\text{ad}}^\infty).
\]

**Proof.** Since the pair \(\text{Ind}(P, P_r^\partial) \oplus g(P_r^\partial) \in K_*(B \oplus C)\) is the image of \([P_r^\partial] \in K_*(A)\), by the exactness of the associated exact sequence \(\partial(\text{Ind}(P, P_r^\partial) \oplus g(P_r^\partial)) = 0\), then the formula is an easy consequence of Lemma \ref{lem:exactness}

**Remark 3.7.** If \(W = X \times [0,1]\) and \(G(W, \partial W) = G \times \Gamma([0,1], \{0,1\})\), then the boundary map in KK-theory associated to the following exact sequence

\[
0 \rightarrow C^*_r(G \times (0,1) \times (0,1)) \rightarrow C^*_r(G \times \Gamma([0,1], \{0,1\})) \rightarrow C^*_r(G \times \mathbb{R} \times \{0\}) \oplus C^*_r(G \times \mathbb{R} \times \{1\}) \rightarrow 0
\]

is given by

\[
\partial(x_0 \oplus x_1) = x_0 \otimes \text{Bott}_0^{-1} + x_1 \otimes \text{Bott}_1^{-1},
\]

where Bott\(_i\) is the Bott element for \(C^*_r(G \times \mathbb{R} \times \{i\})\), defined as the boundary map in Lemma \ref{lem:boundary_map}.

### 3.4 Products

Let \(G \rightrightarrows X\) and \(H \rightrightarrows Y\) be two Lie groupoids. In this section we will define an external product between the C*-algebra of the adiabatic deformation of \(G\) and the C*-algebra of the Lie algebroid of \(H\), valued in the C*-algebra of the adiabatic deformation of \(G \times H\).

Let us build a KK-class \(\alpha \in KK(C^*_r(G_{\text{ad}}^\partial) \otimes C^*_r(\mathfrak{H}(H)), C^*_r((G \times H)^{\text{ad}}))\) in the following way: notice that

\[
\text{id} \otimes \text{ev}_0 \cdot C^*_r(G_{\text{ad}}^\partial) \otimes C^*_r(H_{\text{ad}}) \to C^*_r(G_{\text{ad}}^\partial) \otimes C^*_r(\mathfrak{H}(H))
\]

induces a KK-equivalence; moreover, since \(C^*_r(G_{\text{ad}}^\partial) \otimes C^*_r(H_{\text{ad}}) = C^*_r(G_{\text{ad}}^\partial \times H_{\text{ad}})\), we have a \(\text{Cu}([0,1] \times [0,1])\)-algebra and the restriction the diagonal of \([0,1] \times [0,1]\) induces a KK-element \([\Delta]\) in \(KK(C^*_r(G_{\text{ad}}^\partial \times H_{\text{ad}}), C^*_r((G \times H)^{\text{ad}}))\).

Thus we can define the class \(\alpha\) as the Kasparov product

\[
[i \otimes \text{ev}_0]^{-1} \otimes_{G^\partial \times H_{\text{ad}}} [\Delta] \in KK(C^*_r(G_{\text{ad}}^\partial) \otimes C^*_r(\mathfrak{H}(H)), C^*_r((G \times H)^{\text{ad}})).
\]

**Definition 3.8.** The external product

\[
\mathfrak{E}: KK_1(\mathfrak{C}, C^*_r(G_{\text{ad}}^\partial)) \times KK_j (\mathfrak{C}, C^*_r(\mathfrak{H}(H))) \to KK_{i+j} (\mathfrak{C}, C^*_r((G \times H)^{\text{ad}}))
\]

is defined as the map

\[
x \times y \mapsto (x \otimes_C y) \otimes_D \alpha,
\]

where \(D = C^*_r(G_{\text{ad}}^\partial) \otimes C^*_r(\mathfrak{H}(H))\).

Now we want to investigate the injectivity of the external product with a fixed element \(y \in K_*(C^*_r(\mathfrak{H}(H)))\). To do it, let us construct an element

\[
\beta \in KK(C^*_r((G \times H)^{\text{ad}}), C^*_r(G_{\text{ad}}^\partial) \otimes C^*_r(H)).
\]

Let \(T\) be the restriction of \(G_{\text{ad}}^\partial \times H_{\text{ad}}\) to the triangle

\[
\mathfrak{T} := \{s \geq t \mid (t, s) \in [0,1) \times [0,1]\}.
\]

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Lemma 3.9. Denote the restriction of \( T \) to the diagonal side of \( \Sigma \) by \( \Delta' \). Then it induces a KK-equivalence

\[
[\Delta'] \in KK(C^*_r(T), C^*_r((G \times H)_{ad}^o)).
\]

Proof. Observe that \( C^*_r(T) \) is a \( C_0(\Sigma) \)-algebra and that, for this reason, the restriction to the diagonal gives an exact sequence of reduced C*-algebras. Indeed the kernel of the restriction morphism turns out to be isomorphic to the C*-algebra \( C^*_r(G) \otimes C^*_r(H_{ad}^o) \otimes C[0,1] \), that is KK-contractible. So the only thing to prove is that the restriction admits a completely positive section.

Let \( Q \) be the set \( \{(t,s) \in [0,1]^2 \setminus \{(1,1)\}\} \) and \( Q \) the groupoid over \( X \times Y \times Q \) such that \( Q \) restricted to \( X \times Y \times \{(t,s)\} \) is equal to \( T \) restricted to \( X \times Y \times \{(ts,s)\} \). Let \( \lambda : Q \to T \) be the map that sends \( (t,s) \) to \((ts,s)\). Then, by means of the pull-back through \( \lambda \), \( C^*_r(T) \) is the \(*\)-subalgebra of \( C^*_r(Q) \) of the elements that are constant on \( \{t = 0\} \subset \Sigma \).

Moreover the restriction to the diagonal, corresponds to the restriction to the union of the bottom side \( \mathcal{B} \) and the right side \( \mathcal{R} \) of \( \Sigma \). Now the restriction to \( \mathcal{B} \) of \( Q \) is amenable (it is the product of vector bundles \( \mathfrak{A}H \times \mathfrak{A}G \)), then the \(*\)-homomorphism induced on the reduced C*-algebras admits a completely positive lifting.

The same is true for the restriction to \( \mathcal{R} \) since \( C^*_r(Q) \) contains as ideal \( C^*_r((G \times H)_{ad}^o) \otimes C(0,1] \) and \( \mathcal{R} \) is \( (G \times H)_{ad}^o \times \{1\} \), the map \( \xi \mapsto t \cdot \xi \) is a completely positive section of the restriction to the right side.

Using Lemma 1.17 we obtain a completely positive section for the restriction to \( \mathcal{B} \cup \mathcal{R} \) and then a completely positive section for \( \Delta' \).

Then let us define \( \beta \) as the Kasparov product

\[
(\Delta')^{-1} \otimes_{C^*_r(T)} [ev_{(s=1)}] \in KK(C^*_r((G \times H)_{ad}^o), C^*_r(G_{ad}^o) \otimes C^*_r(H)).
\]

It is easy to verify that

\[
\alpha \otimes_{C^*_r((G \times H)_{ad}^o)} \beta \in KK(C^*_r(G_{ad}^o) \otimes C^*_r(\mathfrak{A}(H)), C^*_r(G_{ad}^o) \otimes C^*_r(H))
\]

is nothing but the class \( id_{C^*_r(G_{ad}^o)} \otimes Ind_H \), where \( Ind_H \in KK(C^*_r(\mathfrak{A}(H), C^*_r(H))) \) is the index KK-class as in the Remark 1.21.

Lemma 3.10. Let \( y \) be a class in \( K_i(\mathfrak{A}(H)) \). Assume that there exists a K-homology class \( \eta \in KK(H, pt) \) such that

\[
y \otimes_{C^*_r(\mathfrak{A}(H))} Ind_H \otimes_{C^*_r(H)} \eta = n \in \mathbb{Z},
\]

with \( n \neq 0 \), then the map \( K_j(G_{ad}^o) \to K_{i+j}((G \times H)_{ad}^o) \) given by

\[
x \mapsto x \boxtimes y
\]

is rationally injective. If \( n = 1 \), then the map is honestly injective.

Proof. From the previous discussion we have that

\[
(x \boxtimes y) \otimes_{C^*_r((G \times H)_{ad}^o)} \beta \otimes_{C^*_r(H)} \eta = x \otimes_{C^*_r} y \otimes_{D} \alpha \otimes_{C^*_r((G \times H)_{ad}^o)} \beta \otimes_{C^*_r(H)} \eta = x \otimes_{C} y \otimes_{C^*_r(\mathfrak{A}(H))} Ind_H \otimes_{C^*_r(H)} \eta = n \cdot x.
\]

So up to invert \( n \) we have that the exterior product with \( y \) is rationally injective and that if \( n = 1 \) it is injective.
3.5 The Signature operator

In [17] the authors give a direct prove of the fact that the K-theory classes of the higher signatures are homotopy invariant. They also prove it in the case of foliations, using a method that can be easily presented in a more abstract way for any Lie groupoid.

Let $G \rightrightarrows X$ be a Lie groupoid such that the rank of its Lie algebra is $n$ and let $\Lambda_c \mathfrak{X}^*(G)$ be the exterior algebra of $\mathfrak{X}^*(G)$. We can construct a right $C^*_r(G)$-module $E(G)$ as the completion of $C^*_r(G, \Lambda_c \ker ds^* \otimes s^*\Omega^2(\mathfrak{A}(G)))$. Furthermore we can define the following $C^*_r(G)$-valued quadratic form putting

$$Q(\xi, \zeta)(\gamma) = m_* \left( p_1^*(\xi^2) \wedge p_2^*(\zeta) \right)(\gamma),$$  \hspace{1cm} (3.6)

where $m, p_k : G^{(2)} \to G$ are such that $m : (\gamma_1, \gamma_2) \mapsto \gamma_1 \cdot \gamma_2$ and $p_k : (\gamma_1, \gamma_2) \mapsto \gamma_k$ (and $i$ is the inversion map of the groupoid). If $T \in \mathcal{L}(E(G))$, let us denote $T'$ its adjoint with respect to the quadratic form $Q$ (i.e. $Q(T\xi, \zeta) = Q(\xi, T'\zeta)$ for any $\xi, \zeta \in E(G)$).

The quadratic form $Q$ is regular in the sense of [17, Definition 1.3], by means of the operator $T$, given by

$$T_\alpha = i^{-\partial(\alpha)(n-\partial(\alpha))} * \alpha,$$

where $*$ is the Hodge operator of $\ker ds^*$ associated to a smooth hermitian structure on it.

Consider the $s$-fiberwise exterior derivative operator on on $C^*_r(G, \Lambda_c \ker ds^* \otimes s^*\Omega^2(\mathfrak{A}(G)))$, that is closable; let us still denote with $d_0$ its closure: it defines a regular operator on $E(G)$. We have that $\text{Im}d_0 \subseteq \text{dom}d_0$ and that $d_0^2 = 0$.

Now put $d_\xi = i^{(k_0)}d_0\xi$. The regular operator $D_G = d + d^*$ is an elliptic and self-adjoint differential $G$-operator on $E(G)$.

**Definition 3.11.** The class $[E(G), D_G] \in K_*(C^*_r(G))$, defined as in [2], is the analytic $G$-signature of $X$.

**Definition 3.12.** Let $G \rightrightarrows X$ and $H \rightrightarrows Y$ be two Lie groupoids. A morphism $\varphi : H \to G$ is a homotopy equivalence if there are a morphism $\psi : G \to H$ and maps $T : X \times [0, 1] \to G$ and $S : Y \times [0, 1] \to H$, such that

- $r \circ T_1(x)$ is constantly equal to $x$ and $T_0 = id_X$;
- for any $x \in X$ we have that $s(T_1(x)) = \varphi \circ \psi(x)$ and $r(T_1(x)) = x$,
- for any $\gamma \in G^2_{\varphi \circ \psi(x)}$ we have that $T_1(x) \cdot (\varphi \circ \psi(\gamma)) = \gamma \cdot T_1(\varphi \circ \psi(x))$,

and similarly for $S$ and $\psi \circ \varphi$.

**Remark 3.13.** This is nothing else than a strong equivalence of groupoids, see [22, 5.4], with natural transformations homotopic to identities. In particular this implies that $H = G^*_c$, see [22, Proposition 5.11].

Of course a homotopy equivalence $\varphi$ between two groupoids gives a Morita equivalence, whose imprimitivity bimodule is given by $\mu_\varphi$ as in the subsection 2.4.

**Theorem 3.14** (Hilsum-Skandalis). Let $H \rightrightarrows Y$ and $G \rightrightarrows X$ be two Lie groupoids, with $X$ and $Y$ compact manifolds, and let $\varphi : H \to G$ be a homotopy equivalence of groupoids. Then there exists a path $DH_{\varphi, T}^H$ from $D_L$ to an invertible operator, where $L = G^*_c \otimes id_X^{\varphi \circ \psi, \mu_\varphi}$. Moreover the existence of this path implies that

$$[E(H), D_H] \otimes \mu_\varphi = [E(G), D_G] \in K_*(C^*_r(G)).$$  \hspace{1cm} (3.7)

**Proof.** Let us notice that by hypotheses $H \cong G^*_c$. Consider the Lie groupoid $L = G^*_c \otimes id_X^{\varphi \circ \psi, \mu_\varphi} \rightrightarrows Y \cup X$ and the $C^*_r(L)$-module $E(L)$, that is the completion of $C^*_r(L, \Lambda_c \ker ds^* \otimes s^*\Omega^2(\mathfrak{A}(L)))$.

We can see an element in $E(L)$ as a $2 \times 2$ matrix in $\left( \begin{array}{cc} E(G) & E(G) \\ E(G^*_c) & E(G^*_c) \end{array} \right)$, where the notation is self-explanatory. Then the $L$-operator $d_L$ given by the exterior derivative is a matrix

$$\begin{pmatrix} d_G & 0 \\ 0 & -d_{G^*_c} \end{pmatrix}.$$
Put \((E_1, Q_1, D_1) = (E(G, \varphi) \otimes E(G), Q, d)\) and \((E_2, Q_2, D_2) = (E(G^\varphi), Q, d_{G^\varphi})\). We want to construct an operator \(T \in \mathcal{L}(E_1, E_2)\) that satisfies the hypotheses of [17, Lemma 2.1].

Let \(E\) be a vector bundle over \(Y\) such that \(E \oplus \varphi^*\mathfrak{A}G = Y \times \mathbb{R}^k\) and let \(\pi: Y \times \mathbb{R}^k \to \varphi^*\mathfrak{A}G\) be the projection. Let \(p: Y \times B^k \to X\) be the map given by

\[
p: (y, \xi) \mapsto \tau(\exp_{\varphi(y)}(\pi(\xi))).
\]

Notice that, since \(\varphi\) is transverse, \(p\) is a submersion and its restriction to \(Y \times \{0\}\) is equal to \(\varphi\). Then the obvious map \(p: G^\varphi \cup G^p \to G_p \cup G\) is a submersion. Moreover \(G_p\) is isomorphic to \(G^\varphi \times B^k\) through the map \((y, \xi, \gamma) \mapsto (\exp_{\varphi(y)}(\pi(\xi))^{-1} \cdot \gamma, \xi, 0)\), and \(G^p\) is isomorphic to \(G^\varphi \times B^k\) in a similar way.

Let \(q: G^\varphi \times B^k \cup G^\varphi \times B^k \to G^\varphi \cup G^\varphi\) be the projection. Let \(v\) be a volume form on \(B^k\) such that \(\int_{B^k} v = 1\) and let \(\omega\) be its pull-back on \(G^\varphi \times B^k\) through the projection onto the ball. Then we can define \(T\) as the operator \(\mathcal{E}_1 \to \mathcal{E}_2\) given by

\[
\xi \mapsto q_*(\omega \wedge p^*(\xi))
\]

and the operator \(\mathcal{Y}\) as in [27, Lemma 2.2]. We can easily verify that \(T\) and \(\mathcal{Y}\) satisfy all the conditions of [17, Lemma 2.1].

As in the proof of [17, Lemma 2.1] we can construct an explicit path \(D^H_{L, t}\) from \(D_L\) to an invertible operator. This in particular means that \([\mathcal{E}(L), D_L]\) = 0 and the following equality

\[
[\mathcal{E}(L), D_L] \otimes \mu_{\varphi, \text{id}_X} = [\mathcal{E}(H), D_H] \otimes \mu_{\varphi} - [\mathcal{E}(G), D_G]
\]

implies \([3, 7]\).

The following set is a generalization of the structure set in Surgery Theory, in which we take in account both the smooth and the groupoid structure of a Lie groupoid \(G\).

**Definition 3.15.** Let \(G \rightrightarrows X\) be a Lie groupoid on a compact smooth manifold. We define the \(G\)-structure set \(\mathcal{S}(G)\) of \(X\) as the set

\[
\{\varphi: H \to G \mid \varphi\text{ is a homotopy equivalence of Lie groupoids }\}/\sim,
\]

where \((H \rightrightarrows Y, \varphi) \sim (H' \rightrightarrows Y', \varphi')\) if there exist

- a cobordism \(W\) with boundary \(Y \cup Y'\),
- a Lie groupoid \(K \rightrightarrows W\), transverse to the boundary
- a morphism \(\Phi: K \to G \times [0, 1] \times [0, 1]\) such that \(\Phi\) is a groupoid homotopy equivalence and, if we restrict it to the boundary, we have that \(\Phi|_Y = \varphi: H \to G\) and \(\Phi|_{Y'} = \varphi': H' \rightrightarrows \mathbb{R} \to G\).

If \(\varphi: H \to G\) is a homotopy equivalence of groupoids, we know that \(H = G^\varphi\). Let \(L\) denote the Lie groupoid \(G^\varphi \wr \text{id}_X\). The Signature operator of \(L_{\text{ad}}\) and \(D^H_{L, t}\), the path from \(D_L\) to an invertible operator given by Theorem 3.14, give an unbounded \(\mathbb{C}\)-\(C^*_r(L_{\text{ad}}^\varphi)\) bimodule. Denote by \([D^H_{L, t}]\) the class \(\vartheta(\sigma(D_L), D^H_{L, t}) \in K^*(\mathbb{C}, C^*_r(G^\varphi))\) given by the Remark 3.4.

**Definition 3.16.** Let us define the secondary invariant \(\rho(\varphi)\) as the class

\[
[D^H_{L, t}] \otimes (\varphi \cup \text{id}_X)|_t^{ad} \in K_n(C^*_r(G^\varphi))
\]

where \(n\) is the rank of \(\varphi^*\mathfrak{A}G\).

**Proposition 3.17.** The map

\[
\rho: \mathcal{S}_G(X) \to K_n(C^*_r(G^\varphi))
\]

is well defined.
Let \( \Phi : K \to G \times [0,1] \times [0,1] \) be a groupoid homotopy equivalence with \( \Phi_1 = \varphi : H \to G \) and \( \Phi_{1'} : H' \to G \). Since \( \Phi \) is a homotopy equivalence of groupoids, \( K(W, \partial W) = G(X \times [0,1], X \times \{0,1\}) \). Let \( \mathcal{L} \cong W \cup X \times [0,1] \) be the pull-back of \( G(X \times [0,1], X \times \{0,1\}) \) through \( \Phi \cup \text{id} \times [0,1] \), let \( L \) and \( L' \) be the restrictions of \( \mathcal{L} \) to \( Y \cup X \times \{0\} \) and \( Y' \cup X \times \{1\} \) respectively. We have to show that

\[
[D_{L,t}^{HS} \circ (\varphi \cup \text{id}_X)]^{ad} = [D_{L',t}^{HS} \circ (\varphi' \cup \text{id}_X)]^{ad} \in K_n \left( C^*_r(G^{(1)}_{ad}) \right).
\]

Thanks to Theorem 3.14, we get a class \([D_{L,t}^{HS} \circ \text{ev}] \in K_n(\mathcal{C}^*_r(\mathcal{L}^{(1)}_{ad}))\). The formula in Proposition 3.6 and the fact that \( \Phi \) is a homotopy equivalence imply that

\[
\partial_C \left( [D_{L,t}^{HS}] \circ [\text{ev}_0] \right) = 0 \in K_n \left( \mathcal{L}_{W \cup X \times \{0,1\}}^{(1)}_{ad} \right).
\]

Let \( \pi_X : X \times [0,1] \to X \) the projection onto the first factor. If we prove that

\[
\partial_C \left( [D_{L,t}^{HS}] \circ [\text{ev}_0] \circ (\Phi_{|W} \cup \text{id}_{X \times \{0,1\}}) \right)^{ad} \circ (\pi_X)!^{ad}
\]

and

\[
[D_{L,t}^{HS} \circ (\varphi \cup \text{id}_X)]^{ad} = [D_{L,t}^{HS} \circ (\varphi' \cup \text{id}_X)]^{ad}
\]

are the same class, we are done. By Proposition 2.22 we get the following equality

\[
\partial_C \left( [D_{L,t}^{HS}] \circ [\text{ev}_0] \circ (\Phi \cup \text{id}_{X \times \{0,1\}}) \right)^{ad} = \partial_C \left( [D_{L,t}^{HS}] \circ [\text{ev}_0] \circ (\Phi_{|W} \cup \text{id}_{X \times \{0,1\}}) \right)^{ad}
\]

where \( \partial_C \) is the boundary map associated to the restriction from the cylinder \( X \times [0,1] \) to the boundary \( X \times \{0,1\} \).

Now, since \([D_{L,t}^{HS}] \circ [\text{ev}_0] = ([D_{L,t}^{HS}] \circ \text{Bott}_0) \circ ([D_{L,t}^{HS}] \circ \text{Bott}_1)\), we get that

\[
\partial_C \left( [D_{L,t}^{HS}] \circ [\text{ev}_0] \circ (\Phi \cup \text{id}_{X \times \{0,1\}}) \right)^{ad} = \partial_C \left( [D_{L,t}^{HS}] \circ [\text{ev}_0] \circ (\Phi_{|W} \cup \text{id}_{X \times \{0,1\}}) \right)^{ad} = \partial_C \left( [D_{L,t}^{HS}] \circ [\text{ev}_0] \circ (\Phi_{|W} \cup \text{id}_{X \times \{0,1\}}) \right)^{ad} \circ (\pi_X)!^{ad} = \partial_C \left( [D_{L,t}^{HS}] \circ [\text{ev}_0] \circ (\Phi_{|W} \cup \text{id}_{X \times \{0,1\}}) \right)^{ad} \circ (\pi_X)!^{ad}
\]

where we used Proposition 2.22 and Remark 3.7.

\[\square\]

### 3.6 The Dirac operator

Let \( G \rightrightarrows X \) be a Lie groupoid over a compact manifold \( X \), with Lie algebroid \( \mathfrak{g}(G) \to X \). Let \( g \) be a metric on \( \mathfrak{g}(G) \), by means of it we can define a \( G \)-invariant metric on \( ker ds \) along the \( s \)-fibers of \( G \). Let \( \nabla \) be the fiber-wise Levi-Civita connection associated to this metric.

**Definition 3.18.** Let \( \text{Cliff} (\mathfrak{g}(G)) \) be the Clifford algebra bundle over \( X \) associated to the metric \( g \). Let \( S \) be a bundle of Clifford modules over \( \text{Cliff} (\mathfrak{g}(G)) \) and \( c(X) \) denotes the Clifford multiplication by \( X \in \text{Cliff} (\mathfrak{g}(G)) \). Assume that \( S \) is equipped with a metric \( g_S \) and a compatible connection \( \nabla_S \) such that:

- Clifford multiplication is skew-symmetric, that is

\[
(c(X)s_1, s_2) + (s_1, c(X)s_2) = 0
\]

for all \( X \in C^\infty (X, \mathfrak{g}(G)) \) and \( s_1, s_2 \in C^\infty (X, S) \);

- \( \nabla_S \) is compatible with the Levi-Civita connection \( \nabla \), namely

\[
\nabla_X^S ((c|_X) s) = c(\nabla_X Y)s + c(Y) \nabla_X^S (s)
\]

for all \( X, Y \in C^\infty (X, \mathfrak{g}(G)) \) and \( s \in C^\infty (X, S) \).
The Dirac operator associated to these data is defined as
\[ D_S : s \mapsto \sum c(e_\alpha) \nabla^S_\alpha(s) \]
for \( s \in C^\infty(X, S) \) and \( \{e_\alpha\}_{\alpha \in A} \) a local orthonormal frame.

With this local expression one can easily prove the analogue of the Weitzenbock formula:
\[ D_S^2 = (\nabla^S)^*\nabla^S + \sum_{\alpha < \beta} c(e_\alpha)c(e_\beta)R(\nabla^S)_{\alpha\beta}, \quad (3.8) \]
where \( R(\nabla^S)_{\alpha\beta} \) denote the terms of the curvature of \( \nabla^S \). Assume that the Lie algebroid \( \mathfrak{A}(G) \) is Spin, namely it is orientable and its structure group \( SO(n) \) can be lifted to the double cover \( \text{Spin}(n) \). Moreover we can consider the spinors bundle \( S \) and denote the associated Dirac operator just by \( D \). In this case the second term in (3.8) is equal to \( \frac{1}{4} \) of the scalar curvature of \( \nabla^S \).

**Remark 3.19.** The above discussion implies that, if the scalar curvature of \( \nabla^S \) is positive everywhere, then the Dirac operator \( D \) is invertible. Hence the operator \( D_{ad} \), defined as in (3.1) and (3.2), is an unbounded multiplier of \( G_{ad} \) that is invertible at 1.

Remember that for the Signature operator we need to perform a homotopy of the operator to an invertible one, whereas in the case of the Dirac operator we already have the invertibility condition at 1 in the adiabatic deformation, thanks to the positivity of the scalar curvature.

*From now on we will assume that \( BG \), the classifying space of \( G \), is a manifold and \( BG \rightrightarrows \text{BG} \) is the Lie groupoid associated to the universal 1-cocycle \( \xi \) (see Appendix \( \text{A} \) for definitions).*

**Remark 3.20.** We can easily generalize what we are going to do to the case of \( B\mathbb{G} \rightrightarrows BG \) is a limit of smooth manifolds.

We want to define a groupoid version of the Stolz sequence
\[
\Omega^\text{spin}_{n+1}(BG) \longrightarrow R^\text{spin}_{n+1}(BG) \longrightarrow \text{Pos}^\text{spin}(BG) \longrightarrow \Omega^\text{spin}_n(BG)
\]
(see for instance [33] for the definition in the case where \( G \) is a group).

**Definition 3.21.** Let \( G \rightrightarrows X \) be a Lie groupoid.

- Let \( \text{Pos}^\text{spin}(BG) \) be the set of bordism classes of triples \((M, f : M \to BG, g)\). Here \( f : M \to BG \) is a smooth transverse map from a smooth closed manifold \( M \) such that: \( f \) is transverse with respect to \( BG; \mathfrak{A}(BG_f) \) is spin of rank \( n \) and it is equipped with a metric \( g \) with positive scalar curvature.
  A bordism between \((M, f : M \to BG, g)\) and \((M', f' : M' \to BG, g')\) is a triple
  \((W, F : W \to BG \times [0, 1], h)\),
  where \( W \) is a compact smooth manifold with boundary \( \partial W = M \sqcup -M' \), a reference map \( F \) (which sends the boundary to the boundary) that restricts to \( f \) and \( f' \) on the boundary and such that \( \mathfrak{A}(BG_{f'}) \) is spin equipped with a metric \( h \) with positive scalar curvature, which has a product structure near the boundary and restricts to \( g \) and \( g' \) on the boundary.

- Let \( R^\text{spin}_{n+1}(BG) \) be the set of bordism classes \((W, f : W \to BG, g)\). Here \( W \) is a compact smooth manifold, possibly with boundary: \( f : W \to BG \) is a smooth transverse map that is transverse with respect to \( BG \) and such that \( \mathfrak{A}(BG_f) \) is spin, of rank \( n \) and equipped with the metric \( g \); the metric \( g \) has positive scalar curvature on the boundary.
  Two triples \((W, f, g)\) and \((W', f', g')\) are bordant if there exists a bordism
  \((N, \varphi : N \to BG, h)\)
between \((\partial W, f_\partial, g_\partial)\) and \((\partial W', f_\partial', g_\partial')\) such that \((\mathfrak{A}(BG_F^0), h)\) is spin with positive scalar curvature and

\[
Y := W \sqcup_{\partial W} N \sqcup_{\partial W'} W'
\]

is the boundary of a manifold \(Z\) such that the reference map \(F = f \sqcup \varphi \sqcup f'\) extends to reference map \(F': Z \to BG \times [0,1] \times [0,1]\) and the Lie algebroid of the associated Lie groupoid is spin.

- Let \(\Omega^{spin}_n(BG)\) be the set of bordism classes \((M, f): M \to BG\). Here \(M\) is a closed smooth manifold; \(f: M \to BG\) is a smooth transverse map that is transverse with respect to \(BG\) and such that \((\mathfrak{A}(BG^+_F), h)\) is spin of rank \(n\). The bordism equivalence between triples is as for \(\text{Pos}^{spin}_n(BG)\), without conditions about the metric.

Thus we obtain a groupoid version of Stolz sequence, as in the classical case, and we want to build a diagram

\[
\begin{array}{cccc}
\Omega^{spin}_{n+1}(BG) & \overset{\beta}{\longrightarrow} & \Omega^{spin}_n(BG) & \longrightarrow \text{Pos}^{spin}_n(BG) & \longrightarrow \Omega^{spin}_n(BG) \\
\beta & & & & 3.9
\end{array}
\]

Such that all the squares are commutative. Let us give the definition of the vertical homomorphisms.

**Definition of \(\beta_0: \Omega^{spin}_n(BG) \to K_n(\mathfrak{A}(BG))\)**

Let \((M, f): M \to BG\) an element of \(\Omega^{spin}_n(BG)\). Then the Lie algebroid \(BG^+_F\) is spin and, as in Definition 3.18 we can define a Dirac operator associated to this spin structure. We will denote it by \(D_f\) and its symbol \(\sigma(D_f) \in \mathcal{M}(C_0(\mathfrak{A}^*(BG^+_F)))\) defines a class in \(K_n(\mathfrak{A}(BG^+_F))\), by Fourier transform. Then \(\beta\) is defined as follows

\[
\beta(M, f) := [\hat{\sigma}(D_f)] \otimes df \in K_n(\mathfrak{A}(BG)).
\]

It is easy to prove that \(\beta\) is well defined. Indeed if \((M, f)\) and \((M', f')\) are bordant through \((W, F)\), let \(E\) denote the dual of the Lie algebroid \(\mathfrak{A}(BG^+_F)\) over \(W\), let \(\partial E = \mathfrak{A}(BG^+_F) \times \mathbb{R} \sqcup \mathfrak{A}(BG^+_F) \times \mathbb{R}\) be its restriction to the boundary of \(W\) and let \(\tilde{E}\) be its restriction to the interior of \(W\). Then the symbol of the Dirac operator \(D_F\) defines a class \(x\) in the group \(K_{n+1}(\mathfrak{A}^*)\). Consider the following commutative diagram:

\[
\begin{array}{cccc}
K_{n+1}(C_0(E^*)) & \overset{ev_0}{\longrightarrow} & K_{n+1}(C_0(\partial E^*)) & \overset{\partial}{\longrightarrow} & K_n(C_0(\tilde{E}^*)) \\
& & \downarrow{d(f \sqcup f') \otimes \text{Bott}} & & \downarrow{d\tilde{F}_1} \\
& & K_n(C_0(\mathfrak{A}^*BG)) & &
\end{array}
\]

where the boundary morphism \(\partial\) is nothing but \(di \otimes \text{Bott}\), with \(i: \partial W \hookrightarrow W\) the inclusion. Then

\[
x \otimes ev_0 \otimes d(f \sqcup f') \otimes \text{Bott} = \beta(M, f) - \beta(M', f').
\]

But

\[
x \otimes ev_0 \otimes d(f \sqcup f') = x \otimes ev_0 \otimes \partial \otimes d\tilde{F}_1 = 0
\]

by exactness of the top row exact sequence. This proves that

\[
\beta(M, F) = \beta(M', f').
\]
Definition of $\text{Ind}_{BG}: R_{n+1}^{\text{pin}}(BG) \to K_n(BG \times (0,1))$

Let us consider an element $(W, f: W \to BG, g) \in R_{n+1}^{\text{pin}}(BG)$. Consider the Dirac operator $\mathcal{D}_f$, since we have positive scalar curvature on the boundary, using [3.1], (3.2) and the KK-equivalence [1.18] we obtain a class

$$y \in K_{n+1}(BG_f(W, \partial W)_ad)$$

(see Definition 1.11). Hence we can define the map $\text{Ind}_{BG}$ in the following way

$$(W, f: W \to BG, g) \mapsto y \otimes ev_1 \otimes \mu_f \otimes \text{Bott} \in K_n(BG \times (0,1)),$$

where

- $ev_1: BG_f(W, \partial W)_ad \to BG_f(\hat{W})$ is the evaluation at $t = 1$ in the adiabatic deformation;
- $\mu_f$ is the Morita equivalence associated to the pull-back construction;
- $\text{Bott}: K_{n+1}(BG \times (0,1) \times (0,1)) \to K_n(BG \times \mathbb{R})$ is the Bott periodicity given by Corollary 1.14.

This map is well-defined on bordism classes: let $(W, f, g)$ and $(W', f', g')$ be two triples in $R_{n+1}^{\text{pin}}(BG)$; let $N$, $(BG_F^p, h)$, $(Y, F)$ and $(Z, F')$ be as in Definition 3.21.

Since $Y$ is a boundary, by Remark 1.10 and [15, Theorem 4.3] the $BG$-index $z \in K_{n+1}(BG_F^p)$ of $\mathcal{D}_F$ is zero. Let $i$ be the inclusion of $W \sqcup W'$ in $Y$. As the scalar curvature on $N$ is positive, we have that $ev_N(z) = 0$ and then $z$ is an element of $K_{n+1}(BG_f(W) \sqcup BG_{f'}(W'))$, that is nothing but $K_{n+1}(BG_f(W)) \oplus K_{n+1}(BG_{f'}(W'))$. This element is the direct sum of $ev_1(y)$ and $-ev_1(g')$ (the sign $-$ is given by the orientation in the pasting process), namely the indices of $\mathcal{D}_f$ and $\mathcal{D}_{f'}$ respectively. By the definition of $F$, it follows that $\mu_i \otimes \mu_F = \mu_{f \sqcup f'}$. Hence

$$\text{Ind}_{BG}(W, f, g) - \text{Ind}_{BG}(W', f', g') = y \otimes ev_1 \otimes \mu_f \otimes \text{Bott} - y' \otimes ev_1 \otimes \mu_{f'} \otimes \text{Bott} =$$

$$= (y \otimes ev_1 \oplus -y' \otimes ev_1) \otimes \mu_{f \sqcup f'} \otimes \text{Bott} =$$

$$= (y \otimes ev_1 \oplus -y' \otimes ev_1) \otimes \mu_i \otimes \mu_F \otimes \text{Bott} =$$

$$= z \otimes \mu_F \otimes \text{Bott} = 0$$

Definition of $\varrho$: $\text{Pos}_{n}^{\text{pin}}(BG) \to K_n(BG_{ad}^o)$

Let $(M, f, g)$ be a triple in $\text{Pos}_{n}^{\text{pin}}(BG)$. In this case, since the algebroid is spin and the scalar curvature is positive, the Dirac operator $\mathcal{D}_f$ defines directly a class $\varrho(\sigma(\mathcal{D}_f), \mathcal{D}_f) \in K_n((BG_f^c)_{ad})$ associated to the path constantly equal to $\mathcal{D}_f$, as in the Remark 3.4. Then we can give the following definition of $\rho$-class:

$$\varrho(M, f, g) := \varrho(\sigma(\mathcal{D}_f), \mathcal{D}_f) \otimes f_i^ad \in K_n(BG_{ad}^o). \quad (3.10)$$

We should check that this map is well-defined, but the proof of this fact is completely analogous to the one of Proposition 3.17. Finally, using the Theorem 3.6 the commutativity of the diagram (3.9) is obvious.

3.7 Product formulas for secondary invariants

Now we would like to apply the product in Definition 3.8 to the $\varrho$ invariant of Definition 3.16 and (3.10).

Proposition 3.22. Let $G \rightrightarrows X$ and $H \rightrightarrows Y$ be two Lie groupoid homotopy equivalent by means of the groupoid morphism $\varphi: H \to G$. Let $J \rightrightarrows Z$ be another Lie groupoid. Consider the secondary invariant $\varrho(\varphi) \in K_r(C^*_r(G_{ad}^o))$ and the symbol class of the $J$-signature operator on $Z$, given by $[\sigma_J] \in K_J(\mathfrak{A}(J))$. 

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Then we have the following product formula

\[ \varrho(\varphi) \boxtimes [\sigma_j] = \varrho(\varphi \times \text{id}_Z) \in K_{i+j}((G \times J)^{o}_{ad}), \]

where \( \varphi \times \text{id}_J \) is a homotopy equivalence between \( H \times J \) and \( G \times J \).

Proof. If \( L = G_{\text{ad}}^{\varphi} \cup \text{id}_X \), then \( \varrho(\varphi) = [D^{HS}_{L,i]} \otimes (\varphi \cup \text{id}_X)^{ad} \). Consequently, following the notations of Definition 3.8, one has that \( \varrho(\varphi) \boxtimes [\sigma_j] \) is equal to

\[ \left( \left[ [D^{HS}_{L,i]} \otimes (\varphi \cup \text{id}_X)^{ad} \right] \otimes_{D} [\text{id} \otimes \text{ev}_0]^{-1} \otimes_{D'} \Delta, \right. \]

where \( D = C^*_r(G_{ad}^{\varphi}) \otimes C^*_r(\mathfrak{A}(J)) \) and \( D' = C^*_r(G_r^{ad}) \otimes C^*_r(J_{ad}) \).

That is equal to

\[ \left( \left[ [D^{HS}_{L,i]} \otimes (\varphi \cup \text{id}_X)^{ad} \right] \otimes_{D} [\text{id} \otimes \text{ev}_0]^{-1} \otimes_{D'} \Delta, \right. \]

Notice that the following equalities holds:

- by Remark 2.24 we have that

\( ((\varphi \cup \text{id}_X)^{ad} \otimes (\text{id}_Z)) \otimes_D [\text{id} \otimes \text{ev}_0]^{-1} = [\text{id} \otimes \text{ev}_0]^{-1} \otimes ((\varphi \cup \text{id}_X)^{ad} \otimes (\text{id}_Z)^{ad}) \);

- moreover it is easy to verify that

\( ((\varphi \cup \text{id}_X)^{ad} \otimes (\text{id}_Z)^{ad}) \otimes \Delta = \Delta \otimes (\varphi \times \text{id}_Z \cup \text{id}_X \times Z)^{ad}. \)

Then it turns out that

\[ \varrho(\varphi) \boxtimes [\sigma_j] = [D^{HS}_{L,i}] \otimes_{\mathfrak{A}} [\sigma_j] \otimes (\varphi \times \text{id}_Z \cup \text{id}_X \times Z)^{ad}. \]

So it only remains to notice that \([D^{HS}_{L,i}] \otimes [\sigma_j] = [D^{HS}_{L,i,j}] \in K_{i+j}((G \times J)^{o}_{ad}). \) And the proposition is proved.

\[ \Box \]

One can similarly prove the analogous result for Dirac operators.

**Proposition 3.23.** Let \( G \rightrightarrows X \) and \( H \rightrightarrows Y \) be two Lie groupoids such that both \( BG \) and \( BH \) are smooth manifolds. Let \( (M, f, g) \) be a triple in \( \text{Pos}^{\text{spin}}_n(BG) \) and let \( (N, f') \) be an element in \( \Omega^m_+ \circ BH \). Then we have that

\[ \varrho(M, f, g) \boxtimes \beta(N, f') = \varrho(M \times N, f \times f', g \oplus h) \in K_{m+n}((BG \times BH)^{o}_{ad}) \]

where \( h \) any metric on \( \mathfrak{A}(BH^r_{f'}) \) such that \( g \oplus h \) on \( \mathfrak{A}(BG^r_f) \oplus \mathfrak{A}(BH^r_{f'}) \) has positive scalar curvature.

**Appendix A Classifying spaces and 1-cocycles**

In this section we are going to recall some basic construction from [9] [10]. Let \( G \rightrightarrows X \) be a topological groupoid.

**Definition A.1.** Let \( Y \) a topological space and \( \{U_i\}_{i \in I} \) an open cover of \( Y \).

- A 0-cocycle with values in \( G \), defined on the cover \( \{U_i\}_{i \in I} \) is the data of a continuous application \( \mu_i: U_i \to G \) for each \( i \in I \), such that \( \mu_i(y) = \mu_j(y) \) for any pair \((i, j) \in I \) and for any \( y \in U_i \cap U_j \). It is actually a global continuous function \( \mu: Y \to G \).
A 1-cocycle with values in $G$, defined on the cover $\{U_i\}_{i \in I}$ is the data of a continuous application

$$\lambda_{ij} : U_i \cap U_j \to G$$

for any pair $(i,j)$, in a such way that, if $y \in U_i \cap U_j \cap U_k$, then $\lambda_{ij}(y)$ is composable with $\lambda_{jk}(y)$ and

$$\lambda_{ik}(y) = \lambda_{ij}(y)\lambda_{jk}(y).$$

A 1-cocycle with values in $G$, defined on the cover $\{U_i\}_{i \in I}$ is the data of a continuous application

$$\lambda_{ij} : U_i \cap U_j \to G$$

for any pair $(i,j)$, in a such way that, if $y \in U_i \cap U_j \cap U_k$, then $\lambda_{ij}(y)$ is composable with $\lambda_{jk}(y)$ and

$$\lambda_{ik}(y) = \lambda_{ij}(y)\lambda_{jk}(y).$$

We will say that two 1-cocycles $\lambda$ and $\lambda'$ are cohomologous if there exists a function $\mu : Y \to G$ such that $\lambda_{ij}(y) = \mu(y)\lambda'_{ij}(y)\mu(y)^{-1}$. Let $H^1(Y, \{U_i\}_{i \in I}, G)$ be the set of the cohomology classes of $G$-valued 1-cocycles associated to the covering $\{U_i\}_{i \in I}$. Finally define $H^1(Y, G)$ as the limit $\lim \{H^1(Y, \{U_i\}_{i \in I}, G)\}$, where $\{U_i\}_{i \in I}$ runs over all open covers of $Y$.

If $\varphi : Y' \to Y$ is a continuous map, then we have a natural map $\varphi^* : H^1(Y, G) \to H^1(Y', G)$ that associates to a 1-cocycle $(\lambda_{ij}, \{U_i\}_{i \in I})$ the pull-back $(\lambda_{ij} \circ \varphi, \{\varphi^{-1}(U_i)\}_{i \in I})$.

**Remark A.2.** If $(\lambda_{ij}, \{U_i\}_{i \in I})$ is a $G$-valued 1-cocycle on $Y$, then $\lambda_{ii}$ takes values in $G^{(0)} = X$ for any $i \in I$ and $\lambda_{ij}(x) = \lambda_{ji}^{-1}(x)$ for any $i,j \in I$ and $x \in U_i \cap U_j$.

Let $(\lambda_{ij}, \{U_i\}_{i \in I})$ be a $G$-valued 1-cocycle on $Y$, then one can canonically construct a groupoid $G^\lambda_X$ over $Y$ in the following way:

- take the disjoint union $\bigsqcup_i U_i$ of all the open sets of the cover;
- consider the map $\Lambda : \bigsqcup_i U_i \to X$ given by $\lambda_{ii}$ on each $U_i$;
- build the pull-back groupoid $G^\lambda_X = \bigsqcup_{i,j} U_i \times X G \times X U_j$;
- finally define $G^\lambda_X$ as the quotient of $G^\lambda_X$ by the following equivalence relation: $(y_i, \gamma, y_j) \sim (y_k, \gamma', y_h)$, with $(y_i, \gamma, y_j) \in U_i \times X G \times X U_j$ and $(y_k, \gamma', y_h) \in U_k \times X G \times X U_h$, if $y_i = y_k \in U_i \cap U_k$, $y_j = y_h \in U_j \cap U_h$ and $\gamma' = \lambda_{jk}(y_i)\gamma\lambda_{ih}(y_j)$.

Of course the isomorphism class of the groupoid $G^\lambda_X \rightrightarrows Y$ depends only on the cohomology class of $\lambda$. Notice that the groupoid $G \rightrightarrows X$ itself is associated to the cocycle $\lambda \in H^1(X, G)$ given by the identity on $X$.

In the literature there are many equivalent definition of the classifying space $BG$ of $G$. In this thesis we will take as definition the following proposition.

**Proposition A.3.** There exists a unique space $BG$ up to homotopy, equipped with a universal 1-cocycle $\xi \in H^1(BG, G)$ such that for any 1-cocycle $\lambda \in H^1(Y, G)$ on a topological space $Y$, there exists a unique function $f : Y \to G$, up to conjugation by 0-cocycles, such that

$$\lambda = f^*\xi \in H^1(Y, G).$$

Let $BG \rightrightarrows BG$ be groupoid associated to the 1-cocycle $\xi \in H^1(BG, G)$. One can easily check that for any $f : Y \to BG$, the groupoid $G^f_{\xi,\xi} \rightrightarrows Y$ is isomorphic to $BG^f_f \rightrightarrows Y$, the pull-back of $BG$ along $f$.

**Remark A.4.** Let $\lambda \in H^1(Y, G)$ be represented by $(\lambda_{ij}, \{U_i\}_{i \in I})$ and let $\{\alpha_i\}_{i \in I}$ a partition of the unity associated to a locally finite cover $\{U_i\}_{i \in I}$. Then the function $f : Y \to BG$ such that $G_\lambda^X \cong BG^f_f$ is given by

$$\sum_{i \in I} \alpha_i \lambda_{ii} : Y \to BG.$$

Now let us assume that $G$ is a Lie groupoid. A smooth 1-cocycle $(\lambda_{ij}, \{U_i\}_{i \in I})$ is transverse if $\lambda_{ii}$ is transverse for any $i \in I$. It is clear that if $\lambda$ is a 1-cocycle smooth and transverse, then $G^\lambda_X$ is a Lie groupoid. In particular if $BG$ is a smooth manifold, then the function $f : Y \to BG$ is a smooth function transverse with respect to $BG$.
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