Separation of variables in Hamilton-Jacobi equation for a charged test particle in the Stackel spaces of type (2.1)

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Abstract

Found all equivalence classes for electromagnetic potentials and space-time metrics of Stackel spaces, provided that the equations of motion of the classical charged test particles are integrated by the method of complete separation of variables in the Hamilton-Jacobi equation. Separation is carried out using the complete sets of mutually-commuting integrals of motion of type (2.1), whereby in a privileged coordinate system the Hamilton-Jacobi equation turns into a parabolic type equation.

Keyword: Hamilton-Jakobi equation, separation of variables, Killings vectors and tensors, integrals of motion.

1 Introduction

The theory of Stackel spaces is one of the consequences of the symmetry theory. A distinctive feature of a Stackel space is the existence of a so-called complete set in it, consisting of Killing vector and tensor fields, mutually commuting, and satisfying some additional conditions. This makes it possible to carry out a complete separation of variables in classical and quantum equations of the test particle motion. (see for example[1] -[11]). Stackel spaces were named after Paul Stäckel, who found the first example of a similar space [2]. Besides Stackel, the main contribution to the construction of the theory was made by Levi-Civita [3], Yarov-Yarovoy [4], and V.V.Shapovalov [5] -[6]. Shapovalov proved the basic theorem of the theory of Stackel spaces. In [7], the theory was generalized to the case of complex privileged coordinate systems.

Shapovalov’s theorem allowed invariant partitioning of a set of Stackel metrics into equivalence classes. According to this partition, the space itself and its complete sets belong to $(N,N_0)$ type, where $N$ is the maximum number of independent Killing vector fields $Y^s_p$ included in the full set; $N_0 = N - \det(Y^s_pY^t_q)$. It can be said in such cases that there is a complete separation of type $(N,N_0)$. In the case of the Lorentz signature, $N - 0 = 0, 1$. If $N - 0 = 0$, the space is called non-null. Otherwise, null. There are seven disjoint sets of Stackel spaces with the given signature; respectively: four non-null ones: (0.0), (1.0), (2.0), (3.0); and three null ones (3.1), (2.1), (1.1). A more detailed description of the theory and a rather detailed bibliography can be found in the works [8]-[11].
Although there are some examples of successful application of non-commutative integration methods for classical and quantum equations of test particles motion (see [12]-[15]), the Stackel spaces, thanks to their rich geometric content remain the most interesting object for research in various sections of theoretical and mathematical physics. They occupy a special place in the theory of gravitation (see for example [16]-[24]) and cosmology, since the Robertson-Walker space belongs to the Stackel class. A high level of space symmetry allows a successful construction of new cosmological models, particularly those which include dark energy or features of modified theories of gravity (see for example [25], [26]), and also allows the use of symmetry theory methods to further justify the choice of models of extended gravity theory (see for example [27]).

Shapovalov’s theorem allows to classify the metrics of the Stackel spaces in the presence of physical fields of different nature. The classification refers to the enumeration of all equivalence classes for metrics and fields with respect to a group of allowable (that is, not violating the conditions of the complete separation of variables) coordinates and potentials transformations. The set of all Stackel spaces is broken down into disjoint subsets containing spaces with metrics of type \( (N.N_0) \). Therefore, classification is carried out separately for each type of space. A large number of works in the theory of gravitation are devoted to the classification problems. (see, for example, the review in [8]). However, they have not been studied until recently for the case where a charged test particle moves in an external electromagnetic field in the absence of any restrictions on that field. For the first time this classification problem has been set and solved in our works: for the spaces of type \((2.0)\) - in [8]; for the spaces of type \((1.0)\) - in [9]. In this paper, the spaces of type \((2.1)\) are studied. Metrics of this type have been investigated in both flat space-time ([28]-[29]) and curved one (see for example [30] -[36])

The Hamilton-Jacobi equation for a charged test particle has the form:

\[
g^{ij}P_iP_j = \tilde{m},
\]

where \( P_i = p_i + A_i = \partial S/\partial u^i + A_i \). Everywhere in the text, lowercase letters with the tilda icon denote constants.

Note that the paper [37] considered the problem of complete separation of variables in the "natural Hamilton-Jacobi equation" containing in addition to the electromagnetic potential a quite arbitrary scalar field. In this case, there are no additional constraints on the Stackel space metric, and the potential is to be set by the Stackel matrix and arbitrary functions, each of which depends on one of the non-ignorable variables.

The complete integral of the equation (1) can be represented in the form:

\[
S = \sum s_i(u^i). \tag{2}
\]

The coordinate system, in which it is possible is called privileged, and it is denoted by the variables: \( u^i, \quad i, j = 0, \ldots, 3 \). Everywhere in the text, there is a summation within the specified limits of index change on repeated upper and lower indexes. By diagonalizing the Killing vector fields \( Y^i_p \) from the full set, let us bring the functions \( s_p \) in (2) to the form: \( s_p = \tilde{\lambda}_p u^p \). Therefore \( u^p \) are ignorable coordinates. Let us denote their coordinate indices with letters \( p, q = 0, 1, \ldots, N - 1 \). Non-ignorable coordinates will have the indexes \( \mu, \nu = N, \ldots, 3 \). Functions that depend only on the variable \( u^2 \) will be denoted by the lowercase Greek letters, and only on the variable \( u^3 \) - by the lowercase Latin letters. Exceptions: Kronecker characters: \( \delta^{ij}, \delta_{ij}, \delta^i_j \); components of metric tensor: \( g^{ij}, g_{ij} ; \quad \varepsilon, \varepsilon_i = +1, -1 \); \( \lambda, \lambda_i = const \); \( h^i_\nu, h^i_\nu, h^i_\nu \) - function of \( u^\nu \).
In these notations, the metric tensor of space, vector, and tensor fields of Killing can be written as:

\[
g_{ij} = X_{ij}^{3} = (\hat{\Phi}^{-1})_{ij}^{3} h^{ij}_{\nu} = \frac{\delta_{p}^{i} \delta_{q}^{j} h^{pq} + \delta_{a}^{i} \delta_{b}^{j} \beta^{ab} + \varepsilon^{3} \delta_{3}^{i} \delta_{3}^{j}}{\Delta},
\]

\[
Y_{p}^{i} = \delta_{p}^{i}, \quad X_{2}^{ij} = f \frac{\delta_{a}^{i} \delta_{b}^{j} \beta^{ab} - \phi \varepsilon^{3} \delta_{3}^{i} \delta_{3}^{j}}{\Delta},
\]

\[
(\hat{\Phi})_{\nu}^{\mu} = \left( \begin{array}{cc}
1 & \phi \\
-1 & f
\end{array} \right), \quad \Delta = \phi + f = \det \hat{\Phi}, \quad h^{pq} = h^{pq}_{3}, \quad \beta^{ab} = h^{ab}_{2},
\]

\[a, b = 0, 1, 2; \quad p, q = 0, 1; \quad \nu, \mu = 2, 3.\]

Since Killing vector fields satisfy the condition:

\[
\det(Y_{p}^{i} Y_{q}^{i}) = 0 \rightarrow \det(g_{pq}) = 0 \rightarrow g^{22} = 0,
\]

the Hamilton-Jacobi equation is a parabolic type equation.

For the Hamilton-Jacobi free equation, the complete set of integrals of motion has the form:

\[
\hat{X}_{\nu} = X_{\nu}^{i} p_{i} p_{j}, \quad \hat{X}_{q} = Y_{q}^{i} p_{i}, \quad (Y_{q}^{i} = \delta_{q}^{i}),
\]

When the classification is implemented, complete sets of integrals of motion for the Hamilton-Jacobi equation in the external electromagnetic field will be found. The full integral in the preferred coordinate system can be reduced to the form:

\[
S = \lambda_{0} u^{0} + \lambda_{1} u^{1} + s_{2}(u^{2}) + s_{3}(u^{3}).
\]

The complete set of integrals of motion allows you to find the complete integral as a solution to the system of equations:

\[
\hat{X}_{q} = Y_{q}^{i} p_{i}, \quad \hat{X}_{\nu} = \lambda_{\nu}, \quad \lambda_{3} = \tilde{m}.
\]

Here \( \hat{X}_{\nu} \) — integrals of motion in case of a charged particle have the form:

\[
\hat{X}_{\nu} = (\hat{\phi}^{-1})_{\nu}^{\mu} \hat{H}_{\mu} = (\hat{\phi}^{-1})_{\nu}^{\mu} (h_{\mu}^{ij} p_{i} p_{j} + 2 h_{\mu}^{i} p_{i} + h_{\mu}),
\]

where

\[
h_{2}^{ij} = \delta_{a}^{i} \delta_{b}^{j} \beta^{ab}, \quad h_{2}^{i} = \delta_{a}^{i} \gamma^{a}, \quad h_{2}^{i} = \rho, \quad h_{3}^{ij} = \delta_{p}^{i} \delta_{q}^{j} \varepsilon^{pq} + \varepsilon^{3} \delta_{3}^{i} \delta_{3}^{j}, \quad h_{3}^{i} = h_{i}, \quad h_{3} = r.
\]

The \( \delta \) relationships allow us to find the contravariant components of the metric tensor and the electromagnetic potential, and also to obtain the condition of separation of variables in the equation \( \Pi \). To do this, equate the coefficients before impulses and their products on the right and left in the equation \( \delta \) when \( \nu = 3 \). As a result, we get:

\[
A^{p} = \frac{\gamma^{p} + h^{p}}{\Delta}, \quad A^{2} = \frac{\sigma}{\Delta}, \quad A^{\nu} = 0, \quad A_{i} A^{i} = \frac{\hat{\omega} + \hat{h}}{\Delta}.
\]

Using gradient transformations of potential as well as the group of allowable privileged variable transformations:

\[
u^{p} \rightarrow \tilde{c}^{p} u^{p} + \int \psi^{p} du^{2}, \quad u^{2} \rightarrow \int \psi du^{2},
\]

\[3\]
bring matrix $\hat{G} = (G^{ab}) = (g^{ab}\Delta)$, and $A^b$ to the form:

$$\hat{G} = \begin{pmatrix} a_0, & a_1, & 1 \\ a_1, & a + \beta, & -\phi \\ 1, & -\phi, & 0 \end{pmatrix} ; \quad A^0 = \frac{h^0}{\Delta}, \quad A^1 = \frac{h^1 + \gamma}{\Delta}, \quad A^2 = \frac{\sigma}{\Delta}, \quad A^3 = 0. \quad (11)$$

Here and further we denote: $a^{00} = a_0, \quad a^{10} = a_1, \quad a^{11} = a, \quad \beta^{11} = \beta$. Functions $\beta^{00}$ and $\beta^{01}$ can become null by the transformation (10). The functions $a_0, \quad a_1, \quad a$ are linearly dependent. In the future, we will need to classify the $\hat{A} = (a^{pq})$ matrices relative to the group (10). Let us list all the non-equivalent classes of $\hat{A}$ matrices.

$$\begin{align*}
\hat{A}_{(1)} &= \begin{pmatrix} a_0 & a_1 \\ a_1 & a \end{pmatrix}, \\
\hat{A}_{(2)} &= \begin{pmatrix} a_0 & a_1 \\ a_1 & -a_0 \end{pmatrix}, \\
\hat{A}_{(3)} &= \begin{pmatrix} a_0 & a_1 \\ a_1 & 0 \end{pmatrix}, \\
\hat{A}_{(4)} &= \begin{pmatrix} a_0 & 0 \\ 0 & a \end{pmatrix}, \\
\hat{A}_{(5)} &= \begin{pmatrix} a & 0 \\ 0 & \varepsilon a \end{pmatrix}, \varepsilon = 1, -1, \\
\hat{A}_{(6)} &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.
\end{align*} \quad (12)$$

We’ll make another point. When considering the resulting conditions of the complete separation of variables in the equation (11), the ratio arises:

$$\det a^{pq} = \hat{a} + \hat{e}_{pq}a^{pq}.$$

Using valid (10) coordinate transformations, it can be simplified and represented as:

$$\det a^{pq} = \hat{a}$$

- for all matrices $\hat{A}_{(\alpha)}$ except $\hat{A}_{(3)}$. Where $\alpha = 3$ we have:

$$- \det a^{pq} = \hat{a} + \varepsilon a_0.$$

In the accepted notations the necessary and sufficient condition for the complete separation of variables in the equation (11):

$$A^i A_i = \frac{\rho + r}{\Delta}$$

- can be transformed to:

$$(\rho + r + a_0\sigma^2 - 2\sigma h^0)(a_0\phi^2 + 2\phi a_1 + a + \beta) =$$

$$= (h^{0\phi} + h^1 + \gamma - (a_0\phi + a_1)\sigma)^2. \quad (13)$$

Let us present the covariant components of the metric tensor and electromagnetic potential in the following convenient form for calculations:

$$\hat{G}^{-1} = (G_{ab}) = (g_{ab}\Delta) = \begin{pmatrix} \frac{\phi^2}{C^2}, & \frac{\phi}{C}, & \frac{1 - \phi B}{C}, \\
\frac{\phi}{C}, & \frac{1}{C}, & \frac{1 - \phi B}{C}, \\
1 - \phi B, & -B, & B^2 - a^0 \end{pmatrix}, \quad (14)$$
\[ A_0 = \phi A_1 + \sigma, \quad A_1 = \frac{h^0 \phi + h^1 + \gamma - B \sigma}{G}, \quad A_2 = h^0 - a_0 \sigma - B A_1, \quad A_3 = 0. \quad (15) \]

It is denoted here by: \( G = -(a_0 \phi_2 + 2 \phi a_1 + a + \beta), \quad B = a_0 \phi + a_1. \)

Solving the functional equation (13) is equivalent to solving two overcrowded systems of the second and third degree algebraic equations. The first system includes functions only from the variable \( u^2 \), the second only from the variable \( u^3 \). As already noted, we will look for solutions in the presence of additional symmetry, for which we will require that the electromagnetic field be external. We call the electromagnetic field external if the potentials contain at least one function independent of the metric tensor. We call such functions free. All other functions included in the electromagnetic potential, we will call related. We also believe that the functions included in (13) are continuous and smooth, therefore there are points on coordinate axes in the neighbourhood of which none of the non-zero functions vanish. In the future, if it is necessary to fix the variables \( u^\nu \) in the process of solving functional equations, we will use exactly such points.

2 Solutions for the case when at least one of the functions \( h^p \) is free.

In this section we find all the solutions for the case when free functions depending on \( u^3 \) exist. Everywhere in the section, all functions from the variable \( u^3 \) except \( h^p \) are related.

I. Both \( h^q \) functions are free.

Let us use the above-mentioned smoothness condition and consider the equation (13) at the fixed point \( \tilde{u}^2 \) on the coordinate axis \( (u^2) \). Since \( G \neq 0 \), the function \( r \) is expressed through the \( h^p \) functions as follows:

\[ r = r_0 + \tilde{h} h^0 + l(\tilde{h}^1 + \tilde{\sigma} h^0 + c)^2, \quad (16) \]

Substituting (16) into (13) one obtains the equation:

\[ b_{pq} h^p h^q + b_p h^p + b = 0. \]

As \( \partial b_{pq} / \partial h^r = \partial b_p / \partial h^r = \partial b_p / \partial h^r = 0 \rightarrow b_{pq}, b_p, b = 0 \). Using the transformation group (10), one gets:

\[ G = t, \quad \phi = t_1 G, \quad \phi^2 = t_0 G \quad \rightarrow \quad \phi = \beta = 0. \]

\[ \gamma - \sigma a^0 = b_0, \quad \sigma = b_1 \quad \rightarrow \quad \gamma = \sigma = 0. \]

We obtain the solution to the equation (13):

\[ \hat{G} = \begin{pmatrix} a_0 & a_1 & 1 \\ a_1 & a & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A^p = \frac{h^p}{\Delta}, \quad A^2 = A^3 = 0. \quad (17) \]

II. \( h^p \) contain one free function.

Substituting (10) into (13), fixing the variable \( u^2 \) in another point of the coordinate axis and solving the resulting equation with respect to \( h^1 \), one gets:

\[ h^1 = w h^0 + u + Q, \quad Q^2 = ph^0 + 2qh^0 + l. \quad (18) \]
Substituting (18) into (13) one obtains the equation:

\[ b_{pq}h^q + b_p h^p + b = 0. \]

As before from \( \partial b_{pq}/\partial h^r = \partial b_p/\partial h^r = \partial b/\partial h^r = 0 \) it follows: \( b_{pq}, b_p, b = 0 \). Let us denote \( z = q^2 - pl \) and consider separately the variants \( z = 0 \) and \( z \neq 0 \). As a result, using the transformation group (10) , we get:

**A) \( z = 0 \).** In this case, (13) contains summands that have four linearly independent functions \( h^0, h^0Q, h_0Q^2 \), as well as a free member. By denoting:

\[ r = r_1 + th^{02} + 2nh^0 + 2m_0h^0Q + 2r_0Q, \]

we get the following systems of equations:

\[
\begin{cases}
(\phi + w)^2 + p = tG, \\
\phi + w = m_0G.
\end{cases}
\]

(19)

\[
\begin{cases}
q + (\phi + w)(\gamma - B\sigma + u) = (n - \sigma)G, \\
u + \gamma - B\sigma = r_0G, \\
(u + \gamma - B\sigma)^2 + l = (\rho + r_1 + a_0\sigma^2)G.
\end{cases}
\]

(20)

One can show that from the (19) system it follows \( \phi = 0 \). Then it’s obvious that \( a = 0 \). Otherwise we get the previous solution (17). The system of equations (19), (20) has a singular non-trivial solution:

\[ a_0 = a_1 = a = \gamma = t = m_0 = w = p = n = r_0 = u = 0, \quad q = \tilde{q}, \quad \sigma = -\tilde{q}/\beta, \quad h_0 = \frac{h^{12}}{2q}. \]

We will write it as:

\[
G = \begin{pmatrix}
0, & 0, & 1 \\
0, & \beta, & 0 \\
1, & 0, & 0
\end{pmatrix}; \quad A^0 = \frac{h^2\tilde{a}}{\Delta}, \quad A^1 = \frac{h}{\Delta}, \quad A^2 = -\frac{1}{2\tilde{a}\beta\Delta}, \quad A^3 = 0. \quad (21)
\]

**B) \( z \neq 0 \).** In this case, from the ratio (18) it follows:

\[ h^p = b^p h + h_0^p, \]

(22)

where \( h \) is a free function. Substitute (22) into (13), and equate to zero the coefficient before the function \( h^2 \). As a result, we get:

\[ (\phi b^0 - b^1)^2 = \tilde{e}(\phi^2 a_0 + 2\phi a_1 + a + \beta), \quad \tilde{e} = 0, 1. \]

(23)

First consider the variant:

1) \( \tilde{e} = 0 \). From (23) it follows \( b^1 = \phi = 0 \). With this taken into account from (13) we get:

\[ (\gamma + h^1)^2 = (\rho + r)(a + \beta), \quad \sigma = 0. \]

(24)

Non-trivial solutions of the equation (24) are:

\[
a) \quad h^1 = \gamma = 0, \quad b) \quad a = h^1 = 0. \]

6
The contravariant components of the metric tensor and potential have the form:

\[
\hat{g} = \begin{pmatrix}
\frac{b_0^2}{\Delta} & \frac{b_0 b_1}{\Delta} & \frac{1}{\Delta} & 0 \\
\frac{b_0 b_1}{\Delta} & \frac{b_1^2}{\Delta} & \frac{-\phi}{\Delta} & 0 \\
\frac{1}{\Delta} & \frac{-\phi}{\Delta} & \Delta & 0 \\
0 & 0 & 0 & \frac{\epsilon}{\Delta}
\end{pmatrix}; \quad A^0 = \frac{h}{\Delta}, \quad A^1 = A^2 = A^3 = 0. \tag{25}
\]

\[
\hat{g} = \begin{pmatrix}
\frac{b_0^2}{\Delta} & \frac{b_0 b_1}{\Delta} & \frac{1}{\Delta} & 0 \\
\frac{b_0 b_1}{\Delta} & \frac{b_1^2}{\Delta} & \frac{\gamma}{\Delta} & 0 \\
\frac{1}{\Delta} & \frac{\gamma}{\Delta} & \Delta & 0 \\
0 & 0 & 0 & \frac{\epsilon}{\Delta}
\end{pmatrix}; \quad A^0 = \frac{h}{\Delta}, \quad A^1 = \frac{\gamma}{\Delta}, \quad A^2 = A^3 = 0. \tag{26}
\]

2) \( c = 1 \). Let us consider the equation (23) in the fixed point \( u^3 = \tilde{u}^3 \). As a result, we get:

\[ \beta = \tilde{a}_0 \phi^2 + 2 \tilde{a}_1 \phi + \tilde{a}, \]

which relative to the \( \ref{10} \) transformations is equivalent to \( \beta = 0 \). The equation (23) takes the form:

\[(a_0 - b^0 b_1^2)\phi^2 + 2(a_1 - b^0 b_1)\phi + a - b_1^2 = 0, \tag{27}\]

and has two solutions:

**a)** \( a_0 = b^0, \quad a_1 = b^0 b_1, \quad a = b^1 \). From the equal-zero in the equation (13) of the coefficient before the function \( h \) it follows:

\[ h_0^1 + h_0^0 \phi + \gamma = \tau(b^1 + b^0 \phi). \]

Solutions are equivalent to the following: \( \gamma = h_0^0 = 0 \). That is why contravariant components of the metric tensor and potential have the form:

\[
(g^{ij}) = \begin{pmatrix}
\frac{b_0^2}{\Delta} & \frac{b_0 b_1}{\Delta} & \frac{1}{\Delta} & 0 \\
\frac{b_0 b_1}{\Delta} & \frac{b_1^2}{\Delta} & \frac{-\phi}{\Delta} & 0 \\
\frac{1}{\Delta} & \frac{-\phi}{\Delta} & \Delta & 0 \\
0 & 0 & 0 & \frac{\epsilon}{\Delta}
\end{pmatrix}; \quad A^p = \frac{b^p h}{\Delta}, \quad A^2 = \frac{\sigma}{\Delta}, \quad A^3 = 0. \tag{28}
\]

**b)** \( a = b^1, \quad \phi = 0 \). From the equality to zero of the coefficient before the function \( h \) in the equation (13) it follows:

\[ (b^0 b_1^1 - a_1)\sigma = t - \gamma. \]

We believe that \( \sigma \neq 0 \), because otherwise we will get the already considered option. Therefore \( a_1 = b^0 b_1^1, \quad \gamma = 0 \). As \( b^1 \neq 0 \rightarrow h_0^1 = 0 \). Taking this into account, the equation (13) will take the form:

\[ (b^0 b_1 - a_0) = \rho + r - 2h_0^0 \sigma. \]

Therefore \( a_0 = b^0, \quad h_0^0 = 0 \). Obtained a particular solution (28).

The variables \( u^2 \) and \( u^3 \) enter the equation unsymmetrically. Therefore, to complete the classification, one must consider separately the case where free functions depend on the \( u^2 \) variable. At the same time, if free functions from \( u^3 \) arise in the process of the solution, the corresponding variants may not be considered.
3 Solutions for the case when both functions $\gamma$ and $\sigma$ are free.

Everywhere in the text, all functions from the variable $u^2$ except $\gamma$ and $\sigma$ are related. Let both $\gamma$ and $\sigma$ are free. Consider the equation (13) at a fixed point $\tilde{u}^2$. Since $G \neq 0$, the function $\rho$ is expressed through the $\gamma$ and $\sigma$ functions as follows:

$$\rho = \rho_0 + \tau_0 \sigma^2 + \tau_1 \gamma^2 - 2\tau \sigma \gamma - 2\mu \sigma + 2\nu \gamma. \quad (29)$$

Substitute this expression back into the equation (13). After the obvious transformations we will get a system of equations:

$$\begin{cases}
\alpha_0 \phi^2 + 2\alpha_1 \phi + a = 0 \quad \rightarrow \quad G = \beta, \\
\alpha_0 \phi^2 + \alpha_1 = \beta \tau, \\
\alpha_0 + \tau_0 = \beta \tau^2, \\
h^1 + \phi h^0 = \beta \nu, \\
h^0 + \mu = \beta \nu \tau, \\
(h^1 + \phi h^0)^2 = \beta (\rho_0 + r). 
\end{cases} \quad (30)$$

From where immediately follows:

$$a = a_0 = a_1 = 0, \quad h^p = 0,$$

The contravariant components of the metric tensor and potential have the form:

$$\mathcal{g} = \begin{pmatrix}
0 & 0 & \frac{1}{\Delta} & 0 \\
0 & \frac{1}{\Delta} & \frac{1}{\Delta} \phi & 0 \\
\frac{-1}{\Delta} \phi & \frac{1}{\Delta} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\Delta}
\end{pmatrix}; \quad A^0 = 0, \quad A^1 = \frac{\gamma}{\Delta}, \quad A^2 = \frac{\sigma}{\Delta}, \quad A^3 = 0. \quad (31)$$

4 Solutions for the case of nonlinear dependence between functions $\gamma$ and $\sigma$.

Substituting the expression (29) into the equation (13). As a result, we will get the following functional equation:

$$(r + \rho_0 + \gamma^2 \pi + (\tau_0 + a_0) \sigma^2 + 2\sigma \gamma \tau - 2(\mu + h^0) \sigma + 2\nu \gamma)G = (H + \gamma - B \sigma)^2, \quad (32)$$

where it is denoted by $H = h^0 \phi + h^1$. Let us solve the equation (32) relative to $\gamma$.

1) $\pi \neq G$.

$$\gamma = \omega_0 \sigma^2 + \gamma_0 + \Sigma, \quad \Sigma^2 = \mu_0 \sigma^2 + 2\mu_1 \sigma + \mu. \quad \mu_1^2 - \mu_1 \mu \neq 0. \quad (33)$$

Substituting (33) into (13) one obtains the equation:

$$\theta_0 + \theta_1 \sigma + \theta_2 \sigma^2 + \theta_3 \Sigma + \theta_4 \sigma \Sigma = 0.$$ 

As $\partial \theta_a/\partial \sigma \neq 0 \rightarrow \theta_a = 0$. From this it follows:

$$\begin{cases}
(\omega_0 - B) = -\xi_0 G, \\
(\omega_0 - B)^2 + \mu_0 = G(\omega + a_0). 
\end{cases} \quad (34)$$
\[
\begin{align*}
\gamma_0 + H &= G\psi_0, \\
(\omega_0 - B)(\gamma_0 + H) + \mu_1 &= (\tau - h^0)G, \\
(\gamma_0 + H)^2 + \mu &= (r + \rho)G.
\end{align*}
\] (35)

Consider the equations of the system in order, using the classification (12).

Let us represent the system (34) as:
\[
\begin{align*}
(\phi a_0 + a_1) - \omega_0 &= \xi_0 (a_0 \phi_2 + 2 \phi a_1 + a + \gamma), \\
\omega_0^2 - 2(\phi a_0 + a_1)\omega_0 + \mu_0 &= \omega (a_0 \phi_2 + 2 \phi a_1 + a + \gamma) + a_0 \beta + \det a^{pq}.
\end{align*}
\] (36)

Consider the system (36) using matrices forms from (12), as well as appropriate ratios for \( \det a^{pq} \).

1) \( \hat{A}(1) \). In this case, it is obvious that the system (36) has no solutions because from the first equation of the system there follows a contradiction, since \( \xi_0 \to a_1 = 0 \), which is impossible. By reasoning similarly, it can be shown that the (36) system is contradictory also for the matrices \( \tilde{A}(2) \), \( \tilde{A}(3) \) and \( \tilde{A}(5) \). Let us consider the remaining options.

2) \( \hat{A}(4) \). \( a \) and \( a_0 \) are linearly independent functions. That is why it follows from (36):
\[
\omega = \omega_0 = \xi_0 = \phi = 0, \quad \mu_0 = \tilde{a}, \quad a = \frac{a}{a_0}.
\]

Substitute these expressions into (35). As a result, we will get the system:
\[
\begin{align*}
(\gamma_0 + h^1) &= (a + \beta)\psi_0, \\
\mu_1 &= (a + \beta)(\tau - h^0), \\
(\gamma_0 + h^1)^2 + \mu &= (a + \beta)(r + \rho).
\end{align*}
\] (37)

From the first equation follows \( \psi_0 = \tilde{\psi} \), which is equivalent to: \( \psi_0 = 0 \to h^1 = \gamma_0 = 0 \). The solution to the second equation is equivalent to the following: \( \mu_1 = \tau = h^0 = \beta = 0 \). The contravariant components of the metric tensor and potential have the form:
\[
\hat{g} = \begin{pmatrix}
\frac{\tilde{a}}{\Delta} & 0 & \frac{1}{\Delta} & 0 \\
0 & \frac{b}{\Delta} & 0 & 0 \\
\frac{1}{\Delta} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\sigma}{\Delta}
\end{pmatrix}; \quad A^0 = 0, \quad A^1 = \sqrt{\frac{\tilde{a} \sigma^2 + b}{\Delta}}, \quad A^2 = \frac{\sigma}{\Delta}, \quad A^3 = 0. \quad (38)
\]

3) \( \hat{A}(6) \). The solution to the system (39) has the form:
\[
\xi_0 = \frac{1}{\phi}, \quad \mu_0 = 0, \quad \omega_0 = -\phi \omega, \quad G = \phi^2 (a_0 + \omega), \quad B = \phi a_0. \quad (39)
\]

By substituting (39) into (35), we find that \( \mu_0 = 0 \). Since from the (39) it follows that \( \mu_0 = 0 \), the condition (33) is broken.

2) \( \pi = G \to \phi (a_0^2 + a_1^2) = 0 \).

As before the function \( \sigma \) can be considered free. The functions \( \gamma \) and \( \rho \) are expressed through it as follows:
\[
\begin{align*}
\gamma &= \omega_0 \sigma + \gamma_0 + \frac{\omega}{\sigma + \sigma_0}, \\
\rho &= \rho_0 + (a_0 - \gamma_0^2)\sigma^2 - 2(h^0 + tau)\sigma - \frac{2\mu_1}{\sigma + \sigma_0} - \frac{\mu_0}{(\sigma + \sigma_0)^2}.
\end{align*}
\] (40)
By substituting (40) into (13) and equating coefficients before linearly independent functions to zero, we get the following system of functional equations:

\[
\begin{align*}
\omega^2 &= \beta \mu_0, \\
(\omega_0 - a_1)^2 &= \beta(\tau + a_0), \\
(a_1 - \omega_0)(\phi h^0 + h^1 + \gamma_0) &= \beta(h^0 + \tau), \\
(\gamma_0 + \phi h^0 + h^1 - \sigma_0(\omega_0 - a_1))\omega &= \beta \mu_1, \\
(\gamma_0 + \phi h^0 + h^1)^2 - 2(a_1 - \omega_0)\omega_0 &= \beta(\tau + \rho_0).
\end{align*}
\]

(41)

It’s easy to show that from \( a^{pq} = 0 \) it follows that \( h^p = 0 \). Therefore \( a_1 \neq 0 \) \( \rightarrow \phi = 0 \).

By extension of the coordinate \( u^1 \) transform \( \beta \) to \( \varepsilon \). The remaining equations give the following solution:

\[
\omega_0 = \gamma_0 = \tau_p = \mu_1 = \rho_0 = 0, \quad \omega = \tilde{\omega}, \quad \sigma_0 = -\tilde{\varepsilon}, \quad h^0 = \varepsilon \tilde{e} a_1^2, \quad h^1 = \tilde{e} a_1.
\]

The contravariant components of the metric tensor and potential have the form:

\[
\hat{g} = \begin{pmatrix}
\frac{\varepsilon^2}{\Delta} & \frac{a}{\Delta} & \frac{1}{\Delta} & 0 \\
\frac{a}{\Delta} & 0 & 0 & 0 \\
\frac{1}{\Delta} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\varepsilon}{\Delta}
\end{pmatrix}; \quad A^0 = \frac{\varepsilon \tilde{e} a_1^2}{\Delta}, \quad A^1 = \frac{(\tilde{c}/(\sigma - \tilde{e}) + \tilde{e} a)}{\Delta}, \quad A^2 = \frac{\sigma}{\Delta}, \quad A^3 = 0.
\]

(42)

5 Solutions for the case of linear dependence between functions \( \gamma \) and \( \sigma \).

Due to the unsymmetric entrance of the functions \( \gamma \) and \( \sigma \) into the electromagnetic potential, in case of their linear dependence, two variants should be considered separately when \( \sigma \) is a free function, and \( \sigma \) is a related function. However, it is not difficult to make sure that the last variant does not give any new solutions. Therefore, we consider \( \sigma \) as a free function. In this case, the \( \gamma \) and \( \rho \) functions can be represented as:

\[
\gamma = \gamma_0 \sigma + \gamma_1, \quad \rho = \rho_0 + \tau_0 \sigma^2 + 2\tau \sigma.
\]

(43)

By substituting (43) into (13), we get the following system of functional equations:

\[
\tilde{a} - \gamma_0^2 + 2\gamma_0(a_0\phi + a_1) + \tau_0(a_0\phi^2 + 2\phi a_1 + a + \beta) + a_0\beta = 0,
\]

(44)

\[
(\gamma_0 - (a_0\phi + a_1))(\gamma_1 + h^0 \psi + h^1) = (\tau - h^0)(a_0\phi^2 + 2\phi a_1 + a + \beta),
\]

(45)

\[
(\gamma_1 + h^0 \phi + h^1)^2 = (\rho_0 + r),
\]

(46)
Consider the (44) equation using the (12) classification.

1) \( \hat{A}_{(1)} \). From the equation (44) it immediately follows:
\[
\tau_0 = \gamma_0 = \beta = \bar{a} = 0 \quad \Rightarrow \quad a^{pq} = \varepsilon b^p b^q.
\]

It is easy to show that from (45) follows:
\[
h^p = h b^p, \quad \gamma_0 = 0.
\]
We receive the solution (28).

2) \( \hat{A}_{(2)} \). This variant is also reduced to (28). Indeed, from the equation (44) follows:
\[
\tau_0 = \bar{b}, \quad \beta = \bar{b}(\phi^2 + 1), \quad \gamma_0 = -\bar{b} \phi, \quad a_0 = \bar{b} \cos 2b, \quad a_1 = \bar{b} \sin 2b, \quad \text{where it is denoted by} \quad \bar{a} = -\bar{b}^2.
\]

Let us make the coordinate transformation:
\[
\begin{align*}
u^0 &\rightarrow u^0 - \frac{1}{2} u^2, \\
u^1 &\rightarrow u^1 + \frac{1}{2} \int \phi du^2,
\end{align*}
\]

Then the matrix \((a^{pq})_{(2)}\) is converted as follows:
\[
(a^{pq})_{(2)} \rightarrow \begin{pmatrix}
\cos b^2 - \sin b^2 & \sin b \cos b \\
\sin b \cos b & \cos b^2 - \sin b^2
\end{pmatrix} \rightarrow \det a^{pq} = \bar{a} = 0.
\]

Therefore \( \tau = \gamma_0 = \beta = 0 \), and the solution is reduced to (28).

3) \( \hat{A}_{(3)} \). As noted, in this case \( a_0 = \varepsilon a_1^2 \). From the equation (44) it follows:
\[
\gamma_0 = -\tau_0 \beta, \quad \tau_0 (\beta - \phi \tau_0) = 0, \quad \beta = \varepsilon -(2 \gamma_0 + \tau_0 \phi) \phi.
\]

Therefore \( \tau_0 = \gamma_0 = 0, \quad \beta = \varepsilon \). From the equation (44) we get:
\[
h^0 = \varepsilon h^1 \rightarrow \text{a particular solution to (28)}.
\]

4) \( \hat{A}_{(4)}, \hat{A}_{(5)} \). Because of the independence of the functions \( a, a_0 \) the equation (44) has no solutions.

5) \( \hat{A}_{(6)} \).

a) First we assume: \( a = 0 \). The equation (44) takes the form:
\[
-\gamma_0^2 + 2 \gamma_0 a_0 \phi + \tau_0 (a_0 \phi^2 + \beta) + a_0 \beta = 0,
\]

Therefore: \( \beta = \tau_0 \phi^2, \quad \gamma_0 = -\tau_0 \phi, \quad G = \phi^2(a_0 + \tau_0) \). Considering this, we will have the equation (44) as the following system:
\[
\begin{align*}
\gamma_1 + h^1 + \tau \phi &= 0 \quad \Rightarrow \quad h^1 = 0, \\
p &= -\tau \phi,
\end{align*}
\]

\[
(\gamma_1 + h^1 + \frac{\tau \phi}{\phi})^2 = \phi^2(\rho_0 + \phi)(a_0 + \tau_0).
\]

Substitute (48) into (49). As a result we get the solution
\[
h^0 - \tau = \varepsilon(a_0 + \tau_0), \quad \text{equivalent to the following:} \quad h^0 = \tau = 0.
\]

The contravariant components of the metric tensor and potential have the form:
\[
\hat{g} = \begin{pmatrix}
\frac{\Delta}{\Delta} & 0 & \frac{1}{\Delta^2} & 0 \\
0 & \frac{\Delta^2}{\Delta - \phi} & -\frac{\Delta}{\Delta} & 0 \\
\frac{1}{\Delta} & \frac{\Delta^2}{\Delta - \phi} & 0 & 0 \\
0 & 0 & 0 & \frac{\Delta}{\Delta}
\end{pmatrix}; \quad A^0 = 0, \quad A^1 = -\frac{\beta \phi \sigma}{\Delta}, \quad A^2 = \frac{\sigma}{\Delta}, \quad A^3 = 0.
\]
b) Let now \( a_0 = 0 \). The equation (44) takes the form:
\[
\gamma_0^2 = \tau_0 (a + \beta) \rightarrow \gamma_0 = \tau_0 = 0.
\]
From (45), (46) follows:
\[
h_0 = \tau = 0, \quad h_1 + \gamma_1 = \tilde{e} (a + \beta),
\]
which is equivalent to the following:
\[
h_1 = \gamma_1 = 0.
\]
The contravariant components of the metric tensor and potential have the form:
\[
\hat{g} = \begin{pmatrix}
0 & 0 & \frac{1}{\Delta} & 0 \\
0 & \frac{a + \beta}{\Delta} & -\frac{\phi}{\Delta} & 0 \\
\frac{1}{\Delta} & -\frac{\phi}{\Delta} & 0 & 0 \\
0 & 0 & 0 & \frac{\Delta}{\Delta}
\end{pmatrix}; \quad A^0 = 0, \quad A^2 = \frac{\sigma}{\Delta}, \quad A^3 = 0.
\]

6 Conclusion.

It is obvious that the classification problem under consideration is non-trivial for separating variables of type \((N, N_0)\), if \( N < 3 \). Hence, to complete the classification we need to consider the only left type \((1, 1)\). Note that the results of this classification in addition to the theoretical can also have an applied value, for example, in the study of the axion field problem. This field associated with an axionic dark matter (see [38] - [41]). In the paper [42] static field configuration in the outer zone of a magnetic star is considered. The configuration is formed by interacting quartet of external fields: axion field, strong intrinsic magnetic field, and axionically induced electric field.

Therefore, in conclusion, we show the acquired results. For each class, the covariant components of the metric tensor and the electromagnetic potential are given. The functions \( s_\nu \) for the complete integral (6) are provided.

1)
\[
ds^2 = \frac{1}{\Delta} \left[ 2 du^0 du^2 + \left( \frac{(du^1 - a_1 du^2)^2}{a} - a_0 du^2 + du^2 \right) \right],
\]
\[
A_0 = 0, \quad A_1 = \frac{h_1}{a}, \quad A_2 = h^0, \quad A_3 = 0.
\]
\[
s_2 = \int \left( \lambda \psi + \lambda_2 \right) \left( \lambda_0 - \frac{\phi}{\lambda_1} \right) du^2,
\]
\[
s_3 = \varepsilon \int \sqrt{\left( \lambda f - \left( \lambda_2 + a_0 \lambda_0^2 + a \lambda_1^2 + 2 a_1 \lambda_0 \lambda_1 + 2 h^0 \lambda_0 + 2 h^1 \lambda_1 \right) du^3}.
\]

2)
\[
ds^2 = \frac{1}{\Delta} \left[ 2 du^0 du^2 + \frac{du^{12}}{\beta} - a_0 du^2 + du^2 \right],
\]
\[
A_0 = -\frac{1}{2 a \beta}, \quad A_1 = \frac{h_1}{\beta}, \quad A_2 = h^2 \tilde{a}, \quad A_3 = 0.
\]
\[
s_2 = \int \left( \lambda \psi + \lambda_2 - \beta \lambda_1^2 \right) \left( \lambda_0 - \frac{\phi}{\theta a^2} \right) du^2,
\]
\[ s^3 = \varepsilon \int \sqrt{(\lambda f - \lambda_2)} du^3. \]

3) \[ ds^2 = \frac{1}{\Delta} [2du^0 du^2 + \frac{(du^1 - a_1 du^2)^2}{a + \beta} - a_0 du^2 + du^3], \quad (54) \]

\[ A_0 = 0 \quad A_1 = 0, \quad A_2 = h, \quad A_3 = 0. \]

\[ s_2 = \int \frac{\lambda \psi + \lambda_2 - (\beta \lambda_1^2)}{2\lambda_0} du^2, \]

\[ s^3 = \varepsilon \int \sqrt{(\lambda f - (\lambda_2 + a\gamma \lambda p \lambda q + 2(h^0 \lambda_0))} du^3. \]

4) \[ ds^2 = \frac{1}{\Delta} [2du^0 du^2 + \frac{(du^1 - a_1 du^2)^2}{\beta} - a_0 du^2 + du^3], \quad (55) \]

\[ A_0 = 0, \quad A_1 = \frac{\gamma}{\beta}, \quad A_2 = h + a_1 \frac{\gamma}{\beta} A_1, \quad A_3 = 0. \]

\[ s_2 = \int \frac{\lambda \psi + \lambda_2 - (\beta \lambda_1^2 + 2\gamma \lambda_1)}{2\lambda_0} du^2, \]

\[ s^3 = \varepsilon \int \sqrt{(\lambda f - (\lambda_2 + a\gamma \lambda p \lambda q + 2h^0 \lambda_0)} du^3. \]

5) \[ ds^2 = \frac{1}{\Delta} \left(2du^0(b_1 du^0 - b_0 du^1) \frac{\phi du^0 + du^1}{(b_1 + \phi b^0)^2 + du^3} + du^3\right), \quad (56) \]

\[ A_0 = \frac{\phi h + b_1 \sigma}{(b_1 + \phi b^0)}, \quad A_1 = \frac{h - a_0 \sigma}{(b_1 + \phi b^0)}, \quad A_2 = 0, \quad A_3 = 0. \]

\[ s_2 = \int \frac{\lambda \psi + \lambda_2}{2(\lambda_0 - \phi \lambda_1 + \sigma)} du^2, \]

\[ s_3 = \varepsilon \int \sqrt{(\lambda f - \lambda_2 - (\lambda_p b^p - h)^2)} du^3. \]

6) \[ ds^2 = \frac{1}{\Delta} [2du^0 du^2 + \frac{(\phi du^0 + du^1)^2}{\beta} + du^3], \quad (57) \]

\[ A_0 = \sigma, \quad A_1 = \frac{\gamma}{\beta}, \quad A_2 = 0, \quad A_3 = 0. \]
\[ s_2 = \int \frac{\lambda \psi + \lambda - (\beta \lambda_1^2 + 2\gamma \lambda_1)}{2(\lambda_0 - \phi \lambda_1 + \sigma)} \, du^2, \]
\[ s_3 = \varepsilon \int \sqrt{\lambda f - \lambda_2} \, du^3. \]
\[ ds^2 = \frac{1}{\Delta} [2du^0du^2 + \frac{(\phi du^0 + du^1)^2}{a + \beta} + du^3], \quad (61) \]

\[ A_0 = \sigma, \quad A_1 = 0, \quad A_2 = 0, \quad A_3 = 0. \]

\[ s_2 = \int \frac{\lambda \psi + \lambda_2 - \beta \lambda_1^2}{2(\lambda_0 - \phi \lambda_1 + \sigma)} du^2, \]

\[ s_3 = \varepsilon \int \sqrt{\lambda f - (\lambda_2 + a \lambda_1^2)} du^3. \]

Acknowledgments This work was supported by the Ministry of Science and Higher Education of the Russian Federation, project FWF-2020-0003.

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