Infinite Order of Growth of Solutions of Second Order Linear Differential Equations

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Abstract. Considering differential equation \( f'' + A(z)f' + B(z)f = 0 \), where \( A(z) \) and \( B(z) \) are entire complex functions, our results revolve around proving all non-trivial solutions are of infinite order taking various restrictions on coefficients \( A(z) \) and \( B(z) \).

1. Introduction and statement of main results

We consider the second order homogeneous linear differential equation,

\[ f'' + A(z)f' + B(z)f = 0 \]  \hspace{1cm} (1)

where \( A(z) \) and \( B(z) \) are entire functions. It is well known result that all solutions of (1) are entire functions (see [8]). All solutions are of finite order if and only if \( A(z) \) and \( B(z) \) are polynomial (see [8]). Thus, if at least one of the coefficients are not polynomial then solutions can be of infinite order. It is wide area of research to find conditions on \( A(z) \) and \( B(z) \) such that solutions are of infinite order. There are many results concerning this problem. Following theorem is the collection of such basic results which provide all non-trivial solutions of infinite order.

**Theorem 1.** Suppose \( A(z) \) and \( B(z) \) are non-constant entire functions, then all non-trivial solutions of (1) are of infinite order, if one of the following holds:

(i) \( \rho(A) < \rho(B) \)

(ii) \( A(z) \) is a polynomial and \( B(z) \) is transcendental

(iii) \( \rho(B) < \rho(A) \leq \frac{1}{2} \).

Consider the equation

\[ w'' + P(z)w = 0 \]  \hspace{1cm} (2)

where \( P(z) \) is a polynomial of degree \( n \). There are several papers in which the authors have considered \( A(z) \) to be a solution of equation (2) and \( B(z) \) to satisfy different conditions so that all non-trivial solutions are of infinite order. The next theorem is the collection of all these results.

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Theorem 2. Suppose $A(z)$ is a solution of (2) and $B(z)$ is a transcendental entire function satisfying one of the conditions mentioned below. Then all non-trivial solutions of (2) are of infinite order.

(i) $\rho(B) < \frac{1}{2}$
(ii) $\mu(B) < \frac{1}{2}$ and $\rho(A) \neq \rho(B)$
(iii) $\mu(B) < \frac{1}{2} + \frac{1}{2(n+1)}$ and $\rho(A) \neq \rho(B)$
(iv) $B(z)$ has Fabry gap and $\rho(A) \neq \rho(B)$.

Motivated by above results we have replaced the conditions on $B(z)$ to be transcendental entire function satisfying $T(r,B) \sim \log M(r,B)$ in a set $E$ of positive upper logarithmic density, where the notation $T(r,B) \sim \log M(r,B)$ means

$$\lim_{r \to \infty} \frac{T(r,B)}{\log M(r,B)} = 1.$$

Theorem 3. Let $A(z)$ be a non-trivial solution of (2), where $P(z) = a_n z^n + \cdots + a_0$, $a_n \neq 0$ and $B(z)$ be transcendental entire function satisfying $T(r,B) \sim \log M(r,B)$ in a set $E$ of positive upper logarithmic density, then all non-trivial solutions of (1) are of infinite order.

Kwon in his paper [16] considered $\rho(A) > 1$ of finite non-integral order with all its zeros fixed in a sector and $0 < \rho(B) < 1/2$. Kumar et. al. [13] assumed that $B(z)$ has Fabry gap. Recall that, for an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ if $\sum_{n=0}^{\infty} \frac{1}{\lambda_n}$ diverges to $\infty$ we say that it has Fabry gap.

Theorem 4. Suppose that $A(z)$ is an entire function of finite non-integral order with $\rho(A) > 1$, and that all the zeros of $A(z)$ lies in the angular sector $\theta_1 < \arg z < \theta_2$ satisfying

$$\theta_2 - \theta_1 < \frac{\pi}{p+1}$$

if $p$ is odd, and

$$\theta_2 - \theta_1 < \frac{(2p-1)\pi}{2p(p+1)}$$

if $p$ is even, where $p$ is the genus of $A(z)$. Let $B(z)$ be an entire function satisfying the conditions mentioned below. Then all non-trivial solutions $f$ of equation (1) are of infinite order.

(i) $0 < \rho(B) < \frac{1}{2}$.
(ii) $B(z)$ has Fabry gap.

Motivated by above results we have replaced $B(z)$ to be a transcendental entire function having a multiply connected Fatou component.

Theorem 5. Suppose that $A(z)$ be an entire function of finite non-integral order with $\rho(A) > 1$, and that all the zeros of $A(z)$ lies in the angular sector $\theta_1 < \arg z < \theta_2$ satisfying

$$\theta_2 - \theta_1 < \frac{\pi}{p+1}$$

if $p$ is odd, and

$$\theta_2 - \theta_1 < \frac{(2p-1)\pi}{2p(p+1)}$$
if \( p \) is even, where \( p \) is the genus of \( A(z) \) and let \( B(z) \) be a transcendental entire function with a multiply connected Fatou component. Then all non-trivial solutions of equation (1) are of infinite order.

In several papers like [12], [14], [15], [21] and [23] authors have considered \( A(z) = h(z)e^{P(z)} \), where \( P(z) \) is a polynomial of degree \( n \) satisfying \( \lambda(A) < \rho(A) \) and \( B(z) \) with various conditions. Here we have exchanged the conditions of \( A(z) \) with \( B(z) \) and prove the following result.

**Theorem 6.** Let \( B(z) = h(z)e^{P(z)} \) be a transcendental entire function satisfying \( \lambda(B) < \rho(B) = n \), where \( P(z) \) is a polynomial of degree \( n \) and \( A(z) \) satisfies the properties mentioned below, then all non-trivial solutions of equation (1) are of infinite order.

(a) \( A(z) \) has Fabry gap
(b) \( A(z) \) satisfies \( T(r,A) \sim \log M(r,A) \) in a set \( E \) of positive upper logarithmic density
(c) \( A(z) \) has multiply connected Fatou component.

Following example shows that if we skip conditions of our theorem then we do get solution of finite order.

**Example 1.** (a) \( f'' - e^z f' + e^z f = 0 \) has a solution \( e^z - 1 \) of finite order 1.
\( T(r,e^z) = \frac{r}{i} \) and \( M(r,e^z) = r \) thus \( T(r,e^z) \not\sim \log M(r,e^z) \)
(i) \( A(z) = -e^z \) is a solution of \( w'' - w = 0 \) and \( B(z) = e^z \) does not satisfy the condition of Theorem 3.
(ii) \( B(z) = e^z \) satisfy \( \lambda(B) < \rho(B) \) but \( A(z) \) does not satisfy the condition of Theorem 6(b)
(b) \( f'' + (e^z - 1) f' + e^z f = 0 \) has a solution \( e^z \) of finite order 1.
\( B(z) = e^z \) satisfy \( \lambda(B) < \rho(B) \) but \( A(z) \) does not satisfies the condition of Theorem 6(a) i.e. it does not has Fabry gap.
(c) \( f'' + Cz^2 \prod_{n=1}^{\infty} \left( 1 + \frac{z}{a_n} \right) f' - Cz \prod_{n=1}^{\infty} \left( 1 + \frac{z}{a_n} \right) f = 0 \) has a solution \( z \) of order 0.
\[ A(z) = Cz^2 \prod_{n=1}^{\infty} \left( 1 + \frac{z}{a_n} \right) \]
where the \( a_n \) satisfy \( 1 < a_1 < a_2 < ... \) and grow so rapidly that \( a_{n+1} < A(a_n) < 2a_{n+1} \), is constructed by Baker [11]. It has multiply connected Fatou component but \( \lambda(B) = \rho(B) \) i.e. it does not satisfy the condition of Theorem 6(c).
(d) \( f'' + Cz^3 \prod_{n=1}^{\infty} \left( 1 + \frac{z}{a_n} \right) f' - Cz^2 \prod_{n=1}^{\infty} \left( 1 + \frac{z}{a_n} \right) f = 0 \) has a solution \( z \) of order 0.
\( B(z) \) has multiply connected Fatou component but \( \rho(A) \leq 1 \) i.e. it does not satisfy the condition of Theorem 5.

2. Preliminary Lemma

Next lemma is due to Gundersen [5] and it gives an estimation of logarithmic derivatives.

**Lemma 1.** [5] Let \( f \) be a transcendental entire function of finite order \( \rho \), let \( \Gamma = \{(k_1,j_1),(k_2,j_2)...(k_m,j_m)\} \) denote finite set of distinct pairs of integers that satisfies \( k_i > j_i \geq 0 \), for \( i = 1,2,...m \), and let \( \epsilon > 0 \) be a given constant. Then the following three statements holds:
(i) there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E_1$, then there is a constant $R_0 = R_0(\psi_0) > 0$ so that for all $z$ satisfying $\text{arg} z = \psi_0$ and $|z| \geq R_0$ and for all $(k, j) \in \Gamma$, we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho-1+\epsilon)} \quad (3)$$

(ii) there exists a set $E_2 \subset (1, \infty)$ that has finite logarithmic measure, such that for all $z$ satisfying $|z|$ does not belong to $E_2 \cup [0, 1]$ and for all $(k, j) \in \Gamma$, the inequality (2) holds.

(iii) there exists a set $E_3 \subset [0, \infty)$ that has finite linear measure, such that for all $z$ satisfying $|z|$ does not belong to $E_3$ and for all $(k, j) \in \Gamma$, we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho+\epsilon)} \quad (4)$$

**Lemma 2.** Let $f(z)$ be a meromorphic function of finite order $\rho$. Given $\xi > 0$ and $\delta$, $0 < \delta < 1/2$, there is a constant $K(\rho, \xi)$ such that for all $r$ in a set $F$ of lower logarithmic density greater than $1 - \xi$ and for every interval $J$ of length $\delta$

$$r \int_{J} \left| \frac{f(\rho e^{i\theta})}{f(\rho e^{i\theta})} \right| d\theta < K(\rho, \xi)(\delta \log \frac{1}{\delta})T(r, f).$$

**Lemma 3.** Let $f(z)$ be an entire function satisfying $T(r, f) \sim \log M(r, f)$ in a set $E$ of positive upper logarithmic density. For given $0 < c < \frac{1}{4}$ and $r \in E$, the set

$$I_r = \{ \theta \in [0, 2\pi) : \log |f(\rho e^{i\theta})| \leq (1 - c) \log M(r, f) \}$$

has linear measure zero.

Next lemma is extracted from proof of the theorem in [18]

**Lemma 4.** Let $f(z)$ be an entire function of finite order satisfying $T(r, f) \sim \log M(r, f)$ in a set $E$ such that $\log \text{dens}(E) > 1 - \xi$, then

$$|f(z)| > (1 - 2c) \log M(r, f),$$

where $0 < c < 1/4$, $r \in E \cap F$ and $\theta \in [0, 2\pi] \setminus I_r$ where

$$I_r = \{ \theta \in [0, 2\pi) : \log |f(\rho e^{i\theta})| \leq (1 - c) \log M(r, f) \}.$$

Let

$$M(r, f) = \max \{|f(z)| : |z| = r\}$$

and $L(r, f) = \min \{|f(z)| : |z| = r\}$, then following lemma gives relation between $M(r, f)$ and $L(r, f)$ when $f$ has at most finite number of poles.

**Lemma 5.** Let $f$ be a transcendental meromorphic function with at most finitely many poles. If $\mathcal{J}(f)$ has only bounded components, then for any complex number $a$, there exist a constant $0 < d < 1$ and two sequences $\{r_n\}$ and $\{R_n\}$ of positive numbers with $r_n \to \infty$ and $R_n/r_n \to \infty(n \to \infty)$ such that

$$M(r, f)^d \leq L(r, f), r \in G$$

where $G = \cup_{n=1}^{\infty} \{ r : r_n < r < R_n \}$. 
3. Proof of the theorems

Before the proof of each of our result, we give some lemmas which will be useful in proving our results.

**Proof of Theorem 3.**

Let $\alpha < \beta$ be such that $\beta - \alpha < 2\pi$, $r > 0$ and $\overline{S}$ denote the closure of $S$. Denote

$$S(\alpha, \beta) = \{ z : \alpha < \arg z < \beta \}$$

and

$$S(\alpha, \beta, r) = \{ z : \alpha < \arg z < \beta \} \cap \{ z : |z| < r \}.$$ 

Let $f$ be an entire function of order $\rho(f) \in (0, \infty)$. For simplicity, set $\rho = \rho(f)$ and $S = S(\alpha, \beta)$. Then $f$ is said to blow up exponentially in $\overline{S}$ if for any $\theta \in (\alpha, \beta)$

$$\lim_{r \to \infty} \frac{\log \log |f(re^{i\theta})|}{\log r} = \rho$$

and decays to zero exponentially in $\overline{S}$ if for any $\theta \in (\alpha, \beta)$

$$\lim_{r \to \infty} \frac{\log \log |f(re^{i\theta})^{-1}|}{\log r} = \rho$$

**Lemma 6.** [9] Let $f$ be a non-trivial solution of $f'' + P(z)f = 0$, where $P(z) = a_nz^n + \cdots + a_0$, $a_n \neq 0$. Set $\theta_j = \frac{2j\pi - \arg(a_{n+1})}{n+2}$ and $S_j = (\theta_j, \theta_{j+1})$, where $j = 0, 1, 2, \cdots, n+1$ and $\theta_{n+2} = \theta_0 + 2\pi$. Then $f$ has the following properties:

(i) In each sector $S_j$, $f$ either blows up or decays to zero exponentially.

(ii) If, for some $j$, $f$ decays to zero in $S_j$, then it must blow up in $S_{j-1}$ and $S_{j+1}$. However, it is possible for $f$ to blow up in many adjacent sectors.

(iii) If $f$ decays to zero in $S_j$, then $f$ has at most finitely many zeros in any closed sub-sector within $S_{j-1} \cup S_j \cup S_{j+1}$.

(iv) If $f$ blows up in $S_{j-1}$ and $S_j$, then for each $\epsilon > 0$, $f$ has finitely many zeros in each sector $\overline{S}(\theta_j - \epsilon, \theta_j + \epsilon)$, and furthermore, as $r \to \infty$,

$$n(\overline{S}(\theta_j - \epsilon, \theta_j + \epsilon), r, f) = (1 + o(1)) \frac{2\sqrt{|a_n|}}{\pi(n+2)} r^{n+2},$$

where $n(\overline{S}(\theta_j - \epsilon, \theta_j + \epsilon), r, f)$ is the number of zeros of $f$ in the region $\overline{S}(\theta_j - \epsilon, \theta_j + \epsilon, r)$.

**Lemma 7.** [6] Let $A(z)$ and $B(z) \neq 0$ be two entire functions such that for real constants $\alpha > 0$, $\beta > 0$, $\theta_1$, $\theta_2$, where $\alpha > 0$, $\beta > 0$ and $\theta_1 < \theta_2$, we have

$$|A(z)| \geq \exp\{ (1 + o(1))\alpha |z|^\beta \},$$

$$|B(z)| \leq \exp\{ (1 + o(1))|z|^\beta \}$$

as $z \to \infty$ in $\overline{S}(\theta_1, \theta_2) = \{ z : \theta_1 \leq \arg z \leq \theta_2 \}$. Let $\epsilon > 0$ be a given small constant, and let $\overline{S}(\theta_1 + \epsilon, \theta_2 - \epsilon) = \{ z : \theta_1 + \epsilon \leq \arg z \leq \theta_2 - \epsilon \}$. If $f$ is a non-trivial solution of (1) with $\rho(f) < \infty$, then the following conclusions hold:

(i) There exists a constant $b(\neq 0)$ such that $f(z) \to b$ as $z \to \infty$ in $\overline{S}(\theta_1 + \epsilon, \theta_2 - \epsilon)$. Furthermore,

$$|f(z) - b| \leq \exp\{ -(1 + o(1))\alpha |z|^\beta \}$$

as $z \to \infty$ in $\overline{S}(\theta_1 + \epsilon, \theta_2 - \epsilon)$.
(ii) For each integer $k > 1$,
\[ |f^{(k)}(z)| \leq \exp\{-1 + \alpha(1)\alpha|z|^\beta\} \quad (8) \]

as $z \to \infty$ in $S(\theta_1 + \epsilon, \theta_2 - \epsilon)$.

PROOF. If $\rho(A) < \rho(B)$, then it is already proved in [6] that all non-trivial solutions of (1) are of infinite order. Now, suppose that $\rho(A) > \rho(B)$.

Assume $\rho(f) < \infty$. Set $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ and $S_j = (\theta_j, \theta_{j+1})$, where $j = 0, 1, 2, \cdots, n + 1$ and $\theta_{n+2} = \theta_0 + 2\pi$. Let $\theta \in S_j$. Since $A(re^{i\theta})$ is a solution of equation (2), thus $A(z)$ either blows up or decays to zero exponentially in each sector $S_j$.

**Case 1:** Suppose $A(z)$ blows up exponentially in each sector $S_j$. Then, we have
\[ \lim_{r \to \infty} \frac{\log \log |A(re^{i\theta})|}{\log r} = \frac{n + 2}{2}. \]

Then for any given constant $\epsilon \in (0, \frac{\pi}{\rho(A)})$ and $\beta \in (0, \frac{\rho(A) - \rho(B)}{3})$, we have
\[ |A(z)| \geq \exp\{|z|^{\frac{n+2}{2} - \beta}\} \]
\[ \geq \exp\{\frac{1}{2}|z|^{\frac{n+2}{2} - \beta} + \frac{1}{2}|z|^{\frac{n+2}{2} - \beta}\} \]
\[ \geq \exp\{\frac{1}{2}|z|^{\frac{n+2}{2} - \beta} + \frac{1}{2}|z|^{\frac{n+2}{2} - \beta}\} \]
\[ \geq \exp\{(1 + |z|^{-\beta})\frac{1}{2}|z|^{\frac{n+2}{2} - \beta}\} \]
\[ \geq \exp\{(1 + o(1))\alpha|z|^{\frac{n+2}{2} - \beta}\} \]

where $\alpha = \frac{1}{2}$.

\[ |B(z)| \leq \exp\{|z|^{\rho(B) + \beta}\} \leq \exp\{|z|^{\rho(A) - 2\beta}\} \leq \exp\{o(1)|z|^{\frac{n+2}{2} - \beta}\} \]

as $z \to \infty$ in $S_j(\epsilon) = \{z : \theta_j + \epsilon < \arg z < \theta_j - \epsilon\}, j = 0, 1, \cdots, n + 1$. Combining above inequalities for $A(z)$, $B(z)$ and Lemma [7] exist corresponding constants $b_j \neq 0$ such that
\[ |f(z) - b_j| \leq \exp\{-1 + o(1)\alpha|z|^{\frac{n+2}{2} - \beta}\} \]

as $z \to \infty$ in $S_j(2\epsilon), j = 0, 1, \cdots, n + 1$. Therefore, $f$ is bounded in the whole complex plane by the Phragmén-Lindelof principle. So $f$ is a nonzero constant in the whole complex plane by Liouville’s theorem. But $f$ cannot be non-constant, which gives rise to a contradiction.

**Case 2:** Suppose $A(z)$ decays to zero in at least one sector $S_j$. Then, we have
\[ \lim_{r \to \infty} \frac{\log \log |A(re^{i\theta})|^{-1}}{\log r} = \frac{n + 2}{2}. \]

Then we get
\[ |A(re^{i\theta})| < \exp(-r^{\frac{n+2}{2} - \xi}) \]

where $r \to \infty$ and $\xi$ is a positive constant. Since $\rho(f) < \infty$ by Lemma [1] we have
\[ \left| \frac{f^{(k)}(z)}{f(z)} \right| \leq |z|^{\rho(f)} \]
for \(|z| \geq R = R(\theta) > 0\) and \(\theta \in [0, 2\pi)/G\), where \(G\) is a set with linear measure 0.

Since we have
\[
T(r, B) \sim \log M(r, B)
\]
in a set \(E\) having positive upper logarithmic density, by using Lemma 4 for \(0 < c < 1/4\) we have
\[
M(r, B)^{1-2c} < |B(z)|
\]
where \(r \in E \cap F\) and \(\theta \in [0, 2\pi) \setminus I_r\) where \(I_r\) and \(F\) as defined in Lemma 4.

From equation 1 we get
\[
|B(z)| \leq \left| \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right| + |A(z)| \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right|
\]
\[
M(r, B)^{1-2c} < |B(z)| \leq (1 + o(1))r^{2\rho(f)}
\]
\[
M(r, B) < (1 + o(1))r^{4\rho(f)}
\]
for \(r > R(\theta); r \in E \cap F \cap G\) and \(\theta \in S_j \setminus I_r\).

But \(M(r, B) < (1 + o(1))r^{4\rho(f)}\) is not possible for transcendental entire function \(B(z)\).

Hence \(\rho(f)\) is infinite.

\(\square\)

**Proof of Theorem 5.**

Following lemma provides the information of the minimum modulus of entire function of non-integral order having zeros in definite sectors.

**Lemma 8.** [16] Let \(f(z)\) be an entire function of finite non-integral order \(\rho\) and of genus \(p > 1\). Suppose that for any given \(\epsilon > 0\), all the zeros of \(f(z)\) have their arguments in the following subset of real numbers:
\[
S(p, \epsilon) = \{\theta : |\theta| \leq \frac{\pi}{2(p + 1)} - \epsilon\}
\]
if \(p\) is odd, and
\[
S(p, \epsilon) = \{\theta : \frac{\pi}{2p} + \epsilon \leq |\theta| \leq \frac{3\pi}{2(p + 1)} - \epsilon\}
\]
if \(p\) is even. Then for any \(c > 1\), there exists a real number \(R > 0\) such that
\[
|f(-r)| \leq \exp(-cr^p)
\]
for all \(r \geq R\).

**Proof.** Let us suppose that solution \(f\) of (11) is of finite order. Using Lemma 1(b), we have
\[
\left| \frac{f^{(k)}(re^{i\theta})}{f(re^{i\theta})} \right| \leq r^{k\rho(f)}
\]
for \(r \notin E \cup [0, 1]\), where \(E\) is a set with finite logarithmic measure and \(r > R_1(\theta)\). Let us rotate the axes of the complex plane, assume that all the zeros of \(A(z)\) have their arguments in the set \(S(p, \epsilon)\) defined in Lemma 8 for some \(\epsilon > 0\). Hence by Lemma 8 there exists a positive real number \(R_2\) such that for all \(r > R_2\), we have
\[
\min_{|z|=r} |A(z)| \leq |A(-r)| \leq \exp(-cr^p) < 1.
\]

By Lemma 5 we have
\[
M(r, B)^d \leq |B(re^{i\theta})|
\]
for $0 < d < 1$ and $r \in G = \bigcup_{n=1}^{\infty} \{ r : r_n < r < R_n \}$. From equation (1) we get,

$$|B(z)| \leq \left| \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right| + |A(z)| \frac{|f'(re^{i\theta})|}{|f(re^{i\theta})|}$$

Using (10), (11) and (12) for $r > \max\{R_1(\theta), R_2\}$ such that $r \in G \setminus E_1 \cup [0, 1]$ and $\theta \in \{ \theta : \min_{|z|=r} |A(z)| = |A(z)| \}$, we have

$$M(r, B)^d < |B(z)| \leq (1 + o(1))r^{2\rho(f)}$$

which is a contradiction for a transcendental entire function.

Proof of Theorem 6.

Combining Theorem 2 from [4] and Lemma 2.2 from [11], we get

**Lemma 9.** Let $g(z) = \sum_{n=0}^{\infty} a_{\lambda_n} z^{\lambda_n}$ be an entire function of finite order with Fabry gap, and let $u(z)$ be an entire function with $\rho(u) \in (0, \infty)$. Then for any given $\epsilon \in (0, \varsigma)$, where $\varsigma = \min(1, \rho(u))$, there exists a set $K \subset (1, \infty)$ satisfying $\log\text{d}ense K \geq \xi$, where $\xi \in (0, 1)$ is a constant such that for all $|z| = r \in K$,

$$\log M(r, u) > r^{\rho(u)-\epsilon}, \quad \log m(r, g) > (1 - \xi) \log M(r, g)$$

where $M(r, u) = \max\{|u(z)| : |z| = r\}$, $m(r, g) = \min\{|g(z)| : |z| = r\}$ and $M(r, g) = \max\{|g(z)| : |z| = r\}$.

**Lemma 10.** [17] Let $f(z)$ be a non-constant entire function. Then there exists a real number $R$ such that for all $r \geq R$ there exists $z_r$ with $|z_r| = r$ satisfying

$$\frac{|f(z_r)|}{f'(z_r)} \leq r.$$  \hfill (13)

**Lemma 11.** [2] Let $A(z) = h(z)e^{P(z)}$ be an entire function with $\lambda(A) < \rho(A) = n$, where $P(z)$ is a polynomial of degree $n$. Then for every $\epsilon > 0$ there exists $E \subset [0, 2\pi)$ of linear measure zero such that

(i) for $\theta \in E^+ \setminus E$ there exists $R > 1$ such that

$$|A(re^{i\theta})| \geq \exp((1 - \epsilon)\delta(P, \theta) r^n)$$

for $r > R$.

(ii) for $\theta \in E^- \setminus E$ there exists $R > 1$ such that

$$|A(re^{i\theta})| \leq \exp((1 - \epsilon)\delta(P, \theta) r^n)$$

for $r > R$.

**Proof.** Suppose $f$ is a solution of finite order of equation (1). Then by Lemma 1 we get

$$\frac{|f''(re^{i\theta})|}{|f'(re^{i\theta})|} \leq r^{\rho(f)}$$

where $\theta \in [0, 2\pi) \setminus E_1$, where $E_1$ is a set of linear measure 0 for $r \geq r_1(\theta) > 0$.

Using Lemma 10 we get

$$\frac{|f(z_r)|}{f'(z_r)} \leq r.$$  \hfill (17)
for \( r \geq r_2 \) such that \(|z_r| = r\).

Since \( \lambda(B) < \rho(B) \), choosing \( \theta \in [0, 2\pi) \setminus E_2 \), where \( E_2 \) is a set of linear measure 0 such that \( \delta(P, \theta) < 0 \) using Lemma 11, we have

\[
|B(re^{i\theta})| \leq \exp((1 - \epsilon)\delta(P, \theta)r^n)
\]  

for \( r > r_3 \). From equation (11), we get

\[
|A(re^{i\theta})| \leq \left| \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right| + |B(re^{i\theta})| \left| \frac{f(re^{i\theta})}{f'(re^{i\theta})} \right|
\]  

(19)

(a) Since \( A(z) \) is a transcendental entire function with Fabry gap, using Lemma 9 there exist a set \( H \subset (1, \infty) \) satisfying \( \log dens H \geq \xi \), where \( \xi \in (0, 1) \) for \( r \in H \) such that

\[
M(r, A)^{(1-\xi)} < |A(z)|.
\]  

(20)

Using (16), (17), (18), (19) and (20) we will get

\[
M(r, A)^{(1-\xi)} \leq r^{\rho(f)} + r \exp((1 - \epsilon)\delta(P, \theta)r^n)
\]

\( r \in H, r > \max\{r_1, r_2, r_3\} \) and \( \theta \in [0, 2\pi) \setminus (E_1 \cup E_2) \),

which is a contradiction for very large \( r \).

(b) Using (9), (16), (17), (18) and (19) we will get

\[
M(r, A)^{(1-2\epsilon)} < r^{2\rho(f)}(1 + o(1)),
\]

for \( r \in E \cap F, r > \max\{r_1, r_2, r_3\} \) and \( \theta \in [0, 2\pi) \setminus (E_1 \cup E_2 \cup I_r) \),

which is a contradiction for very large \( r \).

(c) Using (12), (16), (17), (18) and (19) for , we get for \( 0 < d < 1, r \in G = \bigcup_{n=1}^\infty \{r : r_n < r < R_n\}, r > \max\{r_1, r_2, r_3\} \) and \( \theta \in [0, 2\pi) \setminus E_1 \cup E_2 \),

\[
M(r, A)^d < r^{2\rho(f)}(1 + o(1)),
\]

which is a contradiction for very large \( r \).

\[\square\]

References

[1] I. N. Baker, An entire function which has wandering domains, J. Austral. Math. Soc. Ser. A, 22 (1976), 173-176.

[2] S. Bank, I. Laine and J. Langley, On the frequency of zeros of solutions of second order linear differential equation, Results Math., 10 (1986), 8-24.

[3] M. Frei, Uber die Losungen linearer Differentialgleichungen mit ganzen Funktionen als Koeffizienten, Comment. Math. Helv. 35 (1961), 201-222.

[4] W. Fuchs, Proof of a conjecture of G. Polya concerning gap series, Illinois J. Math., 7 (1963), 661-667.

[5] G. G. Gundersen, Estimates for the logarithmic derivative of a meromorphic function, J. London Math. Soc., 37, 17:1 (1988), 88-104.

[6] G.G. Gundersen, Finite order solutions of second order linear differential equations, Trans. Amer. Math. Soc. 305 (1988), 415-429.

[7] S. Hellerstein, J. Miles and J. Rossi, On the growth of solutions of \( f'' + g f' + h f = 0 \), Trans. Amer. Math. Soc. 324 (1991), 693-706.

[8] H. Herold, Differentialgleichungen im Komplexen, Vandenboeck & Ruprecht, Gottingen, 1975.

[9] E. Hille, Lectures On Ordinary Differential Equations, Addison-Wesley Publishing Company, Reading, Massachusetts-Menlo Park, California-London-Don Mills, Ontario, 1969.
[10] I. Laine, Nevanlinna Theory and Complex Differential Equations, de Gruyter, Berlin, 1993.
[11] K. Ishizaki, K. Tohge, On the complex oscillation of some linear differential equations, J. Math. Anal. Appl. 206 (1997) 503-517.
[12] D. Kumar, S. Kumar and M. Saini, On Solution of Second Order Complex Differential Equation, Bull. Cal. Math. Soc., 111 (2019), No 4, 331-340.
[13] S. Kumar, N. Mehra and M. Saini, Growth of Solutions of Second Order Linear Differential Equations, (communicated)
[14] S. Kumar and M. Saini, On Zeros and Growth of Solutions of Second Order Linear Differential Equation, Commun. Korean Math. Soc. 35 (2020), No. 1, pp. 229–241.
[15] S. Kumar and M. Saini, Order and Hyper-Order of Solutions of Second Order Linear Differential Equations, Bull. Malays. Math. Sci. Soc. https://doi.org/10.1007/s40840-020-00993-w (2020).
[16] K. Kwon, Nonexistence of finite order solutions of certain second order linear differential equations, Kodai Math. J. 19 (1996), 378—387.
[17] K. H. Kwon, On the growth of entire functions satisfying second order linear differential equations, Bull. Korean Math. Soc. 33 (1996), No. 3, pp. 487-496.
[18] K. Kwon and J. Kim, Maximum modulus, characteristic, deficiency and growth of solutions of second order linear differential equations, Kodai Math. J. 24 (2001) 344–351.
[19] J. R. Long, Growth of solutions of second order complex linear differential equations with entire coefficients, Filomat., 32 (2018), 275-284.
[20] J.R. Long, K.E. Qiu, Growth of solutions to a second order complex linear differential equation, Math. Pract. Theory 45 (2) (2015) 243-247.
[21] J. Wang and I. Laine, Growth of solutions of second order linear differential equations, J. Math. Anal. Appl. 342(2008), 39-51.
[22] H. Wittich, Zur Theorie linearer Differentialgleichungen im Komplexen, Ann. Acad. Sci. Fenn. Ser. A I 379 (1966).
[23] X.B. Wu, J.R. Long, J. Heittokangas, K.E. Qiu, On second order complex linear differential equations with special functions or extremal functions as coefficients, Electron. J. Differential Equations (143) (2015) 1-15.
[24] X.B. Wu, P.C. Wu, On the growth of solutions of $f'' + Af' + Bf = 0$, where $A$ is a solution of a second order linear differential equation, Acta Math. Sci. 33 (3) (2013) 46-52.
[25] J.H. Zheng, On multiply-connected Fatou components in iteration of meromorphic functions, J. Math. Anal. Appl. 313 (2006) 24-37.

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