Effective-range signatures in quasi-1D matter waves: sound velocity and solitons

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Abstract
We investigate ultracold and dilute bosonic atoms under strong transverse harmonic confinement using a 1D modified Gross–Pitaevskii equation (1D MGPE), which accounts for the energy dependence of the two-body scattering amplitude within an effective-range expansion. We study sound waves and solitons of the quasi-1D system, comparing the 1D MGPE results with the 1D GPE ones. We find that when the finite-size nature of the interaction is taken into account, the speed of sound and the density profiles of both dark and bright solitons show relevant quantitative changes with respect to predictions given by the standard 1D GPE.

Keywords: ultracold gases, trapped gases, Boson systems, solitons

1. Introduction
The Gross–Pitaevskii equation (GPE), which plays a relevant role in the study of Bose–Einstein condensates (BECs) made of ultracold and dilute alkali-metal atoms, assumes a zero-range interatomic potential [1]. Recently, several experiments [2] employing the Fano–Feshbach resonance technique in cold atomic collisions [3] have shown that it is possible to change the magnitude and the sign of the scattering length \( a_s \) by using an external magnetic field. Thus, by using Fano–Feshbach resonances it is now possible to explore, at fixed density \( n \), regimes where the GPE and its assumptions lose their validity.

In this work, going beyond the Fermi pseudopotential approximation (contact interaction) of the standard GPE, we focus on sound waves and solitons in a BEC of interacting bosons at zero temperature under a strong transverse harmonic confinement. We take into account the dependence on the energy of the two-body scattering amplitude, employing the effective-range expansion illustrated by Fu \textit{et al} in [4] by inserting therein the correction proposed by Collin and co-workers in [5]. These two ingredients allow us to write a modified version of the Gross–Pitaevskii equation (MGPE, as it is named in [4]) which incorporates the finite-range nature of the interatomic interaction. We reduce the dimensionality of the 3D MGPE by integrating out the degrees of freedom in the radial plane and we obtain a 1D MGPE which takes into account both the scattering length and the effective range of the interatomic potential. We model the boson-boson interaction by means of three potentials: hard-sphere potential, van der Waals potential, and square-well potential. We set the s-wave scattering length to a given value and calculate, for this \( a_s \), the effective range of each of the above model potentials. In this way, we find relevant quantitative changes of the atomic cloud properties, i.e. the speed of sound and the width of the dark and bright solitons, with respect to the results provided by the familiar one-dimensional Gross–Pitaevskii equation.

2. The modified Gross–Pitaevskii equation
We consider \( N \) interacting bosons of mass \( m \) confined by an external trapping potential \( V_{\text{trap}}(\vec{r}) \) at zero temperature. The
Hamiltonian is then given by

$$H = \sum_{i=1}^{N} h(\vec{r}_i) + \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} V(\vec{r}_i - \vec{r}_j),$$  \hfill (1)$$

where

$$h(\vec{r}_i) = -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}}(\vec{r}_i)$$  \hfill (2)$$

with \(V(\vec{r}_i - \vec{r}_j)\) describing the interaction between two bosons at positions \(\vec{r}_i\) and \(\vec{r}_j\). The ground state properties of a weakly interacting bosonic gas can be very efficiently described by using the standard Gross–Pitaevskii equation (GPE) [1]. As is well known, one can derive the GPE, minimizing the GP energy functional \(E_{\text{GP}}\). Describing the interatomic potential by the Fermi pseudopotential

$$V_F(\vec{r}_i - \vec{r}_j) = \delta(\vec{r}_i - \vec{r}_j),$$  \hfill (3)$$

where the coupling strength \(g\) is

$$g = \frac{4\pi \hbar^2 a_s}{m},$$  \hfill (4)$$

with \(a_s\) the interparticle s-wave scattering length, the energy functional \(E_{\text{GP}}\) reads:

$$E_{\text{GP}}[\phi, \phi^*] = N \int d^3\vec{r} \phi(\vec{r})^* h(\vec{r}) \phi(\vec{r}) + \frac{g}{2} N (N-1) \int d^3\vec{r} \lvert \phi(\vec{r}) \rvert^4,$$  \hfill (5)$$

where \(\phi(\vec{r})\) is the single-particle wave function (all the \(N\) bosons are in the same single-particle state). We exploit the variational approach, where the functional \(E_{\text{GP}}\) is required to have a minimum with respect to \(\phi(\vec{r})\), obeying the normalization condition:

$$\int d^3\vec{r} \lvert \phi(\vec{r}) \rvert^2 = 1.$$  \hfill (6)$$

For very large \(N\) one can write that \(N(N-1) \sim N\) and by employing the Lagrange multipliers method, one arrives at the standard GPE

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}}(\vec{r}) + g N \lvert \phi(\vec{r}) \rvert^2 \right] \phi(\vec{r}) = \mu \phi(\vec{r}),$$  \hfill (7)$$

where \(\mu\) is the chemical potential.

At this point some considerations about the interatomic potential \((3)\) are in order. Such a potential ignores completely the dependence on the energy of the scattering amplitude. This approximation, however, is valid provided \(na_s^3\) is sufficiently small. However, for stronger confinements and larger values of \(na_s^3\), a better treatment of atomic interactions that preserves much of the structure of the GP theory is possible. This goal can be pursued by introducing an effective interaction potential \(V_{\text{eff}}\) which gives the energy dependence of the scattering amplitude through an effective-range expansion which will also depend on the effective range \(r_e\) of the interatomic potential \([4, 5]\). Specifically, in the following, we use the effective interaction potential

$$V_{\text{eff}} (\vec{r}_i - \vec{r}_j) = V_F(\vec{r}_i - \vec{r}_j) + V_{\text{mod}} (\vec{r}_i - \vec{r}_j),$$  \hfill (8)$$

where

$$V_{\text{mod}} (\vec{r}_i - \vec{r}_j) = \frac{g_2}{2} \left[ \delta(\vec{r}_i - \vec{r}_j) V_{\text{rep}}^2 (\vec{r}_i - \vec{r}_j) + \frac{2}{2} \delta(\vec{r}_i - \vec{r}_j) \right]$$  \hfill (9)$$

and

$$g_2 = \frac{4\pi \hbar^2}{m} a_s^2 \left( \frac{1}{3} a_s - \frac{1}{2} r_e \right).$$  \hfill (10)$$

In this case, from equation \((8)\), it can be deduced that the energy functional has an extra term \(E_{\text{mod}}\), due to \(V_{\text{mod}}\) having the following form:

$$E_{\text{mod}} \left[ \phi^*, \phi \right] \approx \frac{N}{2} \int d^3\vec{r} \int d^3\vec{r}_2 \phi(\vec{r})^* \phi(\vec{r}_2) \times \left( V_{\text{mod}} (\vec{r} - \vec{r}_2) \phi(\vec{r}_2) \phi(\vec{r}) \right) + \frac{N}{2} \int d^3\vec{R} \int d^3\vec{r} \phi^* (\vec{R} + \vec{r} \frac{2}{2} ) \phi^* \times \left( V_{\text{mod}} (\vec{r}) \phi(\vec{R} + \vec{r} \frac{2}{2} ) \phi(\vec{R} - \vec{r} \frac{2}{2} ) \right),$$  \hfill (11)$$

where we have made use of \((N-1) \sim N\) and the second row is a rewriting of the first one in the two body center-of-mass frame \((\vec{r} = \vec{r}_i - \vec{r}_j, \vec{R} = (\vec{r}_i + \vec{r}_j)/2)\). The simplification of \(E_{\text{mod}}\) achieved by doing calculations in the above frame and minimization of the (inclusive-\(E_{\text{mod}}\)) modified Gross–Pitaevskii (MGP) energy functional

$$E_{\text{MGP}} \left[ \phi^*, \phi \right] = \int d^3\vec{r} \phi^* \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}} (\vec{r}) + g N \lvert \phi(\vec{r}) \rvert^2 \right] \phi + \frac{g}{2} \lvert \phi \rvert^2 + \frac{g_2}{2} \nu^2 (\lvert \phi \rvert^2) \right] \phi$$  \hfill (12)$$

with respect to \(\phi^*\) with the constraint \((6)\) provides the following modified Gross–Pitaevskii equation (MGPE)

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}} (\vec{r}) + g N \lvert \phi(\vec{r}) \rvert^2 \right] \phi + \frac{N}{2} g_2 \nu^2 (\lvert \phi(\vec{r}) \rvert^2) \right] \phi = \mu \phi(\vec{r}).$$  \hfill (13)$$

Notice that a similar nonlinear Schrödinger equation has been derived and studied by García-Ripoll, Konotop, Malomed, and Pérez-García [6, 7]. Their investigation starts from the Hartee equation for bosons, which is a nonlocal integral Schrödinger equation (nonlocal GPE) [7], and it is based on a gradient expansion of the nonlocal GPE [6, 7].

3. The one-dimensional MGPE

We assume that the external confinement potential \(V_{\text{trap}}(\vec{r})\) is obtained by superimposing onto a very strong isotropic harmonic confinement in the \(x - y\) (radial) plane a generic
shallow potential along the z (axial) direction, so that 
\[ V_{\text{trap}}(\vec{r}) = \frac{1}{2} m \omega_z^2 (x^2 + y^2) + U(z), \]
where \( \omega_z \) is the trapping harmonic frequency. The spatial degree of freedom in the radial plane is thus frozen and the system can be considered, in practice, one-dimensional (1D) in the axial direction. As suggested by the form (14) of the external trapping potential, we shall use the following Gaussian ansatz for the single-particle wave function \( \phi(\vec{r}) \):
\[ \phi(\vec{r}) = \frac{\varphi(z)}{\sqrt{\pi a_z^2}} e^{-\frac{x^2 + y^2}{2 a_z^2}}, \]
where \( a_z = \sqrt{\hbar/(m \omega_z)} \) is the transverse characteristic length of the ground state of the harmonic potential and \( \int dz |\varphi(z)|^2 = 1 \). This ansatz will be valid when \( g|\varphi|^2/2a_z^2 \ll 2\hbar \omega_z \) [8]. Inserting equations (14) and (15) into equation (12) and then minimizing with respect to \( \varphi^* \) leads to the 1D version of the modified Gross–Pitaevskii equation
\[ \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + U(z) + \gamma |\varphi|^2 + \frac{1}{2} \gamma_2 \frac{d^2}{dz^2} |\varphi|^2 \right] \varphi(z) = \mu \varphi(z), \]
where
\[ \gamma = \frac{1}{2a_z^2} \left( g - \frac{g_2}{a_z^2} \right), \quad \gamma_2 = \frac{g_2}{2a_z^2}, \quad \mu = \mu - \hbar \omega_z. \]

The effective-range effects heralded by equation (16) become clear when the ratio of the absolute value of the effective range \( |\Gamma_z| \) to the interatomic distance (this distance being referred to the 3D system) is of the same order of magnitude as the ratio of this distance to the absolute value of the s-wave scattering length \( a_s \). In this situation the dependence on the energy of the two-body scattering amplitude (see, for example, [4, 11]) cannot be neglected and the usual 1D GPE is not able to describe adequately anymore the physics of our system. Thus, to study the effects of the finite-size nature of the boson-boson interaction on the atomic cloud properties, in our forthcoming 1D MGPE-based studies, \( |\Gamma_z| \) and \( |a_s| \) will be chosen in such a way so as to meet the condition mentioned above. Moreover, note that results from equation (16) are reliable as long as \( N|a_s|/a_z \ll 1 \).

4. Interaction potentials

In this section we present three toy models for the two-body interaction potential between atoms. Then, we use these three potentials in the analysis of the sound velocity and solitonic waves within the system under investigation.

- **Hard-sphere potential.** This model for the description of the boson-boson interaction is defined as follows
\[ V(r) = \begin{cases} \infty & r \leq a_s, \\ 0 & r > a_s. \end{cases} \]

For this potential,
\[ r_c = \frac{2}{3} a_s, \]
and one thus reduces to the standard GPE since \( \gamma_2 = 0 \), as can be seen from the first and second formulas of equation (17) with \( g_2 \) given by equation (10).

- **Square-well potential.** In this case, the two-body collisions are described by a potential well characterized by a finite depth:
\[ V(r) = -V_0 \quad r \leq r_0, \quad \text{while} \quad V(r) = 0 \quad r > r_0 \]
with \( V_0 \) positive. It is possible to show that in the limit of sufficiently small incident wave vector \( q \to 0 \), the s-wave scattering length \( a_s \) is given by
\[ a_s = r_0 \left[ 1 - \frac{\tan \left( \frac{\chi(0)}{r_0} \right)}{\chi(0)} \right], \]
and the effective range \( r_e \) by
\[ r_e = r_0 \left[ 1 - \frac{r_0^2}{3a_s^2} - \frac{1}{\chi(0)^2 a_s r_0} \right], \]
where \( \chi(0)^2 = mV_0/\hbar^2 \).

- **van der Waals potential.** When the interaction is van der Waals-like, the interaction potential may be approximated by a potential well for \( r < r_0 \) (this latter being called the empty-core radius), while, otherwise, by a function of the form \(-C_6/r^6\), that is
\[ V(r) = \begin{cases} \infty & r \leq r_0, \\ -C_6/r^6 & r > r_0. \end{cases} \]

where \( C_6 \) is a parameter which quantifies the interaction strength. Note that the potential above is reminiscent of the Ashcroft pseudopotential used to treat conduction electrons in alkali metals. For the potential (23) the s-wave scattering length \( a_s \) and the effective range \( r_e \) have the following expressions [9]:
\[ a_s = \frac{2\pi}{3} \left( \frac{3}{2} \right) (1 - \tan \Phi) l_{sd}, \]
\[ r_e = \frac{2\pi}{3} \left( \frac{3}{2} \right) (1 - \tan \Phi) l_{sd}, \]
respectively. In the above formulas \( l_{sd} \) is a \( C_6 \)-dependent characteristic length and \( \Phi \) a function depending on the ratio \( l_{sd}^2/r_0^2 \):
\[ l_{sd} = \left( \frac{mC_6}{\hbar^2} \right)^{1/4} \Phi = \frac{l_{sd}^2}{2r_0} - \frac{3\pi}{8}. \]

The forthcoming analysis will be focused on the sound velocity and solitonic density profiles for each of the three boson-boson interaction potential models presented above. We keep fixed the scattering length \( a_s \) and calculate the
effective interaction range $r_c$ by using the formulas above provided, that is, equation (19) for the hard spheres potential (18), equations (21) and (22) for a given $V_0$ in the case of the square-well potential (20), and equations (24) and (25) for a given $C_6$ in the case of the van der Waals potential (23).

5. Sound velocity

We want to gain physical insight into both the spatial and temporal evolution of our system. The theoretical tool which permits us to do this is the time-dependent version of the modified one-dimensional Gross–Pitaevskii equation (16). We suppose that $U(z) = 0$, and scale lengths, times, and energies are in units of $a_\perp$, $1/\omega_\perp$, and $\hbar \omega_\perp$, respectively. We thus use the following adimensional time-dependent 1D MGPE:

$$\frac{\partial}{\partial t} \psi(z, t) = \left[ -\frac{1}{2} \frac{d^2}{dz^2} + \gamma |\psi|^2 + \frac{1}{2} \frac{n}{a_\perp^2} \frac{d^2}{dz^2} |\psi|^2 \right] \psi(z, t),$$  

where, for simplicity of notation, we have denoted the dimensionless quantities by the same symbols used for those with dimensions. We are interested, in particular, in the consequences of a perturbation, with respect to the equilibrium, created at a given spatial point of the system at a given time. We start writing $\psi(z, t)$ as:

$$\psi(z, t) = \sqrt{n(z, t)} e^{iS(z, t)},$$  

with $n(z, t)$ describing the density profile and $S(z, t)$ related to the velocity field $v(z, t)$ via the relation

$$v(z, t) = \frac{\partial}{\partial z} S(z, t).$$

By inserting the two equations above in the time-dependent 1D MGPE (27), one obtains the hydrodynamic equations (HEs)

$$\frac{\partial n}{\partial t} + \frac{d}{dz} \left[ \frac{1}{2} v^2 + \gamma n + \left( \gamma_2 - \frac{1}{4n} \right) \frac{d^2 n}{dz^2} + \frac{1}{8n} \left( \frac{dn}{dz} \right)^2 \right] = 0,$$

$$\frac{\partial n}{\partial t} + \frac{d}{dz} (nv) = 0.$$  

(30)

At this point, let us suppose the system is perturbed with respect to the equilibrium configuration characterized by $n(z, t) = n_0$ and $v(z, t) = v_0 = 0$:

$$n(z, t) = n_0 + \delta n(z, t),$$

$$v(z, t) = v_0 + \delta v(z, t).$$

(31)

We use these formulas in the hydrodynamic equations (30) and assume them to be in the stationary regime, $v_0 = 0$. Under the hypothesis that the perturbation is sufficiently weak so as to retain only the $\delta n$-first-order terms in the HEs, we get

$$\frac{d^2}{dz^2} \delta n = n_0 \gamma \frac{d^2}{dz^2} (\delta n) - n_0 \left( \gamma_2 - \frac{1}{4n_0} \right) \frac{d^4}{dz^4} \delta n = 0.$$  

(32)

If the perturbation is a plane wave, that is $\delta n(z, t) = Ae^{i(kz-\omega t)}$, $Ae^{i(kz-\omega t)}$, the relation of dispersion which characterizes the oscillations associated to the wave induced by the perturbation is

$$\omega = k \sqrt{n_0 \gamma - \left( n_0 \gamma_2 - \frac{1}{4} \right) k^2}$$  

(33)

which depends on the equilibrium density $n_0$ and contains information about two-body collisions via $\gamma$ and $\gamma_2$, see the first two formulas of equation (17), and equations (4) and (10). The perturbation will stabilise with respect to time for real $\omega$, which is always guaranteed when $a_\perp = 2/3 e_c$. If this is the case, the dispersion relation (33) is the usual Bogoliubov dispersion, that is

$$\omega = c_s k$$  

(35)

with the velocity $c_s = \sqrt{n_0 \gamma}$ of sound propagating in the system related to the interaction parameters, equilibrium density, and harmonic trap characteristics. To see more clearly such a dependence we use the standard units of measure so that one has

$$c_s^2 = \frac{n_0}{m a^2_\perp} \left( 1 - \frac{1}{3} \frac{a^2_\perp}{a^2_\perp} + \frac{r_c}{a_\perp^2} \right),$$  

(36)
where we have to take into account the definitions of $\gamma$, $g$ and $g_2$.

As mentioned above, we study the sound velocity $c_s$ as a function of the equilibrium density $n_0$, equation (36), and analyze such a quantity for each of the three interaction potentials previously described.

Figure 1 shows the sound velocity $c_s$ as a function of the axial equilibrium density $n_0$ on varying the shape of the interatomic interaction potential, see section 4. We have fixed the s-wave scattering length $a_s$ and calculated (given $r_0$ and $V_0$ for the potential (20) and $C_0$ and $r_0$ for the potential (23)) the value of $r_c$ for each interatomic potential using equation (19) for the hard-sphere potential, equations (21) and (22) for the square-well potential, and equations (24) and (25) for the van der Waals potential.

For any chosen set of parameters of the interatomic potential under investigation the final result will only depend on the obtained values of $a_s$ and $r_c$. Clearly, except for the case of the hard-core potential, fixing $a_s$ several parameters of the interatomic potential under investigation will give the same $r_c$ and the same sound velocity $c_s$.

We observe that the behavior of the sound velocity, when the type of boson-boson interaction changes, is qualitatively the same. However, at a given $n_0$, by increasing $g_2 > 0$ one gets a larger sound velocity $c_s$.

The solid line of figure 1 represents the sound velocity as a function of the axial equilibrium density when the interaction between the bosonic atoms is described by the hard-sphere potential (19). Since $r_c = 2/\beta a_s$ - equation (19) - $g_2 = 0$ (see equation (10) and the third formula of equation (17)), it reduces to the same behavior as predicted by the 1D GPE with a Dirac-delta interaction characterized by the assigned $a_s$, see equation (27).

For instance, figure 1 compares sound velocity versus density in the three potentials of interest. We can thus conclude that the finite-size nature of the interatomic interaction has the effect of producing quantitative changes in the behavior of the sound velocity $c_s$ with respect to that predicted by the familiar 1D GPE.

6. Solitons

We start by considering the time-dependent 1D MGPE (27). When $g_2 = 0$ this equation reduces to the standard time-dependent one-dimensional Gross–Pitaevskii equation. It is well known that this equation admits the possibility of studying topological configurations of the Bose–Einstein condensate like solitonic solutions (solitary waves preserving their form and propagating with a constant velocity $v$) with positive (repulsive interatomic interaction) or negative (attractive interatomic interaction) s-wave scattering length $a_s$ [10]

$$\varphi(z, t) = f(z - vt) e^{i\gamma z} e^{i(\frac{1}{2}z^2 - \mu vt)}.$$ (37)

The solutions corresponding to $a_s > 0$ are the dark solitons. The axial density $|f|^2$ of these solitons assumes the same finite value when $x = \pm \infty$ (with $x = z - vt$ the comoving coordinate of the soliton) and is characterized by a hole structure with a minimum at $x = 0$. The difference between the phases of the wave function at $\pm \infty$ is finite. For $a_s < 0$ one has the bright solitons that set up when the negative interatomic energy of the BEC balances the positive kinetic energy so that the BEC is self-trapped in the axial direction. In this case $|f|^2$ goes to zero when $x = \pm \infty$ and exhibits a pulse structure with a maximum at $x = 0$. The difference between the phases of the wave function at $\pm \infty$ is zero.

We focus on solitary waves when the the effective-range correction is taken into account, that is with $g_2$ as finite. Proceeding thus from the 1D MGPE, we look for its solutions of the form (37) which, inserted into equation (27), provides the following differential equation:

$$-\frac{1}{2} \frac{d^2 \varphi}{d z^2} + \gamma |\varphi|^2 + \frac{1}{2} g_2 (|\varphi|^2)^2 \varphi = \mu \varphi,$$ (38)

where $\gamma \equiv \frac{\mu^2}{4\beta}$. We observe (see the discussion in the sequel) that the 1D MGPE admits dark (bright) solitonic solutions when the nonlinearity $\gamma$ is positive (negative). Therefore, due to the form of $\gamma$—the first formula of equation (17)—it is possible to have a given type of soliton irrespective of the sign of $a_s$.

6.1. Dark solitons

We study the black solitons that are dark solitons characterized by a vanishing axial density at $x = 0$ and zero velocity $v$ with respect to the condensate. It is possible to achieve a relation which implicitly defines the solution $f$ of the differential equation (38) that reads

$$\sqrt{1 - 2g_2 f(z)^2} \arctan \left( \frac{f(z)}{f_\infty} \right) = \sqrt{2} f_\infty |z|.$$ (39)
with \( f_\infty \) being the absolute value obtained by \( f \) at \( \pm \infty \) and \( \gamma > 0 \). Since \( 0 < |f(z)|^2 < 1 \), the dark solitons solution exists when \(-\infty < \gamma < 1/2\).

The density profile \( f(z)^2 \) can be thus studied as a function of the axial coordinate \( z \) by solving numerically equation (39) when one knows the features of the boson-boson interaction, i.e. both \( \gamma \) and \( \gamma_2 \). To set these two quantities, we have followed the same procedure as that followed to obtain figure 1 (see section 5). We have thus plotted \( f(z)^2 \) versus \( z \), see figure 2.

We observe that when one takes into account the finite-size nature of the interatomic interaction, the width of the solitary wave under investigation is qualitatively the same as one would find using the familiar one-dimensional Gross–Pitaevskii equation (solid line, see the discussion in section 5) but its magnitude meaningfully changes with respect to the latter case.

Actually, the width \( \Delta z \) at half-minimum of the dark soliton can be easily calculated from equation (39), setting \( f_\infty = 1 \), \( f(z) = 1/2 \), and \( z = \Delta z/2 \). In this way, we immediately find

\[
\Delta z = \frac{2}{\arctanh\left(\frac{1}{2}\sqrt{\frac{1 - \gamma_2}{\gamma}}\right)} . \tag{40}
\]

Taking into account the definitions of \( \gamma \) and \( \gamma_2 \), equation (17) with equations (4) and (10), this formula gives the width \( \Delta z \) of dark solitons as a function of the scattering length \( a_s \), effective range \( r_e \), and transverse width \( a_\perp \) of the harmonic confinement.

### 6.2. Bright solitons

We start from equation (38). When \( \gamma < 0 \), the constant of motion for this equation is

\[
K = \frac{1}{2}(f')^2 + \mu f^2 - \frac{1}{2}\mu f^4 - \frac{1}{4}\gamma_2\left[(f')^2\right]^2 . \tag{41}
\]

By requiring that \( f \) and its first derivative tend to zero at \( \pm \infty \), we get \( K = 0 \). By imposing that \( f \) is maximum for \( x = 0 \), we obtain \( \mu = -\frac{1}{2}\gamma_2|f(0)|^2 \), and by defining \( f = \phi(x)^{1/2} \) we get, from equation (41),

\[
\phi' = \pm \frac{1}{\sqrt{8(K - \mu \phi + \frac{1}{2}\gamma_2\phi^2)}} . \tag{42}
\]

Then, by integrating the above expression with + and by using \( K = 0 \) and \( \mu = -1/2\gamma_2|f(0)|^2 \), one has that

\[
2\sqrt{\gamma_2}z = \int_{f(0)^2}^{f(z)^2} dy \sqrt{\frac{1 - 2\gamma_2 y}{y^2(f(0)^2 - y)}} . \tag{43}
\]

The integral on the right hand side of equation (43) can be numerically solved by studying the density profile \( f(z)^2 \) of the soliton as a function of the axial coordinate \( z \), setting both \( \gamma \) and \( \gamma_2 \). Therefore, for the bright solitons too, we have studied the density profile \( f(z)^2 \) as a function of the axial coordinate, varying the boson–boson interaction potential by following the same path as for the black solitons. These results are included in figure 3. From the plots therein, it can be observed—as for the sound velocity and the dark solitons—that the width is quantitatively affected by the nature of the interatomic interaction potential. The width \( \Delta z \) at half-maximum of the bright soliton can be calculated from equation (43), setting \( f(0) = 1 \), \( f(z) = 1/2 \) and \( z = \Delta z/2 \). In this way we immediately find

\[
\Delta z = \frac{1}{\sqrt{|\gamma|}} \int_{1/4}^{1} dy \sqrt{\frac{1 - 2\gamma_2 y}{y^2(1 - y)}} . \tag{44}
\]

This formula is more complex than equation (40), but figure 4 shows that equation (40) has the same behavior as equation (44) once the signs of \( \gamma \) are taken into account.

![Figure 3](image-url)  
**Figure 3.** Axial density profile \( f(z)^2 \) (at \( t = 0 \)) of the bright soliton vs axial coordinate \( z \) for \( a_s = -0.1 \). Solid line: hard-sphere potential (18) (this curve is the same as that provided by the standard 1D GPE). Dot-dashed line: square-well potential (20) \((\sigma_0 = 0.5, \nu_0 = 82.1011)\). Dashed line: van der Waals potential (23) \((C_6 = 0.07, \nu_0 = 0.2492)\). Lengths in units of \( a_s \), energies in units of \( \hbar \omega_\perp C_s \) in units of \( \hbar \omega_\perp a_s^3 \), \( f(z)^2 \) in arbitrary units.

![Figure 4](image-url)  
**Figure 4.** Width \( \Delta z \) of dark solitons (solid line) and bright solitons (dashed line) as a function of the coupling \( \gamma_2 \). We set \( \gamma = 1 \) for dark solitons and \( \gamma = -1 \) for bright solitons. \( \Delta z \) is in units of \( a_s \), \( \gamma \) is in units of \( \hbar \omega_\perp a_s \), \( \gamma_2 \) is in units of \( \hbar \omega_\perp a_s^3 \).
7. Conclusions

We have considered a system of interacting atomic bosons confined in a strong harmonic confinement in the radial plane plus a weak potential along the axial direction at zero temperature. We have carried out our analysis going beyond the Fermi pseudopotential approximation and described the gas evolution by employing a modified one-dimensional Gross–Pitaevskii equation (1D MGPE) in the absence of the axial potential. By using the latter equation we have studied the propagation of sound waves, and that of solitons, in the system under investigation. We have used the 1D MGPE to study the sound velocity versus the axial density and the density profiles of the solitons (black and bright) as a function of the axial coordinate by modeling the boson-boson interaction via a hard-sphere potential, a square-well potential, and a van der Waals potential. We have performed our investigations by fixing the s-wave scattering length, $a_s$, and calculating the effective range, $r_e$, corresponding, for this $a_s$, to each interatomic potential. This analysis has allowed us to conclude that the effective-range signatures are reflected in important quantitative changes (with respect to the results of the familiar 1D GPE) in the speed of sound and solitary waves density profile.

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