Certain minimal varieties are set-theoretic complete intersections

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Abstract We present a class of homogeneous ideals which are generated by monomials and binomials of degree two and are set-theoretic complete intersections. This class includes certain reducible varieties of minimal degree and, in particular, the presentation ideals of the fiber cone algebras of monomial varieties of codimension two.

Keywords: Minimal variety, rational normal scroll, set-theoretic complete intersection, fiber cone.

Introduction and Preliminaries

Let \( K \) be an algebraically closed field, and let \( R \) be a polynomial ring in \( N \) indeterminates over \( K \). Let \( I \) be a proper reduced ideal of \( R \) and consider the variety \( V(I) \) defined in the affine space \( K^N \) (or in the projective space \( \mathbb{P}^{N-1}_K \), if \( I \) is homogeneous and different from the maximal irrelevant ideal) by the vanishing of all polynomials in \( I \). By Hilbert Basissatz there are finitely many polynomials \( F_1, \ldots, F_s \in R \) such that \( V(I) \) is defined by the equations \( F_1 = \cdots = F_s = 0 \). By Hilbert Nullstellensatz this is equivalent to the ideal-theoretic condition

\[
I = \sqrt{(F_1, \ldots, F_s)}.
\]  

Suppose \( s \) is minimal with respect to this property. It is well known that height \( I \leq s \). If equality holds, \( I \) is called a set-theoretic complete intersection (s.t.c.i.) on \( F_1, \ldots, F_s \).

Exhibiting significant examples of s.t.c.i. ideals (or, more generally, determining the minimum number of equations defining given varieties, the so-called arithmetical rank) is one of the most difficult problems in algebraic geometry. The main problem is finding polynomials \( F_1, \ldots, F_s \) fulfilling (1), which need not be part of a minimal generating system for \( I \). This task cannot be accomplished by a general constructive method. There are, however, a few results that allow us to settle several special cases. One of these is due to Schmitt and Vogel.

Lemma 1 [PS, p. 249] Let \( P \) be a finite subset of elements of \( R \). Let \( P_0, \ldots, P_r \) be subsets of \( P \) such that

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(i) $\bigcup_{l=0}^{r} P_l = P$;

(ii) $P_0$ has exactly one element;

(iii) if $p$ and $p''$ are different elements of $P_l$ ($0 < l \leq r$) there is an integer $l'$ with $0 \leq l' < l$ and an element $p' \in P_{l'}$ such that $(pp'')^m \in (p')$ for some positive integer $m$.

We set $q_l = \sum_{p \in P_l} p^{e(p)}$, where $e(p) \geq 1$ are arbitrary integers. We will write $(P)$ for the ideal of $R$ generated by the elements of $P$. Then we get

$$\sqrt{(P)} = \sqrt{(q_0, \ldots, q_r)}.$$

This result, together with its refinements and generalizations established in [4], is especially useful for ideals generated by monomials. An interesting class of s.t.c.i. monomial ideals was introduced by Lyubeznik [21] and studied in [3].

Big classes of s.t.c.i. ideals generated by binomials have been characterized among the toric ideals (see, e.g., [5], [9], [20], and [10], [13], [15], [24] for toric curves), and some cases are also known among the determinantal ideals of two-row matrices. One of these was treated by Robbiano and Valla [20], another one by Bardelli and Verdi [1]. In this paper we generalize Bardelli and Verdi's result by presenting a class of s.t.c.i. ideals generated by some minors and some products of entries of certain blockwise defined matrices, which were considered in [6]. The corresponding varieties include the fiber cones of codimension two monomial varieties, which were studied by Giménez, Morales and Simis in [10] and [17], and, furthermore, they belong to the (larger) class of reducible varieties of minimal degree classified by Xambó [29]. These are all defined by monomials and binomials, but they are not all s.t.c.i. At the end we shall exhibit a counterexample, which is also interesting from another point of view: in positive characteristics its arithmetical rank is strictly greater than its cohomological dimension, which seems to be a rare property.

1 A class of set-theoretic complete intersections

Let $r$ be a positive integer and consider the two-row matrix

$$A = (B_1 \parallel B_2 \parallel \ldots \parallel B_r),$$

where, for all $i = 1, \ldots, r$, $B_i$ is the $2 \times c_i$-matrix

$$B_i = \begin{pmatrix} X_i^1 & X_i^2 & \cdots & X_i^{c_i-1} & X_i^{c_i} \\ X_i^1 & X_i^2 & \cdots & X_i^{c_i} & X_i^{c_i+1} \end{pmatrix}.$$  (3)

Here $X = \{X_{ij}\}$ is a set of $N$ indeterminates over $K$, and $X_i^j \neq X_k^h$ for $(i, j) \neq (h, k)$, with one only possible exception: for every index $i$, $1 \leq i < r$, there is at most one index $i' > i$ such that $X_{i'1}^i = X_i^1$. This kind of matrix was introduced by Giménez [10], and also considered in [6], where $A$ was called a barred matrix and the $B_i$’s were called the big blocks of $A$. 

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**Example 1** An example of matrix of type (2) in the indeterminates \(X_1, \ldots, X_{11}\) is the following:

\[
A = \begin{pmatrix}
X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} \\
X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11}
\end{pmatrix}
\]

Fix one index \(i, 1 \leq i \leq r\). If \(c_i > 1\), for all indices \(j, j', 1 \leq j < j' \leq c_i\) let \(M_{jj'}^i\) be the minor of \(B_i\) formed by the \(j\)-th and the \(j'\)-th column, i.e.,

\[
M_{jj'}^i = \begin{vmatrix}
X_{ij} & X_{ij'} \\
X_{i+1,j} & X_{i+1,j'}
\end{vmatrix} = X_{ij}X_{ij'+1} - X_{ij'}X_{ij+1}.
\]

Let \(I_i \in K[X_1^i, \ldots, X_{c_i+1}^i]\) be the ideal generated by all minors \(M_{jj'}^i\), and consider

\[
F_j = \sum_{k=0}^j (-1)^k \binom{j}{k} (X_{j+2}^i)^{j-k} X_{k+1}^i (X_{j+1}^i)^k, \quad \text{for } j = 1, \ldots, c_i - 1,
\]

The first part of the next theorem can be found in [13], pp. 118ff, the second part is due to Bardelli and Verdi [1] (see [27], Section 2). In order to simplify the notation, we shall omit the index \(i\) in the claim.

**Theorem 1** The ideal \(I\) is prime of height \(c-1\). It is a s.t.c.i. on \(F_1, \ldots, F_{c-1}\).

The ideal \((I_i)\) thus defines an irreducible curve in \(\mathbb{P}^{c-1}_K\), which is known as a rational normal curve, and is a special kind of (rational normal) scroll: we refer to Harris [18] for an introductory treatment of this notion. A matrix of type (3) will be called a scroll matrix. As a consequence of Theorem 1 and Hilbert Nullstellensatz we also have that

\[
\sqrt{(F_1^i, \ldots, F_{c_i-1}^i)} = I_i.
\]

Following [6], we can associate with matrix \(A\) an ideal \(J\) of \(R = K[X]\). This ideal \(J\) is generated by the union of

(i) the set of all minors \(M_{jj'}^i\) with \(i = 1, \ldots, r\), and \(1 \leq j < j' \leq c_i\) (we set \(M_{jj'}^i\) equal to the empty set whenever \(c_i = 1\));

(ii) the set of all products \(X_{ij}^iX_{ij'}^i\), with \(1 \leq i < i' \leq r\), \(1 \leq j \leq c_i\), \(2 \leq j' \leq c_{i'} + 1\). These are the products of one entry of the upper row of \(B_i\) and one entry of the lower row of one of the blocks \(B_{i'}\) following on the right.

Note that

\[
I_i \subset J \quad \text{for all } i = 1, \ldots, r
\]

**Example 2** The ideal \(J\) associated with the matrix \(A\) given in Example 1 is generated by the following 28 elements
We introduce one more piece of notation. We set
\[ G_k = \sum_{i=1}^{r-k} X_i^1 X_{i+k+1}^1, \quad \text{for all } k = 1, \ldots, r - 1. \] (7)

In other words, \( G_k \) is the sum of products of the entry lying in the left upper corner of a block \( B_i \) and the entry lying in the right lower corner of the block \( B_{i+k} \). In particular every summand of \( G_k \) is an element of the set defined in (ii), therefore
\[ G_k \in J \quad \text{for all } k = 1, \ldots, r - 1. \] (8)

We are now ready to state our main result, which generalizes Theorem 1.

**Theorem 2** The ideal \( J \) is reduced of pure height \( \sum_{i=1}^{r} c_i - 1 \). It is a s.t.c.i. on
\[ F_1^1, \ldots, F_{c_1-1}^1, \ldots, F_r^1, \ldots, F_{c_r-1}^r, G_1, \ldots, G_{r-1}. \]

**Proof.** - The first part of the claim is Corollary 1.3 in [6]. We prove the second part. First of all we remark that the number of polynomials \( F_{i,j}, G_k \) is
\[ \sum_{i=1}^{r} (c_i - 1) + r - 1 = \sum_{i=1}^{r} c_i - r + r - 1 = \sum_{i=1}^{r} c_i - 1. \]

Next we have to show that, in \( K^n \), the variety \( V \) defined by \( J \) coincides with the variety \( V' \) defined by the polynomials \( F_{i,j}, G_k \). According to [5], [9] and [8], all these polynomials belong to \( J \), so that we certainly have \( V \subset V' \). For the opposite inclusion, consider a point \( x \in K^n \) such that
\[ F_{i,j}^i(x) = 0 \quad \text{for all } i = 1, \ldots, r, j = 1, \ldots, c_i - 1, \] (9)
\[ G_k(x) = 0 \quad \text{for all } k = 1, \ldots, r - 1. \] (10)

Our claim is that all polynomials in the sets defined in (i) and (ii) vanish at \( x \). By virtue of Theorem [5, 9] implies that, for all \( i = 1, \ldots, r \) and \( 1 \leq j < j' \leq c_i \),
\[ M_{i,j,j'}^i(x) = 0. \] (11)
Thus there remains to prove that, for all \( i, j, i', j' \) with \( 1 \leq i < i' \leq r, 1 \leq j \leq c_i, 2 \leq j' \leq c_{c'_i} + 1, \)
\[
x^i_x x^{i'}_{x'} = 0,
\]
(12)
where \( x^i_x \) is the monomial \( X^i_x \) evaluated at \( x \). Note that, according to (11),
\[
\Gamma_{r-1} = X^1_{1}X^{c'_i+1}_{c'_i+1}.
\]
Let \( 1 \leq k < r - 1 \), and consider the product of two arbitrary summands of \( \Gamma_k \):
\[
\Pi = X^1_{1}X^{i+k}_{i+k+1} \cdot X^{j+k}_{j+k+1},
\]
where \( 1 \leq i < r - k \). Let \( k' \) be such that \( i' + k = i + k' \). Then \( k' > k \), and \( X^i_{1}X^{i+k}_{i+k+1} = X^i_{1}X^{i+k'}_{i+k'+1} \) is a summand of \( \Gamma_k \) which divides \( \Pi \). Let for all \( k = 0, \ldots, r - 2 \), \( P_k \) be the set of all summands of \( \Gamma_{r-1-k} \); then Lemma 1 applies. Thus (10), together with Hilbert Nullstellensatz, implies that
\[
x^i_{x} x^{i+k}_{c_{i+k+1}} = 0, \quad \text{for all } i = 1, \ldots, r - 1, k = 1, \ldots, r - i.
\]
(13)
Now suppose for a contradiction that (12) is false, i.e., that for some \( i, j, j', k \)
with \( 1 \leq i \leq r - 1, 1 \leq k \leq r - i, 1 \leq j \leq c_i, 2 \leq j' \leq c_{i+k} + 1, \)
\[
x^i_x x^{i+k}_{j'} \neq 0.
\]
(14)
For two fixed indices \( i \) and \( k \), let \( j' - j \) be maximal with respect to (14). Then, by (13), necessarily \( j' - j \neq c_{i+k}, \) which means \( j' - j < c_{i+k}, \) i.e., \( 1 \leq j \leq c_i \) or \( 2 \leq j' < c_{i+k} + 1 \). First suppose that \( 1 < j \leq c_i \). Then, by the maximality condition, we have
\[
x^{i}_{j-1} x^{i+k}_{j+1} = 0,
\]
(15)
which, in view of (14), implies that
\[
x^{i}_{j+1} = 0.
\]
(16)
On the other hand, by (11),
\[
M^{i}_{j-1, i}(x) = \begin{vmatrix} x^i_{j-1} & x^i_{j} \\ x^i_{j} & x^i_{j+1} \end{vmatrix} = x^i_{j-1} x^i_{j+1} - (x^i_{j})^2 = 0.
\]
(17)
But (17) and (14) give \( x^i_{j} = 0 \), against our assumption (14). Now suppose that
\[
2 \leq j' < c_{i+k} + 1.
\]
In this case
\[
x^i_x x^{i+k}_{j'+1} = 0,
\]
whence, by (14)
\[
x^{i+k}_{j'+1} = 0.
\]
(18)
Now, by (11),
\[
M^{i+k}_{j'-1, j'}(x) = \begin{vmatrix} x^{i+k}_{j'-1} & x^{i+k}_{j'} \\ x^{i+k}_{j'} & x^{i+k}_{j'+1} \end{vmatrix} = x^{i+k}_{j'-1} x^{i+k}_{j'+1} - (x^{i+k}_{j'})^2 = 0,
\]
which, by (18), yields \( x^{i+k}_{j'} = 0 \). This, again, contradicts (14). Hence (12) is always true. This completes the proof.
Example 3 According to Theorem 2, the ideal \( J \) defined in Example 2 has pure height 7 and is a s.t.c.i. on the following polynomials:

\[
\begin{align*}
F_1^2 &= X_3X_5 - X_4^2 \\
F_1^3 &= X_5X_7 - X_6^2 \\
F_2^2 &= X_5X_8^2 - 2X_6X_7X_8 + X_7^3 \\
F_2^3 &= X_9X_{11} - X_9^2 \\
G_3 &= X_1X_{11} \\
G_2 &= X_1X_8 + X_3X_{11} \\
G_1 &= X_1X_5 + X_3X_8 + X_5X_{11}.
\end{align*}
\]

Computations with CoCoA show that, for all fields \( K \), the least power of \( J \) contained in the ideal \((F_1^2, F_1^3, F_2^2, F_2^3, G_1, G_2, G_3)\) is the 13th.

2 The fiber cone of a monomial variety of codimension two

In this section we show that Theorem 2 applies to a relevant class of polynomial ideals. We first need to recall some preliminary notions from commutative algebra. Let \( I \) be an ideal of \( R \), and let \( t \) be an indeterminate over \( R \). The graded ring \( R[It] = \bigoplus_{i \in \mathbb{N}} Pi^i \) is called the Rees algebra of \( I \). If \( M \) is the ideal of \( R \) generated by the set of indeterminates \( \mathbf{X} \), so that \( R/M \simeq K \), then the quotient ring \( F(I) = R[It]/MR[It] \simeq R[It] \otimes K \) is called the fiber cone algebra of \( I \). Suppose that \( H_1, \ldots, H_m \) form a minimal generating set of \( I \), and introduce a set of \( m \) independent variables over \( R \), say \( T = \{T_1, \ldots, T_m\} \). Consider the ring homomorphism

\[
\phi : R[T] \to R[It]
\]

such that, for all \( i = 1, \ldots, m, \)

\[
\phi(T_i) = H_i t.
\]

It is evidently surjective, so that \( R[It] \simeq R[T]/\ker \phi \). Tensoring with \( K \) yields \( F(I) \cong K[T]/J \), for a suitable presentation ideal \( J \subset K[T] \).

Assume that \( I \subset R = K[X_1, \ldots, X_n, Y_1, Y_2] \) is the defining ideal of a monomial variety of codimension two, i.e., of a variety of \( K^{n+2} \) admitting a parametrization of the following form:

\[
\begin{align*}
x_1 &= u_1^{a_1}, \quad x_2 = u_1^{a_2}, \ldots, \quad x_n = u_1^{a_n}, \quad y_1 = u_1^{b_1}u_2^{b_2} \cdots u_n^{b_n}, \quad y_2 = u_1^{c_1}u_2^{c_2} \cdots u_n^{c_n},
\end{align*}
\]

where, \( a_1, a_2, \ldots, a_n \) are positive integers and, for all \( i = 1, \ldots, n \), the exponents \( b_i, c_i \) are non-negative integers such that \( (b_i, c_i) \neq (0, 0) \), and, moreover, \( (b_1, b_2, \ldots, b_n) \neq (0, \ldots, 0) \) and \( (c_1, c_2, \ldots, c_n) \neq (0, \ldots, 0) \). This is an example of affine toric variety. If the above parametrization is homogeneous, it defines a projective variety of \( \mathbb{P}^{n+1} \). The presentation ideal \( J \) of the fiber cone algebra \( F(I) \) is known to be of the type discussed in Section 1 (see [3], Proposition 3.6). An explicit construction of the barred matrix \( A \) associated with \( J \) can be found in [10] or [17]. We present an example which was considered, from a different point of view, in [3], Example 3.7 (b).
Example 4 Consider the projective monomial curve of $\mathbb{P}^3$ parametrized by

$$x_1 = u_1^{534}, \quad x_2 = u_2^{534}, \quad y_1 = u_1^{245}u_2^{289}, \quad y_2 = u_1^{144}u_2^{390}$$

Its defining ideal $I \subset R = K[X_1, X_2, Y_1, Y_2]$ is minimally generated by the following 6 binomials:

$$
\begin{align*}
P_1 &= y_1^{42} - x_1^{19}x_2^{22}y_2 \\
P_2 &= y_1^{12}x_2^3 - x_1^2y_2^{13} \\
P_3 &= y_1^{30}x_2^{12} - x_1^{17}x_2^{25} \\
P_4 &= y_1^{18}y_2^{25} - x_1^{15}x_2^{28} \\
P_5 &= y_1^{38} - x_1^{13}x_2^{31} \\
P_6 &= y_1^{51} - y_1^{31}x_2^{34},
\end{align*}
$$

and the presentation ideal $J \subset K[T_1, \ldots, T_6]$ of $F(I)$ is associated with the barred matrix

$$
\left( \begin{array}{c|c|c|c}
T_1 & T_3 & T_4 & T_5 \\
T_2 & T_4 & T_5 & T_6 \\
\end{array} \right).
$$

Therefore $J = (T_3T_5 - T_4^2, \ T_1T_4, \ T_1T_5, \ T_1T_6, \ T_3T_6, \ T_4T_6)$, and $J$ is a s.t.c.i. on the following 3 polynomials:

$$
\begin{align*}
F_1 &= T_3T_5 - T_4^2 \\
G_2 &= T_1T_6 \\
G_1 &= T_1T_5 + T_3T_6.
\end{align*}
$$

It can be easily checked that $I^2 \subset (F_1, G_1, G_2)$.

3 On varieties of minimal degree

The classification of projective varieties of minimal degree (i.e., of degree equal to the codimension plus one, see [13] for the details) is due to the contributions of various authors. The irreducible ones are the quadric hypersurfaces and the cones over the Veronese surface in $\mathbb{P}^5$. The reducible case was settled by Xambó, who proved the following

Theorem 3 [29, Section 1] Let $J$ be a reduced homogeneous ideal of $R$, defining a reducible variety $V$ of $\mathbb{P}^n$ of pure dimension $d$ and degree $\delta \leq n - d + 1$. Then $\mathbb{P}^n$ contains linear subspaces $L_1, \ldots, L_r$ and there are $d$-dimensional scrolls $V_i \subset L_i$ such that $V = \bigsqcup_{i=1}^r V_i$ and for each $i = 2, \ldots, r$,

$$V_i \cap (V_1 \cup \cdots \cup V_{i-1}) = L_i \cap (L_1 \cup \cdots \cup L_{i-1}),$$

which is a linear subspace of dimension $d - 1$ (the bar denotes the projective closure). Moreover, $\delta = n - d + 1$. 

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The original version of the theorem contained the assumption of connectivity in codimension one, but this can be dropped, as shown in [8]. A constructive characterization of the defining ideals of the varieties in Theorem 3 was given in [7]. They include all the ideals studied in [6], i.e., not only the ideals of the type discussed in Section 1, but also the ideals associated to a more general kind of barred matrix,

\[ A = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1s_1} & B_{21} & \cdots & B_{2s_2} & \cdots & B_{r1} & \cdots & B_{rs_r} \end{pmatrix}, \]

where every big block \( B_i \) may consist of various small blocks \( B_{ij1}, \ldots, B_{ijl_i} \), which are scroll matrices and have pairwise distinct sets of entries. We show that the s.t.c.i. property established in Theorem 2 extends to some, but not to all ideals associated with this larger class of barred matrices. We preliminarily remark that Theorem 1 is true also in this more general case (see [6], Corollary 1.3).

**Example 5**

Let \( A = \begin{pmatrix} X_1 & X_3 & X_4 \\ X_2 & X_4 & X_5 \end{pmatrix} \).

The associated ideal is

\[ J = (X_1X_4 - X_2X_3, X_1X_5, X_3X_5), \]

and height \( J = 2 \). We show that \( J \) is a s.t.c.i. on the following two polynomials

\[ P_1 = (X_1X_4 - X_2X_3)X_2 + X_1X_5, \]
\[ P_2 = (X_1X_4 - X_2X_3)X_4 + X_3X_5, \]

i.e., that

\[ J = \sqrt{(P_1, P_2)}. \tag{19} \]

It holds:

\[ (X_1X_3)^2 = X_1(X_5 - X_2X_4)P_1 + X_1X_2^2P_2 \]
\[ (X_3X_5)^2 = X_3(X_5 + X_2X_4)P_2 - X_3X_2^2P_1, \]

so that

\[ X_1X_5, X_3X_5 \in \sqrt{(P_1, P_2)}, \tag{20} \]

whence, by definition of \( P_1 \) and \( P_2 \),

\[ (X_1X_4 - X_2X_3)X_2, (X_1X_4 - X_2X_3)X_4 \in \sqrt{(P_1, P_2)}. \]

It follows that \( (X_1X_4 - X_2X_3)^2 \in \sqrt{(P_1, P_2)} \), i.e.,

\[ X_1X_4 - X_2X_3 \in \sqrt{(P_1, P_2)}. \tag{21} \]

Relations (20) and (21) imply (19).
In the next example, the s.t.c.i. property fails to be true.

**Example 6** Let

\[ A = \begin{pmatrix} X_1 & X_3 & X_5 & X_6 \\ X_2 & X_4 & X_6 & X_7 \end{pmatrix}. \]

The associated ideal is

\[ J = (X_1X_4 - X_2X_3, X_3X_6 - X_4X_5, X_1X_6 - X_2X_5, X_1X_7, X_3X_7, X_5X_7), \]

and height \( J = 3 \).

Let \( ara_J \) denote the arithmetical rank of \( J \). We show that \( ara_J > \text{height } J \).

We shall use the following criterion, which is based on étale cohomology and is due to Newstead [23].

**Lemma 2** [11, Lemma 3] Let \( W \subset \tilde{W} \) be affine varieties. Let \( d = \dim \tilde{W} \setminus W \).

If there are \( s \) equations \( F_1, \ldots, F_s \) such that \( W = \tilde{W} \cap V(F_1, \ldots, F_s) \), then

\[ H^d_{\text{et}}(\tilde{W} \setminus W, \mathbb{Z}/r\mathbb{Z}) = 0 \]

for all \( i \geq s \) and for all \( r \in \mathbb{Z} \) which are prime to \( \text{char } K \).

We refer to [22] for the basic notions on étale cohomology. Let \( p \) be a prime such that \( p \neq \text{char } K \). Let \( V = V(J) \subset K^7 \). In view of Lemma [24] for our purpose it suffices to show that

\[ H^0_{\text{et}}(K^7 \setminus V, \mathbb{Z}/p\mathbb{Z}) \neq 0. \] (22)

By Poincaré Duality (see [22], Theorem 14.7, p. 83) we have

\[ \text{Hom}_{\mathbb{Z}/p\mathbb{Z}}(H^0_{\text{et}}(K^7 \setminus V, \mathbb{Z}/p\mathbb{Z}), \mathbb{Z}/p\mathbb{Z}) \cong H^4_{\text{et}}(K^7 \setminus V, \mathbb{Z}/p\mathbb{Z}), \] (23)

where \( H_{\text{et}} \) denotes étale cohomology with compact support. For the sake of simplicity, we shall omit the coefficient group \( \mathbb{Z}/p\mathbb{Z} \) henceforth. Let \( W \) be the subvariety of \( K^7 \) defined by

\[ X_1X_4 - X_2X_3 = 0, \quad X_3X_6 - X_4X_5 = 0, \quad X_1X_6 - X_2X_5 = 0, \quad X_7 = 0. \]

Then \( W \subset V \), and

\[ V \setminus W = \{(x_1, \ldots, x_7) \in K^7 | x_1 = x_3 = x_5 = 0, \ x_7 \neq 0 \} \cong K^3 \times (K \setminus \{0\}). \]

It is well-known that

\[ H^i_c(K^t) \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } i = 2t \\ 0 & \text{else,} \end{cases} \] (24)

and

\[ H^i_c(K^t \setminus \{0\}) \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } i = 1, 2t \\ 0 & \text{else.} \end{cases} \] (25)
Moreover, by the Künneth formula (22, Theorem 22.1),

\[ H^i_c (V \setminus W) \simeq \bigoplus_{h+k=i} H^h_c (K^3) \otimes H^k_c (K \setminus \{0\}), \]

so that, by (23),

\[ H^3_c (V \setminus W) = H^4_c (V \setminus W) = 0. \quad (26) \]

We have a long exact sequence of étale cohomology with compact support:

\[ \cdots \rightarrow H^3_c (V \setminus W) \rightarrow H^4_c (K^7 \setminus V) \rightarrow H^4_c (K^7 \setminus W) \rightarrow H^4_c (V \setminus W) \rightarrow \cdots. \]

By (20) it follows that

\[ H^4_c (K^7 \setminus V) \simeq H^4_c (K^7 \setminus W). \quad (27) \]

In view of (26) and (27) our claim (22) will follow once we have proven that

\[ H^4_c (K^7 \setminus W) \neq 0. \quad (28) \]

This is what we are going to prove next. Note that \( W \) is the variety of \( K^6 \) defined by the vanishing of the 2-minors of a generic \( 2 \times 3 \) matrix of indeterminates. Thus \( \bar{W} = W \setminus \{0\} \) is the set of non zero \( 2 \times 3 \) matrices with proportional rows. The set of such matrices where the first row is zero is a closed subset of \( W \) which can be identified with \( Z = K^3 \), and its complementary set is \( \bar{W} \setminus Z \simeq K^3 \setminus \{0\} \times K \). We thus have a long exact sequence of étale cohomology with compact support:

\[ \cdots \rightarrow H^2_c (Z) \rightarrow H^3_c (\bar{W} \setminus Z) \rightarrow H^3_c (\bar{W}) \rightarrow H^3_c (Z) \rightarrow \cdots, \quad (29) \]

where, according to (24), \( H^2_c (Z) = H^3_c (Z) = 0 \), whereas, by the Künneth formula and (25),

\[ H^3_c (\bar{W} \setminus Z) \simeq H^3_c (K^3 \setminus \{0\}) \otimes H^2_c (K) \simeq \mathbb{Z}/p \mathbb{Z}. \]

It follows that (20) gives rise to an isomorphism

\[ H^3_c (\bar{W}) \simeq \mathbb{Z}/p \mathbb{Z}. \quad (30) \]

On the other hand there is also the following long exact sequence of étale cohomology with compact support:

\[ \cdots \rightarrow H^3_c (K^7 \setminus \{0\}) \rightarrow H^3_c (\bar{W}) \rightarrow H^4_c (K^7 \setminus W) \rightarrow \cdots, \quad (31) \]

where

\[ H^3_c (K^7 \setminus \{0\}) = 0. \quad (32) \]

by (25). Hence, in view of (30) and (31), in (31) the left term is zero, and the middle term is non zero. It follows that the right term is non zero, i.e., claim (28) holds. This proves that \( \text{ara} \ J > 3 \), as desired. It can be easily checked that the variety \( V \) is defined by the following 5 equations:

\[ X_1X_4 - X_2X_3 = 0, \quad X_3X_6 - X_4X_5 = 0, \]
\[ X_1X_6 - X_2X_5 + X_3X_7 = 0, \quad X_1X_7 = 0, \quad X_5X_7 = 0. \]

Hence \( \text{ara} \ J \leq 5 \). We conjecture that \( \text{ara} \ J = 5 \).

Étale cohomology is not the only possible tool for finding a lower bound for the arithmetical rank. In general we have

\[ \text{cd} \ J \leq \text{ara} \ J, \tag{33} \]

where

\[ \text{cd} \ J = \max \{ i \mid H^i_J(R) \neq 0 \}, \]

is called the cohomological dimension of \( J \). Here \( H^i_J \) denotes the \( i \)-th local cohomology group with respect to \( J \); we refer to Huneke [19] for an extensive exposition of this subject. Inequality (33) can be strict, but not many examples of this kind are known so far. One monomial ideal was found by Zhao Yan [30], Example 2 following [28], p. 250, and [21], Example 1, whereas classes of determinantal ideals are described in [11] and [2]. In all these cases (33) is strict in all but one characteristics. We show that for the ideal \( J \) discussed in Example 6 we always have \( \text{cd} \ J < 5 \); if our above conjecture is true, this would provide an example of ideal whose cohomological dimension differs from its arithmetical rank in all characteristics. According to [19], Theorem 2.2, we have the following long exact sequence of local cohomology:

\[ \cdots \rightarrow H^i_{J + (X_7)}(R) \rightarrow H^i_J(R) \rightarrow H^i_{JX_7}(R_{X_7}) \rightarrow \cdots, \tag{34} \]

where, by [19], Proposition 1.10, \( H^i_J(R_{X_7}) \simeq H^i_{JX_7}(R_{X_7}) \). Now

\[
\begin{align*}
J + (X_7) &= (X_1X_4 - X_2X_3, \ X_1X_6 - X_2X_5, \ X_3X_6 - X_4X_5, \ X_7) \subset R, \\
J_{X_7} &= (X_1, \ X_3, \ X_5) \subset R_{X_7}.
\end{align*}
\]

Since \( J + (X_7) \) and \( J_{X_7} \) are generated by sets of at most 4 elements in \( R \) and \( R_{X_7} \) respectively, inequality (33) implies that

\[ H^i_{J + (X_7)}(R) = H^i_{JX_7}(R_{X_7}) = 0, \quad \text{for all} \ i \geq 5, \]

so that, in view of (34),

\[ H^i_J(R) = 0 \quad \text{for all} \ i \geq 5. \]

Hence \( \text{cd} \ J < 5 \), as was to be shown.

**Remark 1** According to [13], Theorem 4.2, every variety fulfilling the assumption of Theorem 3 is Cohen-Macaulay. By [25], Prop. 4.1), this implies that \( \text{cd} \ J = \text{ht} \ J = 3 \). Hence we certainly have that \( \text{cd} \ J < \text{ara} \ J \) in all positive characteristics.
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