Nabla Euler -Lagrange equations in discrete fractional variational calculus within Riemann and Caputo

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Abstract. Different fractional difference types of Euler-Lagrange equations are obtained within Riemann and Caputo by making use of different versions of integration by part forumlas in fractional difference calculus. An example is presented to illustrate part of the results.

Keywords: right (left) delta and nabla fractional sums, right (left) delta and nabla Riemann, dual identity, Euler equation, integration by parts.

1 Introduction

Fractional calculus which deals with integration and differentiation of arbitrary orders attracted the attention of many researchers in the last two decades or so for its widespread applications in different fields of mathematics, physics, engineering, economic and biology. For detailed and sufficient material about this calculus we refer to the books [9, 10, 11]. However, the discrete fractional calculus which is not as old as fractional calculus, was initiated lately in eighty’s of the last century in [8, 18]. Then in the last few years many authors started to investigate the theory and applications of the discrete fractional calculus [1, 3, 4, 5, 6, 7, 12, 20, 21, 22]. Very recently, the authors in [27, 28, 29] have discussed different definitions for fractional differences specially in the right case, under which suitable integration by parts formulae have been initiated. Benefitting from those formulae we continue in this work and apply to discrete fractional variational calculus to obtain different results from those obtained in [14, 6]. In the usual fractional variational case we refer to [13, 23, 24, 25, 26].

The article is organized as follows: In the rest of this section we give basic definitions and preliminary results about nabla fractional sums and differences. In Section 2 we discussed different integration by parts formulae in discrete fractional calculus. In Section 3 we set some discrete variational problems benefitting from the integration by parts formulae obtained in Section 2. Finally, in Section 4 an example of physical interest is given to illustrate our main results.

For the sake of the nabla fractional calculus we have the following definition

Definition 1.1. \( (\mathbb{I}, \mathbb{J}, \mathbb{L}, \mathbb{M}) \)

(i) For a natural number \( m \), the \( m \) rising (ascending) factorial of \( t \) is defined by

\[
\begin{align*}
\n^m t &= \prod_{k=0}^{m-1} (t + k), \quad \n^0 t = 1. \\
\end{align*}
\]

(ii) For any real number the \( \alpha \) rising function is defined by

\[
\n^\alpha t = \frac{\Gamma(t + \alpha)}{\Gamma(t)}, \quad t \in \mathbb{R} - \{..., -2, -1, 0\}, \quad 0^\alpha = 0
\]

Regarding the rising factorial function we observe for example that

\[
\nabla \left( t^\alpha \right) = a t^{a-1}
\]

Notation:

(i) For a real \( \alpha > 0 \), we set \( n = \lceil \alpha \rceil + 1 \), where \( \lceil \alpha \rceil \) is the greatest integer less than or equal to \( \alpha \).

(ii) For real numbers \( a \) and \( b \), we denote \( \mathbb{N}_a = \{a, a+1, ...\} \) and \( \mathbb{B}_b = \{b, b-1, ...\} \).

(iii) For \( n \in \mathbb{N} \) and real \( a \), we denote

\[
\n^\alpha \Delta^n f(t) \triangleq (-1)^n \Delta^n f(t),
\]

where \( \Delta^n f \) is the \( n \) iterating of \( \Delta f(t) = f(t+1) - f(t) \).
Proposition 2.1. \[ \text{Caputo fractional differences can appear.} \]

Then, we proceed to obtain a one more integration by parts formula where both Riemann operators, respectively. Then the (dual) nabla left and right Caputo fractional differences are defined by:

\[ \begin{align*}
\Delta t^a f(t) &= \frac{1}{\Gamma(n-a)} \sum_{s=t}^{t+1} (t - s)^{n-a-1}, \quad t \in \mathbb{N}_{0+1} \\
\nabla t^a f(t) &= \frac{1}{\Gamma(n-a)} \sum_{s=t}^{t+1} (s - t)^{n-a-1}, \quad t \in \mathbb{N}_{0+1}
\end{align*} \]

Definition 1.2. \[ \text{Let } \sigma(t) = t + 1 \text{ and } \rho(t) = t - 1 \text{ be the forward and backward jumping operators, respectively. Then} \]

(i) The (nabla) left fractional sum of order \( \alpha > 0 \) (starting from \( a \)) is defined by:

\[ \nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(n+1)} \sum_{s=a+1}^{t} (t - s)^{n+1}, \quad t \in \mathbb{N}_{a+1}. \]  

(ii) The (nabla) right fractional sum of order \( \alpha > 0 \) (ending at \( b \)) is defined by:

\[ \begin{align*}
\nabla_b^{-\alpha} f(t) &= \frac{1}{\Gamma(n+1)} \sum_{s=a+1}^{t} (s - t)^{n+1}, \quad t \in \mathbb{N}_{a+1} \\
\nabla_b^{-\alpha} f(t) &= \frac{1}{\Gamma(n+1)} \sum_{s=a+1}^{t} (s - t)^{n+1}, \quad t \in \mathbb{N}_{a+1}
\end{align*} \]

Definition 1.3. \[ \text{Let } \alpha > 0 \text{ be noninteger, } n = [\alpha] + 1, a(\alpha) = a + n - 1 \text{ and } b(\alpha) = b - n + 1. \text{ Then the (dual) nabla left and right Caputo fractional differences are defined by} \]

\[ \begin{align*}
\nabla_a^{-\alpha} f(t) &= \nabla_a^{-\alpha} \nabla_a^{-\alpha} f(t), \quad t \in \mathbb{N}_{a+n} \\
\nabla_b^{-\alpha} f(t) &= \nabla_b^{-\alpha} \nabla_a^{-\alpha} f(t), \quad t \in \mathbb{N}_{b-n}
\end{align*} \]

respectively.

Notice that when \( 0 < \alpha < 1 \) we have

\[ \nabla_a^{-\alpha} f(t) = \nabla_a^{-\alpha} f(t) \quad \text{and} \quad \nabla_b^{-\alpha} f(t) = \nabla_a^{-\alpha} f(t). \]

It is important to remark that the two quantities \((\nabla_a^{-\alpha} f^\rho)(t)\) and \((\nabla_a^{-\alpha} f)(\rho(t))\) are different, where \( \rho(t) = t - 1 \). In connection, we state the following properties without proofs.

Proposition 1.1. \[ \text{Let } \rho(t) = t - 1, \sigma(t) = t + 1, \alpha > 0 \text{ and } f \text{ be function defined on } \mathbb{N}_{a} \cap \mathbb{N}_{b} \text{ where } a \equiv b \text{ (mod 1). Then} \]

\[ \begin{align*}
1) & \quad (\nabla_a^{-\alpha} f^\rho)(t) = (\nabla_a^{-\alpha} f)(\rho(t)), \\
2) & \quad (\nabla_b^{-\alpha} f^\rho)(t) = (\nabla_a^{-\alpha} f)(\rho(t)), \\
3) & \quad (\nabla_b^{-\alpha} f^\rho)(t) = (\nabla_a^{-\alpha} f)(\rho(t)), \\
4) & \quad (\nabla_b^{-\alpha} f^\rho)(t) = (\nabla_b^{-\alpha} f)(\rho(t)), \\
5) & \quad (\nabla_b^{-\alpha} f^\rho)(t) = (\nabla_b^{-\alpha} f)(\rho(t)).
\end{align*} \]

2 Integration by parts for fractional sums and differences

In this section we state the integration by parts formulas for nabla fractional sums and differences obtained in \[ \text{27}, \text{whereafter in } \text{29}, \text{delta by parts formulas are obtained by using certain dual identities. Then, we proceed to obtain a one more integration by parts formula where both Riemann and Caputo fractional differences can appear.} \]

Proposition 2.1. \[ \text{For } \alpha > 0, a, b \in \mathbb{R}, f \text{ defined on } \mathbb{N}_{a} \text{ and } g \text{ defined on } \mathbb{N}_{b}, \text{ we have} \]

\[ \sum_{s=a+1}^{b} g(s) \nabla_a^{-\alpha} f(s) = \sum_{s=a+1}^{b} f(s) \nabla_a^{-\alpha} g(s). \]  

(10)
Proposition 2.2. [27] Let $\alpha > 0$ be non-integer and $a, b \in \mathbb{R}$ such that $a < b$ and $b \equiv a \pmod{1}$. If $f$ is defined on $\mathbb{N}$ and $g$ is defined on $\mathbb{N}_a$, then
\[
\sum_{s=a+1}^{b-1} f(s) g(s) = \sum_{s=a+1}^{b-1} g(s) \nabla_{a}^{-\alpha} f(s).
\] (11)

Now by the above nabla integration by parts formulas and using dual identities in [29], the following delta integration by parts formulae were obtained.

Proposition 2.3. Let $\alpha > 0$, $a, b \in \mathbb{R}$ such that $a < b$ and $b \equiv a \pmod{1}$. If $f$ is defined on $\mathbb{N}_a$ and $g$ is defined on $\mathbb{N}_b$, then we have
\[
\sum_{s=a+1}^{b-1} g(s) (\Delta_{a+1}^{-\alpha} f(s + \alpha)) = \sum_{s=a+1}^{b-1} f(s) \Delta_{b-1}^{-\alpha} g(s - \alpha).
\] (12)

Proposition 2.4. Let $\alpha > 0$ be non-integer and assume that $b \equiv a \pmod{1}$. If $f$ is defined on $\mathbb{N}$ and $g$ is defined on $\mathbb{N}_a$, then
\[
\sum_{s=a+1}^{b-1} f(s) \Delta_{a+1}^{-\alpha} g(s - \alpha) = \sum_{s=a+1}^{b-1} g(s) \Delta_{b-1}^{-\alpha} f(s + \alpha).
\] (13)

The following version of integration by parts contains boundary conditions.

Theorem 2.5. [23] Let $0 < \alpha < 1$ and $f, g$ be functions defined on $\mathbb{N}_a \cap \mathbb{N}$ where $a \equiv b \pmod{1}$. Then
\[
\sum_{s=a+1}^{b-1} g(s) C_a^{\alpha} f(s) = f(s) \nabla_{a}^{-\alpha} g(s) \big|_{s=a}^{b-1} - \sum_{s=a+1}^{b-1} f(s) (s - 1) \nabla_{a}^{-\alpha} g(s) - \sum_{s=a+1}^{b-1} g(s) \nabla_{a+1}^{-\alpha} f(s) - \sum_{s=a+1}^{b-1} f(s) \nabla_{a+1}^{-\alpha} g(s)\big|_{s=a}^{b-1} + \sum_{s=a+1}^{b-1} g(s) \nabla_{a}^{-\alpha} f(s) - \sum_{s=a+1}^{b-1} f(s) \nabla_{a}^{-\alpha} g(s)\big|_{s=a}^{b-1} + \sum_{s=a+1}^{b-1} g(s) \nabla_{a+1}^{-\alpha} f(s)
\] (14)

where clearly $\nabla_{a}^{-\alpha} g(a) = 0$.

Similarly, if interchange the role of Caputo and Riemann we obtain the following version of integration by parts for fractional differences.

Theorem 2.6. Let $0 < \alpha < 1$ and $f, g$ be functions defined on $\mathbb{N}_a \cap \mathbb{N}$ where $a \equiv b \pmod{1}$. Then
\[
\sum_{s=a+1}^{b-1} f(s) \nabla_{a}^{-\alpha} g(s) = \big| f(s) \nabla_{a}^{-\alpha} g(s) \big|_{s=a}^{b-1} - \sum_{s=a+1}^{b-1} f(s) \nabla_{a}^{-\alpha} g(s)\big|_{s=a}^{b-1} + \sum_{s=a+1}^{b-1} g(s) \nabla_{a+1}^{-\alpha} f(s) - \sum_{s=a+1}^{b-1} f(s) \nabla_{a+1}^{-\alpha} g(s)\big|_{s=a}^{b-1} + \sum_{s=a+1}^{b-1} g(s) \nabla_{a}^{-\alpha} f(s) - \sum_{s=a+1}^{b-1} f(s) \nabla_{a}^{-\alpha} g(s)\big|_{s=a}^{b-1} + \sum_{s=a+1}^{b-1} g(s) \nabla_{a+1}^{-\alpha} f(s)
\] (15)

where clearly $\nabla_{a}^{-\alpha} g(a) = 0$.

Proof. From the definition of the left Riemann fractional difference, the integration by parts in $\nabla$-difference calculus, Proposition 2.1 noting that $\nabla f(s) = \Delta f(s - 1)$, and the definition of right Caputo fractional difference we can write
\[
\sum_{s=a+1}^{b-1} f(s - 1) \nabla_{a}^{-\alpha} g(s) = \sum_{s=a+1}^{b-1} f(s - 1) \nabla_{a}^{-\alpha} g(s)
\] (16)

where clearly $\nabla_{a}^{-\alpha} g(a) = 0$.

Hence, the proof is completed.
3 Fractional difference Euler-Lagrange Equations

**Theorem 3.1.** Let \( \alpha > 0 \) be non-integer, \( a, b \in \mathbb{R} \), and \( f \) is defined on \( N_a \cap bN \), where \( a \equiv b \pmod{1} \). Assume that the functional

\[
J(f) = \sum_{t=a+1}^{b-1} L(t, f(t), \nabla^\alpha f(t))
\]

has a local extremum in \( S = \{ y : N_a \cap bN \to \mathbb{R} \text{ is bounded}, y(a) = A \} \) at some \( f \in S \), where \( L : (N_a \cap bN) \times \mathbb{R} \to \mathbb{R} \). Then,

\[
[L_1(s) + \nabla^\alpha L_2(s)] = 0, \quad \text{for all } s \in (N_{a+1} \cap b_{-1}N),
\]

where \( L_1(s) = \frac{\partial L}{\partial f}(s) \) and \( L_2(s) = \frac{\partial L}{\partial \nabla^\alpha f}(s) \).

**Proof.** Without loss of generality, assume that \( J \) has local maximum in \( S \). Hence, there exists an \( \epsilon > 0 \) such that \( J(f) - J(f) \leq 0 \) for all \( f \in S \) with \( \| f - f \| = \sup_{t \in N_a \cap bN} |f(t) - f(t)| < \epsilon \). For any \( \tilde{f} \in S \) there is an \( \eta \in H = \{ y : N_a \cap bN \to \mathbb{R} \text{ is bounded}, y(a) = 0 \} \) such that \( \tilde{f} = f + \epsilon \eta \). Then, the \( \epsilon \)-Taylor’s theorem implies that

\[
L(f, \tilde{f}) = L(f, f + \epsilon \eta, \nabla^\alpha f + \epsilon \nabla^\alpha f) = L(f, f, \nabla^\alpha f) + \epsilon [\eta L_1 + \nabla^\alpha f L_2] + O(\epsilon^2).
\]

Then,

\[
J(\tilde{f}) - J(f) = \sum_{t=a+1}^{b-1} \left[ L(t, \tilde{f}(t), \nabla^\alpha \tilde{f}(t)) - L(t, f(t), \nabla^\alpha f(t)) \right] = \epsilon \sum_{t=a+1}^{b-1} [\eta(t)L_1(t) + \nabla^\alpha f(t)L_2(t)] + O(\epsilon^2).
\]

Let the quantity \( \delta J(\eta, y) = \sum_{t=a+1}^{b-1} [\eta(t)L_1(t) + \nabla^\alpha f(t)L_2(t)] \) denote the first variation of \( J \).

Evidently, if \( \eta \in H \) then \( -\eta \in H \), and \( \delta J(\eta, y) = -\delta J(-\eta, y) \). For \( \epsilon \) small, the sign of \( J(\tilde{f}) - J(f) \) is determined by the sign of first variation, unless \( \delta J(\eta, y) = 0 \) for all \( \eta \in H \). To make the parameter \( \eta \) free, we use the integration by parts formula in Proposition 2 together with the fact that \( \nabla^\alpha f(t)(t) = \nabla^\alpha f(t) + \frac{\partial L}{\partial \nabla^\alpha f}(t-a)1^{\alpha-1} \), to reach

\[
\delta J(\eta, y) = \sum_{t=a+1}^{b-1} \eta(t)[L_1(t) + \nabla^\alpha f(t)L_2(t)] + \frac{\eta(a)}{L_2(a)} \sum_{t=a+1}^{b-1} (t-a+1)^{\alpha-1}L_2(t) = 0,
\]

for all \( \eta \in H \), and hence the result follows by taking the special \( \eta \)’s in \{ \epsilon \in (0, \ldots, 1, 0, 0, \ldots) \} in \( t \)-th place:

\[
\epsilon \in N_{a+1} \cap b_{-1}N \quad \text{with } \eta(a) = 0. \]

Note that in the above theorem the Riemann fractional variational difference problem will not require any boundary conditions at the points \( a + 1 \) and \( b - 1 \) if we consider \( \nabla^\alpha f \) instead of \( \nabla^\alpha a \) in the Lagrangian \( L \) and hence the functions \( \eta \) can be taken from \( S \) again without any restrictions.

This is due to that the used integration by parts formula does not contain any boundary conditions. Different boundary conditions can be generated at \( b \) as well, if we terminate the sum at \( b + 1 \).

Next, we develop a discrete Riemann fractional variational problem of order \( 0 < \alpha < 1 \) with different boundary conditions by making use of the integration by part formula in Theorem 2.

**Theorem 3.2.** Let \( 0 < \alpha < 1 \) be non-integer, \( a, b \in \mathbb{R} \), and \( f \) is defined on \( N_a \cap bN \), where \( a \equiv b \pmod{1} \). Assume that the functional

\[
J(y) = \sum_{t=a+1}^{b-1} L(t, f(t), \nabla^\alpha f(t))
\]

has a local extremum in \( S = \{ y : N_a \cap bN \to \mathbb{R} \text{ is bounded} \} \) at some \( f \in S \), where \( L : (N_a \cap bN) \times \mathbb{R} \to \mathbb{R} \). Further, assume either \( \nabla^\alpha f(b-1) = A \) or \( L_2(b) = 0 \). Then,

\[
[L_1(s) + \nabla^\alpha L_2(s)(s-1)] = [L_1(s) + \nabla^\alpha L_2(s)] = 0, \quad \text{for all } s \in (N_{a+1} \cap b_{-1}N),
\]

where \( L_1(s) = \frac{\partial L}{\partial f}(s) \) and \( L_2(s) = \frac{\partial L}{\partial \nabla^\alpha f}(s) \).
Theorem 3.2 above, if we consider for every $\eta$. Namely, let us consider the following fraction al discrete actions,

$$\sum_{\text{actions}}$$

Proof. We proceed as in Theorem 3.1, except when $\nabla_a^{(1-\alpha)} f(b-1)$ is preassigned the function $\eta$ is taken from $H = \{y : \mathbb{N}_a \cap \mathbb{N} \to \mathbb{R} \text{ is bounded}, \nabla_a^{(1-\alpha)} y(b-1) = 0\}$. Then,

$$\delta J(\eta, f) = \sum_{t=a+1}^{b-1} [\eta(t)L_1(t) + \nabla_a^\alpha \eta(t)L_2(t-1)] = 0,$$

for every $\eta \in H$. Then, the integration by parts formula in Theorem 2.5 then implies that

$$\delta J(\eta, f) = \sum_{t=a+1}^{b-1} \eta(t)[L_1(t) + \nabla_a^\alpha L_2(t-1)] + L_2(t)\nabla_a^{-(1-\alpha)} \eta(t)|_{\alpha-1} = 0,$$

for every $\eta \in H$. Finally, the assumption and Proposition 1.1 implies (18) and the proof is finished.

Similar to what applied in Theorem 3.1, we can generate boundary conditions at $a$ as well in Theorem 3.2 above, if we consider $\nabla_a^{\alpha-1}$ instead of $\nabla_a^\alpha$ in the Lagrangian $L$.

Finally, we obtain the Euler-Lagrange equations for a Lagrangian including the Caputo left fractional difference by making use of the integration by parts formula in Theorem 2.5.

**Theorem 3.3.** Let $0 < \alpha < 1$ be non-integer, $a, b \in \mathbb{R}$, and $f$ are defined on $\mathbb{N}_a \cap \mathbb{N}$, where $a \equiv b \pmod{1}$. Assume that the functional

$$J(f) = \sum_{t=a+1}^{b-1} L(t, f(t), C\nabla_a^\alpha f(t))$$

has a local extremum in $S = \{y : \mathbb{N}_a \cap \mathbb{N} \to \mathbb{R} \text{ is bounded}\}$ at some $f \in S$, where $L : (\mathbb{N}_a \cap \mathbb{N}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Further, assume either $f(a) = A$ and $f(b-1) = B$ or the natural boundary conditions $b \nabla_a^{-(1-\alpha)} L_2(a) = b \nabla_a^{-(1-\alpha)} L_2(b-1) = 0$. Then,

$$\left[\nabla_a^\alpha L_2(s) + (b \nabla_a^\alpha L_2)(s)\right] = 0, \text{ for all } s \in (N_{a+1} \cap b-2 \mathbb{N}). \quad (19)$$

Proof. If $f$ is preassigned at $a$ and $b-1$ then the function $\eta$ is taken from $H = \{y : \mathbb{N}_a \cap \mathbb{N} \to \mathbb{R} \text{ is bounded}, y(a) = y(b-1) = 0\}$. Then, we proceed to reach

$$\delta J(\eta, f) = \sum_{t=a+1}^{b-1} [\eta(t-1)L_1(t) + \nabla_a^\alpha \eta(t)L_2(t)] = 0,$$

for every $\eta \in H$. The integration by parts formula in Theorem 2.5 then implies that

$$\delta J(\eta, f) = \sum_{t=a+1}^{b-1} \eta(t-1)[L_1(t) + \nabla_a^\alpha L_2(t-1)] + \eta(t) b \nabla_a^{-(1-\alpha)} L_2(t)|_{\alpha-1} = 0,$$

for every $\eta \in H$. Hence, (19) follows.

We finish this section by remarking that we can obtain a delta analogue of the discussed nabla discrete variational problems in this section by making use of the dual identities studied in [25][29].

### 4 Example

In order to exemplify our results we analyze an example of physical interest under Theorem 3.2 and Theorem 3.3. Namely, let us consider the following fractional discrete actions,

1. $J(y) = \sum_{t=a+1}^{b-1} \left[\frac{1}{2} (\nabla_a^\alpha y(t))^2 - V(y(t))\right]$, where $0 < \alpha < 1$. Assume either $\nabla_a^{-(1-\alpha)} f(b-1) = A$ or $\nabla_a^\alpha y(b) = 0$. Then the Euler-Lagrange equation by applying Theorem 3.2 is

$$\nabla_a^\alpha \nabla_a^\alpha y(s) - \frac{dV}{dy}(s) = 0 \text{ for all } s \in (N_{a+1} \cap b-1 \mathbb{N}).$$

2. $J(y) = \sum_{t=a+1}^{b-1} \left[\frac{1}{2} (C \nabla_a^\alpha y(t))^2 - V(y(p(t)))\right]$, where $0 < \alpha < 1$. Assume either $y(a) = A$ and $y(b-1) = B$ or the natural boundary conditions $b \nabla_a^{-(1-\alpha)} C \nabla_a^\alpha (a) = b \nabla_a^{-(1-\alpha)} C \nabla_a^\alpha (b-1) = 0$. Then the Euler-Lagrange equation by applying Theorem 3.3 is

$$\left( C \nabla_a^\alpha C \nabla_a^\alpha y(s) - \frac{dV}{dy}(s)\right) = 0, \text{ for all } s \in (N_{a+1} \cap b-2 \mathbb{N}).$$

Finally, we remark that it is of interest to deal with the above Euler-Lagrange equations obtained in the above example, where we have composition of left and right fractional differences. In the usual fractional case for such left-right fractional dynamical systems we mention the work done in [2].
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