ON THE EIGENPOINTS OF CUBIC SURFACES

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We show that the eigenschemes of $4 \times 4 \times 4$ symmetric tensors are parametrized by a linear subvariety of the Grassmannian $\text{Gr}(3, \mathbb{P}^{14})$. We also study the decomposition of the eigenscheme into the subscheme associated to the zero eigenvalue and its complement. In particular, we categorize the possible degrees and dimensions.

1. Introduction

The goal of this paper is to study the eigenpoints of $4 \times 4 \times 4$ tensors. The spectral theory of tensors is a multi-linear generalization of the study of eigenvalues, singular values, eigenvectors and singular vectors in the case of matrices. Starting with the works of Qi [7] and Lim [6], there has been steady progress and strong interest in the subject, both theoretically and in the applications to hypergraph theory, data analysis, automatic control, magnetic resonance imaging, higher order Markov chains, or optimization [5, 9].

A $4 \times 4 \times 4$ tensor $T$ and a choice of two axes defines a linear map

$$T : \mathbb{C}^4 \rightarrow \mathbb{C}^4$$

$$x \mapsto T \cdot (x \otimes x)$$

where the dot denotes tensor contraction along the chosen axes. An eigenvector of the tensor $T$, with respect to the chosen directions, is a non-zero vector $x \in \mathbb{C}^4$ such that $T \cdot (x \otimes x) = \lambda x$ for some $\lambda \in \mathbb{C}$, and an eigenpoint is the associated

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equivalence class in $\mathbb{P} C^4$. The *eigenscheme* with respect to the chosen directions is the closed subscheme of such points in $\mathbb{P}^3$.

In this article, we fix the contraction to be along the first two directions and we use the terminology of eigenvector, eigenpoint, eigenscheme with the implicit reference to these axes. However, we stress that there are interesting relations between the eigenschemes associated to different directions of the same tensor; this phenomenon of eigencompatibility was studied in detail by Abo, Seigal, and Sturmfels [1, Section 3].

With a fixed choice of axes, we may assume without loss of generality that $\mathcal{T} \in \text{Sym}^2 C^4 \otimes C^4$, and we refer to elements of $\text{Sym}^2 C \otimes C^4$ as *partially symmetric tensors*. A *symmetric tensor* is an element of $\text{Sym}^3 C^4$. In the setting of symmetric tensors we can interpret the problem in terms of cubic surfaces [10]. Given a symmetric tensor $\mathcal{T}$, one obtains a cubic polynomial in 4 variables by the contraction $\mathcal{T} \cdot (x \otimes x \otimes x)$, where $x = (x_0, x_1, x_2, x_3)$. Conversely, given a homogeneous cubic form $f$ in 4 variables, each of the partial derivatives of $f$ can be viewed as a quadratic form, which in turn can be viewed as an element of $\text{Sym}^2 C^4$; the tuple of four quadratic polynomials $\left(\frac{1}{3} \frac{\partial f}{\partial x_0}, \frac{1}{3} \frac{\partial f}{\partial x_1}, \frac{1}{3} \frac{\partial f}{\partial x_2}, \frac{1}{3} \frac{\partial f}{\partial x_3}\right)$, viewed as four $4 \times 4$ matrices, defines a tensor in $\text{Sym}^3 C^4$. Cubic surfaces (e.g. Figure 1) play an important role in algebraic geometry. The study of the 27 lines that lie on a smooth cubic surface is one of the first major landmarks in the field.

In Section 3, we look at the decomposition of the eigenscheme into the subscheme of eigenpoints with eigenvalue 0 (the *irregular eigenpoints*) and its complement (the *regular eigenpoints*); we study the dimensions of the components in this decomposition. In Section 4 we prove that the degree of the regular eigenscheme can be any value in $\{0, \ldots, 15\}$. In Section 5 we show there is a natural bijection between 3-planes in $\mathbb{P}^{14}$ satisfying linear constraints and eigenschemes of symmetric tensors.
2. Preliminaries

We fix a 3-dimensional projective space $\mathbb{P}^3 = \mathbb{P}^4$, and we will denote by $\mathbb{C}[x_0, \ldots, x_3]$ its coordinate ring. We will also fix $\mathbb{P}^4 := \text{Proj} \mathbb{C}[x_0, \ldots, x_3, \lambda]$. For convenience, we denote $x := (x_0, \ldots, x_3)$.

Similar to how a symmetric tensor $T$ defines a cubic for in 4 variables by the contraction $T \cdot (x \otimes x \otimes x)$, a partially symmetric tensor $T \in \text{Sym}^2 \mathbb{C}^4 \otimes \mathbb{C}^4$ can be viewed as a tuple of 4 quadratic forms $(q_0(x), \ldots, q_3(x))$ given by the contraction $T \cdot (x \otimes x)$. Equivalently, the four quadratic forms are those associated to the 4 slices of $4 \times 4$ symmetric matrices.

**Definition 2.1.** Let $T = (q_0, \ldots, q_3)$ be a partially symmetric tensor. Define the \textit{scheme of eigenpairs of $T$} by

$$\tilde{E}(T) = V(q_0(x) - \lambda x_0, \ldots, q_3(x) - \lambda x_3) \subset \mathbb{P}^4.$$ 

Let $\pi : \mathbb{P}^4 \rightarrow \mathbb{P}^3$ be the projection from $[0, 0, 0, 0, 1]$. The \textit{eigenscheme} of $T$ is the scheme defined by

$$E(T) := \pi \left( \tilde{E}(T) \setminus \{[0, 0, 0, 0, 1]\} \right).$$

Equivalently, $E(T) \subset \mathbb{P}^3$ is the common vanishing set of the $2 \times 2$ minors of

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ q_0 & q_1 & q_2 & q_3 \end{vmatrix}.$$

When $T$ is symmetric, we can consider it as a homogeneous polynomial $f$. In this case we will denote its eigenscheme by $E(f)$.

We see that the local ring of $\tilde{E}(T)$ at the point $[0, 0, 0, 0, 1]$ has Krull dimension 0, as each $q_f(x)$ is either 0 or homogeneous of degree 2. Thus, $[0, 0, 0, 0, 1]$ is always an isolated point of $\tilde{E}(T)$, and the image of $\pi$ is a closed subscheme of $\mathbb{P}^3$. Since it is defined by four equations in $\mathbb{P}^4$, $\tilde{E}(T)$ is never empty. By Bézout’s theorem, when $T$ is generic $E(T)$ consists of 15 reduced points.

Note that 15 distinct fixed points in $\mathbb{P}^3$ are the eigenpoints of a cubic only if the system of equations $\{ \frac{\partial f}{\partial x_i} x_j - \frac{\partial f}{\partial x_j} x_i : 0 \leq i, j \leq 3 \}$, whose indeterminates are the coefficients of $f$, has a solution. These conditions are linear in the coefficients $f$. Thus, the sets of 15 points in $\mathbb{P}^3$ that are the eigenscheme of a cubic are those such that the maximal minors of the $20 \times 90$ Vandermonde type matrix vanish. Although these conditions are complicated, they provide a computational way to check if 15 given points are the points of an eigenscheme.

**Definition 2.2.** Let $T = (q_0, \ldots, q_3)$ be a partially symmetric tensor. Define the \textit{irregular eigenscheme} $\text{Irr}(T) \subset \mathbb{P}^3$ as the largest subscheme of $E(T)$ such that $q_i(x) = 0$ for all $i \in \{0, \ldots, 3\}$ and all closed points $x \in \text{Irr}(T)$. Its complement $\text{Reg}(T)$ is called the \textit{regular eigenscheme}. 
To clarify the terminology, the regular eigenpoints are the points \( p \) such that the rational map \( \mathcal{T} : \mathbb{P}^3 \rightarrow \mathbb{P}^3 \) is both regular at \( p \) and fixes \( p \). When \( \mathcal{T} \) is a symmetric tensor, with associated cubic polynomial \( f \), the regular eigenpoints of \( f \) are the fixed points of the gradient map \( \nabla f : \mathbb{P}^3 \rightarrow \mathbb{P}^3 \) defined by

\[
p \mapsto \left[ \frac{\partial f}{\partial x_0}(p), \ldots, \frac{\partial f}{\partial x_3}(p) \right].
\]

The closed points of the irregular eigenscheme are the singular points of the hypersurface \( V(f) \subset \mathbb{P}^3 \). As a consequence, we can compute the ideal of \( \text{Reg}(f) \) as the saturation

\[
I(\text{Reg}(f)) = I\left(\overline{\text{Reg}(f)}\right) = (I(E(f)) : I(\text{Irr}(f))).
\]

It will be useful for this paper to consider what happens to the eigenscheme under the following group action.

**Definition 2.3.** Let \( U \in \text{GL}_4(\mathbb{C}) \) and let \( \mathcal{T} := (q_0(x), \ldots, q_3(x)) \). We defined the *twisted action* of \( U \) on \( \mathcal{T} \) by

\[
\Psi_U \mathcal{T} := (q_0(xU), \ldots, q_3(xU)) \cdot U^{-1}.
\]

i.e, \( U \) acts on the quadrics by change of coordinates, then \( U^{-1} \) acts by taking linear combinations of the slices of the tensor.

Let \( \rho_{\text{std}} \) be the standard representation of \( \text{GL}_4(\mathbb{C}) \). The action described in Definition 2.3 defines the representation \( \rho_{\text{std}} \otimes_2 \rho_{\text{std}} \). For the subgroup \( \text{SO}_4(\mathbb{C}) \), the action is equivalent to acting by orthogonal change of coordinates.

**Lemma 2.4.** Let \( U \in \text{GL}_4(\mathbb{C}) \) and let \( \mathcal{T} := (q_0(x), \ldots, q_4(x)) \) be a partially symmetric tensor. Then \( E(\Psi_U \mathcal{T}) = U^{-1}E(\mathcal{T}) \).

**Proof.** We have that the equations \( \{ q_i(x) x_j - q_j(x)x_i : 0 \leq i, j \leq 3 \} \) vanish at \( x \) if and only if the minors of

\[
\left( \begin{array}{cccc}
(x_0 & x_1 & x_2 & x_3) \\
(q_0(x) & q_1(x) & q_2(x) & q_3(x)) \cdot U^{-1}
\end{array} \right)
\]

also vanish. Setting \( y := (x_0, x_1, x_2, x_3) \cdot U^{-1} \), we have that the minors of

\[
\left( \begin{array}{cccc}
y_0 & y_1 & y_2 & y_3 \\
(q_0(y) & q_1(y) & q_2(y) & q_3(y)) \cdot U^{-1}
\end{array} \right)
\]

vanish if and only if \( y \in U^{-1}(X) \). The last system of equations defines the eigenscheme of \( \Psi_U \mathcal{T} \). \( \square \)

Lemma 2.4 is a generalization of the fact that taking eigenschemes commutes with orthogonal changes of coordinates [8, Theorem 2.20]. However, in general the \( E \)-characteristic polynomial of a tensor is not invariant under the twisted action by \( \text{GL}_4(\mathbb{C}) \).
3. Dimensions of the regular and irregular eigenschemes of cubics

The eigenscheme of a cubic surface can exhibit a wide range of structure depending on the particular cubic surface. For instance, it can be non-reduced or it can have components of different dimension. In this section, we describe some of the possibilities for the degrees and dimensions of \( \operatorname{Reg}(f) \) and \( \operatorname{Irr}(f) \).

**Proposition 3.1.** Let \( f \) be a cubic in four variables and let \( X = V(f) \). Then:

(a) \( \dim \operatorname{Irr}(f) + 1 \geq \dim \operatorname{Reg}(f) \).

(b) \( \dim \operatorname{Irr}(f) = 2 \) if and only if \( \operatorname{Irr}(f) \) is a plane. In this case \( X \) contains a double plane and \( \operatorname{Reg}(f) \) has either 0, 1, or 2 closed points.

(c) If \( \dim \operatorname{Reg}(f) = 2 \), then \( \dim \operatorname{Irr}(f) = 1 \).

**Proof.** (a) Let \( H \subset \mathbb{P}^4 \) be the hyperplane defined by \( \lambda = 0 \) and let \( \pi : \mathbb{P}^4 \rightarrow \mathbb{P}^3 \) be the projection from the point \( [0,0,0,0,1] \). Since all fibers of \( \pi \) have dimension 1, we have:

\[
\dim \operatorname{Irr}(f) = \dim (\pi^{-1}\operatorname{Irr}(f)) - 1 = \dim (\pi^{-1}E(f) \cap H) - 1 \\
\geq \dim (\pi^{-1}E(f)) - 2 = \dim E(f) - 1 \geq \dim \operatorname{Reg}(f) - 1.
\]

(b) Let \( S \) be the singular subscheme of \( X \). For every pair of points \( s_1, s_2 \in S \) the line \( \langle s_1, s_2 \rangle \) intersects \( X \) with multiplicity at least 4, so it is contained in \( X \) by Bézout’s theorem. If \( \dim S = 2 \), then the surface \( X \) contains the secant variety of a surface, so \( S \) is a plane (with multiplicity). The converse is clear.

We now determine the number of regular eigenpoints in this case. By Lemma 2.4 we may assume up to rotations that \( S = V(x_0^r) \) for some \( r \geq 2 \). We can write \( f = x_0^2(a_0x_0 + \ldots + a_3x_3) \), and the eigenscheme is defined by

\[
\begin{align*}
2x_0(a_0x_0 + \ldots + a_3x_3) + a_0x_0^2 &= \lambda x_0 \\
2x_0^2 &= \lambda x_1, \quad 2x_0^2 = \lambda x_2, \quad 3x_0^2 = \lambda x_3
\end{align*}
\]

If \( \lambda \neq 0 \), then we may assume by scaling \( \lambda = 1 \), so we have

\[
\begin{align*}
2x_0(a_0x_0 + \ldots + a_3x_3) + a_0x_0^2 &= x_0 \\
2x_0^2 &= x_1, \quad 2x_0^2 = x_2, \quad 3x_0^2 = x_3
\end{align*}
\]

We see that \( (a_0^2 + a_2^2 + a_3^2)x_0^3 + 3a_0x_0 - 1 \) \( x_0 = 0 \) by eliminating \( x_1, x_2, x_3 \) from the first relation. The only solution when \( x_0 = 0 \) is \( [0,0,0,0,1] \). The other factor can have 0, 1, or 2 solutions in \( x_0 \), so the claim follows.

(c) From (a), we see \( \dim \operatorname{Irr}(f) \geq 1 \). From part (b), \( \dim \operatorname{Irr}(f) < 2 \). \( \square \)
The bounds from Proposition 3.1 for the dimensions of the strata of the eigenscheme are optimal. Table 1 gives the \((\delta, \epsilon) \in \{-1, 0, 1, 2\}\) such that there exists a cubic \(f\) such that \(\dim \text{Reg}(f) = \delta\) and \(\dim \text{Irr}(f) = \epsilon\). There are partially symmetric tensors whose eigenscheme has dimension 3.

| \(\delta\) | \(\epsilon\) | -1 | 0 | 1 | 2 |
|---------|---------|----|----|----|----|
| -1      | \(0\)   | \(x_0(x_1^2 - x_2^2 - x_3^2) + (\theta x_1 + ix_2 + x_3)^3\) | \(3x_0(x_1^2 + x_2^2) + (x_1 + ix_2)^3\) | \(x_0^2(x_1 + ix_2)\) |
| 0 \(\sum_{j=0}^{3} x_j^3\) | \(x_0^3 + x_1^3 + x_2^3\) | \(x_0^3 + x_1^3\) | \(x_0^3\) |
| 1 \(\sum_{j=1}^{3} x_0 x_j^3 + x_1^3\) | \(x_0(x_1^2 + x_2^2)\) | \(\emptyset\) |
| 2 \(\emptyset\) | \(\emptyset\) | \(x_0(x_1^2 + x_2^2 + x_3^2)\) | \(\emptyset\) |

Table 1: Dimensions of the regular and irregular eigenschemes. Here, \(i\) and \(\theta\) denote elements such that \(i^2 = -1\) and \(\theta^0 = -8/9\).

**Lemma 3.2.** Let \(T = (q_0, \ldots, q_3)\) be a tensor such that every \(x \in \mathbb{C}^4\) is an eigenpoint. Then there is a linear form \(\ell\) such that each \(q_i = \ell x_i\).

**Proof.** If all \(x_i q_j - x_j q_i\) are identically zero, the result follows from the fact that \(\mathbb{C}[x_0, \ldots, x_3]\) is a unique factorization domain. \(\square\)

One can understand the regular eigenscheme of cones in \(\mathbb{P}^3\) by studying the eigenscheme of cubic curves in \(\mathbb{P}^2\). In general, Lemma 3.3 shows how examples in lower dimensions help fill in the classification of possible strata for higher dimensional tensors. If \(V(f) \subset \mathbb{P}^3\) is a cone over a plane cubic curve, then up to rotations we may assume that \(f\) satisfies the hypothesis of Lemma 3.3.

**Lemma 3.3.** Let \(f\) be a cubic form constant in \(x_3\). Then the fixed points of \(\nabla f: \mathbb{P}^3 \to \mathbb{P}^3\) are the fixed points of \(\nabla f: \mathbb{P}^2 \to \mathbb{P}^2\), where \(\mathbb{P}^2\) is identified with the hyperplane \(x_3 = 0\) in \(\mathbb{P}^3\).

**Proof.** By the assumption on \(f\), we have that \(\frac{\partial f}{\partial x_3} = 0\). Let \(C = V(f, x_3) \subset \mathbb{P}^2\), and note that \(C\) is a plane cubic curve. Let \(x = [x_0, x_1, x_2, x_3]\) be a solution of

\[
\text{rank} \begin{pmatrix} \frac{\partial f}{\partial x_0}(x) & \frac{\partial f}{\partial x_1}(x) & \frac{\partial f}{\partial x_2}(x) & 0 \\ x_0 & x_1 & x_2 & x_3 \end{pmatrix} = 1.
\]

If \(x_3 \neq 0\), then the first row is identically zero, so \(x \in \text{Irr}(f)\). Thus, \(x_3 = 0\) for any regular eigenpoint of \(f\). By omitting the last column of the matrix above we see that \([x_0, x_1, x_2]\) is an eigenpoint of \(\nabla f: \mathbb{P}^2 \to \mathbb{P}^2\). \(\square\)
4. Zero dimensional regular eigenschemes

This section is dedicated to the geometric configurations of points in $\mathbb{P}^3$ arising as zero dimensional regular eigenschemes of a cubic. Some special configurations turn up in the survey, such as collinear, triangular (Example 4.2) or tetrahedral (Example 4.3). Theorem 4.1 gives a summary of our investigations.

**Theorem 4.1.** If $\dim \text{Reg}(f) \leq 0$, then $f$ has at most 15 regular eigenpoints. Moreover, for every $n \in \{0, 1, \ldots, 15\}$ there exists a cubic form in $\mathbb{C}[x_0, x_1, x_2, x_3]$ with $n$ regular eigenpoints.

**Proof.** As we have seen in Section 2, the eigenscheme in $\mathbb{P}^4$ of a general cubic consists of 15 points. If there are more, the dimension increases by Bézout’s theorem. To prove the second statement, we exhibit examples in Table 2.

Table 2: For any $0 \leq n \leq 15$, there is a cubic with exactly $n$ regular eigenpoints. The Macaulay2 [4] script to check the examples is available at [2].

| $\text{#Reg}(f)$ | $f$ |
|------------------|-----|
| 0                | $x_0^2(x_1 + i x_2)$ |
| 1                | $x_0^3$ |
| 2                | $x_0^3 x_2$ |
| 3                | $x_0^3 + x_1^3$ |
| 4                | $x_0 x_1 x_2$ |
| 5                | $x_0^3 + x_1^3 x_2$ |
| 6                | $x_0 x_1 x_2 + x_0 x_3^2$ |
| 7                | $x_0^3 + x_1^3 + x_2^3$ |
| 8                | $x_0^2 x_1 + x_2^3 x_3$ |
| 9                | $x_0 x_1 x_2 + x_3^3$ |
| 10               | $x_0 x_1 x_2 + x_0 x_3^3 + x_1 x_2^2$ |
| 11               | $x_0^3 + x_1^3 x_2 + 3 x_3^3$ |
| 12               | $x_0^3 x_1 + x_0^2 x_2 + x_1 x_2^2 + x_0 x_3^2$ |
| 13               | $x_0^3 x_1 + x_0^2 x_2 + x_1 x_2^2 + x_3^3$ |
| 14               | $x_0 x_1 x_2 + x_0 x_3^3 + x_1 x_2^2 + x_2^3 + x_3^3$ |
| 15               | $x_0^3 + x_1^3 + x_2^3 + x_3^3$ |

Example 4.2. The cubic $x_0^3 + x_1^3 + x_2^3$ has exactly 7 eigenpoints:

$[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [1, 1, 0, 0], [1, 0, 1, 0], [0, 1, 0, 0], [1, 1, 1, 0]$. They are coplanar, and they are in the configuration described by Figure 2.

Example 4.3. The quaternary cubic $f = x_0^3 + x_1^3 + x_2^3 + x_3^3$ has 15 eigenpoints:

$[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1], [1, 1, 0, 0], [1, 0, 1, 0], [0, 1, 1, 0], [1, 1, 1, 0], [0, 0, 0, 1], [1, 0, 1, 1], [0, 1, 1, 1], [1, 0, 1, 1], [1, 1, 1, 0], [0, 1, 1, 1], [1, 1, 1, 1]$.

These eigenpoints are not in general position. Each coordinate plane contains exactly 7 of them arranged in the configuration of Figure 2.
5. A Grassmannian as a parameter space of eigenschemes

We now turn our attention to the problem of classifying which sets of points are the eigenpoints of a cubic. We proceed indirectly by first studying the problem for partially symmetric tensors to highlight the underlying geometry, and recover the result for symmetric tensors by specialization.

The main idea is as follows. Let $T = (q_0, \ldots, q_3)$ be a partially symmetric tensor. By construction, the polynomials $q_i - \lambda x_i$ are linearly independent, so

$$H_T := \text{Span}(q_0 - \lambda x_0, \ldots, q_3 - \lambda x_3).$$

is a 3-plane in $\mathbb{P}^{14} = \mathbb{P}(\mathbb{C}[x_0, \ldots, x_3, \lambda]_2)$. Conversely, given a 3-plane in $\mathbb{P}^{14}$, the intersection of the dual 10-plane with the image of the Veronese $\nu_2 : \mathbb{P}^4 \hookrightarrow \mathbb{P}^{14}$ defines a subscheme of $\mathbb{P}^4$, which is generically 0 dimensional and of degree 16. We describe the conditions for when a 3-plane in $\mathbb{P}^{14}$ is of the form (*) in terms of the Plücker coordinates.

**Theorem 5.1.** Fix the monomial basis $\{x_0^2, x_0 x_1, \ldots, x_2, x_0 \lambda, \ldots, x_3 \lambda, \lambda^2\}$ of $\mathbb{C}[x_0, \ldots, x_3, \lambda]_2$. In the Plücker coordinates with respect to this basis, the image of $T \mapsto [H_T] \in \text{Gr}(3, \mathbb{P}^{14})$ is the subscheme of the Grassmanian defined by

- $[***\lambda^2] = 0$ (i.e, $[0,0,0,0,1] \in H_T$) and
- the Plücker coordinate corresponding to the columns of $H_T$ labeled by $\{\lambda x, \lambda y, \lambda z, \lambda w\}$ is non-zero.

If $\nu_2 : \mathbb{P}^4 \hookrightarrow \mathbb{P}^{14}$ is the Veronese embedding, then $\nu_2(\tilde{E}(T)) = H_T^\perp \cap \nu_2(\mathbb{P}^4)$.

**Proof.** If $T = (q_0, q_1, q_2, q_3)$ is a partially symmetric tensor, then the 3-plane $H_T := \text{Span}(q_0 - \lambda x_0, \ldots, q_3 - \lambda x_3)$ satisfies the two conditions in the statement of the theorem.
We now show that the map is a bijection. Consider a 3-plane in \( \text{Gr}(3, \mathbb{P}^{14}) \) that contains the point \([0, 0, 0, 0, 1]\). It is a subspace given by a \( 4 \times 15 \) matrix:

\[
\begin{pmatrix}
x_0^3 & x_0 x_1 & \ldots & x_3 & x_0 \lambda & x_1 \lambda & x_2 \lambda & x_3 \lambda & \lambda^2 \\
m_{1,1} & m_{1,2} & \ldots & m_{1,10} & m_{1,11} & m_{1,12} & m_{1,13} & m_{1,14} & m_{1,15} \\
m_{2,1} & m_{2,2} & \ldots & m_{2,10} & m_{2,11} & m_{2,12} & m_{2,13} & m_{2,14} & m_{2,15} \\
m_{3,1} & m_{3,2} & \ldots & m_{3,10} & m_{3,11} & m_{3,12} & m_{3,13} & m_{3,14} & m_{3,15} \\
m_{4,1} & m_{4,2} & \ldots & m_{4,10} & m_{4,11} & m_{4,12} & m_{4,13} & m_{4,14} & m_{4,15}
\end{pmatrix}
\]

Since the given 3-plane contains the point \([0, 0, 0, 0, 1]\), the entries in the \( \lambda^2 \) column are zero. If the \( 4 \times 4 \) block

\[
\begin{pmatrix}
x_0 \lambda & x_1 \lambda & x_2 \lambda & x_3 \lambda \\
m_{1,11} & m_{1,12} & m_{1,13} & m_{1,14} \\
m_{2,11} & m_{2,12} & m_{2,13} & m_{2,14} \\
m_{3,11} & m_{3,12} & m_{3,13} & m_{3,14} \\
m_{4,11} & m_{4,12} & m_{4,13} & m_{4,14}
\end{pmatrix}
\]

is invertible, which is an open condition on \( \text{Gr}(3, \mathbb{P}^{14}) \), then we can apply the reduced row echelon form to get

\[
\begin{pmatrix}
x_0^3 & x_0 x_1 & \ldots & x_3 & x_0 \lambda & x_1 \lambda & x_2 \lambda & x_3 \lambda & \lambda^2 \\
m_{1,1} & m_{1,2} & \ldots & m_{1,10} & -1 & 0 & 0 & 0 & 0 \\
m_{2,1} & m_{2,2} & \ldots & m_{2,10} & 0 & -1 & 0 & 0 & 0 \\
m_{3,1} & m_{3,2} & \ldots & m_{3,10} & 0 & 0 & -1 & 0 & 0 \\
m_{4,1} & m_{4,2} & \ldots & m_{4,10} & 0 & 0 & 0 & -1 & 0
\end{pmatrix}
\]

This is a 3-plane given by a tensor. The statement about \( \tilde{E}(T) \) is clear.

\[\square\]

**Corollary 5.2.** The parameter space of eigenscheme of symmetric tensors is an open subvariety of a linear subvariety of \( \text{Gr}(3, \mathbb{P}^{14}) \).

**Proof.** Let

\[
\begin{pmatrix}
x_0^3 & x_0 x_1 & \ldots & x_3 & x_0 \lambda & x_1 \lambda & x_2 \lambda & x_3 \lambda & \lambda^2 \\
m_{1,1} & m_{1,2} & \ldots & m_{1,10} & -1 & 0 & 0 & 0 & 0 \\
m_{2,1} & m_{2,2} & \ldots & m_{2,10} & 0 & -1 & 0 & 0 & 0 \\
m_{3,1} & m_{3,2} & \ldots & m_{3,10} & 0 & 0 & -1 & 0 & 0 \\
m_{4,1} & m_{4,2} & \ldots & m_{4,10} & 0 & 0 & 0 & -1 & 0
\end{pmatrix}
\]

be the affine coordinates for a 3-plane coming from a symmetric tensor. The 4 quadrics corresponding to the rows are of the form \( \frac{\partial f}{\partial x_0} - \lambda x_0, \ldots, \frac{\partial f}{\partial x_3} - \lambda x_3 \) if and only if the \( m_{i,j} \), interpreted as coefficients of the quadrics \( q_0, \ldots, q_3 \), satisfy the linear conditions \( \{ \frac{\partial q_i}{\partial x_j} - \frac{\partial q_j}{\partial x_i} = 0 : i < j \} \). These give linear relations of the Plücker coordinates by [3 Proposition 3.1.2].

\[\square\]
As shown in [1, Section 4], the eigendiscriminant is a homogeneous polynomial in the entries of a tensor of degree 96, that vanishes whenever the eigenscheme has a point of multiplicity greater than 2. It is interesting to compare the eigendiscriminant to the Hurwitz form [11] of the image of \( \nu_2 : \mathbb{P}^4 \hookrightarrow \mathbb{P}^{14} \), which is a polynomial of degree 120 in the Steifel coordinates for \( \text{Gr}(3, \mathbb{P}^{14}) \) that vanishes when the codimension-3-plane intersects the Veronese tangentially. As the definitions of these two polynomials are closely related, we make the following conjecture.

**Conjecture 5.3.** Restricted to the 3-planes coming from eigenschemes, the hyperdeterminant divides the Hurwitz form. Moreover, the leftover factor of degree 24 corresponds to 3-planes tangent to the Veronese at \([0,0,0,0,1]\).

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