Shearlet frames with short support

Song Li* and Yi Shen†

Abstract

Compactly supported shearlets have been studied in both theory and applications. In this paper, we construct symmetric compactly supported shearlet systems based on pseudo splines of type II. Specially, using B-splines, we construct shearlet frame having explicit analytical forms which is important for applications. The shearlet systems based on B-splines also provide optimally sparse approximation within cartoon-liked image.

Keywords. Optimal sparsity, frame, shearlets, B-splines, pseudo splines.

AMS subject classification: Prime 42C40; Secondary: 42C15, 65T60, 65T99, 94A08

1 Introduction

Cartoon-liked image are 2-dimensional functions that are $C^2$ except for discontinuities along $C^2$ curves [1]. To find optimally sparse representations of cartoon-like image, several variations of the wavelet scheme have been proposed, such as curvelets [1] and contourlets [8]. Shearlets frame developed in [17] is the first multiscale directional system which also provides almost optimally sparse approximation with cartoon-like images. However, these studies are only concerned band limited generator. Very recently, Kutyniok and Lim presented a complete proof of (almost) optimally sparse approximations of cartoon liked images by using shearlet systems which are generated by compactly supported shearlets under some weak moment conditions [16]. They also constructed a class of compactly supported shearlet frames based on pseudo splines of type I [15]. Hence, excellent spatial localization is achieved. But the shearlet frame still have two disadvantages:

- The shearlet is not symmetric or anti-symmetric;
- The shearlets do not have explicit analytical forms in spatial domain.

These drawbacks motivate us to consider constructing shearlet frame using B-splines. B-splines had a significant impact on the development of the theory of the wavelet analysis. They yield the only wavelets that have explicit analytical forms. All other wavelet bases are defined indirectly through an infinite product in Fourier domain [5, 6]. As Daubechies pointed out in [6], except the Haar wavelet function, there is no compactly supported real-valued symmetric orthonormal wavelet basis in $L_2(\mathbb{R})$. However, it is much easier and more flexible to construct and design compactly

*Department of Mathematics, Zhejiang University Hangzhou, 310027, China
†Corresponding author: sy1133@163.com, Department of Mathematics, Zhejiang University Hangzhou, 310027, China

1
supported wavelet frames or Riesz bases than orthonormal wavelet bases. For example, from any B-spline function of order \( m \), one can construct a symmetric tight wavelet frame with \( m \) generators [18]. Tight wavelet frame from B-splines with high vanishing moments were considered in [2] [7]. The compactly supported Riesz wavelets generated from B-splines were first constructed in [4]. The shortest supported Riesz wavelet with \( m \) vanishing moments from B-spline of order \( m \) were constructed in [12]. The compactly supported wavelet bases from B-splines for Sobolev spaces were investigated in [11] [13].

The rest of the paper is organized as follows. In Section 2, we construct shearlets based on B-splines, then we present some results on the optimally sparse approximations of cartoon-like images. In Section 3, we investigate the lower bounds and the upper bounds of the pseudo splines in Fourier domain. These results not only have their own interests but also have closed relations to the shearlets frame bounds. In sections 4, we presents some examples to illustrate our results.

## 2 Shearlets based on B-splines

In this section, we first introduce the definitions of shearlet frame and B-splines function. Then we construct compactly supported shearlet frame with the generator from B-splines. Finally, we show that these compactly supported shearlet systems provide (almost) optimally sparse approximations of cartoon-like images.

Shearlets are scaled according to a parabolic scaling law encoded in the matrix \( A_{2j} \) or \( \tilde{A}_{2j} \), \( j \in \mathbb{Z} \), and exhibit directionality by parameterizing slope encoded in the matrices \( S_k \), \( k \in \mathbb{Z} \), defined by

\[
A_{2j} = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}, \quad \tilde{A}_{2j} = \begin{pmatrix} 2^{j/2} & 0 \\ 0 & 2^j \end{pmatrix}, \quad S_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.
\]

Now we define discrete shearlet systems in 2D. Let \( c = (c_1, c_2) \in (\mathbb{R}_+)^2 \). For \( \phi, \psi, \tilde{\psi} \in L^2(\mathbb{R}^2) \), the cone-adapted 2D discrete shearlet system \( SH(\phi, \psi, \tilde{\psi}; c) \) is defined by

\[
SH(\phi, \psi, \tilde{\psi}; c) = \Phi(\phi; c_1) \cup \Psi(\psi; c) \cup \tilde{\Psi}(\tilde{\psi}; c),
\]

where

\[
\Phi(\phi; c_1) = \{ \phi(\cdot - cm) : m \in \mathbb{Z}^2 \},
\]

\[
\Psi(\psi; c) = \{ 2^{j/2} \psi(S_k A_{2j} \cdot - cm) : j \geq 0, |k| \leq |2^{j/2}|, m \in \mathbb{Z}^2 \},
\]

\[
\tilde{\Psi}(\tilde{\psi}; c) = \{ 2^{j/2} \tilde{\psi}(S_k^T \tilde{A}_{2j} \cdot - cm) : j \geq 0, |k| \leq |2^{j/2}|, m \in \mathbb{Z}^2 \}.
\]

We partite the frequency plane into \( C_1(\alpha) - C_4(\alpha) \) where

\[
C_l(\alpha) = \begin{cases} 
\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \geq \alpha, |\xi_2/\xi_1| \leq 1 \} : & l = 1 \\
\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_2 \geq \alpha, |\xi_1/\xi_2| \leq 1 \} : & l = 2 \\
\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \geq -\alpha, |\xi_2/\xi_1| \leq 1 \} : & l = 3 \\
\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_2 \geq -\alpha, |\xi_1/\xi_2| \leq 1 \} : & l = 4 
\end{cases}
\]

and a centered rectangle

\[
R(\alpha) = \{ (\xi_1, \xi_2) \in \mathbb{R}^2 : \| (\xi_1, \xi_2) \|_{\infty} < \alpha \}.
\]
The region \( \mathcal{C}_1 \cup \mathcal{C}_3 \) is covered by the frequency support of shearlets in \( \Psi(\psi; c) \). The region \( \mathcal{C}_2 \cup \mathcal{C}_4 \) is covered by the frequency support of shearlets in \( \tilde{\Psi}(\tilde{\psi}; c) \). The region \( \mathcal{R} \) is covered by the frequency support of \( \tilde{\phi} \). Recall \( \{\sigma_i\}_{i \in I} \) form a frame in \( L^2(\mathbb{R}^2) \) if there exist constants \( A, B > 0 \) such that

\[
A\|f\|^2 \leq \sum_{i \in I} |(f, \sigma_i)|^2 \leq B\|f\|^2 \quad \forall \ f \in L^2(\mathbb{R}^2).
\]

The numbers \( A, B \) are called frame bounds. If \( SH(\phi, \psi, \tilde{\psi}; c) \) is a frame for \( L^2(\mathbb{R}^2) \), we refer to \( \psi \) and \( \tilde{\psi} \) as shearlets.

We say that \( \phi \) is a refinable function with mask \( \hat{a}(\xi) \) if \( \hat{\phi}(2\xi) = \hat{a}(\xi)\hat{\phi}(\xi), \xi \in \mathbb{R}^d \). The Fourier transform of a function \( f \in L_1(\mathbb{R}^d) \) is defined as \( \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}dx \) and can be naturally extended to tempered distributions. As an important family of refinable functions, B-spline functions are useful in applications. B-spline with order \( m \) and its mask is defined by

\[
\hat{B}_m(\xi) = e^{-\frac{ij\pi}{2}}\left(\frac{\sin(\xi/2)}{\xi/2}\right)^m \quad \text{and} \quad \hat{a}(\xi) = e^{-\frac{ij\pi}{2}}\cos^m(\xi/2), \quad \xi \in \mathbb{R},
\]

where \( j = 0 \) when \( m \) is even, and \( j = 1 \) when \( m \) is odd. The B-spline function \( B_m \in C^{m-2}(\mathbb{R}) \) is a function of piecewise polynomials of degree less than \( m \), vanishes outside the interval \([0, m]\) and is symmetric about the point \( x = m/2 \). Now we state our first contribution on the shearlet frame in this paper.

**Theorem 2.1.** Let \( B_N \) be the B-spline function of order \( N \) with the mask \( \hat{a}_N(\xi) = (2^{-N})(1+e^{-ix})^N \). Let \( N_1, N_2 \in \mathbb{N} \) be such that \( N_1 > N_2 > 3 \). Define a shearlet \( \psi \in L^2(\mathbb{R}^2) \) by

\[
\psi(2\xi) = 2^{-N_1}e^{-i\xi N_1}(1-e^{-i\xi})^{-N_1}\hat{B}_{N_2}(\xi_1)\hat{B}_{N_2}(\xi_2), \quad \xi = (\xi_1, \xi_2).
\]

For given \( 0 < \alpha < \pi/2 \), there exits a sampling constant \( \hat{c} > 0 \) such that the shearlet system \( \Psi(\psi; c) \) forms a frame for \( \{f \in L_2(\mathbb{R}^2) : \text{supp}\hat{f} \subset C_1(\alpha) \cup C_3(\alpha)\} \) for \( c_2 \leq c_1 \leq \hat{c} \).

**Proof.** The proof is a straightforward consequence of Theorem 3.10. \( \square \)

The cartoon-liked model was first introduced in [1]. The basic idea is to choose a closed boundary curve and then fill the interior and exterior parts with \( C^2 \) functions. For \( \nu > 0 \), the set \( \text{STAR}^2(\nu) \) is defined to be the set of all \( B \subset [0, 1]^2 \) such that \( B \) is a translate of a set

\[
\{x \in \mathbb{R}^2 : |x| \leq \rho(\theta), \ x = (|x|, \theta) \text{ in polar coordinates}\}
\]

which satisfies \( |\rho''(\theta)| < \nu \), \( \rho \leq \rho_0 < 1 \). Then \( \mathcal{S}^2(\nu) \) denotes the set of functions \( f \in L(\mathbb{R}^2) \) of the form

\[
f = f_0 + f_{1XB},
\]

where \( f_0, f_1 \in C^2([0, 1]^2) \) and \( B \in \text{STAR}^2(\nu) \). The bandlimited curvelets, contourlets, and shearlets exhibit (almost) optimally sparse approximation with this model. The first complete proof of (almost) optimally sparse approximations of cartoon-liked images by using compactly supported shearlet frame was given in [16]. Let us now be more precise, and introduce these results. Let \( c > 0 \), and let \( \phi, \psi, \tilde{\psi} \in L_2(\mathbb{R}^2) \) be compactly supported. Suppose that for all \( \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \), the shearlet \( \psi \) satisfies

\[
|\tilde{\psi}(\xi)| \leq C_1 \cdot \min\{1, |\xi_1|^{-\gamma}\} \cdot \min\{1, |\xi_1|^{-\gamma}\} \cdot \min\{1, |\xi_2|^{-\gamma}\}, \quad (2.1)
\]
where \( \alpha > 5, \gamma \geq 4, h \in L_1(\mathbb{R}), \) and \( C_1 \) is a constant, and suppose that the shearlet \( \tilde{\psi} \) satisfies (2.1) and (2.2) with the roles of \( \xi_1 \) and \( \xi_2 \) reversed. Further, suppose that \( SH(c; \phi, \psi, \tilde{\psi}) \) forms a frame for \( L_2(\mathbb{R}^2) \). Denote \((\sigma_i)_{i \in I} = SH(c; \phi, \psi, \tilde{\psi})\). Let \((\tilde{\sigma})_{i \in I}\) be a dual frame of \((\sigma_i)_{i \in I}\). We can take the nonlinear \( N \)-terms approximation

\[
f_N = \sum_{i \in I_N} \langle f, \sigma_i \rangle \tilde{\sigma}_i,
\]

where \((\langle f, \sigma_i \rangle)_{i \in I_N}\) are the \( N \) largest coefficients \( \langle f, \sigma_i \rangle \) in magnitude. Then, for any \( v > 0 \), the shearlet frame \( SH(c; \phi, \psi, \tilde{\psi}) \) provides (almost) optimal sparse approximation of function \( f \in \mathcal{E}^2(v) \) in the sense that there exists some \( C > 0 \) such that

\[
\| f - f_N \|_2^2 \leq C \cdot (\log N)^3 \cdot N^{-2}.
\]

The condition (2.1) and (2.2) can be viewed as a generalization of a second order directional vanishing moment condition, which is crucial for having fast decay of the shearlet coefficients. Following the line of [15], we obtain the following results which is our second contribution in this paper.

**Theorem 2.2.** Let \( B_N \) be the B-spline function of order \( N \) with the mask \( \hat{a}_N(\xi) = (2^{-N})(1 + e^{-i\xi})^N \). Let \( N_1, N_2 \in \mathbb{N} \) be such that \( N_1 > 5, N_2 \geq 4 \) and \( N_1 > N_2 \). Define a shearlet \( \psi_1 \in L^2(\mathbb{R}^2) \) by

\[
\hat{\psi}_1(2\xi) = 2^{-N_1}e^{-i\xi_1}(1 - e^{-i\xi_1})^{N_1}B_{N_2}(\xi_1)B_{N_2}(\xi_2) \quad \xi = (\xi_1, \xi_2).
\]

Let \( \phi(x) = B_{N_2}(x_1)B_{N_2}(x_2) \) and \( \psi_2(x_1, x_2) = \psi_1(x_1, x_2) \). Then there exit sampling constant \( c > 0 \) such that the shearlet system \( SH(\phi, \psi_1, \psi_2; c) \) provides (almost) optimally sparse approximations of function \( f \in \mathcal{E}^2(v) \) in the sense that there exists some constant \( C > 0 \) such that

\[
\| f - f_N \|_2^2 \leq C \cdot (\log N)^3 \cdot N^{-2},
\]

where \( f_N \) is the nonlinear \( N \)-term approximation obtained by choosing the \( N \) largest shearlet coefficients of \( f \).

**Proof.** By Theorem 2.1, there exits a sampling constant \( \hat{c} > 0 \) such that the shearlet system \( \Psi(\psi; c) \) forms a frame for \( \{ f \in L_2(\mathbb{R}^2) : \text{supp} \tilde{f} \in C_1(\alpha) \cup C_3(\alpha) \} \) with \( c = (c_1, c_2) \in (\mathbb{R}_+)^2 \) and \( c_2 \leq c_1 \leq \hat{c} \). With the same argument as in [15], we can prove that \( SH(c; \phi, \psi, \tilde{\psi}) \) forms a frame for \( L_2(\mathbb{R}) \) with \( c = (c_1, c_2) \in (\mathbb{R}_+)^2 \) and \( c_2 \leq c_1 \leq \hat{c} \).

Rewrite \( \hat{\psi}_1(\xi) \) in the following

\[
\hat{\psi}_1(\xi) = 2^{-N_1}e^{-i\xi_1}(1 - e^{-i\xi_1})^{N_1-N_2}B_{N_2}(\xi_1) \cdot (1 - e^{-i\xi_2})^{N_2}B_{N_2}(\xi_2).
\]

We obtain

\[
\left| \frac{\partial \hat{\psi}_1(\xi)}{\partial \xi_2} \right| = \left| 2^{-N_1}e^{-i\xi_1}(1 - e^{-i\xi_1})^{N_1-N_2}B_{N_2}(\xi_1) \cdot (1 - e^{-i\xi_2})^{N_2}B_{N_2}'(\xi_2) \right| \leq \left| 2^{-N_1}e^{-i\xi_1}(1 - e^{-i\xi_1})^{N_1-N_2}B_{N_2}(\xi_1) \cdot |\xi_1|^{N_2} \cdot |B_{N_2}(\xi_2)| \right|.
\]
Set \( h(\xi_1) = 2^{-N_1}e^{-i\xi_1(1-e^{-i\xi_1})}N_1-N_2 \). We have
\[
|h(\xi_1)| \leq C''|\hat{B}_{N_2}(\xi_1)| \leq C''|\xi_1|^{-N_2}.
\]
Therefore, \( h(\xi_1) \in L_1(\mathbb{R}) \).

Let us consider \( |\hat{B}_{N_2}^I(\xi_2)| \). When \( N_2 \) is even, \( \hat{B}_{N_2}^I(\xi_2) = \left( \frac{\sin(\xi_2/2)}{\xi_2/2} \right)^{N_2} \). Then
\[
\hat{B}_{N_2}^I(\xi_2) = N_2 \left( \frac{\sin(\xi_2/2)}{\xi_2/2} \right)^{N_2-1} \left[ \frac{\cos(\xi_2/2)}{4\xi_2} - \frac{\sin(\xi_2/2)}{2\xi_2^2} \right]
\]
\[
\leq C|\xi_2|^{-N_2}.
\]
When \( N_2 \) is odd, \( \hat{B}_{N_2}^I(\xi_2) = e^{-i\xi_2} \left( \frac{\sin(\xi_2/2)}{\xi_2/2} \right)^{N_2} \). Then
\[
|\hat{B}_{N_2}^I(\xi_2)| = \left| -ie^{-i\xi_2} \left( \frac{\sin(\xi_2/2)}{\xi_2/2} \right)^{N_2} + e^{-i\xi_2} N_2 \left( \frac{\sin(\xi_2/2)}{\xi_2/2} \right)^{N_2-1} \left[ \frac{\cos(\xi_2/2)}{4\xi_2} - \frac{\sin(\xi_2/2)}{2\xi_2^2} \right] \right|
\]
\[
\leq \left| \left( \frac{\sin(\xi_2/2)}{\xi_2/2} \right)^{N_2} \right| + \left| N_2 \left( \frac{\sin(\xi_2/2)}{\xi_2/2} \right)^{N_2-1} \left[ \frac{\cos(\xi_2/2)}{4\xi_2} - \frac{\sin(\xi_2/2)}{2\xi_2^2} \right] \right|
\]
\[
\leq C'|\xi_2|^{-N_2}.
\]

We can conclude that there exists a constant \( C \) such that
\[
|\hat{\psi}_1(\xi)| \leq C \min\{1, |\xi_1|^{-N_1}\} \cdot \min\{1, |\xi_1|^{-N_2}\} \cdot \min\{1, |\xi_2|^{-N_2}\},
\]
and
\[
\left| \frac{\partial \hat{\psi}_1(\xi)}{\partial \xi_2} \right| \leq |h(\xi_1)| \left( 1 + \frac{|\xi_2|}{|\xi_1|} \right)^{-N_2},
\]
where \( N_1 > 5, N_2 \geq 4, h \in L_1(\mathbb{R}) \). Hence, the shearlet \( \psi_1 \) satisfies the conditions (2.1) and (2.2), and the shearlet \( \psi_2 \) likewise. The theorem is proved \( \square \)

### 3 Shearlets based on pseudo splines

In this section, we first briefly recall a family of refinable function: pseudo splines. After establishing some useful lemmas, we investigate the lower bound and the decay of the Fourier transform of pseudo splines of type II. Finally, we construct shearlet frame based on pseudo splines.

Pseudo-splines are compactly supported refinable functions in \( L_2(\mathbb{R}) \). For positive integers \( N, l \in \mathbb{N} \) with \( l < N \), denote \( P_{N,l}(x) := \sum_{j=0}^{l} \binom{N-1+j}{j} x^j \). The mask of a pseudo spline of type II with order \( (N, l) \) is defined by
\[
\hat{a}_{N,l}(\xi) := \cos^{2N}(\xi/2)P_{N,l}(\sin^{2}(\xi/2)). \quad (3.1)
\]
The mask of a pseudo spline of type I is defined by \( \hat{a}_{N,l}(\xi) := |\hat{a}_{N,l}(\xi)|^2 \). Hence, \( \hat{a}_{N,l}(\xi) \) with real coefficients is the square root of \( |\hat{a}_{N,l}(\xi)|^2 \) using the Riesz lemma. In general, pseudo splines of type I are neither symmetric nor antisymmetric. To achieve symmetry, compactly supported
complex valued pseudo splines were introduced in [19]. The corresponding pseudo splines can be defined in terms of their Fourier transform, i.e.

$$\hat{k}\phi(\xi) = \prod_{j=1}^{\infty} \hat{k}a_{N,l}(2^{-j}\xi), \quad k = 1, 2.$$  

An important fact is that $\hat{2a_{N,l}}(\xi)$ is defined by the summation of the first $l+1$ terms of the binomial expansion of $$(\cos^2(\xi/2) + \sin^2(\xi/2))^{N+l} = 1$$ i.e.

$$\sum_{j=0}^{l} \binom{N-1+j}{j} \sin^2(\xi/2) = \sum_{j=0}^{l} \binom{N+l}{j} \sin^2(\xi/2) \cos^2(l-j)(\xi/2).$$

(3.2)

The pseudo splines with order $(N,0)$ are B-splines. For the case $l = N-1$, the pseudo splines of type I are orthogonal refinable functions given in [6], and the pseudo splines of type II are interpolatory refinable function given in [10]. The other pseudo splines fill in the gap between the B-spline and orthogonal refinable functions for type I and B-spline and interpolatory refinable function for type II. For positive integers $N,l \in \mathbb{N}$ with $l < N$, let $\hat{a}(\xi)$ be the mask of the pseudo splines of type II with order $(N,l)$. Then $\hat{a}_{N,l}(\xi)$ can be factorized as

$$|\hat{a}_{N,l}(\xi)| = \cos^{2N}(\xi/2)|L(\xi)|, \quad \xi \in [-\pi, \pi].$$

(3.3)

This shows that pseudo splines is the convolution of a B-spline of some order with a distribution. Since $L(\xi)$ is bounded, $L(\xi)$ is actually the mask of a refinable distribution. The regularity of $\phi$ comes from the $\cos^{2N}(\xi/2)$ factor. The distribution part provides some desirable properties for $\phi$, such as orthogonality of its shifts. In [9], Dong and Shen gave a regularity analysis of pseudo splines of both types. The key to regularity analysis is

$$|L(\xi)| \leq \left|L\left(\frac{2\pi}{3}\right)\right|, \quad |\xi| \leq \frac{2\pi}{3},$$

(3.4)

$$|L(\xi)L(2\xi)| \leq |L\left(\frac{2\pi}{3}\right)|^2, \quad \frac{2\pi}{3} \leq |\xi| \leq \pi.$$  

To investigate the bounds and the decay of the pseudo splines in Fourier domain, we establish the following lemmas which have their own interesting. For simplify, we denote the mask of pseudo splines of type II by $\hat{a}(\xi)$.

### 3.1 Lemmas

**Lemma 3.1.** For positive integers $N,l \in \mathbb{N}$ with $l < N$, let $\hat{a}(\xi)$ be the mask of the pseudo splines of type II with order $(N,l)$ and let $L(\xi)$ be defined as in (3.3). Then

$$|\hat{a}(\xi)| \geq 1 - C_1|\xi|^{2l+2},$$

where $C_1 = \frac{\sum_{j=l+1}^{N+l} \binom{N+l}{j}}{2^{2l+2}}$.  


Proof. By (3.2), we obtain

\[
1 - |\hat{a}(\xi)| = \cos^{2N}(\xi/2) \sum_{j=l+1}^{N+l} \binom{N+l}{j} \sin^{2j}(\xi/2) \cos^{2(l-j)}(\xi/2)
\]

\[
\leq \sin^{2l+2}(\xi/2) \sum_{j=l+1}^{N+l} \binom{N+l}{j}
\]

\[
\leq \frac{\sum_{j=l+1}^{N+l} \binom{N+l}{j}}{2^{2l+2}} |\xi|^{2l+2}.
\]

\[\square\]

Lemma 3.2. For positive integers \(N, l \in \mathbb{N}\) with \(l < N\), let \(\hat{a}(\xi)\) be the mask of the pseudo splines of type II with order \((N, l)\). Then

\[|L(\xi)| \leq 1 + C_2 |\xi|^2,
\]

where \(C_2 = \frac{\sum_{j=1}^{l}(N-1+j)}{4} \).

Proof. We obtain

\[
|L(\xi)| - 1 = \sum_{j=0}^{l} \binom{N-1+j}{j} \sin^{2j}(\xi/2) - 1
\]

\[
= \sin^{2}(\xi/2) \sum_{j=1}^{l} \binom{N-1+j}{j} \sin^{2l-2}(\xi/2)
\]

\[
\leq \frac{\sum_{j=1}^{l} \binom{N-1+j}{j}}{4} |\xi|^2.
\]

\[\square\]

Lemma 3.3. For positive integers \(N, l \in \mathbb{N}\) with \(l < N\), let \(\hat{a}(\xi)\) be the mask of the pseudo splines of type II with order \((N, l)\). Let \(b(\xi) = e^{-i\xi} \hat{a}(\xi + \pi)\), then

\[
|\hat{b}(\alpha)| \chi_{[-\beta, -\alpha] \cup [\alpha, \beta]}(\xi) \leq |\hat{b}(\xi)| \leq \min\{1, \frac{\sum_{j=0}^{l} \binom{N+l}{j}}{22N^2} |\xi|^2\}, \quad 0 \leq \alpha < \beta \leq \pi.
\]

(3.5)

Proof. \(|\hat{b}(\xi)|\) is a \(\pi\) shift of the \(2\pi\)-periodic triangle polynomial \(\hat{a}(\xi)\). Note that \(|\hat{a}(\xi)|\) is increasing on \([-\pi, 0]\) and decreasing on \([0, \pi]\). It is easy to see that \(|\hat{b}(\xi)|\) is decreasing on \([-\pi, 0]\) and increasing on \([0, \pi]\). Hence the left side of (3.5) holds. By (3.3), We have

\[
|\hat{b}(\xi)| = \sin^{2N}(\xi/2) \sum_{j=0}^{l} \binom{N+l}{j} \cos^{2j}(\xi/2) \sin^{2(l-j)}(\xi/2)
\]

\[
\leq \frac{\sum_{j=0}^{l} \binom{N+l}{j}}{2^{2N}} |\xi|^{2N}.
\]

\[\square\]
3.2 Regularity

Now we give the lower bound for the pseudo splines of type II in the Fourier domain.

**Theorem 3.4.** Let \( \phi \) be the pseudo-splines of Type II with order \((N, l)\). Let \( K \subset [-\pi, \pi] \), then

\[
|\hat{\phi}(\xi)| \geq C_4 \cdot \chi_K(\xi).
\]

where \( C_4 = \prod_{k=1}^{k_0} |\hat{a}(2^{-k}\xi_0)| \exp(-C_1 2^{-k_0+1} |\xi_0|^{2l+2}) \).

**Proof.** Let \( \xi_0 = \arg \max_{\xi \in K} |\xi| \). Since \( \hat{a}(\xi) \) is decreasing on \([0, \pi]\), we have

\[
|\hat{a}(2^{-k}\xi)| \geq |\hat{a}(2^{-k}\xi_0)|
\]

for \( k \geq 1 \) and \( \xi \in K \). We choose sufficiently large \( k_0 \) so that \( 2^{-k}\xi_0 K |\xi|^{2l+2} < \frac{1}{2} \) if \( \xi \in K \) and \( k > k_0 \).

Using \( 1 - x \geq e^{-2x} \) for \( 0 \leq x \leq 1/2 \), we obtain

\[
|\hat{a}(\xi)| \geq 1 - C_1 2^{-k}\xi |\xi|^{2l+2} \geq \exp(-2^{k+1} C_1 |\xi|^{2l+2}), \quad k > k_0.
\]

Therefore, for \( \xi \in K \)

\[
|\hat{\phi}(\xi)| = \prod_{k=1}^{k_0} |\hat{a}(2^{-k}\xi)| \prod_{k=k_0+1}^{\infty} |\hat{a}(2^{-k}\xi)|
\geq \prod_{k=1}^{k_0} |\hat{a}(2^{-k}\xi_0)| \prod_{k=k_0+1}^{\infty} \exp(-2C_1 2^{-k}|\xi|^{2l+2})
\geq \prod_{k=1}^{k_0} |\hat{a}(2^{-k}\xi_0)| \exp(-C_1 2^{-k_0+1} |\xi_0|^{2l+2}) = C_4.
\]

In the following, we mainly investigate the decay of the pseudo splines of type II in the Fourier domain.

**Lemma 3.5.** For positive integers \( N, l \in \mathbb{N} \) with \( l < N \), let \( \hat{a}(\xi) \) be the mask of the pseudo splines of type II with order \((N, l)\). Let \( \mathcal{L}(\xi) \) be the distribution part. Assume that \( \hat{\phi}_\mathcal{L}(\xi) = \prod_{j=1}^{\infty} \mathcal{L}(2^{-j}\xi) \), then

\[
\hat{\phi}_\mathcal{L}(\xi) \leq \exp(C_2/3) q_1 q_2 q_2^{j-1} |\xi|^{\log_2(q_1^{1/(j-1)} q_2)}, \quad (3.6)
\]

where \( q_1 = \sup_{|\xi| \leq \pi} |\mathcal{L}(\xi)| = \sum_{j=0}^{l} \left( \frac{N-j}{j} \right) \) and \( q_2 = |\mathcal{L}(2\pi/3)| \).

**Proof.** Since \( |\mathcal{L}(\xi)| \leq 1 + C_2 |\xi|^2 \), we have for \( |\xi| \leq 1 \),

\[
\prod_{j=1}^{\infty} |\mathcal{L}(2^{-j}\xi)| = \exp \left( \sum_{j=1}^{\infty} \log(\mathcal{L}(2^{-j}\xi)) \right)
\leq \exp \left( C_2 \sum_{j=1}^{\infty} 2^{-2j} |\xi| \right)
\leq \exp(C_2/3).
\]
Now for $|\xi| > 1$, we have
\[
\prod_{k=1}^{\infty} \mathcal{L}(2^{-j} \xi) = \prod_{k=0}^{\infty} f(2^{-jk} \xi)
\]
where $f(\xi) = \prod_{j=1}^{J} \mathcal{L}(2^{-j} \xi)$. For the given $|\xi| > 1$, there exists a positive integer $k_0$ such that $2^{k_0 J} \leq |\xi| \leq 2^{(k_0 + 1)J}$. Denote $\eta = 2^{-(k_0 + 1)J} \xi$, we obtain $|\eta| < 1$. Define $q_1 = \sup_{|\xi| \leq \pi} |\mathcal{L}(\xi)|$ and $q_2 = |\mathcal{L}(2\pi/3)|$. By (3.4), we obtain
\[
\left| \prod_{j=1}^{J} \mathcal{L}(2^{-j} \xi) \right| \leq q_1 q_2^{J-1}.
\]
Therefore,
\[
\prod_{k=k_0+1}^{\infty} |f(2^{-kJ} \xi)| = \prod_{j=0}^{\infty} |f(2^{-(j+k_0+1)J} \xi)| = \prod_{j=0}^{\infty} |f(2^{-J(j+1)} \xi)| = \prod_{j=0}^{\infty} |f(2^{-j} \xi)|
\]
\[
= \prod_{j=0}^{\infty} |\mathcal{L}(2^{-j} \xi)| \leq \exp(C_2/3).
\]
Moreover,
\[
\prod_{k=0}^{k_0} |f(2^{-kJ} \xi)| \leq \left(q_1 q_2^{J-1}\right)^{k_0+1} \leq q_1 q_2^{J-1} \left(q_1^{1/(J-1)} q_2\right)^{k_0(J-1)}
\]
\[
\leq q_1 q_2^{J-1} \left(q_1^{1/(J-1)} q_2\right)^{\log_2 |\xi|}
\]
\[
\leq q_1 q_2^{J-1} |\xi|^{\log_2(q_1^{1/(J-1)} q_2)}.
\]
We conclude that
\[
\prod_{j=1}^{\infty} \mathcal{L}(2^{-j} \xi) = \exp(C_2/3) q_1 q_2^{J-1} |\xi|^{\log_2(q_1^{1/(J-1)} q_2)}.
\]

\[\square\]

**Theorem 3.6.** For positive integers $N, l \in \mathbb{N}$ with $l < N$, let $\phi$ be the pseudo splines of type II with order $(N, l)$. Then for any given integer $J \geq 1$,

\[
|\hat{\phi}(\xi)| \leq \min\{1, C_3 |\xi|^{-2N+\log_2(q_1^{1/(J-1)} q_2)}\}
\]

where $C_3 = 4^N \exp(C_2/3) q_1 q_2^{J-1}$.

**Proof.** Since
\[
|\hat{a}(\xi)|^2 + |\hat{a}(\xi + \pi)|^2 \leq |\hat{a}(\xi)| + |\hat{a}(\xi + \pi)| \leq 1,
\]
we obtain $|\hat{\phi}(\xi)| \leq 1$.

It is well known that
\[
\prod_{j=1}^{\infty} \cos(2^{-j} \xi) = \frac{\sin(\xi/2)}{\xi/2}.
\]
Hence
\[ |\hat{\phi}(\xi)| = \left( \frac{\sin(\xi/2)}{\xi/2} \right)^{2N} \prod_{j=1}^{\infty} \mathcal{L}(2^{-j}\xi). \]

By (3.6), we have
\[ |\hat{\phi}(\xi)| \leq \left( \frac{2}{\xi} \right)^{2N} \exp(C_2/3)q_1q_2^{J-1}|\xi|^{\log_2(q_1^{1/(J-1)})q_2} \]
\[ = C_3|\xi|^{-2N+\log_2(q_1^{1/(J-1)})q_2}, \]
where \( C_3 = 4^N \exp(C_2/3)q_1q_2^{J-1}. \)

**Corollary 3.7.** For positive integers \( N, l \in \mathbb{N} \) with \( l < N \), let \( \phi \) be the pseudo splines of Type II with order \( (N,l) \). Then
\[ |\hat{\phi}(\xi)| \leq C \min\{1, |\xi|^{-2N+k}\}, \tag{3.8} \]
where \( k = \log(P(3/4))/\log 2. \)

**Proof.** The proof is a straightforward consequence of Theorem 3.6.

**Remark 3.8.** The above results were first proved by Dong and Shen in [9]. The decay of the Fourier transform is optimal. We give estimation (3.7) with explicit constant which is important for the frame upbound of the shearlet frame. Table 1 gives the decay rate of the Fourier transform of pseudo splines of Type II with order \( (N,l) \) for \( 2 \leq N \leq 9 \) and \( 0 \leq l \leq N \). The decay rate of the Fourier transform of pseudo splines of Type II with order \( (N,l) \) is \( \beta_{N,l}/2. \)

### 3.3 Shearlet Frames

In the last subsection, we construct the shearlet frames based on pseudo splines. We give a weaker condition for constructing shearlet frames than Theorem 4.9 in [15].

For function \( \phi, \psi, \tilde{\psi} \in L^2(\mathbb{R}^2) \), we define \( \Theta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) by
\[ \Theta(\xi, \omega) = |\hat{\phi}(\xi)||\hat{\phi}(\xi + \omega)| + \Theta_1(\xi, \omega) + \Theta_2(\xi, \omega), \]
where

\[ \Theta_1(\xi, \omega) = \sum_{j \geq 0} \sum_{|k| \leq [2^{j/2}]} |\hat{\psi}(S_k^T A_{2^{-j}} \xi)||\hat{\psi}(S_k^T A_{2^{-j}} \xi + \omega)| \]

and

\[ \Theta_2(\xi, \omega) = \sum_{j \geq 0} \sum_{|k| \leq [2^{j/2}]} |\hat{\psi}(S_k \tilde{A}_{2^{-j}} \xi)||\hat{\psi}(S_k \tilde{A}_{2^{-j}} \xi + \omega)|. \]

The sufficient conditions for the construction of shearlet frames were given by [15].

**Lemma 3.9.** [Theorem 3.4 in [15]] let \( \phi, \psi \in L^2(\mathbb{R}^2) \) be functions such that

\[ \hat{\phi}(\xi, \eta) \leq C_2 \min\{1, |\xi|^{-\gamma}\} \min\{1, |\eta|^{-\gamma}\} \]

and

\[ |\hat{\psi}(\xi, \eta)| \leq C_1 \min\{1, |\xi|^\alpha\} \min\{1, |\eta|^{-\gamma}\} \min\{1, |\eta|^{-\gamma}\}, \]

for some positive constants \( C_1, C_2 < \infty \) and \( \alpha > \gamma > 3 \). Define \( \tilde{\psi}(x_1, x_2) = \psi(x_2, x_1) \) and let \( L_{\text{inf}} \) be defined by \( L_{\text{inf}} = \inf_{\xi \in \mathbb{R}^2} \Theta(\xi, 0) \). Suppose that there is a constant \( \tilde{L}_{\text{inf}} > 0 \) such that \( 0 < \tilde{L}_{\text{inf}} \leq L_{\text{inf}} \). Then there exist a sampling parameter \( c = (c_1, c_2) \) with \( c_1 = c_2 \) such that \( SH(\phi, \psi, \tilde{\psi}; c) \) forms a frame for \( L^2(\mathbb{R}^2) \).

Now we state the main results in this section.

**Theorem 3.10.** For \( 0 < \alpha < \pi/2 \). Let \( \tilde{a}_{N_1,l_1}(\xi), \tilde{a}_{N_2,l_2}(\xi) \) be the mask of the pseudo splines of type II. For \( l_2 = 0 \), let \( N_1 > N_2 > 2 \). For \( l_2 > 0 \), let \( N_1 > N_2 > 2 \). Let \( \hat{\phi}(\xi) \) be the associated refinable function with \( \tilde{a}_{N_2,l_2}(\xi) \). Define the shearlet \( \psi \in L^2(\mathbb{R}^2) \) by

\[ \hat{\psi}(2\xi) = \hat{b}(\xi) \hat{\phi}(\xi) \hat{\phi}(\xi), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2, \quad (3.9) \]

where \( \hat{b}(\xi_1) = e^{-i\xi_1 \cdot \tilde{a}_{N_1,l_1}(\xi_1 + \pi/2)} \). Then for given \( \alpha \in (0, \pi/2) \), there exits a sampling constant \( \hat{c} > 0 \) such that the shearlet system \( \Psi(\psi; c) \) forms a frame for \( \{f \in L^2(\mathbb{R}^2) : \text{supp} f \in C_1(\alpha) \cup C_3(\alpha)\} \) with \( c = (c_1, c_2) \in (\mathbb{R}_+)^2 \) and \( c_2 \leq c_1 \leq \hat{c} \).

**Proof.** We first give the lower bound of the shearlet \( \hat{\psi}(\xi) \). Let \( 0 < \alpha < \pi/2 \). By Theorem 3.4 we obtain

\[ |\hat{\phi}(\xi)| \geq C_4 |\chi_{[-2\alpha, 2\alpha]}(\xi)| \quad \text{and} \quad |\hat{\phi}(\xi)| \geq C_4 |\chi_{[-\alpha, \alpha]}(\xi)|. \quad (3.10) \]

From (3.5), we have

\[ |\hat{b}(\xi)| \geq |\hat{b}(\alpha)| |\chi_{[-2\alpha, -\alpha]} \cup [\alpha, 2\alpha]|(\xi_1). \quad (3.11) \]

Combine (3.10) and (3.11), we obtain

\[ |\hat{\psi}(\xi)| = |\hat{b}(\xi_1) \hat{\phi}(\xi_1) \hat{\phi}(\xi_2)| \geq |\hat{b}(\alpha)| C_4 C_4 \cdot |\chi_\alpha(\xi)| \]

where \( \Omega = \{\xi = (\xi_1, \xi_2), \quad \xi_1 \in [-2\alpha, -\alpha] \cup [\alpha, 2\alpha], \quad \xi_2 \in [-\alpha, \alpha]\} \). Since

\[ \bigcup_{j=0}^{\infty} \bigcup_{|k| \leq [2^{j/2}]} A_2, S_k^T \Omega = \mathcal{C}, \]

11
we conclude that

$$\Phi(\xi, 0) = \sum_{j,k} |\hat{\psi}(S_k^T A_{2^{-j}} \xi)|^2 \geq \tilde{L}_{\text{inf}} \chi_{\Omega}(S_k^T A_{2^{-j}} \xi) \geq \tilde{L}_{\text{inf}} \quad \text{on} \quad \mathcal{C},$$

where $\tilde{L}_{\text{inf}} = (|\hat{b}(\alpha)|C'_4C_4)^2$.

By (3.5) and (3.8), we obtain the upper bound of the shearlet $\hat{\psi}(\xi)$

$$|\hat{\psi}(\xi)| \leq C \min\{1, |\xi_1|^{2N_1}\} \cdot \min\{1, |\xi_1|^{-2N_2+\kappa}\} \cdot \min\{1, |\xi_2|^{-2N_2+\kappa}\}. \quad (3.12)$$

Let $\beta_{N,l} = 2N - \kappa$, for fixed $N$, $\beta_{N,l}$ decreases as $l$ increases. For fixed $l$, $\beta_{N,l}$ increases as $N$ increases. When $l = N - 1$, $\beta_{N,l}$ increases as $N$ increase [9]. From the table 1, it easy to see that

$$2N_1 > 2N_2 > 3, \quad \text{for} \quad l_2 = 0$$

and

$$2N_1 > 2N_2 - \kappa > 3, \quad \text{for} \quad l_2 > 0.$$

Hence, the condition of Theorem 3.9 holds. The theorem is proved. \(\square\)

\textbf{Remark 3.11.} (3.12) is a standard construction for separable wavelet basis of $L_2(\mathbb{R}^2)$ [6]. Therefore, Theorem 3.10 gives a connection between the shearlet analysis and wavelet analysis and also, we hope, enriches the theory of shearlet analysis.

Since the decay rate of $|\hat{\phi}(\xi)|$ is half of that of $|\hat{\psi}(\xi)|$. The results on pseudo splines of type I follow directly from Theorem 3.10.

\textbf{Theorem 3.12.} For $0 < \alpha < \pi/2$. Let $a_{N_1,l_1}(\xi), \tilde{a}_{N_2,l_2}(\xi)$ be the mask of the pseudo splines of type I. For $l_2 = 0$, let $N_1 > N_2 > 3$. For $l_2 > 0$, let $N_1 \geq N_2 > 8$. Let $\hat{\phi}(\xi)$ be the associated refinable function with $\tilde{a}_{N_2,l_2}(\xi)$. Define a shearlet $\hat{\psi} \in L^2(\mathbb{R}^2)$ by

$$\hat{\psi}(2\xi) = \hat{b}(\xi_1)\hat{\phi}(\xi_1)\hat{\phi}(\xi_2), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,$$

where $\hat{b}(\xi_1) = e^{-i\xi_1}a_{N_1,l_1}(\xi_1 + \pi)$. Then there exits a sampling constant $\hat{c} > 0$ such that the shearlet system $\Psi(\hat{\psi}; c)$ forms a frame for $\{f \in L_2(\mathbb{R}^2) : \text{supp}\hat{f} \in C(\alpha)_1 \cup C(\alpha)_3\}$ for any sampling matrix $M_c$ with $c = (c_1, c_2) \in (\mathbb{R}_+)^2$ and $c_2 \leq c_1 \leq \hat{c}$.

\textbf{Remark 3.13.} In [15], P. Kittipoom, G. Kutyniok, and W.-Q Lim gave a similar construction of the shearlet from pseudo splines of type I with order $(N,l)$ such that $l > 10$ and $\frac{3l}{2} \leq N \leq 3l - 2$. Compared with their results, our construction have smaller support. Moreover, we prove for any given $0 < \alpha < \pi/2$, one can construct shearlet frame. This gives more feasible choice in applications.

\section{Examples}

To illustrate our results, we give two examples in this section.

\textbf{Example 4.1.} Let $B_3$ be the B-spline function of order 3. Define the shearlet

$$\hat{\psi}(2\xi) = 2^{-4}e^{-i\xi_1}(1 - e^{-i\xi_1})^4\tilde{B}_3(\xi_1)\tilde{B}_3(\xi_2), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Then for any given $0 < \alpha < \pi/2$, there exits a sampling constant $\hat{c} > 0$ such that the shearlet system $\Psi(\hat{\psi}; c)$ forms a frame for $\{f \in L_2(\mathbb{R}^2) : \text{supp}\hat{f} \in C_1(\alpha) \cup C_3(\alpha)\}$ for $c_2 \leq c_1 \leq \hat{c}$.
Example 4.2. Let $B_4$ be the B-spline function of order 4. Define the shearlet

$$\hat{\psi}_1(2\xi) = 2^{-6}e^{-i\xi_1}(1 - e^{-i\xi_1})^6\hat{B}_4(\xi_1)\hat{B}_4(\xi_2), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$ 

Let $\phi(x) = B_4(x_1)B_4(x_2)$ and $\psi_2(x_1, x_2) = \psi_1(x_1, x_2)$. Then there exit sampling constant $c > 0$ such that the shearlet system $\Psi(\phi, \psi_1, \psi_2; c)$ provides (almost) optimally sparse approximations of function $f \in \mathcal{E}^2(v)$.

Acknowledgements This work is supported by NSF of China under grant numbers 10771190, 10971189 and Zhejiang Provincial NSF of China under grants number Y6090091.

References

[1] E.J. Candès, D.L. Donoho, New tight frames of curvelets and optimal representations of objects with $C^2$ singularities, Comm. Pure Appl. Math. 56 (2004), 219-266.

[2] C.K. Chui, W. He, Compactely supported tight frames associated with refinable functions, Appl. Comp. Harmon. Anal., 8, (2000), 293-319.

[3] C.K. Chui, W. He, and J. Stöckler, Compactely supported tight and sibling frames with maximum vanishing moments, Appl. Comput. Harmon. Anal., 13, (2002), 224-262.

[4] C.K. Chui, J.Z. Wang, On compactly supported spline wavelets and a duality principle, Trans. Amer. Math. Soc. 330, (1992), 903-915.

[5] C.K. Chui, An introduction to wavelets, Academic press, San Diego, CA, 1992.
Figure 2: $\psi(S_k A_{2^2} \cdot x)$, $k = -2, -1, 0, 1, 2$, in Example 4.2

[6] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Series in Applied Mathematics, SIAM, Philadelphia (1992).

[7] I. Daubechies, B. Han, A. Ron, and Z.W. Shen, *Framelets: MRA-based constructions of wavelet frames*, Appl. Comput. Harmon. Anal. 14, (2003), 1-46.

[8] M.N. Do, M. Vetterli, *The contourlet transform: an efficient directional multiresolution image representation*, IEEE Trans. Image Proc. 14 (2005), 2091-2106.

[9] B. Dong, Z.W. Shen, *Pseudo-splines, wavelets and framelets*, Appl. Comput. Harmon. Anal. 22, (2007), 78-104.

[10] S. Dubuc, *Interpolation through an iterative scheme*, J. Math. Anal. Appl. 114, (1986), 185-204.

[11] B. Han, Z.W. Shen, *Dual Wavelet Frames and Riesz Bases in Sobolev Spaces*, Constr. Approx. 29, (2009) 369-406.

[12] B. Han, Z.W. Shen, *Wavelets with short support*, SIAM J. Math. Anal. 38, (2007), 530-556.

[13] R.Q. Jia, J.Z. Wang, and D.X. Zhou, *Compactly supported wavelet bases for Sobolev spaces*, Appl. Comput. Harmon. Anal. 15, (2003), 224-241.

[14] P. Kittipoom, G. Kutyniok, and W.-Q Lim, *compactly supported shearlet*, preprint.

[15] P. Kittipoom, G. Kutyniok, and W.-Q Lim, *Construction of compactly supported shearlet frames*, preprint, arXiv:1003.5481v2.

[16] G. Kutyniok, W.-Q Lim, *Compactly supported shearlets are optimally sparse*, preprint arXiv:1002.2661v2.
[17] D. Labate, W-Q. Lim, G. Kutyniok, and G. Weiss. Sparse multidimensional representation using shearlets, Wavelets XI (San Diego, CA, 2005), 254-262, SPIE Proc. 5914, SPIE, Bellingham, WA, 2005.

[18] A. Ron, Z.W. Shen, Affine systems in $L_2(\mathbb{R}^d)$: the analysis of the analysis operator, J. Funct. Anal. 148 (2) (1997), 408-447.

[19] Y. Shen, S. Li, and Q. Mo, Complex wavelets and framelets from pseudo splines, J. Fourier Anal. Appl., DOI 10.1007/s00041-009-9095-8