A Characterization of Umbral Calculus Inspired by Fractional Sums

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Abstract

We will use analytic function theory and Fourier analysis to establish a characterization for some classical umbral calculus, which will focus on the generalization of the evaluation function. Although we cannot cover all the umbral calculus people care about, the part about Bernoulli numbers can still answer an open question about fractional sums raised by Müller and Schleicher in 2005 [10] and synthesize some common analysis results. We will only develop the results which are sufficient to serve the purpose of this article, but at the same time, we will briefly mention some possible extensions.

1 Introduction

Umbral calculus is a basic technique in combinatorics, systematized by Gian-Carlo Rota [13]. Generally speaking, for a given sequence \((A_n)_{n \in \mathbb{N}}\), it asks us to treat \(A_n\) as \(A^n\) to simplify some combinatorics deduction and revealed some imperceptible results. [4] gives an appropriate introduction about the applications, and [2] gives a comprehensive survey. Consider Bernoulli numbers \((B_n)_{n \in \mathbb{N}}\), we can write Faulhaber’s formula as

\[
\sum_{k=1}^{n} k^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k} = \frac{(B + n)^{m+1} - B^{m+1}}{m+1}.
\]

Those results strongly hint that we can consider replacing the polynomial \(P\) and the integer \(n\) in (1.1) by function \(f\) and complex number \(z\). However, it requires us to explain \(f(B)\) and \(\sum_{k=1}^{n} \) appropriately, and we need more motivation for doing this.
Fractional sums is a concept first formulated clearly by Müller and Schleicher [10, 11, 12]. They proposed six natural axioms on \( \sum_{k=a}^{b} f(k) \), for appropriate function \( f \) and complex number \( a, b \), those axioms can derive the value of \( \sum_{k=a}^{b} f(k) \) uniquely. Their results can be reformulated as below.

**Definition 1.1** (Müller-Schleicher fractional summation). Suppose \( U \subseteq \mathbb{C} \) satisfied \( U + 1 \subseteq U \) and \( 1 \in U \), \( f \) is a complex value function on \( U \).

If there is a polynomial sequence \( (P_n)_{n \in \mathbb{N}} \) satisfied
1. The degree of the polynomial sequence \( (P_n)_{n \in \mathbb{N}} \) is uniformly bounded;
2. For all \( y + 1 \in U \), the limits
   \[
   \lim_{n \to \infty} \left( \sum_{k=1}^{n} (f(k + y) - f(k)) - P_n(y) \right) = -Q(y)
   \]
   exists;
3. For all \( y + 1 \in U \),
   \[
   Q(y + 1) - Q(y) = f(y + 1),
   \]
then we defined **Müller-Schleicher fractional summation** as
   \[
   (\text{MS}) \sum_{k=x}^{y} f(k) := Q(y) - Q(x - 1), \tag{1.2}
   \]
where \( x, y + 1 \in U \).

For the well-definedness of Müller-Schleicher fractional summation (for short, MS-fractional summation), we referred to [11]. Many classical infinite sum identities can be reformulated in fractional summation and revealed a direct proof [10].

In those reformulations, especially for the Gosper series, which Gosper discussed and proved by the properties of Bessel function [5].

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n + \frac{1}{2}} \sin \left( \frac{\pi b}{2} \sin \left( \frac{\pi (n + 1/2)^2}{2} + \pi \right) \right) = \frac{\pi b}{2b} \quad \tag{1.3}
\]
\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \left( \frac{\pi b}{2} \cos \left( \frac{\pi (n + 1/2)^2}{2} + \pi \right) \right) = \frac{\pi^2}{4} \left( \frac{\sin b}{b} - \cos b \right). \quad \tag{1.4}
\]

They noticed that [10, 11, 12] those series could be proved directly by an interchange of a fractional sum and an infinite series, i.e.

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n + \frac{1}{2}} \frac{\sin \left( \frac{\pi b}{2} \sin \left( \frac{\pi (n + 1/2)^2}{2} + \pi \right) \right)}{\sqrt{b^2 + \pi^2 (n + 1/2)^2}} = (\text{MS}) \sum_{n=1/4}^{-1/4} \frac{1}{2n} \frac{\sin \left( \frac{\pi b}{2} \sin \left( \frac{\pi (2n)^2}{2} + \pi \right) \right)}{\sqrt{b^2 + (2\pi n)^2}}
\]
\[
\overset{?}{=} (\text{MS}) \frac{\sin b}{2b} \sum_{n=1/4}^{-1/4} n^{-1},
\]
and

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \sqrt{b^2 + \pi^2 n^2} = (\text{MS}) - \frac{1}{4} \sum_{n=1}^{1/2} \frac{1}{n^2} \cos \sqrt{b^2 + (2\pi n)^2} \]

\[ \equiv (\text{MS}) - \frac{1}{4} \left( \cos b \sum_{n=1}^{-1/2} n^{-2} - \frac{2\pi^2 \sin b}{b} \sum_{n=1}^{-1/2} \frac{1}{n} \right). \]

Where

\[ \frac{1}{2n} \frac{\sin \sqrt{b^2 + (2\pi n)^2}}{\sqrt{b^2 + (2\pi n)^2}} = \frac{\sin b}{2b} n^{-1} + a_1 n + a_3 n^3 + a_5 n^5 + \cdots \]

\[ \frac{1}{n^2} \cos \sqrt{b^2 + (2\pi n)^2} = \cos b \cdot n^{-2} - \frac{2\pi^2 \sin b}{b} + a_2 n^2 + a_4 n^4 + \cdots, \]

and we have

\[ (\text{MS}) - \sum_{n=1/4}^{-1/4} n^{2k-1} = 0 \quad (\text{MS}) - \sum_{n=1}^{-1/2} n^{2k} = 0 \]

\[ (\text{MS}) - \sum_{n=1/4}^{-1/4} n^{-1} = \pi \quad (\text{MS}) - \sum_{n=1}^{-1/2} n^{-2} = \frac{\pi^2}{3} \quad (\text{MS}) - \sum_{n=1}^{-1/2} 1 = -\frac{1}{2}, \]

for all positive integer \( k \). We have enough motivation to determine the sufficient condition on function for this interchangeability. In [11] they also concerned about the conditions that validated the following formula

\[ (\text{MS}) - \frac{d}{dx} \sum_{k=1}^{x} f(k) = cf + (\text{MS}) - \sum_{k=1}^{x} \frac{d}{dk} f(k). \quad (1.5) \]

Our goal is to justify those speculation.

### 1.1 Framework

Our main idea is generalized [11] to

\[ (\text{MS}) - \sum_{k=1}^{z} f(k) = F(B + z) - F(B), \quad (1.6) \]

where \( F \) is the primitive function of \( f \), and transferred the difficulty of interchangeability to the definition and properties of \( F(B) \), i.e. \( \text{eval}_B(F) \). We achieved this in Theorem [3].

In the next section, we will discuss the evaluation function including a kind of sequence \((A_n)_{n \in \mathbb{N}}\), such as Bernoulli numbers, and developed all the properties we need. This is achieved by the analytic function theory and Fourier transform, i.e.

\[ f(A) := \text{eval}_A(f) := \frac{1}{\sqrt{2\pi}} \int_{-\infty - 1t}^{\infty - 1t} \hat{A}(z)f(iz)dz, \quad (1.7) \]

\[ \text{eval}_A(f) := \frac{1}{\sqrt{2\pi}} \int_{-\infty - 1t}^{\infty - 1t} \hat{A}(z)f(iz)dz, \quad (1.7) \]
where $A(z)$ is the generating function of sequence $(A_n)_{n \in \mathbb{N}}$. In fact, similar ideas can be founded in other places. In \[8, 9\], the author treats $A_n$ as the $n$-th moment of some signed measure $\mu$; in \[14\], the author also noticed connections with Fourier analysis; in \[6, 7\], the author gives an extension of the umbral calculus to certain classes of entire function. Our discussion will follow a different way.

Section 3 will focus on the applications of section 2, Corollary 3.1, 3.2 includes the validity of the speculation mentioned above and determined $c_f$. Finally, in Corollary 3.3, we proved and promoted some conclusions in \[4\] as expected by Müller and Schleicher.

In Section 4 we introduced some possible developments.

2 The Analysis of Umbrae

The Fourier transform of $f \in L^1(\mathbb{R})$, which we take is

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx,$$

and it also induced the Fourier transform on slowly increasing distribution space $S'(\mathbb{R})$. Sometimes we would interpret the integral of the Fourier transform in the meaning of the summation method.

For our purpose, we will generalize the concept of the Fourier transform as below.

**Definition 2.1** (Fourier transform). If $f$ is defined on $\mathbb{R} - it$, then we define the Fourier transform of $f$ is

$$\hat{f}(\xi - is) := \frac{1}{\sqrt{2\pi}} \int_{-i\infty}^{i\infty} f(z)e^{-i(\xi - is)z}dz \quad (2.1)$$

$$= e^{-(\xi - is)t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-sz}f(x - it)e^{-i\xi x}dx.$$ 

It should be emphasized that $f$ could be exponential growth, and $\hat{f}$ could be undefined on $\mathbb{R}$.

**Definition 2.2** (Umbrae). If $A$ is an exponential type analytic function defined on

$$\Omega_{a,b} := \{x - it : x \in \mathbb{R}, t \in (a, b)\},$$

i.e. there exists $s \in \mathbb{R}$ such that for every $t \in (a, b)$,

$$|A(x - it)| \leq C(t)e^{s|x|},$$

where $C$ is locally bounded, then

1. we called $A = (A, \Omega_{a,b})$ is a umbrae;
2. we called $A$ is the **generating function** of $A$.

When $0 \in \Omega_{a,b}$, the umbrae $A$ exactly corresponds to the sequence $(A^{(n)}(0))_{n \in \mathbb{N}}$. $s$ can be treated as dependent on $t$, but we do not need such a detailed treatment here.

**Lemma 2.1.** Suppose $f$ is an exponential type analytic function defined on $\Omega_{a,b}$. $t_1, t_2 \in (a, b)$. If $f$ is Lebesgue integrable on $\mathbb{R} - it_1, \mathbb{R} - it_2$, then

$$
\int_{-\infty - it_1}^{\infty - it_1} f(z) dz = \int_{-\infty - it_2}^{\infty - it_2} f(z) dz.
$$

**Proof.** First notice that $e^{-\varepsilon z^2} f(z)$ locally uniformly tends to 0 when $|z| \to \infty$, therefore

$$
\int_{-\infty - it_1}^{\infty - it_1} e^{-\varepsilon z^2} f(z) dz = \int_{-\infty - it_2}^{\infty - it_2} e^{-\varepsilon z^2} f(z) dz.
$$

By the dominated convergence theorem, only need to let $\varepsilon \to 0$. \qed

This lemma is also valid for larger function classes, but again the exponential type condition is sufficient for our needs.

**Lemma 2.2** ($L^1$-Phragmén–Lindelöf principle). Suppose $f$ is an exponential type analytic function defined on $\Omega_{a,b}$. $t_1, t_2 \in (a, b), t_1 < t_2$. If $f$ is Lebesgue integrable on $\mathbb{R} - it_1, \mathbb{R} - it_2$, then for every $t \in [t_1, t_2]$, $f$ is Lebesgue integrable on $\mathbb{R} - it$, and $\ln \| f(\cdot - it) \|_1$ is a convex function for $t \in [t_1, t_2]$.

**Proof.** See [3] p.479 Theorem 4. \qed

### 2.1 Basic definition and properties

We basically only care about a special class of umbrae.

**Definition 2.3.** Suppose $A = (A, \Omega_{a,b})$ is a umbrae.

1. Denote

$$
\alpha = \inf\{ s \in \mathbb{R} : \exists \text{locally bounded } C : \forall x \geq 0, |f(x - it)| \leq C(t)e^{sx} \}
$$

$$
\beta = \sup\{ s \in \mathbb{R} : \exists \text{locally bounded } C : \forall x \leq 0, |f(x - it)| \leq C(t)e^{sx} \}.
$$

We called $\alpha$ is the **positive index**; $\beta$ is the **negative index**; $(\alpha, \beta)$ is the **index** of the umbrae $A$;

2. If $\alpha \geq \beta$, we called umbrae $A$ is **singular**;

3. If $\alpha < \beta$, we called umbrae $A$ is **regular**, and the open interval $(\alpha, \beta)$ is called the **regular interval** of $A$;

4. We called the open interval $(a, b)$ is the **dominating interval** of $A$. 

The reasons for choosing these terms and signs will become clearer in the following discussion.

**Definition 2.4** (Umbral calculus). Suppose \( A_i = (A_i, \Omega_{a_i, b_i}) \) is a umbrae.

1. For \( r \in \mathbb{R} \), \( rA := (A(rz), r^{-1}\Omega_{a_i, b_i}) \);
2. \( A_1 + A_2 := (A_1, A_2, \Omega_{a_1, b_1} \cap \Omega_{a_2, b_2}) \), \( A_1 - A_2 := A_1 + (-1)A_2 \);
3. For \( n \in \mathbb{N} \), \( 0 \times A := 0 \), \( (n + 1) \times A := (n \times A) + A \);
4. \( A_1[+]A_2 := (A_1 + A_2, \Omega_{a_1, b_1} \cap \Omega_{a_2, b_2}) \);
5. \( A_1[-]A_2 := (A_1 - A_2, \Omega_{a_1, b_1} \cap \Omega_{a_2, b_2}) \).

The scalar multiplication and addition in the above definition come from the classic umbral calculus. They do not satisfy the properties that these operations should normally have, but this definition is convenient for evaluation, and we will verify its well-definedness later.

**Definition 2.5** (Special umbrae).

1. For \( c \in \mathbb{C} \), \( (c) := (e^{cz}, \mathbb{C}) \), the index is \((\text{Re} c, \text{Re} c)\);
2. For \( c \in \mathbb{C} \), \( [c] := (c, \mathbb{C}) \), the index is \((-\infty, \infty)\), \( c \neq 0 \);
3. \( D := (z, \mathbb{C}) \), the index is \((0, 0)\);
4. \( \Delta := (e^z - 1, \mathbb{C}) \), the index is \((1, 0)\);
5. \( B := \left( \frac{e^z}{e^z - 1}, \Omega_{-2\pi, 2\pi} \right) \), the index is \((0, 1)\);
6. \( E := \left( \frac{2e^z}{e^z - 1}, \Omega_{-\frac{\pi}{2}, \frac{\pi}{2}} \right) \), the index is \((-1, 1)\).

In fact, we will identify the complex number \( c \) to the umbrae \((c)\).

For regular umbrae \( A = (A, \Omega_{a, b}) \), we can use Definition 2.1 to calculate the Fourier transform of \( A \). By the knowledge of analysis, it is not difficult to see that \( \hat{A} \) is well-defined, and it also induced a regular umbrae.

**Theorem 1** (Correspondence). If \( A = (A, \Omega_{a, b}) \) is a regular umbrae with regular interval \((\alpha, \beta)\), then \( \hat{A} := (\hat{A}, \Omega_{a, b}) \) is a regular umbrae with regular interval includes \((-b, -a)\).

Fourier transform establishes the correspondence between regular umbrae.

Since we essentially only can deal with the regular umbrae, it is necessary to establish a decomposition theorem for the singular one.

**Theorem 2** (Component). Suppose \( A = (A, \Omega_{a, b}) \) is a singular umbrae with index \((\alpha, \beta)\). If there are regular umbrae pairs \((A_1, A_2)\) inducing the decomposition \( A = A_1[+]A_2 \), then one of the regular umbrae \( A_i \) has positive index \( \alpha \), called the positive component of this decomposition, and the other \( A_j \) has negative index \( \beta \), its opposite \([-1] + A_j \) called the negative component of this decomposition. \((A_i, [-1] + A_j)\) is called the regular decomposition of \( A \).
Theorem 3 (Decomposition). If \( A = (\mathcal{A}, \Omega_{a,b}) \) is a umbrae with index \((\alpha, \beta)\), then there are umbraes \( A^+, A^- \) such that

1. \( A^+ = (\mathcal{A}^+, \Omega_{a,b}) \) is a regular umbrae with index \((\alpha, \infty)\);
2. \( A^- = (\mathcal{A}^-, \Omega_{a,b}) \) is a regular umbrae with index \((-\infty, \beta)\);
3. \( A = A^+[-]A^- \).

Proof. We denoted
\[
\Lambda_+ = \left( \frac{1}{i\sqrt{2\pi}} z^{1/(e^{-z^2})}, \Omega_{0,\infty} \right) = (\rho_+, \Omega_{0,\infty});
\Lambda_- = \left( \frac{1}{i\sqrt{2\pi}} z^{1/(e^{-z^2})}, \Omega_{-\infty,0} \right) = (\rho_-, \Omega_{-\infty,0}).
\]

According to Theorem 1,
1. \( \tilde{\Lambda}_+ = (\tilde{\rho}_+, \Omega_{-\infty,\infty}) \) is a regular umbrae with index \((0, \infty)\);
2. \( \tilde{\Lambda}_- = (\tilde{\rho}_-, \Omega_{-\infty,\infty}) \) is a regular umbrae with index \((-\infty, 0)\);
3. \( \tilde{\Lambda}_+[-]\tilde{\Lambda}_- = 0 \).

Finally, we can take \( A^+ = A + \tilde{\Lambda}_+, A^- = A + \tilde{\Lambda}_- \).

This theorem allows us to transfer the problem of singular umbrae into regular umbraes.

Next, we define a class of functions that the umbrae can be evaluated. This function class will not be broad enough, but it is enough for our purposes.

Definition 2.6. Suppose \( A = (\mathcal{A}, \Omega_{a,b}) \) is a regular umbrae, the index is \((\alpha, \beta)\). \( I \) is an open interval. If \( I \cap (\alpha, \beta) \neq \emptyset \), and the analytic function \( f \) satisfies

1. \( f \) is an exponential type analytic function defined on \( \{ z \in \mathbb{C} : \text{Re} z \in I \} \);
2. For every \( t \in I \cap (\alpha, \beta) \), \( \hat{A}(z)f(iz) \) is Lebesgue integrable on \( \mathbb{R} - it \),

then the class formed by such functions is denoted by \( \mathcal{T}_A(I) \).

Definition 2.7 (Evaluation function). If \( f \in \mathcal{T}_A(I) \), then we defined
\[
f(A) := \text{eval}_A(f(z); z) := \int_{-\infty-it}^{\infty-it} \hat{A}(z)f(iz)dz,
\]
where \( t \in I \cap (\alpha, \beta) \).

Lemma 2.1 ensured the well-definedness of \( f(A) \).

In fact, the argument from Lemma 2.1 can also tell us the following result about summation method.
**Proposition 2.1** (Gauss-Weierstrass summation method). Suppose $A = (\mathcal{A}, \Omega_{a,b})$ is a regular umbrae, the index is $(\alpha, \beta)$. $f$ is an exponential type analytic function defined on $\{z \in \mathbb{C} : \text{Re} z \in I\}$. $I, \bar{I}$ are open intervals satisfied $I \subseteq \bar{I}$.

If $f \in \mathcal{T}_A(I)$, then for every $t \in \bar{I} \cap (\alpha, \beta)$ we have

$$\lim_{\varepsilon \to 0} \int_{-\infty - it}^{\infty - it} e^{-\varepsilon^2 \hat{A}(z)} f(iz) dz = f(A). \quad (2.4)$$

Proof. Because of

$$\int_{-\infty - it}^{\infty - it} e^{-\varepsilon^2 \hat{A}(z)} f(iz) dz = \int_{-\infty - it_0}^{\infty - it_0} e^{-\varepsilon^2 \hat{A}(z)} f(iz) dz;$$

only need to let $\varepsilon \to 0$. \qed

We can also use (2.4) to define the evaluation function, but it will complicate the limit calculus.

### 2.2 Well-definedness

Intuitively we have $f(A_1[-]A_2) = f(A_1) - f(A_2)$, which inspired the following proposition.

**Proposition 2.2.** Suppose $A = (\mathcal{A}, \Omega_{a,b})$ is a singular umbrae with index $(\alpha, \beta)$. $I$ is an open interval. If $I \supseteq [\beta, \alpha]$, and the following conditions are satisfied,

1. $(\mathcal{A}^+_1, \mathcal{A}^-_1), (\mathcal{A}^+_2, \mathcal{A}^-_2)$ is the regular decomposition of $A$;
2. $f \in \left(\mathcal{T}_{\mathcal{A}^+_1}(I) \cap \mathcal{T}_{\mathcal{A}^-_1}(I)\right) \cap \left(\mathcal{T}_{\mathcal{A}^+_2}(I) \cap \mathcal{T}_{\mathcal{A}^-_2}(I)\right)$,

then

1. $f(A^+_1) - f(A^-_1) = f(A^+_2) - f(A^-_2)$;
2. $A_0 = A^+_1[-]A^-_2 = A^+_2[-]A^-_1$ is a regular umbrae;
3. $f \in \mathcal{T}_{A_0}(I)$.

Proof. According to Theorem 2 the regular interval of the component of the regular decomposition always intersects with $I$, and $A_0 = A^+_1[-]A^-_2 = A^+_2[-]A^-_1$ is a regular umbrae.

Notice that

$$f \in \mathcal{T}_{\mathcal{A}^+_1}(I) \cap \mathcal{T}_{\mathcal{A}^-_2}(I) \Rightarrow f \in \mathcal{T}_{A_0}(I \cap (\alpha, \alpha + \varepsilon))$$
$$f \in \mathcal{T}_{\mathcal{A}^-_1}(I) \cap \mathcal{T}_{\mathcal{A}^+_2}(I) \Rightarrow f \in \mathcal{T}_{A_0}(I \cap (\beta - \varepsilon, \beta)).$$

Since $f$ is exponential type, $f(A^+_1) - f(A^-_1) = f(A^+_2) - f(A^-_2)$ holds from Lemma 2.1. Lemma 2.2 ensured that $f \in \mathcal{T}_{A_0}(I)$. \qed
Definition 2.8. Suppose $A = (A, \Omega_{a,b})$ is a singular umbrae with index $(\alpha, \beta)$. $I$ is an open interval includes $[\beta, \alpha]$.

If there is a regular decomposition $(A^+, A^-)$ of $A$ such that $f \in \mathcal{T}_A(I)$, then we defined $f(A) := f(A^+) - f(A^-)$.

Here we are not going to give a complete definition of $f(A)$ for the singular umbrae $A$. The general form of Proposition 2.2 can be derived by Lemma 2.1 and Theorem 3.

Proposition 2.3. Suppose $A_1 = (A_1, \Omega_{a_1,b_1}), A_2 = (A_2, \Omega_{a_2,b_2})$ are regular umbrae with index $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$.

If $f(z_1, z_2)$ satisfied

1. For every $\Re z_2 \in I_2$ we have $f(\cdot, z_2) \in \mathcal{T}_{A_1}(I_1)$;
2. $f(A_1, \cdot) \in \mathcal{T}_{A_2}(I_2)$;
3. For every $\Re z_1 \in I_1$ we have $f(z_1, \cdot) \in \mathcal{T}_{A_2}(I_2)$;
4. $f(\cdot, A_2) \in \mathcal{T}_{A_1}(I_1)$;
5. There exists $t_1 \in I_1 \cap (\alpha_1, \beta_1), t_2 \in I_2 \cap (\alpha_2, \beta_2)$ such that $\hat{A}_1(z_1), \hat{A}_2(z_2) f(iz_1, iz_2)$ is Lebesgue integrable on $(\mathbb{R} - it_1) \times (\mathbb{R} - it_2)$,

then we have

$$\text{eval}_{A_2} \left( \text{eval}_{A_1}(f(z_1, z_2); z_1) z_2 \right) = \text{eval}_{A_1} \left( \text{eval}_{A_2}(f(z_1, z_2); z_2) z_1 \right),$$

(2.5)
i.e. $f(A_1, A_2)$ is well-defined.

Proof. Apply the Fubini theorem.

Proposition 2.4. Suppose $A_1 = (A_1, \Omega_{a,b}), A_2 = (A_2, \Omega_{a,b})$ are regular umbrae with index $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$.

If $g(z_1, z_2) = f(z_1 + z_2)$ satisfies the conditions of Proposition 2.3 and there is an open interval $I \subseteq I_1 + I_2$ such that $f \in \mathcal{T}_{A_1 + A_2}(I)$, then

$$f(A_1 + A_2) = g(A_1, A_2),$$
i.e. $f(A_1 + A_2)$ is well-defined.

Proof. We assume that $s \in (a, b)$. First notice that

$$g(A_1, A_2) = \frac{1}{2\pi} \int_{-\infty - it_1}^{\infty - it_1} \int_{-\infty - it_2}^{\infty - it_2} \hat{A}_1(z_1) \hat{A}_2(z_2) f(i z_1 + i z_2) d_2 d_1$$

$$= \frac{1}{2\pi} \int_{-\infty - i(t_1 + t_2)}^{\infty - i(t_1 + t_2)} \int_{-\infty - i t_2}^{\infty - i t_2} e^{s(z_0 - z_2)} \hat{A}_1(z_0 - z_2) e^{sz_2} \hat{A}_2(z_2) e^{-sz_0} f(i z_0) d_2 d_0$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty - i(t_1 + t_2)}^{\infty - i(t_1 + t_2)} e^{-sz_0} (e^{s(\cdot)} \hat{A}_1 + e^{s(\cdot)} \hat{A}_2)(z_0) f(i z_0) d_0,$$

and then notice that $f$ is an exponential type analytic function defined on $\{z \in \mathbb{C} : \Re z \in I_1 + I_2\}$, so the conclusion followed by Lemma 2.1.
Proposition 2.5. Suppose $A = (\mathcal{A}, \Omega, \alpha, \beta)$ is a regular umbrae with index $(\alpha, \beta)$, $r \in \mathbb{R}$.

If $g(z) = f(rz)$, then we have

$$ f \in T_{rA}(I) \leftrightarrow g \in T_{A}(r^{-1}I), $$

meanwhile $g(A) = f(rA)$ i.e. $f(rA)$ is well-defined.

Finally, we will be concerned about the well-definedness related to some special singular umbrae, such as $(c)$ and $D$. We hope they act like a constant and derivative.

Proposition 2.6. Suppose $A = (\mathcal{A}, \Omega, \alpha, \beta)$ is a regular umbrae with index $(\alpha, \beta)$, $c \in \mathbb{C}$.

If $g(z) = f(z + c)$, then we have

$$ f \in T_{A+(c)}(I) \leftrightarrow g \in T_{A}(I - c), $$

meanwhile $g(A) = f(A + (c))$ i.e. $f(A + c)$ is well-defined.

Unfortunately, the derivative calculus is not completely well-defined. However, understanding this counterexample can also increase our understanding of umbrae calculus. Consider $f(z) = 1, A = (z^{-1}e^{-2\mathbb{i}z}, \Omega_{0, \infty})$, we have $f \in T_{A+D}(\mathbb{R}), f' \in T_{A}(\mathbb{R})$, but $f(A + D) = 1, f'(A) = 0$.

Theorem 4. Suppose $A = (\mathcal{A}, \Omega, \alpha, \beta)$ is a regular umbrae with index $(\alpha, \beta)$.

If $\int_{-\infty}^{\infty} [\hat{A}(z)f(iz + z_0)]dz$ is uniformly bounded for sufficiently small $z_0$, and $f(+z_0) \in T_{A}(I), t \in I$, then we have

1. $f' \in T_{A}(I) \Rightarrow f'(A) = \frac{d}{dz_0}f(A + z_0)(0)$;
2. $f \in T_{A+D}(I) \Rightarrow f(A + D) = \frac{d}{dz_0}f(A + z_0)(0)$,

i.e. $f(A + D)$ is well-defined.

Proof. Suppose that $f' \in T_{A}(I)$. For

$$ g_{\varepsilon}(z_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\varepsilon z^2} \hat{A}(z)f(iz + z_0)dz, $$

we have

1. $\lim_{\varepsilon \to 0} g_{\varepsilon}(z_0) = f(A + z_0)$;
2. Sequence $(g_{\varepsilon})_{\varepsilon > 0}$ is uniformly bounded.

By the Montel’s theorem, there exists a subsequence $(g_{\varepsilon_n})_{n \in \mathbb{N}}$ locally uniformly converges to $f(A + z_0)$. $f'$ is again an exponential type analytic function, by the Leibniz integral rule we have

$$ g'_{\varepsilon}(z_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\varepsilon z^2} \hat{A}(z)f'(iz + z_0)dz. $$
Because \( g'(0) \to f'(A), g'_{\varepsilon_n}(0) \to \frac{d}{dz} f(A + z_0)(0) \).

Suppose that \( f \in T_{A+D}(I) \), and \( t + z_0 \in I \). Only need to notice that

\[
\sqrt{2\pi} \tilde{g}_\varepsilon(z_0) = \int_{-\infty}^{\infty} e^{-\varepsilon(z-z_0)^2} \hat{A}(z) f(iz + z_0) dz = \int_{-\infty}^{\infty} e^{-\varepsilon z^2} \hat{A}(z + iz_0) f(iz) dz = \int_{-\infty}^{\infty} e^{-\varepsilon z^2} \hat{A}(z + iz_0) f(iz) dz,
\]

and we still have

1. \( \lim_{\varepsilon \to 0} \tilde{g}_\varepsilon(z_0) = f(A + z_0) \);
2. Sequence \( (\tilde{g}_\varepsilon)_{\varepsilon > 0} \) is uniformly bounded.

By the Leibniz integral rule again, we have

\[
\tilde{g}'_\varepsilon(z_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\varepsilon z^2} i\hat{A}'(z + iz_0) f(iz) dz,
\]

this completes the proof.

This method seems to be able to replace \( D \) with an umbra \( e \) whose dominating interval is also \( \mathbb{R} \). In fact, Theorem 5 will strengthen the condition to prove more general cases.

### 2.3 Specific calculation

After establishing the required well-definedness, we can see how various conclusions emerge naturally.

**Theorem 5.** Suppose \( (f(iz), \Omega_{\beta, \alpha}) \) is a umbra with index \((b, a)\). If \( A_k = (A_k, \Omega_{\alpha_k, b_k}) \) is a umbra with index \((\alpha_k, \beta_k)\), satisfied

1. \( a_k < a; b < b_k \);
2. \( \beta_k - \alpha_k > - (\alpha - \beta) \),

then we have

1. \( (f(A_k + iz), \Omega_{\beta - \beta_k, \alpha - \alpha_k}) \) is a umbra with index at least \((b \vee a_k, a \wedge b_k)\);
2. If \((a_k, b_k) \cap (a_j, b_j) \neq \emptyset\), and \((\beta_k - a_k) + (\beta_j - a_j) > -(\alpha - \beta)\) then \( f(A_k + A_j + iz) \) is well-defined in the sense of Definition 2.4.
3. If \((a_k, b_k) \cap (a_j, b_j) \neq \emptyset\), and \((\beta - \beta_k, \alpha - \alpha_k) \cap (\beta - \beta_j, \alpha - \alpha_j) \neq \emptyset\) then \( f((A_k[+]A_j) + iz) \) is well-defined, satisfied

\[
f((A_k[+]A_j) + iz) = f(A_k + iz) + f(A_j + iz).
\]
Proof. First assume that $\alpha_k < \beta_k$, and take $a_k < \tilde{a}_k < \tilde{a} < a, b < \tilde{b} < \tilde{b}_k < b$; we have

$$|\tilde{A}(z)| \leq \begin{cases} C_1(-\text{Im} z) \cdot e^{-\tilde{b}_k \text{Re} z} & \text{Re} z \geq 0, \alpha_k < -\text{Im} z < \beta_k \smallskip \cr C_1(-\text{Im} z) \cdot e^{-\tilde{a}_k \text{Re} z} & \text{Re} z \leq 0, \alpha_k < -\text{Im} z < \beta_k \end{cases}$$

$$|f(iz)| \leq \begin{cases} C_2(-\text{Im} z) \cdot e^{\delta \text{Re} z} & \text{Re} z \geq 0, \beta < -\text{Im} z < \alpha \smallskip \cr C_2(-\text{Im} z) \cdot e^{\alpha \text{Re} z} & \text{Re} z \leq 0, \beta < -\text{Im} z < \alpha \end{cases}$$

So we can directly verify that, for $\text{Re} z_0 \in (\beta - \beta_k, \alpha - \alpha_k)$ there exists $t$ such that

$$|f(A + z_0)| \leq \int_{-\infty - it}^{\infty - it} |\tilde{A}(z)| \cdot |f(iz + z_0)|dz \leq e^{\delta \text{Im} z_0} \int_0^{\infty} e^{-(b_k-b)z}dz + e^{\delta \text{Im} z_0} \int_0^{0} e^{(\tilde{b}_k-\tilde{a}_k)z}dz + e^{\alpha \text{Im} z_0} \int_{-\infty}^{\text{Im} z_0} e^{(\tilde{a}_k-\tilde{a})z}dz \leq e^{\delta \text{Im} z_0} \cdot \frac{e^{\delta \text{Im} z_0} - e^{\alpha \text{Im} z_0}}{\tilde{b}_k - \tilde{b}} + e^{\alpha \text{Im} z_0} \cdot \frac{\tilde{b} - \tilde{a}_k}{\tilde{a} - \tilde{a}_k} \text{ Im} z_0 \geq 0.$$

Repeat the same method to estimate the other half and we can conclude that $(f(A + iz), \Omega_{\beta - \beta_k, \alpha - \alpha_k})$ is a umbrae with index $(b \lor a_k, a \land b_k)$. Finally, apply Theorem 3 and Definition 2.8 to $A$.

The condition of the umbrae under Theorem 3 seems to be the most natural, but not enough for our purposes. We can also notice that the performance of the umbrae is more determined by its dominating interval.

In order to be able to perform basic calculations, we need the following theorem.

**Lemma 2.3.** If $(f(iz), \Omega_{\beta, \alpha})$ is a regular umbrae with index $(b, a)$ satisfied $0 \in (b, a)$, then for every $s \in (\beta, \alpha)$ we have $f(s + i\xi) \in \mathcal{S}(\mathbb{R})$.

**Proof.** Apply Theorem 3.

**Lemma 2.4.** Suppose $A = (A, \Omega_{\alpha, b})$ is a regular umbrae with index $(\alpha, \beta)$, and

1. $-\infty < \alpha < \beta$;
2. For every $t \in (a, b)$ we have $e^{-\alpha x}A(x - it) \in \mathcal{S}'(\mathbb{R})$;
3. Denote $h(\xi) := e^{-\xi}f(e^{-\alpha(x - it)}A(x - it))$ in the sense of $\mathcal{S}'(\mathbb{R})$.

If $(f(iz), \Omega_{\beta_0, \alpha_0})$ is a umbrae with index $(b_0, a_0)$ satisfied

$$a < a_0, \quad b_0 < b,$$

then for every $s \in (\beta_0, \alpha_0)$ we have

$$f(A + \alpha - s) = \int_\mathbb{R} e^{i\xi} h(\xi) e^{-i\xi} f(s + i\xi) d\xi.$$
Proof. According to the continuity of Fourier transform on \( S'(\mathbb{R}) \), for every \( t \in (a, b) \) we have
\[
\lim_{\varepsilon \to 0^+} e^{i\varepsilon A}(\xi - i(\alpha + \varepsilon)) = e^{i\varepsilon h}(\xi),
\]
in the sense of \( S'(\mathbb{R}) \).

Apply Theorem 3 on \( f \) to get

1. \( (f^+(iz), \Omega_{b_0, a_0}) \) is a regular umbrae with index \( (b_0, \infty) \);
2. \( (f^-(iz), \Omega_{b_0, a_0}) \) is a regular umbrae with index \( (-\infty, a_0) \);
3. \( f = f^+ - f^- \),

according to Lemma [2.3] for every \( s \in (\beta_0, \alpha_0) \) we have
\[
\int_{\mathbb{R}} A(\xi - i(\alpha + \varepsilon))f^+(s + i\xi)d\xi = \int_{\mathbb{R}} e^{i\xi h}(\xi)e^{-\xi f^+(s + i\xi)}d\xi
\]
\[
\rightarrow \int_{\mathbb{R}} e^{i\xi h}(\xi)e^{-\xi f^+(s + i\xi)}d\xi
\]
\[
= \int_{\mathbb{R}} e^{i\xi h}(\xi)e^{-\xi f^+(s + i\xi)}d\xi;
\]

Repeat the same method to the other half and we can conclude that
\[
\int_{\mathbb{R}} A(\xi - i(\alpha + \varepsilon))f(s + i\xi)d\xi \rightarrow \int_{\mathbb{R}} e^{i\xi h}(\xi)e^{-\xi f(s + i\xi)}d\xi
\]
\[
f(A + (\alpha + s)) \rightarrow f(A + \alpha - s),
\]
where \( a < a_0, b_0 < b, a < t < b \). The continuity of \( f(A + z) \) comes from Theorem 5. \( \square \)

**Theorem 6.** Suppose \( A = (A, \Omega_{\alpha, \beta}) \) is a umbrae with index \( (\alpha, \beta) \), and

1. \( -\infty < \alpha \leq \beta \);
2. For every \( t \in (a, b) \) we have \( e^{-\alpha x}A(x - it) \in S'(\mathbb{R}) \);
3. Denote \( h(\xi) := e^{-\xi F(e^{-\alpha(x-it)}A(x-it))} \) in the sense of \( S'(\mathbb{R}) \),

If \( (f(iz), \Omega_{b_0, a_0}) \) is a umbrae with index \( (b_0, a_0) \) satisfied

1. \( a < a_0; b_0 < b \);
2. \( \alpha \in (\beta_0, \alpha_0) \),

then we have
\[
f(A) = \int_{\mathbb{R}} e^{i\xi h}(\xi)e^{-\xi f(\alpha + i\xi)}d\xi. \quad (2.8)
\]

Proof. For the singular umbrae \( A \), only need to apply Theorem 3. \( \square \)
Corollary 2.1. Suppose $f$ is an exponential type analytic function defined on \( \{ z \in \mathbb{C} : \Re z \in (\beta, \alpha) \} \).

1. \( f((c) + z) = f(c + z) \);
2. \( f(D + z) = f'(z) \);
3. \( f(\Delta + z) = f(z + 1) - f(z) \), where \( \beta - \alpha > 1 \).

Proposition 2.7. Suppose \( A = (\mathcal{A}, \Omega_{a,b}) \) is a umbrae with index \((\alpha, \beta)\).

1. \( e^{Az} = \mathcal{A}(z) \quad z \in \Omega_{a,b} \);
2. \( A^n = \mathcal{A}^{(n)}(0) \quad 0 \in (a, b), n \in \mathbb{N} \);
3. \((A^z, \Omega_{-\infty, \infty})\) is a regular umbrae with index \((-\pi, \pi)\), where \(0 \in (a, b), \beta > 0\).

Proof. For the singular umbrae \( A \), only need to apply Theorem \( \square \).

Just like Carlson’s theorem shows, the function class formed by such \( A^z \) is also determined by the value on natural numbers.

2.4 Limit calculus

Next, we need several important lemmas to help us calculate \( f(A) \) specifically.

Lemma 2.5. Suppose \( f \) is locally integrable, \( b > 0, n \geq 0 \).

If \( e^{-bx}(1 + |x|)^n f(x) \in L^1(0, \infty) \), then \( e^{-bx}(1 + |x|)^n f(t) dt \in L^1(0, \infty) \).

Proof. Only need to notice that

\[
\begin{align*}
\int_0^x |f(t)| dt & \leq \int_0^x e^{-bt}(1 + t)^n |f(t)|(1 + x - t)^n e^{-b(x-t)} dt \\
& = (e^{-b(\cdot)}(1 + \cdot)^n f(\cdot) * e^{-b(\cdot)}(1 + \cdot)^n)(x).
\end{align*}
\]

The result is obtained from Young’s inequality. \( \square \)

The umbrae that satisfy the following properties are good enough to give a more accurate estimate for \( \hat{A} \), and Bernoulli umbrae \( B \) is such a umbrae. Ultimately, it will allow us to do calculus \( f(A) \) on as many functions as possible. In fact, if there is no such fine result, we will not be able to process the Gosper series as we promised.

Lemma 2.6 (Good umbrae). Suppose \( A = (\mathcal{A}, \Omega_{a,b}) \) is a regular umbrae with index \((\alpha, \beta)\).

1. If \( b < \infty \) and \( (\mathcal{A}(z) \prod_{j=1}^m (z - b_j)^{l_j}, \Omega_{a,b+\varepsilon}) \) is a umbrae, then we have

\[
|\hat{A}(\xi + \iota t)| \leq C(t)e^{-b\xi}(1 + |\xi|)^{l-1} \quad \xi \geq 0,
\]

where \( l = \max\{l_1, \ldots, l_m\} \), \(-\Im b_j = b\);
2. If $a > -\infty$ and $(A(z) \prod_{j=1}^{n} (z - a_j)^{k_j}, \Omega_{a,\epsilon,b})$ is a umbrae, then we have
\[
|\hat{A}(\xi - it)| \leq C(t)e^{-\alpha \xi(1 + |\xi|)^{k-1}} \quad \xi \leq 0, \tag{2.10}
\]
where $k = \max\{k_1, \ldots, k_n\}, -\Im a_j = a$.

Proof. Consider the Fourier transform of the singular part of $\hat{A}$ at the pole. It seems that this lemma can be generalized further, but the current form is sufficient for us.

Theorem 7 (Limit calculus). Suppose $A$ is a regular umbrae, $(f_n)_{n \in \mathbb{N}}$ is a sequence in $T_A(I)$, and $f \in T_A(I)$, $t \in I$.

If $\hat{A}(z)f_n(iz)$ tends to $\hat{A}(z)f(iz)$ in the sense of $L^1(\mathbb{R} - it)$, then
\[
\lim_{n \to \infty} f_n(A) = f(A). \tag{2.11}
\]

Corollary 2.2 (Interchangeability). Suppose $A = (A, \Omega_{a,b})$ is a umbrae with index $(\alpha, \beta)$, and satisfy $0 \in (a, b)$.

For $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}$ we have
1. If there exists $r < (-a) \vee b$ such that
\[
\sum_{n=0}^{\infty} |a_n|r^{-n} < \infty, \tag{2.12}
\]
then $f(A) = \sum_{n=0}^{\infty} \frac{a-n^n}{n!}$;
2. If $A$ satisfied one of the conditions in Lemma 2.6 and $f$ satisfies
\[
\sum_{n=0}^{\infty} |b|^{-n}n^{l-1}|a_n| < \infty
\]
\[
\sum_{n=0}^{\infty} |a|^{-n}n^{k-1}|a_n| < \infty,
\]
then $f(A) = \sum_{n=0}^{\infty} \frac{a-n^n}{n!}$.

Proof. For the singular umbrae $A$, only need to apply Theorem 3.

This corollary will eventually provide us the interchangeable results which we care about.

Definition 2.9 (Hierarchy). If $f$ is defined on $\{z \in \mathbb{C} : \Re z > 0\}$ and satisfies the following properties
1. For $\Re t > 0$ we have
\[
\lim_{\Re t \to \infty} \int_{-\infty}^{\infty} |f(t + i\xi)|e^{-\beta \xi(1 + |\xi|)}d\xi = 0,
\]
locally uniformly with respect to $\Im t$.
2. For \( \text{Re} t > 0 \) we have

\[
\lim_{\text{Re} t \to \infty} \int_{-\infty}^{\infty} \left| f(t + i\xi) e^{-a\xi(1 + |\xi|)^k} \right| e^{-a\xi(1 + |\xi|)^k} d\xi = 0,
\]

locally uniformly with respect to \( \text{Im} t \);

3. \( |f(z)| \leq C(\text{Re} z) \cdot e^{\delta |\text{Im} z|} \), where \( C \) is locally bounded,

then the space formed by such functions is denoted by \( T_{a,k,l}^{(p-1)} \).

In fact, we only need \( t \) to go to infinity along the natural number.

**Theorem 8 (Hierarchy).** If \( 0 \in (a, b) \) and \( k, l \geq 0 \), then

\[
f(z) \in T_{a,k,l}^{(p-1)} \Rightarrow \int_1^x f(t) dt \in T_{a,k,l}^{(p)}.
\]

**Proof.** Denote \( F(z) = \int_1^x f(t) dt \). Notice that

\[
\left\| e^{-b\xi (1+|\xi|)^l} \left( F(t + i\xi) - \sum_{k=0}^{p-1} \frac{f^{(k)}(t)}{k!} (i\xi)^k \right) \right\|_{L^1(0, \infty)} =
\]

\[
\left\| e^{-b\xi (1+|\xi|)^l} \int_0^\xi \left( f(t + i\xi_0) - \sum_{k=0}^{p-1} \frac{f^{(k)}(t)}{k!} (i\xi_0)^k \right) d\xi_0 \right\|_{L^1(0, \infty)} \leq
\]

\[
\left\| e^{-b\xi (1+|\xi|)^l} \right\|_{L^1(0, \infty)} \cdot \left\| e^{-b\xi (1+|\xi|)^l} \left( f(t + i\xi) - \sum_{k=0}^{p-1} \frac{f^{(k)}(t)}{k!} (i\xi)^k \right) \right\|_{L^1(0, \infty)} \to 0,
\]

Repeat the same method to estimate the other half and we can conclude that (8) holds.

Similar to Theorem 6 this theorem will help us calculate \( f(B) \) in a more direct way.

**3 Application**

Let’s first redeem the promise and see how the MS-fractional summation related to the Bernoulli umbrae.

Although the selection of \( \Omega_{a,b} \) does not need to include the origin, here we still only care about the Bernoulli umbrae that includes the origin, which allows us to define \( B^n \). Some conclusions can be generalized to other Bernoulli umbras.

**Lemma 3.1.** \( B + \Delta = 1 + D \).

**Theorem 9.** If there exists \( p \in \mathbb{N} \) such that
1. $f \in T_{-2\pi,0;2\pi,0}^{(p-1)}$;

2. \( \lim_{t \to \infty} f^{(p)}(t + i\xi) = 0 \) locally uniformly,

then for every $\Re z > -1$ we have

\[
\text{(MS)} - \sum_{k=1}^{\infty} f(k) = \int_{B} f(k) dk.
\] (3.1)

**Proof.** Denote $F(z) = \int_{1}^{z} f(t) dt$. Notice that the Bernoulli umbrae $B$ satisfies the condition of Lemma 2.6, therefore according to Definition 2.7 and 2.9, $F(t + B)$ is defined for $\Re t > 0$.

By Theorem 8, we have

\[
\lim_{n \to \infty} \left( F(n+B) - \sum_{k=0}^{p} \frac{f^{(k-1)}(n)B_{k}}{k!} \right) = 0.
\]

By the Lagrangian remainder theorem, we have

\[
\sum_{k=0}^{p} \frac{(f^{(k-1)}(t+y) - f^{(k-1)}(t))B_k}{k!} = \sum_{k=0}^{p} a_k(t)y^k + o(1)y^{p+1},
\]

therefore we have polynomial $P_n(y) = \sum_{k=0}^{p} a_k(n)y^k$ such that

\[
\lim_{n \to \infty} \left( \sum_{k=1}^{n} (f(k+y) - f(k)) - P_n(y) \right) = 0.
\]

Finally, by Definition 2.8 and Corollary 2.1, we have

\[
B + \Delta = 1 + D \Rightarrow (B + 1)[-B] = 1 + D
\]

\[
\Rightarrow F(y + ((B + 1)[-B])) = F(y + 1 + D)
\]

\[
\Rightarrow F(y + B + 1) - F(t + B) = F'(y + 1),
\]

which satisfies the conditions in Definition 1.1. \qed

**Corollary 3.1.** If

\[
f(z) = \sum_{m=1}^{M} b_m z^{-m} + \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n
\]

and there exists $p \in \mathbb{N}$ satisfies that

1. $f \in T_{-2\pi,0;2\pi,0}^{(p-1)}$;
2. \( \lim_{t \to \infty} f^{(p)}(t + i\xi) = 0 \) locally uniformly;

3. \( \sum_{n=0}^{\infty} (2\pi)^{-n} |a_n| < \infty, \)

then for every \( \text{Re} \alpha, \text{Re} \beta > -1 \) we have

\[
(\text{MS}) \cdot \sum_{k=\alpha+1}^{\beta} f(k) = \sum_{m=1}^{M} \left( \text{MS} \cdot \sum_{k=\alpha+1}^{\beta} b_m z^{-m} \right) + \sum_{n=0}^{\infty} \left( \text{MS} \cdot \sum_{k=\alpha+1}^{\beta} \frac{a_n}{n!} z^n \right).
\]

(3.2)

**Proof.** Apply Corollary 2.2 and Theorem 9.

**Corollary 3.2.** If there exists \( p \in \mathbb{N} \) satisfies that

1. \( f' \in T_{-2\pi,0:2\pi,0}^{(p-1)} \);

2. \( \lim_{t \to \infty} f^{(p+1)}(t + i\xi) = 0 \) locally uniformly;

3. \( \int_{-\infty}^{\infty} e^{-2\pi|\xi|} |f(t + i\xi)| d\xi \) is locally bounded for \( \text{Re} t > 0 \),

then we have

\[
(\text{MS}) \cdot \frac{d}{dz} \sum_{k=1}^{z} f(k) = f(B) + (\text{MS}) \cdot \sum_{k=1}^{z} \frac{d}{dk} f(k).
\]

**Proof.** Apply Theorem 4 and Lemma 2.5.

In fact, the condition of \( f \) in Corollary 3.1 also satisfies the third condition in Corollary 3.2.

At this point, we justify the speculation from Müller and Schleicher. Next, we will see that these conditions are sufficient to cover the special series they care about.

Finally, it needs to be emphasized that the processing of Bernoulli umbrae is essentially equivalent to applying the Abel-Plana formula. Although we have removed the decay condition of the function when the imaginary part is large, this is actually done by requiring the function to be exponential growth.

### 3.1 Gosper series

Let's consider a sufficiently general example.

Suppose

\[
J(z) = \sum_{n=0}^{\infty} \frac{a_{2n}}{(2n)!} z^{2n}
\]

satisfies that

1. \( |a_n| \leq C \cdot (1 + n)^{\nu_n}; \)
2. $|J(z)| \leq C \cdot e^{|\text{Im}z|} (1 + |z|)^{\nu_J}$  \quad \text{Re} z > 0.$

**Lemma 3.2.**

1. $|\text{Im} \sqrt{z_1 + z_2}| \leq |\text{Im} \sqrt{z_1}| + |\text{Im} \sqrt{z_2}|$;

2. $|\tilde{J}(z)| \leq C \cdot e^{|z|} (1 + |z|)^{\nu_a}$, where $\tilde{J}(z) = \sum_{n=0}^{\infty} \frac{|a_{2n}|}{(2n)!} z^{2n}$.

**Proposition 3.1.** $z^{-n} J(\sqrt{(2\pi z)^2 + b^2})$ satisfies all the conditions of Corollary 3.1, where $n > \max\{\nu_a, \nu_J\} + 1$.

**Proof.** Suppose that

$$z^{-n} J(\sqrt{z^2 + b^2}) = \sum_{m=1}^{M} b_m z^{-m} + \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n,$$

only need to notice that

$$\int_{1}^{\infty} x^{-n} \tilde{J}(\sqrt{x^2 + |b|^2}) e^{-x} dx < \infty \Rightarrow \sum_{n=0}^{\infty} |c_n| < \infty,$$

and for $n > \nu_a + 1$ we have

$$\int_{1}^{\infty} x^{-n} \tilde{J}(\sqrt{x^2 + |b|^2}) e^{-x} dx \leq C \cdot \int_{1}^{\infty} x^{\nu_a - n} e^{\sqrt{x^2 + |b|^2} - x} dx < \infty.$$

When the real part of $z$ is large enough, we have

$$e^{-|\text{Im} z|} |z^{-n} J(\sqrt{z^2 + b^2})| \leq C \cdot e^{|\text{Im} \sqrt{z^2 + b^2}| - |\text{Im} z|} \nu_{\tilde{J}} - n \leq C \cdot |z|^{\nu_{\tilde{J}} - n},$$

and for $n > \nu_{\tilde{J}} + 1$ we have $z^{-n} J(\sqrt{z^2 + b^2}) \in T_{-1}^{(-1)}_{\nu_{\tilde{J}}, 1.0, 1.0}$.

For example, we can take

$$J(z) = z^{-\nu} J_\nu(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2^{\nu + 2n} n!(n + \nu)!}.  \quad (3.4)$$

From the Poisson integral representation, for $\text{Re} \nu > -\frac{1}{2}$ we have

$$J(z) = \frac{1}{2^{\nu}(\nu - \frac{1}{2})! \sqrt{\pi}} \int_{0}^{\pi} e^{iz \cos \theta} \sin^{2\nu} \theta d\theta$$

$$|J(z)| \leq C \cdot e^{|\text{Im} z|},$$

therefore $\nu_a = -(\text{Re} \nu + \frac{1}{2}), \nu_{\tilde{J}} = 0$. In particular, when $\nu = 1/2$ we have $J(z) = \sqrt{\frac{2}{\pi \sin z}}, n > 0$, when $\nu = -1/2$ we have $J(z) = \sqrt{\frac{2}{\pi \cos z}}, n > 1$.

We can improve the related results in [5], that is, generalize $\tilde{b}$ to every complex number.
Corollary 3.3 (Gosper series). If \( J(z) = \sum_{n=0}^{\infty} \frac{a_n z^{2n}}{(2n)!} \) satisfies that

1. \(|a_n| \leq C \cdot (1 + n)^{\nu_a}\);
2. \(|J(z)| \leq C \cdot e^{\text{Im} z} (1 + |z|)^{\nu_J} \quad \text{Re} z > 0, \)

and \( b \in \mathbb{C} \), then

1. For \( \nu_a, \nu_J < 0 \) we have

\[
\text{(MS)} \cdot \sum_{n=1/4}^{-1/4} \frac{J(\sqrt{b^2 + (2\pi n)^2})}{n} = \pi J(b) \quad (3.5)
\]

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n + \frac{1}{2}} J(\sqrt{b^2 + \pi^2 (n + 1/2)^2}) = \frac{\pi}{2} J(b); \quad (3.6)
\]

2. For \( \nu_a, \nu_J < 1 \) we have

\[
\text{(MS)} \cdot \sum_{n=1}^{-1/2} \frac{J(\sqrt{b^2 + (2\pi n)^2})}{n^2} = -\frac{\pi^2 J(b)}{3} - \frac{\pi^2 J'(b)}{b} \quad (3.7)
\]

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2} J(\sqrt{b^2 + \pi^2 n^2}) = -\frac{\pi^2 J(b)}{12} - \frac{\pi^2 J'(b)}{4b}. \quad (3.8)
\]

Proof. By Proposition 3.1, we can apply Corollary 3.1 on \( z^{-n} J(\sqrt{(2\pi z)^2 + b^2}) \).

By applying \(|f(a) - f(b)| \leq M|a - b|\), we are able to generalize further to the situation where \( \sqrt{z^2 + b^2} \) is replaced by \( \sqrt{z^2 + p(z)} \).

The common processing on \( \sum_{n \in \mathbb{Z}} (-1)^n f(n) \) or \( \sum_{n \in \mathbb{Z}} f(n) \) is to apply the Abel-Plana formula, which is essentially equivalent to the processing here.

3.2 Bernoulli umbrae

This subsection will mainly introduce how to use Bernoulli umbrae B to synthesize some common analysis results. This approach can also be easily extended to similar umbrae such as Euler umbrae E.

The proof of the following theorem can actually be given directly by its corollaries, so it is omitted here.

Theorem 10 (Bernoulli umbrae).

1. \( \ln B = -\gamma \);
2. \( B \ln B = \ln \sqrt{\frac{2\pi}{e}} \);
3. \( B^2 \ln B = \frac{1}{4} - 2 \ln A, \) where A is Glaisher-Kinkelin constant;
4. \( B^z = -z \zeta(1 - z) \).

**Lemma 3.3** (Multiplication theorem).

\[
nB[+](nB - 1)[+] \cdots [+](nB - n + 1) = [n] + B.
\]

In fact, this lemma synthesized all the multiplication theorems about special functions.

**Lemma 3.4.** \( \left( \frac{d}{dz} \right)^n B^z = B^z \ln^n B \).

**Proof.** Apply the argument similar to Theorem 4, note that

\[
\int_{-\infty}^{\infty} |\hat{A}(z)| \cdot |(iz)^{z_0} \ln^n(iz)| dz
\]

is locally uniformly bounded for \( z_0 \).

The previous definitions and conclusions such as Theorem 4, 5, 7 have directly justified the validity of each step of the following proofs.

**Corollary 3.4** (\( \ln B \)).

1. \( \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{n} - \ln n \right) = \gamma; \)
2. \( \zeta(1 + s) = \frac{1}{s} - \gamma + o(1); \)
3. \( (x!)'(0) = -\gamma. \)

**Proof.**

\[
\sum_{k=1}^{n} \frac{1}{n} - \ln n = \ln(B + n) + \gamma - \ln n
\]

\[
= \gamma + \ln(1 + \frac{B}{n}) \to \gamma
\]

\[
\zeta(1 + s) = \frac{B^{-s}}{s} = \frac{1 - s \ln B + o(s)}{s}
\]

\[
= \frac{1}{s} - \gamma + o(1)
\]

\[
(\frac{x!}{x})' = \ln(B + x)
\]

\[
(\frac{x!}{x})'(0) = 0! \ln B = -\gamma.
\]

**Corollary 3.5** (\( B \ln B \)).

1. \( n! \sim (\frac{4}{\pi})^n \sqrt{2\pi n}; \)
2. \( \zeta(s) = -\frac{1}{2} - s \ln \sqrt{2\pi} + o(s); \)
3. \((-\frac{1}{2})! = -\frac{\sqrt{\pi}}{2}\).

Proof.

\[
\ln n! = (B + n) \ln(B + n) - n - \ln \sqrt{\frac{e}{2\pi}}
\]
\[
= (B + n) \ln n + (B + n) \ln(1 + \frac{B}{n}) - n - \ln \sqrt{\frac{e}{2\pi}}
\]
\[
= \left(\frac{1}{2} + n\right) \ln n + B^1 + O\left(\frac{1}{n}\right) - n - \ln \sqrt{\frac{e}{2\pi}}
\]
\[
= \left(\frac{1}{2} + n\right) \ln n - n + \left(\frac{1}{2} - \ln \sqrt{\frac{e}{2\pi}}\right) + O\left(\frac{1}{n}\right)
\]
\[
\zeta(s) = \frac{B^{-s} + 1}{s - 1} = -B^1 + s(B^1 - B \ln B) + o(s)
\]
\[
= -\frac{1}{2} - s \ln \sqrt{2\pi} + o(s)
\]
\[
\ln\left(-\frac{1}{2}\right)! = \frac{2B - 1}{2} \ln \frac{2B - 1}{2} - \frac{1}{2} - B \ln B
\]
(4.1)

(by Theorem 5, Lemma 3.3) \((2 \cdot \frac{B}{2} \ln \frac{B}{2} - B \ln B) - \frac{1}{2} - B \ln B
\]
\[
= -B \ln 2 - \frac{1}{2} - B \ln B
\]
\[
= -\frac{1}{2} \ln 2 - \frac{1}{2} - \ln \sqrt{\frac{e}{2\pi}} = \ln \sqrt{\pi}.
\]

4 Some possible developments

If we only apply the Bernoulli umbrae, then what we are actually doing is using that notation to simplify the application of the Abel-Plana formula. So the key points seem to be on the calculus between these umbrae.

For \(A_i = (A_i(z), \Omega_{a,b})\), we can define

\[
A_1 \times A_2 := (e^{\frac{1}{2} \ln \frac{A_1(x)}{A_1(0)} \ln \frac{A_2(x)}{A_2(0)}}, \Omega_{a,b})
\]

where \(A_1\) and \(A_2\) are both invertible respect to the addition. It is again a umbrae, and gives a rare three-level arithmetic, i.e. “\(\times\)” distributes over “\(+\)”, and “\(+\)” distributes over “\(\times\)”.  

Let \(\mu = (\mu_n)_{n \in \mathbb{N}^+}\) be a given sequence. Suppose \(\sum_{n=1}^{\infty} \mu_n z^n\) converges near the origin, such that \(A(z) = z^m \sum_{n=1}^{\infty} \mu_n e^{nz}\) can be continuation to \(\Omega_{a,b}\). Denote

\[
(m \times D) + S(\mu) := (A(z), \Omega_{a,b}).
\]

(4.1)
We can expect that for a suitable $f$, such as $f \in \mathcal{T}_{a,k,b,l}^{(p-1)}$, we have

$$\sum_{n=1}^{\infty} \mu_n f(n) = f(-m)(S(\mu)).$$  \hfill (4.2)

This type of umbrae allows us to promote our previous treatment on Bernoulli umbrae further. For example,

1. $(m \times D) + S(\mu) = B$, where $m = 1$ and $\mu_n = -1$;
2. $(m \times D) + S(\mu) = E$, where $m = 0$ and $\mu_n = 2\chi_1(n)$ where $\chi_1$ is the non-trivial Dirichlet character of modulo 4.

Unfortunately, not like the Dirichlet character, the sequence $\mu$ seems cannot be replaced with a number theory sequence that is too irregular. For example, $\sum_{n=1}^{\infty} z^n n$ is full of singularities on the entire unit circle.

Through the symmetry of the Fourier transformation, we are able to calculate $A^{-n}$ more directly, which is essentially the Ramanujan’s master theorem, i.e.

$$\int_0^{\infty} e^{-Ax} x^n dx = n! A^{-n-1}. \hfill (4.3)$$

It seems there is a chance to achieve more similar results.

We can use Bernoulli umbrae to deal with some series in $\mathbb{P}$. Although the process is relatively complicated, the approach is very straightforward.

**Lemma 4.1.** For suitable $f$, we have

$$\sum_{n=1}^{\infty} \left( n \sum_{k=1}^{n} f(k) \right) = -f^{-3}(B + B + z) + z f^{-2}(B + B + z) + f^{-2}(B + \tilde{B} + z) - \frac{z^2 + z}{2} f^{-1}(B) - f^{-2}(B + \tilde{B}) + f^{-3}(B + B),$$

where $\tilde{B}^n = B_{n+1}$, and the two Bs in $B + B$ should be understood as different umbraes.

**Lemma 4.2.** Suppose $\tilde{B}^n = B_{n+1}, \tilde{B}^n = B_{n+2}$.

1. For suitable $f$, we have $f(\tilde{B}) = B f(B), f(\tilde{B}) = B^2 f(B)$;
2. $B + B = B[+] (B + D)[-](B + D)$;
3. $B + \tilde{B} = (B[+] (B + D)[-](\tilde{B} + D)) + \left\lfloor \frac{1}{2} \right\rfloor$.

Finally, we can also see that if we allow the umbrae to be written as the difference between two analytic functions with disjoint domains, like the Fourier transform of the function $f$, we can eventually cover more general integral transforms.
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