Global action-angle coordinates for completely integrable systems with noncompact invariant submanifolds

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The obstruction to the existence of global action-angle coordinates of Abelian and noncommutative (non-Abelian) completely integrable systems with compact invariant submanifolds has been studied. We extend this analysis to the case of noncompact invariant submanifolds.

I. INTRODUCTION

We consider Abelian and noncommutative (non-Abelian) completely integrable Hamiltonian systems (henceforth CISs) on symplectic manifolds. The Liouville–Arnold (or Liouville–Mineur–Arnold) theorem for Abelian CISs\(^1\)–\(^4\) and the Mishchenko–Fomenko theorem for noncommutative ones\(^5\)–\(^8\) state the existence of action-angle coordinates around a compact invariant submanifold of a CIS. These theorems have been extended to the case of noncompact invariant submanifolds\(^9\)–\(^12\). In particular, this is the case of time-dependent CISs.\(^13\)–\(^14\) Any time-dependent CIS of \(m\) degrees of freedom can be represented as the autonomous one of \(m+1\) degrees of freedom on a homogeneous momentum phase space, where time is a generalized angle coordinate. Therefore, we further consider only autonomous CISs.

If invariant submanifolds of a CIS are compact, a topological obstruction to the existence of global action-angle coordinates has been analyzed.\(^8,15,16\) Here, we aim extending this analysis to the case of noncompact invariant submanifolds (Theorems 3, 4 and 5).

Throughout the paper, all functions and maps are smooth, and symplectic manifolds are real smooth and paracompact. We are not concerned with the real-analytic case because a paracompact real-analytic manifold admits the partition of unity by smooth, not analytic functions. As a consequence, sheaves of modules over real-analytic functions need not be acyclic that is essential for our consideration.

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Definition 1: Let $(Z,\Omega)$ be a $2n$-dimensional connected symplectic manifold, and let $(C^\infty(Z),\{\cdot,\cdot\})$ be the Poisson algebra of smooth real functions on $Z$. A subset $H = (H_1,\ldots,H_k)$, $n \leq k < 2n$, of $C^\infty(Z)$ is called a (noncommutative) CIS if the following conditions hold.

(i) All the functions $H_i$ are independent, i.e., the $k$-form $\wedge^k dH_i$ nowhere vanishes on $Z$. It follows that the map $H : Z \to \mathbb{R}^k$ is a submersion, i.e.,

$$H : Z \to N = H(Z)$$

is a fibered manifold over a connected open subset $N \subset \mathbb{R}^k$.

(ii) There exist smooth real functions $s_{ij}$ on $N$ such that

$$\{H_i, H_j\} = s_{ij} \circ H, \quad i,j = 1,\ldots,k.$$  \hspace{1cm} (2)

(iii) The matrix function $s$ with the entries $s_{ij}$ (2) is of constant corank $m = 2n - k$ at all points of $N$.

In Hamiltonian mechanics, one can think of the functions $H_i$ as being integrals of motion of a CIS which are in involution with its Hamiltonian. Their level surfaces (fibers of $H$) are invariant submanifolds of a CIS.

If $k = n$, then $s = 0$, and we are in the case of an Abelian CIS. If $k > n$, the matrix $s$ is necessarily non-zero, and a CIS is said to be noncommutative.

Note that, in many physical models, the condition (i) of Definition 1 fails to hold. In a general setting, one supposes that the subset $Z_R \subset Z$ of regular points, where $\wedge^k dH_i \neq 0$, is open and dense. Then one considers a CIS on this subset. However, a CIS on $Z_R$ fail to be equivalent to the original one because there is no morphism of Poisson algebras $C^\infty(Z_R) \to C^\infty(Z)$. In particular, canonical quantization of the Poisson algebra $C^\infty(Z_R)$, e.g., with respect to action-angle variables essentially differs from that of $C^\infty(Z)$.$^{17-19}$ For instance, let $M$ be a connected compact invariant manifold of an Abelian CIS through a regular point $z \in Z_R \subset Z$. There exists its open saturated neighborhood $U_M \subset Z_R$ (i.e., a fiber of $H$ through a point of $U_M$ belongs to $U_M$) which is a trivial fiber bundle in tori. By virtue of the above mentioned Liouville–Arnold theorem, $U_M$ is provided with the Darboux action-angle coordinates. Then one treats quantization of the Poisson algebra $C^\infty(U_M)$ with respect to these coordinates as quantization 'around' an invariant submanifold $M$.

Given a CIS in accordance with Definition 1, the above mentioned generalization of the Mishchenko–Fomenko theorem to noncompact invariant submanifolds states the following.$^{12}$

Theorem 1: Let the Hamiltonian vector fields $\vartheta_i$ of the functions $H_i$ be complete, and let the fibers of the fibered manifold $H$ (1) be connected and mutually diffeomorphic. Then the following hold.
(I) The fibers of $H (1)$ are diffeomorphic to a toroidal cylinder

$$\mathbb{R}^{m-r} \times T^r.$$  \hspace{1cm} (3)

(II) Given a fiber $M$ of $H (1)$, there exists an open saturated neighborhood $U_M$ of it which is a trivial principal bundle with the structure group (3).

(III) The neighborhood $U_M$ is provided with the bundle (generalized action-angle) coordinates $(I_\lambda, p_A, q^A, y^\lambda)$, $\lambda = 1, \ldots, m$, $A = 1, \ldots, n - m$, where $(y^\lambda)$ are coordinates on a toroidal cylinder, such that the symplectic form $\Omega$ on $U_M$ reads

$$\Omega = dI_\lambda \wedge dy^\lambda + dp_A \wedge dq^A,$$

and a Hamiltonian of a CIS is a smooth function only of the action coordinates $I_\lambda$.

Theorem 1 restarts the Mishchenko–Fomenko one if its condition is replaced with that the fibers of the fibered manifold $H (1)$ are compact and connected.

The proof of Theorem 1 is based on the following facts.\textsuperscript{8,12} Any function constant on fibers of the fibration $H (1)$ is the pull-back of some function on its base $N$. Due to item (ii) of Definition 1, the Poisson bracket $\{f, f'\}$ of any two functions $f, f' \in C^\infty (Z)$ constant on fibers of $H$ is also of this type. Consequently, the base $N$ of $H$ is provided with a unique coinduced Poisson structure $\{\cdot, \cdot\}_N$ such that $H$ is a Poisson morphism.\textsuperscript{20} By virtue of condition (iii) of Definition 1, the rank of this coinduced Poisson structure equals $2(n - m) = 2\dim N - \dim Z$. Furthermore, one can show the following.\textsuperscript{8,21}

**Lemma 2:** The fibers of the fibration $H (1)$ are maximal integral manifolds of the involutive distribution spanned by the Hamiltonian vector fields of the pull-back $H^* C$ of Casimir functions $C$ of the coinduced Poisson structure on $N$.

In particular, a Hamiltonian of a CIS is the pull-back onto $Z$ of some Casimir function of the coinduced Poisson structure on $N$.

It follows from Lemma 2 that invariant submanifolds of a noncommutative CIS are maximal integral manifolds of a certain Abelian partially integrable system (henceforth PIS).

**Definition 2:** A collection $\{S_1, \ldots, S_m\}$ of $m \leq n$ independent smooth real functions in involution on a $2n$-dimensional symplectic manifold $(Z, \Omega)$ is called a PIS.

Let us consider the map

$$S : Z \to W \subset \mathbb{R}^m.$$  \hspace{1cm} (4)

Since functions $S_\lambda$ are everywhere independent, this map is a submersion onto an open subset $W \subset \mathbb{R}^m$, i.e., $S (4)$ is a fibered manifold of fiber dimension $2n - m$. Hamiltonian vector fields $v_\lambda$ of functions $S_\lambda$ are mutually commutative and independent. Consequently,
they span an $m$-dimensional involutive distribution on $Z$ whose maximal integral manifolds constitute a foliation $\mathcal{F}$ of $Z$. Because functions $S_\lambda$ are constant on leaves of this foliation, each fiber of a fibered manifold $Z \to W$ (4) is foliated by the leaves of the foliation $\mathcal{F}$. If $m = n$, we are in the case of an Abelian CIS, and the leaves of $\mathcal{F}$ are connected components of fibers of the fibered manifold (4). The Poincaré–Lyapounov–Nekhoroshev theorem generalizes the Liouville–Arnold one to a PIS if leaves of the foliation $\mathcal{F}$ are compact. It imposes a sufficient condition which Hamiltonian vector fields $v_\lambda$ must satisfy in order that the foliation $\mathcal{F}$ is a fibered manifold. Extending the Poincaré–Lyapounov–Nekhoroshev theorem to the case of noncompact integral submanifolds, we in fact assumed from the beginning that these submanifolds formed a fibration. Here, we aim to prove the following global variant of Theorem 6 in Ref. [11].

**Theorem 3:** Let a PIS $\{S_1, \ldots, S_m\}$ on a symplectic manifold $(Z, \Omega)$ satisfy the following conditions.

(i) The Hamiltonian vector fields $v_\lambda$ of $S_\lambda$ are complete.

(ii) The foliation $\mathcal{F}$ is a fiber bundle $\mathcal{F} : Z \to N$.

(iii) Its base $N$ is simply connected.

(iv) The cohomology $H^2(N, \mathbb{Z})$ of $N$ with coefficients in the constant sheaf $\mathbb{Z}$ is trivial.

Then the following hold.

(I) The fiber bundle $\mathcal{F}$ is a trivial principal bundle with the structure group (3), and we have a composite fibered manifold

$$S = \zeta \circ \mathcal{F} : Z \to N \to W,$$

where $N \to W$ however need not be a fiber bundle.

(II) The fibered manifold (5) is provided with the adapted fibered (generalized action-angle) coordinates

$$(I_\lambda, x^A, y^\lambda) \to (I_\lambda, x^A) \to (I_\lambda), \quad \lambda = 1, \ldots, m, \quad A = 1, \ldots, 2(n - m),$$

such that the coordinates $(I_\lambda)$ possess identity transition functions, and the symplectic form $\Omega$ reads

$$\Omega = dI_\lambda \wedge dy^\lambda + \Omega^\lambda_A dI_\lambda \wedge dx^A + \Omega_{AB} dx^A \wedge dx^B.$$  

If one supposes from the beginning that leaves of the foliation $\mathcal{F}$ are compact, the condition (i) of Theorem 3 always holds, and the assumption (ii) can be replaced with the requirement that $\mathcal{F}$ is a fibered manifold with mutually diffeomorphic connected fibers. Recall that any fibered manifold whose fibers are diffeomorphic either to $\mathbb{R}^r$ or a compact connected manifold $K$ is a fiber bundle. However, a fibered manifold whose fibers are
diffeomorphic to a product $\mathbb{R}^r \times K$ (e.g., a toroidal cylinder) need not be a fiber bundle (see Ref. [27], Example 1.2.2).

Theorem 3 is proved in Section II. Since $m$-dimensional fibers of the fiber bundle $\mathcal{F}$ admit $m$ complete independent vector fields, they are locally affine manifolds diffeomorphic to a toroidal cylinder (3). Then the condition (iii) of Theorem 3 guarantees that the fiber bundle $\mathcal{F}$ is a principal bundle with the structure group (3). Furthermore, this principal bundle is trivial due to the condition (iv), and it is provided with the bundle action-angle coordinates. Note that conditions (ii) and (iii) of Theorem 3 are sufficient, but not necessary.

If $m = n$, the following corollary of Theorem 3 states the existence of global action-angle coordinates of an Abelian CIS.

**Theorem 4:** Let an Abelian CIS $\{H_1, \ldots, H_n\}$ on a symplectic manifold $(Z, \Omega)$ satisfy the following conditions.

(i) The Hamiltonian vector fields $\vartheta_i$ of $H_i$ are complete.

(ii) The fibered manifold $H(1)$ is a fiber bundle with connected fibers over a simply connected base $N$ whose cohomology $H^2(N, \mathbb{Z})$ is trivial.

Then the following hold.

(I) The fiber bundle $H(1)$ is a trivial principal bundle with the structure group (3).

(II) The symplectic manifold $Z$ is provided with the global Darboux coordinates $(I_\lambda, y^\lambda)$ such that $\Omega = dI_\lambda \wedge dy^\lambda$.

Due to Lemma 2, a manifested global generalization of Theorem 1 is a corollary of Theorem 3 (see Section III).

**Theorem 5:** Given a noncommutative CIS in accordance with Definition 1, let us assume the following.

(i) Hamiltonian vector fields $\vartheta_i$ of integrals of motion $H_i$ are complete.

(ii) The fibration $H(1)$ is a fiber bundle with connected fibers.

(iii) Let $V$ be an open subset of the base $N$ of this fiber bundle which admits $m$ independent Casimir functions of the coinduced Poisson structure on $N$.

(iv) Let $V$ be simply connected, and let the cohomology $H^2(V, \mathbb{Z})$ be trivial.

Then the following hold.

(I) The fibers of $H(1)$ are diffeomorphic to a toroidal cylinder (3).

(II) The restriction $Z_V$ of the fiber bundle $H(1)$ to $V$ is a trivial principal bundle with the structure group (3).

(III) The fiber bundle $Z_V$ is provided with the bundle (generalized action-angle) coordinates $(I_\lambda, x^A, y^\lambda)$ such that the action-angle coordinates $(I_\lambda, y^\lambda)$ possess identity transition functions and the symplectic form $\Omega$ on $Z_V$ reads

$$\Omega = dI_\lambda \wedge dy^\lambda + \Omega_{AB} dx^A \wedge dx^B.$$
Note that, if invariant submanifolds of a CIS are assumed to be connected and compact, condition (i) of Theorem 5 is unnecessary since vector fields $v_\lambda$ on compact fibers of $H$ are complete. In this case, condition (ii) of Theorem 5 also holds because, as was mentioned above, a fibred manifold with compact mutually diffeomorphic fibers is a fiber bundle.

In the case of an Abelian CIS, the coinduced Poisson structure on $N$ equals zero, the integrals of motion $H_\lambda$ are the pull-back of $n$ independent functions on $N$, and Theorem 5 reduces to Theorem 4.

Following the original Mishchenko–Fomenko theorem, let us mention noncommutative CISs whose integrals of motion \{\(H_1, \ldots, H_k\)\} form a $k$-dimensional real Lie algebra $G$ of rank $m$ with the commutation relations

$$\{H_i, H_j\} = c_{ij}^h H_h, \quad c_{ij}^h = \text{const}.$$  

In this case, complete Hamiltonian vector fields $\vartheta_i$ of $H_i$ define a locally free Hamiltonian action on $Z$ of some simply connected Lie group $G$ whose Lie algebra is isomorphic to $G$.\cite{28, 29} Orbits of $G$ coincide with $k$-dimensional maximal integral manifolds of the regular distribution on $Z$ spanned by Hamiltonian vector fields $\vartheta_i$.\cite{30} Furthermore, one can treat $H$ (1) as an equivariant momentum mapping of $Z$ to the Lie coalgebra $G^*$, provided with the coordinates $x_i(H(z)) = H_i(z), z \in Z$.\cite{18, 31} In this case, the coinduced Poisson structure $\{,\}_N$ coincides with the canonical Lie–Poisson structure on $G^*$ given by the Poisson bivector field

$$w = \frac{1}{2} c_{ij}^h x_h \partial^i \wedge \partial^j.$$  

Casimir functions of the Lie–Poisson structure are exactly the coadjoint invariant functions on $G^*$. They are constant on orbits of the coadjoint action of $G$ on $G^*$ which coincide with leaves of the symplectic foliation of $G^*$. Let $V$ be an open subset of $G^*$ which obeys the conditions (iii) and (iv) of Theorem 5. Then the open subset $H^{-1}(V) \subset Z$ is provided with the action-angle coordinates.

II. PROOF OF THEOREM 3

In accordance with the well-known theorem,\cite{28, 29} complete Hamiltonian vector fields $v_\lambda$ define an action of a simply connected Lie group on $Z$. Because vector fields $v_\lambda$ are mutually commutative, it is the additive group $\mathbb{R}^m$ whose group space is coordinated by parameters $s^\lambda$ with respect to the basis $\{e_\lambda = v_\lambda\}$ for its Lie algebra. The orbits of the group $\mathbb{R}^m$ in $Z$ coincide with the fibers of the fiber bundle

$$\mathcal{F} : Z \rightarrow N.$$  

(8)
Since vector fields $v_\lambda$ are independent on $Z$, the action of $\mathbb{R}^m$ on $Z$ is locally free, i.e., isotropy groups of points of $Z$ are discrete subgroups of the group $\mathbb{R}^m$. Given a point $x \in N$, the action of $\mathbb{R}^m$ on a fiber $M_x = F^{-1}(x)$ factorizes as

$$\mathbb{R}^m \times M_x \to G_x \times M_x \to M_x \tag{9}$$

through the free transitive action of the factor group $G_x = \mathbb{R}^m / K_x$, where $K_x$ is the isotropy group of an arbitrary point of $M_x$. It is the same group for all points of $M_x$ because $\mathbb{R}^m$ is a commutative group. Since the fibers $M_x$ are mutually diffeomorphic, all isotropy groups $K_x$ are isomorphic to the group $\mathbb{Z}_r$ for some fixed $0 \leq r \leq m$. Accordingly, the groups $G_x$ are isomorphic to the Abelian group

$$G = \mathbb{R}^{m-r} \times T^r, \tag{10}$$

and fibers of the fiber bundle (8) are diffeomorphic to the toroidal cylinder (10).

Let us bring the fiber bundle (8) into a principal bundle with the structure group (10). Generators of each isotropy subgroup $K_x$ of $\mathbb{R}^m$ are given by $r$ linearly independent vectors $u_i(x)$ of the group space $\mathbb{R}^m$. These vectors are assembled into an $r$-fold covering $K \to N$. This is a subbundle of the trivial bundle

$$N \times \mathbb{R}^m \to N \tag{11}$$

whose local sections are local smooth sections of the fiber bundle (11). Such a section over an open neighborhood of a point $x \in N$ is given by a unique local solution $s^\lambda(x')e_\lambda$ of the equation

$$g(s^\lambda)\sigma(x') = \exp(s^\lambda v_\lambda)\sigma(x') = \sigma(x'), \quad s^\lambda(x)e_\lambda = u_i(x),$$

where $\sigma$ is an arbitrary local section of the fiber bundle $Z \to N$ over an open neighborhood of $x$. Since $N$ is simply connected, the covering $K \to N$ admits $r$ everywhere different global sections $u_i$ which are global smooth sections $u_i(x) = u_i^\lambda(x)e_\lambda$ of the fiber bundle (11). Let us fix a point of $N$ further denoted by $\{0\}$. One can determine linear combinations of the functions $S_\lambda$, say again $S_\lambda$, such that $u_i(0) = e_i$, $i = m - r, \ldots, m$, and the group $G_0$ is identified to the group $G$ (10). Let $E_x$ denote the $r$-dimensional subspace of $\mathbb{R}^m$ passing through the points $u_1(x), \ldots, u_r(x)$. The spaces $E_x$, $x \in N$, constitute an $r$-dimensional subbundle $E \to N$ of the trivial bundle (11). Moreover, the latter is split into the Whitney sum of vector bundles $E \oplus E'$, where $E'_x = \mathbb{R}^m / E_x$. Then there is a global smooth section $\gamma$ of the trivial principal bundle $N \times GL(m, \mathbb{R}) \to N$ such that $\gamma(x)$ is a morphism of $E_0$ onto $E_x$, where $u_i(x) = \gamma(x)(e_i) = \gamma^\lambda e_\lambda$. This morphism is also an automorphism of the group $\mathbb{R}^m$ sending $K_0$ onto $K_x$. Therefore, it provides a group isomorphism $\rho_x : G_0 \to G_x$. 
With these isomorphisms, one can define the fiberwise action of the group $G_0$ on $Z$ given by the law

$$G_0 \times M_x \to \rho_x(G_0) \times M_x \to M_x.$$  \hfill (12)

Namely, let an element of the group $G_0$ be the coset $g(s^\lambda)/K_0$ of an element $g(s^\lambda)$ of the group $\mathbb{R}^m$. Then it acts on $M_x$ by the rule (12) just as the coset $g((\gamma(x)^{-1})^\lambda_\beta s^\beta)/K_x$ of an element $g((\gamma(x)^{-1})^\lambda_\beta s^\beta)$ of $\mathbb{R}^m$ does. Since entries of the matrix $\gamma$ are smooth functions on $N$, the action (12) of the group $G_0$ on $Z$ is smooth. It is free, and $Z/G_0 = N$. Thus, $Z \to N$ (8) is a principal bundle with the structure group $G_0 = G$ (10).

Furthermore, this principal bundle over a paracompact smooth manifold $N$ is trivial as follows. In accordance with the well-known theorem, its structure group $G$ (10) is reducible to the maximal compact subgroup $T^r$, which is also the maximal compact subgroup of the group product $\times GL(1, \mathbb{C})$. Therefore, the equivalence classes of $T^r$-principal bundles $\xi$ are defined as

$$c(\xi) = c(\xi_1 \oplus \cdots \oplus \xi_r) = (1 + c_1(\xi_1)) \cdots (1 + c_1(\xi_r))$$

by the Chern classes $c_1(\xi_i) \in H^2(N, \mathbb{Z})$ of $U(1)$-principal bundles $\xi_i$ over $N$. Since the cohomology group $H^2(N, \mathbb{Z})$ of $N$ is trivial, all Chern classes $c_1$ are trivial, and the principal bundle $Z \to N$ is also trivial. This principal bundle can be provided with the following coordinate atlas.

Let us consider the fibered manifold $S : Z \to W$ (4). Because functions $S_\lambda$ are constant on fibers of the fiber bundle $Z \to N$ (8), the fibered manifold (4) factorizes through the fiber bundle (8), and we have the composite fibered manifold (5). Let us provide the principal bundle $Z \to N$ with a trivialization

$$Z = N \times \mathbb{R}^{m-r} \times T^r \to N,$$  \hfill (13)

whose fibers are endowed with the standard coordinates $(y^\lambda) = (t^a, \varphi^i)$ on the toroidal cylinder (10). Then the composite fibered manifold (5) is provided with the fibered coordinates

$$(J_\lambda, x^A, t^a, \varphi^i),$$

$$\lambda = 1, \ldots, m, \quad A = 1, \ldots, 2(n-m), \quad a = 1, \ldots, m-r, \quad i = 1, \ldots, r,$$  \hfill (14)

where $J_\lambda = S_\lambda(z)$ are coordinates on the base $W$ induced by Cartesian coordinates on $\mathbb{R}^m$, and $(J_\lambda, x^A)$ are fibered coordinates on the fibered manifold $\zeta : N \to W$. The coordinates $J_\lambda$ on $W \subset \mathbb{R}^m$ and the coordinates $(t^a, \varphi^i)$ on the trivial bundle (13) possess the identity transition functions, while the transition function of coordinates $(x^A)$ depends on the coordinates $(J_\lambda)$ in general.
The Hamiltonian vector fields \( v_\lambda \) on \( Z \) relative to the coordinates (14) take the form
\[
v_\lambda = v_\lambda^a(x) \partial_a + v_\lambda^i(x) \partial_i.
\]
Since these vector fields commute (i.e., fibers of \( Z \to N \) are isotropic), the symplectic form \( \Omega \) on \( Z \) reads
\[
\Omega = \Omega_{\beta}^\alpha dJ_\alpha \wedge dy^\beta + \Omega_{\alpha A} dy^\alpha \wedge dx^A + \Omega^\alpha_{\beta} dJ_\alpha \wedge dJ_\beta + \Omega^\alpha_A dJ_\alpha \wedge dx^A + \Omega_{AB} dx^A \wedge dx^B.
\]

Lemma 6: The symplectic form \( \Omega \) is exact.

Proof: In accordance with the well-known Künneth formula, the de Rham cohomology group of the product (13) reads
\[
H^2(Z) = H^2(N) \oplus H^1(N) \otimes H^1(T^r) \oplus H^2(T^r).
\]
By the de Rham theorem, the de Rham cohomology \( H^2(N, \mathbb{R}) \) of \( N \) with coefficients in the constant sheaf \( \mathbb{R} \) is trivial since \( H^2(N, \mathbb{R}) = H^2(N, \mathbb{Z}) \otimes \mathbb{R} \). It is trivial because \( N \) is simply connected. Consequently, \( H^2(Z) = H^2(T^r) \). Then the closed form \( \Omega \) is exact since it does not contain the term \( \Omega^{ij} d\phi^i \wedge d\phi^j \).

Thus, we can write
\[
\Omega = d\Xi, \quad \Xi = \Xi^\lambda(J_\alpha, x^B, y^\alpha) dJ_\lambda + \Xi_i(J_\alpha, x^B) d\phi^i + \Xi_A(J_\alpha, x^B, y^\alpha) dx^A.
\]
Up to an exact summand, the Liouville form \( \Xi \) is brought into the form
\[
\Xi = \Xi^\lambda(J_\alpha, x^B, y^\alpha) dJ_\lambda + \Xi_i(J_\alpha, x^B) d\phi^i + \Xi_A(J_\alpha, x^B, y^\alpha) dx^A,
\]
i.e., it does not contain the term \( \Xi_a dt^a \).

The Hamiltonian vector fields \( v_\lambda \) obey the relations \( v_\lambda \lceil \Omega = -dJ_\lambda \), which result in the coordinate conditions
\[
\Omega_{\beta}^\alpha \partial_\beta = \delta_\lambda^\alpha, \quad \Omega_{A\beta} \partial_\beta = 0.
\]
The first of them shows that \( \Omega_{\beta}^\alpha \) is a nondegenerate matrix independent of coordinates \( y^\lambda \). Then the second one implies \( \Omega_{A\beta} = 0 \).

Since \( \Xi_a = 0 \) and \( \Xi_i \) are independent of \( \phi^i \), it follows from the relations
\[
\Omega_{A\beta} = \partial_A \Xi_\beta - \partial_\beta \Xi_A = 0
\]
that \( \Xi_A \) are independent of coordinates \( t^a \) and at most affine in \( \phi^i \). Since \( \phi^i \) are cyclic coordinates, \( \Xi_A \) are independent of \( \phi^i \). Hence, \( \Xi_i \) are independent of coordinates \( x^A \), and the Liouville form \( \Xi \) reads
\[
\Xi = \Xi^\lambda(J_\alpha, x^B, y^\alpha) dJ_\lambda + \Xi_i(J_\alpha) d\phi^i + \Xi_A(J_\alpha, x^B) dx^A.
\]
Because entries $\Omega^i_j$ of $d\Xi = \Omega$ are independent of $y^\lambda$, we obtain the following.

(i) $\Omega_i^a = \partial^i \Xi_a - \partial_a \Xi^i$. Consequently, $\partial_i \Xi^a$ are independent of $\varphi^i$, and so are $\Xi^i$ since $\varphi^i$ are cyclic coordinates. Hence, $\Omega_i^a = \partial^a \Xi_i$ and $\partial_i | \Omega = -d\Xi_i$. A glance at the last equality shows that $\partial_i$ are Hamiltonian vector fields. It follows that, from the beginning, one can separate $r$ integrals of motion, say $H_i$ again, whose Hamiltonian vector fields are tangent to invariant tori. In this case, the Hamiltonian vector fields $v_\lambda$ (15) read

$$v_a = \partial_a, \quad v_i = v^k_i(x) \partial_k,$$

where the matrix function $v^k_i(x)$ is nondegenerate. Moreover, the coordinates $t^a$ are exactly the flow parameters $s^a$. Substituting the expressions (21) into the first condition (19), we obtain

$$\Omega = dJ_a \wedge ds^a + (v^{-1})^i_k dJ_i \wedge d\varphi^k + \Omega^{\alpha\beta} dJ_\alpha \wedge dJ_\beta + \Omega^A_A dJ_A \wedge dx^A + \Omega^A_B dJ_A \wedge dx^B.$$ 

It follows that $\Xi_i$ are independent of $J_a$, and so are $(v^{-1})^k_i = \partial^k \Xi_i$.

(ii) $\Omega^i_A = -\partial_i \Xi^A = \delta^i_A$. Hence, $\Xi^a = -t^a + E^a(J_\lambda, x^B)$ and $\Xi^i$ are independent of $t^a$.

In view of items (i) – (ii), the Liouville form $\Xi$ (20) reads

$$\Xi = (-t^a + E^a(J_\lambda, x^B)) dJ_a + E^i(J_\lambda, x^B) dJ_i + \Xi_i(J_j) d\varphi^i + \Xi_A(J_\lambda, x^B) dx^A.$$ 

Since the matrix $\partial^k \Xi_i$ is nondegenerate, we can perform the coordinate transformations $I_a = J_a, I_i = \Xi_i(J_j)$ together with the coordinate transformations

$$t^a = -t^a + E^a(J_\lambda, x^B), \quad \varphi^i = \varphi^i - E^i(J_\lambda, x^B) \frac{\partial J_j}{\partial I_i}.$$ 

These transformations bring $\Omega$ into the form (6).

### III. PROOF OF THEOREM 5

Given a fibration $H$ (1), let $V$ be an open subset of its base $N$ which satisfies condition (iii) of Theorem 5, i.e., there is a set $\{C_1, \ldots, C_m\}$ of $m$ independent Casimir functions of the coinduced Poisson structure $\{,\}_N$ on $V$. Note that such functions always exist around any point of $N$. Let $Z_V$ be the restriction of the fiber bundle $Z \to N$ onto $V \subset Z$. By virtue of Lemma 2, $Z_V \to V$ is a fibration in invariant submanifolds of a PIS $\{H^*C_\lambda\}$, where $H^*C_\lambda$ are the pull-back of the Casimir functions $C_\lambda$ onto $Z_V$.

Let $v_\lambda$ be Hamiltonian vector fields of functions $H^*C_\lambda$. Since

$$H^*C_\lambda(z) = (C_\lambda \circ H)(z) = C_\lambda(H_i(z)), \quad z \in Z_V,$$

the Hamiltonian vector fields $v_\lambda$ restricted to any fiber $M$ of $Z_V$ are linear combinations of the Hamiltonian vector fields $\partial_i$ of integrals of motion $H_i$. It follows that $v_\lambda$ are elements
of a finite-dimensional real Lie algebra of vector fields on $M$ generated by the vector fields $\vartheta_i$. Since vector fields $\vartheta_i$ are complete, the vector fields $v_\lambda$ on $M$ are also complete. Consequently, the Hamiltonian vector fields $v_\lambda$ are complete on $Z_V$. Then the conditions of Theorem 3 for a PIS $\{H^*C_\lambda\}$ on the symplectic manifold $(Z_V, \Omega)$ hold.

In accordance with Theorem 3, we have a composite fibered manifold

$$Z_V \xrightarrow{H} V \xrightarrow{C} W,$$

(24)

where $C : V \to W$ is a fibered manifold of level surfaces of the Casimir functions $C_\lambda$. The fibered manifold (24) is provided with the adapted fibered coordinates $(J_\lambda, x^A, y^\lambda)$ (14), where $J_\lambda$ are values of the Casimir functions and $(y^\lambda) = (t^a, \varphi^i)$ are coordinates on a toroidal cylinder. Since $C_\lambda = J_\lambda$ are Casimir functions on $V$, the symplectic form $\Omega$ (16) on $Z_V$ reads

$$\Omega = \Omega^\alpha_\beta dJ_\alpha \wedge dy^\beta + \Omega^A d\varphi^i \wedge dx^A + \Omega_{AB} dx^A \wedge dx^B. \quad (25)$$

In particular, it follows that transition functions of coordinates $x^A$ on $V$ are independent of coordinates $J_\lambda$, i.e., $C : V \to W$ is a trivial bundle.

By virtue of Lemma 6, the symplectic form (25) is exact, i.e., $\Omega = d\Xi$, where the Liouville form $\Xi$ (20) is

$$\Xi = \Xi^\lambda(J_\lambda, y^\lambda) dJ_\lambda + \Xi_i(J_\lambda) d\varphi^i + \Xi_A(x^B) dx^A.$$

It is brought into the form

$$\Xi = (-t^a + E^a(J_\lambda)) dJ_a + E^i(J_\lambda) dJ_i + \Xi_i(J_\lambda) d\varphi^i + \Xi_A(x^B) dx^A.$$

Then the coordinate transformations (22)

$$I_a = J_a, \quad I_i = \Xi_i(J_j),$$

$$t^a = -t^a + E^a(J_\lambda), \quad \varphi^i = \varphi^i - E^j(J_\lambda) \frac{\partial J_j}{\partial I_i}, \quad (26)$$

bring $\Omega$ (25) into the form (7). In comparison with the general case (22), the coordinate transformations (26) are independent of coordinates $x^A$. Therefore, the angle coordinates $\varphi^i$ possess identity transition functions on $V$.

Theorem 5 restarts Theorem 1 if one considers an open subset $U$ of $V$ admitting the Darboux coordinates $x^A$ on the symplectic leaves of $U$.

The proof of Theorem 5 gives something more. Let $H$ be a Hamiltonian of a CIS. It is the pull-back onto $Z_V$ of some Casimir function on $V$. Since $(I_\lambda, x^A)$ are coordinates on $V$, they are also integrals of motion of $H$. Though the original integrals of motion $H_i$ are smooth functions of coordinates $(I_\lambda, x^A)$, the Casimir functions (23)

$$C_\lambda(H_i(I_\mu, x^A)) = C_\lambda(I_\mu)$$

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and, in particular, a Hamiltonian $H$ depend only on the action coordinates $I_\lambda$. Hence, the equations of motion of a CIS take the form

$$\dot{y}^A = \frac{\partial H}{\partial I_\lambda}, \quad I_\lambda = \text{const.}, \quad x^A = \text{const}.$$
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