Minimal non-Iwasawa finite groups

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Abstract

In this note, we describe first the structure of minimal non-Iwasawa finite groups. Then we determine the minimal non-Iwasawa finite groups which are modular. Also, we find connections between minimal non-Iwasawa finite groups and the subgroup commutativity degree, and we give an example of a family of non-nilpotent modular finite groups $G_n$, $n \in \mathbb{N}$, whose subgroup commutativity degree tends to 1 as $n$ tends to infinity.

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1 Introduction

Let $G$ be a finite group and $L(G)$ be the subgroup lattice of $G$. A subgroup $H$ of $G$ is called *permutable* if $HK = KH$, for all $K \in L(G)$, and *modular* if it is a modular element of $L(G)$. Clearly, any normal subgroup is permutable and, by Theorem 2.1.3 of [9], a permutable subgroup is always modular. If all subgroups of $G$ are permutable then we say that $G$ is an *Iwasawa group*, while if all subgroups of $G$ are modular (that is, the lattice $L(G)$ is modular) then we say that $G$ is a *modular group*. The connection between these two classes of groups is very powerful: a finite group $G$ is an Iwasawa group if and only if it is a nilpotent modular group (see e.g. Exercise 3, page 87, [9]). Note that a complete description of the structure of finite Iwasawa groups can be obtained by using Theorems 2.4.13 and 2.4.14 of Schmidt’s book.

All groups considered in this paper are finite.
Given a class of groups $\mathcal{X}$, a group $G$ is said to be a \textit{minimal non-$\mathcal{X}$ group}, or an $\mathcal{X}$-\textit{critical group}, if $G \notin \mathcal{X}$, but all proper subgroups of $G$ belong to $\mathcal{X}$. Many results have been obtained on minimal non-$\mathcal{X}$ groups, for various choices of $\mathcal{X}$. For instance, minimal non-abelian groups were analysed by Miller and Moreno [6], while Schmidt [8] (see also [3, 7]) studied minimal non-nilpotent groups. The latter are now known as Schmidt groups, and their structure is as follows: a Schmidt group $G$ is a solvable group of order $p^m q^n$ (where $p$ and $q$ are different primes) with a unique Sylow $p$-subgroup $P$ and a cyclic Sylow $q$-subgroup $Q$, and hence $G$ is a semidirect product of $P$ by $Q$. Moreover, we have:

- if $Q = \langle y \rangle$ then $y^q \in Z(G)$;
- $Z(G) = \Phi(G) = \Phi(P) \times \langle y^q \rangle$, $G' = P$, $P' = (G')' = \Phi(P)$;
- $|P/P'| = p^r$, where $r$ is the order of $p$ modulo $q$;
- if $P$ is abelian, then $P$ is an elementary abelian $p$-group of order $p^r$ and $P$ is a minimal normal subgroup of $G$;
- if $P$ is non-abelian, then $Z(P) = P' = \Phi(P)$ and $|P/Z(P)| = p^r$.

We also recall the class of minimal non-modular $p$-groups, whose structure has been investigated in [4].

Our main result is the following.

**Theorem 1.** A finite group is a minimal non-Iwasawa group if and only if it is either a minimal non-modular $p$-group or a Schmidt group $G = PQ$ with $P$ modular.

We observe that $A_4$ and $SL(2, 3)$ are examples of Schmidt groups as in Theorem 1 with $P \cong \mathbb{Z}_2^2$ abelian and $P \cong Q_8$ non-abelian, respectively.

By using Theorem 1, we are able to determine those minimal non-Iwasawa groups which are modular.

**Corollary 2.** The class $\mathcal{C}$ of minimal non-Iwasawa groups which are modular consists of all Schmidt groups $G = PQ$ with $P$ cyclic of order $p$.

Note that an interesting subclass of $\mathcal{C}$ is constituted by the non-trivial semidirect products $\mathbb{Z}_3 \rtimes \mathbb{Z}_{2^n}$, $n \in \mathbb{N}^*$. Also, from the proof of Corollary 2, it will follow that:

**Corollary 3.** $\mathcal{C}$ is contained in the class of minimal non-cyclic groups.
2 Proofs of the main results

Proof of Theorem 1. Obviously, a minimal non-modular $p$-group is also a minimal non-Iwasawa group. Let $G = PQ$ be a Schmidt group with $P$ modular. Then $G$ is non-nilpotent, and consequently non-Iwasawa. Since all proper subgroups of $G$ are nilpotent, it suffices to prove that they are modular. Also, we may restrict to maximal subgroups. By looking to the structure of $G$ described in Section 1, we infer that these are $P \times \langle y^q \rangle$ and $\Phi(P)Q_i$, $i = 1, 2, ..., n_q$, where $Q_i$, $i = 1, 2, ..., n_q$, denote the conjugates of $Q$. Being a direct product of modular groups of coprime orders, $P \times \langle y^q \rangle$ is modular. On the other hand, for each $i$ we have

$$\Phi(P)Q_i/Z(\Phi(P)Q_i) = \Phi(P)Q_i/\Phi(P)\langle y^q \rangle \cong Q_i/\langle y^q \rangle \cong \mathbb{Z}_q,$$

implying that $\Phi(P)Q_i$ is abelian. Therefore all maximal subgroups of $G$ are modular, as desired.

Conversely, assume that $G$ is a minimal non-Iwasawa group. We distinguish the following two cases.

Case 1. $G$ is nilpotent.
Then $G$ is not modular. Let $G = \prod_{i=1}^k G_i$ be the decomposition of $G$ as a direct product of Sylow subgroups. Then we must have $k = 1$ because all proper subgroups of $G$ are modular. Thus $G$ is a minimal non-modular $p$-group.

Case 2. $G$ is not nilpotent.
Then $G$ is a Schmidt group, say $G = PQ$ with $P$ and $Q$ as we described in Section 1. Since $P$ is a proper subgroup of $G$, it must be modular by our assumption. This completes the proof.

Proof of Corollary 2. Let $G$ be a group contained in $C$. By Theorem 1, it follows that $G$ is a Schmidt of type $G = PQ$ with $P$ modular. Since $G$ is modular, so is $G_1 = G/Z(G)$. But $G_1$ is again a Schmidt group of order $p^rq$ which can be written as semidirect product of an elementary abelian $p$-group $P_1$ of order $p^r$ by a cyclic group $Q_1$ of order $q$. Suppose that $r > 1$. Then $P_1$ contains a proper non-trivial subgroup, say $P_2$. It is easy to see that the subgroups

$$Z(G), P_1, P_2, Q_1, \text{ and } G_1$$

form a pentagon in $L(G_1)$, a contradiction. So, $r = 1$. This leads to $|P/P'| = p$, i.e. $P$ is abelian, and consequently $|P| = p$. 

\[\Box\]
Proof of Corollary 3. Let $G$ be a group contained in $C$. From the proof of Theorem 1 it follows that all maximal subgroups of $G$ are cyclic. Hence $G$ is a minimal non-cyclic group.

3 Minimal non-Iwasawa finite groups and subgroup commutativity degrees

A notion strongly connected with Iwasawa groups is the subgroup commutativity degree of a finite group $G$, defined in [10] by

$$sd(G) = \frac{1}{|L(G)|^2} |\{(H, K) \in L(G)^2 \mid HK = KH\}|.$$  

This measures the probability that two subgroups of $G$ commute, or equivalently that the product of two subgroups is again a subgroup. Clearly, we have $sd(G) = 1$ if and only if $G$ is an Iwasawa group. $sd(G)$ has been generalized to the relative subgroup commutativity degree of a subgroup $H$ of $G$ (see [11]):

$$sd(H, G) = \frac{1}{|L(H)||L(G)|} |\{(H_1, G_1) \in L(H) \times L(G) \mid H_1G_1 = G_1H_1\}|.$$  

These notions lead to two functions on $L(G)$, namely

$$f, g : L(G) \longrightarrow [0, 1], f(H) = sd(H) \text{ and } g(H) = sd(H, G), \forall H \in L(G),$$

whose study is proposed in [11]. We remark that they are constant on each conjugacy class of subgroups of $G$. On the other hand, we have

$$|Im f| = 1 \Leftrightarrow |Im g| = 1 \Leftrightarrow G = \text{Iwasawa group}.$$  

Having in mind these results, it is an interesting problem to determine the classes $C_f$ and $C_g$ of finite groups $G$ such that $|Im f| = 2$ and $|Im g| = 2$, respectively. We mention that $C_g$ has been studied in [5]. Also, it is clear that the minimal non-Iwasawa groups are contained in $C_f$.

Another interesting problem concerning the subgroup commutativity degree is to find some natural families of groups $G_n$, $n \in \mathbb{N}$, whose subgroup commutativity degree tends to a constant $a \in [0, 1]$ as $n$ tends to infinity. For $a = 0$ many examples of such families are known (see e.g. [1, 2, 10, 12]).
Since minimal non-Iwasawa groups have many commuting subgroups, we expect that they will have large subgroup commutativity degrees. Indeed, this is confirmed for groups in the class $\mathcal{C}$, as shows our following theorem.

**Theorem 4.** Let $p$ and $q$ be two primes such that $p \equiv 1 \pmod{q}$, and $G_n$ be a group of order $pq^n$ contained in $\mathcal{C}$. Then

$$\lim_{n \to \infty} \text{sd}(G_n) = 1.$$ 

**Proof.** Under the notation in the previous sections, we easily infer that $L(G_n)$ consists of $G_n$, of all conjugates $Q_i$, $i = 1, 2, \ldots, n_q = p$, of $Q$, and of all subgroups contained in $P \times \langle y^q \rangle \cong \mathbb{Z}_{pq^{n-1}}$. Then

$$|L(G_n)| = 2n + 1 + p.$$ 

Also, we observe that the non-normal subgroup of $G_n$ are $Q_i$, $i = 1, 2, \ldots, p$. For each $H \leq G_n$, let $C(H) = \{ K \in L(G) \mid HK = KH \}$. Then

$$C(H) = L(G), \forall H \leq G$$

and

$$C(Q_i) = L(G) \setminus \{ Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_p \}, \forall i = 1, 2, \ldots, p.$$ 

One obtains

$$\text{sd}(G_n) = \frac{1}{|L(G)|^2} \sum_{H \leq G} |C(H)| = \frac{1}{|L(G)|^2} \left( \sum_{H \leq G} |C(H)| + \sum_{H \not\leq G} |C(H)| \right)$$

$$= \frac{1}{|L(G)|^2} \left[ (|L(G)| - p)|L(G)| + p(|L(G)| - p + 1) \right]$$

$$= 1 - \frac{p^2 - p}{(2n + 1 + p)^2},$$

which clearly tends to 1 as $n$ tends to infinity. 

**4 Further research**

We end our note by indicating two natural open problems concerning the above results.
Problem 1. Determine the finite groups $G$ containing a unique non-Iwasawa proper subgroup $H$.

Note that in this case $H$ must be a minimal non-Iwasawa group, and also a characteristic maximal subgroup of $G$. Several examples of such groups are the quasi-dihedral group $QD_{16} = \langle x, y \mid x^8 = y^2 = 1, yxy = x^3 \rangle$ and the direct products $S_3 \times \mathbb{Z}_p$, where $p$ is an odd prime.

Problem 2. Give a complete description of the class $C_f$.

Note that $C_f$ contains any direct product between a minimal non-Iwasawa group and an Iwasawa group of coprime orders. Moreover, if $G$ is a group in $C_f$, then by choosing a subgroup $H$ of $G$ which is minimal with the property $1 \neq sd(H) = sd(G)$, it follows that $H$ is minimal non-Iwasawa. So, $C_f$ is strongly connected to the class of minimal non-Iwasawa groups.

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