LARGE POSITIVE AND NEGATIVE VALUES OF HARDY’S
Z-FUNCTION

KAMALAKSHYA MAHATAB

Abstract. Let $Z(t) := \zeta\left(\frac{1}{2} + it\right) \chi^{-\frac{1}{2}} \left(\frac{1}{2} + it\right)$ be Hardy’s function, where
the Riemann zeta function $\zeta(s)$ has the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$. We prove that for any $\epsilon > 0$,
$$\max_{T^{3/4} \leq t \leq T} Z(t) \gg \exp\left(\frac{1}{2} - \epsilon\right) \sqrt{\frac{\log T \log \log \log T}{\log \log T}}$$
and
$$\max_{T^{3/4} \leq t \leq T} Z(t) \gg \exp\left(\frac{1}{2} - \epsilon\right) \sqrt{\frac{\log T \log \log \log T}{\log \log T}}.$$

1. Introduction

The Riemann zeta-function satisfies the following functional equation
$$\zeta(s) = \chi(s)\zeta(1-s) \quad s \in \mathbb{C},$$
where
$$\chi(s) = \frac{\Gamma\left(\frac{1}{2}(1-s)\right)}{\Gamma\left(\frac{1}{2}s\right)} \pi^{s-\frac{1}{2}}.$$

Hardy’s function $Z(t)$, defined by
$$Z(t) = \zeta\left(\frac{1}{2} + it\right) \chi^{-\frac{1}{2}} \left(\frac{1}{2} + it\right) \quad t \in \mathbb{R},$$
is a smooth, real-valued function in $t$ and
$$|Z(t)| = \left|\zeta\left(\frac{1}{2} + it\right)\right|.$$
So the zeros of $\zeta\left(\frac{1}{2} + it\right)$ are the zeros of $Z(t)$. Since $\zeta\left(\frac{1}{2} + it\right)$ has infinitely many
zeros [8, Lemma 2.3], $Z(t)$ often changes sign. In this paper, we will investigate the
sign changes of $Z(t)$ by computing lower bounds for its large positive and negative
values.

Define
$$Z^+(t) := \max(0, Z(t)), \quad Z^-(t) := \max(0, -Z(t)).$$
And denote $\log \ldots \log t$ by $\log^k t$.

Ivić [9] has proved the following lower bounds for large values of $Z^+(t)$ and $Z^-(t)$:
$$\max_{T \leq t \leq T + T^{17/110}} Z^+(t) \gg (\log T)^{1/4},$$
$$\max_{T \leq t \leq T + T^{17/110}} Z^-(t) \gg (\log T)^{1/4}.$$
On the other hand, Balasubramanian and Ramachandra [2, 3] proved that there exists a constant \( B(\sim 0.530) \) such that
\[
\max_{T \leq t \leq 2T} \left| \zeta \left( \frac{1}{2} + it \right) \right| \geq \exp \left( B \sqrt{\frac{\log H}{\log_2 H}} \right).
\]
So either \( Z^+(t) \) or \( Z^-(t) \) is bigger than \( \exp \left( B \sqrt{\frac{\log H}{\log_2 H}} \right) \), which suggests that the above lower bounds for \( Z^+(t) \) and \( Z^-(t) \) can possibly be improved. The lower bound for \( \zeta \left( \frac{1}{2} + it \right) \) has been improved by several authors using the resonance method, while allowing \( t \) to vary on a larger interval. Soundararajan [10] proved that
\[
\max_{T \leq t \leq 2T} \left| \zeta \left( \frac{1}{2} + it \right) \right| \geq \exp \left( (1 + o(1)) \sqrt{\frac{\log T \log_3 T}{\log_2 T}} \right).
\]
Later, Bondarenko and Seip [4, 5] improved this bound significantly
\[
\max_{0 \leq t \leq T} \left| \zeta \left( \frac{1}{2} + it \right) \right| \geq \exp \left( (1 + o(1)) \sqrt{\frac{\log T \log_3 T}{\log_2 T}} \right).
\]
Recently, Bretéché and Tenenbaum [6] optimized the constant in [5] to
\[
\max_{0 \leq t \leq T} \left| \zeta \left( \frac{1}{2} + it \right) \right| \geq \exp \left( \sqrt{2} + o(1) \right) \sqrt{\frac{\log T \log_3 T}{\log_2 T}}.
\]
In this paper, we will use the resonator constructed by Bondarenko and Seip in [4] to prove

**Theorem 1.** For any arbitrarily small \( \epsilon > 0 \) and for sufficiently large \( T \),

(A) \( \max_{T^{3/4} \leq t \leq T} Z^+(t) \gg \exp \left( \frac{1}{2} - \epsilon \right) \sqrt{\frac{\log T \log_3 T}{\log_2 T}} \),

(B) \( \max_{T^{3/4} \leq t \leq T} Z^-(t) \gg \exp \left( \frac{1}{2} - \epsilon \right) \sqrt{\frac{\log T \log_3 T}{\log_2 T}} \).

Our proof is based on the following observation. Suppose we could find a non-negative function \( K(t) \) such that
\[
A(T) \int_{T^{3/4}}^{T} K(t) dt \ll \int_{T^{3/4}}^{T} \left| \zeta \left( \frac{1}{2} + it \right) \right| K(t) dt
\]
and
\[
\int_{T^{3/4}}^{T} Z(t) K(t) dt = o \left( A(T) \int_{T^{3/4}}^{T} K(t) dt \right),
\]
then
\[
\max_{T^{3/4} \leq t \leq T} Z^+(t), \max_{T^{3/4} \leq t \leq T} Z^-(t) \gg A(T).
\]
To show the above, note that there exists a constant \( C_1 > 0 \) such that
\[
C_1 A(T) \int_{T^{3/4}}^{T} K(t) dt \leq \int_{T^{3/4}}^{T} |Z(t)| K(t) dt.
\]
Assume that our claim is not true and \( Z^-(t) \leq C_1 A(T)/3 \). Then
\[
\int_{T^{3/4}}^{T} Z(t) K(t) dt = \int_{T^{3/4}}^{T} |Z(t)| K(t) dt - \int_{T^{3/4}}^{T} 2Z^-(t) K(t) dt \geq \frac{C_1 A(T)}{3} \int_{T^{3/4}}^{T} K(t) dt,
\]
which contradicts to the fact that \( \int_{T/4}^{T} Z(t)K(t)dt = o\left( A(T)\int_{T/4}^{T} K(t)dt \right) \). This proves our claim for \( Z^{-}(t) \). Similarly we can argue for \( Z^{+}(t) \).

We may also note that the lower bound of \( \zeta\left( \frac{1}{2} + it \right) \) in \([6]\) is the optimal bound that can be obtained using the resonance method and the gcd technique, while the lower bounds we obtain for \( Z^{+}(t) \) and \( Z^{-}(t) \) are not necessarily optimal. There are some technical difficulties in our proof that do not allow us to improve our result. In the course of the proof we will observe that if we could improve the upper bound of \( \sum_{m \in \mathcal{M}'} r(m) \) in Lemma \([2]\) to \( T^{\epsilon} \sqrt{T} \), then we can improve the lower bounds in Theorem \([1]\) to \( \exp\left( (1 - \epsilon) \sqrt{\frac{\log T \log \log T}{\log 2}} \right) \). We will explain the notations \( r(m) \) and \( \mathcal{M}' \) in Section \([2]\). Further, if we could find an optimal upper bound for the 4-th moment of \( R(t) \) (where \( R(t) \) is as defined in \([3]\)), then we could improve the bound to \( \exp\left( (\sqrt{2} - \epsilon) \sqrt{\frac{\log T \log \log T}{\log 2}} \right) \).

We can also modify our proof of Theorem \([1]\) by using the resonator defined by Sondararajan \([10]\) to prove a weaker lower bound but with a better localization of \( t \).

**Theorem 2.** For any \( \epsilon > 0 \) and for sufficiently large \( T \),

\[
(A) \quad \max_{T \leq t \leq 2T} Z^{+}(t) \gg \exp\left( \frac{1}{2} - \epsilon \right) \sqrt{\frac{\log T}{\log 2}} T^\epsilon,
\]

\[
(B) \quad \text{and} \quad \max_{T \leq t \leq 2T} Z^{-}(t) \gg \exp\left( \frac{1}{2} - \epsilon \right) \sqrt{\frac{\log T}{\log 2}} T^\epsilon.
\]

Since the proof of Theorem \([2]\) is same as that of Theorem \([1]\) we will skip the proof.

It is possible to compute lower bounds for the Lebesgue measures of the sets where \( Z^{+}(t) \) and \( Z^{-}(t) \) attain the bounds given in Theorem \([1]\) and Theorem \([2]\). But these lower bounds are much weaker in compare to the bounds obtained in \([7]\) and \([9]\). As the gain from these computations are not significant and to keep the paper short, we will not carry out this task.

In Section \([2]\) we will go through the notations from \([4]\) to define the resonator \( R(t) \) and state some related results. In Section \([3]\) we will estimate an integral involving \( Z(t) \) and \( R(t) \). This will be used in Section \([4]\) to prove Theorem \([1]\).

## 2. Construction of The Resonator

The resonator \( R(t) \) constructed by Bondarenko and Seip \([4]\) has the form of a Dirichlet polynomial

\[
R(t) = \sum_{m \in \mathcal{M}'} r(m)m^{-it}.
\]

To define \( r(m) \) and \( \mathcal{M}' \), we need the following notations.

Let \( \gamma = 1 - \varepsilon \), where \( \varepsilon > 0 \) is arbitrarily small. Let \( P \) be the set of primes in the interval \((\varepsilon \log N \log_{2} N, \log N \exp((\log_{2} N)^{3}) \log_{2} N]\). Define

\[
f(p) := \sqrt{\frac{\log N \log_{2} N}{\log_{3} N}} \frac{1}{\sqrt{p} (\log p - \log_{2} N - \log_{3} N)}
\]

for \( p \in P \) and 0 on other primes. We assume that \( f \) is supported on square-free integers and extend the definition of \( f(n) \) as a multiplicative function. For a fixed \( 1 < a < \frac{1}{1 + \varepsilon} \), let \( M_k \) be the set of integers having at least \( k \log_{3} N \) prime divisors
in $P_k$, and let
\[ M := \text{supp}(f) \setminus \bigcup_{k=1}^{[(\log_2 N)']} M_k. \]
Set $N = \lceil T^{1/4} \rceil$. Let $\mathcal{J}$ be the set of integers $j$ such that
\[ \left( (1 + T^{-1})^j, (1 + T^{-1})^{j+1} \right) \cap M \neq \emptyset, \]
and let $m_j$ be the minimum of $[(1 + T^{-1})^j, (1 + T^{-1})^{j+1}) \cap M$ for $j$ in $\mathcal{J}$. Finally, we define
\[ M' := \{ m_j : j \in \mathcal{J} \} \]
and
\[ r(m_j) := \left( \sum_{n \in M, (1-T^{-1})^{j-1} \leq n \leq (1+T^{-1})^{j+2}} f(n)^2 \right)^{1/2} \]
for every $m_j$ in $M'$.
We will also denote $\xi := \sum_{n \in M} f(n)^2$, and $\Phi(t) := e^{-t^2}$.
Now we state some results from [4].

**Lemma 1** (Lemma 2 of [4]). For large $N$,
\[ |M'| \leq |M| \leq N. \]

**Lemma 2.** The sum of the Dirichlet coefficients of the resonator $R(t)$ has the following upper bound
\[ \sum_{m \in M'} r(m) \ll T^{1/8} (\log T)^{3/2} \xi. \]

**Proof.** Using the Cauchy-Schwarz inequality
\[ \left( \sum_{m \in M'} r(m) \right)^2 \leq |M'| \sum_{m \in M'} r(m)^2. \]
In [4] (see proof of Theorem 1), it has been proved that
\[ \sum_{m \in M'} r(m)^2 \ll (\log T)^2 \xi, \]
and by Lemma I, $M' \leq T^{1/4}$. Substituting these bounds in (3) proves the lemma.\[ \square \]

**Lemma 3.** For large $T$,
\[ \int_{T^{3/4}}^T |R(t)|^2 \Phi \left( \frac{t \log T}{T} \right) dt \ll T(\log T)^3 \xi. \]

**Proof.** See (22) of [4].\[ \square \]

**Proposition 1.** For an arbitrarily small $\epsilon > 0$, we have
\[ \int_{T^{3/4}}^T \zeta \left( \frac{1}{2} + it \right) |R(t)|^2 \Phi \left( \frac{t \log T}{T} \right) dt \gg T \exp \left( \left( \frac{1}{2} - \epsilon \right) \sqrt{\frac{T \log \log T}{\log_2 T}} \right). \]

**Proof.** See (25) of [4].\[ \square \]
3. Resonator and Hardy’s Function

Hardy’s function $Z(t)$ has the following approximation formula ((2.3) [8]):

$$Z(t) = 2 \sum_{n \leq \sqrt{t/2\pi}} \frac{1}{\sqrt{n}} \cos (F_n(t)) + O \left( t^{-1/4} \right)$$

(4)

$$= 2 \Re \left( \sum_{n \leq \sqrt{t/2\pi}} \frac{1}{\sqrt{n}} \exp (iF_n(t)) \right) + O \left( t^{-1/4} \right),$$

where

$$F_n(t) = t \log \frac{\sqrt{t/2\pi}}{n} - \frac{t}{n} - \frac{\pi}{8}.$$

We also need the second derivative test to estimate certain integrals involving $Z(t)$

**Lemma 4** (Lemma 2.3 of [8]). Let $F(x)$ be a real, twice differentiable function for $a \leq x \leq b$ such that $|F''(x)| \geq m > 0$. Let $G(x)$ be a positive monotonic function such that $|G(x)| \leq G$ for $x \in [a, b]$. Then

$$\int_a^b G(x) \exp(iF(x))dx \leq \frac{8G}{\sqrt{m}}.$$

Using the above approximation formula for $Z(t)$ and the second derivative test, we prove the following proposition.

**Proposition 2.** Let $R(t)$ be defined as in Section 2. Then for large $T$,

$$\int_{T^{3/4}}^T Z(t)|R(t)|^2 \Phi \left( \frac{t \log T}{T} \right) dt \ll \mathcal{L} T (\log T)^2.$$

**Proof.** Plugging in the expressions of $Z$ and $R$ from [2] and [4] in the above integral, and then exchanging the sums and the integral, we get

$$J := \int_{T^{3/4}}^T Z(t)|R(t)|^2 \Phi \left( \frac{t \log T}{T} \right) dt$$

$$\ll \sum_{m,n \in M'} r(m)r(n) \sum_{k \leq \sqrt{\frac{T}{m}} \pi} \frac{1}{\sqrt{k}} \left| \int_{T^{3/4}}^T \exp(iF_k(t) + i t \log(m/n)) \Phi \left( \frac{t \log T}{T} \right) dt \right|.$$

To apply the second derivative test, observe that

$$\frac{d^2}{dt^2} (F_k(t) + i t \log(m/n)) \gg \frac{1}{t} \gg \frac{1}{T},$$

when $T^{3/4} \leq t \leq T$. So by Lemma 2 and Lemma 4, we have

$$J \ll \sqrt{T} \sum_{m,n \in M'} r(m)r(n) \sum_{k \leq \sqrt{\frac{T}{m}} \pi} \frac{1}{\sqrt{k}}$$

$$\ll T^{3/4} \left( \sum_{m \in M'} r(m) \right)^2 \ll \mathcal{L} T (\log T)^2.$$

□
4. Proof of Theorem 1

We prove Theorem 1 by comparing Proposition 1 and Proposition 2. We will proceed by method of contradiction. So assume that

\[ Z^{-}(t) \leq C_{1} \exp \left( \left( \frac{1}{2} - \epsilon \right) \sqrt{\frac{\log T \log_{3} T}{\log_{2} T}} \right), \]

for all \( t \in [T^{3/4}, T] \) and for some \( C_{1} > 0 \). Let

\[ J_{1} := \int_{T^{3/4}}^{T} \zeta \left( \frac{1}{2} + it \right) |R(t)|^{2} \Phi \left( \frac{t \log T}{T} \right) dt. \]

Then by Proposition 1, for any \( \epsilon > 0 \),

\[ J_{1} \gg \mathcal{L} T \exp \left( \left( \frac{1}{2} - \frac{\epsilon}{2} \right) \sqrt{\frac{\log T \log_{3} T}{\log_{2} T}} \right), \]

Define

\[ J_{2} = J_{1} - 2 \int_{T^{3/4}}^{T} Z^{-}(t)|R(t)|^{2} \Phi \left( \frac{t \log T}{T} \right) dt. \]

We will bound \( J_{2} \) from below using assumption (5) and Lemma 3 as follows

\[ J_{2} \gg \mathcal{L} T \exp \left( \left( \frac{1}{2} - \frac{\epsilon}{2} \right) \sqrt{\frac{\log T \log_{3} T}{\log_{2} T}} \right), \]

as

\[ \int_{T^{3/4}}^{T} Z^{-}(t)|R(t)|^{2} \Phi \left( \frac{t \log T}{T} \right) dt \]

\[ \ll \exp \left( \left( \frac{1}{2} - \epsilon \right) \sqrt{\frac{\log T \log_{3} T}{\log_{2} T}} \right) \int_{T^{3/4}}^{T} |R(t)|^{2} \Phi \left( \frac{t \log T}{T} \right) dt \]

\[ \ll \mathcal{L} T \exp \left( \left( \frac{1}{2} - \frac{2\epsilon}{3} \right) \sqrt{\frac{\log T \log_{3} T}{\log_{2} T}} \right). \]

Since \( |Z(t)| = Z(t) + 2Z^{-}(t) \),

\[ J_{2} \leq \int_{T^{3/4}}^{T} Z(t)|R(t)|^{2} \Phi \left( \frac{t \log T}{T} \right) dt. \]

From Proposition 2 we get

\[ J_{2} \ll \mathcal{L} T (\log T)^{2}, \]

which contradicts (7). So our assumption (5) on \( Z^{-}(t) \) is wrong. This proves (B) of Theorem 1 and the proof of (A) is similar.

References

[1] C. Aistleitner. Lower bounds for the maximum of the Riemann zeta function along vertical lines. Math. Ann., 365(1-2):473–496, 2016.
[2] R. Balasubramanian. On the frequency of Titchmarsh’s phenomenon for \( \zeta(s)-IV. \) Hardy-Ramanujan J., 9:1–10, 1986.
[3] R. Balasubramanian and K. Ramachandra. On the frequency of Titchmarsh’s phenomenon for \( \zeta(s)-III. \) Proc. Indian Acad. Sci., 86:341–351, 1977.
[4] A. Bondarenko and K. Seip. Large greatest common divisor sums and extreme values of the Riemann zeta function. Math. J., 166:1685–1701, 2017.
[5] A. Bondarenko and K. Seip. Extreme values of the Riemann zeta function and its argument. To appear in Mathematische Annalen. Preprint. Available at https://doi.org/10.1007/s00208-018-1663-2
[6] R. de la Bretèche and G. Tenenbaum. Sommes de Gál et applications. Preprint. Available at https://arxiv.org/abs/1804.01629
[7] S. M. Gonek and A. Ivič. On the distribution of positive and negative values of Hardy’s $Z$-function. *J. Number Theory*, 174:189–201, 2017.
[8] A. Ivič. The theory of Hardy’s $Z$-function. *Cambridge University Press*, Cambridge, 245 pp, 2012.
[9] A. Ivič. On large values of Hardy’s function $Z(t)$ and its derivatives. To appear in *Functiones et Approximationes*, Proc. Number Theory Week, Poznan 2017.
[10] K. Soundararajan. Extreme values of zeta and $L$-functions. *Math. Ann.*, 342:467–486, 2008.

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway

E-mail address: accessing.infinity@gmail.com, kanalakshya.mahatab@ntnu.no