Splitting type variational problems with linear growth conditions

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Abstract

Regularity properties of solutions to variational problems are established for a broad class of strictly convex splitting type energy densities of the principal form

\[ f(\xi_1, \xi_2) = f_1(\xi_1) + f_2(\xi_2), \]

with linear growth. As a main result it is shown that, regardless of a corresponding property of \( f_2 \), the assumption

\[ c_1 (1 + |t|)^{-\mu_1} \leq f_1(t) \leq c_2, \quad 1 \leq \mu_1 < 2, \]

is sufficient to obtain higher integrability of \( \partial_1 u \) for any finite exponent. Similar results in the case \( f: \mathbb{R}^n \to \mathbb{R} \) hold with the obvious changes in notation.

1 Introduction

In our paper we discuss variational problems of linear growth with densities which do not belong to the class of \( \mu \)-elliptic energies introduced first in [1].

Guided by linear growth examples of splitting type, which to our knowledge are not systematically studied up to now, we are led to quite general hypotheses which still guarantee some interesting higher regularity properties of generalized solutions.

Before going into details, let us fix the framework of our considerations: in what follows \( \Omega \) denotes a bounded Lipschitz domain in \( \mathbb{R}^n, n \geq 2 \), and we consider a function \( u_0: \Omega \to \mathbb{R} \) such that

\[ u_0 \in W^{1,2}(\Omega) \cap L^\infty(\Omega). \quad (1.1) \]

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2Using a suitable approximation (see, e.g., [2] for more details), it is possible to suppose \( u_0 \in W^{1,1}(\Omega) \cap L^\infty(\Omega) \).
We then are interested in the variational problem
\[
J[u] := \int_{\Omega} f(\nabla u) \, dx \to \min \quad \text{in } u_0 + W^{1,1}_0(\Omega) \tag{1.2}
\]
for a strictly convex energy density \( f : \mathbb{R}^n \to [0, \infty) \) of class \( C^2 \) satisfying
\[
a_1|\xi| - a_2 \leq f(\xi) \leq a_3|\xi| + a_4, \quad \xi \in \mathbb{R}^n, \tag{1.3}
\]
with suitable constants \( a_1, a_3 > 0, a_2, a_4 \geq 0 \).

Condition (1.3) causes the well-known problems concerning the existence and the regularity of solutions to (1.2), which means that (1.2) has to be replaced by a relaxed variant. For the general framework of this approach we refer, e.g., to the monographs [3], [4], [5], [6] and [7], where the reader will find a lot of further references as well as a definition of the underlying spaces such as \( L^p(\Omega) \), \( W^{1,p}(\Omega) \), \( \text{BV}(\Omega) \) and their local variants.

Quoting [6], Theorem 5.47, the natural extension of (1.2) reads as
\[
K[w] := \int_{\Omega} f(\nabla^a w) \, dx + \int_{\Omega} f_{\infty}\left(\frac{\nabla^s w}{|\nabla^s w|}\right) d|\nabla^s w|
+ \int_{\partial \Omega} f_{\infty}((u_0 - w)N) \, d\mathcal{H}^{n-1} \to \min \quad \text{in } \text{BV}(\Omega). \tag{1.4}
\]
Here \( \nabla w = \nabla^a w \mathcal{L}^n + \nabla^s w \) is the Lebesgue decomposition of the vector measure \( \nabla w \) with respect to the \( n \)-dimensional Lebesgue measure \( \mathcal{L}^n \), \( f_{\infty} \) is the recession function of \( f \), i.e.
\[
f_{\infty}(\xi) := \lim_{t \to \infty} \frac{1}{t} f(t\xi), \quad \xi \in \mathbb{R}^n,
\]
\( \mathcal{H}^{n-1} \) is Hausdorff’s measure of dimension \( n - 1 \) and \( N \) denotes the outward unit normal to \( \partial \Omega \).

We summarize some important results concerning the relations between problems (1.2) and (1.4) in the following proposition (compare, e.g., the pioneering work [8] and [9] in the minimal surface case, and, e.g., the papers [10], [11], where the mechanical point of view is represented via the stress tensor as the unique solution of the dual problem).

**Proposition 1.1.** Let (1.1) and (1.3) hold.

i) Problem (1.4) admits at least one solution \( u \in \text{BV}(\Omega) \).

ii) It holds
\[
\inf_{u_0 + W^{1,1}_0(\Omega)} J = \inf_{\text{BV}(\Omega)} K.
\]

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iii) We have

\[
\begin{align*}
  u \in \text{BV}(\Omega) \text{ is } K\text{-minimizing } & \iff \\
  u \in \mathcal{M} := \left\{ v \in L^1(\Omega) : v \text{ is a } L^1(\Omega)\text{-cluster point of some } J\text{-minimizing sequence from } u_0 + W^{1,1}_0(\Omega) \right\}.
\end{align*}
\]

Since Proposition 1.1 in particular guarantees the existence of generalized solutions to problem (1.2), i.e. of functions \( u \in \text{BV}(\Omega) \) solving (1.4), one may ask for their regularity properties.

Here a variety of results is available concerning densities \( f \) of linear growth such that we have in addition

\[
c_1 (1 + |\xi|)^{-\mu}|\eta|^2 \leq D^2 f(\xi)(\eta, \eta) \leq c_2 (1 + |\xi|)^{-1}|\eta|^2
\]

with exponent \( \mu > 1 \) and for constants \( c_1, c_2 > 0 \).

Condition (1.5) is known as the \( \mu \)-ellipticity property of \( f \). Assuming at least \( \mu \leq 3 \), the reader will find regularity results e.g. in the paper [1], in [12] the case of bounded solutions is covered. We also mention the paper [13] and the references quoted therein.

Condition (1.5) is mainly motivated by

\( \triangleright \) energy densities \( f \) of minimal surface type, i.e.

\[
f(\xi) := (1 + |\xi|^k)^{\frac{k}{k}} \quad \text{, } k > 1 ,
\]

\( \triangleright \) or by densities of the form \( f(\xi) = \Phi_\mu(|\xi|) \), where for \( \mu > 1 \) we let \( (t \geq 0) \)

\[
\Phi_\mu(t) := (\mu - 1) \int_0^t \int_0^s (1 + r)^{-\mu} \, dr \, ds ,
\]

\[
= \begin{cases} 
  t - \frac{1}{2-\mu} (1+t)^{2-\mu} + \frac{1}{2-\mu} & \text{if } \mu \neq 2 , \\
  t - \ln(1+t) & \text{if } \mu = 2 .
\end{cases}
\]

(1.7)

Note that recent strain-limiting elastic models with linear growth are strongly related to the class given in (1.6) (see, for instance, [14], [15], [16] and [17]).

Let us have a closer look at the second kind of examples. We carefully have to distinguish between the functions \( f(\xi) = \Phi_\mu(|\xi|) \) defined on \( \mathbb{R}^n \) and the
functions $\Phi_\mu(t)$ depending on one variable in the sense that we have with optimal exponents the inequalities (1.3) for $f$, whereas for all $t \geq 0$

$$c_1(1 + t)^{-\mu} \leq \Phi_\mu'(t) \leq c_2(1 + t)^{-\mu}, \quad \mu > 1,$$

(1.8)

c_1, c_2 > 0. Note that both the exponent occurring in the upper bound of (1.3) and the one of (1.8) are relevant quantities entering the regularity proofs in an essential way.

For instance, in the recent paper [18], the authors benefit from the radial structure of a solution which roughly speaking means that the general ellipticity condition (1.5) can be replaced by the estimate on the right-hand side of (1.8).

However, without using the radial structure of a particular solution, it is no longer possible to benefit that much from the right-hand side of (1.8).

This becomes evident with the following auxiliary lemma which, roughly speaking, states that the right-hand side of (1.3) with exponent $-1$ gives the best possible estimate. Without reducing the problem by, e.g., symmetry properties of the solution, the balancing condition (1.8) may not serve as an additional tool for proving the regularity of solutions.

**Lemma 1.1.** Let $n \geq 2$ and consider a density $f$ of class $C^2$ such that (1.3) and

$$c_1(1 + |\xi|)^{-\mu} |\eta|^2 \leq D^2 f(\xi)(\eta, \eta) \leq c_2(1 + |\xi|)^{-\kappa} |\eta|^2$$

(1.9)

hold for all $\xi, \eta \in \mathbb{R}^n$ with constants $c, c > 0$, and with exponents $\mu > 1$, $\kappa \leq \mu$. Then we have

$$\kappa \leq 1.$$  

The proof of Lemma 1.1 is postponed to the appendix.

Once that $\kappa = 1$ is seen to be the best possible choice in (1.9), the question arises, whether this yields a sufficiently broad class of examples. However, this is not the case if we like to include some kind of splitting structure in our considerations:

**Example 1.1.** For the sake of simplicity let $n = 2$ and consider the energy density of splitting type

$$f(\xi_1, \xi_2) = \Phi_{\mu_1}(|\xi_1|) + \Phi_{\mu_2}(|\xi_2|), \quad \mu_1, \mu_2 > 1.$$  

(1.10)

Then we merely have ($\mu := \max\{\mu_1, \mu_2\}, c_1, c_2 > 0$)

$$c_1(1 + |\xi|)^{-\mu} |\eta|^2 \leq D^2 f(\xi)(\eta, \eta) \leq c_2|\eta|^2, \quad \xi, \eta \in \mathbb{R}^2,$$

(1.11)
and the estimate on the r.h.s. can not be improved as we recognize in the case $|\xi_1| \to \infty$ together with $\xi_2 = \text{const}$, $\eta_1 = 0$.

We emphasize that for this kind of splitting examples the balancing condition (1.8) can not be exploited for improving the condition (1.11), hence instead of $\Phi_\mu$ we may as well consider functions $\psi_\mu$ of the type (adapted to (1.11))

$$c_1 (1 + |t|)^{-\mu} \leq \psi''_\mu(t) \leq c_2, \quad \mu > 1, \quad t \in \mathbb{R},$$

(1.12)

with constants $c_1, c_2 > 0$ still having linear growth. We like to mention that we also do not rely on a $\Delta_2$-condition similar to, e.g., (1.8) of [19].

One may ask whether this generalization is a kind of artefact without providing new relevant examples. The examples sketched in the appendix illustrate that this is not the case.

With (1.11) and (1.12) we are lead to our main theorem on the regularity of solutions.

**Theorem 1.1.** Assume (1.1), let $n = 2$ and consider a density $f$ of the form

$$f(\xi_1, \xi_2) = f_1(\xi_1) + f_2(\xi_2)$$

(1.13)

with strictly convex functions $f_1$, $f_2$ satisfying (1.3). Moreover, suppose that there exist exponents $\mu_i > 1$ such that for $i = 1, 2$

$$c_i (1 + |t|)^{-\mu_i} \leq f''_i(t) \leq \bar{c}_i, \quad t \in \mathbb{R},$$

(1.14)

holds with constants $c_i, \bar{c}_i > 0$.

Let $\mu_1 < 2$. Then there exists a generalized minimizer $u \in \mathcal{M}$ such that

$$\partial_1 u \in L^\infty_{\text{loc}}(\Omega) \quad \text{for any finite } \chi.$$

With the obvious changes in notation, similar results hold in the case $n \geq 3$,

$$f(\xi_1, \ldots, \xi_n) = \sum_{i=1}^n f_i(\xi_i).$$

On one hand, Theorem 1.1 states that higher integrability w.r.t. a particular direction in the splitting case holds provided that the corresponding part of the energy satisfies a sufficient ellipticity condition. No further restriction w.r.t. the second direction is imposed. If we suppose in addition $\mu_2 < 2$, then we expect regular solutions being unique up to a constant. This statement
is formulated in Corollary 1.1 below.

On the other hand, condition (1.11) (with $\mu := \max \{\mu_1, \mu_2\}$ in the splitting case) is very much in the spirit of the ellipticity condition

$$c_1 (1 + |\xi|)^{p-2} |\eta|^2 \leq D^2 f(\xi)(\eta, \eta) \leq c_2 (1 + |\xi|)^{q-2} |\eta|^2, \quad \xi, \eta \in \mathbb{R}^2,$$  
\hspace{1cm} (1.15)

for variational problems with anisotropic superlinear growth conditions, where we have the correspondence $q = 2$ and $\mu = 2 - p$.

Motivated by the famous counterexample of Giaquinta [20] there are a lot of contributions to the regularity theory of solutions which, due to the counterexample, have to impose a suitable relation between the exponents $p$ and $q$. Let us just mention the classical paper [21], the reference [22] on higher integrability or the recent paper [23].

Note that a series of papers is devoted to the splitting case, for instance [20], [24], [25], [26].

In the case of bounded solutions, one suitable relation between $p$ and $q$ implying the regularity of solutions reads as

$$q < p + 2$$

which exactly corresponds to $q = 2$ (i.e. $\kappa = 0$) and $\mu = 2 - p < 2$.

In this spirit we have Corollary 1.2 which in fact is not based on a splitting structure of the problem. We note that a variant discussing mixed linear-superlinear problems is presented in Chapter 6 of [17].

Let us finally remark that the arguments leading to our main Theorem 1.1 may be applied to variational problems of splitting structure with superlinear growth conditions.

**Corollary 1.1.** Suppose that the assumptions of Theorem 1.1 hold together with

$$\mu := \max \{\mu_1, \mu_2\} < 2.$$  

Then the relaxed problem (1.4) admits a solution $u \in C^{1, \alpha}(\Omega)$, $0 < \alpha < 1$. Moreover, this solution is unique up to additive constants.

**Corollary 1.2.** Suppose that we have (1.1). Let $f$ satisfy (1.3) together with

$$c_1 (1 + |\xi|)^{-\frac{2}{2}} |\eta|^2 \leq D^2 f(\xi)(\eta, \eta) \leq c_2 (1 + |\xi|)^{-\kappa} |\eta|^2, \quad \xi, \eta \in \mathbb{R}^2,$$  
\hspace{1cm} (1.16)
for some constants $c_1, c_2 > 0$. Suppose that the exponents $\mu > 1$, $\kappa \leq 1$ satisfy in addition
\[ \mu < 2 + \kappa. \] (1.17)
Then problem (1.4) admits a solution $u \in \mathcal{M}$ being of class $C^{1,\alpha}(\Omega)$ for any $0 < \alpha < 1$. Moreover, this solution is unique up to additive constants, which means $v = u + c$ for any $v \in \mathcal{M}$ with a suitable constant $c \in \mathbb{R}$.

2 Proof of Theorem 1.1

We fix some $0 < \delta < 1$ and let
\[
\begin{align*}
    f_{i,\delta} &:= \frac{\delta}{2} t^2 + f_i(t), \quad t \in \mathbb{R}, \quad i = 1, 2, \\
    f_\delta(\xi) &= f_{1,\delta}(\xi_1) + f_{2,\delta}(\xi_2), \\
    &= \frac{\delta}{2} |\xi|^2 + f(\xi) \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.
\end{align*}
\]
We then consider the regularized minimization problem
\[
J_\delta[w] := \int_{\Omega} f_\delta(\nabla w) \, dx \to \min \quad \text{in} \quad u_0 + W^{1,2}_0(\Omega) \quad (1.2_\delta)
\]
with $u_\delta$ denoting the unique solution of (1.2). Following standard arguments (see [7] and a series of well known references quoted therein) one immediately obtains
\[
u \in W^{2,2}_{\text{loc}}(\Omega) \cap C^1(\Omega)
\]
and passing to a subsequence, if necessary, as $\delta \to 0$
\[
\delta \int_{\Omega} |\nabla u_\delta|^2 \, dx \to 0, \quad u_\delta \to u \quad \text{in} \quad L^1(\Omega) \quad \text{with some} \quad u \in \mathcal{M}.
\]
Note that using (1.1) from now on it is supposed that
\[
\sup_\delta \|u_\delta\|_{L^\infty(\Omega)} < \infty.
\]
We let
\[
\Gamma_{i,\delta} := 1 + |\partial_i u_\delta|^2, \quad i = 1, 2,
\]
differentiate the Euler equation
\[
0 = \int_{\Omega} Df_\delta(\nabla u_\delta) \cdot \nabla \varphi \, dx \quad \text{for all} \quad \varphi \in C_0^{\infty}(\Omega)
\]
in the sense that we insert \( \varphi = \partial_1 \psi \) as test function and obtain for all \( \psi \in C_0^\infty(\Omega) \)
\[
0 = \int_\Omega D^2 f_\delta(\nabla u_\delta)(\nabla \partial_1 u_\delta, \nabla \psi) \, dx. \tag{2.1}
\]

The first main step in the proof of Theorem 1.1 is to show the following Caccioppoli-type inequality.

**Proposition 2.1.** Fix \( l \in \mathbb{N} \) and suppose that \( \eta \in C_0^\infty(\Omega) \), \( 0 \leq \eta \leq 1 \). Then, given the assumptions of Theorem 1.1, the inequality
\[
\int_{\Omega} D^2 f_\delta(\nabla u_\delta)(\nabla \partial_1 u_\delta, \nabla \partial_1 u_\delta) \eta^{2l} \Gamma_{1,\delta}^\alpha \, dx
\]
\[
\leq c \int_{\Omega} D^2 f_\delta(\nabla u_\delta)(\nabla \eta, \nabla \eta) \eta^{2l-2} \Gamma_{1,\delta}^{\alpha+1} \, dx \tag{2.2}
\]
holds for any \( \alpha \geq 0 \), which in particular implies
\[
\int_{\Omega} \eta^{2l} \Gamma_{1,\delta}^{\alpha-1} |\partial_{11} u_\delta|^2 \, dx \leq c \int_{\Omega} |\nabla \eta|^2 \eta^{2l-2} \Gamma_{1,\delta}^{\alpha+1} \, dx. \tag{2.3}
\]

**Proof of Proposition 2.1.** We insert the admissible test function
\[
\psi := \eta^{2l} \partial_1 u_\delta \Gamma_{1,\delta}^\alpha
\]
in (2.1) and obtain
\[
\int_{\Omega} D^2 f_\delta(\nabla u_\delta)(\nabla \partial_1 u_\delta, \nabla \partial_1 u_\delta) \eta^{2l} \Gamma_{1,\delta}^\alpha \, dx
\]
\[
= - \int_{\Omega} D^2 f_\delta(\nabla u_\delta)(\nabla \partial_1 u_\delta, \nabla \Gamma_{1,\delta}^\alpha) \partial_1 u_\delta \eta^{2l} \, dx
\]
\[
- \int_{\Omega} D^2 f_\delta(\nabla u_\delta)(\nabla \partial_1 u_\delta, \nabla (\eta^{2l})) \partial_1 u_\delta \Gamma_{1,\delta}^\alpha \, dx =: S_1 + S_2. \tag{2.4}
\]

For \( S_1 \) we have
\[
S_1 = -\alpha \int_{\Omega} D^2 f_\delta(\nabla u_\delta)(\nabla \partial_1 u_\delta, \nabla |\partial_1 u_\delta|^2) \Gamma_{1,\delta}^{\alpha-1} \partial_1 u_\delta \eta^{2l} \, dx
\]
\[
= -2\alpha \int_{\Omega} D^2 f_\delta(\nabla u_\delta)(\nabla \partial_1 u_\delta, \nabla \partial_1 u_\delta) |\partial_1 u_\delta|^2 \Gamma_{1,\delta}^{\alpha-1} \eta^{2l} \, dx \leq 0
\]
whenever \( \alpha \geq 0 \), hence the left-hand side of (2.4) is bounded by \( |S_2| \).

\( S_2 \) is handled with the help of the Cauchy-Schwarz inequality for \( 0 < \varepsilon \) sufficiently small:
\[
\int_{\Omega} D^2 f_\delta(\nabla u_\delta)(\nabla \partial_1 u_\delta, \nabla \eta) \eta^{2l-1} \Gamma_{1,\delta}^\alpha \partial_1 u_\delta \, dx
\]
\[
\leq \varepsilon \int_{\Omega} D^2 f_\delta(\nabla u_\delta)(\nabla \partial_1 u_\delta, \nabla \partial_1 u_\delta) \eta^{2l} \Gamma_{1,\delta}^\alpha \, dx
\]
\[
+ c(\varepsilon) \int_{\Omega} D^2 f_\delta(\nabla u_\delta)(\nabla \eta, \nabla \eta) \eta^{2l-2} \Gamma_{1,\delta}^\alpha |\partial_1 u_\delta|^2 \, dx
\]
and absorbing the first term we have shown (2.2).

We now benefit from the splitting structure expressed in (1.13), use (1.14) and estimate the left-hand side of (2.2) from below

\[ \int_{\Omega} \eta^2 \Gamma_{1,\delta}^{\alpha - \frac{\mu_1}{2}} |\partial_{11} u_{\delta}|^2 \, dx \]

\[ \leq c \int_{\Omega} f''_1(\partial_1 u_{\delta})|\partial_{11} u_{\delta}|^2 \eta^2 \Gamma_{1,\delta}^{\alpha} \, dx \]

\[ \leq c \int_{\Omega} \left[ f''_1(\partial_1 u_{\delta})|\partial_{11} u_{\delta}|^2 + f''_2(\partial_2 u_{\delta})|\partial_{12} u_{\delta}|^2 \right] \eta^2 \Gamma_{1,\delta}^{\alpha} \, dx \]

\[ \leq c \int_{\Omega} D^2 f_\delta(\nabla \partial_1 u_{\delta}, \nabla \partial_1 u_{\delta}) \eta^2 \Gamma_{1,\delta}^{\alpha} \, dx. \quad (2.5) \]

For the right-hand side of (2.2) we have with \( \delta \leq 1 \) and recalling (1.14)

\[ \int_{\Omega} D^2 f_\delta(\nabla u_{\delta})(\nabla \eta, \nabla \eta) \eta^{2l-2} \Gamma_{1,\delta}^{\alpha+1} \, dx \]

\[ \leq c \int_{\Omega} \left[ f''_1(\partial_1 u_{\delta})|\partial_1 \eta|^2 + f''_2(\partial_2 u_{\delta})|\partial_2 \eta|^2 \right] \eta^{2l-2} \Gamma_{1,\delta}^{\alpha+1} \, dx \]

\[ \leq c \int_{\Omega} |\nabla \eta|^2 (1 + \delta) \eta^{2l-2} \Gamma_{1,\delta}^{\alpha+1} \, dx. \quad (2.6) \]

By combining (2.2), (2.5) and (2.6) we have established the claim (2.3), hence Proposition 2.1. \( \square \)

Now we are going to discuss the second main ingredient of the proof of Theorem 1.1.

**Proposition 2.2.** Given the hypotheses of Theorem 1.1, let

\[ 0 \leq s := \chi - 1, \quad \delta := 1 - \frac{\mu_1}{2} > 0, \quad \alpha := s - \frac{\delta}{2}. \]

Then for \( l \) sufficiently large and a local constant \( c(\eta) \)

\[ \int_{\Omega} \eta^2 \Gamma_{1,\delta}^{\alpha+1} \, dx \leq c \left[ 1 + \int_{\Omega} \eta^2 \Gamma_{1,\delta}^{\alpha+(2+\mu_1)4} \, dx \right]. \quad (2.7) \]

**Proof of Proposition 2.2** We recall that

\[ ||u_{\delta}||_{L^\infty(\Omega)} \leq c \]
Recalling the right-hand side of (2.8):

\[ \int_{\Omega} |\partial_1 u_\delta|^2 \Gamma^{s}_{1,\delta} \eta^{2l} \, dx = \int_{\Omega} \partial_1 u_\delta \partial_1 u_\delta \Gamma^{s}_{1,\delta} \eta^{2l} \, dx = -\int_{\Omega} u_\delta \partial_1 \left[ \partial_1 u_\delta \Gamma^{s}_{1,\delta} \eta^{2l} \right] \, dx \]

\[ \leq c \left[ \int_{\Omega} |\partial_1 u_\delta|^2 \Gamma^{s}_{1,\delta} \eta^{2l} \, dx + \int_{\Omega} |\partial_1 u_\delta|^2 \Gamma^{s-1}_{1,\delta} \eta^{2l} \, dx + \int_{\Omega} \Gamma^{s-1}_{1,\delta} |\partial_1 u_\delta|^2 |\partial_1 u_\delta| \eta^{2l} \, dx \right] \]

\[ \leq c \left[ \int_{\Omega} |\partial_1 u_\delta|^2 \Gamma^{s}_{1,\delta} \eta^{2l} \, dx + \varepsilon \int_{\Omega} |\partial_1 u_\delta|^2 \Gamma^{s}_{1,\delta} \eta^{2l} \, dx + c(\varepsilon) \int_{\Omega} |\nabla \eta|^2 \eta^{2l-2} \Gamma^{s}_{1,\delta} \, dx \right], \tag{2.9} \]

where we may choose \( \varepsilon > 0 \) sufficiently small to absorb the second term on the right-hand side. This means that we have

\[ \int_{\Omega} |\partial_1 u_\delta|^2 \Gamma^{s}_{1,\delta} \eta^{2l} \, dx \leq c \left[ \int_{\Omega} |\partial_1 u_\delta|^2 \Gamma^{s}_{1,\delta} \eta^{2l} \, dx + \int_{\Omega} |\nabla \eta|^2 \eta^{2l-2} \Gamma^{s}_{1,\delta} \, dx \right]. \tag{2.8} \]

Recalling \( \mu_1 < 2 \) and using Young’s inequality, we estimate the first term on the right-hand side of (2.8):

\[ \int_{\Omega} |\partial_1 u_\delta|^2 \Gamma^{s}_{1,\delta} \eta^{2l} \, dx = \int_{\Omega} |\partial_1 u_\delta|^2 \Gamma^\frac{s}{2} \Gamma^\frac{s}{2} \eta^{2l} \, dx \]

\[ \leq c \left[ \int_{\Omega} |\partial_1 u_\delta|^2 \Gamma^{s-\alpha+\frac{\alpha}{2}} \eta^{2l} \, dx + \int_{\Omega} \Gamma^{2s-\alpha+\frac{\alpha}{2}} \eta^{2l} \, dx \right]. \tag{2.9} \]

Here the first term on the right-hand side is handled with the help of the inequality (2.3) given in Proposition 2.1, hence (2.9) implies

\[ \int_{\Omega} |\partial_1 u_\delta|^2 \Gamma^{s}_{1,\delta} \eta^{2l} \, dx \leq c \left[ \int_{\Omega} |\nabla \eta|^2 \eta^{2l-2} \Gamma^{\alpha+1}_{1,\delta} \, dx + \int_{\Omega} \Gamma^{2s-\alpha+\frac{\alpha}{2}} \eta^{2l} \, dx \right]. \tag{2.10} \]

Inserting (2.10) in (2.8) yields

\[ \int_{\Omega} \Gamma^{s+1}_{1,\delta} \eta^{2l} \, dx = \int_{\Omega} \Gamma^{s}_{1,\delta} \eta^{2l} \, dx + \int_{\Omega} |\partial_1 u_\delta|^2 \Gamma^{s}_{1,\delta} \eta^{2l} \, dx \]

\[ \leq c \left[ \int_{\Omega} (\eta^{2l} + |\nabla \eta|^2 \eta^{2l-2}) \Gamma^{s}_{1,\delta} \, dx + \int_{\Omega} |\nabla \eta|^2 \eta^{2l-2} \Gamma^{\alpha+1}_{1,\delta} \, dx \right. \]

\[ + \left. \int_{\Omega} \Gamma^{2s-\alpha+\frac{\alpha}{2}} \eta^{2l} \, dx \right]. \tag{2.11} \]
We now choose \( l \) sufficiently large in order to absorb the first two integrals on the r.h.s. of (2.11): suppose that with some positive real number \( s \)

\[
\gamma_1 = s + 1, \quad \gamma_2 = s, \quad \gamma_2 = \alpha + 1, \quad \tilde{l} \geq l.
\]

Then we have by Young’s inequality for any \( \varepsilon > 0 \)

\[
\int_{\Omega} c(\nabla \eta) \eta^{2 \gamma_1} 1_{\delta} d x = \int_{\Omega} c(\nabla \eta) \eta^{2 \gamma_1} 1_{\delta} d x \leq \varepsilon \int_{\Omega} \eta^{2 \gamma_1} 1_{\delta} d x + c(\varepsilon, \eta) \int_{\Omega} (2 \gamma_1 \varepsilon) d x. \tag{2.12}
\]

We apply (2.12) with the choice \( \gamma_1 = s + 1 \) and \( \gamma_2 = s \), \( \gamma_2 = \alpha + 1 \), respectively, recalling \( \alpha < s \).

Moreover \( \tilde{l} = l - 1 \) shows that \( p \tilde{l} \geq l \) for \( l \) sufficiently large and (2.9) finally gives

\[
\int_{\Omega} \eta^{2 \gamma_1} 1_{\delta} d x \leq c \left[ 1 + \int_{\Omega} \eta^{2 \gamma_2} 1_{\delta} d x \right]. \tag{2.13}
\]

We note that

\[
2s - \alpha + \frac{\mu_1}{2} = s + \frac{\hat{\varepsilon}}{2} + \frac{\mu_1}{2} = s + \frac{2 + \mu_1}{2},
\]

thus with (2.13) we have Proposition 2.2.

To finish the proof of Theorem 1.1 we recall \( \mu_1 < 2 \) and write (2.7) in the form

\[
\int_{\Omega} \eta^{2 \gamma_1} 1_{\delta} d x \leq c \left[ 1 + \int_{\Omega} \eta^{2 \gamma_2} 1_{\delta} d x \right], \quad \gamma_1 > 1. \tag{2.14}
\]

Then the same way of absorbing terms as outlined in (2.12) completes the proof of our main theorem.

### 3 Proof of Corollary 1.1

Clearly we may apply the lines of Theorem 1.1 both for \( \partial_1 u \) and for \( \partial_2 u \) and obtain on account of

\[
\partial_i u_\delta \in L^\chi_{\text{loc}}(\Omega) \quad \text{for } i = 1, 2, \quad \text{for all } \chi \text{ and uniform in } \delta
\]

for any \( \alpha_i \geq 0, \ i = 1, 2 \):

\[
\int_{\Omega} D^2 f_\delta(\nabla \partial_i u_\delta, \nabla \partial_i u_\delta) \Gamma_{\alpha_i} 1_{\delta} d x \leq c. \tag{3.1}
\]
Given (3.1) let us shortly discuss the stress tensor \( \sigma \), i.e. the solution of the dual variational problem. In [27] (see also Section 2.2 of [7]) it is shown by elementary arguments from measure theory, that the dual problem admits a unique solution and this in turn will give the uniqueness of generalized minimizers up to additive constants as it will be outlined below.

We note that as a general hypothesis of [27] it is supposed that

\[
0 \leq D^2 f(\xi)(\eta, \eta) \leq c(1 + |\xi|^2)^{-\frac{1}{2}}|\eta|^2,
\]

where the right-hand side is not given in the setting under consideration.

However, following the proof of [27], condition (3.2) is just needed for showing the uniform local \( W^{1,2} \)-regularity of the regularized sequence \( \sigma_\delta \) which immediately follows from (3.1). As an important consequence, we also have Theorem A.9 of [7].

Now we claim that for any \( \Omega' \subset \Omega \) and uniformly in \( \delta \)

\[
i) \ \| \nabla^2 u_\delta \|_{L^2(\Omega' ; \mathbb{R}^{2\times 2})} \leq c, \quad ii) \ \| \nabla u_\delta \|_{L^\infty(\Omega' ; \mathbb{R}^{2})} \leq c. \quad (3.3)
\]

In fact, we have for arbitrary exponents \( \alpha_1, \alpha_2 > 0 \)

\[
\int_\Omega \left[ \Gamma_{1,\delta}^{\alpha_1-\mu_1,2} |\partial_{11} u_\delta|^2 + \Gamma_{2,\delta}^{\alpha_2-\mu_2,2} |\partial_{22} u_\delta|^2 \right. \\
+ \left. \left[ \Gamma_{2,\delta}^{\mu_2,2} \Gamma_{1,\delta}^{\alpha_1,2} + \Gamma_{1,\delta}^{\mu_1,2} \Gamma_{2,\delta}^{\alpha_2,2} \right] |\partial_{1} \partial_{2} u_\delta|^2 \right] \eta^2 \, dx
\]

\[
\leq \int_\Omega \left[ f''_1(\partial_1 u_\delta) |\partial_{11} u_\delta|^2 + f''_2(\partial_2 u_\delta) |\partial_{22} u_\delta|^2 \right] \Gamma_{1,\delta}^{\alpha_1,2} \eta^2 \, dx
\]

\[
+ \int_\Omega \left[ f''_1(\partial_1 u_\delta) |\partial_{1} \partial_{2} u_\delta|^2 + f''_2(\partial_2 u_\delta) |\partial_{22} u_\delta|^2 \right] \Gamma_{2,\delta}^{\alpha_2,2} \eta^2 \, dx
\]

\[
\leq c \sum_{i=1}^2 \int_\Omega D^2 f_\delta(\nabla u_\delta)(\nabla \partial_i u_\delta, \nabla \partial_i u_\delta) \Gamma_{i,\delta}^{\alpha_i,2} \eta^2 \, dx
\]

and by (3.1) we obtain the first claim of (3.3). For the second claim we refer, for instance, to Theorem 5.22 of [7].

Given (3.3), we pass to the limit, denote \( v = \partial_i u, \ i = 1, 2 \), and obtain

\[
\int_\Omega D^2 f(\nabla u)(\nabla v, \nabla \varphi) \, dx = 0 \quad \text{for all} \ \varphi \in C^1_0(\Omega),
\]
where the coefficients are locally uniformly elliptic. By a standard reasoning (see, e.g., [28], Theorem 8.22) we have Hölder continuity of $v$.

Then, by the “stress-strain relation” for the particular generalized minimizer and the unique dual solution $\sigma$,

$$\sigma = \nabla f(\nabla u),$$

we have continuity of $\sigma$ and $\sigma$ takes values in the set $\text{Im} \nabla f$.

Thus, we may apply Theorem A.9 of [7] to obtain the uniqueness of generalized minimizers up to an additive constant.

4 Appendix

4.1 Proof of Lemma 1.1

Arguing by contradiction we assume the validity of the second inequality in (1.9) with exponent $\kappa > 1$.

W.l.o.g. we assume $n = 2$ since otherwise we replace $f$ by $\tilde{f}: \mathbb{R}^2 \to \mathbb{R}$, $\tilde{f}(p_1, p_2) := f(p_1, p_2, 0, \ldots, 0)$ observing that $\tilde{f}$ satisfies (1.3), (1.9) for $\xi$, $\eta \in \mathbb{R}^2$ with the same exponents $\mu$ and $\kappa$.

Consider an increasing sequence of numbers $c_k > 0$ such that

$$\lim_{k \to \infty} c_k = \infty$$

and let

$$A_k := [f \leq c_k] := \{p \in \mathbb{R}^2 : f(p) \leq c_k\}.$$

Condition (1.3) implies the boundedness of each set $A_k$, moreover, we have strict convexity of $A_k$ on account of (1.9).

Let $\gamma_k$ denote a parametrization by arc length of the closed convex curve $\partial A_k = [f = c_k]$ – in fact, we just need a parametrization inside a small neighborhood of the point $p_k$ considered in (4.6) below. For each $k$ it holds $c_k = f(\gamma_k(t))$, hence

$$0 = \gamma_k''(t),$$

and (4.3) implies by taking the derivative

$$0 = D^2 f(\gamma_k(t)) (\gamma_k'(t), \gamma_k''(t)) + Df(\gamma_k(t)) \cdot \gamma_k''(t).$$
From (4.2) and (4.3) we deduce that \( Df(\gamma_k(t)) \) and \( \gamma_k''(t) \) are proportional for each \( t \), therefore (4.4) implies

\[
|\gamma_k''(t)| Df(\gamma_k(t)) = |\gamma_k''(t) \cdot Df_k(\gamma_k(t))| = D^2 f(\gamma_k(t))(\gamma_k'(t), \gamma_k'(t)).
\]

Combing this equation with (1.9) we find

\[
|\gamma_k''(t)| Df(\gamma_k(t)) = \frac{1}{r_k},
\]

moreover, we have \( |\gamma_k(t_k)| = r_k \). Combining (4.5) and (4.6) we obtain

\[
\frac{1}{|\gamma_k(t_k)|} Df(\gamma_k(t_k)) \leq c_2 (1 + |\gamma_k(t)|)^{-\kappa}.
\]

The convexity of \( f \) implies

\[
Df(\gamma_k(t_k)) \cdot \gamma_k(t_k) \geq b_1 |\gamma_k(t_k)| - b_2
\]

with suitable real numbers \( b_1 > 0, b_2 \geq 0 \) independent of \( k \). Recalling \( f(\gamma_k(t)) = c_k \) as well as our assumption (4.1) concerning the sequence \( (c_k) \), condition (1.3) implies \( |\gamma_k(t_k)| \to \infty \), hence (4.8) shows for \( k \gg 1 \) the validity of

\[
Df(\gamma_k(t_k)) \cdot \gamma_k(t_k) \geq \frac{1}{2} b_1 |\gamma_k(t_k)|.
\]

From (4.7) and (4.9) it finally follows

\[
\frac{1}{|\gamma_k(t_k)|} \leq \frac{2}{b_1} \left| Df(\gamma_k(t_k)) \cdot \gamma_k(t_k) \right| \leq \frac{2}{b_1} c_2 (1 + |\gamma_k(t_k)|)^{-\kappa},
\]

or equivalently

\[
b_1 \leq 2c_2 |\gamma_k(t_k)|(1 + |\gamma_k(t_k)|)^{-\kappa}.
\]

Note that the r.h.s. vanishes as \( k \to \infty \) in case \( \kappa > 1 \) leading to a contradiction and proving the lemma.
4.2 Examples

Even in the one-dimensional setting there is a fundamental difference to the discussion of examples with superlinear growth, where a natural scale is given by

\[ h_p(t) = (1 + t)^p, \quad t \geq 0, \quad p > 1, \]

with growth rate \( p \) and satisfying a balancing condition (1.8) with exponent \( p - 2 = -\mu \) in both directions.

In the case of linear growth problems we have, for instance, the functions \( \Phi_\mu \) satisfying (1.8). However, the growth of \( \Phi_\mu \) is linear, hence not depending on \( \mu \).

In this second part of the appendix we are going to sketch some examples which show that the prototypes \( \Phi_\mu \) leading to energy densities with linear growth are far apart from being universal representatives.

The construction of examples \( h: \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \) of class \( C^2 \) such that \( \hat{h}(t) = h(|t|) \) fits into the above discussion is limited by the two constraints

1. \( h''(t) > 0 \) for all \( t \in \mathbb{R}_0^+ \),
2. \( \int_0^\infty h''(\tau)d\tau < \infty \).

We just mention three examples which are interesting from different point of views.

i) Let us start by considering the family \( \Phi_\mu/ (\mu - 1) \) in limit case \( \mu = 1 \). In this limit case we have

\[
\frac{\Phi_\mu}{\mu - 1} = t \ln(1 + t) + \ln(1 + t) - t,
\]

i.e. the limit case corresponds to an energy density nearly linear growth. We now present an easy example given in an explicit analytic form such that

\[
h''(t) \leq c(1 + t)^{-1} \quad \text{for all} \quad t \in \mathbb{R}_0^+.
\]

with optimal exponent \(-1\) on the right-hand side and such that \( f \) is of linear growth. We let

\[
h(t) = \int_0^t h'(\tau)d\tau, \quad h'(t) = 1 - (1 + t)^{1-\mu(t)},
\]

\[
\mu(t) = 1 + \frac{1}{\ln(1 + t)}, \quad t \gg 1.
\]

Then we have the desired estimate

\[
h''(t) = 2(1 + t)^{-1} \frac{1}{\ln(1 + t)}, \quad t \gg 1.
\]
ii) The second example is of linear growth with a lower bound for the second derivative which is even worse than involving an exponent $-\mu$. We consider the elementary function

$$h(t) = \ln(1 + e^t) \quad \text{for all } t \in \mathbb{R}_0^+.$$ 

An easy calculation shows for all $t \geq 0$

$$h'(t) = \frac{e^t}{1 + e^t}, \quad f''(t) = \frac{1}{4 \cosh^2(t/2)}.$$ 

iii) In view of the above given line for the construction of examples we may also have a countable union of atoms. The example corresponds to an approximation of a convex piecewise affine continuous function of linear growth.

For $i \in \mathbb{N}$ and $\sigma_i > 0$ we consider

$$h''(t) = \sum_{i=1}^{\infty} e^{-\frac{|t-i|^2}{\sigma_i}}, \quad t \geq 0,$$

and for $\sigma_i$ sufficiently small one obtains

$$\int_0^{\infty} h''(t) \, dt \leq c \sum_{i=1}^{\infty} \sigma_i < c.$$ 

The best right-hand side estimate for the second derivative is a constant bound since we have

$$1 \leq h''(i) \leq c, \quad i \in \mathbb{N}.$$ 

We may also consider $h$ as a modulation of a $\mu$-elliptic signal such that (1.14) is the best possible estimate.

Let us finally mention that for the construction of examples with linear growth conditions the boundedness of $h''$ is not needed. We restricted our considerations to this case in order to avoid further technical difficulties. For instance, in the proof of Theorem 1.4 we could rely and the most convenient kind of regularization, i.e. adding a small Dirichlet part to the energy under consideration.
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