A SUBGRID STABILIZING POSTPROCESSED MIXED FINITE ELEMENT METHOD FOR THE TIME-DEPENDENT NAVIER-STOKES EQUATIONS

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ABSTRACT. A postprocessed mixed finite element method based on a subgrid model is presented for the simulation of time-dependent incompressible Navier-Stokes equations. This method consists of two steps: the first step is to solve a subgrid stabilized nonlinear Navier-Stokes system on a coarse grid to obtain an approximate solution \(u_H(x,T)\) at the final time \(T\), and the second step is to postprocess \(u_H(x,T)\) by solving a stabilized Stokes problem on a finer grid or by higher-order finite element elements defined on the same coarse grid. Stability of the method and error estimates of the processing solution are analyzed. Numerical results on an example with known analytic solution and the flow around a circular cylinder are given to verify the theoretical predictions and demonstrate the effectiveness of the proposed method.

1. Introduction. In this paper, we consider the following time-dependent incompressible Navier-Stokes problem:

\[
\begin{align*}
u \frac{\partial u}{\partial t} - \nabla p + (u \cdot \nabla) u &= f, \quad \text{in } \Omega \times (0,T], \\
\nabla \cdot u &= 0, \quad \text{in } \Omega \times (0,T], \\
u \frac{\partial u}{\partial t} &= u_0, \quad \text{in } \Omega, \\
\end{align*}
\]

where \(\Omega \subset \mathbb{R}^d\) \((d = 2, 3)\) is a bounded domain with smooth boundary \(\partial \Omega\), \(u : \Omega \times (0, T] \to \mathbb{R}^d\) represents the velocity vector, \(p : \Omega \times (0, T] \to \mathbb{R}\) the pressure, \(f : \Omega \times (0, T] \to \mathbb{R}^d\) the prescribed body force, \(\nu > 0\) the kinematic viscosity, \(u_0\) the prescribed initial velocity, \(T\) the final time and \(\frac{\partial u}{\partial t} = \frac{du}{dt}\).

The above incompressible Navier-Stokes system is a typical nonlinear system describing the motion of a viscous Newtonian fluid. This system is widely used in weather, ocean current and so on, and its research is very important for people to understand turbulence. However, analytic solutions to the Navier-Stokes system are usually difficult to seek, and hence numerical simulation becomes an important way...
to understand the behaviour of the flow governed by the Navier-Stokes equations. In recent decades, much attention is focused on finite element methods for solving numerically the Navier-Stokes equations (cf. [9, 14, 15, 28, 35]). The idea of post-processing method was originally proposed for Fourier spectral methods in [10,11]. Later, it was studied in mixed finite element case [12]. The postprocessed mixed finite element method consists of two steps. The first step is to solve a nonlinear system on a coarse grid, to obtain an approximate solution $u_H$ at time $T$. And then, the second step is to postprocess the solution $u_H$ by solving a linear Stokes problem [3, 4], or a linear Oseen-type problem [7], or one step of Newton iterations [8] on a finer grid or by higher-order finite elements. From theoretical analysis and numerical experiments, it has been shown that under the conditions of selecting appropriate grid sizes or suitable finite element spaces, the postprocessed finite element method can improve the precision of the mixed finite element solutions and the convergence order by one unit compared with the corresponding methods without postprocessing. In terms of computing time, compared with the standard Galerkin finite element method applied directly on the same fine mesh or in the same higher-order finite element spaces, the postprocessed method costs less CPU time since the time evolution of the nonlinear system is done on the coarse mesh or in low-order finite element spaces, and only at the final time $T$, the solution of the first step is postprocessed by solving a linear problem on a finer grid or by higher-order finite element spaces. These type of postprocessed methods are different from the standard two-level methods (cf. [1, 13, 19–22, 32] ) where each time-evolution step involves computation on both coarse and fine meshes, while here the second level computation is just performed at the last time-evolution step.

However, due to the fact that a completely nonlinear problem is needed to be solved on the coarse grid, it is challenging for the postprocessing finite element method to simulate high Reynolds number flows where instability may occur due to the convection-dominance. For the instability of high Reynolds number flows, researchers have proposed a series of stabilization methods, including the defect-correction methods (cf. [29,31,36]), the variational multiscale methods (cf. [26,27]) and the subgrid stabilization methods (cf. [16–18,30,33,34,37]), among others. In this paper, we combine the postprocessed finite element method [3] with a subgrid stabilization method [33,37] to put forward a subgrid stabilizing postprocessed finite element method for the simulation of convection-dominant time dependent Navier-Stokes equations. We not only derive the error bounds of approximate velocity and pressure, but also give some numerical results to verify the theoretical predictions and demonstrate the effectiveness of the proposed combination method.

The outline of the paper is as follows. In Section 2, we introduce some mathematical preliminaries for the Navier-Stokes equations. We give the subgrid stabilizing postprocessed finite element method in Section 3. In Section 4, we prove the stability of the numerical scheme and derive error estimates for the approximate solutions. Section 5 is devoted to numerical results that verify the theoretical predictions and illustrate the effectiveness of the method. Finally, the article is concluded in Section 6.

2. Preliminaries. Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) of class $C^m, m \geq 3$. And for the mathematical setting of problem (1)-(4), we introduce the following Hilbert spaces

$$H = \{u \in (L^2(\Omega))^d, \ | \ \text{div}(u) = 0, u \cdot n|_{\partial \Omega} = 0\}, V = \{u \in (H^1_0(\Omega))^d, \ | \ \text{div}(u) = 0\},$$
endowed with the inner product of \( L^2(\Omega)^d \) and \( H_0^1(\Omega)^d \). For \( 1 \leq q \leq \infty \) and \( l \geq 0 \), we are going to consider the standard Sobolev spaces \( W^{l,q}(\Omega)^d \) of functions with derivatives up to order \( l \) in \( L^q(\Omega) \), and \( H^l(\Omega)^d = W^{l,2}(\Omega)^d \). We use \( \| \cdot \| \) to denote the norm of the Sobolev space \( H^l(\Omega)^d \), and \( \| \cdot \|_{-l} \) represents the norm of its dual space. The quotient space \( H^l(\Omega)/\mathbb{R} \) is endowed with the norm \( \| p \|_{H^l/\mathbb{R}} = \inf \{ \| p + c \| : c \in \mathbb{R} \} \).

Throughout this article, we shall use the letter \( c \) or \( C \) (with or without subscripts) to denote a generic positive constant, which is independent of the mesh parameters, and may take on different values on different occurrences.

We recall the following Sobolev’s imbeddings (cf. [2]): for \( \forall q \in [1, \infty), q' < \infty \), there exists a constant \( C = C(\Omega, q) \), such that
\[
\| v \|_{L^q(\Omega)^d} \leq C \| v \|_{W^{s,q}(\Omega)^d}, \quad \frac{1}{q} \geq \frac{1}{q'} - \frac{s}{d} > 0, \quad v \in W^{s,q}(\Omega)^d. \tag{5}
\]

In particular, for \( q' = \infty \), it holds with \( \frac{1}{q} < \frac{s}{d} \).

Let \( \Pi : L^2(\Omega)^d \to H \) be a Leray projection, \( A \) the Stokes operator defined in \( \Omega \):
\[
A : D(A) \subset H \to H \quad A = -\Pi \Delta, \quad D(A) = H^2(\Omega)^d \cap V.
\]

If we use Leray’s projection operator on both sides of (1)-(2), we get
\[
u_t + \nu Au + B(u, u) = \Pi f, \quad \text{in} \ \Omega.
\]

where \( B(u, u) = \Pi ((u \cdot \nabla)u) \).

Considering the regularity at time \( t = 0 \), we shall assume that the strong solution \((u, p)\) of (1)-(4) satisfies
\[
\max_{0 \leq t \leq T} (\| u(t) \|_r + \| p(t) \|_{H^{r-1}/\mathbb{R}}) < \infty, \tag{6}
\]
\[
\max_{0 \leq t \leq T} (\| u_t(t) \|_r + \| p_t(t) \|_{H^{r-1}/\mathbb{R}}) < \infty. \tag{7}
\]

Let \( T_h = (\tau_h^i, \phi_h^i)_{i \in I_h}, h > 0 \), be a quasi-uniform grid division of \( \Omega_h \) (where \( \Omega_h \) is an approximation of \( \Omega \)), where \( h \) is the maximum diameter of the elements \( \tau_h^i \in T_h \) and \( \phi_h^i : \tau_h^i \to \tau_h^i \) is injective. When \( r \geq 2 \), based on the grid division \( T_h \), we can define the following finite element spaces
\[
\hat{S}_{h,r} = \{ \chi_h \in C^0(\Omega_h) \mid \chi_{h|\tau_h^i} \circ \phi_h^i \in P_{r-1}(\tau_0) \} \subset H^1(\Omega_h),
\]
\[
\overline{S}_{h,r} = \{ \chi_h \in C^0(\Omega_h) \mid \chi_{h|\tau_h^i} \circ \phi_h^i \in P_{r-1}(\tau_0), \chi_h(x) = 0, \ \forall x \in \partial \Omega_h \} \subset H^1(\Omega_h),
\]
where \( P_{r-1}(\tau_0) \) represents the space of polynomials of degree less than or equal to \( r - 1 \) on \( \tau_0 \).

The following inverse inequality will be used (cf. [2]): for \( \forall \tau = \tau_h^i \in T_h \) with \( \text{diam}(\tau) = h \tau \leq h \), and \( \forall v_h \in (\overline{S}_{h,r})^d \), it holds
\[
\| v_h \|_{W^{m,q}(\tau)^d} \leq C h^{l-m-\frac{d}{2}} \| v_h \|_{W^{l,q'}(\tau)^d}, \quad 0 \leq l \leq m \leq 2, \quad 1 \leq q' \leq q \leq \infty. \tag{8}
\]

In our analysis, we will consider the Hood-Taylor element space \((X_{h,r}, Q_{h,r-1})\), where
\[
X_{h,r} = (\overline{S}_{h,r})^d, \quad Q_{h,r-1} = \hat{S}_{h,r-1} \cap L^2(\Omega_h)/\mathbb{R}, \quad r \geq 3.
\]

And this finite element space \((X_{h,r}, Q_{h,r-1})\) satisfies the following \textit{inf} – \textit{sup} condition (cf. [14]): there exists a constant \( \beta > 0 \), which is independent of the mesh grid
size $h$, such that
\[ \inf_{q_h \in Q_{h,r-1}} \sup_{v_h \in X_{h,r}} \frac{(q_h, \nabla \cdot v_h)}{\|v_h\|_1} \geq \beta. \]  
(9)

For the finite element space $(X_{h,r}, Q_{h,r-1})$, we also need to make the following assumption (cf. [5, 12, 14]).

(A1) For each $(u, p) \in H^r(\Omega)^d \times H^{r-1}(\Omega)$, there exists an approximation $(\pi_h u, \rho_h p) \in X_{h,r} \times Q_{h,r-1}$ such that
\[ \| \nabla (u - \pi_h u) \|_0 \leq ch^s \| u \|_{1+s}, \quad \| p - \rho_h p \|_0 \leq ch^s \| p \|_s, \quad 0 \leq s \leq r - 1. \]  
(10)

Based on this finite element space $(X_{h,r}, Q_{h,r-1})$, we will define a divergence-free space
\[ V_{h,r} = X_{h,r} \cap \{ \chi_h \in H^1_0(\Omega) : \int_{\Omega} q_h \div(\chi_h) = 0, \forall q_h \in Q_{h,r-1} \}. \]

We will frequently use the standard interpolation operator $I_h : C_0(\Omega)^d \to X_{h,r}$, $\forall v \in C_0(\Omega)^d$, and the following relation is established for $\forall v \in H^r(\Omega)^d \cap H^1_0(\Omega)^d$
\[ \| v - I_h(v) \|_{L^2(\Omega)} + h \| v - I_h(v) \|_{H^1(\Omega \setminus \partial \Omega)} \leq C(h^r + \delta(h)) \| v \|_{H^r(\Omega)^d}. \]  
(11)

For the need of theoretical analysis, the interpolation operator $I_h(v)$ is extended by zero in $\Omega \setminus \Omega_h$. Let $\delta(h) = \max_{x \in \partial \Omega_h} \text{dist}(x, \partial \Omega)$, the following relations are established (cf. [5, 14]):
\[ \| v - I_h(v) \|_{L^2(\Omega)^d} + h \| v - I_h(v) \|_{H^1(\Omega \setminus \partial \Omega)} \leq C(h^r + \delta(h)) \| v \|_{H^r(\Omega)^d}, \]  
(12)
\[ \| v - I_h(v) \|_{L^2(\Omega)^d} + h \| v - I_h(v) \|_{H^1(\Omega \setminus \partial \Omega)} \leq C(h^r) \| v \|_{H^r(\Omega)^d}. \]  
(13)

For each fixed time $t \in [0, T]$, it is clear that the solution of the Stokes problem with right side $f - u_t - (u \cdot \nabla)u$ is also the solution $(u, p)$ of the problem (1)-(4). If we use $(s_h, q_h) \in (X_{h,r}, Q_{h,r-1})$ to denote its finite element approximation, then it satisfies
\[ \nu(\nabla s_h, \nabla \phi_h) - (q_h, \nabla \cdot \phi_h) = \nu(\nabla u, \nabla \phi_h) - (p, \nabla \cdot \phi_h) \]
\[ = (f - u_t - (u \cdot \nabla)u, \phi_h), \quad \forall \phi_h \in X_{h,r}, \]  
(14)
\[ (\nabla \cdot s_h, \psi_h) = 0, \quad \forall \psi_h \in Q_{h,r-1}. \]

Let $s_h = S_h(u) : V \to V_{h,r}$ be the discrete Stokes projection of the solution $(u, p)$ of (1)-(4) and satisfies
\[ (\nabla S_h(u), \nabla \chi_h) = (\nabla u, \nabla \chi_h) - (p, \nabla \cdot \chi_h) = (f - u_t - (u \cdot \nabla)u, \chi_h), \quad \forall \chi_h \in V_{h,r}. \]

For $2 \leq m \leq r$, the following bounds hold (cf. [3])
\[ \| u - s_h \|_0 + h \| u - s_h \|_1 \leq Ch^m \| u \|_m + \| p \|_{H^{m-1}/2}], \]  
(15)
\[ \| p - q_h \|_{L^2/2} \leq C \beta h^{m-1} \| u \|_m + \| p \|_{H^{m-1}/2}], \]  
(16)
\[ \| u - s_h \|_{-s} \leq Ch^{r+s} \| u \|_r + \| p \|_{H^{r-1}/2}], \quad 0 \leq s \leq \min(r-2, 1), \]  
(17)

where the constant $C_\beta$ depends on the constant $\beta$ in the inf $f - \sup$ condition (9).

Here and hereafter, the error constant $C$ depends also on the inverse powers of $\nu$.

Let $\Pi_{h,r} : L^2(\Omega)^d \to V_{h,r}$ be the discrete Lévy’s projection satisfying
\[ (\Pi_{h,r} u, \chi_h) = (u, \chi_h), \quad \forall \chi_h \in V_{h,r}, \]  
(18)
\[ \| \Pi_{h,r} u \|_1 \leq C \| u \|_1, \quad \forall u \in V. \]  
(19)
By definition, it is easy to show that $\Pi_{h,r}$ is stable in $L^2$-norm.

Let $A_h$ be the discrete Stokes operator defined as

$$(\nabla v_h, \nabla \phi_h) = (A_h v_h, \phi_h) = (A_h^{1/2} v_h, A_h^{1/2} \phi_h), \quad \forall v_h, \phi_h \in V_{h,r}.$$ 

Since $A_h$ is a discrete self-adjoint operator, there exists a positive constant $C_\gamma$, which is independent of $h$, such that

$$\| A_h^{1/2} e^{-tA_h} \|_0 \leq C_\gamma t^{-\gamma}, \quad 0 \leq \gamma < 1. \quad (20)$$

In the following theoretical analysis, we shall frequently use the following inequalities [3]: for any $f \in L^2(\Omega)^d$

$$\| A_h^{-s/2} \Pi_{h,r} f \|_0 \leq C h^s \| f \|_0 + \| A^{-s/2} \Pi f \|_0, \quad s = 1, 2, \quad (21)$$

$$\| A^{-s/2} \Pi f \|_0 \leq C h^s \| f \|_0 + \| A_h^{-s/2} \Pi_{h,r} f \|_0, \quad s = 1, 2. \quad (22)$$

Due to $\forall v_h \in V_{h,r}, (A_h^{-1/2} \Pi_{h,r} f, v_h) = (f, A_h^{-1/2} v_h)$, it holds

$$\| A_h^{-1/2} \Pi_{h,r} f \|_0 \leq C \| f \|_{-1}. \quad (23)$$

For any $v \in V$, we have $(A^{-1/2} \Pi(\Pi_{h,r} f), v) = (\Pi_{h,r} f, A^{-1/2} v) = (f, \Pi_{h,r} A^{-1/2} v)$, and combining with (19), the following estimation holds

$$\| A^{-1/2} \Pi(\Pi_{h,r} f) \|_0 \leq C \| f \|_{-1}, \quad f \in L^2(\Omega)^2. \quad (24)$$

3. A subgrid stabilizing postprocessed mixed finite element method. According to the definition of nonlinear item $b(\cdot, \cdot, \cdot)$, we define the discrete nonlinear term $b_h(\cdot, \cdot, \cdot)$ as

$$b_h(u_h, v_h, \phi_h) = ((u_h \cdot \nabla)v_h, \phi_h) + \frac{1}{2}(\text{div}(u_h)v_h, \phi_h), \quad \forall u_h, v_h, \phi_h \in X_{h,r} \subset H^1_0(\Omega)^d.$$  

For $\forall u, v \in H^1_0(\Omega)^d$, $F(u, v) = (u \cdot \nabla)v + \frac{1}{2}\text{div}(u)v$ represents the corresponding continuous operator. Extending the definition of $b_h$ into $H^1_0(\Omega)^d$, it follows that [3]

$$b_h(u, v, w) = (F(u, v), w), \quad \forall u, v, w \in H^1_0(\Omega)^d, \quad (25)$$

$$b_h(u, v, w) = -b_h(u, w, v), \quad \forall u, v, w \in H^1_0(\Omega)^d. \quad (26)$$

It is clear that the following relations hold

$$B(u, v) = \Pi F(u, v), \quad \forall u \in V,$$

$$B_h(u, u) = \Pi_{h,r} F(u, u), \quad \forall u, v \in H^1_0(\Omega)^d.$$  

3.1. Subgrid-scale model. Let $P_1$ be the space of polynomials of degree less than or equal to one. The subgrid-scale model is based on an elliptic projection operator $\Pi_h : H^1_0(\Omega)^d \to R_1 = \{ v \in H^1_0(\Omega)^d : v \mid_{\partial \Omega} \in (P_1)^d, \forall \tau_h \in T_h \}$, which is defined as [33, 37]

$$(\nabla \Pi_h u, \nabla v) = (\nabla u, \nabla v), \quad \forall u \in H^1_0(\Omega)^d, v \in R_1, \quad (27)$$

and has the following properties

$$\| \nabla \Pi_h v \|_0 \leq \| \nabla v \|_0, \quad \forall v \in H^1_0(\Omega)^d, \quad (28)$$

$$\| (I - \Pi_h) v \|_k \leq c h^{2-k} \| v \|_2, \quad \forall v \in H^2(\Omega)^d, \quad k = 0, 1. \quad (29)$$

Based on the projection operator $\Pi_h$, the subgrid stabilization term is defined as

$$G(u, v) = \alpha(\nabla (I - \Pi_h) u, \nabla (I - \Pi_h) v), \quad \forall u, v \in H^1_0(\Omega)^d, \quad (30)$$

where $\alpha > 0$ is a stabilization parameter.
According to the definition of the projection operator $\Pi_h$, we also have
\[
G(u, v) = \alpha(\nabla (I - \Pi_h) u, \nabla (I - \Pi_h) v) \\
= \alpha(\nabla u, \nabla v) - \alpha(\nabla \Pi_h u, \nabla v), \quad \forall u, v \in H^1_0(\Omega)^d.
\]

3.2. A subgrid stabilizing postprocessed mixed finite element method.

Now, let us suppose that we are approaching the solution of (1)-(4) at time $T$. For $d = 3$, we assume that $T$ is chosen smaller than $T^*$ with $T^*$ the critical time. Our subgrid stabilizing postprocessed mixed finite element method reads as follows.

**Step 1.** Set $u_h(0) = s_h(0)$, an initial approximation to $u(0)$, find solutions $u_h : [0, T] \rightarrow X_{h,r}$ and $p_h : [0, T] \rightarrow Q_{h,r-1}$ such that
\[
\begin{align*}
(\hat{u}_h, \phi_h) + \nu(\nabla u_h, \nabla \phi_h) + b_h(u_h, u_h, \phi_h) + (\nabla p_h, \phi_h) + G(u_h, \phi_h) \\
= (f, \phi_h), \quad \forall \phi_h \in X_{h,r}, \\
(\nabla \cdot u_h, \psi_h) = 0, \quad \forall \psi_h \in Q_{h,r-1},
\end{align*}
\]
where $b_h(\cdot, \cdot, \cdot)$ is discrete nonlinear term, $\hat{u}_h = \frac{2u_h}{\delta t}$.

**Step 2.** The discrete velocity and pressure solutions $(u_h(T), p_h(T))$ are postprocessed, by finding $(\bar{u}_h, \bar{p}_h) \in (\bar{X}, \bar{Q})$ such that
\[
\nu(\nabla \bar{u}_h, \nabla \bar{\phi}) + (\nabla \bar{p}_h, \bar{\phi}) + G(\bar{u}_h, \bar{\phi}) \\
= (f, \bar{\phi}) - b_h(u_h(T), u_h(T), \bar{\phi}) - (\hat{u}_h(T), \bar{\phi}), \quad \forall \bar{\phi} \in \bar{X},
\]
(\nabla \cdot \bar{u}_h, \bar{\psi}) = 0, \quad \forall \bar{\psi} \in \bar{Q}.
\]

There are two options for the postprocessing elements $(\bar{X}, \bar{Q})$, either

- the same-order Hood-Taylor elements on a finer grid $(\bar{X}, \bar{Q}) = (X_{\bar{h},r}, Q_{\bar{h},r-1})$, $r \geq 3$, $\bar{h} < h$, or
- higher-order Hood-Taylor elements on the same grid $(\bar{X}, \bar{Q}) = (X_{\bar{h},r+1}, Q_{\bar{h},r})$, $r \geq 3$, $\bar{h} = h$.

Using $\bar{V}$ to represent the corresponding discretely divergence-free space that can be either $\bar{V} = V_{h,r}$ or $\bar{V} = V_{h,r+1}$ depending on whether the selected postprocessing element is the same-order or higher-order space, correspondingly, we use $\bar{\Pi}_h$ to represent the discrete Leray’s projection into $\bar{V}$, and $\bar{A}_h$ will denote the discrete Stokes operator acting on functions in $\bar{V}$. The postprocessed solution $\bar{u}_h$ is also the solution of the following pressure-free formulation
\[
\nu(\nabla \bar{u}_h, \nabla \bar{\chi}_h) + G(\bar{u}_h, \bar{\chi}_h) = (f, \bar{\chi}_h) - b_h(u_h(T), u_h(T), \bar{\chi}_h) - (\hat{u}_h(T), \bar{\chi}_h), \quad \forall \bar{\chi}_h \in \bar{V}.
\]

4. Stability analysis and error estimates. Following the framework of [3], we will give stability analysis and error estimates for our present method in this section. Let $(u_h, p_h)$ be the stabilizing mixed finite element approximation to the solution $(u, p)$ of (1)-(4), then $u_h \in V_{h,r}$ is also the solution of the following pressure-free equations
\[
(\hat{u}_h, \chi_h) + \nu(\nabla u_h, \nabla \chi_h) + b_h(u_h, u_h, \chi_h) + G(u_h, \chi_h) \\
= (f, \chi_h), \quad \forall \chi_h \in V_{h,r}.
\]
which can be written in the following abstract form,

\[ \dot{u}_h + \nu A_h u_h + B_h(u_h, u_h) + \alpha A_h u_h - \alpha A_h(\Pi_h u_h) = \Pi_{h,r} f. \] (38)

Accordingly, Stokes projection \( s_h \) also satisfies the following abstract form

\[ \dot{s}_h + \nu A_h s_h + B_h(s_h, s_h) = \Pi_{h,r} f + T_h, \] (39)

where \( T_h \) is the truncation error defined as

\[ T_h(t) = \dot{s}_h - \Pi_{h,r}(u_t) + B_h(s_h, s_h) - B_h(u, u). \] (40)

Let \( v_h : [0, T] \rightarrow V_{h,r} \) be an auxiliary mapping satisfying the following threshold conditions:

\[ \| s_h(t) - v_h(t) \|_0 \leq C_r h^2 \quad \forall t \in [0, t_1], \quad 0 < t_1 \leq T. \] (41)

Its truncation error is defined as

\[ \hat{T}_h(t) = \dot{v}_h + \nu A_h v_h + B_h(v_h, v_h) + \alpha A_h v_h - \alpha A_h(\Pi_h v_h) - \Pi_{h,r} f. \] (42)

**Lemma 4.1.** Let \((u, p)\) be the solution of the Navier-Stokes equations (1)-(4), \( s_h = S_h(u) \) the discrete Stokes projection of the velocity \( u \) and \( v_h : [0, T] \rightarrow V_{h,r} \) satisfies the threshold condition (41). Then, there exists a constant \( K > 0 \), which is independent of \( t_1 \) in (41), such that for all \( t \in [0, t_1] \), there hold

\[ \| F(s_h(t), s_h(t)) - F(v_h(t), v_h(t)) \|_0 \leq K \| s_h(t) - v_h(t) \|_1, \] (43)

\[ \| F(s_h(t), s_h(t)) - F(v_h(t), v_h(t)) \|_1 \leq K \| s_h(t) - v_h(t) \|_0, \] (44)

where the constant \( K = K(c_r, \max_{0 \leq t \leq T}(\| u(t) \|_2 + \| p(t) \|_{H^1(\Omega)})) \).

**Proof.** The proof can be found in [3] (see the proof of Lemma 3.1 in [3]). \qed

**Lemma 4.2.** (Stability). Let \( T > 0 \) be a fixed time, and \((u, p)\) be the solution of the Navier-Stokes equations (1)-(4), \( s_h = S_h(u) \) the discrete Stokes projection of velocity \( u \), \( v_h : [0, T] \rightarrow V_{h,r} \) satisfies (41). Then, there exists a constant \( K_s > 0 \), such that, for all \( t_1 \leq T \), it holds

\[ \max_{0 \leq t \leq t_1} \| s_h(t) - v_h(t) \|_0 \leq e^{K_s t_1} (\| s_h(0) - v_h(0) \|_0 + C \alpha h^{q-1} + c h \]

\[ + \max_{0 \leq t \leq t_1} \| \int_0^t e^{-\nu \alpha (t-s)} A_h[T_h(s) - \hat{T}_h(s)]ds \|_0), \] (45)

where \( T_h(s) \) and \( \hat{T}_h(s) \) are the truncation errors given in (40) and (42), respectively.

**Proof.** We set \( e_h = s_h - v_h \). On the basis of (39) and (42), it follows that

\[ \dot{e}_h(t) + (\nu + \alpha) A_h e_h(t) = B_h(v_h(t), v_h(t)) - B_h(s_h(t), s_h(t)) + T_h(t) - \hat{T}_h(t) \]

\[ - \alpha A_h \Pi_h v_h = B_h(v_h(t), v_h(t)) - B_h(s_h(t), s_h(t)) + T_h(t) - \hat{T}_h(t) \]

\[ - \alpha A_h \Pi_h (v_h - s_h) - \alpha A_h \Pi_h (s_h - u) \]

\[ - \alpha A_h [(\Pi_h - I) u] + \alpha A_h (s_h - u). \]
Then, by integrating the above equation from time 0 up to time \( t \), we have

\[
e_h(t) = e^{-(\nu+\alpha)t} A_h \Pi_{h,r} e_h(0) + \int_0^t e^{-(\nu+\alpha)(t-s)} A_h \Pi_{h,r} [F(v_h, v_h) - F(s_h, s_h)] ds
+ \int_0^t e^{-(\nu+\alpha)(t-s)} A_h \Pi_{h,r} [T_h(s) - \hat{T}_h(s)] ds
- \alpha \int_0^t e^{-(\nu+\alpha)(t-s)} A_h \Pi_{h,r} \Pi_h (v_h - s_h) ds
- \alpha \int_0^t e^{-(\nu+\alpha)(t-s)} A_h \Pi_{h,r} \Pi_h (s_h - u) ds
- \alpha \int_0^t e^{-(\nu+\alpha)(t-s)} A_h \Pi_{h,r} \Pi_h [(\Pi - I) u] ds
+ \alpha \int_0^t e^{-(\nu+\alpha)(t-s)} A_h \Pi_{h,r} \Pi_h (s_h - u) ds. \tag{46}
\]

Since \( \{e^{-(\nu+\alpha)t} A_h \Pi_{h,r}\}_{t>0} \) is a contraction, then \( \| e^{-(\nu+\alpha)t} A_h \Pi_{h,r} e_h(0) \|_0 \leq \| e_h(0) \|_0 \). For the second item on the right side of (46), according to (20), (23) and Lemma 4.1, it holds

\[
\| \int_0^t e^{-(\nu+\alpha)(t-s)} A_h \Pi_{h,r} [F(v_h, v_h) - F(s_h, s_h)] ds \|_0 \\
\leq \frac{C_{1/2}}{\nu + \alpha} \int_0^t \| A_h^{-1/2} \Pi_{h,r} [F(v_h, v_h) - F(s_h, s_h)] \|_0 ds \\
\leq \frac{C_{1/2}}{\nu + \alpha} \int_0^t \| F(v_h, v_h) - F(s_h, s_h) \|^{-1} ds \\
\leq \frac{KC_{1/2}}{\nu + \alpha} \int_0^t \| e_h \|_0 ds.
\]

For the fourth item on the right-hand side of (46), according to (20) and inverse inequality, we get

\[
\alpha \| \int_0^t e^{-(\nu+\alpha)(t-s)} A_h \Pi_{h,r} A_h \Pi_h (v_h - s_h) ds \|_0 \leq \frac{C\alpha}{h\sqrt{\nu + \alpha}} \int_0^t \| e_h \|_0 ds.
\]

And then, combining it with (46), we have

\[
\| e_h(t) \|_0 \leq \| e_h(0) \|_0 + \frac{KC_{1/2}}{\nu + \alpha} \int_0^t \| e_h \|_0 ds + \frac{C\alpha}{h\sqrt{\nu + \alpha}} \int_0^t \| e_h \|_0 ds \\
+ \| \int_0^t e^{-(\nu+\alpha)(t-s)} A_h [T_h(s) - \hat{T}_h(s)] ds \|_0 + \frac{C\alpha}{\sqrt{\nu + \alpha}} \int_0^t \| s_h - u \|_1 ds \\
+ \frac{C\alpha}{\sqrt{\nu + \alpha}} \int_0^t \| A_h^{-1/2} [(\Pi - I) u] \|_0 ds + \frac{C\alpha}{\sqrt{\nu + \alpha}} \int_0^t \| s_h - u \|_1 ds.
\]

Now, according to the generalized Gronwall Lemma ( [25], pp.188-189), it holds

\[
\max_{0 \leq t \leq t_1} \| s_h(t) - v_h(t) \|_0 \leq e^{K_{t_1}} (\| s_h(0) - v_h(0) \|_0 \\
+ \max_{0 \leq t \leq t_1} \| \int_0^t e^{-(\nu+\alpha)(t-s)} A_h [T_h(s) - \hat{T}_h(s)] ds \|_0)
\]
\[ + \frac{C \alpha}{\sqrt{\nu + \alpha}} \max_{0 \leq t \leq T} \| \int_0^t \frac{1}{\sqrt{t-s}} \left( \| s_h - u \|_1 \right) ds \|_0 \]
\[ + \frac{C \alpha}{\sqrt{\nu + \alpha}} \max_{0 \leq t \leq T} \| \int_0^t \frac{1}{\sqrt{t-s}} \left( \| A_h^{1/2} \left( [\Pi_h - I] u \right) \|_0 \right) ds \|_0. \]

Finally, the conclusion can be drawn from (29) and (15).

**Lemma 4.3.** For any \( f \in C([0,T], L^2(\Omega)^d) \), the following estimate holds, for all \( t \in [0,T] \),
\[ \int_0^t \| A_h e^{-\nu(t-s) A_h} \Pi_{h,r} f(s) \|_0 \, ds \leq \frac{C}{\nu} | \log(h) | \max_{0 \leq t \leq T} \| f(t) \|_0. \]

**Proof.** The proof is exactly that for Lemma 3.3 in [3].

**Lemma 4.4.** Let \( v \in (H^2(\Omega))^d \cap V \). Then, there exists a constant \( K = K(\| v \|_2) \), such that for any \( w \in H^1_0(\Omega)^d \), it follows
\[ \| A^{-1} \Pi[F(v, v) - F(w, w)] \|_0 \leq K(\| v - w \|_1 + \| v - w \|_2 + \| v - w \|_0). \]

**Proof.** We refer the readers to the proof of Lemma 3.4 in [3].

**Lemma 4.5.** Let \((u, p)\) be the solution of the Navier-Stokes equations (1)-(4). Then, there exists a constant \( K = K(u, p, \nu) > 0 \) such that, for all \( t \in [0,T] \), the truncation error \( T_h(t) \) defined in (40) satisfies
\[ \| A^{-1}_h T_h(t) \|_0 \leq K h^{r+1}. \]

**Proof.** The proof is the same as that for Lemma 3.5 in [3].

**Theorem 4.6.** (Consistency) Let \((u, p)\) be the solution of the Navier-Stokes equations (1)-(4). Then, there exists a constant \( K = K(u, p, \nu) > 0 \) such that
\[ \max_{0 \leq t \leq T} \| \int_0^t e^{-(\nu + \alpha)(t-s) A_h} T_h(s) ds \|_0 \leq K \frac{h^{r+1}}{\nu + \alpha} | \log(h) |. \]

**Proof.** In Lemma 4.3, we set \( \nu = \nu + \alpha \) and get
\[ \int_0^t \| A_h e^{-(\nu + \alpha)(t-s) A_h} \Pi_{h,r} f(s) \|_0 \, ds \leq \frac{C}{\nu + \alpha} | \log(h) | \max_{0 \leq t \leq T} \| f(t) \|_0, \]
which, combined with Lemma 4.5, leads to
\[ \| \int_0^t e^{-(\nu + \alpha)(t-s) A_h} T_h(s) ds \|_0 \leq \int_0^t \| A_h e^{-(\nu + \alpha)(t-s) A_h} A^{-1}_h T_h(s) \|_0 ds \]
\[ = \int_0^t \| A_h e^{-(\nu + \alpha)(t-s) A_h} \Pi_{h,r} (A^{-1}_h T_h(s)) \|_0 ds \]
\[ \leq \frac{C}{\nu + \alpha} | \log(h) | \max_{0 \leq t \leq T} \| A^{-1}_h T_h(t) \|_0 \]
\[ \leq K \frac{h^{r+1}}{\nu + \alpha} | \log(h) |. \]
Theorem 4.7. (Superconvergence for the velocity) Let \((u, p)\) be the solution of the Navier-Stokes equations (1)-(4), \(s_h\) the Stokes projection of \(u\), and \(u_h\) the stabilizing mixed finite element approximation to \(u\). Then, there exists constants \(K(u, p, \nu) > 0\) and \(h_0 > 0\) such that, for each \(h \in (0, h_0]\), it holds

\[
\max_{0 \leq t \leq T} \| s_h(t) - u_h(t) \|_0 \leq K \frac{h^{r+1}}{\nu + \alpha} | \log(h) | + C\alpha h^{r-1} + h\alpha.
\]

Proof. Since \(u_h(0) = s_h(0)\), according to the proof of Lemma 4.2 (particularly, we take \(v_h = u_h\)), and Theorem 4.6, the result can be easily obtained.

Corollary 1. Let the conditions of Theorem 4.7 hold. Then, there exist constants \(K(u, p, \nu) > 0\) and \(h_0 > 0\) such that, for each \(h \in (0, h_0]\),

\[
\max_{0 \leq t \leq T} \| s_h(t) - u_h(t) \|_1 \leq K \frac{h^r}{\nu + \alpha} | \log(h) | + C\alpha h^{r-2} + \alpha.
\]

Proof. Applying Theorem 4.7 and inverse inequality, we can easily prove the result.

Corollary 2. Under the conditions of Theorem 4.7, there exist constants \(K(u, p, \nu) > 0\) and \(h_0 > 0\) such that, for each \(h \in (0, h_0]\) and \(s = 0, 1\), the following bound holds

\[
\max_{0 \leq t \leq T} \| u(t) - u_h(t) \|_s \leq C h^{r-s} \max_{0 \leq t \leq T} \left( \| u \|_r + \| p \|_{H^{r-1}/\mathbb{R}} \right) + \frac{K}{\nu + \alpha} h^{r+1-s} | \log(h) | + C\alpha h^{r-1-s} + \alpha h^{1-s}.
\]

Proof. From the triangle inequality, we have \(\| u - u_h \|_s \leq \| u - s_h \|_s + \| s_h - u_h \|_s\). For the first term \(\| u - s_h \|_s\), (15) is available for the estimation. While applying Theorem 4.7 and Corollary 1, \(\| s_h - u_h \|_s\) can be directly estimated.

Lemma 4.8. Let \((u, p)\) be the solution of the Navier-Stokes equations (1)-(4), \(u_h : [0, T] \rightarrow V_{h,r}\) the stabilized Hood – Taylor element approximation to \(u\). Then, the following estimates hold

\[
\max_{0 \leq t \leq T} \| u(t) - \hat{u}_h(t) \|_0 \leq \frac{K}{\nu + \alpha} h^{r-1} | \log(h) | + C\alpha h^{r-3} + \alpha h^{-1},
\]

\[
\max_{0 \leq t \leq T} \| A^{-1} \Pi(u(t) - \hat{u}_h(t)) \|_0 \leq \frac{K}{\nu + \alpha} h^{r+1} | \log(h) | + C\alpha h^{r-1} + \alpha h,
\]

\[
\max_{0 \leq t \leq T} \| u(t) - \hat{u}_h(t) \|_{-1} \leq \frac{K}{\nu + \alpha} h^r | \log(h) | + C\alpha h^{r-2} + \alpha.
\]

Proof. We decompose \(u_t - \hat{u}_h = (u_t - \hat{s}_h) + (\hat{s}_h - \hat{u}_h)\). For the first item in the decomposition, we can estimate it by (15) and (17), and then we only need to estimate the second term. By setting \(e_h = s_h - u_h\), we have from (39) and (38) that

\[
\dot{e}_h + (\nu + \alpha) A_h e_h + B_h(u, u) - B_h(u_h, u_h) - \alpha A_h s_h + \alpha A_h (\Pi_h u_h) = \hat{s}_h - \Pi_h r(u_t) \quad (47)
\]

Now we will perform the proof in the following three sub-steps.
For the fourth item on the right side of (48), it follows from (15) that
\[ \| \hat{e}_h \|_0 \leq (\nu + \alpha) \| A_h e_h \|_0 + \| B_h(u_h, u_h) - B_h(u, u) \|_0 \]
+ \alpha \| A_h(\Pi_h u_h - \hat{s}_h) \|_0 + \| \Pi_{h,r}(u_t - \hat{s}_h) \|_0. \] \hspace{1cm} (48)

For the first item on the right side of (48), according to the inverse inequality and Corollary 1, we get
\[ (\nu + \alpha) \| A_h e_h \|_0 = (\nu + \alpha) \| A_h^{1/2} A_h^{1/2} e_h \|_0 \leq (\nu + \alpha) Ch^{-1} \| e_h \|_1 \leq K h^{r-1} | \log(h) | + C(\nu + \alpha) h^{r-3} + c a h^{-1}. \]

For the second item on the right side of (48), according to Lemma 4.1 and Corollary 2, it holds
\[ \| B_h(u_h, u_h) - B_h(u, u) \|_0 \leq K h^{r-1} + K \frac{h^r}{\nu + \alpha} | \log(h) | + C a h^{r-2} + c h. \]

The third item on the right side of (48) can be obtained directly from assumption (A1), the inverse inequality and Corollary 2:
\[ \alpha \| A_h(\Pi_h u_h - \hat{s}_h) \|_0 \leq C a h^{-1} \| A_h^{1/2}(\Pi_h u_h - \hat{s}_h) \|_0 \leq C a h^{-1} \| A_h^{1/2} \Pi_h(u_h - u) \|_0 \]
+ \| A_h^{1/2}((\Pi_h - I)u) \|_0 \leq K a h^{r-2} \left( \max_{0 \leq t \leq T} \| u \|_r + \| P \|_{H^{(-1)}(\overline{\Omega})} \right) + K \frac{a}{\nu + \alpha} h^{r-1} | \log(h) | + C a^2 h^{r-3} + C a h^{r-2} \| u \|_r + \| P \|_{H^{(-1)}(\overline{\Omega})} + c(h + a) \]

For the fourth item on the right side of (48), it follows from (15) that
\[ \| \Pi_{h,r}(u_t - \hat{s}_h) \|_0 \leq C \| u_t - \hat{s}_h \|_0 \leq C h^{r'} \| u_t \|_r + \| P_t \|_{H^{(-1)}(\overline{\Omega})}. \]

Inserting the above four estimations into (48) and applying the triangle inequality, we get the estimation for \( \max_{0 \leq t \leq T} \| u_t(t) - \hat{u}_h(t) \|_0. \)

(2) Next, we estimate \( \max_{0 \leq t \leq T} \| A^{-1}_h \Pi(\hat{u}_t(t) - \hat{u}_h(t)) \|_0. \) It holds that
\[ \| A^{-1}_h \hat{e}_h \|_0 \leq (\nu + \alpha) \| A^{-1}_h \Pi A_h e_h \|_0 + \| A^{-1}_h(\Pi_h(u_h, u_h) - B_h(u, u)) \|_0 \]
+ \alpha \| A^{-1}_h \Pi_A h(\Pi_h u_h - \hat{s}_h) \|_0 + \| A^{-1}_h(\hat{s}_h - \Pi_{h,r} u_t) \|_0. \] \hspace{1cm} (49)

For the first item on the right side of (49), according to the inverse inequality, (22) and Theorem 4.7, it follows
\[ (\nu + \alpha) \| A^{-1}_h \Pi A_h e_h \|_0 \leq C(\nu + \alpha) h^2 \| A_h e_h \|_0 + (\nu + \alpha) \| A^{-1}_h A_h e_h \|_0 \leq K h^{r+1} | \log(h) | + C(\nu + \alpha) h^{r-1} + c a h. \]

For the second item on the right side of (49), applying Lemma 4.1 and Corollary 2, we get
\[ \| A^{-1}_h \Pi(\hat{u}_t(u_h, u_h) - B_h(u, u)) \|_0 = \| A^{-1}_h \Pi_{h,r} (F(u_h, u_h) - F(u, u)) \|_0 \leq \| A^{-1/2}_h \Pi_{h,r} (F(u_h, u_h) - F(u, u)) \|_0 \]
\begin{align*}
&\leq K \| F(u_h, u_h) - F(u, u) \|_1 \\
&\leq K \| u - u_h \|_0 \\
&\leq Kh^r \left( \max_{0 \leq t \leq T} (\| u \|_r + \| p \|_{H^{r-1}/\mathbb{R}}) \right) \\
&\quad + K h^{r+1} \frac{1}{\nu + \alpha} | \log(h) | + C \alpha h^{r-1} + cah.
\end{align*}

For the third item on the right side of (49), from the triangle inequality, (15) and Corollary 2, we have
\begin{align*}
\alpha \| A^{-1} \Pi A_h(\Pi_h u_h - s_h) \|_0 &\leq C \alpha h^2 \| A_h(\Pi_h u_h - s_h) \|_0 \\
&\quad + \alpha \| A_h^{-1} A_h(\Pi_h u_h - s_h) \|_0 \\
&\leq C \alpha \| \Pi_h u_h - s_h \|_0 \\
&\leq C \alpha \| \Pi_h (u_h - u) \|_0 + C \alpha \| u - s_h \|_0 \\
&\quad + C \alpha \| (\Pi_h - I)u \|_0 \\
&\leq C \alpha \| u_h - u \|_0 + C \alpha \| u - s_h \|_0 + C \alpha h \| u \|_2 \\
&\leq C \alpha h^r + K \frac{\alpha}{\nu + \alpha} h^{r+1} | \log(h) | \\
&\quad + C \alpha h^{r-1} + cah.
\end{align*}

For the fourth item on the right side of (49), due to (17), it holds
\begin{align*}
\| A^{-1} \Pi (\dot{s}_h - \Pi_h, u_t) \|_0 &\leq \| A^{-1/2} \Pi (\dot{s}_h - \Pi_h, u_t) \|_0 \leq \| \dot{s}_h - u_t \|_{-1} \\
&\leq C h^{r+1} (\| u_t \|_r + \| p_t \|_{H^{r-1}/\mathbb{R}}).
\end{align*}

Taking the above four estimations into (49) and then applying the triangle inequality, we obtain the required result.

(3) And finally, we are going to estimate \( \max_{0 \leq t \leq T} \| u(t) - \dot{u}_h(t) \|_{-1} \). Due to \( \| \dot{e}_h \|_{-1} \leq C \| A_h^{-1/2} \dot{e}_h \|_0 \) (see page 207 in [6]), we just need to estimate \( \| A_h^{-1/2} \dot{e}_h \|_0 \). According to (47), it holds
\begin{align*}
\| A_h^{-1/2} \dot{e}_h \|_0 &\leq (\nu + \alpha) \| A_h^{-1/2} e_h \|_0 + \| A_h^{-1/2} [B_h(u_h, u_h) - B_h(u, u)] \|_0 \\
&\quad + \alpha \| A_h^{-1/2} (\Pi_h u_h - s_h) \|_0 + \| A_h^{-1/2} \Pi_h (u_t - \dot{s}_h) \|_0 .
\end{align*}

The first item on the right hand side of (50) can be directly estimated by Corollary 1. For the second item, according to Lemma 4.1 and Corollary 2, it holds
\begin{align*}
\| A_h^{-1/2} [B_h(u_h, u_h) - B_h(u, u)] \|_0 &\leq K \| u - u_h \|_0 \\
&\leq Kh^r \left( \max_{0 \leq t \leq T} (\| u \|_r + \| p \|_{H^{r-1}/\mathbb{R}}) \right) \\
&\quad + K h^{r+1} \frac{1}{\nu + \alpha} | \log(h) | + C \alpha h^{r-1} + cah.
\end{align*}

For the third item on the right side of (50), from the triangle inequality, (15) and Corollary 2, we have
\begin{align*}
\alpha \| A_h^{-1/2} (\Pi_h u_h - s_h) \|_0 &\leq C \alpha \| \Pi_h u_h - s_h \|_1 \\
&\leq C \alpha \| \Pi_h (u_h - u) \|_1 + C \alpha \| s_h - u \|_1 \\
&\quad + C \alpha \| (\Pi_h - I)u \|_1
\end{align*}
Proof. The proof is shown in [3].

(\text{Superconvergence for the pressure})\ Let \ Theorem 4.10. \ For any \ Lemma 4.9. \ For any \ \begin{align*}
    \text{Theorem 4.10.} \quad \forall h \in (0, h_0), \quad \max_{0 \leq t \leq T} \| p_h(t) - q_h(t) \|_{L^2(\Omega)/\mathbb{R}} \leq \frac{1}{\beta} \left( \frac{K}{\nu + \alpha} h^r | \log(h) | + Ch^{-1}h^{-2} + ca \right),
\end{align*}
where \( \beta \) is the constant in the inf-sup condition (9).

\textbf{Proof.} \ Subtracting (32) from (14), we get

\begin{align*}
    (p_h - q_h, \nabla \cdot \phi_h) &= (\nu + \alpha)(\nabla(u_h - s_h), \nabla \phi_h) + (F(u_h, u_h) - F(u, u), \phi_h) \\
    &+ \alpha(\nabla(s_h - \Pi_h u_h), \nabla \phi_h) + (\dot{u}_h - \dot{u}_t, \phi_h), \quad \forall \phi_h \in X_{h,r}.
\end{align*}

According to the inf-sup condition (9), (29), \Lemma 4.1, \Corollary 1, \Corollary 2 and triangle inequality, we get

\begin{align*}
    \beta \| p_h - q_h \|_{L^2/\mathbb{R}} &\leq (\nu + \alpha) \| u_h - s_h \|_1 + \| F(u_h, u_h) - F(u, u) \|_{-1} \\
    &+ \| u_t - \dot{u}_h \|_{-1} + \alpha \| \Pi_h (u_h - u) \|_1 \\
    &+ \alpha \| s_h - u \|_1 + \alpha \| (\Pi_h - I) u \|_1 \\
    &\leq (\nu + \alpha) \| u_h - s_h \|_1 + K \| u_h - u \|_0 + \| u_t - \dot{u}_h \|_{-1} \\
    &+ \alpha \| u_h - u \|_1 + \alpha \| s_h - u \|_1 + Ch \\
    &\leq K \frac{h^r}{\nu + \alpha} | \log(h) | + Ch^{-1}h^{-2} + +ca + ca h,
\end{align*}

which finishes the proof. \qed
Corollary 3. Under the conditions of Theorem 4.10, there exist positive constants \( K(u,p,\nu) \) and \( h_0 \) such that, for each \( h \in (0, h_0] \), the following bound holds

\[
\max_{0 \leq t \leq T} \| p(t) - p_h(t) \|_{L^2(\Omega)} \leq C h^{r-1} \max_{0 \leq t \leq T} ( \| u \|_r + \| p \|_{H^{r-1}(\Omega)}) + \frac{K}{\nu + \alpha} h^r | \log(h) | + C \alpha h^{r-2} + c\alpha.
\]

Proof. According to the triangle inequality, we have \( \| p - p_h \|_{L^2(\Omega)} \leq \| p - q_h \|_{L^2(\Omega)} + \| q_h - p_h \|_{L^2(\Omega)} \). Then, applying (16) and Theorem 4.10, the result can be directly obtained.

**Theorem 4.11.** Let \( T > 0 \) be a given time, \( (u,p) \in (H^{r+1}(\Omega)^d) \times H^r(\Omega)/\mathbb{R} \) be the solution of the Navier-Stokes equations (1)-(4), and \( (\tilde{u}_h, \tilde{p}_h) \) the stabilizing postprocessed mixed finite element approximation at time \( T \). Then, there exist constants \( K_1(u,p,\nu) \) and \( K_0(u,p,\nu) \) such that

(1) if the postprocessing elements are the same-order elements \((\tilde{X}, \tilde{Q}) = (X_{\tilde{h},r}, Q_{\tilde{h},r-1})\), then

\[
\| u(T) - \tilde{u}_h \|_1 \leq C (1 + \alpha) h^{r-1} (\| u(T) \|_r + \| p(T) \|_{H^{r-1}(\Omega)}) + \frac{K_1}{\nu + \alpha} h^r | \log(h) | + C \alpha h^{r-2} + c \alpha,
\]

(51)

(2) if at time \( T \), the solution \((u(T), p(T)) \in (H^{r+1}(\Omega)^d) \times H^r(\Omega)/\mathbb{R} \), and the postprocessing elements are the higher-order elements \((\tilde{X}, \tilde{Q}) = (X_{h,r+1}, Q_{h,r})\), then

\[
\| u(T) - \tilde{u}_h \|_1 \leq C (1 + \alpha) h^r (\| u(T) \|_{r+1} + \| p(T) \|_{H^{r}(\Omega)}) + \frac{K_1}{\nu + \alpha} h^r | \log(h) | + C \alpha h^{r-2} + c \alpha,
\]

(53)

\[
\| u(T) - \tilde{u}_h \|_0 \leq C (1 + \alpha) h^{r+1} (\| u(T) \|_{r+1} + \| p(T) \|_{H^{r}(\Omega)}) + \frac{K_0}{\nu + \alpha} h^{r+1} | \log(h) | + C \alpha h^{r-1} + c \alpha h.
\]

(54)

Proof. Let \( \tilde{s}_{\tilde{h}}(u) \in \tilde{V} \) be the Stokes projection of the solution of (1)-(4) at time \( T \), which satisfies

\[
\nu(\nabla \tilde{s}_{\tilde{h}}(u), \nabla \tilde{\chi}_h) = \nu(\nabla u(T), \nabla \tilde{\chi}_h) - (p(T), \nabla \cdot \tilde{\chi}_h) = (f(T) - u(T) - F(u(T), u(T)), \tilde{\chi}_h), \quad \forall \tilde{\chi}_h \in \tilde{V}.
\]

(55)

According to the triangle inequality, we have \( \| u(T) - \tilde{u}_h \|_l \leq \| u(T) - \tilde{s}_{\tilde{h}}(u) \|_l + \| \tilde{s}_{\tilde{h}}(u) - \tilde{u}_h \|_l \), \( l = 0, 1 \). The first item on the right side can be estimated directly using (15) as follows

\[
\| u(T) - \tilde{s}_{\tilde{h}}(u) \|_l \leq \begin{cases} (\tilde{h})^{r-1}(\| u(T) \|_r + \| p(T) \|_{H^{r-1}(\Omega)}), & \tilde{V} = \tilde{V}_{\tilde{h},r}; \\
(C(\tilde{h}))^{r+1}(\| u(T) \|_{r+1} + \| p(T) \|_{H^{r}(\Omega)}), & \tilde{V} = \tilde{V}_{h,r+1}.
\end{cases}
\]

And then, we just need to estimate \( \| \tilde{s}_{\tilde{h}} - \tilde{u}_h \|_1 \) and \( \| \tilde{s}_{\tilde{h}} - \tilde{u}_h \|_0 \).
(1) Firstly, let us estimate \( \| \tilde{s}_h - \tilde{u}_h \|_1 \). According to (55) and (34), we get

\[
(\nu + \alpha)(\nabla(\tilde{u}_h - \tilde{s}_h), \nabla \chi_h) = (u(T) - \tilde{u}_h(T), \chi_h) + \alpha(\nabla \Pi_h(\tilde{u}_h - \tilde{s}_h), \nabla \chi_h) + (F(u(T), u(T)) - F(u_h(T), u_h(T)), \chi_h) + \alpha(\nabla \Pi_h(\tilde{s}_h - u(T)), \nabla \chi_h) + \alpha(\nabla(\Pi_h - I)u, \nabla \chi_h), \quad \forall \chi_h \in \tilde{V}.
\]

Then, by setting \( \chi_h = \tilde{u}_h - \tilde{s}_h \), we have

\[
(\nu + \alpha) \| \nabla(\tilde{u}_h - \tilde{s}_h) \|_0 \leq \alpha \| \nabla(\tilde{s}_h - u(T)) \|_0 + 2\alpha \| \nabla(u(T) - \tilde{s}_h) \|_0 + \alpha \| \nabla(\Pi_h - I)u \|_0 + \| u(T) - \tilde{u}_h(T) \|_{-1} + \| F(u(T), u(T)) - F(u_h(T), u_h(T)) \|_{-1}.
\]

According to (29), Lemma 4.1, Corollary 2 and Corollary 4.8, and noting that \( \tilde{h} < h < 1 \), it holds

\[
\nu \| \nabla(\tilde{u}_h - \tilde{s}_h) \|_0 \leq 2\alpha \| u(T) - \tilde{s}_h \|_1 + \| u(T) - \tilde{u}_h(T) \|_{-1} + K \| u(T) - u_h(T) \|_0 + C\alpha \tilde{h}
\]

\[
\leq 2\alpha \| u(T) - \tilde{s}_h \|_1 + \frac{K}{\nu + \alpha} h^r | \log(h) | + C\alpha h^{r-2} + \alpha.
\]

That is to say \( \| \tilde{s}_h - \tilde{u}_h \|_0 \leq 2\alpha \| u(T) - \tilde{s}_h \|_1 + \frac{K}{\nu + \alpha} h^r | \log(h) | + C\alpha h^{r-2} + \alpha \). Then, we can estimate \( \| u(T) - \tilde{u}_h \|_1 \) by the triangle inequality and the estimation of \( \| u(T) - \tilde{s}_h(u) \|_1 \).

(2) And then, we estimate \( \| \tilde{s}_h - \tilde{u}_h \|_0 \). According to the proof of the previous part, (56) can be written as an abstract operator, which is

\[
(\nu + \alpha) \tilde{A}_h(\tilde{u}_h - \tilde{s}_h) = \alpha \tilde{A}_h \Pi_h(\tilde{u}_h - \tilde{s}_h) + \alpha \tilde{A}_h \Pi_h(\tilde{s}_h - u(T)) + \alpha \tilde{A}_h(u(T) - \tilde{s}_h) + \alpha \tilde{A}_h(\Pi_h - I)u + \tilde{\Pi}_h[F(u(T), u(T)) - F(u_h(T), u_h(T))]
\]

\[
+ \tilde{\Pi}_h[u_h(T) - \tilde{u}_h(T)].
\]

Then, applying \( \tilde{A}_h^{-1} \) to both sides of the above equations, we obtain

\[
(\nu + \alpha) \| \tilde{u}_h - \tilde{s}_h \|_0 \leq \alpha \| \tilde{s}_h - u(T) \|_0 + \alpha \| \tilde{s}_h - u(T) \|_0 + \alpha \| u(T) - \tilde{s}_h \|_0 + \alpha \| (\Pi_h - I)u \|_0 + \| \tilde{A}_h^{-1} \tilde{\Pi}_h[u_h(T) - \tilde{u}_h(T)] \|_0 + \| \tilde{A}_h^{-1} \tilde{\Pi}_h[F(u(T), u(T)) - F(u_h(T), u_h(T))] \|_0.
\]

For the fifth item on the right side of (59), according to (15), (21), the dual theory, assumption (A1), Lemma 4.1, Lemma 4.4, Theorem 4.7 and Corollary 2, it holds

\[
\| \tilde{A}_h^{-1} \tilde{\Pi}_h[F(u(T), u(T)) - F(u_h(T), u_h(T))] \|_0 \leq C(\tilde{h})^2 \| F(u(T), u(T)) - F(u_h(T), u_h(T)) \|_0 + \| A^{-1} \Pi[F(u(T), u(T)) - F(u_h(T), u_h(T))] \|_0
\]

\[
\leq C(\tilde{h})^2 \| u - u_h \|_1 + C(\| u - u_h \|_{-1} + \| u - u_h \|_0) \| u - u_h \|_1
\]

\[
\leq C(\tilde{h})^2 \| u - u_h \|_1 + C(\| u - s_h \|_{-1} + \| s_h - u_h \|_{-1} + \| u - u_h \|_0) \| u - u_h \|_1
\]

\[
\leq C(\tilde{h})^2 \| u - u_h \|_1 + C(\| u - s_h \|_{-1} + \| s_h - u_h \|_0 + \| u - u_h \|_0) \| u - u_h \|_1
\]

\[
\leq \frac{K}{\nu + \alpha} h^{r+1} | \log(h) | + C\alpha h^{r-1} + \alpha.
\]
For the last term on the right side of (59), from (22) and Lemma 4.8, we get
\[
\| \tilde{A}_h^{-1} \Pi_h [u_h(T) - \dot{u}_h(T)] \|_0 \leq C(\tilde{h})^2 \| u_h(T) - \dot{u}_h(T) \|_0 \\
+ \| A^{-1} \Pi [u_h(T) - \dot{u}_h(T)] \|_0 \\
\leq \frac{K}{\nu + \alpha} h^{r+1} \| \log(h) \| + C\alpha h^{r-1} + c\alpha h.
\]
Then, substituting the above estimates into (59) and applying (29), it follows that
\[
\| \tilde{s}_h - \tilde{u}_h \|_0 \leq 2\alpha \| \tilde{s}_h - u(T) \|_0 + \frac{K}{\nu + \alpha} h^{r+1} \| \log(h) \| + C\alpha h^{r-1} + c\alpha h.
\]
Finally, we can estimate \( \| u(T) - \tilde{u}_h \|_0 \) by the triangle inequality and the above estimation of \( \| u(T) - \tilde{s}_h(u) \|_0 \).

**Theorem 4.12.** Let \( T > 0 \) be a fixed time, \((u, p) \in (H^{r+1}(\Omega))^d \times H^r(\Omega)/\mathbb{R}\) the solution of the Navier-Stokes equations (1)-(4), \((\tilde{u}_h, \tilde{p}_h)\) the stabilizing postprocessed mixed finite element approximation at time \( T \). Then, there exists a constant \( K(u, p, \nu) \), such that

1. if the postprocessing elements are the same-order elements \((\tilde{X}, \tilde{Q})=(X_{\tilde{h}, r}, Q_{\tilde{h}, r-1})\), then
   \[
   \| p(T) - \tilde{p}_h \|_{L^2(\Omega)/\mathbb{R}} \leq C(1 + \alpha)(\tilde{h})^{r-1} \| u(T) \|_r + \| p(T) \|_{H^{r-1}/\mathbb{R}} \\
   + K \frac{h^r}{\nu + \alpha} \| \log(h) \| + C\alpha h^{r-2} + c\alpha;
   \]

2. if the postprocessing elements are the higher-order elements \((\tilde{X}, \tilde{Q})=(X_{h, r+1}, Q_{h, r})\), then
   \[
   \| p(T) - \tilde{p}_h \|_{L^2(\Omega)/\mathbb{R}} \leq C(1 + \alpha)h^r \| u(T) \|_{r+1} + \| p(T) \|_{H^r/\mathbb{R}} \\
   + K \frac{h^r}{\nu + \alpha} \| \log(h) \| + C\alpha h^{r-2} + c\alpha.
   \]

**Proof.** Let \( \tilde{q}_h \) be the stabilizing postprocessed mixed finite element approximation to the pressure \( p(T) \) at time \( T \), which is also the solution of Stokes problem (14). According to the triangle inequality, it is obvious that
\[
\| p(T) - \tilde{p}_h \|_{L^2/\mathbb{R}} \leq \| p(T) - \tilde{q}_h \|_{L^2/\mathbb{R}} + \| \tilde{q}_h - \tilde{p}_h \|_{L^2/\mathbb{R}}.
\]
For the first item on the right side, it can be directly estimated by (16),
\[
\| p(T) - \tilde{q}_h \|_{L^2/\mathbb{R}} \leq C(\tilde{h})^{r-1} \| u(T) \|_r + \| p(T) \|_{H^{r-1}/\mathbb{R}}
\]
when \((\tilde{X}, \tilde{Q})=(X_{\tilde{h}, r}, Q_{\tilde{h}, r-1})\),
\[
\| p(T) - \tilde{q}_h \|_{L^2/\mathbb{R}} \leq C h^r \| u(T) \|_{r+1} + \| p(T) \|_{H^r/\mathbb{R}}
\]
when \((\tilde{X}, \tilde{Q})=(X_{h, r+1}, Q_{h, r})\).

So we just have to estimate \( \| \tilde{q}_h - \tilde{p}_h \|_{L^2/\mathbb{R}} \). By subtracting (34) from (14), we get
\[
(\tilde{p}_h - \tilde{q}_h, \nabla \cdot \phi) = (\nu + \alpha)(\nabla (\tilde{u}_h - \tilde{s}_h), \nabla \phi) + \alpha(\nabla \tilde{s}_h, \nabla \phi) - \alpha(\nabla H_{\tilde{h}} \tilde{u}_h, \nabla \phi) \\
+ (F(u_h, u_h) - F(u, u), \phi) + (\dot{u}_h(T) - u_t, \phi), \quad \forall \phi \in \tilde{X}.
\]
According to the \( \inf - \sup \) condition (9) and Lemma 4.1, we deduce
\[
\beta \| \tilde{p}_h - \tilde{q}_h \|_{L^2/\mathbb{R}} \leq (\nu + \alpha) \| \tilde{u}_h - \tilde{s}_h \|_1 + \alpha \| \nabla \Pi_h (\tilde{s}_h - \tilde{u}_h) \|_0 + \alpha \| \nabla (\tilde{s}_h - u) \|_0 + \alpha \| \nabla (\Pi_h - I) u \|_0 + \alpha \| \nabla (\Pi_h - I) u \|_0 \]
\[
+ \alpha \| \nabla \Pi_h (\tilde{s}_h - u) \|_0 + \alpha \| \nabla (\tilde{s}_h - u) \|_0 + \| \tilde{u}_h(T) - u(T) \|_{-1} \leq (\nu + 2\alpha) \| \tilde{u}_h - \tilde{s}_h \|_1 + 2\alpha \| u - \tilde{s}_h \|_1 + \alpha \| \nabla (\Pi_h - I) u \|_0 + K \| u - u_h \|_0 + \| \tilde{u}_h(T) - u(T) \|_{-1} .
\]
Taking (29), Corollary 2 and Theorem 4.11 into account, and noting that \( \tilde{h} < h < 1 \), we infer
\[
\beta \| \tilde{p}_h - \tilde{q}_h \|_{L^2/\mathbb{R}} \leq C\alpha \| u(T) - \tilde{s}_h \|_1 + \frac{K}{\nu + \alpha} h^{\nu} | \log(h) | + C a h^{r-2} + c a.
\]
Finally, by applying the triangle inequality and the estimation of \( \| p(T) - \tilde{q}_h \|_{L^2/\mathbb{R}} \), the result can be easily obtained.

**Remark 4.1.** From Theorems 4.11 and 4.12 we see that, to ensure optimal convergence rate of the presented method, the stabilization parameter \( \alpha \) should be chosen as \( \alpha = \mathcal{O}(\tilde{h}^{r-1}) \) when the postprocessing elements are the same-order elements \((\tilde{X}, \tilde{Q}) = (X_{\tilde{h}}, Q_{\tilde{h}, r-1})\) defined on a finer mesh of size \( \tilde{h} \), or \( \alpha = \mathcal{O}(h^r) \) when the postprocessing elements are the higher-order elements \((\tilde{X}, \tilde{Q}) = (X_{h, r+1}, Q_{h, r})\) defined on the same mesh of size \( h \) as the first step. For example, if we use the second-order Taylor-Hood \( P_2 - P_2 \) elements pair for the spatial discretization on a mesh of size \( h \) in the first step, then \( \alpha = \mathcal{O}(\tilde{h}^2) \) when the postprocessing elements are the same second-order Taylor-Hood \( P_2 - P_2 \) elements on a finer mesh of size \( \tilde{h} \), or \( \alpha = \mathcal{O}(h^3) \) when the postprocessing elements are the third-order Taylor-Hood \( P_3 - P_2 \) elements on the same mesh of size \( h \) as the first step.

5. **Numerical results.** In this section, we shall report some numerical results to verify the theoretical predictions and illustrate the effectiveness of the proposed method. In particular, we present two numerical examples: one is a problem with known analytic solution and the other is the flow around a circular cylinder. The public domain software *Freefem++* [24] is used and the meshes consist of triangular elements. In all the numerical tests, the spatial and temporal discretizations in the first step of our method are performed by the Hood-Taylor \( P_2 - P_1 \) elements and a second-order Crank-Nicolson/Adams-Bashforth scheme (cf. [23]), respectively.

5.1. **Example of analytic solution.** In this test, the true solution of the Navier-Stokes equations defined in \( \Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2 \) is as follows:
\[
 u_1 = 10x^2(x - 1)^2y(y - 1)(2y - 1)\cos(t), \quad (x, y, t) \in \Omega \times [0, T],
\]
\[
 u_2 = -10x(x - 1)(2x - 1)y^2(y - 1)^2\cos(t), \quad (x, y, t) \in \Omega \times [0, T],
\]
\[
 p = 10(2x - 1)(2y - 1)\cos(t), \quad (x, y, t) \in \Omega \times [0, T],
\]
while the right-hand side \( f \) is computed by equations (2).

First, we compute the approximate solutions on meshes of size \( h = (2n)^{-1} \) (\( n = 2, 3, 4, 5, 6, 7, 8 \)) to test the convergence rates of our method, where \( \nu = 0.1, dt = 1/8000, T = 0.1 \), and the higher-order Hood-Taylor \( P_3 - P_2 \) elements are chosen for the postprocessing in the second step of our method. According to Remark 4.1, and for comparison of the stabilizing postprocessed finite element solutions with the
Table 1. Errors of the computed velocities in $L^2$-norm.

| $h$  | $\|u(T) - u_h^N\|_0$ | rate | $\|u(T) - \bar{u}_h^N\|_0$ | rate |
|------|----------------------|------|--------------------------|------|
| 1/4  | 0.00217733           | -    | 0.000992868              | -    |
| 1/6  | 0.000529178          | 3.48866 | 0.00016355             | 4.44792 |
| 1/8  | 0.000202376          | 3.33415 | 4.4057e-05             | 4.45932 |
| 1/10 | 9.95046e-05          | 3.19053 | 1.6059e-05             | 4.52272 |
| 1/12 | 5.64563e-05          | 3.10845 | 4.4057e-05             | 4.46594 |
| 1/14 | 3.52027e-05          | 3.06417 | 1.6059e-05             | 4.52272 |
| 1/16 | 2.34586e-05          | 3.03961 | 1.6059e-05             | 4.54840 |

Table 2. Errors of the computed velocities in $H^1$-norm.

| $h$  | $\|\nabla (u(T) - u_h^N)\|_0$ | rate | $\|\nabla (u(T) - \bar{u}_h^N)\|_0$ | rate |
|------|-----------------------|------|--------------------------|------|
| 1/4  | 0.0508249             | -    | 0.018681                | -    |
| 1/6  | 0.0226069             | 1.99803 | 0.00493605             | 3.28251 |
| 1/8  | 0.0127961             | 1.97827 | 0.0018011              | 3.50444 |
| 1/10 | 0.00823732            | 1.97391 | 0.000813042            | 3.56439 |
| 1/12 | 0.00574502            | 1.97641 | 0.000424348            | 3.56638 |
| 1/14 | 0.00423382            | 1.98007 | 0.000245707            | 3.54467 |
| 1/16 | 0.00324871            | 1.9834  | 0.000153708            | 3.51292 |

First step stabilized solutions, the stabilization parameter in the postprocessing step is set as the same along the time integration in the first step, which is $\alpha = h^3$. The numerical results are listed in Table 1-Table 3, in which $(u_h^N, p_h^N)$ is the stabilizing finite element (S-FEM) solution at time $T$ computed by the first step of our method, while $(\bar{u}_h^N, \bar{p}_h^N)$ is the stabilizing postprocessed finite element (SP-FEM) solution by our postprocessed method.

The numerical results listed in Tables 1-3 support the theoretical predictions very well, which show that the extra postprocessing step (namely, the second step) in our method greatly improves both the accuracy of the approximate solutions and the convergence rates, compared with the standard stabilized finite element method (namely, the first step of present method). In more details, the accuracy of the computed velocities both in $L^2$ and $H^1$-norms is improved by one order of magnitude when the mesh size is smaller than 1/10, while the precision of the computed pressures in $L^2$-norm is improved by two to three orders of magnitude. As for the rate of convergence, both the velocity in $L^2$ and $H^1$-norm and the pressure in $L^2$-norm are obviously improved by one and half unit for the stabilizing postprocessed finite element method compared with the one without postprocessing, which shows a better performance than our theory predicted. Table 4 reports the CPU time by the methods with and without postprocessing, which shows that the CPU time spent by the postprocessing step is really negligible. However, this negligible computational cost leads to a significant improvement of the approximate solutions compared with the method without postprocessing.

Second, to evaluate the efficiency of our proposed method, we compare our present method with the method in [3] where no stabilizations are employed. In this comparison, both the two options for the postprocessing elements are considered, where $T = 0.1$, $dt = 0.005$, $h = 1/40$ and $\nu = 10^{-k}$ ($k = 0, 1, 2, 3, 4, 5, 6$). We firstly choose the postprocessing elements as the higher-order Hood-Taylor $P_3 - P_2$
Table 3. Errors of the computed pressures in $L^2$-norm.

| $h$   | $\| p(T) - p_h^N \|$ | rate | $\| p(T) - \tilde{p}_h^N \|$ | rate |
|-------|----------------|------|----------------|------|
| 1/4   | 0.16059        | -    | 0.00173177     | -    |
| 1/6   | 0.0713664      | 2.00024 | 0.00038025 | 3.7391 |
| 1/8   | 0.0401428      | 2.00007 | 0.000124192 | 3.88971 |
| 1/10  | 0.0256912      | 2.00003 | 5.17408e-05 | 3.92385 |
| 1/12  | 0.0178411      | 2.00001 | 2.55797e-05 | 3.86376 |
| 1/14  | 0.0131077      | 2     | 1.46068e-05 | 3.63484 |
| 1/16  | 0.0100356      | 2     | 9.62881e-06 | 3.12082 |

Table 4. A comparison of computational time in seconds of the methods.

| $h$   | S-FEM   | SP-FEM  |
|-------|---------|---------|
| 1/4   | 7.8146  | 17.1435 |
| 1/6   | 30.0884 | 46.7738 |
| 1/8   | 69.0377 | 93.4098 |
| 1/10  | 93.6512 | 122.664 |
| 1/12  | 122.973 |

elements, and the stabilization parameter is chosen as $\alpha = h^3$ according to Remark 4.1. The numerical results are shown in Table 5, from which we see that when the viscosity $\nu$ is bigger than $10^{-4}$, there is no big difference in accuracy of the computed solutions. However, when the viscosity $\nu$ is smaller than $10^{-4}$, the computed solutions by our present method are much better than those computed by the method in [3]; particularly, when $\nu = 10^{-6}$, the accuracy of both the velocity and pressure computed by our method is better by more than ten times than that of the solutions computed by the method in [3].

We then choose the postprocessing elements as the same-order Hood-Taylor $P_2 - P_1$ elements on a finer mesh of size $\tilde{h} = 1/80$, and set $\alpha = 5\tilde{h}^2$ according to Remark 4.1. The results are listed in Table 6, which show that there is no obvious difference in accuracy of the computed pressures between the two methods for all values of the viscosity being tested. As for the velocity, when the viscosity $\nu$ is big, the two compared methods have near performance in accuracy of the computed velocities; however, as the viscosity becomes smaller (smaller than 0.01 in this test case), our proposed stabilizing postprocessed finite element method performs much better than the postprocessed finite element method without stabilizations in [3], especially, when $\nu = 0.000001$ and $0.000001$, the computed velocities have a better accuracy of more than two orders of magnitude than the ones computed by the postprocessed finite element method without stabilizations in [3]. As expected, the computational time taken by our present method is a little more than that of the method in [3].

5.2. Flow around a circular cylinder. In order to evaluate the performance of our method, we also give a numerical test for the benchmark problem of 2D channel flow around a circular cylinder defined on $\Omega = [0, 2.2] \times [0, 0.41]$. A circle of radius 0.05 centers at $(x, y) = (0.2, 0.2)$. Both the time-dependent inflow and outflow velocity profiles are given by

\[
\begin{align*}
u_1(x, y, t) &= 6 \sin(\frac{\pi t}{8}) 0.41^{-2} y(0.41 - y), & \quad \nu_2(x, y, t) &= 0, \\
x &\in [0, 2.2] & y &\in [0, 0.41], & t &\in [0, 8],
\end{align*}
\]

and no-slip boundary conditions are imposed on the top and bottom of the channel as well as the surface of the cylinder.
Table 5. A comparison of the computed solutions by differential methods with $P_3 - P_2$ elements for the postprocessing.

| Method | $\nu$ | $\| u(T) - \tilde{u}_h \|_0$ | $\| \nabla (u(T) - \tilde{u}_h) \|_0$ | $\| p(T) - \tilde{p}_h \|_0$ | CPU |
|--------|-------|-----------------|-----------------|-----------------|-----|
| Present | 1     | 5.41056e-08    | 6.5595e-06      | 5.55593e-06     | 803.389 |
|         | $10^{-1}$ | 4.25516e-07   | 7.98607e-06     | 5.27302e-06     | 805.513 |
|         | $10^{-2}$ | 4.52353e-06   | 4.51348e-05     | 5.19797e-06     | 820.973 |
|         | $10^{-3}$ | 4.5248e-05    | 0.000441765     | 5.1803e-06      | 800.503 |
|         | $10^{-4}$ | 0.000398027   | 0.00350141      | 5.16923e-06     | 779.926 |
|         | $10^{-5}$ | 0.00179621    | 0.0145128       | 5.17307e-06     | 779.998 |
|         | $10^{-6}$ | 0.00276865    | 0.022324        | 5.172e-06       | 783.388 |
| Ref. [3] | 1     | 5.18142e-08    | 6.5516e-06      | 5.54746e-06     | 718.683 |
|         | $10^{-1}$ | 4.14903e-07   | 7.36316e-06     | 5.25698e-06     | 718.624 |
|         | $10^{-2}$ | 4.51539e-06   | 3.36283e-05     | 5.20106e-06     | 722.98 |
|         | $10^{-3}$ | 4.59266e-05   | 0.000333893     | 5.1757e-06      | 732.069 |
|         | $10^{-4}$ | 0.000460089   | 0.00334075      | 5.1672e-06      | 740.576 |
|         | $10^{-5}$ | 0.00460161    | 0.034164        | 5.16846e-06     | 735.447 |
|         | $10^{-6}$ | 0.0460168     | 0.334242        | 3.85961e-05     | 738.079 |

Table 6. A comparison of the computed solutions by differential methods with $P_2 - P_1$ elements on a finer mesh for the postprocessing.

| Method | $\nu$ | $\| u(T) - \tilde{u}_h \|_0$ | $\| \nabla (u(T) - \tilde{u}_h) \|_0$ | $\| p(T) - \tilde{p}_h \|_0$ | CPU |
|--------|-------|-----------------|-----------------|-----------------|-----|
| Present | 1     | 5.41056e-08    | 6.5595e-06      | 5.55593e-06     | 803.389 |
|         | $10^{-1}$ | 4.25516e-07   | 7.98607e-06     | 5.27302e-06     | 805.513 |
|         | $10^{-2}$ | 4.52353e-06   | 4.51348e-05     | 5.19797e-06     | 820.973 |
|         | $10^{-3}$ | 4.5248e-05    | 0.000441765     | 5.1803e-06      | 800.503 |
|         | $10^{-4}$ | 0.000398027   | 0.00350141      | 5.16923e-06     | 779.926 |
|         | $10^{-5}$ | 0.00179621    | 0.0145128       | 5.17307e-06     | 779.998 |
|         | $10^{-6}$ | 0.00276865    | 0.022324        | 5.172e-06       | 783.388 |
| Ref. [3] | 1     | 5.18142e-08    | 6.5516e-06      | 5.54746e-06     | 718.683 |
|         | $10^{-1}$ | 4.14903e-07   | 7.36316e-06     | 5.25698e-06     | 718.624 |
|         | $10^{-2}$ | 4.51539e-06   | 3.36283e-05     | 5.20106e-06     | 722.98 |
|         | $10^{-3}$ | 4.59266e-05   | 0.000333893     | 5.1757e-06      | 732.069 |
|         | $10^{-4}$ | 0.000460089   | 0.00334075      | 5.1672e-06      | 740.576 |
|         | $10^{-5}$ | 0.00460161    | 0.034164        | 5.16846e-06     | 735.447 |
|         | $10^{-6}$ | 0.0460168     | 0.334242        | 3.85961e-05     | 738.079 |

The Hood-Taylor $P_2 - P_1$ elements are used for the spacial discretization and the higher-order Hood-Taylor $P_3 - P_2$ elements are employed for the postprocessing. The computations are performed for the Reynolds number corresponding to $\nu = 0.001$. The time interval $\Delta t = 0.001$ with final time $T = 8.0$. The mesh size is $h = 0.041$ and the stabilization parameter is set as $\alpha = h^3$ according to Remark 4.1. The mesh is shown in Figure 1. Figure 2 describes the temporal evolution of the drag coefficient, the lift coefficient, and the difference of the pressure between the front and the back of the cylinder $p(0.15, 0.2) - p(0.25, 0.2)$ computed by our present method and the method in [3], which shows that there is no big difference between the two methods. Figures 3-5 plot the contours of computed velocities and pressures.
FIG. 1. The triangulation of the computational domain.

FIG. 2. Temporal evolution of the drag coefficient (left), the lift coefficient (middle), and the difference of the pressure between the front and the back of the cylinder $p(0.15, 0.2) - p(0.25, 0.2)$ (right).

at various time, which agree with those in the literature (cf. [28]) and demonstrate the effectiveness of the proposed method.

6. **Conclusion.** In this paper, we discussed the stability and convergence of a postprocessed finite element method based on a subgrid model for solving numerically the time-dependent Navier-Stokes equations. Error bounds of the stabilizing postprocessed velocity and pressure were derived. Numerical results confirmed the theoretical predictions and verified the effectiveness of the proposed method, which show that compared with the postprocessed finite element method without stabilizations, our stabilizing postprocessed method significantly improves the accuracy of the approximate solutions. Although the error bounds depend on inverse powers of the viscosity, the behaviour of the method in the numerical experiments is better than other non-stabilized methods for small values of the viscosity $\nu$.

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