Statistics of resonances in a one-dimensional chain: a weak disorder limit

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Abstract
We study statistics of resonances in a one-dimensional disordered chain coupled to an outer world simulated by a perfect lead. We consider a limiting case for weak disorder and derive some results which are new in these studies. The main focus of this study is to describe the statistics of the scattered complex energies. We derive compact analytic statistical results for long chains. A comparison of these results has been found to be in good agreement with numerical simulations.

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(Some figures may appear in colour only in the online journal)

1. Introduction
Resonant phenomena have received much attention in atomic and nuclear physics and more recently in chaotic and disordered systems [1–7]. Complex energies, $\tilde{E}_\alpha = E_\alpha - \frac{1}{2} \Gamma_\alpha$, which correspond to poles of the scattering matrix on the unphysical sheet, characterize resonances [8]. Resonances correspond to the long-lived quasi-stationary states which eventually decay to continuum while the distribution of resonance widths, $P(\Gamma)$, determines decay of the corresponding survival probability with time.

In recent years, $P(\Gamma)$ has been the subject of investigations [1, 2] for a simple but much studied, discrete tight-binding one-dimensional random chain which is coupled to a perfect lead at one side. A numerical study [2] shows that in a broad range of $\Gamma$, $P(\Gamma) \sim \Gamma^{-\gamma}$, where the exponent $\gamma$ is very close to 1. Intuitively, the $1/\Gamma$ behavior can be deduced by assuming a uniform distribution for the localization centers of exponentially localized states [3]. However, from the analytic point of view one usually considers an infinitely long chain in which case the average density of resonances (DOR) has a well-defined limit. For a finite size system, the difference between the DOR and $P(\Gamma)$ is the normalization by the system size [4, 5]. Recently, Kunz and Shapiro have derived analytic expression of the DOR for a semi-infinite disordered chain [5]. They have obtained an exact integral representation of the DOR which is valid for arbitrary lead-chain coupling strength. This has been further simplified for small lead-chain coupling strength where a universal scaling formula is found. In this limit, they have proved...
the $1/\Gamma$-behavior of the DOR \cite{4, 5}. Besides, for the continuous limit of this model an integral representation of DOR has been obtained \cite{6}.

Kunz and Shapiro’s work has established a universal $1/\Gamma$ law for arbitrary strength of disorder in a semi-infinite chain. Numerically one can verify $1/\Gamma$ law of the DOR, similar to what has been done by Terraneo and Guarnery \cite{2} in finite samples for $P(\Gamma)$. Such verifications require the localization lengths to be much smaller than the size of the sample. In the case of weak disorder, an analytic result for the localization lengths is particularly useful. It comes from a second-order perturbation theory. It states that the localization length is maximum near the middle of the energy band and is proportional to $W^{-2}$ where $W$ is the width of the disorder \cite{9–11}. On the other hand, this result also leads to an interesting limiting situation where the localization lengths are much longer than the sample size. This is what we refer to as a weak disorder limit in this paper. This limit has scarcely been studied hitherto although it is relevant in the study of localization through resonances. Besides, there has been a belief for some sort of universality in the weak disorder limit. In this paper, we address this limit and derive analytic results which describe the statistics of resonances. Our work probes a fresh area and studies a weak disorder limit which has never been addressed before.

For open systems, instead of studying the scattering matrix in a complex plane, we follow an alternative approach where one solves the Schrödinger equation by describing a particle ejected from the system or equivalently with a boundary condition of outgoing waves (Siegert boundary condition \cite{12}). In this approach, one naturally arrives at the problem of solving a non-Hermitian effective Hamiltonian which admits complex eigenvalues $\tilde{E}_\alpha$ \cite{2, 4, 5, 13}. For details of such non-Hermitian effective Hamiltonians, we refer to a recent study \cite{14} and references therein.

We derive the statistics which describe the scattered complex energies of a disordered chain around those regular ones which correspond to an open chain without any disorder (clean chain). For instance, we derive an average of the square of the absolute values of the shifts in complex energies from the regular ones over all realizations of the set of random site energies. Similarly, we obtain results for the statistics of real and imaginary parts of those shifts. These results lead to compact expressions for long chains. To show the generality of our approach, we also derive these results for the so-called parametric resonances which have been particularly useful in numerical studies \cite{2}. Finally, we give numerical verifications of our analytic results.

The paper is organized as follows. Although the system and its effective Hamiltonian have been nicely explained earlier in \cite{2, 4, 5}, for the sake of completeness of this paper we will describe these briefly in section 2. In the same section, we will also describe the exact and the parametric resonances. In section 3, we will derive the result for resonances in an open-clean chain of finite length, in terms of a polynomial equation. For long chains, we will solve this polynomial equation in the leading order of the inverse of the length. In section 4, we will use perturbation theory to obtain the first- and second-order corrections in the complex energies for a weak disorder. In section 5, we will calculate the statistics of the scattered complex energies. In the same section, we will simplify our results for long chains and obtain compact expressions. In section 6, we will briefly discuss the numerical methods used to calculate complex energies of non-Hermitian effective Hamiltonians and numerically verify our analytical results. This will be followed by the conclusion in section 7.

2. The model and its effective Hamiltonian

A discrete tight-binding one-dimensional chain of length $N$ (shown by positive integers, $n = 1, 2, \ldots, N$, used for indexing the sites of the chain in figure 1) is connected to an
outer world (represented by a perfect lead whose sites are shown by a zero and negative integers, \(n = 0, -1, -2, \ldots\)). Each site of the chain has the site energy \(\epsilon_n\), where \(\epsilon_n\) are statistically independent random variables chosen from some symmetric distribution. Each nearest neighbor site of the chain as well as of the lead is coupled by a hopping amplitude \(t\). The hopping amplitude for the pair \(n = 0\) and \(n = 1\) is \(t'\) which takes values from \(t' = 0\) (closed chain) to \(t' = t\) (fully coupled chain). With this hopping, a particle, which is initially located somewhere in the chain, eventually escapes to the outer world.

Now we write down the Schrödinger equation for the entire system,

\[
-t \psi_{n+1} - t \psi_{n-1} = \tilde{E} \psi_n, \quad \text{for } n < 0, \\
-t \psi_1 - t' \psi_1 = \tilde{E} \psi_0, \quad \text{for } n = 0, \\
-t' \psi_0 - t \psi_2 + \epsilon_1 \psi_1 = \tilde{E} \psi_1, \quad \text{for } n = 1, \\
-t \psi_{n-1} - t \psi_{n+1} + \epsilon_n \psi_n = \tilde{E} \psi_n, \quad \text{for } 2 \leq n \leq N.
\]

In order to avoid cluttering of notations, we always represent quantities corresponding to the disordered system by \textit{script} letters, while quantities for the clean system are represented in usual math notations. Tilde is used to discriminate the open system case from the closed one. Equation (1) is for the lead where \(\epsilon_n = 0\). Equations (2) and (3) describe the lead-chain coupling and equation (4) is for the chain. As in [5], we solve equations (1)–(4) with a boundary condition of an outgoing plane wave in the lead, i.e. \(\psi_n \propto \exp(-i \tilde{k} n)\), where \(0 < \Re(\tilde{k}) < \pi\) and \(\Im(\tilde{k}) < 0\). The condition on \(\Re(\tilde{k})\) ensures that the outgoing wave propagates to left in the lead. The condition on \(\Im(\tilde{k})\) is considered so that the amplitude of the resonance wavefunction grows in the lead. It comes from equation (1) that the complex energy \(\tilde{E}\) is related to the complex wave vector \(\tilde{k}\) via the dispersion relation \(\tilde{E} = -2t \cos(\tilde{k})\). Now we eliminate all \(\psi_n\) for \(n < 1\) from equations (1)–(4) and obtain

\[
-t \psi_{n+1} - t \psi_{n-1} + \tilde{\epsilon}_n \psi_n = \tilde{E} \psi_n,
\]

where

\[
\tilde{\epsilon}_n = \epsilon_n - t \eta \exp(i \tilde{k}) \delta_{n1},
\]

for \(n = 1, 2, \ldots, N\). The parameter \(\eta = (t'/t)^2\) measures the coupling strength to the outside world.

An effective Hamiltonian defined by the equation (5) is non-Hermitian. For instance, if \(\mathcal{H}\) is the \(N \times N\) tridiagonal Hermitian matrix which represents the Hamiltonian of the closed-disordered chain, then one may write the effective Hamiltonian, \(\tilde{\mathcal{H}}\), as

\[
\tilde{\mathcal{H}} = \mathcal{H} - t \eta \lambda(\tilde{k}) P.
\]
Here, $P = |1\rangle\langle 1|$ is the projection for site $n = 1$ and $\lambda = \exp(i\tilde{k})$. The above non-Hermitian effective Hamiltonian was first obtained by Terraneo and Guarnery [2]. The underlying result here is that the same relation (7) is valid for any Hermitian $\mathcal{H}$ representing a (closed) quantum system [14] which has $N$-dimensional state space. Resonances are characterized by the complex eigenvalues, $\tilde{E}_\alpha$, of $\tilde{\mathcal{H}}$. 

Note here the dependency of $\tilde{\mathcal{H}}$ on the complex wave vector $\tilde{k}$ which is related to the complex energies via the dispersion relation mentioned above—this is not a standard eigenvalue problem. To standardize this problem, ‘parametric resonances’ are often used as an alternative. In this approach, the dependence of $\lambda$ on $\tilde{k}$ is typically neglected, thereby reducing the problem of finding the eigenvalues of the effective Hamiltonian at a chosen value of $\tilde{k}$. As expected, parametric resonances yield approximate statistical results which are close to those for the exact resonances in a strongly localized regime [2]. Parametric resonances depend on a chosen parameter, for instance let $\tilde{k} = k_0$ and fix it in the middle of the energy band, $k_0 = \pi/2$.

Writing explicitly $\lambda(\tilde{k}) = \begin{cases} \exp(i\tilde{k}), & \text{for exact resonances} \\ i, & \text{for parametric resonances} \end{cases}$ (8)

From now on, we set the energy scale by taking $t = 1$, denoting the complex variable $\tilde{E}/t$ by $\tilde{Z}$. We denote the Hamiltonian matrix representing the closed-clean chain by $H$. It differs from $\tilde{\mathcal{H}}$ only at the diagonal as, for the clean chain case, all the site energies are zero. The calculation of the eigenvalues of $H$ is a standard exercise where one derives $z_\alpha = -2 \cos[\alpha \pi/(N + 1)]$ for $\alpha = 1, \ldots, N$.

Before going into a detailed treatment of the problem, we should first sketch the outline of our approach. We are interested in a weak disorder regime. Since our approach relies on perturbation theory, we need complex energies of an open-clean chain, i.e. the $\tilde{E}_\alpha$s. So we will begin with calculating the resonances for an open-clean chain of finite length. Then, we will perform the perturbation series expansion up to the second order of strength of the disorder. This will be followed by the derivation of the statistical results. Finally, we will consider the large-$N$ limit of these results.

3. The open-clean chain

We begin with defining the resolvent $\tilde{G}(z) = (z - \tilde{\mathcal{H}})^{-1}$. Using equation (7), we may also write

$$\tilde{G}(z) = (z - \mathcal{H} + \eta \lambda P)^{-1}. \quad (9)$$

For the open-clean chain, we define the resolvent

$$\tilde{G}(z) = (z - H + \eta \lambda P)^{-1} = (1 + \eta \lambda GP)^{-1} G, \quad (10)$$

where we have introduced $G(z) = (z - H)^{-1}$ as the resolvent for the ‘unperturbed’ closed-clean chain. Resonances correspond to the singularities of the matrix $\tilde{G}_{mn}(z)$ or to the roots of the secular equation

$$F(z) = 0 = 1 + \eta \lambda G_{11}(z). \quad (11)$$

where $G_{11}$ is the $[1, 1]$ element of the matrix $G$ in site representation. ($G_{mn}(z) = \langle n| (z - H)^{-1} |m\rangle.$)

To obtain $G_{11}$ for finite $N$, we use the ordinary difference equation (ODE),

$$\psi_{n+1} + \psi_{n-1} + z \psi_n = 0, \quad (12)$$
with the boundary conditions $\psi_0 = \psi_{N+1} = 0$. This equation is obtained from equation (5) by setting all $\epsilon_n = 0$. Next we consider $u_n(z)$ and $v_n(z)$ to be the two linearly independent functions which satisfy the ODE:

$$u_{n+1} + u_{n-1} + z u_n = 0,$$

(13)

$$v_{n+1} + v_{n-1} + z v_n = 0,$$

(14)

where $u_0 = v_{N+1} = 0$. Since norm of $u_n$ and $v_n$ is arbitrary, we fix $u_1 = v_N = 1$. Furthermore, we claim that the resolvent is given by

$$G_{nm} = -\frac{u_n v_m \Theta(m - n) + u_m v_n \Theta(n - m)}{W_n}.$$

(15)

Here, $\Theta(n)$ is the unit-step function and $W_n = u_n v_{n-1} - u_{n-1} v_n$ is the Wronskian. Using equations (13) and (14), it is straightforward to see that the Wronskian is independent of $n$. One can also check that

$$G_{n+1m} + G_{n-1m} + z G_{nm} = \delta_{nm}.$$

(16)

We now set $u_n = v_{N+1-n}$ to match the initial value problems (13) and (14) to the boundary value problem (12). We find

$$G_{11} = -\frac{u_N}{u_{N+1}}.$$

(17)

The ODE (13) is satisfied by the Chebyshev polynomial of the second kind, $U_m(-z/2)$, defined as

$$U_m(x) = \frac{\sin[(m + 1) \cos^{-1}(x)]}{\sin[\cos^{-1}(x)]},$$

(18)

for $U_0(x) = 1$ and $U_1(x) = 2x$. Since we have fixed $u_1 = 1$, therefore $u_n = U_{n-1}$, thus we can write equation (17) as

$$G_{11} = -\frac{U_{N-1}(-z/2)}{U_{N}(-z/2)} = -\frac{\sin[Nk]}{\sin[(N+1)k]}.$$

(19)

Here, the last equality follows from the energy dispersion relation. Using equation (19) in equation (11), we end up with an algebraic equation

$$F(z) = 0 = 1 - \eta \lambda U_{N-1}(-z/2) / U_N(-z/2).$$

(20)

Zeros of $F(z)$ are the roots of a polynomial of order $N$. For exact resonances, equation (20) can be easily transformed into

$$[a(z)]^{2N+1} = \frac{a(z)^{-1} - \eta a(z)}{1 - \eta},$$

(21)

where

$$a(z) = -\exp[ik(z)].$$

(22)

In order to solve equation (21), we propose an ansatz assuming that opening of the system at one end causes $O(N^{-1})$ complex corrections to the $k_\alpha$s. Let

$$\tilde{k}_\alpha = k_\alpha + \frac{\Phi_\alpha}{N},$$

(23)

where $\Phi_\alpha$ is a complex quantity and $k_\alpha = \alpha \pi / (N + 1)$. Inserting this ansatz into equations (21) and (22), we obtain

$$\tilde{k}_\alpha = k_\alpha - \frac{i}{2N} \ln[\Omega(k_\alpha; \eta)] + O\left(\frac{1}{N^2}\right).$$

(24)
Comparison of the result (26) (pluses) with the numerical solution of the polynomial equation (20) (circles, squares and diamonds) for the exact resonances where $\eta = 0.5$, 0.81 and 0.99. We have considered $N = 100$.

where

$$ \Omega(k; \eta) = \frac{1 - \eta e^{2i k \alpha}}{1 - \eta}. $$

Now, up to $O(1/N)$, $\tilde{z}_\alpha$ may be written as

$$ \tilde{z}_\alpha = -2 \cos(k \alpha) - \frac{i \sin(k \alpha)}{N} \ln(\Omega). $$

The same result can be obtained for the parametric resonances, after repeating the similar steps, but with different $\Omega$:

$$ \Omega(k_\alpha; \eta) = \frac{1 - i \eta e^{i k \alpha}}{1 - i \eta e^{-i k \alpha}}. $$

One should bear in mind that there is no resonance for $\eta = 1$, as the system is fully coupled to the lead. However, for parametric resonances, one artificially gets resonances even when $\eta = 1$. Note that the result (26) is symmetric about the imaginary axis for both cases.

In figures 2 and 3, we compare the numerical solutions of the polynomial equation (20) with our results (25–27), for $N = 100$, $\eta = 0.50$, 0.81 and 0.99 and $N = 100$, respectively, for exact and parametric resonances. Equation (20) has been solved by using Newton’s method with the initial guess $k_\alpha = k_\alpha$. These figures show that our result (26) is close to the numerical solution. The agreement gets better as $\eta \to 1$ (not shown separately). However, ansatz (23) is not valid near the band edges. Moreover, the agreement fails for parametric resonances near the middle of the band as $\eta \to 1$; see figure 3 for $\eta = 0.99$.

4. The weak disorder limit

In the next stage of the problem, we switch to a very weak disorder in the chain. From a second-order perturbation theory, we know that for a disordered infinitely long chain the localization length, $\xi(E)$, is maximum at the middle of the band. For small $W$, it is given by [10, 9]

$$ \xi(E) = \frac{24(4t^2 - E^2)}{W^2}, $$

where $t$ is the hopping parameter, and $W$ is the disorder strength.
implying thereby, \( \tilde{\epsilon}(0) = \frac{96t^2}{W^2} \). However, the exact result shows a small deviation at the band center due to the breakdown of the second-order perturbation theory [11]. We consider a limiting situation when \( \tilde{\epsilon}(0)/N \gg 1 \). For instance, in figures 4 and 5, we show the scatter plot (\( \Re(\tilde{z}_\alpha) \) versus \( \Im(\tilde{z}_\alpha) \)) for exact and parametric resonances, respectively. In both cases, we have considered \( N = 100, \eta = 0.81 \) and \( W = 0.015 \) so that \( \tilde{\epsilon}(0) \gg N \). As seen in these figures, complex energies of the disordered chain are scattered around the \( \tilde{z}_\alpha \)s.

We now calculate the corrections to \( \tilde{z}_\alpha \) for such weak disorder case. It is suggestive here to deal with the self-energy. Let \( S_1(\epsilon_2, \ldots, \epsilon_N; z) \) be the self-energy for the first site, defined via

\[
\mathcal{G}_{11}(z) = \frac{1}{z - \epsilon_1 - S_1([\epsilon]; z)}. \tag{29}
\]
Here, \( \{ \epsilon \} \) denotes the set \( \epsilon_2, \ldots, \epsilon_N \) and \( G_{11} \) is the \( [1, 1] \) element of the resolvent \( G(z) = (z - \mathcal{H})^{-1} \), defined for the Hermitian matrix \( \mathcal{H} \). For later convenience, we write

\[
\mathcal{H} = H + W, \tag{30}
\]

where \( W = \sum_{\ell=1}^{N} \epsilon_\ell P_\ell \) and \( P_\ell = | \ell \rangle \langle \ell | \) is the projection for the \( \ell \)th site.

In the rest of the paper, we will work out results only for the exact resonances. For the parametric resonance, these results can be carried out following similar steps, so we skip all the intermediate steps merely by stating the result at the end.

As before in equation (11), for the disordered chain, resonances correspond to the roots of the secular equation

\[
\mathcal{F}(z) = 0 = z - \epsilon_1 - S_1(\{ \epsilon \}; z) + \lambda \eta. \tag{31}
\]

Preserving \( \tilde{z}_\alpha \) as the roots of equation (11), we define \( \tilde{Z}_\alpha \) as the roots of equation (31). Now we expand the roots

\[
\tilde{Z}_\alpha = \tilde{z}_\alpha + (\delta_1 \tilde{Z}_\alpha) + (\delta_2 \tilde{Z}_\alpha),
\]

assuming that \( (\delta_1 \tilde{Z}_\alpha) \) are linear while \( (\delta_2 \tilde{Z}_\alpha) \) are quadratic in the \( \epsilon_j \), for \( j = 1, \ldots, N \). Then, for \( S_1(\{ \epsilon \}; \tilde{Z}_\alpha) \), up to \( \mathcal{O}(\{ \epsilon \}^2) \), we obtain

\[
S_1(\{ \epsilon \}; \tilde{Z}_\alpha) = S_1(\{0\}; \tilde{z}_\alpha) + \sum_{n=2}^{N} \epsilon_n \left( \frac{\partial S_1(\{ \epsilon \}; z)}{\partial \epsilon_n} \right)_{\{ \epsilon \} = 0, z = \tilde{z}_\alpha}
+ (\delta_1 \tilde{Z}_\alpha + \delta_2 \tilde{Z}_\alpha) \left( \frac{\partial S_1(\{ \epsilon \}; z)}{\partial z} \right)_{\{ \epsilon \} = 0, z = \tilde{z}_\alpha}
+ \frac{1}{2} \sum_{n,m=2}^{N} \epsilon_n \epsilon_m \left( \frac{\partial^2 S_1(\{ \epsilon \}; z)}{\partial \epsilon_n \partial \epsilon_m} \right)_{\{ \epsilon \} = 0, z = \tilde{z}_\alpha}
+ \frac{1}{2} (\delta_1 \tilde{Z}_\alpha)^2 \left( \frac{\partial^2 S_1(\{ \epsilon \}; z)}{\partial z^2} \right)_{\{ \epsilon \} = 0, z = \tilde{z}_\alpha}. \tag{32}
\]

We will use this expansion in equation (31). Before that, we evaluate

\[
1 - \left( \frac{\partial S_1(\{ \epsilon \}; z)}{\partial z} \right)_{\{ \epsilon \} = 0, z = \tilde{z}_\alpha} = \frac{\partial}{\partial z} \left. \frac{1}{G_{11}(z)} \right|_{z=\tilde{z}_\alpha}. \tag{33}
\]
and
\[
\frac{\partial S_1(\{\epsilon\}; z)}{\partial \epsilon_n} \bigg|_{\{\epsilon\}=0, z=z_a} = \left. \frac{1}{(G_{11})^2} \frac{\partial G_{11}}{\partial \epsilon_n} \right|_{\{\epsilon\}=0, z=z_a},
\]
for \(n \geq 2\). These equalities come from equation (29). Finally, we calculate derivatives of \(G_{11}\), at \(\{\epsilon\} = 0\) and \(z = z_a\) with respect to \(\{\epsilon\}\) by using equation (30) for the Born-series expansion of \(G(z)\). We find
\[
\frac{\partial G_{11}}{\partial \epsilon_n} \bigg|_{\{\epsilon\}=0, z=z_a} = G_{1n}G_{n1} \bigg|_{z=z_a}.
\]
Grouping all these, for the first-order corrections, we obtain
\[
(\delta_1 \tilde{Z}_a) = \epsilon_1 + \sum_{n=2}^{N} \epsilon_n G_{1n}G_{n1} \bigg|_{z=z_a} + \frac{\delta}{\delta z} \frac{1}{G_{11}(z)} \bigg|_{z=z_a} \exp[i(k_a)]
\]
\[
\times \left[ \frac{d^2 \exp(ik(z))}{d^2 z} \right]^{-1}.
\]
Similarly for the second-order corrections, we obtain
\[
(\delta_2 \tilde{Z}_a) = \sum_{n,m=2}^{N} \epsilon_n \epsilon_m \left( \frac{G_{1n}G_{nm}G_{m1}}{|G_{11}|^2} - \frac{G_{1n}G_{m1}}{|G_{11}|^2} \right)
\]
\[
- \frac{(\delta_1 \tilde{Z}_a)^2}{2} \left( \frac{\partial^2}{\partial^2 z} G_{11} \right) + \eta \left( \frac{d^2 \exp(ik(z))}{d^2 z} \right) \right) \bigg|_{\{\epsilon\}=0, z=z_a}
\]
\[
\times \left[ \frac{d \exp(ik(z))}{d z} \right]^{-1}.
\]
Note that \((\delta_1 \tilde{Z}_a)\) and \((\delta_2 \tilde{Z}_a)\) have been obtained in terms of the resolvent of the closed-clean chain which we already know in terms of Chebyshev polynomials; see equation (15) and the relation between \(u_n\) and \(v_n\) with Chebyshev polynomials.

5. Statistics of the scattered complex energies

We are interested in the statistics of the scattered complex energies. For instance, using the first-order result (36) of the perturbation theory, we calculate the average of the square of the absolute shift in complex energies defined as \(\langle (\Delta \tilde{Z}_a)^2 \rangle = \langle (\tilde{Z}_a - \bar{z}_a)^2 \rangle\). The angular brackets are used here to represent the averaging over many realizations of set of all random site energies \(\{\epsilon_n\}\). This quantity gives a statistical account for the scattered complex energies. We also calculate \(\langle (\Re(\Delta \tilde{Z}_a))^2 \rangle\) and \(\langle (\Im(\Delta \tilde{Z}_a))^2 \rangle\), namely the average of the square of the real and the imaginary part of the shift \((\tilde{Z}_a - \bar{z}_a)\), respectively. To obtain the latter quantities we first need to calculate \(\langle (\Delta \tilde{Z}_a)^2 \rangle\) and \(\langle (\Delta \tilde{Z}_a)^4 \rangle\), since
\[
\langle (\Re(\Delta \tilde{Z}_a))^2 \rangle = \frac{4}{3} \langle (\Delta \tilde{Z}_a)^2 \rangle + \langle (\Delta \tilde{Z}_a)^4 \rangle - 2 \langle (\Delta \tilde{Z}_a)^2 \rangle,
\]
\[
\langle (\Im(\Delta \tilde{Z}_a))^2 \rangle = \frac{2}{3} \langle (\Delta \tilde{Z}_a)^2 \rangle + \langle (\Delta \tilde{Z}_a)^4 \rangle - 2 \langle (\Delta \tilde{Z}_a)^2 \rangle.
\]
Here, we have used \((\cdot)^*\) to represent the complex conjugate (cc).

For all these three statistics, we simplify \((\delta_1 \tilde{Z}_a)\), given in equation (36), in terms of Chebyshev polynomials as
\[
(\delta_1 \tilde{Z}_a) = \sum_{n=1}^{N} \frac{\epsilon_n}{2} \left( U_{n-n}(\bar{z}_a/2) \right)^2 [U_{n-1}(\bar{z}_a/2)T_{n+1}(\bar{z}_a/2) - N]
\]
\[
- \eta \exp[i(k_a)] \sin(k_a) [U_{n-1}(\bar{z}_a/2)]^2 \right)^{-1},
\]
where $T_n(z) = \cos[m \cos^{-1}(z)]$ is the Chebyshev polynomial of the first kind. Further simplifications occur when these polynomials are expressed in their trigonometric forms. For instance, let us calculate $|(\Delta \tilde{z}_a)|^2$, with $\tilde{z}_a / 2 = \cos(\tilde{\theta}_a)$ where $\tilde{\theta}_a = \pi - \bar{k}_a$. We obtain

$$\frac{|(\Delta \tilde{z}_a)|^2}{4} = \frac{\sum_{n,m=1}^{N} e_n e_m \sin^2(n \tilde{\theta}_a) \sin^2(m \tilde{\theta}_a)}{|D(\tilde{z}_a)|^2}. \quad (41)$$

Here, $n'$ and $m'$ are, respectively, $N + 1 - n$ and $N + 1 - m$, and $D(\tilde{z}_a)$ is simply the quantity in the second bracket of equation (40). Averaging releases one of the summation as the $\epsilon_j$’s are statistically independent identically distributed (iid) random variables. We simply have

$$|(\Delta \tilde{z}_a)|^2 = \sigma^2 \sum_{n=1}^{N} 4 \sin^2(n \tilde{\theta}_a) \sin^2(n \tilde{\theta}_a), \quad (42)$$

where $\sigma^2$ is variance of the $\epsilon_j$’s.

Summation in the above equality can be performed by using trigonometric identities. For instance, we first write

$$4 \sin^2(n \theta) \sin^2(n \theta^*) = 1 - \cos(2n \theta) - \cos(2n \theta^*) + \frac{\cos(4n \Im[\theta]) + \cos(4n \Im[\theta])}{2}, \quad (43)$$

and use the summation formula

$$\sum_{n=1}^{N} \cos(n \theta) = 1 \left[ \frac{\sin[(N + 1/2) \theta]}{\sin(\theta/2)} - 1 \right]. \quad (44)$$

It turns out after some trigonometry that one can write the summation in a closed form. We find

$$\sum_{n=1}^{N} 4 \sin^2(n \theta) \sin^2(n \theta^*) = N + \frac{1}{2} - \frac{U_{2N} + U_{2N}^{*}}{2} + \frac{T_{2N+2}^{*} T_{2N} - T_{2N+2} T_{2N}^{*}}{2[T_{2N}^{*} - T_{2N}]} . \quad (45)$$

Here, the argument of the polynomials is $\tilde{z}_a / 2$ and for their complex conjugate it is $\tilde{z}_a^*/2$. Finally, we write down the finite-N result for the average of the absolute square of the shift,

$$|(\Delta \tilde{z}_a)|^2 = \sigma^2 \left[ N + \frac{1}{2} - \frac{U_{2N} + U_{2N}^{*}}{2} + \frac{T_{2N+2}^{*} T_{2N} - T_{2N+2} T_{2N}^{*}}{2[T_{2N}^{*} - T_{2N}]} \right]. \quad (46)$$

We now turn our attention to large-$N$ behavior of result (46). For this purpose, we use ansatz (23) and result (24) for $\bar{k}_a$. Large-$N$ behavior for the Chebyshev polynomials, with argument $\tilde{z}_a$, may be calculated as

$$T_{2N}(\tilde{z}_a / 2) = \frac{\exp[2iN\tilde{\theta}_a] + \exp[-2iN\tilde{\theta}_a]}{2} \approx \frac{\Omega(k_a; \eta) \exp(-2ik_a) + [\Omega(k_a; \eta)]^{-1} \exp(2ik_a)}{2} , \quad (47)$$

$$T_{2N+2}(\tilde{z}_a / 2) = \frac{\Omega(k_a; \eta) + [\Omega(k_a; \eta)]^{-1}}{2} + \mathcal{O}(N^{-1}) , \quad (48)$$

$$U_{2N}(\tilde{z}_a / 2) = \frac{\exp[i(2N + 1)\tilde{\theta}_a] - \exp[-i(2N + 1)\tilde{\theta}_a]}{2} \approx \frac{\Omega(k_a; \eta) \exp(-ik_a) - [\Omega(k_a; \eta)]^{-1} \exp(ik_a)}{2} \exp(-ik_a) - \exp(ik_a) . \quad (49)$$
Finally,
\[ T_2(\tilde{z}_a/2) - T_3(\tilde{z}_a/2) = -\frac{2i \Im\{\Phi_u\}}{N} \tilde{z}_a + \mathcal{O}(N^{-2}) \]
\approx \frac{4i}{N} \cos(k_a) \Im\{\Phi_u\},
(50)
where we have used ansatz (23) in the second-order polynomial \( T_2(z) = 2z^2 - 1 \) and \( \Im\{\Phi_u\} = -\sin(k_a) \ln(|\Omega|) \), as obtained from equations (23) and (24).

We can now plug in these results in equation (46). These asymptotic results give the numerator as \((N + a_1 + a_2/(a3/N))\) where \(a_1, a_2/a3\) are \(\mathcal{O}(N^0)\). Similarly, we obtain the denominator as \((N^2 + b1 N + b2)\) where \(b1\) and \(b2\) are \(\mathcal{O}(N^0)\); see appendix A for details. Thus, in the leading order, we obtain
\[
\langle(\Delta \tilde{Z}_a)^2\rangle = \frac{\sigma^2}{N} \left( 1 + \frac{1}{8} \frac{(|\Omega|^2 - |\Omega|^{-2})}{\ln(|\Omega|)} \right).
(51)
\]

What follows next is the calculation of large-\(N\) results for \(\langle(\Im(\Delta \tilde{Z}_a))^2\rangle\) and \(\langle(\Re(\Delta \tilde{Z}_a))^2\rangle\). Since we need first to calculate \(\langle(\Delta \tilde{Z}_a)^2\rangle\) and \(\langle(\Delta \tilde{Z}_a)^*\rangle^2\) from equation (40), we obtain after averaging
\[
\langle(\Delta \tilde{Z}_a)^2\rangle = \sigma^2 \sum_{n=1}^{N} 4 \sin^4(n\tilde{\theta}_a),
(52)
\]
and
\[
\langle(\Delta \tilde{Z}_a)^*\rangle^2 = \sigma^2 \sum_{n=1}^{N} 4 \sin^4(n\tilde{\theta}_a/\Delta_1).
(53)
\]

For summation, we use formula [15]
\[
\sum_{n=1}^{N} \sin^4(n \theta) = \frac{1}{8} \left[ 3N - \sin(N \theta) \left( 4 \cos[(N + 1) \theta] - \cos[2(N + 1) \theta] \cos(N \theta) \right) \right],
\sum_{n=1}^{N} \sin^4(n \tilde{\theta}_a) = \frac{1}{8} \left[ 3N - 4U_{N-1} T_{N+1} + \frac{T_{2N+2} U_{2N-1}}{\tilde{z}_a} \right],
(54)
\]
where in the second equality the polynomials have argument \(\tilde{z}_a/2\) with \(2 \cos(\tilde{\theta}_a) = \tilde{z}_a\). Similarly for the summation in equation (53), one obtains the polynomials with argument \(\tilde{z}_a/2\). One can now use equality (54) in equations (52) and (53) in order to derive the finite-\(N\) result for \(\langle(\Im(\Delta \tilde{Z}_a))^2\rangle\) and \(\langle(\Re(\Delta \tilde{Z}_a))^2\rangle\). For large-\(N\), we make use of ansatz (23) and calculate the leading order contribution as
\[
\langle(\Im(\Delta \tilde{Z}_a))^2\rangle = \frac{\sigma^2}{2N} \left\{ \frac{5}{2} + \frac{1}{8} \frac{(|\Omega|^2 - |\Omega|^{-2})}{\ln(|\Omega|)} + g(k_a) \right\},
(55)
\]
\[
\langle(\Re(\Delta \tilde{Z}_a))^2\rangle = -\frac{\sigma^2}{2N} \left\{ \frac{1}{2} - \frac{1}{8} \frac{(|\Omega|^2 - |\Omega|^{-2})}{\ln(|\Omega|)} + g(k_a) \right\},
(56)
\]
where
\[
g(k_a) = \frac{1}{8} \left( \frac{e^{-2ik_a \Omega^2} - e^{2ik_a \Omega^{-2}}}{4i \sin(k_a) (2N \cos(k_a) + i \sin(k_a) \ln(|\Omega|))} \right) + \text{cc}.
(57)
\]
Equations (51), (55) and (56) are our main analytical results and they are also valid for parametric resonances with the \(\Omega\) given in equation (27). In figure 6, we verify the asymptotic results (51), (55) and (56) against their finite-\(N\) counterparts, for exact resonances with
N = 100 and \( \eta = 0.81 \). Figure 7 is repeated on the same pattern but for parametric resonances where \( N = 500 \) and \( \eta = 0.81 \). They confirm that the asymptotic results give a good account for the finite-\( N \) results. However, there are some exceptions near the edges (not visible on the scale of the plot) where the ansatz (23) is not valid.

It turns out that in order to calculate the DOR, we need the second-order corrections \((\delta Z_\alpha)\) derived in equation (37). We have followed the method used earlier [17] for the Hatano–Nelson model [18]. However, we have not been able to obtain a closed expression of the DOR. This is discussed in appendix B, where we leave the calculations with a formal expression for the DOR.

6. Numerical methods and verification of the equations (51), (55) and (56)

Numerical simulations for parametric resonance are always cost efficient. The reason being that there one deals with the standard eigenvalue problem for which many fast subroutine
packages are available, for instance, LAPACK. On the other hand, to verify results (51), (55) and (56) for exact resonances, where one needs to obtain numerical solutions of a characteristic polynomial equation of order \(N\) in a complex plane, there is no good algorithm. In this paper, we show results for the exact resonances by calculating roots of the characteristic polynomial where we have used a cost efficient numerical subroutine \(\text{ezero}\). The subroutine is available on the CPC program library. There is one major advantage of using this subroutine over other methods, for instance Newton’s method. This subroutine does not require initial guesses for the roots but only the contour which encloses all the roots of the polynomial. Besides, it also avoids calculating the derivatives which may result into numerical overflow.

In alternative to \(\text{ezero}\), we have used a different approach for calculating the roots. We survey the complex \(\tilde{k}\)-plane for the zeros of the \(\text{Det}[M(\tilde{k})M(\tilde{k})^\dagger]\) where \(M_{rs} = -2\cos(\tilde{k})\delta_{rs} - \tilde{H}_{rs}\) for \(r, s = 1, \ldots, N\). In our system \(-\pi < \Re{\tilde{k}} < \pi\) and \(\Im{\tilde{k}} < 0\). These zeros give the eigenvalues of \(\tilde{H}\). However, in the latter approach it is advisable to disintegrate the complex plane into small cells at first and then at every iteration into smaller one—only for \(N\) cells which contain minima of the lowest eigenvalue and throwing the rest out. In this way, one makes the algorithm faster and obtain the zeros in a reasonable precision. For a tridiagonal matrix, this algorithm consumes a time which roughly grows with \(N^3\). However, while comparing the two methods on a simple machine we find that the method used in \(\text{ezero}\) is much faster than the method described here. We refer to [16] for further details of this subroutine.

In figure 8, we compare asymptotic results with simulation done for the total number of realizations \(L = 2500\), for exact resonances. In figure 9, we compare numerical results obtained for parametric resonances, where \(N = 500\), \(\eta = 0.81\) and \(L = 5000\), with our theory for large-\(N\). Although we have considered only the flat disorder, yet our results are valid for the Gaussian or other symmetric distribution functions. These figures show that our asymptotic results are in fair agreement with the numerical results for almost all \(\alpha\). For instance, near the middle of the energy band it describes reasonably well a dip and a peak, respectively, in the \(\langle (\Im{\Delta\tilde{Z}_\alpha})^2 \rangle\) and \(\langle (\Re{\Delta\tilde{Z}_\alpha})^2 \rangle\). These two opposite effects, however, cancel out in \(\langle |\Delta\tilde{Z}_\alpha|^2 \rangle\).

\(^3\) The method is suggested by H Neuberger and then developed and tested by the author on the system studied herein.
7. Conclusion

In conclusion, we have studied resonances in a one-dimensional discrete tight-binding open chain in a weak disorder limit. In this study, we have calculated complex energies in an open-clean chain of finite length. The result we obtain is a polynomial equation which we have been able to solve for long chains using an ansatz for the solution. To the best of our knowledge, this result has never been derived before. We have used a perturbation theory up to second order where we have derived the first and second-order corrections to the complex energies in terms of Chebyshov polynomials. The first-order corrections have been useful to obtain a closed form of the statistical results for the scattered complex energies. These results have been further simplified for long chains where we obtain compact results. The asymptotic results have been verified against numerics. Evidently, in the weak disorder limit the perturbation theory predicts good statistical results. Our results are new in these studies and they could be useful in further studies of such systems.

It would be interesting to study statistics of resonances in the weak disorder limit for higher dimensional models as well as for the cases when the site energies are not independent random variables but they are correlated with each other [19]. Besides, there has been growing interest for the case when $M$ sites are connected to the outer world where $1 \leq M \leq N$ [20].

We believe that our methods could be useful for the study of such models. Finally, we mention the case where $\xi(0) \sim O(N)$. It requires a separate investigation as our perturbative analysis fails in this limit.

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Appendix A. Large-N behavior of the denominator in (40)

The denominator in equation (40) can be simplified as follows:

\[
D(z_o) = -N + U_{N-1}(z_o/2)T_{N+1}(z_o/2) - i\eta \exp[i(z_o)]\sin[k(z_o)][U_{N-1}(z_o/2)]^2 \\
\approx -N + [-\exp(-ik_o)\Omega(1 - \eta) + 1] \\
+ \exp(ik_o)\Omega^{-1}(1 + \eta \exp(2ik_o)) + 1 - 2\eta)\] 

\[
\text{(A.1)}
\]

Using now

\[
\Omega(1 - \eta) + 1 = 2 - \eta \exp(2ik_o),
\]

and

\[
\Omega^{-1}(1 + \eta \exp(2ik_o)) + 1 = \frac{2 - 3\eta + \eta^2 \exp(2ik_o)}{1 - \eta \exp(2ik_o)},
\]

in (A.1), we obtain

\[
D \approx -N - \frac{1}{1 - \eta \exp(2ik_o)}. 
\]

(A.3)

For \(|D|^2\) this yields

\[
|D|^2 = N^2 + N \frac{1 - \eta \cos(2k_o)}{1 + \eta^2 - 2\eta \cos(2k_o)} + \frac{1}{1 + \eta^2 - 2\eta \cos(2k_o)}. 
\]

(A.4)

Similarly for the parametric resonances, the denominator for large \(N\) is given by

\[
D = -N + U_{N-1}(z_o/2)T_{N+1}(z_o/2) \\
\approx -(N + 1/2) + \frac{\exp(-ik_o)\Omega - \exp(ik_o)\Omega^{-1}}{\exp(ik_o) - \exp(-ik_o)} \\
\approx -(N + 1/2) + \frac{1 + \eta^2}{2[\eta \exp(ik_o) + i]^2}. 
\]

(A.5)

Thus, for \(|D|^2\), we obtain

\[
|D|^2 = N^2 + N \frac{(1 - \eta^4)}{(1 + \eta^2)^2 - 4\eta^2 \sin^2(k_o)} \\
+ \frac{1 + \eta^2}{4(1 + \eta^2)^2 - 4\eta^2 \sin^2(k_o)} \\
\frac{1 + \eta^2}{1 + \eta^2 - 2\eta \sin(2k_o)}. 
\]

(A.6)

Clearly in both cases \(|D|^2\) has a form \(N^2 + b1[N^1] + b2[N^0]\).

Appendix B. Calculation of DOR

We define the average DOR as

\[
\langle \rho(x, y) \rangle = \left\{ \sum_{n=1}^{N} \delta(x - \tilde{x}_n) \delta(y - \tilde{y}_n) \right\}, 
\]

(B.1)

where \(\tilde{x}_n \equiv \Re{\tilde{z}_n}\) and \(\tilde{y}_n \equiv \Im{\tilde{z}_n}\). Next we define \(\tilde{x}_n = \Re{(\tilde{z}_n)}\), \(\tilde{y}_n = \Im{(\tilde{z}_n)}\), \((\delta_1\tilde{x}_n) = \Re{((\delta_1\tilde{z}_n))}\), \((\delta_1\tilde{y}_n) = \Im{((\delta_1\tilde{z}_n))}\), \((\delta_2\tilde{x}_n) = \Re{((\delta_2\tilde{z}_n))}\) and \((\delta_2\tilde{y}_n) = \Im{((\delta_2\tilde{z}_n))}\) and then use the expansion \(\tilde{x}_n = \tilde{x}_n + (\delta_1\tilde{x}_n) + (\delta_2\tilde{x}_n)\) and \(\tilde{y}_n = \tilde{y}_n + (\delta_1\tilde{y}_n) + (\delta_2\tilde{y}_n)\) in (B.1).
We find
\[
\langle \rho(x, y) \rangle = \sum_{\alpha=1}^{N} \delta(x - \bar{x}_\alpha) \delta(y - \bar{y}_\alpha)
\]
\[+ \sum_{\alpha=1}^{N} \left[ \langle \delta_1 \bar{x}_\alpha \delta(y - \bar{y}_\alpha) \rangle \delta'((x - \bar{x}_\alpha) \delta(y - \bar{y}_\alpha) - \langle \delta_2 \bar{x}_\alpha \rangle \delta'(y - \bar{y}_\alpha) \right.
\]
\[= \langle \delta_2 \bar{x}_\alpha \rangle \delta'(y - \bar{y}_\alpha) + \frac{1}{2} \langle \delta_1 \bar{x}_\alpha \delta\rangle \delta''(y - \bar{y}_\alpha) \]
\[+ \frac{1}{2} \langle \delta_1 \bar{x}_\alpha \delta \rangle \delta''(y - \bar{y}_\alpha) \].
\]

Here, \( \delta'(x) = d \delta(x)/dx \) and similarly \( \delta''(x) \) is the second derivative of the Dirac Delta function with respect to the argument. We have already shown that \( \langle \delta_1 \bar{x}_\alpha \rangle \) and \( \langle \delta_2 \bar{x}_\alpha \rangle \) are of \( O(\sigma^2 N^{-1}) \) while \( \langle \delta_1 \bar{x}_\alpha \delta(y - \bar{y}_\alpha) \rangle \) is also \( O(\sigma^2 N^{-1}) \) since
\[
\langle \delta_1 \bar{x}_\alpha \delta \rangle = \langle \delta_1 \bar{x}_\alpha \rangle^2.
\]

Motivated from [17], we calculate the coefficient of \( e_n^2 \) in (37) to obtain \( \langle \delta_2 \bar{x}_\alpha \rangle \) and \( \langle \delta_2 \bar{y}_\alpha \rangle \).

Let us write
\[
\delta_1 \bar{Z}_\alpha = \sum_{n=1}^{N} \epsilon_n c_{n,a},
\]
and
\[
\delta_2 \bar{Z}_\alpha = \sum_{n,m=1}^{N} \epsilon_n \epsilon_m d_{nm,a}.
\]

Expressing the polynomials in (40) in terms of sinusoidal functions, we simply read-off \( c_{n,a} \):
\[
c_{n,a} = \frac{-2 \sin^2(n \theta_a)}{D_a} = \frac{\cos(2n \theta_a) - 1}{D_a}.
\]

Here, \( n' = N + 1 - n \) and \( D_a \) is the denominator inside the brackets of (40). The denominator is \( O(N) \) thus \( c_{n,a} \sim O(N^{-2}) \), hence will be dropped off. Using now our ansatz, for \( d_{nm,a} \) we obtain
\[
d_{nm,a} = \frac{(1 - \cos(2r \Psi_a))}{4 N \sin(\Psi_a/N) \sin(\Psi_a)} \{4 \cos(2r \Psi_a) - \cos[(1 + 2r) \Psi_a] - 3 \cos[(1 - 2r) \Psi_a]\}.
\]

where \( r = n'/N \) and \( \Psi_a = N k_a \) with \( k_a \) given in (24). There is no further simplification of this result to obtain a compact and simple expression for real and imaginary parts of \( d_{nm,a} \) (as the authors of [17] have been able to do for with the result they obtain for the Hatano-Nelson model [18]). So we leave the density formally as
\[
\langle \rho(x, y) \rangle = \rho_0(x, y) - \sigma^2 \sum_{\alpha=1}^{N} \delta'(x - x_\alpha) \delta(y - y_\alpha)
\]
\[\times \sum_{n=1}^{N} \Re\{d_{nm,a}\} + \delta(x - x_\alpha) \delta'(y - y_\alpha) \sum_{n=1}^{N} \Im\{d_{nm,a}\},
\]

where \( \rho_0(x, y) \) is the first term of equation (B.2).


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