The contact mappings of a flat (2,3,5)-distribution

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Abstract
Let Ω and Ω′ be open subsets of a flat (2,3,5)-distribution. We show that a $C^1$-smooth contact mapping $f: Ω \rightarrow Ω′$ is a $C^\infty$-smooth contact mapping. Ultimately, this is a consequence of the rigidity of the associated stratified Lie group. (The Tanaka prolongation of the Lie algebra is of finite type.) The conclusion is reached through a careful study of some differential identities satisfied by components of the Pansu derivative of a $C^1$-smooth contact mapping.

Keywords (2, 3, 5)-distribution · Stratified Lie algebra · Contact mapping · Tanaka prolongation

Mathematics Subject Classification 58A30 · 22E25

1 Introduction

Let $M$ be a connected $C^\infty$-smooth manifold of dimension $n$. A distribution on $M$ is a subbundle $H_M$ of the tangent bundle $T_M$ determined by a collection of $C^\infty$-smooth vector fields $X_1, \ldots, X_d$ via $H_pM = \text{span}(X_1|_p, \ldots, X_d|_p)$. Since $M$ is connected and $H_M$ is a vector bundle, $\dim(H_pM)$ is independent of $p$ and called the rank of the distribution.

Suppose $M$ is a manifold equipped with a distribution $H_M$ as above. If $\Omega \subset M$ is open, then $H_\Omega$ denotes the induced distribution. We write $\mathfrak{V}^k(\Omega)$ for the $C^k$-smooth sections of $T\Omega$ and $\mathfrak{V}^k_H(\Omega)$ for the $C^k$-smooth sections of $H\Omega$. Set $\mathfrak{V}_1 = \mathfrak{V}^\infty_H(M)$ and define by iteration $\mathfrak{V}_k = \mathfrak{V}_k-1 \oplus [\mathfrak{V}_1, \mathfrak{V}_k-1]$. Let $\Gamma^k$ be determined by $\Gamma^k_p = \text{span}\{V|_p : V \in \mathfrak{V}_k\}$. The distribution $H_M$ is called regular if $\Gamma^k$ is a distribution for all $k$.

When $H_M$ is regular, there exists a unique positive integer $s$ such that

$$H_M = \Gamma^1 \subset \Gamma^2 \subset \cdots \subset \Gamma^s$$

and $\Gamma^k = \Gamma^s$ for all $k > s$. If we set $I_1(p) = \Gamma^1_p$ and for $k \geq 2$ define $I_k(p) = \Gamma^k_p/\Gamma^{k-1}_p$, then $I(p) = \bigoplus_{k \geq 1} I_k(p)$ can be given a bracket as in [21, p. 9] making it a stratified Lie algebra of step $s$. See Sect. 2 for the definition of a stratified Lie algebra (of step $s$) and the early sections of [21, 24] for more details on the objects of this paragraph. If there exists
a stratified Lie algebra $I$ such that $I(p)$ is isomorphic to $I$ at every $p \in M$, then the pair $(M, H, M)$ is called of type $I$. Somewhat conversely, if $G$ is a Lie group such that its Lie algebra $I = \bigoplus_{k \geq 1} I_k$ is stratified of step $s$, then $I_1$ determines a regular distribution $H\Omega$ on $G$ and $(\Omega, H\Omega)$ is of type $I$.

Let $M, M'$ be $C^\infty$-smooth manifolds, and let $\Omega \subset M$ be open. We call $f : \Omega \to M'$ a $C^k$-diffeomorphism if $f : \Omega \to f(\Omega)$ is a homeomorphism and both $f$ and $f^{-1}$ are $C^k$-smooth. When $M, M'$ are connected and $H_M, H_{M'}$ are distributions on $M, M'$, respectively, we call $f : \Omega \to M'$ a $(C^k$-smooth) contact mapping if it is a $C^k$-diffeomorphism for some $k \geq 1$ and $Df(H\Omega) = Hf(\Omega)$. This implies $f_*(\nu^\infty_H(\Omega)) \subset \nu^\infty_H(f(\Omega))$. If $f : \Omega \to M'$ is a contact mapping, then so too is $f^{-1} : f(\Omega) \to M$. The pairs $(M, H, M)$ and $(M', H, M')$ are called locally equivalent if for all $p \in M$ and all $q \in M'$ there exist open $\Omega \subset M$ with $p \in \Omega$, open $\Omega' \subset M'$ with $q \in \Omega'$, and a $C^\infty$-smooth contact mapping $f : \Omega \to \Omega'$.

A pair $(M, H, M)$ is called a $(2, 3, 5)$-distribution if $M$ is a connected $C^\infty$-smooth manifold with $\dim(M) = 5$ and $H$ is a regular distribution on $M$ satisfying $\text{rank}(HM) = \text{rank}(\Gamma^1) = 2, \text{rank}(\Gamma^2) = 3, \text{and } \text{rank}(\Gamma^3) = 5$. These were studied by Cartan in his famous “five variables” paper [6]. That work claimed a classification long accepted as complete until a missing case was discovered by Doubrov and Gogorov in 2013 [10]. See also the commentary in [23]. Cartan discussed $(2, 3, 5)$-distributions in the context of space curves of constant torsion. They have since been linked to the symmetries of rolling balls [3] or, if you prefer, a rolling spinor [2].

A $(2, 3, 5)$-distribution is called flat if a certain invariant (quantity preserved under local equivalence) vanishes. Every flat $(2, 3, 5)$-distribution is locally equivalent to the Lie group we study in Sect. 3 commonly called the Cartan group (and we follow suit). The Cartan group, denoted $G$, is an example of a rigid stratified Lie group. The definition of stratified and rigid in this context can be found in Sect. 2. The main purpose of this paper is to prove the following.

**Theorem 1.1** Let $\Omega \subset G$ be open. If $f : \Omega \to G$ is a $C^1$-smooth contact mapping, then $f$ is $C^\infty$-smooth.

This is achieved in Theorem 3.6 of Sect. 3. Our paper can be thought of as a detailed case study of a natural extension to methods making use of the Tanaka prolongation of a stratified Lie algebra. We hope it may provide insight into the way the structure of a rigid group controls the geometry and function theory of that group. In [17] it was shown that if $G$ is any rigid stratified Lie group, $\Omega \subset G$ is open, and $f : \Omega \to G$ is a $C^2$-smooth contact mapping, then $f$ is $C^\infty$-smooth. We are improving on this result in a special case.

The result of [17] is achieved as follows. If $V \in \mathfrak{v}^1(\Omega)$ is contact (for the definition of contact in the context of vector fields, see Sect. 2), then it may be regularized so that the smooth approximations are also contact. A $C^\infty$-smooth contact vector field on a domain of a rigid stratified Lie group is known to have polynomial coefficients, with the maximal degree of these polynomials dependent only on the group structure. Indeed, the $C^\infty$-smooth contact vector fields on a domain of a rigid group form a finite-dimensional Lie algebra. Consequently, the approximating vector fields limit on the original in a finite-dimensional subspace. Thus, the original $C^1$-smooth contact vector field has polynomial coefficients and in fact $V \in \mathfrak{v}^\infty(\Omega)$. This is utilized by taking the vector field generator $V$ of a family of $C^\infty$-smooth contact mappings, left-translations say, and considering $f_*V$ with $f$ a $C^2$-smooth contact mapping on $\Omega$. Since

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we have by definition that \( f_* V \) is a \( C^1 \)-smooth contact vector field on \( f(\Omega) \). The previous argument implies \( f_* V \) is \( C^\infty \)-smooth, and this can be used to deduce that \( f \) itself is smooth. When \( f \) is assumed only \( C^1 \)-smooth, (1) is not justified. The computations of Sect. 3 are a means of circumnavigating this obstacle.

Let us take a closer look at the guiding principle behind those computations. Suppose \( \mathfrak{g} \) is a stratified Lie group with \( I = \text{Lie}(\mathfrak{g}) \). If \( X \in I \), then \( p \mapsto \exp(tX)p \) is left-translation by \( \exp(tX) \), one of a family of \( C^\infty \)-smooth contact mappings indexed by parameter \( t \). Let \( V|_p = \frac{d}{dt}|_{t=0} \exp(tX)p \). With \( f \) a contact mapping on a domain \( \Omega \subset \mathfrak{g} \), we introduce

\[ h_t(q) = f(\exp(tX)f^{-1}(q)). \]

This is a family of contact mappings on \( f(\Omega) \) satisfying \( h_0(q) = q \). When everything is assumed smooth, we have \( h_0(q) = (f_* V)|_q = (Xf)|_{f^{-1}(q)} \). Here, \( X \) is the right-invariant mirror of \( X \). Of course, if \( X \) is in the center of \( I \), then \( X = X \). We are able to rely on such vector fields in Sect. 3.

Suppose \( \mathfrak{g} \) is a rigid group and \( \mathfrak{b}_C(\Omega) \) is the finite-dimensional Lie algebra of \( C^\infty \)-smooth contact vector fields on \( \Omega \) (isomorphic to a Lie algebra independent of \( \Omega \) we will denote \( \mathfrak{b}_C \)). Once a contact mapping \( f \) on \( \Omega \) has been shown to be \( C^\infty \)-smooth, it is found to be induced by an element of the automorphism group of \( \mathfrak{b}_C \). See the paragraphs at the end of Sect. 3 in [17]. In the case of the Cartan group \( \mathfrak{b}_C \) is isomorphic to \( \mathfrak{g}_2 \), the real “split form” of the 14-dimensional exceptional complex simple Lie algebra \( \mathfrak{g}_2^C \). For a painstaking calculation of a basis of vector fields for \( \mathfrak{b}_C(\mathfrak{g}) \) see [18].

A \( C^1 \)-smooth assumption could be said to be natural when defining a contact mapping; however, it leaves something to be desired. Better would be a horizontal Sobolev-type condition. That is, if \( \bigoplus_{I \geq 1} I_1 = \text{Lie}(\mathfrak{g}) \) and \( X_1, \ldots, X_n \) is a basis for \( I_1 \), then we would like to assume only that \( f : \Omega \to f(\Omega) \) is a homeomorphism and that the components of the distributional derivatives \( Xf \) are in \( L^r_{\text{loc}}(\Omega) \) for some \( r \geq 1 \). This setup can then be used to define contact in a weak sense. A sensible value for \( r \) might be the homogeneous dimension of the group, but you can always attempt to work with something smaller. With some algebraic dexterity, our approach might yield a \( C^1 \)-smooth result for all rigid groups\(^1\). Whether a variation on these methods has application in the Sobolev setting remains to be seen, but there is reason to believe it would require some delicate analysis. This is in large part due to the appearance of right-invariant derivatives as mentioned above.

A horizontal Sobolev-type condition is the right assumption in the study of (locally) quasiconformal mappings. It results in equivalence of the analytic definition with the metric definition. For the sake of brevity, we do not pursue this topic in detail here. We should, however, make note of recent work on Xie’s conjecture. This conjecture states that if \( \mathfrak{g} \) is a stratified Lie group equipped with its canonical Carnot–Carthéodory distance function, other than \( \mathbb{R}^n \) or \( \mathbb{H}^n \) (the \( n \)-th Heisenberg group), then every quasiconformal mapping \( f : \Omega \to \mathfrak{g} \) is locally bi-Lipschitz. Moreover, if \( \Omega = \mathfrak{g} \), then \( f \) is bi-Lipschitz. The recent paper [14] proves this in the case of non-rigid Carnot groups. In the case of rigid groups, the conjecture would be verified if it were true that all quasiconformal mappings were \( C^\infty \)-smooth (once coupled with the results of [9]). All geometric mappings such as

\(^1\) After this paper was submitted, [15] was posted to the arXiv which deals with the \( C^1 \) case for all rigid groups. The author, Jona Lelmi, uses a striking new characterization of weakly contact vector fields.
bi-Lipschitz and quasiconformal mappings must be weakly contact. On the other hand, a $C^1$-smooth contact mapping is necessarily locally quasiconformal. In [14] the authors mention that rigid groups will be discussed in a forthcoming article.

If $\mathcal{G}$ is an $H$-type group with center of dimension at least 3 (these groups are rigid), then a quasiconformal mapping $f : \Omega \to \mathcal{G}$ is $C^\infty$-smooth by the regularity results of [4] and [5]. These rely on nonlinear potential theory. We emphasize that our results are achieved using linear operators only. Our program of using vector flow methods in conjunction with linear hypoelliptic operators was begun in collaboration with Jeremy Tyson in [1] and was to some extent influenced by [16] and [19]. These papers are all related to conformal (i.e., 1-quasiconformal) mappings and Liouville-type theorems for them.

2 Contact vector fields on a stratified Lie group

A Lie algebra $\mathfrak{I}$ is called stratified (of step $s$) if $\dim(\mathfrak{I}) < \infty$ and there are vector spaces $\mathfrak{I}_i$ such that

$$\mathfrak{I} = \bigoplus_{i \geq 1} \mathfrak{I}_i, \ [\mathfrak{I}_1, \mathfrak{I}_j] = \mathfrak{I}_{i+1} \text{ for all } i \geq 1, \ I_j \neq \{0\}, \text{ and } I_i = \{0\} \text{ for all } i > s.$$ A Lie group $\mathcal{G}$ is called stratified if it is connected, simply connected, and $\mathcal{G} = \text{Lie}(\mathcal{G})$ is stratified. Nothing essential is lost if a stratified Lie group is regarded as $\mathbb{R}^n$ with a polynomial group law. Indeed, if $\mathcal{G}$ is a stratified Lie group and $\Omega \subset \mathcal{G}$ is open, then we (tacitly) identify $\Omega$ with an open subset of $\mathbb{R}^n$ whenever it is convenient to do so.

If $\mathcal{G}$ is a stratified Lie group and $X_1, \ldots, X_d$ is a basis for $\mathfrak{I}_1$, then $\mathcal{H}\mathcal{G}$ indicates the distribution determined by $\mathcal{H}_p\mathcal{G} = \text{span}(X_1|_p, \ldots, X_d|_p)$. Elements of $\mathfrak{b}^k_\infty(\Omega)$ will be referred to as horizontal vector fields. When $k \geq 1$, a vector field $V \in \mathfrak{b}^k(\Omega)$ is called contact if $[V, \mathfrak{b}^\infty_\infty(\Omega)] \subset \mathfrak{b}^\infty_\infty(\Omega)$. See the opening paragraphs of the introduction for the notation used here.

A simple prolongation of a stratified Lie algebra $\mathfrak{I} = \bigoplus_{i \geq 1} \mathfrak{I}_i$ is a graded Lie algebra $\mathfrak{h} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{h}_i$ satisfying

(i) for all $i \leq -1$, $\mathfrak{h}_i = \mathfrak{I}_{-i}$ and
(ii) for all $i \geq 0$, if $Z \in \mathfrak{h}_i$ is such that $[Z, \mathfrak{h}_{-1}] = \{0\}$, then $Z = 0$.

The Tanaka prolongation of $\mathfrak{I}$ is the simple prolongation $\mathfrak{t}(\mathfrak{I}) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{t}_i(\mathfrak{I})$ such that whenever $\mathfrak{h} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{h}_i$ is another simple prolongation, there is an injective Lie algebra homomorphism $\xi : \mathfrak{h} \to \mathfrak{t}(\mathfrak{I})$ with $\xi(\mathfrak{h}_i) \subset \mathfrak{t}_i(\mathfrak{I})$ for all $i$. See [21, pp. 23-25] for the construction of $\mathfrak{t}(\mathfrak{I})$. A stratified Lie group $\mathcal{G}$ with Lie algebra $\mathfrak{I}$ is called rigid if there exists $N \geq 0$ such that $\mathfrak{t}_i(\mathfrak{I}) = \{0\}$ for all $i \geq N$. The following can be found near the end of Sect. 3 in [17].

**Proposition 2.1** Let $\mathcal{G}$ be a stratified Lie group of dimension $n$, and let $\Omega \subset \mathcal{G}$ be open. If $\mathcal{G}$ is rigid, then there is $N \geq 0$ with the following property: if $Y_1, \ldots, Y_n$ is a basis for $\mathfrak{I} = \text{Lie}(\mathcal{G})$ and $V = \sum_{i=1}^n v_i Y_i \in \mathfrak{b}^1(\Omega)$ is a contact vector field, then the component functions $v_i$ are polynomials of degree no greater than $N$.

Should $\mathfrak{b}^1(\Omega)$ be replaced by $\mathfrak{b}^\infty(\Omega)$, then the statement follows almost immediately from the constructions of Section 2.6 in [17]. Those constructions took inspiration from...
Sect. 2 of [24], which is largely an exposition of Section 6 in [21]. The proof of the proposition uses a smoothing argument which had appeared before in [8, p. 83].

When \( \mathfrak{G} \) is a stratified Lie group and \( \Omega \subset \mathfrak{G} \) is open, we write \( \mathcal{D}'(\Omega) \) for the real-valued distributions on \( \Omega \). Since there is an overlap of vocabulary, we emphasize that distributions is being used here in the sense of generalized functions. They are continuous linear functionals on \( (C^\infty_0(\Omega), \tau) \) with \( \tau \) the usual topology. Convergence in \( \mathcal{D}'(\Omega) \) is with reference to the weak-* topology. With \( \mathfrak{I} = \text{Lie}(\mathfrak{G}) \), define \( \mathfrak{v}^{<1}(\Omega) = \mathcal{D}'(\Omega) \otimes \mathfrak{I} \). We think of \( \mathfrak{v}^{<1}(\Omega) \) as the space of finite formal sums

\[
\left\{ \sum a_i Z_i \middle| \alpha_i \in \mathcal{D}'(\Omega), Z_i \in \mathfrak{I} \right\}
\]

(which it is, modulo the null sums). We sometimes call an element of \( \mathfrak{v}^{<1}(\Omega) \) a generalized vector field (on \( \Omega \)).

Suppose \( \mathfrak{G} \) is a stratified Lie group with \( \mathfrak{I} = \text{Lie}(\mathfrak{G}) \) of step \( s \). Let \( \{ Y_{i,j} \} \) be a basis for \( \mathfrak{I} \) such that \( Y_{i,1}, \ldots, Y_{i,d_i} \) is a basis for \( \mathfrak{I}_i \). For all \( i = 2, \ldots, s, j = 1, \ldots, d_i \), and \( k = 1, \ldots, d_i \) there is a linear combination \( Q_{i,j,k} \) acting on \( d_{i-1} \) objects such that for all open \( \Omega \subset \mathfrak{G} \) and all \( \alpha_1, \ldots, \alpha_{d_{i-1}} \in \mathcal{D}'(\Omega) \) we have

\[
\sum_{j=1}^{d_{i-1}} a_j [Y_{i-1,j}, X_k] = \sum_{j=1}^{d_i} Q_{i,j,k}(\alpha_1, \ldots, \alpha_{d_{i-1}}) Y_{i,j}
\]

in \( \mathfrak{v}^{<1}(\Omega) \) for all \( i = 2, \ldots, s \) and \( k = 1, \ldots, d_i \). A generalized vector field \( Y = \sum_{i=1}^s \sum_{j=1}^{d_i} a_{i,j} Y_{i,j} \) is called contact if for all \( i = 2, \ldots, s, j = 1, \ldots, d_i \), and \( k = 1, \ldots, d_i \) we have

\[
X_k a_{i,j} = Q_{i,j,k}(a_{i-1,1}, \ldots, a_{i-1,d_{i-1}}).
\]

If \( \mathfrak{G} \) is a stratified Lie group, \( \Omega \subset \mathfrak{G} \) is open, and \( \Omega_0 \subset \Omega \) is also open, then \( \alpha \in \mathcal{D}'(\Omega) \) determines \( a_0 \in \mathcal{D}'(\Omega_0) \) via \( \langle a_0, \phi \rangle = \langle \alpha, \phi \rangle \) for all \( \phi \in C^\infty_0(\Omega_0) \subset C^\infty_0(\Omega) \). If \( \Omega_0 \subset \Omega \) and \( \Omega_0 \) is compact (from now on denoted \( \Omega_0 \subset \subset \Omega \), then there exists \( \epsilon_0 > 0 \) with the following property: whenever \( \alpha \in \mathcal{D}'(\Omega) \), there is a family \( \{ \alpha^\epsilon \} | 0 < \epsilon < \epsilon_0 \} \) of \( C^\infty \)-smooth functions defined on \( \Omega_0 \) such that \( \alpha^\epsilon \to a_0 \) in \( \mathcal{D}'(\Omega_0) \) as \( \epsilon \to 0 \). These smooth approximations to \( a_0 \) are achieved by convolving \( \alpha \) with suitable elements of \( C^\infty_0(\Omega) \). For an exposition of this theory see [13, pp. 88-90]. The manner of the regularization implies that \( Z \alpha^\epsilon = (Z \alpha)^\epsilon \) for all \( Z \in \mathfrak{I} \). Moreover, if \( \alpha, \beta \in \mathcal{D}'(\Omega) \), then \( (s\alpha + t\beta)^\epsilon = s\alpha^\epsilon + t\beta^\epsilon \) in \( \mathcal{D}'(\Omega_0) \) for all \( s, t \in \mathbb{R} \).

Suppose \( \alpha^\epsilon \to a_0 \) in \( \mathcal{D}'(\Omega_0) \) as in the previous paragraph and that each \( \alpha^\epsilon \) is a polynomial of degree at most \( N \) (with \( N \) independent of \( \epsilon \)). The distributions that may be identified with polynomials of degree at most \( N \) form a finite-dimensional subspace of \( \mathcal{D}'(\Omega_0) \). Since \( \mathcal{D}'(\Omega_0) \) is a (locally convex and) Hausdorff topological vector space, this subspace is closed. It follows that \( a_0 \) may also be identified with a polynomial of degree at most \( N \).

Every open set \( \Omega \subset \mathfrak{G} \) admits a compact exhaustion. That is, there is a sequence of open sets \( \{ \Omega_k \}_{k=1}^\infty \) such that \( \Omega_k \subset \subset \Omega, \Omega_k \subset \subset \Omega_{k+1} \), and \( \Omega = \bigcup \Omega_k \). If \( a_k \) is the restriction of \( \alpha \) to \( C^\infty_0(\Omega_k) \) (as discussed in the case of \( k = 0 \)), and each \( a_k \) is found to be a polynomial, then the polynomials are the same and \( \alpha \) may be identified with a polynomial on \( \Omega \). If \( \alpha \) is a continuous function to begin with, and \( \alpha \) as distribution may be identified with a polynomial, then \( \alpha \) is that polynomial.
Proposition 2.2 Let $\mathcal{Q}$ be a stratified Lie group with Lie algebra $\mathfrak{l} = \bigoplus_{i \geq 1} \mathfrak{l}_i$. Let $d_j = \dim(\mathfrak{l}_j)$ and let $Y_{i,j}$ with $j = 1, \ldots, d_j$ be a basis for $\mathfrak{l}_i$. Let $\Omega \subset \mathcal{Q}$ be open and suppose $Y = \sum_j \alpha_j Y_{i,j}$ is a contact generalized vector field on $\Omega$. If $\mathcal{Q}$ is rigid, then each $\alpha_{i,j}$ may be identified with a polynomial on $\Omega$.

Proof Let $\Omega_0$ be open such that $\Omega_0 \subset \subset \Omega$. Let $\alpha_{i,j}^\epsilon$ be a sequence of $C^\infty$-smooth functions defined on $\Omega_0$ as discussed above. It follows $Y^\epsilon = \sum_j \alpha_{i,j}^\epsilon Y_{i,j}$ is a $C^\infty$-smooth vector field on $\Omega_0$. Let $X_j = Y_{1,j}$ and suppose $V \in b_1^\infty(\Omega_0)$. It follows $V = \sum_j \sigma_j X_j$ with each $\sigma_j \in C^\infty(\Omega_0)$. Simple rearrangements and a replacement justified by the definition of $Q_{i,j,k}$ alone yield

$$[Y^\epsilon, V] = \sum_{k=1}^{d_1} \sum_{i=1}^{s} \sum_{j=1}^{d_j} [\alpha_{i,j}^\epsilon Y_{i,j}, \sigma_k X_k]$$

$$= \sum_{k=1}^{d_1} \sum_{i=1}^{s} \sum_{j=1}^{d_j} \left( \alpha_{i,j}^\epsilon \sigma_k [Y_{i,j}, X_k] + \alpha_{i,j}^\epsilon (Y_{i,j} \sigma_k) X_k - \sigma_k (X_k \alpha_{i,j}^\epsilon) Y_{i,j} \right)$$

$$= \sum_{k=1}^{d_1} \sum_{i=1}^{s} \sum_{j=1}^{d_j} \alpha_{i,j}^\epsilon (Y_{i,j} \sigma_k) X_k - \sum_{k=1}^{d_1} \sum_{j=1}^{d_j} \sigma_k (X_k \alpha_{i,j}^\epsilon) X_j$$

$$+ \sum_{k=1}^{d_1} \sigma_k \left( \sum_{i=2}^{s+1} \sum_{j=1}^{d_j} \alpha_{i-1,j}^\epsilon [Y_{i-1,j}, X_k] - \sum_{i=2}^{s} \sum_{j=1}^{d_j} (X_k \alpha_{i,j}^\epsilon) Y_{i,j} \right)$$

$$= \sum_{k=1}^{d_1} \sum_{i=1}^{s} \sum_{j=1}^{d_j} \alpha_{i,j}^\epsilon (Y_{i,j} \sigma_k) X_k - \sum_{k=1}^{d_1} \sum_{j=1}^{d_j} \sigma_k (X_k \alpha_{i,j}^\epsilon) X_j$$

$$+ \sum_{k=1}^{d_1} \sigma_k \sum_{i=2}^{s} \sum_{j=1}^{d_j} \left( Q_{i,j,k}(\alpha_{i-1,1}^\epsilon, \ldots, \alpha_{i-1,d_j-1}^\epsilon) - X_k \alpha_{i,j}^\epsilon \right) Y_{i,j}.$$}

Since

$$X_k \alpha_{i,j}^\epsilon = (X_k \alpha_{i,j})^\epsilon = Q_{i,j,k}(\alpha_{i-1,1}^\epsilon, \ldots, \alpha_{i-1,d_j-1}^\epsilon) = Q_{i,j,k}(\alpha_{i-1,1}^\epsilon, \ldots, \alpha_{i-1,d_j-1}^\epsilon)$$

we have that

$$[Y^\epsilon, V] = \sum_{k=1}^{d_1} \sum_{i=1}^{s} \sum_{j=1}^{d_j} \alpha_{i,j}^\epsilon (Y_{i,j} \sigma_k) X_k - \sum_{k=1}^{d_1} \sum_{j=1}^{d_j} \sigma_k (X_k \alpha_{i,j}^\epsilon) X_j.$$}

This is clearly a horizontal vector field on $\Omega_0$; hence, $Y^\epsilon$ is contact in the classical sense. Consequently, there is $N$ such that $\alpha_{i,j}^\epsilon$ is a polynomial of degree at most $N$ for all $i, j$, and suitable $\epsilon$. Our comments above imply that each $\alpha_{i,j}$ may be identified with a polynomial on $\Omega$. \qed

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3 Contact mappings on the Cartan group

The set \( \mathbb{R}^5 \) with group product

\[
(x_1, x_2, y, z_1, z_2)(x'_1, x'_2, y', z'_1, z'_2) = (P_1, P_2, P_3, P_4, P_5),
\]
\[
P_1 = x_1 + x'_1,
\]
\[
P_2 = x_2 + x'_2,
\]
\[
P_3 = y + y' + \frac{1}{5}(x_1x'_2 - x_2x'_1),
\]
\[
P_4 = z_1 + z'_1 + \frac{1}{2}(x_1y' - yx'_1) + \frac{1}{12}((x_1 - x'_1)(x_1x'_2 - x_2x'_1)), \text{ and}
\]
\[
P_5 = z_2 + z'_2 + \frac{1}{2}(x_2y' - yx'_2) + \frac{1}{12}((x_2 - x'_2)(x_1x'_2 - x_2x'_1)),
\]

is a connected, simply connected Lie group we denote \( \mathfrak{C} \). It is a realization of what is sometimes called the Cartan group. We choose the following basis for \( \mathfrak{C} = \text{Lie}(\mathfrak{C}) \):

\[
X_1 = \partial_{x_1} - \frac{1}{2}x_2\partial_y - \frac{1}{2}(y + \frac{1}{6}x_1x_2)\partial_{z_1} - \frac{1}{12}x_2^2\partial_{z_2},
\]
\[
X_2 = \partial_{x_2} + \frac{1}{2}x_1\partial_y + \frac{1}{12}x_1^2\partial_{z_1} - \frac{1}{2}(y - \frac{1}{6}x_1x_2)\partial_{z_2},
\]
\[
Y = \partial_y + \frac{1}{2}x_1\partial_{z_1} + \frac{1}{2}x_2\partial_{z_2},
\]
\[
Z_1 = \partial_{z_1}, \text{ and}
\]
\[
Z_2 = \partial_{z_2}.
\]

On occasion it will be convenient to have the alternative labels

\[
Y_1 = X_1, \quad Y_2 = X_2, \quad Y_3 = Y, \quad Y_4 = Z_1, \quad \text{and} \quad Y_5 = Z_2.
\]

The explicit expressions for these vector fields are achieved by \( Y_j|_{\mathfrak{p}} = D_0L_p(\partial_j) \), with \( \partial_1, \ldots, \partial_5 \) the canonical Euclidean basis at the origin of \( \mathbb{R}^5 \).

It is easily found that

\[
[X_1, X_2] = Y, \quad [X_1, Y] = Z_1, \quad \text{and} \quad [X_2, Y] = Z_2. \tag{2}
\]

All brackets not immediate consequences of these are trivial. Consequently, \( \mathfrak{c} \) is a stratified Lie algebra of step 3 with

\[
\mathfrak{c}_1 = \text{span}(X_1, X_2), \quad \mathfrak{c}_2 = \text{span}(Y), \quad \text{and} \quad \mathfrak{c}_3 = \text{span}(Z_1, Z_2).
\]

Furthermore, \( \mathfrak{c}_1 \) determines a \((2, 3, 5)\)-distribution.

**Remark 3.1** The group product is that arising from the following procedure (in brief). Begin with an abstract Lie algebra \( \mathfrak{c} \) with basis \( \tilde{X}_1, \ldots, \tilde{Z}_2 \) satisfying the relations (2) above (\( \tilde{X}_1 \) replaces \( X_1 \) etc.). There is a connected, simply connected abstract Lie group \( \tilde{\mathfrak{C}} \) with \( \text{Lie}(\tilde{\mathfrak{C}}) = \mathfrak{c} \) and such that \( \exp : \mathfrak{c} \to \tilde{\mathfrak{C}} \) is a \( C^\infty \)-smooth diffeomorphism. Now identify \( (x_1, x_2, y, z_1, z_2) \in \mathbb{R}^5 \) with \( \exp(x_1\tilde{X}_1 + x_2\tilde{X}_2 + y\tilde{Y} + z_1\tilde{Z}_1 + z_2\tilde{Z}_2) \) and use the Baker-Campbell–Hausdorff formula to discover a group law \( \ast \) that makes \((\mathbb{R}^5, \ast)\) isomorphic to \( \tilde{\mathfrak{C}} \). We choose to eliminate \( \ast \) from our notation.

It is well known that \( \mathfrak{C} \) is rigid. Indeed, \( \mathfrak{f}(\mathfrak{c}) \) is isomorphic to the exceptional simple Lie algebra of dimension 14. More details can be found in [21, pp. 29-30].
A basis dual to $X_1, X_2, Y, Z_1, Z_2$ for the 1-forms on $\mathfrak{g}$ is given by
\[ \eta_1 = dx_1, \]
\[ \eta_2 = dx_2, \]
\[ \theta = dy + \frac{1}{2}x_2 dx_1 - \frac{1}{2}x_1 dx_2, \]
\[ t_1 = dz_1 + \left( \frac{1}{2}y - \frac{1}{6}x_1 x_2 \right) dx_1 + \frac{1}{6}x_1^2 dx_2 + \frac{1}{2}x_1 dy, \]
\[ t_2 = dz_2 - \frac{1}{6}x_2^2 dx_1 + \left( \frac{1}{2}y + \frac{1}{6}x_1 x_2 \right) dx_2 - \frac{1}{2}x_2 dy. \]

We will sometimes refer to these by $\theta_1 = \eta_1, \theta_2 = \eta_2, \theta_3 = \theta, \theta_4 = t_1$, and $\theta_5 = t_2$.

Like any stratified Lie group, $\mathfrak{g}$ admits a family $\{\delta_r \mid r \in (0, \infty)\}$ of group automorphisms called homogeneous dilations,
\[ \delta_r(x_1, x_2, y, z_1, z_2) = (rx_1, rx_2, r^2 y, r^3 z_1, r^3 z_2). \]

If $\Omega \subset \mathfrak{g}$ is open and $f : \Omega \to \mathfrak{g}$ is a $C^1$-smooth contact mapping, we define
\[ J_Hf = \det \begin{pmatrix} X_{1f_1} & X_{2f_1} \\ X_{1f_2} & X_{2f_2} \end{pmatrix}. \]

It is sometimes called the horizontal Jacobian of $f$.

We rely on the following result of Warhurst from [22].

**Theorem 3.2** Let $\mathfrak{g}$ be a stratified Lie group, let $\Omega \subset \mathfrak{g}$ be open, and let $f : \Omega \to \mathfrak{g}$ be a $C^1$-diffeomorphism. Then, $f$ is a contact mapping if and only if $f$ is Pansu-differentiable.

Let $\Omega \subset \mathfrak{g}$ be an open subset, and let $f : \Omega \to \mathfrak{g}$ be a $C^1$-smooth contact mapping. These are fixed for the remainder of the section.

By Theorem 3.2 $f$ is Pansu-differentiable. This means for each $p \in \Omega$ there is a graded Lie algebra automorphism $\mathcal{P}_p f$ whose action on $T_0 \mathfrak{g}$ is given by
\[ \mathcal{P}_p f(X|_0) = \lim_{r \to 0} \exp^{-1} \left( \delta_r(f(p)^{-1}f(p\delta_r(\exp X|_0))) \right). \]

Using only the structural properties of $\mathcal{P}$, we find
\[ \mathcal{P}f(Y|_0) = [\mathcal{P}f(X_1|_0), \mathcal{P}f(X_2|_0)] = (J_Hf)Y|_0, \]
\[ \mathcal{P}f(Z_1|_0) = J_Hf \left( (X_{1f_1}) Z_1|_0 + (X_{1f_2}) Z_2|_0 \right), \] and
\[ \mathcal{P}f(Z_2|_0) = J_Hf \left( (X_{2f_1}) Z_1|_0 + (X_{2f_2}) Z_2|_0 \right). \]

Combining these with evaluation of the limit as $r \to 0$ of the component functions of $\exp^{-1} \left( \delta_{1/r}(f(p)^{-1}f(p\delta_r(\exp X|_0))) \right)$ for different choices of $X|_0 \in T_0 \mathfrak{g}$, we discover the following identities:
\[(Pf)_{3,1} = X_1f_3 + \frac{1}{2}(f_2X_1f_1 - f_1X_1f_2) = 0,\]
\[(Pf)_{4,1} = X_1f_4 + \frac{1}{2}(f_2X_1f_1 - f_1X_1f_2) - \frac{1}{6}(f_1(f_2X_1f_1 - f_1X_1f_2)) = 0,\]
\[(Pf)_{5,1} = X_1f_5 + \frac{1}{2}(f_2X_1f_2 - f_2X_1f_2) - \frac{1}{6}(f_2(f_2X_1f_1 - f_1X_1f_2)) = 0,\]
\[(Pf)_{3,2} = X_2f_3 + \frac{1}{2}(f_2X_2f_1 - f_1X_2f_2) = 0,\]
\[(Pf)_{4,2} = X_2f_4 + \frac{1}{2}(f_2X_2f_1 - f_1X_2f_2) - \frac{1}{6}(f_1(f_2X_2f_1 - f_1X_2f_2)) = 0,\]
\[(Pf)_{5,2} = X_2f_5 + \frac{1}{2}(f_2X_2f_2 - f_2X_2f_2) - \frac{1}{6}(f_2(f_2X_2f_1 - f_1X_2f_2)) = 0,\]
\[(Pf)_{3,3} = Yf_3 + \frac{1}{2}(f_2Yf_1 - f_1Yf_2) = JHf,\]
\[(Pf)_{4,3} = Yf_4 + \frac{1}{2}(f_3Yf_1 - f_1Yf_3) - \frac{1}{6}(f_1(f_2Yf_1 - f_1Yf_2)) = 0,\]
\[(Pf)_{5,3} = Yf_5 + \frac{1}{2}(f_3Yf_2 - f_2Yf_3) - \frac{1}{6}(f_2(f_2Yf_1 - f_1Yf_2)) = 0,\]
\[(Pf)_{4,4} = Zf_4 + \frac{1}{2}(f_2Zf_1 - f_1Zf_2) - \frac{1}{6}(f_1(f_2Zf_1 - f_1Zf_2)) = JHf(Xf_1),\]
\[(Pf)_{5,4} = Zf_5 + \frac{1}{2}(f_2Zf_2 - f_2Zf_2) - \frac{1}{6}(f_2(f_2Zf_1 - f_1Zf_2)) = JHf(Xf_2),\]
\[(Pf)_{4,5} = Zf_4 + \frac{1}{2}(f_2Zf_1 - f_1Zf_2) - \frac{1}{6}(f_1(f_2Zf_1 - f_1Zf_2)) = JHf(Xf_1),\]
\[(Pf)_{5,5} = Zf_5 + \frac{1}{2}(f_2Zf_2 - f_2Zf_2) - \frac{1}{6}(f_2(f_2Zf_1 - f_1Zf_2)) = JHf(Xf_2).\]

Here, we have written \((Pf)_{ij}\) for the \((i, j)\)-entry in the matrix \((Pf)\) of \(Pf\) with respect to the basis \(X_1, X_2, Y, Z\). The matrix \((Pf)\) is of block-diagonal form, and it is easily found that \(\det(Pf) = J^2Hf\). Though we do not use the observation, it may be worth noting that for the combinations of \(i, j\) appearing in the above list, \((Pf)_{ij} = \langle f^*\theta, Y_j \rangle\).

Before proceeding further, we require some notation: if \(\mathfrak{g}\) is a stratified Lie group with \(\text{Lie}(I) = \bigoplus_{i \geq 1} \Lambda_i\) and \(\Lambda \subset \mathfrak{g}\) is open, then \(C^1_\mathfrak{g}(\Lambda)\) is the collection of continuous functions \(h : \Lambda \to \mathbb{R}\) such that the classical derivative \(Xh\) exists and is continuous on \(\Lambda\) for all \(X \in I_1\).

The following is Proposition 3.16 of [20, p. 26]. We are grateful to a referee for pointing out that arguably it appears earlier in [11].

**Lemma 3.3** Let \(\mathfrak{g}\) be a stratified Lie group, and let \(\Lambda \subset \mathfrak{g}\) be open. A continuous function \(h : \Lambda \to \mathbb{R}\) belongs to \(C^1_\mathfrak{g}(\Lambda)\) if and only if the distributional derivative \(Xh\) is continuous for all \(X \in I_1\).

Here and later, when we say a distributional derivative is continuous, we mean that its action is given by integration against a function, and that function can be taken to be continuous. The lemma plays an important role in the proof of the next result.

**Proposition 3.4** \((Pf)_{4,4}, (Pf)_{5,4}, (Pf)_{4,5}, (Pf)_{5,5} \in C^1_\mathfrak{g}(\Omega)\).

**Proof** Let \(\Omega_0 \subset \subset \Omega\) be open. There exists \(\varepsilon_0 > 0\) such that for each \(k = 1, \ldots, 5\) there is a family \(\{f^e_k \mid 0 < e < \varepsilon_0\}\) of \(C^\infty\)-smooth functions defined on \(\Omega_0\) with (i) \(f^e_k \to f^e_k|_{\Omega_0}\) locally uniformly as \(e \to 0\) and (ii) \(Yf^e_k \to Yf^e_k|_{\Omega_0}\) locally uniformly as \(e \to 0\) for all \(j = 1, \ldots, 5\). This can be achieved using the underlying Euclidean structure, or using a convolution defined in terms of group operations as developed in [12]. In the following we take \(0 < e < \varepsilon_0\) always.
Define $\mathcal{P}_{ij}^x$ to be $(\mathcal{P}f)_{ij}$ with each instance of the component function $f_k$ replaced with $f_k^e$. For example,

$$
\mathcal{P}_{4,5}^x = Z_2 f_4^e + \frac{1}{2} \left( f_3^e Z_2 f_3^e - f_1^e Z_2 f_3^e \right) - \frac{1}{6} \left( f_1^e f_2^e Z_2 f_3^e - (f_1^e)^2 Z_2 f_2^e \right).
$$

Note, $\mathcal{P}_{ij}^x$ converges to $(\mathcal{P}f)_{ij}$ locally uniformly so $\mathcal{P}_{ij}^x \to (\mathcal{P}f)_{ij}$ in $\mathcal{D}'(\Omega_0)$.

We observe that

$$
X_2 \mathcal{P}_{4,5}^x = X_2 Z_2 f_4^e + \frac{1}{2} \left( X_2 f_3^e Z_2 f_3^e + f_3^e X_2 Z_2 f_3^e - X_2 f_1^e Z_2 f_3^e - f_1^e X_2 Z_2 f_3^e \right)
- \frac{1}{6} \left( f_2^e X_2 f_3^e Z_2 f_3^e - f_3^e X_2 f_1^e Z_2 f_3^e + f_3^e f_2^e X_2 Z_2 f_3^e \right)
+ \frac{1}{6} \left( 2 f_1^e X_2 f_2^e Z_2 f_2^e + (f_1^e)^2 X_2 Z_2 f_2^e \right).
$$

This is nothing but repeated application of the product rule. Now we compute,

$$
Z_2 \mathcal{P}_{4,2}^x = X_2 Z_2 f_4^e + \frac{1}{2} (X_2 f_3^e Z_2 f_3^e + f_3^e X_2 Z_2 f_3^e - X_2 f_1^e Z_2 f_3^e - f_1^e X_2 Z_2 f_3^e)
- \frac{1}{6} (f_2^e X_2 f_3^e Z_2 f_3^e + f_3^e f_2^e X_2 Z_2 f_3^e)
+ \frac{1}{6} \left( 2 f_1^e X_2 f_2^e Z_2 f_2^e + (f_1^e)^2 X_2 Z_2 f_2^e \right).
$$

Here, we have used that $Z_2$ is in the center of $e$ so that $X_2$ and $Z_2$ commute. It follows,

$$
X_2 \mathcal{P}_{4,5}^x - Z_2 \mathcal{P}_{4,2}^x = \frac{1}{2} (X_2 f_3^e Z_2 f_3^e - X_2 f_1^e Z_2 f_3^e - X_2 f_3^e Z_2 f_3^e + X_2 f_3^e Z_2 f_3^e)
- \frac{1}{6} \left( f_1^e X_2 f_3^e Z_2 f_3^e - f_3^e X_2 f_3^e Z_2 f_3^e \right)
+ \frac{1}{6} \left( 2 f_1^e X_2 f_2^e Z_2 f_2^e - 2 f_3^e X_2 f_2^e Z_2 f_2^e \right)
= X_2 f_3^e Z_2 f_3^e - X_2 f_3^e Z_2 f_3^e + \frac{1}{2} \left( f_1^e X_2 f_3^e Z_2 f_3^e - f_3^e X_2 f_3^e Z_2 f_3^e \right).
$$

This last expression involves only first derivatives of the $f_k^e$. Hence,

$$
X_2 (\mathcal{P}f)_{4,5} = \lim_{\epsilon \to 0} \left( X_2 \mathcal{P}_{4,5}^x - Z_2 \mathcal{P}_{4,2}^x \right)
= X_2 f_3 Z_2 f_3 - X_2 f_3 Z_2 f_3 + \frac{1}{2} \left( f_1 X_2 f_3 Z_2 f_3 - f_1 X_2 f_3 Z_2 f_3 \right)
$$
in $\mathcal{D}'(\Omega_0)$. The first equality is true because $(\mathcal{P}f)_{4,2} = 0$.

Already this shows the distributional derivative $X_2 (\mathcal{P}f)_{4,5}$ is continuous. Pushing further, we find it admits a cleaner description. Since $(\mathcal{P}f)_{3,2} = 0$,

$$
X_2 f_3 = \frac{1}{2} (f_1 X_2 f_3 - f_2 X_2 f_3)
$$

and so

$$
X_2 (\mathcal{P}f)_{4,5} = \frac{1}{2} (f_1 X_2 f_3 - f_2 X_2 f_3) Z_2 f_3 - X_2 f_3 Z_2 f_3 + \frac{1}{2} \left( f_1 X_2 f_3 Z_2 f_3 - f_1 X_2 f_3 Z_2 f_3 \right)
= -X_2 f_3 \left( Z_2 f_3 + \frac{1}{2} (f_2 Z_2 f_3 - f_1 Z_2 f_3) \right)
= -X_2 f_3 (f^e \theta, Z_2).
$$

In a similar way, we discover
\(X_1(\mathcal{P}f)_{4,4} = -X_1 f_1 (f^* \theta, Z_1),\)
\(X_2(\mathcal{P}f)_{4,4} = -X_2 f_1 (f^* \theta, Z_1),\)
\(X_1(\mathcal{P}f)_{5,4} = -X_1 f_2 (f^* \theta, Z_1),\)
\(X_2(\mathcal{P}f)_{5,4} = -X_2 f_2 (f^* \theta, Z_1),\)
\(X_1(\mathcal{P}f)_{4,5} = -X_1 f_1 (f^* \theta, Z_2),\)
\(X_2(\mathcal{P}f)_{4,5} = -X_2 f_1 (f^* \theta, Z_2),\)
\(X_1(\mathcal{P}f)_{5,5} = -X_1 f_2 (f^* \theta, Z_2),\) and
\(X_2(\mathcal{P}f)_{5,5} = -X_2 f_2 (f^* \theta, Z_2).\)  \(\tag{3}\)

(We have included the already discussed \(X_3(\mathcal{P}f)_{4,5}\) for the sake of a complete list.)

That \((\mathcal{P}f)_{4,4}, (\mathcal{P}f)_{5,4}, (\mathcal{P}f)_{4,5}, (\mathcal{P}f)_{5,5} \in \mathcal{C}_\mathcal{E}^1(\Omega_0)\) now follows from Lemma 3.3. The classical first horizontal derivatives are given by list (3). As \(\Omega_0\) was an arbitrary open set compactly contained in \(\Omega\), it must be that \((\mathcal{P}f)_{4,4}, (\mathcal{P}f)_{5,4}, (\mathcal{P}f)_{4,5}, (\mathcal{P}f)_{5,5} \in \mathcal{C}_\mathcal{E}^1(\Omega)\) as desired.

Let \(\Omega' = f(\Omega)\) and define \(g : \Omega' \to \Omega\) by \(g = f^{-1}\). Let \(Y = \alpha_{1,1} X_1 + \alpha_{1,2} X_2 + \alpha_2 Y + \alpha_{3,1} Z_1 + \alpha_{3,2} Z_2\) be a generalized vector field on \(\Omega'\). By definition, \(Y\) is contact if
\(X_1 \alpha_2 = -\alpha_{1,2}, \quad X_2 \alpha_2 = \alpha_{1,1},\)
\(X_1 \alpha_{3,1} = \alpha_2, \quad X_1 \alpha_{3,2} = 0, \quad X_2 \alpha_{3,1} = 0, \) and \(X_2 \alpha_{3,2} = \alpha_2.\)

**Proposition 3.5** If
\(\alpha = [(J_H f) \circ g]((X_1 f_1) \circ g)\) and \(\beta = [(J_H f) \circ g]((X_1 f_2) \circ g)\)
or
\(\alpha = [(J_H f) \circ g]((X_2 f_1) \circ g)\) and \(\beta = [(J_H f) \circ g]((X_2 f_2) \circ g),\)
then
\(Y = (X_2 X_1 \alpha) X_1 - (X_1 X_2 \beta) X_2 + (X_1 \alpha) Y + \alpha Z_1 + \beta Z_2\)
is a contact generalized vector field on \(\Omega'\).

**Proof** The form of \(Y\) implies the proof reduces (in either case) to showing both \(X_1 \alpha = X_2 \beta\) and \(X_1 \beta = 0 = X_2 \alpha\) in \(\mathcal{D}'(\Omega')\). We work the case of \(\alpha = [(J_H f) \circ g]((X_2 f_1) \circ g)\) and \(\beta = [(J_H f) \circ g]((X_2 f_2) \circ g)\) in detail.

In the current circumstances, it is basic that \((\mathcal{P}(f \circ g)) = [(\mathcal{P}f) \circ g](Pg)\) and that \((Pg)\) is invertible. These facts allow us to identify that
\(\alpha = -\frac{X_2 g_1}{J_H^2 g} = -\frac{J_H g (X_2 g_1)}{J_H^3 g} = -\frac{(Pg)_{4,5}}{(Pg)_{4,4}(Pg)_{5,5} - (Pg)_{4,5}(Pg)_{5,4}}\)
and
\[\frac{(Pg)_{4,5}}{(Pg)_{4,4}(Pg)_{5,5} - (Pg)_{4,5}(Pg)_{5,4}}\]
\[ \beta = \frac{X_1 g_1}{J^2_{Hg}} = \frac{J^4_{Hg}(X_1 g_1)}{J^3_{Hg}} = \frac{(P_g)_{4,4}}{(P_g)_{4,4}(P_g)_{5,5} - (P_g)_{4,5}(P_g)_{5,4}}. \]

We now start writing \( P_{ij} \) for \( (P_g)_{ij} \). By Proposition 3.4 and (3), we have

\[-X_2 \alpha = \frac{X_2 P_{4,5}(P_{4,4} P_{5,5} - P_{4,5} P_{5,4})}{J^6_{Hg}} - \frac{P_{4,5}((X_2 P_{4,4}) P_{5,5} + P_{4,4}(X_2 P_{5,5}) - (X_2 P_{4,5}) P_{5,4} - P_{4,5}(X_2 P_{5,4}))}{J^6_{Hg}} = \frac{(X_2 P_{4,5}) P_{4,4} P_{5,5}}{J^6_{Hg}} - \frac{(X_2 P_{4,4}) P_{4,5} P_{5,5}}{J^6_{Hg}} - \frac{(X_2 P_{5,5}) P_{4,4} P_{4,5}}{J^6_{Hg}} + \frac{(X_2 P_{5,4}) P_{4,4}^2}{J^6_{Hg}} = -\frac{X_2 g_1(g^* \theta, Z_2) P_{4,4} P_{5,5}}{J^6_{Hg}} + \frac{X_2 g_1(g^* \theta, Z_1) P_{4,5} P_{5,5}}{J^6_{Hg}} + \frac{X_2 g_2(g^* \theta, Z_2) P_{4,4} P_{4,5}}{J^6_{Hg}} - \frac{X_2 g_2(g^* \theta, Z_1) P_{4,5}^2}{J^6_{Hg}} = -\frac{X_1 g_1 X_2 g_1 X_2 g_2(g^* \theta, Z_2)}{J^4_{Hg}} + \frac{(X_2 g_1)^2 X_2 g_2(g^* \theta, Z_1)}{J^4_{Hg}} + \frac{X_1 g_1 X_2 g_1 X_2 g_2(g^* \theta, Z_2)}{J^4_{Hg}} - \frac{(X_2 g_1)^2 X_2 g_2(g^* \theta, Z_1)}{J^4_{Hg}} = 0. \]

Similar calculations lead to

\[ X_1 \beta = \frac{X_1 g_1 X_2 g_1 X_1 g_2(g^* \theta, Z_1)}{J^4_{Hg}} + \frac{(X_1 g_1)^2 X_2 g_2(g^* \theta, Z_2)}{J^4_{Hg}} - \frac{(X_1 g_1)^2 X_2 g_2(g^* \theta, Z_1)}{J^4_{Hg}} = 0. \]

Furthermore, it is straightforward to check that

\[ X_1 \alpha = \frac{(X_1 g_1)^2 X_2 g_2(g^* \theta, Z_2)}{J^4_{Hg}} - \frac{X_1 g_1 X_2 g_1 X_2 g_2(g^* \theta, Z_1)}{J^4_{Hg}} - \frac{X_1 g_1 X_2 g_1 X_2 g_2(g^* \theta, Z_2)}{J^4_{Hg}} + \frac{(X_2 g_1)^2 X_2 g_2(g^* \theta, Z_1)}{J^4_{Hg}} = X_2 \beta. \]

The case of \( \alpha = [(J_{Hf}) \circ g][X_1 f_1] \circ g \) and \( \beta = [(J_{Hf}) \circ g][X_1 f_2] \circ g \) is left to the reader.

\[ \square \]

We are now ready to prove the main result of this paper. In the proof we write \( \Delta_\xi \) for the sub-elliptic Laplacian on \( \xi \), \( \Delta_\xi = X_1 X_1 + X_2 X_2 \).
Theorem 3.6 \( f \in C^\infty(\Omega) \).

**Proof** From Proposition 3.5 and Proposition 2.2, we have that \( X_j g_i / J^2 g \) is a polynomial for each combination of \( i, j = 1, 2 \). Consequently, \( J^{-3} g = \frac{X_1 g_1 X_2 g_2 - X_2 g_1 X_1 g_2}{J^4 g} \) is a polynomial. Since \( 0 < JHG < \infty \) on \( \Omega' \) it follows \( JHG \) is \( C^{\infty} \)-smooth on \( \Omega' \). We now see that each \( X_j g_i, 1 \leq i, j \leq 2 \), is \( C^\infty \)-smooth. This is enough to conclude that \( \Delta g_i \) is \( C^\infty \)-smooth on \( \Omega' \) for each \( i = 1, 2 \). At this point we invoke that \( (P_i)_{3,1}, (P_i)_{4,1}, (P_i)_{5,1}, (P_i)_{3,2}, (P_i)_{4,2}, \) and \( (P_i)_{5,2} \) are all zero to find that \( X_j g_i \) is \( C^\infty \)-smooth for all \( j = 1, 2 \) and \( i = 3, 4, 5 \). Arguing as before, we conclude that \( g_i \) is \( C^\infty \)-smooth for all \( i = 3, 4, 5 \), hence for all \( i = 1, 2, 3, 4, 5 \). Clearly, \( f \) is \( C^\infty \)-smooth by the symmetry of the situation. \( \Box \)

**Remark 3.7** Use of \( \Delta g \) in the proof of Theorem 3.6 could be considered heavy-handed; however, we find it a transparent argument. In any case, it is interesting to note that though \( \Delta g \) is hypoelliptic by Hörmander’s theorem, it is not analytic-hypoelliptic by [7].

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**References**

1. Austin, A.D., Tyson, J.T.: A new proof of the \( C^\infty \) regularity of \( C^2 \) conformal mappings on the Heisenberg group. Colloq. Math. 150(2), 217–228 (2017)
2. Baez, J.C., Huerta, J.: \( G_2 \) and the rolling ball. Trans. Am. Math. Soc. 366(10), 5257–5293 (2014)
3. Bor, G., Montgomery, R.: \( G_2 \) and the rolling distribution. Enseign. Math. 55(1–2), 157–196 (2009)
4. Capogna, L.: Regularity for quasilinear equations and \( \text{Q}-\)harmouc maps in Carnot groups. Math. Ann. 313(2), 263–295 (1999)
5. Capogna, L., Cowling, M.: Conformality and \( \text{Q}-\)harmonicity in Carnot groups. Duke Math. J. 135(3), 455–479 (2006)
6. Cartan, E.: Les systèmes de Pfaff, à cinq variables et les équations aux dérivées partielles du second ordre. Ann. Sci. École Norm. Sup. 3(27), 109–192 (1910)
7. Christ, M.: Nonexistence of invariant analytic hypoelliptic differential operators on nilpotent groups of step greater than two. In: Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991), volume 42 of Princeton Mathematical Series, pp. 127–145. Princeton University Press, Princeton, NJ, (1995)
8. Cowling, M., De Mari, F., Korányi, A., Reimann, H.M.: Contact and conformal maps in parabolic geometry. I. Geom. Dedicata 111, 65–86 (2005)
9. Cowling, M.G., Ottazzi, A.: Global contact and quasiconformal mappings of Carnot groups. Conform. Geom. Dyn. 19, 221–239 (2015)
10. Doubrov, B., Govorov, A.: A new example of a generic 2-distribution on a 5-manifold with large symmetry algebra. arXiv:1305.7297, (2013)
11. Folland, G.B.: Subelliptic estimates and function spaces on nilpotent Lie groups. Ark. Mat. 13(2), 161–207 (1975)
12. Folland, G. B., Stein, E. M.: Hardy Spaces on Homogeneous Groups, volume 28 of Mathematical Notes. Princeton University Press, Princeton (1982)
13. Hörmander, L.: The analysis of linear partial differential operators. I, volume 256 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Berlin, (1983). Distribution theory and Fourier analysis

14. Kleiner, B., Müller, S., Xie, X.: Pansu pullback and rigidity of mappings between carnot groups. arXiv:2004.09271, (2020)

15. Lelmi, J.: On the smoothness of $C^1$-contact maps in $C^\infty$-rigid carnot groups. arXiv:2006.06772, (2020)

16. Liu, Z.: Another proof of the Liouville theorem. Ann. Acad. Sci. Fenn. Math. 38(1), 327–340 (2013)

17. Ottazzi, A., Warhurst, B.: Contact and 1-quasiconformal maps on Carnot groups. J. Lie Theory 21(4), 787–811 (2011)

18. Sachkov, Y.L.: Symmetries of flat rank two distributions and sub-Riemannian structures. Trans. Am. Math. Soc. 356(2), 457–494 (2004)

19. Sarvas, J.: Ahlfors’ trivial deformations and Liouville’s theorem in $R^n$. In: Complex analysis Joensuu 1978, volume 747 of Lecture Notes in Mathematical, pp. 343–348. Springer, Berlin, (1979)

20. Serra Cassano, F.: Some topics of geometric measure theory in Carnot groups. In: Geometry, analysis and dynamics on sub-Riemannian manifolds. Vol. 1, EMS Lecture Notes in Mathematical, pp. 1–121. Zürich, (2016)

21. Tanaka, N.: On differential systems, graded Lie algebras and pseudogroups. J. Math. Kyoto Univ. 10, 1–82 (1970)

22. Warhurst, B.: Contact and Pansu differentiable maps on Carnot groups. Bull. Aust. Math. Soc. 77(3), 495–507 (2008)

23. Wille, T.: Cartan’s incomplete classification and an explicit ambient metric of holonomy $G_2^*$. Eur. J. Math. 4(2), 622–638 (2018)

24. Yamaguchi, K.: Differential systems associated with simple graded Lie algebras. In: Progress in differential geometry, volume 22 of Advanced Studies in Pure Mathematics, pp. 413–494. Japan, Tokyo, (1993)

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