FORMALITY OF FLOER COMPLEX OF THE IDEAL BOUNDARY OF HYPERBOLIC KNOT COMPLEMENT

YOUNGJIN BAE, SEONHWA KIM, YONG-GEUN OH

Abstract. This is a sequel to the authors’ article [BKO]. We consider a hyperbolic knot $K$ in a closed 3-manifold $M$ and the cotangent bundle of its complement $M \setminus K$. We equip a hyperbolic metric $h$ with $M \setminus K$ and the induced kinetic energy Hamiltonian $H_h = \frac{1}{2} |p|^2_h$ and Sasakian almost complex structure $J_h$ with the cotangent bundle $T^*(M \setminus K)$. We consider the conormal $\nu^*T$ of a horo-torus $T$, i.e., the cusp cross-section given by a level set of the Busemann function in the cusp end and maps $u : (\Sigma, \partial \Sigma) \to (T^*(M \setminus K), \nu^*T)$ converging to a non-constant Hamiltonian chord of $H_h$ at each puncture of $\Sigma$, a boundary-punctured open Riemann surface of genus zero with boundary. We prove that all non-constant Hamiltonian chords are transversal and of Morse index 0 relative to the horo-torus $T$. As a consequence, we prove that $\tilde{m}^k = 0$ unless $k \neq 2$ and an $A_\infty$-algebra associated to $\nu^*T$ is reduced to a noncommutative algebra concentrated to degree 0. We then prove that the wrapped Floer cohomology $HW(\nu^*T; H_h)$ with respect to $H_h$ is well-defined and isomorphic to the Knot Floer cohomology $HW(\partial_\infty(M \setminus K))$ that was introduced in [BKO] for arbitrary knot $K \subset M$. We also define a reduced cohomology, denoted by $\tilde{HW}^d(\partial_\infty(M \setminus K))$, by modding out constant chords and prove that if $\tilde{HW}^d(\partial_\infty(M \setminus K)) \neq 0$ for some $d \geq 1$, then $K$ cannot be hyperbolic.

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Date: January 4, 2019.

Key words and phrases. Hyperbolic knots, Knot Floer algebra, horo-torus, formality, totally geodesic triangle.

SK and YO are supported by the IBS project IBS-R003-D1. YO is also partially supported by the National Science Foundation under Grant No. DMS-1440140 during his residence at the Mathematical Sciences Research Institute in Berkeley, California in the fall of 2018. YB was partially supported by IBS-R003-D1 and JSPS International Research Fellowship Program.
1. Introduction

The symplectic idea of constructing knot invariants using the conormal lift of a knot (or link) in $\mathbb{R}^3$ as a Legendrian submanifold in the unit cotangent bundle has been explored in symplectic-contact geometry, especially exploited by Ekholm-Etnyre-Ng-Sullivan [EENS] in their construction of knot contact homology who proved that this analytic invariant recovers Ng’s combinatorial invariants of the knot [Ng]. It has been also observed (see [ENS] for example) that the data of the knot contact homology can be obtained from a version of wrapped Fukaya category on the ambient space, the symplectization of the unit cotangent bundle $ST^*\mathbb{R}^3$ or of an open subset thereof.

In [BKO], the authors considered the knot complement $M \setminus K$ of arbitrary orientable closed 3-manifold $M$ directly, and constructed its associated Fukaya category on it. We emphasize that the base space $N := M \setminus K$ is non-compact. We take a tubular neighborhood $N(K)$ of $K$ and consider its boundary $T := \partial(N(K))$. We define a cylindrical adjustment $g_0$ of the induced metric $g|_{M \setminus K}$ of a smooth metric $g$ of $M$. (See Section 10 for the precise definition thereof.) Then the construction in [BKO] associates an $A_\infty$ algebra $CW_{g_0}(\nu^*T, T^*(M \setminus K)) := CW(\nu^*T, T^*(M \setminus K); H_{g_0})$.

We denote the associated cohomology by $HW_{g_0}(\nu^*T, T^*(M \setminus K)) := HW(\nu^*T, T^*(M \setminus K); H_{g_0})$.

It was shown in [BKO] that this cohomology does not depend on the choices of smooth metric $g$ on $M$, of the tubular neighborhood $N(K)$ but depends on the isotopy class of knot $K$. In particular, we defined the wrapped Floer cohomology as an invariant of the knot $K$. We denote the resulting common graded group by $HW(\partial_\infty(M \setminus K)) = \bigoplus_{d=0}^{\infty} HW^d(\partial_\infty(M \setminus K))$ which is called the knot Floer algebra in [BKO]. Since the group is independent of the choice of tubular neighborhood of $K$, one may regard this group as the Moore homology version (in the horizontal direction) of the wrapped Floer cohomology of the asymptotic boundary $\partial_\infty(M \setminus K)$ of non-compact manifold $M \setminus K$. (See [BKO].)

1.1. Formality of Floer complex $CW(\nu^*T; H_h)$. In this paper, we specialize our focus on the case of hyperbolic knots, i.e., of the knots $K \subset M$ (or links) such that the complement $N$ admits a complete metric of constant curvature $-1$. We exploit the presence of hyperbolic metric $h$ on the complement $M \setminus K$ for the computation of $HW(\partial_\infty(M \setminus K))$, even though the metric $h$ cannot be smoothly extended to $M$ itself. In other words, the wrapping we put in the definition of wrapped Floer cohomology is of different nature from that of [BKO]. However, it is an interesting symplectic topological property of hyperbolic knots (or links) that $T^*(M \setminus K)$ is convex at $\infty$ in the sense that it admits a $J_h$-pluri-subharmonic exhaustion function.
(Proposition 4.2 for the Sasakian almost complex structure $J_h$ associated to the hyperbolic metric $h$, while as already mentioned in the introduction of [BKO], it may not be convex for general knots. The precise construction will be given in Section 3. We utilize the special geometric property in the calculation of the associated $A_{\infty}$ structures, by considering $(CW(\nu^*T), m^k)$ associated to a special choice of the above mentioned tubular neighborhood $N(K)$ so that each component of whose boundary $\partial N(K)$ is given by a horo-torus contained in the cusp-neighborhood of $K$ with respect to the hyperbolic metric. Although we will mostly restrict ourselves to the case of knots for the simplicity of exposition, we would like to emphasize that main results of the present paper also apply to the links whose ramification to the study of links is worthwhile to investigate.

Consider the kinetic energy Hamiltonian $H = \frac{1}{2} |p|^2_h$ of the hyperbolic metric $h$ on $M \setminus K$. We first prove the following general properties of the Hamiltonian chords associated to the conormal $\nu^*T$ and their associated geodesic cords attached to $T$.

**Theorem 1.1** (Theorem 6.3). Let $N$ and $T$ be as above. Then for any geodesic chord $c \in \text{Cord}(T)$, both Morse index and nullity of $c$ vanish. In particular, any non-constant Hamiltonian chord associated to $\nu^*T$ is rigid and nondegenerate.

This enables us to work with the kinetic energy Hamiltonian, without perturbation, in the construction of an $A_{\infty}$ algebra generated by the set $\text{Chord}(H; \nu^*T)$ of Hamiltonian chords of $H$ attached to $L := \nu^*T$. The relevant perturbed pseudoholomorphic equation is nothing but

$$(du - \beta \otimes X_H)^{(0,1)} = 0$$

for a map $u : \Sigma \to T^*N$ satisfying suitable (moving) Lagrangian boundary condition together with asymptotic conditions converging to Hamiltonian chords of the kinetic energy Hamiltonian $H_h$ given above. We especially study the Cauchy-Riemann equation (1.1) with respect to the Sasakian almost complex structure on $T^*N$ of $h$ on $N$; It is given by

$$J_h(X) = X^\flat, \quad J_h(\alpha) = -\alpha^\sharp$$

under the splitting $T(T^*N) \simeq TN \oplus T^*N$ via the Levi-Civita connection of $h$.

The graded Floer chain complex $CW(L; H_h)$ is a free abelian group generated by the Hamiltonian chords:

$$CW(L; H_h) := C^*(T) \bigoplus_{x \in \text{Chord}^*(L; H_h)} \mathbb{Z} \cdot x,$$

where $C^*(T)$ is a cochain complex of $T \cong \mathbb{T}^2$, e.g., the de Rham complex of $T$ or the Morse complex of a Morse function of $T$. Here the grading is given by the grading of the Hamiltonian chords $|x|$. We establish the $C^0$ estimates, especially the horizontal $C^0$ estimate, in Section 8.2 which enables us to directly define the wrapped Floer complex as an $A_{\infty}$ algebra for the hyperbolic metric, without making a cylindrical adjustment unlike in [BKO].

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1 We will follow the terminology adopted by Ng [Ng] the term chord for the Hamiltonian trajectory attached to the conormal and the term cord for the corresponding geodesic attached to the base of the conormal.
We show that \((C^*(T), d)\) forms a sub-complex of \(CW(L; H_h)\) and so can define the reduced complex
\[
\widetilde{CW}^*(L; H_h) = CW(L; H_h)/C^*(T)
\]
and denote its induced \(A_\infty\) operators by \(\widetilde{m}^k\) and its cohomology by \(\widetilde{HW}^*(L; H_h)\).

The following formality of \((\widetilde{CW}^*(L; H_h), \{\widetilde{m}^k\}_{k=1}^{\infty})\) is the first main theorem we prove in the present paper.

**Theorem 1.2 (Theorem 9.1).** Let \(h\) be the hyperbolic metric of \(N\) and \(J = J_h\) be the Sasakian almost complex structure of \(h\) on \(T^*N\), and let \(T\) be a horo-torus as above. Consider the kinetic energy Hamiltonian associated to the metric \(h\)
\[
H_h(q, p) = \frac{1}{2} |p|^2_h
\]
and the associated perturbed Cauchy-Riemann equations (1.1) equipped with some boundary condition associated to the conormal \(\nu^*T\). Let \(\widetilde{m} = \{\widetilde{m}^k\}_{k=1}^{\infty}\) be the corresponding \(A_\infty\) maps. Then we have \(\widetilde{m}^k = 0\) for all \(k \neq 2\).

This theorem is a consequence of the standard Fredholm theory combined with the geometric properties of the hyperbolic metric \(h\) stated in Theorem 1.1 in the study of moduli space of (1.1). (See Section 6 for details.)

The next theorem is the first step towards making an explicit calculation of the map \(\widetilde{m}^2\). To describe the matrix coefficients of \(\widetilde{m}^2\), we will relate solutions of (1.1) to the hyperbolic triangles on \(N\) truncated by \(T\). We recall that the projection \(\pi \circ \gamma^j\) of non-constant Hamiltonian trajectory of \(H\) is a geodesic chord and conversely each geodesic is uniquely lifted to a Hamiltonian chord.

Let \(N\) and \(T\) be as above and \(u\) be any solution to (1.1) with its asymptotic triple \((\gamma^0, \gamma^1, \gamma^2)\) of Hamiltonian chords. Denote by \(c^j = \pi \circ \gamma^j\) the associated geodesic cords for the pair \((N, T)\). For each solution \(u\) of (1.1) in \(T^*N\), we lift it to the universal covering space \(T^*\mathbb{H}^3\) together with the relevant boundary and asymptotic conditions. Then the triple \((c^0, c^1, c^2)\) of geodesic cords determines the triple \((\infty^0, \infty^1, \infty^2)\) of points in \(\partial \mathbb{H}^3\) modulo the action of deck transformations.

We denote by \(\pi : T^*N \to N\) the canonical projection.

**Theorem 1.3.** Denote by
\[
\Delta = \Delta_{(\infty^0, \infty^1, \infty^2)}
\]
the totally geodesic triangle in \(\mathbb{H}^3\) determined by \((\infty^0, \infty^1, \infty^2)\) and let \(\widetilde{u}\) as the unique lift of \(u\) satisfying the asymptotic conditions given by the triple. Then the map \(f : \Sigma \to \mathbb{H}^3\) defined by \(f(\zeta) = \pi \circ \widetilde{u}(\zeta)\) has its image contained in \(\Delta\).

The statements in Theorem 1.3 can be written in terms of \(N = M \setminus K\) as follows.

**Corollary 1.4.** Let \(u\) be any solution to (1.1) with its asymptotic triple \((\gamma^0, \gamma^1, \gamma^2)\) of Hamiltonian chords. Denote by \(c^j = \pi \circ \gamma^j\) the associated geodesics on \(N\). There exists a totally geodesic immersed ideal triangle \(\Delta\) whose ideal edges contain each geodesic triples \((c^0, c^1, c^2)\) such that the map \(f : \Sigma \to N\) defined by
\[
f(\zeta) = \pi \circ u(\zeta)
\]
has its image contained in \(\Delta\).

The following conjecture is an important one to resolve.
Conjecture 1.5. The map \( u \mapsto \pi \circ u \) induces a one-one correspondence between the set of Floer triangles associated to the triple \( (\gamma^0, \gamma^1, \gamma^2) \) and that of geodesic triangle associated to the triple \( (e^0, e^1, e^2) \) with \( e^i = \pi \circ \gamma^i \).

One outcome of Conjecture 1.5 would be that calculation of structure constants of the algebra \( HW(\partial_\infty(M \setminus K)) \) is reduced to a counting problem of geodesic triangles. We hope to come back to investigate validity of this conjecture elsewhere.

1.2. Comparison of \( HW(\nu^*T; H_h) \) with Knot Floer Algebra. Finally we prove a comparison result between \( HW(\nu^*T; H_h) \) with the Knot Floer Algebra introduced in [BKO] which is described in the beginning of this introduction. Combining some known result, Proposition 3.6, on hyperbolic geometry of \( M \setminus K \) relative to its ideal boundary, we obtain

**Theorem 1.6 (Theorem 10.3).** Suppose \( K \) is a hyperbolic knot on \( M \). Then we have an (algebra) isomorphism

\[
HW^d(L; H_h) \cong HW^d(\partial_\infty(M \setminus K))
\]

for all integer \( d \geq 0 \). In particular \( \widetilde{HW}^d(\partial_\infty(M \setminus K)) = 0 \) for all \( d > 0 \) and \( \widetilde{HW}^0(\partial_\infty(M \setminus K)) \) is a free abelian group generated by \( \mathcal{G}_{M \setminus K} \).

We refer to Definition 3.5 for the set \( \mathcal{G}_{M \setminus K} \) of infinite tame geodesics.

It is worthwhile to state the contrapositive of the second statement separately.

**Corollary 1.7.** Let \( K \subset M \) be a knot such that \( \widetilde{HW}^d(\partial_\infty(M \setminus K)) \neq 0 \) for some \( d > 0 \). Then the knot \( K \) cannot be hyperbolic.

We emphasize that in the context of hyperbolic knots which is the case of our main interest in the present paper the given hyperbolic metric \( h \) on \( M \setminus K \) is neither cylindrical at infinity nor smoothly extends to the whole space \( M \). Because of this, we cannot directly use the hyperbolic metric \( h \) defined on \( M \setminus K \) for the calculation of \( HW(\partial_\infty(M \setminus K)) \). The essence of the proof of Theorem 1.6 is to establish the existence of an isomorphism

\[
HW^d(L; H_h) \cong HW^d(L; H_{h_0}) \quad (1.3)
\]

for a cylindrical adjustment \( h_0 \) of \( h \) induced by the homotopy \( s \mapsto (1-s)h_0 + sh \). Since the two metrics \( h_0 \) and \( h \) are not Lipschitz equivalent, the result from [BKO] does not apply to this pair \( h, h_0 \) but should be proved separately. The proof of (1.3) consists of three essential steps in order:

1. The first step is to establish the existence of the continuation map
   \[
   HW^d(L; H_{h_0}) \to HW^d(L; H_h).
   \]
   For this purpose, we take the cylindrical adjustment \( h_0 \) so that \( h_0 \geq h \) and establish a \( C^0 \)-estimate for the relevant non-autonomous perturbed Cauchy-Riemann equation. (See Proposition 9.3 and its proof.)
2. Construct a direct system \( \{(HW^d(L; H_{h_i}), \iota_{ij})\}_{i=0} \) and establish the isomorphism
   \[
   \lim_{i \to \infty} HW^d(L; H_{h_i}) \to HW^d(L; H_h)
   \]
   for a sequence of cylindrical adjustments \( h_1 \geq h_2 \geq \cdots \geq h_i \geq \cdots \) with \( h_i \geq h \). The proof strongly relies on the formality of the complex \( CW(L; H_h) \) (and of \( CW(L; H_{h_0}) \) which relies on the fact that the metric
\( h_0 \) also can be chosen so that it has non-positive curvature everywhere. See Proposition 9.6.)

(3) The final step is to prove that the natural map

\[
HW^d(L; H_{h_0}) \rightarrow \lim_{\rightarrow} HW^d(L; H_{h_1})
\]

is an isomorphism. This step also relies on the formality of the associated complexes.

1.3. Hyperbolic geometry and the Bochner techniques. The main ingredients of the proofs of both theorems mentioned above are various applications of the Bochner-type techniques both in the pointwise version and in the integral version. A brief outline of how we apply these techniques is now in order.

Theorem 1.2 is a consequence of Theorem 1.1 and a degree counting argument. Therefore we will focus on Theorem 1.1 and Theorem 1.3. The proof of Theorem 1.1 is an explicit calculation of the second variation of the energy functional for the paths satisfying the free boundary condition associated to the horo-torus \( T \). Then the explicit formula we obtain manifestly establishes the positivity and nondegeneracy of the second variation, which is thanks to the special geometric property of the horo-torus (See Section 6): it has a constant positive mean curvature relative to the outward unit normal to \( T \) pointing to the cusp direction, which we also compute.

For the proof of Theorem 1.3, we apply some isometry element \( g \in \text{PSL}(2, \mathbb{C}) \), and first reduce the classification problem to that of \( \mathbb{H}^2 \simeq \{ x = 0 \} \).

For these purposes, we exploit

(1) the negative constant curvature property of hyperbolic metric on \( \mathbb{H}^3 \),
(2) the constant mean-curvature property of the horo-torus \( T \) with the correct sign,
(3) a usage of (strong) maximum principle based on rather delicate calculation of the Laplacian of an indicator function and subtle rearrangement of the terms appearing in the computed Laplacian. (See Appendix A.)

1.4. Conventions. In the literature on symplectic geometry, Hamiltonian dynamics, contact geometry and the physics literature, there are various conventions used which are different from one another one way or the other. In the mathematics literature, there are two conventions that have been dominantly appeared, which are summarized in the preface of the book [Oh4]: one is the convention that has been consistently used by the third named author and the other is the one that is called Entov-Polterovich’s convention in [Oh4].

The major differences between the two conventions lie in the choice of the following three definitions:

- **Definition of Hamiltonian vector field:** On a symplectic manifold \((P, \omega)\), the Hamiltonian vector field associated to a function \( H \) is given by the formula

\[
\omega(X_H, \cdot) = dH \quad (\text{resp. } \omega(X_H, \cdot) = -dH),
\]

- **Compatible almost complex structure:** In both conventions, \( J \) is compatible to \( \omega \) if the bilinear form \( \omega(\cdot, J\cdot) \) is positive definite.
• **Canonical symplectic form:** On the cotangent bundle $T^*N$, the canonical symplectic form is given by

$$\omega_0 = \sum_{i=1}^{n} dq^i \wedge dp_i, \quad \text{resp.} \quad \sum_{i=1}^{n} dp_i \wedge dq^i,$$

It appears that in the physics literature (e.g., [AENV] and others) the canonical symplectic form is taken as $dq \wedge dp$ as well as in [Kl] [EENS] and [Oh2], [Oh5]. For the convenience of designating the conventions, let us call the first Convention I and the second Convention II in the paragraph below. Our current convention is consistent with that of [FOOO].

We will utilize various forms of the (strong) maximum principle for the equation

$$(du - \beta \otimes X_{H_h})^{(0,1)} = 0$$

with the *negative* sign in front of $\beta \otimes X_{H_h}$. In the strip coordinate $(\tau, t)$, the equation becomes

$$\frac{\partial u}{\partial \tau} + J \left( \frac{\partial u}{\partial t} - X_{H_h}(u) \right) = 0. \quad (1.4)$$

Applicability of the maximum principle is very sensitive to the choice of conventions and the signs in the relevant equations in general such as (1.1) in the present study. We study the Cauchy-Riemann equation (1.1) on $T^*N$ associated to the triple

$$(\omega_0, J_h, H_h)$$

where $\omega_0$, $H_h$ and $J_h$ are defined on $T^*N$ following Convention I. Under these circumstances, it turns out that it is essential to adopt Convention I to be able to apply the various maximum principles we need for the equation (1.1). (See the calculations provided in Appendix and Subsection 8.2 especially Lemma 8.3 to see how these arise.)

In addition to these, the Floer continuation map is defined over the homotopy of Hamiltonian in the increasing direction. To be able to obtain the necessary energy estimates in the wrapped setting, we consider the action functional associated to Hamiltonian $H$ on $T^*N$ by

$$A_H(\gamma) = -\int \gamma^* \theta + \int_0^1 H(t, \gamma(t)) \, dt$$

which is the negative of the classical action functional. For the kinetic energy Hamiltonian $H = H_g(x)$, we have

$$A_H(\gamma_c) = -E_g(c) \quad (1.5)$$

where $\gamma_c$ is the Hamiltonian chord associated to the geodesic $c$ and $E_g(c)$ is the energy of $c$ with respect to the metric $g$.

While this paper is written as a sequel to [BKO], its content is largely independent of that of [BKO] except that we adopt the same convention as thereof. Except in Section 2 and 10 we directly work with the given hyperbolic metric for the study of perturbed Cauchy-Riemann equation above without taking the cylindrical adjustment. This forces us to establish a new form of horizontal $C^0$ estimates (see Theorem 8.4) directly applicable to the hyperbolic metric without taking a cylindrical adjustment. One important difference of the hyperbolic metric from that of cylindrical metric is that a geodesic issued even inward from $\partial N^\text{cpt}$ for $N^\text{cpt} = M \setminus N(K) \subset M \setminus K$ may go out of the domain $N^\text{cpt}$, get closer to the knot.
and then come back to the domain $N^\text{cpt}$. In Section 10, we compare the wrapped Floer cohomology associated to $(\nu^* T, H_h)$ with the Knot Floer cohomology defined for a general knot, not necessarily a hyperbolic knot, via a cylindrical adjustment of a smooth metric $g$ on $M$ restricted to $M \setminus K$.

Acknowledgement: Y. Bae thanks Research Institute for Mathematical Sciences, Kyoto University for its warm hospitality.

2. Definition of Knot Floer algebra in [BKO]

We first provide the construction of Knot Floer algebra introduced in [BKO] without the details of its construction.

Let $g$ be a smooth Riemannian metric on $M$. Consider a tubular neighborhood $N(K)$ of $K$. We denote its boundary by $T = \partial(N(K))$ and $L = \nu^* T$, the conormal bundle of the torus $T$.

We define a cylindrical adjustment $g_0$ of the metric $g$ on $M$ by

$$g_0 = \begin{cases} g & \text{on } M \setminus N'(K) \\ da^2 \oplus g|_{\partial N(K)} & \text{on } N(K) \setminus K \end{cases}$$

which is suitably interpolated on $N'(K) \setminus N(K)$ and fixed. Then we denote

$$W(K) = T^* N(K) \subset T^* (M \setminus K).$$

We denote by $X(L; H_{g_0}) = X(L,L; H_{g_0})$ the set of Hamiltonian chords of $H_{g_0}$ attached to a Lagrangian submanifold $L$ in general. We have

$$X(L; H_{g_0}) = X_0(L; H_{g_0}) \bigsqcup X_{<0}(L; H_{g_0})$$

where the subindex of $X$ in the right hand side denotes the action of the Hamiltonian chords of $H_{g_0}$. We also define

$$\text{Spec}(L; H_{g_0}) = \{A_{H_{g_0}}(\gamma) \in \mathbb{R} | \gamma \in X(L; H_{g_0})\}$$

and call the action spectrum of the pair $(L; H_{g_0})$. By definition of the kinetic energy Hamiltonian

We note that $X_0(L; H_{g_0}) \cong \mathbb{T}^2$ and the component is clean in the sense of Bott as follows.

**Proposition 2.1.**

1. For any metric $g$, the set of constant Hamiltonian cords of $H_{g_0}$ is the union

$$\bigcup_{\ell > 0} \{\ell\} \times \widehat{T}_\ell$$

where $\widehat{T}_\ell$ is the set of constant paths valued at a point in $\nu^* T$ whose domain is regarded as $[0, \ell]$. Each $T_\ell$ is in one-one correspondence with $\mathbb{T}^2$.

2. Each $\widehat{T}_\ell$ is normally nondegenerate in the path space

$$\Omega_{[0,\ell]}(\nu^* T; T^* N) = \{\gamma : [0, \ell] \to T^* N | \gamma(0), \gamma(\ell) \in \nu^* T\}$$

and diffeomorphic to $T \cong \mathbb{T}^2$, provided $\ell \neq k\pi$ for $k \in \mathbb{Z}$.

3. For a generic choice of metric $g$, all non-constant Hamiltonian cords are non-degenerate and their associated geodesics have lengths $\ell$ not equal to $\neq k\pi$, or equivalently

$$\left\{\frac{k^2 \pi^2}{2} \bigg| k \in \mathbb{Z}_+\right\} \cap \text{Spec}(\nu^* T; H_{g_0}) = \emptyset.$$
Proof: Statement (3) is standard and so omitted. (See [AH] and the discussion and the proofs in [Oh3] Section 3 for the relevant Fredholm setting for such a proof in the more complicated non-exact context.) The statement (1) is also a direct consequence of the boundary condition \( x(0), x(\ell) \in \nu^*T \), since any constant solution of \( \dot{x} = X_{H_{q_0}}(x) \) has zero momentum, i.e., \( p = 0 \).

The remaining proof will be occupied by the proof of Statement (2). Let \( \gamma_q : [0, \ell] \to T^*N \) with \( q \in T \) be a constant Hamiltonian cord valued at \( (q, 0) \in \nu^*T \).

For each pair of vector fields \( \xi_1, \xi_2 \) along \( \gamma_q \) satisfying
\[
\xi_i(0), \xi_i(\ell) \in T_{(q, 0)}(\nu^*T)
\]
a straightforward calculation give rise to the formula for the Hessian of \( \mathcal{A}_{H_k} \) at \( \gamma_q \)
\[
d^2 \mathcal{A}_{H_k}(\gamma_q) (\xi_1, \xi_2) = \int_0^\ell \left( \omega \left( \frac{D\xi_2}{dt} (t), \xi_1(t) \right) + h(\xi_1(t), \xi_2(t)) \right) dt
\]
\[
= \int_0^\ell \omega \left( \frac{D\xi_2}{dt} (t) - J\xi_2(t), \xi_1(t) \right) dt.
\]
(We refer readers to the proof of [Oh3] Proposition 18.28 for the calculation of the first term in the first line, the Hessian of the functional \( \gamma \to \int_0^\ell \theta \).) Therefore a kernel element of \( d^2 \mathcal{A}_{H_k}(\gamma_q) \) is given by the vector field \( \xi \in T_{\gamma_q} \Omega(\nu^*T, T^*N) \) satisfying
\[
\begin{cases}
\frac{D\xi}{dt} (t) - J_{(q, 0)} \xi(t) = 0 \\
\xi(0), \xi(\ell) \in T_{(q, 0)}(\nu^*T)
\end{cases}
\]
This is a first order ODE with constant coefficient matrix \( J_{(q, 0)} \) on the vector space \( T_{(q, 0)}(T^*N) \cong \mathbb{R}^6 \). Therefore the general solution of this ODE is given by
\[
\xi(t) = e^{tJ_{(q, 0)}} \xi_0
\]
for each \( \xi_0 \in T_{(q, 0)}(\nu^*T) \). Furthermore the final condition \( x_i(\pi) \in T_{(q, 0)}(\nu^*T) \) implies
\[
e^{tJ_{(q, 0)}} \xi_0 \in T_{(q, 0)}(\nu^*T).
\]
On the other hand, we have
\[
e^{tJ_{(q, 0)}} = I_{(q, 0)} \cos t + J_{(q, 0)} \sin t
\]
where \( I_{(q, 0)} \) is the identity map on \( T_{(q, 0)}(\nu^*T) \). Therefore we have
\[
\xi(\ell) = \cos \ell \xi_0 + J_{(q, 0)} \sin \ell \xi_0.
\]
By decomposing \( \xi = \xi^\parallel + \xi^\perp \) into the components of \( T_qN \) and \( T_q^*N \) under the decomposition \( T_{(q, 0)}(T^*N) \cong T_qN \oplus T_q^*N \), we rewrite the equation
\[
\xi^\parallel(\ell) + \xi^\perp(\ell) = \cos \ell \left( \xi^\parallel_0 + J_{(q, 0)} \xi^\perp_0 \right) + \sin \ell \left( J_{(q, 0)} \xi^\parallel_0 + J_{(q, 0)} \xi^\perp_0 \right).
\]
Then using the property of Sasakian almost complex structure \( J \), we obtain
\[
\xi^\parallel(\ell) = \cos \ell \xi^\parallel_0 + \sin \ell J_{(q, 0)} \xi^\perp_0
\]
\[
\xi^\perp(\ell) = \cos \ell \xi^\perp_0 + \sin \ell J_{(q, 0)} \xi^\parallel_0.
\]
Finally we aplpy
\[
\cos \ell \xi^\parallel_0 + \sin \ell J_{(q, 0)} \xi^\perp_0 = \xi^\parallel(\ell) \in T_qT
\]
\[
\cos \ell \xi^\perp_0 + \sin \ell J_{(q, 0)} \xi^\parallel_0 = \xi^\perp(\ell) \in \nu_q^*T.
\]
From this, we derive $J_{(q,0)}\xi_0^\perp = 0$ since $J_{(q,0)}\xi_0 = -(\xi_0^\perp)^2 \in N_qT$ provided $\ell \neq k\pi$.
Therefore we have derived $\xi_0^\perp = 0$, i.e.,

$$\xi_0 \in T_qT \oplus \{0\} \subset T_qT \oplus \nu_q^* T \cong T_{(q,0)}(\nu^* T).$$

Conversely, we can check that any constant vector field $\xi(t) \equiv v \in T_qT$ satisfies the above kernel equation. This proves $\ker d^2A_{H_q}(\gamma_q) \cong T_qT$ which finishes the proof. \hfill $\square$

We take

$$CW(\nu^* T; \nu^* T; T^*(M \setminus K); H_{\beta_0}) := C^*(T) \oplus \mathbb{Z}(\mathcal{X}_{>0}(L; H_{\beta_0}))$$  \hspace{1cm} (2.2)

where $C^*(T)$ is a cochain complex of $T$, e.g., $C^*(T) = \Omega^*(T)$ the de Rham complex and associate an $A_{\infty}$ algebra following the construction from [FOOO]. It was shown in [BKO] that $HW_g(T, M \setminus K)$ does not depend on the choice of smooth metric $g$ on $M$ and of the tubular neighborhood $N(K)$ but depends only on the isotopy type of the knot $K$.

**Definition 2.2** (Knot Floer algebra [BKO]). We denote by

$$HW(\partial_{\infty}(M \setminus K)) = HW_g(T, M \setminus K)$$

the resulting common (isomorphism class of the) group and call it the knot Floer algebra of $K$ in $M$.

To facilitate our calculation of this algebra and its comparison with Knot Floer algebra $HW(\partial_{\infty}(M \setminus K)$, we now take the Morse complex model for $C^*(T)$, and realize the model (2.2) as a nondegenerate wrapped Floer complex of a perturbed $\nu^* T$ as follows.

Take a compactly supported smooth function $k : M \setminus K \to \mathbb{R}$ satisfying $k = 1$ in a neighborhood of $T$, and consider the translated conormal $\nu_k^* T \to T$ whose fiber is given by

$$(\nu_k^* T)_q := \{\alpha + dk(q) \in T_q^* N \mid \alpha \in \nu_q^* T\}.$$  \hspace{1cm} (2.3)

Then it is easy to check

$$\nu^* T \cap \nu_k^* T = \{(q, p) \mid q \in dk(q) \in \nu_q^* T\} = \nu^* T \cap \text{Image } dk$$

and the intersection is nondegenerate if and only if $\nu^* T \cap \text{Image } dk$.

We then take a radially cut-off function $\rho : T^* N \to \mathbb{R}$ satisfying $\rho(q, p) = \rho_q(|p|)$ where $\rho_q : \mathbb{R}_+ \to [0, 1]$ is a monotonically increasing function satisfying

$$\rho_q(r) = \begin{cases} 0 & \text{for } r \geq 3\|dk\|_{C^0} \\ 1 & \text{for } r \leq 2\|dk\|_{C^0} \end{cases}$$

and consider the function $f : \nu^* T \to \mathbb{R}$ defined by

$$f(\alpha) = \rho(\alpha)k(\pi(\alpha)).$$

Then we take a Darboux-Weinstein chart $\Phi : V \subset T^*(\nu^* T) \to U \subset T^* N$ of $\nu^* T$ and then consider the exact Lagrangian submanifold

$$\nu_{k, \rho}^* T := \Phi(\text{Image } df) \subset T^* N.$$  \hspace{1cm} (2.5)

In other words, $\nu_{k, \rho}^* T = \text{Image } \iota_{k, \rho}$ for the Lagrangian embedding $\iota_{k, \rho} = \iota_{k, \rho}^\Phi : \nu^* T \to U \subset T^* N$ defined by

$$\iota_{k, \rho}^\Phi(\alpha) = \Phi(\rho k \circ \pi)|_{\alpha}.$$
As usual, we require \( \Phi \) to satisfy

\[
\Phi|_{\partial T^* (\nu^* T')} = id|_{\nu^* T'}, \quad d\Phi|_{\partial T^* (\nu^* T')} = id|_{T^* (\nu^* T)}
\]

under the canonical identifications of \( \partial T^* (\nu^* T) \cong \nu^* T \) and

\[
T(T^*(\nu^* T)) \cong T(\nu^* T).
\]

It is easy to check that \( f \) satisfies \( \iota_{k,\rho}^* \theta = df \) and so \( \nu_{k,\rho}^* T \) is an exact Lagrangian submanifold. We also note that

\[
\nu_{k,\rho}^* T = \begin{cases} 
\nu^* T & \text{for } |p| \geq 3\|dk\|_{C^0} \\
\nu_{k}^* T & \text{for } |p| \leq 2\|dk\|_{C^0}
\end{cases}
\]

and \( f(\beta) = 0 \) for \( |\beta| \geq 3\|dk\|_{C^0} \) and \( f(\beta) = k(\beta) \) for \( |\beta| \leq 2\|dk\|_{C^0} \).

Next we denote by \( \mathfrak{S}_{g_0}(T) \) the energy of the shortest geodesic cord of \( T \) relative to the metric \( g_0 \). Then the following lemma is an immediate consequence of the implicit function theorem.

**Lemma 2.3.** Let \( k \) be the function given above such that \( \nu^* T \cap \text{Image } dk \). Then there exists some \( 0 < \epsilon_0 < \frac{\varphi_{g_0}(T)}{2} \) such that

\[
\mathfrak{X}_{\leq -\epsilon_0}(\nu_{k,\rho}^* T, \nu^* T) = \mathfrak{X}_{\leq -\epsilon_0}(\nu^* T, \nu^* T)
\]

and

\[
\mathfrak{X}_{\geq -\epsilon_0}(\nu_{k,\rho}^* T, \nu_{k}^* T) \cong \text{Image } dk \cap \nu^* T
\]

provided \( \|k\|_{C^2} \geq -\epsilon_0 \). Here \( \cong \) means one-one correspondence.

**Proof.** Both identities then are immediate consequences of Sard-Smale implicit function theorem via the definition of the cut-off function \( \rho \) and \( C^1 \)-smallness of \( df \), and the requirement (2.6). \( \square \)

By the generic transversality proof under the perturbation of Lagrangian boundary from \( \mathfrak{O} [\mathfrak{H}] \), we can choose \( k \) so that \( \mathfrak{X}(\nu_{k,\rho}^* T, \nu_{k,\rho}^* T) \) is nondegenerate. We have one-one correspondence

\[
\mathfrak{X}_{\leq -\epsilon_0}(\nu^* T, \nu^* T) \cong \text{Crit } k
\]

and hence

\[
\mathcal{Z}(\mathfrak{X}_{\leq -\epsilon_0}(\nu^* T, \nu^* T)) \cong \mathcal{Z}(\text{Crit } k).
\]

We denote \( L = \nu_{k,\rho}^* T \). Then we define a Floer chain complex

\[
\text{CW}_g^d(T, M \setminus K) := \text{CW}^d(L, L; T^*(M \setminus K); H_{g_0}).
\]

Here the grading \( d \) is given by the grading of the Hamiltonian chords \( |x| \). Then the construction in \( \mathfrak{B} \mathfrak{K} \mathfrak{O} \) associates an \( A_\infty \) algebra to \( \text{CW}_g(T, M \setminus K) \). We denote the associated cohomology by

\[
\text{HW}_g(T, M \setminus K) = \text{HW}(L, L; T^*(M \setminus K); H_{g_0}).
\]

It follows from the Bott-Morse property of \( (\nu^* T, H_{g_0}) \) that the complex \( \text{CW}_g^d(T, M \setminus K) \) has such a decomposition

\[
\text{CW}_g(T, M \setminus K) := \mathcal{Z}(\mathfrak{X}_{\leq -\epsilon_0}(H_{g_0}; L, L)) \oplus \mathcal{Z}(\mathfrak{X}_{\leq -\varphi_{g_0}(T)/2}(H_{g_0}; L, L)).
\]

Moreover \( \mathcal{Z}(\mathfrak{X}_{\leq -\epsilon_0}(H_{g_0}; L, L)) \) is a subcomplex of \( \text{CW}_g(T, M \setminus K) \) which is isomorphic to the Morse complex \( (C^*(k|_T), d) \) of the Morse function \( k|_T : T \to \mathbb{R} \).
Proposition 2.4. The operator $m^1$ has the matrix form

$$m^1 = \begin{pmatrix} \pm d & 0 \\ \ast & m^1_{<0} \end{pmatrix}$$

with respect to the above decomposition.

Proof. Suppose $\gamma_\pm \in \mathfrak{X}_{\pm -\epsilon_0}(H_{g_0}; L, L)$ and $\gamma_+ \in \mathfrak{X}_{\pm +\epsilon_0}(H_{g_0}; L, L)$. Then we have

$$A_{H_{g_0}}(\gamma_-) \geq -\epsilon_0, \quad A_{H_{g_0}}(\gamma_+) \leq -\mathfrak{G}_{g_0}(T)/2$$

and hence

$$A_{H_{g_0}}(\gamma_+) - A_{H_{g_0}}(\gamma_-) \leq -\mathfrak{G}_{g_0}(T)/2 + \epsilon_0 < 0. \quad (2.9)$$

On the other hand, if there exists a solution $u$ for (1.4) satisfying $u(\pm \infty) = \gamma_\pm$, then

$$0 \leq \int \left| \frac{\partial u}{\partial \tau} \right|^2_{J_{g_0}} = A_{H_{g_0}}(u(\infty)) - A_{H_{g_0}}(u(-\infty)) = A_{H_{g_0}}(\gamma_+) - A_{H_{g_0}}(\gamma_-)$$

which contradicts to (2.9). This finishes the proof. \(\square\)

Definition 2.5 (Reduced Knot Floer complex). Denote by $(\check{CW}_g(T, M \setminus K), \check{m}^1)$ the quotient complex

$$(CW_g(T, M \setminus K)/C^*(T), [m^1_{<0}])$$

We call this complex by the reduced Knot Floer complex.

We also note $m^1_{<0} \circ m^1_{<0} = 0$ and so

$$\left( \mathbb{Z} \left\langle \mathfrak{X}_{< -\mathfrak{G}_{g_0}(T)/2}(H_{g_0}; L, L) \right\rangle, m^1_{<0} \right)$$

is naturally isomorphic to the reduced complex $(\check{CW}_g(T, M \setminus K), \check{m}^1)$.

Therefore the reduced complex is nothing but the complex generated by non-constant Hamiltonian chords.

We emphasize that in the context of hyperbolic knots which is the case of our main interest in the present paper the given hyperbolic metric $h$ on $M \setminus K$ is neither cylindrical at infinity nor smoothly extends to the whole space $M$. Because of this, we cannot directly use the hyperbolic metric $h$ defined on $M \setminus K$ for the calculation of $HW(\partial_\infty(M \setminus K))$. The rest of the paper is occupied by the construction of this wrapped Floer complex associated to the kinetic energy Hamiltonian $H_h$ of the hyperbolic metric on $M \setminus K$ whose injectivity radius is zero.

3. Preliminary on hyperbolic 3-manifold of finite volume

In this section, we briefly review several well-known facts. For a reference, Martelli’s book [M] is readable and enough to know some basics about hyperbolic geometry and 3-manifold theory used in this article.

Let $N$ be a complete hyperbolic 3-manifold. The universal cover $\tilde{N}$ is identified with the hyperbolic 3-space

$$\mathbb{H}^3 = \{ (x, y, z) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}, z \in \mathbb{R}^+ \}$$

and $N$ is isometric to $\mathbb{H}^3/\Gamma$ with a discrete group $\Gamma \cong \pi_1(N)$. If $N$ is orientable, $\Gamma$ consists of orientation preserving isometries. Hence, from now on we identify the hyperbolic space $\mathbb{H}^3$ and the group of orientation preserving isometries Isom$^+(\mathbb{H}^3)$ with a upper half space and $\text{PSL}(2, \mathbb{C})$ respectively. The ideal boundary $\partial_{\text{hyp}}$ of
\[ H^3 \] is identified with \( \hat{\mathbb{C}} := \mathbb{C} \cup \{ \infty \} \) and \( \text{PSL}(2, \mathbb{C}) \) acts on \( \hat{\mathbb{C}} \) and \( H^3 \) as Möbius transformations, and the Poincaré extensions, respectively.

Let \( N \) be a knot complement of a orientable closed 3-manifold \( M \), i.e., \( N = M \setminus K \). Then \( N \) is homeomorphic to the interior of the knot exterior, a compact 3-manifold denoted by \( \overline{N} \), that has a torus boundary of \( N \) and the complete hyperbolic structure should be of finite volume by the torus boundary condition.

### 3.1. \( \varepsilon \)-thick-thin decomposition by the Busemann function.

The first important fact in this situation would be the uniqueness of the hyperbolic metric \( h \) by Mostow-Prasad rigidity, i.e., \( h \) is unique up to isometry. Therefore any hyperbolic metric invariant can be regarded as a topological invariant, as we have a canonical Riemannian metric.

Secondly, we can nicely separate compact part and the non-compact end part of \( N \) as follow.

**Proposition 3.1.** There is a constant \( \varepsilon_0 > 0 \) such that the \( \varepsilon_0 \)-thin part,

\[
N_{(0, \varepsilon_0)} := \{ x \in N \mid \text{inj}_x(N) \leq \varepsilon_0 \},
\]

is homeomorphic to the end of \( N \), which is \( \mathbb{T}^2 \times [0, +\infty) \). Thus the \( \varepsilon \)-thick part \( N_\varepsilon \) is compact and the interior is homeomorphic to \( N \) itself once we take \( \varepsilon \leq \varepsilon_0 \),

\[
N_\varepsilon := N_{(\varepsilon, +\infty)} \approx N.
\]

**Proof.** By the thick-thin decomposition using the Margulis constant \( \varepsilon_0 \), we have two kinds of thin parts, thin-tubes \( S^1 \times D^2 \) and truncated cusp \( \mathbb{T}^2 \times [0, 1) \). When we take \( \varepsilon_0 \) less than smallest injective radius among thin-tubes, then the only possible \( \varepsilon_0 \)-then part is the truncated cusp of the boundary. \( \square \)

By the structural property of the thick-thin decomposition \([M \text{ Section 4}]\), we know that \( N_{(0, \varepsilon]} \) is a truncated cusp and thus \( N_{(0, \varepsilon]} \cap N_\varepsilon = \partial N_{(0, \varepsilon]} = \partial N_\varepsilon \) is a Euclidean torus. We describe \( N_\varepsilon \) by using the Busemann function instead of injective radius.

**Definition 3.2.** Let \( \delta : [0, +\infty) \to N \) be a geodesic ray satisfying

\[
d_h(\delta(t), \delta(t')) = |t - t'|.
\]

The Busemann function \( b_{\delta} : W \to \mathbb{R} \) is defined by

\[
b_{\delta}(q) := \lim_{t \to \infty} (d(q, \delta(t))) - t).
\]

Without loss of generality, we can take a lifting \( \overline{\delta} \) in \( \mathbb{H}^3 \) of \( \delta \),

\[
\overline{\delta}(t) := (0, 0, e^t) \in \mathbb{H}^3,
\]

such that the lifted Busemann function \( \overline{b}_{\delta} \) for \( \mathbb{H}^3 \) is given by

\[
\overline{b}_{\delta}(\overline{(x, y, z)}) \lim_{t \to \infty} (d(\overline{(x, y, z)}, \overline{\delta}(t))) - t) = - \log z,
\]

and the level set of \( \overline{b}_{\delta} \) is a horosphere centered at \( \{ z = +\infty \} \),

\[
\overline{b}_{\delta}^{-1}(t) = \{ (x, y, z) \in \mathbb{H}^3 \mid z = e^{-t} \}.
\]

These are direct consequences of a hyperbolic distance formula in \([F \text{ Section III.4}]\). We remark that there are many other lifts of \( \delta \) which may tend to the other ideal points of \( \partial \mathbb{H}^3 \).
**Proposition 3.3.** We have a decomposition \( N = N_\varepsilon \cup N_{(0,\varepsilon]} \) by a Busemann function \( b_\delta : N \to \mathbb{R} \) where

\[
N_\varepsilon = b_\delta^{-1}([t_0, \infty))
\]

\[
N_{(0,\varepsilon]} = b_\delta^{-1}((-\infty, t_0]) \approx b_\delta^{-1}(t_0) \times \mathbb{R}.
\]

We call \( N_\varepsilon \) a \( \varepsilon \)-thick compact part and \( N_{(0,\varepsilon]} \) a \( \varepsilon \)-thin cusp part, respectively.

**Proof.** It is succinct to use Epstein-Penner decomposition [EP]. Let \( N \) be obtained from a finite union of convex hyperbolic ideal polyhedra with face gluings in \( \text{Isom}^+(\mathbb{H}^3) \). Hence we have a fundamental domain \( D \) whose boundary is made up of totally geodesic convex ideal polygons such that

\[
\mathbb{H}^3 = \bigcup_{g \in \Gamma} g \cdot D,
\]

Without loss of generality, we can assume \( \tilde{\delta}(t) \in D \) and \( \tilde{\delta}(\infty) = \infty \in \tilde{V}_D \subset \tilde{\mathcal{C}} \), where

\[
\tilde{V}_D = \bigcup_{g \in \Gamma} g \cdot V_D
\]

and \( V_D \) is the finite set of ideal vertices of \( D \). We now take a sufficiently large negative number \( t_0 \) such that the horospheres centered at \( \tilde{\delta}(t_0) \) are mutually disjoint, i.e.,

\[
g \cdot b_\delta^{-1}(t_0) \cap g' \cdot b_\delta^{-1}(t_0) = \emptyset,
\]

for any distinct pair of \( g, g' \in \Gamma \). Then the truncated fundamental domain,

\[
D \setminus \bigcup_{i=1, \ldots, n} g_i \cdot b_\delta^{-1}(t_0),
\]

produces \( N_\varepsilon \) by face gluing isometries originally used to make \( N \). The injective radius \( \varepsilon \) is taken at any point in \( b_\delta^{-1}(t_0) \).

\[\square\]

3.2. Infinite tame geodesics and geodesic cords of horo-torus. Denote by \( T = \partial N_\varepsilon \). We call the boundary torus of \( N_\varepsilon \) as a horo-torus \( T \), which is given by a level set \( b_\delta^{-1}(t_0) \) of the Busemann function.

**Definition 3.4.** We define

\[
\text{Cord}(T) = \{ c : [0, 1] \to N \mid \nabla c = 0, c(i) \in T, \dot{c}(i) \perp T, \text{ for } i = 0, 1 \}.
\]

We call an element of \( \text{Cord}(T) \) a geodesic cord of \( T \).

Next we consider an infinite tame geodesic \( \delta : (-\infty, \infty) \to N \).

**Definition 3.5.** An infinite geodesic \( \delta : \mathbb{R} \to N \) is called tame if if there is a continuous map \( \alpha : [0, 1] \to N \setminus N \) such that \( \delta \) and \( \alpha|_{(0, 1]} \) has the same image in \( \overline{N} \). We denote by \( \mathcal{G} = \mathcal{G}_N \) the set of images of all infinite tame geodesics in \( N \).

Now we would like to emphasize the following proposition which gives a crucial intuition for our purpose.

**Proposition 3.6.** Let \( T \) be a horo-torus in hyperbolic knot complement \( N \). Then there is a one-one correspondence between \( \text{Cord}(T) \) and \( \mathcal{G} \).
For the later purpose we discuss about an induced metric on the cotangent bundle.

4.1. The Sasakian almost complex structure on the cotangent bundle.

In this section, we summarize some basic facts on the Riemannian geometry of the cotangent bundle of hyperbolic knot complement.

We remark that some interior points in \( c \) or \( \gamma \) may intersect \( T \) non-perpendicularly and some geodesic cords may not be contained in a fixed \( N_i \) in an exhaustion sequence \( \{ N_i \} \). In fact, only a finite number of geodesic cords can be contained in a fixed \( N_i \).

\section{4. Cotangent bundle of hyperbolic knot complement}

In this section, we summarize some basic facts on the Riemannian geometry of the cotangent bundle of hyperbolic knot complement.

4.1. The Sasakian almost complex structure on the cotangent bundle.

For the later purpose we discuss about an induced metric on the cotangent bundle \( \tilde{\pi} : T^* \mathbb{H}^3 \to \mathbb{H}^3 \) of

\[
\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}
\]

with a complete hyperbolic metric \( h = h_{\mathbb{H}^3} \) given by

\[
(h_{ij})_{1 \leq i, j \leq 3} = \left( \frac{1}{2 \delta_{ij}} \right)_{1 \leq i, j \leq 3}.
\]
Let us start with Levi-Civita connection $\nabla = \nabla^h$ and an induced (co-)frame fields $H_i, V_i$ (and $H^i, V^i$) on $T^*\mathbb{H}^3$ which are given as follows:

\[
    H_i = \partial_{q_i} + p_a \Gamma^a_{ij} \partial_{p_j}, \quad V_i = \partial_{p_i};
\]

\[
    H^i = dq^i, \quad V^i = dp_i - p_a \Gamma^a_{ij} dq^j.
\]

Here $\Gamma^a_{ij}$ are Christoffel symbols for the connection $\nabla$ and we used the Einstein summation convention.

A direct calculation using the definition of Christoffel symbols for the hyperbolic metric $h = dx^2 + dy^2 + dz^2$ on $\mathbb{H}^3$ for the standard coordinate $(x, y, z)$ with $z \geq 0$ gives rise to

\[
    \Gamma^3_{11} = \frac{1}{z}, \quad \Gamma^1_{13} = \Gamma^1_{31} = -\frac{1}{z},
\]

\[
    \Gamma^2_{22} = \frac{1}{z}, \quad \Gamma^2_{33} = \Gamma^3_{22} = -\frac{1}{z},
\]

\[
    \Gamma^3_{33} = -\frac{1}{z}, \quad \text{and all other symbols are zero. (4.1)}
\]

Using this calculation, we explicitly express

\[
    H_1 = \partial_x + \frac{p_z}{z} \frac{\partial}{\partial p_z} - \frac{p_x}{z} \frac{\partial}{\partial p_x};
\]

\[
    H_2 = \partial_y + \frac{p_z}{z} \frac{\partial}{\partial p_y} - \frac{p_y}{z} \frac{\partial}{\partial p_y};
\]

\[
    H_3 = \partial_z - \frac{p_x}{z} \frac{\partial}{\partial p_x} - \frac{p_y}{z} \frac{\partial}{\partial p_y} - \frac{p_z}{z} \frac{\partial}{\partial p_z};
\]

and

\[
    V^1 = dp_x - \frac{p_z}{z} dx + \frac{p_x}{z} dz;
\]

\[
    V^2 = dp_y - \frac{p_z}{z} dy + \frac{p_y}{z} dz;
\]

\[
    V^3 = dp_z + \frac{p_z}{z} dx + \frac{p_y}{z} dy + \frac{p_z}{z} dz.
\]

An induced Riemannian metric $\tilde{h}$ on $T^*\mathbb{H}^3$ with respect to the (co-)frame fields is given by

\[
    h_{ij} dq^i dq^j + h^{ij} \delta p_i \delta p_j
\]

where $(h^{ij})_{i,j}$ is the inverse matrix of $(h_{ij})_{i,j}$ and $\delta p_i = V^i$. In a matrix form we have

\[
    \begin{pmatrix}
        h_{ij} & 0 \\
        0 & h^{ij}
    \end{pmatrix}.
\]

Now we equip $(T^*\mathbb{H}^3, \tilde{h})$ with the canonical symplectic 2-form

\[
    \omega = \sum_{i=1}^3 dq^i \wedge dp_i = \sum_{i=1}^3 H^i \wedge V^i.
\]
and an almost complex structure on $T^*\mathbb{H}^3$ the so called Sasakian almost complex structure $J_h$ associated to the Levi-Civita connections of $h$. First the Levi-Civita connection induces the splitting

$$T_{(q,p)}(T^*\mathbb{H}^3) \simeq T_q\mathbb{H}^3 \oplus T^*_q\mathbb{H}^3$$

at each point $(q,p) \in T^*\mathbb{H}^3$, where the isomorphism is obtained by

$$H_j \mapsto \frac{\partial}{\partial q^j}, \quad V_j \mapsto dq^j.$$  \hfill (4.2)

In terms of the splitting, the Sasakian almost complex Structure $J_h$ on $T^*\mathbb{H}^3$ are defined by

$$J_h(X) = X^\flat, \quad J_h(\alpha) = -\alpha^\sharp$$

for $X \in \mathbb{H}^3$ and $\alpha \in T^*\mathbb{H}^3$ respectively.

In the canonical coordinates, the almost complex structure $J = J_h$ is given by the formulae

$$J_h : T(T^*\mathbb{H}^3) \to T(T^*\mathbb{H}^3);$$

$$H_i \mapsto h_{ij}V_j; \quad V_i \mapsto -h^{ij}H_j,$$

which can be expressed in the following matrix

$$\begin{pmatrix} 0 & -h^{ij} \\ h_{ij} & 0 \end{pmatrix}.$$ 

with respect to the above frame fields. Then the compatibility condition

$$\tilde{h}(\cdot, \cdot, \cdot) = \omega(\cdot, J_h \cdot)$$

between the triple $(\tilde{h}, \omega, J_h)$ can be guaranteed by the following matrix multiplication:

$$\begin{pmatrix} h_{ij} & 0 & 0 \\ 0 & h^{ij} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \delta_{ij} \\ -\delta_{ij} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -h^{ij} \\ h_{ij} & 0 \end{pmatrix}.$$ 

The following proposition indicates some special symplectic topological behavior of hyperbolic knots against general knots.

**Proposition 4.2.** Let $b = b_\delta : N \to \mathbb{R}$ be a Busemann function such that its lift $\tilde{b} = b_\delta \circ p : \mathbb{H}^3 \to \mathbb{R}$ satisfies $e^b \circ p = \frac{1}{z}$ near the ideal boundary. Denote by $\pi : T^*N \to N$ the canonical projection. Then the function $f : T^*N \to \mathbb{R}_+$ defined by

$$f = H_h + e^b \circ \pi,$$

is a strictly $J_h$-pluri-subharmonic exhaustion function ‘at infinity’.

**Proof.** Let $(x, y, z, p_x, p_y, p_z)$ be the canonical coordinates for the $T^*N$ with $N = M \setminus K$. With slight abuse of notations, we also denote the canonical coordinates of $T^*\mathbb{H}^3$ by the same. Then by the hypothesis, we have the formula for the lift $\tilde{f} : \mathbb{H}^3 \to \mathbb{R}$ of $f$

$$\tilde{f} = H_k + \frac{1}{z} = \frac{1}{2z^2} (p_x^2 + p_y^2 + p_z^2) + \frac{1}{z}.$$ 

A straightforward computation leads to

$$-d(dH_k \circ J_h) = \frac{1}{z^2} dz \wedge \theta + \frac{1}{z} \omega_0,$$

$$-d(d(1/z) \circ J_h) = \omega_0.$$
and hence
\[-d(d\tilde{f} \circ J_h) = \frac{1}{z^2} dz \wedge \theta + \left(1 + \frac{1}{z}\right) \omega_0.\]
It is easy to check that
\[-d(d\tilde{f} \circ J_h)(V_i, J V_i) = 1 + \frac{z}{z^3} > 0\]
for each \(V_i = \partial p_i\) with \(i = x, y, z\). Since \(p : \mathbb{H}^3 \to N\) is a local isometry, this implies
\[-d(df \circ J)(X, J X) \geq 0\]
and equality holds only when \(X = 0\) for any \(X \in T(T^* N)\).
This proves that the function \(f\) is \(J_h\)-pluri-subharmonic. \(\square\)

4.2. Horo-tori and conormal Lagrangians. We consider the universal cover \(p : \mathbb{H}^3 \to N\) and then the lifted metric \(p^* g_N\) on \(\mathbb{H}^3\) coincide with \(g_{\mathbb{H}^3}\).

Let us denote the deck-transformation group of the covering \(p\) by \(\Gamma\).

For a given horo-torus \(T\) in \(N\), we consider its lifting \(\tilde{T} := p^{-1}(T)\) to \(\mathbb{H}^3\). It is a disjoint union
\[\tilde{T} = \bigcup_{g \in G} g \cdot \{z = a\}\]
where \(a\) is a sufficiently large positive real number satisfying \(\{z = a\} \cap g \cdot \{z = a\} = \emptyset\) for any \(g \in \Gamma \setminus \{id\}\). For a given submanifold \(S\) in \(N\), we consider its conormal as a (exact) Lagrangian submanifold in \(T^* \mathbb{H}^3\).

Next we derive some geometric property of horospheres in \(\mathbb{H}^3\). By applying the action of \(\text{PSL}(2, \mathbb{C})\), we are reduced to the study of hyperplane
\[\{(x, y, z) \in \mathbb{H}^3 | z = z_0\}\]
for given constant \(z_0 > 0\). The following geometric property, constant mean curvature property with positivity is a crucial ingredient entering in the proof of rigidity and nondegeneracy of Hamiltonian chords attached to the conormal \(\nu^* T\) (Theorem 1.1).

**Proposition 4.3.** Denote by \(N\) be the outward unit normal to the plane \(\{z = z_0\}\) in \(\mathbb{H}^3\). Then the mean-curvature vector \(\vec{H}\) relative to \(N\) is given by
\[\vec{H} = N.\]
In particular the mean-curvature is 1 for all \(z_0 > 0\).

**Proof.** Recall the hyperbolic metric \(h_{ij} = \frac{1}{z} \delta_{ij}\) on the upper half space model of \(\mathbb{H}^3\). Then we have covariant derivative on \(\mathbb{H}^3\) as follows,
\[\nabla_{\partial_{x_*}} \partial_x = \frac{1}{z} \partial_x \quad \text{for } * = x, y; \quad \nabla_{\partial_{y_*}} \partial_y = \nabla_{\partial_y} \partial_x = 0;\]
\[\nabla_{\partial_{z_*}} \partial_z = -\frac{1}{z} \partial_x \quad \text{for } * = x, y, z.\]

Let us consider a horo-sphere \(\bar{N}\) centered at \(\infty\) with \(\{(x, y, z) \in \mathbb{H}^3 | z = t_0\}\).
A unit normal vector \(N\) toward \(\infty\) at \(q \in \bar{N}\), i.e., outward normal, is given by \(N = t_0 \partial_z\) and an orthonormal basis of \(T_p H\) is \(\{t_0 \partial_z, t_0 \partial_y\}\). Let us compute the shape operator \(S_N\) as follows.
\[S_N(\partial_*) = -\nabla_{\partial_*} (N) = \partial_* \quad \text{for } * = x, y.\]
This proves the first statement.
A mean curvature $H$ is computed by the second fundamental form as follows,

$$H = \frac{1}{2} \{ \langle S_{N}(t_{0}\partial_{x}), t_{0}\partial_{x} \rangle_{h} + \langle S_{N}(t_{0}\partial_{y}), t_{0}\partial_{y} \rangle_{h} \} = 1. \quad (4.4)$$

This finishes the proof. \hfill \Box

4.3. $C^0$ bounds of Neumann geodesic cords of horo-torus. In this subsection, we provide the following classification result on the Neumann geodesic cords of a horo-torus $T = \partial N_i$.

**Lemma 4.4.** Consider a geodesic $c : [0, \ell] \to M \setminus K$ with $c(0), c(1) \in T = \partial N_i$. Lift $\tilde{T}$ to a lift to a horo-sphere $S_0 = \{ z = a_0 \}$ in $\mathbb{H}^3$ and $c$ to a geodesic $\tilde{c}$ in $\mathbb{H}^3$ so that $\tilde{c}(0) \in S_0$ that is perpendicular to $S_0$. Let $z_{\ell} = z(\tilde{c}(\ell))$. Consider the function $f(t) = \frac{1}{\cosh(\tilde{c}(t))}$. Then

$$f(t) = a_0 \cosh(\tilde{c}(t)) + b_0 \sinh(\tilde{c}(t))$$

for which $|b_0|$ is bounded by a constant depending only on $S_0$ and $\ell$. In particular

$$\max_{t \in [0, \ell]} |f(t)| \leq a_0 \cosh \ell + |b_0| \sinh \ell. \quad (4.5)$$

**Proof.** We define a function $f$ by $f(t) = \frac{1}{\cosh(\tilde{c}(t))}$. We denote $c(t) = (c_x(t), c_y(t), c_z(t))$ in the standard coordinates $(x, y, z)$ of $\mathbb{H}^3 \subset \mathbb{R}^3$ with the identification $\mathbb{H}^3 = \{(x, y, z) \mid z > 0\}$.

We have

$$f'(t) = -\frac{1}{\cosh^2(\tilde{c}(t))} \dot{c}_x(t)$$

and

$$f''(t) = \frac{2}{\cosh^3(\tilde{c}(t))} |\dot{c}_x(t)|^2 - \frac{1}{\cosh^2(\tilde{c}(t))} \frac{d}{dt} (dz(\dot{c}(t))).$$

Since $c$ is a geodesic, we also have

$$\frac{d}{dt} (dz(\dot{c})) = \nabla_t (dz)(\dot{c}).$$

A straightforward computation using (4.3) gives rise to

$$\nabla_t dz = \frac{1}{z} (-\dot{c}_x dx - \dot{c}_y dy + \dot{c}_z dz)$$

and hence

$$\frac{d}{dt} (dz(\dot{c})) = \frac{1}{z^3(c(t))} (-|\dot{c}_x|^2 - |\dot{c}_y|^2 + |\dot{c}_z|^2).$$

Combining the above calculations, we obtain

$$f''(t) = \frac{1}{\cosh^3(\tilde{c}(t))} (|\dot{c}_x(t)|^2 + |\dot{c}_y(t)|^2 + |\dot{c}_z(t)|^2) = \frac{1}{\cosh^2(\tilde{c}(t))} |\dot{c}(t)|^2 = \ell^2 f(t) \quad (4.6)$$

where $\ell = |\dot{c}(t)|_{\tilde{c}}$ is the length of the geodesic cord $c$. (Recall that geodesic has constant speed.) In other words, $f$ satisfies

$$f''(t) - \ell^2 f(t) = 0.$$

Its general solution is given by

$$f(t) = a \cosh(\ell t) + b \sinh(\ell t).$$

By $f(0) = a_0$, we obtain $a = a_0$. Therefore we obtain

$$|f(t)| \leq a_0 \cosh(\ell) + |b_0| \sinh(\ell)$$
for all $t \in [0, 1]$ noting that both cosh and sinh are increasing for $t > 0$. Finally we note that
\[ f'(0) = \ell b_0 \]
from the above formula. On the other hand, by definition $f(t) = \frac{1}{z(c(t))}$, we also have
\[ f'(0) = -\frac{1}{z(c(0))} c'(0). \]
From this we also have
\[ |f'(0)| = \frac{|c'(0)|}{z(c(0))^2} \leq \frac{1}{z(c(0))} \|c'(0)\|_h = \frac{\ell}{a_0}. \]
Comparing the two, we obtain
\[ |b_0| \leq \frac{\ell}{a_0}. \]
This finishes the proof. \hfill \Box

This lemma completely determines the $z$-coordinate of any geodesic with the initial data on $c(0), c'(0)$.

**Remark 4.5.** Examination of the proof of Lemma 4.4 shows that the same kind of estimate holds for arbitrary admissible Lagrangian submanifolds, not just for the horo-spheres. See [BKO, Definition 4.1] for the definition of admissible Lagrangian submanifolds.

5. THE KINETIC ENERGY HAMILTONIAN AND HYPERBOLIC GEODESICS

We endow $T^*N$, $N = M \setminus K$ with the canonical symplectic structure
\[ \omega_0 = dq \wedge dp \]
and the kinetic energy Hamiltonian function $H = H_h : T^*N \to \mathbb{R}$ defined by
\[ H_h(q, p) = \frac{1}{2} |p|^2_{h^s}, \]
where $h^s$ is the dual metric of $h_N$. It is well-known, see [KL], the Hamiltonian flow of $H$ is nothing but the geodesic flow which is given by
\[ (t, (q, p)) \mapsto (\exp_q(tp^s), (d\exp_q(tp^s)(p^s))^s) \]

**Definition 5.1.** We denote
\[
\text{Chord}(\nu^*T) := \mathfrak{X}_{C0}(\nu^*T, \nu^*T; H_h) = \{ \gamma : [0, 1] \to T^*N \mid \gamma(t) = X_{H_h}(\gamma(t)), \gamma(0), \gamma(1) \in \nu^*T \text{, non-constant} \}.
\]
We call an element of $\text{Chord}(\nu^*T)$ a (non-constant) *Hamiltonian chord* of $\nu^*T$.

We recall from Definition 3.4 the definition
\[ \text{Cord}(T) = \{ c : [0, 1] \to N \mid \nabla_i \dot{c} = 0, c(i) \in T, \dot{c}(i) \perp T, \text{ for } i = 0, 1 \} \]
of geodesic cords of $T$. We have the following one-one correspondence

**Lemma 5.2.** There is a natural one-one correspondence between $\text{Chord}(\nu^*T)$ and $\text{Cord}(T)$. 
Proof. Let \( c \in \text{Cord}(T) \), then \( \gamma_c := (c, \dot{c}) \) defines a Hamiltonian trajectory of \( H \). Furthermore since \( \dot{c}(0), \dot{c}(1) \perp T \) and \( c(1) = d\exp_q(p^\ast)(p^\ast) \), \( (d\exp_q(p^\ast)(p^\ast))^b \in N^*_{\gamma(1)}T \). Therefore the assignment
\[
\Phi : c \mapsto \gamma_c; \quad \gamma_c(t) := \left( \exp_q(tp^\ast), (d(\exp_q(tp^\ast))(p^\ast))^b \right) = (c_t, \dot{c}_t(t)) \quad (5.1)
\]
defines a map \( \text{Cord}(T) \to \text{Chord}(\nu^*T) \). Conversely, we check that any \( \gamma \in \text{Chord}(\nu^*T) \) can be written as
\[
\gamma = \Phi(c_\gamma); \quad c_\gamma = \pi \circ \gamma.
\]
This finishes the proof. \( \square \)

To utilize the presence of hyperbolic structure on \( N \), we lift the Hamiltonian flow to the universal covering \( T^*\mathbb{H}^3 \) of \( T^*N \).

In order to lift the Hamiltonian function to \( T^*\mathbb{H}^3 \), we first recall a symplectomorphism \( T^*g : T^*\mathbb{H}^3 \to T^*\mathbb{H}^3 \) induced by a diffeomorphism \( g : \mathbb{H}^3 \to \mathbb{H}^3 \) which is an element of the deck transformation group \( \Gamma \). Here the map \( T^*g \) is defined by
\[
T^*g(q,p) = (g(q), ((d_qg)^{-1})^* p)
\]
for any \( v \in T_{(g(q))}\mathbb{H}^3 \). By the choice of metric on the universal cover, \( g : \mathbb{H}^3 \to \mathbb{H}^3 \) is an isometry for any \( q \in \Gamma \) and hence we have \( |(g^{-1})^* p_h| = |p_h| \), where \( h^\ast = h^\ast_{\mathbb{H}^3} \) is the dual metric of \( h_{\mathbb{H}^3} \). For the simplicity of notations, we will suppress \( \ast \) from its notation and just denote by \( h \) for both of them. As a consequence, an induced Hamiltonian function \( \tilde{H} : T^*\mathbb{H}^3 \to \mathbb{R} \);
\[
\tilde{H}(q,p) = \frac{1}{2} |\rho_q^2|
\]
satisfies \( \tilde{H} = \tilde{H} \circ g \) for any \( q \in \Gamma \) which implies \( \tilde{H} = H \circ (T^*p) \), where \( T^*p : T^*\mathbb{H}^3 \to T^*N \) is the map defined by
\[
T^*p(q,p) = (p(q), ((d_qp)^{-1})^* p)
\]
over the covering map \( p : \mathbb{H}^3 \to N \). It defines a well-defined map because \( p \) is a covering map and so \( d_qp \) is invertible for all \( q \in \mathbb{H}^3 \). The map \( T^*p \) itself defines a covering map \( T^*\mathbb{H}^3 \to T^*N \) which covers \( p : \mathbb{H}^3 \to N \). In local coordinates \( (x, y, z, p_x, p_y, p_z) \) for \( T^*\mathbb{H}^3 \), \( \tilde{H} \) and its Hamiltonian vector field \( \tilde{X}_{\tilde{H}} \) are expressed by
\[
\tilde{H} = \frac{1}{2} z^2(p_x^2 + p_y^2 + p_z^2); \quad \tilde{X}_{\tilde{H}} = z^2(p_x \partial_x + p_y \partial_y + p_z \partial_z) - z(p_x^2 + p_y^2 + p_z^2) \partial_p_x
\]
where \( \tilde{X}_{\tilde{H}} \) is defined to satisfy \( \omega_0(X_{\tilde{H}}, \cdot) = d\tilde{H} \). Since \( \tilde{H} \) is given by a kinetic energy, it is well-known that the flow of \( \tilde{X}_{\tilde{H}} \) recovers the geodesic flow on the cotangent bundle.

A direct computation shows that
\[
d\tilde{H} = z(p_x^2 + p_y^2 + p_z^2)dz + z^2(p_x dp_x + p_y dp_y + p_z dp_z)
\]
\[
= z^2(p_x V_1 + p_y V_2 + p_z V_3)
\]
\[
d\tilde{H} \circ \tilde{j} = p_x dx + p_y dy + p_z dz = \tilde{\theta}.
\]
Here we recall that our canonical symplectic structure is \( \omega_0 = -d\theta \).
We will consistently denote by $\tilde{\gamma}$ (resp. $\tilde{c}$) the lifting of $\gamma$ (resp. $c$) to $T^*\mathbb{H}^3$ (resp. $\mathbb{H}^3$) in this paper.

6. Second variation and Morse index calculation

The purpose of the present section is to provide the proofs of two results concerning the relationship between the Maslov index of a Hamiltonian chord $\gamma$ attached to the conormal $\nu^*T$ and the Morse index of the associated geodesic cord $c = \pi \circ \gamma$. For this we first need to explain what we mean by the ‘Morse index’ of the geodesic cord $c$ attached to the base manifold $T \subset N$. We recall the path space on which the Floer theory applied is

$$\Omega(\nu^*T) = \Omega(T^*N, \nu^*T) = \{ \gamma : [0, 1] \to T^*N \mid \gamma(0), \gamma(1) \in \nu^*T\}.$$ 

The projection $c = \pi \circ \gamma$ of each element $\gamma$ automatically satisfies the free boundary condition

$$c(0), c(1) \in T,$$

which leads us to considering the space of paths

$$\mathcal{P}(T) = \mathcal{P}(N, T) = \{ c : [0, 1] \to N \mid c(0), c(1) \in T \}$$

and the energy functional

$$E(c) = \frac{1}{2} \int_0^1 |\dot{c}|_h^2 \, dt$$

restricted thereto. The general first variation formula is given by

$$dE(c)(V) = -\int_0^1 \left\langle V(t), \frac{D}{dt} \frac{dc}{dt} \right\rangle \, dt - \left\langle V(0), \frac{dc}{dt}(0) \right\rangle + \left\langle V(1), \frac{dc}{dt}(1) \right\rangle.$$ (6.1)

In particular, standard variational analysis shows that if $c \in \mathcal{P}(T)$ is a critical point, then it satisfies

$$\begin{cases} 
\frac{D}{dt} \frac{dc}{dt} = 0; \\
(c(0), c(1) \in T; \\
\dot{c}(0) \in \nu_{c(0)}T, \quad \dot{c}(1) \in \nu_{c(1)}T.
\end{cases}$$

Next we recall the general second variation formula. (See [dC] p.199 for such a derivation. We warn the readers that the definition of the curvature operator $R(X,Y)$ in [dC] is the negative of the one used here or in [Sp2, KN].)

Lemma 6.1 (Second variation formula). Let $c : [0, 1] \to (N, g)$ be a geodesic on a Riemannian manifold. Let $C : (-\epsilon, \epsilon) \times [0, 1] \to N$ be a variation of $c$ in $\mathcal{P}(T)$ i.e., a map satisfying

$$C(0, t) = c(t), \quad C(s, \cdot) \in \mathcal{P}(T).$$
Denote \( V(s,t) = \frac{\partial C}{\partial s}(s,t) \) and \( c_s := C(s, \cdot) \). The second variational formula restricted to the path space \( P(N,T) \) at a critical point \( c \) is given by
\[
d^2E(c)(V,V) = -\int_0^1 \left\langle \frac{D^2V}{\partial t^2}(0,t) + R(V(0,t),\dot{c})\dot{c},V \right\rangle \ dt - \left\langle \frac{DV}{\partial t}(0,0), V(0,0) \right\rangle + \left\langle \frac{DV}{\partial t}(0,1), V(0,1) \right\rangle
\]
\[
= \int_0^1 \left\langle \frac{DV}{\partial t}(0,t), \frac{DV}{\partial t}(0,t) \right\rangle - \int_0^1 \left\langle R(V(0,t),\dot{c})\dot{c}, V(0,t) \right\rangle \ dt - \left\langle \frac{DV}{\partial s}(0,0), \dot{c}(0) \right\rangle + \left\langle \frac{DV}{\partial s}(0,1), \dot{c}(1) \right\rangle \tag{6.2}
\]

We would like to emphasize that the general second variation formula allowing the end points to move as in the free boundary problem on \( T \) contains the boundary terms appearing in the last line of (6.2) which is the same as
\[
- \left\langle \frac{D}{\partial s}\left( \frac{\partial C}{\partial s}, \frac{\partial C}{\partial t} \right) \right\rangle (0,0) + \left\langle \frac{D}{\partial s}\left( \frac{\partial C}{\partial s}, \frac{\partial C}{\partial t} \right) \right\rangle (0,1).
\]
(These terms will not appear in (6.2) when the variation is fixed at the end \( t = 0, 1 \). See e.g., [Sp2, p.303] or [dC, Remark 2.10] for such a discussion.)

We remark that positivity of the mean-curvature in Proposition 4.3 is consistent with the first variation of area formula since the area of horo-torus decreases along the outward normal direction.

**Proposition 6.2.** Let \( N \) be a hyperbolic manifold and \( g \) be a hyperbolic metric on it. Consider a horo-torus \( T = b^{-1}(r_0) \). Let \( V \) be a variational vector field along \( c \) on \( P(T) \). Then
\[
d^2E(c)(V,V) = \int_0^1 \left( \left| \frac{DV}{\partial t} \right|^2 + \left| \dot{\dot{c}} \right|^2 \right) \ dt + \left| \dot{c}(0) \right| \left| V(0) \right|^2 + \left| \dot{c}(1) \right| \left| V(1) \right|^2 \ dt \tag{6.3}
\]
for all \( V \in T_c P(T) \) where \( \ell \) is the length of the geodesic \( c \).

**Proof.** We first examine the boundary terms of (6.2). Denote by \( N \) the outward unit normal to \( T \). Then we have \( N(c(0)) = -V/\left| V(0,0) \right| \) and \( N(c(1)) = V/\left| V(0,1) \right| \). Therefore by the definition of the shape operator \( S_N \), we have
\[
- \left\langle \frac{DV}{\partial s}(0,0), \dot{c}(0) \right\rangle = -\left| V(0,0) \right| \left\langle S_N(V), V(0,0) \right\rangle = \left| \dot{c}(0) \right| \left| V(0,0) \right|^2
\]
where the last equality comes from Proposition 4.3. Similarly we obtain
\[
\left\langle \frac{DV}{\partial s}(0,1), \dot{c}(1) \right\rangle = \left| \dot{c}(1) \right| \left| V(0,1) \right|^2.
\]
Therefore we have derived
\[
- \left\langle \frac{DV}{\partial s}(0,0), \dot{c}(0) \right\rangle + \left\langle \frac{DV}{\partial s}(0,1), \dot{c}(1) \right\rangle = \left| \dot{c}(0) \right| \left| V(s,0) \right|^2 + \left| \dot{c}(1) \right| \left| V(s,1) \right|^2.
\]
For the second term of (6.2), since \( g \) has constant curvature \(-1\) and so
\[
R(X,Y)Z = \langle Z,X \rangle Y - \langle Z,Y \rangle X \tag{6.4}
\]
and $V \perp \dot{c} = 0$, it becomes

$$-\int_0^1 \langle R(V, \dot{c}) \dot{c}, V \rangle \, dt = \int_0^1 |\dot{c}|^2 |V|^2 \, dt.$$  

Combining all these, we have derived

$$d^2 E(c)(V,V) = \int_0^1 \left( \left| \frac{DV}{dt} \right|^2 + |\dot{c}|^2 |V|^2 \right) \, dt + |\dot{c}(0)||V(0)|^2 + |\dot{c}(1)||V(1)|^2.$$  

Evaluation of this at $s = 0$ gives rise to (6.3). \hfill \Box

An immediate corollary of Proposition 6.2 is the following vanishing result.

**Theorem 6.3.** Let $N$ and $T$ be as in Proposition 6.2. For any geodesic cord $c \in \text{Cord}(T)$, both nullity and Morse index of $c$ vanish.

**Proof.** Vanishing of index follows from the non-negativity of the second variation (6.3).

Next we examine the nullity of $c$, i.e., the dimension of the set of the Jacobi fields $V$ satisfying

$$\frac{D^2 V}{dt^2} - \ell^2 V = 0, \quad V(0), V(1) \in TT$$  

where $\ell = |\dot{c}|$ is the length of $c$. There is a one-one correspondence between this set and the null space of $d^2 E(c)$. In particular each such $V$ satisfies

$$\int_0^1 \left( \left| \frac{DV}{dt} \right|^2 + \ell^2 |V|^2 \right) \, dt = 0.$$  

Therefore we have $\frac{DV}{dt} = 0 = \ell V$. Therefore any kernel element $V$ is covariant constant and so its initial value $V(0) \in Tc(0)T$ uniquely determines $V$. Since $\ell \neq 0$ in addition, $\ell V = 0$ implies $V = 0$.

Summarizing the above discussion, we have finished the proof. \hfill \Box

7. Perturbed Cauchy-Riemann equation and $A_\infty$ structure

We start with the following general property of Sasakian almost complex structure $J_h$ associated to the Riemannian metric $h$.

**Lemma 7.1.** The almost complex structure $J_h$ is of contact type on $T^*N$, i.e.,

$$(-\theta) \circ J_h = dH_h$$

for the kinetic energy Hamiltonian $H_h = \frac{1}{2} |p|^2_h$.

This is a general fact for any Sasakian almost complex structure on the cotangent bundle $T^*N$ associated to the Riemannian metric $g$ on $N$ and its kinetic energy $H_h = \frac{1}{2} |p|^2_h$. For the simplicity of notation, we will just denote $J = J_h$ if there is no confusion.

Recall the conormals $\nu^*T$ are the main object in the current paper and satisfy the admissible Lagrangian conditions given in [BKO, Definition 4.2]. Noting the vanishing $\theta|_{\nu^*T} = 0$ and from our convention of the action functional, the relevant action functional is given by

$$A_{H_h}(x) = -\int_0^1 x^* \theta + \int_0^1 H_h(x(t)) \, dt$$  

(7.1)
on the path space
\[ \Omega(\nu^* T, T^* N) = \{ x \in [0, 1] \to T^* N \mid x(0), x(1) \in \nu^* T \}. \]
We recall the set of time-one Hamiltonian chords
\[ \mathcal{X}(H_h; \nu^* T, \nu^* T) = \{ \gamma : [0, 1] \to T^* N \mid \gamma(t) = X_{H_h}(\gamma(t)), \text{ and } \gamma(0), \gamma(1) \in \nu^* T \} \]
and denoted \text{Chord}(\nu^* T) the subset of non-constant Hamiltonian chords in Definition 5.1. In addition to Theorem 6.3, it follows from Proposition 2.1 that the action functional \( A_{H_h} : \Omega(T^* N, \nu^* T) \to \mathbb{R} \) associated to the hyperbolic metric \( h \) is nondegenerate in the sense of Bott.

The \( L^2 \)-gradient vector field of \( A_H \) is given by
\[ \text{grad} A_H(\gamma) = -J(\dot{\gamma} - X_{H_h}(\gamma)) \]
and hence the relevant (positive) gradient flow equation is given by (1.1).

The relevant energy \( E(u) \) is defined by
\[ E(u) = \int_{\Sigma} u^* \omega - u^* dH \wedge \beta \leq \int_{\Sigma} u^* \omega - u^* dH \wedge \beta - \int_{\Sigma} u^* H \cdot d\beta \]
(7.2)
\[ = \int_{\Sigma} u^* \omega - d(u^* H \cdot \beta), \] (7.3)
where the inequality in (7.2) comes from \( H \geq 0 \) and sub-closedness of \( \beta \). By the Stoke’s theorem with the fixed Lagrangian boundary condition in (7.7) and \( \beta|_{\partial \Sigma} = 0 \) implies that (7.3) becomes
\[ A(x^0) - \sum_{j=1}^{k} A(x^j). \]

Recalling Proposition 2.1 we consider \( L = \nu_{k,\rho}^* T \) defined in (2.5). Then we consider the reduced wrapped Floer complex
\[ C = \overline{CW}^d(T, M \setminus K) := \overline{CW}^d(L, L; T^*(M \setminus K); H_{g_0}). \]
Then the construction in [BKO] associates an \( A_\infty \) algebra to \( CW_{g_0}(T, M \setminus K) \), whose construction we recall below in the context of \( A_\infty \) algebra. We denote the associated cohomology by
\[ HW_g(T, M \setminus K) = HW(L, L; T^*(M \setminus K); H_{g_0}). \] (7.4)
To define the \( A_\infty \)-maps
\[ m^k : C^{\otimes k} \to C^{[2 - k]}, \]
we need a moduli space of perturbed \( J \)-holomorphic curves with respect to the following Floer data.

Now we adopt the Floer data in [BKO] to the current setup. We first briefly recall the construction of one-form \( \beta \) on the domain to be used in writing down the perturbed Cauchy-Riemann equation.
For each given \( k \geq 1 \), let us consider a Riemann surface \((\Sigma, j)\) of genus zero with \((k + 1)\)-ends. Each end admits a holomorphic embedding

\[
\begin{align*}
\ell^0 : Z_- := \{ \tau \leq 0 \} \times [0, 1] &\to \Sigma; \\
\ell^k : Z_+ := \{ \tau \geq 0 \} \times [0, 1] &\to \Sigma, \quad \text{for } i = 1, \ldots, k
\end{align*}
\]

This is isomorphic to the closed unit disk \( \mathbb{D}^2 \) minus \( k + 1 \) boundary points \( z = \{z^0, \ldots, z^k\} \) in the counterclockwise direction. Suppose that the ends are decorated by a weight datum \( w = \{w^0, \ldots, w^k\} \) satisfying the balancing condition

\[
w^0 = w^1 + \cdots + w^k.
\]

It follows from Proposition 2.1 (3) that we can choose the perturbed conormal \( \nu_{k, \rho}T \) so that the weights \( w^\ast \)'s avoid the set \( \{k\pi | k \in \mathbb{Z}\} \subset \mathbb{R} \) so that we can apply Proposition 2.1 (2) to deform \( \nu^*T \) to \( \nu^*_kT \) so that the pair \((\nu^*_k, H_h)\) is nondegenerate and satisfies Lemma 2.3.

Then the one form \( \beta \) on \( \Sigma \) satisfies the following conditions:

\[
\begin{align*}
d\beta &= 0 \\
d(\beta \circ j) &= 0 \\
 i^*\beta &= 0 \quad \text{for the inclusion } i : \partial \Sigma \to \Sigma \\
(\ell^i)^*\beta &= w^j dt \quad \text{on a subset of } Z_k \text{ where } \pm \tau \gg 0.
\end{align*}
\]

We utilize the slit domain for the construction of \( \beta \). Let us consider domains

\[
\begin{align*}
Z^1 &= \{ \tau + \sqrt{-1} t \in \mathbb{C} \mid \tau \in \mathbb{R}, \ t \in [0, w^1] \}; \\
Z^2 &= \{ \tau + \sqrt{-1} t \in \mathbb{C} \mid \tau \in \mathbb{R}, \ t \in [w^1, w^1 + w^2] \}; \\
& \vdots \\
Z^k &= \{ \tau + \sqrt{-1} t \in \mathbb{C} \mid \tau \in \mathbb{R}, \ t \in [w^1 + \cdots + w^k, w^0] \},
\end{align*}
\]

and its gluing along the inclusions of the following rays

\[
R^\ell = \{ \tau + \sqrt{-1} t \in \mathbb{C} \mid \tau \leq s^\ell, \ t = w^1 + \cdots + w^\ell \}; \\
\quad j^\ell_- : R^\ell \hookrightarrow Z^\ell, \quad j^\ell_+ : R^\ell \hookrightarrow Z^{\ell+1},
\]

for some \( s^\ell \in \mathbb{R} \) and for \( \ell = 1, \ldots, k - 1 \). In other words, the glued domain becomes

\[
Z^w(s) = Z^1 \sqcup Z^2 \sqcup \cdots \sqcup Z^k / \sim,
\]

where \( s = \{s^1, \ldots, s^{k-1}\} \), and \( \zeta \sim \zeta' \) means \((\zeta, \zeta') \in Z^\ell \times Z^{\ell+1} \) and \( \zeta = \zeta' \in R^\ell \) for some \( \ell = 1, \ldots, k - 1 \). We may regards

\[
Z^0 \subset \{ \tau + \sqrt{-1} t \in \mathbb{C} \mid \tau \in \mathbb{R}, \ t \in [0, w^0] \},
\]

with \((k - 1)\)-slits

\[
S^\ell = \{ \tau + \sqrt{-1} t_k \in \mathbb{C} \mid \tau \geq s^\ell, \ t_k = w^1 + \cdots + w^\ell \}
\]

where \( \ell = 1, \ldots, k - 1 \). Then for \( k \geq 2 \) there is a unique conformal mapping

\[
\varphi : \Sigma = \mathbb{D}^2 \setminus \{z^0, \ldots, z^k\} \to Z^w(s)
\]
for some rays \( \{ R^\ell \}_{\ell=1,\ldots,k-1} \) with respect to \( s \) satisfying the following asymptotic conditions

\[
\begin{align*}
\lim_{\tau \to -\infty} \hat{\varphi}^{-1}(\{ \tau \} \times (\sum_{j=1}^{\ell-1} w^j, \sum_{j=1}^{\ell} w^j)) &= z^\ell \quad \text{for } \ell = 1, \ldots, k; \\
\lim_{\tau \to \infty} \hat{\varphi}^{-1}(\{ \tau \} \times (0, w^0)) &= z^0.
\end{align*}
\]

For \( k = 1 \), the conformal mapping is unique modulo \( \tau \)-translations. Let us denote such a slit domain by \( Z^w \) or simply \( Z \) if there is no confusion. Then the one form \( \beta \) is defined to be

\[ \beta = \varphi^* dt \in \Omega^1(\Sigma) \]

with \( dt \in \Omega^1(Z) \).

In the rest of this article, we often identify \( \Sigma \) with the slit domain \( Z = Z^w \) via the conformal map \( \varphi \).

We first recall the standard Floer data used in the construction of wrapped Floer homology in general.

**Definition 7.2** (Floer data for \( A_\infty \) map). A Floer datum \( D_m = D_m(\Sigma, j) \) on a stable disk \( (\Sigma, j) \in \overline{M}^{k+1} \) is the following:

1. **Weights**: \( w = (w^0, \ldots, w^k) \) a \((k+1)\)-tuple of positive real numbers which is assigned to \( z = (z^0, \ldots, z^k) \) satisfying the balancing condition (7.5).
2. **One-form**: \( \beta \in \Omega^1(\Sigma) \) constructed above satisfying \( \ell^j_\beta \) agrees with \( w^j dt \).
3. **Almost complex structure**: A map \( J : \Sigma \to J(T^*N) \) whose pull-back under \( \ell^j \) uniformly converges to \( \psi_{w^j}^*, J_i \) near each \( z^j \) for some \( J_i \in J_1 \).
4. **Vertical moving boundary**: \( \eta \in C^\infty(\Sigma, [1, +\infty)) \) which converges to \( w^j \) near each \( z^j \).

In general we need a domain dependent admissible Hamiltonian datum \( H \) which is uniformly converges to \( \frac{H}{w^j} \circ \psi_{w^j} \) near each \( z^j \) for some admissible Hamiltonian \( H \). Since the kinetic Hamiltonian \( H \) is quadratic, it satisfies \( \frac{H}{w^j} \circ \psi_{w^j} = H \) for any positive \( w^j \), and we take domain independent Hamiltonian \( H \).

Following \[BKO\], we make the following specific choice of domain dependent almost complex structures \( J \) defined by

\[ J(\sigma) = \psi_{\eta(\sigma)}^* J_h \]

at each \( \sigma \in \Sigma \).

Then for a given \((k+1)\)-tuple \((\gamma^0, \gamma^j)\) of Hamiltonian chords in \( \mathfrak{X}_i \), we consider the moduli space \( M(\gamma^0, \gamma^j; D_m(\Sigma, j)) \) of maps \( u : \Sigma \to T^*N \) satisfying

\[
\begin{align*}
(du - X_H \otimes \beta)^{(0,1)}_J &= 0; \\
u(z) &\in \psi_{\eta(z)}(L), \quad \text{for } z \in \partial \Sigma; \\
u \circ \ell^j(-\infty, t) &= \psi_{w^j} \circ \gamma^j(t), \quad \text{for } j = 1, \ldots, k; \\
u \circ \ell^0(\infty, t) &= \psi_{w^0} \circ \gamma^0(t).
\end{align*}
\]

Now we consider a parameterized moduli space

\[ M(\gamma^0, \gamma^j) = \bigcup_{\Sigma, j} M(\gamma^0, \gamma^j; D_m(\Sigma, j)). \]

In order to use the moduli space in the definition on \( A_\infty \) structure maps \( m^k \), we need to overcome compactness and the transversality issue. These are dealt in the coming sections.
Here we recall from [BKO] that the transformation
\[ u \mapsto \psi^{-1} \circ u =: v \] (7.8)
transforms (7.7) into the autonomous equation
\[
\begin{cases}
(dv - X_H \otimes \beta)^{(0,1)}_{J_h} = 0, \\
v(z) \in L, \text{ for } z \in \partial \Sigma \\
u \circ e^t(-\infty, t) = x^j(t), \text{ for } j = 1, \ldots, k. \\
u \circ e^0(\infty, t) = x^0(t).
\end{cases}
\] (7.9)

8. The maximum principle and \( C^0 \)-estimates

8.1. Vertical \( C^0 \)-estimates. For the study of compactness properties of the moduli space of (7.7), the following vertical \( C^0 \)-bound is an essential step in the case of noncompact Lagrangian such as the conormal Lagrangian \( L = \nu^* T \). For given \( \gamma^j \in \text{Chord}(H; L) \) with \( j = 0, \ldots, k \), we define
\[
C(H, L; \{ \gamma^j \}_{0 \leq j \leq k}) := \max_{0 \leq j \leq k} \| p \circ \gamma^j \|_{C^0}
\] (8.1)

Proof of the following proposition is a consequence of the strong maximum principle based on the combination of the following
\begin{enumerate}
  \item \( \rho = r \circ u = e^s \circ u \) with \( r = |p|_h \),
  \item the conormal bundle property of \( L \) and
  \item the special form of the Hamiltonian \( H = \frac{1}{2} r^2 \), which is a radial function.
\end{enumerate}

We refer to [BKO, Proposition 5.3] for the proof.

Proposition 8.1 (Proposition 5.3 [BKO]). Let \( (\gamma^0, \tilde{\gamma}) \) be a \((k + 1)\)-tuple of Hamiltonian chords in \( X_i \). Then
\[
\max_{z \in \Sigma} |p \circ u(z)| \leq C(H, L; (\gamma^0, \tilde{\gamma}))
\] (8.2)
for any solution \( u : \Sigma \to T^* N \) of (7.7).

8.2. Horizontal \( C^0 \) estimates. Because the base \( N \) is non-compact, we also need to study the horizontal behavior of solutions \( u \) of (7.7) for the study of compactness property of its moduli space.

We recall from [BKO] that the way how Proposition 8.1 was proved is to exploit the transformation (7.8) and the autonomous equation (7.7) for \( v \). From now on, we will work with \( v \) instead of \( u \) in the rest of the paper, unless otherwise said.

By considering the lift of \( v \) to a map, still denoted by \( v \), \((\Sigma, \partial \Sigma) \to (\mathbb{H}^3, \tilde{L}) \), it is enough to consider maps to \( \mathbb{H}^3 \) whose study is now in order.

Now we consider \( \rho = z^{-1} \circ v : \Sigma \to \mathbb{R} \) and its (classical) Laplacian \( \Delta \rho \), i.e.,
\[
\Delta \rho = \frac{\partial^2 \rho}{\partial \tau^2} + \frac{\partial^2 \rho}{\partial t^2}
\]
in terms of the flat coordinates \((\tau, t)\) of \( \Sigma \). Recall \( *d \rho = d \rho \circ j \). Therefore the geometric Laplacian \(-\Delta \rho \) satisfies
\[
-\Delta \rho dA = \delta(\rho) = -* d * d \rho = -* d (d \rho \circ j)
\]
and so \( d (d \rho \circ j) = \Delta \rho dA \) with \( dA = dA_h \) is the area form associated to the metric \( h \).
Proposition 8.2. We have
\[ \Delta \rho dA \geq d \rho \wedge (\beta - v^* \theta) \]

Proof. We recall the identity
\[ V^3 = dz^{-1} \circ J \]
\[ X_H = z^2(p_x H_1 + p_y H_2 + p_z H_3) \]
\[ J X_H = -p_x V_1 - p_y V_2 - p_z V_3. \]

Using (7.7), we compute
\[ -d \rho \circ j = -d(z^{-1}) \circ dv \circ j \]
\[ = -d(z^{-1})(J du + \beta \circ j \cdot X_H(v) - \beta \cdot J X_H(v)) \]
\[ = -V^3 \circ dv - \beta \circ j \cdot d(z^{-1})(X_H) \]
\[ = -v^* V^3 + p_z(v) \cdot \beta \circ j. \]

The co-closed property of \( \beta \), i.e., \( d(\beta \circ j) = 0 \) implies
\[ dd^\rho = -v^* dV^3 + d(p_z(v)) \wedge \beta \circ j. \]

Using the formula \( V^3 = dp_z + z^{-1} \theta \), we compute
\[ -d V^3 = -d(z^{-1}) \wedge \theta + z^{-1} \omega. \]

Therefore
\[ -v^* dV^3 = pv^* \omega - d \rho \wedge v^* \theta. \] (8.3)

Next we note
\[ d(p_z(v)) \wedge \beta \circ j = -d(p_z(v)) \circ j \wedge \beta. \]

Then using (7.7) on \( \Sigma \), we compute
\[ d(p_z(v)) \circ j = dp_z \circ dv \circ j \]
\[ = dp_z(J dv + \beta \circ j \cdot X_H - \beta \cdot J X_H) \]
\[ = dp_z(J du) + dp_z(X_H(v)) \beta \circ j - dp_z(J X_H(v)) \beta. \]

But using \( \theta \circ J = -dH \), we have
\[ dp_z(J du) = (dp_z \circ J)(dv) = (-d(z^{-1}) - z^{-1} \theta \circ J)(dv) = d \rho - pv^* dH \]

and
\[ dp_z(X_H(v)) = -z(v)(p_x^2 + p_y^2 + p_z^2)(v) = -2 \rho H(v). \]

Combining the above, we have derived
\[ \Delta \rho dA = -d(d^\rho \circ j) =pv^* \omega - d \rho \wedge v^* \theta + d \rho \wedge \beta - pv^* dH \wedge \beta + 2 \rho H(v)(\beta \circ j) \wedge \beta \]
\[ = \rho(v^* \omega - v^* dH \wedge \beta) + 2 \rho H(v) \cdot (\beta \circ j) \wedge \beta + d \rho \wedge (-v^* \theta + \beta). \]

We note that the second form is clearly nonnegative since
\[ (\beta \circ j) \wedge \beta = (\beta_x^2 + \beta_y^2) \partial s \wedge \partial t \]

We now prove the first form is also nonnegative.

Lemma 8.3.
\[ (v^* \omega - v^* dH \wedge \beta) \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = \left| \frac{\partial v}{\partial t} - \beta_t X_H(v) \right|^2 \geq 0. \]
Proof. We evaluate the form against the pair \((\frac{\partial}{\partial \tau}, \frac{\partial}{\partial t})\). A straightforward calculation using (7.7) then gives rise to the inequality. \(\Box\)

Therefore combining the above, we have finished the proof of
\[
\Delta \rho \, dA \geq d\rho \wedge (\beta - v^* \theta).
\]
\(\Box\)

Now we fix an exhaustion sequence of
\[
N = M \setminus K \subset N_1 \subset N_2 \subset \cdots \subset N_i \subset \cdots \hspace{1cm} (8.4)
\]
with \(\partial N_i = T_i\) a horo-torus.

We are now ready to establish the following uniform horizontal \(C^0\) estimates.

**Theorem 8.4.** Let \(L = \nu^* T\) be a horo-torus and \(\ell > 0\) be given real number. Suppose \(\{\gamma^a\}_{a=0}^k\) is a k-tuple of Hamiltonian chords of \(L\) with
\[
-A(\gamma^a) = E(c^a) < \frac{\ell^2}{2}.
\]
Then for any solution \(u\) of (7.7), there exists constant \(j = j(T, \ell)\) independent of \(u\) depending only on \(L\) and \(\ell\) such that
\[
\text{Image } \pi \circ u \subset N_j \hspace{1cm} (8.5)
\]

Proof. Consider the lifts \(\tilde{L}, \tilde{\gamma}^a\) of \(L\) and of \(\gamma^a\) respectively. Using the inequality \(\Delta \rho \, dA \geq d\rho \wedge (\beta - v^* \theta)\) we will apply the maximum principle for the function \(\rho\). First of all this inequality enables us to apply the interior maximum principle and so its supremum must occur either on \(\partial \Sigma\) or along the asymptotic chords \(\gamma^a\) for some \(a = 0, \ldots, k\).

We introduce
\[
C := \max \left\{ z^{-1}(x) \big| x \in \tilde{L} \cup \bigcup_{a=0}^k \tilde{\gamma}^a \right\}. \hspace{1cm} (8.6)
\]
Along the boundary \(\partial \Sigma\), we recall \(\beta|_{\partial \Sigma} = 0\). Furthermore we have
\[
v^* \theta(\frac{\partial}{\partial \tau}) = \theta \left( \frac{\partial v}{\partial \tau} \right)
\]
which vanishes because \(\frac{\partial v}{\partial \tau}\) is tangent to \(\nu^* T\) and \(\theta|_{\nu^* T} \equiv 0\). Therefore we can apply the strong maximum principle and so the maximum \(\rho\) cannot be achieved on \(\partial \Sigma\) either. Therefore the supremum of \(\rho\) must be achieved at some point of \(\bigcup_{a=0}^k \tilde{\gamma}^a\). This proves \(\rho \leq C\).

On the other hand, we also derive the uniform upper bound of \(C\) from Lemma 4.4
\[
C \leq \max_{t \in [0,1]} |f(t)| \leq a_0 \cosh \ell + |b_0| \sinh \ell
\]
where \(a_0\) depends only on \(L\) and \(|b_0|\) depends only on \((T, \ell)\). By noting \(\pi \circ u = \pi \circ v\) and translating this bound on \(\tilde{u}\) to that of \(u\), we have proved that there exists \(j = j(T, \ell) > i\) such that
\[
\text{Image } \pi \circ u = \text{Image } \pi \circ v \subset N_j
\]
for all finite energy solution \(u\) with \(\{\gamma^a\}_{a=0}^k\) as its asymptotic chords and with boundary condition on \(L\). \(\Box\)
It is easy to see that all these $C^0$ estimates can be established for the Lagrangian boundary conditions given by the $k + 1$ tuple of admissible test Lagrangians $(L^0, L^1, \cdots, L^k)$.

(See Remark 4.5 of this paper.)

9. Formality of $A_\infty$ algebra associated to hyperbolic knot

The $C^0$ estimates established in the previous section enables us to directly construct a version of wrapped Fukaya category, denoted by $\mathcal{WF}(M \setminus K, H_h)$ without taking a cylindrical adjustment of $h$ unlike in [BKO].

9.1. $A_\infty$ algebra associated to hyperbolic knot.

Let $L = \nu^*T$ as before and $H = H_h$ be the kinetic energy Hamiltonian associated to the hyperbolic metric $h$. We now consider the conormal $\nu^*T$ of the hori-torus $T$ in $M \setminus K$ as an object in this category.

Then the definition in Section 2 applied to the metric $h$ instead of $g_0$ associates an $A_\infty$ algebra

$$CW_h(T, M \setminus K) := C^*(T) \oplus \mathbb{Z}\langle x_{< -\epsilon_0}(H_h; \nu^*T, \nu^*T) \rangle.$$

We take the perturbed conormal $L = \nu^*_\rho T$ and denote

$$CW^d(L; H_h) = \bigoplus_{x \in \text{Chord}^d(L; H_h)} \mathbb{Z} \cdot x.$$

Here the grading $d$ is given by the grading of the Hamiltonian chords $|x|$. We denote its wrapped Floer cohomology by

$$HW^d(L; H_h).$$

We can also define the reduced Floer chain complex as in Section 2 which we denote by $\tilde{CW}(L; H_h)$. This is the complex generated by the set $\text{Chord}^d_{< \epsilon_0}(\nu^*T; H_h)$ consisting of non-constant Hamiltonian chords. With this mentioned, we will directly work with $\nu^*T$ without taking its perturbation.

In this subsection, we establish a formality result for the complex $\tilde{CW}(\nu^*T; H_h)$.

For given asymptotic data

$$\begin{cases}
x = x^1 \otimes \cdots \otimes x^k \in CW(\nu^*T; H_h)^{\otimes k} \\
x^0 \in CW(L; H_h),
\end{cases}$$

consider the moduli space $\mathcal{M}^{k+1}(x^0; x)$ of maps

$$u : (\Sigma, \partial \Sigma) \to (T^*N, \nu^*T)$$

satisfying the condition [7.7].

Then the map $\tilde{m}^k : CW(\nu^*T; H_h)^{\otimes k} \to \tilde{CW}(\nu^*T; H_h)[2 - k]$ is defined by

$$\tilde{m}^k(x) = \sum_{x^0} |\mathcal{M}^{k+1}(x^0; x)| \cdot x^0,$$

where the sum runs over $x^0 \in \text{Chord}(H; \nu^*T)$ satisfying

$$|x^0| = \sum_{j=1}^k |x^j| + 2 - k,$$
and $|M^{k+1}(x^0; x)|$ denotes the algebraic count of points in the oriented compact 0-dimensional manifold $M^{k+1}(x^0; x)$. As usual, $[d]$ means the grading shifting of a graded module down by $d \in \mathbb{Z}$.

Now the $C^0$ estimates established in Section 7 enables us to study compactified moduli space $M_{k+1}(x^0; x)$ and the standard Fredholm theory proves that the compactified moduli is a smooth manifold with boundary and corners (after making a $C^\infty$-small perturbation of $J_h$, if needed).

Then Theorem 6.3 implies that all the degrees of generators $x^i$ are 0 and so the above dimension formula reduces to $2-k$. Since all the nontrivial matrix coefficients are given by zero dimensional moduli space, only the case of $k = 2$, i.e. $\tilde{m}^k = 0$ for all $k \neq 2$.

Combining the above discussion, we have proved the following theorem. By taking an arbitrarily small tubular neighborhood $N(K)$ of $K$ and setting $T = \partial N(K)$, we may regard it as the ‘ideal boundary’ that appears in the title of the present article.

**Theorem 9.1.** Suppose that $K \subset M$ is a hyperbolic knot and $h$ be its associated hyperbolic metric $h$. Let $\nu^*T$ be the conormal of any horo-torus $T \subset M \setminus K$. The $A_\infty$ structure of $(CW(\nu^*T; H_h), \{\tilde{m}^k\}_{k=1}^{\infty})$ is reduced to an associative algebra whose product is given by $m^2$, i.e., it satisfies $\tilde{m}^k = 0$ unless $k = 2$. In particular $\tilde{HW}_d^d(\nu^*T; H_h) = 0$ for all $d > 0$ and $\tilde{HW}_0^0(\nu^*T; H_h)$ is a free abelian group generated by $\mathcal{G}_{M \setminus K}$. In particular, the rank of $\tilde{HW}_0^0(\nu^*T; H_h)$ is infinity.

We would like to describe the product structure of this algebra in terms of the hyperbolic geometry of the complement $M \setminus K$. In the next section, as the first step towards this goal, we will prove some reduction theorem. A full study of Conjecture 1.5 will be a subject of future study.

**9.2. A step towards comparison with Knot Floer Algebra.** Let $L = \nu_{k,p}^*T$ be the perturbed conormal given in Section 2. We assume

$$\text{supp } k \subset N_0 \subset N$$

so that $L = \nu_{k,p}^*T$ for the region given by $(p, q)$ satisfying

$$q \in N \setminus N_0, \quad |p| \geq 3\|dk\|_{C^0}. \tag{9.2}$$

As the first step towards a comparison result between the wrapped Floer homology $HW^d(L; H_h)$ and the Knot Floer algebra $HW(\partial_\infty(M \setminus K))$, we consider a sequence of cylindrical adjustments $h_j$ of the given metric $h$ associated to the exhaustion (8.4): $h_j$ is defined by

$$h_j = \begin{cases} h & \text{on } N'_j \\ da^2 + h|_{N_j} & \text{on } (M \setminus K) \setminus N_j \end{cases}$$

where $N'_j$ is another subdomain of $N_j$ such that $N'_j \subset N_j$. (See Section 10 for precise details on this definition.)

Utilizing the $C^0$ bound given in Proposition 8.2 and similar bound for $h_i$ obtained in [BKO], we obtain the $A_\infty$ algebras $CW(\nu^*T, h_i)$. Then we consider any pair $i, j$ with $i \leq j$. Note the

$$H_{h_j} \geq H_{h_i} \tag{9.3}$$
for \( j \geq i \), since \( h_j \leq h_i \). We consider a homotopy \( s \mapsto H^s \) associated to the metrics

\[
h^s_{ij} = (1 - s)h_i + sh_j
\]

with \( H^s = H_{h^s_{ij}} \). By the monotonicity (9.3), we have an \( A_\infty \) homomorphism

\[
t_{ij} : CW^d(L, H_i) \to CW^d(L, H_j)
\]

for \( i \leq j \) which induces a (homotopy) direct system

\[
CW^d(L, H_{h^s_{ij}}) \to CW^d(L, H_{h^s_{i+1,j}}) \to \cdots \to CW^d(L, H_{h^s_{ij}}) \to \cdots.
\]

(9.4) (See [FOOO, Section 7.2.12], [Se] for a detailed explanation of such a procedure of taking the limit \( A_\infty \) structures and homomorphisms.)

Next we prove the following existence result on the continuation map. We would like to emphasize that while the vertical \( C^0 \)-estimate for the homotopy of Hamiltonians in the direction of monotonically increasing direction, e.g., for the homotopy \( s \mapsto (1 - s)H_{h^s_i} + sH_h \) can be established and well-known (see [FH], [AS] for example), the horizontal \( C^0 \)-estimate is new.

**Remark 9.2.** The \( C^0 \)-estimates obtained in Subsection 8.2 and in [BKO, Section 11] dealt with the autonomous cases for the hyperbolic metric and for the cylindrical metric respectively. However none of them apply to the current case since we need to establish the horizontal \( C^0 \)-estimate for the non-autonomous equation. The proof will clearly exhibit that existence of such a continuation morphism strongly relies on the direction of the homotopy in terms of the relevant Hamiltonians. This horizontal \( C^0 \)-bound will follow from the elliptic estimates (e.g., [GT, Theorem 3.7]) and the vertical \( C^0 \)-bound established in Proposition 8.1.

**Proposition 9.3.** There exists a natural monotonicity \( A_\infty \) morphism

\[
t_{jh} : CW^d(L, H_{h_j}) \to CW^d(L, H_h)
\]

induced by the linear homotopy \( s \mapsto (1 - s)h_j + sh \).

**Proof.** We consider the non-autonomous version of (7.9) associated to by the linear homotopy \( s \mapsto (1 - s)h_j + sh \) which becomes

\[
\begin{cases}
\frac{\partial v}{\partial \tau} + Jx \left( \frac{\partial v}{\partial \tau} - X_{H^s}(v) \right) = 0 \\
v(z) \in L, \text{ for } z \in \partial \Sigma \\
v \circ e^j(\infty, t) = x^j(t), \text{ for } j = 1, \ldots, k.
\end{cases}
\]

(9.5)

Again it remains to ensure the \( C^0 \)-estimate hold. As mentioned above, the vertical \( C^0 \)-estimate is standard and so omitted. We will focus on the horizontal \( C^0 \)-estimate.

First, we mention that by (9.2) and the remark right afterwards we may just work with the conormal \( \nu^aT \) instead of \( L \). Then, to establish the horizontal \( C^0 \)-estimate, we will follow the approach taken in [BKO] by decomposing the equation into the vertical and the horizontal components in terms of the cylindrical coordinates \((a, x, y)\) and its associated canonical coordinates \((a, x, y, p_a, p_x, p_y)\) on \( T^*(M \setminus K) \) on \( N^{\text{end}}_i \cong [0, \infty) \times T \).

Recall from (3.3) that the function \(-a\) is the lift of the Busemann function on \( N(K) \setminus K \) with \( z = e^a \). The hyperbolic metrics \( h \) on \( N'(K) \) is written in terms of
\( (z, x, y) \) and \((a, x, y)\) as follows,

\[
\begin{align*}
\mathbf{h} &= z^{-2}(dz^2 + dx^2 + dy^2) \\
&= da^2 + e^{-2a}dx^2 + e^{-2a}dy^2
\end{align*}
\]

(9.6)

From this, we consider the exhaustion \(N_i\) such that \(N \setminus N_i = a^{-1}((i, \infty))\) and their cylindrical adjustments \(h_i\)'s for \(i \geq 1\) of \(h\)

\[
\begin{align*}
h_i &= \begin{cases} h & \text{on } N_{i-1/2} \\ da^2 + e^{-2a}dx^2 + e^{-2a}dy^2 & \text{on } N \setminus N_i. \end{cases}
\end{align*}
\]

(9.7)

Then

\[
H_h = \frac{1}{2} \left( p_a^2 + e^{2a}(p_x^2 + p_y^2) \right), \quad H_{h_i} = \frac{1}{2} \left( p_a^2 + e^{2i}(p_x^2 + p_y^2) \right).
\]

Therefore we have

\[
\begin{align*}
\pi T^*[0, \infty)(X_{Hx(\tau)}(a, x, y)) &= p_a \frac{\partial}{\partial a}, \\
\pi T^*[0, \infty)(J_{Hx(\tau)}(a, x, y)) &= (e^{-2i} + \chi(\tau)(e^{-2a} - e^{-2i})) \frac{\partial}{\partial p_a}.
\end{align*}
\]

Recalling \(\beta = dt\), we compute the \((a, p_a)\)-component of (9.5)

\[
\begin{align*}
\left\{ \frac{\partial a(v)}{\partial \tau} - \frac{\partial p_a(v)}{\partial t} = 0, \\
\frac{\partial p_a(v)}{\partial \tau} + \frac{\partial a(v)}{\partial t} - (e^{-2i} + \chi(\tau)(e^{-2a} - e^{-2i})) p_a(v) = 0 \right\}
\end{align*}
\]

(9.8)

A straightforward calculation using these identities proves

**Lemma 9.4.** For any one-form \(\beta\) on \(\Sigma\), we have

\[
\Delta(a(v)) = \frac{\partial (a(v))}{\partial \tau} + \chi(\tau)e^{-2a} \frac{\partial a(v)}{\partial t} - (e^{-2i} + \chi(\tau)(e^{-2a} - e^{-2i})) \frac{\partial p_a}{\partial t}
\]

(9.9)

on \(a^{-1}((-\infty, a_0]) \subset \Sigma\) for any solution \(v\) of (9.5).

Now we define a function \(f : \Sigma \to \mathbb{R}\) by

\[
f(x, y) = (e^{-2i} + \chi(\tau)(e^{-2a} - e^{-2i})) \frac{\partial p_a}{\partial t},
\]

and rewrite the equation (9.9) into

\[
Lv = f, \quad L = \Delta + \frac{\partial}{\partial \tau}
\]

(9.10)

where \(L\) is a uniformly elliptic second-order partial differential operator.

We note

\[
\|f\|_{C^0} \leq \|p_a\|_{C^0} \left\| \frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y} \right\|_{C^0}.
\]

Therefore combining the vertical bound from Proposition 8.1, we obtain a constant \(C_0 > 0\) independent of \(u\) such that

\[
\|f\|_{C^0} \leq C_0
\]
where $C$ depends only on $\{L^i\}$ and $\|d\beta\|_{C^0}$. Then by applying the classical elliptic estimate (see [GT, Theorem 3.7] for example) to $\pm a(v)$ respectively, we prove
\[
\|a(v)\|_{C^0} \leq \|a(u)\|_{C^0} + C_1\|f\|_{C^0} \leq a_0 + C_1C_0
\]
applied on the domain $\Omega = v^{-1}([a_0, \infty)) \subset \Sigma$.

Therefore we can find $n = n(\ell)$ such that the image of $u$ is contained in $W_n(\ell)$ since its asymptotic chords are also assumed to be contained in $W_\ell$.

Once we have this uniform $C^0$-estimate established, construction of the continuation map proceeds as usual. This finishes the proof of the proposition.

The following lemma then immediately follows from the standard construction of the Floer theory once we have Proposition 9.3 in our disposal.

**Lemma 9.5.** The homomorphism $\iota_{jh} : CW^d(L, H_{h_j}) \to CW^d(L, H_h)$ is compatible with the above direct system, i.e., that satisfies
\[
\iota_{ih} \sim \iota_{ij} \circ \iota_{jh}
\]
for all $i \leq j$, where $\sim$ denotes 'being homotopic relative to the ends'.

This induces a natural $A_\infty$ map
\[
\iota_\infty : \lim_{\to} CW^d(L, H_{h_j}) \to CW^d(L, H_h)
\]
(9.11)
which in turn induces a homomorphism
\[
(\iota_\infty)_* : \lim_{\to} HW^d(L, H_{h_j}) \to HW^d(L, H_h).
\]
In fact we have the following extension of Theorem 6.3 to the cylindrical adjustment $h_i$ of $h$.

**Proposition 9.6.** Let $T \subset M \setminus K$ as above. Then we can find $h_i$ so that it has non-positive curvature, i.e., all sectional curvature $K(X,Y) := \langle R(X,Y)Y, X \rangle \leq 0$. In particular for any geodesic cord $c \in \text{Cord}_{h_i}(T)$, both Morse index and nullity of $c$ vanish.

**Proof.** Let us consider a cylindrical adjustment $h_i$ of $h$ of the form
\[
h_i = da^2 + \rho_i^2(dx^2 + dy^2)
\]
with an interpolated function
\[
\rho_i(a) := \begin{cases} 
  e^{-a} & \text{for } a < i - \frac{1}{2} \\
  e^{-i} & \text{for } a \geq i.
\end{cases}
\]
(9.12)

A straightforward calculation gives rise to the following covariant derivatives of the Levi-Civita connection of $h_i$:
\[
\nabla_{\partial_x} \partial_x = \nabla_{\partial_x} \partial_y = \nabla_{\partial_y} \partial_x = 0,
\]
\[
\nabla_{\partial_x} \partial_x = \nabla_{\partial_y} \partial_x = -\rho_i \partial_x,
\]
\[

abla_{\partial_x} \partial_y = \nabla_{\partial_x} \partial_x = \frac{\rho_i}{\rho_x} \partial_y \quad \text{for } * = x, y.
\]

By direct computation, we obtain
\[
K(\partial_x, \partial_y) = -\left( \frac{\rho_i'}{\rho_x} \right)^2, \quad K(\partial_x, \partial_x) = K(\partial_y, \partial_y) = -\frac{\rho_y''}{\rho_x}.
\]
(9.13)
Now we are going to construct $\rho_i$ satisfying (9.12) explicitly. First, let us consider a smooth cut-off function

$$\tau_{0,1}(t) := \begin{cases} 
1 & \text{for } t \leq 0 \\
e^{t/4}/(t+1) & \text{for } 0 < t < 1 \\
0 & \text{for } t \geq 1
\end{cases}$$

and define $\tau_{a,b} := \tau_{0,1}(\frac{t-a}{b-a})$ for a given $a < b$. Then

$$\tau_{a,b}(t) = \begin{cases} 
1 & \text{for } t \leq a \\
0 & \text{for } t \geq b.
\end{cases}$$

In particular, we have the following inequality when $0 < b - a < 1$,

$$0 \geq \tau_{a,b}^2 - \tau_{a,b} \geq \tau_{a,b}'.$$  

(9.14)

Next we consider a smooth function

$$E(t) = 1 + e^{-1/t} \quad \text{for } t > 0$$

and check

$$E^2 \geq E \geq E' \quad \text{for } t > 0.$$  

(9.15)

Now we define a smooth function

$$A_{i,\varepsilon}(t) := \begin{cases} 
1 & \text{for } t \leq i - \frac{1}{2}, \\
E(t - i + \frac{1}{2}) & \text{for } i - \frac{1}{2} < t \leq i - \varepsilon, \\
E(t - i + \frac{1}{2})\tau_{i-\varepsilon, i}(t), & \text{for } i - \varepsilon < t.
\end{cases}$$

Then we can find $0 < \varepsilon_0 < 1$ such that $\int_0^i A_{i,\varepsilon_0}(t)dt = i$ because $\int_0^1 A_{i,1}(t)dt < i$ and $\int_0^i A_{i,0}(t)dt > i$.

Using this function, we define a smooth function

$$B_i(a) := \int_0^a A_{i,\varepsilon_0}dt,$$

which satisfies

$$B_i(a) = \begin{cases} 
a & \text{for } a \leq i - \frac{1}{2}, \\
i & \text{for } a \geq i.
\end{cases}$$

Finally we obtain an interpolated function

$$\rho_i(a) := e^{-B_i(a)}$$

satisfying (9.12) where the second derivative is

$$\rho_i''(a) = (B'_i(a)^2 - B''_i(a))e^{-B_i(a)}.$$  

We can check $B'_i - B''_i = A^2_{i,\varepsilon_0} - A'_{i,\varepsilon_0} \geq 0$ by the inequality (9.14) and (9.15). The non-negativity of the second derivative of $\rho_i$ completes the proof with (9.13). \hfill \square

One immediate consequence of this proposition is the following formality

**Corollary 9.7.** The boundary map $m^i : CW(\nu^*T, H_{h_i}) \to CW(\nu^*T, H_{h_i})$ is zero. Therefore $CW(\nu^*T, H_{h_i}) = \ker m^i$ and the natural map

$$CW(\nu^*T, H_{h_i}) \to HW(\nu^*T, H_{h_i})$$

is an isomorphism.
Therefore we will freely regard the chain map (9.1) as its homological version for the discussion below.

**Theorem 9.8.** The homomorphism $(\iota_\infty)_*$ is an isomorphism for all integer $d \geq 0$.

**Proof.** We first prove surjectivity of the map. Let $a \in HW^d(L, H_h)$ be given. By the formality of $CF(\nu^*T; H_h)$, there exists a unique cycle

$$\alpha = \sum_{m=1}^{k} n_m \gamma_m \in CF(\nu^*T; H_h)$$

representing $a$ with $\delta_m \in \mathcal{G}_M \setminus K$. Here $\gamma_m \in \text{Chord}(\nu^*T)$ is the Hamiltonian chord of $\nu^*T$ associated to the tame geodesic $\delta_m$ via the one-one correspondence established by Proposition 3.6 and Lemma 5.2.

We denote by $c_m = (\delta_m)(T)$ the unique geodesic cord of $T$ corresponding to $\delta_m$ given by Proposition 3.6. Denote $N_0 = \max\{\text{leng}(c_m)\}$. Then it follows from the $C^0$ estimate, Theorem 8.4, that there exists a sufficiently large $i_0 = i_0(N_0)$ such that

$$\text{supp } c_m \subset N_{i_0}, \quad m = 1, \ldots, k$$

and so the chain

$$\sum_{m=1}^{k} n_m \gamma_c c_m$$

becomes a cycle in $CF(\nu^*T; H_h)$ for all $i \geq i_0$. We denote the resulting cycle by $\alpha_i$ for each $i \geq i_0$. Furthermore since $h_i \equiv h$ on $N_{i_0}$, we also have $\iota_{i_0}(\alpha_i) = \alpha$, for all $i \geq i_0$ by the Floer theory construction of $\iota_{i_0}$ and the definition of $\alpha_i$. Then obviously they satisfy the compatibility relation

$$\alpha_{i+1} = \iota_{i+1}(\alpha_i)$$

for all $i \geq i_0$. We extend the sequence $\alpha_i$ all the way up to $i = 1$ by choosing any cycle $\alpha_i$ that satisfies

$$\alpha_{i_0} = \iota_{i_0}(\alpha_i)$$

for $1 \leq i \leq i_0$. Such a cycle $\alpha_i$ always exists because each $\iota_{ji_0}$ is a quasi-isomorphism by the following general lemma (or rather from its proof in Appendix).

**Lemma 9.9.** Let $g, g'$ be any metric on $M \setminus K$ with cylindrical ends. Then we have a quasi-isomorphism

$$CW(L, H_g) \cong CW(L, H_{g'})$$

Furthermore such cycle $\alpha_i$ is unique by the formality given in Proposition 9.6.

Then by construction the sequence $\{\alpha_i\}_{i=1}^{\infty}$ is a compatible sequence of cycles and hence defines an element in $\lim \rightarrow HW^d(L, H_{h_i})$. By construction, we have

$$(\iota_\infty)_*([\alpha_1 \rightarrow \alpha_2 \rightarrow \cdots]) = a$$

which proves surjectivity.
For the proof of injectivity, we recall the energy identity for the continuation equation:

\[
\int \left| \frac{\partial u}{\partial \tau} \right|^2 \, dt \, d\tau = A_{H^+}(z^+) - A_{H^-}(z^-) - \int_{-\infty}^{\infty} \chi'(\tau) \left( \left. \frac{1}{\partial s} \frac{\partial H_s}{\partial s} \right|_{s=\chi(\tau)} (u(\tau,t)) \right) \, d\tau. \tag{9.16}
\]

(See [BKO] (7.15) for the energy formula for the non-autonomous equation in the convention used in the present paper.) Due to the fact \( \partial H_s / \partial s \geq 0 \), we have

\[
A_{H^+}(\gamma_{c^+}) \geq \int \left| \frac{\partial u}{\partial \tau} \right|^2 \, dt \, d\tau + A_{H^-}(\gamma_{c^-})
\]

and so \( A_{H^+}(\gamma_{c^+}) \geq A_{H^-}(\gamma_{c^-}) \). The last inequality is equivalent to

\[
E(c^+) \leq E(c^-)
\]

by (1.5). This bound for \( A_{H^+}(\gamma_{c^+}) \) in particular implies that there are only finitely many such \( c^+ \) for each given \( c^- \).

With this preparation, we proceed with the proof of injectivity. Suppose \( (\iota_\infty)_*(b_\infty) = 0 \). Represent \( b_\infty \in \lim_{\to} HW^d(L, H_{h_j}) \) by a sequence of \( b_j \in HW^d(L, H_{h_j}) \) so that

\[
b_\infty = [b_1 \to b_2 \to b_3 \to \cdots \to b_j \to \cdots].
\]

Using Proposition 9.6, we will also regard \( b_i \) as a chain (cycle) abusing notation. We denote the unique cycle representing \( b_i \) by \( \beta_i \).

Then by the compatibility of the sequence and the formality of \( CW^d(L, H_k) \), we have

\[
0 = \iota_{1k}(\beta_1) = \iota_{2k}(\beta_2) = \cdots
\]

as a chain. Out of this, we will derive \( \beta_i = 0 \) for all sufficiently large \( i \).

We first note that each geodesic \( c_{\ell} \) of \( H_k \) is also a geodesic cord of \( h_j \) whenever \( \text{supp} \, c_{\ell} \subset N_j \) since \( h_j = h \) on \( N_j \). We denote by \( I_j \) the finite subset of \( \mathbb{Z}_+ \) for such \( \ell \)'s and by \( \delta_{s_{\ell}} \) the Hamiltonian chords of \( H_j \) corresponding to such geodesic cord of \( h_j \). In particular we mention that the set \( \{\delta_{s_{\ell}}\}_{\ell \in I_j} \) is linearly independent as chains in \( CW(L, H_{h_j}) \) for each \( j \geq i_1 \).

**Lemma 9.10.** Let \( i_0 \) be given. Express

\[
\iota_{i_0 h}(\beta_{i_0}) = \sum_{\ell \in I} a_{\ell} \delta_{c_{\ell}} \in CW(L; H_k) \quad \tag{9.18}
\]

for a finite index set \( I \subset \mathbb{Z}_+ \) with \( a_\ell \neq 0 \) for \( \ell \in I \). Then there exists \( i_1 > i_0 \) such that

\[
\iota_{i_0 j}(\beta_{i_0}) = \sum_{\ell \in I} a_{\ell} \delta_{s_{k_\ell}}
\]

for all \( j \geq i_1 \).

**Proof.** Since \( h_{i_0} \) is cylindrical outside \( N_{i_0} \), any Hamiltonian chords of \( \nu^* T \) cannot touch the cylindrical region \( N \setminus N_{i_0} \). (See [BKO] Proposition 4.3.) Therefore we can choose a cycle \( \beta_{i_0} \in CW(L, H_{h_{i_0}}) \) representing \( b_{i_0} \) with \( \text{supp} \, \beta_{i_0} \subset \text{Int} W_{i_0} \).
On $W_{i_0}$, we have $H_h = H_{h_{i_0+1}}$. It follows from (9.17) that for all $j \geq i_0$ we have the uniform bound
\[
E(c^+) \leq \ell(\beta_{i_0}) := \max\left\{ E(c^\ell) \mid \beta_{i_0} = \sum_{\ell=1}^n b_\ell \delta_{c^\ell}, \ b_\ell \neq 0 \right\}
\]
for any $c^+ \in \text{Cord}_{h_{i_0}}(T)$ appearing at $\tau = \infty$ for some solution $u$ of (1.1) satisfying $u(\infty) \in \beta_{i_0}$. Since $\text{supp} \beta_{i_0} \subset W_{i_0}$, this energy bound for such $c^+$ forces $\text{supp} c^+$ to be contained in $W_{i_0}$ for a sufficiently large $i_0$ for any $j \geq i_0 + 1$: This is because $\nu^* T \subset W_1$ and $\text{dist}(T, M \setminus N_j) \to \infty$ as $j \to \infty$.

We recall $H_h = H_{h_{i_0}}$ on $W_{i_0}$ for any $j \geq i_0 + 1$ and so $\delta_{i_0}$ for $\ell \in I$ appearing in (9.18) can be also regarded as a cycle of $\text{CW}(L; H_h)$. Then it follows from the uniform energy bound (9.16) that the image of any solution $u$ contributing to the continuation maps $\iota_{i_0 h}$ is contained in $W_{i_1}$ for sufficiently large $i_1$ depending only on $i_0$, $i_0$ but independent of $j \geq i_1 + 1$.

Since $H_h = H_{h_{i_1}}$ on $H_{i_1}$ and so the Cauchy-Riemann equations to solve for the construction of the continuation maps $\iota_{i_0 h}$, $\iota_{i_0 j}$ on $W_{i_1}$ are exactly the same equation, we obtain
\[
\iota_{i_0 j}(\beta_{i_0}) = \sum_{\ell \in I} a_\ell \delta'_{c_\ell}
\]
by the construction of the continuation maps $\iota_{i_0 h}$, $\iota_{i_0 j}$ established in Proposition 9.3. This finishes the proof. \(\square\)

Therefore using this lemma, the formality and the hypothesis $(t_\infty) \ast (b_\infty) = 0$, we derive
\[
0 = \iota_{i_0 h}(\beta_{i_1}) = \iota_{i_0 i_1}(\beta_{i_1})
\]
and so $(\iota_{i_0 i_1}) \ast (b_{i_0}) = 0$. Since $(\iota_{i_0 i_1}) \ast$ is an isomorphism by Lemma 9.9, this implies $b_{i_0} = 0$. Since this holds for $b_{i_0}$ for all given $i_0$, this finishes the proof of injectivity. Hence we have proved that $(t_\infty) \ast$ is an isomorphism. \(\square\)

10. Comparison with Knot Floer cohomology

In this section we would like to compare the Knot Floer algebra constructed in [BKO] with the algebra constructed in Theorem 9.8 directly on $T^* (M \setminus K)$ for $L = \nu^* T$ for a horo-torus $T$ using the hyperbolic metric.

10.1. Comparison between $HW(\nu^* T; H_h)$ and $HW(\partial_{\infty} (M \setminus K))$. The main goal of this section is to study the relationship between the algebra $HW(\nu^* T; H_h)$ constructed in the previous sections via the hyperbolic metric $h$ and the knot Floer algebra $HW(\partial_{\infty} (M \setminus K))$.

For this purpose, we first choose the exhaustion sequence
\[
N_1 \subset N_2 \subset \cdots \subset N_i \subset \cdots
\]
so that its boundary $\partial N_i$ is a horo-torus $\partial N_i = T_i$ with respect to the given hyperbolic metric $h$ on $M \setminus K$. Then we take $W_i = T^* N_i$.

We now express the metric $h$ on $M \setminus K$ in the cylindrical representation
\[
N_{(0, \epsilon)} \cong [0, \infty) \times T\epsilon
\]
with coordinates $(a, q)$ with $q \in T\epsilon$.

Then we prove the following that each $h_i$ is also a cylindrical adjustment of a smooth metric on $M$ restricted to $M \setminus K$. More precisely, we have
Proposition 10.1. We can construct a sequence of smooth metrics $g[i]$ defined on $M$ such that the followings hold:

1. For all $i \geq 1$,
   \[ h_i = g[i]_0 \]  \hspace{1cm} (10.3)
   where $g[i]_0$ is the cylindrical adjustment outside $N_i$.

2. For all $1 \leq i \leq k$,
   \[ g[i]_0 \geq g[k]_0. \]  \hspace{1cm} (10.4)

Proof. Let $N = M \setminus K$ be a hyperbolic knot complement and $N(K) = N_x(K)$ be a sufficient small tubular neighborhood. Let $N_x := N \setminus N(K)$ be the compact thick part and $N'(K)$ be the deleted neighborhood $N(K) \setminus K$. The statement of the proposition is a (uniform) pointwise statement. Therefore we will lift the relevant metrics on $N$ to its universal covering space $\mathbb{H}^3$.

Denote by $p: \mathbb{H}^3 \to M \setminus K$ a universal covering map and put the lifting of the horo-torus $T = \partial N(K)$ to be the horo-sphere of $\{z = 1\}$. Consider a holonomy representation $\rho$ for the hyperbolic structure of $N$. Since $N$ is a knot complement, there is a canonical choice of two generators $m$ and $l$ of $H_1(T)$, called meridian and longitude. Note that $l$ is a closed curve parallel to $K$ and $m$ bounds a disk in the tubular neighborhood $N(K)$.

We can choose an explicit holonomy representative $\rho$ in the conjugacy class $[\rho]$ such that $\rho(m)$ and $\rho(l)$ preserve the horosphere $\{z = 1\}$ and $\rho(m)$ is a translation of only $x$-direction, as follows.

\[ \rho(m) = (x, y) \mapsto (x + m_1, y + m_2) \quad \text{with} \quad m_2 = 0 \]  \hspace{1cm} (10.5)
\[ \rho(l) = (x, y) \mapsto (x + l_1, y + l_2) \]

Here $(m_1, m_2)$ and $(l_1, l_2)$ are linearly independent vectors in the $xy$-plane of $\mathbb{H}^3$, which are determined by the complete hyperbolic structure of $M \setminus K$, the choice of a horo-torus $T$ and the choice of a holonomy $\rho^3$.

Now we have an explicit coordinate of $p^{-1}(N'(K))$ in $\mathbb{H}^3$ as follows.

\[ \{(z, x, y) | z \geq 1 \text{ and } x = \mu m_1 + \lambda l_1, y = \lambda l_2 \text{ for } 0 \leq \mu, \lambda < 1\} \]  \hspace{1cm} (10.6)

Recall that $N'(K)$ is obtained from $p^{-1}(N''(K))$ identified by the holonomies $\rho(m)$ and $\rho(l)$ and hence the coordinate of (10.6) itself can be regarded as a global coordinate of $N'(K)$.

Now we consider cylindrical adjustments. Recall the hyperbolic metrics $h$ on $N'(K)$ is written as

\[ h = z^{-2}(dz^2 + dx^2 + dy^2) = da^2 + e^{-2a}da^2 + e^{-2a}dy^2 \]  \hspace{1cm} (10.7)

in terms of $(z, x, y)$ and $(a, x, y)$, and their cylindrical adjustments $h_i$'s for $i \geq 1$ of $h$ are given by

\[ h_i = \begin{cases} h & \text{on } N_{i-1/2} \\ da^2 + e^{-2i}da^2 + dy^2 & \text{on } N \setminus N_i \end{cases} \]  \hspace{1cm} (10.8)

with respect to the exhaustion $N_i$ such that $N \setminus N_i = a^{-1}((i, \infty))$. We remark that \( h \leq h_i \) for all $i \geq 1$.  \hspace{1cm} (10.9)

\[ \text{The complex number of } \frac{1}{m_1} (l_1 + \sqrt{-1}l_2) \text{ is a hyperbolic knot invariant, called cusp shape.} \]

For the details, there are many texts on hyperbolic knots. For instance, see [M] Section 11,13,14.
Now we consider a metric \( g[i] \) smooth on \( M \) whose cylindrical adjustment coincides with \( h_i \) as follows.

\[
g[i] = \begin{cases} 
  h_i & \text{on } N_i \\
  e^{-2\pi} da^2 + e^{-2\pi} dx^2 + e^{-2\pi} dy^2 & \text{on } N \setminus N_{i+1/4}
\end{cases}
\]  

(10.10)

**Lemma 10.2.** The above metric \( g[i] \) extends smoothly to \( N(K) \), i.e. it is a smooth metric of the ambient manifold \( M \).

**Proof.** To consider smoothly extended metrics on \( N'(K) \), let us look at a standard solid torus \( ST \subset \mathbb{R}^3 \),

\[
ST := D^2 \times S^1 = \{(\cos \varphi, \sin \varphi, 0) + \{r \cos \theta \cos \varphi, r \cos \theta \sin \varphi, r \sin \theta}\} \subset \mathbb{R}^3
\]

with \( 0 \leq r \leq r_0, \; 0 \leq \theta, \varphi < 2\pi \) and a standard polar coordinate as follows,

\[
ST = \{(r, \theta, \varphi) \mid 0 \leq r \leq r_0, \; 0 \leq \theta, \varphi < 2\pi\}.
\]

Here, we also consider a deleted solid torus \( ST' := ST \setminus \{r = 0\} \). We construct an explicit diffeomorphism \( \Phi \) between \((z, x, y)\) of \( N'(K) \) and \((r, \theta, \varphi)\) of \( ST' \),

\[
(z, x, y) = \Phi(r, \theta, \varphi) := \left(\frac{1}{r}, \frac{m_1}{2\pi} \theta + \frac{l_1}{2\pi} \varphi, \frac{l_2}{2\pi} \varphi\right)
\]  

(10.11)

Next, we consider a pullback of the metric \( g[i] \) by \( \Phi \) on \( ST' \),

\[
\Phi^* g[i] = \begin{cases} 
  h_i & \text{on } N_i \\
  \frac{r^2 (m_i^2 \, d\theta^2 + l_i^2 \, d\varphi^2) + e^{-2l_i^2} \, d\varphi^2}{4\pi^2} & \text{on } N \setminus N_{i+1/4}
\end{cases}
\]  

Recall that the standard smooth metric for \( ST \) is given by \( dr^2 + r^2 d\theta^2 + d\varphi^2 \). We can directly verify that \( \Phi^* g[i] \) is also smooth at \( r = 0 \) on \( ST \).

Note that the construction of (10.6) shows that \( \Phi \) between \( N(K)' \) and \( ST' \) extends continuously to a homeomorphism between \( N(K) \) and \( ST \). Therefore the metric completion of \( N'(K) \) with \( g[i] \) also recovers the original \( N(K) \) and it proves the claim. \( \square \)

Now, we can say that a smooth metric \( g[i] \) on \( M \) and a complete hyperbolic metric \( h \) of \( M \setminus K \) has a common cylindrical adjustment \( h_i \).

Moreover, as comparing (10.8) and (10.10) we have an inequality

\[
h_i \geq g[i] \quad \text{for all } i \geq 1.
\]  

(10.12)

Next, let us define cylindrical adjustments \( g[i]_j \) of \( g[i] \) for \( j \geq 0 \) as follows

\[
g[i]_j = \begin{cases} 
  g[i] & \text{on } N_{i+j-1/2} \\
  da^2 + e^{-2i+j} \, dx^2 + e^{-2i+j} \, dy^2 & \text{on } N \setminus N_{i+j}
\end{cases}
\]  

(10.13)

Note that we can put \( g[i]_0 = h_i \). By the direct comparing (10.10) and (10.13), we obtain

\[
g[i]_0 \geq g[k]_0
\]  

(10.14)

for all \( 1 \leq i \leq k \). This finishes the proof. \( \square \)

Now we are ready to prove the following comparison theorem.
**Theorem 10.3.** Suppose $K$ is a hyperbolic knot on $M$. Then we have an (algebra) isomorphism

$$HW^d(\nu^*T; H_h) \cong HW^d(\partial_{\infty}(M \setminus K))$$

for all integer $d \geq 0$. In particular $HW^d(\partial_{\infty}(M \setminus K)) = 0$ for all $d > 0$ and $HW^0(\partial_{\infty}(M \setminus K))$ is a free abelian group generated by $\mathcal{G}(M \setminus K)$.

**Proof.** Let $h_i$ be a cylindrical adjustment of $h$ given in the proof of Theorem 9.8. We first observe

$$h_j \leq h_i \quad \text{for all } j > i. \quad (10.15)$$

Denote by $g[i]$ the smooth metric constructed in the proof of Proposition 10.1 associated to $h_i$, and $g[i]_j$ its cylindrical adjustment with $g[i]_0 = h_i$ associated to $N_i \subset N_{i+1} \subset \cdots$.

Using the monotonicity inequality (10.4), we can find a sequence of metrics

$$g[1]_0 \geq g[2]_0 \cdots \geq g[i]_0 \geq \cdots$$

Then

$$CW(\nu^*T; H_{g[i]_0}) = CW(\nu^*T; H_{h_i})$$

whose cohomology satisfies

$$HW(\nu^*T; H_{g[i]_0}) \cong HW(\partial_{\infty}(M \setminus K)) \quad (10.16)$$

for all $i$ by the definition of the latter in [BKO]. Furthermore we have the monotonicity $A_{\infty}$ homomorphism

$$CW(\nu^*T; H_{g[i]_0}) \rightarrow CW(\nu^*T; H_{g[j]_0}) \quad (10.17)$$

and obtain an $A_{\infty}$ homomorphism (see [FOOO] Section 7.2.12 as before), and hence a homomorphism

$$(\varphi_{ij})_* : HW(\nu^*T; H_{g[i]_0}) \rightarrow HW(\nu^*T; H_{g[j]_0}).$$

(In fact, this is an algebra homomorphism but the property will not be used in the present paper.)

We recall from Proposition 10.1 that $h_i = g[i]_0$ and so $H(g[i]_0) = H(h_i)$ for all $i$. Therefore we have the monotonicity $A_{\infty}$ homomorphism (10.17) coincides with

$$\iota_{ij} : CW(\nu^*T; H_{h_i}) \rightarrow CW(\nu^*T; H_{h_j})$$

since $g[i]_0 = h_i$ by construction of $g[i]$ in Proposition 10.1.

On the other hand, [BKO] Proposition 10.5 and Lemma 9.9 respectively imply that the maps

$$(\iota_{ij})_* = (\varphi_{ij})_* : HW(\nu^*T; H_{g[i]_0}) \rightarrow HW(\nu^*T; H_{g[j]_0})$$

are isomorphisms for all $i \leq j$. Furthermore

$$\lim_{\rightarrow} HW(\nu^*T; H_{h_i}) \cong HW(\nu^*T; H_h)$$

by Theorem 9.8. Therefore the following lemma will finish the proof of the theorem.

**Lemma 10.4.** The canonical map

$$HW(\nu^*T; H_{h_k}) \rightarrow \lim_{\rightarrow} HW(\nu^*T; H_{h_i})$$

is an isomorphism for all $k$. 

Proof. Let $k = i_0$ be fixed. We start with the proof of injectivity. Let $a_{i_0} \in \text{HW}(\nu^* T; H_{h_i})$ be an element satisfying $\iota_i(a_{i_0}) = 0$. In other words, $\iota_i(a_{i_0})$ can be represented by a sequence $a_i \in \text{HW}(\nu^* T; H_{h_i})$ satisfying

$$ (\iota_{i_0})_* (a_{i_0}) = 0 $$

for sufficiently large $j \geq i_0$. Since $(\iota_{i_0})_*$ is an isomorphism, we have $a_{i_0} = 0$. This proves injectivity.

For the surjectivity, let $a \in \lim_{\to} \text{HW}(\nu^* T; H_{h_i})$ be given and let $\{a_i\}$ be a representative thereof. It is enough to show $(\iota_{i_0})_* (a_{i_0}) = a$, i.e., we have

$$ a_j = (\varphi_j)_{i_0} (a_{i_0}) $$

for all sufficiently large $j \geq i_0$. By the energy estimate (9.17), $(\varphi_j)_{i_0} (a_{i_0})$ is eventually stable, i.e., there exists some $i_1 > i_0$ such that

$$ (\varphi_j)_{i_0} (a_{i_0}) = (\varphi_{i_1 j})_{i_0} (a_{i_0}) $$

for all $j \geq i_1$ similarly as in Lemma 9.10.

On the other hand, by compatibility of $\{a_i\}$, we also have

$$ a_j \sim (\iota_{i_0 j})_* (a_{i_0}) $$

and in turn $a_j = (\iota_{i_0 j})_* (a_{i_0})$ for all $j \geq i_0$ by formality. Combining the above discussion, we have shown $a = (\iota_{i_0})_* (a_{i_0})$. Since $k = i_0$ is arbitrarily given, the surjectivity for folds for all $k$ as required. This finished the proof. □

Combining the above, we have finished the proof of Theorem 10.3. □

Combination of Theorem 9.8 and 10.3 gives rise to the following

Corollary 10.5. For any hyperbolic knot $K \subset M$, $\tilde{\text{HW}}^d (\partial_\infty (M \setminus K)) = 0$ for all $d \geq 1$ and the rank of $\text{HW}^0 (\partial_\infty (M \setminus K))$ is infinity. In particular if $\text{HW}^d (\partial_\infty (M \setminus K)) \neq 0$ for some $d \geq 1$, the knot cannot be hyperbolic.

We would like to point out that the knot Floer algebra $\text{HW}(\partial_\infty (M \setminus K))$ is defined for arbitrary knots, while $\text{HW}(L; H_h)$ in the comparison result of Theorem 10.3 should have a description in terms of the hyperbolic geometry of $M \setminus K$. The upshot of Theorem 10.3 is that it enables us to compute the topological invariant $\text{HW}(\partial_\infty (M \setminus K))$ in terms of the hyperbolic geometry of the complement $M \setminus K$ for the case of hyperbolic knots.

11. Reduction of the classification problem to 2 dimension

We note that when all the asymptotic chords are non-constant, solutions for (7.7) exist only for $k + 1 = 3$ by the degree reason: the degree of the $A_\infty$ maps, which is given by $2 - k$, must be zero by Theorem 6.3. Therefore it remains to examine the case $k + 1 = 3$.

We recall the transformation $u \mapsto \psi_\eta^{-1} \circ u =: v$ for which $v$ satisfies the autonomous equation (7.9).
11.1. Rewriting of the Cauchy-Riemann equation lifted to $\mathbb{H}^3$. We consider the lifting $\tilde{\nu} : \Sigma \to T^*\mathbb{H}^3$ of $\nu : \Sigma \to T^*N$ to $\mathbb{H}^3$. With slight abuse of notation, we also denote $\tilde{\nu}$ just by $\nu$ as long as there is no danger of confusion. Then we compute the coordinate expression of the equation of (7.9) lifted to $\mathbb{H}^3$.

Now let $(q, p)$ be the canonical coordinates of $T^*\mathbb{H}^3$ induced by the standard coordinates $q = (x, y, z)$ of $\mathbb{H}^3 \subset \mathbb{C}^3$, and by $p = (p_x, p_y, p_z)$ the associated fiber coordinates. We decompose the derivative $\nu$, a $T(T^*\mathbb{H}^3)$-valued one-form, into the horizontal and vertical components

$$d\nu = d_H\nu + d_V\nu.$$ 

Here

$$d_H\nu = d(x \circ \nu) \otimes H_1(\nu) + d(y \circ \nu) \otimes H_2(\nu) + d(z \circ \nu) \otimes H_3(\nu);$$

$$d_V\nu = \nabla(p_x \circ \nu) \otimes V_1(\nu) + \nabla(p_y \circ \nu) \otimes V_2(\nu) + \nabla(p_z \circ \nu) \otimes V_3(\nu).$$

More explicitly for each $i = 1, 2, 3$, we have

$$\nabla p_i = dp_i - \sum_{j=1}^{3} \sum_{k=1}^{3} p_k \Gamma^k_{ij} dq^j$$

where $q^1 = x$, $q^2 = y$, $q^3 = z$.

Recall that the Levi-Civita connections of $g$ induces the splitting

$$T_{(q, p)}(T^*\mathbb{H}^3) \cong T_q^0\mathbb{H}^3 \oplus T^*_q\mathbb{H}^3$$  \hspace{1cm} (11.1)$$

at each point $(q, p) \in T^*\mathbb{H}^3$. Also recall the lowering and raising operators with respect to $g$ by

$$\flat : T\mathbb{H}^3 \to T^*\mathbb{H}^3, \quad \flat : T^*\mathbb{H}^3 \to T\mathbb{H}^3$$

where $\flat(X) = \langle X, \cdot \rangle_h$ and $\flat$ is its inverse. We may regard $\flat, \sharp$ as operations on $T(T^*\mathbb{H}^3)$ with respect to (11.1). The Sasakian almost complex structure $J_h$ is given by

$$J_h(X) = X^\flat, \quad J_h(\alpha) = -\alpha^\sharp$$

for $X \in T\mathbb{H}^3$ and $\alpha \in T^*\mathbb{H}^3$, respectively, under the identification (11.1).

Here $\nabla p_x, \nabla p_y, \nabla p_z$ are nothing but the coefficients of the covariant derivative of the one-form

$$\alpha = p \circ \nu = p_x(\nu) \, dx|_f + p_y(\nu) \, dy|_f + p_z(\nu) \, dz|_f.$$ 

Here $\alpha$ is considered as a section of $f^*(T^*\mathbb{H}^3) \to \Sigma$ along the map $f := \pi \circ \nu : \Sigma \to \mathbb{H}^3$. In other words, we have

$$\nabla \alpha = \nabla p_x(\nu) \, dx|_f + \nabla p_y(\nu) \, dy|_f + \nabla p_z(\nu) \, dz|_f.$$

We now derive the following coordinate expression of (7.9) for the map $\nu$.

**Lemma 11.1.** Let $J = \psi_{\nu(t, t)}^*(H)$ and let $u$ be a solution to the equation $(du - X_H \otimes \beta)^{(0,1)} = 0$. Let $v$ be the map associated to $u$ as above. Then the coordinate expression of (7.9) with respect to the frame fields $\{\partial_\tau, \partial_i\}$ and $\{H_i, V_j\}_{i, j = 1, 2, 3}$ is given by

$$dq^i(\partial_\tau v) - z^2 \nabla p_i(\partial_\tau v) = 0 \quad \text{for } i = 1, 2, 3;$$

$$\nabla p_i(\partial_\tau v) + \frac{1}{z^2} dq^i(\partial_\tau v) - p_i \circ v = 0 \quad \text{for } i = 1, 2, 3.$$
Proof. Using the above given frame fields, we compute the coordinate expression of \((dv - X_H \otimes \beta_j^{(0,1)}) = 0\) as
\[
(dx(\partial_r v), dy(\partial_r v), dz(\partial_r v), \nabla p_x(\partial_r v), \nabla p_y(\partial_r v), \nabla p_z(\partial_r v)) + J_g(dx(\partial_t v), dy(\partial_t v), dz(\partial_t v), \nabla p_x(\partial_t v), \nabla p_y(\partial_t v), \nabla p_z(\partial_t v))
\]
and deduce the following coordinate expression of \((\ref{eqn:7.9})\)
\[
\begin{align*}
&dq^i(\partial_r v) - z^2 \nabla p_i(\partial_r v) = 0 \quad \text{for } i = 1, 2, 3; \\
&\nabla p_i(\partial_r v) + \frac{1}{z^2} dq^i(\partial_t v) - p_i \circ v = 0 \quad \text{for } i = 1, 2, 3,
\end{align*}
\]
where \(q^1 = x, \ q^2 = y, \ q^3 = z\) and \(p_1 = p_x, \ p_2 = p_y, \ p_3 = p_z\).

The coordinate expression \((\ref{eqn:11.2})\) then admits the following coordinate-free expression with boundary condition:
\[
\begin{align*}
&df - (\nabla \alpha \circ j + \alpha \cdot dt)^2 = 0 \\
&f(z) \in S, \quad \alpha \in \nu_{f(z)} S \quad \text{for } z \in \partial \Sigma,
\end{align*}
\]
where \(S\) is the union of horo-spheres in \(\mathbb{H}^3\) which is the lift of the torus \(T\) in \(N\) to the universal cover. We emphasize that this equation is written purely in terms of the data \((f, \eta)\) which defined in terms of the data of the pull-back bundle \(f^* (T^* N)\) over the base map \(f : \Sigma \rightarrow N\).

11.2. Reduction to 2 dimensional hyperbolic plane \(\mathbb{H}^2\). In this section we will provide a complete description of the set of solutions of \((\ref{eqn:7.7})\) exploiting the following theorem for the equation \((\ref{eqn:11.3})\) on the thrice punctured discs. We will do this first by applying some elementary hyperbolic geometry on \(\mathbb{H}^3\) and reducing the study to the 2 dimensional case \(\mathbb{H}^2\).

**Lemma 11.2.** Let \(N\) be a complete hyperbolic 3-manifold with one cusp and \(T\) is a horo-torus near the cusp. Consider three geodesic cords \(c^0, c^1, c^2\) perpendicular to \(T\). Let \(X\) be a hexagonal domain with edges labelled by \(a^0, b^0, a^1, b^1, a^2, b^2\) counterclockwise. Suppose that there is a continuous map
\[
f : X \rightarrow N,
\]
satisfying \(f(a^j) = c^j\) and \(f(b^j) \subset T\) for \(j = 0, 1, 2\), then the lifted three geodesic cords \(\overline{c}^j \subset \mathbb{H}^3\) are coplanar, i.e. there is an action \(g \in \text{PSL}(2, \mathbb{C})\) sending all \(\overline{c}^j\) into a hyperbolic plane \(\{x = 0\}\).

**Proof.** Consider a lifting of \(f\) into the universal cover \(\mathbb{H}^3\),
\[
\tilde{f} : X \rightarrow \mathbb{H}^3
\]
with the lifted boundary curves denoted by \(\tilde{f}(a^j) =: \tilde{A}^j\) and \(\tilde{f}(b^j) =: \tilde{B}^j\). Each lifted peripheral curve \(\tilde{B}^j\) is contained in a horo-sphere, let say, \(S^j \subset \mathbb{H}^3\) which is a connected component of the lift of \(T\). Then \(\tilde{A}^j\) and \(\tilde{A}^{j+1}\) are geodesics perpendicular to \(S^j\) for \(j = 0, 1, 2 \mod 3\). Because of the fact that all inward geodesics perpendicular to a horosphere goes to the same ideal point called the center of the horosphere, we have an ideal triangle \(\Delta \subset \mathbb{H}^3\) where each ideal vertex is the center of a horosphere \(S^j\). Therefore all \(\overline{c}^j\) are contained in the boundary of a triangle \(\Delta\) and hence coplanar. \(\square\)
The following is again derived using the strong maximum principle applied to the function \( \varphi = \phi \circ v \) with \( \phi(x, y, z) = x/z \).

**Theorem 11.3.** Let \((\gamma^0, \gamma^1, \gamma^2)\) be a triple of three Hamiltonian chords that admit a solution \( v : (\Sigma, \partial \Sigma) \to (T^*\mathbb{H}^3, S) \) to the perturbed \( J \)-holomorphic equation \((7.7)\) where \( \gamma^i \) is the Hamiltonian chord whose projection \( \tilde{\gamma}^i := \pi \circ \gamma^i \) is the geodesic whose image is contained in the plane \( \{ x = 0 \} \). Then the image of \( v \) is also contained in the plane \( \{ x = 0 \} \).

**Proof.** Let us first recall the construction of the Lagrangian \( L \) which is a conormal lifting of \( T \). Since \( T \) is a level set of the Busemann function \( -\log z \),

\[
L := v^*T = \{(x, y, z_0, 0, 0, p_z)\} \subset T^*\mathbb{H}^3
\]

for some \( z_0 \in \mathbb{R}^+ \). Also note that

\[
T_{(q,p)}L = \langle H_1, H_2, V_3 \rangle_{(q,p)} \subset T_{(q,p)}T^*\mathbb{H}^3. \tag{11.4}
\]

For the purpose of classification problem of holomorphic triangles, we will control the behavior of \( w := \pi \circ v : \Sigma \to \mathbb{H}^3 \) in the \( x \)-direction on \( \mathbb{H}^3 \). As the first try, it is natural to attempt to apply it to the coordinate function \( x \) itself. It turns out that this obvious choice of \( x \) does not lead to a favorable formula for the Laplacian of \( x \) in an application of the maximum principle unlike the case of coordinate function \( z \) (or rather \( 1/z \)). What turns out to the right choice is the quotient \( x/z \). With the \( C^0 \) bound of the \( z \)-coordinate away from \( z = 0 \) or \( z = \infty \) already established, the bound of \( x \)-coordinate is equivalent to a bound of \( \frac{x}{z} \) which turns out to be the right quantity to look at for the application of maximum (or strong maximum) principle for the proof of the theorem.

**Proposition 11.4.** Let \( v \) be any solution of \((7.7)\). Define \( \varphi = \phi \circ v \) with \( \phi = x/z \) on \( \mathbb{H}^2 \). Then

\[
\Delta \varphi \, dA = \varphi(v^*\omega - v^*dH \wedge \beta) + 2\varphi H(v)(\beta \circ j) \wedge \beta + d\varphi \wedge (\beta - v^*\theta). \tag{11.5}
\]

Since the precise calculation of \( \Delta(\varphi) \) is rather involved but straightforward, we postpone its derivation till Appendix \( \text{A} \).

We recall from Lemma \( 8.3 \) that the two form \( v^*\omega - v^*dH \wedge \beta \) is a nonnegative form. Therefore the maximum principle (resp. the minimum principle), especially the strong maximum principle, at critical points of positive value (resp. of negative value) applies by the similar arguments used in the proof of Theorem \( 8.4 \).

Because of the asymptotic condition and thanks to the bound on the \( z \)-coordinate \((8.2)\), the maximum (or the minimum) of the function \( (x/z) \circ v : \Sigma \to \mathbb{R} \) is achieved at a point in \( \Sigma \). The interior maximum (and minimum) principle is already done by the equation \((11.5)\), and hence the maximum (or minimum) is achieved on a boundary point in \( \partial \Sigma \). Again, thanks to the bound on the \( z \)-coordinate \((8.2)\), this also implies that the maximum (or the minimum) of the function \( x \circ v : \Sigma \to \mathbb{R} \) is achieved.

Now suppose the maximum of \( z/x \) is achieved at \( m_0 \in \partial \Sigma \) and the maximum value is positive. We choose a complex coordinate \( s + it \) on a neighborhood \( U \) of \( m_0 \) such that

\[
\Sigma \cap U \subset \mathbb{R} + i\mathbb{R} \geq 0 \subset \mathbb{C} \quad \partial \Sigma \cap U \subset \mathbb{R} + i \cdot 0 \subset \mathbb{C}
\]
By the Lagrangian boundary condition \( \tau \mapsto v(\tau + i \cdot 0) \) defines a curve on \( L = \nu^*T \) and

\[
x \circ v |_{\partial \Sigma \cap U} : \mathbb{R} \to \mathbb{R}
\]

\[
s \mapsto x(v(\tau + i \cdot 0))
\]

has a maximum at \( \tau_0 \) where \( m_0 = \tau_0 + i \cdot 0 \). In particular, we have

\[
dx(\partial_{\tau} v(m_0)) = 0
\]

i.e. \( \partial_{\tau} v(m_0) \in \ker H^1_{v(m_0)} \cap T_{v(m_0)} L \). Then \([11.4]\) implies \( \partial_{\tau} v(m_0) \in \langle H_2, V_3 \rangle \) and hence

\[
J \partial_{\tau} v(m_0) \subset (V_2, H_3) v(m_0) \subset \ker dx_v(m_0).
\]

On the other hand, \( J \)-holomorphic equation \([7.7]\) with \( \beta_x = 0 \) on \( \partial \Sigma \) implies

\[
\partial_{\tau} v(m_0) + J \partial_t v(m_0) = \beta_t(m_0) J X_H(v(m_0)).
\]

and so

\[
\partial_{\tau} v(m_0) = J \partial_{\tau} v(m_0) + \beta_t(m_0) X_H(v(m_0)).
\]

Here we recall \( X_H = z^2 (p_3 H_1 + p_2 H_2 + p_1 H_3) \) and \( v(m_0) \in L = \{(x, y, z, 0, 0, p_z)\} \) which implies \( X_H(v(m_0)) \in \langle H_3 \rangle \). Since \( J \partial_x v(m_0) \in \langle J H_2, J V_3 \rangle = \langle V_2, H_3 \rangle \), we have

\[
\partial_{\tau} v(m_0) \in \langle V_2, H_3 \rangle.
\]

Now we claim that \( z \circ v(\sigma) \leq z_0 \) for any \( \sigma \in \Sigma \cap U \). Suppose not, then we may assume that a (local) maximum is achieved at \( \sigma_0 \in \hat{U} \). This assumption is possible because of \( C^1 \)-estimate for the base coordinates \( x, y, z \).

Let us consider an isometry

\[
\psi : \mathbb{H}^3 \to \mathbb{H}^3
\]

that restricts to \( (x_0, y_0, z) \mapsto (x_0, y_0, z^{-1}) \) on the line \( \ell = \{(x, y, z) \mid x = x_0, y = y_0\} \) with \( \sigma_0 = (x_0, y_0, z_0) \) and the induced symplectomorphism

\[
T^*(\psi)^{-1} : T^* \mathbb{H}^3 \to T^* \mathbb{H}^3.
\]

Then \( w := T^*(\psi^{-1}) \circ v : \Sigma \to T^* \mathbb{H}^3 \) again satisfies the \( J \)-holomorphic equation with shifted asymptotic and boundary conditions with respect to \( T^*(\psi^{-1}) \). It is easy to see that \( \psi \circ w : \Sigma \cap U \to \mathbb{R} \) attains its (local) maximum at the interior point \( \sigma_0 \in \hat{U} \).

This is not possible by the estimate in Subsection \([8.2]\). Since \( \partial_t v \in \text{ker} \ T_{m_0}(\Sigma \cap U) \) is an inner normal direction, the above claim

\[
z \circ v(\sigma) \leq z_0 \quad \text{for any } \sigma \in \Sigma \cap U
\]

implies that \( dz(\partial_t v(m_0)) \leq 0 \).

On the other hand, \( \partial_t v(m_0) \in \langle V_2, H_3 \rangle \) and \( dz(\partial_t v(m_0)) \leq 0 \) implies

\[
d\phi(\partial_t v(m_0)) = \frac{zdx - xdz}{z^2} (\partial_t v(m_0)) = - \frac{x}{z} dz(\partial_t v(m_0)) \quad \text{(because } \partial_t v(m_0) \text{ has no } H_1 \text{-factor) } \geq 0.
\]

However this contradicts to the strong maximum principle \( \frac{\partial^2}{\partial r^2} > 0 \) where \( \frac{\partial}{\partial r} \) is the (outward) normal derivative, since \( \frac{\partial}{\partial r} = - \frac{\partial}{\partial z} \) noticing that \( \{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \} \) has the same orientation on \( \Sigma \).

This proves that \( \varphi \) cannot have a boundary maximum point either unless \( \varphi \equiv 0 \) which implies \( \text{Im} \ v \subset \text{Zero}(x/z) = \text{Zero}(x) \). Since \( \lim_{\zeta \to z} x(v(\zeta)) = 0 \), the latter must be the case which finishes the proof.
Based on Theorem 11.3 we examine the case of $\mathbb{H}^2$. An immediate consequence thereof is that for the given triple of geodesics prescribed here with the given boundary horo-circles there is a parametrization $w : \Sigma \rightarrow \mathbb{H}^2$ whose image is the given domain bounded by the triple of geodesics and three horo-circles. We denote this latter domain by $\Theta$. We also denote by $\tilde{\Theta}$ the subset of $\Theta$ with the three boundary geodesics removed.

![Figure 1. Truncated triangle in $\mathbb{H}^2$](image)

In summary, for a given a solution $v : (\Sigma, \partial \Sigma) \rightarrow (T^*\mathbb{H}^3, S)$ satisfying the relevant conditions, its image projected to $\mathbb{H}^2$ is a truncated geodesic triangle $\Theta$. Conjecture 1.5 would be the converse to this theorem, i.e., there should exist a unique solution $v$ for each given truncated geodesic triangle $\Theta$. We leave the study of this conjecture elsewhere but we mention that it is enough to consider the problem in 2-dimensional hyperbolic space $\mathbb{H}^2$: Let $\mathbb{H}^2$ be the hyperbolic two plane with coordinates $(y, z)$ equipped with the metric

$$h_{\mathbb{H}^2} = \frac{dy^2 + dz^2}{z^2}.$$  

It is isometrically embedded into $\mathbb{H}^3$ via the map $(y, z) \rightarrow (0, y, z)$ whose image is totally geodesic with respect to the metric $h_{\mathbb{H}^3}$. This in turn induces a totally geodesic almost Kähler embedding $T^*\mathbb{H}^2 \rightarrow T^*\mathbb{H}^3$ with respect to the Sasakian almost complex structure. These make the perturbed Cauchy-Riemann equations on $\mathbb{H}^2$ whose solutions are described in Theorem 11.3 canonically provide a solution on $\mathbb{H}^3$.

**Appendix A. Computation of $\Delta(x/z)$: Proof of Proposition 11.4**

In this appendix, we prove Proposition 11.4. Let $\Delta$ be the classical Laplacian. We first note

$$-dq^i \circ J = z^2 V^i, \quad i = 1, 2, 3 \quad (A.1)$$

and $-\Delta \varphi \, dA = -d(\varphi \circ j)$. Define the function $\phi$ on $\mathbb{H}^3$ by $\phi(x, y, z) = \frac{x}{z}$ and denote $\varphi = \phi \circ v$. We apply $d\phi$ to (7.7) and derive

$$d\phi(dv + Jdv \circ j - \beta X_H(v) - (\beta \circ j) \cdot JX_H(v)) = 0.$$
By the fact that $JX_H$ is tangent to the fiber of $T^*\mathbb{R}^3$, this becomes
\[ v^*d\phi + d\phi(Jdv \circ j) - \beta \cdot d\phi(X_H(v)) = 0 \]
which is equivalent to
\[ 0 = v^*d\phi \circ j - d\phi \circ J \circ dv - v^*d\phi(X_H) \cdot (\beta \circ j) \]
\[ = v^*d\phi \circ j - v^*(d\phi \circ J) - v^*d\phi(X_H) \cdot (\beta \circ j) \]
By using $\varphi = \phi \circ v$ and taking the differential, we get
\[ -d(d\phi \circ j) = -v^*(d(d\phi \circ J)) - d(v^*d\phi(X_H)) \wedge (\beta \circ j) \quad (A.2) \]
Now we compute the two terms in the right hand side separately. We first compute
\[ -d\phi \circ J = -d\left(\frac{x^2}{z}\right) \circ J = - \left(\frac{zd\theta}{x^2}\right) \circ J \]
\[ = -\left(\frac{zd\theta + x\theta}{x^2}\right) \]
\[ = zV^3 - xV^3 \]
\[ = (zp_x - p_z dx + p_x dz - xdp_z) - \frac{x}{z}\theta \]
\[ = -d(xp_z - zp_x) - \frac{x}{z}\theta. \quad (A.3) \]
Therefore we obtain
\[ -d(d\phi \circ J) = -d\left(\frac{x^2}{z}\theta - \frac{x^2}{z}d\theta\right) = \phi \omega - d\phi \wedge \theta \quad (A.4) \]
and hence
\[ v^*(dd^j \phi) = \phi v^* \omega - d\varphi \wedge v^* \theta. \quad (A.5) \]
For the second, we compute
\[ d\phi(X_H) = \frac{zd\theta}{x^2}(X_H) = \frac{z^3p_x - z^2xp_z}{x^2} = zp_x - xp_z. \]
Therefore
\[ d(v^*(d\phi(X_H))) \wedge (\beta \circ j) = v^*d(xp_z - zp_x) \wedge (\beta \circ j). \]
Using (A.3), we evaluate
\[ v^*d(zp_x - zp_z) \wedge (\beta \circ j) \]
\[ = -v^*(zp_x - zp_z) \circ j \wedge (\beta) \]
\[ = (d\phi \circ J - \phi \theta)(dv \circ j) \wedge (\beta) \]
\[ = (d\phi \circ J - \phi \theta)(Ju + (\beta \circ j) \cdot X_H(v) - \beta \cdot JX_H(v)) \wedge (\beta) \]
\[ = (d\phi \circ J - \phi \theta)(Jdu \wedge (\beta) + (d\phi \circ J - \phi \theta)(X_H(v) \cdot (\beta \circ j)) \wedge (\beta) \]
\[ = (-d(v^*\phi) + \phi v^*dH) \wedge (\beta) + (d\phi(X_H(v)) - \phi(v)\theta(X_H(v))) \cdot (\beta \circ j) \wedge (\beta) \]
\[ = \varphi v^*dH \wedge (\beta) - d\varphi \wedge (\beta) - 2\varphi H(v)(\beta \circ j) \wedge (\beta) \quad (A.6) \]
Subtracting this from (A.2), we have derived the following key identity from (A.2)
This finishes the proof of Proposition 11.4.
**APPENDIX B. PROOF OF LEMMA 9.9**

We consider two metrics $g, g'$, which are both cylindrical outside $N_i$ for some $i$, and their associated kinetic energy Hamiltonian. Consider their convex combination

$$g_\sigma = (1 - \sigma)g + \sigma g'$$

whose associated Hamiltonian satisfies similar identity

$$H_\sigma := (1 - \sigma)H_g + \sigma H_{g'}.$$ 

We note that the metric $g_\sigma$ is cylindrical on $N_j$ which has the form

$$g_\sigma = da^2 \oplus g_\sigma|_{\partial N_j} \quad \text{on } [0, \infty) \times \partial N_j$$

with $g_\sigma|_{\partial N_j} = (1 - \sigma)g|_{\partial N_j} + \sigma g'|_{\partial N_j}$.

For the construction of the quasi-isomorphism

$$\Phi : CW(\nu^*T; H_{g_0}) \to CW(\nu^*T; H_{g'_0}),$$

for given pair $i, j$ with $i \leq j$, we follow the construction given in [BKO] Section 9 over some monotone homotopy $s \mapsto H_s$. For the readers’ convenience, we repeat verbatim with indication of the small change to be made arising from the fact that the hyperbolic metric $h$ does not extend smoothly to $M$.

For notational convenience, we denote the kinetic energy Hamiltonian $H_g$ also by $H(g)$ in this section. We compare $CW(T^*N, H(g_0))$ and $CW(T^*N, H(g_0))$ similarly as in [BKO] Section 7 where the categorical version was constructed in terms of two Riemannian metric $g_0$ and $g_0$ on $M \setminus K$ that arise from smooth metrics $g, g'$ on $M$ as a pair of cylindrical adjustments of their restrictions to $M \setminus K$. They satisfy the inequality

$$\frac{1}{C}g_0 \leq g' \leq Cg_0,$$

for some constant $C = C(g, g') > 1$.

We can verbatim follow the construction of [BKO] in a simpler form to make comparison between the case of metric $g$ coming from a smooth metric on $M$ and the hyperbolic metric $h$ on $M \setminus K$ in the current case of our interest.

Since the Hamiltonian is given by the dual metric, we have

$$\begin{aligned}
H(g_0) &\leq H(\frac{1}{C}g_0); \\
H(g_0) &\leq H(\frac{1}{C}g_0).
\end{aligned}$$

(B.2)

Then there are two $A_\infty$ homomorphisms

$$\begin{aligned}
\Phi & : CW(\nu^*T, H(g_0)) \to CW(\nu^*T, H(\frac{1}{C}g_0)); \\
\Psi & : CW(\nu^*T, H(g_0)) \to CW(\nu^*T, H(\frac{1}{C}g_0)).
\end{aligned}$$

(B.3)

which are defined by the standard $C^0$-estimates for the monotone homotopies.

Now consider the composition of the functors

$$\begin{aligned}
\Psi \circ \Phi & : CW(\nu^*T, H(g_0)) \to CW(\nu^*T, H(\frac{1}{C^2}g_0)); \\
\Phi \circ \Psi & : CW(\nu^*T, H(g_0)) \to CW(\nu^*T, H(\frac{1}{C^2}g_0)).
\end{aligned}$$

These are homotopic to natural isomorphisms induced by the rescaling of metrics

$$\begin{aligned}
\rho_{C^2} & : CW(\nu^*T, H(g_0)) \to CW(\nu^*T, H(\frac{1}{C^2}g_0)); \\
\eta_{C^2} & : CW(\nu^*T, H(g_0)) \to CW(\nu^*T, H(\frac{1}{C^2}g_0)),
\end{aligned}$$

respectively. This finishes the proof.
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