PERFECT MATCHINGS IN HYPERGRAPHS AND THE ERDŐS MATCHING CONJECTURE

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Abstract. We prove a new upper bound for the minimum d-degree threshold for perfect matchings in k-uniform hypergraphs when d < k/2. As a consequence, this determines exact values of the threshold when 0.42k ≤ d < k/2 or when (k, d) = (12, 5) or (17, 7). Our approach is to give an upper bound on the Erdős Matching Conjecture and convert the result to the minimum d-degree setting by an approach of Kühn, Osthus and Townsend. To obtain exact thresholds, we also apply a result of Treglown and Zhao.

1. Introduction

1.1. Perfect matchings via minimum degree conditions. Given k ≥ 2, a k-uniform hypergraph (or a k-graph) H is a pair H = (V, E), where V is a finite vertex set and E is a family of k-element subsets of V. Given a k-graph H and a set S of d vertices in V(H), 0 ≤ d ≤ k − 1, we denote by deg_H(S) the number of edges of H containing S. The minimum d-degree of H then is defined as

\[ \delta_d(H) = \min \left\{ \deg_H(S) : S \in \binom{V(H)}{d} \right\}. \]

Note that \( \delta_0(H) = |E(H)| \) is the number of edges of H.

A matching M in H is a collection of disjoint edges of H. The size of M is the number of edges in M. We say M is a perfect matching if it has size \( \frac{|V|}{k} \). For integers n, k, d, s satisfying 0 ≤ d ≤ k − 1 and 0 ≤ s ≤ n/k, let \( m^*_d(k, n) \) be the smallest integer m such that every n-vertex k-graph H with \( \delta_d(H) \geq m \) has a matching of size s. For simplicity, we write \( m_d(k, n) \) for \( m_d^*(k, n) \). Throughout this note, \( o(1) \) stands for some function that tends to 0 as n tends to infinity. The following conjecture [10, 17] has received much attention in the last decade (see [1, 3, 10, 12, 11, 14, 15, 16, 17, 21, 22, 24, 25, 26, 27, 28] and the recent surveys [23, 29]).

Conjecture 1.1. For 1 ≤ d ≤ k − 1 and k \( \mid n \),

\[ m_d(k, n) = \left( \max \left\{ \frac{1}{2} \left( 1 - \left( 1 - \frac{1}{k} \right)^{k-d} \right) \right\} + o(1) \right) \frac{n-d}{k-d}. \]

We remark that the quantities in the lower bound of the conjecture come from two different constructions. The second term can be seen by the following k-graph. Let G(s) be the k-graph on V whose edges are all k-sets that intersect a fixed S ⊆ V with |S| = s < n/k. Clearly G(n/k − 1) has no perfect matching.

On the other hand, the quantity 1/2 comes from the following parity construction. Given a partition V into non-empty sets A, B, let \( B_{n,k}(A, B) \) (or \( B_{n,k}(A, B) \)) be the k-uniform hypergraph with vertex set V and whose edge set consists of all k-element subsets of V that contains an odd (or even) number of vertices in A. Define \( H_{ext}(n, k) \) to be the collection of all hypergraphs \( B_{n,k}(A, B) \) where |A| = n/k is odd, and all hypergraphs \( B_{n,k}(A, B) \) where |A| − n/k is odd. It is easy to see that no hypergraph in \( H_{ext}(n, k) \) contains a perfect matching (see [3]). Define \( \delta(n, k, d) \) to be the maximum of the minimum d-degrees among all the hypergraphs in \( H_{ext}(n, k) \). Note that \( \delta(n, k, d) = (1/2 + o(1)) \frac{n-d}{k-d} \) but the general formula is unknown (see [20] for more discussion).

Given k ≥ 3, Rödl, Ruciński and Szemerédi [27] showed that \( m_{k-1}(k, n) = \delta(n, k, k-1) + 1 \) for large n. Treglown and Zhao [26, 27] generalized their result and showed that \( m_d(k, n) = \delta(n, k, d) + 1 \) for all
For $d \geq k/2$. For $d < k/2$, Conjecture [13] has been verified [10] [14] [15] [19] [28] for only a few cases, i.e., for $(k, d) \in \{(3, 1), (4, 1), (5, 1), (5, 2), (6, 2), (7, 3)\}$. Moreover, exact values of $m_d(k, n)$ are known for $(k, d) \in \{(3, 1), (4, 1), (5, 2), (7, 3)\}$. In general for $d < k/2$, the following best known upper bound is due to Kühn, Osthus and Townsend [18] Theorem 1.2, which improves earlier results by Hán, Person and Schacht [10], and Markström and Ruciński [21].

**Theorem 1.2.** [18] Let $n$, $1 \leq d < k/2$ be such that $n, k, d, n/k \in \mathbb{N}$. Then

$$m_d(k, n) \leq \left(\frac{k - d}{k} - \frac{k - d - 1}{k^{k-d}} + o(1)\right) \left(\frac{n - d}{k - d}\right).$$

In this paper we show the following new upper bound on $m_d(k, n)$ for $1 \leq d < k/2$.

**Theorem 1.3.** Let $n, k \geq 3$, $1 \leq d < k/2$ be integers and $n \in k\mathbb{N}$. Then

$$m_d(k, n) \leq \max \left\{\delta(n, k, d) + 1, (g(k, d) + o(1)) \left(\frac{n - d}{k - d}\right)\right\},$$

where

$$g(k, d) := 1 - \left(1 - \frac{(k-d)(k-2d-1)}{(k-1)^2}\right) \left(1 - \frac{1}{k}\right)^{k-d}.$$

Here we compare the bounds in Theorems 1.2 and 1.3. First consider the case $d = xk$ for some fixed $x \in (0, 1/2)$. Let $g(x) := \lim_{k \to \infty} g(k, xk)$ and clearly $g(x) = 1 - (3x - 2x^2) e^{x-1}$. Straightforward application of Calculus shows that $g(x) \leq 1 - \frac{2}{3}x \approx 1 - 1.1x$. Note that when $d = xk$ and $k$ tends to infinity, the corresponding coefficient in the bound of Theorem 1.2 becomes $1 - x$. So in this range, when $k$ is sufficiently large, our bound is better than that of Theorem 1.2. Second, by simply plugging in values of $k, d$, one can see that the bound in Theorem 1.2 is better for small values of $k$ or when $d$ is much smaller than $k$.

Theorem 1.3 also provides some new exact values of $m_d(k, n)$.

**Corollary 1.4.** Given $1 \leq d < k/2$, let $n \in k\mathbb{N}$ be sufficiently large. Then $m_d(k, n) = \delta(n, k, d) + 1$ if $0.42k \leq d < k/2$ or $(k, d) \in \{(12, 5), (17, 7)\}$.

**Proof.** For all cases, since $n$ is sufficiently large, by Theorem 1.3 it suffices to show $g(k, d) < 1/2$. The cases when $k \leq 20$ can be verified by hand. For $k \geq 20$, let $d = xk$ for some $x \in (1/4, 1/2)$. Note that $\frac{k-2d}{(k-1)^2} < \frac{k-d}{k}$, then by definition, we have

$$g(k, xk) \leq 1 - \left(1 - (1-x)(1-2x)\right) \left(1 - \frac{1}{k}\right)^{(1-x)k} = 1 - (3x - 2x^2) \left(1 - \frac{1}{k}\right)^{(1-x)k}.$$

Let $h(k, x) := 1 - (3x - 2x^2) (1 - \frac{1}{k})^{(1-x)k}$ and note that for $x \in (1/4, 1/2)$ and $k \geq 2$, $h(k, x)$ is decreasing on $x$ and $k$, respectively. So we are done by noticing that $h(20, 0.42) < 1/2$.

### 1.2. Perfect fractional matchings in hypergraphs.

As shown in [10] [18] [28], to get upper bounds on $m_d(k, n)$, it suffices to study so-called perfect fractional matchings. A fractional matching in a $k$-graph $H = (V, E)$ is a function $w : E \to [0, 1]$, such that for each $v \in V$ we have $\sum_{e \in E} w(e) \leq 1$. The size of $w$ is $\sum_{e \in E} w(e)$ and we say $w$ is a perfect fractional matching if it has size $|V|/k$. For $s \in \mathbb{R}$, let $f^s_d(k, n)$ denote the smallest integer $m$ such that every $n$-vertex $k$-graph $H$ with $\delta_d(H) \geq m$ has a fractional matching of size $s$. Note that the $k$-graph $G(n/k - 1)$ shows that $f^s_d(k, n) \geq \left(1 - \frac{1}{k}\right)^{k-d} + o(1)$ $\left(\frac{n}{k}\right)$.

As the key component of the proof of Theorem 1.3 we show the following upper bound on $f^s_d(k, n)$ for $1 \leq d < k/2$. Let $c^*_{d, k} := \limsup_{n \to \infty} f^s_d(k, n)/\left(\frac{n-d}{k-d}\right)$.

**Theorem 1.5.** Let $n, k \geq 3$, $1 \leq d < k/2$ be integers. Then

$$f^s_d(k, n) \leq (g(k, d) + o(1)) \left(\frac{n-d}{k-d}\right),$$

or, equivalently, $c^*_{d, k} \leq g(k, d)$. 
Now Theorem 2.2 immediately follows from Theorem 1.3 and the following theorem of Treglown and Zhao [28, Theorem 2].

Theorem 2.2. [28] Fix integers $k, d$ with $d \leq k - 1$ and let $n \in k\mathbb{N}$. Then

$$m_d(k, n) = \max \left\{ \delta(n, k, d) + 1, \left( c_{k,d}^* + o(1) \right) \left( \frac{n - d}{k - d} \right) \right\}.$$ 

1.3. The Erdős Matching Conjecture. The following classical conjecture is due to Erdős [4] in 1965. Here we prefer the notation from Extremal Set Theory, where a $k$-uniform family $\mathcal{F} \subseteq \binom{[n]}{k}$ is a collection of $k$-subsets of $[n]$ (so it is a $k$-graph). Given a family $\mathcal{F}$, $\nu(\mathcal{F})$ is the size of the maximum matching in $\mathcal{F}$.

Conjecture 1.7. [4] If $\mathcal{F} \subseteq \binom{[n]}{k}$ and $\nu(\mathcal{F}) = s$ such that $n \geq k(s + 1) - 1$ then

$$|\mathcal{F}| \leq \max \left\{ \left( \frac{k(s + 1) - 1}{k} \right), \left( \frac{n}{k} \right) - \left( \frac{n - s}{k} \right) \right\}$$

holds.

The two quantities in the above conjecture come from the following two simple constructions.

$$\mathcal{A}(k, s) := \left( \left( \frac{k(s + 1) - 1}{k} \right), \mathcal{A}(n, 1, s) := \left\{ A \in \binom{[n]}{k} : A \cap [s] \neq \emptyset \right\}. $$

Note that $\mathcal{A}(n, 1, s)$ is isomorphic to $G(s)$.

The case $s = 1$ is the classical Erdős-Ko-Rado Theorem [9]. For $k = 1$ the conjecture is trivial and for $k = 2$ it was proved by Erdős and Gallai [5]. For general $k \geq 3$, Erdős [4] proved the conjecture for $n \geq n_0(k, s)$. Bollobás, Daykin and Erdős [2] improved $n_0(k, s)$ to $2sk^3$, which was subsequently lowered to $3sk^2$ by Huang, Loh and Sudakov [13]. The best known bound on $n_0$ is $(2k - 1)s + k$ by Frankl [9]. Recently, Conjecture [1.7] is verified for $k = 3$ by Luczak and Mieczkowska [20] for large $n$ and by Frankl [8] for all $n$.

Here we show a result from a different point of view. Instead of looking for exact solutions for smaller values of $n$, we give an upper bound on the size of the family for the unsolved cases. Note that Frankl [7] showed that $|\mathcal{F}| \leq s\left( \frac{n - 1}{k - 1} \right)$ for all $n, k, s$.

Theorem 1.8. Suppose $n, k, s$ are non-negative integers and $\alpha \in (1, 2 - 1/k]$ is a real number such that $n \geq \alpha k(s + 1) + k - 1$. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ and $\nu(\mathcal{F}) = s$ then

$$|\mathcal{F}| \leq \left( \frac{n}{k} \right) - \left( \frac{n - s}{k} \right) + \left( \frac{2 - \alpha}{\alpha k - 1} \right) s\left( \frac{n - s - 1}{k - 1} \right).$$

Note that Theorem 1.8 can be translated into the language of $m_0^s(k, n)$. In fact, for any upper bound $h(n, k, s)$ on $|\mathcal{F}|$ where $\nu(\mathcal{F}) = s \leq n/k - 1$, we immediately have

$$m_0^{s+1}(k, n) \leq h(n, k, s) + 1.$$ 

2. Proof of Theorem 1.8

Our proof of Theorem 1.8 is adapted from the proof of [9, Theorem 1.1]. Let us first recall two results from [9]. For a family $\mathcal{F} \subseteq \binom{[n]}{k}$, its shadow is defined as

$$\partial \mathcal{F} := \left\{ G \in \binom{[n]}{k - 1} : \exists F \in \mathcal{F}, G \subseteq F \right\}.$$ 

Theorem 2.1. [9, Theorem 1.2] If $\mathcal{F} \subseteq \binom{[n]}{k}$ and $\nu(\mathcal{F}) = s$, then

$$s|\partial \mathcal{F}| \geq |\mathcal{F}|.$$ 

The families $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{s+1}$ are called nested if $\mathcal{F}_{s+1} \subseteq \mathcal{F}_s \subseteq \cdots \subseteq \mathcal{F}_1$ holds. The families $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{s+1}$ are called cross-dependent if there is no choice of $F_i \in \mathcal{F}_i$ such that $F_1, \ldots, F_{s+1}$ are pairwise disjoint. Here we use a theorem in [9] in a slightly different form, which follows from the original proof.

Theorem 2.2. [9, Theorem 3.1] Let $\beta \in (0, 1)$ and let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{s+1} \subseteq \binom{[Y]}{t}$, be nested, cross-dependent families, $|Y| \geq t\ell$. Suppose further that $t \geq \beta(2s + 1)$, then

$$|\mathcal{F}_1| + |\mathcal{F}_2| + \cdots + |\mathcal{F}_s| + (s + 1)|\mathcal{F}_{s+1}| \leq \frac{s}{\beta^2} \left( \frac{|Y|}{\ell} \right).$$

It is well known that in proving Theorem 1.8 one can assume that $\mathcal{F}$ is stable. That is, for all $1 \leq i < j \leq n$ and $F \in \mathcal{F}$, the conditions $i \notin F, j \in F$ imply that $F \cup \{i\} \setminus \{j\}$ is in $\mathcal{F}$ as well.

**Proof of Theorem 1.8.** Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a stable family with $\nu(\mathcal{F}) = s, n \geq \alpha k(s+1) + k - 1$. Note that $\alpha > 1$ and thus $n > k(s+1)$. We need to show that

$$|\mathcal{F}| \leq |\mathcal{A}(n, 1, s)| + \frac{(2 - \alpha)k - 1}{\alpha k - 1}s\left(n - s - 1\right) \left(k - 1\right).$$

Let us write $\mathcal{A}$ instead of $\mathcal{A}(n, 1, s)$ throughout the proof. In order to compare $\mathcal{F}$ and $\mathcal{A}$, we partition both families according to the intersection of their edges with $[s+1]$: For a subset $Q \subseteq [s+1]$ define

$$\mathcal{F}(Q) := \{ F \in \mathcal{F} : F \cap [s+1] = Q \}, \quad \mathcal{A}(Q) := \{ A \in \mathcal{A} : A \cap [s+1] = Q \}.$$ 

Let $m := n - s - 1$ and note that for $|Q| \geq 2$, we have $|\mathcal{A}(Q)| = \binom{m}{|Q| - 1}$, which implies $|\mathcal{F}(Q)| \leq |\mathcal{A}(Q)|$.

For $1 \leq i \leq s, |\mathcal{A}(i)| = \binom{m}{k-1}$ and $\mathcal{A}([s+1]) = \emptyset$. Thus it suffices to show

$$|\mathcal{F}(\emptyset)| + \sum_{i=1}^{s+1} |\mathcal{F}(\{i\})| \leq s\left(\frac{m}{k-1}\right) + \frac{(2 - \alpha)k - 1}{\alpha k - 1}s\left(\frac{m}{k-1}\right) = \frac{2k - 2}{\alpha k - 1}s\left(\frac{m}{k-1}\right).$$

Note that $\nu(\mathcal{F}(\emptyset)) \leq s$ and $|\partial(\mathcal{F}(\emptyset))| \leq |\mathcal{F}(\{s+1\})|$, where the latter is because every $H \in \partial(\mathcal{F}(\emptyset))$ satisfies that $H \cup \{s+1\} \in \mathcal{F}(\{s+1\})$. Then by Theorem 2.1 we have

$$|\mathcal{F}(\emptyset)| \leq \nu(\mathcal{F}(\emptyset))|\partial(\mathcal{F}(\emptyset))| \leq s|\mathcal{F}(\{s+1\})|.$$ 

Plugging this into (2.1), we see that it suffices to show

$$|\mathcal{F}(\emptyset)| + \cdots + |\mathcal{F}(\{s\})| + (s+1)|\mathcal{F}(\{s+1\})| \leq \frac{2k - 2}{\alpha k - 1}s\left(\frac{m}{k-1}\right).$$

To apply Theorem 2.2 set $\mathcal{F}_i := \{ F \setminus \{i\} : F \in \mathcal{F}(\{i\}) \}$. Since $\mathcal{F}$ is stable, $\mathcal{F}_1, \ldots, \mathcal{F}_{s+1}$ are nested. Also, since $\nu(\mathcal{F}) = s, \mathcal{F}_1, \ldots, \mathcal{F}_{s+1}$ are cross-dependent. Setting $\ell := k - 1, Y := [s+2, n]$ and thus

$$|Y| = m \geq (\alpha k - 1)(s+1) + k - 1 = \left(\frac{\alpha k - 1}{k} + \frac{1}{2s + 1}\right)(2s + 1) + 1 > \left(\frac{\alpha k - 1}{2k - 2}\right)(2s + 1)$$

So all conditions of Theorem 2.2 are satisfied for $t = \left(\frac{\alpha k - 1}{2k - 2}\right)(2s + 1)$ and $\beta = \frac{\alpha k - 1}{2k - 2}$. Thus (2.2) follows from Theorem 2.2.

\[ \square \]

### 3. Proof of Theorem 1.5

Here we use the approach in [18] as well as two propositions. In fact, we only replace their Theorem 1.8 by our Theorem 1.8.

**Proposition 3.1.** [18] Proposition 4.1 Let $k, d, n$ be integers with $n \geq k \geq 3$, and $1 \leq d \leq k - 2$. Let $a \in [0, 1/k]$. Suppose $H$ is a $k$-uniform hypergraph on $n$ vertices such that $\delta_a(H) \geq f_0^n(k-d, n-d)$, then $H$ has a fractional matching of size an.

**Proposition 3.2.** [18] Proposition 2.3 Suppose that $k \geq 2$ and $a \in (0, 1/k), c \in (0, 1)$ are fixed. Then for every $\epsilon > 0$ there exists $n_0 = n_0(\epsilon, c)$ such that if $n \geq n_0$ and $f_0^n(k, n) \leq c(\epsilon)^{n/2}$ then $f_0^{n+1}(k, n) \leq (c + \epsilon)^{n/2}$.

**Proof of Theorem 1.5.** Let $k' := k - d$ and $n' := n - d$. Let $\alpha := k/k'$ and $s + 1 := \lfloor (n' - k' + 1)/k' \rfloor$, then applying Theorem 1.8 with $n', k', s, \alpha$ implies that

$$m_{0}^{s+1}(k', n') \leq \left(\begin{array}{c} n' \\ k' \end{array}\right) - \left(\begin{array}{c} n' - s \\ k' \end{array}\right) + \frac{k - 2d - 1}{k - 1}\left(\begin{array}{c} s - 1 \\ k' - 1 \end{array}\right) + 1 \leq \left(\begin{array}{c} n' \\ k' \end{array}\right) - \left(1 - \frac{k - 2d - 1}{k - 1}\cdot\frac{k - d}{k' - 1}\right)\left(\begin{array}{c} s - 1 \\ k' \end{array}\right) + 1 = (g(k, d) + o(1))\left(\begin{array}{c} n' \\ d \end{array}\right).$$

Here the last equality is due to that $n' - s = (k - 1 + o(1))s = (1 - 1/k + o(1))n'$, which follows from the definition of $s$. Since $n/k \leq s + 3$, by Proposition 3.2 and the trivial fact that $f_0^{s+1}(k', n') \leq m_{0}^{s+1}(k', n')$, we get

$$f_0^{n/k}(k - d, n - d) = f_0^{n/k}(k', n') \leq m_{0}^{s+1}(k', n') + o(1)\left(\begin{array}{c} n' \\ k' \end{array}\right) \leq (g(k, d) + o(1))\left(\begin{array}{c} n - d \\ k - d \end{array}\right).$$
So Theorem 1.5 follows now from Proposition 3.1.

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