THE LAW OF LARGE NUMBERS FOR LARGE STABLE MATCHINGS

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ABSTRACT. In many empirical studies of a large two-sided matching market (such as in a college admissions problem), the researcher performs statistical inference under the assumption that they observe a random sample from a large matching market. In this paper, we consider a setting in which the researcher observes either all or a nontrivial fraction of outcomes from a stable matching. We establish a concentration inequality for empirical matching probabilities assuming strong correlation among the colleges' preferences while allowing students' preferences to be fully heterogeneous. Our concentration inequality yields laws of large numbers for the empirical matching probabilities and other statistics commonly used in empirical analyses of a large matching market. To illustrate the usefulness of our concentration inequality, we prove consistency for estimators of conditional matching probabilities and measures of positive assortative matching.

KEY WORDS. Two-sided matching, concentration inequality, stable matching, law of large numbers, correlated preferences

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1. Introduction

Large matching markets have long been a focus of study for empirical researchers and econometricians. (See Chiappori and Salanié (2016) for a survey in this literature.) For identification and statistical inference, the literature often explicitly or implicitly assumes that the researcher observes a small fraction of matching outcomes drawn by random or stratified sampling from a single large matching market. For example, the literature on two-sided matching with transferable utility performs identification analysis assuming that we observe the distribution of agents’ types. Such identification analysis implicitly assumes that that the actual observations observed by the researcher are a random sample from this distribution. (See Choo and Siow (2006) and Galichon and Salanié (2021).) In developing identification and estimation in a large one-to-one matching market with nontransferable utility, Menzel (2015) assumes that we observe a random sample of agents from a large limit matching market. Del Boca and Flinn (2014) similarly use an assumption of random sampling from a large market.¹

However, in many settings, it is not uncommon for a researcher to observe a nontrivial fraction of the matching outcomes of agents. In fact, many empirical studies exploit administrative data containing information on a large fraction of the participants in a matching market. For example, in their study of the impact of peer characteristics on student outcomes, Abdulkadiroğlu, Angrist, and Pathak (2014) linked approximately 98 percent of secondary students participating in a Boston high school matching process in the years 2001-2007 to data on their characteristics and outcomes (see Table C.III in Supplemental note). Kirkeboen, Leuven, and Mogstad (2016) were also able to link all the students’ post-secondary applications in Norway from 1998-2004 to socioeconomic characteristics using data from the Norwegian population registry. Hastings, Neilson, and Zimmerman (2013) studied the post-secondary education market in Chile using similarly rich administrative data. The empirical analysis in Fack, Grenet, and He (2019) examines the matching of middle-school students to academic-track public high schools in the Parisian southern district using administrative data containing sufficient information to replicate the 2013-2014 academic year matching (see pp. 27-28 of the online appendix associated with the paper).²

¹To the best of our knowledge, there are two exceptions to this sampling assumption, which use a finite sample inference approach. First, Logan, Hoff, and Newton (2008), Sørensen (2007), Aue, Klein, and Ortega (2020) and He, Sinha, and Sun (2022) adopted a Bayesian approach to estimate the structural parameters in a two-sided matching market. Second, Kim, Schwartz, Song, and Whang (2019) focused on two-sided matching markets with homogeneous preferences on the colleges’ side, and adopted the Monte Carlo inference approach of Dufour (2006) to develop finite sample inference.

²There are many other examples of the use of administrative data to study large matching market. See, for example, Boyd, Lankford, Loeb, and Wyckoff (2013); Abdulkadiroğlu, Agarwal, and Pathak (2017); Agarwal and Somaiini (2018); Luflade (2018); Calsamiglia, Fu, and Güell (2020).
In this paper, we consider a large, many-to-one, two-sided matching market with nontransferable utility, where matching outcomes are generated from a stable matching and the researcher observes all or a nontrivial fraction of the matching outcomes for students. Throughout the paper, we follow Roth and Sotomayor (1990) and refer to the two-sided matching model as a college admissions model, calling one side students and the other side colleges. The analogy of a college admissions model eases the exposition of our paper. However, it is not our goal to develop an empirical model of college admissions or school choice which accommodates institutional details of these market environments in practice.

Here we assume that the matching mechanism behind the data generation is not known to the researcher except that it yields a stable matching under the preferences of agents. Furthermore, we allow the mechanism to receive reports from students and colleges, not necessarily their preferences directly. In a special case, these reports can coincide with their preferences as in the case of truth-telling strategies. However, in some situations, the assumption of truth-telling may be overly restrictive.3 Our framework allows for a wide range of report maps to cover various strategic settings as we explain below.

Our focus in this paper is on the statistical properties of the empirical matching probability and related statistics. More specifically, let \( N = \{1, \ldots, n\} \) and \( M = \{1, \ldots, m\} \) be the sets of students and colleges respectively. We consider statistics of the following form:

\[
\hat{\theta}(\tau) = \frac{1}{n} \sum_{i \in N} \sum_{j \in M} \tau_j(X_i, Z) 1\{i \text{ and } j \text{ are matched}\}, \quad \tau = (\tau_1, \ldots, \tau_m),
\]

where \( X_i, Z_j \) are the observed characteristics of student \( i \) and college \( j \), \( \tau_j \) are maps chosen by the researcher, and \( Z = (Z_1, \ldots, Z_m) \). Then, the empirical matching probability for college \( j \) measures the fraction of students who have their observed characteristic taking a value in a set, say, \( A \), and are matched to college \( j \). This probability is expressed as \( \hat{\theta}(\tau) \) by choosing \( \tau \) with \( \tau_j(X_i, Z) = 1\{X_i \in A, j' = j\}, j' = 1, \ldots, m \).

The statistic \( \hat{\theta}(\tau) \) can be used to capture other empirical features of the matching. One can, for example, explore matching along observed type categories of students and colleges by choosing \( \tau \) with \( \tau_j(X_i, Z) = 1\{X_i \in A, Z_j \in A'\} \), where \( A' \) is a set of values for observed college characteristics. Such statistics are useful for investigating positive assortativity of the matching between students and colleges along their observed characteristics.

3A notable departure from the assumption of truth-telling is to assume that a school is ranked higher under a student’s true preferences whenever the school is ranked higher under the student’s stated preferences, a behavior that has been theoretically supported as an undominated strategy in some environments (Haeringer and Klijn (2009); Fack, Grenet, and He (2019)). Assumptions of relaxed truth-telling are particularly relevant in situations in which students are either limited in the number of schools they can rank or in which students may choose to omit some schools from their stated preferences (see e.g., the above references) and have been used in combination with the stability assumption in empirical studies of school choice and centralized two-sided matching (e.g., Fack, Grenet, and He (2019), Aue, Klein, and Ortega (2020), Combe, Tercieux, and Terrier (2022)).
The main result of this paper establishes a concentration-of-measure phenomenon for \( \hat{\theta}(\tau) \). That is, we establish a finite sample bound \( B(t) \) such that\(^4\)

\[
P \{ \left| \hat{\theta}(\tau) - E[\hat{\theta}(\tau) | Z, \xi] \right| \geq t | Z, \xi \} \leq B(t),
\]

for all \( t > 0 \), where \( \xi \) is the collection of unobserved college characteristics. For example, in the case with \( \tau_j(X_i, Z) = 1\{X_i \in A, j' = j\} \), \( j' = 1, ..., m \), the bound \( B(t) \) measures how much the distribution of the empirical matching probability for college \( j \) concentrates around its conditional expectation given \( Z \) and \( \xi \). The bound can also be used to establish the rate of convergence for a law of large numbers. The conditional expectation \( E[\hat{\theta}(\tau) | Z, \xi] \) here is a finite population quantity that depends on \( n \) and \( m \), which is in contrast with the matching probability in a limit matching market as \( n \to \infty \). This finite population quantity can be viewed as one obtained by averaging the statistic \( \hat{\theta}(\tau) \) across many draws from the distribution of the \( n \) students’ types in a finite matching market.

As we demonstrate in this paper, the result (2) yields a number of empirically relevant applications. For instance, in empirical models of large matching, it is common for the identification analysis to assume that the researcher knows the conditional matching probabilities; that is, the probability that a student matches with a specific college when the student has observed characteristic \( x \) (e.g., Diamond and Agarwal (2017) and He, Sinha, and Sun (2022)). As one application of our concentration inequality, we provide conditions under which a kernel-based estimator of a conditional matching probability is consistent. In another other application, we prove consistent estimation of sorting measures (based on Spearman’s rho) and distributions of students’ characteristics at a college.

There are three main assumptions we rely on to derive the concentration inequality in (2). First, as mentioned before, we assume that the matching mechanism generates a stable matching under the (true) preferences of agents. The assumption of stable matching is widely used in the empirical matching literature (Chiappori and Salanié (2016)) and has been justified in both centralized and decentralized two-sided matching environments with non-transferable utility. Fack, Grenet, and He (2019) discuss the merit of stability in empirical settings in detail. See also the discussions in Menzel (2015) and He, Sinha, and Sun (2022), and Agarwal and Somaini (2020).

\(^4\)We focus on the concentration of \( \hat{\theta}(\tau) \) around its conditional expectation given colleges’ characteristics, \( Z, \xi \), to accommodate a model of the matching market in which the number of the colleges is fixed. In such a case, there is no aggregation over colleges’ characteristics in large samples, and the concentration of measure arises around the conditional expectation of \( \hat{\theta}_j(\tau) \) given \( Z, \xi \).
Second, we assume that the student-specific (both observed and unobserved) types are drawn i.i.d. conditional on the college-specific characteristics. It is not unusual in the literature to assume that the individual utilities are drawn i.i.d. from a certain distribution (e.g., Menzel (2015).)

Third, we assume that the colleges’ preferences over students are strongly correlated while allowing students’ preferences over colleges to be fully heterogeneous. While the assumption that college preferences are strongly correlated limits the scope of our paper, such correlation is not necessarily unreasonable in practice. This is especially true in environments in which the role of college preferences is essentially fulfilled by priority indices. For example, in some school choice markets, school priorities are determined entirely based on common measures of the students’ academic performance that do not differ across colleges.\(^5\) In some college admissions markets, the priorities for different choices of colleges and majors are determined on the basis of a composite index that may differ somewhat across the different options (e.g., Hastings, Neilson, and Zimmerman (2013) and Kirkeboen, Leuven, and Mogstad (2016)). In the market for exam schools in Boston studied in Abdulkadir\(\text{\c{g}}\)lu, Angrist, and Pathak (2014), priorities for students are determined on the basis of a weighted average of the student’s grade-point averages and score on an entrance exam. In the centralized matching of teachers to public secondary schools in France, teachers are ranked by the central administration using a common, points-based system (Terrier (2014)).

The main method we rely on to derive the concentration inequality is to use a conditional version of McDiarmid’s inequality (McDiarmid (1989)). This inequality is useful in our context, as it enables us to derive a bound for the concentration-of-measure for statistics that involve independent random variables in a complex, nonlinear form. The concentration bound is essentially determined by the bounded difference property that shows how sensitively the statistic responds when one of the input random variables is changed. We derive the bounded difference condition for a stable matching by drawing heavily on machinery developed in economic theory. Roth and Vande Vate (1990) proposed a random process through which an arbitrary matching converges to a stable matching with probability one. Blum, Roth, and Rothblum (1997) and Blum and Rothblum (2002) developed a re-stabilization operator that takes a matching and produces a stable matching after a finite number of iterations. Similarly, we rely on a re-stabilization operator that transforms a matching into a stable matching, which we use

\(^5\)Many further examples of school choice markets in which the assumption of school-side alignment of priorities is well justified are listed in Table 1 on page 148 of Fack, Grenet, and He (2019). (In particular, see Teo, Sethuraman, and Tan (2001), Pathak and Sönmez (2013), Ajayi (2021), Pop-Eleches and Urquiola (2013), Artemov, Che, and He (2017), Akyol and Krishna (2017)). In Scotland, medical school graduates are assigned to training programs via a centralized matching procedure in which the priority of candidates at the programs is based on a common score (Irving (2011)).
to obtain a bound for the number of the students affected by one student’s change of preference. Our bounding the number of such students is related to the rejection chain method that Kojima and Pathak (2009) used in their study of strategic proofness of a large student-optimal stable matching (SOSM) mechanism.

Convergence of empirical matching probabilities has drawn attention in the literature. For example, Azevedo and Leshno (2016) and Che, Kim, and Kojima (2019) showed that a sequence of empirical matching probabilities converges to their counterpart in a continuum economy which is populated by a continuum of students. They assume that colleges’ preferences are not over the identities of the students, but over a topological space that the types of students or their distributions take values from. Then a matching is defined between the students’ types and the colleges. This setting is used to construct both a sequence of finite economies and a continuum economy where a stable matching is well-defined. Their convergence result bridges between stable matchings from a continuum economy and those from a finite economy. Unlike these papers, we do not consider a continuum limit market in our paper. In our setting, a matching is defined between the identities of the students and those of the colleges, because this is the way the raw data record matching outcomes in practice.

A result more closely related to ours is found in Menzel (2015) who proved the convergence of empirical matching probabilities as part of his development of an econometric model of a large one-to-one matching market. (See Corollary 3.1 in Menzel (2015). See also Peški (2017) for a related contribution in a large stable roommate problem.) There are a few major differences between his result and ours. First, our paper focuses on a many-to-one matching market, where the number of students is greater than that of colleges. Second, as mentioned before, Menzel (2015) uses the assumption that the sample involved in the empirical matching probability is drawn by a random sampling or a stratified sampling scheme from a large limit matching market. In contrast, we allow the empirical matching probability to be constructed from the whole or a nontrivial portion of the matching outcomes from a large yet finite matching market.

The work of Diamond and Agarwal (2017) is also related to our paper. They studied identification and asymptotic inference in many-to-one matching markets - in particular, discovering the value of many-to-one matchings as opposed to one-to-one matchings in identifying the payoff parameters. The main difference between their setting and ours is that they focused on the case of homogeneous preferences on both sides of the market, where the number of students is fixed in proportion to the number of colleges. Furthermore, their asymptotic inference assumes a sampling process where the observed variables are drawn independently from the conditional distribution given utilities. In our setting, the number of the students is allowed to
be much larger than the number of colleges, and the preferences of the students and colleges are permitted to be heterogeneous.

The remainder of the paper is organized as follows. In the next section, we introduce a two-sided, many-to-one matching market drawing on the analogy of a college admissions model, and introduce assumptions on random preferences that define the scope of our paper. In Section 3, we provide a general concentration inequality on functionals of a large matching. We present examples and various concentration-of-measure results. In Section 4, we conclude. In the appendix, we provide the proof of the main results. In the supplemental note, we introduce the basic settings and results from economic theory of matching markets and prove the bounded difference result that is crucial for establishing our concentration inequality.

2. A College Admissions Model

2.1. Two-Sided Matching

Throughout the paper, we follow Roth and Sotomayor (1990) and refer to a generic many-to-one matching model as a college admissions model, calling one side students and the other side colleges. The analogy of a colleges admissions model eases the exposition of our model. However, the reader does not need to assume that our model is intended to reflect all the details specific to this particular matching environment in practice.

A college admissions model consists of the set $N = \{1, ..., n\}$ of students and the set $M = \{1, ..., m\}$ of colleges. In many situations, colleges are capacity-constrained. For each college $j \in M$, let $q_j$ be a positive integer that represents the quota of college $j$. To accommodate the possibility of unmatched students and colleges with unfilled positions, we denote $N' = N \cup \{0\}$ and $M' = M \cup \{0\}$, so that an unmatched student or an unfilled position at a college is viewed as being matched to 0. A (many-to-one) matching under the capacity constraint $q = (q_j)_{j \in N}$ is defined as a function $\mu: N \to M'$ such that $|\mu^{-1}(j)| \leq q_j$ for each $j \in M$. That is, the matching, $\mu$, is such that the number of students assigned to each college does not exceed the capacity of the college. Throughout the paper, we consider only those matchings that satisfy the capacity constraint. When $\mu^{-1}(j) = \emptyset$ for a college $j$, the college is not matched with any student under $\mu$. Similarly, when $\mu(i) = 0$ for a student $i$, the student is not matched with any college under $\mu$.

In our college admissions model, we let $v_i: M' \to M'$ denote a bijection that represents student $i$’s strict preference ordering over $M'$. Similarly, we let $w_j: N' \to N'$ denote a bijection that represents college $j$’s strict preference ordering over $N'$. For any $j_1, j_2 \in M'$, we write $j_1 \succ_i j_2$ if and only if $v_i(j_1) < v_i(j_2)$, and for any $i_1, i_2 \in N'$, we write $i_1 \succ_j i_2$ if and only if $w_j(i_1) < w_j(i_2)$. If $0 \succ_j i$, this means that college $j$ prefers to be unmatched by any student
than to be matched with student $i$, and in this case we say that student $i$ is \textit{unacceptable} to college $j$. Throughout the paper, we will also maintain the assumption that college preferences over sets of students are responsive. Let $V$ be the set of bijections on $M'$, each of which represents a student's preference over the colleges. Similarly, let $W$ be the set of bijections on $N'$, representing colleges' preferences over the students. Then the collection of preference profiles, $u = (v, w)$, is given by

$$U = V^n \times W^m,$$

where $v = (v_1, ..., v_n) \in V^n$ and $w = (w_1, ..., w_m) \in W^m$.

In modeling the predicted outcomes of matching in economics, it is standard in the literature to focus on (pairwise) stable matchings (Roth and Sotomayor (1990)). We say that a matching $\mu : N \to M'$ is \textit{stable} under the preference profile $u$, if

1. (Individual Rationality) there is no $i \in N$ such that $0 \succ_i \mu(i)$ and no $j \in M$ such that $0 \succ_j i'$ for some $i' \in \mu^{-1}(j)$, and
2. (Incentive Compatibility) there is no $(i, j) \in N \times M$ such that $j \succ_i \mu(i)$ and either (i) $|\mu^{-1}(j)| < q_j$ and $i \succ_j 0$ or (ii) $|\mu^{-1}(j)| = q_j$ and $i \succ_j i'$ for some $i' \in \mu^{-1}(j)$.

Hence, a matching is stable if there is no student who is currently matched with a college that prefers to be unmatched, no college that prefers to unmatch some student currently matched to it, and no student-college pair not currently matched who can improve over their current match by matching with one another. (Such a pair is called a \textit{blocking pair} to the matching.) We call any triple $(N, M, u)$ a \textit{matching market}.\footnote{Our notation here leaves the quotas of colleges implicit, referring to them only when required.}

2.2. Random Preferences with One-Sided Limited Heterogeneity

In empirical modeling of matching markets, it is common to associate preferences with the agents' observed or unobserved types. In this paper, we allow the type of a student to comprise a vector of observables along with a vector of match qualities capturing the student's unobserved taste for each of the colleges. We model the type of a college in a similar way. Formally, we specify the types as follows: for $i \in N$ and $j \in M$,

$$C_j = (Z_j, \xi_j, \eta_{1j}, ..., \eta_{nj}), \text{ and } S_i = (X_i, \epsilon_{i1}, ..., \epsilon_{im}),$$

where $Z_j$ and $X_i$ are observed characteristics specific to college $j$ and student $i$ respectively, and $\eta_{ij}$ refers to college $j$'s match quality with student $i$, $\epsilon_{ij}$ the student $i$'s match quality with college $j$, and $\xi_j$ the vector of unobserved characteristics of college $j$. 
As we make explicit in Assumptions 2.2 and 2.3 below, the preferences of colleges and students are determined entirely by the types, $C_j$’s and $S_i$’s. In empirical modeling, we may take $Z_j$ to be a vector of observable characteristics of college $j$, and $X_i$ a vector of observed characteristics of student $i$. For $i \in N$, we also define

$$
\tilde{S}_i = (S_i, \eta_{i1}, ..., \eta_{im}) = (X_i, \varepsilon_{i1}, ..., \varepsilon_{im}, \eta_{i1}, ..., \eta_{im}),
$$

so that $\tilde{S}_i$ contains both the type of student $i$ as well as the match quality of each college with student $i$. We call $\tilde{S}_i$ the student $i$’s quality. We collect the college-specific quantities $Z$ and $\xi$ and define

$$
\tilde{Z} = (Z, \xi),
$$

where $Z = (Z_1, ..., Z_m)$ and $\xi = (\xi_1, ..., \xi_m)$. Regarding the dependence structure of random quantities, we require only that $\tilde{S}_i$’s be conditionally independent across $i$’s given $\tilde{Z}$, which we formalize as follows.

**Assumption 2.1.** $\tilde{S}_i$ are conditionally i.i.d. across $i$’s given $\tilde{Z}$.

We allow the elements of each $\tilde{S}_i$ to be arbitrarily correlated. For example, it is reasonable for $\eta_{ij}$ to be correlated with $S_i$, since a college’s match quality for a student may depend on the student’s type in general. Since our law of large numbers is conditioned on college-specific characteristics $\tilde{Z}$, we also allow for arbitrary correlation between $\tilde{S}_i$’s and $\tilde{Z}$. This permits, for example, correlation between the student’s match quality at a college $\varepsilon_{ij}$ with the observed characteristics of the college $Z_j$. However, we require that any two random quantities associated with different students (e.g., two random quantities $X_i$ and $X_j$, or $\varepsilon_{ik}$ and $\eta_{jl}$, both pairs associated with two different students $i, j, i \neq j$) be conditionally independent given $\tilde{Z}$.

We relate agents’ types to their preference orderings as follows. For the students’ preferences over colleges, we allow for full heterogeneity. However, for the colleges’ preferences over students, we impose limited heterogeneity in the sense we explain below. We define

$$
S = (S_1, ..., S_n), \text{ and } \tilde{S} = (\tilde{S}_1, ..., \tilde{S}_n).
$$

**Assumption 2.2.** For each student $i \in N$, his preference ordering over the colleges, $v_i : M' \to M'$, is given by

$$
v_i = f_{S_i, \tilde{Z}},
$$

where $f_{S_i, \tilde{Z}} : M' \to M'$ is a stochastic map that is measurable with respect to $\sigma(S_i, \tilde{Z})$, i.e., the $\sigma$-field generated by $(S_i, \tilde{Z})$, such that for any $j_1, j_2 \in M'$,

$$
P\{f_{S_i, \tilde{Z}}(j_1) = f_{S_i, \tilde{Z}}(j_2) \mid \tilde{Z}\} = 0, \text{ whenever } j_1 \neq j_2.
$$
The assumption on the students’ preferences is mild; it assumes that each student’s preference is generated by the student’s own type and the component of the college types that is unrelated to the students. We do not put any further restriction on the preferences of students.

The condition (6) requires that the realized preferences are strict, so that we exclude the case with ties in ranking. While there are research papers which allow for indifferences in preferences in matching markets (e.g., Erdil and Ergin (2008), Erdil and Ergin (2017); Abdulkadiroğlu, Agarwal, and Pathak (2017)), to the best of our knowledge, the assumption of strict preferences remains the most common.\footnote{Here we rely on results that need not hold in the absence of strict preferences, such as the Rural Hospitals Theorem and those results guaranteeing the existence of a unique stable student-optimal matching (see e.g., Roth and Sotomayor (1990)).}

We introduce a model of college preferences that reflect limited heterogeneity. We model each college’s preference to be generated by priority indices over students where each priority index has two components: a vertical component that depends only on students’ types and is common across the colleges, and the horizontal component which is different across the colleges.

**Assumption 2.3.** (i) Each college \( j \in M \) has priority index for a match with student \( i \in N \) as:

\[
\omega_{ij} = \lambda(S_i) + \sigma_n \eta_{ij},
\]

so that \( \omega_{ij} > \omega_{i'j} \) if and only if \( i \succ j \), where \( \eta_{ij} \)'s are continuous random variables, and \( \sigma_n \) is a positive sequence.

(ii) Each college \( j \in M \) has a threshold \( c_j \) such that \( c_j \) is a function of \( \tilde{Z} \) and for each student \( i \in N, c_j > \omega_{ij} \) if and only if \( 0 \succ j i \).

The sequence \( \sigma_n \) in Assumption 2.3(i) determines the degree of heterogeneity in the colleges’ preferences over the students. Note that if \( \sigma_n \) were equal to zero, then the college preferences are fully homogeneous, determined by the rank of students according to \( \lambda(S_i) \). The degree of heterogeneity of colleges’ preferences is captured by the rate at which \( \sigma_n \to 0 \). The continuity of \( \eta_{ij} \)'s is introduced to ensure that the preferences of the colleges are strict with probability one. The priority index generation in (7) allows for a wide range of functional forms for \( \lambda \).

We do not necessarily require that the colleges observe \( \epsilon_{i1}, \ldots, \epsilon_{im} \). When the colleges do not observe \( \epsilon_{i1}, \ldots, \epsilon_{im} \), this is tantamount to imposing a restriction that the map \( \lambda \) does not vary with \( \epsilon_{i1}, \ldots, \epsilon_{im} \). Assumption 2.3(ii) says that each college has a threshold as a function of \( \tilde{Z} \) such that when the priority index of a student is below the threshold, the student is unacceptable to the college.

From here on, we say that the preference profile \( u = (v, w) \) is **generated from** \((\tilde{S}, \tilde{Z})\), if the profile is determined from \( u = (v, w) \) according to (5) in Assumption 2.2 and (7) in...
Assumption 2.3. In this case, the randomness of \((\tilde{S}, \tilde{Z})\) alone is responsible for the randomness of the preference profile \(u = (v, w)\).

The following assumption collects the technical conditions that we require for \(\lambda(S_i)\) and \(\eta_{ij}\).

**Assumption 2.4.**

(i) For all \(i \in N\) and \(j \in M\), and all \(t \geq 0\),

\[
P\{ |\eta_{ij}| > t \mid S, \tilde{Z} \} \leq 2 \exp\left(-\frac{t^2}{2}\right).
\]

(ii) There exists a bounded interval \(B \subset \mathbb{R}\) such that for all \(i \in N\),

\[
P\{ \lambda(S_i) \in B \mid \tilde{Z} \} = 1.
\]

Furthermore, there exists a constant \(C > 0\) such that for all \(i \in N\) and all \(t \geq 0\),

\[
\sup_{c \in \mathbb{R}} P\{ c - t \leq \lambda(S_i) \leq c + t \mid \tilde{Z} \} \leq Ct.
\]

Assumption 2.4(i) is a mild, normalization condition for \(\eta_{ij}\), because the scale of the horizontal preference component in the college preferences is captured by the sequence \(\sigma_n\). Assumption 2.4(ii) requires that \(\lambda(S_i)\) be bounded with probability one, and satisfies an anti-concentration condition. The anti-concentration condition is satisfied if \(\lambda(S_i)\) is continuous and has a bounded density function.

2.3. Generation of Matching Outcomes

The matching outcomes are generated as follows. First, each individual student \(i\) reports her rank order list of colleges to a central decision maker, according to a report map \(\alpha_{s,i} : \tilde{S} \times \tilde{Z} \to \mathcal{R}\), where \(\mathcal{R}\) denotes a set of rank-order lists of colleges, \(\tilde{S}\) denotes the support of \(\tilde{S}\), and \(\tilde{Z}\) denotes the support of \(\tilde{Z}\). Hence each student reports her rank order list to the central decision maker. Our framework allows for a setting where the set \(\mathcal{R}\) admits only rank-order lists that meet a certain length limit. (See Kojima and Pathak (2009) for such restriction’s role in inducing manipulated reports in matching markets.) Similarly, each college \(j\) reports its priority indices of students and a threshold \(c_j\) to a central decision maker, according to a report map \(\alpha_{c,j} : \tilde{S} \times \tilde{Z} \to \Omega^n \times \Xi\), where \(\Omega\) denotes the support of the priority index \(\omega_{ij}\), and \(\Xi\) the support of thresholds \(c_j\). Thus, each college reports their priority indices for the students and a threshold to the central decision maker. We define the map:

\[
\alpha \equiv (\alpha_{s,1}, ..., \alpha_{s,n}, \alpha_{c,1}, ..., \alpha_{c,m}) : \tilde{S} \times \tilde{Z} \to \mathcal{T} \equiv \mathcal{R}^n \times (\Omega^n \times \Xi)^m.
\]

The report profile \(t = (r_1, ..., r_n, \tilde{t}_1, ..., \tilde{t}_m) \in \mathcal{T}\), \(r_i \in \mathcal{R}\) and \(\tilde{t}_j = (\omega_{ij}, c_j) \in \Omega^n \times \Xi\), \(\omega_j = (\omega_{1j}, ..., \omega_{nj})\), denotes the combination of the rank order lists \(r_i\) submitted by the students, the priority index profile \(\omega_j = (\omega_{1j}, ..., \omega_{nj}) \in \Omega^n\) of colleges over the students, and thresholds \(c_j\). Thus, the notation, \(\alpha(\tilde{S}, \tilde{Z}) = t\), with \(t = (r_1, ..., r_n, \tilde{t}_1, ..., \tilde{t}_m) \in \mathcal{T}\), expresses that each student
i with type $S_i$ reports her rank-order list $r_i$, and each college $j$ with type $C_j$ reports its priority index vector $\omega_j$ and the threshold $c_j$. It is important to note that we do not require that the reports truthfully reveal the preferences of the students or colleges.

Second, the central decision maker assigns each student to a college or keeps her unmatched, according to a matching mechanism based on the received reported preferences. Thus, we treat a matching as dependent on the reports. We formalize this by treating a matching as indexed by the reports. More specifically, we define a **matching mechanism** to be a matching $\mu(\cdot; t): N \to M'$, indexed by the report vector $t \in T$. One can view a matching mechanism as a collection of matchings where each matching is determined once the report $t$ is realized.

Throughout the paper, we assume that the pair of the matching mechanism and the report map, $(\mu, \alpha)$, generates a stable matching with probability one. We formalize this into the following assumption.

**Assumption 2.5.** With probability one, $\mu(\cdot; \alpha(\tilde{S}, \tilde{Z}))$ is stable under the preference profile generated from $(\tilde{S}, \tilde{Z})$.

Our framework allows for various information structures. This flexibility is realized through our accommodation of a wide range of report maps, $\alpha$. For each agent, the report map is a map from the agent’s information set to the set of possible reports. In the case of a private information setting for students as in Fack, Grenet, and He (2019), we can place a restriction that each student $i$’s report map $\alpha_{s,i}$ varies only with $S_i$, $Z$, and the priority indices $\omega_{ij}$ by the colleges, $j = 1, \ldots, m$. In the case of a setting where all students observe all the other students’ preferences and the colleges’ priority indices, we may allow each individual student’ report map $\alpha_{s,i}$ to vary with the entire profile of students’ qualities $\tilde{S}$ and $\tilde{Z}$.

Suppose that a matching mechanism $\mu$ and the report map $\alpha$ are given. We assume that the matching $Y_i \in M'$ for each student $i$ is generated as follows:

$$Y_i = \mu(i; \alpha(\tilde{S}, \tilde{Z})).$$

The researcher observes the matching $Y_i$ for each student $i$, where $Y_i$ is a discrete random variable taking values from $M'$.

However, we do not require that the researcher observe the mechanism $\mu$ or the report map $\alpha$.

We introduce an assumption that $(\mu, \alpha)$ is label-free.

\[8\text{Note that although each matching outcome, } Y_i, \text{ depends on the other students and colleges in the market, we choose to omit the dependence of } Y_i \text{ on } n \text{ and } m \text{ from our notation for the sake of readability. We will frequently adopt the same practice when notating other quantities, particularly those defined in terms of the matching outcomes.}\]
Assumption 2.6. For any permutation $\pi$ of $\{1, \ldots, n\}$, we have
\[
\mu(i; \alpha(\tilde{s}, \tilde{Z})) = \mu(\pi(i); \alpha(\tilde{s}, \tilde{Z})), \quad \text{for all } i \in N,
\]
with probability one, where $\tilde{s}_\pi = (\tilde{s}_{\pi(1)}, \ldots, \tilde{s}_{\pi(n)})$.

This assumption is fairly reasonable, especially when for each $i$, $\alpha_{s_i}(\tilde{s}, \tilde{Z}) = \alpha(\tilde{s}, \tilde{Z})$ for a map $\alpha$ that is the same across students $i$. The assumption says that the matching outcome for student $i$ depends only on the value of $\tilde{s}_i$ associated with $i$. For example, suppose that $n = 4$, and $\pi(1) = 2, \pi(2) = 3, \pi(3) = 4, \pi(4) = 1$. Hence $\mu(i; \alpha(\tilde{s}, \tilde{Z}))$ denotes the match of student $i$, when each student $k$'s quality is $\tilde{s}_{\pi(k)}$. Then, (writing $\mu(i; \tilde{s}) = \mu(i; \alpha(\tilde{s}, \tilde{Z}))$ briefly)
\[
\begin{bmatrix}
\mu(1; \tilde{s}_{\pi(1)}, \tilde{s}_{\pi(2)}, \tilde{s}_{\pi(3)}, \tilde{s}_{\pi(4)}) \\
\mu(2; \tilde{s}_{\pi(1)}, \tilde{s}_{\pi(2)}, \tilde{s}_{\pi(3)}, \tilde{s}_{\pi(4)}) \\
\mu(3; \tilde{s}_{\pi(1)}, \tilde{s}_{\pi(2)}, \tilde{s}_{\pi(3)}, \tilde{s}_{\pi(4)}) \\
\mu(4; \tilde{s}_{\pi(1)}, \tilde{s}_{\pi(2)}, \tilde{s}_{\pi(3)}, \tilde{s}_{\pi(4)})
\end{bmatrix}
= \begin{bmatrix}
\mu(1; \tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_1) \\
\mu(2; \tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_1) \\
\mu(3; \tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_1) \\
\mu(4; \tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_1)
\end{bmatrix}
= \begin{bmatrix}
\mu(2; \tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4) \\
\mu(3; \tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4) \\
\mu(4; \tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4) \\
\mu(1; \tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4)
\end{bmatrix}. \]

To see the second equality, note that for example, the student 1 with quality $\tilde{s}_2$ is relabeled as student 2 with quality $\tilde{s}_2$.

An immediate consequence of this assumption is that the observed matches are conditionally exchangeable in $i \in N$ given $\tilde{Z}$. Let $\Pi$ be the set of all permutations of $\{1, \ldots, n\}$.

Lemma 2.1. Suppose that Assumptions 2.1 and 2.6 hold, and for each $i \in N$, let $W_i = (\tilde{s}_i, Y_i)$. Then, the conditional distribution of $(W_{\pi(1)}, \ldots, W_{\pi(n)})$ given $\tilde{Z}$ is the same across $\pi \in \Pi$.

The conditional exchangeability of matches is extremely useful in our context, especially when we consider partial observation of the matching outcomes. Essentially, the conditional exchangeability of matching outcomes allows our results to accommodate a wide range of sampling processes for students in addition to random sampling. We do not require the researcher to know the precise sampling process involved in generating the data. Due to Lemma 2.1, we can obtain a concentration inequality for a population object which does not depend on the
particular sampling process for students used to generate the data. We will give more details later.

3. The Law of Large Numbers for a Large Stable Matching

3.1. The Main Result

In this section, we present the concentration inequality that is the main result of our paper. As the random preference profile is generated from \((\tilde{S}, \tilde{Z})\), the randomness of \(Y_i\) in (8) arises solely from that of \((\tilde{S}, \tilde{Z})\). The main challenge in deriving the law of large numbers for the sum of \(Y_i\)'s over \(i \in N\) is that each \(Y_i\) is a complex function of common random vector \((\tilde{S}, \tilde{Z})\). Our main result establishes a finite-sample concentration-of-measure for a general statistic that involves \(Y_i\)'s. From this, we can derive point-wise or uniform law of large numbers for various statistics as we show below.

In order to accommodate an empirical setting with partially observed matches, we follow the approach of Canen, Schwartz, and Song (2020) and consider a generic sampling process which results in a subset \(N_Z \subset N\) of students and the subset \(M_Z\) of colleges in the sample. For the sets, \(N_Z\) and \(M_Z\), we make the following assumption.

Assumption 3.1 (Sampling Process). (i) \(N_Z\) is \(\sigma(\tilde{Z}, \zeta)\)-measurable and \(M_Z\) is \(\sigma(\tilde{Z})\)-measurable, where \(\zeta\) is a random vector that is conditionally independent of \(\tilde{S}\) given \(\tilde{Z}\).

(ii) Students' sampling indicators, 1\(\{i \in N_Z\}\)'s, are conditionally i.i.d. given \((\tilde{Z}, \tilde{S})\).

The randomness in the sampling process is captured by the random vector \(\zeta\). We require that it is conditionally independent of \(\tilde{S}\) given \(\tilde{Z}\). In other words, the sampling does not depend on the students' individual characteristics. The subset \(N_Z\) of students in the sample can potentially depend on the aggregate characteristics of colleges. As for colleges, we assume that we observe a non-empty subset \(M_Z \subset M'\), where \(M_Z\) is \(\sigma(\tilde{Z})\)-measurable. For example, we may take \(M_Z = \{j \in M : Z_j \in B\}\) for some set \(B\), i.e., the set of colleges whose observed characteristics take values in the set \(B\). It is important to note that the conditional independence of the sampling indicators 1\(\{i \in N_Z\}\) does not imply that the matching outcomes for two students, \(Y_{i_1}\) and \(Y_{i_2}\), are conditionally independent given \(\tilde{Z}\) regardless of whether we condition on that the two students are in the sample or not.

The sampling process we consider is general. It covers the scheme of random sampling. Assumption 3.1 also accommodates the case where we observe the entire set of students and colleges. In this case, Assumption 3.1(ii) is trivially satisfied, because 1\(\{i \in N_Z\}\)'s are simply constants of ones. Let \(n_Z = |N_Z|\), i.e., the number of the students in the sample. Analogously,
we define $m_Z = |M_Z|$. The sampling process accommodates the case with $n_Z/n \to 0$ as $n \to \infty$ and the case with $n_Z = \alpha n$ for all $n \geq 1$ for some $\alpha \in (0, 1]$.

Let us define a generic form of a statistic:

$$
\hat{\theta}(\tau) \equiv \sum_{j \in M_Z} \frac{1}{n_Z} \sum_{i \in N_Z} \tau_j(X_i, Z) 1\{Y_i = j\},
$$

and the target parameter:

$$(10) \quad \theta(\tau; \tilde{Z}) \equiv \sum_{j \in M_Z} \mathbb{E}[\tau_j(X_i, Z) 1\{Y_i = j\} \mid \tilde{Z}], \quad \tau = (\tau_0, \tau_1, \ldots, \tau_m),$$

for some real functions $\tau_j$ such that $\theta(\tau; \tilde{Z})$ exists. As we will see, various statistics involving empirical matching probabilities take the form $\hat{\theta}(\tau)$ for an appropriate choice of $\tau_j$. We emphasize that the population quantity $\theta(\tau; \tilde{Z})$ depends on the finite matching market, and as such, it depends on $n$ and $m$, although the dependence is left implicit in our notation for simplicity.\(^9\)

Due to the conditional exchangeability result in Lemma 2.1 and Assumption 3.1, $\hat{\theta}(\tau)$ is an unbiased estimator of the target parameter $\theta(\tau; \tilde{Z})$.

**Lemma 3.1.** Suppose that Assumptions 2.1, 2.6 and 3.1 hold. Then,

$$
\mathbb{E}[\hat{\theta}(\tau) \mid \tilde{Z}] = \theta(\tau; \tilde{Z}).
$$

As for the functions, $\tau_j$, we make the following assumption.

**Assumption 3.2.** There exists a map $\tau : Z^m \to [1, \infty)$ such that for all $x \in \mathcal{X}$ and $z \in Z^m$,

$$
\max_{j \in M'} |\tau_j(x, z)| \leq \tau(z),
$$

where $\mathcal{X}$ and $Z$ denote the sets from which $X_i$ and $Z_j$ take values respectively.

In many applications, it is not hard to find the bound $\tau(z)$. We illustrate this in the simple example involving individual matching probabilities below. See Section 3.3 for more examples.

**Example 3.1** (Individual Matching Probabilities). We obtain an estimator of the probability of matching with a given college (or being unmatched) as follows. For a fixed $j \in M$, we take

\(^9\)The dependence of the population object on the number of agents in our case can be viewed as arising from a finite population approach. A notable example is the average treatment effect defined as the average of the expected treatment effects, where the expectation is taken with respect to a distribution representing a “superpopulation”. (See, e.g., Imbens and Wooldridge (2009) and Imai, King, and Stuart (2008).) Such a finite population approach is also used in settings with a large network. (See, e.g., Aronow and Samii (2017), Leung (2020), and He and Song (2023).)
$M_z = \{j\}$ and consider

$$\hat{\theta}(\tau) = \frac{1}{n_z} \sum_{i \in N_z} 1\{Y_i = j\}, \tau = (\tau_0, \tau_1, ..., \tau_m),$$

where $\tau_j(X_i, Z) = 1$. Thus, Assumption 3.2 is satisfied with the map $\bar{\tau}(z) = 1$.

Suppose that one is interested in estimating the fraction of students with characteristic $X_i$ being in some set $A$ among those that are matched to college $j$. Then we may consider

$$\hat{\theta}(\tau) = \frac{1}{n_z} \sum_{i \in N_z} 1\{X_i \in A\} 1\{Y_i = j\},$$

where $\tau_j(X_i, Z) = 1\{X_i \in A\}$. Here, Assumption 3.2 is satisfied by taking $\bar{\tau}(z) = 1$. □

The following theorem is our main result.

**Theorem 3.1.** Suppose that Assumptions 2.1-3.2. Then, there exist constants $C > 0$ and $n_0 \geq 2$ which depend only on the constant $\overline{C}$ and the set $B$ in Assumption 2.4 such that whenever $n \geq n_0$, we have for all $t > 0$,

$$P \left\{ \left| n_z \hat{\theta}(\tau) - \mathbb{E}[n_z \hat{\theta}(\tau) | \tilde{Z}] \right| \geq n \pi_Z t | \tilde{Z} \right\} \leq 6 \exp \left( -\frac{Cn(t^2 \wedge t^{3/2})}{\pi^2(Z)(a_n + b_n t)} \right) + 2 \exp \left( -\frac{n \pi_Z t^2}{8}\pi^2(Z)(1 + t) \right),$$

where $\pi_Z = P\{i \in N_z | \tilde{Z}\}$ and, with $\bar{\sigma}_n = \sigma_n \vee n^{-5/6},$

$$a_n = n^2 \bar{\sigma}_n^2 \ln(nm) + 1 \text{ and } b_n = n \sqrt{n} \bar{\sigma}_n.$$

The bound in Theorem 3.1 is a finite sample bound. As we will see below in Corollary 3.1, this theorem can be used to derive the rate of convergence of $\hat{\theta}(\tau) - \theta(\tau; \tilde{Z})$. Due to the finite sample nature of the bound in the theorem, we can see what conditions we require for a sequence of matching markets when we derive the rate of convergence.

**Corollary 3.1.** Suppose that Assumptions 2.1-3.2 hold for all $n, m \geq 1$ and $m = g(n)$ for some function $g$. Furthermore, assume that the following conditions hold.

(i) The constant $\overline{C} > 0$ and the set $B$ in Assumption 2.4 are independent of $n$.

(ii) There exists an absolute constant $C_1 > 0$ such that $\bar{\tau}(Z) < C_1$ for all $n \geq 1$.

(iii) $\bar{\sigma}_n \sqrt{n \ln(nm)} + (n \pi_Z)^{-1/2} \to 0$, as $n \to \infty$.

Then, as $n \to \infty$, we have

$$|\hat{\theta}(\tau) - \theta(\tau; \tilde{Z})| = O_p \left( \bar{\sigma}_n \sqrt{n \ln(nm)} + (n \pi_Z)^{-1/2} \right).$$
In regards to Condition (i), note that the constant $C > 0$ and the set $B$ are concerned only with the distribution of the colleges' and students' types, not with the matching mechanisms, report maps, or the capacities. Condition (ii) is easily checked as it depends on the choice of the map $\tau_j$ in the statistic.

The convergence in (12) shows the rate at which the randomness of $\hat{\theta}(\tau)$, arising from the variations of students' idiosyncratic characteristics, disappears under the conditions stated in the corollary. In other words, when the number of the students in the sample is large enough, and $\sigma_n$ and the number of the colleges in the sample satisfy that

$$\tilde{\sigma}_n \sqrt{n \ln(nm)} + (n\pi_2)^{-1/2} \to 0,$$

we have

$$\hat{\theta}(\tau) - \theta(\tau; \tilde{Z}) = o_P(1),$$

as $n \to \infty$. That is, particular realizations of the students' idiosyncratic characteristics become more and more irrelevant in determining the value of $\hat{\theta}(\tau)$, and the estimated quantity $\hat{\theta}(\tau)$ gets closer to the population quantity $\theta(\tau; \tilde{Z})$.

### 3.2. Discussion

To derive the concentration inequality in Theorem 3.1, we first write

$$n_Z \hat{\theta}(\tau) - E[n_Z \hat{\theta}(\tau) | \tilde{Z}] = n_Z \hat{\theta}(\tau) - E[n_Z \hat{\theta}(\tau) | \tilde{Z}, \tilde{S}]$$

$$+ E[n_Z \hat{\theta}(\tau) | \tilde{Z}, \tilde{S}] - E[n_Z \hat{\theta}(\tau) | \tilde{Z}].$$

The first difference on the right hand side is easy to handle because once we condition on $\tilde{Z}, \tilde{S}$, $n_Z \hat{\theta}(\tau)$ is a weighted sum of independent Bernoulli random variables, $1\{i \in N_Z\}$. We deal with this difference by using a concentration inequality in Chung and Lu (2002). The main challenge is to deal with the second difference, because the match outcomes $Y_i$ are cross-sectionally correlated in a complex form after conditioning on $\tilde{Z}$. To this end, we use a conditional version of McDiarmid’s inequality which is stated as follows:

**Lemma 3.2** (McDiarmid’s Inequality). Suppose that $W_i$’s are random elements which take values in a space $W$ and are conditionally independent given a $\sigma$-field, $F$, and $U$ is a random element that is $F$-measurable, and takes values from a space $U$. Let $g : W^n \times U \to R$ be a map such that for each $i = 1, \ldots, n$, there exists a constant $c_i > 0$ satisfying that for all $w_1, \ldots, w_m$ and $w_i' \in W$, and for all $u \in U$,

$$|g(w_1, \ldots, w_{i-1}, w_i, w_{i+1}, \ldots, w_n, u) - g(w_1, \ldots, w_{i-1}, w_i', w_{i+1}, \ldots, w_n, u)| \leq c_i.$$
Then, for all $t > 0$,

$$
P \{ |g(W_1, ..., W_n, U) - E[g(W_1, ..., W_n, U) | \mathcal{F}]| \geq t \} \leq 2 \exp \left( - \frac{2t^2}{\sum_{i=1}^{n} c_i^2} \right).
$$

A key step in applying McDiarmid’s inequality involves establishing that the matching mechanism of interest obeys the **bounded difference condition** in (15). The bounded difference condition shows how the function $g$ varies as the value of a single argument is arbitrarily perturbed. The inequality shows that the distribution of $g(W_1, ..., W_n, U)$ concentrates more around its conditional mean if the bounds $c_i$ in (15) are small.

In our context, we consider the map

$$
g(W_1, ..., W_n, U) \equiv \sum_{j \in M} \sum_{i \in N} \tau_j(X_i, Z) 1\{\mu(i; \alpha(\tilde{S}, \tilde{Z})) = j\} \pi_Z = E[n_Z \hat{\theta}(\tau) | \tilde{Z}, \tilde{S}],
$$

where $W_i = \tilde{S}_i$ and $U = \tilde{Z}$, and the last equality follows by Assumption 3.1. We take $\mathcal{F}$ to be the $\sigma$-field of $\tilde{Z}$. As a crucial first step, we establish a bounded difference result for a stable matching under the assumption that the colleges’ preferences over students exhibit a form of limited heterogeneity in terms of maximum rank difference. The maximum rank difference measures the degree of preference heterogeneity among the colleges as we explain below.

Given a profile $w = (w_1, ..., w_m)$ of college preferences over $N'$, we define the **maximum rank difference** in $w$ by

$$
h(w) = \max_{j \in M} \max_{(i_1, i_2) \in N(w)} |w_j(i_1) - w_j(i_2)|,
$$

where $N(w)$ denotes the set of pairs of students that at least two colleges disagree on their rankings, i.e.,

$$
N(w) = \{(i_1, i_2) \in N' \times N' : i_1 \prec_{j_1} i_2, \text{ and } i_1 \succ_{j_2} i_2, \text{ for some } j_1, j_2 \in M\}.
$$

(If $N(w) = \emptyset$, we set $h(w) = 0$.) Hence $h(w)$ represents the maximum rank difference between any two students such that there is a disagreement over the ranking of the two students among colleges. If the preferences in $w$ are homogeneous, then $N(w) = \emptyset$, and $h(w) = 0$. On the other hand, if for example, there exist $w_j$ and $w_k$ in $w$ such that $w_j(i_1) > w_j(i_2)$ if and only if $w_k(i_1) < w_k(i_2)$, then we can have $h(w) = n$ in this case. This can occur, for example, if one college ranks as worst a student who another college ranks as best. Thus, the maximum rank difference $h(w)$ measures how “close” the preference orderings in $w$ are to each other. If $h(w) = k$, this means that if any two students have rank difference by more than $k$ in any college’s preference, all other colleges share the same ordering between the two students.

With the preference profile $u$ generated from $(\tilde{s}, \tilde{z})$, we rewrite $h(w)$ as $h(\tilde{s}, \tilde{z})$ from here on.

**Lemma 3.3.** Suppose that $(\tilde{s}, \tilde{z}), (\tilde{s}', \tilde{z}') \in \tilde{S} \times \tilde{Z}$ are chosen to satisfy the following three conditions.
(a) \( \tilde{s} \) and \( \tilde{s}' \) differ by \( \tilde{s}_i \), for at most one student \( i \), for some \( i \in N \).
(b) \( \mu(\cdot; \alpha(\tilde{s}, \tilde{z})) \) and \( \mu(\cdot; \alpha(\tilde{s}', \tilde{z})) \) are stable matchings under preference profiles \( u \) and \( u' \) which are respectively generated from \( (\tilde{s}, \tilde{z}) \) and \( (\tilde{s}', \tilde{z}) \).
(c) \( h(\tilde{s}, \tilde{z}) = h(\tilde{s}', \tilde{z}) = k \), for some \( k \in \{0, 1, \ldots, n\} \).

Then, for any \( j \in M' \),
\[
|\{ i \in N : 1\{\mu(i; \alpha(\tilde{s}, \tilde{z})) = j \} \neq 1\{\mu(i; \alpha(\tilde{s}', \tilde{z})) = j \} \}| \leq 32(k \lor 1) + 1. \tag{19}
\]

The remarkable aspect of Lemma 3.3 is that the bounded difference condition does not impose any restrictions on the report map \( \alpha \) other than requiring the resulting matching to be stable. The proof of Lemma 3.3 is provided in the online supplemental note. It is well known in the literature that under the assumption of strict preferences, the set of two-sided stable matchings has a lattice structure with upper and lower bounds corresponding to two extreme cases of student-optimal and student-worst matchings. Furthermore, it is well known that when the preferences of colleges over students are identical, these two bounds coincide and there is a unique stable matching. Lemma 3.3 shows a finite sample result, explicitly relating the degree of preference heterogeneity among colleges (as expressed by the maximum rank difference \( h(\tilde{s}, \tilde{z}) \)) to the “closeness” of the two bounds. When the maximum rank difference among colleges is bounded by \( k \), the student-optimal and student-worst matchings are different at most for \( (m + 1)(32(k \lor 1) + 1) \) students.\(^{10}\) The role of college preference heterogeneity in our approach are explored further in Example 3.2 below.

By applying McDiarmid’s inequality together with the bounded difference condition established in Lemma 3.3, we obtain a concentration inequality for the second difference on the right hand side of (14) as follows.

**Lemma 3.4.** Suppose that Assumptions 2.1-3.2 hold. Then, for all \( t > 0 \),
\[
P\{ |E[n_Z \hat{\theta}(\tau) \mid \tilde{Z}, \tilde{S}] - E[n_Z \hat{\theta}(\tau) \mid \tilde{Z}] | \geq t \mid \tilde{Z} \} \leq 2 \exp \left( -\frac{t^2}{2n \pi_Z^2 \tilde{\tau}^2(Z)(32\tilde{h}(\tilde{Z}) + 2)^2} \right),
\]
where \( \pi_Z = P\{ i \in N_Z \mid \tilde{Z} \} \) and \( \tilde{h}(\tilde{Z}) = \sup_{\tilde{s} \in S} h(\tilde{s}, \tilde{z}) \lor 1 \).

**Proof:** For each \( i \in N \), let \( \tilde{S}' \) be the same as \( \tilde{S} \) except that \( \tilde{S}_i \) is fixed to be an arbitrary vector \( \tilde{s}_i = (s_{i1}, \tilde{\eta}_i) \) in the support of \( \tilde{S}_i \), with \( s_i = (x_i, \tilde{e}_{i1}, \ldots, \tilde{e}_{im}) \), and \( \tilde{\eta}_i = (\tilde{\eta}_{i1}, \ldots, \tilde{\eta}_{im}) \). First, by Assumption 2.5, \( \mu(\cdot; \alpha(\tilde{S}', \tilde{Z})) \) and \( \mu(\cdot; \alpha(\tilde{S}, \tilde{Z})) \) are stable matchings with probability one. Define a measurable map \( \psi_n \) as
\[
\psi_n(\tilde{Z}, \tilde{S}) = E[n_Z \hat{\theta}(\tau) \mid \tilde{Z}, \tilde{S}].
\]

\(^{10}\)We thank an anonymous referee for pointing out this implication to us.
Let $\bar{Y}_{i'} = \mu(i'; \alpha(\bar{S}', \bar{Z}))$, $i' \in N$. In light of (17), $|\psi_n(\bar{Z}, \bar{S}) - \psi_n(\bar{Z}, \bar{S}')|$ is bounded by

$$2\overline{\pi}(Z)\tau Z + \overline{\pi}(Z) \sum_{j \in M} \sum_{i' \in N} |1\{Y_{i'} = j\} - 1\{\bar{Y}_{i'} = j\}| 1\{Y_{i'} = j\} \tau Z$$

$$+ \overline{\pi}(Z) \sum_{j \in M} \sum_{i' \in N} |1\{Y_{i'} = j\} - 1\{\bar{Y}_{i'} = j\}| 1\{Y_{i'} = j\} \tau Z.$$ 

By Lemma 3.3,

$$\max_{j \in M} \sum_{i' \in N} |1\{Y_{i'} = j\} - 1\{\bar{Y}_{i'} = j\}| \leq 32h(\bar{S}, \bar{Z}) + 1.$$ 

Hence, noting that $\zeta$ is conditionally independent of $\bar{S}$ given $\bar{Z}$, for each $i \in N$,

$$|\psi_n(\bar{Z}, \bar{S}) - \psi_n(\bar{Z}, \bar{S}')| \leq 2\tau Z \overline{\pi}(Z) + 2\tau Z \overline{\pi}(Z) \left(32h(\bar{Z}) + 1\right),$$

by Assumption 3.2 and Lemma 3.3. Since $\bar{S}_i$'s are conditionally independent given $\bar{Z}$ by Assumption 2.2, we obtain the desired bound by Lemma 3.2. □

To obtain the result of Theorem 3.1, we extend Lemma 3.4 to the case where for any pair of students, there may exist colleges whose rankings over the students disagree, though with small probability for many pairs.

As far as our approach of using McDiarmid’s inequality is concerned, some condition for limited heterogeneity in the college preferences appears inevitable. As we show in the example below, when the disagreement among colleges over the ranking of students is too extensive, a change in preference by a single student can alter the match of every student, which can even render the number of students on the right-hand side of (19) to be $n$. Thus, we cannot use the approach based on McDiarmid’s inequality to obtain a useful concentration inequality, if we allow an arbitrary degree of preference heterogeneity for colleges.

**Example 3.2.** Consider a college admissions market with five students, $N = \{i_1, i_2, i_3, i_4, i_5\}$, and three colleges, $M = \{j_1, j_2, j_3\}$ with $q_{i_1} = 1, q_{j_2} = q_{j_3} = 2$. Suppose that the preferences $u = (v, w)$ are given by Table 1. For example, the preference of student $i_2$ is such that the student

| Student Preferences | College Preferences |
|---------------------|---------------------|
| $i_1$ | $(j_1, j_2, j_3)$ | $j_1$ | $(i_1, i_4, i_2, i_3, i_5)$ |
| $i_2$ | $(j_2, j_3, j_1)$ | $j_2$ | $(i_1, i_5, i_2, i_3, i_4)$ |
| $i_3$ | $(j_2, j_3, j_1)$ | $j_3$ | $(i_2, i_3, i_4, i_5, i_1)$ |
| $i_4$ | $(j_3, j_1, j_2)$ |       |       |
| $i_5$ | $(j_3, j_2, j_1)$ |       |       |

| Preferences of Agents for Example 3.2 | |
|--------------------------------------|---------------------|
| $i_1$ | $(j_1, j_2, j_3)$ | $j_1$ | $(i_1, i_4, i_2, i_3, i_5)$ |
| $i_2$ | $(j_2, j_3, j_1)$ | $j_2$ | $(i_1, i_5, i_2, i_3, i_4)$ |
| $i_3$ | $(j_2, j_3, j_1)$ | $j_3$ | $(i_2, i_3, i_4, i_5, i_1)$ |
| $i_4$ | $(j_3, j_1, j_2)$ |       |       |
| $i_5$ | $(j_3, j_2, j_1)$ |       |       |
considers college $j_2$ the best, $j_3$ the second best, and $j_1$ the worst. Consider the following stable matching that is obtained from the deferred acceptance algorithm under the given preferences:

$$(\mu(i_1), \mu(i_2), \mu(i_3), \mu(i_4), \mu(i_5)) = (j_1, j_2, j_2, j_3, j_3).$$

Next, suppose that the preferences of agents are instead given by $u' = (v', w)$, where $v'$ is defined to be identical to $v$, except that we replace the preference of student $i_1$ with the ordering $(j_2, j_3, j_1)$. The student-optimal matching under $u' = (v', w)$ is $\mu' = (j_2, j_3, j_3, j_1, j_2)$. Thus, as students move from matching $\mu$ to matching $\mu'$ due to one student’s preference change, all of the students end up being matched with a different college. ■

The bound in Lemma 3.3 can be tighter when there are vacancies at the colleges. Indeed, Example 3.3 below illustrates how the effects of extensive college-preference heterogeneity can be mitigated by the presence of vacancies at the colleges. Since the bounded difference condition must account for a ‘worst-case scenario’ in which such vacancies are absent at the colleges, this example suggests that the bound in Lemma 3.3 can be conservative in practice when at least some colleges have vacancies.

**Example 3.3.** Consider again the college admissions market introduced in Example 3.2. This time, however, suppose that $q_1 = 1$, $q_2 = 3$, $q_3 = 2$. That is, college $j_2$ has an additional position. As before, the student optimal matching under $u$ is $\mu = (j_1, j_2, j_2, j_3, j_3)$. However, the fact that college $j_2$ has a vacant position at $\mu$ implies that the change of preferences from $v$ to $v'$ (as defined in Example 3.2) would lead to only student $i_1$ changing colleges. That is, the presence of a vacancy at college $j_2$ prevents the ‘cascade’ of changes that occurred in the previous example. By the same logic, if every college at $\mu$ had one vacant position, the change in preferences from $u$ to any profile $u'$ that differed in the preference of one student would lead to at most one student changing college. ■

### 3.3. Examples

3.3.1. **Individual Matching Probabilities.** We revisit Example 3.1. By Lemma 2.1, the conditional distribution of $(X_i, Y_i)$ given $\tilde{Z}$ is identical across $i$’s. As long as (13) is satisfied, by Corollary 3.1, we have

$$\frac{1}{n_Z} \sum_{i \in N_Z} 1 \{ Y_i = j \} = P \{ Y_i = j \mid \tilde{Z} \} + O_p \left( \tilde{\sigma}_n \sqrt{n \ln(nm)} + (n\pi_Z)^{-1/2} \right).$$

Thus, we obtain the rate of convergence for the individual matching probabilities. Similarly, we obtain the same rate of convergence for the second statistic:

$$\frac{1}{n_Z} \sum_{i \in N_Z} 1 \{ X_i \in A, Y_i = j \} = P \{ X_i \in A, Y_i = j \mid \tilde{Z} \} + O_p \left( \tilde{\sigma}_n \sqrt{n \ln(nm)} + (n\pi_Z)^{-1/2} \right).$$
3.3.2. **Matching Probability on Characteristics.** In many situations, it is of interest to estimate the probability of matching on characteristics. Let us assume the specification of $X_i$ and $Z_j$ as in (3), so that $X_i$ and $Z_j$ denote student $i$’s own and college $j$’s own characteristics. One might be interested in measuring the fraction of students being matched with a college $j$ with characteristic $Z_j \in A'$ for a set $A'$, when the students have characteristic $X_i$ in $A$. By taking $\tau_j(X_i, Z) = 1\{X_i \in A, Z_j \in A'\}$ and $M_Z = M'$ in the definition of $\hat{\theta}(\tau)$, we obtain

$$\hat{\theta}(\tau) = \frac{1}{n_Z} \sum_{i \in N_Z} \sum_{j \in M'} 1\{X_i \in A, Z_j \in A'\} 1\{Y_i = j\}. \tag{24}$$

By Corollary 3.1, we have

$$\frac{1}{n_Z} \sum_{i \in N_Z} \sum_{j \in M'} 1\{X_i \in A, Z_j \in A', Y_i = j\} = \sum_{j \in M'} P\{X_i \in A, Z_j \in A', Y_i = j \mid \tilde{Z}\} + O_P\left(\sigma \sqrt{n \ln(nm)} + (n \pi_Z)^{-1/2}\right).$$

3.3.3. **Distribution of Characteristics of Students Matched to a College.** It is often of interest to estimate the distribution of characteristics of students matched with a specific college under a stable matching. Define the conditional CDF of students’ characteristics conditional on that the student is matched with college $j$:

$$\hat{F}_j(x \mid \tilde{Z}) = \frac{\sum_{i \in N_Z} 1\{X_i \leq x, Y_i = j\}}{\sum_{i \in N_Z} 1\{Y_i = j\}} \quad \text{and} \quad F_j(x \mid \tilde{Z}) = P\{X_i \leq x \mid Y_i = j, \tilde{Z}\}, \tag{25}$$

where $X_i \in \mathbb{R}^d$, and the inequality between vectors is element-wise. Suppose that the set of colleges, $M$, is fixed, i.e., does not depend on $n$, so that the number of colleges $m$ is also fixed. We would like to show that $\hat{F}_j(x \mid \tilde{Z})$ and $F_j(x \mid \tilde{Z})$ get closer to each other as $n \to \infty$. Let us make the following assumption.

**Assumption 3.3.** (i) Each random vector, $X_i$, $i \in N$, is either discrete with a finite support that is independent of $n$, or has a continuous conditional distribution function given $\tilde{Z}$.

(ii) There exist $\epsilon > 0$ and a nonempty subset $M(\epsilon) \subset M'$ such that for all $n \geq 1$, and all $j \in M(\epsilon)$, we have $P\{Y_i = j \mid \tilde{Z}\} > \epsilon$.

Then, we can obtain the uniform convergence of $\hat{F}_j(\cdot \mid \tilde{Z})$ to $F_j(\cdot \mid \tilde{Z})$, as $n \to \infty$.

**Corollary 3.2.** Suppose that $M$ is a fixed set not depending on $n$, and that Assumptions 2.1-2.4 and 3.3 hold. Suppose further that (13) holds.
Then for all \( j \in M(\epsilon) \) with the set \( M(\epsilon) \) appearing in Assumption 3.3(ii), as \( n \to \infty \),
\[
\sup_{x \in \mathbb{R}^d} \left| \hat{F}_j(x \mid \tilde{Z}) - F_j(x \mid \tilde{Z}) \right| \to_p 0.
\]

3.3.4. Consistent Estimation of a Measure of Positive Assortative Matching. The result of Corollary 3.2 can be used to prove consistency of a measure of positive assortative matching. For example, one may want to measure the positive (stochastic) assortative matching between students and colleges along two variables \( X_{i,k} \) and \( Z_{Y_i,r} \), where \( X_{i,k} \) denotes the \( k \)-th element of \( X_i \) and \( Z_{Y_i,r} \) the \( r \)-th element of \( Z_i \). One way to measure it is to use a quantity that stems from Spearman’s rho defined as follows.\(^{11}\)
\[
\rho = 12 \int \int_{\mathbb{R}} (F_{X,Z}(t,s) - F_X(t)F_Z(s)) dF_X(t)dF_Z(s),
\]
where
\[
F_{X,Z}(t,s) = \mathbb{P}\{X_{i,k} \leq t, Z_{Y_i,r} \leq s \mid \tilde{Z}, Y_i \neq 0\},
\]
\[
F_X(t) = \mathbb{P}\{X_{i,k} \leq t \mid \tilde{Z}, Y_i \neq 0\}, \text{ and}
\]
\[
F_Z(s) = \mathbb{P}\{Z_{Y_i,r} \leq s \mid \tilde{Z}, Y_i \neq 0\}.
\]

Note that conditional on \( \tilde{Z}, Z_{Y_i} \) is still random, due to the randomness of the students’ preferences that affect the matching outcome \( Y_i \).

We can construct the estimator of \( \rho \) as follows. First, we define \( N_{1,Z} = \{i \in N_Z : Y_i \neq 0\} \), and \( n_{1,Z} = |N_{1,Z}| \). Let
\[
\hat{\rho} = \frac{12}{n_{1,Z}} \sum_{i \in N_{1,Z}} \left( \hat{F}_{X,Z,-i}(X_{i,k}, Z_{Y_i,r}) - \hat{F}_{X,-i}(X_{i,k}) \hat{F}_{Z,-i}(Z_{Y_i,r}) \right),
\]
where
\[
\hat{F}_{X,Z,-i}(t,s) = \frac{1}{n_{1,Z}-1} \sum_{i' \in N_{1,Z}\setminus\{i\}} 1\{X_{i',k} \leq t, Z_{Y_i,r} \leq s\},
\]
\[
\hat{F}_{X,-i}(t) = \frac{1}{n_{1,Z}-1} \sum_{i' \in N_{1,Z}\setminus\{i\}} 1\{X_{i',k} \leq t\}, \text{ and}
\]
\[
\hat{F}_{Z,-i}(s) = \frac{1}{n_{1,Z}-1} \sum_{i' \in N_{1,Z}\setminus\{i\}} 1\{Z_{Y_i,r} \leq s\}.
\]

Let us make the following assumption which is Assumption 3.3(i) for \( X_{i,k} \).

\(^{11}\)The Spearman’s rho has been proposed or used as a measure of positive assortative matching in the literature. See, e.g., Gihleb and Lang (2016), Hagedorn, Law, and Manovskii (2017), and Lochner and Schultz (2021). The quantity \( \rho \) can be viewed as the population version of Spearman’s \( \rho \) between \( X_i \) and \( Z_{Y_i,r} \) after randomly selecting \( i \) from the students matched with some college.
Assumption 3.4. Each random variable, $X_{i,k}$, $i \in N$, is either discrete with a finite support that is independent of $n$, or has a continuous conditional distribution function given $\tilde{Z}$.

Then, using Corollary 3.2, we can show that $\hat{\rho}$ is consistent for $\rho$.

Corollary 3.3. Suppose that $M$ is a fixed set not depending on $n$, and that Assumptions 2.1-2.4 and 3.4 hold. Suppose further that (13) holds, and with probability one,

$$\lim_{n \to \infty} \inf \mathbb{P} \left\{ Y_i \neq 0 \mid \tilde{Z} \right\} > 0. \tag{26}$$

Then, as $n \to \infty$,

$$\hat{\rho} - \rho \to_p 0.$$

The condition (26) is a very mild condition. The failure of this condition means that from some large $n$ on, no student is matched with any college.

3.3.5. Consistent Estimation of Conditional Matching Probabilities. Suppose that we have a fixed set of colleges $M$ which does not depend on $n$. Let us define the conditional probability of a student $i$ with observed characteristic $x$ matched with college $j$ as follows:

$$p(j \mid x, \tilde{Z}) = \mathbb{P} \left\{ Y_i = j \mid X_i = x, \tilde{Z} \right\}. \tag{27}$$

Suppose that $X_i$ is a continuous random vector in $\mathbb{R}^d$. We consider the following local constant estimator of $p(j \mid x, \tilde{Z})$:

$$\hat{p}(j \mid x, \tilde{Z}) \equiv \frac{\sum_{i \in N_Z} 1\{Y_i = j\} \mathcal{K}_h(X_i - x)}{\sum_{i \in N_Z} \mathcal{K}_h(X_i - x)} = \frac{\frac{1}{n \pi_Z} \sum_{i \in N_Z} 1\{Y_i = j\} \mathcal{K}_h(X_i - x)}{\frac{1}{n \pi_Z} \sum_{i \in N_Z} \mathcal{K}_h(X_i - x)}, \tag{28}$$

where $\mathcal{K}_h(\cdot) = \mathcal{K}(\cdot/h)/h^d$ and $\mathcal{K}$ is a multivariate kernel function on $\mathbb{R}^d$, and $h$ is a bandwidth.

When we show the consistency of $\hat{p}(j \mid x, \tilde{Z})$, a new challenge (as compared to the standard nonparametric analysis) arises for dealing with the convergence of the following term in the numerator.

$$\frac{1}{n \pi_Z} \sum_{i \in N_Z} 1\{Y_i = j\} \mathcal{K}_h(X_i - x). \tag{29}$$

In particular, we would like to show the following:

$$\Delta_n(x) \equiv \frac{1}{n \pi_Z} \sum_{i \in N_Z} \left(1\{Y_i = j\} \mathcal{K}_h(X_i - x) - \mathbb{E} \left[ 1\{Y_i = j\} \mathcal{K}_h(X_i - x) \mid \tilde{Z} \right]\right) = o_p(1), \tag{30}$$
as \( n \to \infty \). For this, we take

\[
\hat{\theta}(\tau) = \frac{1}{n^2} \sum_{i \in N_x} 1\{Y_i = j\} \tau_j (X_i, Z), \quad \text{with } \tau_j (X_i, Z) = K_h (X_i - x) h^d,
\]

so that \( \Delta_n(x) = n^2 (\hat{\theta}(\tau) - E[\hat{\theta}(\tau) \mid \hat{Z}])/(n \pi_Z h^d) \). Then, it is not hard to see that Assumption 3.2 is satisfied with \( b(z) = \|K\|_\infty \vee 1 \), where \( \|K\|_\infty = \sup_{x \in R^d} |K(x)| \). Therefore, we find from Theorem 3.1 that

\[
P \left\{ |\Delta_n(x)| > t \mid \hat{Z} \right\} \leq 6 \exp \left( -n \sigma^2 (h^d) \right) + 2 \exp \left( \frac{n \pi_Z (h^d)^2}{4 \sigma^2 (Z) (1 + h^d)} \right).
\]

Hence, if \( \sqrt{n} \sigma n h^{-d} \sqrt{\ln n} + (n \pi_Z)^{-1/2} h^{-d} \to 0 \) as \( n \to \infty \), then we have

\[
\Delta_n(x) = O_p \left( \sqrt{n} \sigma n h^{-d} \sqrt{\ln n} + (n \pi_Z)^{-1/2} h^{-d} \right).
\]

After dealing with the bias part, we can obtain the consistency of the conditional matching probability estimators. Let us present the result formally below.

Let us first introduce smoothness conditions for controlling the bias part.

**Assumption 3.5.** (i) The conditional probability \( P \{Y_i = j \mid X_i = x, \hat{Z} = \hat{z} \} \) and the conditional density function of \( X_i \) given \( \hat{Z} = \hat{z}, f_{\hat{Z}|\hat{z}}(x \mid \hat{z}) \), both as a function of \( x \in R^d \), are twice continuously differentiable with derivatives bounded uniformly over \( j \) and over \( \hat{z} \in \hat{Z} \).

(ii) The conditional density function \( f_{\hat{Z}|\hat{z}}(x \mid \hat{z}) \) is bounded away from zero on an open ball around \( x \) in \( R^d \) uniformly over \( \hat{z} \in \hat{Z} \).

(iii) The kernel function \( K \) is symmetric around zero, vanishes outside a compact set, and takes values in \([-1, 1]\).

Assumptions 3.5(i) and (ii) are an adapted version of standard assumptions used in non-parametric kernel estimation. Using these assumptions and Corollary 3.1, we can show the following

\[
\frac{1}{n \pi_Z} \sum_{i \in N_x} K_h (X_i - x) = f_{\hat{Z}|\hat{z}}(x \mid \hat{Z}) + O_p \left( \frac{1}{\sqrt{n \pi_Z h^d}} + h^2 \right), \quad \text{and}
\]

\[
\frac{1}{n \pi_Z} \sum_{i \in N_x} E[1\{Y_i = j\} K_h (X_i - x) \mid \hat{Z}] = p(j \mid x, \hat{Z}) f_{\hat{Z}|\hat{z}}(x \mid \hat{Z}) + O_p(h^2),
\]

by following the standard arguments. (See Section 2.1 of Li and Racine (2007). Recall that by Assumption 2.3, \( X_i \)'s are conditionally i.i.d. across \( i \)'s given \( \hat{Z} \).

**Corollary 3.4.** Suppose that Assumptions 2.1-2.4 and 3.5 hold. Suppose further that \( M \) is a fixed set not depending on \( n \), and that \( \sqrt{n} \sigma n h^{-d} \sqrt{\ln n} + (n \pi_Z)^{-1/2} h^{-d} \to 0 \), as \( n \to \infty \).
Then, for each $j \in M$ and $x \in \mathbb{R}^d$, as $n \to \infty$,

\begin{equation}
\hat{p}(j \mid x, \tilde{Z}) = p(j \mid x, \tilde{Z}) + O_p\left(\sqrt{n} \tilde{\sigma}_n h^{-d} \sqrt{\ln n} + (n \pi_Z)^{-1/2} h^{-d} + h^{-2}\right).
\end{equation}

Hence, if $\sqrt{n} \tilde{\sigma}_n h^{-d} \sqrt{\ln n} + (n \pi_Z)^{-1/2} h^{-d} \to 0$ and $h \to 0$, as $n \to \infty$, $\hat{p}(j \mid x, \tilde{Z})$ is consistent.

4. Conclusion

This paper considers a large two-sided matching market, where a matching between the two sides is stable. In such a situation, it is a non-trivial matter to establish limit theorems for statistics such as empirical matching probabilities as the number of market participants grows. Using the re-equilibration arguments from economic theory, we derive a concentration inequality for various statistics that involve matching outcomes in this environment.

In order to develop inference that does not require random sampling from a large matching, one needs to take into account the dependence structure of the observations carefully. However, this is challenging in large matching markets. It is left to future research to establish limit distribution theory for the large matching setting, where a complex dependence structure arises naturally due to the interdependence among agents in the underlying market.

5. Appendix: Proofs

**Proof of Lemma 2.1:** We write briefly $\mu(i; \tilde{S}_\pi) = \mu(i; \alpha(\tilde{S}_\pi, \tilde{Z}))$ again. Then, note that

\begin{equation}
\begin{bmatrix}
(\tilde{S}_1, \mu(1; \tilde{S})) \\
\vdots \\
(\tilde{S}_n, \mu(n; \tilde{S}))
\end{bmatrix}
=^d
\begin{bmatrix}
(\tilde{S}_{\pi(1)}, \mu(1; \tilde{S}_\pi)) \\
\vdots \\
(\tilde{S}_{\pi(n)}, \mu(n; \tilde{S}_\pi))
\end{bmatrix},
\end{equation}

where $=^d$ denotes the equality of conditional distributions given $\tilde{Z}$. The distributional equality comes from Assumption 2.1. To see the distributional equality, we take a hyper-retangular set $A = A_1 \times \ldots \times A_n$, where $A_k$ is Borel. Then, we can write

\begin{align*}
P \{(\tilde{S}_1, \mu(1; \tilde{S}_1), \ldots, \tilde{S}_n, \mu(n; \tilde{S}_1, \ldots, \tilde{S}_n)) \in A \mid \tilde{Z}\} \\
= P \{(\tilde{S}_1, \mu(1; \tilde{S}_1, \ldots, \tilde{S}_n)) \in A_1, \ldots, (\tilde{S}_n, \mu(n; \tilde{S}_1, \ldots, \tilde{S}_n)) \in A_n \mid \tilde{Z}\} \\
= P \{(\tilde{S}_1, \ldots, \tilde{S}_n) \in B_1, \ldots, (\tilde{S}_1, \ldots, \tilde{S}_n) \in B_n \mid \tilde{Z}\},
\end{align*}

for some Borel sets $B_1, \ldots, B_n$. As we can approximate each set $B_k$ by a measurable hyper-rectangle with arbitrary accuracy, we can approximate the last probability by

\begin{align*}
P \{(\tilde{S}_1, \ldots, \tilde{S}_n) \in B_{11} \times \ldots \times B_{1n}, \ldots, (\tilde{S}_1, \ldots, \tilde{S}_n) \in B_{n1} \times \ldots \times B_{nn} \mid \tilde{Z}\} \\
= P \{(\tilde{S}_{\pi(1)}, \ldots, \tilde{S}_{\pi(n)}) \in B_{11} \times \ldots \times B_{1n}, \ldots, (\tilde{S}_{\pi(1)}, \ldots, \tilde{S}_{\pi(n)}) \in B_{n1} \times \ldots \times B_{nn} \mid \tilde{Z}\},
\end{align*}
Lemma 5.1. Suppose that Assumptions 2.1-3.2 hold. Then, for all $t$ at most one college. The desired result follows by Lemma 2.1 of Chung and Lu (2002).

Proof: We apply Lemma 2.1 of Chung and Lu (2002) by letting $p_i \equiv P\{i \in N_Z \mid \tilde{Z}, \tilde{S}\} = P\{i \in N_Z \mid \tilde{Z}\} = \pi_Z$, (with the second equality due to the conditional independence of $\zeta$ and $\tilde{S}$ given $\tilde{Z}$) and $a_i = \sum_{j \in M_Z} \tau_j(X_i, Z)1\{Y_i = j\}$, and letting $X_i$ in the lemma be $1\{i \in N_Z\}$ here. Note that

$$
\sum_{i \in N} a_i^2 p_i = \sum_{i \in N} \left( \sum_{j \in M_Z} \tau_j(X_i, Z)1\{Y_i = j\} \right)^2 \pi_Z = \sum_{i \in N} \sum_{j \in M_Z} \tau_j^2(X_i, Z)1\{Y_i = j\}\pi_Z \leq \bar{\tau}^2(Z)n\pi_Z.
$$

The second equality and last inequality are due to the fact that each student is matched with at most one college. The desired result follows by Lemma 2.1 of Chung and Lu (2002).
The lemma below is used to translate the bound in terms of $\tilde{h}(Z)$ into that in terms of $\sigma_n$.

**Lemma 5.2.** Suppose that the preferences of colleges are generated according to Assumption 2.3 with a sequence $\sigma_n > 0$.

Then, for $\tilde{C} > 0$ and the set $B$ in Assumption 2.4(ii), we have that for all $t > 0$,

$$P \left\{ \sigma_n \max_{i \in N, j \in M} |\eta_{ij}| \leq t \text{ and } \tilde{h}(Z) > 12n\tilde{C}t + 2n \sup_{c \in B} R_n(c, 6t) + 1 \right\} = 0,$$

where $\tilde{h}(Z)$ is defined in Lemma 3.4,

$$R_n(c, t) = \frac{1}{n} \sum_{i \in N} \left( f(S_i; c, t) - E \left[ f(S_i; c, t) \mid Z \right] \right),$$

and $f(S_i; c, t) = 1 \{ |\lambda(S_i) - c| \leq t \}$.

**Proof:** For each $c, t \in \mathbb{R}$, define

$$N(S; c, t) = \{ i \in N : |\lambda(S_i) - c| \leq t \}.$$

Let $N_2$ be the set of pairs of students in $N$ who are ranked differently by some colleges. Suppose that $\sigma_n \max_{i \in N, j \in M} |\eta_{ij}| \leq t$. Then, whenever $i_1, i_2 \in N$ are such that $\lambda(S_{i_1}) - \lambda(S_{i_2}) > 2t$, we have $i_1 \succ_j i_2$ for all $j \in M$. Hence,

$$N_2 \subseteq \{ (i_1, i_2) \in N \times N : \left| \lambda(S_{i_2}) - \lambda(S_{i_1}) \right| \leq 2t \}.$$

Take any $(i_1, i_2)$ in the latter set. Then the college $j$’s rank difference between $i_1$ and $i_2$ is bounded by

$$|\{ i \in N \setminus \{i_1, i_2\} : i_1 \succ j i \succ j i_2 \}| 1\{ i_1 \succ j i_2 \} + \left| \{ i \in N \setminus \{i_1, i_2\} : i_2 \succ j i \succ j i_1 \} \right| 1\{ i_2 \succ j i_1 \} + 1.$$

Let us focus on the first term. Note that $i_1 \succ_j i \succ_j i_2$ implies that

$$0 < \lambda(S_{i_1}) - \lambda(S_{i_2}) + \sigma_n (\eta_{i_1 j} - \eta_{i_2 j}) \leq \lambda(S_{i_1}) - \lambda(S_{i_2}) + \sigma_n (\eta_{i_1 j} - \eta_{i_2 j}) \leq 4t,$$

which again implies that $-2t \leq \lambda(S_{i_2}) - \lambda(S_{i_2}) \leq 6t$. Hence, on the event that $\lambda(S_{i_2}) \in B$, with $B$ chosen to be the bounded set in Assumption 2.4(ii),

$$\left| \{ i \in N \setminus \{i_1, i_2\} : i_1 \succ_j i \succ_j i_2 \} \right| \leq \sup_{c \in B} |N(S; c, 6t)|.$$
We can rewrite
\[ |N(S; c, 6t)| = \sum_{i \in N} 1\{|\lambda(S_i) - c| \leq 6t\} \]
\[ = \sum_{i \in N} P\{ |\lambda(S_i) - c| \leq 6t \mid \tilde{Z} \} + nR_n(c, 6t) \leq 6n\tilde{C}t + nR_n(c, 6t), \]
where the last inequality follows by Assumption 2.4(ii). Let us define the event
\[ A_n(t) = \left\{ \sigma_n \max_{i \in N, j \in M} |\eta_{ij}| \leq t, \text{ and } \lambda(S_i) \in B \text{ for all } i \in N \right\}. \]
Thus, we have shown that on the event \( A_n(t) \), for any \((i_1, i_2) \in N_2\), the college \( j \)'s rank difference between \( i_1 \) and \( i_2 \) is bounded by
\[ 12n\tilde{C}t + 2n \sup_{c \in B} R_n(c, 6t) + 1. \]
The last bound does not depend on the particular choice \((i_1, i_2)\) from the set \( N_2 \). Hence on the event \( A_n(t) \), we have
\[ \bar{h}(\tilde{Z}) \leq 12n\tilde{C}t + 2n \sup_{c \in B} R_n(c, 6t) + 1. \]
Since \( \lambda(S_i) \in B \) for all \( i \in N \) with probability one by Assumption 2.4(ii),
\[ P\left\{ \sigma_n \max_{i \in N, j \in M} |\eta_{ij}| \leq t, \text{ and } \bar{h}(\tilde{Z}) > 12n\tilde{C}t + 2n \sup_{c \in B} R_n(c, 6t) + 1 \right\} = 0. \]

Let \( B \) be the bounded interval in Assumption 2.4(ii), and let \( \mathcal{L}(B) \) represent its Lebesgue measure. From here on, without loss of generality, we assume that \( \mathcal{L}(B) \geq 1 \) and \( \tilde{C} \geq 1 \). (If \( \mathcal{L}(B) < 1 \) or \( \tilde{C} < 1 \), we can replace \( \mathcal{L}(B) \) or \( \tilde{C} \) by \( \mathcal{L}(B) \vee 1 \) or \( \tilde{C} \vee 1 \) respectively below.) For any sequence \( t_n > 0 \), define
\[ \mathcal{F}(t_n) = \{ f(\cdot; c, 6t_n) : c \in B \}, \]
where \( f(\cdot; c, t) \) is as defined in Lemma 5.2. Note that if \( t_n > \mathcal{L}(B)/6 \), then \( \mathcal{F}(t_n) \) is the singleton of the constant function one.

**Lemma 5.3.** For each \( 0 < \epsilon \leq \min\{\sqrt{6t_n}, \mathcal{L}^{1/3}(B)\} \), with \( t_n \in (0, \mathcal{L}(B)/6] \), there exist brackets \([f_{L,j}, f_{U,j}]\), \( j = 1, \ldots, n[i](\epsilon) \), that cover \( \mathcal{F}(t_n) \) such that for each integer \( k \geq 2 \), and each \( j = \)
First, we take \( F_{12} \) where \( \lambda \) and define
\( S \) and \( \delta \) to be equal to
\[
(43) \quad E \left[ \left| f_{L,j}(S_i) - f_{U,j}(S_i) \right|^k | \hat{Z} \right] \leq (18\overline{C})^k \varepsilon^2 ,
\]
and
\[
(42) \quad \ln n_{[\varepsilon]}(\delta) \leq 1 + \ln L(B) - 3 \ln \varepsilon .
\]

**Proof:** The proof adapts part of the arguments in the proof of Proposition A.1 of Guerre and Sabbah (2012). First, we take \( \delta > 0 \) such that \( \delta \leq L^{2/3}(B) \), and define
\[
z_\delta(\overline{\lambda}; c) = \left( 1 - \min \left\{ \left( \overline{\lambda} - c - 6t_n \right) / \delta, 1 \right\} \right) \times 1 \left\{ 0 < \overline{\lambda} - c - 6t_n \right\}
\]
\[
+ 1 \left\{ \overline{\lambda} - c - 6t_n \leq 0 \right\}, \quad \overline{\lambda}, c \in \mathbb{R}.
\]

Let
\[
g_{U,\delta}(\overline{\lambda}; c) = z_\delta(\overline{\lambda}; c) - z_\delta(\overline{\lambda} + \delta + 12t_n; c), \quad \text{and}
\]
\[
g_{L,\delta}(\overline{\lambda}; c) = \min \left\{ z_\delta(\overline{\lambda} + \delta; c), z_\delta(-\overline{\lambda} + 2c + \delta; c) \right\}, \quad \overline{\lambda}, c \in \mathbb{R},
\]
and define
\[
f_{U,\delta}(s; c) = g_{U,\delta}(\lambda(s); c), \quad \text{and} \quad f_{L,\delta}(s; c) = g_{L,\delta}(\lambda(s); c),
\]
where \( \lambda \) is a map in Assumption 2.3(i). Then \( f_{U,\delta}(s; c) \) and \( f_{L,\delta}(s; c) \) are Lipschitz in \( c \) with coefficient equal to \( \delta^{-1} \). Furthermore,
\[
(43) \quad E \left[ \left( f_{U,\delta}(S_i; c) - f_{L,\delta}(S_i; c) \right)^2 | \hat{Z} \right] \leq E \left[ \left| f_{U,\delta}(S_i; c) - f_{L,\delta}(S_i; c) \right| | \hat{Z} \right]
\]
\[
\leq P \left\{ c + 6t_n \leq \lambda(S_i) \leq c + 6t_n + \delta | \hat{Z} \right\}
\]
\[
+ P \left\{ c - 6t_n - \delta \leq \lambda(S_i) \leq c - 6t_n | \hat{Z} \right\} \leq 2\overline{C} \delta,
\]
and \( f_{L,\delta}(s; c) \leq f(s; c, 6t_n) \leq f_{U,\delta}(s; c) \). (See Figure 1.) Define
\[
\mathcal{F}_{L,\delta} = \{ f_{L,\delta}(\cdot; c) : c \in B \}, \quad \text{and} \quad \mathcal{F}_{U,\delta} = \{ f_{U,\delta}(\cdot; c) : c \in B \}.
\]

For any real valued measurable map \( f \), we define \( \| f \|_{Z,2} = \sqrt{E[f^2(S_i) | \hat{Z}]} \). We choose \( \varepsilon \) such that \( 0 < \varepsilon \leq \min \{ \sqrt{6t_n}, L^{1/3}(B) \} \). Since \( f_{U,\delta}(s; c) \) and \( f_{L,\delta}(s; c) \) are Lipschitz in \( c \) with coefficient equal to \( \delta^{-1} \) and since \( B \) is a bounded interval, it follows by Theorem 2.7.11 of van der Vaart and Wellner (1996) that for any \( C_2 > 0 \), there exist \( 2C_2 \varepsilon^3 / \delta \)-brackets \( [f_{L,a,j}, f_{L,b,j}, j = 1, \ldots, n_{[\varepsilon]}(\delta)] \) (with respect to \( \| \cdot \|_{Z,2} \)) that cover \( \mathcal{F}_{L,\delta} \), and \( 2C_2 \varepsilon^3 / \delta \)-brackets \( [f_{U,a,j}, f_{U,b,j}, j = 1, \ldots, n_{[\varepsilon]}(\delta)] \) (also with respect to \( \| \cdot \|_{Z,2} \)) that cover \( \mathcal{F}_{U,\delta} \); in other

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12The brackets \( [f_{L,i}, f_{U,i}], j = 1, \ldots, n_{[\varepsilon]}(\delta) \), refer to pairs of functions \( f_{L,i} \) and \( f_{U,i} \) such that \( f_{L,i} \leq f_{U,i} \). We say that they cover \( \mathcal{F}(t_n) \) if for all \( f \in \mathcal{F}(t_n) \), there exists \( j \in \{ 1, \ldots, n_{[\varepsilon]}(\delta) \} \) such that \( f_{L,i} \leq f \leq f_{U,i} \).
words, for any pair \( f_{U,\delta}(\cdot; c) \) and \( f_{L,\delta}(\cdot; c) \), there exists \( j \in \{1, \ldots, n_{[\epsilon]}(\epsilon)\} \) satisfying
\[
(44) \quad f_{L,a,j}(\cdot; c) \leq f_{L,\delta}(\cdot; c) \leq f_{U,a,j}(\cdot) \leq f_{U,b,j}(\cdot; c) \leq f_{U,\delta}(\cdot; c).
\]
Hence, the brackets \([f_{L,j}, f_{U,j}] = [f_{L,a,j}, f_{U,b,j}]\) for \( j = 1, \ldots, n_{[\epsilon]}(\epsilon) \) cover \( F(t_n) \). We then set \( C_2 = (C/2)^{1/2} \) and \( \delta = \epsilon^2 \). Then, we have
\[
n_{[\epsilon]}(\epsilon) = \lfloor \mathcal{L}(B)/(2C)^{1/2}\epsilon^3 \rfloor \leq \lfloor \mathcal{L}(B)/\epsilon^3 \rfloor \text{ (because } C \geq 1),
\]
and obtain
\[
\ln n_{[\epsilon]}(\epsilon) \leq 1 + \ln \mathcal{L}(B) - 3 \ln \epsilon,
\]
due to the requirement that \( \epsilon^2 = \delta \leq \mathcal{L}^{2/3}(B) \). Therefore, we obtain the bound (42).

As for the bound (45), observe that from (44),
\[
(45) \quad \mathbb{E}\left[|f_{L,j}(S_i) - f_{U,j}(S_i)|^k | \tilde{Z}\right] = \mathbb{E}\left[(f_{U,b,j}(S_i) - f_{L,a,j}(S_i))^k | \tilde{Z}\right] 
\leq \mathbb{E}\left[(f_{L,a,j}(S_i) - f_{U,b,j}(S_i))^2 | \tilde{Z}\right],
\]
because \( f_{U,b,j} \) and \( f_{L,a,j} \) are bounded between 0 and 1 and \( k \geq 2 \). The last conditional expectation is bounded by
\[
3\mathbb{E}\left[(f_{L,a,j}(S_i) - f_{L,b,j}(S_i))^2 | \tilde{Z}\right] + 3\mathbb{E}\left[(f_{L,b,j}(S_i) - f_{U,a,j}(S_i))^2 | \tilde{Z}\right] 
\leq 3(2C \epsilon^2 + 2C \epsilon^2 + 2C \epsilon^2),
\]
where the first and third terms \( 2C \epsilon^2 \) are due to the choice of \( (2C)^{1/2} \epsilon \)-brackets and the middle term \( 2C \epsilon^2 \) is from (43) and (44). Because \( |f_{L,j}(S_i) - f_{U,j}(S_i)| \leq 1 \), we find that for all \( k \geq 2 \),
\[
(46) \quad \mathbb{E}\left[|f_{L,j}(S_i) - f_{U,j}(S_i)|^k | \tilde{Z}\right] \leq 18C \epsilon^2 \leq (18C)^{k/2} \epsilon^2,
\]
because \( 18C \geq 1 \). We obtain the desired result. ■
Lemma 5.4. Let \( t_n \) be a \( \sigma(\tilde{Z}) \)-measurable sequence of random variables such that \( t_n > n^{-1} \ln n \), and let \( R_n(c, t) \) be the map defined in Lemma 5.2.

Then, there exist constants \( C > 0 \) that depend only on the constant \( \bar{C} \) and the set \( B \) in Assumption 2.4(ii) such that

\[
P \left\{ \sup_{c \in B} |R_n(c, 6t_n)| \geq C t_n \mid \tilde{Z} \right\} \leq 2 \exp \left( -n t_n \right). \tag{47} \]

Proof: Suppose that \( t_n > \mathcal{L}(B)/6 \). Then, the bound (47) trivially holds, because \( R_n(c, 6t_n) = 0 \) for all \( c \in B \). For the rest of the proof, we assume that \( t_n \leq \mathcal{L}(B)/6 \). Let \( C' = 6\bar{C}M_1^{-2} \) for simplicity, where

\[
M_1 = \max \left\{ 18\bar{C}, \sqrt{2\bar{C}\mathcal{L}(B)} \right\}. \tag{48} \]

By Assumption 2.1, \( S_i \)'s are conditionally i.i.d. across \( i \)'s given \( \tilde{Z} \). We apply Corollary 6.9 of Massart (2007), p.194, to the sum \( M_1^{-1}nR_n(c, 6t_n) \). The corollary is based on Theorem 6.8 there. It suffices to verify two conditions in Theorem 6.8. First, for any \( k \geq 2 \),

\[
E \left[ |f(S_i; c, 6t_n)|^k \mid \tilde{Z} \right] = M_1^{-k}P \left\{ |\lambda(S_i) - c| \leq 6t_n \mid \tilde{Z} \right\} \leq C't_nM_1^{-k+2} \leq C't_n, \]

by Assumption 2.4(ii) and by \( M_1 \geq 1 \). So, the first condition in Theorem 6.8 of Massart (2007) is satisfied by taking \( \sigma^2 = C't_n \) and \( b = 1 \) there. To verify the second condition of the theorem, we invoke our Lemma 5.3. By (48), and choice of \( t_n \leq \mathcal{L}(B)/6 \), we have

\[
\sqrt{C't_n} \leq \sqrt{C'\mathcal{L}(B)/6} \leq \sqrt{0.5}, \tag{49} \]

because \( M_1 \geq \sqrt{2\bar{C}\mathcal{L}(B)} \). Furthermore, \( C't_n \leq \min\{6t_n, \mathcal{L}^{2/3}(B)\} \), by our choice of \( M_1 \) in (48) and because \( \mathcal{L}(B) \geq 1 \). Using (47) and Lemma 5.3, we see that the second condition of Theorem 6.8 of Massart (2007) holds for \( M_1^{-1}nR_n(c, 6t_n) \) with \( b = 1 \) again. We apply Corollary 6.9 of Massart (2007), p.194, (putting \( \varepsilon = 1 \) there) to find that for any positive number \( x \),

\[
P \left\{ M_1^{-1}n \sup_{c \in B} |R_n(c, 6t_n)| \geq nr_n(x) \mid \tilde{Z} \right\} \leq 2 \exp(-x), \]

where

\[
r_n(x) = \frac{27}{\sqrt{n}} \int_0^{\sqrt{C't_n}} \sqrt{\ln n_{[1]}(\varepsilon)} d\varepsilon + \frac{2}{n} \left( 1 + \sqrt{C't_n} \right) \ln n_{[1]}(\sqrt{C't_n}) \]

\[
+ 7 \left( n^{-1/2} \sqrt{2C't_n x + 2n^{-1} x} \right). \tag{50} \]
Proof of Theorem 3.1: First, we bound
\[ r_n(x) \leq \frac{27}{\sqrt{n}} \left( \sqrt{C't_n(1 + \ln \mathcal{L}(B))} + \sqrt{3} \int_{0}^{\sqrt{C't_n}} \sqrt{-\ln \epsilon} \, d\epsilon \right) + \frac{2}{n} \left( 1 + \sqrt{0.5} \right) \left( 1 + \ln \mathcal{L}(B) - 3 \ln \sqrt{C't_n} \right) + 7 \left( n^{-1/2} \sqrt{x} + 2n^{-1}x \right). \]

Since \( \int_{0}^{t} \sqrt{-\ln x} \, dx \leq 3t \sqrt{-\ln t} \) for all \( 0 < t \leq \sqrt{0.5} \), the leading term on the right hand side above is bounded by
\[ \frac{27}{\sqrt{n}} \left( \sqrt{C't_n(1 + \ln \mathcal{L}(B))} + \frac{3\sqrt{3}}{\sqrt{2}} \sqrt{-C't_n(\ln(C') + \ln(t_n))} \right). \]

Taking \( x = nt_n \), and noting that \( t_n \geq n^{-1} \ln n \), we can find \( n_0 \) and constants \( C_1, C_2 > 0 \) which depend only on \( C \) and the set \( B \) such that for all \( n \geq n_0 \),
\[ r_n(x) \leq C_1 \left( n^{-1/2} \sqrt{-t_n \ln(t_n)} - n^{-1} \ln(t_n) + t_n \right) \leq C_2 t_n. \]

Thus, we obtain the desired result. ■

**Proof of Theorem 3.1:** First, we bound
\[
\mathbb{P} \left\{ |n_z \hat{\theta}(\tau) - \mathbb{E}[n_z \hat{\theta}(\tau) | \bar{Z}]| \geq n \pi_z t | \bar{Z} \right\} \\
\leq \mathbb{P} \left\{ |n_z \hat{\theta}(\tau) - \mathbb{E}[n_z \hat{\theta}(\tau) | \bar{Z}, \bar{S}]| \geq \frac{n \pi_z t}{2} | \bar{Z} \right\} \\
+ \mathbb{P} \left\{ |\mathbb{E}[n_z \hat{\theta}(\tau) | \bar{Z}, \bar{S}] - \mathbb{E}[n_z \hat{\theta}(\tau) | \bar{Z}]| \geq \frac{n \pi_z t}{2} | \bar{Z} \right\} = H_{n,1} + H_{n,2}, \text{ say.}
\]

By Lemma 5.1, we have
\[
(51) \quad H_{n,1} \leq 2 \exp \left( -\frac{n \pi_z t^2/4}{2 \tau^2(Z) + (2 \tau(Z)t/3)} \right) \leq 2 \exp \left( -\frac{n \pi_z t^2/4}{2 \tau^2(Z)(1 + t)} \right),
\]

because \( \tau(Z) \geq 1 \).

Let us turn to \( H_{n,2} \). We take \( n_0 \) to be the maximum between 2 and the integer \( n_0 \) in Lemma 5.4. From here on, we take \( n \geq n_0 \). We choose \( t_n \) to be a \( \sigma(\bar{Z}) \)-measurable, positive sequence of random variables. We will determine the precise value of \( t_n \) later. Let \( A_{n,1} \) be the event under which \( \sigma_n | \eta_{ij} | \leq t_n \) for all \( i \in N \) and \( j \in M \), and let \( A_{n,2} \) be the event
\[ \left\{ \sup_{c \in B} |R_n(c, 6t_n) | \leq Ct_n \right\}, \]
for the constant \( C > 0 \) that appears in Lemma 5.4. Let \( A_n = A_{n,1} \cap A_{n,2} \). Also define the event
\[ E_n = \{ \bar{h}(\bar{Z}) \leq (12C + 2C)nt_n + 1 \}. \]
Define $D_n = E[ n z \hat{\theta}(\tau) | Z, \tilde{S}] - E[ n z \hat{\theta}(\tau) | \tilde{Z}]$. By Lemmas 3.4 and 5.2, we have
\begin{equation}
P \left\{ \left| D_n \right| \geq \frac{n \pi Z t}{2} \right\} \cap A_n \cap \tilde{Z} \leq 2 \exp \left( -\frac{C nt^2}{\bar{\tau}^2(Z) (n^2 t_n^2 + 1)} \right),
\end{equation}
where $C > 0$ is a constant that depends only on $C$ and set $B$ in Assumptions 2.4(ii). Without loss of generality, we assume that $C \leq 1$. (If $C > 1$, we simply replace $C$ by $C \land 1$ to obtain the same bound.) On the other hand, by Assumptions 2.4(ii),
\begin{equation}
P \{ A_{n,1} | \tilde{Z} \} \leq 2nm \exp \left( -\frac{t_n^2}{2 \sigma^2_n} \right) = 2 \exp \left( -\left( \frac{t_n^2}{2 \sigma^2_n} - \ln(nm) \right) \right)
\end{equation}
\begin{equation}
\leq 2 \exp \left( -\left( \frac{t_n^2}{2 \sigma^2_n} - \ln(nm) \right) \right),
\end{equation}
where we recall $\sigma_n = \sigma_n \land n^{-5/6}$. Since $\sigma_n \geq n^{-1}, 2\sigma_n^2 \ln(nm)n^2 \geq 1$, because $n \geq n_0, n_0 \geq 2$ and $m \geq 1$. We set $t_n^2$ to be
\begin{equation}
t_n^2 = \frac{2 \sigma_n^2 \ln(nm)n^2 - 1 + \sqrt{\left(2 \sigma_n^2 \ln(nm)n^2 - 1\right)^2 + 4n^2 \left(2 \sigma_n^2 \ln(nm) + n \sigma_n^2 \tilde{\tau}\right)}}{2n^2},
\end{equation}
where $\tilde{\tau} = \sqrt{2\bar{\tau} \bar{t}/\bar{\tau}(Z)}$, so that we have
\begin{equation}
\frac{t_n^2}{2 \sigma^2_n} - \ln(nm) = \frac{\bar{C} nt^2}{\bar{\tau}^2(Z) (n^2 t_n^2 + 1)}.
\end{equation}
Since $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$ for all $a, b \geq 0$, we have
\begin{equation}
t_n^2 \leq \left( t_n^2 \right)^2 = 2 \sigma_n^2 \ln(nm) + \frac{\sqrt{2 \sigma_n \sqrt{\ln(nm)}}}{n} + \bar{\sigma}_n \tilde{\tau}.
\end{equation}
Therefore, from (53) and (55),
\begin{equation}
P \{ A_{n,1} | \tilde{Z} \} \leq 2 \exp \left( -\frac{C nt^2}{\bar{\tau}^2(Z) (n^2 t_n^2 + 1)} \right).
\end{equation}
Since $\sigma_n \geq n^{-5/6}$, we have from (54) that
\begin{equation}
t_n^2 \geq \frac{\bar{\sigma}_n \tilde{\tau}}{\sqrt{n}} = \frac{\sqrt{2\bar{\tau} \sigma_n t}}{\bar{\tau}(Z) \sqrt{n}} \geq \frac{\sqrt{\bar{\tau} t}}{\bar{\tau}^{4/3}(Z)n^{4/3}},
\end{equation}
where the last inequality uses the fact that $\bar{\tau}(Z) \geq 1$. This implies that
\begin{equation}
nt_n \geq \frac{\bar{C}^3 nt^{3/2}}{\bar{\tau}^2(Z)n^2 t_n^2} \geq \frac{\bar{C}^3 nt^{3/2}}{\bar{\tau}^2(Z) (n^2 (t_n^{3/2})^2 + 1)}.
\end{equation}
From (54), we have
\[ t_n \geq \tilde{\sigma}_n \sqrt{\ln(nm)} - n^{-1} \geq n^{-5/6} \sqrt{\ln(nm)} - n^{-1} > n^{-1} \ln n, \]
for all \( n \geq n_0 \), where \( n_0 \) is the first positive integer such that \( n_0^{-5/6} \sqrt{\ln n_0} - n_0^{-1} > n_0^{-1} \ln n_0 \). By Lemma 5.4, for all \( n \geq n_0 \),
\[ P \left\{ A_{n,1} \cap \bar{Z} \right\} \leq 2 \exp(-nt_n) \leq 2 \exp \left( -\frac{\tilde{C}^3 n t_n^{3/2}}{\bar{\tau}^2(Z) \left( n^2(t'_n)^2 + 1 \right)} \right). \]
Hence, we find that
\[ H_{n,2} = P \{ |D_n| \geq t \mid \bar{Z} \} \leq P \{ |D_n| \geq t \cap A_n \mid \bar{Z} \} + P \left\{ A_{n,1} \mid \bar{Z} \right\} + P \left\{ A_{n,2} \mid \bar{Z} \right\} \]
\[ \leq 6 \exp \left( -\frac{\tilde{C}^3 n (t^2 \land t^{3/2})}{\bar{\tau}^2(Z) \left( n^2(t'_n)^2 + 1 \right)} \right), \]
because \( \tilde{C} \leq 1 \). Now, observe that
\[ n^2(t'_n)^2 + 1 = 2n^2 \tilde{\sigma}_n^2 \ln(nm) + \sqrt{2n} \tilde{\sigma}_n \sqrt{\ln(nm)} + n \sqrt{n} \tilde{\sigma}_n \tilde{t} + 1 \]
\[ \leq 4n^2 \tilde{\sigma}_n^2 \ln(nm) + 1 + n \sqrt{n} \tilde{\sigma}_n \tilde{t} \]
\[ \leq 4n^2 \tilde{\sigma}_n^2 \ln(nm) + 1 + \sqrt{2\tilde{C} n \sqrt{n} \tilde{\sigma}_n} t = (4 \lor \sqrt{2\tilde{C}})(a_n + b_n t), \]
where the first inequality follows because
\[ n \tilde{\sigma}_n \leq n^2 \tilde{\sigma}_n^2 (n^{-1} \tilde{\sigma}_n^{-1}) \leq n^2 \tilde{\sigma}_n^2 n^{-5/6} \leq n^2 \tilde{\sigma}_n^2, \]
and the second inequality uses the fact that \( \bar{\tau}(Z) \geq 1 \). We take \( C = \tilde{C}^3 / (4 \lor \sqrt{2\tilde{C}}) \) and obtain the desired result. \( \square \)

**Proof of Corollary 3.1:** First, note that
\[ E[(n_Z - n \pi_Z) \hat{\theta}(\tau) \mid \bar{Z}] = E[n_Z - n \pi_Z \mid \bar{Z}] E[\hat{\theta}(\tau) \mid \bar{Z}] = 0, \]
because \( n_Z - n \pi_Z \) and \( \hat{\theta}(\tau) \) are conditionally independent given \( \bar{Z} \). Hence, we have
\[ |n \pi_Z \hat{\theta}(\tau) - E[n \pi_Z \hat{\theta}(\tau) \mid \bar{Z}]| \leq |n_Z \hat{\theta}(\tau) - E[n_Z \hat{\theta}(\tau) \mid \bar{Z}]| \]
\[ + |n_Z - n \pi_Z| |\hat{\theta}(\tau)| + |E[(n_Z - n \pi_Z) \hat{\theta}(\tau) \mid \bar{Z}]| \]
\[ \leq |n_Z \hat{\theta}(\tau) - E[n_Z \hat{\theta}(\tau) \mid \bar{Z}]| + |n_Z - n \pi_Z| \bar{\tau}(Z), \]
because \( |\hat{\theta}(\tau)| \leq \bar{\tau}(Z) \). We take a large number \( M_z > 0 \) and let
\[ \nu_n = \tilde{\sigma}_n \sqrt{n \ln(nm)} + (n \pi_Z)^{-1/2}. \]
From the previous result, we obtain that
\[ P \left\{ \left| \hat{\theta}(\tau) - \mathbb{E}[\hat{\theta}(\tau) \mid \tilde{Z}] \right| > 2M_2 \nu_n \mid \tilde{Z} \right\} \leq I_{n,1} + I_{n,2}, \]
where
\[ I_{n,1} = P \left\{ n \nu_n \hat{\theta}(\tau) - \mathbb{E}[n \nu_n \hat{\theta}(\tau) \mid \tilde{Z}] > n \pi_Z M_2 \nu_n \mid \tilde{Z} \right\}, \]
and
\[ I_{n,2} = P \left\{ \tilde{\tau}(Z) n \nu_n - n \pi_Z > n \pi_Z M_2 \nu_n \mid \tilde{Z} \right\}. \]

In light of Lemma 3.1, it suffices to show that \( I_{n,1} + I_{n,2} \to 0 \) as \( n \to \infty \) and then \( M_2 \to \infty \). We will show that \( I_{n,1} \to 0 \) and \( I_{n,2} \to 0 \) separately.

As for \( I_{n,1} \), we apply Theorem 3.1 with \( t = M_2 \nu_n \). Since \( \nu_n \to 0 \), we can take a large enough \( n_0 \) such that for all \( n \geq n_0, t \leq 1 \). By Theorem 3.1, for all \( n \geq n_0 \), as \( n \to \infty \) and then \( M_2 \to \infty \),
\[ I_{n,1} \leq 6 \exp \left( - \frac{CM_2^2 n \nu_n^2}{\pi^2(Z)(a_n + b_n M_2 \nu_n)} \right) + 2 \exp \left( - \frac{n \pi_Z M_2^2 \nu_n^2}{8 \pi^2(Z)(1 + M_2 \nu_n)} \right) \to 0. \]

Let us turn to \( I_{n,2} \). By Lemma 2.1 of Chung and Lu (2002), we have
\[ I_{n,2} \leq 2 \exp \left( - \frac{\pi(Z)^2 n \nu_n^2}{2n \pi Z + 2 \left( n \pi_Z M_2 \nu_n / (3 \pi(Z)) \right)} \right) \leq 2 \exp \left( - \frac{n \pi_Z M_2^2 \nu_n^2}{2 \pi^2(Z)(1 + (M_2 \nu_n / 3))} \right) \to 0, \]
again, as \( n \to \infty \) and then \( M_2 \to \infty \).

**Proof of Corollary 3.2:** For simplicity, we focus on the case with \( X_i \in \mathbb{R}, i \in N \). The proof is similar for a general case, involving hyperrectangles in place of intervals. Also, the proof for the case with \( X_i \) being discrete is straightforward. We focus on the case where \( X_i \) has a continuous conditional distribution function given \( \tilde{Z} \). The proof modifies the proof of Lemma 2.11 of van der Vaart (1998). Since
\[ \frac{1}{n_Z} \sum_{i \in N_Z} \left\{ 1 \{ Y_i = j \} - P \{ Y_i = j \mid \tilde{Z} \} \right\} = o_p(1), \]
by Corollary 3.1, it suffices to show that
\[ \frac{1}{n_Z} \sum_{i \in N_Z} \left\{ 1 \{ X_i \leq x, Y_i = j \} - P \{ X_i \leq x, Y_i = j \mid \tilde{Z} \} \right\} = o_p(1), \]
uniformly over \( x \in \mathbb{R} \). For any \( \sigma(\tilde{Z}) \)-measurable random variable \( W \), we have
\[ \frac{1}{n_Z} \sum_{i \in N_Z} \left\{ 1 \{ X_i \leq W, Y_i = j \} - P \{ X_i \leq W, Y_i = j \mid \tilde{Z} \} \right\} = o_p(1), \]
as shown in (23). Since the conditional distribution function of \( X_i \), given \( \tilde{Z} \) is continuous, for any \( \epsilon > 0 \), and any \( \sigma(\tilde{Z}) \)-measurable random variable \( W \), there exists a \( \sigma(\tilde{Z}) \)-measurable random variable \( \eta > 0 \) such that

\[
P \{ W - \eta \leq X_i \leq W + \eta, Y_i = j | \tilde{Z} \} \leq P \{ W - \eta \leq X_i \leq W + \eta | \tilde{Z} \} \leq \epsilon.
\]

Fix \( \epsilon > 0 \) and choose \( \sigma(\tilde{Z}) \)-measurable random variables \( W_1, ..., W_{k-1} \) such that \( -\infty = W_0 < W_1 < W_2 < ... < W_k = \infty \), such that for all \( \ell = 0, 1, 2, ..., k, \)

\[
P \{ X_i \leq W_{\ell+1}, Y_i = j | \tilde{Z} \} - P \{ X_i \leq W_{\ell}, Y_i = j | \tilde{Z} \} \leq \epsilon.
\]

Then, for \( W_{\ell-1} \leq x \leq W_{\ell}, \)

\[
\frac{1}{n_z} \sum_{i \in N_z} (1 \{ X_i \leq W_{\ell-1}, Y_i = j \} - P \{ X_i \leq W_{\ell-1}, Y_i = j | \tilde{Z} \}) - \epsilon \\
\leq \frac{1}{n_z} \sum_{i \in N_z} (1 \{ X_i \leq x, Y_i = j \} - P \{ X_i \leq x, Y_i = j | \tilde{Z} \}) \\
\leq \frac{1}{n_z} \sum_{i \in N_z} (1 \{ X_i \leq W_{\ell}, Y_i = j \} - P \{ X_i \leq W_{\ell}, Y_i = j | \tilde{Z} \}) + \epsilon.
\]

Hence,

\[
\sup_{x \in \mathbb{R}} \left| \frac{1}{n_z} \sum_{i \in N_z} (1 \{ X_i \leq x, Y_i = j \} - P \{ X_i \leq x, Y_i = j | \tilde{Z} \}) \right| \\
\leq \max_{0 \leq \ell \leq k} \left| \frac{1}{n_z} \sum_{i \in N_z} (1 \{ X_i \leq W_{\ell}, Y_i = j \} - P \{ X_i \leq W_{\ell}, Y_i = j | \tilde{Z} \}) \right| + \epsilon = \epsilon + o_p(1),
\]

as \( n \to \infty \), by (61). By sending \( \epsilon \) to zero, we obtain the desired result. ■

**Proof of Corollary 3.3:** We define

\[
\hat{F}_{X,Z}(t,s) = \frac{1}{n_{1,Z}} \sum_{i' \in N_{1Z}} 1 \{ X_{i',k} \leq t, Z_{y_{i',r}} \leq s \}, \quad \text{and}
\]

\[
\hat{F}_{Z}(s) = \frac{1}{n_{1,Z}} \sum_{i' \in N_{1Z}} 1 \{ Z_{y_{i',r}} \leq s \}.
\]

Note that

\[
\hat{F}_{X,Z}(t,s) = \frac{n_z}{n_{1,Z}} \sum_{j \in M} \frac{1}{n_z} \sum_{i' \in N_{1Z}} 1 \{ X_{i',k} \leq t, Z_{j,r} \leq s \} 1 \{ Y_{i'} = j \}.
\]

Furthermore,

\[
\frac{n_{1,Z}}{n_z} = \frac{1}{n_z} \sum_{i \in N_z} 1 \{ Y_i \neq 0 \} = \frac{1}{n_z} \sum_{i \in N_z} P \{ Y_i \neq 0 | \tilde{Z} \} + o_p(1) = P \{ Y_i \neq 0 | \tilde{Z} \} + o_p(1),
\]

(62)
as $n \to \infty$. The second equality follows from (22) because $M$ is independent of $n$. The last equality follows because the conditional distribution of $Y_i$ given $\tilde{Z}$ is identical across $i$’s by Lemma 2.1. On the other hand, by Corollary 3.2, for each $j \in M$,
\[
\frac{1}{n_Z} \sum_{i' \in N_Z} 1\{X_{i',k} \leq t, Y_{i'} = j\} = P\{X_{i',k} \leq t, Y_{i'} = j \mid \tilde{Z}\} + o_p(1),
\]
uniformly over $t \in \mathbb{R}$, as $n \to \infty$. Since, conditional on $\tilde{Z}$, the indicator $1\{Z_{j,r} \leq s\}$ is treated as constant, we have
\[
\sup_{t,s \in \mathbb{R}} \left| \hat{F}_{X,Z}(t,s) - F_{X,Z}(t,s) \right| = o_p(1).
\]
Because $n_Z = n\pi_Z(1 + o_p(1))$ by Lemma 5.1 and $n\pi_Z \to \infty$ as $n \to \infty$ by (13),
\[
\max_{i \in N_z} \sup_{t,s \in \mathbb{R}} \left| \hat{F}_{X,Z,-i}(t,s) - \hat{F}_{X,Z}(t,s) \right| = o_p(1).
\]
Hence,
\[
\max_{i \in N_z} \sup_{t,s \in \mathbb{R}} \left| \hat{F}_{X,Z,-i}(t,s) - F_{X,Z}(t,s) \right| = o_p(1),
\]
as $n \to \infty$. Similarly, $\max_{i \in N_z} \sup_{t,s \in \mathbb{R}} \left| \hat{F}_{X,-i}(t) - F_X(t) \right| = o_p(1)$, as $n \to \infty$. On the other hand,
\[
\sup_{t \in \mathbb{R}} \left| \hat{F}_{Z}(t) - F_Z(t) \right| \leq \sum_{j \in M} \sup_{t} \left| 1\{Z_{j,r} \leq t\} - \frac{1}{n_{1,Z}} \sum_{i' \in N_{1,Z}} \left(1\{Y_{i'} = j\} - P\{Y_{i'} = j \mid \tilde{Z}\}\right) \right|
\]
\[
= \sum_{j \in M} \sup_{t} \left| 1\{Z_{j,r} \leq t\} - \frac{1}{n_{1,Z}} \sum_{i' \in N_{1,Z}} \left(1\{Y_{i'} = j\} - P\{Y_{i'} = j \mid \tilde{Z}\}\right) \right|
\]
where the last equality follows because $j$’s are chosen from $M$. The last term is bounded by
\[
\frac{n_Z}{n_{1,Z}} \sum_{j \in M} \left| \frac{1}{n_{1,Z}} \sum_{i' \in N_{1,Z}} \left(1\{Y_{i'} = j\} - P\{Y_{i'} = j \mid \tilde{Z}\}\right) \right| = o_p(1),
\]
as $n \to \infty$. The last equality follows by (12) and (62), and the condition (26) in Corollary 3.3, and due to $n_Z/n_{1,Z} = O_p(1)$ by (62) and (26). Collecting these results, we conclude that
\[
\max_{i \in N_z} \sup_{t,s \in \mathbb{R}} \left| \hat{F}_{X,Z,-i}(t,s) - \hat{F}_{X,Z}(t,s) - \left(F_{X,Z}(t,s) - F_X(t)F_Z(s)\right) \right| = o_p(1).
\]
Since $|\hat{\rho} - \rho|$ is bounded by 12 times the left hand side term, the desired result follows.

**Proof of Corollary 3.4:** Define
\[
\hat{g}(x) = \frac{1}{n\pi_Z} \sum_{i \in N_Z} 1\{Y_i = j\} K_h(X_i - x).
\]
Also, let
\[ g(x) = p(j \mid x, \bar{Z})f_{X_i \mid \bar{Z}}(x \mid \bar{Z}) \text{ and } \hat{g}(x) = \frac{1}{n\pi_Z} \sum_{i \in N_Z} \mathcal{K}_h(X_i - x). \]

Since \( X_i \)'s are conditionally i.i.d. given \( Z \) by Assumption 2.3, and
\[ \mathbb{E}\left[ \mathcal{K}_h(X_i - x) 1\{i \in N_Z\} \mid \bar{Z} \right] = \mathbb{E}\left[ \mathcal{K}_h(X_i - x) \mid \bar{Z} \right] \pi_Z, \]
(due to the conditional independence of \( \zeta \) and \( \tilde{S} \) given \( \bar{Z} \) in Assumption 3.1), we have
\[ \hat{f}_{X \mid \bar{Z}}(x \mid \bar{Z}) - f_{X \mid \bar{Z}}(x \mid \bar{Z}) = O_p \left( \frac{1}{\sqrt{n\pi_Z h^d}} + h^2 \right), \]
using standard arguments of kernel density estimation. Hence
\[ \hat{p}(j \mid x, \bar{Z}) - p(j \mid x, \bar{Z}) = \frac{\hat{g}(x) - g(x)}{\hat{f}_{X \mid \bar{Z}}(x \mid \bar{Z})} + g(x) \left( \frac{1}{\hat{f}_{X \mid \bar{Z}}(x \mid \bar{Z})} - \frac{1}{f_{X \mid \bar{Z}}(x \mid \bar{Z})} \right) \]
\[ = \frac{\hat{g}(x) - g(x)}{f_{X \mid \bar{Z}}(x \mid \bar{Z})} \left( 1 + o_p(1) \right) + O_p \left( \frac{1}{\sqrt{n\pi_Z h^d}} + h^2 \right). \]

Using the standard arguments in dealing with the bias part, we find that
\[ \hat{g}(x) - g(x) = \Delta_n(x) + O_p(h^2), \]
where \( \Delta_n(x) \) is as defined in (30). The desired result follows from (32), because \( m \) is fixed. ■

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Supplemental Note to “The Law of Large Numbers for Large Stable Matchings”

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The supplemental note is devoted to the proof of the bounded difference result (Lemma 3.3) of Schwartz and Song (2022). Let us present a brief summary of our proof strategy. We first introduce a re-stabilization operator which we use to transform any matching obtained by unmatching one student from a stable matching into another stable matching. Using the re-stabilization operator, we establish a bounded difference condition for a student-optimal stable matching (see Lemma C.3 below), where the bound depends on the maximum rank difference $h(w)$ defined in (18) in Schwartz and Song (2022). Next, using the Rural Hospital Theorem and the fact that the student-optimal stable matching is the college-worst stable matching, we establish a bound for the number of students matched with different colleges between a student-optimal stable matching and any stable matching (see Lemma C.6 below.) As in the case of the first bound, the second is also expressed in terms of $h(w)$. By combining these two bounds, we obtain the desired bounded difference condition for any stable matching in the market.

A. Re-stabilization Operator and Its Properties

We introduce a re-stabilization operator that transforms an unstable matching in one market into a stable matching in another, by repeatedly satisfying a blocking pair. Roth and Vande Vate (1990) showed that given an arbitrary matching, a sequence of matchings in which each is obtained from the previous matching by satisfying blocking pairs is guaranteed to converge to a stable matching when blocking pairs may be chosen randomly at each step in the sequence. Rather than re-stabilizing arbitrary matchings, we focus on small perturbations to the outcomes of stable matchings caused by a change in the type of a single student. Therefore,
we will consider a re-stabilization operator that takes in a matching that is already “close” to being stable.

We begin with some key definitions that we use repeatedly later.

**Definition A.1.** Given a matching \( \mu \), we say that

1. \( \mu \) is **individually rational** if there is no \( i \in N \) or \( j \in M \) such that \( 0 >_{i} \mu(i) \) and \( 0 >_{j} i' \) for some \( i' \in \mu^{-1}(j) \).
2. \( \mu \) is **envy-free** if it is individually rational and any blocking pair, if it exists, involves an unmatched student in \( \mu \).
3. \( \mu \) is **1-envy-free** if it is (i) envy free and (ii) either \( \mu \) is stable or there exists one and only student who belongs to every blocking pair of \( \mu \).

Given a market \((N, M, u)\), let \( S(N, M, u) \) be the set of stable matchings, and let \( E_1(N, M, u) \) be the set of matchings that are 1-envy-free. For each \( i \in N \), with each map \( \mu_i : N \setminus \{i\} \to M' \), we associate a map \( g_i(\mu_i) : N \to M' \) defined by

\[
g_i(\mu_i)(i') = \begin{cases} 
\mu_i(i'), & \text{if } i' \neq i, \\
0, & \text{if } i' = i.
\end{cases}
\]

The map \( g_i(\mu_i) \) is a matching on \( N \) constructed from \( \mu_i \) by matching student \( i \) to 0.

Given a preference profile \( u \), for each student \( i \in N \), we define

\[
u_{-i} = (v_{-i}, w_{-i}),
\]

where \( v_{-i} \) is constructed by removing \( v_i \) from \( v \), and \( w_{-i} \) is constructed by replacing each college’s preference \( w_j \) by the bijection \( w'_j : N' \setminus \{i\} \to N' \setminus \{i\} \) such that \( w'_j(i_1) > w'_j(i_2) \) if and only if \( w_j(i_1) > w_j(i_2) \) for all \( i_1, i_2 \in N' \setminus \{i\} \). In other words, the preference profile \( u_{-i} \) is obtained by “eliminating” the student \( i \) from the market.

Let \((N \setminus \{i\}, M, u_{-i})\) be the matching market derived from \((N, M, u)\) by eliminating the student \( i \). The following remark follows immediately from our definitions.

**Remark A.1.** For every \( u \in \mathbb{U} \) and \( i \in N \), \( \mu_i \in S(N \setminus \{i\}, M, u_{-i}) \) if and only if \( g_i(\mu_i) \in E_1(N, M, u) \).

Given an envy-free matching \( \mu \), a pair \((i, j)\) is a **student-maximal blocking pair** for \( \mu \) if \((i, j)\) is a blocking pair for \( \mu \) and \( j \) is student \( i \)'s most preferred college among those with whom

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\(^{14}\)Closely related definitions have been used. In particular, the notion of doctor quasi-stable matchings discussed in Wu and Roth (2018). The matchings in the second item of the definition can be viewed as a many-to-one version of the matchings studied by Blum and Rothblum (2002) in the one-to-one case.
he can form a blocking pair for \( \mu \). Our re-stablization operator iteratively satisfies student-maximal blocking pairs.\(^{15}\)

Before defining the operator, we introduce some further notation. Define \( B_\mu \) to be the set of all blocking pairs to a matching \( \mu \) for the market \((N, M, u)\). Let the set of student-maximal blocking pairs for a matching \( \mu \) be

\[
B'_\mu = \{(i', j') \in B_\mu : j' \succ_{i'} j'', \text{ for all } (i', j'') \in B_\mu \text{ such that } j' \neq j''\}.
\]

Note that \( B'_\mu = \emptyset \) if and only if \( B_\mu = \emptyset \). In the case that \( \mu \in \mathcal{E}_1(N, M, u) \), \( B'_\mu \) is either empty, or contains exactly one blocking pair. In the case that \( B_\mu \neq \emptyset \), we define \( j^*(i) \) to be a college such that \( B'_\mu = \{(i, j^*(i))\} \).

**Definition A.2.** For any \( \mu \in \mathcal{E}_1(N, M, u) \), we define the operator \( T : \mathcal{E}_1(N, M, u) \rightarrow \mathcal{E}_1(N, M, u) \) as follows.

1. Suppose \( B_\mu = \emptyset \). Then we take \( T(\mu) = \mu \).
2. Suppose \( B_\mu \neq \emptyset \). Then \( B'_\mu \) is a singleton, say, \( \{(i, j^*(i))\} \) and we set \( T(\mu)(i) = j^*(i) \). For each \( i' \neq i \), let us denote \( j' = \mu(i') \). We set \( T(\mu)(i') \) as follows:
   - **Case 1:** \( j' \neq j^*(i) \). Then \( T(\mu)(i') = j' \).
   - **Case 2:** \( j' = j^*(i) \). Then,
     \[
     T(\mu)(i') = \begin{cases} 
     j', & \text{if } |\mu^{-1}(j')| < q_{j'}, \\
     j', & \text{if } |\mu^{-1}(j')| = q_{j'}, \text{ and } i'' \succ_{j'} i' \text{ for some } i'' \in \mu^{-1}(j') \\
     0, & \text{if } |\mu^{-1}(j')| = q_{j'}, \text{ and } i' \succ_{j'} i'' \text{ for all } i'' \in \mu^{-1}(j') \setminus \{i'\}.
     \end{cases}
     \]

From the definition of \( T \), it is clear that any \( \mu \in S(N, M, u) \) satisfies \( T(\mu) = \mu \); i.e., any stable matching is a fixed point of \( T \). The sets \( B_\mu, B'_\mu \), and the operator \( T \) certainly depend on \( u \), but we will often suppress these from our notation for simplicity. It is also true that any fixed point \( \mu \) of \( T \) is a stable matching. Note also that \( T \) maps from \( \mathcal{E}_1(N, M, u) \) to itself, since for any \( \mu \in \mathcal{E}_1(N, M, u) \), \( T(\mu) \) is either stable or blockable by at most one student that is unmatched.

The next result, Lemma A.1, shows that repeated iterations of the operator \( T \) yield a stable matching when the input is a 1-envy-free matching with respect to one student. For this, it is convenient to introduce a partial order \( \succ^*_j \) over matchings. Suppose that \( \succ^*_j \) is college \( j \)'s

\(^{15}\)The operator we propose is a straightforward adaptation of the correcting procedures described in Blum and Rothblum (2002) (itself a special case of Blum, Roth, and Rothblum (1997)), adapted to a many-to-one setup. The approach is also similar to Biró, Cechlárová, and Fleiner (2008), who discuss algorithms for stabilizing matching markets when a single agent is added to a market that is presumed stable in the absence of the additional agent. Note that Wu and Roth (2018) showed that in many-to-one markets in which no students have justified envy, stable matchings can be obtained as fixed points of a lattice operator that generalizes the college-optimal deferred acceptance algorithm.
preference ordering over groups of students. The following definition is from Definition 5.2. on page 128 of Roth and Sotomayor (1990).

**Definition A.3.** The preference relation of a college \( j, \succ^+_j \), over sets of students is responsive to the preference over individual students if, whenever \( \mu^{-1}_1(j) = \mu^{-1}_2(j) \cup \{i_1\} \setminus \{i_2\} \) for \( i_2 \in \mu^{-1}_2(j) \) and \( i_1 \notin \mu^{-1}_2(j) \), then \( \mu^{-1}_1(j) \succ^+_j \mu^{-1}_2(j) \) if and only if \( i_1 \succ_j i_2 \).

We assume that \( \succ^+_j \) is responsive to \( \succ_j \). For any pair of matchings \( \mu_1 \) and \( \mu_2 \), we write \( \mu_1 \succeq \mu_2 \) if and only if for all \( j \in M \), either \( \mu^{-1}_1(j) \succ^+_j \mu^{-1}_2(j) \) or \( \mu^{-1}_1(j) = \mu^{-1}_2(j) \).

**Lemma A.1.** For each \( \mu \in \mathcal{E}_1(N, M, u) \), the following is satisfied.

(i) \( T(\mu) \succeq \mu \).

(ii) There is a finite sequence of matchings, \( \mu_1, \mu_2, ..., \mu_r \), with \( \mu_r \in S(N, M, u) \), where \( \mu_0 = \mu \), and \( \mu_{r'} = T(\mu_{r'-1}) \) for each \( r' = 1, ..., r \).

**Proof:** If \( B'_{\mu} \) is empty, the matching \( \mu \) is stable and we have \( T(\mu) = \mu \), satisfying both (i) and (ii). If \( B'_{\mu} \) is not empty, it contains exactly one blocking pair, say, \((i, j^*(i))\), and \( T \) assigns student \( i \) to college \( j^*(i) \). In the case that \( j^*(i) \) has no vacancies under \( \mu \), its worst student under \( \mu \), say, \( i' \), is made unmatched, and in the case that \( j^*(i) \) has a vacancy under \( \mu \), then all the other students remain in their colleges. Thus, \( T(\mu) \succeq \mu \), since \( T \) either affects no colleges, or leaves exactly one college, \( j^*(i) \), strictly better off while leaving the remaining colleges unaffected. Since there are a finite number of student-college pairs, repeated iterations of \( T \) from any \( \mu \in \mathcal{E}_1(N, M, u) \) are guaranteed to converge to a fixed point which is a stable matching, after finite iterations. ■

Lemma A.1 shows that for any \( \mu \in \mathcal{E}_1(N, M, u) \), repeated iterations of \( T \) lead to a stable matching of the market \((N, M, u)\), in finite iterations. Furthermore, the output of repeated iterations of \( T \) is uniquely determined by the given choice of \( \mu \in \mathcal{E}_1(N, M, u) \), since there is at most one student-maximal blocking pair after each iteration of the operator. It is convenient to develop notation for the stable output of repeated iterations of \( T \) in terms of an input matching. Given any \( \mu \in \mathcal{E}_1(N, M, u) \), we denote \( T^*(\mu) \equiv \mu_r \) where \( \mu_0 = \mu, \mu_1, \mu_2, ..., \mu_r \) is the finite sequence of matchings with \( \mu_r \in S(N, M, u) \), where \( \mu_{r'} = T(\mu_{r'-1}) \) for each \( r' = 1, ..., r \).

The following is a consequence of what is called Rural Hospital Theorem.

**Lemma A.2.** Let \( \mu \in S(N, M, u) \) and \( \mu' \in \mathcal{E}_1(N, M, u) \). Then either of the following two cases must hold.

---

\(^{16}\text{By strictness and responsiveness of college preferences and the fact that } (i, j^*(i)) \text{ is a blocking pair for } \mu, \text{ we have either (i) } |\mu^{-1}(j^*(i))| = q, \text{ and } j^*(i) \text{ strictly prefers } \mu^{-1}(j^*(i)) \cup \{i'\} \setminus \{i\} \text{ to } \mu^{-1}(j^*(i)) \text{ for some } i' \in \mu^{-1}(j^*(i)) \text{ or (ii) } |\mu^{-1}(j^*(i))| < q, \text{ and } j^*(i) \text{ strictly prefers } \mu^{-1}(j^*(i)) \cup \{i\} \text{ to } \mu^{-1}(j^*(i)).\)
Case 1: For every college \( j \in M \),

\[ |\mu^{-1}(j)| = |\mu'^{-1}(j)| \tag{3} \]

Case 2: There exists one and only one college, \( j^* \in M \), such that

\[ |\mu^{-1}(j^*)| = |\mu'^{-1}(j^*)| + 1, \tag{4} \]

and, for every college \( j \in M \setminus \{j^*\} \),

\[ |\mu^{-1}(j)| = |\mu'^{-1}(j)|. \tag{5} \]

**Proof:** Let \( T^*(\mu') \) be the matching obtained by iterations of \( T \) starting from \( \mu' \). Since \( \mu \) is stable and \( T^*(\mu') \) is stable by Lemma A.1, the set of college positions filled under \( \mu \) and \( T^*(\mu') \) is identical by Theorem 5.12 of Roth and Sotomayor (1990). By the arguments in the proof of Lemma A.1, either of the following two cases hold.

Case 1: Every college has the same number of matched students between \( \mu \) and \( \mu' \).

Case 2: One and only one college \( j^* \) fills an additional position as we move from \( \mu' \) to \( \mu \), whereas all the other colleges have the same number of matched students between \( \mu' \) and \( \mu \).

\[ \blacksquare \]

Lemma A.2 immediately implies the following corollary on the cardinality of sets, \( N_{j,1} \) and \( N_{j,0} \) defined as

\[ N_{j,1} \equiv \mu^{-1}(j) \setminus \mu'^{-1}(j) \] \( \text{and} \)
\[ N_{j,0} \equiv \mu'^{-1}(j) \setminus \mu^{-1}(j), \tag{6} \]

where \( \mu \in S(N, M, u) \) and \( \mu' \in E_1(N, M, u) \).

**Corollary A.1.** Let \( \mu \in S(N, M, u) \) and \( \mu' \in E_1(N, M, u) \), and \( N_{j,1} \) and \( N_{j,0} \) be as defined in (6).

If Case 1 holds under Lemma A.2, then for every college \( j \in M \),

\[ |N_{j,1}| = |N_{j,0}|. \tag{7} \]

If Case 2 holds under Lemma A.2, then for each \( j \in M \),

\[ |N_{j,1}| = |N_{j,0}| + 1 \{ j = j^* \}, \tag{8} \]

where \( j^* \) is the college in Lemma A.2.

**Proof:** Let \( j \in M \). Since \( \mu^{-1}(j) \) and \( \mu'^{-1}(j) \) are finite,

\[ |N_{j,1}| = |\mu^{-1}(j) \setminus \mu'^{-1}(j)| = |\mu^{-1}(j)| - |\mu'^{-1}(j) \cap \mu^{-1}(j)|, \text{ and} \tag{9} \]
\[ |N_{j,0}| = |\mu'^{-1}(j) \setminus \mu^{-1}(j)| = |\mu'^{-1}(j)| - |\mu'^{-1}(j) \cap \mu^{-1}(j)|. \tag{10} \]
First, suppose that we are in Case 1 under Lemma A.2. Then, \(|\mu^{-1}(j)| = |\mu'^{-1}(j)|\), so that by (9) and (10) we have \(|N_{j,1}| = |N_{j,0}|\). Next, suppose we are in Case 2 under Lemma A.2. Then \(|N_{j,1}| = |N_{j,0}| + 1\), if \(j = j^*\), whereas if \(j \in M \setminus \{j^*\}\), we have \(|N_{j,1}| = |N_{j,0}|\) as before. ■

### B. Related One-To-One Markets

When preferences of agents are strict (as is the case in our setup under our assumptions), a unique student optimal stable matching exists and can be realized through the Deferred Acceptance (DA) mechanism proposed by Gale and Shapley (1962).

The main result of this section is Lemma B.5 below, which is essentially a many-to-one version of Theorem 5.2 of Blum, Roth, and Rothblum (1997) adapted to our setup.\(^{17}\) It says that after iterations of \(T\) starting from a student-optimal stable matching (SOSM) in a market with one student left out, we obtain the SOSM in the original market with the student included. To proceed, we introduce a notion of one-to-one matching markets that are analogous to the many-to-one matching markets we have dealt with so far. Our definitions follow Section 5.2 of Roth and Sotomayor (1990), but we summarize the main details here for convenience.

Given a many-to-one market \((N,M,u)\), we define the corresponding one-to-one market \((N,\tilde{M},\tilde{u})\) as follows. First, the set of colleges \(\tilde{M}\) in the one-to-one market is obtained by “splitting” each college \(j \in M\) into \(q_j\) positions, \(c_{j,1}, \ldots, c_{j,q_j}\), where each position of \(j\) has the same preferences over students as college \(j\). Students’ preferences over the positions in the one-to-one market are such that each student \(i\) prefers a position of college \(j\) to a position of college \(j'\) in the one-to-one market if and only if \(i\) prefers college \(j\) to college \(j'\) in the many-to-one market. Moreover, when comparing any two positions of the same college \(j\), each student is assumed simply to prefer the position with the smaller index. (So, all students have the same preference ordering between positions in the same college.) For example, each student considers \(c_{j,1}\) the best position of college \(j\), \(c_{j,2}\) to be the second best position of \(j\), and so on.

Next, we define a matching \(\tilde{\mu}\) in a market \((N,\tilde{M},\tilde{u})\) to be a pair of maps \((\tilde{\mu}_N, \tilde{\mu}_M)\), \(\tilde{\mu}_N : N \to \tilde{M} \cup \{0\}\), \(\tilde{\mu}_M : M \to \tilde{N} \cup \{0\}\) such that for all \(i \in N\) and \(j \in M\), \(\tilde{\mu}_N(i) = j\) if and only if \(\tilde{\mu}_M(j) = i\). Under the assumption of strict preferences, we then obtain the following one-to-one correspondence between matchings for the market \((N,M,u)\) and matchings for the market \((N,\tilde{M},\tilde{u})\). A matching \(\mu\) for market \((N,M,u)\) which matches college \(j \in M\) with students \(\mu^{-1}(j)\), corresponds to the matching \(\tilde{\mu} = (\tilde{\mu}_N, \tilde{\mu}_M)\) for market \((N,\tilde{M},\tilde{u})\) in which the students in \(\mu^{-1}(j)\) are matched in the order they occur in the college’s preferences, with the ordered positions of \(j\) that appear in \(\tilde{M}\). Thus, if \(i\) is college \(j\)’s most preferred student in \(\mu^{-1}(j)\) then \(\tilde{\mu}_N(i) = c_{j,1}\) and \(\tilde{\mu}_M(c_{j,1}) = i\), and so on. Following Roth and Sotomayor (1990), we call

\(^{17}\)A special case of the result also appears as the second item of Theorem 2.3 in Blum and Rothblum (2002).
Definition B.1. Given a matching \( \bar{\mu} = (\bar{\mu}_N, \bar{\mu}_M) \) in market \((N, \bar{M}, \bar{u})\), we say that

1. \(\bar{\mu}\) is **individually rational** if there is no \(i \in N\) or \(j \in \bar{M}\) such that \(0 \succ_i \bar{\mu}_N(i)\) or \(0 \succ_j \bar{\mu}_M(j)\),
2. \(\bar{\mu}\) is **simple** if it is individually rational and any \((i, j) \in N \times \bar{M}\) such that \(j \succ_i \bar{\mu}_N(i)\) and \(i \succ_j \bar{\mu}_M(j)\), satisfies that \(\bar{\mu}_N(i) = 0\),\(^1\) and
3. \(\bar{\mu}\) is **1-simple** if it is: (i) simple and (ii) either \(\bar{\mu}\) is stable or there exists one and only one student \(i \in N\) who belongs to every blocking pair of \(\bar{\mu}\).

The following result and its proof are similar to Proposition 2.2 of Wu and Roth (2018).

**Lemma B.1.** Let \(\mu\) be a 1-envy-free matching in \((N, M, u)\) (in the sense of Definition A.1). Then its corresponding matching \(\bar{\mu}\) in \((N, \bar{M}, \bar{u})\) is 1-simple.

**Proof:** First we show that if \(\mu\) is envy-free, then the corresponding matching is simple. Let \(\mu\) be envy-free in \((N, M, u)\). Suppose by contradiction that its corresponding matching \(\bar{\mu}\) is not simple in \((N, \bar{M}, \bar{u})\). We assume that \(\bar{\mu}\) is individually rational, as otherwise the contradiction is immediate. Then there is a blocking pair \((i, j) \in N \times \bar{M}\) for \(\bar{\mu}\) with \(\bar{\mu}_N(i) = j' \neq 0\). Suppose that \(j\) and \(j'\) are positions of distinct colleges in \(M\).\(^2\) Then we obtain the contradiction that \(\mu\) is not envy-free, since \((i, j)\) is a blocking pair for \(\mu\), yet \(i\) is matched to a college under \(\mu\).

Now let \(\mu\) be 1-envy-free in \((N, M, u)\). Suppose by contradiction that its corresponding matching \(\bar{\mu}\) is simple but not 1-simple in \((N, \bar{M}, \bar{u})\). Then there are more than one students forming a blocking pair to \(\bar{\mu}\). Each of those students forms a blocking pair to \(\mu\). This contradicts that \(\mu\) is 1-envy-free. \(\blacksquare\)

Thus, whenever \(\mu\) is stable and \(\mu'\) is 1-envy free, \(\bar{\mu}\) is stable and \(\bar{\mu}'\) is 1-simple in the related market, by Lemma 5.6 of Roth and Sotomayor (1990) and Lemma B.1.

The following lemma is Lemma A.2 of Blum, Roth, and Rothblum (1997), translated into our notation. The lemma is a version of Knuth’s Decomposition Lemma (see Corollary 2.2.1 of Roth and Sotomayor (1990)).

**Lemma B.2.** Let \(\bar{\mu} = (\bar{\mu}_N, \bar{\mu}_M)\) and \(\bar{\mu}' = (\bar{\mu}'_N, \bar{\mu}'_M)\) be the stable and 1-simple matchings in \((N, \bar{M}, \bar{u})\) respectively. Suppose that \(s'\) is a student who does not belong to any blocking pair for \(\bar{\mu}'\), and \(s\) is a student who does not belong to any blocking pair for \(\bar{\mu}\). Then the following statements hold:

---

\(^1\)That is, \(\bar{\mu}\) is simple if it is individually rational and any blocking pair, if it exists, involves an unmatched student in \(\bar{\mu}\).

\(^2\)Note that for such a blocking pair to exist, \(j\) and \(j'\) cannot be positions of the same college under \(M\); if \(j\) is the better position, then the college fills it with a preferred student; if \(j\) is the worse position then \(i\) doesn't prefer \(j\) to \(j'\).
(i) \( \bar{\mu}_N(s') \succ_i \bar{\mu}_N'(s') \equiv c' \) if and only if \( s' \equiv \bar{\mu}'_M(c') \succ_c \bar{\mu}_M(c') \).
(ii) \( \bar{\mu}'_N(s) \succ_i \bar{\mu}_N(s) \equiv c \) if and only if \( s \equiv \bar{\mu}_M(c) \succ_c \bar{\mu}'_M(c) \).

We use this lemma to obtain the following result which could be viewed as an adaptation of Lemma 5.25 of Roth and Sotomayor (1990). The difference is that our setting involves two matchings where one is a 1-envy-free matching rather than a stable matching. This means that the set of positions filled by colleges may not be identical between the two matchings. For the sake of full transparency, we provide detailed arguments in the proof. Let \( \mu \) and \( \mu' \) be matchings with corresponding one-to-one matchings \( \bar{\mu} = (\bar{\mu}_N, \bar{\mu}_M) \) and \( \bar{\mu}' = (\bar{\mu}'_N, \bar{\mu}'_M) \). For any college \( j \in M \) and college position \( c \in \bar{M} \) of \( j \), we write \( \bar{\mu}_M(c) \succeq_j \bar{\mu}'_M(c) \) if either \( \bar{\mu}_M(c) >_j \bar{\mu}'_M(c) \) or \( \bar{\mu}_M(c) = \bar{\mu}'_M(c) \).

**Lemma B.3.** Let \( \mu \in S(N, M, u) \) and \( \mu' \in \mathcal{E}_1(N, M, u) \), and let \( \bar{\mu} = (\bar{\mu}_N, \bar{\mu}_M) \) and \( \bar{\mu}' = (\bar{\mu}'_N, \bar{\mu}'_M) \) be the stable and 1-simple matchings corresponding to \( \mu \) and \( \mu' \) in the related one-to-one market. Suppose that for some college \( j \in M \) and one of its positions \( c \),

\[
\bar{\mu}_M(c) \succ_j \bar{\mu}'_M(c).
\]

Then, \( \bar{\mu}_M(c') \succeq_j \bar{\mu}'_M(c') \) for all the positions \( c' \) of the college \( j \).

**Proof:** Case 1 under Lemma A.2 can be dealt with using the proof of Lemma 5.25 of Roth and Sotomayor (1990) using Lemma B.2 in place of the decomposition lemma.

We focus on Lemma A.2 under Case 2. Then the position vacant under \( \mu' \) and filled under \( \mu \) should be \( c_{j,\ell} \) with \( \ell = |\mu'^{-1}(j)| + 1 = |\mu^{-1}(j)| \), and \( \bar{\mu}_M(c_{j,\ell}) \succ_j \bar{\mu}'_M(c_{j,\ell}) = 0 \) by the individual rationality of \( \bar{\mu} \). Furthermore, if \( |\mu'^{-1}(j)| < q_j \), the positions of college \( j \) must be vacant at indices \( |\mu^{-1}(j)| + 1, ..., q_j \) under both \( \bar{\mu} \) and \( \bar{\mu}' \).

To prove the lemma, we assume that (11) holds for some position \( c \). Without loss of generality, assume that there exists \( i \in \{1, ..., |\mu^{-1}(j)|\} \) such that

\[
\bar{\mu}_M(c_{j,\ell}) = \bar{\mu}'_M(c_{j,\ell}), \text{ for all } \ell = 1, ..., i - 1,
\]
\[
\bar{\mu}_M(c_{j,i}) \succ_j \bar{\mu}'_M(c_{j,i}).
\]

We show that \( \bar{\mu}_M(c_{j,\ell}) \succ_j \bar{\mu}'_M(c_{j,\ell}) \) for all \( \ell = i + 1, ..., |\mu^{-1}(j)| \). For this, we follow the arguments in the proof of Lemma 5.25 of Roth and Sotomayor (1990). Suppose by contradiction that

\[
\bar{\mu}_M(c_{j,\ell}) \succ_j \bar{\mu}'_M(c_{j,\ell}), \text{ for some } \ell = i, ..., |\mu^{-1}(j)| - 1 \text{ and}
\]
\[
\bar{\mu}'_M(c_{j,\ell+1}) \succeq_j \bar{\mu}_M(c_{j,\ell+1}).
\]

The case where \( \bar{\mu}_M(c_{j,\ell}) = \bar{\mu}'_M(c_{j,\ell}) \), for all \( \ell = 1, ..., i - 1 \), and \( \bar{\mu}_M(c_{j,i}) \prec_j \bar{\mu}'_M(c_{j,i}) \) is excluded, because otherwise, we can use the same arguments to prove that \( \bar{\mu}_M(c_{j,\ell}) \prec_j \bar{\mu}'_M(c_{j,\ell}) \) for all \( \ell = i + 1, ..., |\mu^{-1}(j)| \), but this contradicts the assumption that (11) holds for some position \( c \).
Since $c_{j,t}$ with $\ell = i, \ldots, |\mu^{-1}(j)| - 1$ is filled both under $\bar{\mu}$ and $\bar{\mu}'$, we have $s'_\ell \equiv \bar{\mu}'_M(c_{j,t}) \in N$, and hence $s'_\ell$ does not belong to any blocking pairs of $\bar{\mu}'$ (because $\bar{\mu}'$ is 1-simple.)

Since the lower-indexed positions are filled by better students at any 1-simple matching, we have from (13):

$$s'_\ell = \bar{\mu}'_M(c_{j,t}) \succ_j \bar{\mu}'_M(c_{j,t+1}) \succeq_j \bar{\mu}_M(c_{j,t+1}).$$

Next, since (12) implies that $c_{j,t} = \bar{\mu}'_N(s'_\ell) \not\equiv \bar{\mu}_N(s'_\ell)$, we must have either

$$c_{j,t} = \bar{\mu}'_N(s'_\ell) \succ_{s'_\ell} \bar{\mu}_N(s'_\ell) \text{ or } \bar{\mu}_N(s'_\ell) \succ_{s'_\ell} \bar{\mu}'_N(s'_\ell) = c_{j,t}. \tag{15}$$

However, the preference of $s'_\ell$ satisfies the first relation. To see this, suppose instead that it satisfies the second relation in (15). Then by Lemma B.2(i), we must have $s'_\ell \equiv \bar{\mu}'_M(c_{j,t}) \succ_j \bar{\mu}_M(c_{j,t})$, which violates (12). We conclude that $c_{j,t} = \bar{\mu}'_N(s'_\ell) \succ_{s'_\ell} \bar{\mu}_N(s'_\ell) \not\equiv c_{j,t+1}$, where the fact that $\bar{\mu}_N(s'_\ell) \not\equiv c_{j,t+1}$ is by (14). This implies that

$$c_{j,t+1} \succ_{s'_\ell} \bar{\mu}_N(s'_\ell), \tag{16}$$

since $c_{j,t+1}$ immediately follows $c_{j,t}$ in the strict preference of $s'_\ell$ over all the positions in colleges.

From (16) and (14), the student-college pair $(s'_\ell, c_{j,t+1})$ blocks $\bar{\mu}$, contradicting the stability of $\bar{\mu}$, and of $\mu$ via Lemma 5.6 of Roth and Sotomayor (1990).

The following lemma is a many-to-one version of Theorem A6 of Blum and Rothblum (2002).

The proof uses some additional notation. Given two many-to-one matchings $\mu_1$ and $\mu_2$ with corresponding one-to-one matchings $\bar{\mu}_1 = (\bar{\mu}_{1,N}, \bar{\mu}_{1,M})$ and $\bar{\mu}_2 = (\bar{\mu}_{2,N}, \bar{\mu}_{2,M})$, we write $\bar{\mu}_1 \succeq \bar{\mu}_2$ to denote $\bar{\mu}_{1,M}(j) \succ_j \bar{\mu}_{2,M}(j)$ or $\bar{\mu}_{1,M}(j) = \bar{\mu}_{2,M}(j)$ for all colleges $j$.\footnote{Thus $\bar{\mu}_1 \succeq \bar{\mu}_2$ represents the statement that all colleges weakly prefer $\bar{\mu}_1$ to $\bar{\mu}_2$.}

\textbf{Lemma B.4.} For any $\mu' \in \mathcal{E}_1(N, M, u)$, there is no stable matching $\mu''$ for market $(N, M, u)$ satisfying that $\mu'' \not\succeq T^*(\mu')$ and yet

$$T^*(\mu') \succeq \mu'' \succeq \mu'. \tag{17}$$

\textbf{Proof:} Let $\mu = T^*(\mu')$ and let $\mu'$ be the matching in the lemma. Let $\bar{\mu}'$ be the 1-simple matching corresponding to $\mu'$. By Theorem A6 of Blum and Rothblum (2002), the matching $\bar{\mu} = T^*(\bar{\mu}')$ is the college-worst stable matching weakly preferred by the colleges to $\bar{\mu}'$.\footnote{Recall Remark A.1. Note that when $\mu'$ is an 1-simple matching in a market, it is a stable matching for the market once any student that forms a blocking pair is eliminated from the market.} We now argue that $\mu$ is the college-worst stable matching weakly preferred by colleges to $\mu'$. Suppose by contradiction that there is a stable many-to-one matching $\mu''$ satisfying $\mu \succeq \mu'' \succeq \mu'$ with $\mu'' \not\succeq \mu$. We assume that $\mu \not\succeq \mu'$ (as otherwise, the contradiction is immediate). Since $\mu \not\succeq \mu'$ and $\mu$ is stable, it follows by the properties of $T^*$ that $\mu'$ must be an unstable 1-envy free
matching. Hence, \( \mu'' \neq \mu' \). Next, let \( \bar{\mu}'' \) denote the stable one-to-one matching corresponding to \( \mu'' \). Since \( \mu \) and \( \mu'' \) are stable with \( \mu \neq \mu'' \), it follows by Lemma 5.25 of Roth and Sotomayor (1990) that \( \bar{\mu} \geq \bar{\mu}'' \). Similarly, since \( \mu'' \) is stable and \( \mu' \) is 1-envy free with \( \mu'' \succeq \mu' \) and \( \mu'' \neq \mu' \), we obtain \( \bar{\mu}'' \geq \bar{\mu}' \) from Lemma B.3. Hence, \( \bar{\mu} \geq \bar{\mu}'' \geq \bar{\mu}' \), contradicting the requirement that \( \bar{\mu} \) is the college-worst stable matching weakly preferred to \( \bar{\mu}' \).

For the rest of the proofs, we take \( \mu^* : N \times U \to M' \) to be a matching mechanism such that for all \( u \in U \), the matching \( \mu^*(\cdot, u) \) is the SOSM for market \((N, M, u)\). For any \( i \in N \), we take \( \mu^*_i(\cdot, u_{-i}) \) to be the SOSM for market \((N \setminus \{i\}, M, u_{-i})\).

**Lemma B.5.** For any \( u \in U \) and \( i \in N \), we have

\[
\mu^*(\cdot, u) = T^*(g_i(\mu^*_i(\cdot, u_{-i}))).
\]

The proof of Lemma B.5 draws on Corollary A7 of Blum and Rothblum (2002). See also Theorem 4.3 of Blum, Roth, and Rothblum (1997), and Theorem 3.12 of Wu and Roth (2018).

**Proof:** Since \( \mu^*(\cdot, u) \) is the college-worst stable matching by Corollary 5.30 of Roth and Sotomayor (1990), we have

\[
T^*(g_i(\mu^*_i(\cdot, u_{-i}))) \succeq \mu^*(\cdot, u).
\]

On the other hand, by Theorem 5.34 of Roth and Sotomayor (1990), \( \mu^*(\cdot, u) \succeq g_i(\mu^*_i(\cdot, u_{-i})) \), so that

\[
T^*(g_i(\mu^*_i(\cdot, u_{-i}))) \succeq \mu^*(\cdot, u) \succeq g_i(\mu^*_i(\cdot, u_{-i})).
\]

By Lemma B.4, we must have \( \mu^*(\cdot, u) = T^*(g_i(\mu^*_i(\cdot, u_{-i}))) \).

C. Bounded Difference Condition for Stable Matching Mechanisms

C.1. Bounded Difference Condition for SOSM Mechanisms

**Lemma C.1.** Let for \( j \in M \) and \( i \in N \),

\[
N_{j,1} \equiv \mu^{i-1}(j, u) \setminus \mu^{i-1}_i(j, u_{-i}), \quad \text{and} \quad N_{j,0} \equiv \mu^{i-1}_i(j, u_{-i}) \setminus \mu^{i-1}(j, u).
\]

Then for any \( j \) with \( |\mu^{i-1}_i(j, u_{-i})| = q_j \),

\[
|N_{j,1}| = |N_{j,0}|.
\]
Proof: By Lemma A.1, \( |\mu_i^{-1}(j, u_{-i})| = q_j \) implies that \( |\mu_i^{-1}(j, u)| = q_j \), since all students in \( \mu_i^{-1}(j, u_{-i}) \) are acceptable to \( j \) under preferences \( u \) (i.e., a college \( j \) with no vacancy un-matches a student at an iteration of the operator if and only if college \( j \) forms a student-maximal blocking pair with some other student). Thus, the desired result follows. ■

From here on, we will also write \( T(\cdot; u) \) and \( T^*(\cdot; u) \) when the distinction between the preferences in the underlying market is important.

**Lemma C.2.** For each \( j \in M \) and \( i \in N \), we have that for any \( u = (v, w) \in U \) with \( h(w) = k \) for some \( k = 0, 1, \ldots, n \),

\[
|N_{j,1}| \leq 4(k \vee 1), \quad \text{and} \quad |N_{j,0}| \leq 4(k \vee 1),
\]

where \( N_{j,1} \) and \( N_{j,0} \) are defined as in (18) and \( h(w) \) as in (18).

**Proof:** Let \( \mu_0, \mu_1, \ldots, \mu_{r'} \) be matchings, generated by \( \mu_r = T(\mu_{r-1}; u) \) for each \( r = 1, \ldots, r' \), with \( \mu_0 = g_i(\mu_i'(\cdot, u_{-i})) \). By Lemma A.1, \( r' \) is finite and \( \mu_r = T^*(\mu_0; u) \in S(N, M, u) \). However, by Lemma B.5, we also have that \( T^*(\mu_0; u)(\cdot) = \mu^*(\cdot, u) \). Note that by the definition of \( T \), any \( j \) with \( |\mu_i^{-1}(j, u_{-i})| < q_j \) can be involved in at most one iteration of \( T \).\(^{23}\) Hence the first bound holds trivially with the value of one for any college with a vacancy under \( \mu_i'(\cdot, u_{-i}) \). The second bound holds with the value of zero for any such college. Therefore, for the remainder of the proof, we show that for every \( j \in M \) with \( |\mu_i^{-1}(j, u_{-i})| = q_j \),

\[
|N_{j,1}| \leq 4k^* \quad \text{and} \quad |N_{j,0}| \leq 4k^*,
\]

where we write \( k^* \equiv k \vee 1 \). In showing this, we use the following implication of Lemma A.1: for each \( j \in M \),

\[
i_1 > j > i_2, \quad \text{for all } (i_1, i_2) \in N_{j,1} \times N_{j,0}.
\]

First, by Lemma C.1, \( |N_{j,1}| = |N_{j,0}| \). To prove (20), suppose by contradiction that \( |N_{j,0}| = t \) and \( |N_{j,1}| = t \) for some integer \( t > 4k^* \). Let us enumerate

\[
N_{j,1} = \{a_1, a_2, \ldots, a_{t-1}, a_t\}, \quad \text{and} \quad N_{j,0} = \{b_1, b_2, \ldots, b_{t-1}, b_t\},
\]

so that a smaller index indicates that the change to the match of the student with the index took place at an earlier iteration of \( T \). First, we establish the following three facts.

**Fact 1:** For each \( s = 1, \ldots, 4k^* + 1 \), \( a_s > j > b_{4k^*+1} > j \ldots > j b_1 \).

\(^{23}\)Moreover, by Lemma B.5, there is at most one such vacancy-filling college, as any iteration of \( T \) that fills a vacancy is a stable matching, and hence, must be the final iteration of \( T \).
**Proof:** The ordering among the $b_s$’s is by the definition of $T$ (i.e., students less preferred by $j$ are dropped in earlier iterations of $T$).\textsuperscript{24} The fact that college $j$ prefers any of the $a_i$’s to any of the $b_j$’s is by (21). □

For each $s = 1, ..., k^*$, let $c^b_s$ denote the student unmatched from some college (not $j$) which then matches $b_j$, and let $c^a_s$ denote the student matched to some college (not $j$) which then unmatched $a_{s+1}$ (before $a_{s+1}$ matches with college $j$). Then the following fact is a consequence of Lemma A.1.

**Fact 2:** For each $s = 1, ..., k^*$,

(i) $c^b_s$ is not ranked higher than $b_j$ by more than $k^*$ positions under $\succ_j$, and

(ii) $c^a_s$ is not ranked lower than $a_{s+1}$ by more than $k^*$ positions under $\succ_j$.

**Proof:** Recall that by the condition that $h(w) = k$, if the rank difference between two students is more than $k^*$ in some college’s preference, every college agrees with the ranking between the two students. Hence violation of (i) or (ii) implies that for some $s = 1, ..., k^*$, some college $j'$ is made worse off in an iteration of $T$, violating Lemma A.1. □

For notational brevity, we will occasionally write $a \succ_j b \succ_j c$ as $abc$ in the proof of the following fact.

**Fact 3:** For each $s = 1, ..., k^*$,

(i) $c^b_s \neq c^a_s$, and

(ii) $c^a_s b_{3k^*+1} b_{3k^*} ... b_{k^*+s+1} c^b_s$.

**Proof:** Note that by Fact 1, we immediately have $b_{3k^*+1} ... b_2 b_1$. We begin by showing (i). Suppose by contradiction that $c^a_s = c^b_s \equiv c_s$ for some $s = 1, ..., k^*$. We argue that Fact 2 is violated. First, suppose that $c_s \succ_j b_{k^*+s+1}$. Hence,

$$c_s b_{k^*+s+1} b_{k^*+s} ... b_{s+1} b_s.$$

Since $c_s$ is ranked higher than $b_j$ by more than $k^*$ positions under $\succ_j$, this violates Fact 2(i). Now suppose that $b_{k^*+s+1} \succ_j c_s$. Therefore,

$$a_{s+1} b_{4k^*+1} ... b_{k^*+s+1} c_s$$

by Fact 1. However since $a_{s+1}$ is ranked higher than $c_s$ by more than $k^*$, this violates Fact 2(ii). Note also that we must have $c_s \neq b_{k^*+s+1}$, as otherwise, $c_s = b_{k^*+s+1}$, so that $b_{k^*+s+1}$ is the student unmatched by a college $j' \neq j$ that then matches $b_s$. This, however, violates Lemma A.1, since the fact that $b_{k^*+s+1}$ is more than $k^*$ positions higher than $b_s$ under $\succ_j$ implies that college $j'$ is

\textsuperscript{24}Note that by transitivity of college preferences, if a student $i'$ is unmatched by a college $j'$ on some iteration of $T$, $i'$ can never be rematched to $j'$ on a later iteration of $T$. This also implies that once a student $a_s \in N_{j,1}$ is matched to college $j$ on some iteration of $T$, $a_s$ is never unmatched from college $j$ on a subsequent iteration.
made worse off in the iteration of $T$ in which it matches $b_s$ and unmatches $b_{k^*+s+1}$. Since the preferences are strict, we arrive at a contradiction regarding the college $j$’s preference ordering between $b_{k^*+s+1}$ and $c_s$. Hence, the statement (i) in Fact 3 follows.

We now show (ii). Fix any $s = 1, \ldots, k^*$. By Fact 3(i), $c_s \not\succ_j b_{3k^*+1}$, as otherwise $a_{s+1} b_{4k^*+1} \ldots b_{3k^*+1} c_s$ by Fact 1, which violates Fact 2(ii). Lastly, we must have $b_{k^*+s+1} \succ_j c_s$, as otherwise, $c_s b_{k^*+s+1} \ldots b_s$ by Fact 1 which violates Fact 2(i). We also must have $c_s \not\succ b_{k^*+s+1}$, as otherwise we violate Lemma A.1 as we just saw in the proof of (i). Thus we have shown (ii). $\square$

Now we are ready to complete the proof of the lemma. It is convenient to begin the recursive argument that follows by letting $c_{s,0} := c_s$ for each $s = 1, \ldots, k^*$. By Fact 3, we have

\[ \{c_1^a, \ldots, c_{k^*}^a\} b_{3k^*+1} \ldots b_{2k^*+1} \{c_{1,0}, \ldots, c_{k^*,0}\}. \]  

Let $c_{s,0}$ be the worst student among $c_{1,0}, \ldots, c_{k^*,0}$ according to $\succ_j$. Let $c_{s,1}$ denote the student unmatched by the college that then matches $c_{s,0}$. Then we must have $b_{2k^*+1} \succ_j c_{s,1}$ in (23), as otherwise the college that matches $c_{s,0}$ and unmatches $c_{s,1}$ is made worse off, violating Lemma A.1.

Next, we let $c_{s,1} := c_{s,0}$ for all $s \in \{1, \ldots, k^*\} \setminus \{s_1\}$. By (23) and $b_{2k^*+1} \succ_j c_{s,1}$, we must have

\[ \{c_1^a, \ldots, c_{k^*}^a\} b_{3k^*+1} \ldots b_{2k^*+1} \{c_{1,1}, \ldots, c_{k^*,1}\}. \]  

Let $c_{s,1}$ be the worst student among $c_{1,1}, \ldots, c_{1,k^*}$ according to $\succ_j$. Let $c_{s,2}$ denote the student unmatched by the college that matches student $c_{s,1}$. By Lemma A.1, this student must again satisfy $b_{2k^*+1} \succ_j c_{s,2}$.

Now we denote $c_{s,2} := c_{s,1}$ for all $s \in \{1, \ldots, k^*\} \setminus \{s_2\}$. The next displaced student in the sequence must again be below $b_{2k^*+1}$. By continuing to displace students recursively in this fashion, we find that we can never displace any student above $b_{2k^*+1}$. In particular, it follows that for any $s, s' \in \{1, \ldots, k^*\}$, we cannot have that student $c_s^a$ is displaced after student $c_s^b$ under the iterations of $T$ that generate the sets $N_{j,1}$ and $N_{j,0}$ for college $j$.

Now fix any $s = 1, \ldots, k^*$. Consider the iteration of $T$ on which student $a_s$ is matched to college $j$ (so that $a_{s+1}$ is not yet matched to $j$). By Lemma A.1, we must have that either: (i) $b_s$ is unmatched by college $j$ on the same iteration of $T$ on which $a_s$ is matched to $j$, or (ii) $b_s$ is unmatched by college $j$ on a previous iteration of $T$ (before $a_s$ is matched to $j$). In either case, it follows that $c_s^b$ must be displaced before $c_s^a$, so that $a_{s+1}$ can then be matched with college $j$.

By our previous arguments, however, we know that $c_s^b$ cannot be displaced before $c_s^a$ under the iterations of $T$ that ultimately generate the sets $N_{j,1}$ and $N_{j,0}$ for college $j$. Hence, we cannot have $a_{s+1} \in N_{j,1}$ for any $s = 1, \ldots, k^*$. Therefore, we have shown that if $|N_{j,0}| = t$ and $|N_{j,1}| = t$ for some integer $t > 4k^*$, then Lemma A.1 is violated. $\blacksquare$
Corollary C.1. Let \( u = (v, w) \in U \) be the preference profiles with \( h(w) = k \) for some \( k = 0, 1, ..., n \). Then for each \( i \in N \) and \( u \in U \), we have
\[
|\{i' \in N \setminus \{i\} : \mu^*(i', u) \neq \mu^*_i(i', u_{-i})\}| \leq 4m(k \lor 1) + 1.
\]

Proof: By Lemma B.5, \( \mu^*(\cdot, u) \) is obtained from \( \mu^*_i(\cdot, u_{-i}) \) through the iterations of \( T \). Observe that at most one student who is matched to a college under \( \mu^*_i(\cdot, u_{-i}) \) can be left unmatched as a result of the iterations of \( T \) process. That is, the set
\[
A_i = \{i' \in N : \mu^*(i', u) = 0 \text{ and } i' \in \mu^{s-1}_i(j, u_{-i}) \text{ for some } j \in M\},
\]
is either a singleton, say, \( \{i^*\} \), for some \( i^* \in N \setminus \{i\} \), or an empty set. A student \( i' \in N \setminus \{i, i^*\} \) satisfies \( \mu^*(i', u) \neq \mu^*_i(i', u_{-i}) \), if and only if, for exactly one college, say \( j \), the student is a member of \( \mu^{s-1}(j, u) \) but not \( \mu_i^{s-1}(j, u_{-i}) \). Thus for each student \( i' \in N \setminus \{i\} \) we have that
\[
1\{\mu^*(i', u) \neq \mu^*_i(i', u_{-i})\} = \sum_{j \in M} 1\{i' \in \mu^{s-1}(j, u) \setminus \mu_i^{s-1}(j, u_{-i})\} + 1\{i' = i^*\}.
\]
Therefore, summing over \( i' \in N \setminus \{i\} \), we have
\[
|\{i' \in N \setminus \{i\} : \mu^*(i', u) \neq \mu^*_i(i', u_{-i})\}| = \sum_{j \in M} |\mu^{s-1}(j, u) \setminus \mu_i^{s-1}(j, u_{-i})| + \sum_{i' \in N \setminus \{i\}} 1\{i' = i^*\} \leq 4m(k \lor 1) + 1,
\]
where we used Lemma C.2 for the last bound.

Lemma C.3. Choose any \( k = 0, 1, 2, ..., n \), and any \((u', u)\) such that \( u \) and \( u' \) are generated by \((\tilde{s}, \tilde{z}) \in \tilde{S} \times \tilde{Z} \) and \((\tilde{s}', \tilde{z}) \in \tilde{S} \times \tilde{Z} \), where \((\tilde{s}, \tilde{z}) \) and \((\tilde{s}', \tilde{z}) \) satisfy the conditions (a),(b) and (c) in Lemma 3.3 with the chosen \( k \).

Then, for any \( j \in M' \),
\[
\begin{align*}
|\{i \in N : 1\{\mu^*(i, u) = j\} \neq 1\{\mu^*(i, u') = j\}\}| \leq 16(k \lor 1) + 1, \quad \text{and} \\
|\{i \in N : \mu^*(i, u) \neq \mu^*(i, u')\}| \leq 8m(k \lor 1) + 3.
\end{align*}
\]

Proof: We begin by showing the second bound. Choose \((u', u)\) as in the lemma. Then, we have
\[
u_{-i} = u'_{-i},
\]
by Assumption 2.3, because the elimination of student $i$ in the market does not alter the preference ordering between other students by any college. Hence

\[
|i' \in N : \mu^*(i', u) \neq \mu^*(i', u')| = \sum_{i' \in N} 1\{\mu^*(i', u) \neq \mu^*(i', u')\} \\
\leq \sum_{i' \in N} 1\{\mu^*(i', u) \neq \mu^*(i', u'), i' \neq i\} + 1 \\
\leq \sum_{i' \in N} 1\{\mu^*(i', u) \neq \mu^*(i', u_\cdot), i' \neq i\} \\
+ \sum_{i' \in N} 1\{\mu^*_i(i', u_\cdot') \neq \mu^*_i(i', u'), i' \neq i\} + 1.
\]

By Corollary C.1,

\[
\sum_{i' \in N} 1\{\mu^*(i', u) \neq \mu^*_i(i', u_\cdot), i' \neq i\} \leq |\{i' \in N \setminus \{i\} : \mu^*(i', u) \neq \mu^*_i(i', u_\cdot)\}| \\
\leq 4m(k \lor 1) + 1.
\]

Since the above bound is uniform over $u \in U$ and $u_\cdot' = u_\cdot', the same bound applies to the last sum in (26). Thus we conclude that

\[
|i' \in N : \mu^*(i', u) \neq \mu^*(i', u')| \leq 8m(k \lor 1) + 3,
\]

establishing the second bound in the lemma.

As for the first bound in Lemma 3.3, we again choose $(u', u)$ as in the lemma so that $u'$ and $u$ differ by the quality of one student, $i$. Then for any $j \in M'$ we have

\[
\sum_{i' \in N} |1\{\mu^*(i', u) = j\} - 1\{\mu^*(i', u') = j\}|
\leq \sum_{i' \in N} 1\{\mu^*(i', u) = j\}1\{\mu^*(i', u') \neq j\}1\{i' \neq i\}
+ \sum_{i' \in N} 1\{\mu^*(i', u) \neq j\}1\{\mu^*(i', u') = j\}1\{i' \neq i\} + 1.
\]

By multiplying each summand with $i'$ in the first term in (27) by $1 = 1\{\mu^*_i(i', u_\cdot) \neq j\} + 1\{\mu^*_i(i', u_\cdot) = j\}, we bound the first term by

\[
\sum_{i' \in N} 1\{\mu^*(i', u) = j\}1\{\mu^*_i(i', u_\cdot) \neq j\}1\{i' \neq i\}
+ \sum_{i' \in N} 1\{\mu^*(i', u') \neq j\}1\{\mu^*_i(i', u_\cdot) = j\}1\{i' \neq i\} \leq 8(k \lor 1).
\]
Note that if $j \in M$, the last inequality follows immediately by Lemma C.2. If, on the other hand, $j = 0$, then the last inequality follows by Lemma B.5 and the definition of $T$.\footnote{To see this, consider the two terms on the left-hand side of (28) in the case of $j = 0$. The first term has an upper bound of 1, since at most one student $i' \in N \setminus \{i\}$ matched to some college under $\mu^*(\cdot, u_{-i})$ becomes unmatched in the iterations of $T(\cdot, u)$ that yield $\mu^*(\cdot, u)$ from $\mu^*(\cdot, u_{-i})$. The second term is equal to zero, since no student in $i' \in N \setminus \{i\}$ unmatched under $\mu^*(\cdot, u_{-i})$ is matched in the iterations of $T(\cdot, u')$ that yield $\mu^*(\cdot, u')$ from $\mu^*(\cdot, u_{-i})$.}

Since the same bound of $8(k \lor 1)$ also holds for the second term of (27), we conclude that
\[
\left| \{i' \in N : 1\{\mu^*(i', u) = j\} \neq 1\{\mu^*(i', u') = j\}\} \right| \leq 16(k \lor 1) + 1.
\]

\[\blacksquare\]

C.2. Bounding the Distance Between SOSM and an Arbitrary Stable Matching

For $u \in U$, and any matching $\mu : N \to M'$, we define
\[
N^H(\mu) \equiv \{i \in N : \mu(i) \neq \mu^*(i, u)\},
\]
\[
N^*(\mu) \equiv \mu^{-1}(j) \setminus \mu^{-1}(j, u), \text{ and } N^*_{j,0}(\mu) \equiv \mu^{-1}(j, u) \setminus \mu^{-1}(j).
\]
The following is a straightforward consequence of Theorem 5.12 of Roth and Sotomayor (1990).

**Corollary C.2.** (i) For any $u \in U$ and any $\mu \in S(N, M, u)$,

\[
N^H(\mu) = \bigcup_{j, j \in M : j \neq j_2} \left( N^*_{j,0}(\mu) \cap N^*_{j,1}(\mu) \right). \tag{29}
\]

(ii) For any $u \in U$, any $\mu \in S(N, M, u)$, and for any college $j \in M$,

\[
|N^*_{j,1}(\mu)| = |N^*_{j,0}(\mu)|. \tag{30}
\]

**Proof:** Let us first prove (i). Note that since both $\mu(\cdot)$ and $\mu^*(\cdot, u)$ are stable matchings under $u$, the set of students matched to some college must be identical across the two matchings by Theorem 5.12 of Roth and Sotomayor (1990). The desired result comes from this immediately.

We now prove (ii). By Theorem 5.12 of Roth and Sotomayor (1990), the set of filled positions must be identical across the stable matchings, $\mu(\cdot)$ and $\mu^*(\cdot, u)$. Hence,

\[
|\mu^{-1}(j)| = |\mu^{-1}(j, u)| \text{ for all } j \in M.
\]

Thus, (ii) follows as in the proof of Corollary A.1. \[\blacksquare\]

**Corollary C.3.** For any $u \in U$, any $\mu \in S(N, M, u)$, and for any college $j \in M$,

\[
i_1 \succ_j i_2 \text{ for any } i_1 \in N^*_{j,1}(\mu) \text{ and } i_2 \in N^*_{j,0}(\mu). \tag{32}
\]
Proof: Since $\mu^*(\cdot, u)$ is the college-worst stable matching by Corollary 5.30 of Roth and Sotomayor (1990), we have $\mu(\cdot) \succeq \mu^*(\cdot, u)$ for any stable matching $\mu$. Therefore, the result follows from Lemma 5.25 of Roth and Sotomayor (1990).  

Let $u \in U$ and $\mu \in S(N, M, u)$. It is helpful to formalize the notion of “displacement” of one student by another as we move from $\mu^*(\cdot, u)$ to $\mu(\cdot)$. Given a college $j \in M$, a student $i \in N$, and a set $A \subset N$, let 

$$r_{ji}(A) \equiv \sum_{i' \in A} 1\{i' \succ j \, i\} + 1.$$ 

Thus, $r_{ji}(A)$ denotes the rank of student $i$ in the set $A$ according to $\succ_j$. Given two students $a, b \in N^H(\mu)$ and a college $j \in M$, we write $a \succ_j b$ if and only if 

$$a \in N^*_{j,1}(\mu), \ b \in N^*_{j,0}(\mu) \text{ and } r_{ja}(N^*_{j,1}(\mu)) = r_{jb}(N^*_{j,0}(\mu)).$$ 

In this case, we say that student $a$ displaces student $b$ from college $j$ (equivalently, student $b$ is displaced by student $a$ from college $j$) as we move from $\mu^*$ to $\mu$. For any two students $a, b \in N^H(\mu)$, we write $a \triangleright_j b$ if and only if $a \succ_j b$ for some $j \in M$ and say that student $a$ displaces student $b$ (equivalently, student $b$ is displaced by student $a$). For any students $a, b \in N^H(\mu)$ (not necessarily distinct), we also write $a \not\succ_j b$ if $a$ does not displace $b$. Similarly, we write $a \not\triangleright_j b$ if we wish to specify that $a$ does not displace college $b$ from college $j$.

We highlight some properties of $\triangleright$ that are helpful to remember in the exposition that follows. First, $\triangleright$ must satisfy $a \not\succ a$ for any student $a \in N^H(\mu)$, since $N^*_{j,1}(\mu) \cap N^*_{j,0}(\mu) = \emptyset$ for any $j \in M$ from definitions. Hence, $\triangleright$ is an irreflexive binary relation on $N^H$. In addition, we must also have $b \not\succ_j a$ for any distinct pair of students $a, b \in N^H(\mu)$ satisfying that $a \triangleright_j b$, which again follows by $N^*_{j,1}(\mu) \cap N^*_{j,0}(\mu) = \emptyset$. Note also that we must have $a \triangleright_j b$ whenever $a \triangleright_j b$, which is a consequence of Corollary C.3.

Lemma C.4. For any $u \in U$, any $\mu \in S(N, M, u)$, and any $i_1 \in N^H(\mu)$, we have the following.

(i) Student $i_1$ is displaced by one and only one student, say, $i_o$, and student $i_1$ displaces one and only one student, say, $i_2$.

(ii) Let $i_0$ and $i_2$ be the students in (i), and suppose that $i_0 \neq i_2$. Then $i_0, i_1, i_2$ are distinct students.

\footnote{Note that if $\mu(\cdot) = \mu^*(\cdot, u)$ then $N^*_{j,1}(\mu)$ and $N^*_{j,0}(\mu)$ are both empty and (32) holds trivially. If $\mu(\cdot) \neq \mu^*(\cdot, u)$, then $\mu(\cdot) \succeq \mu^*(\cdot, u)$ implies that we must have $\tilde{\mu}_M(c) \succ_j \tilde{\mu}_M'(c)$ at some position $c$ of college $j$, where $\tilde{\mu} = (\tilde{\mu}_N, \tilde{\mu}_M)$ and $\tilde{\mu}' = (\tilde{\mu}'_N, \tilde{\mu}'_M)$ denote the stable-matchings corresponding to $\mu(\cdot)$ and $\mu^*(\cdot, u)$ in the related one-to-one market. Thus, we must have (32), since $\tilde{\mu}_M(c) \succ_j \tilde{\mu}_M'(c)$ at some position $c$ of college $j$ implies that $\tilde{\mu}_M(c') \succeq_j \tilde{\mu}_M'(c')$ for all positions $c'$ of college $j$ by Lemma 5.25 of Roth and Sotomayor (1990).}

\footnote{For example, if $A = \{i_1, i_2, i_3\}$ and $i_1 \succ_j i_3 \succ_j i_2$, then $r_{j,i_1}(A) = 2$.}
in \(N^H\) satisfying

\[(35) \quad i_0 \succ j_1 \succ i_1 \succ j_2, i_2,\]

for some distinct colleges \(j_1, j_2 \in M\).

**Proof:** (i) Since \(i_1 \in N^H(\mu)\), we have by Corollary C.2(i) that \(i_1 \in N_{j_1,0}^*(\mu)\) and \(i_1 \in N_{j_2,1}^*(\mu)\) for two distinct colleges, \(j_1, j_2 \in M\). By Corollary C.2(ii), the sets \(N_{j_1,0}^*(\mu)\) and \(N_{j_2,1}^*(\mu)\) have the same cardinality. Since \(N_{j_1,0}^*(\mu)\) and \(N_{j_2,1}^*(\mu)\) are also disjoint, we conclude that there must be a single student \(i_0 \neq i_1\) belonging to the set \(N_{j_1,1}^*(\mu)\) with the same rank in the set \(N_{j_1,1}^*(\mu)\) according to \(\succ_{j_1}\) that \(i_1\) has in the set \(N_{j_1,0}^*(\mu)\) according to \(\succ_{j_1}\). By applying Corollary C.2(ii) and the same logic, we can also find a single student \(i_2 \neq i_1\) belonging to the set \(N_{j_2,0}^*(\mu)\) with the same rank in \(N_{j_2,0}^*(\mu)\) according to \(\succ_{j_2}\) that \(i_1\) has in the set \(N_{j_2,1}^*(\mu)\) according to \(\succ_{j_2}\). So we have (i).

(ii) Let \(i_0\) and \(i_2\) be the students in (i), so that \(i_0 \succ j_1 \succ i_1 \succ j_2 \succ i_2\) for some colleges \(j_1, j_2 \in M\), where we also assume that \(i_0 \neq i_2\). First, note that \(i_0 \neq i_1\) and \(i_1 \neq i_2\) by the proof of (i). Hence \(i_0, i_1, i_2\) are mutually distinct. Lastly, to see that \(j_1\) and \(j_2\) are distinct, suppose by contradiction that \(j_1 = j_2 \equiv j\). Then since \(i_0 \succ j\), \(i_1 \succ j\), \(i_2\), we must have \(i_1 \in N_{j,0}^*(\mu)\) \(\cap N_{j,1}^*(\mu) = \emptyset\), violating \(i_1 \in N^H(\mu)\). Thus, we must have \(j_1 \neq j_2\). This establishes (ii).

**Lemma C.5.** Let \(u \in \mathbb{U}\) and \(\mu \in S(N, M, u)\). Let \(i_0, i_1, i_2, i_3 \in N^H(\mu)\) be any four distinct students satisfying \(i_0 \succ i_1 \succ i_2 \succ i_3\). Then either of the following two cases holds.

**Case 1:**

\[(36) \quad i_0 \succ i_1 \succ i_2 \succ i_3 \succ i_0,\]

**Case 2:** For some finite \(r \geq 1\), there exist students \(i_4, \ldots, i_{3+r} \in N^H(\mu)\) satisfying that

\[(37) \quad i_0 \succ i_1 \succ i_2 \succ i_3 \succ i_4 \succ \ldots \succ i_{3+r} \succ i_0,\]

where \(i_0, i_1, i_2, i_3, i_4, \ldots, i_{3+r}\) are all distinct students.

**Proof:** Let \(i_0, i_1, i_2, i_3 \in N^H(\mu)\) be any four students satisfying

\[(38) \quad i_0 \succ i_1 \succ i_2 \succ i_3, \text{ with } i_0, i_1, i_2, i_3 \text{ distinct.}\]

Consider the student \(i_3\). By Lemma C.4, there is one and only one student, say \(i_4\), that satisfies \(i_3 \succ i_4\) for \(i_3\). By Lemma C.4 and (38), we have \(i_4 \in N^H(\mu) \{i_1, i_2, i_3\}\).\(^{28}\) If \(i_4 = i_0\), we have (36), and Case 1 is immediately satisfied.

\(^{28}\)To see that \(i_4 \notin \{i_1, i_2, i_3\}\), suppose that \(i_4 = i_1\). Then since \(i_3 \succ i_4\) and \(i_0 \succ i_1\), then we must have \(i_3 = i_0\) because \(i_1\) is displaced by one and only one student. However, this violates our assumption that \(i_3\) and \(i_0\) are distinct. If \(i_4 = i_2\), then \(i_3 = i_1\) (for the same reason as above), this time violating our assumption that \(i_3\) and \(i_1\) are distinct. Finally, by the irreflexivity of \(\succ\), we cannot have \(i_4 = i_3\).
For the remainder of the proof, we will show that Case 2 must hold under the assumption that $i_4 \neq i_0$. So suppose that $i_4 \neq i_0$. By Lemma C.4 and (38) (with $i_3 \triangleright i_4$ and $i_4 \neq i_0$) we must have

\[ i_0 \triangleright i_1 \triangleright i_2 \triangleright i_3 \triangleright i_4, \text{ with } i_0, i_1, i_2, i_3, i_4 \text{ distinct.} \]

By Lemma C.4, there exists one and only one student, satisfying $i_4 \triangleright i_5$ for $i_4$. By Lemma C.4
and (39), we have $i_5 \in N^H(\mu) \setminus \{i_1, i_2, i_3, i_4\}$. If $i_5 \equiv i_0$, then we have (37), so that Case 2 holds with $r = 1$. If $i_5 \neq i_0$, then by Lemma C.4 and (39) we have that

\[ i_0 \triangleright i_1 \triangleright i_2 \triangleright i_3 \triangleright i_4 \triangleright i_5, \text{ with } i_0, i_1, i_2, i_3, i_4, i_5 \text{ distinct.} \]

We then consider whether or not $i_6$, the unique student satisfying $i_5 \triangleright i_6$ for $i_5$ is equal to $i_0$. If $i_6 \equiv i_0$, then we have (37) with all distinct students, and we have Case 2 with $r = 2$. Otherwise, we go on to the next student. Since there are only finitely many students in the set $N$, it follows that sequence of students $i_0, i_1, i_2, i_3, i_4, ...$ (who are all distinct) must eventually terminate with $i_{3+r'} \equiv i_0$ for some $r' \geq 1$. Hence, we conclude that when (38) holds with the student $i_4$ displaced by $i_3$ satisfying $i_4 \neq i_0$, then Case 2 must hold. ■

**Lemma C.6.** Let $\mathbf{u} = (\mathbf{v}, \mathbf{w}) \in \mathbb{U}$ be the preference profiles with $h(\mathbf{w}) = k$ for some $k = 0, ..., n$. Let $\mu \in S(N, M, \mathbf{u})$. Then for each $j \in M$,

\[ |N^*_{j,1}(\mu)| \leq 4(k \lor 1) \text{ and } |N^*_{j,0}(\mu)| \leq 4(k \lor 1). \]

**Proof:** Let $k^* \equiv k \lor 1$. Suppose by contradiction that $|N^*_{j,0}(\mu)| = t$ and $|N^*_{j,1}(\mu)| = t$ for some integer $t > 4k^*$. Let us enumerate

\[ N^*_{j,0}(\mu) = \{b_1, b_2, ..., b_{t-1}, b_t\}, \]

so that a smaller index on the $b_i$’s indicates that the student with the index is worse according to college $j$’s preference. Furthermore, let us enumerate

\[ N^*_{j,1}(\mu) = \{a_2, ..., a_{t-1}, a_t, a_{t+1}\}, \]

so that for each $s = 1, ..., t$, $a_{s+1}$ denotes the unique student satisfying $a_{s+1} \triangleright_j b_s$ for student $b_s$ (which exists by Lemma C.4). Thus $a_{s+1}$ represents the student that displaces $b_s$ from college $j$. By Corollary C.3 and the above ordering convention, we immediately obtain the following fact.

**Fact I:** For each $s = 1, ..., 4k^* + 1$, $a_{s+1} \triangleright_j b_{4k^*+1} \triangleright_j b_{4k^*} \triangleright_j b_{4k^*+1}$. . . $\triangleright_j b_1$. 
Next, for each $s = 1, \ldots, k^*$, let $c_s^b$ denote the unique student satisfying $b_s \succ c_s^b$ and let $c_s^a$ denote the unique student satisfying $c_s^a \succ a_{s+1}$. Thus, for any $s = 1, \ldots, k^*$,

\begin{equation}
    c_s^a \succ a_{s+1} \succ b_s \succ c_s^b.
\end{equation}

Since $a_{s+1}$ displaces $b_s$ from college $j$ we have by Lemma C.4(ii) that $c_s^b$ represents a student that is unmatched by some college (not $j$) which then matches $b_s$, and $c_s^a$ represents a student matched to some college (not $j$) which then unmatches $a_{s+1}$ (before $a_{s+1}$ matches with college $j$).

**Fact II:** For each $s = 1, \ldots, k^*$,

(i) $c_s^b$ is not ranked higher than $b_s$ by more than $k^*$ positions under $\succ_j$, and

(ii) $c_s^a$ is not ranked lower than $a_{s+1}$ by more than $k^*$ positions under $\succ_j$.

**Proof:** The result follows as in the proof of Fact 2 in Lemma C.2, with Corollary C.3 and $\succ$ taking the place of Lemma A.1 and iterations of $T$. $\Box$

**Fact III:** For each $s = 1, \ldots, k^*$,

(i) $c_s^b \not= c_s^a$, and

(ii) $c_s^a b_{3k^*+1} b_{3k^*} \ldots b_{k^*+s+1} c_s^b$.

**Proof:** We have the result by following the argument used in the proof of Fact 3 from Lemma C.2, taking Facts I, II and $\succ$ in place of Facts 1, 2 and iterations of $T$. $\Box$

In light of Fact III, it follows that for each $s = 1, \ldots, k^*$, the students $c_s^b, a_{s+1}, b_s, c_s^a$ are mutually distinct.\(^{29}\) Thus, by Lemma C.5 and (42), we have the following displacement ordering over students for any $s = 1, \ldots, k^*$:

\begin{equation}
    c_s^a \succ a_{s+1} \succ b_s \succ c_s^b \succ c_{s+1}^{i_1} \succ \ldots \succ c_{s+1}^{i_r} \succ c_s^a,
\end{equation}

for some $r_j \geq 1$, with all students distinct. As in the proof of Lemma C.2, we use a recursive argument. Let $c_{s,0} := c_s^b$ for each $s = 1, \ldots, k^*$. By Fact III, we have

\begin{equation}
    \{c_{1,0}, \ldots, c_{k^*,0}\} b_{3k^*+1} \ldots b_{2k^*+1} \{c_{1,0}, \ldots, c_{k^*,0}\}.
\end{equation}

Let $c_{s,0}$ be the worst student among $c_{1,0}, \ldots, c_{k^*,0}$ according to $\succ_j$. Let $c_{s,1}$ denote the unique student displaced by $c_{s,0}$, i.e., satisfying that $c_{s,0} \succ c_{s,1}$. Note that we must have $b_{2k^*+1} \succ_j c_{s,1}$ in (44), as otherwise $c_{s,1} \succ_j b_{2k^*+1}$ means that the rank of $c_{s,1}$ is higher than that of $c_{s,0}$ by more than $k^*$ according to $\succ_j$, as in the proof of Fact II, so that $c_{s,1} \succ_j c_{s,0}$ and yet $c_{s,0} \succ_j c_{s,1}$ for some college $j'$, which violates Corollary C.3.

\(^{29}\)We have $c_s^a \not= a_{s+1}$, $a_{s+1} \not= b_s$, and $b_s \not= c_s^b$, by irreflexivity of $\succ$. By Fact III(i), we have $c_s^b \not= c_s^a$. By Fact III(ii) and Fact I, we have $c_s^a \not= b_s$ and $a_{s+1} \not= c_s^b$. 
Next, we let \( c_{s,1} := c_{s,0} \) for all \( s \in \{1, \ldots, k^*\} \setminus \{s_1\} \). By (44) and \( b_{2k^*+1} \succ_j c_{s,1} \), we must have
\[
\{c^a_1, \ldots, c^a_k\}b_{3k+1} \cdots b_{2k^*+1}\{c_1, \ldots, c_{k^*}\}.
\]
Let \( s_2 \) denote the index \( s \in \{1, \ldots, k^*\} \) satisfying that \( c_{s,1} \) is the worst student among \( c_{1,1}, \ldots, c_{1,k^*} \) according to \( \succ_j \) and \( c_{s,2} \) denote the unique student satisfying \( c_{s,2,1} \succ c_{s,2,2} \). By Corollary C.3, this student must again satisfy \( b_{2k^*+1} \succ_j c_{s,2} \). Now we denote \( c_{s,2} := c_{s,1} \) for all \( s \in \{1, \ldots, k^*\} \setminus \{s_2\} \).

The next displaced student in the sequence must again be below \( b_{2k^*+1} \) in the preference of college \( \succ_j \). By continuing in this fashion, we find that we can never displace any student above \( b_{2k^*+1} \). In particular, we can never displace \( c^a_s \) after \( c^b_s \) for any \( s, s' \in \{1, \ldots, k^*\} \). This violates (43). Thus, we cannot have \( |N_{j,0}(\mu)| = t \) and \( |N_{j,1}(\mu)| = t \) for some integer \( t > 4k^* \).

**Lemma C.7.** Let \( u = (v, w) \in U \) be any preference profile with \( h(w) = k \) for some \( k = 0, \ldots, n \). Let \( \mu \in S(N, M, u) \). Then for any \( j \in M' \):
\[
|\{i \in N : 1{\mu(i) = j} \neq 1{\mu^*(i, u) = j}\}| \leq 8(k \vee 1).
\]

**Proof:** For any \( j \in M' \) we have
\[
\sum_{i \in N} |1{\mu(i) = j} - 1{\mu^*(i, u) = j}| \leq \sum_{i \in N} 1{\mu(i) = j}1{\mu^*(i, u) \neq j} + \sum_{i' \in N} 1{\mu(i) \neq j}1{\mu^*(i, u) = j}.
\]

For \( j \in M \), the sum on the right hand side is bounded by \( 8(k \vee 1) \) by Lemma C.6, and for \( j = 0 \), it is bounded by zero by Theorem 5.12 of *Roth and Sotomayor (1990).*

**C.2.1. Proof of Lemma 3.3.** Let \( \mu^* \) be the SOSM mechanism and let \( \mu(\cdot; \alpha(\tilde{s}, \tilde{z})) \) and \( \mu(\cdot; \alpha(\tilde{s}', \tilde{z})) \) be stable matchings as given in the lemma. Fix any \( k = 0, 1, \ldots, n \) and choose \( j \in M \). The preference profiles \( u \) and \( u' \) are generated from \( (\tilde{s}, \tilde{z}) \) and \( (\tilde{s}', \tilde{z}) \) such that \( h(\tilde{s}, \tilde{z}) = h(\tilde{s}', \tilde{z}) = k \). By the triangle inequality,
\[
\left| \{i \in N : 1{\mu(i; \alpha(\tilde{s}, \tilde{z})) = j} \neq 1{\mu(i; \alpha(\tilde{s}', \tilde{z})) = j}\} \right| \\
\leq \left| \{i \in N : 1{\mu(i; \alpha(\tilde{s}, \tilde{z})) = j} \neq 1{\mu^*(i, u) = j}\} \right| \\
+ \left| \{i \in N : 1{\mu^*(i, u) = j} \neq 1{\mu(i; \alpha(\tilde{s}', \tilde{z})) = j}\} \right|.
\]

By Lemma C.7, the first and the third terms on the right-hand side of the above display are each bounded by \( 8(k \vee 1) \). By Lemma C.3, the second term is bounded by \( 16(k \vee 1) + 1 \).
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