Exploring topological double-Weyl semimetals with cold atoms in optical lattices

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We explore the topological properties of double-Weyl semimetals with cold atoms in optical lattices. We first propose to realize a tight-binding model of simulating the double-Weyl semimetal with a pair of double-Weyl points by engineering the atomic hopping in a three-dimensional optical lattice. We show that the double-Weyl points with topological charges of ±2 behave as sink and source of Berry flux in momentum space connecting by two Fermi arcs and they are stabilized by the \( C_{4h} \) point-group symmetry. By applying a realizable \( C_{4} \) breaking term, we find that each double-Weyl point splits into two single-Weyl points and obtain rich phase diagrams in the parameter space spanned by the strengths of an effective Zeeman term and the \( C_{4} \) breaking term, which contains a topological and a normal insulating phases and two topological Weyl semimetal phases with eight and four single-Weyl points, apart from the double-Weyl semimetal phase. Furthermore, we demonstrate with numerical simulations that (i) the mimicked double- and single-Weyl points can be detected by measuring the atomic transfer fractions after a Bloch oscillation; (ii) the Chern number of different quantum phases in the phase diagram can be extracted from the center shift of the hybrid Wannier functions, which can be directly measured with the time-of-flight imaging; (iii) the band topology of the \( C_{4} \)-symmetric Bloch Hamiltonian can be detected simply from measuring the spin polarization at the high symmetry momentum points with a condensate in the optical lattice. The proposed system would provide a promising platform for elaborating the intrinsic exotic physics of double-Weyl semimetals and the related topological phase transitions.

I. INTRODUCTION

Recently, topological Weyl semimetals have attracted a broad interest due to their wide range of exotic properties that are distinct from those of topological insulators [1–12]. Most importantly, the long-sought Weyl fermions, which are massless spin-1/2 particles in quantum field theory but have never been observed as fundamental particles in nature, can emerge as gapless quasiparticle excitations near band touching points dubbed as Weyl points in Weyl semimetals. The topological nature of the Weyl points in three-dimensional momentum space supports the existence of nontrivial Fermi arc surface states. The Weyl fermions in the bulk and the Fermi arc states in the surfaces are expected to give rise to exotic phenomena in Weyl semimetals, such as anomalous electromagnetic responses [6, 11, 12]. A significant advance has been theoretically and experimentally made for exploring Weyl physics not only in real materials [8–10], but also in some artificial systems, such as photonic and acoustic crystals [13, 14]. Interestingly, a new three-dimensional topological semimetal state, dubbed double-Weyl semimetal, has been theoretically proposed in solids that possess certain point-group symmetries [17–24]. The standard Weyl semimetal has linear dispersion in all the three momentum directions near the single-Weyl points with topological charge of ±1. In contrast, the dispersion of a double-Weyl semimetal is quadratic in two dimensions and linear in the third dimension near the gapless points with topological charge of ±2, which are thus named as double-Weyl points. Several materials have been proposed to be potential candidates of double-Weyl semimetals [17–24], and the double-Weyl points have recently been observed in photonic crystals [16]. Some important properties of this newly predicted topological state are rarely explored, such as the symmetry-breaking effects and the topological phase transition. Thus, other experimentally tunable systems for exploring the exotic topological properties double-Weyl semimetals are highly desirable.

On the other hand, ultracold atoms in optical lattices play an important role in advancing our understanding of condensed matter physics [25]. Remarkably, as recent experimental advances in realizing spin-orbit coupling and artificial gauge field for neutral atoms [26–28], these systems provide a powerful platform with unparalleled controllability towards studying topological states of matter. For instance, the celebrated Harper-Hofstadter [29] model and Haldane model [30] have been realized [31–35] experimentally in optical lattices, where the Chern number has also been successfully probed. The chiral edge states have been experimentally observed in one-dimensional optical lattices subjected to a synthetic magnetic field and an artificial dimension [36, 37]. The topological (geometric) pumping has been demonstrated with cold atoms in optical superlattices [38, 40]. The two-dimensional spin-orbit coupling for Bose-Einstein condensates has been realized in optical lattices and the band...
topology has also been measured \[11\]. Then an important question is raised: can we realize other predicted topological phases that are rare in solid-state materials in the cold atom systems? Several proposals have been suggested to realize exotic topological insulating states \[12,17\] and topological nodal semimetals with single-Weyl points or nodal loops \[18,54\] in optical lattices. Notably, it was proposed to simulate the double-Weyl semimetals with ultracold atoms in optical lattices in the presence of synthetic non-Abelian SU(2) gauge potentials \[24\]. Other feasible schemes for mimicking tunable double-Weyl semimetal states and detecting their intrinsic topological properties in cold atomic systems are still badly awaited.

In this paper, we explore the topological double-Weyl semimetals with cold atoms in optical lattices. We first propose to realize a tight-binding model of simulating the double-Weyl semimetal with tunable double-Weyl points by engineering the atomic hopping in a three-dimensional cubic optical lattice. We show that a pair of double-Weyl points with nontrivial monopole charges behave as sink and source of Berry fluxes in momentum space and they are stabilized by the $C_{3h}$ point-group symmetry. We further investigate the topological properties of the double-Weyl semimetal by calculating $k_z$-dependent Chern number and the gapless edge states. By applying a realized $C_4$ breaking term, we find that each double-Weyl point splits into two single-Weyl points and obtain a rich phase diagram in the parameter space spanned by the strengths of an effective Zeeman potential and the $C_4$ breaking term, which contains a topological insulator phase, a normal band insulator phase, and two topological Weyl semimetal phases with eight and four single-Weyl points apart from the double-Weyl semimetal phase. Finally, we demonstrate with numerical simulations that (i) the analogous double- and single-Weyl points can be detected by measuring the atomic transfer fractions after a Bloch oscillation; (ii) the $k_z$-dependent Chern number of different quantum phases in the phase diagram can be extracted from the center shift of the hybrid Wannier functions, which are based on time-of-flight imaging; (iii) the band topology of the $C_4$-symmetric Bloch Hamiltonian can be detected simply from measuring the spin polarization at the high symmetry momentum points with a condensate in the optical lattice. The proposed cold-atom system provides a promising platform for elaborating the intrinsic exotic physics of double-Weyl semimetals and the related topological phase transitions.

The paper is organized as follows. Section II introduces the tight-binding model and optical-lattice system for realizing double-Weyl semimetals with double-Weyl points. In Section III, with the numerical calculation of the Chern number and the chiral edge states, we elaborate the topological properties of the simulated double-Weyl semimetals and obtain a rich phase diagram containing other topological quantum phases by applying a symmetry breaking term. In Section IV, we propose realistic schemes to detect the simulated Weyl points and the characteristic topological invariant with cold atoms in the optical lattice. Finally, a short conclusion is given in Sec.V.

II. MODEL AND SYSTEM

We consider a non-interacting (pseudo)spin-1/2 degenerate fermionic gas in a three-dimensional cubic optical lattice, where the spins are encoded by two atomic internal states labeled as $| \uparrow \rangle$ and $| \downarrow \rangle$. The tight-binding Hamiltonian of the cold atom system is considered to be

$$
\hat{H} = \frac{t}{2} \sum_{r} \left( \hat{a}_{r+\sigma} \hat{a}_{r} \right) - \frac{i\eta}{4} \sum_{r} \left( \hat{a}_{r+(x+y)} \hat{a}_{r} - \hat{a}_{r+(x-y)} \hat{a}_{r} \right) + \text{H.c.}
\frac{t}{2} \sum_{r,\eta} \left( \hat{a}_{r+\eta} \hat{a}_{r} \right) - \hat{a}_{r-\eta} \hat{a}_{r} + \text{H.c.}
+ m_z \sum_{r} \left( \hat{a}_{r+\sigma} \hat{a}_{r} \right)^{\dagger} - \hat{a}_{r+\sigma} \hat{a}_{r} + \text{H.c.}
\right)
$$

where $\hat{a}_{r,\sigma} (\hat{a}_{r,\sigma}^{\dagger})$ is the annihilation (creation) operator on site $r$ for the fermion with spin $\sigma = \{ \uparrow, \downarrow \}$, $\eta = x, y, z$ denote the hopping directions, $m_z$ is the strength of an effective Zeeman potential, and the hopping strength is set $t = 1$ as the energy unit hereafter. By defining the two-component annihilation operator at site $r$ as $\hat{a}_{r} = (\hat{a}_{r,\uparrow}, \hat{a}_{r,\downarrow})^T$, Hamiltonian (1) can be rewritten as

$$
\hat{H} = \sum_{r,\eta} \left( \hat{a}_{r+\sigma} \hat{U}_{r,\eta} \hat{a}_{r} + \text{H.c.} \right) + m_z \sum_{r} \hat{a}_{r} \sigma_z \hat{a}_{r}
\sum_{r} \left( \hat{a}_{r+\sigma} \hat{U}_{r,\eta} \hat{a}_{r} - \hat{a}_{r+\sigma} \hat{U}_{r,\eta} \hat{a}_{r} + \text{H.c.} \right),
$$

where the hopping matrices along the three axis are $U_x = \frac{1}{2} (\sigma_x - \sigma_z)$, $U_y = -\frac{1}{2} (\sigma_x + \sigma_z)$, and $U_z = \frac{1}{2} \sigma_z$, and along the $xy$ direction is $U_{xy} = -\frac{1}{2} \sigma_y$, with $\sigma_{x,y,z}$ being the Pauli matrices acting on the spin states.

Here the atomic hoppings $U_x$ and $U_{xy}$ between two lattice sites along the corresponding direction can be spin-conserved hopping (the $\sigma_z$ term) or spin-flip hopping (the $\sigma_x$ and $\sigma_y$ terms), which can be achieved by the laser-assisted tunnelling technique with well-designed Raman coupling between the two spin states \[20,23\]. First, one can use a moderate magnetic field to distinguish the spin states with the Zeeman splitting, which allows one to correlate tunnelling in a spatial direction with rotations in internal spin states and state-dependent tunnelling phases. Second, the natural hopping along each direction is suppressed by titling the cubic optical lattice with a homogeneous energy gradient along the $x, y, z$-directions, with the large tilt potential $\Delta \eta \gg t_N$ (such that the hopping probability $(t_N/\Delta \eta)^2$ induced by the natural tunneling is negligible) and $t_N$ denoting the natural tunneling rate. The tilt potential can be created through the natural gravitational field or the gradient of a dc- or ac-Stark
shift, and here we require different linear energy shifts per site $\Delta x \neq \Delta y \neq \Delta z$ in order to distinguish between the tunnelling directed along different directions. Finally, the hopping terms can be restored and engineered by application of two-photon Raman coupling with the laser beams of proper configurations through the laser-frequency and polarization selections [26–28].

In principle, arbitrary $2 \times 2$ hopping matrices including the required $U_{n,xy}$ can be generated in this way with well-designed laser configurations [32–34]. Since several protocols for implementing similar atomic hopping matrices and the tunable Zeeman potential have been theoretically proposed or experimentally realized [40–46] and a different model of double-Weyl semimetals in optical lattices has been presented, here we leave some details of realization out and focus on exploration and detection of their novel topological properties in the following.

For this lattice system under the periodic boundary condition, the tight-binding Hamiltonian can be rewritten as $H = \sum_{k,\sigma,\sigma'} \hat{a}_{k\sigma} \hat{a}_{k\sigma'}^\dagger \mathcal{H}(k)_{\sigma\sigma'} \hat{a}_{k\sigma'}$, where $\hat{a}_{k\sigma}$ is the annihilation operator in momentum space $k = (k_x, k_y, k_z)$, and $\mathcal{H}(k) = \mathcal{H}(k) \cdot \hat{\sigma}$ is Bloch Hamiltonian. Here $\mathcal{H}(k)$ is the lowest-band Berry curvature in the wave function $\chi(v)$ [37]. The energy spectrum of the system is given by $E_{\pm}(k) = \pm |\mathcal{H}(k)|$. The bulk gap closes when $d_x(k) = d_y(k) = 0$. By solving the equations, we can obtain a pair of twofold degenerate points that are double-Weyl points $W_{\pm} = (0, 0, \pm \arccos(m_z - 2))$ for $1 < m_z < 3$ and another pair of double-Weyl points $W_{\pm} = (\pi, \pi, \pm \arccos(m_z + 2))$ for $-3 < m_z < -1$. For instance, the energy spectrum as a function of $k_x$ and $k_z$ with fixed $k_y = 0$ for $m_z = 2$ is shown in Fig. 1(a), where two double-Weyl points locate at $(0, 0, \pm \pi/2)$.

We consider the nodes $W_{\pm} = (0, 0, \pm \arccos(m_z - 2))$ to further show that they are double-Weyl points. Expanding the Bloch Hamiltonian near the two nodes with $q = (q_x, q_y, q_z) \equiv k - W_{\pm}$ yields the low-energy effective Hamiltonian

$$\mathcal{H}_{\pm} = \frac{1}{2} (q_y^2 - q_x^2) \sigma_x + q_x q_y \sigma_y + \chi v_z q_z \sigma_z,$$

where $\chi = \pm 1$ respectively for the two nodes $W_{\pm}$ and $v_z = \sqrt{1 - (m_z - 2)^2}$. The effective Hamiltonian shows that the dispersion near the nodes is quadratic in $k_x$ and $k_y$ and linear in $k_z$. One can rewrite the low-energy effective Hamiltonian as

$$\mathcal{H}_{\pm} = \epsilon \begin{pmatrix} \chi \cos \theta & -\sin \theta e^{i2\varphi} \\ -\sin \theta e^{-i2\varphi} & -\chi \cos \theta \end{pmatrix},$$

where $v_{\parallel} = \frac{1}{2}, q_{\parallel}^2 = q_x^2 + q_y^2, \epsilon = \sqrt{(v_z q_z)^2 + v_{\parallel}^2(q_x^2 + q_y^2)^2}$, $\cos \theta = v_z q_z / \epsilon$, $\sin \theta = q_{\parallel} / \epsilon$, $\sin \varphi = q_x / q_{\parallel}$, and $\cos \varphi = q_y / q_{\parallel}$. The eigenstates of the lowest band near the two nodes with index $\chi = \pm 1$ are respectively given by $|u_{\parallel}\rangle = (\sin \frac{\pi}{2} \epsilon e^{i2\varphi}, \cos \frac{\pi}{2} \epsilon e^{i2\varphi})^T$ and $|u_0\rangle = (\cos \frac{\pi}{2} \epsilon, -\sin \frac{\pi}{2} \epsilon)^T$. The Chern number (topological charge) $C_{\chi}$ of the nodes can thus be computed by integrating the Berry curvature over an arbitrary Fermi sphere $S$ that encloses each node [22]:

$$C_{\chi} = \frac{1}{2\pi} \oint_S dS \cdot F = -2\chi,$$

where the Berry curvature $F = \nabla \times A$ and $A = (A_\theta, A_\varphi)$ is the Berry connection given by $A_\theta = i \langle u_0 | \partial \theta | u_0 \rangle = 0$ and $A_\varphi = i \langle u_0 | \partial \varphi | u_0 \rangle = -2\chi \sin^2 \theta$. The above results reveal that the two nodes have opposite topological charges of $\pm 2$ and quadratic dispersion, in contract to the standard Weyl points in Weyl semimetals that have topological charges of $\pm 1$ and linear dispersion, so they are named double-Weyl points. Thus the system is in the double-Weyl semimetal phase when the Fermi level lies in the vicinity of the double-Weyl points.

In momentum space, the gauge field associated with the Berry curvature near the neighborhood of Weyl node behaves like a magnetic field originating from a magnetic monopole. Here the opposite chirality of the paired double-Weyl points can also be viewed as a monopole-antimonopole pair in the momentum space. To show this point, we calculate the Berry curvature as a function of the momentum $k$: $F(k) = \nabla \times A(k)$ with the Berry connection $A(k) = i \langle u_0 | \nabla_k | u_0 \rangle$ defined by the wave function $|u_0(k)\rangle$ in the lowest band. For the two bands system, the lowest-band Berry curvature in the momentum space is given by [50]

$$F^a = \epsilon_{abc} F^c_{bc} = \epsilon_{abc} \left[ \frac{1}{2d^3} \cdot \left( \frac{\partial d}{\partial k_b} \times \frac{\partial d}{\partial k_c} \right) \right],$$

where the three components are obtained as $F^x = (\sin k_x \cos k_x \cos k_y \sin k_z - \sin k_x \sin k_z) / N(k)$, $F^y = (\sin k_y \sin k_z) / N(k)$, and $F^z = (\cos k_x \sin k_z) / N(k)$.
where $C_4 = e^{-i\frac{\pi}{2} \sigma_z}$ is a point-group operator for the fourfold rotation about the z axis and $P$ is a matrix transferring $(k_x, k_y, k_z)$ to $(k_y, -k_x, -k_z)$. Here we can define $f(k) = d_z(k) - i d_y(k)$ and $\sigma_\pm = (\sigma_x \pm i \sigma_y)/2$, such that the Bloch Hamiltonian of the two-band model can be rewritten as $\mathcal{H}(k) = f(k) \sigma_+ + f^*(k) \sigma_- + d_z(k) \sigma_z$. Thus the transform of $\mathcal{H}(k)$ under $C_4$ leads to $C_4 \mathcal{H}(k) C_4^{-1} = f(k) e^{-i\frac{\pi}{2} \sigma_z} \sigma_+ e^{i\frac{\pi}{2} \sigma_z} + f^*(k) e^{i\frac{\pi}{2} \sigma_z} \sigma_- e^{-i\frac{\pi}{2} \sigma_z} + d_z(k) \sigma_z$. Since the Bloch vectors of the double-Weyl semimetal satisfy $f(Pk) = -f(k)$, $f^*(Pk) = -f^*(k)$ and $d_z(Pk) = d_z(k)$, the system preserves $C_{4h}$ symmetry. When the $C_{4h}$ symmetry are broken, the double-Weyl points will be destroyed and the system is no longer in the double-Weyl semimetal phase. Thus it would be valuable to study the symmetry-breaking effects and the related phase transition in this tunable system.

III. TOPOLOGICAL PROPERTIES OF THE SIMULATED DOUBLE-WEYL SEMIMETALS

To further study the topological properties of this system, we consider the Bloch Hamiltonian with dimension reduction method: considering $k_z$ as a good quantum number and then reduce the three-dimensional system to a set of two-dimensional subsystems with $k_z$ as a parameter. For a fixed $k_z$, the reduced Bloch Hamiltonian $\mathcal{H}_{k_z}(k_x, k_y)$ is given by

$$\mathcal{H}_{k_z}(k_x, k_y) = (cosec k - cosec k_y) \sigma_x + \text{sec k} \text{cosec k} \sigma_y + (M_z - cosec k - cosec k_y) \sigma_z,$$

where $M_z = m_z - cosec k$. The bands of these subsystems with fixed $k_z$ are all gapped when $k_z \neq \pm k_z$ with $k_z^2 = \arccos(m_z - 2)$. Under this condition, $\mathcal{H}_{k_z}(k_x, k_y)$ describes effective two-dimensional Chern insulators since the $k_z$-dependent Chern number is given by

$$C_{k_z} = \frac{1}{4\pi} \int_{-\pi}^{\pi} dk_x \int_{-\pi}^{\pi} dk_y \cdot \hat{d} \cdot \left( \partial_{k_x} \hat{d} \times \partial_{k_y} \hat{d} \right),$$

where $\hat{d} \equiv d/|d|$. In the parameter regime $1 < |m_z| < 3$, we obtain that $C_{k_z} = 2$ for the planes with $-k_z^2 < k_z < k_z^2$ and $C_{k_z} = 0$ for other cases. With similar dimension reduction method, such a two-dimensional Chern insulator can be regarded as a fictitious one-dimensional chain subjected to an external parameter $k_z$, since we can consider $k_z$ as a good quantum number. The tight-binding Hamiltonian of such a one-dimensional chain along the $y$ axis can be written as

$$\hat{H}_y(k_z, k_y) = \frac{1}{2} \sum_{i_y} \left( \hat{a}^\dagger_{i_y \uparrow} \hat{a}_{i_y + 1 \uparrow} + \hat{a}^\dagger_{i_y \downarrow} \hat{a}_{i_y + 1 \downarrow} \right) + H.c.$$

$$-\frac{1}{2} \sum_{i_y} \left( (1 + \sin k_z) \hat{a}^\dagger_{i_y \uparrow} \hat{a}_{i_y + 1 \uparrow} + (1 - \sin k_z) \hat{a}^\dagger_{i_y \downarrow} \hat{a}_{i_y + 1 \downarrow} \right) + H.c.$$
$C_{k_z} = 2$. The two chiral states gradually spread into the bulk when their energies are closer to the bulk bands. The density distributions of some edge modes are shown in the inset in Fig. 2(a) for typical $k_x$. Increasing $|k_z|$ to the critical points $k_z^c$, the two degeneracies of surface states move to the center and then merge at the double-Weyl points at $k_z = \pm k_z^c$, with the energy spectrum of $k_z = k_z^c = 0.5\pi$ being shown in Fig. 2(b). When $k_z$ enters the region $|k_z| > k_z^c$, the energy spectrum is again gapped but there is no chiral edge state since $C_{k_z} = 0$ in this region, with the case of $k_z = 0.6\pi$ shown in Fig. 2(c). The change of topological invariant $C_{k_z}$ from 2 to 0 indicates a double-Weyl point of monopole charge 2.

We further study the Fermi-arc zero modes (with energy $E = 0$) by numerically calculating the energy spectrum $E(k_x, k_z)$ of the surface states, with the results being plotted in Fig. 3(a) for typical parameter $m_z = 1.5$. In this case, the two double-Weyl points locate at $W_{\pm} = (0, 0, \pm \frac{2\pi}{3})$, and they are connecting by two Fermi arcs with $E = 0$ plotted with the black lines. We also find that if $m_z$ are approaching to the critical values ±1 or ±3, the Fermi arcs shrink since the two double-Weyl nodes move to each other, and they vanish entirely at the phase boundaries when the two Weyl points merge.

We proceed to study the effects of symmetry breaking in the double-Weyl semimetals. To this end, we can add a term $H_P = \delta \sigma_z$ to the Bloch Hamiltonian in Eq. (9) to break its $C_4$ symmetry, the resultant Hamiltonian $\mathcal{H} = \mathcal{H} + H_P$ becomes

$$\mathcal{H} = (\cos k_x - \cos k_y + \delta)\sigma_x + \sin k_z \sin k_y \sigma_y$$

$$+ (m_z - \cos k_x - \cos k_y - \cos k_z)\sigma_z.$$  (12)

In the optical lattice, the $H_P$ term corresponds to the tunable in-site coupling between the two spin states described by $H_P = \delta \sum_{\lambda,\mu} \hat{a}_{\lambda,\mu}^\dagger \hat{a}_{\lambda,\mu} + H.c.$, which can be realized by additional Raman coupling. In this case, we find that the bulk gap can be closed if $|\delta| \leq 2$. In particular, for $-m_z < \delta < 3 - m_z$, we can find four single Weyl points located at $W_{1,\pm} = (-\arccos(1 - \delta), 0, \pm \arccos(m_z - 2 + \delta))$ and $W_{2,\pm} = (\arccos(1 - \delta), 0, \pm \arccos(m_z - 2 + \delta))$. For $m_z + 1 < \delta < m_z + 3$, there are also four single Weyl points at $(k_x, k_y, k_z) = (\pi, \pm \arccos(1 - \delta), \pm \arccos(m_z - 2 + \delta))$. Similarly, when $-2 \leq \delta \leq 0$, the single Weyl points are at $(k_x, k_y, k_z) = (0, \pm \arccos(1+\delta), \pm \arccos(m_z - 2 + \delta))$ for $m_z - 3 \leq \delta \leq m_z - 1$ and at $(k_x, k_y, k_z) = (\pm \arccos(-1 - \delta), \pi, \pm \arccos(m_z + 2 + \delta))$ for $-3 - m_z \leq \delta \leq -1 - m_z$.

To reveal the topological nature of these gapless points more clearly, we first expand the Hamiltonian near the four points $W_{\lambda,\mu}$ with $\lambda = 1, 2$ and $\mu = +, -$. We obtain the corresponding low-energy effective Hamiltonian $\mathcal{H}_{\lambda,\mu}$ (the other three cases proceed similarly):

$$\mathcal{H}_{\lambda,\mu} \approx (-1)^\lambda \alpha |q_\lambda \sigma_y - (q_x - q_z)\sigma_z| + \mu |q_\lambda | q_x, q_z = k - W_{\lambda,\mu}.$$  (13)

where $\alpha = \sqrt{1 - (1 - \delta)^2}$ and $\alpha_z = \sqrt{1 - (m_z - 2 + \delta)^2}$ are the effective Fermi velocities, and $q = (q_x, q_y, q_z) = k - W_{1,\pm}$ or $q = k - W_{2,\pm}$ for the four points. Thus, the dispersion near the gapless points is linear along the three momentum directions, indicating that these nodes are single-Weyl points. We then consider the evolution of the Fermi arcs as increasing $\delta$ from 0 to 2, and find that each double-Weyl point of monopole charge $+1(\delta)$ in the double-Weyl semimetal ($1 < m_z < 3$ and $\delta = 0$) splits into two pairs of single-Weyl points with monopole charge $+1(-1)$ connected by two disconnected Fermi arcs. When $\delta = 2$ the two pairs of single-Weyl points merge and the Fermi arcs disappear. For example, there are two Fermi arcs which terminate at the four single-Weyl points for $\delta = 1$ and $m_z = 1.5$, as shown in Fig. 3(b). The corresponding energy spectra on the $k_x - k_z (k_y = 0)$ and $k_x - k_y (k_z = 0)$ planes are respectively shown in Fig. 3(c) and (d), which indicate the linear dispersion of the four single-Weyl points along each momentum direction. Therefore in this parameter region ($1 < m_z < 3$ and $0 < \delta < 2$), the system is in

FIG. 3: (Color online) (a) Fermi arcs connecting the mimicked double-Weyl points $W_{\pm}$ with monopole charges $\mp 2$ denoted by white dots. The black line denotes Fermi arcs formed by gapless zero-energy edge modes. (b) Fermi arcs when $\delta = 1$. The two double-Weyl points split into four single-Weyl points when $\delta \sigma_z$ term is added, and the Fermi arcs (black lines) terminate at four points denoted by white dots. $W_{1,\pm}$ and $W_{2,\pm}$ have monopole charge $-1$, $W_{1,\pm}$ and $W_{2,\pm}$ have monopole charge +1. (c) and (d) The band dispersions $E(k_x, k_z)$ and $E(k_x, k_y)$ for $\delta = 1$ with fixed $k_y = 0$ and $k_z = \frac{2\pi}{3}$, respectively. The other parameter is $m_z = 1.5$. 
the single-Weyl semimetal phase with four single-Weyl points, which can be obtained for the other three parameter regions.

By similar analysis of the gapless points and the topological properties, we obtain the phase diagram for the Hamiltonian in Eq. (12) in the parameter space spanned by \( m_z \) and \( \delta \), as shown in Fig. 4. In the phase diagram, apart from the double-Weyl semimetal phase (denoted by DWSM) and the single-Weyl semimetal phase with four Weyl points (denoted by WSM\(_4\)), there are other three different phases: a normal band insulating phase (denoted by NI) with \( C_{k_z} = 0 \) when \( m_z \) or \( \delta \) is large enough to open a trivial energy gap, a weak topological insulating phase (denoted by TI) with \( C_{k_z} = 2 \) and chiral edge states for all the range of \( k_z \) when \(-m_z - 1 < \delta < 1 - m_z \) and \( m_z - 1 < \delta < m_z + 1 \), and a single-Weyl semimetal phase with eight Weyl points (denoted by WSM\(_8\)). For the WSM\(_8\) phase with \( 1 < \delta < 2 \), the eight (four pairs) single-Weyl points locate at \( W_{1,\pm} = (-\arccos(1 - \delta),0,\pm\arccos(m_z - 2 + \delta)) \), \( W_{2,\pm} = (\arccos(1 - \delta),0,\pm\arccos(m_z - 2 + \delta)) \), \( W'_{1,\pm} = (\pi,\arccos(\delta - 1),\pm\arccos(m_z + 2 - \delta)) \), and \( W'_{2,\pm} = (\pi,\arccos(\delta - 1),\pm\arccos(m_z + 2 - \delta)) \). Figure 5(a) depicts the position and monopole charge of these eight single-Weyl points in momentum space. The corresponding \( k_z \)-dependent Chern number \( C_{k_z} \) as a function of \( k_z \) is plotted in Fig. 5(b).

Finally \( C_{k_z} = 0 \) when \( k_z > k^w \) or \( k_z < -k^{w'} \).

**FIG. 4:** (Color online) The phase diagram of the Hamiltonian in Eq. (12). TI denotes the topological insulating phase (dark red), NI denotes the normal band insulating phase (green), WSM\(_8\) the Weyl semimetal phase with eight single-Weyl points (dark blue), WSM\(_4\) denotes the Weyl semimetal phase with four single-Weyl points (blue), and DWSM is the double-Weyl semimetal phase (yellow lines).

**FIG. 5:** (Color online) (a) Illustration of the eight single-Weyl points. The monopole charges of \( W'_{1,\pm} \), \( W'_{2,\pm} \), \( W_{1,\pm} \), \( W_{2,\pm} \) are -1, and the monopole charges of \( W'_{1,\pm} \), \( W'_{2,\pm} \), \( W_{1,\pm} \), \( W_{2,\pm} \) are +1. (b) The Chern number as function of \( k_z \). Here \( k^w = \arccos(m_z - 2 + \delta) \) and \( k^{w'} = \arccos(m_z + 2 - \delta) \).

### IV. EXPERIMENTAL DETECTION SCHEMES

At this stage, we have introduced the optical lattice system for simulation of the double-Weyl semimetal states and explored the relevant topological properties and the phase diagram. In this section, we propose practical methods for their experimental detection. We first show that the simulated Weyl points can be probed by measuring the atomic Zener tunneling to the excited band after a Bloch oscillation, and then propose two feasible schemes to obtain the \( k_z \)-dependent Chern number from the shift of hybrid Wannier center and from the spin polarization in momentum space, respectively.

#### A. Detection of the Weyl points

Here we propose to use the atomic Bloch-Zener oscillation in the optical lattice to detect the double- and single-Weyl points in this system. One can prepare non-interacting fermionic atoms in the lower band initially and apply a constant force \( F \) along \( \eta \) axis, which push the atoms moving along \( k_{\eta} \) direction in momentum space and gives rise to Bloch oscillations. Then one can obtain the momentum distribution of the transfer fraction in the upper band from time-of-flight measurement after performing a Bloch oscillation. For the system with the double-Weyl points \( W_{\pm} = (0,0,\pm k_{\eta}^z) \), the transfer frac-
FIG. 6: (Color online) (a) The distribution $\xi_x(k_y, k_z)$. Two maximum transfer positions in $k_x$-$k_z$ plane correspond to the positions of the double-Weyl points. (b) The distribution $\xi_y(k_x, k_z)$. The maximum dip inside the ring profile indicates $k_x = k_y = 0$ for the points. In (a) and (b), $\delta = 0$ and $m_z = 2$. (c) $\xi_x(k_z)$ for different parameter $m_z$ with $\delta = 0$ and fixed $k_y = 0$. The maximum transfer positions of $\xi_x(k_z)$ correspond to the expected $k_z$ positions of the paired double-Weyl points, which are denoted by the white dashed line. (d) The distribution $\xi_x(k_x, k_y)$ with two rings. (e) The distribution $\xi_y(k_x, k_z)$ with four maximum peaks. (f) The distribution $\xi'_y(k_x, k_z)$ with two rings. The patterns in (d-f) with $\delta = 0.5$ and $m_z = 2$ reveal the positions of four single-Weyl points ($\pm \pi/3, 0, \pm \pi/3$). The other parameter is $F = 0.2$ in (a-f).

The method is applicable for detecting the single-Weyl points created by the symmetry breaking when $\delta \neq 0$ in this system. We consider the typical case of $\delta = 0.5$ and $m_z = 2$, with four single-Weyl points in $k_x$-$k_z$ plane. In this case, there are two subsequent Landau-Zener transitions along $k_x$ direction and thus the transfer fraction $\xi_x$ in Eq. (14) becomes

$$\xi_x(k_y, k_z) = 2P_{LZ}^{x}(k_y, k_z)[1 - P_{LZ}^{y}(k_y, k_z)],$$

while $\xi_x$ and $\xi_y$ remain the same expressions. The numerical results of $\xi_x$, $\xi_y$ and $\xi'_y$ are respectively shown in Figs. (d,e,f). One can find that both the distributions $\xi_x(k_x, k_y)$ and $\xi'_y(k_x, k_z)$ have two rings and the positions inside each ring with $\xi_x = \xi'_y = 0$ indicate four band crossing points located at $(\pm \pi/3, 0, \pm \pi/3)$ as expected for the single-Weyl points in this case. The four peaks of transfer fraction $\xi_y(k_x, k_z)$ shown in Fig. (e) also reveal the exact positions of the four gapless points. We note that the Bloch-Zener method can not tell the trivial (accidental) gapless points and the non-trivial Weyl points in the bands. However, the double- and single-Weyl points in our model system can be distinguished from the different patterns of $\xi'_y$, as shown in Fig. (f).

To detect the topology of the gapless points, one may further perform the interference between two atomic gases traveling across the points in momentum space revealed by the Bloch-Zener method to extract the Berry phases and thus the Chern numbers. Below we present two different approaches to measure the band topology in our model system.
FIG. 7: (Color online) The hybrid Wannier center in a tight-binding chain of length $L_y = 60$ under the open boundary condition at half filling as a function of the adiabatic pumping parameter $k_z$ for varying $k_x$. (a) The profile of the hybrid Wannier center $\langle n_y(k_x) \rangle$ without trapping potential shows two jumps of one-unit-cell for $k_x$ (outside) within the region $(-0.5\pi, 0.5\pi)$, with two typical examples $k_z = 0$ and $k_z = 0.6\pi$ shown in (b). (c) The profile $\langle n_y(k_x) \rangle$ under a weak harmonic trap with $V_t = 3 \times 10^{-4}$. The parameters are $\delta = 0$ and $m_z = 2$ in (a-c). (d) The profile without trapping potential shows two jumps of $\langle n_y(k_x) \rangle$ for all $k_z$, with two typical examples shown in (e). (f) The profile under a harmonic trap with $V_t = 6 \times 10^{-4}$. The parameters are $\delta = 0$ and $m_z = 0.5$ in (d-f). (g) The profile without trapping potential exhibits two discontinuous jumps of one-unit-cell when $k_z \in (-0.46\pi, -0.20\pi)$ or $k_z \in (0.20\pi, 0.46\pi)$, with two typical examples shown in (h). (i) The profile under a harmonic trap with $V_t = 2 \times 10^{-4}$. The parameters are $\delta = 1.7$ and $m_z = 0.5$ in (g-i). (j) The profile without trapping potential shows no jump of $\langle n_y(k_x) \rangle$ for all $k_z$, with two typical examples shown in (k). (l) The profile under a harmonic trap with $V_t = 3 \times 10^{-4}$. The parameters are $m_z = 3.2$ and $\delta = 0$ in (j-l). The $k_z$-dependent Chern numbers $C_{k_z}$ in (a-l) are also plotted.

B. Detection of the Chern number from the shift of hybrid Wannier center

We now proceed to propose a realistic scheme to directly measure the Chern number of the double-Weyl semimetals and other topological states in optical lattices, based on the particle pumping approach and hybrid Wannier functions in the band theory \cite{38, 40, 60, 62}. With the dimension reduction method, the three-dimensional system can be treated as a collection of $k_z$-modified two-dimensional Chern insulators with the $k_z$-dependent Chern number defined in $k_x$-$k_y$ plane as different slices of out-of-plane quasimomentum $k_z$. Such a two-dimensional insulating subsystem can be further viewed as a fictitious one-dimensional insulator subjected to an external parameter $k_z$. Thus, its Chern number can be defined by the polarization $P(k_x, k_z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(k) dk_y$ for the geometry of the underlying band structure. According to the modern theory of polarization \cite{61, 62}, the Chern number defined in $k_x$-$k_y$ space can be obtained from the change in polarization induced by adiabatically changing the parameter $k_x$ by $2\pi$: $C_{k_z} = \int_{-\pi}^{\pi} \frac{\partial P(k_x, k_z)}{\partial k_x} dk_x$. For measuring $P(k_x, k_z)$, one can use another fact that the polarization can alternatively written as the center of mass of the Wannier function constructed for the single occupied band.

In this system, the polarization $P(k_x, k_z)$ can be expressed by means of the centers of the hybrid Wannier functions, which are localized in the $y$ axis retaining Bloch character in the $k_x$ and $k_z$ dimensions. The variation of the polarization and thus the Chern number are directly related to the shift of the hybrid Wannier center along the $y$ axis in the lattice. The shift of hybrid Wannier center by adiabatically changing $k_z$ is proportional...
to the Chern number, which is a manifestation of topological pumping with $k_z$ being the adiabatic pumping parameter. In this system, the hybrid Wannier center of a one-dimensional insulating chain along $y$ axis described by the Hamiltonian $\hat{H} = \hat{H} + H_P$ can be written as

$$\langle n_y(k_x, k_z) \rangle = \frac{\sum_{i_y} i_y \rho_{i_y}(k_x, k_z)}{\sum_{i_y} \rho_{i_y}(k_x, k_z)},$$

(16)

where $\rho_{i_y}(k_x, k_z)$ denotes the density distribution of the hybrid Wannier function as a function of the parameter $k_x$ and $k_z$ with $i_y$ being the lattice-site index in the one-dimensional chain, and takes the following form

$$\rho_{i_y}(k_x, k_z) = \sum_{\text{occ}} |k_{x_i}(k_x, k_z)|^2,$$

(17)

where $|k_{x_i}(k_x, k_z)|^2$ denotes the hybrid wave function of the system at site $i_y$ and the notation occ denotes the occupied states. In cold atom experiments, the atomic density $\rho_{i_y}(k_x, k_z)$ can be directly measured by the hybrid time-of-flight images, which is referring to an in situ measurement of the density distribution of the atomic cloud in the $y$ direction during free expansion along the $x$ and $z$ directions. In the measurement, the optical lattice is switched off along the $x$ and $z$ directions while keeping the system unchanged in the $y$ direction. One can map out the crystal momentum distribution along $k_x$ and $k_z$ in the time-of-flight images and a real space density resolution in the $y$ direction can be done at the same time. Thus one can directly extract the Chern number from this hybrid time-of-flight images in the cold atom system.

To demonstrate the feasibility of the proposed method, we numerically calculate $\langle n_y(k_x, k_z) \rangle$ in a tight-binding chain of length $L_y = 60$ under the open boundary condition at half filling for some typical parameters, with the results shown in Fig. 7. For $\delta = 0$ and $m_z = 2$ in Figs. 7(a) and 7(b), the system is in the double-Weyl semimetal phase with $k_z^\parallel = \pm \pi/2$, and we find that as $k_z$ changing from $-\pi$ to $\pi$, $\langle n_y(k_x) \rangle$ exhibits two discontinuous jumps of one unit cell within the region $k_z (-0.5\pi, 0.5\pi)$ and the jumps disappear outside this region. To be more clearly, we also plot $\langle n_y(k_x) \rangle$ for $k_z = 0$ and $k_z = 0.6\pi$ as two examples in Fig. 7(b). The double one-unit-cell jumps driven by $k_z$ indicates that two particles are pumped across the system, as expected for $C_{k_z} = 2$. For $\delta = 0$ and $m_z = 0.5$ in Figs. 7(d) and 7(e), we find that two discontinuous jumps of $\langle n_y(k_x) \rangle$ for all $k_z$ as $k_x$ changing from $-\pi$ to $\pi$, indicating that the system is in the topological insulating phase with $C_{k_z} = 2$.

For $m_z = 0.5$ and $\delta = 1.7$ in Figs. 7(g) and 7(h), the system is in the Weyl semimetal phase with eight single-Weyl points and we find that when $k_x \epsilon(-0.46\pi, -0.20\pi)$ and $k_z \epsilon(0.20\pi, 0.46\pi)$, $\langle n_y(k_x) \rangle$ shows two discontinuous jumps of one-unit-cell by varying $k_z$ from $-\pi$ to $\pi$, consistent with $C_{k_z} = 2$ in these $k_z$ regions as shown in Fig. 5.

When the system is in the normal band insulating phase for $m_z = 3.2$ and $\delta = 0$, as expected, there is no jump of the hybrid Wannier center $\langle n_y(k_x) \rangle$ for all $k_z$ by changing adiabatic pumping parameter $k_x$ from $-\pi$ to $\pi$, as shown in Figs. 7(j) and 7(k). We also obtain the results of $\langle n_y(k_x) \rangle$ when the system is in the Weyl semimetal phase with four single-Weyl points, similar with those in Fig. 7(a). This establishes a direct and clear connection between the shift of the hybrid Wannier center and the topological invariant of the system in different phases.

In order to simulate the realistic experiment, we add a weak harmonic trap to this finite-site lattice with the open boundary. The trapping potential in the chain can be effectively described as $\hat{H}_t = V_t \sum_i (y \frac{-4\pi t}{\sqrt{2}})^2 \hat{a}_i^\dagger \hat{a}_i$, where $V_t$ is the trap strength. Within a local density approximation, as long as the lower band is still filled at the center of the trap, the shifts of the hybrid Wannier center can be expected to be nearly the same as those without the trap potential. If the band gap $E_g < V_t (y - \frac{L_x}{2})^2$, the lower band is only partially filled near the two edges and then this pumping argument is no longer well applicable. In practical experiments, one can turn the trap strength to $V_t \sim 4E_g/L^2$ or emphasize the shift of hybrid Wannier center in the central region. With numerical simulations, we demonstrate that the results of $\langle n_y(k_x, k_z) \rangle$ preserve with a deviation less than 2% except the regions near the band crossing points for $V_t = 3 \times 10^{-4}$ in Fig. 7(c) and 7(l), $V_t = 6 \times 10^{-4}$ in Fig. 7(f), and $V_t = 2 \times 10^{-4}$ in Fig. 7(i). They are consistent with the estimates in the local-density analysis.

C. Detection of the band topology from the spin polarization in momentum space

Below we propose an alternative method to probe the band topology of the $C_4$-symmetric Bloch Hamiltonian with $\delta = 0$ from the spin polarization in momentum space, which can be implemented with bosonic atoms in the optical lattice. When the system has $C_4$ symmetry in the $xy$ plane, we can treat $k_z$ as an effective parameter and reduce it to a collection of effective two-dimensional subsystems, whose Chern number $C_{k_z}$ for a fixed $k_z$ can be determined by the following equation

$$e^{i \frac{\pi}{2} C_{k_z}} = \prod_{\text{occ}} \gamma_n(0, 0, k_z) \gamma_n(\pi, \pi, k_z) \chi_n(0, \pi, k_z).$$

(18)

Here $\gamma_n$ and $\chi_n$ are the $C_4$ and $C_2$ eigenvalues on the $n$-th Bloch band at high-symmetry momentum points in $k_x-k_y$ plane, respectively.

For our two bands system, $C_4 = e^{-i \frac{\pi}{2} \sigma_z} = -i \sigma_z$ and $C_2 = C_{1/2}^2 = -1$, such that the term $\chi_n(0, \pi, k_z) = -1$ in Eq. (18) can be dropped from the expression. The Chern number $C_{k_z}$ of the lower band for different $k_z$ can thus be determined by the simple relation

$$e^{i \frac{\pi}{2} C_{k_z}} = S_-(0, 0, k_z) S_-(\pi, \pi, k_z).$$

(19)

where $S_-(0, 0, k_z)$ and $S_-(\pi, \pi, k_z)$ are the eigenvalues of the $\sigma_z$ operator on the lower band. Thus to obtain $C_{k_z}$ for a given $k_z$, one only needs to measure the eigenvalues of...
\(\sigma_z\) in the two high symmetry points in \(k_x\)-\(k_y\) plane \(\Lambda_i = \{\Gamma = (0,0,k_z), M = (\pi, \pi, k_z)\}\). This can simplify the experimental detection of the topological invariant of the Bloch bands. The high symmetry points \(\Lambda_i\) satisfy that \(PA_i = A_i\). Thus, the constraints at these points give \(f(A_i) = -f(A_i)\) and \(f^*(A_i) = -f^*(A_i)\), which imply that \(f(k)\) and \(f^*(k)\) vanish. So at the high symmetry points \(A_i\), the Bloch Hamiltonian can be written as

\[\mathcal{H}(A_i) = d_z(A_i)\sigma_z,\]

where the energy of the two bands \(E_{\pm}(A_i) = \pm |d_z(A_i)|\). Since the Bloch Hamiltonian commutes with the symmetry operator, i.e., \([C_4, \mathcal{H}(A_i)] = 0\), the Bloch states of the two bands \(|u_{\pm}(A_i)\rangle\) are also the eigenstates of \(C_4\). Therefore, one can obtain the Chern number of the lower band for different \(k_z\) by measuring the spin polarization \(\langle \sigma_z \rangle\) near the high symmetry points in momentum space, which can be written as

\[\langle \sigma_z(A_i) \rangle = \frac{n_\uparrow(A_i) - n_\downarrow(A_i)}{n_\uparrow(A_i) + n_\downarrow(A_i)}.\]

Here \(n_{\uparrow,\downarrow}(A_i)\) denotes the atomic density of spin states \(|\uparrow, \downarrow\rangle\) at the high symmetry points in \(k_x\)-\(k_y\) plane for a fixed \(k_z\). Since this detection protocol only requires measurement of the atomic density distribution in momentum space, it can be applied to bosonic atoms, typically Bose-Einstein condensates, in the optical lattice system with the topological bands.

In the experiment with a condensate in the optical lattice, the spin polarization at the two high symmetry momenta can be written as

\[\langle \sigma_z(A_i) \rangle \approx S_-(A_i)f(E_-,T) + S_+(A_i)f(E_+,T),\]

where \(f(E_{\pm},T) = 1/[e^{(E_{\pm}(A_i) - \mu)/k_BT} - 1]\) is the Bose-Einstein statistics with \(\mu\) and \(T\) respectively being the chemical potential and temperature, and \(S_{\pm}(A_i)\) are the eigenvalues of \(\sigma_z\) on the lower and upper bands at \(A_i\). Since \(S_+(A_i) = -S_-(A_i)\), one has \(\langle \sigma_z(A_i) \rangle \approx S_-(A_i)[f(E_-,T) - f(E_+,T)].\) Thus by preparing a cloud of bosonic atoms with the temperature satisfying \(f(E_-(A_i),T) > f(E_+(A_i),T)\), one can obtain

\[\text{sgn}[\langle \sigma_z(A_i) \rangle] = \text{sgn}[S_-(A_i)].\]

Therefore, the spin polarization \(\langle \sigma_z(A_i) \rangle\) can be precisely measured with a condensate at low temperature.

In practical experiments, one can prepare the atoms in the spin-up state and adiabatically load the condensate into the \(A_i\) points. Then one can perform the spin-resolved time-of-flight expansion, which projects the Bloch states onto free momentum states according to the plane-wave expansion with a complete basis of plane waves \(|\psi_{m,n}(A_i), \psi_{p,l}(A_i)\rangle\}. The Bloch state of lowest band can be expressed as \(|u_-(A_i)\rangle = \sum_{m,n} a_{m,n} \psi_{m,n}^\dagger(A_i)|\uparrow\rangle + \sum_{p,l} b_{p,l} \psi_{p,l}^\dagger(A_i)|\downarrow\rangle\), where \(a_{m,n}\) and \(b_{p,l}\) are coefficients. The spin polarization for the Bloch eigenstates of the lower band at high symmetry points is given by \(\langle \sigma_z(A_i) \rangle = \langle u_-(A_i) | \sigma_z | u_-(A_i) \rangle = \sum_{m,n} | a_{m,n} \psi_{m,n}^\dagger(A_i) |^2 - \sum_{p,l} | b_{p,l} \psi_{p,l}^\dagger(A_i) |^2\), which gives rise to the expression in Eq. (21).

Finally one can obtain \(n_{\uparrow,\downarrow}(A_i)\) by the time-of-flight measurement, and thus obtain the \(k_z\)-dependent Chern number of the Bloch bands from Eq. (19) with \(S_-(A)S_-(M) = \text{sgn}[\langle \sigma_z(A) \rangle] \text{sgn}[\langle \sigma_z(M) \rangle]\) in this case.

Finally, the fact that the topology of the \(C_4\)-symmetric bands can be determined by only the Bloch states at the symmetric momenta can greatly simplify the experimental detection of the topological bands. Similar protocol has been implemented to detect the topology of the inversion-symmetric bands with Bose-Einstein condensates in two-dimensional optical lattices [31]. In our three-dimensional system, one can extract \(C_{k_z}\) from the proposed measurements for various \(k_z\) and \(m_{\uparrow,\downarrow}\) corresponding to the line of \(\delta = 0\) in the phase diagram. If all two-dimensional slices have \(C_{k_z} = 0\), the system is in the trivial insulator phase, while it is in the topological insulator phase if \(C_{k_z} = 2\) for all \(k_z\). The change of \(C_{k_z}\) by two along \(k_z\) axis indicates the presence of double-Weyl points and the system is in the double-Weyl semimetal phase.

V. CONCLUSIONS

In summary, we have proposed an optical lattice system for simulation and exploration of double-Weyl semimetals. We have investigated the topological properties of the double-Weyl semimetal phase and obtained a rich phase diagram with several other quantum phases, which include a topological insulator phase and two single-Weyl semimetal phases. Furthermore, with numerical simulations, we have proposed practical methods for the experimental detection of the mimicked Weyl points and the characteristic topological invariants with cold atoms in the lattice system. The proposed system would provide a promising platform for elaborating the intrinsic exotic physics of double-Weyl semimetals and the related topological phase transitions that are elusive in nature.

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