It is shown that the form factors in the semileptonic decay of ground state heavy mesons to light pseudoscalar mesons are dominated by the vector meson pole at all momentum transfers. First, the general approaches to modeling form factors in terms of quark model, pole dominance, or hybrid shapes are reviewed. It is then shown that in the combined limits of heavy quarks, chiral symmetry, and large $N_c$ the form factor is precisely pole-dominated. It is also shown that in a two-dimensional QCD model pole-dominance occurs for any heavy quark mass. Corrections to this limit are discussed and, to illustrate the nature of the approximations, an explicit exact calculation of the approach to the limit in a two-dimensional model is given.
1. Introduction

Weak decays of heavy hadrons directly probe flavor changing interactions. In the standard model of electroweak interactions these decays provide a means of extracting fundamental parameters of the theory: four mixing angles, $|V_{cd}|$, $|V_{cs}|$, $|V_{ub}|$, and $|V_{cb}|$, which determine the decay rates of the heavy hadrons. In practice, however, our inability to compute these rates from first principles limits efforts to extract these fundamental angles. Even when limiting our attention to semileptonic decays, the simpler hadronic matrix elements prove too difficult to compute.

In processes involving the semileptonic decay of a heavy quark to another heavy quark, as in $b \to c \ell \nu$, a systematic expansion in powers of the hadronic scale over the heavy masses allows computation of the hadronic matrix elements involved at one kinematic end-point, namely, the point at which the resulting hadronic system is at rest in the rest frame of the original hadron. This “heavy quark effective theory” (HQET) is therefore of great relevance to the determination of $|V_{cb}|$.

But for decays of a heavy quark into a light one, there is unfortunately little that the HQET has to say. While one can find relations between different measurable decays, it does not seem possible to calculate the matrix elements themselves.

In view of the lack of first-principles calculations of heavy-to-light decay rates, phenomenological estimates depend on hadronic models used. It is no surprise that the results vary considerably.

Present hadronic model calculations of heavy-to-light decay rates attempt first to estimate the hadronic matrix element at one kinematic point and then extrapolate based on some assumed functional shape of the form factor, chosen in a somewhat ad-hoc manner. This ‘shape’ is one of three types:

(i) *Pole-Dominated* shapes assume the form, although not necessarily the residue, of a form factor dominated by the lowest-lying resonance that couples to the weak current, *e.g.*, vector-meson dominance of the $B^*$ in $b \to u$ transitions[1–8].

(ii) *Quark-Model* shapes are suggested by an extrapolation of the non-relativistic constituent quark model from the low recoil region to the relativistic region. The extrapolation is not unique since the non-relativistic computation does not automatically yield a Lorentz-invariant form factor[9–11]. Relativistic wavefunctions have also been used to compute the shape[12–16].
(iii) Two-Component models assume a linear combination of the two preceding shapes\cite{17–20}.

From the practical point of view this situation is quite inadequate; from a theoretical one it is far worse: it leaves us with no insight into the dominant structures and dynamics of these hadrons. More optimistically, detailed knowledge of the correct $q^2$ dependence promises both to help in the experimental determination of fundamental parameters and to bridge the variety of theoretical ideas underlying the present matrix element calculations.

In this paper we argue that there is compelling evidence in favor of the first, pole-dominated, structure for this form factor. We show that the $q^2$ dependence is of the single-pole form in the combined heavy quark, large $N_c$ and chiral limit, where the hadronic matrix element is exactly computed, as we have discussed in a recent letter\cite{21}. We go on to study decays of heavy mesons in two-dimensional planar QCD, where we find striking confirmation of both the general results and the expected approach to the limiting case. While the 't Hooft model is in no sense a controlled approximation to the four-dimensional world, it offers valuable insights. Most importantly it provides a testing grounds for new models since, as we explain, many considerations motivating the modeling of form factors are independent of those details (dimension and number of colors) which make the model solvable.

We have also discovered, in the course of this work, a number of useful relations that permit calculation of some matrix elements in this model for the first time. Thus we expect these results to be interesting as well to those interested in two-dimensional field theory for its own sake.

The paper is organized as follows. In section 2 we discuss physical arguments that suggest the validity of the form factor shapes described above. We pay particular attention to arguments due to Isgur and Wise that use simple power counting to argue in favor of a two-component model\cite{17,18}. In section 3 we prove pole dominance at large $N_c$. This is done first for the simpler two-dimensional case for which spin is absent and the heavy quark limit need not be taken. The four-dimensional generalization is then presented. In section 4 we fleetingly review the 't Hooft model in order to introduce notations and conventions. In section 5 we calculate weak decay form factors and discuss the results in section 6. The principal field-theoretic results of the paper are contained in section 3 and section 5, and the impatient reader may wish to jump there directly.
2. Models of Form Factors

Several possible shapes for the form factors of heavy-to-light decays have been proposed in the literature. Always the idea is to extrapolate (or guess) a solid calculation of a hadronic matrix elements at a single kinematic point to a form factor at other momentum transfers.

*Pole-Dominated* form factors are assumed to have the functional form

\[
f_{\pm}(q^2) = \frac{C_{\pm}}{1 - q^2/\mu^{*2}}. \tag{2.1}
\]

where \( \mu^{*} \) is the mass of the lightest state which couples to the weak current \( V_{\mu} \). The strength of the form factor \( C_{\pm} \) is obtained from by estimating the matrix element at one kinematic point \( q^2 = Q^2 \):

\[
C_{\pm} \equiv f_{\pm}(Q^2) \left( 1 - Q^2/\mu^{*2} \right). \tag{2.2}
\]

In \( B \to \pi \) transitions this pole belongs to the \( B^* \) vector meson. (In two dimensions where spin is absent the corresponding state is the \( B \)-meson itself which couples directly to the vector current).

Pole-dominated form factors also arise in the chiral Lagrangian approach to heavy quark interactions\,[22–26]; however this analysis is necessarily limited to small momentum transfers and applies not to a large range of \( q^2 \) but only to the small-recoil regime in the vicinity of the \( B^* \) pole.

*Quark-Model* forms are suggested by extrapolating a non-relativistic functional shape to the relativistic region. This shape is computed in the low-recoil region using the constituent quark model. The extrapolation is not unique since the non-relativistic computation does not automatically yield a Lorentz-invariant form factor. The computation requires an overlap integral of the non-relativistic wave-functions for the ground state mesons. For example, in the rest frame of the \( B \)-meson

\[
(\mu_B + \mu_\pi) \tilde{f}_+ + (\mu_B - \mu_\pi) \tilde{f}_- = \sqrt{4\mu_B\mu_\pi} \int d\vec{k} \phi_{\pi}^*(\vec{\tilde{p}} + \vec{k}) \phi_B(\vec{k}) \\
(\tilde{f}_+ - \tilde{f}_-)\vec{p} = \sqrt{4\mu_B\mu_\pi} \int d\vec{k} \phi_{\pi}^*(\vec{\tilde{p}} + \vec{k}) \left( \frac{\vec{k}}{2m_b} + \frac{\vec{k} + \vec{\tilde{p}}}{2m_q} \right) \phi_B(\vec{k}), \tag{2.3}
\]

where \( \vec{p} \) is the spatial momentum of the \( \pi \)-meson, \( \vec{\tilde{p}} \equiv (m_q/\mu_\pi)\vec{p} \), \( \phi_X \) is the non-relativistic wave-function of the \( X \)-meson, and the tilde marks form factors which are not Lorentz-invariant. By rotational invariance, \( \tilde{f}_\pm \) are functions of \( |\vec{p}|^2 \) only. One may construct
Lorentz-invariant form factors by writing the momentum dependence in terms of a reasonably chosen—albeit ad-hoc—replacement $|\vec{p}|^2 \to g(q^2)$, such that $g(q^2) \to |\vec{p}|^2$ as $q^2 \to q^2_{\text{max}} \equiv (\mu_B - \mu_\pi)^2$. For example, since in the $B$-rest-frame

$$q^2 = \mu_B^2 + \mu_\pi^2 - 2\mu_B E_\pi,$$

where $E_\pi = \sqrt{|\vec{p}|^2 + \mu_\pi^2}$, one may take

$$g(q^2) \equiv \left(\frac{q^2 - \mu_B^2 - \mu_\pi^2}{2\mu_B}\right)^2 - \mu_\pi^2$$

or, alternatively

$$g(q^2) \equiv (q^2 - (\mu_B + \mu_\pi)^2)^2 \mu_\pi \mu_B$$

It is natural to suspect that these two approaches might be quite limited and that at best each might be applicable to a particular kinematic regime.

Indeed, a dispersion relation may be used to write the form factor as a sum over contributions of resonances

$$f_\pm(q^2) = \frac{f_{B^*} g_{\pi B^n} B^*}{q^2 - \mu_{B^*}^2} + \hat{f}_\pm(q^2),$$

where $\hat{f}_\pm$ is the continuum contribution,* or single poles in the narrow width approximation or large $N_c$ limit:

$$\hat{f}(q^2) = \sum_{n>0} \frac{f_{B^*} g_{\pi B^n} B^n}{q^2 - \mu_n^2}.$$

The pole of course dominates near the kinematic endpoint when $|q^2 - \mu_{B^*}^2|$ is much smaller than the spacing between the $B^*$ and the next resonance that couples to the current $V_\mu$.

By the same token, the lowest term is not expected to dominate in the opposite situation, when $|q^2 - \mu_{B^*}^2|$ is greater than the typical spacing between resonances. In this case many resonances will in general contribute substantially to the form factor. This is a signal that one is not expanding in a useful set of degrees of freedom, and a quark model description may be more adequate in this case.

* The precise nature of the continuum contribution $\hat{f}$ is not important, as we need know only its structure in the complex $q^2$ plane, and the strength of the associated singularity, but not the precise nature of the singularity.
Two-component models attempt to capture both these behaviors particularly when one is interested in a large kinematic range. The character of the actual transition is naturally of great interest.

While suggestive, these simple arguments prove nothing. Pole-dominance fails, for example, if the lowest residue is anomalously small, so that the “nearest singularity” dominates only for values of $|q^2 - \mu_X^2|$ so tiny that they are not in the physical region. It could happen that the higher states, often neglected, actually have large residues which oscillate rapidly. Form factors of this type in fact occur in QCD; they are crucial to understanding transition between effective field theories and quark model physics for $q\bar{q}$ states of a single flavor\[27\]. On the other hand the residue of the lowest lying pole could be so much larger than all of the rest that dominance by that pole alone could be guaranteed for all values of $q^2 < \mu_X^2$.

Isgur and Wise argued for two-component model in heavy-to-light decays\[17\]. This has also been discussed in detail by Burdman and Donoghue\[18\]. They argue that in the combined chiral and heavy-quark limits the form factor is pole-dominated around $q_{\text{max}}^2$ — and only there. Now it is obvious that there is a $B^*$ pole and that this pole plays a special role since the physical value of $|q_{\text{max}}^2 - \mu_{B^*}^2|$ is so small.

Consider the combined limit

$$\mu_\pi \to 0, \quad M_b \to \infty, \quad \bar{\Lambda}^2 \equiv \mu_\pi \mu_B \text{ fixed (2.9)}$$

(the analysis of Ref. \[17\],\[18\], corresponds to $\bar{\Lambda} = 0$). Standard power counting gives $f_{B^*} \propto M_b^{1/2}$ and $g_{\pi BB^*} \propto M_b$. On the other hand, $\mu_{B^*}^2 - \mu_{B}^2 \sim (M_b)^0$, while for higher resonances ($n \geq 1$) one has $\mu_n^2 - \mu_B^2 \propto M_b$. At $q_{\text{max}}^2 = (\mu_B - \mu_\pi)^2$ the denominator in (2.7) scales as $M_b^0$, while the denominators of $\tilde{f}_\pm$ (cf. (2.8)) scale as $M_b$. In the combined limit, the form factors at $q_{\text{max}}^2$ are pole dominated. In fact the single pole dominates over a region that scales like $q_{\text{max}}^2 - q^2 \propto M_b$. This is a small fraction of the physical region, which scales like $M_b^2$. Thus, Isgur and Wise conclude that a two-component form factor is appropriate: the pole term dominates around $q_{\text{max}}^2$ with residue given by $f_{B^*}g_{\pi BB^*}$, and the quark-model term is dominant at smaller $q^2$.

Though suggestive, this scenario rests on crucial assumptions. The key question is over what range the single pole dominates, and this depends on dynamical calculations. We will shortly examine these assumptions critically in the case of a fully calculable toy model which satisfies the same assumptions, yet displays form factors that are truly pole-dominated over a much larger kinematic range. It is tempting to conjecture that this holds in four dimensions. Before describing our exact two-dimensional results, we explain a limit in which the behavior holds exactly in four as well as two dimensions.
3. Pole dominance at large $N_c$

The matrix element can be expressed in terms of two form factors $f_{\pm}$:

$$\langle \pi(p')|V_\nu|\bar{B}(p)\rangle = (p + p')_\nu f_+(q^2) + (p - p')_\nu f_-(q^2). \quad (3.1)$$

where $V_\nu = \bar{q}\gamma_\nu Q$, $q = p - p'$ and throughout the paper $p$ and $p'$ always denote the momentum of the $B$ and $\pi$ mesons, respectively. The mesons $B$ and $\pi$ have quantum numbers that correspond to the valence quarks $Q\bar{q}$ and $q\bar{q}$, respectively.

Let us derive the pole-dominated shape of $\bar{B} \to \pi$ decay in a suitable limit, and then discuss how the approximations might be relaxed.

Consider QCD in the limit of large $N_c$, with one heavy quark $Q$ of mass $M$ and a light quark $q$ of mass $m$. The form factors in (3.1) are saturated by couplings of the flavor changing current $V_\nu$ to the $Q\bar{q}$ resonances in that channel.

To momentarily suppress the complication of spin, we pass to two dimensions where we can write

$$\langle \pi|V_\nu|B\rangle = \sum_n \frac{\langle 0|V_\nu|B_n\rangle \langle \pi B_n|B\rangle}{q^2 - \mu^2_n}. \quad (3.2)$$

For odd parity states $|B_n\rangle$,

$$\langle 0|V_\nu|B_n\rangle = \epsilon_{\nu\lambda} q^\lambda f_n. \quad (3.3)$$

We can describe these interactions conveniently in terms an effective hadron Lagrangian,

$$\mathcal{L} = \sum_n \left( \frac{1}{2} (\partial_\lambda \varphi_n)^2 - \frac{\mu_n^2}{2} \varphi_n^2 \right) + \mathcal{L}_{\text{int}}, \quad (3.4)$$

where a field $\varphi_n(x)$ is introduced for each meson state and $\mathcal{L}_{\text{int}}$ couples the mesons via terms

$$\mathcal{L}_{\text{int}} = \sum_{abc} \hat{g}_{abc}(q^2) \epsilon_{\lambda\nu} \partial_\lambda \varphi^a \partial_\nu \varphi^b \varphi^c. \quad (3.5)$$

Similarly, for even parity,

$$\langle 0|V_\nu|B_n\rangle = q_\nu f_n, \quad (3.6)$$

with couplings

$$\mathcal{L}_{\text{int}} = \sum_{abc} \hat{g}_{abc}(q^2) \varphi^a \varphi^b \varphi^c. \quad (3.7)$$
The form factors of Eq. (3.1) can then be written

\[ f_+(q^2) = \sum_{\text{even parity}} f_n \hat{g}_{\pi Bn}(q^2) \frac{q^2 - \mu_n^2}{q^2} \]

\[ f_-(q^2) = \sum_{\text{odd parity}} f_n \hat{g}_{\pi Bn}(q^2) - \sum_{\text{even parity}} f_n \hat{g}_{\pi Bn}(q^2) \frac{(\mu_B^2 - \mu_{\pi}^2)}{q^2 - \mu_n^2} , \]

which is obtained with the help of the useful two-dimensional formula

\[ \epsilon_{\lambda \nu} q^\nu = \left[-q^2 (p + p')_\lambda + (\mu_B^2 - \mu_{\pi}^2)(p - p')_\nu \right] / 2 \epsilon^{\rho \sigma} p_\rho p'_\sigma . \] (3.9)

Note that the expansions (3.8) have momentum dependent numerators, proportional to the off-shell three point couplings, \( \hat{g}_{\pi Bn}(q^2) \). Using analyticity and assuming suitable convergence, as will be justified later, a contour integral gives

\[ f_+(q^2) = \sum_{\text{even parity}} f_n \hat{g}_{\pi Bn}(\mu_n^2) \frac{\mu_n^2}{q^2} \]

\[ f_-(q^2) = \sum_{\text{odd parity}} f_n \hat{g}_{\pi Bn}(\mu_n^2) \frac{\mu_n^2}{q^2} - \sum_{\text{even parity}} f_n \hat{g}_{\pi Bn}(\mu_n^2) \frac{(\mu_B^2 - \mu_{\pi}^2)}{q^2} , \] (3.10)

in which the numerators are the on-shell, physical couplings.

We make several observations. It is large \( N_c \) which allows us to treat the resonances as stable without continuum couplings in Eq. (3.2). It selects the three point couplings to one-particle intermediate states. In using convergence as \( |q^2| \to \infty \), we make an assumption about the large-momentum behavior of the interactions, information unavailable from a low-energy analysis or standard chiral Lagrangian analysis. The shift of the numerators to the residues at the poles, familiar in dispersion theory, has a simple physical origin: in the “effective” meson field theory there is freedom to make arbitrary field redefinitions without changing the on-shell \( S \)-matrix. Here that freedom is used to replace the momentum dependent couplings (that is, the higher derivative operators) by constants at the expense of shifting the coefficients of higher point functions, which in turn are down by powers of \( 1/N \). We make such a shift. Note the contrast with the use of field redefinitions by Georgi in Ref. [28], in which higher point functions were suppressed instead by powers of the cutoff to yield a related kind of “on-shell effective theory.”

Now in the chiral limit — the light quark mass \( m \to 0 \) and \( \mu_{\pi}^2 \to 0 \) — the decay constants \( f_n \) remain finite while the on-shell three-point elements \( \langle \pi B_n \rangle B \) vanish. This is
a direct consequence of chiral symmetry and leads immediately to a pole-dominated form factor.

To show this in detail, fix the state $B_n$ and consider the matrix element of the light-light current $a_\lambda = \bar{q}\gamma_\lambda\gamma_5q$:

$$\langle B_n(q)|a_\lambda|B(p)\rangle = p'_\lambda F_n(p'^2) = \sum_\ell \frac{(0|a_\lambda|\pi_\ell)\langle\pi_\ell B_n|B\rangle}{p'^2 - \mu_\ell^2}$$

Here the sum is over the tower of $\bar{q}q$ states, $|\pi_\ell\rangle$. The form factors $F_n(p'^2)$ can be written in terms of the off-shell couplings $\hat{g}_{\ell B_n}(p'^2)$ and decay constants $f_\ell$ of the effective Lagrangian, Eqs. (3.5) and (3.7). For parity odd $B_n$-states

$$F_n(p'^2) = \sum_\ell \frac{f_\ell 2\epsilon^{\mu
u\rho}p_\mu p'_\rho \hat{g}_{\ell B_n}(p'^2)}{p'^2 - \mu_\ell^2} \tag{3.12}$$

and for parity even states

$$F_n(p'^2) = \sum_\ell \frac{f_\ell \hat{g}_{\ell B_n}(p'^2)}{p'^2 - \mu_\ell^2} \tag{3.13}$$

Again, given suitable convergence, one may apply Cauchy’s theorem to these form factors and replace the numerators by the corresponding on-shell expressions, i.e., $p'^2 \rightarrow \mu_\ell^2$.

Next, axial current conservation implies $f_\ell = 0$ unless $\mu_\ell = 0$, so the axial current couples only to the massless pion, $\pi = \pi_0$, and only the $\ell = 0$ term persists in Eqs. (3.12) and (3.13) in the chiral limit.

Again applying axial current conservation, $\partial \cdot a = 0$, we have

$$0 = p'^2 F_n(p'^2) \rightarrow \begin{cases} f_\pi \hat{g}_{\pi B_n} & \text{if } n \text{ is parity even;} \\ f_\pi \hat{g}_{\pi B_n}(2\epsilon^{\lambda\sigma}p_\lambda p'_\sigma) & \text{if } n \text{ is parity odd,} \end{cases} \tag{3.14}$$

for all three states on-shell. It immediately follows that

$$\hat{g}_{\pi B_n} = 0, \quad n \neq 0. \tag{3.15}$$

so all the coupling constants in the effective Lagrangian (3.7), (3.7) except to the ground state vanish on-shell. The case $n = 0$ is singled out because, on-shell, the factor $2\epsilon^{\lambda\sigma}p_\lambda p'_\sigma = 0$, so it need not follow that $g_{\pi BB}$ vanishes. Indeed, it does not.
Combining these results and introducing the coupling $g_{\pi BB} \equiv \mu_B^2 \hat{g}_{\pi BB}(\mu_B^2)$ in analogy with the definition that is natural in four dimensions, the form factors are

$$f_+(q^2) = -f_-(q^2) = \frac{f_B \hat{g}_{\pi BB}(\mu_B^2)\mu_B^2}{q^2 - \mu_B^2} = \frac{f_B g_{\pi BB}}{q^2 - \mu_B^2}.$$  \hfill (3.16)

This readily generalizes to four dimensions, remaining still in the limit of large $N_c$. The detailed argument for the limiting case is given in Ref. [21]. The vector and axial-vector currents are of course no longer dual, and the lightest state to couple to the $\pi B$ is the $B^*$ vector meson, not the $B$ itself. The essential point is that $B$ and $B^*$ are mass-degenerate in the heavy quark limit so that the light-quark axial vector current which produces pions can rotate the states into each other.

As above, look in the light-light channel for a fixed state $B_n$ and take the divergence in a frame where $\vec{p}' = 0$:

$$0 = p'^\lambda \langle B_n|a_\lambda|B\rangle = \sum_\ell p'^0 \frac{\langle 0|a_0|\pi\ell\rangle\langle \pi\ell B_n|B\rangle}{p'^2 - \mu^2_\ell} = \frac{\langle \pi B_n|B\rangle f_\pi p'^2}{p'^2 - \mu^2_\pi}$$ \hfill (3.17)

so that chiral symmetry again implies that $\langle \pi B_n|B\rangle = 0$ which in turn implies the vanishing of all $g_{\pi Bn}$ except for $n = 0$. The way in which this comes about is naturally different in four dimensions, reflecting the corresponding spins, symmetries and kinematics. The states $B_n$ must be analyzed according to their spin. Only for spin one is there the possibility of a non-vanishing coupling[21].

One can write

$$\langle 0|V_\lambda|B_n\rangle = f_n \epsilon_\lambda,$$  \hfill (3.18)

where $\epsilon_\lambda$ is the polarization of the $B_n$ and then define couplings by

$$\langle \pi B_n|B\rangle = \epsilon^\lambda(p + p')_\lambda g_+^{(n)} + \epsilon^\lambda(p - p')_\lambda g_-^{(n)}. \hfill (3.19)$$

Nothing is learned about the $g_-$ since $B_n$ is on-shell. It is easy to see in the $B_n$ rest frame $\vec{p} = \vec{p}'$ that

$$g_-^{(n)} \vec{\epsilon} \cdot \vec{p} = 0 \hfill (3.20)$$

Let us restrict attention for now to the exact chiral limit in the heavy quark (infinite) mass limit. Then

$$\vec{\epsilon} \cdot \vec{p} = |\vec{p}| \cos \theta,$$

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where $\theta$ is the angle between the polarization and the momentum vectors, and is generally non-vanishing. But from the kinematics it is also true that

$$|\vec p| = \frac{\mu_n^2 - \mu_B^2}{2\mu_n} = \begin{cases} \Lambda_n^2/2\mu_{B^*} = O(1/M_Q) & \text{for } n = B^* , \\ \Lambda_n + O(1/M_Q) & \text{otherwise}, \end{cases} \tag{3.21}$$

where we introduce the mass difference $\Lambda_n \equiv \mu_n - \mu_B$ for $n \neq B^*$ states, and take the large mass limit in the last equality. Therefore, taking $M \to \infty$, the chiral limit then yields

$$0 = g_+^{(n)} |\vec p| = \begin{cases} 0 & \text{for } n = B^* , \\ g_+^{(n)} \Lambda_n & \text{otherwise}. \end{cases} \tag{3.22}$$

So the couplings to excited states vanish, or $g_+^{(n)}/g_+^{(B^*)} \to 0$.

What can we conclude from this? We see that in four as in two dimensions, the effective Lagrangian in the combined limit of $M \to \infty$, $m \to 0$, and $N_c \to \infty$ has only a single coupling so that the form factors for semileptonic $B \to \pi$ decays are necessarily pole-dominated in all kinematic regions. Moreover, in the large $N_c$ limit the behavior as the light quark mass vanishes is expected to be smooth, so if we are not strictly at the chiral limit, the corrections to the pole-dominated form factor are $O(\mu_\pi^2/\Lambda^2)$.

In order to make this point abundantly explicit, we compute the form factors and effective Lagrangian coupling to all states below in the 't Hooft model and explicitly compute all terms in Eqs. (3.2), (3.5), (3.7). Not only will this illustrate the proof given here, but it allows analysis of the corrections to the limit when $m \neq 0$, and study of the dependence on the heavy mass $M$.

Mesons may be classified as heavy or light in two dimensions much as in four. In four dimensions the QCD Lagrangian is renormalizable and masses are considered according to whether they are large or small compared to the scale $\Lambda_{QCD}$. In two dimensions, QCD is super-renormalizable but the gauge coupling, which has dimensions of mass, plays a role analogous to $\Lambda_{QCD}$ and serves to separate heavy from light.

It is equally clear that these arguments no longer apply when the final state meson is no longer the near-massless ground state. It is of great interest, therefore, to explore the decays $B \to \pi'$, $B \to \rho$. 

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4. The ’t Hooft model

This model has been extensively studied and our work relies on technology pioneered by ’t Hooft\(^29\), Callan, Coote, and Gross\(^30\), and Einhorn\(^31\). In these papers the bound state equations were derived; and it was shown that the scattering amplitudes—and the form factor in particular—can be written entirely in terms of interactions among the meson bound states, with no quarks in the spectrum or in the singularity structure of the amplitudes.

We recall the features of the model which make it solvable, and refer the reader to the original papers for details. The dynamics are defined by the Lagrangian,

\[
\mathcal{L} = \frac{1}{4} \text{tr} \, F^2 + \sum_a \bar{\psi}_a (\gamma^\mu (i\partial_\mu - g_0 A_\mu) - m_a) \psi_a, \tag{4.1}
\]

where \(A_\mu\) is an \(SU(N)\) gauge field, \(F_{\mu\nu}\) is its field strength and \(\psi_a\) is a Dirac fermion of mass \(m_a\). In the large-\(N_c\) limit, the gauge coupling is scaled with \(N_c\): \(g^2 = g_0^2 N\) is held fixed as \(N_c \to \infty\). The label \(a\) runs over two flavors of quark, with bare masses \(m\) and \(M\).

We relate this Yang-Mills Lagrangian \(\mathcal{L}\) to the effective meson theory hadronic Lagrangian \(\mathcal{L}_{\text{meson}}\) by the most pedestrian, concrete method imaginable: by computing the \(S\)-matrix elements of physical states in the model \(\mathcal{L}\) and identifying directly with the coupling functions in \(\mathcal{L}_{\text{meson}}\) which reproduce the same physics.

The main obstacle is that there are no known analytic solutions to the bound state equations, and we have found neither approximation techniques nor limiting cases that adequately serve us in the most interesting regime. We therefore turn to numerical techniques. We compute the bound state wave functions and from them evaluate the form factor as well the precise couplings of the low-energy effective field theory when the quarks are “integrated out.” The techniques we use are those developed in Ref. \[27\].

The theory is most conveniently quantized in light-cone gauge. Because there are no transverse dimensions, setting \(A_- = 0\) eliminates the gluon self-coupling. It also serves to project gamma matrices onto a single component in any Feynman graph that has just gluon vertices and \((-\) component current insertions on fermion lines. The infrared divergence in the gluon propagator, \(i/k^2_\perp\), is regulated by taking the principal value at the pole.

The leading term of the \(1/N_c\) expansion is the sum of planar graphs. The equations for the full propagator and self-energy can be solved exactly, with an extremely simple
result: the net effect of all gluons starting and ending on the same fermion line is just to
renormalize the quark mass appearing in the propagator,

\[ m^2 \to \tilde{m}^2 \equiv m^2 - g^2 / \pi, \quad (4.2) \]

so the full quark propagator is

\[ S(k) = \frac{ik_-}{k^2 - \tilde{m}^2 + i\epsilon}. \quad (4.3) \]

After making this shift, all remaining gluon interactions enter as ladder-type exchanges. Crossings would require either gluon self-couplings, which are absent, or non-planar graphs, which are higher order in \(1/N_c\). The Yang-Mills coupling constant \(g\) has dimensions of mass, and we choose units such that \(g^2 / \pi = 1\), leaving \(m^2\) as the single dimensionless number parameterizing the theory. As is well known, there is a discrete spectrum of free mesons.

\(\Phi(p,q)\) is the full meson-quark vertex, and the wavefunction \(\phi(p,q)\) is defined by

\[ \phi(p,q) = \frac{1}{i\pi} \int d^p+ \Phi(p,q)S(p)S(p-q). \quad (4.4) \]

\(\phi\) and \(\Phi\) only depend on \(p\) through the variable \(x = p_- / q_-\), so we denote \(\phi(x) \equiv \phi(p,q)\) and \(\Phi(x) \equiv \Phi(p,q)q_-\). In terms of \(\phi\), the bound state equation may be written as

\[ \mu_n^2 \phi_n(x) = \left( \frac{M^2 - 1}{x} + \frac{m^2 - 1}{1 - x} \right) \phi_n(x) - \int_0^1 \frac{dy}{(y - x)^2} \phi_n(y). \quad (4.5) \]

Here \(\phi_n(x)\) is a light-cone momentum space wavefunction of the \(n^{th}\) eigenstate, with mass \(\mu_n\), and \(x = p_- / q_-\) is the fraction of light-cone momentum carried by the heavy quark. The \(\phi_n(x)\) vanish at the boundaries, and consistency of (4.5) requires that as \(x \to 0\), \(\phi_n(x) \sim x^\beta\), with

\[ \pi \beta_M \cot \pi \beta_M = 1 - M^2, \quad (4.6) \]

and similarly as \(x \to 1\), as dictated by the boundary behavior of the Hilbert transform. This equation does not have solutions in terms of known functions, but may be readily solved numerically.

The range of \(x\) for the bound states is always in the interval \([0, 1]\), and \(\phi = 0\) outside of this range; but (4.5) determines as well the full meson-fermion-antifermion vertex,

\[ \Phi_n(z) = \int_0^1 \frac{dy}{(y - z)^2} \phi_n(y), \quad (4.7) \]
for values of $z \notin [0, 1]$. This includes $z$ complex, corresponding to the general case where one or more of the lines of the meson-quark vertex is not on-shell in its physical region. $\Phi_n(z)$ is analytic in the complex plane, with a cut on the real axis from 0 to 1. When $x \in [0, 1]$, $\Phi_n(x)$ is defined by the principal value prescription, and

$$\Phi_n(x) = \left( -\mu_n^2 + \frac{M^2 - 1}{x} + \frac{m^2 - 1}{1-x} \right) \phi_n(x),$$

in accordance with (4.8). Since $\phi_n(x)$ is finite, $\Phi_n(x)$ has zeros where the first factor on the right vanishes, and these are the values $x_\pm$ where the quarks would be on-shell. These zeros of the vertex function cancel quark poles in the propagators of loop amplitudes to ensure that no quark singularities appear in gauge-invariant Green functions.

All loop integrations are simplified by the fact that $\phi_n(x)$ is a function of $x = p_-/q_-$ only and is independent of $p_+$. When wave functions and propagators appear in a loop integrals, only the latter depend on $p_+$, so the $\int dp_+$ is over rational functions and can be computed explicitly by contour integration, leaving a single integral over one real variable.

5. Calculating the matrix elements

In this section we calculate the form factors $f_\pm$ as defined in Eq. (3.1). The current matrix element is given by

$$\langle \pi | V_\mu | B \rangle = \frac{2}{\pi} \int d^2 k \Phi_0(k, p) \Psi(k - q, p') \hat{\Phi}_\mu(k, q)$$

$$\times S(k)S(k - q)S(k - p),$$

where $\Phi_0$ is the $B$-meson wavefunction, $\Psi$ denotes the pion wavefunction and $\hat{\Phi}_\mu$ is the full current-quark-quark vertex, including all resummatons of gluon exchange $^3$. This vertex function $\hat{\Phi}_\mu$ has an intuitive meaning in the physical channel. It can be expressed as a sum over a complete set of resonances which couple directly to the current. This is evident from the formula for the Green functions. With $x = k_-/q_-,$

$$\hat{\Phi}_\mu(x, q) = \gamma_\mu - \gamma_- \int \frac{dy dy'}{(y - x)^2} G(y, y', q^2) \left[ g_{\mu+} - \frac{Mm}{2y'(1-y')q_-^2} g_{\mu-} \right]$$

and

$$G(y, y'; q^2) = \sum_n \frac{\phi_n(y)\phi_n(y')}{q^2 - \mu_n^2 + i\epsilon}.$$  

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The identification of the current with the interpolating fields for the mesons will shortly
because obvious.

The asymmetry between plus and minus components, evident in Eq. (5.2), is a conse-
quence of light-cone gauge. This has made it too difficult in previous studies to compute
the plus component of the current directly. We have overcome this obstacle and show
below how both plus and minus components can be used with equal ease — a necessity
for the study of flavor-changing interactions.

For the ‘good’ component of the current we have

\[
\hat{\Phi}^- (x, q) = \gamma^- \left( 1 - \sum_n \frac{f_n \Phi_n(x)}{q^2 - \mu_n^2} \right)
\]

\[
= \gamma^- \left[ \left( q^2 - \frac{M^2 - 1}{x} - \frac{m^2 - 1}{1-x} \right) \sum_n \frac{f_n \phi_n(x)}{q^2 - \mu_n^2} \right]
\]

\[
= \gamma^- \frac{1}{q^2} \left[ \left( q^2 - \frac{M^2 - 1}{x} - \frac{m^2 - 1}{1-x} \right) \left( 1 + \sum_n \frac{f_n \mu_n^2 \phi_n(x)}{q^2 - \mu_n^2} \right) \right]
\]

where the last equality holds for \(0 \leq x \leq 1\). The decay constant \(f_n = \int_0^1 dx \phi_n(x)\), should
properly be multiplied by a factor \(\sqrt{N_c/\pi}\) which we absorb into the overall matrix element
normalization where the \(N_c\)-dependence cancels.

A similar expression of the ‘bad’ plus component of the current has not been presented
before, presumably because of the apparent complications from the associated extra factors
in (5.2). Yet things are not as bad as they look. The bound-state equation and the parity
relation give two important results \(31\),

\[
\int_0^1 dy' \frac{M_m}{y'(1-y')} \phi_n(y') = (-1)^n \mu_n^2 \int_0^1 dy' \phi_n(y') \equiv (-1)^n \mu_n^2 f_n , \quad (5.5)
\]

and

\[
\int_0^1 \frac{dy}{(y-x)^2} G(y, y'; q^2) = \delta(y' - x) - \left( q^2 - \frac{M^2 - 1}{x} - \frac{m^2 - 1}{1-x} \right) G(x, y'; q^2) . \quad (5.6)
\]

Applying these to Eq. (5.2) gives

\[
\hat{\Phi}^+ = \gamma^+ + \gamma^- \sum_n (-1)^n \frac{f_n \mu_n^2}{2q^2_n (q^2 - \mu_n^2)^2} \Phi_n(x) , \quad (5.7a)
\]

and for \(0 \leq x \leq 1\),
\[
\frac{\gamma + \gamma - \frac{1}{2q_+^2} \frac{Mm}{x(1-x)}}{\gamma - \frac{1}{2q_-^2} \left( q^2 - \frac{M^2 - 1}{x} - \frac{m^2 - 1}{1-x} \right)} + \frac{\gamma}{2q_+^2} \left( q^2 - \frac{M^2 - 1}{x} - \frac{m^2 - 1}{1-x} \right) \sum_n (-1)^n \frac{f_n \mu_n^2 \phi_n(x)}{q^2 - \mu_n^2} \quad (5.7b)
\]

Examining the final vertex expressions in Eqs. (5.4) and (5.7b), each contains a weighted sum of pole terms plus some inhomogeneous, non-pole terms. The pole terms are of the same form and are readily interpreted as the contribution from the entire tower of resonant intermediate states. The inhomogeneous terms, however, have no such meaning; happily, they cancel inside matrix elements. In \( \hat{\Phi}_+ \), the \( \gamma_+ \) sandwiched between propagators precisely cancels the second term. In \( \hat{\Phi}_- \), the term unity, which appears in addition to the sum would generally be expected to give rise to a smooth background, quark-model type contribution, as it arises from a coupling of the bare current to the valence quark. However, it drops out of the form factors since it gives a contribution in \( f_\pm \) proportional to

\[
\frac{2}{q^2 \omega - \mu_B^2 \omega} \left[ \frac{1}{1-\omega} \int_0^\omega dv \phi_0(v) \Psi(v \frac{\omega}{1-\omega}) - \frac{1}{\omega} \int_0^1 dv \phi_0(v) \psi(v \frac{\omega}{1-\omega}) \right]. \quad (5.8)
\]

This is zero for all \( q^2 \) since the two integrals cancel each other. To see this, let \( v = t \omega \) in the first and let \( v = \omega + u(1 - \omega) \) in the second, and rewrite each of the \( \Phi(x) = \int dy \phi(y)/(y-x)^2 \) as a Hilbert transform. Then the first integral and minus the second are each equal to

\[
\frac{\omega}{1-\omega} \int_0^1 dt \int_0^1 du \frac{\phi_0(t) \psi(u)}{(u - (t - 1) \frac{\omega}{1-\omega})^2}. \quad (5.9)
\]

Therefore the matrix element of the \((+)\) component of the current involves a sum over the same wavefunctions as for the \((-)\) component, with alternating signs.

It is now easy to combine the currents in the invariant combinations \( q \cdot V \) and \( \varepsilon^{\mu \nu} q_\mu V_\nu \), which couple to states of even and odd parity, or odd and even \( n \), respectively:

\[
\langle \pi | q^\mu V_\mu | B \rangle = \langle \pi(p') | q_+ V_- + q_- V_+ | B(p) \rangle = (\mu_B^2 - \mu_\pi^2) f_+(q^2) + q^2 f_-(q^2) \\
\langle \pi | q^\mu A_\mu | B \rangle = \langle \pi(p') | q_+ V_- - q_- V_+ | B(p) \rangle = (2 \varepsilon^{\mu \nu} p_\mu p'_\nu) f_+(q^2). \quad (5.10)
\]

Using Eqs. (5.4), (5.7b),

\[
q_+ \hat{\Phi}_- \pm q_- \hat{\Phi}_+ = \gamma - \frac{1}{2q_+^2} \left( q^2 - \frac{M^2 - 1}{x} - \frac{m^2 - 1}{1-x} \right) \sum_{n\text{ even}} \frac{f_n \mu_n^2 \phi_n(x)}{q^2 - \mu_n^2} \quad (5.11)
\]

\[
\]
and applying to Eqs. (5.10) and (5.1) we have

\[ f_+(q^2) = \sum_{n \text{ even}} f_n g_{\pi B_n}(q^2) \equiv \sum_{n \text{ even}} \frac{A_n(q^2)}{1 - q^2/\mu_n^2}. \] (5.12)

Here

\[ g_{\pi B_n}(q^2) = \frac{-2\mu_n^2}{q^2 - \mu_B^2} \left[ \frac{1}{1 - \omega} \int_{0}^{\omega} dv \phi_n(v) \phi_0 \left( \frac{v}{\omega} \right) \psi \left( \frac{v - \omega}{1 - \omega} \right) \right. \]

\[ \left. - \frac{1}{\omega} \int_{\omega}^{1} dv \phi_n(v) \Phi_0 \left( \frac{v}{\omega} \right) \psi \left( \frac{v - \omega}{1 - \omega} \right) \right] \] (5.13)

and

\[ A_n(q^2) \equiv -\frac{f_n g_{\pi B_n}(q^2)}{\mu_n^2}. \] (5.14)

The \( g_{\pi B_n}(q^2) \) are the invariant three-point couplings. For \( q^2 \) above threshold \( \omega = p_-/q_- \) is real, between 0 and 1,

\[ \omega(q^2) = \frac{1}{2} \left( 1 + \frac{\mu_B^2 - \mu_\pi^2}{q^2} - \sqrt{1 - 2 \left( \frac{\mu_B^2 + \mu_\pi^2}{q^2} \right) + \left( \frac{\mu_B^2 - \mu_\pi^2}{q^2} \right)^2} \right). \] (5.15)

For even parity,

\[ \langle \pi | q^\mu V_\mu | B \rangle = \sum_{n \text{ odd}} \frac{f_n g_{\pi B_n}(q^2)}{q^2 - \mu_n^2}, \] (5.16)

so the second form factor is given by

\[ f_-(q^2) = \frac{1}{q^2} \left[ \sum_{n \text{ odd}} \frac{B_n(q^2)}{1 - q^2/\mu_n^2} - \sum_{n \text{ even}} \frac{A_n(q^2)}{1 - q^2/\mu_n^2} (\mu_B^2 - \mu_\pi^2) \right] \] (5.17)

where

\[ g_{\pi B_n}(q^2) = -2\mu_n^2 \left[ \frac{1}{1 - \omega} \int_{0}^{\omega} dv \phi_n(v) \phi_0 \left( \frac{v}{\omega} \right) \psi \left( \frac{v - \omega}{1 - \omega} \right) \right. \]

\[ \left. - \frac{1}{\omega} \int_{\omega}^{1} dv \phi_n(v) \Phi_0 \left( \frac{v}{\omega} \right) \psi \left( \frac{v - \omega}{1 - \omega} \right) \right], \] (5.18)

and \( B_n \equiv -f_n g_{\pi B_n}/\mu_n^2 \) for \( n \) odd. The \( g_{\pi B_n} \) have different kinematic prefactors for even and odd parity as expected from the Lorentz-invariant couplings of Eqs. (3.7) and (3.5).

Using Cauchy’s theorem, we may write

\[ f_+(q^2) = \sum_{n \text{ even}} \frac{A_n(q^2)}{1 - q^2/\mu_n^2} = \sum_{n \text{ even}} \frac{A_n(\mu_n^2)}{1 - q^2/\mu_n^2}. \] (5.19)
The needed convergence of the sum is provided by the direct analysis of the wavefunctions in Ref. [31], which can be carried immediately to the present case to give that \( f \sim 1/|q^2|^{1+\beta_m} \), and \( \beta_m > 0 \) is defined in (4.6). The right hand side is far easier to compute, as the residues of the poles are numbers rather than functions. What is their meaning?

The residues of the poles are precisely related to the three-point coupling functions one needs in order to construct an effective meson Lagrangian. As in Eq. (3.4), we introduce a field for each meson state and proceed to write down interaction terms which will reproduce the \( S \)-matrix elements computed from ’t Hooft’s Feynman rules.

The effective substitution of the on-shell values in (5.19) corresponds to making nonlinear field redefinitions of the meson states in order to trade higher-derivative interactions (which arise from Taylor expanding the 3-point functions) for higher-point interactions. But the higher order terms—four-point, five-point, etc.—are already suppressed by extra factors of \( N_c \). We can choose fields, therefore, in such a way that the only interaction terms are cubic couplings with no derivatives in the large \( N_c \) limit. It is interesting to compare these field redefinitions with those studied by Georgi in deriving an on-shell Lagrangian for general low-energy effective field theories [28].
6. Numerical Results

The method used for numerical solution of the three-point couplings was introduced in Ref. [27], to which we refer the reader for details. The wavefunctions were expanded in an appropriate Fourier series and the wave function overlaps computed numerically. The formulas of the preceding section do not hold for the lowest residue of each tower so these may be fit from sum rules along the lines of Ref. [27]: For any chosen $q^2 > (\mu_B + \mu_\pi)^2$, every term in Eq. (5.19), can be computed directly except for $A_0(\mu_0^2)$. Eq. (5.19) can be solved for $A_0(\mu_0^2)$ and selecting different values of $q^2$ provides an arbitrary number of checks on both the accuracy and the sum rule. We find in every case the variation to be in the fourth significant figure at most for $q^2$ is chosen to be of the order of the ground state mass times a factor of order unity.

We have studied two values of the light-quark mass, $m = 0.56$ and $m = 0.1$ — corresponding to $\mu_\pi^2 = 3.09$ and $\mu_\pi^2 = 0.72$ respectively — and numerous values of the heavy-quark mass $M$. The coupling constants $g_{\pi B n}$ depend sensitively on the light-quark mass but weakly on the heavy quark mass.

Figures 1 and 2 show the approach to the chiral limit. $A_n(\mu_n^2)$ is plotted against the light quark mass $m$ for the lowest eight resonances and it is evident that $A_n$ goes to zero for all $n \neq 0$. The ground state coupling, $A_0$, approaches a constant. Recall that $A_n = -g_{\pi B n} f_n / \mu_n^2$. Since $f_n$ and $\mu_n^2$ have negligible $m$ dependence, the shape is identical to that for $g_{\pi B n}$ as well.

These figures also show the lowest residue clearly dominating the higher resonances.

![Figure 1](image_url)

Figure 1. The residue $A_n(\mu_n^2)$ vs. $m$, the light quark mass. The value was computed numerically for
$m = 0.1, 0.56$ and a line connecting the pairs of points drawn to guide the eye. For $n \neq 0$, the $A_n \to 0$ in the chiral limit, $m \to 0$.

Figure 2. The residue $A_n(\mu^2_n)$ vs. $m$, the light quark mass, for more resonant states, $n = 6–18$.

Figure 3 shows how the couplings $\hat{g}_{\pi Bn} \equiv g_{\pi Bn}/\mu^2_n$ reach constant values independent of the heavy quark mass $M$.

3. Approach to the heavy quark limit: $-g_{\pi Bn}/\mu^2_n$ vs. $M$ for $m = 0.56$ ($\mu^2_n = 3.09$) and $M^2 = 25, 2000, 20000, 200000$. Results for $m = 0.1$ are similar.

While the large coupling $g_{\pi BB^*}$ for the $B^*$ is sufficient for it to dominate, the factors
of the meson decay constants, $f_n$, further amplify the effect on the form factor. Figure 4 shows the rapid fall of the $f_n$ with $n$.

Finally, Figures 5 and 6 show the couplings vs. $n$, the state number. This again illustrates that the lowest pole dominates and that the pole dominance is stronger as the pion mass decreases, in agreement with the conclusions of section 3.

![Figure 4](image1.png)

**Figure 4. Typical $n$-dependence of the decay constants $f_n$. (Shown here for $m = 0.56, M^2 = 20000$.)**

![Figure 5](image2.png)

**Figure 5. Typical $n$-dependence of the pole residues $A_n(\mu_n^2)$. (Shown here for $m = 0.56, M^2 = 20000$.)**
Figure 6. Typical $n$-dependence of the pole residues $A_n(\mu_n^2)$. (Shown here for $m = 0.1$, $M^2 = 20000$.)

The $B^*$ pole dominates over a large kinematic region all the way down to $q^2 = 0$, not simply the small neighborhood $q^2 \approx q^2_{\text{max}}$ suggested in Refs. [17], [18].

7. Conclusions

The decay $\bar{B} \to \pi e \bar{\nu}_e$ is dominated by the $B^*$ vector meson pole over a large kinematic range. In four dimensions, the result follows from the combined chiral, heavy-quark, large-$N_c$ limit. This provides a justification for the pole-dominated shape assumed at all $q^2$ by many model calculations. In contrast with previous arguments we find no evidence of a quark-model regime or the need for two-component models of form factors. The reason for the difference may be that these arguments assumed a smooth chiral behavior in which the pion mass could be neglected, whereas we have seen a sensitive dependence on the pion mass which is in any event of order of the splitting between states and shouldn’t be expected to be negligible compared to other relevant quantities.

The next step is to use this pole-dominance result for extracting predictions for the KM mixing angle $V_{ub}$. This requires four-dimensional estimates of the corrections to this limit. Work in this direction is in progress. If the corrections turn out to be small it may be possible to apply these results to semileptonic decays of $D$ mesons, for $D \to \pi$ or even $D \to K$. 

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In two dimensions, we have shown that for any fixed heavy quark mass the form factors for $\bar{B} \to \pi e\bar{\nu}_e$ are given by a single pole at the $B$. The result was derived in the combined large $N_c$ and chiral limit. To study the approach to the chiral limit we wrote the form factors as resonant sums, gave explicit formulae for the residues in terms of overlaps of 't Hooft wave-functions and computed these overlaps numerically. This demonstrated the expected behavior with decreasing pion mass and the expected scaling at large heavy quark mass.

The nature of deviations from the large $N_c$ limit should be more easily addressed in two than in four dimensions. Also, there is no reason to expect that the form factors for $\bar{B} \to \pi^* e\bar{\nu}_e$, where $\pi^*$ is an excited pion resonance, should be pole dominated. Work on these issues is in progress.

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FIGURE CAPTIONS

Fig. 1. The residue $A_n(\mu^2_n)$ vs. $m$, the light quark mass. The value was computed numerically for $m = 0.1, 0.56$ and a line connecting the pairs of points drawn to guide the eye. For $n \neq 0$, the $A_n \rightarrow 0$ in the chiral limit, $m \rightarrow 0$.

Fig. 2. The residue $A_n(\mu^2_n)$ vs. $m$, the light quark mass, for more resonant states, $n = 6–18$.

Fig. 3. Approach to the heavy quark limit: $-g_{\pi B_n}/\mu^2_n$ vs. $M$ for $m = 0.56$ ($\mu^2_n = 3.09$) and $M^2 = 25, 2000, 20000, 200000$. Results for $m = 0.1$ are similar.

Fig. 4. Typical $n$-dependence of the decay constants $f_n$. (Shown here for $m = 0.56, M^2 = 20000$.)

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Fig. 6. Typical $n$-dependence of the pole residues $A_n(\mu^2_n)$. (Shown here for $m = 0.1, M^2 = 20000$.)