LINE ARRANGEMENTS AND \( r \)-STIRLING PARTITIONS

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Abstract. A set partition of \([n] := \{1, 2, \ldots, n\}\) is called \( r \)-Stirling if the first \( r \) letters \( 1, 2, \ldots, r \) lie in distinct blocks. Haglund, Rhoades, and Shimozono constructed graded ring \( R_{n,k} \) depending on two positive integers \( k \leq n \) whose algebraic properties are governed by the combinatorics of ordered set partitions of \([n]\) with \( k \) blocks. We introduce a variant \( R_{n,k}^{(r)} \) of this quotient for ordered \( r \)-Stirling partitions which depends on three integers \( r \leq k \leq n \). We describe the standard monomial basis of \( R_{n,k}^{(r)} \) and use the combinatorial notion of \( \text{coinversion code} \) of an ordered set partition to reprove and generalize some results of Haglund et. al. in a more direct way. Furthermore, we introduce a variety \( X_{n,k}^{(r)} \) of line arrangements whose cohomology is presented as the integral form of \( R_{n,k}^{(r)} \), generalizing results of Pawłowski and Rhoades.

1. Introduction

Given two integers \( r \leq n \), a set partition of \([n] := \{1, 2, \ldots, n\}\) is called \( r \)-Stirling if the first \( r \) letters \( 1, 2, \ldots, r \) belong to distinct blocks. The \( r \)-Stirling number \( \text{Stir}_{n,k}^{(r)} \) counts \( r \)-Stirling partitions of \([n]\) with \( k \) blocks. An ordered \( r \)-Stirling partition is an \( r \)-Stirling partition \( \sigma = (B_1 \mid \cdots \mid B_k) \) equipped with a total order on its blocks. We let \( \mathcal{OP}_n^{(r)} \) denote the family of ordered \( r \)-Stirling partitions of \([n]\) with \( k \) blocks; these are counted by \( |\mathcal{OP}_n^{(r)}| = k! \cdot \text{Stir}_{n,k}^{(r)} \).

An example element of \( \mathcal{OP}_3^{(2)} \) is \((26 \mid 5 \mid 17 \mid 34)\). On the other hand, the ordered set partition \((45 \mid 2 \mid 136 \mid 7)\) fails to be 3-Stirling since 1 and 3 belong to the same block. The symmetric group \( S_n \) acts on ordered set partitions of \([n]\) by letter permutation. Although \( \mathcal{OP}_n^{(r)} \) is not closed under the full action of \( S_n \), it does carry an action of the parabolic subgroup \( S_r \times S_{n-r} \).

When \( r = k = n \), an element of \( \mathcal{OP}_n^{(n)} \) is just a permutation in \( S_n \). The combinatorics of the symmetric group \( S_n \) is well-known to govern both the algebraic structure of the coinvariant ring \( R_n \) and the geometric structure of the flag variety \( \mathcal{F}(n) \).

In the case \( r = 0 \) where \( \mathcal{OP}_n := \mathcal{OP}_n^{(0)} \) is the collection of \( k \)-block ordered set partitions of \([n]\), the Delta Conjecture \( \Box \) in the theory of Macdonald polynomials motivated the definition and study of a generalized coinvariant ring \( R_{n,k} \) and a generalization \( X_{n,k} \) of the flag variety \( \mathcal{F}(n) \) which specialize to their classical counterparts when \( k = n \). The algebraic properties of \( R_{n,k} \) and the geometric properties of \( X_{n,k} \) are governed by combinatorial properties of ordered set partitions in \( \mathcal{OP}_n^{(r)} \).

At a workshop in Montréal in the Summer of 2017, Jeff Remmel asked the authors if it was possible to extend this theory to encapsulate ordered \( r \)-Stirling partitions; in this paper we do exactly that. We consider a quotient ring \( R_{n,k}^{(r)} \) and a variety \( X_{n,k}^{(r)} \) whose properties are controlled by the combinatorics of \( \mathcal{OP}_n^{(r)} \). The quotient \( R_{n,k}^{(r)} \) of \( \mathbb{Q}[x_n] := \mathbb{Q}[x_1, \ldots, x_n] \) (together with its companion quotient \( S_{n,k}^{(r)} \) of \( \mathbb{Z}[x_n] := \mathbb{Z}[x_1, \ldots, x_n] \)) is defined as follows. If \( x_m = (x_1, \ldots, x_m) \) is a list of variables and \( d \geq 0 \), we recall the elementary and homogeneous symmetric polynomials of \( \mathbb{Q}[x_n] \)
Figure 1. A point in $X^{(2)}_{5,3}$.

degree $d$ in the variable set $x_m$:

\[
e_d(x_m) := \sum_{1 \leq i_1 < \cdots < i_d \leq m} x_{i_1} \cdots x_{i_d},
\]

(1.1)

\[
h_d(x_m) := \sum_{1 \leq i_1 \leq \cdots \leq i_d \leq m} x_{i_1} \cdots x_{i_d}.
\]

(1.2)

**Definition 1.1.** For $r \leq k \leq n$, let $I^{(r)}_{n,k} \subseteq \mathbb{Q}[x_n]$ be the ideal

\[
I^{(r)}_{n,k} := \langle e_n(x_n), e_{n-1}(x_n), \ldots, e_{n-k+1}(x_n), h_{k-r+1}(x_r), h_{k-r+2}(x_r), \ldots, h_k(x_r) \rangle
\]

(1.3)

and let $R^{(r)}_{n,k}$ be the corresponding quotient ring:

\[
R^{(r)}_{n,k} := \mathbb{Q}[x_n]/I^{(r)}_{n,k}.
\]

(1.4)

Furthermore, let $J^{(r)}_{n,k} \subseteq \mathbb{Z}[x_n]$ be the ideal in $\mathbb{Z}[x_n]$ with the same generating set as $I^{(r)}_{n,k}$ and let $S^{(r)}_{n,k} = \mathbb{Z}[x_n]/J^{(r)}_{n,k}$ be the corresponding quotient.

When $r = k = n$, the ideal $I_n := I^{(n)}_{n,n}$ is just the classical invariant ideal $(e_1(x_n), e_2(x_n), \ldots, e_n(x_n))$ generated by the $n$ elementary symmetric polynomials. When $r = 0$, the ideal $I_{n,k} := I^{(0)}_{n,k}$ is precisely the ideal considered in [4], and its companion ideal $J_{n,k} := J^{(0)}_{n,k}$ over the ring of integers was considered in [5].

The quotient ring $S^{(r)}_{n,k}$ will be shown to calculate the cohomology (singular, with coefficients in $\mathbb{Z}$) of a natural space $X^{(r)}_{n,k}$ whose geometry is governed by the combinatorics of $\mathcal{O}^{(r)}_{P_{n,k}}$. Let $\mathbb{P}^{k-1}$ be the complex projective space of lines through the origin in $\mathbb{C}^k$, so that $(\mathbb{P}^{k-1})^n$ is the complex algebraic variety of all $n$-tuples $(\ell_1, \ldots, \ell_n)$ of lines through the origin in $\mathbb{C}^k$. We consider the following family of line arrangements:

**Definition 1.2.** Let $r \leq k \leq n$ and define a subset $X^{(r)}_{n,k} \subseteq (\mathbb{P}^{k-1})^n$ by

\[
X^{(r)}_{n,k} := \left\{ (\ell_1, \ell_2, \ldots, \ell_n) \in (\mathbb{P}^{k-1})^n : \ell_1 + \ell_2 + \cdots + \ell_n = \mathbb{C}^k \text{ and } \dim(\ell_1 + \ell_2 + \cdots + \ell_r) = r \right\}.
\]

(1.5)

A typical point in $X^{(r)}_{n,k}$ is an $n$-tuple of lines $(\ell_1, \ldots, \ell_n)$ through the origin in $\mathbb{C}^k$ such that these lines span $\mathbb{C}^k$ and such that the first $r$ of these lines are linearly independent. An example of such a
line arrangement in $X_{5,3}^{(2)}$ is shown in Figure 1; the first two lines $\ell_1$ and $\ell_2$ are linearly independent, and the five lines $\ell_1, \ldots, \ell_5$ together span $\mathbb{C}^3$.

The product group $S_r \times S_{n-r}$ acts on $X_{n,k}^{(r)}$ by line permutation. The set $X_{n,k}^{(r)}$ is a Zariski open subset of $(\mathbb{P}^{k-1})^n$ and is therefore both a variety and a smooth complex manifold.

When $r = k = n$, the space $X_{n,k}^{(r)}$ may be identified with the quotient $G/T$, where $G = GL_n(\mathbb{C})$ is the group of invertible $n \times n$ complex matrices and $T \subseteq G$ is the diagonal torus. If $B \subseteq G$ is the Borel subgroup of upper triangular matrices, the quotient $G/B$ is the classical flag variety $F\ell(n)$ of type $A_{n-1}$ and the canonical projection $G/T \rightarrow G/B$ is a homotopy equivalence. When $r = 0$, the space $X_{n,k} := X_{n,k}^{(0)}$ of $n$-tuples of lines spanning $\mathbb{C}^k$ was defined and studied by Pawlowski and Rhoades as an extension of the flag variety [5].

The remainder of the paper is organized as follows. In Section 2 we will introduce a new statistic on an ordered set partition $\sigma$: the coinversion code code($\sigma$). This will allow us to read off the standard monomial basis of the quotient ring $R_{n,k}^{(r)}$ directly from the combinatorics of $OP_{n,k}^{(r)}$, both extending and making more combinatorial the results regarding $R_{n,k}$ in [4]. In Section 3 we will study the space of line arrangements $X_{n,k}^{(r)}$ and prove that $H^\bullet(X_{n,k}^{(r)}) = S_{n,k}^{(r)}$. We will also describe an affine paving of $X_{n,k}^{(r)}$ with cells indexed by partitions in $OP_{n,k}^{(r)}$, together with formulas for the representatives of these cells in cohomology.

2. Coinversion codes and standard bases

Recall that an inversion of a permutation $w \in S_n$ is a pair $1 \leq i < j \leq n$ such that $i$ appears to the right of $j$ in the one-line notation $w = w_1 \ldots w_n$, so that the inversions of $231 \in S_3$ are the pairs $(1,2)$ and $(1,3)$. Extending this notion to ordered set partitions, if $\sigma = (B_1 \mid \cdots \mid B_k)$ is an ordered set partition of $[n]$ with $k$ blocks, a pair $1 \leq i < j \leq n$ is said to be an inversion of $\sigma$ if

- the block of $i$ is strictly to the right of the block of $j$ in sigma, and
- the letter $i$ is minimal in its block.

We let $\text{inv}(\sigma)$ be the number of inversions of $\sigma$, so that if $\sigma = (25 \mid 1 \mid 34) \in OP_{5,3}^{(r)}$ the inversion pairs are $(1,2), (1,5), \text{and} (3,5)$ so that $\text{inv}(\sigma) = 3$.

We will not be interested in the statistic inv itself, but rather its complementary statistic. For any three integers $r \leq k \leq n$, it is not hard to see that the statistic inv on $OP_{n,k}^{(r)}$ achieves its maximum value at the unique point $\sigma_0 := (k, k+1, \ldots, n-1, n \mid k-1 \mid \cdots \mid 1) \in OP_{n,k}^{(r)}$, and that

$$\text{inv}(\sigma_0) = (n-k)(k-1) + \binom{k}{2}. \tag{2.1}$$

We define the statistic $\text{coinv}$ on $OP_{n,k}^{(r)}$ by the rule

$$\text{coinv}(\sigma) := (n-k)(k-1) + \binom{k}{2} - \text{inv}(\sigma). \tag{2.2}$$

For example, we have

$$\text{coinv}(25 \mid 1 \mid 34) = (5-3)(3-1) + \binom{3}{2} - \text{inv}(25 \mid 1 \mid 34) = 4 + 3 - 3 = 4.$$

It will be convenient to break up the coinversion statistic coinv into a sequence of smaller statistics. Given an ordered set partition $\sigma = (B_1 \mid \cdots \mid B_k) \in OP_{n,k}^{(r)}$, define the coinversion code code($\sigma$) = $(c_1, c_2, \ldots, c_n)$ as follows. Suppose $1 \leq i \leq n$ and $i \in B_j$. Then

$$c_j = \begin{cases} \{|\ell > j : \text{min}(B_\ell) > i|\} & \text{if } i = \text{min}(B_j) \\ \{|\ell > j : \text{min}(B_\ell) > i|\} + (j-1) & \text{if } i \neq \text{min}(B_j). \end{cases} \tag{2.3}$$
The coinversion code of $(25 \mid 1 \mid 34)$ is therefore $\text{code}(\sigma) = (c_1, c_2, c_3, c_4, c_5) = (1, 1, 0, 2, 0)$. The coinversion code breaks the statistic coinv into pieces.

**Proposition 2.1.** Let $\sigma \in \mathcal{P}_{n,k}^{(r)}$ with $\text{code}(\sigma) = (c_1, c_2, \ldots, c_n)$. Then

$$(2.4) \quad \text{coinv}(\sigma) = c_1 + c_2 + \cdots + c_n.$$ 

Which sequences $(c_1, c_2, \ldots, c_n)$ of nonnegative integers can arise as the coinversion code of some element $\sigma \in \mathcal{P}_{n,k}^{(r)}$? When $r = k = n$, these are precisely the sequences $(c_1, c_2, \ldots, c_n)$ which are componentwise $\leq$ the staircase $(n-1, n-2, \ldots, 0)$ of length $n$. To state the answer for general $r \leq k \leq n$, we will need some definitions.

If $S = \{s_1, s_2, \ldots, s_m\}$ is any subset of $[n]$, the *skip composition* $\gamma(S) = (\gamma(S)_1, \ldots, \gamma(S)_n)$ is the sequence given by

$$(2.5) \quad \gamma(S)_i = \begin{cases} 
  i - j + 1 & \text{if } i = s_j \in S \\
  0 & \text{if } i \notin S.
\end{cases}$$

We also let $\gamma(S)^* = (\gamma(S)_n, \ldots, \gamma(S)_1)$ be the reversal of the skip composition. As an example, if $n = 7$ and $S = \{2, 3, 6\}$ then $\gamma(S) = (0, 2, 2, 0, 0, 4, 0)$ and $\gamma(S)^* = (0, 4, 0, 0, 2, 2, 0)$.

**Theorem 2.2.** Let $r \leq k \leq n$. The map $\sigma \mapsto \text{code}(\sigma)$ gives a bijection from $\mathcal{P}_{n,k}^{(r)}$ to the family $(c_1, \ldots, c_n)$ of nonnegative integer sequences such that

- for all $r + 1 \leq i \leq n$ we have $c_i < k$,
- for all $1 \leq i \leq r$ we have $c_i < k - i + 1$, and
- for any subset $S \subseteq [n]$ with $|S| = n-k+1$, the componentwise inequality $\gamma(S)^* \leq (c_1, \ldots, c_n)$ fails to hold.

**Proof.** Let $\mathcal{C}_{n,k}^{(r)}$ be the family of length $n$ sequences of nonnegative integers which satisfy the three conditions in the statement of the theorem. Let $\sigma \in \mathcal{P}_{n,k}^{(r)}$ with $\text{code}(\sigma) = (c_1, \ldots, c_n)$. We show that $(c_1, \ldots, c_n) \in \mathcal{C}_{n,k}^{(r)}$, so that the function $\text{code} : \mathcal{P}_{n,k}^{(r)} \to \mathcal{C}_{n,k}^{(r)}$ is well-defined. This is verified as follows.

- For any $1 \leq i \leq n$, the block $B$ of $\sigma$ containing $i$ cannot contribute to $c_i$, whereas each block $\neq B$ can contribute at most 1 to $c_i$. Consequently, we have $c_i < k$.
- Since $\sigma$ is $r$-Stirling, the letters $1, 2, \ldots, r$ are all minimal in their blocks. In particular, if $1 \leq i \leq r$, the blocks containing $1, 2, \ldots, i-1$ cannot contribute to $c_i$, so that $c_i < k - i + 1$.
- Finally, let $S \subseteq [n]$ satisfy $|S| = n-k+1$. We verify $\gamma(S)^* \not\leq (c_1, \ldots, c_n)$. Working towards a contradiction, suppose $\gamma(S)^* \leq (c_1, \ldots, c_n)$.

Write the reversal $T := \{n-i+1 : i \in S\}$ of $S$ as $T = \{t_1 < \cdots < t_{n-k+1}\}$. Since $\sigma$ has $n$ letters and $k$ blocks, *at least one element of $T$ must be minimal in its block of $\sigma$*. If $t_{n-k+1}$ is minimal in its block of $\sigma$, then

$$(2.6) \quad c_{n-k+1} = |\{\ell > t_{n-k+1} : \ell \text{ is minimal in its block and occurs to the right of } t_{n-k+1} \text{ in } \sigma\}|$$

$$(2.7) \quad \leq |\{t_{n-k+1} + 1, \ldots, n - 1, n\}|$$

$$(2.8) \quad = n - t_{n-k+1}.$$ 

But the term of $\gamma(S)^*$ in position $t_{n-k+1}$ is $n - t_{n-k+1} + 1$. We conclude that $t_{n-k+1}$ is not minimal in its block of $\sigma$. If $t_{n-k}$ were minimal in its block of $\sigma$, then

$$(2.9) \quad c_{n-k} = |\{\ell > t_{n-k} : \ell \text{ is minimal in its block and occurs to the right of } t_{n-k} \text{ in } \sigma\}|$$

$$(2.10) \quad \leq |\{t_{n-k} + 1, \ldots, n - 1, n\} - \{t_{n-k+1}\}|$$

$$(2.11) \quad = n - t_{n-k} - 1,$$
But the term of $\gamma(S)^*$ in position $t_{n-k}$ is $n - t_{n-k}$. We conclude that $t_{n-k}$ is not minimal in its block of $\sigma$. If $t_{n-k-1}$ were minimal in its block of $\sigma$, the same reasoning leads to the contradiction $c_{t_{n-k-1}} < n - t_{n-k-1} - 1$, etc. We see that none of the elements in $T$ are minimal in their block of $\sigma$, a contradiction.

In order to show that code $: OP_{n,k}^{(r)} \rightarrow C_{n,k}^{(r)}$ is a bijection, we construct its inverse. As this inverse will be defined using an insertion procedure, we denote it $\iota : C_{n,k}^{(r)} \rightarrow OP_{n,k}^{(r)}$.

Let $(B_1 | \cdots | B_k)$ be a sequence of of $k$ possibly empty sets of positive integers. We define the coinversion label of the sets $B_1, \ldots, B_k$ by labeling the empty sets with 0, 1, . . . , $j$ from right to left (where there are $j$ empty sets), and then labeling the nonempty sets with $j+1, j+2, \ldots, k−1$ from left to right. An example of conversion labels is as follows, displayed as superscripts:

\[
\begin{array}{c|cccc}
\emptyset^2 & 13^3 & \emptyset^1 & 25^4 & \emptyset^0
\end{array}
\]

By construction, each of the letters 0, 1, . . . , $k−1$ appears exactly once as a coinversion label.

Let $(c_1, \ldots, c_n) \in C_{n,k}^{(r)}$. Then $0 \leq c_i \leq k−1$ for $1 \leq i \leq n$. We define $\iota(m) = (B_1 | \cdots | B_k)$ recursively by starting with the sequence $(\emptyset | \cdots | \emptyset)$ of $k$ copies of the empty set, and for $i = 1, 2, \ldots, n$ inserting $i$ into the unique block with coinversion label $c_i$. Here is an example of this procedure for $(n, k, r) = (9, 4, 3)$ and $(c_1, \ldots, c_9) = (2, 0, 1, 1, 0, 2, 1, 3)$:

| $i$ | $c_i$ | $\sigma$ |
|-----|-------|---------|
| 1   | 2     | $\emptyset^3 | \emptyset^2 | \emptyset^1 | \emptyset^0$ |
| 2   | 0     | $\emptyset^2 | 1^3 | \emptyset^1 | \emptyset^0$ |
| 3   | 1     | $\emptyset^1 | 1^2 | \emptyset^0 | 2^2$ |
| 4   | 1     | $3^1 | 1^2 | \emptyset^0 | 2^3$ |
| 5   | 1     | $34^1 | 1^2 | \emptyset^0 | 2^2$ |
| 6   | 0     | $345^1 | 1^2 | \emptyset^0 | 2^3$ |
| 7   | 2     | $(345^0 | 1^1 | 6^2 | 2^3$ |
| 8   | 1     | $(345^0 | 18^1 | 67^2 | 2^3$ |
| 9   | 3     | $(345^0 | 18^1 | 67^2 | 29^3$ |

We conclude $\iota(2, 0, 1, 1, 0, 2, 1, 3) = (345 | 18 | 67 | 29)$.

We verify that $\iota$ is a well-defined function $C_{n,k}^{(r)} \rightarrow OP_{n,k}^{(r)}$. Let $(c_1, \ldots, c_n) \in C_{n,k}^{(r)}$ and let $\iota(c_1, \ldots, c_n) = (B_1 | \cdots | B_k)$ = $\sigma$. We must show that 1, 2, . . . , $r$ lie in distinct blocks of $\sigma$ and that $\sigma$ does not have any empty blocks.

Suppose there exist $1 \leq i < j \leq r$ such that $i$ and $j$ belong to the same block of $\sigma$. Choose the pair $(i, j)$ to be lexicographically minimal with this property and suppose $i, j \in B_\ell$. Since the sequence $(B_1 | \cdots | B_k)$ consists of $j−1$ singletons and $k − j + 1$ copies of the empty set when $j$ is inserted by $\iota$, the definition of $\iota$ and the fact that $j$ was added to a non-singleton block imply $c_j \geq k − j + 1$, which contradicts the assumption $(c_1, \ldots, c_n) \in C_{n,k}^{(r)}$. We conclude that 1, 2, . . . , $r$ lie in different blocks of $\sigma$.

Now suppose that some of the blocks of $\sigma = (B_1 | \cdots | B_k)$ are empty. This means that at least $n−k+1$ of the letters in $[n]$ are not minimal in their block of $\sigma$. Let $S$ be the lexicographically first set of $n−k+1$ letters in $[n]$ which are not minimal in their blocks. We will derive the contradiction $\gamma(S)^* \leq (c_1, \ldots, c_n)$.

Indeed, write the reversal $T = \{n − i + 1 : i \in S\}$ of $S$ as $T = \{t_1 < \cdots < t_{n-k+1}\}$. Let $1 \leq i \leq n−k+1$. By our choice of $S$, we know that the letters in the set difference

\[
\{ t_i + 1, t_i + 2, \ldots, n \} \setminus \{ t_{i+1}, t_{i+2}, \ldots, t_{n-k+1} \}
\]

are all minimal in their blocks of $\sigma$; this set has $(n−t_i)−(n−k+1−i) = k−t_i+i−1$ elements. Consequently, since $\sigma$ contains at least one empty block, when the $\iota$ inserts $t_i$, there are $\geq k−t_i+i$ empty blocks. This forces $c_{t_i} \geq k−t_i+i+1$. Since $k−t_i+i+1$ is the term of $\gamma(S)^*$ in position
affords an identification of $Y$ points ($Y$). Let Theorem 2.3. Therefore, none of the blocks of $\sigma$ are empty and the function $\iota : c_{n,k}^{(r)} \to OP_{n,k}^{(r)}$ is well-defined. We leave it for the reader to check that code $\iota$ and $\iota$ are mutually inverse.

The code bijection of Theorem 2.2 will have algebraic importance to the theory of Gröbner bases. Recall that a total order $<$ on monomials in $Q[x_n]$ is called a monomial order if

- $1 \leq m$ for any monomial $m$, and
- If $m_1, m_2, m_3$ are monomials with $m_1 < m_2$, we have $m_1 \cdot m_3 < m_2 \cdot m_3$.

In this paper, we will exclusively use the negative lexicographical term order neglex defined by

$$x_1^{a_1} \cdots x_n^{a_n} < x_1^{b_1} \cdots x_n^{b_n} \quad \text{if and only if there exists } 1 \leq i \leq n \text{ such that } a_i < b_i \text{ and } a_{i+1} = b_{i+1}, \ldots, a_n = b_n.$$ If $<$ is any monomial order and $f \in Q[x_n]$ is nonzero, let $\text{in}_<(f)$ be the leading term of $f$. Furthermore, if $I \subseteq Q[x_n]$ is an ideal, the initial ideal is $\text{in}_<(I) := \langle \text{in}_<(f) : f \in I - \{0\} \rangle$. A finite subset $G = \{g_1, \ldots, g_s\} \subseteq I$ is called a Gröbner basis if $\text{in}_<(I) = \langle \text{in}_<(g_1), \ldots, \text{in}_<(g_s) \rangle$. If $G$ is a Gröbner basis for $I$, we necessarily have $I = \langle G \rangle$. Every ideal $I \subseteq Q[x_n]$ has a Gröbner basis (with respect to some fixed monomial order $<$).

Let $I \subseteq Q[x_n]$ be an ideal and fix a monomial order $<$. If $G = \{g_1, \ldots, g_s\}$ is a Gröbner basis for $I$, the set of monomials

$$\{ m : \text{in}_<(f) \nmid m \text{ for all } f \in I - \{0\} \} \cup \{ m : \text{in}_<(g_i) \nmid m \text{ for } 1 \leq i \leq s \}$$

descends to a $Q$-vector space basis for $Q[x_n]/I$. This is called the standard basis of $Q[x_n]/I$. After a monomial order is fixed, any quotient $Q[x_n]/I$ has a unique standard basis. The code map precisely describes the standard basis of $R_{n,k}^{(r)}$, in terms of ordered $r$-Stirling partitions.

Theorem 2.3. Let $r \leq k \leq n$ and consider the set of monomials $M_{n,k}^{(r)}$ given by

$$M_{n,k}^{(r)} = \left\{ x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n} : (c_1, c_2, \ldots, c_n) = \text{code}(\sigma) \text{ for some } \sigma \in OP_{n,k}^{(r)} \right\}.$$ 1. The set $M_{n,k}^{(r)}$ is the standard basis for the $Q$-vector space $R_{n,k}^{(r)}$ with respect to the neglex monomial order.
2. The set $M_{n,k}^{(r)}$ is a $Z$-basis for the $Z$-module $S_{n,k}^{(r)}$.

Proof. 1. We begin by proving the inequality $\dim(R_{n,k}^{(r)}) \geq |OP_{n,k}^{(r)}|$ using a method of Garsia and Procesi [2]. Consider $k$ distinct rational numbers $\alpha_1, \ldots, \alpha_k$ and let $Y_{n,k}^{(r)} \subseteq Q^n$ be the family of points $(y_1, \ldots, y_n)$ such that

- $\{y_1, \ldots, y_n\} = \{\alpha_1, \ldots, \alpha_k\}$, and
- the coordinates $y_1, \ldots, y_r$ are distinct.

It is evident that $Y_{n,k}^{(r)}$ carries an action of the symmetric group product $S_r \times S_{n-r}$, and that this affords an identification of $Y_{n,k}^{(r)}$ with $OP_{n,k}^{(r)}$.

Let $I(Y_{n,k}^{(r)}) \subseteq Q[x_n]$ be the ideal of polynomials in $Q[x_n]$ which vanish on $Y_{n,k}^{(r)}$. We have

$$Q[x_n]/I(Y_{n,k}^{(r)}) \cong Q[Y_{n,k}^{(r)}] \cong Q[OP_{n,k}^{(r)}]$$

as $S_r \times S_{n-r}$-modules. If $f \in I(Y_{n,k}^{(r)})$ is nonzero, let $\tau(f)$ denote the homogeneous component of $f$ of highest degree and set

$$T(Y_{n,k}^{(r)}) := \langle \tau(f) : f \in I(Y_{n,k}^{(r)}) - \{0\} \rangle.$$
We have the further $S_r \times S_{n-r}$-module isomorphism
\begin{equation}
\mathbb{Q}[x_n]/T(Y_{n,k}) \cong \mathbb{Q}[x_n]/I(Y_{n,k}) \cong \mathbb{Q}[Y_{n,k}] \cong \mathbb{Q}[\mathcal{OP}_{n,k}].
\end{equation}

Proving the dimension inequality $\dim(R_{n,k}) \geq |\mathcal{OP}_{n,k}|$ therefore reduces to showing the containment $J_{n,k} \subseteq T(Y_{n,k})$; we do this by considering the generators of $J_{n,k}$.

- Let $1 \leq i \leq n$; we show that the monomial $x_i^k$ lies in $T(Y_{n,k})$. This follows from the fact that $(x_i - \alpha_1)(x_i - \alpha_2) \cdots (x_i - \alpha_k) \in I(Y_{n,k})$.

- We show that $e_n(x_n), e_{n-1}(x_n), \ldots , e_{n-k+1}(x_n) \in T(Y_{n,k})$. Indeed, introduce a new variable $t$ and consider the rational function
\begin{equation}
\frac{(1-x_1 t) \cdots (1-x_i t)}{(1-x_1 t) \cdots (1-x_k t)} = \sum_{i,j} (-1)^i e_i(x_n) h_j(\alpha_1, \ldots , \alpha_k) \cdot t^{i+j}.
\end{equation}

If $(x_1, \ldots , x_n) \in Y_{n,k}$ the factors of the denominator cancel with $k$ factors in the numerator, yielding a polynomial in $t$ of degree $n - k$. If $n - k + 1 \leq i \leq n$, taking the coefficient of $t^i$ on both sides leads to $e_i(x_n) \in T(Y_{n,k})$.

\begin{equation}
\frac{(1-x_1 t) \cdots (1-x_k t)}{(1-x_1 t) \cdots (1-x_r t)} = \sum_{i,j} (-1)^i e_i(\alpha_1, \ldots , \alpha_k) h_j(x_r) \cdot t^{i+j}.
\end{equation}

If $(x_1, \ldots , x_n) \in Y_{n,k}$, the factors in the denominator cancel with $r$ factors in the numerator, yielding a polynomial in $t$ of degree $k - r$. If $k - r + 1 \leq j \leq k$, taking the coefficient of $t^i$ on both sides leads to $h_j(x_r) \in T(Y_{n,k})$.

This completes the proof that $\dim(R_{n,k}) \geq |\mathcal{OP}_{n,k}|$.

Given any subset $S \subseteq [n]$ with reverse skip composition $\gamma(S)^* = (a_1, \ldots , a_n)$, let $x(S)^* := x_1^{a_1} \cdots x_n^{a_n}$ be the associated reverse skip monomial. By \cite[Sec. 3]{4}, we have $x(S)^* \in \text{in}_{<}(I_{n,k})$ whenever $S \subseteq [n]$ satisfies $|S| = n - k + 1$. Furthermore, the identities
\begin{equation}
h_d(x_1, \ldots , x_{i-1}, x_i) - x_i h_{d-1}(x_1, \ldots , x_{i-1}, x_i) = h_d(x_1, \ldots , x_{i-1})
\end{equation}
imply that $x_1^k, x_2^{k-1}, \ldots , x_r^{k-r-1} \in \text{in}_{<}(I_{n,k})$. Finally, we have $x_{r+1}^k, \ldots , x_{n-1}^k, x_n^k \in \text{in}_{<}(I_{n,k})$.

Theorem \ref{2.2} implies that the monomials in $M_{n,k}^{(r)}$ are precisely those monomials in $\mathbb{Q}[x_n]$ which are not divisible by any of the three classes of elements of $\text{in}_{<}(I_{n,k})$ listed above. Again by Theorem \ref{2.2} we have $\dim(R_{n,k}) \geq |\mathcal{OP}_{n,k}| = |M_{n,k}^{(r)}|$, so that $M_{n,k}^{(r)}$ is the standard basis of $R_{n,k}$.

2. From Item 1 of this theorem, we know that the set $M_{n,k}^{(r)}$ descends to a linearly independent subset of $S_{n,k}^{(r)}$; we need only show that $M_{n,k}^{(r)}$ descends to a $\mathbb{Z}$-spanning set of $S_{n,k}^{(r)}$. To this end, let $m$ be any monomial in $\mathbb{Z}[x_n]$. We show inductively that $m + J_{n,k}$ lies in the $\mathbb{Z}$-span of $M_{n,k}^{(r)}$. If $m \in M_{n,k}^{(r)}$ this is obvious. Otherwise, one of the following three things must be true:

1. There exists $1 \leq i \leq r$ such that $x_i^{k+i+1} \mid m$.
2. There exists $r + 1 \leq i \leq n$ such that $x_i^k \mid m$.
3. There exists $S \subseteq [n]$ with $|S| = n - k + 1$ such that $x(S)^* \mid m$.

If (1) holds, Equation \ref{2.20} implies $h_{k+i+1}(x_1, x_2, \ldots , x_i) \in J_{n,k}^{(r)}$. As a consequence, we have
\begin{equation}
x_i^{k+i+1} \equiv \text{a } \mathbb{Z}\text{-linear combination of monomials } x_i^{k+i+1} \text{ in neglex (mod } J_{n,k}^{(r)})
\end{equation}
If we multiply through by the monomial \(m/x_i^{k-i+1}\), we see that

\[
(2.22) \quad m \equiv a \text{ a } \mathbb{Z}\text{-linear combination of monomials } < m \text{ in neglex } \text{(mod } J_{n,k}^{(r)}),
\]

so that inductively we see that \(m + J_{n,k}^{(r)}\) lies in the span of \(\mathcal{M}_{n,k}^{(r)}\).

If (2) holds, then \(m \in J_{n,k}^{(r)}\), so certainly \(m + J_{n,k}^{(r)} = 0\) lies in the \(\mathbb{Z}\)-span of \(\mathcal{M}_{n,k}^{(r)}\).

If (3) holds, let \(\kappa_{\gamma(S)}(x_n) \in \mathbb{Z}[x_n]\) be the Demazure character attached to the reverse skip composition \(\gamma(S)^*\). This is a certain polynomial in the variables \(x_1, \ldots, x_n\) with nonnegative integer coefficients. The precise form of this polynomial is not important for us, but we have (see e.g. [4, Lem. 3.4])

\[
(2.23) \quad \kappa_{\gamma(S)}(x_n) = x(S)^* + a \text{ a } \mathbb{Z}\text{-linear combination of terms } < x(S)^* \text{ in neglex.}
\]

By [4, Lem 3.4] we have \(\kappa_{\gamma(S)}(x_n) \in J_{n,k}^{(r)}\), so that Equation \((2.23)\) implies

\[
(2.24) \quad x(S)^* \equiv a \text{ a } \mathbb{Z}\text{-linear combination of terms } < x(S)^* \text{ in neglex } \text{(mod } J_{n,k}^{(r)}).\]

If we multiply Equation \((2.24)\) through by the monomial \(m/x(S)^*\), we get

\[
(2.25) \quad m \equiv a \text{ a } \mathbb{Z}\text{-linear combination of terms } < m \text{ in neglex } \text{(mod } J_{n,k}^{(r)}).\]

so that inductively we see that \(m + J_{n,k}^{(r)}\) lies in the \(\mathbb{Z}\)-span of \(\mathcal{M}_{n,k}^{(r)}\). \(\square\)

When \(r = 0\), Theorem 2.3 is equivalent to a result of Haglund, Rhoades, and Shimozono [4, Thm. 4.13]. However, the proof of Theorem 2.3 is much more direct that of [4, Thm. 4.13] (and those in [4, Sect. 4] in general); whereas we associate an explicit standard basis element \(x_1^{(r)} \cdots x_n^{(r)}\) to any ordered set partition \(\sigma\), the description of the standard bases in [4] is recursive in nature. We exhibit this link between ordered set partitions and standard basis elements with an example.

**Example 2.4.** To illustrate Theorem 2.3 we give the standard basis of \(R_{4,3}^{(2)}\) with respect to neglex.

| \(\sigma\) | code(\(\sigma\)) | monomial | \(\sigma\) | code(\(\sigma\)) | monomial | \(\sigma\) | code(\(\sigma\)) | monomial |
|---|---|---|---|---|---|---|---|---|
| (1 | 2 | 34) | (2,1,0,2) | \(x_1^2x_2x_4^2\) | (1 | 34 | 2) | (2,0,0,1) | \(x_1^2x_4\) | (34 | 1 | 2) | (1,0,0,0) | \(x_1\) |
| (1 | 24 | 3) | (2,1,0,1) | \(x_1^2x_2x_4\) | (1 | 3 | 24) | (2,0,0,2) | \(x_1^2x_2^2\) | (3 | 14 | 2) | (1,0,0,1) | \(x_1x_4\) |
| (14 | 2 | 3) | (2,1,0,0) | \(x_1^2x_2^2\) | (14 | 3 | 2) | (2,0,0,0) | \(x_1^2\) | (3 | 1 | 24) | (1,0,0,2) | \(x_1x_2x_4\) |
| (1 | 3 | 24) | (2,1,2,0) | \(x_1^2x_2^2x_4^2\) | (1 | 4 | 23) | (2,0,2,0) | \(x_1^2x_2^2\) | (4 | 13 | 2) | (1,0,1,0) | \(x_1x_3\) |
| (13 | 2 | 4) | (2,1,1,0) | \(x_1^2x_2x_4^2\) | (13 | 4 | 2) | (2,0,1,0) | \(x_1^2x_4\) | (4 | 1 | 23) | (1,0,2,0) | \(x_1x_2^3\) |
| (2 | 1 | 34) | (1,1,0,2) | \(x_1x_2x_4^2\) | (2 | 34 | 1) | (0,1,0,1) | \(x_2x_4\) | (34 | 2 | 1) | (0,0,0,0) | 1 |
| (2 | 14 | 3) | (1,1,0,1) | \(x_1x_2x_4\) | (2 | 3 | 14) | (0,1,0,2) | \(x_2x_4^2\) | (34 | 21 | 1) | (0,0,0,1) | \(x_4\) |
| (24 | 1 | 3) | (1,1,0,0) | \(x_1x_2\) | (24 | 3 | 1) | (0,1,0,0) | \(x_2\) | (3 | 2 | 14) | (0,0,0,2) | \(x_4^2\) |
| (2 | 13 | 4) | (1,1,2,0) | \(x_1x_2x_3^2\) | (2 | 4 | 13) | (0,1,2,0) | \(x_2x_3^2\) | (4 | 23 | 1) | (0,0,1,0) | \(x_3\) |
| (23 | 1 | 4) | (1,1,1,0) | \(x_1x_2x_3\) | (25 | 4 | 1) | (0,1,1,0) | \(x_2x_3\) | (4 | 2 | 13) | (0,0,2,0) | \(x_3^2\) |

As an application of Theorem 2.3 we can describe the Hilbert series of \(R_{n,k}^{(r)}\) in terms of the coinv statistic.

**Corollary 2.5.** The Hilbert series of \(R_{n,k}^{(r)}\) is given by

\[
(2.26) \quad \text{Hilb}(R_{n,k}^{(r)}; q) = \sum_{\sigma \in \mathcal{OP}_{n,k}^{(r)}} q^{\text{coinv}(\sigma)}.
\]

As another application of Theorem 2.3 we can describe the ungraded isomorphism type of \(R_{n,k}^{(r)}\) as a module over \(S_r \times S_{n-r}\). When \(r = k = n\), this is Chevalley’s classical result [1] that the coinvariant ring is isomorphic to the regular representation of \(S_n\).

**Corollary 2.6.** We have an isomorphism of ungraded \(S_r \times S_{n-r}\)-modules

\[
(2.27) \quad R_{n,k}^{(r)} \cong \mathbb{Q}[\mathcal{OP}_{n,k}^{(r)}].
\]
It seems that the isomorphism type of $R_{n,k}^{(r)}$ as a graded $S_r \times S_{n-r}$-module can be described in terms of known graded modules by the (graded) tensor product decomposition

$$R_{n,k}^{(r)} \cong R_r \otimes \mathbb{C} \varepsilon_r R_{n,k}.$$  

In the conjectural isomorphism (2.28) of graded $S_r \times S_{n-r}$-modules,

- $R_r = \mathbb{Q}[x_r]/(e_1(x_r), \ldots, e_r(x_r))$ is the classical coinvariant ring in the first $r$ variables $x_r$, with its graded action of $S_r$,
- $R_{n,k} = R_{n,k}^{(0)}$ is the graded $S_n$-module $\mathbb{Q}[x_n]/(x_n^1, \ldots, x_n^k, e_n(x_n), \ldots, e_{n-k+1}(x_n))$, and
- $\varepsilon_r \in \mathbb{Q}[S_n]$ is the group algebra element

$$\varepsilon_r := \sum_{w \in S_r} \text{sign}(w) \cdot w$$

which antisymmetrizes over the subgroup $S_r \subseteq S_n$ (acting on the first $r$ letters), so that $S_{n-r}$ (acting on the last $n-r$ letters) commutes with $\varepsilon_r$ and therefore

- $\varepsilon_r R_{n,k}$ is naturally a $S_{n-r}$-module, and
- the action of the product group $S_r \times S_{n-r}$ on the tensor product is given by

$$\langle w_1 \times w_2, v_1 \otimes v_2 \rangle := (w_1.v_1) \otimes (w_2.v_2).$$

### 3. Line Arrangements and $r$-Stirling Partitions

We shift focus from algebra to geometry and initiate the study of the variety $X_{n,k}^{(r)}$. In order to study the variety $X_{n,k}^{(r)}$, we will need to break it into pieces in a reasonable way. For this we will use the notion of an affine paving (called a cellular decomposition in [5]).

If $X$ is any complex algebraic variety, an affine paving of $X$ is an ordered partition

$$X = C_0 \sqcup C_1 \sqcup \cdots \sqcup C_m$$

such that

- for all $i$, the union $C_0 \sqcup C_1 \sqcup \cdots \sqcup C_i$ is a closed subvariety of $X$, and
- $C_i$ is isomorphic as a variety to the affine space $\mathbb{C}^{n_i}$, for some integer $n_i$.

The $C_i$ are referred to as the cells of the affine paving and we will say that the partition $\{C_0, C_1, \ldots, C_m\}$ induces an affine paving of $X$. In this situation, the classes of the cell closures $\{[C_0], [C_1], \ldots, [C_m]\}$ give a $\mathbb{Z}$-basis for the (singular) cohomology ring $H^\bullet(X_{n,k}^{(r)})$.

The projective space $\mathbb{P}^{k-1}$ has an affine paving induced by the cells $\{C_1, C_2, \ldots, C_k\}$, where

$$C_i = \{[x_1 : x_2 : \cdots : x_k] \in \mathbb{P}^{k-1} : x_1 = \cdots = x_{i-1} = 0 \text{ and } x_i \neq 0\}.$$  

Taking products of these cells gives the standard affine paving of $([k]^{n})$ whose cells are indexed by words $w = w_1 \ldots w_n \in [k]^n$. Following [5], we will consider a different affine paving of $([k]^{n})$ whose cells are again indexed by words in $[k]^n$. In order to describe this paving, we will need some terminology.

Let $\text{Mat}_{k \times n}$ stand for the affine space of all complex $k \times n$ matrices $m$. Let $U_{n,k}^{(r)}$ be the Zariski open subset

$$U_{n,k}^{(r)} := \left\{ m \in \text{Mat}_{k \times n} : \begin{array}{l} \text{the matrix } m \text{ has full rank, no zero columns, and} \\ \text{the first } r \text{ columns of } m \text{ are linearly independent} \end{array} \right\}$$

If we let $T \subseteq GL_n$ be the rank $n$ diagonal torus, then $T$ acts freely on the columns of $U_{n,k}^{(r)}$ and we may identify the orbit space as $U_{n,k}^{(r)}/T = X_{n,k}^{(r)}$. Furthermore, we consider the larger Zariski open set $V_{n,k}$ given by

$$V_{n,k} := \left\{ m \in \text{Mat}_{k \times n} : m \text{ has no zero columns} \right\}.$$
This time we have the identification $V_{n,k}/T = (\mathbb{P}^{k-1})^n$.

Let $w = w_1 \ldots w_n \in [k]^n$ be a word in the letters $1, 2, \ldots, k$ of length $n$. An index $1 \leq j \leq n$ is called initial if $w_j$ is the first occurrence of its letter in $w$; let $\text{in}(w) = \{ j_1 < j_2 < \cdots < j_k \}$ be the set of initial indices in $w$. For example, if $w = 242141 \in [4]^6$ then $\text{in}(w) = \{1, 2, 4\}$. The $k \times n$ pattern matrix $PM(w)$ has entries in the set $\{0, 1, \ast\}$ as follows:

\begin{equation}
PM(w)_{i,j} = \begin{cases}
1 & \text{if } w_j = i \\
0 & \text{if the letter } i \text{ does not appear in } w \\
* & \text{if } j \in \text{in}(w), i < w_j, \text{ and there exists } j' < j \text{ such that } w_{j'} = i \\
0 & \text{if } j \in \text{in}(w) \text{ and } (i > w_j \text{ or there does not exist } j' < j \text{ such that } w_{j'} = i) \\
* & \text{if } j \notin \text{in}(w), i \neq w_j, \text{ and the first occurrence of } i \text{ in } w \text{ is before the first occurrence of } w_j \\
0 & \text{if } j \notin \text{in}(w), i \neq w_j, \text{ and the first occurrence of } i \text{ in } w \text{ is after the first occurrence of } w_j.
\end{cases}
\end{equation}

In our example, $PM(242141) = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & \ast \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & \ast \end{pmatrix}$.

For any word $w = w_1 \ldots w_n \in [k]^n$, let $\hat{C}_w$ be the family of all matrices obtained by replacing the $\ast$’s in $PM(w)$ by complex numbers. Furthermore, if $U \subset GL_k(\mathbb{C})$ denotes the unipotent subgroup of lower triangular matrices with $1$’s on the diagonal, let $U(w) \subset U$ be the subgroup of matrices given by

\begin{equation}
U(w) := \{ (u_{i,j}) \in U : u_{i,j} = 0 \text{ if } i > j \text{ and the letter } j \text{ does not appear in } w \}.
\end{equation}

In our example, matrices in $U(242141)$ have the form

$U(242141) = \begin{Bmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix} \end{Bmatrix}$.

We define $C_w \subseteq (\mathbb{P}^{k-1})^n$ by

\begin{equation}
C_w := \text{image of } U(w) \cdot \hat{C}_w \text{ in } (\mathbb{P}^{k-1})^n.
\end{equation}

**Proposition 3.1.** \([5]\) For any $k \leq n$, the set $\{ C_w : w \in [k]^n \}$ induces an affine paving of $(\mathbb{P}^{k-1})^n$.

The affine paving of Lemma 3.1 induces an affine paving of $(\mathbb{P}^{k-1})^n$. To describe this paving, we define $W_{n,k}^{(r)}$ to be the family of words $w = w_1w_2 \ldots w_n \in [k]^n$ such that the letters $1, 2, \ldots, k$ all appear in $w$ and that the first $r$ letters $w_1, w_2, \ldots, w_r$ of $w$ are distinct.

**Proposition 3.2.** The family of cells $\{ C_w : w \in W_{n,k}^{(r)} \}$ induces an affine paving of the variety $X_{n,k}^{(r)}$.

**Proof.** Let $w \in [k]^n$ be any word and consider the cell $C_w \subset (\mathbb{P}^{k-1})^n$. The definition of the pattern matrix $PM(w)$ implies that $C_w \subset X_{n,k}^{(r)}$ if $w \in W_{n,k}^{(r)}$ and $C_w \cap X_{n,k}^{(r)} = \emptyset$ otherwise. Now observe that the total order on the cells $\{ C_w : w \in [k]^n \}$ inducing the affine paving of Proposition 3.1 may be taken to start with those $w \notin W_{n,k}^{(r)}$ (in some order) and end with those $w \in W_{n,k}^{(r)}$ (in some order). The claim follows. \(\square\)
Our next task is to present the cohomology of $X_{n,k}^{(r)}$ as the quotient $S_{n,k}^{(r)}$ and describe the images of the $\mathbb{Z}$-basis $\{C_w : w \in W_{n,k}^{(r)}\}$ afforded by Proposition 3.2. We being by recalling the standard presentation of the cohomology of $(\mathbb{P}^{k-1})^n$.

The cohomology of $(\mathbb{P}^{k-1})^n$ is presented as
\begin{equation}
H^\bullet((\mathbb{P}^{k-1})^n) = \mathbb{Z}[x_1, \ldots, x_n],
\end{equation}
where $x_i$ represents the Chern class $c_1(\ell^*_i) \in H^2((\mathbb{P}^{k-1})^n)$ of the dual to the $i$th tautological line bundle $\ell_i \to (\mathbb{P}^{k-1})^n$.

Given a word $w \in [k]^n$, a polynomial representative for $C_w \in H^\bullet((\mathbb{P}^{k-1})^n)$ was calculated in [5]. In order to state it, we recall the classical Schubert polynomials attached to permutations in $S_n$.

The Schubert polynomials $\{\mathcal{S}_w : w \in S_n\}$ are defined recursively by
\begin{equation}
\begin{aligned}
\mathcal{S}_{w_0} &= x_1^{n-1}x_2^{n-2} \cdots x_n^0 \\
\mathcal{S}_{ws_i} &= \partial_i \mathcal{S}_w \\
\end{aligned}
\end{equation}
for $w_0 = n(n-1)\ldots 1$ and $w_i < w_{i+1}$. Here $ws_i$ is the permutation whose one-line notation $ws_i = w_1 \ldots w_{i+1} w_i \ldots w_n$ is obtained from that of $w$ by interchanging the letters in positions $i$ and $i+1$ and $\partial_i$ is the divided difference operator
\begin{equation}
\partial_i(f(x_1, \ldots, x_n)) = \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)}{x_i - x_{i+1}}.
\end{equation}

In order to extend Schubert polynomials from permutations in $S_n$ to words in $[k]^n$, we will need some notation. A word $w$ is called convex if it does not have a subword of the form $\ldots i \ldots j \ldots i \ldots$. Any word $w$ has a unique convexification $\text{conv}(w)$ which is characterized by being convex, having the same letter multiplicities as $w$, and having its initial letters appear in the same order from left to right. For example, we have $\text{conv}(242141) = 224411$. Furthermore, let $\sigma(w) \in S_n$ be the unique permutation with a minimal number of inversions which sorts $w$ to convex($w$); in our example $\sigma(242141) = 132546 \in S_6$.

Suppose $w = w_1 \ldots w_n \in [k]^n$ is a convex word with $m$ distinct letters. Let $\{i_1 < i_2 < \cdots < i_{k-m}\}$ be the letters in $[k]$ which do not appear in $w$. We define the standardization $\text{st}(w) = \text{st}(w)_1 \cdots \text{st}(w)_{n+k-m} \in S_{n+k-m}$ to be the permutation obtained from $w$ by fixing the initial letters of $w$, replacing the non-initial letters of $w$ from left to right with $k+1, k+2, \ldots, n+k-m$, and appending the sequence $i_1 i_2 \ldots i_{k-m}$ to the end. For example, if $(n,k) = (7,5)$ and $w = 334411$ then $\text{st}(w) = 364781925 \in S_9$.

Let $w \in [k]^n$ be an arbitrary word of length $n$ in the letters $1,2,\ldots,k$. The word Schubert polynomial $\mathcal{S}_w$ is defined by
\begin{equation}
\mathcal{S}_w := \sigma(w)^{-1} \mathcal{S}_{\text{st(\text{conv}(w))}}.
\end{equation}

Although the permutation $\text{st(\text{conv}(w))}$ will lie in a symmetric group of rank $> n$ when $w$ does not contain all of the letters $1,2,\ldots,k$, the polynomial $\mathcal{S}_w$ depends only on the variables $x_1, x_2, \ldots, x_n$ so that $\mathcal{S}_w \in \mathbb{Z}[x_n]$. Pawlowski and Rhoades proved [5] that the closure of the cell $C_w$ has the is represented by $\mathcal{S}_w$ under the presentation (3.8):
\begin{equation}
\text{conv}(C_w) \text{ is represented by } \mathcal{S}_w \text{ in } H^\bullet((\mathbb{P}^{k-1})^n).
\end{equation}

**Theorem 3.3.** Let $r \leq k \leq n$. The singular cohomology of $X_{n,k}^{(r)}$ may be presented as
\begin{equation}
H^\bullet(X_{n,k}^{(r)}) = S_{n,k}^{(r)}.
\end{equation}

Furthermore, under the presentation (3.13), if $w \in W_{n,k}^{(r)}$ the cell closure $\overline{C_w}$ is represented in $H^\bullet(X_{n,k}^{(r)})$ by $\mathcal{S}_w$. 
Proof. Consider the affine paving \( \{C_w : w \in [k]^n\} \) of \((\mathbb{P}^{k-1})^n\) afforded by Proposition 3.1. If \( w \not\in \mathcal{W}^{(r)}_{n,k} \), we have \( \overline{C_w} \cap X_{n,k} = \emptyset \). By Proposition 3.2, it follows that \( X_{n,k}^{(r)} \) is obtained from \((\mathbb{P}^{k-1})^n\) by excising the union of cell closures \( \bigcup_{w \in [k]^n - \mathcal{W}^{(r)}_{n,k}} C_w \). It follows (see [5]) that the cohomology ring \( H^*(X_{n,k}^{(r)}) \) may be presented as

\[
H^*(X_{n,k}^{(r)}) = H^*((\mathbb{P}^{k-1})^n)/J,
\]

where \( J \subseteq H^*((\mathbb{P}^{k-1})^n) \) is the ideal generated by those \( \{C_w\} \) for which \( w \in [k]^n - \mathcal{W}^{(r)}_{n,k} \). If we use the presentation of \( H^*((\mathbb{P}^{k-1})^n) \) given in (3.8) together with the polynomial representatives (3.12), we can write

\[
H^*(X_{n,k}^{(r)}) = \mathbb{Z}[x_n]/I,
\]

where \( I \subseteq \mathbb{Z}[x_n] \) is the ideal generated by \( x_1^k, x_2^k, \ldots, x_n^k \) together with \( \{S_w : w \in [k]^n - \mathcal{W}^{(r)}_{n,k}\} \).

**Claim:** We have \( J_{n,k}^{(r)} \subseteq I \).

To prove the Claim, we show that every generator of \( J_{n,k}^{(r)} \) lies in \( I \). We handle each type of generator separately.

- The generators \( x_1^k, x_2^k, \ldots, x_n^k \) of \( J_{n,k}^{(r)} \) are also generators of \( I \).

- For the generators \( e_{n-i+1}(x_n) \) (where \( 1 \leq i \leq k \)) of \( J_{n,k}^{(r)} \) we do the following. For \( 1 \leq i \leq k \) let \( w^i \) be the unique weakly increasing word in \([k]^n\) containing exactly the letters \([k]-\{i\}\) and whose first \( k-1 \) letters are distinct. For example, the word \( w^3 \in [5]^7 \) is \( w^3 = 1245555 \). Since \( i \) does not appear in \( w^i \), we have \( w^i \notin \mathcal{W}^{(r)}_{n,k} \), so that \( S_{w^i} \) is a generator of \( I \). Furthermore, we have
  \[
  \text{st}(\text{conv}(w^i)) = 12 \ldots (i-1)(i+1) \ldots n(n+1)i \in S_{n+1}
  \]
  which implies \( S_{w^i} = e_{n-i+1}(x_n) \).

- Finally, we consider the generators \( h_{k-i+1}(x_r) \) (where \( 1 \leq i \leq r \)) of \( J_{n,k}^{(r)} \). These generators are not in general generators of \( I \), but we show that they nevertheless are contained in \( I \).

  For \( 1 \leq i \leq r-1 \), let \( v^i \in [k]^n \) be the following weakly increasing word:
  \[
  v^i = 12 \ldots (i-1)ii(i+1)(i+2) \ldots (k-1)k \ldots k.
  \]
  For example, the word \( v^3 \in [5]^7 \) is \( v^3 = 12334555 \). Since \( k < n \), every letter in \([k]\) appears in \( v^i \). However, since the first \( r \) letters of \( v^i \) are not distinct, we have \( v^i \notin \mathcal{W}^{(r)}_{n,k} \), so that \( S_{v^i} \) is a generator of \( I \). We have
  \[
  \text{st}(\text{conv}(v^i)) = 12 \ldots (i-1)i(k+1)(i+1)(i+2) \ldots n \in S_n
  \]
  which implies \( S_{v^i} = h_{k-i}(x_{i+1}) \).

  The above paragraph shows that
  \[
  h_{k-r+1}(x_r), h_{k-r+2}(x_{r-1}), \ldots, h_{k-1}(x_2) \in I.
  \]
  The variable power \( h_k(x_1) = x_1^k \) also lies in \( I \). The identity

\[
(3.16) \quad h_d(x_1, \ldots, x_{i-1}, x_i) = x_i \cdot h_{d-1}(x_1, \ldots, x_{i-1}, x_i) + h_d(x_1, \ldots, x_{i-1})
\]

together with the fact that \( I \) is an ideal in \( \mathbb{Z}[x_n] \) can be used to show that

\[
(3.16) \quad h_{k-r+1}(x_r), h_{k-r+2}(x_{r-1}), \ldots, h_k(x_r) \in I,
\]

which is what we wanted to show. This completes the proof of the Claim.
By our Claim, we have a canonical surjection of $\mathbb{Z}$-modules
\begin{equation}
S_{n,k}^{(r)} = \mathbb{Z}[x_n]/J_{n,k}^{(r)} \twoheadrightarrow \mathbb{Z}[x_n]/I = H^\bullet(X_{n,k}^{(r)}).
\end{equation}
By Theorem 2.3, the module $S_{n,k}^{(r)}$ is a free $\mathbb{Z}$-module of rank $|OP_{n,k}^{(r)}|$. By Proposition 3.2, the cohomology ring $H^\bullet(X_{n,k}^{(r)})$ is a free $\mathbb{Z}$-module of rank $|W_{n,k}^{(r)}|$. Since we have $|OP_{n,k}^{(r)}| = |W_{n,k}^{(r)}|$ and any surjection between free $\mathbb{Z}$-modules of the same rank must be an isomorphism, we obtain the presentation (3.13) of the cohomology of $X_{n,k}^{(r)}$. The last sentence of the theorem follows from (3.12).

The cohomology representatives of the cell closures in any affine paving of a variety $X$ give rise to a $\mathbb{Z}$-basis for the cohomology ring $H^\bullet(X)$. Theorem 3.3 therefore yields the following immediate corollary.

**Corollary 3.4.** Let $r \leq k \leq n$. The set of polynomials $\{\mathcal{G}_w : w \in W_{n,k}^{(r)}\}$ descends to a $\mathbb{Z}$-basis for $S_{n,k}^{(r)}$.

4. Acknowledgements

B. Rhoades was partially supported by NSF Grant DMS-1500838. A. T. Wilson was partially supported by an NSF Mathematical Sciences Postdoctoral Research Fellowship. We thank Jeff Remmel for his mathematics, mentoring, and friendship.

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