Including resummation in the NLO BK equation

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Outline

Outline of the talk:

- NLO BK equation
- Numerical result: In $r$ divergence
- Double and single log resummation
- Numerical results: resummation only and resummation + finite terms
“Next-to-leading order evolution of color dipoles,” I. Balitsky and G. A. Chirilli, Phys. Rev. D 77 (2008) 014019, [arXiv:0710.4330 [hep-ph]].

“Direct numerical solution of the coordinate space Balitsky-Kovchegov equation at next to leading order,” T. L., H. Mäntysaari, Phys. Rev. D 91 (2015) 074016, [arXiv:1502.02400 [hep-ph]].

“Resumming double logarithms in the QCD evolution of color dipoles,” E. Iancu et al Phys. Lett. B 744 (2015) 293, [arXiv:1502.05642 [hep-ph]].

“Collinearly-improved BK evolution meets the HERA data,” E. Iancu et al Phys. Lett. B 750 (2015) 643, [arXiv:1507.03651 [hep-ph]].

“Next-to-leading order Balitsky-Kovchegov equation with resummation,” T. L., H. Mäntysaari, Phys. Rev. D 93 (2016) 094004, [arXiv:1601.06598 [hep-ph]].
“Next-to-leading order evolution of color dipoles,” I. Balitsky and G. A. Chirilli, Phys. Rev. D 77 (2008) 014019, [arXiv:0710.4330 [hep-ph]].

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Motivation

Many ingredients available for NLO small-$x$ calculations:

- NLO BK equation
- NLO JIMWLK equation
- NLO $\gamma^*$ impact factor for DIS
- NLO single inclusive cross section for forward pA
- ... 

Armed with these, want phenomenology @ NLO!
But first need to solve the evolution equation(s)!
The NLO BK equation
as derived by Balitsky and Chirilli, 2007

**Equation**

\[ \partial_y S(r) = \frac{\alpha_s N_C}{2\pi^2} K_1 \otimes [S(X)S(Y) - S(r)] + \frac{\alpha_s^2 N_F N_C}{8\pi^4} K_f \otimes S(Y) [S(X') - S(X)] \]

\[ + \frac{\alpha_s^2 N_C^2}{8\pi^4} K_2 \otimes [S(X)S(z - z')S(Y') - S(X)S(Y)] \]

**Notations & approximations**

\[ S(x - y) \equiv \langle \text{Tr} \ U^\dagger(x) U(y) \rangle \]

\[ \otimes = \int d^2z / \int d^2z d^2z' \]

▶ Here large \( N_C \) & mean field:

\[ \langle \text{Tr} \ U^\dagger U \text{Tr} \ U^\dagger U \rangle \rightarrow \langle \text{Tr} \ U^\dagger U \rangle \langle \text{Tr} \ U^\dagger U \rangle \]

**Coordinates**
The NLO BK equation
as derived by Balitsky and Chirilli, 2007

**Equation**

\[
\partial_y S(r) = \frac{\alpha_s N_C}{2\pi^2} K_1 \otimes [S(X)S(Y) - S(r)] + \frac{\alpha_s^2 N_F N_C}{8\pi^4} K_f \otimes S(Y)[S(X') - S(X)] \\
+ \frac{\alpha_s^2 N_C^2}{8\pi^4} K_2 \otimes [S(X)S(z - z')S(Y') - S(X)S(Y)]
\]

**Notations & approximations**

\[S(x - y) \equiv \langle \text{Tr} \, U^\dagger(x)U(y) \rangle\]

\[\otimes = \int d^2 z \quad / \quad \int d^2 z \, d^2 z'\]

- Here large \( N_C \) & mean field:

\[\langle \text{Tr} \, U^\dagger \, U \, \text{Tr} \, U^\dagger \, U \rangle \rightarrow \langle \text{Tr} \, U^\dagger \, U \rangle \langle \text{Tr} \, U^\dagger \, U \rangle\]
Kernels

\[ K_1 = \frac{r^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_C}{4\pi} \left( \frac{\beta}{N_C} \ln r^2 \mu^2 - \frac{\beta}{N_C} \frac{X^2 - Y^2}{r^2} \ln \frac{X^2}{Y^2} + \frac{67}{9} - \frac{\pi^2}{3} - \frac{10 N_F}{9 N_C} - 2 \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right) \right] \]

\[ K_2 = -\frac{2}{(Z - Z')^4} + \left[ \frac{X^2 Y'^2 + X'^2 Y^2 - 4r^2(Z - Z')^2}{(Z - Z')^4(X^2 Y'^2 - X'^2 Y^2)} \right] \]

\[ + \frac{r^4}{X^2 Y'^2(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^2}{X^2 Y'^2(Z - Z')^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \]

\[ K_f = \frac{2}{(Z - Z')^4} - \frac{X'^2 Y^2 + Y'^2 X^2 - r^2(Z - Z')^2}{(Z - Z')^4(X^2 Y'^2 - X'^2 Y^2)} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \]
Kernels

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\]
\[
+ \left. \frac{67}{9} - \frac{\pi^2}{3} - \frac{10 N_F}{9 N_C} - 2 \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right) \right]
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+ \frac{r^4}{X^2 Y'^2(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^2}{X^2 Y'^2(z - z')^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2}
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\]

▶ Leading order
Kernels

\[ K_1 = \frac{r^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} \left( \frac{\beta}{N_c} \ln r^2 \mu^2 - \frac{\beta}{N_c} \frac{X^2 - Y^2}{r^2} \ln \frac{X^2}{Y^2} \right.ight. \\
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+ \left. \frac{r^4}{X^2 Y'^2(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^2}{X^2 Y'^2(z - z')^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \\
K_f = \frac{2}{(z - z')^4} - \frac{X'^2 Y^2 + Y'^2 X^2 - r^2(z - z')^2}{(z - z')^4(X^2 Y'^2 - X'^2 Y^2)} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \\

▶ Leading order
▶ Running coupling (Terms with \( \beta \) function coefficient)
Kernels

\[ K_1 = \frac{r^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_C}{4\pi} \left( \frac{\beta}{N_C} \ln r^2 \mu^2 - \frac{\beta}{N_C} \frac{X^2 - Y^2}{r^2} \ln \frac{X^2}{Y^2} \right. \right. \\
\left. \left. + \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9} \frac{N_F}{N_C} - 2 \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right) \right] \]

\[ K_2 = -\frac{2}{(Z - Z')^4} + \left[ \frac{X^2 Y'^2 + X'^2 Y^2 - 4r^2(Z - Z')^2}{(Z - Z')^4(X^2 Y'^2 - X'^2 Y^2)} \right. \\
\left. + \frac{r^4}{X^2 Y'^2(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^2}{X^2 Y'^2(Z - Z')^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \]

\[ K_f = \frac{2}{(Z - Z')^4} - \frac{X'^2 Y^2 + Y'^2 X^2 - r^2(Z - Z')^2}{(Z - Z')^4(X^2 Y'^2 - X'^2 Y^2)} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \]

- **Leading order**
- **Running coupling** (Terms with \( \beta \) function coefficient)
- **Conformal logs** \( \Rightarrow \) vanish for \( r = 0 \) (\( X = Y \) & \( X' = Y' \))
Kernels

\[ K_1 = \frac{r^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_C}{4\pi} \left( \frac{\beta}{N_C} \ln r^2 \mu^2 - \frac{\beta}{N_C} \frac{X^2 - Y^2}{r^2} \ln \frac{X^2}{Y^2} \right. \right. \]
\[ \left. \left. + \frac{67}{9} - \frac{\pi^2}{3} - \frac{10 N_F}{9 N_C} - 2 \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right) \right] \]

\[ K_2 = -\frac{2}{(z - z')^4} + \left[ \frac{X^2 Y'^2 + X'^2 Y^2 - 4r^2(z - z')^2}{(z - z')^4(X^2 Y'^2 - X'^2 Y^2)} \right. \]
\[ \left. + \frac{r^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{r^2}{X^2 Y'^2 (z - z')^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \]

\[ K_f = \frac{2}{(z - z')^4} - \frac{X'^2 Y^2 + Y'^2 X^2 - r^2(z - z')^2}{(z - z')^4(X^2 Y'^2 - X'^2 Y^2)} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \]

- **Leading order**
- **Running coupling** (Terms with \( \beta \) function coefficient)
- **Conformal logs** \( \Rightarrow \) vanish for \( r = 0 \) (\( X = Y \) & \( X' = Y' \))
- **Nonconformal double log** \( \Rightarrow \) blows up for \( r = 0 \)
Running coupling

Absorb the $\beta$-terms into
- “Balitsky” running for LO term
- Parent dipole running for NLO terms

Now:

$$\frac{\alpha_s N_c}{2\pi^2} K_1 = \frac{\alpha_s(r) N_c}{2\pi^2} \left[ \frac{r^2}{X^2 Y^2} + \frac{1}{X^2} \left( \frac{\alpha_s(X)}{\alpha_s(Y)} - 1 \right) + \frac{1}{Y^2} \left( \frac{\alpha_s(Y)}{\alpha_s(X)} - 1 \right) \right]$$

$$+ \frac{\alpha_s(r)^2 N_c^2}{8\pi^3} \frac{r^2}{X^2 Y^2} \left[ \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9} \frac{N_F}{N_c} - 2 \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right]$$

Freeze coupling in the IR to $\alpha_s(r \to \infty) = 0.76$:

$$\alpha_s(r) = \frac{4\pi}{\beta \ln \left\{ \left[ (2.5)^{10} + \left( \frac{4e^{-2\gamma_E}}{r^2 \Lambda_{QCD}^2} \right) \right]^{0.2} \right\}} \sim \frac{4\pi}{\beta \ln \left\{ \left( \frac{4e^{-2\gamma_E}}{r^2 \Lambda_{QCD}^2} \right) \right\}}$$
Initial condition

\[ N(r) = 1 - S(r) = 1 - \exp \left[ -\frac{(r^2 Q_s^2)^\gamma}{4} \ln \left( \frac{1}{r\Lambda_{QCD}} + e \right) \right], \]

2 tunable parameters

- \( \frac{Q_s}{\Lambda_{QCD}} \) \( \to \) basically determines value of \( \alpha_s \) at \( y = 0 \)

- \( \gamma \): anomalous dimension: shape \( N(r) \sim r^{2\gamma} \)
  - LO phenomenology prefers \( \gamma \gtrsim 1 \) at \( y = 0 \)
  - At LO this eventually evolves into \( \gamma \sim 0.8 \) (running \( \alpha_s \))
Large double log

Relative change of $N(r)$ in one step $\frac{dy}{dy}$ diverges for small $r$

Behavior caused by the nonconformal double log term

T.L., H. Mäntysaari 2015
Double log resummation

Integral form of BK: double log $\sim$ shift of rapidity variable

$$\partial_y S_y(r) = \alpha_s \int_z \frac{r^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c}{2\pi} \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right] [S_y(X)S_y(Y) - S_y(r)]$$

$$\iff$$

$$S_y(r) = \alpha_s \int^y dy' \int_z \frac{r^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c}{2\pi} \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right] [S_{y'}(X)S_{y'}(Y) - S_{y'}(r)]$$

$$\iff$$

$$S_y(r) = \alpha_s \int d^2 z \int^{y - \ln z^2/r^2} dy' \frac{r^2}{X^2 Y^2} [S_{y'}(X)S_{y'}(Y) - S_{y'}(r)]$$

For $r \ll z \sim 1/Q_s$. $\Rightarrow$ “Kinematical constraint” Beuf Iancu et al 2015 rewrite as practical rapidity-local form

$$\left[ 1 + \frac{\alpha_s N_c}{2\pi} \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right] \Rightarrow J_1 \left( 2\sqrt{\frac{\alpha_s N_c}{\pi}} \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right)$$

with $\int^y dy'$
Single log resummation

Iancu et al 2015: there is also a single log, with DGLAP anomalous dimension $A_1 = 11/12$:

$$\partial_y S(r) = \alpha_s^2 \int_{zz'} \left[ \frac{2}{(z - z')^4} + \frac{X^2 Y'^2 + X'^2 Y^2 - 4r^2(z - z')^2}{(z - z')^4(X^2 Y'^2 - X'^2 Y^2)} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right]$$

$$\times [S(X)S(z - z')S(Y') - S(X)S(Y)] + \ldots$$

$$\sim \alpha_s^2 A_1 \int_z \frac{r^2}{X^2 Y^2} \ln \frac{\min\{X^2, Y^2\}}{r^2} + \ldots$$

Should be resummed into

$$K_{STL} = \exp \left\{ -\frac{\alpha_s N_c A_1}{\pi} \left| \ln \frac{C_{\text{sub}} r^2}{\min\{X^2, Y^2\} r^2} \right| \right\}$$

- Iancu et al take $C_{\text{sub}} = 1$ and throw other NLO terms away
- This captures leading logs, but leaves other $\alpha_s^2$-terms unknown, and result dependent on arbitrary $C_{\text{sub}}$
Full equation

\[ \partial_y S(r) = \frac{\alpha_s N_C}{2\pi^2} K_1 \otimes [S(X)S(Y) - S(r)] \]

\[ + \frac{\alpha_s^2 N_C^2}{8\pi^4} K_2 \otimes [S(X)S(z - z')S(Y') - S(X)S(Y)] + N_F\text{-part} \]

Now solve resummed equation

\[ \partial_y S(r) = \frac{\alpha_s N_C}{2\pi^2} \left[ K_{DLA} K_{STL} K_{Bal} - K_{sub} + K_{1\text{fin}} \right] \otimes [S(X)S(Y) - S(r)] \]

\[ + \frac{\alpha_s^2 N_C^2}{8\pi^4} K_2 \otimes [S(X)S(z - z')S(Y') - S(X)S(Y)] + N_F\text{-part} \]

- \( K_{DLA} \sim J_1(x)/x \) would give double log in \( K_1 \) if expanded in \( \alpha_s \)
- \( K_{Bal} \) resums \( \beta \)-function terms in \( K_1 \)
- \( K_{STL} = \exp \left\{ -\alpha_s N_C A_1 \# \ln r^2 \right\} \) resums single log
- \( K_{sub} \) subtracts \( \alpha_s \)-part of \( K_{STL} \), which is already in \( K_2 \)
- \( K_{1\text{fin}} \) is rest (nonlog) part in \( K_1 \)
Resummation and rest

\[ \partial_y S(r) = \frac{\alpha_s N_c}{2\pi^2} \left[ K_{DLA} K_{STL} K_{Bal} - K_{sub} + K_1^{\text{fin}} \right] \otimes [S(X)S(Y) - S(r)] + \frac{\alpha_s^2 N_c^2}{8\pi^4} K_2 \otimes [S(X)S(z - z')S(Y') - S(X)S(Y)] + N_F\text{-part} \]

Split this into three parts

\begin{itemize}
    \item \textbf{LO} (running coupling) \( \frac{\alpha_s N_c}{2\pi^2} K_{Bal} \): used so far
    \item \textbf{Resummation} \( \frac{\alpha_s N_c}{2\pi^2} K_{Bal} [K_{DLA} K_{STL} - 1] \): (\( \sim \) used by Iancu et al)
    \item \textbf{Finite NLO} rest: \(-K_{sub} + K_1^{\text{fin}}\) and \(K_2, K_f\)
\end{itemize}

\begin{itemize}
    \item Separation depends on \(C_{sub}\) in
      \[ K_{STL} = \exp \left\{ -\frac{\alpha_s N_c A_1}{\pi} \ln \frac{C_{sub} r^2}{\min\{X^2, Y^2\}} r^2 \right\} \]
    \item Numerically \textbf{choose} \(C_{sub}\) to \textbf{minimize} the finite NLO part:
      result \(C_{sub} = 0.65\)
\end{itemize}
Contribution to evolution speed

\( y = 0 : \text{finite NLO small} \quad y = 10 : \text{finite NLO negligible} \)

Finite NLO terms

- Significant for small \( y \)
  - Phenomenology
- Negligible at \( y \to \infty \)
Anomalous dimension

Recall initial condition: 

\[ N(r) = 1 - e^{-\frac{(r^2 Q_s^2)^\gamma}{4} \ln\left(\frac{1}{r\Lambda_{QCD}} + e\right)}}, \]

Define

\[ \gamma(r) \equiv -\frac{d \ln N(r)}{d \ln r^2} \]

**Geometric scaling?**

- **LO:** fast to \( \gamma \sim 0.8 \)
- **NLO:** stay at initial \( \gamma \)

**Solid:** initial condition

**Dotted:** \( y = 5 \) NLO

**Dot-dashed:** \( y = 5 \) LO (rc)
Anomalous dimension

Recall initial condition: \( N(r) = 1 - e^{-\frac{(r^2 Q_s^2)^\gamma}{4} \ln \left( \frac{1}{r^2 \Lambda_{QCD}} + e \right)} \),

Define
\[
\gamma(r) \equiv -\frac{d \ln N(r)}{d \ln r^2}
\]

Geometric scaling?

- LO: fast to \( \gamma \sim 0.8 \)
- NLO: stay at initial \( \gamma \)

LO \( y = 0 \) to \( y = 5 \)

Solid: initial condition

Dotted: \( y = 5 \) NLO

Dot-dashed: \( y = 5 \) LO (rc)
Conclusions

Numerical solution of NLO BK equation

► Resummation
  ► Resummation of double $\perp$ logs stabilizes equation
  ► Can also resum single $\perp$ logs, ambiguity in constant $C_{\text{sub}}$
  ► Numerically: $C_{\text{sub}} = 0.65$ minimizes finite NLO

► Properties of solution
  ► Finite NLO terms important for realistic $Q_s/\Lambda_{\text{QCD}}$
  ► Geometric scaling: sets in slowly if at all

Outlook:
  ► Fit DIS
  ► Apply to $pA$