Well-posedness and global solutions to the higher order
Camassa-Holm equations with fractional inertia operator in
Besov space

Weikui Ye$^1$ and Zhaoyang Yin$^{2,3}$

$^1$Institute of Applied Physics and Computational Mathematics,
P.O. Box 8009, Beijing 100088, P. R. China

$^2$Department of Mathematics, Sun Yat-sen University,
Guangzhou, 510275, China

$^3$Faculty of Information Technology,
Macau University of Science and Technology, Macau, China

Abstract

In this paper, we study well-posedness and the global solutions to the higher-order Camassa-Holm equations with fractional inertia operator in Besov space. When $a \in \left(\frac{1}{2}, 1\right)$, we prove the existence of the solutions in space $B^{s}_{p,1}(\mathbb{R})$ with $s \geq 1 + \frac{1}{p}$ and $p < \frac{1}{a - \frac{1}{2}}$, the existence and uniqueness of the solutions in space $B^{s}_{p,1}(\mathbb{R})$ with $s \geq 1 + 2a - \min\left\{\frac{1}{p}, \frac{1}{p'}\right\}$, and the local well-posedness in space $B^{s}_{p,1}(\mathbb{R})$ with $s > 1 + 2a - \min\left\{\frac{1}{p}, \frac{1}{p'}\right\}$. When $a > 1$, we obtain the existence of the solutions in space $B^{s}_{p,1}(\mathbb{R})$ with $s \geq a + \max\left\{\frac{1}{p}, \frac{1}{2}\right\}$ and the local well-posedness in space $B^{s}_{p,1}(\mathbb{R})$ with $s \geq 1 + a + \max\left\{\frac{1}{p}, \frac{1}{2}\right\}$. Moreover, we obtain two results about the global solutions.

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1 Introduction and Main results

The two-component higher order Camassa-Holm systems with fractional inertia operator was firstly introduced by Escher and Lyons to describe the geometrical geodesic flow on an appropriate infinite dimensional Lie group [30], especially for the cases that $s$ is a fractional number larger than $1$. It is the following system:

$$
\begin{align*}
    &m_t + 2u_xm + um_x = \beta u_x - \kappa \rho_x, \quad t > 0, \ x \in \mathbb{R}, \\
    &\rho_t + (u\rho)_x = 0, \quad t > 0, \ x \in \mathbb{R}, \\
    &m(t, x) = (1 - \partial_x^2)^s u(t, x), \quad t \geq 0, \ x \in \mathbb{R}, \\
    &u(0, x) = u_0(x), \quad x \in \mathbb{R}, \\
    &\rho(0, x) = \rho_0(x), \quad x \in \mathbb{R},
\end{align*}
$$

(1.1)

where $s \geq 0$ is a constant, $\beta$ is a constant which represents the vorticity of underlying flow, and $\kappa > 0$ is an arbitrary real parameter. This model attracted a lot of attention. In [8], the authors study the local well-posedness in Besov spaces with high-regularity. In [37], the authors established the local-wellposedness in Besov spaces with lower-regularity, they also give the global solutions of (1.1) for the case $a = 2, \kappa > 0$. In [38], the authors study the local and global analyticity and Gevrey regularity of (1.1).

We can take it as the generalization of the shallow water wave systems introduced in the nonlinear dynamics to describe nonlinear phenomena in shallow water.
Indeed, for \( a = 1 \), System \((1.1)\) becomes the two-component Camassa-Holm system:

\[
\begin{align*}
\dot{m} + 2u_x m + um_x &= \beta u_x - \kappa \rho \rho_x, & t > 0, \ x \in \mathbb{R}, \\
\rho_t + (u \rho)_x &= 0, & t > 0, \ x \in \mathbb{R}, \\
m(t, x) &= u(t, x) - u_{xx}(t, x), & t \geq 0, \ x \in \mathbb{R}, \\
u(0, x) &= u_0(x), & x \in \mathbb{R}, \\
\rho(0, x) &= \rho_0(x), & x \in \mathbb{R}.
\end{align*}
\]

The Cauchy problems of \(2\text{CH} \ (1.2)\) with \( \kappa = -1 \) and with \( \kappa = 1 \) have been discussed in \([29]\) and \([19]\), respectively. A new global existence result and several new blow-up results of strong solutions for the Cauchy problem of \((1.2)\) with \( \kappa = 1 \) were obtained in \([34]\). Moreover, the authors studied the existence of global weak solutions and uniqueness of conservative weak solutions to \((1.2)\) in \([35, 46]\), respectively.

In this paper, we consider the case with \( \rho \equiv 0 \) and \( \beta = 0 \) in \((1.1)\), then we get the higher order Camassa-Holm equation with fractional-inertia operator:

\[
\begin{align*}
\dot{m} + 2u_x m + um_x &= 0, & t > 0, \ x \in \mathbb{R}, \\
m(t, x) &= (1 - \partial_x^2)\alpha u(t, x), & t \geq 0, \ x \in \mathbb{R}, \\
u(0, x) &= u_0(x), & x \in \mathbb{R}.
\end{align*}
\]

For \( a = 1 \), Eq. \((1.3)\) becomes the remarkable Camassa-Holm(CH) equation

\[
\begin{align*}
u_t - u_{xxt} + 2u_x(u - u_{xx}) + u(u - u_{xx})_x - \alpha u_x &= 0,
\end{align*}
\]

modeling the unidirectional propagation of shallow water waves over a flat bottom. Here \( u(t, x) \) stands for the fluid velocity at time \( t \) in the spatial \( x \) direction \([6, 22, 28, 40, 41, 42]\). The Camassa-Holm equation is also a model for the propagation of axially symmetric waves in hyperelastic rods \([25, 26]\). It has a bi-Hamiltonian structure \([9, 33]\) and is completely integrable \([6, 11]\). Also there is a geometric interpretation of Eq.(1.1) in terms of geodesic flow on the diffeomorphism group of the circle \([21, 43]\).

One of the main features of the CH equation is that it precesses the peaked wave solution

\[
u(t, x) = ce^{-|x-ct|}
\]

as the solitary weak solution \([7]\). They are orbitally stable and interact like solitons \([2, 24]\). The peaked traveling waves replicate a characteristic for the waves of great height – waves of largest amplitude that are exact solutions of the governing equations for water waves, cf. \([12, 18, 50]\). Recently, it was claimed in the paper \([44]\) that the equation might be relevant to the modeling of tsunami, see also the discussion in \([20]\). The Cauchy problem and initial-boundary value problem for the CH equation have been studied extensively \([13, 15, 27, 31, 32, 45, 47, 48, 53]\). It has been shown that this equation is

1  INTRODUCTION AND MAIN RESULTS
locally well-posed \([13, 15, 27, 45, 47, 48]\) for initial data \(u_0 \in H^r(\mathbb{R}), r > \frac{a}{2}\) and locally ill-posed \([36]\) for initial data \(u_0 \in H^{\frac{3}{2}}(\mathbb{R})\). More interestingly, it has global strong solutions \([10, 13, 15]\) and also finite time blow-up solutions \([10, 13, 15, 17, 27, 47, 48]\). On the other hand, it has global weak solutions in \(H^1(\mathbb{R})\) \([4, 5, 16, 23, 52]\). Uniqueness of the conservative weak solutions to Camassa-Holm equation has been proved in \([3]\). The advantage of the Camassa-Holm equation in comparison with the KdV equation lies in the fact that the Camassa-Holm equation has peaked solitons and models wave breaking \([7, 14]\) (by wave breaking we understand that the wave remains bounded while its slope becomes unbounded in finite time \([51]\)).

In this paper, we first study about the local well-posedness for Eq. \((1.3)\) with initial data \(u_0 \in B^s_{p,1}(\mathbb{R})\) in the Hardmard sense. If \(\frac{1}{2} \leq a < 1\), we get the existence of the solution in space \(B^s_{p,1}(\mathbb{R})\) with \(s \geq 1 + \frac{1}{p}\) and \(p < \frac{1}{a - \frac{1}{2}}\), the existence and uniqueness of the solutions in space \(B^s_{p,1}(\mathbb{R})\) with \(s \geq 1 + 2a - \min\{\frac{1}{p}, \frac{1}{2p}\}\), and the well-posedness in space \(B^s_{p,1}(\mathbb{R})\) with \(s > 1 + 2a - \min\{\frac{1}{p}, \frac{1}{2p}\}\).

When \(a > 1\), we obtain the existence of the solutions in space \(B^s_{p,1}(\mathbb{R})\) with \(s \geq a + \max\{\frac{1}{p}, \frac{1}{2}\}\) and the well-posedness in space \(B^s_{p,1}(\mathbb{R})\) with \(s \geq 1 + a + \max\{\frac{1}{p}, \frac{1}{2}\}\). Moreover, if \(m_0\) satisfies some sign condition, we can obtain two results about global solution. The proof of these local-posedness results give us a method to study the well-posedness of system \((1.1)\), maybe we can a better well-posedness result than that in \([37]\). And we will study this problem in the next paper.

The first result is about the local well-posedness with \(\frac{1}{2} \leq a < 1\) in critical Besov space which can be stated as follow.

**Theorem 1.1.** Let \(u_0 \in B^s_{p,1}(\mathbb{R})\) with \(\frac{1}{2} \leq a < 1\). There exists a \(T\) such that

(1) **Existence:** If \(s \geq 1 + \frac{1}{p}\) and \(p < \frac{1}{a - \frac{1}{2}}\), then the system \((1.3)\) have a solution \(u\) belongs to \(E_p(T) = C([0, T]; B^s_{p,1}(\mathbb{R}))) \cap C^1([0, T]; B^{s-1}_{p,1}(\mathbb{R})))\).

(2) **Uniqueness:** If \(s \geq 1 + 2a - \min\{\frac{1}{p}, \frac{1}{2p}\}\), then the system \((1.3)\) have a unique solution \(u\) belongs to \(E_p(T)\).

(3) **Continuous dependence:** If \(s > 1 + 2a - \min\{\frac{1}{p}, \frac{1}{2p}\}\), then the data-to-solution map \(S_t(u_0)\) is continuous in \(B^s_{p,1}(\mathbb{R})\).

**Theorem 1.2.** Let \(u_0 \in B^s_{p,1}(\mathbb{R})\) with \(a > 1\). There exists a \(T\) such that

(1) **Existence:** If \(s \geq a + \max\{\frac{1}{p}, \frac{1}{2}\}\), then the system \((1.3)\) have a solution \(u\) belongs to \(E_p(T) = C([0, T]; B^s_{p,1}(\mathbb{R}))) \cap C^1([0, T]; B^{s-1}_{p,1}(\mathbb{R})))\).

(2) **Uniqueness and continuous dependence:** If \(s \geq 1 + a + \max\{\frac{1}{p}, \frac{1}{2}\}\), then the system \((1.3)\) have a unique solution \(u\) belongs to \(E_p(T)\). Moreover, the data-to-solution map \(S_t(u_0)\) is continuous in
Theorem 1.3. Let $u_0 \in H^s$ with $s \geq 2a + \frac{1}{2}$. Let $u(t, x)$ be the corresponding local solution of (1.3) with $a > 1$, if the initial data $m_0 := (1 - \partial_{xx})^a u_0 \in L^1$ and $m_0 \geq 0 (or \leq 0)$, then the solution $u(t, x)$ exists globally.

Theorem 1.4. Let $u_0 \in H^s$ with $s \geq 2a + \frac{1}{2}$. Let $u(t, x)$ be the corresponding local solution of (1.3) with $a > 1$, if the initial data $m_0 := (1 - \partial_{xx})^a u_0 \in L^1$ and $m_0(x)$ is an odd function such that $m_0 \leq 0$ when $x \leq 0$, and $m_0 \geq 0$ when $x \geq 0$, then the solution $u(t, x)$ exists globally.

The remainder of the paper is organized as follows. In Section 2 we introduce some preliminaries which will be used in sequel. In section 3, we prove the local-posedness of Eq. eqref1, i.e. Theorems 1.1-1.2. Finally, we give the proof of the existence of global solutions, i.e. Theorems 1.3-1.4.

2 Preliminaries

In this section, we will recall some propositions and lemmas on the Littlewood-Paley decomposition and Besov spaces.

Proposition 2.1. Let $C$ be the annulus $\{\xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{3}{2}\}$. There exist radial functions $\chi$ and $\varphi$, valued in the interval $[0, 1]$, belonging respectively to $D(B(0, \frac{4}{3}))$ and $D(C)$, and such that

$$\forall \xi \in \mathbb{R}^d, \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1,$$

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1,$$

$$|j - j'| \geq 2 \Rightarrow \text{Supp } \varphi(2^{-j} \cdot) \cap \text{Supp } \varphi(2^{-j'} \cdot) = \emptyset,$$

$$j \geq 1 \Rightarrow \text{Supp } \chi(\cdot) \cap \text{Supp } \varphi(2^{-j} \cdot) = \emptyset.$$

The set $\tilde{C} = B(0, \frac{2}{3}) + \tilde{C}$ is an annulus, and we have

$$|j - j'| \geq 5 \Rightarrow 2^j C \cap 2^{j'} \tilde{C} = \emptyset.$$

Further, we have

$$\forall \xi \in \mathbb{R}^d, \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j} \xi) \leq 1,$$

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j} \xi) \leq 1.$$
Definition 2.2. \([\text{[1]}]\) Denote \(\mathcal{F}\) by the Fourier transform and \(\mathcal{F}^{-1}\) by its inverse. Let \(u\) be a tempered distribution in \(\mathcal{S}'(\mathbb{R}^d)\). For all \(j \in \mathbb{Z}\), define

\[
\Delta_j u = 0 \text{ if } j \leq -2, \quad \Delta_{-1} u = \mathcal{F}^{-1}(\chi \mathcal{F} u), \quad \Delta_j u = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} u) \text{ if } j \geq 0, \quad S_j u = \sum_{j' < j} \Delta_{j'} u.
\]

Then the Littlewood-Paley decomposition is given as follows:

\[
u = \sum_{j \in \mathbb{Z}} \Delta_j u \text{ in } \mathcal{S}'(\mathbb{R}^d).
\]

Let \(s \in \mathbb{R}, 1 \leq p, r \leq \infty\). The nonhomogeneous Besov space \(B^s_{p,r}(\mathbb{R}^d)\) is defined by

\[
B^s_{p,r}(\mathbb{R}^d) = \{ u \in \mathcal{S}'(\mathbb{R}^d) : \| u \|_{B^s_{p,r}(\mathbb{R}^d)} = \| (2^{js} \| \Delta_j u \|_{L^r})_j \|_{L^r(\mathbb{Z})} < \infty \}.
\]

Proposition 2.3. \([\text{[1]}]\) (1) For any \(p \in [1, \infty]\), the Besov space \(B^s_{p,1}\) is a Banach algebra. If \(f, g \in B^s_{p,1}\), then

\[
\|fg\|_{B^s_{p,1}} \leq C \|f\|_{B^s_{p,1}} \|g\|_{B^s_{p,1}}.
\]

(2) For any \(s \in \mathbb{R}, p, r \in [1, \infty]\), if \(u \in B^s_{p,r}\), then \(\nabla u \in B^{s-1}_{p,r}\) and \((1 - \Delta)^{-1} u \in B^{s+2}_{p,r}\). Moreover,

\[
\|\nabla u\|_{B^{s-1}_{p,r}} \leq C \|u\|_{B^s_{p,r}}, \quad \|(1 - \Delta)^{-1} u\|_{B^{s+2}_{p,r}} \leq C \|u\|_{B^s_{p,r}}.
\]

In order to prove our main theorem, we have to use the following result about the transport equation

\[
(2.1) \begin{cases}
    f_t + v \cdot \nabla f = g, \\
    f(0, x) = f_0(x),
\end{cases}
\]

Lemma 2.4. \([\text{[1]}]\) Let \(s \in [\max\{-\frac{d}{p}, -\frac{d}{r}\}, \frac{d}{p} + 1]\) (\(s = 1 + \frac{1}{p}, r = 1; s = \max\{-\frac{d}{p}, -\frac{d}{r}\}, r = \infty\)). There exists a constant \(C\) such that for all solutions \(f \in L^\infty([0, T]; B^s_{p,r})\) of \((2.1)\) with initial data \(f_0 \in \dot{B}^s_{p,1}\) and \(g \in L^1([0, T]; B^s_{p,r})\), we have, for a.e. \(t \in [0, T]\),

\[
\|f(t)\|_{B^s_{p,r}} \leq C e^{C_2 V'(t)} \left( \|f_0\|_{B^s_{p,r}} + \int_0^t e^{-C_2 V'(t')} \|g(t')\|_{B^s_{p,r}} \, dt' \right),
\]

where \(V'(t) = \|\nabla v\|_{B^s_{p,1} \cap L^\infty} (\text{if } s = 1 + \frac{1}{p}, r = 1, V'(t) = \|\nabla v\|_{B^s_{p,1}})\).

Lemma 2.5. \([\text{[1]}]\) Let \(1 \leq p < \infty\). Define \(\mathbb{N} = \mathbb{N} \cup \{\infty\}\). Suppose \(f \in L^1([0, T]; B^\frac{d}{p,1})\) and \(a_0 \in B^\frac{d}{p,1}\). For \(n \in \mathbb{N}\), denote by \(a^n \in C([0, T]; B^\frac{d}{p,1})\) the solution of

\[
(2.2) \begin{cases}
    \partial_t a^n + A^n \cdot \nabla a^n = f, \\
    a^n|_{t=0}(x) = a_0(x).
\end{cases}
\]

Assume that \(\sup_{n \in \mathbb{N}} \|A^n\|_{B^\frac{d}{p,1}} \leq \alpha(t)\) with \(\alpha(t) \in L^1(0, T)\). If \(A^n \to A^\infty\) in \(L^1(B^\frac{d}{p,1})\), then \(a^n \to a^\infty\) in \(C([0, T]; B^\frac{d}{p,1})\).

Notation. For simplicity, we drop \(\mathbb{R}\) or \(\mathbb{T}\) in the notation of function spaces if there is no ambiguity.
3 Local well-posedness.

In this section, we will prove the local well-posedness for the system (1.3).

3.1. Uniformly bound and existence.

The proof of Theorem 1.1: Starting with \( u^0 \equiv 0 \), we construct a sequence of the approximate solutions \( u^n \) by solving the following linear transport equations:

\[
\begin{align*}
  u_t^{n+1} + u^n u_{x}^{n+1} &= [(1 - \partial_{xx})^{-a}, u^n \partial_x] m^n - 2(1 - \partial_{xx})^{-a}(u^n_{x} m^n), \\
  m^n &= (1 - \partial_{xx})^{-a} u^n, \\
  u^n |_{t=0}(x) &= S_n u_0(x).
\end{align*}
\]  

(3.1)

Without loss of generality, we consider the low regularity case \( s = 1 + \frac{d}{p} \). Assume that \( u^n \in E_p(T) = C([0, T]; B^{s+\frac{1}{p}}_{p, 1}) \cap C^1([0, T]; B^{s}_{p, 1}) \). Taking advantage of the standard theory of the transport equations, we can deduce that there exist a unique solutions \( u^{n+1} \in E_p(T) \). Using the lemma 2.4, we have

\[
\begin{align*}
  \|u^{n+1}(t)\|_{B^{s+\frac{1}{p}}_{p, 1}} &\leq C \int_0^t \|u^n\|_{B^{s+\frac{1}{p}}_{p, 1}} ds \\
  &+ C \int_0^t \|u^n\|_{B^{s+\frac{1}{p}}_{p, 1}} ds \\
  &\leq C \int_0^t \|u^n\|_{B^{s+\frac{1}{p}}_{p, 1}} ds \\
  &\leq C \int_0^t \|u^n\|_{B^{s+\frac{1}{p}}_{p, 1}} ds,
\end{align*}
\]  

(3.2)

where we use the fact that

\[
[(1 - \partial_{xx})^{-a}, u^n \partial_x] m^n = [(1 - \partial_{xx})^{-a}, T v^n \partial_x] m^n + (1 - \partial_{xx})^{-a} T m^n u^n + T u^n m^n + (1 - \partial_{xx})^{-a} R(u^n, m^n) + R(u^n, u^n_{x}),
\]

by Bony decomposition and Lemma 10.25 in [1] we get the above estimation.

This implies that there exist a \( T \) satisfies that \( 0 < T < \frac{1}{C\|u_0\|_{B^{s+\frac{1}{p}}_{p, 1}}} \) such that

\[
\begin{align*}
  \sup_{t \in [0, T]} \|u^{n+1}\|_{B^{s+\frac{1}{p}}_{p, 1}} &\leq C \|u_0\|_{B^{s+\frac{1}{p}}_{p, 1}}.
\end{align*}
\]

(3.3)

Therefore, \( \{u^n\}_{n \in \mathbb{N}} \) is uniformly bounded in \( L^\infty_T(B^{s+\frac{1}{p}}_{p, 1}) \). From the system (3.1), we can deduce that \( \partial_t u^n \) is uniformly bounded in \( L^\infty_T(B^{s+\frac{1}{p}}_{p, 1}) \). Using an interpolation argument, we obtain that \( u^n \)
is uniformly bounded in \(C([0, T); B^{1+\frac{1}{p}}_{p, 1}) \cap C^\frac{1}{2}([0, T); B^{\frac{1}{p}}_{p, 1})\). Taking advantage of Cantor’s diagonal process and Ascoli’s theorem, we can obtain a function \(u_j\) such that \(\phi_j u^n\) converges to \(u_j\) with \(\phi_j\) is a smooth function with support in the ball \(B(0, j + 1)\). Moreover, we can verify that there exists a function \(u\) such that for all \(\phi \in C_0^\infty\), \(\phi u^n \to \phi u\). (For more details, see the Chapter 10 in [1]). By the Fatou property and the interpolation argument, one can check that \(u \in E_p(T)\) is the solution of the system (1.3).

3.2. Uniqueness.

We will prove the uniqueness of solutions to (1.3) next. Actually, to prove the uniqueness we need more regularity such as \(u \in C_T(B^{2+\frac{1}{p}}_{p, 1})\). Let’s recall (1.3):

\[
\left\{
\begin{array}{ll}
m_t + um_x + 2u_x m = 0, \\
u = (1 - \partial_{xx})^{-a} m, \quad a \in \mathbb{R}^+/\mathbb{Z}^+
\end{array}
\right.
\]

(3.4)

Since the form of (3.4) is more simple than (1.3), one can estimate on (3.4). This need less regularity since \(2 + \frac{1}{p} \leq 1 + 2a - \min\{\frac{1}{p}, \frac{1}{p'}\}\). Suppose that \(m_i = (1 - \partial_{xx})^{-a} u_i\), \(i = 1, 2\) are two solutions of (3.4), then \(u_i \in C_T(B^{1+2a-\min\{\frac{1}{p}, \frac{1}{p'}\}}_{p, 1})\) \((i = 1, 2)\) are two solutions of (3.4). Setting \(w = m_1 - m_2\), we obtain

\[
\partial_t w + m_1 \partial_x w = -wm_{2x} + 2u_{1x} w + 2(1 - \partial_{xx})^{-a} w_x m_2,
\]

By virtue to Lemma 2.4 and the Bony decomposition, we have

\[
\|w(t)\|_{B^{1+\frac{1}{p'}}_{p, \infty}} \leq C_{w_0} \left( \|w(0)\|_{B^{1+\frac{1}{p'}}_{p, \infty}} + \int_0^t \|w_{2x} - 2u_{1x} w + 2(1 - \partial_{xx})^{-a} w_x m_2\|_{B^{1+\frac{1}{p'}}_{p, \infty}} ds \right)
\]

(3.5)

\[
\leq C_{w_0} \left( \|w(0)\|_{B^{1+\frac{1}{p'}}_{p, \infty}} + \int_0^t \|u_1, u_2\|_{B^{1+\frac{1}{p'}}_{p, 1}} ds \right)
\]

Therefore, the uniqueness is obvious in view of (3.4).

3.3. Continuous dependent.

This subsection devote to study about the continuous dependent on initial. Assume that \(u_0^n \to u_0^\infty\) in \(B^{s}_{p, 1}\), \(s > 1 + 2a - \min\{\frac{1}{p}, \frac{1}{p'}\}\) and \(u^n, u^\infty\) are the solutions of (1.3) with the initial data \(u_0^n, u_0^\infty\) respectively. Notice that their corresponding solutions \(u^n, u^\infty\) are uniformly bounded in \(L^\infty_T(B^{1+\frac{1}{p'}}_{p, 1})\).
3 LOCAL WELL-POSEDNESS.

An interpolation argument and (3.5) yield that \( u^n \to u^\infty \) in \( C([0,T);B_p^{s-\varepsilon}) \) for any \( \varepsilon > 0 \). In order to prove that \( u^n \to u^\infty \) in \( C([0,T);B_p^s) \), it is sufficient to show that \( m^s_n \to m^\infty \) in \( C([0,T);B_p^{s-1-2a}) \). Let \( V^n = m^s_n \) for all \( n \in \mathbb{N} \). Split \( V^n \) into \( W^n + Z^n \) with \((W^n,Z^n)\) satisfying that

\[
\begin{align*}
W^n + u^n W^n_k = &-u^\infty k + \partial_k u^\infty := F^\infty, \\
W^n_{t=0}(x) = &V^n_0 = m^\infty_0(x).
\end{align*}
\]

and

\[
\begin{align*}
Z^n + u^n Z^n_k = &F^n - F^\infty, \\
Z^n_{t=0}(x) = &m^n_0 - m^\infty_0.
\end{align*}
\]

By virtue of the lemma (2.5) and \((1 - \partial_{xx})^{-a}u^n_k = m^n_k = V^n = W^n + Z^n\), we verify that

\[
(3.6) \quad W^n \to W^\infty \quad \text{in} \quad C([0,T);B_p^{s-1-2a}).
\]

Using Bony decomposition, we deduce that

\[
\|F^n - F^\infty\|_{B_p^{s-1-2a}} \leq C(\|W^n - W^\infty\|_{B_p^{s-1-2a}} + \|z^n - 0\|_{B_p^{s-1-2a}})\|m^n, m^\infty\|_{B_p^{s-2a}}
\]

\[
\leq C_{u_0}(\|W^n - W^\infty\|_{B_p^{s-1-2a}} + \|z^n - 0\|_{B_p^{s-1-2a}}).
\]

Indeed, the uniqueness of transport equation ensures that \( z^\infty \equiv 0 \). Taking advantage of lemma (2.4), we get

\[
\|z^n\|_{B_p^{s-1-2a}} \leq C(\|u^n_0 - u^\infty_0\|_{B_p^{s+1}+2^a} + \int_0^t \|F^n - F^\infty\|_{B_p^{s-1-2a}} d\tau)
\]

\[
\leq C_{u_0}(\|u^n_0 - u^\infty_0\|_{B_p^{s+1}+2^a} + \int_0^t \|W^n - W^\infty\|_{B_p^{s-1-2a}} + \|z^n - 0\|_{B_p^{s-1-2a}} d\tau).
\]

Using the facts that \( \lim_{n \to \infty} \|u^n_0 - u^\infty_0\|_{B_p^{s+1}} = 0 \), \( \lim_{n \to \infty} \|u^n - u^\infty\|_{B_p^{s-1}} = 0 \), \( \lim_{n \to \infty} \|W^n - W^\infty\|_{B_p^{s-1-2a}} = 0 \), and the Gronwall inequality yields that \( z_n \) tends to 0 in \( C([0,T);B_p^{s-1-2a}) \). Since \((1 - \partial_{xx})^{-a}u^n_k = m^n_k = V^n = W^n + Z^n\), it follows that

\[
\|V^n - V^\infty\|_{B_p^{s+1}} \leq C(\|W^n - W^\infty\|_{B_p^{s-1-2a}} + \|Z^n - Z^\infty\|_{B_p^{s-1-2a}})
\]

\[
\leq C(\|W^n - W^\infty\|_{B_p^{s-1-2a}} + \|Z^n\|_{B_p^{s-1-2a}}) \to 0 \quad \text{as} \quad n \to \infty,
\]

that is \( m^n_k \to m^\infty_k \) in \( C([0,T);B_p^{s-1-2a}) \). Therefore we complete the proof of Theorem (3.1). Similarly, we can obtain Theorem (1.2).
4 Global existence.

In this section we construct a class of special data such that the corresponding solution is global in time when \( a > 1 \).

**Proof of theorem 1.3**

Proof. Set \( T \) be the maximal time of \( m(t, x) \). Firstly, one can easily deduce that \( m(t, x) \geq 0 \) if \( m_0(x) \geq 0 \) by the characteristic method. Since the functional calculus tells us that

\[
u(t, x) = (1 - \partial_{xx})^{-a}m(t, x) = C \int_0^{+\infty} s^{a-1} e^{-s \Delta} m(t, x) ds := (G \ast m)(t, x),\]

we deduce that \( u(t, x) \geq 0 \) if \( m_0(x) \geq 0 \). Moreover, we have \( G(z) = G(-z) \).

Then, by \( m_0 \in L^1(\mathbb{R}) \), one have

\[
\frac{d}{dt} \|m(t)\|_{L^1} = \frac{d}{dt} \int_{\mathbb{R}} m(t, x) dx = -\int_{\mathbb{R}} \partial_x (um)(t, x) dx - \int_{\mathbb{R}} (u_x m)(t, x) dx
\]

\[
= 0 - \langle G \ast m_x, m \rangle
\]

\[
= 0 + \langle m_x, G \ast m \rangle
\]

\[
= 0 + \langle m, G \ast m_x \rangle,
\]

where the last inequality holds by \( G(x) = G(-x) \). So we have \(- \langle G \ast m_x, m \rangle = + \langle m, G \ast m_x \rangle = 0\) and

\[
\frac{d}{dt} \|m(t)\|_{L^1} = 0.
\]

Finally, since

\[
\|u_x\|_{L^\infty} \leq C \|m\|_{B^{1-2a}_{\infty,1}} \leq C \|m\|_{B^{2-2a}_{1,1}} \leq C \|m\|_{L^1} \leq C \|m_0\|_{L^1}, \text{ if } a > 1,
\]

we obtain the global existence by the blow-up criteria and the bootstrap argument.

\[\square\]

Next we prove another global solution of \( (1.3) \) with different form of the initial data.

**Proof of theorem 1.4**

Proof. Set \( T \) be the maximal time of \( m(t, x) \). Firstly, since \( u(t, x) = (G \ast m)(t, x) \) and \( G(z) = G(-z) \), one can easily deduce that \( m(t, x) \) is an odd function if \( m_0(x) \) is an odd function. Similar to the proof of Theorem 1.3 by the characteristic method we deduce that if \( m_0 \leq 0 \) when \( x \leq 0 \), \( m_0 \geq 0 \) when \( x \geq 0 \), then we obtain

\[
m(t, x) \leq 0 \text{ when } x \leq q(t, 0); \quad m_0 \geq 0 \text{ when } x \geq q(t, 0), \quad \forall t \in [0, T),
\]

\[
(4.2)
\]
where \( q(t, \xi) = \xi + \int_0^t m(s, q(s, \xi))ds \) is the characteristic curves.

Next we want to prove that

\[
(4.3) \quad m(t, x) \leq 0 \text{ when } x \leq 0, \quad m_0 \geq 0 \text{ when } x \geq 0, \quad \forall t \in [0, T].
\]

If \( q(t, 0) = 0 \) for any \( t \in [0, T) \), then by (4.2) we immediately get (4.3). Other if \( q(t, 0) > 0 \) (or \( q(t, 0) < 0 \)) for some \( t \in [0, T) \), by (4.2) we deduce that

\[
m(t, x) \leq 0 \text{ when } x \in [-q(t, 0), 0]; \quad m_0 \leq 0 \text{ when } x \in [0, q(t, 0)].
\]

Since \( m(t, x) \) is an odd function such that \( m(t, x) = -m(t, -x), \quad x \in [-q(t, 0), q(t, 0)] \), we obtain

\[
m(t, x) = 0 \text{ when } x \in [-q(t, 0), q(t, 0)].
\]

By (4.2) again we still get (4.3).

Finally, by (4.3) we can easily deduce that \( \frac{d}{dt}\|m(t)\|_{L^1} = 0 \) and \( \|u_x\|_{L^\infty} \leq C\|m_0\|_{L^1} \), for \( a > 1 \). By the blow-up criteria and the bootstrap argument we obtain the global existence.

\[\square\]

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