Hankel and Berezin type operators on weighted Besov spaces of holomorphic functions on polydiscs

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Abstract

Assuming that $S$ is the space of functions of regular variation and $\omega = (\omega_1, \ldots, \omega_n)$, $\omega_j \in S$, by $B_p(\omega)$ we denote the class of all holomorphic functions defined on the polydisk $U^n$ such that

$$\|f\|_{B_p(\omega)}^p = \int_{U^n} |Df(z)|^p \prod_{j=1}^n \frac{\omega_j(1-|z_j|)dm_{2n}(z)}{(1-|z_j|^2)^{2-p}} < +\infty,$$

where $dm_{2n}(z)$ is the $2n$-dimensional Lebesgue measure on $U^n$ and $D$ stands for a special fractional derivative of $f$ defined here.

In this paper we consider the generalized little Hankel and Berezin type operators on $B_p(\omega)$ (and on $L_p(\omega)$) and prove some theorems about the boundedness of these operators.

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1 Introduction and auxiliary constructions

Numerous authors have contributed to holomorphic Besov spaces in the unit disc in $\mathbb{C}$ and in the unit ball in $\mathbb{C}^*$, Arazy-Fisher-Peetre [1], K. Stroethoff [17] O. Blasco [3], A. Karapetyants [10] see K. Zhu [19]. The investigation of holomorphic Besov space on the polydisc is of special interest. The polydisc is a product of $n$ disks and one would expect that the natural generalisations of results from the one-dimensional case would be valid here, but it turns out that this is not true. The case of polydisc is different from the $n = 1$ case and from the case of the $n$-dimensional ball. For example, let us consider the classical theorem of Privalov: if $f \in \text{Lip } \alpha$, then $Kf \in \text{Lip } \alpha$, where $Kf$ is a Cauchy type integral. It is known that the analogue of this theorem for multidimensional Lipschitz classes is not true ([9]), even though the analogue of this theorem for a sphere is valid.

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In many cases, especially, when the class is defined by means of derivatives, the generalisation of functional spaces in the polydisc is different from that on a unit ball. The generalisation of holomorphic Besov spaces on the polydisc see in [8]. Let

\[ U^n = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n, \ |z_j| < 1, \ 1 \leq j \leq n \} \]

be the unit polydisc in the \( n \)-dimensional complex plane \( \mathbb{C}^n \) and

\[ T^n = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n, \ |z_i| = 1, \ 1 \leq i \leq n \} \]

be its torus. We denote by \( H(U^n) \) the set of holomorphic functions on \( U^n \), by \( L^\infty(U^n) \) the set of bounded measurable functions on \( U^n \) and by \( H^\infty(U^n) \) the subspace of \( L^\infty(U^n) \) consisting of holomorphic functions.

Let \( S \) be the class of all non-negative measurable functions \( \omega \) on \((0,1)\), for which there exist positive numbers \( M_\omega, q_\omega, m_\omega, (m_\omega, q_\omega \in (0,1)) \), such that

\[ m_\omega \leq \frac{\omega(\lambda r)}{\omega(r)} \leq M_\omega, \]

for all \( r \in (0,1) \) and \( \lambda \in [q_\omega, 1] \). Some properties of functions from \( S \) can be found in [15]. We set

\[ -\alpha_\omega = \frac{\log m_\omega}{\log q_\omega}; \quad \beta_\omega = \frac{\log M_\omega}{\log q_\omega} \]

and assume that \( 0 < \beta_\omega < 1 \). For example, \( \omega \in S \) if \( \omega(t) = t^\alpha \), where \( -1 < \alpha < \infty \).

Using the results of [15] one can prove that

\[ \omega_j(t) = \exp \left\{ \eta_j(t) + \int_t^1 \frac{\varepsilon_j(u)}{u} \, du \right\}, \]

where \( \eta(u) \), \( \varepsilon(u) \) are bounded measurable functions and \( -\alpha_\omega \leq \varepsilon_j(u) \leq \beta_\omega \) \( (1 \leq j \leq n) \). Without loss of generality we assume that \( \eta(u) = 0 \). Then

\[ t^{\alpha_\omega} \leq \omega_j(t) \leq t^{-\beta_\omega} \]

is always true.

Below, for convenience of notations, for \( \zeta = (\zeta_1, \ldots, \zeta_n) \), \( z = (z_1, \ldots, z_n) \) we set

\[ \omega(1 - |z|) = \prod_{j=1}^n \omega_j(1 - |z_j|), \quad 1 - |z| = \prod_{j=1}^n (1 - |z_j|), \quad 1 - \zeta z = \prod_{j=1}^n (1 - \zeta_j z_j). \]

Further, for \( m = (m_1, \ldots, m_n) \) we set

\[ (m + 1) = (m_1 + 1) \ldots (m_n + 1), \quad (m + 1)! = (m_1 + 1)! \ldots (m_n + 1)!, \]

\[ (1 - |z|)^m = \prod_{j=1}^n (1 - |z_j|)^{m_j}. \]

Throughout the paper let assume \( \omega_j \in S, 1 \leq j \leq n \). The following definition gives the notion of the fractional differential.
Definition 1.1. For a holomorphic function \( f(z) = \sum_{(k) = (0)}^{(\infty)} a_k z^k \), \( z \in U^n \), and for \( \beta = (\beta_1, \ldots, \beta_n) \), \( \beta_j > -1 \) \( (1 \leq j \leq n) \), we define the fractional differential \( D^\beta \) as follows

\[
D^\beta f(z) = \sum_{(k) = (0)}^{(\infty)} \prod_{j=1}^{n} \frac{\Gamma(\beta_j + 1 + k_j)}{\Gamma(\beta_j + 1)\Gamma(k_j + 1)} a_k z^k, \quad k = (k_1, \ldots, k_n), \quad z \in U^n,
\]

where \( \Gamma(\cdot) \) is the Gamma function and \( \sum_{(k) = (0)}^{(\infty)} = \sum_{k_1 = 0}^{\infty} \cdots \sum_{k_n = 0}^{\infty} \).

If \( \beta = (1, \ldots, 1) \) then we put \( D^\beta f(z) \equiv Df(z) \). Hence

\[
Df(z_1, \ldots, z_n) = \frac{\partial^n (f(z_1, \ldots, z_n)z_1 \cdots z_n)}{\partial z_1 \cdots \partial z_n}.
\]

If \( n = 1 \) then \( Df \) is the usual derivative of the function \( zf(z) \).

Let us define the weighted \( L^p \) spaces of holomorphic functions.

Definition 1.2. Let \( 0 < p < +\infty \), \( \beta_j < -1 \) \( (1 \leq j \leq n) \). We denote by \( L^p(\omega) \) the set of all measurable functions on \( U^n \), for which

\[
\| f \|^p_{L^p(\omega)} = \int_{U^n} |f(z)|^p \frac{\omega(1 - |z|)}{(1 - |z|^2)^{2-p}} dm_{2n}(z) < +\infty.
\]

Note that \( L^p(\omega) \) is the \( L^p \)-space with respect to the measure \( \omega(1 - |z|)(1 - |z|^2)^{-2} dm_{2n}(z) \). Using the conditions on \( \omega \) \( (\omega_j \in S) \) we conclude that this measure is bounded.

Now we define holomorphic Besov spaces on the polydisc.

Definition 1.3. Let \( 0 < p < +\infty \) and \( f \in H(U^n) \). The function \( f \) is said to be in \( B^p(\omega) \) if

\[
\| f \|^p_{B^p(\omega)} = \int_{U^n} |Df(z)|^p \frac{\omega(1 - |z|)}{(1 - |z|^2)^{2-p}} dm_{2n}(z) < +\infty.
\]

From the definition of \( Df \) it follows that \( \| \cdot \|_{B^p(\omega)} \) is indeed a norm. (We do not have to add \( |f(0)| \).) This follows from the fact that here \( Df = 0 \) implies \( f = 0 \) for a holomorphic \( f \).

As in the one-dimensional case, \( B^p(\omega) \) is a Banach space with respect to the norm \( \| \cdot \|_{B^p(\omega)} \). For properties of holomorphic Besov spaces see [8].

The investigation of Toeplitz operators are widely known (see for example [5, 6, 18]). Some problems on the Toeplitz operators can be solved by means of Hankel operators and vice versa. In the classical theory of Hardy of holomorphic functions on the unit disk there is only one type of Hankel operator. In the \( B^p(\omega) \) theory they are two: little Hankel operators and big Hankel operators. The analogue of the Hankel operators of the Hardy theory here are little Hankel operators, which were investigated by many authors (see for example [13, 2, 8]).
Let us define the little Hankel operators as follows: denote by \( \overline{B}_p(\omega) \) the space of conjugate holomorphic functions on \( B_p(\omega) \). For the integrable function \( f \) on \( U^n \) we define the generalized little Hankel operator with symbol \( h \in L^\infty(U^n) \) by

\[
h^\alpha_g(f)(z) = \mathcal{P}_\alpha(fg)(z) = \int_{U^n} \frac{(1 - |\zeta|^2)^\alpha}{(1 - \zeta \bar{z})^{\alpha+2}} f(\zeta)g(\zeta)dm_{2n}(\zeta),
\]

\[
\alpha = (\alpha_1, \ldots, \alpha_n), \quad \alpha_j > -1, \quad 1 \leq j \leq n.
\]

For \( n = 1, \alpha = 0 \) this includes the definition of the classical little Hankel operator, see [20]. In Section 2 we consider the boundedness of little Hankel operator on \( B_p(\omega) \). For the case \( 0 < p < 1 \) and for the case \( p = 1 \) we have the following results

**Theorem 1.1.** Let \( 0 < p < 1, f \in B_p(\omega) \) (or \( f \in \overline{B}_p(\omega) \)), \( g \in L^\infty(U^n) \). Then \( h^\alpha_g(f) \in \overline{B}_p(\omega) \) if and only if \( \alpha_j > \alpha_{\omega_j}/p - 2, 1 \leq j \leq n \).

**Theorem 1.2.** Let \( f \in B_1(\omega), g \in L^\infty(U^n) \). Then \( h^\alpha_g(f) \in \overline{B}_1(\omega) \) if and only if \( \alpha_j > \alpha_{\omega_j} - 2, 1 \leq j \leq n \).

The case \( p > 1 \) is different from the cases of \( 0 > p < 1 \) and from the case of \( p = 1 \). Here we have the following

**Theorem 1.3.** Let \( 1 < p < +\infty, f \in B_p(\omega) \) (or \( f \in \overline{B}_p(\omega) \)), \( g \in L^\infty(U^n) \). Then if \( \alpha_j > \alpha_{\omega_j}, 1 \leq j \leq n \) then \( h^\alpha_g(f) \in \overline{B}_p(\omega) \).

The Berezin transform is the analogue of the Poisson transform in the \( A^p(\alpha) \) (respectively, \( (B_p(\omega)) \)) theory. It plays an important role especially in the study of Hankel and Toeplitz operators. In particular, some properties of those operators (for example, compactness, boundedness) can be proved by means of the Berezin transform (see [17] [12] [20]). The Berezin-type operators, on the other hand, are of independent interest.

In the last Section 3 it will be shown, that some properties of Berezin-type operators of the one dimensional classical case also hold in the more general situation. For the integrable function \( f \) on \( U^n \) and for \( g \in L^\infty(U^n) \) we define the Berezin-type operator in the following way

\[
B^\alpha_g f(z) = \frac{(\alpha + 1)}{\pi^n} (1 - |z|^2)^{\alpha+2} \int_{U^n} \frac{(1 - |\zeta|^2)^\alpha}{|1 - z\bar{\zeta}|^{\alpha+2}} f(\zeta)g(\zeta)dm_{2n}(\zeta).
\]

In the case \( \alpha = 0, g \equiv 1 \) the operator \( B^\alpha_g \) will be called the Berezin transform. We have the following results:

1. for the case of \( 0 < p < 1 \) we have

**Theorem 1.4.** Let \( 0 < p < 1, f \in B_p(\omega) \) (or \( f \in \overline{B}_p(\omega) \)), \( g \in L^\infty(U^n) \) and let \( \alpha_j > \alpha_{\omega_j}/p - 2, 1 \leq j \leq n \). Then \( B^\alpha_g(f) \in L^p(\omega) \).

2. the case \( 1 < p < +\infty \) gives the next theorem

**Theorem 1.5.** Let \( 1 < p < +\infty, f \in B_p(\omega) \) (or \( f \in \overline{B}_p(\omega) \)), \( g \in L^\infty(U^n) \) and let \( \alpha_j > (\alpha_{\omega_j}/p - 2, 1 \leq j \leq n \). Then \( B^\alpha_g(f) \in L^p(\omega) \).

3. we consider now the case of \( p = 1 \).
**Theorem 1.6.** Let $f \in B_1(\omega)$ (or $f \in \overline{B}_1(\omega)$), $g \in L^\infty(U^n)$. Then $B^*_g(f) \in L_1(\omega)$ if and only if $\alpha_j > \alpha_{\omega_j}$, $1 \leq j \leq n$.

In general, $h^*_g(f)$ and $B^*_g$ are not bounded.

To prove the main results we need some other notation. The partition of the polydisc into dyadic quadrangles plays an important role (see [4, 16]). Put

\[
\Delta_{k,l,j} = \{ z_j \in U : 1 - \frac{1}{2k_j} \leq |z_j| < 1 - \frac{1}{2^{k_j+1}}, \ \frac{\pi l_j}{2^{k_j}} \leq \arg z_j < \frac{\pi(l_j+1)}{2^{k_j}} \},
\]

where $k = (k_1, \ldots, k_n)$ ($k_j \geq 0$), $l_j$ are some integers such that $-2^{k_j} \leq l_j \leq 2^{k_j+1} - 1$ ($1 \leq j \leq n$) and $2^k = (2^{k_1}, \ldots, 2^{k_n})$.

Then $\Delta_{k,l} = \Delta_{k_1,l_1,j_1} \times \cdots \times \Delta_{k_n,l_n}$ and $\Delta^*_{k,l}$ is defined similarly. The system $\{\Delta_{k,l}\}$ is called the system of dyadic quadrangles.

**Proposition 1.1.** Let $\zeta_{k,l,j}$ be the center of $\Delta_{k,l,j}$, $1 \leq j \leq n$. Then

\[
1 - |\zeta_{k_l,j}| > 1 - |\zeta_j| \ \text{for \ disjoint \ indices} \ \text{are \ disjoint, \ which \ is \ no \ longer \ true \ for} \ \Delta^*_{k,l}.
\]

Note that the partition of the polydisc into dyadic quadrangles is important for obtaining some integral estimates particularly in the case $0 < p \leq 1$. Besides, the system $\{\Delta_{kl}\}$, as well as the system $\{\Delta^*_{kl}\}$, are coverings of $U^n$, and one can observe that the interiors of $\Delta_{kl}$ for disjoint indices are disjoint, which is no longer true for $\Delta^*_{kl}$. On the other hand, $\{\Delta^*_{k,l}\}$ is a finite covering in the sense that any quadrangle $\{\Delta^*_{k_l}\}$ has nonempty intersection only with a finite number of quadrangles from $\{\Delta_{kl}\}$, and this number is independent of $k$ and $l$. Note that this partition for the spaces $A^\alpha_{\omega}$ was used for the first time by F. A. Shamoyan [16] who greatly investigated the theory of weighted classes of functions in the polydisc and unit ball in $\mathbb{C}^n$.

To prove the main results we need the following auxiliary lemmas:

**Lemma 1.1.** Let $m = (m_1, \ldots, m_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_j \geq 0$, $1 \leq j \leq n$. Then If $f \in B_\beta(\omega)$ then

\[
|f(z)| \leq C \left| \int_{U^n} \frac{(1 - |\zeta|^2)^m}{1 - \zeta |^m} |Df(\zeta)| dm_2(\zeta) \right|
\]

where $m_j \geq \alpha_{\omega_j} - 1$ ($1 \leq j \leq n$).

The proof follows from [3, Lemma 2.5].

**Lemma 1.2.** Let $n = 1$. Assume $a + 1 - \beta_\omega > 0$, $b > 1$ and $b - a - 2 > \alpha_\omega$. Then

\[
\int_U \frac{(1 - |\zeta|^2)^a \omega(1 - |\zeta|^2)}{|1 - z\zeta|^b} dm_2(\zeta) \leq \frac{\omega(1 - |z|^2)}{(1 - |z|^2)^{b-a-2}}.
\]

For the proof see [3, Lemma 2].
2 Little Hankel operators on $B_p(\omega)$

We consider the little Hankel operators on $B_p(\omega)(0 < p < +\infty)$. We denote the restriction of $\|\cdot\|_{L^p(\omega)}$ to $\overline{B}_p(\omega)$ by $\|\cdot\|_{\overline{B}_p(\omega)}$. First off all we consider the case $0 < p < 1$.

**Theorem 2.1.** Let $0 < p < 1$, $f \in B_p(\omega)$ (or $f \in \overline{B}_p(\omega)$), $g \in L^\infty(U^n)$. Then $h_g^\alpha(f) \in \overline{B}_p(\omega)$ if and only if $\alpha_j > \alpha_{\omega_j}/p - 2$, $1 \leq j \leq n$.

**Proof.** Let $0 < p < 1$, $f \in B_p(\omega)$ (or $f \in \overline{B}_p(\omega)$), $g \in L^\infty(\omega)$ and $\alpha_j > \alpha_{\omega_j}/p - 2$, $1 \leq j \leq n$. We will show that $h_g^\alpha(f) \in \overline{B}_p(\omega)$. Using the partition of the polydisc, Lemma 3 from [16] and Proposition 1.1 we get

$$I = \int_{U^n} \frac{\omega(1 - |z|)}{(1 - |z|^2)^{-p}} \left( \int_{U^n} \frac{(1 - |\zeta|^2)^{\alpha}}{|1 - \overline{\zeta}|^{\alpha+1}} |f(\zeta)||g(\zeta)| dm_{2n}(\zeta) \right)^p dm_{2n}(z) \leq$$

$$C(g) \int_{U^n} \frac{\omega(1 - |z|)}{(1 - |z|^2)^{-p}} \sum_{k,l} \left( \int_{\Delta_{k,l}} \frac{(1 - |\zeta|)^{\alpha}}{|1 - \overline{\zeta}|^{\alpha+2}} |f(\zeta)||g(\zeta)| dm_{2n}(\zeta) \right)^p dm_{2n}(z) \leq$$

$$C(g) \sum_{k,l} \max_{\zeta \in \Delta_{k,l}} |f(\zeta)|^p |\Delta_{k,l}|^p \omega(1 - |\zeta_{k,l}|) \frac{1 - |\zeta_{k,l}|^{\alpha+2}}{|1 - \overline{\zeta_{k,l}}|^{(\alpha+3)p}} dm_{2n}(z) =$$

where $\zeta_{k,l}$ is the center of $\Delta_{k,l}$. $I = \|h_g^\alpha f\|_{\overline{B}_p(\omega)}$, $C(\alpha, p, \omega) \|g\|_{L^\infty} = C(g)$.

Recalling that $\{\Delta_{k,l}\}$ forms a finite covering of $U^n$, by [2] and Lemma 4 from [16] we obtain

$$I \leq C(g) \sum_{k,l} \max_{\zeta \in \Delta_{k,l}} |f(\zeta)|^p |(1 - |\zeta_{k,l}|)^{-2+2} \omega(1 - |\zeta_{k,l}|) \leq$$

$$C(g) \sum_{k,l} \int_{\Delta_{k,l}} |f(\zeta)|^p \omega(1 - |\zeta|) \frac{1 - |\zeta|^2}{(1 - |\zeta|^2)^2} dm_{2n}(\zeta) \leq$$

$$C(g) \int_{U^n} |f(\zeta)|^p \omega(1 - |\zeta|) \frac{1 - |\zeta|^2}{(1 - |\zeta|^2)^2} dm_{2n}(\zeta)$$

Using (11) we get

$$I \leq C(g) \int_{U^n} \frac{\omega(1 - |z|)}{(1 - |z|^2)^{2}} \left( \int_{U^n} \frac{(1 - |t|^2)^m}{|1 - \overline{t}|^{m+1}} |Df(t)||dm_{2n}(t) | \right)^p dm_{2n}(\zeta) \leq$$

$$C(g) \int_{U^n} \frac{\omega(1 - |z|)}{(1 - |z|^2)^{2}} \sum_{k,l} \left( \int_{\Delta_{k,l}} \frac{(1 - |t|^2)^m}{|1 - \overline{t}|^{m+1}} |Df(t)||dm_{2n}(t) | \right)^p dm_{2n}(\zeta) \leq$$

$$C(g) \sum_{k,l} \max_{t \in \Delta_{k,l}} |Df(t)|^p |\Delta_{k,l}|^p \frac{\omega(1 - |t_{k,l}|)(1 - |t_{k,l}|^2)^{mp}}{|1 - \overline{t_{k,l}}|^{(mp+1)p-2+2}} =$$

$$C(g) \sum_{k,l} \max_{t \in \Delta_{k,l}} |Df(t)|^p |(1 - |t_{k,l}|)^{2p-2+2}.$$
In the last inequality we have used Lemma 2 again. By Lemma 4 from [16] we get

\[ I \leq C(g) \sum_k \sum_l \int_{\Delta_{k,t}} \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} |Df(z)|dm_{2n}(z) \leq \]

\[ \int_{U^n} \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} |Df(z)|dm_{2n}(z) = C(\alpha, p, \omega)\|g\|_{\infty} \|f\|_{B_p(\omega)}^{\rho}, \]

which proves our statement.

Conversely, let \( h_\omega^p(f) \in \overline{B}_p(\omega) \) for all \( g \in L^\infty(U^n) \). For \( r = (r_1, \ldots, r_n) \), \( r_j \in (0, 1) \), \( k = (k_1, \ldots, k_n) \) we take the function

\[ f_r(z) = C_r(1-rz)^{-k}, \quad k_j > (\alpha_{\omega_j} + 2)/p, \quad 1 \leq j \leq n, \]

where \( C_r = (1-r)^{k_\omega - 1/p}(1-r) \). Then we have \( \|f_r\|_{B_p(\omega)} \sim \text{const} \).

We consider the following domains

\[ \tilde{U}_j = \{ z_j \in U, |\arg z_j| < (1-r_j)/2; (4r_j - 1)/3 < |z_j| < (1+2r_j)/3 \} \]

and

\[ \tilde{U}^n = \tilde{U}_1 \times \ldots \times \tilde{U}_n. \]

Take the function \( g_r(\zeta) \) as

\[ g_r(\zeta) = \exp^{-\arg f_r(\zeta)} \]

and a polydisc \( V^n \) centered at \( (r_1, \ldots, r_n) \) with radius of \( (1-r_1)\ldots(1-r_n) \) such that \( V^n \subset \overline{U^n} \) (\( V^n \) is the closure of \( V^n \)), we get

\[ \|h_{g_r} f_r\|_{\overline{B}_p(\omega)} \geq \int_{U^n} \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} \left( \int_{U^n} \frac{(1-|\zeta|)^{\alpha}}{|1-\overline{\zeta}|^{\alpha+3}} |f_r(\zeta)|dm_{2n}(\zeta) \right)^p dm_{2n}(z). \]

Let

\[ \max_{\zeta \in \overline{V^n}} |1-\overline{\zeta}| = |1-\overline{\zeta}|, \]

then

\[ \|h_{g_r} f_r\|_{\overline{B}_p(\omega)} \geq C_1(\alpha, p, \omega) \frac{(1-r)^{\rho p}}{\omega(1-r)} \int_{U^n} \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} \left( \int_{V^n} \frac{dm_{2n}(\zeta)}{|1-\overline{\zeta}|^{\alpha+3}} \right)^p dm_{2n}(z) \]

\[ \geq C_1(\alpha, p, \omega) \frac{(1-r)^{\rho p}}{\omega(1-r)} \int_{U^n} \frac{\omega(1-|z|)dm_{2n}(z)}{|1-\overline{\zeta}|^{(\alpha+3)p}(1-|z|)^{2-p}}. \]

If we assume that \( (\alpha_j + 2)p \leq \alpha_{\omega_j} \) for some \( j \), then for the corresponding integral taking \( \omega_j(t) = t^{\alpha_{\omega_j}} \) we get

\[ \int_{U^n} \frac{\omega(1-|z|)dm_{2n}(z)}{|1-\overline{\zeta}|^{(\alpha+3)p}(1-|z|)^{2-p}} \sim \text{const}, \quad \text{if} \quad (\alpha_j + 2)p < \alpha_{\omega_j} \]

and

\[ \int_{U^n} \frac{\omega(1-|z|)dm_{2n}(z)}{|1-\overline{\zeta}|^{(\alpha+3)p}(1-|z|)^{2-p}} \sim \log \frac{1}{1-|\zeta_j|}, \quad \text{if} \quad (\alpha_j + 2)p = \alpha_{\omega_j} + 2. \]
Consequently,
\[
\frac{(1 - r_j)^{(a_j+2)p}}{\omega_j(1 - r_j)} \rightarrow \infty, \quad \frac{(1 - r_j)^{(a_j+2)p}}{\omega_j(1 - r_j)} \log \frac{1}{1 - r_j} \rightarrow \infty
\]
if \( r_j \rightarrow 1 - 0 \). 

\[\square\]

**Corollary 2.1.** Let \( 0 < p < 1, \; \alpha_j > \alpha_{\omega_j}/p - 2, \; 1 \leq j \leq n, \; g \in L^\infty(U^n) \). Then \( h_g^\alpha \) is bounded on \( B_p(\omega) \), (and on \( \overline{B_p(\omega)} \)). Moreover, \( \|h_g^\alpha\| \leq C\|f\| \cdot \|g\| \)

In the case if \( p = 1 \) we have

**Theorem 2.2.** Let \( f \in B_1(\omega), \; g \in L^\infty(U^n) \). Then \( h_g^\alpha(f) \in \overline{B_1(\omega)} \) if and only if \( \alpha_j > \alpha_{\omega_j} - 2, \; 1 \leq j \leq n \).

**Proof.** Let \( f \in B_1(\omega), \; g \in L^\infty(U^n) \) and \( C(\alpha, \omega)\|g\|_\infty = \tilde{C} \). Then by (1) and (2) we have

\[
\|h_g^\alpha(f)\|_{\overline{B_1(\omega)}} \leq \|g\|_{\infty} \int_{U^n} (1 - |\zeta|^2)^{\alpha} |f(\zeta)| \int_{U^n} \frac{\omega(1 - |z|)dm_{2n}(z)}{(1 - |\zeta|^2)^{\alpha+3}(1 - |z|)} dm_{2n}(\zeta) \leq 
\]

\[
\tilde{C} \int_{U^n} |f(\zeta)| \frac{\omega(1 - |\zeta|^2)}{(1 - |\zeta|^2)^2} dm_{2n}(\zeta) \leq \tilde{C} \int_{U^n} \frac{\omega(1 - |\zeta|)}{(1 - |\zeta|^2)^2} \int_{U^n} \frac{(1 - |t|^2)^m}{(1 - |\zeta|^m+1)Df(t)} \times 
\]

\[
dm_{2n}(t)dm_{2n}(\zeta) = \tilde{C} \int_{U^n} (1 - |t|^2)^m |Df(t)| \int_{U^n} \frac{\omega(1 - |\zeta|)dm_{2n}(\zeta)dm_{2n}(t)}{(1 - |\zeta|^m+1)}. 
\]

Using (1) again we get

\[
\|h_g^\alpha(f)\|_{\overline{B_1(\omega)}} \leq \tilde{C} \int_{U^n} \frac{\omega(1 - |t|)}{(1 - |t|)} |Df(t)|dm_{2n}(t) = \tilde{C}\|f\|_{\overline{B_1(\omega)}}. 
\]

Next, assume that \( h_g^\alpha f \in \overline{B_1(\omega)} \). The proof of the necessity of the condition \( \alpha_j > \alpha_{\omega_j}, \; 1 \leq j \leq n \) is similar to the corresponding proof in Theorem 2.1. We omit the details. This proves the theorem.

\[\square\]

**Corollary 2.2.** Let \( \alpha_j > \alpha_{\omega_j}, \; 1 \leq j \leq n, \; g \in L^\infty(U^n) \). Then \( h_g^\alpha \) is bounded on \( B_1(\omega) \) and \( \|h_g^\alpha\| \leq C\|f\| \cdot \|g\| \).

Now we consider the case of \( 1 < p < +\infty \).

**Theorem 2.3.** Let \( 1 < p < +\infty, \; f \in B_p(\omega) \) (or \( f \in \overline{B_p(\omega)} \)), \( g \in L^\infty(U^n) \). Then if \( \alpha_j > \alpha_{\omega_j}, \; 1 \leq j \leq n \) then \( h_g^\alpha(f) \in \overline{B_p(\omega)} \).

**Proof.** Let \( 1 < p < +\infty, \; f \in B_p(\omega) \) (or \( f \in \overline{B_p(\omega)} \)), \( g \in L^\infty(U^n) \). We show that \( h_g^\alpha(f) \in \overline{B_p(\omega)} \). By Hölder inequality and by (2) we get

\[
|Dh_g^\alpha(f)(\zeta)| \leq \int_{U^n} \frac{(1 - |\xi|^2)^\alpha}{|1 - \xi|^\alpha+3} |f(\xi)| \cdot |g(\xi)| \; dm_{2n}(\xi) \leq 
\]

\[
\|g\|_\infty \int_{U^n} \frac{(1 - |\xi|^2)^\alpha}{|1 - \xi|^\alpha+3} \; dm_{2n}(\xi) \leq \|g\|_\infty \times 
\]

\[
\left( \int_{U^n} \frac{(1 - |\xi|^2)^\alpha |f(\xi)|^p}{|1 - \xi|^\alpha+3} \; dm_{2n}(\xi) \right)^{1/p} \cdot \left( \int_{U^n} \frac{(1 - |\xi|^2)^\alpha dm_{2n}(\xi)}{|1 - \xi|^\alpha+3} \right)^{1/q} \leq 
\]

\[
C(\alpha, q)\|g\|_\infty \left( \int_{U^n} \frac{(1 - |\xi|^2)^\alpha |f(\xi)|^p}{|1 - \xi|^\alpha+3} \; dm_{2n}(\xi) \right)^{1/p} \leq 
\]

\[
(1 - |z|)^{1/q} \left( \int_{U^n} \frac{(1 - |\xi|^2)^\alpha |f(\xi)|^p}{|1 - \xi|^\alpha+3} \; dm_{2n}(\xi) \right)^{1/p}.
\]
Then setting \( C(\alpha, q) \|g\|_\infty = C \) we have
\[
\|h_g^{(\alpha)}(f)\|_{B_p(\omega)} = \int_{U^n} \frac{\omega(1 - |z|)}{(1 - |z|^2)^{2-p}} |Dh_g^{(\alpha)}(f)(z)|^p \, dm_{2n}(z) \leq \\
C \int_{U^n} \omega(1 - |z|) \left( \int_{U^n} \frac{(1 - |\xi|^2)^\alpha |f(\xi)|^p}{|1 - \xi z|^\alpha+3} \right) dm_{2n}(\xi) dm_{2n}(z) \leq \\
C \int_{U^n} |f(\xi)|^p \left( \int_{U^n} \frac{(1 - |\xi|^2)^\alpha}{|1 - \xi z|^\alpha+3(1 - |z|^2)^{2-p}} \right) dm_{2n}(\xi) \leq \\
C_1 \int_{U^n} \frac{(1 - |\xi|^2)^\alpha |f(\xi)|^p \omega(1 - |\xi|)(1 - |\xi|^p)^{2-p/q}}{(1 - |\xi|^2)^{\alpha+1}} \, dm_{2n}(\xi) = \\
\int_{U^n} (1 - |x|^2)^{p-2-p/q} \omega(1 - |\xi|)|f(\xi)|^p \, dm_{2n}(\xi).
\]
In the last inequality we have used (11). On the other hand, by (11) we get
\[
|f(\xi)|^p \leq \left[ \left( \int_{U^n} \frac{(1 - |t|^2)^m}{|1 - t\xi|m+1} |Df(t)| \, dm_{2n}(t) \right)^p \right] \leq \\
\left( \int_{U^n} \frac{(1 - |t|^2)^{m-\delta}(1 - |t|^2)^\delta}{|1 - t\xi|m+1} |Df(t)| \, dm_{2n}(t) \right)^p \leq \\
\int_{U^n} \frac{(1 - |t|^2)^{m-\delta}}{|1 - t\xi|m+1} (1 - |t|^2)^p |Df(t)|^p \, dm_{2n}(t) \cdot \frac{C(m, \delta, q)}{(1 - |\xi|^2)^{(\delta-1)p/q}}
\]
for some \( \delta > 1 \). Then we obtain
\[
\|h_g^{(\alpha)}(f)\|_{B_p(\omega)} \leq C_1 \int_{U^n} (1 - |t|^2)^{m-\delta-\delta p} |Df(t)|^p \int_{U^n} \frac{(1 - |\xi|^2)^{p-3-\delta p/q}}{|1 - t\xi|m+1} \times \\
\omega(1 - |\xi|) \left( \int_{U^n} (1 - |t|^2)^{m-\delta-\delta p} |Df(t)|^p \right) \leq \\
C_2 \int_{U^n} \frac{\omega(1 - |\xi|)(1 - |\xi|^p)^{2-p/q}}{|1 - t\xi|m+1} \, dm_{2n}(\xi) \leq \\
\int_{U^n} (1 - |t|^2)^{m-\delta-\delta p} |Df(t)|^p \omega(1 - |t|) \, dm_{2n}(t) = \\
\int_{U^n} (1 - |t|^2)^{p-2} |Df(t)|^p \omega(1 - |t|) \, dm_{2n}(t) = \|f\|_{B_p(\omega)}.
\]
We have \( \|h_g^{(\alpha)}(f)\|_{B_p(\omega)} \leq C_3 \|f\|_{B_p(\omega)} \|g\|_\infty\), where \( C_3 = C_2 \cdot C^p(m, \delta, q) \).

\[\square\]

**Corollary 2.3.** Let \( \alpha_j > \alpha_{ij}, 1 \leq j \leq n, g \in L^\infty(U^n) \). Then \( h_g^{(\alpha)} \) is bounded on \( B_p(\omega) \) and \( \|h_g^{(\alpha)}\|_{B_p(\omega)} \leq C_3 \|f\|_{B_p(\omega)} \cdot \|g\|_\infty\)

### 3 Berezin-type operators on \( B_p(\omega) \)

In this section we consider the boundedness of the Berezin-type operators. Let us consider first the case \( 0 < p < 1 \).
Theorem 3.1. Let $0 < p < 1$, $f \in \mathcal{B}_p(\omega)$ (or $f \in \overline{\mathcal{B}_p(\omega)}$), $g \in L^\infty(U^n)$ and let $\alpha_j > \alpha_{\omega_j}/p - 2$, $1 \leq j \leq n$. Then $\mathcal{B}_g^p(f) \in L^p(\omega)$.

Proof. Let $f \in \mathcal{B}_p(\omega)$ or $f \in \overline{\mathcal{B}_p(\omega)}$. We will show that $\mathcal{B}_g f \in L_p(\omega)$. To this end we estimate the corresponding integral

$$\int_{U^n} \frac{\omega(1 - |z|)}{(1 - |z|^2)^{a+2}} \left( \int_{U^n} \frac{(1 - |\zeta|^2)^{a}|f(\zeta)||g(\zeta)| \, dm_{2n}(\zeta)}{|1 - z\zeta|^{4+2a}} \right)^p \, dm_{2n}(z) \equiv I$$

Using the partition of the polydisc, we obtain

$$I \leq \|g\|_\infty \int_{U^n} (1 - |z|^2)^{(a+2)p-2}\omega(1 - |z|) \times \sum_{k,l} \left( \int_{\Delta_{k,l}} \frac{(1 - |\zeta|)^a}{|1 - z\zeta|^{4+2a}} |f(\zeta)| \, dm_{2n}(\zeta) \right)^p \, dm_{2n}(z) \leq C(\alpha, \omega, p)\|g\|_\infty \times \int_{U^n} (1 - |z|^2)^{(a+2)p-2}\omega(1 - |z|) \sum_{k,l} \max_{\zeta \in \Delta_{k,l}} |f(\zeta)|^{p(1 - |\zeta_{k,l}|)^{2p}} \, \Delta_{k,l} \leq C(\alpha, \omega, p)\|g\|_\infty \int_{U^n} (1 - |\zeta|^2)^{(a+2)p-2}\omega(1 - |\zeta|) \frac{(1 - |\zeta_{k,l}|)^{2p}}{|1 - \zeta_{k,l}z|^{(4+2a)p}} \, dm_{2n}(\zeta) \times$$

Taking into account that $p(4 + 2\alpha_j) > (\alpha_j + 2)p + \alpha_{\omega_j}$ $(1 \leq j \leq n)$ and (1), we get

$$I \leq C(\alpha, p, \omega)\|g\|_\infty \sum_{k,l} \max_{\zeta \in \Delta_{k,l}} |f(\zeta)|^{p(1 - |\zeta_{k,l}|)^2} \omega(1 - |\zeta_{k,l}|) \leq C(\omega, \alpha, p)\|g\|_\infty \int_{U^n} |f(\zeta)|^{p\omega(1 - |\zeta|)} \frac{(1 - |\zeta|^2)^2}{(1 - |z|^2)} \, dm_{2n}(\zeta).$$

In the last inequality we have used Lemma 4 [16]. Next we estimate the last integral. Using Lemma 1 we obtain

$$I \leq C(\omega, \alpha, p)\|g\|_\infty \int_{U^n} \frac{\omega(1 - |\zeta|)}{(1 - |\zeta|^2)} \left( \int_{U^n} \frac{(1 - |t|^2)^m}{|1 - \zeta_{k,l}t|^{m+1}} |Df(t)| \, dm_{2n}(t) \right)^p \, dm_{2n}(\zeta).$$

Then from

$$\left( \int_{U^n} \frac{(1 - |t|^2)^m}{|1 - \zeta_{k,l}t|^{m+1}} |Df(t)| \, dm_{2n}(t) \right)^p \leq \sum_{k,l} \left( \int_{\Delta_{k,l}} \frac{(1 - |t|^2)^m}{|1 - \zeta_{k,l}t|^{m+1}} |Df(t)| \, dm_{2n}(t) \right)^p \leq \sum_{k,l} \max_{t \in \Delta_{k,l}} |Df(t)|^{p|\Delta_{k,l}|} \frac{(1 - |t_{k,l}|^2)^{mp}}{|1 - t_{k,l}\zeta|^{(m+1)p}},$$

Finally, we have

$$I \leq C(\omega, \alpha, p)\|g\|_\infty \int_{U^n} \frac{\omega(1 - |\zeta|)}{(1 - |\zeta|^2)} \left( \int_{U^n} \frac{(1 - |t|^2)^m}{|1 - \zeta_{k,l}t|^{m+1}} |Df(t)| \, dm_{2n}(t) \right)^p \, dm_{2n}(\zeta).$$
we conclude
\[ I \leq C(\omega, \alpha, p)\|g\|_{\infty} \sum_{k,l} \max_{t \in \Delta_{k,l}} |Df(t)|^p |\Delta_{k,l}|^p (1 - |t_{k,l}|^2)^{mp} \times \int_{U^n} \frac{\omega(1 - |z|)(1 - |\xi|)^{-2}}{|1 - t_{k,l}\xi|^{(m+1)p}} dm_{2n}(z) \leq \]
\[ C(\omega, \alpha, p)\|g\|_{\infty} \sum_{k,l} \max_{t \in \Delta_{k,l}} |Df(t)|^p |\Delta_{k,l}|^p (1 - |t_{k,l}|^2)^{mp} \frac{\omega(1 - |t_{k,l}|)}{(1 - |t_{k,l}|^2)^{(m+1)p}} = \]
\[ C(\omega, \alpha, p)\|g\|_{\infty} \sum_{k,l} \max_{t \in \Delta_{k,l}} |Df(t)|^p \omega(1 - |t_{k,l}|)(1 - |t_{k,l}|^2)^{p} \leq \]
\[ \int_{U^n} |Df(t)|^p \frac{\omega(1 - |t|)}{(1 - |t|^2)^{2-p}} dm_{2n}(t) \]

In the last inequality we have used Lemma 4 again. Next we have
\[ I \leq C(\omega, \alpha, p)\|g\|_{\infty} \|f\|_{B_p(\omega)} \]

\[ \square \]

**Remark 3.1.** The condition \( \alpha_j + 2 > (\alpha_{w_j} + 2)/p \), \((1 \leq j \leq n)\) in Theorem \[\text{3.1}\] is necessary too. Moreover, if \( B_\alpha \) is bounded on \( L^p(\omega) \) then \( \alpha_j + 2 > (\alpha_w + 2)/p \), \((1 \leq j \leq n)\).

The proof is similar to the corresponding part of Theorem \[\text{2.1}\] and we omit it.

**Corollary 3.1.** Let \( 0 < p < 1 \), \( \alpha_j > (\alpha_{w_j} + 2)/p - 2 \), \( 1 \leq j \leq n \), \( g \in L^\infty(U^n) \). Then \( B_\alpha \) is bounded on \( A^p(\omega) \) and on \( A^p(\omega) \).

**Theorem 3.2.** Let \( 1 < p < +\infty \), \( f \in B_p(\omega) \) (or \( f \in B_p(\omega) \)), \( g \in L^\infty(U^n) \) and let \( \alpha_j > (\alpha_{w_j}/p - 2) \), \( 1 \leq j \leq n \). Then \( B_\alpha^\alpha(f) \in L_p(\omega) \).

**Proof.** Let \( f \in B_p(\omega) \) or \( f \in B_p(\omega) \). Our aim is to show that \( B_\alpha f \in L_p(\omega) \). We have
\[ |B_\alpha^\alpha(f)(z)|^p \leq (1 - |z|^2)^{(\alpha+2)p} \frac{C(\alpha, \pi, p)}{(1 - |z|^2)^{(\alpha+2)p/q}} \times \]
\[ \int_{U^n} \frac{(1 - |\xi|^2)^{\alpha} |f(\xi)|^p |g(\xi)|^p}{|1 - z^2|^2} dm_{2n}(\xi) \leq C(\alpha, \pi, p)(1 - |z|^2)^{\alpha+2} \times \]
\[ \int_{U^n} \frac{(1 - |\xi|^2)^{\alpha} |f(\xi)|^p}{|1 - z^2|^2} dm_{2n}(\xi) \leq C(\alpha, \pi, p)(1 - |z|^2)^{\alpha+2} \|g\|_{\infty} \times \]
\[ \int_{U^n} \frac{(1 - |\xi|^2)^{\alpha}}{|1 - z^2|^2} \int_{U^n} \frac{(1 - |t|^2)^{\alpha-\delta}(1 - |t|^2)^{\delta p} |Df(t)|^p}{|1 - t^2|^2} dm_{2n}(t) dm_{2n}(\xi) = \]
\[ C(\alpha, \pi, p)(1 - |z|^2)^{\alpha+2} \|g\|_{\infty} \int_{U^n} (1 - |t|^2)^{\alpha-\delta+\delta p} |Df(t)|^p \times \]
\[ \int_{U^n} \frac{(1 - |\xi|^2)^{\alpha - (\delta-1)p/q}}{|1 - z^2|^2} dm_{2n}(\xi) dm_{2n}(t). \]

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Then
\[ \|B^\alpha_g(f)\|_{L_p(\omega)} = \int_{U^n} (1 - |t|^2)^{m-\delta+p} |Df(t)|^p \int_{U^n} \frac{(1 - \xi^2)^{\alpha-(\delta-1)p/q}}{|1 - \xi|^{m+1}} \times \]
\[ \int_{U^n} \frac{\omega(1 - |t|)(1 - |z|^2)^\alpha}{|1 - \xi|^{2n+4}} dm_{2n}(z) dm_{2n}(\xi) dm_{2n}(t) \leq \int_{U^n} (1 - |t|^2)^{m-\delta+p} \times \]
\[ |Df(t)|^p \int_{U^n} \frac{(1 - |t|^2)^{\alpha-(\delta-1)p/q}}{|1 - \xi|^{m+1}} \frac{\omega(1 - |t|)}{|1 - \xi|^{\alpha+2}} dm_{2n}(\xi) dm_{2n}(t) = \]
\[ \int_{U^n} \frac{\omega(1 - |t|)|Df(t)|^p}{(1 - |t|^2)^{m-1+2(\delta-1)p/q}} dm_{2n}(t) = \|f\|_{B_p(\omega)} \|g\|_{\infty} C(\alpha, \pi, p). \]

\[ \square \]

We consider now the case of \( p = 1 \).

**Theorem 3.3.** Let \( f \in B_1(\omega) \) (or \( f \in \overline{B_1(\omega)} \)), \( g \in L^\infty(U^n) \). Then \( B^\alpha_g(f) \in L_1(\omega) \) if and only if \( \alpha_j > \alpha_{\omega_j}, 1 \leq j \leq n \).

**Proof.** Let \( f \in B_1(\omega) \) or \( f \in \overline{B_1(\omega)} \). Our aim is to show that \( B^\alpha_g f \in L_1(\omega) \). We have
\[ |B^\alpha_g(f)| \leq (1 - |z|^2)^{\alpha+2} \int_{U^n} \int_{U^n} \frac{(1 - |xi|^2)^{\alpha} |f(\xi)| \cdot |g(\xi)| dm_{2n}(\xi)}{|1 - xi|^4+2n} \leq \]
\[ \|g\|_{\infty} (1 - |z|^2)^{\alpha+2} \int_{U^n} \frac{(1 - |xi|^2)^{\alpha} |f(\xi)| dm_{2n}(\xi)}{|1 - xi|^4+2n} \int_{U^n} \frac{\omega(1 - |t|) |Df(t)|}{|1 - t|^\alpha} dm_{2n}(t) dm_{2n}(\xi). \]

Then using (2) we get
\[ \|B^\alpha_g\|_{L_1(\omega)} \leq \int_{U^n} \int_{U^n} \frac{(1 - |\xi|^2)^{\alpha}}{|1 - \xi|^{2n+4}} \int_{U^n} \frac{(1 - |t|^2)^{m} |Df(t)| \omega(1 - |t|)}{|1 - t|^\alpha} dm_{2n}(z) dm_{2n}(t) dm_{2n}(\xi) \]
\[ = \int_{U^n} (1 - |t|^2)^{m} |Df(t)| \int_{U^n} \frac{(1 - |\xi|^2)^{\alpha}_n |Df(t)| \omega(1 - |t|)}{|1 - \xi|^{m+1}} \int_{U^n} \frac{\omega(1 - |z|)}{|1 - z|^4+2n} dm_{2n}(z) dm_{2n}(\xi) dm_{2n}(t) \leq \]
\[ \int_{U^n} (1 - |t|^2)^{m} |Df(t)| \frac{(1 - |t|^2)\omega(1 - |t|)}{(1 - |t|^2)^{m-1+2+\alpha}} dm_{2n}(t) = \|f\|_{B_1(\omega)} \cdot \|g\|_{\infty}. \]

Now using again the described technique of selection of \( f_r \) by (3) for \( p = 1 \) and \( V^n \), taking \( f_r(\zeta) \equiv |f_r(\zeta)| \) we get
\[ \|B_\alpha(f_r)\|_{L^1(\omega)} \geq \int_{U^n} \omega(1 - |z|)(1 - |z|^2)^{\alpha+2} \int_{V^n} \frac{(1 - |\zeta|)^{\alpha}}{|1 - \zeta|^{2n+4}} |f_r(\zeta)| dm_{2n}(\zeta) dm_{2n}(z) \geq \]
\[ C_1(\alpha, \omega) \frac{(1 - r)^{\alpha}}{\omega(1 - r)(1 - r)^2} \int_{U^n} \omega(1 - |z|)(1 - |z|^2)^{\alpha+2} |1 - rz|^{2n+4} dm_{2n}(z). \]
As in the case of little Hankel operators, assumption of the converse results in a contradiction.

\[ \square \]

**Corollary 3.2.** Let \( \alpha_j > \alpha_{\omega_j}, 1 \leq j \leq n, \ g \in L^\infty(U^n) \). Then \( B_g^\alpha \) is bounded on \( L^1(\omega) \).

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