The Final Solutions of Monty Hall Problem and Three Prisoners Problem

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Abstract
Recently we proposed the linguistic interpretation of quantum mechanics (called quantum and classical measurement theory, or quantum language), which was characterized as a kind of meta-physical and linguistic turn of the Copenhagen interpretation. This turn from physics to language does not only extend quantum theory to classical systems but also yield the quantum mechanical world view (i.e., the philosophy of quantum mechanics, in other words, quantum philosophy). And we believe that this quantum language is the most powerful language to describe science. The purpose of this paper is to describe the Monty-Hall problem and the three prisoners problem in quantum language. We of course believe that our proposal is the final solutions of the two problems. Thus in this paper, we can answer the question: "Why have philosophers continued to stick to these problems?" And the readers will find that these problems are never elementary, and they can not be solved without the deep understanding of "probability" and "dualism".

Keywords: Philosophy of probability, Fisher Maximum Likelihood Method, Bayes’ Method, The Principle of Equal (a priori) Probabilities

1 Introduction
1.1 Monty Hall problem and Three prisoners problem
According to ref. [4], we shall introduce the usual descriptions of the Monty Hall problem and the three prisoners problem as follows.

Problem 1 [Monty Hall problem]. Suppose you are on a game show, and you are given the choice of three doors (i.e., “Door A_1”, “Door A_2”, “Door A_3”). Behind one door is a car, behind the others, goats. You do not know what’s behind the doors. However, you pick a door, say “Door A_1”, and the host, who knows what’s behind the doors, opens another door, say “Door A_3”, which has a goat. He says to you, “Do you want to pick Door A_2?” Is it to your advantage to switch your choice of doors?

Problem 2 [Three prisoners problem]. Three prisoners, A_1, A_2, and A_3 were in jail. They knew that one of them was to be set free and the other two were to be executed. They did not know
who was the one to be spared, but the emperor did know. $A_1$ said to the emperor, “I already know that at least one the other two prisoners will be executed, so if you tell me the name of one who will be executed, you won’t have given me any information about my own execution”. After some thinking, the emperor said, “$A_3$ will be executed.” Thereupon $A_1$ felt happier because his chance had increased from $\frac{1}{3}$ to $\frac{1}{2}$. This prisoner $A_1$’s happiness may or may not be reasonable?

The purpose of this paper is to clarify Problem 1 (Monty Hall problem) and Problem 2 (three prisoners problem ) as follows.

(A1) Problem 1 (Monty Hall problem) is solvable, but Problem 2 (Three prisoners problem) is not well posed. In this sense, Problem 1 and Problem 2 are not equivalent. This is the direct consequence of Fisher’s maximal likelihood method mentioned in Section 2.

(A2) Also, there are two ways that the probabilistic property is introduced to both problems as follows:

(A2$_1$) in Problem 1, one (discussed in Section 4) is that the host casts the dice, and another (discussed in Section 6) is that you cast the dice.

(A2$_2$) in Problem 2, one (discussed in Section 4) is that the emperor casts the dice, and another (discussed in Section 6) is that three prisons cast the dice.

In the case of each, the former solution is due to Bayes’ method ( mentioned in Section 2). And the latter solution is due to the principle is equal probabilities ( mentioned in Section 5). And, after all, we can conclude, under the situation (A2), that Problem 1 and Problem 2 are equivalent.

The above will be shown in terms of quantum language (=measurement theory). And therefore, we expect the readers to find that quantum language is superior to Kolmogorov’s probability theory [15].

1.2 Overview: Measurement Theory (= Quantum Language)

As emphasized in refs. [6, 7], measurement theory (or in short, MT) is, by a linguistic turn of quantum mechanics (cf. Figure 1 later), constructed as the scientific theory formulated in a certain $C^*$-algebra $\mathcal{A}$ (i.e., a norm closed subalgebra in the operator algebra $B(H)$ composed of all bounded operators on a Hilbert space $H$, cf. [16,17] ). MT is composed of two theories (i.e., pure measurement theory (or, in short, PMT) and statistical measurement theory (or, in short, SMT). That is, it has the following structure:
(B) MT (measurement theory = quantum language)

\[
\begin{align*}
(B1) & : [PMT] = [(\text{pure measurement})] + [\text{causality}] \\
& \quad \text{(Axiom P) and (Axiom 2)} \\
(B2) & : [SMT] = [(\text{statistical measurement})] + [\text{causality}] \\
& \quad \text{(Axiom S) and (Axiom 2)}
\end{align*}
\]

where Axiom 2 is common in PMT and SMT. For completeness, note that measurement theory (B) (i.e., (B1) and (B2)) is not physics but a kind of language based on “the quantum mechanical world view”. As seen in [8], note that MT gives a foundation to statistics. That is, roughly speaking,

(C1) it may be understandable to consider that PMT and SMT is related to Fisher’s statistics and Bayesian statistics respectively.

When \( \mathcal{A} = B_c(H) \), the \( C^* \)-algebra composed of all compact operators on a Hilbert space \( H \), the (B) is called quantum measurement theory (or, quantum system theory), which can be regarded as the linguistic aspect of quantum mechanics. Also, when \( \mathcal{A} \) is commutative (that is, when \( \mathcal{A} \) is characterized by \( C_0(\Omega) \), the \( C^* \)-algebra composed of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space \( \Omega \) (cf. [16])), the (B) is called classical measurement theory. Thus, we have the following classification:

\[
(C2) \quad \text{MT} \begin{cases} \text{quantum MT} & \text{(when } \mathcal{A} = B_c(H) \text{)} \\ \text{classical MT} & \text{(when } \mathcal{A} = C_0(\Omega) \text{)} \end{cases}
\]

Also, for the position of MT in science, see Figure 1, which was precisely explained in [7, 10].

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**Figure 1: The history of the world-view**

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2 Classical Measurement Theory (Axioms and Interpretation)

2.1 Mathematical Preparations

Since our concern is concentrated to the Monty Hall problem and three prisoners problem, we devote ourselves to classical MT in (C2).
Throughout this paper, we assume that $\Omega$ is a compact Hausdorff space. Thus, we can put $C_0(\Omega) = C(\Omega)$, which is defined by a Banach space (or precisely, a commutative $C^*$-algebra) composed of all continuous complex-valued functions on a compact Hausdorff space $\Omega$, where its norm $\|f\|_{C(\Omega)}$ is defined by $\max_{\omega \in \Omega} |f(\omega)|$.

Let $C(\Omega)^*$ be the dual Banach space of $C(\Omega)$. That is, $C(\Omega)^* = \{\rho \mid \rho$ is a continuous linear functional on $C(\Omega)\}$, and the norm $\|\rho\|_{C(\Omega)^*}$ is defined by $\sup\{|\rho(f)| : f \in C(\Omega) \text{ such that } ||f||_{C(\Omega)} \leq 1\}$. The bi-linear functional $\rho(f)$ is also denoted by $\langle \rho, f \rangle_{C(\Omega)}$, or in short $\langle \rho, f \rangle$. Define the mixed state $\rho \in (C(\Omega)^*)$ such that $\|\rho\|_{C(\Omega)^*} = 1$ and $\rho(f) \geq 0$ for all $f \in C(\Omega)$ such that $f \geq 0$. And put

$$\mathcal{S}^m(C(\Omega)^*) = \{\rho \in C(\Omega)^* \mid \rho \text{ is a mixed state}\}.$$ 

Also, for each $\omega \in \Omega$, define the pure state $\delta_\omega \in (\mathcal{S}^m(C(\Omega)^*))$ such that $C(\Omega)^* \langle \delta_\omega, f \rangle_{C(\Omega)} = f(\omega)$ $(\forall f \in C(\Omega))$. And put

$$\mathcal{S}^p(C(\Omega)^*) = \{\delta_\omega \in C(\Omega)^* \mid \delta_\omega \text{ is a pure state}\},$$

which is called a state space. Note, by the Riesz theorem (cf. [18]), that $C(\Omega)^* \equiv M(\Omega) \equiv \{\rho \mid \rho$ is a measure on $\Omega\}$ and $\mathcal{S}^m(C(\Omega)^*) \equiv M_{m+1}(\Omega) \equiv \{\rho \mid \rho$ is a measure on $\Omega$ such that $\rho(\Omega) = 1\}$. Also, it is clear that $\mathcal{S}^p(C(\Omega)^*) = \{\delta_\omega \mid \delta_\omega \text{ is a point measure at } \omega_0 \in \Omega\}$, where $\int_\Omega f(\omega)\delta_\omega(d\omega) = f(\omega_0)$ $(\forall f \in C(\Omega))$. This implies that the state space $\mathcal{S}^p(C(\Omega)^*)$ can be also identified with $\Omega$ (called a spectrum space or simply, spectrum) such as

$$\mathcal{S}^p(C(\Omega)^*) \ni \delta_\omega \leftrightarrow \omega \in \Omega \ (\text{spectrum})$$

Also, note that $C(\Omega)$ is unital, i.e., it has the identity $I$ (or precisely, $I_{C(\Omega)}$), since we assume that $\Omega$ is compact.

According to the noted idea (cf. [1]) in quantum mechanics, an observable $O := (X, F, F)$ in $C(\Omega)$ is defined as follows:

(D1) \ [Field] \ $X$ is a set, $F(\subseteq 2^X$, the power set of $X$) is a field of $X$, that is, \"$\Xi_1, \Xi_2 \in F \Rightarrow \Xi_1 \cup \Xi_2 \in F$, "$\Xi \in F \Rightarrow X \ \setminus \ \Xi \in F$\".

(D2) \ [Additivity] \ $F$ is a mapping from $F$ to $C(\Omega)$ satisfying: (a): for every $\Xi \in F$, $F(\Xi)$ is a non-negative element in $C(\Omega)$ such that $0 \leq F(\Xi) \leq I$, (b): $F(\emptyset) = 0$ and $F(X) = I$, where $0$ and $I$ is the 0-element and the identity in $C(\Omega)$ respectively. (c): for any $\Xi_1, \Xi_2 \in F$ such that $\Xi_1 \cap \Xi_2 = \emptyset$, it holds that $F(\Xi_1 \cup \Xi_2) = F(\Xi_1) + F(\Xi_2)$.

For the more precise argument (such as countably additivity, etc.), see [8].

### 2.2 Classical PMT in (B1)

In this section we shall explain classical PMT in (A1).

With any system $S$, a commutative $C^*$-algebra $C(\Omega)$ can be associated in which the measurement theory (B) of that system can be formulated. A state of the system $S$ is represented by an element $\delta_\omega \in \mathcal{S}^p(C(\Omega)^*)$ and an observable is represented by an observable $O := (X, F, F)$ in $C(\Omega)$. Also, the measurement of the observable $O$ for the system $S$ with the state $\delta_\omega$ is denoted by $M_{C(\Omega)}(O, S_{[\delta_\omega]})$ (or more precisely, $M_{C(\Omega)}(O := (X, F, F), S_{[\delta_\omega]})$). An observer can obtain a measured value $x \in X$ by the measurement $M_{C(\Omega)}(O, S_{[\delta_\omega]})$. \
The Axiom P 1 presented below is a kind of mathematical generalization of Born’s probabilistic interpretation of quantum mechanics. And thus, it is a statement without reality.

**Axiom P 1** [Classical PMT]. The probability that a measured value \( x \in X \) obtained by the measurement \( M_{C(\Omega)}(O := (X, F, F), S_{[\delta_{\omega_0}]}(\omega_0)) \) belongs to a set \( \Xi(\in F) \) is given by \( \nu_0(F(\Xi)) \) (i.e., \( C(\Omega)^* \cdot (\nu_0, F(\Xi))_{C(\Omega)} \).

Next, we explain Axiom 2 in (B). Let \((T, \leq)\) be a tree, i.e., a partial ordered set such that “\( t_1 \leq t_2 \) and \( t_2 \leq t_3 \)” implies “\( t_1 \leq t_2 \) or \( t_2 \leq t_1 \)”. In this paper, we assume that \( T \) is finite. Assume that there exists an element \( t_0 \in T \), called the root of \( T \), such that \( t_0 \leq t \) (\( \forall t \in T \)) holds. Put \( T^2_\leq := \{(t_1, t_2) \in T^2 \mid t_1 \leq t_2\} \). The family \( \{\Phi_{t_1, t_2} : C(\Omega_{t_2}) \to C(\Omega_{t_1})\}_{(t_1, t_2) \in T^2_\leq} \) is called a causal relation (due to the Heisenberg picture), if it satisfies the following conditions (E1) and (E2).

(E1) With each \( t \in T \), a \( C^* \)-algebra \( C(\Omega_t) \) is associated.

(E2) For every \( (t_1, t_2) \in T^2_\leq \), a Markov operator \( \Phi_{t_1, t_2} : C(\Omega_{t_2}) \to C(\Omega_{t_1}) \) is defined (i.e., \( \Phi_{t_1, t_2} \geq 0 \), \( \Phi_{t_1, t_2}(I_{C(\Omega_{t_2})}) = I_{C(\Omega_{t_1})} \) ). And it satisfies that \( \Phi_{t_1, t_2} \Phi_{t_2, t_3} = \Phi_{t_1, t_3} \) holds for any \( (t_1, t_2) \), \( (t_2, t_3) \in T^2_\leq \).

The family of dual operators \( \{\Phi_{t_1, t_2}^* : \mathcal{S}^m(C(\Omega_{t_1})) \to \mathcal{S}^m(C(\Omega_{t_2}))\}_{(t_1, t_2) \in T^2_\leq} \) is called a dual causal relation (due to the Schrödinger picture). When \( \Phi_{t_1, t_2}^* (\mathcal{S}^p(C(\Omega_{t_1})) \subseteq \mathcal{S}^p(C(\Omega_{t_2})) \) holds for any \( (t_1, t_2) \in T^2_\leq \), the causal relation is said to be deterministic.

Here, Axiom 2 in the measurement theory (B) is presented as follows:

**Axiom 2** [Causality]. The causality is represented by a causal relation \( \{\Phi_{t_1, t_2} : C(\Omega_{t_2}) \to C(\Omega_{t_1})\}_{(t_1, t_2) \in T^2_\leq} \).

For the further argument (i.e., the \( W^* \)-algebraic formulation) of measurement theory, see Appendix in [6].

### 2.3 Classical SMT in (B2)

It is usual to consider that we do not know the state \( \delta_{\omega_0} \) when we take a measurement \( M_{C(\Omega)}(O, S_{[\delta_{\omega_0}]}(\omega_0)) \). That is because we usually take a measurement \( M_{C(\Omega)}(O, S_{[\delta_{\omega_0}]}(\omega_0)) \) in order to know the state \( \delta_{\omega_0} \). Thus, when we want to emphasize that we do not know the state \( \delta_{\omega_0} \), \( M_{C(\Omega)}(O, S_{[\delta_{\omega_0}]}(\omega_0)) \) is denoted by \( M_{C(\Omega)}(O, S_{[\phi_{\omega_0}]}(\omega_0)) \). Also, when we know the distribution \( \nu_0 \in \mathcal{M}^m_{\ominus}(\Omega) = \mathcal{S}^m(C(\Omega^*)) \) of the unknown state \( \delta_{\omega_0} \), the \( M_{C(\Omega)}(O, S_{[\delta_{\omega_0}]}(\omega_0)) \) is denoted by \( M_{C(\Omega)}(O, S_{[\phi_{\omega_0}]}(\nu_0)) \).

The Axiom S 1 presented below is a kind of mathematical generalization of Axiom P 1.

**Axiom S 1** [Classical SMT]. The probability that a measured value \( x \in X \) obtained by the measurement \( M_{C(\Omega)}(O := (X, F, F), S_{[\delta_{\omega_0}]}(\nu_0)) \) belongs to a set \( \Xi(\in F) \) is given by \( \nu_0(F(\Xi)) \) (i.e., \( C(\Omega)^* \cdot (\nu_0, F(\Xi))_{C(\Omega)} \)).

**Remark 1.** Note that two statistical measurements \( M_{C(\Omega)}(O, S_{[\delta_{\omega_1}]}(\nu_0)) \) and \( M_{C(\Omega)}(O, S_{[\delta_{\omega_2}]}(\nu_0)) \) can not be distinguished before measurements. In this sense, we consider that, even if \( \omega_1 \neq \omega_2 \), we can assume that

\[
M_{C(\Omega)}(O, S_{[\delta_{\omega_1}]}(\nu_0)) = M_{C(\Omega)}(O, S_{[\delta_{\omega_2}]}(\nu_0)) = M_{C(\Omega)}(O, S_{[\delta_{\omega}]}(\nu_0)).
\]

### 2.4 Linguistic Interpretation

Next, we have to answer how to use the above axioms as follows. That is, we present the following linguistic interpretation (F) \( [=(F_1)-(F_3)] \), which is characterized as a kind of linguistic turn of so-called Copenhagen interpretation (cf. [6,7]).
That is, we propose:

(F_1) Consider the dualism composed of “observer” and “system (=measuring object)” such as

\[ (\text{observer (I=mind)}) \quad \text{system (matter)} \]

\[ \text{measured value} \leftrightarrow \text{observable} \quad \text{interfere} \quad \text{perceive a reaction} \]

\[ \text{state} \]

Figure 2. Descartes’ figure in MT

And therefore, “observer” and “system” must be absolutely separated.

(F_2) Only one measurement is permitted. And thus, the state after a measurement is meaningless since it can not be measured any longer. Also, the causality should be assumed only in the side of system, however, a state never moves. Thus, the Heisenberg picture should be adopted.

(F_3) Also, the observer does not have the space-time. Thus, the question: “When and where is a measured value obtained?” is out of measurement theory,

and so on. This interpretation is, of course, common to both PMT and SMT.

2.5 Preliminary Fundamental Theorems

We have the following two fundamental theorems in measurement theory:

**Theorem 1** [Fisher’s maximum likelihood method (cf. [8])]. Assume that a measured value obtained by a measurement \( M_{C(\Omega)}(O := (X, F, F), S_*) \) belongs to \( \Xi \ (\in \mathcal{F}) \). Then, there is a reason to infer that the unknown state \([\ast]\) is equal to \( \delta_{\omega_0} \), where \( \omega_0 \ (\in \Omega) \) is defined by

\[
[F(\Xi)](\omega_0) = \max_{\omega \in \Omega} [F(\Xi)](\omega).
\]

**Theorem 2** [Bayes’ method (cf. [8])]. Assume that a measured value obtained by a statistical measurement \( M_{C(\Omega)}(O := (X, F, F), S_*[\nu]) \) belongs to \( \Xi \ (\in \mathcal{F}) \). Then, there is a reason to infer that the posterior state (i.e., the mixed state after the measurement ) is equal to \( \nu_{\text{post}} \), which is defined by

\[
\nu_{\text{post}}(D) = \frac{\int_D [F(\Xi)](\omega)\nu_0(d\omega)}{\int_{\Omega} [F(\Xi)](\omega)\nu_0(d\omega)} \\
(\forall D \in \mathcal{B}_\Omega; \text{Borel field}).
\]

The above two theorems are, of course, the most fundamental in statistics. Thus, we believe in Figure 1, i.e.,

\[
\text{statistics} \quad \longrightarrow \quad \Xi \quad \text{quantum language}
\]
3 The First Answer to Monty Hall Problem [resp. Three prisoners problem] by Fisher’s method

In this section, we present the first answer to Problem 1 (Monty-Hall problem) [resp. Problem 2 (Three prisoners problem)] in classical PMT. The two will be simultaneously solved as follows. The spirit of dualism (in Figure 2) urges us to declare that

\[(G) \text{ "observer } \approx \text{ you" and "system } \approx \text{ three doors" in Problem 1 [resp. "observer } \approx \text{ prisoner } A_1\text{" and "system } \approx \text{ emperor’s mind" in Problem 2}\]

Put \(\Omega = \{\omega_1, \omega_2, \omega_3\}\) with the discrete topology. Assume that each state \(\delta_{\omega_m}(\in \mathcal{G}^p(C(\Omega)^*))\) means

\[
\delta_{\omega_m} \Leftrightarrow \text{ the state that the car is behind the door } A_m
\]

\[\text{[resp. } \delta_{\omega_m} \Leftrightarrow \text{ the state that the prisoner } A_m \text{ will be free }\]

\[(m = 1, 2, 3)\]  \hfill (3)

Define the observable \(O_1 \equiv (\{1, 2, 3\}, 2^{\{1,2,3\}}, F_1)\) in \(C(\Omega)\) such that

\[
[F_1(\{1\}))(\omega_1) = 0.0, \quad [F_1(\{2\}))(\omega_1) = 0.5, \quad [F_1(\{3\}))(\omega_1) = 0.5, \\
[F_1(\{1\}))(\omega_2) = 0.0, \quad [F_1(\{2\}))(\omega_2) = 0.0, \quad [F_1(\{3\}))(\omega_2) = 1.0, \\
[F_1(\{1\}))(\omega_3) = 0.0, \quad [F_1(\{2\}))(\omega_3) = 1.0, \quad [F_1(\{3\}))(\omega_3) = 0.0,\]
\]  \hfill (4)

where it is also possible to assume that \(F_1(\{2\}))(\omega_1) = \alpha, F_1(\{3\}))(\omega_1) = 1 - \alpha (0 < \alpha < 1)\). Thus we have a measurement \(M_{C(\Omega)}(O_1, S_\{s\})\), which should be regarded as the measurement theoretical representation of the measurement that you say "Door \(A_1\)" [resp. "Prisoner \(A_1\)" asks to the emperor\]. Here, we assume that

a) “measured value 1 is obtained by the measurement \(M_{C(\Omega)}(O_1, S_\{s\})\)”

\[\Leftrightarrow \text{ The host says “Door } A_1 \text{ has a goat” [resp. } \Rightarrow \text{ the emperor says “Prisoner } A_1 \text{ will be executed” ]}\]

b) “measured value 2 is obtained by the measurement \(M_{C(\Omega)}(O_1, S_\{s\})\) ”

\[\Leftrightarrow \text{ The host says “Door } A_2 \text{ has a goat” [resp. } \Rightarrow \text{ the emperor says “Prisoner } A_2 \text{ will be executed” ]}\]

c) “measured value 3 is obtained by the measurement \(M_{C(\Omega)}(O_1, S_\{s\})\) ”

\[\Leftrightarrow \text{ The host says “Door } A_3 \text{ has a goat” [resp. } \Rightarrow \text{ the emperor says “Prisoner } A_3 \text{ will be executed” ]}\]

Recall that, in Problem 1 (Monty-Hall problem) [resp. Problem 2 (Three prisoners problem)], the host said “Door 3 has a goat” [resp. the emperor said “Prisoner 3 will be executed”] This implies that you [resp. “Prisoner \(A_1\)] get the measured value “3” by the measurement \(M_{C(\Omega)}(O_1, S_\{s\})\). Note that

\[
[F_1(\{3\}))(\omega_2) = 1.0 = \max\{0.5, \quad 1.0, \quad 0.0\} \\
= \max\{[F_1(\{3\}))(\omega_1), [F_1(\{3\}))(\omega_2), [F_1(\{3\}))(\omega_3)\},\]
\]  \hfill (5)

Therefore, Theorem 1 (Fisher’s maximum likelihood method) says that

(H1) In Problem 1 (Monty-Hall problem), there is a reason to infer that \(* = \delta_{\omega_2}\). Thus, you should switch to Door \(A_2\).
(H2) In Problem 2 (Three prisoners problem), there is a reason to infer that \([\ast] = \delta_{\omega_2}\). However, there is no reasonable answer for the question: whether Prisoner \(A_1\)'s happiness increases. That is, Problem 2 is not a well-posed problem.

4 The Second Answer to Monty Hall Problem \([\text{resp. Three prisoners problem}]\) by Bayes’ method

In order to use Bayes’ method, shall modify Problem 1(Monty Hall problem) and Problem 2(Three prisoners problem) as follows.

4.1 Problems 1' and 2' (Monty Hall Problem \([\text{resp. Three prisoners problem}]\) )

Problem 1' [Monty Hall problem; the host casts the dice]. Suppose you are on a game show, and you are given the choice of three doors (i.e., “Door \(A_1\)”, “Door \(A_2\)”, “Door \(A_3\)”). Behind one door is a car, behind the others, goats. You do not know what’s behind the doors. However, you pick a door, say ”Door \(A_1\)”, and the host, who knows what’s behind the doors, opens another door, say “Door \(A_3\)”, which has a goat. And he adds that

(\(\sharp_1\)) the car was set behind the door decided by the cast of the (distorted) dice. That is, the host set the car behind Door \(A_m\) with probability \(p_m\) (where \(p_1 + p_2 + p_3 = 1, 0 \leq p_1, p_2, p_3 \leq 1\)).

He says to you, “Do you want to pick Door \(A_2\)?” Is it to your advantage to switch your choice of doors?

Problem 2' [Three prisoners problem; the emperor casts the dice]. Three prisoners, \(A_1\), \(A_2\), and \(A_3\) were in jail. They knew that one of them was to be set free and the other two were to be executed. They did not know who was the one to be spared, but they know that

(\(\sharp_2\)) the one to be spared was decided by the cast of the (distorted) dice. That is, Prisoner \(A_m\) is to be spared with probability \(p_m\) (where \(p_1 + p_2 + p_3 = 1, 0 \leq p_1, p_2, p_3 \leq 1\)).

but the emperor did know the one to be spared. \(A_1\) said to the emperor, ”I already know that at least one the other two prisoners will be executed, so if you tell me the name of one who will be executed, you won’t have given me any information about my own execution”. After some thinking, the emperor said, “\(A_3\) will be executed.” Thereupon \(A_1\) felt happier because his chance had increased from \(\frac{1}{3(\text{Num}[\{A_1, A_2, A_3\}])}\) to \(\frac{1}{2(\text{Num}[\{A_1, A_2\}])}\). This prisoner \(A_1\)’s happiness may or may not be reasonable?
Remark 2. In Problem 1′, you may choose "Door A_1" by various ways. For example, you may choose "Door A_1" by the method mentioned in Problem 1′ later.

4.2 The second answer to Problems 1′ and 2′ (Monty Hall Problem [resp. Three prisoners problem]) by Bayes’ method

In what follows we study these problems. Let Ω and O_1 be as in Section 3. Under the hypothesis (♯_1) [resp. (♯_2)], define the mixed state ν_0 (∈ M_{m+1}^m(Ω)) such that:

\[ \nu_0(\{\omega_1\}) = p_1, \quad \nu_0(\{\omega_2\}) = p_2, \quad \nu_0(\{\omega_3\}) = p_3 \]  \hspace{1cm} (6)

Thus we have a statistical measurement \( M_{C(\Omega)}(O_1, S_{[\nu]}(\nu_0)) \). Note that

a) “measured value 1 is obtained by the statistical measurement \( M_{C(\Omega)}(O_1, S_{[\nu]}(\nu_0)) \)”
⇔ the host says “Door A_1 has a goat”
[resp. ⇔ the emperor says “Prisoner A_1 will be executed”]

b) “measured value 2 is obtained by the statistical measurement \( M_{C(\Omega)}(O_1, S_{[\nu]}(\nu_0)) \)”
⇔ the host says “Door A_2 has a goat”
[resp. ⇔ the emperor says “Prisoner A_2 will be executed”]

c) “measured value 3 is obtained by the statistical measurement \( M_{C(\Omega)}(O_1, S_{[\nu]}(\nu_0)) \)”
⇔ the host says “Door A_3 has a goat”
[resp. ⇔ the emperor says “Prisoner A_3 will be executed”]

Here, assume that, by the statistical measurement \( M_{C(\Omega)}(O_1, S_{[\nu]}(\nu_0)) \), you obtain a measured value 3, which corresponds to the fact that the host said “Door A_3 has a goat.” [resp. the emperor said that Prisoner A_3 is to be executed]. Then, Theorem 2 (Bayes’ method) says that the posterior state \( \nu_{\text{post}} \) (∈ M_{m+1}^m(Ω)) is given by

\[ \nu_{\text{post}} = \frac{F_1(\{3\}) \times \nu_0}{\langle \nu_0, F_1(\{3\}) \rangle}. \]  \hspace{1cm} (7)

That is,

\[ \nu_{\text{post}}(\{\omega_1\}) = \frac{p_1}{p_1 + p_2}, \quad \nu_{\text{post}}(\{\omega_2\}) = \frac{p_2}{p_1 + p_2}, \quad \nu_{\text{post}}(\{\omega_3\}) = 0. \]  \hspace{1cm} (8)

Then,
(I1) In Problem 1′,
\[
\begin{cases}
\text{if } \nu_{\text{post}}\left(\{\omega_1\}\right) < \nu_{\text{post}}\left(\{\omega_2\}\right) \text{ (i.e., } p_1 < 2p_2\text{), you should pick Door } A_2 \\
\text{if } \nu_{\text{post}}\left(\{\omega_1\}\right) = \nu_{\text{post}}\left(\{\omega_2\}\right) \text{ (i.e., } p_1 < 2p_2\text{), you may pick Doors } A_1 \text{ or } A_2 \\
\text{if } \nu_{\text{post}}\left(\{\omega_1\}\right) > \nu_{\text{post}}\left(\{\omega_2\}\right) \text{ (i.e., } p_1 > 2p_2\text{), you should not pick Door } A_2
\end{cases}
\]

(II) In Problem 2′,
\[
\begin{cases}
\text{if } \nu_0\left(\{\omega_1\}\right) < \nu_{\text{post}}\left(\{\omega_1\}\right) \text{ (i.e., } p_1 < 1 - 2p_2\text{), the prisoner } A_1's \text{ happiness increases} \\
\text{if } \nu_0\left(\{\omega_1\}\right) = \nu_{\text{post}}\left(\{\omega_1\}\right) \text{ (i.e., } p_1 = 1 - 2p_2\text{), the prisoner } A_1's \text{ happiness is invariant} \\
\text{if } \nu_0\left(\{\omega_1\}\right) > \nu_{\text{post}}\left(\{\omega_1\}\right) \text{ (i.e., } p_1 > 1 - 2p_2\text{), the prisoner } A_1's \text{ happiness decreases}
\end{cases}
\]

5 The Principle of Equal Probability

In this section, according to [4, 6, 11] we prepare Theorem 3 (the principle of equal probability), i.e.,

(J) unless we have sufficient reason to regard one possible case as more probable than another, we treat them as equally probable.

This theorem will be used in the following section.

Put $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_n\}$ with the discrete topology. And consider any observable $O_1 \equiv (X, \mathcal{F}, F_1)$ in $C(\Omega)$.

Define the bijection $\phi_1 : \Omega \to \Omega$ such that
\[
\phi_1(\omega_j) = \begin{cases} 
\omega_{j+1} & (j \neq n) \\
\omega_1 & (j = n)
\end{cases}
\]
and define the observable $O_k \equiv (X, \mathcal{F}, F_k)$ in $C(\Omega)$ such that
\[
[F_k(\Xi)](\omega) = [F_1(\Xi)](\phi_k^{-1}(\omega))
\]
($\forall \omega \in \Omega, k = 1, 2, \ldots, n$)

where $\phi_0(\omega) = \omega(\forall \in \Omega)$ and $\phi_k(\omega) = \phi_1(\phi_{k-1}(\omega))$ ($\forall \omega \in \Omega, k = 1, 2, \ldots, n$).

Let $p_k(k = 1, \ldots, n)$ be a non-negative real number such that $\sum_{k=1}^n p_k = 1$.

(K) For example, fix a state $\delta_{\omega_m}$ ($m = 1, 2, \ldots, n$). And, by the cast of the (distorted) dice, you choose an observable $O_k \equiv (X, \mathcal{F}, F_k)$ with probability $p_k$. And further, you take a measurement $M_{C(\Omega)}(O_k := (X, \mathcal{F}, F_k), S[\delta_{\omega_m}])$.

Here, we can easily see that the probability that a measured value obtained by the measurement (K) belongs to $\Xi(\in \mathcal{F})$ is given by
\[
\sum_{k=1}^n p_k[F_k(\Xi)](\delta_{\omega_m}) = \sum_{k=1}^n p_k[F_k(\Xi)](\omega_m)
\]
which is equal to $\langle F_1(\Xi), \sum_{k=1}^{n} p_k \delta_{\phi_{k-1}(\omega_m)} \rangle$. This implies that the measurement $(K)$ is equivalent to a statistical measurement $M_{C(\Omega)}(O_1 := (X, F, F_1), S_{[\delta_m]}(\sum_{k=1}^{n} p_k \delta_{\phi_{k-1}(\omega_m)}))$. Note that the $(9)$ depends on the state $\delta_m$. Thus, we can not calculate the $(9)$ such as the $(8)$. However, if it holds that $p_k = 1/n$ ($k = 1, ..., n$), we see that $\frac{1}{n} \sum_{k=1}^{n} \delta_{\phi_{k-1}(\omega_m)}$ is independent of the choice of the state $\delta_m$. Thus, putting $\frac{1}{n} \sum_{k=1}^{n} \delta_{\phi_{k-1}(\omega_m)} = \nu_e$, we see that the measurement $(K)$ is equivalent to the statistical measurement $M_{C(\Omega)}(O_1, S_{[\delta_m]}(\nu_e))$, which is also equivalent to $M_{C(\Omega)}(O_1, S_{[\delta_m]}(\nu_e))$ (from the formula (2) in Remark 1).

Thus, under the above notation, we have the following theorem, which realizes the spirit $(J)$.

**Theorem 3** [The principle of equal probability (i.e., the equal probability of selection)]. If $p_k = 1/n$ ($k = 1, ..., n$), the measurement $(K)$ is independent of the choice of the state $\delta_m$. Hence, the $(K)$ is equivalent to a statistical measurement $M_{C(\Omega)}(O_1 := (X, F, F_1), S_{[\delta_m]}(\nu_e))$. It should be noted that the principle of equal probability is not "principle" but "theorem" in measurement theory.

**Remark 3.** In the above argument, we consider the set $B' = \{ \phi_k \mid k = 1, 2, ..., n \}$. However, it may be more natural to consider the set $B = \{ \phi \mid \phi : \Omega \rightarrow \Omega \text{ is a bijection} \}$.

6 The Third Answer to Monty Hall Problem [resp. Three prisoners problem] by the principle of equal probability

6.1 Problems 1” and 2” (Monty Hall Problem [resp. Three prisoners problem])

**Problem 1”** [Monty Hall problem; you cast the dice]. Suppose you are on a game show, and you are given the choice of three doors (i.e., “Door $A_1$”, “Door $A_2$”, “Door $A_3$”). Behind one door is a car, behind the others, goats. You do not know what’s behind the doors. Thus,

(♯₁) you select Door $A_1$ by the cast of the fair dice. That is, you say “Door $A_1$” with probability $1/3$.

The host, who knows what’s behind the doors, opens another door, say “Door $A_3$,” which has a goat. He says to you, “Do you want to pick Door $A_2$?” Is it to your advantage to switch your choice of doors?

---

Door $A_1$ Door $A_2$ Door $A_3$

---

**Problem 2”** [Three prisoners problem; the prisoners cast the dice]. Three prisoners, $A_1$, $A_2$, and $A_3$ were in jail. They knew that one of them was to be set free and the other two were to be executed. They did not know who was the one to be spared, but the emperor did know. Since three prisoners wanted to ask the emperor,

(♯₂) the questioner was decided by the fair die throw. And Prisoner $A_1$ was selected with probability $1/3$
Then, $A_1$ said to the emperor, “I already know that at least one of the other two prisoners will be executed, so if you tell me the name of one who will be executed, you won’t have given me any information about my own execution”. After some thinking, the emperor said, “$A_3$ will be executed.” Thereupon $A_1$ felt happier because his chance had increased from $\frac{1}{3(=\text{Num}[\{A_1, A_2, A_3\}])}$ to $\frac{1}{2(=\text{Num}[\{A_1, A_2\}])}$. This prisoner $A_1$’s happiness may or may not be reasonable?

\[\text{Answer: By Theorem 3 (The principle of equal probability), the above Problems 1}'' \text{ and 2}'' \text{ is respectively the same as Problems 1}' \text{ and 2}' \text{ in the case that } p_1 = p_2 = p_3 = 1/3. \text{ Then, the formulas (6) and (8) say that}
\]

(L1) In Problem 1$''$, since $\nu_{\text{post}}(\{\omega_1\}) = 1/3 < 2/3 = \nu_{\text{post}}(\{\omega_2\})$, you should pick Door $A_2$.

(L2) In Problem 2$''$, since $\nu_0(\{\omega_1\}) = 1/3 = \nu_{\text{post}}(\{\omega_1\})$, the prisoner $A_1$’s happiness is invariant.

7 Conclusions

Although main idea is due to refs. [5, 11], in this paper we simultaneously discussed the Monty Hall problem and the three prisoners problem in terms of quantum language. That is, we gave three answers, i.e.,

(M1) the first answer (due to Fisher’s method) in Section 3,

(M2) the second answer (due to Bayes’ method) in Section 4,

(M3) the third answer (due to Theorem 3 (the principle of equal probability)) in Section 6

We of course believe that our proposal is the final solutions of the two problems. It should be noted that both the Monty Hall problem and the three prisoners problem are never elementary, and they can not be solved without the deep understanding of “probability” and “dualism (G)”. Thus in this paper, we answered the question:

"Why have philosophers continued to stick to these problems?"

We hope that our assertion will be examined from various view points.

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