TRANSFER OF IDEALS AND QUANTIZATION OF SMALL NILPOTENT ORBITS

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Abstract. We introduce and study a transfer map between ideals of the universal enveloping algebras of two members of a reductive dual pair of Lie algebras. Its definition is motivated by the approach to the real Howe duality through the theory of Capelli identities. We prove that this map provides a lower bound on the annihilators of theta lifts of representations with a fixed annihilator ideal. We also show that in the algebraic stable range, transfer respects the class of quantizations of nilpotent orbit closures. As an application, we explicitly describe quantizations of small nilpotent orbits of general linear and orthogonal Lie algebras and give presentations of certain rings of algebraic differential operators. We consider two algebraic versions of Howe duality and reformulate our results in terms of noncommutative Capelli identities.

1. Introduction and the main results

There is a well known relation, going back to Alfredo Capelli, between the centers of the universal enveloping algebras of two Lie algebras forming a reductive dual pair in the sense of Howe, [8, 9, 25]. From the modern viewpoint, it arises by restricting the oscillator representation of the real symplectic group to the members of the dual pair and manifests itself via the so-called Capelli identities, [11, 12, 18]. The case originally considered by Capelli in [3] is now known as the \((GL_n, GL_m)\) duality, [10]. Capelli identities have important applications to the study of the theta correspondence: they determine the correspondence of the infinitesimal characters of the representations matched under the Howe duality, cf [7, 25]. The goal of the present paper is to extend this formalism from central elements to more general ad-invariant subspaces and ideals of universal enveloping algebras. We define a certain transfer map between two-sided ideals of the universal enveloping algebras of the two members of a reductive dual pair, establish its basic properties, and compute it in a special case, for the annihilator of a one-dimensional representation of \(\mathfrak{g}'\), i.e. a codimension one ideal of \(U(\mathfrak{g}')\). If the reductive dual pair is in the stable range with \(\mathfrak{g}'\) the smaller member, this amounts to a noncommutative analogue of the Kraft–Procesi lifting of the zero orbit, [14], and yields an ideal \(I\) of \(U(\mathfrak{g})\) quantizing the closure of a small nilpotent orbit in a classical Lie algebra \(\mathfrak{g}\). We explicitly describe an \(\text{ad}(\mathfrak{g})\)-invariant subspace generating \(I\) when \(\mathfrak{g}\) is a general linear or orthogonal Lie algebra.

Let \(\mathbb{K}\) be a field of characteristic zero, \(W\) a symplectic vector space over \(\mathbb{K}\), \(\mathfrak{sp}(W)\) the symplectic Lie algebra of \(W\) and \((\mathfrak{g}, \mathfrak{g}')\) an irreducible reductive dual pair of
Lie algebras. By definition, this means that \( \mathfrak{g} \) and \( \mathfrak{g}' \) are Lie subalgebras of \( \mathfrak{sp}(W) \) that are full mutual centralizers, \( W \) is completely reducible as a \( \mathfrak{g} \)-module and a \( \mathfrak{g}' \)-module, and the action of \( \mathfrak{g} \oplus \mathfrak{g}' \) on \( W \) is symplectically irreducible. Reductive dual pairs were classified by Howe, see [7]. If \( K \) is algebraically closed then either (1) both \( \mathfrak{g} \) and \( \mathfrak{g}' \) are general linear Lie algebras (type II), or (2) one of them is a symplectic and the other is an orthogonal Lie algebra (type I). The embedding of the pair \( (\mathfrak{g}, \mathfrak{g}') \) into \( \mathfrak{sp}(W) \) is determined uniquely modulo symplectic automorphisms of \( W \). (Later on we will fix a particular realization of the dual pair.) Consider the Weyl algebra of \( W \), denoted \( \mathcal{W} \). Its definition and properties are recalled in Section 2.1 but the salient point is that the universal enveloping algebras of \( \mathfrak{g} \) and \( \mathfrak{g}' \) have canonical homomorphisms into \( \mathcal{W} \), and it follows from Howe’s Double Commutant Theorem, [8], that the images of these homomorphisms are the full mutual centralizers in the even Weyl subalgebra.

**Definition 1.1.** Let \( L : U(\mathfrak{g}') \to \mathcal{W} \) and \( R : U(\mathfrak{g}) \to \mathcal{W} \) be the canonical homomorphisms into the Weyl algebra. For any subspace \( I' \) of \( U(\mathfrak{g}') \), its **transfer** \( \theta(I') \) is defined to be the subspace of \( U(\mathfrak{g}) \) which is the pull-back via \( R \) of the right ideal generated by the image of \( I' \) in \( \mathcal{W} \):

\[
\theta(I') = R^{-1}(L(I') \cdot \mathcal{W}).
\]

The preimage \( R^{-1} \) should be understood in the following way: we consider those elements of \( L(I') \cdot \mathcal{W} \) which belong to \( R(U(\mathfrak{g})) \), or equivalently (by the Double Commutant Theorem), which are \( \mathfrak{g}' \)-invariant, and lift them to \( U(\mathfrak{g}) \). The lifting is well-defined on the level of subspaces and the resulting subspace of \( U(\mathfrak{g}) \) contains \( \ker R \). It is easy to see that \( I = \theta(I') \) is always a right ideal in \( U(\mathfrak{g}) \) and that if \( I' \) is a two-sided ideal of \( U(\mathfrak{g}') \) then \( I \) is a two-sided ideal of \( U(\mathfrak{g}) \). The next theorem motivates this definition: it shows that the transfer map is compatible with the Howe duality correspondence for real Lie groups.

**Theorem 1.3.** Let \((G_0, G_0')\) be an irreducible reductive dual pair of real Lie groups with the complexified Lie algebras \( \mathfrak{g} \) and \( \mathfrak{g}' \). Suppose that \( \rho \) and \( \rho' \) are irreducible admissible representations of the metaplectic covers of \( G_0 \) and \( G_0' \) that correspond to each other under the real Howe duality correspondence, [9]. Denote by \( I \) and \( I' \) the annihilators of \( \rho \) and \( \rho' \) in the universal enveloping algebras \( U(\mathfrak{g}) \) and \( U(\mathfrak{g}') \). Then \( \theta(I') \subseteq I \).

Generalizing Howe and Umeda’s treatment of the classical Capelli identities, in Section 5 we interpret the ideal \( \theta(I') \) as a system of noncommutative Capelli identities. Note that \( \theta(I') \) provides only a lower bound for the annihilators of the representations of the metaplectic cover of \( G_0 \) that are matched under the Howe duality correspondence with representations of the metaplectic cover of \( G_0' \) with a fixed primitive annihilator \( I' \). It is a subtle problem to formulate sufficient conditions for the equality to hold. This question is further discussed at the end of the paper.

The transfer in general is rather complicated. However, we show that ideals of special type, quantizations of nilpotent orbit closures, behave well under the transfer.

**Definition 1.4.** An ideal \( I \) of the universal enveloping algebra \( U(\mathfrak{g}) \) is a quantization of the orbit closure \( \mathcal{O} \) if the associated graded ideal \( \text{gr} I \) is the prime ideal defining \( \mathcal{O} \).
The property of being a quantization of an orbit closure is very strong and it implies that the ideal is completely prime and primitive. When \( g \) is general or special linear Lie algebra, the converse statement had been conjectured by Dixmier and was proved by Moeglin, [17]; it fails for other types.

**Theorem 1.5.** Let \((g, g')\) be an irreducible reductive dual pair of Lie algebras in the stable range with the smaller member \( g' \). Suppose that \( I' \) is a completely prime, primitive ideal of \( U(g') \) quantizing an orbit closure \( \overline{O'} \). Then its transfer \( \theta(I') \) is a completely prime, primitive ideal of \( U(g) \) quantizing an orbit closure \( \overline{O} \), the Kraft–Procesi lifting of \( \overline{O'} \).

We give a short direct proof of this theorem based on the flatness of the moment map \( l \) in the stable range.

The preceding theory is made more explicit in the case of a one-dimensional representation of \( g' \) and its annihilator \( I' \). This ideal has codimension one in \( U(g') \), quantizes the zero orbit, and has an obvious system of generators. We explicitly compute an ad-invariant system of generators of its transfer \( I = \theta(I') \subseteq U(g) \).

By Theorem 1.5 the ideal \( I \) obtained in this way is a quantization of the prime ideal defining an orbit whose Young diagram has exactly two columns (we call such orbits small). It is convenient to state our description of the generators of \( I \) in terms of “noncommutative linear algebra”, using a certain matrix with elements in \( U(g) \) (see Section 4 for the precise formulations, including the notation).

In the general linear case, \( g = \mathfrak{gl}_n \), let us denote by \( E \) the \( n \times n \) matrix over \( U(\mathfrak{gl}_n) \) whose \( ij \)th entry is \( E_{ij} \), the standard generator of \( \mathfrak{gl}_n \) given by the \((i, j)\) matrix unit. In Theorem 4.6 we show that the ideal \( I \) is generated by the trace of \( E \) shifted by a certain constant, the matrix entries of an explicitly given quadratic polynomial in \( E \), and quantum minors of \( E \) introduced in [23], cf Definition 4.9. The main result in the symplectic–orthogonal case, Theorem 4.13, has similar form, with the quantum pfaffians of \([12, 18]\) replacing the quantum minors, cf Definition 4.11. Note that, in contrast with the case of minimal nilpotent orbits considered by Kostant, for a more general small nilpotent orbit there are relations of degree greater than two arising from quantum minors and pfaffians.

It follows from the work of Borho–Brylinski, [2], and Levasseur–Stafford, [15], that the ring of algebraic differential operators on the (projective, nonsingular) Grassmann variety or the (affine, singular) rank variety may be identified with the quotient of \( U(g) \) by the ideal of this type. Therefore, we obtain presentations of these noncommutative rings of differential operators by generators and relations.

## 2. Preliminaries

The goal of this section is to set up the terminology and review the basic properties of Weyl algebras, reductive dual pairs, and nilpotent orbits. We present a formulation of the theory of spherical harmonics in the form needed to compute transfers of ideals. Sections 2.1 and 2.3 follow the approach of Howe’s Schur Lectures [10], where many details may be found.

### 2.1. Weyl algebra

Let \( W \) be a symplectic vector space. The Weyl algebra \( \mathcal{W} \) is a filtered algebra generated by the vector space \( W \) in degree 1, modulo the relations \( w_1 w_2 - w_2 w_1 = \langle w_1, w_2 \rangle \). Its associated graded algebra is the symmetric algebra of
the group $Sp(W)$ acts on $W$ by filtered algebra automorphisms, and $W$ decomposes into a direct sum over non-negative integers $n$ of finite-dimensional $Sp(W)$-modules $W^n \cong S^n(W)$. The components of this decomposition are irreducible and mutually non-isomorphic modules, therefore, there is a canonical $Sp(W)$-invariant linear symbol map between $W$ and $S(W)$. On the elements of degree 2 in $S(W)$, the (inverse) symbol map is given by the formula:

$$s^{-1}(w_1w_2) = \frac{1}{2}(w_1w_2 + w_2w_1).$$

Furthermore, $[W^2, W^n] \subseteq W^n$ and the adjoint action of $W^2$ on $W$ coincides with the infinitesimal action of $sp(W)$, the symplectic Lie algebra of $W$. The direct sum $A$ of all even components $W^{2k}, k \geq 0$, is a subalgebra of $W$ called the even Weyl algebra. It is generated by $W^2$, the lowest positive degree component, and hence $A$ is a quotient of the universal enveloping algebra $U(sp(W))$. Howe’s Double Commutant Theorem states that, given a reductive dual pair of Lie algebras, the subalgebra $R(U(g))$ of the Weyl algebra coincides with the subalgebra of $G'$-invariants in $W$, which also coincides with the space of $ad(g')$-invariants in the even Weyl algebra, i.e., with the commutant of $g'$ or $L(U(g'))$ in $A$.

2.2. Moment maps and the Kraft–Procesi lifting. The Weyl algebra $W$ is a noncommutative filtered algebra, and the maps $L : U(g') \to W$ and $R : U(g) \to W$ become filtered maps if the elements from $g' \subseteq U(g')$ and $g \subseteq U(g)$ are assigned degree two. Taking the associated graded maps, we get a pair of commutative algebra homomorphisms, $\text{gr} L : S(g') \to S(W)$ and $\text{gr} R : S(g) \to S(W)$, and we denote the corresponding morphisms of algebraic varieties by $l : W^* \to g'^*$ and $r : W^* \to g^*$ and refer to them as the moment maps.

A reductive dual pair $(g, g')$ of Lie algebras canonically determines a reductive dual pair $(G, G')$ of algebraic groups over $K$ as follows: $G$ is the pointwise stabilizer of $g'$ in $Sp(W)$ and $G'$ is the pointwise stabilizer of $g$ in $Sp(W)$. If $g$ is a general linear Lie algebra then $G$ is the corresponding general linear group, and if $g$ is the Lie algebra of infinitesimal symmetries of a bilinear or a sesquilinear form then $G$ is the full isometry group of the form ($G$ may be disconnected), and similarly for $G'$. We call $G$ and $G'$ the classical groups of $g$ and $g'$, they form a reductive dual pair in $Sp(W)$. The maps $l$ and $r$ are equivariant with respect to the natural actions of $G$ and $G'$ on $g^*, g'^*$, and $W^*$. Moreover, the First Fundamental Theorem of Classical Invariant Theory for vector representations states that $l$ (respectively, $r$) is the affine factorization map for the action of the group $G$ (respectively, $G'$) on $W$ in the sense of invariant theory.

Nilpotent orbits of a classical Lie group $G$ over an algebraically closed field of characteristic zero with $N$-dimensional defining module are parametrized by certain integer partitions of $N$ whose parts are the sizes of the Jordan blocks of any nilpotent matrix in the orbit. These partitions must satisfy additional parity conditions in the symplectic and orthogonal cases, [4]. We call an orbit $O$ small if the largest part of the corresponding partition is equal to two, equivalently, if its Young diagram consists of two columns. Small nilpotent orbits are classified by their rank, the rank of any matrix in the orbit. The rank is an integer between 1 and $N/2$ equal to the length of the second column of the Young diagram, and must be even if $g = o_N$ is an orthogonal Lie algebra. (In the case $G = O_N, N = 4k$, the small nilpotent $G$-orbit of rank $2k$ is very even, and breaks up into two nilpotent
orbits for the special orthogonal group \(SO_N\). For example, the minimal orbit is small. Small orbits are spherical and form a complete chain in the poset of nilpotent orbits ordered by the inclusions of their closures.

**Definition 2.1.** Let \(\mathcal{O}'\) be a nilpotent coadjoint orbit in \(\mathfrak{g}'^*\) with Zariski closure \(\overline{\mathcal{O}}\). The Kraft–Procesi lifting associates with it the \(G\)-invariant affine subvariety \(\overline{\mathcal{O}} = r(l^{-1}(\overline{\mathcal{O}}))\) of \(\mathfrak{g}'^*\).

The variety \(\overline{\mathcal{O}}\) arising from the Kraft–Procesi lifting is the closure of a nilpotent coadjoint orbit \(\mathcal{O} \subseteq \mathfrak{g}^*\). Moreover, if \((\mathfrak{g}, \mathfrak{g}')\) is in the stable range with \(\mathfrak{g}'\) the smaller member then the Young diagram of \(\mathcal{O}\) is obtained from the Young diagram of \(\mathcal{O}'\) by adding an extra first column of the appropriate length. In particular, if \(\mathcal{O}'\) is the zero orbit then \(\mathcal{O}\) is small. For \((\mathfrak{g}, \mathfrak{g}')\) in the stable range, these facts were proved by Kraft and Procesi, and this is precisely the case we are going to use. More general results appeared in [5].

2.3. Reductive dual pairs and the theory of harmonics. Let \((\mathfrak{g}, \mathfrak{g}')\) be a reductive dual pair of Lie algebras. We say that \((\mathfrak{g}, \mathfrak{g}')\) is in the stable range, with \(\mathfrak{g}'\) the smaller member, if the dimension of the defining module of \(\mathfrak{g}\) is at least twice the dimension of the defining module of \(\mathfrak{g}'\). Thus \((\mathfrak{gl}_n, \mathfrak{gl}_k)\) is in the stable range with \(\mathfrak{gl}_k\) the smaller member if and only if \(n \geq 2k\). More generally, we consider reductive dual pairs with \(\mathfrak{g}'\) the smaller member, defined through comparison of the dimensions of the defining modules.

The **double** of the symplectic vector space \(W\) is the vector space \(W^d = W \oplus W^*,\) with a natural symplectic form making this direct sum decomposition into a lagrangian polarization. We denote by \(\mathfrak{g}^d\) the centralizer of \(\mathfrak{g}'\) in the symplectic Lie algebra of \(W^d\) and call it the double of the Lie algebra \(\mathfrak{g}\). Then \((\mathfrak{g}^d, \mathfrak{g}')\) is a reductive dual pair in \(\mathfrak{sp}(W^d)\). Let \(\mathfrak{g}\) be the centralizer of \(\mathfrak{g}\) in \(\mathfrak{gl}(W)\), the general linear algebra of \(W\). The Weyl algebra \(\mathcal{W}^d\) of the double \(W^d\) acts on \(S(W)\) by polynomial coefficient differential operators in the standard way. The Lie algebra \(\mathfrak{g}^d\) has the following decomposition arising from the polarization of \(W^d\) and the induced decomposition of the second component of \(W^d\) as \(S^2(W) \oplus (W \otimes W^*) \oplus S^2(W^*)\):

\[
\mathfrak{g}^d = \mathfrak{g} \oplus \mathfrak{g}' \oplus \mathfrak{g}^*,
\]

where the three summands correspond to \((2, 0), (1, 1)\), and \((0, 2)\) components of \(\mathfrak{g}^d\) in the terminology of [9]. In the setting of an irreducible reductive dual pair \((\mathfrak{g}, \mathfrak{g}')\) over an algebraically closed field, we have the following possibilities: (1) \(\mathfrak{g} = \mathfrak{gl}_n\) is a general linear algebra and \(\mathfrak{g} = \mathfrak{gl}_n \times \mathfrak{gl}_n\) with \(\mathfrak{g}\) embedded diagonally; (2) \(\mathfrak{g} = \mathfrak{sp}_{2n}\) is a symplectic Lie algebra and \(\mathfrak{g} = \mathfrak{gl}_{2n}\) is the corresponding general linear algebra; (3) \(\mathfrak{g} = \mathfrak{o}_N\) is an orthogonal Lie algebra and \(\mathfrak{g} = \mathfrak{gl}_N\) is the corresponding general linear Lie algebra.

The space \(\mathcal{H}\) of \(G'\)-harmonics is the subspace of \(S(W)\) annihilated by the action of the \(G'\)-invariant constant coefficient differential operators. Equivalently, it is the kernel for the action of the \((0, 2)\)-component of \(\mathfrak{g}^d\) in the decomposition (2.2). The space of harmonics is \(G'\)-invariant and has a direct sum decomposition into isotypic components corresponding to finite-dimensional simple \(G'\)-modules. It was discovered by Gelbart and generalized by Howe that harmonics have very special structure: the space \(\mathcal{H}_\tau\) of \(G'\)-harmonics of type \(\tau\) is an irreducible \(G' \times G\)-module.
of type $\tau \otimes \sigma$ and generates the $\tau$-isotypic component of $S(W)$. Here $\sigma$ is an isomorphism class of simple finite-dimensional $\bar{G}$-modules and is uniquely determined by $\tau$. The correspondence between $\sigma$ and $\tau$ is explicitly described in [10].

Denote the associated graded algebras of $W$ and $U(g')$ by $A$ and $B$. Since $g'$ is the smaller member of the dual pair, the map $grL$ is an injection, and we will pretend that it’s just the identity map, so that $B$ is a subalgebra of $A$ and $U(g')$ is a subalgebra of $W$. The next theorem is a restatement of the decomposition of $A = S(W)$ into invariants and harmonics arising from the reductive dual pair $(g^d, g')$ in $\mathfrak{sp}(W^d)$.

**Theorem 2.3.** Suppose that $\tau$ is an irreducible $g'$-module, $\tau^*$ its contragredient, and $V_\tau$ is an $\text{ad}(g')$-invariant subspace of $B = S(g')$ isomorphic to $\tau$. Let $\sigma$ be the $g'$-module corresponding to $\tau^*$ in the space of harmonics. Then $(V_\tau, A)_{\tau^*}$ is generated as a $S(g)$-module by the subspace $U_\sigma$, the result of the $\text{Ad}(g')$-invariant multiplication pairing between $V_\tau$ and $H_{\tau^*}$, and $U_\sigma$ is an irreducible $\bar{G}$-module of type $\sigma$.

3. **Quantizations of nilpotent orbit closures**

The main goal of this section is to establish Theorem 1.5. Our proof is based on a pair of remarkable properties of the moment map $l : W^* \to g'^*$ in the stable range.

**Proposition 3.1.** Let $(g, g')$ be an irreducible reductive dual pair in stable range with $g'$ the smaller member. Then the moment map $l : W^* \to g'^*$ is flat and its geometric fibers are irreducible. In fact, $S(W)$ is a free $S(g')$-module.

**Proof.** By the method of associated cones and the dimensional criterion of flatness, it is enough to verify that the zero fiber $l^{-1}(0)$ is reduced and irreducible, of dimension $\dim W^* - \dim g'^*$. By [13] [14], the zero fiber is an irreducible complete intersection. The last assertion follows because $S(W)$ is a graded algebra and $S(g')$ is its graded subalgebra. \qed

It is a general fact from commutative algebra that in the situation of Proposition 3.1, the scheme-theoretic preimage of an irreducible subscheme of the base is reduced and irreducible.

**Proposition 3.2.** Let $A$ be an affine algebra over a field $K$, $B$ its affine $K$-subalgebra. Assume that $A$ is flat as a $B$-module and for any maximal ideal $m$ of $B$, the ideal $mA \subseteq A$ is prime. Then for any prime ideal $P$ of $B$, the ideal $PA$ of $A$ is prime.

**Proof.** In an affine algebra, a prime ideal is the intersection of the maximal ideals that contain it. The flatness of $A$ over $B$ implies that intersections are compatible with the “base change” from ideals of $B$ to ideals of $A$, $S \mapsto SA$. Therefore, $PA$ is the intersection of the ideals of the form $mA$ for various maximal $m \supseteq P$, which are prime by assumption. Suppose that $fg \in PA$, then for every such $m$, either $f$ or $g$ is contained in $mA$. Since these are Zariski closed conditions on $m$ and $V(P)$ is irreducible, either $f$ or $g$ is contained in $mA$ for all $m$ containing $P$. It follows that either $f$ or $g$ is contained in $PA$, establishing its primeness. \qed

**Proof of Theorem 1.5.** We show that $I$ has the desired property using Propositions 3.1 and 3.2 and the standard filtered–graded techniques. As in Section 2.3, denote
and \( gr(W) \) by \( A \) and \( B \) and view \( B \) as a subalgebra of \( A \). Then \( U(g') \) is a subalgebra of \( W \). Moreover, by Proposition 3.1 \( A \) is a flat \( B \)-module. Here is the crucial step: the flatness of \( A \) over \( B \) implies by [1], Chapter 2, Theorem 8.6 that

\[
gr(I'W) = (gr\ I')A.
\]

By assumption, the ideal \( P = gr\ I' \) is prime. Proposition 3.2 then shows that the ideal \( PA \) is prime, and the corresponding irreducible subvariety of \( W^* \) is \( l^{-1}(\overline{O}) \). Pulling back \( I'W \) via \( R \) and recalling that \( R(U(g)) \) coincides that with the \( G' \)-invariants of \( W \), we see that \( gr(\theta(I')) = (gr\ R)^{-1}(PA) \) and that the ideal in the right hand side defines the subvariety \( r(l^{-1}(\overline{O})) = \overline{O} \). Thus \( \theta(I') \) is a quantization of \( \overline{O} \), as we claimed. □

Remark 3.4. Our argument can be adapted to analysis of the effect of transfer on the associated cycles of ideals. We hope to return to this question in a future publication.

4. Transfer of codimension one ideals

In this section we find generators for the transfer of a codimension one ideal \( I' \subset U(g') \) in the stable range. First, by an explicit computation, we will exhibit a certain ad-invariant finite-dimensional subspace of \( U(g) \) contained in the ideal \( I = \theta(I') \). Then we use Theorem 2.3 to conclude that this subspace generates \( I \) as a (left or right) \( U(g) \)-module. This step also applies to transfers of more complicated ideals of \( U(g') \).

4.1. General linear case. The Weyl algebra associated with the reductive dual pair \((\mathfrak{g}l_n, \mathfrak{g}l_k)\) may be identified with the algebra of polynomial coefficient differential operators on \( n \times k \) matrices, corresponding to the realization \( W = M_{n,k} \oplus M_{n,k}^* \) for the symplectic vector space. Let \( X \) be the \( n \times k \) matrix with entries \( x_{ia} \), the coordinate functions on \( M_{n,k} \) and \( D \) be the \( n \times k \) matrix with entries \( \partial_{jb} \), the corresponding partial derivatives. Denote the standard generators of \( \mathfrak{g}l_n \) given by the matrix units by \( E_{ij} \), \( 1 \leq i, j \leq n \), and arrange them into a single \( n \times n \) matrix over \( U(\mathfrak{g}l_n) \):

\[
E = \begin{bmatrix}
E_{11} & E_{12} & \cdots & E_{1n} \\
E_{21} & E_{22} & \cdots & E_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
E_{n1} & E_{n2} & \cdots & E_{nn}
\end{bmatrix} \in M_n(U(\mathfrak{g}l_n)).
\]

It turns out that the image of \( E \) under the homomorphism \( R \) into \( W \) can be (almost) factorized as follows: \( R(E) = XD' + \frac{x}{2} I_n \). Similarly, if we denote by \( E' \) the \( k \times k \) matrix whose entries are the standard generators of \( \mathfrak{g}l_k \) then \( L(E') = X'D + \frac{x}{2} I_k \).
In order to simplify some of the formulas that follow, we now modify the definition of $L$ and $R$, so that

\begin{equation}
L(E') = XD', \quad R(E) = XD'.
\end{equation}

These unnormalized homomorphisms $L$ and $R$ have almost all the properties of the original ones, except that the images of $gl_n$ and $gl_k$ no longer belong to $W^2$, the symplectic subalgebra $sp_{2k,n}$ of the Weyl algebra. We will, therefore, consider the “unnormalized transfer” based on the modified maps $L$ and $R$ defined by the formula 1.2

\begin{definition}
Let $I = (i_1, \ldots, i_{k+1})$, $J = (j_1, \ldots, j_{k+1})$ be two $(k+1)$-element sequences of indices from 1 to $n$. The quantum minor $E_{IJ}$ is an element of $U(gl_n)$ given by the following noncommutative determinants (of either row or column type):

\[ E_{IJ} = \sum_\sigma \text{sgn}(\sigma)(E + k)_{i(1)j1} \cdots (E + 1)_{i(2)j2} E_{i(k+1)j(k+1)} \]

\[ = \sum_\sigma \text{sgn}(\sigma)E_{ij_1j(1)}(E + 1)_{i_2j(2)} \cdots (E + k)_{i_{k+1}j(k+1)}, \]

where scalars denote the corresponding scalar multiples of $I_n$ and the sum is taken over all permutations $\sigma \in S_{k+1}$.

Quantum minors of given order quantize ordinary minors of $n \times n$ matrices and span an ad-invariant subspace of $U(gl_n)$. Moreover, we have proved in [23] that the quantum minors of order $k + 1$ strictly generate the ideal $\ker R$, called the rank ideal associated with the reductive dual pair $(gl_n, gl_k)$.

It follows immediately from the definition of transfer that the transfer of any ideal $I'$ of $U(gl')$ contains $\ker R$. Conversely, if we find a set of generators for $L(I')W \cap R(U(gl))$ as an $R(U(gl))$-module then supplanting them with the quantum minors of order $k + 1$, we obtain a set of generators of $\theta(I')$ as a $U(gl)$-module.

\begin{proposition}
The second component $W^2$ of the Weyl algebra decomposes under the adjoint $GL_k \times GL_n$-action as follows:

\begin{equation}
W^2 \simeq S^2(W) = S^2(M_{n,k}) \oplus (M^*_{n,k} \otimes M^*_{n,k}) \oplus S^2(M^*_{n,k}).
\end{equation}

Moreover, the middle summand is isomorphic to $gl_n \otimes gl_k$.

\begin{proof}
The first isomorphism is given by the $GL(W)$-invariant unnormalized symbol map. The decomposition of $S^2(W)$ follows from the multiplicativity of the symmetric power, $S(U \oplus V) = S(U) \otimes S(V)$. The last assertion holds due to the canonical isomorphisms $M_{n,k} \simeq \mathbb{K}^n \otimes \mathbb{K}^k$, $M^*_{n,k} \simeq \mathbb{K}^*n \otimes \mathbb{K}^k$, $gl_n \simeq \mathbb{K}^n \otimes \mathbb{K}^n$, $gl_k \simeq \mathbb{K}^k \otimes \mathbb{K}^k$.
\end{proof}

Let $V'$ be the subspace of $U(gl_k)$ spanned by the entries of the matrix $E' + tI_k$, where $t$ is a fixed scalar, $I'$ be the ideal of $U(gl_k)$ generated $V'$, and let $I = \theta(I')$ be its transfer to $U(gl_n)$. It is clear that $V'$ is isomorphic to $gl_k$ as an $ad(gl_k)$-module and that the element $tr E' + kt$ spans the one-dimensional ad-invariant (central) subspace of $V'$. Since

\[ tr L(E') = \sum_{ia} x_{ia}O_{ia} = tr R(E), \]

we see that $tr E + kt$ is contained in $I$. \hfill \Box
The right hand side is equal to \[ \sum_{t=2}^{k} (t \delta_{ab}) \] where the summation is over \( a, b \) between 1 and \( k \) and over \( l \) between 1 and \( n \). The right hand side is equal to \( \sum_{t} R(E_{il})R(E_{ij}) + (k-n)R(E_{ij}) \), whose preimage under \( R \) is the \( ij \) entry of \( E^2 + (k-n)E \). Replacing \( L(E') \) with \( L(E') + I_k \) has the effect of adding \( t \sum_{a} x_{ia} \partial_{ja} = tR(E_{ij}) \) to both sides. Therefore, the subspace \( V \) of \( U(\mathfrak{gl}_n) \) spanned by \( \text{tr} E + kt \) and the entries of the \( n \times n \) matrix \( E^2 + (k-n+t)E \) is contained in \( I \). This subspace is \( \text{ad}(\mathfrak{gl}_n) \)-invariant and it follows from Theorem 2.3 that its image under \( R \) generates \( \hat{L}(I')W \cap R(U(\mathfrak{g})) \) as a \( U(\mathfrak{g}) \)-module. We summarize the description of the transfer of a codimension one ideal of \( U(\mathfrak{gl}_k) \) in the following theorem.

**Theorem 4.6.** Let \((\mathfrak{g}, \mathfrak{g}') = (\mathfrak{gl}_n, \mathfrak{gl}_k)\) with \( 2k \leq n \) and let \( I' \) be the annihilator of the one-dimensional module \( F_{ab}' \mapsto -t \delta_{ab} \) over \( \mathfrak{gl}_k \). Then its unnormalized transfer \( I \) is a quantization of the closure of the small orbit \( O_k \) of rank \( k \) in \( \mathfrak{gl}_n \). Moreover,

(A) The ideal \( I \) has an \( \text{ad} \)-invariant system of generators spanned by
   - \( \text{tr} E + kt \),
   - the entries of \( p(E) \), where \( p(u) = u^2 + (k-n+t)u \),
   - the quantum minors \( E_{ij} \) of order \( k+1 \).

(B) The corresponding primitive quotient of \( U(\mathfrak{gl}_n) \) may be identified with the ring of differential operators on an algebraic variety in the following cases: for \( t = 0 \), the quotient is isomorphic to the algebra of global differential operators on the Grassmannian \( Gr(k,n) \); for an integer \( t \), \( k < t < n - k \), it is isomorphic to the algebra of the algebraic differential operators on the singular rank variety formed by the \( t \times (n-t) \) matrices of rank at most \( k \).

**Proof.** Part (A) has already been established. To demonstrate Part (B), we will use Theorem 1.3 and Theorem 5.5 whose proofs in Section 5 are logically independent from the rest of the article.

Consider first the case of the rank variety. Levasseur and Stafford have studied in [15] the annihilator \( J_k \) of the unitary highest weight module arising as the unnormalized theta lift of the trivial representation for the reductive dual pair \((U_{p,q}, U_k)\) of compact type, where \( p, q \geq k + 1 \) and proved that the corresponding primitive quotient is the ring of differential operators on the singular affine variety of \( p \times q \) matrices of rank at most \( k \). Let \( p = t, q = n - t \), then the assumption on \( p, q \) holds. By Theorem 1.3 \( I \subseteq J_k \). However, both \( I \) and \( J_k \) have the same associated variety, namely, the closure of the small orbit of rank \( k \). Therefore, these ideals coincide.

The case of the Grassmannian is analogous. Borho and Brylinski proved in [2] that the homomorphism from \( U(\mathfrak{gl}_n) \) into the algebra of global differential operators on the Grassmannian \( Gr(k,n) \) induced by the \( \mathfrak{gl}_n \)-action \( \{1, 2\} \) on \( n \times k \) matrices is surjective and its kernel \( J \) quantizes the closure of the small nilpotent orbit of rank \( k \). By Theorem 5.5 \( J \) contains \( I \), therefore, these ideals coincide. \( \square \)
With applications to theta correspondence in mind, Section 5 we formulate the description of the transfer resulting from using the normalized maps $L$ and $R$ (defined before (4.2)).

**Corollary 4.7.** Let $(\mathfrak{g}, \mathfrak{g}') = (\mathfrak{gl}_k, \mathfrak{gl}_k)$ with $2k \leq n$ and let $I'$ be the annihilator of the one-dimensional module $E_{ab} \mapsto \alpha \delta_{ab}$ over $\mathfrak{gl}_k$. Then its normalized transfer $I \subset U(\mathfrak{gl}_n)$ has an ad-invariant generating set spanned by the following elements:

- $\text{tr } E - k\alpha$,
- the entries of $p(E)$, where $p(u) = (u - k/2)(u - (n - k + \alpha)/2)$,
- the twisted quantum minors $E_{ij}(-k/2)$ of order $k + 1$.

4.2. Symplectic – orthogonal case. The arguments in this case are similar to the general linear case considered above. The role of the matrix $E$ is played by a skew-symmetric matrix $F$ whose entries are the standard generators of the Lie algebra $\mathfrak{o}_N$ in its “canonical” realization by the skew-symmetric $N \times N$ matrices.

Let us briefly recall Howe’s description of the irreducible reductive dual pairs of type I over an algebraically closed field. Let $U$ be a symplectic and $V$ be an orthogonal vector space over $\mathbb{K}$, both finite-dimensional, then $W = U \otimes V$, endowed with the product of the forms on $U$ and $V$, is a symplectic vector space. The isometry groups $Sp(U)$ and $O(V)$ of the forms embed into $Sp(W)$ and form a reductive dual pair in it, and their Lie algebras $\mathfrak{sp}(U)$ and $\mathfrak{o}(V)$ form a reductive dual pair of Lie algebras in $\mathfrak{sp}(W)$.

**Proposition 4.8.** The second component $W^2$ of the Weyl algebra decomposes under the adjoint $Sp(U) \times O(V)$-action as follows:

$$W^2 \cong S^2(W) = S^2(U) \otimes S^2(V) \oplus \Lambda^2(U) \otimes \Lambda^2(V).$$

Moreover, $S^2(U) \cong \mathfrak{sp}(U)$ and $\Lambda^2(V) \cong \mathfrak{o}(V)$ are the adjoint representations of $Sp(U)$ and $O(V)$, while $\Lambda^2(U)$ and $S^2(V)$ contain one-dimensional summands corresponding to the symplectic form on $U$ and the orthogonal form on $V$.

**Proof.** The first isomorphism is given by the $Sp(W)$-equivariant symbol map. The decomposition of $S^2(U \otimes V)$ as a $GL(U) \times GL(V)$ module is given by the Cauchy formula, with the summands corresponding to the partitions $(2)$ and $(1, 1)$ of $2$. The statements concerning the adjoint representations are part of the theory of Cayley transform for classical groups, and the last assertion is obvious. \qed

Let $(\mathfrak{g}, \mathfrak{g}') = (\mathfrak{o}_N, \mathfrak{sp}_{2k})$. We use an identification of the Weyl algebra with the polynomial coefficient differential operators on the space of $N \times k$ matrices and a particular realization of the reductive dual pair with the symplectic vector space completely polarized. Similarly to the general linear case, consider the $N \times k$ matrices $X$ and $D$ consisting of the coordinate functions and the partial derivatives. Let $P$ be the $N \times 2k$ matrix obtained by putting them together, and consider its transpose, its conjugate with respect to the symplectic form with the matrix $J$, and the conjugate transpose, which are explicitly given in the block form as follows:

$$P = [X \ D], \quad P^t = \begin{bmatrix} X^t \\ D^t \end{bmatrix}, \quad P^* = \begin{bmatrix} D^t \\ -X^t \end{bmatrix}, \quad P^{*t} = [-X \ D], \quad J = \begin{bmatrix} 0 & -I_k \\ I_k & 0 \end{bmatrix}.$$

The root generators of $\mathfrak{sp}_{2k}$ are arranged into a $2k \times 2k$ matrix $F'$, and the standard generators of $\mathfrak{o}_N$ are arranged into an $N \times N$ skew-symmetric matrix $F$. The images
of $F'$ and $F$ under the homomorphisms $L$ and $R$ into the Weyl algebra are the following matrices of differential operators on $M_{N,k}$, cf [12]:

\begin{equation}
L(F') = P^t(P^*)^t + \frac{N}{2}I_{2k}, \quad R(F) = PP^* + kI_N.
\end{equation}

**Definition 4.11.** Let $I = (i_1, i_2, \ldots, i_{2k+2})$ be a sequence of indices from 1 to $N$. The *quantum pfaffian* $P_{I}$ of order $k+1$ is the following element of $U(\mathfrak{o}_N)$:

$$P_{I} = \frac{1}{2k+1(k+1)!} \sum \text{sgn}(\sigma)F_{i_{\sigma(1)}i_{\sigma(2)}}\cdots F_{i_{\sigma(2k+1)}i_{\sigma(2k+2)}},$$

where the sum is taken over the permutations $\sigma \in S_{2k+2}$.

Quantum pfaffians are quantizations of the ordinary pfaffians of skew-symmetric $N \times N$ matrices. They are skew-symmetric in their indices and span an ad-invariant subspace of $U(\mathfrak{o}_N)$, [12] [18]. Combining the techniques of [23] with the identities proved in *op cit*, one can demonstrate that the rank ideal ker $R$ is generated by the quantum pfaffians of order $k+1$.

**Proposition 4.12.** Let $p(t) = t^2 - (N/2 - 1)t$. Then the matrix $p(F)$ is symmetric.

**Proof.** From the commutation relations between the entries of the matrix $F$ it follows that

$$\begin{align*}
(F^2)_{ij} - (F^2)_{ji} &= \sum_l F_{il}F_{lj} - F_{jl}F_{li} = \sum_l [F_{il}, F_{lj}] = \\
&= \frac{1}{2} \sum_l (F_{ij} - \delta_{ij}F_{ll} - \delta_{il}F_{lj} + \delta_{jl}F_{li}) = (N-2)F_{ij},
\end{align*}$$

and this is equivalent to the statement we needed to prove. \qed

**Theorem 4.13.** Let $(\mathfrak{g}, \mathfrak{g}') = (\mathfrak{o}_N, \mathfrak{sp}_{2k})$ with $4k \leq N$ and let $I'$ be the annihilator of the trivial representation of $\mathfrak{sp}_{2k}$. Then its transfer $I$ is a quantization of the closure of the small nilpotent orbit $O_{2k}$ of rank $2k$ in $\mathfrak{o}_N$. Moreover,

(A) The matrix $p(F)$, where $p(u) = (u-k)(u-(N/2-k-1))$, is symmetric, its entries span an ad-invariant subspace of $U(\mathfrak{o}_N)$, and together with the quantum pfaffians of order $k+1$, they generate the ideal $I$;

(B) If $N = 2n$ is even and $2k < n-1$ then the corresponding primitive quotient is isomorphic to the ring of the algebraic differential operators on the variety of skew-symmetric $n \times n$ matrices of rank at most $2k$.

**Proof.** We need to determine the result of the $\mathfrak{sp}_{2k}$-invariant multiplication pairing of the subspace of $W$ spanned by the entries of $L(F')$ with the first summand in the decomposition (19). First, let us transform $(P^t(P^*)^t)_{ai} = \sum_b(P^tP^*)_{ab}P_{ib}$:

$$\sum_{b,l} P_{ta}P^*_{bl}P_{ib} = \sum_{b,l} [P_{ta}, P^*_{bl}]P_{ib} + P^*_{bl}[P_{ta}, P_{ib}] + [P^*_{bl}, P_{ib}]P_{ta} + P_{ib}P^*_{bl}P_{ta}.$$

Here and below the indices $a, b$ correspond to the symplectic vector space $U$ and run from 1 to $2k$, and the indices $i, j, l$ correspond to the orthogonal vector space $V$ and run from 1 to $N$. Substituting $[P_{ta}, P^*_{bl}] = -\delta_{ab}[P_{ta}, P_{ib}] = \delta_{il}P_{ta}$ and taking into account that $\sum b_j a P^*_{bl} = P_{ta}$, we get

$$(P^t(P^*)^t)_{ai} = -NP_{ia} + P_{ia} + 2kP_{ta} + \sum (PP^*)_{ti}P_{ta} = (-N+2k+1)P_{ia} + (PP^*)_{ia}.$$
Thus we have established the following convolution formula:

\[(4.14)\]
\[P_t (P \ast t) P_t \] 
\[a_i = \sum_{l} (PP + (2N + 2k + 1)) I_N il P_{la}.\]

Now multiply both sides by \(P_{a_j}^*\) and sum over \(a\) to get

\[(4.15)\]
\[\sum_{ab} L(F')_{ab} P_{ib} P_{a_j}^* = R(p(F))_{ij}, \quad p(u) = (u - k)(u - (\frac{N}{2} - k - 1)).\]

This completes the computation of the \(sp_{2k}\)-invariant multiplication pairing. The subspace of \(W\) spanned by the right hand side of (4.15) is contained in \(I = (I' \cdot W) \cap R(U(g))\) and generates it as a \(R(U(g))\)-module. Indeed, its associated graded is the subspace \(U_{\sigma}\) from Theorem 2.3 that generates \(\text{gr} I\). We conclude that the entries of \(p(F)\) generate \(I = \theta(I')\) modulo the orthogonal rank ideal \(\text{ker} R\). Therefore, the quadratic elements \(p(F)_{ij}\) and the quantum pfaffians of order \(k + 1\) together generate the ideal \(I\).

As in the general linear case, Part (B) follows from [15], where the ring of differential operators on the singular rank variety was identified with the primitive quotient of \(U(o_N)\) by the annihilator of the theta lift of the one-dimensional representation of the symplectic group.

5. Relation between transfer and the Howe duality

In the previous section we have given applications of the transfer map to explicit quantization of the orbits in classical Lie algebras. However, our original motivation for introducing this map came from the theory of reductive dual pairs and the local theta correspondence over \(\mathbb{R}\). In short, the transfer \(\theta(I')\) of an ideal \(I'\) in \(U(g')\) provides a lower bound on the annihilator ideal \(I\) in \(U(g)\) of representations in the Howe duality correspondence with any representation with annihilator ideal \(I'\). We consider two algebraic versions of the Howe duality and use them to prove the lower bound theorem. Then we review the approach to Capelli identities based on the theory of reductive dual pairs, introduce noncommutative Capelli identities and explain how the transfer map may be viewed as a system of identities determined by a reductive dual pair \((g, g')\) and an ideal \(I'\) of \(U(g')\). (A different approach to noncommutative Capelli identities in the general linear case, not involving dual pairs, was developed in [17].)

5.1. Proof of the lower bound on the annihilator ideal. Our proof of Theorem 1.3 establishing a lower bound on the annihilator of a representation arising from Howe correspondence relies on the description of the Howe duality for admissible representations in the real case given in [16], which we now recall. Suppose that \((G_0, G'_0)\) is a reductive dual pair of real Lie groups in the symplectic group \(Sp = Sp(W_0)\), where \(W_0\) is a symplectic real vector space. We denote by \(g\) and \(g'\) the complexified Lie algebras of \(G_0\) and \(G'_0\), which form a reductive dual pair of Lie algebras. The metaplectic two-fold cover \(\tilde{Sp}\) of \(Sp\) acts in the space \(\omega\) of the oscillator representation and this action induces linear representations of the
metaplectic two-fold covers \( \widetilde{G}_0 \) and \( \widetilde{G}'_0 \) of the members of the dual pair. One may choose a maximal compact subgroup \( U \) of \( Sp \) (\( U \) is a unitary group) in such a way that the associated Cartan decomposition of \( Sp \) induces Cartan decompositions of \( G_0 \) and \( G'_0 \). Thus it makes sense to consider restrictions of a Harish-Chandra module of \( Sp \) to \( G_0 \) and \( G'_0 \), and likewise for the metaplectic covers. From now on we replace representations of groups with the corresponding \((g, K)\)-modules of \( K\)-finite vectors. The complexified Weyl algebra \( W \) is identified with the algebra of the endomorphisms of \( \omega \) whose \( Sp \)-conjugates span a finite-dimensional subspace of \( \text{End} \omega \).

**Definition 5.1.** Irreducible admissible representations \( \rho \) and \( \rho' \) of \( \widetilde{G}_0 \) and \( \widetilde{G}'_0 \) correspond to each other under the Howe duality if there exists an \( \text{Ad}(\widetilde{G}_0 \cdot \widetilde{G}'_0) \)-invariant subspace \( \omega_0 \) of \( \omega \) such that the quotient module \( \omega/\omega_0 \) is isomorphic to \( \rho \otimes \rho' \).

The main theorem of [9] asserts that this correspondence is a partial bijection between the sets of equivalence classes of irreducible admissible representations of the metaplectic covers of the members of the reductive dual pair. “Forgetting” about the actions of the maximal compact subgroups on \( \rho \) and \( \rho' \), we can formulate an entirely algebraic version of Howe duality for representations.

**Definition 5.2.** Let \((g, g')\) be a reductive dual pair of Lie algebras and \( A \) be a fixed non-zero module over the corresponding Weyl algebra \( W \). Then a \( g \)-module \( V \) and a \( g' \)-module \( V' \) are in the algebraic Howe duality with each other if there exists an \( \text{Ad}(G) \)-invariant and \( \text{Ad}(G') \)-invariant subspace \( A_0 \) of \( A \) such that the quotient \( A/A_0 \) is isomorphic to \( V \otimes V' \) as a \( g \)-module and a \( g' \)-module.

**Remark 5.3.** The annihilator ideals of the modules \( V \) and \( V' \) in \( U(g) \) and \( U(g') \) can be detected from the quotient \( A/A_0 \). They are the annihilators of this quotient viewed as a \( U(g) \)-module and as a \( U(g') \)-module.

The transfer map \( \theta \) comprises of two steps. The first step takes the ideal \( I' \) of \( U(g') \) into the \( \text{Ad}(G \cdot G') \)-invariant right ideal \( I = L(I') \cdot W \) of \( W \), and the second step replaces \( I \) with its pullback to \( U(g) \), \( R^{-1}(I) = \theta(I') \). This observation forms a basis for our final algebraic approximation to Howe duality.

**Definition 5.4.** Let \((g, g')\) be a reductive dual pair of Lie algebras. Then two-sided ideals \( I \subset U(g) \) and \( I' \subset U(g') \) are in the algebraic Howe duality if there exists an \( \text{Ad}(G) \) and \( \text{Ad}(G') \)-invariant right ideal \( I \) of the Weyl algebra such that \( I \) and \( I' \) are its pullbacks with respect to the maps \( R \) and \( L \) into \( W \).

It is clear that if ideals \( I' \) and \( I \) are in algebraic duality then \( \theta(I') \subset I \), for

\[
\theta(I') = R^{-1}(L(I') \cdot W) \subset R^{-1}(I) = I.
\]

Moreover, if two admissible representations of the metaplectic covers of \( G_0 \) and \( G'_0 \) correspond to each other under the ordinary Howe duality then their underlying \( g \)-module and \( g' \)-module are in the algebraic Howe duality in the sense of Definition 5.2. Therefore, Theorem 1.3 follows from the following more general statement.

**Theorem 5.5.** Let \( V \) and \( V' \) be two modules in the algebraic Howe duality in the sense of Definition 5.2. Denote the annihilator of \( V \) in \( U(g) \) by \( I \) and the annihilator of \( V' \) in \( U(g') \) by \( I' \). Then the ideals \( I \) and \( I' \) are in the algebraic Howe duality.
are polynomial algebras with $k$-thomism between $Z$ this leads to a relation between the centers of their universal enveloping algebras.

From the identity in the algebra of polynomial coefficient differential operators on $M$ Howe and Umeda elucidated the role of the Capelli identity in the proof. The representations of the two groups that occur in it determine each other. In [11], $i \geq C$ Capelli elements of degree $i$ versions of the maps $L$ (The twists are reflected in the difference between the normalized and unnormalized these latter actions arise by restricting the oscillator representation of the symplectic group, realized in the Fock model, to the members of the compact dual pair. (The twists are reflected in the difference between the normalized and unnormalized versions of the maps $L$ and $R$.)

Proof. Let $I$ be the annihilator in $W$ of $A/A_0$. Then $I$ is an $\text{Ad}(G)$-invariant and $\text{Ad}(G')$-invariant right ideal of the Weyl algebra whose pullbacks to $U(g')$ and $U(g)$ via the maps $L$ and $R$ are precisely $I'$ and $I$, establishing the claim.

There is a weak converse to Theorem 5.5, cf Remark 5.3.

**Theorem 5.6.** Let $I$ and $I'$ be two ideals in the algebraic Howe duality in the sense of Definition 5.2. Then there exists a $W$-module $A$ and an $\text{Ad}(G)$-invariant and $\text{Ad}(G')$-invariant subspace $A_0$, such that $A/A_0$ has annihilator $I$ as a $U(g')$-module and annihilator $I'$ as a $U(g')$-module.

Proof. Suppose that $I$ be the right ideal of $W$ realizing the algebraic Howe duality between $I$ and $I'$. Let $A = W$ and $A_0 = I$. Then the conditions that $I = R^{-1}(I)$ and $I' = L^{-1}(I)$ translate into the conditions that the annihilators of $A/A_0$ as a $U(g)$-module and as a $U(g')$-module are $I$ and $I'$.

### 5.2. Noncommutative Capelli identities.

The complex general linear groups $GL_k$ and $GL_n$ act on the vector space $M_{k,n}$ of complex $k \times n$ matrices by matrix left and right multiplication. Thus one obtains a pair of commuting actions of their Lie algebras by polynomial coefficient differential operators. The generators of $\mathfrak{gl}_k$ and $\mathfrak{gl}_n$ act on polynomials via polarization operators of the classical invariant theory. Capelli’s remarkable accomplishment was to prove the full isotypic decomposition of the space of polynomial functions on $M_{k,n}$ under the pair of commuting actions of $GL_k$ and $GL_n$, using the Capelli identity as a tool. This Gordan–Capelli decomposition, also known as the $(GL_n, GL_k)$-duality, is multiplicity-free, and the representations of the two groups that occur in it determine each other. In [11], Howe and Umeda elucidated the role of the Capelli identity in the proof. The two general linear Lie algebras involved form a reductive dual pair $(\mathfrak{gl}_n, \mathfrak{gl}_k)$ and this leads to a relation between the centers of their universal enveloping algebras. Specifically, let us assume that $k \leq n$, then the Capelli identity provides an isomorphism between $Z(\mathfrak{gl}_k)$ and a certain subalgebra/quotient of $Z(\mathfrak{gl}_n)$, both of which are polynomial algebras with $k$ explicitly given generators. This isomorphism arises from the identity

$$R(C_i) = L(C'_i)$$

in the algebra of polynomial coefficient differential operators on $M_{k,n}$ and is determined by the formula $C''_i \mapsto C_i$ for $1 \leq i \leq k$. Here $C''_i$ and $C_i$ are the $i$th (central) Capelli elements of degree $i$ in $U(\mathfrak{gl}_k)$ and $U(\mathfrak{gl}_n)$, $1 \leq i \leq n$, and $C''_i = 0$ for $i \geq k + 1$. Setting $C_i = 0$ for $i \geq k + 1$ exhibits the algebra $\mathbb{K}[C_1, \ldots, C_k]$ as a quotient of $Z(\mathfrak{gl}_n)$. By the theory of highest weight, finite-dimensional simple $\mathfrak{gl}_n$-modules and $\mathfrak{gl}_k$-modules are uniquely determined by their infinitesimal character, therefore, the Capelli identity determines the matching of $\mathfrak{gl}_n$-modules and $\mathfrak{gl}_k$-modules in the Gordan-Capelli decomposition.

The actions of the groups $GL_n$ and $GL_k$ by matrix multiplications are complexifications of the actions of their maximal compact subgroups, the compact unitary groups $U_n$ and $U_k$ forming a reductive dual pair $(U_n, U_k)$ in the real symplectic group $Sp_{2nk}(\mathbb{R})$. Up to twists by certain powers of the determinant characters, these latter actions arise by restricting the oscillator representation of the symplectic group, realized in the Fock model, to the members of the compact dual pair. (The twists are reflected in the difference between the normalized and unnormalized versions of the maps $L$ and $R$.) In this setting, the Capelli identity between
the images of the algebra generators of the centers of $U(g)$ and $U(g')$ completely
describes the matching of representations under the Howe duality. Now, let us
consider a reductive dual pair $(U_{p,q}, U_{r,s})$ of definite or indefinite unitary groups in
$Sp_{2(p+q)}(\mathbb{R})$, and assume that at least one of the groups is non-compact. The
classical Capelli identity leads to a correspondence of the infinitesimal characters
for those representations $\rho$ and $\rho'$ of the metaplectic covers of $G_0$ and $G'_0$ which
coordinate to each other under the Howe duality. However, there is a fundamental
difference between representation theory of compact and noncompact reductive
groups: in the noncompact case, irreducible admissible representations are not de-
determined by the infinitesimal character alone. Thus in this case the Capelli identity
yields only partial information on the Howe duality correspondence. Capelli iden-
tities for other reductive dual pairs, which were established by Molev–Nazarov and
Itoh, [12, 18], involve only the central elements, consequently, they suffer from the
same defect.

The transfer map $I' \mapsto \theta(I')$ serves to sharpen the description of the Howe
duality, by incorporating more refined information on the representations $\rho$ and $\rho'$,
namely, their annihilator (primitive) ideals.

**Definition 5.8.** Given a family of representations of $g$, a **noncommutative Capelli
identity** is an element of the universal enveloping algebra $U(g)$ which acts by 0
on any representation $\rho$ from this family. An identity is **central** if the element
belongs to $Z(g)$. A set of noncommutative Capelli identities is called a **system of
noncommutative Capelli identities** if it spans an ad-invariant subspace of $U(g)$.

Noncommutative Capelli identities form an ideal in $U(gl_n)$ and that a (minimal)
ad-invariant generating subspace produces a (minimal) system of identities, with
others being their formal consequences. Next we present some examples.

1. The theory of rank for ideals of $U(gl_n)$, [23], gave the first examples of
systems of noncommutative Capelli identities that are not central. In this
special case, $I'$ is the zero ideal, $\theta(I') = \ker R$ is the $k$th rank ideal of $U(g)$
and is generated by the (normalized) quantum minors of order $k+1$. The
 corresponding identities are satisfied in any representation of the metaplec-
tic cover of $U_{p,q}$ which appears in the Howe duality with a representation
of $U_{r,s}$, where $p + q = n, r + s = k$.
2. More generally, Theorem 1.3 provides a system $\theta(I')$ of noncommutative
Capelli identities for representations of $g$ that occur in the Howe duality
 correspondence with a representation of $g'$ whose annihilator $I'$ has been
fixed.
3. As a specialization, Corollary 4.7 (in the general linear case) and The-
orem 4.13 (in the symplectic–orthogonal case) described an ad-invariant
generating space for the transfer of a codimension one ideal. It follows
from Theorem 1.3 that we have thus obtained a system of noncommutative
Capelli identities satisfied on the theta lifts of one-dimensional representa-
tions. The degree 2 elements are new.
4. Further examples are obtained from the theory of minimal polynomials and
quantized elementary divisors for $g$-modules, [20, 21].
6. Complements and open questions

6.1. Skew analogue. The theory developed in this paper has a natural “skew” analogue for reductive dual pairs in an orthogonal Lie algebra, with Clifford algebra in place of Weyl algebra, cf \([8, 10, 12]\). Since unlike the oscillator representation, the spin representation is finite-dimensional and decomposes into a direct sum of the tensor products of simple modules for \(G\) and \(G'\), in effect one is dealing with finite-dimensional modules and their annihilators, which completely determine each other when \(G\) and \(G'\) are connected (i.e. excluding the orthogonal – orthogonal case). This results in a streamlined behavior of the transfer map, which becomes a bijection between finite sets consisting of the annihilators of certain finite-dimensional simple modules for \(G\) and \(G'\). On the other hand, in the skew-symmetric case the moment maps and quantization are no longer relevant to the story.

6.2. Effect on primitive ideals. From our algebraic perspective, the most interesting question about transfer concerns the behavior of primitive and completely prime ideals.

**Question 6.1.** Let \((\mathfrak{g}, \mathfrak{g}')\) be an irreducible reductive dual pair of Lie algebras in the stable range with the smaller member \(\mathfrak{g}'\). Suppose that \(I'\) is a primitive ideal of \(U(\mathfrak{g}')\). Is it true that its transfer \(\theta(I')\) is a primitive ideal of \(U(\mathfrak{g})\)? The same question replacing “primitive” with “completely prime”.

More generally, suppose that \((\mathfrak{g}, \mathfrak{g}')\) is a reductive dual pair with \(\mathfrak{g}'\) the smaller member, then one might ask the same questions for sufficiently small ideals \(I'\) of \(U(\mathfrak{g}')\). The precise condition is that the associated variety of \(I'\) has nonempty intersection with a Zariski open subset of \(\mathfrak{g}'^*\) over which the moment map \(l\) is equidimensional with irreducible fibers. For the general linear case this amounts to the restriction that the corresponding nilpotent orbit contains a matrix whose rank is at least \(2k - n\). Partial progress towards answering these questions has been made in \([20, 21]\).

In the general linear case, the set of primitive ideals admits a combinatorial parametrization in terms of Young tableaux that is essentially due to Joseph. In \([22]\), I used it to construct a bijection between the set of primitive ideals of \(U(\mathfrak{gl}_k)\) and the set of primitive ideals of \(U(\mathfrak{gl}_n)\) that have pure rank \(k\). I think that this combinatorially defined bijection describes transfer for the reductive dual pair \((\mathfrak{gl}_n, \mathfrak{gl}_k)\) with \(n \geq 2k\) at the level of Young tableaux, but I have not been able to prove it.

6.3. Is the transfer bound exact? Let \((G_0, G'_0)\) be a reductive dual pair of real groups in stable range, with \(G'_0\) the smaller member, and suppose that irreducible admissible representations \(\rho\) and \(\rho'\) of their metaplectic covers are in Howe duality. It is natural to conjecture that if the representation \(\rho'\) is unitary then \(\theta(\text{Ann } \rho') = \text{Ann } \rho\), in other words, that the transfer bound is sharp. (I had first learned a more qualitative form of this conjecture from P. Trapa.) J.-S. Li’s description of singular unitary representations and T. Przebinda’s results on the behavior of the wave front sets under the Howe duality, \([16, 24]\), constitute strong supporting evidence for the conjecture. In particular, in the stable range, the annihilator of the theta-lift of a one-dimensional representation of the smaller group is generated by the elements from Corollary 4.7 and Theorem 4.13.

Nonetheless, explicit computations of the
Howe duality correspondence indicate that neither the stable range condition nor the unitarity of $\rho'$ could be completely eliminated from the assumptions.

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**References**

[1] J.-E. Björk. *Rings of differential operators*, volume 21 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1979.

[2] W. Borho and J.-L. Brylinski. Differential operators on homogeneous spaces. I. Irreducibility of the associated variety for annihilators of induced modules. *Invent. Math.*, 69(3):437–476, 1982.

[3] A. Capelli. Sur les Opérations dans la théorie des formes algébriques. *Math. Ann.*, 37:1–37, 1890.

[4] D. Collingwood and W. McGovern. *Nilpotent orbits in semisimple Lie algebras*. Van Nostrand Reinhold Co., New York, 1993.

[5] A. Daszkiewicz, W. Kraśkiewicz, and T. Przebinda. Nilpotent orbits and complex dual pairs. *J. Algebra*, 190(2):518–539, 1997.

[6] M. Holland. Quantization of the Marsden-Weinstein reduction for extended Dynkin quivers. *Ann. Sci. École Norm. Sup. (4)*, 32(6):813–834, 1999.

[7] R. Howe. $\theta$-series and invariant theory. In *Automorphic forms, representations and $L$-functions, Part 1*, volume 33 of *Proc. Sympos. Pure Math., XXXIII*, pages 275–285. Amer. Math. Soc., Providence, R.I., 1979.

[8] R. Howe. Remarks on classical invariant theory. *Trans. Amer. Math. Soc.*, 313(2):539–570, 1989.

[9] R. Howe. Transcending classical invariant theory. *J. Amer. Math. Soc.*, 2(3):535–552, 1989.

[10] R. Howe. Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond. In *The Schur lectures (1992) (Tel Aviv)*, Israel Math. Conf. Proc., pages 1–182. Bar-Ilan Univ., Ramat Gan, 1995.

[11] R. Howe and T. Umeda. The Capelli identity, the double commutant theorem, and multiplicity-free actions. *Math. Ann.*, 290(2):565–619, 1991.

[12] M. Itoh. Capelli identities for reductive dual pairs. *Adv. Math.*, 194(2):345–397, 2005.

[13] H. Kraft and C. Procesi. Closures of conjugacy classes of matrices are normal. *Invent. Math.*, 53(3):227–247, 1979.

[14] H. Kraft and C. Procesi. On the geometry of conjugacy classes in classical groups. *Comment. Math. Helv.*, 57(4):539–602, 1982.

[15] T. Levasseur and J. Stafford. Rings of differential operators on classical rings of invariants. *Mem. Amer. Math. Soc.*, 81(412):vi+117, 1989.

[16] J.-S. Li. Singular unitary representations of classical groups. *Invent. Math.*, 97(2):237–255, 1989.

[17] C. Moeglin. Idéaux completement premiers de l’algèbre enveloppante de $gl_n(C)$. *J. Algebra*, 106(2):287–366, 1987.

[18] A. Molev and M. Nazarov. Capelli identities for classical Lie algebras. *Math. Ann.*, 313(2):315–357, 1999.

[19] A. Okounkov. Young basis, Wick formula, and higher Capelli identities. *Internat. Math. Res. Notices*, 17:817–839, 1996.

[20] V. Protsak. Minimal polynomials of simple highest weight modules over classical Lie algebras. In preparation.

[21] V. Protsak. Quantized elementary divisors for ideals of $U(g_{\mathfrak{l}_n})$. In preparation.

[22] V. Protsak. *A Notion of Rank for Enveloping Algebras and Local Theta Correspondence*. PhD thesis, Yale University, 2000.

[23] V. Protsak. Rank ideals and Capelli identities. *J. Algebra*, 273(2):686–699, 2004.

[24] T. Przebinda. Characters, dual pairs, and unitary representations. *Duke Math. J.*, 69(3):547–592, 1993.

[25] T. Przebinda. The duality correspondence of infinitesimal characters. *Colloq. Math.*, 70(1):93–102, 1996.
[26] T. Przebinda. A Capelli Harish-Chandra homomorphism. *Trans. Amer. Math. Soc.*, 356(3):1121–1154 (electronic), 2004.

[27] G. Schwarz. Lifting differential operators from orbit spaces. *Ann. Sci. École Norm. Sup. (4)*, 28(3):253–305, 1995.

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