Intersectional pairs of $n$-knots, local moves of $n$-knots, and their associated invariants of $n$-knots

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Abstract. Let $n$ be an integer $>0$. Let $S_1^{n+2}$ (respectively, $S_2^{n+2}$) be the $(n+2)$-sphere embedded in the $(n+4)$-sphere $S^{n+4}$. Let $S_1^{n+2}$ and $S_2^{n+2}$ intersect transversely. Suppose that the smooth submanifold $S_1^{n+2} \cap S_2^{n+2}$ in $S^{n+4}$ is PL homeomorphic to the $n$-sphere. Then $S_1^{n+2} \cap S_2^{n+2}$ in $S_i^{n+2}$ is an $n$-knot $K_i$. We say that the pair $(K_1, K_2)$ of $n$-knots is realizable.

We consider the following problem in this paper. Let $A_1$ and $A_2$ be $n$-knots. Is the pair $(A_1, A_2)$ of $n$-knots realizable?

We give a complete characterization.

Chapter I is accepted in Mathematical Research Letters, 1998, 5, 577-582. This manuscript is not the published version.

Chapter I is a summary of Chapter II. Chapter II is based on the author’s PhD thesis (University of Tokyo, 1996) University of Tokyo preprint series UTMS 95-50 is a preprint of Chapter II.
1. Introduction

Our first purpose is to discuss the following problem.

Let $S_{1}^{n+2}$ and $S_{2}^{n+2}$ be $(n+2)$-spheres embedded in the $(n+4)$-sphere $S^{n+4}$ $(n \geq 1)$ which intersect transversely. If we assume $M = S_{1}^{n+2} \cap S_{2}^{n+2}$ is PL homeomorphic to the single standard $n$-sphere, we obtain a pair of $n$-knots, $M$ in $S_{1}^{n+2}$ and $M$ in $S_{2}^{n+2}$. We consider which pairs of $n$-knots we obtain as above. That is, let $(K_{1}, K_{2})$ be a pair of $n$-knots. Then we consider whether the pair of $n$-knots $(K_{1}, K_{2})$ is obtained as above.

We give a complete answer to this problem (Theorem 3.1).

In order to get the complete answer, we introduce a local move of $n$-knots $(n \geq 1)$. Furthermore, we show a relation between the local move and some invariants of $n$-knots (Theorem 4.1 and Corollary 4.2).

Our second purpose is to discuss the relation between the local move and the invariants of $n$-knots. In the case of 1-links, there is a great deal known about relations between local moves and knot invariants. (See e.g. [V][Wi][Ka2].) Our discussion is a high dimensional version of this theory.

This research was partially supported by Research Fellowships of the Promotion of Science for Young Scientists.
2. Definitions

An (oriented) (ordered) \( m \)-component \( n \)-(dimensional) link is a smooth, oriented submanifold \( L = \{K_1, ..., K_m\} \) of \( S^{n+2}\), which is the ordered disjoint union of \( m \) connected oriented submanifolds, each PL homeomorphic to the standard \( n \)-sphere. If \( m = 1 \), then \( L \) is called a knot. (This definition is used often. See e.g. [Co],[L1],[L3].)

Let \( L_1 \) and \( L_2 \) be \( n \)-links. \( L_1 \) is said to be equivalent to \( L_2 \) if there exists an orientation preserving diffeomorphism \( h \) of \( S^{n+2} \) such that \( h|L_1 \) is an orientation preserving diffeomorphism from \( L_1 \) to \( L_2 \).

We work in the smooth category.

**Definition** (\( K_1, K_2 \)) is called a pair of \( n \)-knots if \( K_1 \) and \( K_2 \) are \( n \)-knots. \( (K_1, K_2, X_1, X_2) \) is called a \( 4 \)-tuple of \( n \)-knots and \( (n+2) \)-knots or a \( 4 \)-tuple of \( (n, n+2) \)-knots if \( (K_1, K_2) \) is a pair of \( n \)-knots and \( X_1 \) and \( X_2 \) are \( (n+2) \)-knots diffeomorphic to the standard \( (n+2) \)-sphere. \((n \geq 1)\).

**Definition.** A 4-tuple of \((n, n+2)\)-knots \((K_1, K_2, X_1, X_2)\) is said to be realizable if there exists a smooth transverse immersion \( f : S_{1}^{n+2} \sqcup S_{2}^{n+2} \hookrightarrow S^{n+4} \) satisfying the following conditions. \((n \geq 1)\).

1. The intersection \( \Sigma = f(S_{1}^{n+2}) \cap f(S_{2}^{n+2}) \) is PL homeomorphic to the standard \( n \)-sphere.
2. \( f^{-1}(\Sigma) \) in \( S_{i}^{n+2} \) defines an \( n \)-knot \( K_i \) \((i = 1, 2)\).
3. \( f|S_{i}^{n+2} \) is an embedding. \( f(S_{i}^{n+2}) \) in \( S^{n+4} \) is equivalent to \( X_i \) \((i=1,2)\).

A pair of \( n \)-knots \((K_1, K_2)\) is said to be realizable or is called an intersectional pair of \( n \)-knots if there is a realizable 4-tuple of \((n, n+2)\)-knots \((K_1, K_2, X_1, X_2)\).

3. Intersectional pair of \( n \)-knots

Our main theorem is:

**Theorem 3.1.** A pair of \( n \)-knots \((K_1, K_2)\) \((n \geq 1)\) is realizable if and only if \((K_1, K_2)\) satisfies the condition that

\[
\begin{align*}
(K_1, K_2) \text{ is arbitrary} & \quad \text{if } n \text{ is even,} \\
Arf(K_1) = Arf(K_2) & \quad \text{if } n = 4m + 1, \ (m \geq 0). \\
\sigma(K_1) = \sigma(K_2) & \quad \text{if } n = 4m + 3,
\end{align*}
\]

There is a mod 4 periodicity in dimension. It is similar to the periodicity in knot cobordism theory ([L1]) and surgery theory (See e.g. [Br][Wa][CS][We]).

We have the following result on the realization of 4-tuples of \((n, n+2)\)-knots.

**Theorem 3.2.** A \( 4 \)-tuple of \((n, n+2)\)-knots \((K_1, K_2, X_1, X_2)\) is realizable if \( K_1 \) and \( K_2 \) are slice. \((n \geq 1)\). In particular, if \( n \) is even, an arbitrary \( 4 \)-tuple of \((n, n+2)\)-knots \((K_1, K_2, X_1, X_2)\) is realizable.

**Note.** (1) Kervaire proved that all even dimensional knots are slice ([Ke]).

(2) In [O1] the author discussed the case of two 3-spheres in a 5-sphere. In [O2] the author discussed the case of the intersection of three 4-spheres in a 6-sphere.
Problem. Which 4-tuples of \((2n + 1, 2n + 3)\)-knots are realizable \((n \geq 1)\)?

4. HIGH-DIMENSIONAL PASS-MOVES

In order to prove Theorem 3.1, we introduce a new local move for high dimensional knots, the *high dimensional pass-move*. Pass-moves for 1-knots are discussed in p.146 of [Ka].

We define high dimensional pass-moves for \((2k + 1)\)-knots \(\subset S^{2k+3}\).

**Definition** Take a trivially embedded \((2k + 3)\)-ball \(B = B^{2k+2} \times [-1, 1]\) in \(S^{2k+3}\). We define \(J_+, J_- \subset B\) as follows. Refer to Figure 4.1.

In \(\partial B^{2k+2} \times \{0\}\), take trivially embedded \(S^k_1, S^k_2\) such that \(\text{lk}(S^k_1, S^k_2) = 1\). Let \(N(S^k_1)\) be a tubular neighborhood of \(S^k_1\) in \(\partial B^{2k+2} \times \{0\}\).

Let \(h^{k+1}\) be an \((2k + 2)\)-dimensional \((k + 1)\)-handle which is attached to \(\partial B^{2k+2} \times \{0\}\) along \(N(S^k_1)\) with the trivial framing and which is embedded trivially in \(B^{2k+2} \times \{0\}\).

Let \(h^{k+1}_+\) (resp. \(h^{k+1}_-\)) be an \((2k + 2)\)-dimensional \((k + 1)\)-handle which is embedded in \(B = B^{2k+2} \times [0, 1]\) (resp. \(B = B^{2k+2} \times [-1, 0]\)) and which is attached to \(\partial B^{2k+2} \times \{0\}\) along \(N(S^k_2)\) with the trivial framing.

Let \(h^{k+1}_+ \cap h^{k+1}_- = N(S^k_2)\). Let \(h^{k+1}_+ \cap h^{k+1}_- = h^{k+1}_+ \cap h^{k+1}_- = \phi\).

Let \(J_+\) be a submanifold \((\partial h^{k+1}_+) - N(S^k_2) \equiv (\partial h^{k+1}_+) - N(S^k_2)\) in \(B\).

Let \(J_-\) be a submanifold \((\partial h^{k+1}_-) - N(S^k_1) \equiv (\partial h^{k+1}_-) - N(S^k_1)\) in \(B\).

In Figure 4.1, we draw \(B = B^{2k+2} \times [-1, 1]\) by using the projection to \(B^{2k+2} \times \{0\}\).

Figure 4.1.

You can obtain this figure by clicking ‘PostScript’ in the right side of the cite of the abstract of this paper in arXiv (https://arxiv.org/abs/the number of this paper).

You can also obtain it from the author’s website, which can be found by typing his name in search engine.

Let \(K_+, K_-\) be \((2k + 1)\)-knots \(\subset S^{2k+3}\). We say that \(K_+\) is obtained from \(K_-\) by one *high dimensional pass-move* if there is a trivially embedded \((2k + 2)\)-ball \(B \subset S^{2k+3}\) such that \(K_+ \cap B\) is \(J_+\) and \(K_- \cap B\) is \(J_-\).

Let \(K, K'\) be \((2k + 1)\)-knots \(\subset S^{2k+3}\). We say that \(K\) is *pass-move equivalent* to \(K'\) if there are \((2k + 1)\)-knots \(K_1, \ldots, K_\mu\) \((\mu \in \mathbb{N})\) such that \(K_1\) is pass-move equivalent to \(K_{i+1}\).

We prove:

**Theorem 4.1.** For \((2k + 1)\)-knots \(K_1\) and \(K_2\), the following two conditions are equivalent. \((k \geq 1)\)

(1) There exists a \((2k + 1)\)-knot \(K_3\) which is pass-move equivalent to \(K_1\) and cobordant to \(K_2\).
(2) \( K_1 \) and \( K_2 \) satisfy the condition that \[
\begin{cases}
\text{Arf}(K_1) = \text{Arf}(K_2) & \text{when } k \text{ is even} \\
\sigma(K_1) = \sigma(K_2) & \text{when } k \text{ is odd}
\end{cases}
\]

The \( k = 0 \) case of Theorem 4.1 follows from [Ka].

**Corollary 4.2.** Let \( K_1 \) and \( K_2 \) be \((2k + 1)\)-knots \((k \geq 1)\). Suppose that \( K_1 \) is pass-move equivalent to \( K_2 \). Then \( K_1 \) and \( K_2 \) satisfy the condition that
\[
\begin{cases}
\text{Arf}(K_1) = \text{Arf}(K_2) & \text{when } k \text{ is even} \\
\sigma(K_1) = \sigma(K_2) & \text{when } k \text{ is odd}
\end{cases}
\]

**Note.** In [O3] the author proved a relation between another local move of 2-knots and other invariants of 2-knots.

### 5. Proof of Theorem 3.1

We prove the following lemmas by explicit construction.

**Lemma 5.1.** Let \( K \) be an \( n \)-knot. Then the pair of \( n \)-knots \((K, K)\) is realizable \((n \geq 1)\).

**Lemma 5.2.** Let \( K_1 \) and \( K_2 \) be \((2k + 1)\)-knots. Suppose that \( K_1 \) is pass-move equivalent to \( K_2 \). Then the pair of \((2k + 1)\)-knots \((K_1, K_2)\) is realizable \((k \geq 0)\).

**Lemma 5.3.** Let \( K_1, K_2 \) and \( K_3 \) be \( n \)-knots \((n \geq 1)\). Suppose that the pair of \( n \)-knots \((K_1, K_2)\) is realizable and that \( K_2 \) is cobordant to \( K_3 \). Then the pair of \( n \)-knots \((K_1, K_3)\) is realizable.

Theorem 3.1 is deduced from Theorem 4.1 and Lemmas 5.1, 5.2, 5.3.

### 6. Proof of Theorem 3.2

It suffices to prove that a 4-tuple of \((n, n + 2)\)-knots \((K_1, K_2, T, T)\) is realizable, where \( K_1 \) is a slice \( n \)-knot, \( K_2 \) is the trivial \( n \)-knot, \( T \) is the trivial \((n + 2)\)-knot.

Any 1-twist spun knot is unknotted ([Z]). Theorem 3.2 follows from this fact.

### 7. The proof of Theorem 4.1

Every \( p \)-knot \((p > 1)\) is cobordant to a simple knot. (See [L1] for a proof and the definition of simple knots.) By using this fact, we prove that the \( k \geq 1 \) case of Theorem 4.1 can be deduced from Theorem 7.1.

**Proposition 7.1.** For simple \((2k + 1)\)-knots \( K_1 \) and \( K_2 \), the following two conditions are equivalent. \((k \geq 1)\)

1. \( K_1 \) is pass-move equivalent to \( K_2 \).
2. \( K_1 \) and \( K_2 \) satisfy the condition that
\[
\begin{cases}
\text{Arf}(K_1) = \text{Arf}(K_2) & \text{when } k \text{ is even} \\
\sigma(K_1) = \sigma(K_2) & \text{when } k \text{ is odd}
\end{cases}
\]
Proof of Proposition 7.1. (2)⇒(1). $K_1$ bounds a Seifert hypersurface $V_1$ with a handle decomposition (one 0-handle)∪((k+1)-handles). Take a Seifert matrix associated with $V_1$. By using high dimensional pass moves, we can change the Seifert matrix without changing the diffeomorphism type of $V_1$. Thus we obtain a $(2k+1)$-knot $K'_2$ whose Seifert matrix is same as the Seifert matrix of $K_2$ if (2) holds. By the classification theorem of simple knots by [L2], $K'_2$ is equivalent to $K_2$.

(1)⇒(2). Suppose that $(2k+1)$-knots $K_* \subset S^{2k+3}_n$ bounds a Seifert hyper surface $V_*$. Note $V_*$ are $(2k+2)$-manifolds. There is a compact oriented parallelizable $(2k+4)$-manifold $P$ whose boundary is $S^{4k+3}_1 \amalg S^{4k+3}_2$ containing compact oriented $(2k+3)$-manifold $Q$ whose boundary is $V_1 \cup (S^{2k+1}_2 \times [1,2]) \cup V_2$. (Here, $\partial V_*$ is $K_*$ and $S^{2k+1}_2 \times \{*\}$ is $K_*$. ) We use characteristic classes and intersection products to prove (1)⇒(2).

8. Intersectional pair of submanifolds

In §1 suppose $M$ is not PL homeomorphic to the standard sphere. Then we obtain a pair of submanifolds, $M$ in $S^{n+2}_i$ ($i = 1, 2$).

Let $N$ be a closed oriented manifold. $(K_1, K_2)$ is called a pair of submanifolds (diffeomorphic to $N$) if $K_i$ is a submanifold of $S^{n+2}_i$ diffeomorphic to $N$.

Let $(K_1, K_2)$ be a pair of submanifolds diffeomorphic to $M$. We say $(K_1, K_2)$ is an intersectional pair if the submanifold $K_i$ is equivalent to the submanifold $M = S^{n+2}_1 \cap S^{n+2}_2$ in $S^{n+2}_i$ as in §1 ($i = 1, 2$).

It is natural to ask the following problem.

Problem 8.1. Which pairs of submanifolds are intersectional pairs?

The author can prove the following results.

When $n$ is even, not all pair of submanifolds as above are realizable.

When $n = 4m + 3$, we can define the signature as in the knot case and the signature is an obstruction. Therefore not all pairs are realizable. When $n = 3$, $(K_1, K_2)$ is realizable if and only if $\sigma(K_1) = \sigma(K_2)$. When $n \neq 3$, $\sigma(K_1) = \sigma(K_2)$ does not imply $(K_1, K_2)$ is realizable in general.

When $n = 4m + 1$, there is a closed oriented manifold $M$ such that if $K_1$ and $K_2$ are PL homeomorphic to $M$, then $(K_1, K_2)$ is realizable. In other words, there is no invariant corresponding to the Arf invariant as in the knot case. Of course, not all pairs are realizable.

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**Acknowledgement.** The author would like to thank Prof. Levine for his interest in this paper and correcting the author’s English. The author would like to thank the referee and the editor for their reading with patience.

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ON THE INTERSECTION OF SPHERES IN A SPHERE II: 
HIGH DIMENSIONAL CASE

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Abstract. Consider transverse immersions $f : S_{1}^{n+2} II S_{2}^{n+2} \hookrightarrow S^{n+4}$ such that $f|S_{i}^{n+2}$ is an embedding and the intersection $f(S_{1}^{n+2}) \cap f(S_{2}^{n+2})$ is PL homeomorphic to the standard $n$-sphere. $(n \geq 1)$. Then we obtain a pair of $n$-knots, $f^{-1}(f(S_{1}^{n+2}) \cap f(S_{2}^{n+2}))$ in $S_{i}^{n+2}$ $(i = 1, 2)$. We determine which pair of $n$-knots are obtained as above. Roughly speaking, our result is characterized by the Arf invariant and the signature. We find a mod 4 periodicity in the dimension $n$.

1. Introduction and Main results

Let $S_{1}^{n+2}$ and $S_{2}^{n+2}$ be the $(n + 2)$-spheres embedded in the $(n + 4)$-sphere $S^{n+4}$ $(n \geq 1)$ and intersect transversely. Here, the orientation of the intersection $M$ is induced naturally. If we assume $M$ is PL homeomorphic to the single standard $n$-sphere, we obtain a pair of $n$-knots, $M$ in $S_{i}^{n+2}$ $(i = 1, 2)$, and a pair of $(n + 2)$-knots, $S_{i}^{n+2}$ in $S^{n+4}$ $(i = 1, 2)$.

Conversely, let $(K_{1}, K_{2})$ be a pair of $n$-knots. It is natural to ask whether $(K_{1}, K_{2})$ is obtained as above. In this paper we give a complete answer to the above question. Furthermore we discuss somewhat which 4-tuple $(K_{1}, K_{2}, S_{1}^{n+2}, S_{2}^{n+2})$ are realizable.

To state our results we need some definitions.

This research was partially supported by Research Fellowships of the Promotion of Science for Young Scientists.

This research was partially supported by Research Fellowships of the Promotion of Science for Young Scientists.
An (oriented) (ordered) m-component n-(dimensional) link is a smooth, oriented submanifold \( L = \{K_1, ..., K_m\} \) of \( S^{n+2} \), which is the ordered disjoint union of \( m \) manifolds, each PL homeomorphic to the standard n-sphere. (If \( m = 1 \), then \( L \) is called a knot.) We say that m-component n-dimensional links, \( L_0 \) and \( L_1 \), are said to be (link-)concordant or (link-)cobordant if there is a smooth oriented submanifold \( \tilde{C} = \{C_1, ..., C_m\} \) of \( S^{n+2} \times [0, 1] \), which meets the boundary transversely in \( \partial \tilde{C} \), is PL homeomorphic to \( L_0 \times [0, 1] \) and meets \( S^{n+2} \times \{l\} \) in \( L_l \) \((l = 0, 1)\). (See [CO]).

**Definition 1.1.** \((K_1, K_2)\) is called a pair of \( n \)-knots if \( K_1 \) and \( K_2 \) are \( n \)-knots. \((K_1, K_2, X_1, X_2)\) is called a 4-tuple of \( n \)-knots and \((n + 2)\)-knots or a 4-tuple of \((n, n + 2)\)-knots if \( K_1 \) and \( K_2 \) compose a pair of \( n \)-knots \((K_1, K_2)\) and \( X_1 \) and \( X_2 \) are \((n + 2)\)-knots diffeomorphic to the standard \((n + 2)\)-sphere. \((n \geq 1)\).

**Definition 1.2.** A 4-tuple of \((n, n + 2)\)-knots \((K_1, K_2, X_1, X_2)\) is said to be realizable if there exists a smooth transverse immersion \( f : S_1^{n+2} \sqcup S_2^{n+2} \to S^{n+4} \) satisfying the following conditions. \((n \geq 1)\).

1. \( f|S_i^{n+2} \) defines \( X_i \) \((i = 1, 2)\).
2. The intersection \( \Sigma = f(S_1^{n+2}) \cap f(S_2^{n+2}) \) is PL homeomorphic to the standard n-sphere.
3. \( f^{-1}(\Sigma) \) in \( S_i^{n+2} \) defines an \( n \)-knot \( K_i \) \((i = 1, 2)\).

A pair of \( n \)-knots \((K_1, K_2)\) is said to be realizable if there is a realizable 4-tuple of \((n, n + 2)\)-knots \((K_1, K_2, X_1, X_2)\). Then \( f \) is called an immersion to realize \((K_1, K_2, X_1, X_2)\) or \((K_1, K_2)\).

The following theorem characterizes the realizable pair of \( n \)-knots.

**Theorem 1.3.** A pair of \( n \)-knots \((K_1, K_2) \) \((n \geq 1)\) is realizable if and only if \((K_1, K_2)\) satisfies the condition that

\[
\begin{align*}
(K_1, K_2) \text{ is arbitrary} & \quad \text{if } n \text{ is even,} \\
\text{Arf}(K_1) = \text{Arf}(K_2) & \quad \text{if } n = 4m + 1, \ (m \geq 0, m \in \mathbb{Z}). \\
\sigma(K_1) = \sigma(K_2) & \quad \text{if } n = 4m + 3,
\end{align*}
\]

There exists a mod 4 periodicity in dimension. It is similar to the periodicity in the knot cobordism theory and the surgery theory.

We have the following results on the realization of 4-tuple of \((n, n + 2)\)-knots.

**Theorem 1.4.** A 4-tuple of \((n, n + 2)\)-knots \((K_1, K_2, X_1, X_2)\) is realizable if \( K_1 \) and \( K_2 \) are slice. \((n \geq 1)\).

Kervaire proved that all even dimensional knots are slice ([K]). Hence we have:

**Corollary 1.5.** If \( n \) is even, an arbitrary 4-tuple of \((n, n + 2)\)-knots \((K_1, K_2, X_1, X_2)\) is realizable.

The author discussed related topics in [O1], [O2], and [O3].
This paper is organized as follows. In §2 we introduce a new knotting operation, the high dimensional pass-move, and state its relation to the Arf invariant and the signature. In §3 we discuss a sufficient condition for the realization of pair of odd dimensional knots. In §4 we discuss a necessary condition for the realization of pair of \((4m+1)\)-knots. In §5 we discuss a necessary condition for the realization of pair of \((4m+3)\)-knots. In §6 we prove Theorem 1.4 which induces a necessary and sufficient condition for the realization of pair of even dimensional knots. Theorem 1.3 follows from §2-6.

The author would like to thank Professor Takashi Tsuboi for his advise.

2. High dimensional pass-moves

In this section we introduce a new knotting operation for high dimensional knots. The 1-dimensional case of Definition 2.1 is discussed in P. 146 of [Kf].

**Definition 2.1.** Take a \((2k+1)\)-knot \(K\) \((k \geq 0)\). Let \(K\) be defined by a smooth embedding \(g : \Sigma^{2k+1} \hookrightarrow S^{2k+3}\), where \(\Sigma^{2k+1}\) is PL homeomorphic to the standard \((2k+1)\)-sphere. Let \(D^{k+1}_x=\{(x_1, \ldots, x_{k+1})| \Sigma x^2_k < 1\}\) and \(D^{k+1}_y=\{(y_1, \ldots, y_{k+1})| \Sigma y^2_k < 1\}\). Let \(D^{k+1}_x(r)=\{(x_1, \ldots, x_{k+1})| \Sigma x^2_k \leq r^2\}\) and \(D^{k+1}_y(r)=\{(y_1, \ldots, y_{k+1})| \Sigma y^2_k \leq r^2\}\). A local chart \((U, \phi)\) of \(S^{2k+3}\) is called a pass-move-chart of \(K\) if it satisfies the following conditions.

1. \(\phi(U) \cong \mathbb{R}^{2k+3} = (0,1) \times D^{k+1}_x \times D^{k+1}_y\)
2. \(\phi(g(\Sigma^{2k+1} \cap U)) = \left[\left\{(\frac{1}{2}) \times D^{k+1}_x \times \partial D^{k+1}_y(\frac{1}{3})\right\} \sqcup \left\{\left\{(\frac{2}{3}) \times \partial D^{k+1}_x(\frac{1}{3}) \times D^{k+1}_y\right\}\right\}\)

Let \(g_U : \Sigma^{2k+1} \hookrightarrow S^{2k+3}\) be an embedding such that:

1. \(g|\{(\Sigma^{2k+1} \cap g^{-1}(U))\} = g_U|\{(\Sigma^{2k+1} \cap g^{-1}(U))\}\), and
2. \(\phi(g_U(\Sigma^{2k+1} \cap U)) = \left[\left\{(\frac{1}{2}) \times D^{k+1}_x \times \partial D^{k+1}_y(\frac{1}{3})\right\} \sqcup \left\{\left\{(\frac{2}{3}) \times \partial D^{k+1}_x(\frac{1}{3}) \times D^{k+1}_y\right\}\right\}\)

Let \(K_U\) be the \((2k+1)\)-knot defined by \(g_U\). Then we say that \(K_U\) is obtained from \(K\) by the \((high\ dimensional)\ pass-move\ in\ U\). We say that \((2k+1)\)-knot \(K\) and \(K'\) are \((high\ dimensional)\ pass-move\ equivalent\ if\ there\ exist\ \((2k+1)\)-knots \(K=K_1, K_2, \ldots, K_q, K_{q+1}=K'\) and \(K_{i+1}\) is obtained from \(K_i\) by the high dimensional pass-move in a pass-move-chart of \(K_i\) \((i = 1, \ldots, q)\).

High dimensional pass-moves have the following relation with the Arf invariant and the signature of knots. (The case of \(k = 0\) follows from [Kf].) We prove:

**Theorem 2.2.** For \((2k+1)\)-knots \(K_1\) and \(K_2\), the following two conditions are equivalent. \((k \geq 0)\)

1. There exists a \((2k+1)\)-knot \(K_3\) which is pass-move equivalent to \(K_1\) and cobordant to \(K_2\).
2. \(K_1\) and \(K_2\) satisfy the condition \(\text{Arf}(K_1) = \text{Arf}(K_2)\) when \(k\) is even
   \(\sigma(K_1) = \sigma(K_2)\) when \(k\) is odd.
Organization of the proof of Theorem 2.2 is as follows. Obviously Theorem 2.2 is equivalent to the following Claim 2.2.1 and 2.2.2.

**Claim 2.2.1.** If (2) of Theorem 2.2 holds, then (1) of Theorem 2.2 holds.

**Claim 2.2.2.** If (1) of Theorem 2.2 holds, then (2) of Theorem 2.2 holds.

In this section we prove Claim 2.2.1. (We use Claim 2.2.1 in §3.) We use the results of §3, 4 and 5 and prove Claim 2.2.2. (Note. We don’t use Claim 2.2.2 in the proof of §3, 4 and 5.) The proof of Claim 2.2.2 is written in §5.A. after §5.

We begin the proof of Claim 2.2.1. We need the following Lemma 2.3. We prove:

**Lemma 2.3.** If a $(2k+1)$-knot $K$ ($k \geq 0$) satisfy the condition

\[
\begin{align*}
(\ast) \quad & \text{Arf}(K)=0 \quad \text{when } k \text{ is even} \\
& \sigma(K) = 0 \quad \text{when } k \text{ is odd,}
\end{align*}
\]

then there exists a $(2k+1)$-knot $\tilde{K}$ which is pass-move equivalent to the trivial knot and cobordant to the $(2k+1)$-knot $K$.

Before proving Lemma 2.3, we prove:

**Claim.** Lemma 2.3 induces Claim 2.2.1.

Proof. $(-K^*_1)^\sharp K_2$ satisfies the condition (\ast). By Lemma 2.3, there exists a $(2k+1)$ knot $\tilde{K}$ which is pass-move equivalent to the trivial knot and cobordant to $(-K^*_1)^\sharp K_2$ Define $K_3$ to be $K_1^\sharp \tilde{K}$. Then the following (1) and (2) hold. (1)$K_3$ is pass-move equivalent to $K_1$. (2)$K_3 = K_1^\sharp \tilde{K}$ is cobordant to $K_1^\sharp (-K^*_1)^\sharp K_2$ and to $K_2$. □

Before proving Lemma 2.3, we review some definitions. (See [L2] and [Kw] for detail.) We first review on the definition of the Seifert matrix. Let $K$ be a $(2k+1)$-knot and $F$ a Seifert hypersurface. Let $F \times [-1, 1]$ be embedded in $S^{2k+3}$ so that $F \times \{0\}$ coincides with $F$ and the standard orientation of $[-1,1]$ coincides with the orientation of the normal bundle induced from that of $F$ and that of $S^{2k+3}$. For $(k+1)$-cycles $u$ and $v$ in $F$, define $\theta(u, v)$ to be $lk(u, v \times \{1\})$ in $S^{2k+3}$. Let $z_1, ..., z_p$ be $(k+1)$-cycles in $F$ which represent basis of $H_{k+1}(F; \mathbb{Z})/(\text{Torsion part})$. Define the Seifert matrix $A = \{a_{ij}\}$ of $K$ associated with $F$ and $z_i$ to be $a_{ij} = \theta(z_i, z_j)$. Here, recall the following lemma 2.4.

**Lemma 2.4.** (Well-known.)

(1) Let $K$ be a $(2k+1)$-knot ($k \geq 0$) with a Seifert hypersurface $F$. Let $u$ and $v$ be $(k+1)$-cycles in $F \subset S^{2k+3}$. We have:

\[
\theta(u, v) + (-1)^{k+1} \theta(v, u) = u \cdot v,
\]

where $u \cdot v$ is the intersection number in $F$.

(2) For vanishing $(k+1)$-cycles $\mu$ and $\nu$ in $S^{2k+3}$ ($k \geq 0$), where $\mu \cap \nu = \phi$,

\[
lk(\mu, \nu) = (-1)^k \ell k(\nu, \mu).
\]

(3) Let $K$ be a $(2k+1)$-knot with a Seifert matrix $A$ ($k \geq 0$). For appropriate Seifert surfaces, the Seifert matrixes of $-K$, that of $K^*$ and that of $-K^*$ are $(-1)^k \{A\}$, $(-1)^{(k+1)} \{A\}$ and $-A$, respectively.

Recall the following theorem.
Theorem 2.5. ([L1]) \((2k+1)\)-knots \((k \geq 1, k \neq 0)\) \(K_1\) and \(K_2\) are cobordant if and only if for a Seifert matrix \(A_i\) of \(K\(i = 1, 2\), \(\left( \begin{array}{cc} A_1 & O \\ O & -A_2 \end{array} \right)\) is congruent to \(\left( \begin{array}{cc} O & N_1 \\ N_2 & N_3 \end{array} \right)\), where \(N_i\) are same size.

We next review on the definition of the Arf invariant. Let \(K\) be a \((4m + 1)\)-knot. For \((2m + 1)\)-cycles \(x\) in \(F\), define \(q(x) \in \mathbb{Z}_2\) to be \(\theta(x, x)\) mod 2. Let \(x_1, \ldots, x_p, y_1, \ldots, y_p\) be \((2m + 1)\)-cycles in \(F\) which represent symplectic basis of \(H_{2m+1}(F; \mathbb{Z})/\text{(Torsion part)}\), i.e., basis such that (1) \(x_i \cdot y_i = 1\) for all \(i\), (2) \(x_i \cdot x_j = 0, y_i \cdot y_j = 0\) for all \((i, j)\), and (3) \(x_i \cdot y_j = 0\) for \(i \neq j\). Note that symplectic basis always exist. Then define \(\text{Arf}(K) = \sum_{i=1}^p q(x_i)q(y_i) \in \mathbb{Z}_2\).

At the end we review on the definition of the signature of knots. Define the signature \(\sigma(K)\) of \(K\) to be the signature of \(A + ^tA\). Recall that, when \(k\) is odd, \(\sigma(K) = \sigma(F)\).

We now begin the proof of Lemma 2.3.

Proof of Lemma 2.3. The case of \(k = 0\) is induced from [Kf]. We prove the case of \(k \geq 1\).

We first prove:

Claim. Let \(F\) be a Seifert hypersurface for \(K\). We can take \(2p\) \((k + 1)\)-cycles \(x_1, \ldots, x_p, y_1, \ldots, y_p\) in \(F\) which represent basis of \(H_{k+1}(F; \mathbb{Z})/\text{(Torsion part)}\) such that (1) \(x_i \cdot y_i = 1\) for all \(i\), (Hence, \(y_i \cdot x_i = (-1)^{k+1}\) for all \(i\)), (2) \(x_i \cdot x_j = 0, y_i \cdot y_j = 0\) for all \((i, j)\), and (3) \(x_i \cdot y_j = 0\) for \(i \neq j\).

Proof. When \(k\) is even, take symplectic basis. When \(k\) is odd, we need the following.

Sublemma. (See e.g. [S].) A symmetric matrix satisfies the conditions that (1) the elements are integers, (2) the determinant is \(+1\), and (3) the signature is zero, then the matrix is congruent to \(\oplus \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)\).

Since a matrix which represents the intersection products of \(H_{k+1}(F; \mathbb{Z})\) satisfies the condition of the above sublemma, such \(x_i\) and \(y_j\) exist. The proof of the above Claim is completed.

There exists a Seifert matrix \(X\) of \(K\) associated with the basis, \(x_i\) and \(y_j\), and \(F\). The elements of \(X\) are \(\theta(x_i, x_j)\), \(\theta(x_i, y_j)\), \(\theta(y_i, x_j)\), and \(\theta(y_i, y_j)\) \((i, j = 1, \ldots, p)\).

Take an embedding \(f : S^{k+1} \times S^{k+1} \hookrightarrow B^{2k+3}\) so that \(f(S^{k+1} \times S^{k+1})\) is a tubular neighborhood of the standard \((k + 1)\)-sphere embedded trivially in \(B^{2k+3}\). We regard \(S^{k+1}\) as \(D_1^{k+1} \cup D_2^{k+1}\). Let \(A\) denote \(f(S^{k+1} \times S^{k+1} - \text{Int}\{D_2^{k+1} \times D_2^{k+1}\})\). Let \(p_1, \ldots, p_p\) be points in \(D_2^{k+1}\), where \(\mu\) is a large positive integer. Take a neighborhood \(\left\{ \begin{array}{l} U_\alpha \\ V_\beta \end{array} \right\}\) of \(\left\{ \begin{array}{l} D_1^{k+1} \times p_\alpha \\ p_\beta \times D_1^{k+1} \end{array} \right\}\) in \(B^{2k+3}\) such that (1) \(U_\alpha\) and \(V_\beta\) are diffeomorphic to open \((2k + 3)\)-balls, and (2) arbitrary two of them don’t intersect. Let \(q\) be the center of \(D_1^{k+1}\). Let \(x'\) (resp. \(y'\)) denote the homology class which is represented by \(q \times S^{k+1}\) (resp. \(S^{k+1} \times q\)). Note that (1) A cycle represented by \(x'\) intersects with each \(U_\alpha\) at one points and doesn’t intersect any \(V_\beta\), and (2) A cycle representing \(y'\) intersects with each \(V_\beta\) at one points and doesn’t intersect any \(U_\alpha\).

Take disjoint \((2k + 3)\)-balls \(B_i^{2k+3}(i = 1, \ldots, p)\) in \(S^{2k+3}\). Take a copy of \(A\) in each \(B_i^{2k+3}\), say \(A_i\). Take a copy of \(U_\alpha\) (resp. \(V_\beta\)) in \(B_i\), say \(U_{i\alpha}\) (resp. \(V_{i\beta}\)). Take a copy of \(x'\) (resp. \(y'\)) in \(B_i\), say \(x'_i\) (resp. \(y'_i\)). By using \((2k+3)\)-dimensional 1-handles, take the connected sum of \(A_i\) in \(S^{2k+3}\), say \(A_0\). Then \(\partial A_0\) is the trivial \((2k + 1)\)-knot.
There exists a Seifert matrix $X'$ of the trivial knot associated with the basis, $x'_1, y'_1, ..., x'_p, y'_p$, and $A_0$. Obviously, $X'$ is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The elements of $X'$ satisfies that (1) $\theta(x'_i, x'_j)=0$ for all $(i, j)$, (2) $\theta(y'_i, y'_j)=0$ for all $(i, j)$, (3) $\theta(x'_i, y'_j)=0$ for $i \neq j$, (4) $\theta(x'_i, y'_i)=1$ for all $i$, and (5) $\theta(y'_i, x'_i)=0$ for all $i (i, j = 1, ..., p)$.

We make a $(2k+1)$-knot $\tilde{K}$ from the trivial $(2k+1)$-knot by high dimensional pass-moves as in the following paragraphs. Before making $\tilde{K}$, we prove:

**Claim.** If a Seifert matrix of $\tilde{K}$ coincides with that of $K$, then $\tilde{K}$ is what we required, i.e., the proof of Lemma 2.3 is completed.

Proof. By the definition of the construction of $\tilde{K}$, $\tilde{K}$ is pass-move equivalent to the trivial knot. By Theorem 2.5, $\tilde{K}$ is cobordant to $K$.

We take the following pass-move-charts of $\partial A_0$, carry out the pass-moves, and modify the Seifert matrix $X'$ to $X$. For the new knots obtained by the following pass-moves, we can take a Seifert hypersurfaces diffeomorphic to $A$. We can call basis for the new Seifert matrixes $x_i$ and $y_j$, again.

**Step 1.** See $\theta(x_i, x_j)$ $(i > j)$. If $\theta(x_i, x_j)=0$, then $\theta(x'_i, x'_j) = \theta(x_i, x_j)$. If $\theta(x_i, x_j)=\nu \neq 0$, make $|\nu|$ pass-move-charts $U^i_{ijk}$ $(k = 1, ..., |\nu|)$ from $U_{ik}$ and $U_{jk}$ so that pass-moves in $U^i_{ijk}$ let $\theta(x'_i, x'_j) = \theta(x_i, x_j)$.

Here, we have the following. We prove:

**Claim.** Then $\theta(x'_i, x'_j) = \theta(x_i, x_j)$ $(i < j)$.

Proof. Let $i < j$. By Lemma 2.4(1), $\theta(x'_i, x'_j) = x'_i \cdot x'_j + (-1)^k \theta(x'_j, x'_i)$ and $\theta(x_i, x_j) = x_i \cdot x_j + (-1)^k \theta(x_j, x_i)$. By the definition of $x_i$ and $x'_j$, $x_i \cdot x_j = x'_i \cdot x'_j (=0)$. Since $j > i$, $\theta(x'_j, x'_i) = \theta(x_j, x_i)$. Therefore $\theta(x'_i, x'_j) = \theta(x_i, x_j)$.

Here, note that, by the definition of these pass-moves, each pass-move in $U^i_{ijk}$ doesn’t change the value of $\theta(*, \dagger)$ except for $\theta(x'_i, x'_j)$ and $\theta(x'_j, x'_i)$ $(i > j)$.

**Step 2.** See $\theta(y_i, y_j)$ $(i > j)$. If $\theta(y_i, y_j)=0$, then $\theta(y'_i, y'_j) = \theta(y_i, y_j)$. If $\theta(y_i, y_j)=\nu \neq 0$, make $|\nu|$ pass-move-charts $V^i_{ijk}$ $(k = 1, ..., |\nu|)$ from $V_{ik}$ and $V_{jk}$ so that pass-moves in $V^i_{ijk}$ let $\theta(y'_i, y'_j) = \theta(y_i, y_j)$. Here, $\theta(y'_i, y'_j) = \theta(y_i, y_j)$ $(i < j)$ holds by Lemma 2.4(1).

Here, note that, by the definition of these pass-moves, each pass-move in $V^i_{ijk}$ doesn’t change the value of $\theta(*, \dagger)$ except for $\theta(y'_i, y'_j)$ and $\theta(y'_j, y'_i)$ $(i > j)$.

**Step 3.** See $\theta(x_i, y_j)$ for any $(i, j)$. If $\theta(x_i, y_j)=0$, then $\theta(x'_i, y'_j) = \theta(x_i, y_j)$. If $\theta(x_i, y_j)=\nu \neq 0$, make $|\nu|$ pass-move-charts $W^i_{ijk}$ $(k = 1, ..., |\nu|)$ from $U_{ik}$ and $V_{jk}$ so that pass-moves in $W^i_{ijk}$ let $\theta(x'_i, y'_j) = \theta(x_i, y_j)$. Here, $\theta(y'_i, x'_j) = \theta(x_i, y_j)$ $(i < j)$ holds by Lemma 2.4(1).

Here, note that, by the definition of these pass-moves, each pass-move in $W^i_{ijk}$ doesn’t change the value of $\theta(*, \dagger)$ except for $\theta(x'_i, y'_j)$ and $\theta(y'_i, x'_j)$.

Here, note that, by the definition of these pass-moves, each pass-move in $W^i_{ijk}$ doesn’t change the value of $\theta(*, \dagger)$ except for $\theta(y'_i, x'_j)$ and $\theta(y'_j, x'_i)$ $(i > j)$.

Before Step 4 and 5, we prove:
Lemma 3.2. morphic to the standard (2\text{-}k+1)-sphere (k \geq 0). If we have the condition that
\[
\begin{align*}
\text{Arf}(K_1) &= \text{Arf}(K_2) & \text{when } k & \text{ is even} \\
\sigma(K_1) &= \sigma(K_2) & \text{when } k & \text{ is odd},
\end{align*}
\]
then, for a (2k+3)\text{-}knot X_1 \text{ diffeomorphic to the standard (2k+3)-sphere, (K_1, K_2, X_1, X_2) is realizable.}

To prove Proposition 3.1, we need some lemmas.

Lemma 3.2. Let K be an arbitrary n-knot and X_1 and X_2 arbitrary (n+2)-knots diffeomorphic to the standard (n+2)-sphere (n \geq 1). Then (K, K, X_1, X_2) is realizable.
Lemma 3.3. If 4-tuple of \((n, n+2)\)-knots \((K_1, K_2, X_1, X_2)\) is realizable \((n \geq 1)\) and \(K_2\) is cobordant to \(\tilde{K}_2\), then for an \((n+2)\)-knot \(\tilde{X}_1\) diffeomorphic to the standard \((n+2)\)-sphere, \((K_1, \tilde{K}_2, \tilde{X}_1, X_2)\) is realizable. Furthermore, for an arbitrary Seifert hypersurface \(F\) for the \(n\)-knots \(\tilde{K}_2\), there exists an immersion \(\tilde{f}: S_{1}^{n+2} \coprod S_{2}^{n+2} \to S^{n+4}\) to realize \((K_1, \tilde{K}_2, \tilde{X}_1, X_2)\) and a Seifert hypersurface \(\tilde{V}\) for \(\tilde{X}_1 = \tilde{f}(S_{1}^{n+2})\) such that \(\tilde{V} \cap \tilde{f}(S_{2}^{n+2}) = \tilde{V} \cap X_2\) is the Seifert hypersurface \(F\) for \(\tilde{K}_2\).

Lemma 3.4. Let \(K_1\) and \(K_2\) be \((2k + 1)\)-knots and \(X_1\) and \(X_2\) arbitrary \((2k + 3)\)-knots diffeomorphic to the standard \((2k + 3)\)-sphere \((k \geq 0)\). If \(K_1\) and \(K_2\) are pass-move equivalent and there exist pass-move-charts \(U_i\) \((i = 1, \ldots, q)\) of \(K_2\) such that \(U_i \cap U_j = \phi\) \((i \neq j)\) and \(K_1\) is obtained from \(K_2\) by the pass-moves in \(U_i\), then \((K_1, K_2, X_1, X_2)\) is realizable.

We first prove:

Claim. If Claim 2.2.1S, Lemma 3.3, and 3.4 hold, Proposition 3.1 holds.

Proof. By Claim 2.2.1S, (1) there exists a \((2k + 1)\)-knot \(K_3\) which is pass-move equivalent to \(K_1\) and cobordant to \(K_2\) and (2) there exist pass-move-charts \(U_i\) \((i = 1, \ldots, q)\) of \(K_3\) such that \(U_i \cap U_j = \phi\) \((i \neq j)\) and \(K_1\) is obtained from \(K_3\) by the pass-moves in \(U_i\). By Lemma 3.4, \((K_1, K_3, X_2, X_2)\) is realizable. By Lemma 3.3, for a \((2k + 3)\)-knot \(X_1\), \((K_1, K_2, X_1, X_2)\) is realizable.

Note. Lemma 3.2 is used to prove Lemma 3.3 and 3.4. The latter half of Lemma 3.3 is used in §4. The case when \(n\) is even of Theorem 1.3 is induced from Lemma 3.4, obviously. But we prove in §6 Theorem 1.4 stronger than it.

In the rest of this section we prove Lemma 3.2-4.

Proof of Lemma 3.2. Take an embedding \(f_a: S_{1}^{n+2} \coprod S_{2}^{n+2} \to S^{n+4}\) which defines the trivial \((n+2)\)-link. There exists a chart \(U\) of \(S^{n+4}\) with the following properties (1) and (2).

1. \(\phi: U \cong \mathbb{R}^{n+2} \times \{(u, v) | u, v \in \mathbb{R}\} \cong \mathbb{R}^{n+2} \times \mathbb{R}_u \times \mathbb{R}_v\)
2. \(U \cap f_a(S_{1}^{n+2}) = \mathbb{R}^{n+2} \times \{(u, v) | u = 0, v = 0\}\)
   \(U \cap f_a(S_{2}^{n+2}) = \mathbb{R}^{n+2} \times \{(u, v) | u = 3, v = 0\}\)

We modify the embedding \(f_a\) to obtain an immersion \(f_b: S_{1}^{n+2} \coprod S_{2}^{n+2} \to S^{n+4}\) to realize \((K, K, T, T)\), where \(T\) is the trivial knot. We put \(f_b|S_{2}^{n+2} = f_a|S_{2}^{n+2}\). We define \(f_b|S_{1}^{n+2}\) as follows. Take an embedding \(g: \Sigma^{2k+1} \to U \cap f_a(S_{1}^{n+2})\) which defines the \(n\)-knot \(K\) in \(S_{1}^{n+2}\). Let \(F\) be a Seifert hypersurface for \(K\) and a submanifold of \(U \cap f_a(S_{1}^{n+2})\). Let \(N_1(F) = F \times [-1, 1]\) be a submanifold embedded in \(U \cap f_a(S_{1}^{n+2})\) such that \(F = F \times \{0\}\). We define the subsets \(E_1, E_2\) and \(E_3\) of \(N_1(F) \times \mathbb{R}_u \times \mathbb{R}_v\) as follows.

\[E_1 = \{(p, t, u, v) | p \in F, 0 \leq u \leq 1, -1 \leq t \leq 1, v = 0\}\]
\[E_2 = \{(p, t, u, v) | p \in F, 1 \leq u \leq 2, t = k \cdot \cos \frac{\pi (u-1)}{2}, v = k \cdot \sin \frac{\pi (u-1)}{2}, -1 \leq k \leq 1\}\]
\[E_3 = \{(p, t, u, v) | p \in F, 2 \leq u \leq 4, t = 0, -1 \leq v \leq 1\}\]

Then the followings hold by the way of the construction.

\[\Sigma = \{f_a(S_{1}^{n+2}) - N_1(F)\} \cup_{\partial N_1(F)} \{\partial(E_1 \cup E_2 \cup E_3)\} = N_1(F)\]

is a \((n+2)\)-sphere embedded in \(S^{n+4}\).
Since $\{\partial(E_1 \cup E_2 \cup E_3)\} - N_1(F)$ is isotopic to $N_1(F)$ relative to $\partial N_1(F)$, $f_b|S_1^{n+2}$ defines the trivial knot. Both $\Sigma \cap f_b(S_2^{n+2})$ in $\Sigma$ and $\Sigma \cap f_b(S_2^{n+2})$ in $f_b(S_2^{n+2})$ defines $K$. We define $f_b|S_1^{n+2}$ so that $f_b(S_1^{n+2})$ coincides with $\Sigma$. We obtain $f_b$ to realize $(K, K, T, T)$.

By the following Sublemma 3.5, $(K, K, X_1, X_2)$ is realizable. We prove Sublemma 3.5 and complete the proof of Lemma 3.2.

**Sublemma 3.5.** Let $T$ be the trivial $(n+2)$-knot and $X_i$ arbitrary $(n+2)$-knots $(i = 1, 2)$. If $(K, K, T, T)$ is realizable, then $(K, K, X_1, X_2)$ is realizable.

**Proof of Sublemma 3.5.** Let $f' : S_1^{n+2} \sqcup S_2^{n+2} \hookrightarrow S^{n+4}$ realize $(K, K, T, T)$. Let $f_i : S_i^{n+2} \hookrightarrow S^{n+4} (i = 1, 2)$ define $(n+2)$-knots $X_i$. Let $B', B_1$ and $B_2$ be $(n+4)$-balls in $S^{n+4}$ and $B' \cap B_1 = B' \cap B_2 = B_1 \cap B_2 = \phi$. We can take $f'$ and $f_i$ so that $\text{Im} f_i$ in $B_i$ and $\text{Im} f'$ in $B'$. Connect $f_i(S_i^{n+2})$ with $f'(S_i^{n+2})$ by $(n+3)$-dimensional 1-handle $h_i$ embedded in $S^{n+4}$ $(i = 1, 2)$, where $h_1 \cap h_2 = \phi$. Take $f : S_1^{n+2} \sqcup S_2^{n+2} \hookrightarrow S^{n+4}$ so that $f(S_i^{n+2})$ coincides with $f_i(S_i^{n+2}) \neq f'(S_i^{n+2})$. Then $f$ realizes $(K, K, X_1, X_2)$. \(\square\)

**Proof of Lemma 3.3.** Let $f : S_1^{n+2} \sqcup S_2^{n+2} \hookrightarrow S^{n+4}$ be an immersion to realize $(K_1, K_2, X_1, X_2)$ and $V$ a Seifert hypersurface for $f(S_1^{n+2})$. Let $f(S_2^{n+2}) \times D^2 = X_2 \times \{ (x, y) \mid x = r \cdot \cos \theta, y = r \cdot \sin \theta, 0 \leq r \leq 1, 0 \leq \theta < 2\pi \}$ be a tubular neighborhood of $X_2$ in $S^{n+4}$.

See $V \cap \{ f(S_2^{n+2}) \times D^2 \}$. It has the following properties.

1. For any $(x, y), \{ \partial \tilde{V} \} \cap [f(S_2^{n+2}) \times (x, y)]$ in $f(S_2^{n+2}) \times (x, y)$ defines $K_2$.
2. For each $(x, y), \tilde{V} \cap [f(S_2^{n+2}) \times (x, y)]$ is a same Seifert hypersurface $G$.
3. For any $\theta, \{ \partial \tilde{V} \} \cap [f(S_2^{n+2}) \times \{ (x, y) \mid x = r \cdot \cos \theta, y = r \cdot \sin \theta, 0 \leq r \leq 1 \}]$ is diffeomorphic to $K_2 \times [0, 1]$.

Prepare $S^{n+2} \times [0, 1]$. Put $K_2$ and $G$ in $S^{n+2} \times \{ 1 \}$. Put $\tilde{K}_2$ and $F$ in $S^{n+2} \times \{ 0 \}$. Recall $K_2$ and $\tilde{K}_2$ are cobordant. Hence there exists a submanifold $P$ in $S^{n+2} \times [0, 1]$ which meets the boundary transversely in $P$, is diffeomorphic to $K_2 \times [0, 1]$, and meets $S^{n+2} \times \{ 0 \}$ at $K_2$ and $S^{n+2} \times \{ 1 \}$ at $\tilde{K}_2$. By an elementary discussion of the obstruction theory, there exists a compact submanifold $Q$ in $S^{n+2} \times [0, 1]$ such that $\partial Q = F \cup P \cup G$.

We modify $V$ and make the following $\tilde{V}$. Let $\tilde{V}$ be a submanifold of $S^{n+4}$ satisfying the following conditions.

1. $\tilde{V} \cap [S^{n+4} - \{ f(S_2^{n+2}) \times D^2 \}]$ coincides with $V \cap [S^{n+4} - \{ f(S_2^{n+2}) \times D^2 \}]$.
2. $\{ \partial \tilde{V} \} \cap [f(S_2^{n+2}) \times (0, 0)]$ in $f(S_2^{n+2}) \times (0, 0)$ is $K_2$.
3. $\tilde{V} \cap [f(S_2^{n+2}) \times (0, 0)] = F$.
4. For any $\theta, \{ \partial \tilde{V} \} \cap [f(S_2^{n+2}) \times (\cos \theta, \sin \theta)]$ in $[f(S_2^{n+2}) \times (\cos \theta, \sin \theta)]$ is $K_2$.
5. For any $\theta, \{ \partial \tilde{V} \} \cap [f(S_2^{n+2}) \times \{ (x, y) \mid x = r \cdot \cos \theta, y = r \cdot \sin \theta, 0 \leq r \leq 1 \}]$ is diffeomorphic to $K_2 \times [0, 1]$ and $\tilde{K}_2 \times [0, 1]$.
6. For any $\theta, \{ \tilde{V} \} \cap [f(S_2^{n+2}) \times \{ (x, y) \mid x = r \cdot \cos \theta, y = r \cdot \sin \theta, 0 \leq r \leq 1 \}]$ defines the above submanifold $Q$ in $[f(S_2^{n+2}) \times \{ (x, y) \mid x = r \cdot \cos \theta, y = r \cdot \sin \theta, 0 \leq r \leq 1 \}]$.

Note. (i) The above condition (5) holds because $K_2$ is cobordant to $\tilde{K}_2$. (ii) $\partial \tilde{V}$ is diffeomorphic to the standard sphere.

Let $\tilde{f} : S_1^{n+2} \sqcup S_2^{n+2} \hookrightarrow S^{n+4}$ be an immersion such that (1) $\tilde{f}(S_1^{n+2})$ coincides with $\partial \tilde{V}$,
say $\widetilde{X}_1$, and (2) $f(S^{n+2}_2) = f(S^{n+2}_2)$.

By the construction of $\tilde{f}$, the $n$-knot $\tilde{f}(S^{n+2}_1) \cap \tilde{f}(S^{n+2}_2)$ in $\tilde{f}(S^{n+2}_1)$ is equivalent to $\tilde{f}(S^{n+2}_1) \cap \{f(S^{n+2}_2) \times \{(x, y) | x = 1, y = 0\}\}$ in $\tilde{f}(S^{n+2}_1)$, to $\tilde{f}(S^{n+2}_2) \cap \{f(S^{n+2}_2) \times \{(x, y) | x = 1, y = 0\}\}$ in $\tilde{f}(S^{n+2}_1)$, and to $f(S^{n+2}_1) \cap f(S^{n+2}_2)$ in $f(S^{n+2}_1)$, that is, $K_1$.

Then we obtain the immersion $\tilde{f}$ to realize $(K_1, \tilde{K}_2, \widetilde{X}_1, X_2)$ and the required $V$.

**Proof of Lemma 3.4.** Take the pass-move-charts $U_1, ..., U_q$ of $K_2$. In each pass-move-chart $U_i$, take

$$D^{k+1}_{1i} = \left[\frac{7}{18}, \frac{11}{18}\right] \times D^{k+1}_x(0) \times D^{k+1}_y(\frac{1}{2})$$

$$D^{k+1}_{2i} = \left[\frac{5}{9}, \frac{7}{9}\right] \times D^{k+1}_x(\frac{2}{3}) \times D^{k+1}_y(0).$$

Put $S^{k+1}_{1i} = \partial D^{k+1}_{1i}$ and $S^{k+1}_{2i} = \partial D^{k+1}_{2i}$. Note the linking number of $S^{k+1}_{1i}$ and $S^{k+1}_{2i}$ is one. Since $H_{2k+1}(S^{2k+3} - \cup_{i,j}(S^{k+1}_{ji})) = 0$, a Seifert hypersurface $F$ for $K_2$ is included in $S^{2k+3} - \cup_{i,j}(S^{k+1}_{ji})$ ($i = 1, ..., q$, $j = 1, 2$).

Then we have the following.

**Claim.** If we attach $(2k + 4)$-dimensional $(k + 2)$-handles with 0-framing to $U_i$ along $S^{k+1}_{1i}$ and $S^{k+1}_{2i}$ and carry out surgery on $U_i$ (and $S^{2k+3}$), then (1) $U_i$ changes into the $(2k + 3)$-ball again, (2) $S^{2k+3}$ changes into the $(2k + 3)$-sphere again, and (3) the new knot in the new $(2k + 3)$-sphere is the knot obtained from $K_2$ by the pass-moves in $U_i$.

By the above discussion, the followings hold. In all $U_i$ ($i = 1, ..., q$), carry out the above surgeries. Then $S^{2k+3}$ changes into the $(2k + 3)$-sphere again, and $K_2$ in the old $(2k + 3)$-sphere changes into $K_1$ in the new $(2k + 3)$-sphere. There exists a Seifert hypersurface $F$ for $K_2$ such that $S^{2k+3} \cap S^{k+1}_{ji}$ for all $i, j$.

We first construct an immersion to realize $(K_2, K_2, T, T)$ as in the proof of Lemma 3.2. Take $U$, and $\Sigma$ as in the proof of Lemma 3.2. Take the pass-move-charts $U_i$ of $K_2$ in $U$. Take $S^{k+1}_{ji}$ in $U_i \subset U$. We use the Seifert hypersurface $F$ in the previous paragraph as $F$ in the proof of Lemma 3.2. As we see before, we can take a Seifert hypersurface so that it does not intersect with $S^{k+1}_{ji}$ in $U_i$ for all $i, j$. We use the Seifert hypersurface as $F$ in the proof of Lemma 3.2. We use these $U_i$, $\Sigma$ and $F$ and construct an immersion $f_b : S^{2k+3}_1 \coprod S^{2k+3}_2 \hookrightarrow S^{2k+5}$ to realize $(K_2, K_2, T, T)$ as in the proof of Lemma 3.2.

We next construct an immersion $f_c : S^{2k+3}_1 \coprod S^{2k+3}_2 \hookrightarrow S^{2k+5}$ to realize $(K_1, K_2, T, T)$. Let $h_{1i}$ be a tubular neighborhood of

$$[S^{k+1}_{1i} \times \{(u, v) | u = 0, 0 \leq v \leq 1\}] \cup [D^{k+1}_{1i} \times \{(u, v) | u = 0, v = 1\}]$$

in $U_i \times \{(u, v) | u = 0, v \geq 0\}$. Let $h_{2i}$ be a tubular neighborhood of

$$[S^{k+1}_{2i} \times \{(u, v) | u = 0, -1 \leq v \leq 0\}] \cup [D^{k+1}_{2i} \times \{(u, v) | u = 0, v = -1\}]$$

in $U_i \times \{(u, v) | u = 0, v \leq 0\}$. Make a submanifold $\Lambda$ from $\Sigma$ and $h_{ji}$ so that

$$\Lambda = \Sigma - \cup_{i=1}^q (h_{1i} \cap \Sigma) - \cup_{i=1}^q (h_{2i} \cap \Sigma) - \cup_{i=1}^q \delta h_{1i} - (h_{1i} \cap \Sigma) - \cup_{i=1}^q \delta h_{2i} - (h_{1i} \cap \Sigma).$$

Of course, $(h_{ji} \cap \Sigma)$ is $(\delta h_{ji}) \cap \Sigma)$ and is the tubular neighborhood of $S^{k+1}_{ji}$ in $U_i$.

Then the followings hold by the definition of the construction. (1) $\Lambda$ is the trivial $(2k + 3)$-knot. (2) $\Lambda \cap f_b(S^{2k+3})$ in $\Lambda$ is $K_1$. (Because the pass-move is carried out in each pass-
move-chart \( U_i \). (3) \( \Lambda \cap f_b(S_{2}^{2k+3}) \) in \( f_b(S_{2}^{2k+3}) \) is \( K_2 \). Here, we take an immersion \( f_c : S_{1}^{2k+3} \coprod S_{2}^{2k+3} \hookrightarrow S_{2k+5} \) so that \( f_c(S_{1}^{2k+3}) \) coincides with \( \Lambda \) and \( f_c|S_{2}^{2k+3} = f_b|S_{2}^{2k+3} \). Then \( f_c \) is an immersion to realize \((K_1, K_2, T, T)\).

At last, by Sublemma 3.5, \((K_1, K_2, X_1, X_2)\) is realizable. \( \square \)

4. A NECESSARY CONDITION FOR THE REALIZATION OF PAIR OF \((4m+1)\)-KNOTS

In this section we prove:

Proposition 4.1. If a 4-tuple of \((4m+1,4m+3)\)-knots \((K_1, K_2, X_1, X_2)\) is realizable \((m \geq 0)\), then \( \text{Arf}(K_1) = \text{Arf}(K_2) \).

We first prove the following Lemma 4.2.

Lemma 4.2 If 4-tuple of \((n,n+2)\)-knots \((K^{(1)}, K^{(2)}, X^{(1)}, X^{(2)})\) and \((K_{1}^{(1)}, K_{2}^{(1)}, X_{1}^{(1)}, X_{2}^{(2)})\) are realizable \((n \geq 1)\), then \((K^{(1)} \# K^{(2)}, K_{1}^{(1)} \# K_{2}^{(1)}, X^{(1)} \# X^{(2)}, X_{1}^{(1)} \# X_{2}^{(2)})\) is realizable.

Note. Lemma 4.2 includes Sublemma 3.5.

Proof of Lemma 4.2. Take two \((n+4)\)-balls \(B_{1}^{n+4} \sqcup B_{2}^{n+4}\) in \( S^{n+4} \) so that \( B_{1}^{n+4} \cap B_{2}^{n+4} = \phi \). Take \( g_i : S_{1}^{n+2} \sqcup S_{2}^{n+2} \hookrightarrow S^{n+4} \) to realize \((K_{1}^{(i)}, K_{2}^{(i)}, X_{1}^{(i)}, X_{2}^{(i)})\) so that we take \( \text{Im}g_i \) in \( B_{i}^{n+4}(i = 1,2) \). Let \( X_{1}^{(i)} \) denote \( \text{Im}g_i(S_{j}^{n+2}) \) for convenience. Connect \( X_{1}^{(1)} \) with \( X_{1}^{(2)} \) by using \((n+3)\)-dimensional 1-handle \( h_{91} \) embedded in \( S^{n+4} \) to obtain \( X_{1}^{(1)} \sqcup X_{1}^{(2)} \). Connect \( \widetilde{X}_{1} \cap X_{2}^{(1)} \) with \( \widetilde{X}_{1} \cap X_{2}^{(2)} \) by \((n+1)\)-dimensional 1-handle \( h_{92}' \) embedded in \( \widetilde{X}_{1} \). Connect \( X_{2}^{(1)} \) with \( X_{2}^{(2)} \) by using \((n+3)\)-dimensional 1-handle \( h_{g2} = h_{g2}' \times D^2 \) embedded in \( S^{n+4} \) to obtain \( \widetilde{X}_{2} = X_{2}^{(1)} \# X_{2}^{(2)} \). Take \( g : S_{1}^{n+2} \sqcup S_{2}^{n+2} \hookrightarrow S^{n+4} \) so that \( g(S_{i}^{n+2}) \) coincides with \( \widetilde{X}_{i} \). \( \square \)

We prove Proposition 4.1 by the reduction to absurdity. We assume \( \text{Arf}(K_1) \neq \text{Arf}(K_2) \) and induce the absurdity. We may assume that \( \text{Arf}(K_1) = 1 \) and \( \text{Arf}(K_2) = 0 \) without loss of generality.

Note. If \( bP_{4m+2} = \mathbb{Z}_2 \), it is obvious that the proof is easy. Recall that the subgroup \( bP_{4m+2} \subset \Theta_{4m+1} \) is the trivial group for some integers and is \( \mathbb{Z}_2 \) for some integers (See [KM].) It is known that there exist some integers \( m \) such that (i) \( TS^{2m+1} \) is not the trivial bundle, i.e., \( m \neq 0,1,3 \), and (ii) \( bP_{4m+2} \) is the trivial group (See [Br]).

Let \( T \) be the trivial \((4m+1)\)-knot. Let \( K \) be a \((4m+1)\)-knot such that a Seifert hypersurface for \( K \) is the plumbing \( F \) of two copies of the tangent bundle of the standard \((2m+1)\)-sphere and a Seifert matrix associated with \( F \) is
\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]. (See e.g. P. 162 of [LM] for the plumbing.)

Then \( \text{Arf}(K) = 1 \) and for an arbitrary non-vanishing \((2m+1)\)-cycle \( x \) of \( F \), \( \theta(x, x) \) is odd. (See §2 of this paper for the definition of \( \theta( , , ) \).)

We prove:

Claim. The pair of \((4m+1)\)-knots \((T, K)\) is realizable under the hypothesis of the reduction to absurdity.
Proof. By Proposition 3.1, \((-K_1^*, K)\) and \((T, -K_2^*)\) are realizable. By the hypothesis of the reduction to absurdity \((K_1, K_2)\) is realizable. By Lemma 4.2, \((K_1^*\{−K_1^*\}, T, K_2^*\{−K_2^*\})\) is realizable, i.e., \((K_1^*\{−K_1^*\}, K_2^*\{−K_2^*\})\) is realizable. Since \(K_1^*\{−K_1^*\}\) is cobordant to the trivial knot and \(K_2^*\{−K_2^*\}\) is cobordant to \(K, (T, K)\) is realizable by Lemma 3.3. □

Let \(f : S_i^{4m+3} \rightarrow S^4_{m+3} \) be an immersion to realize \((T, K)\). By Lemma 3.3, there exist Seifert hypersurfaces \(V_i\) for \(f(S_i^{4m+3}) \) \((i = 1, 2)\) such that \(f(S_i^{4m+3}) \cap V = (4m + 2)\)-disk \(D \) and \(V \cap f(S_i^{4m+3})\) is the Seifert hypersurface \(F\). Let \(W\) denote \(V \cap V_2\). Then \(\partial W = F \cup D\). The following holds. We prove:

Claim. There exists a non-vanishing \((2m + 1)\)-\(\mathbb{Z}_2\)-cycle \(x\) in \(F\) which is zero cycle in \(W\), i.e., \(x\) bounds a \((2m + 2)\)-\(\mathbb{Z}_2\)-chain \(y\) in \(W\).

Proof. The natural inclusion \(F \hookrightarrow \partial W\) induces \(H_i(F; \mathbb{Z}_2) \cong H_i(\partial W; \mathbb{Z}_2) \) \((i \neq 4m + 2)\). Hence it suffices to prove that \(\text{Ker} \{H_{2m+1}(\partial W; \mathbb{Z}_2) \rightarrow H_{2m+1}(W; \mathbb{Z}_2)\}\) is not the trivial group. Consider the exact sequence: \(H_i(\partial W; \mathbb{Z}_2) \rightarrow H_i(W; \mathbb{Z}_2) \rightarrow H_i(W, \partial W; \mathbb{Z}_2)\). We use the following part \(\ast\) of the exact sequence: \(H_{2m+2}(\partial W; \mathbb{Z}_2) \rightarrow H_{2m+2}(W; \mathbb{Z}_2) \rightarrow H_{2m+2}(W, \partial W; \mathbb{Z}_2) \rightarrow H_{2m+1}(\partial W; \mathbb{Z}_2) \rightarrow H_{2m+1}(W; \mathbb{Z}_2)\). We have \(H_{2m+2}(\partial W; \mathbb{Z}_2) \cong H_{2m}(\partial W; \mathbb{Z}_2) \cong 0\) and \(H_{2m+1}(\partial W; \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2\). By Poincaré duality and the universal coefficient theorem, the followings hold. (1) The \(\mathbb{Z}_2\)-rank of \(H_{2m+2}(W; \mathbb{Z}_2)\) and that of \(H_{2m+1}(W, \partial W; \mathbb{Z}_2)\) are same, put it \(r\). (2) The \(\mathbb{Z}_2\)-rank of \(H_{2m+1}(W; \mathbb{Z}_2)\) and that of \(H_{2m+2}(W, \partial W; \mathbb{Z}_2)\) are same, put it \(s\). Therefore the sequence \(\ast\) becomes as follows: \(0 \rightarrow \oplus^r \mathbb{Z}_2 \rightarrow \oplus^s \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \rightarrow \oplus^r \mathbb{Z}_2 \rightarrow 0\). Therefore the \(\mathbb{Z}_2\)-rank of \(\text{Ker} \{H_{2m+1}(\partial W; \mathbb{Z}_2) \rightarrow H_{2m+1}(W; \mathbb{Z}_2)\}\) = \(s - r = 1\). □

The \((2m + 1)\)-\(\mathbb{Z}_2\)-cycle \(x\) bounds a \((2m + 2)\)-\(\mathbb{Z}_2\)-chain \(z\) in \(S_i^{4m+3}\). Let \(w\) denote the \((2m + 2)\)-\(\mathbb{Z}_2\)-cycle \(y \cup_x z\). Let \(\omega\) denote the homology class of \(w\). We prove:

Claim 4.3. Consider \(\omega \in H_{2m+2}(V_2; \mathbb{Z}_2)\). The \(\mathbb{Z}_2\)-intersection number \(\omega \cdot \omega\) in \(V_2\) is one.

Proof. Let \(W \times \{t\} \leq t \leq 1\) be a tubular neighborhood of \(W\) in \(V_2\). The \((2n+1)\)-cycle \(x \times \{t = 1\}\) in \(S_i^{4m+3}\) bounds a \((2n + 2)\)-chain \(\tilde{z}\) in \(S_i^{4m+3}\). Let \(w_t\) be the cycle \((y \times \{t = 1\}) \cup_{x \times \{t = 1\}} \tilde{z}\).

Let \(\partial V_2 \times \{s \leq 0 \leq 1\}\) be a collar neighborhood of \(\partial V_2\) in \(V_2\). Let \(w_s\) be the cycle \((y \cap \{\partial V_2 \times \{s \leq 0 \leq 1\}\}) \cup_{x \times \{s = 1\}} (z \times \{s = 1\})\).

Then the followings hold. (1) \(w_s\) and \(w_t\) are in \(\partial V_2\) and represent the same homology class \(\omega\). (2) \(w_s\) and \(w_t\) intersect transversely at odd points because \(\theta(x, z)\) is odd. Therefore \(\omega \cdot \omega\) is one. □

On the contrary to the above Claim 4.3, the following Claim 4.4 holds. This is absurdity. We prove Claim 4.4 and complete the proof of Proposition 4.1.

Claim 4.4. The \(\mathbb{Z}_2\)-intersection number \(\omega \cdot \omega\) is zero.

Proof. Since \(V_2\) is a codimension one orientable submanifold in the parallelizable manifold \(S^{4m+5} - \{\text{one point}\}\) and \(\partial V_2 \neq \emptyset\), \(V_2\) is parallelizable. The fact that \(\omega \cdot \omega\) is zero follows from the following elementary Lemma. This lemma is essentially same as in P. 525 of [KM]. In fact, it is proved elementarily without using \(S^4\)-operators.

Lemma. (See e.g. P. 525 of [KM].) Let \(V\) be a compact parallelizable 2k-manifold. For an arbitrary \(k\)-homology class \(\omega \in H_k(V; \mathbb{Z}_2)\) the intersection number \(\omega \cdot \omega\) is zero.
5. A NECESSARY CONDITION FOR THE REALIZATION OF PAIR OF \((4m+3)\)-KNOTS

In this section we prove:

**Proposition 5.1.** If a 4-tuple of \((4m+3, 4m+5)\)-knots \((K_1, K_2, X_1, X_2)\) is realizable \((m \geq 0)\), then \(\sigma(K_1) = \sigma(K_2)\).

**Proof of Proposition 5.1.** Let \(S = \bigcap S^{4m+5}_1 \bigcup S^{4m+7}_1 \to S^{4m+7}\) be an immersion to realize \((K_1, K_2, X_1, X_2)\). We abbreviate \(X_i = f(S^{4m+5}_i)\) to \(S^{4m+5}_i\). There exist compact oriented \((4m+6)\)-manifolds \(V_1, V_2\) such that \(S^{4m+5}_i = \partial V_i \subset V_i \subset S^{4m+7}(i = 1, 2)\) and \(V_1, V_2\) intersect transversely. Let \(W = V_1 \cap V_2\). Then \(\partial W = \partial(V_1 \cap V_2) = (\partial V_1 \cap V_2) \cup (V_1 \cap \partial V_2) = (S^{4m+3}_i \cap V_2) \cup (V_1 \cap S^{4m+3}_i)\). Let \(F_1 = S^{4m+3}_i \cap V_2\) and \(F_2 = V_1 \cap S^{4m+3}_i\). Then \(F_1\) in \(S^{4m+5}_i\) is a Seifert hypersurface for \(K_i\) \((i = 1, 2)\). Therefore \(\sigma(K_1) - \sigma(K_2) = \sigma(K_1) + \sigma(-K_2) = \sigma(F_1) + \sigma(-F_2) = \sigma(\partial W) = 0\). The second equality holds by the definition of the signature. The third equality holds by Novikov additivity. □

5.A. The proof of Claim 2.2.2.

As we state under Theorem 2.2 in §2, we prove Claim 2.2.2 here. We complete the proof of Theorem 2.2.

Since \(K_2\) and \(K_3\) are cobordant, \(\{\begin{array}{ll}
\text{Arf}(K_2) = \text{Arf}(K_3) & \text{when } k \text{ is even} \\
\sigma(K_2) = \sigma(K_3) & \text{when } k \text{ is odd}
\end{array}\)\) There exist \(K'_1 = K_1, K'_2, \ldots, K'_q, K'_{q+1} = K_2\) and pass-move-charts \(U_i\) of \(K'_i\) \((i = 1, \ldots, q)\) and \(K'_{i+1}\) is obtained from \(K'_i\) by the high dimensional pass-move in \(U_i\) \((i = 1, \ldots, q)\). If the equality \((\dagger)\)

\(\{\begin{array}{ll}
\text{Arf}(K'_i) = \text{Arf}(K'_{i+1}) & \text{when } k \text{ is even} \\
\sigma(K'_i) = \sigma(K'_{i+1}) & \text{when } k \text{ is odd}
\end{array}\)

holds for \(i = 1, \ldots, q\), then the proof is completed. By Lemma 3.4, the pair of \((2k+1)\)-knots \((K'_i, K'_{i+1})\) is realizable. Therefore, by Proposition 4.1 and 5.1, the equality \((\dagger)\) holds.

6. A NECESSARY AND SUFFICIENT CONDITION FOR THE REALIZATION OF 4-TUPLE OF EVEN DIMENSIONAL KNOTS

In this section we prove Theorem 1.4.

We need the following Lemma 6.1.

**Lemma 6.1.** Let \(T\) be the trivial \((n+2)\)-knot, \(K'_2\) the trivial \(n\)-knot, and \(K_1\) a slice \(n\)-knot. \((n \geq 1)\). Then \((K_1, K'_2, T, T)\) is realizable.

Before the proof of Lemma 6.1, we prove:

**Claim.** If Sublemma 3.5, Lemma 4.2 and 6.1 hold, Theorem 1.4 holds.

Proof. Let \(K_2\) be a slice \(n\)-knot, \(K'_1\) the trivial \(n\)-knot, \(X_i\) an arbitrary \((n+2)\)-knot diffeomorphic to the standard \((n+2)\)-sphere, and \(T\) the trivial \((n+2)\)-knot \((i = 1, 2)\). By Lemma 6.1, \((K_1, K'_2, T, T)\) and \((K'_1, K_2, T, T)\) are realizable. By Lemma 4.2, \((K'_1, K'_2, K_2, T \# T, T \# T)\) = \((K_1, K_2, T, T)\) is realizable. By Sublemma 3.5, \((K_1, K_2, X_1, X_2)\) is realizable.
We prove Lemma 6.1 to complete the proof of Theorem 1.4.

**Proof of Lemma 6.1.** We define \( f : S^{n+2}_1 \bigsqcup S^{n+2}_2 \hookrightarrow S^{n+4} \) by using the \( k \)-twist spinning in §6 of [Z]. Prepare \( D^{n,n-2} \) in §6 of [Z], and put \( n \) there to be \((n+3)\). As written there, regard \((S^{n+4}, \text{a } (n+2)\)-knot\) as \((\partial D^{n+3,n+1} \times D^2) \cup (D^{n+3,n+1} \times \partial D^2)\). Take \( D^{n+3,n+1} \) as follows.

Recall that \((1)D^{n+3,n+1} \) denote a set of \( D^{n+3} \) and \( D^{n+1} \) embedded in \( D^{n+3} \), \((2) D^{n+3} \cap D^{n+1} = \partial D^{n+1} \) and \( \partial D^{n+1} \) in \( \partial D^{n+3} \) is the trivial n-knot. Regard \( D^{n+3} \) as \( D^{n+2} \times [−1, 1] \).

Let \( D^{n+1} \cap \partial D^{n+3} \subseteq (D^{n+2} × \{-1\}) \). Suppose that \( D^{n+1} \cap (D^{n+2} \times \{0\}) \) in \( (D^{n+2} × \{0\}) \) defines \( K_1 \). Such \( D^{n+3,n+1} \) exists because \( K_1 \) is slice. Define \( f|S^{n+2}_1 \) so that \( f(S^{n+2}_1) \) is the boundary of \( D^{n+2}_s \times [0, 1] \times \{\theta_0\} \), where \( \theta_0 \) is a point in \( \partial D^2 \). Define \( f|S^{n+2}_2 \) so that \( f(S^{n+2}_2) \) coincides with what is made from \( D^{n+1} \) by 1-twist-spinning. Then the following claim holds.

**Claim.** The immersion \( f \) realizes the 4-tuple of \((n, n+2)\)-knots \((K_1, K'_2, T, T)\).

**Proof.** \( f(S^{n+2}_1) \) is the boundary of the \((n+3)\)-ball \( D^{n+2}_s \times [0, 1] \times \{\theta_0\} \). Therefore \( f|S^{n+2}_1 \) defines the trivial knot \( T \). \( f(S^{n+2}_1) \) is a 1-twist spin knot. By [Z] 1-twist spun knots are trivial. Therefore \( f|S^{n+2}_2 \) defines the trivial knot \( T \). By the definition of the construction of \( f \), the \( n \)-knot \( f(S^{n+2}_1) \cap f(S^{n+2}_2) \) in \( f(S^{n+2}_1) \) is \( D^{n+1} \cap (D^{n+2}_s \times \{0\}) \) in \( (D^{n+2}_s) \). Therefore \( f(S^{n+2}_1) \cap f(S^{n+2}_2) \) in \( f(S^{n+2}_1) \) defines \( K_1 \). The \((n+1)\)-disc \( D^{n+1} \cap (D^{n+2} \times [01]) \) is called \( D^{n+1}_1 \). By the definition of the construction of \( f \), \( D^{n+1}_1 \) is in \( f(S^{n+2}_2) \). By the definition of the construction of \( f \), the \( n \)-knot \( f(S^{n+2}_1) \cap f(S^{n+2}_2) \) in \( f(S^{n+2}_2) \) is the boundary of \( D^{n+1}_1 \). Therefore \( f(S^{n+2}_1) \cap f(S^{n+2}_2) \) in \( f(S^{n+2}_2) \) defines the trivial knot \( K'_2 \). Therefore \( f \) is an immersion to realize \((K_1, K'_2, T, T)\). \( \Box \)
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