ON THE FUNDAMENTAL SOLUTION AND ITS APPLICATION IN A LARGE CLASS OF DIFFERENTIAL SYSTEMS DETERMINED BY VOLterra TYPE OPERATORS WITH DELAY

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Abstract. The variation-of-constants formula is one of the principal tools of the theory of differential equations. There are many papers dealing with different versions of the variation-of-constants formula and its applications. Our main purpose in this paper is to give a variation-of-constants formula for inhomogeneous linear functional differential systems determined by general Volterra type operators with delay. Our treatment of the delay in the considered systems is completely different from the usual methods. We deal with the representation of the studied Volterra type operators. Some existence and uniqueness theorems are obtained for the studied linear functional differential and integral systems. Finally, some applications are given.

1. Introduction. The variation-of-constants formula is one of the principal tools of the theory of differential equations (ordinary, functional, partial, etc.) for deriving statements about the qualitative properties of the considered equations. There are many papers dealing with different versions of the variation-of-constants formula and its applications, see e.g. [2], [4], [5], [6], [16], [19], [24].

A very instructive paper on this subject is [16]. The authors studied the inhomogeneous linear system

\[ x'(t) = f(t) + A(t)x(t) + \int_0^t B(t,s)x(s)\,ds, \quad y(0) = x_0, \tag{1} \]

under suitable conditions (here \( A(t) \) and \( B(t,s) \) are \( n \times n \) matrices), and they derived the following variation-of-constants formula: the unique solution \( x \) of the initial value problem (1) can be obtained by

\[ x(t) = R(t,0)x_0 + \int_0^t R(t,s)f(s)\,ds, \quad t \geq 0, \tag{2} \]

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where $R(t,s)$ is the so-called resolvent. They formally defined the resolvent by

$$R(t,s) = I + \int_s^t R(t,u) \Psi(u,s) \, du, \quad 0 \leq s \leq t,$$

where $I$ is the identity matrix and

$$\Psi(t,s) = A(t) + \int_s^t B(t,v) \, dv.$$

We can see the essence of the variation-of-constants formula from (2): the function

$$t \mapsto R(t,0) x_0, \quad t \geq 0,$$

is the solution of the homogeneous linear system

$$y'(t) = A(t) y(t) + \int_0^t B(t,s) y(s) \, ds \quad (3)$$

taking the value $x_0$ at 0, and we are able to generate a particular solution of the inhomogeneous linear system (1) in the form

$$t \mapsto \int_0^t R(t,s) f(s) \, ds, \quad t \geq 0.$$

There is another approach to the problem of giving a variation-of-constants formula for (1) in [4] based on the fundamental matrix solution of the homogeneous linear system (3). This is another way to look at the resolvent $R(t,s)$ and that is from the standpoint of linear systems of ordinary differential equations. The fundamental matrix solution of (3) is the $n \times n$ matrix function $Z(t,s)$ whose columns are linearly independent solutions of (3) and whose value at $t = s$ is the identity matrix $I$. In this case the author obtains (2) in the same form

$$x(t) = Z(t,0) x_0 + \int_0^t Z(t,s) f(s) \, ds, \quad t \geq 0.$$

It is proved in [4] that the fundamental matrix solution and the resolvent are one and the same.

We mention that the conditions for the functions $f$, $A$ and $B$ are different in [16] and [4].

A variation-of-constants formula for linear functional differential equations with delay can be found e.g. in [24] by using the second dual of the phase space, and in [19] by using the theory of semigroups.

Our main purpose here is to give a variation-of-constants formula for inhomogeneous linear functional differential systems with delay determined by general Volterra-type operators. Our treatment of the delay in the considered systems is completely different from the usual methods. The proofs are essentially based on measure- and integral-theoretic considerations.

In Section 2 the terminology and notation is introduced, and the generality of the considered systems is illustrated by comparing them with usually studied systems. We deal with the representation of the studied Volterra-type operators and
give some special realizations. By using a noncanonical approach to functional
differential equations with delay, we are able to establish a variation-of-constant
formula under very general assumptions. In Section 3 some preliminary results are
found, some of them are interesting in their own right. The main part of Section
4 contains existence and uniqueness theorems for inhomogeneous linear functional
integral systems with delay determined by the introduced Volterra-type operators.
As a consequence, we have the basic existence and uniqueness theorems for the
considered differential systems.

The literature on this topic for linear integral and differential equations with
abstract Volterra operators offers a lot of books, papers and results. For equations
without delay, see e.g. the books [7] and [15], and the papers [10], [13], [14] and
[25], and the references therein. Several concrete causal differential and integral
equations with delay are available in the literature, see e.g. the books [9] and [8], and
the papers [12] and [27], and the references therein, but our approach to generate a
causal operator with delay may be new. The usual idea of the proofs of the existence
results is to apply fixed-point theorems. In this paper, the successive approximations
method is applied. Moreover, some differential and integral inequalities are also
studied. In Section 5 we obtain a variation-of-constants formula for the investigated
systems. Our result extends and generalizes the similar results in [16] and [4].

Finally, some applications can be found in Section 6. We give some conditions for
some special homogeneous linear functional differential systems with delay under
which all solutions tend to zero at infinity. We also study the boundedness of the
solution of the corresponding inhomogeneous systems. As a consequence we have
a result about the persistence and permanence of the systems. These results are
closely related to some results in [17] and [18], but in these papers systems are not
studied.

2. Terminology and notation. The set of nonnegative integers will be denoted
by \( \mathbb{N} \). \( \mathbb{R}^d \) stands for the set of all \( d \)-dimensional column vectors with real
components. The norm of a vector \( x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d \) is defined by
\( \|x\| := \left( \sum_{i=1}^{d} x_i^2 \right)^{1/2} \). The vector \( x \) is nonnegative if \( x_i \geq 0 \) \((i = 1, \ldots, d)\). In this case we
write \( x \geq 0 \). A partial ordering is defined on \( \mathbb{R}^d \) by letting \( x \leq y \) if and only if
\( y - x \geq 0 \).

A function \( f = (f_1, \ldots, f_d)^T : U \to \mathbb{R}^d \) is called nonnegative (positive) if
\( f_i(u) \geq 0 \quad (f_i(u) > 0), \quad u \in U, \quad i = 1, \ldots, d \).

Let \( t_0 \in \mathbb{R} \) and \( \tau_0 \geq 0 \) be fixed.

We introduce two function spaces which are needed in the sequel. The general
theoretical background of topological vector spaces can be found, for example, in
the book [20].

From now on the phrase “a function satisfies a property locally” means that
the function satisfies this property on every compact subset of its domain. In the
sequel, if not otherwise explicitly stated, by “measurable” and by “integrable” we
will always mean “Lebesgue measurable” and “Lebesgue measurable and Lebesgue
integrable”, respectively.

A function \( x : [t_0 - \tau_0, \infty[ \to \mathbb{R}^d \) is said to belong to \( B_{t_0 - \tau_0} \) if \( x \) is Borel mea-
surable and locally bounded. We define a family of seminorms on \( B_{t_0 - \tau_0} \) in the
integrable. We obtain a family of seminorms on $L_{t_0}$

$$p_t(x) := \sup_{s \in [t_0 - \tau_0, t]} \|x(s)\|, \quad t \geq t_0.$$

These seminorms generate a locally convex and metrizable topology $\mathcal{T}_p$ on $B_{t_0 - \tau_0}$ (the topology of uniform convergence on compact sets).

The elements of $L_{t_0}$ are those functions $x : [t_0, \infty] \to \mathbb{R}^d$ for which $x$ is locally integrable. We obtain a family of seminorms on $L_{t_0}$ by setting

$$q_t(x) := \|x(t)\|, \quad t \geq t_0.$$

Equipped with this family of seminorms, $L_{t_0}$ becomes a locally convex space. The topology is denoted by $\mathcal{F}$ (the topology of pointwise convergence).

Let $V : B_{t_0 - \tau_0} \to L_{t_0}$ be a linear operator which is of Volterra-type. The late property means the next: if $t \geq t_0$ and $x_1, x_2 \in B_{t_0 - \tau_0}$ such that the restriction of $x_1$ to $[t_0 - \tau_0, t]$ is equal to the restriction of $x_2$ to $[t_0 - \tau_0, t]$, then

$$V(x_1)(t) = V(x_2)(t).$$

Since the family $\{p_t \mid t \geq t_0\}$ of seminorms on $B_{t_0 - \tau_0}$ is increasing, and $V$ is of Volterra-type, the operator $V$ is $\mathcal{T}_p - \mathcal{F}$ continuous if and only if for every seminorm $q_t (t \geq t_0)$ there exists a positive number $M_t$ such that

$$q_t(V(x)) \leq M_t p_t(x), \quad x \in B_{t_0 - \tau_0},$$

or equivalently

$$\|V(x)(t)\| \leq M_t \cdot \sup_{s \in [t_0 - \tau_0, t]} \|x(s)\|, \quad x \in B_{t_0 - \tau_0}.$$ Based on this we say that $V$ is $g$-continuous if there exists a nonnegative and locally integrable function $g : [t_0, \infty] \to \mathbb{R}$ such that

$$\|V(x)(t)\| \leq g(t) \cdot \sup_{s \in [t_0 - \tau_0, t]} \|x(s)\|, \quad t \geq t_0. \quad (4)$$

**Remark 1.** It is not hard to think that if $V$ is nonnegative (this means that $V(x)$ is nonnegative for all nonnegative $x \in B_{t_0 - \tau_0}$) and $\mathcal{T}_p - \mathcal{F}$ continuous, then it is $g$-continuous with $g := \|V(1)\|$, where $1 \in B_{t_0 - \tau_0}$ is the constant function with value 1.

In this paper we study inhomogeneous linear functional differential systems of the form

$$x'(t) = V(x)(t) + \rho(t), \quad t \geq t_0, \quad (5)$$

under the following conditions

(A1) The operator $V : B_{t_0 - \tau_0} \to L_{t_0}$ is linear, of Volterra-type, and $g$-continuous.

(A2) If $(x_n)$ is a sequence of functions from $B_{t_0 - \tau_0}$ such that $(x_n)$ is bounded on every compact subinterval of $[t_0 - \tau_0, \infty]$, and it has a pointwise limit $x$ (obviously $x \in B_{t_0 - \tau_0}$), then

$$\lim_{n \to \infty} V(x_n) = V(x).$$

(A3) The function $\rho$ belongs to $L_{t_0}$.

We say that a function $x \in B_{t_0 - \tau_0}$ is a solution of $(5)$ if $x$ is locally absolutely continuous on $[t_0, \infty]$ and it satisfies $(5)$ almost everywhere on $[t_0, \infty]$.

**Remark 2.** (a) If $V : B_{t_0 - \tau_0} \to L_{t_0}$ is a linear operator such that (A2) holds, then $V$ is obviously $\mathcal{T}_p - \mathcal{F}$ continuous.

(b) Let $C_{t_0 - \tau_0}$ denote the subspace of $B_{t_0 - \tau_0}$ consisting of all functions which are Borel measurable and bounded on $[t_0 - \tau_0, t_0]$ and locally absolutely continuous
on \([t_0, \infty[\). Assume \((A_1)\) and \((A_2)\). We shall show (see Theorem 4.2) that for each \(\rho \in L_{t_0}\) there exists a solution \(x \in C_{t_0-\tau_0}\) of (5), and this is often expressed as follows: by considering equation (5), the pair \((L_{t_0}, C_{t_0-\tau_0})\) is admissible for the operator \(V\) (see [23]).

Some mention should be made here of the generality of the considered systems. This is best understood when viewed as a generalization of the following frequently studied inhomogeneous linear functional differential system

\[
x'(t) = L(x_t) + \rho(t), \quad t \geq t_0,
\]

where \(L : C([-\tau_0,0], \mathbb{R}^d) \to \mathbb{R}^d\) is a bounded linear operator and \(\rho \in C([t_0, \infty[, \mathbb{R}^d)\). As usual, \(\tau_0\) is a fixed nonnegative number, \(C([-\tau_0,0], \mathbb{R}^d)\) denotes the Banach space of continuous functions endowed with the uniform norm topology \(\|\cdot\|_\infty\), and \(x_t\) is defined by \(x_t(s) = x(t+s)\) for \(s \in [-\tau_0,0]\).

For every continuous function \(x : [t_0 - \tau_0, \infty[ \to \mathbb{R}^d\) define the function \(V(x) : [t_0, \infty[ \to \mathbb{R}^d\) by

\[
V(x)(t) = L(x_t), \quad t \geq t_0.
\]

It is easy to think that

(i) \(V\) is a linear operator from \(C([t_0 - \tau_0, \infty[, \mathbb{R}^d)\) to \(C([t_0, \infty[, \mathbb{R}^d)\), \(V\) is continuous in the sense that

\[
\|V(x)(t)\| \leq \|L\| \cdot \|x_t\|_\infty, \quad t \geq t_0,
\]

and if \(t \geq t_0\) and \(x, y \in C([t_0 - \tau_0, \infty[, \mathbb{R}^d)\) with \(x_t = y_t\), then

\[
V(x)(t) = V(y)(t).
\]

(ii) If \((x_n)\) is a sequence of functions from \(C([t_0 - \tau_0, \infty[, \mathbb{R}^d)\) such that \((x_n)\) is bounded on every compact subinterval of \([t_0 - \tau_0, \infty[,\) and it has a pointwise limit \(x \in C([t_0 - \tau_0, \infty[, \mathbb{R}^d)\), then (by the integral representation of \(L\))

\[
\lim_{n \to \infty} V(x_n) = V(x).
\]

It can be seen that the system (5) contains the system (6) as a simple case, and the properties (i) and (ii) are reformulated in the assumptions \((A_1)\) and \((A_2)\) in a very general manner.

The next result shows that the considered operators can be built up from more elementary operators with the same properties. In the left part of this section the dimension of the used space will be indicated explicitly, so we shall write \(B^d_{t_0 - \tau_0}\), \(L^d_{t_0}\), and \(p^d\).

**Lemma 2.1.** The operator \(V : B^d_{t_0 - \tau_0} \to L^d_{t_0}\) satisfies \((A_1)\) and \((A_2)\) if and only if \(V\) can be written in the form

\[
V(x)(t) = \left(\sum_{j=1}^d V_{ij}(x_j)(t), \ldots, \sum_{j=1}^d V_{dj}(x_j)(t)\right)^T, \quad t \geq t_0,
\]

where the operators \(V_{ij} : B^1_{t_0 - \tau_0} \to L^1_{t_0}\) \((i, j = 1, \ldots, d)\) satisfy \((A_1)\) and \((A_2)\) too.

**Proof.** Assume the operator \(V : B^d_{t_0 - \tau_0} \to L^d_{t_0}\) satisfies \((A_1)\) and \((A_2)\). Then for all \(x = (x_1, \ldots, x_d)^T \in B^d_{t_0 - \tau_0}\), and for each \(i, j = 1, \ldots, d\) define the functions \(V_{ij}\) by

\[
V_{ij}(x_j) := p_{ri} \circ V(0, \ldots, 0, x_j, 0, \ldots, 0),
\]
where

\[ pr_i : \mathbb{R}^d \to \mathbb{R}, \quad pr_i (v_1, \ldots, v_d) := v_i. \]

The linearity of \( V \) implies that (8) holds. It is easy to check that \( V_{ij} \) \((i, j = 1, \ldots, d)\) is an operator from \( B^d_{t_0 - \tau_0} \) to \( L^1_{t_0} \) satisfying \((A_1)\) and \((A_2)\).

Conversely, let \( V_{ij} : B^d_{t_0 - \tau_0} \to L^1_{t_0} \) \((i, j = 1, \ldots, d)\) be operators satisfying \((A_1)\) and \((A_2)\) \((V_{ij} \text{ is } g_{ij}-\text{continuous})\), and for all \( x \in B^d_{t_0 - \tau_0} \) define the function \( V(x) : [t_0, \infty[ \to \mathbb{R}^d \) by (8).

Then \( V \) is an operator from \( B^d_{t_0 - \tau_0} \) to \( L^d_{t_0} \), and it is obvious that \( V \) is linear and of Volterra-type.

If \( x \in B^d_{t_0 - \tau_0}, t \geq t_0, \) and \( t_0 - \tau_0 \leq s \leq t, \) then for every \( j = 1, \ldots, d \)

\[ |x_j (s)| \leq \left( \sum_{j=1}^d x_j^2 (s) \right)^{1/2} \leq \sup_{s \in [t_0 - \tau_0, t]} \left( \sum_{j=1}^d x_j^2 (s) \right)^{1/2} = p^d_t (x). \tag{9} \]

Since

\[ |V_{ij} (x_j) (t)| \leq g_{ij} (t) \cdot \sup_{s \in [t_0 - \tau_0, t]} |x_j (s)|, \quad t \geq t_0, \quad i, j = 1, \ldots, d, \]

it follows from (9) that for all \( x \in B^d_{t_0 - \tau_0} \)

\[ \|V (x) (t)\| \leq \left( \sum_{i=1}^d \left( \sum_{j=1}^d g_{ij} (t) \right)^2 \right)^{1/2} \cdot p^d_t (x), \quad t \geq t_0, \]

and hence \( V \) is \( g \)-continuous.

From the foregoing it is clear that \( V \) satisfies \((A_1)\).

It is not hard to see that \( V \) satisfies \((A_2)\) too.

The proof is complete. \( \square \)

Important special cases of elementary Volterra-type operators (from \( B^d_{t_0 - \tau_0} \) to \( L^1_{t_0} \) satisfying conditions \((A_1)\) and \((A_2)\) are described in the next examples.

By a real measure we shall mean a finite signed measure. If \( \mu \) is a real measure, then \( \|\mu\| \) means the total variation of \( \mu \).

**Example 1.** The \( \sigma \)-algebra of Borel subsets of \([t_0 - \tau_0, t]\) is denoted by \( \mathcal{B}_t \) \((t \geq t_0)\).

Assume a real measure \( \mu_t \) on \( \mathcal{B}_t \) is assigned to every \( t \geq t_0 \) such that for all \( x \in B^d_{t_0 - \tau_0} \) the function \( V(x) : [t_0, \infty[ \to \mathbb{R} \) defined by

\[ V(x) (t) := \int_{[t_0 - \tau_0, t]} x(s) d\mu_t (s), \quad t \geq t_0 \tag{10} \]

belongs to \( L^1_{t_0} \).

Clearly the operator \( V \) is linear and of Volterra-type.

The definition of \( V \), and Lebesgue’s dominated convergence theorem show that \((A_2)\) holds for \( V \) too, and hence \( V \) is \( T_p - T_q \) continuous.

Since

\[ |V(x)(t)| \leq \|\mu_t\| \cdot \sup_{s \in [t_0 - \tau_0, t]} \|x(s)\|, \quad x \in B_{t_0 - \tau_0}, \quad t \geq t_0, \]

it follows that if the function

\[ g : [t_0, \infty[ \to \mathbb{R}, \quad g(t) = \|\mu_t\| \]


is locally integrable, then $V$ is $g$-continuous.

Conversely, it follows from Corollary 4 (b) and (c) in the sequel that if $V : B_{t_0 - \tau_0}^1 \to L_{t_0}^1$ is a linear and Volterra-type operator which satisfies (A2), then there exist real measures $\mu_t$ ($t \geq t_0$) on $B_t$ such that (10) holds. If $\mu_t$ is a finite measure for all $t \geq t_0$, then by Remark 1, $V$ is $g$-continuous with $g := |V(1)|$.

**Remark 3.** (a) Since the measures $\mu_t$ ($t \geq t_0$) on $B_t$ are finite, they are Lebesgue-Stieltjes measures.

(b) Let $H$ be a nonmeasurable subset of $[t_0, \infty]$, and let

$$
\mu_t := \begin{cases} 
\varepsilon_t, & t \in H \\
0, & t \notin H
\end{cases}, \quad t \geq t_0
$$

where $\varepsilon_t$ is the unit mass at $t$ on $B_t$.

Then

$$
V(1)(t) := \int_{[t_0 - \tau_0, t]} 1 d\mu_t(s) = \begin{cases} 
1, & t \in H \\
0, & t \notin H
\end{cases}, \quad t \geq t_0
$$

does not belong to $L_{t_0}^1$.

(c) By Example 1, and by (b), it is an interesting problem to characterize those functions

$$
t \mapsto \mu_t, \quad \mu_t \text{ is a real measure on } B_t, \quad t \geq t_0
$$

for which the function defined in (10) belongs to $L_{t_0}^1$ for all $x \in B_{t_0 - \tau_0}$.

We now construct some special operators from $B_{t_0 - \tau_0}^1$ to $L_{t_0}^1$ satisfying (A1) and (A2) in the following three examples. These operators contain frequently used operators. Each of these operators can serve as $V_{ij}$ in (8).

**Example 2.** Let $D_{a,\tau} : B_{t_0 - \tau_0}^1 \to L_{t_0}^1$ be defined by

$$
D_{a,\tau}(x)(t) := a(t) x(t - \tau(t)), \quad t \geq t_0,
$$

where

(B1) The delay $\tau : [t_0, \infty[ \to \mathbb{R}_+$ is nonnegative, measurable and it obeys the inequality

$$
t_0 - \tau_0 \leq t - \tau(t), \quad t \geq t_0.
$$

(B2) The function $a : [t_0, \infty[ \to \mathbb{R}$ is locally integrable.

It is easy to check the fulfilment of properties (A1) and (A2).

**Example 3.** Let $\lambda$ denote the Lebesgue measure on the Borel measurable subsets of $[t_0, \infty[$.

Let $I_{b,\eta,\nu} : B_{t_0 - \tau_0}^1 \to L_{t_0}^1$ be defined by

$$
I_{b,\eta,\nu}(x)(t) := \int_{t_0}^t b(t, s) x(s - \eta(s)) d\nu(s), \quad t \geq t_0,
$$

where

(B3) The delay $\eta : [t_0, \infty[ \to \mathbb{R}_+$ is nonnegative, Borel measurable and it obeys the inequality

$$
t_0 - \tau_0 \leq t - \eta(t), \quad t \geq t_0.
$$

(B4) $\nu$ is a $\sigma$-finite measure on the Borel measurable subsets of $[t_0, \infty[$.

(B5) Let $H := \{(t, s) \in \mathbb{R}^2 | t \geq s \geq t_0, \}$. The function $b : H \to \mathbb{R}$ is Borel measurable, locally $\lambda \times \nu$-integrable, and for each $t \geq t_0$ the sections

$$
s \mapsto b(t, s), \quad t \geq s \geq t_0
$$

in (8).
are \( \nu \)-integrable.

First we show that \( I_{b,\eta,\nu} \) is well defined. Let \( x \in B_{t_0-\tau_0}^1 \) and \( t \geq t_0 \) be fixed. Since the function

\[(u, s) \mapsto b(u, s) x(s - \eta(s)), \quad (u, s) \in H\]

is Borel measurable, and

\[|b(u, s) x(s - \eta(s))| \leq K_t |b(u, s)|, \quad t \geq u \geq s \geq t_0,\]

the integral

\[\int\int_H b(u, s) x(s - \eta(s)) \, d\lambda \times \nu(u, s)\]

exists and finite. The assumption on \( b \) implies that the function

\[u \mapsto \int_{t_0}^u b(u, s) x(s - \eta(s)) \, d\nu(s), \quad t \geq u \geq t_0\]

is real valued. With the help of Fubini’s theorem it follows that this function is \( \lambda \)-integrable on \([t_0, t]\).

It is obvious that \( I_{b,\eta,\nu} \) is linear and of Volterra-type.

Another application of Fubini’s theorem shows that the function

\[g : [t_0 - \tau_0, \infty[ \to \mathbb{R}, \quad g(t) := \int_{t_0}^t \left| b(t, s) \right| \, d\nu(s)\]

is locally integrable, and therefore \( I_{b,\eta,\nu} \) is \( g \)-continuous.

Let \((x_n)\) be a sequence of functions from \( B_{t_0-\tau_0}^1 \) such that \((x_n)\) is uniformly bounded on every compact subinterval of \([t_0 - \tau_0, \infty[\), and it has a pointwise limit \( x \). Lebesgue’s dominated convergence theorem can now be applied to the integral in (11), and it implies that (A2) also holds.

**Example 4.** Let \( \tau_0 = 0 \) and let \( C_{b,\nu} : B_{t_0}^1 \to L_{t_0}^1 \) be defined by

\[C_{b,\nu}(x)(t) := \int_{t_0}^t b(t, s) x(t + t_0 - s) \, d\nu(s), \quad t \geq t_0,\]

where (B4) and (B5) are satisfied.

Similarly to Example 3, we can prove that (A1) and (A2) hold for \( C_{b,\nu} \).

**Remark 4.** An operator of the form

\[\sum_{k=1}^{n_{1,i,j}} D_{a_{1,i,j},\tau_{1,i,j}} + \sum_{k=1}^{n_{2,i,j}} I_{b_{1,i,j},\eta_{1,i,j},\nu_{1,i,j}} + \sum_{k=1}^{n_{3,i,j}} C_{b_{1,i,j},\nu_{1,i,j}}\]

can serve as \( V_{ij} \) in (8). Here \( n_{1,i,j}, n_{2,i,j} \) and \( n_{3,i,j} \) belong to \( \mathbb{N} \), and the value of any empty sum of operators is taken to be zero. This demonstrates the generality of the studied system (5).
3. Preliminary results. Let $t > t_0$ be fixed.

The $\sigma$-algebra of Borel subsets of $[t_0 - \tau_0, t]$ is denoted by $B_t$.

Let $B_d ([t_0 - \tau_0, t])$ be the vector space of all functions $z$ defined on $[t_0 - \tau_0, t]$ and taking their values in $\mathbb{R}^d$, such that they are Borel measurable and bounded. The norm of $z \in B_d ([t_0 - \tau_0, t])$ is defined by

$$\|z\|_\infty := \sup_{s \in [t_0 - \tau_0, t]} \|z(s)\|.$$ 

Then $(B_d ([t_0 - \tau_0, t]), \|\|_\infty)$ is a Banach space.

The dual space of $(B_1 ([t_0 - \tau_0, t]), \|\|_\infty)$ is described in the following basic result.

**Theorem 3.1.** (see [22] and [28]) If $\Lambda$ is a continuous linear functional from $(B_1 ([t_0 - \tau_0, t]), \|\|_\infty)$ to $\mathbb{R}$, then there exists a unique finitely additive real measure $\mu : B_t \to \mathbb{R}$ with finite total variation such that for all $z \in B_1 ([t_0 - \tau_0, t])$ we have

$$\Lambda(z) = \int_{[t_0 - \tau_0, t]} z(u) \, d\mu(u).$$

**Definition 3.2.** A linear operator $T : B_d ([t_0 - \tau_0, t]) \to \mathbb{R}^p$ is called $\sigma$-continuous if it satisfies the following property: if $(z_n)$ is an increasing and bounded sequence of functions from $B_d ([t_0 - \tau_0, t])$ such that $z_n \to z$ pointwise (obviously $z \in B_d ([t_0 - \tau_0, t])$), then

$$\lim_{n \to \infty} T(z_n) = T(z).$$

The next three results are natural extensions of the previous theorem. Being not so easy to find similar results in the literature, and for the sake of completeness, we present the proofs.

**Corollary 1.** If $\Lambda$ is a continuous and $\sigma$-continuous linear functional from the space $(B_1 ([t_0 - \tau_0, t]), \|\|_\infty)$ to $\mathbb{R}$, then there exists a unique real measure $\mu$ on $B_t$ such that for all $z \in B_1 ([t_0 - \tau_0, t])$ we have

$$\Lambda(z) = \int_{[t_0 - \tau_0, t]} z(u) \, d\mu(u). \quad (12)$$

**Proof.** By Theorem 3.1, there exists a unique finitely additive real measure $\mu : B_t \to \mathbb{R}$ with finite total variation such that for all $z \in B_1 ([t_0 - \tau_0, t])$ (12) holds. It is enough to show that $\mu$ is $\sigma$-additive. This is satisfied if and only if for every sequence $(H_n)_{n=1}^\infty$ of pairwise disjoint sets from $B_t$ we have

$$\mu \left( \bigcup_{n=1}^\infty H_n \right) = \sum_{n=1}^\infty \mu(H_n).$$

Denote by $1_{H_n}$ the characteristic function of $H_n$. By using (12) and the $\sigma$-continuity of $\Lambda$, we obtain

$$\mu \left( \bigcup_{n=1}^\infty H_n \right) = \int_{[t_0 - \tau_0, t]} \left( \sum_{n=1}^\infty 1_{H_n} \right) \, d\mu = \Lambda \left( \sum_{n=1}^\infty 1_{H_n} \right)$$

$$= \lim_{n \to \infty} \Lambda \left( \sum_{k=1}^n 1_{H_k} \right) = \lim_{n \to \infty} \sum_{k=1}^n \Lambda(1_{H_k}) = \lim_{n \to \infty} \sum_{k=1}^n \int_{[t_0 - \tau_0, t]} 1_{H_k} \, d\mu$$
Proof. Define the functionals \( \Lambda \) satisfies (A).

\[ \Lambda (z) = \sum_{i=1}^{\infty} \mu (H_i). \] (13)

The series (13) is absolutely convergent, since \( \mu \) has finite total variation.

The proof is complete. \( \square \)

**Corollary 2.** Assume \( d \geq 2 \). If \( \Lambda \) is a continuous and \( \sigma \)-continuous linear functional from \( (B_d ([t_0 - \tau_0, t]), \| \cdot \|_\infty) \) to \( \mathbb{R} \), then there exist real measures \( \mu_i \) on \( \mathcal{B}_t \) (\( i = 1, \ldots, d \)) such that for all \( z = (z_1, \ldots, z_d) \in B_d ([t_0 - \tau_0, t]) \) we have

\[ \Lambda (z) = \sum_{i=1}^{d} \int_{[t_0 - \tau_0, t]} z_i (u) d\mu_i (u). \]

Proof. Define the functionals \( \Lambda_i : (B_1 ([t_0 - \tau_0, t]), \| \cdot \|_\infty) \rightarrow \mathbb{R} \) (\( i = 1, \ldots, d \)) by

\[ \Lambda_i (z) = \lambda (z, 0, \ldots, 0). \]

Then \( \Lambda_i \) is a \( \sigma \)-continuous linear functional (\( i = 1, \ldots, d \)), and therefore Corollary 1 guarantees the existence of a real measure \( \mu_i \) on \( \mathcal{B}_t \) such that for all \( z \in B_1 ([t_0 - \tau_0, t]) \) we have

\[ \Lambda_i (z) = \int_{[t_0 - \tau_0, t]} z (u) d\mu_i (u). \] (14)

Since for all \( z = (z_1, \ldots, z_d) \in B_d ([t_0 - \tau_0, t]) \)

\[ \Lambda (z) = \sum_{i=1}^{d} \Lambda_i (z_i), \]

the result follows from (14).

The proof is complete. \( \square \)

**Corollary 3.** Assume \( d \geq 2 \). If \( T \) is a continuous and \( \sigma \)-continuous linear operator from \( (B_d ([t_0 - \tau_0, t]), \| \cdot \|_\infty) \) to \( \mathbb{R}^d \), then there exist real measures \( \mu_{ij} \) on \( \mathcal{B}_t \) (\( i, j = 1, \ldots, d \)) such that for all \( z = (z_1, \ldots, z_d) \in B_d ([t_0 - \tau_0, t]) \) we have

\[ T (z) = \left( \sum_{j=1}^{d} \int_{[t_0 - \tau_0, t]} z_j (u) d\mu_{1j} (u), \ldots, \sum_{j=1}^{d} \int_{[t_0 - \tau_0, t]} z_j (u) d\mu_{dj} (u) \right)^T. \]

Proof. Apply Corollary 2 to the components of \( T \). \( \square \)

For every \( z \in B_d ([t_0 - \tau_0, t]) \) define the function \( \hat{z} \in B_{t_0 - \tau_0} \) by

\[ \hat{z} (s) = \begin{cases} z (s), & \text{if } s \in [t_0 - \tau_0, t] \\ 0, & \text{if } s > t \end{cases}. \]

With the aid of Corollary 3 we get an essential representation result for the studied operators:

**Corollary 4.** Let \( V : B_{t_0 - \tau_0} \rightarrow L_{t_0} \) be a linear and Volterra-type operator which satisfies (A2). Introduce the operator

\[ T_i : B_d ([t_0 - \tau_0, t]) \rightarrow \mathbb{R}^d, \quad T_i (z) = V (\hat{z}) (t). \]

Then

(a) The operator \( T_i \) is linear and \( \sigma \)-continuous.
(b) There exist real measures $\mu_{ij}^t$ on $\mathcal{B}_t$ ($i, j = 1, \ldots, d$) such that for all $z = (z_1, \ldots, z_d) \in B_d([t_0 - \tau_0, t])$ we have

$$T_t (z) = \left( \sum_{j=1}^d \int_{[t_0 - \tau_0, t]} z_j (u) \, d\mu_{ij}^t (u), \ldots, \sum_{j=1}^d \int_{[t_0 - \tau_0, t]} z_j (u) \, d\mu_{dj}^t (u) \right)^T.$$  

(c) If the restriction of $x \in B_{t_0 - \tau_0}$ to $[t_0 - \tau_0, t]$ is $z$, then

$$T_t (z) = V (x) (t).$$

By using the previous corollary, we obtain the following result which will play an important role in the proof of our variation-of-constants formula.

**Theorem 3.3.** Let $V : B_{t_0 - \tau_0} \to L_{t_0}$ be a linear and Volterra-type operator which satisfies $(A_2)$, and let $v = (v_1, \ldots, v_d) : [t_0 - \tau_0, \infty] \times [t_0, \infty) \to \mathbb{R}^d$ be a Borel measurable and locally bounded function such that

$$v (t, s) = 0, \quad t \in [t_0 - \tau_0, s]. \tag{15}$$

Define the function $w = (w_1, \ldots, w_d) : [t_0 - \tau_0, \infty] \to \mathbb{R}^d$ by

$$w (t) = \begin{cases} 0, & t \in [t_0 - \tau_0, t_0[ \\ \int_{t_0}^t v (t, s) \, ds, & t \geq t_0 \end{cases}. $$

Then the function $s \mapsto V (v (\cdot, s)) (t)$, $s \in [t_0, \infty)$, is Borel measurable for each $t \in [t_0 - \tau_0, \infty]$ and

$$\int_{t_0}^t V (v (\cdot, s)) (t) \, ds = V (w) (t), \quad t \geq t_0.$$

**Proof.** By using some results from the theory of product measure, it is easy to check that $v (\cdot, s) \in B_{t_0 - \tau_0}$ for each $s \in [t_0, \infty]$, and $w \in B_{t_0 - \tau_0}$.

Let $t > t_0$ be fixed. Let

$$H_t := \left\{ (u, s) \in \mathbb{R}^2 \mid u \in [t_0, t], \quad s \in [t_0, u] \right\},$$

and let $\mu_{ij}^t$ ($i, j = 1, \ldots, d$, $t \geq t_0$) be the real measures on $\mathcal{B}_t$ corresponding to $V$ and defined in Corollary 4 (b).

Then $H_t$ is Borel measurable and $v_j$ is $\lambda_t \times \mu_{ij}^t$ and $\lambda_t \times \mu_{ij}^t$ integrable on $H_t$, where $\lambda_t$ is the Lebesgue measure on $\mathcal{B}_t$ and $\mu_{ij}^t$, $\mu_{ij}^t$ are the positive and negative variations of $\mu_{ij}^t$, respectively ($i, j = 1, \ldots, d$). Since $v$ is Borel measurable and locally bounded, we can apply Fubini’s theorem which gives according to (15) that

(i)  

$$\left( \sum_{j=1}^d \int_{H_t} v_j (u, s) \, d\lambda_t \times \mu_{ij}^t, \ldots, \sum_{j=1}^d \int_{H_t} v_j (u, s) \, d\lambda_t \times \mu_{dj}^t \right)^T$$

$$= \left( \sum_{j=1}^d \int_{[t_0, t]} \left( \int_{t_0}^u v_j (u, s) \, ds \right) \, d\mu_{ij}^t (u), \ldots, \sum_{j=1}^d \int_{[t_0, t]} \left( \int_{t_0}^u v_j (u, s) \, ds \right) \, d\mu_{dj}^t (u) \right).$$
and hence

$$\text{Theorem 3.5.} \quad \text{Let}$$

$$\text{Theorem 3.4.} \quad \text{If}$$

$$f$$

conditions.

in many important cases. The following result is a variant of this under weaker

are continuous in one variable and measurable in another) are jointly measurable

is (jointly) measurable.

The proof is complete.

The last equality follows from another application of Corollary (4).

The proof is complete.

We end this section by mentioning two results which will be used later.

It is known (see [1]) that the so-called Charatheodory functions (functions which

are locally integrable for each

are locally absolutely continuous for each

(b) the functions

(c) the partial derivative $\frac{\partial f}{\partial t}$ is locally integrable.
Then the function

\[ F : [t_0, \infty[ \to \mathbb{R}^d, \quad F(t) = \int_{t_0}^{t} f(t,s) \, ds \]

is locally absolutely continuous and

\[ F'(t) = f(t,t) + \int_{t_0}^{t} \frac{\partial f(t,s)}{\partial t} \, ds \quad \text{a.e. on } [t_0, \infty[. \]

Proof. This is an immediate consequence of Theorem 2.7 (b) and (c) in [26]. \( \square \)

4. Existence and uniqueness results. A function \( c : [t_0, \infty[ \to \mathbb{R}^d \) is said to belong to \( B_{t_0} \) if \( c \) is Borel measurable and locally bounded. Equip the vector space \( B_{t_0} \) with the topology of uniform convergence on compact sets which will be denoted by \( T_p \) too.

Let \( B ([t_0 - \tau_0, t_0[) \) denote the vector space of all Borel measurable and bounded functions \( \varphi : [t_0 - \tau_0, t_0[ \to \mathbb{R}^d \).

If \( \varphi \in B ([t_0 - \tau_0, t_0[) \) and \( F \subset B_{t_0} \), then \( x \in F^\varphi \) means that \( x : [t_0 - \tau_0, \infty[ \to \mathbb{R}^d \) for which the restriction of \( x \) to \([t_0, \infty[ \) belongs to \( F \) and \( x(t) = \varphi(t) \) (\( t \in [t_0 - \tau_0, t_0[ \)). Obviously, \( F^\varphi \subset B_{t_0 - \tau_0} \), and it is easy to think that \( F \) is closed in \( (B_{t_0}, T_p) \) if and only if \( F^\varphi \) is closed in \( (B_{t_0 - \tau_0}, T_p) \).

To obtain existence and uniqueness results for the considered differential system (5), we first study more general integral systems of the form

\[ x(t) = c(t) + \int_{t_0}^{t} V(x)(s) \, ds, \quad t \geq t_0, \quad (16) \]

where \( c \in B_{t_0} \) and \((A_1)\) holds.

A solution of (16) is a function \( x \in B_{t_0 - \tau_0} \) that satisfies the equation for all \( t \geq t_0 \).

Let \( \varphi \in B ([t_0 - \tau_0, t_0[) \) and \( F \subset B_{t_0} \). As usual, if a solution of (16) is sought in \( F^\varphi \), then we can say that the initial condition

\[ x(t) = \varphi(t), \quad t \in [t_0 - \tau_0, t_0[ \]

is associated with (16).

Theorem 4.1. Assume \((A_1)\), let \( c \in B_{t_0} \), and let \( \varphi \in B ([t_0 - \tau_0, t_0[) \). Then

(a) The integral equation (16) has at least one solution \( x_\varphi \in B_{t_0}^\varphi \).

(b) Let \( h \in B_{t_0} \) and define the successive approximations \( x_0, x_1, \ldots \) by the formulae

\[ x_0(t) = \begin{cases} \varphi(t), & t_0 - \tau_0 \leq t < t_0 \\ h(t), & t \geq t_0 \end{cases}, \quad (17) \]

\[ x_{n+1}(t) = \begin{cases} \varphi(t), & t_0 - \tau_0 \leq t < t_0 \\ c(t) + \int_{t_0}^{t} V(x_n)(s) \, ds, & t \geq t_0 \end{cases}, \quad n \in \mathbb{N}. \quad (18) \]

Then \( (x_n)_{n=0}^\infty \) is a sequence in \( B_{t_0}^\varphi \) which converges in the topology \( T_p \) (and thus pointwise) to a solution \( x_\varphi \in B_{t_0}^\varphi \) of the integral equation (16).
(c) Let $F$ be a subset of $B_{t_0}$ which is closed in the topology $T_p$. Assume further that for all $x \in F$ the functions
t\mapsto c(t) + \int_{t_0}^{t} V(x)(s) \, ds, \quad t \geq t_0

also belong to $F$. Then $x_\varphi \in F$.

(d) Assume $SB_{t_0-\tau_0}$ is a subspace of $B_{t_0-\tau_0}$ such that $x_\varphi \in SB_{t_0-\tau_0}$, $V$ is nonnegative on $SB_{t_0-\tau_0}$, and for all $x \in SB_{t_0-\tau_0}$ the function
t\mapsto \begin{cases} \varphi(t), & t_0 - \tau_0 \leq t < t_0 \\ c(t) + \int_{t_0}^{t} V(x)(s) \, ds, & t \geq t_0 \end{cases}

belongs to $SB_{t_0-\tau_0}$. If $x \in SB_{t_0-\tau_0}$ such that

\[ x(t) \leq \begin{cases} \varphi(t), & t_0 - \tau_0 \leq t < t_0 \\ c(t) + \int_{t_0}^{t} V(x)(s) \, ds, & t \geq t_0 \end{cases} \]

then $x \leq x_\varphi$.

**Proof.** (a) To prove the uniqueness of the solutions, suppose that $x_1, x_2 \in B_{t_0-\tau_0}$ are solutions of (16) satisfying

\[ x_1(t) = x_2(t) = \varphi(t), \quad t_0 - \tau_0 \leq t < t_0. \]

By the $g$-continuity of $V$,

\[ \|x_1(t) - x_2(t)\| \leq \int_{t_0}^{t} \|V(x_1 - x_2)(s)\| \, ds \]

\[ \leq \int_{t_0}^{t} g(s) \cdot \sup_{u \in [t_0,s]} \|x_1(u) - x_2(u)\| \, ds, \quad t \geq t_0. \]

Since the integral in (22) is increasing,

\[ \sup_{s \in [t_0,t]} \|x_1(s) - x_2(s)\| \leq \int_{t_0}^{t} g(s) \cdot \sup_{u \in [t_0,s]} \|x_1(u) - x_2(u)\| \, ds, \quad t \geq t_0, \]

and therefore an application of the Gronwall’s inequality implies the result.

(b) First we show that the sequence $(x_n)_{n=0}^{\infty}$ is well defined, and the functions $x_n (n \in \mathbb{N})$ belong to $B_{t_0}^\varphi$. By definition, $x_0 \in B_{t_0}^\varphi$. Proceeding inductively, suppose that $x_n \in B_{t_0}^\varphi$ for some $n \in \mathbb{N}$. Since the function

\[ t \mapsto \int_{t_0}^{t} V(x)(s) \, ds, \quad t \geq t_0 \]

is locally absolutely continuous, we have that $x_{n+1} \in B_{t_0}^\varphi$.

Let $t_1 > t_0$ be fixed.
The function $x_0$ is bounded on $[t_0 - \tau_0, t_1]$, the function $c - h$ is bounded on $[t_0, t_1]$, and therefore
\[
K_1 := \sup_{s \in [t_0 - \tau_0, t_1]} \|x_0(s)\| < \infty, \quad K_2 := \sup_{s \in [t_0, t_1]} \|c(s) - h(s)\| < \infty.
\]
By (4),
\[
\|x(t) - x_0(t)\| \leq \|c(t) - h(t)\| + \int_{t_0}^{t} \|V(x(s))\| \, ds \\
\leq K_2 + K_1 \cdot \int_{t_0}^{t} g(s) \, ds, \quad t_0 \leq t \leq t_1.
\]
This and
\[
x_1(t) - x_0(t) = 0, \quad t_0 - r \leq t < t_0
\]
implies that
\[
p_t(x_1 - x_0) \leq K_2 + K_1 \cdot \int_{t_0}^{t} g(s) \, ds, \quad t_0 \leq t \leq t_1. \tag{23}
\]
It follows from the $g$-continuity of $V$ and (23) that
\[
\|x_2(t) - x_1(t)\| = \left\| \int_{t_0}^{t} (V(x_1(s)) - V(x_0(s)) \, ds \right\| \\
\leq \int_{t_0}^{t} \|V(x_1 - x_0)(s)\| \, ds \leq \int_{t_0}^{t} g(s) p_{s}(x_1 - x_0) \, ds \\
\leq K_2 \int_{t_0}^{t} g(s) + K_1 \int_{t_0}^{t} g(s) \left( \int_{t_0}^{s} g(u) \, du \right) \, ds, \quad t_0 \leq t \leq t_1.
\]
Since
\[
x_2(t) - x_1(t) = 0, \quad t_0 - r \leq t < t_0,
\]
we obtain
\[
p_t(x_2 - x_1) \leq K_2 \int_{t_0}^{t} g(s) + K_1 \int_{t_0}^{t} g(s) \left( \int_{t_0}^{s} g(u) \, du \right) \, ds, \quad t_0 \leq t \leq t_1.
\]
By using an induction argument, we can prove similarly that for every $n \geq 1$
\[
\|x_{n+1}(t) - x_n(t)\| = 0, \quad t_0 - r \leq t < t_0,
\]
and
\[
\|x_{n+1}(t) - x_n(t)\| \\
\leq K_2 \int_{t_0}^{t} \left( \int_{t_0}^{s_1} \left( \cdots \left( \int_{t_0}^{s_{n-1}} g(s_1) \, ds_1 \right) \cdots \right) ds_2 \right) \, ds_1 \\
+ K_1 \int_{t_0}^{t} \left( \int_{t_0}^{s_1} \left( \cdots \left( \int_{t_0}^{s_{n+1}} g(s_1) \, ds_1 \right) \cdots \right) ds_2 \right) \, ds_1, \quad t_0 \leq t \leq t_1.
\]
It follows from this (see for example [21]) that for every \( n \geq 1 \)
\[
\|x_{n+1}(t) - x_n(t)\| \leq \frac{K_2}{n!} \left( \int_{t_0}^{t} g(s) \, ds \right)^n
\]
\[+ \frac{K_1}{(n+1)!} \left( \int_{t_0}^{t} g(s) \, ds \right)^{n+1}, \quad t_0 \leq t \leq t_1. \]  
(24)

We obtain from (23) and (24) that
\[
\sum_{n=0}^{\infty} \|x_{n+1}(t) - x_n(t)\| \leq K_2 \exp \left( \int_{t_0}^{t} g(s) \, ds \right)
\]
\[+ K_1 \left( \exp \left( \int_{t_0}^{t} g(s) \, ds \right) - 1 \right) < \infty, \quad t_0 \leq t \leq t_1. \]  
(25)

Since
\[
x_{n+1}(t) = x_0(t) + \sum_{k=0}^{n} (x_{k+1}(t) - x_k(t)), \quad t_0 - \tau_0 \leq t \leq t_1,
\]
and since \( t_1 \) is arbitrarily chosen from \([t_0, \infty[\), we have from (25) and (18) that there exists a function \( x_\varphi: [t_0 - \tau_0, \infty[ \to \mathbb{R}^d \) such that
\[
\lim_{n \to \infty} x_n(t) = x_\varphi(t), \quad t \geq t_0 - \tau_0, \]  
(26)

\[
\|x_\varphi(t)\| \leq (K_1 + K_2) \exp \left( \int_{t_0}^{t} g(s) \, ds \right), \quad t \geq t_0, \]  
(27)
and
\[
x_\varphi(t) = \varphi(t), \quad t_0 - \tau_0 \leq t < t_0. \]  
(28)

Since the functions \( x_n \ (n \in \mathbb{N}) \) belong to \( B_{t_0}^2 \), we conclude from (26), (27) and (28) that \( x_\varphi \in B_{t_0}^2 \).

For every \( n \in \mathbb{N} \) and for every integer \( m \geq n \) we have according to (24) that
\[
\|x_{m+1}(t) - x_n(t)\| \leq \sum_{k=n}^{m} \left( \frac{K_2}{k!} \left( \int_{t_0}^{t} g(s) \, ds \right)^k \right) + \frac{K_1}{(k+1)!} \left( \int_{t_0}^{t} g(s) \, ds \right)^{k+1}
\]
\[ \leq \left( \frac{K_2}{n!} \left( \int_{t_0}^{t} g(s) \, ds \right)^n \right) + \frac{K_1}{(n+1)!} \left( \int_{t_0}^{t} g(s) \, ds \right)^{n+1}
\]
\[ \cdot \exp \left( \int_{t_0}^{t} g(s) \, ds \right), \quad t_0 \leq t \leq t_1, \]
and hence
\[
p_{t_1} (x_\varphi - x_n) \leq \left( \frac{K_2}{n!} \left( \int_{t_0}^{t_1} g(s) \, ds \right)^n \right) + \frac{K_1}{(n+1)!} \left( \int_{t_0}^{t_1} g(s) \, ds \right)^{n+1} \]  
(29)
\[ \exp \left( \int_{t_0}^{t_1} g(s) \, ds \right). \]

This shows \((t_1 \text{ can be chosen arbitrarily})\) that
\[
\lim_{n \to \infty} p_t(x_\varphi - x_n) = 0, \quad t \geq t_0,
\]
which implies that \((x_n)\) converges to \(x_\varphi\) in the topology \(T_p\).

Since
\[
\left\| \int_{t_0}^{t} V(x_n)(s) \, ds - \int_{t_0}^{t} V(x_\varphi)(s) \, ds \right\| \leq \int_{t_0}^{t} \left\| V(x_n - x_\varphi)(s) \right\| \, ds
\]
\[
\leq \int_{t_0}^{t} g(s) \, ds \cdot p_t(x_n - x_\varphi), \quad t \geq t_0,
\]
(30) gives
\[
\lim_{n \to \infty} \int_{t_0}^{t} V(x_n)(s) \, ds = \int_{t_0}^{t} V(x_\varphi)(s) \, ds, \quad t \geq t_0.
\]

It now follows from (18) that \(x_\varphi\) is a solution of (16).

(c) Assume \(h \in F\). By using (19), it is easy to show that the successive approximations \((x_n)_{n=1}^{\infty}\) defined by (18) belong to \(F^\varphi\). By (b) (see (30)), the sequence \((x_n)\) converges to \(x_\varphi\) in the topology \(T_p\). Since \(F\) is closed in \(T_p\), \(x_\varphi \in F^\varphi\).

(d) Since \(SB_{t_0 - \tau_0}\) is a vector space and \(V\) is nonnegative on \(SB_{t_0 - \tau_0}\), \(V\) is monotonic on \(SB_{t_0 - \tau_0}\).

It follows from (20) that the function \(x_0 : [t_0 - \tau_0, \infty[ \to \mathbb{R}^p\) defined by
\[
x_0(t) = \begin{cases} \varphi(t), & t_0 - \tau_0 \leq t < t_0 \\ c(t) + \int_{t_0}^{t} V(x)(s) \, ds, & t \geq t_0 \end{cases}
\]
belongs to \(SB_{t_0 - \tau_0}\). Consider the successive approximations (18) determined by this function. According to (20) \(x_n \in SB_{t_0 - \tau_0}\) for all \(n \in \mathbb{N}\).

Next we show that
\[
x \leq x_n, \quad n \in \mathbb{N}. \tag{31}
\]

By (21), this is true if \(n = 0\). Also, if \(n\) is a nonnegative integer for which (31) holds, then the monotonicity of \(V\) implies that
\[
x_{n+1}(t) = \begin{cases} \varphi(t), & t_0 - \tau_0 \leq t < t_0 \\ c(t) + \int_{t_0}^{t} V(x_n)(s) \, ds, & t \geq t_0 \end{cases}
\]
\[
\geq \begin{cases} \varphi(t), & t_0 - \tau_0 \leq t < t_0 \\ c(t) + \int_{t_0}^{t} V(x)(s) \, ds, & t \geq t_0 \geq x(t), \quad t \geq t_0 - \tau_0, \end{cases}
\]
so that (31) holds for \(n + 1\), and therefore for all \(n \in \mathbb{N}\).
By (b), the sequence \((x_n)_{n=0}^{\infty}\) converges pointwise to \(x_\varphi\), and hence the result follows from (31).

The proof is complete. \qed

Now we give some consequences of Theorem 4.1. To achieve this introduce the following subsets of \(B_{t_0}\):

(a) A function \(x \in B_{t_0}\) is said to belong to \(C_{t_0}\) if \(x\) is continuous. The subspace \(C_{t_0}\) of \(B_{t_0}\) is closed in \(T_p\).

(b) Let \(s \geq t_0\) be fixed, and define the subspace \(B_s\) of \(B_{t_0}\) by

\[
B_s := \{ x \in B_{t_0} \mid x(t) = 0 \text{ for all } t \in [t_0, s]\}.
\]

This subspace is closed in \(T_p\).

(c) A function \(x \in B_{t_0}\) is said to belong to \(N_{t_0}\) if \(x\) is nonnegative. The set \(N_{t_0}\) is closed in \(T_p\) and it is closed under addition.

(d) Let \(K > 0\) be fixed. A function \(x \in B_{t_0}\) is said to belong to \(K_{t_0}\) if \(\|x(t)\| \leq K\) for all \(t \in [t_0, \infty]\). The set \(K_{t_0}\) is closed in \(T_p\).

Corollary 5. Assume (A1).

(a) Let \(F\) denote either \(B_{t_0}\) or \(C_{t_0}\), and let \(c \in F\). Then for every \(\varphi \in B([-t_0 - \tau_0, t_0])\) the integral equation (16) has a unique solution \(x_\varphi \in F_\varphi\) satisfying

\[
x_\varphi(t) = \varphi(t), \quad t_0 - \tau_0 \leq t < t_0.
\]

(b) If \(c \in B_s\), then the integral equation (16) has a unique solution \(x \in B_{t_0-\tau_0}\) satisfying

\[
x(t) = 0, \quad t_0 - \tau_0 \leq t < s.
\]

(c) If the operator \(V\) is nonnegative, \(c \in N_{t_0}\) and \(\varphi \in B([-t_0 - \tau_0, t_0])\) is nonnegative then \(x_\varphi \in N_{t_0}\).

(d) If \(\varphi \in B([-t_0 - \tau_0, t_0]), c \in B_{t_0}, \) the function \(g\) is integrable and

\[
\int_{t_0}^{\infty} g(s) \, ds \leq 1,
\]

and there exists \(K > 0\) such that

\[
\|\varphi(t)\| \leq K, \quad t_0 - \tau_0 \leq t < t_0
\]

and

\[
\|c(t)\| + K \cdot \int_{t_0}^{t} g(s) \, ds \leq K, \quad t \geq t_0,
\]

then \(x_\varphi\) is also bounded.

(e) If \(\varphi \in B([-t_0 - \tau_0, t_0]), \) \(c \in B_{t_0}\) is bounded, the function \(g\) is integrable and

\[
\int_{t_0}^{\infty} g(s) \, ds < 1,
\]

then \(x_\varphi\) is also bounded.

Proof. For every \(x \in B_{t_0-\tau_0}\) define the function (see (19))

\[
\dot{x} : [t_0, \infty] \to \mathbb{R}^d, \quad \dot{x}(t) = \int_{t_0}^{t} V(x(u)) \, du, \quad t \geq t_0.
\]
The function $\hat{x}$ is obviously continuous (moreover locally absolutely continuous), that is $\hat{x} \in C_{l_0}$.

(a) It follows that if $x \in F^\varphi$, then $c + \hat{x}$ belongs to $F$ too.
(b) The operator $V$ is of Volterra-type and hence for every $x \in B_{t_0-\tau_0}$ satisfying (33) we have $c + \hat{x} \in B_s$.
(c) Since the operator $V$ is nonnegative, $c \in N_{t_0}$ and $\varphi$ is also nonnegative, $x \in N_{t_0}^\varphi$ yields $c + \hat{x} \in N_{t_0}$.
(d) For every $x \in K_{t_0}^\varphi$

$$\|c(t) + \hat{x}(t)\| \leq \|c(t)\| + K \cdot \int_{t_0}^{t} g(s) \, ds \leq K, \quad t \geq t_0,$$

and therefore $c + \hat{x} \in K_{t_0}$.

(e) There is $M > 0$ such that $\|\varphi(t)\| \leq M$ for all $t \in [t_0 - \tau_0, t_0]$ and $\|c(t)\| \leq M$ for all $t \in [t_0, \infty[$. It comes from (34) that there exists $K > 0$ such that

$$M + K \cdot \int_{t_0}^{\infty} g(s) \, ds \leq K,$$

and we can follow as in (d).

Now the results follows from Theorem 4.1 (b) and (c).
The proof is complete. \hfill $\Box$

In the next result we consider the inhomogeneous linear functional differential equation (5).

**Theorem 4.2.** Assume $(A_1)$ and $(A_3)$.

(a) Let $\varphi : [t_0 - \tau_0, t_0] \to \mathbb{R}^d$ be Borel measurable and bounded.

(a1) The differential equation (5) has exactly one solution $x_\varphi$ satisfying

$$x_\varphi(t) = \varphi(t), \quad t_0 - \tau_0 \leq t \leq t_0. \quad (35)$$

(a2) If the operator $V$ is nonnegative, $\varphi(t) \geq 0$ for all $t \in [t_0 - \tau_0, t_0]$ and $\rho \geq 0$ a.e. on $[t_0, \infty[$, then the solution $x_\varphi$ is also nonnegative. Provided that $\varphi_i(t_0) > 0$ for some $i = 1, \ldots, d$, the $i$th component of $x_\varphi$ is positive on $[t_0, \infty[$.

(a3) If the function $g$ is integrable and

$$\int_{t_0}^{\infty} g(s) \, ds \leq 1,$$

and there exists $K > 0$ such that the functions $\varphi$ and $c \in B_{t_0}$ defined by

$$c(t) = \varphi(t_0) + \int_{t_0}^{t} \rho(u) \, du \quad t \geq t_0$$

satisfy

$$\|\varphi(t)\| \leq K, \quad t_0 - \tau_0 \leq t \leq t_0$$

and

$$\|c(t)\| + K \cdot \int_{t_0}^{t} g(s) \, ds \leq K, \quad t \geq t_0,$$

then $x_\varphi$ is also bounded.
(a₄) If the operator \( V \) is nonnegative, \( x \in B_{t₀−τ₀} \) such that \( x \) is locally absolutely continuous on \( [t₀, ∞[ \),

\[
x'(t) \leq V(x)(t) + ρ(t), \quad \text{a.e. on } [t₀, ∞],
\]

and \( x(t) \leq φ(t) \) for all \( t \in [t₀ − τ₀, t₀] \), then

\[
x(t) \leq x_φ(t), \quad t ≥ t₀.
\]

(b) Let \( s ≥ t₀ \) be fixed. If \( ρ = 0 \) on \( [t₀, s[ \) and \( c ∈ ℝ^d \), then the inhomogeneous linear functional differential equation

\[
x'(t) = V(x)(t) + ρ(t), \quad t ≥ s
\]

has exactly one solution \( x_s \) satisfying

\[
x_s(t) = \begin{cases} 0, & t₀ − τ₀ ≤ t < s \\
c, & t = s \end{cases}.
\]

Proof. (a) It is easy to check that \( x : [t₀ − τ₀, ∞[ → ℝ^d \) is a solution of the differential equation (5) satisfying (35) if and only if \( x \) is a solution of the integral equation

\[
x(t) = φ(t₀) + \int_{t₀}^{t} ρ(u) \, du + \int_{t₀}^{t} V(x)(u) \, du, \quad t ≥ t₀,
\]

for which \( x(t) = φ(t) \) for all \( t \in [t₀ − τ₀, t₀] \).

Define the function

\[
\hat{ρ} : [t₀, ∞[ → ℝ^d, \quad \hat{ρ}(t) = φ(t₀) + \int_{t₀}^{t} ρ(u) \, du, \quad t ≥ t₀.
\]

To prove (a₁) and (a₂), Corollary 5 (a) and (c) can be applied to the integral equation (39) by considering

(a₁) The function \( \hat{ρ} \) always belongs to \( B_{t₀} \) (moreover, \( \hat{ρ} ∈ C_{t₀} \)).

(a₂) The function \( \hat{ρ} \) belongs to \( N_{t₀} \) if \( φ(t₀) ≥ 0 \) and \( ρ ≥ 0 \) a.e. on \( [t₀, ∞[ \).

(a₃) It follows from Corollary 5 (d).

(a₄) According to (36)

\[
x(t) ≤ φ(t₀) + \int_{t₀}^{t} ρ(u) \, du + \int_{t₀}^{t} V(x)(u) \, du, \quad t ≥ t₀,
\]

and thus Theorem 4.1 (d) can be applied.

(b) Let

\[
c_s : [t₀, ∞[ → ℝ^d, \quad c_s(t) = \begin{cases} 0, & t₀ ≤ t < s \\
c, & t ≥ s \end{cases}.
\]

It is also not hard to see that \( x \) is a solution of the differential equation (37) satisfying (38) if and only if \( x \) is a solution of the integral equation

\[
x(t) = c_s(t) + \int_{t₀}^{t} ρ(u) \, du + \int_{t₀}^{t} V(x)(u) \, du, \quad t ≥ t₀
\]
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for which \( x(t) = 0 \) for all \( t \in [t_0 - \tau_0, t_0] \), or equivalently \( x \) is a solution of the integral equation

\[
x(t) = c + \int_s^t \rho(u) \, du + \int_s^t V(x(u)) \, du, \quad t \geq s
\]

for which \( x(t) = 0 \) for all \( t \in [t_0 - \tau_0, s] \).

Introduce the function

\[
\hat{\rho} : [t_0, \infty] \to \mathbb{R}^d, \quad \hat{\rho}(t) = c_s(t) + \int_{t_0}^t \rho(u) \, du, \quad t \geq t_0.
\]

We can apply Corollary 5 (b), since the function \( \hat{\rho} \) belongs to \( B_s \).

The proof is complete. \( \square \)

5. A variation-of-constants formula. Assume (A1), and consider the homogeneous linear functional differential system

\[
y'(t) = V(y)(t), \quad t \geq t_0.
\]

It follows from Theorem 4.2 (b) that for every \( s \geq t_0 \) and \( i = 1, \ldots, d \) the system

\[
v'(t) = V(v)(t), \quad t \geq s
\]

has a unique solution \( v_i(\cdot, s) : [t_0 - \tau_0, \infty] \to \mathbb{R}^d \) such that

\[
v_i(t, s) = \begin{cases} e_i, & t = s \\ 0, & t_0 - \tau_0 \leq t < s \end{cases}
\]

where \( e_1, \ldots, e_d \) is the standard basis in \( \mathbb{R}^d \). This means that for every \( s \in [t_0, \infty[ \)

\[
\frac{\partial v_i(t, s)}{\partial t} = V(v_i(\cdot, s))(t), \quad \text{a.e. on } [s, \infty[, 
\]

or otherwise

\[
v_i(t, s) = e_i + \int_s^t V(v_i(\cdot, s))(u) \, du, \quad t \geq s
\]

and (43) holds \( (i = 1, \ldots, d) \).

**Definition 5.1.** Define the matrix valued function \( v : [t_0 - \tau_0, \infty[ \times [t_0, \infty] \to \mathbb{R}^{d \times d} \)

in the following way: the \( i \)th column of \( v(t, s) \) is \( v_i(t, s) \) \( (i = 1, \ldots, d) \). The function \( v \) is called the fundamental kernel of (41).

The essential properties of the fundamental kernel are described in the next theorem.

**Theorem 5.2.** Assume (A1) and (A2). Then

(a) The functions \( v_i(t, \cdot) : [t_0, \infty[ \to \mathbb{R}^d \) \( (i = 1, \ldots, d) \) are continuous on \( [t_0, t] \)

for every \( t \in [t_0, \infty[ \).

(b) The functions \( v_i \) \( (i = 1, \ldots, d) \) are (jointly) Borel measurable and locally bounded.

(c) The functions

\[
(t, s) \mapsto V(v_i(\cdot, s))(t), \quad (t, s) \in [t_0 - \tau_0, \infty[ \times [t_0, \infty[ 
\]

(44)

\( (i = 1, \ldots, d) \) are locally integrable.
Proof. We can obviously suppose that \( i = 1 \).

(a) For every \( s \in [t_0, \infty] \) define the functions \( w_n(\cdot, s) : [t_0 - \tau_0, \infty] \to \mathbb{R}^d \) \((n \in \mathbb{N})\) by
\[
 w_0(t, s) = \begin{cases} 
 0, & t_0 - \tau_0 \leq t < s \\
 e_1, & s \leq t 
\end{cases},
\]
and for every \( n \in \mathbb{N} \)
\[
 w_{n+1}(t, s) = \begin{cases} 
 0, & t_0 - \tau_0 \leq t < s \\
 e_1 + \int_{t_0}^{t} V(w_n(\cdot, s))(u) \, du, & s \leq t 
\end{cases}
\]
(45)
where
\[
 e_{1, s}(t) = \begin{cases} 
 0, & t_0 \leq t < s \\
 e_1, & s \leq t 
\end{cases}.
\]

The sequence \( (w_n)_{n=0}^{\infty} \) is well defined (see (40)).

First we show that the functions \( w_n(t, \cdot) \) \((n \in \mathbb{N})\) are continuous on \([t_0, t]\) for every \( t \in [t_0, \infty[ \).

The function \( w_0(t, \cdot) \) is obviously continuous on \([t_0, t]\) for every \( t \in [t_0, \infty[ \). Proceeding inductively, suppose that for some \( n \in \mathbb{N} \) the function \( w_n(t, \cdot) \) is continuous on \([t_0, t]\) for every \( t \in [t_0, \infty[ \).

For a fixed \( s \in [t_0, \infty] \), we have from the definition of \( w_n \) and from the proof of Theorem 4.1 (b) that
\[
 w_n(t, s) = 0, \quad t_0 - \tau_0 \leq t < s,
\]
and
\[
 \|w_n(t, s)\| \leq \exp \left( \int_{t_0}^{t} g(u) \, du \right), \quad s \leq t. \tag{46}
\]
It follows that
\[
 \|w_n(t, s)\| \leq \begin{cases} 
 0, & t_0 - \tau_0 \leq t < t_0 \\
 \exp \left( \int_{t_0}^{t} g(u) \, du \right), & t_0 \leq t, \quad s \geq t_0. 
\end{cases} \tag{47}
\]

The function defined by the right hand side of (47) is obviously locally integrable.

Now we prove that the function \( w_{n+1}(t, \cdot) \) is continuous on \([t_0, t]\) for every \( t \in [t_0, \infty[ \).

Fix \( t \in [t_0, \infty[ \), let \( \hat{s} \in [t_0, \hat{t}] \), and let \( (s_k)_{k=0}^{\infty} \) be a sequence from \([t_0, \hat{t}]\) with limit \( \hat{s} \).

By the induction hypothesis
\[
 \lim_{k \to \infty} w_n(t, s_k) = w_n(t, \hat{s}), \quad t \in [t_0, \infty[. \tag{48}
\]
According to (47), the sequence \( (w_n(\cdot, s_k))_{k=0}^{\infty} \) is uniformly bounded on every compact subinterval of \([t_0 - \tau_0, \infty[ \), and hence (A2) implies that
\[
 \lim_{k \to \infty} V(w_n(\cdot, s_k)) = V(w_n(\cdot, \hat{s})).
\]
It comes from (4) that
\[ \|V(w_n(\cdot, s_k))(t)\| \leq g(t) \cdot \sup_{u \in [t_0 - \tau_0, t]} \|w_n(u, s_k)\| \]
\[ \leq g(t) \cdot \exp \left( \int_{t_0}^{t} g(u) \, du \right), \quad t \in [t_0, \infty[, \quad k \in \mathbb{N}. \]

Lebesgue’s dominated convergence theorem can now be applied to the integral in (45), and it yields that
\[ \lim_{k \to \infty} w_{n+1}((\hat{t}, s)) = \lim_{k \to \infty} \left( e_{1,s_k}(\hat{t}) + \int_{t_0}^{\hat{t}} V(w_n(\cdot, s_k)) (u) \, du \right) \]
\[ = e_{1,s}(\hat{t}) + \int_{t_0}^{\hat{t}} V(w_n(\cdot, s))(u) \, du = w_{n+1}(\hat{t}, s), \]
which gives that \( w_{n+1}(\hat{t}, \cdot) \) is continuous at \( \hat{s} \).

By considering (40), it follows from the proof of Theorem 4.1 (b) that for every \( s \in [t_0, \infty[ \)
\[ \lim_{n \to \infty} w_n(t, s) = v_1(t, s), \quad t \in [t_0 - \tau_0, \infty[. \quad (48) \]

Since for every \( t \geq t_0 \)
\[ \sup_{u \in [t_0 - \tau_0, t]} \|w_0(u, s)\| = 1, \quad t_0 \leq s \leq t, \]
we obtain from (29) (\( K_2 = 0 \)) that
\[ p_t(v_1(\cdot, s) - w_n(\cdot, s)) \leq \frac{1}{(n + 1)!} \left( \int_{t_0}^{t} g(z) \, dz \right)^{n+1} \exp \left( \int_{t_0}^{t} g(z) \, dz \right), \]
and hence for every fixed \( t \geq t_0 \) the sequence \( (w_n(t, \cdot)) \) converges uniformly to \( v_1(t, \cdot) \) in \( s \) on \( [t_0, t] \).

Since the functions \( w_n(t, \cdot) \) \( (n \in \mathbb{N}) \) are continuous on \( [t_0, t] \) for every \( t \in [t_0, \infty[ \), it now follows that \( v_1(t, \cdot) \) is continuous on \( [t_0, t] \).

(b) Define the function \( \hat{v}_1 : [t_0 - \tau_0, \infty[ \times [t_0, \infty[ \to \mathbb{R}^d \) by
\[ \hat{v}_1(t, s) = \left\{ \begin{array}{ll}
\hat{v}_1, & t_0 - \tau_0 \leq t < s \\
v_1(t, s), & t_0 \leq s \leq t
\end{array} \right. . \]

Then \( \hat{v}_1(\cdot, s) \) is (absolutely) continuous for each \( s \in [t_0, \infty[ \), and by (a), \( v_1(t, \cdot) \) is Borel measurable (moreover it is left continuous) for every \( t \in [t_0 - \tau_0, \infty[ \). It follows that \( \hat{v}_1 \) is a Caratheodory function, and therefore (see e.g. [1]) it is (jointly) Borel measurable.

Since the set
\[ H := \{ (t, s) \in \mathbb{R}^2 \mid t \in [t_0, \infty[, \quad s \in [t_0, t] \} \]
is Borel measurable, \( v_1 \) is also Borel measurable.

By (47) and (48), \( v_1 \) is locally bounded.

(c) Let \( \hat{s} \in [t_0, \infty[ \), and let \( (s_k)_{k=0}^{\infty} \) be a sequence from \( [t_0, \hat{s}] \) with limit \( \hat{s} \).
By using (a) and the definition of \( v_1 \), we have that
\[ \lim_{k \to \infty} v_1(t, s_k) = v_1(t, \hat{s}), \quad t \in [t_0 - \tau_0, \infty[. \]
Theorem 5.3. Assume \((A_1), (A_2)\) and \((A_3)\), and let \(\varphi : [t_0 - r, t_0] \to \mathbb{R}\) be Borel measurable and bounded. Denote by \(y(\varphi) \in B_{t_0 - \tau_0}\) the unique solution of
\[
y'(t) = V(y)(t), \quad t \geq t_0, \tag{49}
\]
for which the restriction of \(y(\varphi)\) to \([t_0 - \tau_0, t_0]\) is \(\varphi\).

If \(x \in B_{t_0 - \tau_0}\) is the unique solution of the inhomogeneous linear functional differential equation \((5)\) for which the restriction of \(x\) to \([t_0 - \tau_0, t_0]\) is \(\varphi\), then \(x\) can be obtained by
\[
x(t) = y(\varphi)(t) + \int_{t_0}^{t} v(t, s) \rho(s) ds, \quad t \geq t_0. \tag{50}
\]

Proof. By Theorem 5.2 (a), the integral in \((50)\) exists.

By Theorem 5.2 (c), we can apply Theorem 3.5 which shows that
\[
x'(t) = y'(\varphi)(t) + v(t, t) \rho(t) + \int_{t_0}^{t} \frac{\partial v(t, s)}{\partial t} \rho(s) ds
\]
\[
= V(y(\varphi))(t) + \rho(t) + \int_{t_0}^{t} \sum_{j=1}^{d} \frac{\partial v_j(t, s)}{\partial t} \rho_j(s) ds
\]
\[
= V(y(\varphi))(t) + \rho(t) + \int_{t_0}^{t} \sum_{j=1}^{d} V(v_j(\cdot, s))(t) \rho_j(s) ds, \quad \text{a.e. on } [t_0, \infty[. \tag{51}
\]

Define the function \(w_j : [t_0 - \tau_0, \infty[ \to \mathbb{R} \ (j = 1, \ldots, d)\) by
\[
w_j(t) = \begin{cases} 
0, & t \in [t_0 - \tau_0, 0[ \\
\int_{t_0}^{t} v_j(t, s) \rho_j(s) ds, & t \geq t_0
\end{cases}.
\]

Theorem 3.3 now yields that for every \(j = 1, \ldots, d\)
\[
\int_{t_0}^{t} V(v_j(\cdot, s) \rho_j(s))(t) ds = V(w_j)(t), \quad t \geq t_0.
\]
By using this in (51), we obtain
\[ x'(t) = V(y(\varphi))(t) + \rho(t) + \sum_{j=1}^{d} V(w_j)(t) = V(y(\varphi))(t) + \rho(t) + \left( \sum_{j=1}^{d} w_j \right)(t) = V\left( y(\varphi) + \sum_{j=1}^{d} w_j \right)(t) + \rho(t), \quad \text{a.e. on } [t_0, \infty[ , \]

as desired.

The proof is complete. \( \square \)

6. **Applications.** In this section we study the inhomogeneous linear functional differential system
\[ x'(t) = -\Lambda(t)x(t) + N(x)(t) + \rho(t), \quad t \geq t_0 \quad (52) \]
and the corresponding homogeneous linear functional differential system
\[ y'(t) = -\Lambda(t)y(t) + N(y)(t), \quad t \geq t_0, \quad (53) \]

where

(C1) The functions \( a_i : [t_0, \infty[ \to \mathbb{R} \) \( (i = 1, \ldots, d) \) are locally integrable, and \( \Lambda(t) \) is a diagonal \( d \times d \) matrix whose diagonal entries starting in the upper left corner are \( a_1(t), \ldots, a_d(t) \) for all \( t \geq t_0 \).

(C2) The operator \( N : B_{t_0-\tau_0} \to L_{t_0} \) is linear, of Volterra-type, \( T_p - T_q \) continuous, nonnegative and satisfies (A2).

(C3) The function \( \rho = (\rho_1, \ldots, \rho_d)^T \) belongs to \( L_{t_0} \).

**Remark 5.** Define the operator \( V : B_{t_0-\tau_0} \to L_{t_0} \) by
\[ V(x)(t) = -\Lambda(t)x(t) + N(x)(t), \quad t \geq t_0. \]

It is obvious that \( V \) is linear, of Volterra-type and satisfies (A2).

By Remark 1, \( N \) is \( \|N(1)\| \)-continuous. This implies that \( V \) is \( g \)-continuous with
\[ g = \sum_{i=1}^{d} |a_i| + \|N(1)\| , \]
and hence \( V \) is \( T_p - T_q \) continuous too.

For all \( x \in B_{t_0-\tau_0} \) we denote by \( N_i(x)(t) \) the \( i \)th component of \( N(x)(t) \) \( (t \geq t_0, \ i = 1, \ldots, d) \).

The fundamental kernel of (53) is denoted by \( v : [t_0 - \tau_0, \infty[ \times [t_0, \infty[ \to \mathbb{R}^{d \times d} \)
(see Definition 5.1).

In the first main result of this section we study the behavior of the solutions of the homogeneous system (53).

**Theorem 6.1.** Assume (C1), (C2) and
\[ a_i(t) - N_i(1)(t) \geq 0, \quad t \geq t_0, \quad i = 1, \ldots, d. \quad (54) \]
(a) Every nonnegative solution \( y = (y_1, \ldots, y_d)^T \) of the homogeneous system (53) is bounded. Moreover,

\[
y_i(t) \leq \max_{1 \leq j \leq d} \left( \sup_{s \in [t_0 - \tau_0, t_0]} y_j(s) \right), \quad t \geq t_0, \quad i = 1, \ldots, d.
\]

(b) Assume further that there exists a nonnegative delay \( \tau : [t_0, \infty) \to \mathbb{R}^+ \) such that \( \tau \) is measurable, it obeys the inequality

\[
t_0 - \tau_0 \leq t - \tau(t), \quad t \geq t_0,
\]

and for all \( x \in B_{t_0 - \tau_0} \) and \( t \geq t_0 \) the vector \( N(x)(t) \) depends only on the restriction of \( x \) to \( [t - \tau(t), t] \).

(b1) If there exists a nonnegative and locally integrable function \( \delta : [t_0 - \tau_0, \infty) \to \mathbb{R}^+ \) such that

\[
delta(t) + \exp \left( \int_{t - \tau(t)}^t \delta(s) \, ds \right) \cdot N_i(1)(t) \leq a_i(t), \quad t \geq t_0, \quad i = 1, \ldots, d, \tag{55}
\]

then

\[
y_i(t) \leq \max_{1 \leq j \leq d} \left( \sup_{s \in [t_0 - \tau_0, t_0]} y_j(s) \right) \cdot \exp \left( - \int_{t_0}^t \delta(s) \, ds \right), \quad t \geq t_0, \quad i = 1, \ldots, d.
\]

(b2) If there exists a nonnegative and locally integrable function \( \delta : [t_0 - \tau_0, \infty) \to \mathbb{R}^+ \) such that (55) holds and

\[
\int_{t_0}^{\infty} \delta(s) \, ds = \infty, \tag{56}
\]

then every solution of the homogeneous system (53) tends to zero at infinity.

This at once leads to the question whether the conditions in the previous theorem guarantee the existence of a function \( \delta \) satisfying (55) and (56). To answer this question, we can use the results in [17].

**Theorem 6.2.** (see Theorem 2.7, Theorem 3.3 and Theorem 3.5 in [17]) Assume the functions \( \alpha : [t_0, \infty) \to \mathbb{R} \) and \( \beta : [t_0, \infty) \to \mathbb{R}^+ \) are locally integrable, and

\[
\beta(t) \leq \alpha(t), \quad t \geq t_0.
\]

Assume further that the function \( \tau : [t_0, \infty) \to \mathbb{R}^+ \) is measurable and it obeys the inequality

\[
t_0 - \tau_0 \leq t - \tau(t), \quad t \geq t_0.
\]

(a) There exists a nonnegative and locally integrable function \( \hat{\delta} : [t_0 - \tau_0, \infty) \to \mathbb{R}^+ \) which is unique and satisfies the integral equation

\[
\hat{\delta}(t) + \exp \left( \int_{t - \tau(t)}^t \hat{\delta}(s) \, ds \right) \cdot \beta(t) = \alpha(t), \quad t \geq t_0,
\]
and hence there exists a nonnegative and locally integrable function \( \delta : [t_0 - \tau_0, \infty] \to \mathbb{R}_+ \) such that

\[
\delta(t) + \exp \left( \int_{t_0 - \tau(t)}^t \delta(s) \, ds \right) \cdot \beta(t) \leq \alpha(t), \quad t \geq t_0.
\] (57)

(b) If

\[
\lim_{t \to \infty} (t - \tau(t)) = \infty,
\] (58)

and there is \( q \in (0, 1) \) such that

\[
\limsup_{t \to \infty} \int_{t_0}^t (q\alpha(s) - \beta(s)) \, ds = \infty,
\]

then there exists a nonnegative and locally integrable function \( \delta : [t_0 - \tau_0, \infty] \to \mathbb{R}_+ \) satisfying (57) and (56).

(c) If (58) holds, \( \beta(t) \leq q\alpha(t), \quad t \geq t_0 \) with \( q \in (0, 1) \), and

\[
\int_{t_0}^\infty \alpha(s) \, ds = \infty,
\]

then there exists a nonnegative and locally integrable function \( \delta : [t_0 - \tau_0, \infty] \to \mathbb{R}_+ \) satisfying (57) and (56).

In the second main result of this section the boundedness of the solutions of the system (52) are studied. It is an essential generalization of Theorem 2.1 in [18].

**Theorem 6.3.** Assume (C\(_1\)-C\(_3\)). Assume further that there exists \( t_1 \geq t_0 \) for which

\[
a_i(t) - N_i(1)(t) > 0, \quad t \geq t_1, \quad i = 1, \ldots, d
\] (59)

and every solution of the homogeneous system (53) tends to zero at infinity.

Then for every solution \( x : [t_0 - \tau_0, \infty] \to \mathbb{R}^d \) of the inhomogeneous system (52), we obtain

\[
\min_{1 \leq j \leq d} \liminf_{t \to \infty} \frac{\rho_j(t)}{a_j(t) - N_j(1)(t)} \leq \liminf_{t \to \infty} x_i(t)
\]

\[
\leq \limsup_{t \to \infty} x_i(t) \leq \max_{1 \leq j \leq d} \limsup_{t \to \infty} \frac{\rho_j(t)}{a_j(t) - N_j(1)(t)}, \quad i = 1, \ldots, d.
\]

As an immediate consequence of the previous theorem we have a result about persistence and permanence of the system (52). First we give the definition of these concepts (see e.g. [3] and [11]).

**Definition 6.4.** The system (52) is said to be persistent if any solution \( x \) is bounded away from zero, that is

\[
\liminf_{t \to \infty} x_i(t) > 0, \quad i = 1, \ldots, d;
\]

and uniformly persistent if there is \( k > 0 \) such that

\[
\liminf_{t \to \infty} x_i(t) \geq k, \quad i = 1, \ldots, d.
\]
The system (52) is said to be permanent if there are constants $K > k > 0$ such that, given any solution $x$, there is $t(x) \geq t_0$ such that
\[ k \leq x_i(t) \leq K, \quad t \geq t(x), \quad i = 1, \ldots, d. \]

**Corollary 6.** Assume $(C_1\text{-}C_3)$. Assume further that there exists $t_1 \geq t_0$ for which
\[ a_i(t) - N_i(1)(t) > 0, \quad t \geq t_1, \quad i = 1, \ldots, d \]
and every solution of the homogeneous system (53) tends to zero at infinity.

(a) If
\[ \lim_{t \to \infty} \rho_i(t) > 0, \quad i = 1, \ldots, d, \]
then the system (52) is uniformly persistent.

(b) If
\[ 0 < \min_{1 \leq j \leq d} \lim_{t \to \infty} \frac{\rho_j(t)}{a_j(t) - N_j(1)(t)} \leq \max_{1 \leq j \leq d} \lim_{t \to \infty} \frac{\rho_j(t)}{a_j(t) - N_j(1)(t)} < \infty, \]
then the system (52) is permanent.

To prove these results, we need three lemmas.

**Lemma 6.5.** Assume $(C_1\text{-}C_3)$, and assume further that $\rho$ is nonnegative. Let $x$ be the solution of the inhomogeneous system (52) satisfying
\[ x(t) = \varphi(t), \quad t_0 - \tau_0 \leq t \leq t_0, \]
where $\varphi = (\varphi_1, \ldots, \varphi_d)^T : [t_0 - \tau_0, t_0] \to \mathbb{R}^d$ is nonnegative, Borel measurable and bounded.

(a) If $\varphi(t_0) > 0$, then $x(t) > 0$ for every $t \in [t_0, \infty]$.

(b) If $\varphi(t_0) = 0$, then $x(t) \geq 0$ for every $t \in [t_0, \infty]$.

**Proof.** (a) The solution $x$ is continuous on $[t_0, \infty]$, therefore there are two possibilities:

(i) $x(t) > 0$ for every $t \in [t_0, \infty]$ and then we are ready.

(ii) There exist $t_1 > t_0$ and $i \in \{1, \ldots, d\}$ such that $x_i(t_1) = 0$ and $x(t) > 0$ for every $t \in [t_0, t_1]$.

Since $N$ is of Volterra-type and nonnegative,
\[ N(x)(s) \geq 0, \quad t_0 - \tau_0 \leq s \leq t_1, \]
and hence
\[ x_i(t_1) = \varphi_i(t_0) \exp \left( \int_{t_0}^{t_1} - a_i(s) \, ds \right) \]
\[ + \int_{t_0}^{t_1} (N_i(x)(s) + \rho_i(s)) \exp \left( \int_{s}^{t_1} - a_i(u) \, du \right) \, ds > 0, \]
which is a contradiction.

(b) For any $\varepsilon > 0$ let $x_\varepsilon = (x_{\varepsilon_1}, \ldots, x_{\varepsilon_d})^T : [t_0 - \tau_0, \infty[ \to \mathbb{R}^d$ be the solution of (52) satisfying
\[ x_\varepsilon(t) = \varphi_\varepsilon(t) = \varphi(t) + \varepsilon, \quad t_0 - \tau_0 \leq t \leq t_0. \]

By (a), $x_\varepsilon(t) > 0$ for all $t \in [t_0, \infty[$.

Let $\varepsilon_1 > \varepsilon_2 > 0$. We prove that $x_{\varepsilon_1}(t) > x_{\varepsilon_2}(t)$ for all $t \in [t_0, \infty[$. Suppose this is not satisfied. Since the solutions $x_{\varepsilon_1}$ and $x_{\varepsilon_2}$ are continuous on $[t_0, \infty[$, there
exists \( t_1 > t_0 \) and \( i \in \{1, \ldots, d\} \) such that \( x_{\varepsilon_1}(t_1) = x_{\varepsilon_2}(t_1) \) and \( x_{\varepsilon_1}(t) > x_{\varepsilon_2}(t) \) for every \( t \in [t_0, t_1] \).

Since \( N \) is of Volterra-type and nonnegative,

\[
N(x_{\varepsilon_1})(s) \geq N(x_{\varepsilon_2})(s), \quad t_0 - \tau_0 \leq s \leq t_1,
\]

and therefore

\[
x_{\varepsilon_1}(t_1) = (\varphi_i(t_0) + \varepsilon_1) \exp \left( \int_{t_0}^{t_1} - a_i(s) \, ds \right)
\]

\[
+ \int_{t_0}^{t_1} (N_i(x_{\varepsilon_1})(s) + \rho_i(s)) \exp \left( \int_{s}^{t_1} - a_i(u) \, du \right) \, ds
\]

\[
> (\varphi_i(t_0) + \varepsilon_2) \exp \left( \int_{t_0}^{t_1} - a_i(s) \, ds \right)
\]

\[
+ \int_{t_0}^{t_1} (N_i(x_{\varepsilon_2})(s) + \rho_i(s)) \exp \left( \int_{s}^{t_1} - a_i(u) \, du \right) \, ds = x_{\varepsilon_2}(t_1),
\]

which is a contradiction.

Let \( (\varepsilon_n)_{n=0}^\infty \) be a decreasing sequence of positive numbers. The previous establishment shows that \( (x_{\varepsilon_n}) \) is a decreasing sequence of positive functions on \( [t_0, \infty[ \), and hence

\[
\lim_{n \to \infty} x_{\varepsilon_n}(t) = z(t), \quad t \geq t_0 - \tau_0
\]

for some nonnegative and locally integrable \( z = (z_1, \ldots, z_d)^T : [t_0 - \tau_0, \infty[ \to \mathbb{R}^d \).

Since \( N \) is nonnegative, the sequence \( (N(x_{\varepsilon_n})) \) is nonnegative and decreasing. By \((A_2)\),

\[
\lim_{n \to \infty} N(x_{\varepsilon_n})(t) = N(z)(t), \quad t \geq t_0.
\]

Evidently,

\[
|a_i(s)x_{\varepsilon_n}(s)| \leq |a_i(s)|x_{\varepsilon_n}(s), \quad t \geq t_0, \quad i = 1, \ldots, d.
\]

By using these establishments, we can apply Lebesgue dominated convergence theorem and the monotone convergence theorem in

\[
x_{\varepsilon_n}(t) = (\varphi(t_0) + \varepsilon_n) + \int_{t_0}^{t} \rho_i(s) \, ds - \int_{t_0}^{t} a_i(s)x_{\varepsilon_n}(s) \, ds
\]

\[
+ \int_{t_0}^{t} N_i(x_{\varepsilon_n})(s) \, ds, \quad t \geq t_0, \quad i = 1, \ldots, d, \quad n \geq 0
\]

with

\[
x_{\varepsilon_n}(t) = \varphi_{\varepsilon_n}(t), \quad t_0 - r \leq t < t_0, \quad n \geq 0,
\]

and obtain that

\[
z(t) = \varphi(t_0) + \int_{t_0}^{t} \rho_i(s) \, ds - \int_{t_0}^{t} a_i(s)z_i(s) \, ds
\]
\[
\int_{t_0}^t N_i(z)(s) \, ds, \quad t \geq t_0, \quad i = 1, \ldots, d
\]

with
\[
z(t) = \varphi(t), \quad t_0 - r \leq t < t_0,
\]

which implies that \( z = x \).

The proof is complete. \( \square \)

We now establish some useful properties of the homogeneous system (53).

**Lemma 6.6.** Assume \((C_1)\) and \((C_2)\).

(a) Then every solution of (53) tends to zero at infinity if and only if every non-negative solution of (53) tends to zero at infinity.

(b) The fundamental kernel \( v \) of (53) is nonnegative in the sense that
\[
v_i(t, s) \geq 0, \quad (t, s) \in [t_0 - \tau_0, \infty[ \times [t_0, \infty[., \quad i = 1, \ldots, d.
\]

**Proof.** (a) The “only if” part is trivial.

Conversely, let \( y : [t_0 - \tau_0, \infty[ \to \mathbb{R}^d \) be a solution of (53), and let \( z : [t_0 - \tau_0, \infty[ \to \mathbb{R}^d \) be the solution of (53) satisfying
\[
z(t) = ||y(t)||, \quad t_0 - \tau_0 \leq t \leq t_0.
\]

Since (53) is homogeneous and
\[
z(t) \pm y(t) \geq 0, \quad t_0 - \tau_0 \leq t \leq t_0,
\]

an application of Lemma 6.5 implies that \( z - y \) and \( z + y \) are nonnegative solutions of (53), and hence they tend to zero at infinity.

By using
\[
y = \frac{1}{2} ((z + y) - (z - y)),
\]

the result follows.

(b) By using the definition of \( v \), it follows immediately from Lemma 6.5.

The proof is complete. \( \square \)

**Lemma 6.7.** Assume \((C_1-C_3)\). Assume further that every solution of the homogeneous system (53) tends to zero at infinity.

Fix \( T \geq t_0 \). Then

(a) Every solution of the homogeneous system
\[
y'(t) = -\Lambda(t) y(t) + N(y)(t), \quad t \geq T \quad (60)
\]
tends to zero at infinity too.

(b) If \( \vartheta = (\vartheta_1, \ldots, \vartheta_d) : [t_0, \infty[ \to \mathbb{R}^d \) is locally integrable, then
\[
\lim_{T \to \infty} \int_{t_0}^T v(t, s) \vartheta(s) \, ds = 0
\]

(c) By introducing \( 1_d := (1, \ldots, 1)^T \), we have
\[
\lim_{T \to \infty} \int_{t_0}^T v(t, s) (\Lambda(s) 1_d - N(1)(s)) \, ds = 1_d.
\]
Proof. (a) By Lemma 6.6, it is enough to consider nonnegative solutions of (60).
Let \( y_T \) be a nonnegative solution of (60), and let \( y \) be the solution of (53) satisfying
\[
y_i (t) = 1, \quad t_0 - \tau_0 \leq t \leq t_0, \quad i = 1, \ldots, d. \tag{61}
\]
Since \( y \) is continuous on \([t_0, \infty[\), (61) and Lemma 6.5 (a) imply that there exists \( c \in \mathbb{R}^d, c > 0 \) such that
\[
y (t) > c, \quad t_0 - \tau_0 \leq t \leq T.
\]
It follows from this and from the fact that \( y_T \) is nonnegative and bounded on \([t_0 - \tau_0, T]\) that there is a positive number \( k \) for which
\[
y_T (t) < ky (t), \quad t_0 - \tau_0 \leq t \leq T. \tag{62}
\]
The systems (53) and (60) are homogeneous, and therefore \( ky \) is a solution of both systems.
Since \( ky - y_T \) is also a solution of (60), by (62), and by \((ky - y_T) (T) > 0\), Lemma 6.5 (a) implies that
\[
y_T (t) < ky (t), \quad t_0 - \tau_0 \leq t.
\]
Now the result follows, because \( y_T \) is nonnegative and
\[
\lim_{t \to \infty} ky (t) = 0.
\]
(b) Let \( \hat{\vartheta} : [t_0, \infty[ \to \mathbb{R}^d \) be defined by
\[
\hat{\vartheta} (t) = \begin{cases} \vartheta (t), & t_0 \leq t < T \\ 0, & t \geq T \end{cases}
\]
and let \( z : [t_0 - \tau_0, \infty[ \to \mathbb{R}^d \) be given by
\[
z (t) = \begin{cases} 0, & t_0 - \tau_0 \leq t \leq t_0 \\ \int_{t_0}^{t} v (t, s) \hat{\vartheta} (s) \, ds, & t \geq t_0 \end{cases}.
\]
By Theorem 5.3, \( z \) is the solution of the equation
\[
x' (t) = -\Lambda (t) x (t) + N (x) (t) + \hat{\vartheta} (t), \quad t \geq t_0
\]
satisfying
\[
x (t) = 0, \quad t_0 - \tau_0 \leq t \leq t_0.
\]
According to the definition of \( \hat{\vartheta} \), it can be seen that \( z \) is a solution of (60), and therefore (a) implies the result.
(c) It is obvious that the constant function \( t \mapsto 1_d \) \((t \geq t_0 - \tau_0)\) is the solution of the system
\[
x' (t) = -\Lambda (t) x (t) + N (x) (t) + \Lambda (t) 1_d - N (1) (t), \quad t \geq t_0
\]
satisfying
\[
x (t) = 1_d, \quad t_0 - \tau_0 \leq t \leq t_0.
\]
Since every solution of (53) tends to zero at infinity, Theorem 5.3 gives that
\[
1_d = \int_{t_0}^{t} v (t, s) (\Lambda (s) 1_d - N (1) (s)) \, ds, \quad t \geq t_0. \tag{63}
\]
By using
\[
\int_{T}^{t} v(t,s) (\Lambda(s) 1_d - N(1)(s)) \, ds = \int_{t_0}^{t} v(t,s) (\Lambda(s) 1_d - N(1)(s)) \, ds
\]
\[
- \int_{t_0}^{T} v(t,s) (\Lambda(s) 1_d - N(1)(s)) \, ds, \quad t \geq T,
\]
the result follows from (63) and (b).

The proof is complete.

Finally, we give the proofs of the main results of this section.

Proof of Theorem 6.1. First we prove a useful property of $N$. Let $x = (x_1, \ldots, x_d)^T \in B_{t_0 - \tau_0}$ such that
\[
x_j(t) \leq K, \quad t \in [t_0 - \tau_0, t_K], \quad j = 1, \ldots, d,
\]
where $K > 0$ and $t_K > t_0$. Define the function $z = (z_1, \ldots, z_d)^T \in B_{t_0 - \tau_0}$ by
\[
z_j(t) = \begin{cases} K, & t \in [t_0 - \tau_0, t_K] \\ x_j(t), & t > t_K \end{cases}, \quad j = 1, \ldots, d.
\]

Because $x \leq z$ and $N$ is linear, nonnegative and of Volterra-type, we have
\[
N(x)(t) \leq N(z)(t) = K \cdot N(1)(t), \quad t \in [t_0 - \tau_0, t_K].
\]

(a) Let $y = (y_1, \ldots, y_d)^T$ be a fixed nonnegative solution of (53), and let
\[
K_1 := \max_{1 \leq j \leq d} \left( \sup_{s \in [t_0 - \tau_0, t_0]} y_j(s) \right).
\]

The proof splits into two cases.

(i) Special case: strict inequality holds in (54) that is
\[
a_i(t) - N_i(1)(t) > 0, \quad t \geq t_0, \quad i = 1, \ldots, d.
\]

Assume the result is not true. Then there is a number $K_2 > K_1$ and an index $i \in \{1, \ldots, d\}$ such that $y_i(t) = K_2$ for some $t > t_0$.

The continuity of $y$ on $[t_0, \infty]$ and the definition of $K_1$ show that for each $K \in [K_1, K_2]$ there is at least one indices $i_K \in \{1, \ldots, d\}$ and a number $t_K > t_0$ such that
\[
y_i(t_K) = K, \quad \text{and} \quad y_j(t) < K, \quad t \in [t_0 - \tau_0, t_K[, \quad j = 1, \ldots, d.
\]

For every $K \in [K_1, K_2]$ choose an index $i_K$ satisfying (66).

For every $j = 1, \ldots, d$ let the set $H_j \subset [t_0, \infty]$ consist of all points $t_K$ ($K \in [K_1, K_2]$) for which (66) holds with $i_K = j$. Then $H_j \subset [t_{K_1}, t_{K_2}]$, and
\[
\bigcup_{j=1}^{d} y_j(H_j) = [K_1, K_2].
\]

It is well known that if $f : [a, b] \to \mathbb{R}$ is an absolutely continuous function, then for every set $H \subseteq [a, b]$ with Lebesgue measure 0, $f(A)$ also has Lebesgue measure 0. Since $y$ is absolutely continuous on $[t_0, t_{K_2}]$, it now follows that there exists
i \in \{1, \ldots, d\} \) such that the outer measure of \( y_i (H) \) is positive. This implies that there is \( K \in [K_1, K_2] \) such that \( y_i \) differentiable at \( t_K \) and
\[
y'_i(t_K) = - a_i(t_K) y_i(t_K) + N_i(y)(t_K).
\]
By using (66), (64) with \( x = y \), and (65), we have
\[
0 \leq y'_i(t_K) = - a_i(t_K) y_i(t_K) + N_i(y)(t_K)
\leq - K \cdot a_i(t_K) + K \cdot N_i(1)(t_K) = - K \cdot (a_i(t_K) - N_i(1)(t_K)) < 0,
\]
which is a contradiction.

(ii) The general case: Let \( \varepsilon > 0 \), and let \( \Lambda_t(t) \) be a diagonal \( d \times d \) matrix whose diagonal entries starting in the upper left corner are \( a_1(t) + \varepsilon, \ldots, a_d(t) + \varepsilon \) for all \( t \geq t_0 \). Consider the homogeneous linear functional differential system
\[
y'(t) = - \Lambda_t(t) y(t) + N(y)(t), \quad t \geq t_0,
\]
and denote by \( y_\varepsilon \) that solution of (67) for which
\[
y_\varepsilon(t) = y(t), \quad t_0 - \tau_0 \leq t \leq t_0.
\]
By Lemma 6.5, \( y_\varepsilon \) is also nonnegative.

Since
\[
a_i(t) + \varepsilon - N_i(1)(t) > 0, \quad t \geq t_0, \quad i = 1, \ldots, d,
\]
the first part of the proof and (68) show that
\[
y_\varepsilon(t) \leq K_1, \quad t \geq t_0.
\]
According to Lemma 6.5, the fundamental kernel \( v \) of (53) is nonnegative. We have from Theorem 5.3 that
\[
y_\varepsilon(t) = y(t) - \varepsilon \int_{t_0}^{t} v(t, s) y_\varepsilon(s) \, ds, \quad t \geq t_0.
\]
Since \( v \) is nonnegative, it now follows from (69) that
\[
y(t) \leq K_1 \cdot \left((1, \ldots, 1)^T + \varepsilon \int_{t_0}^{t} v(t, s) (1, \ldots, 1)^T \, ds\right), \quad t \geq t_0.
\]
By Theorem 5.2 (b), the function
\[
t \mapsto \int_{t_0}^{t} v(t, s) (1, \ldots, 1)^T \, ds, \quad t \geq t_0
\]
is locally bounded, and therefore tending \( \varepsilon \) to zero in (70) we obtain the result.

(b1) Let \( y = (y_1, \ldots, y_d)^T \) be a fixed nonnegative solution of (53), and let \( z = (z_1, \ldots, z_d)^T \in B_{t_0 - \tau_0} \) be defined by
\[
z_i(t) = \exp \left( \int_{t_0 - \tau_0}^{t} \delta(s) \, ds \right) \cdot y_i(t), \quad t \geq t_0 - \tau_0, \quad j = 1, \ldots, d.
\]
Then it is easy to check that \( z \) is a nonnegative solution of the homogeneous system
\[
z'_i(t) = -(a_i(t) - \delta(t)) z_i(t) + \exp \left( \int_{t_0 - \tau_0}^{t} \delta(s) \, ds \right)
\]
\[
\cdot N_i \left( v \mapsto \exp \left( - \int_{t_0 - \tau_0}^{v} \delta(s) \, ds \right) \right) \cdot N \left( v \mapsto \exp \left( - \int_{t_0 - \tau_0}^{\tau_0} \delta(s) \, ds \right) \cdot z(v) \right) \right) (t), \quad t \geq t_0, \quad i = 1, \ldots, d. \tag{72}
\]

We know from (a) that \( y \) is bounded, and hence by (71) and (56), it is enough to prove that \( z \) is also bounded.

The functions
\[
\tilde{a}_i : [t_0, \infty[ \to \mathbb{R}, \quad \tilde{a}_i(t) = a_i(t) - \delta(t), \quad i = 1, \ldots, d
\]
are obviously locally integrable.

It is not hard to show that the operator \( \tilde{N} : B_{t_0 - \tau_0} \to L_{t_0} \) defined by
\[
\tilde{N} (x)(t) = \exp \left( \int_{t_0 - \tau_0}^{t} \delta(s) \, ds \right)
\cdot x(t)
\]
is linear, of Volterra-type, \( \| N (1) \| \)-continuous, nonnegative, and satisfies \( (A_2) \).

The previous two establishments insure that the system (72) has the same structure as the system (53), that is \( (C_1) \) and \( (C_2) \) hold for it with \( \tilde{a}_i, \quad i = 1, \ldots, d \) and \( \tilde{N} \).

It now follows from (a) (by applying it to the system (72)) that \( z \) is bounded if the properties
\[
\tilde{a}_i(t) - \tilde{N}_i(1)(t) \geq 0, \quad t \geq t_0, \quad i = 1, \ldots, d
\]
are valid.

Let \( u \geq t_0 \) be fixed, and let \( x_u, z_u \in B_{t_0 - \tau_0} \) be defined by
\[
x_u (t) = \begin{cases} 
\exp \left( - \int_{t_0 - \tau_0}^{u - \tau(u)} \delta(s) \, ds \right) \cdot \left( \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right), & t_0 - \tau_0 \leq t \leq u \\
\exp \left( - \int_{t_0 - \tau_0}^{t} \delta(s) \, ds \right) \cdot \left( \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right), & t > u
\end{cases}
\]
and
\[
z_u (t) = \begin{cases} 
\exp \left( - \int_{t_0 - \tau_0}^{u - \tau(u)} \delta(s) \, ds \right) \cdot \left( \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right), & t_0 - \tau_0 \leq t < u - \tau(u) \\
\exp \left( - \int_{t_0 - \tau_0}^{t} \delta(s) \, ds \right) \cdot \left( \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right), & t \geq u - \tau(u)
\end{cases}.
\]

Since for all \( x \in B_{t_0 - \tau_0} \) and \( t \geq t_0 \) the vector \( N(x)(t) \) depends only on the restriction of \( x \) to \([t - \tau(t), t] \),
\[
\tilde{N}(1)(t) = \tilde{N}(z_u)(t), \quad t \geq u. \tag{73}
\]
The nonnegativity of $N$ implies that
\[
\hat{N} (z_u) (t) \leq N (x_u) (t), \quad t \geq t_0.
\] (74)

Now, by applying (64) with $x := x_u, K := \exp \left( - \int_{t_0 - \tau_0}^{u - \tau(u)} \delta (s) \, ds \right), \quad t_K := \infty$, we obtain that
\[
N (x_u) (t) \leq \exp \left( - \int_{t_0 - \tau_0}^{u - \tau(u)} \delta (s) \, ds \right) \cdot N (1) (t), \quad t \geq t_0.
\]

According to (73) and (74), it now follows that
\[
\hat{N} (1) (t) \leq \exp \left( - \int_{t_0 - \tau_0}^{t - \tau(t)} \delta (s) \, ds \right) \cdot N (1) (t), \quad t \geq u,
\]
and consequently
\[
\hat{N} (1) (t) \leq \exp \left( - \int_{t_0 - \tau_0}^{t - \tau(t)} \delta (s) \, ds \right) \cdot N (1) (t), \quad t \geq t_0.
\]

Finally, this inequality and (55) yield that
\[
\hat{a}_i (t) - \hat{N}_i (1) (t) \geq a_i (t) - \delta (t)
\]
\[
- \exp \left( \int_{t - \tau(t)}^{t} \delta (s) \, ds \right) \cdot N_i (1) (t) \geq 0, \quad t \geq t_0, \quad i = 1, \ldots, d.
\]

(b2) It is an immediate consequence of (b1).

The proof is complete. \( \square \)

**Proof of Theorem 6.3.** The only interesting cases are
\[
-\infty < k := \min_{1 \leq j \leq d} \liminf_{t \to \infty} \frac{\rho_j (t)}{a_j (t) - N_j (1) (t)} \leq \max_{1 \leq j \leq d} \limsup_{t \to \infty} \frac{\rho_j (t)}{a_j (t) - N_j (1) (t)} =: K < \infty
\] (75)

and
\[
k = K = -\infty \text{ or } k = K = \infty.
\] (76)

Suppose first that (75) holds.

Let $x : [t_0 - \tau_0, \infty[ \to \mathbb{R}^d$ be a fixed solution of (52). In addition, fix an index $i \in \{1, \ldots, d\}$.

For every $\varepsilon > 0$ there exists $T_\varepsilon > \max (t_0, t_1)$ such that
\[
k - \varepsilon \leq \frac{\rho_j (t)}{a_j (t) - N_j (1) (t)} \leq K + \varepsilon, \quad t \geq T_\varepsilon, \quad j = 1, \ldots, d.
\]
Let \( y(x) : [t_0 - \tau_0, \infty[ \to \mathbb{R}^d \) be the solution of the homogeneous system (53) satisfying

\[
y(x)(t) = x(t), \quad t_0 - \tau_0 \leq t \leq t_0.
\]

By applying Theorem 5.3, we obtain

\[
x(t) = y(x)(t) + \int_{t_0}^{t} v(t, s) \rho(s) ds, \quad t \geq t_0,
\]

and thus

\[
x_i(t) = y_i(x)(t) + \int_{t_0}^{t} v^i(t, s) \rho(s) ds, \quad t \geq t_0, \tag{77}
\]

where \( v^i(t, s) \) denotes the \( i \)th row of \( v(t, s) \).

We now use the assumption that \( \lim_{t \to \infty} y(x)(t) = 0 \), and we have that

\[
\limsup_{t \to \infty} x_i(t) = \limsup_{t \to \infty} \int_{t_0}^{t} v^i(t, s) \rho(s) ds.
\]

By applying Lemma 6.7 (b) with \( \vartheta = \rho \), we conclude that

\[
\limsup_{t \to \infty} x_i(t) = \limsup_{t \to \infty} \int_{T_\varepsilon}^{t} v^i(t, s) \rho(s) ds, \tag{78}
\]

and

\[
\liminf_{t \to \infty} x_i(t) = \liminf_{t \to \infty} \int_{T_\varepsilon}^{t} v^i(t, s) \rho(s) ds, \tag{79}
\]

Note that for all \( t \geq T_\varepsilon \)

\[
\int_{T_\varepsilon}^{t} v^i(t, s) \rho(s) ds = \int_{T_\varepsilon}^{t} v^i(t, s) \left( \begin{array}{c}
(a_1(t) - N_1(1)(t)) \frac{\rho_1(t)}{\alpha_1(t) - N_1(1)(t)} \\
\vdots \\
(a_d(t) - N_d(1)(t)) \frac{\rho_d(t)}{\alpha_d(t) - N_d(1)(t)}
\end{array} \right) ds.
\]

By Lemma 6.6 (b), \( v^i(t, s) \) is nonnegative, and thus we have

\[
(k - \varepsilon) \int_{T_\varepsilon}^{t} v^i(t, s) (A(s) 1_d - N(1)(s)) ds \leq \int_{T_\varepsilon}^{t} v^i(t, s) \rho(s) ds
\]

\[
\leq (K + \varepsilon) \int_{T_\varepsilon}^{t} v^i(t, s) (A(s) 1_d - N(1)(s)) ds, \quad t \geq T_\varepsilon.
\]

Now the result follows from (78), (79) and from Lemma 6.7 (c).

We can prove similarly if (76) is satisfied.

The proof is complete. \( \square \)
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