Rendezvous on the Line with Different Speeds and Markers that can be Dropped at Chosen Time.

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Abstract

In this paper we introduce a Linear Program (LP) based formulation of a Rendezvous game with markers on the infinite line and solve it. In this game one player moves at unit speed while the second player moves at a speed bounded by $v_{\text{max}} \leq 1$. We observe that in this setting a slow moving player may have interest to rest still instead of moving. This shows that in some conditions the wait-for-mummy strategy is optimal. We observe as well that the strategies are completely different if the player that holds the marker is the fast or slow one. Interestingly, the marker is not useful when the player without marker moves slowly, i.e. the fast moving player holds the marker.

1 Introduction

In this article we introduce variation of the asymmetric Rendezvous problem on the line that was introduced by Alpern and Gal [7]. In the original setup, two players are placed on a line at a known distance $D$ and move on the line to Rendezvous. The player’s strategies may be different and start at the same time and both players move at the same speed $v = 1$. At the start and while moving the players look in a fixed direction, right or left, say. The directions are chosen randomly each with probability $1/2$. It results that players move either in the direction they look to or the other one, i.e. Forward or Backward. A strategy is a succession of Forward and Backward moves. The optimal solution of this problem is shown to be $13D/8$. Many variations have been proposed showing that even a simple topology as the infinite line leads to interesting problems. Among the hardest seems to be the symmetric
Rendezvous on the line where the two players have to play the same strategy. Partial solutions of this problem are obtained. In [1] a strategy is proposed that ensures the Rendezvous time satisfies $R \leq 5D$. Subsequent strategies are proposed in [9] and [10] that use the same technique as in [1] and reduce the Rendezvous time to $R \leq 2.28338D$ and $R \leq 2.2091D$ respectively. [36] generalizes the technique and improves the bound to $R \leq 2.19653D$. The best known bound $R \leq 2.1287D$ is given in [25].

Some papers deal with the problem where the initial distance between the players is unknown, see for instance [11, 4, 5, 33]. Usually, the distance is characterized by a probability function. Note that if the players use strategies tailored for a known distance $D$ with Rendezvous time $R$ to a problem where it is only known that the distance is bounded by $D$ then the Rendezvous time for this problem is bounded by $R$.

The way time enters the game leads to relevant variations as well. The constraint that the players start at the same time may be relaxed and this leads to asynchronous Rendezvous problems [35, 18]. Asynchronous Rendezvous problems may assume that an adversary chooses the starting delay or the clocks are assumed to drift with different speeds. There are relations between problems where clocks drift at different speeds and ones where players move at different speed [18].

A problem where players move on the line and share similarities with the Rendezvous problem is the Group Search Problem on the Line [16]. This problem is motivated by the evacuation problem where players must simultaneously gather at some point. One may imagine that people need to leave a building and are helped with a line drawn on the floor but do not know the right direction to follow. When players move at different speeds, interesting strategies can be found where a fast player move to help slower players.

Problems where players move on a circle share similarities with problems on the line [27, 29, 24, 28, 22]. Compare to the infinite line the ring is a compact topology but symmetry breaking has to be solved as well to ensure Rendezvous. Tokens may be left by players [17, 23, 21]. For Rendezvous problems on the line [12, 30] present results where markers are used by players. With a more robotic and computer science flavor some problems encompass faulty agents [20, 19].

Rendezvous problems are far from being limited to the infinite line or ring. Problems may be stated for agents moving on the plane, on graphs, on a torus, on networks and so on, see for instance [8, 2, 34]. These problems are different from the ones considered in this paper.

Markers can have different effects. For instance, the game may end at the time the marker is found, i.e. Rendezvous occurs or the marker is found. This would be the case if a phone number is written on the marker. With
such a marker, the game would be close to a version of search-and-rescue
game [32]. This may be seen as a mix of Rendezvous and Search games, see
for instance [14, 15, 13] for search games on graphs and [26, 6, 8] for general
references.

2 Our contributions

In this paper we consider the (synchronous) Rendezvous problem on the
line with known initial distance $D$ where players move at different speeds
and where a marker can be left by one of the players. Without loss of the
generality we assume that one player moves at speed 1 while the second player
moves at speed $v \leq 1$. We show that investigations can be conducted with
linear programming techniques to identify optimal strategies. This is not
the conventional approaches in the literature where the results are usually
guessed and optimality is subsequently proved. The reduction of rendezvous
search game to another formalism to be solved appears in the literature,
see for instance [3]. Here, the reduction to parametric linear programming
has the further advantage that the same method can be applied to compute
different measures of optimality. For instance, the optimization of the last
rendezvous time. Actually, any linear combination of the rendezvous times
can be optimized.

In [31] a similar problem with markers is considered by one of the au-
thor of the present article. However, the techniques of proof are completely
different. The parametric linear programming approach of this article leads
to more precise results than the ad-hoc approach of [31]. Moreover, here we
accommodate to players with different speeds.

3 Problem formulation

We begin by presenting the formalization of the problem as given in [7]. Two
players, $I$ and $II$, are placed at distance $D = 1$ apart on the real line, and
faced in random directions which they call “Forward”. Their common aim is
to minimize the expected amount of time required to meet. They each know
the distance 1 but not the direction the other player is facing. It is not a
restriction to assume that player I’s starting point is located at position 0
of the line and his speed is bounded by $v \leq 1$. His position is given by a

\footnote{However, the results depend linearly on the initial distance and are stated for general $D$.}
function \( f(t) \in \mathcal{F}(\alpha) \) where

\[
\mathcal{F}(\alpha) = \{ f : [0, T] \rightarrow \mathbb{R}, \ f(0) = 0, |f(t) - f(t')| \leq \alpha |t - t'| \}, \tag{1}
\]

for some \( T \) sufficiently large so that Rendezvous will have taken place.

What are unknown are the initial position of player II that may be \( \pm 1 \) and the Forward direction of player II that may point to the positive or negative side of the infinite line. Again without restriction of the generality we assume that the speed of player II is bounded by 1. Hence, depending on the initial conditions of player II his position at time \( t \) is given by \( \pm 1 \pm g(t) \) with \( g \in \mathcal{F}(1) = \mathcal{F} \).

The Rendezvous times are defined by

\[
t^1 = \min \{ t : f(t) = 1 + g(t) \}. \tag{2}
\]

when player II is originally located at \( +D \) and his Forward direction points the positive side of the line.

\[
t^2 = \min \{ t : f(t) = 1 - g(t) \}. \tag{3}
\]

when player II is originally located at \( +D \) and his Forward direction points the negative side of the line.

\[
t^3 = \min \{ t : f(t) = -1 + g(t) \}. \tag{4}
\]

when player II is originally located at \( -D \) and his Forward direction points the positive side of the line.

\[
t^4 = \min \{ t : f(t) = -1 - g(t) \}. \tag{5}
\]

when player II is originally located at \( -1 \) and his Forward direction points the negative side of the line.

It is common in the literature to speak of 4 agents (of player II) located at positions \( \pm 1 \) and with Forward direction \( \pm 1 \) and moving concurrently. Player I need to Rendezvous with the four agents to end the game [5]. Concretely,

- agent 1 is located at \( +1 \) with Forward direction \( +1 \) and its Rendezvous time is \( t^1 \),
- agent 2 is located at \( +1 \) with Forward direction \( -1 \) and its Rendezvous time is \( t^2 \),
- agent 3 is located at \( -1 \) with Forward direction \( +1 \) and its Rendezvous time is \( t^3 \),
• agent 4 is located at −1 with Forward direction −1 and its Rendezvous time is \( t^4 \).

The notation \( t_1 \leq t_2 \leq t_3 \leq t_4 \) denote the Rendezvous times in the order they occur and \((o_i, b_i)\) denote the agent with origin \( o_i = \pm 1 \) and Forward direction \( b_i = \pm 1 \). The order of the Rendezvous times is given by the index \( i \), \( t_1 \) is Rendezvous with agent \((o_1, b_1)\), \( t_2 \) is Rendezvous with agent \((o_2, b_2)\), \( t_3 \) is Rendezvous with agent \((o_3, b_3)\), \( t_4 \) is Rendezvous with agent \((o_4, b_4)\). When necessary we use the convention \( t_0 = 0 \)

The Rendezvous value \( R(f, g) \) is defined to be the average value

\[
R(f, g) = \frac{1}{4}(t^1 + t^2 + t^3 + t^4).
\]

Finally, the Rendezvous value of the game is defined by

\[
R = \min \left\{ R(f, g) : f \in \mathcal{F}(v), g \in \mathcal{F} \right\}.
\]  

(6)

A first remark that simplifies the problem is that the functional spaces \( \mathcal{F}(v), \mathcal{F} \) may be reduced to consider only functions \( f \in \mathcal{F}(v), g \in \mathcal{F} \) whose speed is constant between the Rendezvous times. Indeed, if the speed is not constant, moving at the average speed between Rendezvous times leads to the same Rendezvous value. Moreover, similarly to Lemma 5.1 of [7] or Theorem 16.10 of [8] or Proposition 3 of [30] we have the following result for \( g \in \mathcal{F} \).

**Proposition 1.** If \( v \leq 1 \) then for the optimal strategies the function \( g \in \mathcal{F} \) is of constant slope equal at \( \pm 1 \) between the Rendezvous, i.e. the speed of the fast player is always maximal.

**Proof.** Let us assume that player II whose position is given by function \( g \in \mathcal{F} \) and initial position does not move at maximal speed between Rendezvous times \( t_{i-1} < t_i \). This means that player II can reach the Rendezvous position at a time \( t_i - \epsilon \) with \( \epsilon > 0 \). By moving faster it may happen that player II Rendezvous with player I before time \( t_i \) reducing the Rendezvous time \( t_i \). If not we modify the trajectory of player II in the following way. After reaching the Rendezvous point at time \( t_i - \epsilon \) player II continues in the same direction for a period \( \epsilon/2 \) and then goes the other way for a period \( \epsilon/2 \) back to the Rendezvous position at time \( t_i \). At time \( t - \epsilon/2 \) player I must be at a distance less than \( \epsilon v/2 \) to the Rendezvous position and because player II is at a distance \( \epsilon/2 \) and \( \epsilon/2 \geq \epsilon v/2 \) the Rendezvous must occurs before time \( t_i \). To summarize, by moving at full speed player II always reduces the Rendezvous time \( t_i \). After time \( t_i \) player II follows the original strategy and the remaining
Rendezvous times are not changed. In total the modified strategy reduces the Rendezvous value showing that the original strategy is not optimal. We emphasize the fast moving player moves at maximal speed while the slow moving player can move at any speed in $[0, v]$.

Corollary 2. We assume here that the speed of player I is bounded by $v \leq 1$ and the speed of player II by 1, i.e. $f \in \mathcal{F}(v)$, $g \in \mathcal{F}$. The sets of optimal strategies $(f, g)$ for player I and II respectively are given by

$$f(t) = \begin{cases} 
  v_1 \cdot t, & t \in [0, t_1] \\
  v_1 \cdot t_1 + v_2 \cdot (t - t_1), & t \in [t_1, t_2] \\
  v_1 \cdot t_1 + v_2 \cdot (t_2 - t_1) + v_3 \cdot (t - t_2), & t \in [t_2, t_3] \\
  v_1 \cdot t_1 + v_2 \cdot (t_2 - t_1) + v_3 \cdot (t_3 - t_2) + v_4 \cdot (t - t_3), & t \in [t_3, t_4] 
\end{cases}$$

(7)

$$g(t) = \begin{cases} 
  d_1 \cdot t, & t \in [0, t_1] \\
  d_1 \cdot t_1 + d_2 \cdot (t - t_1), & t \in [t_1, t_2] \\
  d_1 \cdot t_1 + d_2 \cdot (t_2 - t_1) + d_3 \cdot (t - t_2), & t \in [t_2, t_3] \\
  d_1 \cdot t_1 + d_2 \cdot (t_2 - t_1) + d_3 \cdot (t_3 - t_2) + d_4 \cdot (t - t_3), & t \in [t_3, t_4] 
\end{cases}$$

(8)

with $v_i \in [-v, v]$ and $d_i = \pm 1$ and $t_1 \leq t_2 \leq t_3 \leq t_4$ are the Rendezvous times.

Proposition 1 and Corollary 2 are not new and constantly used in the literature, see for instance Chapter 17.1 of [8]. We stress that player I having the smallest speed bound may move at a slower speed than the maximal one. Indeed, we will observe that for $v$ small optimal strategy for player I is to not move before $t_2$. The “wait for mummy” strategy is then optimal for starting the game.

We consider that player I has at disposal a marker that may be left at a chosen time. The marker helps player II that stops following the strategy after finding the marker and continues in the same direction at maximal speed until Rendezvousing with player I. The same arguments as the ones in Proposition 1 and Corollary 2 or Proposition 3 of [30] show that player I move at constant velocity before and after dropping the marker. There are four different cases to consider for the formulation of the problem depending on which interval, $[0, t_1], [t_1, t_2], [t_2, t_3], [t_3, t_4]$ player I drops off the marker. This leads to the following Proposition that characterizes the optimal strategies.

Corollary 3. When player I has a marker that can be dropped off at chosen
time $z$, the set of optimal strategies $f$ for player I are given by

$$
f(t) = \begin{cases} 
  v_0 \cdot t, & t \in [0, z] \\
  v_0 \cdot z + v_1 \cdot (t - z), & t \in [z, t_1] \\
  v_0 \cdot z + v_1 \cdot (t - z) + v_2 \cdot (t - t_1), & t \in [t_1, t_2] \\
  v_0 \cdot z + v_1 \cdot (t - z) + v_2 \cdot (t - t_1) + v_3 \cdot (t - t_2), & t \in [t_2, t_3] \\
  v_0 \cdot z + v_1 \cdot (t - z) + v_2 \cdot (t - t_1) + v_3 \cdot (t_3 - t_2) + v_4 \cdot (t - t_3), & t \in [t_3, t_4] 
\end{cases}
$$

(9)

if $z \in [0, t_1]$.

$$
f(t) = \begin{cases} 
  v_1 \cdot t, & t \in [0, t_3] \\
  v_1 \cdot t_1 + v_0 \cdot (t - t_1), & t \in [t_1, z] \\
  v_1 \cdot t_1 + v_0 \cdot (z - t_1) + v_2 \cdot (t - z), & t \in [z, t_2] \\
  v_1 \cdot t_1 + v_0 \cdot (z - t_1) + v_2 \cdot (t - t_1) + v_3 \cdot (t - t_2), & t \in [t_2, t_3] \\
  v_1 \cdot t_1 + v_0 \cdot (z - t_1) + v_2 \cdot (t_3 - t_2) + v_3 \cdot (t_3 - t_2) + v_4 \cdot (t - t_3), & t \in [t_3, t_4] 
\end{cases}
$$

(10)

if $z \in [t_1, t_2]$.

$$
f(t) = \begin{cases} 
  v_1 \cdot t, & t \in [0, t_1] \\
  v_1 \cdot t_1 + v_2 \cdot (t - t_1), & t \in [t_1, t_2] \\
  v_1 \cdot t_1 + v_2 \cdot (t_2 - t_1) + v_0 \cdot (t - t_2), & t \in [t_2, z] \\
  v_1 \cdot t_1 + v_2 \cdot (t_2 - t_1) + v_0 \cdot (z - t_2) + v_3 \cdot (t - z), & t \in [z, t_3] \\
  v_1 \cdot t_1 + v_2 \cdot (t_2 - t_1) + v_0 \cdot (z - t_2) + v_3 \cdot (t_3 - z) + v_4 \cdot (t - t_3), & t \in [t_3, t_4] 
\end{cases}
$$

(11)

if $z \in [t_2, t_3]$.

$$
f(t) = \begin{cases} 
  v_1 \cdot t, & t \in [0, t_1] \\
  v_1 \cdot t_1 + v_2 \cdot (t - t_1), & t \in [t_1, t_2] \\
  v_1 \cdot t_1 + v_2 \cdot (t_2 - t_1) + v_3 \cdot (t - t_2), & t \in [t_2, t_3] \\
  v_1 \cdot t_1 + v_2 \cdot (t_2 - t_1) + v_3 \cdot (t_3 - t_2) + v_0 \cdot (t - t_3), & t \in [t_3, z] \\
  v_1 \cdot t_1 + v_2 \cdot (t_2 - t_1) + v_3 \cdot (t_3 - t_2) + v_0 \cdot (z - t_3) + v_4 \cdot (t - z), & t \in [z, t_4] 
\end{cases}
$$

(12)

if $z \in [t_3, t_4]$. The optimal strategies for player II are still of the form of Equation (6). In any cases the parameters are constrained to $v_i \in [-v, v]$ ($v \leq 1$) and $d_i = \pm 1$ and $t_1 \leq t_2 \leq t_3 \leq t_4$ are the Rendezvous times.

Corollaries 2 and 3 are very useful in making the Rendezvous value of the game given by Equation 6 computable. Indeed, the set of functions $(f, g)$ to
be considered is finite. Notice that for the problem with marker the set is finite provided that \( v \) is fixed.

It is crucial to point out that in Corollary 3 the optimal strategy of player II is of the form of Equation \((8)\), but if the marker is found at some time player II no longer follows the strategy but continues in the same direction thereafter. For instance, if the marker is found in the interval \([0, t_1]\) at time \( t_z \) we must have \( z \leq t_z \) and the condition

\[
o + d_1bt_z = v_o z,
\]

must be satisfied where \( o \) is the original starting point of player II \((o = \pm 1)\) and \( b \) is the Forward direction of player II \((b = \pm 1)\). Indeed, the condition states that player II starting at position \( o \) at time 0 is at the marker’s position at time \( t_z \), i.e. the marker is found. The coefficient \( d_1 \) is given by the strategy followed by player II and \( d_1 \cdot b \) is the effective motion depending on the Forward direction \( b \). Thereafter, player II does not follow the strategy but continues in the same direction, i.e. substitutes \( d_1 \) for \( d_i \) in Equation \((8)\).

Hence, if Rendezvous does not occur in \([0, t_1]\) player II motion is given by

\[
o + d_1bt_z + d_1b(t_1 - t_z) = o + d_1bt_1.
\]

If no Rendezvous occurs in the next time interval \([t_1, t_2]\) the motion of player II is given by

\[
o + d_1bt_1 + d_1b(t_2 - t_1),
\]

(compare with the second line of Equation \((8)\)). The same reasoning applies for the next time intervals \([t_2, t_3]\) and \([t_3, t_4]\).

### 4 Solution of the problem without marker

The optimal strategies of the problem without marker are given in Corollary 2. There are 8 unknowns \( v_i \) and \( d_i \), the Rendezvous time being a consequence of these variables. To reduce the problem to a family of linear programs we rewrite the strategies to remove the products \( v_i \cdot t \) by introducing new variables \( vt_i \) with the bounds \( 0 \leq vt_i \leq v \cdot (t_i - t_{i-1}) \) with \( v \) is the maximal speed of player I. To take into account that the speed of player I may be negative (when player I is going Backward) we introduce new variables \( a_i = \pm 1 \) and the motion is computed relatively to \( a_i \cdot vt_i \). Notice that \( a_i \) is a parameter which is fixed before calling the LP-solver (hence the problem is still linear).

To define the meeting times \( t_i \) it is needed to specify in which order they occur. Ordering the meeting times amounts to choose a permutation \( \sigma \) of \( \{1, 2, 3, 4\} \) such that \( t_i = t^{\sigma(i)} \) where \( t^i \) are defined by equations \((2),(3),(4),(5)\).
For this, we introduce new variables \((o_i, b_i)\) with \(o_i = \pm 1\) and \(b_i = \pm 1\) to refer to specific agents of player II. Concretely,

- agent 1 is referred by \((o_i = +1, b_i = +1)\) and the Rendezvous time \(t^1\) is defined by (2).
- agent 2 is referred by \((o_i = +1, b_i = -1)\) and the Rendezvous time \(t^2\) is defined by (3).
- agent 3 is referred by \((o_i = -1, b_i = +1)\) and the Rendezvous time \(t^3\) is defined by (4).
- agent 4 is referred by \((o_i = -1, b_i = -1)\) and the Rendezvous time \(t^4\) is defined by (5).

Rendezvous always occur in the order \((o_1, b_1), (o_2, b_2), (o_3, b_3), (o_4, b_4)\).

The values of \(o_i\) and \(b_i\):

\[
\begin{align*}
\min_{\Delta_i} & \quad t_1 + t_2 + t_3 + t_4 \nonumber \\
& o_1 + d_1 b_1 t_1 = a_1 v t_1 \nonumber \\
& o_2 + d_1 b_2 t_1 + d_2 b_2 (t_2 - t_1) = a_1 v t_1 + a_2 v t_2 \nonumber \\
& o_3 + d_1 b_3 t_1 + d_2 b_3 (t_2 - t_1) + d_3 b_3 (t_3 - t_2) = a_1 v t_1 + a_2 v t_2 + a_3 v t_3 \nonumber \\
& o_4 + d_1 b_4 t_1 + d_2 b_4 (t_2 - t_1) + d_3 b_4 (t_3 - t_2) + d_4 b_4 (t_4 - t_3) = a_1 v t_1 + \nonumber \\
& \quad a_2 v t_2 + a_3 v t_3 + a_4 v t_4 \nonumber \\
& 0 \leq v t_1 \leq v \cdot t_1 \nonumber \\
& 0 \leq v t_2 \leq v \cdot (t_2 - t_1) \nonumber \\
& 0 \leq v t_3 \leq v \cdot (t_3 - t_2) \nonumber \\
& 0 \leq v t_4 \leq v \cdot (t_4 - t_3) \nonumber
\end{align*}
\]

\(a_i, b_i, o_i, d_i \in \{0, 1\}\) \(\sum o_i = 0, \sum d_i = 0, \ o_i = o_j \Rightarrow d_i \neq d_j,\)

\(t_1 \geq 0, t_2 \geq 0, t_3 \geq 0, t_4 \geq 0\)

For the computation of the solution, we used the variables \(o_i, b_i, d_i, a_i\) as parameters. For each set of values we solve the corresponding linear program. The number of linear programs solved is 1536 which is solved in a few seconds using Python Gurobi library. Notice that by symmetry we fixed \(a_1 = 1\) and \(d_1 = 1\).
Figure 1: Plot of the solution of the Rendezvous problem vs the maximum speed of player I. The computed solution is a plot of 1001 points evaluated at \( n/1000 \), \( n \in [0, 1000] \). The exact solution has two algebraic forms for \( v < (\sqrt{5} - 1)/2 \) (exact <, (13)) or \( v > (\sqrt{5} - 1)/2 \) (exact >, (14)).

The plot of the results are shown on Figure 1, the optimal Rendezvous value is plotted versus the maximal speed of player I for discrete values \( n/1000 \), \( n = 0, 1, \ldots, 1000 \).

Besides the computation of the optimal Rendezvous values we record the corresponding optimal strategy. We observe that for \( v \leq 0.618 \) the optimal strategy is \((a_1 = 0, a_2 = 0, a_3 = 1, a_4 = -1)\) for player I and \((d_1 = 1, d_2 = 1, d_3 = -1, d_4 = -1)\) for player II as illustrated in Figure 2. In terms of the optimal strategy of Corollary 2 the speeds of player I are \( v_1 = 0, v_2 = 0, v_3 = -v, v_4 = v \). Notice that player I can as well play the symmetric strategy \((a_1 = 0, a_2 = 0, a_3 = 1, a_4 = -1)\) which is not illustrated.

It is relevant to observe that until \( t^2 = t^3 \) the wait for mummy strategy is the optimal strategy for player I, motion starts only after. For this strategy the expressions for the Rendezvous times and Rendezvous value of the game
Figure 2: Optimal strategy for \( v \in [0.001, 0.618] \).

are given by:

\[
\begin{align*}
    t_1 &= 1, \\
    t_2 &= 1, \\
    t_3 &= 3 + v, \\
    t_4 &= \frac{v^2 + 8v + 3}{(1 + v)^2},
\end{align*}
\]

\[
R = \frac{4v^2 + 16v + 8}{(1 + v)^2}. \tag{13}
\]

Since \( v = 0.619 \) the optimal strategy is a switch to \( a_1 = 1, a_2 = -1, a_3 = 1, a_4 = -1 \) for player I and \( d_1 = 1, d_2 = 1, d_3 = -1, d_4 = -1 \). In terms of the optimal strategy of Corollary 2 the speeds of player I are \( v_1 = -v, v_2 = v, v_3 = -v, v_4 = v \). The speed of player I is large enough to allow improving the Rendezvous value by moving from the start. The expressions for the Rendezvous times and Rendezvous value of the game are given by:

\[
\begin{align*}
    t_1 &= \frac{1}{v + 1}, \\
    t_2 &= \frac{3v + 1}{(v + 1)^2}, \\
    t_3 &= \frac{5v + 3}{(v + 1)^2}, \\
    t_4 &= \frac{7v^2 + 14v + 3}{(1 + v)^3}, \\
    R &= \frac{16v^2 + 28v + 8}{(1 + v)^3}. \tag{14}
\end{align*}
\]

With direct computations we see that (13) is better that (14) for \( v \leq (\sqrt{5} - 1)/2 \), see Figure 1.

It is stated in [8], Chapter 17.1 that the optimal solution is given by (13) for \( v \leq (\sqrt{5} - 1)/2 \) and by (14) for \( v \geq (\sqrt{5} - 1)/2 \). With our linear programming approach we first conclude that (13) is optimal for \( v = n/1000 \leq \)
Figure 3: Optimal strategy for $v \in [0.619, 0.990]$.

$(\sqrt{5} - 1)/2$ and by (14) for $v = n/1000 \geq (\sqrt{5} - 1)/2$, $n = 0, 1, 2, \ldots$ (discrete values).

However, we can say more. Let us denote $opt(v)$ the function that returns the optimal value of the game when the speed of player I is bounded by $v$ and $nextToopt(v)$ the function that returns the next to optimal value of the game when the speed of player I is bounded by $v$. These two functions are decreasing since a strategy for $v$ is always a strategy for $v' \geq v$. Hence, if we have that $opt(v) < nextToopt(v + dv)$ and the strategy at $opt(v)$ is the same as $opt(v + dv)$ it must be that this strategy is optimal in the interval $[v, v + dv]$. By computing $opt(n/1000)$ and $nextToopt(n/1000)$, $n = 0, \ldots, 1000$ we detect that the condition stated above is satisfied for $v \in [1/1000, 0.618]$ and for $v \in [0.619, 0.990]$. To summarize, we have proved

**Theorem 4.** ([5], Chapter 17.1) For $v \in [1/1000, 0.618]$ the Rendezvous value of the game is given by (13) and the optimal strategy is plotted on Figure 2 and for $v \in [0.619, 0.990]$ the Rendezvous value of the game is given by (14) and the optimal strategy is plotted on Figure 3.

The interval in which the Theorem is stated to be true may be enlarged by computing the numerical solutions on a finer mesh, i.e. increasing the number values of $n$ for which we solve the LP.
5 Solution of the problem with marker held by the slow player

The marker is held by one of the player and may be dropped off at any given time. Once dropped off, the marker is to be found by the other player when it passes at the location. Once found, the player stops following the original strategy and continues in the same direction until Rendezvous occurs. We first assume that the marker is held by player I that moves with the lowest speed bounded by \( v \leq 1 \) and denote \( z \) the dropping time. There are 4 possibilities, \( z \in [0, t_1] \), \( z \in [t_1, t_2] \), \( z \in [t_2, t_3] \), \( z \in [t_3, t_4] \). Each one leading to a family of linear programs to solve. It occurs that only the first case \( z \in [0, t_1] \) is relevant. For all other cases the optimal solutions do not make use of the marker and are given in Section 4.

The strategy of player I is now given by \((a_0, a_1, a_2, a_3, a_4)\) where \( a_i \) indicates whether \( v_i \) is positive or negative, i.e. the speed \( v_i \) are always assumed positive and the motions are depending on the product \( a_i \cdot v_i \). With respect to Section 4 the only novelty is the introduction of \( a_0 \) see Equation 9 (compare with (7) with no marker).

Agents of player II can find the marker or not. Hence, for each agent we must generate two linear programs each one assuming the agent find the marker or not. Actually, for the first agent Rendezvousing (agent \((o_1, d_1)\)) we do not need to make a difference if the marker is found or not, equations are the same. We use the new variable \( k_1 \) to indicate that agent \((o_2, d_2)\) finds the marker \( k_1 = 1 \) or not \((k_1 = 0)\) in the interval \([0, t_1]\). Again if the marker is found by \((o_2, d_2)\) in the interval \([t_1, t_2]\) the equations do not change. The variable \( k_{21}, k_{22} \) indicate that agent \((o_3, d_3)\) finds the marker in the interval \([0, t_1]\) or \([t_1, t_2]\) respectively. And finally, the variable \( k_{31}, k_{32}, k_{33} \) indicate that agent \((o_4, d_4)\) finds the marker in the interval \([0, t_1]\) or \([t_1, t_2]\) or \([t_2, t_3]\) respectively.

This leads to the family of linear programs shown in Equations (15), (16), (17) (we denote \( \Delta t_i = (t_i - t_{i-1}) \) and \( t_{iz} \) the time at which the marker is found by \((o_i, b_i))\).
Rendezvous times. There are four sets of equations, the number given in the Introduction and results are the average of the
be solved. Notice that we minimize the sum of the Rendezvous time while
\[
\min_{\Delta_i} t_1 + t_2 + t_3 + t_4
\]
\[
o_1 + d_1b_1t_1 = a_0vz + a_1vt_1
\]
\[
k_1(o_2 + d_1b_2t_1 + d_2b_2\Delta t_2) +
(1 - k_1)(o_2 + d_1b_2t_{1z} + d_1b_2(t_1 + \Delta t_2 - t_{1z})) = a_0vz + a_1vt_1 + a_2vt_2
\]
\[
(1 - k_1)(o_2 + d_1b_2t_{1z}) = (1 - k_1)a_0vz
\]
\[
k_{21}k_{22}(o_3 + d_1b_3t_1 + d_2b_3\Delta t_2 + d_3b_3\Delta t_3) +
(1 - k_{21})(o_3 + d_1b_3t_{2z} + d_1b_3(t_1 + \Delta t_2 + \Delta t_3 - t_{2z})) +
(1 - k_{22})(o_3 + d_1b_3t_1 + d_2b_3t_{2z} + d_3b_3(\Delta t_2 + \Delta t_3 - t_{2z})) =
a_0vz + a_1vt_1 + a_2vt_2 + a_3vt_3
\]
\[
(1 - k_{21})(o_3 + d_1b_3t_{2z}) + (1 - k_{22})(o_3 + d_1b_3t_1 + d_2b_3t_{2z}) =
(1 - k_{21})(1 - k_{22})a_0vz
\]
\[
(1 - k_{31})(1 - k_{32})(1 - k_{33})(o_4 + d_1b_4t_1 + d_2b_4\Delta t_2 + d_3b_4\Delta t_3 + d_4b_4\Delta t_4) +
(1 - k_{31})(o_4 + d_1b_4t_{3z} + d_1b_4(t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4 - t_{3z})) +
(1 - k_{32})(o_4 + d_1b_4t_1 + d_2b_4t_{3z} + d_3b_4(\Delta t_2 + \Delta t_3 + \Delta t_4 - t_{3z})) +
(1 - k_{33})(o_4 + d_3b_4t_1 + d_2b_4\Delta t_2 + d_3b_4t_{3z} + d_4b_4(\Delta t_3 + \Delta t_4 - t_{3z})) =
a_0vz + a_1vt_1 + a_2vt_2 + a_3vt_3 + a_4vt_4
\]
\[
(1 - k_{31})(o_4 + d_1b_4t_{3z}) + (1 - k_{32})(o_4 + d_1b_4t_1 + d_2b_4t_{3z}) +
(1 - k_{33})(o_4 + d_1b_4t_1 + d_2b_4\Delta t_2 + d_3b_4t_{3z}) =
(1 - k_{31})(1 - k_{32})(1 - k_{33})
\]

In this set of equations, the first one is the minimization problem to be solved. Notice that we minimize the sum of the Rendezvous time while the number given in the Introduction and results are the average of the Rendezvous times. There are four sets of equations, (*), (**) , (***) , (****). Equation (*) is the constraint that player I Rendezvous with agent \((a_1, b_1)\) at time \(t_1\). Agent \((a_1, b_1)\) may find the marker before time \(t_1\). However, in this case the optimal solution continues in the same direction, i.e. the equation would be
\[
k(o_1 + d_1b_1t_1) + (1 - k)(o_1 + d_1* t_{0z}d_1b_1(t_1 - t_{0z})) = a_0vz + a_1vt_1,
\]
where $k = 1$ if player II does not find the marker and $k = 0$ else and $t_oz$ is the time at which player II finds the marker. This equation reduces to $(\ast)$. In $(\ast)$ if player finds the marker in the interval $[t_1, t_2]$ the optimal strategy is to continue the same direction and the marker is useless, and so on for $(\ast \ast \ast)$ and $(\ast \ast \ast \ast)$ if the marker is found in the interval $[t_2, t_3]$ and $[t_3, t_4]$ respectively.

The three remaining sets of equations $(\ast \ast), (\ast \ast \ast), (\ast \ast \ast \ast)$ are composed of two equations. The first one accounts for the Rendezvous of agent $(o_i, b_i)$ with player I and the second one is valid only if the marker is used ($k_1, k_21, k_22, k_31, k_32, k_33$ equal 1) and define the times when the marker is found $t_1z, t_2z, t_3z$.

The next set of equations is composed of the speed constraints. The variable $vz$ is the product of the speed of player I and the time $z$ of dropping the marker, this product is bounded by $v \cdot z$ since the speed of player I is bounded by $v$. In the results we observe that the speed of player I is $v$ (maximal) or 0 but we obtain no solution with $v$ in between.

\begin{align*}
0 \leq vz & \leq v \cdot z \\
0 \leq vt_1 & \leq v \cdot (t_1 - z) \\
0 \leq vt_2 & \leq v \cdot \Delta t_2 \\
0 \leq vt_3 & \leq v \cdot \Delta t_3 \\
0 \leq vt_4 & \leq v \cdot \Delta t_4
\end{align*}

The family of linear programs is generated by assigning values to the parameters of equations (15), (16), (17). These values must satisfy the constraints (the constraints $t_i \geq 0$ are included in all linear programs)

\begin{align*}
a_i, b_i, o_i, d_i & \in \{0, 1\} \sum o_i = 0, \sum d_i = 0, \quad o_i = o_j \Rightarrow d_i \neq d_j, \\
t_1 \geq 0, t_2 \geq 0, t_3 \geq 0, t_4 \geq 0.
\end{align*}

The families of linear programs are solved for maximal speed of player I ranging from 0 to 1 with a step size of 1/1000, i.e. optimal solutions $opt(v)$ are computed for $v = n/1000$, $n = 0, 1, \ldots, 1000$. The result is that the same strategy is used, see Figure 4. The speed of player I is maximal along the trajectory and the best solution is obtained for $z \in [0, t_1]$. With respect to the notation of Corollary 3 the speeds of player I are $(v_0 = -v, v_1 = v, v_2 = -v, v_3 = v)$. The marker reduces the Rendezvous value even when the speed of player I is very slow.
The Rendezvous times are given by
\[ z = \frac{1}{v+3}, \quad t_1 = \frac{3}{v+3}, \quad t_2 = \frac{5v + 3}{(v + 1)(v + 3)}, \]
\[ t_3 = \frac{7v^2 + 12v + 9}{(v + 1)^2(v + 3)}, \quad t_4 = \frac{9v^3 + 27v^2 + 35v + 9}{(v + 1)^3(v + 3)}, \]
(18)
\[ R = \frac{24v^3 + 68v^2 + 76v + 24}{(v + 1)^3(v + 3)}. \]

The optimal solution function \( \text{opt}(v) \) is decreasing because a strategy for \( v \) is a strategy for \( v' \geq v \) as well. Hence, if the value of the next to optimal strategy for speed \( v + dv \), \( \text{nextToopt}(v + dv) \) is larger than \( \text{opt}(v) \) the strategy that leads to \( \text{opt}(v) \) is optimal for speeds in \( [v, v + dv] \). With our mesh size of 1/1000 we numerically observe that this occurs since \( v \geq 17/1000 \). Hence we have a computer assisted proof summarized in the following Theorem.

**Theorem 5.** For \( v \in [17/1000, 1] \) the Rendezvous value of the game is given by (18). The optimal strategy is plotted on Figure 4 and the optimal Rendezvous values on Figure 5.

The interval on which the Theorem is stated to be true may be enlarged by computing the numerical solutions on a finer mesh. It is relevant to point
out that for the optimal strategy the time at which the marker is found is $t_1$, i.e. the same time as the first Rendezvous occurs.

6 Solution of the problem with marker held by the fast player

The family of linear programs to be solved when the marker is held by the fast player (I) is very similar to the one defined by Equations (15), (16), (17). The changes are that the coefficients $a_i$ are no longer multiplied by the maximal speed $v$ as are now the coefficients $d_i$. The system of equations is not reproduced here to save some space.

The results are plotted on Figure 6. We observe that for speeds slower than $v \approx 0.805$ the marker is not useful and the optimal solution is given by the optimal solutions without marker stated in Theorem 4. For speeds faster than $v \approx 0.805$ the marker start to be useful and the strategy is similar to the optimal one when the marker is held by the slow player, see Figure 4. When the fast player holds the marker we obtain that the Rendezvous times
are given by:

\[
\begin{align*}
z &= \frac{1}{3v+1}, \quad t_1 = \frac{3}{3v+1}, \quad t_2 = \frac{3v+5}{(v+1)(3v+1)}, \\
t_3 &= \frac{9v+5}{(v+1)(3v+1)}, \quad t_4 = \frac{3v+7}{(v+1)^2}, \\
R &= \frac{24v^2 + 52v + 20}{(v+1)^2(3v+1)}.
\end{align*}
\]

(19)

Figure 6: Optimal solutions with marker, i.e (13) for \( v \leq (\sqrt{5} - 1)/2 \), (14) for \( v \geq (\sqrt{5} - 1)/2 \), (19) for \( v > 0.805 \), and without marker. The fast player holds the marker.

The optimal Rendezvous value is decreasing with the maximal speed \( v \) because a strategy for maximal speed \( v \) is a strategy for maximal speed \( v' \geq v \). Moreover, if we denote \( \text{opt}(v) \) the optimal Rendezvous value for maximal speed \( v \), and \( \text{nextToopt}(v) \) the next to optimal Rendezvous value, it follows that if \( \text{nextToopt}(v + dv) \geq \text{opt}(v) \) and the strategy leading to \( \text{opt}(v) \) and \( \text{opt}(v + dv) \) is the same \( \Rightarrow \) the strategy is optimal on the entire interval \([v, v + dv]\). By numerical computation and using the two stated observations we obtain the following Theorem.

**Theorem 6.** For \( v \in [1/1000, 0.618] \) the Rendezvous value of the game is given by (13) and the optimal strategy is plotted on Figure 2 and for
$v \in [0.619, 0.805]$ the Rendezvous value of the game is given by (14) and the optimal strategy is plotted on Figure 3 and for $v \in [0.807, 0.966]$ the Rendezvous value of the game is given by (19), the Rendezvous values are plotted on Figure 6 and the optimal strategy is plotted on Figure 7.

Figure 7: Optimal solutions with marker, i.e. (19) for $v > 0.805$, for lower speeds the optimal solutions are the ones without marker (Theorem 4). The fast player holds the marker.

It is relevant to point out the difference between the optimal strategies when the marker is held by the slow (Figure 4) or fast player (Figure 7). After Rendezvous time $t_2$ on Figure 4 the slow player (who holds the marker) turns while on Figure 7 the fast player (who holds the marker) continues on his way. The transition from the two strategies is ‘continuous’ in the sense that when the speeds are equal at time $t_2$ the two remaining agents to be found are at equal distances from player I. Hence, both strategies are optimal (turning or continuing).

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