Efficient numerical method of the fiber Bragg grating synthesis

O.V. Belai,* L.L. Frumin,† E.V. Podivilov,* D.A. Shapiro,*
* Institute of Automation and Electrometry, Siberian Branch, Russian Academy of Sciences, 1 Koptjug Ave, Novosibirsk, 630090 Russia;
† Novosibirsk State University, 2 Pirogov St., Novosibirsk 630090, Russia

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Abstract
A new numerical method is developed for solution of the Gel’fand–Levitan–Marchenko inverse scattering integral equations. The method is based on the fast inversion procedure of a Toeplitz Hermitian matrix and special bordering technique. The method is highly competitive with the known discrete layer peeling method in speed and exceeds it noticeably in accuracy at high reflectance.

1 Introduction

Promising technological applications of fiber Bragg gratings (FBG) [1] stimulate research and development of numerical methods of their synthesis. The propagation of counter-directional waves in single-mode fiber with quasi-sinusoidal refractive index modulation is described by coupled wave differential equations [2]. Calculation of reflection coefficient $r(\omega)$ from given coordinate dependence of the refractive index is the direct scattering problem. The inverse scattering problem consists in recovery of the refractive index from given frequency dependence of the reflection coefficient $r(\omega)$. In mathematical physics the inverse problem for coupled wave equations reduces to
coupled Gel’fand–Levitan–Marchenko (GLM) integral equations [3]. However, the straightforward numerical solution of the GLM equations is usually considered as too complicated for practical FBG synthesis. At first sight it requires $N^4$ operations, where $N$ is the number of discrete points along the grating.

Since, the solution of integral equations seems to be inefficient, other numerical methods of FBG synthesis are elaborated. In particular, iterative methods with $lN^3$ operations are widespread, where $l$ is the number of iterations necessary for convergence. For instance, they are successive kernel approximations by Frangos and Jaggard [4], high-order Born approximations by Peral et al [5] or advanced algorithm by Poladian [6] which uses information about the reflection characteristics from both ends of the grating. Sometimes additional approximations are applied. For example, Song and Shin [7] approximate the reflection spectrum by a rational function or Ahmad and Razzagh [8] approximate the kernel function of integral equations by polynomials.

The alternative approach is the layer peeling method known from quantum mechanics and geophysics and applied for FBG synthesis by Feced et al [9], Poladian [10] and Skaar et al [11]. The method has a clear physical interpretation of the reflected signal as a superposition of impulse responses from different uniform layers or point reflectors placed along the grating. Each thin layer has small reflectivity and can be taken into account within the first Born approximation. Because of high efficiency (of the order of $N^2$ operations) this method becomes widely used. The disadvantage of conventional layer peeling is the exponential decay of accuracy along the grating because of error accumulation during the reconstruction process [12]. The comparable efficiency $N^2$ was demonstrated by Xiao and Yashiro [13] who transformed the GLM integral equations to hyperbolic set of partial differential equations and solved it numerically. This approach have several modifications, in particular, Papachristos and Frangos [14] came to second-order partial differential equations and also solved them numerically.

Better results at high reflectance are demonstrated by combination of the iterations and the layer peeling. It is the integral layer peeling method proposed by Rosenthal and Horowitz [15]. The grating is divided into thin layers, but layers are not assumed to have uniform profile. The profile of each layer is found by iterative solution of GLM equations.

A recent attempt of straightforward numerical solution was made in [16]. The GLM equations was solved with the help of a bordering procedure and
Cholesky decomposition. This approach takes of the order of $N^3$ operations. The aim of present paper is to propose more efficient numerical algorithm with $O(N^2)$ operations. The improvement is possible due to specific symmetry of the matrix in the discrete GLM equations, the Toeplitz symmetry: the elements of any one diagonal are the same. The Toeplitz symmetry leads to considerable decrease in the number of operations, similar to the fast algorithms by Levinson [17], Trench [18] and Zohar [19]. The proposed method utilizes a modified bordering procedure and a second-order approximation of integrands, the Hermitian symmetry is also taken into account.

The paper is organized as follows. In Sec. 2 the GLM equations are reduced to convenient form for numerical calculation. The algorithm based on the specific “inner-bordering” technique and Toeplitz symmetry is described in Sec. 3. Testing numerical calculations and their comparison with the generalized hyperbolic secant (GHS) exactly solvable profile and discrete layer peeling (DLP) results are summarized in Sec. 4.

2 GLM equations

Let us consider the propagation of light through a grating with refractive index $n + \delta n(x)$ consisting of homogeneous background $n = \text{const}$ and weak modulation $\delta n \ll n$. The refractive index modulation is quasi-sinusoidal

$$\frac{\delta n(x)}{n} = 2\alpha(x) \cos (\kappa x + \theta(x)),$$

where $\kappa$ is the spatial frequency, $\alpha(x)$ is the apodization function [2] and $\theta(x)$ is the phase modulation that describes the chirp of the grating, variation of its spatial frequency. These functions are supposed to be slow-varying, that is, $\alpha' \ll \kappa \alpha$, $\theta' \ll \kappa$, where prime denotes the coordinate derivative. The detuning $\omega = k - \kappa/2$ of the light wave with respect to the grating resonance frequency $k_0 = \kappa/2$ is supposed to be small, $\omega \ll \kappa/2$. The wave propagation can be described by the coupled wave equations:

$$\psi'_1 - i\omega \psi_1 = q^* \psi_2, \quad \psi'_2 + i\omega \psi_2 = q \psi_1,$$

where asterisk denotes the complex conjugation, the coupling coefficient $q(x)$ is defined by $q(x) = -i\alpha(x)k_0e^{-i\theta(x)}$.

The inverse problem for coupled wave equations was studied by Zakharov and Shabat [3], see also [7]. The problem was reduced to the Gelfand —
Levitan — Marchenko coupled integral equations

\[ A_1(x, t) + \int_{-\infty}^{x} R(t + y)A_2^*(x, y) \, dy = 0, \]
\[ A_2(x, t) + \int_{-\infty}^{x} R(t + y)A_1^*(x, y) \, dy = -R(x + t), \quad x > t. \]  \tag{2}

Here

\[ R(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r(\omega)e^{-i\omega t} \, d\omega \]  \tag{3}

is the Fourier transform of the left reflection coefficient \( r(\omega) \). For finite grating in the interval \( 0 \leq x \leq L \) kernel functions \( A_{1,2}(x, t) \) are not equal to zero only within triangular domain \(-x < t < x\). Due to the causality principle the impulse respond function equals zero \( R(t) = 0 \) at \( t < 0 \). Integral equations \eqref{2} are closed in triangular domain \(-x < t < x < L\) and allow one to find the kernel functions \( A_{1,2}(x, t) \) from function \( R(t) \) given in interval \( 0 < t < 2L \). The complex coupling coefficient \( q(x) \) can be found from the synthesis relation

\[ q(x) = 2 \lim_{t \to x-0} A_2(x, t). \]  \tag{4}

For numerical analysis let us introduce more convenient variables \( u(x, s) = A_1^*(x, x - s), \quad v(x, \tau) = A_2(x, \tau - x) \). GLM equations \eqref{2} take the form

\[ u(x, s) + \int_{s}^{2x} R^*(\tau - s)v(x, \tau) \, d\tau = 0, \]
\[ v(x, \tau) + \int_{0}^{\tau} R(\tau - s)u(x, s) \, ds = -R(\tau). \]  \tag{5}

Functions \( u(x, \tau), \quad v(x, \tau) \) are determined in domain \( 0 \leq \tau \leq 2x \leq 2L \). The synthesis relation \eqref{4} can be rewritten as

\[ q(x) = 2v(x, 2x - 0). \]  \tag{6}
The integral operator in equations (5) acting in the space of two-component vectors constructed from functions $u, v$ is Hermitian. Note that function $R$ in integrands of Eq. (5) depends on difference of variables only. This property resulting in Toeplitz symmetry of the matrix obtained by discretization of integral operator is exploited in the next section.

3 Numerical procedure

For numerical solution of Eq. (5) let us consider their discrete analogue. Divide interval $0 \leq \tau \leq 2L$, where function $R(\tau)$ is known, by segments of length $h = 2L/N$. Introduce the discrete variables $\tau_n$, $s_k$, $x_m$ in accordance with

$$
\begin{align*}
    s_k &= h \left( k - \frac{1}{2} \right), \quad k = 1, \ldots, m, \\
    \tau_n &= h \left( n - \frac{1}{2} \right), \quad n = 1, \ldots, m, \\
    x_m &= \frac{mh}{2}, \quad m = 1, \ldots, N.
\end{align*}
$$

(7)

Define grid functions $u_n^{(m)} = u(x_m, \tau_n)$, $v_n^{(m)} = v(x_m, \tau_n)$ and $R_n = R(hn)$. The integrals in (5) can be approximated by the simplest rectangular quadrature scheme or more accurate trapezoidal scheme thus being transformed into sums. The accuracy of the algorithm for rectangular approximation is $O(N^{-1})$, for trapezoidal one it is $O(N^{-2})$.

Discrete form of GLM equations for rectangular approximation is

$$
\begin{align*}
    u_k^{(m)} + h \sum_{n=k}^{m} R_{n-k}^* v_n^{(m)} &= 0, \\
    v_n^{(m)} + h \sum_{k=1}^{n} R_{n-k} u_k^{(m)} &= -R_n,
\end{align*}
$$

(8)

$n, k = 1, \ldots, m, \quad m = 1, \ldots, N.$

The synthesis relation for the complex mode coupling coefficient (6) with accuracy $O(N^{-1})$ is

$$
q^{(m)} = 2v_m^{(m)}. 
$$

(9)
The set (8) at fixed index $m$ can be represented as one matrix equation

$$G^{(m)} w^{(m)} = b^{(m)},$$

(10)

where vector $w^{(m)}$ of dimension $2m$ is arranged from the grid functions $u_n^{(m)}$ and $v_n^{(m)}$, namely,

$$w^{(m)} = \begin{pmatrix} u^{(m)} \\ v^{(m)} \end{pmatrix}.$$ 

Vector $b^{(m)}$ is arranged from the zero vector of dimension $m$ and the vector of dimension $m$ with components $-R_n$. Square $2m \times 2m$ matrix $G^{(m)}$ is a block matrix

$$G^{(m)} = \begin{pmatrix} E & hR^\dagger \\ hR & E \end{pmatrix}.$$ 

(11)

Here $E$ is the identity $m \times m$ matrix, $R = R^{(m)}$ is the lower triangular Toeplitz $m \times m$ matrix of the form

$$R = \begin{pmatrix} R_0 & 0 & 0 & \ldots & 0 \\ R_1 & R_0 & 0 & \ldots & 0 \\ R_2 & R_1 & R_0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ R_{m-1} & R_{m-2} & R_{m-3} & \ldots & R_0 \end{pmatrix}.$$ 

(12)

Matrix $R^\dagger$ is the upper triangular Toeplitz $m \times m$ matrix, that is Hermitian conjugate to matrix $R$. Block matrix $G^{(m)}$ is also Toeplitz and Hermitian.

The solution of the algebraic set (10) can be found by the inversion of matrix $G^{(m)}$ using, for example, the Levinson bordering algorithm [17]. However, we should fulfill much simpler task of finding complex mode coupling coefficient $q^{(m)}$ with the help of (9) which requires only the lower element of vector $w_{2m}^{(m)} = v_m^{(m)}$ to be known. Then the lower row of inverse matrix $(G^{(m)})^{-1}$ is interesting for us first of all. It is known that the inverse matrix to Toeplitz matrix is generally not Toeplitz, but it is persymmetric, i.e., symmetric with respect to the secondary diagonal [20]. Therefore, its lower row is the reflection of its left column

$$f^{(m)} = \begin{pmatrix} f_1^{(m)} \\ \vdots \\ f_{2m}^{(m)} \end{pmatrix}.$$ 

6
with respect to its secondary diagonal. The left column in its turn satisfies the relation
\[
G^{(m)} f^{(m)} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\] (13)

The vector-column in the right hand side of (13) is the first column of the identity matrix $2m \times 2m$.

Let us also account for the Hermitian symmetry of matrix $G^{(m)}$. As known, the matrix inverse to Hermitian is also Hermitian. Owing to persymmetry and hermicity of inverse matrix its right column is
\[
\tilde{f}^{(m)} = \begin{pmatrix} f^{* (m)}_{2m} \\ \vdots \\ f^{* (m)}_1 \end{pmatrix}.
\]

Tilde denotes hereafter the inverted numeration of components along with the complex conjugation. The right column of the inverse matrix satisfies the relation
\[
G^{(m)} \tilde{f}^{(m)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.
\] (14)

The last column of the identity matrix enters the right hand side.

Since the unknown vector $w^{(m)}$ is formed from two vectors of dimension $m$, it is convenient for us to present the left column of the inverse matrix $f^{(m)}$ as a merging of two vectors of dimension $m$:
\[
f^{(m)} = \begin{pmatrix} y^{(m)} \\ z^{(m)} \end{pmatrix}.
\]

The same relations, (13) and (14), are valid for left column $f^{(m+1)}$ and right column $\tilde{f}^{(m+1)}$ of the inverse matrix $(G^{(m+1)})^{-1}$ at the next $(m + 1)$-th step.

Similar to Levinson’s algorithm [17], vectors $y^{(m+1)}$ and $z^{(m+1)}$ at the next $(m + 1)$-th step can be found by means of a bordering procedure from the
vectors known at the previous $m$-th step

$$
y^{(m+1)} = c_m \begin{pmatrix} y^{(m)} \\ 0 \end{pmatrix} + d_m \begin{pmatrix} 0 \\ z^{(m)} \end{pmatrix},
$$

$$
z^{(m+1)} = c_m \begin{pmatrix} z^{(m)} \\ 0 \end{pmatrix} + d_m \begin{pmatrix} 0 \\ \tilde{z}^{(m)} \end{pmatrix}.
$$

(15)

Note that the compound structure of the vectors is just what makes the bordering procedure "inner", since extending vectors $y^{(m)}$, $z^{(m)}$ by zeros means inserting of two rows and two columns into matrix $G^{(m)}$ with one row and one column placed in the middle of the matrix. At the first step we find from $2 \times 2$ matrix $G^{(1)}$ that

$$
y_1^{(1)} = \frac{1}{1 - h^2|R_0|^2}, \quad z_1^{(1)} = \frac{-hR_0}{1 - h^2|R_0|^2}.
$$

Unknown coefficients $c_m$, $d_m$ can be obtained from relations (13) and (14)

$$
c_m = \frac{1}{1 - |\beta^{(m)}|^2}, \quad d_m = \frac{\beta^{(m)}}{1 - |\beta^{(m)}|^2},
$$

(16)

with coefficient $\beta^{(m)}$ computed by formula

$$
\beta^{(m)} = -h \left( R_my_1^{(m)} + R_{m-1}y_2^{(m)} + \cdots + R_1y_m^{(m)} \right).
$$

(17)

Then the last component $v_{m+1}^{(m+1)}$ of vector $w^{(m+1)}$ is calculated as the convolution of the last row of the inverse matrix with right hand side $b^{(m+1)}$.

Actually, the last convolution is excessive, since relation $q^{(m+1)} = 2\beta^{(m+1)}/h$ holds. Thus, the number of arithmetic operations at each $(m+1)$-th step is of the order of $m$. Then the total number of required operations is $N^2$ which is approximately the same as in DLP method.

A great advantage of the new algorithm appears when we use the trapezoidal rule [21], i.e., the piecewise linear approximation of functions. The equations in this case remain unchanged except of the right-hand side in (8) that should be replaced by $-(R_n + R_{n-1})/2$ and the main diagonal of matrix $R$ in (12) that should be given with weight $1/2$.

Since the new procedure is based on Toeplitz symmetry of the matrix and the specific procedure putting a column and a row inside the matrix, we call it Toeplitz inner bordering (TIB) method.
Reflectance, dB
0
-10
-20
-30
-40
-50

Frequency, $10^{-4}\omega_0$
-1
-0.5
0
0.5
1

Figure 1: The reflection spectrum of GHS grating for testing examples, $k_0L = 5 \times 10^4$, $F = 3$, $Q = 1$ (dashed line), 2 (dots), 3 (solid).

4 Testing examples

The new method is tested using a specific case of the family of exactly solvable chirped GHS profile of the coupling coefficient [22]

$$q(x) = \frac{Q}{L} \left( \text{sech} \frac{x}{L} \right)^{1-2iF}. \quad (18)$$

It describes a FBG with apodization function

$$\alpha(x) = \frac{\delta n_{\text{max}}}{2n} \text{sech} \frac{x}{L}, \quad (19)$$

and phase modulation

$$\theta(x) = 2F \ln \left( \cosh \frac{x}{L} \right) - \frac{\pi}{2}, \quad (20)$$

where $L$ is the half width of grating apodization profile at level sech(1) = 0.648, parameter $Q = \kappa L \delta n_{\text{max}}/4n$ is the grating strength (the number of strokes through length $L$ multiplied by the modulation depth of the refractive index). Parameter $F$ describes the value of the chirp: the profile has a slowly varying spatial frequency

$$\kappa(x) = \kappa + \frac{d\theta}{dx} = \kappa + \frac{2F}{L} \tanh \frac{x}{L}, \quad (21)$$
that goes smoothly from one constant spatial frequency $\kappa - 2F/L$ to another $\kappa + 2F/L$.

The coupled wave equations (1) have an exact solution that can be expressed via the Gaussian hypergeometric function. It gives the reflection coefficient of the form [22]

$$r(\omega) = -2^{-2iF} Q \frac{\Gamma(d) \Gamma(f_-) \Gamma(f_+)}{\Gamma(d^*) \Gamma(g_-) \Gamma(g_+)},$$  \hspace{1cm} (22)

where arguments of Euler gamma-function [23] are given by relations:

$$d = \frac{1}{2} + i [\omega L - F],$$

$$f_{\pm} = \frac{1}{2} - i \left[ \omega L \pm \sqrt{F^2 + Q^2} \right],$$

$$g_{\pm} = 1 - i \left[ F \pm \sqrt{F^2 + Q^2} \right].$$

The reflection spectrum is expressed in terms of elementary functions

$$|r(\omega)|^2 = \frac{\cosh 2\pi \sqrt{Q^2 + F^2} - \cosh 2\pi F}{\cosh 2\pi \sqrt{Q^2 + F^2} + \cosh 2\pi \omega L}.$$  \hspace{1cm} (23)

For numerical calculations we choose gratings with $L = 5 \times 10^4/k_0$, $F = 3$ and $Q = 1,2,3$, where $k_0 = 2\pi n/\lambda_0$ and the central resonance wavelength

Figure 2: The group delay characteristics of GHS spectrum for testing examples at the same parameters, as in Fig. [1]
Figure 3: Root mean square error $\sigma$ of the first-order (triangles) and second-order (boxes) reconstruction as a function of $1/N$ in logarithmic coordinates for $N = 128 \div 4096$, $Q = 1$, $L = 5 \times 10^4/k_0$, $F = 1$. The straight lines show the least-square linear fitting.

is $\lambda_0 = 1.5 \, \mu\text{m}$. Their maximum reflectances, $|r|^2 = 0.6393, 0.9777, 0.9996$, are referred hereafter as small, medium and high respectively. The reflection spectrum calculated by formula (23) is shown in Fig. 1. The frequency detuning from resonance is shown in units $10^{-4}\omega_0$, where $\omega_0$ is the central frequency of the reflection spectrum. The spectrum is quasi-rectangular with flat top inside the Bragg reflection band. The reflectance increases with optical strength parameter $Q$. The width of the band $\Delta \omega \simeq 2\sqrt{Q^2 + F^2}/L$ increases, too. The group delay characteristics are plotted in Fig. 2. Each curve is close to straight line within the band except of the band edges.

The GLM equations for reflection coefficient (22) are solved using the method described in Sec. 3. As the first step of calculations the fast Fourier transform Eq. (3) is performed at sufficiently long frequency interval and small frequency step $\delta \omega = 2\pi/L_{max}$, where $L_{max} = 35L$, in order to neglect the values outside both the frequency and the coordinate intervals where the reflection spectrum and the grating are defined. The frequency domain for integration is defined as $-\Omega/2 \leq \omega \leq \Omega/2$, where $\Omega = N_\omega \delta \omega$ and $N_\omega$ is the number of discrete points in frequency. While we are going to test the method of solving GLM equations itself, the additional errors produced by the Fourier transform should be minimized. For this purpose the excessively precise
Figure 4: The envelope $\alpha$ as a function of coordinate reconstructed by second-order TIB (solid line) and by DLP (crosses): from the top down $Q = 3, 2, 1$.

determination of function $R(t)$ is made. In order to provide the sufficient accuracy for the second-order method we choose $N_\omega \gg N$, in particular, $N_\omega = 2^{20}$ at $N = 2^{12}$. It does not significantly increase the total number of operations, since the Fourier transform requires $N_\omega \log_2 N_\omega$ operations and done only once.

The inaccuracy of rectangular and that of trapezoidal quadrature formulas are compared. Root mean square error $\sigma$ of the grating reconstruction is shown in Fig. 3 as a function of $1/N$. As evident from the figure the first and second-order algorithms result in different errors. For the first-order method the dependence is linear, whereas for the second order it becomes nearly quadratic. The slopes of fitted straight lines are 1.05 and 2.00, respectively. Moreover, the error of the second order method is significantly less at $N \geq 2^6$. Then the second-order method is applied in all calculations below.

The comparison of the second-order TIB with DLP reconstruction at fixed $N$ reveals that TIB method occurs $2 \div 3$ times faster. The apodization function $\alpha(x)$ at the same parameters, as in Fig. 1 and $N = 8192$ is shown in Fig. 4. For relatively weak grating $Q = 1, 2$ both methods are appropriate, as bottom curves demonstrate, and the resultant curves are in agreement with formula (19). However, for strong grating the DLP calculation gives
Figure 5: Comparison of the second-order TIB method with GHS profile [19]: the deviation of numerical calculations from the analytical formula as a function of coordinate. The number near each curve denotes the value of grating strength $Q$. 

Figure 6: The deviation of the spatial frequency of the grating [21] from $\kappa$ calculated by TIB method (solid line) and DLP (crosses) at $Q = 3$. 
significant error. The reason is probably the error amplification in DLP [12].

The deviation of TIB solution from GHS profile (19) is shown in Fig. 5. The deviation is maximal near the center of the profile and negligible at the ends. The curves are regular at small and medium strength and acquire irregular behavior for strong grating. The maximum relative error of reconstructed apodization function is less than $2.5 \times 10^{-4}$ for all studied parameters.

The phase characteristics of complex coupling coefficient demonstrate the similar features. At $Q = 1, 2$ the phase characteristics calculated by TIB and DLP methods are close. At high optical strength $Q = 3$ the error of DLP grows up towards the right end. The spatial frequency $\theta'$ of reconstructed profile is shown in Fig. 6. The smooth transition between two horizontal asymptotes of analytical expression (21) is reproduced by TIB calculation for $Q = 3$, whereas the DLP gives the deviation at the right side of the curve.

5 Discussion

The discrete layer peeling [11] calculates $q$ at the input end of the grating and then truncates the grating dealing every next step with shorter grating residue. This is the reason of error accumulation throughout the calculation from the input layer to the output one. The TIB method of matrix inversion recovers the complex coupling coefficient $q(x)$ along the whole length at one step. Then it has higher accuracy at comparable efficiency.

It is possible to make TIB even more efficient dividing the length $L$ by segments. After reconstruction of the coupling coefficient in the current segment one could find the amplitudes of opposite waves at the input end of the next segment and repeat the procedure with the next segment. The efficiency could be improved if we choose the optimal number of segments.

The similar combined procedure with indirect iterative solution of GLM equations, known as integral layer peeling (ILP), leads to fast reconstruction of a grating [15]. In that approach the grating is divided by $M$ layers with $m$ intermediate points in each. The total number of points along the grating is $N = mM$. The reconstruction problem in each layer is solved by an iterative procedure applied to GLM integral equations. The reflection coefficient of truncated grating after a peeling step is found with high accuracy. The
computational complexity of ILP is of the order of
\[ n_{\text{total}} \sim \left( lN + \frac{l + 1}{m} N^2 \right) \log_2 N \]
required operations [15], where \( l \) is the number of iterations during the reconstruction of a layer. At \( l = 0 \) and large \( m \) the complexity becomes less than \( N^2 \). However with increasing \( m \) and decreasing \( l \) the accuracy goes down fast.

If we were change the ILP iterations by the proposed TIB technique the complexity of the reconstruction within a layer would be \( N \ln N + m^2 \). We obtain the total number of required operations multiplying it by \( M = N/m \):
\[ n_{\text{total}} \sim \frac{N^2}{m} \log_2 N + mN. \]
This number has a minimum value \( \min n_{\text{total}} \sim N^{3/2}(\ln N)^{1/2} \ll N^2, N \to \infty \) at \( m \sim (N \ln N)^{1/2} \). As long as each layer is sufficiently thin and its optical strength is not large \( (Q \lesssim 1) \), the matrix inversion method shall give the superior result compared to iterations.

For very strong gratings at \( 1 - |r| \to 0 \) all the methods lose their accuracy, since an eigenvalue of GLM equations tends to zero and the problem becomes ill-conditioned. If the grating is strong, then incident light is reflected in the domain close to the input end. Only exponentially small part penetrates far from the input end, then it is almost impossible to reconstruct the profile of the deeper region. Fortunately, it is a formal mathematical problem. For more or less reasonable optical density, for instance, with maximum reflectance up to 99.9%, the proposed TIB method is adequately accurate.

## 6 Conclusions

Thus the new method of the FBG synthesis is proposed. The method is based on direct numerical solution of the coupled GLM equations. The Toeplitz symmetry of the matrix and the inner-bordering procedure provide fast computation, similar to known fast Levinson’s algorithm. The second-order quadrature formula sufficiently improves the accuracy without loss of efficiency. The method is tested using exactly solvable profile of chirped grating. The method does not concede the DLP in speed and at the same time remains more accurate for strong gratings.
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