Linear optical implementation of perfect discrimination between single-bit unitary operations

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Abstract

Discrimination of unitary operations is a fundamental task of quantum information. Assisted by linear optical elements, we experimentally demonstrate perfect discrimination between single-bit unitary operations using the sequential scheme which is proved by Duan \textit{et al} (\textit{Phys. Rev. Lett.} 2007 \textbf{98} 100503). We also make a comparison with another perfect discrimination scheme called the parallel scheme. The complexity and resource consumed are analysed.

For quantum computing and quantum information processing, one important task is the discrimination of quantum states and unitary operations. This is strongly related to quantum nonorthogonality, which is one of the basic features of quantum mechanics. Since the pioneering work of Helstrom [1] on quantum hypothesis testing, the problem of discriminating nonorthogonal quantum states has attracted much attention, both in theory [2, 3] and in experiments [4–8]. Naturally, the concepts of nonorthogonality and distinguishability can also be applied to quantum operations. However, things become very different when we refer to perfect discrimination of quantum operations, which has already had some theoretical work [9–14]. Some pioneering works have been devoted to a good understanding of the exact role of quantum entanglement in the discrimination between unitary operations. It has been proved [9, 10] that perfect discrimination of nonorthogonal unitary operations can always be achieved with a finite number of times of running the unknown unitary operation, by using a suitable entangled state as input. This is contrary to the case of nonorthogonal quantum states, for which perfect discrimination cannot be achieved with a finite number of copies. Recently, Duan \textit{et al} indicated that entanglement is not necessary for discrimination between unitary operations [11]. They show that by taking a suitable state and proper auxiliary unitary operation, nonorthogonal unitary operations can also be perfectly discriminated even without entanglement. This result impacts on the role of quantum entanglement in the context of quantum computing [15], and also makes the experiment much easier, because no \textit{N}-partite entangled state is needed.

First, we briefly explain the basic idea of perfect discrimination between unitary operations [9–11]. Suppose we have an unknown quantum circuit, which is secretly chosen from the two alternatives, \(U\) and \(V\). Here, both \(U\) and \(V\) are unitary operations acting on a \(d\)-dimensional Hilbert space, \(\mathcal{H}_d\). We apply the unknown circuit to a proper initial state \(|\psi_i\rangle\). If the output states, \(|\psi_o\rangle_U\) and \(|\psi_o\rangle_V\), are orthogonal, then perfect discrimination between \(U\) and \(V\) is achieved. For the parallel scheme, shown in figure 1(A), we can perfectly discriminate \(U\) and \(V\) by three steps. First, prepare a proper \(N\)-qudit entangled state \(|\psi_i\rangle\); second, apply the secretly

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is the input state, and $U$ is the output state. A parallel scheme for discriminating two unitary operations, $U$ and $V$. (B) A sequential scheme for discriminating two unitary operations, $U$ and $V$. $X_1$, $X_2$, ..., $X_{N-1}$ are auxiliary unitary operations. $|\psi_i\rangle$ is the input state, and $|\psi_o\rangle$ is the output state. The output states, $|\psi_o\rangle_U$ and $|\psi_o\rangle_V$, need to be orthogonal for perfect discrimination.

Figure 1. Two schemes of perfect discrimination of two unitary operations. $O$ represents one of the unknown circuits, $U$ or $V$. $|\psi_i\rangle$ is the input state, and $|\psi_i\rangle$ is the output state. (A) A parallel scheme for discriminating two unitary operations, $U$ and $V$. (B) A sequential scheme for discriminating two unitary operations, $U$ and $V$. $X_1$, $X_2$, ..., $X_{N-1}$ are auxiliary unitary operations. $|\psi_i\rangle$ is the input state, and $|\psi_o\rangle$ is the output state. The output states, $|\psi_o\rangle_U$ and $|\psi_o\rangle_V$, need to be orthogonal for perfect discrimination.

chosen circuit $N$ times on $|\psi_i\rangle$, where each qudit one time; last, perform a projective measurement on the output states $|\psi_o\rangle_U = U^{\otimes N} |\psi_i\rangle$ and $|\psi_o\rangle_V = V^{\otimes N} |\psi_i\rangle$. If $|\psi_o\rangle_U \perp |\psi_o\rangle_V$ (‘$\perp$’ represents the orthogonality of two states), then two unitary operations are perfectly discriminated. Indeed, we can always find a proper input state to make sure $|\psi_o\rangle_U \perp |\psi_o\rangle_V$ by using the method mentioned in [9, 10].

For the sequential scheme, shown in figure 1(B), we need to prepare a proper one-qudit state, $|\psi_i\rangle \in \mathcal{H}_d$, and perform the unknown circuit sequentially on $|\psi_i\rangle$ for $N$ times. Between each two runs of the unknown circuits, a proper auxiliary unitary operation, $X_i \in \mathcal{U}$ $(i = 1, ..., N - 1)$, is inserted. Here $\mathcal{U}$ represents a set of unitary operations acting on $\mathcal{H}_d$. Perfect discrimination between $U$ and $V$ requires that two output states are orthogonal, such that

$$U X_{N-1} \cdots U X_1 |\psi_i\rangle \perp V X_{N-1} \cdots V X_1 |\psi_i\rangle. \quad (1)$$

As proved in [11], this discrimination task can always be translated to distinguish $U^1 V$ and identity ($I$). Then, relation (1) can be reduced as follows:

$$(U^1 V) X (U^1 V)^{N-1} |\psi_i\rangle \perp X |\psi_i\rangle, \quad (2)$$

where $X_i$, $X$ and $|\psi_i\rangle$ depend on $\Theta(U^1 V)$.

By comparison of these two schemes, the sequential scheme is experimentally much easier than the parallel scheme, especially, in testing the orthogonality of output states. For the sequential scheme, only single-qudit projective measurement is needed while for the parallel scheme non-local measurements should be performed, which is rather difficult with current technology. Furthermore, because of the environmental noise and experimental error, the output states of the parallel scheme are always mixed entangled states which are hardly precisely distinguished by local operation and classical communication (LOCC) [16].

In this paper, we report an experiment that demonstrates perfect discrimination between single-qudit unitary operations using the sequential scheme [11]. Linear optical elements are used to perform unitary operations on photonic qubits in the experiment. At the end, we perform a projective measurement to show the perfect discrimination. The quality of our unitary operation is characterized by the average process fidelity, $F(U)$, [17, 18] between the matrix which is obtained from standard quantum process tomography [19–23] and the theoretical one.

Although the sequential scheme is not limited to single-qubit unitary operations, we choose to discriminate two single-bit unitary operations in our experiment. Because single-qudit unitary operation is a class of fundamental unitary transformations which is widely used in quantum computing and quantum information processing tasks. And with linear optical elements, any single-qudit unitary operation on photonic polarization qubit can be easily implemented using a set of wave plates [24]. For simplicity, we set the run times, $N$, of the unknown circuit to be $N = \lceil \pi/\Theta(U^1 V) \rceil = 2$. And two sets of unitary operations are chosen:

- **Case 1:** $U_1 = \left( \begin{array}{cc} e^{i \pi/2} & 0 \\ 0 & 1 \end{array} \right)$, $V_1 = \left( \begin{array}{cc} e^{i \pi/2} & 0 \\ 0 & 1 \end{array} \right)$, $\Theta(U^1_1 V_1) = \pi/2$. \quad (3)
- **Case 2:** $U_2 = U_1$, $V_2 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$, $\Theta(U^1_2 V_2) = 2\pi/3$. \quad (4)

Case 1: $N = \left\lceil \pi/\Theta(U^1_1 V_1) \right\rceil = 2$, auxiliary operation $X = \left( \begin{array}{cc} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{array} \right)$ and $|\psi_i\rangle = \frac{1}{\sqrt{2}} (|\psi_i\rangle + |\psi_i\rangle_H + |\psi_i\rangle_V)$, where $|\psi_i\rangle_H$ and $|\psi_i\rangle_V$ are the eigenvectors of $X(U^1 V) X(U^1 V)^{N-1}$ [11]. The experimental setup is shown in figure 2. We do a projective measurement to show the results of perfect discrimination. The projective basis that we chose is $|\psi_o\rangle = \frac{1}{\sqrt{2}} (-e^{i \pi/2} |H\rangle + |V\rangle)$.

Photons will only be detected by $D_1$ if their state is $|\psi_o\rangle_H$, or will only be detected by $D_2$ if their state is the orthogonal state of $|\psi_o\rangle_V$. All the operations are achieved by linear optical components. And all the degrees of wave plates are shown in table 1. We do a process tomography of our unitary operations to ensure their validity. The average process fidelities that we obtain are $F(U_1) = 0.985$ and $F(V_1) = 0.975$. After a theoretical calculation, we know that the output state $|\psi_o\rangle$ is $\frac{1}{\sqrt{2}} (-e^{i \pi/2} |H\rangle + |V\rangle)$ when applying $U_1$, while $|\psi_o\rangle$ is $\frac{1}{\sqrt{2}} (-e^{i \pi/2} |H\rangle - |V\rangle)$ when applying $V_1$, and $|\psi_o\rangle = 0$. A polarized beam splitter (PBS) transmits vertical polarizing photons and reflects horizontal polarizing photons. So, with our projective measurement, we get the results that the unknown unitary operation is $U_1$ when photons are only detected by $D_1$, and the unknown unitary operation is $V_1$ when photons are only detected by $D_2$. The results are shown in figure 3.

Case 2: similar to the discussions in case 1, we get $N = \left\lceil \pi/\Theta(U^1_2 V_2) \right\rceil = 2$, the auxiliary operation $X = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} e^{i \pi/2} & -1 \\ 1 & e^{i \pi/2} \end{array} \right)$ and $|\psi_i\rangle = (-0.151 + 0.262 i) |H\rangle + 0.953 |V\rangle$. The projective basis that we chose is $|\psi_o\rangle = \frac{1}{\sqrt{2}} (-e^{i \pi/2} |H\rangle + |V\rangle)$. The experimental setup is not changed, but the settings of wave
Figure 2. Experimental setup for the sequential scheme. Attenu represents the attenuator. P is the polarizer. PBS represents the polarized beam splitter, which transmits vertical polarizing photons and reflects horizontal polarizing photons. IF is the interference filter centered at 780 nm with 4 nm bandwidth. D1 and D2 are photon detectors. Qi (i = 1, 2, ..., 8) and Hj (j = 1, 2, ..., 5) represent quarter wave plates (QWP) and half wave plates (HWP), respectively. Our unitary operations (U, V and X) are all realized by a HWP and two QWPs. At the end, we perform a projective measurement.

Figure 3. Experimental results of perfect discrimination. The above two figures are the results of case 1, and the bottom two figures are the results of case 2. We can directly discriminate the circuits by the counts of the two detectors.

Table 1. Wave plates setting (degree).

| Input | U(V) | X | U(V) | Measurement |
|-------|------|---|------|-------------|
|       | Q1   | H1| Q2   | H2 | Q3 |       | Q4 | H4 | Q5 | H5 | Q6 | H6 | Q7 | H7 |
| U1    | 0    | 22.5 | 45 | 15 | 45 | 45 | 0 | 0 | 0 | 45 | 15 | 45 | 45 | 37.5 |
| V1    | 0    | 22.5 | 45 | 37.5 | 45 | 0 | 0 | 0 | 45 | 37.5 | 45 | 45 | 37.5 |
| U2    | −15  | 42.4 | 45 | 15 | 45 | 27.4 | 45 | 62.6 | 45 | 15 | 45 | 45 | 15 |
| V2    | −15  | 42.4 | 0 | 0 | 0 | 27.4 | 45 | 62.6 | 0 | 0 | 0 | 45 | 15 |

plates are different from the settings in case 1. All the settings of wave plates are shown in table 1. The average process fidelity of the auxiliary operation, X, is $F(X) = 0.996$. We can get two orthogonal output states $|\psi_o\rangle = \frac{1}{\sqrt{2}}(e^{-\frac{\pi}{6}i}|H\rangle + |V\rangle)$ and $|\psi'_o\rangle = \frac{1}{\sqrt{2}}(e^{-\frac{\pi}{6}i}|H\rangle - |V\rangle)$, when applying $U_2$ and $V_2$, respectively. So we know that the unknown unitary operation is $U_2$ when photons are only detected by $D_1$, and the unknown unitary operation is $V_2$ when photons are only detected by $D_2$ which are shown in figure 3.

In our experimental setup, a single-photon source is achieved by attenuating a coherent Ti-sapphire laser to single-photon level. The maximum photon detection rates are attenuated always to less than 200 000 counts per second. Combining with the 76 MHz repetition rate of the Ti-sapphire...
laser, it is obvious that the average photon number in each attenuated pulse is much less than 1. Then, a polarizer (transmitted horizontal polarizing photon), quarter wave plate (QWP), $Q_1$, and half wave plate (HWP), $H_1$, make up the initialization of input states. The set of wave plates with one half wave plate sandwiched between two quarter wave plates can realize any unitary operation on the photonic polarization qubit, which can be represented in terms of three Eulerian angles. Projective measurement consists of QWP ($Q_2$), HWP ($H_2$), PBS (transmitting vertical polarizing photons) and two photon detectors. It is used to verify the orthogonality of output states.

Our results are shown in figure 3. When choosing the input states and projective measurements discussed above, we can discriminate the circuits via the results of the photon detectors. The probabilities of successful discrimination are about 98.0% $^4$ (figure 3(A)), 98.1% (figure 3(B)) of case 1, and 98.3% (figure 3(C)), 98.4% (figure 3(D)) of case 2. In our results, the errors mainly come from the deviations of the angle settings of the wave plates, because there are many wave plates in our experimental setup and their precision is only about 0.2°. This leads to a discrimination probability of less than 1. But it is quite different from other non-perfect discrimination schemes whose probabilities of successful discrimination never achieve 100%, even in theory. One interesting idea to improve our experiment results is to consider using the same actual device to repeat the unknown operation. To achieve this, we can use the multiple passes method that employed in [25]. This improvement will reduce the experiment errors arisen from the difference between the two repetitions of the same unknown operation.

We have experimentally distinguished the two sets of single-bit unitary operations where $N = 2$ using the sequential scheme. If $N > 2$, we need to apply the unknown unitary operation $N$ times, and auxiliary unitary operations $N − 1$ times on the input state. So, one has to perform sequentially at least $2N − 1$ steps of unitary operations. This may lead to long discriminating time; meanwhile, the deviations of the results would be influenced by the additional $N − 1$ auxiliary unitary operations. These are the disadvantages of the sequential scheme compared with the parallel scheme. However, the greatest advantage of the sequential scheme is that no entanglement or joint quantum operations is needed. Instead, in the parallel scheme one needs to prepare an $N$-partite entangled state as the input state. When we have at least $N$ copies of the unknown circuit and a suitable entangled resource, we can complete the discrimination within a single step by applying $N$ copies of the unknown circuit to the input state. This scheme is fast (only one step) but costs entanglement resource. Furthermore, one needs to perform non-local projective measurements on the output $N$-particle entangled states to finally discriminate the unknown unitary operations. This will be very difficult for practical experiments.

$^4$ Suppose the counts of detector $D_1$ and $D_2$ are $c_1$ and $c_2$, respectively. Then the successful discrimination probability can be written as $p = c_1/(c_1 + c_2)$ or $p = c_2/(c_1 + c_2)$ based on the projective basis.

It is worth noting that when we finished our experiments, we noticed a related but independent work by the group at Bristol [26].

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