Effective gravitational mass of the Ayón-Beato and García metric*

A. K. Sinha†, G. K. Pandey†, A. K. Bhaskar§, B. C. Rai¶, A. K. Jha∗∗, S. Kumar††

National Institute of Technology,
Patna 800005, Bihar, India.

S. S. Xulu‡‡

Department of Computer Science, University of Zululand,
3886 Kwa-Dlangezwa, South Africa.

Abstract

In this paper, we calculate the effective gravitational mass of Ayón-Beato and García regular (non-singular) static spherically symmetric asymptotically Minkowskian metric that is a solution to Einstein’s equations coupled with a nonlinear electromagnetic field. The effective gravitational mass is negative, zero, or positive that depends on the ratio of magnitude of electric charge to the ADM mass and the ratio of the radial distance to the ADM mass. As expected, at large value of radial distance, our result gives effective gravitational mass of the Reissner-Nordström metric.

PACS numbers: 04.70 Bw, 04.20.Jb, 04.20.Dw, 04.20.Cv

Keywords: General relativity, Møller’s energy-momentum complex, regular black hole, effective gravitational mass.

---

* Most of this work was done during an International Workshop on Introduction to Research in Einstein’s General Relativity at NIT, Patna (India). Authors’ emails as well as their permanent addresses are mentioned below:
† Department of Physics, College of Commerce, Patna 800020, Bihar, India, Email: ashutosh25june@gmail.com
‡ Patna Science College, Patna University, Patna 80005, Bihar, India, Email: gaurav.golu1@gmail.com
§ Department of Physics, College of Commerce, Patna 800020, Bihar, India, Email: drakbhaskar@gmail.com
¶ Department of Physics, College of Commerce, Patna 800020, Bihar, India, Email: bcraiphy@gmail.com
∗∗ Department of Physics, College of Commerce, Patna 800020, Bihar, India, Email: bcraiphy@gmail.com
†† Department of Physics, College of Commerce, Patna 800020, Bihar, India, Email: drarunkjhacoc@gmail.com
‡‡ Email: ssxulu@pan.uzulu.ac.za
I. INTRODUCTION

In the year 1984, Cohen and de Felice [1] defined effective gravitational mass of a metric at any point in a given space-time at a radial distance $r$ as the energy contained inside the region of $r = \text{constant}$ surface, and used this to explain repulsive nature of timelike singularities. They used Komar energy formula and obtained effective gravitational mass of the Kerr-Newman metric (characterized by mass $M$, electric charge $q$, and rotation parameter $a$):

$$
\mu(r) = M - \frac{q^2}{2r} \left[ 1 + \frac{(a^2 + r^2)}{ar} \arctan \left( \frac{a}{r} \right) \right].
$$

For large values of rotation and/or charge parameters (that is, when these parameters dominate over the mass parameter), the effective gravitational mass becomes negative. For the Reissner-Nordström metric ($a = 0$) in the above result, the effective gravitation mass is

$$
\mu(r) = M - \frac{q^2}{r}.
$$

Obviously, $\mu(r) < 0$ for $r < \frac{q^2}{M}$ and this explains repulsive gravitational effect on an electrically neutral test particle in the gravitational field described by the Reissner-Nordström metric. This fascinating interpretation of energy content in a space-time attracted researchers; however, they soon realized that there is no unique adequate formalism for energy calculation and this had been a difficult problem since the appearance of general theory of relativity. For example, Komar definition cannot be applied to non-static space-times.

Though Einstein’s theory of gravity is a well governing theory of space, time, and gravitation and it is very well experimentally verified, and is, so far, proven to be the most successful theory to describe gravity at small as well as large scales, it does possess some issues. The energy-momentum localization problem is one such important issue in the context of General Relativity. Energy, momentum, and angular momentum are important conserved quantities in Minkowski space-time (when gravitational field is negligibly small.) These have significant role as they provide the first integrals of motion to solve otherwise unmanageable physical problems. However, till today, we do not have a general definition to describe energy-momentum distribution in curved space-times (i.e., in presence of gravitational field) and several difficulties arouse in this direction.

All the attempts to finding this resulted in a large number of different energy-momentum complexes. To solve this problem, Einstein formulated the energy-momentum local conservation law (see in [2]). Then, many physicists including, Landau-Lifshitz [3], Weinberg [4],
Papapetrou⁵, and Bergman and Thomson⁶ proposed various definitions of energy-momentum distribution. But, these definitions of energy-momentum complexes were coordinate dependent, that is, they give meaningful results only when the calculations were done in quasi-Cartesian coordinates. It was suspected that a plethora of different energy-momentum complexes would give acceptable total energy and momentum for isolated systems (i.e., asymptotically flat space-times); however, they would produce different and hence meaningless energy-momentum distributions even in asymptotically flat (Minkowskian) spacetime and would not give any meaningful result at all for asymptotically non-Minkowskian space-times. Virbhadra’s and his collaborators⁷,⁸ seminal work shook this prevailing prejudice and they explicitly showed that different energy-momentum complexes give the same results for many spacetimes (asymptotically Minkowskian as well as non-Minkowskian). Later, Virbhadra and Rosen⁹ (the most famous collaborator of Albert Einstein) studied Einstein-Rosen gravitational waves and for this metric as well different complexes gave same and reasonable results. Further, attracted by these encouraging results, many researchers¹⁰–¹² studied several other metrics and obtained very useful results.

In General Theory of Relativity, majority of well-known exact black holes solutions to the Einstein’s field equations came up with the existence of space-time (curvature) singularities where general relativity theory breaks down. Therefore, it is very desirable to have regular (nonsingular) solutions of Einstein’s equations. To avoid this black hole curvature singularity aspect, researchers came up with a large number of regular black hole models which were referred to as “Bardeen black holes”¹³ because Bardeen was the first to obtain a regular black hole solution. But, the problem with all of them was that none of them were an exact solution to the Einstein’s Field Equations. Moreover, there are no known physical sources pertaining to any of those regular black hole solutions.

Later in 1998, Ayón-Beato and García found out the first singularity free exact black hole solution¹⁴ to the Einstein’s field equations whose source is a non-linear electrodynamic field coupled to gravity and the metric does not possess any curvature singularity. In this paper, we calculate effective gravitational mass of this metric and analyze the results. We use Møller’s ² energy-momentum expression which permits to calculate the energy-momentum distribution in an arbitrary coordinate system. As usual in general relativity papers, we too use the geometrized units ($G = 1, c = 1$). We use Mathematica ¹⁵ software for plots.
II. THE AYÓN-BEATO AND GARCÍA SOLUTION

The Ayón-Beato and García (hereafter referred to as AG) static spherically symmetric asymptotically Minkowskian solution to Einstein’s equations with a source of nonlinear electrodynamic field is given by the line-element

\[ ds^2 = \beta dt^2 - \alpha dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) \] (3)

and the electric field

\[ E = qr^4 \left( \frac{r^2 - 5q^2}{(r^2 + q^2)^4} + \frac{15}{2} \frac{M}{(r^2 + q^2)^{7/2}} \right) , \] (4)

where

\[ \beta = \frac{1}{\alpha} = 1 - \frac{2Mr^2}{(r^2 + q^2)^2} + \frac{q^2r^2}{(r^2 + q^2)^2} . \] (5)

Ayón-Beato and García [14] showed that their solution corresponds to charged non-singular black holes for \( |q| \leq \text{approximately } 0.6M \) and otherwise the solutions are though regular, there are no horizons.

The asymptotic behavior of the AG solution is given by

\[ \alpha = 1 + \frac{2M}{r} + \frac{4M^2 - q^2}{r^2} + O \left( \frac{1}{r^3} \right) , \]
\[ \beta = 1 - \frac{2M}{r} + \frac{q^2}{r^2} + O \left( \frac{1}{r^3} \right) , \] and
\[ E = \frac{q}{r^2} + O \left( \frac{1}{r^3} \right) . \] (6)

Thus the A-G solution behaves asymptotically as the Reissner-Nordström solution to Einstein-Maxwell equations. The symbol \( M \) represents the ADM mass and \( q \) stands for the electric charge. The behavior of the above functions near \( r = 0 \):

\[ \alpha = 1 + r^2 \left( \frac{2M|q|}{q^4} - \frac{1}{q^2} \right) + O \left( r^3 \right) , \]
\[ \beta = 1 - r^2 \left( \frac{2M|q|}{q^4} - \frac{1}{q^2} \right) + O \left( r^3 \right) , \]
\[ E = \left( \frac{15|q|}{2q^6} - \frac{5}{q^5} \right) r^4 + O \left( r^6 \right) . \] (7)

Thus, the metric behaves as de Sitter near \( r = 0 \). As we will need the metric in quasi-
Cartesian coordinate system as well, applying the coordinate transformation

\[
\begin{align*}
    x &= r \sin \theta \cos \phi, \\
    y &= r \sin \theta \sin \phi, \\
    z &= r \cos \theta,
\end{align*}
\]  

(8)

the line element gets the form,

\[
ds^2 = \beta dt^2 - (\gamma x^2 + 1)dx^2 - (\gamma y^2 + 1)dy^2 - (\gamma z^2 + 1)dz^2 - 2\gamma(xydx+yzdy+zdxdy),
\]

(9)

where

\[
\gamma = \frac{A - 1}{x^2 + y^2 + z^2}.
\]

(10)

III. MOLLEL ENERGY-MOMENTUM PRESCRIPTION

The Møller energy-momentum complex is given by [2]

\[
\mathcal{S}_i^{kl} = \frac{1}{8\pi} \chi_i^{kl} \delta_{l},
\]

(11)

which satisfies the local conservation laws as

\[
\frac{\partial \mathcal{S}_i^{kl}}{\partial x^k} = 0,
\]

(12)

where the antisymmetric Møller super-potential \( \chi_i^{kl} \) is

\[
\chi_i^{kl} = -\chi_i^{lk} = \sqrt{-g} [g_{in,m} - g_{im,n}] g^{km} g^{nl}.
\]

(13)

The energy and momentum components are now given by

\[
P_i = \int \int \int \mathcal{S}_i^{0} dx^1 dx^2 dx^3.
\]

(14)

\( P_i \) stands for momentum components \( P_1, P_2, P_3 \), and \( P_0 \) is the energy. Following Cohen and de Felice [1], we interpret the energy function as the effective gravitational mass \( \mu(r) \) of the metric. We apply Gauss’ theorem in above equation and then the total energy and momentum components take the new form:

\[
P_i = \frac{1}{8\pi} \int \int \int \chi_i^{0\delta} n_\delta \ dS,
\]

(15)

where \( n_\delta \) is the outward unit normal vector over an infinitesimal surface element \( dS \).
IV. CALCULATIONS

Møller claimed that his energy-momentum complex can be used in any coordinate system and would produce the same results. In order to obtain the effective gravitational mass of the AG metric and to verify Møller’s claim, we first perform calculations in Schwarzschild coordinates \( \{t, r, \theta, \phi\} \) and then in quasi-Cartesian coordinates \( \{t, x, y, z\} \).

A. Calculations in Schwarzschild coordinates

In Schwarzschild coordinates, the determinant of the covariant metric tensor \( g_{ik} \) for the AG metric given by Eq. (3) is

\[
g = -r^4 \sin^2 \theta \tag{16}
\]

As the metric is represented by a diagonal matrix, \( g_{ik} = \frac{1}{g_{ik}} \forall \) values of indices \( i \) and \( k \). The only non-vanishing component of \( \chi^{kl}_{i} \) which is needed for the energy calculation is

\[
\chi^0_0 = -\chi^0_1 = 2r^3 \sin \theta \left[ M \frac{r^2 - 2q^2}{(r^2 + q^2)^2} - q^2 \frac{r^2 - q^2}{(r^2 + q^2)^3} \right] \tag{17}
\]

while all the other components of \( \chi^{kl}_{i} \) vanishes. Using the above expression in equation (15), the energy distribution is given by

\[
E_S = r^3 \left[ M \frac{r^2 - 2q^2}{(r^2 + q^2)^2} - q^2 \frac{r^2 - q^2}{(r^2 + q^2)^3} \right] \tag{18}
\]

The subscript \( S \) emphasizes that calculations have been performed using Schwarzschild coordinates \( \{t, r, \theta, \phi\} \). All the three momentum components are found to be zero, i.e.,

\[
P_r = P_\theta = P_\phi = 0, \tag{19}
\]

and this is desired result as the AG metric is static.

B. Calculations in quasi-Cartesian Coordinates

The desired covariant components of fundamental metric tensor \( g_{ik} \) is obtained from the equation (9):
\[ g_{ik} = \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & -\gamma x^2 - 1 & -xy\gamma & -xz\gamma \\ 0 & -xy\gamma & -\gamma y^2 - 1 & -yz\gamma \\ 0 & -xz\gamma & -yz\gamma & -\gamma z^2 - 1 \end{pmatrix} \] (20)

The determinant of this metric tensor is,

\[ g = |g_{ik}| = -1. \] (21)

The components of \( g^{ik} \) are given by

\[ g^{ik} = \begin{pmatrix} \frac{1}{\beta} & 0 & 0 & 0 \\ 0 & -\beta (\gamma y^2 + \gamma z^2 + 1) & xy\beta\gamma & xz\beta\gamma \\ 0 & xy\beta\gamma & -\beta (\gamma x^2 + \gamma z^2 + 1) & yz\beta\gamma \\ 0 & xz\beta\gamma & yz\beta\gamma & -\beta (\gamma x^2 + \gamma y^2 + 1) \end{pmatrix}. \] (22)

The components of the super-potential \( \chi^{kl}_{i} \) in quasi-Cartesian coordinates are obtained here:

\[ \chi^{01}_{0} = \frac{\partial \beta}{\partial x}, \]
\[ \chi^{02}_{0} = \frac{\partial \beta}{\partial y}, \]
\[ \chi^{03}_{0} = \frac{\partial \beta}{\partial z}, \]
\[ \chi^{11}_{1} = \chi^{02}_{1} = \chi^{03}_{1} = 0, \]
\[ \chi^{21}_{2} = \chi^{02}_{2} = \chi^{03}_{2} = 0, \]
\[ \chi^{31}_{3} = \chi^{02}_{3} = \chi^{03}_{3} = 0. \] (23)

Using these \( \chi^{jk}_{i} \) into the equation (15), we obtain the energy-momentum components in quasi-Cartesian coordinates,

\[ \mathcal{E}_{QC} = r^3 \left[ M - \frac{r^2 - 2q^2}{(r^2 + q^2)^2} - q^2 \frac{r^2 - q^2}{(r^2 + q^2)^3} \right] \] (24)

The subscript \( QC \) emphasizes that calculations have been performed using quasi-Cartesian coordinates \( \{t, x, y, z\} \). All the three momentum components are found to be zero, i.e.,

\[ P_x = P_y = P_z = 0, \] (25)
Our results, given by equations (18), (19), (24), and (25) for energy and momentum distributions in AG space-time support the claim of Møller that his definition of energy and momentum is applicable to both Schwarzschild coordinates \( \{t, r, \theta, \phi\} \) and quasi-Cartesian coordinates \( \{t, x, y, z\} \).

C. Effective gravitational mass of the AG metric

In the previous sub-section, we showed that Møller’s definition of energy-momentum produces coordinate-independent results. Now, following Cohen and de Felice[1], we interpret the energy distribution as the effective gravitational mass \( \mu(r) \), and then thoroughly analyze the results and explain its importance.

The effective gravitational mass of the AG metric is

\[
\mu(r) = r^3 \left[ M \frac{r^2 - 2q^2}{(r^2 + q^2)^{\frac{3}{2}}} - q^2 \frac{r^2 - q^2}{(r^2 + q^2)^3} \right].
\]

(26)

The asymptotic behavior of the effective gravitational mass is

\[
\mu(r) = M - \frac{q^2}{r} - \frac{9Mq^2}{2r^2} + \frac{4q^4}{r^3} + \frac{75Mq^4}{8r^4} - \frac{9q^6}{r^5} + O \left( \frac{1}{r^6} \right).
\]

(27)

Thus the asymptotic value of the effective gravitational mass is the ADM mass \( M \). It is also clear that at large \( r \), the effective gravitational mass of the AG metric becomes the effective gravitational mass of the Reissner-Nordström metric as obtained by Cohen and de Felice[1] using Komar energy and by Virbhadra[7] using Møller’s complex. \( q = 0 \) in the above equation gives the effective gravitational mass of the Schwarzschild metric.

The behavior of the effective gravitational mass near \( r = 0 \) is

\[
\mu(r) = r^3 \left( \frac{1}{q^2} - \frac{2M|q|}{q^4} \right) + r^5 \left( \frac{6M|q|}{q^6} - \frac{4}{q^4} \right) + O \left( r^7 \right).
\]

(28)

Thus, depending on the values of \( \frac{|q|}{M} \) and \( \frac{r}{M} \), the effective gravitational mass can be negative, zero, or positive. We now make numerous plots to see the behavior of the effective gravitational mass at large as well as small radial distances and also to see the fascinating role of \( \frac{|q|}{M} \) on the effective gravitational mass.

In Figure 1, we plot the ratio of the effective gravitational mass to the ADM mass against the ratio of the radial distance to the ADM mass for different values of \( |q|/M \). The asymptotic value for each case is 1 showing that the total mass of the metric is the ADM
FIG. 1: (color online). In the figure on left side, the ratio of the effective gravitational mass $\mu$ to the ADM mass $M$ is plotted against the ratio of the radial distance $r$ to the ADM mass $M$ for $Q/M = 0.1$ (blue), 0.2 (green), 0.3 (orange), 0.4 (red), and 0.5 (black). Further, the figure on right side, the ratio of the effective gravitational mass $\mu$ to the ADM mass $M$ is plotted against the ratio of the radial distance $r$ to the ADM mass $M$ for $Q/M = 1$ (blue), 2 (green), 3 (orange), 4 (red), and 5 (black). Plots on left and right sides are, respectively, for regular charged massive black holes and for regular charges massive objects with no horizons.

FIG. 2: (color online). The ratio of the effective gravitational mass $\mu$ to the ADM mass $M$ is plotted against the ratio of the radial distance $r$ to the ADM mass $M$ for $|q|/M = 1$ (blue), 2 (green), 3 (orange), 4 (red), and 5 (black) in the vicinity of the center. For any fixed value of $r/M$, the effective gravitational mass is smaller for higher value of $|q|/M$. Thus, the electric charge contributes negatively to the effective gravitational mass. As, for black holes, $|q|/M$ is lower than for charged regular objects with no horizon, for any fixed value of $r/M$ (but outside the even horizon), black holes have higher effective gravitational
FIG. 3: (color online). The ratio of the effective gravitational mass to the ADM mass, i.e., $\mu/M$ is plotted against the ratio of the radial distance to the ADM mass, i.e., $r/M$ and the ratio of the charge to the ADM mass $|q|/M$. The figure on left side is a surface plot whereas the one on right side is the corresponding density plot. This shows behavior of $\mu/M$ at large distances.

masses compared to those without event horizon. In Figure 3, the same effects are shown more clearly through surface and density plots.

In Figure 2, We plot the same quantities as we plotted in Figure 2; however, we plot for no horizon cases only and near the center ($r = 0$). Near the center, unlike the cases of large radial distances, higher value of $\mu/M$ for lower value $|q|/M$ does not always hold. In fact, near $r = 0$, $\mu/M$ could be negative, zero, or positive and that is determined by both $|q|/M$ and $r/M$. The same quantities are plotted in figures 4 and 5 as surface and density plots which show with more clarity the variation of the effective mass to the ADM mass ratio as a function of the radial distance to the ADM mass as well as the ratio of the absolute value of electric charge to the ADM mass.

V. SUMMARY

The AG solution is a static spherically symmetric and asymptotically Minkowskian regular solution of Einstein’s equations coupled with a nonlinear electrodynamic field. The
FIG. 4: (color online). The ratio of the effective gravitational mass to the ADM mass, i.e., $\mu/M$ is plotted against the ratio of the radial distance to the ADM mass, i.e., $r/M$, and the ratio of the charge to the ADM mass $|q|/M$. The figure on left side is a surface plot whereas the one on right side is the corresponding density plot. This plot shows the behavior of $\mu/M$ close to the center.

solution behaves as Reissner-Nordström solution at a large distance and de sitter as $r$ approaches zero. For $|q| \leq$ approximately $0.6M$, this represents black holes and for larger values of $|q|/M$ there is no event horizon.

We calculated the effective gravitational mass of the AG metric and extensively analyzed the results. At large distances, the role of the electric charge $|q|$ is to decrease the effective gravitational mass. However for small value of radial distances, near $r = 0$, the effective gravitational mass dependence on the electric charge is not necessarily similar and the effective gravitational mass could be positive, zero, or negative depending on the radial distance to the ADM mass ratio and relative magnitude of absolute value of electric charge compared to the ADM mass. Our investigation in this work also evidences that Møller’s prescription to find energy and momentum is unaffected by the choice of different coordinate systems.

The study of effective gravitational mass that an electrically neutral test particle experiences predict many physical effects even before calculations. A metric exhibiting negative effective gravitational mass has repulsive effects not only to timelike but also to null geodesics,
FIG. 5: (color online). The ratio of the effective gravitational mass to the ADM mass, i.e., $\mu/M$ is plotted against the ratio of the radial distance to the ADM mass, i.e., $r/M$ and the ratio of the charge to the ADM mass $|q|/M$. The figure on left side is a surface plot whereas the one on right side is the corresponding density plot. This plot exhibits the behavior of $\mu/M$ very close the center.

and hence also affect gravitational lensing phenomenon. Virbhadra’s pioneer research \cite{16} in gravitational lensing as a tool to propose astronomical test to the unproven cosmic censorship was influenced by his works and analysis of the energy distribution (effective gravitational mass) in spacetimes. The knowledge of the effective gravitational mass of AB metric would give more physical insights about the spacetime which might have significant applications to relativistic astrophysics.

VI. ACKNOWLEDGMENTS

The authors heartily thank the National Institute of Technology (NIT), Patna, India for organizing the International Winter Workshop on *Introduction to Research in Einstein’s General Relativity* during which this research work was done. We are also thankful to the people of NIT Patna for their hospitality and the support. SSX thanks the University of Zululand (South Africa) for all support.
[1] J. M. Cohen and F. de Felice, J. Math. Phys., 25 992 (1994); A. Komar, Phys. Rev. 113, 934 (1959).

[2] C. Møller, Ann. Phys. (NY) 4, 347 (1958); 12, 118 (1961).

[3] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields Pergamon Press, 1987 p. 280.

[4] S. Weinberg, Gravitation and Cosmology: Principles and Applications of General Theory of Relativity (John Wiley and Sons, Inc., New York, 1972) p. 165.

[5] A. Papapetrou, Proc. R. Irish. Acad. A52, 11 (1948).

[6] P. G. Bergmann and R. Thomson, Phys. Rev. 89, 400 (1953).

[7] K. S. Virbhadra, Phys. Rev. D41, 1086 (1990); ibid D42 1066 (1990); ibid D42, 2919 (1990); A. Chamorro and K. S. Virbhadra, Pramana-J. Phys. 45, 181 (1995); Int. J. Mod. Phys. D5, 251 (1996); K. S. Virbhadra and J. C. Parikh, Phys. Lett. B317, 312 (1993); ibid B331 302 (1994); K. S. Virbhadra, Int. J. Mod. Phys. A12 4831 (1997); ibid D6 357 (1997); Pramana 44 317 (1995); Pramana 38 31 (1992); Phys.Lett. A157 195 (1991).

[8] K. S. Virbhadra, Phys. Rev. D60, 104041 (1999); J. M. Aguirregabiria, A. Chamorro and K. S. Virbhadra, Gen. Relativ. & Gravit. 28, 1393 (1996).

[9] N. Rosen and K. S. Virbhadra, Gen. Relativ. Gravit. 25, 429 (1993); K. S. Virbhadra, Pramana 45 215 (1995).

[10] S. S. Xulu, Int. J.Theor. Phys. 37 1773 (1998); Int. J.Mod. Phys. D7 773 (1998); Int. J. Mod. Phys. A 15, 2979 (2000); Int. J. Theor. Phys. 39, 1153 (2000); Int. J. Mod. Phys. A 15, 4849 (2000); Mod. Phys. Lett. A 15, 1511 (2000); Astrophys. Space Sci. 283, 23 (2003); S. S. Xulu, Int. J. Theor. Phys. 46, 2915 (2007); Chin. J. Phys. 44, 348 (2006); Found. Phys. Lett. 19, 603 (2006); I. Radinschi, Chin. J. Phys. 39, 393 (2001); Chin. J. Phys. 39, 231 (2001); Int. J. Mod. Phy. D 13, 1019 (2004).

[11] E. C. Vagenas, Mod. Phys. Lett. A 21, 1947 (2006); Int. J. Mod. Phys. D 14, 573 (2005); T. Multamaki, A. Putaja, I. Vilja and E. C. Vagenas,Class. Quant. Grav. 25, 075017 (2008).

[12] V. C. de Andrade, L. C. T. Guillen and J. G. Pereira, Phys. Rev.Lett. 84, 4533 (2000); S. L. Loi and T. Vargas, Chin. J. Phys. 43, 901 (2005); O. Aydogdu, Int. J. Mod. Phys. D 15, 459 (2006); O. Aydogdu and M. Salti, Czech. J. Phys. 56, 8 (2006); S. Aygun and I. Tarhan, Pramana: J. Phys. 78, 531 (2012); P. K. Sahoo, K. L. Mahanta, D. Goit, A. K. Sihna,
U. R. Das, A. Prasad and R. Prasad, [arXiv:1409.6513 [gr-qc]].

[13] J. Bardeen. in Proceedings of GR5, Tiflis, U.S.S.R., 1968.

[14] E. Ayón-Beato and A. García, Phys. Rev. Lett., 80 5056 (1998); Gen. rel. grav., 31 629 (1999).

[15] Mathematica 9.0.

[16] K. S. Virbhadra, D. Narasimha and S. M. Chitre, Astron. Astrophys. 337, 1 (1998); K. S. Virbhadra and G. F. R. Ellis, Phys. Rev. D 62, 084003 (2000); ibid 65, 103004 (2002); C-M. Claudel, K. S. Virbhadra and G. F. R. Ellis, J. Math. Phys. 42, 818 (2001); K. S. Virbhadra and C. R. Keeton, 79, 083004 (2009); 77, 124014 (2008); K. S. Virbhadra, Phys. Rev. D 79, 083004 (2009).