Ground state spin 0+ dominance of many-body systems with random interactions and related topics

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In this talk we shall show our recent results in understanding the spin\textsuperscript{parity} 0+ ground state (0 g.s.) dominance of many-body systems. We propose a simple approach to predict the spin I g.s. probabilities which does not require the diagonalization of a Hamiltonian with random interactions. Some findings related to the 0 g.s. dominance will also be discussed.
1 INTRODUCTION

The ground state spins\textsuperscript{parity} of even-even nuclei are always 0\textsuperscript{+}, which is believed to be a consequence of the attractive short-range interactions between nucleons. However, a predominance of spin\textsuperscript{parity} 0\textsuperscript{+} ground states (0 g.s.) was discovered by Johnson, Bertsch and Dean in 1998 using the two-body random ensemble (TBRE) \cite{1} and was related to a reminiscence of generalized seniority by Johnson, Bertsch, Dean and Talmi in 1999 \cite{2}. These phenomena have been confirmed by many works in different systems \cite{3}. There have been many efforts to understand this interesting and important observation \cite{4, 5, 6, 7, 8, 9}. Further works have been reported on generic collectivity of many-body systems in the presence of random interactions \cite{4, 10}, the behavior of average energies \cite{8, 9}, etc.

2 SIMPLE SYSTEMS WITH A FEW FERMIONS IN SINGLE-\textit{j} SHELLS

2.1 Results obtained by diagonalizing energy matrices with the TBRE

The Hamiltonian that we use for fermions in a single-\textit{j} shell is defined as follows

\[ H = \sum_{J} G_{J} A^{J\dagger} \cdot A^{J} \equiv \sum_{J} \sqrt{2J+1}G_{J} (A^{J\dagger} \times \tilde{A}^{J})^{0}, \]

\[ A^{J\dagger} = \frac{1}{\sqrt{2}} (a_{J}^{\dagger} \times a_{J}^{\dagger})^{J}, \quad \tilde{A}^{J} = -\frac{1}{\sqrt{2}} (\tilde{a}_{J} \times \tilde{a}_{J})^{J}, \quad G_{J} = \langle j^{2}J|V|j^{2}J \rangle. \]

\( G_{J} \)'s are taken as a set of Gaussian-type random numbers with a width being 1 and an average being 0. This two-body random ensemble is referred to as “TBRE”. Hamiltonian of fermions in many-\textit{j} shells or a boson Hamiltonian can be defined in a similar way. The I g.s. probabilities in this paper are obtained by 1000 runs of a TBRE Hamiltonian.

We take very simple systems such as those with four fermions in a single-\textit{j} shell. Many different values of \textit{j} from 7/2 to 33/2 are taken into account. Fig. 1 shows
the 0 g.s. probabilities of 4 fermions in different single-\(j\) shells. From this figure we observe interesting oscillations of the 0 g.s. probabilities \(P(0)\) as a function of \(j\). One also notices easily that the \(P(0)\)'s are the largest, except for a few small \(j\) cases among the \(P(I)\)'s, which are the probabilities of a state with spin \(I\) to be the ground state.

### 2.2 An empirical formula to predict the \(P(I)\)'s

Let us set only one of the \(G_J\)'s equal to \(-1\) and the others to zero, and find the spin \(I\) of the ground state. We repeat this process for all two-body interactions \(G_J\). We can find how many times the ground state has angular momentum \(I\). This number is denoted as \(N_I\) and the values of \(N_I\)'s for four nucleons in a single-\(j\) shell can be easily counted by looking at Table 1. Using the \(N_I\), we can predict the probability that the ground state has angular momentum \(I\) as \(P(I)_{emp} = N_I/N\), where \(N\) is the number of independent two-body matrix elements (\(N = j + 1/2\) for fermions in a single-\(j\) shell).

A nice agreement between our predicted \(P(0)\)'s and those obtained by diagonalizing a TBRE Hamiltonian is shown in Fig. 2. The empirical formula is also used to predict \(P(I)\)'s of systems with four and five nucleons in a \(j = 9/2\) shell. The results are shown in Fig. 3 a) and b). The agreements are again remarkable. For more complicated systems, such as those with four, five, six and seven nucleons in two-\(j\) shells and \(sd\) bosons, the formula works very well, too. To exemplify, Fig. 3c) and 3d) show the cases of seven fermions in a two-\(j\) \((j_1 = 7/2, j_2 = 5/2)\) shell and 10 \(sd\) bosons.

### 3 AVERAGE ENERGIES

#### 3.1 Probabilities of \(I\) g.s. for average energies \(\bar{E}_I\)

Our problem becomes simpler for the complicated systems discussed in the previous sections, if we take a trace of each energy matrix. The average energy \(\bar{E}_I\) can be
expressed in terms of linear combination of \(G_J\)’s:

\[
\bar{E}_I = \sum_J \bar{\alpha}_J^I G_J,
\]

where \(\bar{\alpha}_J^I\) is obtained by averaging

\[
\bar{\alpha}_J^I = \frac{n(n-1)}{2} \sum_{K,\gamma} \left( \langle j^{n-2}K\gamma, j^2J|j^nI\beta \rangle \right)^2
\]

over all \(\beta\)’s. Here \(\langle j^{n-2}K\gamma, j^2J|j^nI\beta \rangle\) are the two-body coefficients of fractional parentage, and \(\beta\) (or \(\gamma\)) refers to additional quantum numbers to define a state of \(n\) (or \(n-2\)) fermions with total angular momentum \(I\) (or \(K\)) uniquely.

We are now ready to apply the same method to predict the probabilities \(\mathcal{P}(I)\)’s of \(\bar{E}_I\) to be the lowest. From Table 2, we can find that \(N_0 = 1\), \(N_2 = 1\), \(N_3 = 2\) and \(N_{12} = 1\) for 4 fermions in a \(j = \frac{9}{2}\) shell. The total number \(N\) of \(G_J\)’s in this shell is 5. Using the empirical formula discussed above, we can predict the \(\mathcal{P}(I)\)’s. The results (labeled by “\(\mathcal{P}^{G_J=-1}(I)\)”) are shown in Table 3 together with \(\mathcal{P}^{exp}(I)\)’s calculated by diagonalizing a TBRE Hamiltonian in 1000 runs.

Comparing the values of \(\mathcal{P}^{exp}(I)\)’s with the ones shown as \(\mathcal{P}^{G_J=-1}(I)\), we see that the agreements are qualitatively good. However, they are not as good as in the cases discussed in the previous section.

One thing, here, should be examined. When the sign of \(G_J\) is changed to be positive, again some \(\bar{E}_I\)’s can be the lowest. Because all \(\bar{E}_I\)’s are pushed up, the lowest ones are those which are least favored by \(G_{J'} = \delta_{J',J}\). Table 2 shows the lowest \(I\)’s which are given by \(G_{J'} = \delta_{J',J}\) (the row labeled by “\(G_J = +1\)”).

For \(G_0\) there are several \(\bar{E}_I\)’s with \(I = 3, 5, 7, 9, 10, 12\) are degenerate. When this kind of degeneracy occurs, we give each of them a weight (one over how many times are degenerate). We then count how many \(G_{J'} = \delta_{J',J}\) and \(G_{J'} = -\delta_{J',J}\) give the lowest energy to an angular momentum \(I\). We divide this number by \(2j + 1\) which is the total number of \(G_{J'} = -\delta_{J',J}\) and \(G_{J'} = \delta_{J',J}\). The probabilities thus calculated are shown as \(\mathcal{P}^{G_J=\pm 1}\). We see much better agreements between the \(\mathcal{P}^{exp}\) calculated using a TBRE and those shown as \(\mathcal{P}^{G_J=\pm 1}\).
We then have to reconsider the empirical way to predict $P(I)'s$, which is discussed in the previous section, where we take into account $G_{J'} = -\delta_{J',J}$ only. The same kind of improvement can be obtained, as shown in Table 5. However, a large number of states have roughly the same energy near the ground states, and therefore the results are not so different from the prediction when only $G_{J'} = -\delta_{J',J}$ is taken into account.

3.2 A trajectory of average energy in terms of $I(I + 1)$

Let $\langle \bar{E}_I \rangle_{\text{min}}$ be a quantity obtained by averaging $\bar{E}_I$ only over those cases where $I$ of the lowest $\bar{E}_I$ is around $I_{\text{min}}$ among the ensemble. We find that $\langle \bar{E}_I \rangle_{\text{min}}$ is nearly proportional to $I(I + 1)$, similar to a rotational spectra. The very same can be found when we replace “min” by “max”. This is because $P(I)'s$ are roughly symmetric about $I_{\text{max}}/2$. Several examples of $\langle \bar{E}_I \rangle_{\text{min}}$ are shown in Fig. 4.

We can explain this finding statistically. The details are described in [9].

4 BY-PRODUCTS

As having been well known for many years, the monopole pairing $G_J = -\delta_{J0}$ gives a ground state with $I = 0$ for an even number of fermions and a ground state with $I = j$ for an odd number of fermions. It is quite interesting to observe that $G_J = -\delta_{J2}$ gives a ground state with $I = n$ ($n=$even) and $I = j-(n-1)/2$ ($n =$odd) in most cases that we have checked [7], although there are some exceptions. Here $n$ is the number of fermions in a single-$j$ shell.

We also observe that the interaction of $G = -\delta_{J(2j-1)}$ gives a large array of eigenvalues which are asymptotic integers. An understanding of this observation is in progress [11].
5 SUMMARY

A simple method is proposed to predict the probability $P(I)$ that the ground state has angular momentum $I$ in many-body systems interacting via a two-body random ensemble. We find and predict that $P(0)$ is always the largest except for a few cases.

We also study the probabilities of average energies with fixed $I$ to be the lowest. It is interesting to find a trajectory of the energy $\langle \bar{E}_I \rangle_{\text{min}}$ (and $\langle \bar{E}_I \rangle_{\text{max}}$) which is nearly proportional to $I(I+1)$.

As by-products, we find: 1) the quadrupole pairing interaction seems to favor the ground states which have angular momentum $n$ ($n$ is even) or $j - (n - 1)/2$ ($n$ is odd) in systems with $n$ fermions in a single-$j$ shell; 2) the highest-multipole pairing interaction presents a large array of asymptotic integer eigenvalues.
Figure 1 The g.s. probabilities of $I=0, 2, 4$, and $I_{max}$ of four fermions in a single-$j$ shell.

Figure 2 A comparison between our predicted $P(0)$'s and those obtained by 1000 runs of a TBRE Hamiltonian.

Figure 3 Several examples of comparison between the empirically predicted $I$ g.s. probabilities and those obtained by diagonalizing TBRE Hamiltonians.

Figure 4 The $I(I+1)$ behavior of $\langle \tilde{E} \rangle_{min}$.
Table 1  The angular momenta $I$’s which give the lowest eigenvalues for 4 fermions in a single-$j$ shell, when $G_J = -1$ and all other parameters are 0.

| $2j$ | $G_0$ | $G_2$ | $G_4$ | $G_6$ | $G_8$ | $G_{10}$ | $G_{12}$ | $G_{14}$ | $G_{16}$ | $G_{18}$ | $G_{20}$ | $G_{22}$ | $G_{24}$ | $G_{26}$ | $G_{28}$ | $G_{30}$ |
|------|-------|-------|-------|-------|-------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 7    | 0     | 4     | 2     | 8     |       |          |          |          |          |          |          |          |          |          |          |          |
| 9    | 0     | 4     | 0     | 0     | 12    |          |          |          |          |          |          |          |          |          |          |          |
| 11   | 0     | 4     | 0     | 4     | 8     | 16       |          |          |          |          |          |          |          |          |          |          |
| 13   | 0     | 4     | 0     | 2     | 12    | 20       |          |          |          |          |          |          |          |          |          |          |
| 15   | 0     | 4     | 0     | 2     | 0     | 16       | 24       |          |          |          |          |          |          |          |          |          |
| 17   | 0     | 4     | 6     | 0     | 4     | 2        | 0        | 20       | 28       |          |          |          |          |          |          |          |
| 19   | 0     | 4     | 8     | 0     | 2     | 8        | 2        | 16       | 24       | 32       |          |          |          |          |          |          |
| 21   | 0     | 4     | 8     | 0     | 2     | 0        | 0        | 20       | 28       | 36       |          |          |          |          |          |          |
| 23   | 0     | 4     | 8     | 0     | 2     | 0        | 10       | 2        | 0        | 24       | 32       | 40       |          |          |          |          |
| 25   | 0     | 4     | 8     | 0     | 2     | 4        | 8        | 10       | 6        | 0        | 28       | 36       | 44       |          |          |          |
| 27   | 0     | 4     | 8     | 0     | 2     | 4        | 2        | 0        | 0        | 4        | 20       | 32       | 40       | 48       |          |          |
| 29   | 0     | 4     | 8     | 0     | 0     | 2        | 6        | 8        | 12       | 8        | 0        | 24       | 36       | 44       | 52       |          |
| 31   | 0     | 4     | 8     | 0     | 0     | 2        | 0        | 8        | 14       | 16       | 6        | 0        | 32       | 40       | 48       | 56       |

Table 2  The spins of the lowest $\bar{E}_I$ for 4 fermions in a single $j=9/2$ shell when only one of $G_J$ is set to $-1$ or $+1$.

| $G_J$ | $G_0$ | $G_2$ | $G_4$ | $G_6$ | $G_8$ |
|-------|-------|-------|-------|-------|-------|
| $G_J = -1$ | 0     | 2     | 3     | 3     | 12    |
| $G_J = +1$ | 3, 5, 7, 9, 10,12 | 12    | 12    | 2     | 3     |

Table 3  $I$ g.s. probabilities for average energies of four fermions in a $j=9/2$ shell.

| $I$ | $0$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $12$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|
| $p^{G_J=-1}(I)$ | 10.2 | 15.4 | 28.9 | 1.7 | 0.6 | 0.3 | 3.2 | 0  | 0   | 8.7  | 31.0 |
| $p^{G_J=+1}(I)$ | 20 | 20 | 40 | 0 | 0 | 0 | 0 | 0 | 0 | 20 |     |
| $p^{G_J=\pm1}(I)$ | 10 | 21.6 | 31.6 | 0 | 1.6 | 1.6 | 1.6 | 1.6 | 1.6 | 31.6 |     |
Table 4  The ground state spins of 4 fermions in a single $j=9/2$ shell when only one of $G_J$ is set to +1 or −1.

| $G_J$ | $G_0$ | $G_2$ | $G_4$ | $G_6$ | $G_8$ |
|-------|-------|-------|-------|-------|-------|
| $G_J = -1$ | 0,2,3,4,5,6,7,8,9,10,12 | 0,12 | 0 | 0 | 0 |
| $G_J = +1$ | 0,2,3,4,5,6,7,8,9,10,12 | 0,12 | 0 | 0 | 0 |

Table 5  I.g.s. probabilities of four fermions in a $j=9/2$ shell.

| $I$ | 0 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 |
|-----|---|---|---|---|---|---|---|---|---|----|----|
| $P^{G_J=1}(I)$ | 66.4 | 3.7 | 0 | 11.8 | 0 | 0 | 0 | 0.2 | 0 | 0 | 17.9 |
| $P^{G_J=-1}(I)$ | 60 | 0 | 0 | 20 | 0 | 0 | 0 | 0 | 0 | 0 | 20 |
| $P^{G_J=\pm1}(I)$ | 65.8 | 0.8 | 0.8 | 11.5 | 0.8 | 1.5 | 0.8 | 0.8 | 0.8 | 0.8 | 15.8 |
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$0$-g.s. probabilities (%) vs. $j$
$I_g.s.$ probabilities (in %)

(a) TBR E, pred.
$j=9/2$ shell with 4 fermions

(b) TRRE, Pred.
$j=9/2$ shell with 5 fermions

(c) 7 fermions in the $j_1=7/2, j_2=5/2$ orbits

(d) 10 sd bosons system

angular momentum $l$
a) $2j=17, n=4$ g.s. are included.

b) $(2j_1, 2j_2)=(5,7)$, $n=4$. $l \leq 3$ g.s. are included.

c) 20 $d$-bosons, $l \leq 5$ g.s. are included.

d) 10 $sd$ bosons, $l \leq 3$ g.s. are included.