Well-posedness of generalized KdV and one-dimensional fourth-order derivative nonlinear Schrödinger equations for data with an infinite $L^2$ norm

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Abstract
We study the Cauchy problem for the generalized Korteweg–de Vries (KdV) and one-dimensional fourth-order derivative nonlinear Schrödinger equations, for which the global well-posedness of solutions with small rough data in certain scaling limit of modulation spaces $\mathcal{M}^{\mu}_{2,1}$ is shown, which contain some data with infinite $L^2$ norm.

KEYWORDS
fourth-order nonlinear Schrödinger equation, KdV, scaling limit of modulation space, well-posedness

1 | INTRODUCTION

In this paper, we consider the generalized KdV and fourth-order derivative nonlinear Schrödinger equations on real line as follows:

\begin{align}
\partial_t u + \partial_x^3 u &= \lambda \partial_x (u^{m+1}), \quad u(0, x) = u_0(x), \quad (1) \\
i \partial_t u + \partial_x^4 u &= \lambda \partial_x (u^{m+1}), \quad u(0, x) = u_0(x). \quad (2)
\end{align}

Here, $u(t, x)$ is a complex-valued function of $(t, x) \in \mathbb{R}^{1+1}$, $i = \sqrt{-1}, \lambda \in \mathbb{C}, \partial_t = \partial/\partial t, \partial_x = \partial/\partial x, \partial_x^n = \partial^n/\partial x^n$ for $n = 3, 4$. $m$ is an integer. We will study the global well-posedness of (1) and (2) with small rough data in scaling limit of modulation spaces $\mathcal{M}^{\mu}_{2,1}$.

The modulation space $\mathcal{M}^{p,q}_2$ is the function space, introduced by Feichtinger \(^1\) in the 1980s using the short-time Fourier transform to measure the decay and the regularity of the function.
differently from the usual $L^p$ Sobolev spaces or Besov–Triebel spaces. Roughly speaking, Besov–Triebel spaces mostly use the dyadic decompositions of the frequency space, while the modulation spaces use the uniform decompositions. Modulation spaces also have many applications in the analysis of partial differential equations. For example, the Schrödinger and wave semigroups, which are not bounded on neither $L^p$ nor $B^s_{p,q}$ for $p \neq 2$, are bounded on $M^s_{p,q}$ (see Ref. 2). Therefore, the modulation space is a good space for initial data of the Cauchy problem for nonlinear dispersive equations (see Refs. 3–8). However, as showed by Sugimoto and Tomita,\(^9\) modulation spaces do not have good scaling properties such as $L^p$ spaces. Therefore, to obtain some kinds of modulation spaces with good scaling property, Bényi and Oh,\(^10\) also Sugimoto and Wang,\(^11\) introduced the scaling limit of modulation spaces $\mathcal{M}^\mu_{p,q}$. They studied the basic properties of these spaces such as the scaling property, the dual space, and the algebraic property. They also got some applications in nonlinear Schrödinger equations. In this paper, we obtain some other applications of the scaling limit of modulation spaces in nonlinear dispersive equations.

Our main results are as follows.

**Theorem 1.** Let $m \geq 8, 0 \leq \mu \leq A$, where

$$A = \frac{3(p - 4)}{2(3p + 2)}, \quad p = \frac{m}{2}. \quad (3)$$

Then there exists $\delta > 0$, such that for any $u_0 \in \mathcal{M}^\mu_{2,1}$, with $\|u_0\|_{\mathcal{M}^\mu_{2,1}} \leq \delta$, then (2) has a unique global solution $u \in C(\mathbb{R}, \mathcal{M}^\mu_{2,1}) \cap X^{\mu-A}(L_6^{p+2/3} L_t^{3p+2})$, where $X^{\mu-A}(L_x^{p+2/3} L_t^{3p+2})$ will be defined in Definition 1. Moreover, the data-to-solution map above is Lipschitz continuous.

**Remark 1.** By the inclusion relation between modulation spaces and the scaling limit of modulation spaces (Proposition 3.2 in Ref. 12), we know that when $0 \leq \mu \leq 3(m - 8)/(6m + 8)$, we have $M_{2,1} \hookrightarrow \mathcal{M}^\mu_{2,1} \hookrightarrow M_{p_0,1}$, where $p_0 = (3m + 4)/14$. Also, by Proposition 1, when $\mu > 0$, $\mathcal{M}^\mu_{2,1} \hookrightarrow L^2$ is not true, which means that there exists some $u_0 \in \mathcal{M}^\mu_{2,1}$ with arbitrarily large $L^2$ norm, such that (2) is global well-posed with initial data $u_0$. Moreover, by Example 1, the $L^2$ norm of $u_0$ could even be infinite.

The fourth-order nonlinear Schrödinger equations (4NLS) have been studied by many authors. For equations without derivative on nonlinear term, the well-posedness theory in $H^s$ has been studied in Refs. 13–16. Hayashi et al\(^17\)–\(^19\) studied the well-posedness and scattering theory of (4NLS) in weighted Sobolev spaces. For equations with first-order derivative on nonlinear terms (d-4NLS) such as (2), the scaling invariant homogeneous Sobolev space is $\tilde{H}^{s_c}$, where $s_c = 1/2 - 3/m$. That is to say, for any solution $u(t, x)$ of (2) with initial data $u_0$, the scaling function $u_\lambda(t, x) = \lambda^{3/m} u(\lambda^4 t, \lambda x)$ is also a solution of (2) with initial data $u_{0,\lambda}(x) = \lambda^{3/m} u_0(\lambda x)$, and satisfies

$$\|u_{0,\lambda}\|_{\tilde{H}^{s_c}} = \|u_0\|_{\tilde{H}^{s_c}}. \quad (4)$$

Wang\(^20\) proved the global well-posedness of (2) with small initial data in $\tilde{H}^{s_c}$ when $m \geq 4$. The cases of $m = 2, 3$ and high-dimensional cases of (d-4NLS) have been studied by Hirayama and Okamoto.\(^21\) The well-posedness theory in weighted Sobolev spaces was studied by Hayashi et al.\(^22\)–\(^23\) For equations with higher-order derivative on nonlinear terms, local well-posedness in
Remark 2. By the results listed above, we could only know the well-posedness of (d-4NLS) with initial data in $H^s$ or $M^s_{2,1}$ for $s > 0$. These spaces are $L^2$ subcritical, which means that they are subspaces of $L^2$. By Remark 1, our result is $L^2$ supercritical. Meantime, the space of scaling limit $M^\mu_{2,1}$ is a nice substitution of the space $M_{2,1}$ for the reason that $M^\mu_{2,1}$ has better scaling properties as showed in Section 5 of Ref. 11. Therefore, by the scaling of the solution $u(t, x) = \lambda^{3/m} u(\lambda^4 t, \lambda x)$, we could also obtain the critical index $\mu_c = 1/2 - 3/m$. We could suspect that the critical space of well-posedness of (d-4NLS) is $M^\mu_{2,1}$, which means that when $\mu \leq \mu_c$, (d-4NLS) is well-posed in $M^\mu_{2,1}$, when $\mu \geq \mu_c$, (d-4NLS) is ill-posed in $M^\mu_{2,1}$. As showed in Theorem 1, we obtain the well-posedness of (d-4NLS) in $M^A_{2,1}$, where $A \leq \mu_c$, which is part of our conjecture.

For the generalized KdV equation (1), our main result is as follows.

**Theorem 2.** Let $m \geq 4$, $A = (m-4)/(2m+2)$, $0 \leq \mu \leq A$. Then, there exists $\delta > 0$ such that for any $u_0 \in M^\mu_{2,1}$ with $||u_0||_{M^\mu_{2,1}} \leq \delta$, then (1) has a unique global solution $u \in C([\mathbb{R}, M^\mu_{2,1}) \cap X^{\mu-A}(L^{m+1}_x L^{2(m+1)}_t)$. Moreover, the data-to-solution map above is Lipschitz continuous.

Remark 3. By the inclusion between modulation spaces and the scaling limit of modulation spaces, we have $M_{2,1} \hookrightarrow M^\mu_{2,1} \hookrightarrow M_{2(m+1)/5,1}$. Therefore, Theorem 2 is a generalization of the corresponding result of Theorem 1.2 in Ref. 29. Meanwhile, the proof of Theorem 2 is similar to the proof of Theorem 1. We omit it for simplicity.

By the same scaling argument as for (2), we know that the critical Sobolev spaces of (1) is $\dot{H}^{\infty}$, $s_c = 1/2 - 2/m$. Recall that Kenig et al\textsuperscript{30} showed the global well-posedness of (1) with small initial data in $\dot{H}^{\infty}$ when $m \geq 4$. Later, they proved the ill-posedness of focusing KdV in $\dot{H}^{s}$ when $s < s_c$.\textsuperscript{31} Molinet and Ribaud\textsuperscript{32} obtained the global well-posedness result in $\dot{B}^{\infty}_{2,\infty}$. Wang and Huang\textsuperscript{29} proved the global well-posedness of (1) in $M_{2,1}$, which was the case of $\mu = 0$ in Theorem 2. For the cases of $m = 1, 2, 3$, many authors have been studied on it. One can refer to Refs. 8, 33–35. For study of the fifth-order KdV equations, one can refer to Refs. 36–38.

The paper is organized as follows. In Section 2, we will give notations and definitions of some function spaces, including the scaling limit of modulation spaces. In Section 3, we will give the characterization of scaling limit of modulation spaces and (Fourier) Lebesgue spaces. The proof of our main result will be given in Sections 4–6.

2 | **PRELIMINARY**

2.1 | **Notation**

We write $\mathcal{S}(\mathbb{R}^d)$ to denote the Schwartz space of all complex-valued rapidly decreasing infinity differentiable functions on $\mathbb{R}^d$, and $\mathcal{S}'(\mathbb{R}^d)$ to denote the dual space of $\mathcal{S}(\mathbb{R}^d)$, called the space of tempered distributions. For simplification, we omit $\mathbb{R}^d$ without ambiguity. The (inverse) Fourier
The short-time Fourier transform \((\mathcal{F}^{-1})\mathcal{F}\) can be defined as follows:

\[
\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix \xi} \, dx,
\]

\[
\mathcal{F}^{-1} f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix \xi} \, d\xi.
\] (5)

For \(\lambda \in \mathbb{R}^+\), denote \(f_\lambda(x) = f(\lambda x)\). For \(1 \leq p < \infty\), we define the \(L^p\) norm:

\[
\|f\|_p = \left(\int_{\mathbb{R}^d} |f(x)|^p \, dx\right)^{1/p},
\] (6)

and \(\|f\|_\infty := \text{ess sup}_{x \in \mathbb{R}^d} |f(x)|\). The Fourier Lebesgue space \(\mathcal{F}L^r\) is defined as follows:

\[
\mathcal{F}L^r = \{ f \in \mathcal{S}' : \|F f\|_{r'} < \infty \}.
\] (7)

We use the notation \(I \lesssim J\) if there is an independent constant \(C\) such that \(I \leq CJ\). Also, we denote \(I \approx J\) if \(I \lesssim J\) and \(J \lesssim I\). For \(1 \leq p \leq \infty\), we denote the dual index \(p'\) with \(1/p + 1/p' = 1\).

For \(a, b \in \mathbb{R}\), denote \(a \vee b = \max\{a, b\}\), \(a \wedge b = \min\{a, b\}\). Let \(W(t) = e^{it^4} = \mathcal{F}^{-1} e^{it^4} \mathcal{F}\), be the fourth-order Schrödinger semigroup; we denote

\[
\mathcal{D} f(t) = \int_0^t W(t-s)f(s)ds.
\] (8)

**Definition 1.** Let \((A, \| \cdot \|_A)\) be a Banach space. For \(\mu \in \mathbb{R}\), denote

\[
X^\mu(A) := \left\{ u \in \mathcal{S}'(\mathbb{R}^2) : u = \sum_{j \leq 0} u_j, \text{ with } \sum_{j \leq 0} 2^{j \mu} \sum_{k \in \mathbb{Z}} \left\| \square^j, k u_j \right\|_A < \infty \right\},
\] (9)

where the norm of this space is as follows:

\[
\|u\|_{X^\mu(A)} := \inf \sum_{j \leq 0} 2^{j \mu} \sum_{k \in \mathbb{Z}} \left\| \square^j, k u_j \right\|_A,
\] (10)

where the infimum is taken over all the decompositions of \(u = \sum_{j \leq 0} u_j \in X^\mu(A)\).

**Remark 4.** By the definition above, we could easily obtain \(X^\mu(A) \hookrightarrow X^{\mu+\varepsilon}(A)\) for any \(\varepsilon > 0\).

### 2.2 Modulation spaces and scaling limit of modulation spaces

Recall that the short-time Fourier transform of \(f\) respect to a window function \(g \in \mathcal{S}\) is defined as (see Ref. 1):

\[
V_g f(x, \xi) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-it \xi} \, dt.
\] (11)
For $1 \leq p, q \leq \infty$, we denote

$$\|f\|_{M_{p,q}} = \left\| \left\| V_g f(x, \xi) \right\|_\ell_p \right\|_{L^q}\xi := \left\| \left\| V_g f(x, \xi) \right\|_{L^q}^q \right\|_\ell_p.$$  \hspace{1cm} (12)

The modulation space $M_{p,q}$ is defined as the space of all tempered distributions $f \in \mathcal{S}'$ for which $\|f\|_{M_{p,q}}$ is finite.

We give another equivalent definition of modulation spaces by uniform decomposition of the frequency space (see Refs. 39, 40).

Let $\sigma$ be a smooth cutoff function adapted to the unit cube $[-1/2, 1/2]^d$ and $\sigma = 0$ outside the cube $[-3/4, 3/4]^d$, we write $\sigma_k = \sigma(\cdot - k)$, and assume that

$$\sum_{k \in \mathbb{Z}^d} \sigma_k(\xi) \equiv 1, \forall \xi \in \mathbb{R}^d. \hspace{1cm} (13)$$

Denote $\Box_k = \mathcal{F}^{-1} \sigma_k \mathcal{F}$, then we have the following equivalent norm of modulation space:

$$\|f\|_{M_{p,q}} = \left\| \left\| \Box_k f \right\|_{L^p} \right\|_{\ell^q} \hspace{1cm} (14)$$

Let us recall the definition of the scaling limit of modulation spaces (see Ref. 11). Denote $\sigma_{j,k}(\xi) = \sigma(2^{-j} \xi - k)$, and $\Box_{j,k} = \mathcal{F}^{-1} \sigma_{j,k} \mathcal{F}$, then we denote

$$\|f\|_{M_{p,q}^{[j]}} := \left\| \left\| \Box_{j,k} f \right\|_{L^p} \right\|_{\ell^q}. \hspace{1cm} (15)$$

The scaling limit of modulation spaces $M_{p,q}^\mu$ is defined as follows:

$$M_{p,q}^\mu = \left\{ f \in \mathcal{S}' : \exists f_j \in M_{p,q}^{[j]} \text{ such that } f = \sum_{j \leq 0} f_j, \sum_{j \leq 0} 2^{j\mu} \|f_j\|_{M_{p,q}^{[j]}} < \infty \right\} \hspace{1cm} (16)$$

and the norm on $M_{p,q}^\mu$ is defined as

$$\|f\|_{M_{p,q}^\mu} = \inf \sum_{j \leq 0} 2^{j\mu} \|f_j\|_{M_{p,q}^{[j]}},$$  \hspace{1cm} (17)

where the infimum is taken over all the decompositions of $f = \sum_{j \leq 0} f_j \in M_{p,q}^\mu$.

Remark 5. By the trivial decomposition $f = f + 0 + \cdots$, we know $M_{p,q} \hookrightarrow M_{p,q}^\mu$. For this reason, we could regard $M_{p,q}^\mu$ as an extension of $M_{p,q}$.

2.3 | Useful lemmas

In this subsection, we gather some useful results.
Lemma 1 (Bernstein’s inequality, Lemma 6.1 in Ref. 41). Let $1 \leq p \leq q \leq \infty, R > 0, \xi_0 \in \mathbb{R}^d$. Denote

$$L^p_{B(\xi_0, R)} = \{ f \in L^p : \text{supp } \hat{f} \subseteq B(\xi_0, R) \}. \quad (18)$$

Then there exists $C > 0$, such that

$$\|f\|_q \leq CR^{d(1/p - 1/q)} \|f\|_p \quad (19)$$

hold for all $f \in L^p_{B(\xi_0, R)}$ and $C$ is independent of $R > 0$ and $\xi_0 \in \mathbb{R}^d$.

Lemma 2 (Christ-Kiselev, Lemma 2 in Ref. 42). Let $T$ be a linear operator defined in space-time functions $f(t, x)$ by

$$Tf(t) = \int_{\mathbb{R}} K(t, s) f(s) ds \quad (20)$$

such that

$$\|Tf\|_{L^{p_1}_{x} L^{q_1}_{t}} \lesssim \|f\|_{L^{p_2}_{x} L^{q_2}_{t}}, \quad (21)$$

where $p_2 \vee q_2 < p_1 \wedge q_1$. Then

$$\left\| \int_0^t K(t, s) f(s) ds \right\|_{L^{p_1}_{x} L^{q_1}_{t}} \lesssim \|f\|_{L^{p_2}_{x} L^{q_2}_{t}}. \quad (22)$$

Lemma 3 (Lemma 4.1 in Ref. 12). Let $1 \leq p, q, r \leq \infty$. Then, $M_{p,q} \hookrightarrow L^r$ if and only if $p \vee q \leq r \leq q'$.

Lemma 4 (Lemma 5.2 in Ref. 12). Let $1 \leq p, q, r \leq \infty$. Then, $M_{p,q} \hookrightarrow \mathcal{F}L^r$ if and only if $p \leq 2, q \leq r' \leq p'$.

3 INCLUSION BETWEEN $M^\mu_{p,q}$ AND $L^r$

Proposition 1. Let $1 \leq p, q, r \leq \infty$, denote

$$a(p, q) = d(1/p + 1/q - 1),$$

$$\sigma(p, q) = 0 \wedge d(1/q - 1/p) \wedge a(p, q). \quad (23)$$

Let $\sigma(p, q) \leq \mu \leq a(p, q), \sigma(p, q) < a(p, q).$ Then, $M^\mu_{p,q} \hookrightarrow L^r$ if and only if

$$p \vee q \leq r \leq q', \mu - d/p \leq -d/r. \quad (24)$$

Proof. Sufficiency: for any $f = \sum_{j \in \mathbb{N}} f_j, f_j \in M^{[j]}_{p,q}$. When $p \vee q \leq r \leq q'$, by Lemma 3, we have $M_{p,q} \hookrightarrow L^r$. Therefore, when $\mu - d/p + d/r \leq 0$, by the triangle inequality, we have

$$\|f\|_r \leq \sum_{j \in \mathbb{N}} \|f_j\|_r \leq \sum_{j \in \mathbb{N}} 2^{j(\mu - d/p + d/r)} \|f_j\|_r.$$
\[
\leq \sum_{j \leq 0} 2^{j(\mu - d/p)} \|f_j\|_r \leq \sum_{j \leq 0} 2^{j(\mu - d/p)} \|f_j\|_{M_{pq}} = \sum_{j \leq 0} 2^{j\mu} \|f_j\|_{M_{pq}}. \tag{25}
\]

Taking the infimum of the decomposition of \(f\), we have \(\|f\|_r \lesssim \|f\|_{M_{pq}}\).

Necessity: if we know \(M_{pq} \hookrightarrow L^r\), then by Remark 5, we have
\[
M_{pq} \hookrightarrow M_{pq} \hookrightarrow L^r. \tag{26}
\]

Then by Lemma 3, we have \(p \vee q \leq r \leq q'\). Meanwhile, when \(\sigma(p, q) < a(p, q)\), we know that \(p, q < \infty\). Then we could use the scaling property of \(M_{pq}\) (see Proposition 8 in Ref. 11).

If we have \(\|f\|_r \lesssim \|f\|_{M_{pq}}\), take \(f = \varphi \lambda\) for some \(\varphi \in \mathcal{S}, 0 < \lambda < 1\). Then, we have
\[
\lambda^{-d/r} \|f\|_r = \|f \lambda\|_r \lesssim \|f \lambda\|_{M_{pq}} \lesssim \lambda^{-d/p} \|f\|_{M_{pq}}. \tag{27}
\]

Thus, we have \(\mu - d/p \leq -d/r\). \(\blacksquare\)

**Example 1.** For \(\mu > 0\), there exists \(u \in M_{2,1} \setminus L^2\).

For any \(j \leq 0, j \in \mathbb{Z}\), denote \(k_j = (j, \ldots, 0) \in \mathbb{Z}^d\). Choose \(\varphi \in \mathcal{S}\) with \(\text{supp } \hat{\varphi} \subseteq [-1/8, 1/8]^d\).

Take
\[
u = \sum_{j \leq 0} 2^{j(d/2-\mu)} e^{i k_j x} \varphi(2^j x). \tag{28}\]

Then, we have
\[
\|u\|_2 \geq \|\square j, k_j u\|_2 \approx \frac{2^{-j\mu}}{j^2}. \tag{29}\]

Let \(j \to -\infty\), we have \(\|u\|_2 = \infty\).

On the other hand, by the definition of \(M_{2,1}\) in Section 2, we have
\[
\|u\|_{M_{2,1}} \leq \sum_{j \leq 0} 2^j \left\| e^{i k_j x} \varphi(2^j x) \right\|_{M_{2,1}} \lesssim \sum_{j \leq 0} \frac{1}{j^2} \leq 1. \tag{30}\]

By a similar method of Proposition 1 with Lemma 4, we could also characterize the inclusion between \(M_{pq}\) and \(FL'\).

**Proposition 2.** Let \(1 \leq p, q, r \leq \infty, \sigma(p, q) \leq \mu \leq a(p, q), \sigma(p, q) < a(p, q)\). Then, \(M_{pq} \hookrightarrow FL'\) if and only if \(q \leq r' \leq p', p \leq 2, \mu - d/p \leq -d/r\).
4  LINEAR ESTIMATES

Lemma 5 (Ref. 43). Denote $D^s f = \mathcal{F}^{-1} |\xi|^s \hat{f}(\xi)$. Then, we have

$$
\|D^{3/2}W(t)u\|_{L_x^\infty L_t^2} \lesssim \|u\|_2^2;
\|W(t)u\|_{L_x^4 L_t^\infty} \lesssim \|u\|_{H^{1/4}}^{1/4}. \tag{31}
$$

We will use Lemma 5 to derive some smoothing effect estimates and maximal estimates in local frequency spaces, as follows.

Lemma 6. For any $p \geq 4, k \in \mathbb{Z}$. We have

$$
\|D^{3/2} \Box_k W(t)u\|_{L_x^\infty L_t^2} \lesssim \|\Box_k u\|_2^2;
\|\Box_k W(t)u\|_{L_x^p L_t^\infty} \lesssim \|\Box_k u\|_{H^{1/p}}. \tag{32}
$$

Proof. Replacing $u$ by $\Box_k u$ in the above estimates, we could get the first and the second estimates with $p = 4$.

By Bernstein’s inequality (Lemma 1) and $L^2$ isometry of $W(t)$, we have

$$
\|\Box_k W(t)u\|_{L_x^\infty} \lesssim \|\Box_k W(t)u\|_{L_t^\infty L_x^2} = \|\Box_k u\|_2. \tag{33}
$$

Taking the interpolation between $p = 4$ and $p = \infty$, we could get the result as desired. \hfill \blacksquare

Proposition 3. For any $p \geq 4, k \in \mathbb{Z}$, we have

$$
\|\Box_k W(t)u\|_{L_x^{p+2/3} L_t^{3p+2}} \lesssim \|\Box_k u\|_2^2;
\|\Box_k W(t)\partial_x u\|_{L_x^{p+2} L_t^{(3p+2)(p+1)}} \lesssim \|\Box_k u\|_2^2. \tag{34}
$$

Proof. Taking the interpolation of the two estimates in Lemma 6, we have

$$
\|D^s \Box_k W(t)u\|_{L_x^a L_t^b} \lesssim \|\Box_k u\|_{H^s}, \tag{35}
$$

when

$$
\begin{cases}
\alpha &= \frac{3(1-\vartheta)}{2} + 0 \cdot \vartheta, \\
\gamma &= 0(1 - \vartheta) + \vartheta / p, \\
\frac{1}{a} &= \frac{1-\vartheta}{\infty} + \frac{\vartheta}{p}, \\
\frac{1}{b} &= \frac{1-\vartheta}{2} + \frac{\vartheta}{\infty},
\end{cases} \tag{36}
$$

which is equivalent to

$$
\begin{cases}
\vartheta &= \frac{p}{q}, \\
\alpha &= \frac{1}{b}, \\
\gamma &= \frac{a}{1}, \\
1 &= \frac{p}{a} + \frac{2}{b},
\end{cases} \tag{37}
$$
Taking $\alpha = s$ into the equations above, we have $a = p + 2/3, b = 3p + 2$, which is the first estimate in the proposition. Taking $\alpha - s = 1$, we can obtain the second estimate as desired. ■

By the duality argument, we could get the following estimate of $\mathcal{A} \partial_x f$.

**Proposition 4.** Let $p \geq 4, k \in \mathbb{Z}$. Then, we have
\[
\|\Box_k \mathcal{A} \partial_x f\|_{L_x^{p+2/3} L_t^{3p+2}} \lesssim \|\Box_k f\|_{L_x^{(3p+2)/(p+1)} L_t^{(3p+2)/(p+1)}}.
\]
\[
= \|\Box_k f\|_{L_x^{(3p+2)/(3p+1)} L_t^{(3p+2)/(2p+1)}}.
\]
\[
(38)
\]

**Proof.** By the Christ-Kiselev lemma (Lemma 2), we only need to prove the following estimate:
\[
\left\| \Box_k \int W(t-s) \partial_x f(s) ds \right\|_{L_x^{p+2/3} L_t^{3p+2}} \lesssim \|\Box_k f\|_{L_x^{(3p+2)/(p+1)} L_t^{(3p+2)/(p+1)}}.
\]
\[
(39)
\]
We prove this by duality.
\[
\left\| \Box_k \int W(t-s) \partial_x f(s) ds \right\|_{L_x^{p+2/3} L_t^{3p+2}} = \sup_{\|g\|_{L_x^{(p+2)/(3p+2)} L_t^{(3p+2)}} \leq 1} \left\langle \Box_k \int W(t-s) \partial_x f(s) ds, g(t, x) \right\rangle_{L_x L_t}
\]
\[
= \sup_{\|g\|_{L_x^{(p+2)/(3p+2)} L_t^{(3p+2)}} \leq 1} \left\langle \Box_k \int W(-s) \partial_x f(s) ds, \sum_{|\ell| \leq 1} \Box_k+\ell \int W(-t) g(t) dt \right\rangle_{L_x}
\]
\[
\leq \sup_{\|g\|_{L_x^{(p+2)/(3p+2)} L_t^{(3p+2)}} \leq 1} \left\| \Box_k \int W(-s) \partial_x f(s) ds \right\|_{L_x^{2}} \sum_{|\ell| \leq 1} \left\| \Box_k+\ell \int W(-t) g(t) dt \right\|_{L_x^{2}}
\]
\[
\lesssim \|\Box_k f\|_{L_x^{(3p+2)/(3p+1)} L_t^{(3p+2)/(2p+1)}} ,
\]
\[
(40)
\]
where we use the dual version of the estimates in Proposition 3 to obtain the last inequality. ■

To obtain the estimates of $\Box_j, k W(t) u, \Box_j, k \mathcal{A} f$ from the estimates of $\Box_k W(t) u, \Box_k \mathcal{A} f$, we need the following scaling lemma.

**Lemma 7.** Let $0 < p, r, p_1, r_1, q \leq \infty, 0 < \alpha < \infty, k \in \mathbb{Z}$. If we have
\[
\|D^{\alpha} \Box_k W(t) u\|_{L_x^{p} L_t^{r}} \lesssim \|\Box_k u\|_{L_x^{p_1} L_t^{r_1}}
\]
\[
\|D^{\alpha} \Box_k \mathcal{A} f\|_{L_x^{p} L_t^{r}} \lesssim \|\Box_k f\|_{L_x^{p_1} L_t^{r_1}}.
\]
\[
(41)
\]
Then for any $j \in \mathbb{Z}$, we have
\[
\left\| D^{\alpha} \Box_{j, k} W(t) u \right\|_{L_x^{p} L_t^{r}} \lesssim 2^{j\delta(p, r, \alpha)} \|\Box_{j, k} u\|_{L_x^{p} L_t^{r}},
\]
\[ \left\| D^\alpha \Box_{j,k} \mathcal{A} f \right\|_{L_x^p L_t^r} \lesssim 2^{j(\rho, p, p_1, r_1, \alpha)} \left\| \Box_{j,k} f \right\|_{L_x^{p_1} L_t^{r_1}}, \] (42)

where

\[ \delta(p, r, q, \alpha) = \alpha - 4/r - 1/p + 1/q, \]
\[ \tau(p, r, p_1, r_1, \alpha) = \alpha - 4 - 4/r - 1/p + 4/r_1 + 1/p_1. \] (43)

**Proof.** The proof is based on scaling. For simplicity, we only give the proof of \( \mathcal{A} f \).

By definitions of \( D^\alpha, \Box_{j,k}, W(t) \), and the change of variables, we have

\[
\begin{align*}
D^\alpha \Box_{j,k} \mathcal{A} f(x, t) &= \int_0^t \mathcal{F}^{-1} \left| \xi \right|^{\alpha} \sigma(2^{-j} \xi - k) e^{\xi(t-s)} \left| \xi \right|^4 \mathcal{F}(\xi, s)(x) ds \\
&= \int_0^t \mathcal{F}^{-1} 2^{j\alpha} \left| \xi \right|^{\alpha} \sigma(\xi - k) e^{2^{j}(t-s)} \left| \xi \right|^4 2^{j} \mathcal{F}(2^j \xi, s)(2^j x) ds \\
&= \int_0^{2^{j} t} \mathcal{F}^{-1} 2^{j\alpha} \left| \xi \right|^{\alpha} \sigma(\xi - k) e^{(2^{j} t-s)} \left| \xi \right|^4 2^{j} 2^{-4} \mathcal{F}(2^j \xi, 2^{-4}s)(2^j x) ds \\
&= 2^{j(\alpha-4)} D^\alpha \Box_{k} \mathcal{A} f_{2^{-j}, 2^{-4}j} (2^j x, 2^j t),
\end{align*}
\] (44)

where \( f_{2^{-j}, 2^{-4}j}(x, t) = f(2^{-j} x, 2^{-4}j t) \). Similarly, by the change of variables, we have

\[ \Box_{k} f_{2^{-j}, 2^{-4}j}(x, t) = \Box_{j,k} f(2^{-j} x, 2^{-4}j t). \] (45)

Therefore, we have

\[
\begin{align*}
\left\| D^\alpha \Box_{j,k} \mathcal{A} f \right\|_{L_x^p L_t^r} &= \left\| 2^{j(\alpha-4)} D^\alpha \Box_{k} \mathcal{A} f_{2^{-j}, 2^{-4}j} (2^j x, 2^j t) \right\|_{L_x^p L_t^r} \\
&= 2^{j(\alpha-4)} 2^{j-4} r^{-1/p} \left\| D^\alpha \Box_{k} \mathcal{A} f_{2^{-j}, 2^{-4}j} \right\|_{L_x^p L_t^r} \\
&\lesssim 2^{j(\alpha-4)} 2^{-4} 2^{j-4} \left\| \Box_{k} f_{2^{-j}, 2^{-4}j} \right\|_{L_x^p L_t^r} \\
&= 2^{j(\alpha-4)} 2^{j} \left\| \Box_{j,k} f \right\|_{L_x^{p_1} L_t^{r_1}} ,
\end{align*}
\] (46)

**Remark 6.** Notice that when \( j \leq 0, \delta, \tau \geq 0 \) or \( j \geq 0, \delta, \tau \leq 0 \), the estimates above are better than the estimates of taking place of \( u \) only by \( \Box_{j,k} u \) in the conditions.

By Propositions 3, 4 and the scaling lemma above, we have

**Proposition 5.** Let \( p \geq 4, j \leq 0, k \in \mathbb{Z} \). We have

\[
\begin{align*}
\left\| \Box_{j,k} W(t) u \right\|_{L_x^{p+2/3} L_t^{3p+2}} &\lesssim 2^{j A} \left\| \Box_{j,k} u \right\|_{L_x^2}, \\
\left\| \mathcal{S} \Box_{j,k} \mathcal{A} f \right\|_{L_x^{p+2/3} L_t^{3p+2}} &\lesssim 2^{j B} \left\| \Box_{j,k} f \right\|_{L_x^{(3p+2)'} L_t^{(3p+2)/(p+1)'}}.
\end{align*}
\] (47)
where

\[ A = \frac{3(p - 4)}{2(3p + 2)}, \quad B = \frac{2(p - 4)}{3p + 2}. \quad (48) \]

Recall the work spaces \( X^\mu(A) \) in Definition 1. By Proposition 5, we could obtain the main estimates of \( W(t) \) as follows.

**Proposition 6.** Let \( p \geq 4, \mu \in \mathbb{R}, A, B \) as in (48). Then we have

\[
\| W(t)u_0 \| _{X^\mu(L_x^{p+2/3}L_t^{3p+2})} \leq \| u_0 \| _{\mathcal{M}^{\mu+A}};
\]

\[
\| \partial_x \partial_t f \| _{X^\mu(L_x^{p+2/3}L_t^{3p+2})} \leq \| f \| _{X^{\mu+B}(L_x^{(p+2)(p+1)})}. \quad (49)
\]

## 5 NONLINEAR ESTIMATES

In this section, we give the nonlinear estimate of the work space \( X^\mu(L_x^pL_t^\gamma) \). First, we need two lemmas.

**Lemma 8.** Let \( \ell \leq j \leq 0, 1 \leq p, \gamma \leq \infty \). Then we have

\[
\sum_{k \in \mathbb{Z}} \| \Box_{j,k} u \| _{L_x^pL_t^\gamma} \leq \sum_{k \in \mathbb{Z}} \| \Box_{\ell,k} u \| _{L_x^pL_t^\gamma}. \quad (50)
\]

**Proof.** For any \( k_1 \in \mathbb{Z} \), denote \( \wedge_{k_1,j,\ell} = \{ k \in \mathbb{Z} : \Box_{\ell,k} \Box_{j,k_1} \neq 0 \} \). Then we know that

\[
\Box_{j,k_1} u = \sum_{k \in \wedge_{k_1,j,\ell}} \Box_{\ell,k} \Box_{j,k_1} u. \quad (51)
\]

Meanwhile, \( \{ \wedge_{k_1,j,\ell} \}_{k_1 \in \mathbb{Z}} \) are finitely overlapped. In fact, if there exists \( k \in \wedge_{k_1,j,\ell} \cap \wedge_{k_2,j,\ell} \), then we have \( |2k_1 - 2k_2| \leq 2j + 2\ell \), which means that \( |k_1 - k_2| \leq 1 \).

By the triangle inequality, we have

\[
\sum_{k_1 \in \mathbb{Z}} \| \Box_{j,k_1} u \| _{L_x^pL_t^\gamma} \leq \sum_{k_1 \in \mathbb{Z}} \sum_{k \in \wedge_{k_1,j,\ell}} \| \Box_{\ell,k} \Box_{j,k_1} u \| _{L_x^pL_t^\gamma} \leq \sum_{k_1 \in \mathbb{Z}} \sum_{k \in \wedge_{k_1,j,\ell}} \| \Box_{\ell,k} u \| _{L_x^pL_t^\gamma} \leq \sum_{k \in \mathbb{Z}} \| \Box_{\ell,k} u \| _{L_x^pL_t^\gamma}. \quad (52)
\]

**Lemma 9.** Let \( 1 \leq p_1 \leq \gamma \leq p \leq \infty, \mu \in \mathbb{R} \). Then we have

\[ X^{\mu+1/p_1}(L_{x}^{p_1}L_{t}^{\gamma}) \hookrightarrow X^{\mu+1/p}(L_{x}^{p}L_{t}^{\gamma}). \quad (53) \]
Proof. For any \( u = \sum_{j \leq 0} u_j \), by Definition 1, Minkowski’s inequality and Bernstein’s inequality (Lemma 1), we have

\[
\|u\|_{X^{\mu+1/p}(L_x^p L_t^\gamma)} \leq \sum_{j \leq 0} 2^{j(\mu+1/p)} \sum_{k \in \mathbb{Z}} \left\| \square_{j,k} u_j \right\|_{L_x^p L_t^\gamma} \leq \sum_{j \leq 0} 2^{j(\mu+1/p)} \sum_{k \in \mathbb{Z}} \left\| \square_{j,k} u_j \right\|_{L_x^p L_t^\gamma}
\]

\[
\leq \sum_{j \leq 0} 2^{j(\mu+1/p_1)} \sum_{k \in \mathbb{Z}} \left\| \square_{j,k} u_j \right\|_{L_x^{p_1} L_t^{\gamma}} \leq \sum_{j \leq 0} 2^{j(\mu+1/p_1)} \sum_{k \in \mathbb{Z}} \left\| \square_{j,k} u_j \right\|_{L_x^{p_1} L_t^{\gamma}}. \tag{54}
\]

Thus, we have

\[
\|u\|_{X^{\mu+1/p}(L_x^p L_t^\gamma)} \leq \|u\|_{X^{\mu+1/p_1}(L_x^{p_1} L_t^{\gamma})}, \tag{55}
\]

which is equivalent to \( X^{\mu+1/p_1}(L_x^{p_1} L_t^{\gamma}) \hookrightarrow X^{\mu+1/p}(L_x^p L_t^\gamma) \). \( \square \)

Proposition 7. Let \( 1 \leq p, p_1, p_2, \gamma, \gamma_1, \gamma_2 \leq \infty, \mu \in \mathbb{R} \), satisfying

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}. \tag{56}
\]

Then, we have

\[
\|u v\|_{X^\mu(L_x^p L_t^\gamma)} \lesssim \|u\|_{X^\mu(L_x^{p_1} L_t^{\gamma_1})} \|v\|_{X^0(L_x^{p_2} L_t^{\gamma_2})} + \|u\|_{X^0(L_x^{p_1} L_t^{\gamma_1})} \|v\|_{X^\mu(L_x^{p_2} L_t^{\gamma_2})}. \tag{57}
\]

Proof. The proof is based on the proof of Lemma 8.8 in Ref. 11.

For any \( u = \sum_{j \leq 0} u_j, v = \sum_{\ell \leq 0} v_\ell \), we have

\[
uv = \sum_{j, \ell \leq 0} u_j v_\ell = \sum_{j \leq 0} \sum_{\ell \leq j} u_j v_\ell + \sum_{\ell \leq 0} \sum_{j < \ell} u_j v_\ell
\]

\[ := I + II. \tag{58}\]

By the triangle inequality, we have

\[
\|uv\|_{X^\mu(L_x^p L_t^\gamma)} \leq \|I\|_{X^\mu(L_x^{p_1} L_t^{\gamma_1})} + \|II\|_{X^\mu(L_x^{p_1} L_t^{\gamma_1})}. \tag{59}
\]

By Definition 1 and the triangle inequality, we have

\[
\|I\|_{X^\mu(L_x^{p_1} L_t^{\gamma_1})} \leq \sum_{j \leq 0} \sum_{\ell \leq j} \left\| \square_{j,k} u_j v_\ell \right\|_{L_x^{\mu} L_t^{\gamma}} \leq \sum_{j \leq 0} \sum_{\ell \leq j} \sum_{k \in \mathbb{Z}} \left\| \square_{j,k} u_j v_\ell \right\|_{L_x^{\mu} L_t^{\gamma}}. \tag{60}
\]

Taking decomposition of \( u_j = \sum_{k_1 \in \mathbb{Z}} \square_{j,k_1} u_j, v_\ell = \sum_{k_2 \in \mathbb{Z}} \square_{j,k_2} v_\ell \), by the orthogonality of \( \square_{j,k}(\square_{j,k_1} u_j \square_{j,k_2} v_\ell) \), we have

\[
(60) \leq \sum_{j \leq 0} \sum_{\ell \leq j} \sum_{k_1, k_2 \in \mathbb{Z}} \left\| \square_{j,k}(\square_{j,k_1} u_j \square_{j,k_2} v_\ell) \right\|_{L_x^{\mu} L_t^{\gamma}}.
\]
\[ \sum_{j \leq 0} 2^{j\mu} \sum_{\ell \leq j} \sum_{k_1, k_2 \in \mathbb{Z}} \sum_{k : |k - k_1 - k_2| \leq 1} \left\| \square_{j,k} (\square_{j,k_1} u_j \square_{j,k_2} v_\ell) \right\|_{L^p_t L^q_x} \]
\[ \leq \sum_{j \leq 0} 2^{j\mu} \sum_{\ell \leq j} \sum_{k_1, k_2 \in \mathbb{Z}} \left\| \square_{j,k} u_j \square_{j,k_2} v_\ell \right\|_{L^p_t L^q_x} \]
\[ (61) \]

By Hölder’s inequality and Lemma 8, we have
\[ (61) \leq \sum_{j \leq 0} 2^{j\mu} \sum_{\ell \leq j} \sum_{k_1, k_2 \in \mathbb{Z}} \left\| \square_{j,k_1} u_j \right\|_{L^{p_1}_{x} L^{\gamma_1}_t} \left\| \square_{j,k_2} v_\ell \right\|_{L^{p_2}_{x} L^{\gamma_2}_t} \]
\[ = \sum_{j \leq 0} 2^{j\mu} \sum_{k_1 \in \mathbb{Z}} \left\| \square_{j,k_1} u_j \right\|_{L^{p_1}_{x} L^{\gamma_1}_t} \sum_{\ell \leq j} \sum_{k_2 \in \mathbb{Z}} \left\| \square_{j,k_2} v_\ell \right\|_{L^{p_2}_{x} L^{\gamma_2}_t} \]
\[ \leq \sum_{j \leq 0} 2^{j\mu} \sum_{k_1 \in \mathbb{Z}} \left\| \square_{j,k_1} u_j \right\|_{L^{p_1}_{x} L^{\gamma_1}_t} \sum_{\ell \leq j} \sum_{k_2 \in \mathbb{Z}} \left\| \square_{\ell,k_2} v_\ell \right\|_{L^{p_2}_{x} L^{\gamma_2}_t}. \]
\[ (62) \]

Therefore, we have
\[ \| I \|_{X^{\mu}(L^{p}_x L^{\gamma}_t)} \lesssim \| u \|_{X^{\mu}(L^{p_1}_{x} L^{\gamma_1}_t)} \| v \|_{X^{0}(L^{p_2}_{x} L^{\gamma_2}_t)}. \]
\[ (63) \]

By the same method, we also have
\[ \| II \|_{X^{\mu}(L^{p}_x L^{\gamma}_t)} \lesssim \| u \|_{X^{0}(L^{p_1}_{x} L^{\gamma_1}_t)} \| v \|_{X^{\mu}(L^{p_2}_{x} L^{\gamma_2}_t)}. \]
\[ (64) \]

Combining these two estimates, we could get the estimate as desired in the proposition. \( \blacksquare \)

By induction, we could obtain the corollary as follows.

**Corollary 1.** Let \( 1 \leq m \in \mathbb{N}, 1 \leq p, p_i, \gamma, \gamma_i \leq \infty, \) for \( i = 1, \ldots, m + 1. \) If
\[ \frac{1}{p} = \sum_{i=1}^{m+1} \frac{1}{p_i}, \quad \frac{1}{\gamma} = \sum_{i=1}^{m+1} \frac{1}{\gamma_i}. \]
\[ (65) \]

Then, we have
\[ \| u^{m+1} \|_{X^{0}(L^{p}_{x} L^{\gamma}_t)} \lesssim \prod_{i=1}^{m+1} \| u \|_{X^{0}(L^{p_i}_{x} L^{\gamma_i}_t)}. \]
\[ (66) \]

Applying this corollary to the work space
\[ X^{0}(L^{(3p+2)'/2}_{x} L^{(3p+2)/(2p+1)}_t) = X^{0}(L^{(3p+2)/(3p+1)}_{x} L^{(3p+2)/(2p+1)}_t), \]
\[ (67) \]

we could get the estimate as follows.

**Proposition 8.** Let \( m = 2p \in \mathbb{N}, p \geq 4. \) Then we have
\[ \| u^{m+1} \|_{X^{0}(L^{(3p+2)/(3p+1)}_{x} L^{(3p+2)/(2p+1)}_t)} \lesssim \| u \|_{X^{0}(L^{p+2/3}_{x} L^{3p+2}_t)}^{m+1}. \]
\[ (68) \]
Proof. Denote
\[ (a, b) = \begin{cases} (n, 2), & m = 2n \in 2\mathbb{N}; \\ (n, 5), & m = 2n + 1 \in 2\mathbb{N} + 1. \end{cases} \] (69)
Then, we have
\[ \frac{3p + 1}{3p + 2} = \frac{3a}{3p + 2} + \frac{b}{2(3p + 2)} + \frac{m + 1 - a - b}{\infty}; \]
\[ \frac{2p + 1}{3p + 2} = \frac{a}{3p + 2} + \frac{b}{3p + 2} + \frac{m + 1 - a - b}{3p + 2}. \] (70)
By Corollary 1, we have
\[ \|u^{m+1}\|_{X_0^0(L_x^3(3p+2)/(3p+1), L_t^3)} \lesssim \|u_0\|_{X_0^0(L_x^{p+2/3}, L_t^3)}^{a} \|u\|_{X_0^0(L_x^{3p+2}, L_t^3)}^{b} \|u\|_{X_0^0(L_x^\infty, L_t^3)}^{m+1-a-b}. \] (71)
By Lemma 9 and Remark 4, we have
\[ \|u\|_{X_0^0(L_x^3(3p+2), L_t^3)} \lesssim \|u\|_{X_0^0(L_x^{3p+2}, L_t^3)} \lesssim \|u\|_{X_0^0(L_x^{p+2/3}, L_t^3)} \lesssim \|u\|_{X_0^0(L_x^{3p+2}, L_t^3)}. \] (72)
Taking these estimates into (71), we could obtain the nonlinear estimate as desired. ■

6 PROOF OF THEOREM 1

Proof. Denote \( p = m/2, A, B \) as in (48), let \( 0 \leq \mu \leq A \). For any \( u_0 \in \mathcal{M}_{2,1}^\mu \), we define the operator
\[ \mathcal{T} : X^{\mu-A}(L_x^{p+2/3}, L_t^{3p+2}) \rightarrow X^{\mu-A}(L_x^{p+2/3}, L_t^{3p+2}) \]
\[ u \rightarrow W(t)u_0 + \partial_x \partial_t u^{m+1}. \] (73)
By Proposition 6, we have
\[ \|\mathcal{T} u\|_{X^{\mu-A}(L_x^{p+2/3}, L_t^{3p+2})} \lesssim \|u_0\|_{X^{\mu}(L_x^{p+2/3}, L_t^{3p+2})} + \|u^{m+1}\|_{X^{\mu-A+B}(L_x^{3p+2}, L_t^{3p+2})}. \] (74)
Notice that \( \mu - A \leq 0, \mu - A + B \geq 0 \), so by Remark 4 and Proposition 8, we have
\[ \|u^{m+1}\|_{X^{\mu-A+B}(L_x^{3p+2}, L_t^{3p+2})} \lesssim \|u^{m+1}\|_{X^{0}(L_x^{3p+2}, L_t^{3p+2})} \]
\[ \lesssim \|u\|_{X^{0}(L_x^{3p+2}, L_t^{3p+2})} \]
\[ \lesssim \|u\|_{X^{0}(L_x^\infty, L_t^3)}^{m+1}. \] (75)
Therefore, by the standard contraction mapping argument, we see that (2) has a unique solution $u \in X^{\mu-A}(L_x^{p+2/3} L_t^{3p+2})$. Moreover, since $W(t)$ is bounded on $M_{2,1}^\mu$ and $M_{2,1}^\mu$ is a multiplication algebra (Proposition 6.2 in Ref. 11), it can be shown that $u \in C(\mathbb{R}, M_{2,1}^\mu)$.

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Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

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