Abstract

We develop a notion of finite order lacunarity for direction sets in $\mathbb{R}^{d+1}$. Given a direction set $\Omega$ that is sublacunary according to this definition, we construct random examples of Euclidean sets that contain unit line segments with directions from $\Omega$ and enjoy analytical features similar to those of traditional Kakeya sets of infinitesimal Lebesgue measure. This generalizes to higher dimensions a planar result due to Bateman [3]. Combined with earlier work of Alfonseca [1], Bateman [3], Parcet and Rogers [21], this notion of lacunarity and Kakeya-type sets also yields a characterization in all dimensions for directional maximal operators to be $L^p$-bounded.

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2010 Mathematics Subject Classification. 28A75, 42B25 (primary), and 60K35 (secondary).
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1 Introduction

1.1 Background

This paper is concerned with a generalization of the classical Euclidean Kakeya set, also called a Besicovitch set. In $\mathbb{R}^{d+1}$, a Kakeya set is one that contains a unit line segment in every direction. Here, we are concerned with sets that contain a unit line segment in every direction of a given subset of directions $\Omega$. For certain “large enough” subsets $\Omega$, the geometric and analytical structure of these sets is remarkably similar to that of traditional Kakeya sets. The bulk of this paper is devoted to making this idea precise.

In the study of Kakeya sets and for quantification purposes, it is often convenient to work with a $\delta$-neighborhood of the set rather than the set itself, where $\delta$ is an infinitesimal positive constant. This neighborhood is therefore a set of small but positive Lebesgue measure built of roughly $\delta^{-d}$ thin tubes of unit length and spherical cross-section of radius $\delta$. For thickenings of Kakeya sets resulting from many concrete classical constructions, these constituent tubes enjoy certain structural properties that have proved to be of considerable analytical and geometric significance [23, Chapter 10], [15, 14]. The present article focuses on one of them (see Definition 1.1 below). We study this property in a context similar but not identical to classical Kakeya sets, investigate its applications in analytical problems of independent interest, and obtain a characterization of direction sets $\Omega$ for which such structure can hold.

**Definition 1.1.** Fix a set of directions $\Omega \subseteq \mathbb{R}^{d+1}$. We say a cylindrical tube is oriented in direction $\omega \in \Omega$ if the principal axis of the cylinder is parallel to $\omega$. If for some fixed constant $A_0 \geq 1$ and any choice of integer $N \geq 1$, there exist

- a number $0 < \delta_N \ll 1$, $\delta_N \downarrow 0$ as $N \nearrow \infty$, and
- a collection of tubes $\{P_t^{(N)}\}$ with orientations in $\Omega$, length at least 1 and cross-sectional radius at most $\delta_N$

obeying

$$\lim_{N \to \infty} \frac{|E_N^*(A_0)|}{|E_N|} = \infty, \quad \text{with} \quad E_N := \bigcup_t P_t^{(N)}, \quad E_N^*(A_0) := \bigcup_t A_0 P_t^{(N)},$$

(1.1)
then we say that $\Omega$ admits Kakeya-type sets. Here, $| \cdot |$ denotes $(d + 1)$-dimensional Lebesgue measure, and $A_0 P_t(N)$ denotes the tube with the same centre, orientation and cross-sectional radius as $P_t(N)$ but $A_0$ times its length. The tubes that constitute $E_N$ may have variable dimensions subject to the restrictions mentioned above. We refer to $\{E_N : N \geq 1\}$ as sets of Kakeya type.

The inadmissibility of Kakeya-type sets is related to the boundedness on Lebesgue spaces of certain maximal averages widely studied in harmonic analysis. Given a set of directions $\Omega \subseteq \mathbb{R}^{d+1}$, we consider the directional maximal operator $D_\Omega$ defined by

$$D_\Omega f(x) := \sup_{\omega \in \Omega} \sup_{h > 0} \frac{1}{2h} \int_{-h}^{h} |f(x + \omega t)| dt,$$

where $f : \mathbb{R}^{d+1} \to \mathbb{R}$ is a function that is locally integrable along lines. We also consider the Kakeya-Nikodym maximal operator $M_\Omega$ defined by

$$M_\Omega f(x) := \sup_{\omega \in \Omega} \sup_{P \ni x} \frac{1}{|P|} \int_P |f(y)| dy,$$

for any locally integrable function $f : \mathbb{R}^{d+1} \to \mathbb{R}$. The inner supremum in the definition (1.3) above is taken over all cylindrical tubes $P$ that contain the point $x$ and are oriented in the direction $\omega$. The tubes are taken to be of arbitrary length $\ell$ and have circular cross-section of arbitrary radius $r$, with $r \leq \ell$.

If $\Omega$ is a set with nonempty interior, then due to the existence of Kakeya sets with $(d + 1)$-dimensional Lebesgue measure zero [5], $D_\Omega$ and $M_\Omega$ are unbounded as operators on $L^p(\mathbb{R}^{d+1})$ for $p \in [1, \infty)$. More generally, if $\Omega$ admits Kakeya-type sets, then both these operators are unbounded on $L^p(\mathbb{R}^{d+1})$ for $p \in [1, \infty)$. Indeed, a standard argument shows that for any tube $P$ of length $\ell$ oriented along a unit vector $\omega$,

$$\frac{1}{2A_0 \ell} \int_{-\ell}^{\ell} 1_P(x + t\omega) dt \geq \text{length of } P \geq \frac{1}{2A_0} \text{ for all } x \in A_0 P.$$

If $P$ is chosen to be one of the tubes that constitute the set $E_N$ defined as in (1.1), the inequality above implies that $M_\Omega 1_{E_N}(x) \geq D_\Omega 1_{E_N}(x) \geq c_0 = (2A_0)^{-1} > 0$ for any $x \in E_N(A_0)$. Hence

$$||M_\Omega||_{p \to p} \geq ||D_\Omega||_{p \to p} \geq c_0 \frac{||1_{E_N(A_0)}||_p}{||1_{E_N}||_p} \geq c_0 \left( \frac{|E_N(A_0)|}{|E_N|} \right)^{\frac{1}{p}}.$$

If $\Omega$ admits Kakeya-type sets, Definition 1.1 ensures that the sets $E_N$ can be chosen so that the right hand side approaches infinity as $N \to \infty$ for $p \in [1, \infty)$. This establishes the claimed unboundedness of both $M_\Omega$ and $D_\Omega$. 

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1.2 Results

All results to date suggest that the direction set $\Omega$ will admit Kakeya-type sets if it is of suitably large size. Of course, the notion of “size” here begs clarification. For us, a direction set will be small when it is sufficiently lacunary. Precise definitions of lacunarity needed in this paper are deferred to Section 2, but the general idea is easy to describe. In one dimension, a relatively compact set $\{a_i\}$ is lacunary of order 1 if there is $a \in \mathbb{R}$ and some positive $\lambda < 1$ such that $|a_{i+1} - a| \leq \lambda |a_i - a|$ for all $i$. Such a set has traditionally been referred to as a lacunary sequence with lacunarity constant (at most) $\lambda$. A lacunary set of order 2 consists of a single (first-level) lacunary sequence $\{a_i\}$, along with a collection of disjoint (second-level) lacunary sequences; a second-level sequence is squeezed between two adjacent elements of $\{a_i\}$. The lacunarity constants of all sequences are uniformly bounded by some positive $\lambda < 1$. Roughly speaking, a set on the real line is lacunary of finite order if there is a decomposition of the real line by points of a lacunary sequence such that the restriction of the set to each of the resulting subintervals is lacunary of lower order. All lacunarity constants implicit in the definition are assumed to be uniformly bounded away from unity. A set is then said to be sublacunary if it does not admit a finite cover by lacunary sets of finite order.

![Figure 1: A direction set in the plane, represented as a collection of unit vectors, with parameters $0 < \gamma < \lambda < 1/2$. The set of angles made by these vectors with the positive horizontal axis is $\{(\lambda^j + \gamma^k) : k \geq j\}$, which is lacunary of order 2.](image)

In higher dimensions, the idea of lacunarity is not immediately clear. For $d \geq 2$, Nagel, Stein, and Wainger [20] considered lacunary sets of the form $\Omega = \{(\theta_1^{m_1}, \ldots, \theta_d^{m_d}) :$
\[ j \geq 1 \}, \text{ where } 0 < m_1 < \cdots < m_d \text{ are fixed constants and } \{\theta_j\} \text{ is a lacunary sequence with lacunarity constant } 0 < \lambda < 1, \text{ i.e., } 0 < \theta_{j+1} \leq \lambda \theta_j. \] For such direction sets \( \Omega \), they showed that the operator \( D_\Omega \) is bounded on all \( L^p(\mathbb{R}^d) \), \( 1 < p \leq \infty \). A two-dimensional extension of this result was obtained by Sjögren and Sjölin [22], where they formulated a generalized notion of lacunarity that was to prove the basis of a body of subsequent work. Carbery [6] considered coordinate-wise lacunary sets of the form \( \Omega = \{(r^{k_1}, \ldots, r^{k_d}) : k_1, \ldots, k_d \in \mathbb{Z}^+ \} \) for some \( 0 < r < 1 \), and showed that the corresponding directional maximal operator is bounded on all \( L^p(\mathbb{R}^d) \), \( 1 < p \leq \infty \).

Very recently, Parcet and Rogers [21] have generalized an almost-orthogonality result of Alfonseca [1], building on previous work of Alfonseca, Soria, and Vargas [2], Carbery [6], Nagel, Stein, and Wainger [20], to recover these results and to extend the \( L^p \)-boundedness of \( D_\Omega \), \( 1 < p \leq \infty \), to sets \( \Omega \) that are lacunary of finite order in a broader sense.

On the other hand, direction sets \( \Omega \) that are sufficiently sublacunary have been shown to admit Kakeya-type sets, and thus lead to unbounded directional maximal operators, per the argument at the end of Section 1.4. There is an extensive body of work in this direction [8, 24, 13, 4], authored in part by Duandikoetxea, Vargas, Bateman and Katz. A fundamental and representative example, examined by Bateman and Katz [4], is a direction set in the plane where the slopes belong to the standard middle-third Cantor set. Combining the aforementioned positive results with strategies developed to treat the reverse direction, the \( L^p \)-boundedness of \( D_\Omega \) and \( M_\Omega \) has been completely characterized in the plane by Bateman [3] and remains one of the most definitive results in the subject. In higher dimensions, the authors [16] have recently constructed Kakeya-type subsets of a curve on the sphere \( S^d, d \geq 2 \). What seems clear is that any attempt to bridge the gap between these negative and positive results in general dimensions will require a precise and appropriate notion of finite order lacunarity. We will provide our definition in Definition 2.7. For now, we state our main results with the precise terminology deferred until Section 2.3.

**Theorem 1.2.** Let \( d \geq 2 \). If the direction set \( \Omega \subseteq \mathbb{R}^{d+1} \) is sublacunary in the sense of Definition 2.7, then \( \Omega \) admits Kakeya-type sets.

Combining this result with others from the literature (most notably [3, 1, 21]), we obtain the following necessary and sufficient condition for Kakeya-type sets to exist.

**Theorem 1.3.** For any dimension \( d \geq 1 \), the following are equivalent:

1. The direction set \( \Omega \subseteq \mathbb{R}^{d+1} \) is sublacunary in the sense of Definition 2.7.
2. The set of directions \( \Omega \) admits Kakeya-type sets in the sense of Definition 1.1.
3. The maximal operators \( D_\Omega \) and \( M_\Omega \) defined in (1.2) and (1.3) are unbounded on \( L^p(\mathbb{R}^{d+1}) \) for every \( p \in (1, \infty) \).

To clarify, the implication (3) \( \implies \) (1) for \( d = 1 \) is in [1], expanding on the work started in [20, 22, 6, 2]. For \( d \geq 2 \), this is due to [21], as we will see in Theorem 2.8. The proof of (1) \( \implies \) (2) is the content of [3] for \( d = 1 \) and of Theorem 1.2 for \( d \geq 2 \).
The implication (2) $\implies$ (3) is established in the argument presented in the paragraph of (1.4) in all dimensions.

Some of the implications above are known to admit stronger variants. For instance, (2) implies (3) even when $p = 1$, as the argument leading to (1.4) shows. Further, it is not necessary to know that the operator $D_\Omega$ is unbounded on all $L^p(\mathbb{R}^{d+1})$, $p \in (1, \infty)$, in order to conclude that $\Omega$ is sublacunary. We will prove in Section 2.4 that the weaker requirement

$$(3') \text{The maximal operator } D_\Omega \text{ is unbounded on } L^p(\mathbb{R}^{d+1}) \text{ for some } p \in (1, \infty),$$

suffices to establish (1). Thus $D_\Omega$ enjoys an interesting dichotomy in that it is either bounded on all or none of the Lebesgue spaces $L^p$ with $p \in (1, \infty)$.

1.3 Structure of the proofs and layout of the paper

The paper is divided into ten main sections, not counting the introduction. In Section 2 we define (admissible) finite order lacunarity and sublacunarity, consider several instructive and famous examples of such sets, and prove the implication (3) $\implies$ (1) of Theorem 1.3.

Section 3 begins the program of proving the implication (1) $\Rightarrow$ (2) of Theorem 1.3; i.e., of constructing Kakeya-type sets over sublacunary direction sets. We begin by reviewing the necessary literature on trees and how they can be used to encode subsets of Euclidean space. The so-called splitting number of a tree, as defined in [3], is then shown to be the critical concept that allows us to recast the notion of (admissible) finite order lacunarity of a set into an equivalent and more tractable form for the purposes of our proof. We use this language of trees in Section 4 to extract a convenient subset of an arbitrary sublacunary direction set, denoted by $\Omega_N$. Section 5 explores the geometry of the intersection of two tubes and the implications of this geometry for the structure of trees encoding the sets of orientations and positions of a given collection of thin $\delta$-tubes.

Section 6 combines results from the previous two sections to describe the actual mechanism we use to assign slopes in $\Omega_N$ to $\delta$-tubes affixed to a prescribed set of points in Euclidean space. Here, we also reformulate the implication (1) $\Rightarrow$ (2) of Theorem 1.3 in terms of quantitative upper and lower bounds on the sizes of a typical Kakeya-type set $E_N$ and its principal dilate $E_N^*(A_0)$ as described in Definition 1.1 (see Proposition 6.2). From here, the paper splits into more or less two disjoint expositions, each one charged with establishing one of these two probabilistic and quantitative bounds.

In Section 7 we prove the quantitative upper bound previously prescribed using an argument similar to [3]. Sections 8, 9, 10, and 11 combine to establish the corresponding lower bound. Section 11 details the actual estimation, utilizing all the smaller pieces developed in Sections 8, 9, and 10. These three sections revolve around a central theme of ideas, notably the structure imposed on the position and slope trees of a collection of two, three, or four $\delta$-tubes, certain pairs of which are required to intersect at a given location in space.

The framework of this paper is the same as in [3, 4], and bears the closest resemblance to [10]. In particular, given a sublacunary direction set, our goal is to construct a family
Figure 2: Diagram illustrating the approximate dependence structure between sections in this paper with respect to the proof of Theorem 1.3. Dotted arrows indicate a dependence in terms of definitions and notation only.

of tubes, all of which originate from (or said to be rooted in) the hyperplane \( \{0\} \times [0, 1)^d \), after an appropriate coordinate transformation. For a given root position, a slope from \( \Omega_N \) is assigned to it using a random mechanism that preserves heights and lineages of both source (root) and image (slope) within their respective trees. The quantitative lower and upper bounds (6.9) and (6.10) formulated in Proposition 6.2 ensure that the random set thus constructed is of Kakeya-type with positive probability. Of the two bounds (6.9) and (6.10), the first is the most significant contribution of this paper. More precisely, the issue is the following. A large lower bound on a union of tubes follows if they do not have significant pairwise overlap among themselves, i.e., if the total size of pairwise intersections is small. In dimension two, a good upper bound on this intersection size was available uniformly in every sticky slope assignment. The counting argument that provided this bound is not transferable to general dimensions, but it is still possible to obtain the desired bound with large probability. A probabilistic statement similar to but not as strong as (6.9) can be derived relatively easily via an estimate on the first moment of the total size of random pairwise intersections. Unfortunately, this is still not sharp enough to yield the disparity in the sizes of the tubes and their translated counterparts necessary to claim the existence of a Kakeya-type set. To strengthen the bound, we need a second moment estimate on the pairwise intersections. Both moment estimates share some common features; for instance,

- Euclidean distance relations between roots and slopes of two intersecting tubes,
- interplay of the above with the relative positions of the roots and slopes within the respective trees that they live in, which affects the slope assignments.

However, the technicalities are far greater for the second moment compared to the first, requiring a study of pairwise intersections among three or four tubes, not just two. The above-mentioned points appear in a somewhat simplified form in [16], where the authors
considered a special case of a direction set $\Omega$ obtained as a Cantor-type subset of a curve. There the direction tree had a richer structure, and as a consequence geometric and probabilistic estimates were simpler. The generality of this paper involved in handling arbitrary sublacunary direction sets gives rise to substantial technical challenges, necessitating the analysis carried out in Sections 8-11.

2 Finite order lacunarity

The concept of finite order lacunarity is ubiquitous, and recognized as fundamental in the study of planar Kakeya-type sets and associated directional maximal operators. It is no surprise that it continues to play a similar central role in this article. The existing literature on the subject embodies several different notions of Euclidean lacunarity both in single and general dimensions, see in particular [3, 6, 21, 22]. The present section is devoted to a discussion of the definitions to be used in the remainder of the paper. The concepts introduced here will be revisited in Section 3.4, using the language of trees. The interplay of these two perspectives is essential to the proofs of Theorems 1.2 and 1.3.

2.1 Lacunarity on the real line

Definition 2.1 (Lacunary sequence). Let $A = \{a_1, a_2, \ldots\}$ be an infinite sequence of points contained in a compact subset of $\mathbb{R}$. Given a constant $0 < \lambda < 1$, we say that $A$ is a lacunary sequence converging to $\alpha$ with constant of lacunarity at most $\lambda$, if

$$|a_{j+1} - \alpha| \leq \lambda |a_j - \alpha| \quad \text{for all } j \geq 1.$$ 

Definition 2.2 (Lacunary sets). In $\mathbb{R}$, a lacunary set of order 0 is a set of cardinality at most 1, i.e., either empty or a singleton. Recursively, given a constant $0 < \lambda < 1$ and an integer $N \geq 1$, we say that a relatively compact subset $U$ of $\mathbb{R}$ is a lacunary set of order at most $N$ with lacunarity constant at most $\lambda$, and write $U \in \Lambda(N; \lambda)$, if there exists a lacunary sequence $A$ with lacunarity constant $\leq \lambda$ with the following properties:

- $U \cap [\sup(A), \infty) = \emptyset$, $U \cap (-\infty, \inf(A)] = \emptyset$,

- For any two elements $a, b \in A$, $a < b$ such that $(a, b) \cap A = \emptyset$, the set $U \cap [a, b) \in \Lambda(N - 1; \lambda)$.

The order of lacunarity of $U$ is exactly $N$ if $U \in \Lambda(N; \lambda) \setminus \Lambda(N - 1; \lambda)$. A lacunary sequence $A$ obeying the conditions above will be called a special sequence and its limit will be termed a special point for $U$.

For any fixed $N$ and $\lambda$, the class $\Lambda(N; \lambda)$ is closed under containment, scalar addition and multiplication; these properties, summarized in the following lemma, are easy to verify and left to the reader.

Lemma 2.3. Let $U \in \Lambda(N; \lambda)$. Then
(i) $V \in \Lambda(N; \lambda)$ for any $V \subseteq U$.

(ii) $c_1U + c_2 \in \Lambda(N; \lambda)$ for any $c_1 \neq 0, c_2 \in \mathbb{R}$.

The sets of interest to us are those that are generated by finite unions of sets of the form described in Definition 2.2.

**Definition 2.4 (Admissible lacunarity of finite order and sublacunarity).** We say that a relatively compact set $U \subseteq \mathbb{R}$ is an admissible lacunary set of finite order if there exist a constant $0 < \lambda < 1$ and integers $1 \leq N_1, N_2 < \infty$ such that $U$ can be covered by $N_1$ lacunary sets of order $\leq N_2$, each with lacunarity constant $\leq \lambda$. If $U$ does not satisfy this criterion, we call it sublacunary.

**2.1.1 Examples**

(a) A standard example of a lacunary set of order 1 and lacunarity constant $\lambda \in (0, 1)$ is $U = \{\lambda^j : j \geq 1\}$, or any nontrivial subsequence thereof. Indeed $U$ is itself a lacunary sequence, and hence its own special sequence.

A general lacunary set of order 1 need not always be a lacunary sequence. For example $\{2^{-2j} \pm 4^{-2j} : j \geq 1\}$ is lacunary of order 1 relative to the special sequence $\{2^{-j} : j \geq 1\}$. Despite this, lacunary sequences are in a sense representative of the class $\Lambda(1; \lambda)$, since any set in $\Lambda(1; \lambda)$ can be written as the union of at most four lacunary sequences with lacunarity constant $\leq \lambda$. By Lemma 2.3, the set $\{a\lambda^j + b : j \geq 1\}$ is lacunary of order at most 1 for any unit vector $(a, b)$.

(b) In general, given an integer $k \geq 1$ and constants $M_1 \leq M_2 \leq \cdots \leq M_k$ with $M_1 \geq \max(2, k - 1)$, the set

$$U = \left\{M_1^{-j_1} + M_2^{-j_2} + \cdots + M_k^{-j_k} : 0 \leq j_1 \leq j_2 \leq \cdots \leq j_k\right\}$$

is lacunary of order $k$ and has lacunarity constant $\leq M_1^{-1}$. The special sequence can be chosen to be $A = \{M_1^{-j} : j \geq 1\}$.

(c) A set that is dense in some nontrivial interval, however small, is sublacunary. For example, dyadic rationals of the form $\{\frac{k}{2^m} : 0 \leq k < 2^m\}$ for a fixed $m$ can be written as a finite union of lacunary sequences with a given lacunarity $\lambda$, but the number of sequences in the union grows without bound as $m \to \infty$. By Lemma 2.3, a set that contains an affine copy of $\{\frac{k}{2^m} : 0 \leq k < 2^m\}$ for every $m$ is sublacunary.

(d) The set $U = \{2^{-j} + 3^{-k} : j, k \geq 0\}$ can be covered by a finite union of sets in $\Lambda(2; \frac{1}{2})$. For instance the two subsets of $U$ where $k \ln 3 \leq (j-1) \ln 2$ and $k \ln 3 \geq j \ln 2$ respectively are each lacunary of order 2, with $\{3^{-k}\}$ and $\{2^{-j}\}$ being their respective special sequences. The complement, where $(j-1) \ln 2 \leq k \ln 3 \leq j \ln 2$, contains at most one $k$ per $j$, and is a finite union of lacunary sets of order 1.
(e) A slight variation of the above example: \( \{ 2^{-j} + (q_j - 2^{-j})3^{-k} : j, k \geq 0 \} \), where \( \{ q_j \} \) is an enumeration of the rationals in \( [\frac{9}{10}, 1) \), leads to a very different conclusion. This set contains \( \{ q_j \} \), and is hence sublacunary, even though the set may be viewed as a special sequence \( \{ 2^{-j} \} \) with collections of lacunary sequences converging to every point of it. This example illustrates the relevance of the requirement that the lower order components of \( \Lambda(N; \lambda) \) lie in disjoint intervals of \( \mathbb{R} \).

(f) Given any \( 0 < \lambda < 1 \) and \( m > 0 \), there is a constant \( C = C(\lambda, m) \) such that for any unit vector \( (a, b) \), the set \( U_{a,b} = \{ a\lambda^j + b\lambda^m : j \geq 1 \} \) can be covered by \( C \) sets in \( \Lambda(1; \lambda) \). We leave the verification of this to the reader, but will provide a general statement along these lines in Section 2.2.1, see example (a).

(g) Given any \( 0 < \lambda < 1 \), \( m \in \mathbb{Q} \cap (0, \infty) \), there is a constant \( C = C(\lambda, m) \) such that for any unit vector \( (a, b) \), the set \( U_{a,b} = \{ u_{jk} = a\lambda^j + b\lambda^m : j, k \geq 1 \} \) can be covered by at most \( C \) lacunary sets of order at most 2. This is clear for \( (a, b) = (1, 0) \) or \( (0, 1) \), with the order of lacunarity being 1. For \( ab \neq 0 \), there are four possibilities concerning the signs of \( a \) and \( b \). We deal with \( a > 0 \) and \( b < 0 \), the treatment of which is representative of the general case. The set \( U_{a,b} \) is decomposed into three parts:

\[
\begin{align*}
V_{a,b} &= \{ u_{jk} \in U : a\lambda^j + b\lambda^m \geq a\lambda^{j+1} \}, \\
W_{a,b} &= \{ u_{jk} \in U : a\lambda^j + b\lambda^m < b\lambda^{m(k+1)} \}, \\
Z_{a,b} &= U_{a,b} \setminus \left[ V_{a,b} \cup W_{a,b} \right].
\end{align*}
\]

Then for every fixed \( j \), the set \( V_{a,b} \cap [a\lambda^{j+1}, a\lambda^j) \) is an increasing lacunary sequence with constant \( \leq \lambda^m \), converging to \( a\lambda^j \). An analogous conclusion holds for \( W_{a,b} \cap [b\lambda^m, b\lambda^{m(k+1)}) \). Thus \( V_{a,b} \) and \( W_{a,b} \) are both lacunary of order 2, with their special sequences being \( A = \{ a\lambda^j \} \) and \( A = \{ b\lambda^m \} \) respectively. For \( u_{jk} \in Z_{a,b} \), the indices \( j \) and \( k \) obey the inequality

\[
-\frac{a}{b}(1 - \lambda) < \lambda^{mk-j} \leq -\frac{a}{b}(1 - \lambda^m)^{-1}.
\]

Since \( m \) is rational, the values of \( mk - j \) range over rationals of a fixed denominator (same as that of \( m \)). The inequality above therefore permits at most \( C \) solutions of \( mk - j \), the constant \( C \) depending on \( \lambda \) and \( m \), but independent of \( (a, b) \). Thus \( Z_{a,b} \) is covered by a \( C \)-fold union of subsets, each consisting of elements \( u_{jk} = \lambda^j(a+b\lambda^{mk-j}) \) for which \( mk - j \) is held fixed at one of these solutions. Each such set is lacunary of order 1 with lacunarity \( \leq \lambda \).

2.1.2 Non-closure of finite order lacunarity under algebraic sums

An important aspect of the class of admissible lacunary sets of finite order is that it is not closed under set-algebraic operations, as we establish in the example furnished
below. This feature, perhaps initially counterintuitive, is the main inspiration for the
definition of higher dimensional lacunarity provided in the next subsection.

Example: Let $N_j \nrightarrow \infty$ be a fast growing sequence, and $M_j = 2^{m_j}$ a slower growing one, so that

$$M_j < N_j - N_{j-1}. \quad (2.1)$$

For instance, $N_j = 2^{j^2}$ and $M_j = 2^j$ will do. For every $j \geq 1$ and $1 \leq k \leq M_j = 2^{m_j}$, set $q_{jk} = 2^{-N_j}(1 + k2^{-m_j})$, and define

$$U_j = \{2^{-N_j+k} + q_{jk} : 1 \leq k \leq M_j\}, \quad U = \bigcup_{j=1}^{\infty} U_j, \quad V = \{-2^{-j} : j \geq 1\}.$$ 

An element of $U_j$ of the form $2^{-N_j+k} + q_{jk}$ lies in the dyadic interval $[2^{-N_j+k}, 2^{-N_j+k+1})$, and for a given $k$, is the only element of $U_j$ in this interval. Further, $U_j \subseteq [2^{-N_j+1}, 2^{-N_j+M_j+1})$, hence by the relation (2.1), $U_j \cap U_{j'} = \emptyset$ if $j \neq j'$. Thus $U \in \Lambda(1; \frac{1}{2})$, since for any $i \geq 1$, the set $U \cap [2^{-i}, 2^{-i+1})$ is either empty or a single point. Clearly $V$ is a lacunary sequence, hence $V \in \Lambda(1; \frac{1}{2})$ as well, being its own special sequence. On the other hand,

$$U + V \supseteq \bigcup_{j=1}^{\infty} \{q_{jk} : 1 \leq k \leq M_j\}.$$ 

In other words, $U + V$ contains an affine copy of the dyadic rationals of the form $\{k2^{-m_j} : 1 \leq k \leq 2^{m_j}\}$ in $[0, 1]$, for every $j$. As discussed in example (c) in Section 2.1.1, $U + V$ is sublacunary.

The counterexample above illustrates the sensitivity of lacunarity on ambient coordinates, and precludes a higher dimensional generalization of this notion that relies on componentwise extension. For instance, the two-dimensional set $U \times V$ (with $U$, $V$ as above) has lacunary coordinate projections in the current system of coordinates, but there are other directions of projection, for instance the line of unit slope, along which the projection of this set is much more dense.

### 2.2 Finite order lacunarity in general dimensions

Let $V$ be a $d$-dimensional affine subspace of an Euclidean space $\mathbb{R}^n$, $n \geq d$. Given a base point $a$ of $V$ and an orthonormal basis $\mathcal{B} = \{v_1, \ldots, v_d\}$ of the linear subspace $V - a$, we define the projection maps

$$\pi_j = \pi_j[a, \mathcal{B}] : V \to \mathbb{R}, \quad \text{via} \quad x = a + \sum_{j=1}^{d} x_j v_j \to x_j = \pi_j(x), \quad 1 \leq j \leq d. \quad (2.2)$$

**Definition 2.5** (Admissible lacunarity and sublacunarity of Euclidean sets). Let $U$ be a relatively compact subset of $V$. 


We say that the set $U$ is admissible lacunary of order at most $N$ (as an Euclidean subset of $V$) with lacunarity constant at most $\lambda < 1$ if there exists an integer $R \geq 1$ satisfying the following property: for any choice of basis $\mathcal{B}$ and base point $a$, and each $1 \leq j \leq d$, the projected set

$$\pi_j(U) = \{\pi_j(x) : x \in U\} \subseteq \mathbb{R}$$

can be covered by $R$ members of $\Lambda(N; \lambda)$, with the class $\Lambda(N; \lambda)$ as described in Definition 2.4. The projection $\pi_j$ depends on $a$ and $\mathcal{B}$ via (2.2). The collection of sets $U$ that obey these conditions for a given choice of $N, \lambda$ and $R$ will be denoted by $\Lambda_d(N, \lambda, R; V)$.

The set $U$ is called sublacunary in $V$ if it is not admissible lacunary of finite order; i.e., if for any $\lambda < 1$ and integers $N, R \geq 1$ there exists a choice of basis $\mathcal{B}$ and an index $1 \leq j \leq d$ such that $\pi_j(U)$ cannot be covered by any $R$-fold union of one-dimensional lacunary sets of order at most $N$ and lacunarity constant at most $\lambda$.

Remarks:

- An equivalent formulation of the definition of $U \in \Lambda_d(N, R, \lambda; V)$ is that for any line $L$ in $V$ (and indeed in $\mathbb{R}^{d+1}$ as we will soon see in Lemma 2.6), the projection of $U$ onto $L$ is coverable by at most $R$ sets in $\Lambda(N; \lambda)$.

- We ask the reader to verify that the choice of base point in $V$ is not important in this definition, since $\pi_j[a, \mathcal{B}](U)$ is a translate of $\pi_j[a', \mathcal{B}](U)$ for any $a, a' \in V$. Thus $\pi_j[a, \mathcal{B}](U) \in \Lambda(N; \lambda)$ if and only if $\pi_j[a', \mathcal{B}](U) \in \Lambda(N; \lambda)$.

- The definition is also invariant under rotation in $\mathbb{R}^n$; if $O$ is an orthogonal transformation of $\mathbb{R}^n$, then $U \in \Lambda_d(N, \lambda, R; V)$ if and only if $O(U) \in \Lambda_d(N, \lambda, R; O(V))$.

- The choice of rotation $\mathcal{B}$ within $V$ is however critical. It is not possible to have necessary and sufficient implications like the ones above for two arbitrary choices of bases $\mathcal{B}$ and $\mathcal{B}'$. We provide examples below. Henceforth, we will refer to the choice of a pair $\varphi = (\mathbf{a}, \mathcal{B})$ as a system of coordinates, with the main focus on $\mathcal{B}$.

Before proceeding to examples, we check the definition for consistency if $U$ is a subset of several affine subspaces.

**Lemma 2.6.** Let $U \subseteq V$ be as above. Then for any choice of $N, R, \lambda$, the set $U \in \Lambda_d(N, R, \lambda; V)$ if and only if $U \in \Lambda_n(N, R, \lambda; \mathbb{R}^n)$.

**Proof.** The “if” implication is clear, so we consider the converse. Without loss of generality, we may choose $V = \{1\} \times \mathbb{R}^{n-1}$. Given any unit vector $\omega = (\omega_1, \cdots, \omega_n) \in \mathbb{R}^n$ with $0 < |\omega_1| < 1$, let $L$ denote the line through the origin in $\mathbb{R}^n$ pointing in the direction of $\omega$. Let $L'$ denote the projection of $L$ on $V$, so that $L' = \{e_1 + s\omega' : s \in \mathbb{R}\}$, where $e_1$ is
the first canonical basis vector in \( \mathbb{R}^n \), and \( \omega' = (0, \omega_2, \cdots, \omega_n) \). The desired conclusion follows from the claim that

\[
\text{the sets } \pi(U) \text{ and } \pi'(U) \text{ are affine copies of each other},
\]

where \( \pi(U) \) and \( \pi'(U) \) denote the scalar projections onto \( L \) and \( L' \), measured from the origin and \( (1,0,\cdots,0) \) respectively. Indeed, Lemma 2.3 then permits us to extend known lacunarity features of the former directly to the latter.

To establish (2.3), it suffices to note that for any \( x \in \mathbb{R}^n \),

\[
\pi(x) = (x \cdot \omega) \omega, \quad \text{and} \quad \pi'(x) = \frac{x \cdot \omega'}{|\omega'|^2} \omega'.
\]

The choice of \( V, \omega \) and \( \omega' \) yield the relations \( (x - y) \cdot \omega = (x - y) \cdot \omega' \) for any \( x, y \in U \), hence the above expressions imply that

\[
|\pi(x) - \pi(y)| = |\omega'| |\pi'(x) - \pi'(y)|,
\]

which is the desired conclusion. \( \square \)

2.2.1 Examples of admissible lacunary and sublacunary sets in \( \mathbb{R}^d \)

(a) A set of the form considered by Nagel, Stein and Wainger [20], such as

\[
U = \{ \gamma(\theta_j) : j \geq 1 \}, \quad \text{where} \quad \gamma(t) = (t^{m_1}, \cdots, t^{m_d})
\]

is admissible lacunary of order 1. Here \( 0 < m_1 < \cdots < m_d \) are fixed constants, and \( 0 < \theta_{j+1} \leq \lambda \theta_j \), for some \( 0 < \lambda < 1 \) and all \( j \). Critical to this verification are the following two properties of \( U \) appearing in [20, Lemma 4]:

- There is a constant \( C_1 = C_1(m_1, \cdots, m_d) \) obeying the following requirement. For any unit vector \( \xi = (\xi_1, \cdots, \xi_d) \) in \( \mathbb{R}^d \), the set \( \mathbb{N} \) of positive integers can be decomposed into \( C_1 \) disjoint consecutive intervals \( \{ N_s \} \); for every \( s \), there exists \( r(s) \in \{ 1, \cdots, d \} \) such that

\[
\max_{1 \leq r \leq d} |\theta_j^{m_r} \xi_r| = |\theta_j^{m_r(s)} \xi_{r(s)}| \quad \text{for all } j \in \mathbb{N}_s.
\]

The composition of \( N_s \) depends on \( \xi \).

- Further for any \( c > 0 \), there is a constant \( C_2 = C_2(c, m_1, \cdots, m_d) \) independent of \( \xi \) and \( N_s \) so that

\[
\max_{r \in \{ 1, \cdots, d \}, r \neq r(s)} |\theta_j^{m_r} \xi_r| < c |\theta_j^{m_r(s)} \xi_{r(s)}|
\]

for all but \( C_2 \) integers \( j \in \mathbb{N}_s \).
Assuming these two facts, the claim of lacunarity is established as follows. Using the definition of \( N_s \) in (2.6), the set \( U \) can be decomposed into \( C_1 \) pieces \( U_s \), where \( U_s = \{ \gamma(\theta_j) : j \in \mathbb{N}_s \} \). Fix a constant \( R \) such that \( 2d\lambda^{m_1 R-1} < 1 \). If \( j' > j \) are two integers in \( \mathbb{N}_s \) that are at least \( R \)-separated and for both of which (2.6) holds with \( c = \frac{1}{2^7} \), then

\[
\sum_{r=1}^{d} |\xi_r \theta_j^{m_r}| \leq d|\xi_{r(s)} \theta_j^{m_r(s)}| \leq d(\lambda^{j'-j})^{m_r(s)} |\xi_{r(s)} \theta_j^{m_r(s)}| \leq 2d(\lambda R)^{m_1} \sum_{r=1}^{d} |\xi_r \theta_j^{m_r}| < \lambda \sum_{r=1}^{d} |\xi_r \theta_j^{m_r}|. \tag{2.7}
\]

Thus each \( U_s \) is the union of at most \( R \) lacunary sequences of lacunarity \( \lambda < \lambda \), together with the \( C_2 \) points where (2.6) fails.

(b) A set of the form considered by Carbery [6], i.e.,

\[
U = \{ \Gamma_k = (\lambda^{k_1}, \ldots, \lambda^{k_d}) : k = (k_1, \ldots, k_d) \in \mathbb{N}^d \} \tag{2.8}
\]
is admissible lacunary of order \( d \). We prove this by induction on \( d \). The initializing step for \( d = 2 \) has been covered in example (g) of Section 2.1.1. For a general \( d \) and after splitting \( U \) into \( d! \) pieces, we may assume that \( k_1 \leq k_2 \leq \cdots \leq k_d \). Given any unit vector \( \xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d \), we write

\[
U = \bigcup_{s=1}^{d} U_s \quad \text{with} \quad U_s = \{ \Gamma_k : k \in \mathbb{N}_s^d \}, \quad \mathbb{N}_s^d = \{ k \in \mathbb{N}^d : |\lambda^{k_1} \xi_s| = \max_{1 \leq r \leq d} |\lambda^{k_r} \xi_r| \}.
\]

Depending on the signs of \( \lambda^{k_s} \xi_s \) and \( \Gamma_k \cdot \xi - \lambda^{k_s} \xi_s \), each \( \mathbb{N}_s^d \) can be decomposed into four parts. Their treatments are similar with trivial adjustments, so we focus on the subset of \( \mathbb{N}_s^d \) where

\[
\lambda^{k_s} \xi_s > 0 \quad \text{and} \quad \sum_{r \neq s} \lambda^{k_r} \xi_r \geq 0,
\]

continuing to call this subset \( \mathbb{N}_s^d \) to ease notational burden. One last splitting is needed; for a constant \( A \) to be specified shortly, we write

\[
\mathbb{N}_s^d = \mathbb{N}_{s,1}^d \cup \mathbb{N}_{s,2}^d, \quad \mathbb{N}_{s,1}^d = \{ k \in \mathbb{N}_{s,1}^d : \lambda^{k_s} \xi_s > A|\lambda^{k_r} \xi_r| \text{ for all } r \neq s \}.
\]

For \( k \in \mathbb{N}_{s,1}^d \),

\[
\lambda^{k_s} \xi_s \leq \Gamma_k \cdot \xi < \lambda^{k_s} \xi_s (1 + dA^{-1}) < \lambda^{k_s-1} \xi_s, \tag{2.9}
\]

where the last inequality follows for a suitable choice of \( A \). We argue that \( \{ \xi_s \lambda^{k_s} : k_s \geq 1 \} \) may be viewed as a special sequence for \( \{ \Gamma_k \cdot \xi : k \in \mathbb{N}_{s,1}^d \} \). Indeed, if \( k_s \) is fixed, then (2.6) shows that

\[
\{ \Gamma_r \cdot \xi : r \in \mathbb{N}_{s,1}^d \} \cap [\xi_s \lambda^{k_s}, \xi_s \lambda^{k_s-1}] = \{ \Gamma_r \cdot \xi : r \in \mathbb{N}_{s,1}^d, \, r_s = k_s \}.
\]
\[ \subseteq \xi_s \lambda^{k_s} + \left\{ \sum_{r \neq s} \lambda^{k_r} \xi_r : k_r \in \mathbb{N}, r \neq s \right\}. \]

By the induction hypothesis, there is a constant \( R \) independent of \( \xi \) such that the set on the right hand side above is coverable by at most \( R \) sets in \( \Lambda(d-1; \lambda) \). Hence \( \{ \Gamma_k : k \in \mathbb{N}^d \} \) is admissible lacunary of order \( d \).

We turn to the complementary set \( \mathbb{N}^d \setminus \mathbb{N}_{s,2} \). After decomposing \( \mathbb{N}^d \setminus \mathbb{N}_{s,2} \) into \( (d-1) \) subsets, we may fix an index \( \ell \) such that
\[
|\lambda^{k_\ell} \xi_\ell| \leq A |\lambda^{k_s} \xi_s| \quad (2.10)
\]
on \( \mathbb{N}^d \setminus \mathbb{N}_{s,2} \). Without loss of generality let \( \ell \geq s \). The number of possible values of \( k_\ell - k_s \) obeying (2.10) is at most a fixed constant \( C \) depending on \( A \), but independent of \( \xi \). Thus \( \mathbb{N}^d \setminus \mathbb{N}_{s,2} \) may be written as the \( C \)-fold union of subsets indexed by \( c \), where the subset identified by \( c \) contains all \( k \in \mathbb{N}^d \setminus \mathbb{N}_{s,2} \) with the property that \( k_\ell - k_s = c \geq 0 \). For \( k \) in such a subset,
\[
\Gamma_k \cdot \xi = (\xi_s + \lambda^c \xi_\ell) \lambda^{k_s} + \sum_{r \neq \ell, s} \lambda^{k_r} \xi_r.
\]

Since the number of summands in the linear combination above is \( (d-1) \), the induction hypothesis dictates that \( \{ \Gamma_k : k \in \mathbb{N}_{s,2}^d \} \) is admissible lacunary of order \( (d-1) \), completing the proof.

(c) A curve in \( \mathbb{R}^d \) is sublacunary. So is a Cantor-like subset of it as considered in [16].

(d) If \( U \) and \( V \) are the lacunary sets of order 1 constructed in Section 2.1.2, the set \( U \times V \) is sublacunary. Indeed, after a rotation of angle \( \frac{\pi}{4} \) one of the coordinate projections turns out to be a constant multiple of \( U + V \). We have seen in Section 2.1.2 that this last set is sublacunary on \( \mathbb{R} \).

2.3 Finite order lacunarity for direction sets

Given two sets \( \Omega_1, \Omega_2 \subseteq \mathbb{R}^{d+1} \setminus \{0\} \), we say that \( \Omega_1 \sim \Omega_2 \) if
\[
\left\{ \frac{\omega}{|\omega|} : \omega \in \Omega_1 \right\} = \left\{ \frac{\omega}{|\omega|} : \omega \in \Omega_2 \right\}.
\]
The binary relation \( \sim \) is clearly an equivalence relation among sets in \( \mathbb{R}^{d+1} \setminus \{0\} \). An equivalence class of \( \sim \) is, by definition, a direction set. By a slight abuse of nomenclature, we will refer to a set \( \Omega \subseteq \mathbb{R}^{d+1} \setminus \{0\} \) as a direction set to mean the equivalence class of \( \sim \) that contains \( \Omega \). Clearly the maximal operators \( D_\Omega \) and \( M_\Omega \), as well as the admittance of Kakeya-type sets (as in Definition 1.1), remain unchanged for all members of this equivalence class.

Certain modifications are necessary to extend the notion of lacunarity from Euclidean sets to direction sets, in view of the latter’s scale invariance. Given a direction set \( \Omega \subseteq \mathbb{R}^{d+1} \setminus \{0\} \), we denote by \( C_\Omega \) the cone generated by this set of directions, namely
\[
C_\Omega := \{ r\omega : r > 0, \omega \in \Omega \}. \quad (2.11)
\]
Definition 2.7. Let $\Omega \subseteq \mathbb{R}^{d+1} \setminus \{0\}$ be a direction set, with $C_\Omega$ as in (2.11).

(i) Given an integer $N$ and a positive constant $\lambda < 1$, we say that $\Omega$ is admissible lacunary as a direction set with order at most $N$ and lacunarity at most $\lambda$ if there exists an integer $R$ such that $U \in \Lambda_d(N, \lambda, R; V)$ in the sense of Definition 2.5, for every hyperplane $V$ at unit distance from the origin and every relatively compact subset $U$ of $C_\Omega \cap V$.

(ii) A direction set $\Omega \subseteq \Omega_0$ failing this property is termed a sublacunary direction set. Thus $\Omega$ is sublacunary as a direction set if for any choice of integers $N, R$ and positive constant $\lambda < 1$ there is a tangential hyperplane $V$ of the unit sphere, a relatively compact subset $U$ of $C_\Omega \cap V$ and a line $L$ in $V$ such that the projection of $U$ along $L$ cannot be covered by any $R$-fold union of sets in $\Lambda(N; \lambda)$.

2.3.1 Examples of admissible lacunary and sublacunary direction sets

(a) A direction set $\Omega$ of the form considered by Nagel, Stein and Wainger [20],

$$\Omega = \{u_j = (\gamma(\theta_j), 1) : j \geq J\}$$

is admissible lacunary of order 1. Here the function $\gamma$ and the sequence $\theta_j$ are as described in example (a) of Section 2.2.1. Thus $\Omega$ is parameterized by the positive constants $m_1 < m_2 < \cdots < m_d$. We set $m_{d+1} = 0$. To verify the claim, we choose $V = \{x \in \mathbb{R}^{d+1} : x \cdot \eta = 1\}$ for some unit vector $\eta$, so that $C_\Omega \cap V \subseteq \left\{v_j = \frac{u_j}{u_j \cdot \eta} : u_j \in \Omega\right\}$.

Fix a unit vector $\omega = (\omega', \omega_{d+1}) \in \mathbb{R}^{d+1}$, and let $\pi_\omega$ denote the scalar projection onto $\omega$; i.e., $\pi_\omega(v) = v \cdot \omega$. As required by Definitions 2.7 and 2.5 and in view of Lemma 2.6 we aim to show that there is a large constant $R$ (independent of $V$) for which any relatively compact subset of $\pi_\omega(C_\Omega \cap V)$ can be covered by $R$ members of $\Lambda(1; \lambda)$. By the property (2.12) of $\Omega$, we first decompose the integers into a bounded number $C_1$ of disjoint intervals ($C_1$ independent of $\omega$ and $\eta$), on each of which there exists an index $1 \leq r \leq d+1$ such that

$$\max_{1 \leq i \leq d+1} |\theta_{ij}^\omega \eta_i| = |\theta_{jr}^\omega \eta_r|.$$  \hfill (2.12)

Let us denote by $N_r[\eta]$ one of the subintervals for which (2.12) holds. For $j \in N_r[\eta],$

$$\pi_\omega(v_j) - \frac{\omega_r}{\eta_r} = \frac{\xi \cdot u_j}{\eta_r (\eta \cdot u_j)},$$

where $\xi = (\xi_1, \ldots, \xi_{d+1}) \in \mathbb{R}^{d+1}$  \hfill (2.13)

with $\xi_k = \omega_k \eta_r - \omega_r \eta_k$, so that $\xi_r = 0$. Our goal is to show that for $j \in N_r[\eta]$, the sequence on the right hand side above can be covered by an $R$-fold union of lacunary sequences converging to 0.
Using (2.5) again, we decompose \( N_r[\eta] \) into at most \( C_1 \) pieces, of the form \( N_r[\eta, \xi] = N_r[\eta] \cap N_s[\xi] \). Since \( \xi_r = 0 \), we conclude that \( N_r[\xi] = \emptyset \); hence \( s \neq r \). By property (2.6), for every \( c > 0 \), there are at most a bounded number \( C_2 = C_2(c) \) indices \( j \in N_{rs}[\eta, \xi] \) for which at least one of the inequalities

\[
\max_{i \neq r} |\theta_j^{m_i} \eta_i| < c |\theta_j^{m_r} \eta_r|, \quad \max_{i \neq s} |\theta_j^{m_i} \xi_i| < c |\theta_j^{m_s} \xi_s| \tag{2.14}
\]

fails.

First suppose \( s > r \). Choosing two integers \( j, j' \in N_{rs}[\eta, \xi] \) with \( j' - j \geq R \) for both of which the constraints in (2.14) hold, we follow the steps laid out in (2.7), obtaining from (2.13)

\[
\left| \pi_\omega(v_{j'}) - \frac{\omega_r}{\eta_r} \right| \left| \pi_\omega(v_j) - \frac{\omega_r}{\eta_r} \right|^{-1} = \frac{\xi \cdot u_{j'}}{\xi \cdot u_j} \cdot \frac{\eta \cdot u_j}{\eta \cdot u_{j'}} \leq \frac{d |\xi_s| |\theta_j^{m_s}|}{\frac{d}{2} |\xi_s| |\theta_j^{m_r}|} \cdot \frac{d |\eta_r| |\theta_j^{m_r}|}{\frac{d}{2} |\eta_r| |\theta_j^{m_s}|} \leq 4d^2 \left( \frac{\theta_j^{m_r}}{\theta_j^{m_s}} \right)^{m_s - m_r} \leq 4d^2 \lambda R(m_s - m_r).
\]

If \( R \) is selected large enough to satisfy \( 4d^2 \lambda R(m_s - m_r) < \lambda \), then for \( j \in N_{rs}[\eta, \xi] \) the sequence on the right hand side of (2.13) can be covered by the union of \( R \) lacunary sequences converging to zero, excluding the \( C_2 \) points where (2.14) fails. For \( s < r \), the same calculation above can be replicated for \( j' < j \) with \( j' - j < -R \). Thus in this case the sequence in (2.13) grows as \( j \) increases, and hence has to be finite by the assumption of relative compactness. Nonetheless, this finite sequence is still coverable by a lacunary sequence going to zero, this time in reverse order of \( j \). In either event, we have decomposed the set \( \{ \pi_\omega(v_j) : j \in N_{rs}[\eta, \xi] \} \) into \( R \) lacunary sequences of lacunarity \( \lambda \), proving the claim.

(b) A direction set of the type studied in [6], namely

\[
\Omega = \{(\Gamma_k, 1) : 0 \leq k_1 \leq k_2 \leq \cdots \leq k_d \},
\]

(with \( \Gamma_k \) as in (2.8)) is admissible lacunary of order \( d \). This is proved along lines similar to the example above, using methods already explained in examples (2) and (5) of Section 2.1.1 and 2.2.1 respectively; we omit the details here.

(c) A curve in \( \mathbb{R}^{d+1} \) is sublacunary as a direction set.

(d) For sets \( U, V \) as constructed in Section 2.1.2 the direction set \( \Omega = \{1\} \times U \times V \) is sublacunary, since \( U \times V \) is sublacunary as an Euclidean set (see example (4) in Section 2.2.1).
(e) Let \( \{ q_\ell : \ell \geq 1 \} \) be an enumeration of the rationals on any nontrivial interval, say on \([\frac{1}{2}, \frac{2}{3}]\). A direction set of the type considered by Parcet and Rogers [21, Example 1 on page 4], such as

\[ \Omega = \{(q_\ell 2^{-\ell}, 2^{-\ell}, 1) : \ell \geq 1 \} \]

is sublacunary, even though the one-dimensional coordinate projections in the current coordinate system are lacunary of order at most 1. Choosing \( V = \{ x_2 = 1 \} \), we find that

\[ C_{\Omega} \cap V = \{(q_\ell, 1, 2^\ell) : \ell \geq 1 \}. \]

The order of lacunarity of the \( x_1 \)-projection grows without bound as we choose increasingly large compact subsets of \( C_{\Omega} \cap V \).

(f) We also mention another example considered by Parcet and Rogers [21, Example 2 on page 4]. Given the canonical orthonormal basis \( \{ e_1, e_2, e_3 \} \) of \( \mathbb{R}^3 \), let us fix another orthonormal basis \( \{ e_1', e_2', e_3' \} \) with \( \text{span} \{ e_2, e_3 \} = \text{span} \{ e_2', e_3' \} \) and \( e_3' \) lying in the first quadrant determined by \( e_2 \) and \( e_3 \). The direction set under consideration is \( \Omega = \{ u_\ell : \ell \geq 1 \} \), where \( u_\ell \) is a sequence of vectors satisfying \( u_\ell \cdot e_2' = q_\ell u_\ell \cdot e_1 \) for some enumeration of rationals \( \{ q_\ell \} \) in an interval. The last condition does not completely specify \( u_\ell \), hence the direction set so defined is not unique (further restrictions are imposed in [21]), but regardless of any subsequent choice \( \Omega \) is sublacunary. Choosing \( V = \{ x_1 = 1 \} \), we observe that

\[ C_{\Omega} \cap V = \left\{ \frac{u_\ell}{u_\ell \cdot e_1} : \ell \geq 1 \right\}. \]

Projecting \( C_{\Omega} \cap V \) in the direction \( e_2' \), we find that the projected set is \( \{ q_\ell : \ell \geq 1 \} \), which is not lacunary of finite order.

### 2.4 Boundedness of directional maximal operators

To confirm that our definition of directional lacunarity of finite order agrees with similar notions existing in the literature, we are able to use the result of Parcet and Rogers [21] to establish the \( L^p \)-boundedness of directional and Kakeya-Nikodym maximal operators associated to such direction sets \( \Omega \subseteq \mathbb{R}^{d+1} \). Incidentally, this also proves the implication “(3) \( \implies \) (1)” in Theorem 1.3. Let us recall from (1.2) and (1.3) the relevant definitions.

**Theorem 2.8.** Given positive integers \( N, R \), a positive constant \( \lambda < 1 \) and any exponent \( p \in (1, \infty] \), there exists a positive finite constant \( C_p = C_p(N, \lambda, R) \) with the following property. Any admissible lacunary direction set \( \Omega \subseteq \mathbb{R}^{d+1} \) of finite order that obeys Definition 2.7(i) with the prescribed values of \( N, \lambda \) and \( R \) also satisfies

\[ ||M_\Omega||_{p \rightarrow p} \leq C_p \quad \text{and} \quad ||D_\Omega||_{p \rightarrow p} \leq C_p. \] (2.15)

**Proof.** We first argue that the boundedness of \( D_\Omega \) on any \( L^p(\mathbb{R}^{d+1}) \) implies the same for \( M_\Omega \). Without loss of generality, we may assume that \( \Omega \subseteq (-\epsilon, \epsilon)^d \times \{1\} \) for some small constant \( \epsilon > 0 \). Let us define for any \( x \in \mathbb{R}^{d+1} \) the vectors

\[ v_j(x) = x_{d+1} e_j - x_j e_{d+1}, \quad 1 \leq j \leq d, \]

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where \( \{e_1, \ldots, e_{d+1}\} \) denotes the canonical orthonormal basis in \( \mathbb{R}^{d+1} \). For \( \omega = (\omega_1, \ldots, \omega_d, 1) \in \Omega \), the collection \( \{v_1(\omega), \ldots, v_d(\omega)\} \) spans \( \omega^\perp \). Then

\[
M_\Omega f(x) \leq C_d \sup_{\omega \in \Omega} \sup_{0 < r \leq h} \frac{1}{r^d} \int_{|t| \leq h} \left| f(x - t\omega - \sum_{j=1}^{d} s_j v_j(\omega)) \right| dt ds
\]

\[
\leq C_d \sup_{\omega \in \Omega} \sup_{r > 0} \frac{1}{r^d} \int D_\Omega f(x - \sum_{j=1}^{d} s_j v_j(\omega)) ds
\]

\[
\leq C_d \sup_{\omega \in \Omega} \sup_{r > 0} \frac{1}{r^{d-1}} \int D_{\Omega_1} \circ D_\Omega f(x - \sum_{j=2}^{d} s_j v_j(\omega)) ds_2 \cdots ds_d
\]

\[
\leq \cdots \leq C_d D_{\Omega_d} \circ D_{\Omega_{d-1}} \circ \cdots \circ D_{\Omega_1} \circ D_{\Omega} f(x),
\]

where \( \Omega_j = \{v_j(\omega) : \omega \in \Omega\}, 1 \leq j \leq d \). The relation

\[
\frac{v_j(\omega) \cdot \xi}{v_j(\omega) \cdot \eta} = \frac{\omega \cdot v_j(\xi)}{\omega \cdot v_j(\eta)} \quad \text{for all } \xi, \eta \in \mathbb{R}
\]

implies that if \( \Omega \) is admissible lacunary of order at most \( N \) as a direction set, then so is \( \Omega_j \) for every \( j \). Thus a bound on the \( L^p \) operator norm of \( M_\Omega \) would follow if the second conclusion (for directional maximal operators) in \( 2.15 \) is known to hold for all such direction sets. We will henceforth concentrate only on \( D_{\Omega} \), with \( \Omega \) being admissible lacunary of finite order.

As mentioned before, the \( L^p \)-boundedness of \( D_{\Omega} \) is a restatement of the main result in \( 21 \); we merely supply the connecting details. The proof is by induction. The quantity that forms the basis for induction is related but not identical to the order of lacunarity of the direction set as prescribed in Definition \( 2.7 \). To set up the induction parameter and hypothesis, we need a few preparatory steps. Without loss of generality and by a generic rotation if necessary, we may assume that \( \Omega \), which is admissible lacunary of order \( N = N(\Omega) \) as a direction set, is contained in a fixed small spherical cap in the first orthant that stays away from the coordinate hyperplanes. For each index \( 1 \leq j \leq d + 1 \), we set \( \mathcal{V}_j := \{ x \in \mathbb{R}^{d+1} : x_j = 1 \} \) and define

\[
\Theta_j(\Omega) = C_\Omega \cap \mathcal{V}_j = \left\{ \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_d \\ \omega_{d+1} \\ \omega_j \end{pmatrix} : \omega = (\omega_1, \ldots, \omega_{d+1}) \in \Omega \right\}.
\]

Then \( \Theta_j(\Omega) \in \Lambda_d(N, \lambda, R; \mathcal{V}_j) \), according to Definition \( 2.7 \). Appealing to Definition \( 2.5 \) let \( N_{kj} = N_{kj}(\Omega) \leq N \) and \( R_{kj} = R_{kj}(\Omega) \leq R \) be the smallest non-negative integers such that

\[
\pi_k[\Theta_j(\Omega)] = \left\{ \frac{\omega_k}{\omega_j} : \omega \in \Omega \right\}, \quad k \neq j
\]

is coverable by at most \( R_{kj} \) members of \( \Lambda(N_{kj}; \lambda) \). Here \( \pi_k \) denotes the projection onto the \( k \)th coordinate axis in the ambient coordinate system. By decomposing \( \Omega \) into at most \( Rd^2 \) pieces if necessary, we may assume that \( R_{kj} = 1 \) for all \( k \neq j \). Set

\[
\Sigma = \{(k, j) : 1 \leq k < j \leq d + 1\} \quad \text{and} \quad \Sigma^* = \Sigma^*(\Omega) = \{(k, j) \in \Sigma : N_{kj} \geq 1\}.
\]

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For a generic rotation mentioned at the beginning of this proof, $\Sigma = \Sigma^*$.

The induction is based on

$$N_0(\Omega) = N(\Omega) + \sum_{\sigma = (k, j) \in \Sigma^*} N_{kj}(\Omega).$$  \hfill (2.16)

The induction hypothesis is that the second inequality in (2.15) holds for all $\Omega$ with $N_0(\Omega) \leq N_0$. The initializing step $N_0 = 0$ follows from the one-dimensional Hardy-Littlewood maximal theorem. For a direction set $\Omega$ with $N_0(\Omega) = 0$ and any $\sigma = (k, j) \in \Sigma^*$, let $\{\theta_{\sigma, i} : i \geq 1\}$ be a lacunary (without loss of generality decreasing) sequence with lacunarity constant $\leq \lambda$ that serves as a special sequence for $\pi_k [\Theta_j(\Omega)]$ (see Definition 2.1). As in [21], we set

$$\Omega_{\sigma, i} = \left\{ \omega \in \Omega : \theta_{\sigma, i+1} \leq \frac{\omega_k}{\omega_j} \leq \theta_{\sigma, i} \right\},$$

and observe that $\Omega_{\sigma, i} \subseteq \Omega$ is admissible lacunary of order at most $N$ as a direction set with the same parameters $R$ and $\lambda$ as before. In particular, $N(\Omega_{\sigma, i}) \leq N(\Omega)$, and $N_{k'j'}(\Omega_{\sigma, i}) \leq N_{k'j'}(\Omega)$ for all $(k', j') \in \Sigma$. The result of [21] states that

$$||D_{\Omega_{\sigma, i}}||_{p \to p} \leq C \sup_{\sigma \in \Sigma^*} \sup_{i \geq 1} ||D_{\Omega_{\sigma, i}}||_{p \to p}.$$  \hfill (2.17)

(In fact, [21] addresses the generic and nontrivial case of $\Sigma^* = \Sigma$, but the proof goes through with trivial modifications after a reduction to lower dimensions even when $\Sigma^* \subset \Sigma$, i.e., if $N_{kj} = 0$ for certain pairs $(k, j) \in \Sigma$). From the definition of $\Omega_{\sigma, i}$, we conclude that

$$\pi_k [\Theta_j(\Omega_{\sigma, i})] = \pi_k [\Theta_j(\Omega)] \cap (\theta_{\sigma, i+1}, \theta_{\sigma, i}] \in \Lambda(N_{kj} - 1, \lambda)$$

for any $\sigma = (k, j) \in \Sigma^*$; hence $N_{kj}(\Omega_{\sigma, i}) \leq N_{kj} - 1$. It now follows from (2.16) that

$$N_0(\Omega_{\sigma, i}) \leq N_0(\Omega) - 1,$$

allowing us to carry the induction forward. \hfill \square

3 Rooted, labelled trees

As in [4], [3] and [16], the language of rooted, labelled trees continues to be the vehicle of choice for construction of Kakeya-type sets. We recall the basic terminology of trees and state the relevant facts in Sections 3.1 and 3.3 below, referring the reader to our previous work [16] for a more detailed discussion of some of the stated results, and to [19] for a comprehensive treatise on the subject.

3.1 The terminology of trees

A tree is defined to be a connected undirected graph with no cycles. A rooted, labelled tree $T$ is one whose vertex set is a nonempty collection of finite sequences of nonnegative integers such that if $\langle i_1, \ldots, i_n \rangle \in T$, then
(i.) for any \( k, 0 \leq k \leq n \), \( \langle i_1, \ldots, i_k \rangle \in \mathcal{T} \), where \( k = 0 \) corresponds to the empty sequence, and

(ii.) for every \( j \in \{0, 1, \ldots, i_n\} \), we have \( \langle i_1, \ldots, i_{n-1}, j \rangle \in \mathcal{T} \).

We say that \( \langle i_1, \ldots, i_{n-1} \rangle \) is the parent of \( \langle i_1, \ldots, i_{n-1}, j \rangle \) and that \( \langle i_1, \ldots, i_{n-1}, j \rangle \) is the \( (j+1) \)th child of \( \langle i_1, \ldots, i_{n-1} \rangle \). A parent-child pair is an edge, and a sequence of connected edges \( (e_1, e_2, \ldots) \) is a ray, where by convention we require that the child vertex of \( e_i \) agree with the parent vertex of \( e_{i+1} \) for all \( i \geq 1 \). The empty sequence \( \emptyset \) is the designated root of the tree \( \mathcal{T} \) and all vertices of the form \( \langle i_1 \rangle \in \mathcal{T} \) are children of this root. We let \( \partial \mathcal{T} \) denote the collection of all rays in \( \mathcal{T} \) of maximal (possibly infinite) length. For a fixed vertex \( v \in \mathcal{T} \), we also define the subtree of \( \mathcal{T} \) generated by the vertex \( v \) to be the maximal subtree of \( \mathcal{T} \) with \( v \) as the root.

The height of the tree is taken to be the supremum of the lengths of all the sequences in the tree. Further, we define the height \( h(\cdot) \) of a vertex to be the length of its identifying sequence. If the height of a vertex \( v \) is equal to \( k \), we say that \( v \) is a \( k \)th generation vertex of the tree. The height of the root is always taken to be zero. If \( u \) and \( v \) are two vertices in \( \mathcal{T} \) that lie along the same ray, with \( h(u) > h(v) \), then we say \( u \) is a descendant of \( v \) (or that \( v \) is an ancestor of \( u \)), and we write \( u \subset v \). The youngest common ancestor of \( u \) and \( v \), denoted by \( D(u, v) \), is the vertex of maximal height that any ray passing through \( u \) has in common with any ray passing through \( v \).

If \( \mathcal{T} \) is a tree and \( n \in \mathbb{Z}^+ \), the truncation of \( \mathcal{T} \) to height \( n \), denoted \( \mathcal{T}_n \), is the subtree of \( \mathcal{T} \) consisting of all vertices with height no more than \( n \). A tree is called locally finite if its truncation to every level is finite; i.e. consists of finitely many vertices. All of our trees will have this property. In the remainder of this article, when we speak of a tree we will always mean a locally finite, rooted, labelled tree.

The following definition will be very important for us later.

**Definition 3.1.** Let \( \mathcal{T} \) and \( \mathcal{T}' \) be two trees with equal (possibly infinite) heights. A map \( \sigma : \mathcal{T} \to \mathcal{T}' \) is called sticky if

- for all \( v \in \mathcal{T} \), \( h(v) = h(\sigma(v)) \), and
- \( u \subset v \) implies \( \sigma(u) \subset \sigma(v) \) for all \( u, v \in \mathcal{T} \).

We often say that \( \sigma \) is sticky if it preserves heights and lineages.

A one-to-one and onto sticky map between two trees, when it exists, is said to be an isomorphism and the two trees are said to be isomorphic. Two isomorphic trees can and will be treated as essentially identical objects.

**3.2 Encoding bounded subsets of Euclidean space by trees**

The language of rooted, labelled trees is especially convenient for representing bounded sets in Euclidean spaces. This connection is well-studied in the literature. We refer the interested reader to [19] for more information.
Fix any integer \( M \geq 2 \). For any nonnegative integer \( i \) and positive integer \( k \) such that \( i < M^k \), there exists a unique representation
\[
  i = i_1 M^{k-1} + i_2 M^{k-2} + \cdots + i_{k-1} M + i_k,
\]
where the integers \( i_1, \ldots, i_k \) take values in \( \mathbb{Z}_M := \{0, 1, \ldots, M-1\} \). These are the digits of the \( M \)-adic expansion of \( i \). An easy consequence of (3.1) is that there is a one-to-one correspondence between \( M \)-adic rationals in \([0,1)\) of the form \( i/M^k \) and finite integer sequences \( \langle i_1, \ldots, i_k \rangle \) of length \( k \) with \( i_j \in \mathbb{Z}_M \) for each \( j \). More generally, for any \( i = \langle j_1, \ldots, j_d \rangle \in \mathbb{Z}_M^d \) such that \( i \cdot M^{-k} \in [0,1)^d \), we can apply (3.1) to each component of \( i \) to obtain
\[
  \frac{i}{M^k} = \frac{1}{M^k} \langle j_1, \ldots, j_d \rangle = \frac{i_1}{M} + \frac{i_2}{M^2} + \cdots + \frac{i_k}{M^k},
\]
with \( i_j \in \mathbb{Z}_M^d \) for all \( j \). In this way, we identify \( i \) with \( \langle i_1, \ldots, i_k \rangle \). Let \( \phi : \mathbb{Z}_M^d \to \{0, 1, \ldots, M^d-1\} \) be an enumeration of \( \mathbb{Z}_M^d \). We refer to
\[
  \mathcal{T}([0,1)^d; M, \phi) = \{ \langle \phi(i_1), \ldots, \phi(i_k) \rangle : k \geq 0, \ i_j \in \mathbb{Z}_M^d \}
\]
as the full \( M \)-adic tree in dimension \( d \). Every vertex of the full \( M \)-adic tree has exactly \( M^d \) children; therefore there are exactly \( M^{kd} \) vertices of the \( k \)th generation. For our purposes, it will suffice to fix \( \phi \) to be the lexicographic ordering, and so we will omit the notation for \( \phi \) in (3.3), writing simply, and with a slight abuse of notation,
\[
  \mathcal{T}([0,1)^d; M) = \{ \langle i_1, \ldots, i_k \rangle : k \geq 0, \ i_j \in \mathbb{Z}_M^d \}. 
\]
We will refer to the tree in (3.3) by the notation \( \mathcal{T}([0,1)^d) \) once the base \( M \) has been fixed.

Each vertex \( v = \langle i_1, \ldots, i_k \rangle \) of \( \mathcal{T}([0,1)^d; M) \) at height \( k \) represents the unique \( M \)-adic cube in \([0,1)^d \) of sidelength \( M^{-k} \), containing \( 1 \cdot M^{-k} \), of the form
\[
  Q = \left[ \frac{j_1}{M^k}, \frac{j_1 + 1}{M^k} \right) \times \cdots \times \left[ \frac{j_d}{M^k}, \frac{j_d + 1}{M^k} \right).
\]
Here \( \langle i_1, \ldots, i_k \rangle \) is related to \( \langle j_1, \ldots, j_d \rangle \) by (3.2). Consequently, any \( x \in [0,1)^d \) can be realized as the intersection of a nested sequence of \( M \)-adic cubes. Thus, we view the tree in (3.3) as an encoding of the set \([0,1)^d \) with respect to base \( M \). Any subset \( E \subseteq [0,1)^d \) then corresponds to a subtree \( \mathcal{T}(E; M) \) of \( \mathcal{T}([0,1)^d; M) \). The vertices on the tree \( \mathcal{T}(E; M) \) represent \( M \)-adic cubes that have nontrivial intersection with \( E \). As a result, an infinite ray in \( \mathcal{T}(E; M) \) identifies a point in \( E \) or its closure. Needless to say, the tree representation of the set \( E \) is coordinate-sensitive. Indeed trees representing the same set in two systems of coordinates may possess widely different features - a property that we will need to take into account shortly.

In light of the discussion above and for simplicity, we will henceforth identify the vertex \( v = \langle i_1, i_2, \cdots, i_k \rangle \in \mathcal{T}([0,1)^d) \) with the corresponding cube \( Q \) as in (3.3) lying on \([0,1)^d \). With this understanding, the notation \( u \subset v \) stands both for set containment as well as tree ancestry.
3.3 The splitting number of a tree

There are many ways to quantify the “size” or “spread” of a tree (see [19]). Of these, the concept of a splitting number proved to be the most relevant in the planar characterization of directions that admit Kakeya-type sets [3]. Not surprisingly, it will turn out to be equally important for us. One of its applications is the explicit restatement of finite order lacunarity of a set Ω in terms of the structure of the tree encoding Ω. We define the notion of splitting number below, then collect some fundamental results about this quantity that will allow us to prove Theorem 1.2, which is also the first forward implication in Theorem 1.3.

We say that a vertex \( v \in T \) splits in \( T \) if it has at least two children in \( T \). When it is clear to which tree we are referring, we will just say that \( v \) splits, and we will call \( v \) a splitting vertex. Define split\(_T\)(R), the splitting number of a ray \( R \) in \( T \) to be the number of splitting vertices in \( T \) along that ray. The splitting number of a vertex \( v \) with respect to a tree \( T \) is defined to be

\[
\text{split}_T(v) := \max_{S_v \subseteq T} \min_{R_v \in \partial S_v} \text{split}_v(R_v),
\]

where the maximum is taken over all subtrees \( S_v \subseteq T \) rooted at \( v \), and the minimum is taken over all rays \( R_v \) in \( S_v \) that originate at the vertex \( v \). Finally, the splitting number of the tree \( T \) is defined as

\[
\text{split}(T) := \max_{v \in T} \text{split}_T(v).
\]

3.3.1 Examples

(a) If \( \Omega = \{2^{-j} : j \geq 1\} \), then \( \text{split}(T(\Omega; 2)) = 1 \).

(b) If \( \Omega_m = \{\frac{k}{2^m} : 0 \leq k < 2^m\} \), then \( \text{split}(T(\Omega_m; 2)) = m \). As a result, the tree depicting all dyadic rationals has infinite splitting number.

(c) Let \( U \) and \( V \) be the sets constructed in Section 2.1.2. Then \( \text{split}(T(U \times V); 2) = 2 \), while \( \text{split}(T(\varphi(U \times V); 2)) = \infty \) for the coordinate transformation \( \varphi(u, v) = (u + v, u - v) \).

3.3.2 Preliminary facts about splitting numbers

Our first result about splitting numbers (of vertices) says that they are monotone non-increasing in lineages.

**Lemma 3.2.** Let \( u, v \in T \) with \( u \subseteq v \). Then \( \text{split}_T(u) \leq \text{split}_T(v) \).

**Proof.** Let \( S_u \) be a subtree of \( T \) rooted at \( u \). Define \( S_{v \to u} \) to be the union of the tree \( S_u \) with the path in \( T \) connecting \( v \) to \( u \). This is a subtree of \( T \) rooted at \( v \). Since \( v \) does not split in \( S_{v \to u} \) and there are no splitting vertices in \( S_{v \to u} \) between \( v \) and \( u \), we find that for any ray \( R \) in \( S_u \),

\[
\text{split}_{S_u}(R) = \text{split}_{S_{v \to u}}(R_v),
\]

where the maximum is taken over all subtrees \( S_u \subseteq T \) rooted at \( u \), and the minimum is taken over all rays \( R_v \) in \( S_{v \to u} \) that originate at the vertex \( v \). Finally, the splitting number of the tree \( T \) is defined as

\[
\text{split}(T) := \max_{v \in T} \text{split}_T(v).
\]
where $R_v$ is the ray in $S_{v\to u}$ rooted at $v$ obtained by extending $R$ to $v$. Conversely, if $R_v$ is a ray in $S_{v\to u}$, then (3.8) holds for $R = R_v \cap S_u$. Maximizing over all subtrees $S \subseteq T$ rooted at $u$, we have that

$$\text{split}_T(u) = \max_{S_u \subseteq T} \min_{R_v \in \partial S_u} \text{split}_{S_u}(R_v)$$

$$= \max_{S_{v\to u} \subseteq T} \min_{R_v \in \partial S_{v\to u}} \text{split}_{S_{v\to u}}(R_v)$$

$$\leq \text{split}_T(v).$$

The last inequality is a consequence of (3.6), since the class of subtrees of the form $S_{v\to u}$ is a subcollection of trees rooted at $v$.

An immediate consequence of Lemma 3.2 is that $\text{split}(T) = \text{split}_T(v_0)$, where $v_0$ is the root of $T$. Our next result says that splitting numbers of trees are also monotonic in an appropriate sense.

**Lemma 3.3.** Let $S \subseteq T$. Then $\text{split}(S) \leq \text{split}(T)$.

**Proof.** By Lemma 3.2, $\text{split}(S) = \text{split}_S(v_0)$, where $v_0$ is the root of $S$. Since $v_0 \in S \subseteq T$ and any subtree of $S$ is also a subtree of $T$, we find that

$$\text{split}_S(v_0) = \max_{S_{v_0} \subseteq S} \min_{R_{v_0} \in \partial S_{v_0}} \text{split}_{S_{v_0}}(R_{v_0})$$

$$\leq \max_{S_{v\to u} \subseteq T} \min_{R_{v_0} \in \partial S_{v\to u}} \text{split}_{S_{v\to u}}(R_{v_0})$$

$$\leq \text{split}_T(v_0)$$

$$\leq \text{split}(T),$$

where the last two inequalities are implied by (3.6) and (3.7) respectively. Lemma 3.3 follows. □

A feature of trees with finite splitting number, originally observed in [3, Lemma 5], is that all vertices with largest split occur along a ray. This specialized ray will turn out to be critical in the detection of lacunary limits.

**Lemma 3.4.** Let $T$ be a tree with $\text{split}(T) = N$. Then there exists a ray $R$ in $T$ (of finite or infinite length) such that a vertex $v$ lies on $R$ if and only if $\text{split}_T(v) = N$, provided the latter collection contains more than one element.

**Proof.** We prove by contradiction. Suppose there are two vertices $u, v \in T$ with $\text{split}_T(u) = \text{split}_T(v) = N$, $u \not\subseteq v$, $v \not\subseteq u$. Then their youngest common ancestor $D(u, v)$ is neither $u$ nor $v$. By Lemma 3.2, we know that $\text{split}_T(D(u, v)) \geq N$. Since $u \neq v$, the vertex $D(u, v)$ is actually a splitting vertex. Therefore, $\text{split}_T(D(u, v)) \geq N + 1$. But this contradicts the requirement that $\text{split}(T) = N$, establishing our claim. □
3.3.3 A reformulation of Theorem 1.2

The dichotomy between trees with finite versus infinite splitting number will prove to be our main distinction of interest. Roughly speaking, a tree that has infinite splitting number in some coordinate system must encode a “large” subset of Euclidean space, the threshold of size being determined by sublacunarity. However, as we have seen in example (c) of Section 3.3.1, the splitting number of a tree encoding a set is sensitive to the coordinates used to represent the set. More strongly, even the finiteness of the splitting number could be affected by the choice. This consideration features prominently in the restatement of Theorem 1.2 that we are about to set down.

Our proof of Theorem 1.2 will follow a two-step route.

Proposition 3.5. Fix a dimension $d \geq 2$ and an integer $M \geq 2$. If a direction set $\Omega \subseteq \mathbb{R}^{d+1} \setminus \{0\}$ is sublacunary (in the sense of Definition 2.7), then

$$\sup_{V} \sup_{W_{\Omega}} \sup_{\varphi} \text{split}(T(\varphi(W_{\Omega}); M)) = \infty. \quad (3.9)$$

Here $V$ ranges over the collection of all hyperplanes at unit distance from the origin. For a fixed $V$, the set $W_{\Omega}$ ranges over all relatively compact subsets of $C_{\Omega} \cap V$, and the innermost supremum is taken over all coordinate choices $\varphi = (a, B)$ on $V$, where $a \in \mathbb{V}$ is the point closest to the origin and $B = \{v_1, \ldots, v_d\}$ is any orthonormal basis of $V - a$. In other words, $\varphi$ represents a rotation in $V$ centred at $a$, with

$$\varphi(C_{\Omega} \cap V) = \{(x_1, \ldots, x_d) : x = a + \sum_{j=1}^{d} x_j v_j \in C_{\Omega} \cap V\}.$$ 

Thus for every $N \geq 1$, there exists a hyperplane $V_N$, a relatively compact subset $W_N$ of $C_{\Omega} \cap V_N$, and a coordinate system $\varphi_N$ on $V_N$ such that

$$\text{split}(T(\varphi_N(W_N); M)) > N. \quad (3.10)$$

Proposition 3.6. If a direction set $\Omega$ obeys (3.9) for some $M \geq 2$, then $\Omega$ admits Kakeya-type sets.

Proposition 3.6 will be the subject of the main body of our paper (Sections 7 – 11). We prove Proposition 3.5 in Section 3.4 below.

We end this section with a natural question: how does the splitting number of a tree $T(\Omega; M)$ change if $\Omega$ is re-encoded as a tree with respect to a different base? It is not difficult to see that the number itself is not invariant under change of base. For example, if $\Omega = \{\frac{k}{4} : 0 \leq k < 4^N\}$ for some integer $N \geq 1$, then $\text{split}(T(\Omega; 2)) = 2N$, whereas $\text{split}(T(\Omega; 4)) = N$. On the other hand, no consistent notion of “size” of a set should be dependent on the choice of base we use to encode that set. The appropriate base-invariant concept here turns out not to be the value of the quantity in (3.9), but whether it is finite or not. Indeed for any two choices of base integers $M, M' \geq 2$, the corresponding expressions in (3.9) are either both finite or both infinite. We do not need
this stronger conclusion, but observe that Proposition 3.5 combined with Theorem 1.3 gives an aposteriori proof of this fact. This all serves to remind the reader that in the Kakeya-type construction, the choice of base used to encode the direction set as a tree is purely utilitarian and non-central to the proof.

3.4 Lacunarity on trees

A distinctive feature in the planar characterization of Kakeya-type sets [3] is the observation that the lacunarity of a set is reflected in the structure of its tree. Following the ideas developed there, we recast the concept of finite order lacunarity of a one-dimensional set using the structure of the splitting vertices of its tree. This provides a tool of convenience in the proof of Proposition 3.5, the main objective of this section.

Lemma 3.7. For any $M \geq 2$, $N \geq 1$, there is a constant $C = C(N, M)$ with the following property. If a relatively compact set $U \subseteq \mathbb{R}$ is such that $\text{split}(T(U; M)) = N$, then $U$ can be covered by the $C$-fold union of sets in $\Lambda(N; M^{-1})$ as described in Definition 2.2.

The proof of this lemma will be presented later in this section. Assuming this, the proof of the proposition is completed as follows.

Proof of Proposition 3.5. We prove the contrapositive, starting with the assumption that

$$\sup_{V} \sup_{\varphi} \sup_{W_\Omega} \text{split}(T(\varphi(W_\Omega; M))) = N < \infty.$$  \hfill (3.11)

Fix an arbitrary coordinate system $\varphi = (a, B)$ of $\mathbb{V}$ and let $\pi_j$ denote the projection maps defined in (2.2) with respect to this choice. For the remainder of this proof, we will assume that $\mathbb{V}$ is represented in these coordinates, so that $\pi_j$ may be thought of as the coordinate projections. Let $W = W_\Omega$ be an arbitrary relatively compact subset of $C_\Omega \cap \mathbb{V}$. Since the tree encoding a set matches that of its closure, we may suppose without loss of generality that $W = W_\Omega$ is compact in $V$.

For any $1 \leq j \leq d+1$, we create a subset $W_j \subseteq W$ that contains for every $x_j \in \pi_j(W)$ a unique point $x \in W$ with $\pi_j(x) = x_j$. For concreteness, $x$ could be chosen to be minimal in $\pi_j^{-1}(x_j) \cap W$ with respect to the lexicographic ordering. In other words, $\pi_j$ restricted to $W_j$ is a bijection onto $\pi_j(W)$. We claim that

$$\text{split}(T(\pi_j(W); M)) \leq \text{split}(T(W_j; M)).$$  \hfill (3.12)

Assuming this for the moment, we obtain from the hypothesis (3.11) and Lemma 3.5 that $\text{split}(T(\pi_j(W); M)) \leq \text{split}(T(W; M)) \leq N$. Applying Lemma 3.7 to $U = \pi_j(W)$, we see that there is a constant $C$ (uniform in $\mathbb{V}$, $\varphi$, $j$ and $W$) such that the projections $\pi_j(W)$ can be covered by the $C$-fold union of one-dimensional lacunary sets of order $\leq N$ and lacunarity $\leq M^{-1}$. Thus, $W = W_\Omega$ is admissible lacunary of order at most $N$ according to Definition 2.5. Hence $\Omega$ is admissible lacunary of finite order as a direction set by Definition 2.7.
It remains to establish (3.12). Any infinite ray \( R = R(x_j) \) in \( \partial T(\pi_j(W); M) \) corresponds to a point \( x_j \in \pi_j(W) \). Let \( R^* = R^*(x) \in \partial T(W_j; M) \) denote the ray that represents \( \pi_j^{-1}(x_j) = x \). This establishes a bijection between the collection of rays in the two trees. Let \( v_0 \) and \( v'_0 \) denote the roots of the trees \( T(\pi_j(W); M) \) and \( T(W_j; M) \) respectively, so that \( \pi_j(v'_0) = v_0 \). If \( S \) is a subtree of \( T(\pi_j(W); M) \) rooted at \( v_0 \), let us denote by \( S^* \) the subtree of \( T(W_j; M) \) rooted at \( v'_0 \) generated by all rays \( R^* \) such that \( R \) is a ray of \( S \). It is clear that if a vertex \( v \) on \( R(x_j) \) splits in \( S \), then there are two points \( x_j \neq x'_j \) in \( \pi_j(W) \) lying in distinct children of \( v \). This implies that \( x = \pi_j^{-1}(x_j) \) and \( x' = \pi_j^{-1}(x'_j) \) lie in distinct children of \( v^* \), which denotes the vertex of height \( h(v) \) on \( R^*(x) \). This makes \( v^* \) a splitting vertex of \( S^* \). Thus every splitting vertex of \( S \) lying on \( R \) generates a splitting vertex of \( S^* \) lying on \( R^* \) at the same height. As a result, \( \text{split}_S(R) \leq \text{split}_{S^*}(R^*) \). Combining these facts with the definition of the splitting number of a tree, we obtain

\[
\text{split}(T(\pi_j(W); M)) = \max_{S} \min_{x \in \partial S} \text{split}_S(R) \\
\leq \max_{S} \min_{x \in \partial S} \text{split}_{S^*}(R^*) \\
\leq \text{split}(T(W_j; M)).
\]

In view of Lemma 3.2, the maxima in the first and second lines above are taken over all subtrees \( S \) and \( S^* \) rooted at \( v_0 \) and \( v'_0 \) respectively. This completes the proof of (3.12) and hence of Proposition 3.5.

We now turn to the proof of the lemma on which the argument above was predicated.

**Proof of Lemma 3.2.** We apply induction on \( N \). The base case \( N = 1 \) will be treated momentarily in Lemma 3.8. Proceeding to the induction step, let \( R^* \) denote an infinite ray of the tree \( T = T(U; M) \) that contains all the vertices \( \{v^* : \text{split}_T(v^*) = N\} \). The existence of such a ray has been established in Lemma 3.4. For every vertex \( v \) in \( T(U; M) \) which does not lie on \( R^* \) but whose parent does, we define a set \( U_v \) as follows: \( T_v = T(U_v; M) \), where \( T_v \) denotes the maximal subtree of \( T \) rooted at \( v \). The definition of the ray \( R^* \) dictates that each \( U_v \) has the property that \( \text{split}(T(U_v; M)) \leq N - 1 \). By the induction hypothesis, there exists a constant \( C = C(N - 1, M) \) such that each \( U_v \) is covered by the \( C \)-fold union of sets in \( \Lambda(N - 1; M^{-1}) \). The set \( U \) can therefore be covered by the \( C \)-fold union of sets \( U^{[i]} \), where each \( U^{[i]} \) shares a tree structure similar to \( U \); it contains the point identified by \( R^* \), with the additional feature that now \( U^{[i]}_v \in \Lambda(N - 1; M^{-1}) \) for every \( v \in U^{[i]} \), where

\[
V^{[i]} := \{ v \in T(U^{[i]}; M) : v \notin R^* \text{ but parent of } v \text{ is in } R^* \}.
\]

For every vertex \( v \in V^{[i]} \), let \( a_v \) denote the left hand endpoint of the \( M \)-adic interval represented by \( v \). The tree encoding the collection of points \( A = \{ a_v : v \in V^{[i]} \} \) contains the ray \( R^* \); indeed the only splitting vertices of \( T(A; M) \) lie on \( R^* \). Therefore \( \text{split}(T(A; M)) = 1 \). Hence, by Lemma 3.8 \( A \) is at most a \( C \)-fold union of monotone
lacunary sequences with lacunarity $M^{-1}$, each converging to the point identifying $R^*$. Let us continue to denote by $A$ one such monotone (say decreasing) sequence. If $a = a_v$ and $b$ are two successive elements of this sequence with $a < b$, then $U_i^{[j]} \cap [a, b] = U_i^{[j]}$, which is in $\Lambda(N - 1; M^{-1})$. Thus $U_i^{[j]}$ is in $\Lambda(N; M^{-1})$ according to Definition 2.2 completing the proof.

**Lemma 3.8.** Fix $M \geq 2$, and let $A \subseteq \mathbb{R}$ be a relatively compact set with the property that $\text{split}(\mathcal{T}(A; M)) = 1$. Then $A$ can be written as the union of at most $6M$ lacunary sequences (defined in Definition 2.2) each with lacunarity constant $\leq M^{-1}$.

**Proof.** The argument here closely follows the line of reasoning in [3, Remark 2, page 60]. By Lemma 3.1, there is a ray $R^*$ in $\mathcal{T}(A; M)$ of infinite length such that all the splitting vertices of $\mathcal{T}(A; M)$ lie on it. The ray $R^*$ uniquely identifies a point in $\mathbb{R}$, say $a^* = \alpha(R^*)$. Any ray that is not $R^*$ but is rooted at a vertex of $R^*$ is therefore non-splitting. Thus for every $j = 0, 1, 2, \cdots$ there exists at most $M - 1$ rays $R_j$ in $\mathcal{T}(A; M)$ whose $M$-adic distance from $R^*$ is $j$. In other words, if $a_j = \alpha(R_j)$ is the point in $A$ identified by $R_j$, then there are at most $M - 1$ distinct points $a_j \neq a^*$ such that

$$h(D(a^*, a_j)) = h(D(\alpha(R^*), \alpha(R_j))) = j. \quad (3.13)$$

We define two subsets $A_\pm$ of $A$, containing respectively points $a \geq a^*$ and $a \leq a^*$. This decomposes $\mathcal{T}(A; M)$ into two subtrees $\mathcal{T}(A_\pm; M)$. Let us focus on $\mathcal{T}(A_+; M)$, the treatment for the other tree being identical. We decompose $\mathcal{T}(A_+; M)$ as the union of at most $M$ trees $\mathcal{T}(A_i_+; M)$, $i \in \mathbb{Z}_M$, constructed as follows. The tree $\mathcal{T}(A_i_+; M)$ contains the ray $R^*$, and for every vertex $v$ in $R^*$ the ray in $\mathcal{T}(A_i_+; M)$, if any, descended from the $i$th child of $v$. In view of the discussion in the preceding paragraph, if there exists an integer $j$ for which a ray $R_j$ in $\mathcal{T}(A_i_+; M)$ obeys (3.13), then such a ray must be unique.

We now fix $i \in \mathbb{Z}_M$ and proceed to cover $A_i_+$ by a threefold union of lacunary sequences converging to $a^*$. Let $\{n_1 < n_2 < \cdots\}$ be the subsequence of integers with the property that $R_j \in \mathcal{T}(A_i_+; M)$ if and only if $j = n_k$ for some $k$. The important observation is that if $n_{k+2}$ is a member of this subsequence, then

$$a_{n_k} - a^* \geq \frac{1}{M^{n_k+2}}. \quad (3.14)$$

We will return to the proof of this statement in a moment, but a consequence of it and (3.13) is that for any $k \geq 0$ and fixed $\ell = 0, 1, 2$,

$$a_{n_{3(k+1)+\ell}} - a^* \leq M^{-n_{3k+3+\ell}} = M^{-n_{3k+3+\ell}+n_{3k+2+\ell}}M^{-n_{3k+2+\ell}} \leq M^{-1}(a_{n_{3k+\ell}} - a^*).$$

Thus for every fixed $\ell = 0, 1, 2$, the sequence $\mathcal{A}_\ell = \{a_{n_{3k+\ell}} : k \geq 0\}$ is covered by a lacunary sequence with constant $\leq M^{-1}$ converging to $a^*$. Since $A_i_+$ is the union of $\{\mathcal{A}_\ell : \ell = 0, 1, 2\}$, the result follows.

It remains to settle (3.14), which is best explained by Figure 3. If $I_j$ is the $M$-adic interval of length $M^{-n_j}$ containing $a^*$, then $I_{k+2}$ cannot share a right endpoint with $I_{k+1}$, since this would prevent the existence of a point $a_{n_{k+1}} \geq a^*$ obeying (3.13) with
Figure 3: A figure explaining inequality (3.14) when $M = 2$ and $n_k = k$.

$j = n_{k+1}$. Thus $a^*$ (in $I_{k+2}$) and $a_{n_k}$ (which is to the right of $I_{k+1}$) must lie on opposite sides of $J$, the rightmost $M$-adic subinterval of length $M^{-n_{k+2}}$ in $I_{k+1}$. This implies $a_{n_k} - a^* \geq |J|$, which is the conclusion of (3.14).

4 Pruning of the slope tree

We now fix a base integer $M \geq 2$ and a sublacunary direction set $\Omega \subseteq \mathbb{R}^{d+1}$ (obeying the conclusion of Proposition 3.5), and turn our attention to the proof of Proposition 3.6. We will also fix an absolute constant $C_0 \geq 1$, which will remain unchanged for the rest of the proof, and whose value will be specified later ($C_0 = 10$ will do). Given any integer $N$ however large, Proposition 3.5 (see (3.10)) supplies a hyperplane $V_N$ at unit distance from the origin, a coordinate system $\varphi_N$ on $V_N$, and a relatively compact subset $W_N \subseteq C_\Omega \cap V_N$ for which $\text{split}(T(\varphi_N(W_N); M)) > (N + 1)(2C_0 + 1)^d$. The choice of $N$, and hence $V_N$, $W_N$ and $\varphi_N$ will stay fixed during the analysis in Sections 5-11. The existence of Kakeya-type sets, which is the goal of Proposition 3.6, relies on the ability to conduct this analysis for arbitrarily large $N$. The constant $C_0$, on the other hand, does not change with $N$.

Without loss of generality we will assume that $V_N = \{1\} \times \mathbb{R}^d$ and that $\varphi_N$ is the ambient coordinate system in $V_N$ (and hence in all hyperplanes parallel to $V_N$). The use of $\varphi_N$ will be dropped in the sequel, and we will simply write $\text{split}(T(W_N; M)) > (N + 1)(2C_0 + 1)^d$. We will also assume that $W_N \subseteq \{1\} \times [0, 1)^d$; indeed if $W_N \subseteq \{1\} \times [0, M^L)^d$ for some large $L$, then we scale by a factor of $M^{-L}$ in directions perpendicular to $e_1 = (1, 0, \cdots, 0)$, leaving the direction $e_1$ unchanged. The tree corresponding to the scaled version of $W_N$ has the same splitting number as the original tree. Further, a union $E_N$ of tubes pointing in the scaled directions can be rescaled back to tubes with orientations in $W_N$, with the ratio $|E_N^*|/|E_N|$ (as explained in (1.1)) unchanged. From this point onwards, our direction set will be an appropriately chosen subset of $W_N \subseteq \{1\} \times [0, 1)^d$ for a fixed $N$. We rename $W_N$ as $\Omega$, since this will not cause any confusion in the sequel.

An important preparatory step in the construction of Kakeya-type sets is the extraction of a subset of the direction set $\Omega$, whose representative tree with respect to base
$M$ enjoys special structural properties, in terms of $M$-adic and Euclidean distance between certain vertices. The essential features of this trimming process and the modified direction set are summarized below in the main result of this section.

**Proposition 4.1.** Let $M \geq 2$ be a base integer, $C_0 \geq 1$ a fixed constant, and $N \gg 1$ a large parameter as described above. Let $\Omega \subseteq \{1\} \times [0, 1]^d$ be a direction set obeying the hypothesis split$(T(\Omega; M)) > (N + 1)(2C_0 + 1)^d$. Then there exist

- a finite subset $\Omega_N \subseteq \Omega$ of cardinality $2^N$, and
- an integer $J = J(\Omega, N) \geq N$

such that the following properties hold for the tree $T_J(\Omega_N; M)$ of height $J$ encoding $\Omega_N$:

(i) Every ray in $T_J(\Omega_N; M)$ splits exactly $N$ times.

(ii) Every splitting vertex in $T_J(\Omega_N; M)$ has exactly two children.

(iii) Let $v$ be any splitting vertex of $T_J(\Omega_N; M)$ and let $w_1, w_2$ be its two children as specified by part (ii). If $v_i \subseteq w_i$ denotes the first splitting descendant of $w_i$ for $i = 1, 2$, then the Euclidean distance between the cubes $v_1$ and $v_2$ is at least $C_0 M^{-h}$, where $h = \min\{h(v_i) : i = 1, 2\}$.

The integer $J$ can be chosen to ensure that the following additional condition is met:

(iv) $C_0 M^{-J} \leq \min\{|\omega - \omega'| : \omega \neq \omega', \omega, \omega' \in \Omega_N\}$.

The pruning process leading to the outcome claimed in the proposition is based on an iterative algorithm. The building block of the iteration is contained in Lemma 4.3 below, with Lemma 4.2 supplying an easy but necessary intermediate step.

**Lemma 4.2.** Fix integers $r \geq 0$ and $C_0 \geq 1$. A collection of cubes of cardinality $\geq (2C_0 + 1)^d + 1$ consisting of $M$-adic cubes of sidelength $M^{-r}$ and must contain at least two cubes whose Euclidean separation is $\geq C_0 M^{-r}$.

**Proof.** We first treat the case $r = 0$. The cube $Q_0 = [0, 2C_0 + 1]^d$ contains exactly $(2C_0 + 1)^d$ subcubes of unit sidelength with vertices in $\mathbb{Z}^d$. The central subcube $Q$ maintains a minimum distance of $C_0$ from the boundary of $Q_0$. Rephrasing this after a translation, any cube $Q$ with vertices in $\mathbb{Z}^d$ and of sidelength 1 admits at most $(2C_0 + 1)^d$ similar cubes whose distance from itself is $\leq C_0$. The case of a general $r \geq 0$ follows by scaling $Q_0$ by a factor of $M^{-r}$. $\square$

**Lemma 4.3.** Fix a constant integer $C_0 \geq 1$, an integer $N_0 \geq (2C_0 + 1)^d$ and a vertex $v_0$ of the full $M^d$-adic tree $T(\{1\} \times [0, 1]^d; M)$. Let $T_{[0]}$ rooted at $v_0$ be a subtree with the property that every ray in $T_{[0]}$ splits at least $N_0$ times. Then there exist an integer $k \geq 1$ and a subtree $T_{[1]}$ of $T_{[0]}$ rooted at $v_0$ and of height $k$ such that:

(i) The root $v_0$ has exactly two descendants $v_1$ and $v_2$ of height $k$ in $T_1$. 
(ii) The Euclidean separation between the cubes $v_1$ and $v_2$ is given by $\text{dist}(v_1, v_2) \geq C_0 M^{-k}$.

(iii) If $\mathcal{T}_{[0]}(v_i)$ is the maximal subtree of $\mathcal{T}_{[0]}$ rooted at $v_i$ then each ray in $\mathcal{T}_{[0]}(v_i)$ splits at least $N_0 - (2C_0 + 1)^d$ times.

Proof. Each ray in $\mathcal{T}_{[0]}$ splits at least $N_0$ times, so there exists a generation in this tree consisting of at least $2^{N_0}$ vertices. Since $2^{N_0} \gg (2C_0 + 1)^d$, let us define $k$ to be the smallest height in $\mathcal{T}_{[0]}$ such that the number of vertices at that height exceeds $(2C_0 + 1)^d$. By Lemma 4.2 there exist vertices $v_1$ and $v_2$ at height $k$ such that $\text{dist}(v_1, v_2) \geq C_0 M^{-k}$.

The subtree $\mathcal{T}_1$ of height $k$ rooted at $v_0$ and generated by $v_1$, $v_2$ clearly obeys conditions (i) and (ii) stated in Lemma 4.3. To complete the proof, let us recall that the number of elements of $\mathcal{T}_{[0]}$ at height $k - 1$ is $\leq (2C_0 + 1)^d$. Thus any ray of $\mathcal{T}_{[0]}$ rooted at $v_0$ contains at most $(2C_0 + 1)^d - 1$ splitting vertices of height $\leq k - 2$, since each splitting vertex of height $\leq k - 2$ gives rise to at least one new element (different among themselves and distinct from the terminating vertex of the ray) at height $k - 1$. Since every ray of $\mathcal{T}_{[0]}$ contained at least $N_0$ splitting vertices to begin with, at most $(2C_0 + 1)^d$ of which may be lost by height $k - 1$, we are left with at least $N_0 - (2C_0 + 1)^d$ splitting vertices per ray rooted at $v_i$, which is the conclusion claimed in (iii).

With the preliminary steps out of the way, we are ready to prove the main proposition.

Proof of Proposition 4.4. We know that $\text{split}(\mathcal{T}(\Omega; M)) > (N + 1)(2C_0 + 1)^d$. Given any $N \geq 1$, we can therefore fix a subtree $\overline{\mathcal{T}}$ of $\mathcal{T}(\Omega; M)$ of infinite height in which every ray splits at least $(N + 1)(2C_0 + 1)^d$ times. The pruning is executed on the subtree $\overline{\mathcal{T}}$ as follows.

In the first step we apply Lemma 4.3 with

$$\mathcal{T}_{[0]} = \overline{\mathcal{T}}, \quad v_0 = \{1\} \times [0, 1]^d \quad \text{and} \quad N_0 = (N + 1)(2C_0 + 1)^d.$$ 

This yields a subtree $\mathcal{T}_{[1]}$ rooted at $\{1\} \times [0, 1]^d$ of height $i_0$ consisting of two vertices $w_1$ and $w_2$ at the bottom-most level with $\text{dist}(w_1, w_2) \geq C_0 M^{-i_0}$. Every ray in $\mathcal{T}_{[1]}$ splits exactly once. Let us denote by $\overline{\mathcal{T}}(w_i)$ the maximal subtree of $\overline{\mathcal{T}}$ rooted at $w_i$. By Lemma 4.3 any ray of $\overline{\mathcal{T}}(w_i)$ splits at least $N(2C_0 + 1)^d$ times. Set $W_{i_1} := \{w_1, w_2\}$.

At the second step we invoke Lemma 4.3 twice, resetting the parameters in that lemma to be

$$\mathcal{T}_{[0]} = \overline{\mathcal{T}}(w_i), \quad v_0 = w_i, \quad N_0 = N(2C_0 + 1)^d$$

for $i = 1, 2$ respectively, and obtaining two subtrees as a consequence. Appending these two newly pruned subtrees of $\overline{\mathcal{T}}(w_i)$ to $\mathcal{T}_{[1]}$ from the previous step, we arrive at a tree $\mathcal{T}_{[2]}$ rooted at $\{1\} \times [0, 1]^d$ of finite height but with rays of possibly variable length, in which every ray splits exactly twice. If $v_1$ and $v_2$ are the first two splitting descendants of $\{1\} \times [0, 1]^d$ in this tree, then $v_i \subseteq w_i$. Hence

$$\text{dist}(v_1, v_2) \geq \text{dist}(w_1, w_2) \geq C_0 M^{-i_0} \geq C_0 M^{-h} \quad \text{where} \quad h = \min_{i=1,2} h(v_i) \geq i_0,$$
verifying the requirements (i)-(iii) of Proposition 4.1 for \( N = 2 \). Let us denote by \( \mathcal{W}_2 \) the collection of four vertices of maximal lineage in \( T_{[2]} \) obtained at the conclusion of this step. For any \( w \in \mathcal{W}_2 \), every ray of the tree \( \mathcal{T}(w) \) (defined as before as the maximal subtree of \( \mathcal{T} \) rooted at \( w \)) contains at least \( (N - 1)(2C_0 + 1)^d \) splitting vertices. Further \( \mathcal{W}_2 \) can be written as

\[
\mathcal{W}_2 = \bigcup \{ \mathcal{W}_2(w') : w' \in \mathcal{W}_1 \},
\]

where \( \mathcal{W}_2(w') \) consists of the two vertices in \( \mathcal{W}_2 \) descended from \( w' \). For fixed \( w' \in \mathcal{W}_1 \), Lemma 4.3 asserts that the vertices \( v, v' \) in \( \mathcal{W}_2(w') \) have the same height \( i_{w'} \), with

\[
\text{dist}(v, v') \geq C_0 M^{-i_{w'}}.
\]

In general at the end of the \( k \)th step we have a tree \( T_{[k]} \) of finite height, but with rays of potentially variable length, obeying the requirements (i)-(iii) for \( N = k \). The collection of vertices of highest lineage in \( T_{[k]} \) is termed \( \mathcal{W}_k \). We have that \( \#(\mathcal{W}_k) = 2^k \).

The collection \( \mathcal{W}_k \) can be decomposed as

\[
\mathcal{W}_k = \bigcup \{ \mathcal{W}_k(w') : w' \in \mathcal{W}_{k-1} \}, \quad \text{where} \quad \mathcal{W}_k(w') = \{ w_1(w'), w_2(w') \}
\]

consists of the two descendants of \( w' \) that lie in \( \mathcal{W}_k \). Lemma 4.3 ensures that

\[
h(w_1(w')) = h(w_2(w')) =: i_{w'} > h(w'),
\]

and that

\[
Figure 4: \ An illustration of the procedure generating the forced Euclidean separation between the descendants \( v_1 \) and \( v_2 \) of \( v \in \mathcal{T} \), in \( \mathbb{R}^2 \) when \( M = 2 \).
\[ \text{dist}(w_1(w'), w_2(w')) \geq C_0 M^{-i w'}. \] (4.1)

Any ray in \( T_{[k]} \) splits exactly \( k \) times, and for any \( w \in \mathbb{W}_k \) each ray of \( \mathcal{T}(w) \) splits at least \((N - k + 1)(2C_0 + 1)^d\) times.

In the \((k+1)\)th step, Lemma 4.3 is applied \( 2^k \) times in succession. In each application, the values of \( T_{[0]} \), \( v_0 \), \( N_0 \) are reset to

\[ T_{[0]} = \mathcal{T}(w), \quad v_0 = w, \quad N_0 = (N - k + 1)(2C_0 + 1)^d \]

respectively for some \( w \in \mathbb{W}_k \). The resulting tree \( T_{[k+1]} \), obtained by appending the \( 2^k \) newly constructed trees to \( T_{[k]} \) at the appropriate roots, clearly obeys (ii) and also (i) with \( N = k + 1 \). Part (iii) only needs to be verified for the splitting vertices \( v_1(w') \) and \( v_2(w') \) descended from \( w' \in \mathbb{W}_{k-1} \), since the splitting vertices of older generations have been dealt with in previous steps. But

\[ v_i(w') \subseteq v_i(w') \text{ for } i = 1, 2, \text{ so } \min_{i=1,2} h(v_i(w')) = h \geq i w'. \]

Combining this with (4.1), we obtain

\[ \text{dist}(v_1(w'), v_2(w')) \geq \text{dist}(w_1(w'), w_2(w')) \geq C_0 M^{-i w'} \geq C_0 M^{-h}. \] (4.2)

In view of the number of splitting vertices per ray in the original subtree \( \mathcal{T} \), the process described above can be continued at least \( N \) steps. The tree \( T_{[N]} \) of finite height but variable ray lengths obtained at the conclusion of the \( N \)th step satisfies the conditions (ii)-(iii). We pick from every vertex of maximal lineage in \( T_{[N]} \) exactly one point of \( \Omega \), calling the resulting collection of \( 2^N \) chosen points \( \Omega_N \). Set \( \delta := \min\{|\omega - \omega'| : \omega, \omega' \in \Omega_N, \omega \neq \omega'\} > 0 \). The rays in \( T_{[N]} \) are now extended as rays representing the points in \( \Omega_N \) (and hence without introducing any further splits) to a uniform height \( J \) that satisfies \( M^{-J} \leq C_0^{-1} \delta \), thereby meeting the criterion in part (iv).

4.1 Splitting and basic slope cubes

The pruned slope tree \( T_J(\Omega_N; M) \) produced by Proposition 4.1 looks like an elongated version of the full binary tree of height \( N \). Rays in this tree may have long segments with no splits. However only the splitting vertices of \( T_J(\Omega_N; M) \) and certain other vertices related to these are of central importance to the subsequent analysis. With this in mind and to aid in quantification later on, we introduce the class of splitting vertices

\[ G = G(\Omega_N) := \bigcup_{j=1}^{N} G_j(\Omega_N), \text{ where for every } 1 \leq j \leq N \] (4.2)

\[ G_j(\Omega_N) := \left\{ \gamma : \text{ there exists } v \in \Omega_N \text{ such that } \gamma \text{ is the } j \text{th splitting vertex on the ray identifying } v \text{ in } T_J(\Omega_N; M) \right\}. \] (4.3)
The vertices in $G_j(\Omega_N)$ will be termed the $j$th splitting vertices. As dictated by the pruning mechanism, such vertices $\gamma$ may occur at different heights of the tree $T_J(\Omega_N; M)$, and hence could represent $M$-adic cubes of varying sizes. Thus the index $j$, which encodes the number of splitting vertices on the ray leading up to and including $\gamma$, should not be confused with the height of $\gamma$ in $T_J(\Omega_N; M)$. Given $\gamma \in G(\Omega_N)$, we write

$$\nu(\gamma) = j \quad \text{if} \quad \gamma \in G_j(\Omega_N),$$

and refer to $\nu(\gamma)$ as the splitting index of $\gamma$. Indeed $N - \nu(\gamma)$ is the splitting number of $\gamma$ with respect to $T_J(\Omega_N; M)$, defined as in (3.6). Note that $G_1(\Omega_N)$ consists of a single element, namely the unique splitting vertex of $T(\Omega_N; M)$ of minimal height. In general $\#(G_j(\Omega_N)) = 2^{j-1}$, i.e., there are exactly $2^{j-1}$ splitting vertices of index $j$. We declare $G_{N+1}(\Omega_N) \equiv \Omega_N$.

Another related quantity of importance is the one mentioned in part (iii) of Proposition 4.1. In view of its ubiquitous occurrence in the sequel, we set up the following notation. For $\gamma \in G_j(\Omega_N)$, $1 \leq j \leq N - 1$,

$$\lambda(\gamma) = \lambda_j(\gamma) := \min\{h(\gamma') : \gamma' \subseteq \gamma, \gamma' \in G_{j+1}(\Omega_N)\}.$$ (4.5)

Thus $\lambda_j(\gamma) > h(\gamma)$ is the height of the first splitting vertex of index $(j + 1)$ descended from $\gamma$. We refer to an element of $\{\lambda_j(\gamma) : \gamma \in G_j(\Omega_N)\}$ as a $j$th fundamental height of $\Omega_N$. There could be at most $2^{j-1}$ such heights. The collection of all fundamental heights will be denoted by $\mathcal{R}$; it will play a vital role in the remainder of the article, specifically in the random construction outlined in Section 6. The two descendants of $\gamma \in G_j(\Omega_N)$ at height $\lambda_j(\gamma)$, at least one (but not necessarily both) of which corresponds to a $(j + 1)$th splitting vertex, are called the $j$th basic slope cubes. The entirety of $j$th
basic slope cubes as \( \gamma \) ranges over \( \mathcal{G}_j(\Omega_N) \) is termed \( \mathcal{H}_j(\Omega_N) \). More precisely,

\[
\begin{align*}
\mathcal{H}_j(\Omega_N) := \left\{ \theta : \text{there exists } \omega \in \Omega_N \text{ and } \gamma_j \in \mathcal{G}_j(\Omega_N) \right. \\
\left. \text{such that } \omega \in \theta \subseteq \gamma_j \text{ and } h(\theta) = \lambda(\gamma_j) \right\}. 
\end{align*}
\]

(4.6)

Note that every \( j \)th basic slope cube \( \theta \) is either itself a \((j + 1)\)th splitting vertex \( \gamma_{j+1} \in \mathcal{G}_{j+1}(\Omega_N) \), or uniquely identifies such a vertex in the sense that there exists a non-splitting ray in the slope tree rooted at \( \theta \) that terminates at \( \gamma_{j+1} \). In either event, we say that \( \gamma_{j+1} \in \mathcal{G}_{j+1}(\Omega_N) \) is identified by \( \theta \in \mathcal{H}_j(\Omega_N) \). Since every \( \gamma \in \mathcal{G}_j(\Omega_N) \) contributes exactly two cubes to \( \mathcal{H}_j(\Omega_N) \), it follows that \( \#(\mathcal{H}_j(\Omega_N)) = 2^j \). We declare \( \mathcal{H}_0(\Omega_N) = \mathcal{G}_1(\Omega_N) \) and \( \mathcal{H}_N(\Omega_N) = \Omega_N \).

One of the important features of the pruning mechanism outlined in Proposition 4.1 is an Euclidean separation condition between the two \( j \)th basic slope cubes descended from a common splitting vertex \( \gamma \in \mathcal{G}_j(\Omega_N) \). The following implication of this condition will be convenient for later use.

**Corollary 4.4.** Given a splitting vertex \( \gamma \) of \( \mathcal{T}_J(\Omega_N) \), define

\[
\rho_\gamma := \sup\{|a - b| : a \in \gamma_1 \cap \Omega_N, \ b \in \gamma_2 \cap \Omega_N\}, \quad (4.7)
\]

\[
\delta_\gamma := \inf\{|a - b| : a \in \gamma_1 \cap \Omega_N, \ b \in \gamma_2 \cap \Omega_N\}, \quad (4.8)
\]

where \( \gamma_1 \) and \( \gamma_2 \) are the two children of \( \gamma \) in \( \mathcal{T}_J(\Omega_N; M) \). Then, the two quantities \( \rho_\gamma \) and \( \delta_\gamma \), both of which are trivially bounded by \( \text{diam}(\gamma) = \sqrt{dM^{-h(\gamma)}} \), are comparable, i.e., \( \delta_\gamma \leq \rho_\gamma \leq (1 + 2\sqrt{dC_0^{-1}})\delta_\gamma \).
Proof. Using part (iii) of Proposition 4.1 and the notation set up in (4.5), we observe that \( \gamma_i \cap \Omega_N \subseteq v_i \) where \( v_i \) is the first splitting descendant of \( \gamma_i \), so that \( \delta_\gamma \geq C_0 M^{-\lambda(\gamma)} \). Let \( a_i, b_i \) be points in the closures of \( \gamma_i \cap \Omega_N, i = 1, 2 \) such that \( \delta_\gamma = |a_1 - b_1|, \rho_\gamma = |a_2 - b_2| \). Then

\[
\rho_\gamma = |a_2 - b_2| \leq |a_1 - b_1| + |a_2 - a_1| + |b_2 - b_1| \\
\leq |a_1 - b_1| + \text{diam}(v_1) + \text{diam}(v_2) \\
\leq |a_1 - b_1| + 2\sqrt{\delta M^{-\lambda(\gamma)}} \\
\leq \delta_\gamma + 2\sqrt{\delta C_0^{-1}} \delta_\gamma \leq C_2 \delta_\gamma,
\]

where the third inequality above follows from the fact that \( v_i \) is either itself a cube of sidelength \( M^{-\lambda(\gamma)} \) or is contained in one. \( \square \)

4.2 Binary representation of \( \Omega_N \)

The classes of basic slope cubes \( \mathcal{H}_j(\Omega_N) \) allow us to represent each element in \( \Omega_N \) in terms of a unique \( N \)-long binary sequence as follows. Since every splitting vertex of \( T_j(\Omega_N) \) has exactly two children, one of them must be larger (or older) than the other in the lexicographic ordering. Let us agree to call the older (respectively younger) child of a vertex \( v \) its 0th (respectively 1st) offspring. For \( 1 \leq j \leq N \), we define a bijective map \( \Psi_j : \{0,1\}^j \rightarrow \mathcal{H}_j(\Omega_N) \) inductively as follows. For \( j = 1 \),

\[
\Psi_1(i) := \begin{cases} 
\text{the unique element of } \mathcal{H}_1(\Omega_N) \\
\text{descended from the } i \text{th child of } \gamma_1,
\end{cases}
\]

where \( i = 0, 1 \), and \( \gamma_1 \) is the single element in \( \mathcal{H}_0(\Omega_N) = \mathcal{G}_1(\Omega_N) \). In general if \( \Psi_j \) has been defined, then for \( \bar{e} \in \{0,1\}^j \) and \( i = 0, 1 \), we set

\[
\Psi_{j+1}(\bar{e}, i) := \begin{cases} 
\text{the unique element of } \mathcal{H}_{j+1}(\Omega_N) \\
\text{descended from the } i \text{th child of } \gamma_{j+1},
\end{cases}
\]

where \( \gamma_{j+1} \) is the unique element of \( \mathcal{G}_{j+1}(\Omega_N) \) identified by \( \Psi_j(\bar{e}) \).

The map \( \Psi_N \) provides the claimed bijection of \( \{0,1\}^N \) onto \( \Omega_N \). In fact, the discussion above yields the following stronger conclusion, the verification of which is straightforward and left to the reader.

**Proposition 4.5.** Let \( \mathcal{H}_j(\Omega_N) \) be as in (4.8):

(i) The collection of vertices

\[
\mathcal{H}(\Omega_N) := \bigcup_{j=1}^{N} \{ \theta_1, \ldots, \theta_j \} \bigcup \{ \omega \in \Omega_N, \text{ such that } \omega \in \theta_k, 1 \leq k \leq j \} \bigcup \{ \gamma_1 \} \tag{4.11}
\]

is a tree rooted at \( \gamma_1 \in \mathcal{H}_0(\Omega_N) \) of height \( N \), in which \( \theta_1, \ldots, \theta_j, \theta_{j+1} \) is a vertex of height \( j + 1 \) and a child of \( \theta_1, \ldots, \theta_j \). Every element \( \theta_j \in \mathcal{H}_j(\Omega_N) \) identifies a vertex \( \theta_1, \ldots, \theta_j \) of the \( j \)th generation in this tree.
(ii) Let \( \mathcal{B}_N \) denote the full binary tree of height \( N \), namely the tree \( T_N([0,1);2) \). The map \( \Psi : \mathcal{B}_N \rightarrow \mathcal{H}(\Omega_N) \) defined by
\[
\Psi(\emptyset) = \text{the unique element } \gamma_1 \in \mathcal{H}_0(\Omega_N), \\
\Psi(\bar{\epsilon}) = \Psi_j(\bar{\epsilon}) \text{ if } \bar{\epsilon} \in \{0,1\}^J, 1 \leq j \leq N,
\]
with \( \Psi_j \) as in (4.10) is a tree isomorphism in the sense of Definition 3.1.

Although we will not need to use it, an analogous argument shows that the class of splitting vertices \( \mathcal{G}(\Omega_N) \) is isomorphic to \( \mathcal{B}_N-1 \).

5 Families of intersecting tubes

The finite set of directions \( \Omega_N \) created in Proposition 4.1 forms the basis of the construction of Kakeya-type sets. Indeed the sets of interest that verify the conclusion of Theorem 1.2 will be the union of a family of tubes, with each tube assigned a slope from \( \Omega_N \). Each tube is based on a suitably fine subcube of the \( d \)-dimensional unit cube \( \{0\} \times [0,1)^d \), hereafter referred to as the root hyperplane. The tree depicting the root hyperplane, more precisely the full \( M \)-adic tree of dimension \( d \) and height \( J \) will be termed the root tree. For \( 0 \leq k \leq J \), let \( Q(k) \) be the collection of all vertices of height \( k \) in the root tree, i.e.,
\[
Q(k) := \left\{ Q : Q \in T([0\} \times [0,1)^d;M), h(Q) = k \right\}.
\]

Geometrically, and in view of the discussion in Section 3.3, a member \( Q \) of \( Q(k) \) is an \( M \)-adic cube of sidelenight \( M^{-k} \) of the form
\[
Q = \{0\} \times \prod_{\ell=1}^{d} \left[ \frac{j_\ell}{M^k} : \frac{j_{\ell+1}}{M^k} \right], \text{ where } (j_1, j_2, \cdots, j_d) \in \{0,1,\cdots,M^k-1\}^d,
\]
so that \( \#(Q(k)) = M^{kd} \). In view of the above, and for the purpose of distinguishing vertices of the root and the slope trees, a vertex in the root tree is termed a spatial cube. For reasons to be made clear in a moment, an element of \( Q(J) \) (i.e., a youngest vertex of the root tree) is of added significance and will be called a root cube. Given a fixed constant \( A_0 \geq 1 \), and for \( t \in Q(J) \), \( \omega \in \Omega_N \), we define a tube rooted at \( t \) with orientation \( \omega \) to be the set
\[
P_{t,\omega} := \widetilde{Q}_t + [0,10A_0]\omega = \{ s + r\omega : s \in \widetilde{Q}_t, 0 \leq r \leq 10A_0 \}.
\]

Here \( \widetilde{Q}_t \) denotes the \( c_d \)-dilate of the cube \( t \); i.e., the cube with the same centre as \( t \) but with \( c_d \) times its sidelenight, for a small positive constant \( c_d \) soon to be specified in Corollary 5.2. For instance, the choice \( c_d = d^{-2d} \) will suffice. Thus \( P_{t,\omega} \) is essentially a \((d+1)\)-dimensional cylinder of constant length and with cubical cross-section of sidelenight \( c_dM^{-J} \) perpendicular to the \( x_1 \)-axis. An algorithm \( \sigma \) that assigns to every root
$t \in Q(J)$ a slope $\sigma(t) \in \Omega_N$ produces, according to the prescription (5.3), a family of tubes of cardinality $M^{Jd}$, and a corresponding set

$$K(\sigma) = K(\sigma; N, J) := \bigcup \{ P_{t,\sigma(t)} : t \in Q(J) \}. \quad (5.4)$$

While this definition is quite general, in our applications the slope assignment map $\sigma$ will always be chosen to be sticky in the sense of Definition 3.1 and as a mapping between the trees representing roots and slopes respectively; specifically,

$$\sigma : T_J(\{0\} \times [0,1)^d; M) \to T_J(\Omega_N; M).$$

Random sticky slope assignment algorithms will be prescribed in the next section, but for now we record some properties of arbitrary tubes and features of general sets of the form $K(\sigma)$.

5.1 Intersection of two tubes

**Lemma 5.1.** For $v, v' \in \Omega_N$ and $t, t' \in Q(J)$, $t \neq t'$, let $P_{t,v}$ and $P_{t',v'}$ be the tubes defined as in (5.3). If there exists $x = (x_1, \cdots, x_{d+1}) \in P_{t,v} \cap P_{t',v'}$, then the inequality

$$|\text{cen}(t') - \text{cen}(t) + x_1(v' - v)| \leq 2c_d \sqrt{d}M^{-J} \quad (5.5)$$

holds, where $\text{cen}(t)$ denotes the centre of the cube $t$.

**Proof.** If $x \in P_{t,v} \cap P_{t',v'}$, then there exist points $y, y' \in \tilde{Q}_t, \tilde{Q}_{t'}$ such that $x = y + x_1v = y' + x_1v'$, i.e., $x_1(v' - v) = y - y'$. The inequality (5.5) follows since both $|y - \text{cen}(t)|$ and $|y' - \text{cen}(t')|$ are bounded above by $c_d \sqrt{d}M^{-J}$.

**Corollary 5.2.** If the constant $c_d$ is chosen sufficiently small, then under the hypotheses of Lemma 5.1

$$|x_1|v - v'| \geq \frac{1}{2}M^{-J}. \quad (5.6)$$

**Proof.** Since $t \neq t'$, we know that $|\text{cen}(t') - \text{cen}(t)| \geq M^{-J}$. The inequality in (5.5) therefore implies that

$$|x_1|v - v'| \geq |\text{cen}(t') - \text{cen}(t)| - 2c_d \sqrt{d}M^{-J} \geq (1 - 2c_d \sqrt{d})M^{-J} \geq \frac{1}{2}M^{-J},$$

provided $c_d$ is chosen to satisfy $2c_d \sqrt{d} \leq \frac{1}{2}$. \hfill $\Box$

Lemma 5.1 provides an intersection criterion for two tubes in the form of an algebro-geometric inequality. We will also need to know the size of this intersection. This estimate is by now standard in the literature, dating back to the work of Córdoba [7]. The result below is easily verifiable, but the reader may consult [11, Lemma 10.3.6, p. 374] as a reference.

**Lemma 5.3.** If $P_{t,v}$ and $P_{t',v'}$ are any two intersecting tubes of the form (5.3), then

$$|P_{t,v} \cap P_{t',v'}| \leq \frac{C_dM^{-J(d+1)}}{M^{-J} + |v - v'|},$$

where $C_d$ is a dimension-dependent constant.
5.2 Tubes and a point

A crucial component of the proof of Proposition 3.6, amplified in Section 7, is to identify when a given point \( x \) belongs to a union of tubes of the form (5.4). In our applications, the set \( K(\sigma) \) in (5.4) will be probabilistically generated by random sticky maps, and we will need to estimate the likelihood of such an inclusion. But many major ingredients of the argument pertain to general sets \( K(\sigma) \) generated by an arbitrary sticky \( \sigma \). We discuss these features here.

**Lemma 5.4.** Let \( x \in \mathbb{R}^{d+1}, A_0 \leq x_1 \leq 10A_0 \). If the parameter \( C_0 \) used in the pruning of the slope tree \( T(\Omega; M) \) (see Proposition 4.1) is chosen sufficiently large relative to the constant \( A_0 \) in (5.3), then the following property holds: for any \( t \in Q(J) \), there exists at most one \( v(t) \in \Omega_N \) such that \( x \in P_t,v(t) \).

**Proof.** If there exist slopes \( v, v' \in \Omega_N \) such that \( x \in P_t,v \cap P_t,v' \), then the points \( x-x_1v \) and \( x-x_1v' \) must both lie in \( t \). In other words,

\[
|x_1(v-v')| = |(x-x_1v) - (x-x_1v')| \leq \sqrt{dM-J}.
\]

Since \( x_1 \geq A_0 \), this implies that \( |v-v'| \leq A_0^{-1}\sqrt{dM-J} \), which is \( \leq C_0 M^{-J} \) for a choice of \( C_0 \) sufficiently large. Comparing with part (iv) of Proposition 4.1, we find this is possible in \( \Omega_N \) only if \( v = v' \). \( \square \)

The lemma above motivates the following definition: for \( x \in \mathbb{R}^{d+1} \) with \( A_0 \leq x_1 \leq 10A_0 \),

\[
\text{Poss}(x) := \left\{ t \in Q(J) : \text{there exists } v(t) = v(t; x) \in \Omega_N, \text{ such that } x \in P_{t,v(t)} \right\}.
\]

(5.7)

**Lemma 5.5.** The set \( \text{Poss}(x) \) introduced in (5.7) can also be characterized as follows:

\[
\text{Poss}(x) = \left\{ t \in Q(J) : t \cap (x-x_1\Omega_N) \neq \emptyset \right\}.
\]

(5.8)

Thus \( \text{Poss}(x) \) is contained in an \( O(M^{-J}) \)-neighborhood of an affine copy of \( \Omega_N \) in the root hyperplane \( \{0\} \times [0,1)^d \).

**Proof.** If \( t \in \text{Poss}(x) \), it follows from the definition (5.3) of a tube and the description (5.7) of \( \text{Poss}(x) \) that \( x-x_1v(t) \in t \) for some \( v(t) \in \Omega_N \). Thus the left hand side of (5.8) is contained in the right hand side. Conversely, if \( x-x_1v \in t \) for some \( v \in \Omega_N \), then \( x + t \cap x_1v \subseteq P_{t,v} \). This means that \( t \in \text{Poss}(x) \), and the result follows. \( \square \)

The mapping

\[
v : \text{Poss}(x) \to \Omega_N \quad \text{which sends } \ t \mapsto v(t) \quad \text{with } \ x \in P_{t,v(t)}
\]

(5.9)

is uniquely defined by Lemma 5.4. It captures for every \( t \in \text{Poss}(x) \) the “correct slope” that ensures that a tube rooted at \( t \) with that slope contains \( x \). A purely deterministic object driven by \( \Omega_N \), this map has a certain structure that is critical to the subsequent
analysis. To formalize this property, let us recall the definitions of $G_j(\Omega_N)$ and $H_j(\Omega_N)$ from (4.3) and (4.8). We denote for every $\omega \in \Omega_N$ and $1 \leq j \leq N$,

$$\eta_j(\omega) := h(\theta) \quad \text{where} \quad \omega \subseteq \theta \in H_j(\Omega_N).$$

(5.10)

In other words, $\eta_j(\omega)$ is the height of the $j$th basic slope cube on the ray identifying $\omega$ in $T_j(\Omega_N; M)$. We note that $\eta_N(\omega) \equiv J$ for all $\omega \in H_N(\Omega_N) = \Omega_N$.

The quantity $\eta_j$ is used to define the following objects:

$$N_x := \{\Phi_j(t) : t \in \text{Poss}(x), \ 0 \leq j \leq N\},$$

(5.11)

$$M_x := \{\Theta_j(t) : t \in \text{Poss}(x), \ 0 \leq j \leq N\}, \quad \text{where}$$

(5.12)

$$\Phi_j(t) := \begin{cases} \{0\} \times [0, 1)^d & \text{for } j = 0 \\ (Q_i^1(t), \cdots, Q_i^j(t)) & \text{for } j \geq 1, \quad \text{and} \end{cases}$$

(5.13)

$$\Theta_j(t) := \begin{cases} \{1\} \times [0, 1)^d & \text{for } j = 0 \\ (\theta_1(t), \cdots, \theta_j(t)) & \text{for } j \geq 1. \end{cases}$$

(5.14)

Here for $j \geq 1$, the cube $Q_i^j(t)$ is a cube in the root hyperplane containing $t$. In contrast, $\theta_j(t)$ is a vertex in $H_j(\Omega_N)$, hence a cube in $\{1\} \times [0, 1)^d$, containing the point $v(t) \in \Omega_N$. Furthermore, both cubes are located at the same height in their respective trees and obey the defining properties

$$t \subseteq Q_i^j(t), \quad v(t) \in \theta_j(t), \quad \text{and} \quad h(Q_i^j(t)) = h(\theta_j(t)) = \eta_j(v(t)).$$

(5.15)

We pause briefly to clarify the definitions (5.13) and (5.14) (see Figure 7). Given any $t \in \text{Poss}(x)$, we pick on the ray identifying $t$ the vertices that lie at the same height as the basic slope cubes of $v(t)$. The entries of the vector $\Phi_j(t)$ are the first $j$ chosen vertices on this ray. On the other hand, $\Theta_j(t)$ consists of the first $j$ basic slope cubes containing $v(t)$. The vectors $\Phi_N(t)$ and $\Theta_N(t)$ identify $t$ and $v(t)$ respectively. For reasons to emerge shortly in Lemma 5.7, we view the collection $N_x$ as a tree, in which $\Phi_j(t)$ is a vertex of height $j$, and $\Phi_{j+1}(t)$ is a child of $\Phi_j(t)$. As we have already noted, the set Poss($x$), and hence the youngest generation of $N_x$, contains all possible roots that could support tubes with directions in $\Omega_N$ containing $x$. For an arbitrary $\sigma$, it is therefore natural to phrase a necessary criterion for the inclusion $x \in K(\sigma)$ in terms of $N_x$. For this reason we choose to call $N_x$ the reference tree, and its defining cubes $Q_i^j(t)$ as reference cubes. The collection $M_x$ should be thought of as the “image” of $N_x$ on the slope side, and hence a tree as well, with $\Theta_j(t)$ being a vertex of the $j$th generation and the parent of $\Theta_{j+1}(t)$. In fact, $M_x$ is a subtree of $H(\Omega_N)$ defined as in (4.11). In view of Proposition 4.5, any vertex $\Theta_j(t)$ of height $j \geq 1$ in $M_x$ is identified with the $j$-long binary sequence $\Psi^{-1}(\Theta_j(t))$.

Given the constraints of our pruning mechanism in Proposition 4.1, the “correct slope” map $t \mapsto v(t)$ need not be sticky as a mapping from $T_j(\text{Poss}(x); M)$ to $T_j(\Omega_N; M)$. It does however possess a weak variant of the stickiness property that we specify in the next lemma. As we will see in Lemma 5.7, this milder substitute is able to achieve two goals that are of fundamental relevance to this study. First, it assigns a tree structure to $N_x$ and $M_x$. Second, it is strong enough to lift $v$ as a sticky map from $N_x \rightarrow M_x$. 41
Lemma 5.6. There is a sufficiently large choice of the parameter $C_0$ in Proposition 4.1 for which the following conclusion holds. Let $x \in \mathbb{R}^{d+1}$ with $A_0 \leq x_1 \leq 10A_0$, $t, t' \in \text{Poss}(x)$ and $u = D(t, t')$. Set $w = D(v(t), v(t'))$, so that $w \in \mathcal{G}(\Omega_N)$, the class of splitting vertices defined in (4.2). Then

$$h(u) < \lambda(w),$$

with $\lambda$ defined as in (4.5).

Remark: If $v$ defined in (5.9) was indeed a sticky map, one would have access to the inequality $h(u) \leq h(w)$. We know however that $\lambda(w) > h(w)$, and hence (5.16) should be viewed as a weak version of stickiness.

Proof. If $x \in P_t,v(t) \cap P_{t'},v(t')$, then by the inequality (5.5) in Lemma 5.1,

$$A_0|v(t) - v(t')| \leq |x_1||v(t) - v(t')| \leq |\text{cen}(t') - \text{cen}(t)| + 2\sqrt{d}M^{-J} \leq 2\sqrt{d}M^{-h_0} + 2\sqrt{d}M^{-J} \leq 4\sqrt{d}A_0^{-1}M^{-h_0},$$

and thus $|v(t) - v(t')| \leq 4\sqrt{d}A_0^{-1}M^{-h_0}$. (5.17)

On the other hand, $v(t)$ and $v(t')$ each lie in distinct children of $w$, which must be a splitting vertex of $T_j(\Omega_N; M)$. If $w \in \mathcal{G}_j(\Omega_N)$ and if $\gamma, \gamma'$ denote the $(j + 1)$th splitting vertex
vertices descended from \( w \), then each of \( \gamma \) and \( \gamma' \) contains exactly one of \( v(t) \) and \( v(t') \).

By Proposition 4.1(iii),
\[
|v(t) - v(t')| \geq \text{dist}(\gamma, \gamma') \geq C_0 M^{-\lambda_j(w)}.
\]

Combining (5.17) and (5.18) we obtain
\[
C_0 M^{-\lambda_j(w)} \leq 4\sqrt{d} A_0^{-1} M^{-h(u)}.
\]

If the constant \( C_0 \) is chosen larger than \( 4\sqrt{d} A_0^{-1} \), then the inequality above implies (5.16), as claimed.

**Lemma 5.7.** The collection of vertex tuples \( N_x, M_x \) defined in (5.11), (5.12) are well-defined as trees rooted at \( \{0\} \times [0,1]^d \) and \( \{1\} \times [0,1]^d \) respectively, with the ancestry relation as described in the discussion leading up to Lemma 5.6. More precisely, the map \( v \) defined in (5.9) meets the following consistency requirements:

(i) Let \( t, t' \in \text{Poss}(x) \), \( u = D(t,t') \). If the index \( j \) satisfies \( \eta_j(v(t)) \leq h(u) \) then we also have \( \eta_j(v(t')) \leq h(u) \), in which case \( \Phi_j(t) = \Phi_j(t') \) and \( \Theta_j(t) = \Theta_j(t') \).

(ii) The map from \( N_x \to M_x \) that sends \( \Phi_j(t) \mapsto \Theta_j(t) \) is well-defined and sticky.

**Proof.** Let \( \gamma_j(t) \in G_j(\Omega_X) \) denote the \( j \)th splitting vertex on the ray identifying \( v(t) \). Then \( \eta_j(v(t)) = \lambda_j(\gamma_j(t)) \). If \( \eta_j(v(t)) = \lambda_j(\gamma_j(t)) \leq h(u) \), then Lemma 5.6 implies that \( \eta_j(v(t')) = \lambda_j(\gamma_j(t')) < \lambda(w) \), where \( w = D(v(t),v(t')) \). Unravelling the implication of this inequality, we see that the height of the \( j \)th splitting descendant of \( \gamma_j(t) \) is strictly smaller than the corresponding quantity for \( w \). Since both \( \gamma_j(t) \) and \( w \) are splitting vertices lying on the ray of \( v(t) \), this means that \( \gamma_j(t) \) is an ancestor of \( w \) of strictly lesser height. In other words, \( w \leq \gamma_j+1(t) \). Since the rays for \( v(t) \) and \( v(t') \) agree up to and including height \( h(u) \), we conclude that their first \( (j+1) \) splitting vertices are identical; i.e.,
\[
\gamma_k(t) = \gamma_k(t') \quad \text{for} \quad k \leq j + 1.
\]

Hence \( \eta_k(v(t)) = \lambda_k(\gamma_k(t)) = \lambda_k(\gamma_k(t')) = \eta_k(v(t')) \) for all such \( k \), implying one of the desired conclusions in part (i). Since
\[
h(w) \geq h(\gamma_{j+1}(t)) = h(\gamma_{j+1}(t')) \geq \eta_j(v(t)) = \eta_j(v(t')),
\]
the vectors \( v(t) \) and \( v(t') \) must agree at height \( \eta_j \). Thus \( \Theta_j(t) = \Theta_j(t') \). Of course if \( \eta_j(v(t)) = \eta_j(v(t')) \leq h(u) \), then \( \Phi_j(t) = \Phi_j(t') \). This completes the proof of the first part of the lemma.

Part (ii) is essentially a restatement of the result in part (i). To ascertain that the map is well-defined we choose \( t, t' \in \text{Poss}(x) \) with \( u = D(t,t') \) and \( \Phi_j(t) = \Phi_j(t') \) and aim to show that \( \Theta_j(t) = \Theta_j(t') \). The hypothesis \( \Phi_j(t) = \Phi_j(t') \) implies that \( \eta_j(v(t)) = \eta_j(v(t')) \leq h(u) \), and part (i) implies that the images match. Stickiness is a by-product of the definitions. \qed
Lemma 5.7 permits the unambiguous assignment of an “ideal image” (namely an edge in $\mathcal{M}_x$) to every edge of the tree $\mathcal{N}_x$, in the following sense: if every edge in the ray leading up to $\Phi_i(t)$ receives its ideal image, then $x \in P_{t, \sigma(t)}$. To make this quantitatively precise, let us define the reference slope function $\kappa$ as follows: for every edge $e$ in $\mathcal{N}_x$ joining the vertices $\Phi_j(t)$ to $\Phi_{j+1}(t)$, we define a binary counter $\kappa(e)$ through the defining equation

$$\Psi^{-1} \circ \Theta_{j+1}(t) = (\Psi^{-1} \circ \Theta_j(t), \kappa(e))$$

(5.20)

where $\Psi$ is the tree isomorphism defined in Proposition 4.5. In other words, $\kappa(e)$ is zero (respectively one) if and only if the ray identifying $\Theta_{j+1}(t)$ in $T_J(\Omega_N)$ passes through the 0th (respectively 1st) child of the $(j+1)$th splitting vertex identified by $\Theta_j(t)$.

**Corollary 5.8.** The reference slope function $\kappa$ described in (5.20) is well-defined, and assigns to each edge of $\mathcal{N}_x$ a unique value of 0 or 1.

**Proof.** If there exist $t \neq t'$ in $\text{Poss}(x)$ such that the terminating vertex of $e$ could be represented either as $\Phi_{j+1}(t)$ or as $\Phi_{j+1}(t')$, then Lemma 5.7 guarantees that $\Theta_k(t) = \Theta_k(t')$ for all $k \leq j + 1$, proving that $\kappa(e)$ given by (5.20) is a well-defined function on the edge set of $\mathcal{N}_x$.

The reader may find it helpful to visualize the edges of the reference tree $\mathcal{N}_x$ with an overlay of model binary values assigned by $\kappa$, against which any other slope assignment will be tested. This intuition is made precise in the following lemma. Given a fixed point $x$ and a union of tubes $\mathbb{K}(\sigma)$ of the form (5.4) generated by a sticky slope map $\sigma$, the result offers a criterion governed by the reference slope function $\kappa$ for verifying whether $x \in \mathbb{K}(\sigma)$. Indeed for such $\sigma$, we can define $\mathcal{N}_x(\sigma)$ and $\mathcal{M}_x(\sigma)$ akin to (5.11) and (5.12), but using the given slope map $t \mapsto \sigma(t)$ instead of the naturally generated $\nu$ given by (5.9). More precisely, we set

$$\mathcal{N}_x(\sigma) := \{ \Phi_j(t; \sigma) : t \in \text{Poss}(x), 0 \leq j \leq N \},$$

(5.21)

$$\mathcal{M}_x(\sigma) := \{ \Theta_j(t; \sigma) : t \in \text{Poss}(x), 0 \leq j \leq N \},$$

(5.22)

where for $j \geq 1$, both $\Phi_j(t; \sigma)$ and $\Theta_j(t; \sigma)$ are $j$-long vectors whose $i$th components are $M$-adic cubes of identical size, containing $t$ in the root hyperplane and $\sigma(t)$ in the slope tree respectively. For $\Theta_j(t; \sigma)$, the $i$th entry is required to lie in $H_i(\Omega_N)$, which uniquely specifies both vectors. In light of the preceding results in this section, it is not surprising that the collections (5.21) and (5.22) are trees and that $\sigma$ extends to a map between these trees.

**Lemma 5.9.** The following conclusions hold:

(i) The collections $\mathcal{N}_x(\sigma)$ and $\mathcal{M}_x(\sigma)$ as in (5.21) and (5.22) are well-defined as trees rooted respectively at $\{0\} \times [0,1)^d$ and $\{1\} \times [0,1)^d$. The tuples $\Phi_j(t; \sigma)$ and $\Theta_j(t; \sigma)$ are deemed vertices of generation $j$, and parents of $\Phi_{j+1}(t; \sigma)$ and $\Theta_{j+1}(t; \sigma)$ respectively. The map $\Phi_j(t; \sigma) \mapsto \Theta_j(t; \sigma)$ from $\mathcal{N}_x(\sigma) \to \mathcal{M}_x(\sigma)$ is well-defined and sticky.

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(ii) If \(e\) denotes the edge connecting \(\Phi_j(t;\sigma)\) and \(\Phi_{j+1}(t;\sigma)\) in \(\mathcal{N}_x(\sigma)\), then the quantity \(\iota_\sigma(e)\) defined by

\[
\Psi^{-1} \circ \Theta_{j+1}(t;\sigma) = (\Psi^{-1} \circ \Theta_j(t;\sigma), \iota_\sigma(e))
\]

gives rise to a well-defined binary function on the edge set of \(\mathcal{N}_x(\sigma)\).

(iii) If \(x \in \mathbb{K}(\sigma)\), then there exists \(t \in \text{Poss}(x)\) such that \(\Theta_N(t;\sigma) = \Theta_N(t)\). In particular, this implies that

\[
\Phi_j(t;\sigma) = \Phi_j(t) \quad \text{for all} \quad 1 \leq j \leq N,\]

and hence that \(\mathcal{N}_x\) and \(\mathcal{N}_x(\sigma)\) share a common ray \(R\) identifying \(t\) with the property

\[
\iota_\sigma(e) = \kappa(e) \quad \text{for every edge} \ e \ \text{in} \ R.
\]

**Proof.** Despite the obvious similarity of the statement with that of Lemma 5.7, the distinction in the proofs should be noted. The assumed stickiness of \(\sigma\) simplifies the proof of part (i), compared to that of Lemma 5.7 where \(v\) was not known to be sticky. Indeed if \(t \neq t'\) are such that \(\Phi_j(t;\sigma) = \Phi_j(t';\sigma)\), then as before \(\eta_j(\sigma(t)) = \eta_j(\sigma(t')) \leq h(u)\). By stickiness of \(\sigma\), we have \(h(u) \leq h(z)\), where \(z = D(\sigma(t),\sigma(t'))\). This implies that the rays identifying \(\sigma(t)\) and \(\sigma(t')\) agree up to and including height \(\eta_j(\sigma(t))\), i.e., \(\Theta_j(t;\sigma) = \Theta_j(t';\sigma)\). Part (ii) is an easy consequence of part (i) and follows exactly the same way as Corollary 5.8 was deduced from Lemma 5.7. Finally, if \(x \in \mathbb{K}(\sigma)\), then there is some \(t \in \text{Poss}(x)\) such that \(\sigma(t) = v(t)\). Since the chain of basic slope cubes containing any \(v \in \Omega_N\) is unique, this implies that \(\Theta_N(t;\sigma) = \Theta_N(t)\), and hence \(\Phi_j(t;\sigma) = \Phi_j(t)\) for all \(1 \leq j \leq N\). The last equality says that \(t\) is identified by the same sequence of vertices and hence the same ray in both \(\mathcal{N}_x\) and \(\mathcal{N}_x(\sigma)\). If \(e_1, e_2, \cdots, e_N\) are the successive edges in this ray, with \(e_{j+1}\) connecting \(\Phi_j(t)\) with \(\Phi_{j+1}(t)\), then a consequence of the definitions (5.20), (5.23) of \(\kappa\) and \(\iota_\sigma\) is that

\[
(\iota_\sigma(e_1), \cdots, \iota_\sigma(e_N)) = \Psi^{-1} \circ \sigma(t) = \Psi^{-1} \circ v(t) = (\kappa(e_1), \cdots, \kappa(e_N)),
\]

where \(\Psi\) is the tree isomorphism defined in Proposition 4.5 and part (iii) follows. \(\square\)

We end this section with a bound on the number of vertices of the reference tree at a given height, a result that will be useful for probability computations later. In view of the characterization (5.8) of \(\text{Poss}(x)\) given in Lemma 5.5 and our construction of \(\Omega_N\), this is intuitively clear.

**Lemma 5.10.** There exists a positive constant \(C\) depending on \(d\) and \(A_0\) but uniform in \(x \in [A_0, A_0 + 1] \times \mathbb{R}^d\) such that the number of vertices of height \(j\) in \(\mathcal{N}_x\) is bounded above by \(C2^j\).

**Proof.** Let \(n_j(x)\) denote the number of vertices of height \(j\) in \(\mathcal{N}_x\). In view of the relations (5.11) and (5.15) defining \(\mathcal{N}_x\), the cardinality \(n_j(x)\) equals the number of spatial cubes in the collection

\[
\{Q^*_j(t) : t \in \text{Poss}(x), t \subseteq Q^*_j(t), h(Q^*_j(t)) = \eta_j(v(t))\},
\]

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so we proceed to count the number of such cubes $Q_j^*(t)$. Let us recall from the definition \((5.7)\) of Poss$(x)$ that $x \in P_{t,v(t)}$. This implies that $x - x_1v(t) \in t$, and hence for $\theta_j(t)$ as in \((5.15)\),

$$
|\text{cen}(Q_j^*(t)) - x + x_1\text{cen}(\theta_j(t))| \\
\leq |\text{cen}(Q_j^*(t)) - \text{cen}(t)| + |x_1||\text{cen}(\theta_j(t)) - v(t)| + |\text{cen}(t) - x + x_1v(t)| \\
\leq \sqrt{d}M^{-\eta_j(v(t))} + (A_0 + 1)\sqrt{d}M^{-\eta_j(v(t))} + \sqrt{d}M^{-J} \\
\leq 4A_0\sqrt{d}M^{-\eta_j(v(t))}.
$$

Let us unravel the geometric implications of the inequality above. For a given $\theta_j(t)$ containing $v(t)$, there are at most a constant number $C(d,A_0)$ of $M$-adic cubes of sidelength same as $\theta_j(t)$ (hence candidates for $Q_j^*(t)$) whose centres are within distance $4A_0\sqrt{d}M^{-\eta_j(v(t))}$ of $x - x_1\text{cen}(\theta_j(t))$. On the other hand, each $\theta_j(t) \in \mathcal{H}_j(\Omega_N)$, and hence the total number of possible $\theta_j(t)$ as $t$ ranges over Poss$(x)$ is at most $\#(\mathcal{H}_j(\Omega_N)) = 2^j$, by Proposition\([4.5]\). Since $n_j(x)$ is the cardinality of the collection in \((6.26)\), the observations above lead to the bound $n_j(x) = O(2^j)$ as claimed. \hfill \Box

6 Random construction of Kakeya-type sets

Motivated by the generalities laid out in the previous section, specifically Lemmas\([5.7]\) and\([5.9]\) we now proceed to describe a randomized algorithm for generating a class of sticky slope assignments $\sigma$. Let us recall the class $\mathcal{R}$ of fundamental heights of $\Omega_N$ defined in \((4.5)\) and the discussion thereafter.

We start with a collection of independent and identically distributed Bernoulli$(\frac{1}{2})$ random variables

$$
X := \{X_Q : Q \in \mathcal{Q}(k), \ k \in \mathcal{R}\},
$$

(6.1)

with $\mathcal{Q}(k)$ defined as in \((5.1)\). The collection $X$ therefore assigns, for every fundamental height $k$ an independent binary random variable to every $M$-adic cube of sidelength $M^{-k}$ in the root hyperplane. We use $X$ as the randomization source for our construction.

Let $h_0$ denote the height of the single element $\theta_0 \in \mathcal{G}_1(\Omega_N) = \mathcal{H}_0(\Omega_N)$, in other words, the first splitting vertex of $\mathcal{T}_j(\Omega_N;M)$. We define $\sigma(Q_0) \equiv \theta_0$ for all $Q_0 \in \mathcal{Q}(h_0)$. At the first step of the randomization process, each $Q_0 \in \mathcal{Q}(h_0)$ is decomposed into subcubes $Q_1$ of sidelength $M^{-h_1}$ where $h_1 = \lambda_1(\theta_0) > h_0$. We call these subcubes the first basic spatial cubes. Each first basic spatial cube $Q_1$ receives from the Bernoulli collection $X$ defined in \((6.1)\) a value of $X_{Q_1}$, which is either zero or one. Recalling from \((4.9)\) that

$$
\Psi_1(X_{Q_1}) \in \mathcal{H}_1(\Omega_N), \quad \text{and that} \quad h(\Psi_1(X_{Q_1})) = h_1,
$$

we define

$$
\sigma(Q_1) = \sigma_X(Q_1) = \Psi_1(X_{Q_1})
$$

for any first basic spatial cube $Q_1$. Each element of $\mathcal{H}_1(\Omega_N)$, and hence each $\sigma(Q_1)$, is either a second splitting vertex of $\Omega_N$ or the identifier of one. If the root cube $Q_1$ already
maps into a second splitting vertex under \( \sigma \), no further action is needed for it in step one. Now, suppose there exists \( \gamma \in G_2(\Omega_N) \) such that \( h(\gamma) > h_1 \) (there could be at most one such \( \gamma \)). Then for any cube \( Q_1 \) for which \( \Psi_1(X_{Q_1}) \) is the unique ancestor of \( \gamma \) at height \( h_1 \), we decompose \( Q_1 \) into subcubes \( Q'_1 \) of sidelength \( M^{-h(\gamma)} \) and set \( \sigma(Q'_1) = \gamma \) for all such \( Q'_1 \subseteq Q_1 \). Thus, at the end of the first step,

(a) we have obtained a partition of the root hyperplane into first basic spatial cubes, and randomly assigned each such cube a first basic slope cube in \( H_1(\Omega_N) \) of the same height, namely \( \lambda_1(\theta_0) = h_1 \).

(b) If the vertices in \( G_2(\Omega_N) \) occur at different heights, then predicated on the random assignment in part (a) certain first basic spatial cubes could subdivide further to generate a different partition of the root hyperplane, say \( \{Q_1(\gamma) : \gamma \in G_2(\Omega_N) \} \).

Each cube \( Q'_1 \in Q_1(\gamma) \) is of height \( h(\gamma) \) and is mapped to \( \gamma \). We will refer to \( Q'_1 \) as a spatial cube of second splitting height. Thus a first basic spatial cube is either itself a spatial cube of second splitting height, or is uniformly partitioned into a disjoint union of such cubes.

\[ \gamma \in G_j(\Omega_N) \]
\[ \gamma_{j+1} = \theta_1 \]
\[ \theta_2 \]
\[ \sim \gamma_{j+1} \]
\[ Q'_{j-1} \in Q_{j-1}(\gamma) \]

\[ Q_j \]
\[ Q'_j \]

**Figure 8:** A pictorial representation of the basic slope and root cubes and a typical slope assignment. Vertices \( Q_j \) for which \( X_{Q_j} = 0 \) are indicated by a circle and assigned \( \theta_1 \); others are indicated by squares and assigned \( \theta_2 \). For the squared vertices, a further slope assignment is made at a finer level.

In general, the \( j \)th step of the construction generates a random and possibly non-uniform partition of the root hyperplane into spatial cubes \( Q'_j \) of \( (j + 1) \)th splitting height. Each \( Q'_j \) is the terminal member of a descending chain

\[ Q'_j \subseteq Q_j \subseteq Q'_{j-1} \subseteq Q_{j-1} \subseteq \cdots \subseteq Q'_1 \subseteq Q_1, \quad (6.2) \]

where for every \( k \leq j \), \( Q_k \) is a \( k \)th basic spatial cube, and \( Q'_k \) is a spatial cube of \( (k + 1) \)th splitting height. Each \( Q_k \) is mapped by \( \sigma \) to a \( k \)th basic slope cube in \( H_k(\Omega_N) \), whereas
Q′_j is mapped to a splitting vertex in G_{k+1}(Ω_N). All such assignments preserve heights and lineages; in other words, for a sequence of cubes as in (6.2),

\[ \sigma(Q_j') \subseteq \sigma(Q_j) \subseteq \sigma(Q_{j-1}') \subseteq \cdots \subseteq \sigma(Q_1') \subseteq \sigma(Q_1), \]
\[ h(\sigma(Q_j)) = h(Q_j), \quad h(\sigma(Q_j')) = h(Q_j'). \]

(6.3)

The spatial cubes at \((j + 1)\)th splitting height can therefore be classified as follows:

\[ Q_j(\gamma) := \{ Q_j' : \sigma(Q_j') = \gamma \}, \quad \gamma \in G_{j+1}(Ω_N). \]

(6.4)

At the \((j + 1)\)th step each \(Q_j'\) from the collection \(Q_j(\gamma)\) is decomposed into subcubes \(Q_{j+1}\) of height \(\lambda_{j+1}(\gamma) > h(\gamma)\). These are the \((j + 1)\)th basic spatial cubes. Each spatial cube \(Q_{j+1}\) is assigned the binary value \(X_{Q_{j+1}}\) from the Bernoulli collection \(X\) in (6.1). Combined with the random assignments that the basic ancestors of \(Q_{j+1}\) have received, this produces an image of \(Q_{j+1}\) under \(\sigma\):

\[ \sigma_X(Q_{j+1}) := \Psi_{j+1}(X_{Q_1}, \cdots, X_{Q_{j+1}}) \in \mathcal{H}_{j+1}(Ω_N), \quad Q_{j+1} \subsetneq \cdots \subsetneq Q_1. \]

(6.5)

Each \(\sigma(Q_{j+1})\) is the unique identifier of some \(\gamma \in G_{j+2}(Ω_N)\). We decompose \(Q_{j+1}\) into subcubes \(Q_{j+1}'\) of height \(h(\gamma)\) (in some cases no further decomposition may be needed) and set \(\sigma(Q_{j+1}') = \gamma\). This results in a newer and finer partition of the root hyperplane into spatial cubes \(Q_{j+1}'\) of \((j + 1)\)th splitting height, producing an analogue of (6.4) for the \((j + 2)\)th step and allowing us to carry the induction forward.

Continuing the procedure described above for \(N\) steps, we obtain a decomposition of the root hyperplane into a family of basic cubes of order \(N\), each of which is of sidelength \(M^{-J}\), and hence is by definition a root cube. Every such cube \(t = Q_N(t)\) is contained in a unique chain of basic spatial cubes of lower order:

\[ t = Q_N(t) \subseteq Q_{N-1}(t) \subseteq \cdots \subseteq Q_2(t) \subseteq Q_1(t) \]

(6.6)

and is assigned a slope \(\sigma_X(t) = \Psi_N(X_{Q_1}, \cdots, X_{Q_N})\) in \(\mathcal{H}_N(Ω_N) = Ω_N\). We will shortly expand on further structural properties of the slope map \(t \mapsto \sigma_X(t)\), but first observe that it gives rise to a random set

\[ K_N(X) := K(\sigma_X; N, J) \]

(6.7)

according to the prescription (6.4).

### 6.1 Features of the construction

We pause briefly to summarize the important features of the construction above:

- Randomization only occurs for cubes in the root hyperplane that correspond to the fundamental heights, though all cubes of a given fundamental height need not receive a random assignment.
- The only cubes that receive a random binary assignment from $X$ are by definition the basic spatial cubes. Unlike the basic slope cubes that constitute $H_j(\Omega_N)$, a basic spatial cube $Q_j$ is a random quantity. For instance, the size of a $j$th basic spatial cube $Q_j$ always ranges in the set $\{h(\theta) : \theta \in H_j(\Omega_N)\} \subseteq \mathbb{R}$, but the exact value of the size depends on the binary assignment $X_{Q_1}, \ldots, X_{Q_{j-1}}$ received by its basic ancestors. Similarly, a spatial cube $Q'_j$ of $j$th splitting height is random, though of course a splitting vertex in $G_j(\Omega_N)$ is not.

- On the other hand, the random variable $X_{Q_j}$ that a basic cube $Q_j$ receives is independent of all random variables used in previous or concurrent steps of the process, by virtue of our choice of (6.1). In other words,

$$\text{The collection of random variables } \{X_{Q_j} : Q_j \text{ basic}\} \text{ is independent.} \quad (6.8)$$

This fact is vital in computing slope assignment probabilities in Sections 7 and 8.

- Thus far, $\sigma$ has been prescribed only for basic cubes and their subcubes of splitting heights. Having achieved this, it is not difficult to extend $\sigma$ as a sticky map between the root tree and the slope tree. We address this in the next lemma.

**Lemma 6.1.** For every realization of $X$, there exists a sticky map

$$\sigma_X : T_J(\{0\} \times [0,1)^d, M) \rightarrow T_J(\Omega_N; M)$$

that agrees with the slope assignment algorithm prescribed in (6.5).

**Proof.** For every $1 \leq j \leq N$, the root hyperplane is partitioned into $j$th basic spatial cubes. Any $M$-adic cube $Q$ is therefore either a basic spatial cube or contained in one. Thus there exists for every $Q$ a unique index $\bar{j} = \bar{j}_X(Q)$ and a nested sequence of $k$th basic spatial cubes $Q_k$ such that

$$Q_N \subset \cdots Q_{j+1} \subset Q \subset Q_j \subset Q_{j-1} \subset \cdots \subset Q_1.$$  

Recalling that $\sigma(Q_k)$ has been defined for all $1 \leq k \leq N$ obeying the requirements (6.3) of preserving height and lineage, we set

$$\sigma_X(Q) := \left\{\begin{array}{l}
\text{unique vertex in } T_J(\Omega_N; M) \text{ of height } \\
h(Q) \text{ on the ray identifying } \sigma(Q_{j+1}) \end{array}\right\}.$$  

Then $\sigma_X$ is well-defined, sticky, and consistent with the prescriptions made in (6.5). \hfill \square

### 6.2 Theorem 1.2 revisited

We will now invest our efforts into proving that with positive probability the sets $K_N(X)$ just created in (6.7) are of Kakeya type.
Proposition 6.2. There exist positive absolute constants \( c = c(d, M) \) and \( C = C(d, M) \) obeying the property described below. For every \( N \geq 1 \) and \( \Omega_N \) as in Proposition 4.1, the random set \( K_N(\mathcal{X}) \) defined in \((6.7)\) satisfies the following inequalities:

\[
\Pr\left( \mathcal{X} : |K_N(\mathcal{X}) \cap [0, 1] \times \mathbb{R}^d| \geq c \frac{\sqrt{\log N}}{N} \right) \geq \frac{3}{4}, \tag{6.9}
\]

\[
\mathbb{E}_{\mathcal{X}} |K_N(\mathcal{X}) \cap [A_0, A_0 + 1] \times \mathbb{R}^d| \leq C. \tag{6.10}
\]

The proof of the proposition will occupy the remainder of the paper, with the estimates \((6.10)\) and \((6.9)\) established in Sections 7 and 11 respectively. Before launching into them, let us observe that these two estimates combine to generate the Kakeya-type set whose existence is claimed in Theorem 1.2 and subsequently reformulated in Proposition 3.6.

Corollary 6.3. Given Proposition 6.2, the statement of Proposition 3.6 follows. Specifically, for every \( N \geq 1 \) there exists a realization of \( \mathcal{X} \) for which the union of tubes defined by

\[
E_N := K_N(\mathcal{X}) \cap [A_0, A_0 + 1] \times \mathbb{R}^d \quad \text{obeys} \quad \frac{|E_N^*(2A_0 + 1)|}{|E_N^*|} \to \infty. \tag{6.11}
\]

In other words, \( \Omega \) admits Kakeya-type sets.

Proof. The proof is identical to that of [16, Proposition 2.1], so we briefly sketch the outline. The bound \((6.10)\) on the expected value implies that the estimate

\[
|K_N(\mathcal{X}) \cap [A_0, A_0 + 1] \times \mathbb{R}^d| \leq 4C \tag{6.12}
\]

holds with probability at least \( \frac{3}{4} \), by Markov’s inequality. Combined with \((6.9)\), this lets us conclude that both \((6.12)\) and

\[
|K_N(\mathcal{X}) \cap [0, 1] \times \mathbb{R}^d| \geq c \frac{\sqrt{\log N}}{N} \tag{6.13}
\]

must hold with probability at least \( \frac{1}{2} \). Since \( E_N^*(2A_0 + 1) \supset K_N(\mathcal{X}) \cap [0, 1] \times \mathbb{R}^d \) for \( E_N \) defined as in \((6.11)\), any \( K_N(\mathcal{X}) \) obeying both \((6.12)\) and \((6.13)\) yields

\[
\frac{|E_N^*(2A_0 + 1)|}{|E_N^*|} \geq c \sqrt{\log N} \to \infty,
\]

as claimed. \( \square \)

7 Proof of the upper bound \((6.10)\)

Proposition 7.1. There exists a positive constant \( C \) possibly depending on \( d \) and \( M \) but uniform in \( x \in [A_0, A_0 + 1] \times \mathbb{R}^d \) such that the probability \( \Pr(x) := \Pr(x \in K_N(\mathcal{X})) \) obeys the estimate

\[
\Pr(x) \leq \frac{C}{N}. \tag{7.1}
\]

As a consequence, \((6.10)\) holds.
Proof. The proof of \((7.1)\) is a consequence of the three lemmas stated and proved below in this section. In Lemma 7.3 and following the direction laid out in [4, 3], we establish that \(\Pr(x)\) is bounded above by the probability that the reference tree \(N_x\) survives a Bernoulli(\(\frac{1}{2}\)) percolation, as described in the appendix (Section 12). The details of the specific percolation criterion that permit this correspondence are described in Lemma 7.2. Using general facts about percolation collected in Section 12 and information on \(N_x\) observed in Section 5, we compute in Lemma 7.4 a bound on the survival probability that is uniform in \(x\) to obtain the claimed estimate \((7.1)\).

Given \((7.1)\), the upper bound in \((6.10)\) follows easily. Since \(\Omega_N \subseteq \{1\} \times [0, 1)^d\), any tube, and hence \(K_{N}(\mathcal{X})\), is contained in the compact set \([0, 10A_0]^{d+1}\). Thus

\[
K_N(\mathcal{X}) \cap [A_0, A_0 + 1] \times \mathbb{R}^d = K_N(\mathcal{X}) \cap [A_0, A_0 + 1] \times [0, 10A_0]^d, 
\]

and hence

\[
\mathbb{E}_{\mathcal{X}} |K_N(\mathcal{X}) \cap [A_0, A_0 + 1] \times \mathbb{R}^d| = \mathbb{E}_{\mathcal{X}} \int_{[A_0, A_0 + 1] \times [0, 10A_0]^d} 1_{K_N(\mathcal{X})}(x) \, dx \\
= \int_{[A_0, A_0 + 1] \times [0, 10A_0]^d} \mathbb{E}_{\mathcal{X}}(1_{K_N(\mathcal{X})}(x)) \, dx \\
= \int_{[A_0, A_0 + 1] \times [0, 10A_0]^d} \Pr(x) \, dx \\
\leq \frac{C}{N},
\]

completing the proof.

Much of the groundwork for Lemma 7.3 has already been established in Section 5.2. In particular, let us recall the definition of the reference tree \(N_x\) and reference cubes \(Q_j^*(t)\) from \((5.11), (5.13), and (5.15)\). We will also need the reference slope function \(\kappa\) as in \((5.20)\) defined on the edges of \(N_x\). Motivated by Lemma 5.9(iii), we define a random variable for each edge of \(N_x:\)

\[
Y_e = Y_e(\mathcal{X}) := \begin{cases} 
1 & \text{if } X_{Q_{j+1}^*(t)} = \kappa(e), \\
0 & \text{otherwise},
\end{cases} \quad (7.2)
\]

where as usual \(e\) denotes the edge in \(N_x\) joining \(\Phi_j(t)\) and \(\Phi_{j+1}(t)\). As described in Section 12, we use \(Y_e\) to determine whether to retain or to remove the edge \(e\) in \(N_x\), the value zero corresponding to removal. We emphasize that a reference cube \(Q_{j+1}^*(t)\) is a deterministic vertex of the tree representing the root hyperplane, and need not in general coincide with the \((j + 1)\)th basic spatial cube \(Q_{j+1}(t)\) described in \((6.7)\). The important point, as we will see in Lemma 7.3, is that if \(x \in K_N(\mathcal{X})\), then these two cubes do match for some \(t\) and for every \(j\).

**Lemma 7.2.** The retention-removal criterion described in \((7.2)\) gives rise to a well-defined Bernoulli(\(\frac{1}{2}\)) percolation on \(N_x\).

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Proof. Since $Q^*_j(t)$ identifies the terminating vertex of the edge $e$, any two representations $\Phi_j(t) = \Phi_j(t')$ of this vertex gives rise to $Q^*_j(t) = Q^*_j(t')$. So $X_{Q^*_j(t)}$ is consistently defined on the edges. We have already seen in Corollary 5.8 that $\kappa$ is a well-defined function on the edge set of $N_x$, hence so is $Y_e$. The probability that $Y_e$ equals one is clearly $1/2$ since it is given by the Bernoulli$(\frac{1}{2})$ random variable $X_{Q^*_j(t)}$. Finally, any two distinct edges $e$ and $e'$ must have distinct terminating vertices, and therefore end in distinct reference cubes. The random variable assignments for such cubes are independent by our assumption on $X$. Hence the events of retention and removal are independent for different edges, and the result follows.

Lemma 7.3. Let $x$ be a point in $[A_0, A_0 + 1] \times \mathbb{R}^d$. If $x \in K_N(X)$, then there is at least one ray of full length in $N_x$ all of whose edges are retained after the percolation described by $Y_e(X)$. As a result, the probability $Pr(x)$ defined in Proposition 7.1 admits the bound

\[ Pr(x) \leq p^*(x), \]  

where $p^*(x)$ denotes the survival probability of $N_x$ under the Bernoulli$(\frac{1}{2})$ percolation given in (7.2).

Proof. If $x \in K_N(X)$, then by Lemma 5.9(iii) there exists $t \in \text{Poss}(x)$ such that the ray identifying $t$ is common to $N_x$ and $N_x(\sigma_X)$. Restating (5.24), this means that $\Phi_j(t) = \Phi_j(t; \sigma)$ for all $1 \leq j \leq N$. But the left hand side of the preceding equality identifies the (deterministic) $j$th reference cube containing $t$, whereas the right hand side represents the (random) $j$th basic spatial cube containing $t$. In other words, we find that $Q_j(t; X) = Q^*_j(t)$ for all $1 \leq j \leq N$, and hence

\[ \iota_{\sigma}(e) = X_{Q^*_j(t; X)} = X_{Q^*_j(t)}. \]

Combined with (7.2) and (5.25), this implies the existence of an entire ray in $N_x$ (namely the one identifying $t$) that survives the percolation given by $Y_e$. Summarizing, we obtain that

\[ \{ X : x \in K_N(X) \} \subseteq \left\{ X : N_x \text{ survives the Bernoulli}(1/2) \text{ percolation dictated by } Y_e(X) \right\}, \]

from which (7.3) follows.

Lemma 7.4. There is a positive constant $C$ that is uniform in $x \in [A_0, A_0 + 1] \times \mathbb{R}^d$ such that the survival probability $p^*(x)$ of $N_x$ under Bernoulli$(\frac{1}{2})$ percolation is $\leq C/N$.

Proof. In view of Corollary 12.3, $p^*(x)$ is bounded above by

\[ \left[ \sum_{j=1}^{N} \frac{2^j}{n_j(x)} \right]^{-1} \]

where $n_j(x)$ = number of vertices in $N_x$ of height $j$.

But Lemma 5.10 gives that $n_j(x) \leq C2^j$, which leads to the stated bound.
8 Probability estimates for slope assignments

We now turn to (6.9), where we need to establish that with high probability, the volume of space close to the root hyperplane is much more widely populated by the random set \( K_N(X) \) than away from it. As indicated in Section 1.3, the proof requires detailed knowledge of the probability that a given subset of root cubes receives prescribed slope assignments. We establish the necessary probabilistic estimates in this section for easy reference in the proof of (6.9), which is presented in Section 11.

8.1 A general rule

To get started, let us recall from Section 6 that a slope assigned to a root is not completely arbitrary and has to obey the requirement of stickiness. The definition below, introduced to avoid vacuous root-slope combinations, draws attention to this constraint.

Definition 8.1. Let \( A \) be a collection of root cubes and \( \Gamma_A = \{ \alpha(t) : t \in A \} \subseteq \Omega_N \) a collection of slopes indexed by \( A \). We say that the collection of root-slope pairs

\[
\{(t, \alpha(t)) : t \in A \subseteq Q(J), \alpha(t) \in \Gamma_A \subseteq \Omega_N\}
\]

is sticky-admissible if there exists a realization of \( X \) as in (6.1) for which the sticky map \( \sigma_X \) described in Section 6 has the property that

\[
\sigma_X(t) = \alpha(t) \quad \text{for all } t \in A.
\]

Given a sticky-admissible collection (8.1), we first prescribe a general algorithm for computing the probability of the event (8.2). Preparatory to stating the result, let us define two collections consisting of tuples of vertices from the root tree and the slope tree respectively:

\[
N(A; \alpha) := \{ \Phi_j(t; \alpha) : t \in A, 0 \leq j \leq N \},
\]

\[
M(A; \alpha) := \{ \Theta_j(t; \alpha) : t \in A, 0 \leq j \leq N \}.
\]

These objects are analogous to the trees (5.21) and (5.22) introduced earlier, with the usual interpretation of \( \Phi_j(t; \alpha) \) and \( \Theta_j(t; \alpha) \) following those definitions. Namely, for \( j \geq 1 \), the element \( \Theta_j(t; \alpha) \) is a vector with \( j \) entries, whose \( i \)th component represents the \( i \)th basic slope cube in \( T_j(\Omega_N) \) containing \( \alpha(t) \). The vector \( \Phi_j(t; \alpha) \) is also a \( j \)-long sequence. Its \( i \)th entry represents the unique cube containing \( t \) located at the same height as the \( i \)th entry of \( \Theta_j(t; \alpha) \). This common height is \( \eta_i(\alpha(t)) \) defined as in (5.10).

Not surprisingly, for a choice \( A \) and \( \alpha \) that gives rise to a sticky-admissible collection (8.1), the collections \( N(A; \alpha) \) and \( M(A; \alpha) \) are indeed trees (with the 0th generations removed) that contain the information required for computing the probability of the event (8.2). This is the content of Lemma 8.2 below, which forms the computational framework for all the probability estimates in this section.

Lemma 8.2. Let \( A \subseteq Q(J) \) and \( \Gamma_A = \{ \alpha(t) : t \in A \} \subseteq \Omega_N \) be sets for which the collection given in (8.1) is sticky-admissible. Then the following conclusions hold.
(i) The collections $\mathbb{N}(A; \alpha)$ and $\mathbb{M}(A; \alpha)$ defined in (8.3) and (8.4) are well-defined trees in which $\Phi_j(t; \alpha)$ and $\Theta_j(t; \alpha)$ are deemed vertices of height $j$, and parents of $\Phi_{j+1}(t; \alpha)$ and $\Theta_{j+1}(t; \alpha)$ respectively.

(ii) If $n(A; \alpha)$ denotes the total number of vertices in $\mathbb{N}(A; \alpha)$ not counting the root, then

$$\Pr(\sigma_X(t) = \alpha(t) \text{ for all } t \in A) = 2^{-n(A; \alpha)}. \quad (8.5)$$

Proof. The proof of the first claim bears a close resemblance with that of Lemma 5.9. To check that the trees are well-defined, we pick root cubes $t \neq t'$ with $u = D(t, t')$ and aim to show that

$$\Phi_j(t; \alpha) = \Phi_j(t', \alpha) \quad \text{and} \quad \Theta_j(t; \alpha) = \Theta_j(t', \alpha)$$

for all $j$ such that $\eta_j(\alpha(t)) \leq h(u)$. But the collection (8.1) is sticky-admissible by hypothesis, hence there is a sticky map $\sigma$ such that (8.2) holds. By the property of stickiness,

$$h(D(\alpha(t), \alpha(t'))) = h(D(\sigma(t), \sigma(t'))) \geq h(u).$$

Since $\alpha(t)$ and $\alpha(t')$ agree up to height $h(D(\alpha(t), \alpha(t')))$, it follows that $\Theta_j(t; \alpha)$ and $\Theta_j(t'; \alpha)$ must match for $\eta_j(\alpha(t)) < h(u)$. This in turn implies that $\Phi_j(t; \alpha) = \Phi_j(t'; \alpha)$.

We turn now to the proof of (8.5). Let us write

$$\Phi_j(t; \alpha) = (Q^*_1(t; \alpha), \cdots, Q^*_j(t; \alpha)) \quad \text{and} \quad \Theta_j(t; \alpha) = (\theta_1(t; \alpha), \cdots, \theta_j(t; \alpha)). \quad (8.6)$$

In order to describe the event of interest, we need to recall from (6.6) the definition of basic spatial cubes $Q_j(t)$ containing $t$, their role in the random construction as explained in Section 6 and also the definition of the maps $\Psi_j$ and $\Psi$ from (4.10) and Proposition 4.5. Putting these together we find that

$$\{ \sigma_X(t) = \alpha(t) \text{ for all } t \in A \}
= \{ \sigma_X(Q_j(t)) = \theta_j(t; \alpha) \text{ for all } 1 \leq j \leq N \text{ and all } t \in A \}
= \{ \Psi_N(X_{Q_1(t)}, \cdots, X_{Q_N(t)}) = \alpha(t) \text{ for all } t \in A \}
= \bigcap_{j=1}^N \bigcap_{t \in A} \{ X_{Q_j(t)} = \pi_j \circ \Psi^{-1} \circ \alpha(t) \}
= \bigcap_{j=1}^N \bigcap_{t \in A} \{ Q_j(t) = Q^*_j(t; \alpha) \text{ and } X_{Q_j(t; \alpha)} = \pi_j \circ \Psi^{-1} \circ \alpha(t) \}. \quad (8.7)$$

Here $\pi_j$ denotes the projection onto the $j$th component of an input sequence. In the first two steps of the string of equations above, we have used the definition (6.5) of $\sigma$ and its stickiness as ensured by Lemma 6.1. To justify the last step we observe that

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$Q_1(t) = Q^*_1(t; \alpha)$ is non-random; further if it is given that $Q_\ell(t) = Q^*_\ell(t; \alpha)$ for all $\ell \leq j$, then the additional requirement

$$X_{Q_j(t)} = \pi_j \circ \Psi^{-1} \circ \alpha(t) \quad \text{implies} \quad Q_{j+1}(t) = Q^*_j(t; \alpha),$$

leading to the conclusion in (8.7). By virtue of our assumption of sticky-admissibility, the event described above is of positive probability; in particular the value assignment to the random variables in $X$ as prescribed in (8.7) is consistent, i.e., for $t \neq t'$,

$$\pi_j \circ \Psi^{-1} \circ \alpha(t) = \pi_j \circ \Psi^{-1} \circ \alpha(t') \quad \text{whenever} \quad Q^*_j(t; \alpha) = Q^*_j(t'; \alpha).$$

In view of our assumption (6.1) on the distribution of $X$, the probability of the event in (8.7) is half raised to a power that equals the number of distinct cubes in the collection \{ $Q^*_j(t; \alpha); 1 \leq j \leq N, t \in A$ \}, in other words $n(A; \alpha)$. 

### 8.2 Root configurations

Application of Lemma 8.2 requires explicit knowledge of the structure of the trees $N(A; \alpha)$ and $M(A; \alpha)$, from which $n(A; \alpha)$ can be computed. These objects depend in turn on the trees depicting $A$ and $\Gamma_A$. We now proceed to compute $n(A; \alpha)$ in some simple situations where $\#(A) \leq 4$. On one hand, the small size of $A$ permits the classification of possible root configurations into relatively few categories, each of which gives rise to a specific $n(A; \alpha)$. On the other hand, these cases cover all the probabilistic estimates that we will need in Section 11.

While each root configuration requires distinct consideration, it is recommended that the first-time reader focus on the cases when $\#(A) = 2$, and when $\#(A) = 4$ with the four roots in what we call a type 1 configuration (see Definition 8.7). These cases contain many of the main ideas needed to push through the proof of the lower bound on the size of a typical $K_N(X)$ claimed in (6.9), Proposition 6.2 A thorough treatment of all distinct cases when $\#(A) \leq 4$ is needed to completely establish Proposition 6.2 but focusing on the two recommended cases should make the arguments far easier to absorb upon a first reading. When $\#(A) = 2$ in particular, the reader may focus attention on Lemmas 8.3, 9.3, 10.1 and 10.2 and the application of these lemmas in the proof of Proposition 11.1 The treatment of the case of four distinct roots in type 1 configuration has been carried out on Lemmas 8.8, 9.6, 10.1 and 10.2, with the application of these lemmas occurring in the proof of Proposition 11.2, for which this is the generic case.

### 8.3 Notation

Throughout this section the following notation will be used, in conjunction with the terminology of root hyperplane, root tree and root cube already set up in Section 5. Since any vertex $\Phi_j(t; \alpha)$ in $N(A; \alpha)$ is uniquely identified by its last component $Q^*_j(t; \alpha)$ defined as in (8.6), we write

$$\Phi_j(t; \alpha) \cong Q^*_j(t; \alpha), \quad (8.8)$$
often opting to describe the left hand side by the right. In particular if \( j = N \), then \( \Phi_N(t; \alpha) \equiv Q_N^*(t; \alpha) = t \), in which case the latter notation is used instead of the (more cumbersome) former.

Given a vertex \( u \) in the root tree, a vertex \( \omega \in T_J(\Omega_N) \) and a positive integer \( k \) that is no larger than either \( h(\omega) \) or \( h(u) \), we also define

\[
\theta(\omega, k) := \text{the basic slope cube containing } \omega \text{ of maximal height } \leq k, \quad \text{and} \quad (8.9)
\]

\[
\mu(\omega, k) := j \text{ if } \theta(\omega, k) \in H_j(\Omega_N)
\]

\[
= \text{number of basic slope cubes of height } \leq k \text{ that contain } \omega, \quad \text{and} \quad (8.10)
\]

\[
Q_u[\omega, k] := \text{ancestor of } u \text{ in the root tree at height } \mu(\omega, k).
\]

Figure 9 on page 56 depicts these quantities. If \( \omega' \subseteq \omega \) and/or \( u' \subseteq u \), then it follows from the definitions above that

\[
\theta(\omega, k) = \theta(\omega', k), \quad \mu(\omega, k) = \mu(\omega', k), \quad \text{and} \quad (8.11)
\]

\[
Q_u[\omega, k] = Q_{u'}[\omega, k] = Q_u[\omega', k] = Q_{u'}[\omega', k].
\]

These facts will be frequently used in the sequel without further reference.

\[
\omega_j \in G_j(\Omega_N)
\]

\[
\theta(\omega, k) = \theta(\omega', k) \in H_j(\Omega_N)
\]

\[
\mu(\omega, k) = \mu(\omega', k)
\]

\[
Q_u[\omega, k] = Q_u[\omega', k]
\]

\[
\omega \in T_J(\Omega_N)
\]

**Figure 9:** Given \( \omega \in \Omega_N \) and a set of heights \( k_i, i = 1, 2, 3 \), the basic slope cubes \( \theta(\omega, k_i) \) are identified. Here \( \mu(\omega, k_1) = \mu(\omega, k_2) = j \) and \( \mu(\omega, k_3) = j + 1 \). All vertices depicting basic slope cubes are circled.
8.3.1 The case of two roots

We start with the simplest case when \( A \) consists of two root cubes.

**Lemma 8.3.** Let \( A = \{t_1, t_2\} \) be two distinct root cubes and \( \Gamma_A = \{(\alpha(t_1) = v_1, \alpha(t_2) = v_2)\} \subseteq \Omega_N \) be a subset of (not necessarily distinct) slopes such that \( \{(t_1, v_1), (t_2, v_2)\} \) is sticky-admissible. If \( u = D(t_1, t_2), \omega = D(v_1, v_2) \) and \( k = h(u) \), then \( k \leq h(\omega) \), and

\[
\Pr(\sigma(t_1) = v_1, \sigma(t_2) = v_2) = \left(\frac{1}{2}\right)^{2N - \mu(\omega, k)}.
\]  

**(8.12)**

*Proof.* Since there exists a sticky map \( \sigma \) such that \( \sigma(t_i) = v_i \) for \( i = 1, 2 \), we see that

\[
h(\omega) = h(D(v_1, v_2)) \geq h(D(t_1, t_2)) = h(u) = k.
\]  

**(8.13)**

In order to establish \((8.12)\), we invoke Lemma \(8.2\). The tree \( N(A; \alpha) \) consists of two rays terminating at \( \Phi_N(t_1; \alpha) \cong t_1 \) and \( \Phi_N(t_2; \alpha) \cong t_2 \) respectively, according to the notational rule prescribed in \((8.8)\). Letting \( u_N = D_N(t_1, t_2) \) denote the youngest common ancestor of \( t_1 \) and \( t_2 \) in \( N(A; \alpha) \), we observe that \( u_N \cong Q_u[\omega, k] \), with \( Q_u[\omega, k] \) defined as in \((8.11)\). Thus \( u_N \) lies at height \( \mu(\omega, k) \) in \( N(A; \alpha) \). This allows us to compute \( n(A; \alpha) \) as follows: \( n(A; \alpha) = \mu(\omega, k) + 2(N - \mu(\omega, k)) = 2N - \mu(\omega, k) \).

\[\square\]

8.3.2 The case of three roots

Next we turn to the slightly more complex event where three distinct root cubes receive prescribed slopes. Here for the first time we observe the dependence of slope assignment probabilities on configuration types of the roots.

**Definition 8.4.** Let \( t_1, t_2, t'_2 \) be three distinct root cubes. We say that the ordered tuple \( \Gamma = \{(t_1, t_2); (t_1, t'_2)\} \) with

\[
u = D(t_1, t_2), \quad u' = D(t_1, t'_2), \quad u' \subseteq u
\]  

**(8.14)**

is in type 1 configuration if exactly one of the following conditions hold:

(a) \( u' \subseteq u \), or
(b) \( u = u' = D(t_2, t'_2) \).

A tuple \( \Gamma \) that obeys \((8.14)\) but is not of type 1 is said to be of type 2. Thus for \( \Gamma \) of type 2, one must have \( u = u' \) and additionally \( t = D(t_2, t'_2) \) satisfies \( t \subseteq u \). If \( \Gamma = \{(t_1, t_2); (t_1, t'_2)\} \) with the same definitions of \( u \) and \( u' \) does not meet the containment relation required by \((8.11)\), i.e., if \( u \not\subseteq u' \), then we declare \( \Gamma \) to be of the same type as \( \Gamma' = \{(t_1, t'_2); (t_1, t_2)\} \).

The different structural possibilities are shown in Figure \(10\)

As in Lemma \(8.3\), the quantity \( \mu \) defined in \((8.11)\) when evaluated at certain vertices of the slope tree dictated by \( A = \{t_1, t_2, t'_2\} \) provides the value of \( n(A; \alpha) \) necessary for estimating the probability in \((8.5)\).
### Lemma 8.5

Let \( A = \{t_1, t_2, t'_2\} \) be three distinct root cubes such that the ordered tuple \( I = \{(t_1, t_2); (t_1, t'_2)\} \) obeys \((8.14)\) and is of type 1. Set

\[
 k = h(u), \quad k' = h(u').
\]

Suppose that \( \Gamma_A = \{\alpha(t_1) = v_1, \alpha(t_2) = v_2, \alpha(t'_2) = v'_2\} \subseteq \Omega_N \) is a subset of (not necessarily distinct) directions such that the collection \( \{(t_1, v_1); (t_2, v_2); (t'_2, v'_2)\} \) is sticky-admissible. Then the vertices defined by

\[
\omega = D(v_1, v_2), \quad \omega' = D(v_1, v'_2)
\]

must satisfy the height relations

\[
k \leq h(\omega), \quad k' \leq h(\omega')
\]

and the following equality holds:

\[
\Pr(\sigma(t_1) = v_1, \sigma(t_2) = v_2, \sigma(t'_2) = v'_2) = \left(\frac{1}{2}\right)^{3N - \mu(\omega, k) - \mu(\omega', k')}
\].

**Proof.** The inequalities in \((8.15)\) are proved exactly as in Lemma 8.3; we omit these. The probability is again computed using Lemma 8.2, via counting \( n(A; \alpha) \). The tree \( N = N(A; \alpha) \) now consists of three rays, terminating at \( \Phi_N(t_1; \alpha), \Phi_N(t_2; \alpha) \) and \( \Phi_N(t'_2; \alpha) \), which are identified with \( t_1, t_2 \) and \( t'_2 \) respectively. Let us recall from the proof of Lemma 8.3 that \( u_N = D_N(t_1, t_2) \) denotes the \( M \)-adic cube specifying the youngest common ancestor of \( t_1 \) and \( t_2 \) in \( N(A; \alpha) \). The vertex \( u'_N = D_N(t_1, t'_2) \) is defined similarly. Then using the notation \((8.8)\),

\[
u'_N \cong Q_{u'}[\omega', k'] = Q_{t_1}[v_1, k'], \quad \text{and} \quad u_N \cong Q_u[\omega, k] = Q_{t_1}[v_1, k].
\]
Since $k \leq k'$, it follows from (8.16) above that $u_N' \subseteq u_N$. If $h_N(\cdot)$ denotes the height of a vertex within the tree $N(A; \alpha)$, then (8.16) also yields

$$h_N(u_N) = \mu(\omega, k) \quad \text{and} \quad h_N(u_N') = \mu(\omega', k'),$$

so that $\mu(\omega, k) \leq \mu(\omega', k')$.

Using these relations and referring to Figure 10, we compute $n(A; \alpha)$ as follows,

$$n(A; \alpha) = h_N(u_N) + \left[ N - h_N(u_N) \right] + \left( h_N(u_N') - h_N(u_N) \right) + 2\left[ N - h_N(u_N') \right]$$

vertices on the ray of $t_2$ in $N$

$$= \mu(\omega, k) + \left[ N - \mu(\omega, k) \right] + \left( \mu(\omega', k') - \mu(\omega, k) \right) + 2\left[ N - \mu(\omega', k') \right]$$

vertices between $u_N$ and $u_N'$

$$= 3N - \mu(\omega, k) - \mu(\omega', k'),$$

vertices below $u_N'$

which leads to the desired probability estimate by Lemma 8.2. \hfill \Box

**Lemma 8.6.** Let $A = \{t_1, t_2, t_2'\}$ be three distinct root cubes such that the ordered tuple $I = \{(t_1, t_2); (t_1, t_2')\}$ obeys (8.14) and is of type 2. Set

$$k = h(u) = h(u'), \quad \text{and} \quad \ell = h(t), \quad \text{where} \quad t = D(t_2, t_2') \subseteq u = u'.$$

If $\{(t_1, v_1); (t_2, v_2); (t_2', v_2')\}$ is a sticky-admissible collection, then the vertices

$$\omega = D(v_1, v_2), \quad \omega' = D(v_1, v_2'), \quad \theta = D(v_2, v_2')$$

must satisfy the relations

$$k \leq \min\{h(\omega), h(\omega')\}, \quad \ell \leq h(\theta), \quad \mu(\omega, k) = \mu(\omega', k),$$

and the following equality holds:

$$Pr(\sigma(t_1) = v_1, \sigma(t_2) = v_2, \sigma(t_2') = v_2') = \left( \frac{1}{2} \right)^{3N - \mu(\omega, k) - \mu(\omega', \ell)}.$$ (8.18)

**Proof.** The first two inequalities in (8.17) are consequences of stickiness, since there exists a sticky map $\sigma$ that assigns $\sigma(t_1) = v_1$, $\sigma(t_2) = v_2$, $\sigma(t_2') = v_2'$. Thus the first inequality in (8.17) is proved as in (8.14), while the second one also follows a similar route:

$$h(\theta) = h(D(v_2, v_2')) = h\left( D(\sigma(t_2), \sigma(t_2')) \right) \geq h(D(t_2, t_2')) = h(t) = \ell.$$

For the last identity in (8.17), we observe that both $\omega$ and $\omega'$ lie on the ray identifying $v_1$. Thus $\theta(\omega, k) = \theta(\omega', k)$ and hence $\mu(\omega, k) = \mu(\omega', k)$ by the first inequality in (8.17).

We now turn to the counting of $n(A; \alpha)$, which leads to the probability estimate (8.18) via Lemma 8.2. Using the notation introduced in the proof of Lemma 8.5, the pairwise youngest common ancestors of the last generation vertices in $N(A; \alpha)$ are seen to satisfy the following:

$$u_N = D_N(t_1, t_2) \cong Q_{1, N}(v_1, k) = Q_{1, N}(v_2, k),$$

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\[ t_N = D_N(t_2, t'_2) \cong Q_t[\vartheta, \ell] = Q_{t_2}[v_2, \ell]. \]

Since the type of \( \mathbb{I} \) guarantees that \( k < \ell \), the relations above imply
\[ t_N \subseteq u_N \quad \text{and hence} \quad h_N(u_N) = \mu(\omega, k) \leq h_N(t_N) = \mu(\vartheta, \ell). \]

This enables us to compute, with the aid of Figure 10,
\[ n(A; \alpha) = \mu(\omega, k) + \left[ N - \mu(\omega, k) \right] + \left[ \mu(\vartheta, \ell) - \mu(\omega, k) \right] + 2 \left[ N - \mu(\vartheta, \ell) \right] \]
\[ = 3N - \mu(\omega, k) - \mu(\vartheta, \ell). \]

This is the exponent claimed in (8.18).

### 8.3.3 The case of four roots

Finally we turn our attention to four point root configurations. Depending on the relative positions of root cubes within the root tree, we can classify the configuration types as follows. Let \( \mathbb{I} = \{(t_1, t_2); (t'_1, t'_2)\} \) be an ordered tuple of four distinct root cubes, for which
\[ u = D(t_1, t_2) \quad \text{and} \quad u' = D(t'_1, t'_2) \]
\[ \text{obey} \quad h(u) \leq h(u'). \quad \text{(8.19)} \]

Then exactly one of the following conditions must hold:
\[ u \cap u' = \emptyset, \quad \text{(8.20)} \]
\[ u = u' = D(t_i, t'_j) \quad \text{for all} \quad i, j = 1, 2, \quad \text{(8.21)} \]
\[ u' \subset u, \quad \text{(8.22)} \]
\[ u = u', \quad \text{and} \quad \exists \text{ indices } 1 \leq i, j \leq 2 \quad \text{such that} \quad D(t_i, t'_j) \subset u. \quad \text{(8.23)} \]

**Definition 8.7.** For an ordered tuple \( \mathbb{I} = \{(t_1, t_2); (t'_1, t'_2)\} \) of four distinct root cubes meeting the requirement of (8.19), we say that \( \mathbb{I} \) is of
(a) type 1 if exactly one of (8.20) or (8.21) holds,
(b) type 2 if (8.22) holds, and
(c) type 3 if (8.23) holds.

If \( \mathbb{I} \) does not meet the height relation in (8.19), then \( \mathbb{I}' = \{(t'_1, t'_2); (t_1, t_2)\} \) does, and the type of \( \mathbb{I} \) is said to be the same as that of \( \mathbb{I}' \).

Several different structural possibilities for the root quadruple exist within the confines of a single type, excluding permutations within and between the pairs \( \{t_1, t_2\} \) and \( \{t'_1, t'_2\} \). These have been listed in Figure 11. We note in passing that the type definition above is slightly different from that in [16]. Here, the main motivation for the nomenclature is the classification of the unconditional probabilities of slope assignment as exemplified in (8.5), whereas in [16] a simpler analysis involving conditional probabilities only was possible.

We now proceed to analyze how the configuration types affect the slope assignment probabilities.
**Lemma 8.8.** Let \( A = \{t_1, t_2, t'_1, t'_2\} \) be a collection of four distinct root cubes such that \( I = \{(t_1, t_2); (t'_1, t'_2)\} \) obeys (8.19) and is of type 1. Let \( \Gamma_A = \{v_1, v_2, v'_1, v'_2\} = \{\alpha(t_i) = v_i, \alpha(t'_i) = v'_i, i = 1, 2\} \subseteq \Omega_N \) be a choice of slopes such that the collection \( \{(t_i, v_i); (t'_i, v'_i); i = 1, 2\} \) is sticky-admissible. Set \( z = D(u, u') \), \( k = h(u) \), \( k' = h(u') \), \( \ell = h(z) \), so that

\[
\begin{align*}
\{ & u, u' \subset z, \text{ and hence } \ell < k \leq k' \text{ if (8.20) holds,} \\
& u = u' = z, \text{ and hence } \ell = k = k' \text{ if (8.21) holds.}
\end{align*}
\]
The proofs of the height relations may be reproduced verbatim from the previous sections. Let 
\[ \omega = D(v_1, v_2) , \quad \omega' = D(v_1', v_2') , \quad v = D(\omega, \omega') , \]
must satisfy \( k \leq h(\omega) , \; k' \leq h(\omega') \) and \( \ell \leq h(v) \), and the following equality holds:

\[
Pr(\sigma(t_i) = v_i , \; \sigma(t_i') = v_i' , \; i = 1, 2) = \left( \frac{1}{2} \right)^{4N - \mu(\omega, k) - \mu(\omega', k') - \mu(v, \ell)} .
\] (8.24)

**Proof.** The proofs of the height relations may be reproduced verbatim from the previous lemmas in this section, so we focus only on the probability estimate. As before,

\[
u_N = D_N(t_1, t_2) \equiv Q_u[\omega, k] , \quad u'_N = D_N(t_1', t_2') \equiv Q_u'[\omega', k']
\]

\[
z_N = D_N(u_N, u'_N) \equiv Q_z[v, \ell] = Q_u[\omega, \ell] , \quad v = D(\omega', k') , \quad h_N(z_N) = \mu(\omega', k') .
\] (8.25)

Since \( \ell \leq k \leq k' \), (8.25) implies

\[
u_N \cup u'_N \subseteq z_N , \quad \text{and thus} \quad \mu(v, \ell) \leq \min[\mu(\omega, k), \mu(\omega', k')] .
\]

It is important to keep in mind that \( N(A; \alpha) \) need not inherit the same type of structure as \( A \). For example, if (8.20) holds, it need not be true that \( u_N \cap u'_N = \emptyset \); indeed the vertices \( u_N, u'_N \) and \( z_N \) could be distinct or (partially) coincident depending on the structure of the slope tree. Nonetheless the information collected above is sufficient to compute the number of vertices in \( N(A; \alpha) \) (see Figure 11):

\[
n(A; \alpha) = \mu(v, \ell) + \left[ \mu(\omega, k) - \mu(v, \ell) \right] + \left[ \mu(\omega', k') - \mu(v, \ell) \right] + 2[N - \mu(\omega, k)] + 2[N - \mu(\omega', k')].
\]

Combined with Lemma 8.2, this leads to (8.21).

**Lemma 8.9.** Let \( A = \{t_i, t'_i ; i = 1, 2\} \) be a collection of four distinct root cubes such that \( I = \{(t_1, t_2); (t_1', t_2')\} \) obeys (8.19) and is of type 2. Suppose that \( \Gamma_A = \{\alpha(t_i) = v_i, \alpha(t'_i) = v'_i ; i = 1, 2\} \) is a choice of slopes such that the collection \( \{(t_i, v_i); (t'_i, v'_i); i = 1, 2\} \) is sticky-admissible. Set

\[
\omega = D(v_1, v_2) , \quad \omega' = D(v_1', v_2') , \quad k = h(u) , \quad k' = h(u') ,
\]
such that \( k < k' \). Then the following inequalities hold: \( k \leq h(\omega) , \; k' \leq h(\omega') \). Further, there exist permutations \( \{i_1, i_2\} \) and \( \{j_1, j_2\} \) of \( \{1, 2\} \) for which the quantities

\[
\vartheta = D(v_{i_1}, v'_{j_2}) , \quad t = D(t_{i_2}, t'_{j_2}) , \quad \ell = h(t)
\]

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The definition of the configuration type dictates that we find that

\[ \Pr(\sigma(t_i) = v_i, \sigma(t'_i) = v'_i \text{ for } i = 1, 2) = \left( \frac{1}{2} \right)^{4N - \mu(\omega, k) - \mu(\omega', k') - \mu(\vartheta, \ell)} \]  \hspace{1in} (8.26)

Proof. The definition of the configuration type dictates that \( u' \) is strictly contained in \( u \), but depending on other properties of the ray joining \( u \) and \( u' \) we are led to consider several cases. If there does not exist any vertex in the root tree that is strictly contained in \( u \) and also contains \( t_i \) for some \( i = 1, 2 \), then any permutation of the root pairs \( \{ t_1, t_2 \} \) and \( \{ t'_1, t'_2 \} \) works. In particular, it suffices to choose \( i_1 = j_1 = 1, i_2 = j_2 = 2 \). In this case \( t = u \), hence \( \ell = k \). In particular this implies

\[ \theta(\omega, k) = \theta(v_2, k) = \theta(v_2, \ell) = \theta(\vartheta, \ell), \hspace{0.5cm} \text{hence } \mu(\omega, k) = \mu(\vartheta, \ell). \] \hspace{1in} (8.27)

Further

\[ u'_N = Q_u[\omega', k'] \subseteq Q_u[t_1', k] = Q_u[\omega, k] = u_N, \text{ and } \]
\[ h_N(u_N) = \mu(\omega, k), \hspace{0.5cm} h_N(u'_N) = \mu(\omega', k'). \]

Referring to Figure **11** we find that

\[ n(A; \alpha) = \mu(\omega, k) + 2[N - \mu(\omega, k)] + [\mu(\omega', k') - \mu(\omega, k)] + 2[N - \mu(\omega', k')] \]
\[ = 4N - 2\mu(\omega, k) - \mu(\omega', k') \]
\[ = 4N - \mu(\omega, k) - \mu(\omega', k') - \mu(\vartheta, \ell), \]

where the last step uses one of the equalities in (8.27).

Suppose next that the previous case does not hold, and also that none of the descendants of \( u' \) lying on the rays of \( t'_1, t'_2 \) is an ancestor of \( t_1 \) or \( t_2 \). Then there is a vertex, let us call it \( t \), such that \( u' \subseteq t \subseteq u \), and \( t \) is of maximal height in this class subject to the restriction that it is an ancestor of some \( t_i \), which we call \( t_{i_2} \). Thus \( t_i \) is the unique element in \( \{ t_1, t_2 \} \) that is not a descendant of \( t \). In this case, any permutation of \( \{ t'_1, t'_2 \} \) works, and we can keep \( j_1 = 1, j_2 = 2 \). Then \( t = D(t_{i_1}, t'_{j_2}), k < \ell \leq k' \), and

\[ u'_N = D_N(t'_1, t'_2) \equiv Q_u[\omega', k'] = Q_u[t'_{j_2}, k'], \]
\[ t_N = D_N(t_{i_1}, t'_{j_2}) \equiv Q_u[t'_{j_2}, \ell] = Q_u[t'_{j_2}, \ell], \text{ and } \]
\[ u_N = D_N(t_1, t_2) \equiv Q_u[\omega, k] = Q_u[\omega_0, k] = Q_u[t'_{j_2}, k], \]

where the last line uses the fact that \( u = D(t_1, t_2, t'_1, t'_2), \) so that the second equality in that line holds \( \omega_0 = D(v_1, v_2, v'_1, v'_2). \) These relations imply that

\[ u'_N \subseteq t_N \subseteq u_N \text{ with } h_N(u_N) = \mu(\omega, k), h_N(u'_N) = \mu(\omega', k'), h_N(t_N) = \mu(\vartheta, \ell). \] \hspace{1in} (8.28)

Using this, we compute \( n(A; \alpha) \) as follows,

\[
n(A; \alpha) = \mu(\omega, k) + [N - \mu(\omega, k)] + [\mu(\vartheta, \ell) - \mu(\omega, k)] + [N - \mu(\vartheta, \ell)]
\]

\[
= \text{vertices of } t_{i_1} \text{ in } N \hspace{0.5cm} + \hspace{0.5cm} \text{vertices of } t_{i_2} \text{ in } N \text{ below } u_N
\]
\[
\begin{align*}
&= 4N - \mu(\omega, k) - \mu(\omega', k') - \mu(\partial, \ell),
\end{align*}
\]
which is the required exponent.

The last case, complementary to the ones already considered is when there exists a pair of indices, denoted \(i_2, j_2 \in \{1, 2\}\) such that \(t = D(t_{i_2}, t'_{j_2}) \subseteq u'\). In this case we leave the reader to verify by the usual means that

\[ t_{3N} \subseteq u'_{2N} \subseteq u_N, \]

with their heights given by the same expressions as in (8.25). Accordingly,

\[
n(A; \alpha) = \sum_{\text{vertices of } t_{i_1}} N - \mu(\omega, k) + \left[ N - \mu(\omega, k) \right] + \left[ \mu(\partial, \ell) - \mu(\omega', k') \right] + 2 \left[ N - \mu(\partial, \ell) \right]
\]

\[
= 4N - \mu(\omega, k) - \mu(\omega', k') - \mu(\partial, \ell).
\]

Thus, despite structural differences, all the cases give rise to the same value of \(n(A; \alpha)\) that agrees with the exponent in (8.25), completing the proof.

We pause for a moment to record a few properties of the youngest common ancestors of the roots and slopes that emerged in the proof of Lemma 8.9.

**Corollary 8.10.** Let \(A\) and \(\Gamma_A\) be as in Lemma 8.9.

(i) The possibly distinct vertices \(u, u'\) and \(t\), as described in Lemma 8.9, are linearly ordered in terms of ancestry, i.e., there is some ray of the root tree that they all lie on. Depending on \(A\), the vertex \(t\) may lie above or below \(u'\), but always in \(u\).

(ii) The splitting vertices \(\omega, \omega', \partial\) in the slope tree also obey certain inclusions; namely, for each of the pairs \((\omega, \partial)\) and \((\omega', \partial)\), one member of the pair is contained in the other.

**Proof.** Both \(u'\) and \(t\) lie on the ray identifying \(t'_{j_2}\), by definition, and \(u\) lies on the ray of \(u'\) by the assumption on the type of the root configuration. This establishes the first claim. The definitions also imply that \(v_{i_2} \subseteq \omega \cap \partial\) and \(v'_{j_2} \subseteq \omega' \cap \partial\), hence both intersections are non-empty. The second conclusion then follows from the nesting property of \(M\)-adic cubes.

**Lemma 8.11.** Let \(A = \{t_i, t'_i; i = 1, 2\}\) be a collection of four distinct root cubes such that \(I = \{(t_1, t_2); (t'_1, t'_2)\}\) is of type 3. Suppose that \(\Gamma_A = \{\alpha(t_i) = v_i, \alpha(t'_i) = v'_i ; i = 1, 2\}\) is a choice of slopes such that the collection \(\{(t_i, v_i); (t'_i, v'_i); i = 1, 2\}\) is sticky-admissible. Set \(\omega = D(v_1, v_2), \omega' = D(v'_1, v'_2), k = h(u) = h(u')\).
Then the following relations must hold: \( k \leq h(\omega), \ k \leq h(\omega') \). Further, there exist permutations \( \{i_1, i_2\} \) and \( \{j_1, j_2\} \) of \( \{1, 2\} \) such that the quantities
\[
s_1 = D(t_{i_1}, t'_{j_1}), \quad s_2 = D(t_{i_2}, t'_{j_2}), \quad \ell_1 = h(s_1),
\]
\[
\vartheta_1 = D(v_{i_1}, v'_{j_1}), \quad \vartheta_2 = D(v_{i_2}, v'_{j_2}), \quad \ell_2 = h(s_2)
\]
satisfy
\[
s_1 \subseteq u, \quad s_2 \subseteq u, \quad k \leq \ell_1 \leq \ell_2, \quad \ell_i \leq h(\vartheta_i) \text{ for } i = 1, 2, \quad (8.29)
\]
and for which
\[
Pr(\sigma(t_i) = v_i, \sigma(t'_{i}) = v'_i \text{ for } i = 1, 2) = \left(\frac{1}{2}\right)^{4N - \mu(\omega, k) - \mu(\vartheta_1, \ell_1) - \mu(\vartheta_2, \ell_2)}.
\]

Proof. Since \( I \) is of type 3, \( u = u' \) is the youngest common ancestor of the four elements in \( I \). If \( \omega_0 \) is the youngest common ancestor of the slopes \( \{v_i, v'_i : i = 1, 2\} \), then
\[
h(\omega_0) \geq h(u) = k
\]
by sticky admissibility. Thus \( \theta(\omega, k) = \theta(\omega_0, k) = \theta(\omega', k) \), and therefore \( \mu(\omega, k) = \mu(\omega', k) \), as claimed.

We turn to \( (8.29) \) and the probability estimate. The configuration type dictates that there exist indices \( (i, j) \in \{1, 2\}^2 \) such that \( D(t_i, t'_j) \subseteq u \). Among all such pairs \((i, j)\), we pick one for which \( D(t_i, t'_j) \) is of maximal height. Let us call this pair \((i_2, j_2)\), so that \( h(D(t_2, t'_{j_2})) \geq h(D(t_i, t'_j)) \) for all \( 1 \leq i, j \leq 2 \). The first three relations in \( (8.29) \) are now immediate. The last one follows from sticky admissibility and is left to the reader.

It remains to compute \( n(A; \alpha) \). The structure of \( N(A; \alpha) \) gives that
\[
\begin{align*}
    u_N = u'_N &= D_N(t_1, t_2) = D_N(t'_1, t'_2), \\
    u_N &= Q_u[\omega, k] = Q_{u'}[\omega', k] = Q_{t_1}[v_1, k] = Q_{t_2}[v_2, k], \\
    s_{iN} &= D_N(t_i, t'_i) = Q_{s_i}[\vartheta_i, \ell_i] = Q_{t_i}[v_i, \ell_i], \\
    s_{iN} \subseteq u_N &= D_N(s_{1N}, s_{2N}) \text{ for } i = 1, 2, \text{ so that} \\
    h_N(u_N) &= \mu(\omega, k) \leq h_N(s_{iN}) = \mu(\vartheta_i, \ell_i), \quad i = 1, 2.
\end{align*}
\]

Putting these together, the number of vertices in \( N(A; \alpha) \) is obtained as follows,
\[
n(A; \alpha) = \mu(\omega, k) + \sum_{i=1}^{2} \frac{\mu(\vartheta_i, \ell_i) - \mu(\omega, k)}{\mu(\vartheta_i, \ell_i)} + \sum_{i=1}^{2} \frac{2[N - \mu(\vartheta_i, \ell_i)]}{\mu(\vartheta_i, \ell_i)}
\]
\[
= 4N - \mu(\vartheta_1, \ell_1) - \mu(\vartheta_2, \ell_2) - \mu(\omega, k).
\]

The probability estimate claimed in \( (8.30) \) now follows from Lemma \( 8.2 \).

**Corollary 8.12.** Let \( \omega, \omega', \vartheta_1, \vartheta_2 \) be as in Lemma \( 8.9 \). Then each of the pairs \( (\omega, \vartheta_1), (\omega, \vartheta_2), (\omega', \vartheta_1) \) and \( (\omega', \vartheta_2) \) has the property that one member of the pair is contained in the other.

Proof. Since \( v_{i_1} \subseteq \omega \cap \vartheta_1, v_{i_2} \subseteq \omega \cap \vartheta_2, v'_{j_1} \subseteq \omega' \cap \vartheta_1 \) and \( v'_{j_2} \subseteq \omega' \cap \vartheta_2 \), all four intersections are nonempty, and the desired conclusion follows from the nesting property of \( M \)-adic cubes.
As the reader has noticed, the classification of probability estimates in this section is predicated on the configuration types of the roots, not the slopes. Of course, such definitions of type apply equally well to slope tuples \( \{(v_1, v_2); (v'_1, v'_2)\} \). Indeed, a point worth noting is that configuration types are not preserved under sticky maps; see for example the diagram in Figure 12 below, where a four tuple of roots of type 1 maps to a sticky image of type 3. In view of these considerations, we shall refrain for the most part from using any type properties of slopes. In the rare instances where structural properties of slopes are relevant, a case in point being Section 11.2.3, we need to consider all possible configurations.

![Diagram](image.png)

**Figure 12:** An example of a four tuple of roots of type 1 mapping to a sticky image of type 3. Notice that \( D(\sigma(t_1), \sigma(t_2)) = D(\sigma(t'_1), \sigma(t'_2)) \).

9 Tube counts

A question of considerable import, the full significance of which will emerge in Section 11 is the following: what is the maximum possible cardinality of a sticky-admissible collection of tube tuples that admit certain pairwise intersections in a pre-fixed segment of space? The answer depends, among other things, on the size and configuration type of the roots of the tubes. In this section, we discuss these size counts for collections that are simple enough in the sense that an element in the collection is either a pair, a triple or at most a quadruple of tubes, so that the configuration type of the roots has to fall in one of the categories described in Section 8.2.

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9.1 Collections of two intersecting tubes

Let us start with the case where the collection consists of pairs of tubes. To phrase the question above in more refined terms we define a collection of root-slope tuples \( \mathcal{E}_2[u, \omega; \varrho] \), where \( u \) is a vertex of the root tree, \( \omega \) is a splitting vertex of the slope tree, and \( \varrho \in [M^{-J}, 10A_0] \) is a constant that represents the (horizontal) distance from the root hyperplane to where intersection takes place.

\[
\mathcal{E}_2[u, \omega; \varrho] := \left\{ \left( t_1, t_2 \right) \in \mathcal{Q}(J), u = D(t_1, t_2), t_1 \neq t_2, \left( t_1, v_1 \right), \left( t_2, v_2 \right) \in \Omega_N, \omega = D(v_1, v_2), P_{t_1, v_1} \cap P_{t_2, v_2} \cap \left[ \varrho, C, 1, \varrho \right] \times \mathbb{R}^d \neq \emptyset \right\}.
\]

In this context the question at the beginning of this section can be restated as: what is the cardinality of \( \mathcal{E}_2[u, \omega; \varrho] \)? We answer this question in Lemma 9.1 of this section, splitting the necessary work between two intermediate lemmas whose content will also be used in later counting arguments. To be specific, Lemma 9.1 obtains a uniform bound on a \( t_2 \)-slice of \( \mathcal{E}_2[u, \omega; \varrho] \) for fixed \( t_1, v_1 \) and \( v_2 \). The cardinality of the projection of \( \mathcal{E}_2[u, \omega; \varrho] \) onto the \( t_1 \) coordinate is obtained in Lemma 9.2.

**Lemma 9.1.** Let \( \mathcal{E}_2[u, \omega; \varrho] \) be the collection defined in (9.1), and let \( \rho_\omega = \sup \{ |a - b| : a, b \in \Omega_N, D(a, b) = \omega \} \) be the quantity defined in (4.7).

(i) If \( \mathcal{E}_2[u, \omega; \varrho] \) is nonempty, then \( 2C_1 \rho_\omega \geq M^{-J} \).

(ii) Given a constant \( C_1 > 0 \) used to define \( \mathcal{E}_2[u, \omega; \varrho] \), there exists a constant \( C_2 = C_2(d, M, C_0, A_0, C_1) > 0 \) with the following property. For any fixed choice of \( t_1 \in \mathcal{Q}(J) \) and \( v_1, v_2 \in \Omega_N \) the following estimate holds:

\[
\# \left\{ t_2 \in \mathcal{Q}(J) : \left( t_1, v_1 \right), \left( t_2, v_2 \right) \in \mathcal{E}_2[u, \omega; \varrho] \right\} \leq C_2 \rho_\omega M^{J}.
\]

**Proof.** If \( \{ (t_1, v_1), (t_2, v_2) \} \) is a tuple that lies in \( \mathcal{E}_2[u, \omega; \varrho] \), then there exists \( x \in [\varrho, C, 1, \varrho] \times \mathbb{R}^d \) such that \( x \) also belongs to \( P_{t_1, v_1} \cap P_{t_2, v_2} \). By Lemma 5.1, an appropriate version of inequality (5.3) must hold, i.e., there exists \( x_1 \in [\varrho, C, 1, \varrho] \) such that

\[
|\text{cen}(t_2) - \text{cen}(t_1) + x_1(v_2 - v_1)| \leq 2c_d \sqrt{d} M^{-J}.
\]

In conjunction with Corollary 5.2 this leads to the inequality

\[
M^{-J} \leq |\text{cen}(t_2) - \text{cen}(t_1)| \leq |x_1||v_2 - v_1| + 2c_d \sqrt{d} M^{-J} \leq 2|x_1||v_2 - v_1| \leq 2C_1 \rho_\omega,
\]

which is the conclusion of part (ii). The inequality (6.3) also implies that \( \text{cen}(t_2) \) is constrained to lie in a \( O(M^{-J}) \) neighborhood of the line segment

\[
\text{cen}(t_1) - s(v_2 - v_1), \quad \varrho \leq s \leq C_1 \varrho.
\]

The length of this segment is at most \( C_1 \varrho |v_2 - v_1| \leq C_1 \varrho \rho_\omega \), since \( v_1 \) and \( v_2 \) must lie in distinct children on \( \omega \). In view of part (ii), the number of possible choices for \( M^{-J} \)-separated points \( \text{cen}(t_2) \), and hence for \( t_2 \), lying within this neighborhood is \( O(\varrho \rho_\omega M^{J}) \), as claimed in part (ii). \( \square \)
Lemma 9.2. Given $C_1 > 0$, there exists a positive constant $C_2 = C_2(d, M, A_0, C_1)$ with the following property. For any $\mathcal{E}_2[u, \omega; \varrho]$ defined as in (4.1), the following estimate holds:

$$
\# \left\{ t_1 \in \mathcal{Q}(J) \mid \exists t_2 \in \mathcal{Q}(J) \text{ and } v_1, v_2 \in \Omega_N \text{ such that } \{(t_1, v_1); (t_2, v_2)\} \in \mathcal{E}_2[u, \omega; \varrho] \right\} \leq C_2 \varrho \rho \omega M^{-(d-1)h(u)+dJ},
$$

(9.5)

where $\rho \omega$ is as in (4.7).

Proof. If $\{(t_1, v_1), (t_2, v_2)\} \in \mathcal{E}_2[u, \omega; \varrho]$, then there exists $x = (x_1, \cdots, x_{d+1}) \in P_{t_1, v_1} \cap P_{t_2, v_2}$ with $\varrho \leq x_1 \leq C_1 \varrho$. Combining inequality (9.3) obtained from Lemma 5.1 along with Corollary 5.2 as we did in Lemma 9.1, we obtain

$$
|\text{cen}(t_2) - \text{cen}(t_1)| \leq |x_1||v_2 - v_1| + 2c_d \sqrt{d} M^{-J} \\
\leq (1 + 4c_d \sqrt{d}) |x_1||v_1 - v_2| \\
\leq C_1(1 + 4c_d \sqrt{d}) \varrho \omega = C \varrho \omega,
$$

(9.6)

where the last step follows from the definition of $\omega$. This means that $\text{cen}(t_1)$ and $\text{cen}(t_2)$ must be within distance $C \varrho \omega$ of each other. On the other hand, it is known as part of
the definition of $\mathcal{E}_2[u, \omega; \varrho]$ that $u = D(t_1, t_2)$, so cen($t_1$) and cen($t_2$) must lie in distinct children of $u$. This forces the location of cen($t_1$) to be within distance $C\varrho\omega$ of the boundary of some child of $u$, to allow for the existence of a point cen($t_2$) contained in a different child and obeying the constraint of (9.6). In other words, cen($t_1$) belongs to the set
\[
A_u = \{ s \in u : \text{dist}(s, \text{bdry}(u')) \leq C\varrho\omega \text{ for some child } u' \text{ of } u \}, \tag{9.7}
\]
which is the union of at most $dM$ parallelepipeds of dimension $d$, with length $M^{-h(u)}$ in $(d-1)$ “long” directions and $C\varrho\omega$ in the remaining “short” direction. Note that $\varrho\omega \leq M^{-h(\omega)} \leq M^{-h(u)}$ by sticky-admissibility, hence $\varrho\omega = O(M^{-h(u)})$, which justifies this description.

Since $C\varrho\omega \geq M^{-J}$ by Lemma [9.1][1], the constituent parallelepipeds of $A_u$ as described above are thick relative to the finest scale $M^{-J}$ in all directions. The volume of $A_u$ is then easily computed as
\[
|A_u| \leq C\varrho\omega M^{-(d-1)h(u)}.
\]
Therefore the number of $M^{-J}$ separated points cen($t_1$), and hence the number of possible root cubes $t_1$, contained in $A_u$ is at most $C_2\varrho\omega M^{-(d-1)h(u)+dJ}$, as claimed. \hfill \Box

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure14.png}
\caption{Proof of Lemma 9.2 illustrated, with $d = 2$ and $M = 3$. The outermost square is $u$, and the smallest squares depict the root cubes in $A_u$.}
\end{figure}
Lemma 9.3. Let \( \mathcal{E}_2[u, \omega; \varrho] \) be the collection of pairs of tubes defined in (9.1). Then
\[
\#(\mathcal{E}_2[u, \omega; \varrho]) \leq C(\varrho \omega)^2 2^{(N - \nu(\omega))} M^{-(d-1)h(u)+(d+1)J}
\]
Here \( \nu(\omega) \) denotes the index of the splitting vertex \( \omega \), as defined in (1.4).

Proof. We combine the counts from Lemmas 9.1 and 9.2. For fixed \( t_1, v_1 \) and \( v_2 \), the number of possible \( t_2 \) such that \( \{(t_1, v_1), (t_2, v_2)\} \in \mathcal{E}_2[u, \omega; \varrho] \) is bounded above by the quantity on the right hand side of (9.2). The number of possible \( t_1 \) is at most the right hand side of (9.5), whereas the number of possible \( v_1 \), hence also \( v_2 \), is \( 2^{N - \nu(\omega)} \) due to the binary nature of \( \Omega_N \) as discussed in Section 4.2. The claimed size estimate of \( \mathcal{E}_2[u, \omega; \varrho] \) is simply the product of all the quantities mentioned above. \( \square \)

9.2 Counting slope tuples

Variations of the arguments presented in Section 9.1 also apply to more general collections. For the proof of the lower bound in (9.3), we will need to estimate, in addition to the above, the sizes of collections consisting of tube triples and tube quadruples with certain pairwise intersections. The collections of tube tuples whose cardinalities are of interest are analogues of \( \mathcal{E}_2[u, \omega; \varrho] \) of greater complexity, and their constructions share the common feature that the probability of slope assignment for any tube tuple within a collection is constant and falls into one of the categories classified in Section 8. As we have seen in that section, the probability depends, among other things, on certain splitting vertices of the slope tree occurring as pairwise youngest common ancestors. In particular, which subset of pairwise youngest common ancestors has to be considered, whether for root or slope, is dictated by the root configuration type. An important component of tube-counting is therefore to estimate how many possible slope tuples can be generated from a given set of such splitting vertices. Before moving on to the main counting arguments in this section presented in Sections 9.3 and 9.4, we observe a few facts that help in counting tuples of slopes, given some information about their ancestry.

Lemma 9.4. (i) Given any \( \Gamma \subseteq \Omega_N \), \( \#(\Gamma) \leq 4 \), there exist at most three distinct vertices \( \{\varpi_i : i = 1, 2, 3\} \subseteq \mathcal{G}(\Omega_N) \) with the properties
\[
h(\varpi_1) \leq h(\varpi_2) \leq h(\varpi_3), \quad \varpi_2, \varpi_3 \subseteq \varpi_1,
\]
such that \( D(w, w') \in \{\varpi_i : i = 1, 2, 3\} \) for any \( w \neq w' \), \( w, w' \in \Gamma \).

(ii) Suppose now that we are given \( \{\varpi_i : i = 1, 2, 3\} \), possibly distinct splitting vertices of the slope tree obeying (9.8). Define
\[
m = m[\varpi_1, \varpi_2, \varpi_3] := \begin{cases} 2(\nu(\varpi_3) + \nu(\varpi_2)) & \text{if } \varpi_3 \not\subseteq \varpi_2, \\ 2\nu(\varpi_3) + \nu(\varpi_2) + \nu(\varpi_1) & \text{if } \varpi_3 \subseteq \varpi_2. \end{cases}
\]
Fix three distinct pairs of indices \( \{(i_k, j_k) : i_k \neq j_k, 1 \leq k \leq 3\} \subseteq \{1, 2, 3, 4\}^2 \) with the property that \( \bigcup\{i_k, j_k : k = 1, 2, 3\} = \{1, 2, 3, 4\} \). Then
\[
\#\{(w_1, w_2, w_3, w_4) \in \Omega_N^4 : D(w_{i_k}, w_{j_k}) = \varpi_k, 1 \leq k \leq 3\} \leq C 2^{4N - m}
\]
provided the collection is nonempty.

Proof. If \( \Gamma \) is given, we arrange all the pairwise youngest common ancestors of \( \Gamma \), i.e., the vertices in \( D_\Gamma := \{ D(w, w') : w \neq w', w, w' \in \Gamma \} \), in increasing order of height, where distinct vertices of the same height can be arranged in any way, say according to the lexicographic ordering. We define \( w_3 \) to be a vertex of maximal height in \( D_\Gamma \), and \( w_2 \) to be a vertex of maximal height in \( D_\Gamma \setminus \{ w_3 \} \). Due to maximality of height and the binary nature of the slope tree as ensured by Proposition 4.1, \( w_3 \) has exactly two descendants in \( \Gamma \), say \( w_1 \) and \( w_2 \).

If \( w_3 \not\subseteq w_2 \), then there is no overlap among the descendants of these two vertices. Thus the two descendants \( w_3 \) and \( w_4 \) of \( w_2 \) must be distinct from \( w_1, w_2 \), thus accounting for all the elements of \( \Gamma \). In this case the conclusion of the lemma holds with \( w_1 = D(w_2, w_3) \). If \( w_3 \subseteq w_2 \), then again by maximality of height \( w_2 \) can contribute exactly one member of \( \Gamma \) that is neither \( w_1 \) nor \( w_2 \). Let us call this new member \( w_3 \). If \( \#(\Gamma) = 3 \), then the proof is completed by setting \( w_1 = D(w_2, w_3) = w_2 \). If \( \#(\Gamma) = 4 \), we call the remaining child \( w_4 \), which is not descended from \( w_2 \), and set \( w_1 = D(w_2, w_4) \). This selection meets (9.8), and also accounts for all the pairwise youngest common ancestors of \( \Gamma \), as required by part (i) of the lemma.

A very similar argument can be used to prove part (ii). Since the total number of slopes in \( \Omega_N \) generated by \( w_3 \) is exactly \( 2^{N - \nu(w_3)} + 1 \), this is the maximum number of possible choices for each of \( w_{i_3} \) and \( w_{j_3} \). If \( w_3 \not\subseteq w_2 \), then \( \{i_2, j_2\} \cap \{i_3, j_3\} = \emptyset \). Since each of \( w_{i_2} \) and \( w_{j_2} \) admits at most \( 2^{N - \nu(w_2) + 1} \) possibilities by the same reasoning, the size of possible four tuples \( (w_1, w_2, w_3, w_4) \) in this case is at most 2 raised to the power \( 2(N - \nu(w_3)) + 1 \), which gives the claimed estimate. If \( w_3 \subseteq w_2 \subseteq w_1 \), then by our assumptions on \( i_k, j_k \), there exist indices \( \ell_2 \in \{i_2, j_2\} \setminus \{i_3, j_3\} \) and \( \ell_1 \in \{i_1, j_1\} \setminus \{i_3, j_3, \ell_2\} \). Since \( i_3, j_3, \ell_1, \ell_2 \) are distinct indices and the number of possible choices of \( w_{i_3}, w_{j_3}, w_{\ell_1} \), and \( w_{\ell_2} \) are at most \( 2^{N - \nu(w_3)} \), \( 2^{N - \nu(w_3)} \), \( 2^{N - \nu(w_1)} \) and \( 2^{N - \nu(w_2)} \) respectively, the result follows.

Minor modifications of the argument above yield the following analogue for slope triples. The proof is left to the reader.

**Lemma 9.5.**

(i) Given a collection \( \Gamma \subseteq \Omega_N \), \( \#(\Gamma) \leq 3 \), it is possible to rearrange the collection of vertices \( \{D(w, w') : w \neq w', w, w' \in \Gamma \} \) as \( \{w_1, w_2\} \) with \( w_2 \subseteq w_1 \).

(ii) Given a pair \( \{w_1, w_2\} \subseteq \mathcal{G}(\Omega_N) \) with \( w_2 \subseteq w_1 \), define

\[
\tilde{m} = \tilde{m}[w_1, w_2] := 2\nu(w_2) + \nu(w_1). \tag{9.10}
\]

Let \( (i_1, j_1) \neq (i_2, j_2) \) be two pairs of indices such that \( \{i_1, j_1, i_2, j_2\} = \{1, 2, 3\} \). Then the following estimate holds:

\[
\#\{(w_1, w_2, w_3) : D(w_1, w_{j_1}) = w_1, D(w_{i_2}, w_{j_2}) = w_2\} \leq 2^{3N - \tilde{m}}.
\]

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9.3 Collections of four tubes with at least two pairwise intersections

9.3.1 Four roots of type 1

We start with the simplest and generic situation, when the root quadruple is of type 1. Motivated by the expression of the probability obtained in (8.24), let us first fix two vertex triples \((u, u', z)\) and \((\omega, \omega', v)\) in the root tree and slope tree respectively that satisfy the height and containment relations prescribed in Lemma 8.8. For such a selection and with \(\varrho \in [M^{-J}, 10A_0]\), we define a collection \(\mathcal{E}_{41} = \mathcal{E}_{41}[u, u', z; \omega, \omega', v; \varrho]\) of sticky-admissible tube quadruples of the form \(\{(t_1, v_1), (t_2, v_2), (t'_1, v'_1), (t'_2, v'_2)\}\), obeying the following restrictions:

\[
\begin{align*}
\mathbb{I} &= \{(t_1, t_2); (t'_1, t'_2)\} \text{ is of type 1, } t_1 \neq t_2, t'_1 \neq t'_2, u = D(t_1, t_2), \\
u' &= D(t'_1, t'_2), \quad z = D(u, u'), \quad \omega = D(v_1, v_2), \quad \omega' = D(v'_1, v'_2), \quad v = D(\omega, \omega'), \\
P_{t_1, v_1} \cap P_{t_2, v_2} \cap [\varrho, C_1 \varrho] \times \mathbb{R}^d \neq \emptyset, \quad P_{t'_1, v'_1} \cap P_{t'_2, v'_2} \cap [\varrho, C_1 \varrho] \times \mathbb{R}^d \neq \emptyset.
\end{align*}
\]

The result below provides a bound on the size of \(\mathcal{E}_{41}\).

**Lemma 9.6.** There exists a constant \(C > 0\) such that

\[
\#(\mathcal{E}_{41}) \leq C \left( \varrho^{2 \nu(D)} \right)^2 2^{4N-2(\nu(u)+\nu(\omega'))} M^{-(d-1)\left[h(u)+h(u')\right]} + 2(d+1)J.
\]

**Proof.** Since the intersection and ancestry conditions imply that

\[
\mathcal{E}_{41}[u, u', z; \omega, \omega', v; \varrho] \subseteq \mathcal{E}_2[u, \omega; \varrho] \times \mathcal{E}_2[u', \omega'; \varrho],
\]

the stated size bound for \(\mathcal{E}_{41}\) is the product of the sizes of the two factors on the right. These are obtained from Lemma 9.3 in Section 9.1 applied twice. \(\square\)

9.3.2 Four roots of type 2

The treatment of this case follows a similar route, though with certain important variations. The main distinction from Section 9.3.1 is that the intersection and type requirements place greater constraints on the selection of the roots and slopes, and hence on the number of tube quadruples. Thus better bounds are possible, compared to the trivial ones exploited in Lemma 9.3.

Let \((u, u', t)\) and \((\omega, \omega', \vartheta)\) be vertex triples in the root tree and slope tree respectively that meet the requirement of Corollary 8.10. In other words, the vertices \(u, u', t\) are linearly ordered in terms of ancestry, and obey \(u' \subseteq u\), while \(\omega \cap \vartheta \neq \emptyset\) and \(\omega' \cap \vartheta \neq \emptyset\). Holding these fixed, define \(\mathcal{E}_{42} = \mathcal{E}_{42}[u, u', t; \omega, \omega', \vartheta; \varrho]\) to be the collection of all sticky-admissible tuples of the form \(\{(t_i, v_i), (t'_i, v'_i) : i = 1, 2\}\) obeying the properties:

\[
\begin{align*}
\mathbb{I} &= \{(t_1, t_2); (t'_1, t'_2)\} \text{ is of type 2, } u' = D(t'_1, t'_2) \subseteq u = D(t_1, t_2), \\
t &= D(t_2, t'_2), \quad \omega = D(v_1, v_2), \quad \omega' = D(v'_1, v'_2), \quad \vartheta = D(v_2, v'_2), \\
P_{t_1, v_1} \cap P_{t_2, v_2} \cap [\varrho, C_1 \varrho] \times \mathbb{R}^d \neq \emptyset, \quad P_{t'_1, v'_1} \cap P_{t'_2, v'_2} \cap [\varrho, C_1 \varrho] \times \mathbb{R}^d \neq \emptyset.
\end{align*}
\]

The vertex triple \((\omega, \omega', \vartheta)\) obeys the hypothesis of Lemma 9.4(ii), permitting the application of this lemma in the counting argument presented in Lemma 9.8.
Lemma 9.7. If the vertex pairs \((\omega, \vartheta)\) and \((\omega', \vartheta)\) both have the property that one member of the pair is contained in the other, then there exists a rearrangement of \(\{\omega, \omega', \vartheta\}\) as \(\{\omega_1, \omega_2, \omega_3\}\) that meets the requirement (9.8).

Proof. If \(\omega \cap \omega' = \emptyset\), then \(\vartheta\) must contain both \(\omega\) and \(\omega'\). In this case, we rename \(\vartheta\) as \(\omega_1\) and call \(\omega_3\) the element of \(\{\omega, \omega'\}\) with greater height. If \(\omega \cap \omega' \neq \emptyset\), then the inclusion requirements imply that there must be a ray which contains all three vertices. Since the vertices are linearly ordered, we rename them based on height. \(\square\)

Lemma 9.7 above allows us to define the quantity \(m\) as in (9.9), which by a slight abuse of notation we denote by \(m[\omega, \omega', \vartheta]\). We are now in a position to state the main result of this subsection, namely the size estimate for \(\mathcal{E}_{42}\). The location of \(t\) relative to \(u, u'\) affects the size estimate of \(\mathcal{E}_{42}\), even though we have seen that the probability estimate in (8.10) remains unchanged with respect to this property.

Lemma 9.8. The following conclusions hold:

(i) If \(u' \subseteq t \subseteq u\), then \(\mathcal{E}_{42}\) is non-empty only if \(\text{dist}(t, \text{bdry}(u_*)) \leq C \rho_{\omega}\). Here \(u_*\) is defined to be the unique child of \(u\) containing \(t\) if \(t \subseteq u\) and is set to be equal to \(u\) if \(t = u\). In either case,

\[
\#(\mathcal{E}_{42}) \leq C \left( \rho^3 \rho_{\omega}^2 \rho_{\omega'} \right) \min \left[ \rho_{\omega}, M^{-h(t)} \right] 2^{4N-m[\omega, \omega', \vartheta]} \times M^{-(d-1) \left( h(t)+h(u') \right)+2(d+1)J}.
\]

(ii) If \(t \subseteq u' \subseteq u\), then \(\mathcal{E}_{42}\) is non-empty only if

\[
\text{dist}(t, \text{bdry}(u_*)) \leq C \rho_{\omega}\quad \text{and} \quad \text{dist}(t, \text{bdry}(u'_*)) \leq C \rho_{\omega'},
\]

where \(u_*, u'_*\) are the children of \(u, u'\) respectively that contain \(t\). In this case,

\[
\#(\mathcal{E}_{42}) \leq C \left( \rho^3 \rho_{\omega} \rho_{\omega'} \right) 2^{4N-m[\omega, \omega', \vartheta]} \min \left[ \rho_{\omega}, M^{-h(t)} \right] \times \min \left[ \rho_{\omega}, M^{-h(t)} \right] M^{-2(d+1)h(t)+2(d+1)J}.
\]

Proof. Both statements in the lemma involve similar arguments. We only prove part (i) in detail, and leave a brief sketch for the other part. The argument here follows the basic structure of Lemma 9.3 since we still have the trivial containment

\[
\mathcal{E}_{42}[t, t; \omega, \omega', \vartheta; \vartheta] \subseteq \mathcal{E}_2[u, u; \vartheta] \times \mathcal{E}_2[u', u'; \vartheta],
\]

but with a few modifications resulting from the more refined information about the roots and slopes available from \(t\) and \(\vartheta\). For instance, combining the defining assumptions that \(t_2 \subseteq t\) and \(u = D(t_1, t_2)\) with the intersection inequality \(|\text{cen}(t_2) - \text{cen}(t_1)| \leq 2C_1 \rho_{\omega'}\) derived from (9.3) in Lemma 9.1, we deduce that \(t\) has to lie within distance \(2C_1 \rho_{\omega'}\) of the boundary of \(u_*\). This is the first conclusion of part (i). For the size bound, we

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reason as follows. By Lemma 9.1(ii), the number of \( t_1 \) and \( t'_1 \), if everything else is held fixed, is \( \leq C(q\rho_\omega M^J)(q\rho_\omega M^J) \leq Cg^2\rho_\omega\rho_\omega'M^{2J} \). Turning to slope counts, we apply Lemma 9.4(ii), the use of which has already been justified in Lemma 9.7 to deduce that the number of possible slope quadruples \((v_1, v_2, v'_1, v'_2)\) is \( 2^{4N-m} \). It remains to compute the size of the \( t_2 \) and \( t'_2 \) projections of \( \mathcal{E}_{42} \). In view of (9.13), a bound on the size of the \( t'_2 \) projection is given by the right hand side of (9.5) with \( u \) replaced by \( u' \). On the other hand, \( t_2 \) is restricted to lie within \( t \) and within distance \( 2C_1\rho_\omega \) from the boundary of \( t \) if \( t \subset u \). This places a nontrivial spatial restriction on \( t_2 \) only if \( 2C_1\rho_\omega < M^{-h(t)} \). If \( t = u \), the argument leading up to (9.5) shows that \( t_2 \) lies in \( A_u \) defined in (9.7). In either event the volume of the region where \( t_2 \) can range is at most \( C_{\min(\rho_\omega, M^{-h(t)})} M^{-(d-1)h(t)} \), hence the cardinality of the \( t_2 \) projection is at most \( M^{dJ} \) times this quantity (see Figure 15). Combining all these counts yields the bound on the size of \( \mathcal{E}_{42} \) given in part (ii).

![Figure 15: Illustration of the spatial restriction on \( t_2 \) imposed by the conditions \( u' \subset t \subset u \), \( t_2 \subset t \), \( \text{dist}(t_1, t_2) \leq 2C_1\rho_\omega < M^{-h(t)} \). Here, \( t_2 \) must lie within the shaded region along the boundary of \( t \), with \( t_1 \) falling just outside this boundary in the unshaded thatched region.](image)

For part (iii), the size estimate of \( \mathcal{E}_{42} \) is a product of a number of factors analogous to the ones already considered, the origins of which are indicated below.

\[
\begin{align*}
\#(t_1 \text{ given } v_1, v_2, t_2) &\leq C\rho_\omega M^J, \\
\#(t'_1 \text{ given } v'_1, v'_2, t'_2) &\leq C\rho_\omega'M^{J}, \\
\#(t_2) &\leq C_{\min(\rho_\omega, M^{-h(t)})} M^{-(d-1)h(t)+dJ}, \\
\#(t'_2) &\leq C_{\min(\rho_\omega', M^{-h(t)})} M^{-(d-1)h(t)+dJ}, \\
\#(v_1, v_2, v'_1, v'_2) &\leq 2^{4N-m[\omega,\omega',\theta]} \quad \text{(from Lemma 9.4(ii)).}
\end{align*}
\]

We omit the details. \( \square \)
9.3.3 Four roots of type 3

To complete the discussion of size for collections consisting of intersecting tube quadruples, it remains to consider the case where the root configuration is of type 3. Motivated by the conclusions of Lemma 8.11 and Corollary 8.12, we first fix two vertex tuples \((u, s_1, s_2)\) and \((\omega, \omega', \vartheta_1, \vartheta_2)\) in the root tree and the slope tree respectively, with the properties that \(s_1, s_2 \subseteq u\), \(h(u) \leq h(s_1) \leq h(s_2)\), \(\omega \cap \vartheta_1 \neq \emptyset\), and \(\omega' \cap \vartheta_i \neq \emptyset\) for \(i = 1, 2\). For such a selection, we define \(\mathcal{E}_{43}[u, s_1, s_2; \omega, \omega', \vartheta_1, \vartheta_2; \varrho]\) to be the collection of all sticky-admissible tuples \(\{(t_i, v_i), (t'_i, v'_i) : i = 1, 2\}\) that satisfy the list of conditions below:

\[
\begin{align*}
\mathcal{I} = \{(t_1, t_2); (t'_1, t'_2)\} & \text{ is of type 3, } u = D(t_1, t_2) = D(t'_1, t'_2), \\
\omega' = D(v'_1, v'_2) & \subseteq \omega = D(v_1, v_2), \\
\vartheta_i = D(t_i, t'_i), & \varrho_i = D(v_i, v'_i), i = 1, 2, \\
P_{t_1, v_1} \cap P_{t_2, v_2} & \subseteq [\varrho, C_1 \varrho] \times \mathbb{R}^d \neq \emptyset, \\
P_{t'_1, v'_1} \cap P_{t'_2, v'_2} & \subseteq [\varrho, C_1 \varrho] \times \mathbb{R}^d \neq \emptyset.
\end{align*}
\]

Since \(\mathcal{I}\) is of type 3, interchanging \((t_1, t_2)\) and \((t'_1, t'_2)\) leaves \(u\) unchanged. Hence we may assume without loss of generality that \(\varrho_\omega \leq \varrho_{\omega'}\). Further, Lemma 9.11 dictates that for \(\mathcal{E}_{43}\) to be non-empty, at most three out of the four vertices \(\omega, \omega', \vartheta_1, \vartheta_2\) can be distinct. We leave the reader to verify that Lemma 9.7 can be applied to any triple of these four vertices. Thus for any choice of an eligible tuple \{\(\omega, \omega', \vartheta_1, \vartheta_2\)\}, there exists a rearrangement of its entries as \(\{\varpi_1, \varpi_2, \varpi_3\}\) obeying the hypothesis and hence the conclusion of Lemma 9.4[\ref{9.4}]. This permits an unambiguous definition of the quantity \(m[\omega, \omega', \vartheta_1, \vartheta_2]\) as in \(9.9\)\), which we use in the statement of the lemma below.

**Lemma 9.9.** If \(s_i \subseteq u\), let \(u_i\) denote the child of \(u\) that contains \(s_i\). Set \(\Delta := \min[\varrho_\omega, \varrho_{\omega'}]\).

(i) The collection \(\mathcal{E}_{43}\) is nonempty only if

\[
\sum_{i=1}^2 \mathrm{dist}(s_i, \mathrm{bdry}(u_i)) \leq C\Delta,
\]

where \(\mathrm{dist}(s_i, \mathrm{bdry}(u_i))\) is defined to be zero if \(u = s_1\).

(ii) If \(\Delta \leq M^{-h(s_1)}\) and \(\mathcal{E}_{43}\) is nonempty, then in addition to \(9.15\), one of the following two conditions must hold:

1. \(s_2 \subseteq s_1 = u\), in which case \(s_2\) lies within distance \(C\Delta\) of the boundary of some child of \(s_1 = u\).
2. \(s_2 \cap s_1 = \emptyset\), in which case \(\mathrm{dist}(s_2, \mathrm{bdry}(s_1)) \leq C\Delta\).

In either case, \(s_2\) is constrained to lie in the union of at most \(2^d M\) slab-like parallelepipeds, each with \((d-1)\) “long” directions of sidelength \(M^{-h(s_1)}\) and one “short” direction of sidelength \(\Delta\).
(iii) If \( \Delta \geq M^{-h(s_1)} \) and \( \mathcal{E}_{43} \) is nonempty, then in addition to (9.15), \( s_2 \) has to lie within a thin tube-like parallelepiped of length \( \varrho \min(M^{-h(\omega)}, M^{-h(\omega')}) \) in one “long” direction and thickness \( CM^{-h(s_1)} \) in the remaining \( (d-1) \) “short” directions; more precisely, both the following inequalities must hold:

\[
|\text{cen}(s_2) - \text{cen}(s_1) + x_1(\text{cen}(\omega \cap \vartheta_2) - \text{cen}(\omega \cap \vartheta_1))| \leq CM^{-h(s_1)}, \quad \text{and} \quad (9.16)
\]

\[
|\text{cen}(s_2) - \text{cen}(s_1) + x'_1(\text{cen}(\omega' \cap \vartheta_2) - \text{cen}(\omega' \cap \vartheta_1))| \leq CM^{-h(s_1)} \quad (9.17)
\]

for some \( x_1, x'_1 \in [\varrho, C_1 \varrho] \). Here \( \text{cen}(t) \) denotes the centre of the cube \( t \).

(iv) In all cases, if \( \mathcal{E}_{43} \) is nonempty,

\[
\#(\mathcal{E}_{43}) \leq C 2^{4N - m[\omega, \omega', \vartheta_1, \vartheta_2]} M^{-2(d-1)h(s_2) + 2(d+1)J} \times \prod_{i=1}^{2} \left[ \min[\varrho \rho_\omega, M^{-h(s_1)}] \min[\varrho \rho_{\omega'}, M^{-h(s_1)}] \right].
\]

**Proof.** Let us fix a tuple \( \{(t_i, v_i); (t'_i, v'_i) : i = 1, 2\} \) in \( \mathcal{E}_{43} \). As in previous proofs such as Lemma 9.2 (applications of which have appeared in the counting arguments of Lemma 9.3 and 9.8), the key elements are the inequalities

\[
|\text{cen}(t_2) - \text{cen}(t_1)| \leq C \varrho \rho_\omega \quad \text{and} \quad |\text{cen}(t'_2) - \text{cen}(t'_1)| \leq C \varrho \rho_{\omega'} \quad (9.18)
\]

They are proved exactly in the same way as (9.6) follows from (9.3), resulting from the nontrivial intersection conditions that define \( \mathcal{E}_{43} \). Combined with the set inclusion relations \( u = D(t_1, t_2) = D(t'_1, t'_2) \) and \( t_1, t'_1 \subseteq s_1 \) and \( t_2, t'_2 \subseteq s_2 \) that are guaranteed by the type assumption on the roots, this yields that

\[
\text{dist}(s_1, \text{bdry}(u_i)) \leq \inf \left[ \text{dist}(t_i, \text{bdry}(u_i)), \text{dist}(t'_i, \text{bdry}(u_i)) \right]
\]

\[
= \inf \left[ \text{dist}(t_i, u_i), \text{dist}(t'_i, u_i) \right]
\]

\[
\leq \min \left[ \text{dist}(t_1, t_2), \text{dist}(t'_1, t'_2) \right]
\]

\[
\leq C \min[\varrho \rho_\omega, \varrho \rho_{\omega'}] = C \Delta
\]

leading to the distance constraints in (9.14). Incidentally, the inequalities (9.18) also prove the relation in part (iii) if \( s_2 \cap s_1 = \emptyset \). On the other hand, if \( s_2 \subseteq s_1 \), then \( s_1 = u \) and the desired inequality is simply a restatement of the one in (9.13). For part (iii), we refer again to the intersection inequality (9.3), using it to deduce that

\[
|\text{cen}(s_2) - \text{cen}(s_1) + x_1(\text{cen}(\omega \cap \vartheta_2) - \text{cen}(\omega \cap \vartheta_1))|
\]

\[
\leq \sum_{i=1}^{2} \left| \text{cen}(s_i) - \text{cen}(t_i) \right| + |x_1| \sum_{i=1}^{2} \left| \text{cen}(\omega \cap \vartheta_i) - v_i \right| + |\text{cen}(t_2) - \text{cen}(t_1)| + x_1(v_2 - v_1)
\]

\[
\leq \sqrt{d} \sum_{i=1}^{2} M^{-h(s_i)} + C_1 \varrho \sqrt{d} \sum_{i=1}^{2} M^{-h(\theta_i)} + 2c_d \sqrt{d} M^{-J} \leq C M^{-h(s_1)}.
\]

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Here we have also used the height and inclusion relations associated with the root configuration type established in Lemma 8.11, namely,

\[ t_i \subseteq s_i, \quad v_i \in \omega \cap \vartheta_i, \quad h(s_i) \leq h(\vartheta_i), \quad h(s_1) \leq h(s_2), \quad i = 1, 2. \]

The inequality above implies that \( s_2 \) has to lie within distance \( O(M^{-h(s_1)}) \) of a line segment of length at most \( \varrho[\text{cen}(\omega \cap \vartheta_2) - \text{cen}(\omega \cap \vartheta_1)] \leq \varrho M^{-h(\omega)} \). The inequality (9.17) is proved in an identical manner, using \( t'_1, v'_1, \omega' \) instead of \( t_1, v_1, \omega \). The first statement in part (iii) is a consequence of both these inequalities.

The bound on the size of \( E_{g_3} \) uses the same machinery as in the proof of Lemma 9.8, so we simply indicate the breakdown of the contributions from the different sources:

\[
\#(E_{g_3}) \leq C \min_{\#(t_1) \text{ with } t_2, v_1, v_2 \text{ fixed}} \times C \min_{\#(t'_1) \text{ with } t'_2, v'_1, v'_2 \text{ fixed}} \times \min_{\#(t_2-\text{projection})} \times \min_{\#(t'_2-\text{projection})}
\]

which leads to the stated estimate.

\[ \Box \]

### 9.4 Collections of three tubes with at least two pairwise intersections

For the sake of completeness and book-keeping, we record in this section the cardinality of collections consisting of intersecting tube triples. No new ideas are involved in the proofs, which are in fact simpler than the ones in Section 9.3. These are left to the interested reader.

Using the notation set up in Lemmas 8.5 and 8.6, we define the collections \( E_{31} = E_{31}[u, u'; \omega, \omega'; \varrho] \) and \( E_{32} = E_{32}[u, t; \omega, \omega', \vartheta; \varrho] \) in exactly the same way \( E_{31} \) were defined. Namely, \( E_{31} \) consists of all sticky-admissible tuples of the form \( \{(t_1, v_1), (t_2, v_2), (t'_1, v'_1)\} \) such that \( I = \{t_1, t_2, t'_2\} \) is of type \( i \) and

\[ P_{t_1, v_1} \cap P_{t_2, v_2} \cap [\varrho, C_1] \times \mathbb{R}^d \neq \emptyset, \quad P_{t_1, v_1} \cap P_{t'_2, v'_2} \cap [\varrho, C_1] \times \mathbb{R}^d \neq \emptyset. \]

In addition, the members of \( E_{31} \) must satisfy

\[ u = D(t_1, t_2), \quad u' = D(t_1, t'_2), \quad \omega = D(v_1, v_2), \quad \omega' = D(v_1, v'_2), \]

with \( u = u', \quad t = D(t_2, t'_2) \) and \( \vartheta = D(v_2, v'_2) \) if \( i = 2 \). We also define the quantities \( \tilde{m}[, \omega', \vartheta] \) for \( E_{31} \) and \( \tilde{m}[\omega, \omega', \vartheta] \) for \( E_{32} \); both are expressed using the formula (9.10), where \( \varrho_1, \varrho_2 \) with \( \varrho_2 \subseteq \varrho_1 \) is a rearrangement of \( \{\omega, \omega'\} \) for \( E_{31} \) and of \( \{\omega, \omega', \vartheta\} \) for \( E_{32} \), by virtue of Lemma 9.5. With this notation in place, the size estimates on \( E_{31} \) are as follows.

**Lemma 9.10.**

(i) Set \( \Delta := \min(gp_\omega, gp_\omega') \). Then

\[ \#(E_{31}) \leq C \Delta \varrho^2 \rho_\omega \rho_\omega' 2^{3N-\tilde{m}[\omega, \omega']} M^{-(d-1)(h(\omega)+h(\omega'))+(2d+1)J}. \]

(ii) With the same definition of \( \Delta \) as in part (i),

\[ \#(E_{32}) \leq C \Delta \min(gp_\omega, M^{-h(t)}) \min(gp_\omega, M^{-h(t)}) 2^{3N-\tilde{m}[\omega, \omega', \vartheta]} M^{-2(d-1)h(t)+(2d+1)J}. \]
10 Sums over root and slope vertices

A recurrent feature of the proof of (6.9), as we will soon see in Section 11, is the use of certain sums over specific subsets of vertices in the root and slope trees. We record the outcomes of these summation procedures in this section for easy reference later.

Lemma 10.1. Fix a vertex \( \varpi_0 \in \mathcal{G}(\Omega_N) \), i.e. \( \varpi_0 \) is a splitting vertex of the slope tree. Then the following estimates hold.

(i) For any \( \alpha \in \mathbb{R} \),

\[
\sum_{\varpi \in \mathcal{G}(\Omega_N)} 2^{-\alpha \nu(\varpi)} \leq \begin{cases} 
C_\alpha 2^{-\alpha \nu(\varpi_0)} & \text{if } \alpha > 1, \\
N 2^{-\nu(\varpi_0)} & \text{if } \alpha = 1, \\
C_\alpha 2^{-\alpha \nu(\varpi_0)+N(1-\alpha)} & \text{if } \alpha < 1.
\end{cases}
\]

(ii) For \( M \geq 2, \beta > 0 \) and \( \alpha \geq 1 \),

\[
\sum_{\varpi \in \mathcal{G}(\Omega_N)} M^{-\beta h(\varpi)} 2^{-\alpha \nu(\varpi)} \leq C_{\alpha,\beta} M^{-\beta h(\varpi_0)} 2^{-\alpha \nu(\varpi_0)}.
\]

Proof. By Proposition 4.5, the number of splitting vertices descended from \( \varpi_0 \) with the property that \( \nu(\varpi) = \nu(\varpi_0) + j \) is \( 2^j \). Since \( j \) can be at most \( N \), we see that

\[
\sum_{\varpi \in \mathcal{G}(\Omega_N)} 2^{-\alpha \nu(\varpi)} \leq \sum_j 2^{-\alpha (\nu(\varpi_0)+j)} 2^j \leq 2^{-\alpha \nu(\varpi_0)} \sum_{j=1}^N 2^{j(1-\alpha)},
\]

from which part (i) follows. On the other hand, if \( \nu(\varpi) = \nu(\varpi_0) + j \), then \( h(\varpi) - h(\varpi_0) \geq \nu(\varpi) - \nu(\varpi_0) = j \). Thus, a similar computation shows that

\[
\sum_{\varpi \in \mathcal{G}(\Omega_N)} M^{-\beta h(\varpi)} 2^{-\alpha \nu(\varpi)} = \sum_j M^{-\beta (h(\varpi_0)+j)} 2^{-\alpha (\nu(\varpi_0)+j)} 2^j \leq M^{-\beta h(\varpi_0)} 2^{-\alpha \nu(\varpi_0)} \sum_{j=1}^{\infty} M^{-\beta j} 2^{-(\alpha-1)j}.
\]

The last sum in the displayed expression is convergent, establishing the desired conclusion in part (ii).

Lemma 10.2. Fix a vertex \( y \) in the root tree and a splitting vertex \( \varpi \) in the slope tree such that \( h(y) \leq h(\varpi) \). Given a constant \( \beta \), one of the following estimates holds for

\[
s(\beta) := \sum_{\varpi} M^{-\beta h(\varpi)} 2^{\nu(\varpi,h(\varpi))},
\]

where the sum \( \sum' \) takes place over all vertices \( z \) of the root tree such that \( z \subseteq y \) and \( h(z) \leq h(\varpi) \).
(i) If $\beta < d$, then $g(\beta) \leq C_\beta 2^{\nu(\varpi)} M^{(d-\beta)h(\varpi)-dh(y)}$.

(ii) If $\beta = d$, then $g(d) \leq C 2^{\nu(\varpi)} h(\varpi) M^{-dh(y)}$.

(iii) If $\beta > d$, then $g(\beta) \leq C_\beta 2^{\nu(\varpi)} M^{-\beta h(y)}$.

(iv) If $\beta > d$ is large enough so that $2M^d < M^\beta$, then $g(\beta) \leq C_\beta M^{-dh(y)}$.

Proof. Since $\varpi$ is a splitting vertex of the slope tree, there exists an integer $j \in [1,N]$ such that $\varpi \in G_j(\Omega_N)$, i.e., $\nu(\varpi) = j$. By definition, every $j$th splitting vertex is either itself a $(j-1)$th basic slope cube or is contained in one. Let $\varpi \in H_\ell(\Omega_N)$ be the $\ell$th slope cube that contains $\varpi$, so that

$$\varpi_1 \supseteq \varpi_2 \supseteq \cdots \supseteq \varpi_{j-1} \supseteq \varpi.$$ 

If $z$ is a vertex of the root tree such that $h(\varpi_{\ell-1}) \leq h(z) < h(\varpi_{\ell})$ for some $\ell \leq j-1$, then $\mu(\varpi, h(z)) = \ell - 1$; on the other hand, if $h(\varpi_{j-1}) \leq h(z) \leq h(\varpi)$, then $\mu(\varpi, h(z)) = j-1$. This suggests decomposing the sum defining $g(\beta)$ according to the heights of the slope cubes containing $\varpi$. Implementing this and recalling that $\#\{z : z \subset y, h(z) = k\} = M^{dk-h(y)}$, we obtain

$$g(\beta) = \sum_{\ell=1}^{j-1} \sum_{k=h(\varpi_{\ell-1})}^{h(\varpi)} 2^{\ell-1} \sum_{z : h(z) = k} M^{-\beta k} + \sum_{k=h(\varpi_{j-1})}^{h(\varpi)} 2^{j-1} \sum_{z : h(z) = k} M^{-\beta k}$$

$$\leq C \left[ \sum_{\ell=1}^{j} \sum_{k=h(y)}^{h(\varpi)} 2^{\ell-1} M^{-\beta k} M^{dk-h(y)} \right]$$

$$\leq CM^{-dh(y)} \sum_{\ell=1}^{j} \sum_{k=h(y)}^{h(\varpi)} 2^{\ell-1} M^{(d-\beta)k}$$

$$\leq CM^{-dh(y)} 2^j \sum_{k=h(y)}^{h(\varpi)} M^{(d-\beta)k}$$

$$\leq C 2^{\nu(\varpi)} M^{-dh(y)} \begin{cases} M^{(d-\beta)h(\varpi)} & \text{if } \beta < d, \\ h(\varpi) & \text{if } \beta = d, \\ M^{-(\beta-d)h(y)} & \text{if } \beta > d. \end{cases}$$

Upon simplification, these are the estimates claimed in parts (ii)-(iii) of the lemma. Part (iv) follows from the observation that $\mu(\varpi, h(z)) \leq h(z)$, hence

$$g(\beta) \leq \sum_{z} 2^{\nu(\varpi)} M^{-\beta h(z)} 2^{h(z)} \leq \sum_{k} 2^{k} M^{-\beta k + d(k-h(y))}$$

$$\leq M^{-dh(y)} \sum_{k} \left( \frac{2M^d}{M^\beta} \right)^k \leq C_\beta M^{-dh(y)}.$$

$\square$
In view of spatial constraints on the ancestors of root cubes as encountered in Lemmas 9.8 and 9.9, occasionally the sums that we consider take place over more restricted ranges of vertices than the one in Lemma 10.2, even though the summands may retain the same form. The next result makes this quantitatively precise. Let \( \mathcal{R} \) be a splitting vertex of the slope tree, and \( R \) a fixed parallelepiped in the root hyperplane with sidelength \( \beta \) in \((d - r)\) directions and \( \gamma \) in the remaining \( r \) directions, where \( 1 \leq r \leq d - 1 \) and \( \beta \geq \gamma \geq M^{-J} \). Given constants \( \epsilon \geq M^{-h(\infty)} \) and \( \alpha \in \mathbb{R} \), we define

\[
\mathcal{s}_\pm = \mathcal{s}_\pm(\alpha, \epsilon, \mathcal{R}, \infty) := \sum_{z \in \mathcal{Z}_\pm} M^{-\varphi(z)} \mu(\omega, h(z)),
\]

where the index sets \( \mathcal{Z}_\pm \) are collections of vertices of the root tree defined as follows:

\[
\mathcal{Z} := \{ z \subseteq \mathcal{R} : h(z) \leq h(\infty), \ M^{-h(z)} \leq \epsilon \},
\]

\[
\mathcal{Z}_+ := \mathcal{Z} \cap \{ z : M^{-h(z)} \geq \gamma \},
\]

\[
\mathcal{Z}_- := \mathcal{Z} \cap \{ z : M^{-h(z)} \leq \gamma \}.
\]

**Lemma 10.3.** The following estimates hold for \( \mathcal{s}_\pm \) defined in (10.1).

(i) If \( \alpha > d - r \) and \( \epsilon \geq \gamma \) then \( \mathcal{s}_+ \leq C2^{\varphi(\infty)} \beta^{d-r} e^{\alpha-d+r} \).

(ii) If \( \alpha > d \), then \( \mathcal{s}_- \leq C2^{\varphi(\infty)} \beta^{d-r} \gamma^r \) \( \min(\epsilon, \gamma)^{\alpha-d} \).

**Proof.** We have already established in the proof of Lemma 10.2 that \( \mu(\infty, h(z)) \leq \nu(\infty) - 1 \). Further if \( M^{-k} \geq \gamma \), then there can be at most a constant number of possible choices of \( k \)th generation \( M \)-adic cubes \( z \) that are contained in \( \mathcal{R} \) and intersect with a slice of \( \mathcal{R} \) that fixes coordinates in the \((d - r)\) long directions. Thus we only need to count the number of possible \( z \) in the long directions, obtaining

\[
\# \{ z \in \mathcal{Q}(k) : z \subseteq \mathcal{R} \} \leq C_r \beta^{d-r} M^{(d-r)k}.
\]

Taking this into account, we obtain

\[
\mathcal{s}_+ \leq 2^{\varphi(\infty)} \sum_{k : \gamma \leq M^{-k} \leq \epsilon} M^{-ak} \beta^{d-r} M^{(d-r)k} \leq C2^{\varphi(\infty)} \beta^{d-r} \epsilon^{\alpha-d+r},
\]

as claimed in part (i). Part (ii) follows in an identical manner; the only difference is that now all directions of \( \mathcal{R} \) are thick relative to the scale of \( z \), hence (10.2) has to be replaced by

\[
\# \{ z \in \mathcal{Q}(k) : z \subseteq \mathcal{R} \} \leq C \gamma^r \beta^{d-r} M^{dk}.
\]
11 Proof of the lower bound \((6.9)\)

We are now in a position to complete the proof of Proposition \((5.2)\) by verifying the probabilistic statement on the lower bound of \(K_N(\mathcal{X})\) claimed in \((6.9)\). The two propositions stated below are the main results of this section and allow passage to this final step.

**Proposition 11.1.** Fix integers \(N\) and \(R\) with \(N \gg M\) and \(10 \leq R \leq \frac{1}{10} \log M N\).

Define

\[
P^*_t,\sigma,R := P_{t,\sigma(t)} \cap [M^{-R}, M^{-R+1}] \times \mathbb{R}^d,
\]

where \(\sigma = \sigma_\mathcal{X}\) is the randomized sticky map described in Section 6. Then there exists a constant \(C = C(M, d) > 0\) such that

\[
\mathbb{E}_\mathcal{X} \left[ \sum_{t_1 \neq t_2} |P^*_{t_1,\sigma,R} \cap P^*_{t_2,\sigma,R}| \right] \leq CNM^{-2R}.
\]

(11.1)

**Proposition 11.2.** Under the same hypotheses as Proposition 11.1,

\[
\mathbb{E}_\mathcal{X} \left[ \left( \sum_{t_1 \neq t_2} |P^*_{t_1,\sigma,R} \cap P^*_{t_2,\sigma,R}| \right)^2 \right] \leq C (NM^{-2R})^2.
\]

(11.2)

Propositions 11.1 and 11.2 should be viewed as the direct generalizations of \([16]\), Propositions 8.2 and 8.3] for arbitrary direction sets. These are proved below in Sections 11.1 and 11.2 respectively. Of the two results, Proposition 11.2 is of direct interest, since it leads to \((6.9)\), as we will see momentarily in Corollary 11.3. Proposition 11.1, while not strictly speaking relevant to \((6.9)\), nevertheless provides a context for presenting the core arguments within a simpler framework.

**Corollary 11.3.** Proposition 11.2 implies \((6.9)\).

*Proof.* The argument here is identical to \([16]\) Corollary 8.4], and is briefly sketched. The estimate \((11.2)\) implies that for any fixed integer \(R \in [10, \frac{1}{10} \log N]\), the event

\[
\sum_{t_1 \neq t_2} |P^*_{t_1,\sigma,R} \cap P^*_{t_2,\sigma,R}| > CN \sqrt{\log NM^{-2R}}
\]

(11.3)

holds with probability at most \((C \log N)^{-1}\), by Markov’s inequality. Choosing a constant \(c > 0\) sufficiently small, one can ensure that the probability of occurrence of the event \((11.3)\) for some \(R \in [c \log N, 2c \log N]\) cannot exceed \(\frac{1}{10}\). Thus for an appropriate choice of small but positive \(c\), the revised estimate

\[
\sum_{t_1, t_2} |P^*_{t_1,\sigma,R} \cap P^*_{t_2,\sigma,R}| \leq CN \sqrt{\log NM^{-2R}} + C M^{-R} M^{-dJ} M^{dJ}
\]

\[
\leq CN \sqrt{\log NM^{-2R}}
\]

holds with probability at most \((C \log N)^{-1}\). This completes the proof of Proposition 11.2.
We first recast the sum on the left hand side of (11.1) and obtain $R$ available with high probability for all essentially disjoint, since they lie in disjoint horizontal strips. Since these bounds are the intersection criterion in (11.1) where $\sum_{t} = \emptyset$, $\sum_{t} = \emptyset$. Unravelling the implications of the intersection we find that:

$$|K_N(X) \cap [0, 1] \times \mathbb{R}^d| \geq \sum_{R = c \log N} \left| \bigcup_{t} P^*_{t, \sigma, R} \right| \geq C^{-1} \frac{1}{N \sqrt{\log N}}.$$ 

which is the statement (11.3). \hfill \square

### 11.1 Proof of Proposition 11.1

Proof. We first recast the sum on the left hand side of (11.1) in a form that brings into focus its connections with the material in Sections 8 and 9. By Lemma 5.3:

$$\sum_{t_1 \neq t_2} \left| P^*_{t_1, \sigma, R} \cap P^*_{t_2, \sigma, R} \right| \leq \sum_{1} \frac{C_dM^{-(d+1)J}}{2|\sigma(t_1) - \sigma(t_2)|} + M^{-J}, \quad (11.4)$$

where $\sum_{1}$ denotes the sum over all root pairs $(t_1, t_2)$ such that $t_1 \neq t_2$ and $P^*_{t_1, \sigma, R} \cap P^*_{t_2, \sigma, R} \neq \emptyset$. Unravelling the implications of the intersection we find that:

$$\left\{ (t_1, t_2) : t_1 \neq t_2, P^*_{t_1, \sigma, R} \cap P^*_{t_2, \sigma, R} \neq \emptyset \right\} \subseteq \left\{ (t_1, t_2) \left\| \exists \text{ a unique pair } (v_1, v_2) \in \Omega_2^2 \text{ such that} \right. \right. \left. \left. P_{t_1, v_1} \cap P_{t_2, v_2} \cap [M^{-R}, M^{-R+1}] \times \mathbb{R}^d \neq \emptyset \right\}.$$

(11.5)

For a given root pair $(t_1, t_2)$, there may exist more than one slope pair $(v_1, v_2)$ that meets the intersection criterion in (11.5). But only one pair will also satisfy, for a given $\sigma$, the requirement $\sigma(t_1) = v_1, \sigma(t_2) = v_2$, which explains the uniqueness claim in (11.5). Using this, the expression on the right hand side of (11.4) can be expanded as follows:

$$\sum_{t_1 \neq t_2} \left| P^*_{t_1, \sigma, R} \cap P^*_{t_2, \sigma, R} \right| \leq \sum_{1} \frac{C_dM^{-(d+1)J}}{|\sigma(t_1) - \sigma(t_2)|}$$

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\[
\begin{align*}
&\leq \sum_{2} \frac{C_d M^{-(d+1)J}}{|v_1 - v_2|} T((t_1, v_1), (t_2, v_2)) \\
&\leq \sum_{u, \omega} \frac{C_d M^{-(d+1)J}}{\delta_\omega} \sum \delta \sum T((t_1, v_1), (t_2, v_2)), 
\end{align*}
\]

where the notation \(\sum_2\) in the second step denotes summation over the collection in (11.5), and \(T((t_1, v_1), (t_2, v_2))\) is a binary (random) counter given by

\[
T((t_1, v_1), (t_2, v_2)) = \begin{cases} 
1 & \text{if } \sigma(t_1) = v_1 \text{ and } \sigma(t_2) = v_2, \\
0 & \text{otherwise.}
\end{cases}
\]

In the last step (11.6) of the string of inequalities above, we have rearranged the sum in terms of the youngest common ancestors \(u = D(t_1, t_2)\) and \(\omega = D(v_1, v_2)\) in the root tree and in the slope tree respectively. The summation \(\sum_3\) takes place over all sticky-admissible tube pairs \((t_1, v_1), (t_2, v_2)\) in the deterministic collection \(E_2[u, \omega; \vartheta]\) defined in (9.1), with \(\varrho = \varrho_R = M^{-R}\) and \(C_1 = M\). Incidentally, the requirement of sticky-admissibility restricts \(u\) and \(\omega\) to obey the height relation \(h(u) \leq h(\omega)\). The quantity \(\delta_\omega\) has been defined in (11.8), and is therefore \(\leq |v_1 - v_2|\).

With this preliminary simplification out of the way, we proceed to compute the expected value of the expression in (11.6), combining the geometric facts and counting arguments from Section 9 with appropriate probability estimates from Section 8. Accordingly, we get

\[
\begin{align*}
\mathbb{E}_X \left[ \sum_{t_1 \neq t_2} \left| P^*_{t_1, \sigma, R} \cap P^*_{t_2, \sigma, R} \right| \right] &\leq \sum_{u, \omega} \frac{C_d M^{-(d+1)J}}{\delta_\omega} \sum_{E_2[u, \omega; \vartheta]} \mathbb{E}_X[T((t_1, v_1), (t_2, v_2))] \\
&\leq \sum_{u, \omega} \frac{C_d M^{-(d+1)J}}{C_2 \rho_\omega} \sum_{E_2[u, \omega; \vartheta]} \Pr(\sigma(t_1) = v_1, \sigma(t_2) = v_2) \\
&\leq CM^{-(d+1)J} \sum_{u, \omega} \rho^{-1}_\omega \#(E_2[u, \omega; \vartheta]) \left( \frac{1}{2} \right)^{2N - \mu(\omega, h(u))} \quad \text{Lemma 6.3} \\
&\leq CM^{-(d+1)J} \sum_{u, \omega} \rho^{-1}_\omega \left( \varrho \rho_\omega \right)^{2(2N - \nu(\omega))} M^{-(d-1)h(\omega) + (d+1)J} \times \left( \frac{1}{2} \right)^{2N - \mu(\omega, h(u))} \quad \text{Lemma 9.3} \\
&\leq CM^{2R} \sum_{u, \omega} M^{-h(\omega) - (d-1)h(\omega) + 2\mu(\omega, h(u)) - 2\nu(\omega)},
\end{align*}
\]

where the last step uses the fact that \(\rho_\omega \leq \text{diam}(\omega) = \sqrt{d} M^{-h(\omega)}\). To establish the conclusion claimed in (11.1), it remains to show that the last expression in the displayed
steps above is bounded by $CN$. This follows from a judicious use of the summation results proved in Section 10, namely,

$$\sum_{u,\omega} M^{-h(\omega)-(d-1)h(u)}2^{2\nu(\omega)-2\nu(\omega)} = \sum_{u,\omega} M^{-h(\omega)}2^{-2\nu(\omega)} \sum_{u} M^{-(d-1)h(u)}2^{\nu(\omega)}$$

$$\leq C \sum_{u,\omega} M^{-h(\omega)}2^{-2\nu(\omega)} \left[2^{\nu(\omega)} M^{h(\omega)}\right]$$

$$\leq C \sum_{\omega \in \mathcal{G}} 2^{-\nu(\omega)} \leq CN,$$

where the second and last steps are consequences, respectively, of Lemma 10.2 with $\omega = \omega$ and $\beta = d - 1$ and of Lemma 10.1 with $\alpha = 1$. In both applications, $y$ and $\omega_0$ have been chosen to be the unit cube, in the root tree and the slope tree respectively. 

11.2 Proof of Proposition 11.2

We are now ready to prove the main Proposition 11.2.

**Proof.** As in the proof of Proposition 11.1, an initial processing of the sum on the left hand side of (11.2) is needed before embarking on the evaluation of the expectation. Accordingly, we decompose and simplify the quantity of interest as follows,

$$\left[\sum_{t_1 \neq t_2} |P_{t_1,\sigma,R}^* \cap P_{t_2,\sigma,R}^*| \right]^2 = \sum_{t_1 \neq t_2} |P_{t_1,\sigma,R}^* \cap P_{t_2,\sigma,R}^*| \times |P_{t_1',\sigma,R}^* \cap P_{t_2',\sigma,R}^*|$$

$$= \mathcal{G}_2 + \mathcal{G}_3 + \mathcal{G}_4,$$

where for $i = 2, 3, 4$,

$$\mathcal{G}_i := \sum_{\mathcal{J}_i} |P_{t_1,\sigma,R}^* \cap P_{t_2,\sigma,R}^*| \times |P_{t_1',\sigma,R}^* \cap P_{t_2',\sigma,R}^*|,$$

and

$$\mathcal{J}_i := \left\{ I = \{(t_1, t_2); (t_1', t_2')\} \mid t_1, t_2, t_1', t_2' \in \mathcal{Q}(J), t_1 \neq t_2, t_1' \neq t_2' \right\}.$$

Without loss of generality, by interchanging the pairs $(t_1, t_2)$ and $(t_1', t_2')$ if necessary, we may assume that $h(D(t_1, t_2)) \leq h(D(t_1', t_2'))$ for all quadruples $I \in \mathcal{J}_i$. We will continue to make this assumption for the treatment of all the terms $\mathcal{G}_i$.

The claimed inequality in (11.2) is a consequence of the three main estimates below:

$$\mathbb{E}_X(\mathcal{G}_2) \leq CNM^{-2R-dJ},$$

$$\mathbb{E}_X(\mathcal{G}_3) \leq CNM^{-3R-J},$$

$$\mathbb{E}_X(\mathcal{G}_4) \leq CN^2M^{-4R}.$$ (11.10)

We will prove (11.10) in full detail, since this clearly makes the primary contribution among the three terms mentioned above. The other two estimates follow analogous and
in fact simpler routes using the machinery developed in Sections 8 and 9. We leave their verification to the reader.

The configuration type of the quadruple \( I = \{ (t_1, t_2); (t'_1, t'_2) \} \) of distinct roots, as introduced in Section 8.3.3, plays a decisive role in the estimation of (11.10). Recalling the type definitions from that section, we decompose \( I \) as

\[
I = \bigcup_{i=1}^{3} I_{4i} \quad \text{where} \quad I_{4i} := \left\{ \bar{I} \in I \mid \bar{I} \text{ is of type } i \text{ in the sense of Definition 8.3.1} \right\}.
\]

This results in a corresponding decomposition of \( S_4 \):

\[
S_4 = S_{41} + S_{42} + S_{43}, \quad \text{where} \quad S_{4i} = \sum_{I_{4i}} |P_{t_1, \sigma, R}^{*} \cap P_{t_2, \sigma, R}^{*}| \times |P_{t'_1, \sigma, R}^{*} \cap P_{t'_2, \sigma, R}^{*}|.
\]

We will prove in Sections 11.2.1, 11.2.3 below that

\[
\mathbb{E}_{\bar{X}}[S_{4i}] \leq CN^2M^{-4R} \quad \text{for } i = 1, 2, 3.
\]

11.2.1 Expected value of \( S_{41} \)

We start with \( S_{41} \), simplifying it initially along the same lines as in Proposition 11.1. As before, a summand in \( S_{41} \) is nonzero if and only if the tuple \( \{ (t_1, t_2); (t'_1, t'_2) \} \) lies in the set

\[
\left\{ \{ (t_1, t_2); (t'_1, t'_2) \} \in I_{41} \mid P_{t_1, \sigma, R}^{*} \cap P_{t_2, \sigma, R}^{*} \neq \emptyset \right\}.
\]

which in turn is contained in

\[
\left\{ \{ (t_1, t_2); (t'_1, t'_2) \} \in I_{41} \mid \exists \text{ a unique tuple } (v_1, v_2, v'_1, v'_2) \in \Omega_N \ni
\begin{align*}
& P_{t_1, v_1} \cap P_{t_2, v_2} \cap [M^{-R}, M^{-R+1}] \times \mathbb{R}^d \neq \emptyset, \\
& P_{t'_1, v'_1} \cap P_{t'_2, v'_2} \cap [M^{-R}, M^{-R+1}] \times \mathbb{R}^d \neq \emptyset, \\
& \sigma(t_1) = v_1, \sigma(t'_1) = v'_1, \quad i = 1, 2
\end{align*}
\right\}.
\]

Incorporating this information into the simplification of the sum, we obtain

\[
S_{41} \leq \sum_{I_{41}} \frac{C_d M^{-(d+1)J}}{|\sigma(t_1) - \sigma(t_2)|} \times \frac{C_d M^{-(d+1)J}}{|\sigma(t'_1) - \sigma(t'_2)|}
\]

\[
\leq CM^{-2(d+1)J} \sum_{v_1, v_2} T((t_1, v_1), (t_2, v_2)) \times T((t'_1, v'_1), (t'_2, v'_2))
\]

\[
\leq CM^{-2(d+1)J} \sum_{v_1, v_2} \frac{1}{|v_1 - v_2|} \sum_{v'_1, v'_2} T((t_1, v_1), (t_2, v_2)) T((t'_1, v'_1), (t'_2, v'_2)).
\]
where the summations $\sum_1$ and $\sum_2$ range over the root quadruples in (11.12) and (11.13) respectively. The notation $T((t_1, v_1), (t_2, v_2))$ and $\delta_\omega$ represent the same quantities as they did in Proposition 11.11, with their definitions in (11.4) and (11.8) respectively. Following the same reasoning that led to (11.5), in the last step we have stratified the sum in terms of the root vertices $u = D(t_1, t_2)$, $u' = D(t'_1, t'_2)$, $z = D(u, u')$ and the (splitting) slope vertices $\omega = D(v_1, v_2)$, $\omega' = D(v'_1, v'_2)$, $v = D(\omega, \omega')$, so that the summation $\sum_3$ takes place over the tube tuples in the collection $E_{41} = E_{41}[u, u', z; \omega, \omega, v; \varrho]$ defined in (9.11), with $\varrho = M^{-R}$, $C_1 = M$. We are now in a position to compute the expected value of $\mathfrak{S}_{41}$.

**Lemma 11.4.** The estimate in (11.11) holds for $i = 1$.

**Proof.** Let us refer to the bound $\mathfrak{S}_{41}$ on $\mathfrak{S}_{41}$ defined by (11.14) that we obtained from the preliminary simplification. Assembling the various components of the estimation from the previous sections, the expected value of $\mathfrak{S}_{41}$ is estimated as follows,

\[
E_X(\mathfrak{S}_{41}) \leq E_X(\mathfrak{S}_{41})
\]

\[
\leq CM^{-2(d+1)J} \sum_{u, u', z} \frac{1}{\rho_\omega \rho_{\omega'}} \sum_3 \Pr(\sigma(t_1) = v_1, \sigma(t'_1) = v'_1, i = 1, 2)
\]

\[
\leq CM^{-2(d+1)J} \sum_{u, u', z} \#(E_{41}) \left(1 + 2^{4N-2\mu(\omega, h(u)) - \mu(\omega', h(u')) - \mu(v, h(z))}
\right)
\]

\[
\leq CM^{-2(d+1)J} \sum_{u, u', z} \left(2^{2\mu(\omega, h(u)) + \mu(\omega', h(u')) + \mu(v, h(z))}
\right)
\]

\[
\leq CM^{-4R} \mathfrak{S}_{41}^*, \text{ where}
\]

\[
\mathfrak{S}_{41}^* := \sum_{\omega, \omega', v} 2^{-(d-1)(h(u) + h(u'))} M^{-[h(u) + h(u')]} \sum_{u, u', z} 2\mu(\omega, h(u)) + \mu(\omega', h(u')) + \mu(v, h(z))
\]

It remains to use the appropriate summation results in Section 10 to show that $\mathfrak{S}_{41}^*$ is bounded above by a constant multiple of $N^2$. We start with the inner sum.

\[
\sum_{u, u', z} M^{-(d-1)(h(u) + h(u'))} 2\mu(\omega, h(u)) + \mu(\omega', h(u')) + \mu(v, h(z))
\]

\[
\leq \sum_{z} 2\mu(v, h(z)) \left[ \sum_{u, u', z} M^{-(d-1)(h(u) + h(u'))} \right] \left[ \sum_{u, u', z} M^{-(d-1)(h(u) + h(u'))} \right]
\]

apply Lemma 10.24, $\beta = d - 1$ apply the same lemma again

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\[
\begin{align*}
&\leq \sum_{z} 2^\mu(v, h(z)) \left[ M^{-dh(z)+h(\omega)} 2^\nu(\omega) M^{-dh(z)+h(\omega')} 2^\nu(\omega') \right] \\
&\leq CM^{h(\omega)+h(\omega')} 2^\nu(\omega)+\nu(\omega') \left[ \sum_{z} 2^\mu(v, h(z)) M^{-2dh(z)} \right] \\
&\leq CM^{h(\omega)+h(\omega')} 2^\nu(\omega)+\nu(\omega'). \quad (11.16)
\end{align*}
\]

Note that Lemma 11.25 applies with \( \beta = 2d \) since \( 2M^d < M^{2d} \) for \( M \geq 2 \) and \( d > 2 \).

Inserting the expression in (11.16) into the inner sum of (11.15), we proceed to complete the outer sum in \( \mathcal{S}_{41} \).

\[
\mathcal{S}_{41} \leq C \sum_{\omega, \omega', v \in \mathcal{G}} M^{-h(\omega)+h(\omega')} 2^{-2(\nu(\omega)+\nu(\omega'))} M^{h(\omega)+h(\omega')} 2^\nu(\omega)+\nu(\omega') \\
\leq C \sum_{\omega, \omega', v \in \mathcal{G}} 2^{-\nu(\omega)-\nu(\omega')} \\
\leq C \sum_{v \in \mathcal{G}} \left[ \sum_{\omega \in \mathcal{G}, \omega \subseteq v} 2^{-\nu(\omega)} \right] \times \left[ \sum_{\omega' \in \mathcal{G}, \omega' \subseteq v} 2^{-\nu(\omega')} \right] \\
\leq C \sum_{v \in \mathcal{G}} [N2^{-\nu(v)}]^2 \leq CN^2 \sum_{v \in \mathcal{G}} 2^{-2\nu(v)} \leq CN^2,
\]

where at the last step we have again used Lemma 11.16 with \( \alpha = 2 \), and \( \nu(\omega_0) = 0 \).

This completes the proof of the lemma.

### 11.2.2 Expected value of \( \mathcal{S}_{42} \)

We turn to \( \mathcal{S}_{42} \) next. After the usual preliminary simplification similar to that of \( \mathcal{S}_{41} \), we find that \( \mathcal{S}_{42} \) is bounded by a sum \( \mathcal{S}_{42} \) of the form (11.14), where

\[
\mathcal{S}_{42} := CM^{-2(d+1)d} \sum_{\delta \omega, \delta \omega'} 1 \sum_{3} T((t_1, v_1), (t_2, v_2)) T((t_1', v_1'), (t_2', v_2')). \quad (11.17)
\]

In view of Lemma 11.9 we may assume, after a permutation of \( (t_1, t_2) \) and of \( (t_1', t_2') \) if necessary, that the outer sum \( \sum' \) in (11.17) is over all vertex tuples \( (u, u', t) \) and \( (\omega, \omega', \theta) \) in the root tree and the slope tree respectively, such that \( u, u', t \) lies on a single ray with \( u' \subseteq u \), while \( \omega, \omega', \theta \in \mathcal{G}(\Omega_N) \), \( \omega \cap \theta = \emptyset \), \( \omega' \cap \theta = \emptyset \). The inner sum \( \sum_{3} \) in \( \mathcal{S}_{42} \) ranges over the collection \( \mathcal{E}_{42} = \mathcal{E}_{42}[u, u', t; \omega, \omega', \theta; \theta] \) defined in (9.12) with the usual \( \theta = M^{-R} \) and \( C_1 = M \).

**Lemma 11.5.** The estimate in (11.11) holds for \( i = 2 \).

**Proof.** As in Lemma 11.4 the evaluation of the expectation requires a combination of the appropriate probabilistic estimate from Section 8.2 (specifically Lemma 8.9), size 87
estimate of $\mathcal{E}_{42}$ from Section 9.3.2 (specifically Lemma 9.8) and the summation results from Section 10. Putting these together, we obtain

$$
\mathbb{E}_X(\mathcal{G}_{42}) \leq \mathbb{E}_X(\mathcal{G}_{42})
$$

$$
\leq CM^{-2(d+1)J} \sum_{\rho_\omega \rho_{\omega'}} \sum_{i=1,2} \Pr(\sigma(t_i) = v_i, \sigma(t'_i) = v'_i, i = 1, 2)
$$

$$
\leq CM^{-2(d+1)J} \frac{\#(\mathcal{E}_{42})}{\rho_\omega \rho_{\omega'}} \left( \frac{1}{2} \right)^{4N - \mu(\omega, h(u)) - \mu(\omega', h(u')) - \mu(\vartheta, h(t))}
$$

$$
\leq CM^{-4R}[\mathcal{G}_{42}^* + \mathcal{G}_{42}^0],
$$

where the closed form expressions for $\mathcal{G}_{42}^*$ and $\mathcal{G}_{42}^0$ at the last step are obtained from the count on the size of $\mathcal{E}_{42}$ from Lemma 9.8 and reflect the two complementary cases considered therein. To be precise,

$$
\mathcal{G}_{42}^* := g^{-1} \sum_{u' \subseteq t \subseteq u} \rho_\omega \rho_{\omega'} \min[\varrho_\rho_\omega, M^{-h(t)}] M^{-(d-1)\left(h(t) + h(u')\right)}
$$

$$
\times 2^{\mu(\omega, h(u)) + \mu(\omega', h(u')) + \mu(\vartheta, h(t)) - m[\omega, \omega', \vartheta]}, and
$$

$$
\mathcal{G}_{42}^0 := g^{-2} \sum_{t \subseteq u' \subseteq u} \min[\varrho_\rho_\omega, M^{-h(t)}] \min[\varrho_\rho_{\omega'}, M^{-h(t')}][M^{-2(d-1)h(t)}]
$$

$$
\times 2^{\mu(\omega, h(u)) + \mu(\omega', h(u')) + \mu(\vartheta, h(t)) - m[\omega, \omega', \vartheta]},
$$

where the notation $\sum_{\rho}$ indicates the summation of $\sum_{\rho}'$ subject to the additional requirement $P$. These two quantities are estimated via the usual channels. Lemma 9.8 places certain restrictions on the spatial location of $t$, but for a large part of the proof the full strength of these statements will not be needed. For instance, replacing $\min(\varrho_\rho_\omega, M^{-h(t)})$ in (11.18) by $\varrho_\rho_\omega$, we arrive at the following bound for $\mathcal{G}_{42}^*$:

$$
\mathcal{G}_{42}^* \leq \sum_{u' \subseteq t \subseteq u} \rho_\omega \rho_{\omega'} M^{-(d-1)(h(t) + h(u'))} 2^{\mu(\omega, h(u)) + \mu(\omega', h(u')) + \mu(\vartheta, h(t)) - m[\omega, \omega', \vartheta]}
$$

$$
\leq \sum_{\omega, \omega', \vartheta} \rho_\omega \rho_{\omega'} 2^{-m[\omega, \omega', \vartheta]} \mathcal{G}_{42}^0 (\text{inner}), \quad (11.20)
$$

where the inner expression $\mathcal{G}_{42}^0 (\text{inner})$ is a sequence of three summations in root vertices, the computation of each requiring a suitable form of Lemma 10.2. Precisely,

$$
\mathcal{G}_{42}^0 (\text{inner}) := \sum_{u, u', t, u' \subseteq t \subseteq u} M^{-(d-1)\left(h(t) + h(u')\right)} 2^{\mu(\omega, h(u)) + \mu(\omega', h(u')) + \mu(\vartheta, h(t))}
$$

$$
= \sum_{u \subseteq t} M^{-(d-1)h(t)} 2^{\mu(\omega, h(u)) + \mu(\vartheta, h(t))} \left[ \sum_{u' \subseteq t} M^{-(d-1)h(u')} 2^{\mu(\omega', h(u'))} \right].
$$

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\[ \leq C \sum_{(t,u) \subseteq u} M^{-(d-1)h(t)2\mu(\omega,h(u))+\mu(\bar{\vartheta},h(t))} \left[ 2^{\nu(\omega')} M^{-dh(t)+h(\omega')} \right] \]

from Lemma [10.23], \( \beta = d - 1 \)

\[ \leq C 2^{\nu(\omega')} M^{h(\omega')} \sum_u 2\mu(\omega,h(u)) \left[ M^{-(2d-1)h(t)2\nu(\bar{\vartheta})} \right] \]

from Lemma [10.23], \( \beta = 2d - 1 \)

\[ \leq C 2^{\nu(\omega')} M^{h(\omega')} \sum_u 2\mu(\omega,h(u)) M^{-(2d-1)h(u)} \]

\[ \leq C 2^{\nu(\omega')} + \nu(\bar{\vartheta}) + \nu(\omega) M^{h(\omega')}, \quad (11.21) \]

where the summation in \( u \) in the last step also follows from Lemma [10.23], since \( \beta = 2d - 1 > d \). Inserting the estimate (11.21) of \( \Theta_{42}^s \) (inner) into (11.20), we proceed to simplify the outer sum. Let us recall from Lemma [9.7] that \( \{\omega, \omega', \bar{\vartheta}\} \) can be rearranged as \( \{\omega_1, \omega_2, \omega_3\} \) satisfying (9.8), and that \( m[\omega, \omega', \bar{\vartheta}] \) is defined as in (9.9). Since the definition of \( m \) involves two possibilities, we write \( \sum^a \) and \( \sum^b \) to denote the sum over vertex triples \( (\omega, \omega', \bar{\vartheta}) \) for which \( \omega_3 \not\subseteq \omega_2 \) and \( \omega_3 \subseteq \omega_2 \) respectively. This means that

\[ \Theta_{42}^s \leq C \sum_{\omega,\omega',\bar{\vartheta}} \rho_{\omega} \rho_{\omega'} 2^{-m[\omega,\omega',\bar{\vartheta}]} \left[ 2^{\nu(\omega')} + \nu(\bar{\vartheta}) + \nu(\omega) M^{h(\omega')} \right] \]

\[ \leq C \left[ \sum^a + \sum^b \right] \rho_{\omega} \rho_{\omega'} 2^{-m[\omega,\omega',\bar{\vartheta}]} \left[ 2^{\nu(\omega')} + \nu(\bar{\vartheta}) + \nu(\omega) M^{h(\omega')} \right]. \]

Using the trivial bounds

\[ \rho_{\omega'} M^{h(\omega')} \leq C \quad \text{and} \quad \rho_{\omega} \leq C M^{-h(\omega)} \leq C M^{-h(\omega_1)}, \]

the estimation is completed as follows,

\[ \sum^a \rho_{\omega} \rho_{\omega'} 2^{-m[\omega,\omega',\bar{\vartheta}]} \left[ 2^{\nu(\omega')} + \nu(\bar{\vartheta}) + \nu(\omega) M^{h(\omega')} \right] \]

\[ \leq C \sum_{\omega_1} M^{-h(\omega_1)} 2^{\nu(\omega_1)} \left[ \sum_{\omega_2 \subseteq \omega_1} 2^{-\nu(\omega_2)} \right] \times \left[ \sum_{\omega_3 \subseteq \omega_1} 2^{-\nu(\omega_3)} \right] \]

\[ \leq C \sum_{\omega_1} M^{-h(\omega_1)} 2^{\nu(\omega_1)} \left( N2^{-\nu(\omega_1)} \right)^2 \]

\[ \text{Lemma [10.31] twice} \]

\[ \leq CN^2 \sum_{\omega_1} M^{-h(\omega_1)} 2^{-\nu(\omega_1)} \leq CN^2. \]

The same bound holds for \( \sum^b \), and is proved along similar lines:

\[ \sum^b \rho_{\omega} \rho_{\omega'} 2^{-m[\omega,\omega',\bar{\vartheta}]} \left[ 2^{\nu(\omega')} + \nu(\bar{\vartheta}) + \nu(\omega) M^{h(\omega')} \right] \]

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This completes the estimation of \( \mathcal{S}_{42}^* \).

We briefly remark on the analysis of \( \mathcal{S}_{42}^0 \). For \( d \geq 3 \), replacing the minima in \( \mathcal{S}_{42}^* \) by the trivial bounds \( \varrho_{\omega, \omega'} \) results in an expression analogous to that of \( \mathcal{S}_{42}^* \):

\[
\mathcal{S}_{42}^0 \leq \sum_{t \leq u' \subseteq u} 2^{\mu(\omega, h(u)) + \mu(\omega', h(u')) + \mu(\vartheta, h(t)) - m[\omega, \omega', \vartheta]} \times \begin{cases} 
\varrho_{\omega, \omega'} M^{-2h(t)} & \text{if } M^{-h(t)} \geq \varrho_{\omega, \omega'}, \\
\varrho_{\omega} M^{-3h(t)} & \text{if } M^{-h(t)} < \varrho_{\omega, \omega'}. 
\end{cases}
\]  

\[ (11.22) \]

Further, Lemma \( \ref{9.8} \) prescribes that \( t \) cannot be arbitrarily placed inside \( u' \), but must lie within the union of at most 2M thin rectangles of dimension \( \varrho_{\omega, \omega'} \times M^{-h(u')} \) each. Using this information, we sum the expression \( (11.22) \) in \( t \) as follows: if \( \sum_1 \) and \( \sum_2 \) denote the summations in \( t \) with \( t \subseteq u' \) and \( \mathcal{E}_{42} \neq \emptyset \) subject to the conditions \( M^{-h(t)} \geq \varrho_{\omega, \omega'} \) and \( M^{-h(t)} < \varrho_{\omega, \omega'} \) respectively, then

\[
\rho_{\omega, \omega'} \sum_1 M^{-2h(t)} 2^{\mu(\vartheta, h(t))} + \varrho^{-1} \rho_{\omega} \sum_2 M^{-3h(t)} 2^{\mu(\vartheta, h(t))} \leq C \rho_{\omega, \omega'} 2^{\mu(\vartheta)} M^{-2h(u')} + C \varrho^{-1} \rho_{\omega} 2^{\mu(\vartheta)} M^{-h(u')} \min[\varrho_{\omega, \omega'}, M^{-h(u')}],
\]

where both sums have been evaluated using Lemma \( \ref{10.3} \) with \( d = 2, r = 1, \varpi = \vartheta, \beta = \epsilon = M^{-h(u')} \) and \( \gamma = \varrho_{\omega, \omega'} \). In particular, \( \sum_1 \) appeals to part \( \text{ii} \) of this lemma with
We pause for a moment to explain these steps. In the first application of Lemma 10.2 in (11.24) above we have used, in addition to $h(u') \leq h(\omega')$, the fact that

$$h(u') = h(D(t_1', t_2')) \leq h(t) = h(D(t_2', t_2')) \leq h(D(v_2, v_2')) = h(\vartheta),$$

which is a consequence of stickiness. Since one of $\omega'$ and $\vartheta$ is contained in the other, this implies that $\mu(\omega', h(u')) = \mu(\vartheta, h(u'))$. Hence Lemma 10.2, applied once with $\omega = \omega'$ and again with $\vartheta = \vartheta$, yields

$$\sum_{u' \leq u} M^{-2h(u')} 2^{\omega'(u')} \leq CM^{-2h(u)} [2^{\mu(\vartheta)} h(\vartheta), 2^{\mu(\omega')} h(\omega')]$$

$$\leq C h(\omega') 2^{\omega'} M^{-2h(u)}.$$
11.2.3 Expected value of $\mathcal{G}_{43}$

**Lemma 11.6.** The estimate in (11.11) holds for $i = 3$.

**Proof.** After the usual initial processing of $\mathcal{G}_{43}$ which we omit, we reduce to the following estimate:

$$
\mathbb{E}_{\mathcal{X}} (\mathcal{G}_{43}) \leq CM^{-2(d+1)J} \sum_{\rho_\omega \rho_{\omega'}} \frac{\#(\mathcal{E}_{43})}{\rho_\omega \rho_{\omega'}} \left(\frac{1}{2}\right)^{4N-\mu(\omega, h(u)) - \mu(\theta_1, h(s_1)) - \mu(\theta_2, h(s_2))}
$$

$$
\leq C \sum_{\rho_\omega \rho_{\omega'}} \left(\rho_\omega \rho_{\omega'}\right)^{-1} \left(\rho_{\omega'}\right)^2 M^{-2(d-1)h(s_2)} \prod_{i=1}^2 \left[ \min[\rho_{\omega'}, M^{-h(s_1)}] \right]
$$

$$
\times 2^{-m[\omega, \omega', \theta_1, \theta_2] + \mu(\omega, h(u)) + \mu(\theta_1, h(s_1)) + \mu(\theta_2, h(s_2))}
$$

$$
\leq CM^{-4R[\mathcal{G}_{43}^* + \mathcal{G}_{43}^0]},
$$

where $\sum'$ denotes the sum over all tuples $(u, s_1, s_2)$ in the root tree and $(\omega, \omega', \theta_1, \theta_2)$ in the slope tree such that $s_1, s_2 \subseteq u$, $h(u) \leq h(s_1) \leq h(s_2)$, $\rho_\omega \leq \rho_{\omega'}$ and for which $\mathcal{E}_{43}$ is nonempty. The second inequality displayed above uses the estimate on $\#(\mathcal{E}_{43})$ obtained in Lemma 9.3 with an additional simplification resulting from $\min[\rho_{\omega'}, M^{-h(s_1)}] \leq \rho_{\omega'}$. The quantities $\mathcal{G}_{43}^*$ and $\mathcal{G}_{43}^0$ refer to the subsum of $\sum'$ under the additional constraints of $M^{-h(s_1)} \geq \rho_{\omega'}$ and $M^{-h(s_1)} < \rho_{\omega'}$ respectively. Thus

$$
\mathcal{G}_{43}^* = \varrho^{-1} \sum_{M^{-h(s_1)} \geq \rho_{\omega'}} \rho_{\omega'} \min[\rho_{\omega'}, M^{-h(s_2)}] M^{-2(d-1)h(s_2)}
$$

$$
\times 2^{-m[\omega, \omega', \theta_1, \theta_2] + \mu(\omega, h(u)) + \mu(\theta_1, h(s_1)) + \mu(\theta_2, h(s_2))},
$$

$$
= \varrho^{-1} \sum_{\omega, \omega', \theta_1, \theta_2} \rho_{\omega'} 2^{-m[\omega, \omega', \theta_1, \theta_2]} \mathcal{G}_{43}^*(\text{inner}), \quad \text{and} \quad (11.27)
$$

$$
\mathcal{G}_{43}^0 = \varrho^{-2} \sum_{M^{-h(s_1)} < \rho_{\omega'}} \rho_{\omega} \min[\rho_{\omega'}, M^{-h(s_2)}] M^{-2(d-1)h(s_2)}
$$

$$
\times 2^{-m[\omega, \omega', \theta_1, \theta_2] + \mu(\omega, h(u)) + \mu(\theta_1, h(s_1)) + \mu(\theta_2, h(s_2))},
$$

$$
= \varrho^{-2} \sum_{\omega, \omega', \theta_1, \theta_2} \rho_{\omega'} 2^{-m[\omega, \omega', \theta_1, \theta_2]} \mathcal{G}_{43}^0(\text{inner}). \quad (11.28)
$$

For the purpose of simplifying $\mathcal{G}_{43}^*(\text{inner})$, we recall from Lemma 9.3 that $s_2 \subseteq u$ has sidelength no more than $M^{-h(s_1)}$, and moreover, is constrained to lie in the union of at most $2dM$ parallelepipeds with $(d - 1)$ long directions and one short direction, of dimensions $M^{-h(s_1)}$ and $\rho_{\omega'}$ respectively. Denoting by $\sum_{s_2}^*$ the summation over all such cubes $s_2$, we find that

$$
\sum_{s_2}^* \rho_{\omega}(\theta_2, h(s_2)) M^{-2(d-1)h(s_2)} \min[\rho_{\omega'}, M^{-h(s_2)}]
$$
\[ \leq \varrho_{\varpi} \sum_{M^{-h(s_2)} \geq \varrho_{\varpi}} M^{-2(d-1)h(s_2)} 2^{\mu(\varpi, h(s_2))} + \sum_{M^{-h(s_2)} < \varrho_{\varpi}} M^{-2(d-1)h(s_2)} 2^{\mu(\varpi, h(s_2))} \]
\[
\leq \varrho_{\varpi} s_\pm + s_- \\
\leq C \left[ \varrho_{\varpi} 2^{\mu(\varpi_2)} M^{-2(d-1)h(s_1)} + 2^{\mu(\varpi_2)} (\varrho_{\varpi})^d M^{-2(d-1)h(s_1)} \right] \\
\leq C \varrho_{\varpi} 2^{\mu(\varpi_2)} M^{-2(d-1)h(s_1)}, \quad (11.29)
\]
where \( s_\pm \) are defined as in (10.1), and estimated according to Lemma [10.3] with the parameters being set at \( \epsilon = \beta = M^{-h(s_1)}, \gamma = \varrho_{\varpi}, \varpi = \varpi_2 \) for both. The value of \( \alpha \) is \( 2(d - 1) \) for \( s_+ \) and \( (2d - 1) \) for \( s_- \). A similar argument applies for the summation in \( s_1 \) with \( M^{-h(s_1)} \geq \varrho_{\varpi} \). According to Lemma [10.4], \( s_1 \) has to lie in \( u \) and within a distance at most \( C \Delta \) from the boundary of some child of \( u \). Hence the range of \( s_1 \) lies within the union of at most \( dM \) parallelepipeds, each of dimension \( M^{-h(u)} \) in \( (d - 1) \) directions and \( C \Delta \) in the remaining one. Denoting by \( \sum_{s_1} \) the relevant sum, and applying Lemma [10.3] again with \( \alpha = 2(d - 1), \epsilon = \beta = M^{-h(u)}, \gamma = \varrho_{\varpi}, \varpi = \varpi_1, \)
\[
\sum_{s_1} M^{-2(d-1)h(s_1)} 2^{\mu(\varpi_1, h(s_1))} \leq s_+ \leq 2^{\mu(\varpi_1)} M^{-2(d-1)h(u)}. \quad (11.30)
\]
Inserting the estimates (11.29) and (11.30), we arrive at the following bound on \( \mathcal{G}_{43}^* \) (inner):
\[
\mathcal{G}_{43}^* \text{(inner)} = \sum_u \sum_{s_1} 2^{\mu(\varpi, h(u)) + \mu(\varpi_1, h(s_1))} \\
\times \left[ \sum_{s_2} 2^{\mu(\varpi_2, h(s_2))} M^{-2(d-1)h(s_2)} \min \left[ \varrho_{\varpi_2}, M^{-h(s_2)} \right] \right] \\
\leq C \sum_{u, s_1} 2^{\mu(\varpi, h(u)) + \mu(\varpi_1, h(s_1))} \left[ 2^{\mu(\varpi_2)} \varrho_{\varpi} M^{-2(d-1)h(s_1)} \right] \\
\leq \varrho_{\varpi} 2^{\mu(\varpi_2)} \sum_u 2^{\mu(\varpi, h(u))} \sum_{s_1} M^{-2(d-1)h(s_1)} 2^{\mu(\varpi_1, h(s_1))} \\
\leq \varrho_{\varpi} 2^{\mu(\varpi_2)} \sum_u 2^{\mu(\varpi, h(u))} \left[ M^{-2(d-1)h(u)} 2^{\mu(\varpi_1)} \right] \\
\leq \varrho_{\varpi} 2^{\mu(\varpi_2) + \nu(\varpi_1)} \sum_u 2^{\mu(\varpi, h(u))} M^{-2(d-1)h(u)} \\
\leq \varrho_{\varpi} 2^{\mu(\varpi_2) + \mu(\varpi_1) + \nu(\varpi_1)} h(\varpi_1), \quad (11.31)
\]
where \( \varpi_1 \) is the youngest common ancestor of \( \omega, \omega', \vartheta_1, \vartheta_2, \) and hence \( h(\varpi_1) \geq h(u) \).

The last estimate follows from Lemma [10.2], invoking part [11] if \( d \geq 3 \) and part [10] if \( d = 2 \). An analogous sequence of steps, the details of which are left to the reader, can be executed to estimate \( \mathcal{G}_{43}^* \) (inner), the only distinction being that the space restrictions are now dictated by Lemma [9.9][11], so that the summation in \( s_2 \) invokes Lemma [10.3].
with \( r = d - 1, \beta = \varrho \min(M^{-h(\omega)}, M^{-h(\omega')}), \gamma = M^{-h(s_1)}. \) The outcome of this is that
\[
\mathcal{S}_{43}^0(\text{inner}) \leq \varrho^2 \rho_\omega \min(M^{-h(\omega)}, M^{-h(\omega')}) 2^{\nu(\vartheta_2) + \nu(\vartheta_1)} h(\omega_1). \tag{11.32}
\]
Substituting (11.31) into (11.27) and (11.32) into (11.28) leads to the following simpler sum over slope vertices:
\[
\mathcal{S}_{43}^* + \mathcal{S}_{43}^0 \leq C \sum_{\omega, \omega', \vartheta_1, \vartheta_2} M^{-h(\omega) - h(\omega')} 2^{-m[\omega, \omega', \vartheta_1, \vartheta_2] + \nu(\omega_1) + \nu(\vartheta_1) + \nu(\vartheta_2)}.
\]
In order to complete the summation, let us recall that the sum, ostensibly over four parameters, in fact ranges over at most three vertices \( \{\vartheta_1, \vartheta_2, \vartheta_3\} \), which is a rearrangement of the quadruple \( \{\omega, \omega', \vartheta_1, \vartheta_2\} \) satisfying (9.8). However, it is not apriori possible to assign a unique correspondence between these two sets of vertices. Indeed, as already indicated in the last paragraph of Section 8, the configuration type of the slopes (which does not in general mimic the configuration type of the roots) dictates which vertex or vertices of the quadruple \( \{\omega, \omega', \vartheta_1, \vartheta_2\} \) represents \( \omega_1 \) after the rearrangement. A careful analysis of the possible structures of \( \omega, \omega', \vartheta_1, \vartheta_2 \), as depicted in Figure 16, shows that
\[
M^{-h(\omega) - h(\omega')} h(\omega_1) 2^{-m[\omega, \omega', \vartheta_1, \vartheta_2] + \nu(\omega_1) + \nu(\vartheta_1) + \nu(\vartheta_2)} \leq M^{-2h(\omega_1)} h(\omega_1) \times \begin{cases} 2^{-\nu(\omega_3)} & \text{if } \omega_3 \not\subseteq \omega_2 \\ 2^{-\nu(\omega_3)} & \text{if } \omega_3 \subseteq \omega_2. \end{cases}
\]
The expression on the right hand side is of the type already considered in the estimation of \( \mathcal{S}_{42}^* \) and \( \mathcal{S}_{42}^0 \). In particular, it is summable in \( \omega_1, \omega_2, \omega_3 \) using repeated applications of Lemma 10.1 and yields the desired bound of \( CN^2 \).

![Figure 16](image.png)

Figure 16: A partial list of 4-slope configurations for 4 roots of type 3, with distinct \( \{\omega_1, \omega_2, \omega_3\} \). Other configurations (where partial coincidences may arise) are possible after permutation of \( \{v_1, v'_1, v_2, v'_2\} \) in these diagrams.
12 Appendix: Percolation on trees

As in [4,3,16], the argument of Section 7 requires the use of a special probabilistic process on certain trees called a (bond) percolation. More precisely, given some tree $T$ with edge set $E$, we define an edge-dependent Bernoulli (bond) percolation process to be a collection of independent random variables $\{X_e : e \in E\}$, where $X_e$ is Bernoulli($p_e$) with $p_e < 1$. If the random variables $\{X_e : e \in E\}$ are mutually independent and identically distributed Bernoulli($p$) random variables, with a constant $p < 1$ independent of the edge $e$, then the process is called a standard Bernoulli($p$) percolation. We are concerned with only standard Bernoulli($\frac{1}{2}$) percolation in this paper. The interested reader may consult [12] for a discussion of percolation processes in much greater generality.

Given a percolation process on a tree $T$, we think of the event $\{X_e = 0\}$ as the event that we remove the edge $e$ from the edge set $E$, and the event $\{X_e = 1\}$ as the event that we retain this edge. Thus, for a given edge $e \in E$, we think of $p = \Pr(X_e = 1)$ as the probability that we retain this edge after percolation. Survival of the tree is defined to be the event that at least one ray remains from the root of the tree to its bottommost level. The probability of this event is referred to as the survival probability of the corresponding percolation process. This probability can be estimated by visualizing percolation on a tree as a certain electrical network, as first imagined by Lyons in [17]. The natural electrical network is defined as follows: we truncate the tree $T$ at height $N$ and place the positive node of a battery at the root of $T_N$. Then, for every ray in $\partial T_N$, there is a unique terminating vertex; we connect each of these vertices to the negative node of the battery. A resistor is placed on every edge $e$ of $T_N$ with resistance $R_e$ defined by

$$\frac{1}{R_e} = \frac{1}{1 - p_e} \prod_{e' \in E \atop v(e) \subseteq v(e')} p_{e'},$$

where $v(e)$ is the vertex in $T$ at which $e$ terminates. Notice that the resistance for the edge $e$ is essentially the reciprocal of the probability that a path remains from the root of the tree to the vertex $v(e)$ after percolation. For standard Bernoulli($\frac{1}{2}$) percolation, we have

$$R_e = 2^{h(v(e))} - 1.$$  \hfill (12.2)

A seminal result of Lyons [18, Theorem 2.1], says that for quite general trees the total resistance of an electrical network is comparable to the inverse of the survival probability of the associated Bernoulli percolation process. For our purposes, a reasonable upper bound on the survival probability of Bernoulli($\frac{1}{2}$) percolation on a rooted labelled subtree of the $M$-adic tree suffices. We state the necessary result in a form convenient for our usage.

**Proposition 12.1** (Lyons [18]). Let $M \geq 2$ and let $T_N$ be a subtree of height $N$ of the full $M$-adic tree of the same height in dimension $d$. For a Bernoulli($\frac{1}{2}$) percolation process defined on $T$, let $R(T_N)$ be the total resistance of the electrical network on $T_N$ defined via (12.1). If $\Pr(T_N)$ denotes the survival probability after percolation on $T_N$,
then
\[ Pr(T_N) \leq \frac{2}{1 + R(T_N)}. \] (12.3)

See [3] or [16, Proposition 5.3] for a proof of this result. In light of Proposition 12.1, we see that to bound the survival probability after Bernoulli percolation it is sufficient to bound the resistance of the network from below. To accomplish this, we need the useful fact that connecting any two vertices at a given height by an ideal conductor (i.e. one with zero resistance) only decreases the overall resistance of the circuit.

**Proposition 12.2.** Let \( T_N \) be a truncated tree of height \( N \) with corresponding electrical network generated by a standard Bernoulli \((\frac{1}{2})\) percolation process. Suppose at height \( k < N \) we connect two vertices by a conductor with zero resistance. Then the resulting electrical network has a total resistance no greater than that of the original network.

For a proof of this fact, see [16, Proposition 5.1]. The main consequence of this observation that we draw upon in Lemma 7.4 is given by the following corollary.

**Corollary 12.3.** Given a subtree \( T_N \) of height \( N \) contained in the full \( d \)-dimensional \( M \)-adic tree, let \( R(T_N) \) denote the total resistance of the electrical network that corresponds to standard Bernoulli \((\frac{1}{2})\) percolation on this tree, in the sense of the theorem of Lyons as given in Proposition 12.1. Then
\[ R(T_N) \geq \sum_{k=1}^{N} \frac{2^{k-1}}{n_k}, \] (12.4)
where \( n_k \) denote the number of its \( k \)th generation vertices in \( T_N \).

**Proof.** To show this, we construct an auxiliary electrical network from the one naturally associated to our tree \( T_N \), as follows. For every \( k \geq 1 \), we connect all vertices at height \( k \) by an ideal conductor to make one node \( V_k \). Call this new circuit \( E \). The resistance of \( E \) cannot be greater than the resistance of the original circuit, by Proposition 12.2.

Fix \( k, 1 \leq k \leq N \), and let \( R_k \) denote the resistance in \( E \) between \( V_{k-1} \) and \( V_k \). The number of edges between \( V_{k-1} \) and \( V_k \) is equal to the number \( n_k \) of \( k \)th generation vertices in \( T_N \), and each edge is endowed with resistance \( 2^{k-1} \) by (12.2). Since these resistors are in parallel, we obtain
\[ \frac{1}{R_k} = \sum_{i=1}^{n_k} \frac{1}{2^{k-1}} = \frac{n_k}{2^{k-1}}. \]

This holds for every \( 1 \leq k \leq N \). Since the resistors \( \{R_k\}_{k=1}^{N} \) are in series, \( R(T_N) \geq R(E) = \sum_{k=1}^{N} R_k \), establishing inequality (12.4). \( \square \)
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