ON THE ABSOLUTELY CONTINUOUS SPECTRUM IN 
A MODEL OF IRREVERSIBLE QUANTUM GRAPH

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Abstract. A family $A_\alpha$ of differential operators depending on a real parameter $\alpha \geq 0$ is considered. This family was suggested by Smilansky as a model of an irreversible quantum system. We find the absolutely continuous spectrum $\sigma_{a.c.}$ of the operator $A_\alpha$ and its multiplicity for all values of the parameter. The spectrum of $A_0$ is purely a.c. and admits an explicit description. It turns out that for $\alpha < \sqrt{2}$ one has $\sigma_{a.c.}(A_\alpha) = \sigma_{a.c.}(A_0)$, including the multiplicity. For $\alpha \geq \sqrt{2}$ an additional branch of absolutely continuous spectrum arises, its source is an auxiliary Jacobi matrix which is related to the operator $A_\alpha$. This birth of an extra-branch of a.c. spectrum is the exact mathematical expression of the effect which was interpreted by Smilansky as irreversibility.

1. Introduction

In this paper we study the spectrum of a family $A_\alpha$ of differential operators in the space $L^2(\mathbb{R}^2)$. Each operator $A_\alpha$ is defined by the same differential expression

\begin{equation}
A U = -U_{xx} + \frac{1}{2}(-U_{qq} + q^2 U).
\end{equation}

The parameter $\alpha \in \mathbb{R}$ appears in the “transmission condition” across the line $x = 0$:

\begin{equation}
U_x'(0+, q) - U_x'(0-, q) = \alpha q U(0, q), \quad q \in \mathbb{R}.
\end{equation}

As we shall show in Theorem 4.1 for any $\alpha$ the operator $A_\alpha$ has a unique natural self-adjoint realization. The replacement $\alpha \mapsto -\alpha$ corresponds to the change of variables $q \mapsto -q$ which does not affect the spectrum. For this reason, below we shall discuss only non-negative $\alpha$.

The family $A_\alpha$ was suggested by the physicist Smilansky in [12] as a model of an irreversible quantum system. He carried out a formal computation of the scattering matrix for the pair $(A_0, A_\alpha)$ and showed...
that this matrix is unitary only if $\alpha < \sqrt{2}$. The loss of unitarity of the scattering matrix for large values of $\alpha$ was interpreted in [12] as irreversibility of the system.

First rigorous mathematical results on the family $A_\alpha$ were obtained in the paper [13], inspired by [12]. The present paper can be considered as the second part of [13], but it can be read independently.

The family $A_\alpha$ exhibits many unusual features, partly revealed in [12] and [13]. The most important of them is a phase transition at the point $\alpha = \sqrt{2}$: the spectral properties of $A_\alpha$ for $\alpha < \sqrt{2}$ and for $\alpha > \sqrt{2}$ are quite different. In what follows we refer to the values $\alpha < \sqrt{2}$ as “small” and to $\alpha > \sqrt{2}$ as “large”.

The spectrum of the operator $A_0$ can be easily described via separation of variables. It is absolutely continuous, fills the half-line $[1/2, \infty)$, and its multiplicity function is given by eq. (2.4) in Section 2. It seems natural to study the spectrum $\sigma(A_\alpha)$ for $\alpha > 0$ with the help of the perturbation theory of quadratic forms. The standard assumption in this type of problems is relative compactness (see [11]) of the perturbation with respect to the unperturbed quadratic form, i.e. to the one of the operator $A_0$. However, in our case this property is violated: the perturbation is only form-bounded but not form-compact. This was shown in [13].

As a rule, for the form-bounded perturbations the quadratic form approach does not give much information. Nevertheless, a rather complete description of the essential spectrum and of the point spectrum of $A_\alpha$ for small $\alpha$ was obtained in [13] and [14] by means of this approach. In particular, it was shown in [14] that for any $\alpha < \sqrt{2}$ the spectrum of the operator $A_\alpha$ below the threshold $1/2$ consists of a finite number of eigenvalues. This number grows indefinitely as $\alpha \nearrow \sqrt{2}$ and has regular asymptotics of the order $O((\sqrt{2} - \alpha)^{-1/2})$. If $\alpha \geq \sqrt{2}$, then the point spectrum of $A_\alpha$ is empty (see Theorem 5.2).

Our goal in this paper is to study the absolutely continuous spectrum $\sigma_{a.c.}$ of the operators $A_\alpha$ for all values of the parameter $\alpha$. This problem was not dealt with in [13]. A certain Jacobi operator in the space $\ell^2$ is involved in the description of $\sigma_{a.c.}(A_\alpha)$, namely

$$
J_0(\mu) : \{C_n\} \mapsto \{d_{n+1}C_{n+1} + (2n + 1)\mu C_n + d_nC_{n-1}\},
$$

$$
d_n = n^{1/2}(n^2 - 1/4)^{1/4},
$$
Here $\mu > 0$ is an auxiliary parameter. It is often convenient to use it along with $\alpha$. Our main result, Theorem 5.1, states that

$$\sigma_{a.c.}(A_{\alpha}) = \sigma_{a.c.}(A_0) \cup \sigma_{a.c.}(J_0(\sqrt{2}/\alpha)), \quad (1.3)$$

$$m_{a.c.}(E; A_{\alpha}) = m_{a.c.}(E; A_0) + m_{a.c.}(E; J_0(\sqrt{2}/\alpha)), \quad a.e. \ E \in \mathbb{R}. \quad (1.4)$$

Here the symbol $m_{a.c.}$ stands for the multiplicity function of the absolutely continuous spectrum.

The spectrum of $J_0(\mu)$ is discrete for $\mu > 1$ and is purely absolutely continuous for $\mu \leq 1$. Moreover, we show in Theorem 3.1 that

$$\sigma(J_0(\mu)) = (-\infty, \infty) \text{ for } \mu < 1, \quad \sigma(J_0(1)) = [0, \infty);$$

$$m_{a.c.}(E; J_0(\mu)) = 1 \quad a.e. \text{ on } \sigma(J_0(\mu)).$$

Thus, the equalities (1.3) and (1.4) give the complete description of the absolutely continuous spectrum of the operators $A_{\alpha}$. Namely, for small $\alpha$ it coincides with $\sigma_{a.c.}(A_0)$, including equality of the multiplicities. For large $\alpha$ a new branch of the absolutely continuous spectrum of multiplicity 1 adds to $\sigma_{a.c.}(A_0)$, its source is the Jacobi matrix $J_0(\mu)$. This birth of an additional branch of the a.c. spectrum is the exact mathematical expression of the effect which was interpreted in [12] as irreversibility.

The family $A_\alpha$ is a striking example of a problem which, in spite of its seeming simplicity, exhibits many unexpected effects. This refers to both the point spectrum and the absolutely continuous spectrum. For this reason we believe that the detailed analysis of this family is of general interest. Note that one important question remains unanswered. Namely, our method does not check whether $\sigma(A_\alpha)$ has singular continuous component.

In the course of the proof of the equalities (1.3) and (1.4) we use the tools coming from different parts of the spectral theory. In Section 6 we obtain a convenient representation of the operator

$$(A_{\alpha} - L)^{-1} - (A_0 - L)^{-1}. \quad (1.5)$$

This representation involves some matrix-valued function $J_0(L; \mu)$ which arises in a natural way when looking for formal (i.e. not necessarily lying in $L^2(\mathbb{R}^2)$) solutions of the equation $A_{\alpha}V = LV$. The function $J_0(L; \mu)^{-1}$ is close, in an appropriate sense, to the resolvent of the Jacobi operator $J_0(\mu)$ involved in (1.3) and (1.4). This allows us to find a connection between the boundary behaviour of these two matrix-valued functions as $L$ approaches the real line. Technically, this is the most
difficult part of the paper. Here we make use of theory of analytic operator-valued functions.

In Section 7 we present Theorem 7.1 which relates the a.c. spectrum of a self-adjoint operator, and also its multiplicity function, to the jump of its "bordered resolvent" across the real line. We could not find this result in its full generality in the literature. For the reader’s convenience, we present its proof in Appendix B.

The above mentioned representation of the operator (1.5) leads to an equality which expresses the jump of the bordered resolvent of the operator $A_\alpha$ through the similar characteristics of $\mathcal{J}_0(\mu)$. As soon as this is done, Theorem 7.1 applies and gives the equality (1.4). This scheme is especially transparent for the values $E < 1/2$. In order to include the values $E > 1/2$, we need an additional technical trick.

Below we briefly describe the structure of the paper. Sections 2 – 4 contain the necessary technical material. Our main result on the absolutely continuous spectrum $\sigma_{a.c.}(A_\alpha)$ for different values of $\alpha$, Theorem 5.1, is formulated in Section 5. Its proof is given in Sections 6 – 8. We also present Theorem 5.2 on the point spectrum of $A_\alpha$. The latter material is mostly borrowed from the papers [13] and [14].

In Section 9 we discuss the possibility to extend the results to operators on quantum graphs. As a matter of fact, the family $A_\alpha$ was suggested by Smilansky in [12] for this, more general situation.

Two appendices are devoted to the proofs of Theorem 3.1 and Theorem 7.1.

The notation used in the paper is mostly standard. We denote $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. The symbol $(s)$-lim stands for the strong limit of operators. Abbreviation ”a.e.” always means almost everywhere with respect to the Lebesgue measure. The symbols $H^l$ stand for Sobolev spaces. Other necessary notations are introduced in the course of presentation.

2. Reduction to an infinite system of ODE

Equation (1.1) involves the harmonic oscillator in the variable $q$. For this reason, it is convenient to represent the functions $U \in \mathcal{H}$ as

$$U(x, q) = \sum_{n \in \mathbb{N}_0} u_n(x) \chi_n(q),$$

where $\chi_n$ are Hermite functions, normalized in $L^2(\mathbb{R})$. We often identify a function $U(x, q)$ with the sequence $\{u_n(x)\}$ and write $U \sim \{u_n\}$. This identification is a unitary mapping of the space $L^2(\mathbb{R}^2)$ onto the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}_0, L^2(\mathbb{R}))$ or, equivalently, onto the tensor product $\ell^2 \otimes$
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$L^2(\mathbb{R})$ with the natural Hilbert space structure. For $U \sim \{u_n\}$ we have

$$(2.1) \quad AU \sim \{(L + n)u_n\}, \quad (Lu)(x) = -u''(x) + u(x)/2, \; x \neq 0.$$ 

The condition (1.2) gives

$$\sum_{n \in \mathbb{N}_0} \chi_n(q)(u_n'(0^+) - u_n'(0^-)) = \alpha \sum_{n \in \mathbb{N}_0} q\chi_n(q)u_n(0).$$

Taking into account the recurrence equation for the Hermite functions:

$$\sqrt{n + 1}\chi_{n+1}(q) - \sqrt{2} q\chi_n(q) + \sqrt{n}\chi_{n-1}(q) = 0,$$

we come to the system of matching conditions

$$(2.2) \quad \mu\left(u_n'(0^+) - u_n'(0^-)\right) = \sqrt{n + 1} u_{n+1}(0) + \sqrt{n} u_{n-1}(0)$$

where $\mu = \sqrt{2}/\alpha$.

The operator $A_0$ admits separation of variables, which leads to the complete description of its spectrum. Let $H$ stand for the operator in $L^2(\mathbb{R})$, defined as

$$(2.3) \quad H = -d^2/dx^2 + 1/2, \quad Dom H = H^2(\mathbb{R}).$$

One should distinguish between the self-adjoint operator $H$ and the differential expression (formal differential operator) $L$ defined in (2.1). The operator $A_0$ splits into the orthogonal sum of the operators $H + n$, $n \in \mathbb{N}_0$. An element $U \sim \{u_n\}$ belongs to the domain $D_0 := Dom A_0$ if and only if $u_n \in H^2(\mathbb{R})$ for each $n \in \mathbb{N}_0$ and

$$\sum_{n \in \mathbb{N}_0} \|(H + n)u_n\|^2 < \infty.$$ 

Here and in what follows, unless otherwise explicitly stated, the symbols $(\cdot, \cdot), \|\cdot\|$ without indication of the space stand for the scalar product and the norm in $L^2(\mathbb{R})$.

The spectrum $\sigma(H)$ is absolutely continuous of multiplicity 2 and coincides with the half-line $[1/2, \infty)$. As a consequence, the spectrum of $A_0$ is also absolutely continuous, and

$$(2.4) \quad m_{a.c.}(E; A_0) = 2n \quad \text{for} \; E \in (n - 1/2, n + 1/2), \; n \in \mathbb{N}. $$
3. JACOBI MATRICES \( \mathcal{J}(L; \mu) \) AND \( \mathcal{J}_0(\mu) \)

3.1. Preliminaries. Our main goal in this section is to define the Jacobi matrix \( \mathcal{J}_0(\mu) \) involved in (1.3), (1.4), and another matrix \( \mathcal{J}(L; \mu) \), depending on two parameters \( L \in \mathbb{C} \) and \( \mu > 0 \). This latter matrix appears when analyzing the homogeneous equation

\[
\mathcal{A} V = L V
\]

under the matching condition (1.2). In the representation \( V \sim \{v_n\} \) equation (3.1) reduces to the infinite system

\[
-v''_n(x) + \left( n + 1/2 - L \right) v_n(x) = 0, \quad x \neq 0; \quad n \in \mathbb{N}_0
\]

under the matching conditions (2.2). Our immediate task is to describe all the formal solutions of this system. This means the following. Introduce the linear space

\[
W(\mathbb{R}) = \{ u \in C(\mathbb{R}) : u \upharpoonright \mathbb{R}_\pm \in H^2(\mathbb{R}_\pm) \}.
\]

Note that for any \( u \in W(\mathbb{R}) \) the left-hand side in (2.2) is well-defined. We seek the solutions \( V \sim \{v_n\} \) such that \( v_n \in W(\mathbb{R}) \) for each \( n \), but not necessarily \( V \in \mathcal{F} \).

Denote

\[
\zeta_n := \zeta_n(L) = \sqrt{n + 1/2 - L}.
\]

We take the branch of the square root which is analytic in the domain \( \Omega_n := \mathbb{C} \setminus [n + 1/2, \infty) \) and such that \( \zeta_n(L) > 0 \) for \( L = L < n + 1/2 \), then

\[
\text{Re } \zeta_n(L) > 0, \quad \text{Im } \zeta_n(L) \cdot \text{Im } L < 0, \quad L \in \Omega_n.
\]

The subspace of continuous \( L^2 \)-solutions of each equation (3.2) is one-dimensional, it is generated by the function

\[
\eta_n(x; L) = (n + 1/2)^{1/4} e^{-\zeta_n(L)|x|}.
\]

We choose such normalization of the vector-valued functions \( \eta_n(\cdot; L) \), that each of them is analytic in \( \Omega_n \) and

\[
c_1(L) \leq \|\eta_n(\cdot; L)\|^2 \leq c_2(L), \quad \forall n \in \mathbb{N}_0.
\]

We have

\[
\begin{cases}
\eta_n(0; L) = (n + 1/2)^{1/4}, \\
\eta_n'(0+; L) - \eta_n'(0-; L) = -2(n + 1/2)^{1/4} \zeta_n(L).
\end{cases}
\]
From (3.2) we obtain $v_n(x) = C_n \eta_n(x; L)$, and by (3.7) the matching conditions (2.2) reduce to the recurrence system
\[(n + 1)^{1/2}(n + 3/2)^{1/4}C_{n+1} + 2\mu(n + 1/2)^{1/4}\zeta_n(L)C_n + n^{1/2}(n - 1/2)^{1/4}C_{n-1} = 0, \quad n \in \mathbb{N}_0.\]

Taking into account our further needs, we multiply both sides of this equality by $(n + 1/2)^{1/4}$. As a result, we find that equation (3.1) is equivalent to the system
\[
V \sim \{C_n \eta_n(x; L)\};
\]
\[
d_{n+1}C_{n+1} + 2\mu y_n(L)C_n + d_n C_{n-1} = 0
\]
where
\[
d_n = n^{1/2}(n^2 - 1/4)^{1/4}, \quad y_n(L) = (n + 1/2)^{1/2}\zeta_n(L).
\]

The Jacobi matrix $J(L; \mu)$ which corresponds to equation (3.9) is one of our main objects. In our notations we do not distinguish between a Jacobi matrix and the operator which it defines in $\ell^2(\mathbb{N}_0)$. Now we write the operator $J(L; \mu)$ in a more convenient form. Given a number sequence $\{\omega_n\}_{n \in \mathbb{N}_0}$, let $D\{\omega_n\}$ stand for the diagonal operator in $\ell^2(\mathbb{N}_0)$ acting as
\[
D\{\omega_n\} : \{r_0, r_1, \ldots\} \mapsto \{\omega_0 r_0, \omega_1 r_1, \ldots\}.
\]

Let, in particular,
\[
D = D\{d_n\}, \quad Y(L) = D\{y_n(L)\}.
\]

Denoting by $S$ the operator of the forward shift in $\ell^2(\mathbb{N}_0)$,
\[
S : \{r_0, r_1, \ldots\} \mapsto \{0, r_0, r_1, \ldots\},
\]
we can re-write the operator $J(L; \mu)$ as
\[
J(L; \mu) = DS + S^*D + 2\mu Y(L).
\]

We also let
\[
J_0(\mu) = DS + S^*D + 2\mu Y_0, \quad Y_0 = D\{n + 1/2\}.
\]

The operator $J_0(\mu)$, defined initially on the set of all sequences with a finite number of non-zero elements, is essentially self-adjoint in $\ell^2$, and we denote by the same symbol $J_0(\mu)$ its unique self-adjoint extension. We do not need the explicit description of its domain $\text{Dom } J_0(\mu)$. The next result describes the spectral properties of $J_0(\mu)$ depending on $\mu$.

**Theorem 3.1.** For $\mu > 1$ the operator $J_0(\mu)$ is positive definite and its spectrum is discrete.
For \( \mu \leq 1 \) the spectrum of \( J_0(\mu) \) is purely absolutely continuous and

\[
\sigma(J_0(1)) = [0, \infty); \quad \sigma(J_0(\mu)) = \mathbb{R} \text{ for } \mu < 1.
\]

The proof is given in Appendix A.

The difference \( J(\mu) - (J_0(\mu) - \mu L) \) is a compact operator. This follows from the equality

\[
\Psi(L; \mu) := J(L; \mu) - (J_0(\mu) - \mu L) = \mu \mathcal{D}\{\psi_n(L)\}
\]

where

\[
\psi_n(L) = y_n(L) - 2(n + 1/2 - L/2) = -L^2(4y_n(L) + 4(n + 1/2 - L/2))^{-1} = O(n^{-1}).
\]

Hence, the operator \( J(L; \mu) \) is closed on the domain \( \text{Dom } J(L; \mu) = \text{Dom } J_0(\mu) \). Moreover,

\[
\text{Im } J(L; \mu) = 2\mu \text{ Im } Y(L).
\]

This implies an important property:

\[
\text{Im } J(L; \mu) < 0 \text{ for } L \in \mathbb{C}_+; \quad \text{Im } J(L; \mu) > 0 \text{ for } L \in \mathbb{C}_-.
\]

Besides, the operator-valued function \( J(L; \mu) \) forms a holomorphic family of type (A) in the variable \( L \), see [7], §VII.2.

It follows from (3.10) that there is a constant \( c > 0 \) such that

\[
\text{Im } y_n(-i\tau) \geq c\sqrt{\tau}, \quad \forall n \in \mathbb{N}_0, \quad \tau > 1.
\]

Hence, for \( \tau > 1 \) we have

\[
\text{Im}(J(-i\tau; \mu) - 2i\mu c\sqrt{\tau}) \geq 0.
\]

By a well known estimate for the dissipative operators, see e.g. [5], Theorem IV.4.1, this implies a useful inequality

\[
(3.15) \quad \|J(-i\tau; \mu)^{-1}\| \leq (2\mu c\sqrt{\tau})^{-1}, \quad \tau > 1.
\]

3.2. Birkhoff – Adams theorem. We base our analysis of the system (3.9), and also the proof of Theorem 3.1 on the classical result due to Birkhoff and Adams, see e.g. the book [3], Theorem 8.36. For the reader’s convenience, we reproduce the formulation of the part of this theorem which we need below. It concerns the general recurrence system

\[
(3.16) \quad C(n + 1) + p_1(n)C(n) + p_2(n)C(n - 1) = 0
\]
where the functions \( p_1(n) \) and \( p_2(n) \) have asymptotic expansions of the form
\[
(3.17) \quad p_1(n) \sim \sum_{j=0}^{\infty} a_j n^{-j}, \quad p_2(n) \sim \sum_{j=0}^{\infty} b_j n^{-j}, \quad b_0 \neq 0.
\]

Let \( l_\pm \) stand for the roots of the equation
\[
l^2 + a_0 l + b_0 = 0.
\]

**Proposition 3.2.** (a) Let \( l_+ \neq l_- \), then the system (3.16) has two linearly independent solutions \( \{C^\pm\} \) with the asymptotics
\[
C^\pm(n) \sim l_\pm n^{d_\pm}, \quad d_\pm = \frac{a_1 l_\pm + b_1}{a_0 l_\pm + 2b_0}.
\]
(b) Let \( l_+ = l_- = l \) but \( 2b_1 \neq a_0 a_1 \). Then the system (3.16) has two linearly independent solutions \( \{C^\pm(n)\} \) with the asymptotics
\[
(3.18) \quad C^\pm(n) \sim l^n e^{\pm \delta \sqrt{n} \kappa}, \quad \delta = 2 \sqrt{\frac{a_0 a_1 - 2b_1}{2b_0}}, \quad \kappa = \frac{1}{4} + \frac{b_1}{2b_0}.
\]

Actually, Theorem 8.36 in [3] describes the complete asymptotic expansions of the solutions \( C^\pm(n) \) of the system (3.9), but we need only their leading terms. (Note that there is an evident misprint in eq. (8.6.7) in [3], whose part is reproduced above in (3.18). In (3.18) this misprint is corrected.)

We also need an identity for solutions of recurrence equations with Jacobi matrices, of the type
\[
(3.19) \quad Q_{n+1} C_{n+1} + P_n C_n + Q_n C_{n-1} = 0, \quad n \in \mathbb{N}_0,
\]
with \( Q_n \) real and \( Q_0 = 0 \). Namely,
\[
(3.20) \quad \sum_{n=0}^{N} |C_n|^2 \text{Im} P_n = -Q_{N+1} \text{Im}(C_{N+1} C_N^\ast), \quad \forall N \in \mathbb{N}.
\]

The proof is straightforward and we skip it.

### 3.3. Solutions of the system (3.9).

Here we apply Proposition 3.2 to the system (3.9) that is actually equivalent to equation (3.1). The system (3.9) can be re-written in the form (3.16), with the functions \( p_1(n), p_2(n) \) admitting the asymptotic expansions of the type (3.17), where in particular
\[
(3.21) \quad a_0 = 2\mu, \quad a_1 = -\mu(1 + L); \quad b_0 = 1, \quad b_1 = -1.
\]

The following Lemma is a direct consequence of Proposition 3.2.
Lemma 3.3. Let $\mu > 0$ and $L \in \Omega_0$. Then the system (3.9) has two linearly independent solutions whose asymptotic behaviour is given by

$$C_n^\pm \sim \begin{cases} 
(\mu + i \sqrt{1 - \mu^2})^{\pm n} n^{-\frac{1}{2} + i \frac{\mu}{2\sqrt{1 - \mu^2}}}, & \mu < 1; \\
(-1)^n e^{\pm 2\sqrt{-Ln} n^{-1/4}}, & \mu = 1; \\
(-\mu + \sqrt{\mu^2 - 1})^{\pm n} n^{-\frac{1}{2} + \frac{\mu}{2\sqrt{\mu^2 - 1}}}, & \mu > 1.
\end{cases}$$

We repeatedly use this Lemma in our further exposition.

4. The self-adjoint operator $A_\alpha$, $\alpha > 0$

Here we describe the domain on which the operator given by equations (1.1) – (1.2) or equivalently, by (2.1) – (2.2), is self-adjoint in $\mathcal{H}$. Consider a linear subset $D_\alpha \subset \mathcal{H}$: an element $U \sim \{u_n\}$ belongs to $D_\alpha$ if and only if each component $u_n$ lies in $W(\mathbb{R})$ (see (3.3)), the conditions (2.2) are satisfied, and

$$\sum_{n \in \mathbb{N}_0} \|(L + n)u_n\|^2 < \infty.$$  

Along with $D_\alpha$, we need its subset $D_\alpha^0$ consisting of all elements $U \in D_\alpha$ which have only a finite number of non-zero components. Taking each component equal zero in a vicinity of the point $x = 0$, we obtain a dense subset in $\mathcal{H}$. Hence, both $D_\alpha^0$ and $D_\alpha$ are dense in $\mathcal{H}$.

Define the operators $A_\alpha$ and $A_\alpha^0$ as

$$A_\alpha U = A U \sim \{(L + n)u_n\}, \quad \text{Dom } A_\alpha = D_\alpha;$$

$$A_\alpha^0 = A_\alpha \upharpoonright D_\alpha^0.$$  

Evidently, the operator $A_\alpha^0$ is symmetric in $\mathcal{H}$. Our goal is to prove the following result.

Theorem 4.1. The operator $A_\alpha$ is self-adjoint and coincides with the closure of $A_\alpha^0$.

Note that for $\alpha \neq \sqrt{2}$ the result is covered by [13], Theorem 5.1. Nevertheless, below we give the full proof of Theorem. We do this mostly in order to illustrate the usage of Proposition 3.2. In [13] another, more sophisticated technical tools were used for the proof.

Proof. First of all, we show that

$$A_\alpha = (A_\alpha^0)^*.$$  

The inclusion “$\subset$” in (4.2) can be easily checked by the direct inspection. To prove the reverse inclusion, suppose that $V \sim \{v_n\} \in$
$\text{Dom} \left( (A_0^\alpha)^* \right)$ and $(A_0^\alpha)^* V = W \sim \{w_n\}$. According to the definition of the adjoint operator, this means that for any $U \sim \{u_n\} \in \mathcal{D}_0^\alpha$ we have

$$
\sum_{n \in \mathbb{N}_0} \int_R (L + n)u_n \overline{v_n} \, dx = \sum_{n \in \mathbb{N}_0} \int_R u_n \overline{w_n} \, dx.
$$

Take a function $f \in \mathcal{W}(\mathbb{R})$ such that $f(x) = 0$ in a vicinity of $x = 0$ and fix a number $n_0 \in \mathbb{N}_0$. The element $U \sim \{u_n\}$, such that $u_{n_0} = f$ and $u_n = 0$ for $n \neq n_0$, belongs to $\mathcal{D}_0^\alpha$. Applying the identity (4.3) to all such $U$, we conclude that if $V \in \text{Dom} \left( (A_0^\alpha)^* \right)$, then $v_n \in \mathcal{W}(\mathbb{R})$ and $w_n = (L + n)v_n$ for all $n \in \mathbb{N}_0$. Hence, for $\{v_n\}$ the condition (4.1) is satisfied. It remains to check that the matching conditions (2.2) are also fulfilled. To this end, we fix a number $n_0 \in \mathbb{N}_0$ and choose an element $U \sim \{u_n\}$ as follows: $u_n(x) \equiv 0$ for $|n - n_0| > 1$; the functions $u_{n_0}, u_{n_0+1, -1}$ are supported in a vicinity of the $x = 0$ and in some smaller vicinity are given by $u_{n_0}(x) = 1; u_{n_0+1}(x) = h_\pm |x|$ where $h_\pm$ are some appropriate numbers. Then $U \in \mathcal{D}_0^\alpha$, provided that

$$
2\mu h_+ = \sqrt{n_0 + 1}, \quad 2\mu h_- = \sqrt{n_0}.
$$

Now (2.2) for $n = n_0$ is implied by (4.3) for the element $U$ constructed. So, the equality (4.2) is justified.

The statement of Theorem is equivalent to the fact that both deficiency indexes of the operator $A_0^\alpha$ are equal to zero. Since all the coefficients in the equation and in the matching conditions are real, it is enough to prove that the only solution $V \in \mathcal{H}$ of the equation

$$
A_\alpha V - i V = 0
$$

is $V \equiv 0$. Equation (4.4) is nothing but (3.1) for $L = i$. Using the representation (3.8), we reduce the equation to the form (3.9) and can apply Lemma 3.3.

If $\mu < 1$, then according to (3.22) the system has a pair of linearly independent solutions $\{C_n^\pm\}$ such that

$$
|C_n^\pm|^2 \sim n^{-1 \pm \mu / \sqrt{1 - \mu^2}}.
$$

In view of (3.6), only the sequence $\{C_n^-\}$ may generate a solution $V \in \mathcal{H}$ of equation (4.1).

Equation (3.9) is of the form (3.19). Now we use for it the identity (3.20), with $C_n = C_n^-$. This gives

$$
\sum_{n=0}^N |C_n^-|^2 \text{Im} \zeta_n = -\sqrt{N + 1} \text{Im}(C_{N+1}^- \overline{C_N^-}).
$$
By \((1.3)\), the right-hand side of \((4.6)\) vanishes as \(N \to \infty\). Since \(\text{Im} \zeta_n > 0\) for each \(n\), we conclude from \((4.6)\) that \(V \equiv 0\). This shows that for \(\alpha > \sqrt{2}\) the operator \(A_\alpha\) is self-adjoint.

If \(\alpha < \sqrt{2}\), then \(\mu > 1\) and according to \((3.22)\) one of the solutions of the system \((3.9)\) exponentially grows and another exponentially decays. Only the latter may give rise to the solution \(V \in \mathcal{H}\) of equation \((4.4)\). Again, using the identity \((3.20)\) we conclude that this solution is identically zero.

Finally, let \(\alpha = \sqrt{2}\), then \(\mu = 1\) and the formula \((3.22)\) gives
\[
C_n^\pm \sim e^{\pm \sqrt{2}(1+i)n^{-1/4}}.
\]
Only the sequence \(\{C_1^-\}\) lies in \(\ell^2\) and again, we conclude from \((3.20)\) that the deficiency indexes are equal to 0. □

5. Spectrum of the operators \(A_\alpha\)

5.1. Results. The following theorem is the central result of the paper.

**Theorem 5.1.** Let \(\alpha > 0\), \(\mu = \sqrt{2}/\alpha\) and let \(J_0(\mu)\) be the Jacobi matrix (operator), defined in \((3.12)\). Then the a.c. spectrum of the operator \(A_\alpha\) and its multiplicity are described by the equalities \((1.3)\) and \((1.4)\).

In particular, for \(\alpha < \sqrt{2}\)
\[
\begin{align*}
\sigma_{a.c.}(A_\alpha) &= [1/2, \infty), \\
m_{a.c.}(E; A_\alpha) &= m_{a.c.}(E; A_0), \quad \text{a.e. } E \geq 1/2.
\end{align*}
\]

Further,
\[
\sigma_{a.c.}(A_{\sqrt{2}}) = [0, \infty); \quad \sigma_{a.c.}(A_\alpha) = \mathbb{R}, \; \alpha > \sqrt{2},
\]
and for \(\alpha \geq \sqrt{2}\) and \(E \in \sigma_{a.c.}(A_\alpha)\) we have
\[
m_{a.c.}(E; A_\alpha) = m_{a.c.}(E; A_0) + 1.
\]

The equalities \((5.1)\), \((5.2)\) and \((5.3)\) immediately follow from the relations \((1.3)\), \((1.4)\) and Theorem 3.1. So, our main goal for the rest of the paper is to prove \((1.3)\) and \((1.4)\).

To make the picture more complete, we present also the result concerning the point spectrum \(\sigma_p(A_\alpha)\). Within minor detail, these results were proved in the papers \([13], [14]\).

**Theorem 5.2.** 1. For any \(\alpha \geq 0\) the operator \(A_\alpha\) has no eigenvalues \(E \geq 1/2\), and \(\sigma_p(A_\alpha) = \emptyset\) for \(\alpha \geq \sqrt{2}\).

2. For \(0 < \alpha < \sqrt{2}\) the operator \(A_\alpha\) is positive definite. Its point spectrum is always non-empty and finite, and the number \(N(1/2; A_\alpha)\)
of its eigenvalues (counted with multiplicities) satisfies the asymptotic formula
\[ N(1/2; A_\alpha) \sim \frac{1}{4\sqrt{2(\mu(\alpha) - 1)}}, \quad \alpha > \sqrt{2} \quad (\mu(\alpha) = \frac{\sqrt{2}}{\sqrt{\alpha}}). \]

It follows from Theorems 5.1 and 5.2 that the essential and the absolutely continuous spectra of the operator \( A_\alpha \) coincide as sets. However, our approach does not show that \( A_\alpha \) has no singular continuous spectrum.

5.2. Outline of proof of Theorem 5.2. 1. If \( E \geq 1/2, V \sim \{v_n\} \) and \( A_\alpha V = EV \), then each component \( v_n \) lies in \( \mathcal{W}(\mathbb{R}) \) and satisfies equation (3.2) with \( \mu = E \). For \( n \leq E - 1/2 \) this yields \( v_n \equiv 0 \). This can be interpreted as the equality \( C_n = 0 \) for the coefficients in (3.9). Then these equations imply that \( C_n = 0 \) also for all \( n > E - 1/2 \).

The absence of eigenvalues \( E < 1/2 \) for the operator \( A_\alpha \) with \( \alpha > \sqrt{2} \) was proved in [13], Theorem 7.1. The possibility to use Proposition 3.2 simplifies the proof, and also allows one to include the borderline case \( \alpha \geq \sqrt{2} \). We leave this to the reader.

The statement 2 is covered by [13], Theorem 6.2 and [14], formula (3.10).

6. Representation of the resolvent

6.1. Auxiliary considerations. In this section we derive a convenient representation of the operator
\[ (A_\alpha - L)^{-1} - (A_0 - L)^{-1}, \quad L \notin \mathbb{R}. \]
The equality (6.6) which we establish in Theorem 6.1 can be interpreted in terms of the extension theory of symmetric operators. However, formally we do not use this theory in our construction.

Given an element \( F \sim \{f_n\} \in \mathcal{F} \), denote
\[ U_\alpha \sim \{u_{\alpha,n}\} = (A_\alpha - L)^{-1}F, \quad \alpha \geq 0. \]
The functions \( u_{0,n} \) satisfy the equation
\[ -u''_{0,n} + (n + 1/2 - L)u_{0,n} = f_n \]
and lie in \( H^2(\mathbb{R}) \). Solving this equation, we find that
\[ 2\zeta_n(L)u_{0,n}(x) = \int_{\mathbb{R}} e^{-\zeta_n(L)|x-t|}f_n(t)dt. \]
Denote
\[
(6.1) \quad J_n = \int_{\mathbb{R}} \eta_n(t; L) f_n(t) dt = (f_n, \eta_n(\cdot; L)),
\]
then
\[
(6.2) \quad 2\zeta_n(L)(n + 1/2)^{1/4}u_{0,n}(0) = J_n, \quad n \in \mathbb{N}_0.
\]
Recall that the numbers \(\zeta_n(L)\) and the functions \(\eta_n(\cdot; L)\) were defined in (3.4) and (3.5).

Let now \(U_\alpha - U_0 \sim \{v_n\}\). Each function \(v_n\) satisfies the homogeneous equation (3.2) and belongs to \(W(\mathbb{R})\) and hence, \(v_n(x) = C_n \eta_n(x; L)\). The coefficients \(C_n\) are determined by the matching conditions for \(U_\alpha\). Since in view of (6.2)
\[
(6.3) \quad (n + 1/2)^{1/4}C_n = v_{n}(0) = u_{\alpha,n}(0) - u_{0,n}(0)
\]
and the derivative \(u'_{\alpha,n}\) is continuous at \(x = 0\), we get
\[
\begin{align*}
  u'_{\alpha,n}(0^+) - u'_{\alpha,n}(0^-) &= v'_n(0^+) - v'_n(0^-) \\
  &= -2\zeta_n(L)(n + 1/2)^{1/4}C_n = (n + 1/2)^{-1/4}J_n - 2\zeta_n(L)u_{\alpha,n}(0).
\end{align*}
\]

It is convenient for us to denote
\[
X_n = (n + 1/2)^{-1/4}u_{\alpha,n}(0),
\]
then
\[
(6.4) \quad u'_{\alpha,n}(0^+) - u'_{\alpha,n}(0^-) = (n + 1/2)^{-1/4}J_n - 2\zeta_n(L)(n + 1/2)^{1/4}X_n
\]
and the matching conditions (2.2) reduce to
\[
\begin{align*}
  \mu((n + 1/2)^{-1/4}J_n - 2\zeta_n(L)(n + 1/2)^{1/4}X_n) \\
  &= (n + 1)^{1/2}(n + 3/2)^{1/4}X_{n+1} + n^{1/2}(n - 1/2)^{1/4}X_{n-1},
\end{align*}
\]
or
\[
(6.5) \quad d_{n+1}X_{n+1} + 2\mu y_n(L)X_n + d_nX_{n-1} = \mu(f_n, \eta_n(\cdot; \widetilde{L})).
\]
This is the non-homogeneous counterpart of the recurrence system (3.9). It can be re-written in terms of the matrix \(\mathcal{J}(L; \mu)\) introduced in (3.11):
\[
(6.6) \quad \mathcal{J}(L; \mu)X = \mu(f_n, \eta_n(\cdot; \widetilde{L})), \quad X = \{X_n\}.
\]
6.2. **Basic formula.** For $L \notin \mathbb{R}$ consider the operator

$$T(L) : \ell^2(N_0) \to \mathfrak{S}, \quad T(L) \{X_n\} \sim \{X_n \eta_n(., L)\},$$

by (6.6) it is bounded and has bounded inverse. Its adjoint acts from $\mathfrak{S}$ to $\ell^2$ as

$$T(L)^* F = \{\int_{\mathbb{R}} f_n(x) \eta_n(x; L) dx\}, \quad F \sim \{f_n\}.$$
exist a.e. and belong to $S_1$, see e.g. [1]. Hence, the jump also exists a.e. (with respect to the Lebesgue measure). For a given operator $A$, the jump of its bordered resolvent may exist also for a wider class of borderings.

The following statement of a rather general nature plays the key role in our analysis. Its proof is given in Appendix B.

**Theorem 7.1.** Let $A$ be a self-adjoint operator in a separable Hilbert space $H$ and $\Delta \subset \mathbb{R}$ a given Borelian subset. Let $G$ be a bounded operator with the dense range, such that the operator

$$V_{A,G}(E) = (s)\lim_{L \to E+i0} G^* \left( (A - L)^{-1} - (A - T)^{-1} \right) G$$

is well-defined a.e. on $\Delta$. Then

$$m_{a.c.}(E; A) = \text{rank} V_{A,G}(E) \quad \text{a.e. on} \, \Delta.$$

**Corollary 7.2.** Let the assumptions of Theorem 7.1 be fulfilled for an interval $\Delta \subseteq \mathbb{R}$. If $\text{rank} V_{A,G}(E) > 0$ a.e. on $\Delta$, then $\Delta \subset \sigma_{a.c.}(A)$, and if $\text{rank} V_{A,G}(E) = 0$ a.e. on $\Delta$, then $\sigma_{a.c.}(A) \cap \Delta = \emptyset$.

7.2. Bordered resolvent of the operator $A_\alpha$. Along with the operator $Y_0 = \mathcal{D}\{n + 1/2\}$ acting in $\ell^2$, let us define $\mathcal{Y}_0 = I \otimes Y_0$. The latter operator acts in the space $\mathcal{H}$ interpreted as the tensor product $\ell^2 \otimes L^2(\mathbb{R})$, cf. Section 2. The powers $Y_0^{-\gamma}$, $\gamma > 0$ are compact operators in $\ell^2$, while the powers $\mathcal{Y}_0^{-\gamma}$ are only bounded operators. If $U \sim \{u_n\} \in \mathcal{H}$, then $\mathcal{Y}_0^{-\gamma} U \sim \{(n + 1/2)^{-\gamma} u_n\}$. Note that

$$\mathcal{Y}_0^{-\gamma} T(L) = T(L) Y_0^{-\gamma}, \quad T(L)^* \mathcal{Y}_0^{-\gamma} = Y^{-\gamma} T(L)^*.$$

We shall show that the boundary limits of the bordered resolvent $G^*(A_\alpha - L)^{-1} G$ do exist if we take $G = \mathcal{Y}_0^{-\gamma}$ with $\gamma > 1/4$, though $G$ is non-compact, and even the operator $Y_0^{-\gamma}$ lies in $S_2$ only if $\gamma > 1/2$.

Let us consider the operator-valued functions

$$Z_\gamma(L; \mu) = \mathcal{Y}_0^{-\gamma} (A_\alpha - L)^{-1} \mathcal{Y}_0^{-\gamma}, \quad \alpha = \sqrt{2}/\mu,$$

$$\widetilde{Z}_\gamma(L; \mu) = Y_0^{-\gamma} J(L; \mu)^{-1} Y_0^{-\gamma},$$

$$\mathcal{Z}_{0,\gamma}(L; \mu) = Y_0^{-\gamma} (J_0(\mu) - \mu L)^{-1} Y_0^{-\gamma}.$$

In (7.4) $J_0(\mu)$ is the Jacobi matrix introduced in (3.12).

By (6.6) and (7.3),

$$Z_\gamma(L; \mu) = \mathcal{Y}_0^{-\gamma} (A_0 - L)^{-1} \mathcal{Y}_0^{-\gamma} + T(L) \left( \mu \widetilde{Z}_\gamma(L; \mu) - \mathcal{Y}_0^{-\gamma} (2Y(L))^{-1} \mathcal{Y}_0^{-\gamma} \right) T(L)^*.$$
The jumps at a point $E \in \mathbb{R}$ of all but one terms in the right-hand side are evidently equal to zero, provided that $E < 1/2$, and we get, at least formally,

\begin{equation}
(7.5) \quad [Z_{\gamma}(L; \mu)]_{L \to E+i0} = \mu T(E) [\tilde{Z}_{\gamma}(L; \mu)]_{L \to E+i0} T(E)^*, \quad E < 1/2.
\end{equation}

Taking into account the equalities (3.13) and (3.14), it is natural to expect that the jumps $[\tilde{Z}_{\gamma}(.; \mu)](E)$ and $[\tilde{Z}_{0,\gamma}(.; \mu)](E)$ are close to each other. Together with (7.5), this would allow us to express the quantity $[Z_{\gamma}(.; \mu)](E)$ through $[\tilde{Z}_{0,\gamma}(.; \mu)](E)$, which makes it possible to use Theorem 7.1. Now we proceed to the successive realization of this program.

We start with the following lemma.

**Lemma 7.3.** Let $\mu > 0$, $L \neq L \in \mathbb{C}$ and $\gamma > 1/4$. Then $\tilde{Z}_{\gamma}(L; \mu) \in \mathcal{S}_1$. Besides, the strong non-tangential limits $\tilde{Z}_{\gamma}(E \pm i0; \mu)$ exist for almost all $E < 1/2$.

The same results are valid for $\tilde{Z}_{0,\gamma}(L; \mu)$.

**Proof.** For definiteness, we consider $L \in \mathbb{C}_+$. The operator-valued function $\tilde{Z}_{\gamma}(L; \mu)$ is analytic in $L$ in the half-plane $\mathbb{C}_+$. Since $\text{Im} \theta(L; \mu) \leq 0$, we conclude that $\text{Im} \tilde{Z}_{\gamma}(L; \mu) \geq 0$, i.e. the values of this function are dissipative operators. By a result due to S.Naboko, see \[10\], Remark (1) to Theorem 2.2, any such function admits the representation

\begin{equation}
(7.6) \quad \tilde{Z}_{\gamma}(L; \mu) = A + BL + R^*(I + L\mathcal{L})(\mathcal{L} - L)^{-1}R,
\end{equation}

where $\mathcal{L}$ is a self-adjoint operator in an auxiliary Hilbert space $\mathcal{H}_0$, $A = A^*$ and $B = B^* \geq 0$ are bounded operators in $\ell^2$, and $R$ is a bounded operator from $\ell^2$ to $\mathcal{H}_0$. It immediately follows from (3.15) that in this representation $B = 0$. The operators $\mathcal{L}, A$ and $R$ may depend on $\mu$ but we do not reflect this dependence in our notations.

Taking in (7.6) $L = i$, we find that

\begin{equation}
(7.7) \quad \tilde{Z}_{\gamma}(i; \mu) = A + iR^*R.
\end{equation}

Further, let us show that for any $\gamma > 0$ one has $\tilde{Z}_{\gamma}(i; \mu) \in \mathcal{S}_2$. To this end, consider the inverse matrix $\{G_{n,k}\} := \theta(i; \mu)^{-1}$. This matrix can be expressed through the solutions of the homogeneous system (3.9) for $L = i$. Namely, it is symmetric (i.e. $G_{n,k} = G_{k,n}$) and its entries for $k \geq n$ are

\begin{equation}
G_{n,k} = Cg^{(1)}(k)g^{(2)}(n)
\end{equation}
where \( C \) is a constant, \( \{g^{(1)}(k)\} \) is the decaying solution of the system, and \( \{g^{(2)}(n)\} \) is some other solution, a certain linear combination of two basic solutions. The matrix \( \widetilde{Z}_\gamma(i; \mu) \) has the entries

\[
(n + 1/2)^{-\gamma}G_{n,k}(k + 1/2)^{-\gamma},
\]

and the standard calculation shows that \( \widetilde{Z}_\gamma(i; \mu) \in \mathcal{S}_2 \) for any \( \gamma > 0 \) and an arbitrary \( \mu > 0 \). This immediately yields that \( \widetilde{Z}_\gamma(i; \mu) \in \mathcal{S}_1 \), provided that \( \gamma > 1/4 \).

Now we conclude from (7.7) that \( R \in \mathcal{S}_2 \). Hence, \( \widetilde{Z}_\gamma(L; \mu) \in \mathcal{S}_1 \) for all \( L \in \mathbb{C}_+ \). Besides, by a result of [1], Lemma 2.4, this implies the existence a.e. of the non-tangential strong limits of the bordered resolvent \( R^*(\mathcal{L} - L)^{-1}R \). The equality (7.6) shows that such limits do exist also for the operator-valued function \( Z_\gamma(L; \mu) \).

For the operator-valued function \( \widetilde{Z}_{0,\gamma} \) the proof is simpler, since we need not the representation (7.6). Otherwise, the argument remains the same.

As a result of Lemma 7.3 the equality (7.5) is justified.

7.3. Jumps of \( \widetilde{Z}_\gamma \) and of \( \widetilde{Z}_{0,\gamma} \). A subset \( \Delta \subset (-\infty, 1/2) \) of full measure can be selected, such that both these jumps are well-defined for \( E \in \Delta \). According to the Hilbert identity and using the equality (3.13), we find that

\[
\begin{align*}
\widetilde{Z}_\gamma(L; \mu) - \widetilde{Z}_{0,\gamma}(L; \mu) &= Y_0^{-\gamma}(\partial(L; \mu)^{-1} - (\partial_0(\mu) - \mu L)^{-1})Y_0^{-\gamma} \\
&= -Y_0^{-\gamma}\partial(L; \mu)^{-1}\Psi(L; \mu)(\partial_0(\mu) - \mu L)^{-1}Y_0^{-\gamma} \\
&= -\widetilde{Z}_\gamma(L; \mu)\Phi(L; \mu)\widetilde{Z}_{0,\gamma}(L; \mu)
\end{align*}
\]

where

\[
(7.9) \quad \Phi(L; \mu) = Y_0^\gamma\Psi(L; \mu)Y_0^{-\gamma}.
\]

More exactly, \( \Phi(L; \mu) \) is the extension by continuity of this operator, defined originally on the set \( \text{Ran} \ Y_0^\gamma \).

The calculations below are carried through for a fixed value of \( \mu \), and we drop this parameter from our notations. It follows from (3.14) that for \( \gamma < 1/2 \) the operator-valued function \( \Phi(\cdot) \) is analytic in \( \mathbb{C}_+ \) and its values are compact operators in \( \ell^2 \). Below we always assume \( 1/4 < \gamma < 1/2 \), then both operator-valued functions \( \widetilde{Z}_\gamma(L) \) and \( \widetilde{Z}_{0,\gamma}(L) \) have boundary limits as \( L \to E \pm i0 \) a.e. on \( (-\infty, 1/2) \).
The equality (7.8) and the similar equality for $L$ imply that
\[ \mathcal{Z}_{0,\gamma}(L) - \mathcal{Z}_{0,\gamma}(\bar{L}) = \mathcal{Z}_{\gamma}(L) - \mathcal{Z}_{\gamma}(\bar{L}) \]
\[ + (\mathcal{Z}_{\gamma}(L) - \mathcal{Z}_{\gamma}(\bar{L})) \Phi(L) \mathcal{Z}_{0,\gamma}(L) + \mathcal{Z}_{\gamma}(\bar{L}) (\Phi(L) - \Phi(\bar{L})) \mathcal{Z}_{0,\gamma}(L) \]
\[ + \mathcal{Z}_{\gamma}(\bar{L}) \Phi(\bar{L})(\mathcal{Z}_{0,\gamma}(L) - \mathcal{Z}_{0,\gamma}(\bar{L})). \]

Letting $L \to E + i0$ non-tangentially, we obtain for a.e. $E < 1/2$: 
\[ [\mathcal{Z}_{0,\gamma}](E) = [\mathcal{Z}_{\gamma}](E) + [\mathcal{Z}_{\gamma}](E) \Phi(E) [\mathcal{Z}_{0,\gamma}](E + i0) \]
\[ + \mathcal{Z}_{\gamma}(E - i0) \Phi(E) [\mathcal{Z}_{0,\gamma}](E). \]

Let us consider the operator-valued functions
\[ \mathcal{G}_+(L) = I + \Phi(L) \mathcal{Z}_{0,\gamma}(L), \quad L \in \mathbb{C}_+, \]
\[ \mathcal{G}_-(L) = I - \mathcal{Z}_{\gamma}(L) \Phi(L), \quad L \in \mathbb{C}_-. \]

The function $\mathcal{G}_+(L)$ can be represented in a different way: using the definitions (7.9) and (7.12), we find that
\[ \mathcal{G}_+(L) = I + Y_0^\gamma (\beta(L) - (\beta_0 - \mu L)) Y_0^{-\gamma} (\beta_0 - \mu L)^{-1} Y_0^{-\gamma} \]
\[ = Y_0^\gamma \beta(L)(\beta_0 - \mu L)^{-1} Y_0^{-\gamma}. \]

This calculation shows also that the right-hand side in (7.13) is well-defined as a bounded operator in $\ell^2$. For the "minus" sign, it is more convenient to deal with the adjoint operator, and we get
\[ \mathcal{G}_-(L)^* = Y_0^\gamma (\beta_0 - \mu L) \beta(\bar{L})^{-1} Y_0^{-\gamma}. \]

It follows from the definitions (7.11) and (7.12) and from compactness of $\Phi(L)$ that the functions $\mathcal{G}_\pm(L)$ are analytic in $\mathbb{C}_\pm$ respectively, and for each $L \in \mathbb{C}_\pm$ the operator $\mathcal{G}_\pm(L) - I$ is compact. Hence, the image of $\mathcal{G}_\pm(L)$ is a closed subset in $\ell^2$.

Now, we need the following Lemma.

**Lemma 7.4.** For any $L \in \mathbb{C}_\pm$ the operator $\mathcal{G}_\pm(L)$ has bounded inverse.

**Proof.** Throughout the proof, $(\cdot, \cdot)$ stands for the scalar product in $\ell^2$. By (7.13), we have for any $f \in \ell^2$ and any $g \in \text{Dom } Y_0^\gamma$:
\[ (\mathcal{G}_+(L)f, g) = (\beta(L)(\beta_0 - \mu L)^{-1} Y_0^{-\gamma} f, Y_0^\gamma g). \]

Suppose that $f \in \text{Ker } \mathcal{G}_+(L)$ for some $f \neq 0$ and $L \in \mathbb{C}_+$. Since $\text{Ran } Y_0^\gamma = \ell^2$, we conclude from (7.13) that then $\text{Ker } \beta(L)$ is non-trivial which contradicts the dissipativity of $\beta(L)$.

In a similar way, we derive from (7.14) that for $f \in \text{Dom } Y_0^\gamma$ and any $g \in \ell^2$
\[ (\mathcal{G}_-(L)f, g) = (Y_0^\gamma f, (\beta_0 - \mu L) \beta(\bar{L})^{-1} Y_0^{-\gamma} g). \]
Suppose that \( \text{Ran} \mathcal{G}_-(L) \neq \ell^2 \) for some \( L \in \mathbb{C}_- \). Then there exists an element \( g \in \ell^2 \), such that \( (\mathcal{G}_-(L)f, g) = 0 \) for all \( f \in \ell^2 \). This would imply

\[
(\mathcal{G}_0 - \mu L)\mathcal{G}(L)^{-1}Y_0^{-\gamma}g = 0.
\]

However, for \( g \neq 0 \) this is impossible, since \( \mathcal{G}(L)^{-1} \) and \( Y_0^{-\gamma} \) are inverse operators and hence, have trivial kernels. Further, \( \text{Ker}(\mathcal{G}_0 - \mu L) = \{0\} \) for \( L \notin \mathbb{R} \), since the operator \( \mathcal{G}_0 \) is self-adjoint.

Now we use the following statement, see [9].

**Proposition 7.5.** Let \( X \in \mathbb{C}_\pm \) be a domain, such that \( \overline{X} \cap \mathbb{R} \) contains an interval \( \Delta \). Let \( \mathcal{G}(L) \) be an analytic operator-valued function in \( X \), such that \( \mathcal{G}(L) - I \) are compact operators in a Hilbert space \( \mathcal{H} \). Suppose that for almost all \( E \in \Delta \) the function \( \mathcal{G} \) is non-tangentially bounded at \( E \) and has a strong non-tangential limit \( \mathcal{G}(E + i0) \), such that \( \mathcal{G}(E + i0) - I \in \mathcal{S}_\infty \). Suppose also that for at least one point \( L_0 \in \mathbb{C}_+ \) the operator \( \mathcal{G}(L_0) \) has bounded inverse.

Then for almost all \( E \in \Delta \) the operator \( \mathcal{G}(E + i0) \) has bounded inverse.

It follows from Lemmas [7.8] and [7.3] that the assumptions of Proposition are fulfilled for the operator-valued functions \( \mathcal{G}_\pm(L) \) defined in (7.11) and (7.12). This allows us to conclude from (7.5) and (7.10) that for a.e. \( E < 1/2 \)

\[
(7.16) \quad [Z_{\gamma}](E) = T(E)\mathcal{G}_-(E - i0)[\widetilde{Z}_{0,\gamma}](E)(\mathcal{G}_+(E + i0))^{-1}T(E)^*.
\]

The equalities (7.35) and (7.40) yield that

\[
\text{rank } [Z_{\gamma}](E) = \text{rank } [\widetilde{Z}_{0,\gamma}](E), \quad \text{a.e. } E < 1/2.
\]

According to Theorem 7.1 it follows that

\[
\text{mea.c.}(E; A_\alpha) = \text{mea.c.}(E; \mathcal{G}_0(\mu)) \quad \text{a.e. on } (-\infty, 1/2).
\]

Since \( \text{mea.c.}(E; A_0) = 0 \) for \( E < 1/2 \), the equality (1.4) for such \( E \) is justified.

Now, we conclude from Corollary 7.2 that for \( \alpha < \sqrt{2} \) the operator \( A_\alpha \) has no a.c. spectrum below the point 1/2. For \( \alpha > \sqrt{2} \) we have \( (-\infty, 1/2) \subset \sigma_{a.c.}(A_\alpha) \), and \([0, 1/2) \subset \sigma_{a.c.}(A_{\sqrt{2}})\).

8. PROOF OF THEOREM 5.1: \( E \geq 1/2 \)

The proof extends to \( l > 1/2 \) with the help of a simple technical trick. It is based on the passage to the subspace \( \mathfrak{F}_m = \ell^2(\mathbb{N}_m, L^2(\mathbb{R})) \), with \( m \) large enough. Here \( \mathbb{N}_m = \{m, m+1, \ldots\} \), so that, in particular, \( \mathbb{N}_1 = \mathbb{N} \). The subspace \( \mathfrak{F}_m \) is not invariant for the operator \( A_\alpha \), however
it is invariant for an appropriate operator $\hat{A}_\alpha^{(m)}$, that can be obtained from $A_\alpha$ by the perturbation of its resolvent by a finite rank operator. For the operator $\hat{A}_\alpha^{(m)}$ the scheme developed in Section 7 works for $E < m + 1/2$, and return to the original operator $A_\alpha$ does not change the absolutely continuous spectrum and its multiplicity function.

8.1. Operators $A_\alpha^{(m)}$ and $\hat{A}_\alpha^{(m)}$. Denote $\mathbb{N}_m = \{m, m + 1, \ldots\}$, so that, in particular, $\mathbb{N}_1 = \mathbb{N}$. Let us define an operator $A_\alpha^{(m)}$, acting in the space $\mathcal{H}_m = \ell^2(\mathbb{N}_m, L^2(\mathbb{R}))$. Namely, its domain $\mathcal{D}_\alpha^{(m)}$ consists of the elements $U \sim \{u_n\}_{n \geq m}$, such that each component $u_n$ lies in $\mathcal{W}(\mathbb{R})$ (see (3.23)), the matching conditions

\[
 u'_n(0^+) - u'_n(0-) = \begin{cases} \alpha \sqrt{m + 1}u_{m+1}(0), & n = m, \\ \alpha \sqrt{n + 1}u_{n+1}(0) + \sqrt{n}u_{n-1}(0), & n > m \end{cases}
\]

are satisfied, and $\sum_{n \geq m} \| (L + n) u_n \|^2 < \infty$. For $U \in \mathcal{D}_\alpha^{(m)}$ we let

\[
 A_\alpha^{(m)} U \sim \{(L + n) u_n\}_{n \geq m},
\]

cf. (2.1). Evidently, $A_\alpha^{(0)} = A_\alpha$. Each operator $A_\alpha^{(m)}$ is self-adjoint, the proof is the same as for $m = 0$.

Along with $A_\alpha^{(m)}$, we need also the operators $\hat{A}_\alpha^{(m)}$, acting in the original Hilbert space $\mathcal{H}$. Namely, we let

\[
 \hat{A}_\alpha^{(m)} = \left( \sum_{n < m} \oplus (H + n) \right) \oplus A_\alpha^{(m)}
\]

where $H$ is the operator in $L^2(\mathbb{R})$, defined in (2.3). Note that $\hat{A}_0^{(m)} = A_0$ for any $m$.

Consider also two sequences of Jacobi matrices, $J_0^{(m)}(\mu)$ and $\mathcal{J}^{(m)}(L; \mu)$. Each $J_0^{(m)}(\mu)$ is a sub-matrix of the matrix $J_0(\mu)$ given by (3.12), it is obtained from $J_0(\mu)$ by removing its first $m$ rows and $m$ columns. The matrix $\mathcal{J}^{(m)}(L; \mu)$ is obtained in the same way from the matrix (3.11).

We also define a sequence $Y^{(m)}(L)$ of diagonal operators in $\ell^2(\mathbb{N}_m)$, namely $Y^{(m)}(L) = \mathcal{D}\{y_n(L)\}_{n \geq m}$. It is clear that

\[
 J_0^{(0)}(\mu) = J_0(\mu), \quad J^{(0)}(L; \mu) = \mathcal{J}(L; \mu), \quad Y^{(0)}(L) = Y(L).
\]

Let $T^{(m)}(L) : \ell^2(\mathbb{N}_m) \to \mathcal{H}_m$ be the natural restriction of the operator $T(L)$ defined in (5.3). The equality

\[
 (A_\alpha^{(m)} - L)^{-1} - (A_0^{(m)} - L)^{-1} = T^{(m)}(L)(\mu\mathcal{J}^{(m)}(L; \mu)^{-1} - (2Y^{(m)}(L))^{-1}) T^{(m)}(L)^* \]
can be justified in the same way as (6.6). It is important that all the operator-valued functions appearing in (8.2) are analytic in the domain \( \Omega_m = \mathbb{C} \setminus [m + 1/2, \infty) \).

It follows from (8.2) that the operator
\[
(\hat{A}_\alpha^{(m)} - L)^{-1} - (A_0 - L)^{-1}
\]
is equal to the orthogonal sum of the null operator acting in the space \( \ell^2([0, \ldots, m - 1], L^2(\mathbb{R})) \) and the operator
\[
T^{(m)}(L)(\mu \delta^{(m)}(L; \mu)^{-1} - (2Y^{(m)}(L))^{-1})T^{(m)}(L)^*.
\]
The next statement is an immediate consequence of the above reasonings.

**Lemma 8.1.** For any \( \alpha > 0 \) and \( m \in \mathbb{N} \) and for any \( L \notin [1/2, \infty) \) we have
\[
\text{rank}
\bigg((A_\alpha - L)^{-1} - (\hat{A}_\alpha^{(m)} - L)^{-1}\bigg) = m
\]
and therefore,
\[
\sigma_{a.c.}(\hat{A}_\alpha^{(m)}) = \sigma_{a.c.}(A_\alpha), \quad m_{a.c.}(E; \hat{A}_\alpha^{(m)}) = m_{a.c.}(E; A_\alpha) \text{ a.e.}
\]

Now it is easy to conclude the proof of Theorem 5.1 for \( E \geq 1/2 \). Take \( m \in \mathbb{N} \) such that \( E < m + 1/2 \). Theorem 5.1 evidently applies to the matrices \( J^{(m)}_0 \), and the scheme developed in Section 7 works for the operator \( \hat{A}_\alpha^{(m)} \) without any change. One only has to keep in mind that now in the corresponding version of Lemma 7.3 one can take \( E < m + 1/2 \). As a result, we obtain that
\[
\sigma_{a.c.}(A_\alpha^{(m)}) = \sigma_{a.c.}(A_0^{(m)}) \cup \sigma_{a.c.}(J^{(m)}_0(\mu)),
\]
\[
m_{a.c.}(E; A_\alpha^{(m)}) = m_{a.c.}(E; A_0^{(m)}) + m_{a.c.}(E; J^{(m)}_0(\mu))
\]
Taking into account the equalities (8.1) and (8.3), we arrive at the desired result.

Note that for \( \alpha < \sqrt{2} \) it is easy to prove Theorem 5.1 by means of the quadratic form approach, cf. proof of Theorem 6.2 in [13]. However, in the present paper we decided to give a unified exposition for all values of the parameter.

9. Concluding remarks

In the model suggested by Smilansky in [12] the operators \( A_\alpha \) act in the space \( L^2(\Gamma \times \mathbb{R}) \) where \( \Gamma \) is a metric graph (in another terminology, a quantum graph). The model is interpreted as “harmonic oscillator, attached to a graph”. In order to describe the setting of this, more general problem, consider first the case when \( \Gamma = \Gamma_m \) is a star graph...
with \( m \) bonds, each of infinite length. More precisely, \( \Gamma_m \) is the union of \( m \) half-lines \( B_1, \ldots, B_m \), emanating from the common vertex \( o \), the root of the tree. Let \( t \in [0, \infty) \) stand for the coordinate along each bond. The value \( t = 0 \) corresponds to the root \( o \). Each function \( f \) on \( \Gamma_m \) can be viewed as a family of \( m \) functions \( f_j = f \mid_{B_j} \) defined on \([0, \infty)\). If each \( f_j \) has the derivative at \( t = 0 \), we set

\[
[f'](o) = \sum_{j=1}^{m} f'_j(0).
\]

In the case considered, the operator \( A_\alpha \) in \( L^2(\Gamma_m \times \mathbb{R}) \) is defined by the differential expression (1.1) for \((x,q) \in B_j \otimes \mathbb{R}, j = 1, \ldots, m\) and the matching condition

\[
[U'_x(\cdot, q)](o) = \alpha q U(o,q), \quad q \in \mathbb{R}.
\]

The real axis \( \mathbb{R} \) with the marked point \( o = 0 \) can be identified with the graph \( \Gamma_2 \), and in this case the condition (9.1) turns into (1.2).

All the results of the present paper extend to the star graphs \( \Gamma_m \) with an arbitrary \( m > 0 \), with only minor changes: 1) the equality in (2.4) has to be replaced by \( m_{a.c.}(E; A_0) = mn \); 2) the borderline point between the small and the large values of \( \alpha \) is now \( m/\sqrt{2} \), and the expression for \( \mu \) becomes \( \mu = m(\alpha \sqrt{2})^{-1} \). The equalities (1.3) and (1.4) survive. The proofs basically remain the same, but the technical calculations sometimes become rather lengthy. This was the only reason, why we restricted ourselves to the case \( m = 2 \) in the main part of this paper.

In a similar way, the case when \( \Gamma \) is a general star graph with \( m \) bonds can be considered. Some of the bonds (say, \( m_0 \) where \( 0 \leq m_0 \leq m \)) are supposed to be of infinite, and other of finite length. The Dirichlet boundary condition is imposed at the ends of the finite bonds. The point spectrum and the essential spectrum of the operators \( A_\alpha \) were considered in [13] for this case. The a.c. spectrum can be analyzed by means of the same approach as in the present paper, but somewhat more serious changes in the formulations are necessary. They stem from the fact that now in the analog of (2.4) we have \( m_{a.c.}(E; A_0) = m_0 n \). In particular, if \( m_0 = 0 \), that is if the graph is compact, then the spectrum of \( A_0 \) is discrete. It remains discrete for \( A_\alpha \) with \( \alpha < m/\sqrt{2} \), but its absolutely continuous component fills the whole of \( \mathbb{R} \) for \( \alpha \geq m/\sqrt{2} \).

The case of an arbitrary metric graph with a finite number of bonds can be also analyzed, but this requires a bit more advanced technical
tools. Still, the main ideas remain the same. This material will be presented elsewhere.

**Appendix A. Proof of Theorem 3.1**

The proof is based upon the Gilbert – Pearson theory [4] of subordinate solutions, or more exactly upon the version of this theory for Jacobi matrices, see [8].

First of all, we have to consider the homogeneous equation

\[ J_0(\mu)h = zh, \]

or

(A.1) \[ d_{n+1}C(n+1) + ((2n+1)\mu - z)C(n) + d_nC(n-1) = 0. \]

This equation is similar to (3.9), it can be written in the form (3.16), and moreover, for the coefficients in the decompositions (3.17) we have

\[ a_0 = 2\mu, \quad a_1 = -(\mu + z); \quad b_0 = 1, \quad b_1 = -1. \]

Comparing this with (3.21), we conclude that the asymptotic formulas (3.22) apply to the system (A.1) if we take \( L = z/\mu \).

1. If \( \mu > 1 \), the formula (3.22) shows that for any \( E \in \mathbb{R} \) equation (A.1) has a subordinate solution. Namely, this is the solution with the sign “+” in the exponent. It follows that the spectrum of \( J_0(\mu) \) is discrete. It is easy to show that for \( \mu > 1 \) the matrix \( J_0(\mu) \) is positive definite, hence its eigenvalues tend to \( +\infty \).

2. If \( \mu < 1 \), the formula (3.22) shows that for any \( E \in \mathbb{R} \) equation (A.1) has no subordinate solution, since \( |C^\pm(n)| \sim n^{-1/2} \).

As for any self-adjoint Jacobi matrix, the spectrum of \( J_0(\mu) \) is simple, and the element \( e_0 = \{1, 0, 0 \ldots\}^\top \) can be taken as the generating vector. Let \( \mathcal{E} \) stand for the spectral measure of the operator \( J_0(\mu) \), then there exists a non-negative function \( \tau \in L^1(\mathbb{R}) \) such that

\[ (\mathcal{E}(\Delta)e_0, e_0)_{\ell^2} = \int_{\Delta} \tau(l)dl \]

for an arbitrary Borelian set \( \Delta \subset \mathbb{R} \). Our aim is to show that \( \tau(l) \neq 0 \) a.e. on \( \mathbb{R} \).

Let \( m_\infty(z) \) be the Weyl function for the equation \( J(\mu)h - zh = 0 \). It is a Herglotz function, and therefore it has boundary limits \( m_\infty(E + i0) \) for a.e. \( E \in \mathbb{R} \). It follows from [8], Theorem 1 that \( \text{Im} m(E + i0) \neq 0 \) a.e. According to the formula (5) from [8]

\[ (J_0(\mu) - z)^{-1}e_0, e_0 = (\mu - z + d_1m_\infty(z))^{-1}. \]

Recall that by (3.10) \( d_1 = (3/4)^{1/4} \).
By the Spectral Theorem,

\[
(J_0(\mu) - z)^{-1}e_0, e_0 = \int_{\mathbb{R}} \tau(l) dl / (l - z).
\]

The last two equalities imply (when \( z = E + i\varepsilon \) and \( \varepsilon \to +0 \)) that

\[
\text{Im}(\mu - z + d_1 m_\infty(z))^{-1} \to \frac{-d_1 \text{Im} m_\infty(E + i0)}{|\mu - E + d_1 m_\infty(E + i0)|^2} \neq 0.
\]

This limit is equal to \( \pi \tau(E) \). Therefore, \( \tau(E) > 0 \) a.e. which shows that \( m(E; J_0(\mu)) = 1 \) a.e.

3. The matrix \( J_0(1) \) is non-negative, therefore its spectrum lies on \([0, \infty)\). The formula (3.22) shows that for \( E > 0 \) the equation \( J_0(1)h = Eh \) has no subordinate solution. The equality \( m_a.c.(E; J_0(1)) = 1 \) a.e. on \( \mathbb{R}_+ \) can be proved in the same way as in the previous case.

This concludes the proof of Theorem 3.1.

Appendix B. Proof of Theorem 7.1

B.1. Remarks on the a.c. spectrum and on its multiplicity.

Before giving the proof, we present some remarks of a rather general nature, concerning the notion of a.c. spectrum. We consider this useful because of the dual nature of the a.c. spectrum. Indeed, it combines some features coming from the measure theory with another ones, coming from topology of the real line.

Let \( K \) be a self-adjoint operator whose spectrum is purely a.c., of multiplicity one. This means that \( K \) is unitary equivalent to the operator of multiplication, \( u(l) \mapsto lu(l) \), in the space \( L^2(\mathcal{X}; dl) \) where \( \mathcal{X} \subset \mathbb{R} \) is some Borelian set and \( dl \) is the Lebesgue measure. The spectrum of \( K \) is the closure \( \overline{\mathcal{X}} \) of the set \( \mathcal{X} \), and it may happen that \( \text{meas}(\overline{\mathcal{X}} \setminus \mathcal{X}) > 0 \). For this reason, characterization of the a.c. spectrum of \( K \) has to include description of both the set \( \sigma_{a.c.}(K) \) and the multiplicity function \( m_{a.c.}(l; K) \). Say, in the above example \( \sigma_{a.c.}(K) = \overline{\mathcal{X}} \) and the multiplicity function is equal to one a.e. on \( \mathcal{X} \) and to zero a.e. outside \( \mathcal{X} \). It is clear, how this extends to the case of spectrum of higher, or of varying multiplicity.

Note that any Borelian set \( \mathcal{X} \subset \mathbb{R} \), such that \( m_{a.c.}(l; K) > 0 \) a.e. on \( \mathcal{X} \) and \( m_{a.c.}(l; K) = 0 \) a.e. on \( \mathbb{R} \setminus \mathcal{X} \), in the book [15] is called the core of \( \sigma_{a.c.}(K) \).

B.2. Preparatory material. We need some facts from the general theory of spectral measure. Let \( A \) be a self-adjoint operator in a separable Hilbert space \( \mathcal{H} \) and \( \mathcal{E} \) be its spectral measure. For elements \( \phi, \psi \in \mathcal{H} \), we denote by \( \rho_{\phi, \psi} \) the scalar complex-valued measure
The Radon-Nikodym derivative with respect to the Lebesgue measure. This derivative is defined a.e. on $\mathbb{R}$. It is equal to zero if either of the elements $\phi, \psi$ is orthogonal to the absolutely continuous subspace of the operator $A$.

For any $\phi, \psi \in \mathcal{H}$ there exists a subset $\mathcal{X}_{\phi, \psi} \subset \mathbb{R}$ of zero Lebesgue measure, such that

$$\lim_{L \to E+i0} ((A - L)^{-1} - (A - \overline{L})^{-1}) \phi, \psi = 2\pi i \rho'_{\phi, \psi}(E), \quad E \notin \mathcal{X}_{\phi, \psi}. \quad \text{(B.1)}$$

In such cases we do not use the notation $[\cdot](E)$, since the exceptional set $\mathcal{X}_{\phi, \psi}$ depends on the chosen elements. The set $\mathcal{X}_{\phi, \psi}$ can be chosen in such a way that the relation (B.1) is satisfied for the pairs $\{\phi, \phi\}$, $\{\psi, \psi\}$ and $\{\phi, \psi\}$ simultaneously, then it is satisfied also for any pair from the linear hull of the elements $\phi$ and $\psi$.

Suppose now that $G$ is a bounded linear operator, such that the jump (7.1) exists a.e. on $\Delta$ where $\Delta \subset \mathbb{R}$ is a given Borelian set. This implies that for any pair $\phi, \psi \in \mathcal{H}$ the limit

$$\lim_{L \to E+i0} ((A - L)^{-1}G\phi - (A - \overline{L})^{-1}G\phi, G\psi) = (V_{A,G}(E)\phi, \psi)$$

does exist for $E \notin \mathcal{X}_G$ where $\mathcal{X}_G \subset \Delta$ is a set of Lebesgue measure 0. Unlike (B.1), the set $\mathcal{X}_G$ does not depend on the choice of $\phi$ and $\psi$. It follows from (B.1) that necessarily

$$\lim_{L \to E+i0} ((A - L)^{-1} - (A - \overline{L})^{-1}) G\phi, G\psi = (V_{A,G}(E)\phi, \psi) \quad a.e. \text{ on } \Delta. \quad \text{(B.2)}$$

Again, here the exceptional set may depend on $\phi$ and $\psi$. However, it is important that according to our assumption, for $E \notin \mathcal{X}_G$ the expression in the left-hand side of (B.2), and thus the one in the right-hand side, is the sesqui-linear form of a bounded operator.

Take a dense countable set $\{\phi_n\}$ of elements in $\mathcal{H}$, then there exists a subset $\mathcal{X}' \subset \Delta$ of the Lebesgue measure 0, such that for $E \in \Delta \setminus \mathcal{X}'$ the equality (B.2) is satisfied for all pairs $\{\phi_n, \phi_m\}$, and therefore for any $\phi, \psi$ from the linear hull $\mathcal{M}$ of the system $\{\phi_n\}$. In other words, there exists a dense linear subspace $\mathcal{M} \subset \mathcal{H}$, such that (B.2) is satisfied for all $E \in \Delta \setminus \mathcal{X}'$ and for all $\phi, \psi \in \mathcal{M}$ simultaneously.

**B.3. Proof of the theorem.** Taking an appropriate partition of the original set, we may assume that $m_{a.e.}(E; \mathcal{A}) = \nu = \text{const}$ a.e. on $\Delta$. Suppose first that $\nu < \infty$. According to the general theory of spectral measure, see e.g. [2], there is a subspace $\mathcal{H}_\Delta \subset \mathcal{H}$ invariant with respect to $A$, isometric to $L^2(\Delta, \ell^2; dx)$ and such that on $\mathcal{H}_\Delta$ the operator $A$ acts as multiplication by $x$. More precisely, let $\Pi : \mathcal{H}_\Delta \to$
L²(∆, ℓ²ν; dx) stand for the above isometry. Extending it by zero to the orthogonal complement of ℋ∆, we obtain a partially isometric operator Π : ℋ → L²(∆, ℓ²ν; dx). If φ ∈ Dom A ∩ ℋ∆, then Aφ ∈ ℋ∆ and

\[(ΠAφ)(x) = x(Πφ)(x).\]

For any φ, ψ ∈ ℋ∆ and any Borelian subset ∂ ⊂ ∆ we have

\[ρ_{φ,ψ}(∂) = \int_∂ ((Πφ)(x), (Πψ)(x))_ℓ²ν dx\]

and

\[ρ'_{φ,ψ}(E) = ((Πφ)(E), (Πψ)(E))_ℓ²ν, \quad a.e. \ on \ ∆.\]

Moreover, ρ'_{φ,ψ}(E) = 0 a.e. on ∆, provided that either of the elements φ, ψ is orthogonal to ℋ∆. Now, the formula (B.2) and the above remarks imply that there exists a dense linear subset M ⊂ ℋ, such that for all φ, ψ ∈ M the equality

(B.3) \[ (V_{A,G}(E)φ, ψ) = 2πi ((Πφ)(E), (Πψ)(E))_ℓ²ν \]

is satisfied a.e. on ∆, and the exceptional subset does not depend on the choice of φ, ψ. The expression in the right-hand side is the sesquilinear form of an operator in ℓ²ν. Necessarily, its rank does not exceed ν. Therefore, the restriction of the operator V_{A,G}(E) to M also has rank no greater than ν. Since the linear set M is dense in ℋ and the operator V_{A,G}(E) is bounded, we find that rank V_{A,G}(E) ≤ ν.

In order to obtain the reverse inequality, take elements f₁, ..., fν ∈ ℋ, such that

\[(Πf_j)(x) = χ∆(x)e_j, \quad j = 1, ..., ν\]

where χ∆ is the characteristic function of the set ∆ and the vectors \{e_j\} form the natural basis in ℓ²ν. Since the range of G is assumed dense, we can choose elements φ_j ∈ ℋ in such a way that

\[∥Gφ_j - f_j∥_ℓ²ν < ε, \quad j = 1, ..., ν,\]

where ε > 0 is arbitrarily small. Then

\[\int_∆ ∥(Πφ_j)(x) - χ∆(x)e_j∥_ℓ²ν dx = ∥Πφ_j - f_j∥_ℓ²ν < ε².\]

This yields that the vectors (Πφ_j)(E) ∈ ℓ²ν are linearly independent for E lying outside a subset \(\mathcal{X}_ε\) of a small measure η(ε), and η(ε) → 0 as ε → 0. For any E ∈ ∆ \ \(\mathcal{X}_ε\) we have rank V_{A,G}(E) = ν. Letting ε → 0, we conclude from (B.3) that rank V_{A,G} = ν a.e. on ∆.

The equality (7.2) is justified for any ν < ∞. Therefore, it remains valid also if ν = ∞.
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