Abstract. We investigate (possibly uncountable) graphs equipped with an action of a groupoid and a measure invariant under this action. Examples include periodic graphs, fractal graphs and graphings. Making use of Connes’ non-commutative integration theory we construct a Zeta function and present a determinant formula for it. We show that our construction is compatible with convergence of graphs in the sense of Benjamini-Schramm. As an application we show that the Zeta function can be calculated by an approximation procedure if the groupoid in question is a sofic group. As special cases we recover all earlier corresponding convergence results. Along our way we also present a rather careful study of colored graphs in our setting. This study is based on inverse-semigroups.

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INTRODUCTION

The theory of Zeta functions of finite graphs is a well-established topic connecting various branches of mathematics, see e.g. the monograph by Terras [36]. In contrast, Zeta functions on infinite graphs are much less understood. In fact, for general infinite graphs it is not even clear how to define a Zeta function in the first place.

Recent years have seen quite some interest in Zeta functions on infinite graphs. Indeed, for certain periodic graphs an ad-hoc definition of the Zeta function has been given by Clair / Mokhtari-Sharghi in [4] and for certain specific examples it has been investigated how to define a Zeta function via suitable approximations by Grigorchuk / Zuk [15], Clair / Mokhtari-Sharghi [5] and Guido / Isola / Lapidus [17, 18]. The authors of [17] note as a main motivation for their study that there are only very few infinite graphs for which a Zeta function is defined. Also, for the two dimensional integer lattice a Zeta function has been defined and computed by Clair in [6] and for general regular graphs with a transitive group action a Zeta function has been defined and studied in its connection to heat kernels in Chinta / Jorgenson / Karlsson [3]. A recent approach for a class of infinite weighted graphs can be found in [9]. However, so far, there is no general procedure on how to associate a Zeta function to a graph or how to approximate it and there is no closed formula for a Zeta function on a general graph. This is the starting point for our paper. Our main aims are the following:

• To associate a Zeta function to rather general graphs.
• To provide a closed formula for the Zeta function via determinants on von Neumann algebras.
• To study approximation by Zeta functions on finite graphs.

The corresponding results are all new in the general context provided here. At the same time they give a systematic and unified foundation for the works mentioned above. Moreover, they allow us to introduce a rather general class of graphs on which the Zeta function can be approximated viz graphs periodic with respect to a free action of a sofic group.

To achieve the mentioned aims we put forward the concept of a measure graph \((G, M)\). Here, \(G\) is a measurable graph and \(M\) is a measure on its vertex set. This measure needs to satisfy a certain invariance property. In order to formulate this property we will need the action of a groupoid \(G\) on \(G\). The assumptions required for this action lead us to the concept of a graph over a groupoid. All of this is discussed in Section 1. Our framework may be of interest for other questions as well. For example it may be useful for dealing with random Schrödinger operators on graphs. Our considerations are phrased within the measurable category. However, in prominent classes of examples we often have some additional topological information at hand. This leads us to a discussion of a topological setting in Subsection 1.4.

In the subsequent Subsection 1.5 we discuss various classes of examples.

Let us already here mention two further classes of examples studied later in the paper: Colored graphs are discussed at quite some length in the second part of the paper in Section 7. In fact, we present a new systematic approach to such graphs via inverse semigroups. Another popular class of examples are graphings as discussed in Section 8. As a consequence of our considerations we obtain in our context a systematic perspective on results of Elek [11] on convergence of sequences of finite graphs towards a graphing. Our corresponding results on colored graphs and graphings will provide the crucial background for our result on continuity of the Zeta function later on.

After having presented our framework in Section 1 we then use it and introduce the concept of the Ihara Zeta function of a measure graph \((G, M)\) in Section 2. The Zeta function
is introduced as the exponential of a power series. The coefficients of the power series are determined via Connes’ non-commutative integration theory. In this way we effectively obtain these coefficients as integrals over the space of vertices. This can be seen as a crucial new insight of our study.

Non-commutative integration theory also allows us to introduce von Neumann algebras associated to measure graphs. This is discussed in Section 4 and may, again, be of independent interest in further studies as well. These tools serve us to prove that the Zeta function can be calculated via a determinant of an operator on the vertices. Specifically, with notation introduced below Theorem 5.3 gives for each measure graph $G$ the following.

\[ \text{Determinant formula. } Z_{(G,M)}(u)^{-1} = (1 - u^2)^{-\chi(G,M)} \det(I - uA_G + u^2Q_G). \]

Of course, the use of determinants of non-positive operators requires some care. Here, we essentially use the determinant provided in [17, 18]. For positive invertible operators, this notion coincides with the famous Fuglede-Kadison determinant [14]. For possibly non-invertible operators, one has to deal with singularities. Results for such elements in a von Neumann algebra associated with some countable, amenable group have recently been proven by Li / Thom in [26].

In Section 6 we have a look at the case that the vertex degree is constant. In this case the determinant formula can be considerably simplified. It can then be expressed via the so-called integrated density of states. This is the content of Theorem 6.3. The formula proven in the theorem has been used in [17] to define a Zeta function. It also has recently been obtained in [3] via an analysis of Bessel functions and heat kernels.

Our setting can be used to investigate continuity of the Zeta function in the underlying graph. We obtain various precise versions of this continuity. The basic result, Theorem 3.2 gives continuity in the underlying measure. Combined with the systematic perspective on colored graphs and graphings developed in Section 7 and Section 8 we then rather easily obtain the following result (Section 9):

\[ \text{Continuity result. } \text{If the graphs } G_n \text{ converge to the graph } G \text{ in the sense of Benjamini-Schramm then } Z_{G_n} \text{ converge to } Z_G \text{ compactly around zero.} \]

As discussed in Section 9 this result covers all the earlier results on convergence of graph Zeta functions given in [5, 15, 17, 18]. It even strengthens them by providing an interpretation of the limit as the Zeta function of a graph. More importantly, the result can be used to provide a (rather large) class of new examples on which the Zeta function can be obtained via approximation. These are the graphs allowing for the action of a sofic group. This is discussed in Section 10.

Our approach can be applied to further class of examples, in particular, random ensembles of graphs. This will be discussed elsewhere.

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List of Essential Pieces of Notation

Here we present a list of the main pieces of notation together with a short explanation and the number of the pages on which they are introduced.

- $G = (V,E)$: graph with vertex set $V$ and edge set $E$ (Page 4).
- $V(2)$: set of pairs of vertices in the same connected component (Page 5).
- $a_G$: the adjacency matrix of $G$ (Page 5).
- $B^G_r(x)$ rooted graph with root $x$ induced from the $r$-ball around $x$ (Page 5).
- $G$: groupoid (Page 7).
- $\nu$: transversal function on $G$ (Page 8).
- $\eta$: the canonical random variable assigning each vertex the mass one (Page 8).
- $\Omega$: units of the groupoid (Page 8).
- $u$: averaging function i.e. function satisfying $\nu \ast u = 1$ (Page 11).
- $\mathcal{M}^u_\nu(G)$: the set of all $\nu$-invariant measures on $\Omega$ with $m \circ \eta(u) = 1$ (Page 11).

1. The Framework of Measure Graphs

In this section we present the notation and concepts used throughout the paper. In particular, we introduce our concept of measure graphs over groupoids. These consists of a (not necessarily countable) measurable graph together with a measure satisfying some invariance property. The invariance property is phrased via a groupoid. More specifically, the basic pieces of data used in our setting are the following:

- A measurable graph $G = (V,E)$.
- A measurable groupoid $\mathcal{G}$ such that the graph is a space over the groupoid in the sense of Connes.
- A measure $m$ which is invariant with respect to the groupoid.
- An averaging function $u$ providing a connection between the groupoid and the graph.

These four pieces of data are discussed in subsequent subsections.

1.1. Graphs. Here we introduce the concept of graphs used in the sequel. These will be undirected graphs with uniform bounded vertex degree and without loops.

By a graph we mean a tuple $G = (V,E)$ consisting of a set of vertices $V \neq \emptyset$ and a set of edges $E \subseteq V \times V$ such that the following holds:

- Whenever $(x,y)$ belongs to $E$ then so does $(y,x)$.
- There is no $x \in V$ such that $(x,x)$ belongs to $E$.
- There is a $D > 0$ such that the cardinality of $\{y \in V : (x,y) \in E\}$ is bounded by $D$ for any $x \in V$.

Remark 1.1. We emphasize that we do not put any restrictions on the cardinality of $V$ nor $E$.

Let $G = (V,E)$ be a graph. For given $x \in V$ we call the pair $(G,x)$ a rooted graph with root $x$. If $(x,y) = e \in E$ we write $x \sim y$ and call $x = o(e)$ the origin and $y = t(e)$ the terminal vertex of $e$. For an edge $e = (o(e),t(e))$ we define the reversed edge via $\bar{e} = (t(e),o(e))$. Two edges $e,f$ are called incident if $\{t(e),o(e)\} \cap \{t(f),o(f)\}$ consists of exactly one element.

The vertex degree at $x$ is the number of edges with origin $x$. It will be denoted by $\deg(x)$. In this way, $\deg$ becomes a function from $V$ to the non-negative integers.
A path is a finite sequence of edges \((e_1, \ldots, e_n)\), such that \(a(e_{i+1}) = t(e_i)\) for each \(i = 1, \ldots, n - 1\). The number of edges occurring in a path \(P\) is called its length and is denoted by \(\ell(P)\). Two vertices \(x, y \in V\) are said to be connected, if there exists a path \((e_1, \ldots, e_n)\), such that \(a(e_1) = x \) and \(t(e_n) = y\). If \(x, y \in V\) are connected their combinatorial distance, \(d(x, y)\), is the length of the shortest path connecting them. If \(x\) and \(y\) are not connected we set \(d(x, y) = \infty\).

A connected component in a graph is a maximal set of vertices such that the combinatorial distance between any two elements of this set is finite. For an \(x \in V\) the connected component containing \(x\) is the set \(V(x)\) of all vertices which are connected with \(x\). We denote the induced subgraph by \(G(x) = (V(x), E \cap [V(x) \times V(x)])\). For vertices \(x, y \in V\), we will write \(x \sim y\) if \(x\) and \(y\) belong to the same connected component.

Further, for \(r \in \mathbb{N}\), we let \(B^G_r(x)\) denote the graph with root \(x\) which is induced by \(G\) when restricting the vertex set to the combinatorial \(r\)-ball around \(x\). Note that by assumption on the uniform boundedness of the degree the graph \(G(x)\) is at most countable and \(B^G_r(x)\) is finite.

The radius, \(\rho(G, x)\) of a finite connected graph \(G\) with root \(x\) is the maximal distance of a vertex from the root, i.e.

\[
\rho(G, x) = \max\{d(y, x) : y \in V(G)\}.
\]

Any graph comes naturally with a certain product space and a canonical function on it. This is discussed next. Let \(G\) be a graph. Then, we define

\[
V^{(2)} := V^{(2)}_G := \{(x, y) \in V \times V : G(x) = G(y)\} \subset V \times V.
\]

On \(V^{(2)}\) there is the canonical function, called adjacency matrix of \(G\), defined via

\[
a_G : V^{(2)} \to \{0, 1\}, \quad a_G(x, y) = 1 \text{ if } x \sim y \text{ and } a_G(x, y) = 0 \text{ else.}
\]

So, in particular, we have that

\[
E = a_G^{-1}(1) \subset V^{(2)}.
\]

We denote the restriction of \(d\) to \(V^{(2)}\) by \(d\) again.

Whenever \(V\) carries a \(\sigma\)-algebra, then \(V^{(2)}\) becomes a measurable space with the \(\sigma\)-algebra induced by the product \(\sigma\)-algebra on \(V \times V\) and so does its subset \(E\).

The real numbers and the complex numbers (and subsets thereof) will always be equipped with the Borel-\(\sigma\)-algebra generated by the open subsets. Moreover, we will need the extended positive half-axis \([0, \infty) = [0, \infty) \cup \{\infty\}\). It will be equipped with the \(\sigma\)-algebra generated by the Borel-\(\sigma\)-algebra on \([0, \infty)\).

Two graphs \(G_1 = (V_1, E_1)\) and \(G_2 = (V_2, E_2)\) are called isomorphic if there exists a bijective map \(\varphi : V_1 \to V_2\) with \(x \sim y\) if and only if \(\varphi(x) \sim \varphi(y)\). This map is then called a graph isomorphism. Two finite rooted graphs are called isomorphic if there exists an isomorphism between them which maps the root of one into the root of the other graph. Obviously, isomorphism is an equivalence relation on all finite rooted graphs. For \(r \geq 0\), we denote by \(A^D_r\) the set of all equivalence classes of finite connected rooted graphs with vertex degree bounded by \(D\) and radius not exceeding \(r\). By \(A^D\) we denote the union over \(r \geq 0\) of all \(A^D_r\). Due to the boundedness assumption on the degree, this is a countable set. We will equip it with the discrete topology and the induced \(\sigma\)-algebra (both of which agree with the power set). For any \(r > 0\) we let \(\pi_r\) be the map

\[
\pi_r : V \to A^D, \quad x \mapsto [B^G_r(x)],
\]
where $[\cdot]$ denotes the class of a rooted graph modulo isomorphy.

We will be interested in graphs carrying a $\sigma$-algebra so that the graph is locally constant in a certain sense.

**Definition 1.2 (Measurable graph).** A pair $(G, B)$ consisting of a graph $G = (V, E)$ and a $\sigma$-algebra $B$ on $V$ is called a **measurable graph** if the following conditions are satisfied:

- For any $r > 0$ the map $\pi_r : V \to A^D$, $x \mapsto [B^G_r(x)]$, is measurable.
- The adjacency matrix $a_G : V^{(2)} \to [0, \infty)$ is measurable.
- For any two measurable $a, b : V^{(2)} \to [0, \infty]$, the matrix product

  \[ a \ast b : V^{(2)} \to [0, \infty], \quad (a \ast b)(x, y) := \sum_{z > x} a(x, z)b(z, y) \]

  is measurable.

**Remark 1.3.** We do not require the measurability of $V^{(2)}$ as a subset of $V \times V$ (equipped with the product $\sigma$-algebra). We rather work directly with the $\sigma$-algebra induced on $V^{(2)}$ by the product $\sigma$-algebra. In some sense this is a main achievement of our definition. In fact, it is exactly this part of the definition that allows us to cope with the ’curse of symmetry’ (see e.g. [27] for discussion of this issue).

In a measurable graph certain basic quantities are automatically measurable. This is collected in the next proposition. It will be used tacitly in the sequel.

**Proposition 1.4 (Measurability of basic quantities).** Let $(G, B)$ be a measurable graph. Then, the following assertions hold:

- (a) The combinatorial distance $d : V^{(2)} \to [0, \infty)$ is measurable.
- (b) The diagonal $\{(x, x) : x \in V\} \subset V^{(2)}$ is measurable.
- (c) For any $r, s, t \geq 0$ and $\alpha, \beta \in A^D$ the set

  \[ \{ (x, y) \in V^{(2)} : \pi_r(x) = \alpha, \pi_s(y) = \beta, d(x, y) \leq t \} \]

  is measurable.

**Proof.** Clearly, (b) is a direct consequence of (a). Similarly, (c) is a direct consequence of (a) and the measurability of the $\pi_u$, $u \geq 0$. Thus, it suffices to show (a): By measurability of the adjacency matrix $a_G$ and the matrix product all powers $a^n_G$, $n \in \mathbb{N}$, (defined inductively via $a^1_G := a_G$ and $a^{n+1}_G := a_G \ast a^n_G$) are measurable. Now, clearly, $d(x, y) = 1$ holds if and only if $a_G(x, y) = 1$ holds and, for $n \geq 1$, $d(x, y) = n$ holds if and only if both $a^n_G(x, y) \neq 0$ and $a^k_G(x, y) = 0$, $k = 1, \ldots, n - 1$ hold. This shows measurability of the sets $\{(x, y) \in V^{(2)} : d(x, y) = n\}$ for $n \in \mathbb{N}$. This, then implies measurability of

\[ \{(x, y) \in V^{(2)} : d(x, y) = 0\} = V^{(2)} \setminus \bigcup_{n \in \mathbb{N}} \{(x, y) \in V^{(2)} : d(x, y) = n\}. \]

This finishes the proof. \square

**Proposition 1.5 (Measurability of product of matrix with a function).** Let $(G, B)$ be a measurable graph. Then, for any measurable $a : V^{(2)} \to [0, \infty]$ and any measurable $F : V \to [0, \infty]$ the map

\[ a \ast F : V \to [0, \infty], \quad x \mapsto \sum_{y > x} a(x, y)F(y) \]

is measurable. In particular, the vertex degree $\deg : V \to [0, \infty)$ is measurable.
Proof. As the $\sigma$-algebra on $V^{(2)}$ is the restriction of the product $\sigma$-algebra the maps

$$p_1 : V^{(2)} \to V, (x,y) \mapsto x, \text{ and } j : V \to V^{(2)}, x \mapsto (x,x)$$

are measurable. Hence,

$$\tilde{a} \cdot F = (a * (F \circ p_1)) \circ j$$

is measurable as a composition of measurable functions. Now, the last statement follows from $\deg = aG\tilde{1}$. □

Any measurable graph $(G, \mathcal{B})$ comes naturally with a canonical family of measures. More precisely, we define on such a graph for any $x \in V$ the measure $\tilde{\eta}^x$ on $V$ by

$$\tilde{\eta}^x := \sum_{y \approx x} \delta_y,$$

Here $\delta_y$ denotes the unit point mass at $y$ (i.e. $\delta_y(A) = 1$ if $y \in A$ and $\delta_y(A) = 0$ otherwise). We refer to the family $\tilde{\eta}^x, x \in V$, as the canonical component measures. It has a certain measurability property.

**Proposition 1.6** (Measurability of $\tilde{\eta}$). Let $(G, \mathcal{B})$ be a measurable graph. Then the canonical component measures $\tilde{\eta}^x, x \in V$, are measurable in the sense that for any measurable $F : V \to [0, \infty]$ the map

$$\tilde{\eta}(F) : V \to [0, \infty], x \mapsto \tilde{\eta}^x(F) = \sum_{y \approx x} F(y),$$

is measurable.

**Proof.** This is a direct consequence of the preceding proposition with $a \equiv 1$. □

1.2. Graphs over groupoids and invariant measures. Here we discuss groupoids, invariant measures on groupoids and spaces over groupoids. In the measurable setting this can be found e.g. in Connes’ lecture notes [7]. Based on these lecture notes a discussion of random Schroedinger operators in this context was then given in [25]. There, a specific situation is singled out and studied in some detail. This situation is called ‘admissible setting’ there. Here, we basically present a graph version of the corresponding considerations centered around the admissible setting in [25].

**Notation.** As usual we will denote the set of all measurable functions on a measurable space by $\mathcal{F}(X)$. The set of non-negative measurable functions is then denoted by $\mathcal{F}^+(X)$. The set of all measures on $X$ is denoted by $\mathcal{M}(X)$.

A concise definition of a groupoid is that it is a small category in which every morphism is an isomorphism. A more detailed definition can then be given as follows, see e.g. [32].

**Definition 1.7.** A triple $(G, \cdot, ^{-1})$ consisting of a set $G$, a partially defined multiplication $\cdot$, and an inverse operation $^{-1} : G \to G$ is called a groupoid if the following conditions are satisfied:

- $(g^{-1})^{-1} = g$ for all $g \in G$,
- If $g_1 \cdot g_2$ and $g_2 \cdot g_3$ exist, then $(g_1 \cdot g_2) \cdot g_3$ and $g_1 \cdot (g_2 \cdot g_3)$ exist as well and they are equal,
- $g^{-1} \cdot g$ exists always and $g^{-1} \cdot g \cdot h = h$, whenever $g \cdot h$ exists,
- $h \cdot h^{-1}$ exists always and $g \cdot h \cdot h^{-1} = g$, whenever $g \cdot h$ exists.
A given groupoid $\mathcal{G}$ comes along with the following standard objects. The subset
$$\Omega := \mathcal{G}^{(0)} := \{g \cdot g^{-1} \mid g \in \mathcal{G}\}$$
is called the set of units.

For $g \in \mathcal{G}$ we define its range $r(g)$ by $r(g) = g \cdot g^{-1}$ and its source by $s(g) = g^{-1} \cdot g$. Moreover, we set $\mathcal{G}^\omega = r^{-1}(\{\omega\})$ for any unit $\omega \in \mathcal{G}^0$. One easily checks that $g \cdot h$ exists if and only if $r(h) = s(g)$.

The groupoids under consideration will always be measurable, i.e., they possess $\sigma$-algebras such that all relevant maps are measurable. More precisely, we require that $\cdot : \mathcal{G}^{(2)}_G \to \mathcal{G}$, $\cdot : \mathcal{G}^{-1} \to \mathcal{G}$, $s, r : \mathcal{G} \to \mathcal{G}^0$ are measurable, where
$$\mathcal{G}^{(2)}_G := \{(g_1, g_2) \mid s(g_1) = r(g_2)\} \subset \mathcal{G}^2$$and $\mathcal{G}^0 \subset \mathcal{G}$ are equipped with the induced $\sigma$-algebras. Furthermore, we assume that singletons $\{\omega\}$ with $\omega \in \mathcal{G}^0$ are measurable as subsets of $\mathcal{G}^0$. In this way, $\mathcal{G}^\omega \subset \mathcal{G}$ become measurable sets (and thus measurable spaces).

**Definition 1.8** (Graph over $\mathcal{G}$). Let $\mathcal{G}$ be a measurable groupoid with the previously introduced notations. A triple $(\mathcal{G}, \pi, J)$ consisting of a measurable graph $\mathcal{G}$ with vertex set $V$ and maps $\pi$ and $J$ is called a graph over $\mathcal{G}$ if the following properties are satisfied.

- The map $\pi : V \to \Omega$ is measurable.
- For any $\omega \in \Omega$ the set
  $$V^\omega := \pi^{-1}(\{\omega\})$$is a connected component of $\mathcal{G}$.
- The map $\eta : \Omega \to \mathcal{M}(V)$, $\eta^\omega := \sum_{y \in \pi^{-1}(\omega)} \delta_y$, is measurable in the sense that for any measurable $F : V \to [0, \infty]$, the map $\Omega \to [0, \infty]$, $\omega \to \eta^\omega(F)$ is measurable.
- The map $J$ assigns, to every $g \in \mathcal{G}$, a graph isomorphism $J(g) : V^{s(g)} \to V^{r(g)}$ with the properties $J(g^{-1}) = J(g)^{-1}$ and $J(g_1 \cdot g_2) = J(g_1) \circ J(g_2)$ if $s(g_1) = r(g_2)$.

The map $\eta$ is called the canonical random variable.

**Remark 1.9.** We obviously have
$$\eta \circ \pi = \tilde{\eta}.$$
Thus, if $\Omega$ carries the $\sigma$-algebra induced by $\pi$ then measurability of $\eta$ is immediate from the already shown measurability of $\tilde{\eta}$.

**Notation.** To simplify notation, we will often write $gh$ respectively $gx$ for $g \cdot h$ respectively $J(g)x$.

Our next aim is to exhibit natural measures on these objects. The first step in this direction is the definition of a transverse function.

**Definition 1.10** (Transversal function). Let $\mathcal{G}$ be a measurable groupoid and with the notation given above. A transversal function $\nu$ of $\mathcal{G}$ is a map $\nu : \Omega \to \mathcal{M}(\mathcal{G})$ with the following properties:

- The map $\omega \mapsto \nu^\omega(f)$ is measurable for every $f \in F^+(\mathcal{G})$.
- $\nu^\omega$ is supported on $\mathcal{G}^\omega$, i.e., $\nu^\omega(\mathcal{G} - \mathcal{G}^\omega) = 0$. 
• $\nu$ satisfies the following invariance condition
\[
\int_{G^s(g)} f(g \cdot h) d\nu^s(g)(h) = \int_{G^r(g)} f(k) d\nu^r(g)(k)
\]
for all $g \in G$ and $f \in F^+(G^r(g))$.

In the next definition we introduce appropriate measures on the base space $\Omega$ of an abstract groupoid $G$.

**Definition 1.11 (Invariant measure).** Let $G$ be a measurable groupoid with a transversal function $\nu$. A measure $m$ on the base space $(\Omega, B_\Omega)$ of units is called $\nu$-invariant (or simply invariant, if there is no ambiguity in the choice of $\nu$) if
\[
m \circ \nu = (m \circ \nu)^\sim,
\]
where $(m \circ \nu)(f) = \int_\Omega \nu^\omega(f) dm(\omega)$ and $(m \circ \nu)^\sim(f) = (m \circ \nu)(f)$ with $\tilde{f}(g) = f(g^{-1})$.

Analogously to transversal functions on the groupoid, we introduce a corresponding fiber-wise consistent family $\alpha$ of measures on the graph over the groupoid.

**Definition 1.12 (Random variable in the sense of Connes).** Let $G$ be a measurable groupoid and $G$ a graph over $G$ with vertex set $V$. A choice of measures $\xi : \Omega \to M(V)$ is called a random variable with values in $G$ (in the sense of Connes) if it has the following properties:

- The map $\omega \mapsto \xi^\omega(f)$ is measurable for every $f \in F^+(V)$,
- $\xi^\omega$ is supported on $G^\omega$, i.e., $\xi^\omega(V - V^\omega) = 0$,
- $\xi$ satisfies the following invariance condition
\[
\int_{V^s(g)} f(J(g)x) d\xi^s(g)(x) = \int_{V^r(g)} f(y) d\xi^r(g)(y)
\]
for all $g \in G$ and $f \in F^+(V^r(g))$.

**Remark 1.13.** Let $\xi$ be a random variable.

- By considering (positive) linear combinations of functions of the form
\[
F : V^2 \to [0, \infty], (x, y) \mapsto f(x)g(y),
\]
with measurable $f, g : V \to [0, \infty]$ and using standard monotone class arguments, we can easily obtain measurability of
\[
V \times \Omega \to [0, \infty], (x, \omega) \mapsto \xi^\omega(F(x, \cdot)),
\]
for any measurable $F : V \times V \to [0, \infty]$.

- If furthermore $V^{(2)}$ is a measurable subset of $V \times V$, then any measurable function $F$ on $V^{(2)} \to [0, \infty]$ can be extended (by zero) to a measurable function on $V \times V$. Thus, in this case we obtain measurability of $V \times \Omega \to [0, \infty], (x, \omega) \mapsto \xi^\omega(F(x, \cdot))$ for any measurable $F : V^{(2)} \to [0, \infty]$.

We can actually provide a large supply of random variables.

**Proposition 1.14 (Generating random variables).** Let $G$ be a measurable groupoid and $G$ a graph over $G$ with vertex set $V$. Then, the canonical random variable $\eta$ is a random variable.
Moreover, for any measurable $H : V \to [0, \infty]$ with $H(gx) = H(x)$ for all $x \in V$ and $g \in G$ with $\pi(x) = s(g)$ the map

$$\xi_H : \Omega \to \mathcal{M}(V), \xi_H^\omega := \sum_{y \in \pi^{-1}(\omega)} H(y) \delta_y$$

is a random variable.

Proof. The canonical random variable $\eta$ is a random variable by the very definition of a graph over a groupoid. Now, consider a measurable $H : V \to [0, \infty]$. We have to show a measurability and an invariance property of $\xi_H$. Now, whenever $F : V \to [0, \infty]$ is measurable, then so is $FH : V \to [0, \infty]$. Hence,

$$\omega \mapsto \xi_H^\omega = \eta^\omega(HF)$$

is measurable by the assumption on $\eta$. The invariance of $\xi_H$ is clear from the invariance property of $H$. \hfill \Box

Remark 1.15. If the diagonal $\{(x, x) : x \in V\}$ is a measurable subset of $V \times V$, then any random variable arises in this way. Indeed, in the case the previous remark applied with $F$ the characteristic function of the diagonal, gives easily that for any random variable $\xi$ the function $H(x) = \xi^\pi(x)(F(x, \cdot))$ is measurable and invariant. By construction $H$ is the density of $\xi$ with respect to $\eta$.

Whenever $G$ is a graph over a groupoid $G$ with transverse $\nu$, we use the following notation for the convolution of a $w \in \mathcal{F}^+(V)$ with respect to $\nu$

$$\nu * w(x) := \int_{G^\pi(x)} w(g^{-1}x) d\nu^\pi(g) \quad \text{for } x \in V.$$

A crucial fact about the integration of random variables is given in the following lemma, essentially contained in (the proof of) Lemma III.1 in [7], see [23] as well.

Lemma 1.16. Let $G$ be a measurable groupoid with transversal function $\nu$ and $\nu$-invariant measure $m$. Let $G$ be graph over $G$.

(a) For a given transversal function $\nu$ on $G$, the integral $\int_{\Omega} \nu^\omega(f) \, dm(\omega)$ does not depend on $f \in \mathcal{F}^+(G)$, provided $f$ satisfies $\nu(\widehat{f}) \equiv 1$.

(b) For a given random variable $\xi$ with values in $G$ the integral $\int_{\Omega} \xi^\omega(u) \, dm(\omega)$ does not depend on $u$, provided $u \in \mathcal{F}^+(V)$ satisfies $\nu * u \equiv 1$.

The previous lemma gives the possibility to define an integral over a random variable.

Definition 1.17 (Integral of a random variable). Let $G$ be a measurable groupoid with transversal function $\nu$ and $\nu$-invariant measure $m$. Let $G$ be a graph over $G$ with vertex set $V$ such that there exists a $u \in \mathcal{F}^+(V)$ satisfying $\nu * u \equiv 1$. Then, the integral over the random variable $\xi$ is denoted as $\int \xi$ and defined via

$$\int \xi = \int_{\Omega} \xi^\omega(u) \, dm(\omega),$$

where $u$ is any (not necessarily strictly positive) element of $\mathcal{F}^+(V)$ satisfying $\nu * u \equiv 1$.

The functions $u$ appearing in the previous definition deserve a special name.
Definition 1.18 (Averaging function). Let \( \mathcal{G} \) be a measurable groupoid with transversal function \( \nu \) and \( \nu \)-invariant measure \( m \). Let \( G \) be a graph over \( \mathcal{G} \) with vertex set \( V \). Then, any function \( u \in \mathcal{F}^+(V) \) satisfying \( \nu \ast u \equiv 1 \) is called an averaging function.

Following the corresponding considerations in [23] we note that existence of an averaging function is quite essential to our approach. In fact, it is only via these averaging functions that the integration of random variables could be defined above. On the conceptual level the existence of an averaging function \( u \) can be understood as giving a way to relate \( G \) and \( G \) via a map from \( \mathcal{F}(\mathcal{G}) \) to \( \mathcal{F}(V) \). Namely, every averaging function \( u \) gives rise to the fiberwise defined map

\[
q = q_u : \mathcal{F}(\mathcal{G}) \to \mathcal{F}(V)
\]

with

\[
q(f)(x) := \int_{G^{\pi(x)}} u(g^{-1}x) f(g) \, db^{\pi(x)}(g)
\]

for all \( x \in V \). Note the defining property of \( u \) implies that the map \( q \) satisfies \( q(1_G) = 1_G \).

1.3. Measure graphs. After all these preparations we can now introduce the main concept of our study. We will need one further assumption on countability of generators of the \( \sigma \)-algebras in question in order to apply the integration theory developed in [7].

Definition 1.19 (Measure graph over a groupoid). A pair \((G, M)\) consisting of a measurable graph \( G \) and a measure \( M \) on the vertices of \( G \) is called a measure graph over the groupoid \( \mathcal{G} \) or just measure graph for short if the following properties hold:

- \( G \) is a measurable groupoid and \( G \) is a graph over \( \mathcal{G} \).
- \( G \) admits a transversal measure \( \nu \) together with an invariant measure \( m \) and \( M = m \circ \eta \) (where \( \eta \) is the canonical random variable).
- The \( \sigma \)-algebras of both \( G \) and \( \Omega \) possess a countable basis of sets, all of which have finite measure w.r.t. \( M \) (respectively w.r.t. \( m \)).
- There exists a strictly positive averaging function.
- The canonical random variable \( \eta \) is integrable, i.e. \( \int \eta = \int \eta^\omega(u) \, dm(\omega) = \int_V u(x) \, dM(x) < \infty \).

The measure \( m \) will be called the invariant measure underlying the measure \( M \) and \( \nu \) will be called the underlying transversal on the groupoid.

Definition 1.20 (Invariant normalized measures). Let \( G \) be a graph over the measurable groupoid \( \mathcal{G} \) and let \( \nu \) be a transversal on \( \mathcal{G} \) and \( u \) an averaging function with respect to \( \nu \). Then, we denote by \( \mathcal{M}_u^\nu(G) \) the set of all measures \( m \) on \( \Omega \), which are invariant with respect to \( \nu \) and satisfy

\[
\int_{\Omega} \eta^\omega(u) \, dm(\omega) = 1.
\]

We refer to elements of the set \( \mathcal{M}_u^\nu(G) \) as invariant normalized measures.

For our further considerations two features of measure graphs will be crucial (see below):

- Any measure graph comes with a natural Hilbert space \( L^2(V, M) \). We will be concerned with operators on this Hilbert space.
- Any measure graph naturally allows for integration of random variable as it possesses by its very definition an averaging function.
1.4. **Topological measure graphs.** We have presented the above theory in the measurable category and not in the topological category. In this way, the presented theory is more general and has better points of contact with the existing theory of measurable graphs. However, it turns out that in concrete examples often more structure is given as the objects in question carry topologies and the measurable structure comes from the induced Borel-$\sigma$-algebras. For this reason we shortly present here a framework of what could be called a topological measure graph.

**Definition 1.21** (Topological graph over a groupoid). A graph $G$ over the measurable groupoid $G$ with transversal $\nu$ on the groupoid is called a topological graph over $G$ if the following properties hold:

- The $\sigma$-algebra of $\Omega$ is the Borel-$\sigma$-algebra of a topology generated by countably many sets making $\Omega$ into a compact space.
- The $\sigma$-algebra on $V$ is the Borel-$\sigma$-algebra of a topology generated by countably many sets $B_n$, $n \in \mathbb{N}$, such that the function $\Omega \to [0, \infty], \omega \mapsto \eta^\nu(B_n)$ is bounded on $\Omega$ for all $n \in \mathbb{N}$.
- The map $\pi : V \to \Omega$ is continuous.
- There exists a strictly positive continuous averaging function $u$ such that for any continuous bounded $F : V \to \mathbb{C}$ the function

$$\Omega \to \mathbb{C}, \omega \mapsto \eta^\nu(uF),$$

is continuous. We call a function $u$ as above a topological averaging function.

**Definition 1.22** (Topological measure graph). A pair $(G, M)$ consisting of a topological graph over the measurable groupoid $G$ with underlying transversal $\nu$ and a measure $M$ on $G$ is called a topological measure graph if there exists an $\nu$-invariant measure $m$ on $\Omega$ with $m(\Omega) < \infty$ and $M = m \circ \eta$.

**Remark 1.23.** By its very definition a topological graph over a groupoid is a graph over the groupoid. Similarly, a topological measure graph over a groupoid is a measure graph over a groupoid. In fact, the assumptions on the generators of the corresponding $\sigma$-algebras are satisfied as the algebras are Borel-$\sigma$-algebras of suitably generated topologies.

For later use we collect the following two features.

**Proposition 1.24.** Let $G$ be a topological graph over the measurable groupoid $G$ with underlying transversal $\nu$. Then, there exists a $C \geq 0$ such that any measure in $M^\nu$ satisfies $m(\Omega) \leq C$.

**Proof.** As $u$ is strictly positive, so is the continuous function $\omega \mapsto \eta^\nu(u)$. As $\Omega$ is compact, there exists then a $c > 0$ with $\eta^\nu(u) \geq c$ for all $\omega \in \Omega$. From

$$1 = \int_{\Omega} \eta^\nu(u)dm(\omega) \geq cm(\Omega)$$

we directly obtain the desired statement. \(\square\)
Proposition 1.25. Let $G$ be a topological graph over the groupoid $G$. Let $f : V \to \mathbb{C}$ be continuous with compact support. Then, the map

$$\Omega \to \mathbb{C}, \omega \mapsto \eta^\omega(f),$$

is continuous.

Proof. Let $S$ be the compact support of $f$. The topological averaging function $u$ attains its minimum $c$ on the compact set $S$. As $u$ is strictly positive, we have $c > 0$. We now define the function $v$ to be the minimum of $1/u$ and $1/c$. Then, $v$ is continuous and bounded and so is then $\tilde{f} = vf$. Moreover, by construction $u\tilde{f} = uvf = f$. Hence, we infer continuity of $\omega \mapsto \eta^\omega(f) = \eta^\omega(u\tilde{f})$. □

While it will not be used in the sequel, we note that even more is true under suitable topological assumptions on $V^{(2)}$. This is the content of the next proposition.

Proposition 1.26. Let $G$ be a topological graph over the groupoid $G$ and assume that the topology is Hausdorff and $V^{(2)}$ is closed in $V \times V$. Then, the matrix product $a * b : V^{(2)} \to \mathbb{C}$ is continuous for any continuous $a, b : V^{(2)} \to \mathbb{C}$ with compact support.

Proof. By our assumptions on the topology, we can extend $a, b$ (by zero) to continuous functions with compact support on the whole of $V \times V$. Thus, it suffices to show that

$$V \times V \to \mathbb{C}, (x, y) \mapsto \sum_{z \approx x} a(x, z)b(z, y)$$

is continuous for all continuous $a, b$ on $V \times V$ with compact support. The previous proposition easily shows that

$$V \ni x \mapsto \eta^{\pi(x)}(l) = \sum_{z \approx x} l(z)$$

is continuous for any continuous $l : V \to \mathbb{C}$ with compact support. Thus, the desired statement follows immediately for all functions $a, b$ of the form $a(x, y) = f(x)g(y)$ and $b(x, y) = h(x)k(y)$ with $f, g, h, k$ continuous function on $V$ with compact support. A straightforward approximation (via e.g. Stone-Weierstrass theorem) finishes the proof. □

1.5. Some examples of (topological) measure graphs. In this section we discuss some examples of (topological) measure graphs over groupoids. Special roles among such examples are played by colored graphs and graphings. These classes of examples will be discussed in later sections.

1.5.1. Finite graphs. Let $G$ be a finite connected graph with vertex set $V$. Equip $V$ with the discrete topology. Then the Borel-$\sigma$-algebra agrees with the topology and is finite. Let $G$ be the trivial groupoid (consisting only of one element $e$) acting on $G$ as the identity. Then, $\Omega = \{e\}$ consists of only one point and is therefore a compact space. Furthermore, $\pi : V \to \{e\}$ is continuous and the canonical random variable applied to any function is continuous. Thus, $G$ is a topological graph over $G$. Let $\nu$ be the point measure giving value 1 to the set $\{e\}$. Then, the measure $m$ given by the point measure on $\{e\}$ is invariant and the constant function $u$ with value 1 satisfies $\nu * u \equiv 1$. The measure $M$ on $G$ is then just the counting measure on the vertex set. In this way, $(G, M)$ is then a topological measure graph over $G$. 

THE IHARA ZETA FUNCTION FOR INFINITE GRAPHS
1.5.2. Periodic graphs. Let us first recall the notion of a periodic graph.

**Definition 1.27** (Periodic graph). A pair \((G, \Gamma)\) consisting of a countable infinite connected graph \(G = (V,E)\) and a countable subgroup \(\Gamma\) of the automorphism group of \(G\) is called periodic graph, if

- \(\Gamma\) acts co-finitely on \(V\), i.e. \(G/\Gamma = \{\Gamma x : x \in V\}\) is finite,
- \(\Gamma\) acts discretely on \(V\), i.e. for each \(x \in V\) the stabilizer \(\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}\) is finite.

A minimal subset \(F \subseteq V\) for which \(\Gamma F = V\) will be called a fundamental domain.

Let \((G, \Gamma)\) be a periodic graph. We equip its vertex set \(V\) with the discrete topology. Then the Borel-\(\sigma\)-algebra agrees with this topology and is countably generated. Let \(G\) be given by \(\Gamma\) with the corresponding action on \(G\). Then, \(\Omega\) consists exactly of the neutral element \(e\) of \(\Gamma\) and is compact. The map \(\pi : V \to \{e\}\) is continuous and the canonical transversal \(\nu\), i.e. the counting measure applied to any function, is continuous. In this way, \(G\) becomes a topological graph over \(\Gamma\). Then, the measure \(m\) giving mass 1 to the set \(\{e\}\) is invariant. Fix a fundamental domain \(F\) of the action of \(\Gamma\) and define the function \(u_0\) on \(V\) by

\[
u_0(x) := \frac{1}{|\Gamma_x|} 1_F(x),
\]

where \(|\Gamma_x|\) denotes the number of elements of the stabilizer of \(x\) and \(1_F\) is the characteristic function of \(F\). Then, a short calculation shows that \(\nu \ast u_0(x) = 1\) for all \(x \in V\). Thus, the function \(u_0\) is an averaging function. It is not strictly positive, though. However, for any \(\gamma \in \Gamma\), the function \(u_\gamma(\cdot) := u_0(\gamma \cdot)\) is clearly an averaging function as well. So, if we chose a sequence \((c_\gamma)_{\gamma \in \Gamma}\) of positive numbers with \(\sum c_\gamma = 1\), then

\[
u := \sum_{\gamma \in \Gamma} c_\gamma u_\gamma
\]

will be a strictly positive averaging function. Hence, we are again in the setting of a measure graph over a groupoid. The measure \(M\) on \(G\) is then just the counting measure on the vertex set. In this way, \((G, M)\) is then a topological measure graph over \(\Gamma\).

1.5.3. Limit graphings of weakly convergent graph sequences. Limit graphings of weakly convergent graph sequence have attracted quite some attention (see e.g. [11]). They also fall within our framework. This will be discussed later in the context of colored graphs.

1.5.4. Fractal graphs. Another example falling into our framework by giving rise to a limit graphing are fractal graphs which were studied in [18]. See Remark 9.6 for a further discussion.

2. The Ihara Zeta function of a measure graph

In this section we discuss how a general measure graph over a groupoid gives immediately rise to a Zeta function. We will call this the Ihara Zeta function. For finite and for periodic graphs it will be shown to agree with the usual Zeta function.

Before introducing the Zeta function we need some further graph theoretic notions. Let a graph \(G = (V,E)\) be given. A circle or closed path is a path \((e_1, \ldots, e_n)\) such that \(t(e_n) = o(e_1)\).

**Definition 2.1** (Reduced paths). Let \(G = (V,E)\) be a graph.
(a) A closed path \((e_1,\ldots,e_n)\) has a **backtrack** if \(e_{i+1} = \bar{e}_i\) for some \(i \in \{1,\ldots,n-1\}\). A path with no backtracking is said to be **proper**.

(b) We call a circle **primitive** if it is not obtained by going \(n \geq 2\) times around a shorter closed path.

(c) A closed path \((e_1,\ldots,e_n)\) has a **tail** if there is a number \(k \in \mathbb{N}\) such that \(\bar{e}_j = e_{n-j+1}\) for \(1 \leq j \leq k\).

(d) A closed path without backtracking or tail is called **reduced**.

**Definition 2.2 (Counting of closed path).** Whenever we are given a graph \(G = (V,E)\), we let \(N_j^G(x)\) be the number of reduced closed paths of length \(j\) starting at a given vertex \(x \in V\).

Similarly, \(P_j^G(x)\) will denote the number of primitive reduced circles of length \(j\) starting at \(x\). If the graph is understood from the context, we will suppress the superscript \(G\) and just write \(N_j(x)\) and \(P_j(x)\).

**Lemma 2.3 (Averaged circle counting).** Let \(\mathcal{G}\) be a groupoid and let \((G,M)\) be a measure graph over \(\mathcal{G}\). Then, for any natural number \(j\) the functions \(\xi_{N_j}\) and \(\xi_{P_j}\) given by

\[
\Omega \ni \omega \mapsto \xi_{N_j}^\omega := \sum_{x \in \pi^{-1}(\omega)} N_j(x)\delta_x \in \mathcal{M}(V),
\]

and

\[
\Omega \ni \omega \mapsto \xi_{P_j}^\omega := \sum_{x \in \pi^{-1}(\omega)} P_j(x)\delta_x \in \mathcal{M}(V)
\]

are random variables. In particular, the numbers

\[
\overline{N}_j := \int \xi_{N_j} = \int_{\Omega} \xi_{N_j}^\omega(u)dm(\omega) = \int_V N_j(x)u(x)dM(x),
\]

and

\[
\overline{P}_j := \int \xi_{P_j} = \int_{\Omega} \xi_{P_j}^\omega(u)dm(\omega) = \int_V P_j(x)u(x)dM(x)
\]

exist and do not depend on the choice of \(u\), provided \(\nu \ast u = 1\).

**Proof.** The functions \(N_j\) and \(P_j\) only depend on the isomorphism class of a finite ball around the corresponding points. Hence, they are clearly measurable by the definition of a measurable graph. Moreover, they are invariant under the action of the groupoid as this action results in graph isomorphisms and hence preserves the isomorphism classes of finite balls around the corresponding points. Thus, by Proposition 1.14 the functions \(\xi_{N_j}\) and \(\xi_{P_j}\) are random variables. Now, the remaining statements follow from the considerations concerning the integration of random variables in the previous section and specifically from Definition 1.17.

**Definition 2.4 (The Zeta function of a measure graph).** Let \((G,M)\) be a measure graph over a groupoid. The function

\[
Z(u) := Z_{(G,M)}(u) := \exp \left( \sum_{j \geq 1} \overline{N}_j u^j \right)
\]

is called **Ihara Zeta function** of \((G,M)\). The numbers \(\overline{N}_j, j \in \mathbb{N}\) appearing in its definition are called the **coefficients** of the Zeta function.
Let $(G, M)$ be a measure graph over a groupoid. $Z = Z_{(G, M)}$ is a holomorphic function on the disc $\{u \in \mathbb{C} \mid |u| < (D - 1)^{-1}\}$, where $D$ is the maximal vertex degree of $G$. Furthermore, $Z$ has an Euler product representation, that is, for each $|u| < (D - 1)^{-1}$ the equality

$$Z(u) = \prod_{j \geq 1} (1 - u^j)^{-\mathcal{P}_j}$$

holds.

**Proof.** The first assertion is a direct consequence of the observation $N_j(x) \leq D(D - 1)^j - 1$ and the finiteness of the measure $wdM$. For the second statement let us note that for each $j$ and each $x \in V$ the equality

$$N_j(x) = \sum_{i \mid j} P_i(x)$$

holds, where $i \mid j$ means that $i$ divides $j$. Thus, using an integrated version of this identity we obtain

$$\sum_{j \geq 1} \frac{N_j}{j} u^j = \sum_{j \geq 1} \sum_{i \mid j} \frac{P_i}{j} u^{ij}$$

(interchanging summation) $= \sum_{i \geq 1} \sum_{j \geq 1} \frac{P_i}{ij} u^{ij}$

(logarithmic series) $= \sum_{i \geq 1} -\log(1 - u^i) \frac{\mathcal{P}_i}{i}$.

Note that for $|u| < (D - 1)^{-1}$ all the above series are absolutely convergent. Taking exponentials yields the claim. \qed

**Remark 2.6.** We emphasize that the validity of the Euler product representation of $Z$ only owes to the validity of

$$N_j(x) = \sum_{i \mid j} P_i(x).$$

We finish this section by discussing the Zeta functions of finite and periodic graphs. It is quite instructive to compute the quantities $\overline{N}_j$ and $\overline{P}_j$ for these examples. The exponent $\overline{P}_j$ in the Euler product representation can be further resolved in these situations and the product then be taken over certain geometric quantities. First, let us recall some additional graph theoretic notions.

**Definition 2.7** (Cycles).

(a) Two closed paths $(e_1, \ldots, e_m)$ and $(f_1, \ldots, f_m)$ are said to be equivalent if there is some $l$ such that $e_i = f_{i+l}$ for all $1 \leq i \leq m$. Here we use the convention $f_i = f_{i+m}$. The corresponding equivalence classes of closed paths are called *cycles*. The equivalence class of a closed path $C$ is denoted by $[C]$.

(b) Cycles with their representatives being reduced and primitive are called *prime cycles*. The set of all prime cycles is denoted by $\mathcal{P}$. 
2.1. Finite graphs. Let \( G = (V, E) \) be finite. As discussed at the end of the previous section \( G \) is a measure graph over the trivial groupoid. \( M \) is the counting measure on \( V \) and we can take \( u \equiv 1 \) as an averaging function to obtain

\[
\overline{N}_j = \sum_{x \in V} N_j(x) \quad \text{and} \quad \overline{P}_j = \sum_{x \in V} P_j(x).
\]

Thus, in this situation \( \overline{N}_j \) is the total number of reduced closed paths of length \( j \) and \( \overline{P}_j \) is the total number of primitive reduced closed paths of length \( j \).

**Proposition 2.8.** Let \( G = (V, E) \) be a finite graph and let \( M \) be the counting measure. Then the Zeta function satisfies

\[
Z_{(G,M)}(u) = \prod_{[P] \in \mathcal{P}} \left( 1 - u^{\ell([P])} \right)^{-1},
\]

where \( \mathcal{P} \) is the set of prime cycles and \( \ell([P]) \) is the length of \([P]\).

**Proof.** Since \( M \) is the counting measure on \( V \) the quantity \( \overline{P}_j \) coincides with the number of primitive reduced closed paths of length \( j \). Thus, we have \( \overline{P}_j = j \cdot |P_j| \), with \( P_j \) being the set of prime cycles of length \( j \). Using this fact and reordering the product of Proposition 2.5 yields the claim. \( \square \)

**Remark 2.9.** For a finite graph the Ihara Zeta function is 'usually' defined via the product of the previous proposition (see e.g. [36]). Thus, for finite graphs our definition of Zeta function coincides with the 'usual' one.

Our approach suggests to also consider the Zeta function coming from the normalized measure \( M_1 \) giving the mass \( 1/|V| \) to any vertex. We call the arising Zeta function the normalized Zeta function of the finite graph and denote it as \( Z_{(\text{norm})}(G) \). Obviously \( Z_{(\text{norm})}^{|V|}(G) = Z_{(G,M)} \) holds.

2.2. Periodic graphs. Let us first fix some notation. For a group \( \Upsilon \) acting on some space \( X \) we let \( \Upsilon x = \{ \gamma x : \gamma \in \Upsilon \} \) be the orbit and \( \Upsilon_x = \{ \gamma \in \Upsilon : \gamma x = x \} \) be the stabilizer of \( x \in X \). Further, we denote by \( X/\Upsilon = \{ \Upsilon x : x \in X \} \) the collection of orbits.

Let \( (G, \Gamma) \) be a periodic graph (see Definition 1.27). Recall that \( M = m \circ \eta \) is the counting measure on the vertices and \( (G, M) \) is a measure graph over the groupoid \( G = \Gamma \). As seen before, \( x \mapsto u(x) = \frac{1}{|\Gamma_x|} F(x) \) with \( F \) being a fundamental domain of the \( \Gamma \) action on \( V \) is an averaging function. We obtain

\[
\overline{N}_j = \sum_{x \in F} \frac{N_j(x)}{|\Gamma_x|} \quad \text{and} \quad \overline{P}_j = \sum_{x \in F} \frac{P_j(x)}{|\Gamma_x|}.
\]

As these quantities (and thus \( Z_{(G,M)} \)) only depend on \( G \) and \( \Gamma \) we shall write \( Z_{(G,\Gamma)} \) for the corresponding zeta function.

The group \( \Gamma \) naturally acts on closed paths and on cycles of \( G \). Namely, for a closed path \( C = (e_1, \ldots, e_m) \) and \( \gamma \in \Gamma \) we let \( \gamma C = (\gamma e_1, \ldots, \gamma e_m) \). This action is compatible with passing from closed paths to cycles. Thus, for a cycle \([C]\) we can set \( \gamma [C] = [\gamma C] \).
Proposition 2.10. Let \((G, \Gamma)\) be a periodic graph. Suppose \(M\) and \(u\) are given as above. Then the Zeta function satisfies
\[
Z_{(G, \Gamma)}(u) = \prod_{\Gamma|P| \in \mathcal{P}} \left(1 - u^{\ell(P)}\right)^{-|\Gamma|},
\]
where \(\mathcal{P}\) is the set of all prime cycles and \(\ell(P)\) is the length of \([P]\).

Proof. We first note that the cardinality of the stabilizer \([\Gamma|P]\) and the length \(\ell(P)\) are independent of the chosen representative of \([\Gamma|P]\). Now the statement follows from Proposition 2.5 by reordering the product once we show the equality
\[
\frac{\mathcal{T}_j}{j} = \sum_{\{\Gamma|P| \in \mathcal{P}/\Gamma : \ell(P) = j\}} \frac{1}{|\Gamma|}.
\]
Multiplying both sides with \(j\) and passing from cycles to closed paths we have to show
\[
\mathcal{T}_j = \sum_{\{\Gamma|P| \in \mathcal{P}/\Gamma : \ell(P) = j\}} \frac{1}{|\Gamma|},
\]
where \(\mathcal{P}\) denotes the set of all primitive reduced closed paths. Now fix a fundamental domain \(F \subset V\). For each orbit \(\Gamma|P| \in \mathcal{P}/\Gamma\) there is a unique \(x \in F\) such that there exist paths in \(\Gamma|P|\) starting in \(x\). Thus, we say \(\Gamma|P|\) starts in \(x\) if one of the paths in \(\Gamma|P|\) starts in \(x\). We obtain
\[
\sum_{\{\Gamma|P| \in \mathcal{P}/\Gamma : \ell(P) = j\}} \frac{1}{|\Gamma|} = \sum_{x \in F} \sum_{\{\Gamma|P| \in \mathcal{P}/\Gamma : \ell(P) = j and \Gamma|P| starts in x\}} \frac{1}{|\Gamma|}.
\]
In order to sum over paths starting in \(x \in F\) (as required to compute \(\mathcal{T}_j\)) instead of orbits of paths we need to count how many different paths in \(\Gamma|P|\) actually do start in \(x\). It is easy to see that the group \(\Gamma_x/\Gamma|P|\) acts transitively and freely on the paths of \(\Gamma|P|\) that start in \(x\). Hence, for each \(x \in F\) we obtain \(|\{Q \in \Gamma|P| : Q\) starts in \(x\}\| = |\Gamma_x/\Gamma|P|| = |\Gamma_x|/|\Gamma|\). These considerations yield
\[
\sum_{x \in F} \sum_{\{\Gamma|P| \in \mathcal{P}/\Gamma : \ell(P) = j and \Gamma|P| starts in x\}} \frac{1}{|\Gamma|} = \sum_{x \in F} \frac{|\Gamma_x|}{|\Gamma_x|} \frac{P_j(x)}{|\Gamma_x|} = \mathcal{T}_j.
\]
This finishes the proof. \(\square\)

Remark 2.11. • In periodic graphs were the first class of examples of infinite graphs for which Zeta functions were defined with the above product representation serving as a definition. Thus, our concept of Zeta function of a periodic graph coincides with the one previously studied.
• The previous proposition can be seen as an analogue of Theorem 2.2 (iii) of [17] which states that
\[
\prod_{\Gamma|P| \in \mathcal{P}/\Gamma} \left(1 - u^{\ell(P)}\right)^{-|\Gamma|} = \exp \left(\sum_{j \geq 1} \frac{N_j^\Gamma}{j} u^j\right),
\]
for a sequence of numbers \(N_j^\Gamma\) that depend on geometric quantities of the graph. The crucial new insight is that the coefficients \(N_j^\Gamma\) equal \(\overline{N}_j\) and can thus be expressed
as integrals over functions on the vertices of the graph. Indeed, this is the main observation that lies at the heart of our paper.

3. Topological measure graphs and continuity of the Zeta function

In this section, we turn to a certain continuity property of the Zeta function. In order to state it we equip the space of holomorphic functions on

\[ B_{(D-1)^{-1}} := \{ u \in \mathbb{C} : |u| < (D - 1)^{-1} \} \]

with the topology of uniform convergence on compact subsets.

**Proposition 3.1.** Let \( G \) be a graph over the measurable groupoid \( G \). Let \( \eta \) be its canonical random variable and let \( \nu \) be a transversal on \( G \) and \( u \) an averaging function with respect to \( \nu \). Let \( (m_n) \) be a sequence in the set \( \mathcal{M}_\nu^u(G) \) and define \( M_n := m_n \circ \eta \). Let furthermore \( m \in \mathcal{M}_\nu^u(G) \) with associated measure \( M := m \circ \eta \) be given. If

\[
\int_V N_j(x)u(x)dM_n(x) \to \int_V N_j(x)u(x)dM(x), \quad n \to \infty
\]

holds for any \( j \in \mathbb{N} \), then

\[ Z_{(G,M_n)} \to Z_{(G,M)}, \]

uniformly on compact subsets of \( B_{(D-1)^{-1}} \).

**Proof.** By assumption we have convergence of the coefficients

\[ \overline{N}_j^{(n)} := \int_V N_j(x)u(x)dM_n(x) \to \int_V N_j(x)u(x)dM(x) =: \overline{N}_j. \]

Moreover, by the normalization of the measures and some apriori bounds, we obtain uniform bound on these coefficients. This easily gives the desired convergence. \( \square \)

As a consequence of the previous we obtain continuity of the Zeta functions in the topological setting. Recall that in the topological setting \( \Omega \) is a compact space and any invariant measure is finite (by Proposition 1.24). Hence, we can equip the set of \( \mathcal{M}_\nu^u(G) \) with the topology of weak convergence, i.e. the smallest topology making all maps

\[ \mathcal{M}_\nu^u(G) \to \mathbb{C}, \quad m \mapsto \int_\Omega f(\omega)dm(\omega), \]

continuous for all continuous functions \( f \) on \( \Omega \).

Assume that we are given a topological graph \( G \) with canonical random variable \( \eta \) over the groupoid \( G \) with transversal \( \nu \) and a topological averaging function \( u \) with respect to \( \nu \). Then, we can associate a Zeta function to any invariant measure \( m \) on \( \Omega \). We will be interested in the dependence of the Zeta function on the invariant measure \( m \). For this reason we then define for any \( m \in \mathcal{M}_\nu^u(G) \)

\[ Z_m := Z_{(G,m\circ \eta)}. \]

**Theorem 3.2** (Continuity of the Zeta function). Let \( G \) be a topological graph over the groupoid \( G \) together with a transversal \( \nu \) and a topological averaging function. Then, the map

\[ \mathcal{M}_\nu^u(G) \to \text{Holomorphic functions on } B_{(D-1)^{-1}}, \quad m \mapsto Z_m \]

is continuous.
Proof. We have
\[ N_j = \int_V N_j(x)u(x)dM(x) = \int_\Omega \eta^\omega(N_ju)dm. \]
As \( N_j \) is clearly continuous in the topological setting, we obtain continuity of \( \omega \mapsto \eta^\omega(N_ju) \) from the defining properties of the topological setting. Now, the desired continuity follows directly from the previous proposition and the definition of the topology on the set of measures.

\[ \square \]

4. THE VON NEUMANN ALGEBRA OF A MEASURE GRAPH AND ITS TRACE

Any measure graphs comes naturally with a von Neumann algebra. In fact, a measure graph is a special case of the setting of a proper space over a groupoid in the sense of the non-commutative integration theory developed by Connes in [7]. It even belongs to the somewhat more special situation discussed in [25] under the name of 'admissible setting'. Here, we give a more special role will be played by those operators which respect the bundle structure of \( G \). Let the unitary operator \( U \) be given by
\[ U_g : \ell^2(V, s) \to \ell^2(V, r), \]
\[ U_g f(x) := f(g^{-1}x). \]
A family \((A_\omega)_{\omega \in \Omega}\) of bounded operators \( A_\omega : \ell^2(V^\omega, \eta^\omega) \to \ell^2(V^\omega, \eta^\omega) \) is called a bounded random operator if it satisfies:
- \( \omega \mapsto \langle g_\omega, A_\omega f_\omega \rangle \) is measurable for arbitrary \( f, g \in L^2(V, M) \).
- There exists a \( C \geq 0 \) with \( \|A_\omega\| \leq C \) for almost all \( \omega \in \Omega \).
- The equivariance condition \( A_{r(g)} = U_g A_{s(g)} U_g^* \) is satisfied for all \( g \in G \).
Two bounded random operators \((A_\omega), (B_\omega)\) are called equivalent, \((A_\omega) \sim (B_\omega)\) if \(A_\omega = B_\omega\) for \(m\)-almost every \(\omega \in \Omega\). Each equivalence class of bounded random operators \((A_\omega)\) gives rise to a bounded operator \(A\) on \(L^2(V, M)\) by \(Af(x) := A_{\pi(x)} f(x)\). This allows us to identify the class of \((A_\omega)\) with the bounded operator \(A\). The set of classes of bounded random operators is denoted by \(\mathcal{N}(G, G)\). It is obviously a an algebra (under the usual pointwise addition and multiplication on the level of representatives). Moreover, it carries a norm given by

\[
\| (A_\omega)_{\omega \in \Omega} \| := \inf \{ C \geq 0 : \| A_\omega \| \leq C \text{ for } m\text{-almost every } \omega \in \Omega \}.
\]

(It is not hard to see that this is indeed well-defined and a norm.) The following is the main theorem on the structure of the space of random operators (see \[7, \text{Thm. V.2}\])

**Theorem 4.2.** The set \(\mathcal{N}(G, G)\) of classes of bounded random operators is a von Neumann algebra.

### 4.2. The canonical trace

The admissible setting of \[25\] always gives a canonical weight on the von Neumann algebra in question. In this section we show that this weight is actually a trace.

Let a measure graph \((G, M)\) over a groupoid \(G\) be given and \(\mathcal{N}(G, G)\) be the associated von Neumann algebra. Let \(\mathcal{N}^+(G, G)\) denote the set of non-negative self-adjoint operators in \(\mathcal{N}(G, G)\). We will show that every operator \(A \in \mathcal{N}^+(G, G)\) gives rise to a new random variable \(\beta_A\). Integrating this random variable, we obtain a weight on \(\mathcal{N}(G, G)\). This weight will be shown to be a trace. We start by associating a transversal function as well as a random variable with each element in \(\mathcal{N}^+(G, G)\). For a nonnegative function \(f\) on \(G^\omega\) we will denote by \(q_\omega(f)\) the function on \(\pi^{-1}(\omega)\) given by

\[
q_\omega(f)(x) = \int_{G^\omega} u(g^{-1} x) f(g) d\nu^\omega(g)
\]

for all \(x \in V\). Thus, \(q_\omega(f)\) is the restriction of \(q(f)\) as defined at the end of Section 1.2 to the fiber \(V^\omega\).

**Lemma 4.3.** \[25\] Let \(A \in \mathcal{N}^+(G, G)\) be given.
(a) Then \(\varphi_A\), given by \(\varphi_A^\omega(f) := \text{tr}(A_\omega M_{q_\omega(f)})\), \(f \in \mathcal{F}(G)\), defines a transversal function.
(b) Then \(\xi_A\), given by \(\xi_A^\omega(f) := \text{tr}(A_\omega M_f)\), \(f \in \mathcal{F}(V)\), is a random variable.

Let us recall the following definitions. A *weight* on a von Neumann algebra \(\mathcal{N}\) is a map \(\tau : \mathcal{N}^+ \to [0, \infty]\) satisfying \(\tau(A + B) = \tau(A) + \tau(B)\) and \(\tau(\lambda A) = \lambda \tau(A)\) for arbitrary \(A, B \in \mathcal{N}^+\) and \(\lambda \geq 0\). The weight is called *normal* if \(\tau(A_n)\) converges to \(\tau(A)\) whenever \(A_n\) is an increasing sequence of operators (i.e. \(A_n \leq A_{n+1}\), \(n \in \mathbb{N}\) converging strongly to \(A\). It is called *faithful* if \(\tau(A) = 0\) implies \(A = 0\). It is called *semifinite* if \(\tau(B) = 0\) implies \(B = 0\). If a weight \(\tau\) satisfies \(\tau(C^* C) = \tau(C^* C)\) for arbitrary \(C \in \mathcal{N}\) (or equivalently \(\tau(U A U^*) = \tau(A)\) for arbitrary \(A \in \mathcal{N}^+\) and unitary \(U \in \mathcal{N}\), cf. \[10\] Cor. 1 in 1.6.1]), it is called a *trace*.

In our situation we have a canonical candidate for a weight at our disposal. This is introduced in the next proposition.

**Proposition 4.4** (Introducing \(\tau\)). Let \((G, M)\) be a measure graph over \(G\). For \(A \in \mathcal{N}^+(G, G)\), the expression

\[
\tau(A) := \int_\Omega \text{tr}(A_{\omega} M_{\omega} A_{\omega}^*) d\nu(\omega) = \int_\Omega \text{tr}(M_{\omega} A_{\omega} M_{\omega}^*) d\nu(\omega) \in [0, \infty]
\]
does not depend on $u \in \mathcal{F}^+(G)$ provided $\nu \ast u \equiv 1$.

Proof. This follows directly from Lemma 1.16 and Lemma 4.3.

We are going to show that the map

$$\tau : \mathcal{N}^+(G,G) \to [0, \infty], A \mapsto \tau(A),$$

is a faithful, semifinite normal trace on $\mathcal{N}(G,G)$. In order to do so we will present some additional pieces of information. These may also be interesting on their own right.

An operator $K$ on $L^2(V,M)$ is called a Carleman operator (cf. [37] for further details) if there exists a $k \in \mathcal{F}(V^{(2)})$ with

$$k(x,\cdot) \in \ell^2(V^{\pi(x)},\eta^{\pi(x)})$$

for all $x \in V$ such that for any $f \in L^2(V,M)$

$$Kf(x) = \int_{V^{\pi(x)}} k(x,y)f(y)d\eta^{\pi(x)}(y) = \sum_{y \in V^{\pi(x)}} k(x,y)f(y) =: K_{\pi(x)}f_{\pi(x)}(x)$$

in the sense of $L^2(V,M)$. This $k$ is called the kernel of $K$. Obviously, $K = \int_{\Omega} K_\omega dm$. Let $\mathcal{K}$ be the set of all Carleman operators satisfying for all $g \in G$

$$k(gx,gy) = k(x,y) \text{ for } \eta^{\pi(x)} \times \eta^{\pi(y)} \text{ almost all } x,y.$$ \hspace{1cm} (4.3)

For a Carleman operator $K$ the expressions $\tau(KK^*)$ and $\tau(K^*K)$ can directly be calculated.

Proposition 4.5 ([25]). Let $(G,M)$ be a measure graph over $G$. Let $K \in \mathcal{K}$ with kernel $k$ be given. Then we have

$$\tau(K^*K) = \int_{\Omega} \int_{V^{\pi(x)}} u(y)|k(x,y)|^2d\eta^{\pi(x)}(x)d\eta^{\pi(x)}(y)d\mu(\omega) = \tau(KK^*),$$

for any $u \in \mathcal{F}^+(V)$ satisfying $\nu \ast u \equiv 1$.

In our situation all elements of the von Neumann algebra are Carleman operators.

Proposition 4.6 ($\mathcal{N}(G,G) = \mathcal{K}$). Let $(G,M)$ be a graph over $G$. Then, any element of $\mathcal{N}(G,G)$ is a Carleman operator.

Proof. As shown in [25] the set $\mathcal{K}$ is a right ideal in $\mathcal{N}(G,G)$. Now, obviously, the identity is a Carleman operator (with kernel given by $k(x,y) = 1$ if $x = y$ and $k(x,y) = 0$ otherwise). Hence, the desired statement follows. \hspace{1cm} \Box

After these preparations we can now state (and prove) the main result of this subsection.

Theorem 4.7. Let $(G,M)$ be a measure graph over $G$. Then, the map

$$\tau : \mathcal{N}^+(G,G) \to [0, \infty]$$

is a faithful, semifinite, normal trace. Furthermore, this trace is finite, i.e. $\tau$ gives a finite value when applied to the identity. Thus, $\tau$ can be uniquely extended to a continuous map on all of $\mathcal{N}(G,G)$. 
Proof. It follows from [7] (see [23] as well) that $\tau$ is a faithful, semifinite weight. From the preceding two propositions it follows immediately that $\tau$ is a trace. Now, a direct calculation shows that for the identity $I$ we obtain

$$\tau(I) = \int_V u(x)dM(x).$$

By definition of a measure graph the above integral is finite. Now, the last statement follows by standard theory. \hfill $\square$

We finish this section by introducing two special Carleman operators.

**Proposition 4.8** (The adjacency and the degree operator). Let $a_G$ be the adjacency matrix of the graph $G$ and $\deg$ its degree function. Then $a_G$ is the kernel of a Carleman operator which belongs to $N(G,G)$. The associated operator will be denoted by $A_G$ and called the adjacency operator. Furthermore, multiplication by $\deg$ provides a Carleman operator belonging to $N(G,G)$ which will be denoted by $\Deg_G$ and called the degree operator.

**Proof.** We first show the measurability statements of the corresponding kernels. The function $a_G$ is measurable on $V^{(2)}$ by definition of a measurable graph. The kernel of $\Deg_G$ is given by

$$V^{(2)} \to \mathbb{R}, \ (x,y) \mapsto 1_{\{(z,z) : z \in V\}}(x,y) \deg(y).$$

The measurability of this function follows from the Propositions 1.4 and 1.5 and the definition of the $\sigma$-algebra on $V^{(2)}$. The invariance of both kernels under the action of $G$ is clear as $G$ acts upon $V$ by graph isomorphisms. \hfill $\square$

5. **A determinant formula for the Zeta function**

In this section we provide a determinant formula for the Ihara Zeta function in terms of the adjacency and the degree operator. It will show that in a small neighborhood around 0 the function $Z(G,M)$ is the reciprocal of a holomorphic function up to some multiplicative factor. For the proof we will follow the lines of [18] and give details only when they need to be adapted to our situation. For the case of finite graphs this approach towards proving the determinant formula was first established in [35].

In order to state the theorem, we need a holomorphic determinant on certain operators of $N(G,G)$. A natural way to obtain such a functional is to set

$$\det_\tau(T) := \exp \circ \tau \circ \log(T).$$

Here, the logarithm of an operator $T$ with $\|I - T\| < 1$ is defined via the power series expansion of the main branch of the logarithm around 1. Namely we let

$$\log(T) = -\sum_{k=1}^{\infty} \frac{1}{k} (I - T)^k.$$ 

Note that this determinant is holomorphic in $T$ since the trace $\tau$ is continuous in the norm topology.

**Remark 5.1.** The above definition of a determinant functional suffices for the purposes of this paper. As discussed in [17] one can use holomorphic functional calculus to extend the definition of the determinant to operators whose convex hull of the spectrum does not contain 0. See also Remark 6.6 for an application of this extended determinant.
Lemma 5.5. The rest follows by induction on \( j \) given by the family \((A_j)\) where \( A_0 = I \), \( A_1 = A_G \), \( A_2 = A_G^2 - Q_G - I \) and \( A_j = A_{j-1}A_G - A_{j-2}Q_G \), for \( j \geq 3 \). Let \( a_j \) denote the corresponding kernels. The following lemma explains the importance of the \( A_j \)'s.

Lemma 5.4. For each \( p, q \in V \) the kernel entry \( a_j(p, q) \) is equal to the number of proper paths of length \( j \) starting in \( p \) and ending in \( q \). Furthermore, for each \( j \) the norms of the corresponding operators satisfy \( \|A_j\| \leq R^j \), where \( R := \frac{D + \sqrt{D^2 + 4D}}{2} \) and \( D \) is the maximal vertex degree of \( G \).

Proof. Note that multiplying Carleman operators is essentially done via matrix multiplication with the corresponding kernels. Thus, the first statement can be deduced as in the proof of Lemma 5.2 in [18]. For the statement on the norm let us note that the operator \( A_G \) is given by the family \((A_G, \omega)\), where \( A_G, \omega \) is the adjacency operator on the fiber \( V^\omega \). Thus, for \( f = (f_\omega) \in L^2(V, M) \) we obtain

\[
\|A_Gf\|^2 = \int_\Omega \|A_G,\omega f_\omega\|^2 \omega(\omega) \leq D^2 \int_\Omega \|f_\omega\|^2 \omega(\omega) = D^2 \|f\|^2,
\]

showing \( \|A_1\| = \|A_G\| \leq D \). As \( Q_G + I \) is multiplication by \( \text{deg} \) we obtain \( \|A_2\| \leq D^2 + D \).

The rest follows by induction on \( j \), using the inequality \( \|A_j\| \leq D(\|A_{j-1}\| + \|A_{j-2}\|) \).

The functions \( a_j \) count the number of proper paths. As such may still have tails we need to introduce some further quantity. Following the notation in [18] we let \( t_j(x) \) be the number of proper closed paths with tail of length \( j \) starting in \( x \). By the definition of a measure graph the function \( V \ni x \mapsto t_j(x) \) is measurable and we set

\[
t_j := \int_V t_j(x)u(x)dM(x).
\]

Lemma 5.5. (a) For each \( j \geq 0 \) we have \( \overline{N}_j = \tau(A_j) - t_j \).
(b) \( t_1 = t_2 = 0 \), and, for \( j \geq 3 \), \( t_j = t_{j-2} + \tau((Q - I)A_{j-2}) \).

(c) For each \( j \geq 0 \) the equality \( t_j = \tau\left((Q - I)\sum_{i=1}^{\left\lfloor \frac{j+1}{2} \right\rfloor} A_{j-2i}\right) \) holds.
Proof. By Lemma 4.4 we have $N_j(x) = a_j(x, x) - t_j(x)$. Integrating with respect to the measure $u dM$ yields statement (a). Next we turn to proving statement (b). Whenever $(x, y)$ is an edge we let $\tilde{t}_j(x, y)$ be the number of reduced closed paths of length $j$ with tail starting with the edge $(x, y)$. If $(x, y)$ is not an edge we set $\tilde{t}_j(x, y) = 0$. The function $\tilde{t}_j$ is the kernel of an operator belonging to $\mathcal{N}(G, G)$. Indeed, the required measurability of $\tilde{t}_j$ follows from Proposition 4.4 (c) and the invariance under the action of the groupoid is clear, as it acts by graph isomorphisms. We obtain

$$
\int_V t_j(x) u(x) d\mu(x) = \int_V u(x) \sum_{\{y : y \sim x\}} \tilde{t}_j(x, y) d\mu(x)
$$

$$
= \int_\Omega \sum_{x \in V} \sum_{y \in V} u(x) \tilde{t}_j(x, y) d\mu(\omega)
$$

$$
= \int_\Omega \sum_{y \in V} \sum_{x \in V} u(y) \tilde{t}_j(x, y) d\mu(\omega)
$$

$$
= \int_V u(y) \sum_{\{x : x \sim y\}} \tilde{t}_j(x, y) d\mu(y),
$$

where the third equality is due to the invariance of $\tilde{t}_j(x, y)$ under the action of the groupoid and the invariance of $m$ (use Proposition 4.4 with the kernel $k(x, y) = \tilde{t}_j(x, y)^{1/2}$). Now (b) can be obtained from the equation

$$
\sum_{\{x : x \sim y\}} \tilde{t}_j(x, y) = t_{j-2}(y) + (\deg(y) - 2) a_{j-2}(y, y),
$$

which is a consequence of the following observation: Each closed path with tail of length $j$ can be shortened to a closed path of length $j - 2$ by removing its first and last edge. Thus, we have to count the number of ways in which one can extend reduced closed paths of length $j - 2$ starting in $y$ to reduced closed paths with tail starting in neighbors of $y$. There are two possibilities: Either such a path of length $j - 2$ has a tail or not. There are exactly $t_{j-2}(y)$ reduced closed paths with tail starting in $y$. If one wants to avoid backtracking, each of these can be extended in $\deg(y) - 1$ ways to neighbors of $y$. There are $a_{j-2}(y, y) - t_{j-2}(y)$ reduced closed paths without tails. To avoid backtracking there are $\deg(y) - 2$ possibilities to extend each of them. Thus, we obtain

$$
\sum_{\{x : x \sim y\}} \tilde{t}_j(x, y) = (\deg(y) - 1) t_{j-2}(y) + (\deg(y) - 2) (a_{j-2}(y, y) - t_{j-2}(y))
$$

and the claim follows. Assertion (c) is an immediate consequence of (b). \qed

Lemma 5.6. For $j \geq 0$ let $B_j := A_j - (Q_G - I) \sum_{i=1}^{[j/2]} A_{j-2i}$. Then

(a)

$$
\tau(B_j) = \begin{cases} 
N_j - \tau(Q_G - I), & j \text{ even} \\
N_j, & j \text{ odd}
\end{cases}
$$

(b)

$$
\tau\left(\sum_{j \geq 1} B_j u^j\right) = \tau\left(-u \frac{d}{du} \log(I - A_G u + Q_G u^2)\right), \ |u| < R^{-1}.
$$
Proof. The algebraic manipulations required to prove (a) are contained in the proof of Lemma 7.2 of [18]. To carry them out in our setting one only needs the identities of Lemma 5.5. Assertion (b) is given by Corollary 7.4 of [18] whose proof involves algebraic manipulations of power series that require the estimate on the norm of the \( A_j \) of Lemma 5.4. □

Proof of Theorem 5.3. Since \( \tau \) is norm continuous we obtain by Lemma 5.6 (a) that the equality

\[
\tau \left( \sum_{j \geq 1} B_j u^j \right) = \sum_{j \geq 1} \tau(B_j) u^j = \sum_{j \geq 1} \mathbb{N}_j u^j - \sum_{j \geq 1} \tau(Q_G - I) u^{2j}
\]

holds. This computation together with 5.6 (b) yields

\[
u \frac{d}{du} \log Z(G,M) = \sum_{j \geq 1} \mathbb{N}_j u^j \]

\[
= \tau \left( -u \frac{d}{du} \log(I - A_G u + Q_G u^2) \right) - \frac{u}{2} \frac{d}{du} \log(1 - u^2) \tau(Q_G - I).
\]

Dividing by \( u \) for \( u \neq 0 \), integrating from 0 to \( u \) and taking exponentials yield the claim. □

6. Zeta function and integrated density of states on essentially regular graphs

In this section we discuss properties of the Ihara Zeta function for essentially regular graphs. As for regular graphs in the finite case, their Zeta function is closely related to the spectrum of the adjacency operator.

Definition 6.1 (Essentially regular graph). Let \((G,M)\) be a measure graph over a groupoid. Then, \(G\) is called essentially \((r + 1)\)-regular if \(\deg(x) = r + 1\) for \(M\)-almost all \(x \in V\).

Let \((G,M)\) be a measure graph over a groupoid \(G\). Recall that each self-adjoint operator \(T \in \mathcal{N}(G,G)\) possesses a spectral measure \(\mu_T\) on the spectrum \(\sigma(T)\) of \(T\) which is uniquely determined by the identity

\[
\tau(\varphi(A)) = \int_{\sigma(T)} \varphi(\lambda) d\mu_T(\lambda), \text{ for each } \varphi \in C(\sigma(T)).
\]

Remark 6.2. The measure \(\mu_T\) is sometimes called the abstract integrated density of states of \(T\), see Chapter 5 and 6 of [25] for a detailed discussion.

We let \(\mu_G := \mu_{A_G}\) be the spectral measure of the adjacency operator of \((G,M)\). The following is the main observation of this paragraph. It connects the spectral theory of \(A_G\) and the Zeta function \(Z(G,M)\).

Theorem 6.3. Let \((G,M)\) be a measure graph over a groupoid \(G\). Assume that \((G,M)\) is essentially \((r + 1)\)-regular. Then, for \(|u| < R^{-1}\) its Zeta function satisfies

\[
Z(G,M)(u)^{-1} = (1 - u^2)^{(r-1)\tau(I)/2} \exp \left( \int_{\sigma(A_G)} \log(1 - u\lambda + u^2r) d\mu_G(\lambda) \right),
\]
where \(2R = (r+1)+\sqrt{(r+1)^2 + 4(r+1)}\) and log is the main branch of the complex logarithm on the sliced plane \(\mathbb{C}_- := \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}\).

**Proof.** To prove the theorem we will compute the quantities involved in the determinant formula of Theorem 6.3. As \(G\) is essentially \((r+1)\)-regular, the operator \(Q_G = \text{Deg}_G - I\) is multiplication by the constant \(r\). Thus, we obtain
\[
\chi(G,M) = \frac{1}{2} \tau(I - Q_G) = \frac{(1-r)\tau(I)}{2}.
\]
Using the inclusion \(\sigma(A_G) \subseteq [-r+1, r+1]\) and \(R < r\) an elementary computation shows \(1 - \lambda u + u^2 r \in \mathbb{C}_-\), whenever \(|u| < R^{-1}\) and \(\lambda \in \sigma(A_G)\). Thus, for all \(|u| < R^{-1}\) the function \(\lambda \mapsto \psi(\lambda) := \log(1 - u\lambda + ru^2)\) is continuous on \(\sigma(A_G)\). We obtain
\[
\log(1 - uA_G + u^2 Q_G) = \log(1 - uA_G + u^2 rI) = \psi(A_G).
\]
Hence, the definition of \(\mu_G\) yields
\[
\tau(\log(1 - uA_G + u^2 Q_G)) = \int_{\sigma(A_G)} \psi(\lambda) d\mu_G(\lambda).
\]
Now the claim follows from Theorem 5.3. \(\square\)

**Remark 6.4.** In [15] the previous theorem serves as a definition for the Ihara Zeta function of certain infinite regular graphs. There, the measure \(\mu_G\) is given by the integrated density of states (the KNS-spectral measure) of a weakly convergent graph sequence. For vertex-transitive (regular) graphs the above formula recovers Theorem 1.3 of [3] which was shown there by different means via an analysis of Bessel functions and heat kernels. (In both cases the authors consider spectral measures associated with the Laplacian \(L = \text{Deg}_G - A_G\) instead of measures corresponding to \(A_G\). However, for essentially regular graphs the operator \(\text{Deg}_G\) is a constant multiple of the identity and, thus, the Laplacian and the adjacency matrix are essentially the same operators up to a shift by a constant.)

**Corollary 6.5.** Let \((G,M)\) be a measure graph over a groupoid \(G\). Assume that \((G,M)\) is essentially \((r+1)\)-regular. Then its Zeta function can be continued to a holomorphic function on the open set
\[
\mathcal{O} := \{z \in \mathbb{C} : |z| < r^{-1/2}\} \setminus \{x \in \mathbb{R} : r^{-1} \leq |x| \leq 1\}.
\]

**Proof.** Theorem 6.3, the inclusion \(\sigma(A_G) \subseteq [-r+1, r+1]\) and the elementary fact that \(1 - \lambda u + u^2 r \in \mathbb{C}_-\) for each \(u \in \mathcal{O}\) and each \(\lambda \in [-r+1, r+1]\) show the claim. \(\square\)

**Remark 6.6.** With some additional effort one can actually obtain a holomorphic extension of the Zeta function of an essentially \((r+1)\)-regular graph to the set
\[
\mathcal{O}' := \{z \in \mathbb{C} : |z| \neq r^{-1/2}\} \setminus \{x \in \mathbb{R} : r^{-1} \leq |x| \leq 1\},
\]
provided \(\tau(I) = 1\). This can either be done using the holomorphic determinant of [17] or by carefully choosing different logarithm functions in the representation of Theorem 6.3. The extension then satisfies certain functional equations under the transformation \(u \mapsto 1/ur\). Note however, that \(\mathcal{O}'\) is not connected. Thus, in general, the relation of the Zeta function in the bounded region of \(\mathcal{O}'\) to the part in the unbounded region of \(\mathcal{O}'\) is not so clear. We refrain from giving a detailed discussion of this rather subtle subject and refer the reader to [16], whose results can more or less literally be transferred to our situation. See also [6] for the situation on the infinite grid, where the continuation of the Zeta function actually becomes a multi-valued function on a Riemannian surface.
7. Colored graphs and their Zeta functions

A particular convincing (and non-trivial) application of our theory is to colored graphs. This is discussed in this section. The graphs we discuss are Schreier type graphs. We will introduce what could be called the universal groupoid of all colored graphs. Certainly, this groupoid has featured in various contexts already. Here, we provide a new and rather structural approach via inverse semigroups. This will give us a direct and rather quick way to introduce the groupoid and its topology. This approach may be interesting in other contexts as well.

7.1. Basics on colored graphs. Let \( G = (V, E) \) be a graph. Let an \( N \in \mathbb{N} \) be given. An edge coloring (or coloring for short) of \( G \) with \( N \)-colors is a mapping \( C : E \to \{1, \ldots, N\} \) satisfying

- \( C(e) = C(\overline{e}) \) for all \( e \in E \) and
- \( C(e) \neq C(f) \) whenever \( e \) and \( f \) are incident.

A graph equipped with some edge coloring will be called colored graph.

Notation. We will write a graph \( G \) with a coloring \( C \) as a pair \((G, C)\). However, we will often suppress the \( C \) from the notation if it is clear from the context.

We will not only need rooted graphs but also graphs in which finitely many points have been distinguished. If \( x_1, \ldots, x_n \) are vertices of the graph \( G \) then tuple \((G, x_1, \ldots, x_n)\) is called a multi-rooted graph.

Given two multi-rooted colored connected graphs \( G_1 = (V_1, E_1) \) with coloring \( C_1 \) and roots \( x_1, \ldots, x_n \), and \( G_2 = (V_2, E_2) \) with coloring \( C_2 \) and roots \( y_1, \ldots, y_n \) we say that \( G_1 \) injects into \( G_2 \), if there exists an injective map \( \varphi : V_1 \to V_2 \), such that

- for each \( x, y \in V_1 \) we have \( \varphi(x) \sim \varphi(y) \) if \( x \sim y \),
- \( C_1(e) = C_2(\varphi(o(e)), \varphi(t(e))) \) for all \( e \in E \),
- \( \varphi(x_j) = y_j, j = 1, \ldots, n \).

Such a map \( \varphi \) will then be called an injection from \((G_1, x_1, \ldots, x_n)\) into \((G_2, y_1, \ldots, y_n)\) and we will say that \((G_2, y_1, \ldots, y_n)\) extends \((G_1, x_1, \ldots, x_n)\) or that \((G_2, y_1, \ldots, y_n)\) is an extension of \((G_1, x_1, \ldots, x_n)\). An injection is called an embedding if for each \( x, y \in V_1 \) we have \( \varphi(x) \sim \varphi(y) \) if and only if \( x \sim y \). In this case we write \((G_1, x_1, \ldots, x_n) \hookrightarrow (G_2, y_1, \ldots, y_n)\) and say that \((G_1, x_1, \ldots, x_n)\) embeds into \((G_2, y_1, \ldots, y_n)\). If, furthermore, \( \varphi \) is bijective we call \((G_1, x_1, \ldots, x_n)\) and \((G_2, y_1, \ldots, y_n)\) isomorphic and write \((G_1, x_1, \ldots, x_n) \cong (G_2, y_1, \ldots, y_n)\).

Remark 7.1. Note that the only difference between an injection and an embedding is the "only if".

Notation. If the roots are clear from the context, we will suppress them in the notation.

The next proposition contains a simple observation for colored graphs. It will be very crucial to our considerations.

Proposition 7.2 (Uniqueness of injections). Let connected rooted colored graphs \((G, x_1, \ldots, x_n)\) and \((H, y_1, \ldots, y_n)\) be given. Then, there is at most one embedding from \( G \) into \( H \).
Proof. A simple induction (on \( k \)) shows that the restriction of \( \varphi \) to the \( k \)-ball around any root is unique. As the graphs are connected this yields the desired statement. \( \square \)

7.2. The inverse semigroup of finite colored graphs and its groupoid. In this section we will discuss how the set of finite connected colored graphs gives rise to a canonical inverse semigroup. The universal groupoid of this inverse semigroup will then be a key object of our interest.

Recall that a set \( S \) equipped with an associative multiplication
\[
S \times S \rightarrow S, (a, b) \mapsto ab,
\]
is called an inverse semigroup with zero if for any \( a \in S \) there exists an element \( a^* \) with \( a^* a a^* = a^* \) and \( a a^* a = a \) and there exists an element \( 0 \in S \) with \( 0a = a0 = 0 \) for all \( a \in S \) (see e.g. [21]). In this situation \( a^* \) is unique and will henceforth be denoted by \( a^{-1} \) and called the inverse of \( a \). The multiplication gives rise to a partial order \( \prec \) on \( S \) with \( a \prec b \) if and only if \( ab^{-1} = aa^{-1} \).

Now, fix \( N \in \mathbb{N} \) and consider the set \( S(N) \) consisting of all classes of finite double rooted connected graphs which are colored with \( N \) colors and an additional element \( 0 \). We will often rather work with double rooted graphs than with classes. This does not pose a problem as isomorphisms are unique. Define a multiplication on \( S(N) \) in the following way: If there exists a graph \((L, z_1, z_2, z_3)\) such that \((G, x_1, x_2)\) embeds into \((L, z_1, z_2)\) and \((H, y_1, y_2)\) embeds into \((L, z_2, z_3)\) we let \((L^*, z_1^*, z_2^*, z_3^*)\) the smallest such graph and define the product \((G, x_1, x_2)(H, y_1, y_2)\) as \((L^*, z_1^*, z_2^*, z_3^*)\). If such a graph does not exist we define the product \((G, x_1, x_2)(H, y_1, y_2)\) as \(0\). It is straightforward to see that \( S(N) \) with this multiplication gives an inverse semigroup with zero given by \(0\). The inverse of \((G, x_1, x_2)\) is given by \((G, x_2, x_1)\) and \((G, x_1, x_2) \prec (H, y_1, y_2)\) holds if and only if \((G, x_1, x_2)\) extends \((H, y_1, y_2)\).

**Definition 7.3.** Let \( N \in \mathbb{N} \) be given. Then, the inverse semigroup \( S(N) \) is called the inverse semigroup of colored graphs with \( N \) colors.

A filter \( F \) in an inverse semigroup with zero is a subset of \( S \) satisfying the following conditions:
- For any \( a, b \in F \) there exists a \( c \in F \) with \( c \prec a, b \).
- If \( c \) belongs to \( F \) and \( a \) satisfies \( c \prec a \) then \( a \) belongs to \( F \).
- The element \( 0 \) does not belong to \( F \).

On the set of filters on \( S \) together with the set \( S \) we define a multiplication in the following way. For filters \( F, H \subseteq S \), set
\[
FH := \{ c \in S : \text{there exist } a \in F, b \in H \text{ with } ab \prec c \}
\]
and an involution by
\[
F^* := \{ c^{-1} : c \in F \}.
\]
Note that \( FH = S \) if and only if there exist \( a \in F \) and \( b \in H \) with \( ab = 0 \). In particular, \( FH \neq S \) if \( F^* F = HH^* \) (as \( 0 \) does not belong to \( F^* F \)).

For any \( a \in S \) we denote by \( U_a \) the set of all filters \( F \) with \( F \prec a \) in the sense that there exists a \( c \in F \) with \( c \prec a \). Moreover, for \( a \in S \) and \( a_1, \ldots, a_n \prec a \) we define
\[
U_{a; a_1; \ldots; a_n} := U_a \setminus (U_{a_1} \cup \ldots \cup U_{a_n}).
\]
Theorem 7.4 (Universal groupoid of an inverse semigroup). Let $S$ be an inverse semigroup with zero and $G(S)$ the set of its filters. Then, $G(S)$ together with the partially defined multiplication

$$F * H := FH$$

whenever $FF^* = H^*H$ and the involution $F^{-1} := F^*$ is a groupoid. The sets $U_{a,a_1,\ldots,a_n}$ for $a \in S$, $a_1,\ldots,a_n \prec a$, form a basis of a topology making both multiplication and inversion continuous and in this topology they are compact open sets.

Proof. In a slightly different formulation this has been shown in [24] (see [20] as well). More specifically, [24] does not deal with filters but with classes of directed sets. That these classes are in one-to-one correspondence to filters has been discussed in [23].

□

Remark 7.5. As discussed in [24] the groupoid given above is just the universal groupoid of $S$ introduced by Paterson in [28], see also [22] for further study.

Definition 7.6 (Groupoid of colored graphs). Let $S(N)$ be the inverse semigroup of colored graphs. Then, we denote by $G(N)$ the groupoid associated to $S(N)$ according to the previous theorem and refer to it as the groupoid of colored graphs with $N$ colors. Its unit space $G(N)^{(0)}$ will be denoted as $\Omega(N)$.

It is possible to describe the elements of $G(N)$ via classes of double rooted graphs. We denote the set of all classes (modulo isomorphism) of double rooted connected graphs with $N$ colors with countably many vertices by $\text{DR}(N)$. Note that by Proposition 7.2 isomorphisms between double rooted graphs are unique. We denote the class of $(G,x,y)$ by $[(G,x,y)]$.

Proposition 7.7 (Elements of $G(N)$ as double rooted graphs). Let $N \in \mathbb{N}$ be given. Then, the map

$$\text{DR}(N) \ni [(G,x,y)] \mapsto \{(H,p,q) : (H,p,q) \text{ injects into } (G,x,y)\} \in G(N),$$

is a bijection. The units of $G(N)$ correspond to (classes of) graphs in which both roots agree.

Proof. This is straightforward. □

Conventions. In the sequel we will tacitly identify elements of $G(N)$ with classes of double rooted graphs. As isomorphism between double rooted graphs are uniquely determined we can even sometimes work with double rooted graphs instead of classes of double rooted graphs.

The elements of $\Omega(N)$ are graphs with two roots both of which agree. In the sequel we will often tacitly identify such graphs with graphs with only one root. In particular, we will write $(H,x)$ instead of $(H,x,x)$. We will then also write $U_{[H,x]}$ instead of $U_{[(H,x,x)]}$.

Proposition 7.8. Let $N \in \mathbb{N}$ be given and $\Omega(N)$ the unit space of the associated groupoid. Then, $\Omega(N)$ is a compact metrizable Hausdorff space.

Proof. Hausdorffness is clear as the topology is defined via taking classes of double rooted graphs. Compactness and metrizability follow as there are only finitely many connected graphs of a fixed diameter. □

In the remainder of this section we have a look at the invariant measures on the universal groupoid. Thus, let $N \in \mathbb{N}$ be given and let $G(N)$ be the associated topological groupoid of colored graphs with $N$ colors and $\Omega(N)$ its unit space. We note that there is a canonical
transversal function \( \nu \) given by the counting measure on each fiber, i.e. for \( \omega = (G, x) \) we have

\[
\nu^\omega = \sum_{y \in V(G)} \delta_{(G, x, y)}.
\]

There are some canonical local homeomorphism on \( \Omega(N) \): More specifically, we define involutions \( I_k : \Omega(N) \to \Omega(N) \), for \( k \in \{1, \ldots, N\} \), via

\[
I_k(G, x) = \begin{cases} (G, y), & \text{if there exists } y \sim x \text{ such that } C((x, y)) = k \\ (G, x), & \text{else.} \end{cases}
\]

Here, comes our characterization of invariant measures.

**Theorem 7.9** (Characterization of invariant measures on \( G(N) \)). Let \( N \in \mathbb{N} \) be given and let \( G(N) \) be the associated topological groupoid of colored graphs with \( N \) colors and \( \Omega(N) \) its unit space. Let \( m \) be a finite Borel measure on \( \Omega \). Then, the following assertions are equivalent:

(i) The measure \( m \) is invariant, i.e. \( (m \circ \nu)^\omega = m \circ \nu \).

(ii) For any finite connected colored graph \( H \) and arbitrary \( x, y \in V(H) \) the equality

\[
m \circ \nu(U_{(H,x,y)}) = m \circ \nu(U_{(H,y,x)})
\]

holds.

(iii) For any finite connected colored graph \( H \) and arbitrary \( x, y \in V(H) \) the equality

\[
m(U_{(H,x)}) = m(U_{(H,y)})
\]

holds.

(iv) For any \( k \in \{1, \ldots, N\} \) and any measurable set \( S \) in \( V \) we have \( m(I_k S) = m(S) \).

**Proof.** (i) \( \iff \) (ii): First note that \( \tilde{U}_{(H,x,y)} = U_{(H,y,x)} \). Now, the collection of all \( U_{(H,x,y)} \), for \( H \) finite connected graph and \( x, y \in V(H) \), is a countable generator of the \( \sigma \)-algebra, which is closed under taking finite intersections and has finite \( m \circ \eta \) measure. Thus, the desired equivalence follows from a standard uniqueness theorem of measure theory, see e.g. Theorem 5.4 of [1].

(ii) \( \iff \) (iii): This follows directly from \( m \circ \nu(U_{(H,x,y)}) = m(U_{(H,x)}) \) for all finite connected \( H \) and \( x, y \in V(H) \).

(iii) \( \iff \) (iv): As the collection of all \( U_{(H,x)} \), for \( H \) finite connected graph and \( x \in V(H) \), is a countable generator of the \( \sigma \)-algebra, which is closed under taking finite intersections, validity of (iv) for all measurable \( S \) is equivalent to validity of (iv) for \( S \) of the form \( U_{(H,x)} \). This, however, can easily be seen to be equivalent to (iii). \( \square \)

**Remark 7.10.** The finiteness assumption on the measure in the above theorem may be weakened. We refrain from giving details.

**Proposition 7.11.** Let \( N \in \mathbb{N} \) be given. Then, the set of invariant probability measures is compact with respect to the topology of weak convergence of measures.

**Proof.** As \( \Omega(N) \) is compact and metrizable the set of probability measures on it is compact with respect to the topology of weak convergence. Thus, it suffices to show that the set of invariant measures is closed. This, however, is easy. \( \square \)
7.3. The universal colored graph. In this section we show that the groupoid $G(N)$ gives rise to a graph $G(N)$ over $G(N)$. This graph can be seen as a kind of universal colored graph. Any invariant measure on $\Omega(N)$ makes it into a measure graph and gives rise to Zeta functions in this way.

Let $N \in \mathbb{N}$ be given. Then, we define the universal graph with $N$ colors, $G(N)$, as follows: The set $V(N)$ of vertices of $G(N)$ is just given by $G(N)$. In particular, $V(N)$ inherits the topology of $G(N)$. We now turn to the set of edges. There is an edge with color $k$ between $(G_1, x_1, x_2)$ and $(G_2, y_1, y_2)$ if and only if there exists a three rooted graph $(G, z_1, z_2, z_3)$ such that $(G_1, x_1, x_2)$ is isomorphic to $(G, z_1, z_2)$ and $(G_2, y_1, y_2)$ is isomorphic to $(G, z_1, z_3)$ and $z_2$ and $z_3$ are connected by an edge of color $k$ in $G$.

**Theorem 7.12** ($G(N)$ as a topological graph over $G(N)$). Let $N \in \mathbb{N}$ be given. Then, $G(N)$ is a topological graph over $G(N)$.

**Proof.** This is rather straightforward. We show existence of a (strictly) positive averaging function. We define

$$u_0 : V(N) \to [0, \infty), u_0((G, x, y)) = \delta_{x,y}.$$  

Then, a short calculation shows that $u_0$ is an averaging function. It is not strictly positive though. This will be taken care of next. Define for $v = (v(1), \ldots, v(n)) \in \{1, \ldots, N\}^n$ the (local homeomorphism)

$$I_v : V(N) \to V(N), I_v(x) = I_{v(1)} \cdots I_{v(n)}x.$$  

Then, a short calculation shows that $u_v$ with $u_v(G, x, y) = u_0(G, x, I_v y)$ is an averaging function as well. Then,

$$u := \sum_v c_v u_v$$  

with $v$ running over all $v \in \{1, \ldots, N\}^n$ for all natural numbers $n$ and $c_v > 0$ with $\sum_v c_v = 1$ is an averaging function with the desired properties. \qed

**Convention.** As the proof of the theorem shows that there is a canonical averaging function $u$ in our context with $u((G, x, y)) = 1$ if $x = y$ holds and $u((G, x, y)) = 0$ otherwise. If not noted otherwise we will always use this averaging function in the context of colored graphs.

In Definition 1.20 we have defined the concept of normalized invariant measures by referring to a normalization with respect to an averaging function. In the context of colored graphs this normalization just amounts to considering probability measures as shown next.

**Proposition 7.13.** Let $m$ be an invariant measure on $\Omega(N)$. Then, $m$ is normalized if and only if it is a probability measure.

**Proof.** Obviously $\eta^\omega(u) = 1$ for all $\omega \in \Omega(N)$. This gives the desired statement. \qed

The previous theorem immediately gives the following consequence.

**Corollary 7.14.** Let $m$ be a finite invariant measure on $\Omega(N) = G(N)^{(0)}$ and define $M := m \circ \eta$. Then, $(G(N), M)$ is a topological measure graph over $G(N)$.

By the previous corollary and the material developed in Section 1.14 we can associate to any finite invariant measure $m$ on $\Omega(N)$ a Zeta function and this Zeta function depends continuously on $m$ by Theorem 3.2.
Definition 7.15 (Zeta function of invariant measure). Whenever $m$ is an invariant measure on $\Omega(N)$, we denote by $Z_m$ the Zeta function associated to the measure $m \circ \eta$.

7.4. The Zeta function of arbitrary colored graphs. In this section we show how general colored graphs fit into our framework and use this to associate a Zeta function to them. In this context, the basic idea is that invariant measures on colored graphs give rise to invariant measures on $\Omega(N)$.

Let $C = \{1, \ldots, N\}$ be given. We denote by $A_C^r$ the set of all equivalence classes of finite connected colored (with elements from $C$) graphs with one root and all of whose vertices have distance at most $r$ to the root. Furthermore, we let $A_C^n = \bigcup_{r=0}^{\infty} A_C^r$. Note that any element of $A_C^r$ must be connected (as otherwise the distance to the root could not be finite for all vertices). Let $G = (V, E)$ be a colored graph and consider the maps $\pi_C^r : V \rightarrow A_C^r, x \mapsto [B^G_r(x)]$, where the bracket means taking isomorphisms respecting the colors of the edges. Here, we use the superscript $C$ in the definition of the $\pi_C^r$ to distinguish them from the maps $\pi_r$ defined at the very beginning of the article.

Definition 7.16. Let $G$ be a colored graph. Then, the smallest topology on $G$ making all $\pi_r, r \in \mathbb{N} \cup \{0\}$ continuous is called the local topology.

Remark 7.17. (a) The local topology is well-known. For the non-colored situation a detailed treatment can be found in Chapter 18.1 of [27].

(b) A colored graph is not necessarily Hausdorff in the local topology. Indeed, there are two sources of failure of Hausdorffness of the topology:
- the ‘same’ connected component can appear several times in the graph.
- there are some non-trivial automorphisms acting on the graph (‘curse of symmetry’, cf. [27]).

(c) A map $a : X \times X \rightarrow [0, \infty)$ is a pseudo-ultrametric on $X$ if it satisfies both
- $a(x, y) = a(y, x)$,
- $a(x, y) \leq \min\{a(x, z), a(z, y)\}$
for all $x, y, z \in X$. The space $X$ with the topology induced from $a$ is then called a Cantor set. Now, the local topology can also be generated by a pseudo-ultrametric. Denote by $C_G$ the color map for $G$. Set
\[ c(x, y) = \sup\{r \in \mathbb{N}_0 \mid (B^G_r(x), C_G) \simeq (B^G_r(y), C_G)\} \]
and $a(x, y) = 2^{-c(x, y)}$. Then $a$ is a pseudo-ultrametric that generates $\mathcal{T}_G$. Therefore, the vertex set equipped with this particular topology is a Cantor set.

(d) The subsequent considerations will work for any topology, which is finer than the local topology. We refrain from giving further details.

Let now a colored graph $G = (V, E)$ be given. Then, there are two alternative ways of describing the local topology: For a finite connected graph $H$ and a vertex $x$ in $H$ define
\[ V_{(H, x)} := \{ x \in V : (H, x) \text{ embeds into } (G(x), x) \}. \]
Then, it is not hard to see that the topology on $G$ is generated by the family
\[ V_{(H, x)} \setminus (V_{(H_1, x_1)} \cup \ldots \cup V_{(H_n, x_n)}), \]
where $H$ is running over all finite connected graphs $H$ and $x$ is running over all vertices in $H$ and $(H_1,x_1), \ldots, (H_n,x_n)$ are extensions of $(H,x)$. Also, it is not hard to see that the topology is generated by the family of sets 

$$W_{\alpha,r} := \{ x \in V : \pi_r^C(x) = \alpha \} = (\pi_r^C)^{-1}(\alpha)$$

with $r \in \mathbb{N}$ and $\alpha$ running over all all elements from $A^C$.

Similarly to the discussion on the universal colored graph any colored graph allows for certain canonical involutions. More specifically, there are involutions $J_k : V \to V$, for $k \in \{1, \ldots, N\}$, via

$$J_k x = \begin{cases} y, & \text{if there exists } y \sim x \text{ such that } C((x,y)) = k \\ x, & \text{else.} \end{cases}$$

Whenever $H$ is a finite connected graph and $x, y$ are different vertices in $H$ with $J_k x = y$ for a $k \in \{1, \ldots, N\}$ then $J_k V(H,x) = V(H,y)$ holds. Thus, the alternative description of the topology gives that the involutions $J_k$, $k = 1, \ldots, N$, are actually homeomorphisms.

**Definition 7.18.** Let $G = (V,E)$ be a colored graph. Let $V$ be equipped with the Borel-$\sigma$-algebra of the local topology. Then a measure $\mu$ on $V$ is called invariant if $\mu(J_k U) = \mu(U)$ holds for all measurable $U$ and all $k \in \{1, \ldots, N\}$.

We will now transfer colored graphs to subsets of $\Omega(N)$. The tool to do this is the map $\pi_\infty$ introduced via

$$\pi_\infty : V \to \Omega(N), x \mapsto [(G(x), x)].$$

By definition of $\pi_\infty$ we clearly have $\pi_\infty(V(H,x)) = U(H,x) \cap \pi_\infty(V)$. Thus, the map $\pi_\infty$ is continuous and open. Note that the map $\pi_\infty$ just identifies those vertices which have the same rooted connected component (up to isomorphism). The map $\pi_\infty$ is well compatible with the involutions. In fact, it clearly satisfies

$$\pi_\infty \circ J_k = I_k \circ \pi_\infty$$

for all $k = 1, \ldots, N$. Whenever $\mu$ is a measure on $V$ (equipped with the Borel-$\sigma$-algebra) then we define the induced measure $m := \pi_\infty(\mu)$ on $\Omega(N)$ in the usual way via

$$m(U) := \mu(\pi_\infty^{-1}(U)).$$

**Lemma 7.19.** Let $G = (V,E)$ be a graph colored by $N$ colors. Let $\mu$ be a measure on $V$ and let $m := \pi_\infty(\mu)$ be the measure induced on $\Omega(N)$. Then, the following assertions are equivalent:

(i) The measure $\mu$ is invariant.

(ii) The measure $m = \pi_\infty(\mu)$ is invariant.

**Proof.** The invariance of $m = \pi_\infty(\mu)$ can be characterized via the involutions $I_k$, $k = 1, \ldots, N$ by Theorem [26]. The invariance of $\mu$ in turn is defined via the involutions $J_k$, $k = 1, \ldots, N$. As $I_k \pi_\infty = \pi_\infty J_k$ the equivalence follows. □

The previous lemma opens up the possibility to define a Zeta function for any pair $(G, \mu)$ consisting of a colored graph $G$ and an invariant measure $\mu$ by defining

$$Z(G,\mu) := Z_m$$
with $m := \pi_\infty(\mu)$. The coefficients appearing in this Zeta function can directly be calculated as

$$\overline{N}_j = \int_{\Omega(N)} N_j(x)u(x)dM(x) = \int_{\Omega(N)} N_j(x)dm(x) = \int_{\Omega} N^G_j(y)d\mu(y),$$

where we used $N_j(\pi_\infty(x)) = N^G_j(x)$. Of course, a similar formula holds for the $\overline{P}_j$.

If the colored graph $G = (V,E)$ is compact and Hausdorff in the local topology, it gives naturally rise to a measure graph over a groupoid by taking suitable restrictions: In fact, in this case the map $\pi_\infty$ is a homeomorphism. Hence, $V$ can be identified with its range $\Omega(N)_V := \pi_\infty(V)$ and $G(N)_V := \pi^{-1}(\Omega(N)_V)$ is a topological graph over the topological groupoid $r^{-1}(\Omega(N)_V) = s^{-1}(\Omega(N)_V)$ and $m = \pi_\infty(\mu)$ is an invariant measure on $\Omega(N)_V$. Thus, we obtain a measure graph $(G(N)_V, m \circ \eta)$. The Zeta function of this measure graph can easily be seen to agree with the above defined $Z_{(G,\mu)}$ (as both Zeta functions clearly have the same coefficients).

### 8. Graphings as measure graphs

In this section we shortly discuss how topological graphings can be considered as measure graphs. This will allow us to associate a Zeta function to any topological graphing by the general theory developed above. Along the way it will also clarify the relationship between the topological graphs of the last section and graphings. Roughly speaking one can say that graphings (even topological ones) can be seen as a measurable version of the colored graphs of the last section.

Recall that a topological graphing, $(X, \mu, \iota_1, \ldots, \iota_N)$, consists of a compact Hausdorff space $X$ together with involutions $\iota_1, \ldots, \iota_N$ and a probability measure $\mu$ on $X$ invariant under the $\iota_j$, $j = 1, \ldots, N$.

In the context of colored graphs graphings appear quite naturally. More specifically, whenever $m$ is an invariant probability measure on $\Omega$, then the support supp$(m)$ of $m$ will be a compact subset of $\Omega(N)$ which is invariant under the action of the $I_k$, $k = 1, \ldots, N$. Hence, (supp$(m)$, $m, I_1, \ldots, I_N$) will be a topological graphing. We call such a graphing a graphing with basis $\Omega(N)$.

Given a graphing, it will be convenient to call an element $w = (w_1, \ldots, w_k) \in \{1, \ldots, N\}^k$ for some $k \in \mathbb{N}$ a word and to define for such a word the map $\iota_w$ by $\iota_w = \iota_{w_1} \cdots \iota_{w_k}$. Obviously, $\iota_w$ is measurable.

Clearly, the set $X$ can be considered as a colored graph in the following way: The elements of $X$ are the vertices and there is an edge of color $k$ between $x$ and $y$ if and only if $x \neq y$ and $x = \iota_k y$. The measurability of the $\iota_k$ easily gives that this graph is indeed a measurable graph. Thus, the most direct way of associating a Zeta function to a graphing is to define it using the coefficients

$$\overline{N}_j^X := \int_X N_j^X(x)d\mu(x),$$

where $N_j^X$ denotes the counting in the graph just described. This graph does not fit directly in the framework we have developed above. The reason is that the graph ‘does not fiber’ over connected components. In fact, the space of connected components may be very ugly. We will overcome this problem by giving two alternative ways of arriving at this Zeta function and both of these ways fit into our framework.
Any graphing gives rise to an equivalence relation $\mathcal{R}(X)$ on $X$ consisting of all $(x, y)$ such that there exists a word $w$ with $\iota_w x = y$. Then, clearly with the diagonal $\operatorname{diag}(X) = \{(x, x) : x \in X\}$ the equivalence relation is given by

$$\mathcal{R}(X) = \{(x, y) : \text{there exists a word } w \text{ with } x = \iota_w y\} = \bigcap_w (\operatorname{id} \times \iota_w)^{-1}\operatorname{diag}(X)$$

and is hence measurable, i.e. a measurable subset of $X \times X$. As a measurable equivalence relation $\mathcal{R}(X)$ is also a measurable groupoid.

This equivalence relation itself can now be considered as a colored graph $G(X)$. More specifically, the set of vertices $V(X)$ of the graph associated to this equivalence relation is given by $\mathcal{R}(X)$ and there is an edge between $(x, y)$ and $(x, z)$ of color $k$ if and only if $y = \iota_k z$. Clearly, the connected component of $(x, x)$ in this graph (with root $(x, x)$) is isomorphic to the connected component of $x$ in the graph $X$ (with root $x$).

Moreover, the whole 'local structure' of the graph is encoded in the measurable maps $\iota$:

$$\iota : (x, y) \mapsto (x, \iota_k y).$$

This easily shows that the arising graph is a measurable graph. As the set of vertices of this graph agrees with the groupoid $\mathcal{R}(X)$, it is clearly a graph over this groupoid.

Thus, we are in the setting of a graph over a groupoid. The unit space $\Omega = \{(x, x) : x \in X\}$ can easily be identified with $X$. This will be done tacitly in the sequel. Also, it is not hard to see that the measure $\mu$ on $\Omega = X$ is invariant. (In fact, an arbitrary measure on $\Omega$ can be seen to be invariant if and only if it is invariant under the $\iota_k$, $k = 1, \ldots, N$.) Thus, a graphing gives rise to a measure graph $(G(X), \mu \circ \eta)$ over $\Omega$. The coefficients of the associated Zeta function $Z_{(G(X), \mu \circ \eta)}$ can easily be seen to be given by

$$\int_{\Omega} N_j^{G(X)}((x, x))d\mu(x) = \int_{\Omega} N_j^X(x)d\mu(x).$$

Thus, this Zeta function agrees with the Zeta function presented at the beginning of this section.

It is rather enlightening to discuss a somewhat alternative way to arrive at this Zeta function. This is done next. Whenever we are given a graphing we have a natural map

$$\Pi : X \to \Omega(N),$$

where $\Pi(x)$ is the class of rooted colored graph with vertex set

$$V_x := \{\iota_w x : w = (w_1, \ldots, w_k) \in \{1, \ldots, N\}^k, k \in \mathbb{N}_0\}$$

and an edge between $p$ and $q$ of color $k$ if and only if $q \neq p$ and $\iota_k p = q$ and the root being given by $x$. It is not hard to see that the map $\Pi$ is measurable and satisfies

$$\Pi \circ \iota_k = I_k \circ \Pi.$$

In particular, by Theorem 7.9 the invariant measure $\mu$ induces an invariant measure $m = \Pi(\mu)$ on $\Omega(N)$ via

$$\Pi(\mu)(U) := \mu(\Pi^{-1}(U)).$$

We can therefore associate a Zeta function to any graphing via

$$Z_{(X, \mu \iota_1, \ldots, \iota_N)} := Z_{\Pi(\mu)}.$$

By construction the coefficients of this Zeta function are given by the formulae

$$\overline{N}_j = \int_{\Omega(N)} N_j(p)dm(p) = \int_X N_j(\Pi(x))d\mu(x).$$
for all \( j \in \mathbb{N} \). Now, by the very definition of the coefficients \( N^G_j(X) \) the equality \( N_j \circ \Pi = N^G_j(X) \) hold. Thus, \( N_j = N^\xi_j \) holds and the equality \( Z(X, \mu, \iota_1, \ldots, \iota_N) = Z(G(X), \mu \circ \eta) \) follows.

**Remark 8.1.** The above considerations rise the question for the relationship between graphings and colored graphs. In this context we note that \( \Pi \) is continuous if and only if the graphing has the additional property

(AP) Whenever \( x \in X \) satisfies \( \iota_w x = x \) for some \( w = (w_1, \ldots, w_k) \in \{1, \ldots, N\}^k \), then \( \iota_w y = y \) for all \( y \) sufficiently close to \( x \).

9. **A result on convergence**

In this section we provide a general result on convergence of Zeta functions. More specifically, we show that weak convergence of graphs implies convergence of their Zeta functions in the compact topology. The concept of weak convergence for graph sequences has been introduced in [2]. Comprising Følner and sofic approximations for finitely generated groups, it is a very general notion for convergent geometries. While not noted explicitly, particular cases of weak convergence of graphs are at the core of all the earlier attempts to provide a Zeta function for graphs via approximation [5, 15, 17, 18]. Therefore, our convergence theorem generalizes and unifies the results of the mentioned literature. We discuss this in more detail at the end of this section.

We first define weakly convergent graph sequences. As a preparation, we need a notion to measure the empirical occurrence frequency of isomorphism classes in a finite, connected graph \( G = (V, E) \) with \( \deg(V) \leq D \). So for \( \alpha \in \mathcal{A}^D \) with \( \varrho(\alpha) = r \) define

\[
p(G, \alpha) := \frac{|\{v \in V | \pi_r(v) = \alpha\}|}{|V|}.
\]

Note that we deal here with uncolored graphs.

**Definition 9.1 (Weakly convergent graph sequences).** Let \((G_n) = (V_n, E_n)\) be a sequence of finite, connected graphs with uniform vertex degree bound \( D \in \mathbb{N} \) and \( \lim_{n \to \infty} |V_n| = \infty \). Then \((G_n)\) is weakly convergent if for each \( \alpha \in \mathcal{A}^D \), the frequency of \( \alpha \)

\[
p(\alpha) := \lim_{n \to \infty} p(G_n, \alpha)
\]

exists.

**Remark 9.2.** It is not hard to come up with examples. For instance, considering regular graphs, one obtains such sequences from sofic approximations of groups. Further details including a precise definition for soficity of groups are given in Section 10.

For general sequences \((G_n)\), one cannot expect to obtain a unique countable graph that can be interpreted as the limit of \((G_n)\). One rather has to deal with graphings. In fact, by a result of Elek [11] any weakly convergent sequence of graphs can be shown to converge to a graphing in a suitable sense. In the context of universal graphs with \( N \) colors, we can very easily provide a proof for a variant of the result. This is done next. Our result provides a slight strengthening of the result of [11] in that we show that one can actually obtain a graphing with base space \( \Omega(N) \).
Proposition 9.3 (Limits for weakly convergent graph sequences). Assume that \((G_n)\) is a weakly convergent graph sequence with uniform vertex degree bound \(D\) and let \(N = 2D - 1\). Then, there exists an invariant probability measure \(m\) on \(\Omega(N)\) with
\[
\lim_{n \to \infty} p(G_n, \alpha) = m(\pi^{-1}_r(\alpha))
\]
for all \(\alpha \in \mathcal{A}^D\) with \(q(\alpha) = r\).

Proof. By Vizing’s Theorem on colorings we can actually color the graphs \(G_n\) by \(N\) colors. Having done this, we then consider the measures
\[
m_n := \frac{1}{|V(G_n)|} \sum_{x \in V(G_n)} \delta_{(G_n, x, x)}
\]
on \(\Omega(N)\). These are obviously invariant probability measures. By Proposition 7.11 there exist then a convergent subsequence \((m_{n_k})_k\) and an invariant probability measure \(m\) such that \(m_{n_k}\) converges in the weak topology to \(m\). Now, clearly,
\[
m_n(U_\alpha) = p(G_n, \alpha)
\]
holds. This implies
\[
\lim_{n \to \infty} p(G_n, \alpha) = \lim_{k \to \infty} p(G_{n_k}, \alpha) = m(\{x \in \Omega(N) \mid B_{\rho(\alpha)}(x, x) \simeq \alpha\})
\]
and the proof is finished. \(\square\)

Well in line with the considerations of the previous and the next section we make the following definition.

Definition 9.4 (Graphings and sofic approximation). In the situation of the previous proposition, \((G_n)\) is called a sofic approximation for \(m\) and \(m\) is called a limit graphing of \((G_n)\).

Let now \((G_n)\) be a sofic approximation to \(m\). Then, following the previous considerations, there is an Ihara Zeta function \(Z_m\). It is given as
\[
Z_m(u) := \exp \left( \sum_{j \geq 1} \frac{N_j}{j} u^j \right), \quad |u| \leq (D - 1)^{-1},
\]
where
\[
N_j := \int_{G(N)} N_j(x) u(x) dM(x) = \int_{\Omega(N)} N_j(x) dm(x).
\]
Here, \(M = m \circ \eta\) and \(u\) is the canonical averaging function. It is possible to further compute \(N_j\) in terms of the \(p(\alpha) = m(\pi^{-1}_{\rho(\alpha)}(\alpha))\). In fact, for each radius \(r\) the set \(\Omega(N)\) can be decomposed into the disjoint union
\[
\Omega(N) = \bigcup_{\alpha \in \mathcal{A}^D : \rho(\alpha) = r} \pi^{-1}_r(\alpha) \cup \mathcal{A}^D_{r-1}.
\]
It is not hard to see that limit graphings of weakly convergent graph sequences give measure 0 to each finite rooted graph. In particular, \(m(\mathcal{A}^D_{r-1}) = 0\) holds. As the number of circles of length \(j\) starting in \(x\) only depends on the combinatorial \(j/2 + 1\) neighborhood around \(x\), these considerations show
for every $L \geq j/2 + 1$, where $N_j(\alpha)$ is the number of reduced closed paths of length $j$ in $\alpha$ starting at its root.

We now turn to the main result of this section. In fact, we show that the Zeta function is continuous with respect to sofic approximation. We state the result here for limit graphings with base $\Omega(N)$. However, the proof virtually carries over to give the results for any limiting graphing (in the sense of [11]) of the sequence $(G_n)$.

Recall that any finite graph $H$ comes with a normalized Zeta function $Z_{(\text{norm})}(H)$ as discussed above on Page 17.

**Theorem 9.5** (Approximation of the Ihara Zeta function). Let $(G_n) = (V_n, E_n)$ be a weakly convergent graph sequence and let the invariant measure $m$ on $\Omega(N)$ be a limit graphing of $(G_n)$. Then

$$\lim_{n \to \infty} Z_{(\text{norm})}(G_n) = Z_m$$

in the topology of uniform convergence on compact sets in $B_{(D-1)}(0)$.

**Proof.** As in the proof of the previous proposition, we can - after coloring - consider the measures

$$m_n := \frac{1}{|V(G_n)|} \sum_{x \in V(G_n)} \delta(G_n, x, x)$$

on $\Omega(N)$. Obviously, the coloring is irrelevant for the definition of the Zeta functions of finite graphs and we easily obtain

$$Z_{(\text{norm})}(G_n) = Z_{m_n}.$$ 

Thus, we have to show that the $Z_{m_n}$ converge to $Z_m$ for $n \to \infty$. To do so it suffices to show that

- any subsequence of $Z_{m_n}$ has a converging subsequence and
- the limit of each converging subsequence of $Z_{m_n}$ is equal to $Z_m$.

The first point is clear as any subsequence of $(m_n)$ has a converging subsequence by compactness (Proposition 7.11) and the Zeta function is continuous as a map from invariant measures to holomorphic functions (Theorem 3.2). The second point is clear as the Zeta function of any limiting measure $m$ can be expressed by the $p(\alpha)$ (see considerations just in front of the theorem) and hence is the same for all $m$ arising as limits of the $(G_n)$.

**Remark 9.6.** The above theorem extends the approximation results in the literature: For amenable, periodic graphs (with a discrete, amenable group acting freely and co-finitely as automorphisms), an approximation theorem for the Ihara Zeta function has been shown in [17]. In [5, 15], the authors show the existence of the Ihara Zeta function for certain infinite regular graphs via a convergence statement along covering sequences $(G_n)$ of finite, regular graphs. One way to see the existence of the limit is to use a theorem of Serre [34] on equidistribution of eigenvalues of Markov operators. The convergence along Følner type subgraphs of amenable, self-similar graphs has been proven in [18]. In all the mentioned papers, the graph sequences under consideration can easily be seen to be weakly convergent. Thus, for every approximating sequence given or constructed in the works [5, 15, 17, 18], Theorem 9.5 shows the convergence towards the Zeta function associated to the limit graphing.
of the sequence. By comparing coefficients, it is not hard to see that in all those cases, this function coincides with the Ihara Zeta function of the original (infinite) graph. (For the periodic case, we verify this explicitly in the proof of Corollary 10.4.) Thus, the above theorem generalizes and unifies the approximation results in the literature.

10. Actions of sofic groups on graphs

In this section we will investigate approximation of Zeta functions on graphs which are periodic with respect to the action of a sofic group. As the class of sofic groups is rather large we obtain a quite general result. In fact, our result extends all the previous results provided in [5, 17, 15] (compare remark at the end of this section). The key step of our investigation is the construction of a weakly convergent graph sequence for any periodic graph with a free sofic group action in Theorem 10.3. This will then be combined with the result on weak convergence of graphs from the previous section.

There are various equivalent definitions for soficity of a group. For our purposes, it will be convenient to work with the concept of almost homomorphisms. To give a precise definition, we need some preparation. For \( n \in \mathbb{N} \), we denote by \( \text{Sym}(n) \) the symmetric group over \( \{1, \ldots, n\} \) with unit element \( \text{Id}_n \). This group is naturally endowed with the normalized Hamming distance \( d_H \), defined as

\[
d_H(\sigma, \tau) := \frac{\# \{ a \in \{1, \ldots, n\} | \sigma(a) \neq \tau(a) \}}{n}
\]

for \( \sigma, \tau \in \text{Sym}(n) \). Note that \( d_H \) is a metric on \( \text{Sym}(n) \), see e.g. [29].

**Definition 10.1 (Sofic groups).** A group \( \Gamma \) with unity \( e \) is called sofic if for every finite set \( T \subseteq \Gamma \) and for each \( \varepsilon > 0 \), there exist \( n \in \mathbb{N} \), as well as a mapping \( \sigma : T \to \text{Sym}(n) : s \mapsto \sigma_s \) such that

(i) if \( s, t, st \in T \), then \( d_H(\sigma_s \sigma_t, \sigma_{st}) < \varepsilon \),
(ii) if \( e \in T \), then \( d_H(\sigma_e, \text{Id}_n) < \varepsilon \),
(iii) if \( s, t \in T \) with \( s \neq t \), then \( d_H(\sigma_s, \sigma_t) \geq 1 - \varepsilon \).

If for \( T \) and \( \varepsilon \), there is some map \( \sigma \) satisfying (i) and (ii), then we say that \( \sigma \) is an almost homomorphism for \((T, \varepsilon)\).

**Remark 10.2.** Sofic groups have been invented by Gromov, cf. [19]. The name was given by Weiss in [38], where the author defined the notion for finitely generated groups. The class of all sofic groups is large. In fact, it is not known whether all groups satisfy this property. Sofic groups have become a flourishing subject in various fields of mathematics, such as geometric group theory, ergodic theory and symbolic dynamics. A survey can be found in [29].

Periodic graphs are introduced in Definition 1.27.

**Theorem 10.3.** Let \((G, \Gamma)\) be a periodic graph with vertex degree bounded by \( D \in \mathbb{N} \) and assume that the group \( \Gamma \) is sofic and acts freely on the vertices. Then there exists a weakly convergent sequence of finite graphs \((G_n)\) such that for all \( \alpha \in \mathcal{A}^D \),

\[
\lim_{n \to \infty} p(G_n, \alpha) = \frac{|F_\alpha|}{|F|},
\]
where $F$ is a fundamental domain and $F_\alpha = \{ f \in F : B^{G}_{\rho(\alpha)}(f) \simeq \alpha \}$.

**Proof.** Fix an arbitrary $r \in \mathbb{N}$, $\delta > 0$ and a fundamental domain of vertices $F \subset V$. The corresponding covering map is denoted by $\pi : V \to F$. We construct a finite graph $G_r$ such that

$$|p(G_r, \alpha) - |F_{\alpha}|/|F|| < \delta$$

holds for any $\alpha \in \mathcal{A}_D$. Since the action of $\Gamma$ is free, for each $x \in V$ there exists a unique $\gamma_x \in \Gamma$ such that $x = \gamma_x \pi(x)$ holds. We let

$$T := \{ \gamma_x : x \in B^G_r(F) \}$$

denote the collection of the $\gamma_x$ corresponding to elements in the $r$-Ball around $F$. Note that due to the freeness of the action, the set $T$ is finite. We set

$$\tilde{T} := TT \cup (TT)^{-1} \cup T^{-1}T \cup TT^{-1}.$$  

Note that $T \cup T^{-1} \subseteq \tilde{T}$ since $e \in T$. Now choose $\varepsilon := \delta/(2|\tilde{T}|^2)$. By soficity of the group $\Gamma$ there exists an $N \in \mathbb{N}$ and a mapping $\sigma : \tilde{T} \to \text{Sym}(N)$ satisfying properties (i) to (iii) of Definition 10.1. With the chosen $\varepsilon$. We will now use this map $\sigma$ to construct the graph $G_r$. Namely, we set $V_r := F \times \{ 1, \ldots, N \}$ and connect two vertices $(f,i),(g,j)$ by an edge if and only if there exist $\gamma, \gamma' \in T$ such that $\gamma f \sim \gamma' g$ and $\sigma_\gamma(i) = \sigma_{\gamma'}(j)$ holds. To show the desired statement, for each $1 \leq i \leq N$ we introduce the maps

$$\varphi_i : B^G_r(F) \to V_r, x \mapsto (\pi(x), \sigma_{\gamma_x^{-1}}(i)).$$

Inequality (10) will be an immediate consequence of the following claim.

**Claim:** For at least $(1 - \delta)N$ numbers $i$ between 1 and $N$ the map $\varphi_i$ is a graph isomorphism onto its image.

**Proof of the claim:** Using soficity we remove a set of 'bad' indices of small cardinality from $\{ 1, \ldots, N \}$ to prove the claim. We let $S$ be the set of numbers $i$ which simultaneously satisfy the following properties.

(a) For each $\gamma, \gamma' \in T \cup T^{-1}$, the equality $\sigma_\gamma \sigma_{\gamma'}(i) = \sigma_{\gamma \gamma'}(i)$ holds. We remind the reader at this point that $T \cup T^{-1} \subseteq \tilde{T}$.

(b) For each $\gamma, \gamma' \in \tilde{T}$ the equality $\sigma_\gamma(i) = \sigma_{\gamma'}(i)$ implies $\gamma = \gamma'$.

By (i) of Definition 10.1 there are at most $\varepsilon|\tilde{T}|^2 N$ indices violating property (a) and by property (iii) there are at most $\varepsilon|\tilde{T}|^2 N$ numbers violating property (b). Thus, we obtain

$$|S| \geq (1 - 2\varepsilon|\tilde{T}|^2) N \geq (1 - \delta)N,$$

where the second inequality follows from the choice of $\varepsilon$. Now let $i \in S$. We show that $\varphi_i$ is a graph isomorphism onto its image. Its injectivity is an immediate consequence of the definition of $\varphi_i$ and property (b). It remains to show that $\varphi_i$ and its inverse respect the edge relation. To see this, suppose $(\pi(x), \sigma_{\gamma_x^{-1}}(i)) \sim (\pi(y), \sigma_{\gamma_y^{-1}}(i))$. By the definition of $G_r$ it is equivalent to the existence of $\gamma, \gamma' \in T$; such that

$$\gamma \pi(x) \sim \gamma' \pi(y) \text{ and } \sigma_\gamma \sigma_{\gamma_x^{-1}}(i) = \sigma_{\gamma'} \sigma_{\gamma_y^{-1}}(i).$$

Furthermore, by property (a) this in turn is equivalent to the existence of $\gamma, \gamma' \in T$ satisfying

$$\gamma \pi(x) \sim \gamma' \pi(y) \text{ and } \sigma_{\gamma \gamma_x^{-1}}(i) = \sigma_{\gamma' \gamma_y^{-1}}(i),$$
which by property (b) is equivalent to the existence of $\gamma, \gamma' \in T$ satisfying

$$\gamma \pi(x) \sim \gamma' \pi(y)$$

and $\gamma \gamma x^{-1} = \gamma' \gamma y^{-1}$. Now $\pi(x) = \gamma x^{-1} x$ and $\pi(y) = \gamma y^{-1} y$. Since the element $\gamma \gamma x^{-1} = \gamma' \gamma y^{-1}$ is a graph isomorphism, the previous statement is in fact equivalent to $x \sim y$. This proves the claim.

To finish the proof of the theorem fix $\alpha \in A^D$. Then, the previous claim implies

$$(1 - \delta)|F_\alpha| \leq |\{(f,i) : B^{G_n}_{\rho(\alpha)}((f,i)) \simeq \alpha\}| \leq |F_\alpha| + \delta|F|\epsilon$$

showing inequality (V). This finishes the proof. \hfill \Box

In Section 2.2 we have defined the Ihara Zeta function for periodic graphs $(G, \Gamma)$. When $\Gamma$ acts freely on the vertices, the coefficients of the corresponding Zeta function satisfy

$$N_j(\Gamma) = \sum_{\alpha \in A^D : \rho(\alpha) = L} |F_\alpha| N_j(\alpha),$$

where $N_j(\alpha)$ denotes the number of reduced closed paths of length $j$ starting at the root of $\alpha$ and $L \geq j/2 + 1$. The following corollary now is an easy consequence.

**Corollary 10.4.** Let $(G, \Gamma)$ be a periodic graph with vertex degree bounded by $D$. Assume further that the group $\Gamma$ is sofic and acts freely on the vertices. Then there is a weakly convergent sequence of finite graphs $(G_n) = (V_n, E_n)$ such that

$$\lim_{n \to \infty} Z_{(\text{norm})}(G_n)^{|F|} = Z_{(G, \Gamma)},$$

in the topology of compact convergence on $B_{(D-1)^{-1}} := \{u \in C : |u| < (D - 1)^{-1}\}$. Here $F$ is a fundamental domain of the $\Gamma$-action.

**Proof.** Take a weakly convergent graph sequence $(G_n)$ as in the previous Theorem 10.3. By Theorem 11.3 there exists a limit graphing $m$ on $\Omega(N)$ such that $Z_{(\text{norm})}(G_n) \to Z_m$ uniformly on compact subsets of $B_{(D-1)^{-1}}$. It suffices to show $Z_m^{|F|} = Z_{(G, \Gamma)}$. Therefore, let $N_j, N_j(G_n)$ and $N_j(\Gamma)$ be the coefficients of $Z_m, Z_{(\text{norm})}(G_n)$ and $Z_{(G, \Gamma)}$, respectively. For arbitrary $L \geq j/2 + 1$ we obtain

$$N_j = \lim_{n \to \infty} N_j(G_n) = \lim_{n \to \infty} \sum_{\alpha : \rho(\alpha) = L} p(G_n, \alpha) N_j(\alpha) = \sum_{\alpha : \rho(\alpha) = L} |F_\alpha| N_j(\alpha) = \frac{N_j(\Gamma)}{|F|},$$

where for $\alpha \in A^D$, the quantity $N_j(\alpha)$ is equal to the number of reduced closed paths of length $j$ starting at the root of $\alpha$. This finishes the proof. \hfill \Box

**Remark 10.5.** (a) The above corollary is a direct extension of the approximating theorems of [1] dealing with residually finite groups acting freely on a regular graph and [13] dealing with limits of covering sequences of finite, regular graphs. Further, it is the natural extension of the approximation result in [17] which is concerned with amenable graphs $G$ and associated Følner subgraphs $(F_n)$ exhausting $G$.

(b) For graphs with positive Cheeger constant, finite exhaustions are not weakly convergent. Thus, the approximation through induced subgraphs fails in general in non-amenable situations. However, sofic approximation can still be found in many situations.

(c) The weakly converging sequence of finite graphs constructed in Theorem 10.3 for periodic graphs with a sofic group action can also be used for spectral approximation of suitable self-adjoint operators. In fact, whenever a sequence of finite graphs converges weakly then
weak convergence of the normalized empirical spectral distributions of corresponding operators to a limit follows and this limit can be expressed through a trace on the von Neumann algebra associated with the limit graphing (see [12] [13] [31] for detailed explanations). In this way, Theorem 10.3 could be used to recover parts of the results of [33]. (The results of [33] are more general in that they allow for unbounded operators and include some randomness.) For hyperfinite graphs even uniform convergence of the normalized empirical spectral distributions can be shown, cf. [12] [31] [30].

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