ALGEBRAIC COHOMOLOGY OF THE MODULI SPACE
OF RANK 2 VECTOR BUNDLES ON A CURVE

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Abstract. Let $\mathcal{N}_C$ be the moduli space of stable holomorphic vector bundles of rank 2 and fixed determinant of odd degree, over a smooth projective curve $C$. This paper identifies the algebraic cohomology ring $H^*_A(\mathcal{N}_C)$, i.e., the subring of the rational cohomology ring $H^*(\mathcal{N}_C; \mathbb{Q})$ spanned by the fundamental classes of algebraic cycles, in terms of the algebraic cohomology ring of the Jacobian $J_C$.

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1. Introduction

Let $C$ be a smooth projective complex curve of genus $g \geq 2$ and $\xi$ a line bundle on $C$ of odd degree $d$. Let $\mathcal{N}_C$ be the moduli space of rank 2 stable holomorphic vector bundles on $C$ with determinant $\xi$. This notation is justified by the fact that the isomorphism class of the moduli space is independent of $\xi$. In addition, let $J_C$ be the Jacobian of $C$.

For any smooth projective variety $X$, the (rational) algebraic cohomology group

$$H^i_A(X) \subseteq H^{2i}(X; \mathbb{Q})$$

is the subspace spanned by the fundamental classes of algebraic cycles of (complex) codimension $i$ on $X$. (Note: unless otherwise stated, all cohomology groups in this paper, including Chow groups, have rational coefficients.) The algebraic Poincaré polynomial of $X$ is then defined by

$$P_A(X; t) = \sum_{i=0}^{\infty} \dim H^i_A(X)t^i.$$

In this paper we shall relate the algebraic cohomology of $\mathcal{N}_C$ to that of $J_C$. Indeed, we shall show that $H^*_A(\mathcal{N}_C)$ bears essentially the same relationship to $H^*_A(J_C)$ as the ordinary cohomology $H^*(\mathcal{N}_C)$ does to $H^*(J_C)$. This latter relationship derives from the fact that there is a natural isomorphism

$$H^*(J_C) \cong \Lambda^*(H^3(\mathcal{N}_C)). \quad (1.1)$$

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and that $H^*(\mathcal{N}_C)$ is generated by $H^3(\mathcal{N}_C)$ together with two other algebraic classes $\alpha \in H^2(\mathcal{N}_C)$ and $\beta \in H^4(\mathcal{N}_C)$ (see [Ne2]). Thus, there is a surjective ring homomorphism

$$\nu : \mathbb{Q}[\alpha, \beta] \otimes H^*(J_C) \rightarrow H^*(\mathcal{N}_C).$$

(1.2)

Furthermore (see [Ha]), the ordinary Poincaré polynomial of $\mathcal{N}_C$ is

$$P(\mathcal{N}_C; t) = \frac{P(J_C; t^3) - t^g P(J_C; t)}{(1 - t^2)(1 - t^4)}.$$  

(1.3)

The main point of this paper is to show that the analogues of (1.2) and (1.3) hold for algebraic cohomology.

**Theorem 1.**

$$P_A(\mathcal{N}_C; t) = \frac{P_A(J_C; t^3) - t^g P_A(J_C; t)}{(1 - t)(1 - t^2)}.$$  

**Theorem 2.**

$$H^*_A(\mathcal{N}_C) = \nu(\mathbb{Q}[\alpha, \beta] \otimes H^*_A(J_C)).$$  

Note that the differences in powers of $t$ between (1.3) and Theorem 1 are simply due to the difference in grading between ordinary and algebraic cohomology. We prove Theorem 1 by using the technique of Thaddeus [Th1] to relate a projective bundle over $\mathcal{N}_C$ to a projective space, through a chain of ‘smooth flips’ whose centres are all symmetric powers of the curve. Theorem 2 then follows from Theorem 1 and the fact that $\nu$ takes algebraic classes (in $H^*(J_C)$) to algebraic classes.

One immediate consequence (Corollary 4.1) is that numerical and homological equivalence coincide for $\mathcal{N}_C$. In addition, the arguments of this paper may be repeated to show that Theorems 1 & 2 are equally valid with $H^i_A$ replaced by the Hodge cohomology $H^{i,j}(X) \cap H^{2i}(X; \mathbb{Q})$. Thus the Hodge conjecture for $\mathcal{N}_C$ would be implied by the Hodge conjecture for $J_C$. For a general curve $C$, the Hodge conjecture is known to hold for $J_C$; in this case, recent work of Biswas & Narasimhan [BN] shows directly that it also holds for $\mathcal{N}_C$ and indeed for a large class of smooth moduli spaces over $C$. On the other hand, over all curves of small genus and for moduli spaces of low rank bundles (i.e. $g \leq 4$, $r = 2$ and $g = 2$, $r = 3$), the Hodge conjecture has been verified in [Bal1].

The paper is laid out as follows. In §2 we describe how the algebraic cohomology transforms under a smooth flip. In §3 we prove the generalisation of Macdonald’s formula (3.1) for the algebraic Poincaré polynomial of the symmetric powers of $C$. In §4 we prove Theorem 1 by applying the results of §2 and §3 to Thaddeus’ chain of flips. In §5 we describe some of the consequences for a general curve. In §6 we deduce Theorem 2 from Theorem 1. In §7 we discuss how far the results can be extended to the Chow ring.

The work in this paper — in particular (5.1) — was the inspiration for a further investigation by the second two authors [KN] into the structure of the ordinary cohomology ring $H^*(\mathcal{N}_C)$. While the two papers refer to each other to clarify various points, there is no strict logical dependence between them.

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2. Through a flip.

In this section, we describe how the algebraic cohomology groups transform under the simplest type of ‘flip’, in the sense of [Th2]. More precisely, we will say that a birational map \( X_- \to X_+ \) is a ‘smooth flip of type \((\lambda, \mu)\) with centre \( S \)’ if we have the following commutative diagram of smooth projective varieties

\[
\begin{array}{c}
\tilde{X} \\
| f \\
| \downarrow j \\
X_- & Z_- & Z_+ & X_+ \\
| \downarrow g_- & \downarrow i_- & \downarrow h_- & h_+ & \downarrow i_+ \\
E & S & \end{array}
\]

in which the central square is Cartesian, the other two are blowup diagrams, \( Z_+ \) has codimension \( \lambda \) in \( X_+ \) and \( Z_- \) has codimension \( \mu \) in \( X_- \). Thus \( g_+ \) and \( h_- \) are projective bundles associated to vector bundles of rank \( \lambda \) and \( g_- \) and \( h_+ \) are projective bundles associated to vector bundles of rank \( \mu \). Note that the case \( \lambda = 1 \) is just that of a usual (smooth) blowup.

Recall ([Fu] Prop. 6.7(e)) that for a blowup diagram

\[
\begin{array}{ccc}
E & \xrightarrow{j} & \tilde{X} \\
\downarrow g & & \downarrow f \\
Z & \xrightarrow{i} & X
\end{array}
\]

in which \( Z \) has codimension \( \lambda \) in \( X \), the Chow groups are related by the fact that the map

\[
\begin{pmatrix} f^* & j_* \\ 0 & g_* \end{pmatrix} : A^k(X) \oplus A^{k-1}(E) \longrightarrow A^k(\tilde{X}) \oplus A^{k-\lambda}(Z)
\]

is an isomorphism. The analogous result is true for ordinary cohomology and the class map \( A^k \to H^{2k} \) is natural with respect to both pull-back and push-forward. Hence we also have

\[
H^k_A(X) \oplus H^{k-1}_A(E) \cong H^k_A(\tilde{X}) \oplus H^{k-\lambda}_A(Z)
\]

and thus

\[
P_A(\tilde{X}) - tP_A(E) = P_A(X) - t^\lambda P_A(Z)
\]

Applying this to both blowup diagrams in the flip diagram (2.1), and using the fact that \( Z_+ \) and \( Z_- \) are projective bundles over \( S \), we obtain

\[
P_A(X_+) - P_A(X_-) = \frac{t^\lambda - t^\mu}{1 - t} P_A(S)
\]
3. Symmetric powers.

In this section we prove the following formula, which implicitly gives the algebraic Poincaré polynomials of $S^kC$ for all $k$. This is a direct analogue of Macdonald’s formula for the ordinary Poincaré polynomials ([Mac] (4.3)). It also turns out to be a convenient way to use the information.

$$\sum_{k=0}^{\infty} P_A(S^kC; t)s^k = \frac{P_A(JC; s^2t)}{(1-s)(1-st)}$$  

(3.1)

From Collino’s description of the Chow ring of $S^kC$ ([Co] Theorem 3), one may immediately see that the algebraic cohomology ring $H^*_A(S^kC)$ is generated by $q_k^*(H^*_{\mathfrak{q}}(JC))$, where $q_k : S^kC \to JC$ is the Abel-Jacobi map, and the class $x \in H^*_A(S^kC)$ represented by any of the canonical embeddings $S^{k-1}C \hookrightarrow S^kC$. Thus we have a natural surjection of rings

$$\phi_k : \mathbb{Q}[x] \otimes H^*_A(JC) \to H^*_A(S^kC).$$

We can deduce (3.1) directly from the following.

**Proposition 3.1.** The restriction of $\phi_k$ to

$$V^k := \bigoplus_{i,j \geq 0 \atop i+2j \leq k} x^iH^j_A(JC).$$

(3.2)

is an isomorphism.

**Proof.** We first prove the proposition for $k \geq 2g$, when $q_k : S^kC \to JC$ is a projective bundle of fibre dimension $k - g$ and $x$ is its relative hyperplane class. In this case, the restriction of $\phi_k$ to

$$\bigoplus_{i,j \geq 0 \atop i \leq k-g} x^iH^j_A(JC).$$

(3.3)

is an isomorphism. The summands of (3.2) coincide with those of (3.3) when the degree $i + j \leq k/2$. However, we also know that $x$ is ample (c.f. [Mac]) and hence, by the Hard Lefschetz Theorem, the multiplication map $x^{k-2d} : H^d_A(S^kC) \to H^{k-d}_A(S^kC)$ is injective when $d \leq k/2$. Thus $\phi_k$ is at least injective when restricted to $V^k$. But now, $JC$ has dimension $g$ and $H^*_A(JC)$ satisfies numerical Poincaré duality ([Lieb]). Hence, (3.2) and (3.3) have the same dimension and so $\phi_k$ is also surjective when restricted to $V^k$.

For $k < 2g$, the result follows by ‘backwards induction’ based on the fact from [Co] that

$$f \in \ker\phi_k \iff xf \in \ker\phi_{k+1}.$$  

More precisely, suppose that the result is true for $\phi_{k+1}$. First observe that $xV^k \subseteq V^{k+1}$, so that $(\ker\phi_{k+1}) \cap V^{k+1} = 0$ implies that $(\ker\phi_k) \cap V^k = 0$. Secondly recall that the restriction map $H^*_A(S^{k+1}C) \to H^*_A(S^kC)$ is surjective, so that we at least know that $\phi_k(V^{k+1}) = H^*_A(S^kC)$. But now the inductive hypothesis implies that, if $\beta \in H^*_A(JC)$ and $i + 2j = k + 1$, then there is a relation in $\ker\phi_{k+1}$ of the form $x^{i+1}\beta + \cdots$, where “…” involves only higher powers of $x$. Thus we may divide by $x$ to obtain a relation in $\ker\phi_k$ of the similar form $x^i\beta + \cdots$, and hence $\phi_k(V^k) = \phi_k(V^{k+1})$.  \[\square\]
4. Proof of Theorem 1.

We now use Thaddeus’ chain of flips and the formulae from the previous two sections to prove Theorem 1.

Recall from [Th1] the following diagram (modified slightly to improve the symmetry)

\[
\begin{array}{cccc}
X_0 & \xrightarrow{pt} & X_1 & \cdots & X_{w-1} & \xleftarrow{pt} & X_w \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tilde{X}_1 & & \cdots & & \tilde{X}_w & & \\
\end{array}
\]

for the moduli space $\mathcal{N}_C$ of bundles of degree $d = 2w + 1 \geq 4g - 3$. Here the diagonal maps are all birational and the two vertical maps $X_0 \to pt$ and $X_w \to \mathcal{N}_C$ are projective bundles associated to vector bundles of ranks $m = d + g - 1$ and $n = d - 2g + 2$ respectively. (This is, in part, a perverse way of saying that $X_0 \cong \mathbb{P}^{m-1}$.) The birational map $X_{k-1} \to X_k$ is a smooth flip of type $(\lambda, \mu) = (k, m - 2k)$ with centre $S^kC$, in the sense of §2.

Thus, repeated application of (2.3) yields

\[
P_A(X_w) = P_A(X_0) + \sum_{k=1}^{w} \frac{t^k - t^{m-2k}}{1-t} P_A(S^kC)
\]

and then using the fact that $X_0$ and $X_w$ are projective bundles gives

\[
(1 - t^n)P_A(\mathcal{N}_C) = \sum_{k=0}^{w} (t^k - t^{m-2k}) P_A(S^kC)
\]

(4.1)

Bringing in formula (3.1) and again exploiting the fact that $S^kC \to J_C$ is a projective bundle for $k \geq 2g - 1$, we may write, for $w \geq 2g - 2$,

\[
\sum_{k=0}^{w} P_A(S^kC; t)s^k = \frac{P_A(J_C; s^2t)}{(1 - s)(1 - st)} - P_A(J_C; t) \sum_{k=w+1}^{\infty} \left( \frac{1 - t^{k-g+1}}{1-t} \right) s^k
\]

\[
= \frac{P_A(J_C; s^2t)}{(1 - s)(1 - st)} - \frac{P_A(J_C; t)}{1-t} \left( \frac{s^{w+1}}{1-s} - \frac{s^{w+1}t^{w-g+2}}{1-st} \right)
\]

Applying this to the right hand side of (4.1) yields

\[
(1 - t^n)P_A(\mathcal{N}_C) = \frac{P_A(J_C; t^3) - P_A(J_C; t^{-3})t^{m+3}}{(1-t)(1-t^2)} - \frac{P_A(J_C; t)}{1-t} \left( \frac{t^{w+1} - t^{m-g-w+1}}{1-t} - \frac{t^{2w-g+3} - t^{m-2w}}{1-t^2} \right)
\]

Now using Poincaré duality for $J_C$ to make the substitution

\[
P_A(J_C; t^{-3})t^{3g} = P_A(J_C; t^3),
\]

together with $m = n + 3g - 3$ and $2w = n + 2g - 3$, we obtain a factor of $1 - t^n$ on the right hand side which cancels to leave

\[
P_A(\mathcal{N}_C; t) = \frac{P_A(J_C; t^3) - t^2P_A(J_C; t)}{(1-t)(1-t^2)}(4.2)
\]

thereby proving Theorem 1. \qed
One immediate consequence is

**Corollary 4.1.** Numerical and homological equivalence coincide for $\mathcal{N}_C$.

*Proof.* By [Lieb] Theorem 1, this is equivalent to the statement that Poincaré duality holds numerically for the algebraic cohomology of $\mathcal{N}_C$, which follows from (4.2), because the same is true for $J_C$, by [Lieb] Theorem 3. $\square$

In contrast, algebraic and homological equivalence do not coincide for $\mathcal{N}_C$, for certain curves $C$ (see [Bal2]).

5. The general curve.

We use Theorem 1 to deduce some stronger statements about $\mathcal{N}_C$ for the general curve, i.e. for all curves $C$ lying in the complement of a countable union of proper closed subvarieties in the moduli space $M_g$ of curves.

We require first the following well-known fact about the general Jacobian.

**Proposition 5.1.** For a general curve $H^*_A(J_C)$ is generated by the $\theta$ divisor.

*Proof.* (We give a sketch here, having not found a suitable reference.) Any algebraic class is in $H^{p,p}$. The subalgebra of $H^*(J_C)$ consisting of classes that are in $H^{p,p}$ for all curves is invariant under the monodromy action of the mapping class group, which factors through the obvious action of the symplectic group $Sp(H^1(C;\mathbb{Z}))$. However, the only symplectically invariant subalgebra which is small enough to be contained in $\bigoplus_p H^{p,p}(J_C,\mathbb{C})$ is the one generated by $\theta$. On the other hand, because the Hodge filtration depends holomorphically on $M_g$, the condition that a given class in $H^{2p}(J_C,\mathbb{Q})$ is in $H^{p,p}$ determines a closed analytic subvariety of $M_g$ (or strictly Teichmüller space). $\square$

Hence, for a general curve $C$,

$$P_A(J_C) = \frac{1 - t^{g+1}}{1 - t}$$

and thus Theorem 1 yields

$$P_A(\mathcal{N}_C) = \frac{(1 - t^g)(1 - t^{g+1})(1 - t^{g+2})}{(1 - t)(1 - t^2)(1 - t^3)}$$

(5.1)

The above proof of Proposition 5.1 actually shows that the Hodge conjecture is true for the general Jacobian and hence, as observed in the Introduction,

**Corollary 5.2.** For general $C$, the Hodge conjecture is true for $\mathcal{N}_C$.

In addition (5.1) can be used to deduce
Corollary 5.3. For a general curve, the cohomology ring $H^*_A(N_C)$ is generated by $\alpha$, $\beta$ and $\gamma$ (in the notation of [Ne2]).

Proof. First note that $\alpha$, $\beta$ and $\gamma$ are certainly algebraic classes. From [Ne1] and [Ne2] (c.f. [KN] Proposition 2.1) one may see that, for $2n \leq 3g - 3$, the monomials

$$\alpha^i \beta^j \gamma^p \quad i + 2j + 3p = n, \quad i + 2p < g$$

are independent in $H^n_T$. Multiplying by $\alpha^{3g-3-2n}$ we may obtain an equal number of independent monomials in $H^{3g-3-n}_T$. As in [KN] Remark 2.3, the number of such monomials in degree $n$ is the coefficient of $t^n$ in

$$\sum_{p=0}^{[\frac{n}{3}]} \frac{(1-t^{g-2p})(1-t^{2g-4p})}{(1-t)(1-t^2)} t^{3p}$$

which (c.f. [KN] (2.8)) is equal to (5.1).

Alternatively, the proof of Proposition 5.1 may easily be adapted to prove Corollaries 5.2 and 5.3 directly. The much harder task of extending this argument to all smooth moduli spaces of plain and parabolic bundles has been carried out in [BN].

6. Proof of Theorem 2.

So far, we have only found the size of the algebraic cohomology ring $H^*_A(N_C)$ and not identified it as a subring of the full cohomology ring $H^*(N_C)$. However, Theorem 1 does indicate that there may be a natural relationship between the algebraic cohomology of $J_C$ and that of $N_C$, and this turns out to be the case. Recall from §1 the definition of

$$\nu : \mathbb{Q}[\alpha, \beta] \otimes H^*(J_C) \to H^*(N_C).$$

Proposition 6.1. The map $\nu$ takes algebraic classes on $J_C$ to algebraic classes on $N_C$, i.e.

$$\nu(1 \otimes H_A(J_C)) \subseteq H_A(N_C).$$

Proof. We start by recalling how the isomorphism (1.1) is defined. Let $\mathcal{L}$ be a universal bundle on $C \times J_C$ and $\phi$ the $(1,1)$ Künneth component of $c_1(\mathcal{L})$. Similarly, let $\mathcal{U}$ be a universal bundle on $C \times N_C$ and $\psi$ the $(1,3)$ Künneth component of $c_2(\mathcal{U})$. Note that, while there is an ambiguity in the choice of $\mathcal{L}$ and $\mathcal{U}$, this does not affect $\phi$ and $\psi$. Note also that $\phi$ and $\psi$ are both algebraic classes, because they differ from $c_1(\mathcal{L})$ and $c_2(\mathcal{U})$ respectively by obviously algebraic classes.

Now $\phi$ and $\psi$ induce two correspondences

$$H^1(C) \to H^1(J_C) : \omega \mapsto \int_C \omega \phi$$

$$H^1(C) \to H^3(N_C) : \omega \mapsto \int_C \omega \psi$$

(6.1)
which are both isomorphisms and which can then be combined to give the isomorphism $H^1(J_C) \cong H^3(N_C)$. This in turn induces the isomorphism (1.1), which determines the map

$$\nu : 1 \otimes H^*(J_C) \to H^*(N_C).$$

To prove the proposition it is sufficient to show that this map is a correspondence induced by an algebraic class on $J_C \times N_C$. If we define

$$\Delta = -\frac{1}{2} \int_C (\phi - \psi)^2,$$

which is clearly an algebraic class, then we claim that

$$\nu(1 \otimes \omega) = \int_{J_C} \omega \frac{\Delta^g}{g!}.$$  \hspace{1cm} (6.2)

To verify the claim by direct calculation, we introduce a basis $e_1, \ldots, e_{2g}$ for $H^1(C; \mathbb{Z})$ and let $e_1^\vee, \ldots, e_{2g}^\vee$ be the dual basis with respect to the symplectic structure given by the intersection form, using the convention that $\int_C e_i e_i^\vee = 1$. It is most convenient to choose a symplectic basis so that

$$e_i^\vee = \begin{cases} e_{i+g} & i \leq g \\ -e_{i-g} & i > g \end{cases}$$

We may use the isomorphisms (6.1) to define bases $\phi_1, \ldots, \phi_{2g}$ and $\phi_1^\vee, \ldots, \phi_{2g}^\vee$ of $H^1(J_C)$, and $\psi_1, \ldots, \psi_{2g}$ and $\psi_1^\vee, \ldots, \psi_{2g}^\vee$ of $H^3(N_C)$. With respect to these bases, we have

$$\phi = \sum_{i=1}^{2g} e_i^\vee \phi_i \quad \psi = \sum_{i=1}^{2g} e_i^\vee \psi_i$$

and thus

$$\Delta = \sum_{i=1}^{g} \phi_i \phi_i^\vee + \sum_{i=1}^{2g} \phi_i^\vee \psi_i + \sum_{i=1}^{g} \psi_i \psi_i^\vee.$$ 

Observe that the factor of $-\frac{1}{2}$ in the definition of $\Delta$ is required because it is

$$\theta = -\frac{1}{2} \int_C \phi^2 = \sum_{i=1}^{g} \phi_i \phi_i^\vee$$

which is the ample generator of $H^1(J_C; \mathbb{Z})$ and has the key property

$$\int_{J_C} \frac{\theta^g}{g!} = 1.$$ 

It is then fairly straightforward to verify (6.2) on monomials. \hspace{1cm} $\Box$
To complete the proof of Theorem 2, observe that it follows from results in [Ne1] and [Ne2] (c.f. [KN] Proposition 2.1 & Remark 2.2) that the map \( \nu \) induces an isomorphism

\[
\bigoplus_{(i,j,k) \in S} \alpha^i \beta^j H^k(J_C) \cong H^*(\mathcal{N}_C)
\]

for some subset \( S \subseteq \mathbb{N}^3 \). Moreover (c.f. [KN] Remark 2.3), the identity of Poincaré polynomials

\[
\sum_{(i,j,k) \in S} t^{2i+4j+3k} \dim H^k(J_C) = \frac{P(J_C; t^3) - t^{2g} P(J_C; t)}{(1 - t^2)(1 - t^4)}
\]

depends only on the fact that \( \dim H^k(J_C) = \dim H^{2g-k}(J_C) \), i.e. that \( H^*(J_C) \) satisfies Poincaré duality numerically.

Now Proposition 6.1 implies that \( \nu(\mathbb{Q}[\alpha, \beta] \otimes H^*_A(J_C)) \subseteq H_A(\mathcal{N}_C) \) and thus \( \nu \) also embeds

\[
\bigoplus_{(i,j,2k) \in S} \alpha^i \beta^j H^k_A(J_C)
\]

as a subspace of \( H^*_A(\mathcal{N}_C) \). However, we may deduce as above that

\[
\sum_{(i,j,2k) \in S} t^{i+2j+3k} \dim H^k_A(J_C) = \frac{P_A(J_C; t^3) - t^g P_A(J_C; t)}{(1 - t)(1 - t^2)}
\]

because \( H^*_A(J_C) \) also satisfies Poincaré duality numerically (by [Lieb] Theorem 3). By Theorem 1, the right hand side is \( P_A(\mathcal{N}_C) \), and so the subspace is equal to \( H^*_A(\mathcal{N}_C) \) and Theorem 2 is proved.

Remark 6.2. Proposition 3.3(iii) of [KN] identifies a natural choice for \( S \) and thereby shows that \( \nu \) induces an isomorphism

\[
H^*_A(\mathcal{N}_C) \cong \bigoplus_{i+2k < g \atop j+2k < g} \alpha^i \beta^j H^k_A(J_C).
\]
7. Chow groups.

In this section, we extend the results in Section 2 to describe how to relate the Chow groups \( A^k(X_+) \) and \( A^k(X_-) \), when \( X_+ \) and \( X_- \) are related by a smooth flip. We use the notation of Section 2 and in addition let \( \xi \in A^1(Z) \) be the relative hyperplane class for \( h_\pm \).

Proposition 7.1. Let

\[
B^k_+ = \bigoplus_{s=0}^{\mu-2} (\xi_+)^s \cdot (h_+)^{s} A^{k-\lambda-s}(S) = \ker (h_+)_* \subseteq A^{k-\lambda}(Z_+)
\]
\[
B^k_- = \bigoplus_{r=0}^{\lambda-2} (\xi_-)^r \cdot (h_-)^{r} A^{k-\mu-r}(S) = \ker (h_-)_* \subseteq A^{k-\mu}(Z_-)
\]

Then \( (i_\pm)_* : B^k_\pm \rightarrow A^k(X_\pm) \) is an injection and there is a canonical isomorphism

\[
A^k(X_+)/ (i_+)_* B^k_+ \cong A^k(X_-)/ (i_-)_* B^k_-
\]

Proof. First observe that we may rewrite (2.2) as

\[
A^k(\tilde{X}) = f^* A^k(X) \oplus j_* B^k,
\]

where \( j_* \) is injective on

\[
B^k = \bigoplus_{r=0}^{\lambda-2} \zeta^r \cdot g^* A^{k-r-1}(Z) = \ker g_* \subseteq A^{k-1}(E)
\]

and \( \zeta \in A^1(E) \) is the relative hyperplane class of \( g \).

Further recall the ‘key formula’ ([Fu] Prop. 6.7(a)) that, for any \( z \in A^{k-\lambda}(Z) \),

\[
f^* i_* z = j_* (\gamma \cdot g^* z)
\]

where

\[
\gamma = \sum_{i=0}^{\lambda-1} \zeta^{\lambda-1} \cdot g^* c_i(N).
\]

and \( N \) is the normal bundle of \( Z \) in \( X \).

We now use the flip diagram (2.1) to write \( A^k(\tilde{X}) \) in two different ways. From the right hand blowup diagram and the fact that \( \zeta_+ = g^*_+ \zeta_- \), we have

\[
A^k(\tilde{X}) = f_+^* A^k(X_+) \oplus j_* \bigoplus_{r=0}^{\lambda-2} g^-_* \xi^r \cdot g^+_* A^{k-r-1}(Z_+)
\]

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However, $h_+$ is also a projective bundle and $\gamma_+ = g_+^* \xi_+^{\mu - 1} + \cdots$. Hence, writing

$$B_0^k = \bigoplus_{r=0}^{\lambda-2} \bigoplus_{s=0}^{\mu-2} g_+^* \xi_-^r \cdot g_+^* \xi_-^s \cdot g_+^* h_+^* A^{k-r-s-1}(S) = \ker(g_+)_* \cap \ker(g_-)_* \subseteq A^{k-1}(E),$$

we have

$$A^k(\tilde{X}) = f_+^* A^k(X_+) \oplus j_* B_0^k \oplus \bigoplus_{r=0}^{\lambda-2} g_+^* \xi_-^r \cdot g_+^* h_+^* A^{k-\mu-r}(S)$$

$$= f_+^* A^k(X_+) \oplus j_* B_0^k \oplus f_+^* (i_*)_* \bigoplus_{r=0}^{\mu-2} \xi_-^r \cdot h_+^* A^{k-\lambda-s}(S)$$

using the ‘key formula’. Note that $(i_-)_*$ is an injection here because $j_*$ is above.

Similarly, from the left hand blowup diagram, we obtain

$$A^k(\tilde{X}) = f_-^* A^k(X_-) \oplus j_* B_0^k \oplus \bigoplus_{s=0}^{\mu-2} \xi_+^s \cdot h_+^* A^{k-\lambda-s}(S)$$

Dividing $A^k(\tilde{X})$ by the part common to both expressions completes the proof.

Proposition 7.1 (with the above proof) is also valid with “$A$” replaced by “$H_A$” and we may regard this as an enhanced version of (2.3).

We may use this proposition to identify the first two Chow groups

$$A^1(\mathcal{N}_C) \cong \mathbb{Z}$$

$$A^2(\mathcal{N}_C) \cong \begin{cases} A^1(C) & g = 2 \\ A^1(C) \oplus \mathbb{Z} & g > 2 \end{cases}$$

The first case is well-known (c.f. [Ra] Prop. 3.4), because $A^1 = Pic$. The second case is closely related to the isomorphism $J_C \cong J^2(\mathcal{N}_C)$ proved in [MN], where

$$J^2(\mathcal{N}_C) = H^3(\mathcal{N}_C; \mathbb{R})/H^3(\mathcal{N}_C; \mathbb{Z})$$

is the Weil-Griffiths intermediate Jacobian. Indeed, a small modification of the argument in [BV] shows that the Abel-Jacobi map $A^2_H(\mathcal{N}_C) \to J^2(\mathcal{N}_C)$ is an isomorphism, where $A^*_H \subseteq A^*$ is the ideal of cycles homologically equivalent to 0.

It is reasonable to hope that a closer analysis would yield complete information about the Chow ring of $\mathcal{N}_C$, at least modulo information about the Chow ring of $J_C$. 


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