Almost Sure Invariance Principle for Random Distance Expanding Maps with a Nonuniform Decay of Correlations

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Abstract We prove a quenched almost sure invariance principle for certain classes of random distance expanding dynamical systems which do not necessarily exhibit uniform decay of correlations.

1 Introduction

The aim of this note is to establish an almost sure invariance principle (ASIP) for certain classes of random dynamical systems. More precisely, similarly to the setting introduced in [21], the dynamics is formed by compositions

\[ f^n_\omega := f_{\sigma^{n-1} \omega} \circ \ldots \circ f_{\sigma \omega} \circ f_\omega, \ \omega \in \Omega \]

of locally distance expanding maps \( f_\omega \) satisfying certain topological assumptions which are driven by an invertible, measure preserving transformation \( \sigma \) on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Then, under suitable assumptions and for Hölder continuous observables \( \psi_\omega : X \to \mathbb{R}, \ \omega \in \Omega \) we establish a quenched ASIP. Namely, we prove that for \( \mathbb{P}\)-a.e. \( \omega \in \Omega \), the random Birkhoff sums \( \sum_{j=0}^{n-1} \psi_{\sigma^j \omega} \circ f^j_\omega \) can be approximated in the strong sense by a sum of Gaussian independent random variables \( \sum_{j=0}^{n-1} Z_j \) with the error being negligible compared to \( n^{3/2} \). In comparison with the previous results dealing with the ASIP for random or sequential dynamical systems, the main novelty of our work is that we do not require that our dynamics exhibits uniform (with respect to \( \omega \)) decay of correlations.

In a more general setting and under suitable assumptions, Kifer proved in [13] a central limit theorem (CLT) and a law of iterated logarithm (LIL). As Kifer remarks, his arguments (see [13, Remark 4.1]) also yield an ASIP when there is an underlying

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random family of $\sigma$-algebras which are sufficiently fast well mixing in an appropriate (random) sense (i.e. in the setup of [13, Theorem 2.1]). In the context of random dynamics, Kifer’s results can be applied to random expanding maps which admit a (random) symbolic representation. One of the main ingredients in [13] is a certain inducing argument, an approach that we also follow in the present paper. The main idea is that an ASIP for the original system will follow from an ASIP for a suitably constructed induced system.

For some classical work devoted to ASIP, we refer to [3, 19]. In addition, we stress that there are quite a few works whose aim is to establish ASIP for deterministic dynamical systems. In this direction, we refer to the works of Field, Melbourne and Török [8], Melbourne and Nicol [16, 17], and more recently to Korepanov [14, 15]. In [9], Gouëzel developed a new spectral technique for establishing ASIP, which was applied to certain classes of deterministic dynamical systems with the property that the corresponding transfer operator exhibits a spectral gap.

Gouëzel’s method was also used in [1] to obtain the annealed ASIP for certain classes of piecewise expanding random dynamical systems. In [6] the authors proved for the first time (we recall that Kifer in [13] only briefly commented that his methods also yield an ASIP) a quenched ASIP for piecewise expanding random dynamical systems, by invoking a recent ASIP for (reverse) martingales due to Cuny and Merlevède [5] (which was also applied in many other deterministic and sequential setups; see for example [12]). While the type of maps $f_\omega$ considered in [6] is more general than the ones considered in the present paper, in contrast to [6] in the present paper we do not assume a uniform decay of correlations. Moreover, the methods used in this paper can be extended to vector-valued observables $\psi_\omega$ (see Remark 1). On the other hand, it is unclear if the techniques in [6] can be extended to the vector-valued case since the results in [5] deal exclusively with the scalar-valued observables. Finally, we mention our previous work [7], where we have obtained a quenched ASIP for certain classes of hyperbolic random dynamical systems. In addition, we have improved the main result from [6]. However, the classes of dynamics we have considered again exhibit uniform decay of correlations.

Our techniques for establishing ASIP (besides the already mentioned inducing arguments), rely on a certain adaptation of the method of Gouëzel [9] which is of independent interest. Indeed, we first need to modify Gouëzel’s arguments and show that they yield an ASIP for non-stationary sequences of random variables, which are not necessarily bounded in some $L^p$ space.

We stress that our error term in ASIP is of order $n^{1/4+O(1/p)}$, where $p$ comes from certain $L^p$-regularity conditions we impose for the induced system. This is rather close to the $n^{1/4}$ rate for deterministic uniformly expanding systems [9], when $p \to \infty$ (although this rate was significantly improved by Korepanov [15]).
2 Random distance expanding maps

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space. Furthermore, let \(\sigma : \Omega \to \Omega\) be an invertible \(\mathbb{P}\)-preserving transformation such that \((\Omega, \mathcal{F}, \mathbb{P}, \sigma)\) is ergodic. Moreover, let \((X, \rho)\) be a compact metric space normalized in size so that \(\text{diam} X \leq 1\) together with the Borel \(\sigma\)-algebra \(\mathcal{B}\), and let \(E \subset \Omega \times X\) be a measurable set (with respect to the product \(\sigma\)-algebra \(\mathcal{F} \times \mathcal{B}\)) such that the fibers
\[ E_\omega = \{ x \in X : (\omega, x) \in E \}, \quad \omega \in \Omega \]
are compact. Hence (see [4, Chapter III]), it follows that the map \(\omega \mapsto E_\omega\) is measurable with respect to the Borel \(\sigma\)-algebra induced by the Hausdorff topology on the space \(\mathcal{K}(X)\) of compact subspaces of \(X\). Moreover, the map \(\omega \mapsto \rho(x, E_\omega)\) is measurable for each \(x \in X\). Finally, the projection map \(\pi_\Omega(\omega, x) = \omega\) is measurable and it maps any \(\mathcal{F} \times \mathcal{B}\)-measurable set to an \(\mathcal{F}\)-measurable set (see [4, Theorem III.23]).

Let \(f_\omega : E_\omega \to E_{\sigma^\omega}\), \(\omega \in \Omega\) be a family of surjective maps such that the map \((\omega, x) \mapsto f_\omega(x)\) is measurable with respect to the \(\sigma\)-algebra \(\mathcal{P}\) which is the restriction of \(\mathcal{F} \times \mathcal{B}\) on \(E\). Consider the skew product transformation \(F : E \to E\) given by
\[ F(\omega, x) = (\sigma \omega, f_\omega(x)). \] (1)

For \(\omega \in \Omega\) and \(n \in \mathbb{N}\), set
\[ f_\omega^n := f_{\sigma^{n-1} \omega} \circ \ldots \circ f_\omega : E_\omega \to E_{\sigma^n \omega}. \]

Let us now introduce several additional assumptions for the family \(f_\omega, \omega \in \Omega\). More precisely, we require that:

- **(topological exactness)** there exist a constant \(\xi > 0\) and a random variable \(\omega \mapsto n_\omega \in \mathbb{N}\) such that for \(\mathbb{P}\)-a.e. \(\omega \in \Omega\) and any \(x \in E_\omega\) we have that
  \[ f_\omega^{n_\omega}(B_\omega(x, \xi)) = E_{\sigma^{n_\omega} \omega}, \] (2)
  where \(B_\omega(x, r)\) denotes an open ball in \(E_\omega\) centered in \(x\) with radius \(r\);

- **(pairing property)** there exist random variables \(\omega \mapsto \gamma_\omega > 1\) and \(\omega \mapsto D_\omega \in \mathbb{N}\) such that for \(\mathbb{P}\)-a.e. \(\omega \in \Omega\) and for any \(x, x' \in E_{\sigma \omega}\) with \(\rho(x, x') < \xi\) (\(\xi\) comes from the previous assumption), we have that
  \[ f_\omega^{-1}(\{x\}) = \{y_1, \ldots, y_k\}, \quad f_\omega^{-1}(\{x'\}) = \{y_1', \ldots, y'_k\}, \] (3)
  \[ k = k_{\omega, x} = |f_\omega^{-1}(\{x\})| \leq D_\omega \]
  and
  \[ \rho(y_i, y'_i) \leq (\gamma_\omega)^{-1}\rho(x, x'), \quad \text{for } 1 \leq i \leq k. \] (4)

The above assumptions were considered in [10], and they hold true in the setup of distance expanding maps considered in [21]. We note that all the results stated in
[21] hold true under these assumptions (see [21] Chapter 7) and not only under the assumptions from [21] Section 2. For \( \omega \in \Omega \) and \( n \in \mathbb{N} \), set

\[
\gamma_{\omega,n} := \prod_{i=0}^{n-1} \gamma_{\omega,i} \quad \text{and} \quad D_{\omega,n} := \prod_{i=0}^{n-1} D_{\omega,i}.
\]  

(5)

By induction, it follows from the pairing property that for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) and for any \( x, x' \in \mathcal{E}_{\omega} \) with \( \rho(x, x') < \xi \), we have that

\[
(f^n_\omega)^{-1}(\{x\}) = \{y_1, \ldots, y_k\} \quad \text{and} \quad (f^n_\omega)^{-1}(\{x'\}) = \{y'_1, \ldots, y'_k\},
\]

(6)

where

\[
k = k_{\omega,x,n} = |(f^n_\omega)^{-1}(\{x\})| \leq D_{\omega,n},
\]

and

\[
\rho(f^n_\omega y_i, f^n_\omega y'_j) \leq (\gamma_{\omega,i}\omega,n-j)^{-1}\rho(x, x'), \quad \text{for } 1 \leq i \leq k \text{ and } 0 \leq j < n.
\]

(7)

Let \( g : \mathcal{E} \to \mathbb{C} \) be a measurable function. For any \( \omega \in \Omega \), consider the function \( g_\omega := g(\omega, \cdot) : \mathcal{E}_\omega \to \mathbb{C} \). For any \( 0 < \alpha \leq 1 \), set

\[
v_{\alpha,\xi}(g_\omega) := \inf \{R > 0 : |g_\omega(x) - g_\omega(x')| \leq R \rho^\alpha(x, x') \text{ if } \rho(x, x') < \xi\},
\]

and let

\[
\|g_\omega\|_{\alpha,\xi} = \|g_\omega\|_\infty + v_{\alpha,\xi}(g_\omega),
\]

where \( \|\cdot\|_\infty \) denotes the supremum norm and \( \rho^\alpha(x, x') := (\rho(x, x'))^\alpha \). We emphasize that these norms are \( \mathcal{F} \)-measurable (see [10] p. 199).

Let \( \mathcal{H}^{\alpha,\xi}_\omega = (\mathcal{H}^{\alpha,\xi}_\omega, \|\cdot\|_{\alpha,\xi}) \) denote the space of all \( h : \mathcal{E}_\omega \to \mathbb{C} \) such that \( \|h\|_{\alpha,\xi} < \infty \). Moreover, let \( \mathcal{H}^{\alpha,\xi}_\omega \) be the space of all real-valued functions in \( \mathcal{H}^{\alpha,\xi}_\omega \).

Take a random variable \( H : \Omega \to (1, \infty) \) such that

\[
\int_\Omega \ln H_\omega \, d\mathbb{P}(\omega) < \infty,
\]

where \( H_\omega := H(\omega) \). Moreover, let \( \mathcal{H}^{\alpha,\xi}(H) \) be the set of all measurable functions \( g : \mathcal{E} \to \mathbb{C} \) satisfying \( v_{\alpha,\xi}(g_\omega) \leq H_\omega \) for \( \omega \in \Omega \). Furthermore, for \( \omega \in \Omega \) set

\[
\mathcal{H}^{\alpha,\xi}_\omega(H) := \{g : \mathcal{E}_\omega \to \mathbb{C} : g \text{ measurable and } v_{\alpha,\xi}(g) \leq H_\omega\}
\]

and

\[
Q_\omega(H) = \sum_{j=1}^\infty H_{\sigma^{-j}\omega}(\gamma_{\sigma^{-j}\omega,j})^{-\alpha}.
\]

(8)

Since \( \omega \mapsto \ln H_\omega \) is integrable, we have (see [21] Chapter 2) that \( Q_\omega(H) < \infty \) for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \). The following simple distortion property is a direct consequence of [7].
Lemma 1. Take $\omega \in \Omega$, $n \in \mathbb{N}$ and $\varphi = (\varphi_0, \ldots, \varphi_{n-1})$, where $\varphi_i \in \mathcal{H}_{\sigma(\omega)}^{\alpha, \xi}(H)$ for $0 \leq i \leq n - 1$. Set

$$S_n^\omega \varphi := \sum_{j=0}^{n-1} \varphi_j \circ f_j.$$

Furthermore, take $x, x' \in E_{\sigma(\omega)}$ such that $\rho(x, x') < \xi$ and let $y_i, y'_i, 1 \leq i \leq k$ be as in (6). Then, for any $1 \leq i \leq k$ we have that

$$|S_n^\omega \varphi(y_i) - S_n^\omega \varphi(y'_i)| \leq \rho(x, x')Q_{\sigma(\omega)}(H).$$

2.1 Transfer operators

Let us take an observable $\psi : E \to \mathbb{R}$ such that $\psi \in \mathcal{H}_{\sigma(\omega)}^{\alpha, \xi}(H)$. We consider the associated random Birkhoff sums

$$S_n^\omega \psi = \sum_{i=0}^{n-1} \psi_{\sigma^i(\omega)} \circ f_i,$$ for $n \in \mathbb{N}$ and $\omega \in \Omega$.

Furthermore, suppose that $\phi : E \to \mathbb{R}$ also belongs to $\mathcal{H}_{\sigma(\omega)}^{\alpha, \xi}(H)$. For $\omega \in \Omega$, $z \in \mathbb{C}$ and $g : E_{\sigma(\omega)} \to \mathbb{C}$, we define

$$L_z^\omega g(x) = \sum_{y \in E_{\omega}^{-1}(\{x\})} e^{\phi(y) + z\psi(y)} g(y).$$ (9)

It follows from [10, Theorem 5.4.1.] that $L_z^\omega : \mathcal{H}_{\sigma(\omega)}^{\alpha, \xi} \to \mathcal{H}_{\sigma(\omega)}^{\alpha, \xi}$ is a well-defined and bounded linear operator for each $\omega \in \Omega$ and $z \in \mathbb{C}$. Moreover, the map $z \mapsto L_z^\omega$ is analytic for each $\omega \in \Omega$.

Let us denote $L_0^\omega$ simply by $L_\omega^\omega$. It follows from [21, Theorem 3.1.] that for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists a triplet $(\lambda_\omega, h_\omega, \nu_\omega)$ consisting of a positive number $\lambda_\omega > 0$, a strictly positive function $h_\omega \in \mathcal{H}_{\sigma(\omega)}^{\alpha, \xi}$ and a probability measure $\nu_\omega$ on $E_{\sigma(\omega)}$ so that

$$L_\omega^\omega h_\omega = \lambda_\omega h_{\sigma(\omega)}^\omega, \quad (L_\omega^\omega)^* \nu_{\sigma(\omega)} = \lambda_\omega \nu_\omega, \quad \nu_\omega(h_\omega) = 1,$$

and that maps $\omega \mapsto \lambda_\omega$, $\omega \mapsto h_\omega$ and $\omega \mapsto \nu_\omega$ are measurable. We can assume without any loss of generality that $\lambda_\omega = 1$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ (since otherwise we can replace $L_\omega^\omega$ with $L_\omega^\omega/\lambda_\omega$). For $\mathbb{P}$-a.e. $\omega \in \Omega$, let $\mu_\omega$ be a measure on $E_{\omega}$ given by $d\mu_\omega := h_\omega d\nu_\omega$. We recall (see [21, Lemma 3.9]) that these measures satisfy the so-called equivariant property, i.e. we have that

$$f_\omega^* \mu_\omega = \mu_{\sigma(\omega)}, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$ (10)

Moreover, these measures give rise to a measure $\mu$ on $\Omega \times E$ with the property that for any $A \in \mathcal{F} \times \mathcal{B}$,
\[
\mu(A) = \int_{\Omega} \mu_\omega(A_\omega) d\mathbb{P}(\omega),
\]
where \(A_\omega = \{ x \in E_\omega; (\omega, x) \in A \}\). Then, \(\mu\) is invariant for the skew-product transformation \(F\) given by (1). Moreover, \(\mu\) is ergodic.

For \(t = (t_0, \ldots, t_{n-1}) \in \mathbb{R}^n\), set
\[
L^n_\omega : = \mathcal{L}^{t_0, \sigma \circ t_1, \omega} \circ \cdots \circ \mathcal{L}^{t_{n-1}, \omega}.
\]
Moreover, let \(L^n_\omega : = L^n_\omega \), where \(\mathfrak{F} = (0, \ldots, 0) \in \mathbb{R}^n\). Note that
\[
\| L^n_\omega 1 \|_\infty \leq (\deg f^n_\omega ) \cdot e^{\| S^n_\omega \phi \|_\infty} \leq D_{\alpha, n} e^{\| S^n_\omega \phi \|_\infty} < \infty,
\]
where \(1\) is the function taking constant value 1 and 
\[
\deg f^n_\omega : = \sup_{x \in E_{\alpha, \omega}} |(f^n_\omega)^{-1}(\{x\})|.
\]

**Lemma 2.** For any \(\mathbb{P}\text{-a.e. } \omega \in \Omega\) we have that for any \(n \in \mathbb{N}\), \(T > 0\), \(\mathcal{F} = (t_0, \ldots, t_{n-1}) \in [-T, T]^n\) and \(g \in H^\alpha_{\mathcal{F}, \omega}\),
\[
\nu_{\alpha, \epsilon}(L^n_\omega g) \leq \| L^n_\omega 1 \|_\infty (\nu_{\alpha, \epsilon}(g)(y_{\alpha, n})^{-\alpha} + 2Q_{\sigma^n \omega}(H)(1 + T)\| g \|_\infty).
\]
Consequently,
\[
\| L^n_\omega g \|_{\alpha, \epsilon} \leq \| L^n_\omega 1 \|_\infty (\nu_{\alpha, \epsilon}(g)(y_{\alpha, n})^{-\alpha} + (1 + 2Q_{\sigma^n \omega}(H))(1 + T)\| g \|_\infty). \quad (11)
\]

**Proof.** The proof is similar to the proof of [10, Lemma 5.6.1.], but for reader’s convenience all the details are given. The idea is to apply Lemma[11] for \(\varphi = (\varphi_0, \ldots, \varphi_{n-1})\) given by
\[
\varphi_j : = \phi_{\sigma \circ t_j, \omega} + i_j \psi_{\sigma \circ t_j, \omega}, \quad \text{for } 0 \leq j \leq n - 1.
\]
Set \(A^n_\omega = \sum_{j=0}^{n-1} i_j \psi_{\sigma \circ t_j, \omega} \circ f_j^1\). Firstly, by the definition of \(L^n_\omega\) we have
\[
\| L^n_\omega g \|_\infty \leq \| g \|_\infty \| L^n_\omega 1 \|_\infty. \quad (12)
\]
In order to complete the proof of the lemma we need to approximate \(\nu_{\alpha, \epsilon}(L^n_\omega g)\). Let \(x, x' \in E_{\sigma^n \omega}\) be such that \(\rho(x, x') < \xi\) and let \(y_1, \ldots, y_k\) and \(y'_1, \ldots, y'_k\) be the points in \(E_{\omega}\) satisfying (3) and (4). We can write
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In order to estimate $I_1$, observe that for any $1 \leq q \leq k$,

$$|e^{iA_{\sigma n}^q(y_q)}g(y_q) - e^{iA_{\sigma n}^q(y'_q)}g(y'_q)| \leq |g(y_q)| \cdot |e^{iA_{\sigma n}^q(y_q)} - e^{iA_{\sigma n}^q(y'_q)}| + |g(y_q) - g(y'_q)| =: J_1 + J_2.$$ 

By the mean value theorem and then by Lemma 1,

$$J_1 \leq 2T\|g\|_{\infty}Q_{\sigma^n,\omega}(H)|\rho^\alpha(x, x'),$$

while by (7),

$$J_2 \leq \nu_{\alpha,\xi}(g)|\rho^\alpha(y_q, y'_q) \leq \nu_{\alpha,\xi}(g)(\gamma_{\omega, n})^{-\alpha}\rho^\alpha(x, x').$$

It follows that

$$I_1 \leq L_{\infty}^n1(x)(2T\|g\|_{\infty}Q_{\sigma^n,\omega}(H) + \nu_{\alpha,\xi}(g)(\gamma_{\omega, n})^{-\alpha})\rho^\alpha(x, x').$$

Next, we estimate $I_2$. By the mean value theorem and Lemma 1

$$|e^{S^n_{\sigma n}(y_q)} - e^{S^n_{\sigma n}(y'_q)}| \leq Q_{\sigma^n,\omega}(H) \cdot \max\{e^{S^n_{\sigma n}(y_q)}, e^{S^n_{\sigma n}(y'_q)}\}\rho^\alpha(x, x')$$

and therefore

$$I_2 \leq \|g\|_{\infty}(L_{\infty}^n1(x) + L_{\infty}^n1(x'))Q_{\sigma^n,\omega}(H)\rho^\alpha(x, x') \leq 2\|g\|_{\infty}L_{\infty}^n1(\|Q_{\sigma^n,\omega}(H)\rho^\alpha(x, x'),$$

yielding the first statement of the lemma and (11) follows from (12), together with the first statement.

By Lemma 2 together with the observation that $(\gamma_{\omega, n})^{-\alpha} \leq 1$, we conclude that there exists a random variable $C : \Omega \to [1, \infty)$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $n \in \mathbb{N}$ and for any $\tilde{t} = (t_0, t_1, \ldots, t_{n-1}) \in [-1, 1]^n$, we have that

$$\|\mathcal{L}_{\sigma n}^\omega\|_{\alpha, \xi} \leq C(\sigma^n, \omega)\|\mathcal{L}_{\sigma n}^\omega\|_{\infty},$$

(13)
where \( \| L_{1,n}^{\tau} \|_{\alpha,\xi} \) denotes the operator norm of \( L_{1,n}^{\tau} \) when considered as a linear operator from \( \mathcal{H}_{\omega}^{\alpha,\xi} \) to \( \mathcal{H}_{\omega}^\mathcal{Q}_{\omega}^{\alpha,\xi} \). Note that we can just take \( C(\omega) = 4(1 + Q_{\omega}) \). For \( \mathbb{P}\text{-a.e. } \omega \in \Omega \), we define \( \hat{L}_{\omega}^{\tau} : \mathcal{H}_{\omega}^{\alpha,\xi} \rightarrow \mathcal{H}_{\omega}^{\alpha,\xi} \) by

\[
\hat{L}_{\omega}^{\tau}g = L_{\omega}(gh_{\omega})/h_{\sigma^{\omega}}, \quad g \in \mathcal{H}_{\omega}^{\alpha,\xi}.
\]

Moreover, for \( n \in \mathbb{N} \), set

\[
\hat{L}_{\omega}^{n} := \hat{L}_{\sigma^{n-1}\omega} \circ \ldots \circ \hat{L}_{\omega}.
\]

Clearly,

\[
\hat{L}_{\omega}^{n}g = L_{\omega}^{n}(gh_{\omega})/h_{\sigma^{n}\omega}, \quad g \in \mathcal{H}_{\omega}^{\alpha,\xi} \quad \text{and} \quad n \in \mathbb{N}.
\]

We need the following result which is a direct consequence of \([21, \text{Lemma 3.18}].\)

**Lemma 3.** There exist \( \lambda > 0 \) and a random variable \( K : \Omega \rightarrow (0, \infty) \) such that

\[
\| \hat{L}_{\omega}^{n} g \|_{\infty} \leq \max(1, 1/Q_{\omega})K(\sigma^{n}\omega)e^{-\lambda n}\| g \|_{\alpha,\xi},
\]

for \( \mathbb{P}\text{-a.e. } \omega \in \Omega, n \in \mathbb{N} \) and \( g \in \mathcal{H}_{\omega}^{\alpha,\xi} \) such that \( \int L_{\omega}g \, d\mu_{\omega} = 0 \).

Applying Lemma 3 with the function \( g = 1/h_{\omega} - 1 \), and taking into account that \( L_{\omega}^{n}h_{\omega} = h_{\sigma^{n}\omega} \) (since \( \lambda_{\omega} = 1 \)), it follows from (13) that for \( \mathbb{P}\text{-a.e. } \omega \in \Omega, n \in \mathbb{N} \) and for any \( \tilde{t} = (t_{0}, t_{1}, \ldots, t_{n-1}) \in [-1, 1]^{n} \),

\[
\| L_{\omega}^{\tilde{t},n} \|_{\alpha,\xi} \leq (1 + U(\omega))K(\sigma^{n}\omega)C(\sigma^{n}\omega) \tag{14}
\]

where \( C(\omega) = C(\omega)\| h_{\omega} \|_{\infty} \) and \( U(\omega) = \max(1, 1/Q_{\omega}) \cdot (1 + \| 1/h_{\omega} \|_{\alpha,\xi}) \).

### 3 A refined version of Gouëzel’s theorem

In this section we present a more general version of Gouëzel’s almost sure invariance principle for non-stationary processes \([9, \text{Theorem 1.3}].\) This result will then be used in the next section to obtain the almost sure invariance principle for random distance expanding maps.

Let \( (A_{1}, A_{2}, \ldots) \) be an \( \mathbb{R}\text{-valued process on some probability space} (\Omega, \mathcal{F}, \mathbb{P}) \). We first recall the condition that we denote (following \([9] \)) by (H): there exist \( \epsilon_{0} > 0 \) and \( C, c > 0 \) such that for any \( n, m > 0, b_{1} < b_{2} < \ldots < b_{n+m+k}, k > 0 \) and \( t_{1}, \ldots, t_{n+m} \in \mathbb{R} \) with \( |t_{j}| \leq \epsilon_{0} \), we have that

\[
\left| \mathbb{E} \left( e^{i \sum_{j=1}^{b_{j+1}} A_{t_{j}}} \right) - \mathbb{E} \left( e^{i \sum_{j=1}^{b_{j+1}} \epsilon_{0}} \right) \right| \leq C(1 + \max |b_{j+1} - b_{j}|)^{C(n+m)} e^{-ck}.
\]
**Theorem 1.** Suppose that \((A_1, A_2, \ldots)\) is an \(\mathbb{R}\)-valued centered process on the probability space \((\Omega, \mathcal{F}, P)\) that satisfies \((H)\). Furthermore, assume that:

1. there exist \(u > 0\) and \(L \in \mathbb{N}\) such that for any \(n, m \in \mathbb{N}, m \geq L\) we have that
   \[
   \text{Var} \left( \sum_{j=n+1}^{n+m} A_j \right) \geq um; \quad (15)
   \]
2. there exist constants \(p \geq 6\) and \(a, C > 0\) such that for any \(n \in \mathbb{N}\) we have
   \[
   \|A_n\|_{L^p} \leq an^{1/p}. \quad (16)
   \]

In addition, for any \(n, m \in \mathbb{N}\) the finite sequence \((A_i/(n + m)^{1/p})_{n+1 \leq i \leq n+m}\) also satisfies condition \((H)\) with the same constants \(c_0, C\) and \(c\).

Then for any \(\delta > 0\), there exists a coupling between \((A_i)\) and a sequence \((B_i)\) of independent centered normal random variables such that

\[
\sum_{j=1}^{n} (A_j - B_j) = o(n^{a_p+\delta}) \quad \text{a.s.}, \quad (17)
\]

where

\[
a_p = \frac{p}{4(p-1)} + \frac{1}{p}.
\]

Moreover, there exists a constant \(C > 0\) such that for any \(n \in \mathbb{N}\),

\[
\left\| \sum_{j=1}^{n} A_j \right\|_{L^2} - Cn^{a_p+\delta} \leq \left\| \sum_{j=1}^{n} B_j \right\|_{L^2} \leq \left\| \sum_{j=1}^{n} A_j \right\|_{L^2} + Cn^{a_p+\delta}. \quad (18)
\]

Finally, there exists a coupling between \((A_i)\) and a standard Brownian motion \((W_t)_{t \geq 0}\) such that

\[
\sum_{j=1}^{n} A_j - W_{\sigma_n^2} = o(n^{a_p+\delta}) \quad \text{a.s.},
\]

where

\[
\sigma_n = \left\| \sum_{j=1}^{n} A_j \right\|_{L^2}.
\]

**Remark 1.** The above result (together with its proof) is similar to [9, Theorem 1.3]. However, we stress that [9, Theorem 1.3] requires that the process \((A_1, A_2, \ldots)\) is bounded in \(L^p\), while the above Theorem works under the assumption that \((16)\) holds. Consequently, the estimate for the error term in \((17)\) is different from that in [9, Theorem 1.3].

Note also that our condition \((15)\) replaces condition \((1.3)\) in [9, Theorem 1.3]. This, of course, makes it impossible to get a precise formula for the variance of
the approximating Gaussian random variables $\sum_{j=1}^{n} B_j$, as in [9]. However, in our context we have the estimate (18). Observe that (18) together with (15) ensures that
\[
\lim_{n \to \infty} \frac{\|\sum_{j=1}^{n} B_j\|_{L^2}}{\|\sum_{j=1}^{n} A_j\|_{L^2}} = 1.
\]

Therefore, Theorem 11 yields a corresponding almost sure version of the CLT for the sequence $\frac{1}{n} \sum_{j=1}^{n} A_j$, where $a_n = \|\sum_{j=1}^{n} A_j\|_{L^2}$. As we have mentioned, a precise formula for the variance of the approximating Gaussian random variables in the context of [9] Theorem 1.3 was obtained in [9] Lemma 5.7. Hence, in our modification of the proof of [9] Theorem 1.3] we will not need an appropriate version of [9] Lemma 5.7] (and instead we will prove (18) directly).

We also note that our modification of the arguments in [9] also yields a certain convergence rate for $p \in (4, 6)$, but in order to keep our exposition as simple as possible we have formulated the results only under the assumption that $p \geq 6$.

Finally, we remark that like in [9] we can consider processes taking values in $\mathbb{R}^d$ and that Theorem 1 holds in this case also. We prefer to work with processes in $\mathbb{R}$ to keep our exposition as simple as possible.

Proof (Proof of Theorem 11). We follow step by step the proof of [9] Theorem 1.3] by making necessary adjustments. Firstly, applying [9] Proposition 4.1 with the finite sequence $(A_j/(n + m)^{1/p})_{n+1 \leq j \leq n+m}$, we get that for each $\eta > 0$ there exists $C > 0$ such that
\[
\left\| \sum_{j=n+1}^{n+m} A_j \right\|_{L^{p-\eta}} \leq C n^{2}(n + m)^{1/p}, \quad \text{for } m, n \geq 0. 
\] (19)

We note that although [9] Proposition 4.1] was formulated for an infinite sequence, the proof for a finite sequence proceeds by using the same arguments. We consider the so-called big and small blocks as introduced in [9] p.1659. Fix $\beta \in (0, 1)$ and $\varepsilon \in (0, 1 - \beta)$. Furthermore, let $f = f(n) = [\beta n]$. Then, Gouëzel decomposes $[2^n, 2^{n+1}]$ into a union of $F = 2^{f}$ intervals $(I_{n,j})_{0 \leq j < F}$ of the same length, and $F$ gaps $(J_{n,j})_{0 \leq j < F}$ between them. In other words, we have
\[
[2^n, 2^{n+1}] = \bigcup_{j=0}^{F-1} I_{n,j} \cup \bigcup_{j=0}^{F-1} J_{n,j} \cup \bigcup_{j=0}^{F-1} J_{n,F-1} \cup I_{n,F-1}.
\]

Let us outline the construction of this decomposition. For $1 \leq j < F$, we write $j$ in the form $j = \sum_{k=0}^{f-1} \alpha_k(j) 2^k$ with $\alpha_k \in \{0, 1\}$. We then take the smallest $r$ with the property that $\alpha_r(j) \neq 0$ and take $2^{[cn]} 2^r$ to be the length of $J_{n,j}$. In addition, the length of $J_{n,0}$ is $2^{[cn]} 2^f$. Finally, the length of each interval $I_{n,j}$ is $2^{n-f} - (f + 2)2^{[cn]-1}$.

In addition, we need some notations from [9] which we will also use. We define a partial order on $\{(n, j) : n \in \mathbb{N}, 0 \leq j < F(n)\}$ by writing $(n, j) < (n', j')$ if the interval $I_{n,j}$ is to the left of $I_{n',j'}$. Observe that a sequence $((n_k, j_k))_k$ tends to infinity if and only if $n_k \to \infty$. Moreover, let
\[
X_{n,j} := \sum_{\ell \in I_{n,j}} A_{\ell}
\]
and
\[
I := \bigcup_{n,j} I_{n,j} \quad \text{and} \quad J := \bigcup_{n,j} J_{n,j}.
\]

The rest of the proof will be divided (following again [9]) into six steps.

**First step:** We first prove the following version of [9, Proposition 5.1].

**Proposition 1.** There exists a coupling between \((X_{n,j})\) and \((Y_{n,j})\) such that, almost surely, when \((n,j)\) tends to infinity,

\[
\left| \sum_{(n',j') < (n,j)} X_{n',j'} - Y_{n',j'} \right| = o(2^{\beta n + \varepsilon n/2}).
\]

Here, \((Y_{n,j})\) is a family of independent random variables such that \(Y_{n,j}\) and \(X_{n,j}\) are equally distributed.

Before we outline the proof of Proposition 1, we will first introduce some preparatory material. Let \(\hat{X}_{n,j} = X_{n,j} + V_{n,j}\), where the \(V_{n,j}\)'s are independent copies of the random variable \(V\) constructed in [9, Proposition 3.8], which are independent of everything else (enlarging our probability space if necessary). Write \(X_{n} = (X_{n,j})_{0 \leq j < F(n)}\) and \(\hat{X}_{n} = (\hat{X}_{n,j})_{0 \leq j < F(n)}\). Then, we have the following version of [9, Lemma 5.2].

**Lemma 4.** Let \(\tilde{Q}_{n}\) be a random variable distributed like \(\hat{X}_{n}\), but independent of \((\hat{X}_{1}, \ldots, \hat{X}_{n-1})\). We have

\[
\pi((\hat{X}_{1}, \ldots, \hat{X}_{n-1}, \hat{X}_{n}), (\hat{X}_{1}, \ldots, \hat{X}_{n-1}, \tilde{Q}_{n})) \leq C 4^{-n},
\]

where \(\pi(\cdot, \cdot)\) is the Prokhorov metric (see [9, Definition 3.3]) and \(C > 0\) is some constant not depending on \(n\).

**Proof (Proof of Lemma 4).** The proof is carried out by repeating the proof of [9, Lemma 5.2] with one slight modification. For reader’s convenience we provide a complete proof.

The random process \((X_{1}, \ldots, X_{n})\) takes its values in \(\mathbb{R}^{D}\), where \(D = \sum_{m=1}^{n} F(m) \leq C 2^{\beta n}\). Moreover, each component in \(\mathbb{R}\) of this process is one of the \(X_{n,j}\), hence it is a sum of at most \(2^{n}\) consecutive variables \(A_{\ell}\). On the other hand, the interval \(J_{0,0}\) is a gap between \((X_{j})_{j=n}\) and \(X_{n}\), and its length \(k\) is \(C^{\leq 2^{\varepsilon n} + \beta n}\). Let \(\phi\) and \(\gamma\) denote the respective characteristic functions of \((X_{1}, \ldots, X_{n-1}, X_{n})\) and \((X_{1}, \ldots, X_{n-1}, \tilde{Q}_{n})\), where \(\tilde{Q}_{n}\) is distributed like \(X_{n}\) and is independent of \((X_{1}, \ldots, X_{n-1})\). The assumption (H) ensures that for Fourier parameters \(t_{m,j}\) all bounded by \(\varepsilon_{0}\), we have

\[
|\phi - \gamma| \leq C(1 + 2^{n})^{CD} e^{-ck} \leq C e^{-c2^{\varepsilon n + \beta n}},
\]

if \(n\) is large enough. Let \(\tilde{\phi}\) and \(\tilde{\gamma}\) be the characteristic functions of, respectively, \((\hat{X}_{1}, \ldots, \hat{X}_{n})\) and \((\hat{X}_{1}, \ldots, \hat{X}_{n-1}, \tilde{Q}_{n})\): they are obtained by multiplying \(\phi\) and \(\gamma\) by...
the characteristic function of \( V \) is each variable. Since this function is supported in \( \{ |t| \leq \varepsilon_0 \} \), we obtain, in particular, that
\[
|\hat{\phi} - \hat{\gamma}| \leq C e^{-c 2^{2n+\varepsilon n}}.
\]

We then use [9] Lemma 3.5] with \( N = D \) and \( T' = e^{2\varepsilon n/2} \) to obtain that
\[
\pi((\tilde{X}_1, \ldots, \tilde{X}_n), (\tilde{X}_1, \ldots, \tilde{X}_{n-1}, \tilde{Q}_n)) \leq \sum_{m \geq n} \sum_{j \in F(m)} \mathbb{P}(|\tilde{X}_{m,j}| \geq e^{2\varepsilon n/2}) + e^{CD 2^{2n/2}} e^{-c 2^{2n+\varepsilon n}}.
\]

So far our arguments were identical to those in the proof of [9 Lemma 5.2]. In the rest of the proof we will introduce the above mentioned modification of the arguments from [9]. Using the Markov inequality, we obtain that
\[
\mathbb{P}(|\tilde{X}_{m,j}| \geq e^{2\varepsilon n/2}) \leq e^{-2\varepsilon n/2} \mathbb{E}|\tilde{X}_{m,j}|.
\]

However, since \( \|A_l\|_{L^p} \leq a l^{1/p} \) for every \( l \in \mathbb{N} \) (and for some constant \( a > 0 \)), we have that \( \mathbb{E}|\tilde{X}_{m,j}| \leq C 2^{n+\varepsilon n} \). Summing the resulting upper bounds for \( \mathbb{P}(|\tilde{X}_{m,j}| \geq e^{2\varepsilon n/2}) \), we obtain the desired result.

The following result follows from Lemma 4 exactly in the same way as [9 Corollary 5.3] follows from [9 Lemma 5.2].

**Corollary 1.** Let \( \tilde{R}_n = (\tilde{R}_{n,j})_{j \in F(n)} \) be distributed like \( \tilde{X}_n \) and such that the \( \tilde{R}_n \) are independent of each other. Then there exist \( C > 0 \) and a coupling between \( (\tilde{X}_1, \tilde{X}_2, \ldots) \) and \( (\tilde{R}_1, \tilde{R}_2, \ldots) \) such that for all \((n, j)\),
\[
\mathbb{P}(|\tilde{X}_{n,j} - \tilde{R}_{n,j}| \geq C^{-n}) \leq C^{-n}.
\]

We also need the following version of [9 Lemma 5.4].

**Lemma 5.** For any \( n \in \mathbb{N} \), we have
\[
\pi((\tilde{R}_{n,j})_{0 \leq j < F(n)}, (\tilde{P}_{n,j})_{0 \leq j < F(n)}) \leq C 4^{-n}
\]
where \( \tilde{Y}_{n,j} = Y_{n,j} + V_{n,j} \).

**Proof (Proof of Lemma 5).** We follow the proof of [9 Lemma 5.4]. We define \( \tilde{Y}^i_{n,j} \) for \( 0 \leq i \leq f \) as follows: for \( 0 \leq k < 2^f-i \), the random vector \( \tilde{Y}^i_{n,k} : = (\tilde{Y}^i_{n,j})_{k 2^i \leq j < (k+1)2^i} \) is distributed as \( (\tilde{X}_{n,j})_{k 2^i \leq j < (k+1)2^i} \), and \( \tilde{Y}^i_{n,k} \) is independent of \( \tilde{Y}^i_{n,k'} \) when \( k \neq k' \). Set \( \tilde{Y}^i = (\tilde{Y}^i_{n,j})_{0 \leq j < F} \), for \( 0 \leq i \leq f \). By [9 (5.7)], we have that
\[
\pi(\tilde{Y}^i, \tilde{Y}^{i-1}) \leq \sum_{k=0}^{2^f-i-1} \pi(\tilde{Y}^i_{n,k}, (\tilde{Y}^{i-1}_{n,2k}, \tilde{Y}^{i-1}_{n,2k+1})),
\]
(21)
for \(1 \leq i \leq f\). As in the proof of [9] Lemma 5.4, as a consequence of the condition (H), the difference between the characteristic functions of \(\tilde{Y}_{n,k}^i\) and \((\tilde{Y}_{n,2k}^{i-1}, \tilde{Y}_{n,2k+1}^{i-1})\) is at most \(C e^{-c^2 \epsilon n^{q+i}}\) for \(n\) large enough. Hence, by applying [9] Lemma 3.5] with \(N = 2^i\) and \(T' = e^{x n^{q/2}}\) we obtain that

\[
\pi(\tilde{Y}_{n,k}^i, (\tilde{Y}_{n,2k}^{i-1}, \tilde{Y}_{n,2k+1}^{i-1}))^\prime \leq \sum_{j=k}^{(k+1)2^i-1} \mathbb{P}(|\tilde{X}_{n,j}| \geq e^{x n^{q/2}}) + C e^{2x n^{q+i}} - e^{c^2 \epsilon n^{q+i}}.
\]

By estimating \(\mathbb{P}(|\tilde{X}_{n,j}| \geq e^{x n^{q/2}})\) as in the proof of Lemma 3, we conclude that

\[
\pi(\tilde{Y}_{n,k}^i, (\tilde{Y}_{n,2k}^{i-1}, \tilde{Y}_{n,2k+1}^{i-1})) \leq C e^{-2\epsilon n^{q}}.
\]

(22)

for some \(\delta > 0\). The conclusion of the lemma now follows from (21) and (22) by summing over \(i\) and noting that the process \((\tilde{Y}_{n,j})_{0 \leq j < F}\) coincides with \((\tilde{R}_{n,j})_{0 \leq j < F}\) and that \((\tilde{Y}_{n,j}^0)_{0 \leq j < F}\) coincides with \((\tilde{Y}_{n,j})_{0 \leq j < F}\).

Finally, relying on Corollary 1 and Lemma 5, the proof of Proposition 1 is completed exactly as in [9]. \(\square\)

**Second step:** We now establish the version of [9] Lemma 5.6. We first recall the following result (see [22] Corollary 3) or [9] Proposition 5.5).

**Proposition 2.** Let \(Y_0, \ldots, Y_{b-1}\) be independent centered \(\mathbb{R}^d\)-valued random vectors. Let \(q \geq 2\) and set \(M = \left( \sum_{j=0}^{b-1} \mathbb{E}[Y_j^q] \right)^{1/q}\). Assume that there exists a sequence \(0 = m_0 < m_1 < \ldots < m_s = b\) such that with \(\zeta_k = Y_{m_k} + \ldots + Y_{m_{k+1}-1}\) and \(B_k = \text{Cov}(\zeta_k)\), for any \(v \in \mathbb{R}^d\) and \(0 \leq k < s\) we have that

\[
100M^2|v|^2 \leq B_k v \cdot v \leq 100CM^2|v|^2,
\]

(23)

where \(C \geq 1\) is some constant. Then, there exists a coupling between \((Y_0, \ldots, Y_{b-1})\) and a sequence of independent Gaussian random vectors \((S_0, \ldots, S_{b-1})\) such that \(\text{Cov}(S_j) = \text{Cov}(Y_j)\) for each \(j \in \mathbb{N}\) and

\[
\mathbb{P}\left( \max_{0 \leq i < b-1} \left| \sum_{j=0}^{i} Y_j - S_j \right| \geq Mz \right) \leq C'z^{-q} + \exp(-C'z),
\]

(24)

for all \(z \geq C' \log s\). Here, \(C'\) is a positive constant which depends only on \(C\), \(d\) and \(q\).

**Lemma 6.** Suppose that \(p > 2 + 2/\beta\). Then for any \(n \in \mathbb{N}\), there exists a coupling between \((Y_{0n}, \ldots, Y_{n,F(n)-1})\) and \((S_{0n}, \ldots, S_{n,F(n)-1})\), where the \(S_{n,j}\)’s are independent centered Gaussian random variables with \(\text{Var}(S_{n,j}) = \text{Var}(Y_{n,j})\), such that
where $k$ of [9, Lemma 5.6] such that (23) holds. We can continue by forming blocks, namely

Proof (Proof of Lemma 6). Take $q \in (2, p)$. By (19), we have that
\[
\|Y_{n,j}\|_{L^q} \leq C2^{(1-\beta)n/2+n/p},
\]
where we have used that the right end point of each $I_{n,j}$ does not exceed $2^{n+1}$ and that $X_{n,j}$ and $Y_{n,j}$ are equally distributed. It follows from (20) that
\[
M := \left( \sum_{j=0}^{F-1} \|Y_{n,j}\|_{L^q}^q \right)^{1/q}
\]
satisfies
\[
M \leq C2^{n/p+\beta n/q+(1-\beta)n/2}.
\]
Therefore, if $q$ is sufficiently close to $p$ then $M^2$ is much smaller than $2^n$, where we have used that $p > 2 + 2/\beta$. On the other hand, by (15) we have
\[
\text{Var}(Y_{n,j}) = \text{Var}(X_{n,j}) \geq \alpha 2^{1-\beta n}
\]
for some constant $\alpha > 0$ which does not depend on $n$ and $j$. Here we have taken into account that the length of each $I_{n,j}$ is of magnitude $2^{1-\beta n}$. By (27) we have
\[
\text{Var} \left( \sum_{j=0}^{F-1} Y_{n,j} \right) = \sum_{j=0}^{F-1} \text{Var}(Y_{n,j}) \geq \epsilon 2^n,
\]
where $\epsilon > 0$ is some constant.

Next, set $v_j = v_{n,j} = \text{Var}(Y_{n,j})$. Then $v_j \leq \|Y_{n,j}\|_{L^q}^2 \leq M^2$. Let $u_1$ be the largest index such that
\[
v_0 + \ldots + v_{u_1-1} \geq 100M^2.
\]
Such index exists since $\sum_{j=0}^{F-1} v_j$ is much larger than $M^2$ (see (28)). Notice now that
\[
v_0 + \ldots + v_{u_1-1} \leq v_0 + \ldots + v_{u_1-2} + M^2 \leq 101M^2.
\]
This gives us the first block $\{Y_{n,0}, \ldots, Y_{n,u_1-1}\}$ of consecutive $Y_{n,j}$’s from the proof of [9, Lemma 5.6] such that (23) holds. We can continue by forming $k+1$ consecutive blocks, namely
\[
\{Y_{n,0}, \ldots, Y_{n,u_k-1}\}, \ldots, \{Y_{n,u_k}, \ldots, Y_{n,u_{k+1}-1}\},
\]
where $k$ is the first step in the construction such that
\[
v_{u_{k+1}} + \ldots + v_F < 100M^2.
\]
Then, we add \( Y_{n_k n+1}, \ldots, Y_{n_k, F} \) to the last block \( \{ Y_{n_k, 1}, \ldots, Y_{n_k, n_k-1} \} \) we have constructed. This means that we can always assume that the sum of the variances of the random variables \( Y_i = Y_{n, j} \) along successive blocks is not less than \( 100 M^2 \) and that it doesn’t exceed \( 201 M^2 \). The statement of the lemma now follows by applying Proposition \( 2 \) with \( z = 2^m/2 \), taking into account that the number of blocks is trivially bounded by \( F = F(n) \).

**Third step:** It follows from the previous two steps of the proof that, when \( p > 2 + 2/\beta \) there exists a coupling between \( (A_n)_{n \in I} \) and a sequence \( (B_n)_{n \in I} \) of independent centered normal random variables so that when \( (n, j) \) tends to infinity, we have

\[
\sum_{\ell < I_{n, j}, \ell \in I} (A_{\ell} - B_{\ell}) = o(2^{(\beta+1)/2} + 2^{((1-\beta)/2+\beta+1)/p+1/2}) \quad \text{a.s.}
\]

where \( I_{n, j} \) denotes the smallest element of \( I_{n, j} \). We note that we have also used the so-called Berkes-Philipp lemma (see [3, Lemma A.1] or [9, Lemma 3.1]).

**Fourth step:** We now establish the version of [9, Lemma 5.8]. However, before we do that we need the following result, which is a consequence of [18, Theorem 1] (see also [20, Corollary B1]).

**Lemma 7.** Let \( Y_1, \ldots, Y_d \) be a finite sequence of random variables. Let \( \nu > 2 \) be finite and assume that there exist constants \( C_1, C_2 > 0 \) such that \( \| Y_i \|_{L^\nu} \leq C_1 \) for every \( i \in \{1, \ldots, d\} \). Moreover, assume that for any \( a, n \in \mathbb{N} \) satisfying \( a + n \leq d \), we have that

\[
\| S_{a, n} \|_{L^\nu} \leq C_2^2 n^{\frac{1}{\nu}},
\]

where

\[
S_{a, n} = \sum_{i=a+1}^{a+n} Y_i.
\]

Then, there exists a constant \( K > 0 \) (depending only on \( C_1, C_2 \) and \( \nu \) ) such that for any \( a \) and \( n \),

\[
\| M_{a, n} \|_{L^\nu} \leq Kn^{\frac{1}{\nu}},
\]

where

\[
M_{a, n} = \max \{|S_{a, 1}|, \ldots, |S_{a, n}|\}.
\]

The following is the already announced version of [9, Lemma 5.8].

**Lemma 8.** We have that as \( (n, j) \to \infty \),

\[
\max_{m < |I_{n, j}|} \left| \sum_{\ell \in I_{n, j}} A_{\ell} \right| = a(2^{(1-\beta)/2+\beta/2+1/p+1/2}) \quad \text{a.s.}
\]

**Proof (Proof of Lemma 8).** Let \( q \in (2, p) \). Consider the finite sequence

\[
Y_k = A_k / (|I_{n, j}|)^{1/p}, \quad k \in I_{n, j}.
\]
Then, by (16) there exists a constant $C_1 > 0$ which does not depend on $n$ and $j$ so that $\|Y_i\|_{L^q} \leq C_1$, for any $k \in I_{n,j}$. Moreover, by (19), there exists a constant $C_2 > 0$ which does not depend on $n$ and $j$ so that for any relevant $a$ and $b$,

$$\left\| \sum_{k=a+1}^{a+b} Y_k \right\|_{L^q} \leq C_2 b^2.$$

Using the same notation as in statement of Lemma 7, we observe that it follows from (29) that

$$\|M_{n,b}\|_{L^q} \leq K b^2,$$

for some constant $K > 0$ (which depends only $C_1$, $C_2$ and $q$).

In particular, by setting $v = (1 - \beta)/2 + \beta/p + \varepsilon/2$, we have that

$$\mathbb{P}(M_{n,j}|I_{n,j}| \geq 2^{vn}) \leq \|M_{n,j}|I_{n,j}|\|_{L^q}^{q/2} 2^{vn}q \leq K|I_{n,j}|^{q/2} 2^{vn}q.$$

Moreover, observe that

$$\sum_{n,j} |I_{n,j}|^{q/2} 2^{vn}q \leq \sum_n 2^{2\beta n} 2^{(1-\beta)nq/2-vnq}.$$

Notice that the above sum is finite if $q$ is sufficiently close to $p$. Applying the Borel-Cantelli lemma yields that, as $(n, j) \to \infty$,

$$\max_{m<|I_{n,j}|} \left| \sum_{\ell=m}^{m+|I_{n,j}|} Y_\ell \right| = o(2^{(1-\beta)/2+\beta/p+\varepsilon)n),$$

which implies that (30) holds (since the right end point of $I_{n,j}$ does not exceed $2^{n+1}$).

**Fifth step:** By combining the last two steps, we derive that when $k$ tends to infinity,

$$\left| \sum_{\ell<k, \ell \in I} (A_\ell - B_\ell) \right| = o(k^{(\beta+\varepsilon)/2} + k^{(1-\beta)/2+\beta+1/p+\varepsilon})$$

assuming that $p > 2 + 2/\beta$.

**Sixth step:** Fix some $n$ and consider the finite sequence $Y_i = A_i/n^{1/p}$ where $i \in \{1, \ldots, n\}$. It follows from our assumptions that $(Y_i)_i$ satisfies property (H) (with constants that do not depend on $n$). Applying [9, Lemma 5.9] with the finite sequence $(Y_i)$ (instead of $A_i$ there), we see that for any $\alpha > 0$, there exists $C = C_\alpha$ (which does not depend on $n$) such that for any interval $J \subset [1, n]$ we have

$$n^{-2/p} \mathbb{E} \left[ \sum_{\ell \in J \cap \mathcal{F}} A_\ell \right]^2 = \mathbb{E} \left[ \sum_{\ell \in J \cap \mathcal{F}} Y_\ell \right]^2 \leq C |J \cap \mathcal{F}|^{1+\alpha}.$$

(31)
We recall the following version of the Gal-Koksma law of large numbers, which is a direct consequence of [13, Theorem 3] together with some routine estimates (as those given in the proof of [18, Theorem 6]). We also note that the lemma can be proved by an easy adaptation of the arguments in the proof of [19, Theorem A1].

**Lemma 9.** Let $Y_1, Y_2, \ldots$ be a sequence of random variables such that with some constants $\sigma \geq 1$, $C > 0$, $p > 1$ and for any $m, n \in \mathbb{N}$ we have that

$$\left\| \sum_{j=m+1}^{m+n} Y_j \right\|_{L^2}^2 \leq C((n + m)^\sigma - m^\sigma) \cdot (n + m)^{\frac{p}{2}}.$$  

Then, for any $\delta > 0$ we have that $\mathbb{P}$-a.s. as $n \to \infty$,

$$\sum_{j=1}^{n} Y_j = o(n^{\sigma/2+1/p} \ln^{3/2+\delta} n).$$

Relying on [31] and Lemma 9 one can now repeat the arguments appearing after the statement of [9, Lemma 5.9] with the finite sequence $(A_i/k^n)_{1 \leq i \leq k}$ (instead of $(A_i)$), and conclude that

$$\sum_{\ell<k, \ell \in [j]} A_\ell/k^n = o(k^{B/2+\epsilon}).$$

**Finalizing the proof:** Combining the estimates from the previous steps we get a coupling of $(A_\ell)$ with independent centered normal random variables $(B_\ell)$ such that

$$\left\| \sum_{\ell<k} (A_\ell - B_\ell) \right\|_{L^2} = o(k^{B/2+\epsilon} + k^{1/2+\beta(\beta+1)/p} + k^2), \quad \text{a.s.}$$

Taking $\beta = \rho/(2\rho - 2)$, we obtain (17). Observe that for this choice of $\beta$ we have $p > 2 + 2/\beta$ since $p \geq 6$. When $4 < p < 6$ we can make a different choice of $\beta$ and obtain a slightly less attractive rate. To complete the proof of Theorem 11 it remains to estimate the variance of the approximating Gaussian $G_n = \sum_{j=1}^{n} B_j$. Firstly, by applying [7, Proposition 9] with the finite sequence $(A_i/k^{n+1})_{1 \leq i \leq n}$ replacing $(A_i)$, we obtain that

$$\left\| \sum_{(n',j') < (n,j)} X_{n',j'} - Y_{n',j'} \right\|_{L^2} \leq C 2^{Bn/2+n/p},$$

where $(Y_{n',j'})$ are given by Proposition 11. Since $Y_{n',j'}$ and $S_{n',j'}$ have the same variances, we conclude that

$$\left\| \sum_{(n',j') < (n,j)} X_{n',j'} \right\|_{L^2} - \left\| \sum_{(n',j') < (n,j)} S_{n',j'} \right\|_{L^2} \leq C 2^{Bn/2+n/p}.$$  \hspace{1cm} (32)
Take \( n \in \mathbb{N} \), and let \( N_n \) be such that \( 2^{N_n} \leq n < 2^{N_n+1} \). Furthermore, let \( j_n \) be the largest index such that the left end point of \( I_{n, j_n} \) is smaller than \( n \). In the case when \( n \in I_{N_n, j_n} \) we have

\[
\sum_{i=1}^{n} A_i - \sum_{(n', j') \in (N_n, j_n)} X_{n', j'} = \sum_{(n', j') \in (N_n, j_n)} \sum_{i \in J \cap (N_n, j_n)} A_i + \sum_{i \in I_{n, j_n}} A_i + \sum_{i \in I_{N_n, j_n}} A_i,
\]

where we have used that for our specific choice of \( N_n \) is at most \( Cn^{\beta+3/2} \) (where \( C \) denotes a generic constant independent of \( n \)). Using \( ||I_1||_{L^2} \leq C n^{1/p+\beta/2+\epsilon} \).

On the other hand, applying (19) we obtain that

\[
||I_2||_{L^2} \leq C|I_{N_n, j_n}|^{2N_n/p} \leq Cn^{N_n(1-\beta)/2+\beta/p} \leq Cn^{\beta/2+1/p}.
\]

where we have used that for our specific choice of \( \beta \) we have \((1-\beta)/2 = \beta/2 - \beta/p < \beta/2 \). We conclude that there exists a constant \( C' > 0 \) so that for any \( n \geq 1 \),

\[
\left\| \sum_{j=1}^{N_n} A_j - \sum_{(n', j') \in (N_n, j_n)} X_{n', j'} \right\|_{L^2} \leq C' n^{\beta/2+\beta/2+1/p}.
\]

The proof of (13) in the case when \( n \in I_{N_n, j_n} \) is completed now using (32). The case when \( n \not\in I_{N_n, j_n} \) is treated similarly. We first write

\[
\sum_{i=1}^{n} A_i - \sum_{(n', j') \in (N_n, j_n)} X_{n', j'} = \sum_{j \in J, j \leq n} A_i + X_{N_n, j_n} := I_1 + I_2.
\]

Then the \( L^2 \)-norms of \( I_1 \) and \( I_2 \) are bounded exactly as in the case when \( n \in I_{N_n, j_n} \), and the proof of (18) is complete. Finally, the last conclusion in the statement of the theorem follows directly from (17), (18) together with [11, Theorem 3.2A], [3, Lemma A.1] (see also [9, Lemma 3.1]) and the so-called Strassen-Dudley theorem [2, Theorem 6.9] (see also [9, Theorem 3.4]).
4 Main result

The goal of this section is to establish the quenched almost sure invariance principle for random distance expanding maps satisfying suitable conditions. This is done by applying Theorem 1.

Without any loss of generality, we can suppose that our observable \( \psi : E \rightarrow \mathbb{R} \) is fiberwise centered, i.e. that \( \int_{E} \psi_{\omega} d\mu_{\omega} = 0 \) for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \). Indeed, otherwise we can simply replace \( \psi \) with \( \tilde{\psi} \) given by

\[
\tilde{\psi}_{\omega} = \psi_{\omega} - \int_{E} \psi_{\omega} d\mu_{\omega}, \quad \omega \in \Omega.
\]

In what follows, \( E_{\omega}(\varphi) \) will denote the expectation of a measurable \( \varphi : E_{\omega} \rightarrow \mathbb{R} \) with respect to \( \mu_{\omega} \). The proof of the following result can be obtained by repeating the arguments from [6, Lemma 12.] and [6, Proposition 3.] (see also [13, Theorem 2.3.])

**Proposition 3.** We have the following:
1. there exists \( \Sigma^2 \geq 0 \) such that
   \[
   \lim_{n \to \infty} \frac{1}{n} E_{\omega} \left( \sum_{k=0}^{n-1} \psi_{\sigma^k \omega} \circ f_{\omega}^k \right)^2 = \Sigma^2, \quad \text{for } \mathbb{P} \text{-a.e. } \omega \in \Omega; \quad (33)
   \]
2. \( \Sigma^2 = 0 \) if and only if there exists \( \varphi \in L^2_{\mu}(E) \) such that
   \[
   \psi = \varphi - \varphi \circ F.
   \]

From now on we shall assume that \( \Sigma^2 > 0 \). For any integer \( L \geq 1 \) consider the set

\[
A_L = \left\{ \omega \in \Omega : \frac{1}{n} E_{\omega} \left( \sum_{k=0}^{n-1} \psi_{\sigma^k \omega} \circ f_{\omega}^k \right)^2 \geq \frac{1}{2} \Sigma^2, \quad \forall n \geq L \right\}.
\]

Then \( A_L \subset A_{L'} \) if \( L \leq L' \) and the union of the \( A_L \)'s has probability 1. Due to measurability of \( Q_{\omega}, C(\omega), K(\omega) \), and \( \omega \mapsto h_{\omega} \), for any \( C_0 > 0 \) and \( L \in \mathbb{N} \) the set

\[
E := \left\{ \omega \in \Omega : \max\{C(\omega), K(\omega), \|h_{\omega}\|, \|1/h_{\omega}\|, 1/Q_{\omega} \} \leq C_0 \right\} \cap A_L \quad (34)
\]

is measurable, and when \( C_0 \) and \( L \) are sufficiently large we have that \( \mathbb{P}(E) > 0 \). Fix some large enough \( C_0 \) and \( L \), and for \( \omega \in \Omega \), let

\[
m_1(\omega) := \inf\{n \in \mathbb{N} : \sigma^n \omega \in E\}.
\]

For \( k > 1 \) we inductively define

\[
m_k(\omega) := \inf\{n > m_{k-1}(\omega) : \sigma^n \omega \in E\}.
\]
Due to ergodicity of \( \mathbb{P} \), we have that \( m_k(\omega) \) is well-defined for \( \mathbb{P} \text{-a.e. } \omega \in \Omega \) and every \( k \in \mathbb{N} \). Let us consider the associated induced system \((E, \mathcal{F}_E, \mathbb{P}_E, \iota)\), where \( \mathcal{F}_E = \{ A \cap E : A \in \mathcal{F} \} \), \( \mathbb{P}_E(A) = \frac{\mathbb{P}(A)}{\mathbb{P}(E)} \) and \( \iota(\omega) = \sigma^{m_k(\omega)} \omega \) for \( \omega \in E \). We recall that \( \mathbb{P}_E \) is invariant for \( \iota \) and in fact ergodic.

It follows from Birkhoff’s ergodic theorem that

\[
\lim_{n \to \infty} \frac{k_n(\omega)}{n} = \mathbb{P}(E) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \tag{35}
\]

where

\[
k_n(\omega) := \max \{ k \in \mathbb{N} : m_k(\omega) \leq n \}.
\]

Moreover, Kac’s lemma implies that

\[
\lim_{n \to \infty} \frac{m_n(\omega)}{n} = \frac{1}{\mathbb{P}(E)} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.
\]

By combining the last two equalities, we conclude that

\[
\lim_{n \to \infty} \frac{m_{k_n(\omega)}(\omega)}{n} = 1, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.
\]

For \( \mathbb{P} \text{-a.e. } \omega \in \Omega \), set

\[
\Psi_\omega := \sum_{j=0}^{m_1(\omega)-1} \psi_{\sigma^j\omega} \circ f^j_\omega.
\]

We assume that there exists \( p \geq 6 \), so that

the map \( \omega \mapsto A(\omega) := \| \Psi_\omega \|_\infty \) belongs to \( L^p(\Omega, \mathcal{F}, \mathbb{P}) \). \tag{36}

Finally, let \( L_\omega := f^m(\omega) \) and \( F_\omega := f^{m(\omega)}_\omega \), for \( \omega \in \Omega \).

We are now in a position to state the main result of our paper (recall our assumption that \( \Sigma^2 > 0 \)).

**Theorem 2.** For \( \mathbb{P} \text{-a.e. } \omega \in \Omega \) and arbitrary \( \delta > 0 \), there exists a coupling between \( (\psi_{\sigma^i\omega} \circ f^i_\omega) \) considered as a sequence of random variables on \((E_\omega, \mu_\omega)\), and a sequence \((Z_k)_k\) of independent centered (i.e. of zero mean) Gaussian random variables such that

\[
\left| \sum_{i=1}^n \psi_{\sigma^i\omega} \circ f^i_\omega - \sum_{i=1}^n Z_i \right| = o(n^{2p_+ + \delta}), \quad \text{a.s.,} \tag{37}
\]

where

\[
a_p = \frac{p}{4(p-1)} + \frac{1}{p}.
\]

Moreover, there exists \( C = C(\omega) > 0 \) so that for any \( n \geq 1 \),
asymptotically deterministic. More precisely, it follows from (33) and (38) that
\[ m \leq \sum_{i=1}^{n} \psi_{\sigma_{i} \omega} \circ f_{\omega} \leq \sum_{i=1}^{n} Z_{i} \leq \sum_{i=1}^{n} \psi_{\sigma_{i} \omega} \circ f_{\omega} + Cn^{p + \delta}. \] (38)

Finally, there exists a coupling between \((\psi_{\sigma_{i} \omega} \circ f_{\omega})_{i}\) and a standard Brownian motion \((W_{t})_{t \geq 0}\) such that
\[ \sum_{i=1}^{n} \psi_{\sigma_{i} \omega} \circ f_{\omega} - W_{\sigma_{i} \omega, n} = o(n^{p + \delta}) \text{ a.s.} \]
where
\[ \sigma_{\omega, n} = \left\| \sum_{i=1}^{n} \psi_{\sigma_{i} \omega} \circ f_{\omega} \right\|_{L_{2}}. \]

Remark 2. Observe that \(a_{p} \to \frac{1}{2}\) as \(p \to \infty\). We note that our proof also yields convergence rate when \(4 < p < 6\), which has a slightly less attractive form in terms of \(p\). In addition, we emphasize that \(\left\| \sum_{i=1}^{n} Z_{i} \right\|_{L_{2}}\) depends on \(\omega\) but that it is asymptotically deterministic. More precisely, it follows from (33) and (38) that
\[ \lim_{n \to \infty} \frac{\left\| \sum_{i=1}^{n} Z_{i} \right\|_{L_{2}}^{2}}{n\Sigma^{2}} = 1. \]

Proof (Proof of Theorem 2). Our strategy proceeds as follows. Firstly, we will apply Theorem 1 to establish the invariance principle for the induced system. Secondly, we extend the invariance principle to our original system. Throughout the proof, \(C > 0\) will denote a generic constant independent on \(\omega\) and other parameters involved in the estimates.

For \(\omega \in E\) (recall that \(E\) is given by (34)), set \(A_{n} = \Psi_{e_{i} \omega} \circ F_{\omega}^{n}, n \in \mathbb{N}\). Obviously, \(A_{n}\) depends also on \(\omega\) but in order to make the notation as simple as possible, we do not make this dependence explicit.

Observe that it follows from (36) and Birkhoff’s ergodic theorem that there exists a random variable \(R: E \to (0, \infty)\) such that:
\[ \|A_{n}\|_{L_{p}} \leq R(\omega)n^{1/p} \text{ for } \mathbb{P}\text{-a.e. } \omega \in E \text{ and } n \in \mathbb{N}. \] (39)

It follows easily from (10) and (34) that for any \(k \in \mathbb{N}, n \geq L\) and \(\omega \in E\),
\[ \frac{1}{n} \text{Var} \left( \sum_{j=0}^{n-1} A_{j+k} \right) \geq \frac{1}{2}\Sigma^{2}, \] (40)
where we have used that \(m_{n}(\ell^{k}(\omega)) \geq n\). We conclude from (39) and (40) that the processes \((A_{n})_{n \in \mathbb{N}}\) satisfies (16) and (15), respectively.

Hence, in order to apply Theorem 1 we need to show that \((A_{n})_{n \in \mathbb{N}}\) satisfies property (H) and, in addition, that for any \(n < m\) the finite sequence \((A_{i}/(n + m)^{1/p})_{n+1 \leq i \leq n+m}\) also satisfies (H) (with uniform constants). In fact, we will prove
Finally, for \( \omega \) it is easy to verify that \( \hat{L}_\omega^\varepsilon g = \hat{L}_\omega(g e^{\varepsilon \phi_\omega})/h_\sigma \omega, \) for \( g \in \mathcal{H}_\omega^{\alpha, \xi}. \)

Furthermore, for \( z \in \mathbb{C} \) and \( n \in \mathbb{N} \), set

\[
\hat{L}_\omega^{z, n} := \hat{L}_\omega^{z, n-1} \circ \ldots \circ \hat{L}_\omega^{z}.
\]

It is easy to verify that

\[
\hat{L}_\omega^{z, n} g = L_\omega^n(g e^{\varepsilon \phi_\omega})/h_\sigma \omega = L_\omega^n(g h_\omega)/h_\sigma \omega.
\]

Finally, for \( \omega \in \Omega, n \in \mathbb{N} \) and \( \mathbf{T} = (t_0, t_1, \ldots, t_{n-1}) \in \mathbb{R}^n \), let

\[
\hat{L}_\omega^{T_n} g = \hat{L}_\omega^{i_1, t_1, m_1(\omega) - m_1(\omega)} \circ \ldots \circ \hat{L}_\omega^{i_n, t_n, m_n(\omega)}(g h_\omega)/h_\sigma \omega.
\]

Observe that

\[
\hat{L}_\omega^{T_n} g = \hat{L}_\omega^{i_1, t_1, m_1(\omega) - m_1(\omega)} \circ \ldots \circ \hat{L}_\omega^{i_n, t_n, m_n(\omega)}(g h_\omega)/h_\sigma \omega.
\]

for any \( g \in \mathcal{H}_\omega^{\alpha, \xi}. \) It follows from \((44), (34)\) and the above formula that for \( n \in \mathbb{N} \) and \( \mathbf{T} \in [-1, 1]^n \), we have that

\[
\| L_\omega^{T_n} \|_{\alpha, \xi} \leq C. \tag{41}
\]

For \( \omega \in \Omega \) and \( g \in \mathcal{H}_\omega^{\alpha, \xi} \), set

\[
\Pi_\omega g := \left( \int_{E_\omega} g \, d\mu_\omega \right) \mathbf{1}
\]

where \( \mathbf{1} \) denotes the function which takes the constant value 1, regardless of the space on which it is defined. Since \( L_\omega^{T_k} = \hat{L}_\omega^{m_k(\omega)} \) and \( m_k(\omega) \geq k \), it follows from Lemma \(3\) and \( \Xi \) that

\[
\| (L_\omega^{T_k} - \Pi_\omega) g \|_{\infty} \leq C e^{-\kappa k} \| g \|_{\alpha, \xi}, \tag{42}
\]

for \( \omega \in E, g \in \mathcal{H}_\omega^{\alpha, \xi} \) and \( k \in \mathbb{N} \).

Take now \( n, m, k \in \mathbb{N}, b_1 < b_2 < \ldots < b_{n+m+k} \) and \( t_1, \ldots, t_{n+m} \in \mathbb{R} \) with \( |t_j| \leq 1 \).

We have that

\[
\mathbb{E}_{\mu_\omega}(e^{i \sum_{j=1}^{n+m+k-1} (t_j \Sigma_{r=b_j}^{b_{j+1}-1} B_\ell + i \sum_{j=1}^{m+n} (t_j \Sigma_{r=b_j}^{b_{j+1}-1} B_\ell)}) = \mathbb{E}_{\mu_\omega^a}(L_\omega^{T_{b_n+m+1-b_n+1}} L_\omega^{i_{b_n+m+1} - b_n+1} \mathbf{1})
\]

where \( B_n = a_n A_n \).
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\[ \mathcal{T} = (a_{b_1} t_1, \ldots, a_{b_2} - t_1, a_{b_2} t_2, \ldots, a_{b_3} - t_2, \ldots, a_{b_n} t_n, \ldots, a_{b_{n+1} - 1} t_n), \]

and

\[ \mathcal{T} = (a_{b_{n+1} + k} t_{n+1}, \ldots, a_{b_{n+2} + k - 1} t_{n+1}, \ldots, a_{b_{n+m} + k} t_{n+m}, \ldots, a_{b_{n+m+1} + k - 1} t_{n+m}). \]

Consequently,

\[
\mathbb{E}_{\omega} \left( e^{i \sum_{j=1}^{n} t_j (\sum_{r=1}^{b_{j+1} - 1} B_r + i \sum_{r=1}^{n+m} t_j (\sum_{s=1}^{b_{j+k} - 1} B_r))} \right) \\
= \mathbb{E}_{\omega} \left( e^{i \sum_{j=1}^{n} t_j (\sum_{r=1}^{b_{j+1} - 1} B_r)} \left( L_{\mathcal{T}}^{b_{n+m+1} - b_{n+1}} (L_{\mathcal{T}}^{b_{n+1} + \omega} - \Pi_{b_{n+1} + \omega}) L_{\mathcal{T}}^{b_{n+1} - b_1} \mathbf{1} \right) + \mathbb{E}_{\omega} \left( e^{i \sum_{j=1}^{n} t_j (\sum_{r=1}^{b_{j+k} - 1} B_r)} \left( L_{\mathcal{T}}^{b_{n+m+1} - b_{n+1}} \Pi_{b_{n+1} + \omega} L_{\mathcal{T}}^{b_{n+1} - b_1} \mathbf{1} \right) \right) \\
= : I_1 + I_2.
\]

We claim next that

\[ |I_1| \leq C e^{-\lambda k}. \quad (43) \]

Indeed, set

\[ A := L_{\mathcal{T}}^{b_{n+m+1} - b_{n+1}}, \quad B := L_{\mathcal{T}}^{b_{n+1} + \omega} - \Pi_{b_{n+1} + \omega} \quad \text{and} \quad g := L_{\mathcal{T}}^{b_{n+1} - b_1} \mathbf{1}. \]

Then,

\[ ||A||_{\infty} := \sup_{f: ||f||_{\infty} = 1} \|Af\|_{\infty} \leq \|L_{\mathcal{T}}^{b_{n+m+1} - b_{n+1}} \mathbf{1}\|_{\infty} = 1, \]

and therefore

\[ |I_1| \leq \|A(Bg)\|_{\infty} \leq ||A||_{\infty} \cdot \|Bg\|_{\infty} \leq ||Bg||_{\infty}. \]

Applying (41) we have

\[ \|g\|_{a, \varepsilon} \leq C, \]

and thus it follows from (42) that

\[ |I_1| \leq \|Bg\|_{\infty} \leq C e^{-\lambda k}. \]

We conclude that (43) holds.

On the other hand,

\[ I_2 = \mathbb{E}_{\omega} \left( e^{i \sum_{j=1}^{n} t_j (\sum_{r=1}^{b_{j+k} - 1} B_r)} \right) \cdot \mathbb{E}_{\omega} \left( e^{i \sum_{j=1}^{n+m} t_j (\sum_{s=1}^{b_{j+k} - 1} B_s)} \right). \]

We conclude that the process \((B_n)_{n \in \mathbb{N}}\) satisfies property (H) with constants that do not depend on the sequence \((a_l)\). Thus, Theorem 11 yields the almost sure invariance principle for the process \((\Psi_{\omega} \circ F_{\omega}^n)_{n \in \mathbb{N}}\).

It remains to observe that the conclusion of the Theorem 2 now follows from the BerkesâŠPhilipp lemma (see [3] Lemma A.1] or [9] Lemma 3.1] and the following lemma which together with (35), ensures that (38) holds true.

Lemma 10. There exists a random variable \(U: \Omega \rightarrow (0, \infty)\) such that
for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n \in \mathbb{N}$.

Proof (Proof of the lemma). If $n = m_{k_n(\omega)}(\omega)$ then there is nothing to prove, and so we assume that $m_{k_n(\omega)}(\omega) < n$. Observe that

$$
\left\| \sum_{j=0}^{n-1} \psi^{j}_\sigma(\omega) \circ f^j_\omega - \sum_{j=0}^{k_n(\omega)-1} \psi^{j}_\sigma(\omega) \circ F^j_\omega \right\|_{\infty} = \sum_{j=m_{k_n(\omega)}(\omega)}^{m_{k_n(\omega)+1}(\omega)-1} \psi^{j}_\sigma(\omega) \circ f^j_\omega - \sum_{j=n}^{m_{k_n(\omega)+1}(\omega)-1} \psi^{j}_\sigma(\omega) \circ f^j_\omega
$$

and thus

$$
\left\| \sum_{j=0}^{n-1} \psi^{j}_\sigma(\omega) \circ f^j_\omega - \sum_{j=0}^{k_n(\omega)-1} \psi^{j}_\sigma(\omega) \circ F^j_\omega \right\|_{\infty} \leq \left\| \psi^{m_{k_n(\omega)}(\omega)}(\omega) \right\|_{\infty} + \left\| \psi^{m_{k_n(\omega)+1}(\omega)}(\omega) \right\|_{\infty},
$$

where we have used that $\sigma^j(\omega) \not\in E$ when $m_{k_n(\omega)}(\omega) < j < m_{k_n(\omega)+1}(\omega)$. Hence, the conclusion of the lemma follows directly from Birkhoff’s ergodic theorem, (35) and (36).

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