ON THE CONTINUITY OF PARTIAL ACTIONS OF HAUSDORFF GROUPS
ON METRIC SPACES

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Abstract. We provide a sufficient condition for a topological partial action of a Hausdorff group
on a metric space is continuous, provided that it is separately continuous.

1. Introduction

The notion of partial action of a group is a weakening of the classical concept of group action
and was introduced in [6] and [11], and was then developed in [1] and [10], in which the authors
provided examples in different guises. Since then partial actions have been an important tool in
$C^*$-algebras and dynamical systems, and in the developing of new cohomological theories [4], [5]
and [12]. Every partial action of a group $G$ on a set $X$ can be obtained, roughly speaking, as a
restriction of a global action (see [1] and [10]) on some bigger set $X_G$, called the enveloping space
of $X$. Nevertheless, in the category of topological spaces, when $G$ acts partially on a space $X$, the
superspace $X_G$ does not necessarily inherit its topological properties; for instance the globalization
of a partial action of a group on a Hausdorff space is not in general Hausdorff (see e.g. [1, Example
1.4], [1, Proposition 1.2]).

On the other hand, actions of Polish groups have important connections with many areas of
mathematics (see [2], [8] and the references therein). Recall that a Polish space is a topological
space which is separable and completely metrizable, and a Polish group is a topological group
whose topology is Polish. Partial actions of Polish groups have been recently considered in the
works [9, 13, 14].

It is known that an action of a Polish group $G$ on a metric space $X$ is continuous provided
that it is separately continuous (see [8, Theorem 3.1.4]). In this note, under a mild restriction, we
generalize [8, Theorem 3.1.4] in two directions, first we only assume that $G$ is Hausdorff and Baire,
and second we only assume that $G$ acts partially on $X$.

2. The notions

Let $G$ be a topological group with identity element $1$ and $X$ a topological space. A partially
defined map $G \times X \to X$ is a map whose domain is a subset of $G \times X$. Let $m: G \times X \to X$, $(g, x) \mapsto
m(g, x) = g \cdot x \in X$ be a partial map, that is $m$ is a map whose domain is contained in $G \times X$.
We write $\exists g \cdot x$ to mean that $(g, x)$ is in the domain of $m$. Then, following [1, 10] $m$ defines a (set
theoretic) partial action of $G$ on $X$, if for all $g, h \in G$ and $x \in X$ the following assertions hold:

(PA1) $\exists g \cdot x$ implies $\exists g^{-1} \cdot (g \cdot x)$ and $g^{-1} \cdot (g \cdot x) = x$,
(PA2) $\exists g \cdot (h \cdot x)$ implies $\exists (gh) \cdot x$ and $g \cdot (h \cdot x) = (gh) \cdot x$,
(PA3) $\exists 1 \cdot x$, and $1 \cdot x = x$.
Let $G \times X = \{(g, x) \in G \times X \mid \exists g \cdot x\}$ be the domain of $m$, set $X_{g^{-1}} = \{x \in X \mid g \cdot x\}$, and $m_g : X_{g^{-1}} \ni x \mapsto g \cdot x \in X_g$. Then a partial action $m : G \times X \to X$ induces a family of bijections \( \{m_g : X_{g^{-1}} \to X_g\}_{g \in G} \), and we denote $m = \{m_g : X_{g^{-1}} \to X_g\}_{g \in G}$.

**Proposition 2.1.** [15, Lemma 1.2] A partial action $m$ of $G$ on $X$ is a family $m = \{m_g : X_{g^{-1}} \to X_g\}_{g \in G}$, where $X_g \subseteq X$, $m_g : X_{g^{-1}} \to X_g$ is bijective, for all $g \in G$, and such that:

(i) $X_1 = X$ and $m_1 = \text{Id}_X$;

(ii) $m_g(X_{g^{-1}} \cap X_h) = X_g \cap X_{gh}$;

(iii) $m_g m_h : X_{h^{-1}} \cap X_{h^{-1}g^{-1}} \to X_g \cap X_{gh}$, and $m_g m_h = m_{gh}$ in $X_{h^{-1}} \cap X_{h^{-1}g^{-1}}$;

for all $g, h \in G$.

Notice that conditions (ii) and (iii) in Lemma 2.1 say that $m_{gh}$ is an extension of $m_g m_h$ for all $g, h \in G$. We consider $G \times X$ with the product topology and the subset $G \times X$ of $G \times X$ inherits the subspace topology.

**Definition 2.2.** A topological partial action of the group $G$ on the topological space $X$ is a partial action $m = \{m_g : X_{g^{-1}} \to X_g\}_{g \in G}$ on the underlying set $X$, such that each $X_g$ is open in $X$, and each $m_g$ is a homeomorphism, for any $g \in G$. If $m : G \times X \to X$ is continuous, we say that the partial action is continuous.

**Example 2.3.** Induced partial action: Let $G$ be a topological group, and $Y$ a topological space and let $u : G \times Y \to Y$ be a continuous action of $G$ on $Y$ and $X \subseteq Y$ be an open set. For $g \in G$, set $X_g = X \cap u_g(X)$ and let $m_g = u_g |_{X_g}$ (the restriction of $u_g$ to $X_g$). Then $m : G \times X \ni (g, x) \mapsto m_g(x) \in X_g$ is a topological partial action of $G$ on $X$.

The interested reader may consult others examples of topological partial actions in [1, Example 1.2, Remark 1.1, Example 1.3, Example 1.4] and [10, p. 108].

3. **Topological partial actions of Hausdorff groups on metric spaces**

Let $m : G \times X \to X$ be a topological partial action. For $x \in X$, we set $G^x = \{g \in G \mid (g, x) \in G \times X\}$, then by (PA3) $1 \in G^x$, we also set $m^x : G^x \ni g \mapsto m(g, x) \in X$, $m$ is called separately continuous if the maps $m^x$ are continuous, for all $x \in X$.\(^1\) It is known that group actions of Polish groups on metric spaces are continuous, if and only if, they are separately continuous. We provide a mild condition on $G^x$, $x \in X$, for which separately continuous partial actions of Hausdorff-Baire groups are continuous, the proof we present is inspired by the one given in [8, Theorem 3.1.4], for classical group actions.

**Theorem 3.1.** Let $G$ be a Hausdorff group, $(X, d)$ a metric space, and $m$ a topological partial action of $G$ on $X$. Suppose that $G$ is Baire $G^x$ is open in $G$, for any $x \in X$. Then $m$ is continuous, if and only if, it is separately continuous.

**Proof.** It is clear that continuous partial actions are separately continuous. For the converse, suppose that $m$ is separately continuous and let $(g_0, x_0) \in G \times X$, we check that $m$ is continuous at $(g_0, x_0)$. Let $l, n \in \mathbb{N}$ and set

$$\quad F_{n,l} = \{g \in G^{x_0} \mid \forall x \in X_{g^{-1}}(d(x, x_0) < 2^{-n} \Rightarrow d(m(g, x), m(g, x_0)) \leq 2^{-l})\}.$$

We shall check that $F_{n,l}$ is a closed subset of $G^{x_0}$. Indeed, let $g \in G^{x_0}$ and $\{g_i\}_{i \in I}$ a net with $g_i \to g$ such that $g_k \in F_{n,l}$, for all $k \in \mathbb{N}$. Then

$$(\forall k \in \mathbb{N}) \left( \forall x \in X_{g_i^{-1}} \left( d(x, x_0) < \frac{1}{2^n} \Rightarrow d(m(g_i, x), m(g_i, x_0)) \leq \frac{1}{2^l} \right) \right).$$

\(^1\)Notice that for any $g \in G$ the map $m_g$ is continuous by definition of topological partial action.
Let $x \in X_{g^{-1}}$, that is $g \in G^x$, since $G^x$ is open one may assume that $(g_i)_{i \in I} \subseteq G^x$, and $x \in X_{g^{-1}} \cap X_{g_i^{-1}}$, for all $i \in I$. Thus, by the continuity of $m^x$ and $m^{x_0}$, we have that $m(g_i, x) \to m(g, x)$ and $m(g_i, x_0) \to m(g, x_0)$, which gives
\[(\forall x \in X_{g^{-1}}) \left( d(x, x_0) < \frac{1}{2^n} \Rightarrow d(m(g, x), m(g, x_0)) \leq \frac{1}{2^n} \right),\]
and we conclude that $g \in F_{n,l}$. Now, we check that
\[(3.1) \quad G^{x_0} = \bigcap_{l=1}^{n} F_{n,l}.\]
It is clear that $G^{x_0} \supseteq \bigcap_{l=1}^{n} F_{n,l}$. Conversely, take $g \in G^{x_0}$ and $l \in \mathbb{N}$. Since $m_g$ is continuous at $x_0$, for $\varepsilon = \frac{1}{2^n}$, there exists $\delta > 0$ such that
\[(\forall x \in X_{g^{-1}}) \left( d(x, x_0) < \delta \Rightarrow d(m(g, x), m(g, x_0)) \leq \frac{1}{2^n} \right),\]
Let $n \in \mathbb{N}$ with $\frac{1}{2^n} < \delta$, then $g \in F_{n,l}$. Thus $G^{x_0} \subseteq \bigcap_{l=1}^{n} F_{n,l}$, and we have (3.1).

Since $F_{l,n}$ is closed in $G^{x_0}$, for all $n, l$; the set $D = \bigcup_{l=1}^{n} (F_{n,l} \setminus \text{int}(F_{n,l}))$ is meager. But $G^{x_0}$ is a non-empty open set, since $G$ is Baire, there is $g_1 \in G^{x_0} \setminus D$. We shall check that $m$ is continuous at $(g_1, x_0)$. Indeed, let $\{(h_\alpha, y_\alpha)\}_{\alpha \in \Lambda} \subseteq G^X \times \mathbb{R}$ be a net converging to $(g_1, x_0)$. Take $\varepsilon > 0$ and $l \in \mathbb{N}$ such that $\frac{1}{2^{l-1}} < \varepsilon$. By (3.1), there exists $n \in \mathbb{N}$ such that $g_1 \in F_{n,l}$. Since $g_1 \notin D$ then $g_1 \in \text{int}(F_{n,l})$. The fact that $h_\alpha \to g_1$, implies that there is $\alpha_1 \in \Lambda$ such that $h_\alpha \in \text{int}(F_{n,l})$, for all $\alpha \geq \alpha_1$. Also, since $y_\alpha \to x_0$, there exists $\alpha_2 \in \mathbb{N}$ such that $d(y_\alpha, x_0) < \frac{1}{2^n}$, for all $\alpha \geq \alpha_2$. Additionally, by the continuity of $m^{x_0}$, we have that $m(h_\alpha, x_0) \to m(g_1, x_0)$. Thus, there exists $\alpha_3 \in \Lambda$ such that $d(m(h_\alpha, x_0), m(g_1, x_0)) < \frac{1}{2^l}$, for all $\alpha \geq \alpha_3$.

Let $\alpha \geq \max \{\alpha_1, \alpha_2, \alpha_3\}$. Then $h_\alpha \in F_{n,l}$ and $d(y_\alpha, x_0) < \frac{1}{2^n}$, hence
\[d(m(h_\alpha, y_\alpha), m(g_1, x_0)) \leq d(m(h_\alpha, y_\alpha), m(h_\alpha, x_0)) + d(m(h_\alpha, x_0), m(g_1, x_0)) < \frac{1}{2^{l-1}} < \varepsilon.\]
Thus, $m$ is continuous at $(g_1, x_0)$.

Now, since $x_0 \in X_{g_1^{-1}} \cap X_{g_0^{-1}}$, we have, by Proposition 2.1(ii), that $g_1 \cdot x_0 \in X_{g_1} \cap X_{g_0^{-1}}$, then $(g_0g_1^{-1}) \cdot (g_1 \cdot x_0)$ is defined and $(g_0g_1^{-1}) \cdot (g_1 \cdot x_0) = g_0 \cdot x_0$, by (PA2). That is
\[(3.2) \quad m(g_0, x_0) = m(g_0g_1^{-1}, m(g_1, x_0)).\]

Finally take a net $\{(h_j, y_j)\}_{j \in J}$ in $G^X \times \mathbb{R}$ converging to $(g_0, x_0)$. Then $g_0g_1^{-1} \cdot h_j \to g_1 \in G^{x_0}$, and since $G^{x_0}$ is open we assume that the net $\{(g_0g_1^{-1}h_j)\}_{j \in J}$ is contained in $G^{x_0}$, thus $x_0 \in X_{(g_0g_1^{-1}h_j)^{-1}}$, for all $j \in J$. But $y_j \to x_0$, then there is $j_0 \in J$ such that $y_j \in X_{(g_0g_1^{-1}h_j)^{-1}}$, for all $j \geq j_0$. Now $y_j \in X_{(g_0g_1^{-1}h_j)^{-1}} \cap X_{h_j^{-1}}$, then $h_j \cdot y_j \in X_{(g_0g_1^{-1})^{-1}}$ (by Proposition 2.1(ii)). By (PA2) we get
\[(g_0g_1^{-1}h_j) \cdot y_j = (g_0g_1^{-1}) \cdot (h_j \cdot y_j) \in X_{g_0^{-1}}.\]
Thus, by (PA1),
\[h_j \cdot y_j = (g_0g_1^{-1}) \cdot [(g_0g_1^{-1}) \cdot y_j].\]
Notice that \((g_1g_0^{-1}h_j, y_j) \rightarrow (g_1, x_0)\). Then by continuity of \(m_{g_0g_1^{-1}}\) and the continuity of \(m\) at \((g_1, x_0)\) one gets
\[
m(h_j, y_j) = (g_0g_1^{-1}) \cdot [(g_1g_0^{-1}h_j) \cdot y_j] \rightarrow (g_0g_1^{-1}) \cdot (g_1 \cdot x_0) = m(g_0g_1^{-1}, m(g_1, x_0)),
\]
and by (3.2) we obtain \(m(h_j, y_j) \rightarrow m(g_0, x_0)\). \(\square\)

**Corollary 3.2.** Let \(G\) be a topological Hausdorff group, and \(a: G \times X \rightarrow X\) an action of \(G\) on a metric space \(X\). If \(G\) is Baire, then \(a\) is continuous, if and only if, \(a\) is separately continuous.

**Corollary 3.3.** Let \(G\) be a countable discrete group and \(m\) a topological partial action of \(G\) on a metric space \(X\). Then \(m\) is continuous, if and only if, \(m\) is separately continuous.

**Example 3.4.** M"{o}bius transformations \([3, p. 175]\) The group \(G = GL(2, \mathbb{R})\) is Polish and acts partially on \(\mathbb{R}\) by setting
\[
g \cdot x = \frac{ax + b}{cx + d}, \quad \text{where} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.
\]

Notice that for all \(g \in G\) the set \(\mathbb{R}_g = \{x \in \mathbb{R} \mid cx + d \neq 0\}\) is open, and \(m = \{m_g: \mathbb{R}_g \ni x \rightarrow g \cdot x \in \mathbb{R}_g\}_{g \in G}\) is a topological partial action. For \(x \in \mathbb{R}\) let \(t_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\), then for \(y \in \mathbb{R}\) one has that \(t_y - t_x \cdot x = y\). Since
\[
G^0 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : d \neq 0 \right\}
\]
is open, and
\[
G^x = G^{t_x - t_0} = G^0 t_x^{-1} = G^0 \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix},
\]
then \(G^x\) is open. Finally, since
\[
m^x: G^x \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{ax + b}{cx + d} \in \mathbb{R},
\]
is continuous, then \(m\) is continuous.

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