Abstract. The $J^k$ space of $k$-jets of a real function of one real variable $x$ admits the structure of a subRiemannian manifold, which then has an associated Hamiltonian geodesic flow, and it is integrable. As in any Hamiltonian flow, a natural question is the existence of periodic solutions. Does $J^k$ have periodic geodesics? This study will find the action-angle coordinates in $T^*J^k$ for the geodesic flow and demonstrate that geodesics in $J^k$ are never periodic.

1. Introduction

This paper is the first attempt to prove that Carnot groups do not have periodic subRiemannian geodesics; Enrico Le Donne made this conjecture. Here, we will establish the first case we found, which also has a simple and elegant proof.

This work is the continuation of [1, 2], in [1] $J^k$ was presented as subRiemannian manifold, the subRiemannian geodesic flow was defined, and its integrability was verified. In [2], the subRiemannian geodesics in $J^k$ were classified, and some of their minimizing properties were studied. The main goal of this paper is to prove:

Theorem A. $J^k$ does not have periodic geodesics.

Following the classification of geodesics from [2] (see pg. 5), the only candidates to be periodic are the ones called $x$-periodic (the other geodesics are not periodic on the $x$-coordinate); so we are focusing on the $x$-periodic geodesics.

An essential tool during this work is the bijection made by Monroy-Perez and Anzaldo-Meneses [3, 4, 5], also described in [2] (see pg. 4), between geodesics on $J^k$ and the pair $(F, I)$ (module translation $F(x) \to F(x - x_0)$), where $F(x)$ is a polynomial of degree bounded by $k$ and $I$ is a closed interval called Hill interval. Let us formalize its definition.

Key words and phrases. Carnot group, Jet space, integrable system, Goursat distribution, sub Riemannian geometry, Hamilton-Jacobi, periodic geodesics.
Definition 1. A closed interval $I$ is called Hill interval of $F(x)$, if for each $x$ inside $I$ then $F(x) < 1$ and $F(x) = 1$ if $x$ is in the boundary of $I$.

By definition, the Hill interval $I$ of a constant polynomial $F(x) = c^2 < 1$ is $\mathbb{R}$, while the Hill interval $I$ of the constant polynomial $F(x) = \pm 1$ is a single point. Also, $I$ is compact, if and only if, $F(x)$ is not a constant polynomial; in this case, if $I$ is in the form $[x_0, x_1]$, then $F(x_1) = F(x_0) = 1$. This terminology comes from celestial mechanics, and $I$ is the region where the dynamics governed by the fundamental equation (3.5) take place.

Geodesics corresponding to constant polynomials are called horizontal lines since their projection to $(x, \theta_0)$ planes are lines. In particular, geodesic corresponding to $F(x) = \pm 1$ are abnormal geodesics (see [6], [7] or [8]). Then this work will be restricted to geodesics associated with non-constant polynomials. $x$-periodic geodesics correspond to the pair $(F, [x_0, x_1])$, where $x_0$ and $x_1$ are regular points of $F(x)$, which implies they are simple roots of $1 - F^2(x)$.

Outline of the paper. In Section 2, Proposition 1 is introduced and Theorem A is proved. The main purpose of Section 3 is to prove Proposition 1. In sub-Section 3.1, the subRiemannian structure and the subRiemannian Hamiltonian geodesic function are introduced. In sub-Section 3.2, a generating function is presented and a canonical transformation from traditional coordinates in $T^*J^k$ to action-angle coordinates $(\mu, \phi)$ for the Hamiltonian systems are shown. In sub-Section 3.3, Proposition 1 is proved.

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2. Proof of theorem A

Throughout the work the alternate coordinates $(x, \theta_0, \cdots, \theta_k)$ will be used, the meaning of which meaning is introduced in the Section 3 and described in more detail in [3, 4] or [2]. $x$-periodic geodesics have the property that the change undergone by the coordinates $\theta_i$ after one $x$-period is finite and does not depend on the initial point. We summarize the above discussion with the following proposition.
Proposition 1. Let $\gamma(t) = (x(t), \theta_0(t), \ldots, \theta_k(t))$ in $J^k$ be an $x$-periodic geodesic corresponding to the pair $(F, I)$. Then the $x$-period is

$$L(F, I) = 2 \int_I \frac{dx}{\sqrt{1 - F^2(x)}},$$

Moreover, it is twice the time it takes for the $x$-curve to cross its Hill interval exactly once. After one period, the changes $\Delta \theta_i := \theta_i(t_0 + L) - \theta_i(t_0)$ for $i = 0, 1, \ldots, k$ undergone by $\theta_i$ are given by

$$\Delta \theta_i(F, I) = 2 \frac{i!}{i} \int_I x^i F(x) dx \sqrt{1 - F^2(x)}.$$

In [2], a subRiemannian manifold $\mathbb{R}^3_F$, called magnetic space, was introduced and a similar statement like Proposition 1 was proved, see Proposition 4.1 from [2] (pg. 13), with an argument of classical mechanics, see [9] page 25 equation (11.5).

1 implies that a $x$-periodic geodesic $\gamma(t)$ corresponding to the pair $(F, I)$ is periodic if and only if $\Delta \theta_i(F, I) = 0$ for all $i$.

Because that period $L$ from equation (2.1) is finite, we can define an inner product in the space of polynomials of degree bounded by $k$ in the following way:

$$< P_1(x), P_2(x) >_F := \int_I \frac{P_1(x)P_2(x)dx}{\sqrt{1 - F^2(x)}}.$$

This inner product is non-degenerate and will be the key to the proof of theorem A.

2.1. **Proof of Theorem A.**

*Proof. We will proceed by contradiction. Let us assume $\gamma(t)$ is a periodic geodesic on $J^k$ corresponding to the pair $(F, I)$, where $F(x)$ is not constant, then $\Delta \theta_i(F, I) = 0$ for all $i$ in $0, \ldots, k$.

In the context of the space of polynomials of degree bounded by $k$ with inner product $<, >_F$, the condition $\Delta \theta_i(F, I) = 0$ is equivalent to $F(x)$ being perpendicular to $x^i$ ($0 = \Delta \theta_i(F, I) \Rightarrow x^i, F(x) >_F$), so $F(x)$ being perpendicular to $x^i$ for all $i$ in $0, 1, \ldots, k$. However, the set $\{x^i\}$ with $0 \leq i \leq k$ is a base for the space of polynomials with degree bounded by $k$, then $F(x)$ is perpendicular to any vector, so $F(x)$ is zero since the inner product is non-degenerate. Being $F(x)$ equals 0 contradicts the assumption that $F(x)$ is not a constant polynomial. □

Coming work: The proof of the conjecture in the meta-abelian group $\mathbb{G}$, that is, $\mathbb{G}$ is such that $0 = [[\mathbb{G}, \mathbb{G}], [\mathbb{G}, \mathbb{G}]].$
3. Proof of Proposition 1

3.1. $J^k$ as a subRiemannian manifold. The subRiemannian structure on $J^k$ will be here briefly described. For more details, see [1, 2]. We see $J^k$ as $\mathbb{R}^{k+2}$, using $(x, \theta_0, \ldots, \theta_k)$ as global coordinates, then $J^k$ is endowed with a natural rank 2 distribution $D \subset TJ^k$ characterized by the $k$ Pafaffian equations

\begin{equation}
0 = d\theta_i - \frac{1}{i!} x^i d\theta_0, \quad i = 1, \ldots, k.
\end{equation}

$D$ is globally framed by two vector fields

\begin{equation}
X_1 = \frac{\partial}{\partial x}, \quad X_2 = \sum_{i=0}^{k} \frac{x^i}{i!} \frac{\partial}{\partial \theta_i}.
\end{equation}

A subRiemannian structure on $J^k$ is defined by declaring these two vector fields to be orthonormal. In these coordinates the subRiemannian metric is given by restricting $ds^2 = dx^2 + d\theta^2_0$ to $D$.

3.1.1. Sub-Riemannian geodesic flow. Here it is emphasized that the projections of the solution curves for the Hamiltonian geodesic flow are geodesics, that is, if $(p(t), \gamma(t))$ is a solution for the Hamiltonian geodesic flow then $\gamma(t)$ is a geodesic on $J^k$.

Let $(p_x, p_{\theta_0}, \ldots, p_{\theta_k}, x, \theta_0, \ldots, \theta_k)$ be the traditional coordinates on $T^*J^k$, or in short way as $(p, q)$. Let $P_1, P_2 : T^*J^k \to \mathbb{R}$ be the momentum functions of the vector fields $X_1, X_2$, see [6] 8 pg or see [10], in terms of the coordinates $(p, q)$ are given by

\begin{equation}
P_1(p, q) := p_x, \quad P_2(p, q) := \sum_{i=0}^{k} p_{\theta_i} \frac{x^i}{i!}.
\end{equation}

Then the Hamiltonian governing the geodesic on $J^k$ is

\begin{equation}
H_{SR}(p, q) := \frac{1}{2} (P_1^2 + P_2^2) = \frac{1}{2} p_x^2 + \frac{1}{2} \left(\sum_{i=0}^{k} p_{\theta_i} \frac{x^i}{i!}\right)^2.
\end{equation}

It is noteworthy that $h = 1/2$ implies that the geodesic is parameterized by arc-length. It can be noticed that $H$ does not depend on $\theta_i$ for all $i$, then $p_{\theta_i}$’s define a $k+1$ constants of motion.

**Lemma 1.** The subRiemannian geodesic flow in $J^k$ is integrable, if $(p(t), \gamma(t))$ is a solution then

\[
\dot{\gamma}(t) = P_1(t)X_1 + P_2(t)X_2, \quad (P_1(t), P_2(t)) = (p_x(t), F(x(t))),
\]

where $p_{\theta_i} = i!a_i$ and $F(x) = \sum_{i=0}^{k} a_i x^i$. 
Proof. \( H \) does not depend on \( t \) and \( \theta \) for all \( i \), so \( h := H_{sR} \) and \( p_{\theta_i} \) are constants of motion, thus the Hamiltonian system is integrable. First equation form the Lemma 1 is consequence that \( P_1 \) and \( P_2 \) are linear in \( p_x \) and \( p_{\theta_i}'s \). We denote by \((a_0, \cdots, a_k) \) the level set \( i!a_i = p_{\theta_i}, \) then by definition of \( P_1 \) and \( P_2 \) given by equation 3.3. \( \square \)

3.1.2. **Fundamental equation.** The level set \((a_0, \cdots, a_k) \) defines a fundamental equation

\[
H_F(p_x, x) := \frac{1}{2} p_x^2 + \frac{1}{2} F^2(x) = H|_{(a_0, \cdots, a_k)}(p, q) = \frac{1}{2}.
\]

Here \( H_F(p_x, x) \) is a Hamiltonian function in the phase plane \((p_x, x)\), where the dynamic of \( x(s) \) takes place in the Hill region \( I = [x_0, x_1] \) and its solution \((p_x(t), x(t)) \) with energy \( h = 1/2 \) lies in an algebraic curve or loop given by

\[
\alpha_{(F, I)} := \{(p_x, x) : \frac{1}{2} = \frac{1}{2} p_x^2 + \frac{1}{2} F^2(x) \text{ and } x_0 \leq x \leq x_1\},
\]

and \( \alpha_{(F, I)} \) is close and simple.

**Lemma 2.** \( \alpha(F, I) \) is smooth if and only if \( x_0 \) and \( x_1 \) are regular points of \( F(x) \), in other words, \( \alpha(F, I) \) is smooth if and only if the corresponding geodesic \( \gamma(t) \) is \( x \)-periodic.

Proof. A point \( \alpha = (p_x, x) \) in \( \alpha(F, I) \) is smooth if and only

\[
0 \neq \nabla H_F(p_x, x)|_{\alpha(F, I)} = (p_x, F(x)F'(x)),
\]

then \( \alpha \) is smooth for all \( p_x \neq 0 \), the points \( \alpha(F, I) \) such that \( p_x = 0 \) correspond to endpoints of the Hill interval \( I \), since the condition \( p_x = 0 \) implies \( F^2(x) = 1 \), the point \( \alpha = (0, x_0) \) is smooth if \( F'(x_0) \neq 0 \), as well as, the point \( \alpha = (0, x_1) \) is smooth if \( F'(x_1) \neq 0 \). Then \( \alpha(F, I) \) is smooth if and only \( x_0 \) and \( x_1 \) are regular points of \( F(x) \). Also, \( \alpha(F, I) \) is smooth is equivalent to \( H_F(p_x, x)|_{\alpha(F, I)} \) is never zero, which is equivalent to the Hamiltonian vector field is never zero on \( \alpha(F, I) \). \( \square \)

3.1.3. **Arnold-Liouville manifold.** The Arnold-Liouville manifold \( M|_F \) is given by

\[
M_F := \{(p, q) \in T^*J^k : \frac{1}{2} = H_F(p_x, x), \ p_{\theta_i} = i!a_i\}.
\]

In the case \( \gamma(t) \) is \( x \)-periodic, \( M_F \) is diffeomorphic to \( S^1 \times \mathbb{R}^{k+1} \), where \( S^1 \) is the simple closed and smooth curve \( \alpha(F, I) \).

\( \alpha(F, I) \) has two natural charts using \( x \) as coordinates and given by solve the equation \( H_F = 1/2 \) with respect of \( p_x \), namely, \((p_x, x) = (\pm \sqrt{1 - F^2(x)}, x)\). Having this in mind,
Lemma 3. Let \( d\phi_t \) be the close one-form on \( M_F \subset T^*J^k \) give by
\[
(3.7) \quad d\phi_h := \frac{p_x}{\Pi(F,I)}|_{M_F} = \frac{\sqrt{1 - F^2(x)}}{\Pi(F,I)} dx,
\]
where \( \Pi(F,I) \) is the area enclosed by \( \alpha(F,I) \). Then,
\[
\int_{\alpha(F,I)} d\phi_h = 1 \quad \frac{\partial}{\partial h} \Pi(F,I) = L(F,I).
\]
as a consequence exist the inverse function \( h(\Pi) \).

Proof. Let \( \Omega(F,I) \) be the closed region by \( \alpha(F,I) \), then \( d\phi_h \) can be extended to \( \Omega(F,I) \) and Stokes’ Theorem implies
\[
(3.8) \quad \Pi(F,I) := \int_{\alpha(F,I)} p_x dx = \int_{\Omega(F,I)} dp_x \wedge dx = 2 \int_I \sqrt{2h - F^2(x)}|_{h=\frac{1}{2}} dx.
\]
This tell that \( \int_{\alpha(F,I)} d\phi_h = 1 \), thus \( d\phi_h \) is not exact.

\( \Pi(F,I) \) is a function of \( h \), so
\[
(3.9) \quad \frac{\partial}{\partial h} \Pi(F,I) = \frac{\partial}{\partial h} \int_I d\phi_h = \int_I \frac{2dx}{\sqrt{1 - F^2(x)}}.
\]
\( \square \)

\( \Pi(F,I) \) is also called an adiabatic invariant see [11] pg 297. We will use \( \Pi \) when we use it as a variable and \( \Pi(F,I) \) for the adiabatic invariant.

3.2. Action-angle variables in \( T^*J^k \). We will consider the actions \( \mu = (\Pi, a_0, \cdots, a_k) \) and find its angle coordinates \( \phi = (\phi_h, \phi_0, \cdots, \phi_k) \), such the set \( (\mu, \phi) \) of coordinates are an action-angle coordinates in \( T^*J^k \).

Lemma 4. There exist a canonical transformation \( \Phi(p,q) = (\mu, \phi) \), where \( \phi_h \) is the local function defined by the close form \( d\phi_h \) from Lemma 3 and
\[
\phi_i = -\int^x \frac{\tilde{x}^iF(\tilde{x})d\tilde{x}}{\sqrt{1 - F^2(\tilde{x})}} + i!\theta_i \quad x \in I \text{ and } i = 0, \cdots, k.
\]

To construct the canonical transformation \( \Phi(p,q) \), we will look for its generating function \( S(\mu,q) \), of the second type that satisfies the three following conditions.
\[
(3.10) \quad p = \frac{\partial S}{\partial q}, \quad \phi = \frac{\partial S}{\partial \mu}, \quad H(\frac{\partial S}{\partial q}, q) = h(\Pi) = \frac{1}{2},
\]
where $h(\Pi)$ is the function defined in Lema 3. For more detail on the definition of $S(\mu, q)$, see [11] Section 50 or [9].

To find $S(\mu, q)$, we will solve the subRiemannian Hamilton-Jacobi equation associated with the subRiemannian geodesic flow. For more details about the definition of this equation in subRiemannian geometry and its relations with the Eikonal equation, see [6] 8 pg or [2].

**Proof.** The subRiemannian Hamilton-Jacobi equation is given by

\[
(3.11) \quad h|_{1/2} = \frac{1}{2} \left( \frac{\partial S}{\partial x} \right)^2 + \frac{1}{2} \left( \sum_{i=0}^{k} x^i \frac{\partial S}{\partial \theta_i} \right)^2.
\]

Take the ansatz

\[
S(\mu, q) := f(x) + \sum_{i=0}^{k} i \lambda_i \theta_i,
\]

as a solution. The equation (3.11) becomes equation (3.5), then the generating function is given by

\[
(3.12) \quad S(\mu, q) = \int_{x_0}^{x} \sqrt{2h(\Pi) - F^2(\tilde{x})} d\tilde{x} + \sum_{i=0}^{n} i \lambda_i \theta_i.
\]

Here, $h(\Pi) = 1/2$ and $S(\mu, q)$ is a local function, since $x$ must lay in the Hill region $I$, that is, $S(\mu, q)$ is defined in the sub-set $\mu \times I \times \mathbb{R}^{k+1}$.

We can see that conditions 1 and 3 of equation (3.10) are satisfied: $p(\mu, q) = \partial S/\partial q$ and $H(p(\mu, q), q) = h$. To find the new coordinates $\phi$, we use the condition 2:

\[
\frac{\partial S}{\partial h} = \int_{x_0}^{x} \frac{d\tilde{x}}{\sqrt{1 - F^2(\tilde{x})}} = \phi_h,
\]

\[
\frac{\partial S}{\partial a_i} = - \int_{x_0}^{x} \frac{\tilde{x}^i F(\tilde{x}) d\tilde{x}}{\sqrt{1 - F^2(\tilde{x})}} + i \lambda_i = \phi_i.
\]

\[ \square \]

Note: In [2] a projection $\pi_F : J^k \to \mathbb{R}^3_F$ was built and the solution to the subRiemannian Hamilton-Jacobi equation on the magnetic space $\mathbb{R}^3_F$ was found. The solution given by equation (3.12) is the pull-back by $\pi_F$ of the solution previously found it in $\mathbb{R}_F$, where $\pi_F$ is in fact, a subRiemannian submersion.

**Corollary 1.** $(\mu, \phi)$ are action-angle coordinates.

**Proof.** Using the Hamilton equations for the new coordinates $(\mu, \phi)$, we have $\phi_t = t$ and $\phi_i = const.$ \[ \square \]
Note: that $h$ and $\phi_t$ are action-angles coordinates for the Hamiltonian $H_F$.

3.2.1. Horizontal derivative. A horizontal derivative $\nabla_{\text{hor}}$ of a function $S: J^k \to \mathbb{R}$ is the unique horizontal vector field that satisfies; for every $q$ in $J^k$,

$$< \nabla_{\text{hor}} S, v >_q = dS(v), \text{ for } v \in D_q,$$

where $<,>_q$ is the subRiemannian metric in $D_q$. For more detail see [6] pg 14-15 or [10].

**Lemma 5.** Let $\gamma(t)$ be a geodesic parameterized by arc-length corresponding to the pair $(F, I)$ and $S_F$ the solution given by equation (3.12), then

$$dS_F(\dot{\gamma})(t) = 1.$$

**Proof.** Let us prove that $\dot{\gamma}(t) = (\nabla_{\text{hor}} S_F)_{\gamma(t)}$, which is just a consequence that $S_F$ is solution to the Hamilton-Jacobi equation, that is,

$$X_1(S_F)|_{\gamma(t)} = \frac{\partial S}{\partial x}|_{\gamma(t)} = p_x(t),$$

but, Lemma 1 implies that $P_1(t) = p_x(t)$, so $P_1(t) = X_1(S_F)|_{\gamma(t)}$. As well,

$$X_2(S_F)|_{\gamma(t)} = \sum_{i=0}^{k} \frac{x^i(t)}{i!} \frac{\partial S}{\partial \theta_i}|_{\gamma(t)} = \sum_{i=0}^{k} a_i x^i(t) = F(x(t)),$$

also, Lemma 1 implies that $P_2(t) = F(x(t))$, so $P_2(t) = X_2(S_F)|_{\gamma(t)}$. As a consequence;

$$\nabla_{\text{hor}} S|_{\gamma(t)} := X_1(S_F)|_{\gamma(t)} X_1 + X_2(S_F)|_{\gamma(t)} X_2 = P_1(t) X_1 + P_2(t) X_2,$$

Lemma 1 implies $P_1(t) X_1 + P_2(t) X_2 = \dot{\gamma}(t)$. Thus, $\nabla_{\text{hor}} S = \dot{\gamma}(t)$ and $dS_F(v)|_q = < \nabla_{\text{hor}} S, v >$ for all $D_q$. In particular,

$$dS_F(\dot{\gamma}) = < \dot{\gamma}(t), \dot{\gamma}(t) > = 1,$$

since $t$ is arc-length parameter.

3.3. Proof of Proposition 1.

**Proof.** It is well-known that the fundamental system system $H_F$ with energy $1/2$ has period $L(F, I)$ given by equation (2.1) and the relation between $\Pi(F, I)$ and $L(F, I)$ is given by Lemma 3, see [11] pg 281. Let $\gamma(t)$ be a $x$-periodic corresponding to $(F, I)$, we are interested in seeing the change suffered by the coordinates $\theta_i$ after one $L(I, F)$. For that,
we consider the change in $S(\mu, q)$ after $\gamma(t)$ travel form $t$ to $t + L(F, I)$, in other words,

$$L(F, I) = \int_t^{t + L(F, I)} dS(\dot{\gamma}(t)) dt = \Pi(F, I) + \sum_{i=0}^{n} i!a_i \Delta \theta_i(F, I).$$  \hspace{1cm} (3.14)$$

On the left side of the equation is a consequence of Lemma 5, and the right side is the integration term by term. The derivative of equation (3.14) with respect to $a_i$ to find $-\frac{\partial}{\partial a_i} \Pi(F, I) = i! \Delta \theta_i$, which is equivalent to equation (2.2).

We differentiate $\Delta \theta_i := \theta_i(t + L) - \theta_i(t)$ respect to $t$, to see that $\Delta \theta_i(F, I)$ is independent of the initial point. The derivative is

$$\frac{x^i(t + L)F(x(t + L))}{\sqrt{1 - F^2(x(t + L))}} - \frac{x^i(t)F(x(t))}{\sqrt{1 - F^2(x(t))}},$$

but $x(t + L) = x(t)$.

**References**

[1] Alejandro Bravo-Doddoli. Higher elastica: Geodesics in the jet space. *European Journal of Mathematics*.
[2] Alejandro Bravo-Doddoli and Richard Montgomery. Geodesics in Jet Space. *Regular and Chaotic Dynamics*, 2021.
[3] A. Anzaldo-Meneses and F. Monroy-Perez. Goursat distribution and sub-riemannian structures. *Journal of Mathematical Physics*, 44(12):6101–6111.
[4] A. Anzaldo-Meneses and F. Monroy-Perez. Integrability of nilpotent sub-riemannian structures.
[5] Felipe Monroy-Pérez and A. Anzaldo-Meneses. Optimal control on nilpotent lie groups. *Journal of Dynamical and Control Systems*, 8:487–504, 2002.
[6] Richard Montgomery. *A tour of subriemannian geometries, their geodesics and applications*. Number 91. American Mathematical Soc.
[7] Robert Bryant and Hsu Lucas L. Rigidity of integral curves of rank 2 distributions. *Inventiones mathematicae*, 114(2):435–462, 1993.
[8] Richard Montgomery and Michail Zhitomirskii. Geometric approach to goursat flags. volume 18, pages 459–493.
[9] LD Landau and EM Lifshitz. Mechanics third edition: Volume 1 of course of theoretical physics. *Elsevier Science*, 1976.
[10] Andrei Agrachev, Davide Barilari, and Ugo Boscain. *A Comprehensive Introduction to Sub-Riemannian Geometry*. Cambridge University Press, 2019.
[11] Vladimir Igorevich Arnol’d. *Mathematical methods of classical mechanics*. Springer Science.
[12] Ben Warhurst. Jet spaces as nonrigid carnot groups. *Journal of Lie Theory*, 15(1):341–356, 2005.

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