Ashkin-Teller transition of Rydberg atoms with two-site blockade

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(Dated: January 22, 2020)

Motivated by recent experiments on Rydberg atoms in a 1D trap, we study the phase transition out of the $Z_4$ crystalline phase in the context of a hard-boson model with two-site blockade. Using extensive DMRG simulations, we show that the transition along the commensurate line is in the Ashkin-Teller universality class with an exponent $\nu \simeq 0.78$, large enough for chiral perturbations to be relevant. This is further confirmed by our numerical results along the transition line which show that the transition immediately leaves the Ashkin-Teller universality class, and which are consistent with a small range of chiral transition followed by the opening of a floating phase.

The problem of commensurate-incommensurate transitions has a long history that goes back to the physics of adsorbed monolayers on surfaces\textsuperscript{1}, and that has been revived recently in the context of the 1D quantum problem of Rydberg atoms in an optical trap\textsuperscript{2,3}. At low enough temperature for 2D classical systems, or at small enough chemical potential for the 1D quantum analog, crystalline phases with a broken $Z_p$ symmetry develop. They correspond to periodic arrangements of domain walls or of particles. The melting of these crystalline phases is a very subtle problem that has not yet received a full solution. For $p = 2$, the transition is generically Ising, while for $p \geq 5$ it has to be a 2-step process through an algebraic phase (called a floating phase in the context of adsorbed monolayers) if it is not first order. The difficult cases are $p = 3$ and $p = 4$. In these cases, the order-disorder transition along the commensurate line of the disordered phase can be continuous in the 3-state Potts universality class for $p = 3$, and in the Ashkin-Teller universality class for $p = 4$. Away from this line, the disordered phase is incommensurate. As pointed out by Ostlund\textsuperscript{4} and Huse\textsuperscript{5}, this introduces a chiral perturbation, and the problem is to understand the effect of this chiral perturbation on the transition.

For $p = 3$, the chiral perturbation is always relevant, and the question is whether it immediately opens a floating phase away from the Potts point, or whether the transition remains direct and continuous for a while, but in a new chiral universality class, as suggested by Huse and Fisher\textsuperscript{6}. Numerical\textsuperscript{7,8} and experimental evidence\textsuperscript{9,10} in favour of this possibility has been obtained in the eighties and early nineties in the context of adsorbed layers, and very recently in the context of Rydberg atoms\textsuperscript{2,3,11,12}. It turned out to be very difficult however to definitely exclude the presence of an extremely narrow floating phase.

For $p = 4$, the situation is even richer. As we shall discuss below, the chiral perturbation is not always relevant\textsuperscript{1}. In particular, it is not if the transition is in the 4-state Potts universality class, a special case of Ashkin-Teller. In that case, a line of continuous transition in the Ashkin-Teller universality class can be expected. If however the critical exponents along the commensurate line are such that the chiral perturbation is relevant, then the situation is similar to the $p = 3$ case, with the possibility of a chiral transition before a floating phase appears.

In this Letter, we investigate this problem in the context of a quantum hard-boson model with two-site blockade. We show that the transition along the commensurate line is sufficiently far from the 4-state Potts point to ensure that the chiral perturbation is relevant, and we provide evidence in favour of a small region of chiral...
transition before a floating phase develops.

The Hamiltonian of hard-boson with \( r \)-site blockade is defined by:

\[
H = \sum_i -w(d_i^\dagger + d_i) + U n_i + V n_{i}n_{i+r+1},
\]

where \( d_i \) (\( d_i^\dagger \)) is an annihilation (creation) operator that acts in a constrained Hilbert space:

\[
n_i(n_i - 1) = n_i n_{i+1} = \ldots = n_i n_{i+r} = 0.
\]

When \( r = 1 \), the model reduces to the original hard-boson model introduced by Fendley et al.\[15\]. This family of constrained models is of direct relevance to quantum simulators based on Rydberg atoms trapped with optical tweezers. Indeed, the van der Waals interaction between Rydberg states scales with the distance as \( 1/r^6 \). The very fast decay leads to an effective Rydberg blockade - vanishing probability to find two Rydberg states within a certain distance defined by the Rabi frequency. The chemical potential \( U \) can be controlled experimentally by laser detuning, while the coupling constant \( V \) between neighbors at distance \( r + 1 \) is the leading-order correction beyond the blockade. In the present paper, we will concentrate on \( r = 2 \).

Let us start by discussing the phase diagram. Our numerical results have been obtained with a state-of-the-art density matrix renormalization group (DMRG) algorithm\[16-19\] that explicitly implements the constraints\[16-19\] and are summarized in Fig.1. There are three main phases: a disordered phase with incommensurate short-range correlations, and two ordered commensurate phases with period 3 and 4 respectively.\[21\] There are also small floating phases close to the ordered phases. In particular, for large and negative values of \( U \), there are two floating phases at the boundaries of the period-three and period-four phases that come closer and create an area of extremely high correlation length. It is therefore probable that the disordered phase eventually disappears and that, for some parameter range, the two ordered phases are connected through a single floating phase, as suggested in Ref.\[2\]22]. Due to the exponential growth of the correlation length at the Kosterlitz-Thouless\[23\] phase transition, an accurate investigation of this scenario would require simulations far beyond our current limitations.

The transition out of the period-three phase in Fig.1 is expected to take place either through an intermediate floating phase or through a direct transition in the Huse-Fisher chiral universality class upon approaching the 3-state Potts point, which is pushed to \( V = -\infty \) since the wave-vector \( q > 2\pi/3 \) cannot be realized at finite \( V \) because of the two-site blockade. Since the nature of the \( p = 3 \) transition has already been investigated in the context of the \( r = 1 \) model, we will not discuss this transition further.

The transition out of the period-four phases is the main focus of the rest of the paper. Our first task is to locate the commensurate line, and to study the universality class of the transition along this line.\[24\] The equal-\( q \) lines are depicted in Fig.2(a). The commensurate line corresponds to \( q = \pi/2 \). It enters the period-four phase at \( U \simeq -3.186 \). An accurate estimate of the second coordinate has been obtained by a finite-size scaling of the order parameter. Indeed, open boundary conditions favor a boson on the first and last sites. This effectively acts as a fixed boundary condition at the critical point and induces Friedel oscillations in the local boson density. According to the boundary CFT the profile of these oscillations on a finite-size chain is given by \( \propto [N \sin(\pi j/N)]^{-d} \), where the scaling dimension \( d \simeq 1/8 \) for the Ashkin-Teller model\[25\]. By scanning \( V \) for \( U = -3.186 \), we identify a separatrix in the log-log scaling at \( V = 2.5678 \) as shown in Fig.2(b). The slope corresponds to \( d \simeq 0.124 \), in excellent agreement with the scaling dimension \( d = 1/8 \). As a further check that this is a critical point, we have extracted the central charge by fitting the profile of the reduced entanglement entropy to the Calabrese-Cardy formula\[20\], leading to a central charge \( c \simeq 0.94 \), within 6% of the CFT prediction \( c = 1 \).

At that stage, it is useful to review some properties of the Ashkin-Teller model. The Ashkin-Teller transition is a family of universality classes tuned by an asymmetry parameter \( \lambda \). The quantum version of the Ashkin-Teller model is defined by the following Hamiltonian:

\[
H_{AT} = -\sum_{j=1}^{N} (\sigma_j^+ + \tau_j^+ + \lambda \sigma_j^\dagger \tau_j^\dagger) - \beta \sum_{j=1}^{N-1} (\sigma_j^+ \sigma_{j+1}^+ + \tau_j^+ \tau_{j+1}^+ + \lambda \sigma_j^\dagger \tau_j^\dagger \sigma_{j+1}^\dagger \tau_{j+1}^\dagger),
\]
where $\sigma^{x,z}$ and $\tau^{x,z}$ are Pauli matrices. The model is critical along $\beta = 1$. At $\lambda = 0$ the model is known as the four-state clock model and corresponds to two decoupled transverse-field Ising chains. At $\lambda = 1$ the Hamiltonian is identical to the four-state Potts model. For our purpose, the most important characteristics of this model is that the energy cost $|U|$: one particle less $A...A...A
A...AB...BA...A
A...AC...CA...A
A...AD...DA...A$
nomore $\nu$ of the correlation length. Indeed, according to Schulz[20], the cross-over exponent $\phi$ of the chiral perturbation for the Ashkin-Teller model is given by

$$\phi = \frac{3\nu}{2} - \frac{1}{4} - \frac{\nu^2}{2\nu - 1}$$

The chiral perturbation is relevant if $\phi > 0$, i.e. if $\nu > \nu_c = (1 + \sqrt{3})/4 = 0.683...$, irrelevant otherwise. Now, the exponent $\nu$ is known exactly as a function of $\lambda$[27,28]:

$$\nu = \frac{1}{2 - \frac{\pi}{4}(\arccos(-\lambda))^{-1}}$$

For the 4-state Potts model ($\lambda = 1$), $\nu = 2/3 < \nu_c$: The chiral perturbation is irrelevant. The critical value of $\lambda$ below which the chiral perturbation becomes relevant is given by

$$\lambda_c = -\cos \frac{\pi(\sqrt{3} + 1)}{4(\sqrt{3} - 1)} \approx 0.9779...$$

The correlation length of the hard-boson model can be simply obtained by fitting correlations, a straightforward task along the commensurate line. The resulting correlation diverges at the critical point with an exponent $\nu \approx 0.78$. This is the first indication that $\lambda$ must be significantly smaller than 1. This is actually quite natural. Indeed, when $\lambda = 1$, the model corresponds to the four-states Potts model with the same amplitude for all flipping processes, while for $\lambda < 1$ two processes are favoured over the third one by the transverse field term. Such an asymmetry naturally appears in the hard-boson model due to the two-site blockade. From Fig.3 one can see that domains B and D shifted by one site with respect to the bulk A cost less energy than the domain C shifted by two sites. ne can see that domains B and D shifted by one site with respect to the bulk A cost less energy than the domain C shifted by two sites.

To estimate $\lambda$ numerically, we have computed the spectrum by targeting several states (up to 11) at every DMRG iteration[29]. In Fig.4(a) we show the energy spectrum for $N = 60$ with fixed A-A boundary conditions[20]. We compare these results with the spectrum of the hard-boson model with $N = 201$ sites. Since the velocity is a non-universal constant, one cannot compare the absolute values of the gap. However we find that the structure of the spectrum in the hard-boson model corresponds to the structure of the Ashkin-Teller spectrum at $\lambda \approx 0.57$ (red line) (see 20 for details). In Fig.4(c) we further compare the finite-size scaling for hard boson (red) and Ashkin-Teller model at $\lambda = 0.57$ (green) and at $\lambda = 1$ (4-state Potts, blue) and the agreement is quite good (for spectra with other boundary conditions, see 20).

So we reach the first important conclusion of this paper: The transition along the commensurate line is in the universality class of the Ashkin-Teller model with $\lambda \approx 0.57$ and $\nu \approx 0.78$. At that point, the chiral transition is relevant, with a crossover exponent $\phi \approx 0.33$. This means in particular that, away from that point, the transition cannot be a standard continuous transition in the Ashkin-Teller universality class. Either a floating phase opens, or the transition becomes chiral.

Quite generally, the incommensurate wave-vector $q$ is expected to approach the commensurate value $\pi/2$ with a critical exponent called $\beta$. To the best of our knowledge the exact value of this critical exponent is not known for the Ashkin-Teller model, but Huse and Fisher[14] argue that $\beta \approx \nu$. This implies that the product $\xi \times |\pi/2 - q|$ decays to zero upon approaching the Ashkin-Teller transition. By contrast, if the transition is chiral, the equality $\beta = \nu$ should hold, and $\xi \times |\pi/2 - q|$ is expected to go to some finite value[14]. When the transition is Ashkin-
In the disordered phase, the correlation length is fitted either with a power-law with critical exponent $\nu$. In the disordered phase and in the ordered phase, the correlation length diverges as a power law with similar exponents on both sides of the transition, but, by contrast to the Ashkin-Teller point, the critical exponent $\beta$ is much smaller than one, a clear indication that the chiral perturbation changes the physics immediately away from the Ashkin-Teller point. Its value is comparable to $\nu$ and $\nu'$, and accordingly, even if it increases slightly towards the transition, the product $\xi \times |\pi/2 - q|$ seems to remain finite. The absence of divergence of the product $\xi \times |\pi/2 - q|$ is a clear indication that the transition is probably in the Huse-Fisher universality class. However, as in the case of $p = 3$ chiral universality class, an extremely narrow floating phase cannot be excluded.

Further away from the commensurate point, the inverse of the correlation length decays in a very asymmetric way, as we show for the horizontal cut at $V = 7$ in Fig. 5(g)-(i). The numerically extracted critical exponent $\nu'$ is in reasonable agreement with the Pokrovsky-Talapov value 1/2, while the product $\xi \times |\pi/2 - q|$ clearly diverges towards the transition. The physics is very similar on the other side of the Ashkin-Teller line (see [24] for more data).

To summarize, we have shown that there is only one conformal critical point along the boundary of the period-four phase of the 1D hard-boson model with two-site blockade. It is in the Ashkin-Teller universality class with an exponent $\nu \approx 0.78$ which, according to Schulz’s crossover theory, implies that the chiral perturbation is relevant. Accordingly, the transition away from this point should be either a Huse-Fisher chiral transition, or a two-step process through a floating phase. The numerical evidence points to a narrow range of chiral transition, but a very narrow floating phase cannot be excluded. To make contact with experiments, it would be very interesting to perform a similar analysis for the model with $1/r^6$ long-range interactions. This model has already been studied by Rader and Läuchli, but the emphasis was put on the floating phase, and the immediate vicinity of the tip of the period-4 lobe has not been looked at in detail. Finally, we note that the experimental results on Rydberg atoms[23] are compatible with a continuous transition, with a Kibble-Zurek exponent $\mu \approx 0.25$. This exponent is related to $\nu$ by the relation $\mu = \nu/(1 + \nu z)$, where $z$ is the dynamical exponent. Since between the clock model ($\lambda = 0$) and the Potts model ($\lambda = 1$) the exponent $\nu$ decreases from 1 to 2/3, the Kibble-Zurek exponent should be between 1/2 and 2/5 if the dynamical exponent was equal to 1. So, according to experiments, the dynamical exponent has to be larger than 1, consistent with the chiral Huse-Fisher transition suggested by our results in the neighbourhood of the Ashkin-Teller model. Work is in progress to track this exciting physics in the model with van der Waals interactions.
ACKNOWLEDGMENTS

This work has been supported by the Swiss National Science Foundation. The calculations have been performed using the facilities of the University of Amsterdam.

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[21] Note that these three main phases have been accessed in recent Rydberg atom experiments.[2][3].
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SUPPLEMENTAL MATERIAL FOR:
“ASHKIN-TELLER TRANSITION OF RYDBERG ATOMS WITH TWO-SITE BLOCKADE”

Implementation details

The size of the Hilbert space for a model with two-site Rydberg blockade can be calculated using a recursive relation $\mathcal{H}(N) = \mathcal{H}(N - 1) + \mathcal{H}(N - 3)$, with the first three elements of the sequence $\mathcal{H}(1) = 2$, $\mathcal{H}(2) = 3$ and $\mathcal{H}(3) = 4$. So the growth of the Hilbert space with the system size $\mathcal{H}(N) \propto 1.466^N$ is much slower than $\mathcal{H}(N) \propto 2^N$ for an unconstrained model. In order to fully profit from the restricted Hilbert space we implement the blockade explicitly into the DMRG. Recently it has been shown that the hard-boson model with $r = 1$ can be rigorously mapped onto a quantum dimer model on a two-leg ladder \cite{2003PhRvL..90o0401K} that provides a simple and intuitive way to encode the constraint into DMRG. Although this mapping is not valid for $r > 1$, we can rely on the idea of auxiliary quantum numbers that would preserve the block-diagonal structure of the local tensors. This is achieved by a rigorous mapping onto an effective model that spans the local Hilbert space over three consecutive sites on the original lattice as shown in Fig.6(b). The new local Hilbert space contains four states listed in Fig.6(c). Because of the overlap, the three possible states of two shared sites can be used as a quantum label for the auxiliary bond between two consecutive sites of the new model. By adding a site, for example by increasing the left environment, one can change the quantum labels according to the fusion graph shown in Fig.6(d). The fusion graph for the right environment can be obtained by inverting the arrows. An example of the label assignment is provided in Fig.6(e).

![Fig. 6.](image)

At the next step, one has to rewrite the hard-boson model given by Eq.1 of the main text in terms of new local variables $|h_i\rangle$. For example, the boson occupation number operator $n_i$, which is also equal to $(1 - n_{i-1})n_i(1 - n_{i+1})$, can be written in the new local Hilbert space as a $4 \times 4$ matrix $\hat{n}_i$ with the only non-zero element $\hat{n}_i(3, 3) = 1$. The term $V n_{i-1} h_{i+2}$ can be written in the new Hilbert space as a nearest-neighbor interaction $\hat{V} \hat{p}_i \hat{q}_{i+1}$ where the only non-zero matrix elements of the operators $\hat{p}$ and $\hat{q}$ are given by $\hat{p}(4, 4) = 1$ and $\hat{q}(2, 2) = 1$. Finally the constrained hopping term

$$w(1 - n_{i-2})(1 - n_{i-1})(d^+_i + d_i)(1 - n_{i+1})(1 - n_{i+2})$$

can be rewritten as a three-site operator

$$w(\hat{a}_{i-1} \hat{b}_i \hat{c}_{i+1} + h.c.),$$

where the only non-zero matrix elements of the operators $\hat{a}$, $\hat{b}$, and $\hat{c}$ are given by $\hat{a}(1, 2) = 1$, $\hat{b}(1, 3) = 1$, and $\hat{c}(1, 4) = 1$.

With these definitions, the matrix product operator in the bulk takes the following simple form:

$$U \hat{\Pi} \hat{V} \hat{p}_i \hat{q}_{i+1} \hat{w} \hat{w}^\dagger \hat{I}$$

Close to the edges one has to carefully modify the MPO to properly encode the boundary terms. This requires the definition of local operators slightly different from those used in the bulk.

There is yet another crucial point that we want to mention. The labels that we have introduced split the Hilbert space into blocks or sectors and therefore correspond to some conserved quantity. For the hard-boson model with a single-site blockade, the quantum labels correspond to the parity of the domain walls. In the present case, the physical meaning of this quantity is not as obvious. However, the only relevant information for us is that the conservation of this abstract quantity requires at least three sites. In other words, by acting with any term (read hopping term) on a two-site MPS, one necessary changes one of the out-going labels, while the hopping term applied on three consecutive MPS keeps all external labels fixed. As a consequence, neither single- nor two-site DMRG routines are compatible with the presented constraint implementation, and one has to go for at least three-site updates. At a glance this might look costly with a local Hilbert space of dimension 4 since it leads in principle to an MPO operator of size $7 \times 7 \times 64 \times 64$. However, taking into account all the constraints on three sites, the projected three-site MPO is only of size $7 \times 7 \times 9 \times 9$. 
The explicit implementation of two-sites blockade allows us to reach systems with up to \(N = 3001\) sites systematically (and \(N = 4801\) sites occasionally), keeping up to 2000 states.

### Calabrese-Cardy formula

According to Calabrese and Cardy\cite{31} the entanglement entropy in finite-size chain with open boundary conditions scales with the block size \(l\) as:

\[
S_L(l) = \frac{c}{6} \ln d(l) + s_1 + \log g, 
\]

where \(d(l) = \frac{2L}{\pi} \sin \left(\frac{\pi l}{L}\right)\) is the conformal distance; \(s_1\) and \(\log g\) are non-universal constants. The presence of Friedel oscillations caused by the fixed boundary conditions is also reflected in the entanglement entropy profile. In order to remove the oscillations we follow Ref.\cite{32} and construct the reduced entanglement entropy:

\[
\tilde{S}_N(l) = S_N(l) - \zeta(n_{l-1}n_{l+2}), 
\]

where \(\zeta\) is a non-universal constant in front of the leading local correlations between nearest allowed neighbors adjusted to best remove the oscillations.

### FURTHER DETAILS ON ASHKIN-TELLER POINT

In the main text we justify our selection of \(\lambda = 0.57\) by a qualitative comparison of the Ashkin-Teller and hard-boson spectra. In Fig.\[7\] we show how this can be done in a more systematic way. In both the Ashkin-Teller and hard-boson models, the velocity of sound is a non-universal constant. So a quantitative comparison of these two spectra is only possible when the velocity is removed. This we achieve by re-scaling both spectra with respect to the lowest excited state, using a pre-factor 2 for a reason explained below. The crossing point of the two spectra gives the estimate of \(\lambda \approx 0.57\) as shown in Fig.\[7\].

We compute the energy spectrum in a chain with open and fixed boundary conditions. There are two reasons for that. First, DMRG is well known to be more efficient for open boundary conditions than for periodic ones. Secondly, the number of conformal towers of states that appears in the spectra of periodic or anti-periodic chains are usually larger than the number of towers selected by fixed boundary conditions. However, we have to establish the correspondence between the different boundary conditions in the hard-boson model and in the original Ashkin-Teller model. In the hard-boson model, the simplest way to fix the boundary is to force the first and the last sites to be occupied. In the \(p = 4\) phase every fourth site is occupied by a boson. So each of the ground states, let us call them A, B, C and D, corresponds to the location of the occupied site mod 4. If the total number of sites is \(4k + 1\) the same state is favored at each edge, corresponding to the \(A - A\) boundary condition in the Ashkin-Teller model. If the total number of sites is \(4k\) or \(4k + 2\), we expect A-B and A-D boundary conditions. They are expected to give the same spectrum (assuming that states B and D have equal weight in the transverse field applied on A, while C has a factor \(\lambda\)). Finally, if the total number of sites is \(4k + 3\), we expect to observe the spectrum of the \(A - C\) boundary condition.

We further test the extracted value of \(\lambda\) by looking at the finite-size scaling of the conformal towers of states. The results for A-A boundary conditions have been presented in the main text. In Fig.\[8\] we present the results for A-B (same as A-D) and A-C boundary condition. Note that here we do not use the pre-factor 2 for the re-scaled spectrum. This is related to the fact that for A-A boundary condition we expect the lowest energy states to be described by the identity conformal tower \(I\) for which the \(n = 1\) excitation is missing. According to the Fig.4(a) of the main text this seems to be the case everywhere for \(0 \leq \lambda \leq 1\). To the best of our knowledge the boundary-filed correspondence has not been worked out in CFT for the Ashkin-Teller model, so the operator content for A-B, A-C and A-D boundary conditions is not known. However by analogy with other minimal models one can expect an equally spaced spectrum for the primary conformal tower, so the lowest state will be
n = 1.

We extract the critical exponent correlation length along the commensurate line which, close to the transition, is given by \( V = -0.3645U + 1.4125 \). Since we expect a direct transition the critical exponent has to be the same on both sides of the critical point. However, the pre-factor is non-universal. We therefore fit our numerical data with:

\[
|x - U_c|^\nu \times \left[a\theta(x - U_c) + b\theta(U_c - x)\right],
\]

where \( a, b, U_c \), and \( \nu \) are fitting parameters; and \( \theta(x) \) is the Heaviside function: \( \theta(x) = 1 \) if \( x > 0 \) and zero otherwise. The results are presented in Fig.9.

We compare the values of \( \lambda \) and \( \nu \) obtained to fit the hard-boson model with the CFT result of Kohmoto et al.\cite{27, 28} in Fig.10.

**Extraction of the correlation length and of the wave-vector**

In order to extract the correlation length and the wave-vector \( q \), we fit the boson-boson correlation function to the Ornstein-Zernicke form\cite{33}:

\[
C_{OZ,i,j} \propto \frac{e^{-|i-j|/\xi}}{|i-j|^{\nu \phi_0}},
\]

where the correlation length \( \xi \), the wave vector \( q \), and the initial phase \( \phi_0 \) are fitting parameters. In order to extract the correlation length and the wave-vector with a sufficiently high precision, we fit the correlation function in two steps. First, we discard the oscillations and fit the main slope of the decay as shown in Fig.11. This allows us to perform a fit in a semi-log scale \( \log C(x = |i-j|) \approx c - x/\xi - \log(x)/2 \), that in general provides more accurate estimates of the correlation length on a long scale. Second we define a reduced correlation function

\[
\tilde{C}_{i,j} = C_{i,j} \frac{\sqrt{|i-j|}}{e^{-|i-j|/\xi + c}}
\]

and fit it with a cosine \( \tilde{C}_{i,j} \approx a \cos(q|i-j| + \phi_0) \) as shown in Fig.11(b). The agreement is almost perfect: The DMRG data (blue dots) are almost completely behind the fit (red dots).

**Numerical data along a few selected cuts**

In the main text we probe the nature of the phase transition to the \( p = 4 \) phase based on numerical results for the correlation length \( \xi \) and the wave-vector \( q \) across three selected cuts. The location of these cuts is shown on the phase diagram in Fig.12 by red dashed lines. In Fig.13 we provide data for five more cuts that cross the \( p = 4 \) phase boundary at different places indicated in Fig.12 by green solid lines.

The data for \( U = -4 \) and for \( V = 5 \) point towards an intermediate floating phase. The data for \( V = 2.4 \) just below the Ashkin-Teller point is consistent with a chiral transition, similar to the data for \( V = 2.7 \) just above
Figure 11. Example of fit of the correlation function to the Ornstein-Zernicke form. In the first step (a), we extract the correlation length discarding the oscillations. In the second step (b), we fit the reduced correlation function to extract the wave-vector $q$.

Figure 12. Position of the cuts across the transition to the $p = 4$ phase along which we have investigated the properties of the transition in detail. For clarity the length of each cut is enlarged. Solid black lines indicate the phase boundaries, the open green circle states for the symmetric Ashkin-Teller point. Dotted lines between $p = 3$ and $p = 4$ phases are equal-$\xi$ lines with $\xi = 50$ (yellow), 100 (purple), 200 (green). Red dashed lines mark the cuts considered in the main text. Data across solid green lines are provided in Fig.13. Data along the cuts marked by blue dash-dotted lines are shown in Fig.14.

the Ashkin-Teller point (see main text). The data for $U = -3.5$ and $V = 3$ are not conclusive: the product $|q\pi - 1/2| \times \xi$ for a few selected horizontal (a-d) and vertical (e-j) cuts across $p = 4$ phase transition below (a-f) and above (g-j) the symmetric Ashkin-Teller point. Different symbols correspond to the system sizes listed in (a).

by blue dash-dotted lines. We see that the correlation length remains very large between the two Pokrovskyy-Talapov transitions: for $U = -5$ it never goes below $\xi = 120$. Shortly beyond $U = -5$ the finite-$\xi$ region cannot be resolved with our algorithm. Also, for the selected parameter range $U \geq -7$, we were not able to resolve intermediate plateaux (e.g. at $q = 3\pi/5$), as predicted in Ref. [22] for the model with van der Waals interactions.

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Figure 14. Inverse of the correlation length (left) and wave-vector $q$ (right) across two vertical cuts from $p = 3$ to $p = 4$ phase. In (a) and (c) levels with $\xi = 50$ (yellow), $\xi = 100$ (purple) and $\xi = 200$ (green) are shown for reference.

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