Stability of QED

M. P. Fry
School of Mathematics, University of Dublin, Trinity College, Dublin 2, Ireland
(Dated: February 1, 2013)

It is shown for a class of random, time-independent, square-integrable, three-dimensional magnetic fields that the one-loop effective fermion action of four-dimensional QED increases faster than a quadratic in $\mathbf{B}$ in the strong coupling limit. The limit is universal. The result relies on the paramagnetism of charged spin-1/2 fermions and the diamagnetism of charged scalar bosons.

PACS numbers: 12.20.Ds, 11.10.Kk, 11.15.Tk

I. INTRODUCTION

Integrating out the fermion fields in four-dimensional QED continued to the Euclidean metric results in the measure for the gauge field integration

$$d\mu(A) = Z^{-1}e^{-\int d^4x(\mathcal{L} + \text{gauge fixing})} \times \det_{\text{ren}}(1 - eS\mathcal{A}) \prod_{x,\mu} dA_{\mu}(x), \quad (1.1)$$

where $\det_{\text{ren}}$ is the renormalized fermion determinant defined in Sec. II: $S$ is the free fermion propagator, and $Z$ is chosen so that $\int d\mu(A) = 1$. In the limit $e = 0$ the Gaussian measure for the potential $A_{\mu}$ is chosen to have mean zero and covariance

$$\int d\mu(A)A_{\mu}(x)A_{\nu}(y) = D_{\mu\nu}(x - y), \quad (1.2)$$

where $D_{\mu\nu}$ is the free photon propagator in some fixed gauge. Naively, integration over the fermion fields produces the ratio of determinants $\det(\mathcal{P} + e\mathcal{A} + m)/\det(\mathcal{P} + m)$ which is not well-defined; $\det_{\text{ren}}$ makes sense of this ratio. It is gauge invariant and depends only on the field strength $F_{\mu\nu}$ and invariants formed from it.

We have chosen to introduce this paper with an abrupt intrusion of definitions in order to emphasize the central role of $\det_{\text{ren}}$ in QED: it is everywhere. It is the origin of all fermion loops in QED. If there are multiple charged fermions then $\det_{\text{ren}}$ is replaced by a product of renormalization determinants, one for each species. For our purpose here it is sufficient to consider one fermion.

The nonperturbative calculation of $\det_{\text{ren}}$ reduces to finding the eigenvalues of $S\mathcal{A}$,

$$\int d^4y S(x - y)\mathcal{A}(y)\psi_n(y) = \frac{1}{\epsilon_n}\psi_n(x). \quad (1.3)$$

There are at least two complications. Firstly, $S\mathcal{A}$ is not a self-adjoint operator, and so many powerful theorems from analysis do not apply. And secondly, since $A_{\mu}$ is part of a functional measure, it is a random field, making the task of calculating the $\epsilon_n$ for all admissible fields impossible. What can be done is to expand $\ln \det_{\text{ren}}$, the one-loop effective action, in a power series in $e$. Then the functional integration can be done term-by-term to obtain textbook QED.

The first nonperturbative calculation of $\det_{\text{ren}}$ was done by Heisenberg and Euler [1] seventy five years ago for the special case of constant electric and magnetic fields. Their paper gave rise to a vast subfield known as quantum field theory under the influence of external conditions. A comprehensive review of this body of work relevant to $\det_{\text{ren}}$ is given by Dunne [2].

An outstanding problem is the strong field behavior of $\det_{\text{ren}}$ that goes beyond constant fields or slowly varying fields or special fields rapidly varying in one variable [2, 3]. That is, what is the strong field behavior of $\det_{\text{ren}}$ for a class of random fields $F_{\mu\nu}$ on $\mathbb{R}^4$? What if $\ln \det_{\text{ren}}$ increases faster than a quadratic in $F_{\mu\nu}$ for such fields? Is $\det_{\text{ren}}$ integrable for any Gaussian measure in this case? This is a question with profound implications for the stability of QED in isolation. Of course, QED is part of the standard model, thereby making the overall stability question a much more intricate one. Nevertheless, the stability of QED in isolation remains unknown and deserves an answer.

In this paper we consider the case of square integrable, time-independent magnetic fields $\mathbf{B}(x)$ defined on $\mathbb{R}^3$. There are additional technical conditions on $\mathbf{B}$ introduced later. The magnetic field lines are typically tangled loops. We find that

$$\lim_{e \to \infty} \frac{\ln \det_{\text{ren}}}{e^2 \ln e} = \frac{||\mathbf{B}||^2 T}{24\pi^2}, \quad (1.4)$$

where $||\mathbf{B}||^2 = \int d^3x \mathbf{B} \cdot \mathbf{B}(x)$, and $T$ is the size of the time box. Since $e$ always multiplies $\mathbf{B}$, this means that $\ln \det_{\text{ren}}$ is growing faster than a quadratic in $\mathbf{B}$. In the

---

1 We note here progress in scalar QED since the review [2] in going beyond these fields. Using the multidimensional worldline instanton technique the vacuum pair production rate has been calculated from the one loop effective action of a charged scalar particle in selected two and three-dimensional electric fields [3]. These fields have to be sufficiently regular in order to define a formal functional semiclassical expansion of the quantum mechanical path integral representation of the effective action. The extension of this technique to spinor QED has not been done yet.
The fermion determinant in (1.5) is defined in Sec.II. The diamagnetic inequality is known to be true for lattice formulations of QED$_3$ obeying reflection positivity and using Wilson fermions [3, 4]. Since Wilson fermions are CP invariant there is no Chern-Simons term to interfere with the uniqueness of det$_{QED}_3$. And since det$_{QED}_3$ is gauge invariant there are no divergences when the lattice spacing for the fermions is sent to zero. As stated by Seiler [7], (1.5) is more an obvious truth than a conjecture. Since det$_{QED}_3 |_{e=0} = 1$ and det$_{QED}_3$ has no zeros in $e$ for real values of $e$ when $m \neq 0$ [3], (1.5) can be rewritten as

$$0 < \text{det}_{QED}_3 \leq 1.$$  

(1.6)

An inspection of Eq. (2.4) below indicates that (1.6) is a reflection of the tendency of an external magnetic field to lower the energy of a charged fermion. Therefore, the historic heading of (1.5) and (1.6) as "diamagnetic" inequalities is a misnomer; paramagnetic inequalities would be a more accurate designation. The detailed justification for going from (1.5) to (1.6) is given in Sec.II.

The second general principle underlying (1.4) is the diamagnetism of charged spin-0 bosons in an external magnetic field. This is encapsulated in one of the versions of Kato’s inequality discussed in Sec. III. As the foregoing remarks indicate, QED$_3$ is central to the derivation of (1.4), and it is to the connection between QED$_3$ and QED$_4$ that we now turn.

II. QED$_3$ AND QED$_4$

A. The connection

The connection has been dealt with previously [10]. In order to make this paper reasonably self-contained we will review the relevant definitions and results. The upper bound on $\text{det}_{ren}$ obtained in [10] is not optimal; it will be optimized here.

The renormalized and regularized fermion determinant in Wick-rotated Euclidean QED$_4$ with on-shell renormalization, $\text{det}_{ren}$, may be defined by Schwinger’s proper time representation [11]

$$\ln \text{det}_{ren}(1 - eSA) = \frac{1}{2} \int_0^\infty \frac{dt}{t} \{ \text{Tr} \left( e^{-P^2} - \exp \left[ -(D^2 + \frac{e}{2} \sigma_{\mu\nu} F_{\mu\nu}) t \right] \right) + \frac{e^2 ||F||^2}{24\pi^2} \} e^{-tm^2},$$  

(2.1)

where $D_\mu = P_\mu - eA_\mu$, $\sigma_{\mu\nu} = (1/2i) [\gamma_\mu, \gamma_\nu]$, $\gamma_\mu = -\gamma_\mu$, $||F||^2 = \int d^4xF_{\mu\nu}^2(x)$, and $e$ is assumed to be real. We choose the chiral representation of the $\gamma$-matrices so that $\sigma_{ij} = (-\sigma_k \ 0 \ -\sigma_k)$, $i, j, k = 1, 2, 3$ in cyclic order. Since we will consider time-independent magnetic fields we set $A_\mu = (\text{0, } A(x))$ with $x$ in $\mathbb{R}^3$. Then (2.1) reduces to

$$\ln \text{det}_{ren} = \frac{T}{2} \int_0^\infty \frac{dt}{t} \left\{ \frac{2}{(4\pi t)^{1/2}} \text{Tr} \left( e^{-P^2} - \exp \left[ -[(P - eA) - e\sigma \cdot B] t \right] \right) + \frac{e^2 ||B||^2}{12\pi^2} \right\} e^{-tm^2},$$  

(2.2)

where $T$ is the dimension of the time box, and the factor 2 is from the partial spin trace. Clearly we must have $B \in L^2(\mathbb{R}^3)$. If $A$ is assumed to be in the Coulomb gauge $\nabla \cdot A = 0$, then by the Sobolev-Talenti-Aubin inequality [12]

$$\int d^3x B(x) \cdot B(x) \geq \left( \frac{27\pi^4}{16} \right)^{1/3} \sum_{i=1}^3 \left( \int d^3x |A_i(x)|^6 \right)^{1/3}.$$  

(2.3)

So we must also have $A \in L^6(\mathbb{R}^3)$.

In analogy with $\text{det}_{ren}$ in (2.1), without the charge renormalization subtraction, $\text{det}_{QED}_3$ may be defined by

$$\ln \text{det}_{QED}_3(m^2) = \frac{1}{2} \int_0^\infty \frac{dt}{t} \left\{ \text{Tr} \left( e^{-P^2} - \exp \left[ -[(P - eA) - e\sigma \cdot B] t \right] \right) \right\} e^{-tm^2}.$$

(2.4)

This definition and regularization of $\text{det}_{QED}_3$ is parity conserving and gives no Chern-Simons term. Substituting (2.4) in (2.2) and, noting that $\pi^{-1} \int_0^\infty dE e^{-tE^2} = (4\pi t)^{-1/2}$, we obtain [10].
**B. Justification of (1.6)**

Continuing our review of previous work we turn to the derivation of the upper bound on \( \ln \det_{\text{ren}} \) in (1.4). Since the degrees of divergence of the first, second and third-order contributions to \( \ln \det_{\text{QED}} \) are 2, 1 and 0, respectively, these must be dealt with separately. Their definition is obtained from the expansion of (2.4) through \( O(\varepsilon^4) \), resulting in

\[
\ln \det_{\text{ren}} = \frac{2T}{\pi} \int_0^\infty dE \left( \ln \det_{\text{QED}} (E^2 + m^2) + \frac{e^2 |B|^2}{24 \pi \sqrt{M^2 - m^2}} \ln \det_{\text{QED}} (M^2) + \frac{e^2 |B|^2}{24 \pi \sqrt{M^2}} \right).
\]

Result (2.5) will be referred to repeatedly in what follows.

**Theorem 1** [6, 13, 14]. Let the operator \( S \in \mathcal{H} \) be a bounded operator on \( \mathcal{H} \). Theorem 1 [6, 13, 14]. Let the operator \( S \in \mathcal{H} \) be a bounded operator on \( \mathcal{H} \). Theorem 1 [6, 13, 14]. Let the operator \( S \in \mathcal{H} \) be a bounded operator on \( \mathcal{H} \). Then \( \det_n (1 + A) = \det \left( (1 + A) \exp \left( \sum_{k=1}^{n-1} (-1)^k A^k / k \right) \right) \).

**Theorem 2** [13, 17]. Define the regularized determinant

\[
\det_n (1 + A) = \det \left( (1 + A) \exp \left( \sum_{k=1}^{n-1} (-1)^k A^k / k \right) \right).
\]

Then \( \det_n \) can be expressed in terms of the eigenvalues of \( A \in \mathcal{H} \) for \( n \geq 2 \).

Accordingly, \( \det_4 \) is defined and can be represented as

\[
\det_4 (1 - eS \Lambda) = \prod_{n=1}^\infty \left( 1 - e \epsilon_n \right) \exp \left( \sum_{k=1}^3 \left( \frac{e}{\epsilon_n} \right)^k / k \right).
\]

The reality of \( \det_4 \) for real \( e \) and \( C \)-invariance require that the eigenvalues \( \epsilon_n \) appear in the complex plane as quartets \( \pm \epsilon_n, \pm \epsilon_n^* \) or as imaginary pairs when \( m \neq 0 \).

We have established that \( \det_4 |_{e=0} = 1 \) and that \( \det_4 \) has no zeros for real values of \( e \). Therefore, by (2.4), \( \det_4 < 0 \) for all real \( e \), thereby allowing one to go from (1.5) to (1.6). It might be objected that this is obvious, but we will need the detailed information introduced about \( \det_4 \) in the sequel.

The determinant \( \det_4 \) is an entire function of \( e \) considered as a complex variable, meaning that it is holomorphic in the entire complex \( e \)-plane. Since \( \sum_{n=1}^\infty |\epsilon_n|^{-3-\epsilon} < \infty \) for \( \epsilon > 0 \), its order is at most 3 [13]. This means that for any complex value of \( e \), and positive constants \( A, K \), \( |\det_4| < A(e) \exp(K(e)|e|^{3+\epsilon}) \) for any \( \epsilon > 0 \).

From (1.6), (2.4) and for real values of \( e \)

\[
\ln \det_4 \leq \frac{e^2}{4\pi} \int \frac{d^3 k}{(2\pi)^3} |B(k)|^2 \int_0^1 dz \frac{z(1-z)}{|z(1-z)k^2 + m^2|^{1/2}}.
\]

This is a truly remarkable inequality. Referring to (2.5), \( \det_4 \)'s growth is slower on the real \( e \)-axis than its potential growth in other directions. We also note that \( \det_4 \) is largely unknown. Even the reduction of the fourth-order term in its expansion to an explicitly gauge invariant form involving only \( B \)-fields requires a huge effort when the fields are not constant [19]. The sixth-order reduction has not been completed as far as the author knows.

**C. Upper bound on \( \det_{\text{ren}} \)**

Insert (2.4) in (2.4) and get

\[
\ln \det_{\text{ren}} = \frac{e^2 T}{4\pi^2} \int \frac{d^3 k}{(2\pi)^3} |B(k)|^2 \int_0^\infty dz \frac{z(1-z)}{|z(1-z)k^2 + m^2|^{1/2}}.
\]
\[
\ln(\frac{z(1-z)}{m^2}) + \frac{T}{\pi} \int_{m^2}^\infty \frac{dM^2}{\sqrt{M^2 - m^2}} \ln(\det_4(M^2)).
\]

(2.10)

The objective here is to obtain the behavior of \(\ln_{\text{ren}}\) when the log of determinant is large. Since \(e\) always multiplies \(B\) we introduce the scale parameter \(\mathcal{B} = \max |B|\), which has the dimension of \(M^2\). Why \(\mathcal{B}\) is finite will be explained in Sec. III.B. Then the integral in (2.10) is broken up into \(\int \) and \(\int\).

Substitution of (2.9) into the lower range integral gives

\[
\ln(\det_{\text{ren}}) \leq \frac{e^2 T}{4\pi^2} \int \frac{d^3k}{(2\pi)^3} |B(k)|^2 \int_0^1 dz (1-z) \\
\times \ln \left( \frac{4eB + 2z(1-z)k^2 - 2m^2}{m^2} \right) \\
+ \frac{T}{\pi} \int_{eB}^\infty \frac{dM^2}{\sqrt{M^2 - m^2}} \ln(\det_4(eB, M^2)).
\]

(2.11)

We have simplified the argument of the logarithm using \(\sqrt{2m^2} \leq x + y\) for \(x, y \geq 0\). Then for \(eB \gg m^2\)

\[
\ln(\det_{\text{ren}}) \leq \frac{e^2 T |B|^2}{24\pi^2} \ln \left( \frac{4eB}{m^2} \right) \\
+ \frac{T}{\pi} \int_{eB}^\infty \frac{dM^2}{\sqrt{M^2 - m^2}} \ln(\det_4(eB, M^2)) \\
+ O \left( \frac{eT |B| \cdot \nabla B}{\mathcal{B}} \right).
\]

(2.12)

The integral in (2.12) can be estimated by making a large mass expansion of \(\ln(\det_4)\). This is facilitated by inserting (2.10) in (2.2) and examining the small \(t\) region of \(\ln(\det_4)\)'s resulting proper time representation. The details of this expansion are in Sec. 3B of [10], and give the result

\[
\ln(\det_4(eB, M^2)) = \frac{1}{2} \lim_{M \to \infty} \int_0^\infty \frac{dt}{t} (4\pi t)^{-3/2} e^{-tM^2} \\
\times \int d^3x \left[ \frac{2}{45} e^4 t^4 (B \cdot B)^2 + O(e^4 t^2 (B \cdot BB \cdot \nabla B)) \right] \\
= \frac{e^4}{480\pi M^3} \int (B \cdot B)^2 + O \left( \frac{e^4 (B \cdot BB \cdot \nabla B)}{M^7} \right).
\]

(2.13)

In the first line of (2.13) it is assumed that the heat kernel expansion is an asymptotic expansion in \(t\) in the strict sense of its definition, namely [20].

\[
\sim e^{-t([P - eA]^2 - e\sigma B])} |\phi > - (4\pi t)^{-3/2} \sum_{n=0}^N a_n(x)t^n
\]

(2.14)

This must hold for every \(N\). A necessary condition for (2.14) is that \(B \) be infinitely differentiable to ensure that each coefficient \(a_n\) is finite. As far as the author knows it is not known yet if this is a sufficient condition. So (2.14) is an assumption that may require additional conditions on \(B\). Only coefficients \(a_n\) of \((e^{32n})\), \(n \geq 2\) are present in \(\ln(\det_4)\)'s expansion.

The \(t\)-integration in (2.13), although extending to infinity, is limited to small \(t\) since \(M \to \infty\) due to the parameter \(e\mathcal{B}\) in (2.10). Substituting (2.10) in (2.12) results in

\[
\ln(\det_{\text{ren}}) \leq \frac{e^2 |B|^2 T}{24\pi^2} \ln \left( \frac{4eB}{m^2} \right) + O \left( \frac{e^2 T |B| \cdot \nabla B}{\mathcal{B}} \right).
\]

(2.15)

or

\[
\lim_{e \to \infty} \frac{\ln(\det_{\text{ren}})}{e^2 \ln e} \leq \frac{|B|^2 T}{24\pi^2},
\]

(2.16)

consistent with (1.4). This bound is independent of the charge renormalization subtraction point. If the subtraction were made at photon momentum \(k^2 = \mu^2\) instead of \(k^2 = 0\) then the \(ln m^2\) terms in (2.10) and (2.11) would be replaced with \(\ln((1-z)\mu^2 + m^2)\), which has nothing to do with strong coupling.

The scaling procedure used here is designed to obtain the least upper bound on \(\ln(\det_{\text{ren}})\). In [10] we chose to break up the \(M\)-integral as \(\int \frac{dM^2}{\sqrt{M^2 - m^2}}\). This resulted in a fast \(1/e^4\) falloff of the \(\ln(\det_4)\) terms compared to \(e^2\), but gave a weaker upper bound on \(\ln(\det_{\text{ren}})\), namely

\[
\lim_{e \to \infty} \frac{\ln(\det_{\text{ren}})}{e^2 \ln e} \leq \frac{|B|^2 T}{6\pi^2}.
\]

(2.17)

We mention that the coefficient \(1/960\) in (3.10) in [10] should be \(1/360\). Here we might have chosen a more general scaling such as \(e^{\alpha(\ln e)^\beta B}\) or \(e^{\alpha(\ln e)^\beta B} \cdot \mathcal{B}\), etc., with \(\alpha \geq 1, \beta > 0\). Then the right-hand side of (2.19) would have been replaced with \(\alpha|B|^2 T/24\pi^2\). The case \(\alpha < 1\) causes the first remainder term in (2.15) to be no longer subdominant. Therefore, our scaling \(e\mathcal{B}\) is an optimal one.

III. LOWER BOUND ON \(\det_{\text{ren}}\)

A. Fundamentals

On referring to (2.5) the lower bound on \(\det_{\text{ren}}\) will come from operations on \(\ln(\det_{\text{QED}_3})\). We begin with
the operator identity (A2) in Appendix A applied to $\ln \det_{\text{QED}_3}$ in (2.4). Letting $X = (P - eA)^2$ and $Y = -e\sigma \cdot B$ we obtain

$$
\ln \det_{\text{QED}_3} = -\frac{1}{2} \int_0^\infty d\tau \left( e^{\int_0^\tau ds e^{-(t-s)}(P - eA)^2} \times \sigma \cdot B e^{-s(P - eA)^2} e^{s(P - eA)^2} \right) + \frac{1}{2} \int_0^\infty d\tau \left( e^{-(P - eA)^2} - e^{-(P - eA)^2} \right) e^{-t^2 m^2}.
$$

The spin trace in the first term is zero, and the last term, after tracing over spin, is the one-loop effective action of scalar QED$_3$.

$$
\ln \det_{\text{QED}_3} = \int_0^\infty \frac{dt}{t} e^{-tm^2} \text{Tr} \left( e^{-P^2 t} - e^{-(P - eA)^2 t} \right).
$$

Thus,

$$
\ln \det_{\text{QED}_3} = \ln \det_{\text{QED}_3} - \frac{e^2}{2} \int_0^\infty \frac{dt}{t} e^{-tm^2} \text{Tr} \left( \int_0^t ds_1 \right.
\left. \times \int_0^{t-s_1} ds_2 e^{-(t-s_1-s_2)}(P - eA)^2 - e\sigma \cdot B \right)
\left. \times \sigma \cdot B e^{-s_2(P - eA)^2} \sigma \cdot B e^{-s_1(P - eA)^2} \right),
$$

(3.3)

remembering that the factor 1/2 in the last term of (3.1) is cancelled by the spin trace.

Let $\Delta_A = [(P - eA)^2 + m^2]^{-1}$. In Appendix B it is shown that $\Delta_A^{1/2} \sigma \cdot B \Delta_A^{1/2} \in L^2$; that is, it is a Hilbert-Schmidt operator provided $B \in L^2$ and $m \neq 0$. Then (2.7) gives

$$
\ln \det_2 \left( 1 - e\Delta_A^{1/2} \sigma \cdot B \Delta_A^{1/2} \right) = \int_0^\infty \frac{dt}{t} e^{-tm^2} \text{Tr} \left( e^{-t(P - eA)^2} - e^{-(P - eA)^2} \sigma \cdot B \right) e^{-t(P - eA)^2}.
$$

(3.4)

In going from the penultimate to the last line in (3.4) use was again made of the identity (A2). Substituting (3.4) in (3.3) gives

$$
\ln \det_{\text{QED}_3} = \frac{1}{2} \ln \det_2 \left( 1 - e\Delta_A^{1/2} \sigma \cdot B \Delta_A^{1/2} \right) + \ln \det_{\text{SQED}_3}.
$$

(3.5)

As in $\ln \det_{\text{QED}_3}$ and $\ln \det_2$ are well-defined by our choice of fields, so is $\ln \det_{\text{SQED}_3}$ in (3.5). What has been accomplished here is to isolate the Zeeman term $\sigma \cdot B$ in $\ln \det_2$. Since $\Delta_A^{1/2} \sigma \cdot B \Delta_A^{1/2}$ is Hilbert-Schmidt and self-adjoint, $\ln \det_2$ is susceptible to extensive analytic analysis.

Substitute (3.5) in (2.6):

$$
\ln \det_{\text{ren}} = \int \frac{dM^2}{M^2} \left( \frac{1}{2} \ln \det_2 \left( 1 - e\Delta_A^{1/2} \sigma \cdot B \Delta_A^{1/2} \right) \right.
\left. + \ln \det_{\text{SQED}_3} + \frac{e^2|B|^2}{24\pi\sqrt{M^2}} \right).
$$

(3.6)

We now introduce two central inequalities. The first relies on the diamagnetism of charged scalar bosons as expressed by Kato’s inequality in the form $[20, 21]$

$$
\text{Tr} \left( e^{-\Delta_A^{1/2} \sigma \cdot B \Delta_A^{1/2}} \right) \leq \text{Tr} e^{-P^2 t}.
$$

(3.7)

This implies that on average the energy eigenvalues of such bosons rise in a magnetic field and hence by (3.2) that $[22]$

$$
\ln \det_{\text{QED}_3} \geq 0.
$$

(3.8)

The second inequality is introduced beginning with the penultimate line of (3.3). Noting that the spin trace of the $\Delta_A \sigma \cdot B$ term is zero, then

$$
\ln \det_2 \left( 1 - e\Delta_A^{1/2} \sigma \cdot B \Delta_A^{1/2} \right) = \int_0^\infty \frac{dt}{t} e^{-tm^2} \text{Tr} \left( e^{-t(P - eA)^2} - e^{-(P - eA)^2} \sigma \cdot B \right) e^{-t(P - eA)^2}.
$$

(3.9)

By the Bogoliubov-Peierls inequality $[23, 24]$ and Sec. 2.1.8 of [23]

$$
\text{Tr} e^{-t|P - eA|^2 - e\sigma \cdot B}|t| \geq \text{Tr} e^{-t|P - eA|^2} e^{-t\sigma \cdot B}.
$$

(3.10)

where

$$
\sigma \cdot B = \frac{\text{Tr} \left( \sigma \cdot B e^{-t(P - eA)^2} \right)}{\text{Tr} e^{-t(P - eA)^2}} = 0.
$$

Hence,

$$
\ln \det_2 \left( 1 - e\Delta_A^{1/2} \sigma \cdot B \Delta_A^{1/2} \right) \leq 0.
$$

(3.12)

consistent with (3.5) when combined with (1.6) and (3.8).
There is another reason why (3.12) holds. Let \( C = e^{\Delta_A^{1/2} \sigma \cdot B \Delta_A^{1/2}} \). Since \( C \) is Hilbert-Schmidt,
\[
\ln \det_2 (1 - C) = \ln \det [(1 - C)e^C] = \text{Tr} [\ln(1 - C) + C] = \frac{1}{2} \text{Tr} \ln(1 - C^2) = \frac{1}{2} \sum_{n=1}^{\infty} \ln(1 - \lambda_n^2).
\]

The third line of (3.13) follows from the second since the trace over spin eliminates all odd powers of \( C \). In the last line we introduced the real eigenvalues \( \lambda_n \) of \( e^{\Delta_A^{1/2} \sigma \cdot B \Delta_A^{1/2}} \). Since \( \ln \det_2 \) is real and finite then \( |\lambda_n| < 1 \) for all \( n \), giving (3.12). Because \( \Delta_A^{1/2} \sigma \cdot B \Delta_A^{1/2} \in \mathcal{F}_2 \), it is a compact operator, and so the \( \lambda_n \) are countable and of finite multiplicity.

Now consider
\[
\frac{\partial}{\partial m^2} \ln \det_2(m^2) = \int_0^\infty dt \ e^{-tm^2} \times \text{Tr} \left( e^{-[(P - eA)^2 - e\sigma \cdot B]t} - e^{-[(P - eA)^2]t} \right) \geq 0,
\]
by (3.9)-(3.11). Therefore, \( \det_2 \) is a monotonically increasing function of \( m^2 \).

Next, break up the \( M \)-integral in (3.6) as in Sec.II.C:
\[
\ln \det_{ren} = \frac{T}{\pi} \int_{e^{R^2}} \frac{dM^2}{\sqrt{M^2 - m^2}} \left( \frac{1}{2} \ln \det_2 (1 - e^{\Delta_A^{1/2} \sigma \cdot B \Delta_A^{1/2}}) + \ln \det_{SQED_3} + \frac{e^2||B||^2}{24\pi \sqrt{M_2}} \right) + \frac{T}{\pi} \int_{e^{R^2}} \frac{dM^2}{\sqrt{M^2 - m^2}} \left( \ln \det_{QED_3} + \frac{e^2||B||^2}{24\pi \sqrt{M_2}} \right),
\]

where we reinserted (3.5) into the upper-range \( M \)-integral. By (3.14)
\[
\int \frac{dM^2}{\sqrt{M^2 - m^2}} \ln \det_2(M^2) \geq \ln \det_2 \bigg|_{M^2 = m^2} \int \frac{dM^2}{\sqrt{M^2 - m^2}} = 2 \ln \det_2 \left( 1 - e^{\Delta_A^{1/2} \sigma \cdot B \Delta_A^{1/2}} \right)_{M^2 = m^2}.
\]

Hence, (3.8) and (3.13) result in (3.13) becoming
\[
\ln \det_{ren} \geq \frac{T}{\pi} \sqrt{e^{R^2} - m^2} \ln \det_2 \left( 1 - e^{\Delta_A^{1/2} \sigma \cdot B \Delta_A^{1/2}} \right)_{M^2 = m^2} + \frac{e^2T||B||^2}{24\pi^2} \ln \left( \frac{e^{R^2}}{m^2} \right) + \frac{e^2T}{12\pi^2} ||B||^2 \ln \left( 1 + \sqrt{1 - m^2} / e^{R^2} \right) + \frac{T}{\pi} \int_{e^{R^2}} \frac{dM^2}{\sqrt{M^2 - m^2}} \left( \ln \det_{QED_3} + \frac{e^2||B||^2}{24\pi \sqrt{M^2}} \right).
\]

We now turn to the strong coupling behavior of \( \ln \det_2 \).

### B. Strong coupling behavior of \( \ln \det_2 \)

The eigenvalues \( \lambda_n \) in (3.13) are obtained from
\[
e^{\Delta_A^{1/2} \sigma \cdot B \Delta_A^{1/2}} \varphi_n = \lambda_n \varphi_n,
\]
for \( \varphi_n \in L^2 \) following the remark under (B3) in Appendix B. Letting \( \Delta_A^{1/2} \varphi_n = \psi_n \) gives
\[
\left( (P - eA)^2 - \frac{e \sigma \cdot B}{\lambda_n} \right) \psi_n = -m^2 \psi_n,
\]
where \( \psi_n \in L^2 \) provided \( m \neq 0 \). This follows from (B5) and Young’s inequality (B7). The requirement that \( m \neq 0 \) follows from the role of the eigenvalues \( \{ \lambda_n \}_n=1 \) as adjustable coupling constants whose discrete values result in bound states with energy \( -m^2 \) for a fixed value of \( e \). Since the operator \( (P - eA)^2 - e \sigma \cdot B \geq 0 \), such bound states are impossible unless \( |\lambda_n| < 1 \) for all \( n \), which is the physical reason why (3.12) is true. Inspection of (3.19) suggests that as \( e \) increases \( |\lambda_n| \) likewise increases for fixed \( n \) to maintain the bound state energy at \( -m^2 \). This is illustrated by the constant field case that is excluded from our analysis:
\[
|\lambda_n| = \frac{|eB|}{(2n+1)|eB| + m^2}, \quad n = 0, 1, \ldots
\]

Because the operator \( \Delta_A^{1/2} \sigma \cdot B \Delta_A^{1/2} \) is Hilbert-Schmidt the eigenfunction \( \varphi_n \) has finite multiplicity, and the \( \lambda_n \) in (3.13) are counted up to this multiplicity. To estimate the multiplicity note that the eigenfunctions \( \varphi_n \) and \( \psi_n \) are in one-to-one correspondence. Next, note that for \( \psi \in L^2(\mathbb{R}^4; C^2) \) and a generic \( \lambda \) with \( |\lambda| < 1 \),
\[
(\psi, [(P - eA)^2 - \frac{e \sigma \cdot B}{\lambda} \psi] \psi) \geq (\psi, (P - eA)^2 - \frac{|e|}{\lambda} |B| \psi).
\]

Thus the Hamiltonian on the left, \( H_+ \), dominates that on the right, \( H_- \). Let \( N_{-m^2}(H) \) denote the dimension of the spectral projection onto the eigenstates of Hamiltonian \( H \) with eigenvalues less than or equal to \( -m^2 \). Because \( H_+ \geq H_- \) then \( N_{-m^2}(H_+) \leq N_{-m^2}(H_-) \). \( N_{-m^2}(H_+) \) is an overestimate of the number of the bound states of \( H_+ \) at \( -m^2 \) for a fixed value of \( \lambda \) but satisfactory for our purpose here.

By the Cwinkel-Lieb-Rozenblum bound in the form
\[
N_{-m^2}(H_-) \leq C \int d^3x \left[ \frac{e}{\lambda} |B(x)| - m^2 \right]^{3/2},
\]
where \( a_+ = \max(a, 0) \) and \( C = 2 \times 0.1156 \). The factor 2 accounts for the additional spin degrees of freedom in the present estimate. Since \( |\lambda_n| = O(1) \), we are confident that the degeneracy/multiplicity associated with each \( \lambda_n \) in (3.13) does not exceed \( e|e|^{3/2} \int d^3x |B|^{3/2} \), where \( e \geq 0.2312 \) is another finite constant. This estimate has to be modified for values of \( n > N \) beyond
which $\lambda_n$ assumes its asymptotic form as discussed below. Therefore, for $n \leq N$ we will estimate the sum in (3.13) by factoring out the common maximal degeneracy $c(e^{-3/2} \int d^3x |B|^{3/2} )$ and treat each $\lambda_n$ in the factored sum as having multiplicity equal to one. Those $\lambda_n$, if any, that vanish as $e \to \infty$ give a subdominant contribution to $\ln \det_2$ in (3.13) since by inspection their contribution grows at most as $\lambda_n^2/e/\lambda_n^{3/2}$.

We now turn to the large $e$ dependence of $\lambda_n$. From here on we assume that $\psi_n$ is normalized to one. By C-invariance we may assume $e > 0$. Now consider the expectation value of (3.19):

$$< n | (P - eA)^2 | n > = \frac{e}{\lambda_n} < n | \sigma \cdot B | n > = -m^2. \quad (3.23)$$

From (3.23) if $< n | \sigma \cdot B | n > > 0$ then $\lambda_n > 0$ and vice versa. Therefore, we need only consider $\lambda_n > 0$ and write

$$\lambda_n = \left( \frac{< n | (P - eA)^2 | n >}{e} < n | \sigma \cdot B | n > + \frac{m^2}{e} < n | \sigma \cdot B | n > \right)^{-1}, \quad (3.24)$$

where $< n | \sigma \cdot B | n > \equiv 0$ as (3.23) must be satisfied. The case $\lambda_n = 0$ for some $n$ corresponding to $< n | \sigma \cdot B | n > = 0$ can be ignored as $\lambda_n = 0$ contributes nothing to $\ln \det_2$ in (3.13). An easy estimate gives

$$| (\psi_n, \sigma \cdot B \psi_n) | \leq (\psi_n, |B| \psi_n) \leq \max_x |B(x)|. \quad (3.25)$$

Because $B \in L^2$ and is assumed infinitely differentiable then $\max_x |B|$ is finite. Hence, $< n | \sigma \cdot B | n >$ is a bounded function of $e$ and $n$.

Now consider the ratio $R_n = < n | (P - eA)^2 | n > /e < n | \sigma \cdot B | n >$ in (3.24). The case $R_n \to 0$ is ruled out since this implies $\lambda_n \to \infty$. The case $R_n \to \infty$ implies $\lambda_n \to 0$, which gives a subdominant contribution to (3.13) as discussed above. The final possibility is $1 \leq R_n \leq \infty$ for $e \to \infty$. The case $R_n \to 1$ for $e \to \infty$ happens if $< n | (P - eA)^2 | n > > e < n | \sigma \cdot B | n >$. Since $\psi_n \in L^2(\mathbb{R}^3)$, $< n | (P - eA)^2 - e \sigma \cdot B | n > = 0$ implies $\sigma \cdot (P - eA) \psi_n = 0$. Now this may happen for the $B$-fields considered so far. But if we exclude zero-mode supporting $B$ fields from our analysis it cannot. By so doing we can exclude the case $\lambda_n = 1 - \delta_n(e), \delta_n(\infty) = 0$. We will see below why this is necessary.

We proceed to estimate the strong coupling limit of $\ln \det_2$ in (3.13). First, consider the sum for $n \leq N$. We need only consider $0 < |\lambda_n| < 1$ for all $e$, including $e = \infty$ as concluded above. Hence, on factoring out the common maximal multiplicity of the $\lambda_n$ we get

$$\lim_{e \to \infty} \frac{1}{N} \sum_{n=1}^{N} \ln(1 - \lambda_n^2) \leq c_1 e^{3/2} \int d^3x |B|^{3/2}, \quad (3.26)$$

where $c_1$ is a constant and noting again that the eigenvalues $\lambda_n \to 0$ are subdominant.

Since $\lambda_n \to 0$ for $n \to \infty$ and $1/2 \leq \ln(1 - \lambda_n^2)/\lambda_n^2 \leq 3/2$ for $\lambda_n^2 < 1/2$ the absolute convergence of the series in (3.13) requires $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$. Consider this sum for $n > N$ and indicate the degeneracy factors $\mu_n$ explicitly:

$$S \equiv \sum_{n>N} \mu_n(e) \lambda_n^2(e). \quad (3.27)$$

We estimated from (3.22) that $\mu_n \leq c/e/\lambda_n^{3/2} \int d^3x |B|^{3/2}$. So

$$S \leq c e^{3/2} \int d^3x |B|^{3/2} \sum_{n>N} |\lambda_n|^{1/2} < \infty. \quad (3.28)$$

This implies that for $n > N$

$$|\lambda_n(e)| = \frac{C_n(e)}{n^{2+\epsilon}}, \quad (3.29)$$

where $e > 0$ and $C_n$ is a bounded function of $n$ and $e$ with $\lim_{e \to \infty} C_n(e) < \infty$. Otherwise $|\lambda_n| < 1$ for any $n$ cannot be satisfied. Accordingly, the series in (3.28) is uniformly convergent in $e$ by the Weierstrass M-test and so

$$\lim_{e \to \infty} \left| \sum_{n>N} \ln(1 - \lambda_n^2) /e^{3/2} \right| \leq c_2 \int d^3x |B|^{3/2}, \quad (3.30)$$

where $c_2$ is a constant. From (3.13), (3.20) and (3.30) we conclude

$$\lim_{e \to \infty} \left| \ln \det_2(1 - e\Delta_A^{1/2} \sigma \cdot B \Delta_A^{1/2}) /e^{3/2} \right| \leq c_3 \int d^3x |B|^{3/2}, \quad (3.31)$$

where $c_3$ is another constant.

As a check on (3.31) refer to (3.5). For $B \in L^{3/2}(\mathbb{R}^2)$ we found

$$\ln \det_\text{QED}_3 \geq - \frac{Z e^{3/2}}{6\pi} \int d^2x |B(x)|^{3/2}, \quad (3.32)$$

for $B(x) \geq 0$ or $B(x) \leq 0, x \in \mathbb{R}^2$. $Z$ is the dimension of the remaining space box. We know that $\ln \det_2 \leq 0$ and $\ln \det_\text{QED}_3 \geq 0$ in (3.3). Specializing (3.31) to these $B$ fields it is seen that the strong coupling growth of $\ln \det_2$ is consistent with (3.32).

Finally, if zero mode supporting $B$ fields were allowed we would have obtained $\ln \det_2 = O(e^{3/2} \ln \rho(e)), \rho(e) \to 0$, when $|\lambda_n| \sim 1 - \delta_n(e), \delta_n(\infty) = 0$, the logarithm in (3.13) gives an additional factor $\ln \delta_n$. As will be seen below the limit (1.3) requires

$$\lim_{e \to \infty} \ln \det_2/e^{3/2} = \text{finite (zero)}.$$
C. Strong coupling limit of (3.17)

It remains to estimate the large coupling limit of the last term in (3.17),

\[
I = \frac{T}{\pi} \int_0^\infty \frac{dM^2}{\sqrt{M^2 - m^2}} \left( \ln \text{det}_{\text{QED}_3} + \frac{e^2 \|B\|^2}{24\pi\sqrt{M^2}} \right)
\]

\[
= \frac{T}{\pi} \int_0^\infty \frac{dM^2}{\sqrt{M^2 - m^2}} \left( -\frac{e^2}{4\pi} \int \frac{d^4k}{(2\pi)^4} \|B(k)\|^2 \right)
\]

\[
\times \int_0^1 dz \frac{z(1-z)}{[z(1-z)k^2 + M^2]^{\frac{3}{2}}} + \frac{e^2 \|B\|^2}{24\pi\sqrt{M^2}} + \ln \text{det}_4(1 - eS\mathcal{A}),
\]

(3.33)

where we substituted (2.0) for \(\ln \text{det}_{\text{QED}_3}\). Calculation of the first two terms in (3.33) is straightforward. The last term has already been estimated in Sec. II and is given by the second term in (2.15). Hence,

\[
I = O \left( \frac{e^2 T \int_B B \cdot B}{\mathcal{B}^2} \right) + O \left( eT \int_B V^2 B \right).
\]

(3.34)

Taking into account (3.31) and (3.34) we obtain from (3.17)

\[
\lim_{e \to \infty} \frac{\ln \text{det}_{\text{ren}}}{e^2 \ln e} \geq \frac{\|B\|^2 T}{24\pi^2}.
\]

(3.35)

Equations (2.10) and (3.35) therefore establish (1.4).

IV. SUMMARY

The two assumptions underlying (1.4) are first that the continuum limit of the lattice diamagnetic inequality coincides with (1.3), and second that the heat kernel expansion of the Pauli operator in (2.1) is an asymptotic series. These assumptions can and should be proven or falsified.

In addition, the result (1.4) assumes that the vector potential and magnetic field satisfy the following conditions:

- \(B \in L^2(\mathbb{R}^3)\) to define \(\ln \text{det}_{\text{ren}}\) in (2.5) and to ensure that \(\Delta_x^{1/2} \sigma \cdot B \Delta_x^{1/2} \in \mathscr{S}_2\) following Appendix B. In addition \(B \in L^{3/2}(\mathbb{R}^4)\) in order that the degeneracy estimate in (3.22) is defined. To ensure that the bound in (3.31) holds, zero mode supporting \(B\) fields are excluded. Also \(B\) must be infinitely differentiable \((C^\infty)\) to ensure that the expansion coefficients in (2.1) are finite.

- If \(A\) is assumed to be in the Coulomb gauge then by (2.3) \(A \in L^6(\mathbb{R}^3)\). If \(B \in L^{3/2}(\mathbb{R}^3)\) then \(A \in L^3(\mathbb{R}^3)\) by the Sobolev-Talenti-Aubin inequality [12]. In order to define \(\text{det}_{\text{QED}_3}\) it is necessary to assume \(A \in L^r(\mathbb{R}^3), r > 3\), following the discussion under (2.0). If \(A \in L^3(\mathbb{R}^3)\) and \(L^6(\mathbb{R}^3)\), then \(A \in L^r(\mathbb{R}^3), 3 < r < 6\) also. This follows from Hölder’s inequality [28].

Finally, the obvious generalization of (1.4) for an admissible class of fields on \(\mathbb{R}^4\) is

\[
\lim_{e \to \infty} \frac{\ln \text{det}_{\text{ren}}}{e^2 \ln e} \geq \frac{\|B\|^2 T}{48\pi^2},
\]

(4.2)

There is no chiral anomaly term since \(F_{\mu\nu}\) falls off faster than \(1/|x|^2\) and \(\int d^4x \tilde{F}_{\mu\nu} F_{\mu\nu} = \int d^4x \partial_\alpha (\epsilon_{\alpha\beta\mu\nu} A_\beta F_{\mu\nu}) = 0\), where \(\tilde{F}_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}\). Equation (1.2) remains to be verified.

If (1.2) and (1.4) do indeed indicate instability then they are yet another reason why QED should not be considered in isolation.

APPENDIX A

The operator identity on which (5.1) is based is obtained as follows. Let (30)

\[
F_t = e^{-t(X+Y)} e^{tX}.
\]

Then

\[
\frac{dF_t}{dt} = -e^{-t(X+Y)} Ye^{tX}.
\]

Integrating gives

\[
e^{-t(X+Y)} - e^{-tX} = -\int_0^t ds \, e^{-(t-s)(X+Y)} Ye^{-sX},
\]

(A1)

known as Duhamel’s formula. Iterating once gives the required identity:

\[
e^{-t(X+Y)} - e^{-tX} = -\int_0^t \int_0^s ds \, e^{-(t-s)(X+Y)} Ye^{-sX}.
\]
\[ (Kf)(x) = \int K(x,y)f(y)d^3y, \quad f \in L^2, \quad (B1) \]

where

\[ K(x,y) = \langle x|\Delta_{A}^{1/2}\sigma \cdot B\Delta_{A}^{1/2}|y \rangle, \quad (B2) \]

and \( K \in L^2(\mathbb{R}^3 \times \mathbb{R}^3; d^3x \times d^3y) \). Moreover,

\[ ||K||_2^2 = \int ||K(x,y)||^2d^3xd^3y. \quad (B3) \]

If it can be shown that \( K \in L^2 \) then it trivially follows that \( K \) maps \( L^2 \) into itself. So consider

\[ ||K||_{L^2} = 2\sum_i \int d^3xd^3y B_i(x)\Delta_A(x,y)B_i(y)\Delta_A(y,x) \leq 2\sum_i \int d^3xd^3y ||B_i(x)||||\Delta_A(x,y)||||B_i(y)||||\Delta_A(y,x)||. \quad (B4) \]

A form of Kato’s inequality \[8, 21, 31\] asserts that the interacting scalar propagator is bounded by the free propagator

\[ |\Delta_A(x,y)| \leq \Delta(x-y), \quad (B5) \]

where \( \Delta(x) = (4\pi|x|)^{-1}e^{-m|x|} \) in three dimensions. Then

\[ ||K||_{L^2} \leq \frac{1}{8\pi^2} \sum_i \int d^3xd^3y |B_i(x)| \frac{1}{|x-y|^2}e^{-2m|x-y||B_i(y)|}. \quad (B6) \]

By Young’s inequality in the form \[23\]

\[ \left| \int d^3xd^3y f(x)g(x-y)h(y) \right| \leq ||f||_p||g||_q||h||_r, \quad (B7) \]

where \( p^{-1} + q^{-1} + r^{-1} = 2, \quad p, q, r \geq 1, \) and \( ||f||_p = (\int d^3x f(x)^p)^{1/p} \) etc., obtain from (B6) with \( p = r = 2, \quad q = 1 \)

\[ ||K||_2 \leq \frac{1}{8\pi^2} \sum_i ||B_i||^2\int d^3xe^{-2m|x|}/x^2 = ||B||^2(4\pi m)^{-1}. \quad (B8) \]

Therefore, by the theorem that began this appendix \( \Delta_{A}^{1/2}\sigma \cdot B\Delta_{A}^{1/2} \in \mathcal{S}_2 \) when \( m \neq 0 \) and \( B \in L^2 \). We mention that this can be proved even when \( m = 0 \) provided \( B \in L^{3/2} \).

---

[1] W. Heisenberg and H. Euler, Z. Phys. 98, 714 (1936), English translation at physics/0605038
[2] G. V. Dunne, in Ian Kogan Memorial Collection From Fields to Strings: Circumnavigating Theoretical Physics, edited by M. Shifman, A. Vainshtein, and J. Wheeler, Vol. 1 (World Scientific, Danvers, MA, 2005), p.445.
[3] Z. Haba, Phys. Rev. D 29 (1984) 1718.
[4] G. V. Dunne and Q.-h. Wang, Phys. Rev. D 74, 065015 (2006).
[5] D. Brydges, J. Fröhlich, and E. Seiler, Ann. Phys. (N.Y.) 121, 227 (1979).
[6] E. Seiler, Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics, Lecture Notes in Physics Vol. 159 (Springer, Berlin/Heidelberg/New York, 1982).
[7] E. Seiler, in Gauge Theories: Fundamental Interactions and Rigorous Results, Proceedings of the International Summer School of Theoretical Physics, Poiana Brasov, Romania, 1981, edited by P. Dita, V. Georgescu, and R. Purice, Progress in Physics Vol. 5 (Birkhäuser, Boston, 1982), p. 263.
[8] E. Seiler (private communication).
[9] S. L. Adler, Phys. Rev. D 16, 2943 (1977).
[10] M. P. Fry, Phys. Rev. D 54, 6444 (1996).
[11] J. Schwinger, Phys. Rev. 82, 664 (1951).
[12] E. H. Lieb and W. E. Thirring, in Studies in Mathematical Physics, Essays in Honor of Valentine Bargmann, edited by E. H. Lieb, B. Simon, and A. S. Wightman (Princeton University Press, Princeton, NJ, 1976) p.269.
[13] E. Seiler and B. Simon, Commun. Math. Phys. 45, 99 (1975).
[14] B. Simon, Trace Ideals and their Applications, London Mathematical Society Lecture Note Series 35 (Cambridge University Press, Cambridge, England, 1979).
[15] N. Dunford and J. Schwartz, Linear Operators Part II: Spectral Theory (Interscience, New York, 1963).
[16] I. C. Gohberg and M. G. Kreĭn, Introduction to the Theory of Nonselfadjoint Operators, Translations of Mathematical Monographs Vol. 18 (American Mathematical Society, Providence, RI, 1969).
[17] B. Simon, Adv. Math. 24, 244 (1977).
[18] B. Ja. Levin, Distribution of Zeros of Entire Functions, Translations of Mathematical Monographs Vol. 5 (American Mathematical Society, Providence, RI, 1964).
[19] R. Karplus and M. Neuman, Phys. Rev. 80 (1950) 380.
[20] C. M. Bender and S. A. Orszag, Advanced Mathematical Methods for Scientists and Engineers (Springer, New York, 1999).
[21] T. Kato, Israel J. Math. 13, 135 (1972); B. Simon, Math. Z. 131, 361 (1973); M. Schechter, J. Func. Anal. 20, 93
10

(1975); H. Hess, R. Schrader and D. A. Uhlenbrock, Duke Math. J. 44, 893 (1977); B. Simon, Indiana Math. J. 26, 1067 (1977); J. Func. Anal. 32, 97 (1979).

[22] R. Schrader and R. Seiler, Commun. Math. Phys. 61, 169 (1978).

[23] E. H. Lieb, Rev. Mod. Phys. 48, 553 (1976).

[24] H. Hogreve, R. Schrader and R. Seiler, Nucl. Phys. B 142, 525 (1978).

[25] W. Thirring, Lehrbuch der Mathematischen Physik 4: Quantenmechanik grosser Systeme (Springer, Vienna, New York, 1980).

[26] E. H. Lieb, M. Loss and J. P. Solovej, Phys. Rev. Lett. 75 (1995) 985.

[27] M. Loss and H.-T. Yau, Commun. Math. Phys. 104 (1986) 283; C. Adam, B. Muratori and C. Nash, Phys. Rev. D 60 (1999) 125001; ibid. 62 (2000) 085026; ibid. Phys Lett. B 485 (2000) 314; D. Elton, J. Phys. A 33 (2000) 7297; L. Erdös and J. P. Solovej, Rev. Math. Phys. 13 (2001) 1247; A. Balinsky and W. D. Evans, J. Func. Anal. 179 (2001) 120; ibid. Bull. London Math. Soc., 34 (2002) 236; D. Elton, Commun. Math. Phys. 229 (2002) 121; G. V. Dunne and H. Min, Phys. Rev. D 78 (2008) 067701.

[28] M. Reed and B. Simon, Functional Analysis (Academic, New York, 1972).

[29] E. Seiler, Commn. Math. Phys. 42, 163 (1975).

[30] H. L. Cycon, R. G Froese, W. Kirsch and B. Simon, Schrödinger Operators (Springer, Berlin/Heidelberg, 1987).

[31] C. Vafa and E. Witten, Nucl. Phys. B 234, 173 (1984).