Singular elliptic equation involving the GJMS operator on the standard unit sphere.

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Abstract

Given a Riemannian compact manifold \((M, g)\) of dimension \(n \geq 5\), we have proven in [1] under some conditions that the equation:

\[ P_g(u) = Bu^{2^*-1} + \frac{A}{u^{2^*+1}} + C u^p \]  

(1)

where \(P_g\) is GJMS-operator, \(n = \dim(M) > 2k\) (\(k \in \mathbb{N}^*\)), \(A, B\) and \(C\) are smooth positive functions on \(M\), \(p > 1\) and \(2^* = \frac{2n}{n-2k}\) denotes the critical Sobolev of the embedding \(H^2_k(M) \subset L^{2^*}(M)\), admits two distinct positive solutions. The proof of this result is essentially based on the given smooth function \(\varphi > 0\) with norm \(\|\varphi\|_{P_g} = 1\) fulfilling some conditions (see Theorem 3 in [1]). In this note we construct an example of such function on the unit standard sphere \((S^n, h)\). Consequently the conditions of the Theorem are improved in the case of \((S^n, h)\).

1 Construction of the function \(\varphi\)

Inspired by the work of F. Robert. (see [4]), we construct an example of a smooth function \(\varphi > 0\) on the Euclidean sphere \((S^n, h)\) with norm \(\|\varphi\|_{P_h} = 1\). Indeed let \(\lambda > 0\) and \(x_0 \in S^n\). To a rotation, we may assume that \(x_0\) is the north pole i.e. \(x_0 = (0, ... 0, 1)\). We consider the transformation

\[ \phi_\lambda : S^n \rightarrow S^n \]

defined by \(\phi_\lambda(x) = \psi^{-1}_{x_0}(\lambda^{-1} \psi_{x_0}(x))\) if \(x \neq x_0\) and \(\phi_\lambda(x_0) = x_0\) where \(\psi_{x_0}\) is the stereographic projection of \(x_0\) given by

\[ \psi_{x_0} : \left(S^n \setminus \{x_0\}, h\right) \rightarrow \left(\mathbb{R}^n, \xi\right), \]
for any \( a = (\eta_1, \ldots, \eta_n, \zeta) \) associates \( \psi_{x_0}(a) = \left( \frac{\eta_1}{1 - \zeta}, \ldots, \frac{\eta_n}{1 - \zeta} \right) \) and

\[
\delta_{\lambda}: (\mathbb{R}^n, \xi) \to (\mathbb{R}^n, \xi)
\]

\[x \mapsto \delta_{\lambda}(x) = \frac{1}{\lambda} x\]

is the homothetic mapping. \( h \) is the canonical metric on \( \mathbb{S}^n \) and \( \xi \) is the Euclidean one on \( \mathbb{R}^n \).

Note that \( \psi_{x_0} \) is a conformal, mapping more precisely we have

\[
(\psi_{x_0}^{-1})^* h = U^\frac{4}{2n - 2k} \xi
\]

where \( U(x) = \left( \frac{1 + ||x||^2}{2} \right)^{k - \frac{n}{2}} \). Hence \( \phi_{\lambda} \) is conformal i.e.

\[
\phi_{\lambda}^* h = u_{x_0, \beta}^{n - 2k} h \quad \text{where} \quad \beta = \frac{1 + \lambda^2}{\lambda^2 - 1}
\]

and

\[
u_{x_0, \beta}(x) = \left( \frac{\sqrt{\beta^2 - 1}}{\beta - \cos d_h(x, x_0)} \right)^{\frac{n - 2k}{2}} \forall x \in \mathbb{S}^n \text{ with } \beta > 1.
\]

In particular we have

\[
\int_{\mathbb{S}^n} u_{x_0, \beta}^{2t} dv_h = \omega_n \tag{2}
\]

where \( \omega_n > 0 \) is the volume of the unit standard sphere \((\mathbb{S}^n, h)\).

By the conformal invariance of the operator \( P_h \) on \((\mathbb{S}^n, h)\), we obtain that

\[
P_h(u_{x_0, \beta}) = \frac{n - 2k}{2} Q_h u_{x_0, \beta}^{2t - 1} \tag{3}
\]

where \( Q_h \) denotes the \( Q \)-curvature of \((\mathbb{S}^n, h)\) which expresses by the Gover’s formula as:

\[
Q_h = \frac{2}{n - 2k} P_h(1) = \frac{2}{n - 2k} (-1)^k \prod_{l=1}^k (c_l \cdot Sc)
\]

where \( c_l = \frac{(n + 2l - 2)(n - 2l)}{4n(n - 1)} \), \( Sc = n(n - 1) \) (the scalar curvature of \((\mathbb{S}^n, h)\)).

So the \( Q_h \) is a positive constant.

Multiplying the two sides of (3) by \( u_{x_0, \beta} \) and integrating on \( \mathbb{S}^n \) we get:

\[
\int_{\mathbb{S}^n} u_{x_0, \beta} P_h(u_{x_0, \beta}) dv_h = \frac{n - 2k}{2} Q_h \int_{\mathbb{S}^n} u_{x_0, \beta}^{2t} dv_h.
\]
And since
\[ \int_{S^n} u_{x_0,\beta} P_h(u_{x_0,\beta}) dv_h = \| u_{x_0,\beta} \|_{P_h}^2 \]
(3) writes
\[ \| u_{x_0,\beta} \|_{P_h}^2 = \frac{n - 2k}{2} Q_h \omega_n. \]
Hence by putting
\[ \varphi = \left( \frac{n - 2k}{2} \omega_n Q_h \right)^{\frac{1}{n}} u_{x_0,\beta} \]
we obtain a function satisfying the conditions of Theorem 3 in [1] i.e. \( \varphi > 0 \)
smooth on \((S^n, h)\) such that \( \| \varphi \|_{P_g} = 1 \)

### 2 Existence results on the sphere

On the standard unit sphere \((S^n, h)\), if we take the function \( \varphi \) of Theorem 3 in [1] equals \( \left( \frac{n - 2k}{2} \omega_n Q_h \right)^{\frac{1}{n}} u_{x_0,\beta} \), we obtain

**Theorem 1** Let \( (S^n, h) \) be the unit standard unit sphere of dimension \( n > 2k, \ k \in \mathbb{N}^* \). There is a constant \( C(n, p, k) > 0 \) depending only on \( n, p, k \) such that

\[ \frac{1}{2^p} \left( \frac{n - 2k}{2} \omega_n Q_h \right)^{\frac{3}{2}} \int_{S^n} A(x) u_{x_0,\beta}^2 dv_h \leq C(n, p, k) \left( \max_{x \in S^n} B(x) \right)^{\frac{2 + p^2}{2 - 2^p}} \]
(4)

and

\[ \frac{1}{p - 1} \left( \frac{n - 2k}{2} \omega_n Q_h \right)^{\frac{p - 1}{2}} \int_{S^n} C(x) u_{x_0,\beta}^{p - 1} dv_h \leq C(n, p, k) \left( \max_{x \in S^n} B(x) \right)^{\frac{p + 1}{2 - 2^p}} \]
(5)

where
\[ u_{x_0,\beta}(x) = \left( \frac{\sqrt{\beta^2 - 1}}{\beta - \cos d_h(x, x_0)} \right)^{\frac{n - 2k}{2}} \forall x \in S^n \text{ and } \beta > 1. \]

then the equation \([1]\) admits a solution of class \( C^\infty(S^n) \). If moreover for any \( \varepsilon \in ]0, \lambda^*[, \ \lambda^* \) is a positive constant the two following conditions are satisfied

\[ \frac{2}{a} \left( \int_{S^n} \sqrt{A(x)} dv_h \right)^2 \left( \frac{1}{l_0 a_1} \right)^{2^2} > 2^2 k \frac{l_0^2}{4n} (2 - a) \]

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and
\[
\left(\frac{2}{a}\right)^{\frac{p-1}{2}} \left(\int_{S^n} \sqrt{C(x)} \, dv_h\right)^2 \left(\frac{1}{l_0 a_2}\right)^{p-1} > (p - 1)k_{l_0}^2 \frac{t_0^2}{4n}(2 - a)
\]

where \(a_1, a_2\) are positive constants, \(2^\sharp = \frac{2n}{n - 2k}\), \(3 < p < 2^\sharp + 1\). Then the equation (1) admits a second solution.

Note that since
\[
\left(\beta - 1\right)_{\frac{n-2k}{\beta + 1}} \leq u_{x_0,\beta}(x) \leq \left(\frac{\beta + 1}{\beta - 1}\right)_{\frac{n-2k}{\beta + 1}}
\]
we can improve the conditions (4) and (5) of Theorem 1. Indeed, from (??) we deduce that
\[
\varphi(x) \geq \left(\frac{\beta - 1}{\beta + 1}\right)^{\frac{n-2k}{\beta + 1}} \left(\frac{n - 2k}{2} \omega_n Q_h\right)^{-\frac{1}{2}}.
\]

Consequently
\[
\frac{\|\varphi\|^{2^\sharp}}{2^\sharp} \int_{S^n} A(x) \, dv_h = \frac{1}{2^\sharp} \int_{S^n} \frac{A(x)}{\varphi^{2^\sharp}} \, dv_h \leq \frac{1}{2^\sharp} \left(\frac{\beta + 1}{\beta - 1}\right)^{\frac{n}{2}} \left(\frac{n - 2k}{2} \omega_n Q_h\right)^{\frac{2^\sharp}{\beta}} \int_{S^n} A(x) \, dv_h
\]

So, if
\[
\frac{1}{2^\sharp} \left(\frac{\beta + 1}{\beta - 1}\right)^{\frac{n}{2}} \left(\frac{n - 2k}{2} \omega_n Q_h\right)^{\frac{2^\sharp}{\beta}} \int_{S^n} A(x) \, dv_h \leq C(n, p, k) \left(S_{\max} B(x)\right)^{\frac{2^\sharp + 2^\sharp}{2 - 2^\sharp}}
\]

then the condition (4) is fulfilled. Likewise if
\[
\frac{1}{p - 1} \left(\frac{n - 2k}{2} \omega_n Q_h\right)^{\frac{p-1}{2}} \left(\frac{\beta + 1}{\beta - 1}\right)^{\frac{n(p-1)}{2^\sharp}} \int_{S^n} C(x) \, dv_h \leq C(n, p, k) \left(S_{\max} B(x)\right)^{\frac{p+1}{2 - 2^\sharp}}
\]

the condition (5) is also true and we deduce the following result:

**Corollary 2** Let \((S^n, h)\) be the unit standard unit sphere of dimension \(n > 2k, k \in \mathbb{N}^*\). There is a constant \(C(n, p, k) > 0\) depending only on \(n, p, k\) such that
\[
\frac{1}{2^\sharp} \left(\frac{\beta + 1}{\beta - 1}\right)^{\frac{n}{2}} \left(\frac{n - 2k}{2} \omega_n Q_h\right)^{\frac{2^\sharp}{\beta}} \int_{S^n} A(x) \, dv_h \leq C(n, p, k) \left(S_{\max} B(x)\right)^{\frac{2^\sharp + 2^\sharp}{2 - 2^\sharp}}
\]
and
\[
\frac{1}{p-1} \left( \frac{n-2k}{2} \omega_n. Q_n h \right)^{p-1} \left( \beta + 1 \right) \frac{n(p-1)}{2} \int_{S^n} C(x) dv_h \leq C (n, p, k) \left( \max_{x \in M} B(x) \right)^{p+1} \]
\[(8)\]

where \( \beta > 1 \). Then the equation (1) admits a solution of class \( C^\infty(S^n) \).

If moreover for any \( \varepsilon \in ]0, \lambda^*[, \) where \( \lambda^* \) is a positive constant, the two following assumptions are satisfied

\[
\frac{2}{a} \left( \int_{S^n} \sqrt{A(x) dv_h} \right)^2 \left( \frac{1}{\lambda_0 a_1} \right)^{2^\varepsilon} > 2^\varepsilon k \frac{\lambda_0^2}{4n} (2-a) \]
\[(9)\]

and

\[
\left( \frac{2}{a} \right)^{p-1} \left( \int_{S^n} \sqrt{C(x) dv_h} \right)^2 \left( \frac{1}{\lambda_0 a_2} \right)^{p-1} > (p-1)k \frac{\lambda_0^2}{4n} (2-a) \]
\[(10)\]

where \( a_1, a_2 \) are positive constants, \( 2^\varepsilon = \frac{2n}{n-2k}, 3 < p < 2^\varepsilon + 1 \). Then the equation (1) admits a second solution.

References

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