On the existence of positive solutions for a quasilinear Schrödinger equation*

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Abstract

This paper is concerned with the quasilinear Schrödinger equation

\[-\Delta u + V(x)u - \Delta (u^2)u = h(u), \quad \text{in} \ \mathbb{R}^N,\]

where \(N \geq 3\). Under appropriate assumptions on \(V\) and \(h\), we establish the existence of positive solutions. The main novelty is that, unlike most other papers on this problem, we do not assume \(h\) is 4-superlinear at infinity.

Keywords: Quasilinear Schrödinger equation, radial potential, well potential, positive solution.

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1 Introduction and main results

In this paper, we consider the quasilinear Schrödinger equation

\[-\Delta u + V(x)u - \Delta (u^2)u = |u|^{p-2}u, \quad \text{in} \ \mathbb{R}^N,\] (1.1)

where \(N \geq 3, \ 2 < p < 2 \cdot 2^*, \ 2^* = \frac{2N}{N-2}\) is the critical Sobolev exponent and \(V\) is a continuous function. It is known that, via the ansatz \(z(t, x) = e^{-iEt}u(x)\), solutions of problem (1.1) correspond to stationary waves of

\[i\partial_t z = -\Delta z + W(x)z - \Delta (|z|^2)z - |z|^{p-2}z, \quad \text{in} \ \mathbb{R} \times \mathbb{R}^N,\]

where \(W = V + E\) is a new potential. Quasilinear Schrödinger equations of this type arise in plasma physics, see e.g. [15, 16] for details on the physical background.

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The natural energy functional corresponding to (1.1) is given by

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx + \int_{\mathbb{R}^N} u^2|\nabla u|^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p \, dx,$$

which is not well defined in $H^1(\mathbb{R}^N)$. Due to this fact, the usual variational methods can not be applied directly. This difficulty makes problems like (1.1) interesting and challenging. Indeed, during the last ten years, there have been a considerable amount of researches on such problems. Many existence and multiplicity results were proved by different approaches, such as minimizations [19, 24], change of variables [6, 10, 17], Nehari method [18], and perturbation method [22]. In [17], by a suitable change of variables, the quasilinear problem (1.1) was reduced to a semilinear one and existence results were given in the cases of bounded, radial or coercive potential in an Orlicz space framework. By similar change of variables, Colin and Jeanjean [6] investigated the new functional in $H^1(\mathbb{R}^N)$. They established the existence of solutions of (1.1) with $V(x) \equiv 1$ and the general nonlinearity introduced by Berestycki and Lions [3]. Moreover, under the following variant Ambrosetti-Rabinowitz condition (AR) there exists $\mu > 4$ such that $0 < \mu \int_0^t h(s) \, ds \leq h(t)t$ for all $t \in \mathbb{R}^+$, the existence result was also obtained for the well potential and the power nonlinearity $|u|^{p-2}u$ replaced by $h$.

It is worth pointing out that most of these results are based on the condition $4 \leq p < 2 \cdot 2^*$. As observed in [18], the number $2 \cdot 2^*$ behaves as a critical exponent for problem (1.1). In fact, nonexistence result can be formulated when $p \geq 2 \cdot 2^*$ by a Pohozaev type identity.

To the best of our knowledge, very few results are known about problem (1.1) with $p \in (2, 4)$. We are only aware of the papers [6, 11, 19, 24, 25]. In [19, 24], an unknown Lagrange multiplier appears in the equation. In [6], an existence result was given for constant potential. In [11], semiclassical solution of (1.1) was studied and it was shown that there exists a positive solution which concentrates at a local minimum of the potential. Recently, Ruiz and Siciliano [25] proved the existence of a positive ground state solution of (1.1) with $2 < p < 2 \cdot 2^*$. The proof is based on a constrained minimization procedure. We remark that a concavity hypothesis was imposed on the potential, which is technique and important in their arguments. For more references related to (1.1), we refer the reader to [8, 9, 12, 20, 21, 23, 26, 29].

Inspired by [6, 25], we consider the following equation

$$- \Delta u + V(x)u - \Delta(u^2)u = h(u), \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$ and $V \in C^1(\mathbb{R}^N, \mathbb{R})$ satisfies the following conditions:

(V$_1$) $V(x) = V(|x|)$ and $0 < V_0 \leq V(x) \leq V_1 < \infty$ for all $x \in \mathbb{R}^N$;
$(V'_1)$ $0 < V_0 \leq V(x) \leq V_\infty := \lim_{|x| \to \infty} V(x) < \infty$ for all $x \in \mathbb{R}^N$;

$(V_2)$ there exists $\alpha \in [1, 2)$ such that

$$(\nabla V(x) \cdot x)^+ \in L^{2^*'}(\mathbb{R}^N),$$

where $(\nabla V(x) \cdot x)^+ = \max\{\nabla V(x) \cdot x, 0\}$.

For the nonlinearity $h$, we assume:

$(h_1)$ $h \in C(\mathbb{R}, \mathbb{R})$, $h(t) = 0$ for $t \leq 0$ and $h(t) = o(t)$ as $t \to 0^+$;

$(h_2)$ there exists $C > 0$ and $q \in (2, 2 \cdot 2^*)$ such that

$$|h(t)| \leq C(t + t^{q-1}), \text{ for all } t \in \mathbb{R}^+;$$

$(h_3)$ $\lim_{t \to +\infty} \frac{h(t)}{t} = +\infty$;

$(h_4)$ $H(t) = \int_0^t h(s)ds \geq 0$ for all $t \in \mathbb{R}^+$.

The first two results of this paper are the following theorems.

**Theorem 1.1.** Suppose that $(V_1)$, $(V_2)$ and $(h_1) - (h_4)$ hold. Then problem $(1.2)$ has at least a positive solution.

**Theorem 1.2.** Suppose that $(V'_1)$, $(V_2)$ and $(h_1) - (h_4)$ hold. Then problem $(1.2)$ has at least a positive solution.

**Remark 1.1.** (1) Compared with [25], the current paper deals with a more general nonlinearity and conditions on the potential are different. Especially, we do not need concavity hypothesis which is essential in their arguments. Thus our results can be regarded as complements of Theorem 1.1 in [25].

(2) Condition $(V_2)$ shall be used to prove the boundedness of a special Palais-Smale sequence. Similar conditions can be found in [2, 14]. It should be mentioned that, due to the well properties of the transformation (see Lemma 2.1 and Corollary 2.2), we only need a weaker condition than the one in [14].

(3) We point out that $(h_4)$ is assumed for the sake of simplicity and it can be dropped. In fact, by $(h_1)$ and $(h_3)$, we can choose a large positive constant $M$ such that $h(t) + Mt \geq 0$ for all $t \in \mathbb{R}^+$. Then, applying the arguments in this paper to the equation

$$-\Delta u + (V(x) + M)u - \Delta (u^2)u = h(u) + Mu, \text{ in } \mathbb{R}^N,$$

we obtain a positive solution of $(1.2)$.

(4) In Section 6, we apply our methods to problem $(1.2)$ with a nonlinearity of Berestycki and Lions type [3], see Theorem 6.1.
The second part of this paper is devoted to problem (1.2) with a parameter

\[- \Delta u + V(x)u - \Delta(u^2)u = \mu h(u), \quad \text{in } \mathbb{R}^N,\]  

where \( N \geq 3 \) and \( V \in C^1(\mathbb{R}^N, \mathbb{R}) \) satisfies (\( V_1 \)) and

(\( V_2 \)) \( \nabla V(x) \cdot x \leq 0 \) for all \( x \in \mathbb{R}^N \).

For the nonlinearity \( h \), we only assume the following conditions near the origin:

(\( h'_1 \)) \( h \in C(\mathbb{R}, \mathbb{R}) \), \( h(t) = 0 \) for \( t \leq 0 \) and there exists \( q \in (2, 2^* ) \) such that

\[ \limsup_{t \to 0^+} \frac{h(t)t}{t^q} < +\infty; \]

(\( h'_2 \)) there exists \( p \in (2, 2^* ) \) such that

\[ \liminf_{t \to 0^+} \frac{H(t)}{t^p} > 0. \]

**Theorem 1.3.** Suppose that (\( V_1 \)), (\( V'_2 \)) and (\( h'_1 \)) - (\( h'_2 \)) hold. If

\[ p - q < \frac{2(p - 2)(2^* - p)}{2^*(2^* - 2) - 2(p - 2)}, \]

then there exists \( \mu_0 > 0 \) such that, for any \( \mu > \mu_0 \), problem (1.3) has at least a positive solution.

**Remark 1.2.** (1) From (\( h'_1 \)) and (\( h'_2 \)), it is easy to see that \( q \leq p \). Moreover, assumption (1.4) holds if \( q \) is close to \( p \).

(2) In Theorem 1.3, there is no condition assumed on the nonlinearity near infinity. Similar assumptions were used in [5, 7] for the semilinear elliptic problems on a bounded domain.

To prove Theorems 1.1 and 1.2, we are faced with several difficulties. On one hand, due to the presence of \( \Delta(u^2)u \) and growth condition on the nonlinearity, the natural energy functional is not well defined in \( H^1(\mathbb{R}^N) \). Thus we can not apply variational methods directly. To overcome this difficulty, we employ an argument developed in [6, 17] and make a change of variables to reformulate the problem into a semilinear one.

On the other hand, it will be shown later that the functional \( I \) associated to equivalent semilinear problem possesses the Mountain Pass geometry (see Lemma 3.2) and so there exists a Palais-Smale sequence. However, the boundedness of Palais-Smale sequence seems hard to verify. Our strategy is applying Jeanjean’s monotonicity trick [13], which can be traced back to [28], to find a bounded Palais-Smale sequence for \( I \). More precisely, we will take the following three steps. Firstly, we define a family of
functionals $I_{\lambda}$, $\lambda \in [\frac{1}{2}, 1]$, such that $I_1 = I$. By an abstract result in [13], for almost every $\lambda \in [\frac{1}{2}, 1]$, there is a bounded Palais-Smale sequence for $I_{\lambda}$. Secondly, restricting in the subspace of radially symmetric functions if $V$ is a radial potential or using a version of global compactness lemma due to Adachi and Watanabe [1] when $V$ is a well potential, we obtain a nontrivial critical point $v_{\lambda}$ of $I_{\lambda}$ for almost every $\lambda \in [\frac{1}{2}, 1]$. Finally, choosing $\lambda_n \to 1$, we have a sequence of $\{v_{\lambda_n}\}$ being the critical points of $I_{\lambda_n}$.

The proof of Theorem 1.3 is based on the conclusion of Theorem 1.1 and a priori estimate. Firstly, we modify $h$ to a new nonlinearity $\tilde{h}$ which satisfies $(h_1) - (h_4)$. In view of Theorem 1.1 the modified problem has a positive solution. Secondly, it will be shown that the solution obtained converges to zero in $L^\infty$-norm as $\mu \to \infty$. Thus, for $\mu$ large, it is a solution of original problem. This method is borrowed from Costa and Wang [7]. But here we have to analyse carefully the effect of the term $\Delta(u^2)u$ and the transformation $f$.

The paper is organized as follows. In Section 2, following the method in [6, 17], we reformulate (1.2) into a semilinear problem. We give the proofs of Theorems 1.1−1.3 in Sections 3−5 respectively. The last section is devoted to a generalized result.

**Notations:** In the sequel, $C$ and $C_i$ represent variant positive constants. The standard norms of $L^p(\mathbb{R}^N)$ ($p \geq 1$) and $H^1(\mathbb{R}^N)$ are denoted by $|\cdot|_p$ and $||\cdot||$ respectively. Set $H^1_r(\mathbb{R}^N) = \{ u \in H^1(\mathbb{R}^N) \mid u$ is radially symmetric $\}$.

## 2 Equivalent variational problem

The natural energy functional associated to (1.2) is

$$ J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx + \int_{\mathbb{R}^N} u^2|\nabla u|^2 \, dx - \int_{\mathbb{R}^N} H(u) \, dx, $$

which is not well defined for all $u \in H^1(\mathbb{R}^N)$. To apply variational methods, we employ an argument developed in [6, 17] and make a change of variables.

Let $f$ be defined by

$$ f'(t) = \frac{1}{\sqrt{1 + 2f^2(t)}} \quad \text{and} \quad f(0) = 0 $$

on $[0, +\infty)$ and by $f(t) = -f(-t)$ on $(-\infty, 0]$. Then $f$ is uniquely defined, smooth and invertible. In next lemma, we summarize some properties of $f$ which have been proved in [6, 17].

**Lemma 2.1.** (1) $|f'(t)| \leq 1$ for all $t \in \mathbb{R}$;
(2) $|f(t)| \leq |t|$ for all $t \in \mathbb{R}$;
(3) $|f(t)| \leq 2^{\frac{1}{2}}|t|^{\frac{1}{2}}$ for all $t \in \mathbb{R}$;
(4) \( \lim_{t \to 0} \frac{f(t)}{t} = 1 \);
(5) \( \lim_{t \to +\infty} \frac{f(t)}{t} = 2 \frac{1}{6} \);
(6) \( \frac{1}{2} f(t) \leq tf'(t) \leq f(t) \) for all \( t \in \mathbb{R}^+ \);
(7) \( \frac{1}{2} f^2(t) \leq f(t) f'(t) \leq f^2(t) \) for all \( t \in \mathbb{R} \);
(8) there exists a positive constant \( C \) such that
\[
|f(t)| \geq \begin{cases} 
C|t|, & \text{if } |t| \leq 1, \\
C|t|^{\frac{1}{2}}, & \text{if } |t| \geq 1.
\end{cases}
\]

As a consequence of Lemma 2.1 we have

**Corollary 2.2.** Let \( \alpha \in [1, 2) \), then \( |f(t)| \leq 2 \frac{1}{6} |t|^{\frac{k}{6}} \) for all \( t \in \mathbb{R} \).

Set \( v = f^{-1}(u) \), then we obtain
\[
I(v) := J(f(v)) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)f^2(v)) \, dx - \int_{\mathbb{R}^N} H(f(v)) \, dx,
\]
which is well defined in \( H^1(\mathbb{R}^N) \) and belongs to \( C^1 \) under our assumptions. It is well known that critical points of \( I \) are weak solutions of semilinear elliptic equation
\[
-\Delta v = h(f(v)) f'(v) - V(x)f(v)f'(v), \quad \text{in } \mathbb{R}^N.
\]
Moreover, if \( v \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N) \) is a critical point of \( I \), then \( u = f(v) \) is a classical solution of problem (1.2).

### 3 Proof of Theorem 1.1

As stated in the introduction, since we do not assume \( h \) is 4-superlinear at infinity, it seems hard to prove the boundedness of Palais-Smale sequence. We will use the following abstract result [13] to construct a special Palais-Smale sequence.

**Proposition 3.1.** Let \( X \) be a Banach space equipped with a norm \( \| \cdot \|_X \) and let \( J \subset \mathbb{R}^+ \) be an interval. We consider a family \( \{ \Phi_\lambda \}_{\lambda \in J} \) of \( C^1 \)-functionals on \( X \) of the form
\[
\Phi_\lambda(v) = A(v) - \lambda B(v), \quad \text{for all } \lambda \in J,
\]
where \( B(v) \geq 0 \) for all \( v \in X \) and either \( A(v) \to +\infty \) or \( B(v) \to +\infty \) as \( \|v\|_X \to \infty \). Assume that there exist two points \( v_1, v_2 \in X \) such that
\[
c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\lambda(\gamma(t)) > \max\{\Phi_\lambda(v_1), \Phi_\lambda(v_2)\}, \quad \text{for all } \lambda \in J,
\]
where \( \Gamma = \{ \gamma \in C([0,1], X) \mid \gamma(0) = v_1, \gamma(1) = v_2 \} \). Then, for almost every \( \lambda \in J \), there exists a sequence \( \{v_n(\lambda)\} \subset X \) such that
(1) \( \{v_n(\lambda)\} \) is bounded in \( X \);
(2) \( \Phi_\lambda(v_n(\lambda)) \to c_\lambda \);
(3) \( \Phi_\lambda(v_n(\lambda)) \to 0 \) in \( X^* \), where \( X^* \) is the dual space of \( X \).
Furthermore, the map \( \lambda \mapsto c_\lambda \) is continuous from the left.
To apply Proposition 3.1, we set \( X = H^1_0(\mathbb{R}^N) \) and introduce a family of functionals

\[
I_\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)f^2(v)) \, dx - \lambda \int_{\mathbb{R}^N} H(f(v)) \, dx, \quad v \in X,
\]

where \( \lambda \in [\frac{1}{2}, 1] \).

Define \( A(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)f^2(v)) \, dx \) and \( B(v) = \int_{\mathbb{R}^N} H(f(v)) \, dx \). Then \( I_\lambda(v) = A(v) - \lambda B(v) \). Next lemma ensures that \( I_\lambda \) satisfies all assumptions of Proposition 3.1.

**Lemma 3.2.** Assume that \((V_1)\) and \((h_1) - (h_4)\) hold. Then

1. \( B(v) \geq 0 \) for all \( v \in X \);
2. \( A(v) \to \infty \) as \( \|v\| \to \infty \);
3. there exists \( v_0 \in X \), independent of \( \lambda \), such that \( I_\lambda(v_0) < 0 \) for all \( \lambda \in [\frac{1}{2}, 1] \);
4. for all \( \lambda \in [\frac{1}{2}, 1] \), it holds

\[
c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(0), I_\lambda(v_0)\},
\]

where \( \Gamma = \{ \gamma \in C([0,1], X) \mid \gamma(0) = 0, \gamma(1) = v_0 \} \).

**Proof.** (1) is a direct consequence of \((h_4)\). Now we prove (2). By Lemma 2.1, we deduce

\[
\|v\|^2 = \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \int_{\{x: |v(x)| \leq 1\}} v^2 \, dx + \int_{\{x: |v(x)| > 1\}} v^2 \, dx
\]

\[
\leq \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + C \int_{\{x: |v(x)| \leq 1\}} f^2(v) \, dx + \int_{\{x: |v(x)| > 1\}} |v|^{2*} \, dx
\]

\[
\leq \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + C_1 \int_{\mathbb{R}^N} V(x)f^2(v) \, dx + C_2 \left( \int_{\mathbb{R}^N} |\nabla v|^2 \, dx \right)^{2*} \right)^{\frac{2}{2}}
\]

\[
\leq C_3 \left( A(v) + A(v)^{\frac{2*}{2}} \right),
\]

which implies the coercivity of \( A \).

In order to prove (3), we set

\[
J_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx + \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx - \lambda \int_{\mathbb{R}^N} H(u) \, dx.
\]

Let us fix some nonnegative radially symmetric function \( u \in C^\infty_0(\mathbb{R}^N) \) \( \setminus \{0\} \). Then, for \( t > 0 \), we have

\[
J_{1/2}(tu(x/t)) = \frac{t^N}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{t^{N+2}}{2} \int_{\mathbb{R}^N} V(tx)u^2 \, dx
\]

\[
+ t^{N+2} \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx - \frac{t^N}{2} \int_{\mathbb{R}^N} H(tu) \, dx
\]

\[
\leq \frac{t^{N+2}}{2} \left[ \frac{1}{t^2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} V_1 u^2 \, dx
\]

\[
+ 2 \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} \frac{H(tu)}{t^2} \, dx \right].
\]
By assumption \((h_3)\), it is easy to see that \(J_{1/2}(tu(x/t)) < 0\) for \(t\) large. Thus there exists \(v_0 = f^{-1}(u_0) \in X\) (independent of \(\lambda \in [\frac{1}{2}, 1]\)) such that \(I_\lambda(v_0) = J_\lambda(u_0) \leq J_{1/2}(u_0) < 0\) for all \(\lambda \in [\frac{1}{2}, 1]\).

It remains to prove (4). Define \(\tilde{H}(t) = -\frac{V_0}{2} t^2 + H(f(t))\). Using \((h_1), (h_2)\) and Lemma 2.1, we obtain

\[
\lim_{t \to 0} \frac{\tilde{H}(t)}{t^2} = -\frac{V_0}{2} \quad \text{and} \quad \lim_{t \to \infty} \frac{\tilde{H}(t)}{|t|^{2^*}} = 0.
\]

Thus there exists \(C > 0\) such that

\[
\tilde{H}(t) \leq -\frac{V_0}{4} t^2 + C|t|^{2^*}, \quad \text{for all } t \in \mathbb{R}.
\]

It follows that

\[
I_\lambda(v) \leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V_0 f^2(v) \, dx - \int_{\mathbb{R}^N} H(f(v)) \, dx
\]
\[
\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{V_0}{4} \int_{\mathbb{R}^N} v^2 \, dx - C \int_{\mathbb{R}^N} |v|^{2^*} \, dx
\]
\[
\geq \min\left\{\frac{1}{2}, \frac{V_0}{4}\right\} \|v\|^2 - C\|v\|^2^*.
\]

From this, we get \(c_\lambda > 0\) and the proof is complete.

By Lemma 3.2 and Proposition 3.1 there exists \(\mathcal{J}_1 \subset [\frac{1}{2}, 1]\) with \(\text{meas}(\mathcal{J}_1) = 0\) such that, for any \(\lambda \in [\frac{1}{2}, 1] \setminus \mathcal{J}_1\), there is a sequence \(\{v_n\} \subset X\) satisfying

\[(i) \{v_n\} \text{ is bounded in } X, \quad (ii) I_\lambda(v_n) \to c_\lambda, \quad (iii) I'_\lambda(v_n) \to 0 \text{ in } X^*.
\]

**Lemma 3.3.** Up to a subsequence, \(\{v_n\}\) converges to a positive critical point \(v_\lambda\) of \(I_\lambda\) with \(I_\lambda(v_\lambda) = c_\lambda\).

**Proof.** Without loss of generality, we can suppose that \(q \in (4, 2 \cdot 2^*)\) in condition \((h_2)\). Since \(\{v_n\} \subset X\) is bounded, up to a subsequence, we have

\[v_n \rightharpoonup v_\lambda \text{ in } X, \quad v_n \to v_\lambda \text{ in } L^{\frac{4}{3}}(\mathbb{R}^N), \quad v_n \to v_\lambda \text{ a.e. in } \mathbb{R}^N,
\]

for some \(v_\lambda \in X\). It is easy to check that \(I'_\lambda(v_\lambda) = 0\). Next we prove \(v_n \to v_\lambda\) in \(X\). First of all, setting \(G(x,t) = \frac{1}{2} V(x)t^2 - \frac{1}{2} V(x)f^2(t) + \lambda H(f(t))\), we can rewrite \(I_\lambda\) as

\[
I_\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)v^2) \, dx - \int_{\mathbb{R}^N} G(x, v) \, dx.
\]

Let \(g(x,t) = \frac{d}{dt}G(x,t)\). By \((h_1), (h_2)\) and Lemma 2.1 for any \(\varepsilon > 0\), there exists \(C(\varepsilon) > 0\) such that

\[|g(x,t)| \leq \varepsilon |t| + C(\varepsilon)|t|^{\frac{2^*}{2}}, \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}.
\]

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Lemma 3.6. The sequence \( v_n \) is bounded. By the assumptions of Theorem 3.4, using Theorem 1.1, there exist \( \{\lambda_n\} \subset \left[ \frac{1}{2}, 1 \right] \) and \( \{v_n\} \subset X \setminus \{0\} \) such that \( \lim_{n \to \infty} \lambda_n = 1 \), \( v_n > 0 \), \( I_{\lambda_n}(v_n) = c_{\lambda_n} \leq c_{1/2} \) and \( I'_{\lambda_n}(v_n) = 0 \).

Next we show that the sequence \( \{v_n\} \) obtained in Lemma 3.4 is bounded. For this purpose, we shall use the following Pohozaev type identity. Since the proof is standard, we omit it.

Lemma 3.5. If \( v \in X \) is a critical point of \( I_{\lambda} \), then
\[
\frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{N}{2} \int_{\mathbb{R}^N} V(x)f^2(v) \, dx + \frac{1}{2} \int_{\mathbb{R}^N} \nabla V(x) \cdot x f^2(v) \, dx - \lambda N \int_{\mathbb{R}^N} H(f(v)) \, dx = 0.
\]

Lemma 3.6. The sequence \( \{v_n\} \) obtained in Lemma 3.4 is bounded in \( X \).

Proof. In view of Lemma 3.2, it is enough to prove that \( \int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)f^2(v_n)) \, dx \) is bounded. By \( I_{\lambda_n}(v_n) \leq c_{1/2} \), Lemma 3.5, Hölder inequality and Sobolev inequality, we have
\[
\int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx \leq \frac{1}{2} \int_{\mathbb{R}^N} \nabla V(x) \cdot x f^2(v_n) \, dx + Nc_{1/2}
\]

\[
\leq \frac{1}{2} \left| (\nabla V(x) \cdot x)^+ \right|_{p^* - \alpha}^{\frac{\alpha}{p^* - \alpha}} \left( \int_{\mathbb{R}^N} f^{\frac{2p}{p^*}}(v_n) \, dx \right)^{\frac{p}{p^*}} + Nc_{1/2}
\]

\[
\leq C \left( \int_{\mathbb{R}^N} |v_n|^2 \, dx \right)^{\frac{p}{2}} + Nc_{1/2}
\]

\[
\leq C \left( \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx \right)^{\frac{p}{2}} + Nc_{1/2},
\]

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} (g(x, v_n) - g(x, v_\lambda))(v_n - v_\lambda) \, dx = 0.
\]

Hence
\[
o(1) = \langle I'_{\lambda}(v_n) - I'_{\lambda}(v_\lambda), v_n - v_\lambda \rangle \]

\[
= \int_{\mathbb{R}^N} (|\nabla (v_n - v_\lambda)|^2 + V(x)(v_n - v_\lambda)^2) \, dx - \int_{\mathbb{R}^N} (g(x, v_n) - g(x, v_\lambda))(v_n - v_\lambda) \, dx
\]

\[
\geq \min\{1, V_0\} \|v_n - v_\lambda\|^2 + o(1),
\]

which implies \( v_n \to v_\lambda \) in \( X \). Therefore \( v_\lambda \) is a nontrivial critical point of \( I_{\lambda} \) with \( I(v_\lambda) = c_{\lambda} \). The positivity of \( v_\lambda \) follows by a standard argument.

At this point, for almost every \( \lambda \in \left[ \frac{1}{2}, 1 \right] \), we obtain a positive critical point \( v_\lambda \) of \( I_{\lambda} \). In general, it is not known whether it is true for \( \lambda = 1 \). However we have

Lemma 3.4. Under the assumptions of Theorem 1.1, there exist \( \{\lambda_n\} \subset \left[ \frac{1}{2}, 1 \right] \) and \( \{v_n\} \subset X \setminus \{0\} \) such that \( \lim_{n \to \infty} \lambda_n = 1 \), \( v_n > 0 \), \( I_{\lambda_n}(v_n) = c_{\lambda_n} \leq c_{1/2} \) and \( I'_{\lambda_n}(v_n) = 0 \).
where we used assumption \((V_2)\) and Corollary 2.2. Since \(\alpha \in [1,2)\), we obtain the boundedness of \(\int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx\).

Next we prove that \(\int_{\mathbb{R}^N} V(x) f^2(v_n) \, dx\) is bounded. By \((h_1)\), \((h_2)\) and Lemma 2.1 we get

\[
\lim_{t \to 0} \frac{|h(f(t)f'(t)t|}{f^2(t)} = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{|h(f(t)f'(t)|}{|t|^{2^*}} = 0.
\]

Thus, for any \(\varepsilon > 0\), there exists \(C(\varepsilon) > 0\) such that

\[
|h(f(t)f'(t)|t| \leq \varepsilon f^2(t) + C(\varepsilon)|t|^{2^*}, \quad \text{for all } t \in \mathbb{R}.
\]  \hfill (3.2)

Then we have, using \(\langle I'_{\lambda_n}(v_n), v_n \rangle = 0\) and Lemma 2.1

\[
\begin{align*}
&\int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v_n) \, dx \\
&\leq \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + \int_{\mathbb{R}^N} V(x)f(v_n)f'(v_n)v_n \, dx \\
&= \lambda_n \int_{\mathbb{R}^N} h(f(v_n))f'(v_n)v_n \, dx \\
&\leq \varepsilon \int_{\mathbb{R}^N} |f(v_n)|^2 \, dx + C(\varepsilon) \int_{\mathbb{R}^N} |v_n|^{2^*} \, dx \\
&\leq \frac{\varepsilon}{V_0} \int_{\mathbb{R}^N} V(x) f^2(v_n) \, dx + C'(\varepsilon) \left( \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx \right)^{\frac{2^*}{2}}.
\end{align*}
\]

Choosing \(\varepsilon > 0\) small enough, we complete the proof. \hfill \Box

**Proof of Theorem 1.1** (completed). By Lemmas 3.3 and 3.6 there exist \(\{\lambda_n\} \subset \left[\frac{1}{2},1\right]\) and a bounded sequence \(\{v_n\} \subset X \setminus \{0\}\) such that

\[
\lim_{n \to \infty} \lambda_n = 1, \quad I_{\lambda_n}(v_n) = c_{\lambda_n}, \quad I'_{\lambda_n}(v_n) = 0.
\]

Then

\[
\lim_{n \to \infty} I(v_n) = \lim_{n \to \infty} \left( I_{\lambda_n}(v_n) + (\lambda_n - 1) \int_{\mathbb{R}^N} H(f(v_n)) \, dx \right) = \lim_{n \to \infty} c_{\lambda_n} = c_1,
\]

where we used the fact that the map \(\lambda \mapsto c_{\lambda}\) is continuous from the left. Similarly, \(I'(v_n) \to 0\) in \(X^*\). That is, \(\{v_n\}\) is a bounded Palais-Smale sequence for \(I\) satisfying \(\lim_{n \to \infty} I(v_n) = c_1\). Using Lemma 3.3 again, we obtain a positive critical point \(v\) of \(I\). \hfill \Box

## 4 Proof of Theorem 1.2

This section is devoted to the case of well potential. In what follows, we always assume that \(V(x) \neq V_\infty\) (otherwise Theorem 1.1 gives the conclusion). We first recall some known results of “limit” functional

\[
I^\infty_{\lambda}(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V_\infty f^2(v)) \, dx - \lambda \int_{\mathbb{R}^N} H(f(v)) \, dx.
\]
Define
\[ m_\lambda^\infty = \inf \{ I_\lambda^\infty(v) \mid v \in H^1(\mathbb{R}^N) \setminus \{0\}, (I_\lambda^\infty)'(v) = 0 \}. \]

The following proposition [6] presents the results on least energy solutions for autonomous problems which are crucial to ensure the compactness of bounded Palais-Smale sequences.

**Proposition 4.1.** Under assumptions \((h_1)-(h_3)\), \(m_\lambda^\infty > 0\) and is achieved by some positive function \(w_\lambda^\infty \in H^1(\mathbb{R}^N)\). Moreover, we can find a path \(\gamma \in C([0,1], H^1(\mathbb{R}^N))\) such that \(\gamma(t)(x) > 0\) for all \(x \in \mathbb{R}^N\) and \(t \in (0,1]\), \(\gamma(0) = 0\), \(I_\lambda^\infty(\gamma(1)) < 0\), \(w_\lambda^\infty \in \gamma([0,1])\) and
\[
\max_{t \in [0,1]} I_\lambda^\infty(\gamma(t)) = I_\lambda^\infty(w_\lambda^\infty).
\]

**Lemma 4.2.** Assume \((V'_1)\) and \((h_1)-(h_3)\) hold. Define \(c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t))\), where \(I_\lambda\) is given in Section 3 and \(\Gamma = \{ \gamma \in C([0,1], H^1(\mathbb{R}^N)) \mid \gamma(0) = 0, I_\lambda(\gamma(1)) < 0 \}\). Then \(c_\lambda < m_\lambda^\infty\) for any \(\lambda \in [\frac{1}{2},1]\).

**Proof.** Let \(w_\lambda^\infty\) and \(\gamma\) be chosen as in Proposition 4.1. Then
\[ I_\lambda(\gamma(t)) < I_\lambda^\infty(\gamma(t)), \text{ for all } t \in (0,1], \]
and it follows that
\[ c_\lambda \leq \max_{t \in [0,1]} I_\lambda(\gamma(t)) < \max_{t \in [0,1]} I_\lambda^\infty(\gamma(t)) = m_\lambda^\infty, \]
which completes the proof. \(\square\)

Since \(m_\lambda^\infty > 0\), we have the following decomposition of bounded Palais-Smale sequences, which was proved in [1].

**Proposition 4.3.** Suppose that \((V'_1)\) and \((h_1)-(h_2)\) are satisfied. Let \(\{v_n\} \subset H^1(\mathbb{R}^N)\) be a bounded Palais-Smale sequence for \(I_\lambda\). Then there exists a subsequence of \(\{v_n\}\), denoted also by \(\{v_n\}\), an integer \(l \in \mathbb{N} \cup \{0\}\), sequences \(\{y_k^n\} \subset \mathbb{R}^N\), \(w_k \in H^1(\mathbb{R}^N)\) for \(1 \leq k \leq l\), such that

1. \(|y^n_k| \to \infty\) and \(|y^n_k - y^n_{k'}| \to \infty\) as \(n \to \infty\), for \(k \neq k'\),
2. \(v_n \to v_0\) in \(H^1(\mathbb{R}^N)\) with \(I_\lambda'(v_0) = 0\),
3. \(w_k \neq 0\) and \((I_\lambda^\infty)'(w_k) = 0\) for \(1 \leq k \leq l\),
4. \(\left\| v_n - v_0 - \sum_{k=1}^l w_k(\cdot - y_k^n) \right\| \to 0\),
5. \(I_\lambda(v_n) \to I_\lambda(v_0) + \sum_{k=1}^l I_\lambda^\infty(w_k)\),

where we agree that in the case \(l = 0\) the above holds without \(w_k\) and \(\{y_k^n\}\).
Using Lemma 4.2 and Proposition 4.3 we can prove

**Lemma 4.4.** Assume that \((V'_1)\) and \((h_1)-(h_3)\) hold. Let \(\{v_n\} \subset H^1(\mathbb{R}^N)\) be a bounded Palais-Smale sequence for \(I_\lambda\) satisfying \(\limsup_{n \to \infty} I_\lambda(v_n) \leq c_\lambda\) and \(\|v_n\| \to 0\) as \(n \to \infty\). Then, up to a subsequence, \(\{v_n\}\) converges weakly to a nontrivial critical point \(v_\lambda\) of \(I_\lambda\) with \(I_\lambda(v_\lambda) \leq c_\lambda\).

**Proof.** By Proposition 4.3 up to a subsequence, there exist \(l \in \mathbb{N} \cup \{0\}\) and \(v_\lambda \in H^1(\mathbb{R}^N)\) such that \(v_n \rightharpoonup v_\lambda\) in \(H^1(\mathbb{R}^N)\), \(I_\lambda(v_\lambda) = 0\) and

\[
I_\lambda(v_n) \to I_\lambda(v_\lambda) + \sum_{k=1}^{l} I_\lambda^\infty(w^k_\lambda),
\]

where \(\{w^k_\lambda\}_{k=1}^{l}\) are nontrivial critical points of \(I_\lambda^\infty\).

If \(I_\lambda(v_\lambda) < 0\), then the proof is complete. If \(I_\lambda(v_\lambda) \geq 0\), then we claim that \(l = 0\). Otherwise,

\[
c_\lambda \geq \lim_{n \to \infty} I_\lambda(v_n) = I_\lambda(v_\lambda) + \sum_{k=1}^{l} I_\lambda^\infty(w^k_\lambda) \geq m_\lambda^\infty,
\]

which contradicts Lemma 4.2. Thus \(v_n \to v_\lambda\) in \(H^1(\mathbb{R}^N)\) and \(I_\lambda(v_\lambda) \leq c_\lambda\). Since \(\|v_n\| \to 0\) as \(n \to \infty\), \(v_\lambda\) is a nontrivial critical point of \(I_\lambda\). This completes the proof.

**Lemma 4.5.** Under the assumptions of Theorem 1.2, there exists \(\sigma > 0\) (independent of \(\lambda \in [\frac{1}{2},1]\)) such that, if \(v\) is a nontrivial critical point of \(I_\lambda\), then \(\|v\| \geq \sigma\).

**Proof.** It follows from \(\langle I'_\lambda(v), v \rangle = 0\) that

\[
\int_{\mathbb{R}^N} \left( |\nabla v|^2 + V(x)f(v)f'(v)v \right) \, dx = \lambda \int_{\mathbb{R}^N} h(f(v))f'(v)v \, dx.
\]

By Lemma 2.1 and (3.2), we have

\[
\int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)f^2(v) \, dx \leq \frac{1}{4} \int_{\mathbb{R}^N} V(x)f^2(v) \, dx + C \int_{\mathbb{R}^N} |v|^{2^*} \, dx,
\]

which implies

\[
\int_{\mathbb{R}^N} \left( |\nabla v|^2 + V(x)f^2(v) \right) \, dx \leq C \int_{\mathbb{R}^N} |v|^{2^*} \, dx \leq C \left[ \int_{\mathbb{R}^N} \left( |\nabla v|^2 + V(x)f^2(v) \right) \, dx \right]^\frac{2^*}{2^* - 2}.
\]

Since \(v \neq 0\), we obtain

\[
\int_{\mathbb{R}^N} \left( |\nabla v|^2 + V(x)f^2(v) \right) \, dx \geq \sigma_0
\]

for some positive constant \(\sigma_0\). Then the conclusion follows immediately from \((V'_1)\) and \(|f(t)| \leq |t|\) for all \(t \in \mathbb{R}\).
Proof of Theorem 1.2. It is easy to see that, under hypotheses of Theorem 1.2, all assumptions in Proposition 3.1 are satisfied. Then there exists $J_1 \subset [\frac{1}{2}, 1]$ with $\text{meas}(J_1) = 0$ such that, for any $\lambda \in [\frac{1}{2}, 1] \setminus J_1$, there is a bounded Palais-Smale sequence $\{v_n\} \subset H^1(\mathbb{R}^N)$ for $I_\lambda$ satisfying $\lim_{n \to \infty} I_\lambda(v_n) = c_\lambda$. Since $c_\lambda > 0$, we know $\|v_n\| \to 0$ as $n \to \infty$. Using Lemmas 4.4 and 4.5 for any $\lambda \in [\frac{1}{2}, 1] \setminus J_1$, we obtain a nontrivial critical point $v_\lambda$ of $I_\lambda$ with $I_\lambda(v_\lambda) \leq c_\lambda$ and $\|v_\lambda\| \geq \sigma > 0$.

Choosing $\lambda_n \subset [\frac{1}{2}, 1] \setminus J_1$ such that $\lambda_n \to 1$ as $n \to \infty$, we obtain a sequence $\{v_n\} \subset H^1(\mathbb{R}^N)$ satisfying

$$\|v_n\| \geq \sigma > 0, \quad I_{\lambda_n}(v_n) \leq c_{\lambda_n} \leq c_{1/2}, \quad I'_{\lambda_n}(v_n) = 0.$$ 

By Lemma 3.6, $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Then

$$\limsup_{n \to \infty} I(v_n) = \limsup_{n \to \infty} \left( I_{\lambda_n}(v_n) + (\lambda_n - 1) \int_{\mathbb{R}^N} H(f(v_n)) \, dx \right) \leq \lim_{n \to \infty} c_{\lambda_n} = c_1,$$

and $I'(v_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$. That is, $\{v_n\}$ is a bounded Palais-Smale sequence for $I$ satisfying $\limsup_{n \to \infty} I(v_n) \leq c_1$ and $\|v_n\| \to 0$ as $n \to \infty$. Using Lemma 4.4 again, we obtain a nontrivial critical point $v$ of $I$. A standard argument can show that $v > 0$. \qed

5 Proof of Theorem 1.3

The goal of this section is to prove Theorem 1.3. To this end, we use the same idea as in [7]. Firstly, since there is no assumption on $h$ near infinity, we need to modify the nonlinearity to a new one which satisfies $(h_1) - (h_4)$. Thanks to Theorem 1.1, the modified problem has a positive solution. Secondly, we shall prove that the solution obtained converges to zero in $L^\infty$-norm as $\mu \to \infty$. Thus, for $\mu$ large, it is in fact a positive solution of original problem (1.3).

By $(h'_1)$ and $(h'_2)$, there exist two positive constants $K_0$ and $K_1$ such that

$$h(t) \leq \frac{1}{q} K_1 t^{q-1} \quad \text{and} \quad K_0 t^p \leq H(t) \leq K_1 t^q$$

for $t > 0$ sufficiently small. Choose $\delta > 0$ such that (5.1) holds for $0 < t \leq 2\delta$. Let $\xi$ be a cut-off function satisfying $0 \leq \xi \leq 1$, $\xi(t) = 1$ for $t \leq \delta$, $\xi(t) = 0$ for $t \geq 2\delta$ and $|\xi'(t)| \leq 2/\delta$ for $\delta \leq t \leq 2\delta$. Define

$$\tilde{H}(t) = \xi(t) H(t) + (1 - \xi(t)) K_1 |t|^q$$

and $\tilde{h}(t) = \tilde{H}'(t)$. Then it is easy to verify that $\tilde{h}$ satisfies $(h_1) - (h_4)$. Moreover, we have

**Lemma 5.1.** (1) There exists $C > 0$ such that

$$\tilde{h}(t) \leq Ct^{q-1}, \quad \text{for all } t > 0,$$  

(5.2)
and
\[
\tilde{h}(t) \leq \varepsilon t + C\varepsilon^{\frac{q-2}{q-1}}t^{2^{*}-1}, \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \tag{5.3}
\]

(2) For any \( T > 0 \), there exists \( C(T) > 0 \) such that
\[
\tilde{H}(t) \geq C(T)t^p, \quad \text{for all } t \in [0, T]. \tag{5.4}
\]

Now we consider the modified problem
\[
-\Delta u + V(x)u - \Delta (u^2)u = \mu \tilde{h}(u), \quad \text{in } \mathbb{R}^N, \tag{5.5}
\]
with the natural energy functional given by
\[
\tilde{J}_\mu(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx + \int_{\mathbb{R}^N} u^2|\nabla u|^2 \, dx - \mu \int_{\mathbb{R}^N} \tilde{H}(u) \, dx.
\]
As in Section 2, setting \( v = f^{-1}(u) \), we obtain
\[
\tilde{I}_\mu(v) := \tilde{J}_\mu(f(v)) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)f^2(v)) \, dx - \mu \int_{\mathbb{R}^N} \tilde{H}(f(v)) \, dx.
\]

In order to show that solutions of the modified problem (5.5) are in fact solutions of the original problem (1.3), we need the following \( L^\infty \)-estimate by Moser iteration. We give the proof for completeness.

**Lemma 5.2.** If \( v \) is a positive critical point of \( \tilde{I}_\mu \), then \( v \in L^\infty(\mathbb{R}^N) \) and there exists \( C > 0 \) (independent of \( \mu \)) such that
\[
|v|_\infty \leq C (\mu \|v\|^{q-2})^{\frac{1}{2-q}} \|v\|.
\]

**Proof.** For each \( k > 0 \), we define
\[
v_k = \begin{cases} v, & \text{if } v \leq k, \\ k, & \text{if } v \geq k. \end{cases}
\]
Set \( w_k = vv_k^{2(\gamma-1)} \) and \( \tilde{w}_k = vv_k^{\gamma-1} \), where \( \gamma > 1 \) is to be determined later. It follows from \( \langle \tilde{I}_\mu'(v), w_k \rangle = 0 \), (5.2), and Lemma 2.1 that
\[
\int_{\mathbb{R}^N} v_k^{2(\gamma-1)}|\nabla v|^2 \, dx \leq \mu \int_{\mathbb{R}^N} \tilde{h}(f(v))f'(v)vv_k^{2(\gamma-1)} \, dx
\]
\[
\leq C\mu \int_{\mathbb{R}^N} f^{q-1}(v)vv_k^{2(\gamma-1)} \, dx
\]
\[
\leq C\mu \int_{\mathbb{R}^N} v^{q-1}v_k^{2(\gamma-1)} \, dx
\]
\[
= C\mu \int_{\mathbb{R}^N} v^{q-2}\tilde{w}_k^2 \, dx.
\]
Combining this with Gagliardo-Nirenberg-Sobolev inequality yields

\[
\left( \int_{\mathbb{R}^N} \tilde{w}_k^{2^*} \, dx \right)^{\frac{2}{2^*}} \leq C \int_{\mathbb{R}^N} |\nabla \tilde{w}_k|^2 \, dx
\]

\[
\leq C \int_{\mathbb{R}^N} \left( v_k^{2(\gamma - 1)} |\nabla v|^2 + (\gamma - 1)^2 v_k^{2(\gamma - 1) - 2} |\nabla v_k|^2 \right) \, dx
\]

\[
\leq C\gamma^2 \int_{\mathbb{R}^N} v_k^{2(\gamma - 1)} |\nabla v|^2 \, dx
\]

\[
\leq C\gamma^2 \mu \int_{\mathbb{R}^N} v^{q - 2} \tilde{w}_k^2 \, dx,
\]

where we have used the facts that \( v^2 |\nabla v_k|^2 \leq v_k^2 |\nabla v|^2 \) and \( 1 + (\gamma - 1)^2 < \gamma^2 \) for \( \gamma > 1 \). By Hölder inequality and Sobolev inequality,

\[
\left( \int_{\mathbb{R}^N} (v v_k^{\gamma - 1})^{2^*} \, dx \right)^{\frac{2}{2^*}} \leq \left( \int_{\mathbb{R}^N} v_{\gamma}^{2^*} \, dx \right)^{\frac{2}{2^*}} \leq C\gamma^2 \mu \int_{\mathbb{R}^N} v^{q - 2} \tilde{w}_k^2 \, dx
\]

\[
\leq C\gamma^2 \mu \left( \int_{\mathbb{R}^N} v^{2^*} \, dx \right)^{\frac{q - 2}{2^*}} \left( \int_{\mathbb{R}^N} \tilde{w}_k^{2^* - q + 2} \, dx \right)^{\frac{2^* - q + 2}{2^*}}
\]

\[
\leq C\gamma^2 \mu \|v\|^{q - 2} \left( \int_{\mathbb{R}^N} v^{2^* - q + 2} \, dx \right)^{\frac{2^* - q + 2}{2^*}}.
\]

Denote \( \alpha_0 = \frac{2^* - q + 2}{2^* - q + 2} \). Choosing \( \gamma = \frac{2^* - q + 2}{2^* - q + 2} \), we have \( \frac{2^* - q + 2}{2^* - q + 2} = 2^* \) and so

\[
\left( \int_{\mathbb{R}^N} (v v_k^{\gamma - 1})^{2^*} \, dx \right)^{\frac{2}{2^*}} \leq C\gamma^2 \mu \|v\|^{q - 2} |v|_{\gamma_0}^{2\gamma}.
\]

Letting \( k \to \infty \), by Fatou’s lemma, we obtain

\[
|v|_{\gamma, 2^*} \leq \left( C\gamma^2 \mu \|v\|^{q - 2} \right)^{\frac{1}{2\gamma}} |v|_{\gamma_0}.
\]

For \( m = 0, 1, \cdots \), set \( \gamma_m = \gamma^{m + 1} \). Repeating the above arguments for \( \gamma_1 \), we have

\[
|v|_{\gamma_1, 2^*} \leq \left( C\gamma_1^2 \mu \|v\|^{q - 2} \right)^{\frac{1}{2\gamma_1}} |v|_{\gamma_0}
\]

\[
\leq \left( C\gamma_1^2 \mu \|v\|^{q - 2} \right)^{\frac{1}{2\gamma_1}} \left( C\gamma^2 \mu \|v\|^{q - 2} \right)^{\frac{1}{2\gamma}} |v|_{\gamma_0}
\]

\[
= (C\mu \|v\|^{q - 2})^{\frac{1}{2\gamma_1} + \frac{1}{2\gamma}} (\gamma_1)^{\frac{1}{2\gamma_1}} (\gamma)^{\frac{1}{2\gamma}} |v|_{2^*}.
\]

By iteration, it follows that

\[
|v|_{\gamma_m, 2^*} \leq (C\mu \|v\|^{q - 2})^{\frac{m}{2\gamma} \sum_{i=0}^{m} \gamma_i^{-i}} (\gamma)^{\frac{m}{2\gamma} \sum_{i=0}^{m} \gamma_i^{-i}} \gamma_i^{\frac{1}{2\gamma_1}} \gamma_i^{\frac{1}{2\gamma}} |v|_{2^*}.
\]
Since \( \gamma > 1 \), the series \( \sum_{i=0}^{\infty} \gamma^{-i} \) and \( \sum_{i=0}^{\infty} i \gamma^{-i} \) are convergent. Taking \( m \to \infty \), we get \( v \in L^\infty(\mathbb{R}^N) \) and 
\[
|v|_\infty \leq C (\mu \|v\|^{q-2})^{\frac{1}{2(q-2)}} \|v\|.
\]
This completes the proof. \(\Box\)

**Lemma 5.3.** Let \( \mu > \frac{V_0}{4} \). If \( v \) is a critical point of \( \tilde{I}_\mu \) with \( \tilde{I}_\mu(v) = d_\mu \), then there exists \( C > 0 \) (independent of \( \mu \)) such that
\[
\|v\|^2 \leq C \left( d_\mu + d_\mu^{\frac{q^*}{q-2}} + \mu^\frac{q^*}{q-2} d_\mu^{\frac{q^*}{q}} \right).
\]

**Proof.** By \( \tilde{I}_\mu(v) = d_\mu \), Lemma 3.5 and (5.3), we obtain
\[
\int_{\mathbb{R}^N} |\nabla v|^2 \, dx \leq Nd_\mu. \tag{5.6}
\]
Next we estimate the term \( \int_{\mathbb{R}^N} V(x)f^2(v) \, dx \). It follows from \( \langle \tilde{I}_\mu'(v), v \rangle = 0 \), Lemma 2.1 and (5.3) that
\[
\frac{1}{2} \int_{\mathbb{R}^N} V(x)f^2(v) \, dx \leq \mu \int_{\mathbb{R}^N} \tilde{h}(f(v))f(v) \, dx
\leq \mu \varepsilon \int_{\mathbb{R}^N} f^2(v) \, dx + C \mu \varepsilon^{\frac{q^*}{q}} \int_{\mathbb{R}^N} |v|^{2^*} \, dx
\leq \frac{\mu \varepsilon}{V_0} \int_{\mathbb{R}^N} V(x)f^2(v) \, dx + C \mu \varepsilon^{\frac{q^*}{q}} \left( \int_{\mathbb{R}^N} |\nabla v|^2 \, dx \right)^{\frac{q^*}{2}}
\]
Taking \( \varepsilon = \frac{V_0}{4\mu} \) and using (5.6), we have
\[
\int_{\mathbb{R}^N} V(x)f^2(v) \, dx \leq C \mu^{\frac{q^*}{q-2}} d_\mu^{\frac{q^*}{q}}. \tag{5.7}
\]
Then desired conclusion follows from (5.6), (5.7) and the proof of Lemma 3.2. \(\Box\)

Now we are ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** Observing that the functional \( \tilde{I}_\mu \) has the Mountain Pass geometry, we can define
\[
d_\mu = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \tilde{I}_\mu(\gamma(t)) > 0,
\]
where \( \Gamma = \{ \gamma \in C([0,1], H^1_r(\mathbb{R}^N)) \mid \gamma(0) = 0, \tilde{I}_\mu(\gamma(1)) < 0 \} \). Since \( \tilde{h} \) satisfies \( (h_1)-(h_4), \) Theorem 1.1 implies that there is a positive critical point \( v_\mu \) of \( \tilde{I}_\mu \) with \( I_\mu(v_\mu) = d_\mu \).
Let \( v_0 \in C_c^\infty(\mathbb{R}^N) \setminus \{0\} \) be a nonnegative radially symmetric function such that \( \tilde{I}_\mu(v_0) < 0 \). Then, by (5.4) with \( T = |v_0|_\infty \) and properties of \( f \), we have
\[
d_\mu \leq \max_{t \in [0, 1]} \tilde{I}_\mu(t v_0)
\leq \max_{t \in [0, 1]} \left( \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla v_0|^2 + V_1 v_0^2) \, dx - \mu \int_{\mathbb{R}^N} \tilde{H}(f(t v_0)) \, dx \right)
\leq \max_{t \in [0, 1]} \left( \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla v_0|^2 + V_1 v_0^2) \, dx - C \mu t^p \int_{\mathbb{R}^N} v_0^p \, dx \right)
\leq C \mu^{\frac{2}{p-2}}.
\]

Combining this with Lemmas 5.2 and 5.3, we have
\[
|v_\mu|_\infty \leq C \mu^{\frac{(p-q)(2^*-2)-(p-2)}{2(p-2)(q-2)(2^*-q)}}
\]
for \( \mu \) sufficiently large. Consequently, by (1.4), we obtain a positive solution \( u_\mu = f(v_\mu) \) of problem (5.5) with \( |u_\mu|_\infty \leq |v_\mu|_\infty < \delta \) for \( \mu \) large enough. Then \( u_\mu \) is a positive solution of problem (1.3).

6 Generalized result

In this section, we apply our methods to (1.2) with a general nonlinearity of Berestycki and Lions type [3]. Throughout this section, we assume that \( V \in C^1(\mathbb{R}^N, \mathbb{R}) \) satisfies (V1), (V2) and
\[
(V_3) \quad \lim_{|x| \to \infty} V(x) = V_0.
\]
The nonlinearity \( h \) satisfies (h1) and
\[
(h_2) \quad \lim_{t \to +\infty} \frac{h(t)}{t^{2^*-2}} = 0;
\]
\[
(h_3) \quad \text{there exists } \zeta > 0 \text{ such that } H(\zeta) > \frac{V_0}{2} \zeta^2.
\]

Theorem 6.1. Suppose that \( V \) satisfies (V1), (V2), (V3) and \( h \) satisfies (h1), (h2), (h3). Then problem (1.2) has at least a positive solution.

Define \( h_1 = \max\{h, 0\} \) and \( h_2 = \max\{-h, 0\} \), then \( h = h_1 - h_2 \). By (h3), there exists \( \tilde{u} \in H^1_+(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) (see [3]) such that
\[
\int_{\mathbb{R}^N} H_1(\tilde{u}) \, dx - \int_{\mathbb{R}^N} H_2(\tilde{u} + \frac{V_0}{2} \tilde{u}^2) \, dx = \int_{\mathbb{R}^N} \left( H(\tilde{u}) - \frac{V_0}{2} \tilde{u}^2 \right) \, dx > 0,
\]
where $H_i(t) = \int_0^t h_i(s) ds$, $i = 1, 2$. Thus, for some $\bar{\lambda} \in (0, 1)$, we have

$$\bar{\lambda} \int_{\mathbb{R}^N} H_1(\bar{u}) \, dx - \int_{\mathbb{R}^N} \left( H_2(\bar{u}) + \frac{V_0}{2} \bar{u}^2 \right) \, dx > 0. \tag{6.1}$$

As in Section 3, we introduce a family of functionals

$$I_\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)f^2(v)) \, dx + \int_{\mathbb{R}^N} H_2(f(v)) \, dx - \lambda \int_{\mathbb{R}^N} H_1(f(v)) \, dx,$$

where $\lambda \in [\bar{\lambda}, 1]$. Set

$$A(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)f^2(v)) \, dx + \int_{\mathbb{R}^N} H_2(f(v)) \, dx$$

and

$$B(v) = \int_{\mathbb{R}^N} H_1(f(v)) \, dx,$$

then $I_\lambda(v) = A(v) - \lambda B(v)$. Similar to Lemma 3.2, we have

**Lemma 6.2.** Assume that $(V_1)$, $(V_3)$, $(h_1)$, $(\bar{h}_2)$ and $(\bar{h}_3)$ hold. Then

1. $B(v) \geq 0$ for all $v \in H^1_0(\mathbb{R}^N)$;
2. $A(v) \to \infty$ as $\|v\| \to \infty$;
3. there exists $v_0 \in H^1_0(\mathbb{R}^N)$, independent of $\lambda$, such that $I_\lambda(v_0) < 0$ for all $\lambda \in [\bar{\lambda}, 1]$;
4. for all $\lambda \in [\bar{\lambda}, 1]$, it holds

$$c_\lambda = \inf_{\gamma \in \Gamma, t \in [0,1]} \max I_\lambda(\gamma(t)) > \max \{I_\lambda(0), I_\lambda(v_0)\},$$

where $\Gamma = \{\gamma \in C([0, 1], H^1_0(\mathbb{R}^N)) \mid \gamma(0) = 0, \gamma(1) = v_0\}$.

**Proof.** We only prove (3). Set

$$J_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx + \int_{\mathbb{R}^N} u^2|\nabla u|^2 \, dx + \int_{\mathbb{R}^N} H_2(u) \, dx - \lambda \int_{\mathbb{R}^N} H_1(u) \, dx.$$

For any $t > 0$, we have

$$J_\lambda(\bar{u}(x/t)) = \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 \, dx + \frac{t^N}{2} \int_{\mathbb{R}^N} V(tx)\bar{u}^2 \, dx$$

$$+ \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} \bar{u}^2|\nabla \bar{u}|^2 \, dx + t^N \int_{\mathbb{R}^N} H_2(\bar{u}) \, dx - \bar{\lambda} t^N \int_{\mathbb{R}^N} H_1(\bar{u}) \, dx$$

$$= \frac{t^{N-2}}{2} \left[ \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 \, dx + 2 \int_{\mathbb{R}^N} \bar{u}^2|\nabla \bar{u}|^2 \, dx \right]$$

$$+ \frac{1}{2} \int_{\mathbb{R}^N} V(tx)\bar{u}^2 \, dx + \int_{\mathbb{R}^N} H_2(\bar{u}) \, dx - \bar{\lambda} \int_{\mathbb{R}^N} H_1(\bar{u}) \, dx \right].$$

By $(V_3)$ and (6.1), it is easy to see that $J_\lambda(\bar{u}(x/t)) < 0$ for $t$ large. Then desired result follows. \hfill \Box
Arguing as the proof of Brezis-Lieb lemma [4], we have

**Lemma 6.3.** Assume that $g \in C(\mathbb{R}, \mathbb{R})$ satisfies
\[ |g(t)| \leq C(|t| + |t|^{p-1}), \quad 2 < p < 2^*. \]
If $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $v_n \to v$ a.e. in $\mathbb{R}^N$, then
\[ \int_{\mathbb{R}^N} (G(v_n) - G(v_n - v) - G(v)) \, dx \to 0, \quad \text{as } n \to \infty, \]
where $G(t) = \int_0^t g(s) \, ds$.

The following lemma was due to Strauss [27] (see also [3]).

**Lemma 6.4.** Let $P$ and $Q$ be two continuous functions satisfying $P(t)/Q(t) \to 0$ as $t \to \infty$. If $\{v_n\}$ is a sequence of measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that
\[ \sup_n \int_{\mathbb{R}^N} |Q(v_n)| \, dx < \infty \]
and $P(v_n) \to v$ a.e. in $\mathbb{R}^N$, then one has
\[ \int_{\Omega} |P(v_n) - v| \, dx \to 0, \quad \text{as } n \to \infty, \]
for any bounded Borel set $\Omega$. Moreover, if one assumes also that $P(t)/Q(t) \to 0$ as $t \to 0$ and $\sup_n |v_n(x)| \to 0$ as $|x| \to \infty$, then
\[ \int_{\mathbb{R}^N} |P(v_n) - v| \, dx \to 0, \quad \text{as } n \to \infty. \]

**Lemma 6.5.** Assume that $\{v_n\} \subset H^1_0(\mathbb{R}^N)$ is a bounded Palais-Smale sequence for $I_\lambda$ satisfying $\lim_{n \to \infty} I_\lambda(v_n) = c_\lambda$. Then, up to a subsequence, $\{v_n\}$ converges to a positive critical point $v_\lambda$ of $I_\lambda$ with $I_\lambda(v_\lambda) = c_\lambda$.

**Proof.** First of all we may assume $v_n \to v_\lambda$ in $H^1_0(\mathbb{R}^N)$ and $v_n \to v_\lambda$ a.e. in $\mathbb{R}^N$. By Lebesgue dominate theorem and Lemma 6.4, we conclude that $I_\lambda'(v_\lambda) = 0$. Set $w_n = v_n - v_\lambda$. Using Lemma 6.3 and Lemma 6.4 with $Q(t) = t^2 + |t|^{2^*}$, we deduce
\[ \langle I_\lambda'(v_n), v_n \rangle - \langle I_\lambda'(v_\lambda), v_\lambda \rangle - \langle I_\lambda'(w_n), w_n \rangle = o(1) \]
and so $\langle I_\lambda'(w_n), w_n \rangle = o(1)$. Therefore
\[
\begin{align*}
&\limsup_{n \to \infty} \left( \int_{\mathbb{R}^N} |\nabla w_n|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(w_n) \, dx \right) \\
&\leq \limsup_{n \to \infty} \left( \int_{\mathbb{R}^N} |\nabla w_n|^2 \, dx + \int_{\mathbb{R}^N} V(x) f(w_n) f'(w_n) w_n \, dx \right) \\
&= \limsup_{n \to \infty} \left( \lambda \int_{\mathbb{R}^N} h_1(f(w_n)) f'(w_n) w_n \, dx - \int_{\mathbb{R}^N} h_2(f(w_n)) f'(w_n) w_n \, dx \right) \\
&= 0.
\end{align*}
\]
Combining this with Lemma 3.2, we obtain \( w_n \to 0 \) in \( H^1_r(\mathbb{R}^N) \). Consequently, \( v_n \to v_\lambda \) in \( H^1_r(\mathbb{R}^N) \). Hence \( v_\lambda \) is a nontrivial critical point of \( I_\lambda \) with \( I(v_\lambda) = c_\lambda \). A standard argument can show that \( v_\lambda > 0 \) in \( \mathbb{R}^N \).

\[ \square \]

Proof of Theorem 6.1. It is similar to the proof of Theorem 1.1

\[ \square \]

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