Spiky Strings and Spin Chains

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Abstract
We determine spectral curves for the known spiky string solutions in $AdS$ space in the limit of large angular momentum. We also construct generic multi-spike solutions in this limit and compute the corresponding spectral data. The resulting spectral curves precisely match those of the classical spin chain describing the dual operators in one-loop gauge theory. Our results confirm the map between string theory and gauge theory degrees of freedom proposed in arXiv:0805.4387 [hep-th].
1 Introduction

The AdS/CFT correspondence predicts an exact equivalence between the operator dimensions of planar $\mathcal{N} = 4$ SUSY Yang-Mills and the energy levels of free string theory on $AdS_5 \times S^5$. This matching is complicated by the fact that the two theories involved are only tractable at small and large values of the ‘t Hooft coupling, $\lambda = g^2 N$ respectively. Nevertheless, as we vary $\lambda$ from small to large values, each local operator of definite scaling dimension in perturbative gauge theory should go over to a particular string state on $AdS_5 \times S^5$. The emergence of integrability on both sides of the correspondence \cite{1, 2, 3} leads to the hope that we can describe this interpolation exactly for generic states. So far this has only been accomplished definitively for operators with very large $R$-charge where a description of the spectrum in terms of asymptotic states can be obtained on both sides. Analytic results for the dispersion relation \cite{4, 5} and scattering matrix \cite{5, 6, 7, 8} of these states interpolate smoothly between weakly coupled gauge theory and semiclassical string theory, providing a complete description of the spectrum in this limit. In a recent paper \cite{9}, one of the present authors proposed a similarly precise matching between states in the limit of large conformal spin (denoted $S$). The character of this limit is quite different from the case of large $R$-charge where gauge theory operators are described by a quantum spin chain of infinite length. Instead, large $S$ leads to a classical spin chain of fixed length on the gauge theory side. In \cite{9}, the spectrum of semiclassical string theory in this limit was shown to exactly match that of the corresponding sector of one-loop gauge theory up to a single overall normalisation. A precise map between string theory and gauge theory degrees of freedom was also proposed. In this paper we will expand on the proposal of \cite{9} and perform several detailed tests. In particular we will construct explicit string solutions in the large-$S$ limit and check the correspondence to one-loop gauge theory directly. The main results are described in the remainder of this introductory section.

As in \cite{9}, we focus on single-trace operators of the form,

$$\hat{O} \sim \text{Tr}_N \left[ D_+^{s_1} Z D_+^{s_2} Z \ldots D_+^{s_J} Z \right]$$

(1)
having total Lorentz spin $S = \sum_{l=1}^{J} s_l$ and twist (or equivalently $R$-charge) equal to $J$. Here $D_+$ is a covariant derivative with conformal spin plus one and $Z$ is one of the three complex adjoint scalar fields of the $\mathcal{N} = 4$ theory. These operators belong to the non-compact $sl(2)$ sector of the $\mathcal{N} = 4$ theory. In this sector, the one-loop dilatation operator is equivalent to the Hamiltonian of an integrable $SL(2,\mathbb{R})$ spin chain of length $J$ \cite{1,2,10} which can be diagonalised exactly using the Bethe ansatz. Here, we are primarily interested in the large spin limit $S \to \infty$ with fixed $J$, where the non-compact spins at each site are highly excited and can be replaced by corresponding classical variables \cite{11,12,13}. The phase space of the resulting classical spin chain splits into sectors labelled by a positive integer $K \leq J$. In each sector the phase space is characterised by a spectral curve,

$$\Gamma_K : \quad t + \frac{1}{t} = 2 + \frac{q_2}{u^2} + \frac{q_3}{u^3} + \ldots + \frac{q_K}{u^K}$$

which defines a Riemann surface of genus $K - 2$. The moduli, $q_j$, $j = 2,\ldots,K$, of the curve correspond to the higher conserved charges of the spin chain. The one-loop anomalous dimensions of operators in this sector are given as,

$$\Delta - S - J = \frac{\lambda}{4\pi^2} \left( \log q_K + C_{1\text{-loop}} + O\left(\frac{1}{\log^2 S}\right) \right)$$

where $C_{1\text{-loop}}$ is an undetermined constant which may depend on $K$ and $J$ but not on the charges $q_k$. As $q_K \sim S^K$, this formula exhibits the characteristic $K \log S$ scaling of the anomalous dimensions of twist $K$ operators \cite{14,15}. As reviewed in Section 2 below, semiclassical quantisation leads to a discrete spectrum for the charges $q_k$ and hence also for the anomalous dimensions.

For large 't Hooft coupling, operators of the form (1) are dual to semiclassical strings moving on an $AdS_3 \times S^1$ submanifold of $AdS_5 \times S^5$. Here the spin $S$ corresponds to angular momentum on $AdS_3$ and twist $J$ to angular momentum on $S^1$. The large-$S$ limit of the string spectrum was studied in \cite{9} and found to be identical to the semiclassical spectrum of the spin chain described above up to a single coupling-dependent normalisation. The equality of the two spectra was demonstrated using the finite-gap formalism for classical string theory developed in \cite{21,22} in which each classical solution of string theory in $AdS_3 \times S^1$ has an associated spectral curve which encodes the values of the higher conserved charges of

\[\text{For other relevant work on this limit see }\text{\cite{16,17,18,19,20}}\]
the worldsheet sigma model. It was shown that the spectral curve of the “$K$-gap” string solutions reduces to the gauge theory curve $\Gamma_K$ in the large-$S$ limit. The symplectic form and Hamiltonian were also found to coincide with their gauge theory counterparts in this limit. In particular, the leading semiclassical spectrum of string energies is given by,

$$\Delta - S - J = \frac{\sqrt{\lambda}}{2\pi} \left( \log q_K + C_{\text{string}} + O \left( \frac{1}{\log^2 S} \right) \right)$$  \hspace{1cm} (4)$$

where $C_{\text{string}}$ is a constant which we will determine below. A more specific proposal for how gauge theory and string theory degrees of freedom are related was also made in [9]. The key idea was that generic $K$-gap string solutions should develop $K$ cusps which approach the boundary as $S \to \infty$. Here we will make these ideas more concrete by constructing the limiting string solutions explicitly and checking their correspondence to the gauge theory spin chain.

We begin by studying the known large-spin solutions of string theory on $AdS_3$ (motion on $S^1$ can be neglected in this limit). These are the multiply folded spinning string of Gubser, Klebanov and Polyakov (GKP) and the spiky string of Kruczenski. Both these solutions have cusps which approach the boundary as $S \to \infty$. In this limit, the charge density corresponding to the spin $S$ becomes $\delta$-function localised at the spikes. We calculate the monodromy matrix of each solution and thereby determine the corresponding spectral curve. The resulting curves for the $K$ spike solutions correspond to particular points in the moduli space of the gauge theory curve $\Gamma_K$. For both types of solution, these turn out to be special points in the moduli space where $\Gamma_K$ degenerates from genus $K - 2$ to genus zero. The filling numbers of Bethe roots for the dual gauge theory operator are evaluated from the curve in each case. The two types of solution correspond to particular single-cut configurations in gauge theory.

In the next part of the paper we construct a $(K-1)$-parameter generalisation of the spiky string solution of Kruczenski [16]. In the solution of [16], the angular separation between each pair of adjacent cusps is the same. Here, we construct a limiting solution at large-$S$ in which the angular separations $\Delta \theta_j = \theta_j - \theta_{j-1}$, with $j = 1, 2, \ldots, K$, between adjacent cusps can be adjusted subject to the constraints,

$$0 < \Delta \theta_j \leq \pi \quad \sum_{j=1}^{K} \Delta \theta_j = 2\pi n$$

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Figure 1: The general multi-spike solution.

for positive integer \( n \) (see Figure 1). The original solution \([16]\) corresponds to the choice \( \theta_j = 2\pi/K \) for all \( j = 1, 2, \ldots, K \) where the spikes lie at the vertices of a regular \( K \)-gon. The \( N \)-folded string of GKP also arises as a special case corresponding to the choice \( \Delta \theta_j = \pi \) for all \( j = 1, 2, \ldots, K \) with \( K = 2N \).

The next step is to evaluate the monodromy matrix and spectral curve of the limiting solution described above. Once again the curve precisely matches the gauge theory result (2). The conserved charges \( q_k \) are related to the initial angular positions \( \theta_i \) of the cusps as,

\[
q_k = \left( \frac{-2S}{K} \right)^k \sum_{1 \leq j_1 < \ldots < j_k \leq K} \prod_{l=1}^{k} \sin \left( \frac{\theta_{j_{l+1}} - \theta_{j_l}}{2} \right)
\]

for \( k = 1, 2, \ldots, K \). We also evaluate the string energy which obeys the expected relation (4) with constant part given as,

\[
C_{\text{string}} = K \left[ \log \left( \frac{8\pi}{\sqrt{\lambda}} \right) - 1 \right] + \log(-1)^{K+n}
\]
All the solutions studied in this paper exhibit the localisation of angular momentum density predicted in [9]. In each case the worldsheet charge density \( j_\tau \) at large \( S \) has the form,

\[
j_\tau(\sigma, \tau) \approx \frac{8\pi}{\sqrt{\lambda}} \sum_{k=0}^{K-1} L_k \delta(\sigma - \sigma_k)
\]

for some \( su(1,1) \)-valued variables \( L_k \), where \( \sigma = \sigma_k \) are the locations of the \( K \) cusps of the string. In [9], it was proposed that these variables should be identified directly with the classical spin variables of the one-loop spin chain of length \( K \). The agreement of the string theory and gauge theory curves described above provides further evidence for this identification. We also calculate the values of the variables \( L_k \) for the general spiky string.

2 The gauge theory spin chain

We consider the one-loop anomalous dimensions of operators in the non-compact rank one subsector of planar \( \mathcal{N} = 4 \) SUSY Yang-Mills (also known as the \( sl(2) \) sector). Generic single-trace operators in this sector are labelled by their Lorentz spin \( S \) and \( U(1)_R \) charge \( J \) and have the form (1). The classical dimension of each operator is \( \Delta_0 = J + S \) and its twist (classical dimension minus spin) is therefore equal to \( J \).

The one-loop anomalous dimensions of operators in the \( sl(2) \) sector are determined by the eigenvalues of the Hamiltonian of the Heisenberg XXX \(-\frac{1}{2}\) spin chain with \( J \) sites. Each site of this chain carries a representation of \( SL(2, \mathbb{R}) \) with quadratic Casimir equal to minus one half. Our discussion of the chain in this section follows that of [11, 13] (See in particular Section 2.2 of [13]).

Here we will focus on the large-spin limit of the chain: \( S \to \infty \) with \( J \) fixed. This is effectively a semiclassical limit where \( 1/S \) plays the role of Planck’s constant \( \hbar \) [11, 13]. In this limit the quantum spins are replaced by the classical variables \( \mathcal{L}_k^\pm, \mathcal{L}_0^k \), for \( k = 1, 2, \ldots, J \), introduced above. The commutators of spin operators are replaced by the Poisson brackets

\[
\{ \mathcal{L}_k^+, \mathcal{L}_{k'}^- \} = 2i\delta_{kk'}\mathcal{L}_k^0 \quad \{ \mathcal{L}_k^0, \mathcal{L}_{k'}^\pm \} = \pm i\delta_{kk'}\mathcal{L}_k^\pm
\]
of these classical spins. As the quadratic Casimir equal to $-1/2$ is negligible in the $S \to \infty$ limit, the classical spins at each site obey the relation

$$L_k^+ L_k^- + (L_k^0)^2 = 0$$

(6)

up to $1/S$ corrections. We will restrict our attention to states obeying the highest weight condition,

$$\sum_{k=1}^J L_k^\pm = 0$$

(7)

Integrability of the classical spin chain starts from the existence of a Lax matrix,

$$\mathbb{L}_k(u) = \begin{pmatrix} u + i L_k^0 & i L_k^+ \\ i L_k^- & u - i L_k^0 \end{pmatrix}$$

(8)

where $u \in \mathbb{C}$ is a spectral parameter. A tower of conserved quantities are obtained by constructing the monodromy,

$$t_J(u) = \text{tr}_2 [\mathbb{L}_1(u) \mathbb{L}_2(u) \ldots \mathbb{L}_J(u)]$$

$$= 2u^J + q_2 u^{J-2} + \ldots + q_{J-1} u + q_J$$

(9)

At large-$S$ we find $q_2 = -S^2$ up to corrections of order $1/S$. One may check starting from the Poisson brackets (5) that the conserved charges, $q_j$, $j = 2, 3, \ldots J$ are in involution: \{q_j, q_k\} = 0 $\forall$ $j, k$. Taking into account the highest-weight constraint (7), this is a sufficient number of conserved quantities for complete integrability of the chain.

The one-loop spectrum of operator dimensions at large-$S$ is determined from the semi-classical spectrum of the spin chain. It has different branches, labelled by an integer $K \leq J$, corresponding to the highest non-zero conserved charge \[13\],

$$q_K \neq 0 \quad q_j = 0 \quad \forall \ j > K$$

The one-loop anomalous dimensions in this sector are given as,

$$\Delta - S - J = \frac{\lambda}{4\pi^2} \left( \log q_K + C_{1\text{-loop}} + O\left(\frac{1}{\log^2 S}\right) \right)$$

(3)

where $C_{1\text{-loop}}$ is an undetermined constant which is independent of the moduli $q_j$. We call the branch with $K = J$ the highest sector. For each $K < J$ there is also a sector of states isomorphic to the highest sector of a shorter chain with only $K$ sites. In the limit of large $S$, the conserved charge $q_j$ scales as $S^j$ for $j = 2, \ldots, K$. 

At the classical level, the conserved charges $q_j$ vary continuously. The leading large-$S$ form of the discrete spectrum arises from imposing appropriate Bohr-Sommerfeld quantisation conditions. These conditions are formulated in terms of the spectral curve,

$$\Gamma_K : \quad t + \frac{1}{t} = 2 + \frac{q_2}{u^2} + \frac{q_3}{u^3} + \ldots + \frac{q_K}{u^K}$$

which corresponds to a double cover of the $u$ plane with $2K - 2$ branch points at $u = u_1, \ldots, u_{2K-2}$ as shown in Figure 2. We also define $K - 1$ one-cycles $\alpha_j$, $j = 1, \ldots, K - 1$ as shown in the Figure 2.

The Bohr-Sommerfeld conditions are expressed in terms of the periods of a certain meromorphic differential on $\Gamma_K$,

$$\frac{1}{2\pi} \oint_{\alpha_j} u \frac{dt}{t} = l_j \in \mathbb{Z}^+ \quad (10)$$

for $j = 1, 2, \ldots, K - 1$. The positive integers $l_j$ which label the states are known as filling numbers. As discussed below they correspond to the total number of Bethe roots associated with the corresponding cut. Imposing these conditions leads to a discrete spectrum for the conserved charges,

$$q_j = q_j [l_1, l_2, \ldots, l_{K-1}] \quad (11)$$

and therefore also for the anomalous dimensions (3).

The semiclassical limit of large spin can also be studied as a limit of the Bethe Ansatz which provides the exact solution of the corresponding quantum spin chain. Exact eigenstates of

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2To state the main results of [11, 13], we will not need a to introduce a full basis of cycles on $\Gamma_K$. 

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the spin chain are characterised by a set of magnon rapidities \( \{ u_a \} \), \( a = 1, 2, \ldots, S \) also known as Bethe roots. These roots solve the Bethe Ansatz equations (BAE),

\[
\left( \frac{u_a + i/2}{u_a - i/2} \right)^J = \prod_{b \neq a}^{S} \left( \frac{u_a - u_b - i}{u_a - u_b + i} \right)
\]

(12)

for each \( a = 1, 2, \ldots, S \). The conserved momentum associated with each Bethe root is,

\[
p_a = \frac{1}{i} \log \left( \frac{u_a - \frac{i}{2}}{u_a + \frac{i}{2}} \right) + 2\pi n_a
\]

(13)

Here \( n_a \) is an integer known as the mode number of the Bethe root \( u_a \) which is naturally defined modulo \( J \). It is believed that the BAE (12) has a unique solution corresponding to each set of mode numbers \( \{ n_a \} \). Thus a state of the quantum spin chain is completely specified by its set of mode numbers \( \{ n_a \} \). Equivalently, the state is also characterised by its filling numbers \( l_j \) introduced above. In the context of the Bethe ansatz these correspond to the number of roots with mode number equal (modulo \( J \)) to \( j \). Hence they obey \( \sum_j l_j = S \). For each solution of the BAE, the formula for the one loop anomalous dimension is,

\[
\Delta - S - J = \frac{\lambda}{8\pi^2} \sum_{a=1}^{S} \frac{1}{u_a^2 + \frac{1}{4}}
\]

(14)

In the large-\( S \) limit the roots condense to form cuts in the complex \( u \) plane and the resulting double-cover of the \( u \) plane is precisely the spectral curve \( \Gamma_K \) introduced above. Here \( K \) corresponds to the number \( \leq J \) of filling numbers \( l_j \) which scale linearly with \( S \) as \( S \to \infty \). In this limit the exact spectrum of one-loop anomalous dimensions specified by equations (12,14) goes over to the semiclassical result specified by (3,10,11).

### 3 Classical string theory in \( \text{AdS}_3 \)

#### 3.1 Preliminaries and Conventions

3.1.1 Coordinates, string equations of motion and Virasoro constraints

\( \text{AdS}_3 \) space is a 3-dimensional hyperboloid embedded in \( \mathbb{R}^{2,2} \) defined by the following constraint:

\[
X_\mu X^\mu = -X_0^2 - X_1^2 + X_2^2 + X_3^2 = -1
\]
The $\sigma$-model action, in conformal gauge, is then defined in terms of the embedding coordinates $X_\mu$ as:

$$I = -\frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau [G_{\mu\nu} \partial_a X^\mu \partial^a X^\nu + \Lambda (X_\mu X^\mu + 1)]$$  \hspace{1cm} (15)$$

where $\lambda = g^2 N$ is the t’Hooft coupling, $G_{\mu\nu} = \text{diag}(-1, -1, 1, 1)$ is the $\mathbb{R}^{2,2}$ metric, and the worldsheet indices are contracted with the 2-dimensional Minkowski metric $\eta_{ab} = \text{diag}(-1, 1)$.

Once we eliminate the Lagrange multiplier $\Lambda$, the equations of motion for this action become:

$$\partial_+ \partial_- X_\mu - (\partial_+ X^\nu \partial_- X_\nu) X_\mu = 0$$

where we have introduced the light-cone worldsheet coordinates, $\sigma^\pm = (\tau \pm \sigma)/2$. The string must also obey the Virasoro constraints which read:

$$\partial_\pm X^\mu \partial_\pm X_\mu = 0$$

$AdS_3$ can also be parametrised by global coordinates $(t, \rho, \phi)$, which are related to the embedding coordinates as follows:

$$X_0 = \cosh \rho \cos t$$
$$X_1 = \cosh \rho \sin t$$
$$X_2 = \sinh \rho \cos \phi$$
$$X_3 = \sinh \rho \sin \phi$$

In terms of these coordinates, the $AdS_3$ line element is:

$$ds^2 = -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\phi^2$$ \hspace{1cm} (16)$$

We also notice that the points $(t, -\rho, \phi)$ and $(t, \rho, \phi + \pi)$ correspond to the same point of the hyperboloid in the embedding space $\mathbb{R}^{2,2}$. Since we want our coordinate chart to be 1-1, we then need to demand that $\rho \geq 0$.

Finally, we introduce the complex coordinates $Z_i$:

$$Z_1 = X_0 + iX_1 = \cosh \rho \, e^{it}$$
$$Z_2 = X_2 + iX_3 = \sinh \rho \, e^{i\phi}$$  \hspace{1cm} (17)$$
In terms of these coordinates, the $AdS_3$ constraint, the equations of motion and the Virasoro constraints are respectively rewritten as:

\[
|Z_1|^2 - |Z_2|^2 = 1 \tag{18}
\]

\[
\partial_+ \partial_- Z_i - \text{Re}(-\partial_+ Z_i \partial_- \bar{Z}_1 + \partial_+ Z_2 \partial_- \bar{Z}_2)Z_i = 0 \quad i = 1, 2 \tag{19}
\]

\[
-\partial_\pm Z_1 \partial_\pm \bar{Z}_1 + \partial_\pm Z_2 \partial_\pm \bar{Z}_2 = 0 \tag{20}
\]

### 3.1.2 Conserved charges, SU(1,1) and the Monodromy Matrix

The action (15) is invariant under global time translations $t \rightarrow t + a$, and rotations $\phi \rightarrow \phi + b$. By Noether’s theorem, the associated conserved charges are the energy $\Delta$ and the angular momentum $S$, given by:

\[
\Delta = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma \text{Im}(\bar{Z}_1 \partial_{\tau} Z_1) \tag{21}
\]

\[
S = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma \text{Im}(\bar{Z}_2 \partial_{\tau} Z_2) \tag{22}
\]

where the integrals are carried out over the entire range of $\sigma$ (e.g. $[0, 2\pi]$ for a closed string).

We now recall the fact that it is possible to associate each point in $AdS_3$ with an element $g$ of the group $SU(1,1)$ as follows:

\[
g = \begin{pmatrix} Z_1 & Z_2 \\ Z_2 & \bar{Z}_1 \end{pmatrix} \tag{23}
\]

The matrix $g$ satisfies the $SU(1,1)$ properties:

\[
g^\dagger M g = M, \quad M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

\[
\det g = 1
\]

by virtue of the $AdS_3$ constraint (18).

The corresponding Lie algebra $\mathfrak{su}(1,1)$ is defined as the space of $2 \times 2$ matrices $m$ satisfying:

\[
\text{tr} m = 0, \quad m^\dagger = -M m M
\]

We choose the set of generators $(s^1, s^2, s^3) = (-i\sigma_3, \sigma_1, -\sigma_2)$ (where $\sigma_1$, $\sigma_2$ and $\sigma_3$ are the Pauli matrices) obeying:

\[
[s^A, s^B] = -2\epsilon^{ABC} \eta_{CD} s^D
\]

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and:
\[ \text{tr}(s^A s^B) = 2\eta^{AB} \]  

(24)

where \( \eta_{AB}/2 \), with \( \eta = \text{diag}(-1, 1, 1) \), is the metric on the Lie algebra.

A generic Lie algebra valued quantity \( V \) is expressed as:
\[ V = V_A s^A = \frac{1}{2} \eta_{AB} V^A s^B = \frac{1}{2} \left( \begin{array}{cc} iV^0 & V^1 + iV^2 \\ V^1 - iV^2 & -iV^0 \end{array} \right) \]  

(25)

In particular we define the \( \mathfrak{su}(1, 1) \)-valued right current \( j \):
\[ j_a = g^{-1} \partial_a g = \frac{1}{2} \eta_{AB} j_a^A s^B \]  

(26)

in terms of which the \( \sigma \)-model action (15) becomes:
\[ I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \frac{1}{2} \text{tr}(-j_a j^a) \]

which is the action of the \( SL(2, \mathbb{R}) \) Principal Chiral model.

In this formulation the theory is invariant under left and right multiplication by a constant \( SU(1, 1) \) group element \( U_L/U_R \):
\[ g \rightarrow U_L g \ , \quad g \rightarrow g U_R \]  

(27)

The associated conserved currents are:
\[ J^a_L = -\frac{\sqrt{\lambda}}{4\pi} j^a , \quad J^a_R = -\frac{\sqrt{\lambda}}{4\pi} j^a \]

where \( j^a \) is defined in (26) above and
\[ l_a = (\partial_a g) g^{-1} \]  

(28)

The equations of motion imply the conservation conditions:
\[ \partial_+ j_- + \partial_- j_+ = -2 \partial^a j_a = 0 , \quad \partial_+ l_- + \partial_- l_+ = -2 \partial^a l_a = 0 \]

and the currents also obey flatness conditions:
\[ \partial_+ j_- - \partial_- j_+ - [j_-, j_+] = 0 , \quad \partial_+ l_- - \partial_- l_+ + [l_-, l_+] = 0 \]

The corresponding conserved charges are:
\[ Q_L = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma l_\tau , \quad Q_R = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma j_\tau \]  

(29)
The components of these charges lying in the Cartan subalgebra generated by $s^0$ can be related to the string energy (21) and AdS$_3$ angular momentum (22):

$$Q^0_L = \Delta - S, \quad Q^0_R = \Delta + S$$  \hspace{1cm} (30)

Another important point is the fact that:

$$-\frac{1}{2} \text{tr} j^2_a = \det j_a = \partial_a Z_1 \partial_a Z_1 - \partial_a Z_2 \partial_a Z_2$$  \hspace{1cm} (31)

(where the first equality is just a consequence of the general $\mathfrak{su}(1,1)$ matrix structure (25), and $j^2_a$ indicates the matrix square of $j_a$) and thus it is possible to express the Virasoro constraints (20) in terms of the current:

$$-\frac{1}{2} \text{tr} j^2_{\pm} = \det j_{\pm} = 0$$  \hspace{1cm} (32)

Integrability of the string $\sigma$-model follows from the existence of the Lax connection:

$$J_{\tau}(x, \sigma, \tau) = \frac{1}{2} \left( \frac{\dot{j}_+}{1-x} + \frac{\dot{j}_-}{1+x} \right)$$

$$J_{\sigma}(x, \sigma, \tau) = \frac{1}{2} \left( \frac{\dot{j}_+}{1-x} - \frac{\dot{j}_-}{1+x} \right)$$

which is a one-parameter family of $\mathfrak{su}(1,1)$ connections labelled by the spectral parameter $x$. The flatness of the Lax connection, for all $x \in \mathbb{R}$ is equivalent to the string equations of motion given above. This flatness condition in turn implies that the monodromy matrix:

$$\Omega[x; \sigma_0, \tau_0] = \mathcal{P}\exp \left[ \int_{\gamma(\sigma_0, \tau_0)} d\sigma^a (-J_a) \right] \in SL(2, \mathbb{R})$$  \hspace{1cm} (33)

evolves by conjugation in the group. Here $\mathcal{P}\exp$ indicates the path-ordered exponential and the integral is calculated along an arbitrary closed path $\gamma$ with base point $(\sigma_0, \tau_0)$. In the following we specialize to a path which winds once around the string at constant worldsheet time $\tau = \tau_0$. In this case

$$\Omega[x; \tau] = \mathcal{P} \exp \left[ \frac{1}{2} \int_0^{2\pi} d\sigma \left( \frac{\dot{j}_+}{x-1} + \frac{\dot{j}_-}{x+1} \right) \right] \in SL(2, \mathbb{R})$$  \hspace{1cm} (34)

and the corresponding eigenvalues $t_{\pm} = \exp(\pm i p(x))$ are $\tau$-independent for all values of the spectral parameter $x$. It is convenient to consider the analytic continuation of the monodromy matrix $\Omega[x; \tau]$ and of the quasi-momentum $p(x)$ to complex values of $x$. In this case $\Omega$ will take values in $SL(2, \mathbb{C})$ and appropriate reality conditions must be imposed to recover the physical case.
The eigenvalues $t_{\pm}(x)$ are two branches of an analytic function defined on the spectral curve,

$$
\Sigma_\Omega : \quad t + \frac{1}{t} = \text{tr} \Omega[x; \tau] = 2\cos p(x) \quad t, \ x \in \mathbb{C}
$$

(35)

This curve corresponds to a double cover of the complex $x$-plane with branch points at the simple zeros of the discriminant $D = 4\sin^2 p(x)$. In the following sections we will determine this curve for a variety of solutions in the limit of large angular momentum.

### 3.2 The $N$-folded string

#### 3.2.1 General Properties

The $N$-folded GKP string is a simple generalisation of the solution that was first described in [23]. We start with the following ansatz\(^3\): $t = \tilde{\tau}$, $\rho = \rho(\tilde{\sigma})$, $\phi = \phi_0 + \omega\tilde{\tau}$. The string equations of motion in conformal gauge then reduce to:

$$(\partial_{\tilde{\sigma}} \rho)^2 = \cosh^2 \rho - \omega^2 \sinh^2 \rho
$$

(36)

which we can integrate as:

$$
\tilde{\sigma} = \int_0^{\rho(\tilde{\sigma})} \frac{dy}{\sqrt{\cosh^2 y - \omega^2 \sinh^2 y}}
$$

(37)

with an appropriate choice of sign for the square root in the integrand. As we want the integrand to be real, we have to restrict $\rho$ to the range $\rho(\tilde{\sigma}) \in [0, \rho_1]$, where $\coth \rho_1 = \omega$ with $\omega > 1$. In the limit $\omega \to 1$, we have $\rho_1 \to +\infty$ and the bound is lifted. The solution to (37) is:

$$
\rho(\tilde{\sigma}) = -i \text{am}(i\tilde{\sigma}|\sqrt{1-\omega^2})
$$

(38)

which yields:

$$
\coth \rho(\tilde{\sigma}) = \frac{\omega}{\text{sn} \left( \omega \tilde{\sigma} \left| \frac{1}{\omega} \right. \right)}
$$

(39)

This expression is periodic in $\tilde{\sigma}$ with period $(4/\omega)K(1/\omega) \equiv 4\tilde{L}$. However, to impose the condition $\rho \geq 0$ we must restrict the solution to the first half-period $\tilde{\sigma} \in [0, 2\tilde{L}]$. In this

---

\(^3\)In the following we reserve the notation $\tau$ and $\sigma$ for worldsheet coordinates with periodicity $\sigma + 2\pi$. These are related to the present worldsheet coordinates $\tilde{\tau}$ and $\tilde{\sigma}$ by a rescaling we will describe below.
interval \( \rho \) increases from zero to its maximum value \( \rho_1 \) at \( \bar{\sigma} = \tilde{L} \) and then returns to zero at \( \bar{\sigma} = 2\tilde{L} \). The snapshot of the string at fixed global time \( t \) consists of two straight segments of string connecting the origin of the global coordinates with the point \((\rho_1, \phi(t))\). We can add a second pair of segments, stretching out in the opposite direction, by gluing a similar solution, with \( \bar{\sigma} \) replaced by \( \bar{\sigma} - 2\tilde{L} \) (translations of \( \bar{\sigma} \) are clearly a symmetry of the differential equation (36)) and \( \phi(t) \) replaced by \( \phi(t) + \pi \) (this only amounts to a change in the value of \( \phi_0 \), which is arbitrary). In this way we obtain the composite solution,

\[
\begin{align*}
\rho &= \rho(\bar{\sigma}) \quad \phi = \phi_0 + \omega \bar{\tau} \quad \text{for } \bar{\sigma} \in [0, 2\tilde{L}] \\
\rho &= \rho(\bar{\sigma} - 2\tilde{L}) \quad \phi = \phi_0 + \omega \bar{\tau} + \pi \quad \text{for } \bar{\sigma} \in [2\tilde{L}, 4\tilde{L}]
\end{align*}
\]

where now \( \bar{\sigma} \) runs over a full period \( 4\tilde{L} \) but, because of the gluing, we preserve the condition \( \rho \geq 0 \). The solution corresponds to a folded, closed string rotating about its midpoint which lies at the origin. The string has two cusps at the points \( \bar{\sigma} = \tilde{L} \) and \( \bar{\sigma} = 3\tilde{L} \).

The above solution can be easily generalised by allowing \( \bar{\sigma} \) to range over \( N \) periods:

\[
\begin{align*}
\rho &= \rho(\bar{\sigma} - l(\bar{\sigma})(2\tilde{L})) \quad \text{where } l(\bar{\sigma}) = \left\lfloor \frac{\bar{\sigma}}{2\tilde{L}} \right\rfloor \\
\phi &= \phi_1 + \omega \bar{\tau} = \begin{cases} 
\phi_0 + \omega \bar{\tau} & \text{if } l(\bar{\sigma}) \text{ is even} \\
\phi_0 + \omega \bar{\tau} + \pi & \text{if } l(\bar{\sigma}) \text{ is odd}
\end{cases}
\end{align*}
\]

were now \( \bar{\sigma} \in [0, 4N\tilde{L}] \). The image of the string in the target space is unchanged, but the increased range of the coordinate \( \bar{\sigma} \) corresponds to a string folded \( N \) times upon itself. We define:

\[ L = 4N\tilde{L} = \frac{4N}{\omega} K \]

as the upper extremum of the range of \( \bar{\sigma} \), and:

\[ \bar{\sigma}_* = \bar{\sigma} - l(\bar{\sigma})(2\tilde{L}) \]

where we have introduced the shorthand notation \( E \equiv E(1/\omega) \) and \( K \equiv K(1/\omega) \), which we will use throughout the rest of this section for the elliptic integrals of the first and second kind.

\footnote{Here \([X] \) denotes the greatest integer less than \( X \).}
We note that this solution has $K = 2N$ cusps, that is two for each fold of the string, located at the tips $\rho = \rho_1$ of the line segment, which we can identify with the following worldsheet positions:

$$\tilde{\sigma}_m = (2m + 1)\tilde{L}, \quad m = 0, \ldots, K - 1$$

(41)

Cusps are points along the string where the unit normalised tangent vector has a discontinuity. The only way in which this can happen without compromising the smoothness of the worldsheet is that all components of the tangent vector vanish at the point, so that its direction is actually allowed to change discontinuously even though the vector itself varies smoothly. Therefore, spikes are points at which the derivatives with respect to $\tilde{\sigma}$ of all target space coordinates vanish. Here, we can easily see that $\partial_{\tilde{\sigma}}Z_1 = 0 = \partial_{\tilde{\sigma}}Z_2$ at $\tilde{\sigma} = \tilde{\sigma}_m$, for all $m$, as required. As $\omega \to 1$, $\rho_1$ tends to $+\infty$, and hence the spikes touch the boundary of $AdS_3$ and the string becomes infinitely long.

### 3.2.2 Energy, angular momentum and large angular momentum behaviour

The energy and the angular momentum of the solution (40) can straightforwardly be computed from (21) and (22):

$$\Delta = 2N\frac{\sqrt{\lambda}}{2\pi} \int_0^{2\tilde{L}} d\tilde{\sigma} \frac{1}{\text{dn}^2(\omega\tilde{\sigma}_*|\frac{1}{\omega})} = K\frac{\sqrt{\lambda}}{\pi} \frac{\omega}{\omega^2 - 1}$$

$$S = 2N\frac{\sqrt{\lambda}}{2\pi} \int_0^{2\tilde{L}} d\tilde{\sigma} \frac{\text{sn}^2(\omega\tilde{\sigma}_*|\frac{1}{\omega})}{\text{dn}^2(\omega\tilde{\sigma}_*|\frac{1}{\omega})}$$

$$= K\frac{\sqrt{\lambda}}{\pi} \left[ \frac{\omega^2}{\omega^2 - 1} E - K \right]$$

As we could expect due to the periodicity of the integrands, the values are just $N$ times the original GKP values, as they appear in [24].

We are interested in the large $S$ behaviour of this solution, which corresponds to the limit $\omega \to 1$. Both $\Delta$ and $S$ diverge in this limit and, if we define:

$$\omega = 1 + \eta$$

then their respective behaviours are given by:

$$\Delta = K\frac{\sqrt{\lambda}}{2\pi\eta} - K\frac{11\sqrt{\lambda}}{32\pi} \log \eta - K\frac{\sqrt{\lambda}}{64\pi} [13 - 44 \log(2\sqrt{2})] + O(\eta \log \eta)$$

(42)

$$S = K\frac{\sqrt{\lambda}}{2\pi\eta} + K\frac{5\sqrt{\lambda}}{32\pi} \log \eta + K\frac{\sqrt{\lambda}}{64\pi} [19 - 20 \log(2\sqrt{2})] + O(\eta \log \eta)$$

(43)

Thus each spike contributes $\sqrt{\lambda}/(2\pi\eta)$ to the leading order term.
From these equations, we can deduce the leading behaviour of the anomalous dimension:

\[
\Delta - S = \frac{K \sqrt{\lambda}}{2\pi} \log \left( \frac{2\pi S}{K \sqrt{\lambda}} \right) + \frac{K \sqrt{\lambda}}{2\pi} (3 \log 2 - 1) + O(\eta \log \eta)
\]

which exhibits the same logarithmic growth found in gauge theory.

### 3.2.3 Spectral curve for large S

In this section we will compute the monodromy matrix and spectral curve of the \(N\)-folded string solution discussed above. As a starting point, we need to compute the time-like component of the right current \(j\). Using the ansatz \(\rho = \rho(\tilde{\sigma})\) we obtain:

\[
j^0_\tau = 2[(\partial_\tau t) \cosh^2 \rho + (\partial_\tau \phi) \sinh^2 \rho] \\
j^1_\tau + ij^2_\tau = 2i \sinh \rho \cosh \rho e^{i(\phi - t)}[(\partial_\tau t) + (\partial_\tau \phi)]
\]

Evaluating this on the \(N\)-folded string solution we obtain,

\[
j^0_\tau(\tau, \sigma) = \frac{2K\pi K}{\pi \omega} \frac{1 + \frac{1}{\omega} \text{sn}^2 \left( \omega \sigma_s \left| \frac{1}{\omega} \right. \right)}{\text{dn}^2 \left( \omega \sigma_s \left| \frac{1}{\omega} \right. \right)} \\
j^1_\tau(\tau, \sigma) + ij^2_\tau(\tau, \sigma) = i \frac{2K\pi K}{\pi} \frac{\text{sn} \left( \omega \sigma_s \left| \frac{1}{\omega} \right. \right)}{\omega^2} e^{i[\phi_1 + (\omega - 1) \frac{\pi}{\omega} \tau]} \text{dn}^2 \left( \omega \sigma_s \left| \frac{1}{\omega} \right. \right)
\]

Here we have introduced rescaled worldsheet coordinates \((\tau, \sigma)\) with \(\sigma \in [0, 2\pi]\):

\[
(\tau, \sigma) = \frac{2\pi}{L}(\tilde{\tau}, \tilde{\sigma}) = \frac{\pi \omega}{K}\pi(\tilde{\tau}, \tilde{\sigma})
\]

and \(\sigma_s\) is just \(\tilde{\sigma}_s\) written as a function of \(\sigma\). We can also express the worldsheet positions of the cusps \((\Pi)\) in terms of the rescaled coordinate \(\sigma\) as:

\[
\sigma_m = (2m + 1) \frac{\pi}{K}, \quad m = 0, \ldots, K - 1
\]

The next step is to take the limit \(\omega \to 1\) so that the angular momentum \(S\) of the solution diverges. The key point here is that, as \(S \to \infty\), the charge density is dominated by the vicinity of the cusp points \(\sigma = \sigma_m\). To demonstrate this we expand around the \(m\)-th cusp point setting:

\[
\sigma = \sigma_m + \hat{\sigma}, \quad \text{with } |\hat{\sigma}| < \frac{\pi}{K}
\]
which is equivalent to \( \sigma_* = \tilde{L} + \hat{\sigma} K \mathbb{K} / (\pi \omega) \) for each \( m \). We can then use the quarter-period transformation formulae for the elliptic functions appearing in (46) to get:

\[
\begin{align*}
\text{sn} \left( \omega \sigma_* \left| \frac{1}{\omega} \right. \right) &= \text{sn} \left( \mathbb{K} + \frac{K \mathbb{K}}{\pi} \hat{\sigma} \left| \frac{1}{\omega} \right. \right) = \text{cd} \left( \frac{K \mathbb{K}}{\pi} \hat{\sigma} \left| \frac{1}{\omega} \right. \right) \\
\text{dn} \left( \omega \sigma_* \left| \frac{1}{\omega} \right. \right) &= \text{dn} \left( \mathbb{K} + \frac{K \mathbb{K}}{\pi} \hat{\sigma} \left| \frac{1}{\omega} \right. \right) = \sqrt{1 - \frac{1}{\omega^2}} \text{nd} \left( \frac{K \mathbb{K}}{\pi} \hat{\sigma} \left| \frac{1}{\omega} \right. \right)
\end{align*}
\]

As we are interested in the limit \( \omega \to 1 \) we can use the standard series expansions for \( \text{cd}(z|k) \) and \( \text{nd}(z|k) \) in powers of \((k - 1) \) [25], with

\[
z = \frac{K \mathbb{K}}{\pi} \hat{\sigma}
\]

and \( k = 1/\omega \). Note however that \( \mathbb{K} \sim -(1/2) \ln(\omega - 1) \) for \( \omega \to 1 \). This implies we must consider a limit where not only \( k \to 1 \), but also \( z \to \pm \infty \), depending on the sign of \( \hat{\sigma} \). However, since \(|z| = K \mathbb{K} |\hat{\sigma}|/\pi < \mathbb{K} \), i.e. \( z \) is always within the first quarter-period in both directions, one can check that higher order terms remain suppressed. We thus consider only the lowest order terms in these series, which give the leading behaviour of \( j_\tau \) near each spike as:

\[
\begin{align*}
J^0_\tau(\tau, \sigma) &\approx \frac{2K \mathbb{K}}{\pi} \frac{1}{\eta \cosh^2 \left( \frac{K \mathbb{K}}{\pi} \hat{\sigma} \right)} \\
J^1_\tau(\tau, \sigma) + ij^2_\tau(\tau, \sigma) &\approx (-1)^m i \frac{2K \mathbb{K}}{\pi} \frac{e^{i\phi_0}}{\eta \cosh^2 \left( \frac{K \mathbb{K}}{\pi} \hat{\sigma} \right)}
\end{align*}
\]

Here the factor \((-1)^m\) comes from the extra \( \pi \) which is added to \( \phi \) every other period, as specified in (40), and consequently affects the contribution of every other cusp.

As the constant \( \mathbb{K} \) diverges like \( \log \eta \) for \( \omega \to 1 \), the resulting expression for \( j_\tau \) is singular in this limit. To obtain a finite result, we first define the normalised charge density:

\[
\mu^A(\tau, \sigma) = \lim_{\omega \to 1} \frac{\sqrt{\lambda}}{8\pi S} J^A_\tau(\tau, \sigma)
\]

which satisfies:

\[
\int_0^{2\pi} d\sigma \tilde{\mu}(\tau, \sigma) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

on highest weight states. Using (48) and (43), we can compute the contribution to the normalised charge density from the region near the \( m \)-th cusp and then sum it over \( m = 17 \).
\[
\mu^0(\tau, \sigma) = \frac{1}{K} \sum_{m=0}^{K-1} \delta(\sigma - \sigma_m)
\]

\[
\mu^1(\tau, \sigma) + i\mu^2(\tau, \sigma) = \frac{e^{i(\phi_0 + \Phi)}}{K} \sum_{m=0}^{K-1} (-1)^m \delta(\sigma - \sigma_m)
\]

where we have used the identity:

\[
\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \frac{1}{\cosh^2 \left( \frac{\epsilon}{\tau} \right)} = \delta(x)
\]

and we have eliminated \(\hat{\sigma}\) in favour of \(\sigma\) according to (47). It is now easy to see that the normalisation condition (50) is indeed satisfied. Finally, using (49), we can obtain the large-\(S\) asymptotics of the current \(j_\tau\) in the form:

\[
j^A_\tau(\tau, \sigma) \to \frac{8\pi}{\sqrt{\lambda}} \sum_{m=0}^{K-1} \bar{L}_m^A \delta(\sigma - \sigma_m)
\]

with:

\[
\bar{L}_m = \frac{S}{K} \begin{pmatrix}
1 \\
(-1)^{m+1} \sin \phi_0 \\
(-1)^m \cos \phi_0
\end{pmatrix}
\]

m = 0, \ldots, K - 1

The \textbf{su}(1,1)-valued quantities \(L_m\) played an important role in the proposal of [9]. In particular, for generic large-\(S\) solutions, they can be mapped onto the classical spins of the dual gauge theory spin chain. For this particular solution, we can easily verify the properties:

\[
\sum_{m=0}^{K-1} \bar{L}_m = \begin{pmatrix} S \\ 0 \\ 0 \end{pmatrix}, \quad \eta_{AB} L_m^A L_m^B = 0 \quad \text{for} \quad m = 0, \ldots, K - 1
\]

which are significant in this context as the string theory counterparts of the relations (6) and (7). The first equality is just a rephrasing of the normalisation property (50), while the second can be seen as a consequence of the Virasoro constraints (32). In particular, as \(j_\sigma(\tau, \sigma_m) = 0, \forall m\), we have \(j_\pm(\tau, \sigma_m) = j_\tau(\tau, \sigma_m)\), and the Virasoro constraints at the spikes become:

\[
-\frac{1}{2} \lim_{\sigma \to \sigma_m} \text{tr}[j^2_\pm(\tau, \sigma)] = -\frac{1}{2} \lim_{\sigma \to \sigma_m} \text{tr}[j^2_\tau(\tau, \sigma)] = 0
\]

If we now also take the limit \(\omega \to 1\), substitute in equation (52), decompose \(j_\tau\) onto the generators \(s^A\) as in (25) and use property (24), we see that the only way in which this condition can hold is that the spin vectors satisfy the second equation (54).
Our next goal is to compute the limiting form of the monodromy matrix (34) in the large spin limit. This calculation was described for the $N = 1$ case in [9] and we will follow the same steps here. To keep the exponent of the monodromy matrix (34) finite as $S \to \infty$ we are forced to scale the spectral parameter as $x \sim S$. The limiting form of the monodromy matrix then becomes:

$$\Omega[x; \tau] \simeq \mathcal{P} \exp \left[ \frac{1}{x} \int_0^{2\pi} d\sigma j_\tau(\tau, \sigma) \right]$$

We can now replace $j_\tau$ by its limiting form (52). The resulting sum of $\delta$-functions in the integrand converts the path-ordered exponential into a finite ordered product of exponentials:

$$\Omega[x; \tau] \simeq \prod_{m=0}^{K-1} \exp \left[ \frac{4\pi}{\sqrt{\lambda}} x L_m^A s^B \eta_{AB} \right]$$

where we have also expressed $L_m$ in terms of the $\mathfrak{su}(1,1)$ generators, according to (25). We then observe that:

$$(\eta_{AB} L_m^A s^B)^2 = \frac{1}{2} \eta_{AB} L_m^A L_m^B = 0$$

as a consequence of the fact that the generators $s^A = (-i\sigma_3, \sigma_1, -\sigma_2)^A$ satisfy:

$$\{s^A, s^B\} = 2\eta^{AB}$$

and of the second property (54) of the spin vectors. Therefore, the series expansion for the exponential in (55) actually truncates at the linear term:

$$\Omega[x; \tau] \simeq \prod_{m=0}^{K-1} \left[ 1 + \frac{1}{u} \eta_{AB} L_m^A s^B \right]$$

where we have defined a rescaled spectral parameter $u = x\sqrt{\lambda}/(4\pi)$. Using the explicit form (53) for the spin vectors $L_m$ we obtain:

$$\Omega[x; \tau] \simeq \frac{1}{u^n} \prod_{m=0}^{K-1} \mathbb{L}_m(u)$$

where:

$$\mathbb{L}_m(u) = \begin{pmatrix} u + \frac{iS}{K} & (-1)^m \frac{iS}{K} e^{i\phi_0} \\ (-1)^m \frac{iS}{K} e^{-i\phi_0} & u - \frac{iS}{K} \end{pmatrix}$$

As explained in [9], the matrices $\mathbb{L}_m$ are the string theory analogues of the Lax matrices (3) of the gauge theory spin chain. In the present case, it is easily seen that the matrices $\mathbb{L}_m(u)$
only depend on the parity of \( m \). Therefore, if we define \( \mathbb{L}(u) = \mathbb{L}_0(u)\mathbb{L}_1(u) \), we can write the monodromy matrix as:

\[
\Omega[x; \tau] \approx \frac{1}{u^m}[\mathbb{L}(u)]^\frac{K}{2}
\]

It follows that the eigenvalues of \( \Omega \) can be expressed in terms of the eigenvalues of the matrix \( \mathbb{L}(u) \), which can be evaluated explicitly as:

\[
\kappa_{\pm} = u^2 - \frac{2S^2}{K^2} \pm 2\sqrt{\frac{S^4}{K^4} - \frac{u^2 S^2}{K^2}}
\]

Finally we can write the trace of the monodromy matrix as:

\[
\text{tr } \Omega[x] = \frac{\kappa_{\pm} + \kappa_{-}}{u^K}
\]

\[
= \left(1 - \frac{2S^2}{K^2 u^2} + i\sqrt{\frac{4S^2}{K^2 u^2} - \frac{4S^4}{K^4 u^4}}\right)^\frac{K}{2} + \left(1 - \frac{2S^2}{K^2 u^2} - i\sqrt{\frac{4S^2}{K^2 u^2} - \frac{4S^4}{K^4 u^4}}\right)^\frac{K}{2}
\]

\[
= 2T_{\frac{K}{2}} \left(1 - \frac{2S^2}{K^2 u^2}\right) = 2T_{K} \left(\sqrt{1 - \frac{S^2}{K^2 u^2}}\right) = 2 \cos \left[K \sin^{-1} \left(\frac{S}{Ku}\right)\right]
\]

where \( T_k(y) \) is the Chebyshev polynomial of the first kind:

\[
T_k(y) = \frac{1}{2} \left[ (y + i\sqrt{1-y^2})^k + (y - i\sqrt{1-y^2})^k \right] = \cos(k \arccos y)
\]

This is a polynomial in \( y \) of degree \( k \). We recall from the discussion of the monodromy matrix at the end of section 3.1.2 that its eigenvalues are \( t_{\pm} = \exp(\pm ip(u)) \), so that:

\[
\text{tr } \Omega[x] = 2 \cos p(u)
\]

which, together with (58), yields the following expression for the quasi-momentum associated with the N-folded GKP solution:

\[
p(u) = K \sin^{-1} \left(\frac{S}{Ku}\right)
\]

In summary, the string theory spectral curve (35) for the N-folded GKP solution takes the explicit form:

\[
\Sigma_{\Omega} : \quad t + \frac{1}{t} = 2 \cos \left[K \sin^{-1} \left(\frac{S}{Ku}\right)\right]
\]

We may now compare this directly with the gauge theory spectral curve \( \Gamma_K \) which can be written in the form:

\[
\Gamma_K : \quad t + \frac{1}{t} = \hat{p}_K \left(\frac{1}{u}\right) = 2 + \frac{q_2}{u^2} + \frac{q_3}{u^3} + \ldots + \frac{q_k}{u^k}
\]
We then find that the limiting string theory curve corresponds to a particular point in the moduli space of the gauge theory curve where the conserved charges $q_k$ take the particular values:

$$q_k = \left(\frac{2S}{K}\right)^k \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq K} \prod_{l=1}^{k} \sin \left[\frac{\pi}{2}(j_{i+1} - j_i)\right], \quad k = 2, 4, \ldots, K$$

(63)

where $j_{k+1} \equiv j_1$ and we notice that $q_k = 0$ for odd $k$ (the normalisation $q_2 = -S^2$ is checked in appendix B).

To characterise more precisely the curve corresponding to the $N$-folded string solution it is useful to determine the pattern of the branch points of the spectral curve (61), which coincide with the simple zeros of the discriminant $D = 4 \sin^2 \Delta(u)$:

$$D(u) = 4 \left[1 - T^2_\frac{K^2}{\pi} \left(1 - \frac{2S^2}{K^2 u^2}\right)\right] = 4 \left(\frac{4S^2}{K^2 u^2} \left(1 - \frac{4S^4}{K^4 u^4}\right)\right) \frac{U^2_{k-1}}{\pi-1} \left(1 - \frac{2S^2}{K^2 u^2}\right)$$

(64)

where $U_k(y)$ is the Chebyshev polynomial of the second kind:

$$U_k(y) = \frac{1}{2i\sqrt{1-y^2}} \left[\left(y + i\sqrt{1-y^2}\right)^{k+1} - \left(y - i\sqrt{1-y^2}\right)^{k+1}\right]$$

$$= \frac{\sin[(k+1)\arccos y]}{\sin \arccos y}$$

The zeros of the discriminant can then be determined as:

$$u^{-1} = \pm \frac{K}{S} \quad \text{simple}$$

$$u^{-1} = 0 \quad \text{double}$$

$$u^{-1} = \pm \frac{K}{S} \sin \left(\frac{k\pi}{K}\right), \quad k = 1, \ldots, \frac{K}{2} - 1 \quad \text{double}$$

(65)

which all lie in the interval $u^{-1} \in [-K/S, K/S]$. The presence of $K-1$ double zeros indicates the degeneration of the spectral curve to genus zero. In fact, the Abelian integral $\Delta(u)$ from (60) is analytic on the complex plane, except for the following singularities: a logarithmic branch point at $u = 0$ and two square root branch points at $u = \pm S/K$. We can make $\Delta(u)$ single-valued by introducing a single branch cut connecting these three points along the real axis. Correspondingly, the differential $d\Delta(u)$ is given by:

$$d\Delta(u) = -\frac{S \, du}{u \sqrt{u^2 - \frac{S^2}{K^2}}}$$

and it has a simple pole at $u = 0$, two square root branch points at $u = \pm S/K$ and the corresponding two square root branch points at infinity. It can be made single-valued by introducing the same cut along the real axis. This single cut defines a Riemann surface of genus zero.
From the gauge theory point of view, as explained in [13], the same degenerate curve arises as a limiting case of a two-cut solution, where the two cuts collide and merge into one at the origin. In particular we can calculate the filling fractions $l_1$ and $l_2$ for the two cuts using the general formula of Section 2:

$$l_j = \frac{1}{2\pi} \oint_{\alpha_j} u \frac{dt}{t}$$

(66)

where $\alpha_j$ is a closed contour encircling the $j$-th cut and no other singularities. In the present case the two cuts correspond to the intervals $[-S/K, 0]$ and $[0, S/K]$ in the $u$ plane and the filling fractions are given as:

$$l = \frac{S}{\pi i} \int_{-\frac{S}{K}}^{0} \frac{du}{\sqrt{u^2 - \frac{S^2}{k^2}}} = \frac{S}{2} \quad \tilde{l} = \frac{S}{\pi i} \int_{0}^{\frac{S}{K}} \frac{du}{\sqrt{u^2 - \frac{S^2}{k^2}}} = \frac{S}{2}$$

and thus turn out to be equal.

Finally, to compare the string energy with the dimension of the corresponding gauge theory operator (3) we compute the highest conserved charge $q_K$ from (63):

$$q_K = (-1)^{\frac{K}{2}} \left(\frac{2S}{K}\right)^K$$

(67)

and then compare (44) with the gauge theory prediction from equation (3):

$$\Delta - S = \frac{\sqrt{\lambda}}{2\pi} \log q_K + C_{\text{string}}(K)$$

(68)

where we have omitted subleading terms in the limit $\omega \to 1$. We obtain:

$$\Delta - S = \frac{K \sqrt{\lambda}}{2\pi} \log S + \frac{\sqrt{\lambda}}{2\pi} \left(K \log 2 - K \log K + \log(-1)^{\frac{K}{2}} + C_{\text{string}}(K)\right)$$

which suggests:

$$C_{\text{string}}(K) = K \left[\log \left(\frac{8\pi}{\sqrt{\lambda}}\right) - 1\right] - \log(-1)^{\frac{K}{2}}$$

(69)

We would also like to remark that all calculations concerning the N-folded GKP string reduce to the standard GKP results for $N = 1$ (i.e. $K = 2$), as listed in [9, 23, 24].

3.3 The symmetric spiky string

3.3.1 General properties

The Kruczenski spiky string was first discovered [16] as a solution to the equations of motion generated by the Nambu-Goto action. More recently, its conformal gauge version was found
In appendix A we verify directly that the solution presented in [24] really is gauge equivalent to the original spiky string solution of [16]. Here, we will only be interested in the conformal gauge version of the solution. The ansatz made in this case is \[ t = \tilde{\tau} + f(\tilde{\sigma}), \quad \phi = \phi_0 + \omega \tilde{\tau} + g(\tilde{\sigma}), \quad \rho = \rho(\tilde{\sigma}), \] and it leads to the following solution to the equations of motion and Virasoro constraints:

\[
\partial_{\tilde{\sigma}} f(\tilde{\sigma}) = \frac{\omega \sinh 2\rho_0}{2 \cosh^2 \rho}, \quad \partial_{\tilde{\sigma}} g(\tilde{\sigma}) = \frac{\sinh 2\rho_0}{2 \sinh^2 \rho} \]

\[
[\partial_{\tilde{\sigma}} \rho(\tilde{\sigma})]^2 = \frac{(\cosh^2 \rho - \omega^2 \sinh^2 \rho)(\sinh^2 2\rho - \sinh^2 2\rho_0)}{\sinh^2 2\rho} \]  

(70)

Again we impose \( \rho_0 \leq \rho \leq \rho_1 \), with \( \coth \rho_1 = \omega \), so that both factors in the numerator of \([\partial_{\tilde{\sigma}} \rho(\tilde{\sigma})]^2\) are positive. These equations can be integrated to give:

\[
\rho(\tilde{\sigma}) = \frac{1}{2} \cosh^{-1} [\cosh 2\rho_1 \cosh^2(v|k) + \cosh 2\rho_0 \sinh^2(v|k)]
\]

(71)

where

\[
v \equiv \sqrt{\frac{\cosh 2\rho_1 + \cosh 2\rho_0}{\cosh 2\rho_1 - 1}}, \quad k \equiv \sqrt{\frac{\cosh 2\rho_1 - \cosh 2\rho_0}{\cosh 2\rho_1 + \cosh 2\rho_0}}
\]

(72)

and:

\[
f(\tilde{\sigma}) = \frac{\sqrt{2} \omega \sinh 2\rho_0 \sinh \rho_1}{(\cosh 2\rho_1 + 1)\sqrt{\cosh 2\rho_1 + \cosh 2\rho_0}} \Pi \left( \frac{\cosh 2\rho_1 - \cosh 2\rho_0}{\cosh 2\rho_1 + 1}, x, k \right)
\]

\[
g(\tilde{\sigma}) = \frac{\sqrt{2} \sinh 2\rho_0 \sinh \rho_1}{(\cosh 2\rho_1 - 1)\sqrt{\cosh 2\rho_1 + \cosh 2\rho_0}} \Pi \left( \frac{\cosh 2\rho_1 - \cosh 2\rho_0}{\cosh 2\rho_1 - 1}, x, k \right)
\]

(73)

where \( x = \text{am}(v|k) \) \( (0 \leq k \leq 1) \). For simpler notation, we introduce \( w \equiv \cosh 2\rho \), \( w_0 \equiv \cosh 2\rho_0 \), \( w_1 \equiv \cosh 2\rho_1 \) and define:

\[
n_\pm \equiv \frac{w_1 - w_0}{w_1 \pm 1}, \quad K \equiv K(k), \quad E \equiv E(k)
\]

(74)

which will be used throughout the rest of this section.

In order to understand the shape of this solution, we first need to observe that \( \rho(v(\tilde{\sigma})) \) is periodic of period \( 2K \), starting off at \( \rho(0) = \rho_1 \), then decreasing to \( \rho(K) = \rho_0 \) at half the period and finally going back to \( \rho(2K) = \rho_1 \). Therefore, for the string to be closed, we impose \( v(\tilde{\sigma}) \in [0, 2K] \), which corresponds to \( \tilde{\sigma} \in [0, L] \), with:

\[
L = 2K \sqrt{\frac{w_1 - 1}{w_1 + w_0}} \equiv 2K \tilde{L}
\]

As in the previous section \( \tilde{\sigma} \) denotes the worldsheet coordinate prior to a rescaling which normalises its periodicity to \( 2\pi \).
\[ f(\tilde{\sigma} + 2m\tilde{L}) = f(\tilde{\sigma}) + \sqrt{2} \omega \sinh 2\rho_0 \sinh \rho_1 \frac{2m\Pi(n_+, k)}{(w_1 + 1)\sqrt{w_1 + w_0}} \]
\[ g(\tilde{\sigma} + 2m\tilde{L}) = g(\tilde{\sigma}) + \sqrt{2} \sinh 2\rho_0 \sinh \rho_1 \frac{2m\Pi(n_-, k)}{(w_1 - 1)\sqrt{w_1 + w_0}} \]

due to the pseudo-periodicities of the amplitude function and of the incomplete elliptic integral of the third kind.

Now, in order to have a closed string at constant global time \( t \), we need to substitute \( \tau = t - f(\tilde{\sigma}) \) into the original ansatz for \( \phi \), thus finding \( \phi(t, \tilde{\sigma}) = \omega t + g(\tilde{\sigma}) - \omega f(\tilde{\sigma}) \), and then to impose \( \phi(t, L) = \phi(t, 0) + 2n\pi \), for \( n \in \mathbb{Z} \). By the pseudo-periodicity, we can easily see that \( \phi(t, L) - \phi(t, 0) = 2K\Delta\phi \), where:
\[
\Delta\phi = \frac{\sqrt{2} \sinh 2\rho_0 \sinh \rho_1}{\sqrt{w_1 + w_0}} \left[ \frac{\Pi(n_-, k)}{w_1 - 1} - \frac{\omega^2 \Pi(n_+, k)}{w_1 + 1} \right]
= \frac{\sqrt{2} \sinh 2\rho_0}{\sqrt{2} \sinh \rho_1 \sqrt{w_1 + w_0}} [\Pi(n_-, k) - \Pi(n_+, k)] \tag{75}
\]
The closedness constraint then becomes:
\[
\Delta\phi = \frac{n}{K}\pi \tag{76}
\]
The resulting plot is shown in Fig. 3 and consists of \( K \) arcs of equal angular separation \( \Delta\theta = 2\Delta\phi \); a cusp is present at the joining point between each pair of consecutive arcs, where \( \rho = \rho_1 \). As global time varies, the string rigidly rotates.

We can also easily check that the cusp condition is satisfied for \( \rho = \rho_1 \). Remembering that we’re interested in the plot at constant \( t \), from (70) we immediately see that \( \partial_{\tilde{\sigma}}\rho = 0 \) and \( \partial_{\tilde{\sigma}}\phi(t, \tilde{\sigma}) = 0 \) for \( \rho = \rho_1 \). This also implies \( \partial_{\tilde{\sigma}}X_\mu = 0 \) at constant \( t \) for \( \rho = \rho_1, \mu = 0, \ldots, 3 \), with \( X_\mu \) now representing the embedding coordinates. Therefore, we can deduce that the \( n \) cusps are located at:
\[
\tilde{\sigma}_m = 2m\tilde{L}, \quad m = 0, \ldots, K - 1
\]
As far as the behaviour of the solution in conformal gauge as \( \omega \rightarrow 1 \) is concerned, we find that, analogously to the GKP case, \( \rho_1 \rightarrow +\infty \), the spikes touch the boundary and, as we’ll see shortly, the energy and angular momentum diverge. Note that, when considering this limit, \( \rho_0 \) is not fixed, since it depends on \( \rho_1 \) through equations (75) and (76). Instead, it changes so that \( \Delta\phi \) remains constant.
Figure 3: The Kruczenski spiky string in the \((\rho, \phi)\)-plane at \(t = 0\), with \(\rho_0 = 0.882663\) and \(\rho_1 = 2\).
It is also possible to compute a solution to the equations of motion and Virasoro constraints which holds for \( \omega = 1 \):

\[
\rho(\tilde{\sigma}) = \frac{1}{2} \cosh^{-1}(w_0 \cosh 2\tilde{\sigma})
\]

\[
t(\tilde{\tau}, \tilde{\sigma}) = \tilde{\tau} + \arctan \left[ \frac{\coth 2\rho_0 e^{2\tilde{\sigma}} + \frac{1}{\sinh 2\rho_0}}{\coth 2\rho_0 e^{2\tilde{\sigma}} - \frac{1}{\sinh 2\rho_0}} \right]
\]

\[
\phi(\tilde{\tau}, \tilde{\sigma}) = \tilde{\tau} + \arctan \left[ \frac{\coth 2\rho_0 e^{2\tilde{\sigma}} + \frac{1}{\sinh 2\rho_0}}{\coth 2\rho_0 e^{2\tilde{\sigma}} - \frac{1}{\sinh 2\rho_0}} \right]
\]  

(77)

This solution is obtained simply by integrating \((70)\) after substituting in \(\omega = 1\). It describes a single arc which has its endpoints on the boundary of \(AdS\), reached for \(\tilde{\sigma} \to \pm \infty\) (Fig. 4). What we see is the result of “blowing up” one of the interconnecting arcs located between two consecutive cusps in the original \(\omega > 1\) solution. In the process, the spikes are “pushed away” into the region in which \(\tilde{\sigma}\) becomes infinite, ultimately disappearing from the worldsheet. Other than from the plot, we can also see this from the fact that now we have \(\partial_\tilde{\sigma} \rho(\tilde{\sigma})]^2 = (\sinh^2 2\rho - \sinh^2 2\rho_0)/\sinh^2 2\rho\), and thus the first derivative of \(\rho(\tilde{\sigma})\) does not vanish any longer at the endpoints \(\rho \to +\infty\).

The reason why this solution is particularly helpful is that it allows us to obtain the relationship between the angular separation \(\Delta \theta = 2\Delta \phi\) at constant \(t\) of the arcs in the original solution and the parameter \(\rho_0\) in the limit \(\omega \to 1\). Since \((77)\) describes one of these arcs at \(\omega = 1\), all we have to do is to compute \(\Delta \theta\) from it:

\[
\Delta \theta = \left( \lim_{\tilde{\sigma} \to +\infty} - \lim_{\tilde{\sigma} \to -\infty} \right) \phi(t, \tilde{\sigma}) = 2\arctan \frac{1}{\sinh 2\rho_0}
\]

Therefore, we deduce that the expression defined in \((75)\) has the following behaviour:

\[
\Delta \theta \simeq 2\arctan \frac{1}{\sinh 2\rho_0}, \quad \text{as } \omega \to 1
\]  

(78)

Note that \(\Delta \theta \in (0, \pi)\), since \(0 < \rho_0 < +\infty\); this is true for any \(\omega > 1\), due to the fact that, as we remarked earlier, \(\Delta \theta\) is always fixed at a constant value by the closedness constraint \((76)\). This implies that \(n < K/2\). This result will be helpful later, when computing the monodromy matrix for large \(S\), since it shows that \(\rho_0\) always approaches a constant non-zero value as \(\rho_1\) diverges, and therefore it always behaves as \(O(1)\) in the limit \(\omega \to 1\).

We can recognise the GKP \(N\)-folded string solution as a special case of the spiky string. In particular \((78)\), shows that, when \(\Delta \theta \to \pi\), we have \(\rho_0 \to 0\) and we recover a folded string solution, which passes through the origin \(\rho = 0\).
Figure 4: The Kruczenski spiky string in conformal gauge, for $\omega = 1$ and $\rho_0 = 0.9$. 
3.3.2 Energy, angular momentum and large angular momentum behaviour

By using (71), (21) and (22), we can easily compute the energy and the angular momentum of this solution:

\[
\Delta = K \sqrt{\frac{\lambda}{\pi}} \int_0^{2\tilde{L}} d\tilde{\sigma} \cosh^2 \rho(\tilde{\sigma})
\]
\[
= K \sqrt{\frac{\lambda}{\pi}} \sqrt{\frac{w_1 - 1}{w_1 + w_0}} \left[ \frac{1}{2} (w_1 + w_0) E - \sinh^2 \rho_0 K \right]
\]

\[
S = K \omega \sqrt{\frac{\lambda}{\pi}} \int_0^{2\tilde{L}} d\tilde{\sigma} \sinh^2 \rho(\tilde{\sigma})
\]
\[
= K \omega \sqrt{\frac{\lambda}{\pi}} \sqrt{\frac{w_1 - 1}{w_1 + w_0}} \left[ \frac{1}{2} (w_1 + w_0) E - \cosh^2 \rho_0 K \right]
\]

both of which are just \(K\) times the contribution of a single arc, due to the periodicity of \(\rho(\tilde{\sigma})\).

Again, we consider the limit as \(\omega \to 1\), with \(\omega = 1 + \eta, \eta \gtrsim 0\), and expand these two quantities:

\[
\Delta = K \sqrt{\frac{\lambda}{2\pi \eta}} - K \sqrt{\frac{\lambda}{32\pi}} (8 + 3w_0) \log \eta
\]
\[
+ K \frac{\sqrt{\lambda}}{64\pi} \left[ -13w_0 + (32 + 12w_0) \log \frac{2\sqrt{2}}{\sqrt{w_0}} \right] + O(\eta \log \eta) \quad (79)
\]

\[
S = K \frac{\sqrt{\lambda}}{2\pi \eta} + K \frac{\sqrt{\lambda}}{32\pi} (8 - 3w_0) \log \eta
\]
\[
+ K \frac{\sqrt{\lambda}}{64\pi} \left[ 32 - 13w_0 + (-32 + 12w_0) \log \frac{2\sqrt{2}}{\sqrt{w_0}} \right] + O(\eta \log \eta) \quad (80)
\]

and we still find that each spike contributes \(\sqrt{\lambda}/(2\pi \eta)\) to the leading behaviour in both cases. We also compute the \(O(1)\) correction to the anomalous dimension:

\[
\Delta - S = K \frac{\sqrt{\lambda}}{2\pi} \log \left( \frac{2\pi S}{K \sqrt{\lambda}} \right) + K \frac{\sqrt{\lambda}}{2\pi} \left[ 3 \log 2 - 1 + \log \left( \frac{\Delta \theta}{2} \right) \right] + O(\eta \log \eta) \quad (81)
\]

where we have used the relation:

\[
\frac{1}{w_0} \simeq \sin \frac{\Delta \theta}{2} \quad \text{as} \ \omega \to 1
\]

(82)

which is easily obtained from (82). Again we find the usual logarithmic growth \(\Delta - S \sim K \log(S)\) which is characteristic of the gauge theory anomalous dimensions for operators of twist \(K\).

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3.3.3 Spectral curve for large S

We’ll now repeat the calculation carried out in section 3.2.3 for the Kruczenski spiky string in conformal gauge.

As before, we start by computing the components of the right current, by substituting (71) into (45), and then re-expressing them in terms of the rescaled worldsheet coordinates $(\tau, \sigma)$, defined so that $\sigma \in [0, 2\pi]$:

$$(\tau, \sigma) = \frac{2\pi}{L} (\tilde{\tau}, \tilde{\sigma}) = \pi K K \sqrt{\frac{w_1 + w_0}{w_1 - 1}} (\tilde{\tau}, \tilde{\sigma})$$

The charge density is then given by:

$$j^0_\tau(\tau, \sigma) = \frac{K K}{\pi} \sqrt{\frac{w_1 - 1}{w_1 + w_0}}$$

$$\times \left\{ (\omega + 1)[w_1 \text{cn}^2(v|k) + w_0 \text{sn}^2(v|k)] + 1 - \omega \right\}$$

$$j^1_\tau(\tau, \sigma) + i j^2_\tau(\tau, \sigma) = \frac{i K K}{\pi} \sqrt{\frac{w_1 - 1}{w_1 + w_0}} (\omega + 1) e^{i(\phi - t)}$$

$$\times \sqrt{[w_1 \text{cn}^2(v|k) + w_0 \text{sn}^2(v|k)]^2 - 1}$$

where now:

$$v \equiv \sqrt{\frac{w_1 + w_0}{w_1 - 1}} = \frac{K K}{\pi} \sigma$$

In terms of the new coordinates, the cusps are located at:

$$\sigma_m = 2m \frac{\pi}{K}, \quad m = 0, \ldots, K - 1$$

(84)

As in the GKP case, the leading order of the charge density will be dominated by the contributions coming from the cusps, which we compute individually by setting $\sigma = \sigma_m + \hat{\sigma}$, with $|\hat{\sigma}| < \pi/K$, and then expanding (83) as $\omega \to 1$, obtaining:

$$j^0_\tau(\tau, \sigma) \simeq \frac{2K K}{\pi} \frac{1}{\eta \cosh^2 \left(\frac{K K}{\pi} \hat{\sigma}\right)}$$

which is the same we found in the N-folded GKP case (c.f. equation 48), except for the fact that $K \equiv K(1/\omega)$ for GKP, whereas now we have $K \equiv K(k)$ (the leading behaviour $K \simeq -(1/2) \log \eta$ is however identical).

For the remaining two components of the charge density we find:

$$j^1_\tau(\tau, \sigma) + ij^2_\tau(\tau, \sigma) \simeq i \frac{2K K}{\pi} \frac{1}{\eta \cosh^2 \left(\frac{K K}{\pi} \hat{\sigma}\right)} e^{i(\phi_0 + m \frac{2\pi}{K})}$$
We can now use definition (49), together with the leading behaviour of $S$ as $\omega \to 1$, which can be extracted from (80), and with identity (51), to calculate the contribution to the normalised charge density from the $m$-th cusp. Then, we only have to sum over all values of $m$ to obtain:

$$
\mu^0(\tau, \sigma) = \frac{1}{K} \sum_{m=0}^{K-1} \delta(\sigma - \sigma_m)
$$

$$
\mu^1(\tau, \sigma) + i \mu^2(\tau, \sigma) = i \frac{1}{K} \sum_{m=0}^{K-1} e^{i(\phi_0 + m \frac{2\pi}{K})} \delta(\sigma - \sigma_m)
$$

From these equations, we can calculate the spin vector at each spike according to (52):

$$
\vec{L}_m = S \begin{pmatrix}
1 \\
- \sin \left( \phi_0 + m \frac{2\pi}{K} \right) \\
\cos \left( \phi_0 + m \frac{2\pi}{K} \right)
\end{pmatrix}
$$

As in the GKP case, these vectors satisfy the properties listed in (54), so that the Kruczenski solution is also a highest weight state.

Then, we can evaluate the monodromy matrix as we did in the folded string case, leading to (56), where in this case the matrix $\mathbb{L}_m(u)$ reads:

$$
\mathbb{L}_m(u) = \begin{pmatrix}
u + \frac{iS}{K} & \frac{iS}{K} e^{i(\phi_0 + m \frac{2\pi}{K})} \\
-i \frac{iS}{K} e^{-i(\phi_0 + m \frac{2\pi}{K})} & - \frac{iS}{K}
\end{pmatrix}
$$

(85)

We notice that this time no simplification occurs, i.e. the product of two consecutive matrices $\mathbb{L}_m(u)$ and $\mathbb{L}_{m+1}(u)$ still depends on $m$, and therefore we can’t proceed as we did earlier. Instead, we introduce the following sequence of matrices:

$$
S_m = \begin{pmatrix} ce^{im \frac{2\pi}{K}} & de^{im \frac{2\pi}{K}} \\
\bar{d} e^{-im \frac{2\pi}{K}} & \bar{c} e^{-im \frac{2\pi}{K}} \end{pmatrix}, \quad \text{with } |c|^2 - |d|^2 = 1
$$

for $m = 0, \ldots, K$ (where $c$ and $d$ are arbitrary, apart from the constraint on their absolute values), and notice that it makes the product $S_m^{-1} \mathbb{L}_m(u) S_{m+1} \equiv \mathbb{M}(u)$ independent of $m$. We also observe that:

$$
S_n = \begin{pmatrix} ce^{in \pi} & de^{i\pi} \\
\bar{d} e^{-in \pi} & \bar{c} e^{-i\pi} \end{pmatrix} = (-1)^n \begin{pmatrix} c & d \\
\bar{d} & \bar{c} \end{pmatrix} = (-1)^n S_0
$$

(86)
We can now compute the trace of the monodromy matrix, by inserting copies of the identity matrix, in the form of the products $S_m S_m^{-1}$, between consecutive matrices $L_m(u)$:

$$\text{tr} \Omega[x] = \frac{1}{u^K} \text{tr} \prod_{m=0}^{K-1} L_m(u)$$

$$= \frac{1}{u^K} \text{tr} \left[ S_0 S_0^{-1} \prod_{m=0}^{K-1} L_1(u) \right]$$

$$= \frac{(-1)^n}{u^K} \text{tr} \left[ S_0^{-1} \prod_{m=0}^{K-1} L_1(u) S_K^{-1} \right]$$

where in obtaining the third line we have used (86) and the cyclicity property of the trace.

The rest of the calculation proceeds as in the GKP case. We first determine the eigenvalues of $M(u)$:

$$\kappa_\pm = u \cos \frac{n\pi}{K} - \frac{S}{K} \sin \frac{n\pi}{K} \pm \sqrt{-2S u \sin \frac{n\pi}{K} \cos \frac{n\pi}{K} + \left( \frac{S^2}{K^2} - u^2 \right) \sin^2 \frac{n\pi}{K}}$$

and then deduce:

$$\text{tr} \Omega[x] = \frac{1}{u^K} \left( \kappa_+^K + \kappa_-^K \right)$$

$$= (-1)^n 2T_K \left( \cos \frac{n\pi}{K} - \frac{S \sin \frac{n\pi}{K}}{Ku} \right)$$

$$= 2 \cos \left[ n\pi + K \cos^{-1} \left( \cos \frac{n\pi}{K} - \frac{S \sin \frac{n\pi}{K}}{Ku} \right) \right]$$

where again we have used (59). Hence, we obtain the following expression for the quasimomentum:

$$p(u) = n\pi + K \cos^{-1} \left( \cos \frac{n\pi}{K} - \frac{S \sin \frac{n\pi}{K}}{Ku} \right)$$

which then yields the expression for the spectral curve for the Kruczenski solution:

$$\Sigma_{\Omega} : \quad t + \frac{1}{t} = 2 \cos \left[ n\pi + K \cos^{-1} \left( \cos \frac{n\pi}{K} - \frac{S \sin \frac{n\pi}{K}}{Ku} \right) \right]$$

We see that this again corresponds to a point in the moduli space of the curve $\Gamma_K$ appearing in equation (62) where the conserved charges take the following values:

$$q_k = \left( -\frac{2S}{K} \right)^k \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq K} \prod_{l=1}^{k} \sin \left[ \frac{n\pi}{K} (j_{l+1} - j_l) \right], \quad k = 2, \ldots, K$$

where $j_{k+1} \equiv j_1$ (the normalisation $q_2 = -S^2$ is checked in appendix B).
We can now compute the discriminant $D = 4 \sin^2 p(u)$:

$$D(u) = 4 \left[ 1 - T^2_K \left( \cos \frac{n\pi}{K} - \frac{S \sin \frac{n\pi}{K}}{Ku} \right) \right] =$$

$$= -4 \sin \frac{n\pi}{K} \left( - \sin \frac{n\pi}{K} - \frac{2S}{Ku} \cos \frac{n\pi}{K} + \frac{S^2}{K^2 u^2 \sin \frac{n\pi}{K}} \right) \times U^2_{K-1} \left( \cos \frac{n\pi}{K} - \frac{S \sin \frac{n\pi}{K}}{Ku} \right)$$

which we use to determine the pattern of branch points for the spectral curve (88):

$$u^{-1} = \frac{K}{S \sin \frac{n\pi}{K}} \left( \cos \frac{n\pi}{K} \pm 1 \right) \quad \text{simple}$$

$$u^{-1} = \frac{K}{S \sin \frac{n\pi}{K}} \left( \cos \frac{n\pi}{K} - \cos \frac{j\pi}{K} \right), \quad \text{for } j = 1, \ldots, K - 1 \quad \text{double}$$

As a side remark, we observe that, for the special case of even $K$, these zeros can be rewritten in a form which is very similar to that of the GKP zeros (65):

$$u^{-1} = \frac{K}{S \sin \frac{n\pi}{K}} \left( \cos \frac{n\pi}{K} \pm 1 \right) \quad \text{simple}$$

$$u^{-1} = \frac{K}{S \sin \frac{n\pi}{K}} \cos \frac{n\pi}{K} \quad \text{double}$$

$$u^{-1} = \frac{K}{S \sin \frac{n\pi}{K}} \left( \cos \frac{n\pi}{K} \pm \sin \frac{l\pi}{K} \right), \quad \text{for } l = 1, \ldots, \frac{K}{2} - 1 \quad \text{double}$$

these are exactly the same values appearing in (65), shifted by $\cos \frac{n\pi}{K}$ and rescaled by $K/(S \sin \frac{n\pi}{K})$.

As before, we have $K - 1$ double zeros and thus the spectral curve degenerates to genus zero. Accordingly, the quasi-momentum (87) is again an analytic function on the complex plane, with a logarithmic branch point at $u = 0$ and two square root branch points at:

$$u = \frac{S \sin \frac{n\pi}{K}}{K \left( \cos \frac{n\pi}{K} \pm 1 \right)} \equiv u_\pm$$

which always satisfy $u_- < 0 < u_+$. In order to make it single-valued, we introduce the usual branch cut connecting $u_\pm$ and the origin along the real axis. The differential $dp(u)$ is given by:

$$dp(u) = -\frac{K \sin \frac{n\pi}{K} du}{u \sqrt{(\frac{Ku}{S} \sin \frac{n\pi}{K} + \cos \frac{n\pi}{K})^2 - 1}}$$

and displays a simple pole at $u = 0$, two square root branch points at $u = u_\pm$ and the other two corresponding square root branch points at infinity. As in the GKP case, the same cut we introduced for $p(u)$ also makes $dp(u)$ single-valued.
On the gauge theory side, this spectral curve can again originate from a two-cut solution with cuts colliding at the origin $u = 0$. These cuts correspond to the intervals $[u_-, 0]$ and $[0, u_+]$, and, according to equation (66), their filling fractions are:

$$l = \frac{1}{\pi i} \int_{u_-}^{0} \frac{K \sin \frac{n\pi u}{K} du}{\sqrt{(\frac{Ku}{S} \sin \frac{n\pi u}{K} + \cos \frac{n\pi u}{K})^2 - 1}} = S \left( 1 - \frac{n}{K} \right)$$

$$\tilde{l} = \frac{1}{\pi i} \int_{0}^{u_+} \frac{K \sin \frac{n\pi u}{K} du}{\sqrt{(\frac{Ku}{S} \sin \frac{n\pi u}{K} + \cos \frac{n\pi u}{K})^2 - 1}} = S \frac{n}{K}$$

We observe that, differently from the GKP case, there is an asymmetry in the filling fractions, which is clearly due to the fact that the two square root branch points in $p(u)$ are no longer symmetric with respect to the origin.

Finally, as we did in the previous case, we compare the string energy with the dimension of the corresponding gauge theory operator (3), by computing the highest conserved charge:

$$q_K = (-1)^{K+n} \left( \frac{2S}{K} \right)^K \left( \sin \frac{n\pi}{K} \right)^K$$

Thus, the gauge theory prediction (68) for $\Delta - S$ gives the following expression:

$$\Delta - S = \frac{K \sqrt{\lambda}}{2\pi} \log S + \frac{\sqrt{\lambda}}{2\pi} \left[ K \log 2 - K \log K + \log(-1)^{K+n} + K \log \left( \sin \frac{n\pi}{K} \right) \right]$$

where again we have omitted terms which are subleading as $\omega \to 1$. Comparison with (81) (we recall that $\Delta \theta = 2\Delta \phi = 2(n/K)\pi$) yields:

$$C_{\text{string}}(K) = K \left[ \log \left( \frac{8\pi}{\sqrt{\lambda}} \right) - 1 \right] - \log(-1)^{K+n}$$

Furthermore, we would like to observe that it is possible to obtain all results for the GKP $N$-folded string from the Kruczenski string in conformal gauge, if we assume, of course, that the two solutions have the same number of cusps. This is done by interpreting the GKP configuration as a set of $K = 2N$ spikes with angular separation between consecutive cusps equal to $\pi$, or, in other words, a set of 2 spikes for each turn around the origin in $AdS$ space, which can then be described by a Kruczenski-type solution with $K$ even and $n = K/2$ (which implies $\Delta \theta = \pi$).

First of all, we can substitute this into (81) to recover (44). (79) and (80) are also seen to reduce to (42) and (43) respectively, by noticing that $w_0 = 1$ in the limit $\omega \to 1$ for $\Delta \theta = \pi$, which is easily deduced from (82). Moreover, all the conserved charges listed in equations (89) and (63) also match. Lastly, (92) agrees with (69).
4 The general patched solution

4.1 General properties

In this section, we’re going to discuss a generalised version of Kruczenski’s solution in conformal gauge, which allows for arcs with arbitrary individual angular separations $\Delta \theta_j$, $j = 1, \ldots, K$, which however are still subject to the constraint $\Delta \theta_j \in (0, \pi)$ (as seen at the end of section 3.3.1; this is an intrinsic property of the solution found by Jevicki and Jin).

The idea is to use different versions of (71) and (73) to describe each single arc, and then to patch all the arcs together by gluing them at the endpoints. In this way, we are going to construct an approximate solution, which becomes exact in the large angular momentum limit $\omega \to 1$.

We start by considering equation (75), which determines the angular separation between two consecutive cusps, as a function of the two parameters $\rho_0$ and $\rho_1$. We keep $\rho_1$ fixed, and define $K$ parameters $\rho_0^{(j)}$, $j = 1, \ldots, K$ by imposing the following constraints:

$$\Delta \theta_j = \frac{\sqrt{2} \sinh 2 \rho_0^{(j)}}{\sinh \rho_1 \sqrt{w_1 + w_0^{(j)}}} \left[ \Pi(n_-^{(j)}, k^{(j)}) - \Pi(n_+^{(j)}, k^{(j)}) \right], \quad \text{for } j = 1, \ldots, K \tag{93}$$

where $n_\pm^{(j)}$ and $k^{(j)}$ (and also $v^{(j)}$, which we’ll use later) are defined in (72) and (74), with $\rho_0$ replaced by $\rho_0^{(j)}$. Each pair $(\rho_0^{(j)}, \rho_1)$ defines a different version of the solution given in (71) and (73), with different fundamental half-period $K_j \equiv K(k^{(j)})$ (we also define $E_j \equiv E(k^{(j)})$) and angular separation $\Delta \theta_j$, but with the same radial position of the spikes $\rho = \rho_1$. We also define:

$$\tilde{L}_j = K_j \sqrt{\frac{w_1 - 1}{w_1 + w_0^{(j)}}}, \quad \tilde{\sigma}_j = 2 \sum_{k=1}^{j} \tilde{L}_k, \quad L = 2 \sum_{j=1}^{K} \tilde{L}_j \tag{94}$$

In order to glue these different solutions together, we let $\tilde{\sigma}$ run in the interval $[0, L]$: for $0 = \tilde{\sigma}_0 \leq \tilde{\sigma} \leq \tilde{\sigma}_1$ we want the patched solution to describe the first period of the $(\rho_0^{(1)}, \rho_1)$ spiky string, for $\tilde{\sigma}_1 \leq \tilde{\sigma} \leq \tilde{\sigma}_2$ we want it to describe the first period of the $(\rho_0^{(2)}, \rho_1)$ spiky string, and so on until we see the first period of the $(\rho_0^{(K)}, \rho_1)$ spiky string for $\tilde{\sigma}_{K-1} \leq \tilde{\sigma} \leq L$. 
This is achieved by the following definition:

\[
\begin{align*}
\rho(\tilde{\sigma}) &= \rho(\tilde{\sigma} - \tilde{\sigma}_{j-1}, \rho_0^{(j)}) \\
f(\tilde{\sigma}) &= f(\tilde{\sigma} - \tilde{\sigma}_{j-1}, \rho_0^{(j)}) + \sum_{k=1}^{j-1} f(2\tilde{L}_k, \rho_0^{(k)}) \\
g(\tilde{\sigma}) &= g(\tilde{\sigma} - \tilde{\sigma}_{j-1}, \rho_0^{(j)}) + \sum_{k=1}^{j-1} g(2\tilde{L}_k, \rho_0^{(k)}) ,
\end{align*}
\]

for \( \tilde{\sigma}_{j-1} \leq \tilde{\sigma} \leq \tilde{\sigma}_j \) \( (95) \)

where \( \rho(\tilde{\sigma}, \rho_0) \), \( f(\tilde{\sigma}, \rho_0) \) and \( g(\tilde{\sigma}, \rho_0) \) are given by equations (71) and (73). In obtaining this, we have used our freedom to shift \( \tilde{\sigma} \), \( f \) and \( g \) by a constant (these are all symmetries of equations (70)).

We note that, although (95) clearly satisfies the equations of motion and Virasoro constraints in each interval \( \tilde{\sigma}_{j-1} < \tilde{\sigma} < \tilde{\sigma}_j \), it is not smooth at the junction points. In particular, \( \rho(\tilde{\sigma}) \) is \( C^1 \), whereas \( f(\tilde{\sigma}) \) and \( g(\tilde{\sigma}) \) are only \( C^0 \). In the case of \( \rho \), we can see this by considering equation (70):

\[
\partial_\tilde{\sigma} \rho(\tilde{\sigma}) = \pm \sqrt{h(\rho)} , \quad h(\rho) = h_1(\rho)h_2(\rho) \\
h_1(\rho) = \cosh^2 \rho - \omega^2 \sinh^2 \rho , \quad h_2(\rho) = 1 - \frac{\sinh^2 2\rho^{(j)}}{\sinh^2 2\rho} 
\]

for \( \tilde{\sigma}_{j-1} \leq \tilde{\sigma} \leq \tilde{\sigma}_j \) \( (96) \)

where the sign is plus or minus depending on which half of the \( j \)-th arc we are considering (\( \rho \) is an increasing function of \( \tilde{\sigma} \) along one half of every arc and it is instead decreasing on the other half). Clearly, at the junction points \( \rho = \rho_1 \) we have \( \partial_\tilde{\sigma} \rho(\tilde{\sigma}) = 0 \), independently of \( j \), due to the fact that \( h_1(\rho_1) = 0 \). However, when we turn our attention to the second derivative of \( \rho \), we obtain:

\[
\partial_\tilde{\sigma}^2 \rho(\tilde{\sigma}) = \frac{h'(\rho)}{2} , \quad \text{where} \quad ' \equiv \partial_\rho 
\]

and it is very easy to see that \( h'(\rho) \) contains a term which does not vanish at \( \rho = \rho_1 \) and which depends on \( \rho_0^{(j)} \):

\[
h'(\rho_1) = h'_1(\rho_1)h_2(\rho_1) = (1 - \omega^2) \sinh 2\rho_1 \left( 1 - \frac{\sinh^2 2\rho^{(j)}}{\sinh^2 2\rho_1} \right)
\]

This term generates a discontinuity in \( \partial_\tilde{\sigma}^2 \rho(\tilde{\sigma}) \) at the junction points, since the value of \( \rho_0 \) jumps from \( \rho_0^{(j)} \) to \( \rho_0^{(j+1)} \) there.
Similarly:

\[ \partial_\bar{\sigma} g(\bar{\sigma}) = \sinh 2 \rho_0^{(j)} l(\rho), \quad l(\rho) = \frac{1}{\sinh^2 \rho} \]

explicitly depends on \( \rho_0^{(j)} \) at \( \rho = \rho_1 \) and thus is discontinuous at the junction points. The same clearly applies to \( \partial_\bar{\sigma} f(\bar{\sigma}) \).

Therefore, the patched version of Kruczsenki’s solution is not a proper closed string solution for fixed \( \omega > 1 \). However, the situation changes as \( \omega \to 1 \). In fact, in this limit, the function \( \rho(\bar{\sigma}) \) from equation (71) displays the following leading behaviour near the cusp located at \( \bar{\sigma} = 0 \):

\[ \rho(\bar{\sigma}) = -\frac{1}{2} \log \eta + \frac{1}{2} \log(2 \text{sech}^2 \bar{\sigma}) + O(\eta) \]

where, in our usual notation, \( \omega = 1 + \eta \). Clearly, the situation is identical near any other cusp, due to the periodicity of \( \rho(\bar{\sigma}) \). Thus, \( \rho(\bar{\sigma}) \) has a universal profile near the cusps, which is independent of \( \rho_0 \), so that it is no longer sensitive to jumps in that parameter as we move across the junction points.

We now study \( h_1(\rho) \) in more detail:

\[ h_1(\rho_1) = 0 \]

\[ \partial^k_\rho h_1(\rho_1) = 2^{k-1}(1 - \omega^2) \begin{cases} \sinh 2 \rho_1 & k \text{ even} \\ \cosh 2 \rho_1 & k \text{ odd} \end{cases} \]

It is easy to check that \( w_1 = \cosh 2 \rho_1 = 1/\eta + O(1) \) and \( \sinh 2 \rho_1 = 1/\eta + O(1) \), which then implies \( \partial^k_\rho h_1(\rho_1) = O(1) \), \( \forall k > 0 \). Of course, no discontinuities arise from this factor. Next, we consider the troublesome function \( h_2(\rho) \):

\[ h_2(\rho_1) = 1 - \frac{\sinh^2 2 \rho_0^{(j)}}{\sinh^2 2 \rho_1} \]

\[ \partial_\rho h_2(\rho_1) = 2 \sinh^2 2 \rho_0^{(j)} \frac{\cosh 2 \rho_1}{\sinh^3 2 \rho_1} \]

\[ \partial^k_\rho h_2(\rho_1) = 2^k \sinh^2 2 \rho_0^{(j)} \frac{P_k(2 \rho_1)}{\sinh^{k+2} 2 \rho_1}, \quad \text{for } \bar{\sigma}_{j-1} \leq \bar{\sigma} \leq \bar{\sigma}_j \]

where \( P_k(2 \rho_1) \) is a polynomial of degree \( k \) in \( \cosh 2 \rho_1 \) and \( \sinh 2 \rho_1 \). It is now easy to deduce that \( h_2(\rho_1) \to 1 \) and \( \partial^k_\rho h_2(\rho_1) \to 0 \) as \( \omega \to 1 \), which means that \( h_2(\rho) \) becomes smooth at the junction points in this limit. Taking the behaviour of \( h_1(\rho) \) into account we can then deduce:

\[ \partial^k_\rho h(\rho_1) = \sum_{m=0}^k \partial^m_\rho h_1(\rho_1) \partial^{k-m}_\rho h_2(\rho_1) \to \partial^k_\rho h_1(\rho_1) = O(1), \quad \text{as } \omega \to 1 \]
and therefore \( h(\rho) \) also becomes smooth as \( \omega \to 1 \). This immediately shows that \( \partial_\tilde{\sigma}^2 \rho(\tilde{\sigma}) \) from (97) is continuous in the same limit. Now, working from that equation, we see that, in general, \( \partial_\tilde{\sigma}^k \rho(\tilde{\sigma}) \) is a sum of products of derivatives of \( h(\rho) \), up to order \( k - 1 \), and of derivatives of \( \rho \), up to order \( k - 2 \). The latter can all be re-expressed in terms of lower derivatives of \( h(\rho) \) through (97) and (96), so that, in the end, we’re only left with derivatives of \( h(\rho) \) which all become smooth in the limit considered (in particular, notice that there are never any diverging factors involved, so that the exponential suppression of the discontinuities as \( \omega \to 1 \) is never undone). Hence, \( \partial_\tilde{\sigma} \rho(\tilde{\sigma}) \) becomes a smooth function as \( \omega \to 1 \).

If we now consider \( l(\rho) \), we easily see that:

\[
\partial_\rho^k l(\rho_1) = \frac{P_k(\rho_1)}{\sinh^{k+2} \rho_1} \to 0, \quad \text{as } \omega \to 1
\]

which then implies that all derivatives of \( g(\tilde{\sigma}) \) at the junction points vanish as these points approach the boundary, since they are given by sums of products of derivatives of \( l(\rho) \) and of \( \rho(\tilde{\sigma}) \) (these then reduce to derivatives of \( h(\rho) \) through \( (\partial_\tilde{\sigma} \rho)^2 = h(\rho) \) and \( \partial_\tilde{\sigma}^2 \rho = h'(\rho)/2 \); the former vanish, while the latter do not diverge. Thus, \( g(\tilde{\sigma}) \) and, similarly, \( f(\tilde{\sigma}) \) both become smooth as \( \omega \to 1 \).

Consequently, the patched Kruczenski string is an approximate solution which only becomes exact in the limit of large angular momentum, as the spikes approach the boundary of \( AdS_3 \).

It is also important to notice that the \( \omega = 1 \) solution (77), representing a single arc with endpoints on the boundary, displays a different type of behaviour, since, by definition, it satisfies \( \partial_\tilde{\sigma} \rho(\tilde{\sigma}) = h_2(\rho) \), and thus now the first derivative of \( \rho(\tilde{\sigma}) \) no longer vanishes at the endpoints, where the spikes should be located. The reason is that in this solution we see the extreme consequences of the \( \omega \to 1 \) limit: the cusps are “pushed away” at infinity and eventually disappear from the worldsheet, which is now infinitely long. It is hence necessary to maintain \( \omega > 1 \) and then to study the type of limit we used just above, in order to keep track of the spikes.

We now go back to the analysis of the general properties of the patched string. By construction, this solution, when plotted at constant \( t \), has \( K \) arcs of angular separation \( \Delta \theta_j \), for \( j = 1, \ldots, K \), and hence the closedness condition is:

\[
\sum_{j=1}^{K} \Delta \theta_j = 2n\pi
\]
Clearly $\rho(\tilde{\sigma}_j) = \rho_1$, and thus we have $K$ cusps located at $\tilde{\sigma} = \sigma_j$, for $j = 0, \ldots, K - 1$ (the analysis of section 3.3.1 still applies, and thus we have $\partial_y(t, \rho, \phi) = (0, 0, 0)$ at each cusp). We denote their angular positions by $\tilde{\phi}_j \equiv \omega \tilde{\tau} + \theta_j$, where, without loss of generality, we can assume $\theta_0 = 0$. We also define $\theta_K \equiv 2n\pi - \theta_0 = 2n\pi$, so that $\Delta \theta_j = \theta_j - \theta_{j-1}$ and $\sum_{j=1}^m \Delta \theta_j = \theta_m$. The plot at constant time $t$ is shown in Fig. 5.

As usual, we will be interested in the limit of this solution as the spikes touch the boundary of $AdS_3$, i.e. $\omega \to 1$, in which $\rho_1 \to +\infty$ and $\rho_0^{(j)}$ approaches the value which satisfies the following equation, coming from (78):

$$\Delta \theta_j \simeq 2 \arctan \frac{1}{\sinh 2\rho_0^{(j)}}, \quad \text{as } \omega \to 1$$

### 4.2 Energy, angular momentum and large angular momentum behaviour

The energy and angular momentum are computed from (93), (21) and (22). They clearly reduce to the sum of the usual Kruczenski-type contributions from each individual arc:

$$\Delta = \frac{\sqrt{\lambda}}{2\pi} \sum_{j=1}^K \int_0^{2\tilde{L}_j} d\tilde{\sigma} \cosh^2 \rho(\tilde{\sigma}, \rho_0^{(j)})$$

$$\begin{align*}
&= \sum_{j=1}^K \frac{\sqrt{\lambda}}{\pi} \sqrt{\frac{w_1 - 1}{w_1 + w_0^{(j)}}} \left[ \frac{1}{2} (w_1 + w_0^{(j)}) E_j - \sinh^2 \rho_0^{(j)} K_j \right] \\
&= \omega \frac{\sqrt{\lambda}}{2\pi} \sum_{j=1}^K \int_0^{2\tilde{L}_j} d\tilde{\sigma} \sinh^2 \rho(\tilde{\sigma}, \rho_0^{(j)}) \\
&= \sum_{j=1}^K \frac{\omega \sqrt{\lambda}}{\pi} \sqrt{\frac{w_1 - 1}{w_1 + w_0^{(j)}}} \left[ \frac{1}{2} (w_1 + w_0^{(j)}) E_j - \cosh^2 \rho_0^{(j)} K_j \right]
\end{align*}$$

where $\rho(\tilde{\sigma}, \rho_0)$ again represents the original solution (71).
Figure 5: The patched Kruczenski spiky string, with 7 spikes, angular separations equal to \(\pi/3, \pi/6, \pi/10, \pi/2, 2\pi/5, 5\pi/18, 2\pi/9\) and corresponding parameters given by \(\rho_1 = 2, \rho_0^{(1)} = 0.638475, \rho_0^{(2)} = 0.965792, \rho_0^{(3)} = 1.18657, \rho_0^{(4)} = 0.43007, \rho_0^{(5)} = 0.546767, \rho_0^{(6)} = 0.727608, \rho_0^{(7)} = 0.833672\).
As \( \omega \to 1 \), with \( \omega = 1 + \eta \), we have:

\[
\Delta = K \frac{\sqrt{\lambda}}{2\pi \eta} - \frac{\sqrt{\lambda}}{32\pi} \sum_{j=1}^{K} (8 + 3w_0^{(j)}) \log \eta \\
+ \frac{\sqrt{\lambda}}{64\pi} \sum_{j=1}^{K} \left[ -13w_0^{(j)} + (32 + 12w_0^{(j)}) \log \frac{2\sqrt{2}}{\sqrt{w_0^{(j)}}} \right] + O(\eta \log \eta)
\]

\[
S = K \frac{\sqrt{\lambda}}{2\pi \eta} + \frac{\sqrt{\lambda}}{32\pi} \sum_{j=1}^{K} (8 - 3w_0^{(j)}) \log \eta \\
+ \frac{\sqrt{\lambda}}{64\pi} \sum_{j=1}^{K} \left[ 32 - 13w_0^{(j)} + (-32 + 12w_0^{(j)}) \log \frac{2\sqrt{2}}{\sqrt{w_0^{(j)}}} \right] + O(\eta \log \eta)
\]

(98)

and we can, as usual, compute the \( O(1) \) correction to the typical logarithmic growth of the anomalous dimension:

\[
\Delta - S = \frac{K \sqrt{\lambda}}{2\pi} \log \left( \frac{2\pi S}{K \sqrt{\lambda}} \right) + \frac{\sqrt{\lambda}}{2\pi} \left[ K(3 \log 2 - 1) + \sum_{j=1}^{K} \log \left( \sin \frac{\Delta \theta_j}{2} \right) \right] + O(\eta \log \eta) \tag{99}
\]

where we have used:

\[
\frac{1}{w_0^{(j)}} \simeq \sin \frac{\Delta \theta_j}{2} \quad \text{as} \quad \omega \to 1
\]

which generalises (82).

### 4.3 Spectral curve for large \( S \)

As always, we introduce the rescaled worldsheet coordinates, defined as:

\[
(\tau, \sigma) = \frac{2\pi}{L} (\tilde{\tau}, \tilde{\sigma}) = \frac{\pi}{\sum_{j=1}^{K} L_j} (\tilde{\tau}, \tilde{\sigma})
\]

in terms of which the cusp positions become:

\[
\sigma_j = \frac{4\pi}{L} \sum_{k=1}^{j} \tilde{L}_k = 2\pi \frac{\sum_{k=1}^{j} \tilde{L}_k}{\sum_{k=1}^{K} L_k} \neq \frac{2j\pi}{K}, \quad \text{for} \quad j = 0, \ldots, n - 1 \tag{100}
\]
We then substitute (95) into (45) and write everything in terms of \((\tau, \sigma)\), thus obtaining the rescaled charge density:

\[
\frac{L}{2\pi} \left\{ (\omega + 1) \left[ w_1 \cosh^2 \left( v^{(j)} \left( \frac{L}{2\pi} (\sigma - \sigma_{j-1}) \right) \right) \right] k^{(j)} \right. \\
+ w_0 \left. \sinh^2 \left( v^{(j)} \left( \frac{L}{2\pi} (\sigma - \sigma_{j-1}) \right) \right) k^{(j)} \right] + 1 - \omega \}
\]

\[
\frac{L}{2\pi} (\omega + 1) e^{i(\phi - t)}
\]

\[
\times \left\{ \left[ w_1 \cosh^2 \left( v^{(j)} \left( \frac{L}{2\pi} (\sigma - \sigma_{j-1}) \right) \right) \right] k^{(j)} \right. \\
+ \left. w_0 \sinh^2 \left( v^{(j)} \left( \frac{L}{2\pi} (\sigma - \sigma_{j-1}) \right) \right) k^{(j)} \right] \right)^2 - 1 \}
\]

for \(\sigma \in [\sigma_{j-1}, \sigma_j]\) (101)

As before, we express \(\sigma\) near the \(m\)-th spike as \(\sigma = \sigma_m + \hat{\sigma}\), where \(\hat{\sigma}\) is never allowed to reach one half of the distance to the nearest cusp in both directions. For the patched Kruczenski solution, this translates into an asymmetric condition on \(\hat{\sigma}\) (since the fundamental periods \(\tilde{L}_j\) are in general different from each other), which, however, reduces to the usual \(|\hat{\sigma}| \leq \pi/K\) as \(\omega \to 1\) (since all \(\tilde{L}_j\) become identical in this limit).

Then, it is only a matter of tedious algebra to carry out the usual expansion of the elliptic functions and integrals as \(\omega \to 1\) and obtain:

\[
\frac{L}{2\pi} \left\{ (\omega + 1) \left[ w_1 \cosh^2 \left( v^{(j)} \left( \frac{L}{2\pi} (\sigma - \sigma_{j-1}) \right) \right) \right] k^{(j)} \right. \\
+ w_0 \left. \sinh^2 \left( v^{(j)} \left( \frac{L}{2\pi} (\sigma - \sigma_{j-1}) \right) \right) k^{(j)} \right] + 1 - \omega \}
\]

\[
\times \left\{ \left[ w_1 \cosh^2 \left( v^{(j)} \left( \frac{L}{2\pi} (\sigma - \sigma_{j-1}) \right) \right) \right] k^{(j)} \right. \\
+ \left. w_0 \sinh^2 \left( v^{(j)} \left( \frac{L}{2\pi} (\sigma - \sigma_{j-1}) \right) \right) k^{(j)} \right] \right)^2 - 1 \}
\]

where \(K \equiv \sum_{j=1}^{K} K_j\). As in the previous cases, we can use this result to compute the normalised charge density:

\[
\mu^0(\tau, \sigma) = \frac{1}{K} \sum_{m=0}^{K-1} \delta(\sigma - \sigma_m)
\]

\[
\mu^1(\tau, \sigma) + i\mu^2(\tau, \sigma) = i \frac{1}{K} \sum_{m=0}^{K-1} e^{i\sum_{j=1}^{m} \Delta\theta_j} \delta(\sigma - \sigma_m)
\]

(102)

The corresponding spin vectors are:

\[
\vec{L}_m = \frac{S}{K} \begin{pmatrix}
- \sin \left( \sum_{j=1}^{m} \Delta\theta_j \right) \\
\cos \left( \sum_{j=1}^{m} \Delta\theta_j \right)
\end{pmatrix} = \frac{S}{K} \begin{pmatrix}
1 \\
- \sin \theta_m
\end{pmatrix}
\]

(103)
(we recall that \( \theta_0 = 0 \)). These vectors satisfy the second property listed in (54), but not the first, i.e. the highest weight condition (in particular, the integrals over \( \sigma \) of \( \mu^1 \) and \( \mu^2 \) don’t vanish, whereas the integral of \( \mu^0 \) is still equal to 1). It is however possible to obtain the highest weight state by performing a right \( SU(1,1) \) rotation of the patched Kruczenski solution, as described in (27). It is easy to see from the definition that the right current \( j \) transforms as: \( j_a \rightarrow j'_a = U_R^{-1} j_a U_R \). In general, such a rotation can modify the first component of \( j_a \), and it is possible to show that the type of rotation required in order to find the highest weight state will do so, and thus \( \Delta' + S' \neq \Delta + S \). On the contrary, the left current \( l \) is invariant under this rotation, and hence we have \( \Delta' - S' = \Delta - S \), which implies both \( \Delta' \neq \Delta \) and \( S' \neq S \). Consequently, a different spin \( S' \) appears in the definition (49) of the new normalised charge density \( \mu' \), which can be expressed in terms of \( \mu \) as follows:

\[
\mu'(\tau, \sigma) = U_R^{-1} \mu(\tau, \sigma) U_R \lim_{\omega \to 1} \frac{S}{S'}
\]

We can now impose that the integrals of \( \mu'^1 \) and \( \mu'^2 \) over \( \sigma \) vanish (the integral of \( \mu'^0 \) is automatically 1 due to the new normalisation factor, which contains \( S' \) instead of \( S \)) and solve for the parameters of the rotation. As a consequence of their definition (52), the spin vectors (which are \( su(1,1) \) matrices), transform as:

\[
L'_m = U_R^{-1} L_m U_R
\]

and it is then easy to see from the definition \( L_m(u) = \mathbb{1} u + \eta_{AB} L^A_m s^B / S \) that the same applies to the matrix \( L_m(u) \):

\[
L'_m(u) = U_R^{-1} L_m(u) U_R \tag{104}
\]

It is now clear from this last equation that the trace of the monodromy matrix, as a function of the variable \( u \), is also invariant, and consequently the same is true of the spectral curve and of all the conserved charges \( q_k \) associated with the spin chain. Therefore, rather than explicitly performing a \( \{ \Delta \theta_j \} \)-dependent \( SU(1,1) \) rotation, we’ll continue working with the current version of the patched Kruczenski spiky string, and all results will equally apply to the case of the highest weight state.

We can now proceed along the usual path and calculate the matrix \( L_m(u) \) from (103):

\[
L_m(u) = \begin{pmatrix}
u + \frac{iS}{K} & iS e^{i\theta_m} \\
-iS e^{-i\theta_m} & u - \frac{iS}{K}
\end{pmatrix}
\]

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As expected, this matrix coincides with the one computed for the original Kruczenski solution\(^6\) if we choose $\Delta \theta_j = 2n\pi/K$, \(\forall j\), and therefore, under this condition, all subsequent results will reduce to those we obtained for that solution.

Due to the arbitrariness of the angular separations $\Delta \theta_j$, the procedure we used in section 3.3.3 in order to calculate $\text{tr} \, \Omega[x]$ is no longer effective. Nonetheless, it is possible, through a tedious calculation which however only involves elementary reasoning, to show that the conserved charges $q_k$ (we recall that $\hat{P}_K(1/u) = \text{tr} \, \Omega[x]$ is related to these by (62)) can be expressed as:

$$q_k = 2 \left( \frac{S}{K} \right)^k \sum_{r=0}^{\left\lfloor \frac{k}{2} \right\rfloor} (-1)^r \sum_{d_1, \ldots, d_r = 0, \ldots, K-2r, D \leq K-2r} C(k, K, r, D) \times \sum_{j_1, \ldots, j_r = 2, \ldots, K} \text{Re} \left[ k e^{-i \sum_{l=1}^r (\theta_{j_l+d_l}-\theta_{j_l-1})} \right]$$ \hspace{1cm} (105)

where we define:

$$D \equiv \sum_{l=1}^r d_l,$$

$$C(k, K, r, D) \equiv \sum_{j = \max\{k-D,0\}, \ldots, \min\{D,k-2r\}} (-1)^j \binom{K-2r-D}{k-2r-j} \binom{D}{j}$$

Since the method we used in obtaining (105) defines $q_k$ as the coefficient of $1/u^k$ in $\text{tr} \, \Omega[x]$, this expression also correctly reproduces the known term, $q_0 = 2$, and the missing linear term in $1/u$, $q_1 = 0$.

A less unwieldy expression for $q_k$ can be obtained by identifying the string theory equivalents of the spin chain variables $z_k$ and $p_k$ introduced in [26], for $k = 1, \ldots, K$ (where $K$ is the number of spins in the chain, which matches the number of cusps). As described in

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\(^6\)In order to have a perfect match with equation (85), we could introduce an arbitrary shift $\phi_0$ on the angular coordinate, without changing anything in the previous discussion. However, by explicitly performing the calculation, it is easy to see that $\phi_0$ would quickly disappear from the result anyway, as it did in the previous cases.
These parameters are related to the individual spin vectors $L_k$ at each site of the chain:

$$\mathcal{L}_k^0 = iz_k p_k, \quad \mathcal{L}_k^+ = iz_k^2 p_k, \quad \mathcal{L}_k^- = -i p_k$$

According to [9], we relate these to the spin vectors at each cusp (103) as follows:

$$L_k^0 = L_k^0, \quad iL_k^± = L_k^1 ± iL_k^2$$

It is now straightforward to obtain:

$$p_k = -\frac{iS}{K} e^{-iθ_k}, \quad z_k = e^{iθ_k}$$

Then, the $k$-th conserved charge is given by:

$$q_k = \sum_{1 ≤ j_1 < j_2 < ... < j_k ≤ K} z_{j_1 j_2} z_{j_2 j_3} ... z_{j_{k-1} j_k} z_{j_k j_1} p_{j_1} p_{j_2} ... p_{j_k}, \quad k = 2, ..., K \quad (106)$$

where $z_{ab} = z_a - z_b$. After some algebra, we can recast this expression into the following form:

$$q_k = \left(-\frac{2S}{K}\right)^k \sum_{1 ≤ j_1 < j_2 < ... < j_k ≤ K} \prod_{l=1}^{k} \sin \left(\frac{θ_{j_{l+1}} - θ_{j_l}}{2}\right), \quad k = 2, ..., K \quad (107)$$

where we have defined $j_{k+1} = j_1$. A tedious but straightforward calculation shows that this expression equals (102) for $k ≥ 2$, and thus yields the conserved charges from tr $Ω$. By using this result, we show, in appendix B, that $q_2$ has a complicated dependence on the angular separations $Δθ_j$ and that, in general, $q_2 \neq -S^2$, unless all the $Δθ_j$ are equal, which, as we can see from (102), is precisely the condition for the patched solution to be a highest weight state. This is exactly the behaviour we expected from the general finite gap picture: $q_2$ only equals $-S^2$ when the string considered is a highest weight state.

As usual, we’re interested in the highest conserved charge, which is given by:

$$q_K = \left(-\frac{2S}{K}\right)^K \prod_{l=1}^{K} \sin \left(\frac{θ_{l+1} - θ_l}{2}\right) = \left(-\frac{2S}{K}\right)^K \sin \left(\frac{Δθ_1 - 2nπ}{2}\right) \prod_{l=1}^{K-1} \sin \left(\frac{Δθ_{l+1}}{2}\right) = \left(-\frac{2S}{K}\right)^K (-1)^n \prod_{l=1}^{K} \sin \left(\frac{Δθ_l}{2}\right) \quad (108)$$

7In their notation, $L_3 ≡ \sum_{k=1}^{N} \mathcal{L}_k^0$, $L_+ ≡ \sum_{k=1}^{N} \mathcal{L}_k^+$ and $L_- ≡ \sum_{k=1}^{N} \mathcal{L}_k^-$. 

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where we have used \( \theta_1 - \theta_K = - \sum_{m=2}^{K} \Delta \theta_m = -2n\pi + \Delta \theta_1 \) in deriving the second line. We can now substitute this into (68) in order to obtain the gauge theory prediction for \( \Delta - S \):

\[
\Delta - S = \frac{K \sqrt{\lambda}}{2\pi} \log S + \frac{\sqrt{\lambda}}{2\pi} \left[ K \log 2 - K \log K + \log(-1)^{K+n} + \sum_{j=1}^{K} \log \left( \sin \frac{\Delta \theta_j}{2} \right) + C_{\text{string}}(K) \right]
\]

(where, as usual, we have omitted subleading terms as \( \omega \to 1 \)) which, by comparison with (99), implies:

\[
C_{\text{string}}(K) = K \left[ \log \left( \frac{8\pi}{\sqrt{\lambda}} \right) - 1 \right] - \log(-1)^{K+n}
\]

This expression agrees with those obtained in the previous two cases, (92) and (69) (in order to obtain the latter, we must set \( K \) even and \( n = K/2 \), as we remarked at the end of section 3.3.3).

Finally, we’d like to observe that, as we noticed earlier in this section, it is possible to obtain all the previous results concerning the N-folded GKP string and the Kruczenski string from this generalised version. For instance, it is easy to check that the highest conserved charge (108) reduces to the expression (91) valid in the Kruczenski case if we set \( \Delta \theta_j = \frac{2n\pi}{K}, \forall j \).

5 Results and Interpretation

In this paper we have constructed explicit string solutions with large angular momentum whose spectral curves and energies agree precisely with those of the classical spin chain arising in the dual gauge theory. This is consistent with the exact equality of the semiclassical spectra of the two theories observed in [9]. In principle one can reconstruct the gauge theory operator corresponding to any large-\( S \) string solution. In particular the filling fractions for Bethe roots, which define an eigenstate of the dilatation operator at one-loop can be extracted from the curve using the semiclassical formulae (11).

The results reported in this paper are consistent with the explicit identification between string theory and gauge theory proposed in [9]. For all the solutions constructed above, the large-\( S \) limit of the angular momentum density \( j_\tau \) is \( \delta \)-function localised at the spikes and
has the form,

\[ j_i(\sigma, \tau) \sim \frac{8\pi}{\sqrt{\lambda}} \sum_{k=0}^{K-1} L_k \delta(\sigma - \sigma_k) \]

for some \(\mathfrak{su}(1,1)\)-valued variables \(L_k\). The proposal of [9] was that these variables should be identified directly with the classical spin variables of the one-loop spin chain introduced in Section 2 according to \(L^0_k = L^0_k\) and \(iL^\pm_k = L^1_k \pm iL^2_k\). As explained in Section 6 of [9], this identification implies the agreement of the gauge theory and string theory curves. Conversely the agreement of the curves found above provides a non-trivial test of the proposed identification. In the previous section we also obtained an explicit formula (103) for the variables \(L_k\) evaluated on the general multi-spike solution.

An interesting perspective on the results of this paper is obtained by invoking the standard correspondence between local operators of the \(\mathcal{N} = 4\) theory in Minkowski space and states of the same theory defined on a three-sphere. In particular the operators (1) correspond to states comprising of \(J\) massless quanta of the adjoint scalar field with angular momenta \(s_j, j = 1, 2, \ldots J\), on \(S^5\). In the large spin limit, \(K \leq J\) particles have large angular momentum and therefore can be assigned to classical orbits around a great circle on \(S^3\). Each particle is a source for chromoelectric flux, and successive particles are joined by flux lines which spread out on the sphere reflecting the non-abelian Coulomb phase of the \(\mathcal{N} = 4\) theory. It is interesting to compare this state with the corresponding solution in string theory on \(S^3\) which corresponds to a string with \(K\) cusps approaching the boundary. The cusps correspond to localised regions of energy/angular momentum density in the boundary theory while the arcs of string drooping into the interior of \(AdS_3\) correspond in the usual way to flux spreading out on the boundary. The two pictures are in surprisingly good agreement. One interesting feature of this agreement is the correspondence between cusps and the massless adjoint quanta of the field theory [27]. At least for large spin, it seems that the partonic nature of the gauge-invariant state created by the operators (1) persists at strong coupling.

Finally there are several interesting extensions of this work that could be studied. The first is to consider more general operators of the \(\mathcal{N} = 4\) theory including different covariant derivatives and scalars as well as fermions. This requires a treatment of the string theory on \(AdS_5 \times S^5\). Another interesting direction would be to quantize the dynamics of the spikes.
Finally, it would be interesting to consider the semi-classical limit of the large-$N$ QCD spin chain of [28] and try to interpret it in terms of a semiclassical string theory.

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A Gauge transformation for the Kruczenski solution

The Kruczenski spiky string is described by the following ansatz: $t = \tau, \rho = \rho(\sigma), \phi = \omega \tau + \sigma$, which guarantees that all equations of motion from the Nambu-Goto action are satisfied if $\rho(\sigma)$ solves the following:

$$
\rho' = \pm \frac{1}{2} \frac{\sinh 2\rho \sqrt{\sinh^2 2\rho - \sinh^2 2\rho_0}}{\sinh 2\rho_0 \sqrt{\cosh^2 \rho - \omega^2 \sinh^2 \rho}}
$$

(109)

where $\rho_0$ is an integration constant. The requirement of reality placed upon $\rho$ forces $\rho_0 \leq \rho \leq \rho_1$, where $\coth \rho_1 = \omega$. From now on, we will refer to the function which solves equation (109) as $\hat{\rho}(\sigma)$. It is possible to integrate (109) to get the inverse function $\sigma(\hat{\rho})$:

$$
\sigma = \pm \frac{\sinh 2\rho_0}{\sqrt{2} \sqrt{w_0 + w_1 \sinh \rho_1}} \left\{ \Pi \left( \frac{w_1 - w_0}{w_1 - 1}, \beta, p \right) - \Pi \left( \frac{w_1 - w_0}{w_1 + 1}, \beta, p \right) \right\}
$$

(110)

where:

$$
p \equiv \sqrt{\frac{w_1 - w_0}{w_1 + w_0}}, \quad \sin \beta \equiv \sqrt{\frac{w_1 - w(\hat{\rho})}{w_1 - w_0}}
$$

(111)

($\beta \in [0, \pi/2]$) and we define $w(x) \equiv \cosh(2x), w_0 \equiv \cosh 2\rho_0$ and $w_1 \equiv \cosh 2\rho_1$.

We can construct a spiky string from this object by taking (110) with the plus sign and then replacing $\beta$ with the new coordinate $\sigma'$:

$$
\sigma = \frac{\sinh 2\rho_0}{\sqrt{2} \sqrt{w_0 + w_1 \sinh \rho_1}} \left\{ \Pi \left( \frac{w_1 - w_0}{w_1 - 1}, \sigma', p \right) - \Pi \left( \frac{w_1 - w_0}{w_1 + 1}, \sigma', p \right) \right\}
$$

(112)

While (112) implies that $\sigma$ is an increasing function of $\sigma'$ (with $\sigma(\sigma' = 0) = 0$), (111) allows us to express $\hat{\rho}$ as a function of $\sigma'$:

$$
\sinh^2 \hat{\rho} = \sinh^2 \rho_1 \cos^2 \sigma' + \sinh^2 \rho_0 \sin^2 \sigma'
$$

From (112), we see that, for each increase of $\pi/2$ in $\sigma'$, $\sigma$ and consequently $\phi$ increase by:

$$
\Delta \phi = \frac{\sinh 2\rho_0}{\sqrt{2} \sqrt{w_0 + w_1 \sinh \rho_1}} \left\{ \Pi \left( \frac{w_1 - w_0}{w_1 - 1}, p \right) - \Pi \left( \frac{w_1 - w_0}{w_1 + 1}, p \right) \right\}
$$

(113)
Thus, for the string to be closed at fixed \( t = \tau \), we allow \( \sigma' \) to vary in \([0, K\pi]\), \( K \in \mathbb{N} \) (i.e. \( \sigma \in [0, 2K\Delta \phi] \)), and then demand that the corresponding total increase in \( \phi \) be an integer multiple of \( 2\pi \): \( 2K\Delta \phi = 2n\pi \). Since (113) matches (75), the closedness conditions for these two solutions are actually the same.

We are now ready to discuss the worldsheet coordinate transformation which maps this solution onto the corresponding conformal gauge version (71), (73). In order to find it, we just need to impose the equality of the global coordinates \((t, \rho, \phi)\) specified by the two different versions of the ansatz, which leads to the following set of relations:

\[
\tilde{\tau} + f(\tilde{\sigma}) = \tau, \quad g(\tilde{\sigma}) - \omega f(\tilde{\sigma}) = \sigma, \quad \rho(\tilde{\sigma}) = \hat{\rho}(\sigma)
\]  

These are actually three conditions on two unknown functions \( \tau(\tilde{\tau}, \tilde{\sigma}), \sigma(\tilde{\tau}, \tilde{\sigma}) \), and we easily see that they give two potentially conflicting expressions for \( \sigma(\tilde{\tau}, \tilde{\sigma}) \). For the transformation to exist, these must coincide:

\[
g(\tilde{\sigma}) - \omega f(\tilde{\sigma}) = \hat{\rho}^{-1}(\rho(\sigma))
\]

We already have the inverse of \( \hat{\rho} \) from (110). We can then compute \( w(\rho(\tilde{\sigma})) = \cosh 2\rho(\tilde{\sigma}) \) from (71) and then use it to find:

\[
\sin^2 \sigma' = \frac{w_1 - w(\rho(\tilde{\sigma}))}{w_1 - w_0} = \sin^2(v|k)
\]

Remembering that \( \sigma' \in [0, K\pi] \), it is natural to identify \( \sigma' = \text{am}(v|k) \). Therefore, by substituting this into (112), we get:

\[
\hat{\rho}^{-1}(\rho(\sigma)) = \frac{\sinh 2\rho_0}{\sqrt{2} \sinh \rho_1 \sqrt{w_1 + w_0}} \{ \Pi(n-, \text{am}(v|k), k) - \Pi(n+, \text{am}(v|k), k) \}
\]

It is now only a matter of simple algebra to show that this expression matches \( g(\tilde{\sigma}) - \omega f(\tilde{\sigma}) \), i.e. that the last two conditions in (114) are equivalent, and thus that the coordinate transformation exists. Its explicit form is the following:

\[
\tau = \tilde{\tau} + \frac{\sqrt{2} \omega \sinh 2\rho_0 \sinh \rho_1}{(w_1 + 1) \sqrt{w_0 + w_1}} \Pi(n+, \text{am}(v|k), k) \\
\sigma = \frac{\sinh 2\rho_0}{\sqrt{2} \sqrt{w_0 + w_1} \sinh \rho_1} \{ \Pi(n-, \text{am}(v|k), k) - \Pi(n+, \text{am}(v|k), k) \}
\]  

(115)

It is also possible to determine the worldsheet metric \( h_{ab} \) from Kruczenski’s parametrization and then show that (115) brings it to the 2-dimensional Minkowski metric, up to a
conformal transformation. We recall that the Nambu-Goto action is obtained from the general \( \sigma \)-model action by substituting the equations of motion for \( h_{ab} \) into it:

\[
\partial_\mu X_a \partial^\mu X_b = \frac{1}{2} h_{ab} h^{cd} \partial_\mu X_c \partial^\mu X_d
\]

We can then invert these equations to find \( h_{ab} \) as a function of \( \partial_\mu X_a \partial^\mu X_b \), up to an overall rescaling factor (the combination \( h_{ab} h^{cd} \) is clearly conformally invariant):

\[
h_{ab} = a(\tau, \sigma) \begin{pmatrix}
\dot{X}^2 & \dot{X}^\mu X'_\mu \\
\dot{X}^\mu X'_\mu & X'^2
\end{pmatrix}
\]

Now, if a worldsheet coordinate transformation \((\tau, \sigma) \rightarrow (\tilde{\tau}, \tilde{\sigma})\) brings this metric to conformal gauge, i.e. if it makes it diagonal and traceless (the overall scaling factor can then be eliminated by a conformal transformation), then it must satisfy the following set of conditions:

\[
0 = h_{00} \left[ \left( \frac{\partial \tau}{\partial \tilde{\tau}} \right)^2 + \left( \frac{\partial \tau}{\partial \tilde{\sigma}} \right)^2 \right] + 2h_{01} \left[ \left( \frac{\partial \tau}{\partial \tilde{\tau}} \right) \left( \frac{\partial \sigma}{\partial \tilde{\tau}} \right) + \left( \frac{\partial \tau}{\partial \tilde{\sigma}} \right) \left( \frac{\partial \sigma}{\partial \tilde{\sigma}} \right) \right]
\]

\[
0 = h_{11} \left[ \left( \frac{\partial \sigma}{\partial \tilde{\tau}} \right)^2 + \left( \frac{\partial \sigma}{\partial \tilde{\sigma}} \right)^2 \right]
\]

All the required derivatives can be obtained from the first two equations (114), and then it is just a matter of algebra to check that, as expected, these equations are verified.

At this point, it is also interesting to notice that the standard string closedness constraint

\[ X_\mu(\tau, \sigma + \sigma_0) = X_\mu(\tau, \sigma), \]

where \( \sigma_0 \) is some period, is not gauge invariant. In fact, this solution is a clear example of this, since condition (76) ensures that the string is closed at constant global time \( t \) in both gauges. While we have \( t = \tau \) in Kruczenski’s gauge and thus standard closedness holds, this is not the case for Jevicki’s and Jin’s solution, due to the presence of \( f(\tilde{\sigma}) \). The gauge-invariant object here is \( t \) and thus if we impose the standard closedness constraint in static gauge \( t = \tau \), we automatically have closedness at constant \( t \) in all gauges, but this translates into closedness at constant \( \tau \) only in those gauges in which \( t = \text{const} \) implies \( \tau = \text{const} \).
B Computing $q_2$ for the N-folded GKP and the Kruczenski solutions

The easiest way of computing $q_2$ in both cases is by using equation (107), which yields all conserved charges for the patched Kruczenski solution. This solution is discussed in detail in section 4; here we simply recall that it allows arbitrary angular separations $0 < \Delta \theta_j < \pi$ between each pair of consecutive cusps. As we previously observed in section 4.3, all results concerning the spectral curve of this generalised solution reduce to those obtained for the Kruczenski spiky string if we set $\Delta \theta_j = 2n\pi/K$, $\forall j$, where $n$ is a natural number counting how many times the Kruczenski string winds around the centre of $AdS_3$. Furthermore, we also saw at the end of section 3.3.3 that, by setting $n = K/2$, these results in turn reduce to those associated with the N-folded GKP case, where $2N = K$. Therefore, we can compute the conserved charge $q_2$ by specialising the general expression (107) to the desired simpler case.

We start by evaluating it for $k = 2$:

$$q_2 = \frac{4S^2}{K^2} \sum_{1 \leq j_1 < j_2 \leq K} \sin \left(\frac{\theta_{j_2} - \theta_{j_1}}{2}\right) \sin \left(\frac{\theta_{j_1} - \theta_{j_2}}{2}\right)$$

$$= -\frac{4S^2}{K^2} \sum_{1 \leq j_1 < j_2 \leq K} \sin^2 \left(\frac{1}{2} \sum_{l=j_1+1}^{j_2} \Delta \theta_l\right)$$

(116)

where we have used $\theta_m \equiv \sum_{j=1}^{m} \Delta \theta_j$.

We now specialise to the Kruczenski case, by setting $\Delta \theta_j = 2n\pi/K$, $\forall j$, which implies:

$$\sum_{l=j_1+1}^{j_2} \Delta \theta_l = \frac{2n\pi}{K}(j_2 - j_1)$$

By substituting this into (116) and introducing the new index $m = j_2 - j_1$, which replaces $j_2$, we obtain:

$$q_2 = -\frac{4S^2}{K^2} \sum_{j_1=1}^{K-1} \sum_{m=1}^{K-j_1} \sin^2 \left(\frac{n\pi}{K} m\right) = -\frac{2S^2}{K^2} \sum_{j_1=1}^{K-1} \sum_{m=1}^{K-j_1} \left[ 1 - \cos \left(\frac{2n\pi}{K} m\right) \right]$$

$$= -\frac{2S^2}{K^2} \sum_{j_1=1}^{K-1} \left[ K - j_1 - \sum_{m=1}^{K-j_1} \cos \left(\frac{2n\pi}{K} m\right) \right]$$

We now use the general result for the Dirichlet kernel:

$$1 + 2 \sum_{k=1}^{n} \cos(kx) = \frac{\sin \left[\left(n + \frac{1}{2}\right)x\right]}{\sin \left(\frac{x}{2}\right)}$$

(117)
to calculate the last remaining sum over \( m \):

\[
q_2 = -\frac{2S^2}{K^2} \sum_{j_1=1}^{K-1} \left\{ K - j_1 - \frac{1}{2} \left[ \frac{\sin \left( \left( K - j_1 + \frac{1}{2} \right) \frac{2n\pi}{K} \right)}{\sin \left( \frac{n\pi}{K} \right)} - 1 \right] \right\}
\]

\[
= -\frac{2S^2}{K^2} \sum_{j_1=1}^{K-1} \left\{ K - j_1 + \frac{1}{2} - \frac{1}{2} \frac{\sin \left( \frac{n\pi}{K} - j_1 \frac{2n\pi}{K} \right)}{\sin \left( \frac{n\pi}{K} \right)} \right\}
\]

\[
= -\frac{2S^2}{K^2} \sum_{j_1=1}^{K-1} \left\{ K - j_1 + \frac{1}{2} - \frac{1}{2} \left[ \cos \left( j_1 \frac{2n\pi}{K} \right) - \cot \left( \frac{n\pi}{K} \right) \sin \left( j_1 \frac{2n\pi}{K} \right) \right] \right\}
\]

\[
= \frac{S^2}{K^2} \left[ -2 \left( K + \frac{1}{2} \right) (K - 1) + (K - 1)K + \sum_{j_1=1}^{K-1} \cos \left( j_1 \frac{2n\pi}{K} \right) \right]
\]

\[
- \cot \left( \frac{n\pi}{K} \right) \sum_{j_1=1}^{K-1} \sin \left( j_1 \frac{2n\pi}{K} \right)
\]

At this point, we can evaluate the sums over \( j_1 \) by using (117) and the analogous result:

\[
\sum_{k=1}^{n} \sin(kx) = \frac{\sin x + \sin(nx) - \sin[(n+1)x]}{2(1 - \cos x)}
\]

thus obtaining:

\[
q_2 = -S^2 + \frac{S^2}{2K^2} \left\{ \frac{\sin \left( \left( K - \frac{1}{2} \right) \frac{2n\pi}{K} \right)}{\sin \left( \frac{n\pi}{K} \right)} - 1 \right\}
\]

\[
- S^2 \cot \left( \frac{n\pi}{K} \right) \frac{\sin \left( \frac{2n\pi}{K} \right) + \sin \left( (K - 1) \frac{2n\pi}{K} \right) - \sin(2n\pi)}{2K^2 \left[ 1 - \cos \left( \frac{2n\pi}{K} \right) \right]}
\]

\[
= -S^2
\]

Since the result is independent of \( n \), it also holds for the N-folded GKP case.

One may wonder whether a similar relation exists for the patched Kruczenski solution, but the answer is negative: as a counter-example, we study the case \( K = 3 \), with \( \Delta\theta_1 = 5\pi/6 \), \( \Delta\theta_2 = 2\pi/3 \), \( \Delta\theta_3 = \pi/2 \) and consequently \( n = 1 \). It is easy to check from (116) that:

\[
q_2 = -S^2 \frac{4}{9} \left[ \sin^2 \left( \frac{\Delta\theta_2}{2} \right) + \sin^2 \left( \frac{\Delta\theta_2 + \Delta\theta_3}{2} \right) + \sin^2 \left( \frac{\Delta\theta_3}{2} \right) \right] = -S^2 \frac{7 + \sqrt{3}}{9}
\]

The only property that continues to hold for the patched solution is the fact that \( q_2 < 0 \), as we can easily see from (116).
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