A Note on Projective Klingenberg Planes over Rings of Plural Numbers

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Abstract—This paper deals with a certain class of projective Klingenberg planes over the local ring \( F[\eta]/<\eta^m> \) with \( F \) an arbitrary field, known as the plural algebra of order \( m \). In particular addition and multiplication of points on a line is defined geometrically and interpreted algebraically, by using the coordinate ring.

Index Terms—plural algebra, local ring, projective Klingenberg plane, geometric addition and multiplication.

I. INTRODUCTION

Klingenberg in [13] introduced real plural algebras as an example of an H-ring without using the name “plural numbers”. Jukl, in [8], studied the real plural algebra of order \( m \) and investigated linear forms on a line is defined geometrically and interpreted algebraically, by using the coordinate ring. Jukl continued to study free finite dimensional modules in [9]. In [5], Erdogan et. al. investigated some properties of the modules constructed over the real plural algebra and later, in [6], Ciftci and Erdogan obtained an \( n \)-dimensional projective coordinate space associated with the \((n+1)\)-dimensional free module over this real plural algebra. For more detailed information on modules, see [14]. For the algebraic and linear algebraic notions that will be used throughout this paper, we refer to [7] and [15].

In this paper we will study a class of projective Klingenberg (PK) planes coordinatized by the plural algebra \( A:=F+\eta F+\eta^2 F+\cdots F\eta^m \) such that \( \eta^m=0 \) for \( \eta \notin F \). Consider \( A:=F[\eta]=F[\eta^m] \) with componentwise addition and multiplication modulo \( \eta^m \). Then \( A \) is a (unital, commutative and associative) local ring with the maximal ideal \( \mathbf{I}=A\eta \) of non-units. Also, the local ring \( A \) can be considered as plural \( F \)-algebra of order \( m \) with a basis \( \{1,\eta^2,\cdots,\eta^m-1\} \). Note that the algebra can be seen as quotient ring of the polynomial ring \( F[\eta] \) by the principal ideal \( \langle \eta^m \rangle \). For more detailed information about quotient rings, it can be seen to [16]. If we choose the field of real numbers instead of \( F \) then we have the real plural algebra of order \( m \) (see [8, Def. 1.1]).

It is clear that an element \( x \) of \( A \) is of the form \( x=a_0+1 \eta+a_2 \eta^2+\cdots+a_{m-1} \eta^{m-1} \) where \( a_{m-1} \notin F \) for \( 0 \leq i \leq m-1 \).

Now we can consecutively state the following two results, analogues of Proposition 1.3 and 1.5 given in [8], without proof.

Proposition 1.
An element \( x=a_0+a_1 \eta+a_2 \eta^2+\cdots+a_{m-1} \eta^{m-1} \in A \) is a unit if and only if \( a_0 \neq 0 \).

Proposition 2.
\( A \) is a local ring with maximal ideal \( \eta A \). The subsets \( \eta^j A \), \( 1 \leq j \leq m \), are all ideals in \( A \).

From [2] we recall the following:

Definition 3.
Let \( M=(P.L,\sim) \) consist of an incidence structure \((P.L,\sim)\) (points, lines, incidence) and an equivalence relation \( \sim \) (neighbour relation) on \( P \) and on \( L \). Then \( M \) is called a projective Klingenberg plane (PK-plane), if it satisfies the following axioms:

(PK1) If \( P,Q \) are non-neighbour points, then there is a unique line \( PQ \) through \( P \) and \( Q \).

(PK2) If \( g,h \) are non-neighbour lines, then there is a unique point \( g \cap h \) on both \( g \) and \( h \).

(PK3) There is a projective plane \( M^\sim=(P^\sim,\sim,\sim,\varepsilon) \) and an incidence structure epimorphism \( \Psi: \mathcal{M} \rightarrow \mathcal{M}^\sim \), such that the conditions \( \Psi(P)\sim\Psi(Q) \Rightarrow P \sim Q, \Psi(g)\sim\Psi(h) \Rightarrow g \sim h \) hold for all \( P,Q \in \mathcal{P}, g,h \in \mathcal{L} \).

Let \( \mathbf{R} \) be a local ring. Then \( \mathcal{M}(\mathbf{R})=(P.L,\sim) \) is the

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incidence structure with neighbour relation defined as follows:

\( \mathcal{P} = \{(x,y) | x, y \in \mathbb{R} \} \cup \{(1,y,z) | y \in \mathbb{R}, z \in \mathbb{I} \} \cup \{(w,1,z) | w \in \mathbb{I} \}, \)

\( \mathcal{L} = \{(m,k) | m, k \in \mathbb{R} \} \cup \{(1,n,p) | p \in \mathbb{I}, n \in \mathbb{E} \} \cup \{(q,n,1) | q, n \in \mathbb{I} \}, \)

\( [1, m, k] = \{(x, m + k, 1) | x \in \mathbb{R} \} \cup \{(1, z, k + m, z) | z \in \mathbb{I} \}, \)

\( [1, n, p] = \{(y, n + p, 1) | y \in \mathbb{R} \} \cup \{(z + p, n, 1, z) \in \mathbb{I} \}, \)

\( [q, n, 1] = \{(y, y + q, 1) | y \in \mathbb{R} \} \cup \{(w, 1, wq + n) | w \in \mathbb{E} \}. \)

\( P = \{(x_1, x_2, x_3) | (y_1, y_2, y_3) \} = Q \iff x_1 - y_1 \in (1, 2, 3), \forall P \in \mathcal{Q} \mathcal{P}; \)

\( g = \{(x_1, x_2, x_3) | (y_1, y_2, y_3) \} = h \iff x_1 - y_1 \in (1, 2, 3), \forall g, h \in \mathcal{L}. \)

From [2] we recall the following theorem.

**Theorem 4.**

\( M(\mathbb{R}) \) is a PK-plane, and each desarguesian PK-plane is isomorphic to some \( M(\mathbb{R}) \).

For more detailed information about desarguesian PK-plane, it can be seen to the papers of [1, 10]. By Theorem 4 it is obvious that \( M(\mathbb{A}) \) is a PK-plane.

An n-tuple (\( n \geq 3 \)) of pairwise non-neighbour points is called an (ordered) n-gon if no three of its elements are on neighbour lines.

Baker et. al., [2], use \( O=(0,0,1), U=(1,0,0), V=(0,1,0), \)

\( S=(1,1,1) \) as a coordinatization 4-gon of a PK-plane.

Finally, we give the definition of addition and multiplication of points on the line \( OU \) of \( M(\mathbb{A}) \) in the sense of [4].

**Definition 5.**

Let \( A \) and \( B \) be non-neighbour points on the line \( OU=\{0,1,0\} \) of \( M(\mathbb{A}) \). Then

(i) \( A+B \) is defined as the intersection point of the lines \( LV \) and \( OU \) where \( L=KU=BS=\{1,1,-b\} \) and \( KU=\{0,1,a\} \), we get the intersection point as \( K=(a,a,1) \). Also, for the lines \( BS=\{1,1,-b\} \) and \( KU=\{0,1,a\} \), we get the intersection point as \( L=(a+b,a,1) \). Finally

\[ A+B = LV \land OU \]

\[ = (1,0,a+b) \land OU \]

\[ = (a+b,0,1) \]

is obtained.

If \( B=(1,0,z) \), that is, \( B \sim U \), then for the lines \( AV=\{1,0,a\} \) and \( OS=\{1,1,0\} \), we have the intersection point as \( K=(a,a,1) \). Also, for the lines \( BS=\{z,-z,1\} \) and \( KU=\{0,1,a\} \) we get the intersection point as \( L=(1,z-(1+a-z)^{-1},a,z-(1+a-z)^{-1}) \). Finally

\[ A+B = LV \land OU \]

\[ = [z-(1+a-z)^{-1},0,1] \land [0,1,0] \]

\[ = (1,0,z-(1+a-z)^{-1}) \]

\[ = (1,0,z) \sim B \]

is obtained.

(ii) Since \( A,B \sim O \) we know that a and b are units of \( \mathbb{A} \). For the lines \( IS=\{1,1,-1\} \) and \( BV=\{1,0,b\} \) we have the intersection point as \( M=(b,b,1) \). Also, for the lines \( AS=\{1,1,-a\} \) and \( OM=\{1-b^*1,1,0\} \) we get the intersection point as \( N=(a-b,(a-b)-a,1) \). Finally

\[ A \cdot B = V \land OU \]

\[ = [1,0,a-b] \land [0,1,0] \]

\[ = (a-b,0,1) \]

is obtained.

If \( B=(1,0,z) \), that is, \( B \sim U \), then for the lines \( IS=\{1,1,-1\} \) and \( BV=\{z,0,1\} \) we have the intersection point as \( M=(1,1-z,z) \). Also, for the lines \( AS=\{1,1,-a\} \) and \( OM=\{1-z,z-a^{-1}\} \) we get the intersection point as \( N=(1,1-z,z-a^{-1}) \). Finally

\[ A \cdot B = V \land OU \]

\[ = [z \cdot (1+a-z)^{-1},0,1] \land [0,1,0] \]

\[ = (1,0,z \cdot (1+a-z)^{-1}) \]

\[ = (1,0,z) \sim \]

is obtained.

In the next section, we will give the main results.

**III. THE MAIN RESULTS**

We immediately start with giving the following proposition which is analogue of a result given in [4]. The calculations in the proof of the proposition are based on similar calculations used in the coordinatization procedure for general PK-planes due to Keppens [11, 12].

**Proposition 6.**

The addition and multiplication of two non-neighbour points \( A \) and \( B \) on the line \( OU \) in \( M(\mathbb{A}) \) as defined geometrically in Definition 5 can be calculated algebraically using the ring operations in the coordinatizing plural F-algebra.

**Proof.** Let \( A=(a,0,1) \) and \( B=(b,0,1) \) be non-neighbour points on the line \( OU=\{0,1,0\} \) where

\[ a=a_0 + a_i \eta + a_2 \eta^2 + \cdots + a_{m-1} \eta^{m-1} \in \mathbb{A} \]

and

\[ b=b_0 + b_1 \eta + b_2 \eta^2 + \cdots + b_{m-1} \eta^{m-1} \in \mathbb{A} \]

i) For the lines \( AV=\{1,0,a\} \) and \( OS=\{1,1,0\} \), we have the intersection point as \( K=(a,a,1) \). Also, for the lines \( BS=\{1,1,-b\} \) and \( KU=\{0,1,a\} \), we get the intersection point as \( L=(a+b,a,1) \). Finally

\[ A+B = LV \land OU \]

\[ = [1,0,a+b] \land OU \]

\[ = (a+b,0,1) \]

is obtained.

If \( B=(1,0,z) \), that is, \( B \sim U \), then for the lines \( AV=\{1,0,a\} \) and \( OS=\{1,1,0\} \), we have the intersection point as \( K=(a,a,1) \). Also, for the lines \( BS=\{z,-z,1\} \) and \( KU=\{0,1,a\} \) we get the intersection point as \( L=(1,z-(1+a-z)^{-1},a,z-(1+a-z)^{-1}) \). Finally

\[ A+B = LV \land OU \]

\[ = [z \cdot (1+a-z)^{-1},0,1] \land [0,1,0] \]

\[ = (1,0,z-(1+a-z)^{-1}) \]

\[ = (1,0,z) \sim \]

is obtained.

As a corollary of Proposition 6, we can state the following:

**Corollary 7.**

The point \( S=(1,1,0) \) in Definition 5 may be replaced by any point \( S \) on UV with \( S \sim U \). Hence, the definition of the addition and multiplication of points on the line \( OU \) is independent of the choice of the point \( S \).

**Proof.** If \( S \) is an arbitrary point on the line \( UV \) non-neighbour to \( V \) then, let \( S=(1,s,0) \).
where \( S = s_0 + s_1 \eta + s_2 \eta^2 + \cdots + s_{m-1} \eta^{m-1} \in \mathbb{A} \) is a unit since \( S^{-1} \in \mathbb{U} \). By similar calculations we replace \( S \) by \( S^{-1} \) in the proof of Proposition 6. Then,

i) For the lines \( AV = [1,0,a] \) and \( OS = [s,1,0] \) we have the intersection point as \( K = (a,a^{-1},a) \). Also, for the lines \( BS = [s,1,-(b-s)] \) and \( KU = [0,1,a-s] \), we get the intersection point as \( L = (a+b,a,s) \). Finally,

\[
A + B = LVAOU = [1,0,a+b]A[0,1,0] = (a+b,0,1)
\]

is obtained.

If \( B = (1,0,z) \), that is, \( B \sim U \), then for the lines \( AV = [1,0,a] \) and \( OS = [s,1,0] \), we have the intersection point as \( K = (a,a,s,1) \). Also, for the lines \( BS = [z,-(s^{-1} z),1] \) and \( KU = [0,1,a,s] \), we get the intersection point as \( L = (1,z-(1+a-z)^{-1}(a),z) \). Finally,

\[
A + B = LVAOU = [z-(1+a-z)^{-1},0,1]A[0,1,0] = (1,0,z^{-1}) = B^{-1}
\]

is obtained.

ii) For the lines \( IS = [s,1,-s] \) and \( BV = [1,0,b] \) we have the intersection point as \( M = (b,b^{-1},s,1) \). Also, for the lines \( AS = [s,1,-(a-s)] \) and \( OM = [s-(z^{-1},1,0) \) where \( b \in \mathbb{A} \) is a unit since \( B \in \mathbb{O} \), we get the intersection point as \( N = (a,b,a^{-1}b,a^{-1}b^{-1},s,1) \). Finally

\[
A - B = VNAOU = [1,0,a-b]A[0,1,0] = (a-b,0,1)
\]

is obtained.

If \( B = (1,0,z) \), that is, \( B \sim U \), then for the lines \( IS = [s,1,-s] \) and \( BV = [z,0,1] \), we have the intersection point as \( M = (1,s-(z,s),z) \). Also, for the lines \( AS = [s,1,-(a-s)] \) and \( OM = [s-(z,s),1,0] \), we get the intersection point as \( N = (1,s-(z,s),z,a^{-1}) \) where \( a \in \mathbb{A} \) is a unit since \( A \in \mathbb{O} \). Finally,

\[
A - B = VNAOU = [z-a^{-1},0,1]A[0,1,0] = (1,0,z^{-1}) = B^{-1}
\]

is obtained.

As an immediate consequence of Proposition 6, addition and multiplication of points on the line \( OU \) corresponds to addition and multiplication of elements of the local ring \( \mathbb{A} \) of plural numbers over a field. This means that \( OU,+,-,\cdot \) itself has the structure of a local ring. The situation generalizes the one valid in an ordinary desarguesian (affine or projective)

eplane over a field \( F \) where the points on a line can also be added and multiplied in such a way that one obtains a field isomorphic to \( F \) (see [3, Chapter 3]). Also, in [4], a similar result was obtained for PK-planes over a local ring of dual numbers (over a field or even over a quaternion skewfield).

REFERENCES

[1] P.Y. Bacon, Desarguesian Klingenberg Planes. Trans. Amer. Math. Soc., 241, 1978, 343-355.
[2] C.A. Baker, N.D. Lane, J.W. Lorimer, A coordinatization for Moufang-Klingenberg Planes. Simon Stevin, 65, 1991, 3-22.
[3] M.K. Bennett, Affine and Projective Geometry, New York: Wiley & Sons Inc., 1995.
[4] B. Celik, F.O. Erdogan, On Addition and Multiplication of points in a certain class of projective Klingenberg planes. Journal of Inequalities and Applications, 230, 1, 2013.
[5] F.O. Erdogan, S. Ciftci, A. Akpmar, On Modules over Local Rings. Analele Univ. "Ovidius" din Constanta, Math Series, 24(1), 2016, 217-230.
[6] S. Ciftci, F.O. Erdogan, On projective coordinate spaces. Filomat, 31(4), 2017, 941-952.
[7] T.W. Hungerford, Algebra, New York: Holt, Rinehart and Winston, 1974.
[8] M. Jukl, Linear forms on free modules over certain local rings. Acta Univ. Palacki. Olomuc. Fac. Rerum Natur. Math., 32, 1993, 49-62.
[9] M. Jukl, Grassmann formula for certain type of modules. Acta Univ. Palacki. Olomuc. Fac. Rerum Natur. Math., 34, 1995, 69-74.
[10] M. Jukl, Desargues theorem for Klingenberg projective plane over certain local ring. Acta Univ. Palacki. Olomuc. Mathematica, 36, 1997, 33-39.
[11] D. Keppens, Coordinatization of projective Klingenberg planes, part 2, Simon Stevin, 62, 1988, 163-188.
[12] D. Keppens, Coordinatization of projective Klingenberg planes, part 3, Simon Stevin, 63, 1989, 117-140.
[13] W. Klingenberg, Projektive und affine Ebenen mit Nachbarelementen. Math. Z., 60, 1954, 384-406.
[14] B.R. McDonald, Geometric Ebenen mit Nachbarelementen. Math. Z., 60, 1954, 384-406.
[15] K. Nomizu, Fundamental of Linear Algebra, New York: Marcel Dekker, 1976.
[16] J. Rotman, Galois Theory, New York: Springer, 1998.