Branes as solutions of gauge theories in gravitational field

A. A. Zheltukhin

$^a$ Kharkov Institute of Physics and Technology, 1, Akademicheskaya St., Kharkov, 61108, Ukraine
$^b$ Nordita, the Nordic Institute for Theoretical Physics KTH Royal Institute of Technology and Stockholm University Roslagstullsbacken 23 SE 106 91 Stockholm Sweden

Abstract

Proposed is a new set of gauge models in gravitational fields possessing exact solutions. The latter are presented by the first-order Gauss-Codazzi PDEs treated as gauge constraints. The use of the tetrade postulate shows that these exact solutions describe minimal hypersurfaces and Dirac p-branes embedded into Minkowski spaces.

1 Introduction

The integrability problem of the nonlinear PDEs describing membranes attracts much attention [1-13]. An important role of the string geometric approach [14-17] in studying this problem was shown in [18], [19], where the connection of the membrane equations with the pendulum and generalized Abel nonlinear differential equations was established. These results were generalized to p-branes embedded into higher dim. Minkowski spaces [20].

Therefore, the geometric approach seems to be a powerful tool for the investigation of integrability of nonlinear brane equations. A reformulation of the geometric approach to strings in terms of gauge field theories based on

*e-mail: aaz@physto.se
the use of the Cartan’s moving coordinate frames [21] and ideas [22, 23] was
developed in [24]. This gauge treatment allowed to reveal isomorphism be-
 tween relativistic string and a closed sector of states of the exactly integrable
two-dimensional SO(1,1)×SO(D-2) invariant gauge model. As a result, the
classical string description in the geometric approach was mapped onto a
chain of the two-dimensional exactly integrable equations.

So, it is interesting to generalize the mentioned gauge approach to the case
of p-branes. This implies construction of a (p+1)-dimensional SO(1,p)×SO(D-
p-1) invariant gauge model possessing the solution associated with p-brane.
The problem finds its positive solution in the present paper.

2 Hypesurfaces in the Minkowski space

A time-like (p + 1)-dimensional hypersurface Σ_{p+1} embedded into the D-
dimensional Minkowski spacetime with the signature \( \eta_{mn} = (+, -, \ldots, -) \) is
described by its radius vector \( \mathbf{x}(\xi^\mu) \) parametrized by the coordinates \( \xi^\mu = (\tau, \sigma^r), (r = 1, 2, \ldots, p) \). Using the local orthonormal frame \( \mathbf{n}_A(\xi^\mu) = (\mathbf{n}_i, \mathbf{n}_a) \) with
\( A = (i, a) \), attached to \( \Sigma_{p+1} \), one can expand the infinitesimal displace-
ments \( d\mathbf{x}(\xi^\mu) \) and \( d\mathbf{n}_A(\xi^\mu) \) in the basis \( \mathbf{n}_A(\xi^\mu) \)

\[
d\mathbf{x} = \omega^i n_i, \quad \omega^a = 0, \tag{1}
d\mathbf{n}_A = -\omega_A^B \mathbf{n}_B, \tag{2}
\]

with the frame vectors \( \mathbf{n}_i, (i, k = 0, 1, \ldots, p) \) tangent and \( \mathbf{n}_a, (a, b = p+1, p+2, \ldots, D - p - 1) \) - normal to the hypersurface. The fixation \( \omega^a = 0 \) of the
normal displacement of \( \mathbf{x} \) breaks down the local Lorentz group \( SO(1, D-1) \)
of the moving frame to its local subgroup \( SO(1,p) \times SO(D-p-1) \). As a
result, the antisymmetric matrix generators \( \omega_{AB} = -\omega_{BA} \) of \( SO(1,D-1) \)
split into three irreducible representations of \( SO(1,p) \times SO(D-p-1) \)

\[
\omega_A^B = \begin{pmatrix} A_{\mu i}^k \quad W_{\mu a}^b \\ W_{\mu i}^k \quad B_{\mu a}^b \end{pmatrix} \ d\xi^\mu, \tag{3}
\]

where \( A_{\mu i}^k \) and \( B_{\mu a}^b \) may be treated as \( SO(1,p) \) and \( SO(D-p-1) \) gauge
fields with their strengths \( F_{\mu i}^k \) and \( H_{\mu a}^b \) given by

\[
F_{\mu i}^k \equiv [D^\mu, D_i]^k \equiv (\partial_{[\mu} A_{\nu]} + A_{[\mu} A_{\nu]})_i^k, \tag{4}
H_{\mu a}^b \equiv [D^\mu, D^a_{\nu}]^b = (\partial_{[\mu} B_{\nu]} + B_{[\mu} B_{\nu]})_a^b. \tag{5}
\]
and $W_{\mu}^b$ is a charged vector multiplet with the covariant derivative

$$(D_{\mu}W_{\nu})_i^a = \partial_{\mu}W_{\nu i}^a + A_{\mu}^k W_{\nu k}^a + B_{\mu}^a b W_{\nu i}^b$$

including the $SO(1, p) \times SO(D - p - 1)$ gauge fields. The integrability conditions for the PDEs (1) and (2)

$$d \wedge \omega_A + \omega_A B \wedge \omega_B = 0,$$

$$d \wedge \omega_A B + \omega_A C \wedge \omega_C B = 0,$$

generates the well-known Maurer-Cartan structure equations prescribing for the torsion and curvature forms of the Minkowski space to be equal to zero.

After the break-up of the matrix indices in the tangent and normal components to $\Sigma_{p+1}$ Eqs. (8) take the form of the field constraints

$$F_{\mu\nu i}^k = -(W_{[\mu}W_{\nu]})_i^k,$$

$$H_{\mu\nu a}^b = -(W_{[\mu}W_{\nu]})_a^b,$$

$$(D_{[\mu}W_{\nu]})_i^a = 0,$$

where $\hat{W}_{[\mu}\hat{W}_{\nu]} \equiv \hat{W}_{\mu}\hat{W}_{\nu} - \hat{W}_{\nu}\hat{W}_{\mu}$. Equations (7) take the form

$$D_{[\mu}^{||} \omega_{\nu]}^i = 0,$$

$$\omega_{[\mu}^{||} W_{\nu]}^a i = 0,$$

where $D_{\mu}^{||}$ is the covariant derivative associated with the local Lorentz group $SO(1, p)$ acting in the local planes tangent to $\Sigma_{p+1}$

$$D_{\mu}^{||} \omega_{\nu}^i = \partial_{\mu} \omega_{\nu}^i + A_{\mu}^k \omega_{\nu}^k.$$

The object $\omega_{\mu}^i$ has a geometric sense as the $(p+1)$-bein of $\Sigma_{p+1}$ which connects the orthonormal moving frame $n_i$ with the natural frame $e_{\mu}$. Then the metric tensor $G_{\mu\nu}(\xi^\rho)$ of the hypersurface is presented as

$$e_{\mu} = \omega_{\mu}^i n_i, \quad G_{\mu\nu} = \omega_{\mu}^i \eta_{ik} \omega_{\nu}^k, \quad \omega_{\mu}^i \omega_{\nu}^k = \delta_{\nu}^i.$$

The solution of the constraints (13) introduces the second fundamental form $l_{\mu\nu}^a$ of the hypersurface $\Sigma_{p+1}$

$$W_{\mu}^a = -l_{\mu\nu}^a \omega_{\nu}^i, \quad l_{\mu\nu}^a := n^a \partial_{\mu} x.$$

The presented gauge reformulation [24] of the Regge-Lund geometric approach for strings revealed their new description by the closed exactly solvable sector of states of the two-dimensional $SO(1,1) \times SO(D-2)$ gauge invariant model. The model includes a massless scalar multiplet originating from the independent components of $l_{\mu\nu}^a$ of the string worldsheet $\Sigma_2$ in a fixed gauge.
3 \ SO(1,p) \times \ SO(D-p-1) gauge invariant model

Further let us consider a \ SO(1,p) \times \ SO(D-p-1) \ invariant \ gauge \ model \ in
a curved \ (p+1)-dimensional \ space \ with \ the \ space-time \ coordinates \ \xi^\mu \ and \ a
given \ pseudo-Riemannian \ metric \ g_{\mu\nu}.

\[ S = \gamma \int d^{p+1} \xi \sqrt{|g|} \mathcal{L}, \]  \hspace{1cm} (17)

\[ \mathcal{L} = \frac{1}{4} Sp(F_{\mu\nu} F^{\mu\nu}) - \frac{1}{4} Sp(H_{\mu\nu} H^{\mu\nu}) + \frac{1}{4} \hat{\nabla}_\mu W^{ia}_\nu \hat{\nabla}^{\{\mu} W^{\nu\}}_{ia} - \hat{\nabla}_\mu W^{ia}_\nu \hat{\nabla}_\nu W^{\nu}_{ia} + V, \]  \hspace{1cm} (18)

where \ \hat{\nabla}_\mu W^{ia}_\nu \equiv \hat{\nabla}_\mu W^{ia}_\nu + \hat{\nabla}_\mu \hat{\nabla}_\nu W_{ia}, \ and \ the \ potential \ term \ V \ describes \ generally \ and \ gauge \ invariant \ nonlinear \ (self)interactions \ of \ the \ vector \ multiplet \ W^{ia}_\mu. \ The \ generally \ and \ gauge \ covariant \ derivative \ \hat{\nabla}_\mu \ is \ defined \ by

\[ \hat{\nabla}_\mu W^{ia}_\nu = \partial_\mu W^{ia}_\nu - \Gamma^{\rho}_{\mu\nu} W^{ia}_\rho + A^{k}_{\mu} W^{ia}_\nu + B^{b}_{\mu} W^{ia}_\nu \]  \hspace{1cm} (19)

and \ it \ differs \ from \ the \ usual \ general \ covariant \ derivative

\[ \nabla_\mu W^{ia}_\nu = \partial_\mu W^{ia}_\nu - \Gamma^{\rho}_{\mu\nu} W^{ia}_\rho, \hspace{1cm} \nabla_\mu g_{\nu\rho} = 0 \]  \hspace{1cm} (20)

which \ contains \ only \ the \ metric \ compatible \ with \ the \ Levi-Chivita \ connection \ \Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}).

Our \ objective \ is \ to \ study \ the \ equations \ of \ motion \ of \ this \ model \ and \ their\ exact \ analytical \ solutions.

The \ variation \ of \ S \ (17) \ in \ the \ gauge \ and \ vector \ fields \ results \ in \ the \ generalized \ Maxwell \ and \ Euler-Lagrange \ equations \ for \ W^{ia}_\mu:

\[ \hat{\nabla}_\mu F^{i\nu}_{\mu k} = - \hat{\nabla}_\mu (W^{i[\mu}_{\nu} W^{\nu a]}_{k}) - \frac{1}{2} W^{i[a}_{\mu} \hat{\nabla}^{\nu} W^{\mu a]}_{k}], \]  \hspace{1cm} (21)

\[ \hat{\nabla}_\mu H^{i\nu}_{ab} = - \hat{\nabla}_\mu (W^{i[\mu}_{\nu} W^{\nu b]}_{a}) - \frac{1}{2} W^{i[a}_{\mu} \hat{\nabla}^{\nu} W^{\mu b]}_{a}], \]  \hspace{1cm} (22)

\[ \hat{\nabla}_\mu \hat{\nabla}^{\{\mu W^{\nu\}}_{ia} = 2 \hat{\nabla}^{\nu} \hat{\nabla}_\mu W^{\nu i}_{a} + \frac{\partial V}{\partial W_{ia}}. \]  \hspace{1cm} (23)

Using \ the \ shifted \ gauge \ field \ strengths \ \mathcal{F}^{i\nu}_{\mu k} \ and \ \mathcal{H}^{i\nu}_{ab}:

\[ \mathcal{F}^{i\nu}_{\mu k} = (F_{\mu\nu} + W_{[\mu W_{\nu]}^{i\nu}_{k}]}, \]  \hspace{1cm} (24)

\[ \mathcal{H}^{i\nu}_{ab} = (H_{\mu\nu} + W_{[\mu W_{\nu]}^{i\nu}_{ab}]}, \]  \hspace{1cm} (25)
one can present Eqs. (21–23) in the compact form

\[ \hat{\nabla}_\mu \mathcal{F}^{\mu \nu}_{ik} = -\frac{1}{2} W_{\mu[i} W^{\nu]}_{a[k]}, \] (26)

\[ \hat{\nabla}_\mu \mathcal{H}^{\mu \nu}_{ab} = -\frac{1}{2} W_{\mu[a] i} \hat{\nabla}^{[\nu W^\mu]}_{b]}, \] (27)

\[ \hat{\nabla}_\mu \hat{\nabla}^{[\mu W^\nu]}_{ia} = -2 [\hat{\nabla}^{\mu}, \hat{\nabla}^{\nu}] W^\mu_{\mu} + \frac{\partial V}{\partial W_{\nu ia}}. \] (28)

The next step is to apply the generalized first Bianchi identity

\[ [\hat{\nabla}_\mu, \hat{\nabla}_\nu] = \hat{\nabla}^{\mu \nu}_\lambda V^\lambda =: (\partial_{[\mu} \Gamma^\gamma_{\nu] \lambda} + \Gamma^\gamma_{[\mu | \rho} \Gamma^\rho_{| \nu] \lambda}) V^\lambda. \] (29)

Then one can present the EOM (28) for \( W^\mu_{ia} \) in the form

\[ \frac{1}{2} \hat{\nabla}_\mu \hat{\nabla}^{[\mu W^\nu]}_{ia} - \mathcal{F}^{\mu \nu}_{k W^k a} - \mathcal{H}^{\mu \nu a b} W^\mu_{ib} = \frac{1}{2} \frac{\partial V}{\partial W_{\nu ia}} + ([W^\mu, W^\nu], W^\mu_{i})^{ia} - R^\mu_{\nu \lambda} W^\mu_{ia}, \] (31)

where \( R^\mu_{\nu \lambda} := R^\mu_{\nu \mu \lambda} \) is the Ricci tensor. From the relation

\[ \frac{1}{4} \frac{\partial}{\partial W_{\nu ia}} (W^\mu_{i} W^\nu_{i} W^\rho_{j} W^\rho_{j})^{i} = (\partial_{[\mu} \Gamma^\gamma_{\nu] \lambda} + \Gamma^\gamma_{[\mu | \rho} \Gamma^\rho_{| \nu] \lambda}) V^\lambda, \] (32)

including the commutators of \( \hat{W}_\mu \) one can introduce the shifted potential \( \mathcal{V} \)

\[ \mathcal{V} = V + \frac{1}{2} S_{\rho} (W^\mu_{\nu}, W^\rho_{\nu}), \] (33)

where the trace \( S_{\rho} (W^\mu_{\nu}, W^\rho_{\nu}) \equiv (W^\mu_{\nu}, W^\rho_{\nu})^{i} \).

Then EOM (26), (27) and (31) of the model can be rewritten as

\[ \hat{\nabla}_\mu \mathcal{F}^{\mu \nu}_{ik} = -\frac{1}{2} W_{\mu[i} W^{\nu]}_{a[k]}, \] (34)

\[ \hat{\nabla}_\mu \mathcal{H}^{\mu \nu}_{ab} = -\frac{1}{2} W_{\mu[a] i} \hat{\nabla}^{[\nu W^\mu]}_{b]}, \] (35)

\[ \frac{1}{2} \hat{\nabla}_\mu \hat{\nabla}^{[\mu W^\nu]}_{ia} + \mathcal{F}^{\mu \nu}_{k W^k a} + \mathcal{H}^{\mu \nu a b} W^\mu_{ib} = \frac{1}{2} \frac{\partial \mathcal{V}}{\partial W_{\nu ia}} - R^\mu_{\nu \lambda} W^\mu_{ia}. \] (36)
It is easily seen that the first-order PDEs
\[ \mathcal{F}_{\mu \nu} = 0, \quad \mathcal{H}^{\mu \nu} = 0, \quad \hat{\nabla}^{[\mu} W_{\nu]\alpha} = 0, \quad (37) \]
form a particular solution of Eqs. (34-36), provided that the condition
\[ \frac{1}{2} \frac{\partial \mathcal{V}}{\partial W_{\nu\alpha}} - R^{\mu \nu} W_{\nu\alpha} = 0 \quad (38) \]
is satisfied. Using arbitrariness of \( \mathcal{V} \), and consequently of \( \mathcal{V} \), one can choose the latter in the form
\[ \mathcal{V} = R^{\mu \nu} W_{\nu\alpha} W_{\nu\alpha} \quad (39) \]
which solves Eq. (38) if \( R^{\mu \nu} \) does not depend on \( W_{\nu\alpha} \).

Thus, we find that the model (17) with the Lagrangian density
\[ \mathcal{L} = \frac{1}{4} \text{Sp}(F_{\mu \nu} F^{\mu \nu}) - \frac{1}{4} \text{Sp}(H_{\mu \nu} H^{\mu \nu}) + \frac{1}{2} \hat{\nabla}_{\mu} W^{\nu \alpha} \hat{\nabla}_{\nu} W_{\nu\alpha} - \frac{1}{2} R^{\mu \nu} W_{\nu\alpha} W_{\nu\alpha} - \frac{1}{2} S_{\mu \nu} W_{\nu\alpha} W_{\nu\alpha} \quad (40) \]
produces the following nonlinear Euler-Lagrange equations
\[ \hat{\nabla}_{\mu} F_{\nu k} = - \frac{1}{2} W_{\mu[i]} \hat{\nabla}^{[\nu} W_{\mu]a[k]}, \quad (41) \]
\[ \hat{\nabla}_{\mu} H^{\mu \nu} = - \frac{1}{2} W_{\mu[j]} \hat{\nabla}^{[\nu} W_{\mu]i}, \quad (42) \]
\[ \frac{1}{2} \hat{\nabla}_{\mu} \hat{\nabla}^{[\mu} W^{\nu]a} + \mathcal{F}^{\mu \nu}_{k} W^{k \alpha}_{\mu} + \mathcal{H}^{\mu \nu}_{ab} W^{\nu}_{\mu} = 0 \quad (43) \]
for the gauge \( A_{\mu k} \), \( B_{\mu a} \) and the vector \( W_{\nu\alpha} \) fields in an external gravitational field \( g_{\mu \nu}(\xi^\rho) \). These equations have the particular solution (37) which coincides with the Maurer-Cartan Eqs. (9), (10) and (11) called the Gauss-Codazzi (G-C) equations in the classical differential geometry of surfaces.

One can weaken the demand for the gravitational field to be external and to consider it in the equal rights with the dynamical gauge and vector fields. Then EOM (41-43) have to be completed by the variational eqs. with respect to \( g_{\mu \nu} \). These equations will connect the Ricci tensor \( R_{\mu \nu} \) with the
gauge strengths, $W_{\mu i\alpha}$ and its covariant derivatives. However, solution of this equation even on the Gauss-Codazzi shell needs in an additional investigation and is not studied in this paper. It is also interesting that the model (17) with the Lagrangian (40) and a dynamical gravitational field $g_{\mu\nu}$ seems to be a natural generalization of the Dirac scale-invariant gravity theory with the four-dimensional action $W$ (25) (see also [26])

$$W = \int d^4x \sqrt{|g|} \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + 6 \beta^\mu \beta_\mu - \beta^2 R + c \beta^4 \right),$$ (44)

where $\beta$ is the dilaton field.

Below we shall study another interesting possibility to treat the proposed action (17) as a gauge-invariant action associated with some hypersurfaces and branes embedded into the higher dimensional Minkowski space.

### 4 Branes as solutions of the gauge model

The observation that the Gauss-Codazzi equations give a particular solution of the gauge model in an external gravitational field points to the possibility to consider branes as solutions of gauge theories. To develop this conjecture one can treat the considered $(p+1)$-dimensional pseudo-Riemannian space-time of the gauge model (17), (18) as a $(p+1)$-dimensional world hypersurface swept by a p-brane in D-dim. Minkowski space.

To realize this conjecture one has to take into account the remaining M-C equations (12), (13). Here we observe that Eqs. (12) have the solution

$$\nabla^\parallel_{\mu} \omega^i_{\nu} \equiv \partial_\mu \omega^i_{\nu} - \Gamma^p_{\mu \nu} \omega^i_p + A^i_{\mu k} \omega^k_{\nu} = 0$$ (45)

that generalizes the well-known tetrad postulate of the general relativity to higher dimensions and identifies the metric connection $\Gamma^p_{\mu \nu}$, with $A^i_{\mu k}$ by means of the gauge transformation

$$\Gamma^p_{\nu \lambda} = \omega^p_i A_{\nuik} \omega^k_\lambda + \partial_\nu \omega^k_\lambda \omega^i_k \equiv \omega^p_i D^\parallel_{\nu} \omega^i_\lambda.$$ (46)

The transformation function $\omega^p_i$ in (46) coincides with the $(p+1)$-bein (13) of the brane hypersurface equipped with the metric $G_{\mu\nu} = g_{\mu\nu}$.

Using the expressions (30) and (4) for the Riemann tensor $R_{\mu\nu} \gamma^\lambda_\chi$ and the strength $F_{\mu i\nu k}$ we find their connection

$$R_{\mu \nu \gamma \lambda} = \omega^\gamma_i \ F_{\mu i\nu k} \omega^k_\lambda, \quad R_{\nu \lambda} = \omega^\mu_i \ F_{\mu i\nu k} \omega^k_\lambda.$$ (47)
Then, with the help of the Gauss-Codazzi condition for $F_{ik}^\mu$ we obtain
\[ R_{\nu\lambda} = -\omega_i^\mu (W^{[\mu} W^{\nu]})_k \omega_k^\lambda \]  
which represents the Ricci tensor of the hypersurface on the G-C shell.

As a result, the term $R_{\nu\lambda} W^a_{\nu a}$ in Eq. (38) becomes the function of $\hat{W}_\lambda$
\[ \frac{1}{2} \frac{\partial \mathcal{V}}{\partial W_{\nu a}} = -\omega_i^\mu (W^{[\mu} W^{\nu]})_j \omega^k_j \omega^k_{\nu a} \]  
and the potential $\mathcal{V}$ is to be found as the solution of this equation.

Moreover, Eq. (49) has to take into account still remaining M-C equations (13) that show that not all of the components of $W^a_{\nu a}$ are independent DOF. This follows from the relation
\[ \omega_i^\mu W^a_{\mu k} \omega^k_{\lambda j} = W^a_{\lambda j} \]  
resulting from the symmetry $l_{\mu\nu}^a = l_{\nu\mu}^a$ for the second fundamental form components. Taking into account the linear dependence (50) gives
\[ R_{\nu\lambda} W^a_{\nu a} = -\omega_i^\mu W^a_{j c} W^\lambda_{ci} - W^a_{j c} W^\lambda_{c i} \]  
which has the solution
\[ \mathcal{V} = -\frac{1}{2} Sp(W_{\nu} W_{\nu}) Sp(W^\nu W^\nu), \quad Sp(l^a) = 0. \]  

The constraint $Sp(l^a) = 0$ is the well-known minimality condition for the $(p+1)$-dim. worldvolume equivalent to the brane EOM
\[ \Box^{(p+1)} \mathbf{x} = 0, \]  
where $\Box^{(p+1)} := \frac{1}{\sqrt{|G|}} \partial_\alpha \sqrt{|G|} G^{\alpha\beta} \partial_\beta$ is the reparametrization invariant Laplace-Beltrami operator on $\Sigma_{p+1}$ [20]. Eq. (55) follows from the Dirac action
\[ S = T \int \sqrt{|G|} d^{p+1} \xi, \]  
where $G$ is the determinant of the induced metric $G_{\alpha\beta} := \partial_\alpha \mathbf{x} \partial_\beta \mathbf{x}$.

So, EOM of gauge model (17) are compatible with the generalized tetrade anzats and have the solution describing the Dirac p-brane.
5 Reduction to SO(D-p-1) invariant gauge model

Having revealed the brane solutions one can use the tetrade postulate (45) directly in the gauge action (17) together with the transition from $W_{\mu i}^a$ to $l_{\mu \nu}^a = -\omega_i^a W_{\mu i}^a$ owing to solution (16) of the first Maurer-Cartan equations. Then the G-C eqs. (9-11) take the form

$$R_{\mu \nu}^{\gamma \lambda} = l_{\mu}^{\gamma a} l_{\nu \lambda}^a, \quad (57)$$

$$H_{\mu \nu}^{ab} = l_{\mu}^{\gamma a} l_{\nu \gamma}^b, \quad (58)$$

$$\nabla_{[\mu} l_{\nu \rho a]} = 0, \quad (59)$$

where $\nabla_{\mu} l_{\nu}^a := \partial_\mu l_{\nu}^a - \Gamma^\lambda_{\mu \nu} l_{\lambda}^a - \Gamma^\lambda_{\mu \rho} l_{\nu}^a + B_{\mu a}^{ab} l_{\nu \rho b}$. As is seen the relation (57) is the generalization of the Gauss theorem a egregium to a (p+1)-dimensional hypersurface embedded into the D-dimensional Minkowski space.

The corresponding action which solutions are the G-C eqs. (57-59) is

$$S = \gamma \int d^{p+1} \sqrt{|g|} \mathcal{L},$$

$$\mathcal{L} = -\frac{1}{4} R_{\mu \rho \lambda \rho} R^{\mu \rho \lambda \rho} - \frac{1}{4} Sp(H_{\mu \nu} H^{\mu \nu}) + \frac{1}{2} \nabla_{\mu} l_{\nu \rho a} \nabla_{\nu \rho a} - \nabla_{\mu} l_{\nu \rho a} \nabla_{\rho a}$$

$$- \frac{1}{2} Sp(l_{a b}) S p(l^a l^b) + Sp(l_{a b} l^a l^b) - Sp(l_{a} l_{b} l_{b}). \quad (60)$$

To prove this we consider the action (17) with $l_{\mu \nu}^a$ substituted for $W_{\mu i}^a$

$$S = \gamma \int d^{p+1} \sqrt{|g|} \mathcal{L},$$

$$\mathcal{L} = -\frac{1}{4} R_{\mu \rho \lambda \rho} R^{\mu \rho \lambda \rho} - \frac{1}{4} Sp(H_{\mu \nu} H^{\mu \nu})$$

$$+ \frac{1}{2} \nabla_{\mu} l_{\nu \rho a} \nabla_{\nu \rho a} - \nabla_{\mu} l_{\nu \rho a} \nabla_{\rho a} + V. \quad (61)$$

Variation of $S$ (61) in the dynamical fields $l_{\mu \nu}^a$, $B_{\mu}^{ab}$ gives their EOM

$$\nabla_{\nu} H_{\mu \rho a}^{\nu\mu} = \frac{1}{2} l_{\rho \mu} \partial_{[\nu} l_{\rho \mu]}^{\nu\mu, a}, \quad (62)$$

$$\frac{1}{2} \nabla_{\mu} \nabla_{\nu} l_{\rho a} = -[\nabla_{\nu}, \nabla_{\mu}] l_{\rho a} + \frac{1}{2} \partial_{\nu} V, \quad (63)$$

where $H_{\mu \nu}^{ab} := H_{\mu \nu}^{ab} - l_{[\mu}^{\gamma a} l_{\nu \gamma] b}$. Eqs. (62-63) have G-C eqs. (58-59)

$$H_{\mu \nu}^{ab} = 0, \quad \nabla_{[\mu} l_{\nu]}^{\rho a} = 0. \quad (64)$$
their particular solution provided the condition

\[
\frac{1}{2} \frac{\partial V}{\partial l_{\nu \rho a}} = [\nabla^\bot, \nabla^\bot] l^\mu_{\rho a}. \tag{65}
\]

Using the G-C constraints (57-59) and generalized Bianchi identity

\[
[\nabla^\gamma, \nabla^\nu] l^\mu_{\rho a} = R_{\gamma \nu}^{\rho \lambda a} l^\mu_{\lambda a} + R_{\gamma \nu}^{\rho a} l^\mu_{\rho a} + H_{\gamma \nu}^{\rho a} b^{\mu \rho b} \tag{66}
\]
one can transform Eq. (65) to the following form

\[
\frac{1}{2} \frac{\partial V}{\partial l_{\nu \rho a}} = (l^a l^b)^{\rho \nu} Sp(l_b) + (2 l^b l^a l^b - l^a l^b l^b - l^b l^a l^a)^{\rho \nu} - l^{\rho \nu} b l^a Sp(l_b). \tag{67}
\]

Eq. (67) fixes the potential term \(V\) and the trace of \(l^{\mu \alpha}\) to be equal to

\[
V = -\frac{1}{2} Sp(l_a l_b) Sp(l^a l^b) + Sp(l_a l^a l^b) - Sp(l^a l_a l_b), \quad Sp(l_a) = 0. \tag{68}
\]

We see that Dirac p-brane (56) is the solution of Eqs. (62-63) for \(S\) (60) which is presented by the M-C conditions (7-8). The substitution of an arbitrary function \(f(R, \nabla R)\) of the background Riemannian tensor for the \(R^2\) term in \(S\) (60) preserves the above conclusion.

Variation of \(S\) (60) and its \(f(R)\) extension with respect to \(g_{\mu \nu}\) will produce generalized G-E equations for the dynamical gravitational field associated with p-branes, or and the \(\Sigma_{\mu+1}\) diffeomorphism constraints if \(f = \text{const.}\)

6 Summary

A new gauge reformulation of the geometric Regge-Lund approach for the relativistic string is generalized to the case of relativistic p-branes imbedded into D-dimensional Minkowski space. There is constructed a set of SO(1,p) \(\times\) SO(D-p-1) invariant gauge models including massless vector multiplets in gravitational field and possessing exact solutions. The latter are presented by the first-order Gauss-Codazzi PDEs rewritten as gauge constraints.

It is shown that the generalized tetrad postulate allows to treat the exact solutions as associated with hypersurfaces, or branes embedded into higher dimensional Minkowski spaces. The treatment is accompanied with the transition from the vector multiplets to the massless tensor fields.
It has been surprising that the SO(1,p) × SO(D-p-1) gauge-invariant model \cite{17} turns out to be a generalization of the Dirac scale invariant gravity theory \cite{25} with the massless vector multiplet $W^a_{\mu i}$ or the tensor $l^{\mu \nu a}$ substituted for the Dirac dilaton field $\beta$. This observation needs in further study.

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