BANACH PROPERTY (T) FOR SLₙ(Z) VIA RELATIVE BANACH PROPERTY (T)

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Abstract. We prove a new result of relative Banach property (T) for the uni-upper-triangular group in SL₃(Z). This allows us to prove a uniform version of Banach property (T) for SLₙ(Z), SLₙ(R) with respect to all super-reflexive Banach spaces.

Consequences of this result are: First, for every \( n \geq 5 \), the groups SLₙ(Z) and SLₙ(R) have the Banach fixed point property with respect to any super-reflexive space. Second, we settle a long standing open problem and show that the Margulis expanders of Cayley graphs of SLₙ(Z/ₘZ) for a fixed \( n \geq 3 \) are super-expanders. Last, we deduce Banach property (T) with respect to all super-reflexive Banach spaces for a large family of higher rank algebraic groups.

1. Introduction

Property (T) was introduced by Kazhdan in [Kaz67] as a tool to prove compact generation. Since then it was found useful for a wide range of applications in various different areas of mathematics (see [BdlHV08] and the introduction of [BFGM07], and reference therein). We mention two such applications that are relevant in the context of this paper: First, property (T) for a group \( G \) is equivalent (under some mild assumptions on \( G \)) to property (FH) which states that every continuous isometric affine action of \( G \) on a real Hilbert space admits a fixed point. Second, Margulis gave the first explicit construction of expanders using property (T).

More recently, Bader, Furman, Gelander and Monod [BFGM07] defined a Banach version of property (T) (and its connection to fixed point properties with respect to Banach spaces). They conjectured that higher rank algebraic group should have this form of Banach property (T) with respect to the class of all super-reflexive Banach spaces. Roughly simultaneously to the work of [BFGM07], V. Lafforgue [Laf09] proved that groups of the form SL₃(F) where \( F \) is a non-Archimedian local field have a strong form of Banach property (T) for large classes of Banach spaces. In particular, his work corroborates the conjecture of [BFGM07], since as a result of his work the groups SL₃(F) where \( F \) is a non-Archimedian local field have Banach property (T) (and the fixed point property) with respect to all super-reflexive Banach spaces. Later, Liao [Lia14] extended the work of Lafforgue and proved the strong version of Banach property (T) holds for every connected almost \( F \)-simple algebraic group whose \( F \)-split rank

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is $\geq 2$, where $F$ is a non-Archimedean local field. In his work \cite{Laf09}, Lafforgue also showed how to use his result to construct super-expander families, i.e., families of graphs that are expanders with respect to every super-reflexive Banach space (see exact definition below).

Unlike in the non-Archimedean case, much less is known regarding Banach property (T) for algebraic groups over $\mathbb{R}$ (and their lattices). In the paper of Bader, Furman, Gelnader and Monod \cite{BFGM07} they show that higher rank algebraic groups have Banach property (T) (and fixed point properties) for $L^p$ spaces. For general super-reflexive spaces (that are not $L^p$ spaces), partial results were proven by de Laat, Mimura and de la Salle in various collaborations \cite{dLdlS15, Sal16, dLMdlS16, dLdlS18, dlS19}. However, none of these works cover all super-reflexive spaces even for the group $\text{SL}_3(\mathbb{R})$.

In this paper, we make a major breakthrough regarding Banach property (T) for $\text{SL}_n(\mathbb{Z})$ and $\text{SL}_n(\mathbb{R})$, namely, we prove that for $n \geq 3$, these groups have (a uniform version) of Banach property (T) for all super-reflexive Banach spaces. This has several striking consequences: First, it allows us to prove that for every $n \geq 5$, $\text{SL}_n(\mathbb{Z})$ and $\text{SL}_n(\mathbb{R})$ have fixed point properties with respect to all super-reflexive Banach spaces. Second, we settle a long standing open problem and show that the Margulis expanders arising from quotients of $\text{SL}_n(\mathbb{Z})$ are super-expanders. Last, we show that a large family of connected $\mathbb{R}$-almost simple (higher rank) algebraic groups have Banach property (T) with respect to all super-reflexive Banach spaces.

Our method for proving Banach property (T) is also novel. The works of prior works de Laat, Mimura and de la Salle mentioned tried to generalize the work of Lafforgue for strong (Hilbert) property (T) of $\text{SL}_3(\mathbb{R})$ to the Banach setting. Our approach is very different: We prove a relative version of Banach property (T) for the uni-triangular matrices in $\text{SL}_3(\mathbb{Z})$ with respect to super-reflexive Banach spaces. We note that this proof is new even in the Hilbert setting. After that, we use a bounded generation argument à la Shalom to deduce Banach property (T) for $\text{SL}_3(\mathbb{Z})$ for all super-reflexive spaces and then by induction (using bounded generation again) for $\text{SL}_n(\mathbb{Z}), n \geq 3$ for all super-reflexive spaces. The version of Banach property (T) we work with is inherited from lattices, and thus it readily follows that $\text{SL}_n(\mathbb{R}), n \geq 3$ have Banach property (T) for all super-reflexive spaces.

1.1. (Relative) Banach property (T). We give a definition of (relative) Banach property (T) with respect to a class of representations that is inspired by the work of V. Laffourge \cite{Laf08}, \cite{Laf09}. Before we give the definition, we introduce some notations. Let $G$ be a locally compact group with Haar measure $\mu$. We denote $C_c(G)$ to be the compactly supported continuous functions $f : G \to \mathbb{C}$ with the convolution product. We further denote $\text{Prob}_c(G) \subseteq C_c(G)$ to be functions $f : G \to [0, \infty)$ such that $\int_G f(g) d\mu(g) = 1$. Given a continuous representation $\pi : G \to B(\mathcal{E})$ where $\mathcal{E}$ is a Banach space, we define for every $f \in C_c(G)$ an operator $\pi(f)$ via the Bochner integral

$$\pi(f)\xi = \int_G f(g)\pi(g)\xi d\mu(g), \forall \xi \in \mathcal{E}.$$
Let $F$ be a class of continuous representations of $G$ such that $\sup_{(\pi, E) \in F} \|\pi(f)\|_{B(E)}$ is bounded on compact sets of $G$. For such a class, we define a norm $\|\cdot\|_F$ on $C_c(G)$ by

$$\|f\|_F = \sup_{\pi \in F} \|\pi(f)\|,$$

and denote $C_F(G)$ to be the completion of $C_c(G)$ with respect to this norm. We note that for every $f \in C_F(G)$ and every $\pi \in F$, the operator $\pi(f) \in B(E)$ is well-defined as a limit of operators $\pi(f_n)$ with $f_n \in C_c(G)$.

**Definition 1.1.** Let $G$ be a locally compact group, $H < G$ be a subgroup and $F$ be a class of continuous representations of $G$ such that $\sup_{(\pi, E) \in F} \|\pi(f)\|_{B(E)}$ is bounded on compact sets of $G$. We will say that $(G, H)$ has relative property $(T_F)$ if there is a sequence $h_n \in \text{Prob}_c(G)$ that converges to $f \in C_F(G)$ (with respect to the norm $\|\cdot\|_F$) such that for every $(\pi, E) \in F$, $\text{Im}(\pi(f)) \subseteq E^{\pi(H)}$. We will say that $G$ has property $(T_F)$ if $(G, G)$ has property $(T_F)$.

**Remark 1.2.** Assume that $G$ has property $(T_F)$ as defined above, i.e., there is $f \in C_F(G)$ such that for every $(\pi, E) \in F$, $\text{Im}(\pi(f)) \subseteq E^{\pi(G)}$. Note that this $f$ is always a projection in the sense that for every $\pi \in F$, it holds that $\pi(f)^2 = \pi(f)$. Indeed, for every $h \in C_c(G)$, every $(\pi, E) \in F$ and every $\xi \in E$ it holds that $\pi(h)\pi(f)\xi = \pi(f)\xi$ and thus $f^*f = f$. Because of this, such $f$ is sometimes referred to as a *Kazhdan's projection*.

However, we note that for a pair $(G, H)$ that has relative property $(T_F)$, the function $f \in C_F(G)$ given in the Definition above need not be a projection on $E^{\pi(H)}$ for $(\pi, E) \in F$.

**Definition 1.3.** Let $G$ be a locally compact group, $H < G$ be a subgroup and $F$ be a class of Banach spaces. Denote $U(G, E)$ to be the class of all continuous isometric linear representations $(\pi, E)$ where $E \in \mathcal{E}$. When $G$ is obvious from the context, we will denote $U(\mathcal{E}) = U(G, \mathcal{E})$. We will say that $(G, H)$ has relative (Banach) property $(T_E)$ if it has relative property $(T_{U(\mathcal{E})})$. We will say that $G$ has (Banach) property $(T_E)$ if $(G, G)$ has property $(T_E)$.

**Remark 1.4.** The Definition given above for property $(T_E)$ is usually stronger than the definition of Banach property $(T)$ given by Bader, Furman, Gelander and Monod in [BFGM07]. Namely, As noted in [LdL18] Section 5 and in Remark 3.4 below, for a class of uniformly convex spaces $\mathcal{E}$, property $(T_E)$ implies property $(T_{\mathcal{E}})$ of [BFGM07] for every $E \in \mathcal{E}$, but not vice-versa. Even for a single uniformly convex Banach space $E$, our property $(T_{\{E\}})$ (i.e., property $(T_E)$ as defined above for $\mathcal{E} = \{E\}$) is stronger than property $T_E$ of [BFGM07]: In Remark 3.4 below we note that that property $T_{\{E\}}$ of [BFGM07] is a non-uniform version of our property $(T_{\{E\}})$. Thus in some sources (see [DN19]) our property $(T_{\{E\}})$ is referred to as “uniform property $(T_E)$”.

### 1.2. Uniformly convex Banach spaces.

A Banach space $E$ is called uniformly convex if there is a function $\delta : (0,2) \to (0,1]$ called the *modulus of convexity* such that for every $0 < \varepsilon \leq 2$ and every $\xi, \eta \in E$ with $\|\xi\| = 1$, $\|\eta\| = 1$, if $\|\xi - \eta\| \geq \varepsilon$, then $\frac{\|\xi + \eta\|}{2} \leq 1 - \delta(\varepsilon)$.

In [BFGM07] Proposition 2.3 it was proven that if a group acts by uniformly bounded linear transformations on a super-reflexive space $E$, then there is a compatible uniformly
convex norm on $E$ and the group acts by isometries with respect to that norm. Thus below we will only consider isometric action on uniformly convex Banach spaces in lieu of isometric action on super-reflexive Banach spaces.

We will be interested in classes of uniformly convex Banach spaces defined as follows: Let $\delta_0 : (0, 2] \to (0, 1]$ be a function. Denote $E_{uc}(\delta_0)$ to be the class of all uniformly convex Banach spaces with a modulus of convexity bounded by $\delta_0$, i.e., a Banach space $E$ with a modulus of convexity $\delta : (0, 2] \to (0, 1]$ is in $E_{uc}(\delta_0)$ if for every $0 < \varepsilon \leq 2$ it holds that $\delta(\varepsilon) \geq \delta_0(\varepsilon)$.

1.3. **Relative Banach property (T) for uni-triangular in $\text{SL}_3(\mathbb{Z})$.** Let $\text{UT}_3(\mathbb{Z})$ and $\text{LT}_3(\mathbb{Z})$ denote the subgroups of uni-upper-triangular and uni-lower-triangular matrices in $\text{SL}_3(\mathbb{Z})$, i.e.,

\[
\text{UT}_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\},
\]

and

\[
\text{LT}_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 0 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.
\]

With the definition of relative Banach property (T) given above, we prove the following:

**Theorem 1.5.** For any function $\delta_0 : (0, 2] \to (0, 1]$, the pairs $(\text{SL}_3(\mathbb{Z}), \text{UT}_3(\mathbb{Z}))$ and $(\text{SL}_3(\mathbb{Z}), \text{LT}_3(\mathbb{Z}))$ both have relative property $(T_{E_{uc}(\delta_0)})$.

We note that the proof of this Theorem is new even in the classical Hilbert setting. Moreover, our proof is completely elementary in contrast with the more classical proofs of relative property (T), e.g., the proof that $(\text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ has relative property (T) in [BdlHV08, Theorem 4.2.2] requires Fourier analysis and projections valued measures while our proof requires neither.

1.4. **Banach property (T) for $\text{SL}_n(\mathbb{Z})$.** Theorem 1.5 coupled with a bounded generation argument à la Shalom yields the following:

**Theorem 1.6.** Let $\delta_0 : (0, 2] \to (0, 1]$ and $n \geq 3$. The group $\text{SL}_n(\mathbb{Z})$ has property $(T_{E_{uc}(\delta_0)})$. In particular, for every uniformly convex Banach space $E$ and every $n \geq 3$, the group $\text{SL}_n(\mathbb{Z})$ has property $(T_{E(\varepsilon)})$.

Since property $(T_{E_{uc}(\delta_0)})$ is inherited from lattices, this readily implies the following:

**Corollary 1.7.** Let $\delta_0 : (0, 2] \to (0, 1]$ and $n \geq 3$. The group $\text{SL}_n(\mathbb{R})$ has property $(T_{E_{uc}(\delta_0)})$. In particular, for every uniformly convex Banach space $E$ and any $n \geq 3$, the group $\text{SL}_n(\mathbb{R})$ has property $(T_{E(\varepsilon)})$.

1.5. **Applications.**
Banach fixed point properties. Given a Banach space $E$, a topological group $G$ is said to have property ($F_E$) if every affine (continuous) isometric action of $G$ on $E$ admits a fixed point. In [Mim11], Mimura also defined the notion of property ($FF_E$) (see exact definition in §8.1 below) that is a Banach version of property (TT) defined by Monod in [Mon01]. For a uniformly convex Banach space $E$ (or more generally for a reflexive Banach space $E$), it follows from Ryll-Nardzewski fixed-point Theorem that property ($FF_E$) implies property ($F_E$). A result of de Laat, Mimura and de la Salle allows one to deduce property ($FF_E$) for the groups $\text{SL}_n(Z)$, $\text{SL}_n(R)$ from property ($T_E$) for the groups $\text{SL}_n(Z)$ and $\text{SL}_n(R)$. Thus, we can deduce the following:

**Corollary 1.8.** For every $n \geq 5$ and every uniformly convex Banach space, the groups $\text{SL}_n(Z), \text{SL}_n(R)$ have property ($FF_E$) and in particular property ($F_E$).

Super-expanders. A family of uniformly bounded degree graphs is called a super-expander family if it does not coarsely embed in any uniformly convex Banach space (see exact definition in §8.2). The first examples of super-expanders were constructed by Lafforgue in [Laf08] as a consequence of his work on strong Banach property (T) for $\text{SL}_3(F)$, where $F$ is a non-Archimedean local field. Since Lafforgue’s work there have been several constructions of super-expanders using several different techniques: Namely, the work of Mendel and Naor on spectral calculus [MN14] which gave a zig-zag construction for super-expanders and, in a different direction, constructions using warped cones of groups actions arising from groups with strong Banach property (T) [Vig19, dLdS18, FNvL19, Saw20]. However, it was asked in several places (see [Obe18, Problem 5], [dLdS18], [Mim13, Remark 5.3]) if for a fixed $n \geq 3$ the Cayley graphs of $\text{SL}_n(Z/iZ)$ (known also as the Margulis expanders) form a super-expander family. Partial results in this direction were achieved by de Laat and de la Salle in [dLdS18], but up until our work, the problem remained open. As a consequence of our Theorem 1.6 we settle this problem to the affirmative and prove the following:

**Theorem 1.9.** Let $n \geq 3$ and $S$ be a finite generating set of $\text{SL}_n(Z)$ (e.g., $S = \{e_{i,j}(\pm1) : 1 \leq i, j \leq n, i \neq j\}$). Also, let $\phi_i : \text{SL}_n(Z) \to \text{SL}_n(Z/iZ)$ be the natural surjective homomorphism for every $i \in \mathbb{N}$. Then the family of Cayley graphs of $\{(\text{SL}_n(Z/iZ), \phi_i(S))\}_{i \in \mathbb{N}}$ is a super-expander family.

Banach property (T) for algebraic groups. A (now standard) argument using the Mautner’s Lemma, shows how property (T) of $\text{SL}_3(R)$ implies property (T) of almost simple real algebraic groups whose Lie algebra contains $\text{sl}_3(R)$ as a Lie sub-algebra. This argument passes almost verbatim to the Banach setting and applying it with our Corollary 1.7 above yields:

**Theorem 1.10.** Let $\delta_0 : (0, 2] \to (0, 1]$ be a function and let $G$ be a connected, $\mathbb{R}$-almost simple algebraic group with a Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ contains $\text{sl}_3(\mathbb{R})$ as a Lie sub-algebra, then $G$ and any lattice $\Gamma < G$ have property ($T_{E_{\text{uc}}}(\delta_0)$).

This corroborates a conjecture stated in [BFGM07] in which it was conjectured that all higher rank have property ($T_E$) for every uniformly convex Banach space $E$ (see [BFGM07, Remark 2.28]).
Structure of this paper.} This paper is organized as follows: In §2 we cover some needed preliminaries. In §3 we give we detail some facts regarding our definition of Banach property $(T)$. In §4 we show how, in our setting, bounded generation and relative Banach property $(T)$ imply Banach property $(T)$. In §5 we prove some bounds on the norms of averaging operations for the Heisenberg group that are needed for our relative Banach property $(T)$ Theorem. In §6 we prove Theorem 1.5 stated above. In §7 we prove Theorem 1.6 and Corollary 1.7 stated above. Last, in §8 we prove the applications stated above.

2. Preliminaries

2.1. Uniformly convex Banach spaces. Below, we will state some needed facts regarding uniformly convex spaces.

**Proposition 2.1.** Let $E$ be a uniformly convex Banach space with a modulus of convexity $\delta : (0,2] \to (0,1]$ and denote $O(E)$ to be the group of invertible linear isometries of $E$. Then for every $0 < \varepsilon \leq 2$, every $S,T \in O(E)$ such that $TS = ST$ and every $\xi \in E$, if $\|(I - S)\xi\| \geq \varepsilon \|\xi\|$, then

$$\frac{1}{2} \left\| \frac{I + T}{2} \xi \right\| + \frac{1}{2} \left\| \frac{I + TS}{2} \xi \right\| \leq \max \left\{ 1 - \frac{1}{2} \delta (\varepsilon), 1 - \frac{1}{4} \delta (\varepsilon) \right\} \|\xi\|,$$

$$\frac{1}{2} \left\| \frac{I + T}{2} \xi \right\| + \frac{1}{2} \left\| \frac{I + TS^{-1}}{2} \xi \right\| \leq \max \left\{ 1 - \frac{1}{2} \delta (\varepsilon), 1 - \frac{1}{4} \delta (\varepsilon) \right\} \|\xi\|.$$

This Proposition is probably well-known and we give the proof for completeness:

**Proof.** We will start by proving the first inequality.

Fix $\xi \in E$ and $0 < \varepsilon \leq 2$, and assume that $\|(I - S)\xi\| \geq \varepsilon \|\xi\|$. If $\|(I - T)\xi\| \geq \varepsilon (\varepsilon)$, then

$$\frac{1}{2} \left\| \frac{I + T}{2} \xi \right\| + \frac{1}{2} \left\| \frac{I + TS}{2} \xi \right\| \leq \frac{1}{2} \left( 1 - \varepsilon (\varepsilon) \right) \|\xi\| + \frac{1}{2} \|\xi\| = \left( 1 - \frac{1}{2} \varepsilon (\varepsilon) \right) \|\xi\|,$$

as needed.

Otherwise, $\|\frac{I - T}{2} \xi\| \leq \frac{\varepsilon (\varepsilon)}{2}$ and

$$\frac{1}{2} \left\| \frac{I + T}{2} \xi \right\| + \frac{1}{2} \left\| \frac{I + TS}{2} \xi \right\| \leq \frac{1}{2} \|\xi\| + \frac{1}{2} \left\| \frac{I + S}{2} \xi \right\| + \frac{1}{2} \left\| \frac{S(T - I)}{2} \xi \right\| \leq \frac{1}{2} \|\xi\| + \frac{1}{2} \left( 1 - \varepsilon (\varepsilon) \right) \|\xi\| + \frac{1}{4} \varepsilon (\varepsilon) \|\xi\| = \left( 1 - \frac{1}{4} \varepsilon (\varepsilon) \right) \|\xi\|.$$

For the second inequality, observe that $\|(I - S^{-1})\xi\| = \|(I - S)\xi\|$ and thus we can apply the first inequality substituting $S$ with $S^{-1}$.

□

For classes of Banach space of the form $E_{uc}(\delta_0)$ introduced above, we state an immediate corollary of Proposition 2.1.
Corollary 2.2. Let $\delta_0 : (0, 2] \to (0, 1]$ be a function and $0 < \varepsilon \leq 2$ be a constant. There is $r_0 = r_0(\delta_0, \varepsilon)$, $0 \leq r_0 < 1$ such that for every $E \in E_{uc}(\delta_0)$ and every two commuting operators $S, T \in O(E)$ it holds for every $\xi \in E$ that if $\|(I - S)\xi\| \geq \varepsilon \|\xi\|$, then

$$\frac{1}{2} \left\| \frac{I + T}{2} \xi \right\| + \frac{1}{2} \left\| \frac{I + TS}{2} \xi \right\| \leq r_0 \|\xi\|,$$

$$\frac{1}{2} \left\| \frac{I + T}{2} \xi \right\| + \frac{1}{2} \left\| \frac{I + TS^{-1}}{2} \xi \right\| \leq r_0 \|\xi\|.$$

Last, we will need the following Theorem:

Theorem 2.3. \cite{Zac11} Corollary 4.9] Let $\delta_0 : (0, 2] \to (0, 1]$ be a function. There is a function $\delta'_0 : (0, 2] \to (0, 1]$ such that every finite or infinite $l_2^2$ sum of spaces from $E_{uc}(\delta_0)$ is contained in $E_{uc}(\delta'_0)$.

2.2. Steinberg relations in $\text{SL}_3(\mathbb{Z})$. For $1 \leq i, j \leq 3, i \neq j$ and $m \in \mathbb{Z}$, denote $e_{i,j}(m)$ to be the elementary matrix with 1’s along the main diagonal, $m$ in the $(i,j)$-entry and 0 in all other entries. Using the convention $[a, b] = a^{-1}b^{-1}ab$, the group $\text{SL}_3(\mathbb{Z})$ has the following relations that are called the Steinberg relations:

1. For every $1 \leq i, j \leq 3, i \neq j$ and every $m_1, m_2 \in \mathbb{Z}$,

$$e_{i,j}(m_1)e_{i,j}(m_2) = e_{i,j}(m_1 + m_2).$$

2. For every $1 \leq i, j, k \leq 3, \{i, j, k\} = \{1, 2, 3\}$ and every $m_1, m_2 \in \mathbb{Z}$,

$$[e_{i,j}(m_1), e_{j,k}(m_2)] = e_{i,k}(m_1m_2).$$

3. For every $1 \leq i, j, k \leq 3, \{i, j, k\} = \{1, 2, 3\}$ and every $m_1, m_2 \in \mathbb{Z}$,

$$[e_{i,j}(m_1), e_{i,k}(m_2)] = [e_{j,i}(m_1), e_{k,i}(m_2)] = I.$$

The group $\text{SL}_3(\mathbb{Z})$ has other relations that do not stem from the Steinberg relations. Forgetting the other relations of $\text{SL}_3(\mathbb{Z})$ yields the Steinberg group $\text{St}_3(\mathbb{Z})$. Explicitly, the Steinberg group $\text{St}_3(\mathbb{Z})$ is the group generated by the set $S = \{x_{i,j} : 1 \leq i, j \leq 3, i \neq j\}$ with the following relations: For every $m \in \mathbb{Z}$, denote $x_{i,j}(m) = x_{i,j}^m$. With this notation, the relations defining $\text{St}_3(\mathbb{Z})$ are:

1. For every $1 \leq i, j \leq 3, i \neq j$ and every $m_1, m_2 \in \mathbb{Z}$,

$$x_{i,j}(m_1)x_{i,j}(m_2) = x_{i,j}(m_1 + m_2).$$

2. For every $1 \leq i, j, k \leq 3, \{i, j, k\} = \{1, 2, 3\}$ and every $m_1, m_2 \in \mathbb{Z}$,

$$[x_{i,j}(m_1), x_{j,k}(m_2)] = x_{i,k}(m_1m_2).$$

3. For every $1 \leq i, j, k \leq 3, \{i, j, k\} = \{1, 2, 3\}$ and every $m_1, m_2 \in \mathbb{Z}$,

$$[x_{i,j}(m_1), x_{i,k}(m_2)] = [x_{j,i}(m_1), x_{k,i}(m_2)] = I.$$


2.3. The Heisenberg group $H_3(\mathbb{Z})$. The group Heisenberg group $H_3(\mathbb{Z})$ is the group

$$ H_3(\mathbb{Z}) = \langle x, y, z | [x, y] = z, [x, z] = e, [y, z] = e \rangle, $$

where $[a, b]$ is the commutator defined as $[a, b] = a^{-1}b^{-1}ab$. Below, we will use the following relations for the Heisenberg group that hard not hard to verify: for every $k, m \in \mathbb{Z}$ it holds that $y^{-k}x^m y^k = x^m z^{km}$ and $x^{-k}y^m x^k = y^m z^{-km}$.

In the sequel, we will use the fact that $SL_3(\mathbb{Z})$ (and $St_3(\mathbb{Z})$) contain several copies of $H_3(\mathbb{Z})$. Explicitly, for every $\{i, j, k\} = \{1, 2, 3\}$, if denote $\tilde{x} = e_{i,j}(1), \tilde{y} = e_{j,k}(1), \tilde{z} = e_{i,k}(1) \in SL_3(\mathbb{Z})$, then $\langle \tilde{x}, \tilde{y}, \tilde{z} \rangle < SL_3(\mathbb{Z})$ is isomorphic to $H_3(\mathbb{Z})$ (by the Steinberg relations) via the isomorphism $x \mapsto \tilde{x}, y \mapsto \tilde{y}, z \mapsto \tilde{z}$.

3. (Uniform) Banach property (T)

In this section, we gather some facts regarding our Definition of Banach property $(T_\mathbb{E})$ given above.

We recall the following result of [BFGM07]:

**Proposition 3.1.** [BFGM07, Proposition 2.6] Let $\mathbb{E}$ be a uniformly convex Banach space, $G$ be a topological group and $\pi : G \to O(\mathbb{E})$ be a continuous linear isometric representation. There is a subspace $\mathbb{E}'(\pi) \subseteq \mathbb{E}$ such that $\mathbb{E} = \mathbb{E}''(\pi) \oplus \mathbb{E}'(\pi)$.

With this Proposition, [BFGM07] gave the following definition of Banach property $(T)$ for uniformly convex Banach spaces:

**Definition 3.2.** [BFGM07, Remark 2.11] Let $\mathbb{E}$ be uniformly convex space and $G$ be a topological group. The group $G$ has property $(T_\mathbb{E})$ if for every continuous linear isometric representation $\pi : G \to (\mathbb{E})$, the restricted representation $\pi' : G \to B'(\pi)$ does not have almost invariant vectors, i.e., there is a Kazhdan pair $(K, \varepsilon)$ (that depends on $\pi$) where $K \subseteq G$ is compact and $\varepsilon > 0$ such that for every vector $\xi \in B'(\pi)$ it holds that

$$ \max_{g \in K} \| \pi'(g) \xi - \xi \| \geq \varepsilon \| \xi \|. $$

Due and Nowak [DN19] showed the following characterization of (our version of) property $(T_\mathbb{E})$ in terms of almost invariant vectors:

**Theorem 3.3.** [DN19, Theorem 4.6] Let $\mathbb{E}$ be a class of uniformly convex Banach spaces and $G$ a locally compact group. The group $G$ has property $(T_\mathbb{E})$ if and only if there is a Kazhdan pair $(K, \varepsilon)$ where $K \subseteq G$ is compact and $\varepsilon > 0$ such that for every representation $(\pi, \mathbb{E}) \in U(\mathbb{E})$, the restricted representation $\pi' : G \to B'(\pi)$ does not have $(K, \varepsilon)$-almost invariant vectors, i.e., for every $\xi \in B'(\pi)$,

$$ \max_{g \in K} \| \pi'(g) \xi - \xi \| \geq \varepsilon \| \xi \|. $$

**Remark 3.4.** Comparing Definition 3.2 to this Theorem shows the difference between our definition and Definition 3.2. Even for a single uniformly convex Banach space $E$, in the Definition 3.2 for property $(T_{E})$, the Kazhdan pair $(K, \varepsilon)$ depends on the representation and in our definition of $(T_{E})$ the same Kazhdan pair $(K, \varepsilon)$ applies for all isometric representations.
For classes of uniformly convex Banach spaces that are closed under passing to subspaces, Theorem 3.3 yields the following convenient characterization of property \((T_E)\):

**Corollary 3.5.** Let \(E\) be a class of uniformly convex Banach spaces that is closed under passing to subspaces and \(G\) a locally compact group.

The group \(G\) has property \((T_E)\) if and only if there is a compact set \(K \subseteq G\) and a constant \(\varepsilon > 0\) such that for every representation \((\pi, E) \in U(E)\) with \(\mathbb{E}^{\pi(G)} = \{0\}\) does not have \((K, \varepsilon)\)-almost invariant vectors, i.e., for every \(\xi \in E\),

\[
\max_{g \in K} \|\pi(g)\xi - \xi\| \geq \varepsilon \|\xi\|.
\]

Equivalently, the group \(G\) has property \((T_E)\) if and only if there is a compact set \(K \subseteq G\) and a constant \(\varepsilon > 0\) such that for every representation \((\pi, E) \in U(E)\),

\[
\left( \exists \xi \in E, \max_{g \in K} \|\pi(g)\xi - \xi\| < \varepsilon \|\xi\| \right) \Rightarrow \left( \mathbb{E}^{\pi(G)} \neq \{0\} \right).
\]

**Proof.** By the fact that \(E\) is closed under passing to subspaces, it follows that for every \((\pi, E) \in U(E)\), the restricted representation \((\pi', E'(\pi))\) is also in \(U(E)\) and thus the Corollary follows directly from Theorem 3.3. \(\Box\)

Lafforgue showed that property \((T_E)\) is inherited by lattices:

**Proposition 3.6.** [Laf09, Proposition 5.3] Let \(G\) be a locally compact group, \(\Gamma \lhd G\) be a lattice and \(E\) be a class of Banach spaces. If \(G\) has property \((T_E)\), then \(\Gamma\) has property \((T_E)\).

The other direction is also true for classes of uniformly convex spaces that are closed under subspaces:

**Proposition 3.7.** Let \(G\) be a locally compact group, \(\Gamma \lhd G\) be a lattice and \(E\) be a class of uniformly convex Banach spaces that is closed under passing to subspaces. If \(\Gamma\) has property \((T_E)\), then \(G\) has property \((T_E)\).

Using the characterization of property \((T_E)\) given in Corollary 3.5, the proof of the above Proposition is basically the same proof given in the Hilbert setting in [BdlHV08, Theorem 1.7.1]. We include the proof for completeness and claim no originality here.

**Proof.** Let \(G, \Gamma, E\) as above.

We assume that \(\Gamma\) has property \((T_E)\) and thus, by Theorem 3.3 there are \(K_0 \subseteq \Gamma\) compact and \(1 > \varepsilon_0 > 0\) such that for every continuous representation \(\pi \in U(\Gamma, E)\), the restricted representation \((\pi', E'(\pi))\) does not have \((K_0, \varepsilon_0)\)-almost invariant vectors.

Let \(\mu_{G/\Gamma}\) be an invariant probability measure on \(G/\Gamma\) and \(p : G \to G/\Gamma\) the canonical projection. Denote \(\varepsilon = \frac{\varepsilon_0}{10}\) and choose a compact subset \(K \subseteq G\) with \(K_0 \subseteq K\) and

\[
\mu_{G/\Gamma}(p(K)) \geq \frac{\varepsilon_0 + 9}{10}.
\]

By the assumption that \(E\) is closed under passing to subspace, thus by Corollary 3.5 it is enough to prove the following: For \(K\) and \(\varepsilon\) as above it holds for every \((\pi, E) \in U(G, E)\)
with $\mathbb{E}^\pi(G) = \{0\}$ and every unit vector $\xi \in \mathbb{E}$ that

$$\max_{g \in K} \| \pi_G(g) \xi - \xi \| \geq \varepsilon.$$ 

Fix $(\pi, \mathbb{E}) \in \mathcal{U}(G, \mathcal{E})$ with $\mathbb{E}^\pi(G) = \{0\}$. Assume towards contradiction that there is a unit vector $\xi \in \mathbb{E}$ such that

$$\max_{g \in K} \| \pi_G(g) \xi - \xi \| < \varepsilon = \varepsilon_0.$$ 

Restricting $\pi$ to $\Gamma$, denote $\mathbb{E} = \mathbb{E}^\pi(\Gamma) \oplus \mathbb{E}^\prime_{|\Gamma}$ to be the decomposition given in Proposition 3.1 for $\pi|_{\Gamma}$. By this decomposition, there are $\xi_0 \in \mathbb{E}^\pi(\Gamma), \xi_1 \in \mathbb{E}^\prime_{|\Gamma}$ such that $\xi = \xi_0 + \xi_1$. Note that $\frac{\varepsilon_0}{4} > \max_{g \in K} \| \pi_G(g) \xi - \xi \| \geq \max_{g \in K_0} \| \pi_G(g) \xi - \xi \| = \max_{g \in K_0} \| \pi_G(g) \xi_1 - \xi_1 \| \geq \varepsilon_0 \| \xi_1 \|$.

Thus $\| \xi_1 \| < \frac{1}{4}$ and $\frac{3}{4} < \| \xi_0 \| < \frac{5}{4}$. Note that the mapping $G/\Gamma \to \mathbb{E}$ defined by $g\Gamma \mapsto \pi_G(g)\xi_0$

is well-defined, continuous and bounded. Define

$$\eta = \int_{G/\Gamma} \pi_G(g)\xi_0 d\mu_{G/\Gamma}(g\Gamma).$$

The $G$-invariance of $\mu_{G/\Gamma}$ implies that $\eta \in \mathbb{E}^\pi(G)$: Indeed, for every $g' \in G$,

$$\pi(g')\eta = \int_{G/\Gamma} \pi_G(g'g)\xi_0 d\mu_{G/\Gamma}(g\Gamma) = \int_{G/\Gamma} \pi_G(g')\xi_0 d\mu_{G/\Gamma}(g'g\Gamma) = \eta.$$

In order to derive a contradiction, we will show that $\| \eta \| > 0$ (which contradicts the assumption that $\mathbb{E}^\pi(G) = \{0\}$). We showed that $\| \xi_1 \| < \frac{1}{4}$ and thus for every $g \in K$,

$$\| \pi_G(g) \xi_0 - \xi_0 \| \leq \| \pi_G(g) \xi_1 - \xi_1 \| + \| \pi_G(g) \xi - \xi \| < \frac{1}{2} + \frac{\varepsilon_0}{4}.$$ 

Therefore,

$$\| \eta - \xi_0 \| = \left\| \int_{G/\Gamma} (\pi_G(g)\xi_0 - \xi_0) d\mu_{G/\Gamma}(g\Gamma) \right\| \leq$$

$$\int_K \| \pi_G(g)\xi_0 - \xi_0 \| d\mu_{G/\Gamma}(g\Gamma) + 2\| \xi_0 \| (1 - \mu_{G/\Gamma}(K)) < \| \xi_0 \| \left( \frac{1}{2} + \frac{\varepsilon_0}{4} + \frac{10}{4} \right) = \frac{3}{4}.$$ 

This calculation shows that $\| \eta - \xi_0 \| < \frac{3}{4}$ and recall that $\| \xi_0 \| > \frac{3}{4}$ and thus $\| \eta \| > 0$ as needed. \qed
4. Bounded generation and Banach property (T)

In this section, we adapt a bounded generation argument of Shalom [Sha99] to our setting and show that for classes of uniformly convex Banach spaces that are closed under passing to subspaces, and show that relative property (T) combined with bounded generation imply Banach property (T).

**Definition 4.1.** Let $G$ be a group with subgroups $H_1, ..., H_k$. We say that $H_1, ..., H_k$ boundedly generate $G$ if there is a number $\nu = \nu(H_1, ..., H_k) \in \mathbb{N}$ such that every element $g \in G$ can be written by at most $\nu$ elements of $H_1 \cup ... \cup H_k$.

**Theorem 4.2.** Let $G$ be a locally compact group and $\mathcal{E}$ be a class of uniformly convex Banach spaces that is closed under passing to subspaces. If there are subgroups $H_1, ..., H_k < G$ that boundedly generate $G$ and $(G, H_1), ..., (G, H_k)$ has relative property (T$_\mathcal{E}$), then $G$ has property (T$_\mathcal{E}$).

**Proof.** Denote $\nu = \nu(H_1, ..., H_k) \in \mathbb{N}$ as in the definition above. By assumption, there are $h_1, ..., h_k \in \text{Prob}_\nu(G)$ such that for every $i = 1, ..., k$, and every $(\pi, \mathcal{E}) \in \mathcal{U}(\mathcal{E})$ and every unit vector $\xi \in \mathcal{E}$,

$$\min_{\eta \in \mathcal{E}^{\pi(H_i)}} \|\pi(h_i)\xi - \eta\| \leq \frac{1}{4(\nu + 1)}.$$

Denote $K = \bigcup_{i=1}^{k} \text{supp}(h_i)$ and $\epsilon = \frac{1}{4(\nu + 1)}$. By assumption, $\mathcal{E}$ is closed under passing to subspaces and thus by Corollary 3.5 it is enough to prove the following: For $K$ and $\epsilon$ as above it holds for every $(\pi, \mathcal{E}) \in \mathcal{U}(\mathcal{E})$,

$$\left( \exists \xi \in \mathcal{E}, \max_{g \in K} \|\pi(g)\xi - \xi\| < \epsilon \|\xi\| \right) \Rightarrow \left( \mathcal{E}^{\pi(G)} \neq \{0\} \right).$$

Fix $(\pi, \mathcal{E}) \in \mathcal{U}(\mathcal{E})$ such that there is a unit vector $\xi \in \mathcal{E}$ such that

$$\max_{g \in K} \|\pi(g)\xi - \xi\| < \frac{1}{4(\nu + 1)}.$$

We will show that $\mathcal{E}^{\pi(G)} \neq \{0\}$.

By the choice of $h_i$’s, for every $i = 1, ..., k$ there is $\eta_i \in \mathcal{E}^{\pi(H_i)}$ such that

$$\|\pi(h_i)\xi - \eta_i\| \leq \frac{1}{4(\nu + 1)}.$$

Thus, for every $i = 1, ..., k,$

$$\|\xi - \eta_i\| \leq \|\xi - \pi(h_i)\xi\| + \|\pi(h_i)\xi - \eta_i\| \leq \left\| \int_{G} h_i(g)(\xi - \pi(g)\xi) d\mu(g) \right\| + \frac{1}{4(\nu + 1)} \leq$$

$$\int_{G} h_i(g) \|\xi - \pi(g)\xi\| d\mu(g) + \frac{1}{4(\nu + 1)} \leq \max_{g \in \text{supp}(h_i)} \|\xi - \pi(g)\xi\| + \frac{1}{4(\nu + 1)} \leq \frac{1}{2(\nu + 1)}.$$

Let $g \in G$ arbitrary. By the assumption of bounded generation, there is a function $s : \{1, ..., \nu\} \to \{1, ..., k\}$ and elements

$$g_1 \in H_{s(1)}, ..., g_\nu \in H_{s(\nu)}$$
such that $g = g_1 \ldots g_\nu$. Then
\[
\|\pi(g)\xi - \xi\| \leq \|\pi(g_1 \ldots g_\nu)(\xi - \eta_{s(\nu)})\| + \|\pi(g_1 \ldots g_\nu)\eta_{s(\nu)} - \xi\| \leq \frac{1}{2(\nu + 1)} + \|\pi(g_1 \ldots g_{\nu-1})\eta_{s(\nu)} - \xi\| \leq \nu \frac{1}{2(\nu + 1)} + \|\eta_{s(1)} - \xi\| \leq \frac{1}{2}.
\]
This holds for every $g \in G$ and thus the orbit of $\xi$ under the action of $G$ is contained in a closed ball of radius $\frac{1}{2}$ around $\xi$. Denote $C$ to be the closure of the convex hull of the orbit of $\xi$. Recall that $\xi$ is a unit vector and thus $0 \notin C$. By Ryll-Nardzewski fixed-point Theorem (which applies since every uniformly convex Banach space is reflexive) it follows that $C \cap E^{\pi(G)} \neq \emptyset$. Thus we showed that $E^{\pi(G)} \neq \{0\}$ as needed. $\square$

5. Averaging operations on $H_3(\mathbb{Z})$

In this section, we will prove norm bounds on averaging operations on the Heisenberg group that are needed in our proof of relative Banach property (T) stated in the introduction.

For every $k \in \mathbb{N} \cup \{0\}$ define $X_k, Y_k, Z_k \in \text{Prob}_c(H_3(\mathbb{Z}))$ by
\[
X_k = \frac{e + x^{2k}}{2}, Y_k = \frac{e + y^{2k}}{2}, Z_k = \frac{e + z^{2k}}{2}.
\]

Observation 5.1. For any $m, k \in \mathbb{N} \cup \{0\},$
\[
\prod_{a=k}^{k+m} X_a = \frac{1}{2^{m+1}} \sum_{a=0}^{2^{m+1}-1} x^{a2k},
\]
\[
\prod_{b=k}^{k+m} Y_b = \frac{1}{2^{m+1}} \sum_{b=0}^{2^{m+1}-1} y^{b2k},
\]
\[
\prod_{c=k}^{k+m} Z_c = \frac{1}{2^{m+1}} \sum_{c=0}^{2^{m+1}-1} z^{c2k}.
\]

For $d \in \mathbb{N}$ further define $X^d, Y^d, Z^d \in \text{Prob}_c(H_3(\mathbb{Z}))$ by
\[
X^d = \prod_{a=0}^{d-1} X_a, Y^d = \prod_{b=0}^{d-1} Y_b, Z^d = \prod_{c=0}^{d-1} Z_c.
\]

Lemma 5.2. Let $d \in \mathbb{N}$ be a constant and $A, B \subseteq \{0, \ldots, d-1\}$ be sets such that $(\max A)(\max B) \leq d-2$, then for every Banach space $E$ and every isometric linear representation $\pi : H_3(\mathbb{Z}) \to O(E)$ it holds that
\[
\|\pi\left(\left(\prod_{a \in A} X_a\right)\left(\prod_{b \in B} Y_b\right) - \left(\prod_{b \in B} Y_b\right)\left(\prod_{a \in A} X_a\right)\right) Z^d\| \leq 8 \left(\frac{1}{2}\right)^{d - \max A - \max B}.
\]
In particular, for \( d_1, d_2, d_3 \in \mathbb{N} \cup \{0\} \), if \( d_1 + d_2 \leq d_3 - 2 \), then for any class of Banach spaces \( \mathcal{E} \) it holds that

\[
\left\| \left( X^{d_1} Y^{d_2} - Y^{d_2} X^{d_1} \right) Z^d \right\|_{\mathcal{U}(\mathcal{E})} \leq 8 \left( \frac{1}{2} \right)^{d_3 - (d_1 + d_2)}.
\]

Proof. We note that

\[
\left( \prod_{a \in A} X_a \right) \left( \prod_{b \in B} Y_b \right) - \left( \prod_{b \in B} Y_b \right) \left( \prod_{a \in A} X_a \right) = \frac{1}{2|A|+|B|} \sum_{f \in \{0,1\}^A, h \in \{0,1\}^B} x^{\sum_{k \in A} f(k) 2^k} y^{\sum_{l \in B} h(l) 2^l} - y^{\sum_{l \in B} h(l) 2^l} x^{\sum_{k \in A} f(k) 2^k} = \frac{1}{2|A|+|B|} \sum_{f \in \{0,1\}^A, h \in \{0,1\}^B} x^{\sum_{k \in A} f(k) 2^k} y^{\sum_{l \in B} h(l) 2^l} \left( e - z^{\sum_{k \in A} f(k) 2^k} (\sum_{l \in B} h(l) 2^l) \right).
\]

Note that for every \( f \in \{0,1\}^A, h \in \{0,1\}^B \) it holds that

\[
(\sum_{k \in A} f(k) 2^k)(\sum_{l \in B} h(l) 2^l) \leq 2^{\max A + \max B + 2}.
\]

Thus, it is enough to show that for every \( 1 \leq m \leq 2^{\max A + \max B + 2} \) it holds that

\[
\|\pi \left( (e - z^m) Z^d \right) \| \leq \frac{2m}{2^d} \leq \left( \frac{1}{2} \right)^{d - \max A - \max B - 3},
\]

but this follows immediately from the fact that

\[
(e - z^m) Z^d = \frac{1}{2^d} \left( \sum_{k=0}^{m-1} z^k - \sum_{k=2^d}^{2^d+m-1} z^k \right).
\]

\[ \square \]

**Proposition 5.3.** For every \( m, k, l \in \mathbb{Z} \),

\[
\begin{align*}
e + x^k z^m e + y^l &= \frac{1}{2} \left( e + x^k z^m \right) + \frac{y^l}{2} \left( e + x^k z^{m+kl} \right), \\
e + y^l z^m e + x^k &= \frac{1}{2} \left( e + y^l z^m \right) + \frac{x^k}{2} \left( e + y^l z^{m-kl} \right).
\end{align*}
\]

Proof. The proofs of both the identities are similar and we will prove only the first one:

\[
\begin{align*}
e + x^k z^m e + y^l &= \frac{1}{2} \left( e + x^k z^m \right) + \frac{1}{2} \left( y^l + x^k z^{m+l} \right) = \\
&= \frac{1}{2} \left( e + x^k z^m \right) + \frac{y^l}{2} \left( e + x^k z^{m+kl} \right) = \\
&= \frac{1}{2} \left( e + x^k z^m \right) + \frac{y^l}{2} \left( e + x^k z^{m+kl} \right).
\end{align*}
\]
Proposition 5.4. Let $\mathbb{E}$ be a Banach space and $\pi : H_\beta(\mathbb{Z}) \rightarrow O(\mathbb{E})$ be a linear isometric representation. Then for every $a, b \in \mathbb{N} \cup \{0\}$, every $n \in \mathbb{N}$, every $1 \leq m \leq 2^{n-2}$ and every $\zeta \in \mathbb{E}$ the following holds:

\[
\| \pi \left( X_a \left( \prod_{j=1}^{n} Y_{b+j} \right) \right) \zeta \| \\
\leq \frac{m}{2^n-1} \| \zeta \| + \max_{0 \leq i \leq \left\lfloor \frac{2n-1}{m} \right\rfloor - 1} \left( \frac{1}{2} \| \pi \left( e + x^{2a} z^{(l(2m)+i)2^{n+b+1}} \right) \zeta \| \right) + \\
\frac{1}{2} \| \pi \left( e + x^{2a} z^{(l(2m)+i)2^{n+b+1}} z^{m2^{n+b+1}} \right) \zeta \| ,
\]

and

\[
\| \pi \left( Y_b \left( \prod_{j=1}^{n} X_{a+j} \right) \right) \zeta \| \\
\leq \frac{m}{2^n-1} \| \zeta \| + \max_{0 \leq i \leq \left\lfloor \frac{2n-1}{m} \right\rfloor - 1} \left( \frac{1}{2} \| \pi \left( e + y^{2b} z^{-(l(2m)+i)2^{n+b+1}} \right) \zeta \| \right) + \\
\frac{1}{2} \| \pi \left( e + y^{2b} z^{-(l(2m)+i)2^{n+b+1}} z^{-m2^{n+b+1}} \right) \zeta \| .
\]

Proof. The proofs of both inequalities are similar and we will prove only the first one. We will start by proving that

\[
\| \pi \left( X_a \left( \prod_{j=1}^{n} Y_{b+j} \right) \right) \zeta \| \leq \frac{1}{2^n} \sum_{t=0}^{2^n-1} \| \pi \left( e + x^{2a} z^{2^{n+b+1}} \right) \zeta \| .
\]

The proof is by induction on $n$. For $n = 1$, by Proposition 5.3

\[
\| \pi (X_a Y_{b+1}) \zeta \| = \| \pi \left( \frac{1}{2} \left( e + x^{2a} \right) + y^{2b+1} \left( e + x^{2a} z^{2^{n+b+1}} \right) \right) \zeta \| \\
\leq \frac{1}{2} \| \pi \left( e + x^{2a} \right) \zeta \| + \frac{1}{2} \| \pi \left( e + x^{2a} z^{2^{n+b+1}} \right) \zeta \| .
\]
as needed. Assume that the inequality holds for \( n \), then

\[
\left\| \pi \left( X_a \left( \prod_{j=1}^{n+1} Y^c_{b+j} \right) \right) \right\| \leq \frac{1}{2^n} \sum_{t=0}^{2^n-1} \left\| \pi \left( \frac{e + x^{2a} z^{2^t+0+b+1}}{2} \right) \right\|
\]

as needed.

Next, fix \( n \) and \( m \) as above. Then

\[
\left\| \pi \left( X_a \left( \prod_{j=1}^{n} Y^c_{b+j} \right) \right) \right\| \leq \frac{1}{2^n} \sum_{t=0}^{2^n-1} \left\| \pi \left( \frac{e + x^{2a} z^{2^t+0+b+1}}{2} \right) \right\|
\]

as needed.

\[
= \text{Proposition } 5.3 \quad \frac{1}{2^n} \sum_{t=0}^{2^n-1} \left\| \pi \left( \frac{e + x^{2a} z^{2^t+0+b+1}}{2} \right) \right\|
\]

as needed.

\[
\leq \frac{1}{2^n} \sum_{t=0}^{2^n-1} \left\| \pi \left( \frac{e + x^{2a} z^{2^t+0+b+1}}{2} \right) \right\|
\]

as needed.
Lemma 5.5. Let \( \delta_0 : (0, 2] \to (0, 1] \) be a function. There is a constant \( 0 \leq r_0 < 1 \) such that for every \( E \in E_{uc}(\delta_0) \), every linear isometric representation \( \pi : H_3(\mathbb{Z}) \to O(\mathbb{E}) \), every \( \zeta \in E \) and every \( a, b, n, m \in \mathbb{N} \) such that \( 1 \leq m \leq 2^{n^2} \) if \( \|\pi(e - z^{m^2b+1})\zeta\| \geq \frac{1}{2}\|\zeta\| \), then

\[
\left\| \pi \left( X_a \left( \prod_{j=1}^{n} Y_{b+j} \right) \right) \zeta \right\| \leq \frac{m}{2^{n-1}}\|\zeta\| + r_0\|\zeta\|, \\
\left\| \pi \left( Y_b \left( \prod_{j=1}^{n} X_{a+j} \right) \right) \zeta \right\| \leq \frac{m}{2^{n-1}}\|\zeta\| + r_0\|\zeta\|.
\]

Proof. The proofs of both the inequalities are similar and we will prove only the first one. Let \( r_0 = r_0(\delta_0, \frac{1}{2}) \) of Corollary 2.2. By Proposition 5.4 it is enough to show that for every \( 0 \leq l \leq \lfloor \frac{2^n-1}{m} \rfloor - 1, 0 \leq i \leq m - 1 \) it holds that

\[
\frac{1}{2}\left\| \left( I + \pi(z^{l(2m+i)2^{n+b+1}}) \right) \zeta \right\| + \frac{1}{2}\left\| \left( I - \pi(z^{l(2m+i)2^{n+b+1}}) \right) \zeta \right\| \leq r_0\|\zeta\|,
\]

and this follows immediately from Corollary 2.2 with \( T = \pi(z^{l(2m+i)2^{n+b+1}}) \) and \( S = \pi(z^{m^{2n+b+1}}) \).

\[\square\]

Theorem 5.6. Let \( \delta_0 : (0, 2] \to (0, 1] \) be a function. There are constants \( 0 \leq r_1 < 1, C > 0 \) such that for every \( E \in E_{uc}(\delta_0) \), every linear isometric representation \( \pi : H_3(\mathbb{Z}) \to O(\mathbb{E}) \) and every \( d_1, d_2, d_3 \in \mathbb{N} \) such that \( d_1, d_2 \leq d_3 \) and \( d_1 + d_2 - d_3 \geq 100 \), it holds that

\[
\left\| \pi \left( X^{d_1} Y^{d_2} Z^{d_3} (e - Z_{d_3+1}) \right) \right\| \leq Cr_1^{d_1+d_2-d_3},
\]

and

\[
\left\| \pi \left( Y^{d_2} X^{d_1} Z^{d_3} (e - Z_{d_3+1}) \right) \right\| \leq Cr_1^{d_1+d_2-d_3}.
\]

Proof. The proofs of both inequalities are similar and we will prove only the first one. Fix \( E, \pi, d_1, d_2, d_3 \) as above. We note that by our assumptions \( \lfloor \frac{1}{4}d_1 + d_2 - d_3 \rfloor \geq 2 \) and thus below we will not suffer from problems of rounding to 0.

Let \( r_0 = r_0(\delta_0, \frac{1}{2}) \) be the constant of Corollary 2.2 and denote \( r_1 = \max\{r_0, \frac{1}{\sqrt{2}}\} \).

We define the following sets:

\[
A_0 = \{d_1 - i \lfloor \sqrt{d_1 + d_2 - d_3} \rfloor : 0 \leq i \leq \lfloor \sqrt{d_1 + d_2 - d_3} \rfloor \}, B_0 = \{0, \ldots, d_2 \}
\]

and for \( 1 \leq k \leq \lfloor \sqrt{d_1 + d_2 - d_3} \rfloor \),

\[
A_k = A_{k-1} \setminus \{d_1 - (k-1) \lfloor \sqrt{d_1 + d_2 - d_3} \rfloor \}, \\
B_k = B_{k-1} \setminus \left\{d_3 - d_1 + (k-1) \lfloor \sqrt{d_1 + d_2 - d_3} \rfloor + j : 1 \leq j \leq \left\lfloor \frac{1}{2} \sqrt{d_1 + d_2 - d_3} \right\rfloor \right\}.
\]

We note that be our assumptions on \( d_1, d_2, d_3 \) it holds for every \( 0 \leq k \leq \lfloor \sqrt{d_1 + d_2 - d_3} \rfloor - 1 \) that

\[
\{d_1 - (k-1) \lfloor \sqrt{d_1 + d_2 - d_3} \rfloor \} \subseteq A_{k-1} \subseteq \{0, \ldots, d_1 \},
\]
and
\[ \left\{ d_3 - d_1 + (k - 1)\left[ \sqrt{d_1 + d_2 - d_3} \right] + j : 1 \leq j \leq \left\lfloor \frac{1}{2}\sqrt{d_1 + d_2 - d_3} \right\rfloor \right\} \subseteq B_{k-1} \subseteq \{0, \ldots, d_2\}. \]

We observe that
\[ \left\| \pi \left( \prod_{a \in \{0, \ldots, d_1\} \setminus A_0} X_a \right) \right\| \leq \left\| \pi \left( \prod_{a \in A_0} X_a \right) \right\| \leq \left\| \prod_{a \in \{0, \ldots, d_1\} \setminus A_0} X_a \right\| \]

Thus it is enough to prove that there is some \( C' \) such that for \( r_1 \) defined as above it holds for every \( 0 \leq k \leq \left\lfloor \sqrt{d_1 + d_2 - d_3} \right\rfloor - 1 \) and every \( \xi \in E \) with \( \|\xi\| = 1 \) that
\[
\left\| \pi \left( \prod_{a \in A_k} X_a \right) \left( \prod_{b \in B_k} Y_b \right) Z^{d_3 (e - Z_{d_3+1})} \xi \right\| \leq C' r_1 \left\| \prod_{a \in A_{k+1}} X_a \right\| \left( \prod_{b \in B_{k+1}} Y_b \right) Z^{d_3 (e - Z_{d_3+1})} \xi.
\]

Indeed, (1) would imply that for every \( \xi \in E \) with \( \|\xi\| = 1 \) it holds that
\[
\left\| \pi \left( \prod_{a \in A_0} X_a \right) \right\| \leq C' r_1 \left( \prod_{a \in A_{k+1}} X_a \right) \left( \prod_{b \in B_{k+1}} Y_b \right) Z^{d_3 (e - Z_{d_3+1})} \xi
\]

and we can take \( C = (\frac{C'}{1 - r_1} + 2)^\frac{1}{r_1} \).

We will complete the proof of the first inequality by proving (1). Fix \( 0 \leq k \leq \left\lfloor \sqrt{d_1 + d_2 - d_3} \right\rfloor - 1 \) and \( \xi \in E \) with \( \|\xi\| = 1 \). Note that \( \max A_{k+1} = d_1 - (k+1)\left\lfloor \sqrt{d_1 + d_2 - d_3} \right\rfloor \) thus by Lemma
and note that

\[ \frac{1}{2} \leq \frac{1}{\sqrt{2}} \leq 1. \]

Thus, it is enough to prove that there is some \( C \) such that

\[
\left\| \pi \left( \left( \prod_{a \in A_{k+1}} X_a \right) \left( \prod_{j=1}^{d_3} Y_{d_3-j+1+k[d_1+d_2-d_3]+j} \right) \right) \right\| \leq 8 \left( \frac{1}{2} \right)^{d_3-(d_1-(k+1)[d_1+d_2-d_3]+d_3-d_1+k[d_1+d_2-d_3]+[\frac{1}{2}\sqrt{d_1+d_2-d_3}]} \leq 16 \left( \frac{1}{\sqrt{2}} \right)^{d_3} \leq 16r_1[d_1+d_2-d_3].
\]

Thus,

\[
\left\| \pi \left( \left( \prod_{a \in A_k} X_a \right) \left( \prod_{b \in B_k} Y_b \right) \right) \right\| \leq 32r'[d_1+d_2-d_3] \left( \frac{1}{2} \right)^{d_3-(d_1-(k+1)[d_1+d_2-d_3]+d_3-d_1+k[d_1+d_2-d_3]+[\frac{1}{2}\sqrt{d_1+d_2-d_3}]} \leq 16r_1[d_1+d_2-d_3].
\]

Denote

\[
\eta = \pi \left( \left( \prod_{a \in A_{k+1}} X_a \right) \left( \prod_{b \in B_{k+1}} Y_b \right) \right) \xi,
\]

and note that \( \|\eta\| \leq 1. \)

Thus it is enough to prove that there is some \( C'' \) such that

\[
\left\| \pi \left( X_{d_1-k[d_1+d_2-d_3]} \left( \prod_{j=1}^{d_3} Y_{d_3-j+1+k[d_1+d_2-d_3]+j} \right) (e-Z_{d_3+1}) \right) \right\| \leq C''r_1[d_1+d_2-d_3] + r_1 \|\pi(e-Z_{d_3+1})\eta\|.
\]

If \( \|\pi(e-Z_{d_3+1})\eta\| \leq 4\frac{1}{\sqrt{2}}[d_1+d_2-d_3] \|\eta\|, \) then

\[
\left\| \pi \left( \left( \prod_{a \in A_{k+1}} X_a \right) \left( \prod_{j=1}^{d_3} Y_{d_3-j+1+k[d_1+d_2-d_3]+j} \right) (e-Z_{d_3+1}) \right) \right\| \leq \left\| \pi(e-Z_{d_3+1})\eta\right\| \leq 4\frac{1}{\sqrt{2}}[d_1+d_2-d_3] \|\eta\| \leq 4r_1[d_1+d_2-d_3],
\]

and we are done.
Applying Lemma 5.5 with

Thus there is \(1 \leq \eta \leq 4^{1/2}\|\pi\|\), then (using Observation 5.1)

\[
\frac{1}{2^{1/2}\sqrt[n]{d_1+d_2-d_3}} \sum_{k=0}^{2^{1/2}\sqrt[n]{d_1+d_2-d_3} - 1} \| \pi \left( \left( e - z^{k\eta^{d_3+1}} \right) (e - Z_{d_3+1}) \right) \eta \|
\]

\[
\geq \pi \left( \left( e - \frac{z^{2\eta^{d_3+1}}}{2} \right) \sum_{c=0}^{2^{1/2}\sqrt[n]{d_1+d_2-d_3} - 1} \frac{1}{z^{2\eta^{d_3+1}c}} \right) \eta
\]

\[
= \pi \left( e - \frac{z^{2\eta^{d_3+1}}} {2^{1/2} e^{d_1+d_2-d_3}} \right) \eta
\]

\[
\geq \pi \left( e - \frac{1}{2^{1/2} e^{d_1+d_2-d_3}} \right) \eta
\]

\[
\geq \frac{1}{2} \| \pi(e - Z_{d_3+1}) \eta \|.
\]

Thus there is \(1 \leq m \leq 2^{1/2} e^{d_1+d_2-d_3} - 1\) such that

\[
\| \pi \left( e - z^{m\eta^{d_3+1}} \right) (e - Z_{d_3+1}) \eta \| \geq \frac{1}{2} \| \pi(e - Z_{d_3+1}) \eta \|.
\]

Applying Lemma 5.5 with \(a = d_1 - k\|\sqrt[d_1+d_2-d_3]\|, b = d_3 - d_1 + k\|\sqrt[d_1+d_2-d_3]\|, n = \frac{1}{2} \sqrt[d_1+d_2-d_3], \zeta = \pi \left( e - Z_{d_3+1} \right) \eta \) and \(m\) as above yields

\[
\| \pi \left( X_{d_1-k\|\sqrt[d_1+d_2-d_3]\|} \left( \prod_{j=1}^{m} Y_{d_3-d_1+k\|\sqrt[d_1+d_2-d_3]\|+j} \right) (e - Z_{d_3+1}) \right) \eta \|
\]

\[
\leq \frac{m}{2^{1/2} \sqrt[d_1+d_2-d_3]} \| \pi(e - Z_{d_3+1}) \eta \| + r_1 \| \pi(e - Z_{d_3+1}) \eta \|
\]

\[
\leq 4 \frac{2^{1/2} \sqrt[d_1+d_2-d_3]} {2^{1/2} \sqrt[d_1+d_2-d_3]} \| \pi(e - Z_{d_3+1}) \eta \|
\]

\[
\leq 4 \left( \frac{1}{\sqrt[d_1+d_2-d_3]} \right) \| \pi(e - Z_{d_3+1}) \eta \|
\]

\[
\leq 4r_1 \| \pi(e - Z_{d_3+1}) \eta \|
\]

and this completes the proof of (2) and thus of the Theorem. \(\square\)
Corollary 5.7. Let $\delta_0 : (0, 2] \to (0, 1]$ be a function. There are constants $0 \leq r < 1, C > 0$ such that for every $E \in \mathcal{E}_{uc}(\delta_0)$, every linear isometric representation $\pi : H_3(\mathbb{Z}) \to O(E)$ and every $d_1, d_2, d_3, k \in \mathbb{N}$ such that $d_3 \geq 400$, $d_1, d_2 \geq \frac{1}{4}d_3$ and $d_1 + d_2 - d_3 - k \geq \frac{1}{4}d_3$, it holds that
\[
\left\| X^{d_1} Y^{d_2} \left( Z^{d_3} - Z^{d_3+k} \right) \right\|_{\mathcal{U}(\mathcal{E}_{uc}(\delta_0))} \leq kC_r \sqrt{d_3},
\]
and
\[
\left\| Y^{d_2} X^{d_1} \left( Z^{d_3} - Z^{d_3+k} \right) \right\|_{\mathcal{U}(\mathcal{E}_{uc}(\delta_0))} \leq kC_r \sqrt{d_3}.
\]

Proof. The proofs of both inequalities are similar and we will prove only the first one.

Fix $(\pi, E) \in \mathcal{U}(\mathcal{E}_{uc}(\delta_0))$. Let $r_1, C$ be the constants of Theorem 5.6 and take $r = \sqrt{r_1}$. We note that it is enough to prove that for any $0 \leq j \leq k - 1$ it holds that
\[
\left\| \pi \left( X^{d_1} Y^{d_2} \left( Z^{d_3+j} - Z^{d_3+j+1} \right) \right) \right\| \leq C \sqrt{d_3}.
\]
If $d_1, d_2 \leq d_3 + j$ this inequality follow immediately from Theorem 5.6. Otherwise, either $d_2 > d_3 + j$ or $d_1 \geq d_3 + j$ (or both) and thus
\[
\left\{ d_1, d_3 + j \right\} + \left\{ d_2, d_3 + j \right\} - d_3 - j \geq \frac{1}{4}d_3.
\]
In this case, we apply Theorem 5.6 replacing $d_i$ with $\min\{d_i, d_3 + j\}$ for $i = 1, 2$:
\[
\left\| \pi \left( X^{d_1} Y^{d_2} \left( Z^{d_3+j} - Z^{d_3+j+1} \right) \right) \right\| \leq C \sqrt{d_3},
\]
as needed. \hfill \Box

6. Relative Banach property (T) for $(\text{SL}_3(\mathbb{Z}), \text{UT}_3(\mathbb{Z}))$ and $(\text{SL}_3(\mathbb{Z}), \text{LT}_3(\mathbb{Z}))$

In this section we will prove our main relative Banach property (T) result stated in the introduction.

For any $1 \leq i, k \leq 3, i \neq k$, we denote
\[
\text{H}_{i,k} = \langle e_{i,j}(1), e_{j,k}(1), e_{i,k}(1) \rangle < \text{SL}_3(\mathbb{Z}),
\]
\[
\tilde{H}_{i,k} = \langle x_{i,j}(1), x_{j,k}(1), x_{i,k}(1) \rangle < \text{St}_3(\mathbb{Z}),
\]
For example, $\text{H}_{1,3}$ is the group of uni-upper-triangular matrices $\text{UT}_3(\mathbb{Z})$ that appeared in the introduction.

We will prove the following Theorem that appeared in the introduction (Theorem 1.5):

Theorem 6.1. For any function $\delta_0 : (0, 2] \to (0, 1]$ and any for any $1 \leq i, k \leq 3, i \neq k$, the pair $(\text{SL}_3(\mathbb{Z}), \text{H}_{i,k})$ and the pair $(\text{St}_3(\mathbb{Z}), \tilde{\text{H}}_{i,k})$ has relative property $(T_{\mathcal{E}_{uc}(\delta_0)})$.

In particular, For any function $\delta_0 : (0, 2] \to (0, 1]$, for the pairs $(\text{SL}_3(\mathbb{Z}), \text{UT}_3(\mathbb{Z}))$ and $(\text{SL}_3(\mathbb{Z}), \text{LT}_3(\mathbb{Z}))$ have relative property $(T_{\mathcal{E}_{uc}(\delta_0)})$. 

Below, we will prove this Theorem only for the pair \((SL_3(\mathbb{Z}), UT_3(\mathbb{Z}))\). The proof will only use the Steinberg relations of \(SL_3(\mathbb{Z})\) and thus it applies verbatim to the pair \((St_3(\mathbb{Z}), \tilde{H}_{1,3}(\mathbb{Z}))\) (replacing each \(e_{i,j}\) with \(x_{i,j}\)). The proof for any other \(H_{i,k}\) (or \(\tilde{H}_{i,k}\) in the case of the Steinberg group) follows from the proof of the case \(UT_3(\mathbb{Z}) = H_{1,3}\) after permuting the indices.

In order to prove this Theorem, we define the following: Let \(1 \leq i, k \leq 3, i \neq k\) and \(d \in \mathbb{N}\). Define \(X^d_{i,k} \in \text{Prob}_c(SL_3(\mathbb{Z}))\) by

\[
X^d_{i,k} = \frac{1}{2d} \sum_{a=0}^{2d-1} e_{i,k}(a).
\]

**Lemma 6.2.** Let \(\delta_0 : (0, 2] \to (0, 1]\) be some function. Then there are constants \(0 \leq r < 1\) and \(L > 0\) such that for every \(d \in \mathbb{N}, d \geq 400\) it holds that

\[
\left\| X^d_{1,2}X^{10d}_{1,3}X^{9d}_{2,3}X^{10d}_{2,1}X^{10d}_{3,1}X^{9d}_{3,2} - X^{4(d+1)}_{2,3}X^{10(d+1)}_{1,3}X^{9(d+1)}_{1,2}X^{10(d+1)}_{3,2}X^{4(d+1)}_{3,1}X^{4(d+1)}_{2,1} \right\|_{\mathcal{U}(\mathcal{E}_{uc}(\delta_0))} \leq Ld^r \sqrt{d}.
\]

The idea of the proof of this Lemma is to identify copies of \(H_3(\mathbb{Z})\) is \(SL_3(\mathbb{Z})\) and apply Lemma 5.2 and Corollary 5.7 to bound the norm of certain moves permuting the order of the \(X\)'s and changing the value of the \(d\)'s.

**Proof.** We will prove only the \(d+1\) case - the proof of the \(d-1\) case is similar. Let \(r, C\) be the constants of Corollary 5.7.

Denote \(H_{i,k} = \langle e_{i,j}(1), e_{j,k}(1), e_{i,k}(1) \rangle < SL_3(\mathbb{Z})\) and note that each of these subgroups is isomorphic to \(H_3(\mathbb{Z})\) such that \(e_{i,k}(1) \in H_{i,k}\) is mapped by the isomorphism to \(z \in H_3(\mathbb{Z})\).

Denote \(T = X^4_{2,3}X^{10d}_{1,3}X^9_{1,2}X^{9d}_{3,2}X^{10d}_{3,1}X^{4d}_{3,1}X^{4d}_{2,1}\). In this proof we will consider

\[
\left\| X^d_{1,2}X^{10d}_{1,3}X^{9d}_{2,3}X^{10d}_{2,1}X^{4d}_{3,1}X^{9d}_{3,2} - T \right\|_{\mathcal{U}(\mathcal{E}_{uc}(\delta_0))}
\]

and change it using moves with a small “norm cost” until we will reach \(\|T - T\|_{\mathcal{U}(\mathcal{E}_{uc}(\delta_0))} = 0\).

By Corollary 5.7 applied to \(H_{1,3}\) and \(H_{3,1}\) it holds that

\[
\left\| X^d_{1,2}X^{10d}_{1,3}X^{9d}_{2,3} - X^d_{1,2}X^{10d}_{1,3}X^{9d}_{2,3} \right\|_{\mathcal{U}(\mathcal{E}_{uc}(\delta_0))} \leq 10Cr \sqrt{d},
\]

\[
\left\| X^d_{2,1}X^{10d}_{3,1}X^{3d}_{3,2} - X^d_{2,1}X^{10d}_{3,1}X^{3d}_{3,2} \right\|_{\mathcal{U}(\mathcal{E}_{uc}(\delta_0))} \leq 10Cr \sqrt{d}.
\]

Thus,

\[
\left\| X^d_{1,2}X^{10d}_{1,3}X^{9d}_{2,3}X^{10d}_{2,1}X^{4d}_{3,1}X^{9d}_{3,2} - T \right\|_{\mathcal{U}(\mathcal{E}_{uc}(\delta_0))} \leq 20Cr \sqrt{d} + \left\| X^d_{1,2}X^{10d}_{1,3}X^{9d}_{2,3}X^{10d}_{2,1}X^{4d}_{3,1}X^{9d}_{3,2} - T \right\|_{\mathcal{U}(\mathcal{E}_{uc}(\delta_0))}.
\]

By Corollary 5.7 applied to \(H_{2,3}\),

\[
\left\| X^{10d}_{1,3}X^{9d}_{2,3}X^{9d}_{2,1}X^{10d}_{2,1}X^{4d}_{3,1}X^{9d}_{3,2} \right\|_{\mathcal{U}(\mathcal{E}_{uc}(\delta_0))} \leq 5dCr \sqrt{d}.
\]
Thus,
\[
\|X_{1,2}^{4d} X_{2,3}^{10(d+1)} X_{2,1}^{9d} X_{3,1}^{10(d+1)} X_{3,2}^{4d} - T\|_{U(\mathcal{U}_c(\delta_0))} \leq 5dC r^{\sqrt{d}} + \|X_{1,2}^{4d} X_{2,3}^{10(d+1)} X_{2,1}^{9d} X_{3,1}^{10(d+1)} X_{3,2}^{4d} - T\|_{U(\mathcal{U}_c(\delta_0))}
\]

By Corollary 5.7 applied for $H_{2,1}$,
\[
\|X_{1,2}^{4d} X_{2,3}^{10(d+1)} X_{2,1}^{9d} X_{3,1}^{10(d+1)} X_{3,2}^{4d} - T\|_{U(\mathcal{U}_c(\delta_0))} \leq 5dC r^{\sqrt{d}}.
\]

Thus,
\[
\|X_{1,2}^{4d} X_{2,3}^{10(d+1)} X_{2,1}^{9d} X_{3,1}^{10(d+1)} X_{3,2}^{4d} - T\|_{U(\mathcal{U}_c(\delta_0))} \leq 8d r^{\sqrt{d}}.
\]

By Lemma 5.2
\[
\|X_{1,2}^{4d} X_{2,3}^{10(d+1)} X_{2,1}^{9d} X_{3,1}^{10(d+1)} X_{3,2}^{4d} - T\|_{U(\mathcal{U}_c(\delta_0))} \leq 8d r^{\sqrt{d}}.
\]

Thus,
\[
\|X_{1,2}^{4d} X_{2,3}^{10(d+1)} X_{2,1}^{9d} X_{3,1}^{10(d+1)} X_{3,2}^{4d} - T\|_{U(\mathcal{U}_c(\delta_0))} \leq 16 r^{\sqrt{d}}.
\]

Finally, arguing as in (1), (2) but for the groups $H_{1,2}, H_{3,2}$ yields
\[
\|X_{2,3}^{4d} X_{1,3}^{10(d+1)} X_{1,2}^{9d} X_{3,2}^{10(d+1)} X_{2,1}^{4d} - T\|_{U(\mathcal{U}_c(\delta_0))} \leq 12d C r^{\sqrt{d}} + \|X_{2,3}^{4d} X_{1,3}^{10(d+1)} X_{1,2}^{9d} X_{3,2}^{10(d+1)} X_{2,1}^{4d} - T\|_{U(\mathcal{U}_c(\delta_0))}
\]
\[
= 12d C r^{\sqrt{d}} + \|T - T\|_{U(\mathcal{U}_c(\delta_0))} = 12d C r^{\sqrt{d}}.
\]

Combining (3), (4), (5), (6), (7) yields the needed inequality. \(\square\)

After this, we can prove Theorem 6.1:

**Proof.** Fix $\delta_0 : (0, 2] \to (0, 1]$.

Define $h_n \in \text{Prob}_{c}(\text{SL}_3(\mathbb{Z}))$ by
\[
h_n = \begin{cases} X_{1,2}^{4d(n+1)} X_{1,3}^{10d(n+1)} X_{2,1}^{9d(n)} X_{3,1}^{10d(n)} X_{3,2}^{4d(n)} & n \text{ is odd} \\ X_{2,3}^{4n} X_{1,3}^{10n} X_{1,2}^{9n} X_{3,2}^{10n} X_{2,1}^{4n} & n \text{ is even} \end{cases}
\]
By Lemma 6.2 there are $L > 0, 0 < r < 1$ such that for every large enough $n$

$$\|h_n - h_{n+1}\|_{U(E_{\text{uc}}(\delta_0))} \leq nLr\sqrt{n}.$$ 

Thus $\{h_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_{U(E_{\text{uc}}(\delta_0))}$ and it has a limit that we will denote $f \in C_{U(E_{\text{uc}}(\delta_0))}$. We note that for every odd $n$,

$$\|(e - e_{1,2}(1))h_n\|_{U(E_{\text{uc}}(\delta_0))} \leq \|(e - e_{1,2}(1))X_{1}^{4n}\|_{U(E_{\text{uc}}(\delta_0))} \leq \frac{1}{24n(2^{4n})},$$

Therefore $\|(e - e_{1,2}(1))f\|_{U(E_{\text{uc}}(\delta_0))} = 0$. This implies that for every $(\pi, E) \in U(E_{\text{uc}}(\delta_0))$, $\pi(e_{1,2}(1)f) = \pi(f)$ and thus $\text{Im}(\pi(f)) \subseteq \mathbb{E}^{\pi(e_{1,2}(1))}$. Similarly, for every even $n$,

$$\|(e - e_{2,3}(1))h_n\|_{U(E_{\text{uc}}(\delta_0))} \leq \|(e - e_{2,3}(1))X_{2,3}^{4n}\|_{U(E_{\text{uc}}(\delta_0))} \leq \frac{1}{24n(2^{4n})},$$

and thus for every $(\pi, E) \in U(E_{\text{uc}}(\delta_0))$, $\text{Im}(\pi(f)) \subseteq \mathbb{E}^{\pi(e_{2,3}(1))}$. It follows that for every $(\pi, E) \in U(E_{\text{uc}}(\delta_0))$, $\text{Im}(\pi(f)) \subseteq \mathbb{E}^{\pi(e_{1,2}(1), e_{2,3}(1))} = \mathbb{E}^{\pi(UT_3(\mathbb{Z}))}$ as needed. □

7. Banach property (T) for $\text{SL}_n(\mathbb{Z})$ and $\text{SL}_n(\mathbb{R})$

In this section, we will prove our main result regarding the Banach property (T) of $\text{SL}_n(\mathbb{Z})$ and $\text{SL}_n(\mathbb{R})$.

We start by proving the $\text{SL}_n(\mathbb{Z})$ has property $(T_{E_{\text{uc}}(\delta_0)})$ for every $n \geq 3$ and every function $\delta_0 : (0, 2] \to (0, 1]$.

We fix the following terminology: an elementary subgroup of $\text{SL}_n(\mathbb{Z})$ is a subgroup of the form $E_{i,j} = \{e_{i,j}(a) : a \in \mathbb{Z}\}$ for some $1 \leq i, j \leq n, i \neq j$. A theorem Carter and Keller is that these subgroups boundedly generate $\text{SL}_n(\mathbb{Z})$:

**Theorem 7.1.** [CK83 Main Theorem] Let $n \geq 3$. The group $\text{SL}_n(\mathbb{Z})$ is boundedly generated by all the elementary subgroups.

This allows us to prove the following Theorem that appeared in the introduction (Theorem 1.6):

**Theorem 7.2.** Let $\delta_0 : (0, 2] \to (0, 1]$ and $n \geq 3$. The group $\text{SL}_n(\mathbb{Z})$ has property $(T_{E_{\text{uc}}(\delta_0)})$. In particular, for every uniformly convex Banach space $E$ and every $n \geq 3$, the group $\text{SL}_n(\mathbb{Z})$ has property $(T_{\mathbb{E}})$.

**Proof.** The proof is by induction on $n$. For $n = 3$, denote $\text{UT}_3(\mathbb{Z})$ and $\text{LT}_3(\mathbb{Z})$ be the subgroups of uni-upper-triangular and uni-lower-triangular matrices defined above.

By Theorem 7.1, $\text{UT}_3(\mathbb{Z})$ and $\text{LT}_3(\mathbb{Z})$ boundedly generate $\text{SL}_3(\mathbb{Z})$ and by Theorem 6.1 ($\text{SL}_3(\mathbb{Z}), \text{UT}_3(\mathbb{Z})$) and ($\text{SL}_3(\mathbb{Z}), \text{LT}_3(\mathbb{Z})$) both have relative property $(T_{E_{\text{uc}}(\delta_0)})$. Thus, by Theorem 4.2 $\text{SL}_3(\mathbb{Z})$ has property $(T_{E_{\text{uc}}(\delta_0)})$. 

Next, we assume that $\text{SL}_n(\mathbb{Z})$ has property $(T_{\mathcal{E}_{uc}(\delta_0)})$ and prove that $\text{SL}_{n+1}(\mathbb{Z})$ has property $(T_{\mathcal{E}_{uc}(\delta_0)})$. Define $H_1, H_2 < \text{SL}_{n+1}(\mathbb{Z})$ as follows:

$$H_1 = \langle E_{i,j} : 1 \leq i, j \leq n, i \neq j \rangle = \begin{pmatrix} 0 \\ \text{SL}_n(\mathbb{Z}) \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$H_2 = \langle E_{i,j} : 2 \leq i, j \leq n + 1, i \neq j \rangle = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \text{SL}_n(\mathbb{Z}) \end{pmatrix}.$$ 

Both $H_1, H_2$ are isomorphic to $\text{SL}_n(\mathbb{Z})$ and thus by the induction assumption have property $(T_{\mathcal{E}_{uc}(\delta_0)})$ and in particular $(\text{SL}_{n+1}(\mathbb{Z}), H_1)$ and $(\text{SL}_{n+1}(\mathbb{Z}), H_2)$ both have relative property $(T_{\mathcal{E}_{uc}(\delta_0)})$. We note that for every $\{i, j\} \neq \{1, n+1\}$ it holds that $E_{i,j} \subseteq H_1 \cup H_2$. Also, by that by the Steinberg relations, every element in $E_{1,n+1}$ and $E_{n+1,1}$ is a product of 4 elements in $H_1 \cup H_2$. Thus, by Theorem 4.2, $H_1, H_2$ boundedly generate $\text{SL}_{n+1}(\mathbb{Z})$ and by Theorem 7.2 $\text{SL}_{n+1}(\mathbb{Z})$ has property $(T_{\mathcal{E}_{uc}(\delta_0)})$.

**Corollary 7.3.** Let $\delta_0 : (0, 2] \to [0, 1]$ and $n \geq 3$. The group $\text{SL}_n(\mathbb{R})$ has property $(T_{\mathcal{E}_{uc}(\delta_0)})$. In particular, for every uniformly convex Banach space $E$ and any $n \geq 3$, the group $\text{SL}_n(\mathbb{R})$ has property $(T_{\mathcal{E}_{uc}(\delta_0)})$.

**Proof.** The class $\mathcal{E}_{uc}(\delta_0)$ is closed under passing to subspaces and thus we can apply Proposition 3.7. 

---

8. Applications

8.1. Banach fixed point properties. Let $E$ be a Banach space and $G$ be a topological group. An affine isometric action of $G$ on $E$ is a continuous homomorphism $\rho : G \to \text{Isom}_{aff}(E)$, where $\text{Isom}_{aff}(E)$ denotes the group of affine isometric automorphisms of $E$. It is not hard to see that every such $\rho$ is of the form

$$\rho(g)\xi = \pi(g)\xi + c(g), \forall \xi \in E$$

where $\pi : G \to O(G)$ is an isometric linear representation and $c : G \to E$ is a 1-cocycle into $\pi$, i.e., for every $g, h \in G$,

$$c(gh) = c(g) + \pi(g)c(h).$$

The group $G$ is said to have property $(F_E)$ if every affine isometric action on $E$ admits a fixed point. When $E$ is reflexive it follows for the Ryll-Nardzewski fixed-point Theorem that $G$ has property $(F_E)$ if and only if for every isometric linear representation $\pi : G \to O(G)$ it holds that every 1-cocycle into $\pi$ is bounded. This lead to the stronger notion of property $(FF_E)$ defined by Mimura [Mim11] as a Banach version of Monod’s [Mon01] property (TT):
Given a continuous isometric linear representation \( \pi : G \to O(G) \), a quasi-1-cocycle into \( \pi \) is a map \( c : G \to E \) such that
\[
\sup_{g,h \in G} \|c(gh) - (c(g) + \pi(g)c(h))\| < \infty.
\]

A group \( G \) is said to have property \((FF_Z)\) if for every continuous isometric linear representation \( \pi : G \to O(G) \) it holds that every quasi-1-cocycle into \( \pi \) is bounded. As noted above, if \( E \) is reflexive, then property \((FF_Z)\) implies property \((FZ)\) and in particular, for every uniformly convex Banach space \( E \), property \((FF_Z)\) implies property \((FZ)\).

The following result of de Laat, Mimura and de la Salle allows one to deduce property \((FF_Z)\) from property \((T_Z)\):

**Theorem 8.1.** [dLMdLS16] Section 5] Let \( n \geq 3 \) and \( E \) be a uniformly convex Banach space. For \( R = \mathbb{Z}, \mathbb{R} \), if \( \text{SL}_n(R) \) has property \((T_Z)\), then \( \text{SL}_{n+2}(R) \) has property \((FF_Z)\) and in particular property \((FZ)\).

Combining this Theorem with Theorem 7.2 and Corollary 7.3 yields the following Corollary that appeared in the introduction (Corollary 1.8):

**Corollary 8.2.** For every \( n \geq 5 \) and every uniformly convex Banach space, the groups \( \text{SL}_n(\mathbb{Z}), \text{SL}_n(\mathbb{R}) \) have property \((FF_Z)\) and in particular property \((FZ)\).

### 8.2. Super-expanders.
Let \( E \) be a Banach space and \( \{(V_i, E_i)\}_{i \in \mathbb{N}} \) be a sequence of finite graphs with uniformly bounded degree, such that \( \lim_i |V_i| = \infty \). We say that \( \{(V_i, E_i)\}_{i \in \mathbb{N}} \) has a uniform coarse embedding in \( E \) if there are functions \( \phi_i : V_i \to X \) and functions \( \rho_- : \mathbb{N} \to \mathbb{R} \) such that \( \lim_n \rho_-(n) = \infty \) and
\[
\forall i \in \mathbb{N}, \forall u, v \in V_i, \rho_-(d_i(u, v)) \leq \|\phi_i(u) - \phi_i(v)\| \leq \rho_+(d_i(u, v)),
\]
where \( d_i(u, v) \) is the graph distance in \( (V_i, E_i) \) between \( u \) and \( v \).

If \( \{(V_i, E_i)\}_{i \in \mathbb{N}} \) has no uniform coarse embedding in \( E \), we shall say that \( \{(V_i, E_i)\}_{i \in \mathbb{N}} \) is a \( E \)-expander family. A super-expander family as a sequence of graphs \( \{(V_i, E_i)\}_{i \in \mathbb{N}} \) as above that is a \( E \)-expander family for every uniformly convex Banach space \( E \).

For Cayley graphs, the following Proposition gives a relation between \( E \)-expansion of Cayley graphs and Banach property \((T)\):

**Proposition 8.3.** Let \( \Gamma \) be a finitely generated discrete group and let \( \{N_i\}_{i \in \mathbb{N}} \) be a sequence of finite index normal subgroups of \( \Gamma \) such that \( \bigcap_i N_i = \{1\} \). Also let \( E \) be a Banach space and let \( E \) be a class of Banach spaces that contain all the finite \( \ell^2 \) sums of \( E \). If \( \Gamma \) has property \((T_Z)\), then for every fixed finite symmetric generating set \( S \), the family of Cayley graphs of \( \{(G/N_i, S/N_i)\}_{i \in \mathbb{N}} \) is an \( E \)-expander family.

A proof of this Proposition can be found in [Opp17] Appendix] (the assumptions in [Opp17] are slightly different, but the adaptation to our case is straight-forward). A corollary of this Proposition and Theorem 7.2 implies the following Theorem that appeared in the introduction (Theorem 1.9):

**Theorem 8.4.** Let \( n \geq 3 \) and let \( S \) be a finite generating set of \( \text{SL}_n(\mathbb{Z}) \) (e.g., \( S = \{e_{i,j}(\pm 1) : 1 \leq i, j \leq n, i \neq j\} \)). Let \( \phi_i : \text{SL}_n(\mathbb{Z}) \to \text{SL}_n(\mathbb{Z}/i\mathbb{Z}) \) be the natural surjective homomorphism. Then the family of Cayley graphs of \( \{\text{(SL}_n(\mathbb{Z}/i\mathbb{Z}), \phi_i(S))\}_{i \in \mathbb{N}} \) is a super-expander family.
Theorem 8.6. Let simple algebraic group with a Lie algebra $g$ in $h$ be a Banach space and $G$ and any lattice $\Gamma$. By our assumption on $\delta_0 : (0, 2] \to (0, 1]$ such that $E_{uc}(\delta_0)$ contains all the finite $\ell^2$-sums of $K$. By Theorem 7.2, the group $SL_n(\mathbb{Z})$ has property $(T_{E_{uc}(\delta_0)})$ and thus by Proposition 8.3 the family $\{(SL_n(\mathbb{Z}/i\mathbb{Z}), S)\}_{i \in \mathbb{N}}$ is an $E$-expander family. 

8.3. Banach property (T) for algebraic groups. Using Mautner’s Lemma (see below) we are able to pass Banach property (T) (for classes of uniformly convex spaces) from $SL_3(\mathbb{R})$ to $\mathbb{R}$-almost simple algebraic groups whose Lie algebra contain $sl_3(\mathbb{R})$. The proof is verbatim the proof given in the classical case for (Hilbert) property (T) in [BdlHV08, Theorem 1.6.1]. We give the proof for completeness and claim no originality.

**Lemma 8.5 (Mautner’s Lemma).** [BFGM07, Lemma 9.3] Let $G$ be a topological group, $E$ a Banach space and $\pi : G \to O(E)$ a continuous representation of $G$. Suppose $\{a_n\}_{n \in \mathbb{N}}$ and $h$ in $G$ satisfy $\lim_{n \to \infty} a_n^{-1} h a_n$. If $\xi \in E$ such that for every $n$, $\pi(a_n) \xi = \xi$, then $\pi(h) \xi = \xi$.

**Theorem 8.6.** Let $\delta_0 : (0, 2] \to (0, 1]$ be a function and let $G$ be a connected, $\mathbb{R}$-almost simple algebraic group with a Lie algebra $g$. If $g$ contains $sl_3(\mathbb{R})$ as a Lie sub-algebra, then $G$ and any lattice $\Gamma < G$ have property (T) $E_{uc}(\delta_0)$.

**Proof.** By our assumption on $g$, the group $G$ contains a subgroup $H$ whose simply connected covering is isomorphic to $SL_3(\mathbb{R})$. Observe that property $(T_{E_{uc}(\delta_0)})$ is preserved under passing to quotients and thus the group $H$ has property $(T_{E_{uc}(\delta_0)})$. By Corollary 3.5, there is a compact set $K \subseteq H$ and $\varepsilon > 0$ such that for every $E \in E_{uc}(\delta_0)$ and every continuous isometric representation $\pi : H \to O(E)$, if there is a unit vector $\xi \in E$ with $\max_{g \in K} \|\pi(g) \xi - \xi\| < \varepsilon$, then $E^{\pi(H)} \neq \{0\}$.

Again by Corollary 3.5 it is enough to show that for $K, \varepsilon$ as above the following holds: For every $E \in E_{uc}(\delta_0)$ and every continuous isometric representation $\pi : G \to O(E)$, if there is a unit vector $\xi \in E$ with $\max_{g \in K} \|\pi(g) \xi - \xi\| < \varepsilon$, then $E^{\pi(G)} \neq \{0\}$.

Fix $E \in E_{uc}(\delta_0)$ and fix a continuous isometric representation $\pi : G \to O(E)$. Assume that there is a unit vector $\xi \in E$ such that $\max_{g \in K} \|\pi(g) \xi - \xi\| < \varepsilon$.

Restricting $\pi$ to $H$, it follows that there is $\xi_0 \in E^{\pi(H)}$. Take $a \in H, a \neq e$ to be $a = \exp X$, such that $\text{ad}X \in \text{End}(g)$ is diagonalizable over $\mathbb{R}$. Let

$$g = \bigoplus_{\lambda \in \mathbb{R}} g^\lambda$$

to be the eigenspace decomposition of $g$ under $\text{ad}X$. Denote

$$g^+ = \bigoplus_{\lambda \in \mathbb{R}, \lambda > 0} g^\lambda, \quad g^- = \bigoplus_{\lambda \in \mathbb{R}, \lambda < 0} g^\lambda.$$

A general argument given in [BdlHV08]. Proof of Theorem 1.6.1] shows that $g^+ \cup g^-$ generate $g$. Note that it holds for every $Y \in g^\lambda$ that

$$a \exp Y a^{-1} = \exp((\text{Ad}(a)) Y) = \exp(e^\lambda Y).$$

Thus, for every $Y \in g^+$,

$$\lim_n a^{-n} \exp Y a^n = e.$$
and for every $Y \in g^+$,
\[
\lim_{n} a^n \exp Y a^{-n} = e.
\]

Note that $\xi_0 \neq 0$ is fixed by $H$ and thus for every $n$, it holds that $\pi(a^n)\xi_0 = \pi(a^{-n})\xi_0 = \xi_0$. It follows from Mautner’s Lemma stated above that for every $Y \in g^+ \cup g^-$, $\pi(\exp(Y))\xi_0 = \xi_0$ and hence for every $g \in G$, $\pi(g)\xi_0 = \xi_0$, since $g^+ \cup g^-$ generate $g$.

To conclude, we showed for every $Y \in g^+ \cup g^-$, $\pi(\exp(Y))\xi_0 = \xi_0$ and hence for every $g \in G$, $\pi(g)\xi_0 = \xi_0$, since $g^+ \cup g^-$ generate $g$.

To conclude, we showed for every $Y \in g^+ \cup g^-$, $\pi(\exp(Y))\xi_0 = \xi_0$ and hence for every $g \in G$, $\pi(g)\xi_0 = \xi_0$, since $g^+ \cup g^-$ generate $g$.

By Proposition 3.6, if $G$ is non-compact, then every lattice in $G$ has property ($T_{\text{uc}}(\delta_0)$).

□

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