SYMMETRIES AND MARTINGALES IN A STOCHASTIC MODEL FOR THE NAVIER-STOKES EQUATION

RÉMI LASSALLE AND ANA BELA CRUZEIRO

Abstract: A stochastic description of solutions of the Navier-Stokes equation is investigated. These solutions are represented by laws of finite dimensional semi-martingales and characterized by a weak Euler-Lagrange condition. A least action principle, related to the relative entropy, is provided. Within this stochastic framework, by assuming further symmetries, the corresponding invariances are expressed by martingales, stemming from a weak Noether’s theorem.

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Several stochastic models for the Navier-Stokes equation have been proposed in the literature. Some refer to random perturbations of the fluid velocity. This is not the case here: we are interested in stochastic Lagrangian paths whose (mean) velocity, or drift, represent the deterministic velocity of the fluid. Different studies of stochastic Lagrangian paths in fluid dynamics and in particular in turbulence can be found in a collection of works, from which we refer to [2], [15] and [16] as examples. Also representation formulae in terms of different random processes were given in [5], [6], [8], among others.

Concerning the derivation of solutions of Navier-Stokes equations from (stochastic) variational principles, after the early articles [17] and [20], such principles were developed in [3] and subsequent works. We mention also [11] and [9] for different, unrelated approaches to the same kind of problems.

In [7] and [13] a weak description of a stochastic deformation of mechanics has been investigated: the Euler-Lagrange condition extends as a condition on laws of stochastic processes. The main originality of
this approach, with respect to other weak deformations of mechanics, is to handle problems in a functional-analytic framework, in a full consistency with the deterministic case.

In this paper, within the framework of \cite{7} and \cite{13}, we develop a specific case, from an example of \cite{7}, which provides a stochastic description for solutions to the Navier-Stokes equation. Then we investigate several symmetries, whose associated invariances correspond in this setup to martingales.

Section 1 fixes the framework and notations of the paper; the weak Euler-Lagrange condition of \cite{13} and \cite{7} is recalled. Under conditions, in Section 2, a map

\[ P : u \rightarrow P_u, \]

associates laws of \( \mathbb{R}^d \)-valued semi-martingales to divergence free vector fields. Solutions of the Navier-Stokes equation are shown to be divergence free vector fields \( u \), whose associated probability \( P_u \) satisfies a weak Euler-Lagrange condition (Proposition 1); Corollary 2 characterizes those solutions as critical points of the stochastic action

\[ S^p(\nu) := E_{\nu} \left[ \int_0^1 \left( \frac{|v_s|^2}{2} - p(1-t, W_t) \right) dt \right], \]

where \( \nu \) denotes the law of specific continuous semi-martingales, and \( (v_t^s) \) the characteristic drift of \( \nu \), as stated accurately below. The function \( p \) is a smooth pressure field which is assumed to be given.

The action functional above is related to the relative entropy with respect to a reference law \( \mu_p \) induced by the pressure field.

Finally, within this stochastic model, Section 3 investigates invariances, stemming from symmetries, by the weak Noether’s theorem of \cite{7}; within this stochastic framework martingales on the canonical space play the rôle of constants of motion in classical mechanics.

1. The weak stochastic Euler-Lagrange condition.

The weak stochastic Euler-Lagrange condition, recalled below, was introduced in \cite{7}, \cite{13}. It embeds in probability measures, specifically in a set of laws of semi-martingales, the classical condition. Thus, it provides a functional analytic approach to tackle stochastic variational problems; in particular, in contrast with usual diffusion approaches, it directly embeds the not stochastic case. In this context the extension of Noether’s theorem becomes natural. Moreover, as stated in \cite{7}, another specificity of this framework is that it provides critical conditions to semi-martingale optimal transportation problems. The latter, introduced in \cite{18}, correspond to a relaxation of a specific dynamical Schrödinger problems (see \cite{14}) by allowing, in particular, the characteristic dispersion to be not-trivial. Finally, as it is expected of optimization over a subset of Borel probabilities on a Polish space, one crucial advantage of this framework is that compactness is rather simple to obtain.

1.1. Admissible trajectories. Trajectories of infinitely small passive tracers in fluids can be described by elements in the space \( W := C([0, 1], \mathbb{R}^d) \), namely the set of continuous \( \mathbb{R}^d \)-valued paths (where we consider the norm \( |.|_W \) of uniform convergence), endowed with the Borel sigma-field \( \mathcal{B}(W) \). In particular, trajectories of finite energy can be described by the subset

\[ H := \left\{ h \in W, h := \int_0^1 \dot{h}_s ds, \int_0^1 |\dot{h}_s|^2 ds < \infty \right\}, \]

of absolutely continuous paths with square integrable derivatives.

Let \( (W_t)_{t \in [0, 1]} \) denote the evaluation process

\[ (t, \omega) \in [0, 1] \times W \rightarrow W_t(\omega) := \omega(t) \in \mathbb{R}^d. \]

We consider \( (\mathcal{F}_t^W) \) the natural (past) filtration and denote by \( P_W \) the set of Borel probabilities on \( W \).
1.2. **A weak description of random trajectories.** In order to avoid measurability issues, we consider $(\mathcal{F}_t^\nu)$, the $\nu$--usual augmentation of the filtration $(\mathcal{F}_t^\nu)$ under $\nu \in \mathcal{P}_W$. The latter naturally models random trajectories, since any $\nu \in \mathcal{P}_W$ is the law of the evaluation process, on the completed probability space $(W, \mathcal{B}(W)^\nu, \nu)$. We define $\mathcal{S}$ to be the subset of $\nu \in \mathcal{P}_W$ such that there exists a $(\mathcal{F}_t^\nu)$--martingale $(M_t^\nu)$ on $(W, \mathcal{B}(W)^\nu, \nu)$, which satisfies

$$W_t = W_0 + M_t^\nu + \int_0^t v_s^\nu \, ds,$$

for all $t \in [0, 1]$, $\nu$--a.s., where $(v_s^\nu)$ is a $(\mathcal{F}_t^\nu)$--predictable process on the same space, and where the predictable covariation process of $(M^\nu)$ is of the specific form

$$\langle (M_t^\nu)^i, (M_t^\nu)^j \rangle = \int_0^t (\alpha_s^\nu)^{ij} \, ds,$$

for a predictable process $(\alpha_s^\nu)$; subsequently, by abuse of language, we refer to $(v_t^\nu, \alpha_t^\nu)$ as the characteristics of $\nu$. In the whole paper, notations are those of [7].

1.3. **A weak Euler-Lagrange condition.** Given a smooth Lagrangian function

$$(1.1) \quad \mathcal{L} : (t, x, v, a) \in [0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \times (\mathbb{R}^d \otimes \mathbb{R}^d) \to \mathcal{L}_t(x, v, a) \in \mathbb{R},$$

the classical Euler-Lagrange condition naturally extends to $\mathcal{S}$ (see [13]). A semi-martingale $\nu \in \mathcal{S}$ satisfies the Euler-Lagrange condition if there exists a $(\mathcal{F}_t^\nu)$ càdlàg martingale $(N_t^\nu)$, such that

$$\partial_t \mathcal{L}_t(W_t, v_t^\nu, \alpha_t^\nu) - \int_0^t \partial_a \mathcal{L}_s(W_s, v_s^\nu, \alpha_s^\nu) \, ds = N_t^\nu \lambda \otimes \nu - \text{a.e.},$$

with $\partial_t \mathcal{L}$ and $\partial_a \mathcal{L}_t$ denoting the respective gradients (in the first and in the second variables, respectively).

**Remark 1.** Similar conditions were considered in [1] for arbitrary semi-martingales $U$ on abstract stochastic basis $(\Omega, A, (A_t), \mathcal{P})$. On the contrary, condition (1.2) imposes constraints on laws of processes. A semi-martingale $U$ on an arbitrary stochastic basis whose law satisfies (1.2) exhibits very precise properties, depending on the Lagrangian; for instance, in a specific case, it is associated to systems of coupled stochastic differential equations; the latter are not satisfied, in general, when $U$ verifies the critical condition of [1]. Moreover, the associated variational principles of [1] do not contain the optimum criticality for semi-martingale optimal transportation problems either.

2. **Navier-Stokes equation and the weak Euler-Lagrange condition**

Henceforth, and until the end of the paper, $p$ denotes a smooth map

$$(2.3) \quad p : (t, x) \in [0, 1] \times \mathbb{R}^d \to p(t, x) \in \mathbb{R}^+,$$

which is further assumed to be bounded, with bounded derivatives; $p$ models the pressure field. This map being given, we provide a stochastic model for solutions of the equation

$$\partial_t u + (u, \nabla) u = -\nabla p + \frac{\Delta u}{2} ; \quad \text{div } u = 0.$$  

For sake of clarity, we focus on the case where the **divergence free velocity vector field**, involved in the Navier-Stokes equation, belongs to

$$C^{1,2}_{\text{div}}([0, 1] \times \mathbb{R}^d) := \{ u \in C^{1,2}([0, 1] \times \mathbb{R}^d; \mathbb{R}^d) : \text{div } u(t, \cdot) = 0, \text{ for all } t \in [0, 1] \}.$$
2.1. Description of dissipative flows by laws of semi-martingales. Given \( u \in C_{b, \text{d}t}^{1,2}([0, 1] \times \mathbb{R}^d) \), we define \( P_u \) to be the probability measure, which is equivalent with respect to the Wiener measure \( \mu \in \mathcal{P}_W \) (the law of standard Brownian motion) by
\[
\frac{dP_u}{d\mu} := \exp \left( - \int_0^1 u(1 - t, W_t) dW_t - \frac{1}{2} \int_0^1 |u(1 - t, W_t)|^2 dt \right).
\]
By the Girsanov theorem (see for example [10]), we obtain a map
\[
P : u \in C_{b, \text{d}t}^{1,2}([0, 1] \times \mathbb{R}^d) \rightarrow P_u \in \mathbb{S},
\]
such that \( \lambda \otimes P_u \text{ a.e., } \alpha^P_u = I_{\mathbb{S}_d} \), and \( v^P_u = -u(1 - t, W_t) \), \( \lambda \) denoting the Lebesgue measure. Itô’s formula yields the following:

**Proposition 1.** A time-dependent vector field \( u \in C_{b, \text{d}t}^{1,2}([0, 1] \times \mathbb{R}^d) \) satisfies the Navier-Stokes equation (2.4) if and only if \( P_u \) satisfies (2.2) for
\[
\mathcal{L}^u_p(x, v, a) := \frac{|v|^2}{2} - p(1 - t, x).
\]

2.2. Stochastic action and relative entropy. Define the stochastic action associated to the Lagrangian \( \mathcal{L}^u \) of (2.4) by
\[
S^u : \nu \in \mathbb{S} \rightarrow S^u(\nu) := E_{\nu} \left[ \int_0^1 \mathcal{L}^u_p(W_s, v^\nu_s, \alpha^\nu_s) ds \right] \in \mathbb{R} \cup \{+\infty\}.
\]
As \( \mathcal{L}^u \) and \( P_u \) satisfy the assumptions of Theorem 5.1 of [7], for all \( p \) as above and \( u \in C_{b, \text{d}t}^{1,2}([0, 1] \times \mathbb{R}^d) \), the \( \mathbb{S} \)-functional \( S^u \) is differentiable in the sense considered in [7]. Thus, we obtain the following result:

**Proposition 2.** A vector field \( u \in C_{b, \text{d}t}^{1,2}([0, 1] \times \mathbb{R}^d) \) is a solution of the equation (2.4) if and only if
\[
\delta S_{P_u}[h] = 0,
\]
for all \( (\mathcal{F}^P_t) \)-adapted process \( (h_t) \) of finite energy, such that
\[
h_0 = h_1 = 0 \text{ } P_u \text{ a.s.,}
\]
where \( P_u \in \mathbb{S} \) is given by (2.5) and \( \delta S_{P_u} \) denotes the \( \mathbb{S} \)-differential of (11) at \( P_u \).

2.3. Least action principle and relative entropy. Subsequently, assuming \( \nu \in \mathcal{P}_W \) to be absolutely continuous with respect to a reference law \( \eta \in \mathcal{P}_W \), the relative entropy of \( \nu \) w.r.t. \( \eta \) is defined as
\[
\mathcal{H}(\nu|\eta) := E_{\nu} \left[ \ln \frac{d\nu}{d\eta} \right].
\]
By the representation formula in [12], we obtain
\[
S^u(P_u) = \mathcal{H}(P_u|\mu_p) + \ln Z_p,
\]
for all \( u \in C^{1,2}([0, 1] \times \mathbb{R}^d) \), where \( P_u \) is defined by (2.3), and where \( \mu_p \) is the absolutely continuous probability with respect to the standard Wiener measure \( (W_0 = 0, \mu - \text{a.s.}) \), whose density is given by
\[
\frac{d\mu_p}{d\mu} := \exp \left( \int_0^1 p(1 - s, W_s) ds \right) Z_p,
\]
with \( Z_p \) a normalization constant. Whence we obtain the following ersatz of Proposition 2.
Proposition 3. A time-dependent vector field \( u \in C^{1,2}_{b,\text{div}}([0,1] \times \mathbb{R}^d) \) is a solution of the equation (2.4) if and only if
\[
\delta \mathcal{H}(\mu_p) P_u [h] = 0
\]
for all \((\mathcal{F}_t^u)\)-adapted process \((h_t)\) of finite energy, such that
\[
h_0 = h_1 = 0 \quad P_u \text{ a.s.}
\]

3. INVARIANCES AND THE STOCHASTIC NOETHER THEOREM.

Within this section we consider the case \( d = 3 \) and we denote by \((e_1, e_2, e_3)\) the canonical orthogonal basis of \( \mathbb{R}^3 \). We further assume that \( u \in C^{1,2}_{b,\text{div}}([0,1] \times \mathbb{R}^3) \) is a solution of (2.4) for a given smooth function \( p \). By Proposition 1, the associated law of the continuous semi-martingale \( P_u \) (c.f. (2.5)) satisfies (1.2) for \( L^p \) defined by (2.6).

The next subsections investigate different symmetries and compute the related local martingales, stemming from the weak Noether Theorem 6.1. of [7]. In each particular case considered below the symmetries are expressed through a condition on the pressure field \( p \). Given the associated family of transformations \( (h^t) \), subsequently, the symmetry condition on \( p \) yields that \((h^t)\) is a smooth family of \( S \)-invariant transformations for \( L^p \), in the sense considered in [7].

We recall this symmetry condition on \( S \). First, by setting
\[
\Gamma^\epsilon : \omega \in W \to \Gamma^\epsilon(\omega) \in W,
\]
where
\[
\Gamma^\epsilon(\omega) := h^t(t, \omega(t)),
\]
for all \( t \in [0,1], \omega \in W \), \((h^t)\) induces a family \((\Gamma^\epsilon)\) of transformations of \( W \). Given \( \eta \in S \), for all \( \epsilon \), \((\Gamma^\epsilon)\) defines a stochastic process on the probability space \((W, \mathcal{B}(W)^n, \eta)\). Thus, by Itô’s formula on the probability space \((W, \mathcal{B}(W)^n, \eta)\), for all \( \epsilon \in \mathbb{R}, \) the transformation \( h^\epsilon \) of the state space \( \mathbb{R}^3 \) is lifted to a transformation
\[
\eta \in S \to \Gamma^\epsilon(\eta) \in S
\]
of \( S \), by pushforward. The symmetry condition considered in [7] consists in the relation
\[
(3.9) \quad \mathcal{L}_{\Gamma^\epsilon}^p(W_t, v^\eta_t, \alpha^\eta_t) = \mathcal{L}_{\Gamma^\epsilon}^p(\Gamma^\epsilon_t, v^\eta_t \circ \Gamma^\epsilon_t, \alpha^\eta_t \circ \Gamma^\epsilon_t),
\]
holding a.e., for all \( \eta \in S \) in the domain of the map defined in (2.7), and for all \( \epsilon \in \mathbb{R} \). Here \( \circ \) denotes the pullback of the \((\Gamma^\epsilon \eta-\text{equivalence class of}) \) map(s) \( v^\eta_t, \alpha^\eta_t : W \to \mathbb{R}^3 \) with the \((\eta- \text{equivalence class of}) \) map(s) \( \Gamma^\epsilon : W \to W \).

Consider a smooth Lagrangian \( \mathcal{L} \) and assume that \( \nu \in S \) satisfies the weak Euler-Lagrange condition for this Lagrangian. The stochastic weak Noether’s Theorem in [7] associates to a family \((h_\epsilon)\) of \( S \)-invariant transformations of \( L \) local martingales on the probability space \((W, \mathcal{B}(W)^\nu, \nu)\). These local martingales, that we denote by \((\mathcal{L}_t)_{t \in [0,1]} \), are in fact explicitly given:

\[
(3.10) \quad \mathcal{L}_t := \left< \frac{d}{dt} I_{t=0} h^t(W_t), \phi^\nu_t >_{\mathbb{R}^d} - \sum_i \left[ \frac{d}{dt} I_{t=0} \langle h^t(W_t)^i, \phi^{\nu, i} \rangle \right]_t + \int_0^t \theta_s ds \right.
\]
where \([.,.]\) stands for the quadratic co-variation process of \( \text{càdlàg} \) semi-martingales, \((\phi^\nu_t)\) denotes a \( \text{càdlàg} \) modification of the process \( \partial_{\nu} \mathcal{L}_s(W_s, v^\nu_s, \alpha^\nu_s) \), and
\[
\theta_s := \sum_{i,j} \kappa^\nu_{ij} \frac{\partial^2 \mathcal{L}}{\partial \alpha^\nu_{ij}}(W_s, v^\nu_s, \alpha^\nu_s),
\]
where \((\kappa_s(\omega))\) is the \(\mathcal{M}_d(\mathbb{R})\)-valued process defined by

\[
\kappa_s(\omega) := \alpha_{s} \left( \left( \nabla \frac{d}{d\epsilon} h^{\epsilon}|_{\epsilon=0} \right)(s, W_{s}) \right) + \left( \left( \nabla \frac{d}{d\epsilon} h^{\epsilon}|_{\epsilon=0} \right)(s, W_{s}) \right) \alpha_{s}.
\]

### 3.1. Symmetry by translation and the momentum process.

Assume the symmetry by translation along \(e_3\) of the pressure, that is

\[
p(t, x + ae_3) = p(t, x),
\]

for all \(a \in \mathbb{R}\), \(t \in [0, 1]\), \(x \in \mathbb{R}^3\). To check that Noether’s theorem yields the expected result, set

\[
h^{\epsilon}( \cdot, x) : [0, 1] \times \mathbb{R}^3 \rightarrow h^{\epsilon}( t, x) := x + \epsilon e_3 \in \mathbb{R}^3.
\]

By proposition 3.2. of [7], (3.9) is trivially satisfied, so that, by Theorem 6.1 of [7], we obtain \(\langle v_{l}^{P_{u}}|_{\eta}, e_3 \rangle\), as the related \((F_{l}^{P_{u}})\)-local martingale.

### 3.2. Symmetry by rotation and the kinetic momentum process.

Assume the symmetry by rotation along the axis \(e_3\) of the pressure; that is,

\[
p(t, R^{\epsilon}x) = p(t, x),
\]

for all \((t, x) \in [0, 1] \times \mathbb{R}^3\) and \(\epsilon \in \mathbb{R}\), where \(R^{\epsilon} : \mathbb{R}^3 \rightarrow \mathbb{R}^3\) denotes the operator of rotation, along the axis \(e_3\), with angle \(\epsilon\). We consider the family of space transformations

\[
h^{\epsilon}( \cdot, x) : [0, 1] \times \mathbb{R}^3 \rightarrow h^{\epsilon}( t, x) := R^{\epsilon}x \in \mathbb{R}^3.
\]

Applying Lemma 3.2. of [7], with \(h^{\epsilon}\) given by (3.14), we compute the characteristics of \(\Gamma_{\epsilon}^{\eta}\) and we obtain the relation

\[
\langle u_{l}^{P_{u}}|_{\eta} \circ \Gamma_{\epsilon}^{\eta}|_{\mathbb{R}^3} = |u|_{\mathbb{R}^3} \lambda \otimes \eta - a.e.,
\]

for any \(\epsilon \in \mathbb{R}\). Whence, the symmetry condition (3.9) is satisfied, from (3.13). Define the stochastic process \((l_{t})\), on the complete probability space \((W, \mathcal{B}(W)^{P_{u}}, P_{u})\), by

\[
l_{t} := \langle W_{t}, e_1 \rangle_{\mathbb{R}^3} < v_{l}^{P_{u}}, e_2 \rangle_{\mathbb{R}^3}^{2} - \langle W_{t}, e_2 \rangle_{\mathbb{R}^3} < v_{l}^{P_{u}}, e_1 \rangle_{\mathbb{R}^3}
\]

for \(t \in [0, 1]\), the stochastic counterpart to the kinetic momentum along \(e_3\). Further denoting by

\[
\text{rot } u_{l} : \mathbb{R}^3 \rightarrow \mathbb{R}^3,
\]

the rotational of \(u(t,.)\), the weak Noether Theorem 6.1 of [7] implies that the corresponding process \((\mathcal{I}_{l})\), defined by

\[
\mathcal{I}_{l} := l_{t} + \int_{0}^{t} \langle \text{rot } u_{l-1}(W_{s}), e_3 \rangle \, ds,
\]

is the local martingale associated to this symmetry by rotation; otherwise stated, the stochastic kinetic momentum process \(l_{t}\) along \(e_3\) is a semi-martingale on the canonical filtered probability space of \(\nu\), whose finite variation term is determined by the rotational of \(u\) along the symmetry axis. The latter expresses the dissipation, modeled through the martingale part of \(P_{u}\), which is involved in the covariation process in the expression of \(\mathcal{I}_{l}\).
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(Rémi Lassalle) CEREMADE Université Paris-Dauphine Place du Maréchal de Lattre de Tassigny 75016 Paris
E-mail address: lassalle@ceremade.dauphine.fr

(Ana Bela Cruzeiro) GFMUL and Departamento Matemática Instituto Superior Técnico, Univ. de Lisboa, Av. Rovisco Pais, 1049-001 Lisbon, Portugal
E-mail address: abcruz@math.tecnico.ulisboa.pt