Ratchet transport with subdiffusion

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Abstract. We introduce a model which incorporate the subdiffusive dynamics and the ratchet effect. Using a subordination ideology, we show that the resulting directed transport is sublinear, \( \langle x(t) \rangle \sim J t^\beta, \beta < 1 \). The proposed model may be relevant to a phenomenon of saltatory microbiological motility.

Thermal fluctuations alone cannot create a steady transport in an unbiased system. Luckily, microbiological realm operates far from equilibrium [1], where directed motion can appear under nonequilibrium conditions [2]. The corresponding ratchet effect has been proposed as a physical mechanism of a microbiological motility [3]. The nonequilibrium conditions might induce another intriguing peculiarity of microbiological transport; namely, the anomalous diffusion [4]. The quasi-random wandering at the molecular scale can be characterized by a mean square displacement (msd), \( \sigma(t) = \langle x^2(t) \rangle - \langle x(t) \rangle^2 \), which, in many cases, follows a power law, \( \sigma(t) \sim t^\alpha, \alpha \neq 1 \), rather than the linear time dependence for a Brownian particle [5,6]. \( \alpha > 1 \) corresponds to enhance or superdiffusion, while \( \alpha < 1 \) to subdiffusion [4,7].

Directed current and diffusion are generated by the same trajectory, \( x(t) \), and characterized by the first and the second moments of the same process being therefore strongly conjugated. In the case of the normal diffusion and the superdiffusion, due to finiteness of all statistical moments, a directed transport, if any, corresponds to a linear grow of the mean displacement, \( \langle x(t) \rangle \sim t \), with the corresponding current \( J = \lim_{t \to \infty} x(t)/t \) [8].

For the subdiffusion a situation is less obvious. In this regime the motion is made up of periods of sticking events separated by fast jumps to a new position (see, f.e., Ref. [6]). Formally, the broad power-law distribution of sticking times, \( \psi(t) \sim t^{-1-\beta}, 0 < \beta < 1 \), (1) leads to the subdiffusion with an exponent \( \alpha = \beta [7] \). The saltatory molecular motor’s transport [9], rapid bursts of directed movements interrupted by pauses of variable duration, has been tracked within a cell by using microinjected fluorescent beads [10]. The abovementioned observations call for a study of a ratchet transport mediated by anomalous trapping events.

In this paper we propose a simple model which naturally incorporates both mechanisms, subdiffusion and ratchet effect, thus bridging two research lines that were so far basically disconnected from one another. Using subordinated ideology [11] we show that the subdiffusion in a ratchet potential results in the sublinear directed transport, \( \langle x(t) \rangle \sim J t^\beta \). The subordinate formalism enables us to reformulate the problem within the circle map’s theory and to derive necessary and sufficient conditions for the directed current appearance.

We start with the model which describes a dynamics of the overdamped particle exposed to the shot-noise, \( \dot{x} = \sum a_j(x) \delta(t - t_j) \) (2) where \( a_j \) gives the length and the direction of the corresponding step taking place at the time instant \( t_j \). We assume that the length \( a_j \) depends locally on a periodic potential \( U(x) \),

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$U(x + L) = U(x)$, and a noise $\xi$, such that

$$a_j = -U'(x(t_j)) + \xi(t_j).$$  \hspace{1cm} (3)

The model (2,3) can be treated as the overdamped limit of the standard model [12]. The corresponding process, $x(t)$, can be considered not as a function of time $t$, but rather as a function of the number of steps, $n$. The dynamics of the system (2) can be represented as the noised circle map [12],

$$x_{n+1} = x_n + f(x_n) + \xi_n,$$  \hspace{1cm} (4)

where $f(x) = -U'(x)$ stands for the acting force. The process $x(t)$ is subordinated [11] to the map $[4]$, such that the time is governed by the linear map,

$$t_{n+1} = t_n + \Delta t_n.$$  \hspace{1cm} (5)

In addition, here we assume that (i) the dispersion is finite, $\langle a_j^2 \rangle < \infty$, and (ii) the time between steps, $\Delta t_j = t_{j+1} - t_j$, is a random stationary process with the probability density function (pdf) $\psi(\Delta t)$. If $\psi(\Delta t)$ has a finite first moment, $\langle \Delta t \rangle < \infty$, and $a_j$ is independent on $x$ and has a symmetrical distribution (Gaussian, Poissonian, etc) then for the time scale $t \gg \langle \Delta t \rangle$ we get normal Gaussian diffusion, $<x^2(t)> \sim t$, which can be described by an ordinary Langevin equation [11]. If $\psi(t)$ has a divergent first moment, which is the case of the distribution [11], then we deal with the subdiffusion, where the corresponding msd’s exponent is $\alpha = \beta = 1/2$.

The role of the sticking time reduces to the fact that the actual number of steps made up to the time instant $t$ fluctuates, so the operational time $n$ is a random function of the physical time $t$. This function, however, is monotonously nondecaying with $t$ and thus allows a causal ordering of the events. For the pdf $p(x,t)$ one has [11]

$$p(x,t) = \sum_n W(x,n) \chi_n(t),$$  \hspace{1cm} (6)

where $W(x,n)$ is the pdf for the iterated process [4], and $\chi_n(t)$ is the probability to make exactly $n$ steps up to time $t$.

The asymptotic transport is $\langle x(t) \rangle = J \langle n(t) \rangle$, where $\langle n(t) \rangle = \sum_{n=0}^{\infty} n \chi_n(t)$, and the current value $J$ follows from transport properties of the map [4], $\langle x(n) \rangle \approx J n$. By using the Laplace transform in the time domain, it can be shown that $\langle \tilde{n}(s) \rangle = \psi/s(1 - \tilde{\psi})$, where $\psi(t)$ is the pdf for sticking time. For the Poissonian process, $\psi(s) = \nu/(s + \nu)$, one can easily get $\langle n(t) \rangle = \nu t$. For the power-law pdf $\psi(t)$ follows that $\langle \tilde{n}(s) \rangle \approx \tau^{-\beta} s^{-1-\beta}$ and, finally, $\langle n(t) \rangle \approx \tau^{-\beta} s^{-1-\beta}$, so that

$$\langle x(t) \rangle \approx \frac{J}{\tau^\beta \Gamma(1+\beta)} t^\beta,$$  \hspace{1cm} (7)

where $\Gamma(x)$ is the Gamma function.

The subordination approach allows us to separate transport properties, precisely the value of the generalized current $J$, which follows from the map [4], from the sublinear asymptotic [7], which is governed by the sticking time pdf $\psi(t)$.

As an illustrative example we consider here the two-harmonics potential force,

$$f(x) = E_1 \sin(2\pi x) + E_2 \sin(4\pi x + \theta),$$  \hspace{1cm} (8)

which transforms the system [4] into a ratchet version of a climbing circle map [12]. Without loss of generality, we chose here the Gaussian noise $\xi$ with the dispersion $\eta^2$.

Certain symmetries of the Eqs. (2,3) need to be broken in order to fulfill necessary conditions for a directed transport appearance [11]. Suppose that there is a transformation which leaves the equation (2) invariant and changes either the sign of $x$, $x \rightarrow -x$, or invert time, $t \rightarrow -t$ (but not both operations simultaneously!). Such a transformation maps a given trajectory into another
Fig. 1. (a) The bifurcation diagram and (b) the current value $J$ as functions of $E_2$ for the map (4-8) ($E_1 = 0.5$). Black line corresponds to deterministic case, $\eta^2 = 0$, and the grey line corresponds to noised case, $\eta^2 = 0.1$. Note that the current value in the last case is scaled by the factor 40.

one, with the opposite velocity. If at least one of such symmetries exists, then contributions of the trajectory and its symmetry-related counterpart will cancel each other and the asymptotic current $J$ will be equal to zero [15].

In order to fulfill the necessary condition for the dc-current appearance, the following symmetries of the potential force should be violated:

$$f(x) = -f(-x) \quad \text{(or } U(x) = U(-x)), \quad (9)$$
$$f(x) = -f(x + L/2) \quad \text{(or } U(x) = -U(x + L/2)), \quad (10)$$

which are the reflection- and the shift-symmetry following to Ref.[15] and termed as ”symmetry” and ”supersymmetry” in Ref.[16]. Both the symmetries are violated when $E_2 \neq 0$ and $\theta \neq k\pi$.

The symmetry violation is the necessary condition for the current appearance. The current value is determined by microscopic dynamical mechanisms [17]. It is reasonable then to start the analysis of map’s transport properties from the deterministic limit, $\xi = 0$. Fig.1 shows the bifurcation diagram and the current value as functions of the second harmonic amplitude, $E_2$. There is an evident relationship between multiple current reversals and different kinds of bifurcations, which is a general property of underdamped deterministic ratchets [18]. Here, at the overdamped limit, this relationship can be explained qualitatively. The current value is equal to the average force,

$$J = \langle f(x) \rangle = \int_0^L dx f(x) \hat{P}(x), \quad (11)$$

where $\hat{P}(x) = \sum_{n=-\infty}^{\infty} P(x + nL)$, $n \in \mathbb{Z}$, is the reduced invariant density for the map [18]. Since any critical bifurcation, like a tangent bifurcation (transition from a limit cycle to a chaotic regime) [12], always accompanied by drastic changes of the invariant density $P(x)$, such a bifurcation leads to a ”jump” of the current value.

The addition of a noise changes the rectification dynamics, but still the system generates a non-zero current (gray curve in Fig.1b). However, the invariant density even at a weak noise limit converges to the Boltzmann pdf [19], $\hat{P}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-U(x)/\sigma^2)$, so that the integral in rhs of Eq.(11) goes to zero. Thus, the current decays rapidly with the increasing of the noise strength (Fig.1b).
Fig. 2. The evolution of the ensemble mean displacement $\langle x(t) \rangle$ versus $t$ for the different values of subdiffusive exponent: $\alpha = 0.5$ ($\bullet$), $\alpha = 0.7$ ($\blacksquare$), and $\alpha = 1.2$ ($\blacklozenge$). Lines correspond to asymptotics from Eq. (12). The parameters are $N = 10^5$, $E_1 = 0.5$, $E_2 = 0.8$, $\theta = -2.24\pi$, and $\eta^2 = 0.02$.

We furthermore assume that a time between subsequent kicking events is a random stationary process with the pdf (1) \cite{11}. The asymptotic displacement can be written as $\langle x(t) \rangle = J \cdot \langle N(t) \rangle$, where the current value $J$ follows from transport properties of the map (4-8), $\langle x(n) \rangle \approx Jn$. Formally, we get

$$\langle x(t) \rangle \approx Jt^{\alpha}.$$ (12)

We consider now a large ensemble of noninteracting ratchets. The dynamics of each particle in the operational time frame is governed by the same map (4-8), but the physical time $t$ is different for different particles. Nevertheless, the subordination formalism allows to make casual ordering of events \cite{11}. Thus we can calculate the displacement $\langle x(t) \rangle$ by using an ensemble averaging. In Fig.2 we shown the evolution of the mean displacement, $\langle x(t) \rangle$, for different values of the waiting time exponent $\alpha$.

Spatial distributions for different times are shown on Fig.3. For the case of normal diffusion (Fig.3a), $\alpha > 1$, the evolution follows an universal Gaussian scaling, which has been found for weakly underdamped ratchets \cite{20}. This Galilei invariant Brownian process, $p(x,t) \approx g(x-Jt\sqrt{t})$, where $g(x)$ stands for the Gaussian pdf, is very different from the Galilei variant subdiffusive ratchet regime \cite{7}. The corresponding pdf is asymmetric with respect to its cusp-like maximum which stays fixed at the origin, and the plume stretches more and more into transport direction (Fig.3b). This behavior is reminiscent of a subdiffusive dispersive transport under a constant tilting force \cite{21}.

Summing up, we have considered the new model which yields anomalous ratchet dynamics. Transport properties, such as direction and value of the generalized current stem from the ratchet-like periodic potential. The anomalous character of a kinetics is governed by the waiting time pdf. The proposed model provides further contribution to the studies of the microbiological transport. It sets up the link between ratchets and a power-stroke approach to a microbiological transport, still existing dichotomy \cite{22}. The model can also be useful for an analysis of experimentally detected saltatory motility in a cell by using of microinjected fluorescent beads \cite{10}.

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1 For the generation of the random variable $\Delta t$ with the pdf (11) we have used the random variable $\xi$ with the uniform distribution on the unit interval, [0, 1], and the transformation $\Delta t = \Delta t_c \xi^{-1/3}$.
Fig. 3. Spatial distributions for the ensemble of $N = 10^6$ single ratchets for (a) $\alpha = 1.2$ and (b) $\alpha = 0.5$ for different times $t$. The inset shows single particle’s trajectory. Other parameters are as in Fig.2.

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