A NOTE ON EXPONENTIAL-TYPE SOLUTIONS FOR
THE LINEAR, DELAYED HEAT PARTIAL
DIFFERENTIAL EQUATION

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Abstract. We construct a class of exponential type solutions for the linear, delayed heat equation. These representations may be used to provide a priori ansatzes for certain boundary and/or initial-value problems arising in heat transfer. Several of the important mathematical properties of the representations are examined, including a discussion of the dependence on the delay parameter.

1. Introduction

The purpose of this Note is to construct a special class of exact solutions to the linear, delayed heat partial differential equation (PDE) ([1], [5], [6], [9])

\begin{equation}
(1.1) \quad u_t (x, t + \tau) = Du_{xx} (x, t),
\end{equation}

where \((x, t)\) are respectively space and time variables, \(\tau \geq 0\) is time delay, \(D > 0\). The importance of this equation is indicated not only by its application to a broad range of phenomena in the physical and engineering sciences, but also the number of papers devoted to investigating the nature of its solutions; see the above indicated references, along with that of Tzou [9] and the works cited within them.

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Our major result is that the delayed heat equation (DHE) has a special class of solutions which take the form
\begin{equation}
(1.2)
 u(x, t, \tau, a, b) = e^{ax+bt},
\end{equation}
where \( a \) depends on \( b \), i.e.
\begin{equation}
(1.3)
 a = a(\tau, D, b).
\end{equation}
Further since the DHE is linear, general solutions may be constructed by linear combination, i.e.
\begin{equation}
(1.4)
 u(x, t, \tau) = \sum_{b} f(b) e^{a(b+bt)} db
\end{equation}
where \( f(b) \) is such that this mathematical expression is defined. Note that the explicit dependence on \( D \) has been dropped and the symbol in front on the right side of \( (1.4) \) indicates “integration” over \( b \) for continuous values of \( b \) and “sums” for discrete values of \( b \). Generally, which occurs depends on the particular initial- and/or boundary-values considered for a given problem.

Of interest is that for boundary-value problems, the so-called Lambert \( W \) function ([1], [10]) makes an appearance. Previously an instability was uncovered in the analysis of the heat equation with time delay when solved on a finite spatial interval [5], by making use of the \( W \)–function.

The next section is devoted to the explicit construction of these exponential solutions. This is followed by an examination of the mathematical structure of the solutions, along with a brief discussion of some possible next steps.

2. Exponential solutions

From \( (1.2) \) it follows that
\begin{align}
(2.1)
 u(x, t + \tau) &= e^{b\tau} (\ldots), \\
(2.2)
 u_t(x, t + \tau) &= be^{b\tau} (\ldots), \\
(2.3)
 u_{xx}(x, t) &= a^2 (\ldots)
\end{align}
where
\begin{equation}
(2.4)
 (\ldots) = e^{ax+bt}.
\end{equation}
Therefore, substituting these expressions into eqn. \( (1.1) \) and cancelling a common factor gives
\begin{equation}
(2.5)
 be^{b\tau} = Da^2,
\end{equation}
\[ a^2 = \left( \frac{1}{D} \right) b e^{\nu \tau}. \]

There are two nontrivial cases to consider.

\( b > 0 : \)  
This case gives

\[ a = (\pm) a_+ = (\pm) \left( \sqrt{\frac{b}{D}} \right) e^{\nu \tau/2} \tag{2.7} \]

\( b < 0 : \)  
For this situation, we have

\[ a = (\pm) ia_- = (\pm) i \left( \sqrt{\frac{|b|}{D}} \right) e^{-|b|\tau/2}, \tag{2.8} \]

where \( i = \sqrt{-1}. \)

Making use of the Euler relations

\[ \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \tag{2.9} \]
\[ \cosh \theta = \frac{e^\theta + e^{-\theta}}{2}, \quad \sinh \theta = \frac{e^\theta - e^{-\theta}}{2}, \tag{2.10} \]

we obtain, after some algebraic manipulations, the results

\[ u(x, t, \tau, b > 0) \equiv u_+(x, t, \tau, b), \tag{2.11} \]
\[ u(x, t, \tau, b < 0) \equiv u_-(x, t, \tau, b). \tag{2.12} \]

In detailed, explicit form, we have

\[ u_+(x, t, \tau, b) = e^{bt} \left[ A_1(b) \cosh (a_+x) + A_2(b) \sinh (a_+x) \right] \tag{2.13} \]

and

\[ u_-(x, t, \tau, b) = e^{-|b|t} \left[ B_1(b) \cos (a_-x) + B_2(b) \sin (a_-x) \right], \tag{2.14} \]

where \((a_+, a_-)\) are given respectively, by (2.7) and (2.8) and \(A_1, A_2, B_1, B_2\) are arbitrary functions of \(b\).

Using the fact that the DHE is linear allows for the construction of more general solutions through the formulation of linear combination of solutions. Doing this gives the result

\[ u(x, t, \tau) = \sum_{b > 0} u_+(x, t, \tau, b) \, db + \sum_{b < 0} u_-(x, t, \tau, b) \, db. \tag{2.15} \]

Inspection of eqn. (2.15) indicates that the first term on the right-side, gives rise to solutions unbounded in time, if given finite initial and boundary conditions at \(t = 0\). Thus, for bounded solutions, only the second term on the right side should be considered.
3. Discussion

First, observe that from eqns. (2.13), (2.14) and (2.15) that

\begin{equation}
\lim_{\tau \to 0} u(x, t, \tau) = w(x, t),
\end{equation}

where \( w(x, t) \) is a solution to

\begin{equation}
w_t(x, t) = Dw_{xx}(x, t),
\end{equation}

which is the standard heat equation without delay. In this limit, \( \tau \to 0 \)

\begin{align*}
a_+(b, \tau) & \to \sqrt{\frac{b}{D}}, b > 0; \\
a_-(b, \tau) & \to \sqrt{\frac{|b|}{D}}, b < 0.
\end{align*}

Consequently, the exponential type solutions of eqn. (1.1) reduce to

the correct, corresponding solutions of (3.2).

Second, for the initial-value/boundary-value problem \( 0 \leq x \leq L \)

\begin{align*}
&u(x, t, \tau) = f(x) = \text{given}, \quad -\tau \leq t \leq 0 \\
u(0, t, \tau) = 0, \quad u(L, t, \tau) = 0, \quad t > 0
\end{align*}

the following condition is obtained

\begin{equation}
a_-(b, \tau) = \frac{n\pi L}{D}, (n = 1, 2, 3, ...),
\end{equation}

and this can be rewritten to the form

\begin{equation}
\left(\frac{|b|}{D}\right) e^{-|b|\tau} = \frac{n^2\pi^2}{L^2}
\end{equation}

Let \( z = -|b| \tau \), then we have

\begin{equation}
ze^z = -\tau D \left(\frac{n\pi}{L}\right)^2.
\end{equation}

Note that this equation may be written in the form of the Lambert \( W \) function \([\text{I}]\).

However, without the use of this function, we have from a traditional knowledge of transcendental functions that all the roots of

\( ze^z + q = 0 \), for \( q \) real,

have negative real parts if and only if

\( 0 < q < \frac{\pi}{2} \).

Applied to this case this means

\( \tau D \left(\frac{n\pi}{L}\right)^2 < \frac{\pi}{2} \).
This inequality will not hold for \( n \) sufficiently large.

Comment-1. The extensive work of Pedro Jordan clearly indicates the perils of using eqn. (1.1) as a mathematical model for heat transfer; see [5]. Further, the exponential solution is related to his work. For example the solution given by (1.2)

\[
 u(x, t, \tau, a, b) = e^{ax+bt},
\]

is an alternative to Jordan’s separation of variables ansatz [5]

\[
 u(x, t, \tau, a, b) = X(x)T(t).
\]

The major difference of the two procedures is in the appearance of the Lambert \( W \)-function, now occurring in calculating solutions for \( T(t) \). This illustrates the fact that exponential and separation of variables methods are related to each other. In particular the exponential ansatz is a special case of the separation of variables method.

Comment-2. Using a somewhat different methodology, Ismagilov et al [2] came to same conclusion as Jordan et al [5] regarding the stability of solutions for the heat equation with delay. In [2], a similar conclusion was reached for the wave equation with delay

\[
 u_{tt}(x, t + \tau) = u_{xx}(x, t).
\]

See also the paper [8].

Comment-3. Polynain and Zhurov [7] have constructed a set of procedures for calculating solutions to non-linear, delay, reaction-diffusion partial differential equations. It might prove of value to see if this methodology can also be usefully applied to the linear, delayed heat equation. Similarly, one could apply the generating function technique (GFT) created by Robert Jackson [3] to this equation to see if new types of solutions can be constructed.

Comment-4. The above methodology can also be applied to the linear delayed advection-diffusion PDE

\[
 u_t(x, t + \tau) + \varepsilon u_x(x, t) = Du_{xx}(x, t),
\]

where \((\varepsilon, D)\) are non-negative parameters.

For this case, we have

\[
 u(x, t) = e^{ax+bt}
\]

with

\[
 Da^2 - \varepsilon a - be^{b\tau} = 0.
\]

Solving for \( a \) gives

\[
 a_\pm = \frac{1}{2D} \left[ \varepsilon \pm \sqrt{\varepsilon^2 + 4bDe^{b\tau}} \right]
\]
where
\begin{equation}
(3.16) \quad a_- < 0 < a_+, \quad a_+ > |a_-|.
\end{equation}
Therefore,
\begin{equation}
(3.17) \quad u(x, t) = \sum \hat{u}(b, x, t) \, db
\end{equation}
with
\begin{equation}
(3.18) \quad \hat{u}(b, x, t) = e^{bt} \left[ A_1(b) e^{a_+x} + A_2(b) e^{-|a_-|x} \right],
\end{equation}
where $A_1(b)$ and $A_2(b)$ are functions of $b$ such that for particular application, eqn. (3.15) exists.

Finally it should be mentioned that the major reason for the interest in the delayed heat equation (DHE) is that it provides a model for heat transfer for a broad range of phenomena. However, there are a variety of investigations indicating that this model and approximation to it may not be valid (\cite{4}, \cite{5}, \cite{6}, \cite{9}). A major problem is that the microscopic, i.e. atomic structure of matter, is not included in the mathematical formulation. How to overcome this difficulty is a goal to fulfill by extending the results of the current research efforts.

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