HEIGHT ESTIMATES FOR EQUIDIMENSIONAL DOMINANT RATIONAL MAPS

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Abstract. Let $\varphi : W \rightarrow V$ be a dominant rational map between quasi-projective varieties of the same dimension. We give two proofs that $h_V(\varphi(P)) \gg h_W(P)$ for all points $P$ in a nonempty Zariski open subset of $W$. For dominant rational maps $\varphi : \mathbb{P}^n \rightarrow \mathbb{P}^n$, we give a uniform estimate in which the implied constant depends only on $n$ and the degree of $\varphi$. As an application, we prove a specialization theorem for equidimensional dominant rational maps to semiabelian varieties, providing a complement to Habegger’s recent theorem on unlikely intersections.

Introduction

A fundamental property of Weil heights [7, B.3.2(b)] is functoriality for morphisms $\varphi : W \rightarrow V$ of (normal) projective varieties:

$$h_{V,D}(\varphi(P)) = h_{W,\varphi^*D}(P) + O(1).$$

(1)

Functoriality breaks down quite badly for rational maps, as shown by simple examples such as

$$\varphi : \mathbb{P}^2 \rightarrow \mathbb{P}^2, \quad \varphi([X,Y,Z]) = [X^2, Y^2, XZ],$$

(2)

which is a map of degree two having fixed points $[a,a,b]$.

When $D$ is ample, a simple triangle inequality argument shows that even for rational maps, there is an upper bound

$$h_{V,D}(\varphi(P)) \leq h_{W,\varphi^*D}(P) + C.$$

Our first result is a lower bound which, although not as strong as (1), is sufficiently nontrivial to have interesting applications.

Theorem 1. Let $\varphi : W \rightarrow V$ be a dominant rational map between quasi-projective varieties, all defined over $\mathbb{Q}$. Assume further that $\dim(V) = \dim(W)$. Fix height functions $h_V$ and $h_W$ on $V$ and $W$, respectively.
respectively, corresponding to ample divisors. Then there are constants $C_1 > 0$ and $C_2$ and a nonempty Zariski open set $U \subset W$ such that
\[ h_V(\varphi(P)) \geq C_1 h_W(P) - C_2 \quad \text{for all } P \in U(\mathbb{Q}). \tag{3} \]
The constants $C_1$ and $C_2$ depend on $V$, $W$, $\varphi$, and the choice of $h_V$ and $h_W$, but are independent of the point $P$.

We will give two proofs of Theorem 1, the first a short proof that relies on a numerical criterion involving nef and big line bundles, the second a direct proof using only elementary properties of height functions.

Theorem 1 says that there is a nonempty open set on which the ratio $h_V(\varphi(P))/h_W(P)$ is bounded below by a positive constant. This prompts the following definition.

**Definition.** Let $\varphi : W \dasharrow V$ be a rational map between quasi-projective varieties, all defined over $\overline{\mathbb{Q}}$. Fix height functions $h_V$ and $h_W$ on $V$ and $W$, respectively, corresponding to ample divisors. The height expansion coefficient of $\varphi$ (relative to the chosen height functions $h_V$ and $h_W$) is the quantity
\[ \mu(\varphi) = \sup_{\emptyset \neq U \subset W} \liminf_{P \in U(\mathbb{Q}), h_W(P) \to \infty} \frac{h_V(\varphi(P))}{h_W(P)}, \]
where the sup is over all nonempty Zariski open subsets of $W$. (Note that as we make $U$ smaller, the liminf becomes larger, so we may restrict attention to sets $U$ such that $\varphi$ is defined at every point of $U$.)

Theorem 1 is equivalent to the assertion that if $\varphi$ is equidimensional and dominant, then $\mu(\varphi) > 0$. Our second result gives a uniform bound for dominant self-maps of projective space.

**Theorem 2.** Let $n \geq 1$ and $d \geq 1$ be integers. There are constants $C_i = C_i(d, n)$ with $C_1 > 0$ such that for all dominant rational maps $\varphi : \mathbb{P}^n \dasharrow \mathbb{P}^n$ defined over $\overline{\mathbb{Q}}$ there is a nonempty Zariski open set $U_\varphi \subset \mathbb{P}^n$ such that
\[ h(\varphi(P)) \geq C_1 h(P) - C_2 h(\varphi) - C_3 \quad \text{for all } P \in U_\varphi(\mathbb{Q}). \]
(N.B. The constants $C_1$, $C_2$, and $C_3$ depend only on $d$ and $n$, and are independent of the map $\varphi$ and the point $P$. See Section 3 for the exact definition of the height $h(\varphi)$ of a rational map $\varphi$.)

Theorem 2 implies that for dominant maps $\varphi : \mathbb{P}^n \dasharrow \mathbb{P}^n$ of degree $d$, the height expansion coefficient $\mu(\varphi)$ is bounded below by a constant.
that depends only on $n$ and $d$. This prompts the definition

$$\overline{\mu}_d(\mathbb{P}^n) \overset{\text{def}}{=} \inf_{\phi: \mathbb{P}^n \to \mathbb{P}^n \text{ dominant}, \deg \phi = d} \mu(\phi).$$

We note that Theorem 2 implies that $\overline{\mu}_d(\mathbb{P}^n) > 0$.

It would be interesting to know the exact value of $\overline{\mu}_d(\mathbb{P}^n)$. It is clear that $\overline{\mu}_d(\mathbb{P}^1) = d$, since every rational self-map of $\mathbb{P}^1$ is a morphism. In Section 4 we give examples of maps on $\mathbb{P}^n$ for which we can compute, or estimate, the value of $\mu(\phi)$. In particular, we prove that

$$\overline{\mu}_d(\mathbb{P}^n) \leq d^{-(n+1)}$$

for all $n \geq 2$ and $d \geq 2$.

We also show that certain automorphisms $\phi: V \to V$ of K3 surfaces satisfy $\mu(\phi^n) \leq (2 + \sqrt{3})^{-n}$, so even for automorphisms of varieties, the height expansion coefficient can be arbitrarily small.

For further properties of $\mu(\phi)$, and for a lower bound for $\mu(\phi)$ that is more closely tied to the geometry of the map $\phi$, see [13, 12].

Acknowledgements. The author would like to thank Marc Hindry and David Masser for their assistance, David Cox for suggesting a method of proving Proposition 6, and Chong Gyu Lee for a suggestion regarding Proposition 9.

1. An algebro-geometric proof of Theorem 1

We use the following numerical criterion for bigness due to Siu.

**Theorem 3.** (Siu [20], [11, Theorem 2.2.15]) Let $V$ be a projective variety of dimension $n$, and let $D$ and $E$ be nef divisors on $V$. Assume that

$$(D^n) > n(D^{n-1} \cdot E).$$

Then $D - E$ is big.

We recall that ample divisors are nef, that the nef property is preserved under pull-back by morphisms, and that a divisor is big if some multiple defines a rational embedding into projective space. See [11, §§1.4,2.2] for basic definitions and properties of nef and big divisors.

**Proof of Theorem 1.** Without loss of generality, we may replace $V$ and $W$ by normal projective varieties, since the statement of the theorem applies only to points in some Zariski open subset of $W$.

The map $\phi: W \to V$ is only assumed to be rational, so we resolve the indeterminacy by finding a projective variety $X$ and morphisms $\psi: X \to V$ and $\pi: X \to W$ such that $\pi$ is a birational morphism and the following diagram commutes [5, II.7.17.3]:
Let \( n = \dim(V) = \dim(W) \). We let \( D \in \text{Div}(W) \) and \( E \in \text{Div}(V) \) be ample divisors associated to the Weil height functions \( h_W = h_{W,D} \) and \( h_V = h_{V,E} \). The fact that \( D \) is ample implies that its self-intersection \( (D^n) \) is positive, and then the projection formula tells us that

\[
(\pi^*D^n) = (D^n) \geq 1.
\]

Hence we can find an integer \( m \geq 1 \) satisfying

\[
m((\pi^*D)^n) > n((\pi^*D)^{n-1} \cdot \psi^*E).
\]

Multiplying by \( m^{n-1} \) yields

\[
(m\pi^*D)^n > n((m\pi^*D)^{n-1} \cdot \psi^*E).
\]

This allows us to apply Siu’s theorem (Theorem 3) to the divisors \( m\pi^*D \) and \( \psi^*E \) to conclude that \( m\pi^*D - \psi^*E \) is big. (We are using the facts that ample divisors are nef and that the pull-back of a nef divisor by a morphism is nef.) In particular, there is an integer \( k \geq 1 \) such that \( km\pi^*D - k\psi^*E \) is effective. It follows from a standard property of height functions \([7, B.3.2(e)]\) that there is a nonempty Zariski open set \( U \subset X \) such that

\[
h_{X, km\pi^*D - k\psi^*E}(x) \geq O(1) \quad \text{for all } x \in U(\overline{\mathbb{Q}}).
\] (4)

(More precisely, we may take \( U \) to be the complement of the base locus of the divisor \( km\pi^*D - k\psi^*E \).) Functorial properties of height functions \([7, B.3.2(b,c)]\) tell us that

\[
h_{X, km\pi^*D - k\psi^*E}(x) = kmh_{X, \pi^*D}(x) - kh_{X, \psi^*E}(x) + O(1)
= kmh_{W,D}(\pi(x)) - kh_{V,E}(\psi(x)) + O(1).
\] (5)

Combining (4) and (5) yields

\[
h_{W,D}(\pi(x)) \geq \frac{1}{m} h_{V,E}(\psi(x)) + O(1) \quad \text{for all } x \in U(\overline{\mathbb{Q}}).
\]

Using the facts that \( \pi \) is surjective, that \( \psi = \varphi \circ \pi \), and that \( \varphi \) is defined on an open subset of \( W \), we conclude that there is an open subset \( U' \subset W \) such that

\[
h_{W,D}(P) \geq \frac{1}{m} h_{V,E}(\varphi(P)) + O(1) \quad \text{for all } P \in U'(\overline{\mathbb{Q}}).
\]
This concludes the proof of Theorem 1.

2. An elementary height-based proof of Theorem 1

In this section we use basic properties of height functions to give an alternative proof of Theorem 1. The key estimate is the standard inequality relating the height of the roots of a polynomial to the height of its coefficients. We begin with an elementary result that will be needed for the proof.

Lemma 4. Let

\[ \varphi : W \rightarrow V \]

be a rational map of projective varieties defined over \( \overline{\mathbb{Q}} \), and let \( Z_\varphi \) be the indeterminacy locus of \( \varphi \), so \( \varphi \) is well-defined on \( V \setminus Z_\varphi \). Fix height functions \( h_V \) and \( h_W \) corresponding to ample divisors on \( V \) and \( W \), respectively. Then there are constants \( C_1 > 0 \) and \( C_2 \), such that

\[ h_V(\varphi(P)) \leq C_1 h_W(P) + C_2 \quad \text{for all } P \in W(\overline{\mathbb{Q}}) \setminus Z_\varphi. \]

Proof. This is a standard triangle inequality estimate, but lacking a suitable reference, we sketch the proof. Replacing \( h_V \) and \( h_W \) by multiples, we may assume that they correspond to embeddings \( V \subset \mathbb{P}^n \) and \( W \subset \mathbb{P}^m \). Extending \( \varphi \) to a rational map \( \varphi : \mathbb{P}^n \rightarrow \mathbb{P}^m \), this reduces the lemma to the case that \( V \) and \( W \) are projective spaces, in which case [7, B.2.5(a)] completes the proof. (More precisely, we take a finite number of extensions of \( \varphi \) in order to cover all of \( V \setminus Z_\varphi \).) \( \square \)

Proof of Theorem 1. The assumptions that \( \dim(V) = \dim(W) \) and that \( \varphi \) is a dominant rational map imply that \( \overline{\mathbb{Q}}(W) \) is a finite algebraic extension of \( \varphi^*\overline{\mathbb{Q}}(V) \). Let \( f \in \overline{\mathbb{Q}}(W) \) be a rational function on \( W \). Then \( f \) is a root of a polynomial

\[ X^d + A_1 X^{d-1} + \cdots + A_{d-1} X + A_d \quad \text{with } A_1, \ldots, A_d \in \varphi^*\overline{\mathbb{Q}}(V). \]

Let \( U = U_f \) be a nonempty open subset of \( W \) such that \( f \) and all of the functions \( A_1, \ldots, A_d \) are defined at every \( P \in U(\overline{\mathbb{Q}}) \). Then for every \( P \in U(\overline{\mathbb{Q}}) \), the number \( f(P) \in \overline{\mathbb{Q}} \) is a root of the polynomial

\[ X^d + A_1(P) X^{d-1} + \cdots + A_{d-1}(P) X + A_d(P). \]

A standard estimate [18, VIII.5.9] relating the heights of the roots and the coefficients of a polynomial gives

\[ h(f(P)) - d \log 2 \leq h([1, A_1(P), A_2(P), \ldots, A_d(P)]) \]

for all \( P \in U(\overline{\mathbb{Q}}) \). (6)
Since \( A_1, \ldots, A_d \in \varphi^* \bar{\mathcal{Q}}(V) \), there are functions \( B_i \in \bar{\mathcal{Q}}(V) \) such that \( A_i = \varphi^* B_i = B_i \circ \varphi \). We define rational maps
\[
\alpha = [1, A_1, \ldots, A_d]: W \longrightarrow \mathbb{P}^d, \quad \beta = [1, B_1, \ldots, B_d]: V \longrightarrow \mathbb{P}^d.
\]
With this notation, the estimate (6) becomes
\[
\frac{h(f(P))}{-d} \log 2 \leq h(\alpha(P)) \quad \text{for all } P \in U(\bar{\mathcal{Q}}). \tag{7}
\]
Applying the elementary triangle inequality estimate described in Lemma 4 to the rational map \( \beta \) gives
\[
h(\beta(Q)) \leq c_1 h_V(Q) + c_2 \tag{8}
\]
for all \( Q \in V(\bar{\mathcal{Q}}) \) at which \( B_1, \ldots, B_d \) are defined. (Here and in what follows, the constants \( c_i = c_i(W, V, \varphi, h_W, h_V, f) > 0 \) are independent of \( P \in U(\bar{\mathcal{Q}}) \).) Applying (8) with \( Q = \varphi(P) \) for \( P \in U(\bar{\mathcal{Q}}) \) and combining it with (7) yields
\[
h(f(P)) \leq c_3 h(\alpha(P)) + c_4 \quad \text{from (7),}
\]
\[
= c_5 h(\beta \circ \varphi(P)) + c_6 \quad \text{since } \alpha = \beta \circ \varphi, \tag{9}
\]
\[
\leq c_7 h_V(\varphi(P)) + c_8 \quad \text{from (8) with } Q = \varphi(P).
\]

The height \( h_W \) is relative to an ample divisor, so taking a multiple of \( h_W \), we may assume that it is associated to a projective embedding \( \psi: W \hookrightarrow \mathbb{P}^n \), i.e., \( h_W(P) = h(\psi(P)) \). The map \( \psi \) is given by rational functions, say
\[
\psi = [1, f_1, \ldots, f_n] \quad \text{with } f_1, \ldots, f_n \in \bar{\mathcal{Q}}(W). \tag{10}
\]
Applying (9) to each of \( f_1, \ldots, f_n \), we find that
\[
h_W(P) = h(\psi(P)) \quad \text{from the choice of } \psi,
\]
\[
= h([1, f_1(P), \ldots, f_n(P)]) \quad \text{from (10),}
\]
\[
\leq \sum_{i=1}^n h(f_i(P)) \quad \text{elementary height estimate,}
\]
\[
\leq c_9 h_V(\varphi(P)) + c_{10} \quad \text{from (9).}
\]
This completes the proof of Theorem 1.

\[ \square \]

Remark 5. The open subset \( U \) in Theorem 1 is necessary. To see this, consider the map
\[
\varphi: \mathbb{P}^2 \longrightarrow \mathbb{P}^2, \quad \varphi = [X^2, Y^2, XZ].
\]
Then \( \varphi \) is a dominant rational map, but
\[
\varphi([0, \alpha, \beta]) = [0, 1, 0] \quad \text{for all } \alpha \neq 0.
\]
Thus
\[ h(\varphi([0, \alpha, \beta])) = h([0, 1, 0]) = 0, \]
which is certainly not larger than a multiple of \( h([0, \alpha, \beta]) \). For this example, one can check that
\[ h(\varphi(P)) \geq h(P) \quad \text{for all } P \in U = \{ X \neq 0 \}. \]

3. A uniform height estimate for rational self-maps of \( \mathbb{P}^n \)

We use Theorem 1 to prove Theorem 2, which says that there is a uniform lower bound for heights on \( \mathbb{P}^n \) relative to dominant rational self-maps. Before starting the proof, we briefly describe the universal parameter space of dominant rational degree \( d \) self-maps of \( \mathbb{P}^n \). To ease notation, let
\[ N = \binom{n + d}{n} \]
be the number of monomials of degree \( d \) in \( n + 1 \) variables. A rational map \( \varphi : \mathbb{P}^n \to \mathbb{P}^n \) of degree \( d \) has the form \( \varphi = [\varphi_0, \ldots, \varphi_n] \), where each \( \varphi_i \) is a homogeneous polynomial of degree \( d \). Taking the coefficients of \( \varphi_0, \ldots, \varphi_n \) as coordinates of a point in projective space, the map \( \varphi \) corresponds to a point \( A_\varphi \in \mathbb{P}^{(n+1)N - 1} \). We define subsets of \( \mathbb{P}^{(n+1)N - 1} \) as follows:
\[
\begin{align*}
\text{Rat}_n^d &= \{ A_\varphi : \text{\varphi has degree } d \}, \\
\text{Dom}_n^d &= \{ A_\varphi : \text{\varphi is dominant of degree } d \}, \\
\text{Mor}_n^d &= \{ A_\varphi : \text{\varphi is a morphism} \}.
\end{align*}
\]

We note that
\[ \text{Mor}_n^d \subset \text{Dom}_n^d \subset \text{Rat}_n^d \subset \mathbb{P}^{(n+1)N - 1}. \]

We use these inclusions to define the height of a rational map \( \varphi \in \text{Rat}_d^n(\overline{\mathbb{Q}}) \) to be the Weil height of the corresponding point in projective space,
\[ h(\varphi) = h_{\mathbb{P}^{(n+1)N - 1}}(A_\varphi). \]
Similarly, for a point
\[ P = (x, A_\varphi) \in \mathbb{P}^{n}_\text{Rat}_d^n = \mathbb{P}^{n} \times \text{Rat}_d^n \subset \mathbb{P}^{n} \times \mathbb{P}^{(n+1)N - 1}, \]
we define
\[ h(P) = h_{\mathbb{P}^{n}}(x) + h(\varphi) \]
to be the Weil height relative to the line bundle \( \mathcal{O}_{\mathbb{P}^{n} \times \mathbb{P}^{(n+1)N - 1}}(1, 1) \).

The set \( \text{Mor}_d^n \) is a nonempty Zariski open subset of \( \mathbb{P}^{(n+1)N - 1} \); more precisely, \( \text{Mor}_d^n \) is an affine variety, since it is the complement of a single polynomial, the Macaulay resultant [8]. Similarly, \( \text{Rat}_d^n \) is a nonempty
Zariski open subset of $\mathbb{P}^{(n+1)N-1}$, since $A_ϕ \not\in \text{Rat}^n_d$ if and only if there is a non-constant homogeneous polynomial $ψ$ dividing all of $ϕ_0, \ldots, ϕ_n$. Again, elimination theory says that for each fixed degree of $ψ$, the set of such $A_ϕ$ is a Zariski closed set.

The fact that Dom$^n_d$ is quasi-projective is perhaps less clear, and although undoubtedly well known, for lack of a suitable reference we sketch the proof.

**Proposition 6.** Over any field of characteristic zero, the parameter space Dom$^n_d$ of dominant degree $d$ rational self-maps of $\mathbb{P}^n$ is a (nonempty) Zariski open subset of $\mathbb{P}^{(n+1)N-1}$.

**Proof.** We write $ϕ(\mathbb{P}^n)$ for the image of $ϕ$, i.e., $ϕ(\mathbb{P}^n)$ is the Zariski closure in $\mathbb{P}^n$ of $ϕ(\mathbb{P}^n \setminus Z_ϕ)$, where $Z_ϕ$ is the locus of indeterminacy of $ϕ$. Then

$$A_ϕ \in \text{Dom}_d^n \iff ϕ(\mathbb{P}^n) = \mathbb{P}^n.$$  

Equivalently, $A_ϕ \not\in \text{Dom}_d^n$ if and only if $ϕ(\mathbb{P}^n)$ is a proper Zariski closed subset of $\mathbb{P}^n$.

Let $P \in \mathbb{P}^n \setminus Z_ϕ$ be a point at which $ϕ$ is defined, and consider the map on the cotangent spaces,

$$ϕ^*_P : Ω_{ϕ(\mathbb{P}^n) \setminus ϕ(P)} \rightarrow Ω_{\mathbb{P}^n \setminus P}.$$  

Our characteristic zero assumption means that we do not have to worry about inseparability, so we have:

- If $\dim ϕ(\mathbb{P}^n) = n$, then $ϕ^*_P$ is injective for almost all $P$.
- If $\dim ϕ(\mathbb{P}^n) < n$, then $ϕ^*_P = 0$ for all $P$.

Hence letting

$$J_ϕ = \det(∂f_i/∂x_j)_{0≤i,j≤n}$$  

be the Jacobian determinant, we obtain the characterization

$$A_ϕ \in \text{Rat}^n_d \setminus \text{Dom}_d^n \iff J_ϕ = 0.$$  

The Jacobian $J_ϕ$ consists of a certain number of monomials in $x_0, \ldots, x_n$ whose coefficients are polynomials in the coefficients of $ϕ_0, \ldots, ϕ_n$. The ideal generated by those coefficients is the ideal that defines the complement $\text{Rat}^n_d \setminus \text{Dom}_d^n$, which completes the proof that Dom$^n_d$ is a quasi-projective subvariety of $\text{Rat}^n_d$.  

**Proof of Theorem 2.** The idea is to use the universal family over the parameter space Dom$^n_d$ of dominant degree $d$ rational self-maps of $\mathbb{P}^n$. Directly from the definition of Dom$^n_d$, we have a rational map $Φ$ as in the following diagram
with the property that for all \( A_\varphi \in \text{Dom}_d^n \), the restriction of \( \Phi \) to the fiber above \( A_\varphi \) is the map \( \varphi \). We first consider the pull-back of this diagram to a subvariety \( S \subset \text{Dom}_d^n \).

**Lemma 7.** Let \( S \subset \text{Dom}_d^n \) be a irreducible subvariety, so we have a commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}^n_{\text{Dom}_d^n} & \overset{\Phi}{\longrightarrow} & \mathbb{P}^n_{\text{Dom}_d^n} \\
\pi \downarrow & & \pi \downarrow \\
\text{Dom}_d^n & & \text{Dom}_d^n
\end{array}
\]

Then there are constants \( C_1(S) > 0 \) and \( C_2(S) \geq 0 \) and a nonempty Zariski open subset \( U_S \subset \mathbb{P}^n_S \) such that

\[
h(\Phi(P)) \geq C_1(S)h(P) - C_2(S) \quad \text{for all} \quad P \in U_S. \]

**Proof.** Apply Theorem 1 to the rational map \( \Phi : \mathbb{P}^n_S \to \mathbb{P}^n_S \). \( \square \)

We are going to apply Lemma 7 inductively on the dimension of \( S \). For a given irreducible \( S \subset \text{Dom}_d^n \), we find constants \( C_1(S) \) and an open set \( U_S \subset \mathbb{P}^n_S \) as in the lemma. The complement \( \mathbb{P}^n_S \setminus U_S \) is a proper Zariski closed subset, so it is a finite union of irreducible subvarieties, say \( T_1 \cup \cdots \cup T_r \). We separate the \( T_i \)'s into two cases, depending on whether they are horizontal or vertical. (We say that \( T \subset \mathbb{P}^n_S \) is horizontal if \( \pi(T) = S \), and vertical otherwise.) Let \( T \) be any one of the \( T_i \)'s.

If \( T \) is horizontal, then its intersection with every fiber of \( \pi \) is a proper closed subset of the fiber. So horizontal \( T \) delineate exceptional sets on each fiber. We let \( \mathcal{H}_S \) denote the set of horizontal \( T_i \).

If \( T \) is vertical, then \( \pi(T) \) is a proper closed subvariety of \( S \), so in particular it is a closed subvariety of \( \text{Dom}_d^n \) satisfying \( \dim(\pi(T)) < \dim(S) \). (Note that this is a strict inequality.) We let \( \mathcal{V}_S \) denote the set of \( \pi(T_i) \) such that that \( T_i \) is vertical.
We now start the induction with $S = S_0 = \text{Dom}_d^n$. This gives us a set of horizontal subvarieties $\mathcal{H}_{S_0}$, which we put aside for later, and a set of $\mathcal{V}_{S_0}$ consisting of proper closed subvarieties of $S_0$ associated to vertical subvarieties. To ease notation, we denote these sets by $\mathcal{H}_0$ and $\mathcal{V}_0$. For each variety $S \in \mathcal{V}_0$, we apply Lemma 7 to obtain sets of horizontal and vertical subvarieties, and we write

$$\mathcal{H}_1 = \bigcup_{S \in \mathcal{V}_0} \mathcal{H}_S \quad \text{and} \quad \mathcal{V}_1 = \bigcup_{S \in \mathcal{V}_0} \mathcal{V}_S.$$  

Repeating this process, we inductively obtain two sequences of finite sets of varieties by the rule

$$\mathcal{H}_{k+1} = \bigcup_{S \in \mathcal{V}_k} \mathcal{H}_S \quad \text{and} \quad \mathcal{V}_{k+1} = \bigcup_{S \in \mathcal{V}_k} \mathcal{V}_S \quad \text{for} \quad k = 0, 1, 2, \ldots.$$  

By construction, the dimension of the varieties in $\mathcal{V}_k$ are strictly decreasing as $k$ increases, so there is a $K$ such that $\mathcal{V}_k = \emptyset$ for $k > K$. More precisely, we can take $K = \dim(\text{Dom}_d^n) = (n+1)N - 1$. We now let

$$\mathcal{H} = \bigcup_{k=0}^{K} \mathcal{H}_k \quad \text{and} \quad \mathcal{V} = \bigcup_{k=0}^{K} \mathcal{V}_k.$$  

Associated to each $S \in \mathcal{V}$ are constants $C_1(S) > 0$ and $C_2(S) \geq 0$, and we set

$$\overline{C}_1 = \min_{S \in \mathcal{V}} C_1(S) \quad \text{and} \quad \overline{C}_2 = \max_{S \in \mathcal{V}} C_2(S).$$  

We also let

$$W = \bigcup_{T \in \mathcal{H}} T \subset \mathbb{P}_d^n.$$  

Note that $W$ is a proper algebraic subset of $\mathbb{P}_d^n$ with the property that $W$ contains no entire fibers of the fibration $\pi: \mathbb{P}_d^n \to \text{Dom}_d^n$.

By construction and from the inequality in Lemma 7, we have

$$h(\phi(P)) \geq \overline{C}_1 h(P) - \overline{C}_2 \quad \text{for all} \quad P \in \mathbb{P}_d^n \setminus W.$$  

We now observe that a point $P \in \mathbb{P}^n_{\text{Rat}_d}$ is really a pair $P = (x, A_\varphi)$, and the map $\phi$ is given by $\phi(P) = (\varphi(x), A_\varphi)$. Further, as noted earlier, the height of $P = (x, A_\varphi)$ is simply the sum $h(P) = h(x) + h(\varphi)$. Hence (11) may be rewritten as

$$h(\varphi(x)) + h(\varphi) \geq \overline{C}_1 (h(x) + h(\varphi)) - \overline{C}_2 \quad \text{for all} \quad (x, A_\varphi) \in \mathbb{P}_d^n \setminus W.$$  

For any point $A_\varphi \in \text{Dom}_d^n$, let

$$W_\varphi = \pi^{-1}(A_\varphi) \cap W \subset \pi^{-1}(A_\varphi) \cong \mathbb{P}^n.$$
By construction, the set $W_\varphi$ is a proper Zariski closed subset of $\mathbb{P}^n$. We have proven that
\[ h(\varphi(x)) \geq C_1 h(x) - h(\varphi) - C_2 \quad \text{for all } A_\varphi \in \text{Dom}_d^n \text{ and } x \in \mathbb{P}^n \setminus W_\varphi. \]

The constants $C_1$ and $C_2$ depend only on $d$ and $n$, which completes the proof of Theorem 2.

4. Height expansion coefficients for $\mathbb{P}^n$

We recall that the height expansion coefficient of a rational map $\varphi : V \to W$ is defined to be the quantity
\[ \mu(\varphi) = \sup_{\emptyset \neq U \subset W} \liminf_{P \in U(\mathbb{Q})} \frac{h_V(\varphi(P))}{h_W(P)} \cdot \]

The value of $\mu(\varphi)$ clearly depends on the choice of height functions $h_V$ and $h_W$, which in turn depend on the choice of ample divisors $D \in \text{Div}(V)$ and $E \in \text{Div}(W)$. More precisely, it follows from [7, B.3.2(f)] that $\mu(\varphi; D, E)$ depends only on the algebraic equivalence classes of $D$ and $E$.

In the special case that $W = V$, which is of interest for example in dynamics, it is natural to take $D = E$, or equivalently $h_W = h_V$. If further NS$(V)$ has rank one, as happens for example when $V = W = \mathbb{P}^n$, then $\mu(\varphi)$ is defined independent of the choice of $h_W = h_V$.

In this section we investigate the height expansion coefficient for dominant rational self-maps of $\mathbb{P}^n$. For example, if $\varphi : \mathbb{P}^n \to \mathbb{P}^n$ is a nonconstant morphism, then
\[ h(\varphi(P)) = (\deg \varphi) h(P) + O(1), \]

so $\mu(\varphi) = \deg \varphi$. On the other hand, the degree two rational map $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ described in the introduction (2) satisfies $\mu(\varphi) = 1$.

We recall the definition
\[ \overline{\mu}_d(\mathbb{P}^n) \overset{\text{def}}{=} \inf_{\varphi : \mathbb{P}^n \to \mathbb{P}^n} \text{dominant } \deg \varphi = d \mu(\varphi). \]

Theorem 2 tells us that $\overline{\mu}_d(\mathbb{P}^n) > 0$. For $n = 1$, every rational map $\mathbb{P}^1 \to \mathbb{P}^1$ is a morphism, so $\overline{\mu}_d(\mathbb{P}^1) = d$. The value of $\overline{\mu}_d(\mathbb{P}^n)$ for $n \geq 2$ is less clear. We are going to prove that
\[ \overline{\mu}_d(\mathbb{P}^n) \leq \frac{1}{d^{n-1}}. \]

Question 8. If $\varphi : \mathbb{P}^n \to \mathbb{P}^n$ is a rational map that is not a morphism, is it true that $\mu(\varphi) < \deg \varphi$ (strict inequality)?
Let \( \psi : \mathbb{P}^n \rightarrow \mathbb{P}^n \) be a rational map. We write \( Z_\psi \) for the locus of indeterminacy of \( \psi \). We also recall the elementary (triangle inequality) height estimate [7, B.2.5(a)],
\[
h(\psi(P)) \leq (\deg \psi) h(P) + O(1) \quad \text{for all } P \in \mathbb{P}^n(\bar{\mathbb{Q}}) \setminus Z_\psi.
\] (12)

A birational map \( \phi : \mathbb{P}^n \rightarrow \mathbb{P}^n \) is called a regular affine automorphism if it is not a morphism, if it restricts to an automorphism \( \mathbb{A}^n \rightarrow \mathbb{A}^n \), and if \( Z_\phi \cap Z_{\phi^{-1}} = \emptyset \).

**Proposition 9.** Let \( n \geq 2 \) and let \( \phi : \mathbb{P}^n \rightarrow \mathbb{P}^n \) be a regular affine automorphism. Then
\[
\mu(\phi) = \frac{1}{(\deg \phi)^{\frac{1}{1+\dim Z_{\phi^{-1}}}}}. 
\]

In particular, there exist regular affine automorphisms of \( \mathbb{P}^n \) of every degree \( d \geq 2 \) satisfying \( \mu(\phi) = d^{-(n-1)} \). Hence
\[
\overline{\mu}_d(\mathbb{P}^n) \leq \frac{1}{d^{n-1}}.
\]

**Proof.** To ease notation, we let
\[
d_1 = \deg(\phi), \quad d_2 = \deg(\phi^{-1}).
\]

We make use of Kawaguchi’s theory of canonical heights for regular affine automorphisms; see [9] or [19, Exercises 7.17–7.22]. Kawaguchi constructed canonical heights under the assumption that \( \phi \) satisfies the following height estimate:
\[
\frac{h(\phi(P))}{d_1} + \frac{h(\phi^{-1}(P))}{d_2} \geq \left( 1 + \frac{1}{d_1d_2} \right) h(P) + O(1) \quad \text{for all } P \in \mathbb{A}^n(\bar{\mathbb{Q}}).
\] (13)

This estimate was proven for \( n = 2 \) by Kawaguchi [9] and by Chong Gyu Lee [12] in general.

Thus there are canonical height functions \( \hat{h}^+ \) and \( \hat{h}^- \) such that for all \( P \in \mathbb{A}^n(\bar{\mathbb{Q}}) \),
\[
\hat{h}^+(\phi P) = d_1 \hat{h}^+(P), \quad \hat{h}^+(P) \leq h(P) + O(1), \\
\hat{h}^-(\phi^{-1}P) = d_2 \hat{h}^-(P), \quad h(P) \leq \hat{h}^+(P) + \hat{h}^-(P) + O(1).
\]

We now fix a point \( P \in \mathbb{A}^n(\bar{\mathbb{Q}}) \) having Zariski dense orbit under iteration of \( \phi \). (It is not hard to see that this is true for most points.) For each \( k \geq 0 \) we let \( Q_k = \phi^{-k}P \) and we compute
\[
h(\phi Q_k) \leq \hat{h}^+(\phi Q_k) + \hat{h}^-(\phi Q_k) + O(1)
\]
\[
= d_1 \hat{h}^+(Q_k) + d_2^{-1} \hat{h}^-(Q_k) + O(1) \quad \text{from properties of } \hat{h}^\pm,
\]
\[ d_1 \hat{h}^+ + d_2^{-1} \hat{h}^- (Q_k) + O(1) \quad \text{since } Q_k = \varphi^{-k} P, \]
\[ = d_1^{-k} \hat{h}^+ (P) + d_2^{-1} \hat{h}^- (Q_k) + O(1) \quad \text{since } \hat{h}^+ \circ \varphi^{-1} = d_1^{-1} \hat{h}^+, \]
\[ \leq d_1^{-k} \hat{h}^+ (P) + d_2^{-1} h(Q_k) + O(1) \quad \text{since } \hat{h}^- \leq h + O(1). \]

Hence
\[ \mu(\varphi) \leq \lim_{k \to \infty} \frac{h(\varphi Q_k)}{h(Q_k)} \leq \lim_{k \to \infty} \frac{d_1^{-k} \hat{h}^+ (P) + O(1)}{h(Q_k)} + d_2^{-1} \leq \frac{1}{d_2}. \]

For the other direction, we compute
\[ \left(1 + \frac{1}{d_1 d_2}\right) h(P) \leq \frac{h(\varphi(P))}{d_1} + \frac{h(\varphi^{-1}(P))}{d_2} + O(1) \quad \text{from (13)}, \]
\[ \leq \frac{h(\varphi(P))}{d_1} + h(P) + O(1) \quad \text{from (12)}. \]

Now a little bit of algebra yields
\[ \frac{1}{d_2} \leq \frac{h(\varphi(P))}{h(P)} + O\left(\frac{1}{h(P)}\right). \]

This holds for all \( P \in \mathbb{A}^n(\bar{\mathbb{Q}}) \), so
\[ \frac{1}{d_2} \leq \sup_{\theta \neq U \subset \mathbb{P}^n} \liminf_{P \in U(\bar{\mathbb{Q}})} \frac{h(\varphi(P))}{h(P)} = \mu(\varphi). \]

This completes the proof that \( \mu(\varphi) = 1/d_2 \). In order to express this bound in terms of the degree of \( \varphi \), we let
\[ \ell_1 = 1 + \dim Z_{\varphi}, \quad \ell_2 = 1 + \dim Z_{\varphi^{-1}}, \]
and use the relations [15, Proposition 2.3.2]
\[ \ell_1 + \ell_2 = n \quad \text{and} \quad d_1^{\ell_1} = d_2^{\ell_2}. \]

Thus
\[ d_2 = d_1^{\ell_2/\ell_1} = d_1^{n/\ell_1 - 1}, \]

so \( \mu(\varphi) = d_2^{-1} = d_1^{-(n/\ell_1 - 1)} \).

Finally, for any \( n \geq 2 \) and \( d \geq 2 \) it is easy to write down regular affine automorphisms of \( \mathbb{P}^n \) of degree \( d \) for which \( Z_{\varphi} \) has dimension zero, and for such maps we have \( \mu(\varphi) = d^{-(n-1)}. \) \( \square \)

We next compute the height expansion coefficient of a rational map that is not an automorphism.
Proposition 10. The dominant rational map $\varphi: \mathbb{P}^n \to \mathbb{P}^n$ defined by 

$$\varphi([X_0, \ldots, X_n]) = [X_0^{-1}, \ldots, X_n^{-1}]$$

has height expansion ratio

$$\mu(\varphi) = \frac{1}{n} = \frac{1}{\deg \varphi}.$$ 

Proof. From the definition, it is clear that $\varphi$ is dominant and satisfies 

$$\varphi(\varphi(X)) = X.$$ 

Fix $\epsilon > 0$, let $T$ be a large number, and choose integers $a_0, \ldots, a_n \in \mathbb{Z}$ satisfying 

$$T^{1-\epsilon} < a_i < T \text{ for all } i \quad \text{and} \quad \gcd(a_0, \ldots, a_n) = 1.$$ 

Consider the point 

$$P = \varphi([a_0, \ldots, a_n])$$

and it’s image $\varphi(P) = [a_0, \ldots, a_n]$. (We are using the fact that $\varphi \circ \varphi$ is the identity map.) The height of $\varphi(P)$ is given by 

$$H(\varphi(P)) = H([a_0, \ldots, a_n]) = \max_{0 \leq i \leq n} |a_i| \leq T. \quad (14)$$

Next we observe that the coordinates of 

$$P = [\ldots, a_0 \cdots a_{j-1}a_{j+1} \cdots a_n, \ldots]$$

are relatively prime integers, so 

$$H(P) = \max_{0 \leq j \leq n} |a_0 \cdots a_{j-1}a_{j+1} \cdots a_n| \geq T^{(1-\epsilon)n}. \quad (15)$$

Combining (14) and (15) and taking logarithms yields 

$$h(\varphi(P)) \leq \frac{1}{(1-\epsilon)n} h(P).$$

The set of points for which this is valid is Zariski dense, so 

$$\mu(\varphi) \leq \frac{h(\varphi(P))}{h(P)} \leq \frac{1}{(1-\epsilon)n}.$$ 

This holds for every $\epsilon > 0$, which gives the upper bound $\mu(\varphi) \leq 1/n$.

To prove a lower bound for $\mu(\varphi)$, we note that for any rational map $\psi: \mathbb{P}^n \to \mathbb{P}^n$, the triangle inequality gives an elementary upper bound 

$$h(\psi(P)) \leq (\deg \psi) h(P) + O(1).$$

Our map $\varphi$ has degree $n$, so $h(\varphi(P)) \leq nh(P) + O(1)$. Replacing $P$ with $\varphi(P)$ and using the fact that $\varphi^2(P) = P$ yields 

$$h(P) \leq nh(\varphi(P)) + O(1),$$
\[ \mu(\varphi) = \lim \inf \frac{H(\varphi(P))}{h(P)} \geq \lim \inf \left\{ \frac{1}{n} + O\left(\frac{1}{h(P)}\right) \right\} = \frac{1}{n}. \]

This completes the proof that \( \mu(\varphi) = n. \)

**Question 11.** Let \( \varphi : \mathbb{P}^n \to \mathbb{P}^n \) be a dominant rational map. Based on the examples in Propositions 9 and 10, it is natural to ask if \( \mu(\varphi) \) always has the form \( d \kappa \) for some rational number \( \kappa \).

**Remark 12.** If \( \varphi, \psi : V \to V \) are rational self-maps of a variety \( V \), it is natural to ask if there is a relation between \( \mu(\varphi) \), \( \mu(\psi) \), and \( \mu(\varphi \circ \psi) \). In particular, for applications to dynamics it would be interesting to relate \( \mu(\varphi^n) \) to \( \mu(\varphi) \). For example, is \( \mu(\varphi^2) \leq \mu(\varphi) ? \) The map described in Proposition 10 shows that the answer is no, since that map satisfies \( \varphi^2(P) = P \), so \( \mu(\varphi) = 1/n \) and \( \mu(\varphi^2) = 1. \) For further discussion of the relation between \( \mu(\varphi) \), \( \mu(\psi) \), and \( \mu(\varphi \circ \psi) \), see [12].

As noted earlier, if \( \varphi : \mathbb{P}^n \to \mathbb{P}^n \) is a morphism of degree \( d \), then \( \mu(\varphi) = d \), so nonconstant morphisms \( \mathbb{P}^n \to \mathbb{P}^n \) never have small height expansion coefficients. It turns out that self-morphisms of other types of varieties may have height expansion coefficients that are arbitrarily small, as in the following example.

**Proposition 13.** Let \( V \subset \mathbb{P}^2 \times \mathbb{P}^2 \) be a non-singular variety of type (2,2), so \( V \) is a K3 surface with noncommuting involutions \( \iota_1 \) and \( \iota_2 \), and let \( \varphi = \iota_1 \circ \iota_2 : V \to V \) be their composition, so \( \varphi \) is an automorphism of \( V \). Further, let \( D_1, D_2 \in \text{Div}(V) \) be the pull-backs to \( V \) of divisors of the form \( H \times \mathbb{P}^2 \) and \( \mathbb{P}^2 \times H \), respectively, and let \( D \in \mathbb{Z}D_1 + \mathbb{Z}D_2 \subset \text{Div}(V) \) be an ample divisor in the linear span of \( D_1 \) and \( D_2 \). Then for all \( n \geq 1 \), the height expansion coefficient of \( \varphi^n \) relative to the divisor \( D \) equals

\[ \mu(\varphi^n) = (2 + \sqrt{3})^{-2n}. \]

**Proof.** For basic properties of the K3 surface \( V \), see [17] or [19, §7.4]. To ease notation, we let \( \alpha = 2 + \sqrt{3} \), and we define divisors \( E^+, E^- \in \text{Div}(V) \otimes \mathbb{R} \) by

\[ E^+ = -D_1 + \alpha D_2 \quad \text{and} \quad E^- = \alpha D_1 - D_2. \]

We write the given divisor \( D \) as \( D = aE^+ + bE^- \) and note that the ampleness of \( D \) is equivalent to \( a > 0 \) and \( b > 0 \); see [17].

There are canonical heights \( \hat{h}^+ \) and \( \hat{h}^- \) associated to \( \varphi \), with properties similar to the canonical heights on \( \mathbb{P}^2 \) described in the proof of
Proposition 9. More precisely, as described in [17] and [19, §7.4], we have

\[
\hat{h}^+(\varphi P) = \alpha^2 \hat{h}^+(P), \quad \hat{h}^+(P) = h_{E^+}(P) + O(1), \\
\hat{h}^-(\varphi P) = \alpha^{-2} \hat{h}^-(P), \quad \hat{h}^-(P) = h_{E^-}(P) + O(1).
\]

We fix a point \( P \in V(\bar{\mathbb{Q}}) \) with Zariski dense \( \varphi \)-orbit, and for each \( k \geq 0 \) we let \( Q_k = \varphi^{-k}(P) \). Then for all \( n \geq 0 \) we have

\[
h_D(\varphi^n(Q_k)) = a\hat{h}^+(\varphi^n(Q_k)) + b\hat{h}^-(\varphi^n(Q_k)) + O(1) \\
= a\alpha^{2n-2k}\hat{h}^+(P) + b\alpha^{-2n}\hat{h}^-(Q_k) + O(1).
\]

Hence

\[
\mu(\varphi^n) \leq \lim_{k \to \infty} h_D(\varphi^n(Q_k)) = b\alpha^{-2n} \lim_{k \to \infty} \frac{\hat{h}^-(Q_k)}{h_D(Q_k)} = b\alpha^{-2n} \lim_{k \to \infty} \frac{\alpha^{2k}\hat{h}^-(P)}{a\alpha^{2k}\hat{h}^+(P) + b\alpha^{2k}\hat{h}^-(P) + O(1)} = \alpha^{-2n}.
\]

For the other inequality, we note that

\[
h_D(\varphi^n(P)) = \alpha^{-2n}h_D(P) \\
= (a\alpha^{2n}\hat{h}^+(P) + b\alpha^{-2n}\hat{h}^-(P)) - \alpha^{-2n}(a\hat{h}^+(P) + b\hat{h}^-(P)) + O(1) \\
= a(\alpha^{2n} - \alpha^{-2n})\hat{h}^+(P) + O(1).
\]

Hence

\[
\mu(\varphi^n) = \lim_{h_D(P) \to \infty} \frac{h_D(\varphi^n(P))}{h_D(P)} \\
\geq \alpha^{-2n} + \lim_{h_D(P) \to \infty} \frac{a(\alpha^{2n} - \alpha^{-2n})\hat{h}^+(P) + O(1)}{h_D(P)} \geq \alpha^{-2n},
\]

since \( a > 0 \) and \( \alpha^{2n} - \alpha^{-2n} \geq 0 \). \( \square \)

Remark 14. We observe that Proposition 13 provides an example in which the Néron–Severi group \( \text{NS}(V) \) has rank greater than one, but the height expansion coefficient is independent of the divisor class associated to the chosen height function. In general, for a given variety \( V \) and map \( \varphi : V \to V \), it might be interesting to study the association

\[
\text{NS}(V) \otimes \mathbb{R} \to \mathbb{R}, \quad [D] \mapsto \mu(\varphi; h_D).
\]
5. An application to specialization maps

We apply Theorem 1 to prove a specialization result. Specialization theorems over one-dimensional bases are known for families of abelian varieties [16] and for products of multiplicative groups [1]. Harbegger [3, 4] has recently proven strong results for intersections $X^\text{oa} \cap G^{[\dim X]}$, where $G$ is a torus or an abelian variety, $G^{[n]}$ is the set of codimension $n$ subgroups of $G$, and $X^\text{oa}$ is the nonanomalous part of $X$, and he has announced a forthcoming work dealing with the case that $G$ is a semiabelian variety. An immediate application of Theorem 1 is a complementary specialization result for dominant rational maps to semiabelian varieties.

**Corollary 15.** Let $G/\mathbb{Q}$ be a semiabelian variety, let $W/\mathbb{Q}$ be a projective variety, and let $\varphi: W \rightarrow G$ be a dominant rational map. Assume further that $\dim(W) = \dim(G)$. Then there is a nonempty Zariski open subset $U \subset W$ such that

$$\{ P \in U(\mathbb{Q}) : \varphi(P) \text{ is a torsion point} \}$$

is a set of bounded height.

**Proof.** We let $U \subset W$ be as in Theorem 1 for the map $\varphi$, so

$$h_G(\varphi(P)) \geq C_1 h_W(P) - C_2 \quad \text{for all } P \in U(\mathbb{Q}).$$

It is well-known that the height of torsion points on tori and on abelian varieties are bounded, and the same is true more generally for semiabelian varieties; see for example [2, appendix]. Thus there is a $C_3$ such that $h_G(Q) \leq C_3$ for all $Q \in G(\mathbb{Q})_{\text{tors}}$. Hence

$$P \in U(\mathbb{Q}) \text{ and } \varphi(P) \in G_{\text{tors}} \quad \implies h_W(P) \leq C_1^{-1} \left( h_G(\varphi(P)) + C_2 \right) \leq C_1^{-1}(C_3 + C_2).$$

**Remark 16.** We mention that a version of Corollary 15 remains valid when $\varphi$ is not dominant. More precisely, if the image of $\varphi$ is not contained in the translate of a subgroup of $G$, then $\varphi(W) \cap G_{\text{tors}}$ is not Zariski dense in $\varphi(W)$, so $\varphi^{-1}(G_{\text{tors}})$ is not Zariski dense in $W$. This follows immediately from a general version of the Manin–Mumford conjecture for semiabelian varieties proven by Hindry [6], generalizing the proof for tori by Laurent [10] and for abelian varieties by Raynaud [14].

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