Addressing Graph Products and Distance-Regular Graphs

Sebastian M. Cioabă*, Randall J. Elzinga†, Michelle Markiewitz‡, Kevin Vander Meulen§, and Trevor Vanderwoerd¶

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Abstract

Graham and Pollak showed that the vertices of any connected graph $G$ can be assigned $t$-tuples with entries in $\{0, a, b\}$, called addresses, such that the distance in $G$ between any two vertices equals the number of positions in their addresses where one of the addresses equals $a$ and the other equals $b$. In this paper, we are interested in determining the minimum value of such $t$ for various families of graphs. We develop two ways to obtain this value for the Hamming graphs and present a lower bound for the triangular graphs.

Keywords: distance matrix, spectrum, triangular graphs, Hamming graphs, graph addressing.

1 Graph Addressings

A $t$-address is a $t$-tuple with entries in $\{0, a, b\}$. An addressing of length $t$ for a graph $G$ is an assignment of $t$-addresses to the vertices of $G$ so that the distance between two vertices is equal to the number of locations in the addresses at which one of the addresses equals $a$ and the other address equals $b$. For example, we have a 3-addressing of a graph in Figure 1. Graham and Pollak [13] introduced such addressings, using symbols $\{*, 0, 1\}$ instead $\{0, a, b\}$, in the context of loop switching networks.

![Figure 1: A graph addressing](image)

We are interested in the minimum $t$ such that $G$ has an addressing of length $t$. We denote such a minimum by $N(G)$. Graham and Pollak [13, 14] showed that $N(G)$ equals the biclique partition number of the distance multigraph of $G$. Specifically, the distance multigraph of $G$, $D(G)$, is the multigraph with the same vertex set as $G$ where the multiplicity of any edge $uv$ equals the distance in $G$ between vertices $u$ and $v$. The biclique partition number $bp(H)$ of a multigraph $H$ is the minimum number of complete bipartite subgraphs (bicliques) of $H$ whose edges partition the edge-set of $H$. This parameter and its close covering variations have been studied by several researchers and appear in different contexts such as computational complexity or geometry (see for example, [8, 13, 14, 15, 16, 19, 20, 25]). Graham and Pollak deduced that $N(G) \leq r(n - 1)$ for any connected $G$ of order $n$ and diameter $r$ and conjectured that $N(G) \leq n - 1$ for any connected graph $G$ of order $n$. This conjecture, also known as the squashed cube conjecture, was proved by Winkler [24].

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*Department of Mathematical Sciences, University of Delaware, Newark, DE 19716-2553, USA; cioaba@udel.edu. Research partially supported by NSF grant DMS-1600768.

†Department of Mathematics, Royal Military College, Kingston, ON K7K 7B4, Canada; rjelzinga@gmail.com. Current address: Info-Tech Research Group, London, ON, N6B 1Y8, Canada.

‡Department of Mathematical Sciences, University of Delaware, Newark, DE 19716-2553, USA; mmark@udel.edu. Research supported by the Summer Scholars Undergraduate Program at the University of Delaware.

§Department of Mathematics, Redeemer University College, Ancaster, ON L9K 1J4, Canada; kvanderm@redeemer.ca. Research supported by NSERC Discovery Grant 203336.

¶Department of Mathematics, Redeemer University College, Ancaster, ON L9K 1J4, Canada; tvandervoerd@redeemer.ca.
To bound \( N(G) \) below, Graham and Pollak used an eigenvalue argument on the adjacency matrix of \( D(G) \). Specifically, if \( M \) is a symmetric real matrix, let \( n_+(M), n_-(M), \) and \( n_0(M) \) denote the number of eigenvalues of \( M \) (including multiplicity) that are positive, negative and zero, respectively. The inertia of \( M \) is the triple \((n_+(M), n_0(M), n_-(M))\). The adjacency matrix of \( D(G) \) will be denoted by \( D(G) \); we will also refer to \( D(G) \) as the distance matrix of \( G \). The inertia of distance matrices has been studied by various authors for many classes of graphs \([3, 17, 18, 26]\). Witsenhausen (cf. \([13, 14]\)) showed that equality holds in (1) for all choices of \( G \).

Letting \( J_n \) denotes the all one \( n \times n \) matrix and \( I_n \) denotes the \( n \times n \) identity matrix, and noting that \( n_-(D(K_n)) = n_-(J_n - I_n) \), Graham and Pollak \([13, 14]\) used the bound (1) to conclude that

\[
N(K_n) = n - 1. 
\]

Graham and Pollak \([13, 14]\) also determined \( N(K_{n,m}) \) for many values of \( n \) and \( m \). The determination of \( N(K_{n,m}) \) for all values of \( n \) and \( m \) was completed by Fuji and Sawa \([11]\). A more general addressing scheme, allowing the addresses to contain more than two different nonzero symbols, was recently studied by Watanabe, Ishii and Sawa \([23]\). The parameter \( N(G) \) has been determined when \( G \) is a tree or a cycle \([14]\), as well as one particular triangular graph \( T_4 \) \([25]\), described in Section 5. For the Petersen graph \( P \), Elzinga, Gregory and Vander Meulen \([10]\) showed that \( N(P) = 6 \). To the best of our knowledge, these are the only graphs \( G \) for which addressings of length \( N(G) \) have been determined. We will say a \( t \)-addressing of \( G \) is optimal if \( t = N(G) \). An addressing is eigensharp \([19]\) if equality is obtained in (1).

In this paper, we study optimal addressings of Cartesian graph products and the distance-regular graphs known as triangular graphs. Let \( H(n,q) \) is the Hamming graph whose vertices are the \( n \)-tuples over an alphabet with \( q \) letters with two \( n \)-tuples being adjacent if and only if their Hamming distance is \( 1 \). We give two different proofs showing that \( N(H(n,q)) = n(q-1) \). This generalizes the Graham-Pollak result (2) since \( H(1,q) = K_q \). We determine that the triangular graphs are not eigensharp.

### 2 Addressing Cartesian Products

Suppose that \( G_i, i = 1, \ldots, k \) are graphs and that each graph \( G_i \) has vertex set \( V(G_i) \) and order \( n_i = |V(G_i)| \). The Cartesian product \( G_1 \square G_2 \square \cdots \square G_k \) of \( G_1,G_2, \ldots, G_k \) is the graph with vertex set \( V(G_1) \times V(G_2) \times \cdots \times V(G_k) \), order \( n = n_1 n_2 \cdots n_k \), and with vertices \( x = (x_1, \ldots, x_k) \) and \( y = (y_1, \ldots, y_k) \) adjacent if for some index \( j \), \( x_j \) is adjacent to \( y_j \) in \( G_j \) while \( x_i = y_i \) for all remaining indices \( i \neq j \). Thus, if \( d \) and \( d_i \) denote distances in \( G \) and \( G_i \), respectively, then

\[
d(x,y) = \sum_{i=1}^{k} d_i(x_i, y_i)
\]

It follows that if each \( G_i, i = 1, \ldots, k \) is given an addressing, then each vertex \( x \) of \( G \) may be addressed by concatenating the addresses of its components \( x_i \). Therefore, the parameter \( N \) is subadditive on Cartesian products; that is, if

\[
G = G_1 \square \cdots \square G_k
\]

then

\[
N(G) \leq N(G_1) + \cdots + N(G_k)
\]

Note that \( N(G_1) + \cdots + N(G_k) \leq \left( \sum_{i=1}^{k} n_i \right) - k \leq \left( \prod_{i=1}^{k} n_i \right) - 1 = n - 1 \). Thus (5) can improve on Winkler’s upper bound of \( n - 1 \) when \( G \) is a Cartesian product.

**Question 2.1.** Must equality holds in (5) for all choices of \( G_i \)? Remark 3.4 might provide a possible counterexample.
3 Distance Matrices of Cartesian Products

If \( v_1, \ldots, v_n \) denote the vertices of a connected graph \( G \), the distance matrix \( D(G) \) of \( G \) is the \( n \times n \) matrix with entries \( D(G)_{ij} = d(v_i, v_j) \). Because \( G \) is connected, its adjacency matrix \( A(G) \) and its distance matrix \( D(G) \) are irreducible symmetric nonnegative integer matrices and by the Perron-Frobenius Theorem (see [5, Proposition 3.1.1] or [12, Theorem 8.8.1]), the largest eigenvalue of each of these matrices has multiplicity 1. We call this largest eigenvalue the Perron value of the matrix and often denote it by \( \rho \).

To obtain a formula for the distance matrix of a Cartesian product of graphs, we will use an additive analogue of the Kronecker product of matrices. Recall first that if \( A \) is an \( n \times n \) matrix and \( B \) an \( m \times m \) matrix, with \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^m \), then the Kronecker products \( A \otimes B \) and \( x \otimes y \) are defined by

\[
A \otimes B = \begin{bmatrix}
a_{11}B & a_{12}B & \cdots & a_{1m}B \\
a_{21}B & a_{22}B & \cdots & a_{2m}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1}B & a_{m2}B & \cdots & a_{mm}B
\end{bmatrix}
\quad \text{and} \quad x \otimes y = \begin{bmatrix} x_1y \\ x_2y \\ \vdots \\ x_ny \end{bmatrix}
\]

For the additive analogue, we use the symbol \( \diamond \) and define \( A \diamond B \) and \( x \diamond y \) by

\[
A \diamond B = \begin{bmatrix}
a_{11} + B & a_{12} + B & \cdots & a_{1m} + B \\
a_{21} + B & a_{22} + B & \cdots & a_{2m} + B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} + B & a_{m2} + B & \cdots & a_{mm} + B
\end{bmatrix}
\quad \text{and} \quad x \diamond y = \begin{bmatrix} x_1 + y \\ x_2 + y \\ \vdots \\ x_n + y \end{bmatrix}
\]

If \( G = G_1 \square G_2 \square \cdots \square G_k \), then the additive property (3) implies that

\[
D(G) = D(G_1) \diamond D(G_2) \diamond \cdots \diamond D(G_k)
\]

(8)

Observe that

\[
A \diamond B = A \otimes J_m + J_n \otimes B \quad \text{and} \quad x \diamond y = x \otimes I_m + I_n \otimes y
\]

(9)

where \( I_n \in \mathbb{R}^n \) denotes the column vector whose entries are all one. Let \( 0_n \in \mathbb{R}^n \) denote the column vector with all zero entries. The following two lemmas are due to D.A. Gregory.

**Lemma 3.1.** Let \( A \) and \( B \) be \( n \times n \) and \( m \times m \) real matrices respectively. If \( Ax = \lambda x \) and \( x^\top I_n = \sum x_i = 0 \), then \( (A \diamond B)(x \otimes 0_m) = m\lambda(x \diamond 0_m) \). Also, if \( I_m^\top y = 0 \) and \( By = \mu y \), then \( (A \diamond B)(0_n \otimes y) = n\mu(0_n \otimes y) \).

**Proof.** We use properties of Kronecker products:

\[
(A \diamond B)(x \otimes 0) = (A \otimes J_m + J_n \otimes B)(x \otimes I_m + I_n \otimes 0)
\]

\[
= Ax \otimes J_m I_m + J_n x \otimes B I_m + A I_n \otimes J_m 0 + J_n \otimes B 0
\]

\[
= Ax \otimes J_m I_m = \lambda m(x \otimes I_m)
\]

\[
= \lambda m(x \otimes I_m + I_n \otimes 0) = \lambda m(x \diamond 0)
\]

A similar argument work the vector \( (0_n \otimes y) \).

Throughout we will say a square matrix is \( k \)-regular if it has constant row sum \( k \).

**Lemma 3.2.** If \( A \) is \( \rho_A \)-regular and \( B \) is \( \rho_B \)-regular then \( A \diamond B \) is \( (m\rho_A + n\rho_B) \)-regular.

**Proof.** Using properties of Kronecker products,

\[
(A \diamond B)(I_n \otimes I_m) = (A \otimes J_m + J_n \otimes B)(I_n \otimes I_m)
\]

\[
= AI_n \otimes J_m I_m + J_n I_n \otimes B I_m
\]

\[
= \rho_A m(I_n \otimes I_m) + n\rho_B(I_n \otimes I_m)
\]

\[
= (\rho_A m + n\rho_B)(I_n \otimes I_m).
\]

Thus \( (A \diamond B)I = (\rho_A m + n\rho_B)I \).

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\(^{1}\)This approach was suggested by the late David A. Gregory.
Lemma 3.3. If $G = G_1 \square G_2 \square \cdots \square G_k$ and each $D(G_i), i = 1, \ldots, k$ is regular then

(a) $n_-(D(G)) \geq \sum_i n_-(D(G_i))$, and

(b) $n_+(D(G)) \geq 1 + \sum_i (n_+(D(G_i)) - 1)$.

Proof. Because $D(G_i)$ regular, $D(G_i)I_n = \rho_i I_n$, where $\rho_i$ is the Perron value of $D(G_i)$. Then $I_n$ is a $\rho_i$-eigenvector of $D(G_i)$ and $\mathbb{R}^{n_i}$ has an orthogonal basis of eigenvectors of $D(G_i)$ that includes $I_n$, as a member. Thus, Lemma 3.1 with $A = D(G_i)$ and $B = D(\square_{j \neq i} G_j)$ implies that the $n_i - 1$ eigenvectors of $D(G_i)$ in the basis other than $I_n$ contribute $n_i - 1$ orthogonal eigenvectors to the matrix $D(G)$.

An eigenvector of $D(G_i)$ with eigenvalue $\lambda \neq \rho_i$ contributes an eigenvector of $D(G)$ with eigenvalue $\lambda(n_1 n_2 \cdots n_k)/n_i = \lambda/n_i$. This eigenvalue has the same sign as $\lambda$ if $\lambda \neq 0$. Also, if $i \neq j$, then each of the $n_i - 1$ eigenvectors contributed to $D(G)$ by $D(G_i)$ is orthogonal to each of the analogous $n_j - 1$ eigenvectors contributed to $D(G)$ by $D(G_j)$. Thus, the inequality (a) claimed for $n_-$ follows. Also, by Lemma 3.1, $I_n$ is an eigenvector of $D(G)$ with a positive eigenvalue $\rho$, so the inequality (b) for $n_+$ follows.

Remark 3.4. (Observed by D.A. Gregory) The inequality in Lemma 3.3 need not hold if the regularity assumption is dropped. For example, suppose $G = G_1 \square G_1$ where $G_1$ is the graph on 6 vertices obtained $K_{2,4}$ by inserted an edge incident to the two vertices in the part of size 2. Then $n_-(D(G_1)) = 5$ but $n_-(D(G)) = 9 < 5 + 5$. Also, $N(G_1) = 5$, so $9 \leq N(G) \leq 10$ by (1) and (5). An affirmative answer to Question 2.1 would imply $N(G) = 10$.

If each $D(G_i)$ in (8) is regular, then Lemma 3.3 accounts for $1 + \sum_i (\text{rank } D(G_i) - 1) = k - 1 + \sum_i \text{rank } D(G_i)$ of the rank $D(G)$ nonzero eigenvalues of $D(G)$. The following results imply that if each $D(G_i)$ is regular then all of the remaining eigenvalues must be equal to zero. Equivalently, the results will imply that if each $D(G_i)$ in (8) is regular, then equality must hold in Lemma 3.3(a) and (b).

The next result (proved by D.A. Gregory) is obtained by exhibiting an orthogonal basis of $\mathbb{R}^{mn}$ consisting of eigenvectors of $A \circ B$ when $A$ and $B$ are symmetric and regular.

Theorem 3.5. Let $A$ be a regular symmetric real $n \times n$ matrix with $A I_n = \rho_A I_n$ with $\rho_A > 0$ and let $B$ be a regular symmetric matrix of order $m$ with $B I_m = \rho_B I_m$ with $\rho_B > 0$. Then

(a) $n_-(A \circ B) = n_-(A) + n_-(B)$,

(b) $n_+(A \circ B) = n_+(A) + n_+(B) - 1$, and

(c) $n_0(A \circ B) = nm - n - m + 1 + n_0(A) + n_0(B)$.

Proof. As in Lemma 3.3, Lemma 3.1 can be used to provide eigenvectors that imply that $n_-(A \circ B) \geq n_-(A) + n_-(B)$ and $n_+(A \circ B) \geq n_+(A) + n_+(B) - 1$. It remains to exhibit an adequate number of linearly independent eigenvectors of $A \circ B$ for the eigenvalue 0.

If $I_n^T x = 0$ and $I_m^T y = 0$, then

$$(A \circ B)(x \otimes y) = (A \otimes J_m + J_n \otimes B)(x \otimes y) = Ax \otimes 0_m + 0_n \otimes By = 0_{nm}$$

This accounts for at least $(n - 1)(m - 1) = nm - n - m + 1$ orthogonal eigenvectors of $A \circ B$ with eigenvalue 0. Moreover, if $Au = 0$ then $I_n^T u = 0$ and hence, by Lemma 3.1, $(A \circ B)(u \otimes 0_m) = 0_{nm}$. Likewise, if $Bv = 0$ then $I_m^T v = 0$ and by Lemma 3.1, $(A \circ B)(0_n \otimes v) = 0_{nm}$. If each set of vectors $x$, each set of vectors $y$, each set of vectors $u$ and each set of vectors $v$ that occur above are chosen to be orthogonal, then the resulting vectors $x \otimes y, u \otimes I_m, I_n \otimes v$ will be orthogonal. Thus, $n_0(A \circ B) \geq nm - n - m + 1 + n_0(A) + n_0(B)$. Adding the three inequalities obtained above, we get

$$nm = n_-(A \circ B) + n_+(A \circ B) + n_0(A \circ B) \geq n_-(A) + n_-(B) + n_+(A) + n_+(B) - 1 + nm - n - m + 1 + n_0(A) + n_0(B) = nm.$$ 

Thus equality holds in each of the three inequalities.

Corollary 3.6. If $G = G_1 \square G_2 \square \cdots \square G_k$ and each $D(G_i), i = 1, \ldots, k$ is regular then

(a) $n_-(D(G)) = \sum_i n_-(D(G_i))$, and
his thesis (see also [22, Theorem 30.1]).

Remark 3.7. In the proof of Theorem 3.5, whether or not \( A \) and \( B \) are symmetric and regular, we always have \((A \circ B)(x \otimes y) = 0_{nm}\) whenever \( I_n^x x = 0 \) and \( I_m^y y = 0 \). Thus,

\[
\text{Nul}(A \circ B) \geq (n-1)(m-1)
\]

for all square matrices \( A \) and \( B \) of orders \( n \) and \( m \), respectively.

In order to apply Lemma 3.3 to the Cartesian product (4), it would be helpful to have conditions on the graphs \( G_i \) that imply that the distance matrices \( D(G_i) \) are regular. The following remark gives a few examples of graphs whose distance matrix has constant row sums.

Remark 3.8. (Regular distance matrices)

1. If \( G \) is either distance regular or vertex transitive, then \( D(G) \) is \( \rho \)-regular where \( \rho \) is equal to the sum of all the distances from a particular vertex to each of the others.

2. If \( G \) is a regular graph of order \( n \) and the diameter of \( G \) is either one or two, then \( D(G) \) is \( \rho \)-regular with \( \rho = 2(n-1) - \rho_A \) where \( \rho_A \) is the Perron value of the adjacency matrix \( A \) of \( G \). For if \( A \) is the adjacency matrix of \( G \), then \( D(G) = A + 2(J_n - I_n - A) = 2(J_n - I_n) - A \). This holds, for example, when \( G \) is the Petersen graph or \( G = K_n \) (the complete graph on \( n \) vertices) or when \( G = K_{m,m} \) (the complete balanced bipartite graph on \( n = 2m \) vertices).

Question 3.9. What are other conditions on a graph that imply that its distance matrix is regular?

Theorem 3.10. Let \( G = G_1 \square G_2 \square \cdots \square G_k \). If \( D(G_i) \) is regular and \( N(G_i) = n_-(D(G_i)) \) for \( i = 1, \ldots, k \), then \( N(G) = \sum_{i=1}^k N(G_i) \).

Proof. By the lower bound (1) and the subadditivity property (5), \( \sum_i N(G_i) \geq N(G) \geq n_-(D(G)) \), where by Lemma 3.3(a), \( n_-(D(G)) \geq \sum_i n_-(D(G_i)) = \sum_i N(G_i) \).

Example 3.11. The Cartesian product of complete graphs, \( G = K_n \square K_{n_2} \square \cdots \square K_{n_k} \) is also known as a Hamming graph. By (2) and Theorem 3.10, it follows that \( N(G) = \sum_{i=1}^k (n_i - 1) \). In the next section, we explore this result using a different description of the Hamming graphs.

4 Optimal Addressing of Hamming Graphs

Let \( n \geq 1 \) and \( q \geq 2 \) be two integers. The vertices of the Hamming graph \( H(n,q) \) can be described as the words of length \( n \) over the alphabet \( \{1, \ldots, q\} \). Two vertices \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_n)\) are adjacent if and only if their Hamming distance is 1. If \( n = 1 \), \( H(1,q) \) is the complete graph \( K_q \). The following result, can be derived from Example 3.11, but we provide another interesting and constructive argument.

Theorem 4.1. If \( n \geq 1 \) and \( q \geq 2 \), then \( N(H(n,q)) = n(q-1) \).

Proof. We first prove that the length of any addressing of \( H(n,q) \) is at least \( n(q-1) \). For \( 0 \leq k \leq n \), let \( A_k \) denote the distance \( k \) adjacency matrix of \( H(n,q) \). The adjacency matrix of the distance multigraph of \( H(n,q) \) is \( D(H(n,q)) = \sum_{k=1}^n k A_k \). The graph \( H(n,q) \) is distance-regular and therefore, \( A_1, \ldots, A_n \) are simultaneously diagonalizable. The eigenvalues of the matrices \( A_1, \ldots, A_n \) were determined by Delsarte in his thesis [9] (see also, [22, Theorem 30.1]).

Proposition 4.2. Let \( k \in \{1, \ldots, n\} \). The eigenvalues of \( A_k \) are given by the Krawtchouk polynomials:

\[
\lambda_{k,x} = \sum_{i=0}^{k} (-q)^i (q-1)^{k-i} \binom{n-i}{k-i} \binom{x}{i} \tag{10}
\]

with multiplicity \( \binom{n}{k} (q-1)^x \) for \( x \in \{0, 1, \ldots, n\} \).
One can check easily that if \( H B \) the biclique of \( H \) and \( B \) where the first row contains the distinct eigenvalues of \( D \) that \( uv \) containing the edge with multiplicity \( k \). Thus, max(\( N \)) address of vertex \( T \) K. The triangular graph \( T_n \) has spectrum \( \lambda_k = \sum_{i=0}^{n} \sum_{k=0}^{n} (-q)^i (q-1)^{k-i} \binom{n}{i} \left( \begin{array}{c} x \\ i \end{array} \right) \),

\[
\mu_x = \sum_{k=1}^{n} k \lambda_{k,x} = \sum_{k=1}^{n} k \sum_{i=0}^{n} (-q)^i (q-1)^{k-i} \binom{n-i}{k-i} \left( \begin{array}{c} x \\ i \end{array} \right)
= \sum_{i=0}^{n} (-q)^i \left( \begin{array}{c} x \\ i \end{array} \right) \sum_{k=1}^{n} (q-1)^{k-i} \binom{n-i}{k-i} = \sum_{i=0}^{n} (-q)^i \left( \begin{array}{c} x \\ i \end{array} \right) \sum_{l=0}^{n-i} (i+t)(q-1)^t \binom{n-i}{t}
= q^{n-1} \sum_{i=0}^{n} \left( \begin{array}{c} x \\ i \end{array} \right) (-1)^i = \begin{cases} -q^{n-1} & \text{if } x = 1 \\ 0 & \text{if } x \geq 2. \end{cases}
\]

with multiplicity \( \binom{n}{x} (q-1)^x \) for \( 1 \leq x \leq n \). Thus, the spectrum of \( D(H(n,q)) \), with multiplicities, is

\[
\begin{pmatrix} nq^{n-1}(q-1) & -q^{n-1} \\ 1 & n(q-1) \end{pmatrix}
\]

where the first row contains the distinct eigenvalues of \( D(H(n,q)) \) and the second row contains their multiplicities. Thus, \( \max(n-(D(H(n,q))), n+(D(H(n,q)))) = n(q-1) \) and Witsenhausen’s inequality (1) implies that \( N(H(n,q)) \geq n(q-1) \).

To show \( n(q-1) \) is the optimal length of an addressing of \( H(n,q) \), we describe a partition of the edge-set of the distance multigraph of \( H(n,q) \) into exactly \( n(q-1) \) bicliques. For \( 1 \leq i \leq n \) and \( 1 \leq t \leq q-1 \), define the biclique \( B_{i,t} \) whose color classes are

\[
\{(x_1, \ldots, x_n) : x_i = t\}
\]

and

\[
\{(x_1, \ldots, x_n) : x_i \geq t+1\}.
\]

One can check easily that if \( u \) and \( v \) are two distinct vertices in \( H(n,q) \), there exactly \( d_H(u,v) \) bicliques \( B_{i,t} \) containing the edge \( uv \). Thus, the \( n(q-1) \) bicliques \( B_{i,t} \) partition the edge set of the distance multigraph of \( H(n,q) \) and \( N(H(n,q)) \leq n(q-1) \). This finishes our proof.

We remark here that the spectrum of the distance matrix of \( H(n,q) \) was also computed by Indulal [17].

5 Triangular Graphs

The triangular graph \( T_n \) is the line graph of the complete graph \( K_n \) on \( n \) vertices. When \( n \geq 4 \), the triangular graph \( T_n \) is a strongly regular graph with parameters \( \binom{n}{2}, 2(n-2), n-2, 4 \). The adjacency matrix of \( T_n \) has spectrum

\[
\begin{pmatrix} 2(n-2) & n-4 & -2 \\ 1 & n-1 & \binom{n}{2} - n \end{pmatrix}
\]

and therefore, the distance matrix \( D(T_n) \) has spectrum

\[
\begin{pmatrix} (n-1)(n-2) & 2-n & 0 \\ 1 & n-1 & \binom{n}{2} - n \end{pmatrix}
\]

Witsenhausen’s inequality (1) implies that \( N(T_n) = b p(D(T_n)) \geq n-1 \) for \( n \geq 4 \).

The problem of addressing \( T_4 \) is equivalent to determining the biclique partition number of the multigraph obtained from \( K_6 \) by adding one perfect matching. This formulation of the problem was studied by Zaks [25] and Hoffman [16] (see also Section 6). Zaks proved that \( N(T_4) = 4 \) and hence \( T_4 \) is not eigensharp. We will reproduce the lower bound of Zaks [25] in Lemma 5.2 using a technique of [10]. The argument of Lemma 5.2 will then be used to show that \( T_n \) is not eigensharp for any \( n \geq 4 \) in Theorem 5.3.

The addressing matrix of a \( t \)-addressing is the \( n \times n \) matrix \( M(a,b) \) where the \( i \)-th row of \( M(a,b) \) is the address of vertex \( i \). \( M(a,b) \) can be written as a function of \( a \) and \( b \):

\[
M(a,b) = aX + bY,
\]
where $X$ and $Y$ are matrices with entries in $\{0, 1\}$. Elzinga et al. [10] use the addressing matrix, along with results from Brandenburg et al. [4] and Gregory et al. [15], to create the following theorem:

**Theorem 5.1.** [10] Let $M(a, b)$ be the address matrix of an eigensharp addressing of a graph $G$. Then for all real scalars $a, b$, each column of $M(a, b)$ is orthogonal to the null space of $D(G)$. Also, the columns of $M(1, 0)$ are linearly independent, as are the columns of $M(0, 1)$.

In [10, Theorem 3], Elzinga et al. use Theorem 5.1 to show that the Petersen graph does not have an eigensharp addressing. We will use a similar approach on triangular graphs.

**Lemma 5.2.** The triangular graph $T_4$ is not eigensharp, that is, $N(T_4) \geq 4$.

**Proof.** Suppose $T_4$ is eigensharp. Let $D = D(T_4)$. By Theorem 5.1, all vectors in the null space of $D$ are orthogonal to the columns of a $6 \times 3$ addressing matrix $M(a, b)$.

We can construct null vectors of $D$ in the following manner, referring to the entries of the null vector as labels: choose any two non-adjacent vertices, and label them with zeroes. The remaining four vertices form a 4-cycle, which will be alternatingly labelled with 1 and $-1$, as in Figure 2.

![Figure 2: $T_4$ with a $D(T_4)$ null-vector labelling](image)

Let $w(a, b)$ be any column of $M(a, b)$. We claim that $w(a, b)$ has at least three $a$-entries, and at least three $b$-entries. For convenience, we’ll refer to vertices corresponding to the $a$-entries of $w(a, b)$ as $a$-vertices. If there are no $a$-vertices, then since $M(a, b) = aX + bY$, one of the columns of $X$ is the zero vector. It would follow that the columns of $M(1, 0)$ are linearly dependent, contradicting Theorem 5.1. Thus $w(a, b)$ has at least one $a$ and at least one $b$ entry.

Suppose $w(a, b)$ has at most 2 $a$-entries. There are three cases we will consider: there are two adjacent $a$-vertices, there are two non-adjacent $a$-vertices, or there is exactly one $a$-vertex. In each case, we will construct a null vector $x$ of $D$ which is not orthogonal to $u = w(1, 0)$, contradicting Theorem 5.1. We will use the labelling in Figure 2.

Suppose $w(a, b)$ has two adjacent $a$-vertices. Label one of the $a$-vertices with a zero and the adjacent $a$-vertex with 1. Then $x^T u = 1 \neq 0$. Suppose $w(a, b)$ has two non-adjacent $a$-vertices. Label the two $a$-vertices with 1. Then $x^T u = 2 \neq 0$. Suppose there is only one $a$-vertex in $w$. Label the $a$-vertex with 1. Then $x^T u = 1 \neq 0$.

Therefore, at least three positions of $w(a, b)$ have the value $a$. Similarly, at least three positions of $w(a, b)$ must have value $b$.

Since each column of $M(a, b)$ has at least three $a$ and three $b$-entries, there are at least nine $a, b$ pairings corresponding to each column. Since $M(a, b)$ has three columns, there are $27$ $a, b$ column-wise pairs in total. However the number of column-wise $a, b$ pairs in the addressing matrix $M(a, b)$ is simply the number of edges in $D(T_4)$, namely 18. This contradiction implies that $T_4$ is not eigensharp.

**Theorem 5.3.** The triangular graph $T_n$ is not eigensharp for any $n \geq 4$, that is, $N(T_n) \geq n$ for all $n \geq 4$.

**Proof.** Note that $T_4$ is an induced subgraph of $T_n$ since $K_4$ is an induced subgraph of $K_n$. Let $T$ be an induced subgraph of $T_n$ isomorphic to $T_4$.

Suppose $T_n$ is eigensharp. Let $M(a, b)$ be an eigensharp addressing matrix of $T_n$. By Theorem 5.1, the columns of $M(a, b)$ are orthogonal to any null vector of $D(T_n)$. Let $w$ be one of the columns of $M(a, b)$. We can construct a null vector $y$ of $D(T_n)$ by labelling the vertices corresponding to $T$ as described in Figure 2 and labelling the remaining vertices of $T_n$ with zeroes. In [10], it is described that the columns of an addressing matrix correspond to bicliques that partition the edgeset of $D(T_n)$. Every biclique decomposition
of $\mathcal{D}(T_n)$ induces a decomposition of $\mathcal{D}(T)$, an induced subgraph of $\mathcal{D}(T_n)$. Lemma 5.2 tells us that at least 4 bicliques are needed to decompose $\mathcal{D}(T)$. Therefore, there must be at least four columns of $M(a, b)$ whose 6 entries corresponding to $T$ have at least one $a$ and one $b$. The proof of Lemma 5.2 guarantees that each of these 4 vectors, restricted to the vertices of $T$, has at least three $a$ entries and three $b$ entries. Since $\mathcal{D}(T)$ is an induced subgraph of $\mathcal{D}(T_n)$, there are the same number of edges between the corresponding vertices in the two graphs. However, a contradiction occurs: the eigensharp addressing implies that there are at least 36 edges in $\mathcal{D}(T)$, but there are in fact 18. Therefore $T_n$ is not eigensharp.

For the triangular graph $T_5$ (the complement of the Petersen graph), the following six bicliques partition the edge set of $\mathcal{D}(T_5)$:

$$
\{12, 13, 14, 15\} \cup \{23, 24, 25, 34, 35, 45\}
\{12, 25\} \cup \{13, 14, 34, 35, 45\}
\{23, 24\} \cup \{15, 25, 34, 35, 45\}
\{13, 23, 35\} \cup \{14, 24, 45\}
\{15\} \cup \{12, 13, 14, 34\}
\{34\} \cup \{25, 35, 45\}.
$$

Thus, by Theorem 5.3, we know that $5 \leq N(T_5) \leq 6$.

6 Complete multipartite graphs $K_{2,\ldots,2}$

We note here that finding an optimal addressing of the complete multipartite graph $K_{2,\ldots,2}$ with $m$ color classes of size 2 is a highly non-trivial open problem. It is equivalent to finding the biclique partition number of the multigraph obtained from the complete graph $K_{2m}$ by adding a perfect matching. Motivated by questions in geometry involving nearly-neighborly families of tetrahedra, this problem was studied by Zaks [25] and Hoffman [16]. The best current results for $N(K_{2,\ldots,2}) = \text{bp}(\mathcal{D}(K_{2,\ldots,2}))$ are due to these authors (the lower bound is due to Hoffman [16] and the upper bound is due to Zaks [25]):

$$m + \lfloor 2\sqrt{m} \rfloor - 1 \leq N(K_{2,\ldots,2}) \leq \begin{cases} 
3m/2 - 1 & \text{if } m \text{ is even} \\
(3m - 1)/2 & \text{if } m \text{ is odd}.
\end{cases} \quad (14)$$

7 Open Problems

We conclude this paper with some open problems.

1. Must equality hold in (5) for all choices of $G_i$ ?

2. It is known that determining $\text{bp}(G)$ for a graph $G$ is an NP-hard problem (see [19]). This problem is NP-hard even when restricted to graphs $G$ with maximum degree $\Delta(G) \leq 3$ (see [7]). What is the complexity of finding $N(G)$ for general graphs $G$ ? How about graphs with $\Delta(G) > 3$, or other families of graphs ?

3. What is $N(T_n)$ for $n \geq 5$ ?

4. The triangular graph $T_n$ is a special case of a Johnson graph. For $n \geq m \geq 2$, the Johnson graph $J(n, m)$ has as its vertex set the $m$-subsets of $[n]$ with two $m$-subsets being adjacent if and only if their intersection has size $m - 1$. The Johnson graph is distance-regular and its eigenvalues were determined by Delsarte in his thesis [9] (see also [22, Theorem 30.1]). Atik and Panigrahi [3] computed the spectrum of the distance matrix $D(J(n, m))$:

$$(s \begin{array}{cc} 0 & -\frac{s}{n-1} \\ 1 & \binom{n}{m} - n \end{array})$$

(15)

where $s = \sum_{j=1}^{m} j \binom{m}{j} \binom{n-m}{j}$. Inequality (1) implies that $N(J(n, m)) \geq n - 1$. What is $N(J(n, m))$ ?
5. The Clebsch graph is the strongly regular graph with parameters $(16,5,0,2)$ that is obtained from the 5-dimensional cube by identifying antipodal vertices. The eigenvalue bound gives $N \geq 11$ and the connection with the 5-dimensional cube might be useful to find a good biclique decomposition of the distance multigraph of this graph.

6. What is $N(G)$ if $G$ is a random graph? Winkler’s work [24], Witsenhausen inequality 1 and the Wigner semicircle law imply that $n - 1 \geq N(G) \geq n/2 - c\sqrt{n}$ for some positive constant $c$. Recently, Chung and Peng [6] (see also [1, 2]) have shown for a random graph $G \in G_{n,p}$ with $p \leq 1/2$ and $p = \Omega(1)$, almost surely

$$n - o((\log n)^{3+\epsilon}) \leq h_{p}(G) \leq n - 2\log_{b} n \quad (16)$$

for $b = 1/p$ and any positive constant $\epsilon$. Here $G_{n,p}$ is the Erdős-Rényi random graph model.

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