QUANTITATIVE $C_p$ ESTIMATES FOR CALDERÓN-ZYGMUND OPERATORS

JAVIER CANTO

Abstract. We prove an appropriate quantitative reverse Hölder inequality for the $C_p$ class of weights from which we obtain as a limiting case the sharp reverse Hölder inequality for the $A_\infty$ class of weights [12, 13]. We use this result to provide a quantitative weighted norm inequality between Calderón-Zygmund operators and the Hardy-Littlewood maximal function, precisely

$$\|Tf\|_{L^p(w)} \leq c_{n,p,q} [w]_{C_p}(1 + \log^+ [w]_{C_p}) \|Mf\|_{L^p(w)},$$

for $w \in C_q$ and $q > p > 1$ improving Sawyer’s theorem [24].

Contents

1. Introduction and main results 1
2. Preliminaries 5
3. Proof of Theorem 2.13 7
4. Recovering $A_\infty$ from $C_p$ 13
5. A quantitative weighted norm inequality 15
References 21

1. INTRODUCTION AND MAIN RESULTS

One of the main principles of the classical Calderón-Zygmund theory is that one can control singular integral operators by suitable maximal operators. An example of this principle is a classical inequality by Coifman and Fefferman [7]. It states that for a Calderón-Zygmund operator $T$ and a weight $w \in A_\infty$, the following weighted inequality holds for $1 < p < \infty$,

$$\int_{\mathbb{R}^n} (T^* f(x))^p w(x) dx \leq c \int_{\mathbb{R}^n} (M f(x))^p w(x) dx.$$

(1.1)

Here $M$ denotes the Hardy-Littlewood maximal operator, and $T^*$ the maximal truncated singular integral operator. We refer to section 5 for the precise definitions.

2010 Mathematics Subject Classification. Primary: 42B20; Secondary: 42B25.

Key words and phrases. Weighted inequalities, $A_\infty$, $C_p$, Calderón-Zygmund operator, Hardy-Littlewood maximal operator.

The author is supported by the Basque Government through the program Programa Predoctoral de Formación de Personal Investigador No Doctor.
The constant $c$ in (1.1) depends on the exponent $p$, on the operator $T$, and on the weight $w$. More precisely, $c$ depends on the regularity of the kernel of $T$.

The classical proof of inequality (1.1) uses a good-$\lambda$ inequality between the operators $T^*$ and $M$. If the kernel of $T$ is not regular enough, there is in general no good-$\lambda$ inequality and even inequality (1.1) can be false, as is shown in [17].

There are ways of proving inequality (1.1) without using the good-$\lambda$ inequality. For example, the proof given in [1] uses a pointwise estimate involving the sharp maximal function. Another proof can be found in [9], where the main tool is an extrapolation result that allows to obtain estimates like (1.1) for any $A_\infty$ weight from the smaller class $A_1$ (see also [11]).

Inequality (1.1) is a very important inequality in the classical theory of Calderón-Zygmund operators, as it is used in the proof of many other weighted norm inequalities. The first, and probably most important consequence of (1.1) is the boundedness of $T^*$ in $L^p(w)$ for any weight $w \in A_p$, $1 < p < \infty$, namely

$$\int_{\mathbb{R}^n} (T^* f)^p w \leq c \int_{\mathbb{R}^n} |f|^p w.$$ 

This comes as a direct corollary of Muckenhoupt’s theorem [18].

Another consequence of inequality (1.1), though not as direct as the previous one, is the following inequality, obtained in [22]. For any weight $w$ it holds

$$\left\| T^* f \right\|_{L^p(w)} \leq c \| f \|_{L^p(M^{[p]+1} w)},$$

where $[p]$ denotes the integer part of $p$ and $M^k$ denotes the $k$-fold composition of $M$. This result is sharp since $[p] + 1$ cannot be replaced by $[p] + 1$. This is saying that inequality (1.1) encodes a lot of information. Very recently, this result was extended in [16] to the non-smooth case kernels, more precisely to the case case of rough singular operators $T_{\Omega}$ with $\Omega \in L^\infty(\mathbb{S}^{n-1})$, by proving inequality (1.1) for these operators. The proof of this result is quite different from the classical situation since there is no good-$\lambda$ estimate involving these operators and it is a consequence of a sparse domination result for $T_{\Omega}$ obtained in [8] combined with the $A_{\infty}$ extrapolation theorem mentioned above in [9].

Norm inequalities similar to (1.1) are true for other operators, for instance in [20] (fractional integrals) or [26] (square functions). Also, in the context of multilinear harmonic analysis one can find other examples, for example, it was shown in [15] an analogue for multilinear Calderón-Zygmund operators $T$, namely

$$\| T(f_1, \ldots, f_m) \|_{L^p(w)} \leq c \| M(f_1, \ldots, f_m) \|_{L^p(w)},$$

for $w \in A_{\infty}$ extending (1.1). We refer to [15] for the definition of the operator $M$. The proof for the multilinear setting is in the spirit of the proof of inequality (1.1) given in [1]. There are also inequalities for (1.1) for more singular operators like the case of commutators of Calderón-Zygmund operators with BMO functions, as was proved in [23]. In this case, the result is, for $w \in A_{\infty}$,

$$\|[b, T] f\|_{L^p(w)} \leq c \| M^2 f \|_{L^p(w)},$$
where \([b, T]f = bTf - T(bf)\) and \(M^2 = M \circ M\). The result is false for \(M\), because the commutator is not of weak type \((1,1)\) and it would then contradict the extrapolation result from [9].

All of the inequalities mentioned above are true for the class \(A_{\infty}\) of weights, but some of them are also true for a larger class of weights. In an attempt to characterize the class of weights for which inequality (1.1) is true, Muckenhoupt showed in [19] that \(A_{\infty}\) is not a necessary condition. In that article, he gave a necessary condition which he named the \(C_p\) condition. Later on, Sawyer [24] proved a sufficient condition, namely \(w \in C_{p+\eta}\) for some \(\eta > 0\) in the range \(p \in (1, \infty)\). It is still not clear if \(C_p\) is a sufficient condition.

Recently, Cejas, Li, P´erez and Rivera-R´ıos [6] extended Sawyer’s result to a wider class of operators than Calder´on-Zygmund operators, including some pseudo-differential operators and oscillatory integrals. They used a technique of [25], which is based on the sharp maximal function of Fefferman-Stein. This approach allowed them to obtain a better result, since they obtained (1.1) for the expected range of exponents \(p \in (0, \infty)\) and for weights \(w \in C_{\max(1,p)+\eta}\).

The results discussed above for \(C_p\) weights are purely qualitative, in the sense that none of them specify the dependence of the implicit constants on the weight \(w\). Probably, the first result of this sort was obtained in [14] where the following quantitative weighted inequality was obtained, for \(1 \leq q < \infty\),

\[
\|T^*f\|_{L^q(w)} \leq c_{T^*}[w]_{A_q} \|Mf\|_{L^q(w)}.
\]

This result was a central step to derive the main result from [14]. Here \([w]_{A_q}\) denotes the Muckenhoupt constant, defined for \(q > 1\) by

\[
[w]_{A_q} = \sup_Q \left( \frac{1}{Q} \right) \left( \int_Q w^{1-q} \right)^{-1},
\]

and the supremum is taken over all cubes with sides parallel to the coordinate axes. This was done by combining the improved version of the good-lambda inequality in [7] obtained by Buckley in [5] where an exponential decay was obtained instead of a linear decay. A similar result can be obtained for general the range \(p \in (0, \infty)\), combining these ideas with the sharp Reverse Hölder Inequality (RHI) for \(A_{\infty}\) weights from [13]. One can prove for \(0 < p < \infty\)

(1.2)

\[
\|T^*f\|_{L^p(w)} \leq c_{p,T}[w]_{A_{\infty}} \|Mf\|_{L^p(w)},
\]

where \([w]_{A_{\infty}}\) denotes the \(A_{\infty}\) constant

\[
[w]_{A_{\infty}} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q).
\]

Inequality (1.2), and many other quantitative weighted inequalities were obtained in [21], including inequalities concerning commutators, multilinear Calderón-Zygmund operators and vector valued extensions. Following the proof of (1.2), it is easy to obtain the same inequality for the weak \(A_{\infty}\) class, namely

\[
\|T^*f\|_{L^p(w)} \leq c_{p,T}[w]_{A_{\infty}^{\text{weak}}} \|Mf\|_{L^p(w)}.
\]

We refer to [2] for details on the weak \(A_{\infty}\) class.
The goal of this article is to improve these results by obtaining a similar quantitative result for weights in the class $C_p$. In order to do that, we first have to define an appropriate constant for this class, in the same way that $[w]_{A_\infty}$ is to $A_\infty$. This will allow us to quantify the weights in this class and obtain weighted inequalities with explicit dependence on the weight.

For a non-zero weight $w$, we define

$$[w]_{C_p} := \sup_Q \frac{1}{|Q|} \int_Q |M(\chi_Q)|^p w,$$

where the supremum is taken over all cubes $Q$ with sides parallel to the axes.

Once the $C_p$ constant is defined, we obtain a quantitative version of the RHI for $C_p$, which we believe to be sharp in the dependence on the constant. We combine arguments from [3] and [13] to prove the following.

**Theorem (Quantitative RHI for $C_p$ weights).** Let $1 < p < \infty$ and let $w$ be a weight such that $0 < [w]_{C_p} < \infty$. Then $w \in C_p$ and $w$ satisfies, for $\delta = \frac{1}{B_{n,p} \max([w]_{C_p},1)}$,

$$\left( \int_Q w^{1+\delta} \right)^{\frac{1}{\delta}} \leq \frac{4}{|Q|} \int_{\mathbb{R}^n} |M(\chi_Q)|^p w.$$

Taking advantage of the connection between the classes $A_\infty$ and $C_p$, we are able to obtain the sharp RHI for $A_\infty$ weights as a consequence of the RHI for $C_p$ weights. In this way, we know that the dependence of the $C_p$ constant is sharp.

Finally, we obtain a quantification on the weighted inequalities between the Hardy-Littlewood maximal operator and Calderón-Zygmund operators. See Section 5 for precise definitions.

**Theorem.** Let $T$ be a Calderón-Zygmund operator and let $q > p > 1$. Then, if $w \in C_q$ and $f \in C^\infty_c(\mathbb{R}^n)$, then the following estimate holds

$$\||T^*f||_{L^p(w)} \leq c_{n,T,p,q}([w]_{C_q} + 1) \log(e + [w]_{C_q}) \|Mf\|_{L^p(w)}.$$

It is not clear if the proof given in [6] would work to prove this theorem and we use instead the original scheme in [24] with some variants. In particular, the quantitative RHI for $C_p$ weights above and the use of the good-$\lambda$ inequality with exponential decay of Buckley [5] rather than the linear decay of Coifman-Fefferman [7] will play a main role in the argument

We note that the logarithm appears as a consequence of the non-local nature of the $C_p$ condition, but we conjecture that the correct dependence should be linear.

**Conjecture.** Let $T$ and $q, p$ as in the theorem. Then

$$\||T^*f||_{L^p(w)} \leq c_{n,T,p,q}([w]_{C_q} + 1) \|Mf\|_{L^p(w)}.$$

Since $\lim_{p \to \infty} [w]_{C_p} = [w]_{A_\infty}$ as shown in Section 4 we should get (1.2) as a limiting case when $q \to \infty$.

This article is organized as follows. In Section 2 we present the basic definitions and we state the quantitative RHI, which we prove in Section 3. In Section 4 we explain how to obtain the RHI for $A_\infty$ weights as a corollary of the $C_p$ RHI. In
Section 5 we give a quantified version of the Coifman-Fefferman weighted norm inequality for Calderón-Zygmund operators.

2. Preliminaries

We start by fixing the basic notation. By a weight we mean a non-negative locally integrable function in \( \mathbb{R}^n \). Weights will be denoted by the symbol \( w \). For a measurable set \( E \), \( \chi_E \) denotes the characteristic function of \( E \). \( M \) will denote the Hardy-Littlewood maximal operator

\[
Mf(x) := \sup_Q \frac{\chi_Q(x)}{|Q|} \int_Q |f|
\]

where the supremum is taken over all cubes with sides parallel to the coordinate axes. For a weight \( w \) and a measurable set \( E \), \( w(E) \) denotes \( \int_E w(x)dx \). Also we will be using the notation,\( \frac{1}{|E|} \int_E w = \frac{1}{|Q|} \int_E w \) when \( E \) is of finite measure.

We present the definition of \( C_p \) as given in [19] and [24].

Definition 2.1 (\( C_p \) weights). Let \( 1 < p < \infty \). We say that a weight \( w \) is of class \( C_p \), and we write \( w \in C_p \), if there exist \( C, \varepsilon > 0 \) such that for every cube \( Q \) and every measurable \( E \subset Q \) we have

\[
|w(E)| \leq C \left( \frac{|E|}{|Q|} \right)^\varepsilon \int_{\mathbb{R}^n} (M\chi_Q)^p w(x)dx.
\]

It is clear, and this is a key point, that the \( A_\infty \) class of weights is contained in \( C_p \) for any \( p \in (1, \infty) \).

We call the quantity \( \int_{\mathbb{R}^n} (M\chi_Q)^p w \) the \( C_p \)-tail of \( w \) at \( Q \). A weight has either finite \( C_p \)-tails at every cube or infinite \( C_p \)-tails at every cube.

Example 2.3 ([4], Chapter 7). Let \( w \in A_p \) and \( g \) a non-negative bounded convexely contoured function. Then \( gw \in C_p \). The weights in \( C_p \) are non-doubling, and they may even vanish in a set of positive measure.

The weights in this class also satisfy a non-local weak Reverse Hölder Inequality, as stated in the following proposition. We shall call this property Reverse Hölder Inequality (RHI) for \( C_p \)-weights, though it is not actually a proper RHI.

Proposition 2.4 (Reverse Hölder Inequality for \( C_p \) weights). A weight \( w \) belongs to the class \( C_p \) if and only if there exist \( C, \delta > 0 \) such that for every cube \( Q \)

\[
\left( \int_Q w^{1+\delta} \right)^{\frac{1}{1+\delta}} \leq C \left( \frac{1}{|Q|} \right) \int_{\mathbb{R}^n} (M\chi_Q)^p w.
\]

Moreover, we have that \( \delta \) in (2.5) and \( \varepsilon \) in (2.2) are equivalent up to a dimensional constant.

We present the sharp reverse Hölder inequality for \( A_\infty \) weights. Using the notation in [13], we define for a positive weight \( w \)

\[
[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q),
\]
where the supremum is taken over all cubes with sides parallel to the axes. It is known that \( w \in A_\infty \) if and only if \( [w]_{A_\infty} < \infty \).

**Theorem 2.6** (Sharp Reverse Hölder Inequality for \( A_\infty \) weights, [13]). Let \( w \in A_\infty \) and let \( Q \) be a cube. Then

\[
\left( \frac{\int_Q w^{1+\delta}}{\delta} \right)^{\frac{1}{1+\delta}} \leq 2 \int_Q w,
\]

for any \( \delta > 0 \) such that \( 0 < \delta \leq \frac{1}{2n+1} \).

When we compare Proposition 2.4 and Theorem 2.6, we notice that \( (M \chi_Q) w \) in (2.5) plays the role of \( w(Q) \) in (2.7). Keeping this similarity in mind, we define the \( C_p \) constant.

**Definition 2.8** (\( C_p \) constant). For an arbitrary non-zero weight \( w \), we define

\[
[w]_{C_p} := \sup_Q \frac{1}{\int_{\mathbb{R}^n} (M \chi_Q)^p w} \int_Q M(\chi_Q w),
\]

where the supremum is taken over all cubes \( Q \) with sides parallel to the axes.

Notice that if \( w \) is not identically zero, the quantity on the denominator is always strictly greater than zero.

**Remark 2.9.** A weight \( w \) has infinite \( C_p \)-tails if and only if \( [w]_{C_p} = 0 \). Indeed, if \( w \) has infinite \( C_p \)-tails then the denominator equals infinity and we have \( [w]_{C_p} = 0 \). Conversely, if \( [w]_{C_p} = 0 \) we have that for every cube \( Q \),

\[
\frac{1}{\int_{\mathbb{R}^n} (M \chi_Q)^p w} \int_Q M(\chi_Q w) = 0.
\]

This means that either \( \int_Q (M \chi_Q w) = 0 \) or \( \int_{\mathbb{R}^n} M(\chi_Q)^p w = \infty \) for every cube \( Q \). In the latter case, \( w \) has infinite \( C_p \)-tails. If \( \int_Q (M \chi_Q w) = 0 \) for every cube, then \( w \) must be zero almost everywhere.

By Proposition 2.4 we have that a weight \( w \) is in the class \( C_p \) if and only if \( 0 \leq [w]_{C_p} < \infty \).

**Remark 2.10.** Let \( w \) be a weight with \( [w]_{C_p} = 0 \). Then \( \int_{\mathbb{R}^n} (M f)^p w = \infty \) for every non-zero function \( f \). In particular, inequality (1.1) is true for any constant \( c > 0 \).

**Example 2.11.** For \( p > 1 \) and small \( \varepsilon \), for \( w_\varepsilon(x) = |x|^{n(p-1-\varepsilon)} \) we have \( [w_\varepsilon]_{C_p} \lesssim \varepsilon \).

This can be shown by direct computation.

This is the main difference between the \( A_\infty \) and \( C_p \) constants, since \( [w]_{A_\infty} \geq 1 \) for an arbitrary weight \( w \).

**Remark 2.12.** For any weight \( w \) we have the following relation between the different constants for \( q \leq p \), \( [w]_{C_q} \leq [w]_{C_p} \leq [w]_{A_\infty} \).

We now restate the quantitative RHI for \( C_p \) weights we mentioned on the introduction.
Theorem 2.13 (Quantitative RHI for $C_p$ weights). Let $1 < p < \infty$ and let $w$ be a weight such that $0 \leq [w]_{C_p} < \infty$. Then $w \in C_p$ and $w$ satisfies, for $\delta = \frac{1}{B\max\{[w]_{C_p},1\}}$, with

$$B = \frac{2^{2n+3n}(20)^n}{1 - 2^{-n(p-1)}},$$

$$\left(\int_Q w^{1+\delta}\right)^{1/\delta} \leq \frac{4}{|Q|} \int_{\mathbb{R}^n} (M\chi_Q)^p w,$$

(2.14)

Remark 2.15. Notice that $B$ depends on the dimension and on $p$. Moreover, we have $B \to \infty$ whenever $p$ tends to either $\infty$ or 1.

Remark 2.16. The quantification in terms of the parameters $\epsilon$ and $C$ in (2.2) is $C = 2$ and

$$\epsilon = \frac{1 - 2^{-n(p-1)}}{2^{np+3n}(20)^n} \min(1, [w]^{-1}_{C_p}).$$

In particular, we have that both $\epsilon$ and $\delta$ are smaller than one.

3. Proof of Theorem 2.13

We may assume that $w$ has finite $C_p$–tails, that is, $[w]_{C_p} > 0$. Indeed, if $[w]_{C_p} = 0$ then the right side of (2.14) equals infinity and the theorem is trivially true.

The proof follows a remark from [3], section 8.1, keeping track of the dependence on the constant of the weight combined with the proof given in [13] of the RHI for $A_\infty$ weights.

We now introduce a functional over cubes that serves as a discrete analogue for the $C_p$–tails. Define, for a cube $Q$

$$a_{C_p}(Q) := \sum_{k=0}^{\infty} 2^{-n(p-1)k} \int_{2^k Q} w.$$

We note that $\alpha = \sum_{k \geq 0} 2^{-n(p-1)k} = (2^{n(p-1)})^\gamma < \infty$ only depends on $n$ and $p$. In the following lemma we prove that the discrete and continuous $C_p$–tails are equivalent.

Lemma 3.2. Let $\beta = \sum_{l=0}^{\infty} 2^{-nl}$. Then, for every weight $w$ and every cube $Q$, we have

$$\frac{1}{\beta} a_{C_p}(Q) \leq \frac{1}{|Q|} \int_{\mathbb{R}^n} (M\chi_Q)^p w \leq \frac{4np}{\beta} a_{C_p}(Q).$$

As a corollary of this, we have that $a_{C_p}(Q) < \infty$ for every cube $Q$ whenever $w$ has finite $C_p$–tails.

Proof. Observe that $\beta = \sum_{l=0}^{\infty} 2^{-nl} = (2^{np})^\gamma$ and hence $\beta < 2$. Note that for $x \in 2^k Q \setminus 2^{k-1} Q$ we have $2^{-kn} \leq M\chi_Q(x) \leq 2^{-n(k-2)}$. Then

$$\frac{1}{|Q|} \int_{\mathbb{R}^n} (M\chi_Q)^p w = \int_Q w + \sum_{k=1}^{\infty} \frac{1}{|Q|} \int_{2^k Q \setminus 2^{k-1} Q} (M\chi_Q)^p w,$$
so we actually have
\[
\int_Q w + \sum_{k=1}^{\infty} \frac{2^{-npk}}{|Q|} w(2^k Q \setminus 2^{k-1} Q) \leq \frac{1}{|Q|} \int_{\mathbb{R}^n} (M\chi_Q)^p w \\
\leq \int_Q w + \sum_{k=1}^{\infty} \frac{2^{-npk}}{|Q|} w(2^k Q \setminus 2^{k-1} Q) \\
\leq 4^p \left( \int_Q w + \sum_{k=1}^{\infty} \frac{2^{-npk}}{|Q|} w(2^k Q \setminus 2^{k-1} Q) \right)
\]

Now we rewrite (3.1) in the following way
\[
\sum_{k=0}^{\infty} 2^{-(p-1)k} \int_{2^k Q} w = \int_Q w + \sum_{k=1}^{\infty} \frac{2^{-npk}}{|Q|} \left( \int_Q w + \sum_{j=1}^{k} \int_{2^{j-1} Q} w \right) \\
= \beta \int_Q w + \frac{1}{|Q|} \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} 2^{-npj} \int_{2^{j-1} Q} w \\
= \beta \left( \int_Q w + \frac{1}{|Q|} \sum_{j=1}^{\infty} 2^{-npj} \int_{2^{j-1} Q} w \right).
\]

This finishes the proof of (3.3). □

**Proposition 3.4.** Let \( w \) be a weight and \( p > 1 \). Suppose that there exists a constant \( 0 < \gamma < \infty \) such that for every cube \( Q \)
\[
\int_Q M(\chi_Q w) \leq \gamma a_{C_p}(Q) < \infty.
\]
Then there exists \( 0 < \delta \leq \frac{1}{A_{\max(\gamma,1)}} \), with
\[
A = 20^p \frac{2^{1+3n}}{1-2^{-mp-1}},
\]
such that for every cube \( Q \),
\[
\int_Q M(\chi_Q w)^{1+\delta} \leq 2^{1+3n(2p+3)} \gamma a_{C_p}(Q)^{1+\delta}.
\]

Note that the infimum of the constants \( \gamma \) such that (3.5) holds is equivalent to the \( C_p \) constant of \( w \), because of Lemma 3.2. In this case we will have \( 0 < [w]_{C_p} < \infty \).

**Proof.** Fix a cube \( Q = Q(x_0, R) \), that is, the cube centred at the point \( x_0 \) and with side length \( 2R \) (\( Q(x, R) \) is just a ball with the \( l^\infty \) distance in \( \mathbb{R}^n \)). The proof will be carried out following some steps.

**Step 1.** Let \( r, \rho > 0 \) and \( l \in \mathbb{Z} \) be numbers that satisfy \( R \leq r < \rho \leq 2R \) and \( 2^{(\rho - r)} = R \). This in particular implies \( l \geq 0 \).

We define a new maximal operator
\[
\tilde{M}v(x) := \sup_{k \in \mathbb{Z}} \int_{Q(x, 2^k(\rho - r))} |v|.
\]
We have the following pointwise bounds between the different maximal functions
\[
\bar{M}v \leq Mv \leq \kappa \bar{M}v,
\]
where \(\kappa\) does not depend on \(\rho - r\). In particular, we can choose \(\kappa = 4^n\). For \(t \geq 0\) and a function \(F\) we define \(F_t = \min(F, t)\). Now fix \(m > 0\) with the intention of letting \(m \to \infty\) in the end. Call \(Q_r = Q(x_0, r)\) and \(Q_\rho = Q(x_0, \rho)\).

We then have
\[
\int_{Q_r} (M(\chi_{Q_r}w))_m^{1+\delta} \leq \int_{Q_r} (\bar{M}(\chi_{Q_r}w))_m^{1+\delta} \leq \int_{Q_r} (\bar{M}(\chi_{Q_r}w))_m^{1+\delta} \leq \int_{Q_r} (\bar{M}(\chi_{Q_r}w))_m^{1+\delta} \leq \kappa^{1+\delta} \int_0^m \lambda^{δ-1} u(Q_r \cap [u > \lambda]) d\lambda,
\]
where \(u = \bar{M}(\chi_{Q_r}w)\). To state it in a separate line, we have
\[
(3.6) \quad \int_{Q_r} (M(\chi_{Q_r}w))_m^{1+\delta} \leq \kappa^{1+\delta} \int_0^m \lambda^{δ-1} u(Q_r \cap [u > \lambda]) d\lambda.
\]

**Step 2.** Now we pick \(\lambda_0 := 2^{n(l+1)}ac_p(2Q)\) (which is finite by hypothesis). It is easy to see that for \(x \in Q_r\) and \(k \geq 0\), by the choice of \(\lambda_0\) we have
\[
(3.7) \quad \int_{Q(x, 2^k(\rho-r))} \chi_{Q_r}w \leq \lambda_0.
\]
Indeed, we have that \(Q_\rho \subset 2Q\), so we can make
\[
\int_{Q(x, 2^k(\rho-r))} \chi_{Q_r}w \leq \int_{Q(x, 2^k(\rho-r))} \chi_{2Q}w
= \frac{|2Q|}{|Q(x, 2^k(\rho-r))|} \int_{Q} w
\leq 2^{n(l+1-k)}ac_p(2Q) \leq 2^{n(l+1)}ac_p(2Q).
\]

This completes the proof of (3.7) when \(x \in Q_r\) and \(k \geq 0\).

Let \(\lambda > \lambda_0\) and \(x \in Q_r \cap [u > \lambda]\). As \(u(x) = \bar{M}(\chi_{Q_r}w)(x) > \lambda > \lambda_0\), (3.7) and the fact \(Q(x, 2^k(\rho-r)) \subset Q_\rho\) when \(k < 0\) imply
\[
u(x) = \sup_{k < 0} \int_{Q(x, 2^k(\rho-r))} \chi_{Q_r}w = \sup_{k < 0} \int_{Q(x, 2^k(\rho-r))} w.
\]
For such an \(x\), let \(k_x = \max(k : \int_{Q(x, 2^k(\rho-r))} w > \lambda)\). Trivially, we have
\[
Q_r \cap [u > \lambda] \subset \bigcup_{x \in Q_r \cap [u > \lambda]} Q(x, \frac{1}{2}2^{k_x(\rho-r)}).
\]
We use the Vitali covering lemma for infinite sets and choose a countable collection of \(x_i \in Q_r \cap [u > \lambda]\) so that the family of cubes \(Q_i = Q(x_i, 2^{k_x}(\rho-r))\) satisfy the following properties:

- \(Q_r \cap [u > \lambda] \subset \bigcup_i Q_i\),
- the cubes \(\frac{1}{2}Q_i\) are pairwise disjoint,
\[ \frac{\int_{Q_{i}} w > \lambda}{\int_{Q_{i}} w < \lambda}, \text{ for any } k \geq 1 \]

\[ \frac{\int_{Q_{i}} w \leq \lambda}{\int_{Q_{i}} w}, \quad Q_{i} \subseteq Q_{p}. \]

We make the following claim. If we denote \( Q_{i}' = 2Q_{i} \) then for all \( x \in Q_{i} \cap Q_{r} \),
\[ u(x) \leq 2^n M(\chi_{Q_{i}'} w)(x). \]

Indeed, fix \( x \in Q_{i} \cap Q_{r} \) and \( k < 0 \). If \( k \geq k_{x} \) then by the stopping time we get
\[ \int_{Q_{i}(x,2^{(\rho-r)})} w \leq \frac{|Q(x,2^{k+1}(\rho-r))|}{|Q(x,2^{k}(\rho-r))|} \int_{Q_{i}(x,2^{k+1}(\rho-r))} w \leq 2^n \lambda \leq 2^n \int_{Q_{i}} w \leq 2^n M(\chi_{Q_{i}'} w)(x). \]

In the other case, namely \( k < k_{x} \) we have \( Q(x,2^{k}(\rho-r)) \subseteq Q_{i}' \cap Q_{r} \), and hence
\[ \int_{Q(x,2^{(\rho-r)})} w \leq M(\chi_{Q_{i}'} w)(x), \]
and thus the claim is proved.

**Step 3.** We use now this claim together with the stopping time and the hypothesis \((3.5)\) to see
\[ u(Q_{r} \cap \{ u > \lambda \}) \leq \sum_{i} u(Q_{i} \cap Q_{r}) \leq \sum_{i} \int_{Q_{i} \cap Q_{r}} u \leq 2^n \sum_{i} \int_{Q_{i} \cap Q_{r}} M(\chi_{Q_{i}'} w) \leq 2^n \sum_{i} |Q_{i}'| M(\chi_{Q_{i}'} w) \leq 2^n \gamma \sum_{i} |a_{C_{r}}(Q_{i}')| \]

But, using the properties of \( Q_{i} \) we get
\[ a_{C_{r}}(Q_{i}') = \sum_{k=0}^{\infty} 2^{-n(k-1)} \int_{2^{k+1} Q_{i}} w \leq \lambda \alpha, \]
so we have
\[ u(Q_{r} \cap \{ u > \lambda \}) \leq 2^n \gamma \sum_{i} |Q_{i}'| \alpha \lambda \leq (20)^n \gamma \alpha \sum_{i} |Q_{i}| \lambda, \]
where in the last inequality we have used that \( \frac{1}{2} \) \( Q_{i} \) are disjoint. Since each one of the cubes \( Q_{i} \subseteq Q_{p} \) and \( \lambda < \int_{Q_{r}} w \) we have \( \cup_{i} Q_{i} \subseteq Q_{p} \cap \{ M(\chi_{Q_{p}} w) > \lambda \} \) so we have obtained for \( \lambda > \lambda_{0} \)
\[ u(Q_{r} \cap \{ u > \lambda \}) \leq (20)^n \gamma \alpha \lambda |Q_{p} \cap \{ M(\chi_{Q_{p}} w) > \lambda \}|. \]

Plugging everything on what we had in \((3.6)\) we have
\[ \int_{Q_{p}} (M(\chi_{Q_{p}}))^{1+\delta}_{m} \leq \kappa^{1+\delta}_{0} u(Q_{r}) + \kappa^{\delta+1}(20)^n \gamma \alpha \delta \int_{\lambda_{0}}^{\lambda} \lambda^{\delta} |Q_{p} \cap \{ M(\chi_{Q_{p}} w) > \lambda \}| d\lambda. \]

**Step 4.** We define
\[ \varphi(t) = \int_{Q_{p}} (M(\chi_{Q_{p}} w))^{1+\delta}_{m}, \quad t > 0. \]
Observe that $\varphi(t) < \infty$ for any $t > 0$. We claim that,

$$\varphi(r) \leq c_1 \gamma(Q) 2^{n\delta} \left( a_{C_p}(Q) \right)^{1+\delta} + \delta \kappa^{1+1}(20)^{\gamma} a \varphi(\rho). \quad (3.8)$$

Indeed, combining what we obtained before in the following way:

$$\varphi(r) \leq c_1 \gamma(Q) 2^{n\delta} \left( a_{C_p}(Q) \right)^{1+\delta} + \kappa^{1+1}(20)^{\gamma} a \varphi(\rho),$$

where $c_1 = 2^{n(p+1)\delta + 1}$, and where we have used

$$u(Q_r) = \int_{Q_r} M(\chi_{Q_r} w) \leq |2Q| \int_{2Q} M(\chi_{2Q} w) \leq 2^n |Q| \gamma a_{C_p}(2Q) \leq 2^{n+1} |Q| \gamma a_{C_p}(Q),$$

since

$$a_{C_p}(2Q) \leq 2^{n(p+1)} a_{C_p}(Q).$$

This yields the claim.

**Step 5.** Now we present an iteration scheme starting from claim (3.8). Remember that $l \geq 0$ was an integer such that $2^l(\rho - r) = R$. Set

$$t_0 = R,$$

$$t_{i+1} = t_i + 2^{-(i+1)} R = \sum_{j=0}^{i+1} 2^{-j} R, \quad i \geq 0.$$ 

Clearly, $t_i \to 2R$ as $i \to \infty$. This way, $2^{l+1} (t_{i+1} - t_i) = R$ and we can use them as $\rho = t_{i+1}$, $t_i = r$, and $l = i + 1$ in (3.8).

In other words, we have the estimate for $\varphi(t_i)$ in terms of $\varphi(t_{i+1})$:

$$\varphi(t_i) \leq c_2 2^{n\delta} + c_3 \varphi(t_{i+1}),$$

where $c_2 = c_1 2^{n\delta} \gamma(Q) (a_{C_p}(Q))^{1+\delta}$, $c_3 = \kappa^{1+1}(20)^{\gamma} \delta$. So, iterating this last inequality $i_0$ times we get

$$\varphi(R) = \varphi(t_0) \leq c_2 \sum_{j=0}^{i_0-1} (c_3 2^{n\delta})^j + c_3^{i_0} \varphi(t_{i_0}) \leq c_2 \sum_{j=0}^{i_0-1} (c_3 2^{n\delta})^j + (c_3)^{i_0} \varphi(2R)$$

We have to choose $\delta > 0$ so that we have the relation

$$c_3 2^{n\delta} = 2^{n+1} \kappa^{1+1} \gamma a \delta < 1/2. \quad (3.9)$$

We may suppose $\delta < 1$. Once we have (3.9), we can take the limit $i_0 \to \infty$ and the sum is bounded by 2 and the second term goes to zero since $\varphi(2R) < \infty$. Hence

$$\varphi(R) \leq 2c_2 = 2^{1+n\delta+n(\delta+1)(p+1)} \gamma(Q) (a_{C_p}(Q))^{1+\delta} < 2^{1+n(2p+3)} \gamma(Q) (a_{C_p}(Q))^{1+\delta},$$

and then

$$\frac{1}{|Q|} \int_Q M(\chi_{Q} w)_{m}^{1+\delta} \leq 2^{1+n(2p+3)} \gamma (a_{C_p}(Q))^{1+\delta}.$$

Now, letting $m \to \infty$ and using the Fatou lemma we can conclude the proof.
To finish the proof, we make the choice of $\delta$ as follows. Coming back to (3.9) we see that, since we have $\delta$ in the exponent and $\gamma$ can be arbitrarily small, we have to choose $\delta = \frac{1}{A \max(1, \gamma)}$ with

$$A = 2k^2(20)^n 2^n \alpha = (20)^n \frac{2^{1+3n}}{1 - 2^{-m(p-1)}}. \quad \Box$$

We are ready to finally prove the theorem.

**Proof of Theorem 2.13.** Fix a cube $Q$. Let $M_{d,Q}$ denote the maximal operator with respect to the dyadic children of $Q$, that is

$$M_{d,Q}v(x) = \sup_{R \in D(Q)} \frac{1}{|R|} \int_{R} |v|, \quad x \in Q.$$

We argue as in [13], Theorem 2.3. By the Lebesgue differentiation theorem,

$$\int_{Q} w^{1+\delta} \leq \int_{Q} (M_{d,Q}w)^{\delta} w.$$

Call now $\Omega_{\lambda} = \{ x \in Q : M_{d,Q}w(x) > \lambda \}$. For $\lambda \geq w_{Q}$ we make the Calderón-Zygmund decomposition of $w$ at height $\lambda$ to obtain $\Omega_{\lambda} = \cup_{j} Q_{j}$ with $Q_{j}$ pairwise disjoint and

$$\lambda < \frac{1}{|Q_{j}|} \int_{Q_{j}} w \leq 2^{n} \lambda.$$

Multiplying by $|Q_{j}|$ and summing on $j$ this inequality chain becomes

$$\lambda |\Omega_{\lambda}| \leq w(\Omega_{\lambda}) \leq 2^{n} \lambda |\Omega_{\lambda}|.$$

Then we have

$$\int_{Q} (M_{d,Q}w)^{\delta} w = \frac{1}{|Q|} \int_{0}^{\infty} \delta \lambda^{\delta-1} w(\Omega_{\lambda}) d\lambda$$

$$\leq w_{Q}^{\delta+1} + \frac{1}{|Q|} \int_{w_{Q}}^{\infty} \delta \lambda^{\delta-1} w(\Omega_{\lambda}) d\lambda$$

$$\leq w_{Q}^{\delta+1} + \frac{2^{n}}{|Q|} \int_{w_{Q}}^{\infty} \lambda^{\delta} |\Omega_{\lambda}| d\lambda$$

$$\leq w_{Q}^{\delta+1} + 2^{n} \frac{\delta}{\delta + 1} |Q| \int_{Q} (M_{d,Q}w)^{1+\delta}.$$

Now we apply Proposition 3.4. We have $[w]_{c_{p}} \leq \beta \gamma \leq 4^{np}[w]_{c_{p}}$, so we need $\delta \leq \beta/A (\max(1, [w]_{c_{p}})$, with $\beta$ as in Lemma 3.2. So we get

$$\int_{Q} (M_{d,Q}w)^{\delta} w \leq (1 + 2^{1+n(2p+4)} \frac{\delta}{\delta + 1} \gamma) (w_{c_{p}}(Q))^{1+\delta}$$

$$\leq (1 + 2^{1+n(2p+4)} \frac{\delta}{\delta + 1} [w]_{c_{p}} \frac{4^{np}}{\beta} \left( \frac{\beta}{|Q|} \int_{Q} (M_{X,Q}w)^{p} \right)^{1+\delta},$$

$$\leq (1 + 2^{1+n(2p+4)} \frac{\delta}{\delta + 1} [w]_{c_{p}} \frac{4^{np}}{\beta} \left( \frac{\beta}{|Q|} \int_{Q} (M_{X,Q}w)^{p} \right)^{1+\delta},$$
where we have used Lemma 3.2. Now, since we have $2^{4np}/\beta$ multiplying $\delta$, we have to change the choice of $\delta$ slightly and make

$$\delta \leq \frac{2^{4np}}{\beta} \frac{\beta}{A \max(1, [w]_{C_p})} = \frac{1}{B \max(1, [w]_{C_p})}.$$ 

This finishes the proof of the theorem. \(\square\)

4. Recovering $A_\infty$ from $C_p$

For a cube $Q$, it is clear that $M_{\chi_Q}$ equals 1 on the cube and is smaller than 1 outside the cube. Therefore $(M_{\chi_Q})^p$ converges to $\chi_Q$ a.e. when $p \to \infty$. Moreover, for a weight $w \in C_{p_0}$, by the dominated convergence theorem we have

$$\lim_{p \to \infty} \int_{\mathbb{R}^n} (M_{\chi_Q})^p w = w(Q).$$

For any weight $w \in A_\infty$, we have by the definition of the constant $[w]_{A_\infty}$ that for any cube $Q$

$$\int_Q M(w_{\chi_Q}) \leq [w]_{A_\infty} w(Q) \leq [w]_{A_\infty} a_{C_p}(Q),$$

where $a_{C_p}(Q) = \sum_{k \geq 0} 2^{-m(p-1)k} \int_{2^k Q} w$ is the discrete $C_p$-tail introduced in the previous section.

If we modify slightly the proof of Proposition 3.4 and Theorem 2.13 and add some extra hypothesis, we can recover the RHI for $A_\infty$ weights. We explain how to do this in this section.

Fix a number $s > 1$. This will be the dilation parameter, which was $s = 2$ in the previous section. We plan on letting $t$ tend to one in the end. We introduce the corresponding discrete $C_p$-tail with respect to $t$,

$$a_{C_p,s}(Q) = \sum_{k \geq 0} s^{-m(p-1)k} \int_{s^k Q} w.$$ 

Note that for any weight $w \in C_{p_0}$ we have $\lim_{p \to \infty} a_{C_p,s}(Q) = w_Q$ for any $s > 1$. Also, for a fixed $s > 1$ we introduce the corresponding discrete $C_p$ constant

$$[w]_{C_p,s} := \sup_Q \frac{\int_Q M(\chi_{Qw})}{a_{C_p,s}(Q)}.$$ 

Remark 4.1. For a weight $w \in A_\infty$ and any $s > 1$ we have $\lim_{p \to \infty} [w]_{C_p,s} \leq [w]_{A_\infty}$.

Theorem 4.2. Fix $2 \geq s > 1$ and $1 < p < \infty$. For a weight $w \in C_p$ and $\delta = \frac{1}{A_{p,\max(1, [w]_{C_p,s})}}$ and every cube $Q$, with

$$A_{p,p} = \frac{s^n 2^{1+5n}}{1 - s^{-m(p-1)}},$$

we have

$$\left(\frac{1}{|Q|} \int_Q w^{1+\theta}\right)^{\frac{1}{\theta}} \leq (2^n + 1) a_{C_p,s}(sQ).$$
Before we prove this theorem, we give a proof of Theorem 2.6 as a corollary. Let \( w \in A_{\infty} \). By Remark 4.1, we can let \( p \to \infty \) in equation (4.3) and we obtain
\[
(4.4) \quad \left( \frac{1}{|Q|} \int_Q w^{1+\delta_{\infty}} \right)^{\frac{1}{1+\delta_{\infty}}} \leq (2^n + 1) w_Q,
\]
where
\[
\delta_{\infty} = \lim_{p \to \infty} \frac{1 - s^{-n(p-1)}}{c_n \max(1, [w]_{C^{p,s}})} = \frac{1}{c_n [w]_{A_{\infty}}}.
\]
Now we let \( s \to 1 \) in (4.4) and obtain
\[
(4.5) \quad \left( \frac{1}{|Q|} \int_Q w^{1+\delta_{\infty}} \right)^{\frac{1}{1+\delta_{\infty}}} \leq (2^n + 1) w_Q,
\]
which is in fact the reverse Hölder inequality for \( A_{\infty} \) weights.

**Remark 4.5.** The dimensional constants are bigger from those in Theorem 2.6, but the dependence on the weight is essentially the same. Because of this, we obtain that the dependence on \( w \) in Theorem 2.13 is sharp.

**Proof of Theorem 4.2.** We repeat the first three steps of the proof of Proposition 3.4, with the following modifications. This time, \( r, \rho, l \) will satisfy \( s'(\rho - r) = R \) and \( R \leq r < \rho \leq R \). Also, now we will use the maximal operator \( Mv(x) = \sup_{k \in \mathbb{Z}} \int_{Q(k,s^{(\rho-r)})} u \), and some other trivial changes. For the fourth step, we leave \( a_{C^{p,s},s}(sQ) \) in the equation, so we get
\[
\varphi(r) \leq 2^{\gamma_0} \rho |Q|^{\gamma_0} (a_{C^{p,s},s}(sQ))^{1+\delta} + (k^{1+\delta}(5\gamma)\nu \alpha_s) \delta \varphi(r),
\]
where \( \alpha_s = \sum_{k \geq 0} \gamma_0 \rho^{s-k(p-1)} = (1 - \rho^{-n(p-1)})^{-1} \). We make a similar iteration scheme, namely \( t_0 = R \) and \( t_{i+1} = t_i + s^{-(i+1)}R \leq sR \). Now the condition for \( \delta \) translates to \( \delta \leq A_{s,p} \nu \max(1, \gamma) \) where
\[
A_{s,p} = \frac{5\gamma_0}{1 - s^{-n(p-1)}}.
\]
The main difference is that now we get
\[
\frac{1}{|Q|} \int_Q (M(x)w)^{1+\delta} \leq 2^{1+5\gamma} (a_{C^{p,s},s}(sQ))^{1+\delta},
\]
where the right part stays bounded whenever \( p \to \infty \). Now we use Fatou lemma and make \( m \to \infty \) to get
\[
(4.6) \quad \frac{1}{|Q|} \int_Q M(x)w)^{1+\delta} \leq 2^{1+5\gamma} (a_{C^{p,s},s}(sQ))^{1+\delta}.
\]

Now we make the argument in the proof of Theorem 2.13 and combine it with (4.6). We get,
\[
\int_Q w^{1+\delta} \leq (\rho w)^{1+\delta} + 2^n \frac{\delta}{1+\delta} \frac{1}{|Q|} \int_Q (Mw)^{1+\delta}
\]
\[
\leq (\rho w)^{1+\delta} + 2^n \frac{\delta}{1+\delta} 2^{1+5\gamma} (a_{C^{p,s},s}(sQ))^{1+\delta}
\]
\[
\leq (2^n + \delta 2^{1+6\gamma} (a_{C^{p,s},s}(sQ))^{1+\delta}
\]
\[
\leq (2^n + 1) \left( a_{C_p'}(xQ) \right)^{1+\delta},
\]
whenever \( \delta \leq \frac{1}{2^{n+1}} \), which is true by the choice of \( \delta \). This finishes the proof. \( \square \)

5. A quantitative weighted norm inequality

We define now the Calderón-Zygmund operators in a similar way as in [7]. We will need a kernel \( K \) defined away from the diagonal \( x = y \) of \( (\mathbb{R}^n)^2 \) that satisfies the size condition

\[
|K(x, y)| \leq \frac{A}{|x-y|^n}
\]

for some \( A > 0 \) and every \( x \neq y \). Furthermore, we require the following regularity conditions for some \( \varepsilon > 0 \)

\[
|K(x, y) - K(x', y)| \leq A \frac{|x-x'|^\varepsilon}{|x-y|^{n+\varepsilon}}
\]

whenever \( 2|x-x'| \leq |x-y| \), and the symmetric condition

\[
|K(x, y) - K(x, y')| \leq A \frac{|y-y'|^\varepsilon}{|x-y|^{n+\varepsilon}}
\]

whenever \( 2|y-y'| \leq |x-y| \).

A Calderón-Zygmund operator associated to a kernel \( K \) satisfying the above conditions is a linear operator \( T : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n) \) that satisfies

\[
T f(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy,
\]

for \( f \in C_c^\infty(\mathbb{R}^n) \) and \( x \notin \text{supp}(f) \). Additionally, we will require that \( T \) is bounded in \( L^2 \).

Now we define the maximal truncated singular integral operator \( T^* \) as follows

\[
T^* f(x) = \sup_{\varepsilon > 0} \left| \int_{|y| > \varepsilon} K(x, y) f(y) dy \right|.
\]

We state the main theorem, which is a quantification of Theorem B from [24] and Theorem 16 from [6].

**Theorem 5.1.** Fix \( q > p > 1 \). For all Calderón-Zygmund operator \( T \), all bounded \( f \) with compact support and all weights \( w \in C_q \) we have

\[
\|T^* f\|_{L^q(w)} \leq c_{n,T} \left( q + \frac{qp^2}{q-p} \right) \Phi(\max(1, |w|_{C_p}, 1)) \|Mf\|_{L^p(w)},
\]

where \( \Phi(t) = t \log(e + t) \).

The rest of the section is dedicated to the proof of the theorem. We begin with a few lemmas, which correspond to Lemmas 2-4 in [24]. We include most of the details concerning the quantification of the weight for the sake of completion.

**Lemma 5.2.** Let \( w \in C_q \). Fix \( R \geq 2 \) and \( \delta > 0 \). Then for every cube \( Q \) and any collection of pairwise disjoint cubes \( Q_j \subset Q \) we have

\[
\sum_j (M\chi_{Q_j}(x))^p w(x) dx \leq \frac{1}{ae} \log \frac{c R^{nq}}{e\delta} w(RQ) + \delta \int_{\mathbb{R}^n} M\chi_Q(x)^q w(x) dx,
\]

where
where \(a, c\) are dimensional constants and \(\varepsilon\) is the parameter for \(w\) in (2.2). Hence, we have

\[
(5.4) \quad \int \sum_j (M \chi_{Q_j}(x)) q w(x) dx \leq c a^{q} \frac{1}{\varepsilon} \int_\mathbb{R}^n (M \chi_{Q}(x)) q w(x) dx.
\]

**Proof.** For \(\lambda > 0\), we will call

\[
E_\lambda = \{x \in \mathbb{R}^n : \sum_j M \chi_{Q_j}(x) > \lambda\}.
\]

Since the cubes are pairwise disjoint, we have \(\sum_j \chi_{Q_j} \in L^\infty\). Then by the exponential inequality from [10] we have \(|E_\lambda| \leq c_n e^{-a \lambda |RQ|}\), where \(c_n\) and \(a\) are positive dimensional constants. Then, applying the \(C_q\) condition (2.2) we get

\[
\[w(E_\lambda) \leq 2 \left( \frac{|E_\lambda|}{|RQ|} \right)^{\varepsilon} \int_\mathbb{R}^n (M \chi_{RQ}(x)) q w(x) dx \]
\]

Now we compute

\[
\int_{RQ} \sum_j (M \chi_{Q_j}(x)) q w(x) dx = \int_0^\infty w(E_\lambda) dt = \lambda w(E_\lambda) + \int_\lambda^\infty w(E_\lambda) dt
\]

\[
\leq \lambda w(RQ) + c_n R^{qn} \frac{1}{a e^{\varepsilon \lambda}} \int_\mathbb{R}^n (M \chi_{Q}(x)) q w(x) dx.
\]

We can choose \(\lambda\) big enough so that

\[
c_n R^{qn} \frac{1}{a e^{\varepsilon \lambda}} \leq \delta,
\]

and we get (5.3). In order to get (5.4), choose \(R = 2, \delta = \frac{1}{e}\) and use \(\sum_j M \chi_{Q_j} \leq 2^{aq} M \chi_{Q}\) almost everywhere outside of \(2Q\). \(\square\)

**Lemma 5.5 (Whitney covering lemma).** Given \(R \geq 1\), there is \(C = C(n, R)\) such that if \(\Omega\) is an open subset in \(\mathbb{R}^n\), then \(\Omega = \bigcup_j Q_j\) where the \(Q_j\) are disjoint cubes satisfying

\[
5R \leq \frac{\text{dist}(Q_j, \mathbb{R}^n \setminus \Omega)}{\text{diam} Q_j} \leq 15R,
\]

\[
\sum_j \chi_{RQ_j} \leq C \chi_{Q}.
\]

We now define an auxiliary function considered in [24]. This operator will be used to intuitively represent the integral of the function \(h\) to the power \(p\) after we apply the \(C_q\) condition.

**Definition 5.6.** Let \(h\) be a positive lower-semicontinuous function on \(\mathbb{R}^n\). Let \(\Omega_k = \{h(x) > 2^k\} = \bigcup_j Q_j^k\), as in the Whitney covering lemma. We define the function

\[
(5.7) \quad M_{p,q} h(x) = \sum_{k,j} 2^{kp} (M \chi_{Q_j^k}(x)) q.
\]

We need lower-semicontinuity in this definition to ensure that we can apply Whitney’s decomposition theorem. In the practice, we will apply this operator to \(M f\) and to \(T^* f\), which are always lower-semicontinuous.
Lemma 5.8. For a bounded, compactly supported function $f$ and a weight $w \in C_q$ with $q > p$, we have

$$
\int_{\mathbb{R}^n} (M_{p,q} f(x))^p w(x) dx \leq \left( c_n 2^{\frac{np}{n-p}} \frac{1}{e} \log \frac{1}{\epsilon} \right) \int_{\mathbb{R}^n} (M f(x))^p w(x) dx,
$$

where $M_{p,q}$ denotes the Marcinkiewicz integral operator as defined in (5.7).

Proof. Let $\Omega_k = \{ M f > 2^k \} = \cup_i Q_i^k$ as in the Whitney decomposition lemma. Let $N$ be a positive integer to be chosen later and fix a cube $Q_i^{k-N}$ from the $k-N$ generation. We have, as in [24],

$$
|\Omega_k \cap 5Q_i^{k-N}| \leq C 2^{-N} |Q_i^{k-N}|,
$$

where $C$ depends only on the dimension $n$.

Now let $S(k) = 2^{kp} \sum_j \int_{\mathbb{R}^n} (M_{X_Q} f)^q w$ and $S(k; i, N) = 2^{kp} \sum_j \int_{\mathbb{R}^n} (M_{X_Q} f)^q w$, where the last sum is taken over those $j$ for which $Q_j^k \cap Q_i^{k-N} \neq \emptyset$. But because of the Whitney decomposition, $Q_j^k \cap Q_i^{k-N} \neq \emptyset$ implies $Q_j^k \subset 5Q_i^{k-N}$ for large $N$, so we have

$$
S(k; N, i) \leq \int_{\mathbb{R}^n} 2^{kp} \sum_{j: Q_j^k \subset 5Q_i^{k-N}} (M_{X_Q} f)^q w
$$

$$
= \int_{10Q_i^{k-N}} + \int_{(10Q_i^{k-N})^c} = I + II \quad \text{for large } N.
$$

Now, by (5.3), for any $\eta > 0$, which will be chosen chosen later, and for $R = 10$ we get

$$
I \leq 2^{kp} \frac{1}{ae} \frac{1}{\eta \epsilon} w(10Q_i^{k-N}) + \eta 2^{kp} \int_{\mathbb{R}^n} (M_{X_{Q_i^{k-N}}} f)^q w.
$$

Standard estimates for the maximal function of characteristics of cubes show that if $X_{Q_i^{k-N}}$ is the centre of the cube $Q_i^{k-N}$ then

$$
II \leq c_n 2^{kp} \int_{10Q_i^{k-N}} \frac{\sum |Q_j|^q}{|x - X_{Q_i^{k-N}}|^{pq}} w(x) dx
$$

$$
\leq c_n 2^{kp} \int_{10Q_i^{k-N}} \frac{|\Omega_k \cap Q_i^{k-N}|^q}{|x - X_{Q_i^{k-N}}|^{pq}} w(x) dx
$$

$$
\leq c_n 2^{kp} \int_{10Q_i^{k-N}} \frac{2^{-qN} |Q_i^{k-N}|^q}{|x - X_{Q_i^{k-N}}|^{pq}} w(x) dx
$$

$$
\leq c_n 2^{N(p-q)+(k-N)p} \int_{\mathbb{R}^n} (M_{X_{Q_i^{k-N}}} f)^q w,
$$

where we have used (5.10) on the third inequality. Thus we have, by the Whitney decomposition theorem, for $N$ large,

$$
S(k) \leq \sum_i S(k; N, i)
$$

$$
\leq \frac{1}{ae} \frac{1}{\eta \epsilon} \int_{\mathbb{R}^n} \sum_i \left( X_{10Q_i^{k-N}} \right) w + (\eta 2^{Np} + c_n 2^{N(p-q)}) S(k - N)
$$
Thus, with Lemma 5.12. One can prove as in [18], p. 260, we have that
\[ S(k) = c_n 2^{Np} \left( q c_n + \log \frac{1}{\epsilon} + c_n \frac{pq}{q-p} \right) 2^{p(k-N)} w(\Omega_{k-N}) + \frac{1}{2} S(k-N), \]
Now, since \( q > p \), we can chose \( N \) so that \( c_n 2^{N(p-q)} < \frac{1}{4} \), that is, \( N \geq c_n \frac{q-p}{q-p} \), and \( \eta \) so that \( \eta 2^{Np} < \frac{1}{4} \).

Now, exactly as in [18], p. 260, we have that \( S_M < \infty \) and since it is clear that \( \sup_M S_M = \int_{\mathbb{R}^n} (M_{p,q}(Mf))^p w \),
we conclude the proof of the lemma.

Remark 5.11. The important part of the dependence of the constant on the exponents \( p \) and \( q \) is that the lemma will fail to be true for \( p = q \), with this kind of blowup.

Lemma 5.12. Under the same assumptions of Theorem 5.1 we have
\[ \int_{\mathbb{R}^n} (M_{p,q}T^* f(x))^p w(x)dx \leq \left( c_n 2^{p} \frac{1}{aE} \log \frac{c_n 10^p q 2p^2 + 2}{\epsilon} \right) \int_{\mathbb{R}^n} (T^* f(x))^p w(x)dx \\
+ \left( c_n 2^{q(p-q)/p-q} \frac{1}{\epsilon^2} \log \frac{1}{\epsilon} \right) \int_{\mathbb{R}^n} (M f(x))^p w(x)dx. \]

Proof. Let \( \Omega_k = \{ x \in \mathbb{R}^n : T^* f(x) > 2^k \} = \cup_j Q_{j,k} \) as in the Whitney decomposition lemma. One can prove as in [7] the following inequality. Let \( 5Q_{j,k} \in \{ M f > 2^{-N} \} \) for some \( N \geq 1 \), then
\[ ||x \in 5Q_{j,k}^{-1}, T^* f > 2^k || \leq C_f 2^{-N} |Q_{j,k}^{-1}|. \]

Let \( \{ M f > 2^k \} = \cup_j Q_j^k \) as in the Whitney decomposition lemma. We observe that for each cube \( Q_{j,k} \) there are two cases (for a fixed \( N \) that we will chose later).

Case (a). \( 5Q_{j,k}^{-1} \in \{ M f > 2^{-N} \} \) in which case \( 5Q_{j,k}^{-1} \subset c_n R_{j,k}^{-N} \) for some \( l \).

Case (b). \( 5Q_{j,k}^{-1} \not\in \{ M f > 2^{-N} \} \) in which case (5.13) implies
\[ \sum_{Q_{j,k} \subset 5Q_{j,k}^{-1}} |Q_j^k| \leq c_f 2^{-N} |Q_{j,k}^{-1}|. \]

Now let
\[ S(k) = \sum_j 2^{jk} \int_{\mathbb{R}^n} (M X Q_j^k)^p w \]
and

\[ S(k; i) = \sum_{j : Q_j^{(l)} \cap Q_i^{(l)} \neq \emptyset} 2^{kp} \int_{\mathbb{R}^n} (M \chi_j)^g w \leq \sum_{j : Q_j^{(l)} \cap Q_i^{(l)} \neq \emptyset} 2^{kp} \int_{\mathbb{R}^n} (M \chi_j)^g w. \]

This last inequality follows from the Whitney decomposition. Thus,

\[ S(k; i) \leq \sum_{j : Q_j^{(l)} \subseteq Q_i^{(l)}} 2^{kp} \int_{\mathbb{R}^n} (M \chi_j)^g w = \int_{10Q_i^{(l)}} + \int_{10Q_i^{(l)} \setminus I} = I + II. \]

By (5.3) with \( R = 10 \) we have

\[ I \leq c_n \frac{1}{ae} \log \frac{c_n 10^{pq}}{en} 2^{kp} w(5Q_i^{(k-1)}) + \eta 2^{kp} \int_{\mathbb{R}^n} (M \chi_j)^g w, \]

where \( \eta > 0 \) is a positive number at our disposal and if \( x_i^{(k-1)} \) denotes the centre of the cube \( Q_i^{(k-1)} \) then, as in the previous lemma one can show

\[ II \leq c_n 2^{kp-Nq} \int_{\mathbb{R}^n} (M \chi_j)^g w. \]

Combining estimates for \( I \) and \( II \) we obtain, for every case \( (b) \) cube \( Q_i^{(k-1)} \),

(5.14) \[ S(k; i) \leq c_n \frac{1}{ae} \log \frac{c_n 10^{pq}}{en} 2^{kp} w(5Q_i^{(k-1)}) + (\eta + c_n 2^{-Nq}) 2^{kp} \int_{\mathbb{R}^n} (M \chi_j)^g w. \]

Thus

\[ S(k) \leq \sum_{i : Q_{i}^{(l)} \cap \Omega \neq \emptyset} S(k; i) + \sum_{i : Q_{i}^{(l)} \subseteq \Omega} S(k; i) = III + IV. \]

Now, since each of the \( Q_j^{(l)} \) of type \( (a) \) intersects at most \( c \) of the \( Q_i^{(k-1)} \), (yet again due to the Whitney decomposition), we have

\[ III \leq \sum_{l} \sum_{Q_j^{(l)} \cap \Omega \neq \emptyset} 2^{kp} \int_{\mathbb{R}^n} (M \chi_j)^g w \leq c_n \frac{1}{e} \sum_{l} 2^{kp} \int_{\mathbb{R}^n} (M \chi_j)^g w, \]

where we have used (5.4) and \( M \chi_{\Omega_{j}} \leq c_n M \chi_{\Omega} \) (for two different \( c_n \) of course). For the remaining part we have by (5.14)

\[ IV \leq c_n \frac{1}{ae} \log \frac{c_n 10^{pq}}{en} 2^{kp} \int_{\mathbb{R}^n} (M \chi_j)^g w(\Omega_{k-1}) + (\eta 2^p + c_n 2^{-Nq}) 2^{(k-1)p} \int_{\mathbb{R}^n} (M \chi_j)^g w \]

\[ \leq c_n 2^p \frac{1}{ae} \log \frac{c_n 10^{pq}}{en} \int_{\mathbb{R}^n} (M \chi_j)^g w(\Omega_{k-1}) + \frac{1}{2} S(k - 1), \]

if we choose \( \eta \) small enough and \( N \) big enough. This means \( \eta = 2^{-(p+2)} \) and \( N \geq c_n \frac{p+q}{q} \). Combining now estimates for \( III \) and \( IV \) we get

\[ S(k) \leq \frac{1}{2} S(k - 1) + \left( c_n 2^p \frac{1}{ae} \log \frac{c_n 10^{pq}}{en} \right) 2^{(1-k)p} \int_{\mathbb{R}^n} (M \chi_j)^g w(\Omega_{k-1}) \]

\[ + \left( c_n 2^p \frac{1}{ae} \log \frac{c_n 10^{pq}}{en} \right) \sum_{l} 2^{(k-N)p} \int_{\mathbb{R}^n} (M \chi_j)^g w. \]

Set \( S_M = \sum_{k \leq M} S(k) \) and sum the previous inequality over \( k \leq M \) to obtain

\[ S_M \leq \frac{1}{2} S_M + \left( c_n 2^p \frac{1}{ae} \log \frac{c_n 10^{pq}}{en} \right) \int_{\mathbb{R}^n} (T^* f)^g w. \]
+ \left( c_n^2 e^{q2(p+q)} \frac{1}{e} \right) \int_{\mathbb{R}^n} (M_{p,q}(Mf))^p w \\
\leq \frac{1}{2} S_M + \left( c_n 2^p \frac{1}{ae} \log \frac{c_n 10^q 2^{p+2}}{e} \right) \int_{\mathbb{R}^n} (T^* f)^p w \\
+ \left( c_n^2 e^{q2(p+q)} \frac{1}{e} \right) \left( c_n 2^p \frac{1}{ae} \log \frac{1}{e} \right) \int_{\mathbb{R}^n} (Mf)^p w,

by (5.9). It can be shown (cf. [24], p.262) that $S_M < \infty$, so taking it to the left and then taking the supremum over all $M$ we obtain the desired result. \hfill \Box

**Proof of theorem 5.1.** Using the exponential decay from [5], we know that if we write $\{T^* f > 2^k\} = \cup_j Q_j$ as in the Whitney decomposition theorem, we have

\begin{equation}
\|x \in Q_j : T^* f(x) > 2^k, Mf(x) \leq \gamma \|x\| \leq c e^{-\frac{r}{\gamma}} |Q_j|,
\end{equation}

for any $\gamma > 0$. We call $E_j$ to the set in the left side of (5.15). Then, if we call $r$ to the exponent $1 + \delta$ in Theorem 2.13, we get

\begin{align*}
w(E_j) &= |E_j| \left( \frac{1}{|E_j|} \int_{E_j} w \right)^{\frac{1}{r}} \\
&\leq |E_j|^{\frac{1}{p}} |Q_j|^{\frac{1}{p}} \left( \frac{1}{|Q_j|} \int_{Q_j} w^r \right)^{\frac{1}{p}} \\
&\leq (Mx_Q)^p w \leq c e^{-\frac{r}{\gamma}} \int_{\mathbb{R}^n} (Mx_Q)^p w.
\end{align*}

We use the standard good-$\lambda$ techniques as in [24] combined with Lemma 5.12 to get

\begin{align*}
\int_{\mathbb{R}^n} (T^* f)^p w &\leq \left( \frac{2}{\gamma} \right)^p \int_{\mathbb{R}^n} (Mf)^p w + c e^{-\frac{r}{\gamma}} \int_{\mathbb{R}^n} (M_{p,q} T^* f)^p w \\
&\leq \left( 2^p \gamma^{-p} + e^{-\frac{r}{\gamma}} \left( c_n^2 e^{q2(p+q)} \frac{1}{e} \log \frac{1}{e} \right) \right) \int_{\mathbb{R}^n} (Mf)^p w \\
&+ c e^{-\frac{r}{\gamma}} \left( c_n 2^p \frac{1}{ae} \log \frac{c_n 10^q 2^{p+2}}{e} \right) \int_{\mathbb{R}^n} (T^* f)^p w,
\end{align*}

Choosing $\gamma^{-1} \approx c_n (q + \frac{p^2 q}{q - p}) \frac{1}{e} \log \frac{1}{e}$ we can make

\begin{equation}
e^{-\frac{r}{\gamma}} \left( c_n^2 e^{q2(p+q)} \frac{1}{e} \log \frac{1}{e} \right) < \frac{1}{2}
\end{equation}

and

\begin{equation}
ne^{-\frac{r}{\gamma}} \left( c_n 2^p \frac{1}{ae} \log \frac{c_n 10^q 2^{p+2}}{e} \right) < \frac{1}{2}
\end{equation}

and taking the term to the left side (which is possible since it is finite, see [24]) we obtain

\begin{align*}
\int_{\mathbb{R}^n} (T^* f)^p w &\leq c_n^p \left( (c_n (q + \frac{p^2 q}{q - p}) \frac{1}{e} \log \frac{1}{e}) \right)^p \int_{\mathbb{R}^n} (Mf)^p w.
\end{align*}

\hfill \Box

**Remark 5.16.** We conjecture that the first $q$ in the constant should not be there. That way $\lim_{q \to \infty} c_q < \infty$. We think this should be the case because whenever $w \in C_q$ and $q$ is bigger, we have more information. This way we could recover a weighted inequality for the $A_\infty$ class, though it would be a worse one than the
one we mention in the introduction. For this very reason, we conjecture that the dependence on the $C_q$ constant is not sharp in this sense.

Note that Lemma 5.8 does not involve the operator $T^*$. This lemma is used in the proof of the following inequality by Yabuta [25], if $1 < p < \infty$ and $w \in C_{p+\eta}, \eta > 0$.

$$\int (M^s f)^p w \leq c \int (M^{\sharp} f)^p w,$$

where $M^s$ denotes the sharp maximal function of Fefferman and Stein. This proof uses the non-quantitative version of Lemma 5.8 given in [24], combined with a good-$\lambda$ inequality between $M^\sharp$ and $M$. If this good-$\lambda$ had an exponential decay, a variation of the proof of Theorem 5.1 would yield

$$\|Mf\|_{L^p(w)} \leq c_{n, p, q}(1 + [w]_{C_q}) \log(1 + [w]_{C_q}) \|M^{\sharp} f\|_{L^p(w)},$$

for any weight $w \in C_q$ and $q > p$. This inequality (without the $C_q$ dependence of the weight) is a key point in the proof of many of the results in [6], so we would immediately improve all those results to a quantitative version.

References

[1] J. Álvarez, C. Pérez, Estimates with $A_\infty$ weights for various singular integral operators, Boll. U. M. I. (7) 8-A (1994), 123-133.
[2] T. Anderson, T. Hytönen and O. Tapiola, Weak $A_\infty$ weights and weak reverse Hölder property in a space of homogeneous type, J. Geom. Anal. 27 (2017), no. 1, 95–119.
[3] P. Auscher, S. Bortz, M. Egert, O. Saari, Non-local Ghering Lemmas, preprint, arXiv:1707.02080v1.
[4] S. M. Buckley, Harmonic analysis on weighted spaces. ProQuest LLC, Ann Arbor, MI, 1990. Thesis (Ph.D.)The University of Chicago.
[5] S. M. Buckley, Estimates for operator norms on weighted spaces and reverse Jensen inequalities, Trans. Amer. Math. Soc. 340 (1993), no. 1, 253-272.
[6] E. Cejas, K. Li, C. Pérez, I. P. Rivera-Ros, Vector-valued operators, optimal weighted estimates and the $C_p$ condition, to appear Science China Mathematics.
[7] R. R. Coifman, C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, Studia Mathematica, 51 (1974), 241-250.
[8] J. M. Conde-Alonso, A. Culiuc, F. Di Plinio, and Y. Ou, A sparse domination principle for rough singular integrals, Anal. PDE 10 (2017), no. 5, 1255–1284.
[9] D. Cruz-Uribe, J. M. Martell, C. Pérez, Extrapolation from $A_\infty$ weights and applications, J. Func. Anal., 213 (2004), 412-439.
[10] C. Fefferman and E. M. Stein, Some maximal inequalities Amer. J. Math., 93 (1971), no. 1, 107-115.
[11] G.P. Curbera, J. García-Cuerva, J.M. Martell and C. Pérez, Extrapolation with weights, Re-arrangement Invariant Function Spaces, modular inequalities and applications to Singular Integrals, Advances in Mathematics, 203 (2006) 256-318.
[12] T. Hytönen, C. Pérez, Sharp weighted bounds involving $A_\infty$, Anal. PDE, 6 (2013), 777-818.
[13] T. Hytönen, C. Pérez and E. Rela, Sharp reverse Hölder property for $A_\infty$ weights on spaces of homogeneous type, Jour. of Func. Anal. 263 (2012) no. 12, 3883-3899.
[14] A. Lerner, S. Ombrosi, C. Pérez, $A_1$ bounds for Calderón-Zygmund operators related to a problem of Muckenhoupt and Wheeden, Math. Res. Lett. 16 (2009) no.1, 149-156.
[15] A. Lerner, S. Ombrosi, C. Pérez, R. Torres, R. Trujillo-Gonzalez, New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory, Adv. Math., 220 (2009) 1222-1264.
[16] K. Li, C. Pérez, L. Roncal, I. P. Rivero-Ros, Weighted norm inequalities for rough singular integral operators, J. Geom. Anal., 2018.
[17] J. M. Martell, C. Pérez, R. Trujillo-González, *Lack of natural weighted estimates for some singular integral operators*, Trans. Amer. Math. Soc., 357 (2004), no. 1, 385-396.

[18] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc., 165 (1972), 207-226.

[19] B. Muckenhoupt, *Norm inequalities relating the Hilbert transform to the Hardy-Littlewood maximal function*, in: Functional analysis and approximation (Oberwolfach 1980), Internat. Ser. Number. Math. 60 (Birkhäuser, Basel-Boston, Mass., 1981) pp. 219–231.

[20] B. Muckenhoupt and R. L. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. **191** (1974), 261–274.

[21] C. Ortiz-Caraballo, C. Pérez, E. Rela, *Improving Bounds for Singular Integral Operators via Sharp Reverse Hölder Inequality for $A_{\infty}$*, Operator Theory: Adv. App., 229 (2013) 303-321.

[22] C. Pérez, *Weighted norm inequalities for singular integral operators*, J. London. Math. Soc., 49 (1994), 296-308.

[23] C. Pérez, *Sharp estimates for commutators of singular integrals via iterations of the Hardy-Littlewood maximal function*, J. Fourier Anal. App., 3 (1997) no. 6, 743-756.

[24] E. T. Sawyer, *Norm inequalities relating singular integrals and the maximal function*, Studia Math. **75** (1983), no. 3, 253–263.

[25] K. Yabuta, *Sharp maximal function and $C_p$ condition*, Arch. Math. 55 (1990), 151-155.

[26] J. M. Wilson, *Weighted Littlewood-Paley Theory and Exponential-Square Integrability* (Springer-Verlag, Lecture Notes in Mathematics) **1924** (2007).

**Basque Center for Applied Mathematics,** 48009 Bilbao, Spain

_E-mail address: jcanto@bcamath.org_