A solution of the maximality problem for one-parameter dynamical systems

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Abstract We prove a maximality theorem for one-parameter dynamical systems that include W*- , C*- and multiplier one-parameter dynamical systems. Our main result is new even for one-parameter actions on commutative multiplier algebras including the algebra $C_b(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$ acted upon by translations. The methods we develop and use in our characterization of maximality include harmonic analysis, topological vector spaces and operator algebra techniques.

1 Introduction

The current paper grew out of an effort to find a solution to the general problem of maximality of subalgebras of analytic elements associated to various dynamical systems: W*-dynamical systems, C*-dynamical systems and, as it will follow from the present work, multiplier dynamical systems as defined below (Section 2). The study of maximality of analytic subalgebras associated with C*- or W*-dynamical systems has a history of over six decades. It started with Wermer’s maximality theorem [27]. Motivated by some earlier problems about approximation of continuous functions, Wermer showed that if $X = C(T)$ is the C*-algebra of all continuous functions on $T = \{ t \in \mathbb{C} : |t| = 1 \}$, then the norm closed subalgebra $A$ of all functions $f \in C(T)$ that have an analytic extension to the unit disc $D = \{ s \in \mathbb{C} : |s| < 1 \}$ is a maximal norm-closed subalgebra of $X$. Hoffman and Singer [9] and Simon [23] obtained some generalizations of Wermer’s theorem for the case of compact groups with archimedean-linearly ordered Pontryagin duals.

A significant and far reaching generalization of Wermer’s maximality theorem was obtained by Forelli [7] using the seminal concepts and results from his paper [6]. In [7], Forelli considered a minimal action of $\mathbb{R}$ on a locally compact Hausdorff space $S$, that is a homomorphism $\tau$ of $\mathbb{R}$ into the group of homeomorphisms of $S$ onto itself such that the mapping $t \to \tau_t(s)$ is continuous for every $s \in S$ and every orbit $\{ \tau_t(s) : t \in \mathbb{R} \}, s \in S$ is dense in $S$. If $f \in C_0(S)$ and $t \in \mathbb{R}$, denote $\alpha_t(f) = f \circ \tau_t$. Then $\alpha$ is a homeomorphism of $\mathbb{R}$ into the group of automorphisms of the C*-algebra $X = C_0(S)$ such that the mapping $t \to \alpha_t(f)$ is continuous from $\mathbb{R}$ to $C_0(S)$ endowed with the uniform norm topology for every $f \in C_0(S)$. Forelli proved that the subalgebra $X^{\alpha}(\mathbb{R})$ consisting of all functions $f \in C_0(S)$ such that the $\alpha$-spectrum of $f$, $sp_{\alpha}(f)$, as defined in the next section, is contained in $[0, \infty)$, is a maximal norm
closed subalgebra of $X$. An equivalent description of the algebra $X^\alpha([0, \infty))$ for $X = C_0(S)$ and $\alpha$ as defined above is the following: $f \in X^\alpha([0, \infty))$ if and only if the mapping $t \to \alpha_t(f)(s) = (f \circ \tau_t)(s)$ has a bounded analytic extension to the upper half plane for every $s \in S$. In what follows a system $(X, G, \alpha)$ consisting of a C$^*$ algebra $X$, a locally compact group $G$ and a homeomorphism $\alpha$ from $G$ into the group $Aut(X)$ of all automorphisms of $X$ such that the mapping $t \to \alpha_t(x)$ is continuous from $G$ to $X$ with the norm topology for every $x \in X$, will be called a C$^*$-dynamical system. When $G = \mathbb{R}$ the system will be called a one-parameter C$^*$-dynamical system. If $X$ does not contain any non trivial norm-closed $\alpha$-invariant ideal, the system will be called $\alpha$-simple. This is the case with Forelli system $(C_0(S), \mathbb{R}, \alpha)$ if $\tau$ is a minimal flow as defined above.

Another direction of study of maximal subalgebras is the following: If $(X, \mathbb{R}, \alpha)$ is a W$^*$-dynamical system consisting of a von Neumann algebra $X$ and an action of $\mathbb{R}$ on $X$ such that the mapping $t \to \alpha_t(x)$ is continuous from $\mathbb{R}$ to $X$ endowed with the $w^*$-topology, for every $x \in X$, when is $X^\alpha([0, \infty))$ a maximal $w^*$-closed subalgebra of $X$? In [21] Sarason noticed that Hoffman and Singer [10] basically proved the first result in this direction: $H^\infty(T)$ is a maximal $w^*$-closed subalgebra of $L^\infty(T)$, where $H^\infty(T) \subset L^\infty(T)$ is the subalgebra of all $f \in L^\infty(T)$ that have an analytic extension to the unit disk $D$. Further in [14] Muhly showed that if $(X, \mathbb{R}, \alpha)$ is an ergodic W$^*$-dynamical system with $X$ an abelian von Neumann algebra, then the analytic subalgebra, $X^\alpha([0, \infty))$ as defined below is a maximal $w^*$-closed subalgebra of $X$.

Some extensions of Wermer’s result to the more general case of C$^*$- and W$^*$-dynamical systems were obtained in [13] for W$^*$-crossed products and in [18] for C$^*$-crossed products. In [24] Solel has found necessary and sufficient conditions for maximality of subalgebras of analytic elements associated with periodic $\sigma$-finite W$^*$-dynamical systems and, further, in [25] he has considered the maximality for the particular case of an $\alpha$-simple one-parameter $\sigma$-finite W$^*$-dynamical system, i.e. the case when $X$ has no non-trivial $w^*$-closed $\alpha$-invariant ideals (or equivalently the center of $X$ contains no non-trivial $\alpha$-invariant projections).

In [19] we found a necessary and sufficient condition, that we called Condition (S) (S from the word spectrum), for maximality of the analytic subalgebra associated with a periodic C$^*$-dynamical system. Soon after, Kishimoto [12] considered the case of one-parameter C$^*$-dynamical systems $(X, \mathbb{R}, \alpha)$. He proved that if the crossed product C$^*$-algebra of the system is a simple C$^*$-algebra, then $X^\alpha([0, \infty))$ is a maximal norm-closed subalgebra of $X$.

In this paper we will find a spectral characterization of maximality for general one-parameter dynamical systems, not necessarily separable or $\sigma$-finite (Condition 4.3.). This characterization is equivalent to our Condition (S) for periodic C$^*$-dynamical systems [19] and with Solel characterization of maximality for $\sigma$-finite W$^*$-dynamical systems [24]. It characterizes also maximality for one-parameter multiplier dynamical systems (as defined in Remark 2.2.). The methods we develop and use include harmonic analysis, topological vector spaces and operator algebra techniques. Our main result (Theorem 4.18.) contains and improves on all of the above mentioned results. In particular it contains the fol-
ollowing result that was not considered so far even for commutative dynamical systems: Let \( C_b(\mathbb{R}) \) be the C*-algebra of bounded continuous functions on \( \mathbb{R} \) and \( \alpha_t(f)(s) = f(s - t) \) be the translation of \( f \) by \( t \in \mathbb{R} \). Then \( C_b(\mathbb{R}) \) is the multiplier C*-algebra of \( C_0(\mathbb{R}) \) and for each \( f \in C_b(\mathbb{R}) \) the mapping \( t \to \alpha_t(f) \) is continuous from \( \mathbb{R} \) to \( C_b(\mathbb{R}) \) endowed with the strict topology. A consequence of our main result is that \( H^\infty(\mathbb{R}) \cap C_b(\mathbb{R}) \) is a maximal strictly closed subalgebra of \( C_b(\mathbb{R}) \).

This paper is organized as follows. In Section 2 we discuss the concept of dual pairs, \((X, F)\) of Banach spaces ([1], [5], [28]) and representations of locally compact groups on a Banach space \( X \) endowed with the \( F \)-topology. We will define the spectral subspaces associated with these representations and define the \( F \)-dynamical systems. We also define the concept of strong \( F \)-Connes spectrum of such systems and we show that our notion of strong \( F \)-Connes spectrum contains both the original concept of Connes spectrum for \( W^* \)-dynamical systems and the strong Connes spectrum defined by Kishimoto for \( C^* \)-dynamical systems. The strong \( F \)-Connes spectrum is a new concept for multiplier dynamical systems. In Section 3 we obtain some results about periodic \( F \)-dynamical systems that will be used in Section 4. These results extend to the case of \( F \)-dynamical systems results by Connes [3] for \( W^* \)-dynamical systems and Olesen, Pedersen and Stormer [16] for \( C^* \)-dynamical systems. In Section 4 we prove our main result about maximality of subalgebras of analytic elements of an \( F \)-dynamical system. I am grateful to Laszlo Zsido for several useful discussions and suggestions during the completion of this work. I also thank Akitaka Kishimoto for enlightening discussions about his work on the subject.

2 Notations and preliminary results

In this section we will set up the framework for future discussions and will establish the notations that will be used in the rest of the paper. We will also state and prove some basic preliminary results.

2.1. Definition. A dual pair of Banach spaces is, by definition a pair \((X, F)\) of Banach spaces together with a bilinear functional
\[
(x, \varphi) \in X \times F \to \langle x, \varphi \rangle = \varphi(x) \in \mathbb{C}
\]
such that
1) \( \|x\| = \sup_{\varphi \in F, \|\varphi\| \leq 1} |\langle x, \varphi \rangle|, x \in X \)
2) \( \|\varphi\| = \sup_{x \in X, \|x\| \leq 1} |\langle x, \varphi \rangle|, \varphi \in F \)
3) The convex hull of every relatively \( F \)-compact subset of \( X \) is relatively \( F \)-compact
4) The convex hull of every relatively \( X \)-compact subset of \( F \) is relatively \( X \)-compact.

If \( X \) is a \( C^* \)-algebra we will assume in addition that the following conditions hold
5) The involution of $X$ is $\mathcal{F}$-continuous and the multiplication in $X$ is separately $\mathcal{F}$-continuous.

Clearly, if $(X, \mathcal{F})$ is a dual pair of Banach spaces as in the above definition, then $X$ is naturally embedded in $\mathcal{F}^*$ and $\mathcal{F}$ is naturally embedded in $X^*$.

2.2. Remark

i) If $X$ is a $C^*$-algebra, and $\mathcal{F} = X^*$ is its dual then the pair $(X, X^*)$ satisfies conditions 1)-5)

ii) If $X$ is a $W^*$-algebra and $\mathcal{F} = X_*$ is its predual, then the pair $(X, X_*)$ satisfies conditions 1)-5)

iii) If $X = M(Y)$ is the multiplier algebra of a $C^*$-algebra $Y$ and $\mathcal{F} = Y^*$ is the dual of $Y$, then the pair $(X, Y^*)$ satisfies conditions 1)-5). In addition, in this case, the duality is compatible with the strict topology on $X = M(Y)$.

Proof. Parts i) and ii) are discussed in [1]. Part iii) follows from [4] and [26].

Notice that in all three cases i), ii) and iii) of the above Remark 2.2., the dual $\mathcal{F}^*$ of $\mathcal{F}$ is a von Neumann algebra and $X$ is naturally embedded in $\mathcal{F}^*$.

2.3. Remark

i) Let $(X, \mathcal{F})$ be a dual pair of Banach spaces with $\mathcal{F} = X^*$. If $Y \subset X$ is a norm closed subspace of $X$, then $Y$ is $\mathcal{F}$-closed in $X$.

ii) Let $(X, \mathcal{F})$ be a dual pair of Banach spaces. If $Y \subset X$ is an $\mathcal{F}$-closed subspace, then $Y$ is norm closed.

Proof. Part i) follows from the Hahn Banach separation theorem. Part ii) is trivial since the $\mathcal{F}$-topology is weaker than the norm topology.

2.4. Notation

Let $(X, \mathcal{F})$ be a dual pair of Banach spaces. If $Y, Z$ are subsets of $X$ denote:

a) $\text{lin}\{Y\}$ is the linear span of $Y$.

b) If $X$ is a Banach space with involution, in particular a $C^*$-algebra, denote $Y^* = \{y^* : y \in Y\}$.

c) If $X$ is a $C^*$-algebra, denote $YZ = \text{lin}\{yz : y \in Y, z \in Z\}$.

d) $\overline{Y}^\mathcal{F}$ = $\mathcal{F}$-closure of $Y$ in $X$.

e) $\overline{Y}^\|\| = \text{norm closure of } Y$.

f) $\overline{Y}^w$ = the $w^*$-closure of $Y$ in $\mathcal{F}^*$.

Suppose that $(X, \mathcal{F})$ is a dual pair of Banach spaces with $X$ a $C^*$-algebra, so Definition 2.1. 1)-5) is satisfied.

g) If $\phi \in \mathcal{F}$, denote by $\phi^* \in X^*$ the adjoint of $\phi$, defined by $\phi^*(x) = \overline{\phi(x^*)}$, $x \in X$, where $\overline{\phi(x^*)}$ is the complex conjugate of $\phi(x^*)$.

h) If $\phi \in \mathcal{F}$ and $y \in X$ denote by $L_y \phi \in X^*$ the functional defined by $(L_y \phi)(x) = \phi(yx), x \in X$.

Suppose that $(X, \mathcal{F})$ is a dual pair of Banach spaces satisfying conditions 1)-4) of the Definition 2.1. and $G$ a locally compact abelian group.
2.5. Definition [1] A representation of $G$ on $X$ is a homomorphism of $G$ into the group of all invertible bounded $\mathcal{F}$-continuous linear operators on $X$ such that $\sup_{t \in G} \| \alpha_t \| < \infty$ and for every $x \in X$, and $\varphi \in \mathcal{F}$, the map $t \rightarrow \varphi(\alpha_t(x))$ is continuous. The triple $(\mathcal{F}, G, \alpha^*)$ is called the dual representation of $G$ on $\mathcal{F}$.

If $X$ is a C*-algebra and $\alpha_t, t \in G$ are $\mathcal{F}$-continuous automorphisms of $X$, the triple $(X, G, \alpha)$ will be called an $\mathcal{F}$-dynamical system. If $G = \mathbb{R}$, the system $(X, \mathbb{R}, \alpha)$ will be called a one-parameter $\mathcal{F}$-dynamical system.

2.6. Notation. If $X$ is a C*-algebra, we will also use the following additional notations

- $g) \mathcal{H}_\alpha^g(X)$ the set of all non-zero globally $\alpha$-invariant $\mathcal{F}$-closed hereditary C*-subalgebras of $X$.
- $h) \mathcal{H}_\alpha(X)$ the set of all non-zero globally $\alpha$-invariant hereditary C*-subalgebras of $X$.
- $i) \mathcal{H}(X)$ the set of all non-zero $\mathcal{F}$-closed hereditary C*-subalgebras of $X$.
- $j) \mathcal{H}(X)$ the set of all non-zero norm-closed hereditary C*-subalgebras of $X$.

2.7. Remark

i) If $\mathcal{F} = X^*$ then the above concept of $\mathcal{F}$-dynamical system coincides with the concept of C*-dynamical system. Indeed in this case, every automorphism of $X$ is $\mathcal{F}$-continuous and the continuity of $g \rightarrow \varphi(\alpha_g(a))$ for every $\varphi \in X^*$ and $a \in X$ is equivalent with the continuity of $g \rightarrow \alpha_g(a)$ for every $a \in X$, in the norm of $X$ [8, p.306].

ii) If $X$ is a $W^*$-algebra and $\mathcal{F} = X^*$, then the concept of $\mathcal{F}$-dynamical system coincides with the usual notion of $W^*$-dynamical system.

2.8. Lemma Let $(X, \mathcal{F})$ be a dual pair of Banach spaces with $X$ a C*-algebra. If $A \subset X$ is a subset then, $A^{\mathcal{F}}\mathcal{A}_A^{\mathcal{F}}$ is an $\mathcal{F}$-closed hereditary C*-subalgebra of $X$.

**Proof.** Since the involution is $\mathcal{F}$-continuous and the multiplication is separately $\mathcal{F}$-continuous (Definition 2.1. 5)), it follows that $X^{\mathcal{F}}$ is a C*-algebra. We have to prove that $X^{\mathcal{F}}$ is a hereditary C*-subalgebra of $X$. Let $y \in X^{\mathcal{F}}, y \geq 0$ and $x \in X$ be such that $0 \leq x \leq y$. Since $X^{\mathcal{F}}$ is a C*-subalgebra of $X$, $y^\dagger \in X^{\mathcal{F}}$. By [17, Proposition 1.4.5.] there exists $u \in X$ such that $x^\dagger = u y^\dagger = y^\dagger u^\dagger$. Thus $x = y^\dagger u^\dagger u y^\dagger$. Applying again Definition 2.1. 5), it follows that $x \in X^{\mathcal{F}}$. But, as $Y$ is a hereditary C*-subalgebra of $X$, we have $Y X Y \subset Y$. Therefore $x \in Y^{\mathcal{F}}$.  

2.9. Lemma Let $(X, \mathcal{F})$ be a dual pair of Banach spaces with $X$ a C*-algebra. If $A \subset X$ is a subset then, $A^{\mathcal{F}}\mathcal{A}_A^{\mathcal{F}}$ is an $\mathcal{F}$-closed hereditary subalgebra of $X$ which coincides with $A^{\mathcal{F}}A A^{\mathcal{F}} A^{\mathcal{F}}$. Moreover, if $Y$ is an $\mathcal{F}$-closed hereditary subalgebra of $X$ such that $A^* A \subset Y$, then $A^{\mathcal{F}}\mathcal{A}_A^{\mathcal{F}} \subset Y$. 

5
Proof. If \( p \) is the support projection of \( A \) in \( X^{**} \) then, clearly, \( p \) is the range projection of \( A^* \) and therefore the support and range projection of \( A^*A \). Hence \( \| A^*A \| = \| A^*AXA^* \| \), so the first part of the Lemma is proven. To prove the second part, notice that if \( A^*A \subset Y \), then \( A^*AXA^*A \subset Y \). Hence \( \| A^*X \| = \| A^*AXA^* \| \), so the first part of the Lemma is proven. To prove the second part, notice that if \( A^*A \subset Y \), then \( A^*AXA^*A \subset Y \). Hence \( \| A^*X \| = \| A^*AXA^* \| \), so the first part of the Lemma is proven. To prove the second part, notice that if \( A^*A \subset Y \), then \( A^*AXA^*A \subset Y \). Hence \( \| A^*X \| = \| A^*AXA^* \| \), so the first part of the Lemma is proven. To prove the second part, notice that if \( A^*A \subset Y \), then \( A^*AXA^*A \subset Y \). Hence

\[
\varphi(y) = \int_{G} f(t)\varphi(\alpha_t(\gamma))dt.
\]

for every \( \varphi \in F \) and the operator \( x \to \alpha_f(x) \) is \( F \)-continuous [1, Proposition 1.4]. If \( x \in X \), define the Arveson spectrum of \( x \)

\[
sp_{\alpha}(x) = \left\{ \gamma \in \hat{G} : \widehat{f}(\gamma) = 0 \text{ for all } f \in L^1(G) \text{ with } \alpha_f(x) = 0 \right\}.
\]

where \( \widehat{f} \) is the Fourier transform of \( f \). Then, it is clear that

\[
sp_{\alpha}(x^*) = \{-\gamma : \gamma \in sp_{\alpha}(x)\}
\]

The spectral subspaces of the system \((X, G, \alpha)\) are defined as follows ([1], [17]).

If \( F \) is a closed subset of \( \hat{G}\), denote

\[
X^{\alpha}(F) = \{ x \in X : sp_{\alpha}(x) \subset F \}.
\]

Then \( X^{\alpha}(F) \) is an \( F \)-closed \( \alpha \)-invariant subspace of \( X \). Using the observation above about \( sp_{\alpha}(x^*) \) we have

\[
X^{\alpha}(F)^* = X^{\alpha}(-F)
\]

where \(-F = \{-\gamma : \gamma \in F\}\). If \( F = \{\gamma\} \), \( \gamma \in \hat{G} \) we will denote

\[
X^{\alpha}(\{\gamma\}) = X_{\gamma}.
\]

In particular for \( \gamma = 0 \) we denote

\[
X_0 = X^{\alpha}
\]

Then, we have the following consequence of the above discussions

2.10. Corollary Let \((X, G, \alpha)\) be an \( F \)-dynamical system and \( \gamma \in \hat{G} \). We have

i) If \( x \in G \) is such that \( sp(x) = \{\gamma\} \), then \( \alpha_t(x) = \langle t, \gamma \rangle x \) for all \( t \in G \).

If in addition \( G \) is compact we also have

ii) \( X_{\gamma} = \{ \int \langle t, \gamma \rangle \alpha_t(x) dt : x \in X \} \).
iii) The mapping $P^n_\gamma : x \to x_\gamma = \int (t, \gamma) \alpha_t(x) dt$ from $X$ to $X_\gamma$ is an $\mathcal{F}$-continuous linear projection of $X$ onto $X_\gamma$.

**Proof.** Part i) follows from [17, Cor. 8.1.8.]. Part ii) is an easy consequence of the definition of $sp(x)$.

iii) Since obviously $x_\gamma = \alpha_f(x)$ where $f(t) = (t, \gamma)$, this part follows from the discussion above. ■

If $U$ is an open subset of $\hat{G}$, denote

$$X^\alpha(U) = \text{lin} \left\{ \alpha_f(x) : x \in X, f \in L^1(G) \text{ and } \text{supp}(\hat{f}) \subset U, \text{ compact} \right\},$$

where the notation $\text{supp}(\hat{f})$ stands for the support of the Fourier transform of $f$. Certainly, the above spectral subspaces can be defined similarly for the pair $(\mathcal{F}, X)$ with the dual representation $\alpha^*$ of $G$. The next lemma is a collection of known facts about spectral subspaces. These facts are true for every locally compact abelian group $G$.

In the next lemma, the notations $U, U_i$ stand for open subsets and $F, F_i$ for closed subsets of $G$.

**2.11. Lemma** Let $(X, \mathcal{F})$ be a a dual pair of Banach spaces and $\alpha$ a representation of $G$ on $X$. Then

i) $X^\alpha(U) = \sum X^\alpha(F_i)$ if $U = \bigcup F_i = \bigcup F_i^o$.

ii) $X^\alpha(F) = \bigcap X^\alpha(U_i)$ if $F = \bigcap U_i = \bigcap U_i^\circ$.

iii) $\sum X^\alpha(U_i) = X^\alpha(\bigcup U_i)$.

iv) $X^\alpha(\bigcap F_i) = \bigcap X^\alpha(F_i)$.

If $X$ is a $C^*$-algebra and $(X, G, \alpha)$ an $\mathcal{F}$-dynamical system, then we also have

v) $X^\alpha(F_1, F_2) \subset X^\alpha(F_1 + F_2)$, in particular $X^\alpha(F_1)X^\alpha(F_2) \subset X^\alpha(F_1 + F_2)$ if $F_1, F_2$ are compact.

vi) $X^\alpha(U_1)X^\alpha(U_2) \subset X^\alpha(U_1 + U_2)$ if $U_1, U_2$ are open sets.

vii) Let $x \in X$ be such that $sp(x) = F$ is compact. Then the map $t \to \alpha_t(x)$ is norm continuous.

viii) Let $F$ be a closed subset of $\hat{G}$. Then

$$X^\alpha(F) = \left\{ x \in X : \alpha_f(x) = 0 \text{ if } \text{supp} \hat{f} \subset \hat{G} - F \right\}$$

**Proof.** Parts i), ii), iii) and iv) follow from [17, Theorem 8.1.4. viii), vii), iii) and ii)] since, under our conditions 1)-4), the action $\alpha$ is integrable as noticed in [29, page 214]. Part v) follows from [29, Corollary 2.3. ii)] for closed $F_1, F_2$ and, if $F_1, F_2$ are compact, then clearly, $F_1 + F_2$ is compact in $G$, so closed.

vi) To prove that part vi) follows from i) and v), notice that for every open subset $U \subset G$ there exists a family $\{F_i\}$ of compact subsets of $G$ such that $U = \bigcup F_i = \bigcup F_i^o$. Hence $U_1 = \bigcup F_{1i} = \bigcup F_{1i}^o$ and $U_2 = \bigcup F_{2i} = \bigcup F_{2i}^o$ where $F_{1i}$,
and $F_{2\nu}$ are compact. Moreover, since $F_{1\nu}^\circ + F_{2\nu}^\circ \subset (F_{1\nu} + F_{2\nu})^\circ \subset F_{1\nu} + F_{2\nu}$, we have

$$U_1 + U_2 = \cup (F_{1\nu}^\circ + F_{2\nu}^\circ) = \cup (F_{1\nu} + F_{2\nu}) = \cup (F_{1\nu} + F_{2\nu})^\circ.$$  
and we can apply i) and v).

vii) By [20, Theorem 2.6.2.], there exists a function $g \in L^1(G)$ such that $g = 1$ on an open set, $U$, containing $sp(x)$. Then, if $f \in L^1(G)$, and $supp \hat{f}$ is a compact subset of $U$ it follows that $g \hat{f} = g \ast \hat{f} = \hat{f}$. Therefore, $\alpha_g(\alpha_f(z)) = \alpha_f(z)$ for all $z \in X$. Since the set

$$\{ \alpha_f(z) : z \in X, f \in L^1(G), \text{ with } supp \hat{f} \text{ a compact subset of } U \}$$

is, by definition a total subset of $X^\alpha(U)$ and $F \subset U$ it follows, in particular, that $\alpha_g(x) = x$. Therefore

$$\|\alpha_t(x) - x\| = \|\alpha_t(\alpha_g(x)) - \alpha_g(x)\| = \|\alpha_{g_t}(x) - \alpha_g(x)\| \leq \|g_t - g\| \|x\|.$$  
where $g_t$ is the $t$-translate of $g$. Since $\lim_{t \to 0} \|g_t - g\| = 0$, we are done.

viii) This follows from [2, the discussion after Definition 3.2.].

Suppose that $X$ is a C*-algebra and $(X, G, \alpha)$ an $\mathcal{F}$-dynamical system. We will define the Arveson and Connes spectra of the action $\alpha$. The Arveson spectrum of $\alpha$ is the following subset of the dual $\Gamma = \hat{G}$ of $G$

$$sp(\alpha) = \{ \gamma \in \Gamma : X^\gamma(F) \neq (0) \text{ for every closed neighborhood } F \text{ of } \gamma \}.$$  
Clearly, $sp(\alpha)$ is a closed subset of $\Gamma$ and, since, as noticed above, $X^\alpha(F)^* = X^\alpha(-F)$, it follows that if $\gamma \in sp(\alpha)$, then $-\gamma \in sp(\alpha)$. Define the strong $\mathcal{F}$-Arveson spectrum of the action $\alpha$

$$\tilde{sp}_\mathcal{F}(\alpha) = \{ \gamma \in \Gamma : X^\gamma(F)^* X X^\gamma(F) = X \text{ for every closed neighborhood } F \text{ of } \gamma \}.$$  
It follows immediately that $\tilde{sp}_\mathcal{F}(\alpha)$ is a semigroup and a closed subset of $\Gamma$.

Now let $(X, G, \alpha)$ be an $\mathcal{F}$-dynamical system. We define the strong $\mathcal{F}$-Connes spectrum of $\alpha$ by

$$\overline{\Gamma}_\mathcal{F}(\alpha) = \cap \{ \tilde{sp}_\mathcal{F}(\alpha)|_Y : Y \in \mathcal{H}_\sigma^\alpha(X) \}.$$  
where $\mathcal{H}_\sigma^\alpha(X)$ is the set of all globally $\alpha$-invariant $\mathcal{F}$-closed hereditary C*-subalgebras of $X$. Using the above observation, it follows that $\overline{\Gamma}_\mathcal{F}(\alpha)$ is a closed semigroup.

The next result shows that the concept of strong $\mathcal{F}$-Connes spectrum defined above coincides with the Connes spectrum for $\mathcal{W}^*$-dynamical systems (i.e if $\mathcal{F} = X_\nu$), [3] and with the strong Connes spectrum for C*-dynamical systems (i.e if $\mathcal{F} = X^*$) as defined by Kishimoto [11]. A different concept of Connes
spectra that are related to the current paper has been defined for C*-dynamical systems by Olesen [15].

2.12. Proposition i) If $X$ is a von Neumann algebra and $\mathcal{F} = X_\sigma$, then $\tilde{\Gamma}_{\mathcal{F}}(\alpha)$ coincides with the Connes spectrum as defined by Connes [3].

ii) If $X$ is a C*-algebra and $\mathcal{F} = X^*$, then $\tilde{\Gamma}_{\mathcal{F}}(\alpha)$ equals the strong Connes spectrum as defined by Kishimoto [11].

Proof. i) Recall that the definition of the Connes spectrum for a W*-dynamical system $(X, G, \alpha)$ [3] is the following:

$$\Gamma(\alpha) = \cap_{p \in X_\alpha} \text{sp}(\alpha|_{pXp}),$$

where $p$ is an $\alpha$-invariant projection.

Notice that in the case of W*-dynamical systems every $\alpha$-invariant $\mathcal{F}$-closed hereditary subalgebra of $X$ is of the form $pXp$ with $p \in X_\alpha$, projection. Clearly, $\tilde{\Gamma}_{\mathcal{F}}(\alpha) \subset \Gamma(\alpha)$. Now let $\gamma \in \hat{G}$. Suppose that $\gamma \notin \tilde{\Gamma}_{\mathcal{F}}(\alpha)$. Then, there exists an $\alpha$-invariant projection $p \in X_\alpha$ and a closed neighborhood, $F$, of $\gamma$ such that $pX^\alpha(F)pXpX^\alpha(F)^*p \neq pXp$. Since $pX^\alpha(F)pXpX^\alpha(F)^*p$ is an $\alpha$-invariant $w^*$-closed hereditary W*-subalgebra, there exists an $\alpha$-invariant projection $q \in X$, $q < p$ such that

$$pX^\alpha(F)pXpX^\alpha(F)^*p = qXq.$$

It follows that $(p - q)X^\alpha(F)(p - q)X(p - q)X^\alpha(F)^*(p - q) = (0)$, so $\gamma \notin \text{sp}(\alpha|_{(p - q)X(p - q)})$. Hence $\gamma \notin \Gamma(\alpha)$. Therefore, $\Gamma(\alpha) \subset \tilde{\Gamma}_{\mathcal{F}}(\alpha)$ and part i) is proven.

ii) Suppose that $X$ is a C*-algebra and $\mathcal{F} = X^*$. In this case, by Remark 2.3, i) the set of all $\mathcal{F}$-closed hereditary C* subalgebras of $X$ coincides with the set of all norm closed hereditary C* subalgebras of $X$ and our claim follows from the definition of the strong Connes spectrum for C*-dynamical systems [11, Section 2].

3 \mathcal{F}\text{-dynamical systems associated with compact abelian groups}

Let $(X, G, \alpha)$ be an $\mathcal{F}$-dynamical system with $G$ compact abelian. The next results (Theorems 3.2. and 3.4.) that will be used in our study of maximality (Section 4), extend some results of Connes [3, Prop. 2.2.2. b) and Theorem 2.4.1.] for compact abelian groups to the case of $\mathcal{F}$-dynamical systems, as well as the result of Olesen, Pedersen and Stormer [16, Theorem 2, i$\Leftrightarrow$ii)].

3.1. Lemma Let $(X, G, \alpha)$ be an $\mathcal{F}$-dynamical system with $G$ compact abelian. If $J$ is a two sided ideal of $X_\alpha$, and $\gamma_0 \in \hat{G}$, then

$$(XJX^\sigma)_{\gamma_0} = \lim{X_{\gamma_1}JX_{\gamma_2} : \gamma_1, \gamma_2 \in \text{sp}(\alpha) \text{ and } \gamma_1\gamma_2 = \gamma_0}.$$
In particular

\[(XJX^*)^\alpha = \overline{\text{lin} \{ X_{\gamma_1}X_{\gamma_2}^* : \gamma \in \text{sp}(\alpha) \}}^\sigma\]

Proof. Clearly,

\[\text{lin} \{ X_{\gamma_1}X_{\gamma_2} : \gamma_1, \gamma_2 \in \text{sp}(\alpha), \gamma_1\gamma_2 = \gamma_0 \} \subset (XJX)_{\gamma_0} \subset (XJX^*)_{\gamma_0}\]

and therefore,

\[\overline{\text{lin} \{ X_{\gamma_1}X_{\gamma_2} : \gamma_1, \gamma_2 \in \text{sp}(\alpha), \gamma_1\gamma_2 = \gamma_0 \}}^\sigma \subset (XJX^*)_{\gamma_0}\]

If \(x \in X\), by Lemma 2.11, i), there is a net \(\{ x_i \} \subset \text{lin} \{ X_\gamma \gamma \in \text{sp}(\alpha) \}\) such that \(x_i \to x\) in the \(F\)-topology. Now, let \(x_{\gamma_1} \in X_{\gamma_1}\) for some \(\gamma_1 \in \text{sp}(\alpha)\). Clearly, since by Definition 2.1, 5) the multiplication is separately \(F\)-continuous, it follows that \(x_jx_{\gamma_1} \to x_jx_{\gamma_1}\) in the \(F\)-topology for all \(j \in J\). By Corollary 2.10, iii) \((x_jx_{\gamma_1})_{\gamma_0} \to (x_jx_{\gamma_1})_{\gamma_0}\) in the \(F\)-topology. But, it is obvious that \((x_jx_{\gamma_1})_{\gamma_0} = (x_j)_{\gamma_0} : jx_{\gamma_1}\), so

\[(XJX_{\gamma_1})_{\gamma_0} \subset \overline{\text{lin} \{ X_{\gamma_1}X_{\gamma_2} : \gamma_1, \gamma_2 \in \text{sp}(\alpha), \gamma_1\gamma_2 = \gamma_0 \}}^\sigma\]

for every \(\gamma_1 \in \text{sp}(\alpha)\). It follows that

\[(XJ(\text{lin} \{ X_\gamma \gamma \in \text{sp}(\alpha) \}))_{\gamma_0} \subset \overline{\text{lin} \{ X_{\gamma_1}X_{\gamma_2} : \gamma_1, \gamma_2 \in \text{sp}(\alpha), \gamma_1\gamma_2 = \gamma_0 \}}^\sigma\]

for every finite subset \(F \subset \hat{G}\). Using again the separate \(F\)-continuity of the multiplication, we get

\[(XJX)_{\gamma_0} \subset \overline{\text{lin} \{ X_{\gamma_1}X_{\gamma_2} : \gamma_1, \gamma_2 \in \text{sp}(\alpha), \gamma_1\gamma_2 = \gamma_0 \}}^\sigma\]

Applying again Corollary 2.10, iii) we have

\[(XJX^*)_{\gamma_0} = P_{\gamma_0}(XJX^*) \subset P_{\gamma_0}(XJX) = (XJX)_{\gamma_0} \subset \overline{\text{lin} \{ X_{\gamma_1}X_{\gamma_2} : \gamma_1, \gamma_2 \in \text{sp}(\alpha), \gamma_1\gamma_2 = \gamma_0 \}}^\sigma\]

and we are done. \(\blacksquare\)

The next result is an extension of a result of Connes [3, Proposition 2.2.2, b)] to the framework of \(F\)-dynamical systems \((X, G, \alpha)\) with \(G\) compact abelian. In order to state the result we make the following notation: if \(J \subset X^\alpha\) is an \(F\)-closed ideal, let \(X_J = JXJ^\sigma\). Then \(X_J\) is an \(F\)-closed hereditary C*-subalgebra of \(X\) and obviously \(X_J\) is \(\alpha\)-invariant so \(X_J \in \mathcal{H}_\sigma^\alpha(X)\).

3.2. Theorem \(\hat{\Gamma}_F(\alpha) = \cap \{ \text{sp}_F(\alpha | X_J) : J \subset X^\alpha, F - \text{closed ideal} \}\).

Proof. Clearly, \(\hat{\Gamma}_F(\alpha) \subset \cap \{ \text{sp}_F(\alpha | X_J) : J \subset X^\alpha \text{ an } F \text{- closed ideal} \}\). Now let \(\gamma \in \cap \{ \text{sp}_F(X_J) : J \subset X^\alpha, F \text{- closed ideal} \}\) and \(Y \in \mathcal{H}_\sigma^\alpha(X)\). We will prove
that \( \gamma \in \tilde{sp}_F(\alpha|Y) \) that is \( Y_\gamma Y_\gamma^\ast = Y^{\alpha} \), so \( \gamma \in \tilde{\Gamma}_F(\alpha) \). Denote by \( J \) the following ideal of \( X^{\alpha} \)

\[
J = X^{\alpha}Y^{\alpha}X^{\alpha}
\]

If \( j \in J^+ \), then since \( J \subset X^{\alpha} \) and obviously \( j = j^\ast j j^\ast j \) we have \( j \in JX^{\alpha}J \), so \( J^+ \subset JX^{\alpha}J \). Thus, \( J = JX^{\alpha}J \). Therefore, since \( Y \in H^{\alpha}_F(X) \), so \( Y^{\alpha} \in H_F(X^{\alpha}) \), and the multiplication is separately \( \sigma \)-continuous, we have

\[
J = JX^{\alpha}J = X^{\alpha}(Y^{\alpha}X^{\alpha}Y^{\alpha})X^{\alpha} = X^{\alpha}Y^{\alpha}X^{\alpha}
\]

and

\[
Y^{\alpha}X^{\alpha}Y^{\alpha}X^{\alpha} = Y^{\alpha}
\]

Using the notation above, let \( X_J = JXJ \). Since \( \gamma \in \tilde{sp}_F(\alpha|X_J) \), it follows that

\[
(X_J)_\gamma(X_J)_\gamma^\ast = (JXJ)_\gamma^\ast(JXJ)_\gamma = J
\]

Therefore, since \( Y \in H^{\alpha}_F(X) \), so \( Y^{\alpha} \in H_F(X^{\alpha}) \), and the multiplication is separately \( \sigma \)-continuous, we have

\[
J^+ \subset JX^{\alpha}J
\]

and

\[
J = JX^{\alpha}J = X^{\alpha}Y^{\alpha}X^{\alpha}
\]

I will prove next that \( Y_\gamma = Y^{\alpha}Y_\gamma Y^{\alpha} = Y^{\alpha}X_\gamma Y^{\alpha} \). Actually, for the the proof of the theorem, we need only the obvious equality \( Y_\gamma = X^{\alpha}Y_\gamma Y^{\alpha} \), but I believe that the equality without the closure is worth proving. Clearly

\[
Y^{\alpha}X_\gamma Y^{\alpha} \subset Y_\gamma
\]

Now let \( x \in Y_\gamma \). Denote \( a = x^\ast x \in Y^{\alpha}, b = xx^\ast \in Y^{\alpha} \). Applying [17, Proposition 1.4.5.] and its proof, for any \( s > 0, s < \frac{1}{2} \), fixed, we have

\[
x^\ast = vb^s \text{ where } v = (norm) \lim_{n \to \infty} x^\ast \left( \frac{1}{n} + b \right)^{-\frac{1}{2}} b^{1-s}
\]

so

\[
x = b^sv^\ast = b^s(norm) \lim_{n \to \infty} b^{1-s} \left( \frac{1}{n} + b \right)^{-\frac{1}{2}} x
\]

and

\[
x = ua^s \text{ where } u = (norm) \lim_{n \to \infty} x \left( \frac{1}{n} + a \right)^{-\frac{1}{2}} a^{1-s}
\]

Therefore,

\[
x = b^s[(norm) \lim_{n \to \infty} (b^{1-s} \left( \frac{1}{n} + b \right)^{-\frac{1}{2}} x \left( \frac{1}{n} + a \right)^{-\frac{1}{2}} a^{1-s})]a^s
\]

and, since

\[
(norm) \lim_{n \to \infty} (b^{1-s} \left( \frac{1}{n} + b \right)^{-\frac{1}{2}} x \left( \frac{1}{n} + a \right)^{-\frac{1}{2}} a^{1-s}) \in Y_\gamma
\]
the equality is proven. So, in particular, it follows that

$$Y_\alpha Y_\gamma Y_\sigma = Y_\gamma$$

Hence

$$JX_\gamma J' = JY_\gamma J'$$

Therefore

$$JX_\gamma J' JX_\gamma J' = JY_\gamma J' JY_\gamma J' \subset X_\alpha (Y_\gamma Y_\sigma) X_\alpha$$

and thus

$$(X_\gamma Y_\gamma J') = X_\alpha (Y_\gamma Y_\sigma) X_\alpha$$

By multiplying the above equality by $Y_\alpha$ to the left and to the right, and taking into account the separate F-continuity of the multiplication, we get

$$Y_\gamma Y_\sigma = Y_\alpha$$

Therefore, $\gamma \in \tilde{sp}_F(\alpha|_Y)$ and we are done.

Let $(X, F)$ be a dual pair of Banach spaces with $X$ a C*-algebra. $X$ is said to be $F$-simple if $X$ does not contain any non-trivial $F$-closed two sided ideal.

3.3. Remark If $X$ is $F$-simple and $Y \subset X$ is an $F$-closed hereditary C*-subalgebra, then $Y$ is $F$-simple.

Proof. Indeed, let $Y \subset X$ be an $F$-closed hereditary C*-subalgebra. Then, it follows that

$$Y = Y XY$$

Indeed, since, in particular, $Y$ is a hereditary C*-subalgebra of $X$ it follows that $Y = Y XY$. Hence

$$Y XY \subset Y = Y XY \subset Y XY.$$ 

Since $Y$ is $F$-closed the claim follows. Now, if $I \subset Y$ is an $F$-closed ideal of $Y$ then $XIX$ is an ideal of $X$. Since $X$ is $F$-simple, we have $XIX = X$. Therefore

$$YXIXY' = Y$$

On the other hand, since $I$ is an ideal of $Y$

$$YXIXY' = YXIXY' = I$$

Thus $I = Y$ and $Y$ is $F$-simple.

Let $(X, G, \alpha)$ be an $F$-dynamical system. $X$ is called $\alpha$-simple if $X$ does not contain any non-trivial $F$-closed $\alpha$-invariant two sided ideal. The following result extends [3, Théorème 3.4.1.] and [16,Theorem 2, i)\iff ii)] to the case of
\( \mathcal{F} \)-dynamical systems. It will be used in the characterization of maximality for periodic \( \mathcal{F} \)-dynamical systems (Proposition 4.5. below).

### 3.4. Theorem
Let \((X, G, \alpha)\) be an \( \mathcal{F} \)-dynamical system with \( G \) compact abelian. Then the following are equivalent:

i) \( X^\alpha \) is \( \mathcal{F} \)-simple

ii) \( X \) is \( \alpha \)-simple and \( \tilde{\Gamma}_\mathcal{F}(\alpha) = sp(\alpha) \).

**Proof.** i)⇒ii) Suppose that \( X^\alpha \) is \( \mathcal{F} \)-simple. Then it is immediate from the definitions that \( sp_\mathcal{F}(\alpha) = sp(\alpha) \). Let \( \gamma \in sp(\alpha) \) be arbitrary. Applying Theorem 3.2. it follows that \( \gamma \in \tilde{\Gamma}_\mathcal{F}(\alpha) \), so \( \tilde{\Gamma}_\mathcal{F}(\alpha) = sp(\alpha) \). Let us prove that \( X \) is \( \alpha \)-simple. If \( J \) is an \( \mathcal{F} \)-closed \( \alpha \)-invariant ideal of \( X \), then \( J \alpha \) is an ideal of \( X \alpha \). Since \( X \alpha \) is \( \mathcal{F} \)-simple, it follows that \( J \alpha = X \alpha \), so \( J = X \).

ii)⇒i) Suppose that \( X \) is \( \alpha \)-simple and that \( \tilde{\Gamma}_\mathcal{F}(\alpha) = sp(\alpha) \). Let \( J \) be an \( \mathcal{F} \)-closed ideal of \( X \alpha \). We will prove first that, for every \( \gamma \in sp(\alpha) \) we have \( X_\gamma JX_\gamma^* \subset J \). Then, by using Lemma 3.1. we have that \( (XJX^{\sigma})^\alpha \subset J \) and, since \( X \) is assumed to be \( \alpha \)-simple it will follow \( J = X^\alpha \). Let now \( \gamma \in sp(\alpha) = \Gamma_\mathcal{F}(\alpha) \). If we denote \( Y = \frac{JXJ^{\sigma}}{\sigma} \), then, since \( JXJ^{\sigma} \) is a \( C^* \)-hereditary subalgebra of \( X \), from Lemma 2.8. it follows that \( Y \in \mathcal{H}_\mathcal{F}^\alpha(X) \). It is immediate that \( Y_\gamma = \frac{JX_\gamma J^{\sigma}}{\sigma} \). Since \( \gamma \in \Gamma_\mathcal{F}(\alpha) \), we have

\[
Y^\alpha = Y_\gamma Y_\gamma^{*\sigma} = \frac{JX_\gamma J^{\sigma} J^{*\sigma}}{\sigma} \subset X^\alpha JX^\alpha \subset J
\]

It follows that

\[
X_\gamma^* JX_\gamma \subset J
\]

for every \( \gamma \in sp(\alpha) \). By Lemma 3.1.

\[
(XJX^{\sigma})^\alpha = \text{lin} \{ X_\gamma JX_\gamma^{*\sigma} | \gamma \in \sigma(\alpha) \}
\]

Hence

\[
(XJX^{\sigma})^\alpha \subset J
\]

Since \( X \) is \( \alpha \)-simple, it follows that \( J = X^\alpha \), so \( X^\alpha \) is \( \mathcal{F} \)-simple.

### 4 Maximaliy of subalgebras of analytic elements associated with one-parameter \( \mathcal{F} \)-dynamical systems

This section contains our main result. Let \((X, \mathbb{R}, \alpha)\) be a non trivial one-parameter \( \mathcal{F} \)-dynamical system. Then, according to Lemma 2.11. v), \( X^\alpha([0, \infty)) \)
Examples of such dual pairs were given in Remark 2.2. Throughout this section we will assume that Definition 2.1., 1)-5) is satisfied. To C*-algebras. C*-algebra techniques, including the use of irreducible representations, specific to von Neumann algebras and those used by Kishimoto [12] are mostly the translation of \( C \)-subalgebra of \( X \) among all \( F \)-closed subalgebras of \( X \) is characterized by a spectral condition (Condition 4.3. below). Our result contains all the special cases considered previously ([27], [10], [7], [13], [14], [18], [19], [24], [25], [12]). In addition our result contains also the case of multiplier dynamical systems that was not considered before. In particular, as we mentioned in the Introduction, if \( C_b(\mathbb{R}) \) is the C*-algebra of bounded continuous functions on \( \mathbb{R} \) and \( \alpha_i(f)(s) = f(s-t) \) is the translation of \( f \) by \( t \in \mathbb{R} \), then \( H^\infty(\mathbb{R}) \cap C_b(\mathbb{R}) \) is a maximal strictly closed subalgebra of \( C_b(\mathbb{R}) \). I mention that the methods used by Solel [24], and [25] are specific to von Neumann algebras and those used by Kishimoto [12] are mostly C*-algebra techniques, including the use of irreducible representations, specific to C*-algebras.

Let \( (X,F) \) be a dual pair of Banach spaces such that \( X \) is a C*-algebra. Throughout this section we will assume that Definition 2.1., 1)-5) is satisfied. Examples of such dual pairs were given in Remark 2.2.

### 4.1. Proposition

Suppose that \( (X,F) \) is a dual pair of Banach spaces such that \( X \) is a C*-algebra, so Definition 2.1., 1)-5) holds. Then the dual \( F^* \) of \( F \) is a von Neumann algebra. Actually, there exists a central projection \( q \in X^{**} \) such that \( F^* = qX^{**} \).

**Proof.** We will prove first that \( F \) is closed to involution and to translations with elements of \( X \). Indeed let \( \varphi \in F \). If \( \{x_\alpha\} \subset X \), is such that \( x_\alpha \to x_0 \in X \) in the \( F \)-topology, then since the involution in \( X \) is \( F \)-continuous, it follows that \( x^*_\alpha \to x^*_0 \) in the \( F \)-topology of \( X \), so \( \varphi(x^*_\alpha) \to \varphi(x^*_0) \). Hence \( \varphi^*(x_\alpha) = \varphi(x^*_\alpha) \to \varphi(x^*_0) = \varphi^*(x_0) \) and therefore \( \varphi^*(x_\alpha) \to \varphi^*(x_0) \). It follows that \( \varphi^* \) is \( F \)-continuous functional on \( X \). Similarly, if \( y \in X \), since the multiplication in \( X \) is separately \( F \)-continuous, it follows that \( (Ly\varphi)(x_\alpha) = \varphi(yx_\alpha) \to \varphi(yx_0) = (Ly\varphi)(x_0) \), so \( Ly\varphi \) is \( F \)-continuous linear functional on \( X \). Then, according to a well known fact about dual pairs of topological vector spaces ([22, Chapter IV, 1.2.]), we have that \( \varphi^* \in F \) and \( Ly\varphi \in F \). Using the facts proven above, one can immediately infer that the annihilator \( F^\perp \) of \( F \) in \( X^{**} \) is a two sided ultraweakly closed ideal of \( X^{**} \), so \( F^\perp = pX^{**} \) for some central projection \( p \in X^{**} \) and therefore the dual \( F^* \) of \( F \), which is isomorphic with the quotient \( X^{**}/F^\perp = (1-p)X^{**} \), is a von Neumann algebra with identity \( q = 1 - p \).

We can state the following

### 4.2. Remark

Suppose that \( (X,F) \) is a dual pair of Banach spaces such that \( X \) is a C*-algebra. Then

i) \( X^{**} = F^* \)

ii) If \( B \in H^a_\sigma(X) \), then there exists a projection \( p \in F^* \) such that \( \overline{B}^{**} = (pF^*p) \), so \( B = (pF^*p) \cap X \) and, in particular,

iii) If \( Y \) is an \( F \)-closed ideal of \( X \), then there exists a central projection \( q \in F^* \) such that \( Y = (qF^*q) \cap X \), so \( Y^{**} = qF^*q \).
iv) If $p \in \mathcal{F}^*$ is a projection which is the strong limit of an increasing net $\{e_\lambda\}$ of positive elements of $X$, then $p\mathcal{F}^*p \cap X$ is an $\mathcal{F}$-closed hereditary $C^*$-subalgebra of $X$ that contains $\{e_\lambda\}$.

v) Let $(X, \mathbb{R}, \alpha)$ be a one-parameter $\mathcal{F}$-dynamical system and $\delta > 0$. Let $\{e_\lambda\}_\lambda$ be an approximate identity of $X$ and $f \in L^1(\mathbb{R}), f \geq 0$ such that

$$\int_G f(p) dp = 1 \text{ and supp}\hat{f} \subset (-\delta, \delta).$$

Then $\{d_\lambda\}_\lambda$ where

$$d_\lambda = \alpha_f(e_\lambda) = \int_G f(p)\alpha_p(e_\lambda) dp \in X^{\alpha}((-\delta, \delta)).$$

is an approximate identity of $X$.

**Proof.** Part i) follows from well known facts about dual pairs of topological vector spaces [22, Chapter IV, 1.3.]. To prove ii) notice that from Lemma 2.8 applied to the pair $(\mathcal{F}^*, \mathcal{F})$ it follows that $\overline{\mathcal{B}}^w$ is an $\mathcal{F}$-closed hereditary subalgebra of the von Neumann algebra $\mathcal{F}^*$ and therefore there exists a projection $p \in \mathcal{F}^*$ such that $\overline{\mathcal{B}}^w = p\mathcal{F}^*p$. Since the pair $(X, \mathcal{F})$ is a dual pair of Banach spaces, the Hahn Banach separation theorem implies $\overline{\mathcal{B}}^w \cap X = B$. Part iii) is a consequence of ii). The proof of iv) is straightforward. Next, we will prove v). Such a function, $f$, exists (for example a Fejer kernel). Since $\int_\mathbb{R} f(p) dp = 1$ it follows immediately that $\|d_\lambda\| = \sup \{\|\varphi(d_\lambda)\| : \varphi \in \mathcal{F}, \|\varphi\| \leq 1\} \leq \int f(p) \|\alpha_p(e_\lambda)\| dp \leq 1$, for all $\lambda$. Let $x \in X$ and $p \in G$. Since $\{e_\lambda\}_\lambda$ is an approximate identity of $X$, we have, for every $p \in G$

$$(\text{norm}) \lim_\lambda x\alpha_p(e_\lambda) = (\text{norm}) \lim_\lambda \alpha_p(\alpha_{p^{-1}}(x)e_\lambda) = x.$$

and

$$(\text{norm}) \lim_\lambda \alpha_p(e_\lambda)x = (\text{norm}) \lim_\lambda \alpha_p(e_\lambda\alpha_{p^{-1}}(x)) = x.$$

Using the definition of $\alpha_f$, the assumption that $f \geq 0$, the Lebesgue dominated convergence theorem and taking into account that $\int_G f(p) dp = 1$ the result follows.

In the rest of this paper we will assume that the $\mathcal{F}$-dynamical system $(X, \mathbb{R}, \alpha)$ is non trivial, that is $sp(\alpha) \neq \{0\}$.

The next Condition will be proven to be necessary and sufficient for maximality of subalgebras of analytic elements of non trivial $\mathcal{F}$-dynamical systems.

### 4.3. Spectral Condition for one-parameter $\mathcal{F}$-dynamical systems

Let $(X, \mathbb{R}, \alpha)$ be a non trivial $\mathcal{F}$-dynamical system. Suppose that

a) Either $sp(\alpha) = \{-\gamma_0, 0, \gamma_0\}$ for some $\gamma_0 > 0$ and $J = \overline{X_{\gamma_0}X^*_{\gamma_0}}$ is an $\mathcal{F}$-simple ideal of $X^\alpha$.

or

b) There exists an $\mathcal{F}$-closed $\alpha$-simple two sided ideal $Y \subset X$ such that

1) $Y^\alpha((0, \infty)) = X^\alpha((0, \infty))$, and so $Y^\alpha((-\infty, 0)) = X^\alpha((-\infty, 0))$ as well.

b2) $sp(\alpha|_Y) = \Gamma_f(\alpha|_Y)$.

b3) $Y + X^\alpha([0, \infty)) = X$. 

15
Suppose that \((X, \mathbb{R}, \alpha)\) is a periodic \(F\)-dynamical system, that is \(sp(\alpha) \subset \mathbb{Z} \gamma_1\) for some \(\gamma_1 > 0\). Then the quotient group \(\mathbb{R}/\mathbb{Z} \gamma_1\) is a compact group and \(\hat{G} = \mathbb{Z} \gamma_1\). We will prove that for periodic \(F\)-dynamical systems the Condition 4.3. is equivalent to the following condition:

\[(S)\] There exists an \(F\)-simple ideal \(J \subset X^\alpha\) such that for every \(\gamma \in sp(\alpha)\), \(\gamma \neq 0\), \(X_\gamma X'_\gamma = J\).

Condition \((S)\) above was considered by us [19] in the particular case of periodic \(C^*\)-dynamical systems. The proof of the equivalence of the Condition 4.3. and condition \((S)\) for periodic \(F\)-dynamical systems will be given below. This equivalence is necessary in proving that if \((X, \mathbb{R}, \alpha)\) is a periodic \(F\)-dynamical system such that \(X^\alpha([0, \infty))\) is a maximal \(F\)-closed subalgebra of \(X\), then Condition 4.3. is satisfied (Lemma 4.10. below).

Let \((X, \mathbb{R}, \alpha)\) be a one-parameter \(F\)-dynamical system. In what follows, we will need the following notations: If there exists \(\gamma_1 \in sp(\alpha), \gamma_1 > 0\) such that \(sp(\alpha) \subset \mathbb{Z} \gamma_1\), then the annihilator \((sp(\alpha))_\perp \subset \mathbb{R}\) is a discrete subgroup of \(\mathbb{R}\) that equals \((\mathbb{Z} \gamma_1)_\perp = \mathbb{Z} \frac{\mathbb{Z}}{\gamma_1}\). In this case we can consider the \(F\)-dynamical system \((X, G, \tilde{\alpha})\) with the compact group \(G = \mathbb{R}/(\mathbb{Z} \gamma_1)_\perp\). If \(h: \mathbb{R} \to G\) is the canonical mapping, then \(\tilde{\alpha}_h(t)(x) = \alpha_t(x)\). Clearly, \(sp(\tilde{\alpha}) = sp(\alpha) \subset \mathbb{Z} \gamma_1\). For every \(\gamma \in sp(\tilde{\alpha}) = sp(\alpha)\) let \(P^\alpha_\gamma: X \to X_\gamma\) be the mapping defined above

\[P^\alpha_\gamma(x) = \int_G \overline{(t, \gamma)\tilde{\alpha}_t(x)} dt\]

where \(dt\) is the normalized Haar measure on \(G\) and the integral is taken in the \(F\)-topology as discussed above. The next lemma is an extension of [19, Proposition 10] to the case of \(F\)-dynamical systems.

### 4.4. Lemma

Suppose that \((X, \mathbb{R}, \alpha)\) is a periodic \(F\)-dynamical system and Condition \((S)\) holds. Then, either there exists \(\gamma_1 > 0\), such that \(sp(\alpha) = \{-\gamma_1, 0, \gamma_1\}\) and \(X_{-\gamma_1} X^\alpha_{\gamma_1}\) is a simple ideal of \(X^\alpha\), or there exists \(\gamma_1 > 0\) such that \(sp(\alpha) = \mathbb{Z} \gamma_1\), so, in this case, \(sp(\alpha)\) is a discrete subgroup of \(\mathbb{R}\).

**Proof.** We will prove first that, if Condition \((S)\) holds and \(\gamma, \gamma' \in sp(\alpha) \cap (0, \infty)\), then \(\gamma - \gamma' \in sp(\alpha)\). From the definition of spectral subspaces it follows that \(X_{-\gamma'} X_{\gamma} \subset X_{-\gamma -\gamma'}\), and thus if we prove that \(X_{-\gamma'} X_{\gamma} \neq \{0\}\), then it will follow that \(\gamma - \gamma' \in sp(\alpha)\). Suppose to the contrary that

\[X_{-\gamma'} X_{\gamma} = \{0\}\]

By multiplying the above equality to the left by \(X_{\gamma'}\) and to the right by \(X_{-\gamma}\) we get

\[X_{\gamma'} X_{-\gamma'} X_{\gamma} X_{-\gamma} = \{0\}\]

Taking into account that the multiplication in \(X\) is -continuous it follows that

\[J = JJ = \{0\}\]
which contradicts the assumptions that $\gamma, \gamma' \in sp(\alpha)$ so the claim is proven. Now suppose that $sp(\alpha)$ contains more than three points. Let $\gamma_1$ be the smallest positive element of $sp(\alpha)$. Denote $\gamma_2 = \min \{\gamma \in sp(\alpha): \gamma > \gamma_1\}$. Since $sp(\alpha)$ contains more than three points and the system is periodic, $\gamma_2 > \gamma_1$. Then $\gamma_2 - \gamma_1 \in sp(\alpha)$. From the definition of $\gamma_1$ and $\gamma_2$ it immediately follows that $\gamma_2 = 2\gamma_1$. Therefore

$$X_{2\gamma_1}X_{-2\gamma_1} = X_{\gamma_1}X_{-\gamma_1} = J$$

By multiplying the above double equality to the left by $X_{-\gamma_1}$ and to the right by $X_{\gamma_1}$ and taking into account that, as shown above, $X_{-\gamma_1}, X_\gamma \neq \{0\}$ for every $\gamma, \gamma' \in sp(\alpha)$, we get that $\{0\} \neq X_{-\gamma_1}X_{\gamma_1} \subset J$. Since $J$ is $\mathcal{F}$-simple, and clearly $X_{-\gamma_1}X_{\gamma_1}$ is an ideal, it follows that

$$X_{-\gamma_1}X_{\gamma_1} = J$$

This equality and the definition of $J$ implies

$$X_{\gamma_1} = X_{\gamma_1}J$$

and

$$X_{\gamma_1}JX_{-\gamma_1} = J$$

It follows that

$$X_{\gamma_1}X_{2\gamma_1}X_{-2\gamma_1}X_{-\gamma_1} = J$$

Hence $X_{\gamma_1}X_{2\gamma_1} \neq \{0\}$ and thus $X_{3\gamma_1} \neq \{0\}$, so $3\gamma_1 \in sp(\alpha)$. By induction it follows that $sp(\alpha) = \mathbb{Z}\gamma_1$ and the lemma is proven. $\blacksquare$

4.5. Proposition Let $(X, \mathbb{R}, \alpha)$ be a periodic $\mathcal{F}$-dynamical system. Then Condition 4.3. and Condition (S) are equivalent.

Proof. Suppose Condition 4.3. holds. If $sp(\alpha)$ contains only three points we have nothing to prove. Suppose that $sp(\alpha)$ contains more than three points. By Condition 4.3. b2), there exists $\gamma_1 \in \mathbb{R}, \gamma_1 > 0$ such that $sp(\alpha) = \mathbb{Z}\gamma_1$. Let $Y \subset X$ be the $\alpha$-simple ideal of $X$ as in Condition 4.3. b). Clearly, since $Y$ is an ideal of $X$, it follows that $Y^\alpha$ is an ideal of $X^\alpha$. Then, using Condition 4.3. b2) and Theorem 3.4. it follows that $Y^\alpha$ is $\mathcal{F}$-simple. Therefore, if $\gamma \in sp(\alpha)$, since $\overline{Y\gamma Y^\alpha}$ is an $\mathcal{F}$-closed ideal of $Y^\alpha$, it follows that $\overline{Y\gamma Y^\alpha} = Y^\alpha$, so Condition (S) is satisfied.

Now suppose that Condition (S) holds true. If Condition 4.3., a) does not hold then, by the previous Lemma 4.4., $sp(\alpha)$ is a discrete subgroup of $\mathbb{R}$. Then, as noticed in the proof of Lemma 4.4., $X_\gamma X_\gamma = J$ for every $\gamma \in sp(\alpha), \gamma \neq 0$. Let $Y = \overline{XJX}$ and $\gamma_0 \in sp(\alpha), \gamma_0 \neq 0$. Applying Lemma 3.1. and taking into account Condition (S), we get

$$X_{\gamma_0} = JX_{\gamma_0} = X_{\gamma_0}JX_{-\gamma_0}X_{-\gamma_0} \subset Y_{\gamma_0} \subset X_{\gamma_0}.$$
so $Y_{\gamma_0} = X_{\gamma_0}$ and therefore Condition 4.3. b1) is satisfied. Applying again Lemma 3.1. and Condition (S), it follows that $Y^\alpha = J$. Since $Y^\alpha$ is $\mathcal{F}$-simple, Theorem 3.4. implies that $Y$ is $\alpha$-simple and $\Gamma_\mathcal{F}(\alpha|_Y) = sp(\alpha|_Y)$. By Lemma 2.11. i), Condition 4.3. b3) is satisfied and we are done. ■

In the next Proposition we will discuss the Condition 4.3. a). We will describe the structure of the systems satisfying Condition 4.3. a) and will prove that Condition 4.3. a) implies the maximality of $X^\alpha([0, \infty))$.

4.6. Proposition Suppose that Condition 4.3. a) is satisfied. Then

i) $I = X_{\gamma_0} X_{\gamma_0}^\sigma$ is an $\mathcal{F}$-simple ideal of $X^\alpha$ and $IJ = \{0\}$.

ii) $Z = (I + J)X(I + J)^\sigma$ is an $\mathcal{F}$-simple $C^*$-algebra and if $p_1$ and $p_2$ are the projections in $\mathcal{F}$ corresponding to the hereditary $C^*$-subalgebras $IIXJ$ and $TT'X$ of $X$ as in Remark 4.2. Then $X^\alpha([0, \infty)) = \{x \in X : p_2xp_1 = 0\}$.

iii) $X^\alpha([0, \infty))$ is a maximal $\mathcal{F}$-closed subalgebra of $X$.

Proof. i) Let $I_1$ be an $\mathcal{F}$-closed ideal of $I$. Then $X_{\gamma_0} I_1 X_{\gamma_0}$ is an $\mathcal{F}$-dense ideal of $J$. Since $J$ is an $\mathcal{F}$-simple $C^*$-subalgebra of $X^\alpha$ it follows that $X_{\gamma_0} I_1 X_{\gamma_0} = J$ and therefore, by multiplying this equation to the left by $X_{\gamma_0}$ and to the right by $X_{\gamma_0}$ it follows that $I_1 = X_{\gamma_0} I = I$ and therefore $I$ is $\mathcal{F}$-simple.

ii) We will prove first that $Z = (p_1 + p_2)\mathcal{F}^\alpha(p_1 + p_2) \cap X$ is $\alpha$-simple. Notice that $Z = (I + J)X(I + J)^\sigma$ and $Z^\alpha = T + I$. Then, if $K \subset Z$ is a $\sigma$-closed $\alpha$-invariant ideal, it follows that $K^\alpha$ is an ideal of $T + I$. If $K^\alpha J = \{0\}$ then, if $x \in K_{\gamma_0}$ we have $xx^* \in K \cap J$, so $x = 0$. Therefore, in this case, $K = K^\alpha = I$.

But $I$ cannot be an ideal of $Z$ since $IX_{\gamma_0} = X_{\gamma_0} \not\subseteq I$. Similarly, it can be shown that the situation $K^\alpha I = \{0\}$ cannot occur. Therefore $K^\alpha J \neq \{0\}$ and $K^\alpha I \neq \{0\}$. Since $I, J$ are $\mathcal{F}$-simple ideals of $X^\alpha$ and $K^\alpha$ is an ideal of $T + I$ it follows that $K^\alpha = T + K^\alpha$, so $K = Z$. Hence $Z$ is $\alpha$-simple. We will prove next that $Z$ is $\mathcal{F}$-simple. Since $R$ is connected and $sp(\alpha)$ is finite, by [17, 8.11.2] it follows that $\alpha$ is uniformly continuous. On the other hand, it is clear that the spectrum of the action $\alpha^*$ induced by $\alpha$ on $\mathcal{F}$ is also compact (and equals $\{-\gamma_0, 0, \gamma_0\}$) as is the spectrum of $\alpha^{**}$ the action on $\mathcal{F}^*$ induced by $\alpha^*$. Therefore, $\alpha^{**}$ is uniformly continuous. According to [17, 8.5.3.], there exists a uniformly continuous unitary representation $t \to u_t$ of $R$ into $Z^{**} \subset \mathcal{F}^*$ such that

$$\alpha_t^{**}(x) = u_t xu_t^*$$

for all $t \in R, x \in Z^{**}$.

It follows, in particular, that every central projection of $Z^{**}$ is $\alpha^{**}$-invariant. This means that every $\mathcal{F}$-closed ideal of $Z$ is (globally) $\alpha$-invariant. Since by the previous arguments $Z$ is $\alpha$-simple, it follows that $Z$ is $\mathcal{F}$-simple. The fact that $X^\alpha([0, \infty)) = \{x \in X : p_2xp_1 = 0\}$ is straightforward.

iii) Let $W$ be an $\mathcal{F}$-closed subalgebra of $X$ such that $X^\alpha([0, \infty)) \subseteq W$. Then, by [19, Lemma 4 and Corollary 6], $W$ is $\alpha$-invariant. If $X^\alpha([0, \infty)) \not\subseteq W$, then $W_{-\gamma_0} = \{0\}$. It is easy to check that $W_{-\gamma_0} W_{-\gamma_0}^*$ is an ideal of $I$. Therefore, since $I$ is $\mathcal{F}$-simple, $\overline{W_{-\gamma_0} W_{-\gamma_0}^*}$ is $\mathcal{F}$-dense in $I$. Since, obviously, $W_{-\gamma_0} W_{-\gamma_0}^*$ is a
norm dense two sided ideal of $W_{-\gamma_0}W_{-\gamma_0}^*$, we can apply [5, Proposition 1.7.2]. Therefore, there exists an approximate identity $(e_\lambda)$ of $W_{-\gamma_0}W_{-\gamma_0}^*$ contained in $W_{-\gamma_0}W_{-\gamma_0}^*$. It then follows that $w^* - \lim e_\lambda = p_2$ in $\mathcal{F}^*$. On the other hand, since $IX_{-\gamma_0} = X_{-\gamma_0}$, it follows that $(e_\lambda)$ is a left approximate identity of $X_{-\gamma_0}$ in the $\mathcal{F}$-topology of $X$. So, if $x \in X_{-\gamma_0}$ we have $e_\lambda x \in W_{-\gamma_0}$ and $e_\lambda x \to x$ in the $\mathcal{F}$-topology of $X$. Since the spectral subspace $W_{-\gamma_0}$ is $\mathcal{F}$-closed it follows that $x \in W$. Hence $W = X$. ■

Notice that the projections $p_1$ and $p_2$ may not belong to the multiplier algebra of $X$ as claimed in [12, Theorem 3.7. i)].

4.7. Lemma Let $Y$ be a $C^*$-algebra of operators on a Hilbert space $H$ whose weak closure, $\overline{Y}^w$ is a von Neumann algebra, i.e. contains the identity 1 of $B(H)$, where $B(H)$ is the algebra of all linear bounded operators on $H$. Let $T \subset Y$ be a norm closed subspace of $Y$ such that the norm-closed hereditary $C^*$-subalgebra $C = \overline{TT^*}^\|\|$ of $Y$ is w-dense in $Y$. Then there exists a directed family $\{e_\lambda\} \subset \overline{alg(TT^*)}^\| \cap Y^1_+$ such that $\lim_\lambda e_\lambda = 1$ in the strong operator topology of $\overline{Y}^w$, where $\overline{alg(TT^*)}^\|$ denotes the norm closed subalgebra of $Y$ generated by $TT^*$.

Proof. Let $\Lambda$ be the collection of all finite subsets $\{x_i\}_{i=1}^n \subset T, n \in \mathbb{N}$. If $\lambda \in \Lambda$, set $a_\lambda = \sum_{i=1}^n x_i x_i^*$ and $e_\lambda = a_\lambda(\frac{1}{n} + a_\lambda)^{-1}$. It is clear that $e_\lambda \in \overline{alg(TT^*)}^\| \cap Y^1_+$ and by Lemma 2.9., $e_\lambda \in C$. The next arguments are inspired by [5, proof of Proposition 1.7.2], even if that result is different from the statement of our current Lemma 4.7. One can easily check that for each $i = 1, \ldots, n$.

$$|| (1 - e_\lambda) x_i x_i^* (1 - e_\lambda) || \leq \sum_{j=1}^n || (1 - e_\lambda) x_j x_j^* (1 - e_\lambda) || = \frac{1}{n^2} \left| a_\lambda \left( \frac{1}{n} + a_\lambda \right)^{-2} \right|.$$ 

and, since as noticed in [5, Proof of Proposition 1.7.2], $t(\frac{1}{n} + t)^{-2} \leq \frac{4}{n}$ for all $n \in \mathbb{N}, t \in \mathbb{R}^+$, we have

$$|| (1 - e_\lambda) x_i x_i^* (1 - e_\lambda) || \leq \frac{1}{4n}.$$ 

Therefore, $\lim_\lambda || (1 - e_\lambda) x || = 0$ for all $x \in T$. Hence, $\lim_\lambda || (1 - e_\lambda) x y z^* || = 0$ for all $x, z \in T, y \in Y$. It then follows that $\lim_\lambda || (1 - e_\lambda) z || = 0$ for all $z \in \overline{TYT^*}^\| = C$. Then $\overline{C}^w = \overline{Y}^w \subset B(H)$ and certainly, 1 is the unit of $\overline{C}^w$. Let $\xi \in H$. Then, for every $\epsilon > 0$ there exists $z \in C$ such that $|| z \xi - \xi || < \frac{\epsilon}{3}$. Since $\{e_\lambda\}$ is an approximate identity of $C$, there exists $\lambda_0$ such that $|| e_\lambda z - z || < \frac{\epsilon}{3}$ for all $\lambda \gg \lambda_0$. Therefore

$$|| e_\lambda \xi - \xi || \leq || e_\lambda (\xi - z \xi) || + || (e_\lambda z - z) \xi || + || z \xi - \xi || < \epsilon.$$ 

for every $\lambda \gg \lambda_0$. ■

19
The next lemma is an extension of [12, Lemma 3.1.] to the case of dual $\mathcal{F}$-dynamical systems and general locally compact groups $G$.

**4.8. Lemma** Let $(Y, G, \alpha)$ be an $\alpha$-simple $\mathcal{F}$-dynamical system with $G$ a locally compact abelian group such that $\text{sp}(\alpha) = \hat{\Gamma}_F(\alpha)$. If $C \in H_\sigma^\infty(Y)$ and $\gamma \in \text{sp}(\alpha)$, then $\overline{Y^\alpha(V) CY^\alpha(V)^*} = Y$ for every open neighborhood, $V$, of $\gamma$.

**Proof.** Since $\text{sp}(\alpha) = \hat{\Gamma}_F(\alpha)$, it follows that $\hat{\Gamma}_F(\alpha)$ is a closed subgroup of $\Gamma = \hat{G}$. Let $U$ be a fixed neighborhood of $0 \in \Gamma$ so $V = U + \gamma$ is a fixed neighborhood of $\gamma$. Further, let $\{\gamma_i\}_{i \in I}$ be a dense subset of $\hat{\Gamma}_F(\alpha)$ and, for each $i \in I$, let $U_i$ be an open neighborhood of $\gamma_i$ such that $U_i - U_i \subset U$. Since $\Gamma_F(\alpha) \subset \cup U_i$, by Lemma 2.11. i) it follows that $Y^\alpha(\hat{\Gamma}_F(\alpha)) = Y \subset \sum_{i \in I} Y^\alpha(U_i)$. So

$$\sum_{i \in I} Y^\alpha(U_i)^* = Y.$$

Therefore

$$\sum_{i \in I} Y^\alpha(U_i) \sigma C = YC.$$

and

$$C \sum_{i \in I} Y^\alpha(U_i)^* = CY.$$ 

On the other hand, since $C \in H_\sigma^\infty(Y)$ and $\hat{\Gamma}_F(\alpha)$ is a group, for each $i \in I$ we have $\gamma_i - \gamma \in \hat{\Gamma}_F(\alpha)$, so $\overline{C^\alpha(-U_i + \gamma) CC^\alpha(-U_i + \gamma)^*} = C$. Therefore

$$Y^\alpha(U_i)C = Y^\alpha(U_i)C \overline{C^\alpha(-U_i + \gamma) CC^\alpha(-U_i + \gamma)^*} \subset \overline{Y^\alpha(U_i)CC^\alpha(-U_i + \gamma) CC^\alpha(-U_i + \gamma)^*} \subset \overline{Y^\alpha(U + \gamma)C^\gamma}.$$

Similarly, for each $\kappa \in I$

$$CY^\alpha(U_\kappa) \subset \overline{CY^\alpha(U + \gamma)^*}.$$ 

Since the multiplication in $Y$ is separately $\mathcal{F}$-continuous, it follows that for every $i, \kappa \in I$

$$Y^\alpha(U_i)CY^\alpha(U_\kappa) \subset \overline{Y^\alpha(U + \gamma) CY^\alpha(U + \gamma)^*}$$

so

$$YCY \subset \overline{Y^\alpha(U + \gamma) CY^\alpha(U + \gamma)^*}.$$

Since $Y$ is $\alpha$-simple, the proof is complete. \[\blacksquare\]

**4.9. Theorem** Let $(X, \mathbb{R}, \alpha)$ be a one-parameter $\mathcal{F}$-dynamical system satisfying Condition 4.3. Then $X^\alpha([0, \infty))$ is a maximal $\mathcal{F}$-closed subalgebra of $X$. 20
Proof. If Condition 4.3. a) is satisfied, the conclusion follows from Proposition 4.6. iii). Suppose that Condition 4.3. b) is satisfied and let \( Y \) be the ideal considered in that condition. By Condition 4.3. b3), \( sp(\alpha|Y) = \overline{\Gamma_F}(\alpha|Y) \) and therefore \( sp(\alpha) \) is a closed subgroup of \( \mathbb{R} \). It follows that either \( sp(\alpha) \) is discrete, i.e. there exists \( \gamma \in \mathbb{R} \) such that \( sp(\alpha) = \{ n\gamma : n \in \mathbb{Z} \} \) or \( sp(\alpha) = \mathbb{R} \). Let \( Z \) be an \( \mathcal{F} \)-closed subalgebra of \( X \) that contains properly \( X^\alpha([0, \infty)) \). According to [19, Lemma 4 and Corollary 6], \( Z \) is \( \alpha \)-invariant. Since \( X^\alpha([0, \infty)) \subseteq Z \), there exists \( \gamma \in sp(\alpha|Z) - [0, \infty) \). We will prove that \( n\gamma \in sp(\alpha|Z) \) for all \( n \in \mathbb{N} \). This fact will be proved separately for each of the two possible cases:

\[
sp(\alpha) = \mathbb{R} \text{ or } sp(\alpha) \text{ is a discrete subgroup of } \mathbb{R}.
\]

Suppose first that \( sp(\alpha) = \mathbb{R} \), and let \( r > 0 \) such that

\[
Z^\alpha((\gamma - r, \gamma + r)) \subset X^\alpha((-\infty, 0)) = Y^\alpha((-\infty, 0))
\]

Then, there exists \( x \in Z \) with \( sp(x) \subset (\gamma - \frac{r}{3}, \gamma + \frac{r}{3}) \). Let \( S = \{ \alpha_t(x) : t \in \mathbb{R} \} \). Consider the hereditary subalgebra \( W \in \mathcal{H}_F(Y) \), where \( Y \) is as in Condition 4.3. b)

\[
W = SY^\alpha S^\ast.
\]

Since \( sp(\alpha) = \overline{\Gamma_F}(\alpha) = \mathbb{R} \), it follows that \((0, \frac{r}{3}) \cap \overline{\Gamma_F}(\alpha) \neq \emptyset \), so

\[
Y = Y^\alpha(0, \frac{r}{3})YY^\alpha(0, \frac{r}{3})^\ast.
\]

Therefore

\[
W = SY^\alpha(0, \frac{r}{3})YY^\alpha(0, \frac{r}{3})^\ast S^\ast.
\]

The last equality above follows from the assumption of separate continuity of the multiplication of \( X \) (Definition 2.1. 5)). Applying Lemma 4.8., it follows that

\[
Y = Y^\alpha(0, \frac{r}{3})SY^\alpha(0, \frac{r}{3})YY^\alpha(0, \frac{r}{3})^\ast S^\ast Y^\alpha(0, \frac{r}{3})^\ast.
\]

Clearly

\[
Y^\alpha(0, \frac{r}{3})SY^\alpha(0, \frac{r}{3}) \subset Z^\alpha((\gamma - \frac{r}{3}, \gamma + r)) \subset Z^\alpha((\gamma - r, \gamma + r)).
\]

Therefore

\[
Y = Z^\alpha((\gamma - r, \gamma + r))Y Z^\alpha((\gamma - r, \gamma + r))^\ast.
\]

We will prove now that \( n\gamma \in sp(\alpha|Z) \) for all \( n \in \mathbb{N} \). Multiplying the above to the left by \( Z^\alpha((\gamma - r, \gamma + r)) \) and to the right by \( Z^\alpha((\gamma - r, \gamma + r))^\ast \), it follows that

\[
\{0\} \neq Z^\alpha((\gamma - r, \gamma + r))Z^\alpha((\gamma - r, \gamma + r))^\ast \subset Z^\alpha((2\gamma - 2r, 2\gamma + 2r)).
\]

Since \( r > 0 \) is arbitrarily small, we have that \( 2\gamma \in sp(\alpha|Z) \). By induction, it follows that

\[
n\gamma \in sp(\alpha|Z) \text{ for all } n \in \mathbb{N}.
\]
Now suppose that $sp(\alpha) = \mathbb{Z}_{\gamma_0}$ for some $\gamma_0 \in \mathbb{R}$ and let $\gamma \in sp(\alpha|_Z) - [0, \infty)$. By taking $V = \{\gamma\}$ in Lemma 4.8., we obtain similarly that $n\gamma \in sp(\alpha|_Z)$ for all $n \in \mathbb{N}$.

By Remark 4.2. iii) there exists a central projection $q \in \mathcal{F}^*$ such that $\mathcal{Y}^w = q\mathcal{F}^*$. Let $y \in Y$ be an element such that $sp(y)$ is compact. Then, since as shown above, $sp(\alpha|_Z)$ is unbounded, there exists $\gamma \in sp(\alpha|_Z) - [0, \infty)$ and $r > 0$ such that $-\gamma + sp(y) \subset [r, \infty)$, so

$$sp(y) \subset [\gamma + r, \infty)$$

. Notice that, since $Z^\alpha((\gamma - r, \gamma + r))YZ^\alpha((\gamma - r, \gamma + r))^* \subset \mathcal{F}^*$ is $\mathcal{F}$-dense in $Y$, it follows that $\mathcal{Y}^w \subset (\mathcal{F}^*\mathcal{F})^w = \mathcal{Y}^w$ and therefore

$$C = \mathcal{Y}^w \subset (\mathcal{F}^*\mathcal{F})^w = \mathcal{F}^w$$

is strongly dense in $\mathcal{Y}^w = q\mathcal{F}^*$. By Lemma 4.7. there exists a directed set

$$\{e_\lambda\} \subset \overline{alg}(Z^\alpha((\gamma - r, \gamma + r))Z^\alpha((\gamma - r, \gamma + r))^*)$$

such that $\lim_\lambda e_\lambda = q$ in the strong operator topology of $\mathcal{Y}^w = q\mathcal{F}^*$. Therefore $\lim_\lambda e_\lambda y = y$ in the strong operator topology of $q\mathcal{F}^*$. But, since

$$e_\lambda \in \overline{alg}(Z^\alpha((\gamma - r, \gamma + r))Z^\alpha((\gamma - r, \gamma + r))^*)$$

and, by the choice of $\gamma, r$ and $y \in Z$,

$$Z^\alpha((\gamma - r, -\gamma + r))y \subset X^\alpha([0, \infty)) \subset Z$$

it follows that

$$alg(Z^\alpha((\gamma - r, \gamma + r))Z^\alpha((\gamma - r, -\gamma + r))y \subset Z$$

Since $Z$ is, in particular, norm closed, it follows that $e_\lambda y \in Z$ for all $\lambda \in \Lambda$. Since $Z$ is $\mathcal{F}$- closed in $X$, we have that $y \in Z$. Applying Lemma 2.11. i) it follows in particular that $Y \subset Z$. Since $X^\alpha([0, \infty)) \subset Z$, we have $Y + X^\alpha([0, \infty)) \subset Z$.

By Condition 4.3. b3), $Y + X^\alpha([0, \infty))$ is $\mathcal{F}$-dense in $X$. Hence $Z = X$. ■

To prove the converse of the above Theorem 4.9. we need to prove a series of lemmas. We start with the following

**4.10. Lemma** Let $(X, \mathbb{R}, \alpha)$ be a one-parameter $\mathcal{F}$-dynamical system. If $X^\alpha([0, \infty))$ is a maximal $\mathcal{F}$-closed subalgebra of $X$, then there exists an $\mathcal{F}$-closed, $\alpha$-simple ideal, $Y$, of $X$ such that

i) $Y^\alpha((0, \infty)) = X^\alpha((0, \infty))$, and so $Y^\alpha((\infty, 0)) = X^\alpha((\infty, 0))$ as well.

ii) $Y + X^\alpha([0, \infty))$ is $\mathcal{F}$-dense in $X$. 

22
Proof. Let $J_1$ be an $\alpha$-invariant ideal of $X$ such that $J_1 \not\subseteq X^\alpha$. Since $X^\alpha([0, \infty))$ is a maximal $\mathcal{F}$-closed subalgebra of $X$ and $J_1 + X^\alpha([0, \infty))$ is an $\mathcal{F}$-closed subalgebra properly containing $X^\alpha([0, \infty))$, it follows that $J_1 + X^\alpha([0, \infty)) = X$. We will prove that $J_1^\alpha((0, \infty)) = X^\alpha((0, \infty))$. Indeed, let $x \in X^\alpha((0, \infty))$. Then, $x = \mathcal{F} - \lim(x_\lambda + j_\lambda)$ where $x_\lambda \in X^\alpha[0, \infty)$ and $j_\lambda \in J_1$. Now let $f \in L^1(\mathbb{R})$ such that $\text{supp}(f) \subseteq (-\infty, 0)$. Since $\alpha_f$ is $\mathcal{F}$-continuous, it follows that $\alpha_f(x) = \mathcal{F} - \lim(\alpha_f(x_\lambda + j_\lambda))$. Since $x_\lambda \in X^\alpha([0, \infty))$, by Lemma 2.11. viii, we have $\alpha_f(x_\lambda) = 0$ and so, since $J_1$ is $\mathcal{F}$-closed and $\alpha$-invariant,

$$\alpha_f(x) \mathcal{F} - \lim(\alpha_f(j_\lambda)) \in J_1.$$  

Hence $X^\alpha((\infty, 0)) \subset J_1^\alpha((-\infty, 0))$ so $J_1^\alpha((-\infty, 0)) = X^\alpha((-\infty, 0))$ and consequently, $J_1^\alpha((0, \infty)) = X^\alpha((0, \infty))$. Let $J$ be the intersection of all such $J_1$. Then $J$ is an $\mathcal{F}$-closed $\alpha$-invariant ideal of $X$ such that $J^\alpha((-\infty, 0)) = X^\alpha((-\infty, 0))$. In order to prove the existence of the ideal $Y$ in the statement of the lemma, notice that every $\alpha$-invariant ideal $I_1$ of $J$ must be included in the fixed point algebra $X^\alpha$. Let $I$ be the $\mathcal{F}$-closed linear span of all the ideals of $X$ that are contained in $X^\alpha$. Denote

$$Y = \{y \in J : yI = \{0\}\}$$

Then

$$Y = \{y \in J : yI = \{0\}\}$$

Indeed, if $iy \neq 0$ for some $i \in I, y \in J$ with $yI = \{0\}$, then $iyy^{-1} \neq 0$ but this contradicts the fact that $yI = \{0\}$. Then $Y$ is an $\mathcal{F}$-closed $\alpha$-invariant $\alpha$-simple ideal of $X$. The fact that $Y$ is $\mathcal{F}$-closed and $\alpha$-invariant is obvious from the definition of $Y$. To prove that $Y$ is $\alpha$-simple let $K \subseteq Y$ be a non-zero $\alpha$-invariant ideal of $Y$. By the definition of $Y$ it follows that $K \not\subseteq X^\alpha$. Therefore $Y \subset J \subset K$ and thus $K = Y = J$. Clearly, $Y = J$ satisfies the conditions of the Lemma. 

We will prove first the converse of Theorem 4.9. in the special case when $0$ is an isolated point of $sp(\alpha)$.

4.11. Lemma Let $(X, \mathbb{R}, \alpha)$ be a one-parameter $\mathcal{F}$-dynamical system. Suppose that $X^\alpha([0, \infty))$ is a maximal $\mathcal{F}$-closed subalgebra of $X$. If $0$ is an isolated point of $sp(\alpha)$, then Condition 4.3. is satisfied.

Proof. We will prove first that there exists $\gamma_1 \in sp(\alpha)$ such that $sp(\alpha) \subset \mathbb{Z}\gamma_1$. Let $\gamma_1 = \inf \{\gamma \in sp(\alpha) : \gamma > 0\}$. Then, since $sp(\alpha)$ is closed and 0 is isolated in $sp(\alpha)$, we have that $\gamma_1 > 0$ and $\gamma_1 \in sp(\alpha)$. Since 0 is isolated in $sp(\alpha)$, we have $X^\alpha((-\epsilon, \epsilon)) = X^\alpha$ for every $0 < \epsilon < \gamma_1$. We will prove next that $\gamma_1$ is an isolated point of $sp(\alpha)$. Suppose that there exists $\gamma_2 \in sp(\alpha)$ such that $\gamma_1 < \gamma_2 < 2\gamma_1$. Let $0 < \epsilon < \min \{\frac{2\gamma_1}{3}, \frac{2\gamma_2}{3}\}$ and denote

$$J_{\gamma_1}^\epsilon = X^\alpha((\gamma_1 - \epsilon, \gamma_1 + \epsilon))X^\alpha((-\gamma_1 - \epsilon, -\gamma_1 + \epsilon)) \subset X^\alpha$$
Then, it is immediate that \( J_{\gamma_1} \) is a two sided ideal of \( X^\alpha \). Consider the following \( \mathcal{F} \)-closed \( \alpha \)-invariant subspace of \( X \)

\[
\mathcal{M} = X^\alpha((-\infty, 0)) J_{\gamma_1}^\epsilon + X^\alpha([0, \infty)).
\]

and let

\[
W = \{ x \in X : x \mathcal{M} \subset \mathcal{M} \}.
\]

Then, \( W \) is an \( \mathcal{F} \)-closed \( \alpha \)-invariant subalgebra of \( X \). Taking into account that 0 is isolated in \( sp(\alpha) \), and that by Lemma 2.11. i) we have

\[
X^\alpha([0, \infty)) = X^\alpha + \sum_{k \in \mathbb{N}} X^\alpha((k \gamma_1 - \epsilon, k \gamma_1 + \epsilon)) + \sum_{m \in \mathbb{N}} X^\alpha((m \gamma_1 + \frac{\epsilon}{2}, (m + 1) \gamma_1 - \frac{\epsilon}{2})^\sigma).
\]

and

\[
X^\alpha((-\infty, 0)) = \sum_{k \in \mathbb{N}} X^\alpha((-k \gamma_1 - \epsilon, -k \gamma_1 + \epsilon)) + \sum_{m \in \mathbb{N}} X^\alpha((-m + 1) \gamma_1 + \frac{\epsilon}{2}, -m \gamma_1 - \frac{\epsilon}{2})^\sigma.
\]

it can be checked that \( X^\alpha([0, \infty)) \subset W \). On the other hand, since the ideal \( J_{\gamma_1}^\epsilon \) of \( X^\alpha \) contains a right approximate identity for \( X^\alpha((-\gamma_1 - \epsilon, -\gamma_1 + \epsilon)) \), it follows that \( X^\alpha((-\gamma_1 - \epsilon, -\gamma_1 + \epsilon)) \subset W \). Therefore,

\[
X^\alpha([0, \infty)) \subseteq W.
\]

Since \( X^\alpha([0, \infty)) \) is maximal, it follows that \( W = X \). Let \( y \in X^\alpha((\gamma_2 - \epsilon, \gamma_2 + \epsilon)), y \neq 0 \). Then \( y^* \in W = X \). So, in particular, \( y^* x^\alpha = y^* x^\alpha((-\epsilon, \epsilon)) \subset W \). By Remark 4.2. v) \( X^\alpha((-\epsilon, \epsilon)) \) contains a Banach algebra approximate identity of \( X \), so \( y^* x^\alpha \neq \{0 \} \). Since

\[
y^* x^\alpha \subset X^\alpha((-\infty, 0)).
\]

we must have

\[
y^* x^\alpha \subset \mathcal{M}^\alpha((-\infty, 0)) = X^\alpha((-\infty, 0)) J_{\gamma_1}^\epsilon \sigma.
\]

Therefore

\[
y^* x^\alpha J_{\gamma_1}^\epsilon = y^* J_{\gamma_1}^\epsilon \neq \{0 \}.
\]

So, in particular

\[
y^* x^\alpha((\gamma_1 - \epsilon, \gamma_1 + \epsilon)) \neq 0.
\]

But, according to Lemma 2.11. vi) we have

\[
\{0 \} \neq y^* x^\alpha((\gamma_1 - \epsilon, \gamma_1 + \epsilon)) \subset x^\alpha((\gamma_1 - \gamma_2 + 2\epsilon, \gamma_1 - \gamma_2 + 2\epsilon)) \subset X^\alpha((-\gamma_1 - \gamma_2 + 2\epsilon, \gamma_1 - \gamma_2 + 2\epsilon)).
\]

The choice of \( \gamma_2 \) and \( \epsilon \) imply \( (\gamma_1 - \gamma_2 - 2\epsilon, \gamma_1 - \gamma_2 + 2\epsilon) \subset (-\gamma_1, 0) \). This is a contradiction with the assumption that that 0 is an isolated point of \( sp(\alpha) \) and \( \gamma_1 = \)
inf \{ \gamma \in sp(\alpha) : \gamma > 0 \} = \min \{ \gamma \in sp(\alpha) : \gamma > 0 \}. Hence \( sp(\alpha) \cap (\gamma_1, 2\gamma_1) = \emptyset \).

Similarly, one can show that \( sp(\alpha) \cap (k\gamma_1, (k + 1)\gamma_1) = \emptyset \) for every \( k \in \mathbb{Z} \). It follows that \( sp(\alpha) \subset \mathbb{Z}\gamma_1 \). We will prove next that Condition 4.3. is satisfied. Since as shown above, \( sp(\alpha) \subset \mathbb{Z}\gamma_1 \) where \( \gamma_1 = \min \{ \gamma \in sp(\alpha) : \gamma > 0 \} \), from Proposition 4.5. it follows that Condition 4.3. is equivalent in this case with Condition (S). We will show that Condition (S) is satisfied. Let \( J = \overline{X_{\gamma_1}X_{-\gamma_1}}^{-} \) and let \( I \) be a non zero \( F \)-closed ideal of \( J \). Let

\[
\mathcal{M} = \sum_{\gamma \in sp(\alpha), \gamma > 0} X_{-\gamma} I + X^\alpha([0, \infty))
\]

Then \( \mathcal{M} \) is an \( \alpha \)-invariant \( F \)-closed linear subspace of \( X \). Let

\[
W = \{ x \in X : x\mathcal{M} \subset M \}
\]

Then \( W \) is an \( F \)-closed subalgebra of \( X \). Clearly \( X^\alpha([0, \infty)) \subset W \). On the other hand, since \( I \neq \{0\} \), it follows that \( X_{-\gamma_1} I \neq \{0\} \) and therefore there exists \( y \in X_{-\gamma_1} \) and \( i_0 \in I \) such that \( z = yi_0 \neq 0 \). Since \( \gamma_1 = \min \{ \gamma \in sp(\alpha) : \gamma > 0 \} \), it follows that \( z \in W \). Therefore, since \( z \in X - X^\alpha([0, \infty)) \) and \( X^\alpha([0, \infty)) \) is maximal, it follows that \( W = X \). Hence, in particular \( X_{-\gamma_1} X^\alpha \subset \mathcal{M} \). By the definition of \( \mathcal{M} \) this means that \( X_{-\gamma_1} X^\alpha \subset (\mathcal{M})_{-\gamma_1} = \overline{X_{-\gamma_1} X^\alpha}^{-} \). It follows that \( I = J \), so \( J \) is \( F \)-simple and

\[
\mathcal{M} = \sum_{\gamma \in sp(\alpha), \gamma > 0} X_{-\gamma} J + X^\alpha([0, \infty))
\]

Now let \( \gamma \in sp(\alpha), \gamma \geq \gamma_1 \). Since \( W = X \), it follows that in particular

\[
X_{-\gamma} X^\alpha \subset (\mathcal{M})_{-\gamma} = \overline{X_{-\gamma} J}^{-}.
\]

By multiplying the previous relation to the left by \( X_\gamma \) we get

\[
X_\gamma X_{-\gamma} X^\alpha = X_\gamma \overline{X_{-\gamma} J}^{-} \subset X_\gamma X_{-\gamma} J^+ = J.
\]

Hence \( \overline{X_{-\gamma} X^\alpha}^{-} = J \), so Condition (S) and therefore Condition 4.3. is satisfied.

Next we will study the case when 0 is an accumulation point of \( sp(\alpha) \).

Recall that all dynamical systems considered in this Section are supposed to be non trivial, that is \( sp(\alpha) \neq \{0\} \).

4.12. Remark Let \((X, \mathbb{R}, \alpha)\) be an one-parameter \( F \)-dynamical system such that \( X^\alpha([0, \infty)) \) is a maximal \( F \)-closed subalgebra of \( X \) and let \( Y \subset X \) be as in Lemma 4.10. If

\[
\overline{X^\alpha((0, \infty))X X^\alpha((0, \infty))}^{-} \subset Y^\alpha.
\]

Then

\[
\overline{X^\alpha((0, \infty))X X^\alpha((-\infty, 0))}^{-} \subset Y^\alpha.
\]
and conversely.

**Proof.** First notice that $X^\alpha((0,\infty))^2 = \{0\}$. Indeed, let $x,y \in X^\alpha((0,\infty))$ be such that $xy \neq 0$. Then $x^*xy \neq 0$. But, according to the hypotheses, we have on the one hand

$$x^*xy = x^*(xy) \in X^\alpha((-\infty,0))X^\alpha((0,\infty)) \subset X^\alpha.$$  

and on the other hand

$$x^*xy = (x^*x)y \in X^\alpha((0,\infty)) \subset X^\alpha((0,\infty)).$$

Since $X^\alpha((0,\infty)) \cap X^\alpha = \{0\}$, this shows that $xy = 0$. Suppose that $Z_1 = X^\alpha((0,\infty))$ is maximal. Hence

$$Z_1 \not\subseteq X^\alpha((0,\infty)).$$

Notice that

$$X^\alpha([0,\infty))Z_1 \subset Z_1.$$  

Since, obviously, $Z_1Z_1 \subset Z_1$, and, as noticed above $Z_1 \not\subseteq X^\alpha([0,\infty))$, it follows that the $F$-closed subalgebra $W$ of $X$ defined by

$$W = \{x \in X : xZ_1 \subset Z_1\}.$$  

strictly contains $X^\alpha([0,\infty))$ and therefore $W = X$. Thus $XZ_1 \subset Z_1$ and, as $Z_1$ is a C*-subalgebra, we also have $Z_1X \subset Z_1$, so $XZ_1X \subset Z_1$. But since $Y$ is $\alpha$-simple and $Z_1 \subset Y$, it follows that $XZ_1X = Y \subset Z_1$ so $Z_1 = Y$. On the other hand, since as shown above, $X^\alpha((0,\infty))^2 = \{0\}$, we have

$$X^\alpha((0,\infty))Z_1 = \{0\}.$$  

Thus

$$X^\alpha((0,\infty))Y = \{0\}.$$  

Since $X^\alpha((0,\infty)) = Y^\alpha((0,\infty))$, it follows that $X^\alpha((0,\infty)) = \{0\}$ which is a contradiction with our standing assumption $sp(\alpha) \neq \{0\}$. □

**4.13. Lemma** Suppose that $(X, R, \alpha)$ is a one-parameter $F$-dynamical system such that $sp(\alpha)$ contains more than three points and $X^\alpha([0,\infty))$ is maximal. We have

$$Z_1 = X^\alpha((-\infty,0))X^\alpha((0,\infty))' \not\subseteq Y^\alpha.$$  

and, therefore, by Remark 4.12.

$$Z_2 = X^\alpha((0,\infty))X^\alpha((-\infty,0))' \not\subseteq Y^\alpha.$$  

26
Proof. Suppose to the contrary that

\[ X^\alpha((-\infty,0))X^\alpha((0,\infty)) \subseteq Y^\alpha. \]

so, in particular

\[ X^\alpha((-\infty,0))X^\alpha((0,\infty)) \subseteq Y^\alpha. \]

Let \( \gamma_1, \gamma_2 \in sp(\alpha), 0 < \gamma_1 < \gamma_2 \) Then for every \( \epsilon > 0 \), sufficiently small we have

\[ X^\alpha((-\gamma_1 - \epsilon, -\gamma_1 + \epsilon))X^\alpha((\gamma_1 - \epsilon, \gamma_1 + \epsilon)) \subseteq Y^\alpha. \]

It follows that, for every \( 0 < \epsilon < \frac{\gamma_1}{2} \), the set

\[ J^\epsilon_{\gamma_1} = X^\alpha((-\gamma_1 - \epsilon, -\gamma_1 + \epsilon))X^\alpha((\gamma_1 - \epsilon, \gamma_1 + \epsilon)) \]

is an \( \mathcal{F} \)-closed ideal of \( Y^\alpha \). Let

\[ M = J^\epsilon_{\gamma_1}X^\alpha((-\infty,0)) + X^\alpha((0,\infty)). \]

Then \( M \) is an \( \mathcal{F} \)-closed \( \alpha \)-invariant subspace of \( X \). Denote

\[ W = \{ x \in X : xM \subseteq M \}. \]

Clearly, \( W \) is an \( \mathcal{F} \)-closed subalgebra of \( X \). Next, we will show that \( X^\alpha((0,\infty)) \subseteq W \). Clearly

\[ X^\alpha((0,\infty))X^\alpha((0,\infty)) \subseteq X^\alpha([0,\infty)) \subseteq M. \]

so we have to prove that

\[ X^\alpha([0,\infty))J^\epsilon_{\gamma_1}X^\alpha((-\infty,0)) \subseteq M. \]

Indeed, by Lemma 2.11. ii)

\[ X^\alpha([0,\infty)) = \cap_{\epsilon > 0} X^\alpha((-\epsilon,\infty)). \]

and by Lemma 2.11. iii)

\[ X^\alpha((-\epsilon',\infty)) = X^\alpha((-\epsilon',\epsilon')) + X^\alpha((\epsilon'/2,\infty)). \]

for every \( \epsilon' > 0 \). Hence, in order to prove that \( X^\alpha([0,\infty))J^\epsilon_{\gamma_1}X^\alpha((-\infty,0)) \subseteq M \) it is sufficient to prove that

\[ X^\alpha((-\epsilon,\epsilon))J^\epsilon_{\gamma_1}X^\alpha((-\infty,0)) \subseteq M \text{ and } X^\alpha((\frac{\epsilon}{2},\infty))J^\epsilon_{\gamma_1}X^\alpha((-\infty,0)) \subseteq M \]

for all \( \epsilon > 0 \) sufficiently small. By Remark 4.12. we have

\[ X^\alpha((\frac{\epsilon}{2},\infty))J^\epsilon_{\gamma_1}X^\alpha((-\infty,0)) \subseteq X^\alphaX^\alpha((\gamma_1-\epsilon,\gamma_1+\epsilon))X^\alpha((-\infty,0)) \subseteq X^\alpha \subseteq M. \]

On the other hand

\[ X^\alpha((-\epsilon,\epsilon))J^\epsilon_{\gamma_1}X^\alpha((-\infty,0)) = \]
\[ X^\alpha((-\epsilon, \epsilon)) X^\alpha((-\gamma_1 - \epsilon, -\gamma_1 + \epsilon)) X^\alpha((\gamma_1 - \epsilon, \gamma_1 + \epsilon)) J^\epsilon_{\gamma_1} X^\alpha((-\infty, 0)) \subset \]
\[ \subset X^\alpha((-\gamma_1 - 2\epsilon, -\gamma_1 + 2\epsilon)) X^\alpha((\gamma_1 - \epsilon, \gamma_1 + \epsilon)) J^\epsilon_{\gamma_1} X^\alpha((-\infty, 0)) \subset \]
\[ \subset X^\alpha J^\epsilon_{\gamma_1} X^\alpha((-\infty, 0)) \subset J^\epsilon_{\gamma_1} X^\alpha((-\infty, 0)) \subset \mathcal{M}. \]

We have thus proven that \( X^\alpha([0, \infty)) \subset W \). Now we will show that this inclusion is strict. Indeed, let \( 0 \neq x \in X^\alpha((-\gamma_1 - \epsilon, -\gamma_1 + \epsilon)) \) and denote \( y = xx^*x \in J^\epsilon_{\gamma_1} X^\alpha((-\gamma_1 - \epsilon, -\gamma_1 + \epsilon)) \subset X^\alpha((-\gamma_1 - \epsilon, -\gamma_1 + \epsilon)). \) Then, \( y \neq 0 \) and \( y \notin X^\alpha((0, \infty)). \) We have

\[ y J^\epsilon_{\gamma_1} X^\alpha((-\infty, 0)) = xx^*x J^\epsilon_{\gamma_1} X^\alpha((-\infty, 0)) \subset J^\epsilon_{\gamma_1} X^\alpha((-\infty, 0))^2 = \{0\}. \]

We show next that \( y X^\alpha([0, \infty)) \subseteq \mathcal{M}. \) As noticed above, we have

\[ X^\alpha([0, \infty)) = \cap_{\epsilon' > 0} X^\alpha((-\epsilon', \infty)). \]

and

\[ X^\alpha((-\epsilon', \infty)) = X^\alpha((-\epsilon', \epsilon')) + X^\alpha((\epsilon', \infty)). \]

Clearly, if \( 0 < \epsilon' < \frac{2\epsilon}{\alpha} \) we have

\[ y X^\alpha((-\epsilon', \epsilon')) \subset J^\epsilon_{\gamma_1} X^\alpha((-\gamma_1 - \epsilon - \epsilon', -\gamma_1 + \epsilon + \epsilon')) \subset J^\epsilon_{\gamma_1} X^\alpha((-\infty, 0)) \subset \mathcal{M}. \]

and

\[ y X^\alpha((\frac{\epsilon'}{2}, \infty)) = (xx^*x)x X^\alpha((\frac{\epsilon'}{2}, \infty)) \subset J^\epsilon_{\gamma_1} X^\alpha((-\infty, 0)) X^\alpha((0, \infty)) \subset X^\alpha \subset \mathcal{M}. \]

So \( X^\alpha([0, \infty)) \not\subset W. \) Since \( X^\alpha([0, \infty)) \) is maximal, it follows that \( W = X. \) In order to show that the assumption

\[ X^\alpha((-\infty, 0)) X^\alpha((0, \infty)) \subset Y^\alpha \]

leads to a contradiction, we will use the hypothesis that there is \( \gamma_2 \in sp(\alpha) \) such that \( 0 < \gamma_1 < \gamma_2. \) Let \( 0 < \epsilon < \min \left\{ \frac{\epsilon'}{2}, \frac{\epsilon'}{2\alpha} \right\}. \) Since \( W = X \) we have

\[ X^\alpha((-\gamma_2 - \epsilon, -\gamma_2 + \epsilon)) \mathcal{M} \subset \mathcal{M}. \]

In particular

\[ X^\alpha((-\gamma_2 - \epsilon, -\gamma_2 + \epsilon)) X^\alpha \subset \mathcal{M}. \]

so

\( \{0\} \neq X^\alpha((-\gamma_2 - \epsilon, -\gamma_2 + \epsilon)) X^\alpha((\gamma_2 - \epsilon, \gamma_2 + \epsilon)) X^\alpha((-\gamma_2 - \epsilon, -\gamma_2 + \epsilon)) \subset \mathcal{M}. \)

If we denote \( J^\epsilon_{\gamma_2} = \overline{X^\alpha((-\gamma_2 - \epsilon, -\gamma_2 + \epsilon)) X^\alpha((\gamma_2 - \epsilon, \gamma_2 + \epsilon))} \) then \( J^\epsilon_{\gamma_2} \) is an ideal of \( X^\alpha \) and

\[ J^\epsilon_{\gamma_2} X^\alpha((-\gamma_2 - \epsilon, -\gamma_2 + \epsilon)) \subset \mathcal{M} \]
At the same time, since $\mathcal{M}$ is an $\mathcal{F}$-closed $\alpha$-invariant subspace of $X$, and $J_{t_2}^* X^\alpha((\gamma_2 - \epsilon, -\gamma_2 + \epsilon)) \subset X^\alpha((\infty, 0))$, it follows that

$$J_{t_2}^* X^\alpha((\gamma_2 - \epsilon, -\gamma_2 + \epsilon)) \subset \mathcal{M}^\alpha((\infty, 0)) = J_{t_1}^* X^\alpha((\infty, 0))$$

Multiplying to the right the above equality by $X^\alpha((\gamma_2 - \epsilon, \gamma_2 + \epsilon))$ we get

$$J_{t_2}^* \subset J_{t_1}^*$$

so, since $J_{t_2}^*$ and $J_{t_1}^*$ are ideals of $X^\alpha$, it follows that $J_{t_2}^* J_{t_1}^* = J_{t_2}^* \neq \{0\}$. Using the definition of the ideals $J_{t_2}^*$ and $J_{t_1}^*$ we get

$$X^\alpha((\gamma_2 - \epsilon, \gamma_2 + \epsilon)) X^\alpha((\gamma_1 - \epsilon, -\gamma_1 + \epsilon)) \neq \{0\}$$

and at the same time

$$X^\alpha((\gamma_2 - \epsilon, \gamma_2 + \epsilon)) X^\alpha((\gamma_1 - \epsilon, -\gamma_1 + \epsilon)) \subset X^\alpha$$

This is a contradiction since $X^\alpha((\gamma_2 - \epsilon, \gamma_2 + \epsilon)) X^\alpha((\gamma_1 - \epsilon, -\gamma_1 + \epsilon)) \subset X^\alpha((\gamma_2 - \epsilon, \gamma_2 + \epsilon)) X^\alpha((\gamma_1 - \epsilon, -\gamma_1 + \epsilon)) \subset X^\alpha((0, \infty))$ and the proof is completed. 

**4.14. Lemma** Let $(X, \mathbb{R}, \alpha)$ be a one-parameter $\mathcal{F}$-dynamical system such that $sp(\alpha)$ contains more than three points and let $Z \in H^2_{\mathcal{F}}$. If $X^\alpha([0, \infty))$ is a maximal $\mathcal{F}$-closed subalgebra of $X$, then,

$$X^\alpha((\gamma, \infty)) Z X^\alpha((\infty, \gamma)) = Y.$$

and

$$X^\alpha((\gamma, \infty)) Z X^\alpha((\infty, \gamma)) = Y.$$

for every $\gamma \in \mathbb{R}, \gamma \geq 0$ such that $sp(\alpha) \cap (\gamma, \infty) \neq \emptyset$ where $Y$ is the $\alpha$-simple ideal of $X$ whose existence was established in Lemma 4.10.

**Proof.** Let $\gamma \in \mathbb{R}, \gamma \geq 0$ be such that $sp(\alpha) \cap (\gamma, \infty) \neq \emptyset$, so $X^\alpha((\gamma, \infty)) \neq \{0\}$. Consider first the case when $Z = X$. Let $Z_1 = X^\alpha((\gamma, \infty)) X X^\alpha((\infty, \gamma))$. Since $Y$ is $\alpha$-simple and $Z_1$ is $\alpha$-invariant, we have that $Y_{Z_1} Y_{Z_1} = Y$, so, if $Z_1 \subset Y$, we have $Y_{Z_1} Y_{Z_1} \subset Y$. Therefore, if $Z_1 \subset Y$, then, there exist $y \in Y$ and $z \in Z_1$ such that $yz \notin Z_1$. Denote

$$W = \{x \in X : xZ_1 \subset Z_1\}.$$

Then, $W$ is a $\mathcal{F}$-closed subalgebra of $X$ and, by the above argument, if $Z_1 \subset Y$, we have that $W \neq X$. Let us prove that

$$X^\alpha([0, \infty)) \subset W.$$

By Lemma 2.11. i), $X^\alpha((\gamma, \infty)) = \sum_{n \in \mathbb{N}} X^\alpha([\gamma + \frac{1}{n}, \infty))$. Applying Lemma 2.11. v), for every $n \in \mathbb{N}$, we have

$$X^\alpha([0, \infty)) X^\alpha([\gamma + \frac{1}{n}, \infty)) \subset X^\alpha([\gamma + \frac{1}{n}, \infty)).$$
Therefore,

\[ X^\alpha([0, \infty)) \sum_{n \in \mathbb{N}} X^\alpha([\gamma + \frac{1}{n}, \infty)) \subset \sum_{n \in \mathbb{N}} X^\alpha([\gamma + \frac{1}{n}, \infty)). \]

So

\[ X^\alpha([0, \infty))X^\alpha((\gamma, \infty)) \subset X^\alpha((\gamma, \infty)). \]

and therefore

\[ X^\alpha([0, \infty))Z_1 \subset Z_1. \]

and hence

\[ X^\alpha([0, \infty)) \subset W. \]

On the other hand, clearly, \( Z_1 \subset W \) and, since \( sp(\alpha) \) contains at least five points, by the previous Lemma 4.13. it follows that \( Z_1 \not\subset Y^\alpha \), so \( sp(\alpha|_{Z_1}) \neq \{0\} \). Since \( Z_1 \) is an \( \alpha \)-invariant (hereditary) \( C^* \)-subalgebra there exists \( \gamma \in sp(\alpha|_{Z_1}), \gamma < 0 \), so \( Z_1 - X^\alpha([0, \infty)) \neq \emptyset \). Therefore, if \( z \in Z_1 - X^\alpha([0, \infty)) \), then \( z \in W - X^\alpha([0, \infty)) \). We have thus proved that if \( Z \not\subset Y \), then there exists an \( \mathcal{F} \)-closed subalgebra \( W \) of \( X \) such that

\[ X^\alpha([0, \infty)) \subset W \]

which contradicts the maximality of \( X^\alpha([0, \infty)) \). Hence \( Z_1 = Y \). To prove that

\[ \overline{X^\alpha((\gamma, \infty))XX^\alpha([\gamma, \infty))} = Y. \]

let \( Z_1 = \overline{X^\alpha((\gamma, \infty))XX^\alpha([\gamma, \infty))} \). If we denote

\[ W = \{ x \in X : Z_1 x \subset Z_1 \}. \]

then, by similar arguments it can be proven that \( Z_1 = Y \). Now let \( Z \in \mathcal{H}^\alpha_\omega(Y) \) be arbitrary and denote

\[ Z_1 = \overline{X^\alpha((\gamma, \infty))ZX^\alpha((\infty, -\gamma))}. \]

Then, notice that \( Z_1 \neq \{0\} \). Indeed if \( Z_1 = \{0\} \), then \( X^\alpha((0, \infty))Z = \{0\} \). By the first part of the proof, it follows that \( YZ = \{0\} \), so \( Z = \{0\} \), which contradicts the fact that \( Z \in \mathcal{H}^\alpha_\omega(Y) \). We notice also that \( Z_1 \not\subset Y^\alpha \). Indeed, suppose to the contrary that \( Z_1 \subset Y^\alpha \). Then, since \( X^\alpha((0, \infty))Z_1 \subset Z_1 \subset Y^\alpha \), and, on the other hand, \( X^\alpha((0, \infty))Z_1 \subset Y^\alpha((0, \infty)) \), it follows that \( X^\alpha((0, \infty))Z_1 = \{0\} \). Therefore

\[ X^\alpha((\infty, 0))XX^\alpha((0, \infty))Z_1 \neq \emptyset \]

so \( YZ_1 = \{0\} \) and thus \( Z_1 = \{0\} \), contradiction. Using these two facts about \( Z_1 \) for arbitrary \( Z \in \mathcal{H}^\alpha_\omega(Y) \) and the arguments above for the particular case \( Z = X \) the proof is completed.

The next lemma is probably known, but I include its proof below.
**4.15. Lemma** Let $Y$ be a $C^*$-algebra and $D \subset Y$ a hereditary $C^*$-subalgebra of $Y$. Let $\{A_i\}_{i \in I}$ be a collection of subsets of $Y$ such that $(\sum A_i)^*Y(\sum A_i) \subset D$ (Here, the symbol $\sum A_i$ denotes the set of all finite sums of $A_i$’s).

**Proof.** Let $i, j \in I, i \neq j$ be arbitrary. If we prove that $A_i^*Y A_j \subset D$, it will follow that $(\sum A_i)^*Y (\sum A_i) \subset D$. Let $a \in A_i, y \in Y$ and $b \in A_j$. Then, using the well known and easy to prove inequality

$$(a^*y^* \pm b^*y^*)(ya \pm yb) \leq 2(a^*y^*ya + b^*y^*yb)$$

and the hypotheses that $a^*y^*ya \in D, b^*y^*yb \in D$ and that $D$ is a hereditary $C^*$-subalgebra of $Y$, it follows that $a^*y^*yb, \pm b^*y^*ya \in D$ so $A_i^*Y A_j \subset D$ (where $Y^+$ denotes the set of all non negative elements of $Y$) and therefore $A_i^*Y A_j \subset D$. 

**4.16. Lemma** Let $(X, \mathbb{R}, \alpha)$ be an $F$-dynamical system. Suppose that $0$ is an accumulation point of $\text{sp}(\alpha)$ and $X^\alpha([0, \infty))$ is a maximal $F$-closed subalgebra of $X$. Then Condition 4.3. b) holds.

**Proof.** Let $Y$ be the $\alpha$-simple ideal of $X$ from Lemma 4.10. We will prove this lemma in three steps:

**Step 1.** Let $\gamma_0 \in \text{sp}(\alpha)$ and $\epsilon > 0$. Then

$$S_{\gamma_0} = X^{\alpha((\gamma_0 - \epsilon, -\gamma_0 + \epsilon))}X^{\alpha((\gamma_0 - \epsilon, -\gamma_0 + \epsilon))}\sigma = Y.$$  

and

$$T_{\gamma_0} = X^{\alpha((\gamma_0 - \epsilon, -\gamma_0 + \epsilon))}X^{\alpha((\gamma_0 - \epsilon, -\gamma_0 + \epsilon))}\sigma = Y.$$  

where $S_{\gamma_0}$ and $T_{\gamma_0}$ are notations for the corresponding $F$-closed hereditary subalgebras of $Y$. We will prove the first equality and then show how to obtain the second one. Obviously, it is sufficient to prove the equality for $0 < \epsilon < 2\gamma_0$. Denote

$$M = \overline{XS_{\gamma_0}\sigma} + X^{\alpha((\gamma_0 - \frac{\epsilon}{2}, \infty))}\sigma$$  

and

$$W = \{x \in X : xM \subset M\}.$$  

Then, $M$ is an $F$-closed subspace of $X$ and $W$ is an $F$-closed subalgebra of $X$. Using Lemma 2.11. i) and v) as in the proof of Lemma 4.14. it follows immediately that $X^{\alpha([0, \infty))} \subset W$. Next, we will show that $X^{\alpha([0, \infty))}$ is a proper subset of $W$. Indeed, since $0$ is a point of accumulation of $\text{sp}(\alpha)$, there
exists $0 \neq x \in X^\alpha((0, \frac{1}{2})) = Y^\alpha((0, \frac{1}{2}))$. We will show that $x^* \in W$. Since clearly $x^* \notin X^\alpha([0, \infty))$ the above claim will be proven. By Lemma 2.11. iii) we have

$$X^\alpha((\gamma_0 - \frac{\epsilon}{2}, \infty)) = X^\alpha((\gamma_0 - \frac{\epsilon}{2}, \gamma_0 + \epsilon)) + X^\alpha((\gamma_0 + \frac{3\epsilon}{4}, \infty))^\sigma.$$ 

Taking into account that $Y$ is an $\alpha$-simple ideal of $X$ it follows that

$$x^*X^\alpha((\gamma_0 - \frac{\epsilon}{2}, \gamma_0 + \epsilon)) \subset Y^\alpha((\gamma_0 - \frac{3\epsilon}{4}, \gamma_0 + \epsilon)) \subset Y^\alpha((\gamma_0 - \epsilon, \gamma_0 + \epsilon)) \subset \subset \overline{YX^\alpha((\gamma_0 - \epsilon, \gamma_0 + \epsilon))} = YX^\alpha((\gamma_0 - \epsilon, \gamma_0 + \epsilon))^\sigma \subset \subset YS_{\gamma_0}^\sigma.$$

and

$$x^*X^\alpha((\gamma_0 + \frac{3\epsilon}{4}, \infty)) \subset X^\alpha((\gamma_0 - \frac{\epsilon}{2}, \infty)).$$

Therefore $x^* \in W - X^\alpha([0, \infty))$, so $X^\alpha([0, \infty)) \not\subset W$. Since $X^\alpha([0, \infty))$ is a maximal $\mathcal{J}$-closed subalgebra of $X$ it follows that $W = X$. We will prove that $S_{\gamma_0} = Y$ by contrapositive. Suppose that $S_{\gamma_0} \not\subset Y$. We will produce an element $x \neq 0$ such that $x^* \in X - W$. Since $(\gamma_0 - \frac{\epsilon}{2}, \infty) = \cup_{\gamma > \gamma_0 - \frac{\epsilon}{2}} (\gamma, \gamma_0 - \frac{\epsilon}{2})$, from Lemma 2.11. iii) it follows that

$$X^\alpha((\gamma_0 - \frac{\epsilon}{2}, \infty)) = \sum_{\gamma > \gamma_0 - \frac{\epsilon}{2}} X^\alpha((\gamma, \gamma_0 + \frac{\epsilon}{2}))^\sigma.$$ 

By Lemma 4.14.

$$X^\alpha((\gamma_0 - \frac{\epsilon}{2}, \infty))^* YX^\alpha((-\infty, -\gamma_0 + \frac{\epsilon}{2})) = Y.$$ 

Hence

$$(\sum_{\gamma > \gamma_0 - \frac{\epsilon}{2}} X^\alpha((\gamma, \gamma_0 + \frac{\epsilon}{2}))^* Y(\sum_{\gamma > \gamma_0 - \frac{\epsilon}{2}} X^\alpha((\gamma, \gamma_0 + \frac{\epsilon}{2}))) = Y.$$

Since we are assuming that $S_{\gamma_0} \not\subset Y$, Lemma 4.15. implies that there exists $\gamma > \gamma_0 - \frac{\epsilon}{2}$ such that

$$X^\alpha((\gamma, \gamma_0 + \frac{\epsilon}{2}))^* YX^\alpha((\gamma, \gamma_0 + \frac{\epsilon}{2}))^* \not\subset S_{\gamma_0}.$$

Since $X^\alpha((\gamma, \gamma_0 + \frac{\epsilon}{2}))$ is a subspace of $X$, there exists $x \in X^\alpha((\gamma, \gamma_0 + \frac{\epsilon}{2}))$ such that $x^*x \not\in S_{\gamma_0}$. Then $x^*x \not\in YS_{\gamma_0}$ since otherwise $(x^*x)^2 \in S_{\gamma_0} YS_{\gamma_0} \subset S_{\gamma_0}$, so $x^*x \in S_{\gamma_0}$. On the other hand

$$x^*x \in Y^\alpha((-\gamma_0 + \frac{\epsilon}{2}, 0)), \gamma_0 + \frac{\epsilon}{2}),$$

so $x^*x \notin X^\alpha((\gamma_0 - \frac{\epsilon}{2}, \infty))$. Notice that both $YS_{\gamma_0}^\sigma$ and $X^\alpha((0, \gamma_0 + \frac{\epsilon}{2}))$ are $\alpha$-invariant subspaces, so if $z \in YS_{\gamma_0}^\sigma$ (respectively $y \in X^\alpha((\gamma_0 - \frac{\epsilon}{2}, \infty))$ and
$f \in L^1(\mathbb{R})$ we have $\alpha_f(z) \in YS_{\gamma_0}$ (respectively $\alpha_f(y) \in X^\alpha((\gamma_0 - \frac{s}{2}, \infty))$).
Next, we will prove that $x^* x \notin M$, so $x^* \notin W$. Indeed if $x^* x \in M$, then there exists a net \{ $z_i + y_i : z_i \in YS_{\gamma_0}, y_i \in X^\alpha((\gamma_0 - \frac{s}{2}, \infty))$ \} such that $x^* x = \lim(z_i + y_i)$. By [20, Theorem 2.6.2.] there exists a function $f \in L^1(\mathbb{R})$ such that $\hat{f} = 1$ on an open set containing $sp(x^* x)$ whose closure is included in $(-\gamma_0 + \frac{s}{2}, \gamma_0 - \frac{s}{2})$ and $\hat{f} = 0$ outside $(-\gamma_0 + \frac{s}{2}, \gamma_0 - \frac{s}{2})$. Hence, since $\alpha_f$ is $\mathcal{F}$-continuous, we have
$$\alpha_f(x^* x) = x^* x = \mathcal{F} - \lim \alpha_f(z_i + y_i) = \mathcal{F} - \lim \alpha_f(z_i) \in YS_{\gamma_0}^-.$$ contradiction. Hence if $S_{\gamma_0} \subseteq Y$, then $W \subseteq X$ so $X^\alpha([0, \infty))$ is not maximal.

To prove that $T_{\gamma_0} = Y$, replace the above $M$ and $W$ by
$$M = X\{x \in X : Mx \subseteq M\}.$$ and use similar arguments.

**Step 2.** Let $\gamma_0 \in sp(\alpha)$ and $0 < \epsilon < \frac{s}{2}$. If $Z \in \mathcal{H}_0^\alpha(Y)$, then
$$S_{\gamma_0}^\alpha(Z) = X^\alpha((-\gamma_0 - \epsilon, -\gamma_0 + \epsilon))ZX^\alpha((\gamma_0 - \epsilon, \gamma_0 + \epsilon)) = Y.$$ and
$$T_{\gamma_0}^\alpha(Z) = X^\alpha((\gamma_0 - \epsilon, \gamma_0 + \epsilon))ZX^\alpha((-\gamma_0 - \epsilon, -\gamma_0 + \epsilon)) = Y.$$ We will prove only the first equality and then describe how to obtain the second one. As in Step 1. denote
$$M = XS_{\gamma_0}(Z)^\alpha + X^\alpha((\gamma_0 - \epsilon, \gamma_0 + \epsilon))^-.$$ and
$$W = \{x \in X : xM \subseteq M\}.$$ Then $M$ is an $\mathcal{F}$-closed subspace of $X$ and $W$ is an $\mathcal{F}$-closed subalgebra of $X$. Clearly $X^\alpha([0, \infty)) \subseteq W$. The fact that this inclusion is strict is the only difference between the case of general $Z \in \mathcal{H}_0^\alpha(Y)$ and the case when $Z = X$ considered in Step 1. Since 0 is an accumulation point of $sp(\alpha)$, there exists $\gamma_1 \in sp(\alpha), \gamma_1 > 0$ such that $\gamma_1 < \frac{s}{2}$. By Step 1, we have
$$X^\alpha((-\gamma_1 - \epsilon', -\gamma_1 + \epsilon'))YX^\alpha((\gamma_1 - \epsilon', \gamma_1 + \epsilon')) = Y.$$ and
$$X^\alpha((\gamma_1 - \epsilon', \gamma_1 + \epsilon'))YX^\alpha((-\gamma_1 - \epsilon', -\gamma_1 + \epsilon')) = Y.$$ for every $\epsilon' > 0$. Hence $S_{\gamma_1}^\alpha(Z) \neq \{0\}$, since, otherwise $YZY = \{0\}$, hence $Z = \{0\}$ which is not possible since $Z \in \mathcal{H}_0^\alpha(Y)$, so $Z \neq \{0\}$ by the definition of $\mathcal{H}_0^\alpha(Y)$. In addition, for every $\delta > 0$, and $\epsilon' > 0$ we have
$$X^\alpha((-\gamma_0 - \epsilon', -\gamma_0 + \epsilon'))X^\alpha((-\delta, \delta))X^\alpha((\gamma_0 - \epsilon', \gamma_0 + \epsilon')) \neq \{0\}.$$
for the same reason and the fact that $0 \in \text{sp}(\alpha|_Z)$. Since $\gamma_1 < \frac{\epsilon}{7}$ there exists $n \in \mathbb{N}, n > 4$ such that $\frac{\epsilon}{2n} \leq \gamma_1 < \frac{\epsilon}{8n}$. Then, $\{0\} \neq Z^\alpha(-\frac{\epsilon}{2n}, \frac{\epsilon}{8n})X^\alpha(\gamma_1 - \frac{\epsilon}{2n}, \gamma_1 + \frac{\epsilon}{8n}) \subset X^\alpha(0, \frac{\epsilon}{7})$. Let $x \in Z^\alpha(-\frac{\epsilon}{2n}, \frac{\epsilon}{8n})X^\alpha(\gamma_1 - \frac{\epsilon}{2n}, \gamma_1 + \frac{\epsilon}{8n}), x \neq 0$. Clearly, $x^* \notin X^\alpha([0, \infty))$. The proof that $x^* \in W$ is very similar with the corresponding proof in Step 1. and we will omit it. The rest of the proof is a verbatim repetition of the arguments in Step 1.

**Step 3.** Proof of Lemma. Applying Remark 4.2. v) to $(Z, \mathbb{R}, \alpha)$ and $\epsilon > 0$, it follows that

$$Z^\alpha((-\epsilon, \epsilon))Z^\alpha((-\epsilon, \epsilon))^\sigma = Z \quad (1)$$

and therefore, since $Z$ is a hereditary C*-subalgebra of $Y$

$$Z^\alpha((-\epsilon, \epsilon))YZ^\alpha((-\epsilon, \epsilon))^\sigma = Z \quad (2)$$

Let $\gamma_0 \in \text{sp}(\alpha) - \{0\}$ and $\epsilon > 0$ as in Step 2. By replacing $Y$ in equation (2) above by $X^\alpha(-\gamma_0 - \epsilon, -\gamma_0 + \epsilon)Z^\alpha((-\epsilon, \gamma_0 + \epsilon))$ as in Step 2 and then replacing $Z$ by $Z^\alpha((-\epsilon, \epsilon))Z^\alpha((-\epsilon, \epsilon))^\sigma$ as in relation (1) above, we get

$$Z^\alpha((-\gamma_0 - 3\epsilon, -\gamma_0 + 3\epsilon))Z^\alpha((-\gamma_0 - 3\epsilon, -\gamma_0 + 3\epsilon))^\sigma = Z$$

Therefore

$$Z^\alpha((-\gamma_0 - \delta, -\gamma_0 + \delta))Z^\alpha((-\gamma_0 - \delta, -\gamma_0 + \delta))^\sigma = Z$$

for all $\delta > 0$ and consequently $\text{sp}(\alpha|_Y) = \tilde{\Gamma}_F(\alpha|_Y)$.

**4.17. Corollary** Let $(X, \mathbb{R}, \alpha)$ be a one-parameter $\mathcal{F}$-dynamical system. and such that $X^\alpha([0, \infty))$ is a maximal $\mathcal{F}$-closed subalgebra of $X$. Then Condition 4.3. is satisfied.

**Proof.** Follows from Lemmas 4.11. and 4.16. ■

Now we can state the main result of this paper. This result contains and improves on all the previous results about maximality of the algebra of analytic elements associated with a C*- or W*-one-parameter dynamical system. Moreover it also answers the maximality question for multiplier one-parameter dynamical systems.

**4.18. Theorem** Let $(X, \mathbb{R}, \alpha)$ be a one-parameter $\mathcal{F}$-dynamical system. Then $X^\alpha([0, \infty))$ is a maximal $\mathcal{F}$-closed subalgebra of $X$ if and only if the Spectral Condition 4.3. is satisfied.

**Proof.** Follows from Theorem 4.9. and Corollary 4.17. ■

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References

[1] W. B. ARVESON, On groups of automorphisms of operator algebras, *J. Funct. Anal.* 15 (1974), 217-243.

[2] W. B. ARVESON, The harmonic analysis of automorphism groups, *Operator Algebras and Applications, Part I, Kingston, Ontario 1980, Proc. Symposia Pure Math.*, Vol. 38, AMS, Providence, RI 1982, 199-269.

[3] A. CONNES, Une classification des facteurs de type III, *Ann. Sci. École Norm. Sup.* 6 (1973), 133–252.

[4] C. D’ANTONI and L. ZSIDO, Groups of linear isometries on multiplier C*-algebras, *Pacific J. Math.* 193 (2000), 279–306.

[5] J. DIXMIER, Les C*-algèbres et leurs représentations, Gauthier-Villars, Paris, 1964.

[6] F. FORELLI, Analytic and quasi-invariant measures, *Acta Math.* 118 (1967), 33–59.

[7] F. FORELLI, A maximal algebra, *Math. Scand.* 30 (1972), 152–158.

[8] E. HILLE and R. PHILLIPS, Functional Analysis and Semi-groups, AMS, 1957.

[9] K. HOFFMAN and I. M. SINGER, Maximal subalgebras of C(Γ), *Amer. J. Math.* 79 (1957), 295-305.

[10] K. HOFFMAN and I. M. SINGER, Maximal algebras of continuous functions, *Acta Math.* 103 (1960), 217-241.

[11] A. KISHIMOTO, Simple crossed products of C*-algebras by locally compact abelian groups, *Yokohama Math. J.* 28 (1980), 69–85.

[12] A. KISHIMOTO, Maximality of the analytic subalgebras of C*-algebras with flows, *J. Korean Math. Soc.* 50 (2013), 1333–1348.

[13] M. MCASEY, P. MUHLY and K.-S. SAITO, Nonsselfadjoint crossed products (invariant subspaces and maximality), *Trans. Amer. Math. Soc.* 248 (1979), 381-409.

[14] P. MUHLY, Function algebras and flows, *Acta Sci. Math. (Szeged)* 35 (1973), 111–121.
[15] D. OLESEN, Inner*-automorphisms of simple $C^*$-algebras, Comm. Math. Phys. 44 (1975), 175–190.

[16] D. OLESEN, G. K. PEDERSEN and E. STORMER, Compact abelian groups of automorphisms of simple $C^*$-algebras, Invent. Math. 39 (1977), 55–64.

[17] G. K. PEDERSEN, $C^*$-algebras and their automorphism groups, Academic Press, 1979.

[18] C. PELIGRAD and S. RUBINSTEIN, Maximal subalgebras of $C^*$-crossed products, Pacific J. Math. 110 (1984), 325-333.

[19] C. PELIGRAD and L. ZSIDO, Maximal subalgebras of $C^*$-algebras associated with periodic flows, J.Funct. Anal. 262 (2012), 3626-3637.

[20] W. RUDIN, Fourier Analysis on groups, Interscience, New York, 1962.

[21] D. SARASON, Algebras of functions on the unit circle, Bull. Amer. Math. Soc. 79 (1973), 286–299.

[22] H. SCHAEFER, Topological vector spaces, Springer Verlag New York Heidelberg Berlin 1971.

[23] A. B. SIMON, On the maximality of vanishing algebras, Amer. J. Math. 81 (1959), 613-616.

[24] B. SOLEL, Algebras of analytic operators associated with a periodic flow on a von Neumann algebra, Canad. J. Math. 37 (1985), 405–429.

[25] B. SOLEL, Maximality of analytic operator algebras, Israel J. Math. 62 (1988), 63–89.

[26] D. C. TAYLOR, The strict topology for double centralizer algebras, Trans. Amer. Math. Soc., 150 (1970), 633–643.

[27] J. WERMER, On algebras of continuous functions, Proc. Amer. Math. Soc. 4, (1953). 866–869.

[28] L. ZSIDO, Spectral and ergodic properties of the analytic generators, J. Approximation Theory 20 (1977), 77–138.

[29] L. ZSIDO, On spectral subspaces associated to locally compact abelian groups of operators, Adv. in Math. 36 (1980), 213–276.