Abstract. We present reasons for developing a theory of forcing notions which satisfy the properness demand for countable models which are not necessarily elementary submodels of some \((\mathcal{H}(\chi), \in)\). This leads to forcing notions which are “reasonably” definable. We present two specific properties materializing this intuition: nep (non-elementary properness) and snep (Souslin non-elementary properness). For this we consider candidates (countable models to which the definition applies), and the older Souslin proper. A major theme here is “preservation by iteration”, but we also show a dichotomy: if such forcing notions preserve the positiveness of the set of old reals for some naturally define c.c.c. ideals, then they preserve the positiveness of any old positive set. We also prove that (among such forcing notions) the only one commuting with Cohen is Cohen itself.

Annotated Content.

Section 0: Introduction  We present reasons for developing the theory of forcing notions which satisfy the properness demand for countable models which are not necessarily elementary submodels of some \((\mathcal{H}(\chi), \in)\). This will lead us to forcing notions which are “reasonably” definable.

Section 1: Basic definitions  We present two specific properties materializing this intuition: nep (non-elementary properness) and snep (Souslin non-elementary properness). For this we consider candidates (countable models to which the definition applies), and the older Souslin proper.

Section 2: Connections between the basic definitions  We point out various implications (snep implies nep, etc.). We also point out how much the properties are absolute.

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Section 3: There are examples

We point out that not just the reasonably definable forcing notions in use fit our framework, but that all the general theorems of Roslanowski–Shelah [14], which prove properness, actually prove the stronger properties introduced earlier.

Section 4: Preservation under iteration: first round

First we address a point we ignored earlier (it was not needed, but is certainly part of our expectations). In the definition of “q is \((N, Q)\)-generic” predensity of each \(I \in \text{pd}(N, Q)\) was originally designed to enable us to say things on \(N[G_Q]\), i.e. \(N[G_Q] \cap \mathcal{H}(\chi)^V = N\), but we should be careful saying what we intend by \(N[G_Q]\) now, so we replace it by \(N(G_Q)\). The preservation theorem 5.5 says that CS iterations of nep forcing notions have the main property of nep. For this we define \(p^{\langle \langle N \rangle \rangle}\) if \(N \models p \in \text{Lim}(\bar{Q})\). We also define and should consider (5.2) the “\(K\)-absolute nep”.

Section 5: True preservation theorems

We consider two closure operations of nep forcing notions \((\text{cl}_1, \text{cl}_2)\), investigate what is preserved and what is gained and prove a general preservation theorem (6.13). This is done for the “straight” version of nep.

Section 6: When a real is \((Q, \eta)\)-generic over \(V\)

We define the class \(\mathcal{K}\) of pairs \((Q, \eta)\), in particular when \(\eta\) is the generic real for \(Q\), and how nice is the subforcing \(Q'\) of \(Q\) generated by \(\eta\).

Section 7: Preserving a little implies preserving much

We are interested in the preservation of the property (of forcing notions) “retaining positiveness modulo the ideal derived from a c.c.c. nep forcing notion”, e.g. being non-null (by forcing notions which are not necessarily c.c.c.). In [17, Ch.VI, §1, §2, §3, Ch.XVIII, §3] this is dealt with but mainly in the limit case. Our main aim is to show that for “nice” enough forcing notion we have a dichotomy (which implies preservation under e.g. CS iterations (of proper forcing) of the property above) retaining the positiveness of \(\omega^\omega\) (or in general every positive Borel set) implies retaining the positiveness of any \(X \subseteq \omega^\omega\).

Section 8: Non-symmetry

We start to investigate for c.c.c. nep forcing: when does “if \(\eta_0\) is \((Q_0, \eta_0)\)-generic over \(N\) and \(\eta_1\) is \((Q_1, \eta_1)\)-generic over \(N[\eta_0]\) then \(\eta_1\) is \((Q_0, \eta_0)\)-generic over \(N[\eta_1]\)”? This property is known for Cohen reals and random reals above.

Section 9: Poor Cohen commute only with himself

We prove that commuting with Cohen is quite rare. In fact, c.c.c. Souslin forcing which adds \(\eta\) which is (absolutely) nowhere essentially Cohen does not commute with Cohen. So such forcing makes the set of old reals meagre.
Section 10: Some c.c.c. nep forcing notions are not nice

We define such forcing notions which are not essentially Cohen as long as $\aleph_1$ is not too large in $L$. This shows that “c.c.c. Souslin” cannot be outright replaced by “absolutely c.c.c. nep”.

Section 11: Preservation of “no dominating real”

We would like to strengthen the main conclusion of §7, (that retaining of positiveness is preserved by composition) of nice forcing notions (i.e. if each separately has it, then so does its composition) to additional natural ideals, mainly the one mentioned in the title, which does not flatly fall into the context of §7. Though 11.2 contains a counterexample, we prove it for “nice” enough forcing notions.

Section 12: Open problems

We formulate several open questions.

0. Introduction. The thema of [18], [17] is:

**Thesis 1.1.** It is good to have general theory of forcings, particularly for iterated forcing.

Some years ago, Judah asked me a question (on inequalities on cardinal invariants of the continuum). Looking for a forcing proof we arrived to the following question:

**Question 1.2.** Will it not be nice to have a theory of forcing notions $Q$ such that:

$$(\oplus) \text{ if } Q \in N \subseteq (H(\chi), \in), N \text{ a countable model of ZFC}^- \text{ and } p \in N \cap Q, \text{ then there is } q \in Q \text{ which is } (N, Q)\text{-generic.}$$

Note the absence of $\prec$ (i.e. $N$ is just a submodel of $(H(\chi), \in)$), which is the difference between this property and “properness”, and is alluded to in the name of this paper. This evolved to “Souslin proper forcing” (see 2.10) in Judah and Shelah [13], which was continued in Goldstern Judah [11].

There are still some additional desirable properties (absent there):

(a) many “nicely defined” forcing notions do not satisfy “Souslin proper”, in fact not so esoteric ones: the Sacks forcing, the Laver forcing;

(b) actual preservation by CS iteration was not proved, just the desired conclusion $(\oplus)$ hold for $P_\alpha$ when $\langle P_i, Q_j : i \leq \alpha, j < \alpha \rangle$ is a countable support iteration and $i < \alpha \Rightarrow \Vdash_{P_i} "Q_i \text{ is a Souslin proper forcing notion}"$;

(c) to prove for such forcing notions better preservation theorems when we add properties in addition to properness.
Martin Goldstern asked me some years ago on the inadequacy of Souslin
proper from clause (a). I suggested a version of the definitions here, and
this was preliminarily announced in Goldstern [10].

The intention here is to include forcing notions with “nice definition” (not
ones constructed by diagonalization like Baumgartner’s “every $\aleph_1$-dense sets
of reals are isomorphic” $\mathtt{R}$ or the forcing notions constructed for the oracle
c.c.c., see [8, Ch.IV], or forcing notions defined from an ultrafilter).

Note that our treatment (nep/snep) in a sense stands between [17] and
Roslanowski Shelah [14]. In [17] we like to have theorems on iterations
$\mathcal{Q}$, mainly CS, getting results on the whole $\text{Lim}(\mathcal{Q})$ from assumptions on
each $\mathcal{Q}_i$, but with no closer look at $\mathcal{Q}_i$ – by intention, as we would like to
cover as much as we can. In Roslanowski Shelah [14] we deal with forcing
notions which are quite concrete, usually built from countably many finite
“creatures” (still relative to specific forcing this is quite general).

Here, our forcing notions are definable but not in so specific way as in
[14], which still provides examples (all are included), and the theorems are
quite parallel to [17]. So we are solving the “equations”

\[ \text{“theory of manifolds” = theory of proper forcing [18, 17]/general topology = theory of forcing based on creatures [14}/theory of manifolds in $\mathbb{R}^3$.} \]

\textbf{Thesis 1.3.} “Nice” forcing notions which are proved to be proper, normally
satisfy (even by same proof) the stronger demands defined in the next section.

\textbf{History:} The paper is based on the author’s lectures in Rutgers University in Fall 1996, which results probably in too many explanations. Answering Goldstern’s question was mentioned above. A version of §8 (on non-symmetry) was done in Spring of ’95 aiming at the symmetry question, and the rest in the Summer and Fall of ’96. I thank the audience of the lectures for their remarks and mainly Andrzej Roslanowski for correcting the paper.

\textbf{Notation:} We try to keep our notation standard and compatible with
that of classical textbooks on Set Theory (like Bartoszyński Judah [3] or
Jech [13]). However in forcing we keep the older tradition that a stronger
condition is the larger one.

For a regular cardinal $\chi$, $\mathcal{H}(\chi)$ stands for the family of sets which are
hereditarily of size less than $\chi$. The collection of all sets which are heredi-
tarily countable relatively to $\kappa$ is denoted by $\mathcal{H}_{<\aleph_1}(\kappa)$.

$\text{Tc}^{\text{ord}}(x)$ is defined by induction on $\text{rk}(x) = \gamma$ as follows:

- if $\gamma = 0$ then $\text{Tc}^{\text{ord}}(x) = x \cup \{x\}$,
if \( \gamma > 0 \) then \( \text{Te}^\text{ord}(x) = x \cup \bigcup \{ \text{Te}^\text{ord}(y) : y \in x, y \text{ not an ordinal} \} \cup \{ x \} \).

So \( \mathcal{H}_{< \kappa_1}(\kappa) = \{ x \in \mathcal{H}(\kappa) : \text{Te}^\text{ord}(x) \text{ is countable} \} \).

We say that a set \( M \subseteq \mathcal{H}(\chi) \) is ord–transitive if
\[
x \in M \text{ \& \ } x \text{ is not an ordinal} \implies x \subseteq M.
\]

**Notation 1.4.** We will keep the following rules for our notation:

1. \( \alpha, \beta, \gamma, \delta, \xi, \zeta, i, j \ldots \) will denote ordinals,
2. \( \theta, \kappa, \lambda, \mu, \chi \ldots \) will stand for cardinal numbers, \( \theta \leq \kappa \) if not said otherwise,
3. a tilde indicates that we are dealing with a name for an object in forcing extension (like \( x \))
4. a bar above a name indicates that the object is a sequence, usually \( \bar{X} = \langle X_i : i < \ell g(\bar{X}) \rangle \), where \( \ell g(\bar{X}) \) denotes the length of \( \bar{X} \),
5. For two sequences \( \eta, \nu \) we write \( \nu \rhd \eta \) whenever \( \nu \) is a proper initial segment of \( \eta \), and \( \nu \rhd \eta \) when either \( \nu \rhd \eta \) or \( \nu = \eta \).
6. The length of a sequence \( \eta \) is denoted by \( \ell g(\eta) \).

1. **Basic definitions.** Let us try to analyze the situation. Our intuition is that: looking at \( Q \) inside \( N \) we can construct a generic condition \( q \) for \( N \), but if \( N \not\in (\mathcal{H}(\chi), \in) \), \( Q \cap N \) might be arbitrary. So let \( Q \) be a definition. What is the meaning of, say, \( N \models "r \in Q" \)? It is \( N \models "\varphi_0(-)" \) for a suitable \( \varphi_0 \). It seems quite compelling to demand that inside \( N \) we can say in some sense “\( r \in Q \)”, and as we would like to have
\[
q \models "G_Q \cap Q^N \text{ is a subset of } Q^N \text{ generic over } N "
\]
we should demand

\((*)_1\) \(N \models \langle r \in Q \rangle \) implies \(V \models \langle r \in Q \rangle \).

So \(\varphi_0\) (the definition of the set of members of \(Q\)) should have this amount of absoluteness. Similarly we would like to have:

\((*)_2\) if \(N \models \langle p_1 \leq Q p_2 \rangle \) and \(p_2 \in G_Q\) then \(p_1 \in G_Q\).

So we would like to have a \(\varphi_1\) (or \(\langle <_{\varphi_1} \rangle\)) (the definition of the partial order of \(Q\)) and to have this upward absoluteness for \(\varphi_1\).

But before we define this notion of properness without elementarity, we should define the class of models to which it applies.

We may have put in this section the “straight nep” (see 6.11) and/or “absolute nep” (see 5.2). Advice: The reader may concentrate on the case of local correct explicit simply good and nep forcing notions which are normal (see Definitions 2.1, 2.3(11), 2.3(2), 2.3(5), 2.3(1),(4), 2.11(3), 2.11(4), respectively).

When we consider “preservation by iteration”, it is natural to define the following:

**Definition 2.1.**

1. Let \(\text{ZFC}^{-}_\ast\) be a fixed version of set theory e.g. \(\text{ZFC}^{-} + \langle \exists \beth_7 \rangle\) which may speak on \(\mathfrak{C}\) (or see more in the end of this section), and let \(\mathcal{C}\) be a fixed model with countable vocabulary (say \(\subseteq \mathcal{H}(\mathbb{N}_0)\)) and universe an ordinal \(\alpha = \alpha_\ast(\mathfrak{C})\) and let \(\Delta\) be a fixed set of first order formulas in the vocabulary of \(\mathfrak{C}\) (closed under subformulas normally). Let \(\mathfrak{B}\) denote another such model (not fixed) but we may allow the universe to satisfy \(\kappa(\mathfrak{B}) \subseteq |\mathfrak{B}| \subseteq \mathcal{H}_{< \mathfrak{N}_1}(\kappa(\mathfrak{B}))\) for some cardinal \(\kappa(\mathfrak{B})\).

2. We say that \(N\) is a class \((\mathfrak{B}, p, \theta)\)-candidate if:

(a) \(N \subseteq (\mathcal{H}(\chi), \epsilon)\) for some \(\chi\),

(b) \(N\) is countable,

(c) \(N\) is a model of \(\text{ZFC}^{-}_\ast\),

(d) \(\mathcal{C} \subseteq N, p \subseteq N, \mathfrak{B} \subseteq N\) (but see below),

(e) \(\mathfrak{B} \models \langle \mathfrak{C} \lessdot \mathfrak{B} \rangle\) for transparency we treat \(\mathfrak{B}\) as relations of \(N\) and \(\mathfrak{B} \models \langle \mathfrak{C} \lessdot \mathfrak{B} \rangle\) are their interpretations in \(N\); so we allow \(|\mathfrak{B}| \cap |N| \setminus |\mathfrak{B} \cap N| \neq \emptyset\), and \(\tau(\mathfrak{B})\), the vocabulary of \(\mathfrak{B}\), belongs to \(N\), but \(N \models \langle \mathfrak{C} \models \langle \mathfrak{B}_G \rangle \rangle \Rightarrow x \in \mathfrak{B}\). Similarly for \(\mathfrak{C}\) (this is less essential),

\(1\) We do not fix the order between \(\alpha_\ast(\mathfrak{C})\) and \(\kappa(\mathfrak{B})\), but there is no loss if we assume that \(\theta \geq \alpha_\ast(\mathfrak{C}), \theta \geq \kappa(\mathfrak{B})\).

\(2\) so if \(\Delta = \Delta_1 = \{\exists \psi : \psi \text{ is q.f.}\}, \mathcal{C}, \mathfrak{B}\) have Skolem functions, we have \(\langle e' \rangle\) \(\mathcal{C} \models \langle \mathfrak{C} \cap N \lessdot \mathfrak{B} \rangle\); \(\langle e' \rangle\) can use \(\langle e' \rangle\) instead of \(\langle e \rangle\)
(f) if $N \models \"\alpha \text{ is an ordinal} < \theta, \text{ or } \leq |\mathcal{C}| \text{ (that is } \alpha_*(\mathcal{C}) \text{ or } \leq \kappa(\mathcal{B})\")$
then $\alpha$ is an ordinal,
(g) if $N \models \"x \text{ is countable}\"$ then $x \subseteq N$,
(h) if $N \models \"x \text{ is an ordinal}\"$ then $x$ is an ordinal.
3. We omit the “class” if additionally
(i) $p = \overline{\varphi}$ is a tuple of formulas, $\varphi_0 = \varphi_0(x)$ and in $N$, $\varphi_0(x)$ defines
a set.
4. We add the adjective “semi” if we omit clause (b) (the countability
demand).
5. If $p$ is absent (or clear from context) we may omit it, similarly $\theta$ when
$\theta = \aleph_0$ or clear from the context. We tend to “forget” to mention $\mathcal{C}$
e.g. demand $\mathcal{B}$ expands it).
6. We say that a formula $\varphi$ is upward absolute for (or from) class $(\mathcal{B}, p, \theta)$–
candidates when: if $N_1$ is a class $(\mathcal{B}, p, \theta)$–candidate, $N_1 \models \varphi[\overline{x}]$, and
$N_2$ is a class $(\mathcal{B}, p, \theta)$–candidate or is $(\mathcal{H}(\chi), \in)$ for $\chi$ large enough,
and $N_1$ is a set or just a class of $N_2$, then $N_2 \models \varphi[\overline{x}]$.
We say above “through (class) $(\mathcal{B}, p, \theta)$–candidates” if $N_2$ is de-
manded to be a (class) $(\mathcal{B}, p, \theta)$–candidate. Note that we can omit
$\mathcal{H}(\chi)$ in the correct case (see 2.3(11)).
If $\mathcal{B}, p, \theta$ are clear from the context, we may forget to say “for class
$(\mathcal{B}, p, \theta)$–candidates”.
7. We say that $\varphi$ defines $X$ absolutely through $(\mathcal{B}, p, \theta)$–candidates if
$\alpha$ holds then we add “weakly”.
(\alpha) $\varphi = \varphi(x)$ is upward absolute through $(\mathcal{B}, p, \theta)$–candidates,
(\beta) $X = \bigcup\{X^N : N \text{ is a } (\mathcal{B}, p, \theta)$–candidate\}, where $X^N = \{x \in
N : N \models \varphi(x)\}$.
If only clause (a) holds then we add “weakly”.

Discussion 2.2. Should we prefer $|\mathcal{B}| = \alpha$ an ordinal or $|\mathcal{B}| \subseteq \mathcal{H}_{<\aleph_1}(\alpha)$? The former is more convenient when we “collapse $N$ over $\kappa \cup \theta$” (see 3.3). Also then we can fix the universe whereas $|\mathcal{B}| = \mathcal{H}_{<\aleph_1}(\alpha)$ is less reasonable as it is less absolute. On the other hand, when we would like to prove preservation by iteration the second is more useful (see §5). To have the best of both we adopt the somewhat unnatural meaning of $\mathcal{B} \downarrow N \prec_\Delta \mathcal{B}$ in clause (e) of Definition 2.1.
We may have forgotten sometimes to write $|\mathcal{B}|$ instead of $\kappa = \kappa(\mathcal{B})$.
In some cases, we may omit the demand (h) in the definition 2.1 of
$(\mathcal{B}, p, \theta)$–candidates (and then calling them “impolite candidates”), but still
we should demand then that

$$N \models \"x \text{ is an ordinal from } \mathcal{B} \text{ or } \mathcal{C}\" \Rightarrow x \text{ is an ordinal,}$$

3This is normally the forcing notion $\mathbb{Q}$.
and we should change “ordinal collapse” appropriately. However, there is no reason to attend the “impolite” company here.

This motivates:

Definition 2.3. 1. Let \( \bar{\varphi} = \langle \varphi_0, \varphi_1 \rangle \) and \( \mathcal{B} \) be a model as in [2.1], \( \kappa = \kappa(\mathcal{B}) \), and of countable vocabulary, say \( \subseteq \mathcal{H}(\aleph_0) \). We say that \( \bar{\varphi} \) or \( (\bar{\varphi}, \mathcal{B}) \) is a temporary \((\kappa, \theta)\)-definition, or \((\mathcal{B}, \theta)\)-definition, of a nep-forcing notion\(^4\) \( Q \) if, in \( V \):

(a) \( \varphi_0 \) defines the set of elements of \( Q \) and \( \varphi_0 \) is upward absolute from \( (\mathcal{B}, \bar{\varphi}, \theta) \)-candidates,

(b) \( \varphi_1 \) defines the partial (or quasi) ordering of \( Q \), also in every \( (\mathcal{B}, \bar{\varphi}, \theta) \)-candidate, and \( \varphi_1 \) is upward absolute from \( (\mathcal{B}, \bar{\varphi}, \theta) \)-candidates,

(c) if \( N \) is a \( (\mathcal{B}, \bar{\varphi}, \theta) \)-candidate and \( p \in \mathcal{Q}_N \), then there is \( q \in \mathcal{Q} \) such that \( p \leq Q q \) and

\[
q \Vdash \text{"} G_\mathcal{Q} \cap \mathcal{Q}_N \text{ is a subset of } \mathcal{Q}_N \text{ generic over } N \text{ "}
\]

where, of course, \( \mathcal{Q}_N = \{ p : N \models \varphi_0(p) \} \).

2. We add the adjective “explicitly” if \( \bar{\varphi} = \langle \varphi_0, \varphi_1, \varphi_2 \rangle \) and additionally

(b)\(^+\) we add: \( \varphi_2 \) is an \((\omega + 1)\)-place relation, upward absolute through \( (\mathcal{B}, \bar{\varphi}, \theta) \)-candidates and \( \varphi_2((p_i : i \leq \omega)) \Rightarrow \{p_i : i \leq \omega\} \subseteq \mathcal{Q} \) and \( \{p_i : i < \omega\} \) is predense above \( p_\omega \), not just in \( V \) but in every \( Q \)-candidate (which, if \( Q \) is correct, implies the case in \( V \)); in this situation we say: \( \{p_i : i < \omega\} \) is explicitly predense above \( p_\omega \),

(c)\(^+\) we add: if \( N \models \text{"} \mathcal{I} := Q \text{ is dense open} \text{"} \) (or just predense) (so \( \mathcal{I} \in N \) then for some list \( \langle p_i : i < \omega \rangle \) of \( \mathcal{I} \cap N \) we have \( \varphi_2((p_i : i < \omega)) \rightarrow (q) \)).

3. For a class \( (\mathcal{B}, \bar{\varphi}, \theta) \)-candidate \( N \) we let \( \text{pd}(\mathcal{N}, \mathcal{Q}) = \text{pd}_\mathcal{Q}(\mathcal{N}) = \{ \mathcal{I} : \mathcal{I} \subseteq \mathcal{N} \text{ is a class of } \mathcal{N} \text{ (i.e. defined in } \mathcal{N} \text{ by a first order formula with parameters from } \mathcal{N} \text{) and is a predense subset of } \mathcal{Q}_N \} \). If \( N \) is a candidate, it is \( \{ \mathcal{I} \in \mathcal{N} : N \models \text{"} \mathcal{I} \text{ is predense} \text{"} \} \).

4. We replace “temporary” by \( K \) if the relevant proposition holds not only in \( V \) but in any forcing extension of \( V \) by a forcing notion \( \mathbb{P} \in K \). If \( K \) is understood from the context (normally: all forcing notions we will use in that application) we may omit it.

5. We say that \( (\bar{\varphi}, \mathcal{B}) \) is simply \( \text{[explicitly]} \) \( K-(\kappa, \theta) \)-definition of a nep-forcing notion \( Q \), if:

(a) \( (\bar{\varphi}, \mathcal{B}) \) is \( \text{[explicitly]} \) \( K \)-definition of a nep-forcing notion \( Q \),

(b) \( Q \subseteq \mathcal{H}_{<\aleph_1}(\theta) \); i.e. \( \mathbb{P} \in K \) implies \( \models_{\mathbb{P}} \text{"} \varphi_0(x) \text{ then } x \in \mathcal{H}_{<\aleph_1}(\theta) \text{"} \),

(c) \( \mathcal{B}, \kappa, \theta \) are the only parameters of \( \bar{\varphi} \) (meaning there are no others, but even \( \mathcal{B}, \kappa, \theta \) do not necessarily appear).

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\(^4\) so in the normal case (see [2.11(4), 2.13], \( \bar{\varphi} \) defines \( Q \)}
6. We add “very simply” if in addition:
   (δ) \( Q \subseteq \omega \theta \).

7. We may say “\( Q \) is a nep-forcing notion”, “\( N \) is a \( Q \)-candidate” abusing notion. If not clear, we write \( Q^\theta \) or \( (Q^\theta)^V \). If not said otherwise, \( \Delta \) is the set of first order formulas. Inversely, we write \( (\mathfrak{B}, \bar{\varphi}, \theta) = (\mathfrak{B}^Q, \bar{\varphi}^Q, \theta^Q) \) and ZFC\(^Q \) for the relevant ZFC\(^Q \).

8. We say “\( I \subseteq Q^N \) is explicitly predense over \( p_\omega \)” if \( \phi_2(\langle p_i : i \leq \omega \rangle) \) for some list \( \{p_i : i < \omega \} \) of \( I \).

9. We add the adjective “class” if we allow ourselves (in clauses (b), (c) of part (1) and (c)\(^+\) of part (2)) class \( (B, \bar{\varphi}, \theta) \)-candidates \( N \); so in clauses (c), (c)\(^+\), \( I \) is a class of \( N \); i.e. first order definable with parameters from \( N \), and use the weak version of absoluteness.

   If we use \( (B, \bar{p}, \theta) \) we mean \( \bar{\varphi} \) is an initial segment of \( p \).

10. We say \( (B, \bar{\varphi}, \theta) \) (or abusing notation, \( Q \)) is class=set if every class \( (B, \bar{\varphi}, \theta) \)-candidate is a \( (B, \bar{\varphi}, \theta) \)-candidate.

11. In 2.3(1) we add the adjective “correctly” (and we say that \( (B, \bar{\varphi}, \theta) \) is correct) if, for a large enough regular cardinal \( \chi \):
   (a) the formula \( \phi_0 \) defines the set of members of \( Q \) absolutely through \( (B, \bar{\varphi}, \theta) \)-candidates, that is
   \[
   Q = \bigcup \{ Q^N : N \text{ is a } (B, \bar{\varphi}, \theta)\text{-candidate} \},
   \]
   \[
   Q^N = \{ x : N \models \phi_0(x) \},
   \]
   (b) the formula \( \phi_1 \) defines the quasi order of \( Q \) absolutely through \( (B, \bar{\varphi}, \theta) \)-candidates, that is
   \[
   \leq_Q = \bigcup \{ \leq_N^Q : N \text{ is a } (B, \bar{\varphi}, \theta)\text{-candidate} \}, \leq_N^Q = \{ (p, q) : N \models \phi_1(p, q) \}.
   \]
   Similarly when we add “explicitly”.

   So in those cases we can ignore \( \mathcal{H}(\chi) \models \phi_\ell(x) \) and just ask for satisfaction in suitable candidates. (Note: correct is less relevant to snep.)

**Convention:** We may say “\( Q \) is . . .” when we mean “(\( B, \bar{\varphi}, \theta \)) is . . .” or “(\( B, \bar{p}, \theta \)) is . . .”.

**Remark:** The main case for us is candidates (not class ones), etc; still mostly we can use the class version of nep. Also we can play with various free choices.

**Discussion 2.4.** 1) Note: if \( x \in I \in N, N \models “I \subseteq Q“ \), possibly \( x \notin Q \) so those \( x \) are not relevant (e.g. though \( \alpha < \kappa(\mathfrak{B}) \) have a special role).

2) We think of using CS iteration \( \bar{Q} = \langle P_i, Q_i : i < \delta \rangle \), each \( Q_i \) has a definition \( \varphi^i \) and we would like to prove things on \( P_\alpha \) for \( \alpha \leq \delta \). So the relevant family \( K_i \) of forcing notions we really should consider for \( \varphi^i \) is.
\( \{ P_\beta / P_i : \beta \in [i, \delta) \} \), at least this holds almost always (maybe we can look as help in other extensions).

3) Note that a significant fraction of iterated forcing of proper forcing related to reals are forcing notions called “nice” above. The proof that they are proper usually gives more and we think that they will be included even by the same proof.

4) If \( K \) is trivial, (i.e. has only the trivial forcing notion as a member) this means we can replace it by “temporarily”.

5) See also for “\( K \)-absolutely”.

6) Note a crucial point in Definition 2.3, the relation \( \{ p_n : n < \omega \} \) is predense above \( p \) is not demanded to be absolute; only a “dense” family of cases of it is demanded (we also allow other basic relations; e.g. \( q \notin Q \) to be non-absolute but those are less crucial). This change may seem technical, but is central being the difference between including not few natural examples and including all those we have in mind.

7) Note that in clause (c) of 2.3(1) we mean: \( G \cap Q^N \) is directed (by \( \leq Q^N \), not only by \( \leq Q \)) and \( G \cap N \cap I \neq \emptyset \) for \( I \in \text{pd}(N, Q) \).

8) Note that the demand described in 7) above almost implies “incompatibility is upward absolute from \( N \)”, but not quite.

Let us consider a more restrictive class, where the absoluteness holds because of more concrete reasons, the usual ones for upward absoluteness, the relations are \( \Sigma_1^1 \), or more generally, \( \kappa \)-Souslin.

**Definition 2.5.** 1. We say that \( \bar{T} \) is a temporary \( (\kappa, \theta) \)-definition of a snap-forcing notion \( Q \) if:

(a) \( \bar{T} = (T_0, T_1) \) where \( T_0 \subseteq \omega^>(\kappa \times \kappa) \) and \( T_1 \subseteq \omega^>(\kappa \times \kappa \times \kappa) \) are trees (i.e. closed under initial segments, non-empty) and \( \theta \leq \kappa \),

(b) the set of elements of \( Q \) is

\[
\text{proj}_0(T_0) \overset{\text{def}}{=} \{ \nu \in \omega^\theta : \text{ for some } \eta \in \omega \kappa \text{ we have } \nu \ast \eta \overset{\text{def}}{=} (\langle \nu(n), \eta(n) \rangle : n < \omega) \in \text{lim}(T_0) \},
\]

(c) the partial order of \( Q \), \( \{ (p_0, p_1) : Q \models p_0 \leq p_1 \} \) is

\[
\text{proj}_1(T_1) \overset{\text{def}}{=} \{ (\nu_0, \nu_1) : \nu_0, \nu_1 \in Q \text{ and for some } \eta \in \omega \kappa \text{ we have } \nu_0 * \nu_1 * \eta \overset{\text{def}}{=} (\langle \nu_0(n), \nu_1(n), \eta(n) \rangle : n < \omega) \in \text{lim}(T_1) \},
\]

(d) for a large enough regular cardinal \( \chi \), if \( N \subseteq (H(\chi), \in) \) is a \( (\mathcal{B}_T, \bar{T}, \theta) \)-candidate and \( \kappa \in N \), \( \bar{T} \in N \), \( p \in Q^N \) then there is \( q \in Q \) such that \( p \leq Q q \) and

\[
q \models " G_Q \cap Q^N \text{ is a generic subset of } Q^N \text{ over } N \”,
\]
where \( B_T \) is the model with universe \( \kappa \) and the sequence relations
\[ T_0 \cap n(\kappa \times \kappa), \quad T_1 \cap n(\kappa \times \kappa \times \kappa) \text{ for } n < \omega. \]

2. We add “explicitly” if \( \bar{T} = \langle T_0, T_1, T_2 \rangle \) and we add

(a)\(^+\) also \( T_2 \subseteq \omega \rangle (\theta \times \theta \times \kappa) \) and we let

\[
\text{proj}_2(T_2) \overset{\text{def}}{=} \{ \langle \nu_i : i \leq \omega \rangle : \text{for some } \eta \in \omega \kappa \text{ we have } \nu \ast \nu_\omega \ast \eta \in \lim(T_2) \]
\[
\text{where } \nu = \text{code}(\langle \nu_\ell : \ell < \omega \rangle) \text{ is the member of } \omega \theta \text{ satisfying } \nu(\ell + k + 1) + \ell = \nu_\ell(k) \}
\]

and \( \langle \nu_i : i \leq \omega \rangle \in \text{proj}_2(T_2) \) implies \( \{ \nu_i : i \leq \omega \} \subseteq Q \) (even in candidates; the natural case is that witnesses are coded).

(d)\(^+\) we add: \( q \) is \( \bar{T} \)-explicitly \((N, Q)\)-generic, which means that

\[ \text{if } N \models "I \text{ is a dense open subset of } Q\" \]

\[ \text{then for some list } \langle p_n : n < \omega \rangle \text{ of } I \cap N \text{ we have } \langle p_n : n < \omega \rangle \subseteq \text{proj}_2(T_2), \]

(e)\(^+\) if \( \nu_i \in Q \) for \( i \leq \omega \) and for some \( \eta \in \omega \kappa \) we have code(\( \nu_0, \nu_1, \ldots \)) \ast \nu_\omega \ast \eta \in \lim(T_2) \) then \( \{ \nu_0, \nu_1, \ldots \} \subseteq Q \) is predense above \( \nu_\omega \) (and this holds in candidates too).

3. We will also say “\( Q \) is a snep-forcing notion”, “\( N \) is a \( Q \)-candidate”, etc.

4. We say \( \eta \) is a witness for \( \nu \in Q \) if \( \nu \ast \eta \in \lim(T_0) \); similarly for \( T_1, T_2 \).

We say that \( I \) is explicitly predense over \( p_\omega \) if code(\( \nu_i : i \leq \omega \)) \in \text{proj}_2(T_2) \) for some list \( \{ p_i : i < \omega \} \) of \( I \).

Remark 2.6. In clause (a)\(^+\) we would like the proj\(_2\)(\( T_2 \)) to be an \( (\omega + 1) \)-place relation on \( Q \), but we do not like the first coordinate to give too much information so we use the above coding, but it is in no way special. Note: we do not want to have one coordinate giving \( \langle \varphi_\ell(0) : \ell < \omega \rangle \).

Another possible coding is code(\( \nu_0, \nu_1, \ldots \)) \equiv \( \langle \nu_\ell | i : \ell \leq i : i < \omega \rangle \), so \( T \subseteq \omega \rangle (\omega \rangle (\omega \rangle \theta) \times \theta \times \kappa) \).

Proposition 2.7. Assume that \( \bar{T} \) is in \( V \) a temporary \((\kappa, \theta)\)-definition of a snep forcing notion which we call \( Q \). Let \( V' \) be a transitive class of \( V \) containing \( \bar{T} \). Then:

1. also in \( V' \), \( Q \) is snep,
2. if \( V' \models "p \in Q" \) then \( V \models "p \in Q" \),
3. if \( V' \models "p \leq \bar{Q} q" \) then \( V' \models "p \leq \bar{Q} q" \),
4. if in \( V' \), the model \( N \) is a \((\mathfrak{B}_T, \varphi_T, \kappa_T)\)-candidate then also in \( V \), \( N \) is a \((\mathfrak{B}_T, \varphi_T, \kappa_T)\)-candidate. \]

Definition 2.8. 1. Let \( Q \) be explicitly snep. We add the adjective “local” if in the “properness clause i.e. 2.5\(^2\)(d)\(^+\)” we can add:
(⊗) the witnesses for “\(q \in \mathbb{Q}\)”, “\((\mu_n^\mathbb{Q} : n < \omega)\) is \(\mathbb{Q}\)-explicitly predense above \(q\)” are from \(\omega(N \cap \kappa)\).

2. Let \(\mathbb{Q}\) be explicitly nep. We add the adjective “\(K\)-local” if in the “properness clause i.e. \(2.3(2)(b)^+\)” we can add: for each candidate \(N\) which is ord–transitive we have
\[
(\oplus) \text{ for some } K\text{–extension } N^+ \text{ of } N, \text{it is a } \mathbb{Q}\text{–candidate (in particular a model of ZFC}^-)\text{ and } N^+ \models “Q^N \text{ is countable}” \text{ and } q \in N^+, N^+ \models “p \leq^\mathbb{Q} q \text{ and for each } I \in \text{pd}(N, \mathbb{Q}), I^N \text{ is explicitly predense over } q”.
\]
(Note that \(B \upharpoonright N^+ = B \upharpoonright N\).)

If \(K\) is the family of set forcing notions, or constant understood from the context, we may omit \(K\).

Discussion 2.9. 1) Couldn’t we fix \(\theta = \omega\)? Well, if we would like to have the result of “the limit of a CS iteration \(\bar{\mathbb{Q}}\) of such forcing notions is such a forcing notion”, we normally need \(\theta \geq \ell g(\bar{\mathbb{Q}})\). Also \(\kappa > \aleph_0\) is good for including \(\Pi^1_2\)–relations.

2) In “Souslin proper” (starting with [12]) the demands were

**Definition 2.10.** A forcing notion \(\mathbb{Q}\) is Souslin proper if it is proper and: the relations “\(x \in \mathbb{Q}\)”, “\(x \leq^\mathbb{Q} y\)” are \(\Sigma^1_1\) and “the notion of incompatibility in \(\mathbb{Q}\)” is \(\Sigma^1_1\) (where, of course, the compatibility relation is \(\Sigma^1_1\)).

This makes “\(\{p_n : n < \omega\}\) is predense over \(p_\omega\)” a \(\Pi^1_2\)–property, hence an \(\aleph_1\)-Souslin one. So we can get the “explicitly” cheaply, however possibly increasing \(\kappa\). Note that for a Souslin proper forcing notion \(\mathbb{Q}\), also \(p \in \mathbb{Q}^N \iff p \in \mathbb{Q} \& p \in N\) and similarly for \(p \leq^\mathbb{Q} q\).

\* \* \* \* \* \*

If you like to be more pedant on the ZFC\(^-\), look at the following definition. Normally there is no problem in having ZFC\(^-\) as required.

**Definition 2.11.** 1. We say \(\text{ZFC}^-\) is a \(K\)–good version [with parameter \(\mathcal{C}\), possibly “for \((\mathcal{B}, p, \theta)\)” for \(\mathcal{B}, p, \theta\) as in \(2.3\) from the relevant family] if:

(a) it contains ZC\(^-\); i.e. Zermelo set theory without power set, [and the axioms may speak on \(\mathcal{C}\)]

(b) \(\mathcal{C}\) is a model with countable vocabulary (given as a well ordered sequence, so \(\mathcal{C}\) is an individual constant in the theory \(\text{ZFC}^-\)) and universe \(|\mathcal{C}|\) is an ordinal \(\alpha(\mathcal{C})\),

(c) for every \(\chi\) large enough, if \(X \subseteq \mathcal{H}(\chi)\) is countable then for some countable \(N \subseteq (\mathcal{H}(\chi), \epsilon), N \models \text{ZFC}^-\), \(X \subseteq N\) and
\[
x \in N \& N \models “|x| = \aleph_0” \quad \Rightarrow \quad x \subseteq N
\]
and $C \in \mathbb{N}$ and $C \upharpoonright (N \cap |C|) \prec \mathbb{C}$ (can be weakened to a submodel or $\prec_\Delta$, we do not loose much as we can expand by Skolem functions); in the “for $(\mathcal{B}, p, \theta)$” version we add “$N$ is a $(\mathcal{B}, p, \theta)$–candidate”,

d) $\text{ZFC}^-$ satisfies the forcing theorem (see e.g. [7, Ch. I]) at least for forcing notions in $K$,

e) those properties are preserved by forcing notions in $K$ (if $P \in K$, $G \subseteq P$ generic over $V[G]$ then $K^{V[G]}$ will be interpreted as $\{Q[G] : P \ast Q \in K\}$).

2. If $K$ is the class of all (set) forcing notions, we may omit it.

3. We say $\text{ZFC}^-$ is normal if for $\chi$ large enough any countable $N \prec (H(\chi), \in)$ to which $C$ belongs is O.K. (for clause (1)(c) above).

4. We say $\text{ZFC}^-$ is semi-normal for $(\mathcal{B}, p, \theta)$ if for $\chi$ large enough, for any countable $N \prec (H(\chi), \in)$ (to which appropriate $p, C, \mathcal{B}, \theta(\in H(\chi))$ belong), for some $Q \in N$ such that $N \models \text{“}Q$ is a forcing notion” we have:

\begin{itemize}
  \item[(\ast)] if $N'$ is countable $N \subseteq N' \subseteq (H(\chi), \in)$, $N' \cap \chi = N \cap \chi$ and
  \begin{align*}
    (\forall x)[N' \models \text{“}x \text{ is countable } \Rightarrow x \subseteq N'],
  \end{align*}
  
  \noindent and $N'$ is a generic extension of $N$ for $Q^N$ then $N'$ is $(\mathcal{B}, p, \theta)$–candidate and $Q^{N'} \upharpoonright N = Q \upharpoonright N$, $\varphi^{N'} \upharpoonright N = \varphi^N \upharpoonright N$.

  We say “$K$–semi-normal” if we demand $N \models Q \in K$.

5. We say $\text{ZFC}^-$ is weakly normal for $(\mathcal{B}, p, \theta)$ if clause (c) of part (1) holds.

6. In parts (4), (5) we can replace $(\mathcal{B}, p, \theta)$ by a family of such triples meaning $N$ is a candidate for all of them.

7. In parts (4), (5), (6) if $(\mathcal{B}, p, \theta) = (\mathcal{B}^Q, \varphi^Q, \theta^Q)$ we may replace $(\mathcal{B}, p, \theta)$ by $Q$.

Discussion 2.12. 1) What are the points of parameters? E.g. we may have $\kappa^*$ an Erdős cardinal, $C$ codes every $A \in H(\chi)$ for each $\chi < \kappa^*$, $\text{ZFC}^- = \text{ZFC}^- + \text{“}\kappa^*$ is an Erdős cardinal $C$ as above”, $K$ = the class of forcing notions of cardinality $< \kappa^*$. Then we have stronger absoluteness results to play with.

2) On the other hand, we may use $\text{ZFC}^- = \text{ZFC}^- + (\forall r \in \omega^2)(r^# \text{ exists}) + \text{“}\exists \mathbb{R}$ exists”. This is a good version if $V \models (\forall r \in \omega^2)(r^# \text{ exists})$ so we can e.g. weaken the definition snep (or Souslin-proper or Souslin-$\mathbf{c.c.c}$).

\[\text{for } 8.8 \text{ we need: if } P, Q \text{ are forcing notions, } G \text{ is a } P \text{–name for a subset of } Q \text{ such that } \mathbf{I} \vdash \text{“}G \text{ is a generic subset of } Q \text{”}, \text{ and } q \in Q \Rightarrow \mathbf{P} \ast q \notin G \text{ then for some } Q \text{–name } \mathbb{R} \text{ of a forcing notion, } Q \ast \mathbb{R}, P \text{ are equivalent.}\]
3) What is the point of semi-normal? E.g. if we would like \( \text{ZFC}^- \vdash \text{CH} \), whereas in \( V \) the Continuum Hypothesis fails. But as we have said in the beginning, the normal case is usually enough.

**Proposition 2.13.** 1. Assume \( \text{ZFC}^- \) is \( \{\emptyset\} \)-good. Then the clause \((c)^+\) of 2.3(2) is equivalent to clause \((c) + (*)\), where

\[(*) \text{ if } p \in Q \text{ and } I_n \text{ is predense over } p \text{ (for } n < \omega) \text{, each } I_n \text{ is countable, then for some } q, p \leq q \in Q, \text{ and for some } p_n^\ell \in I_n \text{ for } n < \omega, \ell < \omega \text{ we have } \varphi_2(p_n^\ell : \ell < \omega) \neg q) \]

(this is an obvious abusing of notation, we mean that this holds in some candidate).

2. If \( \text{ZFC}^- \) is normal for \((\mathcal{B}, p, \theta)\) then in Definition 2.3(1),(2) there is no difference between “absolutely through” and “weakly absolutely”.

**Proposition 2.14.** 1. Assume \( V_1 \subseteq V_2 \) (so \( V_1 \) is a transitive class of \( V_2 \) containing the ordinals, \( \mathcal{C}, \mathcal{B}, \theta, p, \in V_1 \)). If \( \text{ZFC}^- \) is temporarily good then also in \( V_1 \) it is temporarily good.

2. If co-(\(\kappa + \theta\))-Souslin relations are downward absolute (from \( V_2 \) to \( V_1 \)) then also inverse holds.

**Proof** By Shoenfield–Levy absoluteness.

2. *Connections between the basic definitions.* We first give the most transparent implications: we can omit “explicitly” and we can replace snap by nep (this is 3.1) and the model \( \mathcal{B} \) can be expanded, \( \kappa, \theta \) increased, (see 3.2). Then we note that if \( \kappa \geq \theta + \aleph_1 \) and we are in the simple nep case, we can get from nep to snap because saying “there is a countable model \( N \subseteq (\mathcal{H}(\chi), \in) \) such that . . .” can be expressed as a \( \kappa \)-Souslin relation (see 3.3) and comment on the non-simple case. Then we discuss how the absoluteness lemmas help us to change the universe (in 3.4), to get the case with a class \( K \) from the case of temporarily (3.5) and to get explicit case from snap or from Souslin proper (in 3.6).

**Proposition 3.1.** 1. If \((\bar{\varphi}, \mathcal{B})\) is explicitly a \( K \)-definition of a nep-forcing notion \( Q \), then \( \bar{\varphi} \upharpoonright 2 \) is a \( K \)-definition of a nep-forcing notion \( Q \).

2. If \( \bar{T} \) is explicitly a \( K \)-definition of a snap-forcing notion \( Q \), then \((\bar{T} \upharpoonright 2)\) is a \( K \)-definition of an snap-forcing notion \( Q \).

3. If \( \bar{T} \) is [explicitly] a \( K-(\kappa, \theta) \)-definition of a snap-forcing notion \( Q \), and \( \mathcal{B} \) any model with universe \( \kappa \) coding the \( T_\ell \)'s and \( \varphi_\ell \) is defined as \( \text{proj}_\ell(T_\ell) \), then \((\bar{\varphi}, \mathcal{B})\) is very simply [explicitly] \( K-(\kappa, \theta) \)-definition of a nep forcing notion \( Q \) (and let \( \mathcal{B} = \mathcal{B}_{\bar{T}}, \bar{\varphi} = \bar{\varphi}_{\bar{T}} \).
Proof. Read the definitions.

Proposition 3.2. 1. If $(\bar{\varphi}, \mathcal{B})$ is [explicitly] a K-definition of a nep-forcing notion and $\mathcal{B}$ is definable in $\mathcal{B}'$ (and $\Delta$ is $L_{\omega, \omega}$, or change $\Delta$ accordingly to the interpretation), then $(\bar{\varphi}, \mathcal{B}')$ is [explicitly] a K-definition of a nep-forcing notion; moreover, if $\mathcal{B}$ is the only parameter of the $\varphi_\ell$, we can replace it by $\mathcal{B}'$ (changing trivially the $\varphi_\ell$’s).

2. Similarly we can increase $\kappa$ and $\theta$ and add “simply” (to the assumption and the conclusion); we may also add “very simply”.

Proof. Straight.

A converse to 3.1(1)+(2) is

Proposition 3.3. 1. Assume that $\kappa' = \kappa + \theta + \aleph_1 + \|\mathcal{B}\|$ and

\[
(\oplus) \quad (\bar{\varphi}, \mathcal{B}) \text{ is a correct very simple [explicit] K-} (\kappa, \theta) \text{-definition of a nep forcing notion } Q.
\]

Then some $T$ is an [explicitly] K-$(\kappa', \theta)$-definition of a nep forcing notion $Q$ (the same $Q$).

2. If $\kappa = \theta = \aleph_0$ we get a similar result with the $\varphi_\ell$ being $\Pi^1_1$-sets.

3. If in clause $(\oplus)$ of 3.3(1) we replace very simple by simple (so we weaken $Q \subseteq \kappa \theta$ to $Q \subseteq H_{\aleph_1}(\theta)$), then part (1) still holds for some $Q'$ isomorphic to $Q$.

Proof. 1) This is, by now, totally straight; still we present the case of $\varphi_0$ for part (1) for completeness. If in Definition 3.1(2), clause (e) we use $\prec$, let $\bar{\psi}_n^1(y, x_0, \ldots, x_{n-1}) : n < \omega$ list the first order formulas in the vocabulary of $\mathcal{B}$ in the variables $\{y, x_\ell : \ell < \omega\}$, (so in $\psi_n^1$ no $x_\ell$, $\ell \geq n$ appears, but some $x_\ell$, $\ell < n$ may not appear); if we use $\preceq_\Delta$ let it list subformulas of members of $\Delta$. Similarly $\bar{\psi}_n^2(y, x_0, \ldots, x_{n-1}) : 4 \leq n < \omega$ for the vocabulary of set theory. Let us define $T_0$ by defining a set of $\omega$-sequences $Y_0$, and then we will let $T_0 = \{\rho \upharpoonright n : \rho \in Y_0 \text{ and } n < \omega\}$. For $\alpha < \omega_1$ let $\{\beta_{n, \ell} : \ell < \omega\}$ list $\{\beta : \beta \leq \alpha\}$.

Now let $Y_0$ be the set of $\omega$-sequences $\rho \in (\theta \times \kappa')$ such that for some $(\mathcal{B}, \bar{\varphi}, \theta)$-candidate $N \subseteq (\mathcal{H}(\chi), \in)$ (so $\mathcal{B}, \theta, \kappa$ belong to $N$) and some list $\langle a_n : n < \omega \rangle$ of the member of $N$ we have: $\rho = \nu \star \eta$; i.e. $\rho(n) = (\nu(n), \eta(n))$ and

- (i) $a_0 = \mathcal{B}$, $a_1 = \theta$, $a_2 = \kappa$, $a_3 = \nu$,
- (ii) $\{n : N \models a_n \in \kappa'\} = \{\eta(8n + 1) : 0 < n < \omega\}$,
- (iii) every $\eta(8n + 2)$ is a countable ordinal such that:

\[N \models \text{ "rk}(a_n) < \text{rk}(a_m)" \text{ iff } \eta(8n + 2) < \eta(8m + 2) < \kappa_1 \leq \kappa',\]

- (iv) if $\mathcal{B} \models (\exists y)\psi_n^1(y, a_0, \ldots, a_{n-1})$ then $\mathcal{B} \models \psi_n^1[\eta(8n+1+3), a_0, \ldots, a_{n-1}]$,.
(v) \( N \models \varphi_0[\nu] \); i.e. \( N \models \varphi_0[a_3] \).

(vi) \( N \models "a_\ell \in a_m" \) \iff \( \eta(8({\ell + m + 1 \over 2}) + \ell) + 4 = 0 \),

(vii) if \( n \geq 4 \) and \( N \models (\exists y)\psi^2_n(y, a_0, \ldots, a_{n-1}) \)

then \( N \models \psi^2_n[a_q(8n+5), a_0, \ldots, a_{n-1}] \) and \( \eta(8n + 6) = 1 \),

(viii) if \( N \models "a_n \) is a countable ordinal” and \( a_k = \beta_{a_n, \ell} \)

then \( \eta(8(\ell + n + 1) + \ell) + 7) = k \).

Let \( T_0 = \{ \rho \mid n : \rho \in Y_0, n < \omega \} \).

**Claim 3.3.1.** 1. \( Y_0 \) is a closed subset of \( ^\omega(\theta \times \kappa) \).

2. \( \mathbb{Q} = \{ \nu \in \omega : (\exists \eta)(\eta \in \omega(\kappa') \& \nu \ast \eta \in Y_0(= \text{lim}(T_0))) \} = \text{proj}_0(T_0) \).

**Proof of the claim:** 1) Given \( \nu \ast \eta \in \text{lim}(T_0) \) we can define a model \( N' \) with set of elements say \( \{ a'_n : n < \omega \} \) by clause (vi), it is a model of \( \text{ZFC}^- \) by clause (vii) (and the demand \( N \models \text{ZFC}^- \)), it is well founded by clause (iii) (and the earlier information).

We start to define an embedding \( h \) of \( N' \) into \( \mathcal{H}(\chi) \) and we put \( h(a'_0) = \mathfrak{B}, h(a'_1) = \theta, h(a'_2) = \kappa \) and \( h(a'_n) = \eta(8n + 1) \) if \( N' \models a'_n \in a'_2, n > 0 \). Then let \( h(a'_3) \in \omega \theta \) be such that \( h(a'_3)(\ell) = \gamma \) if letting \( n \) be such that \( \psi^2_n \equiv [y = x_3(\ell)] \), so necessarily \( N' \models "a'_3(\ell) = a'_n(8n+5)" \), we have \( \eta(8(n+5) + 1) = \gamma \) (see clause (vii)).

Lastly we define \( h(a''_n) \) for the other \( a''_n \) by induction of \( \text{rk}^{N'}(a''_n) \), note that we can give then dummy elements to relation \( \text{Rang}(h) \cap \kappa = \{ \eta(8(n+1)+1) : n < \omega \} \).

The model \( h[N'] \) above should be built in such a way that it is ord–transitive. This (and clause (viii)) will ensure that the clause (g) of the demand \( \mathbb{B} \) (2) is satisfied.

Note that, actually, the coding (of candidates) which we use above does not change when passing to the ord–collapse.

2) Should be clear from the above noting: \( p \in \mathbb{Q} \) iff for some \( N \) as above, \( N \models \varphi_0(p) \) [as \( \models \) holds by the definition and \( \Rightarrow \) holds as there are countable \( N \prec (\mathcal{H}(\chi), \in) \) to which \( p, \mathfrak{B}, \theta, \kappa \) belong].

This finishes the proof of the claim and so the first part of the proposition.

(2), (3) \ Easy.

What if in \( \mathbb{B} \) we omit “the only parameters of \( \overline{\varphi} \) are \( \mathfrak{B}, \theta, \kappa \)”, so what do we do? Well, the role of \( \mathfrak{B} \) is assumed by the transitive closure of \( \langle \overline{\varphi}, \mathfrak{B}, \theta, \kappa \rangle \), which we can then map onto some \( \kappa^* \geq \kappa \).

**Proposition 3.4.** 1. Assume \( \text{ZFC}^- \) is \( \theta \)-normal for \( (\mathfrak{B}, \overline{\varphi}, \theta) \), and, in \( V \), \( \overline{\varphi} \) is a \( (\mathfrak{B}, \theta) \)-definition of an [explicit] nep forcing notion. Then we get “correctly”.
2. Assume $\bar{\varphi}$ is a $K-(\mathfrak{B}, \theta)$–definition of a nep-forcing notion $Q$ (the “nep” part is not really needed). Let $V'$ be a transitive class of $V$ such that

(i) $\bar{\varphi}$ and $\mathfrak{B}$ belong to $V'$ (and of course $\mathcal{C}$),
(ii) the family of $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidates is unbounded in $V'$, moreover
(iii) for $\chi$ large enough, in $V$ (or just in $V'$) the set
\[ \{ N \subseteq (\mathcal{H}(\chi), \in) : N \text{ is a } (\mathfrak{B}, \bar{\varphi}, \theta)-\text{candidate} \} \]

is stationary, or at least
\[
\text{(iii') } V' \models "\varphi(\bar{x})" \text{ implies that for unboundedly many } (\mathfrak{B}, \bar{\varphi}, \theta)-\text{candidates } N \text{ in } V, \text{ or just in } V', \text{ in } V \text{ or in } V' \text{ the set } N | = "\varphi_0(p)" \text{ then } V | = "\varphi_0(p)", \\
\text{if } V' \models "p \leq Q q" \text{ then } V | = "p \leq Q q", \\
\text{if in } V', N \text{ is a } (\mathfrak{B}, \bar{\varphi}, \theta)-\text{candidate then also in } V, N \text{ is a } (\mathfrak{B}, \bar{\varphi}, \theta)-\text{candidate.}
\]

3. If in (2) we add “explicitly” then
\[
\text{(d) } \text{if } V' \models \varphi_2((p_i : i \leq \omega)) \text{ then } V \models \varphi_2((p_i : i \leq \omega)), \\
\text{(e) } \text{if } N \text{ is a } (\mathfrak{B}, \bar{\varphi})-\text{candidate and } q \text{ is explicitly } (N, Q)\text{-generic then this holds in } V.
\]

4. If in (2) we add “$\bar{\varphi}$ is a temporary explicit correct $(\mathfrak{B}, \theta)$–definition of a nep forcing notion” (in $V$) then also in $V'$, $\bar{\varphi}$ is a temporary explicit correct $(\mathfrak{B}, \theta)$–definition of a nep-forcing notion, $\text{(3.4) } \kappa = \theta = \aleph_0$ or $\kappa = \aleph_0$ and $[\theta]^{\leq \aleph_0}V$ is cofinal in $([\theta]^{\leq \aleph_0}V_1$, or (there are large enough cardinals to guarantee) any co-$(\kappa+\theta+N_1)$–Souslin relation in $V'$ is upward absolute to $V_1$.

5. If in (2) we add (3.4) below and we add “local” to the assumption, then also in $V'$, $\bar{\varphi}$ is a temporary explicit $(\mathfrak{B}, \theta)$–definition of a local nep-forcing notion, where $\text{(3.4) } ([\kappa \cup \theta]^{\leq \aleph_0})V'$ is cofinal in $(([\kappa \cup \theta]^{\leq \aleph_0}V, \subseteq)^V$.

**Proof**

1) Straight.

2) There are two implications implicit in (3.4) concerning the versions of clause (iii). Let
\[ S_\chi \overset{\text{def}}{=} \{ N : N \in V, N \text{ is a countable submodel of } (\mathcal{H}(\chi), \in)^V \text{ and } N \text{ is a } (\mathfrak{B}, \bar{\varphi}, \theta)-\text{candidate} \} \]
and let
\[ S'_\chi \overset{\text{def}}{=} \{ N : N \in V', N \text{ is a countable submodel of } (\mathcal{H}(\chi), \in)^V \text{ and } N \text{ is a } (\mathfrak{B}, \bar{\psi}, \theta)\text{-candidate } \}. \]

Let \( <_\chi \in V \) be a well ordering of \( \mathcal{H}(\chi) \) and let
\[ C \overset{\text{def}}{=} \{ N : N \in V, N \text{ is a countable elementary submodel of } (\mathcal{H}(\chi)^V, \in, V \cap \mathcal{H}(\chi), <^*_\chi) \text{ to which } (\mathfrak{B}, \bar{\psi}, \theta) \text{ belongs } \}. \]

If clause (iii) for \( V \) then clause (iii) for \( V' \). Why? Just observe that
\((\star)_1\) in \( V \): \( C \) is a club of \( [\mathcal{H}(\chi)]^{\leq \aleph_0} \) and \( \{ N \cap V' : N \in C \} \) is a club of \( [\mathcal{H}(\chi)^V]^{\leq \aleph_0} \).

Now suppose that, in \( V' \), \( C' \) is a club of \( \mathcal{H}(\chi)^V \) and we should prove \( C' \cap S'_\chi \neq \emptyset \) (say for some model \( \mathfrak{B} \in V' \) with countable vocabulary, the universe \( \mathcal{H}(\chi)^V \) and Skolem functions, \( C' = \{ N : N \prec \mathfrak{B} \text{ countable } \} \)).

As \( S'_\chi \) is stationary in \( V \), also \( C_1 = \{ N \in C : N \cap V' \in C' \} \) is club of \( [\mathcal{H}(\chi)^V]^{\leq \aleph_0} \) in \( V \). Hence there is \( N \in S_\chi \cap C_1 \). Now, \( N \cap V' \) is almost a member of \( C' \cap S'_\chi \), it satisfies the requirements in the definitions of \( C' \) and \( S'_\chi \). But \( N \cap V' \) is a countable subset of \( \mathcal{H}(\chi)^V \), so by Shoenfield–Levy absoluteness it exists.

If clause (iii) for \( V' \) then clause (iii) for \( V \). Work in \( V' \). So let \( x \in Q \) and let \( \chi \) be large enough such that \( (\mathcal{H}(\chi), \in) \prec_{\Sigma_n} V' \) for \( n \) large enough. The set
\[ C^* \overset{\text{def}}{=} \{ N : N \in V, N \text{ is a countable elementary submodel of } \mathcal{H}(\chi) \text{ to which } x, \mathfrak{B}, \bar{\psi}, \theta, \mathfrak{C} \text{ belong } \}. \]
is a club of \( [\mathcal{H}]^{\leq \aleph_0} \), hence has non-empty intersection with any stationary subset of \( [\mathcal{H}]^{\leq \aleph_0} \). In particular, by the assumption, there is a \( (\mathfrak{B}, \bar{\psi}, \theta)\text{-candidate } N \in C^* \). So \( N < (\mathcal{H}(\chi), \in), x, \mathfrak{B}, \bar{\psi}, \theta, \mathfrak{C} \in N \). So
\[ V' \models \varphi_0(x) \Rightarrow (\mathcal{H}(\chi), \in) \models \varphi_0(x) \Rightarrow N \models \varphi_0(x) \Rightarrow x \in Q^N. \]

3) Straight.

4) Suppose that
\[ V' \models " N \text{ is a } (\bar{\psi}, \mathfrak{B})\text{-candidate and } p \in Q^N " . \]

In \( V' \), let \( \langle I_n : n < \omega \rangle \) list the \( I \) such that \( N \models \langle I \rangle \) is a predense subset of \( Q^n \). We know (by \[\text{[1.4]}(2)(c)\]) that \( N \) is a candidate in \( V_1 \). Hence, in \( V_1 \), there are \( q, \langle p^\ell_n : \ell < \omega, n < \omega \rangle \) such that:
(i) \( \langle p^\ell_n : \ell < \omega \rangle \) lists \( I_n \cap N \),
(ii) \( p \leq^Q q \in Q \),
(iii) \( \varphi_2(\langle p^\ell_n : \ell < \omega \rangle \setminus q) \) for each \( n < \omega \).
So there is a \((\mathcal{B}, \varphi, \theta)\)-candidate \(N_1\) such that \(N \in N_1, (p^n_\ell : \ell < \omega) : n < \omega\), \(q\) and \(\langle I_n : n < \omega \rangle\) belong to \(N_1\), and \(N_1 \models " p \subseteq Q q \)”, and \(N_1 \models \varphi_2((p^n_\ell : \ell < \omega) \Rightarrow q)\) for \(n < \omega\) (by “correct”). It is enough to find such \(N_1 \in V'\), which follows from 2.13.

(We use an amount of downward absoluteness which holds as \(V'\) is a transitive class including enough ordinals).

5) Similar proof.

**Proposition 3.5.** 1. Assume \(\bar{T}\) is an explicit temporary \((\kappa, \theta)\)-definition of a snep-forcing notion \(Q\). For any extension \(V_1\) of \(V\), this still holds if (*) of 3.4(4) above holds. So we can replace “temporary” by \(K =\) class of all set forcing notions.

2. Assume \((\bar{\phi}, \mathcal{B})\) is a simple explicit temporary \((\kappa, \theta)\)-definition of a nep-forcing notion \(Q\). For any extension \(V_1\) of \(V\) this still holds in \(V_1\) if (*) of 3.4(4) holds. So we can add/replace “temporary” by the class \(K\) of all forcing notions preserving “\((\theta \leq \aleph_0) V\) is cofinal in \((\theta \leq \aleph_0, \subseteq) V_1\)”.

3. Assume \((\bar{\phi}, \mathcal{B})\) is a local explicit temporary \((\kappa, \theta)\)-definition of a nep forcing notion \(Q\). Then for any extension \(V_1\) of \(V\) this still holds, provided that:

\((*)_4 \quad ([\kappa \cup \theta] \leq \aleph_0) V\) is cofinal in \(([\kappa \cup \theta] \leq \aleph_0, \subseteq) V_1\).

**Proof** 1), 2) Left to the reader (and similar to the proof of part (3)).

3) Let \(\theta_1 = \kappa + \theta\), and let \(a \in [\theta_1]^{\aleph_0}\), and consider the statement

\(\exists_\alpha \text{ if } N \text{ is a } (\mathcal{B}, \varphi, \theta)-\text{candidate satisfying } N \cap \theta_1 \subseteq a \text{ and } p \in Q^N \) (i.e. \(N \models \varphi_0[p]\)),

then there are \(N'\), a generic extension of \(N\) (so have the same ordinals and \(N\) is a class of \(N'\)) which is a \((\mathcal{B}, \varphi, \theta)-\text{candidate such that } N' \models " P(\theta)^N \text{ is countable} " \) and

\(N' \models " (\exists q) [q \in Q \& q \text{ is explicitly } (N \cap P(Q), Q)-\text{generic} ] \) ”.

Note: for \([x \in N' \& N' \models "x \text{ is countable} \) \(\Rightarrow x \subseteq N]\) just use a suitable collapse.

Now, only \((N', c)_{c \in N}/ \cong \) and \((N, \alpha)_{\alpha \in a}/ \cong \) and \((N', N, \alpha)_{\alpha \in a}\) are important and we can code \(N\) as a subset of \(a\) (as all three are countable). Thus the statement is essentially

\((\forall N)[(N \text{ is not well founded or } \not\exists \mathcal{B} | (N \cap a) \prec_{\Delta} \mathcal{B}, \text{ etc.}) \lor \nu(\exists N')(N' \text{ as above})]\).

So it is \(\Pi^1_2\), hence it is absolute from \(V\) to \(V_1\). Now, both in \(V'\) and in \(V_1\) the statement “\(Q\) is simply, locally, explicitly nep” is equivalent to
\((\forall a \in [\theta_1]^{\aleph_0})\exists a\), which is equivalent to \(S = \{a \in [\theta_1]^{\aleph_0} : \exists a\}\) is cofinal in \([\theta_1]^{\aleph_0}\). But by the previous paragraph \(S[V] \subseteq S[V_1]\). Now \((*)_4\) gives the needed implication. \(\Box\)

**Proposition 3.6.** 1. Assume \(\bar{T}\) is a temporarily \((\kappa, \theta)\)-definition of a snep-forcing notion \(Q\). If \((*)\) below holds, then we can find a tree \(T_2 \subseteq \omega^\omega(\theta \times \theta \times \kappa')\) such that \(\bar{T}^{-1}(T_2)\) is an explicit temporary \((\kappa, \theta)\)-definition of a snep-forcing notion \(Q\), where 
\((*)\) \(\kappa = \theta = \aleph_0, \kappa' = \aleph_1\) or enough absoluteness. 
2. If \(Q\) (i.e. \((\varphi_0, \varphi_1)\)) is a Souslin proper forcing notion (see \(2.10\)) and \(B\) codes the parameter (so has universe \(\kappa = \aleph_0\) and let \(\theta = \aleph_0\)), then \((\bar{\varphi}, B)\) is a simple explicit temporary \((\kappa, \theta)\)-definition of the nep-forcing notion \(Q\).

**Proof** 1) The question is to express \(\{p_n : n < \omega\}\) is predense above \(q\) which is equivalent to 
\[(\forall \nu \in \omega^\omega)(\varphi_0(\nu) \land \varphi_1(q, \nu) \Rightarrow (\exists \nu' \in \omega^\omega)(\bigvee_n \varphi_1(p_n, \nu') \land \varphi_1(\nu, \nu'))].\]

So, as \(\kappa = \theta = \aleph_0\), this is a \(\Pi_2^1\)-formula and hence it is \(\aleph_1\)-Souslin.
2) Similarly (for \(\varphi_2\) being upward absolute note that the relation is now \(\Pi_1^1\) and \(\Pi_1^1\) formulas are upward absolute). \(\Box\)

**Definition 3.7.** Assume that \((\bar{\varphi}, B)\) is a temporary \((\kappa, \theta)\)-definition of a nep forcing notion \(Q\), and \(N\) is a \(Q\)-candidate. We say that a condition \(q' \in Q\) is essentially explicitly \((N, Q)\)-generic if for some candidate \(N'\), \(N \subseteq N', N \in N', \) \(q'\) is explicitly \((N', Q)\)-generic and for some \(q_0 \in Q^{N'}\), 
\(q_0 \leq Q q'\) and \(N' \models q_0\) is \((N, Q)\)-generic”.

Note: if \(Q\) is a snep-forcing for \(\bar{T}\), this relation is \((\kappa + \theta + \aleph_1)\)-Souslin, too.

**Proposition 3.8.** Assume \(Q\) is a correct explicitly nep-forcing notion, say by \((\bar{\varphi}, B)\). If \(q\) is \((N, Q)\)-generic, then for some \(q'\) we have 
\(q \leq q' \in Q\) and \(q'\) is essentially explicitly \((N, Q)\)-generic.

**Proof** Let \(\varphi_2((p^T_n : n < \omega), q)\) hold for some list \((p^T_n : n < \omega)\) of \(I \in \text{pd}(N, Q)\). For \(I \in \text{pd}(N, Q)\) let \(N_I\) be a \(Q\)-candidate such that \(N_I \models \varphi_2((p^T_n : n < \omega), q)\).

Let \(N' \subseteq H(\chi)\) be a countable \(Q\)-candidate satisfying
\[\{N, q\} \cup \{N_I : I \in \text{pd}(N, Q)\} \in N'.\]
By our assumptions there is $q'$ such that: $q \leq q' \in \mathbb{Q}$ and $q'$ is explicitly $(N', \mathbb{Q})$-generic.

**Proposition 3.9.** 1. If $N$ is a $\mathcal{B}$-candidate, so in particular

$$[N \models \left. \alpha < \kappa \lor \alpha < \theta \right] \Rightarrow \alpha \in \kappa \lor \alpha \in \theta,$$

and $|\mathcal{B}|$ is an ordinal, then there is a unique $N' = \text{MosCol}_{\mathcal{B}}(N)$ and $f$ such that

(a) $f$ is an isomorphism from $N$ onto $N'$,
(b) $f \upharpoonright (N \cap \kappa) = \text{id}$, $f(\kappa) = \kappa$ and $f \upharpoonright (N \cap \theta) = \text{id}$, $f(\theta) = \theta$,
(c) if $x \in N \setminus (\kappa + 1) \setminus (\theta + 1)$ then $f(x) = \{f(y) : N \models \left. y \in x \right\}$,
(d) $N'$ is a $\mathcal{B}$-candidate.

2. Note that if $N \models \left. x \in H^{\kappa, \theta} \right|$ then $f(x) = x$.

3. If $|\mathcal{B}|$ is not an ordinal (so $\kappa \subseteq |\mathcal{B}| \subseteq H^{\kappa, \theta}$), then $N'$ is still a $(\mathcal{B}, \bar{\varphi}, \theta)$-candidate, using the “but” of clause (e) of Definition 2.1(2).

**Fact 3.10.** In the definition of nep (or snep) in the “properness” clause, it is enough to restrict ourselves to a family $I$ of predense subsets of $\mathbb{Q}^N$ such that:

if $I \in \text{pd}(N, \mathbb{Q})$

then for some $J \in I$ we have ($\forall p \in I \cap N)(\exists q \in J)(N \models p \leq \mathbb{Q} q)$.

**Proposition 3.11.** 1. Assume $\bar{T}$ defines an explicit $(\kappa, \theta)$-snep forcing notion. Let $\bar{\varphi} = \varphi_T$, $\mathcal{B} = \mathcal{B}_{\bar{T}}$ (see 3.1(3)). If $\mathbb{Q}_{\bar{T}}$ is local then $\mathbb{Q}_{\bar{\varphi}}$ is local, in fact in Definition 2.8(2).

2. If ($\text{ZFC}^*_{\mathcal{B}}$ is $K$-good and) $\text{ZFC}^*$ says that $(\mathcal{B}, \bar{\varphi}, \theta)$ is explicitly nep, and $\bar{\varphi}$ is correct then $(\mathcal{B}_{\bar{\varphi}}, \bar{\varphi}, \theta)$ is explicitly nep and local.

Moving from nep to snep (and inversely) we may ask what occurs to “local”. It is usually preserved.

3. **There are examples.** In this section we show that a large family of natural forcing notions satisfies our definition. Later we will deal with preservation theorems but to get nicer results we better “doctor” the forcing notions, but this is delayed to the next section.

In fact all the theorems of Roslanowski Shelah [14], which were designed to prove properness, actually give one notion or another from §1 here (confirming the thesis 1.3 of §0). We will state them without giving the definitions from [14] and give a proof of (hopefully) well known specific cases, indicating why it works.
Lemma 4.1 (Roslanowski Shelah [14]).

1. Suppose that \( Q \) is a forcing notion of one of the following types:
   
   (a) \( Q_{\text{tree}}^e(K, \Sigma) \) for some finitary tree-creating pair \((K, \Sigma)\), where \( e = 1 \) and \((K, \Sigma)\) is 2-big or \( e = 0 \) and \((K, \Sigma)\) is t-omittory (see [14, §2.3]; so e.g. this covers the Sacks forcing notion),
   
   (b) \( Q_{\text{*}}^\infty(K, \Sigma) \) for some finitary creating pair \((K, \Sigma)\) which is growing, condensed and of the AB–type or omittory, or the \( AB^{+} \)–type and satisfies \( \oplus_0, \oplus_3 \) of [14, 4.3.8] (see [14, §3.4]; this captures the Blass–Shelah forcing notion of [4]),
   
   (c) \( Q_{\text{*}}^w\infty(K, \Sigma) \) for some finitary creating pair which captures singletons (see [14, §2.1])
   
   (d) \( Q_{\text{f}}^\ast(K, \Sigma) \) for some finitary, 2-big creating pair \((K, \Sigma)\) with the Halving Property which is either simple or gluing and an \( H \)-fast function \( f : \omega \times \omega \rightarrow \omega \) (see [14, §2.2]).

Then \( Q \) is an explicit \( \aleph_0 \)-snep forcing notion, moreover, it is local.

2. Assume that \( Q \) is a forcing notion of one of the following types:
   
   (a) \( Q_{\text{tree}}^e(K, \Sigma) \) for \( e < 3 \) and a tree-creating pair \((K, \Sigma)\), which is bounded if \( e = 2 \) (see [14, §2.3]; this includes the Laver forcing notion),
   
   (b) \( Q_{\text{*}}^\infty(K, \Sigma) \) for a finitary growing creating pair \((K, \Sigma)\) (see [14, §2.1]; this covers the Mathias forcing notion).

Then \( Q \) is an explicit \( \aleph_0 \)-nep forcing notion, moreover, it is local.

Proof

Let \( N \) be a \( Q \)-candidate and \( p \in \dot{Q}^N \). Let \( \langle J_n : n < \omega \rangle \) list \( \{ J : N \models \text{\"} J \subseteq Q \text{ is open dense\"} \} \). Then there is a sequence \( \langle p_n, I_n : n < \omega \rangle \) such that \( p_n, I_n \in N \), \( N \models p_n \leq p_{n+1} \), \( I_n \subseteq J_n \) is a countable set, \( \langle p_n : n < \omega \rangle \) has an upper bound in \( Q \) and \( I_n \) is predense above \( p_{n+1} \), moreover, in an explicit way as described below (see the respective subsections in [14]). Moreover,

in part (1) cases (a)+(c), \( I_n \) is finite and moreover, we can say “\( I_n \) is predense above \( p_{n+1} \)” in a Borel way.

For the Sacks forcing notion: for some \( k < \omega \), \( I_n = \{ p_{n+1}^{[\eta]} : \eta \in p_{n+1}, \ell g(\eta) = k \} \), so \( I_n \) corresponds to a front of \( p_{n+1} \), which necessarily is finite. This property serves as \( \varphi_2 \) (compare with more detailed description for the Laver forcing below).

In part (1) case (b) (e.g. the Blass–Shelah forcing notion) \( I_n \) is countable. We do not know which level will be activated, but if use \( n \), then we get into \( I_n \), so \( I_n \) countable but the property is Borel not \( \Pi^1_1 \).

Now, in part (2), \( I_n \) is countable and again it corresponds to some front \( A \) of \( p_{n+1} \) in an appropriate sense. So \( I_n = \{ p_{n+1}^{[\eta]} : \eta \in A \} \), but to say “\( A \) is
a front" is $\Pi_1$ (in some instances of 2(a) we have $e$-thick antichains instead of fronts, but the complexity is the same).

Recall that for a subtree $T \subseteq \omega^\omega$, $A \subseteq T$ is a front of $T$ if
\[(\forall \eta \in \text{lim}(T))(\exists n)(\eta \upharpoonright n \in A)\]
(usually members of $A$ are pairwise incomparable).

Specifically, for the Laver forcing notion, we can guarantee $I_n = \{p_{[\eta]} : \eta \in A\}$, where $A$ is a front of $p_{n+1}$. Now being a front is a $\Pi_1$–sentence (see the definition above) which is upward absolute and this is our choice for $\varphi_2$.

Let us write this formula in a more explicit way (for the case of the Laver forcing notion):
\[
\varphi_2(\langle p_i : i \leq \omega \rangle) \equiv \text{each } p_i \text{ is a Laver condition and } \\
\bigwedge_{i \in \omega} (\exists! \eta)(\eta \in p_\omega & p_{2i} = p_{[\eta]}) \\
[\text{call this unique } \eta \text{ by } \eta_i] \text{ and } \\
\bigwedge_{i \neq j} (\eta_i \not\equiv \eta_j \text{ (incomparable)} \& (\forall \rho \in \text{lim}(p_\omega))(\bigvee_{n \in \rho} n = \eta_m) \\
[\text{this is: } \{p_i : i \in \omega\} \text{ is explicitly predense above } p_\omega].
\]
So it is $\Pi_1$ (of course, $\Sigma_2$ is okay, too.)

Note that even for the Sacks forcing notion, “$p, q$ are incompatible” is complete $\Pi_1$. So “$\{p_n : n \in \omega\}$ is predense above $p$” will be $\Pi_2$. For Laver forcing we cannot do better. Now, generally $\Pi_2$ is not upward absolute from countable submodels, whereas $\Pi_1$ is.

**Proposition 4.2.** All the forcing notions $Q$ defined in [14], [15], are correct, and we can use $\text{ZFC}_* = \text{ZC}^*$ which is good and normal (see 2.7). Also the relation “$p, q$ are incompatible members of $Q$” is upward absolute from $Q$–candidates (as well as $p \in Q$, $p \not\in Q$, $p \leq q$, and “$p, q$ are compatible”).

**Proof** Check.

**4. Preservation under iteration: first round.** We give here one variant of the preservation theorem, but for it we need some preliminary clarification. We have said “there is $q$ which is $(N, Q)$-generic”; i.e. $q \Vdash_{Q} G_Q \cap Q^N$ is a generic subset of $Q^N$ over $N$”. Note that we have said $Q^N$ and not $Q \cap N$ as we intended to demand $N \models “r \in Q” \Rightarrow V \models “r \in Q”$ rather than $r \in N \Rightarrow [N \models “r \in Q” \iff V \models “r \in Q”]$ (the version we use is, of course, weaker and so better). Now, to use Definition 2.3(1) we usually use $N[G_Q]$ (e.g. when iterating).
But what is $N[G]$ here? In fact, what is the connection between $N \models “\tau$ is a $\mathbb{Q}$–name” and $V \models “\tau$ is a $\mathbb{Q}$–name”? Because $[x \in Y \in N \Rightarrow x \in N]$, none of the implications holds.

For our purpose, the usual $N[G] = \{ \tau[G] : \tau \in N \text{ is a } \mathbb{Q}\text{-name}\}$ is not appropriate as it is not clear where being a $\mathbb{Q}$-name is defined. We use $N\langle G \rangle$ which is $N[G \cap \mathbb{Q}^N]$ when we disregard objects in $V \setminus N$. Of course, if the models are $\subseteq \mathcal{H}_\kappa$ life is easier; but we may lose $N \models \text{ZFC}^-$.

We then prove (in 5.5) the first version of preservation by CS iteration. We aim at proving only that $\mathbb{P}_\alpha = \text{Lim}(\bar{\mathbb{Q}})$ satisfies the main clause, i.e. clause (c) of Definition 2.3 (but did not say that $\mathbb{P}_\alpha$ is nep itself). For this we need again to define what is $N\langle G \rangle$ for $N$ which is not necessarily a candidate. The second treatment (in 5.5) depends just on Definition 5.1 from this section.

**Definition 5.1.** 1. Assume $N \models “\mathbb{Q}$ is a nep-forcing notion” and $G \subseteq \mathbb{Q}^N$ is generic over $N$. We define $N\langle G \rangle = N\langle G \cap \mathbb{Q}^N \rangle$ “ignoring $V$” and letting $\mathfrak{B}^{N\langle G \rangle} = \mathfrak{B}^N$ for the relevant $\mathfrak{B}$. In details, $N\langle G \rangle \overset{\text{def}}{=} \{ \tau^N\langle G \rangle : N \models “\tau$ is a $\mathbb{Q}$–name”$\}$, where $\tau^N\langle G \rangle$ is defined by induction on $\text{rk}^N(\tau)$ (see e.g. [17, Ch.I]):

(a) if for some $p \in G \cap \mathbb{Q}^N$ and $x \in N$ we have $N \models [p \Vdash_{\mathbb{Q}} “\tau = x”]$ then $\tau^N\langle G \rangle = x$,

(b) if not (a) then necessarily $N \models “\tau$ has the form $\{(p_i, \tau_i) : i < i^*\}$, $p_i \in \mathbb{Q}$, $\tau_i$ a $\mathbb{Q}$–name of rank $< \text{rk}(\tau)$; now we let $\tau^N\langle G \rangle = \{(\tau^*)^N\langle G \rangle : \tau^* \in N$ and for some $p \in G \cap \mathbb{Q}^N$ we have $(p, \tau^*) \in \tau\}$.

2. If $N \models “\tau$ is a $\mathbb{Q}$–name” we define a $\mathbb{Q}$–name $\tau^{(N)}$ as follows:

(a) if $N \models “\tau = \check{x}”$, $x \in N$, we let $\tau = \check{x}$ (see e.g. [17, Ch.I]),

(b) if $N \models “\tau = \{(p_i, \tau_i) : i < i^*\}$, where $p_i \in \mathbb{Q}$, $\tau_i$ a $\mathbb{Q}$–name of rank $< \text{rk}(\tau)$” then $\tau^{(N)} = \{(p, (\tau^*)^{(N)}) : N \models “(p, \tau^*) \in \tau\}$.

3. We say “$q$ is $(N, \mathbb{Q})$–generic” if $q \Vdash_{\mathbb{Q}} “\mathbb{G}_\mathbb{Q} \cap \mathbb{Q}^N$ is a subset of $(\mathbb{Q}^N, <_\mathbb{Q})$ generic over $N$”.

**Definition 5.2.** 1. In Definition 2.3(1) replacing “temporarily” by “$K$–absolutely” means

(a) if $V_1$ is a $K$–extension of $V$ (i.e. a generic extension of $V$ by a forcing notion from $K\langle V \rangle$) then

(i) $V \models “x \in \mathbb{Q}_2^\omega$ => $V_1 \models “x \in \mathbb{Q}_2^\omega$,

(ii) $V \models “x <_\mathbb{Q}^\omega y” => $V_1 \models “x <_\mathbb{Q}^\omega y$,
(iii) in the explicit case we have a similar demand for \( \varphi_2 \); otherwise, if \( N \) is a \( Q^\varphi \)-candidate in \( V \), \( q \in Q^\varphi \) is \( \langle N, Q \rangle \)-generic (see \( \ref{5.1}(3) \)) in \( V \) then \( q \) is \( \langle N, Q \rangle \)-generic in \( V_1 \).

(b) if \( V_1 \) is a \( K \)-extension, then the relevant part of Definition 2.3 and clause (a) here holds in \( V_1 \).

(c) if \( V_{\ell+1} \) is a \( K \)-extension of \( V_\ell \) for \( \ell \in \{0, 1, 2\} \), \( V_0 = V \) then \( V_3 \) is a \( K \)-extension of \( V_1 \).

2. We omit \( K \) when we mean: any set forcing.

Note that (a)(i) + (ii) is automatic for explicitly snep, also (a)(iii). One can make “absolutely nep” to the main case.

The following is natural to assume.

**Definition 5.3.**

1. We say \( \text{ZFC}^- \) is nice to \( \chi_1 \) if \( \chi_1 \) is a constant in \( C \), \( \text{ZFC}^- \) says \( \chi_1 \) is strong limit and \( \text{ZFC}^- \) is preserved by forcing by forcing notions of cardinality \( < \chi_1 \).

2. We say \( Q \) is nice (or \( \text{ZFC}^- \) nice to \( Q \)) if for some \( \chi_1 \), \( \text{ZFC}^- \) is nice to \( \chi_1 \) and it says \( Q \in H(\chi_1) \).

**Proposition 5.4.** If \( N \) is a \( Q \)-candidate, \( Q \) is a nep-forcing notion, \( G_Q \subseteq Q \) is generic over \( V \) and \( G_Q \cap N \) is generic over \( N \) then:

(a) \( N \models “T \text{ is a } Q \text{-name}” \) implies \( T^N(\langle G \rangle) = T^{\langle N \rangle}[G] \),

(b) \( N(\langle G \rangle) \) is a model of \( \text{ZFC}^- \) and moreover it is a \( Q \)-candidate and is a forcing extension of \( N \), provided that the forcing theorem applies, i.e. \( \text{ZFC}^- \) is \( K \)-good, \( Q \in K \) (see Definition 2.11),

(c) \( N(\langle G \rangle) \cap \kappa = N \cap \kappa, N(\langle G \rangle) \cap \theta = N \cap \theta \).

**Remark:** It seems that usually (but not in general) we have:

\[
(\mathcal{H}_{<\aleph_1}(\kappa))^V[G] \cap N(\langle G \rangle) = \mathcal{H}_{<\aleph_1}(\kappa)^V \cap N \quad \text{and} \quad (\mathcal{H}_{<\aleph_1}(\theta))^V[G] \cap N(\langle G \rangle) = (\mathcal{H}_{<\aleph_1}(\theta))^V \cap N.
\]

**Proposition 5.5.** Assume

(a) \( \mathcal{Q} = \{ \mathbb{P}_i, \mathbb{Q}_j : i \leq \alpha, j < \alpha \} \) is a CS iteration,

(b) for each \( i < \alpha \)

\( \mathbb{P}_i (“\mathfrak{P}_i, \mathbb{B}_i”) \) is a temporary \( (\kappa_i, \theta_i) \)-definition of a nep-forcing notion \( \mathbb{Q}_i \),

and the only parameter of \( \mathfrak{P}_i \) is \( \mathbb{B}_i \), so we are demanding \( (\mathfrak{P}_i, \mathbb{B}_i) : i < \alpha \) \( \in V \),

(c) \( \mathbb{B} \) is a model with universe \( \alpha^* \), or including \( \alpha^* \) and included in \( \mathcal{H}_{<\aleph_1}(\alpha^*) \), where \( \alpha^* \geq \alpha \), \( \alpha^* \geq \kappa_i = \kappa(\mathbb{B}_i) \), \( \mathbb{B} \) codes \( (\mathbb{B}_i, \mathfrak{P}_i) : i < \alpha \) and the functions \( \alpha - 1, \alpha + 1 \).
We can use a vocabulary $\subseteq \{P_{n,m} : n, m < \omega\}$ where $P_{n,m}$ is an $n$-place predicate to code $\langle \mathfrak{B}^i : i < \alpha \rangle$: let $P_{n+1,2m}^\mathfrak{B} = \{ \langle i, x_1, \ldots, x_n \rangle : \langle x_1, \ldots, x_n \rangle \in P_{n,m} \}$, $P_{2,1} = \{ (\alpha, \alpha+1): \alpha+1 < \alpha^* \}$ (and $\Delta$ is the set of first order formulas).

Then: if $N \subseteq (\mathcal{H}(\chi), \in)$ is a $\mathfrak{B}$–candidate, $p \in \mathbb{P}_\alpha \cap N$, then for some condition $q$, $p \leq q \in \mathbb{P}_\alpha$ and $q$ is $\langle N, \mathbb{P}_\alpha \rangle$–generic (in particular $\mathbb{P}_\alpha$ is defined from $\mathfrak{B}$) which is defined below.

**Definition 5.6.** Under the assumptions of 5.4 in $N$ we have a definition of the countable support iteration $\bar{Q} = \langle \mathcal{P}_i, \mathcal{Q}_j : i \leq \alpha, j < \alpha \rangle$. We define by induction on $j \in N \cap (\alpha + 1)$ when $q \in \mathbb{P}_j$ is $(N, \mathbb{P}_j)$–generic:

(⊕) if $q \in G_j \subseteq \mathbb{P}_j$ and $G_j$ is generic over $\mathbb{V}$ then $G_j^{(N)}$ is a generic subset of $\mathbb{P}_j^N$ over $N$, where

$$G_j^{(N)} \overset{def}{=} \{ p : N \models \text{"}p \in \mathbb{P}_j\text{"} \text{ and } p^{(N)} \in G_j \},$$

where $p^{(N)}$ is a function with domain $\text{Dom}(p)^N$, and $p(\gamma)$ is the following $\mathbb{Q}_\gamma$–name: if $p(\gamma)^{N(G_j \cap N)} \in \mathbb{Q}_\gamma$, then it is $p(\gamma)$; if not, then it is $\emptyset_{\mathbb{Q}_\gamma}$.

**Remark 5.7.** The major weakness is that $\mathbb{P}_\alpha$ is not proved to be in some of our classes (nep or snep). We get the “original property” without the “support team”, i.e. the $\mathbb{Q}_i$ are nep, but on $\mathbb{P}_\alpha$ we just say it satisfies the main part of nep. A minor one is that $\mathfrak{B}_i$ is not allowed to be a $\mathbb{P}_i$–name in any way. In the later theorems, we use $\mathbb{P}_\alpha'$ consisting of “hereditarily countable” names.

Note: inside $N$, if “$N \models p \in \mathbb{P}_\alpha$” then $\text{Dom}(p_\alpha) \subseteq [\alpha]^{<\aleph_0}$ in $N$’s sense hence (see Definition 2.3(2)), $\text{Dom}(p_\alpha) \subseteq N$ and similarly the names are actually from $N$, members outside $N$ do not count, they may not be in $\mathbb{P}_\alpha$ at all.

**Proof of 5.5** We imitate the proof of the preservation of properness. So we prove by induction on $j \in (\alpha + 1) \cap N$ that:

(*) if $i \in j \cap N$, $q$ is $(N, \mathbb{P}_i)$–generic, and $q \models \text{"}(p \upharpoonright i)^{(N)} \in G_{P_i}\text{"}$ then we can find a condition $r \in \mathbb{P}_j$ such that $r \upharpoonright i = q$, and $r \models \text{"}(p \upharpoonright j)^{(N)} \in G_{P_j}\text{"}$, $r$ is $(N, \mathbb{P}_j)$–generic, and $\text{Dom}(r) \cap i \subseteq N$.

**Case 0:** $j = 0$.

Left to the reader.

**Case 1:** $j = j_1 + 1$.

So $j_1 \in N$ (why? use $P_{2,1}$ and 2.3(2)(e)), and by the inductive hypothesis
and the form of the conclusion without loss of generality \( i = j_1 \). Let \( q \in G_i \subseteq \mathbb{P}_i \), \( G_i \) generic over \( V \). So \( N\langle G_i^{(N)} \rangle \cap \alpha^+ = N \cap \alpha^+ \) (by 5.4), and hence \( B \upharpoonright N\langle G_i^{(N)} \rangle = B \upharpoonright N \prec B \). But \( i \in N \), so this applies to \( \mathcal{B}_i \), too. So \( V[G_i] \models "N(G_i^{(N)})\) is a \( \mathcal{B}_i \)-candidate"). Also \( N\langle G_i^{(N)} \rangle \models "p(i)\langle N(G_i^{(N)}) \rangle \in \mathcal{Q}_i\) because \( G_i^{(N)} \) is a generic subset of \( P_i^N = \{ x : x \in N, N \models "x \in \mathbb{P}_i\)\} over \( N \) and use the property of \( \mathcal{Q}_i \).

**Case 3:** \( j \) is a limit ordinal.
As in the proof for properness (see [17, Ch.III, 3.2]).

**Remark:** Note that if \( N \models "w \) is a subset of \( \alpha \)" then we can deal with \( \mathbb{P}_w \), as in §5.

**5. True preservation theorems.** Let us recall that \( \mathcal{Q} \) is nep if \( "p \in \mathcal{Q}\)”, \( "p \leq q \) are defined by upward absolute formulas for models \( N \) which are \( (\mathcal{B}^\theta, \mathcal{P}^\theta, \theta^Q) \)-candidates; i.e. \( N \subseteq (\mathcal{H}(\chi), \mathcal{E}) \) countable, \( \mathcal{B}^Q \in N \) a model on some \( \kappa \), \( \mathcal{B}^Q \upharpoonright N \prec \Delta \mathcal{B}^Q \), \( N \) model of ZFC\(^- \) and for each such model we have the properness condition. Usually \( \mathcal{Q} \subseteq \omega_\theta \), or \( \mathcal{H} < \kappa \) (\( \theta \)) or so. We would like to prove that CS iteration preserves “being nep”, but CS may give “too large” names of conditions (of \( \mathcal{Q}_i \), \( i > 0 \)) depending say on large maximal antichains (of \( \mathbb{P}_i \)). Note: if \( \mathcal{Q}_0 \) is not c.c.c. normally it has maximal antichain which is not absolutely so; start with a perfect set of pairwise incompatible elements and extend it to a maximal antichain. Then whenever a real is added, the maximality is lost. Finally, c.c.c. is normally lost in \( \mathbb{P}_\omega \). So we will revise our iteration so that we consider only hereditarily countable names.

But in the iteration, trying to prove a case of properness for a candidate \( N \) and \( p \in \mathbb{P}_\alpha^{N+1} \), considering \( q \in \mathbb{P}_\alpha \) which is \( \langle N, \mathbb{P}_\alpha^N \rangle \)-generic, we know that in \( V[G_{\mathbb{P}_\alpha}] \) (if \( q \in G_{\mathbb{P}_\alpha} \)), there is \( q' \in Q_\alpha \langle G_{\mathbb{P}_\alpha} \rangle \) which is \( \langle N[G_{\mathbb{P}_\alpha}], Q_\alpha[G_{\mathbb{P}_\alpha}] \rangle \)-generic. But under present circumstances, we have no idea where to look for \( q' \), so no way to make a name of it, \( q' \), which is hereditarily countable, without increasing \( q \in \mathbb{P}_\alpha \). Except when \( \mathcal{Q} \) is local (see 2.8), of course; it is not unreasonable to assume it but we prefer not to and even then, we just have to look for it in, essentially, a copy of the set of reals. The solution is to increase \( \mathcal{Q}_i \) insubstantially so that we will exactly have the right element \( q' \):

\[
p(\alpha) \& \bigwedge_{I \in \text{pd}(\alpha)} \bigvee_{p \in I \cap N} p,
\]

as explained below. We give two variants.
Notation 6.1. Let $\text{pd}_Q(N) = \text{pd}(N, Q) = \{ I : N \models "I \text{ is a predense subset of } Q" \}$ and $\mathcal{I}[N] = \mathcal{I}^N = \mathcal{I} \cap N$.

Definition 6.2. Let $Q$ be an explicitly nep-forcing notion. Then we define $Q' = \text{cl}(Q)$ as follows:

(a) the set of elements is

\[
\{ \underbrace{p} & \underbrace{\land_{I \in \text{pd}_Q(N)} \lor_{r \in I \cap N} r : \ p \in Q^N \text{ and } N \text{ is a } Q-\text{candidate}}_{\text{we are assuming no incidental identification}} \}
\]

(b) the order $\leq^{Q'}$ is given by $q_1 \leq^{Q'} q_2$ if and only if one of the following occurs:

(α) $q_1, q_2 \in Q$, $q_1 \leq^Q q_2$,

(β) $q_1 \in Q$, $q_2 = p \land \left( \land_{I \in \text{pd}_Q(N)} \lor_{r \in I \cap N} r \right)$ and $q_1 \leq^Q p$,

(γ) $q_1 = p \land \left( \land_{I \in \text{pd}_Q(N)} \lor_{r \in I \cap N} r \right)$ and $q_2 \in Q$, $p \leq^Q q_2$ and if $I \in \text{pd}(N, Q)$ then

\[
(\exists q' \in Q)(\exists(p_n : n \in \omega))(q' \leq^Q q_2 \land \varphi^Q_q(\ldots, p_n, \ldots, q') \land \{p_n : n < \omega \} \text{ lists } I \cap N)
\]

(δ) $q_\ell = p_\ell \land \left( \land_{I \in \text{pd}_Q(N, Q)} \lor_{r \in I \cap N} r \right)$ (for $\ell = 1, 2$) and: $q_1 = q_2$ or $q_1 \leq p_2$ by clause (γ).

Remark: In [17], for a hereditarily countable name, instead of

\[
p \land \bigvee_{I \in \text{pd}_Q(N)} r \land_{r \in I \cap N}
\]

we use the first member of $Q_i$, which forces this. Simpler, but when we ask whether this guy is $\leq q$ (for some $q \in Q$) we run into uncountable antichains.

Proposition 6.3. 1. Assume $Q$ is explicitly nep. Then:

(a) in Definition 6.2, $Q'$ is a (quasi) order,

(b) $\leq^{Q'} \upharpoonright Q = \leq^Q$,

(c) $Q$ is a dense subset of $Q'$.

2. Assume in addition:

(2) $Q$ is explicitly nep in every $Q$-candidate.
Then:
(d) if $N$ is a $\mathcal{Q}$–candidate, $N \models \exists p \in \mathcal{Q}’$, then for some $q \in N$ we have $N \models \exists p \leq^\mathcal{Q} q \& q \in \mathcal{Q}$,
(e) $\mathcal{Q}’$ is explicitly nep (with the same $\mathcal{B}^\mathcal{Q}$ and parameters).

3. Assume in addition
$(\exists_3)$ for any $\mathcal{Q}$–candidate $N$, if $N’$ is a generic extension of $N$ for the
forcing notion $\text{Levy}(\mathcal{R}_0, |P(\mathcal{Q})|^N)$, then $N’$ is a $\mathcal{Q}$–candidate.

Then we can add
$(e)^+ \mathcal{Q}’$ is explicitly local nep (see Definition 2.8).

4. We can replace above (in the assumption and conclusion) nep by snep,
or nep by simple nep.

Remark 6.4. The definition of “local” (in 2.8) and the statement $(\exists_3)$ in 6.3(3) can be handled a little differently. We can (in 2.8(2)) demand less on $N’$ (it is not a $\mathcal{Q}$–candidate), just have some of its main properties and in $\exists_3$ of 6.3(3), ZFC$^-$ says that $H(\theta)$ is a set (so has a cardinality) and is a $\mathcal{Q}$–candidate. So we may consider having ZFC$^-$ for several $\ell$’s, ZFC$^+\ell$ speaks on $\chi_0 > \ldots > \chi_{\ell-1}$, and the generic extensions of a model of ZFC$^+\ell+1$ for Levy$(\mathcal{R}_0, \chi_\ell)$ is a model of ZFC$^-\ell$. Similar remarks hold for §7. But, as we can deal with the nice case (see Definition 5.3), we may start with a countable $\mathcal{N} \prec (H(\mathcal{M}), \in)$ (or even better $(H(\mathcal{M}), \in)$ so that “countable depth can be absorbed”), we ignore this in our main presentation.

Does $(\exists_3)$ of 6.3(3) occur at all? Let $G$ be a subset of Levy$(\mathcal{R}_0, |P(\mathcal{Q})|^N)$
generic over $N$. Then $N’ \stackrel{\text{def}}{=} N(G)$ is a $\mathcal{Q}$–candidate.

Proof of 6.3. 1) Clause (a): Assume $q_1 \leq q_2 \leq q_3$; we have $2^3 = 8$
cases according to truth values of $q_i \in \mathcal{Q}$:

Case (A): $q_1, q_2, q_3 \in \mathcal{Q}$.

Trivial.

Case (B): $q_1, q_2 \in \mathcal{Q}$, $q_3 \notin \mathcal{Q}$.

Check.

Case (C): $q_1 \notin \mathcal{Q}$, $q_2, q_3 \in \mathcal{Q}$.

Check.

Case (D): $q_1 \in \mathcal{Q}$, $q_2 \notin \mathcal{Q}$, $q_3 \in \mathcal{Q}$.

Then $q_2 = p_2 \& \bigwedge_{I \in \text{pd}(N, \mathcal{Q})} \bigvee_{r \in I \cap N} r$ and $q_1 \leq^\mathcal{Q} p_2$ (by 6.2(b)(\gamma)) and $p_2 \leq^\mathcal{Q} q_3$
(by 6.2(b)(\gamma)). Hence $q_1 \leq^\mathcal{Q} q_3$ follows.

Case (E): $q_1 \in \mathcal{Q}$, $q_2 \notin \mathcal{Q}$, $q_3 \notin \mathcal{Q}$.

Let $q_\ell = p_\ell \& \bigwedge_{I \in \text{pd}(N, \mathcal{Q})} \bigvee_{r \in I \cap N} r$ for $\ell = 2, 3$. So $q_1 \leq^\mathcal{Q} p_2$ (see 6.2(b)(\beta))
and $p_2 \leq^Q p_3$ (see 6.2(b)(γ), (δ)). Hence $q_1 \leq^Q p_2$ (as $\leq^Q$ is transitive) and so $q_1 \leq q_3$ (see 6.2(b)(β)).

**Case (F):** $q_1 \not\in Q, q_2 \not\in Q, q_3 \in Q$.
Let $q_\ell = p_\ell \& \bigwedge_{I \in \text{pd}(N,Q)} \bigvee_{r \in I \cap N} r$ for $\ell = 1, 2$ and suppose that $q_1 \neq q_2$ (otherwise trivial). Then, by 6.2(b)(δ), $q_1 \leq p_2$ and by 6.2(b)(γ), $p_2 \leq q_3$ so by the previous case (C), $q_1 \leq q_3$ as required.

**Case (G):** $q_1 \not\in Q, q_2 \in Q, q_3 \not\in Q$.
Let $q_\ell = p_\ell \& \bigwedge_{I \in \text{pd}(N,Q)} \bigvee_{r \in I \cap N} r$ for $\ell = 1, 3$. Now, by 6.2(b)(β), $q_2 \leq p_3$ and by the previous case (C), $q_1 \leq p_3$ and hence, by 6.2(b)(δ), $q_1 \leq q_3$ as required.

**Case (H):** $\bigwedge \ell q_\ell \not\in Q$.
Let $q_\ell = p_\ell \& \bigwedge_{I \in \text{pd}(N,Q)} \bigvee_{r \in I \cap N} r$. If $q_1 = q_2$ or $q_2 = q_3$ then the conclusion is totally trivial. So assume not. Thus

\[
q_1 \leq p_2 \quad \text{ (by clause (δ) a case defined in (γ))}
\]
\[
q_2 \leq p_3 \quad \text{ (by clause (δ)).}
\]

Hence $p_2 \leq p_3$ (see clause (γ)), so “a previous case” applies. This finishes the proof of the clause (a).

**Clause (b):** Totally trivial.

**Clause (c):** Let $q \in Q’$; if $q \in Q$ then there is nothing to do; otherwise for some $Q$-candidate $N$ we have $q = p \& \bigwedge_{I \in \text{pd}(N,Q)} \bigvee_{r \in I \cap N} r$ and use nep (i.e. clause (c) of 2.3(1)) on the $Q$-candidate $N$.

2) Assume ($\exists_2$).
**Clause (d):** Proved inside the proof of clause (e).

**Clause (e):** More pedantically we have to define

\[
\varphi_0^{Q’}, \varphi_1^{Q’}, \varphi_2^{Q’}, \exists_2^{Q’}, \theta^{Q’}
\]

and then prove the required demands for a $Q’$-candidates. We let $\exists_2^{Q’} = \exists_2^Q, \theta^{Q’} = \theta^Q$, the formulas will be different, but with the same parameters.

So the $Q’$-candidates are the $Q$-candidates. What is $\varphi_0^{Q’}$? It is

\[
\varphi_0^{Q}(x) \lor "x \text{ has the form } p \& \bigwedge_{I \in \text{pd}(q(M))} \bigvee_{r \in I \cap M} r, \text{ where } M \text{ is a } Q\text{-candidate (so countable) and } \varphi_0^Q(p) ".
\]
Clearly \( \varphi_Q^0 \) defines \( Q' \) through \( Q' \)-candidates. Note that if \( N \) is a \( Q' \)-candidate and \( N \models \text{“} M \text{ is a countable } Q \text{-candidate} \text{”} \), then we have \( M \subseteq N \), and if \( M \models \text{“} x \text{ is countable} \text{”} \), then \( x \subseteq M \subseteq N \); so \( M \) is really a \( Q \)-candidate. Consequently, \( \varphi_Q^0 \) is upward absolute for \( Q' \)-candidates and it defines \( Q' \). So clause (a) of Definition 2.3(1) holds.

Now we pay our debt proving clause (d). Let \( N \) be a \( Q' \)-candidate and \( N \models \text{“} p \in Q' \text{”} \), i.e. \( N \models \varphi_Q^0 (p) \). By the definition of \( Q' \), either \( N \models \text{“} p \in Q \text{”} \) and we are done, or for some \( p', M \in N \) we have

\[
N \models \text{“} M \text{ is a } Q' \text{-candidate, } p' \in Q^M, \text{ and } p = (p' \& \bigwedge_{I \in \text{pd}(M,Q)} \bigvee_{r \in I \cap M} r) \text{”}.
\]

By the assumption (\( \boxtimes_2 \)), for some \( q \in Q^N \) we have \( N \models \text{“} q \text{ is explicitly } \langle M, Q \rangle \text{-generic} \text{”} \) and \( N \models \text{“} p' \leq Q q \text{”} \). Then for some \( \langle \langle r_{I, \ell} : \ell < \omega \rangle : I \in \text{pd}(M,Q) \rangle \in N \) we have: \( N \models \text{“} \{ r_{I, \ell} : \ell < \omega \} \text{ enumerates } I \cap M \text{”} \) and \( N \models \text{“} \varphi_Q^0 (r_{I,0}, r_{I,1}, \ldots, q) \text{”} \). Now it follows from the definition of \( Q' \) that \( N \models \text{“} p \leq Q' q \text{”} \), so \( q \) is as required.

What is \( \varphi_Q^1 \)? Just write the definition of \( p \leq Q q \) from clause (b) of 5.2. Clearly also \( \varphi_Q^1 \) is upward absolute for \( Q' \)-candidates and it defines the partial order of \( Q' \) (even in \( Q' \)-candidates). So clause (b) of Definition 2.3(1) holds.

What is \( \varphi_Q^2 \)? Let it be:

\[
\varphi_Q^2 (p_0, p_1, \ldots, p_\omega) \overset{\text{def}}{=} \text{“} \text{there are } M, p, q \text{ such that: } M \text{ is a } Q' \text{-candidate and } p \in Q^M \text{ and } q = (p \& \bigwedge_{I \in \text{pd}(M,Q)} \bigvee_{r \in I \cap M} r) \text{ and } q \leq Q p_\omega \text{ and for some } J \in \text{pd}(M,Q) \text{, if } r \in J \cap M \text{ then there is } \ell \text{ such that } p_\ell \leq Q r \text{”}.
\]

To show that \( \varphi_Q^2 \) is upward absolute for \( Q' \)-candidates suppose that \( N \) is a \( Q' \)-candidate and \( N \models \varphi_Q^2 (p_0, p_1, \ldots, p_\omega) \) and let \( M, p, q \) witness it. Then, in \( N \), \( M \) is a \( Q' \)-candidate, so \( p \in Q \) and for some \( J \in \text{pd}(M,Q) \) we have:

\[
\text{if } r \in J \cap M \text{, then there is } \ell \text{ such that } p_\ell \leq Q r.
\]

By the known upward absoluteness all those statements hold in \( V \), too. Assume now that \( \varphi_Q^2 (p_0, p_1, \ldots, p_\omega) \) holds as witnessed by \( M, p, q \) and \( J \in \text{pd}(M,Q) \). Suppose \( q' \geq p_\omega \) and we may assume that \( q' \in Q \) (by (1)(c)). Then \( q \leq q' \) and (by clause (\( \gamma \)) of the definition of \( \leq Q' \)) we have \( q'' \leq q' \) and \( \varphi_Q^2 (r_0, r_1, \ldots, q'') \) for some list \( \{ r_n : n < \omega \} \) of \( J \cap M \). Thus \( J \cap M \) is predense (in \( Q \)) above \( q'' \) and we find \( r \in J \cap M \) such that \( r, q' \) are compatible. But now, there is \( \ell < \omega \) such that \( p_\ell \leq Q r \), so
necessarily \( p, q' \) are compatible (in \( Q' \)). This shows \( 2.3(2)(b)^+ \). Let us turn
to clause (c)\(^+\) of Definition 2.3(2). So suppose that \( N \) is a \( Q'\)-candidate and
\( p \in Q' \cap N \). By clause (d), there is \( p' \) such that \( N \models " p \leq Q' p' \land p' \in Q' \). Let \( q = p' \land \bigwedge_{I \in pd(N, Q')} r \in I \cap N \), clearly \( q \in Q' \) and \( Q' \models " p' \leq q \). Hence,
by 6.3(1)(a), we know \( Q' \models " p \leq q \). For \( J \in pd(N, Q') \) let
\[
J' = \{ q \in Q^N : N \models " q \text{ is above some member of } J \text{ in } Q' \}
\]

Note that if \( J \in pd(N, Q') \) then \( J' \in pd(N, Q) \), and so \( J' \cap N \) is predense
above \( q \). Moreover, \(( \forall r \in J \cap N)(\exists r' \in J' \cap N)(r \leq Q' r') \). So let \( J \in pd(N, Q') \) and let \( \langle p_n : n < \omega \rangle \) be an enumeration of \( J \cap N \). It should be
clear that \( \varphi_2 (p_0, p_1, \ldots , q) \) holds as witnessed by \( N, p', q \) and \( T' \).

3) Compared to (e) of 6.3(2) we have also to prove (e)\(^+\), i.e. strengthen the
clause (c)\(^+\) of Definition 2.3(1) by \((*)\) of Definition 2.8(2).

Let \( N^+ \) be a generic extension of a \( Q'\)-candidate \( N \) by the forcing notion
Levy(\( \aleph_0, |\mathcal{P}(Q)|^N \)). Clearly for every \( p \in Q^N \), the condition
\[
p \land \bigwedge_{I \in pd(N, Q')} r \in I \cap N
\]
belongs to \( N^+ \). So by the proof of clause (c)\(^+\) of Definition 2.3(1) in the
proof of (e) above, we are done.

4) Similar proof.

Discussion 6.5. If we would like not to use 6.3, we may like to try the
following Definition 6.6. Note that there: cl\(_1\)\( (Q) \) cannot serve as a forcing
notion as it contains “false”, cl\(_2\)\( (Q) \) is the reasonable restriction, and cl\(_3\)\( (Q) \)
has the same elements but more “explicit” quasi order. We do not define
a quasi order on cl\(_1\)\( (Q) \), but it is natural to use the one of cl\(_2\)\( (Q) \) adding:
\( \psi \leq \varphi \) if \( \varphi \in \text{cl}_1(Q) \setminus \text{cl}_2(Q) \). No harm in allowing in the definition of cl\(_1\)\( (Q) \)
also \( \neg \) (the negation). The previous cl\( (Q) \) is close to cl\(_3\)\( (Q) \).

Definition 6.6. Let \( Q \) be a forcing notion.

1. Let cl\(_1\)\( (Q) \) be the closure of the set \( Q \) by conjunctions and disjunctions
over sequences of members of length \( \leq \omega \) [we may add: and \( \neg \) (the
negation)]; wlog there are no incidental identification and \( Q \subseteq \text{cl}_1(Q) \).
2. For a generic \( G \subseteq Q \) over \( V \) and \( \psi \in \text{cl}_1 (Q) \) let \( \psi[G] \) be the truth
value of \( \psi \) under \( G \) where for \( \psi = p \in Q, \psi[G] \) is the truth value of
\( p \in G \). (We will use \( t \) for “truth”.)
3. \( \hat{Q} = \text{cl}_2(Q) = \{ \psi \in \text{cl}_1(Q) : \text{for some } p \in Q \text{ we have } p \models \psi[G_Q] = t^* \} \), ordered by:

\[
\psi_1 \leq_{\hat{Q}} \psi_2 \iff (\forall p \in Q)[p \models \psi[G_Q] = t^* \Rightarrow p \models \psi[G_Q] = t^*].
\]

4. Let \( Q \) be explicitly nep. We let \( \text{cl}_3(Q) \) be the following forcing notion:

(a) the set of elements is \( \text{cl}_2(Q) \),

(b) the order \( \leq_{\hat{Q}} = \leq_{\text{cl}_3(Q)} = \leq_{\text{cl}_1(Q)} \) is the transitive closure of \( \leq_{\hat{Q}} \) which is defined by

\[
\psi_1 \leq_{\text{cl}_3} \psi_2 \text{ iff one of the following occurs}
\]

(i) \( \psi_1, \psi_2 \in Q \) and \( \psi_1 \leq_{\hat{Q}} \psi_2 \),

(ii) \( \psi_1 \) is a conjunct of \( \psi_2 \) (meaning: \( \psi_1 = \psi_2 \) or \( \psi_2 = \bigwedge_{n<\alpha} \psi_{2,n} \), and \( \psi_1 \in \{ \psi_{2,n} : n < \alpha \} \)),

(iii) \( \psi_2 \in Q \) and there is a \( Q \)-candidate \( M \) such that \( p, \psi_1 \in M, p \in Q^M, p \leq_{\hat{Q}} \psi_2, \psi_2 \) is explicitly \( (M, Q) \)-generic and \( M \models \text{"p \models } \psi_1[G_Q] = t^* \) and if \( q \in Q \) is a conjunct of \( \psi_1 \) then \( M \models \text{"} q \leq_{\hat{Q}} p \).

Proposition 6.7.  

1. \( Q \subseteq \hat{Q}, \leq_{\hat{Q}} \) is a quasi order, and \( \leq_{\hat{Q}} \models Q = \{ (p, q) : 
q \models \text{"p \in G_Q } \} \), so if \( Q \) is separative then \( \leq_{\hat{Q}} \models Q = \leq_{\hat{Q}}, \) and \( Q \) is a dense subset of \( \hat{Q} \).

2. Assume \( Q \) is temporarily explicitly nep. Then:

(a) \( Q \subseteq \text{cl}_3(Q) \) and \( \leq_{\text{cl}_3} \models \leq_{\hat{Q}} \) and \( \leq_{\text{cl}_3} \subseteq \leq_{\hat{Q}} \),

(b) \( Q \) is a dense subset of \( \text{cl}_3(Q) \).

3. Assume in addition

(\( \oplus_3 \)) \( Q \) is correctly explicitly nep in \( V \) and in every \( Q \)-candidate.

Then

(d) if \( N \) is a \( Q \)-candidate and \( N \models \text{"p \in cl}_3(Q) \)”

then for some \( q \in N \) we have \( N \models \text{"p \leq}_{\text{cl}_3} q \) & \( q \in Q \),

(e) \( Q' \) is explicitly nep and correct.

4. Assume in addition

(\( \oplus_4 \)) for any \( Q \)-candidate \( N \), if \( N' \) is a generic extension of \( N \) for the forcing notion \( \text{Levy}(R_0, |\mathcal{P}(Q)|^N) \), then \( N' \) is a \( Q \)-candidate.

Then we can add

(e\(^+\)) \( \text{cl}_3(Q) \) is explicitly local nep (see Definition 2.3).

Proof: Straight, e.g.

(2) Clause (b): Assume \( \psi \in \text{cl}_3(Q) \), so \( \psi \in \text{cl}_2(Q) \) and for some \( p \in Q \) we have \( p \models \psi[G_Q] = t^* \). There is a \( Q \)-candidate \( M \) to which \( p \) and \( \psi \) belong (as \( \text{ZFC^*} \) is \( \emptyset \)-good). Let \( q \) be explicitly \( (M, Q) \)-generic, and \( Q \models p \leq q \).

So, by clause (iii) of \( (\oplus_4)(b) \), we have \( \text{cl}_3(Q) \models \text{"} \psi \leq q^* \text{"} \), as required.
(3) Clause (e): Let $\varphi_0^3(x)$ say that there is a $Q$--candidate $M$ such that $M \models "x \in \text{cl}_3(Q)"$. Let $\varphi_1^3(x,y)$ say the definition of $\leq^6_3$. Lastly, $\varphi_2^3(\langle x_i : i \leq \omega \rangle)$ says that for some $\langle y_i : i \leq \omega \rangle$ we have

$$\varphi_2^3(\langle y_i : i \leq \omega \rangle), \; y_\omega \leq^3_3 x_\omega \quad \text{(i.e. } \varphi_1^3(y_\omega, x_\omega)) \text{ and } \bigwedge_{i<\omega} \bigvee_{j<\omega} x_j \leq^3_3 y_i.$$ 

Remark 6.8. Instead of using $\text{cl}(Q)$ from 6.2 below we can have in $\bar{\varphi}$, a function which from an $\omega$–list of the elements of $N$ and from $p$ computes an element of $Q$ having the role of $p \& \bigwedge_{I \in \text{pd}(N,Q)} \bigvee_{r \in I \cap N} r$. The choice does not seem to matter.

Definition 6.9. For a forcing notion $P$ and a cardinal (or ordinal) $\kappa$, we define what is an hc-$\kappa$-$P$–name (here hc stands for hereditarily countable), and for this we define by induction on $\zeta < \omega_1$ what is such a name of depth $\leq \zeta$.

$\zeta = 0$: It is $\alpha$, that is $\bar{\alpha}$, for some $\alpha < \kappa$.

$\zeta > 0$: It has the form $\tau = \{\langle p_i, I_i \rangle : i < i^* \}$, where $i^* < \omega_1$, $p_i \in \text{cl}_1(P)$ from Definition 6.6(1) and $\tau_i$ an hc-$\kappa$-$P$–name of some depth $< \zeta$; that is for $G \subseteq P$ generic over $V$, we let $\tau[G] = \{\tau_i[G] : p_i[G] = t\}$.

An hc-$\kappa$-$P$–name is an hc-$\kappa$-$P$–name of some depth $< \omega_1$. An hc-$\kappa$-$P$–name $\tau$ has depth $\zeta$ if it has depth $\leq \zeta$, but not $\leq \xi$ for $\xi < \zeta$.

Remark: Why did we use $p \in \text{cl}_1(Q)$ and not $p \in \text{cl}_3(Q)$? As the membership in $\text{cl}_1(Q)$ is easier to define.

Proposition 6.10. 1. If $\tau$ is an hc-$\kappa$-$P$–name and $G \subseteq P$ is generic over $V$ then $\tau[G] \in \mathcal{H}_{<\kappa_1}(\kappa)$. If in addition $P \subseteq \mathcal{H}_{<\kappa_1}(\kappa)$ then $\tau \in \mathcal{H}_{<\kappa_1}(\kappa)$.

2. Let $\varphi(x_0, \ldots, x_{n-1})$ be a first order formula and $\tau_0, \ldots, \tau_{n-1}$ be hc-$\kappa$-$P$–names. Then there is $p \in \text{cl}_1(P)$ such that for every $G \subseteq P$ generic over $V$:

$$\left( \bigcup_{\ell < n} \text{Te}^{\text{ord}}(\tau_{\ell}[G]), \varepsilon \right) \models \varphi(\tau_0[G], \ldots, \tau_{n-1}[G]) \iff p[G] = t.$$

3. The set of hc-$\kappa$-$P$–names is closed under the following operations:

(a) difference,

(b) union and intersection of two, finitely many and even countably many,
(c) definition by cases: for $p_n \in \text{cl}_1(\mathbb{P})$ and hc-$\kappa$-$\mathbb{P}$-names $\tau_n$ (for $n < \omega$) there is a hc-$\kappa$-$\mathbb{P}$-name $\tau$ such that for a generic $G \subseteq \mathbb{Q}$ over $\mathbb{V}$ we have

$$\tau[G] \begin{cases} \tau_n[G] & \text{if } p_n[G] = t \& \bigwedge_{\ell \leq n} \neg p_\ell[G] = t \\ \emptyset & \text{if } \bigwedge_{\ell < \omega} \neg p_\ell[G] = t. \end{cases}$$

**Definition 6.11.** 1. A forcing notion $\mathbb{Q}$ (or $\varphi$) is temporarily, explicitly straight $(\kappa, \theta)$-nep for $\mathfrak{B}$ if: old conditions from Definition 2.3(1),(2) (for explicitly $(\kappa, \theta)$-nep) but possibly $\mathfrak{B} \subseteq H_{\aleph_1}(\kappa)$; and

(d) $\mathbb{Q} \subseteq H_{\aleph_1}(\theta)$ (i.e. $\mathbb{Q}$ is simple) and $\aleph_1 + \theta \leq \kappa$, 

(e) for $\ell < 3$ the formula $\varphi^Q_i(\bar{x})$ is of the form

$$\exists t[\bar{x} \in H_{\aleph_1}(\kappa) \& t = \text{To}(t) \& t \cap \omega_1 \text{ is an ordinal} \& \psi^Q_i(\bar{x}, t)],$$

where in the formula $\psi^Q_i$ the quantifiers are of the form $(\exists s \in t)$ and the atomic formulas are “$x \in y$”, “$x$ is an ordinal” and those of $\mathfrak{B}^Q$.

2. In clause (e) of part (1), we call such $t$ an explicit witness for $\varphi^Q_i(\bar{x})$. We call $t$ a weak witness, if for every $\mathbb{Q}$-candidate $N$, $\bar{x} \in N$, if $t \in N$ then $N \models \varphi^Q_i(\bar{x})$. We call it a witness if:

(i) $\ell = 0$ and it is an explicit witness, or

(ii) $\ell = 1$ (so $\bar{x} = \langle x_0, x_1 \rangle$) and $t$ gives $k, y_0, \ldots, y_k, t_0, \ldots, t_{k-1}$, $s_0, \ldots, s_k$ such that: $s_\ell$ explicitly witnesses $\varphi_0(y_\ell)$, $t_\ell$ explicitly witnesses $y_\ell \leq^Q y_{\ell+1}$ and $y_0 = x_0$, $y_k = x_1$ (so $y_\ell \in t$, $s_\ell \in t$, $x_\ell \in t$),

(iii) $\ell = 2$ (so $\bar{x} = \langle x_i : i < \omega \rangle$) and $t$ gives $\langle y_i : i \leq \omega\rangle$,

$$\langle s_i : i \leq \omega + 1 \rangle$$

such that $s_\omega$ is a witness to $y_\omega \leq x_\omega$, $s_{\omega+1}$ is an explicit witness to $\varphi^Q_2(\langle y_i : i \leq \omega \rangle)$, $s_i$ is a witness to $x_i \leq^Q y_{k_i}$ (so also they all belong to $t$, as well as witnesses to $x_i, y_j \in \mathbb{Q}$).

**Proposition 6.12.** 1. Assume $\mathbb{Q}$ is temporarily explicitly straight $(\kappa, \theta)$-nep for $\mathfrak{B}$. Then $\mathbb{Q}$ is temporarily simply explicitly $(\kappa, \theta)$-nep for $\mathfrak{B}$. Sufficient conditions for “$K$–absolutely” are as in $\S2$.

2. Assume $\mathbb{Q}$ is temporarily correctly simply explicitly $(\kappa, \theta)$-nep for $\mathfrak{B}$ and $\theta + \aleph_1 \leq \kappa$. Then $\mathbb{Q}$ is temporarily straight explicitly $(\kappa, \theta)$-nep for $\mathfrak{B}$ and is correct.

[Nevertheless, “simple” and “straight” are distinct as properties of $(\mathfrak{B}, \varphi, \theta)$, i.e. the point is changing $\varphi$.]

**Proof** Straight.
Definition/Theorem 6.13. By induction on \( \alpha \) we define and prove the following situations:

(A) [Definition] \( \bar{Q} = \langle (P_i, Q_i, \bar{\varphi}_i, B_i, r_i, \theta_i) : i < \alpha \rangle \) is nep-CS-iteration.

(B) [Definition] \( \kappa^\bar{Q} = \kappa[Q] \), in short \( \kappa^\alpha \) abusing notation.

(C) [Definition] We define \( B^\alpha = B^\bar{Q} \).

(D) [Definition] \( \text{Lim}(\bar{Q}) = P_\alpha \) and \( P_\alpha \) for any set \( w \) of ordinals \( < \alpha \) for \( \bar{Q} \) as above.

(E) [Claim] If \( \bar{Q} \) is a nep-CS-iteration, and \( \alpha = \ell g(\bar{Q}) \), then \( \bar{Q} \upharpoonright \beta \) is a nep-CS-iteration (for \( \beta < \alpha \)), \( \text{Lim}(\bar{Q} \upharpoonright \beta) = P_\beta \) and \( P_\beta \subseteq \mathcal{H}_{\kappa^\alpha}(\kappa_\beta) \).

(F) [Claim] For \( \bar{Q} \) as in (A), a \( B^\alpha \)-candidate \( N, \gamma \leq \beta \leq \alpha \) and \( p, q \in P_\beta \):

(a) \( p \) is a function with domain a countable subset of \( \beta \) (pedantically see clause (D)),

(b) \( P_\beta \) is a forcing notion (i.e. a quasi order) satisfying (d) + (e) of \ref{2.3} and (a), (b), (b) of \ref{2.3}(1),(2),

(c) \( p \gamma \in P_\gamma \) and \( P_\beta \models \gamma \leq p^\gamma \),

(d) \( P_\gamma \models \gamma \leq q \) implies \( P_\beta \models \gamma \leq (q \cup P \upharpoonright [\gamma, \beta]) \),

(e) \( P_\gamma \subseteq P_\beta \) and even \( P_\gamma \prec P_\beta \),

(f) \( p \in P_\beta \) iff \( p \) a function with domain \( \in [\beta]^{<\aleph_0} \) and

\[ \zeta \in \text{Dom}(p) \quad \Rightarrow \quad p \zeta \in P_{\zeta + 1}. \]

(G) [Definition] For a \( B^\alpha \)-candidate \( N \) and \( w, \beta, \gamma \) such that \( N \models \gamma \leq w \leq \alpha \), \( \gamma < \beta \leq \alpha \) and \( \beta, \gamma \in (w \cup \{\alpha\}) \cap N \), and \( q \in P_\beta, p \in N \) such that \( N \models \gamma \leq p \in P_\beta \) and \( q \gamma \) is \( (N, P_\beta, w) \)-generic we define when \( q \) is \( [\beta, \gamma) \)-canonically \( (N, P_\beta, w) \)-generic above \( p \).

(H) [Theorem] If \( q \in N \) is a \( [\beta, \gamma) \)-canonically \( (N, P_\beta, w) \)-generic above \( p \), then \( q \) is \( (N, P_\beta) \)-generic and \( p \leq q \).

(I) [Theorem] \( P_\alpha \) is explicitly straight correct \( \kappa^\alpha \)-nep for \( \bar{\varphi}, B^\alpha \).

(J) [Theorem] For any \( \kappa \geq \kappa^\alpha \),

\[ \mathbb{P}_{\kappa^\alpha} \models (\mathcal{H}_{\kappa^\alpha}(\kappa))^{\mathbb{V}[\mathbb{P}]_\alpha} = \{ \tau[F_{\kappa^\alpha}] : \tau \text{ is an hc-} \kappa^{\mathbb{P}_{\kappa^\alpha} \text{-name}} \} \].

Let us carry out the clauses one by one.

Clause (A), Definition: \( \bar{Q} = \langle (P_i, Q_i, \bar{\varphi}_i, B_i, r_i, \theta_i) : i < \alpha \rangle \) is a nep-CS-iteration if:

(a) \( \beta < \alpha \quad \Rightarrow \quad \bar{Q} \upharpoonright \beta \) is a nep-CS-iteration,

(\beta) if \( \alpha = \beta + 1 \) then

(i) \( P_\beta = \text{Lim}(\bar{Q} \upharpoonright \beta) \) (use clause (D))

(ii) \( \bar{\varphi}_\beta = \langle \varphi_{\beta, \ell} : \ell < 3 \rangle \) is formally as in the definition of nep (the substantial demand in (v) below, but the parameter \( B_\beta \) is a name!)

(iii) \( \kappa_\beta, \theta_\beta \) are infinite cardinals (or ordinals)
Case 2: If Case 1: If Clause (D)

So in any case \( \kappa \) assuming the course, if the result is an ordinal we can replace it by its cardinality, coding \( \kappa \subseteq H \times \kappa \). 

Clause (B), Definition: We define \( \kappa^\alpha = \sup\{ \kappa_i : i < \alpha \} \cup \{ \alpha \} \) (of course, if the result is an ordinal we can replace it by its cardinality, coding it assuming the \( \kappa_i \)'s are cardinals; remember that \( \kappa_i \geq \theta_i \)).

Clause (C), Definition: We define \( \mathcal{B}^\alpha = \mathcal{B}^{\tilde{Q}_\alpha} \), a model with universe \( \subseteq \mathcal{H}_{\mathcal{N}_1}(\kappa^\alpha) \) or write \( \kappa^\alpha \) and the usual vocabulary such that

(*) \( \mathcal{B}^\alpha \) codes (by its relations) \( \alpha, \{ (\beta, \tilde{\varphi}, \kappa, \theta) : \beta < \alpha \} \) and \( \mathcal{B}^{\mathcal{P}_\beta} \) \( \beta < \alpha \); i.e. for every atomic formula in the vocabulary \( \tau_0 \) (so is of \( \mathcal{B}^{\mathcal{P}_\beta} \), \( \psi = \psi(x_0, \ldots, x_{n-1}) \) for some function symbol \( F_\psi \) we have: if \( \alpha_\ell < \kappa_\beta \) for \( \ell < n \) then \( F_\psi(\beta; \alpha_0, \ldots, \alpha_{n-1}) = p^{\mathcal{P}_\beta}_{\psi(\alpha_0, \ldots, \alpha_{n-1})} \) (see clause (A)(iv)) and

if the \( \mathcal{B}^{\mathcal{P}_\beta} \)'s are on \( \kappa \), we have also \( F_{\psi, \ell} \), functions of \( \mathcal{B}^\alpha \) such that:

if \( \alpha_\ell < \kappa_\beta \) for \( \ell < n \) then \( \{ F_\psi(\beta, \ell; \alpha_0, \ldots, \alpha_{n-1}) : \ell < \omega \} \) lists the ordinals in \( \mathcal{T}_{\mathcal{P}_\beta}^{\text{card}}(p^{\mathcal{P}_\beta}_{\psi(\alpha_0, \ldots, \alpha_{n-1})}) \) (the condition in \( \mathcal{P}_\beta \) saying...) and \( F_\psi^{\ell} \) codes how \( p \) was gotten from them (so we need \( \kappa^\alpha \geq \omega_1 \)).

So in any case

(**) if \( N \) is a \( \mathcal{B}^\alpha \)-candidate, and \( \beta \in \alpha \cap N \) then \( N \) is a \( \mathcal{B}^{\mathcal{P}_\beta} \)-candidate.

Clause (D), Definition:

Case 1: If \( \alpha = 0 \) then \( \mathcal{P}_\alpha = \{ \emptyset \} \).

Case 2: If \( \alpha = \beta + 1 \) then

\( \mathcal{P}_\alpha = \{ p : p \) is a function, \( \mathcal{D}(p) \subseteq \alpha, p \mathcal{P}_\beta \subseteq \mathcal{P}_\beta \) and if \( \beta \in \mathcal{D}(p) \) then for some \( r = r_{p, \beta} \in \mathcal{C}(\mathcal{P}_\beta) \) determined by \( p \) we have: \( p(\beta) \) is defined by cases:

if \( r[G_{\mathcal{P}_\beta}] = t \), it is in \( \mathcal{C}(\mathcal{Q}_\beta) \), and an explicit witness is provided (say \( p[\beta] \) codes it and having \( r[G_{\mathcal{P}_\beta}] = t \) says so),

if not \( p(\beta) \) is \( \emptyset = \emptyset_{\mathcal{Q}_\beta} = \min(\mathcal{Q}_\beta) \).
Pedantically, $p \in P_\alpha$ if and only if $p$ has the form $p' \cup \{ \langle \beta, \ell, x_i \rangle : \ell < 3 \}$ where $p' \in P_\beta$, $x_0 \in \text{cl}(1(\langle \beta, x_i \rangle))$, $x_1, x_2$ are $\kappa$-$\mathcal{P}_\beta$-names of members of $\mathcal{H}(\theta_\beta)$ and $x_0$ is the truth value of “$x_2[G_\beta]$ is a witness to $x_1[G_\beta] \in \theta_\beta$.”

**Case 3:** If $\alpha$ is limit, then $\mathbb{P}_\alpha = \{ p : p $ is a function, $\text{Dom}(p) \in [\alpha]^{\leq \kappa_0}$ and $\beta \leq \alpha \implies p \upharpoonright \beta \in \mathbb{P}_\beta \}$.

**The order:**
For $\alpha = 0$ nothing to do.
For $\alpha$ limit: $p \leq q$ if and only if $\bigwedge_{\beta < \alpha} \mathbb{P}_\beta \models \left[ \langle \beta \rangle \leq q \right]$ (equivalently: $\bigwedge_{\beta < \alpha} \mathbb{P}_{\beta+1} \models \left[ p \upharpoonright (\beta + 1) \leq q \upharpoonright (\beta + 1) \right]$), (see (C)).

For $\alpha = \beta + 1$: the order is the transitive closure of the following cases:

(a) $p \in \mathbb{P}_\beta$, $q \in \mathbb{P}_\alpha$, $\mathbb{P}_\beta \models \left[ p \upharpoonright \beta \leq q \right]$.

(b) $p(\beta) = q(\beta)$ and $\mathbb{P}_\beta \models \left[ p \upharpoonright \beta \leq q \right]$.

(c) $p \upharpoonright \beta = q \upharpoonright \beta$ and there is a $\mathfrak{B}^\alpha$-candidate $N$ such that $q \upharpoonright \beta$ is a $[0, \beta] \cap \mathfrak{B}^\alpha$-canonical $(N, \mathbb{P}_\beta)$-generic above $p' \upharpoonright \beta$, $\mathbb{P}_\beta \models \left[ p' \upharpoonright \beta \leq q \right]$, $p' \in \mathbb{P}_N$ and $N \models \left[ p' \upharpoonright \beta \Vdash \text{cl}(Q_\beta) \models p(\beta) \leq p'(\beta) \text{ and } p'(\beta) \in Q_\beta \right]$ and

$q(\beta)$ is canonically generic for $(Q_\beta, N[G_\beta])$ above $p$, i.e. is

$p'(\beta) \& \bigwedge_{I \in \text{pd}([N, \mathbb{P}_\alpha])} \bigvee_{r(\beta) \in Q_\beta} \{ r(\alpha) : N \models \left[ r \in I \right] \text{ and } r \upharpoonright \beta \in G_\beta \}$

if $q \upharpoonright \beta \in G_{\mathbb{P}_\beta}$ and $p'(\beta)$ if $q \upharpoonright \beta \notin G_{\mathbb{P}_\beta}$.

Lastly, $\mathbb{P}_{\alpha, w} = \{ p \in \mathbb{P}_\alpha : \text{Dom}(p) \subseteq w \}$ for $w \subseteq \alpha$ (note that if $w \subseteq \beta \leq \alpha$ we get the same forcing notion).

**Clause (E), Claim:** Trivial.

**Clause (F), Claim:** Subclauses (a) and (c)–(f) are trivial.

**Subclause (b):** Here we should be careful as we do not ask just that the order is forced but there is a hc witness; as we ask for a witness and not explicit witness (see Definition 7.11) this is okay. See more in the proof of clause (I).

**Clause (G), Definition:**

**Case 1:** For $\beta < \alpha$ note that $N$ is also a $\mathfrak{B}^\beta$-candidate and use the definition for $Q \upharpoonright \beta$.

**Case 2:** If $\gamma = \beta = \alpha$ – trivial.

**Case 3:** For $\beta = \alpha$, $\alpha = 0$ – trivial.
Case 4: For $\gamma < \beta = \alpha$ and $\beta = \beta' + 1$, $\beta' \notin w$ - trivial.

Case 5: Suppose $\gamma < \beta = \alpha$, $\alpha = \beta' + 1$, $\beta' \in w$.
Then: $q \upharpoonright \beta'$ is $[\gamma, \beta', \gamma_\beta]$-canonically $(N, P_\beta, w)$-generic and for some $\tau$,

$$N \models \tau \text{ is a hc-}\kappa_\beta-P_\beta-\text{name of a member of } Q_\beta$$

which is above $p(\beta)$ (which is in $\text{cl}(Q_\beta)$) and is in $N \langle G_{P_\beta \cap w} \rangle$.

and

$$q(\beta) = \tau \& \bigwedge_{I \in \text{pd}(N, P_\alpha, w)} \bigwedge_{r(\beta) \in \bigwedge \big\{r(\beta) : N \models \tau \text{ and } r \upharpoonright \beta \in G_{P_\beta} \big\}}.$$

Case 6: $\gamma < \beta = \alpha$, $\beta$ a limit.
Say that diagonalization was used.

Clause (H), Theorem: Prove by induction.

Clause (I), Theorem: We have defined $B_\alpha$ and $\kappa_\alpha$ (so $\theta_\alpha = \kappa_\alpha$). The formulas $\varphi_{P_\alpha}^{\ell}$ ($\ell < 3$) are implicitly defined (in the induction).

Why $\varphi_0^{P_\alpha}$ is absolute enough? As the demand on $p(\beta)$ above says that $r_p(\beta+1)$, the witness for $p(\beta) \in \text{cl}(Q)$, is such that $r_p = \tau$ gives all the required information.

Why $\varphi_1^{P_\alpha}$ is absolute enough? Because the canonical genericity is about $\varphi_2$ and the properness requirement, see clause (G), fit.

Now one proves by induction on $\beta \leq \alpha$:

(5) if $N$ is a $B_\alpha$-candidate, $w \in N$, $N \models \text{"}w \subseteq \alpha\text{"}$, $\gamma_0 \leq \gamma_1 \leq \beta$,

$$\{\gamma_0, \gamma_1, \beta\} \subseteq (\alpha + 1) \cap N \cap w, p \in P_\beta^N, q \in P_\gamma, p \upharpoonright \gamma_1 \leq q, q \text{ is } [\gamma_0, \gamma_1]$$

canonically $(N, P_{\gamma_1}, w)$-generic

then we can find $q^+$ such that:

$$\text{(a) } q^+ \in P_\beta, q^+ \upharpoonright \gamma = q,$$

$$\text{(b) } p \leq^+, q^+ \upharpoonright \gamma = q,$$

$$\text{(c) } q^+ \text{ is } [\gamma, \beta] \text{--canonically } (N, P_\beta, w) \text{--generic.}$$

Clause (I), Theorem: Straight.

Proposition 6.14. The iteration in 6.13 is equivalent to the CS iteration.

More formally, assume

$$\bar{Q} = \langle (P_i, Q_i, \varphi_i, B_i, \kappa_i, \theta_i) : i < \alpha \rangle \text{ is an CS-nep iteration.}$$

We can define $\bar{Q}' = \langle P'_i, Q'_i : i < \alpha \rangle$ and $\langle F_i : i < \alpha \rangle$ such that

(a) $\bar{Q}'$ is a CS iteration,

(b) $F_i$ is a mapping from $P_i$ into $P'_i$,

(c) $j < i \Rightarrow F_j = F_i \upharpoonright P_j$. 


(d) $F_i$ is an embedding of $P_i$ into $P'_i$ with dense range,
(e) $Q_i$ is mapped by $F_i$ to $Q'_i$.

**Proof** Straight.

**Proposition 6.15.** In the context of 6.14:
1. Assume that each $B_{i\beta}$ is essentially a real; i.e. $\kappa_{\beta} = \omega$ and if $R$ is in the vocabulary of $B_{i\beta}$ then $R^{B_{i\beta}} \subseteq n(R)\omega$. If $\alpha < \omega_1$ then so is the $B_{i\alpha}$.
   (If $\alpha \geq \omega_1$ we get weaker results).
2. Assume that $P_{i\beta}$ “the universe of $B_{i\beta}$ is $\kappa_{\beta}$”. Then we can make “$B_{i\beta}$ has universe $\kappa_{\beta}$”, coding the $p_{i\beta}^\psi$’s.

**Proof** Left to the reader.

**Remark 6.16.** 1) Note that 6.13, 6.14 (and 6.15(4)) say something even for $\alpha = 1$ so it speaks on $cl(Q_0) = P_1$ (or $cl_3(Q_0) = P_1$).
2) Concerning 6.15 note that if $\kappa(B) \geq \omega_1$, the difference between nep and smp is not large, so the case $\alpha < \omega_1$ has special interest.
3) In 6.13, 6.14, we can replace the use of $Q' = cl(Q)$ from 6.2 by $cl_3(Q)$ from Definition 6.6 (using 5.7).
4) We can derive a theorem on local in 6.14, but for strong enough ZFC$^-$, then any follows.

Of course, we can get forcing axioms.

**Proposition 6.17.** 1. Assume for simplicity that $V \models 2^{\aleph_0} = \aleph_1$ & $2^{\aleph_1} = \aleph_2$. Then for some proper $\aleph_2$–c.c. forcing notion $P$ of cardinality $\aleph_2$ we have in $V^P$:

$(\oplus)$ $\text{Ax}_{\omega_1}[(\aleph_1, \aleph_1)\text{–nep}]:$ if $Q$ is a $(\kappa, \theta)$–nep forcing notion, $\kappa, \theta \leq \aleph_1$ and $I_i$ is a dense subset of $Q$ for $i < \omega_1$ and $S_i$ as a $Q$–name of stationary subset of $\omega_1$ for $i < i(*) \leq \omega_1$,

then for some directed $G \subseteq Q$ we have: $i < \omega_1 \Rightarrow G \cap I_i \neq \emptyset$ and $S_i[G] \overset{\text{def}}{=} \{ \zeta < \omega_1 : \text{for some } q \in G \text{ we have } q \Vdash \check{\zeta} \in S_i \}$

is a stationary subset of $\omega_1$.

2. We can demand that $P$ is explicitly $(\aleph_2, \aleph_2)$–nep provided that in $(\oplus)$ we add “explicitly simply” to the requirements on $Q$.

3. In parts 1) and 2), we can strengthen $(\oplus)$ to $\text{AX}_{\omega_1}\text{[nep]}$.

**Proof** Straight (as failure of “$Q$, i.e. $\check{\varphi}$, is nep” is preserved when extending the universe).

**Proposition 6.18.** We can generalize the definitions and claims so far by:
(a) a forcing notion \( \mathbb{Q} \) is \((\mathbb{Q}, \leq, \leq_{pr}, \emptyset_\mathbb{Q})\), where \( \leq_{pr} \) is a quasi order, \( p \leq_{pr} q \Rightarrow p \leq q \) and \( \emptyset_\mathbb{Q} \) the minimal element;
(b) in the definition of nep in addition to \( \varphi_1 \) we have \( \varphi_{1,pr} \) defining \( \leq_{pr} \), which is upward absolute from \( \mathbb{Q} \)–candidates, and in Definition 7.3(2)(c) we strengthen \( p \leq q \) to \( p \leq_{pr} q \);
(c) the definition of CS iteration \( \langle \mathbb{P}_i, \mathbb{Q}_i : i < \alpha \rangle \) is modified in one of the following ways:
\[(a) \mathbb{P}_i = \{ p : p \text{ is a function, } \text{Dom}(p) \text{ is a countable subset of } i, j \in \text{Dom}(p) \Rightarrow p(j) \in \mathbb{Q}_j \text{ } \text{ and the set } \{ j \in \text{Dom}(p) : q \upharpoonright p, \emptyset_\mathbb{Q}_j \leq_{pr} p(j) \} \text{ is finite } \}, \]
with the order
\[ p \leq q \text{ if and only if } j \in \text{Dom}(p) \Rightarrow q \upharpoonright j \models p(j) \leq_{\mathbb{Q}_j} q(j) \text{ and the set } \{ j \in \text{Dom}(p) : q \upharpoonright j \models \emptyset_\mathbb{Q}_j \leq_{pr} p(j) \} \text{ is finite; } \]
\[(b) \mathbb{P}_i = \{ p : p \text{ is a function, } \text{Dom}(p) \text{ is a countable subset of } i, j \in \text{Dom}(p) \Rightarrow p(j) \in \mathbb{Q}_j \} , \]
with the order
\[ p \leq q \text{ if and only if } j \in \text{Dom}(p) \Rightarrow q \upharpoonright j \models p(j) \leq_{\mathbb{Q}_j} q(j) \text{ and } \{ j \in \text{Dom}(p) : q \upharpoonright j \models \emptyset_\mathbb{Q}_j \leq_{pr} p(j) \} \text{ is finite; } \]
(d) similarly for the CS-nep iteration.

**Proof** Left to the reader. ■

6. When a real is \( (\mathbb{Q}, \eta) \)–generic over \( V \).

**Definition 7.1.**

1. We say that \( (\mathbb{Q}, \hat{W}) \) is a temporary \((\mathcal{B}, \theta, \sigma, \tau)\)–pair if for some \( \mathbb{Q} \)–name \( \eta \) the following conditions are satisfied:
   (a) \( \mathbb{Q} \) is a nep-forcing notion for \( (\mathcal{B}, \hat{\varphi}, \theta) \); possibly \( \mathcal{B} \) expands \( \mathcal{B}^\mathbb{Q} \),
   (b) \( \Vdash^\mathbb{Q} \eta \in \sigma^\tau \),
   (c) \( \hat{W} = \{ W_n : n < \sigma \} \),
   (d) for each \( n < \sigma, W_n \subseteq \{ (p, \alpha) : p \in \mathbb{Q}, \alpha < \sigma \} \),
   (e) if \((p_\ell, \alpha_\ell) \in W_n \) for \( \ell = 1, 2 \) and \( \alpha_1, \alpha_2 \) are not equal, then \( p_1, p_2 \) are incompatible in \( \mathbb{Q} \),
   (f) for each \( n < \sigma \) the set \( \mathcal{I}_n = \mathcal{I}_n[\hat{W}] \overset{\text{def}}{=} \{ p : (\exists \alpha)[(p, \alpha) \in W_n] \} \) is a predense subset of \( \mathbb{Q} \),
   (g) so \( \tau = \tau[\hat{W}] = \tau[\mathbb{Q}, \hat{W}] \) and (abusing notation) let \( \sigma = \sigma[\hat{W}] = \sigma[\mathbb{Q}, \hat{W}] \).
2. For \((\mathbb{Q}, W)\) as above, \( \eta = \eta[\hat{W}] = \eta[\mathbb{Q}, \hat{W}] \) is the \( \mathbb{Q} \)–name
   \[ \bigcup \{ (p, (n, \alpha)) : (\exists p \in G_\mathbb{Q})((p, (n, \alpha)) \in W_n) \} \text{, so } n < \sigma \].
3. We replace the temporary by \( K \) if this (specifically the demand \((f)\)) holds in any \( K \)-extension.

4. We may write \((Q, \eta), W \) instead of \( W \) abuse notation. If we omit \( B \) we mean \( B = B_Q \). If \( \tau = \aleph_0 \) we may omit it; if \( \tau = \sigma = \aleph_0 \) we may omit them, if \( \theta = \sigma = \tau = \aleph_0 \), we may write \( \kappa \).

5. We say that \( \eta[Q, W] \) is a temporarily generic real (or function) for \( Q \) if for no distinct \( G_1, G_2 \) generic over \( V \) do we have \( \eta[G_1] = \eta[G_2] \).

6. Instead \((Q, W)\) we may write \(((B_Q, \bar{\varphi}, \theta_Q), W)\) (or with \( \bar{\eta} \) instead \( \bar{W} \)).

**Definition 7.2.**
1. Let \( K_{\kappa, \theta, \sigma, \tau} \) be the class of all \((Q, \eta)\) which are temporary \((B, \theta, \sigma, \tau)\)-pairs for some \( B \) with \( \kappa(B) \leq \kappa, \| B \| \leq \kappa \).

2. Let \((Q, \eta)\) be a temporary \((\kappa, \theta)\)-pair (actually more accurately write \((B, \bar{\varphi}, \theta), W)\); so \( \sigma = \tau = \aleph_0 \).

Let \( N \) be a \( Q \)-candidate and \( \eta \in \omega^\omega \). We say that \( \eta \) is a \((Q, \eta)\)-generic real over \( N \) if for some \( G \subseteq Q \) which is generic over \( N \) we have \( \eta = \eta[G] \).

3. We say that \( \eta \) is hereditarily countable if each \( \omega_n \) is countable (note: the generic reals of the forcing notions from \([14]\) are like that, but for our purpose just “absolute enough” suffices).

**Definition 7.3.**
1. \((Q, W)\) is a temporary explicitly \((B, \theta, \sigma, \tau)\)-pair (or nep pair) if for some \( Q \)-name \( \eta \) we have:
   a. \( Q \) is an explicit nep forcing notion for \((B, \bar{\varphi}, \theta)\),
   b. \( \vdash_{Q} \eta \in \sigma^\tau \),
   c. \( W = \langle \psi_{\alpha, \zeta} : \alpha < \sigma, \zeta < \tau \rangle \),
   d. \( \psi_{\alpha, \zeta} \in cl_1(Q) \) for \( \alpha < \sigma, \zeta < \tau \),
   e. \( \vdash_{Q} \eta(\alpha) = \zeta \iff \psi_{\alpha, \zeta}[G_{\tilde{Q}}] = t \)."

2. In this case \( \eta = \eta[W] = \eta[Q, W] \) is the \( Q \)-name above (it is unique). Abusing notation we may write \((Q, \bar{\eta})\) instead \((Q, W)\) and then let \( \bar{W} = W[\bar{\eta}] = \bar{W}[Q, \bar{\eta}] \).

3. We introduce the notions from \([7,3]\)–(6) for the current case with almost no changes.

**Definition 7.4.** \( K^\text{ex}_{\kappa, \theta, \sigma, \tau} = \{(Q, \eta) \in K_{\kappa, \theta, \sigma, \tau} : (Q, \eta) \) is temporarily explicitly \((B, \theta, \sigma, \tau)\)-pair for some model \( B \) with \( \kappa(B) \leq \kappa, \| B \| \leq \kappa \}. \)

**Proposition 7.5.** Assume that:
   a. \( Q \) is an explicitly nep forcing notion which satisfies the c.c.c.
   b. \( \vdash Q \eta \in \omega^\omega \) and \( (\alpha < \sigma \text{ and } m < \omega) \psi_{\alpha, m} \in cl_1(Q) \) are such that \( \vdash_{Q} \eta(\alpha) = m \iff \psi_{\alpha, m}[G_{\tilde{Q}}] = t \)."
(c) \( Q' \) \( \overset{\text{def}}{=} B_2(Q, \eta) \) is the following suborder of \( \text{cl}_2(Q) \):
\[
\{ p \in \text{cl}_2(Q) : \ p \text{ is generated by the } \psi_{\alpha,m} \text{ i.e. it belongs to the closure of } \{ \psi_{\alpha,m} : \alpha < \sigma, m < \omega \} \text{ under } \neg, \bigwedge_{i<\gamma} \text{ for } \gamma < \omega_1 \text{ in } \text{cl}_2(Q) \}
\]
(i.e. it is the quasi order \( \leq^Q_2 \) restricted to this set).

Then:

1. \( Q' \triangleleft \text{cl}_2(Q) \) and \( \eta \in \sigma \omega \) is a generic function for \( Q' \).
2. Assume additionally that
   
   (\ast) if \( M \) is a \( Q \)-candidate, \( M \models \text{\"I is a maximal antichain of } Q\text{\"} \),
   then \( M^I \) is a maximal antichain of \( Q \).
   
   Then we also have
   
   (α) \( Q' \) is \( (\kappa, \theta) \)-nep c.c.c. forcing notion,
   (β) if \( Q \) is simple, then \( Q' \) is simple,
   (γ) if \( Q \) is \( K \)-local, then \( Q' \) is \( K \)-local.

**Proof** Straight.

Now the hypothesis (\ast) in 7.5(2) is undesirable, so we use \( B_3(Q, \eta) \) (see 7.6(c) below), which has a suitable quasi order.

**Proposition 7.6.** Assume that:

(a) \( Q \) is explicitly nep forcing notion which satisfies the c.c.c.
(b) \( \vdash^Q \eta \in \sigma \omega \) and \( \psi_{\alpha,m} \in \text{cl}_2(Q) \) are such that
\[
\vdash^Q \eta(\alpha) = m \iff \psi_{\alpha,m}[G^Q] = t,
\]
(c) \( Q' \) \( \overset{\text{def}}{=} B_3(Q, \eta) \) is a forcing notion defined as follows:
the set of elements is like \( B_2(Q, \eta) \); i.e. it is the closure of \( \{ \psi_{\alpha,m} : \alpha < \sigma, m < \omega \} \text{ under } \neg, \bigwedge_{i<\gamma} \text{ for } \gamma < \omega_1 \text{ inside } \text{cl}_2(Q) \);
the quasi order \( \leq^Q_3 \overset{\text{def}}{=} \leq^3 \) is \( \leq^3 \) restricted to \( B_3(Q, \eta) \),
(d) the statements (\( \otimes_3 \)) and (\( \otimes_4 \)) of 6. hold.

Then:

(α) \( Q' \) is essentially a suborder of \( \text{cl}_2(Q) \); i.e. \( \psi \in Q' \Rightarrow \psi \in \text{cl}_2(Q), \) and for \( \psi_1, \psi_2 \in Q' \) we have: \( \psi_1 \leq_3 \psi_2 \iff \psi_1 \leq_{\text{cl}_2(Q)} \psi_2 \),
(β) \( \eta \) is a \( Q' \)-name, \( \vdash^Q \eta \in \sigma \omega \) and \( \eta \) is a generic function for \( Q' \),
(γ) \( Q' \) is explicitly nep c.c.c. forcing notion with \( \mathbf{B}_Q \overset{\text{def}}{=} \mathbf{B}, \varphi^Q = \varphi_{B_3(Q, \eta)}, g^Q = g^Q, \)
(γ) \( ^+ \) each forcing extension of \( V \) which preserves the assumption (a) (hence also (b)) preserves (γ),
(δ) if \( Q \) is simple (or straight) then \( Q' \) is simple (or straight),
(ε) if $\mathbb{Q}$ is $K$–local, then $\mathbb{Q}'$ is $K$–local.

**Proof** Straight. 

**Proposition 7.7.** In 7.1–7.6 above, we can replace $\mathbb{Q}$ by $\mathbb{Q}|\{p \in \mathbb{Q} : p \geq q\}$ preserving the properties of $(\mathbb{Q}, \eta)$.

**Fact 7.8.** If $\mathbb{Q}$ is simply correctly nep for $K$, $\mathbb{Q}$ is in $\mathbb{V}$, and $\mathbb{V}_1$ is a $K$–extension of $\mathbb{V}$ then

(i) in $\mathbb{V}_1$, $\mathbb{Q}^V \leq_{ic} \mathbb{Q}^{V_1}$ (see [17, Ch.IV]), i.e. for $p, q \in V_0$, “$p \leq q$”, “$(p \leq q)$”, “$p, q$ compatible”, “$p, q$ compatible” are preserved from $\mathbb{V}$ to $\mathbb{V}_1$,

(ii) for $p, p_n \in V$ the statements “$p \notin \mathbb{Q}$” and “$\mathcal{I} = \{p_n : n < \omega\}$ is predense above $p$ in $\mathbb{Q}$” are preserved from $\mathbb{V}$ to $\mathbb{V}_1$,

(iii) if $\mathbb{Q}$ satisfies the c.c.c. then in clause (ii) above we can omit the countability of $\mathcal{I}$.

**Proof** Straight, for example:

“$p, q$ are incompatible” iff there is no $\mathbb{Q}$–candidate $M$ such that

$M \models \text{“} p, q \text{ have a common } \leq_{\mathbb{Q}} \text{–upper bound”}.$

So by Shoenfield–Levy absoluteness, if this holds in $\mathbb{V}$, it holds in $\mathbb{V}_1$.

(ii) Similarly.

(iii) Follows (and repeated in 8.11).

**Proposition 7.9.** Let $(\mathbb{Q}, \eta)$ be temporarily explicitly nep pair. Assume $N$ is a $\mathbb{Q}$–candidate. If $N \models \text{“} \eta^* \text{ is } (\mathbb{Q}, \eta)\text{–generic over a } \mathbb{Q} \text{–candidate } M \text{”}$, then $\eta^*$ is a $(\mathbb{Q}, \eta)$–generic for $M$.

**Proof** Straight.

**Proposition 7.10.** Assume that:

(a) $\mathbb{Q}$ is explicitly nep,

(b) $\mathbb{Q}$ is c.c.c. moreover it satisfies the c.c.c. in every $\mathbb{Q}$–candidate,

(c) incompatibility in $\mathbb{Q}$ is upward absolute from $\mathbb{Q}$–candidates (but see 7.8),

(d) $\eta$ is a $hc_{\kappa(\mathbb{B}^\mathbb{Q})}$–$\mathbb{Q}$–name of a member of $\omega^\omega$ defined from $\mathbb{B}^\mathbb{Q}$ (so we demand this in every $\mathbb{Q}$–candidate).

Furthermore, suppose that

(A) $N_1, N_2$ are $\mathbb{Q}$–candidates, $N_2$ is a generic extension of $N_1$ for a forcing notion $\mathbb{R}$, (so $\mathbb{B}^{N_2} = \mathbb{B}^{N_1}$ and $N_1 \models \text{“} \mathbb{R} \text{ is a forcing notion”}$),

(B) $N_1 \models \text{“} for every countable } X \subseteq \mathbb{Q} \text{ and } n < \omega \text{ there is a } \mathbb{Q}\text{-candidate } N_0 \prec_{\Sigma_n} N_1 \text{ to which } X \text{ and } \mathbb{R} \text{ belong”}.$
(C) \( \eta^* \in \omega \omega \) is a \((Q, \eta)\)-generic real over \( N_2 \).

Then \( \eta^* \in \omega \omega \) is a \((Q, \eta)\)-generic real over \( N_1 \).

**Remark 7.11.**

1. In (B), we can replace \( X \) by “a maximal antichain of \( Q, \eta \).”

2. Clearly we can replace “maximal antichain” by “predense set” or “pre-dense set over \( p \)” (note \( I^{N_2} = I^{N_1} \) as \( N_2 = N_1^R \)).

3. We can weaken “\( N_0 \prec_{\Sigma_n} N_1 \)” in clause (B).

**Proof of 7.10**

Clearly it suffices to prove that (assuming (a)-(d), (A), (B) and (C)):

\((*)\) if \( N_1 \models “I \text{ is a maximal antichain of } Q” \),

then \( I^{N_1} = I^{N_2} \) and \( N_2 \models “I \text{ is a maximal antichain of } Q” \).

Assume that this fails for \( I \). Then some \( r \in R \) forces this failure (in \( N_1 \)). By assumption (b), in \( N_1 \) the set \( I^{N_1} \) is countable so let \( N_1 \models “I = \{ p_n : n < \alpha \}” \), where \( \alpha \leq \omega \). Let \( n < \omega \) be large enough. By clause (B) in \( N_1 \) there is a \( Q \)-candidate \( N_0 \) to which \( I \) and \( r \) and \( R \) belong and \( N_0 \prec_{\Sigma_n} N_1 \).

Since

\[ N_1 \models “(\exists r \in R)[r \force_R “I \text{ is not a maximal antichain of } Q \text{ (and } N_1[G_R] \text{ is a } Q \text{-candidate”)”}” \],

there is \( r_0 \in R \cap N_0 \) such that

\[ N_0 \models “[r_0 \force_R “I \text{ is not a maximal antichain of } Q \text{ (and } N_0[G_R] \text{ is a } Q \text{-candidate”)”}” \).

Now, as \( N_1 \) satisfies enough set theory and \( N_1 \) “thinks” that \( N_0 \) is countable and \( R^{N_0} \) is a forcing notion in \( N_0 \), there is in \( N_1 \) a subset \( G_R^r \) of \( R \cap N_0 = R^{N_0} \) generic over \( N_0 \) to which \( r_0 \) belongs. So in \( N_0[G_R^r] \) there is \( p \in Q^{N_0[G_R^r]} \) incompatible (in \( Q^{N_0[G_R^r]} \)) with each \( p_n \). By the assumption (c) this holds in \( N_1 \), contradiction to the choice of \( I \) (see \((*)\)).

**Definition 7.12.**

1. We say that \( \tilde{\varphi} \) or \((\tilde{\varphi}, \mathcal{B})\) is a temporary \((\kappa, \theta)\)-definition of a strong c.c.c.-nep forcing notion \( Q \) if:

   (a) \( \varphi_0 \) defines the set of elements of \( Q \) and \( \varphi_0 \) is upward absolute from \((\mathcal{B}, \tilde{\varphi}, \theta)\)-candidates,

   (b) \( \varphi_1 \) defines the partial ordering of \( Q \) (even in \((\mathcal{B}, \tilde{\varphi}, \theta)\)-candidates) and \( \varphi_1 \) is upward absolute from \((\mathcal{B}, \tilde{\varphi}, \theta)\)-candidates,

   (c) for any \((\mathcal{B}, \tilde{\varphi}, \theta)\)-candidate \( N \), if \( N \models “I \subseteq Q \text{ is predense”} \), then also in \( V, I^N \) is a predense subset of \( Q \).

2. We say that \( \tilde{\varphi} \) or \((\tilde{\varphi}, \mathcal{B})\) is a temporarily [explicitly] \((\kappa, \theta)\)-definition of a c.c.c.-nep forcing notion \( Q \) if
(α) it is a temporary [explicitly] \((\kappa, \theta)\)-definition of a nep forcing notion,
(β) for every \(Q\)-candidate \(N\) we have \(N \models \text{“}Q\text{ satisfies the c.c.c.”}\).
3. The variants are defined as usual.

**Proposition 7.13.** 1. If \(Q\) is strongly c.c.c.–nep forcing notion and \(N_1 \subseteq N_2\) are \(Q\)-candidates, then every \(\eta\) which is \((Q, \eta)\)–generic over \(N_2\) is also \((Q, \eta)\)–generic over \(N_1\).
2. If \(ZFC^-\) is normal and \(Q\) is temporarily c.c.c.–nep then \(Q\) satisfies the c.c.c.

**Comment 7.14.** We can spell out various absoluteness, e.g.
1. If \(Q\) is simple nep, c.c.c. and “\(\langle p_n : n < \omega \rangle\) is predense” has the form 
   \((\exists t \in H_{\leq \aleph_1}((\kappa+\theta)))\mathbb{V} \models t = \ldots \}\) (e.g. \(\kappa^Q = \omega\) and it is \(\Pi^1_2\)) then predensity of countable sets is preserved in any forcing extension.
2. Note that strong c.c.c.–nep (from 7.13(1)) does not imply c.c.c.–nep (from 7.13(2)). But if \(ZFC_{**}^- \vdash ZFC_{**}^-\) and \(ZFC_{**}^-\) says that \(ZFC_{**}^-\) is normal and \(Q\) is strong c.c.c.–nep for \(ZFC_{**}^-\), then \(Q\) is c.c.c.–nep for \(ZFC_{**}^-\).

7. **Preserving a little implies preserving much.** Our main intention is to show that, for example if a “nice” forcing notion \(P\) satisfies \(\mathbb{V} \models \text{“}P \text{“}(\omega^2)\text{ is not null}\)”, then it preserves “\(X \subseteq \omega^2 (X \in \mathbb{V})\) is not null”.

By Goldstern Shelah ([17, Ch.XVIII, 3.11]) if a Souslin proper forcing preserves “\((\omega, \omega)\text{ is non-meagre}\) then it preserves “\(X \subseteq \omega, \omega\text{ is non-meagre}\) and more (in a way suitable for the preservation theorems there).

The main question not resolved there was: is it special for Cohen forcing (which is a way to speak on non-meagre), or it holds for nice c.c.c. forcing notions in general, in particular does a similar theorem hold for “non-null” instead of “non-meagre”. Though there have been doubts about it, we succeed to do it here. In fact, even for a wider family of forcing notions but we have to work more in the proof.

See §11 on a generalization. The reader may concentrate on the case that \(Q\) is strongly c.c.c. nep and \(P, Q\) are explicitly \(\aleph_0\)–nep and simple. It is natural to assume that \(\eta\) is a generic real for \(Q\) but we do not ask for it when not used.

**Convention 8.1.** 1. \(Q\) is an explicitly nep forcing notion.
2. \(\eta \in \omega \omega\) is a hereditarily countable \(Q\)-name which is \(\mathcal{B}\)-definable.

We would like to preserve something like: “\(x\) is \(Q\)-generic over \(N\)”.
Definition 8.2. 1. $I_{(Q,η)}^{\text{ex}} \overset{\text{def}}{=} \{ A \in \text{Borel}(\omega, \omega) : \not\models_{Q} "η \notin A" \}$ (it is an ideal on the Boolean algebra of Borel subsets of $\omega, \omega$).

2. $I_{(Q,η)}^{\text{ex}}$ is the ideal generated by $I_{(Q,η)}$ on $\mathcal{P}(\omega, \omega)$. (So for $A \in \text{Borel}(\omega, \omega)$ we have: $A \in I_{(Q,η)}$ if and only if $A \in I_{(Q,η)}^{\text{ex}}$). Let

$I_{(Q,η)}^{dx} \overset{\text{def}}{=} \{ X \subseteq \omega, \omega : \text{for a dense set of } q \in Q, \text{for some Borel set } B \subseteq \omega, \omega, \text{ we have } X \subseteq B \text{ and } q \not\models_{Q} "η \notin B" \}.$

3. For an ideal $I$ (on Borel sets, respectively), the family of $I$–positive (Borel, respectively) sets is denoted by $I^{+}$.

(Thus, for a Borel subset $A$ of $\omega, \omega$, $A \in I_{(Q,η)}^{+}$ iff there is $q \in Q$ such that $q \not\models_{Q} "η \in A"$.

Definition 8.3. 1. A forcing notion $\mathbb{P}$ is $I_{(Q,η)}$–preserving if for every Borel set $A$

$$A \in (I_{(Q,η)})^{+} \implies \not\models_{\mathbb{P}} "A^{\mathbb{V}} \in (I_{(Q,η)}^{\text{ex}})^{+}".$$  

($A^{\mathbb{V}}$ means: the same set, i.e. $A \cap \mathbb{V}$).

2. $\mathbb{P}$ is strongly $I_{(Q,η)}$–preserving if for all $X \subseteq \omega, \omega$ (i.e. not only Borel sets)

$$X \in (I_{(Q,η)}^{dx})^{+} \implies \not\models_{\mathbb{P}} "X \in (I_{(Q,η)}^{\text{ex}})^{+}".$$  

[See \S 3 (7) for $Q$ which is c.c.c.]

3. We say that a forcing notion $\mathbb{P}$ is weakly $I_{(Q,η)}$–preserving if $\not\models_{\mathbb{P}} "(\omega, \omega)^{\mathbb{V}} \in (I_{(Q,η)}^{\text{ex}})^{+} \".$

4. $\mathbb{P}$ is super–$I_{(Q,η)}$–preserving if for all $X \subseteq \omega, \omega$ we have:

$$X \in (I_{(Q,η)}^{dx})^{+} \implies \not\models_{\mathbb{P}} X^{\mathbb{V}} \in (I_{(Q,η)}^{\text{ex}})^{+}.$$  

Proposition 8.4. 1. $I_{(Q,η)}$ is an $\aleph_{1}$–complete ideal (in fact, if $\langle A_{i} : i \leq \alpha \rangle \in \mathbb{V}$, each $A_{i} \in \text{Borel}(\omega, \omega)$ and $\not\models_{Q} "\bigcup_{i<\alpha} A_{i}^{\mathbb{V}} \subseteq \bigcup_{i<\alpha} A_{i}^{\mathbb{V}} \"$ and $A_{i} \in I_{(Q,η)}$ for $i < \alpha$ then $A_{\alpha} \in I_{(Q,η)}$).

2. If $(Q,η)$ is not trivial (i.e. $\not\models_{Q} "(\omega, \omega)^{\mathbb{V}} \"$, then singletons belong to $I_{(Q,η)}$.

3. $\omega, \omega \notin I_{(Q,η)}$.

4. Assume (ZFC* is K–good and) $\mathbb{Q}$ is correct. If in $\mathbb{V}$, $X \in I_{(Q,η)}^{\text{ex}}$ and $\mathbb{P} \in K$, then in $\mathbb{V}^{\mathbb{P}}$ still $X \in I_{(Q,η)}^{\text{ex}}$ (but see later).
5. Assume \( \text{ZFC}_\small{Q}^- \) is \( K \)-good, particularly (c) of \( \text{ZFC}_\small{Q}^+ \) and \( Q \) is correct. If, in \( V \), \( B \) is a Borel subset of \( \omega \omega \), then \( I_{(Q,\eta)} \) and \( V_1 = V^P \) then also \( V_1 \models \text{"}B \in I_{Q,\eta}\text{"} \).

6. \( I_{(Q,\eta)}^{ex} \), \( I_{(Q,\eta)}^{dx} \) are ideals of \( \mathcal{P}(\omega \omega) \) and

\[
I_{(Q,\eta)}^{ex} \upharpoonright (\text{the family of Borel sets}) = I_{(Q,\eta)}^{dx} \upharpoonright (\text{the family of Borel sets}).
\]

7. If \( Q \) satisfies the c.c.c. then \( I_{(Q,\eta)}^{dx} \) is generated by \( I_{(Q,\eta)}^{ex} \), so equal to \( I_{(Q,\eta)}^{ex} \).

8. \( I_{(Q,\eta)}^{ex} \) is \( \aleph_1 \)-complete.

9. If for some stationary \( S \subseteq [\chi]^{\aleph_0} \), \( Q \) is \( S \)-proper then \( I_{(Q,\eta)}^{dx} \) is \( \aleph_1 \)-complete.

**Proof** We will prove parts 5) and 4) only, the rest is left to the reader.

5) First work in \( V^P \). If the conclusion fails then for some \( q \in Q \) we have \( q \Vdash \text{"}\eta \in B\text{"} \). So there is a \( Q \)-candidate \( M \) to which \( q, B \) (i.e. the code of \( B \)) belong. There is \( q' \) such that \( q \leq q' \) and \( q' \) is \( (M,Q) \)-generic. Now for every \( G \subseteq Q \) generic on \( V^P \), \( \eta[G] \in B^{V^P[G]} \). By absoluteness, also \( M(G) \models \eta(G \cap Q^M) \in B^{M(G)} \) and hence (by the forcing theorem) for some \( p \in G \cap Q^M \) we have \( M \models [p \Vdash \text{"}\eta \in B\text{"}] \). Now, returning to \( V \), by Shoenfield–Levy absoluteness there are such \( M', p' \) in \( V \). Let \( p'' \) be \( (M',Q) \)-generic, \( p' \leq Q \). So similarly to the above, \( p'' \Vdash Q \text{"}\eta \in B\text{"} \).

4) As \( X \in I_{(Q,\eta)}^{ex} \), clearly for some Borel set \( B \in I_{(Q,\eta)}^{dx} \) we have \( X \subseteq B \).

By part (5), also in \( V^P \) we have \( B \in I_{(Q,\eta)}^{ex} \) and trivially \( X \subseteq B^{V} \subseteq B^{V^P} \).

---

**Proposition 8.5.**

1. If a forcing notion \( \mathbb{P} \) is \( I_{(Q,\eta)}^{ex} \)-preserving, then \( \mathbb{P} \) is weakly \( I_{(Q,\eta)}^{ex} \)-preserving.

2. If \( \mathbb{P} \) is strongly \( I_{(Q,\eta)}^{ex} \)-preserving, then \( \mathbb{P} \) is \( I_{(Q,\eta)}^{ex} \)-preserving.

3. Assume that \( Q \) satisfies the c.c.c. and \( (Q,\eta) \) is homogeneous (see \( (*) \) below). Then: \( \mathbb{P} \) is \( I_{(Q,\eta)}^{ex} \)-preserving iff \( \mathbb{P} \) is weakly \( I_{(Q,\eta)}^{ex} \)-preserving, where

\[
(*) \quad (Q,\eta) \text{ is homogeneous if:}
\]

for any (Borel) sets \( B_1, B_2 \in (I_{(Q,\eta)})^+ \) we can find a Borel set \( B'_1 \subseteq B_1, B_1 \in (I_{(Q,\eta)})^+ \) and a Borel function \( F \) from \( B'_1 \) into \( B_1 \) such that

\[
(\alpha) \text{ for every Borel set } A \in I_{(Q,\eta)}, F^{-1}[A \cap B_2] \in I_{(Q,\eta)},
\]

\[
(\beta) \text{ this is absolute (or at least it holds also in } V^P \).
\]
Proof 3) By part (1) it suffices to show “non–preserving” assuming “not weakly preserving”. So there are \( p, B^*, A, q \) such that \( B \in (I_{(\mathcal{Q}, \eta)})^+ \) is a Borel subset of \( \omega^\omega \) and

\[
p \Vdash \text{“}(a) \text{ \( A \) is a Borel set (b) \( q \in Q \) (in \( V^P \)) (c) \( q \) witnesses \( A \in I_{(\mathcal{Q}, \eta)} \), that is \( q \Vdash \text{“} \eta \notin A \text{”} \) (d) \( \nu \in A \) for every \( \nu \in (B^*)^V \). “}
\]

Let \( \mathcal{J} = \{ B : B \in (I_{(\mathcal{Q}, \eta)})^+ \), so a Borel subset of \( \omega^\omega \), and for some Borel one-to-one function \( F \) from \( B \) to \( B^* \) we have \( F \) is absolutely \((I_{(\mathcal{Q}, \eta)})^+–\)preserving \}

Choose a maximal family \( \{ B_i : i < i^* \} \subseteq \mathcal{J} \) such that \( i \neq j \Rightarrow B_i \cap B_j \in I_{(\mathcal{Q}, \eta)} \). As \( Q \) satisfies the c.c.c. necessarily \( i^* < \omega_1 \), so wlog \( i^* \leq \omega \). By the assumption, \( \omega^\omega \setminus \bigcup_{i < i^*} B_i \in I_{(\mathcal{Q}, \eta)} \). Let \( F_i \) witness that \( B_i \in \mathcal{J} \). Let

\[
A_i = \{ \eta \in \omega^\omega : \eta \in B_i \text{ and } F_i(\eta) \in A \}.
\]

Then \( A_i \) is a Borel subset of \( \omega^\omega \) and \( p \Vdash \text{“} A_i \in I_{(\mathcal{Q}, \eta)} \text{”} \) as \( p \Vdash \text{“} A \in I_{(\mathcal{Q}, \eta)} \text{”} \). Hence

\[
p \Vdash \text{“} \bigcup_{i < i^*} A_i \cup (\omega^\omega \setminus \bigcup_{i < i^*}) \in I_{(\mathcal{Q}, \eta)} \text{”}
\]

(call this set \( A^* \)). Now

\[
p \Vdash \text{“} (\omega^\omega)^V = (\omega^\omega \setminus \bigcup_{i < i^*} B_i)^V \cup \bigcup_{i < i^*} B_i^V \subseteq (\omega^\omega \setminus \bigcup_{i < i^*} B_i) \cup \bigcup_{i < i^*} A_i \in I_{(\mathcal{Q}, \eta)} \text{”}
\]

so we are done. ■

Comment: 1) It is easy to find a forcing notion \( \mathbb{P} \) which is \( I_{(\mathcal{Q}, \eta)} \)–preserving, but not strongly \( I_{(\mathcal{Q}, \eta)} \)–preserving, e.g. for \( Q = \text{Cohen} \) (see §8.10 below). However, for sufficiently nice forcing notion \( \mathbb{P} \), “\( I_{(\mathcal{Q}, \eta)} \)–preserving” and “strongly \( I_{(\mathcal{Q}, \eta)} \)–preserving” coincide, as we will see in §8.8 (Parallel to the phenomenon that for “nice” sets, CH holds).

2) It is even easier to find a weakly \( I_{(\mathcal{Q}, \eta)} \)–preserving forcing notion \( \mathbb{P} \) which is not \( I_{(\mathcal{Q}, \eta)} \)–preserving.

Assume that for \( \ell < 2 \) we have \( (\mathcal{Q}_\ell, \eta_\ell) \) as in §5.1, e.g. \( \mathcal{Q}_0 \) is Cohen forcing, \( \mathcal{Q}_1 \) is random real forcing. Let \( Q = \{ \emptyset \} \cup \bigcup_{\ell < 2} \{ \ell \} \times \mathcal{Q}_\ell \), \( \emptyset \) minimal, \( (\ell_1, q_1) \leq
(ℓ₂, q₂) iff ℓ₁ = ℓ₂ and ℚₗ |= q₁ ≤ q₂. We define a ℚ-name η by defining for a generic G ⊆ ℚ over V:

η[G] is  \begin{align*}
(0) \sim (η₀[G₀]) & \quad \text{if } \{0\} × ℚ₀ ∩ G \neq \emptyset, \text{ and } G₀ = \{q ∈ ℚ₀ : (0, q) ∈ G\}, \\
(1) \sim (η₁[G₁]) & \quad \text{if } \{1\} × ℚ₁ ∩ G \neq \emptyset, \text{ and } G₁ = \{q ∈ ℚ₁ : (1, q) ∈ G\}.
\end{align*}

Then usually (and certainly for our choice) we get a counterexample.

Proposition 8.6. Assume that A is a Borel subset (better: a definition of a Borel subset) of ω, M is a ℚ-candidate (so η ∈ M, i.e. \(ψ_{α,m} : α < ω, m < ω\) ∈ M) and \(A ∈ M\) (i.e. the definition). Further, suppose that \(q ∈ ℚ\) is such that \(q \|_{Q} \text{ “} η ∈ A \text{”} \). Then

(a) \(M \models “q \|_{Q} η ∈ A \” \),

(b) there is \(η ∈ A \) which is a \((ℚ, η)\)-generic real over M.

Proof As for (a), if it fails then for some \(q' ∈ ℚ\), we have

\(M \models “q ≤ Q q' \text{ and } q' \|_{Q} η ∉ A \” \),

and let \(r ∈ ℚ\) be \(⟨M, ℚ⟩\)-generic above \(q'\). So if \(G\) is a subset of \(ℚ\) generic over \(V\) to which \(r\) belongs then \(q' ∈ G\) and \(G ∩ ℚ\) is a subset of \(ℚ\) generic over \(M\) to which \(q'\) belongs. Hence \(M(G) \models “η[G ∩ ℚ] ∉ A \” \) and \(G[G ∩ ℚ] ∈ ωω\). By absoluteness also \(V(G) \models η[G ∩ ℚ] ∉ A \) and \(η[G ∩ ℚ] ∈ ωω\). But as \(η ∈ M\) clearly \(η[G ∩ ℚ] = η[G]\) and as \(q' ∈ G\) also \(q ∈ G\), so we get contradiction to \(q \|_{Q} η ∈ A \” \).

By clause (a) clause (b) is easy: we can find a subset \(G ∈ V\) of \(ℚ\) to which \(q\) belongs which is generic over \(M\). So \(η[G] ∈ ωω\) and it belongs to \(A\) as \(M \models “q \|_{Q} η ∈ A \” \). ☐

Proposition 8.7. Assume \(ℚ\) is correct and satisfies the c.c.c. The following conditions are equivalent for a set \(X ⊆ ωω\):

(A) \(X ∈ ℙ^∞(Q, η)\),

(B) for some \(ρ ∈ ω2\), for every ℚ-candidate \(N\) to which \(ρ\) belongs there is no \(η ∈ X \) which is \((ℚ, η)\)-generic over \(N\),

(C) for every \(p ∈ ℚ\) for some ℚ-candidate \(N\) such that \(p ∈ ℚ\) \(N\), there is no \(η ∈ X \) which is \((ℚ, η)\)-generic over \(N\).

Proof (A) ⇒ (B): So assume (A), i.e. \(X ∈ ℙ^∞(Q, η)\). Then for some Borel set \(A ∈ ℙ(Q, η)\) we have \(X ⊆ A\). Let \(ρ ∈ ω2\) code \(A\). Since \(A \models “η ∉ A \V[Gq] \” \), it follows from S.8 that

(*) for any ℚ-candidate \(N\) to which \(ρ\) belongs there is no \((ℚ, η)\)-generic real \(η\) over \(N\) which belongs to \(X\) (or even just to \(A\)).
(B) ⇒ (C): Easy as \( Q \) is correct.
(C) ⇒ (A): Assume (C). Let
\[
\mathcal{I} = \{ p \in \mathbb{Q} : \text{ for some Borel subset } A = A_p \text{ of } \omega_\omega \\
\text{we have } p \models \langle \eta \notin A_p \rangle \text{ and } X \subseteq A_p \}.
\]
Suppose first that \( \mathcal{I} \) is predense in \( \mathbb{Q} \). Clearly it is open, so we can find a maximal antichain \( \mathcal{J} \) of \( \mathbb{Q} \) such that \( \mathcal{J} \subseteq \mathcal{I} \). As \( \mathbb{Q} \) satisfies the c.c.c., necessarily \( \mathcal{J} \) is countable. So \( A \overset{\text{def}}{=} \bigcap_{p \in \mathcal{J}} A_p \) is a Borel subset of \( \omega_\omega \) (as \( \mathcal{J} \) is countable) and it includes \( X \) (as each \( A_p \) does). Moreover, since \( \mathcal{J} \) is a maximal antichain of \( \mathbb{Q} \) (and \( p \in \mathcal{J} \Rightarrow p \in \mathcal{I} \Rightarrow p \models \langle \eta \notin A_p \rangle \Rightarrow p \models \langle \eta \notin A_p \rangle \langle \eta \notin X \rangle \) we have \( p \models \langle \eta \notin A_p \rangle \). Consequently (A) holds.

Suppose now that \( \mathcal{I} \) is not predense in \( \mathbb{Q} \) and let \( p^* \in \mathbb{Q} \) exemplifies it, i.e. it is incompatible with every member of \( \mathcal{I} \). Let \( N \) be a \( \mathbb{Q} \)-candidate to which belongs some \( p \) given by the assumption (C) for \( p^* \). Thus \( p^* \in \mathbb{Q}^N \) and no \( \eta \in X \) is \( \langle \mathbb{Q}, \eta \rangle \)-generic over \( N \). Let \( q \) be a member of \( \mathbb{Q} \) which is above \( p^* \) and is \( \langle N, \mathbb{Q}^N \rangle \)-generic (i.e. \( q \models \langle G^P \cap \mathbb{Q}^N \rangle \text{ is generic over } N \) ). Let
\[
A \overset{\text{def}}{=} \{ \eta \in \omega_\omega : \text{ is not } \langle \mathbb{Q}, \eta \rangle \text{-generic over } N \}.
\]
Now
(a) \( A \) is a Borel subset of \( \omega_\omega \) and \( X \subseteq A \)
(why? as \( N \) is countable),
(b) \( q \models \langle \eta \notin A \rangle \)
(why? by the definition of \( A \)),
(c) \( q \in \mathcal{I} \)
(why? by (a)+(b)).
Thus \( p^* \leq q \in \mathcal{I} \) and we get contradiction to the choice of \( p^* \).

**Theorem 8.8.** Assume that:
(a) \( \mathbb{Q}, \eta \) are as above (see 8.4), and \( \mathbb{Q} \) is correct,
(b) \( P \) is nep-forcing notion with respect to our fixed version \( \text{ZFC}^*_\ast \),
(c) \( P \) is \( I(Q, \eta) \)-preserving,
(d) \( \text{ZFC}^*_\ast \) is a stronger version of set theory including clauses (i)–(v) below for some \( \chi_1 < \chi_2 \),
   (i) \( (H(\chi_2), \in) \) is a (well defined) model of \( \text{ZFC}^*_\ast \),
   (ii) \( (a), (b) \) and \( (c) \) (with \( B^P, B^Q, \eta \) as individual constants),
   (iii) \( Q, P \in H(\chi_1) \) and \( (H(\chi_2), \in) \) is a semi-\( P \)-candidate and a semi-\( Q \)-candidate with \( (B^P) \) interpreted as \( (B^P)^N \upharpoonright H(\chi_2)^N \) and similarly for \( Q \), so (natural to assume) \( B^P, B^Q \in H(\chi_2) \),
(remember, “semi” means omitting the countability demand)
(iv) forcing of cardinality \( \chi_1 \) preserves the properties (i), (ii), (iii), and \( \chi_1 \) is a strong limit cardinal,
(v) forcing by $\mathbb{P}$ preserves “$\mathcal{I}$ is a predense subset of $\mathbb{Q}$” (follows if $\mathbb{Q}$ satisfies the c.c.c. by 7.8(ii)).

Then:

(α) if, additionally,

(c) ZFC$^{\ast\ast}$ is normal (see Definition 2.14(3))

then $\mathbb{P}$ is strongly $I_{(\mathbb{Q},\bar{\eta})}$-preserving,

(β) if $N$ is a $\mathbb{P}$–candidate (and $\mathbb{Q}$–candidate) and moreover it is a model of ZFC$^{\ast\ast}$ and $N \models \text{“} p \in \mathbb{P} \text{” and } \eta^* \text{ is } (\mathbb{Q},\bar{\eta})$–generic over $N$,

then for some $q$ we have:

(i) $p \leq q$ and $q \in \mathbb{P}$,

(ii) $q$ is $\langle N,\mathbb{P} \rangle$–generic; i.e. $q \Vdash \text{“} G^\mathbb{P} \cap \mathbb{P}^N \text{ is generic over } N \text{” (see 5.1)}$,

(iii) $q \Vdash \text{“} \eta^* \text{ is } (\mathbb{Q},\bar{\eta})$–generic over $N \lbrack \mathbb{P}^N \cap G^\mathbb{P} \rbrack \text{”}$.

(α)$^+$ We can strengthen the conclusion of (α) to “$\mathbb{P}$ is super–$I_{(\mathbb{Q},\bar{\eta})}$–preserving”.

Remark 8.9. 1) We consider, for a nep forcing notion $\mathbb{Q}$

(*)$_1$ $\mathbb{Q}$ satisfies the c.c.c.

We also consider

(*)$_2$ being a predense subset (or just a maximal antichain) of $\mathbb{Q}$ is $K$–absolute.

By results of the previous section, (*)$_1$ $\Rightarrow$ (*)$_2$ under reasonable conditions. You may wonder whether (*)$_2$ $\Rightarrow$ (*)$_1$, but by the examples in section 11 the answer is not.

2) Note that in (α), (α)$^+$ we can use the weak normality if $\mathbb{Q}$ satisfies the c.c.c., see 8.11. We do not use “$\mathbb{P}$ is explicitly nep” so we do not demand it.

Before we prove the theorem, let us give an example for a forcing notion failing the conclusion and see why many times we can simplify assumptions.

Example 8.10. Start with $\mathbb{V}_0$. Let $\bar{s} = \langle s_i : i < \omega_1 \rangle$ be a sequence of random reals, forced by the measure algebra on $\omega_1(\omega_2)$. Let $\mathbb{V}_1 = \mathbb{V}_0[\bar{s}]$, $\mathbb{V}_2 = \mathbb{V}_1[r]$, $r$ a Cohen over $\mathbb{V}_1$ and

$\mathbb{V}_3 = \mathbb{V}_2[\bar{t}]$ where $\bar{t} = \langle t_i : i < \omega_1 \rangle$ is a sequence of random reals forced by the measure algebra.

Then in $\mathbb{V}_3$ (in fact, already in $\mathbb{V}_2$), $\{s_i : i < \omega_1\}$ is a null set, whereas $\{t_i : i < \omega_1\}$ is not null. But $\bar{t}$ is also generic for the measure algebra over $\mathbb{V}_1$. So $\mathbb{V}_2' = \mathbb{V}_1[\bar{t}]$ is a generic extension of $\mathbb{V}_1$. We have $\mathbb{V}_3 = \mathbb{V}_2'[r]$, where $r$ is generic for some algebra, more specifically for

$\mathbb{R} \overset{\text{def}}{=} (\text{Cohen} \ast \text{measure algebra adding } \bar{t})/\bar{t}$. 

So in $\mathbf{V}'_2$ the sets $t$ and $s$ are not null and $\mathbb{R}$ makes $s$ null, but not $t$.

How can $\mathbb{R}$ do that? $\mathbb{R}$ uses $\langle t_i : i < \omega_1 \rangle$ in its definition, so it is not “nice” enough.

**Remark** In the proof of 8.8 of course, we may assume $N < (\mathbb{H}(\chi, \varepsilon))$ if $(\mathbb{H}(\chi, \varepsilon)) \models \text{ZFC}_{\text{ns}}$, as this normally holds. In (α) the use of such $N$ does not matter. In (β) it slightly weakens the conclusion. Now, (α) is our original aim. But (β) both is needed for (α) and is a step towards preserving them (as in [17]). So typically $N$ is an elementary submodel of appropriate $\mathbb{H}(\chi)$.

**Proof of 8.8** Clause (α): To prove (α) we will use (β). So let $X \subseteq \omega_\omega, X \in (I_{(\omega, \omega)})^+$. Then there is a condition $q^* \in \mathbb{Q}$ such that

\[ (\ast)_1 \text{ for no Borel subset } B \text{ of } \omega_\omega \text{ do we have: } \quad X \subseteq B \text{ and } q^* \not\models \text{“}\eta \notin B” \text{.} \]

Let $\chi$ be large enough. We can find $N \subseteq (\mathbb{H}(\chi, \varepsilon))$ as in (β), moreover $N \succ (\mathbb{H}(\chi, \varepsilon))$ a model of ZFC$_{\text{ns}}$ (and so a $\mathbb{P}$–candidate and a $\mathbb{Q}$–candidate) if exists because by clause (e) of the assumptions, ZFC$_{\text{ns}}$ is normal so for $\chi$ large enough any countable $N \prec (\mathbb{H}(\chi, \varepsilon))$ to which $\mathfrak{B}^\mathbb{Q}, \mathfrak{B}^\mathbb{P}$ belong is a model of ZFC$_{\text{ns}}$ and is a $\mathbb{P}$–candidate and a $\mathbb{Q}$–candidate, so as required.

Towards a contradiction, assume $p^* \in \mathbb{P}$ and $p^* \forces X \in I_{(\omega, \omega)}$. So for some $\mathbb{P}$–name $A$ we have

\[ p^* \not\models \text{“} A \text{ is a Borel subset of } \omega_\omega, X \subseteq A \text{ and } A \in I_{(\omega, \omega)} \text{, i.e. } \not\models \text{“}\eta \notin A” \text{.} \]

Without loss of generality the name $A$ is hereditarily countable and $A, p^*, q^*$ belong to $N$. In $\mathbf{V}$, let

\[ B = \{ \eta \in \omega_\omega : \theta \text{ is a } (\mathbb{Q}^{\ge q^*}, \eta) \text{–generic real over } N, \text{ which means: } \eta = \eta[G] \text{ for some } G \subseteq \mathbb{Q}^N \text{ generic over } N \text{ such that } q^* \in G \} \]

Clearly, it is an analytic set (if $\eta$ was generic real then Borel; both holds as “$\eta$ is a generic real for $Q$” follows from ZFC$_{\text{ns}}^\mathbb{Q}$). So $B = \bigcup_{i<\omega_1} B_i$, each $B_i$ is Borel. Let $q \in \mathbb{Q}$ be $(N, \mathbb{Q})$–generic and $q^* \le q$. Then $q \not\models \text{“}\eta \in B”$ and hence wlog for some $i < \omega_1$ we have $q \not\models \text{“}\eta \in B_i”$. Since $q \not\models \text{“}\eta \notin (\omega_\omega \setminus B_i)”$ (as $q^* \le q, q^* \not\models \text{“}\eta \in B_i”$), we may apply $(\ast)_1$ to the set $\omega_\omega \setminus B_i$ to conclude that $X \subseteq \omega_\omega \setminus B_i$. Take $\eta^* \in X \cap B_i$ (so it is $(\mathbb{Q}, \eta)$–generic over $N$). So by clause (β) (proved below), there is a condition $p \in \mathbb{P}, p \ge p^*$ which is $(\mathbb{N}, \mathbb{P})$–generic (i.e. it forces that $G_p \cap \mathbb{P}^N$ is generic over $N$, not necessarily $G_p \cap N$) and such that

\[ p \forces \text{“} \eta^* \text{ is } (\mathbb{Q}, \eta) \text{–generic over } N[G_p \cap \mathbb{P}^N] \text{”} \]
Choose \( G_\mathbb{P} \subseteq \mathbb{P} \), generic over \( \mathbb{V} \), such that \( p \in G_\mathbb{P} \). In \( \mathbb{V}[G_\mathbb{P}] \), \( N[G_\mathbb{P} \cap \mathbb{P}^N] \) is a generic extension of \( N \) (for \( \mathbb{P}^N \)), a \( \mathbb{Q} \) candidate, and \( \eta^* \) is \((\mathbb{Q}, \eta)\)-generic over it. As \( p^* \leq p \in G_\mathbb{P} \), clearly if \( G_\mathbb{Q} \subseteq \mathbb{Q}^\mathbb{V}[G_\mathbb{P}] \) is generic over \( \mathbb{V}[G_\mathbb{P}] \) then \( \eta[G_\mathbb{Q}] \notin A[G_\mathbb{P}] \). But \( N[G_\mathbb{P} \cap \mathbb{P}^N] \prec (\mathcal{H}(\chi)^{\mathbb{V}[G_\mathbb{P}]}, \in) \), so \( N[G_\mathbb{P} \cap \mathbb{P}^N] \) satisfies the parallel statement. Since \( \eta^* \) is \((\mathbb{Q}, \eta)\)-generic over \( N[G_\mathbb{P} \cap \mathbb{P}^N] \), it cannot belong to \( A[G_\mathbb{P} \cap \mathbb{P}^N] \). But easily \( A[G_\mathbb{P} \cap \mathbb{P}^N] = A[G_\mathbb{P}] \) and hence, by absoluteness, \( \eta^* \in X \subseteq A[G_\mathbb{P}] \), a contradiction. This ends the proof of 8.8 clause (α).

Clause (α)^+:

Like the proof of clause (α). We start like there but now we choose functions \( r^*, A^*, \mathcal{I} \) such that

\[ (*)_2 \text{ Dom}(r^*) = \text{Dom}(\mathcal{I}) \text{ is the set of all hereditarily countable canonical } \mathbb{P}\text{-names for elements of } \mathbb{Q} \text{ (so it is a member of } \mathcal{H}_{<\aleph_1}(\kappa(\mathbb{P}) + \kappa(\mathbb{Q})), \text{ and} \]

\[ \text{Dom}(A^*) = \{(p,q) : p \in \mathcal{I}(q), q \in \text{Dom}(r^*)\}, \]

\[ (*)_3 \text{ for each } q \in \text{Dom}(r^*) = \text{Dom}(\mathcal{I}), \mathcal{I}(q) \text{ is a predense subset of } \mathbb{P} \text{ such that for each } p \in \mathcal{I}(q) \text{ we have:} \]

\[ \begin{align*}
    p \Vdash_{\mathbb{P}} & \text{“ } A^*(q) \text{ is a Borel subset of } \omega^\omega \text{”}, \\
    p \Vdash_{\mathbb{P}} & \text{“ } r^*(q) \Vdash_{\mathbb{Q}} \text{“ } \eta \notin A^*(q) \text{”}, \\
    p \Vdash_{\mathbb{P}} & \text{“ } X \subseteq A^*(q) \text{”}. 
\end{align*} \]

Without loss of generality, the set \( X \), and the functions \( r^*, A^*, \mathcal{I} \) belong to \( N \). We choose conditions \( q \in \mathbb{Q} \), \( p \in \mathbb{P} \) and a real \( \eta^* \in X \) and a generic filter \( G_\mathbb{P} \subseteq \mathbb{P} \) over \( \mathbb{V} \) in a similar manner as in clause (α). We note that

\[ q \in \text{Dom}(r^*) \cap N \implies N \cap \mathcal{I}(q) \cap G_\mathbb{P} \neq \emptyset, \]

so say \( p[q] \in G_\mathbb{P} \cap N \). Since \( \eta^* \) is \((\mathbb{Q}, \eta)\)-generic over \( N[G_\mathbb{P} \cap \mathbb{P}^N] \), there is \( G^* \subseteq \mathbb{Q}^N[G_\mathbb{P} \cap \mathbb{P}^N] \) generic over \( N \) such that \( \eta^* = \eta[G^*] \). By the choice of \( r^*, A^* \) there is \( q \in N \cap \text{Dom}(r^*) \) such that \( r^*[q][G^*] \cap \mathbb{P}^N \in G^* \). Now, \( A = A^*(p[q], q) \in N[G_\mathbb{P} \cap \mathbb{P}^N] \) is a Borel subset of \( \omega^\omega \) and \( N[G_\mathbb{P} \cap \mathbb{P}^N] \models \text{“} \forall x \in A \text{”} \), hence \( N[G_\mathbb{P} \cap \mathbb{P}^N] \models \text{“} \exists \eta[G^*] \notin A \text{”} \). But

\[ N[G_\mathbb{P}] = N[G_\mathbb{P} \cap \mathbb{P}^N] \models \text{“ } X \setminus A = \emptyset \text{”}, \]

contradicting \( \eta^* = \eta[G^*] \in X \setminus A \).

Clause (β):

So \( N, \eta^*, \mathbb{Q}, \mathbb{P}, p \) are given. Let \( N_1 = N[G^*] \) be a generic extension of \( N \) by a subset \( G^* \) of \( \mathbb{Q}^N \) generic over \( N \) and such that \( \eta^* = \eta[G] \) (see 7.2). Now choose (in \( N \)) a model \( M \prec (\mathcal{H}(\chi_2), \in)^N \) such that

(i) \( \mathbb{P}, \mathbb{Q}, \eta, p \in M \),
(ii) \( \mathbb{Q}^N \subseteq M \) and \( \mathbb{P}^N \subseteq M \),
(iii) the family of maximal antichains of $P$ and of $Q$ from $N$ are included in $M$.
(iv) $M \in N$, moreover $M \in H(\chi_2)^N$.
(v) $M \models$ “forcing by $P$ preserves predensity of subsets of $Q$”

[Why is clause (v) possible? As $N \upharpoonright H(\chi_1)$ inherits clause (v) of (d) of the assumptions].

Hence, by assumption (d),

(\otimes) $M$ is a $P$-candidate and a $Q$-candidate and

$N \models$ “$M$ is a semi $P$-candidate and semi $Q$-candidate”.

Let $R = \text{Levy}(\aleph_0, |M|)$. In $V$ let $G_R \subseteq R$ be generic over $N_1 = N[G^*]$ (note that as $N_1$ is countable, clearly $G_R$ exists) and let $N_2 = N_1[G_R]$ (note that it too is a $P$-candidate and a $Q$-candidate).

Note: $\eta^*$ is $(Q, \eta)$-generic over $M$ too and $G^*$ is a subset of $Q^M$ generic over $M$ (by clauses (ii) + (iii)) and $Q^N = Q^M$, $P^N = P^M$ (note that in $N_2$ the model $M$ is countable).

Now we ask the following question:

Is there $q \in P^{N_2}$ such that

$N_2 \models$ “$p \leq^P q$, $q$ is $(M, P^M)$-generic and $q \Vdash_P “ \eta^*$ is $(Q, \eta)$-generic over $M[G_P \cap P^M]” ”?

Depending on the answer, we consider two cases.

Case 1: The answer is “yes”.

Choose $q^* \in P$, $q^* \geq q$, $q^*$ is $(N_2, P^{N_2})$-generic. Then we have

$q^* \Vdash_P “$ in $V[G_P]$, $G_P \cap P^{N_2}$ is generic over $N_2$, $p, q \in G_P$, and

in $N_2[G_P \cap P^{N_2}]$, $\eta^*$ is $(Q, \eta)$-generic over $M[G_P \cap P^M]$, hence also over $N[G_P \cap P^N]$ ”.

[Why does $q^*$ force this? As:
(A) “$G_P \cap P^{N_2}$ is generic over $N_2$” holds because $q^*$ is $(N_2, P^{N_2})$-generic;
(B) “$p, q \in G_P$” holds as $p \leq q \leq q^* \in G_P$ (forced by $q^*$);
(C) “in $N_2[G_P \cap P^{N_2}]$, $\eta^*$ is $(Q, \eta)$-generic over $M[G_P \cap P^M]$” holds because of the choice of $q$ (i.e. the assumption of the case ad as $q \in G_P$);
(D) “$\eta^*$ is $(Q, \eta)$-generic over $N[G_P \cap P^N]$ for $Q^N$ holds by clause (C) above and clause (iii) of the choice of $M$.]

By absoluteness we can omit the “in $N_2[G_P \cap P^M]”$, i.e. $q^* \Vdash_P “ \eta^*$ is $(Q, \eta)$-generic over $N[G_P \cap P^N]$”.

So $q^*$ is as required.

Case 2: The answer is “no”.

Let $\psi(x)$ be the following statement:
there is no $q$ such that:

\[ q \in \mathbb{P}, \, \mathbb{P} \models \text{"}p \leq q\text{"}, \quad q \text{ is } (M, \mathbb{P}^M)-\text{generic and } q \forces \text{"} x \text{ is a (Q, } \eta )-\text{generic real over } M[G_\mathbb{P} \cap \mathbb{P}^M]\text{".}\]

So $\psi$ is a first order formula in set theory, all parameters are in $N_1 = N[G^*] \subseteq N_2 = N[G^*][G_{\mathbb{R}}]$, and by the assumption of the case

\[ N[G^*][G_{\mathbb{R}}] \models \psi[\eta^*]. \]

Since $\mathbb{R}$ is homogeneous we may assume that $r = \emptyset$. As $G_\mathbb{R} \subseteq \mathbb{R}$ is generic over $N[G^*]$ for $\mathbb{R}$, necessarily (by the forcing theorem), for some $r \in G_\mathbb{R}$

\[ N[G^*] \models \text{"} r \forces \psi[\eta^*] \text{".} \]

So necessarily, for some $q \in G^* \subseteq Q^\mathbb{R} = Q^\mathbb{M}$ we have

\[ N \models (q \forces [r \forces \psi(\eta[G_Q]))). \]

Now $\mathbb{R} \in N$ (as it members are finite sets of pairs of ordinals) so

\[ (\otimes) \quad N \models ((q, r) \forces_{Q \times \mathbb{R}} \psi(\eta[G_Q])). \]

Next, $N[G_{\mathbb{R}}]$ is a generic extension by a “small” forcing of $N$ which is a model of $\text{ZFC}_{**}$, so $N[G_{\mathbb{R}}]$ satisfies (i), (ii) and (iii) of the clause (d) of the assumptions. Note that $N \models \text{"} M$ is a semi $\mathbb{Q}$-candidate and a semi $\mathbb{P}$-candidate”, see clause (d)(iii) of the assumptions and the choice of $M$, so also $N[G_{\mathbb{R}}]$ satisfies this. Moreover, $N[G_{\mathbb{R}}] \models \text{"} M$ is countable”, so $N[G_{\mathbb{R}}] \models \text{"} M$ is a $\mathbb{Q}$-candidate and a $\mathbb{P}$-candidate”. Hence by assumption (d)(ii) there are $p_1, \eta^\otimes, G_Q^\otimes \in N[G_{\mathbb{R}}]$ such that:

\[ N[G_{\mathbb{R}}] \models \text{"} p_1 \in \mathbb{P}, \, p \leq^\mathbb{P} p_1, \, p_1$ is $(M, \mathbb{P}^M)$-generic and $p_1 \forces \text{"} \eta^\otimes$ is a $(Q, \eta)$-real over $M[G_\mathbb{P} \cap \mathbb{P}^M]$ satisfying $q$] moreover, $p_1 \forces \text{"} \eta^\otimes = \eta[G_Q^\otimes]$"

\[ G_Q^\otimes \subseteq Q^M[G_\mathbb{R} \cap \mathbb{P}^M]$ is a generic set over $M$ such that $q \in G_Q^\otimes$.\]

[Here we use the following: if $G \subseteq \mathbb{P}^N[G_{\mathbb{R}}]$ is generic over $N[G_{\mathbb{R}}]$ then $N[G_{\mathbb{R}}][G]$ is a $\mathbb{Q}$-candidate (apply clause (iv) of the assumption (d) to $\mathbb{R} \times \mathbb{P}$).] It follows from clause (d)(v) of the choice of $M$ that

\[ G_Q^\otimes \cap Q^M \text{ is generic over } M. \]

Let $p^1, \eta^\otimes, G_Q^\otimes \in N$ be $\mathbb{R}$-names such that $\eta^\otimes[G_{\mathbb{R}}] = \eta^\otimes, G_Q^\otimes = G_Q^\otimes[G_{\mathbb{R}}]$ and $p^1[G_{\mathbb{R}}] = p_1$, and without loss of generality in $N$ we have

\[ r \forces \text{"} \eta^\otimes \text{ is a } (Q, \eta )-\text{generic real over } M \text{ satisfying } q \text{ and } p^1 \in \mathbb{P} \text{ and } p^1 \text{ forces } (\forces_{\mathbb{P}}) \text{ that } \eta^\otimes \text{ is } (Q, \eta )-\text{generic over } M[G_\mathbb{P} \cap \mathbb{P}^M], G_Q^\otimes \text{ is a subset of } Q^M \text{ generic over } V \text{".} \]
Let $G'_R \subseteq \mathbb{R}$ be generic over $N[G_R]$, to which $r$ belongs, so $N[G_R][G'_R]$ is a forcing extension of $N[G_R]$ so both are generic extensions of $N$ by a small forcing.

Now $G''_R$ is essentially a complete embedding of $\mathbb{Q} \upharpoonright (\geq q)$ into $\mathbb{R}$ (by basic forcing theory, see the footnote to 8.11(1)(d); and we can use the value for 0 of the function $\{f : f \in G_R\}$ to choose $q'$, $q \leq q' \in \mathbb{Q}^N$). Hence, for some $\mathbb{Q}$–name $\mathbb{R}^*$ we have $(\mathbb{Q} \upharpoonright (\geq q)) \ast \mathbb{R}^*$ is $\mathbb{R}$, so $G_R = G''_R \ast G_R^*$ for some $G_R^* \in V[G_R]$, where $\mathbb{R}^* = \mathbb{R}^*[G''_R]$, $\mathbb{R}^*$ a $\mathbb{Q}$–name. So we can represent $N[G_R][G'_R]$ also as $N^3 \overset{\text{def}}{=} N[G''_R][G_R^*];$ i.e. forcing first with $\mathbb{Q} \upharpoonright (\geq q)$, then with $\mathbb{R}$, lastly with $\mathbb{R}^*[G''_R]$. Now let $N^2 \overset{\text{def}}{=} N[G''_R][G_R^*]$, so $N^2$ is a generic extension of $N$ and $N^3$ is a generic extension of $N^2$ (both by “small” forcing), and in $N^3$ we have $p^1$ and $\eta^\circ$ and $G''_R$. But $G''_R \times G_R^*$ is a generic subset of $(\mathbb{Q}^N \upharpoonright (\geq q)) \times \mathbb{R}$ over $N$, so essentially a generic (over $N$) subset of $\mathbb{Q}^N \times \mathbb{R}$ to which $(q, r)$ belongs, hence (by (⊗) above) $N^2 \models \psi[\eta[G''_R]]$. Therefore there is no $p' \in N^2$ such that:

$$\underset{\text{(Ω)}}{\forall p' \in \mathbb{P}, p \leq p', p' \Vdash \mathbb{P} \mathbb{M} \mathbb{R}^*[G''_R]} \text{ is } (\mathbb{Q}, \eta)\text{–generic over } M[G_\mathbb{P} \cap \mathbb{P}^M] \mathbb{N} \text{.}$$

In $N^2$ we can find a countable $M' \overset{\text{def}}{=}(\mathcal{H}(\chi_2)^N, \in)$ to which $\mathbb{P}$, $\mathbb{Q}$, $\eta^\circ$, $p^1$, $\mathbb{R}^*$, and $G''_R, G_R^*, M$ belong (so $N^2 \models \text{“} M' \text{ is countable and is a } \mathbb{P}\text{–candidate and a } \mathbb{Q}\text{–candidate”} \text{) and } M'' = M' \upharpoonright N \in N$. In $N^2$ we can find $G''_R[G''_R] \subseteq \mathbb{R}^*[G''_R] \cap M'$ which is generic over $M'$. In $M^3 = M''[G''_R][G'^*_R[G''_R]]$, again by the forcing theorem, there is $p'$ as required in (Ω) above. So $M^3$ is a $\mathbb{P}$–candidate inside $N^2$, hence there is $p'_1$ such that $N^2 \models \text{“}p' \leq p'_1 \text{ and } p'_1 \text{ is } (M^3, \mathbb{P})\text{–generic} \text{”}$. By the amount of absoluteness we require (moving up from $M^3$ to $N^2$) this $p'_1$ can serve in $N^2$ for (Ω), contradiction to the previous assertion.

\textbf{Proposition 8.11.} Assume (a),(b),(c) and (d) of \textbf{8.7} and

(c) $\text{ZFC}^\ast$ is weakly normal,

(f) $\mathbb{Q}$ is c.c.c. and simple (for simplicity) and correct.

Then $\mathbb{P}$ is strongly $I(\mathbb{Q}, \eta)$–preserving.

\textbf{Proof} First note that $I^\text{ex}_{\mathbb{Q}, \eta} = I^\text{dx}_{\mathbb{Q}, \eta}$. Now, if the conclusion fails as witnessed by a set $X$, then, by \ref{8.7}, the statements (A), (B), (C) of \ref{8.7} fail.

\textsuperscript{6}Of course, we can use a weaker demand on $G''_R$
Hence, by \( -(A) \), \( X \in (I^{\exists}_{(\mathcal{Q},\eta)})^+ \) and \( p \in \mathbb{P} \) and a \( \mathbb{P} \)-name \( y \) such that
\[
p \vdash \mathbb{P} \text{ " } y \in \mathcal{H}_{<\aleph_1}(\mathbb{Q}) \text{ and for no } Q \text{-candidate } M \text{ such that } y \in M \text{ there is } \nu \in X \text{ which is } (\mathbb{Q},\eta)\text{-generic over } N[\mathbb{G}_\mathbb{P}] \text{ "}.
\]

As we can increase \( p \), without loss of generality \( y \) is a hereditarily countable \( \mathbb{P} \)-name. As \( \text{ZFC}^{\omega_1}_{\omega} \) is weakly normal we can find a model \( N \) of \( \text{ZFC}^{\omega_1}_{\omega} \) which is a \( \mathbb{P} \)-candidate and a \( Q \)-candidate and to which \( p, y \) belongs. Let \( \eta^* \in X \subseteq \omega_1 \omega \) (in \( V \)) be \( (\mathbb{Q},\eta)\)-generic over \( N \) (exists by the negation of (B) of §8.7). By (\( \beta \)) of §8.8 there is \( q \in \mathbb{P} \) such that \( p \leq q \), \( q \) is \( (N,\mathbb{P})\)-generic and \( q \vdash \mathbb{P} \text{ " } \eta^* \text{ is } (\mathbb{Q},\eta)\text{-generic over } N[\mathbb{G}_\mathbb{P}] \text{ "}, a contradiction. \)

**Proposition 8.12.** Assume (a), (b), (c) of §8.8. Let \( \mathbb{P}, \mathbb{Q} \) be normal and forcing with \( \mathbb{P} \) preserves “\( \mathcal{I} \subseteq \mathbb{P} \) is predense”. Then
\[(\alpha)^{\prime} \mathbb{P} \text{ is strongly } I_{(\mathcal{Q},\eta)}\text{-preserving},\]
\[(\beta) \text{ for } \chi \text{ large enough, if } N \prec (\mathcal{H}(\chi),\in) \text{ is countable (and } \mathcal{E}, \mathcal{B}^\mathbb{Q}, \mathcal{B}^\mathbb{P}, \eta \in N) \text{ and } N \models \text{" } p \in \mathbb{P} \text{ " and } \eta^* \text{ is } (\mathbb{Q},\eta)\text{-generic over } N \text{ then for some } q \text{ we have}\]
\[(i) \ p \leq q, \ q \in \mathbb{P},\]
\[(ii) \ q \text{ is } (N,\mathbb{P})\text{-generic; i.e. } q \vdash \mathbb{P} \text{ " } \mathcal{G}_\mathbb{P} \cap \mathbb{P}^N \text{ is generic over } N \text{ "},\]
\[(iii) \ q \vdash \mathbb{P} \text{ " } \eta^* \text{ is } (\mathbb{Q},\eta)\text{-generic over } N[\mathbb{P}^N \cap \mathbb{G}_\mathbb{P}] \text{ "}.\]

**Proof** Let \( \chi_1 \) be a large enough strong limit, and \( \chi_2 = \Sigma_\omega(\chi_1), \chi = \sum_\omega(\chi_2) \), and repeat the proof of §8.8 using \( N \prec (\mathcal{H}(\chi_3),\in) \) to which \( \mathcal{E}, \mathcal{B}^\mathbb{Q}, \mathcal{B}^\mathbb{P} \) and \( \mathbb{P}, \mathbb{Q}, \theta, \chi_1, \chi_2 \) belong. \( \square \)

**Proposition 8.13.** Assume (a),(b),(c) of §8.8 and
\[(d)^{\prime} \text{ZFC}^{\omega_1}_{\omega} \text{ is a version of set theory including, for some } \chi_1 < \chi_2\]
\[(i) \ (\mathcal{H}(\chi_2),\in) \text{ is a (well defined) model of } \text{ZFC}^{\omega_1}_{\omega},\]
\[(ii) \ (a) \text{ and } (b) \text{ (with } \mathcal{B}^\mathbb{P}, \mathcal{B}^\mathbb{Q}, \eta \text{ as individual constants) and (c),}\]
\[(iii) \ Q, \mathbb{P} \in \mathcal{H}(\chi_1) \text{ and } (\mathcal{H}(\chi_2),\in) \text{ is a } \mathbb{P} \text{-candidate and a } Q \text{-candidate with } \mathcal{B}^\mathbb{P} \text{ interpreted as } (\mathcal{B}^\mathbb{P})^N \upharpoonright \mathcal{H}(\chi_2) \text{ and similarly for } Q, \text{ so}\]
\[\text{" } \mathcal{B}^\mathbb{P}, \mathcal{B}^\mathbb{Q} \in \mathcal{H}(\chi_2) \text{ "},\]
\[(iv) \text{ forcing with } \mathbb{P} \text{ preserves being a } Q \text{-candidate,}\]
\[(v) \ Q \text{ satisfies the c.c.c. and being incompatible in } Q \text{ is upward absolute from } Q \text{-candidates.}\]

Then for models of \( \text{ZFC}^{\omega_1}_{\omega} \), forcing with \( \mathbb{P} \) preserves “\( \mathcal{I} \) is a predense subset of \( Q \)” (i.e. (\( d \))(\( v \)) of §8.8).
Remark: 1) When ZFC\(\neg\neg\neg\) is normal, this applies to \(11.6\).

2) Compare with \(7.8\)(iii). Here we have redundant assumptions as we have a use for \(8.8\) in mind.

Proof Let \(N\) be a model of ZFC\(\neg\neg\neg\) and let \(N \models \exists \mathcal{I}\) is a predense subset of \(\mathbb{Q}\). As \(N \models \exists \mathcal{J} \subseteq \{q : N \models \exists p \in \mathcal{I}(p \leq q \in \mathbb{Q})\}\) such that

\[
N \models \text{\(\mathcal{J}\) is countable, say } \{p_n : n < \omega\}, \text{ and \(\mathcal{J}\) is predense in \(\mathbb{Q}\)}.
\]

Toward contradiction assume \(p^* \in \mathbb{Q}^N\) and \(\tilde{r} \in N\) are such that

\[
N \models \text{\(p^* \Vdash \tilde{r} \in \mathbb{Q}\) is incompatible with every } p_n.
\]

We can replace \(N\) by \(H(\chi_2)^N\). Let \(M \in N\) be such that

\[
N \models \text{\(M \prec (H(\chi_2)^N, \epsilon)\) is countable and}
\]

\[
\mathbb{P}, \mathbb{Q}, \mathcal{B}^\mathbb{P}, \mathcal{B}^\mathbb{Q}, \langle p_n : n < \omega, p^*, \tilde{r} \rangle \in M.
\]

In \(N\) we can find \(G \subseteq \mathbb{P}^N\) generic over \(M\), so \(M\) inherits from \(H(\chi_2)\) the property \(M[G]\) is a \(\mathbb{Q}\)-candidate and also \(M[G] \models \tilde{r}[G], p_n\) are incompatible in \(\mathbb{Q}\). So \(\tilde{r}[G] \in \mathbb{Q}^N\) contradicts the choice of \(\mathcal{J}\).

Conclusion 8.14. For \(8.8(\beta)\) to hold, we can omit clause (v) of (e) there if we add:

(g) \(\mathbb{Q}\) satisfies the c.c.c. in \(\mathbb{Q}\)-candidates and being incompatible in \(\mathbb{Q}\) is upward absolute from \(\mathbb{Q}\)-candidates.

We can conclude (phrased for simplicity for strongly c.c.c. nep).

Conclusion 8.15. Assume that

(a) \(\mathbb{Q}\) is strongly c.c.c. explicitly nep (see Definition \(7.12\)) and simple and correct,

(b) \(\eta \in \omega^\omega\) generic for \(\mathbb{Q}\), a hereditarily countable \(\mathbb{Q}\)-name.

If \(\mathbb{P}_0\) is nep, \(I(\mathbb{Q}, \eta)^{\mathbb{P}_0}\)-preserving and \(\Gamma\models_{\mathbb{P}_0}\text{\(\mathbb{P}_1\) is nep, } I(\mathbb{Q}, \eta)^{\mathbb{P}_1}\text{-preserving}"

then \(\mathbb{P}_0 \ast \mathbb{P}_1\) is (nep and) \(I(\mathbb{Q}, \eta)^{\mathbb{P}_1}\)-preserving.

The reader may ask: what about \(\omega\) limits (etc)? We shall address these problems in the continuation \([21]\).

8. Non-symmetry. The following hypothesis \(9.1\) will be assumed in this and the next section, though for the end (including the main theorems \(10.11\), \(10.13\)) we assume snep (i.e. \(9.2\)).

Hypothesis 9.1. \(\mathbb{Q}\) is correct c.c.c. simple, strongly c.c.c. nep, \(\eta\) is a hereditarily countable name of a generic real, i.e. \((\mathbb{Q}, \eta) \in \mathcal{K}\) (see Definition \(7.2\)).
for ZFC\textsuperscript{−} and ZFC\textsuperscript{−}\# (and the properties above) are preserved by a forcing of cardinality $< \bar{\chi}$, $|Q|^{|\aleph_0|} < \bar{\chi}$, for $Q$-candidates.

**Hypothesis 9.2.** Like 9.1 with snep.

**Definition 9.3.** Let $Q, \eta$ be as in 9.1 and let $\alpha$ be an ordinal.

1. Let $Q[\alpha] = P_\alpha$, where $\langle P_i, Q_j : i \leq \alpha, j < \alpha \rangle$ is a FS iteration and $Q_j = Q[V[P_j]]$.
2. We let $\eta[\alpha] = \langle \eta_\ell : \ell < \alpha \rangle$, where $\eta_\ell$ is $\eta$ “copied to $Q_\ell$” (see 9.4(1) below).
3. $(Q^{(\alpha)}, \eta^{(\alpha)})$ is defined similarly as an FS product.
4. For a finite set $u \subseteq \alpha$ we define $F = F^{\alpha,u}_Q : Q \rightarrow Q[\alpha]$ by $F(p) = \bar{p}$, where $\bar{p} = \langle p_\ell : \ell < \alpha \rangle$, $p_\ell = p$ if $\ell \in u$ and $p_\ell = 0_Q$ otherwise.
5. The FS iteration $\bar{Q} = \langle P_i, Q_j, \eta_j : i \leq \alpha, j < i \rangle$ of neps means $(Q_j, \eta_j) \in K$.

We write $\eta$ to mean $\eta_j = F^{\alpha,u}_Q(\eta)$.

**Proposition 9.4.**

1. In Definition 9.3(4), for finite $u \subseteq \alpha$, $F = F^{\alpha,u}_Q$ is a complete ($\circ$) embedding, as “$p \leq q$”, “$p,q$ compatible”, “$p,q$ incompatible”, “$\{p_n : n < \omega \}$ is predense set above $q$” are upward absolute from $Q$–candidates (holds as $Q$ is strongly c.c.c. by 9.1). So $\eta_\ell = F^{\alpha,\{\ell\}}_Q(\eta)$ if $\alpha \in u$.
2. $Q[\alpha]$ satisfies the c.c.c.
3. Same holds for $Q^{(\alpha)}$.
4. $(Q^{[\alpha]}, \eta^{[\alpha]})$ for $\alpha < \omega_1$ are as in 9.4, too.

**Proof** For example:

3) It is enough to prove it for finite $\alpha$, and this we prove by induction on $\alpha$ for $\alpha = n + 1$. For the c.c.c. use “incompatibility is absolute” for forcing by $Q^{(n)}$, so we can use the last phrase in 9.1.

4) The main point here is the strong c.c.c., so let $N$ be a $Q$-candidate (and $\alpha + 1 \subseteq N$) and $N \models "I \subseteq Q[\alpha]"$ is predense “.

Let $G^{[\alpha]} \subseteq Q[\alpha]$ be generic over $V$ and for $\beta \leq \alpha$, $G^{[\beta]} = G^{[\alpha]} \cap Q^{[\beta]}$. Show by induction on $\beta$ that $G^{[\beta]} \cap (Q^{[\beta]}|^N \cap Q^{[\beta]}|^N)$ is a generic subset of $(Q^{[\beta]}|^N \cap Q^{[\beta]}|^N)$ over $N(G^{[\beta]}).$

**Definition 9.5.** 1. We say that $Q$ is $[n]$–symmetric if:
if \( \eta^*_\ell : \ell < n \) is generic for \( \langle \mathcal{P}_\ell, \mathcal{Q}_\ell, \eta_\ell : \ell < n \rangle \) and \( \sigma \) is a permutation of \( \{0, \ldots , n - 1\} \)
then \( \langle \eta_{\sigma(\ell)} : \ell < n \rangle \) is generic for \( \langle \mathcal{P}_\ell, \mathcal{Q}_\ell, \eta : \ell < n \rangle \).

2. If \( (\mathcal{Q}', \eta'), (\mathcal{Q}'', \eta'') \) are as in 1., we say that they commute if:
if \( r' \) is \( (\mathcal{Q}', \eta') \)–generic over \( \mathcal{V} \) and \( r'' \) is \( (\mathcal{Q}'', \eta'') \)–generic over \( \mathcal{V}[r'] \)
then \( r'' \) is \( (\mathcal{Q}', \eta') \)–generic over \( \mathcal{V}[r''] \)
(note that \( \eta'' \) is \( (\mathcal{Q}'', \eta'') \)–generic over \( \mathcal{V} \) is always true by 1.).

3. For \( (\mathcal{Q}', \eta'), (\mathcal{Q}'', \eta'') \) we say that they weakly commute if
\( (\mathcal{Q}' \upharpoonright (\geq q'), \eta'), (\mathcal{Q}'' \upharpoonright (\geq q''), \eta'') \) commute for some \( q' \in \mathcal{Q}' \) and \( q'' \in \mathcal{Q}'' \).

**Proposition 9.6.**

1. “Commute” is a commutative relation.

For \( n \geq 2 \) we have:
\( \mathcal{Q} \) is \([n]–symmetric \) iff
\( \mathcal{Q}, \mathcal{Q}^{[n-1]} \) commute and \( \mathcal{Q} \) is \([n-1]–symmetric \) iff
\( \mathcal{Q} \) is \([2]–symmetric \).

2. If \( \mathcal{P}, \mathcal{Q} \) commute, \( m \leq n \) then \( \mathcal{P}, \mathcal{Q}^{[m]} \) commute. Similarly, if \( \mathcal{P}, \mathcal{Q} \)
commute and \( \mathcal{Q} \prec \mathcal{P} \), the \( \mathcal{P}, \mathcal{Q} \) commute.

3. In part 2) we can replace \( [-] \) by \( \langle - \rangle \).

4. If \( (\mathcal{Q}', \eta'), (\mathcal{Q}'', \eta'') \) weakly commute and \( \mathcal{Q}', \mathcal{Q}'' \) are homogeneous, then
they commute.

**Proof**

1) Let \( (\mathcal{Q}', \eta'), (\mathcal{Q}'', \eta'') \) as in 1. Then “\( (\mathcal{Q}', \eta'), (\mathcal{Q}'', \eta'') \)
commute” says \( \mathcal{Q}' \ast \mathcal{Q}'' = \mathcal{Q}'' \ast \mathcal{Q}' \), which is symmetric.

2) For the second “iff”, use “the permutations \( \pi_\ell = (\ell, \ell + 1) \) for \( \ell < n \)
generate the group of permutations of \( \{0, \ldots , n - 1\} \)”.

**Proposition 9.7.**

1. If \( \mathcal{Q}^{[\omega]} \) and Cohen do not commute, then for some
\( n < \omega, Q^{[n]} \) and Cohen do not commute.
(The inverse holds by 9.4(3), second phrase.)

2. If \( \mathcal{Q}^{[\omega]} \) and Cohen do not commute, then for some \( n < \omega, Q^{[n]} \) and Cohen do not commute.

**Proof**

1) Since Cohen and \( Q^{[\omega]} \) do not commute, there is a \( Q^{[\omega]}–name \)
\( \mathcal{I} \) of a dense open subset of Cohen (i.e. of \( (\omega > 2, <) \)) such that for some
condition \( (p, q) \in \text{Cohen} \ast Q^{[\omega]} \) we have
\( (p, q) \models " \eta \text{Cohen has no initial segment in } \mathcal{I} " \).
Without loss of generality for some \( n^* < \omega \) we have \( p \models \text{Cohen} \upharpoonright \text{Dom}(q) \subseteq \{0, \ldots , n^* - 1\} " \).
Let \( \mathcal{I} \) be the \( Q^{[n^*]}–name \) for the following set:
\( \{ \eta \in \omega^\omega : \text{for some } p \in Q^{[\omega]} \), \( p \upharpoonright n^* \in G_{Q^{[n^*]}} \) and \( p \models Q^{[\omega]} \ " \eta \in \mathcal{I} " \).
It should be clear that $\models_{Q[\omega]} \text{“} T \text{”}$ is a dense open subset of $(\omega^2, <)$. Now we ask the following question.

Does $(p, q) \models_{\text{Cohen} \ast Q[\omega]} \text{“} \eta^{\text{Cohen}} \upharpoonright n \notin T \text{”}$ for each $n < \omega$?

If yes, we have gotten the desired conclusion (i.e. Cohen and $Q[\omega]$ do not commute). If not, for some $(p', q')$ such that $(p, q) \leq (p', q') \in \text{Cohen} \ast Q[\omega]$ and for some $n < \omega$ we have:

$$(p', q') \models_{\text{Cohen} \ast Q[\omega]} \text{“} \eta^{\text{Cohen}} \upharpoonright n = \eta \in T \text{”}.$$ 

Without loss of generality, for some $p \in (Q[\omega])^V$ we have $(p', q') \models \text{“} p \upharpoonright n^* \in G_{\text{Cohen} \ast Q[\omega]} \text{”}$ and $p \models \text{“} \eta \in T \text{”}$. Then $(p', q' \cup p \upharpoonright [n^*, \omega))$ forces (in Cohen $\ast Q[\omega]$) that $\eta^{\text{Cohen}} \upharpoonright n = \eta \in T$, a contradiction.

2) Similarly.

**Proposition 9.8.** 1. If $Q[\omega]$ and Cohen do not commute (as before), then $Q$ and Cohen do not commute (both “absolute”).

2. The following conditions are equivalent:
   (i) $Q$ commutes with Cohen,
   (ii) $\models_Q \text{“}(\omega^2)^V \text{ is not meagre”}$,
   (iii) $(\forall A)[V \models \text{“} A \subseteq (\omega^2) \text{ non-meagre”} \Rightarrow \models_Q \text{“} A \text{ is non-meagre”}]$
   (all “absolutely”, i.e. not only in the present universe but in its generic extensions too).

3. We can replace Cohen by others to which 8.8 applies and are homogeneous (see 8.4).

**Proof** 1) Assume toward contradiction that $Q$ and Cohen commute (absolutely). Let $\eta \in \omega^2$ be a Cohen real over $V$. Let $G_\ell \subseteq QV[\check{G}_0, \ldots, \check{G}_{\ell-1}, \eta]$ be generic over $V[\check{G}_0, \ldots, \check{G}_{\ell-1}, \eta]$ for $\ell < n$, and let $\eta_\ell = \eta[G_\ell]$. We now prove by induction on $\ell$, that $\eta$ is a Cohen real over $V[\check{G}_0, \ldots, \check{G}_{\ell-1}]$.

The induction step is by the assumption “$Q$ and Cohen commute”. The net result is that $\eta$ is a Cohen real over $V[\check{\eta}_0, \ldots, \check{\eta}_{n-1}]$, contradicting the assumption.

2) The second clause implies the third by 8.8. The third clause implies the second trivially.

Let us argue that the implication (i) $\Rightarrow$ (ii) holds. Add $\aleph_1$ Cohen reals $\{\eta_i : i < \omega_1\}$ and then force by $Q$. Let $G_Q \subseteq QV[(\check{\eta}_i : i < \omega_1)]$ be generic over $V$, and $\eta = \eta_Q[G_Q]$. Then (i) implies that for every $j < \omega_1$ we have: $\eta_j$ is Cohen over $V[(\check{\eta}_i : i < \omega_1, i \neq j), \eta]$. Hence in $V[(\check{\eta}_i : i < \omega_1), \eta] = V[(\check{\eta}_i : i < \omega_1)]G_Q]$, the set $\{\eta_i : i < \omega_1\}$ is not meagre and consequently (ii) holds.
Lastly, assume (iii) and let $\nu \in {}^\omega |Q|$ be generic for $\text{Levy}(\mathcal{N}_0, |Q|)$. Let $\eta$ be $(Q, \eta)$-generic real over $V[\nu]$. By (ii), we can find in $V[\nu]$ a real $\rho \in \omega^2$ which is in no meagre set from $V[\eta]$ (note that there are countably many such meagre sets from the point of view of $V[\nu]$). Now we easily finish.

3) Same proof.

9. Poor Cohen commutes only with himself.

**Definition 10.1.**
1. We say a $Q$-name $x$ of a subset of some countable $a^* \in V$ is [somewhere] essentially Cohen if $B_2(Q, x)$ is [somewhere] essentially countable; i.e. [above some $p$] has countable density.
2. We say $(Q, \eta) \in \mathcal{K}^c$ (a non-Cohen pair) if:
   (a) $(Q, \eta)$ is as in 9.2,
   (b) $(Q, \eta)$ (see Definition 7.5) is nowhere essentially Cohen (i.e. above every condition).

**Hypothesis 10.2.** $\chi$ is regular large enough cardinal, and $(Q, \eta) \in \mathcal{K}^c$ will be fixed as in 10.1, and $\text{ZFC}^*$ is normal (see Definition 2.11).

**Definition 10.3.**
1. $\mathcal{D} = \mathcal{D}_{<\mathcal{N}_0}(\mathcal{H}(\chi))$ is the filter of clubs on $[\mathcal{H}(\chi)]^{\le\mathcal{N}_0}$.
2. $C_0 = \{ a : a \prec (\mathcal{H}(\chi), \in) \text{ is countable, and } (Q, \eta) \in a \text{ (i.e. their definitions) so is a } Q \text{-candidate} \}$.

**Definition 10.4.** We say that $q \in Q$ is strong on $a \in C_0$ if:

$(\mathbb{1})_{a,q}$ the set $\{ p \in a \cap Q : p, q \text{ are incompatible in } Q \}$ is dense in the (quasi) order $Q \cap a$.

**Proposition 10.5.**
1. For every $a \in C_0$ there is $q \in Q$ which is strong on $a$.
2. Moreover, for every $p \in Q$ and $a \in C_0$ there is $q$ strong on $a$ such that $p \le_Q q$.

**Proof** Clearly $G_Q \cap a$ is a $Q$-name of a countable subset of an old set $Q \cap a$, so it can be considered as a real. Note that $^1 \text{G}_Q \cap a$ is not somewhere essentially Cohen.

Why? We can restrict ourselves to be above some fix $p \in Q$. From $G_Q \cap a$ we can compute $\eta$ (as $\eta \in a$, i.e. the relevant maximal antichains belong to $a$), so $\eta$ can be considered a $B_2[Q, G_Q \cap a]$-name. But “any (name of) a real in an essentially Cohen forcing notion is essentially Cohen itself”, so $\eta$ is essentially Cohen $Q$-name, contradicting Hypothesis 10.2.
Consequently, $\mathbb{I} \models \text{"} G_{\mathbb{Q}} \cap a \text{ is not a generic subset of } \mathbb{Q} \upharpoonright a \text{ (over } \mathbb{V}) \text{"}$ and hence $p \mathbb{I} \models \text{"} G_{\mathbb{Q}} \cap a \text{ is not a generic subset of } \mathbb{Q} \upharpoonright a \text{ (over } \mathbb{V}) \text{"}$. Thus there are $q$ and $\mathcal{I}$ such that:

(i) $p \leq q \in \mathbb{Q}$,
(ii) $\mathcal{I} \subseteq \mathbb{Q} \cap a$ is a dense open subset of $\mathbb{Q} \upharpoonright a$,
(iii) $q \mathbb{I} \models \text{"} G_{\mathbb{Q}} \text{ is disjoint to } \mathcal{I} \text{"}$.  

But this means that

$(\ast)_{2}$ $q$ is incompatible with every $r \in \mathcal{I}$.  
[Why? Otherwise $q \not\models \text{r} \notin G_{\mathbb{Q}}$].

So $\{r \in a \cap \mathbb{Q} : q, r \text{ incompatible (in } \mathbb{Q})\}$ is a subset of $\mathbb{Q} \cap a$ including $\mathcal{I}$ hence it is dense in $\mathbb{Q} \upharpoonright a$.  

\[10.7\]

**Choice 10.6.** We choose $\bar{p} = \langle p_a : a \in C_0 \rangle$ such that $p_a \in \mathbb{Q}$ is strong on $a$ (possible by 10.5).

**Definition 10.7.**
1. For $R \subseteq \mathbb{Q}$ let $A[R] \overset{\text{def}}{=} \{a \in C_0 : p_a \in R\}$.
2. $D_{\bar{p}} = D_{\bar{p}, \mathbb{Q}} \overset{\text{def}}{=} \{R \subseteq \mathbb{Q} : A[R] \in \mathcal{D}\}$.
   The family of $D_{\bar{p}}$-positive sets will be denoted $D_{\bar{p}}^+$ (so for a set $S \subseteq \mathbb{Q}$, $S \in D_{\bar{p}}^+$ iff $R \cap S \neq \emptyset$ for each $R \in D_{\bar{p}}$).
3. For $R \subseteq \mathbb{Q}$ and $q \in \mathbb{Q}$ let $R[q] \overset{\text{def}}{=} \{p \in R : p, q \text{ are incompatible in } \mathbb{Q}\}$ (so $R[q]$ is in a sense the orthogonal complement of $q$ inside $R$).

**Fact 10.8.**
1. $D_{\bar{p}}$ is an $\aleph_1$-complete filter on $\mathbb{Q}$.
2. For $R \subseteq \mathbb{Q}$ we have $R \in D_{\bar{p}}^+ \iff A[R] \in \mathcal{D}^+$.

**Proposition 10.9.** If $R \in D_{\bar{p}}^+$ then the set

$$R^\otimes = \{q \in \mathbb{Q} : R[q] \in D_{\bar{p}}^+\}$$

is dense in $\mathbb{Q}$.

**Proof** Assume not, so for some $q^* \in \mathbb{Q}$ we have

$(\ast)_1$ there is no $q \in \mathbb{Q}$ such that $q^* \leq q \in \mathbb{Q}$ & $R[q] \in D_{\bar{p}}^+$.

Thus

$$q^* \leq q \in \mathbb{Q} \Rightarrow R[q] = \emptyset \text{ mod } D_{\bar{p}} \Rightarrow A[R[q]] = \emptyset \text{ mod } \mathcal{D}$$

$$
\Rightarrow \text{ for some club } C_q \subseteq C_0 \text{ of } [\mathcal{H}(\chi)]^{\leq \aleph_0} \text{ we have }\\
(\forall a \in C_q)[p_a \notin R[q], \text{ i.e. } p_a, q \text{ are compatible}].
$$

Let $C^* = \{a \in C_0 : q^* \in a \text{ and } (\forall q)[q^* \leq q \in a \cap \mathbb{Q} \Rightarrow a \in C_q]\}$. As each $C_q$ is a club of $[\mathcal{H}(\chi)]^{\leq \aleph_0}$ clearly $C^*$ (as a diagonal intersection) is a club of
$[\mathcal{H}(\chi)]^{\aleph_0}$, i.e. $\mathcal{C}^* \in \mathcal{D}$. Since $R \in D^+_\bar{p}$ we have $A[R] \in \mathcal{D}^+$, so together with the previous sentence we know that there is $a^* \in A[R] \cap \mathcal{C}^*$. By the choice of $\bar{p}$ (see \ref{10.6}, and Definition \ref{10.3} as $q^* \in a^* \cap \mathcal{Q}$ (see the choice of $\mathcal{C}^*$) for some $q$ we have:

$$q^* \leq q \in a^* \quad \text{and} \quad p_{a^*}, q \text{ are incompatible.}$$

Now this contradicts “$a^* \in \mathcal{C}_q$”.

**Definition 10.10.** Assume $\chi_1 = (2^\chi)^+$ (so $\mathcal{H}(\chi) \in \mathcal{H}(\chi_1)$) and $N$ is a countable elementary submodel of $(\mathcal{H}(\chi_1),\in)$ to which $\{\chi, \mathcal{Q}, \bar{p}\}$ belong (so $\mathcal{D}^+_{\bar{p}} \in \mathcal{N}$). Further, assume that $\mathcal{Q}$ is snep.

1. We let Cohen$_N = \text{Cohen}_{N,\mathcal{Q}}$ be $(\mathcal{D}^+_{\bar{p},\mathcal{Q}}, \supseteq) \restriction N$ (so this is a countable atomless forcing notion and hence equivalent to Cohen forcing).

2. If $G_N \in \text{Gen}(N, \mathcal{D}^+_{\bar{p},\mathcal{Q}})$ define $\{G : G \subseteq \text{Cohen}_N \text{ is generic for } (N, (\mathcal{D}^+_{\bar{p},\mathcal{Q}}, \supseteq) \restriction N)\}$ (possibly in a universe $\mathcal{V}'$ extending $\mathcal{V}$) then let $p_N[G]$ be the sequence (i.e. in $\omega_\omega$ or just member of $\omega(I))$ such that for each $\ell < \omega$ and $\gamma$

\[
(p_N[G_N])(\ell) = \gamma \iff (\exists R \in G)(\forall p \in R)[p(\ell) = \gamma].
\]

**Proposition 10.11.** Assume \ref{10.3} and, additionally, $\mathcal{Q}$ is Souslin c.c.c. (i.e. the incompatibility relation is $\Sigma^1_1$). If $\chi_1, N$ and $G \in \text{Gen}(N, \mathcal{D}^+_{\bar{p}})$ are as in \ref{10.11} (so $G$ is possibly in some generic extension $\mathcal{V}_1$ of $\mathcal{V}$ but Cohen$_N$ is from $\mathcal{V}$) then

(a) $p_N[G]$ is an $\omega$-sequence (i.e. for each $\ell$ there is one and only one $\gamma$),

(b) $p_N[G] \in \mathcal{Q}$,

(c) $p_N[G]$ is strong for $N \restriction \mathcal{H}(\chi)$ (which belongs to $\mathcal{C}_0$).

**Proof.** For every $p \in \mathcal{Q}$ there is $\nu_p \in \omega_\omega$ which witnesses $p \in \mathcal{Q}$, i.e. $p \ast \nu_p \in \lim(T^Q_0)$. So choose such a function $p \mapsto \nu_p$. Now in $\mathcal{V}$, for $n < \omega$ the function $p_n \mapsto (p_n \restriction n, \nu_{p_n} \restriction n)$ is a mapping from $\{p_a : a \in \mathcal{C}_0\} \in \mathcal{D}_{\bar{p}}$ with countable range. Since $\mathcal{D}_{\bar{p}}$ is $\aleph_1$-complete

\[(*)_1 \text{ in } \mathcal{V}, \text{ if } R \in \mathcal{D}^+_{\bar{p}} \text{ and } n < \omega \text{ then for some } R' \subseteq R \text{ and } (\eta^n, \nu^n) \text{ we have}
\]

\[R' \in \mathcal{D}^+_{\bar{p}} \quad \text{and} \quad (\forall p \in R')[(p \restriction n, \nu_p \restriction n) = (\eta^n, \nu^n)].
\]

This is inherited by $N$, hence $p_N[G]$ satisfies clauses (a), (b) (in fact $\nu[G] = \bigcup\{\nu^* : \text{ for some } n < \omega \text{ and } R \in G \text{ we have } (\forall p \in R)[\nu_p \restriction n = \nu^*] \}$ is a witness for $p_N[G] \in \mathcal{Q}$). Also for each $q \in \mathcal{Q} \cap N$ the set

\[\mathcal{J}_q = \{R \in \mathcal{D}^+_{\bar{p}} : \text{ for some } q' \in \mathcal{Q} \text{ stronger than } q \text{ we have:}
\]

\[(\forall p \in R)[p, q' \text{ are incompatible (in } \mathcal{Q})]\} \]
is a dense subset of $(D^+_p, \supseteq)$ (remember $p_a$ is strong on $a$; use Fodor lemma). Clearly it belongs to $N$, so by the demand on $G$ we know that $G \cap \mathcal{J}_q \neq \emptyset$. Choose $R_q \in G \cap \mathcal{J}_q$ and let $q' \in Q \cap N$ witness it, so

$$R_q \in D^+_p \cap N \quad \text{and} \quad (\forall p \in R_q)[p, q' \text{ are incompatible}].$$

Now “incompatible in $Q$” is a $\Sigma^1_1$-relation (belonging to $N$) hence as above, $p_N[G], q'$ are incompatible. As $q$ was any member of $Q \cap N$ we have finished proving clause (c). \cite{10.1}

**Proposition 10.12.** Assume \cite{9.2} and let $Q$ be Souslin c.c.c. Then $Q^{[\omega]}$ (see \cite{9.3}) and Cohen do not commute.

**Proof** Assume that $Q^{[\omega]}$ and Cohen do commute. Let $\chi$ be large enough, $N \prec (H(\chi), \in)$ be countable such that $(Q, \eta) \in N$ (as in \cite{10.10}). Now we can interpret a Cohen real $\nu$ (over $V$) as a subset of $D^+_p \cap N$ called $g_\nu$. Thus it is Cohen$_N, Q$–generic over $V$ so $p_N[g_\nu]$ is well defined, and it belongs to $Q \backslash Q^{[\nu]}$ (by \cite{10.11}). Moreover, in $V[\nu]$ we have:

$$\{q \in Q^N : q, p_N(g_a) \text{ are incompatible } \} \text{ is dense in } Q^N.$$ 

Let $\langle \eta_\ell : \ell < \omega \rangle$ be generic for $(Q^{[\omega]}, \eta^{[\omega]})$ and let $\nu$ be Cohen generic over $V[\langle \eta_\ell : \ell < \omega \rangle]$. For each $\ell$, clearly $\eta_\ell$ is $(Q, \eta)$–generic over $V$, so let $\eta_\ell = \eta[G_\ell]$, where $G_\ell \subseteq Q$ is generic over $V$. Clearly $G_\ell \cap N$ is a subset of $Q^N$ generic over $V$ (by “$Q$ is strongly c.c.c.”). So $\langle G_\ell \cap N, g_\nu \rangle$ is a subset of $Q^N \ast (D^+_p \cap N, \supseteq)$ generic over $N$. By \cite{10.11} for any $q \in Q^N$ and $R \in (D^+_p \cap N)$, for some $R' \subseteq R$ and $q'$ we have $R' \in (D^+_p \cap N)$, $N \models \langle \forall a \in R' \rangle (p_a, q', q' \text{ are incompatible}).$

So look at the set

$$\{(q, R) \in Q^N \times (D^+_p \cap N) : (\forall a \in R')(p_a, q \text{ are incompatible})\}$$

-- there is $(q, R) \in (G \cap N) \times g_\nu$ which belongs to it. Hence, as in \cite{10.11}, for each $\ell$, $p_N[g_\nu]$ is incompatible with some $q \in G_Q[\eta_\ell]$. By the assumption that the forcing notions commute we know that $\langle \eta_\ell : \ell < \omega \rangle$ is generic for $(Q^{[\omega]}, \eta^{[\omega]})$ over $V(\nu)$. Necessarily (by FS + genericity) for some $\ell$ we have $F_Q^{\omega, \ell}(p_N(g_\eta)) \in G_Q[\langle \eta_\ell : \ell < \omega \rangle];$ a contradiction. \cite{10.12}

**Conclusion 10.13.** Assume \cite{9.2} and let $Q$ be Souslin c.c.c. Then $(Q, \eta)$ does not commute with Cohen (even above any $q \in Q$).
PROOF If we restrict ourselves above \( q_0 \in \mathbb{Q} \), the Hypothesis 10.2 still holds so we can ignore this. By 10.12 we have \((Q^{[\omega]}, \eta^{[\omega]})\) does not commute with Cohen. So by 9.7 we have that, for some \( n \), \((Q^{[n]}, \eta^{[n]})\) does not commute with Cohen and by 9.8 we finish.

**Proposition 10.14.** If \( Q \) is Souslin c.c.c. then for suitable \( \text{ZFC}_{*} \), \( Q \) satisfies \( 9.3 \).

**Proof** Let \( \rho \in \omega^2 \) be the real parameter in the definition of \( Q \). Let \( \text{ZFC}_{*} \) say:

(a) ZC (i.e. the axioms of Zermelo satisfied by \( (\mathcal{H}(\omega), \in) \)),

(b) \( Q \) (defined from \( \rho \) which is an individual constant) satisfies the c.c.c.

(c) for each \( n < \omega \), generic extensions for forcing notions of cardinality \( \leq \omega_\omega \) preserve (b) (and, of course (a)).

Now the desired properties are easy.

**Conclusion 10.15.** If \( Q \) is a Souslin c.c.c. forcing notion which is not \( \omega^\omega \)-bounding (say \( p \Vdash \text{"there is an unbounded } \eta \in \omega^\omega \text{"} \)), but adds an essentially non-Cohen real then \( Q \) does not commute with itself.

**Proof** By 21, \( Q \) adds a Cohen real; now by the assumptions, for some \( Q \)-name \( \eta \), \((Q, \eta) \in K^-c \). By 10.13 we know that \( Q \) and Cohen do not commute, so by 9.6(3) we are done.

**Conclusion 10.16.** If \( Q \) is a Souslin c.c.c. forcing notion adding a non-Cohen real, then the forcing by \( Q \) makes the old reals meagre.

**10. Some c.c.c. nep forcing notions are not nice.** We may wonder can we replace the assumption “\( Q \) is Souslin c.c.c.” by weaker one in \( \S 8 \) and in 20. We review limitations and then see how much we can weaken it.

**Proposition 11.1.** Assume that \( \eta^* \in \omega^2 \) and \( \aleph_1 = \aleph_1^L[\eta^*] \). Then there is a definition of a forcing notion \( Q \) (i.e. \( \varphi \)) such that

(a) the definition is \( \Sigma_1^1 \) (with parameter \( \eta^* \)), so \( p \in Q \), \( p \leq Q q, \text{ “} p,q \text{ incompatible”}, \{ p_n : n < \omega \} \subseteq a \text{ is a maximal antichain of } Q \text{” are preserved by forcing extensions},

(b) \( Q \) is c.c.c. (even in a forcing extension; even \( \sigma \)-centered),

(c) there is \( Q \)-name \( \eta \) of a generic for \( Q \),

(d) \( \eta \) is not essentially Cohen (preserved by extensions not collapsing \( \aleph_1 \)), in fact has cardinality \( \aleph_1 \),

(e) \( Q \) commutes with Cohen,

(f) \( Q \) is nep (though not Souslin c.c.c.).
Proof  A condition $p$ in $Q$ is a quadruple $(E_p, X_p, u_p, w_p)$ consisting of:
a 2-place relation $E_p$ on $\omega$ and subset $X_p$ of $\omega$ and a finite subset $u_p$ of $X_p$
and a finite subset $w_p$ of $\omega$ such that:

$N_p \overset{\text{def}}{=} (\omega, E_p)$ is a model of ZFC$^- + V = L$ (let $\mathcal{N}^*_p$ be the
canonical ordering of $N_p$, we do not require well foundedness)
such that:

$(N_p, X_p) \models " (\alpha) every \ x \in X_p \ is \ an \ infinite \ subset \ of \ \omega,$
$(\beta) if \ x \neq y \ are \ from \ X_p \ then \ x \cap y \ is \ finite,$
$(\gamma) if \ x \in X \ then \ there \ is \ no \ y \ satisfying$
y $<^*_p x \ \text{&} \ (\forall z \in X_p)(z <^*_p x \Rightarrow z \cap y \ finite) \ \text{&} \ y \ an \ infinite \ subset \ of \ \omega,$
$(\delta) \ \bigwedge_{n<\omega} (\forall z_1 \ldots z_n \in X_p)(\bigwedge_{\ell=1}^n z_\ell <^*_p x \Rightarrow (\exists^\infty m < \omega)(m \notin x \cup \bigcup_{\ell=1}^n z_\ell) ".$

The order is defined by:  \( p \leq q \) if and only if one of the following occurs:

(A)  $p = q$,

(B)  there are $Y \subseteq \omega$ and $a \in N_q$ and $f \in Y \ \omega$ such that:
    (i)  $[x \in Y \ \& \ N_p \models y \in x] \ \Rightarrow \ y \in Y,$
    (ii)  $[N_p \models \text{"rk}(x) = y", y \in Y] \ \Rightarrow \ x \in Y,$
    (iii)  $N_p \models \text{"}a \text{ is a transitive set"},$
    (iv)  the set $\{x : N_p \models \text{"}x \text{ an ordinal"}, x \notin Y\}$ has no first element,
    (v)  $N_q \models \text{"}a \text{ is a transitive set"},$
    (vi)  $f$ is an isomorphism from $N_p \models Y \text{ onto } N_q \models \{b : N_q \models b \in a\},$
    (vii)  $f \text{ maps } X_p \text{ onto } X_q \models \text{Rang}(f),$  
    (viii)  $f \text{ maps } u_p \text{ onto } u_q \text{ onto } \text{Rang}(f),$  
    (ix)  $w_p \subseteq w_q,$
    (x)  if $n \in w_q \setminus w_p$ and $x \in f(u_p)$ then $N_q \models \text{"the } n \text{-th natural number}$
does not belong to $x$".

The reader can now check (note that $w = \bigcup\{w^p : p \in \mathcal{G}_Q\}$ is forced to be
an infinite subset of $\omega$ almost disjoint to every $A \in X^*$, $X^*$ a reasonably
defined MAD family in $L$); see more details in the proof of \[11.3\] \[11.3\]

Proposition 11.2. Assume $V = L$. There is $Q = Q_0 \ast Q_1$ such that:

(a)  $Q_0$ is nep c.c.c. not adding a dominating real,
(b)  if $Q_0 \ast Q_1$ is nep c.c.c. (even Souslin c.c.c.) not adding a dominating real",
(c)  $Q$ adds a dominating real,
(d)  in fact, $bQ_0$ is the Cohen forcing (so in any $V_1$ it is c.c.c. strongly
c.c.c., correct, very simple nep (and snep), and it is really absolute,
i.e. it is the same in $V_1$ and $V$, and its definition uses no parameters),
(e) moreover, $Q_1$ is defined in $L$, really absolute, and in any $V_1$ it is c.c.c., strongly c.c.c. nep (and even snep). In $V_1$, $Q_1$ adds a dominating real iff $(\omega')^L$ is a dominating family in $V_1$.

**Proof** Let $Q_0$ be Cohen. We shall define $Q_1$ in a similar manner as $Q$ in the proof of [11.1.]

A condition in $Q_1$ is a triple $(E_p, u_p, w_p)$ such that $E_p$ is a 2-place relation on $\omega$, $u_p$ is a finite subset of $\omega$ and $w_p$ is a finite function from a subset of $\omega$ to $\omega$ and:

$$N_p = (\omega, E_p)$$

where $N_p$ is a model of ZFC$^-$ $+$ $V = L$ (let $N_p$ be the canonical ordering of $N_p$, we do not require well-foundedness); so in formulas we use $\in$.

[What is the intended meaning of a condition $p$? Let]

$$M_p = N_p \upharpoonright \{x : (\text{trans}(x))^{N_p}, E_p \upharpoonright \text{trans}(x)^{N_p} \text{ is well founded}\},$$

where $\text{trans}(x)$ is the transitive closure of $x$. Let $M'_p$ be the Mostowski collapse of $M_p$, $h_p : M_p \longrightarrow M'_p$ be the isomorphism. Now, $p$ gives us information on the function $w = \bigcup \{w_p : p \in G\}$ from $\omega$ to $\omega$, it says: $w$ extends the function $w_p$ and if $x \in M_p \cap u_p$ is a function from $\omega$ to $\omega$ then for every natural number $n \notin \text{Dom}(w_p)$ we have $x(n) \leq w(n)$. Note that $h_p(x)$ is a function from $\omega$ to $\omega$ iff $M_p \models \text{“} x \text{ is a function from } \omega \text{ to } \omega \text{”}$ iff $N_p \models \text{“} x \text{ is a function from } \omega \text{ to } \omega \text{”}$]

The order is defined by: $p \leq q$ if and only if one of the following occurs:

(A) $p = q$,

(B) there are $Y \subseteq \omega$ and $a \in N_q$ and $f \in Y_\omega$ such that

(i) $[x \in Y \Rightarrow N_p \models y \in x] \Rightarrow y \in Y$,

(ii) $[N_p \models \text{rk}(x) = y \Rightarrow y \in Y] \Rightarrow x \in Y$,

(iii) $N_p \models Y$ is a model of ZFC$^- \ + \ V = L$,

(iv) the set $\{x : N_p \models \text{“} x \text{ is an ordinal”, } x \notin Y \}$ has no first element (by $E_p$),

(v) $N_q \models \text{“} a \text{ is a transitive set”},$

(vi) $f$ is an isomorphism from $N_q \upharpoonright Y$ onto $N_q \upharpoonright \{b : N_q \models b = a\}$,

(vii) $f$ maps $u_p \cap Y$ into $u_q \cap \text{Rang}(f)$,

(viii) $w_p \subseteq w_q$,

(ix) if $n \in \text{Dom}(w^q) \setminus \text{Dom}(w^p)$ and $x \in u_p$, $N_p \models \text{“} x \text{ is a function from the natural numbers to the natural numbers”}$ and $x^* = f(x)$ then $N_q \models \text{“} y \text{ is the } n\text{-th natural number then } w^q(y) \geq x(y)\text{”}$.

Clearly $Q$ is equivalent to $Q' = \text{(the Hechler forcing)}^L$, just let us define, for $p \in Q_1$, $g(p) = (w^p, F^p)$ where $F^p = \{h_p(x) : x \in M_p\}$. Now, $g$ is onto
The order is given by:

\[ Y \]

Claim 11.3.1. \( Q \) is a quasi order.

The rest is left to the reader.

**Proposition 11.3.** 1. Assume that:

(a) \( \bar{\varphi} = (\varphi_0(x), \varphi_1(x,y)) \) defines, in any model of \( \text{ZFC}^- \), a forcing notion \( Q_{\bar{\varphi}} \) with parameters from \( L_{\omega_1} \),

(b) for every \( \beta < \omega_1 \) such that \( L_{\beta} \models \text{ZFC}^- \), for every \( x,y \in L_{\beta} \) we have:

\[ [x \in Q_{\bar{\varphi}}^\beta \iff x \in Q_{\bar{\varphi}}^{L_{\beta}}, \quad \text{and} \quad [x < y \in Q_{\bar{\varphi}}^\beta \iff x < y \in Q_{\bar{\varphi}}^{L_{\beta}}], \]

(c) for unboundedly many \( \alpha < \omega_1 \) we have \( L_{\alpha} \models \text{ZFC}^- \),

(d) any two compatible members of \( Q_{\bar{\varphi}}^{L_{\omega_1}} \) have a lub,

(e) like (c) for compatibility and for existence of lub.

Then there is an \( N_0 \)-snef forcing notion \( Q \) equivalent to \( Q_{\bar{\varphi}}^{L_{\omega_1}} \): the \( \Sigma_1^1 \) (i.e. Souslin) relations have just the real parameters of \( \bar{\varphi} \).

2. We can use a real parameter \( \rho \) and replace \( L_{\alpha} \) by \( L_{\alpha}[\rho] \).

**Proof** It is similar to the proof of 11.1. Let \( Q \) be the set of quadruples \( p = (E_p, n_p, a_p, \bar{a}_p) \) such that:

(a) \( E_p \) is a two-place relation on \( \omega \),

(b) \( N_p \overset{\text{def}}{=} (\omega, E_p) \) is a model of \( \text{ZFC}^- + V = L \),

(c) for some \( n = n_p \) we have

\[ \bar{a}_p = \langle a_{p,\ell} : \ell < n \rangle, \quad \bar{a}_p = \langle a_{p,\ell} : \ell < n \rangle, \]

(d) \( N_p \models \langle a_{p,\ell} : \ell < n \rangle \) is an ordinal, \( a_{p,\ell} \in L_{\alpha_{p,\ell}}, \ L_{\alpha_{p,\ell}} \models \text{ZFC}^- \), and for \( k \leq \ell < n \) we have \( L_{\alpha_{p,\ell}} \models \varphi_0(a_{p,k}), \varphi_1(a_{p,k}) \),

\[ (e) \text{ if } m \leq k \leq \ell < n \text{ then } N_p \models \varphi_1(a_p, a_m, a_{p,a_k}) \]

The order is given by: \( p_0 \leq p_1 \) if and only if \( (p_0 \models p_1) \) and for some \( Y_0, Y_1 \subseteq \omega \) and \( f \) we have:

(i) for \( \ell = 0,1 \): \( Y_\ell \) is an \( E_p \)-transitive subset of \( N_p \),

\[ (\forall x \in N_{p_0})(x \in Y_\ell \iff \text{rk}^{N_{p_0}}(x) \in Y_\ell), \]

(ii) \( f \) is an isomorphism from \( N_{p_0} | Y_0 \) onto \( N_{p_1} | Y_1 \),

(iii) \( f \) maps \( \{x \in N_{p_0} : N_{p_0} \models \text{“}x \text{ is an ordinal} \} \) there is no \( E_p \)-minimal element,

(iv) \( f \) maps \( \{a_{p_0,\ell} : \ell < n \} \cap Y_0 \) into \( \{a_{p_1,\ell} : \ell < n_{p_1} \} \cap Y_1 \),

(v) if \( f(a_{p_0,\ell}) = a_{p_1,\ell} \) and then \( N_{p_1} \models \varphi_1(f(a_{p_0,\ell}), a_{p_1,\ell}) \).

**Claim 11.3.1.** \( Q \) is a quasi order.
Proof of the claim: Check.

Now define \( M_p, h_p, M'_p \) as in the proof of \( \ref{11.2} \).

Claim 11.3.2. The set
\[
\mathcal{Q}' \triangleq \{ p \in \mathcal{Q} : N_p \text{ is well founded, } n_p > 0 \}
\]
is dense in \( \mathcal{Q} \).

Proof of the claim: Check.

Define \( g : \mathcal{Q}' \rightarrow \mathcal{Q}^L_{\omega_1} \) by \( g(p) = h_p(a_{p,n_p-1}) \).

Claim 11.3.3. \( g \) is really a function from \( \mathcal{Q}' \) onto \( \mathcal{Q}^L_{\omega_1} \) and
\[
p_0 \leq \mathcal{Q} p_1 \Rightarrow \mathcal{Q}^L_{\omega_1} \models g(p_0) \leq g(p_1) \Rightarrow \]
\[
\text{[if } p_1 \leq \mathcal{Q} p_2 \text{ then for some } p_3 \text{ we have } p_2 \leq \mathcal{Q} p_3 \text{ and } p_0 \leq \mathcal{Q} p_3 \text{].}
\]

Proof of the claim: The first implication is immediate (by clause (v) in the definition of \( \leq \mathcal{Q} \)). For the second implication assume \( \mathcal{Q}^L_{\omega_1} \models g(p_0) \leq g(p_1) \) and let \( p_1 \leq \mathcal{Q} p_2 \). For \( \ell = 0, 1, 2 \) let
\[
n_\ell = \min\{ n : n = n_p \text{ or } n < n_{p_\ell} \text{ and } \alpha_{p_\ell,n} \notin M_p \}.
\]
Let \( p_3 \) be defined as follows: \( M_p = L_\gamma, L_\gamma \models ZFC^\_ - \), and \( \gamma > M_{p_0}^\_ \cap \omega_1, M_{p_1}^\_ \cap \omega_1, M_{p_2}^\_ \cap \omega_1 \). Let \( g^\_ \) be the isomorphism from \( M_{p_\ell} \) onto \( L_{\gamma_\ell}, \gamma_\ell < \gamma, \) and let \( w = \{ f_\ell(\alpha_{p_\ell,m}) : m < n_\ell, \ell < 2 \} \). List it as \( \{ \alpha_{p_\ell,k} : k < n_{p_\ell} \} \) (increasing enumeration) and let \( Y = \{ f_\ell(a_{p_\ell,m}) : m < n_\ell, \ell < 2 \} \). Now, \( f_2(a_{p_2,n_2-1}) \) is a \( \leq \mathcal{Q}^L_{\omega_1} \)-upper bound of \( Y \). Consequently, by clauses (d) and (e) of the assumptions, we can define \( a_{p_3,m} \) as required.

Proposition 11.4. Assume that \( \varphi = \varphi(x,y) \) is such that

(i) \( ZFC^\_ - \vdash \) for every infinite cardinal \( x \in X \triangleq \{ \alpha : \alpha = \omega \text{ or } \omega^\alpha = \alpha \text{ (ordinal exponentiation)} \} \), there is a unique \( A_x \), an unbounded subset of \( x \) of order type \( x \) such that \( \varphi(x,A_x) \), and \( \psi(\cdot) \) defines a set \( S \subseteq X \) not reflecting,

(ii) \( ZFC^\_ - \vdash \) if \( \mu_1 < \mu_2 \) are from \( X \) then \( A_{\mu_1} \nsubseteq A_{\mu_2} \),

(iii) \( \omega_1 = \sup\{ \alpha : L_\alpha \models ZFC^\_ - \}, \) and the truth value of \( \beta \in A_\gamma, \beta \in S \) is the same in \( L_\alpha \) for every \( \alpha < \omega_1 \) for which \( L_\alpha \models ZFC^\_ - \),

(iv) the set \( S \), i.e. \( \{ \beta < \omega_1 : (\exists \alpha)(L_\alpha \models ZFC^\_ - \& \psi(\beta)) \} \), is a stationary subset of \( \omega_1 \)

[a kind of \( \forall Y^1 \] is below first ineffable of \( L \) and is not weakly compact].

Then for some \( \tilde{\varphi} \) as in the assumptions of \( \ref{11.3} \) and \( \eta \) we have:
(a) $\mathbb{Q}_{\omega}$ is a c.c.c. forcing notion,

(b) $\eta \in \omega^2$ is a generic real of $\mathbb{Q}_{\omega}$, and is nowhere essentially Cohen,

(c) $\mathbb{Q}_{\omega}$ commute with Cohen.

**Proof** Let $pr(\alpha, \beta) = (\alpha + \beta)(\alpha + \beta) + \alpha$, it is a pairing function. By coding, without loss of generality (e.g. letting

\[ A_\alpha = \{ pr^+(n, pr_n(\beta_1, \ldots, \beta_n)) : n < \omega, \{ \beta_1, \ldots, \beta_n \} \subseteq B_\alpha \}, \]

where $pr_1(\beta) = \beta$, $pr_{n+1}(\beta_1, \ldots, \beta_{n+1}) = pr(pr_n(\beta_1, \ldots, \beta_n), \beta_{n+1})$

(ii) if $x, x_1, \ldots, x_n$ are distinct cardinals in $L_{\omega_1}$, then $A_\alpha \not\subseteq \bigcup_{\ell=1}^n A_{x^\ell}$.

For $\delta \in X$ let $f_0(\delta) = \min(X \setminus (\delta + 1))$ and let $f_0^1$ be the first (in the canonical well ordering of $L$) one-to-one function from $f_0(\delta)$ onto $\delta$. Let $C_\delta$ be the first club of $\delta$ disjoint to $S$. For $\alpha \in [\omega, \omega_1)$, let $\delta_\alpha = \max(X \cap \delta)$ and let

\[ B^*_\alpha = \{ pr_3(\varepsilon, \zeta, \xi) : \varepsilon \in C_\delta, \zeta = f_0^1(\alpha), \xi \in A_{\delta_\alpha} \} \text{ and } \varepsilon > \zeta, \xi > \eta. \]

Note that

(*) $B^*_\alpha$ is an unbounded subset of $\delta_\alpha$ such that

(a) $\beta \in S \cap \alpha \Rightarrow \beta > \sup(B^*_\alpha \cap \beta)$,

(b) if $\alpha_1, \ldots, \alpha_n \in [\omega, \omega_1) \setminus \{ \alpha \}$ then $B_\alpha \setminus \bigcup_{\ell=1}^n B^*_\alpha$ is unbounded in $\delta_\alpha$.

[Why? For (a), suppose that $\beta \in S \cap \alpha$. Trivially, $\min(B^*_\alpha) > \min(C_\alpha)$, so $\gamma = \sup(C_\delta \cap \beta)$ is well defined. Now,

\[ B^*_\alpha \cap \beta \subseteq \{ pr_3(\varepsilon, \zeta, \xi) : \varepsilon, \zeta, \xi \leq \gamma \} \subseteq (\gamma + \gamma + \gamma)^3 < \beta \]

(the last inequality follows from the fact that $\beta \in X$). To show (b) suppose that $\gamma_0 < \delta_\alpha$ and choose $\xi \in B^*_\alpha \setminus \bigcup_{\ell=1}^n B^*_\alpha$. Let $\xi = f_0^1(\alpha)$ and let $\varepsilon \in C_\delta$ be large enough. So $pr_3(\varepsilon, \zeta, \xi) \in B^*_\alpha$ (by definition) and $pr_3(\varepsilon, \zeta, \xi) \not\in B^*_\alpha$ (use the third coordinate) and $pr_3(\varepsilon, \zeta, \xi) > \varepsilon > \gamma_0$.]

Let $I_\alpha$ be the ideal of subsets of $B^*_\alpha$ generated by

\[ \{ B^*_\alpha \cap B^*_\beta : \omega \leq \beta < \omega_1, \beta \neq \alpha \} \cup \{ B^*_\alpha \cap \beta : \beta < \delta_\alpha \}. \]

Let $\mathbb{Q}$ be the set of finite functions $p$ from $\omega_1 \setminus \omega$ to $\{0, 1, 2\}$ ordered by:

$p \leq q$ if and only if:

- if $\alpha \in \text{Dom}(p)$, $\beta \in \text{Dom}(q) \cap A_\alpha \setminus \text{Dom}(p)$
- then $q(\beta) = p(\alpha)$ and $\beta > \sup(\delta_\alpha \cap \text{Dom}(p)) \lor q(\beta) = 2$.

**Claim 11.4.1.** $\mathbb{Q}$ is a partial order.
Claim 11.4.2. For each \( \alpha \in [\omega, \omega_1) \) the set \( I_\alpha = \{ p : \alpha \in \text{Dom}(p) \} \) is dense in \( Q \).

Proof of the claim: Let \( p \in Q \) and suppose that \( \alpha \notin \text{Dom}(p) \). Let 
\[ q = p \cup \{ (\alpha, 2) \}. \]
Let \( \bar{f} \) be the \( Q \)-name defined by \( \models \bar{f} = \bigcup G_Q \).

Claim 11.4.3. For \( \alpha \in [\omega, \alpha) \),
\[ \models_{Q} \text{"for some } \ell < 3, \text{ for any } m < 3 \text{ we have } \{ \beta \in B^*_\alpha : f(\beta) = m \} \neq \emptyset \text{ mod } I_\alpha \text{ iff } \bar{f} \in \{ \ell, \ell \} \". \]

Proof of the claim: Take \( p \in G_Q \) such that \( \alpha \in \text{Dom}(p) \) and let 
\[ B = \bigcup \{ B^*_\alpha \cap B_\gamma : \gamma \in \text{Dom}(p) \setminus \{ \alpha \} \}, \]
so \( B \in I_\alpha \). Clearly, \( p \models \text{"if } \beta \in B^*_\alpha \setminus B \text{ then } f(\beta) \in \{ 2, p(\alpha) \} \" \).

Now, if \( B' \in I_\alpha \), and \( p \leq Q q \) then there is \( \gamma \in A_\alpha \setminus B' \setminus \bigcup \{ B_\gamma^* : \gamma \in \text{Dom}(p) \setminus \{ \alpha \} \} \) such that \( q \cup \{ (\gamma, 2) \} \) and \( q \cup \{ (\gamma, p(\alpha)) \} \) are in \( Q \) above \( q \).
Reflecting we are done.

Claim 11.4.4. One can define \( f \) from \( \bar{f} \restriction \omega \in \omega^3 \).

Proof of the claim: Define \( f \restriction \alpha \) by induction on \( \alpha \in X \) using 11.4.3.

Claim 11.4.5. The forcing notion \( Q \) is nowhere essentially Cohen.

Proof of the claim: For every \( \alpha^* < \omega_1 \) and for every large enough \( \gamma < \omega_1 \) and for \( \ell \in \{ 0, 1, 2 \} \), the condition \( q_\ell = \{ (\gamma, \ell) \} \) is compatible with every \( q \in Q \) such that \( \text{Dom}(p) \subseteq \alpha^* \).

Claim 11.4.6. The \( Q \)-name \( f \) (for a real) is nowhere essentially Cohen.

Proof of the claim: By 11.4.4, 11.4.3.

Claim 11.4.7. The forcing notion \( Q \) satisfies the demands in 11.4.

Proof of the claim: Check.

Claim 11.4.8. The forcing notion \( Q \) satisfies the c.c.c.

Proof of the claim: Use "\( S \subseteq \omega_1 \) is stationary".

Remark 11.5. 1. Of course, such forcing can make \( \aleph_1 \) to be \( \aleph_1^{L[\eta]} \). But it seems that we can have such forcing which preserves the \( L_{\omega_1} \)-cardinals (and even their being "large" in suitable senses). For this it should be like "coding the universe by a real" of Jensen Beller Welch [1], and see Shelah Stanley [21].
2. Instead of coding $\aleph_1$–Cohen we can iterate adding dominating reals or whatever.

**Definition 11.6.** 1. We say that forcing notions $Q_0, Q_1$ are equivalent if their completions to Boolean algebras $(BA(Q_0), BA(Q_1))$ are isomorphic.

2. Forcing notions $Q_0, Q_1$ are locally equivalent if
   
   (i) for each $p_0 \in Q_0$ there are $q_0, q_1$ such that
   
   $p_0 \leq q_0 \in Q_0 \& q_1 \in Q_1 \& BA(Q_0 \mid (\geq q_0)) \cong BA(Q_1 \mid (\geq q_1))$,
   
   (ii) for every $p_1 \in Q_1$ there are $q_0, q_1$ such that
   
   $q_0 \in Q_0 \& p_1 \leq q_1 \in Q_1 \& BA(Q_0 \mid (\geq q_0)) \cong BA(Q_1 \mid (\geq q_1))$.

Now we may phrase the conclusions of 11.3, 11.4.

**Proposition 11.7.** 1. Assume $\bar{\varphi}_1 = (\varphi_1^0, \varphi_1^1)$ and $\bar{\varphi}_2 = (\varphi_2^0, \varphi_2^1)$ are as in 11.3. Then we can find $\bar{\varphi}_3$ as there, only with the parameters of $\bar{\varphi}_1, \bar{\varphi}_2$ and such that:

   (a) if in $L_{\omega_1}$ there is a last cardinal $\mu$ (i.e. $\aleph^V_1$ is a successor cardinal in $L$), then $Q^L_{\bar{\varphi}_3}$ is locally equivalent to

   $$\bigcup\{Q^L_{\bar{\varphi}_3} : \mu < \alpha, L_\alpha \models \text{is the last cardinal}\},$$

   (b) if in $L_{\omega_1}$ there is no last cardinal (i.e. $\aleph^V_1$ is a limit cardinal in $L$), then $Q^L_{\bar{\varphi}_3}$ is locally equivalent to

   $$\bigcup\{Q^L_{\bar{\varphi}_3} : L_{\omega_1} \models \alpha \text{ a cardinal}\}.$$

2. In 11.3, 11.4(1) we can replace $L_{\omega_1}$ by $L_{\omega_1}[\eta^*], \eta^* \in \omega_1$.

3. In 11.3, 11.4(1) we can replace $L_{\omega_1}$ by $L_{\omega_1}[A]$ where $A \subseteq \omega_1$ but have $\aleph_1$–snep instead of $\aleph_0$–snep.

**Proof**  Let $\varphi_{3,0}(x)$ say

   (i) $x = \langle \bar{a}^x, \bar{\beta}^x, \bar{\alpha}^x, \bar{\beta}^x \rangle$, $\bar{a}^x = \langle \alpha^x_\ell : \ell \leq n^x \rangle$, $\langle \beta^x_\ell : k \leq k^x_\ell \rangle \leq n^x$, $\bar{a}^x = \langle \alpha^x_\ell : \ell \leq n^x \rangle$, $\bar{\beta}^x = \langle \beta^x_\ell : \ell < n^x \rangle$,

   (ii) $\alpha^x_\ell < \beta^x_\ell \leq \beta^x_\ell \leq \beta^x_\ell$, $L_{\beta^x_\ell} \models \alpha^x_\ell$ is the last cardinal”,

   (iii) $L_{\beta^x_\ell} \models \varphi_{1,0}(\beta^x_\ell)$, $L_{\alpha^x_\ell+1} \models \varphi_{2,0}(\alpha^x_\ell)$ for $\ell < n^x$,

   (iv) $L_{\beta^x_\ell} \models \alpha^x_\ell$ is a cardinal”,

   (v) $L_{\alpha^x_\ell+1} \models \varphi_{3,1}(\alpha^x_\ell, \alpha^x_\ell+1)$.

Let $\beta(x) = x$. Let $\varphi_{3,1}(x, y)$ say:

   (a) $\beta^x_\ell \leq \beta^y_\ell$,

   (b) $\{\alpha^x_\ell : \ell \leq n^x \text{ and } L_{\beta^x_\ell} \models \alpha^x_\ell \text{ is a cardinal}\}$ is a subset of $\{\alpha^y_\ell : \ell \leq n^y\}$,
Thesis 12.1. Nep forcing notions do not discern sets then diagonalization, say between $X, Y$

Now check.

11. Preservation of “no dominating real”. The main result of §7: (for homogeneous c.c.c. $\mathbb{Q}$) if a nep forcing $\mathbb{P}$ preserves $\langle \omega^\omega \rangle^V \in (I^+_{(\mathbb{Q}, \eta)})^+$ then it preserves $X \in (I^+_{(\mathbb{Q}, \eta)})^+$ (see §8) is a case of the following

**Thesis 12.1.** Nep forcing notions do not discern sets $X \subseteq \omega^\omega$ built by diagonalization, say between $X, Y \subseteq \omega^\omega$ which are generic enough.

But there are interesting cases not covered by §8, most prominent is:

**Question 12.2.** If a nep forcing notion preserves “$F \subseteq \omega^\omega$ is unbounded” for some (unbounded) $F \subseteq \omega^\omega$ then does it preserve this for every (unbounded) $F' \subseteq \omega^\omega$?

**Definition 12.3.** 1. For a Borel 2-place relation $\mathcal{R}$ on $\omega^\omega$ let $I_\mathcal{R}$ be the $\mathcal{R}_1$-complete ideal on $\omega^\omega$ generated by the sets of the form $A_\nu = \{ \eta \in \omega^\omega : \neg (\eta R \nu) \}$.

2. We say that $\nu$ is $\mathcal{R}$-generic over $N$ if $\eta \in N \cap \omega^\omega \Rightarrow \eta R \nu$.

3. A forcing notion $\mathbb{P}$ is weakly $\mathcal{R}$-preserving if for any $\eta_0, \eta_1, \ldots , \eta_n, \ldots \in (\omega^\omega)^V$ there is $\nu \in (\omega^\omega)^V$ such that $n < \omega \Rightarrow \eta_n R \nu$ (i.e. $\Vdash_{\mathbb{P}} (\omega^\omega)^V \in I_\mathcal{R}^+$).

4. We say that a forcing notion $\mathbb{P}$ is $\mathcal{R}$-preserving if for any Borel subset $B$ of $\omega^\omega$ from $V$ which is in $I_\mathcal{R}^+$, for any $\eta_0, \eta_1, \ldots , \eta_n, \ldots \in (\omega^\omega)^V$ there is $\nu \in B^V$ such that $n < \omega \Rightarrow \eta_n R \nu$ (i.e. $\Vdash_{\mathbb{P}} B^V \in I_\mathcal{R}^+$).

5. We say that a forcing notion $\mathbb{P}$ is strongly $\mathcal{R}$-preserving if for any $X \in I_\mathcal{R}^+$ (in $V$) we have $\Vdash_{\mathbb{P}} "X \in I_\mathcal{R}^+"$.

6. We say that a forcing notion $\mathbb{P}$ is super $\mathcal{R}$-preserving as witnessed by $(\mathcal{B}, \text{ZFC}^-)$ if every $(\mathcal{B}, \text{ZFC}^-)$-candidate is a $\mathbb{P}$-candidate, for any $(\mathcal{B}, \text{ZFC}^-)$-candidate $N$ such that $p \in N$ and for any $\nu \in \omega^\omega$ which is $\mathcal{R}$-generic over $N$, there is $q$ such that $p \leq q \in \mathbb{P}$, $q$ is $(N, \mathbb{P})$-generic and $q \Vdash "\nu$ is $\mathcal{R}$-generic over $N \langle G^\mathbb{P} \rangle = N \langle G^\mathbb{P} \cap \mathcal{P}^N \rangle$".

\[11.3\]
Proposition 12.4. 1. If $P$ is super $R$–preserving, then $P$ is strongly $R$–preserving (also for $P$ nep).

2. If $P$ is strongly $R$–preserving, then $P$ is $R$–preserving.

3. If $P$ is $R$–preserving, then it is weakly $R$–preserving.

Proof Easy.

Proposition 12.5. A sufficient condition for “$P$ is super $R$–preserving as witnessed by $(\mathcal{B}, \text{ZFC}^{++})$” is that for some nep forcing notion $Q$ and hc-$Q$-name $\tilde{\eta}^*$ and a Borel relation $R_1$ we have

(a) every $(\mathcal{B}, \text{ZFC}^{++})$–candidate is a $Q$–candidate and $\tilde{\eta}^* \in N$ and also it is a $P$–candidate,

(b) if $N$ is a $(\mathcal{B}, \text{ZFC}^{++})$–candidate (so countable) and $\nu$ is $R$–generic over $N$ then for some $G_Q \subseteq Q^N$ generic over $N$, in $V$ we have

$$\forall x (x R_1 \tilde{\eta}^*|G_Q| \Rightarrow x R_1 \nu)$$

and for every $G_Q \subseteq Q^N$, generic over $N$, and $p \in P^N$ there is $q$ such that $p \leq q \in \mathbb{P}$ and $q$ is $(N, \mathbb{P})$–generic and

$$q \models “ \tilde{\eta}^*|G_Q| \text{ is } R_1 \text{–generic over } N(G_P \cap P^N) “.$$ 

Proof Straight.

Remark 12.6. In [12.7] below we phrase a sufficient condition. Note that clause $(\delta)$ can be naturally phrased as “an appropriate sentence $\psi$ follows from $\text{ZFC}^{++}$; this is slightly stronger as possibly $\psi$ holds only for all $(\mathcal{B}, \text{ZFC}^{++})$–candidates but not for some (e.g. non-well founded) models of $\text{ZFC}^{++}$ (this does not matter).

Proposition 12.7. Assume that:

(a) $P, Q$ are nep forcing notions, $\tilde{\eta}^*$ is a hc-$Q$–name, $R, R_1$ are Borel relations,

(b) every $(\mathcal{B}, \text{ZFC}^{++})$–candidate $N$ is a $Q$–candidate and $P$–candidate, $\tilde{\eta}^* \in N$,

(c) for every $(\mathcal{B}, \text{ZFC}^{++})$–candidate $N$ and $\nu \in \omega_\omega$ which is $R$–generic over $N$ and $r \in Q^N$ we can find $G_Q \subseteq Q^N$ generic over $N$ such that

$$r \in G_Q \quad \text{and} \quad \forall x (x R_1 \nu \Rightarrow x R_1 \tilde{\eta}^*|G_Q|),$$

(d) if $N$ is a $(\mathcal{B}, \text{ZFC}^{++})$–candidate then for every $G_Q \subseteq Q^N$ generic over $N$ and $G_R \subseteq \text{Levy}(\aleph_0, 2^{[P]} + 2^{[Q]})^N$ generic over $N[G_Q]$ we have

$$N[G_Q][G_R] \models “ \text{there are } G_Q \text{ and } q, p \leq q \in \mathbb{P} \text{ such that } q \text{ is explicitly } (N, \mathbb{P})\text{–generic and } G_Q \text{ is a generic over } N \text{ subset of } Q^N \text{ and } q \models “ \tilde{\eta}^*|G_Q| \text{ is } R_1 \text{–generic over } N[G_Q, G_P] “.$$
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Then “\( \mathbb{P} \) is super \( \mathcal{R} \)-preserving as witnessed by ZFC\(_s^s \)”.

**Proof** Clause (a) of \[12.3\] holds by clause (b) here. Next, the first demand in clause \[12.3\](b) (“for some \( G_Q \subseteq Q^N \) generic over \( N^N \)”)

Then suppose that \( G_Q \subseteq Q^N \) is generic over \( N \), equivalently, \( \nu^* \) is a \( Q \)-generic real. Let \( G \subseteq \text{Levy}(\mathbb{N}_0, |2^{|P|}|^N) \) be generic over \( N \), equivalently over \( N[\nu^*] \). In \( N \), by clause (e) of the assumptions, in \( N[G] \), there is a semi \( \mathbb{P} \)-candidate \( M \), \( \mathcal{P}(\mathbb{P}^N)^N = \mathcal{P}(\mathbb{P}M)^M \). So in \( N[G], M[G] \) is a \( \mathbb{P} \)-candidate. So there is \( q \in \mathbb{P}^N[G] \) such that \( N[G] \models “p \leq q \) and \( q \) is \( (M, \mathbb{P}^M) \)-generic”. As above possibly increasing \( q \),

\( (\oplus) \) \( N[G] \models [q \mathbb{P} “ \nu \) is \( \mathcal{R} \)-generic over \( M[G\mathbb{P}] “ \) and \( \nu \) is Cohen over \( M \).

So for some \( \nu \),

\[
N \models “ \nu, q \) are \( \text{Levy}(\mathbb{N}_0, |2^{|P^0|}|^N) \)-names of a Cohen real and a member of \( \mathbb{P} \), respectively, and some \( r \in \text{Levy}(\mathbb{N}_0, (|2^{|P^0|}|)^N) \) forces the statement \((\oplus)\) above on \( q, \nu \” .
\]

Now we can find \( G' \subseteq \text{Levy}(\mathbb{N}_0, (|2^{|P^0|}|)^N) \) generic over \( N \) to which \( r \) belongs and \( \nu[G'] =^* \nu \) (i.e. they are equal except for finitely many coordinates).

Let \( q' \in \mathbb{P} \) be \( \geq \) \( q[G'] \) and be \( (N(G'), \mathbb{P}^{N(G')}) \)-generic, so we are done. \( \blacksquare \)

**Theorem 12.8.** Assume that:

(a) \( \mathcal{R} \) is: \( fRg \) iff \( g \) is non-decreasing and \( (\exists \omega^n)(f(n) \geq g(n)) \);

(b) \( \mathcal{Q} = (\eta \in \omega > \omega : \eta \) non-decreasing\}), \( \eta^* \) is the generic real \( \bigcup G_Q \) (so really \( \mathcal{Q} \) is the Cohen forcing),

(c) \( \mathbb{P} \) is nep,

(d) every \( (\mathcal{B}, \text{ZFC}^s) \)-candidate \( N \) is a \( \mathbb{P} \)-candidate (and easily it is a \( Q \)-candidate),

(e) \( \text{ZFC}^s \) says: \( “\mathbb{P} \) is nep, \( \mathcal{P}(\mathbb{P}) \in \mathcal{H}(\chi), \mathcal{H}(\chi) \) is a semi \( \mathbb{P} \)-candidate and after forcing with \( \text{Levy}(\mathbb{N}_0, |2^{\mathbb{P}}| + 2^{\mathbb{Q}}) \) still is, and forcing with \( \mathbb{P} \) does not add a dominating real”.

Then the conditions (a)–(d) of \[12.4\] hold.

**Proof** Let \( N \) be a \( \mathbb{P} \)-candidate and let \( q \in \mathbb{Q} \). Now, for any \( \nu \in \omega \omega \)

there are \( g_1, g_2 \) such that

(a) \( g_1 \) is a subset of \( \mathbb{Q}^N \) generic over \( N \) to which \( q \) belongs; and let \( \eta^*_q = \eta^*[g_2] \in \omega \omega \),

(b) \( m \in [g(q), \omega) \Rightarrow \eta_1(m) = \nu(m) \lor \eta_2(m) = \nu(m) \lor \nu(m) = 0 \).

[Why? Quite easy, letting \( \langle I_k : k < \omega \rangle \) list the dense open subsets of \( \mathbb{Q}^N \) in \( N \), we choose inductively \( m_k \) and \( (\eta_1^k, \eta_2^k) \) such that \( \eta_1^k \in (m_k)_{\omega}, \eta_2^k \in (m_k)_{\omega}, \)


$\eta_0^1 = \eta_0^2 = q$, $\eta_k^1 < \eta_{k+1}^1$, $\eta_k^2 < \eta_{k+1}^2$, $\eta_{2k+1}^1 \in \mathcal{I}_k$, $\eta_{2k+2}^2 \in \mathcal{I}_k$ and the demand in (b) is satisfied and $\eta_1 \overset{\text{def}}{=} \bigcup_{k < \omega} \eta_k^1$, $\eta_2 \overset{\text{def}}{=} \bigcup_{k < \omega} \eta_k^2$ induce $g_1, g_2$ respectively.

Next, $
abla_2$ if $\nu$ is $\mathcal{R}$–generic over $N'$ (any $\mathbb{P}$–candidate), then $\eta_2^*$ is $\mathcal{R}$–generic over $N'$ or $\eta_1^*$ is $\mathcal{R}$–generic over $N'$.

[Why? Assume this fails, as $\eta_1^*$ is not $\mathcal{R}$–generic over $N'$ then some $f_1 \in (\omega, \omega)^M$ dominates $\eta_1^*$, and as $\eta_2^*$ is not $\mathcal{R}$–generic over $N'$ some $f_2 \in (\omega, \omega)^M$ dominates $\eta_2^*$, so $f^* = \max\{f_1 + 1, f_2 + 1\} \in N'$ dominates $\nu$ (i.e. $f^*(m) = \max\{f_1(m) + 1, f_2(m) + 1\}$).]

$\otimes_3$ if $\nu \in \omega, \omega$ is non-decreasing $\mathcal{R}$–generic over $N$ (so it is not dominated by $N$ and is non-decreasing), $q \in \mathbb{Q}$, then there is $G_\mathbb{Q} \subseteq \mathbb{Q}^N$ generic over $N$, $q \in G_\mathbb{Q}$ and $\nu^*[G_\mathbb{Q}] <^* \nu$.

[Why? Let $\langle \mathcal{I}_n : n < \omega \rangle$ list the dense open subset of $\mathbb{Q}$ in $N$. We choose by induction on $n$, $q_n \in k_n, \omega \subseteq \mathbb{Q}$ such that $q_0 = 1, q_n \leq q_{n+1}$, and $q_n \upharpoonright [k_0, k_n) \leq \nu \upharpoonright [k_0, k_n)$ and $q_{n+1} \in \mathcal{I}_n$. For $n = 0$ trivial, for $n + 1$ choose in $N$ by induction on $\ell$, $m_n, \ell, \rho_n, \ell$ such that $m_n, 0 = k_n$ and

$$m_{n, \ell} < m_{n, \ell + 1}, \quad \rho_{m, \ell} \in [m_{n, \ell}, m_{n, \ell + 1}) \omega, \quad q_n \cup 0 \cup [k_0, m_{n, \ell}] \cup \rho_{m, \ell} \in \mathcal{I}_n.$$ This is easy and $\langle \rho(m_{n, \ell}, \rho_{m, \ell}) : \ell < \omega \rangle \in N$. Now define $\rho^* \in \omega, \omega$ by:

$$\rho^*[m_{n, \ell}, m_{n, \ell + 1}) \text{ is constantly } \max\left( \bigcup_{i \leq \ell + 1} \text{Rang}(\rho_i) \cup \text{Rang}(q_n) \right).$$

So $(\exists j < \omega)(\rho^*(j) < \nu(j))$ hence for some $\ell$ and some $m \in [m_{n, \ell}, m_{n, \ell + 1})$ we have $\rho^*(m) < \nu(m)$. So

$$(\forall m')(m_{n, \ell + 1} \leq m' < m_{n, \ell + 2} \Rightarrow \rho_{m, \ell + 1}(m') < \nu(m)), \quad \text{but } \nu(m) < \min\{\nu(j) : m_{n, \ell + 1} \leq j < m_{n, \ell + 2}\}, \text{ so we are done.}$$

Now we have to check the conditions in [27, 7] so obviously clauses (a), (b) hold. Also clause (c) there holds by $\otimes_3$. So let us prove clause (d). Let $N$ be a $(\mathfrak{D}, \text{ZFC}_-\omega)$–candidate and $p \in \mathbb{P}^N$. Let $q$ be $\langle N, \mathbb{P} \rangle$–generic, $p \leq q$ by $\otimes_1 + \otimes_2$. Let $G \subseteq \text{Levy}(\mathcal{R}_0, [2^\omega]^N)$ be generic over $N$ (equivalently over $\mathbb{P}^{\omega}$). In $N$, by clause (e) of the assumptions, in $N[G]$, there is a semi $\mathbb{P}$–candidate $M$, $\mathcal{P}(\mathbb{P}^N)^N = \mathcal{P}(\mathbb{P}^M)^M$. Then in $N[G]$, $M[G]$ is a $\mathbb{P}$–candidate. So there is $q \in \mathbb{P}^{\omega N[G]}$ such that $N[G] \models \langle p \leq q \text{ and } q \text{ is } (M, \mathbb{P}^M)\text{–generic} \rangle$. As above possibly increasing $q$,

$$N[G] = \langle q \upharpoonright p \quad \text{”} \nu \text{ is } \mathcal{R}\text{–generic over } M[G \upharpoonright p] \text{”} \text{ and } \nu \text{ is Cohen over } M \rangle.$$
Remark 12.9. Clearly this proof is similar to §7, so we can replace “Cohen” by more general $Q$. More exactly, the point is that in §7 the demand was $\forall \alpha \in \omega \eta \in Q$–generic over $N$. Here we replace it by other demands.

Conclusion 12.10. For any Souslin proper forcing notion $P$, if $P$ add no dominating real, then forcing with $P$ adds no member of $\omega^\omega$ dominating some $F \subseteq \omega^\omega$ from $V$ not dominated there.

Proof. By 12.8, 12.6, 12.7.

Conclusion 12.11. 1. Suppose that
(a) $P$ is a forcing notion adding no dominating real,
(b) $Q \bar{\eta}$ is a $P$–name for a Souslin proper forcing notion not adding a
    dominating real.
Then $P * Q \bar{\eta}$ adds no dominating real.
2. $P * Q \bar{\eta}$ adds no real dominating an old undominated family if both $P$
   and $Q \bar{\eta}$ satisfy this and are Souslin proper.

Proof. By 12.10.

12. Open problems.

Problem 13.1. 1. Can we in [20] weaken the assumptions (from Souslin
    c.c.c.) to “$Q$ is nep and c.c.c.”?
2. Similarly in the symmetry theorem.
3. Similarly other problems here have such versions too.

Problem 13.2. 1. (von Neumann) Is it consistent that every c.c.c. $\omega^\omega$
    bounding atomless forcing notion is a measure algebra? We may now
    rephrase: is the non-existence consistent?
2. (Velickovic) Is it consistent that every c.c.c. forcing notion adding new
    reals adds a real $f \in \omega^\omega$ such that
    if $S \in \prod_{n<\omega} [\omega]^n \cap V$ then $(\exists \infty n \in \omega)(f(n) \notin S(n))$.
    [Note that [20] answers a relative of 13.2(2): there is no such Souslin
    c.c.c. forcing notion.]

A relative of the von Neumann problem is a problem which Fremlin [3]
stresses and has many equivalent versions (see [3] on its history). Half way
between them and our context is the following.

Problem 13.3. Assume $Q$ is a Souslin c.c.c. $\omega^\omega$–bounding forcing notion.
Is it random forcing?
Problem 13.4. 1. Is it consistent that every c.c.c. forcing notion adding an unbounded real adds a Cohen real? (See Błaszczyk Shelah [3] for a proof of the $\sigma$-centered version).

2. If $P$ satisfies [21, 1.5], does it imply $P$ adds a Cohen real?

Problem 13.5. Are there any symmetric (or $<\omega$-symmetric) c.c.c. Souslin forcing notions in addition to Cohen forcing and random forcing?

[“Yes” here implies “no” to 13.3 so not of present interest.]

Problem 13.6 (Gitik Shelah [3, 4]). 1. Assume $I$ is an $\aleph_1$–complete ideal on $\kappa$ such that $P/I$ is atomless. Can $I^+$ (as a forcing notion) be a c.c.c. Souslin forcing generated by a real.

2. Replace Souslin by “definable in an $(\mathcal{H}_{<\delta}(\theta), \in, \mathcal{B})$, $\mathcal{B}$ has universe $\kappa$ or $\mathcal{H}_{<\delta}(\kappa)$, and $I$ is $(\theta + \kappa)^+$–complete (see [3]).

3. Generalize the results of the form “if $P(\kappa)/I$ is the measure algebra with Maharam dimension $\mu$ (or is the adding of $\mu$ Cohen reals) then $\lambda$ is large enough”, see [3, 7] for those results.

4. Combine (2) and (3).

Problem 13.7 (Judah). Can a Souslin c.c.c. forcing notion add a minimal real? (Note: this is of interest only if the answer in 13.3 is NO and/or the answer to 13.11 is NO.)

Problem 13.8. Give examples of a Souslin forcing notion which is only temporarily c.c.c. and/or proper (L) (see §10).

Problem 13.9. Do iterations (CS,FS) of Souslin c.c.c. forcing notions not adding a dominating real have this property? Is each almost $\omega^\omega$–bounding? [Maybe §8 answers need better: replace $\eta^*$ is generic real for $(N, \mathbb{Q}, \mathcal{L})$ by less]. See §11 + §10.

Problem 13.10. 1. Is there a pair $(Q, r)$ such that:

(a) $\vdash_Q "r \in \omega^2"$ is new,

(b) if $P$ is a Souslin c.c.c. forcing notion with no $P$–name $r'$ of a real such that the forcing notion $B_P(r')$ is $\omega^\omega$–bounding but $P$ adds a nowhere essentially Cohen real

then forcing with $P$ adds a $(Q, r)$ real,i.e. for some $P$–name $r''$ for a real we have $\vdash_P "for some G'' \subseteq V^Q$ generic over $V$, $r''[G_P] = r[G'']"$.

2. As above $P$ is $\sigma$–centered.
3. If $\mathbb{P}$ is a Souslin c.c.c. forcing notion adding new reals but not adding a real $r'$ with $\mathcal{B}_\mathbb{P}(r')$ being $\omega$-bounding, then forcing with $\mathbb{P}$ adds a new real $r''$ such that $\mathcal{B}_\mathbb{P}(r'')$ is $\sigma$–centered.

**Problem 13.11.** 1. Let $\mathbb{Q}$ be a Souslin c.c.c. forcing notion and $\Vdash_{\mathbb{Q}} \forall r \in \omega^\omega$. Is $\mathcal{B}(r)$ also a Souslin c.c.c. forcing notion?

2. Similarly for nep c.c.c.

**Problem 13.12.** Assume $\mathbb{Q}$ is a Souslin c.c.c. forcing notion which is nep and even “$x \in \mathbb{Q}$”, “$x \leq_{\mathbb{Q}} y$”, “$\{p_n : n < \omega\}$ predense above $q$” are $\Sigma^1_1$ relations. Does $\mathbb{Q}$ add Cohen or random real?

**Problem 13.13.** Develop the theory of “definable forcing notions” when we allow an ultrafilter on $\omega$ as a parameter.

**Problem 13.14.** Does nep $\neq$ snep? (the case $\theta = \kappa = \aleph_0$, of course).

**Problem 13.15.** Try to generalize our present context to $\lambda$–complete forcing notions (Baumgartner’s Axiom: [19], [16]).

**Problem 13.16.** When $\mathbb{Q}^V \not\leq \mathbb{Q}^\mathbb{P}$?

**Problem 13.17.** Does $Ax_{\omega_1}[(\aleph_1, \aleph_1)-\text{nep}]$ imply $2^{\aleph_0} = \aleph_2$?

Or does $Ax_{\omega_1}[\text{nep}]$ imply $2^{\aleph_0} = \aleph_2$?

[The parallel question for Souslin proper was formulated in xxx]

**References**

[1] Jensen, R. B., Beller, A. and Welch, P., *Coding the universe*. Number 47 in London Mathematical Society Lecture notes series. Cambridge University Press, 1982.

[2] Bartoszyński, T. and Judah, H., *Set Theory: On the Structure of the Real Line*. A K Peters, Wellesley, Massachusetts, 1995.

[3] Baumgartner, J., *All $\aleph_1$-dense sets of reals can be isomorphic*, Fundamenta Math., 79 (1973), 101–106.

[4] Blass, A. and Shelah, S., *There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed*, Annals of Pure and Applied Logic, 33 (1987), 213–243.

[5] Blaszczyk, A. and Shelah, S., *Complete $\sigma$-centered Boolean Algebra not adding Cohen reals*, preprint.

[6] David Fremlin, *Problem list*, circulated notes (1994).

[7] Gitik, M. and Shelah, S., *More on real-valued measurable cardinals and forcing with ideals*, Israel Journal of Mathematics, submitted.

[8] Gitik, M. and Shelah, S., *Forcings with ideals and simple forcing notions*, Israel Journal of Mathematics, 68 (1989), 129–160.
[9] Gitik, M and Shelah, S., More on simple forcing notions and forcings with ideals. Annals of Pure and Applied Logic, 59 (1993), 219–238.
[10] Goldstern, M., Tools for your forcing construction, In Set Theory of the Reals, volume 6 of Israel Mathematical Conference Proceedings, 1993, 305–360.
[11] Goldstern, M. and Judah, H., Iteration of Souslin Forcing, Projective Measurability and the Borel Conjecture, Israel Journal of Mathematics, 78 (1992), 335–362.
[12] Ihoda, J. (Judah, H.) and Shelah, S., Souslin forcing. The Journal of Symbolic Logic, 53 (1988), 1188–1207.
[13] Jech, T., Set theory. Academic Press, New York, 1978.
[14] Roslanowski, A. and Shelah, S., Norms on possibilities I: forcing with trees and creatures. Memoirs of the AMS, accepted.
[15] Roslanowski, A. and Shelah, S., Norms on possibilities II: more ccc ideals on $2^{\omega}$, Journal of Applied Analysis, 3 (1997), 103–127.
[16] Shelah, S., Iteration of $\lambda$-complete forcing not collapsing $\lambda^+$, in preparation.
[17] Shelah, S., Proper and improper forcing. Springer-Verlag, 1997.
[18] Shelah, S., Proper forcing, volume 940 of Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1982.
[19] Shelah, S., Diamonds, uniformization. The Journal of Symbolic Logic, 49 (1984), 1022–1033.
[20] Shelah, S., How special are Cohen and random forcings i.e. Boolean algebras of the family of subsets of reals modulo meagre or null. Israel Journal of Mathematics, 88 (1994), 159–174.
[21] Shelah, S. and Stanley, L., A combinatorial forcing for coding the universe by a real when there are no sharps, Journal of Symbolic Logic, 60 (1995), 1–35.
[21] Shelah, S., More on non-elementary proper forcing notions, in preparation.

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