The abelian projection revisited

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Abelian projection is reanalysed in the frame of the $SU(N)$ Higgs model. The extension to QCD is discussed. It is shown that dual superconductivity of the vacuum is an intrinsic property independent of the choice of the abelian projection.

1. Introduction

The mechanism of color confinement by dual superconductivity of the vacuum\textsuperscript{[12]} requires the identification in QCD of a magnetic $U_M(1)$ gauge field, which has to be a color singlet if magnetic charges have to condense in the vacuum and preserve the color gauge symmetry.

For $T < T_c$ $U_M(1)$ has to be Higgs broken (dual superconductivity), for $T > T_c$ it must be restored (normal ground state). $U_M(1)$ is defined by a procedure known as “Abelian Projection”\textsuperscript{[3]}.

In sect.2 we reanalyse the Abelian Projection in the $SU(N)$ Higgs model. In sect.3 we discuss the extension to QCD and we show that dual superconductivity (or not) is an intrinsic property independent of the particular choice of the projection.

2. $SU(N)$ Higgs model.

Consider the $SU(N)$ Higgs model. In the usual notation

$$L = Tr \left\{ D_\mu \phi \Gamma D_\mu \phi \right\} - \frac{1}{4} Tr \left\{ G_{\mu\nu} G^{\mu\nu} \right\} - V(\phi)$$

$\phi = \sum_{i=1}^{N^2-1} \phi_i T^i$ is a scalar field in the adjoint representation. In the Higgs phase $\langle \phi^a \rangle = \phi^a \neq 0$, $\phi^a$ being a nontrivial minimum of $V(\phi)$.

Consider the field strength tensor\textsuperscript{[2]}

$$F^a_{\mu\nu} = \partial_\mu \phi^a \Gamma_{\nu\rho} - \frac{i}{g} \partial_\rho \phi^a \left[ D_\mu \phi^a, D_\nu \phi^a \right]$$

$F^a_{\mu\nu}$ is a color singlet and gauge invariant, and such are separately the two terms of eq.(1). The following theorem has been proved in ref.\textsuperscript{[5]}.

\textbf{Theorem.} Bilinear terms in $A_\mu A_\nu$ cancel between the two terms on the right side of eq.(1) and $F^a_{\mu\nu}$ obeys Bianchi identities ($\partial_\mu F^a_{\mu\nu} = 0$) iff

$$\phi^a = U(x) \phi^a_{\text{diag}} U(x)^\dagger$$

with $U(x)$ a generic gauge transformation and

$$\phi^a_{\text{diag}} = \text{diag} \left( \begin{array}{cc} \frac{a}{N}, & \cdots, & \frac{a}{N} \\ \frac{N-a}{N}, & \cdots, & \frac{N-a}{N} \end{array} \right)$$

The invariance group of $\phi^a_{\text{diag}}$ is $g = SU(a) \otimes SU(N-a) \otimes U(1)$. It identifies a symmetric space\textsuperscript{[2]}, in the sense that if $L_0$ is the Lie algebra of the subgroup $g$, $L$ the algebra of $SU(N)$ and $L_1 = L - L_0$, then $[L_0, L_0] \subset L_0$, $[L_0, L_1] \subset L_1$, $[L_1, L_1] \subset L_0$.

Viceversa there is a conjecture by Michel\textsuperscript{[7,8]} that if the Higgs field belongs to the adjoint representation any $\phi^a$ identifies a symmetric space, or has the form eq.(2), eq.(3).

If $\phi^a$ is of the form of eq.(2), eq.(1) reduces identically to the form

$$F^a_{\mu\nu} = \partial_\mu Tr \left\{ \phi^a A_\nu \right\} - \partial_\nu Tr \left\{ \phi^a A_\mu \right\} - \frac{i}{g} Tr \left\{ \phi^a \left[ \partial_\mu \phi^a, \partial_\nu \phi^a \right] \right\}$$

\textsuperscript{*}Talk presented by A. Di Giacomo.
which is again gauge invariant and color singlet. In the gauge \( \phi^a = \phi^a_{\text{diag}} \) (unitary gauge) \( F^a_{\mu \nu} \) reduces to the abelian form

\[
F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu
\]

with \( A^a_\mu = Tr \{ \phi^a A_\mu \} \), whence the name “abelian projection” of the gauge transformation to the unitary gauge.

\( A^a_\mu \) is defined by expanding diagonal part of \( A_\mu \) in terms of roots \( A_{\mu, \text{diag}} = \sum_i \alpha^i A^a_\mu \)

\[
\alpha^i = \text{diag}(0, 0, 0 \ldots 1, -1, 0 \ldots 0)
\]

\[
Tr \{ \phi^a_{\text{diag}} \alpha^k \} = \delta^{ab}
\]

\[
\frac{1}{g} \frac{\vec{r}}{r^3} + \text{Dirac string}
\]

In the Higgs phase \( \phi^a \neq 0 \) monopoles exist as solitons. The construction is the same as in ref.\(^4\) for \( SU(2) \), in the \( SU(2) \) subgroup of \( SU(N) \) spanned by the \( i, i+1 \) elements of eq.\(^5\). For these solutions the electric field \( F^a_{\mu \nu} = 0 \), the magnetic field at large distances \( H^a_\mu = \frac{1}{2} \epsilon_{ijk} F^j_{\mu k} \) is that of a Dirac monopole of charge \( 1/g \)

\[
\vec{H} = \frac{1}{g} \frac{\vec{r}}{r^3} + \text{Dirac string}
\]

However \( F^a_{\mu \nu} \), eq.\(^1\), can also be defined in the Coulomb phase, where monopoles do not exist as solitons, by assuming \( \phi^a \) of the form eq.\(^2\), transforming in the adjoint representation, with \( U(x) \) an arbitrary gauge transformation. \( U(x) \) identifies the abelian projected system, or the system in which \( \phi^a \) is diagonal: \( U(x) \) can be taken as the operator which diagonalizes \( \phi \), the Higgs field, but any other choice is legitimate.

The order parameter for a possible dual superconductivity will be the vev of an operator which creates a magnetic charge.

Such an operator exists\(^10\) and has the form

\[
\mu^a(\vec{x}, t) = e^i \int d^3y Tr(\phi^a(\vec{y}, t)E(\vec{y}, t)) \vec{b}_\perp(\vec{x} - \vec{y})
\]

\[
\vec{\nabla} \vec{b}_\perp = 0 \quad \vec{\nabla} \times \vec{b}_\perp = \frac{2\pi \vec{r}}{g r^3} + \text{Dirac string}
\]

\( \mu^a \) is gauge invariant and color singlet.

In the abelian projected gauge \( \phi^a = \phi^a_{\text{diag}} \),

\[
tr \{ \phi^a E \} = E^a \text{ (component of the field along the root } \alpha^a \text{, and}
\]

\[
\mu^a(\vec{x}, t) = \exp \left\{ i \int d^3y \vec{E}_\perp^a(\vec{y}, t) \vec{b}_\perp(\vec{x} - \vec{y}) \right\}
\]

Only \( \vec{E}_\perp^a \) survives in the convolution with \( \vec{b}_\perp \). In any quantization procedure \( \vec{E}_\perp^a \) is the conjugate momentum to \( A_\perp^a \), and the operator eq.\(^9\) is the translation operator of \( A_\perp^a \), or, in the Schrödinger representation

\[
\mu^a(\vec{x}, t)|\vec{A}_\perp^a(\vec{y}, t)\rangle = |\vec{A}_\perp^a(\vec{y}, t) + \vec{b}_\perp(\vec{x} - \vec{y})\rangle
\]

i.e. \( \mu^a(\vec{x}, t) \) creates a monopole. Of course \( \mu^a \) depends on \( \phi^a(x) \), i.e. on the abelian projection \( U(x) \).

3. QCD

In QCD there is no Higgs field, but \( F^a_{\mu \nu} \) can be defined anyhow by eq.\(^5\), with \( \phi^a = U(x)\phi^a_{\text{diag}} U^\dagger(x) \) a scalar field in the adjoint representation belonging to the orbit of \( \phi^a_{\text{diag}} \). \( U(x) \) identifies the representation in which \( \phi^a \) is diagonal (abelian projection), which can be chosen to coincide with the one in which any local operator \( O(x) \) in the adjoint representation is diagonal.\(^3\) The construction is independent of the properties of \( O(x) \) under Lorentz group or discrete transformations\(^11\).

The corresponding operator \( \mu^a \), eq.\(^9\), by use of the cyclic invariance of the trace can then be written\(^12\)

\[
\mu^a(\vec{x}, t) = e^i \int d^3y Tr(\phi^a_{\text{diag}} U^\dagger(\vec{y}, t)E(\vec{y}, t)U(\vec{y}, t)\vec{b}_\perp(\vec{x} - \vec{y})
\]

If \( U(x) \) is independent of the field configuration it can be reabsorbed by changing variables in the Feynman integral by a gauge transformation, with Jacobian equal to one, when computing correlators of \( \mu^a \). To all effects

\[
\mu^a(\vec{x}, t) = e^i \int d^3y Tr(\phi^a_{\text{diag}} \vec{E}_\perp(\vec{y}, t)\vec{b}_\perp(\vec{x} - \vec{y})
\]

and correlations do not depend on \( U(x) \).

If \( U(x) \) depends on the gauge fields the Jacobian can be non trivial and the effective lagrangean of the order parameter may depend on it.

However if the bulk density of monopoles is finite the gauge transformation between two different abelian projections will be continuous except at a finite number of points at a given time, i.e. at the location of monopoles and will preserve topology. \( \mu^a \), eq.\(^10\) will then create a monopole in all abelian projections. \( \langle \mu^a \rangle \neq 0 \) signals then
dual superconductivity in all abelian projections. This is confirmed by numerical simulations on the lattice, showing that $\langle \mu \rangle$ is independent on the abelian projection. The density of monopoles is indeed finite as shown in fig.1 where the distribution of the difference of eigenvalues of the Polyakov line is displayed. The number of sites where two of them coincide is zero. This happens for many choices of the operator and different values of the lattice spacing. See also re. [14] for the density of monopoles in the maximal abelian gauge.

Dual superconductivity is an intrinsic property independent of the choice of the abelian projection.

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