COBORDISMS BETWEEN SYMPLECTIC FIBRATIONS

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Abstract. We discuss the existence and non-existence of cobordisms between symplectic surface bundles over the circle.

1. Introduction

In [4] Eliashberg showed that any weak filling of a contact 3-manifold can be embedded into a closed symplectic 4-manifold. His result is based on the construction of a symplectic cap. One part of that cap is a cobordism from the given contact manifold to a symplectic surface bundle over $S^1$. Such cobordisms also play a role in [2], which led us to look at the rigidity or flexibility inherent in the construction of cobordisms between symplectic fibrations.

Our first result is a rigidity statement for spherical fibrations.

Theorem 1. Let $(W, \Omega)$ be a compact symplectic 4-manifold with each boundary component a surface bundle over $S^1$ on whose fibres $\Omega$ restricts to an area form. If one of these surface bundles has spherical fibres, then so do all the others.

In particular, there can be no symplectic round handle construction for symplectic fibrations, in contrast with the situation for contact manifolds [1, 3].

For cobordisms between surface bundles of genus greater than zero, on the other hand, the situation is completely flexible. In order to formulate the result, we need to be a little more specific about the concept of a symplectic fibration and cobordisms between them, cf. [4]. We prefer to speak simply of ‘cobordisms’ rather than ‘symplectic cobordisms’, since the latter is commonly used for cobordisms inducing contact structures on the boundary.

Definition. A symplectic fibration is a pair $(V, \omega)$ consisting of a closed, connected, oriented 3-manifold $V$ fibred over the circle $S^1$ and a closed 2-form $\omega$ that restricts to an area form on each fibre. The genus of the fibration is the genus of its fibre.

The assumption on $V$ being connected is made merely for convenience. The cobordism-theoretic results below have fairly straightforward analogues when we allow several boundary components at one or the other end of the cobordism.

The 2-form $\omega$ defines an orientation of the fibres, and the orientation of $V$ then determines an orientation on the 1-dimensional characteristic foliation of $V$ determined by the kernel of $\omega$, which is transverse to the fibres. Since $\omega$ is closed, any flow along this characteristic foliation preserves $\omega$ by Cartan’s formula for the Lie derivative. This allows one to define an area-preserving holonomy diffeomorphism.
from one fixed fibre to itself, which determines the bundle up to a fibre-preserving diffeomorphism.

**Definition.** Let \((V_i, \omega^i), i = 0, 1\) be two symplectic fibrations. A **cobordism** from \((V^0, \omega^0)\) to \((V^1, \omega^1)\) is a compact symplectic 4-manifold \((W, \Omega)\) with

\[
\partial(W, \Omega) = (V^1, \omega^1) \sqcup (-V^0, \omega^0).
\]

Here \(-V_0\) stands for \(V_0\) with the reversed orientation. By the expression for \(\partial(W, \Omega)\) we mean that with \(W\) oriented by \(\Omega \wedge \Omega\), the oriented boundary \(\partial W\) equals \(V^1 \sqcup -V^0\), and \(\Omega|_{TV_i} = \omega^i\) for \(i = 0, 1\).

According to the symplectic neighbourhood theorem for hypersurfaces [14, Exercise 3.36], the restriction \(\Omega|_{TV_i}\) determines \(\Omega\) in a neighbourhood of \(V_i\). In fact, up to symplectomorphism, \(\Omega\) looks like \(\omega^i + d(t \alpha^i)\) on a neighbourhood of \(V_i\), where \(\alpha^i\) is a 1-form that does not vanish on the characteristic foliation, and \(t\) is the collar parameter.

This makes the above notion of cobordism both reflexive and transitive, but it is not clear, a priori, that it is symmetric. (Observe that the holonomy diffeomorphisms of \((V, \omega)\) and \((-V, \omega)\) are inverses of each other.) However, provided at least one boundary component is a symplectic fibration of genus greater than zero, symmetry of the cobordism relation is one of the consequences of Theorem 1 and the following flexibility result.

**Theorem 2.** Let \((V_g, \omega_g)\) and \((V_{g'}, \omega_{g'})\) be two symplectic fibrations of respective genus \(g, g' > 0\). Then there is a cobordism from one to the other.

The situation for symplectic fibrations of genus zero is discussed in Section 4.

Symplectic fibrations are a special case of odd-symplectic manifolds in the sense of Ginzburg [9]. So our results are in some sense dual to his. Ginzburg considers cobordisms carrying odd-symplectic forms inducing given symplectic forms on the boundary, whereas here we deal with cobordisms carrying symplectic forms inducing symplectic fibrations on the boundary. One could investigate the more general cobordism relation where the boundary condition is weakened to odd-symplectic, also in higher dimensions.

Our cobordisms are always supposed to have two non-empty boundary components. Without that assumption, any symplectic fibration would be null-cobordant by [4].

## 2. Proof of Theorem 1

Let \((W, \Omega)\) be a compact symplectic 4-manifold satisfying the assumptions of Theorem 1. By the results in Section 3 of [4], the boundaries of \(W\) can be capped off, i.e. \((W, \Omega)\) embeds symplectically into a closed symplectic 4-manifold \((\tilde{W}, \tilde{\omega})\).

Arguing by contradiction, we assume that \((\tilde{W}, \tilde{\omega})\) has a spherical symplectic fibration as one boundary component, and a further boundary component which is a symplectic bundle over \(S^1\) with fibre a closed, orientable surface \(\Sigma_g\) of genus \(g > 0\). Then \((\tilde{W}, \tilde{\omega})\) contains a symplectically embedded copy \(S\) of \(S^2\) and a symplectically embedded copy \(\Sigma\) of \(\Sigma_g\), where \(S\) is disjoint from \(\Sigma\), and each has self-intersection number zero.

We next construct two further symplectic 4-manifolds containing a symplectically embedded \(S^2\) or \(\Sigma_g\), respectively, with self-intersection number zero. By fibre
connected sum (also called ‘symplectic sum’) in the sense of Gompf [11] we shall then build a symplectic 4-manifold with contradictory properties.

2.1. The building blocks. Our first building block will be a symplectic 4-manifold with a single convex boundary component and a symplectically embedded surface $\Sigma'$ of some genus $g' > 0$ with self-intersection number zero.

Start with a compact symplectic 4-manifold with two boundary components $M, M'$, both of which are supposed to be strongly convex boundaries, so that they carry induced contact structures $\xi, \xi'$ cf. [11 Chapter 5]. Examples of such manifolds have been constructed by McDuff [13] and the first author [6]. The contact structure $\xi'$ is supported (in the sense of Giroux [10]) by an open book of some genus $g'$; by stabilising the open book, if necessary, we may assume $g' \geq 1$. (In fact, Etnyre [5] has shown that any contact structure induced on a boundary component of a symplectic 4-manifold with disconnected convex boundary can only be supported by an open book of genus at least 1.) Eliashberg’s capping construction [4] applied to the boundary component $(M', \xi')$ then produces a symplectic 4-manifold $(W_{g'}, \Omega_{g'})$ with convex boundary $(M, \xi)$. This symplectic manifold contains, inside the cap, a symplectically embedded $\Sigma'$ having the desired properties.

Our second building block is described in the following proposition.

**Proposition 3.** Given $g, g' > 0$, there is a closed symplectic 4-manifold $(X_{g}^{g'}, \Omega_{g}^{g'})$ containing disjoint symplectically embedded copies of $\Sigma_{g}$ and $\Sigma_{g'}$, each of self-intersection zero.

**Proof.** Start with the product $\Sigma_{g} \times \Sigma_{g'}$, equipped with a product symplectic structure. Let $T \subset \Sigma_{g} \times \Sigma_{g'}$ be a Lagrangian torus given as the product of homotopically non-trivial curves in $\Sigma_{g}$ and $\Sigma_{g'}$, so that the homology class $[T]$ is non-trivial in $H_{2}(\Sigma_{g} \times \Sigma_{g'})$. As explained in [11 Lemma 1.6], there is a symplectic form on $T \subset \Sigma_{g} \times \Sigma_{g'}$ for which $T$ is symplectic. This new symplectic form can be chosen arbitrarily close to the product form we started with; this allows us to assume that $\Sigma_{g} \times \{\ast\}$ and $\{\ast\} \times \Sigma_{g'}$ are still symplectic surfaces.

Now take two copies of this manifold, and perform a symplectic sum as in [11] along the two copies of $T$, which have zero self-intersection. This produces a closed symplectic 4-manifold $(X_{g}^{g'}, \Omega_{g}^{g'})$ containing disjoint symplectically embedded copies of $\Sigma_{g}$ and $\Sigma_{g'}$, coming from a surface $\Sigma_{g} \times \{\ast\}$ in the first summand and a surface $\{\ast\} \times \Sigma_{g'}$ in the second summand; we only have to ensure that the respective point $\ast$ is chosen away from the curves that define $T$. \[\Box\]

Notice that for the symplectic summing we always assume implicitly that the symplectic forms on the two summands have been scaled such that they induce area forms on the relevant surfaces of equal total area; in this situation, the construction from [11] is applicable.

2.2. The symplectic manifold $(W', \Omega')$. We define $(W', \Omega')$ as the symplectic 4-manifold obtained by symplectically summing $(\tilde{W}, \tilde{\omega})$, $(X_{g}^{g'}, \Omega_{g}^{g'})$ and $(W_{g'}, \Omega_{g'})$, where the sum is taken along $\Sigma \subset \tilde{W}$ and $\Sigma_{g} \subset X_{g}^{g'}$, as well as along $\Sigma_{g'} \subset X_{g}^{g'}$ and $\Sigma' \subset W_{g'}$.

Observe that $(W', \Omega')$ has convex boundary $(M, \xi)$, and it contains a symplectically embedded 2-sphere $S$ of self-intersection zero in the $(\tilde{W}, \tilde{\omega})$ summand. McDuff [13 Theorem 5.1] showed that such a symplectic manifold cannot exist, cf. [8]. By
analysing the moduli space of holomorphic spheres in $W'$ — with respect to an almost complex structure $J$ tamed by $\Omega'$ for which $S$ is holomorphic and $M$ is $J$-convex — one would find a holomorphic sphere through every point on a path joining $S$ to $M$, contradicting the maximum principle at the convex boundary $M$.

This contradiction proves Theorem 1.

2.3. Discussion of an alternative argument. One might try to prove Theorem 1 via a more direct route, by establishing an argument on the basis of McDuff’s classification [12] of ruled symplectic 4-manifolds.

Consider the manifold pair $(\tilde{W}, S)$ constructed at the beginning of the proof of Theorem 1, together with the symplectic surface $\Sigma \subset \tilde{W} \setminus S$. By blowing down any potential exceptional spheres in $\tilde{W} \setminus S$, i.e. symplectic spheres of self-intersection $-1$, we may assume that the pair $(\tilde{W}, S)$ is minimal in the sense of [12].

Blowing down an exceptional sphere $E$ amounts to taking a fibre connected sum of $(\tilde{W}, E)$ with $(\mathbb{C}P^2, \mathbb{C}P^1)$. Write $\nu E$ for a closed tubular neighbourhood of $E$ in $\tilde{W}$; likewise we write $\nu \mathbb{C}P^1$. Then blowing down $E$ means that we replace $\nu E$ by the 4-ball $D^4 = \mathbb{C}P^2 \setminus \text{Int}(\nu \mathbb{C}P^1)$. The $S^1$-fibres of $\partial(\nu \mathbb{C}P^1)$ are the Hopf fibres on the boundary of the complementary 4-ball $D^4$. Hence, if $\Sigma$ intersects $E$ transversely, then the effect on $\Sigma$ of blowing down $E$ is to replace the disjoint discs $\nu E \cap \Sigma$ by the discs in $D^4$ bounded by the Hopf fibres $\partial(\nu E) \cap \Sigma$. We write $\Sigma^*$ for the transformed surface (the ‘proper transform’) in the blown-down manifold.

Observe that $\Sigma^*$ has the same genus as $\Sigma$, but its self-intersection number will have changed. Any pair of points in $\Sigma \cap E$ (including pairs made up of twice the same point) will add $\pm 1$ to the self-intersection number of $\Sigma^*$, depending on whether the intersection points have the same sign or not. This follows from the observation that any two Hopf fibres bound holomorphic discs that intersect positively in a single point.

If $\Sigma^*$ were still symplectic, we would again have a contradiction as follows. Choose an almost complex structure for which $S$ and $\Sigma^*$ are holomorphic; by [15, Corollary 3.3.4] this almost complex structure will be regular for the homology class $[S]$. Then the proof of [12, Proposition 4.1] shows that $S$ is a fibre in a holomorphic ruling of $\tilde{W}$. The surface $\Sigma^*$ has to intersect one of the spherical fibres geometrically, but the homological intersection is zero, since $\Sigma^*$ is disjoint from the fibre $S$. By positivity of intersection, $\Sigma^*$ would have to coincide with a spherical fibre, contradicting the assumption that $\Sigma$ (and hence $\Sigma^*$) has positive genus.

Unfortunately, the proper transform $\Sigma^*$ will not, in general, be symplectic. Here is an example (with $\Sigma$ of genus 0). Start with the symplectic manifold $\mathbb{C}P^2 \# \mathbb{C}P^2$, the blow-up of $\mathbb{C}P^2$ in a single point. Let $E$ be the exceptional divisor and $\Sigma$ a transverse symplectic copy of $E$, so that the intersection numbers are $\Sigma \cdot E = -1$ and $\Sigma \cdot \Sigma = -1$. After blowing down $E$, we have $\Sigma^* \subset \mathbb{C}P^2$ with self-intersection number 0, so it cannot be realised as a symplectic submanifold in $\mathbb{C}P^2$ for cohomological reasons.

3. Proof of Theorem 2

Theorem 2 will be an obvious consequence of the two lemmata we prove in this section, together with the transitivity of the cobordism relation.
Lemma 4. Let \((V^i, \omega^i), i = 0, 1\), be two symplectic fibrations of the same genus \(g\) with
\[
\int_{\Sigma_g} \omega^0 = \int_{\Sigma_g} \omega^1.
\]
Then there is a cobordism from \((V^0, \omega^0)\) to \((V^1, \omega^1)\).

In particular, given any symplectic fibration of genus \(g\), there is a cobordism both from and to a trivial symplectic fibration \(\Sigma_g \times S^1\) with symplectic form pulled back from an area form on \(\Sigma_g\) of the appropriate total area.

Proof of Lemma 4. By [4, Theorem 3.1] there are compact symplectic 4-manifolds \((W^i, \Omega^i), i = 0, 1\), with \(\partial(W^1, \Omega^1) = (V^1, \omega^1)\) and \(\partial(W^0, \Omega^0) = (V^0, \omega^0)\). Each \(W^i\) contains a copy of \(\Sigma_g\) embedded symplectically in the interior, with trivial normal bundle, and both of the same total area. Hence, we can perform a fibre connected sum to produce the desired cobordism.

Lemma 5. For any \(g, g' > 0\), there is a cobordism between any trivial symplectic fibration of genus \(g\) to any such fibration of genus \(g'\).

Changing the orientation of the \(S^1\)-factor in the trivial symplectic fibration defines an orientation-reversing diffeomorphism of this fibration to itself, so we can ignore issues of orientation in the following proof.

Proof of Lemma 5. Let \((V_g = \Sigma_g \times S^1, \omega_g)\) and \((V_{g'} = \Sigma_{g'} \times S^1, \omega_{g'})\) be two trivial symplectic fibrations, where we think of \(\omega_g\) as both the positive area form on \(\Sigma_g\) and the 2-form on \(V_g\), likewise for \(\omega_{g'}\).

Choose an area form \(\omega_{T^2}\) on \(T^2 \setminus \text{Int}(D^2)\), the 2-torus with an open disc removed. Equip \(\Sigma_g \times (T^2 \setminus \text{Int}(D^2))\) with the product symplectic form \(\omega_g \oplus \omega_{T^2}\). Now let \(T = a \times b\) be a Lagrangian torus in this symplectic manifold as in the proof of Proposition 3 i.e. choose a simple closed curve \(a\) in \(\Sigma_g\) representing a generator of \(H_1(\Sigma_g)\), and a simple closed curve \(b\) in \(T^2 \setminus \text{Int}(D^2)\) representing a generator of the relative homology group \(H_1(T^2 \setminus \text{Int}(D^2), \partial D^2)\). As before we now appeal to [11, Lemma 1.6]. Since that lemma is formulated for closed manifolds only, we are a little more explicit. There are closed 1-forms \(\alpha, \beta\) supported near the dual curve of \(a, b\) on \(\Sigma_g, T^2 \setminus \text{Int}(D^2)\), respectively, with
\[
\int_a \alpha = \int_b \beta = 1, \text{ hence } \int_T \alpha \wedge \beta = 1.
\]
We think of \(\alpha\) and \(\beta\) as 1-forms on the symplectic 4-manifold by pulling them back under the projection to one of the two factors.

Let \(\eta\) be any area form on \(T\) of total area 1. With \(j\) denoting the inclusion of \(T\) in the symplectic 4-manifold, the 2-form \(\eta - j^*(\alpha \wedge \beta)\) integrates to zero over \(T\) and hence equals an exact 1-form \(d\gamma\) on \(T\). Extend \(\gamma\) to a 1-form on the whole 4-manifold, supported near \(T\). Then, for any small \(\varepsilon > 0\), the 2-form \(\omega_g \oplus \omega_{T^2} + \varepsilon(\alpha \wedge \beta + d\gamma)\) will be symplectic, and it pulls back to \(\varepsilon \eta\) on \(T\), which makes that torus symplectic.

Now perform the same construction starting from \((\Sigma_{g'}, \omega_{g'})\). If we choose the same \(\varepsilon\) in both instances, we can then perform a fibre connected sum along the respective copies of \(T\). This results in the desired cobordism.  

□
4. Spherical fibrations

It remains to discuss the existence of cobordisms between two symplectic fibrations of genus zero. The following proposition is a special case of Lemma 4, but we include a direct proof, since the case $g = 0$ requires considerably less machinery.

**Proposition 6.** There is a cobordism between any two spherical symplectic fibrations, provided the fibres in the two fibrations have the same total area.

**Proof.** The total space of any spherical symplectic fibration is $\mathbb{S}^1 \times \mathbb{S}^2$. Any two product symplectic fibrations on this space, where the symplectic form is pulled back from an area form on $\mathbb{S}^2$, are diffeomorphic by the usual Moser argument, provided they have the same total area on $\mathbb{S}^2$.

By the transitivity of the cobordism relation, it suffices to show that given any spherical symplectic fibration $(\mathbb{S}^1 \times \mathbb{S}^2, \omega)$, we can find cobordisms both to and from a product symplectic fibration with the same total area on the fibres.

The construction of such cobordisms is essentially given in the proof of [4, Lemma 3.5]. Define $\mathbb{S}^2_0 = \{0\} \times \mathbb{S}^2$, where we identify $\mathbb{S}^1$ with $\mathbb{R}/2\pi \mathbb{Z}$, and set $\omega_0 = \omega|_{T\mathbb{S}^2_0}$. Since all symplectomorphisms of $\mathbb{S}^2$ are Hamiltonian, the holonomy of $(\mathbb{S}^1 \times \mathbb{S}^2, \omega)$, regarded as a symplectomorphism of $(\mathbb{S}^2_0, \omega_0)$, is given as the time-$2\pi$ flow of a $2\pi$-periodic time-dependent Hamiltonian $H_t: \mathbb{S}^2_0 \to \mathbb{R}$; the corresponding time-dependent Hamiltonian vector field $X_t$ is given by $i_{X_t} \omega_0 = -dH_t$. We may assume that there are constants $m$ and $M$ with $0 < m < H_t < M$. Define an embedding $f: \mathbb{S}^1 \times \mathbb{S}^2_0 \to \mathbb{C} \times \mathbb{S}^2_0$ by

$$(t, x) \mapsto (\sqrt{H_t(x)} e^{-it}, x).$$

Then the split form $\Omega_0 = \omega_0 + d(r^2 d\varphi)$ pulls back to $\omega$. Indeed, we have

$$f^* \Omega_0 = \omega_0 - d(H_t(x) dt) = \omega_0 + iX_t \omega_0 \wedge dt,$$

which implies that the characteristic foliation of $f^* \Omega_0$ is spanned by $X_t + \partial_t$, as desired.

It follows that the restriction of $\Omega_0$ to

$$\{(z, x) \in \mathbb{C} \times \mathbb{S}^2_0: \sqrt{m} \leq |z| \leq \sqrt{H_t(x)}\}$$

and

$$\{(z, x) \in \mathbb{C} \times \mathbb{S}^2_0: \sqrt{H_t(x)} \leq |z| \leq \sqrt{M}\}$$

defines a cobordism between $(\mathbb{S}^1 \times \mathbb{S}^2, \omega)$ and a product symplectic fibration, in one or the other direction. \hfill \Box

If a cobordism exists between two spherical symplectic fibrations of different fibre area, it must be rather wild, for the following reason. By capping off such a purported cobordism we would obtain a closed symplectic 4-manifold containing two symplectic spheres $S, S'$ of self-intersection zero and different area. We can blow down all exceptional spheres in the complement of $S \cup S'$. If all remaining exceptional spheres intersect $S'$, say, we would have again a minimal manifold pair as in Section 2.3 and the presence of two symplectic spheres of different area would contradict the results of McDuff [12]. So necessarily the capped-off cobordism must contain exceptional spheres that intersect $S$ but not $S'$, and exceptional spheres that cut $S'$ but not $S$. 
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