Logarithmic Relaxation in Glass-Forming Systems

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Within the mode-coupling theory for ideal glass transitions, an analysis for the correlation functions of glass-forming systems for states near higher-order glass-transition singularities is presented. It is shown that the solutions of the equations of motion can be asymptotically expanded in polynomials of the logarithm of time \( t \). In leading order, an \( \ln(t) \)-law is obtained, and the leading corrections are given by a fourth-order polynomial. The correlators interpolate between three scenarios. First, there are planes in parameter space where the dominant corrections to the \( \ln(t) \)-law vanish, so that the logarithmic decay governs the structural relaxation process. Second, the dynamics due to the higher-order singularity can describe the initial and intermediate part of the \( \alpha \)-process thereby reducing the range of validity of von Schweidler’s law and leading to strong \( \alpha \)-relaxation stretching. Third, the \( \ln(t) \)-law can replace the critical decay law of the \( \beta \)-process leading to a particularly large crossover interval between the end of the transient and the beginning of the \( \alpha \)-process. This may lead to susceptibility spectra below the band of microscopic excitations exhibiting two peaks. Typical results of the theory are demonstrated for models dealing with one and two correlation functions.

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I. INTRODUCTION

Within the mode-coupling theory for ideal glass transitions (MCT), the dynamics of an amorphous system of strongly interacting spherical particles is described by \( M \) functions of time \( t \), \( \phi_q(t) \), \( q = 1, 2, \ldots, M \). These are autocorrelation functions of density fluctuations with wave-vector modulus \( q \) chosen from a grid of \( M \) values. The theory is based on a closed set of coupled nonlinear equations of motion for the \( \phi_q(t) \). The coupling coefficients in these equations are given in terms of the equilibrium structure functions. The latter are assumed to be known smooth functions of the control parameters of the system like, e.g., the packing fraction \( \varphi \). The MCT equations exhibit fold bifurcations at certain critical values of the control parameters, say at \( \varphi = \varphi_c \), describing a transition from ergodic liquid dynamics for \( \varphi < \varphi_c \) to non-ergodic glass dynamics for \( \varphi > \varphi_c \). The transition is accompanied by the evolution of a slow stretched dynamics that was suggested as the explanation of structural relaxation observed in glass-forming liquids. The leading-order asymptotic solutions of the equations for parameters approaching the transition provide predictions for the universal properties of glassy dynamics. These predictions have been tested extensively against experimental data and molecular-dynamics-simulation results. The outcome of these tests qualifies MCT as a candidate for a theory of structural relaxation in glass-forming systems.

It was shown that schematic MCT models exhibit also higher-order-bifurcation singularities like the cusp and swallow-tail bifurcation. The accompanying dynamics is utterly different from the one for the fold bifurcation. For example, in certain parameter regions, the leading order result reads \( \phi(t) \propto -\ln(t/\tau) \). This logarithmic decay is equivalent to a susceptibility spectrum that is independent of frequency \( \omega \), \( \chi''(\omega) \propto \omega^0 \), or to a \( 1/f \)-noise fluctuation spectrum. There are corrections to this leading-order result which alter qualitatively the straight \( \phi(t) \)-versus-\( \ln t \) lines or the plateaus of the \( \chi''(\omega) \)-versus-log\( (\omega) \) plots. One needs to understand these corrections if one intends to get an overview of the relaxation scenarios for parameters near the higher-order-bifurcation points. It is the goal of this paper to provide such understanding by construction of a general theory for the logarithmic-relaxation law and its leading corrections.

For parameters at a cusp singularity of schematic \( M = 1 \) models, the leading-order long-time decay follows the law \( \phi(t) - f \propto 1/\ln^2 t \). This law has been embedded in a leading-order description of the dynamics near the singularities in terms of multi-parameter scaling laws. It was shown by L. Sjögren that dielectric-loss spectra for certain polymers could be interpreted by this scaling-law description, and further work extended his analysis. However, it was also demonstrated that the cited decay laws at the critical points have to be complemented by their leading corrections in order to describe the numerical solutions of MCT equations within a time regime relevant for data analysis. But, so far it was not possible to evaluate the corrections for the mentioned scaling laws. The results of this paper will be obtained along a different route of asymptotic expansion of the MCT solutions than followed in Ref. 7.

Logarithmic decay of correlations for glassy systems has been observed, for example, in Monte-Carlo-simulation results for a spin-glass model, for photon-correlation data from a dense colloidal suspension, and for optical Kerr-effect data for a van der Waals liquid. But the present work is motivated by three recent discoveries. First, density correlators \( \phi_q(t) \) measured by photon-correlation spectroscopy for colloids of micellar particles demonstrated logarithmic decay within time windows of two orders of magnitude in size.
Second, the MCT equations for a system whose structure was described by Baxter’s model for sticky hard spheres exhibit cusp bifurcations [7, 13]. These findings have been corroborated by a comprehensive analysis of the glass transitions of a square-well system [14]. Third, logarithmic decay extending over three decades in time was found in a molecular-dynamics simulation for a system with an interaction given by a strong repulsion complemented by a short-ranged attraction [15].

One concludes that higher-order bifurcation singularities are crucial step in the derivation of MCT is the application of Kawasaki’s factorization approximation in order to express \( m_q(t) \) as functionals \( F_q \) of the \( M \) correlators \( \phi_k(t) \) called mode-coupling functionals. For simple systems, they are quadratic polynomials, whose coefficients are given in terms of the equilibrium structure functions \([1, 22]\). The functionals depend smoothly on, say, \( N \) control parameters to be combined to a control parameter vector \( V = (V_1, \ldots, V_N) \). In conventional liquids, the packing fraction and the temperature may be the control parameters. In colloids, one of the control parameters may be the attraction strength, which can be changed by modifying the solvent. Let us note the functionals by \( F_q[V, f_k] \). For \( 0 \leq |f_k| \leq 1, k = 1, 2, \ldots, M \), they can be written as Taylor series with non-negative coefficients. Thus, the second set of MCT equations is

\[
m_q(t) = F_q[V, \phi_k(t)] .
\]  

Specifying the functionals \( F_q \), the regular kernels \( M_{q}^{\text{reg}}(t) \), and the frequencies \( \Omega_q \) or \( \nu_q \), Eqs. (1a) or Eqs. (1b) together with Eq. (2) are closed. In the present paper, a topologically stable singularity of Eqs. (1) and (2) shall be discussed in full generality. Therefore, microscopic details are not of concern.

It will be convenient to discuss dynamics in the domain of complex frequencies \( z, \text{Im} z > 0 \). This can be achieved by Laplace transformation of functions of time, say \( F(t) \), to functions of \( z \) denoted by \( \mathcal{L}[F(t)](z) = i \int_0^\infty \exp(izt)F(t)dt \). Equations (1a) and (1b) together with the initial conditions are equivalent to a fraction representation of \( \phi_q(z) = \mathcal{L}[\phi_q(t)](z) \) in terms of \( M_{q}^{\text{reg}}(z) = \mathcal{L}[M_{q}^{\text{reg}}(t)](z) \) and \( m_q(z) = \mathcal{L}[m_q(t)](z) \),

\[
\phi_q(z) = -1/\{z - \Omega^2_q/ [z + M_{q}^{\text{reg}}(z) + \Omega^2_q m_q(z)] \} .
\]  

The analogue for the colloid dynamics is derived from Eq. (1b):

\[
\phi_q(z) = -1/\{z - \Omega^2_q/ [i\nu_q + M_{q}^{\text{reg}}(z) + \Omega^2_q m_q(z)] \} .
\]  

There are two possibilities for the solutions of the preceding equations. Either, all long-time limits of the correlators vanish as expected for an ergodic system. States...
with such control parameters $V$ are referred to as liquids. Or, there may be nonvanishing long-time limits $f_q = \phi_q(t \to \infty), 0 < f_q \leq 1$, as expected for non-ergodic systems. States with such control parameters $V$ are referred to as glasses, and $f_q$ is called the glass-form factor. Changing $V$, it may happen that there are values $V^c$ where one changes from a liquid to a glass—these are the ideal liquid-glass transitions discussed within MCT [21].

For glass states, $\phi_q(z)$ exhibits a zero-frequency pole, $\phi_q(z \to 0) \sim -f_q/z$. Because of Eq. (2), a similar state-

\[
\phi_q(z) = -1/[z - 1/m_q(z)] .
\]  

This equation exhibits most clearly the fraction representation of correlators, which is the essence of the Zwanzig-Mori theory. It shows that there is a one-to-one correspondence between density fluctuations and force fluctuations. There is no separation between the time scales for the particle motion within cages and for the particles forming the cage. Therefore, the correlators $\phi_q(t)$ and the kernels $m_q(t)$ have to be calculated self-consistently — this is the essence of MCT. Equation (4) is scale invariant. With $\phi_q(t)$, also $\phi_q^s(x-t) = \phi_q(x-t)$ is a solution for any $x > 0$. The scale for the high frequency dynamics is determined by the transient motion and this is governed by $M_q^\nu(z)$ and $\Omega_q$ or $\nu_q$. But these quantities do not occur anywhere in Eq. (4). Thus, Eq. (4) can fix the solution only up to some time scale.

Equations (3a, b) simplify to

\[
\phi_q(z) = -1/[z - 1/m_q(z)] .
\]  

This equation may have other solutions, say $\tilde{f}_q$. The glass-form factor is distinguished by the maximum theorem: $\tilde{f}_q \leq f_q, q = 1, \ldots, M$ [22].

Let $V^c$ denote some reference state. The long-time limits of the correlators for this state shall be denoted by $f_q^c$. The correlators shall be written in terms of new functions $\tilde{\phi}_q(t)$

\[
\phi_q(t) = f_q^c + (1 - f_q^c)\tilde{\phi}_q(t) .
\]  

The functional $F_q[V, \phi_q(t)]$ can be rewritten as a Taylor series in $\tilde{\phi}_q(t)$, using the coefficients

\[
A_{qk_1\ldots k_n}^{(n)}(V) = \frac{1}{n!} (1 - f_q^c) \{ \partial^n F_q[V, \tilde{f}_k] / \partial f_{k_1} \cdots \partial f_{k_n} \} \times (1 - f_{k_1}^c) \cdots (1 - f_{k_n}^c) .
\]  

These shall be split in the values for the reference state,

\[
A_{qk_1\ldots k_n}^{(n)}(V) = A_{qk_1\ldots k_n}(V^c) ,
\]  

and remainders:

\[
A_{qk_1\ldots k_n}^{(n)}(V) = A_{qk_1\ldots k_n}^{(nc)} + A_{qk_1\ldots k_n}^{(nc)}(V) .
\]  

Let us consider a path in control-parameter space given by $V(\epsilon) = (V_1(\epsilon), \ldots, V_N(\epsilon))$. The $N$ components of $V(\epsilon)$ are smooth functions of the path parameter $\epsilon$, and the tangent vector $dV(\epsilon)/d\epsilon$ must not vanish. We choose $V(\epsilon = 0) = V^c$, so that $\epsilon$ can be considered as a distance parameter specifying the neighborhood of $V^c$. One gets $V(\epsilon) = T\epsilon + O(\epsilon^2)$, with $T = dV(0)/d\epsilon$ being the tangent vector of the path at $V^c$. The mode-coupling functional is a smooth function of $V$, i.e.

\[
\tilde{A}_{qk_1\ldots k_n}^{(n)}(V) = O(\epsilon) .
\]  

The details of the $\epsilon$-dependence of the coefficients are not important. The parameter $\epsilon$ is introduced mainly as a means for bookkeeping in the following expansions in $V - V^c$. Expanding the left hand side of Eq. (4) in powers of $S[\tilde{\phi}_q(t)]$, one can rewrite this equation in the form

\[
\delta_{qk} - A_{qk}^{(1)c} \right) S[\tilde{\phi}_q(t)](z) = J_q(z) ,
\]  

where $\delta_{qk}$ denotes the Kronecker delta.
Here and in the following, summation over pairs of equal labels \( k \) is implied. These equations are equivalent to Eqs. (4) and (6) for the structural relaxation. The small quantities to be used for the asymptotic solution are the coefficients \( \hat{A}^{(n)}_{q_k} \) and the functions \( \hat{\phi}_q(t) \) or \( S[\hat{\phi}_q(t)](z) \), respectively.

Specializing Eqs. (10) to the \( z \to 0 \) limit, one gets the equation for \( \tilde{f}_q = \hat{\phi}_q(t \to \infty) \):

\[
\begin{align*}
[\delta_{q_k} - A^{(1)c}_{q_k}] \tilde{f}_k - A^{(0)}(V) + A^{(1)}(V) \tilde{f}_k & \\
+ \sum_{n=2}^{\infty} \{ A^{(n)}_{q_k} \} \tilde{f}_{k_1} \cdots \tilde{f}_{k_n} - \tilde{f}_q \} \end{align*}
\]

This is a rewriting of Eq. (10) so that small deviations of \( f_q \) from \( f_q^{(0)} \) and \( V \) from \( V^c \) are explicit. The \( M \) by \( M \) matrix \( [\delta_{q_k} - A^{(1)c}_{q_k}] \) is the Jacobian of the set of implicit equations (9) for the reference solution \( f_q^{(0)} \) at \( V = V^c \). This Jacobian consists of a unit matrix \( \delta_{q_k} \) and a matrix \( A^{(1)c}_{q_k} \) of positive elements. The Frobenius theorems imply that, generically, this matrix has no non-degenerate maximum eigenvalue \( E^c > 0 \). All other eigenvalues have a modulus smaller than \( E^c \). It is a subtle property of MCT that \( E^c \leq 1 \) [22]. If \( E^c < 1 \), the implicit-function theorem guarantees that all states \( V \) for sufficiently small \( \epsilon \) are states whose long-time limits \( f_q = f_q^{(0)} + (1 - f_q^{(0)}) \tilde{f}_q \) depend smoothly on \( \epsilon \). In the following, the reference state \( V^c \) shall be specialized so that

\[ E^c = 1. \]

In this case, \( V^c \) is a bifurcation point of Eq. (6). The \( f_q \) are singular functions of \( \epsilon \) for \( \epsilon \to 0 \), and therefore \( V^c \) is referred to as a glass-transition singularity. Since \( E^c \) is non-degenerate, the possible bifurcations are from the so-called cuspoid family \( A_l, l = 2, 3, \ldots \). The bifurcation singularity \( A_l \) is topologically equivalent to that for the zeroes of a real polynomial of degree \( l \) [24]. The \( A_2 \), also called fold bifurcation, is the generic singularity obtained by varying a single control parameter. The liquid-glass transition of MCT is of this type. In this paper, the dynamics near a higher order singularity \( A_l, l \geq 3 \), shall be analyzed.

### III. Relaxation Described by One-Component Models

It will be shown in Sec. V that each iteration step of the asymptotic solution of the equations of motion splits into two parts. First, one has to reduce the problem of calculating \( M \) correlators to the one of calculating the projection of the correlators on the dangerous eigenvector of the above defined Jacobian. Second, one has to solve the equation for the projection. In this section, the second problem will be studied, which is equivalent to a discussion of \( M = 1 \) models.

#### A. Classification of glass-transition singularities

One component models deal with a single correlator \( \phi(t) \), a single glass-form factor \( f \), etc. All matrix indices can be dropped in the formulas of Sec. II. The one-by-one matrix \( A^{(1)c}_{q_k} \) is identical with its maximum eigenvalue \( E^c \). Because of Eq. (12), the left-hand side of Eq. (11) vanishes. The equation for \( \tilde{f} \) reads

\[
\mu_1 = 1 - A^{(1)c}, \quad \epsilon_n(V) = \hat{A}^{(n-1)}(V), \quad n = 1, 2, \ldots .
\]

The singularity exhibited by \( \tilde{f} \) for \( \epsilon \) tending to zero depends on the number of successive vanishing coefficients \( \mu_n \). A singularity of index \( l, l \geq 2 \), shall be defined by

\[
\mu_1 = \mu_2 = \cdots = \mu_{l-1} = 0, \quad \mu_l \neq 0.
\]

The equation for \( \tilde{f} \) reads

\[
\mu_l \tilde{f} = \epsilon_{l-1}(V) \tilde{f}^{l-2} + \epsilon_{l-2}(V) \tilde{f}^{l-3} + \cdots + \epsilon_1(V) + \{ \epsilon_l(V) \tilde{f}^{l-1} + \epsilon_{l+1}(V) \tilde{f}^l + \sum_{n \geq l+1} A^{(n)}(V) - 1 \} \tilde{f}^n \}
\]

The implicit-function theorem can be used to show that there is a smooth invertible transformation of the \( l \) variables \( (\epsilon_1, \epsilon_2, \ldots, \epsilon_l, \tilde{f}) \) which eliminates the curly bracket in Eq. (13). Thus, the singularities described by this equation are topologically equivalent to the ones described by the first line, i.e. by the zeros of a polynomial of degree \( l \). In Arnol’d’s terminology [2] such singularity is referred to as \( A_l \). Because of Eq. (6), the \( \epsilon_n(V) \) are of order \( \epsilon \) and shall be referred to as separation parameters.

The simplest glass-transition singularity is the \( A_2 \). In this case, there is only one relevant control parameter \( \epsilon_1(V) \). One infers from Eq. (11) that there is a discontinuous change of \( \tilde{f} \) at the surface specified by \( \epsilon_1(V) = 0 \). The bifurcation dynamics is characterized by power-law decay and there appear power-law dependencies of the relaxation scales on [\( \epsilon_1(V) \)]. All exponents in these laws are to be calculated from \( \lambda = 1 - \mu_2 \), which is called the exponent parameter [22]. The transition surface has a boundary that is determined by \( \lambda = 1 \), i.e. by \( \mu_2 = 0 \). These endpoints are the higher-order singularities. The \( A_3 \) and \( A_4 \) are also referred to as cusp and swallow-tail singularities, respectively.
B. Equations for an asymptotic solution

Let us specialize Eqs. (10a) and (10b) for $M = 1$. Let us also express the coefficients $A^{(n)}(V)$ in terms of $\mu_n$ and $\epsilon_n(V)$:

$$0 = \epsilon_1(V) + (1 - \mu_2)S[\hat{\phi}^2(t)](z) - S[\hat{\phi}(t)]^2(z) + \epsilon_2(V)S[\hat{\phi}(t)](z) + (1 - \mu_3)S[\hat{\phi}^3(t)](z) - S[\hat{\phi}(t)]^3(z) + \epsilon_3(V)S[\hat{\phi}^3(t)](z) + (1 - \mu_4)S[\hat{\phi}^4(t)](z) - S[\hat{\phi}(t)]^4(z) + \cdots .$$

(16)

This suggests an expansion of the solution in powers of $|\epsilon|^{1/2}$. With $G^{(n)}(t) = O(|\epsilon|^{n/2})$, let us write

$$\hat{\phi}(t) = G^{(1)}(t) + G^{(2)}(t) + G^{(3)}(t) + \cdots .$$

(17)

The first line of Eq. (16) is of order $|\epsilon|$ and it provides a nonlinear integral equation for $G^{(1)}(t)$. The contributions to this line which are of order $|\epsilon|^{3/2}$ together with the leading terms of the second line provide a linear integral equation for $G^{(2)}(t)$, etc. This procedure will yield the desired asymptotic expansion provided the indicated integral equations define meaningful solutions. This is indeed the case as shall be demonstrated below by explicit construction of the $G^{(n)}(t)$. To proceed, the following discussion shall be restricted to higher-order singularities by requiring

$$\mu_2 = 0 .$$

(18)

C. The leading contribution

The equation for the leading contribution to the correlator at a glass transition $A_1$ with $l \geq 3$ reads

$$\epsilon_1(V) + S[G^{(1)}^2(t)](z) - S[G^{(1)}(t)]^2(z) = 0 .$$

(19)

The formulas for the Laplace transforms of the logarithm and its square imply $S[\ln(t)](z) = \ln(i/z) - \gamma$ and $S[\ln^2(t)](z) = \ln^2(i/z) - 2\gamma \ln(i/z) + \gamma^2 + (\pi^2/6)$, where $\gamma = 0.577 \ldots$ is Euler’s constant (cf. Appendix A, Eq. (A1)). Hence, Eq. (13) is solved by $-B \ln(t)$ if $\epsilon_1(V) + (B^2 \pi^2/6) = 0$. Since the correlators are monotonically decreasing functions of $t$, one must require $B > 0$. One concludes that a solution is given by

$$G^{(1)}(t) = -B \ln(t) , \quad B = \sqrt{-6\epsilon_1(V)/\pi^2} ,$$

(20)

provided the control parameters $V$ obey

$$\epsilon_1(V) < 0 .$$

(21)

A more general solution is $G^{(1)}(t) = G^{(1)}(t) + c$, where $c$ can be any real constant. Introducing $x = \exp(-c/B)$, one gets $G^{(1)}(t) = G^{(1)}(xt)$. Thus, the generalization is the one implied by the scale invariance of the basic Eq. (1). It shall not be considered here. Rather it shall be accounted for at the end of all calculations by rescaling $t$ to $t/\tau$. Ignoring corrections of order $|\epsilon|$, one derives from Eqs. (8), (17), and (20) the leading approximation for the correlator $f$:

$$\phi(t) = f^c - (1 - f^c)B \ln(t/\tau) .$$

(22)

Let us anticipate that the smooth function $\epsilon_1(V)$ is generic for $V$ near $V_c$ and has a nonvanishing gradient. Then, $\epsilon_1(V) = 0$ defines a smooth surface through $V_c$ in the control-parameter space. It separates the neighborhood of the glass-transition singularity $V_c$ into a strong-coupling side where $\epsilon_1(V) > 0$ and a weak-coupling side where $\epsilon_1(V) < 0$. The results of this paper refer to the latter regime.

D. The leading correction

In order to solve Eq. (16) up to order $|\epsilon|^{3/2}$, one has to incorporate from the first line the contribution $2S[G^{(1)}(t)]^2(z) - 2S[G^{(1)}(z)]S[G^{(2)}(t)](z)$, one has to evaluate the second line with $\hat{\phi}$ replaced by $G^{(1)}(t)$, and one can ignore all other terms. Hence, the equation for the leading correction $G^{(2)}(t)$ can be written in the form

$$\mathcal{T} \left[ G^{(2)}(t) \right] (z) = f^{(2)}(z) .$$

(23)

Here, the linear integral operator $\mathcal{T}$ is defined by

$$\mathcal{T} [G(t)](z) = S[\ln(t)G(t)](z) - S[\ln(t)](z)S[G(t)](z) ,$$

(24)

and the inhomogeneity of Eq. (23) reads

$$f^{(2)}(z) = -\left\{ \epsilon_2(V)S[G^{(1)}(t)](z) - \mu_3 S[G^{(1)}^3(t)](z) + 2\zeta \{ S[G^{(1)}^3(t)](z) - S[G^{(1)}](z)^3(z) \} \right\} / (2B) .$$

(25)

A factor $2\zeta$ has been introduced for later convenience. For the study of $M = 1$ models, one has to substitute $\zeta = 1/2$.

The solution of Eq. (16) was built on the equations $\mathcal{T}[\epsilon](z) = 0$, $\mathcal{T}[\ln(t)](z) = \pi^2/6$. These formulas are generalized in Appendix A by constructing polynomials $p_n(x)$ of degree $n \geq 1$ with the properties:

$$p_n(x) = b_{n,1}x + b_{n,2}x^2 + \cdots + b_{n,n-1}x^{n-1} + x^n ,$$

(26a)

$$\mathcal{T} [p_n(\ln(t))] (z) = n(\pi^2/6) \ln^{n-1}(i/z) .$$

(26b)

These polynomials are a convenient tool to solve the equation

$$\mathcal{T} [g(t)](z) = f(z)$$

(27a)
for inhomogeneities \( f(z) \) which are polynomials in \( \ln(i/z) \),

\[
f(z) = \sum_{n=0}^{m} a_n \ln^n(i/z) . \tag{27b}
\]

Obviously,

\[
g(t) = \sum_{n=1}^{m+1} \left[ a_{n-1}/(n\pi^2/6) \right] p_n(\ln(t)) . \tag{27c}
\]

Using Eq. (24) and applying Eqs. (A1) and (A2) for the evaluation of the transformations of the powers of \( \ln(t) \), one can write \( f^{(2)}(z) \) in the form of Eq. (27a) for \( m = 3 \). The coefficients are linear functions of \( \epsilon_1(V) \) and \( \epsilon_2(V) \):

\[
a_0 = \frac{[6\zeta/\pi^2](\Gamma_3 - \Gamma_1^3)}{(3\mu_3/\pi^2)\Gamma_3} \epsilon_1(V) - (\Gamma_1/2)\epsilon_2(V) . \tag{28a}
\]

\[
a_1 = \frac{[3\zeta - (9\mu_3/\pi^2)\Gamma_3]}{(3\mu_3/\pi^2)\Gamma_3} \epsilon_1(V) - (1/2)\epsilon_2(V) , \tag{28b}
\]

\[
a_2 = -(9\mu_3/\pi^2)\Gamma_1 \epsilon_1(V) , \quad a_3 = -(3\mu_3/\pi^2)\epsilon_1(V) . \tag{28c}
\]

Here, \( \Gamma_k = d^k\Gamma(1)/dx^k \) denotes the \( k \)th derivative of the gamma function at unity. One concludes that \( G^{(2)}(t) = g(t) \), where Eq. (27a) is to be used with \( m = 3 \):

\[
G^{(2)}(t) = \sum_{j=1}^{4} B_j \ln^j(t) . \tag{29a}
\]

The coefficients are derived with Eqs. (A3–c):

\[
B_1 = (0.44425\zeta - 0.065381\mu_3)\epsilon_1(V) - 0.22213\epsilon_2(V) , \tag{29b}
\]

\[
B_2 = (0.91189\zeta + 0.068713\mu_3)\epsilon_1(V) - 0.15198\epsilon_2(V) , \tag{29c}
\]

\[
B_3 = -0.13504\mu_3\epsilon_1(V) , \quad B_4 = -0.046197\mu_3\epsilon_1(V) . \tag{29d}
\]

Dropping corrections of order \(|\epsilon|^{3/2}\), the solution up to next-to-leading order reads

\[
\phi(t) - f^c = (1 - f^c) \left[ (-B + B_1) \ln(t/\tau) + B_2 \ln^2(t/\tau) \right. \\
\left. + B_3 \ln^3(t/\tau) + B_4 \ln^4(t/\tau) \right] . \tag{30}
\]

A singularity \( A_1 \) with \( \lambda \geq 4 \) implies \( \mu_\lambda = 0 \). In this case, the formula simplifies because \( B_3 = B_4 = 0 \).

The described procedure can be continued. To solve Eq. (16) up to order \( e^2 \), one derives the analogue to Eq. (23): \( T[G^{(3)}(t)](z) = f^{(3)}(z) \). Function \( f^{(3)}(z) \) has the form of Eq. (27a) with \( m = 6 \), where the coefficients \( a_j \) depend on the parameters \( \epsilon_1(V), \epsilon_2(V), \epsilon_3(V), \mu_3 \) and \( \mu_4 \). As a result, one gets

\[
G^{(3)}(t) = \sum_{j=1}^{7} C_j \ln^j(i/z) , \tag{31}
\]

where \( C_j = O(|\epsilon|^{3/2}) \).

IV. RESULTS FOR A ONE-COMPONENT MODEL

The simplest example for a generic cusp bifurcation is provided by an \( M = 1 \) model with the mode-coupling functional \( \mathcal{F}[V, f] = v_1f + v_3f^3 \). This model was derived originally within a microscopic theory of spin-glass transitions [23]. It shall be used here in order to demonstrate several implications of our theory. Let us use the model with a Brownian microscopic dynamics so that Eqs. (11) and (4) specialize to

\[
\tau_1 \partial_t \phi(t) + \phi(t) + \int_0^t m(t - t') \partial_{tt'} \phi(t') dt' = 0 , \tag{32a}
\]

\[
m(t) = v_1 \phi(t) + v_3 \phi^3(t) . \tag{32b}
\]

The two coupling constants \( v_1 \geq 0 \) and \( v_3 \geq 0 \) are considered as the components of the control-parameter vector \( V = (v_1, v_3) \).

Figure 1 reproduces the phase diagram [2, 22]. It is obtained from the largest of the solutions for \( f^c \) of Eq. (5), i.e., \( v_1f^c + v_3f^c = f^c/(1 - f^c) \), and Eq. (12), i.e., \( v_1 + 3v_3f^c = 1/(1 - f^c)^2 \), \( 0 \leq f^c \leq 1 \). There are two transition lines. The first one is the straight horizontal line of degenerate \( A_2 \) bifurcations: \( v_1^* = 1, 0 \leq v_3^* \leq 4, f^c = 0 \). Crossing this line by increasing \( v_1, f = \phi(t \to \infty) \) increases continuously. The second one is the smooth curve of \( A_2 \) singularities \( V^{(2)} \) shown as a heavy full line. It starts at \( v_1^{(2)} = 0, v_3^{(2)} = 27/4 \). \( f^{(2)} = 2/3 \). With decreasing \( v_3^{(2)} \), \( f^{(2)} \) decreases along the line. For \( v_1^{(2)} = 1, v_3^{(2)} = 4 \) one gets \( f^{(2)} = 1/2 \). Decreasing \( f^{(2)} \) further, the line reaches the endpoint that is marked by a circle. This is the \( A_3 \) singularity \( V^c \) specified by

\[
v_1^c = 9/8 , \quad v_3^c = 27/8 , \quad f^c = 1/3 , \quad \mu_3 = 1/3 . \tag{33}
\]

The two separation parameters are obtained from Eqs. (34) and (43) as linear functions of the parameters differences \( \hat{v}_{1,3} = v_{1,3} - v_3 \):

\[
\epsilon_1(V) = (2/81)[9\hat{v}_1 + \hat{v}_3] , \quad \epsilon_2(V) = (4/27)[3\hat{v}_1 + \hat{v}_3] . \tag{34}
\]

These formulas determine the coefficient \( B \) in Eq. (24) and \( B_1 - B_4 \) in Eqs. (29). The scales \( \tau \) for the results in Eqs. (22), (30), and (34) are determined as the time where the correlator crosses the critical form factor: \( \phi(\tau) = f^c \).

The dominant deviation of the correlators from the logarithmic-decay law, Eq. (22), is caused by the term \( B_2 \ln^2(t/\tau) \) in Eq. (30). Thus, the logarithmic-decay law is exhibited best for states \( V \) with \( B_2 = 0 \). This line is shown dashed-dotted in Fig. 1. Figure 2 demonstrates the evolution of the dynamics upon shifting states on this line towards the \( A_3 \) singularity. The \( \ln(t/\tau) \) interval, where Eq. (22) or (34) describe the correlators within an error margin of 5% is marked by closed or open symbols, respectively. For \( n \geq 2 \), these intervals increase with decreasing \( V - V^c \).
and triangles with label $n$ and $n'$ denote states discussed in Figs. 2 and 3 respectively.

There are two peculiarities concerning the range of applicability of the asymptotic expansions. First, it can happen that for sufficiently large $\epsilon$ the range shrinks if one proceeds from the leading approximation to the next-to-leading one as is demonstrated in Fig. 4 for the $n = 1, 2$ results. This is caused by a cancellation of errors due to neglecting the $B_1$ correction in the prefactor of the $\ln(t/\tau)$ term in Eq. (30) and due to neglecting the terms proportional to $B_3$ and $B_4$. This peculiarity would disappear if the tolerated error margin were decreased sufficiently below the 5% used. Second, for small $V - V_c$, the interval of decay for $\phi(t)$ below the critical form factor $f_c$ that is described by the asymptotic expansion shrinks with decreasing separation. This is inferred by comparing the $n = 3$ with the $n = 4$ results. The reason is the following. The correlator $\phi(t)$ decreases monotonically towards its long-time limit $f$. But the interval $f_c - f = -\hat{f}$ shrinks for $\epsilon \to 0$, since Eq. (13) implies $-\hat{f} = (-\epsilon_1(V)/\mu_3)^{1/3}[1 + O(\epsilon^{1/3})].$

Figure 3 demonstrates that the transient regime extends to about $t/\tau_1 = 1$. For vanishing mode-coupling functional, the correlator describes a Debye process: $\phi(t) = \exp(-t/\tau_1)$. Mode-coupling effects cause a slower decay for $t/\tau_1 \geq 1$. But for $V$ close to $V_c$, the transient dynamics is rather insensitive to changes of the coupling constants. There is a crossover interval, say $\tau_1 < t < \tau^*$, before the decay of $\phi(t)$ towards $f_c$ can be described by the $\ln(t/\tau)$ law. The beginning $\tau^*$ of the range of validity of Eq. (30) is indicated by the filled symbols. There are two subtleties demonstrated for $n \geq 2$. First, the time $\tau^*$ increases upon approaching the $A_3$ singularity, and therefore the decay interval $\phi(\tau^*) - f_c$ which is described by the logarithmic law shrinks with decreasing separation parameters. The control-parameter sensitive structural relaxation is governed by the two time scales $\tau^*$ and $\tau$. Both times become large, but $\tau/\tau^*$ becomes large as well for $\epsilon \to 0$. Second, the beginning of the range of applicability of the leading correction Eq. (30) is control-parameter insensitive, as is shown by the open symbols on the short-time part of the decay curves.

The explanation of the findings in the preceding paragraph is based on the fact that for every fixed finite time interval, the MCT solutions are smooth functions of the control parameters $\mu_3$. Therefore, for $\epsilon$ tending to zero, the correlator for state $V$ has to approach the correlator for state $V_c$, the so-called critical correlator. The latter is shown as the heavy full line with label $c$ in Fig. 5. Thus, for every time interval $0 \leq t \leq t_{\text{max}}$ and every error margin, there exists an $\epsilon^*$, so that $\phi(t)$ agrees with the critical correlator within the error margin for all $|\epsilon| < \epsilon^*$ and all $0 \leq t \leq t_{\text{max}}$. This feature is demonstrated in Fig. 5 by the two curves with label $n = 4$ and $\epsilon_1 \geq 0$. They refer to states with $v_1 - v_2 = \pm 0.9298/4^4$ and $v_3 - v_4 = \pm 3.3750/4^4$. These correlators are very close to the critical one for $t \leq t_{\text{max}}$; $t_{\text{max}} \approx 325$ is indicated by open circles in Fig. 5. The critical correlator does not exhibit an $\ln(t/\tau)$-part. Thus, the time $\tau^*$ for the onset of the description by Eq. (30) has to increase beyond any bound if $\epsilon$ tends to zero. The asymptotic expansion in Sec. II was based on $|\phi(t) - f_c| / t$ to be small. This condition is fulfilled for the critical correlator if the time is sufficiently large, since $\phi(V_c,t)$ decreases monotonically to $f_c$. Hence, for $\epsilon \to 0$ there must appear the increasing time interval $\tau_1 < t < \tau^*$ where the asymp-

![Figure 1](image1.png)

**FIG. 1:** Phase diagram for the one-component model defined in Sec. I. The horizontal light full line marks the liquid-glass-transition curve connected with a continuous variation of the glass-form factor. The heavy full line presents the set of $A_2$ singularities, that ends at the $A_3$ singularity marked by a circle. The dashed straight line describes the points of vanishing separation parameter $v_1$ and the dashed-dotted one the points of vanishing coefficients $B_2$. Crosses with labels $n$ and triangles with label $n'$ denote states discussed in Figs. 2 and 3 respectively.

![Figure 2](image2.png)

**FIG. 2:** Correlators $\phi(t)$ for the one-component model defined in Sec. I. The states are located on the line $B_2 = 0$ with coupling constants: $v_1 - v_1 = 0.9298/4^n$, $v_3 - v_3 = 3.3750/4^n$, $n = 1, \ldots, 4$, marked by crosses in Fig. 1. The full lines are the solutions of Eqs. (12), b with $\tau_1 = 1$ as unit of time as also in the following figures. The dotted straight lines exhibit the leading approximation, Eq. (22), the dashed lines the leading correction, Eq. (20). The filled and open symbols, respectively, mark the times where these approximations deviate from the solution by 5%. The dotted line marked by D is the Debye law $\exp[-t/\tau_1]$. The horizontal line shows the critical form factor $f_c = 1/3$. The explanation of the findings in the preceding paragraph is based on the fact that for every fixed finite time interval, the MCT solutions are smooth functions of the control parameters $\mu_3$. Therefore, for $\epsilon$ tending to zero, the correlator for state $V$ has to approach the correlator for state $V_c$, the so-called critical correlator. The latter is shown as the heavy full line with label $c$ in Fig. 5. Thus, for every time interval $0 \leq t \leq t_{\text{max}}$ and every error margin, there exists an $\epsilon^*$, so that $\phi(t)$ agrees with the critical correlator within the error margin for all $|\epsilon| < \epsilon^*$ and all $0 \leq t \leq t_{\text{max}}$. This feature is demonstrated in Fig. 5 by the two curves with label $n = 4$ and $\epsilon_1 \geq 0$. They refer to states with $v_1 - v_2 = \pm 0.9298/4^4$ and $v_3 - v_4 = \pm 3.3750/4^4$. These correlators are very close to the critical one for $t \leq t_{\text{max}}$; $t_{\text{max}} \approx 325$ is indicated by open circles in Fig. 5. The critical correlator does not exhibit an $\ln(t/\tau)$-part. Thus, the time $\tau^*$ for the onset of the description by Eq. (30) has to increase beyond any bound if $\epsilon$ tends to zero. The asymptotic expansion in Sec. II was based on $|\phi(t) - f_c| / t$ to be small. This condition is fulfilled for the critical correlator if the time is sufficiently large, since $\phi(V_c,t)$ decreases monotonically to $f_c$. Hence, for $\epsilon \to 0$ there must appear the increasing time interval $\tau_1 < t < \tau^*$ where the asymp-

There are two peculiarities concerning the range of applicability of the asymptotic expansions. First, it can happen that for sufficiently large $\epsilon$ the range shrinks if one proceeds from the leading approximation to the next-to-leading one as is demonstrated in Fig. 4 for the $n = 1, 2$ results. This is caused by a cancellation of er-
The heavy full curve with label c is the solution of Eqs. (24a, b) at the $A_3$ singularity, $V = V^*$, called the critical correlator. The thin full curves with labels $n = 1$ and 4, $\epsilon_1 < 0$ reproduce two of the solutions discussed in Fig. 3. The dashed lines reproduce from Fig. 3 the corresponding approximations by the next-to-leading order, Eq. (30). The dashed-dotted curves extend the asymptotic expansion by including the second order corrections, Eq. (31). The horizontal lines mark the critical form factor $f'' = 1/3$. The full curve marked with $n = 4$, $\epsilon_1 \geq 0$ deviate by 2% from the critical one.

The asymptotic expansion describes the critical correlator. The $\epsilon$-dependence due to the separation parameters $\epsilon_n(V)$ and the $\epsilon$-dependence due to the time scale $\tau$ cancel to produce the critical correlator outside the transient regime and prior to the onset of the $\ln(t/\tau)$ law. This is shown most clearly by the dashed-dotted lines in Fig. 3. They exhibit the result of the asymptotic expansion up to the second correction given by Eq. (31). They describe the plateau for $t/\tau_1 > 1$ except for the final exponential approach towards $f = \phi(t \to \infty)$; and this for states as far from the critical point as given by the one with label $n = 1$ (cf. Fig. 3).

Figure 4 exhibits the dynamical susceptibilities $\chi''(\omega) = 1 - \mathcal{S}[\phi(t)][\omega + i\delta] = \chi'(\omega) - i\chi''(\omega)$ for the states discussed above. Without interaction effects, the susceptibility spectrum was a Debye peak $\chi''_D = C_D\omega\tau_D/[1 + (\omega\tau_D)^2]$ with $C_D = 1$, $\tau_D = \tau_1$. Such Lorentzian spectrum is added to the upper panel as dotted line with label D, where $C_D$ and $\tau_D$ are fitted to the maximum of the critical susceptibility spectrum. This shows that the spectral peaks near $\omega = 1$, in particular their high-frequency wings, are due to the transient dynamics. However, the low-frequency wings of the peaks are enhanced relative to the Debye spectrum and they are stretched to lower frequencies due to the critical relaxation within the interval $1/\tau' < \omega < 1/\tau$. It is the structural relaxation near the $A_3$ singularity which causes the skewed shape of the spectral peaks. The leading approximation, Eq. (22), implies constant-loss plateaus: $\chi''(\omega)/(1 - f^c) = \pi B/2$. However, this formula describes the plateau only for $n \geq 3$ and then for a frequency interval that is considerably smaller than the time interval for which Eq. (22) describes the correlators in Fig. 3. The leading corrections in Eq. (34) are much more important for an adequate description of the spectra than for the approximation of the correlators. The second correction, Eq. (31), is necessary to describe the plateau for the $n = 2$ state within a 5% error margin. It is also necessary to describe the crossover of the spectrum from the plateau towards the critical one for $n = 3, 4$.

To understand the range of validity of Eq. (22) and its Fourier transform, one has to compare it with Eq. (30) and its Fourier transform, respectively. This amounts to comparing polynomials in $\ln(t/\tau)$ and $\ln(i/\omega\tau)$. Let us restrict ourselves to the dominant terms for the model to grasp the essence. Then one can write $(\phi(t) - f^c)/(1 - f^c) = -B\ln(t/\tau)[1 - (B_4/B)\ln^2(t/\tau)]$. Thus, within the error margin $\delta$, the leading linear-in-$\ln(t/\tau)$ approximation holds for $|\ln(t/\tau)| < \sqrt{\delta(B/B_4)}$. For the spectrum one gets from Eq. (31) in leading order $\chi''(\omega)/(1 - f^c) = B(\pi/2)[1 - 4(B_4/B)\ln^2(1/\omega\tau)]$. Hence,

FIG. 3: The heavy full curve with label c is the solution of Eqs. (24a, b) at the $A_3$ singularity, $V = V^*$, called the critical correlator. The thin full curves with labels $n = 1$ and 4, $\epsilon_1 < 0$ reproduce two of the solutions discussed in Fig. 3. The dashed lines reproduce from Fig. 3 the corresponding approximations by the next-to-leading order, Eq. (30). The dashed-dotted curves extend the asymptotic expansion by including the second order corrections, Eq. (31). The horizontal line marks the critical form factor $f'' = 1/3$. The full curve marked with $n = 4$, $\epsilon_1 \geq 0$ deviate by 2% from the critical one.

FIG. 4: Susceptibility spectra $\chi''(\omega)$ and reactive parts of the dynamical susceptibility $\chi'(\omega)$ for the one-component model defined in Sec. IV. The full lines with labels n and c correspond to the correlators with the same labels shown in Figs. 2 and 3. The dotted straight lines, the dashed lines, and the dashed-dotted lines are the leading approximation, Eq. (22), the leading correction, Eq. (30), and the second correction, Eq. (31), respectively. The filled, open, and half-filled symbols mark the frequencies where the corresponding approximation deviates from the spectrum by 5%. The dotted line with label D exhibits a Debye spectrum $C_D(\omega\tau_D)/(1 + (\omega\tau_D)^2)$ with $C_D = 0.2503$ and $\tau_D/\tau_1 = 0.670$ fitted to the maximum of the critical spectrum. The horizontal lines in the lower panel exhibit the static susceptibility $\chi'(\omega \to 0) = 1 - f$ for the states with labels $n = 2, 3, 4$, and c, respectively.
the spectrum is at the plateau within a deviation \( \delta \) for
\(|\ln(1/\omega \tau)| \leq \sqrt{\delta(B/4\beta)}\). As a result, the range of applicability on a logarithmic axis shrinks by a factor \( \sqrt{\delta} \)
if one transforms from the time domain to the frequency domain.

Because of Eq. (A11), \( S[\ln^a(t/\tau)](z) = \ln^a(i/z\tau) - a\gamma \ln^{a-1}(i/z\tau) + \cdots \). In leading approximation for \( t \to \infty \)
and \( z \to 0 \), one finds \( \gamma(\omega) \propto 1 - \phi(1/\omega) \) whenever \( \phi(t) \) is a polynomial in \( \ln(t/\tau) \). This explains the lower panel
of Fig. 2 as a different representation of Figs. 3 and 3. In particular, the linear-ln(\( \omega \)) parts in Fig. 4 are of a similar size as the linear-ln(t/\( \tau \)) parts in Fig. 2.

![FIG. 5: Correlators \( \phi(t) \) for the one-component model defined in Sec. IV for states located on the line \( \epsilon_1 = -0.0182 \). The states with labels \( n' = 1-4 \) have the coordinates \((v_1, v_3) = (1.1169, 2.7141), (1.0669, 3.1641), (1.0169, 3.6141), (0.9669, 4.0641)\), respectively, and they are marked by triangles in Fig. 4. The state labeled \( n' = 2 \) is identical with the state discussed in Figs. 3 and 3 with label \( n = 2 \). The states with labels 5, 6, and 7 have the coordinates \((0.9599, 4.1271), (0.9569, 4.1541), (0.9549, 4.1721)\), respectively. The straight line through the states 1–7 crosses the liquid-glass-transition curve at the state \( V^{(2)c} = (0.95466, 4.17407) \), where the critical glass-form factor has the value \( f^{(2)c} = 0.520 \). The exponent parameter for the \( A_2 \)-glass-transition singularity is \( \lambda = 0.719 \) implying a critical exponent \( \alpha = 0.318 \) and a von Schweidler exponent \( b = 0.608 \). The critical decay law \( \phi(t) = f^{(2)c} \propto t^{-\alpha} \) and von Schweidler’s law \( \phi(t) - f^{(2)c} \approx -t^b \) are shown by dotted lines labeled a and b, respectively; the constants of proportionality are fitted to curve 7. The dashed-dotted curve extends the von Schweidler expansion for curve 7 to \( \phi(t) = f^{(2)c} - (t/\tau)^b + 1.48(t/\tau)^{2b} \). The horizontal lines mark the critical glass-form factors \( f^{(2)c} \) and \( f^c \), respectively. The dotted straight lines and the dashed curves are the leading asymptotic laws, Eq. (22), and the leading correction, Eq. (30), respectively.

Let us consider the states labeled \( n' = 1-3 \) and shown by triangles in Fig. 3 in order to analyze the implications of the correction term in Eq. (30) proportional to \( B_2 \). These states are chosen on the line \( \epsilon_1 = -0.0182 \) and state \( n' = 2 \) is identical with state \( n = 2 \) considered in Fig. 3 as example for \( B_2 = 0 \). Figure 3 exhibits the correlators together with their approximations. For \( B_2 > 0 \), the \( \phi(t) \)-versus-log(t) diagram is concave for all times outside the transient, since a parabola with positive curvature is added to the leading linear variation described by Eq. (22). The formula with the leading correction describes the complete structural relaxation, except for the very last piece for the approach to the long-time limit \( f \), as shown by curve \( n' = 1 \). This observation also holds for cases with \( B_2 < 0 \) as is demonstrated for state \( n' = 3 \). However, for negative \( B_2 \), the \( \phi(t) \)-versus-log(t) curve exhibits two inflection points because \( \phi(t) \) crosses the critical form factor \( f^c \) with negative curvature. Since the \( \phi(t) \)-versus-log(t) curve is convex for \( \phi(t) \approx f^c \), it has to have an inflection point for \( \phi(t) < f^c \) in order to approach the exponential, i.e. concave, long-time asymptote. It has to exhibit an inflection point also for \( \phi(t) > f^c \) in order to approach the concave critical correlator for short times. The described alternation of convex and concave parts is identical to the behavior discussed earlier for the MCT correlators for states near an \( A_2 \)-bifurcation [20, 21]. But, contrary to the characteristic decay pattern found for the MCT liquid-glass transition, curve \( n' = 3 \) does not show a two step relaxation scenario, even though there is a huge stretching of the dynamics. For the decay from 0.80 to 0.05 a dynamical window of 5 orders of magnitude is required. Within this large window, the correlator follows closely the law \( \phi(t) \approx \ln(t/\tau_{eff}) \).

The qualitative features described above for state \( n' = 3 \) are more pronounced for state \( n' = 4 \), since \( B_2 \) is decreased to larger negative values. The relaxation curve 4 has the form expected for states near a liquid-glass transition. To corroborate this statement, further states 5–7 are considered on the line \( \epsilon_1 = -0.0182 \) between the state 4 and the intersection \( V^{(2)c} \) of this line with the liquid-glass transition curve. The transition point \( V^{(2)c} \) is characterized by a critical glass-form factor \( f^{(2)c} \approx f^c \). The decay of the correlator from the value \( f^{(2)c} \) to zero is the corresponding \( \alpha \)-process. Its initial part is described by von Schweidler’s power law, as indicated in Fig. 4 for curve \( n' = 7 \) by the dotted line. In this case, von Schweidler’s law accounts for the decay from \( f^{(2)c} \) to about 0.45, i.e. for about 15% of the \( \alpha \)-relaxation.

The analytical description of the \( \alpha \)-process can be expanded by using the extension of von Schweidler’s law, \( \phi(t) = f^{(2)c} - (t/\tau)^b + B(t/\tau)^{2b} \), as shown by the dashed-dotted line. Asymptotically, the \( \alpha \)-process obeys the superposition principle: \( \phi(t) = \phi(t/\tau_0) \), where \( \phi \) is the control-parameter-independent shape function. The \( \phi(t) \)-versus-log(t) curves for the \( \alpha \)-process can be superimposed by rescaling the time, i.e. by shifts parallel to the log(\( t \))-axis. The reader can check by herself that the curves \( n' = 4-7 \) have the same shape for \( \phi(t) < f^{(2)c} \). Outside the transient for \( \phi(t) > f^{(2)c} \), the correlator follows the critical decay law for the fold bifurcation \( \phi(t) = f^{(2)c} \approx 1/t^a \), as is also demonstrated for curve 7. The results for states 4–7 exemplify the well understood scenario for the evolution of structural relaxation near a liquid-glass transition. The formula (30) provides...
an accurate description of 60% of the $\alpha$-process.

Comparison of the results for states $n' = 1-3$ with the second-correction formula based on Eq. (31) yields the same conclusions as discussed above in connection with Fig. 3. The second-correction formula does not alter seriously the fit quality for the long-time part of the curves $n' = 4-6$ in Fig. 3. However, for $\phi(t) \approx f(2)c$, the extended formula yields slightly worse results than Eq. (30). This is so, since for $\phi(t) \geq f(2)c$, the dynamics is governed by the $A_2$ singularity $V^{(2)c}$ whose existence is ignored in the expansions near the higher-order singularity $V^c$. The number $f(2)c - f^c$ marks the limit where the expansion in the small parameter $\phi(t) - f^c$ makes sense. The opposite conclusion holds for the description of the $\alpha$-process for states $n' = 4-6$. Von Schweidler’s law results from an expansion for states $V$ near $V^{(2)c}$ in terms of the small parameter $f^{(2)c} - \phi(t)$. This number becomes too large if $\phi(t) \approx f^c$. It is the dynamics dominated by the higher order glass-transition singularity $V^c$ that ruins the relevance of the expansion resulting in the von Schweidler law. The stretching of the $\alpha$-process connected with the transition of $V^{(2)c}$ is larger than estimated by von Schweidler’s law, because of the logarithmic-decay effects.

![FIG. 6: Susceptibility spectra $\chi''(\omega)$ for the correlators in Fig. 3 for the labels $n' = 1-6$. The dashed lines are obtained from the leading-correction formula, Eq. (24). The dotted line with label a is the critical spectral proportion to $\omega^n$ and dotted line with label b is the von Schweidler law $\chi''(\omega) \propto -\log(\omega)$ for state $n' = 6$. The dotted horizontal line corresponds to the spectrum of the leading approximation, Eqs. (24), $\chi''(\omega) = (1 - f^c)\sqrt{-6\epsilon_1}$, that is ignored by all states $n' = 1-6$.](image)

Figure 3 exhibits the susceptibility spectra calculated from correlators discussed in Fig. 3. The results for the states $n' = 1$, 2, and 3 exhibit the evolution of an $\alpha$-peak if the states cross the line $B_2 = 0$ in the phase diagram of Fig. 3. The leading-correction formula, Eq. (30), describes this scenario qualitatively. The spectra for states $n' = 4$, 5, and 6 exhibit the superposition principle for the $\alpha$-peak of the susceptibility spectra. The $A_2$-dynamics causes a high-frequency wing of the $\alpha$-peak following closely a linear variation with the logarithm of the frequency: $\chi''(\omega) \propto -\ln(\omega\tau)$. This phenomenon is described well by Eq. (31) and it causes a strong $\alpha$-process stretching. The $\alpha$-peak width at half of the $\alpha$-peak height is about 2.5 decades. Von Schweidler’s asymptotic law is irrelevant for the description of the $\alpha$-peak for states $n' = 4-6$.

V. RELAXATION FORMULAS FOR STATES NEAR HIGHER-ORDER GLASS-TRANSITION SINGULARITIES

In this section, the generalizations of Eqs. (23) and (31) shall be derived for the asymptotic expansion of the solutions dealing with an arbitrary number $M$ of correlators. In particular, the general formulas for $\mu_2, \mu_3, \zeta$ and for the separation parameters $\epsilon_1(V)$ and $\epsilon_2(V)$ shall be obtained. The starting formulas are Eqs. (10a) and (10b). The subtlety of the problem is the treatment of the singular $M$ by $M$ matrix $[\delta_{qk} - A_{qk}^{(1)c}]$.

A. Asymptotic expansion of the equations of motion

The left and right eigenvectors of matrix $A_{qk}^{(1)c}$ for the maximum eigenvalue $E^c = 1$ shall be denoted by $a^c_k$ and $a^c_k$, $k = 1,..., M$, respectively. According to the Frobenius theorems [23], one can require $a^c_k \geq 0$ and $a^c_k \geq 0$. It will be convenient to fix the eigenvectors uniquely by the conditions $\sum_a q^a k a q = 1$ and $\sum_q a^c q a q^2 = 1$. The solubility condition of Eq. (10b) reads

$$\sum_q a^c q J_q(z) = 0 ,$$

and its general solution can be written as

$$\tilde{\phi}_q(t) = a_q \phi(t) + \tilde{\phi}_q(t) .$$

The splitting of $\tilde{\phi}_q(t)$ in two terms is unique if one imposes the condition $\sum_q a^c q \tilde{\phi}_q(t) = \tilde{\phi}_q(t)$.

The part $\tilde{\phi}_q(t)$ can be expressed by means of the reduced resolvent $R_{qk}$ of $A_{qk}^{(1)c}$:

$$S[\tilde{\phi}_q(t)](z) = R_{qk} J_k(z) .$$

It is an elementary task to evaluate from matrix $A_{qk}^{(1)c}$ the vectors $a^c_k$, $a^c_k$, and the matrix $R_{qk}$ [23].

Equations (10a) and (10b) suggest an expansion of $\phi(t)$ as Eq. (17) and

$$\tilde{\phi}_q(t) = G_q^{(2)}(t) + G_q^{(3)}(t) + \cdots , \quad G_q^{(n)}(t) = O(|\epsilon|^{n/2}) ,$$

$$J_q(t) = J_q^{(2)}(t) + J_q^{(3)}(t) + \cdots , \quad J_q^{(n)}(t) = O(|\epsilon|^{n/2}) .$$
Here, for example,

$$J^{(2)}_q(z) = \hat{A}^{(0)}_q(V) + A^{(2)c}_{q,k_1,k_2}a_{k_1}a_{k_2}S[G^{(1)}(t)](z) - a^3_q\hat{S}[G^{(1)}(t)]^3(z) , \quad (37a)$$

$$J^{(3)}_q(z) = 2\{A^{(2)c}_{q,k_1,k_2}a_{k_1}a_{k_2}S[G^{(1)}(t)G^{(2)}(t)](z) - a^3_q\hat{S}[G^{(1)}(t)]^3(z)S[G^{(2)}(t)](z)\} + \hat{A}^{(1)}_{q,k}(V)a_kS[G^{(1)}(t)](z) + 2\{A^{(2)c}_{q,k_1,k_2}a_{k_1}S[G^{(1)}(t)G^{(2)}(t)](z) - a^3_q\hat{S}[G^{(1)}(t)]^3(z)S[G^{(2)}(t)](z)\} + A^{(3)c}_{q,k_1,k_2}a_{k_1}a_{k_2}a_{k_3}S[G^{(1)}^3(t)](z) - a^3_q\hat{S}[G^{(1)}(t)]^3(z) . \quad (37b)$$

The justification of the preceding expansions shall be given by demonstrating how the equations can be solved recursively.

### B. The leading-order contribution

The leading-order contribution to the solubility condition is obtained by substituting Eq. (37a) into Eq. (35a). One arrives at: $\epsilon_1(V) + \lambda S[G^{(1)}^2(t)](z) - S[G^{(1)}(t)]^2(z) = 0$. Here $\lambda = \sum a_q^\ast A_q^{(2)c}a_{k_1}a_{k_2}$ is the expression for the exponent parameter \[22, 26\] and

$$\epsilon_1(V) = \sum a_q^\ast \hat{A}^{(0)}_q(V) . \quad (38)$$

The $z = 0$ limit leads to $\epsilon_1(V) + (\lambda - 1)f^{(1)} = 0$. Comparison with Eq. (35a) yields the conclusion that

$$\mu_2 = 1 - \sum a_q^\ast A_q^{(2)c}a_{k_1}a_{k_2} . \quad (39)$$

This parameter has to be zero according to Eq. (18) in order for $V^{\ast}$ to be a higher-order singularity. For $\mu_2 = 0$, $\lambda = 1$, and the equation found for $G^{(1)}(t)$ is identical with Eq. (19). Thus, $\epsilon_1(V)$ is the first separation parameter and Eqs. (21) and (22) remain valid.

Introducing the critical amplitude $h_q$ by the same formula as in the theory for the $A_2$-singularity \[22, 26\]

$$h_q = (1 - f^c_q)a_q , \quad (40)$$

the leading approximation for the correlators is

$$\phi_q(t) = f^c_q + h_q [-B\ln(t/\tau)] . \quad (41)$$

Here, $B = \sqrt{-6\epsilon_1(V)/\pi} = \mathcal{O}(\epsilon^{1/2})$. Equation (41) describes the dynamics up to errors of order $\epsilon_1$; it is the generalization of the logarithmic decay law \[8\] to arbitrary MCT models.

Equation (41) is the factorization theorem of MCT. In leading order, $\phi_q(t) - f^c_q$ factorizes in two terms. The factor $h_q$ is time- and control-parameter-independent and it characterizes via its $q$-dependence the specific correlator. The other factor is the function $G^{(1)}(t) = -B\ln(t/\tau)$. This factor is shared by all correlators. It describes via $B$ and $\tau$ the control-parameter dependence and via $\ln(t)$ the complete time dependence. Within the range of validity of Eq. (13), the rescaled correlators $\hat{\phi}_q(1) = [\phi_q(t) - f^c] / h_q$ are the same for all $q$. Let us emphasize that Eq. (41) is an exact limit result for the solutions of Eqs. (1) and (2):

$$\lim_{\epsilon \to 0}[\phi_q(t/\tau) - f_q^c]/\sqrt{-\epsilon_1(V)} = -\sqrt{6/\pi^2}h_q\ln(t) . \quad (42)$$

The interval of rescaled times $\hat{\tau} = t/\tau$, where $[\phi_q(t) - f^c_q] / \sqrt{-\epsilon_1(V)}$ becomes close to the r.h.s. of Eq. (12), expands beyond any bound if $V$ approaches $V^{\ast}$ arbitrarily close. It will be shown below, how the leading corrections for $\phi_q(t)$ describe violations of the factorization theorem.

Substitution of Eq. (37a) into Eq. (35a) yields the leading order contribution to $\hat{\phi}_q(t)$, i.e. the function $G^{(2)}_q(t)$ in Eq. (36a). Equation (14) is used to express $S[G^{(1)}(t)]^2(z)$ in terms of $S[G^{(1)}^2(t)](z)$ so that

$$G^{(2)}_q(t) = X_qG^{(1)}_q(t) + \hat{Y}_q(V) . \quad (43a)$$

The amplitude $X_q$ is independent of $\epsilon$,

$$X_q = R_{q,k} \left[ A^{(2)c}_{q,k_1,k_2}a_{k_1}a_{k_2} - A_{q,k}^2 \right] . \quad (43b)$$

$$\hat{Y}_q(V) = \mathcal{O}(\epsilon) \quad \text{and reads}$$

$$\hat{Y}_q(V) = R_{q,k} \left[ \hat{A}^{(0)}_{q,k}(V) - \epsilon_1(V)a^2_{q,k} \right] . \quad (43c)$$

### C. The leading correction

If one substitutes Eq. (43a) into Eq. (37b), one gets an expression for $J^{(3)}_q(z)$ in terms of the known $G^{(1)}(t)$ and the unknown $G^{(2)}(t)$. Therefore, the solubility condition, Eq. (35a), evaluated up to order $\epsilon^3/2$, yields an equation for $G^{(2)}(t)$. The latter has the form of Eq. (23), where also the inhomogeneity is given by Eq. (27). This holds with the formula

$$\epsilon_2(V) = \sum a_q^\ast \hat{A}^{(1)}_{q,k}(V)a_k + 2\epsilon_1(V) \sum a_q^\ast a_q X_q + 2 \sum a_q^\ast \left[ A^{(2)c}_{q,k_1,k_2}a_{k_1}a_{k_2} - a_q^\ast \hat{Y}_q(V) \right] \quad (44)$$

for the second separation parameter, and the constants

$$\zeta = \sum a_q^\ast [a_q X_q + a_q^3/2] , \quad (45)$$
\[ \mu_3 = 2\zeta - \sum_q a_q \left[ A_{q_1 k_1 k_2 k_3} a_k a_k a_k + 2 A_{q_1 k_2 k_3} a_k X_{k_2} \right]. \]  
(46)

As a result, Eqs. (23) for the function \( G^{(2)}(t) \) remain valid.

Combining the results for \( G^{(1)}(t) \), \( G^{(2)}(t) \) and \( G^{(3)}(t) \) with Eq. (30a), and this with Eq. (8), one obtains the main result of this paper. It describes the correlators up to errors of order \(|\epsilon|^{3/2}\):

\[ \phi_q(t) = (f_q^c + f_q) + h_q \left( -(B + B_1) \ln(t/\tau) + (B_2 + K_2 B^2) \ln^2(t/\tau) + B_3 \ln^3(t/\tau) + B_4 \ln^4(t/\tau) \right) \]  
(47)

Here

\[ f_q = (1 - f_q^c) \tilde{Y}_q \]  
(48)

is a renormalization of the glass-form factor of order \( \epsilon \) to be calculated from Eq. (35c). The critical amplitude \( h_q \) is defined by Eq. (41). The parameter \( B_1 \) from Eq. (29d) is a renormalization of order \( \epsilon \) of the prefactor of the logarithmic decay laws. The three terms proportional to \( B_2, B_3, \) and \( B_4 \), respectively, describe leading deviations from the logarithmic decay. They are of order \( \epsilon \) and follow from Eqs. (29c) and (29d). The relative size of these deviations is the same for all correlators. This means, that these terms imply a modification of the factorization theorem, \( \phi_q(t) - (f_q^c + f_q) = h_q G(t) \), in the sense that \( G(t) = -B \ln(t/\tau) \) in Eq. (41) is to be generalized by the bracket on the r.h.s of Eq. (40). It is solely the contribution proportional to \( B^2 = -6\epsilon_1/\pi^2 = \mathcal{O}(\epsilon) \) which describes a violation of the factorization theorem. It enters with the correction amplitude

\[ K_q = X_q/a_q. \]  
(49)

Its \( q \)-dependence expresses the fact that the size of the leading corrections depends on the chosen correlator. Thus, the range of validity of the universal Eq. (41) is not universal. The correction amplitude is to be calculated from Eq. (13d). The formula for \( K_q \) is the same as discussed in the theory for the \( A_2 \) singularity [26].

VI. RESULTS FOR A TWO-COMPONENT MODEL

The simplest example exhibiting a generic swallow-tail bifurcation is given by an \( M = 2 \) model with the mode-coupling functionals \( \mathcal{F}_1[V, f_k] = v_1 f_1^2 + v_2 f_2^2, \mathcal{F}_2[V, f_k] = v_3 f_1 f_2 \). This model was motivated originally as truncation of microscopic equations of motion for a symmetric molten salt [27]. The model shall be used here in order to demonstrate implications of our theory which could not be demonstrated for the \( M = 1 \) model studied in Sec. [4].

Using Brownian microscopic dynamics, the equations of motion (11) and (2) read for \( \phi_q(t), q = 1, 2 \):

\[ \tau_q \partial_t \phi_q(t) + \phi_q(t) + \int_0^t m_q(t - t') \partial_t \phi_q(t') dt' = (80a) \]

\[ m_1(t) = v_1 \phi_1^2(t) + v_2 \phi_2^2(t), \]  
(50b)

\[ m_2(t) = v_3 \phi_1(t) \phi_2(t). \]  
(50c)

The three coupling constants \( v_n \geq 0 \) shall be considered as the components of the control-parameter vector \( V = (v_1, v_2, v_3) \).

Let us note convenient equations for the discussion of the phase diagram [22], restricting ourselves to \( v_3 > 4 \). Equation (6) for the second form factor implies \( f_2 = [v_3 f_1 - 1]/(v_3 f_1) \), and this result can be used to eliminate \( f_2 \) in the following expressions. Thus, Eq. (6) for the first form factor, \( f_1/(1 - f_1) = v_1 f_1^2 + v_2 f_2^2 \), is a linear equation for \( (v_1, v_2) \) with coefficients that are nonlinear in \( f_1 \) and \( v_3 \). The same statement holds for Eq. (12) for a singularity which is equivalent to \( f_1^{(2c)}/(1 - f_1^{(2c)})^2 = 2v_1^{(2c)} f_1^{(2c)} + 2v_2^{(2c)} f_2^{(2c)}(1 - f_2^{(2c)}) \). These equations can be used to express \( v_1^{(2c)} \) and \( v_2^{(2c)} \) in terms of \( v_3^{(2c)} \) and \( f_1^{(2c)} \). To ease the notation, variables \( x \) and \( y \) shall be introduced as

\[ v_3^{(2c)} = x, \quad f_1^{(2c)} = y. \]  
(51a)

One gets

\[ v_1^{(2c)} = \frac{3 - (2 + x) y}{2(1 - y)^2 y (2 - xy)}, \]  
(51b)

\[ v_2^{(2c)} = \frac{x^2 y (2 - 2y^3)}{2(1 - y)^2(x^2 y^2 - 3xy + 2)}. \]  
(51c)

These equations define the surface of bifurcation singularities of Eq. (6) in the 3-dimensional parameter space. The variables \( x \) and \( y \) with \( 4 < x \) and \( 1/2 \leq y \leq 3/(2 + x) \) serve as surface parameters. The exponent parameter \( \lambda = 1 - \mu_2 \) is determined by

\[ \mu_2 = \frac{(3x^2 + 6x)y^3 - (x^2 + 18x + 8)y^2 + (6x + 18)y - 6}{(2x^2 + 4x)y^3 - 12xy^2 + (2x + 4)y}. \]  
(51d)

The maximum theorem, mentioned above in connection with Eq. (6), has to be used to identify among the points \( (v_1^{(2c)}, v_2^{(2c)}, v_3^{(2c)}) \) those that are glass-transition singularities.

Figure [26] exhibits three cuts through the parameter space. The cut \( v_3 = 20 \) is typical for sufficiently small values of \( x \). The cut through the bifurcation surface yields a smooth curve of \( A_2 \) glass-transition singularities. The bifurcation surface for such \( v_3^{(2c)} \) deals solely with the generic scenario for liquid-glass transitions.

The cut shown for \( v_3 = 45 \) is representative for sufficiently large values of \( x \). In this case, the cubic numerator polynomial in Eq. (51d) has two zeros \( y_1(x) < y_2(x) \)
deals with liquid-glass transitions. Also the third line between the crossing point and the parameter-boundary at $v_1 = 0$ deals with liquid-glass transitions. Both lines are characterized by a discontinuous increase of the correlators’ long-time limits from zero to the positive critical glass-form factors $f_q(2)^c > 0$.

Decreasing $x$ from large values to smaller ones, the two cusp values $y_1(x)$ and $y_2(x)$ approach each other. The corresponding parameter vectors $V = (v_1(x), v_2(x), v_3(x))$ form curves that approach each other with decreasing $x$ and join at a certain value $x^*: y_1(x^*) = y_2(x^*) = y^*$. The pair $(x^*, y^*)$ defines the $A_4$ singularity for the model. The parameters for this singularity are obtained, if the derivative of the numerator polynomial in Eq. (51a) is zero for $\mu_2 = 0$. This leads to $(x^* - 2)/(x^* - 4)(x^* - 30x^3 + 136x^2 - 168x + 88) = 0$. The elementary solution for the zeros of the quartic polynomial [28] determines the coordinates of the swallow-tail singularity $x^* = 24.779392\ldots, y^* = 0.24266325\ldots$. The cut through the transition surface for $v_3 = x^*$ is shown in Fig. 7 as a pair of light full lines joining at the $A_4$ singularity which is indicated by a star.

above some $y_0$, they can be evaluated elementarily [28]. The transition line consists of several pieces. The first one, obtained for $1/2 \geq y > y_2(x)$, is shown in Fig. 7 as heavy full line. It starts at $v_1^{(2)c} = 4, v_2^{(2)c} = 0$ and ends at the $A_3$ singularity marked by a circle. The second piece describes bifurcations with $\mu_2 < 0$ for $y_2(x) > y > y_1(x)$. It connects the mentioned $A_3$ singularity with a second $A_2$ singularity of Eq. (7) that is shown as a shaded circle. This piece of the line is shown dotted. Decreasing $y$ further, one gets a curve with $\mu_2 > 0$ that joins the second $A_2$ singularity with the point $v_1^{(2)c} = 0, v_2^{(2)c} = 3/(2 + x)$. This line exhibits a crossing point with the first line piece mentioned above, that is shown as diamond. The part between the second $A_3$ singularity and the crossing point is shown dotted, and the final piece is shown as light full line. The dotted bifurcation lines and the second $A_3$ singularity are excluded from the set of glass-transition singularities because of the maximum theorem. These items have been added to the figure merely in order to allow the reader to recognize the familiar swallow-tail scenario [3]. The crossing point organizes three lines of fold singularities. Between the $A_3$ singularity and the crossing point, there is a line of glass-glass transitions. The continuation of the line to the boundary of the admissible parameter range $v_2 = 0$
Figure 8 demonstrates the validity of the factorization theorem for states close enough to a cusp singularity $V^c$ and its violation for states sufficiently away from it. For the $A_3$ singularity with $v_1^c = 45$, the correction amplitudes calculated from Eq. (19) are quite different for the two correlators: $K_1 = 0.06857$, $K_2 = -2.049$. Therefore, the lines for vanishing dominant correction, i.e. the cut of the surfaces $B_2(q) = B_2 + K_q B^2 = 0$, $q = 1, 2$, with the plane $v_1 = 45$ are quite different as well, as shown by the dashed-dotted lines in Fig. 8. The four states discussed in Fig. 8 are chosen on the surface $B_2(1) = 0$. Thus, the scenario for the evolution of the $\ln(t/\tau)$ law shown for $\phi_1(t)$ is in qualitative agreement with the one discussed in Fig. 2. The states with labels $n = 3$ and 4 are so close to the singularity, that the correction term in Eq. (17) proportional to $B_2(2) = O(\epsilon)$ is not important. As a result, the rescaled functions $(\phi_q(t) - f_q^c)/h_q$, $q = 1$ and 2 agree for the states $n = 3$ and 4, and the same holds for the corresponding approximations. However, for the states with labels $n = 1$ and 2, the negative coefficient $B_2(2)$ is so large that the $\phi_2(t)$-versus-$\ln(t)$ curve does not exhibit the straight line obtained for $\phi_1(t)$-versus-$\ln(t)$ diagram. Rather, the correlator $\phi_2(t)$ exhibits changes of the curvature and inflection points as explained above in Fig. 2 for state $n' = 3$.

Figure 8 also exemplifies a problem concerning the choice of the time scale $\tau$. The complete solution of Eqs. (3) and (5) is unique up to the choice of a control-parameter independent time scale. The nonlinear coupling of the correlators of different index $q$ requires scale universality. However, if a time scale like $\tau$ is deduced from some approximation to the equation of motion, the error of the approximation will result in violations of the scale universality for the approximate solutions. Constructing the approximate solutions in Fig. 8 — and also in the upper panel of Fig. 8 — the time $\tau$ was fixed for the leading approximation from $\phi_1(\tau) = f_1^c$ and for the leading correction from $\phi_1(\tau) = f_1^c + f_1$. The errors explained lead to offsets for the second correlator: $\phi_2(\tau) \neq f_2^c + f_2$, respectively, for the two approximations studied. This explains, e.g., that the dashed line for $\phi_2(t)$ for the state $n = 1$ does not coincide with the full one. One could also choose $\tau$ differently, e.g. by requesting $\phi_2(\tau) = f_2^c + f_2$ as done in the lower panel of Fig. 8.

The transition line which is shown in Fig. 8 by the light full and almost horizontal curve for the cut $v_3 = 45$ intersects the line $B_2(1) = 0$ at some glass-transition singularity $V^{(2)c} = (2.94, 0.130, \ldots, 45.0)$. For states on the line $B_2(1) = 0$ that are close enough to this singularity, one gets the standard liquid-glass transition scenario, e.g., the evolution of a plateau of the $\phi_q(t)$-versus-$\ln(t)$ diagram at the critical glass-form factor $f_q^{(2)c}$ and an $\alpha$-process for the decay below this plateau. The universal bifurcation results for an $A_2$ singularity require that the plateau values are below the critical form factors of the nearby $A_3$ singularity: $f_q^{(2)c} < f_q^c$. For the example under discussion, one gets $f_1^{(2)c} = 0.0747$, $f_2^{(2)c} = 0.7027$ and $f_3^{(2)c} = 0.3125$, $f_4^{(2)c} = 0.9289$. The precursor of the liquid-glass transition at $V^{(2)c}$ explains the stretched tail exhibited in Fig. 8 for the decay of $\phi_1(t)$ below 0.1 for the state $n = 1$.

To corroborate the discussion of the preceding paragraph, the correlators with label $n = 1$ are reproduced as curves with label $n' = 1$ in Fig. 8. Two further curves with labels $n' = 2$ and 3 are added. They refer to states between state 1 and the transition point $V^{(2)c}$ as noted in Fig. 8 by triangles. The diagrams for $\phi_1(t)$ for states 2 and 3 exhibit the two-step-relaxation scenario characteristic for an $A_2$ bifurcation. The decay for $\phi_q(t) < f_q^{(2)c}$ demonstrates the superposition principle for the $\alpha$-process, and its initial part can be de-
scribed by von Schweidler’s power law. The decay towards the plateaus $f_q^{(2)c}$ for $t > 1000$ follows the critical law for the $A_2$ singularity $V^{(2)c}$. The universal laws for the dynamics near a fold bifurcation imply that the correlators follow the asymptote of the critical law, $(\phi_q(t) - f_q^{(2)c})/\eta_q = (t/t_0)\,^{-\alpha}$, for short times down to about one decade above the end of the transient dynamics, i.e. until about $t = 10$. In particular, for small times, the correlator for state $n' = 2$ should approach the one for state $n' = 3$. However, these features are not exhibited in Fig. 10. Rather, the $t^{-\alpha}$ law becomes irrelevant for the description of the dynamics below times around $10^3$, where the $A_2$ critical curve crosses the curves describing the logarithmic laws for the $A_2$ singularity. As a result, there appears a window between the end of the transient and the beginning of the description by the $A_2$-singularity results where the correlators are described by Eq. (46). This window deals with an increase in time over about two orders of magnitude. In this window, the logarithmic decay processes destroy the manifestation of the $t^{-\alpha}$ law.

The lower panel of Fig. 10 demonstrates a further implication of $V^c$ dynamics on the precursors of the liquid-glass transition dynamics. Even though the time scale for the $\alpha$-process for states $n' = 1$ or 2 exceeds the one for the transient by factors $10^4$ and $10^6$, respectively, the correlator $\phi_2(t)$ does not exhibit the two-step scenario for these states. Rather, there is a large time interval where the approach towards the plateau $f_2^{(2)c}$ follows closely the law $(\phi_2(t) - f_2^{(2)c}) \propto \ln(t/\tau_{\phi_2})$. This is due to cancellation of two effects: The asymptotes for the $V^c$ dynamics and for the $V^{(2)c}$ dynamics yield a positive curvature, while the onset of the $\alpha$-process causes a negative one. The resulting nearly linear-$\log(t)$ variation must not be mistaken as the true asymptotic logarithmic law given by Eq. (11).

The destruction of the critical decay law of the liquid-glass-transition dynamics by the presence of a higher-order glass-transition singularity nearby alters the familiar pattern of the susceptibility spectra, as shown in Fig. 10. The Debye-peak for the transient dynamics deals with the spectra for $\omega > 0.1$, as shown by the peak around $\omega \approx 1$ for $\chi''_1^c(\omega)$. This peak is strongly suppressed and shifted to higher frequencies for $\chi''_2^c(\omega)$. There is the large frequency regime $-4 \leq \log_{10} \omega \leq -1$, where the $\omega^\alpha$-law is irrelevant for the description of the structural relaxation spectrum. Rather, the critical relaxation spectrum of the cusp singularity leads to a high spectral enhancement of $\chi''_2(\omega)$ relative to the $\omega^\alpha$-spectrum; it leads to a second structural relaxation peak near $\omega \approx 0.01$ in addition to the low-frequency $\alpha$-peak. It was discussed in connection with states $n' = 3$ and 4 in Figs. 10b and 10c, that the winding of the $\phi(t)$-versus-$\log(t)$ curve around an effective $\ln(t)$-law for states with $B_2 < 0$ is a precursor phenomenon of a nearby $A_2$-transition singularity. Indeed, the spectrum $\chi''_3(\omega)$ exhibits the $\alpha$-peak of the mentioned transition with a maximum for $\log \omega \approx -3$.

Thus, because of $B_2(2) < 0$, the susceptibility for the second correlator exhibits two $\alpha$-peaks, referring to the two parts of the liquid-glass-transition lines discussed in Fig. 10. The low frequency $\alpha$-peaks shift strongly with changes of $n' = 1, 2,$ and 3, since the states are shifted towards the transition singularity $V^{(2)c}$ on one of the lines. The high frequency $\alpha$-peak does not change significantly since the distance of the states from the other transition line is almost unaltered. As explained in connection with Fig. 10, the leading correction formula, Eq. (17), describes the high-frequency wing of the second $\alpha$-peak.

VII. CONCLUSIONS

Describing the states of a system by a vector $V$ of control parameters, the neighborhood of a glass transition singularity $V^c$ was characterized by a sequence of separation parameters $\epsilon_1(V), \epsilon_2(V), \ldots$. These are smooth functions of $V$ that vanish at $V^c$, and are considered as small of order $\epsilon$. The glass-transition singularities $V^c$ are the bifurcation points of the glass-form factors $f_q(V) = \phi_q(t \to \infty)$, i.e. of the long-time limits of the correlators $\phi_q(t)$. These bifurcations are of the cuspoid family $A_l, l = 2, 3, \ldots$. The $\epsilon_1, \ldots, \epsilon_{l-1}$ are the relevant small coefficients specifying the polynomial of degree $l$ whose largest zero determines $f_q(V)$ for small $\epsilon$. The major result is the proof that the solution of the MCT equations can be asymptotically expanded in polynomials $P(x)$ of the logarithm of time, $x = \ln(t/\tau)$, for states close to an $A_l, l \geq 3$, and $\epsilon_1(V) < 0$. The leading term of order $|\epsilon|^{l/2}$ yields the $\ln(t/\tau)$ law, Eq. (11). The prefactor for this polynomial of degree 1 is given solely by the first separation parameter $\epsilon_1(V)$. The leading correc-
tion adds a polynomial of degree 4. The coefficients are of order $|\epsilon|$, and they are determined by $\epsilon_1(V)$ and $\epsilon_2(V)$, Eqs. (29) and (17). The second correction adds a polynomial of order 7; the coefficients are of order $|\epsilon|^{3/2}$ and determined by $\epsilon_1(V)$, $\epsilon_2(V)$, $\epsilon_3(V)$, and so on.

Several relaxation scenarios have been identified that are utterly different from the MCT scenario for the liquid-glass transition. The latter is described by an $A_2$ singularity.

There are distinguished surfaces in parameter space, where the prefactor of the $x^2$ monomial in the polynomial $P(x)$ vanishes. For this case, the $\ln(t/\tau)$ law dominates the dynamics for such times where $\phi_q(t) \approx f_q(Vc')$. This law may describe the complete decay except for the transient and for the final exponential approach of $\phi_q(t)$ towards its long-time limit, Fig. 4.

States near the mentioned surface exhibit slight deformations of the straight $\phi(t)$-versus-$\log(t)$ curve. There is a concave behavior on one side of the surface and a winding around the straight line with alternating convex and concave parts on the other side, as shown for the states $n' = 1$–3 in Fig. 5.

The corrections to the leading-order asymptotic results depend on the correlator under consideration. The surfaces of dominant $\ln(t/\tau)$ behavior are different for different correlation functions, as explained in connection with Fig. 3.

Every higher-order glass-transition singularity $V^c$ is an endpoint of a surface of fold bifurcation points $V^{(2)c}$ with $f_q(V^c) = f_q^c < f_q^{(2)c} = f_q(V^{(2)c})$. For states sufficiently close to $V^{(2)c}$, one finds the standard transition scenario with two-step relaxation described by the interplay of $\alpha$- and $\beta$-scaling laws. The correlators for $\phi_q(t) \approx f_q^c$ are a part of the $\alpha$-process. Therefore, the logarithmic decay laws as formulated by Eq. (12) describe the $\alpha$-relaxation master functions.

Generically, near a higher-order singularity $V^c$, there is a further surface of fold bifurcations that crosses the transition surface discussed in the preceding paragraph, Fig. 3. As a result, there is the scenario for transition singularities of type $A_2$, but now with critical form factors smaller than the ones at $V^c$: $f_q^{(2)c} < f_q^c$. Consequently, the logarithmic decay laws are a part of the relaxation towards the plateau $f_q^{(2)c}$. They reduce the range of applicability of the critical decay and introduce a large crossover interval for structural relaxation between the end of the transient and the beginning of the transition dynamics caused by $V^{(2)c}$, as is demonstrated in Figs. 9 and 10 for the correlator $\phi_1(t)$. In particular, there can be a crossover from the transient to a simple $\ln(t/\tau)$ law followed by a crossover to a von Schweidler power law. This scenario was demonstrated by the experiments reported in Ref. 14 and by numerical solutions of MCT equations for the square-well liquid in Ref. 15.

There can also be a susceptibility spectrum for structural relaxation consisting of two peaks, as shown for $\chi_2''(\omega)$ in Fig. 10.

The asymptotic expansion also describes the critical correlator of the higher-order glass-transition singularity outside the transient, Fig. 3. These correlators deal with the decay towards $f_q^c$ for control parameters at the singularity. For states with $\epsilon_1 > 0$, $\phi_q(t)$ follows the critical decay until close to its intersection with the long-time asymptote $\phi_q(t \to \infty) = f_q > f_q^c$. Here it crosses over rapidly to the glass-form factor $f_q$. Summarizing, the formulas of this paper provide a qualitative understanding of the decay of the correlators provided the state $V$ of the system is close to a higher-order glass-transition singularity and the correlator $\phi_q(t)$ is close to the glass-form factor $f_q^c$ at this singularity.

Let $L_t$ denote the length of the log($t$) interval where an approximation by one of the polynomials $P(\ln(t/\tau))$ describes the solution for the correlator $\phi_q(t)$. Let $L_w$ denote the length of the log($\omega$) interval where the Fourier-cosine transform of $P(\ln(t/\tau))$ leads to a description of comparable accuracy for the susceptibility spectrum $\chi_2''(\omega)$. It was explained in connection with Fig. 10 that $L_w$ is considerably smaller than $L_t$. This phenomenon for glassy dynamics was discussed earlier for the liquid-glass transition [23], but it is more pronounced for the higher-order singularities. This feature for stretched relaxation is the reason, why it is more difficult to test asymptotic MCT formulas with data for spectra than it is with data for correlators in the time domain.

If there is a higher-order glass-transition singularity in a disordered system, there is no generic path for the evolution of the structural relaxation. Only a parameter surface can be generic for the description of the dynamics near a cusp singularity. One has to vary two independent parameters to identify an $A_3$ singularity, three parameters to identify an $A_4$ bifurcation, and so on. We hope that the demonstration of all basic scenarios for the dynamics near an $A_3$ singularity will be of use to identify such singularities in colloids, if there are any. In such case, the derived formulas are elementary enough for data fitting. Such fitting might lead to a judgment on the relevance of the subtle implications of mode-coupling theory for the discussion of glass-forming systems.

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APPENDIX A: LAPLACE TRANSFORMS OF LOGARITHMS

The modification of the Laplace transform, introduced in Eq. (6), shall be used to map invertibly functions $F(t)$
of time to functions of the complex frequency \( z \). The functions are defined for \( t > 0 \) and \( \text{Im} \, z > 0 \), respectively. Euler’s second integral for the gamma function \( \Gamma(y) \) implies \( S[f^2](z) = (i/z)^{y} \Gamma(1+x) \) if \( x > -1 \). Differentiating this identity \( n \) times for \( x = 0, n = 0, 1, 2, \ldots \), one arrives at the formula:

\[
S[\ln^n(t)](z) = \sum_n \frac{n!}{k^n} \Gamma_k \ln^{n-k}(i/z). \quad (A1)
\]

Here \( \binom{n}{k} = n!/[k!(n-k)!] \) and \( \Gamma_k = d^k \Gamma(x = 1)/dx^k \). One gets in particular \( \Gamma_0 = 1 \) and \( \Gamma_1 = -\gamma \), where \( \gamma \) is Euler’s constant. If \( \psi(y) \) denotes the digamma function, one can write \( \Gamma'(y) = \Gamma(y) \psi(y) \). Iterating this formula, one can express \( \Gamma_k \) in terms of the first \( (k-1) \) derivatives of \( \psi(y) \) for \( y = 1 \). The latter are given by the tabulated values of the zeta function \( \zeta(k) \) \( ^{23} \): for example, \( \Gamma_2 = \zeta(2) = \pi^2/6 \). Implications of Eq. \( (A1) \) read with \( n > 1, n_1, n_2 > 1, n_3 \geq 1 \):

\[
S[\ln^n(t)](z) - S[\ln(t)]^n(z) = \frac{\pi^2}{12} n(n-1) \ln^{n-2}(i/z) + \sum_{k=3}^{n} \frac{n!}{k^n} \Gamma_k - \Gamma^k \ln^{n-k}(i/z). \quad (A2)
\]

Thus, the sequence of \( p_n(x) \) can be constructed recursively in terms of the coefficients \( \Gamma_{n,k} \). To derive Eqs. \( (A2-b) \), one needs:

\[
p_2(x) = 2.6160x + x^2, \quad p_3(x) = -2.1482x + 3.9239x^2 + x^3, \quad p_4(x) = -12.813x - 4.2964x^2 + 5.2319x^3 + x^4. \quad (A7c)
\]

Specializing Eq. \( (A3) \) to \( n_1 = n \) and \( n_2 = 1 \) and using the definition of the linear operator \( T \) from Eq. \( (24) \), one gets

\[
T[\ln^n(t)](z) = (\pi^2/6)[n \ln^{n-1}(i/z) + \frac{n}{k=2} (n - k + 1) \Gamma_{n,k} \ln^{n-k}(i/z)] \quad (A4)
\]

where the coefficients are

\[
\Gamma_{n,k} = \left( \binom{n}{k} \Gamma_{k+1} - \Gamma_{k+1}\right)/\left[ (\pi^2/6)(n - k + 1) \right]. \quad (A5)
\]

Let us construct polynomials \( p_n(x) \) of degree \( n = 1, 2, \ldots \) obeying Eqs. \( (26) \). Specializing Eq. \( (A3) \) to \( n = 1 \) shows that one can choose \( p_1(x) = x \). Assuming the polynomials for degree \( l < n \) to be known, Eq. \( (A4) \) provides the formula for the degree \( n \)

\[
p_n(x) = x^n - \frac{1}{k=2} \Gamma_{n,k} p_{n+1-k}(x). \quad (A6)
\]
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