Asymptotic multipartite version of the Alon-Yuster theorem

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Abstract

In this paper, we prove the asymptotic multipartite version of the Alon-Yuster theorem, which is a generalization of the Hajnal-Szemerédi theorem: If \( k \geq 3 \) is an integer, \( H \) is a \( k \)-colorable graph and \( \gamma > 0 \) is fixed, then, for every sufficiently large \( n \) and for every balanced \( k \)-partite graph \( G \) on \( kn \) vertices with each of its corresponding \( \binom{k}{2} \) bipartite subgraphs having minimum degree at least \((k - 1)n/k + \gamma n\), \( G \) has a subgraph consisting of \( \lfloor n/|V(H)| \rfloor \) vertex-disjoint copies of \( H \).

The proof uses the Regularity method together with linear programming.

Keywords: tiling, Hajnal-Szemerédi, Alon-Yuster, multipartite, regularity, linear programming

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1. Introduction

1.1. Motivation

One of the celebrated results of extremal graph theory is the theorem of Hajnal and Szemerédi on tiling simple graphs with vertex-disjoint copies of a given complete graph \( K_k \) on \( k \) vertices. Let \( G \) be a simple graph with vertex-set \( V(G) \) and edge-set \( E(G) \). We denote by \( \deg_G(v) \), or simply \( \deg(v) \), the degree of a vertex \( v \in V(G) \) and we denote by \( \delta(G) \) the minimum degree of the graph \( G \).

For a graph \( H \), we say that \( G \) has a perfect \( H \)-tiling (also a perfect \( H \)-factor or perfect \( H \)-packing) if there is a subgraph of \( G \) that consists of \( \lfloor |V(G)|/|V(H)| \rfloor \) vertex-disjoint copies of \( H \).

The theorem of Hajnal and Szemerédi can be then stated in the following way:

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Theorem 1 (Hajnal, Szemerédi [12]). If \( G \) is a graph on \( n \) vertices and \( \delta(G) \geq (k-1)n/k \), then \( G \) has a perfect \( K_k \)-tiling.

The case of \( k = 3 \) was first proven by Corrádi and Hajnal [6] before the general case. The original proof in [12] was relatively long and intricate. A shorter proof was provided later by Kierstead and Kostochka [18].

The question of finding a minimum-degree condition for the existence of a perfect \( H \)-tiling in the case when \( H \) is not a clique and \( n \) obeys some divisibility conditions was first considered by Alon and Yuster [2]:

Theorem 2 (Alon, Yuster [2]). Let \( H \) be a graph with chromatic number \( k \) and let \( \gamma > 0 \). If \( n \) is large enough and \( G \) is a graph on \( n \) vertices with \( \delta(G) \geq (k-1)n/k + \gamma n \), then \( G \) has a perfect \( H \)-tiling.

Komlós, Sárközy and Szemerédi [22] removed the \( \gamma n \) term from the minimum degree condition and replaced it with a constant that depends only on \( H \).

Kühn and Osthus [25] determined that \( (1 - 1/\chi^*(H))n + C \) was the necessary minimum degree to guarantee an \( H \)-tiling in an \( n \)-vertex graph for \( n \) sufficiently large, and they also showed that this was best possible asymptotically. The constant \( C = C(H) \) depends only on \( H \) and \( \chi^* \) is an invariant related to the so-called critical chromatic number of \( H \), which was introduced by Komlós [20].

1.2. Background

In this paper, we consider the multipartite variant of Theorem 2. Before we can state the problem, we need several a few definitions.

A \( k \)-partite graph \( G = (V_1, \ldots, V_k; E) \) is balanced if \( |V_1| = \cdots = |V_k| \). The natural bipartite subgraphs of \( G \) are those induced by the pairs \( (V_i, V_j) \), and which we denote by \( G[V_i, V_j] \). For a \( k \)-partite graph \( G = (V_1, \ldots, V_k; E) \), we define the minimum bipartite degree, \( \hat{\delta}_k(G) \), to be the smallest minimum degree among all of the natural bipartite subgraphs of \( G \), that is,

\[
\hat{\delta}_k(G) = \min_{1 \leq i < j \leq k} \delta(G[V_i, V_j]).
\]

Now we can state the conjecture that inspired this work, a slightly weaker version of which appeared in [7].

Conjecture 3. Fix an integer \( k \geq 3 \). If \( G \) is a balanced \( k \)-partite graph on \( kn \) vertices such that \( \hat{\delta}(G) \geq (k-1)n/k \), then either \( G \) has a perfect \( K_k \)-tiling or both \( k \) and \( n/k \) are odd integers and \( G \) is isomorphic to the fixed graph \( \Gamma_k(n) \).

The exceptional graphs \( \Gamma_k(n) \), where \( n \) is an integer divisible by \( k \), are due to Catlin [4] who called them “type 2 graphs”. The graph \( \Gamma_k(k) \) has vertex set \( \{h_{ij} : i, j \in \{1, \ldots, k\}\} \) and \( h_{ij} \) is adjacent to \( h_{i'j'} \) if \( i \neq i' \) and either \( j = j' \in \{k-1, k\} \) or \( j \neq j' \) and at least one of \( j, j' \) is in \( \{1, \ldots, k-2\} \). For \( n \) divisible by \( k \), we obtain the graph \( \Gamma_k(n) \) by replacing each vertex with an independent set of \( n/k \) vertices and each edge with a copy of the complete bipartite graph \( K_{n/k, n/k} \).
We notice that if $G$ satisfies the minimum bipartite degree condition in Conjecture 3, then its minimum degree $\delta(G)$ can still be as small as $(k-1) \left(\frac{k-1}{k}\right) n = \left(\frac{k-1}{k}\right)^2 (kn)$, which is not enough to apply Theorem 1 directly.

The case of $k = 2$ of Conjecture 3 is an immediate corollary of the classical matching theorem due to König [24] and Hall [13]. Fischer [9] observed that if $G$ is a balanced $k$-partite graph on $kn$ vertices with $\hat{\delta}(G) \geq (1 - 1/2(k - 1)) n$, then $G$ has a perfect $K_k$-tiling. If there were necessary and sufficient conditions, à la Hall’s condition for $k = 2$, that we could use to determine whether a tripartite graph has a $K_3$-tiling, then $P$ would equal co-NP (see, e.g. [10]). Hence, finding such conditions is unlikely. Consequently, we look for the smallest $\hat{\delta}(G)$ that would guarantee a perfect $K_k$-tiling.

Some partial results were obtained, for $k = 3$, by Johansson [15] and, for $k = 3, 4$, by Fischer [9]. The case of $k = 3$ was settled for $n$ sufficiently large by Magyar and the first author [27], and the case of $k = 4$ was settled for $n$ sufficiently large by Szemerédi and the first author [28]. The results in [27, 28] each have as a key lemma a variation of the results from Fischer. However, it seems that such techniques are impossible for $k \geq 5$. An interesting result toward proving Conjecture 3 for general $k$ is due to Csaba and Mydlarz [7] who proved that if $G$ is a balanced $k$-partite graph on $kn$ vertices, $\hat{\delta}(G) \geq \frac{q_k}{k^{q_k + 1}} n$ and $n$ is large enough, then $G$ has a perfect $K_k$-tiling. Here, $q_k := k - 3 + \frac{1}{2} \sum_{i=1}^{k} \frac{1}{i} = k + O(\log k)$.

Recently, Keevash and Mycroft [10] proved that, for any $\gamma > 0$, if $n$ is large enough, then $\hat{\delta}(G) \geq (k-1)n/k + \gamma n$ guarantees a perfect $K_k$-tiling in a balanced $k$-partite graph $G$ on $kn$ vertices. Their result is a consequence of a more general theorem on hypergraph matching, proof of which uses the hypergraph regularity method and a hypergraph version of the Blow-up Lemma. Very shortly thereafter, Lo and Markström [26] proved the same result using methods from linear programming and the so-called “absorbing method”. This effort culminated in [17], in which Keevash and Mycroft proved Conjecture 3.

In this paper, we are interested in more general problem of tiling $k$-partite balanced graphs by a fixed $k$-colorable graph $H$. More precisely, if $H$ is a $k$-colorable graph and $n$ obeys certain natural divisibility conditions, we look for a condition on $\hat{\delta}(G)$ to ensure that every balanced $k$-partite graph $G$ on $kn$ vertices satisfying this condition has a perfect $H$-tiling.

Zhao [34] found that the minimum degree required to perfectly tile a balanced bipartite graph on $2n$ vertices with copies of $K_{h,h}$ ($h$ divides $n$) is $n/2 + C(h)$, where $C(h)$ differs sharply as to whether $n/h$ is odd or even. Zhao and the first author [29, 30] showed similar results for tiling with $K_{h,h,h}$. Hladký and Schacht [14] and then Czygrinow and DeBasio [8] improved the results of [34] by finding the minimum degree for copies of $K_{s,t}$, where $s + t$ divides $n$. Bush and Zhao [3] proved a Kühn-Osthus-type result by finding asymptotically best-possible minimum degree condition in a balanced bipartite graph on $2n$ vertices in order to ensure its perfect $H$-tiling, for any bipartite $H$. All results are for $n$ sufficiently large.
1.3. Main Result

We prove a multipartite version of the Alon-Yuster theorem (Theorem 2). Let $K_{k}^{h}$ denote a $k$-partite graph with $h$ vertices in each partite set. For example, the complete bipartite graph $K_{h,h}$ would be denoted $K_{2}^{h}$. Since the partite sets can be rotated, it is easy to see that any $k$-chromatic graph $H$ of order $h$ perfectly tiles the graph $K_{k}^{h}$. Hence, the following theorem gives a sufficient condition for a perfect $H$-tiling.

**Theorem 4.** Fix an integer $k \geq 2$, an integer $h \geq 1$ and $\gamma \in (0,1)$. There exists an $n_{4} = n_{4}(k,h,\gamma)$ such that if $G$ is a balanced $k$-partite graph on $kn$ vertices with $\hat{\delta}(G) \geq \left(\frac{k-1}{k} + \gamma\right) n$ and $n \geq n_{4}$, then $G$ has a perfect $K_{k}^{h}$-tiling.

Our proof relies on the regularity method for graphs and linear programming and it differs from approaches in [16, 26].

1.4. Structure of the Paper

In Section 2, we prove a fractional version of the multipartite Hajnal-Szemerédi theorem. This is the main tool in proving Theorem 4. Section 3 is the main proof and Section 4 gives the proofs of the supporting lemmas. We finish with Section 5, which has some concluding remarks.

2. Linear Programming

In this section, we shall prove a fractional version of Conjecture 3.

**Definition 5.** For any graph $G$, let $T_{k}(G)$ denote the set of all copies of $K_{k}$ in $G$. The **fractional $K_{k}$-tiling number** $\tau_{k}^{\ast}(G)$ is defined as:

$$\tau_{k}^{\ast}(G) = \max \sum_{T \in T_{k}(G)} w(T) \quad \text{(1)}$$

subject to $\sum_{T \in T_{k}(G)} w(T) \leq 1$, $\forall v \in V(G)$,

$$w(T) \geq 0, \quad \forall T \in T_{k}(G).$$

From the Duality Theorem of linear programming (see Section 7.4 in [31]), we obtain that

$$\tau_{k}^{\ast}(G) = \min \sum_{v \in V(G)} x(v) \quad \text{(2)}$$

subject to $\sum_{v \in V(T)} x(v) \geq 1$, $\forall T \in T_{k}(G)$,

$$x(v) \geq 0, \quad \forall v \in V(G).$$

Let $w^{\ast}$ be a function that achieves an optimal solution to (1). If there exists a vertex $v \in V(G)$ such that $\sum_{T \in T_{k}(G), V(T) \ni v} w^{\ast}(T) < 1$, then we call $v$ a
slack vertex or just say that \( v \) is slack. Similarly, if \( x^* \) is a function that achieves an optimal solution to (2) and there exists a \( T \in \mathcal{T}(G) \) such that \( \sum_{v \in V(G), v \in v(T)} x^*(v) > 1 \), then we say that \( T \) is slack.

**Remark 6.** Consider an optimal solution to (2), call it \( w^* \). We may assume that \( w^*(T) \) is rational for each \( T \in \mathcal{T}(G) \). To see this, observe that the set of feasible solutions is a polyhedron for which each vertex is the solution to a system of equations that result from setting a subset of the constraints of the program to equality. (For more details, see Theorem 18.1 in [11] ) Since the objective function achieves its maximum at such a vertex (See Section 3.2 of [11] ) we may choose an optimal solution \( w^*(T) \) with rational entries.

Now we can state and prove a fractional version of the multipartite Hajnal-
Szemerédi Theorem.

**Theorem 7.** Let \( k \geq 2 \). If \( G \) is a balanced \( k \)-partite graph on \( kn \) vertices such that \( \delta_k(G) \geq (k - 1)n/k \), then \( \tau_k^*(G) = n \).

**Proof.** Setting \( x(v) = 1/k \) for all vertices \( v \in V(G) \) gives a feasible solution \( x \) to (2), and so \( \tau_k^*(G) \leq \sum_{v \in V(G)} x(v) = n \). We establish that \( \tau_k^*(G) \geq n \) by induction on \( k \).

**Base Case.** \( k = 2 \). This case follows from the fact that Hall’s matching condition implies that a balanced bipartite graph on \( 2n \) vertices with minimum degree at least \( n/2 \) has a perfect matching. Setting \( w(e) \) equal to 1 if edge \( e \) is in the matching and equal to 0 otherwise, gives a feasible solution to (1), thus establishing that \( \tau^*(G) \geq n \).

**Induction step.** \( k \geq 3 \). Now we assume \( k \geq 3 \) and suppose, for any balanced \((k - 1)\)-partite graph \( G' \) on a total of \((k - 1)n'\) vertices with \( \delta_{k - 1}(G') \geq \frac{k - 2}{k - 1}n' \), that \( \tau^*(G') \geq n' \).

Let \( w^* \) be an optimal solution to (1). Let \( x^* \) be an optimal solution corresponding to (2) such that \( x^*(z) = 0 \) whenever vertex \( z \) is slack. Denote by \( S \) the set of slack vertices, and, for \( i \in [k] \), set \( S_i = S \cap V_i \). We may assume that every \( S_i \) is non-empty. As in the base case, if \( V_i \) has no slack vertices, then \( \tau_k^*(G) = n \).

Denote \([k] := \{1, \ldots, k\} \). For every \( i \in [k] \), fix some \( z_i \in S_i \), choose exactly \( n' := \left\lfloor \frac{k - 1}{k}n \right\rfloor \) neighbors of \( z_i \) in each \( V_j \), \( j \in [k] - \{i\} \), and denote by \( G_i \) the subgraph of \( G \) induced on these \((k - 1)n' \) neighbors.

Observe that the set of weights \( \{x^*(v) : v \in V(G_i)\} \) must be a feasible solution to the minimization problem defined by the \((k - 1)\)-partite graph \( G_i \). This is because every copy of \( K_{k - 1} \) in \( G_i \) extends to a copy of \( K_k \) in \( G \) containing the vertex \( z_i \) and the sum of the weights of the vertices on that \( K_{k - 1} \) must be at least 1. Hence, we have that \( \sum_{v \in V(G_i)} x^*(v) \geq \tau_{k - 1}^*(G_i) \).
Each vertex of $G_i$ has at most $n - n'$ neighbors outside of $V(G_i)$ in each of its classes. Thus,

$$\delta(G_i) \geq n' - (n - n') = \frac{k-2}{k-1} n' + \left( \frac{k}{k-1} n' - n \right) \geq \frac{k-2}{k-1} n'.$$

So, for every $i$, we may apply the inductive hypothesis to $G_i$ and conclude that $	au_{k-1}^*(G_i) = n'$.

Combining the previous two observations with the fact that each vertex $v$ is in at most $k - 1$ of the subgraphs $G_i$, we get

$$(k-1)\tau_{k}^*(G) = (k-1) \sum_{v \in V(G)} x^*(v) \geq \sum_{i=1}^{k} \sum_{v \in V(G_i)} x^*(v) \geq \sum_{i=1}^{k} \tau_{k-1}^*(G_i) = kn'.$$

So, $\tau_{k}^*(G) \geq \frac{k}{k-1} n' = \frac{k}{k-1} \left[ \frac{k-1}{k} n \right] \geq n$. This concludes the proof of Theorem 7. 

\[\square\]

3. Proof of Theorem 4

First, we will have a sequence of constants and the notation $a \gg b$ means that the constant $b$ is sufficiently small compared to $a$. We fix $k \geq 2$ and $h \geq 1$ and let

$$\gamma \gg d \gg \varepsilon' \gg \varepsilon = (\varepsilon')^{5/16} \gg \zeta, \quad (3)$$

3.1. Applying the Regularity Lemma

We are going to use a variant of Szemerédi’s Regularity Lemma. Before we can state it, we need a few basic definitions. For disjoint vertex sets $A$ and $B$ in some graph, let $e(A, B)$ denote the number of edges with one endpoint in $A$ and the other in $B$. Further, let the \textit{density} of the pair $(A, B)$ be $d(A, B) = e(A, B) / |A||B|$. The pair $(A, B)$ is \textit{$\varepsilon$-regular} if $X \subseteq A$, $Y \subseteq B$, $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ imply $|d(X, Y) - d(A, B)| \leq \varepsilon$.

We say that a pair $(A, B)$ is $(\varepsilon, \delta)$-\textit{super-regular} if $X \subseteq A$, $Y \subseteq B$, $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ imply $d(X, Y) \geq \delta$ and $\text{deg}_B(a) \geq \delta |B|$ for all $a \in A$ and $\text{deg}_A(b) \geq \delta |A|$ for all $b \in B$.

The degree form of Szemerédi’s Regularity Lemma (see, for instance, [23]) is sufficient here, modified for the multipartite setting.

\textbf{Theorem 8.} For every integer $k \geq 2$ and every $\varepsilon > 0$, there is an $M = M(k, \varepsilon)$ such that if $G = (V_1, \ldots, V_k; E)$ is a balanced $k$-partite graph on $kn$ vertices and $d \in [0, 1]$ is any real number, then there is an integer $\ell$, a subgraph $G' = (V_1, \ldots, V_k; E')$ and, for $i = 1, \ldots, k$, partitions of $V_i$ into clusters $V_i^{(0)}, V_i^{(1)}, \ldots, V_i^{(\ell)}$ with the following properties:

\begin{itemize}
  \item[(P1)] $\ell \leq M$,
  \item[(P2)] $\ldots$,
  \item[(P3)] $\ldots$.
\end{itemize}
We omit the proof of Theorem 8, which follows from the proof given in [33].

In sum, programs is only dependent only on reduced graphs is only dependent on Gorem (Theorem 7) to 3.2. Partitioning the clusters

We first apply the fractional version of the k-partite Hajnal-Szemerédi Theorem (Theorem 7) to G_r and obtain that the value of \( \tau^*_k(G_r) \) is equal to \( \ell \). Consider a corresponding optimal solution \( w^* \) to the linear program (1) as it is applied to \( G_r \). By Remark 6 we may fix a corresponding solution \( w^* \) that is rational for every \( T \in \mathcal{T}_k(G_r) \). We will call this \( w^* \) a rational-entry solution for \( G_r \) and denote by \( D(G_r) \) the common denominator of all the entries of \( w^* \). Note that \( D(G_r) \) depends only on \( k \) and \( \ell \leq M(k, \varepsilon) \).

Since the linear program (1) depends only on \( G_r \) and the number of such reduced graphs is only dependent on \( M(k, \varepsilon) \), the number of possible linear programs is only dependent only on \( k \) and \( \varepsilon \). For each possible linear program we fix one rational-entry solution.

Therefore, the least common multiple of all of the common denominators \( D(G_r) \) for these reduced graphs is a function only of \( k \) and \( \varepsilon \). Call it \( D = D(k, \varepsilon) \). In sum, \( D \) has the property that for every reduced graph \( G_r \), there is a rational-entry solution \( w^* \) of the linear program (1) such that \( D \cdot w^*(T) \) is an integer for every \( T \in \mathcal{T}_k(G_r) \).

The next step is to randomly partition each set \( V_i^{(j)} \) into \( D \) parts of size \( h[L/(D\ell)] \) and put the remaining \( L - D\ell[L/(D\ell)] \) < \( D\ell \) vertices into the
corresponding leftover set, \( V_i^{(0)} \). The resulting leftover set, \( \tilde{V}_i^{(0)} \), has size less than \( \varepsilon n + D\ell^2 < 2\varepsilon n \).

Thus, for \( L' = \lfloor L/D \rfloor \), we obtain \( k\ell' \) clusters \( \tilde{V}_i^{(j)}, i \in [k], j \in [D\ell] \), such that each of them has size exactly \( L' \). This new partition has the following properties:

\((P1')\) \( \ell' = D\ell \),

\((P2')\) \( |\tilde{V}_i^{(0)}| \leq 2\varepsilon n \) for \( i \in [k] \),

\((P3')\) \( |\tilde{V}_i^{(j)}| = L' = \lfloor L/D \rfloor \) for \( i \in [k] \) and \( j \in [\ell'] \),

\((P4')\) \( \deg_{G'}(v, V_i) \geq \deg_{G}(v, V_i) - (d + \varepsilon)n \) for all \( i, i' \in [k], i \neq i', v \in V_i \) and

Now we prove that a property similar to property \((P5)\) holds.

\((P5')\) all pairs \( (\tilde{V}_i^{(j)}, \tilde{V}_i^{(j')}) \), \( i, i' \in [k], i \neq i', j, j' \in [\ell'] \) are \( \varepsilon' \)-regular in \( G' \), each with density either 0 or exceeding \( d' := d - \varepsilon \).

Recall from \((3)\) that \( \varepsilon \leq (\varepsilon')^5/16 \) and, consequently, \( \varepsilon' \geq (16\varepsilon)^{1/5} \).

The upcoming Lemma \([10]\) a slight modification of a similar lemma by Csaba and Mydlarz \([7]\), implies that, in fact, \((P5)\) holds with probability going to 1 as \( n \to \infty \).

**Lemma 10 (Random Slicing Lemma).** Let \( 0 < d < 1/3, 0 < \varepsilon < d/4 \) and \( D \) be a positive integer. There exists a \( C_{\text{reg}} = C_{\text{reg}}(\varepsilon, D) > 0 \) such that the following holds: Let \( (X, Y) \) be an \( \varepsilon \)-regular pair of density \( d \) with \( |X| = |Y| = L = DL' \). If \( X \) and \( Y \) are randomly partitioned into sets \( A_1, \ldots, A_D \), and \( B_1, \ldots, B_D \), respectively, each of size \( L' \), then, with probability at least \( 1 - \exp\{-C_{\text{reg}}L\} \), all pairs \((A_i, B_j)\) are \((16\varepsilon)^{1/5}\)-regular with density at least \( d - \varepsilon \).

Consequently, with probability at least \( 1 - \binom{k}{2}^2 \ell^2 \exp\{-C_{\text{reg}}L\} \), the property \((P5)\) holds. Since \( \ell \leq M = M(k, \varepsilon) \) and \( L \geq n(1 - \varepsilon)/M \), then for every sufficiently large \( n \), a partition satisfying \((P1)-(P5)\) exists (with high probability).

We fix one such a partition.

To understand this new partition, we define its reduced graph \( G'_r \) with vertex set \( \bigcup_{i=1}^k \{v_1^{(i)}, \ldots, v_i^{(\ell')}\} \). The vertex \( u_i^{(j)} \) corresponds to the cluster \( \tilde{V}_i^{(j)} \). The vertices \( u_i^{(j)} \) and \( u_i^{(j')} \) are adjacent in \( G'_r \) if and only if the pair \( (\tilde{V}_i^{(j)}, \tilde{V}_i^{(j')}) \) is \( \varepsilon' \)-regular with density at least \( d' \). The graph \( G'_r \) clearly has the following properties:

- \( G'_r \) is \( k \)-partite and balanced on \( k\ell' \) vertices. We denote its partite sets \( U_i' = \{u_1^{(i)}, \ldots, u_i^{(\ell')}\}, i \in [k] \).
- \( \delta(G'_r) \geq (\frac{k-1}{k} + \gamma/2) \ell' \).

The usefulness of \( G'_r \) is that it has a \( K_k \)-tiling, which is derived from the fractional \( K_k \)-tiling of \( G_r \):
Fact 11. The reduced graph $G'_r$ has a perfect $K_k$-tiling.

Proof of Fact 11. Observe first that, in fact, $G'_r$ is simply a blow-up of $G_r$; that is, each vertex $v$ in $V(G_r)$ is replaced with a set $U_v$ of $D$ vertices and each edge $\{v_1, v_2\}$ in $G_r$ is replaced with a copy of $K_{D,D}$ with partite sets $U_{v_1}$ and $U_{v_2}$. Let $w^*$ be the previously-chosen rational-valued solution to the linear program $[1]$ as it is applied to $G_r$.

Consider some $T \in \mathcal{T}_k(G_r)$ with vertices $\{v_1, \ldots, v_k\}$. Observe that, by the definition of $D$, $Dw^*(T)$ is an integer. Then, we take $Dw^*(T)$ of the vertices from $U_{v_1}$, $Dw^*(T)$ of the vertices from $U_{v_2}$ and so on. This process generates $Dw^*(T)$ vertex-disjoint copies of $K_k$ in $G'_r$.

By the constraint inequalities in [1], the total number of vertices used from $U_v$ is

$$\sum_{T \in \mathcal{T}_k(G_r), V(T) \ni v} Dw^*(T) \leq D = |U_v|,$$

hence the process never fails. The total number of vertex-disjoint $K_k$-s that are created in this way is $\sum_{T \in \mathcal{T}_k(G_r)} Dw^*(T) = D\ell = \ell'$. This uses each of the $k\ell'$ vertices of $G'_r$. \hfill $\square$

Since $G'_r$ has a perfect tiling, we may re-index its vertices so that vertices of $G'_r$ (hence, sub-clusters of $G$) with the same upper-index are in the same copy of the tiling from Fact [11]. More precisely,

- for $j = 1, \ldots, \ell'$, the $k$-tuple $(u_1^{(j)}, \ldots, u_k^{(j)})$ forms a $K_k$ in $G'_r$. We refer to the $k$-tuples $(\tilde{V}_1^{(j)}, \ldots, \tilde{V}_k^{(j)})$ as columns.[2]

3.3. Making the cliques super-regular

In preparation for using the Blow-up Lemma (Lemma [18] below), we need to make each $k$-tuple $(\tilde{V}_1^{(j)}, \ldots, \tilde{V}_k^{(j)}), j \in \{1, \ldots, \ell'\}$ pairwise super-regular by placing some vertices from the corresponding clusters into the respective leftover set. This is easy to do by a simple fact:

Fact 12. Let $\varepsilon' > 0$ and $\varepsilon' < d'/(2(k+1))$ let $(A_1, \ldots, A_k)$ be a $k$-tuple that is pairwise $\varepsilon'$-regular of density at least $d'$ with $|A_1| = \cdots = |A_k| = L'$. There exist subsets $A'_i \subset A_i$ for $i \in [k]$ such that $|A'_i| = h[(1 - (k-1)\varepsilon')L'/h]$ and each pair of $(A'_1, \ldots, A'_k)$ is $(2\varepsilon', d' - k\varepsilon')$-super-regular (with density at least $d' - \varepsilon'$).

Fact [12] follows from well-known properties of regular pairs. We apply it to each $k$-tuple $(\tilde{V}_1^{(j)}, \ldots, \tilde{V}_k^{(j)}), j \in \{1, \ldots, \ell'\}$. We do not rename the sets $\tilde{V}_i^{(j)}$ since they only shrink in magnitude only by $(k - 1)\varepsilon'L'$. Consequently,

- the leftover sets $\tilde{V}_i^{(0)}, 1 \leq i \leq k$, are of size $2\varepsilon n + (k - 1)\varepsilon'L'\ell' < k\varepsilon'n$.

---

[2] We visualize the vertex sets $V_i$ as being horizontal, like rows in a matrix, so it is natural to think of these $k$-tuples as columns.
each pair \((\tilde{V}_i^{j(i)}, \tilde{V}_i^{j(i)})\), \(i \neq i'\), is \((2\varepsilon', d'/2)\)-super-regular, and

- each pair \((\tilde{V}_i^{j(i)}, \tilde{V}_{i'}^{j(i')})\) is \(2\varepsilon'\)-regular with density either 0 or at least \(d' - \varepsilon'\), regardless of whether or not \(j = j'\).

If we use the Blow-up Lemma (Lemma \[18\]) at this point, we would obtain a \(K_k^h\)-tiling that covers every vertex of \(G\) except those in the leftover sets. The remainder of the proof is to establish that we can, in fact, ensure that the leftover vertices can be absorbed by the clusters and we can obtain a \(K_k^h\)-tiling that covers all the vertices of \(G\).

### 3.4. Preparing for absorption

In order to insert the vertices from the leftover sets, we need to prepare some copies of \(K_k^h\) throughout \(G\) that may be included in the final \(K_k^h\)-tiling. Their purpose is to ensure that, after vertex-insertion, the number of vertices in each of the clusters will be balanced so that the Blow-up Lemma (Lemma \[18\]) can be used. These copies of \(K_k^h\) will be specially designated and colored either red or blue according to their role.

The Reachability Lemma (Lemma \[13\]) is how we transfer the imbalance of the sizes of one column to the first column.

**Lemma 13 (Reachability Lemma).** Let \(G'_r\) be a balanced \(k\)-partite graph with partite sets \(U_i' = \{u_i^{(j)} : j \in \ell'\}, i \in [k]\). Let \(\delta(G'_r) \geq \frac{k-1}{2} \ell' + 2\). Then, for each \(i \in [k]\) and \(j \in \{2, \ldots, \ell'\}\), there is a pair \((T_1, T_2)\) of copies of \(K_k\) such that their symmetric difference is \(\{u_i^{(1)}, u_i^{(j)}\}\) and \(T_1\) and \(T_2\) contain no additional vertices from \(\{u_1^{(1)}, \ldots, u_k^{(1)}, u_1^{(j)}, \ldots, u_k^{(j)}\}\).

![Diagram for \(T_1\) and \(T_2\) formed in reaching \(u_1^{(1)}\) from \(u_1^{(j)}\).](image)

**Proof of Lemma \[13\]** Without loss of generality, it suffices to show that \(u_1^{(1)}\) can be reached from \(u_1^{(\ell')}\). The vertices \(u_1^{(1)}\) and \(u_1^{(\ell')}\) have at least \(\ell' - 2\delta(G'_r) \geq \)
\[ t' - 2 \left( t' - \frac{k-2}{k} t' - 2 \right) = \left( \frac{k-2}{k} \right) t' + 4 \] common neighbors in each of \( U'_2, \ldots, U'_{k}. \) Hence, one can choose a sequence \( w_2, \ldots, w_k \) of vertices so that, for \( i = 2, \ldots, k, \) \( w_i \) is in \( U'_i - \{ u_1^{(1)}, u_1^{(\ell)} \} \) and is a common neighbor of \( u_1^{(1)}, u_1^{(\ell)}, w_2, \ldots, w_{i-1}. \) Note that at each stage, the number of available choices for \( w_i \) is at least

\[
\left( \frac{k-2}{k} t' + 4 \right) - (i - 2) \left( t' - \frac{k-1}{k} t' - 2 \right) - 2 = \frac{k-i}{k} t' + 2k - 2.
\]

This quantity is positive since \( i \leq k \) and \( k \geq 2. \)

In preparation to insert the vertices, we create a set of special vertex-disjoint copies of \( K^k_h. \)

**Lemma 14.** There exist disjoint sets \( X_i(j) \subset \hat{V}_i^{(j)}, i \in [k], j \in [t'], \) such that for every \( i \in [k], j \in [t'], \)

1. \( |X_i(j)| = 3h\zeta n, \) and
2. for every \( v \in X_i(j), \) there exist two vertex-disjoint copies of \( K^k_h, \) call them \( R(v) \) and \( B(v), \) such that
   1. \( R(v) \) contains \( v, \)
   2. \( R(v) \) contains \( h - 1 \) vertices from \( \hat{V}_i^{(j)} \) and \( B(v) \) contains \( h \) vertices from \( \hat{V}_i^{(j)} \) and
   3. for every \( i' \neq i, \) there exists a \( j' \notin \{1, j\} \) such that both \( R(v) \) and \( B(v) \) each have \( h \) vertices from \( \hat{V}_i^{(j')} \).

**Proof of Lemma 14** The proof will proceed as follows: We will have some arbitrary order on the pairs \( \{(i, j) : i \in [k], j \in [t'] - \{1\}\} \) and dynamically define

\[ X = \bigcup_{(i', j') \in \{(i, j) : i \in [k], j \in [t'] - \{1\}\}} \left( V(R(v)) \cup V(B(v)) \right). \]

That is, \( X \) is the set of all vertices belonging to a \( R(v) \) or a \( B(v) \) for all \( (i', j') \) that precede the current \( (i, j). \)

We will show that, for all \( v \in X_i(j) \) the vertex-disjoint \( R(v) \) and \( B(v) \) can be found among vertices not in \( X, \) as long as \( |X| \leq \zeta^{1/2} L'. \)

Fix \( i \in [k] \) and \( j \in [t']. \) Let \( (T_1, T_2) \) be a pair of \( K_k \)-s in \( G'_c \) from Lemma 13 for these values of \( i \) and \( j. \) Consider the subgraph \( F \) of \( G'_c \) induced on the clusters \( \hat{V}_i^{(j')} \) such that \( u_i^{(j')} \) form \( V(T_2). \) Since \( T_2 \) is a \( K_k \) in the reduced graph \( G'_c, \) every pair of clusters in this subgraph is \( \varepsilon' \)-regular with density at least \( d'. \) Since \( |X| \leq \zeta^{1/2} L', \) \( |V_i^{(j')} - X| \geq \frac{1}{2} |V_i^{(j')}| \) and it follows from the definition of regularity that each pair of clusters of \( F - X \) is \( 2\varepsilon' \)-regular with density at least \( d' - \varepsilon'. \) By the Key Lemma (Lemma 2.1 from [23]), \( F - X \) contains at least \( 3h\zeta n \) vertex-disjoint copies of \( K^k_h \) as long as \( 3h\zeta n \ll \varepsilon' L'. \) This is satisfied because

\[
3h\zeta n \leq 3h\zeta \frac{\varepsilon' L'}{1 - 2\varepsilon} \leq 4h\zeta \varepsilon' L' = 4h\zeta dDL' \leq 4h\zeta MDL' \ll \varepsilon' L'.
\]
In the above inequality, we use the fact that $\zeta \ll \varepsilon'$ and that $\varepsilon \ll 1$. Also, $\ell' = \ell \cdot D \leq M \cdot D$ and $M$ and $D$ depend only on $k$ and $\varepsilon \ll \varepsilon'$. We refer to these $3h\zeta n$ copies of $K_h^k$ as blue copies of $K_h^k$ and we add their vertices to $X$.

In a similar fashion, let $F$ now be the graph induced on the clusters $V_i^{(j)}$ such that $u_i^{(j)}$ is in $V(T_1) \cup V(T_2)$. The graph $F - X$ also satisfies the assumptions of the Key Lemma and therefore we can find $3h\zeta n$ copies of $K_h^k$ in such a way that each copy has one vertex in $\tilde{X}$ which have a total of $2hk$ vertices of the Key Lemma and therefore we can find $3h\zeta n$ copies of $K_h^k$ as red copies of $K_h^k$ and add their vertices to $X$. For each red copy of $K_h^k$, we put its unique vertex in $V_i^{(1)}$ into $X_i(j)$ and call this copy $R(v)$. For each $v \in X_i(j)$, let $B(v)$ be a distinct blue copy of $K_h^k$ as found above.

For this process to work, we need to ensure that $|X| \leq \zeta^1/2L'$ at each step. This is true because each member of each $X_i(j)$ corresponds to two $K_h^k$-s which have a total of $2hk$ vertices and, hence, $|X| \leq 2hk \sum_{i,j} |X_i(j)| = 2hk(\ell' \cdot (3h\zeta n) = 6h^2k^2\ell'\zeta n \ll \zeta^1/2L'$. \hfill $\square$

We color the vertices of each $R(v)$ red and the vertices of each $B(v)$ blue.

### 3.5. Nearly-equalizing the sizes of the clusters

Let us summarize where we are: We have a designated first column (we call the first column the receptacle column and its clusters receptacle clusters) with each cluster of size $L'$ and each cluster having the same number of red vertices, which is at most $\zeta^1/2L'$. Each such red vertex is in a different vertex-disjoint red copy of $K_h^k$. In the remaining columns, each cluster has $L'$ original vertices, of which at most $\zeta^1/2L'$ are colored red and at most $\zeta^1/2L'$ are colored blue. The total number of red vertices in each $V_i$ is the same multiple of $h$. Moreover, in every column, every pair of clusters is $(2\varepsilon', d'/2)$-super regular. Finally, for each $i \in [k]$, there is a leftover set, $V_i^{(0)}$, of size at most $ke'n$.

We shall now re-distribute the vertices from leftover sets $V_i^{(0)}$, $i \in [k]$, to non-receptacle clusters in such a way that the size of leftover sets becomes $O(\zeta n)$ and each non-receptacle cluster will contain exactly $h \left\lceil (1 - d'/4) (L'/h) \right\rceil$ non-red vertices. These two properties will be essential for our procedure for finding perfect $K_h^k$-tiling to work.

We say that a vertex $v$ belongs in the cluster $\tilde{V}_i^{(j)}$ if $v \in V_i$ and $v$ is adjacent to at least $(d'/2)L'$ vertices in each of the other clusters $\tilde{V}_i^{(j)}, i \neq i'$, in the $j$-th column.

**Fact 15.** For every $i \in [k]$, we partition the leftover set $V_i^{(0)}$ into subsets $Y_i^{(2)}, \ldots, Y_i^{(\ell')}$ so that the members of $Y_i^{(j)}$ belong in cluster $\tilde{V}_i^{(j)}$. The number of vertices each cluster receives is no more than

$$\frac{k\varepsilon'n}{(1/k + \gamma/2)\ell'} \leq k^2\varepsilon'L'.$$
The number of red vertices in each cluster may vary, but it is always less than $\zeta^{1/2}L'$. Hence, after applying Fact 15, the number of non-red vertices in each cluster is in the interval $\left((1 - \zeta^{1/2})L', (1 + k'\epsilon)L'\right)$.

Next, we wish to remove copies of $K^k_{h}$ in such a way that the number of non-red vertices in each non-receptacle cluster is the same and there are new leftover sets of size $O(\zeta n)$. This is accomplished via Lemma 16.

**Lemma 16.** For each $i \in [k]$, there exist disjoint vertex sets $\hat{V}_i^{(0)}, \hat{V}_i^{(1)}, \ldots, \hat{V}_i^{(\ell')}$ in $V_i$ such that the following occurs:

- $|\hat{V}_i^{(0)}| \leq 3h\zeta n$,
- $\hat{V}_i^{(1)} = \tilde{V}_i^{(1)}$, has exactly $3h\zeta n$ red vertices and exactly $L'$ vertices total,
- for $j \in \{2, \ldots, \ell'\}$, $\hat{V}_i^{(j)} \subset \tilde{V}_i^{(j)}$ and $\hat{V}_i^{(j)}$ contains all red and blue vertices of $\tilde{V}_i^{(j)}$,
- for $j \in \{2, \ldots, \ell'\}$, $\hat{V}_i^{(j)}$ contains exactly $h\left\lceil \left(1 - \frac{d'}{4}\right)\frac{L'}{k}\right\rceil$ non-red vertices, and
- the graph induced by $V(G_{r'}) - \bigcup_{i=1}^{k} \bigcup_{j=0}^{\ell'} \hat{V}_i^{(j)}$ is spanned by the union of vertex-disjoint copies of $K^k_{h}$.

**Proof of Lemma 16.** In this proof, we will remove some copies of $K^k_{h}$ to thin the graph so that the clusters satisfy the conditions above. We shall do this by taking the reduced graph $G_{r'}$ and creating an auxiliary graph $A_{r'}$ and then we apply Theorem 7 to $A_{r'}$. The resulting fractional $K^k_{h}$-tiling in $A_{r'}$ will produce a family of vertex-disjoint $K^k_{h}$-s in $G$ that we shall remove.

From Section 3.2 recall that $D = D(k, \epsilon)$ was the least common multiple of a common denominator of a rational-valued solution to linear program (1) over all balanced $k$-partite graphs with at most $kM = M(k, \epsilon)$ vertices. In a similar way, we may define $D_0 = D_0(k, \epsilon, \zeta)$ to be the least common multiple of the common denominator of a rational-valued solution to linear program (1) over all balanced $k$-partite graphs with at most $\frac{d'}{3\zeta}L' \leq \frac{1}{3\zeta}D(k, \epsilon)M(k, \epsilon)$ vertices in each class.

Now we will define the auxiliary reduced graph $A_{r'}$ by blowing up the vertices and edges of the subgraph of $G'_{r'}$ induced by $V(G'_{r'}) - \{u_1^{(1)}, u_2^{(1)}, \ldots, u_k^{(1)}\}$. The number of copies of each vertex, however, will not be the same. For $i \in [k]$ and $j \in \{2, \ldots, \ell'\}$, define $\nu(u_i^{(j)})$ to be the number of non-red vertices in cluster $\tilde{V}_i^{(j)}$.

For $V(A_{r'})$, replace each vertex $u_i^{(j)}$ with the following number of copies: either the ceiling or the floor of

$$\frac{\nu(u_i^{(j)}) - \left\lceil (1 - d'/4)L'\right\rceil}{hD_0[\zeta L'/D_0]} - 1.$$
The choice of ceiling or floor is made arbitrarily, but only to ensure that the resulting graph is balanced. This is always possible because \( \sum_{j=2}^{\ell'} \nu(u_i^{(j)}) \) is the same for all \( i \in [k] \). For \( E(A_r) \), we replace each edge in \( G'_r \) by a complete bipartite graph and each nonedge by an empty bipartite graph.

First, we need to check that the number of vertices of \( A_r \) is not too large. Since \( (1 - \xi^{1/2})L' \leq \nu(u_i^{(j)}) \leq (1 + k^2\xi')L' \), the number of vertices in each partite set of \( A_r \) is at most

\[
\sum_{j=2}^{\ell'} \left\lfloor \frac{\nu(V_i^{(j)}) - [(1 - d'/4)L']}{hD_0 (\xi L'/D_0)} - 1 \right\rfloor \leq (\ell' - 1) \left[ \frac{(1 + k^2\xi')L' - (1 - d'/4)L'}{h\xi L'} - 1 \right] < (\ell' - 1) \frac{d'/4 + k^2\xi'}{h\xi}. \tag{4}
\]

This quantity is at most \( \frac{d'}{32}\ell' \) because \( \zeta \ll \varepsilon' \ll d' \).

Second, we need to check that each vertex \( A_r \) has sufficiently large degrees in order to apply Theorem 7. We observe that if \( u \) were adjacent to \( u_i^{(j)} \) in \( G'_r \), then every copy of \( u \) in \( V(A_r) \) is adjacent to at least

\[
\left\lfloor \frac{(1 - \zeta^{1/2})L' - [(1 - d'/4)L']}{hD_0 (\xi L'/D_0)} - 1 \right\rfloor \geq \frac{d'/4 - 2\zeta^{1/2}}{h\zeta},
\]

copies of \( u_i^{(j)} \) in \( V(A_r) \). So, each vertex in \( V(A_r) \) is adjacent to at least

\[\left( \left( \frac{k-1}{k} + \frac{\gamma}{2} \right) \ell' - 1 \right) \frac{d'/4 - 2\zeta^{1/2}}{h\zeta} \geq (\ell' - 1) \left( \frac{k-1}{k} + \frac{\gamma}{3} \right) \frac{d'}{4h\zeta},\]

vertices in each of the other partite sets of \( V(A_r) \). By (4), every partite set of \( V(A_r) \) has size at most \( (\ell' - 1) \frac{d'/4 + k^2\xi'}{h\xi} \). Using (3), the proportion of neighbors of a vertex in \( V(A_r) \) in any other vertex class is at least

\[
\frac{(\ell' - 1) \left( \frac{k-1}{k} + \frac{\gamma}{3} \right) \frac{d'}{4h\xi}}{(\ell' - 1) \frac{d'/4 + k^2\xi'}{h\xi}} \geq \frac{k-1}{k}.
\]

So, we can apply Theorem 7 to the auxiliary reduced graph \( A_r \) and obtain an optimal solution to linear program (1) with the property that \( D_0 w(T) \) is an integer for every \( T \in \mathcal{T}_k(A_r) \).

As in Fact 11, this implies that the blow-up graph \( A_r(D_0) \) must have a perfect \( K_k \)-tiling. For each \( K_k \) in this tiling, we will remove \( \lceil \xi L'/D_0 \rceil \) vertex-disjoint copies of \( K^k_k \) from the uncolored vertices of the corresponding clusters.
of $G_r^i$. So, the total number of vertices removed from cluster $V_i^{(j)}$ is

$$hD_0 \left\lceil \frac{\zeta L'/D_0}{L'} \right\rceil \times \left\lceil \nu(V_i^{(j)}) - \left\lceil \frac{(1 - d'/4)L'}{hD_0}\right\rceil - 1 \right\rceil,$$

where $\lceil \cdot \rceil$ is either the floor or ceiling of its argument.

Removing these copies of $K_h^k$ has the effect of making the number of uncolored vertices in each cluster nearly identical, within $h\zeta L'$ of each other, ignoring ceilings. For $i \in [k]$, place into the new leftover set $V_i$ at most $hD_0 \left\lceil \frac{\zeta L'/D_0}{L'} \right\rceil - 1$ uncolored vertices from each cluster to ensure that every cluster retains either $\left\lceil (1 - d'/4)L' \right\rceil$ or $\left\lceil (1 - d'/4)L' \right\rceil + hD_0[\zeta L'/D_0]$ uncolored vertices, depending on whether the ceiling or floor function was chosen for rounding. In the latter case, place an additional $hD_0 \left\lceil \frac{\zeta L'/D_0}{L'} \right\rceil$ uncolored vertices from the cluster to the leftover set.

Summarizing:

- We removed at most $2hD_0[\zeta L'/D_0]$ vertices from each cluster, so each new leftover set $\tilde{V}_i^{(0)}$ has a size of at most $\ell' \cdot 2hD_0[\zeta L'/D_0] \leq 3h\zeta n$.
- The sets $\tilde{V}_i^{(1)}$ are unchanged.
- For $j \in \{2, \ldots, \ell'\}$, $\tilde{V}_i^{(j)}$ is formed by removing uncolored vertices from $\tilde{V}_i^{(j)}$.
- For $j \in \{2, \ldots, \ell'\}$, the number of non-red vertices in $\tilde{V}_i^{(j)}$ is explicitly prescribed to be $h \left\lceil (1 - d'/4)(L'/h) \right\rceil$.
- The vertices that are removed are all in vertex-disjoint copies of $K_{h}^k$.

\[\blacksquare\]

3.6. Inserting the leftover vertices and construction of perfect $K_h^k$-tiling

We first insert the leftover vertices from $\bigcup_{i=1}^{k} \tilde{V}_i^{(0)}$ to non-receptacle clusters in such a way that we shall be able to find a perfect $K_h^k$-tiling in every column using the Blow-up Lemma. That is, each cluster in the column will have the same number of vertices (divisible by $h$) and each pair of clusters will be super-regular.

---

*It is easy to find such vertex-disjoint copies of $K_h^k$ in a $k$-tuple. Observe that every cluster has at most $k^2 \epsilon L'$ uncolored vertices added to the cluster. Moreover, a set of $hD_0[\zeta L'/D_0]$ vertices will be removed from a cluster at most $d'/2(2\zeta)$ times as long as $\epsilon' \ll d'$. So, there will always be at least $(1 - d'/4)L' - k^2 \epsilon L' - hdL'/2 \geq (1 - d')L'$ uncolored vertices from the original cluster. Using the Slicing Lemma (Fact 19), any pair of them form a $2(2\epsilon')$-regular pair. As long as $\zeta \ll \epsilon' \ll d'$, we could apply, say, the Key Lemma from 23 to ensure the existence of at most $[\zeta L'/D_0]$ vertex-disjoint copies of $K_h^k$ in the $k$-tuple.*
Suppose that vertex \( w \in \hat{V}_i^{(0)} \) belongs to cluster \( \hat{V}_i^{(j)} \), \( j \in \{2, \ldots, \ell\} \). We then take any \( v \in X_i^{(j)} \) and the red and blue copies \( R(v), B(v) \) of \( K^h_h \) guaranteed by Lemma 14. We uncolor the vertices of \( R(v) \), remove the vertices of \( B(v) \) from their respective clusters and place \( B(v) \) aside to be included in the final tiling of \( G \). We also add \( w \) to the cluster \( \hat{V}_i^{(j)} \) and remove \( v \) from \( X_i^{(j)} \).

Each time this procedure is undertaken, the number of non-red vertices in each non-receptacle cluster does not change and it is equal to \( h \lfloor (1 - d'/4) (L'/h) \rfloor \).

After doing this procedure for every vertex in the leftover sets, we remove all the remaining (unused) red copies of of \( K^h_h \) and place them aside to be included in the final tiling of \( G \). This ensures that the clusters in the first (receptacle) column have the same number of non-red vertices as each other and the number of non-red vertices in each receptacle cluster has the same congruency modulo \( h \) as \( n \) does. That is, if we remove \( n - h \lfloor n/h \rfloor \) non-red vertices from each receptacle cluster, the remaining number of non-red vertices is divisible by \( h \).

The non-red vertices in each receptacle cluster form pairwise \((4\varepsilon', d'/4)\)-super-regular pairs, this follows from the Slicing Lemma (Fact 19) because no vertices were added to these clusters. So we focus on the non-receptacle clusters.

Throughout this proof, in every non-receptacle cluster, at most \( \varepsilon' L' \) vertices were colored red and at most \( \varepsilon' L' \) red vertices will be uncolored (i.e., they become non-red). In addition, the non-red vertices in any non-receptacle cluster will have cardinality exactly \( h \lfloor (1 - d'/4) L'/h \rfloor \). Recall that the original clusters formed \((2\varepsilon', d')\)-super-regular pairs in each column. There were at most \( k^2 \varepsilon' L' \) new vertices added to each cluster, each of which were adjacent to at least \( (d'/2) L' \) vertices in each of the original clusters of the column. The next lemma will imply that the non-red vertices in every non-receptacle column will form super-regular pairs.

**Fact 17.** Let \((A, B)\) be an \((\varepsilon_1, \delta_1)\)-super-regular pair. Furthermore, let

- at most \( \varepsilon_2 |A| \) vertices be added to \( A \) such that each has at least \( \delta_2 |B| \) neighbors in \( B \) and
- at most \( \varepsilon_2 |B| \) vertices be added to \( B \) such that each has at least \( \delta_2 |A| \) neighbors in \( A \).

The resulting pair \((A', B')\) is \((\varepsilon_0, \delta_0)\)-super-regular, where \( \varepsilon_0 = \varepsilon_1 + \varepsilon_2 \) and \( \delta_0 = \frac{\min(\delta_1, \delta_2)}{1 + \varepsilon_2} \).

We apply Fact 17 with \( \varepsilon_1 = 2\varepsilon', \delta_1 = d', \varepsilon_2 = k\varepsilon' \) and \( \delta_2 = d'/2 \). Consequently, we use \( \varepsilon_1 + \varepsilon_2 \leq (k + 2)\varepsilon' \leq \sqrt{\varepsilon'} \) and \( \frac{\min(\delta_1, \delta_2)}{1 + \varepsilon_2} = \frac{d'/2}{(1 + k\varepsilon')^3} \geq \frac{d'/3}{2} \) to conclude that the augmented pairs in each column are \((\sqrt{\varepsilon'}, d'/3)\)-super-regular.

Finally, to finish the tiling, apply the Blow-up Lemma to non-red vertices in each non-receptacle column (recall that the number of such vertices is the same and is divisible by \( h \)). In the receptacle column, we can apply the Blow-up Lemma to all but \( n - h \lfloor n/h \rfloor \) of the non-red vertices.
Lemma 18 (Blow-up Lemma, Komlós-Sárközy-Szemerédi [21]). Given a graph $R$ of order $r$ and positive parameters $\delta, \Delta$, there exists an $\varepsilon_18 > 0$ such that the following holds: Let $N$ be an arbitrary positive integer, and let us replace the vertices of $R$ with pairwise disjoint $N$-sets $V_1, V_2, \ldots, V_r$ (blowing up). We construct two graphs on the same vertex-set $V = \bigcup V_i$. The graph $R(N)$ is obtained by replacing all edges of $R$ with copies of the complete bipartite graph $K_{N,N}$ and a sparser graph $G$ is constructed by replacing the edges of $R$ with some $(\varepsilon_18, \delta)$-super-regular pairs. If a graph $H$ with maximum degree $\Delta(H) \leq \Delta$ can be embedded into $R(N)$, then it can be embedded into $G$.

Our $K_h^k$-tiling consists of

1. the copies of $K_h^k$ that are outside of the sets $\hat{V}_i^{(j)}$, as established in Lemma 16,
2. the red copies of $K_h^k$ that were not used in absorbing vertices from the leftover sets $\hat{V}_i^{(0)}$ to non-receptacle clusters, and
3. the copies of $K_h^k$ found by applying the Blow-up Lemma to the non-red vertices in each column.

This is the tiling of $G$ with $\lfloor n/h \rfloor$ copies of $K_h^k$.

What remains to show is that we can choose our constants to satisfy (3) so that all inequalities in our proof will be satisfied for sufficiently large $n$. Indeed, for given $\gamma > 0$ and $h$, we let $d = \gamma/4$. We also set $R = K_k$, $r = k$, $\Delta = (k - 1)h$ and $\delta = \gamma/12$ and apply Lemma 18 to obtain $\varepsilon_18$. Now we define $\varepsilon' = \min\{\delta, d/(12k^2)\}$ and we let $\varepsilon = \min\{\varepsilon'^2/16, d/4(k + 2)\}$. Finally, we set $\zeta = 1/(12h^2k^2M(k, \varepsilon)2D(k, \varepsilon)^2)$, where $M(k, \varepsilon)$ comes from the Regularity Lemma (Theorem 8) and $D(k, \varepsilon)$ is defined in Section 3.2. This concludes the proof of Theorem 4.

4. Proofs of Facts

For convenience, we restate the facts to be proven.

**Fact 15.** For every $i \in [k]$, we partition the leftover set $\hat{V}_i^{(0)}$ into subsets $Y_i^{(2)}, \ldots, Y_i^{(\ell')}$. The number of vertices each cluster receives is no more than $k\varepsilon' n \leq k^2\varepsilon' L'$.  

**Proof of Fact 15.** First, we show that each vertex belongs in at least $(1/k + \gamma/2)\ell'$ clusters. To see this, let $x$ be the number of clusters in $V_i'$, $i' \neq i$ such that $v$ is adjacent to less than $(d'/2)L'$ vertices of that cluster. Then,

$$x \frac{d'}{2} L' + (\ell' - x)L' + (n - \ell'L') \geq \left(\frac{k - 1}{k} + \gamma\right)n.$$

From this it is easy to derive that with $d', \varepsilon'$ small enough relative to $\gamma$, it is the case that $x < (1/k - \gamma/2)\ell'$. By a simple union bound, the number of clusters
in which \( v \) belongs is greater than \( \ell' - (k-1)(1/\kappa - \gamma/2)\ell' \geq (1/k + \gamma/2)\ell' \).
Hence, there are at least \((1/k + \gamma/2)\ell'\) clusters outside of the receptacle column in which \( v \) belongs.

Sequentially and arbitrarily assign \( v \in \tilde{V}_i^{(0)} \) to \( Y_i^{(j)} \) if both \( v \) belongs in \( \tilde{V}_i^{(j)} \) and \(|Y_i^{(j)}| < \frac{k\varepsilon'n}{(1/k + \gamma/2)\ell'}\). Since the size of \( \tilde{V}_i^{(0)} \) is at most \( k\varepsilon'n \), we can always find a place for \( v \).

**Fact 12.** Let \( \varepsilon' > 0 \) and \( \varepsilon' < d'/(2(k+1)) \) let \( (A_1, \ldots, A_k) \) be a \( k \)-tuple that is pairwise \( \varepsilon' \)-regular of density at least \( d' \) with \( |A_1| = \cdots = |A_k| = L' \). There exist subsets \( A_0 i \in [k] \) such that \( |A_j| = h[(1 - (k-1)\epsilon')L'/h] \) and each pair of \((A_1', \ldots, A_k')\) is \((2\epsilon', d' - k\epsilon')\)-super-regular (with density at least \( d' - \epsilon' \)).

**Proof of Fact 12.** We use the so-called Slicing Lemma, which is Fact 19 in Fact 1.5 in [23].

**Fact 19 (Slicing Lemma).** Given \( \varepsilon, \alpha, d \) such that \( 0 < \varepsilon < \alpha < 1 \) and \( d, 1 - d > \max\{2\varepsilon, \varepsilon/\alpha\} \). Let \( (A, B) \) be an \( \varepsilon \)-regular pair with density \( d \), \( A' \subset A \) with \( |A'| \geq \alpha |A| \) and \( B' \subset B \) with \( |B'| \geq \alpha |B| \). Then \( (A', B') \) is \( \varepsilon_0 \)-regular with \( \varepsilon_0 = \max\{2\varepsilon, \varepsilon/\alpha\} \) and density in \( [d - \varepsilon, d + \varepsilon]\).

It follows from the \( \varepsilon' \)-regularity of \( (A_i, A_j) \) that all but at most \( \varepsilon'|A_i| \) vertices of \( A_i \) have at least \( (d' - \varepsilon')|A_j| \) neighbors in \( A_j \). So, there is a set \( A'_i \subset A_i \) of size \( (1 - (k-1)\varepsilon')|A_i| \) such that each vertex of \( A'_i \) has at least \( (d' - \varepsilon')|A_j| \) neighbors in \( A_j \) for every \( j \neq i \) and, consequently, at least \( (d' - \varepsilon')|A_j| - (k - 1)\varepsilon'|A_j| = (d' - k\varepsilon')|A'_j| \) neighbors in \( A'_j \) for every \( j \neq i \). Since \( \varepsilon' < d'/(2(k+1)) \), \( 1 - (k-1)\varepsilon') > 1/2 \) and the Slicing Lemma (Fact 19)

Using Fact 19 with \( \alpha = 1/2 \) and \( \varepsilon' < d'/(2k+1) \), we conclude that each pair \((A'_i, A'_j)\) is \((2\varepsilon', d' - k\varepsilon')\)-super-regular. \( \square \)

**Fact 17.** Let \( (A, B) \) be an \( (\varepsilon_1, \delta_1) \)-super-regular pair. Furthermore, let at most \( \varepsilon_2 |A| \) vertices be added to \( A \) such that each has at least \( \delta_2 |B| \) neighbors in \( B \) and at most \( \varepsilon_2 |B| \) vertices be added to \( B \) such that each has at least \( \delta_2 |A| \) neighbors in \( A \). The resulting pair \((A', B')\) is \((\varepsilon_0, \delta_0)\)-super-regular, where \( \varepsilon_0 = \varepsilon_1 + \varepsilon_2 \) and \( \delta_0 = \frac{\min \{ \delta_1, \delta_2 \} }{(1 + \varepsilon_2)^2} \).

**Proof of Fact 17.** First we establish the minimum degree condition. Each of the original vertices in \( A \) is adjacent to at least \( \delta_1 |A| = \delta_1 \frac{|A|}{|A'|} |A'| \) vertices in \( A' \).

Each of the new vertices in \( A \) adjacent to at least \( \delta_2 |A| = \delta_2 \frac{|A|}{|A'|} |A'| \) neighbors in \( A' \). Similar conditions hold for vertices in \( B' \).

Since

\[
\delta_0 \leq \frac{\min \{ \delta_1, \delta_2 \} }{(1 + \varepsilon_2)^2} \leq \min \left\{ \delta_1 \frac{|A|}{|A'|}, \delta_2 \frac{|A|}{|A'|}, \delta_1 \frac{|B|}{|B'|}, \delta_2 \frac{|B|}{|B'|} \right\},
\]

each vertex \( a \in A' \) has at least \( \delta_0 |B'| \) neighbors in \( B' \) and each vertex \( b \in B' \) has at least \( \delta_0 |A'| \) neighbors in \( A' \).
Now, consider any $X' \subseteq A'$ and $Y' \subseteq B'$ such that $|X'| \geq \varepsilon_0 |A'|$ and $|Y'| \geq \varepsilon_0 |A'|$. Consider $X = X' - (A' - A)$ and $Y = Y' - (B' - B)$. Note that

$$|X| \geq |X'| - \varepsilon_2 |A| \geq \varepsilon_0 |A'| - \varepsilon_2 |A| \geq \varepsilon_1 |A|.$$  

Similarly, $|Y| \geq \varepsilon_1 |B|$ and so $d(X, Y) \geq \delta_1$.

Consequently,$$d(X', Y') \geq d(X, Y) \frac{|X| |Y|}{|X'||Y'|} \geq \frac{\delta_1}{(1 + \varepsilon_2)^2} \geq \delta_0,$$

and the pair is $(\varepsilon_0, \delta_0)$-super-regular. □

5. Concluding Remarks

The common denominator $D = D(k, \varepsilon)$ used in Section 3.2 can, in principle, be astronomically large, as it is the common denominator of values of rational-valued solutions for all balanced $k$-partite graphs on at most $M = M(k, \varepsilon)$ vertices. We chose this value for the convenience of the proof. Indeed, the constant $M$ is quite large itself and so $D$ is not so large, relatively speaking.

We could utilize a much smaller integer value of $D$ by choosing a $D$ such that if $w^*$ is the rational-valued solution of (1), then for every $v \in V(G_r)$ and every $T \in \mathcal{T}_k(G_r)$ for which $V(T) \ni v$, we assign $\lfloor Dw^*(T) \rfloor$ vertices of $G'_r$ to copies of $T$. Because $Dw^*(T)$ is not necessarily an integer, we end up with $D - \sum_{V(T) \ni v} \lfloor Dw^*(T) \rfloor$ unused vertices. Choose $D$ large enough to ensure that this is always small ($O(\varepsilon M^{k-1})$ suffices), and they can be placed in the leftover set.

We should also note that, asymptotically, Conjecture 3 is stronger than the Hajnal-Szemerédi Theorem. That is, if $G$ is a graph on $kn$ vertices with minimum degree at least $(\frac{k-1}{k} + \gamma) kn$, then a random partition of the vertex set into $k$ equal parts gives a $k$-partite graph $\tilde{G}$ with $\hat{\delta}(\tilde{G}) \geq (\frac{k-1}{k} + \gamma) kn - O(\sqrt{k} \log n)$ and applying Conjecture 3 would give a $K_k$-tiling in $\tilde{G}$ and, hence, $G$ itself.

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Appendix A. Proof of the Random Slicing Lemma

Suppose that \((X, Y)\) is an \(\varepsilon\)-regular pair such that \(|X| = |Y| = L\). We will show that if \(X = A_1 \cup \cdots \cup A_m \cup A_0\) and \(Y = B_1 \cup \cdots \cup B_m \cup B_0\) is a random equipartition (that is, \(|A_1| = \cdots = |A_m| = |B_1| = \cdots = |B_m| = L'\) and \(|A_0| = |B_0| < m\), then, with high probability, each pair \((A_i, B_j)\) is \(2\varepsilon^{1/5}\)-regular, provided \(L' = \omega(\sqrt{L \log L})\).

First we cite the relevant theorem of Kohayakawa and Rödl [19]. In order to do so, for a vertex set \(Y\) and vertices \(x, x' \in X\), we denote \(\deg_Y(x, x') = |N_Y(x) \cap N_Y(x')|\).

**Theorem 20.** Let \(\varepsilon\) be a constant such that \(0 < \varepsilon < 1\). Let \(G\) be a graph with \((X, Y)\) a pair of nonempty disjoint vertex sets with \(|X| \geq 2/\varepsilon\). Set \(d = d(X, Y) = e(X, Y)/|X||Y|\). Let \(D\) be the collection of all pairs \(\{x, x'\}\) of vertices of \(X\) for which

\[
(i) \quad \deg_Y(x, x') > (d - \varepsilon)|Y|,
(ii) \quad \deg_Y(x, x') < (d + \varepsilon)^2|Y|.
\]

Then if \(|D| > (1/2) (1 - 5\varepsilon)|X|^2\), the pair \((X, Y)\) is \((16\varepsilon)^{1/5}\)-regular.

We prefer to use ordered pairs, so we modify the statement.

**Corollary 21.** Let \(\varepsilon\) be a constant such that \(0 < \varepsilon < 1\). Let \(G\) be a graph with \((X, Y)\) a pair of nonempty disjoint vertex sets with \(|X| \geq 2/\varepsilon\). Set \(d = d(X, Y) = e(X, Y)/|X||Y|\). Let \(P\) be the collection of all ordered pairs \((x, x')\) of vertices of \(X\) for which

\[
(i) \quad \deg_Y(x, x') > (d - \varepsilon)|Y|,
(ii) \quad \deg_Y(x, x') < (d + \varepsilon)^2|Y|.
\]

If \(|P| > (1 - 4.5\varepsilon)|X|^2\), then the pair \((X, Y)\) is \((16\varepsilon)^{1/5}\)-regular. Furthermore, if \(\varepsilon \leq 1/9, 4\varepsilon \leq d \leq 1 - 4\varepsilon\) and \(|P| > (1 - 5\varepsilon)|X|^2\), then \((X, Y)\) is \((16\varepsilon)^{1/5}\)-regular.

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Proof of Corollary 21. The number of ordered pairs that can satisfy the conditions is \( P \leq 2D + |X| \) because the pairs \((x, x)\) can be included.

\[
D \geq P/2 + |X|/2 \\
= P/2 + 1/(2|X|)|X|^2 \\
> (1/2)(1 - 4.5\varepsilon)|X|^2 + (\varepsilon/4)|X|^2,
\]
giving the main result. The second result follows if we can prove that the pairs \((x, x)\) cannot exist. This is true as long as

\[
(d - \varepsilon) \geq (d + \varepsilon)^2. \tag{A.1}
\]

This, however, is true as long as \(\varepsilon \leq 1/9\) and \(4\varepsilon \leq d \leq 1 - 4\varepsilon\) because

\[
\frac{1 - 2\varepsilon - \sqrt{1 - 8\varepsilon}}{2} \leq 4\varepsilon \leq 1 - 4\varepsilon \leq \frac{1 - 2\varepsilon + \sqrt{1 - 8\varepsilon}}{2}.
\]

The upper and lower bounds are the solutions to the quadratic equation resulting from (A.1). □

A lemma that we need is a direct result from martingales:

Lemma 22. Let \(Y\) be a set of size \(L\) and \(S \subseteq Y\). If \(B \subset Y\) is chosen uniformly at random from all sets of size \(L'\), then with probability at most \(\exp\{\varepsilon^2(L')^2/(2L)\}\), it is the case that \(|S \cap B| \leq |S|L'/2 - \varepsilon L'\) and with probability at most \(\exp\{\varepsilon^2(L')^2/(2L)\}\), it is the case that \(|S \cap B| > |S|L'/2 + \varepsilon L'\).

Proof of Lemma 22. We use a simple construction of a Doob martingale, which might be thought of as an “element-exposure martingale.” We refer the reader to Chapter 7 of Alon and Spencer [1]. We order the members of \(Y = \{y_1, \ldots, y_L\}\) and define a sequence of random variables \(Z_0, \ldots, Z_L\) as follows:

\[
Z_i(C) = E[|B \cap S| \mid B \cap \{y_1, \ldots, y_i\} = C \cap \{y_1, \ldots, y_i\}, 1 \leq j \leq i].
\]

So, \(Z_L(C) = |C \cap S|\) and \(Z_0(C) = E_{B \in \{Y\}}[|B \cap S|].\)

The sequence of random variables \(Z_0(C), Z_1(C), \ldots, Z_L(C)\) forms a martingale because \(E[Z_{i+1}(C) \mid Z_i(C), \ldots, Z_0(C)] = Z_i(C)\). In fact, this is an example of a so-called Doob Martingale. Furthermore, the random variables satisfy a Lipschitz condition. That is, \(|X_{i+1}(C) - X_i(C)| \leq 1\) for \(i = 1, \ldots, L - 1\). Using this, Azuma’s inequality gives that

\[
\Pr\left[|B \cap S| - E[|B \cap S|] > \lambda \sqrt{L}\right] < e^{-\lambda^2/2}
\]

\[
\Pr\left[|B \cap S| - E[|B \cap S|] < -\lambda \sqrt{L}\right] < e^{-\lambda^2/2}
\]

and our result follows. □
We say that a pair \((x, x') \in X^2\) is good in \(Y\) if \(\deg_Y(x), \deg_Y(x') > (d-\varepsilon)|Y|\) and \(\deg_Y(x, x') < (d+\varepsilon)^2|Y|\). For some \(B \subset Y\), with \(|B| = L'\), we say that the pair is good in \(B\) if \(\deg_Y(x), \deg_Y(x') > (d-2\varepsilon)|Y|\) and \(\deg_Y(x, x') < (d+2\varepsilon)^2|Y|\). Proposition 23 shows that a pair that’s good in \(Y\) is good in all parts of the partition, if chosen uniformly.

**Proposition 23.** Fix \(0 < \varepsilon \leq 1/3\). If \((x, x') \in X^2\) is a pair that is good in \(Y\), then the pair \((x, x')\) is good in each of \(B_1, \ldots, B_m\) where \((B_1, \ldots, B_m)\) is chosen uniformly from among all equipartitions of \(Y\) into \(m\) sets of size \(L'\), with probability at most

\[
2m \exp\{-\varepsilon^2(L')^2/(2L)\} + m \exp\{-9\varepsilon^4(L')^2/(2L)\} \leq 3m \exp\{-9\varepsilon^4(L')^2/(2L)\}.
\]

**Proof of Proposition 23.** Let \(d\) be the density of \((X, Y)\). Let \(B\) be chosen uniformly at random from \(\binom{Y}{L'}\). Let \(\mu_1 = E[|N_Y(x) \cap B|]\).

\[
\Pr[|N_Y(x) \cap B| < (d-2\varepsilon)|B|] = \Pr[|N_Y(x) \cap B| - \mu_1 < (d-2\varepsilon)|B| - \mu_1]
\]

\[
= \Pr\left[|N_Y(x) \cap B| - \mu_1 < (d-2\varepsilon)|B| - |N_Y(x)|\frac{|B|}{|Y|}\right] \leq \Pr\left[|N_Y(x) \cap B| - \mu_1 < (d-2\varepsilon)|B| - (d-\varepsilon)|Y|\frac{|B|}{|Y|}\right] = \Pr\left[|N_Y(x) \cap B| - \mu_1 < -\varepsilon \frac{|B|}{\sqrt{|Y|}} \sqrt{|Y|}\right] < \exp\left\{\frac{-1}{2} \varepsilon^2 \frac{|B|^2}{|Y|}\right\},
\]

where the last inequality is Azuma’s inequality. Symmetrically,

\[
\Pr[|N_Y(x) \cap B| < (d-2\varepsilon)|B|] < \exp\left\{\frac{-1}{2} \varepsilon^2 \frac{|B|^2}{|Y|}\right\}.
\]

Let \(\mu_2 = E[|N_Y(x) \cap B|].

\[
\Pr[|N_Y(x, x') \cap B| > (d+2\varepsilon)^2|B|] = \Pr\left[|N_Y(x, x') \cap B| - \mu_2 > (d+2\varepsilon)^2|B| - \mu_2\right] \leq \Pr\left[|N_Y(x, x') \cap B| - \mu_2 > (d+2\varepsilon)^2|B| - (d+\varepsilon)^2|Y|\frac{|B|}{|Y|}\right] = \Pr\left[|N_Y(x, x') \cap B| - \mu > (2\varepsilon + 3\varepsilon^2) \frac{|B|}{\sqrt{|Y|}} \sqrt{|Y|}\right] < \exp\left\{\frac{-1}{2} (2d + 3\varepsilon) \varepsilon^2 \frac{|B|^2}{|Y|}\right\},
\]

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where the last inequality is Azuma’s inequality.

For fixed $i \in \{1, \ldots, m\}$, the probability that a good pair $(x, x')$ is not good in $B_i$ is at most $2 \exp\{-\varepsilon^2(L')^2/(2L)\} + \exp\{-(2d + 3\varepsilon)^2\varepsilon^2(L')^2/(2L)\}$. The union bound gives that the probability that a good pair is not good in at least one member of $\{B_1, \ldots, B_m\}$ is at most $2m \exp\{-\varepsilon^2(L')^2/(2L)\} + m \exp\{-(2d + 3\varepsilon)^2\varepsilon^2(L')^2/(2L)\}$ and a simple computation gives the statement of the proposition. □

The proof will be finished by showing that, with high probability, each of the sets $A_1, \ldots, A_m$ have at most $6\varepsilon(L')^2$ pairs that are not good in $Y$.

**Proposition 24.** Fix $0 < \varepsilon < 1$. If $(X, Y)$ is an $\varepsilon$-regular pair with $|X| = |Y| = L$, then each set $A_1, \ldots, A_m$ has at least $(1 - 8\varepsilon)(L')^2$ good pairs, where $(A_1, \ldots, A_m)$ is chosen uniformly from among all equipartitions of $X$ into $m$ sets of size $L'$, with probability at most

$$4m \exp\{-\varepsilon^2(L')^2/(2L)\}.$$ 

**Proof of Proposition 24.** Let $A$ be chosen uniformly at random from $\binom{X}{L'}$. Let $S_1$ consist of the pairs $(x, x') \in A^2$ for which $\text{deg}_Y(x) \leq (d - \varepsilon)|Y|$, let $S_2$ consist of the pairs $(x, x') \in A^2$ for which $\text{deg}_Y(x') \leq (d - \varepsilon)|Y|$ and let $S_3$ consist of the pairs $(x, x') \in A^2$ for which $\text{deg}_Y(x), \text{deg}_Y(x') > (d - \varepsilon)|Y|$ and $\text{deg}_Y(x, x') \geq (d + \varepsilon)^2|Y|$.

Let $X' = \{x \in X : \text{deg}_Y(x) \leq \varepsilon L\}$. By $\varepsilon$-regularity, $|X'| \leq \varepsilon|X|$.

$$\Pr[|S_1| > 2\varepsilon(L')^2] = \Pr[|X' \cap A| > 2\varepsilon L']$$

$$\leq \Pr\left[|X' \cap A| - E[|X' \cap A|] > 2\varepsilon L' - \varepsilon|X||A|/|X|\right]$$

$$= \Pr\left[|X' \cap A| - E[|X' \cap A|] > \varepsilon L'\right]$$

$$= \Pr\left[|X' \cap A| - E[|X' \cap A|] > \varepsilon L' \sqrt{L}\right]$$

$$< \exp\left\{-\frac{1}{2}\varepsilon^2(L')^2/L\right\},$$

where the last inequality is Azuma’s inequality. Symmetrically,

$$\Pr[|S_2| > 2\varepsilon(L')^2] < \exp\left\{-\frac{1}{2}\varepsilon^2(L')^2/L\right\}.$$ 

Now we address $S_3$. By $\varepsilon$-regularity, there are at most $\varepsilon|X|$ elements $x \in X$ for which $\text{deg}_Y(x) \geq (d + \varepsilon)|Y|$. Moreover, if $\text{deg}_Y(x) < (d + \varepsilon)|Y|$ then there are at most $\varepsilon|X|$ elements $x' \in X$ for which $\text{deg}_Y(x, x') \geq (d + \varepsilon)^2|Y|$. If $x \in X$ has small degree, then we compute the probability that

$$\Pr[(x, x') \text{ is not good in } A] \leq \Pr[\text{deg}_Y(x) \geq (d + \varepsilon)|Y|]$$

$$+ \Pr[\text{deg}_Y(x, x') \geq (d + \varepsilon)^2|Y| \mid \text{deg}_Y(x) < (d + \varepsilon)|Y|]$$

$$\leq 2 \exp\left\{-\frac{1}{2}\varepsilon^2(L')^2/L\right\},$$

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using similar calculations to those used to calculate $S_1$ and $S_2$. \hfill \Box

This gives us the solution. As long as $\varepsilon \leq 1/3$, the probability that any pair $(A_i, B_j)$ has fewer than $(1 - 8\varepsilon)(L')^2$ good pairs is at most

$$8m \exp\left\{-9\varepsilon^4 (L')^2 / (2L)\right\}.$$  \hfill (A.2)

(Note that $\varepsilon$ can be allowed to be as large as 1, but the expression $9\varepsilon^4$ in (A.2) changes to $\min\{\varepsilon^2, 9\varepsilon^4\}$.)

Since $m \leq L/L'$, straightforward computations verify that, if $L' \geq \sqrt{L \ln L} / (3\varepsilon^2)$ and $L$ is large enough, then the probability in (A.2) is at most $1/2$. Moreover, if $L' = \omega(\sqrt{L \ln L})$, then the probability in (A.2) goes to zero as $L \to \infty$. 

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