Particle creation, renormalizability conditions and the mass-energy spectrum in gravity theories of quadratic Lagrangians

K Kleidis and D B Papadopoulos
Department of Physics
Section of Astrophysics, Astronomy and Mechanics
Aristotle University of Thessaloniki
54006 Thessaloniki, Greece

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Abstract
Massive scalar particle production, due to the anisotropic evolution of a five-dimensional spacetime, is considered in the context of a quadratic Lagrangian theory of gravity. Those particles, corresponding to field modes with non-vanishing momentum component along the fifth dimension, are created mostly in the neighbourhood of a singular epoch where only their high-frequency behaviour is of considerable importance. At the one-loop approximation level, general renormalizability conditions on the physical quantities relevant to particle production are derived and discussed. Exact solutions of the resulting Klein-Gordon field equation are obtained and the mass-energy spectrum, attributed to the scalar field due to the cosmological evolution, is being further investigated. Finally, analytic expressions regarding the number and the energy density of the created particles at late times, are also derived and discussed.

1. Introduction
Gravitational Lagrangian densities, which include higher-order curvature terms in connection to the Einstein-Hilbert (EH) one, are suggested by superstring theories (Candelas et al. 1985, Green et al. 1987) and by the one-loop approximation of quantum gravity (Birrell and Davies 1982, Barrow and Ottewill 1983). Quadratic Lagrangians, in particular, have been used to yield renormalizable theories of gravity coupled to matter (Stelle 1977). They can also help us to improve the semiclassical approximation, where quantized matter fields interact with a classical gravitational field (Utiyama and De Witt 1962). In fact, renormalization of the energy-momentum
tensor for a quantum field in four dimensions, indicates that the presence of quadratic terms in the gravitational action is a priori expected (Stelle 1977, 1978, Horowitz and Wald 1978).

However, every quadratic combination of curvature terms is not physically accepted, since its introduction into the gravitational action leads to differential equations of the fourth order with respect to the metric (Farina-Busto 1988) and those higher-derivative terms are associated with ghost particles (Weinberg 1979, Zwiebach 1985). A ghosts-free, non-linear Lagrangian theory of gravity was formulated by Lovelock (Lovelock 1971). He proposed that the most general gravitational Lagrangian is of the form

$$\mathcal{L} = \sqrt{-g} \sum_{m=0}^{n/2} \lambda_m \mathcal{L}^{(m)}$$

where \(\lambda_m\) are arbitrary constants, \(n\) denotes the spacetime dimensions, \(g\) is the determinant of the metric tensor and \(\mathcal{L}^{(m)}\) are functions of the Riemann curvature tensor \(\mathcal{R}_{\mu\nu\kappa\lambda}\) and its contractions \(\mathcal{R}_{\mu\nu}\) and \(\mathcal{R}\), of the form

$$\mathcal{L}^{(m)} = \frac{1}{2m} \delta^{\beta_1\ldots\beta_{2m}}_{\alpha_1\ldots\alpha_{2m}} \mathcal{R}^{\alpha_1\alpha_2} \ldots \mathcal{R}^{\alpha_{2m-1}\alpha_{2m}}_{\beta_2\ldots\beta_{2m}}$$

where \(\delta^{\beta_1\ldots\beta_{2m}}_{\alpha_1\ldots\alpha_{2m}}\) is the generalized Kronecker symbol. In Eq. (2), \(\mathcal{L}^{(1)} = \frac{1}{2} \mathcal{R}\) is the EH Lagrangian, while \(\mathcal{L}^{(2)}\) is a particular combination of the quadratic terms, known as the Gauss-Bonnett (GB) combination, which in four dimensions satisfies the functional relation (Kobayashi and Nomizu 1969)

$$\frac{\delta}{\delta g^{\mu\nu}} \int \sqrt{-g} \left( \mathcal{R}^2 - 4 \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\kappa\lambda} \mathcal{R}^{\mu\nu\kappa\lambda} \right) d^4x = 0$$

An important aspect of a ghosts-free, higher-order gravitational theory arises from the combination of Eqs. (1) and (3). Accordingly, for a gravitational Lagrangian with curvature terms of orders higher or equal than the second \((m \geq 2)\), one needs to have a spacetime of more than four dimensions.

The idea that the spacetime may have more than four dimensions was introduced by Kaluza and Klein in their efford to unify gravity and electromagnetism (Kaluza 1921, Klein 1926). Recently, higher-dimensional theories have been studied as an attractive way to unify all gauge interactions with gravity, in a supergravity scenario (Freund and Rubin 1980, Englert 1982), while established as unavoidable in superstring theories (Forgacs and Horvath 1979, Green et al. 1987). In most higher-dimensional theories of gravity the extra dimensions are assumed to form, at the present epoch, a compact manifold (internal space) of very small size compared to that of the three-dimensional visible space (external space) (Applequist et al. 1987). This size is directly related to the fundamnetal constants and consequently must be stable (Accetta et al. 1986). In this context, it has been proposed that compactification of the internal space may arise as a result of the cosmological evolution (Abbott et al. 1984, Kolb et al. 1984). Recent developments in this topic indicate
that, contraction of the extra dimensions in the EH cosmology, could lead to particle production in the visible space (Maeda 1984a, 1984b, Garriga and Verdaguer 1989).

In the present article we discuss creation of massive scalar particles in the external space, as a quantum consequence of the cosmological evolution in a quadratic, higher-dimensional theory of gravity. In the search for general renormalizability conditions regarding the quantities relevant to particle production, we find that in a $C^\infty$, globally hyperbolic, $n$-dimensional spacetime the total probability ($P$) to produce a particle pair over the entire history of the Universe is a positive, covariant and (what’s most important) finite quantity, provided that the spacetime dimension is an odd number. The same is also true for the vacuum expectation value of the field’s energy-momentum tensor in curved spacetime, since the divergences of $P$ are the same as those that afflict $<0_{\text{out}}|T^{\mu\nu}|0_{\text{in}}>$ (Hartle and Hu 1980, Birrell and Davies 1982). The most simple extrapolation of the four-dimensional spacetime to an odd-dimensional one is a curved background of five dimensions. Accordingly, as a model to perform our computations, we consider a five-dimensional spacetime in which the internal space is reduced to a static size as $t \to \infty$, accompanied by an isotropic expansion of the visible space.

Once the appropriate background has been determined, particle production may be performed explicitly through the solution of the wave equation for a scalar field in curved spacetime under consideration. No general exact solutions of the equation governing the propagation of a quantum field in curved spacetime have been derived so far, within the context of quadratic theories of gravity (Mijic et al. 1985). Nevertheless, in the present paper, we solve explicitly the corresponding Klein-Gordon equation, to obtain exact massive mode solutions in terms of hypergeometric functions. Accordingly, two major aspects of the theory are revealed: (i) The mass-energy spectrum, attributed to the scalar field due to the cosmological evolution, is discrete (and hence the energy is quantized), as a result of compactification of the extra dimension and (ii) the energies of the produced particles are strongly related to the inverse radius of the internal space. Finally, using a high-frequency approximation, which, as we show, corresponds to creation of particles in the neighbourhood of a singular epoch, we arrive at analytic expressions regarding the “observable” number and energy density of the particles produced in four dimensions.

2. The model Universe

We look for solutions of the cosmological field equations which may be obtained, through Hamilton’s principle, from a five-dimensional action with gravitational part of the form $(h = 1 = c)$ (Kleidis and Papadopoulos 1997b)

$$S_{GR} = \frac{1}{L_5} \int \sqrt{-g} \left[ \frac{1}{2\kappa} R + \alpha \left( R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\kappa\lambda} R^{\mu\nu\kappa\lambda} \right) \right] d^5x \quad (4)$$

In Eq. (4), $\kappa = 8\pi G$, where $G$ is the gravitational constant in four dimensions, $\alpha$ is a dimensionless constant and $L_5 = 2\pi R_5$ is a normalization constant, corresponding
to the physical size of the internal space, once it can be considered static (Farina-Busto 1988). A simple cosmological solution with the properties required for particle production has already been obtained in the context of the quadratic theory under consideration (Kleidis and Papadopoulos 1997b). It consists of a five-dimensional spacetime with large scale anisotropic evolution between the two subspaces (the external space and the internal one). This model arises as an exact solution of the gravitational field equations when the minimum value of the external scale function is zero and the matter that fills the Universe has the following equation of state

\[ p_{\text{ext}} = \frac{1}{3} \rho , \quad p_{\text{int}} = 0 \]  

where \( p_{\text{ext}} \) and \( p_{\text{int}} \) are the pressures of the external and the internal space, respectively and \( \rho \) is the five-dimensional matter-energy density. The corresponding line-element is written in the form

\[ ds^2 = -dt^2 + t^2 \left[ \frac{dr^2}{1 + r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] + \left( 1 - \frac{t_s^2}{t^2} \right)^2 (dx^5)^2 \]  

where \( t_s \) is a constant of dimensions \( T \). The model (6) is a Ricci flat solution, i.e. \( \mathcal{R} = 0 \). In this solution, the hyperbolic \((\kappa = -1)\) external space exhibits isotropic expansion, governed by the scale function

\[ g_{ii}(t) = R_{ii}^2(t) = t^2 , \quad i = 1, 2, 3 \]  

Thus, the visible Universe coincides with the asymptotic Milne phase of the open Friedmann-Robertson-Walker (FRW) models. As regards the internal space, it is assumed to be compact, i.e. closed and bounded. Therefore, at each time, the values of the fifth coordinate are restricted to a certain range

\[ 0 \leq x^5 \leq 2\pi R_5 \]  

The time evolution of the extra dimension, governed by the scale function

\[ g_{55}(t) = S_{55}^2(t) = \left( 1 - \frac{t_s^2}{t^2} \right)^2 \]  

consists of an initial contraction, up to a minimum size at \( t = t_s \), followed by a subsequent expansion (at a much lower rate) towards a static size, thus achieving spontaneous compactification as \( t \to \infty \) (see Fig. 1).

For \( t \neq 0 \) the determinant of the metric tensor under consideration vanishes at \( t_s \) and therefore, the volume of the spacelike hypersurface of simultaneity \( t = t_s \) is zero. To determine the nature of the "anomalous" point \( t_s \), we consider the curvature invariants \( R_{\mu\nu}R^{\mu\nu} \) and \( R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} \) (Ryan and Shepley 1975). For \( t = t_s \) we obtain

\[ R_{\mu\nu}R^{\mu\nu} = \infty = R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} \]  

and hence the associated hypersurface corresponds to an essential physical singularity (D'Inverno 1993). In contrast to the initial one, at \( t = 0 \), this singularity exists in the future of the spacetime background. Nevertheless, at \( t = t_s \), the external Hubble parameter remains finite and the corresponding scale function differs from zero.
3. Renormalizability conditions

3.1. The effective action method

The geometrical features of the model (6) are relevant to particle creation due to the cosmological evolution of the extra dimension. Nevertheless, before computing the macroscopic quantities arising from quantum particle production at the one-loop approximation (such as the expectation value of the energy-momentum tensor \( < T_{\mu\nu} > = < 0_{\text{out}} | T_{\mu\nu} | 0_{\text{in}} > \) for the created matter), we need, by a general method, to establish whether these quantities are well-defined and finite (or at least renormalizable) or not. To do so, we consider the effective action method within the context of the path-integral quantization procedure in a \( C^\infty \), \( n \)-dimensional, globally hyperbolic spacetime. The differentiability conditions ensure the existence of differential equations and the global hyperbolicity ensures the existence of Cauchy hypersurfaces.

One way to define the effective action \( (W) \) in a curved spacetime, is by using the identity

\[
\frac{2}{\sqrt{-\nabla}} \nabla W = < 0_{\text{out}} | T_{\mu\nu} | 0_{\text{in}} > \cdot < 0_{\text{out}} | 0_{\text{in}} >
\]

which is the quantum analogue of the classical variational principle, at the one-loop approximation (Birrell and Davies 1982). In this case, one needs only to examine whether \( W \) is finite or not, since the divergences in \( W \) are, of course, the same as those that afflict \( < T_{\mu\nu} > \). In what follows, we treat the number of the spacetime dimensions \( (n) \) as a variable which can be analytically continued throughout the complex plane (dimensional regularization).

The central quantity in the effective action method is the effective action generating functional in the absence of external source currents (free field in a curved spacetime). This gives the vacuum persistence amplitude in a curved background (Fischetti et al. 1979, Hartle and Hu 1979, Hartle and Hu 1980, Birrell and Davies 1982)

\[
Z[0] = < 0_{\text{out}} | 0_{\text{in}} > = \int \mathcal{D}[\Phi] e^{i[S_G(g_{\mu\nu}) + S_m(\Phi)]}
\]

In Eq. (12), \( S_G(g_{\mu\nu}) \) is the gravitational action of the background field \( g_{\mu\nu}(x) \), \( S_m(\Phi) \) is the matter action of the quantum field and \( \mathcal{D}[\Phi] \) is an appropriate functional measure. In the simple theory we are considering, the matter action is no more than quadratic in the quantum field.

In computing the functional integral (12) we may set the background field equal to a solution of the classical field equations, since the error we make in doing so is of two-loop order (Jackiw 1974). Then Eq. (12) is written in the form

\[
Z[0] = e^{iS_G} \int \mathcal{D}[\Phi] e^{-\frac{1}{2} \int d^n y \sqrt{-g(y)} \int d^n x \sqrt{-g(x)} K_{xy} \Phi_x \Phi_y}
\]

In Eq. (13), \( K_{xy} \) denotes the differential operator

\[
K_{xy} = \left[ \Box_x + m^2 - i\epsilon + \xi R \right] \delta^n(x - y) \frac{1}{\sqrt{-g(y)}}
\]

In Eq. (14),
where $m$ is the mass of the field’s quanta and the infinitesimal factor $i\epsilon$ (which is related to Feynman boundary conditions on $\Phi$) is used to make the functional integral convergent. Gaussian functional integration (Ryder 1985) of Eq. (13), yields

$$Z[0] = (2\pi i)^{n/2} e^{iS_G(g_{\mu\nu})} \left[ \text{det} K^{1/2} \right]^{-1} \quad (15)$$

In what follows we omit the numerical factor $(2\pi i)^{n/2}$, since it is constant and it may be absorbed into the measure, a thing that simply amounts to a redefinition of $\mathcal{D}[\Phi]$. In any case, that term is metric-independent and therefore irrelevant to particle production.

The effective action, $W(\Phi)$, is defined by (Fischetti et al. 1979, Hartle and Hu 1979, Hartle and Hu 1980, Birrell and Davies 1982)

$$Z[0] = < 0_{\text{out}} | 0_{\text{in}} > = e^{iW(\Phi)} \Rightarrow W(\Phi) = -i \ln < 0_{\text{out}} | 0_{\text{in}} > \quad (16)$$

Then, from Eq. (15) we obtain

$$W(\Phi) = S_G(g_{\mu\nu}) + \frac{i}{2} tr \ln K \quad (17)$$

The first term appearing in Eq. (17) is the classical gravitational action. The second term represents the one-loop quantum correction to the classical action and is relevant to particle production. Indeed, according to the definition of particle production probability (Itzykson and Zuber 1985), from Eq. (16) we find that the total probability to produce a particle pair over the entire history of the Universe ($P$), is given by the formal expression

$$P = 2 \text{Im} W \quad (18)$$

However, in Eq. (17) the action term $S_G(g_{\mu\nu})$ must be real, since we have considered only classical solutions of the equation that governs the propagation of the gravitational field (Hartle and Hu 1980). Therefore, we obtain

$$P = i \text{tr} \ln K \quad (19)$$

where the operator $K$ is taken to act on a space of vectors, $|x>$, normalized by

$$<x|x'> = \delta^n(x-x') \frac{1}{\sqrt{-g(x)}} \quad (20)$$

According to Eq. (17), if there are any divergences included in the effective action (and therefore in $< 0_{\text{out}} | T_{\mu\nu} | 0_{\text{in}} >$ also), those are contained into $P$. Therefore, it is essential to examine under which conditions this quantity may become either finite or renormalizable, in the one-loop approximation.
3.2. Renormalizability criteria

Variation of both sides of Eq. (19) leads to
\[ \delta P = i \text{tr} \left[ K^{-1} \delta K \right] \]  
(21)

Now, the identity
\[ K^{-1} = i \int_0^\infty ds \ e^{-iKs} \]  
(22)

can be used, to obtain
\[ \delta P = -\text{tr} \int_0^\infty ds \ e^{-iKs} \delta K \]  
(23)

which, since both \( \delta \) and \( K \) are independent of \( s \), leads to
\[ \delta P = -i \delta \left[ \text{tr} \int_0^\infty \frac{ds}{s} e^{-iKs} \right] \]  
(24)

Accordingly, Eq. (24) leads to the definition
\[ P = -i \text{tr} \int_0^\infty \frac{ds}{s} e^{-iKs} \]  
(25)

up to an arbitrary integration constant, which in any case may be fixed by renormalization and therefore can be ignored. The trace of the operator \( e^{-iKs} \), which acts on the space of vectors given by Eq. (20), is defined by (Birrell and Davies 1982)
\[ \text{tr} e^{-iKs} = \int d^n x \sqrt{-g} \ < x | e^{-iKs} | x > \]  
(26)

Now, in order to make sense of the formal expression (25) [or (19)] we need a representation for the operator \( e^{-iKs} \). To do so, we recall that the combination of Eq. (14) with the normalization condition
\[ \int d^n y \frac{1}{\sqrt{-g(y)}} K_{xy} K_{yz}^{-1} = \delta^n(x-z) \frac{1}{\sqrt{-g(z)}} \]  
(27)

and the equation for the Feynman propagator \( G_F(x, x') \) in a curved spacetime
\[ \left[ \square_x + m^2 - i\epsilon + \xi \mathcal{R} \right] G_F(x, x') = -\frac{1}{\sqrt{-g(x)}} d^n(x-x') \]  
(28)

(Birrell and Davies 1982) lead to
\[ K_{x,x'}^{-1} = G_F(x, x') \]  
(29)

In this case, it is the existence of the \( i\epsilon \) factor into Eq. (28), which ensures that \( G_F(x, x') \) represents the expectation value, in some set of states, of a time-ordered product of fields
\[ G_F(x, x') = < x | G_F | x' > \]  
(30)
If we, furthermore, use Eqs (29) and (30) together with the identity (22), we obtain

\[ G_F(x, x') = -i \int_0^\infty ds < x|e^{-iKs}|x' > \] (31)

For small geodesic distances between two events, the Feynman Green function \( G_F(x, x') \) in curved spacetime, has a nice asymptotic representation, called the De Witt-Schwinger representation (De Witt 1975, Bunch and Parker 1979)

\[ G_F(x, x') = \frac{1}{(4\pi)^{n/2}} \Delta^{1/2}(x, x') \int_0^\infty (is)^{-n/2} e^{-im^2s + \frac{\sigma^2}{2m}} F(x, x'; is) \, ds \] (32)

where \( \sigma = \frac{1}{2} \xi_\alpha \xi^\alpha s^2 \) is half of the square of the proper distance between \( x \) and \( x' \) (\( \xi^\alpha \) is the unit vector tangent to the geodesic between \( x \) and \( x' \), at the point \( x' \)) and \( \Delta(x, x') \) is the Van Vleck determinant, defined by

\[ \Delta(x, x') = -det \{ \partial_\mu \partial_\nu \sigma(x, x') \} \frac{1}{\sqrt{g(x)g(x')}} \] (33)

Notice that, in normal coordinates (\( y^\mu = \xi^\mu s \)) around \( x' \), \( \Delta \) reduces to \( [g(x)]^{-1/2} \). The function \( F(x, x'; is) \) is given by the asymptotic expansion (De Witt 1975)

\[ F(x, x'; is) = \sum_{j=0}^{\infty} \alpha_j(x, x') (is)^j \] (34)

with

\[ \alpha_0(x, x') = 1 \]
\[ \alpha_1(x, x') = \left( \frac{1}{6} - \xi \right) R - \frac{1}{2} \left( \frac{1}{6} - \xi \right) R_{\mu\nu} y^\mu y^\nu - \frac{1}{3} \alpha_{\mu\nu} y^\mu y^\nu \]
\[ \alpha_2(x, x') = \frac{1}{2} \left( \frac{1}{6} - \xi \right)^2 R^2 + \frac{1}{3} \alpha_0^2 \] (35)

where

\[ \alpha_{\mu\nu} = \frac{1}{2} \left( \frac{1}{6} - \xi \right) R_{\mu\nu} - \frac{1}{120} R_{\mu\rho\nu\sigma} - \frac{1}{30} \alpha_0^2 \] (36)

and the rest of the \( \alpha_j \)'s (for \( j > 2 \)) are given by certain recursion relations (De Witt 1975, Christensen 1976). Now, using Eqs. (31) and (32) we obtain

\[ < x|e^{-iKs}|x' > = \frac{i}{(4\pi)^{n/2}} \Delta^{1/2}(x, x') (is)^{-n/2} e^{-im^2s + \frac{\sigma^2}{2m}} F(x, x'; is) \] (37)
and the combination of Eqs. (25), (26) and (37) leads to

\[ P = \lim_{x \to x'} \int d^n x \sqrt{-g} \left( \frac{1}{(4\pi)^{n/2}} \Delta^{1/2}(x, x') \right) \int_0^\infty (is)^{-n/2} e^{-im^2 s + \frac{m^2}{s}} \mathcal{F}(x, x'; is) \frac{ds}{s} \]  

(38)

If the number of the spacetime dimensions can be treated as a variable which can be analytically continued throughout the complex plane, then we may take the \( x \to x' \) limit, to obtain

\[ P = i \int d^n x \sqrt{-g} \left( \frac{1}{(4\pi)^{n/2}} \right) \sum_{j=0}^\infty \alpha_j(x) \int_0^\infty (is)^{j-\frac{n}{2}} e^{-im^2 s} ds \]  

(39)

which yields

\[ P = \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^\infty m^{n-2j} \Gamma(j - \frac{n}{2}) \int d^n x \sqrt{-g} \alpha_j(x) \]  

(40)

where, now, the quantities \( \alpha_j \) become

\[ \alpha_j(x) = \alpha_j(x, x')|_{s=0} \sim \mathcal{R}_j(x) \]  

(41)

i.e. polynomials in scalar combinations of the Riemann curvature tensor and its contractions. Therefore, the particle production probability is manifestly positive and covariant under real coordinate transformations. Nevertheless, from Eq. (40) it is evident that \( P \) may diverge, because of poles in the \( \Gamma \)-functions. Notice, for instance, that for \( n = 4 \) the first three terms in Eq. (40) are divergent.

The question that arises now is, whether there exists some special fixing of the parameters involved, in order to make the particle production probability finite for every value of the index \( j \) or at least, as far as we are concerned, up to \( j = 2 \) i.e. for a gravity theory quadratic in the curvature tensor. We conclude that the simplest fixing of this sort is that the spacetime dimension is an odd number. Then, the r.h.s. of Eq. (40) is not only renormalizable, but actually finite. According to Eqs. (17) and (11), this is also true for both the vacuum persistence amplitude and the vacuum expectation value of the energy-momentum tensor for a scalar field in a \( C^\infty \), globally hyperbolic, \( n \)-dimensional spacetime. Since in a curved background \( \left| 0_{\text{out}} \right> \neq \left| 0_{\text{in}} \right> \) even in the absence of external current sources, we are forced to consider an odd-dimensional spacetime in order to deal with a finite, well-defined probability of particle production, without using any renormalization technique. The most simple extrapolation of the four-dimensional spacetime to an odd-dimensional one, is a curved background of five dimensions.

Notice that, for \( n \) odd, in Eq. (40) we have \( P \neq 0 \) only as long as \( m \neq 0 \) and hence, no massless particles are actually produced. This result is probably a mathematical template, related to the short-distance approximation used in the derivation of of the representation (32) for Feynman’s Green function in curved spacetime (De Witt 1975, Christensen 1976, Bunch and Parker 1979). Nevertheless, short distances are probed by the high-frequency modes of a quantum field (Birrell and Davies 1982).
and a high-frequency approximation may become quite accurate as regards particle production in the neighbourhood of the singular epoch \( t = t_s \).

Indeed, when a particle is created at a proper distance \( d_s \) from a singularity, it should "fit" inside the corresponding area (Nesteruk 1991) and therefore its wavelength should satisfy the condition \( \lambda \leq d_s \). Accordingly, for the metric (6) we obtain

\[
\omega^2 \geq 4\pi^2 \frac{t_s^4}{(t/ t_s - 1)^2} \tag{42}
\]

In the immediate vicinity of the singularity, that is when \( t \to t_s \), Eq. (42) results to \( \omega^2 \gg 1 \) and therefore, we are led to examine only the high-frequency behaviour of the quantum field under consideration. It is therefore probable that no massless scalar quanta are created in the neighbourhood of the singular epoch \( t = t_s \).

On the other hand, the fact that no particles are produced in the external space of the metric (6) when \( m = 0 \), is in complete agreement with a corresponding result obtained using thermodynamical considerations (Kleidis and Papadopoulos 1997a). Indeed, the metric (6) is an exact solution of the classical gravitational field equations when the pressure of the internal (one-dimensional) space equals to zero (\( p_{int} = 0 \)). According to Kleidis and Papadopoulos (1997a), as regards the thermodynamical description of two subspaces (the external space and the internal one), this condition corresponds to an adiabaticity criterion and therefore, no radiation (massless particles) is allowed to be transferred from one space to the other. Both subspaces represent closed thermodynamical systems (see also Prigogine et al 1989). However, when \( m \neq 0 \), we do not have exchange of energy (radiation) between the two subspaces, but exchange of matter (particles) where the total number of particles in each space is no longer constant (\( \delta N \neq 0 \)). Then, the problem is reduced to that of the open thermodynamical systems, which allows for particle production in the visible space (Prigogine 1961, Prigogine et al. 1989, Kleidis and Papadopoulos 1997a).

In what follows, we solve explicitly the resulting Klein-Gordon wave equation, to obtain exact mode solutions for the scalar field in the five-dimensional spacetime under consideration, in order to study massive particle creation in four dimensions due to the cosmological evolution of the fifth dimension. It has been recently shown that the four-dimensional external space of the model (6) is stable against perturbations in the energy density (Kleidis et al. 1996) and hence its dynamics will be driven by other factors than just particle production. Accordingly, we shall ignore backreaction of the created particles on the external background.

### 4. Particle creation

#### 4.1. Quantum field in curved spacetime

At the one-loop level, the (semiclassical) interaction between a massive quantum scalar field \( \Phi(t, \vec{x}) \) and a classical gravitational one (the five-dimensional spacetime
under consideration), may be determined through Hamilton’s principle, involving the action scalar (Calan et al. 1970, Birrell and Davies 1982)

\[ S_m(\Phi) = \frac{1}{2} \int \sqrt{-g(x)} \left[ g^{\mu \nu}(x) \Phi,_{\mu}(x) \Phi,_{\nu}(x) - \left( m^2 + \xi \mathcal{R}(x) \right) \Phi^2(x) \right] d^5x \]  

(43)

where \( m \) is the five-dimensional mass of the field’s quanta (Kleidis and Papadopoulos 1997a) and \( \xi \) is a dimensionless coupling constant. Euler variation of Eq. (43) leads to the wave equation that governs the propagation of the massive scalar field in a five-dimensional spacetime (Friedlander 1975).

The quantization of the field \( \Phi(t, \vec{x}) \) is performed by imposing canonical commutation relations on the hypersurface \( t = \text{constant} \) (Isham 1975, 1981)

\[ [\Phi(t, \vec{x}) , \Phi(t, \vec{x}')] = [\pi(t, \vec{x}) , \pi(t, \vec{x}')] = 0 \]

\[ [\Phi(t, \vec{x}) , \pi(t, \vec{x}')] = i\delta^{(3)}(\vec{x} - \vec{x}') \]  

(44)

where \( \pi(t, \vec{x}) \) is the momentum canonically conjugated to the field \( \Phi(t, \vec{x}) \). Once an appropriate definition of the ”in” and the ”out” vacuum states is given (Fulling 1979), the quantum field operator, \( \Phi(t, \vec{x}) \), may be expanded in terms of creation (\( A_{k,k_5}^\dagger \)) and annihilation (\( A_{k,k_5} \)) operators (Birrell and Davies 1982, Garriga and Verdaguer 1989), as follows

\[ \Phi(t, \vec{x}) = \frac{1}{4\pi^2} \sum_{k_5=0}^{\infty} \int d\mu(k) \left[ A_{k,k_5} u_{k,k_5}(t, \vec{x}) + A_{k,k_5}^\dagger u^*_{k,k_5}(t, \vec{x}) \right] \]  

(45)

where \( u_{k,k_5} \) is a complete, orthonormal set of field modes (the solutions of the equation of propagation) and \( d\mu(k) \) an appropriate measure depending on the FRW model considered for the external space (Grib et al. 1976, 1980). In Eq. (45) \( k, k_5 \) are quantum numbers labeling a particular mode. The resulting spectrum is continuous in \( k \) (\( 0 \leq k < \infty \)) and discrete in \( k_5 \). In fact, since the internal space is assumed compact, we have

\[ k_5 = \frac{\ell}{R_5} \]  

(46)

where \( R_5 \) is the ”radius” of the internal space and \( \ell \) an integer. In what follows, we focus attention on modes with \( k_5 \neq 0 \) which may be associated to massive particles in the four-dimensional sense (e.g. see Garriga and Verdaguer 1989).

To interpret the quantized field \( \Phi(t, \vec{x}) \) in terms of particles, we need to give an appropriate definition of the positive frequency modes, \( u_{k,k_5}(x) \), in the ”in” and the ”out” vacuum states (Birrell and Davies 1982). For most cosmological models it is difficult to find a natural candidate for the in-modes (Isham 1981), due to the ill-defined nature of the particle concept in a curved spacetime (Wald 1984). This difficulty is usually circumvented by assuming that before some initial time \( t_{in} \), larger than the Planck time \( t_{Pl} \), the spacetime is matched to be static (Hu 1974). This is justified by the fact that at the Planck epoch, the semiclassical approach which
was taken into account, must break down. Therefore, on mathematical grounds, in order to discuss particle creation we need to exclude the influence of the period $0 < t < t_{pl} \sim 10^{-43} \, \text{sec}$ and to give an unambiguous definition for the positive frequency modes $u_{k,k_5}^{in}$. Hence, it seems reasonable to assume a static in-metric. Then, the in-vacuum is equivalent to the Minkowski vacuum (Candelas and Dowker 1979, Birrell and Davies 1982). As regards the out-modes we note that, as $t \to \infty$ (out-region), the internal scale function becomes static, $S^2(t) \to 1$, while the external one varies at a very low rate

$$\frac{d\ell}{dt} \left( \frac{\dot{R}}{R} \right) \to 0 \quad , \quad \ell \geq 0$$ (47)

and therefore, the visible space evolves adiabatically (Fulling et al 1974, Davies and Fulling 1977, Bunch et al 1978). Then, the out-vacuum state corresponds to an "adiabatic vacuum", where the positive frequency modes are well-defined and no particles are created (see also Birrell and Davies 1982, Garriga and Verdaguer 1989).

Those two vacuum states are related by means of a Bogolubov transformation (Bogolubov 1958)

$$u_{k,k_5}^{in} = \alpha_{k,k_5}(t) u_{k,k_5}^{out} + \beta_{k,k_5}(t) u_{-k,-k_5}^{out}$$ (48)

where the functions $\alpha_{k,k_5}(t)$ and $\beta_{k,k_5}(t)$ are the Bogolubov coefficients. Initially (in the in-region), we have $\alpha_{k,k_5}(t_{in}) = 1$ and $\beta_{k,k_5}(t_{in}) = 0$ (Zel’dovich and Starobinsky 1972, Hu and Parker 1978). As $t \to \infty$ (in the out-region), we expect that these functions reduce to constant values, different from those at $t = t_{in}$, which give us the linear combination of positive and negative frequency out-modes that makes up a positive frequency in-mode. Then, the number of particles created in the mode $(k, k_5)$ is given by (Birrell and Davies 1982)

$$N_{k,k_5} = |\beta_{k,k_5}|^2$$ (49)

Therefore, in order to calculate the number of particles created in the spacetime (6), we need to determine the Bogolubov coefficient $\beta_{k,k_5}(t)$ at $t = \infty$. By virtue of Eq. (48), this may be performed by solving the equation of propagation for the quantum scalar field in the curved spacetime under consideration.

Minimizing the variation of the action (43) with respect to $\Phi(t, \vec{x})$, we are led to the following equation of propagation for the quantum field

$$\Box u_{k,k_5}(x) + \left[ m^2 + \xi R(x) \right] u_{k,k_5}(x) = 0$$ (50)

To separate the variables in Eq. (50), we substitute

$$u_{k,k_5}(\tau, \vec{x}, \vec{x}^5) = \frac{1}{R(\tau)S^{1/2}(\tau)} Y_k(\vec{x}) \chi_{k,k_5}(\tau) e^{ik_5 x^5}$$ (51)

where $\vec{x} = (r, \theta, \phi)$ and

$$\tau = \int_{t_s}^t \frac{dt}{R(t)} = \ln \left( \frac{t}{t_s} \right)$$ (52)
is the conformal time, with its origin placed at $t = t_s$. In Eq. (51), the quantities $\mathcal{Y}_k(\vec{x})$ are the eigenfunctions of the operator $\Delta^{(3)}$:

$$\Delta^{(3)} \mathcal{Y}_k(\vec{x}) = -(k^2 + 1) \mathcal{Y}_k(\vec{x}) \quad (53)$$

which corresponds to the Laplacian associated with the hyperbolic spatial metric of the external space, in the model (6). The harmonics $\mathcal{Y}_k(\vec{x})$ are normalized according to

$$\int d^3x \sqrt{g} \mathcal{Y}_k(\vec{x}) \mathcal{Y}_{k'}^*(\vec{x}) = \delta(k, k') \quad (54)$$

where $\sqrt{g}$ is the determinant associated to the three-dimensional external metric and $\delta(k, k')$ is the $\delta$-function with respect to the measure $\mu(k)$

$$\int d\mu(k) f(k') \delta(k, k') = f(k) \quad (55)$$

In this case, the eigenfunctions of the three-dimensional Laplacian are given by (Parker and Fulling 1974)

$$\mathcal{Y}_k(\vec{x}) = \Pi^{(-)}_{k,J}(r) Y^M_J(\theta, \phi) \quad (56)$$

where $k = (k, J, M)$ and

$$0 \leq k < \infty$$

$$J = \quad 0, 1, 2, \ldots$$

$$M = \quad -J \ldots 0 \ldots J \quad (57)$$

In Eq. (56), the functions $Y^M_J(\theta, \phi)$ are spherical harmonics. As regards the functions $\Pi^{(-)}_{k,J}(r)$, they are defined by (Lifshitz and Khalatnikov 1963, Bander and Itzykson 1966, Ford 1976)

$$\Pi^{(-)}_{k,J}(r) = \left\{ \frac{1}{2} \pi k^2 (k + 1)^2 \cdots [k^2 + (2 + J)^2] \right\}^{-1/2} \sinh \chi \left( \frac{d}{d \cosh \chi} \right)^{1+J} \cosh k \chi \quad (58)$$

where $\chi = \sinh r^{-1}$. Finally, the measure $\mu(k)$ is defined as follows

$$\int d\mu(k) = \int_0^\infty \sum_{J,M} \quad (59)$$

4.2. Particle creation and the mass-energy spectrum

Taking into account the above definitions and introducing Eq. (51) into (50), we obtain the following equation for the time-dependent part of the modes $\chi_{k,k_5}(\tau)$

$$\chi''_{k,k_5}(\tau) + \left[ k^2 + R^2(\tau) \left( \frac{k_5^2}{S^2(\tau)} - m^2 \right) + \frac{1}{(e^{2\tau} - 1)^2} \right] \chi_{k,k_5}(\tau) = 0 \quad (60)$$
where $R^2(\tau)$ and $S^2(\tau)$ are the scale functions of the external and the internal space respectively, in terms of the conformal time and a prime denotes differentiation with respect to $\tau$. Notice that, since the metric (6) is Ricci flat, the propagation of the scalar field in the curved spacetime under consideration is no longer dependent on the value of the coupling constant $\xi$. The orthonormality of the modes $u_{k,k_5}(x)$, according to the scalar product in curved spacetime (Birrell and Davies 1982), in connection to normalization conditions (54) and (55), result to the following Wronskian relation on $\chi_{k,k_5}(\tau)$

$$\chi_{k,k_5}' \chi_{k,k_5} - \chi_{k,k_5}' \chi_{k,k_5}^* = i$$  

(61)

Eq. (60) is essentially five-dimensional. In order to study massive particle creation in the visible space of the metric (6), at first, we have to determine the four-dimensional analogue of the five-dimensional mass for the field’s quanta. It has been recently suggested that, in the context of a $1 + 3 + D$-dimensional spacetime (where a $D$-dimensional, compact internal space is present), the matter-energy content of a volume element in three dimensions is given by (Kleidis and Papadopoulos 1997a)

$$\mathcal{E}_3 = \mathcal{E}_{3+D} S^D$$  

(62)

where $\mathcal{E}_{3+D}$ is the corresponding matter-energy content in $3 + D$ dimensions and $S^D$ is the proper volume of the (closed and bounded) internal space. In this case, the overall matter-energy content of a $3 + D$-dimensional volume element is being projected onto its three-dimensional counterpart (see also Kolb et al. 1984, Farina-Busto 1988), which is in complete agreement with the definition of the stress-energy tensor on a hypersurface (Misner et al. 1973). In the absence of the extra dimensions ($D = 0$) we obtain $\mathcal{E}_3 = \mathcal{E}_{3+D}$, corresponding to the matter-energy content of a volume element in a four-dimensional spacetime. Accordingly, we consider the volume element occupied by a subatomic particle in five-dimensions ($D = 1$) and its four-dimensional projection. Then, Eq. (62) leads to the following relation between the mass-energy of a particle in a five-dimensional spacetime ($m$) and its four-dimensional counterpart ($m_4$)

$$m_4 = m S$$  

(63)

In this case, $m_4$ may be identified as the phenomenological mass of a particle in four dimensions, when a compact, one-dimensional, time-dependent internal space is present (in this respect, see also Wesson 1992, Mashhoon et al 1994). Now, inserting Eq. (63) into Eq. (60), we obtain

$$\chi_{k,k_5}'''(\tau) + \left[ k^2 + \frac{R^2}{S^2} \left( k_5^2 - m_4^2 \right) + \frac{1}{(e^{2\tau} - 1)^2} \right] \chi_{k,k_5}(\tau) = 0$$  

(64)

The solutions of the wave equation (64) describe the exitation of four-dimensional, massive scalar modes, which takes place as a result of the large scale anisotropic cosmological evolution between the internal and the external space. This is due to the fact that these modes possess an additional, non-vanishing momentum component
along the fifth dimension \((k_5 \neq 0)\). Then, at times \(-\infty < \tau < 0\) where the internal space contracts at high rate, the \textit{physical frequency} of the scalar field along the extra dimension

\[
\omega_5(k_5) = \frac{k_5}{S(\tau)} \tag{65}
\]

is blueshifted, to bring about particle creation at high energies, in the neighbourhood of the singular epoch \(\tau = 0\) \((t = t_s)\) [e.g. see Eq. (42)]. The energies of the created particles are subsequently decreased in the region \(0 < \tau < +\infty\), where the internal space expands at a low rate (Fig. 1) and the fifth-dimensional frequency component of the modes is accordingly redshifted.

Eq. (64) is reminiscent of the classical equation of motion for a harmonic oscillator, where the coefficient of the linear term \(\chi_{k,k_5}(\tau)\) may be considered as representing a time-dependent frequency parameter. At \(\tau \to -\infty\) (the in-region), as well as at \(\tau \to +\infty\) (the out-region), this parameter must settle down to non-negative constant values, since no mode excitation takes place in these regions. As \(\tau \to \pm\infty\), the modes of the quantum field are the well-defined positive frequency modes and no particles are created (Birrell and Davies 1982). Accordingly, in the search for solutions of Eq. (64) that exhibit a nice-behaviour in both the ”in” and the ”out” vacuum states, we demand

\[
k_5 - m_4 = 0 \tag{66}
\]

or in physical units

\[
m_4c^2 = \hbar c k_5 \tag{67}
\]

which, by virtue of Eq. (46), is written in the form

\[
m_4c^2 = \ell \frac{\hbar c}{R_5}, \quad \ell = 1, 2, ... \tag{68}
\]

thus, resulting in

\[
m_4c^2 = \ell \times 3.1638 \times 10^{-17} \frac{1}{R_5} \text{ (erg)} \tag{69}
\]

or

\[
m_4c^2 = \ell \times 1.977375 \times 10^{-11} \frac{1}{R_5} \text{ (MeV)} \tag{70}
\]

where, in both cases, \(R_5\) is measured in \(cm\).

Eq. (68) represents the energy spectrum attributed to the scalar field in four dimensions as a consequence of the cosmological evolution of the five-dimensional spacetime under consideration. We see that, because of condition (66), two major aspects of the theory are revealed: \(i\) The spectrum is discrete (and therefore the energy is quantized) as a result of compactification of the extra dimension. \(ii\) The mass-energy of the particles produced (the quanta of the scalar field) is related to the inverse radius of the internal space and therefore it is crucially dependent on the exact value of \(R_5\).
By virtue of condition (66), the equation which describes the creation of massive particles in the visible space of the spacetime (6), is written in the form

$$
\chi''_{k,k_5}(\tau) + \left[ k^2 + \frac{1}{(e^{2\tau} - 1)^2} \right] \chi_{k,k_5}(\tau) = 0 \quad (71)
$$

Notice that the corresponding analysis is no longer dependent on the four-dimensional mass parameter ($m_4$). Eq. (71) may be written in a more convenient form, as follows

$$
\chi''_{k,k_5}(\tau) + \left[ k^2 + \frac{1}{4} (1 - \coth \tau)^2 \right] \chi_{k,k_5}(\tau) = 0 \quad (72)
$$

To solve Eq. (72), we perform a substitution in the dependent variable, of the form

$$
\chi_{k,k_5}(\tau) = e^{-i\Omega(\tau)} F(\tau) \quad (73)
$$

where

$$
\Omega(\tau) = \omega_+ \tau + \omega_- \ln (\sinh \tau) \quad (74)
$$

and

$$
\omega_+ = \frac{1}{2} \left( \sqrt{k^2 + 1 + k} \right) \quad (75)
$$

$$
\omega_- = \frac{1}{2} \left( \sqrt{k^2 + 1 - k} \right) \quad (76)
$$

together with the following substitution in the independent variable

$$
z = \frac{1}{2} (1 + \coth \tau) \quad (77)
$$

Then, Eq. (72) is reduced to a hypergeometric equation for $F(z)$ of the form

$$
z(1 - z) \frac{d^2 F}{dz^2} + [c - (a + b + 1)z] \frac{dF}{dz} - abF = 0 \quad (78)
$$

where

$$a = \frac{1}{2} + i\omega_- = b \quad (79)
$$

and

$$c = 1 - ik \quad (80)
$$

This equation can be solved analytically. Since the function $\coth \tau$ consists of two irrelative branches (one for $\tau < 0$ and the other for $\tau > 0$), the corresponding analysis is accordingly separated in two cases, one for each sign of $\tau$. Then, the time-dependent part of the normalized modes which behave like the positive frequency Minkowski modes in the remote past ($\tau < 0$ and $\tau \to -\infty$), is

$$
\chi^{in}_{k,k_5}(\tau) = \frac{1}{\sqrt{2\omega_-}} e^{-i[\omega_+ \tau + \ln (\sinh \tau)]} \\
\times {}_2F_1 \left( 1 + i\omega_-, \frac{1}{2} + i\omega_- ; 1 - i\omega_- ; \frac{1}{2} [1 + \coth \tau] \right) \quad (81)
$$
where \( \omega_0 = k \). On the other hand, the time-dependent part of the modes which behave like positive frequency modes in the remote future (\( \tau > 0 \) and \( \tau \to \infty \)), is found to be

\[
\chi_{k,k_5}^{out}(\tau) = \frac{1}{\sqrt{2\omega_+}} e^{-i[\omega_+\tau + \ln(\sinh\tau)]} \\
\times {}_2F_1\left( \frac{1}{2} + i\omega_-, \frac{1}{2} + i\omega_- ; 1 + i\omega_+ ; \frac{1}{2}[1 - \coth\tau] \right)
\]

(82)

where \( \omega_+ = \sqrt{k^2 + 1} \). Clearly, the solutions (81) and (82) are not equal, which means that the Bogolubov coefficient \( \beta_{k,k_5}(\tau) \) in Eq. (48) is non-vanishing. To determine its value at \( \tau \to \infty \) we use the linear transformation properties of hypergeometric functions (Abramowitz and Stegun 1970). As regards the modes (81) and (82), we verify that Eq. (48) holds, provided that

\[
\alpha_{k,k_5} = \sqrt{\frac{\omega_+}{\omega_-}} \frac{\Gamma(1 - i\omega_-) \Gamma(-i\omega_+)}{\Gamma\left(\frac{1}{2} - i\omega_+\right)^2}
\]

(83)

and

\[
\beta_{k,k_5} = \sqrt{\frac{\omega_+}{\omega_-}} \frac{\Gamma(1 - i\omega_-) \Gamma(i\omega_+)}{\Gamma\left(\frac{1}{2} + i\omega_-\right)^2}
\]

(84)

Furthermore, using the properties of the Gamma function in a complex plane (Abramowitz and Stegun 1970), we find that the number of massive particles created in the \textit{out-region}, in the mode denoted by \((k, k_5)\), is

\[
N_{k,k_5} = |\beta_{k,k_5}|^2 = \frac{\cosh^2(\pi\omega_-)}{\sinh(\pi\omega_-) \sinh(\pi\omega_+)}
\]

(85)

and that the following normalization condition holds

\[
|\alpha_k|^2 - |\beta_k|^2 = 1
\]

(86)

as a consequence of Eq. (61).

Since the quantum state chosen for \( \tau \to -\infty \) corresponds to the \textit{in-vacuum}, in the \textit{out-region} the number density per unit proper volume and the corresponding energy density (in physical units) of the massive particles created in the ordinary space, in a particular mode \( k_5 \), are (Grib et al 1976, 1980, Birrell and Davies 1980, 1982, Anderson and Parker 1987, Garriga and Verdaguer 1989, Laciana 1993)

\[
\eta = \frac{1}{2\pi^2 c^3 R^3} \int_0^\infty k^2 |\beta_{k,k_5}|^2 dk
\]

(87)

and

\[
\rho = \frac{\hbar}{2\pi^2 c^3 R^4} \int_0^\infty k^3 |\beta_{k,k_5}|^2 dk
\]

(88)
where, in our case, $R(t)$ is given by Eq. (7).

Analytic evaluation of the integrals (87) and (88), to estimate the corresponding physical quantities, may be carried out only in the ultraviolet limit, where $k^2 \gg 1$. According to Eq. (42), this approximation corresponds to creation of particles in the immediate neighbourhood of the singularity at $t = t_s$ ($\tau = 0$). In this case, the number of massive particles created in the mode $(k, k_5)$, may be written in the form

$$N_{k,k_5} = |\beta_{k,k_5}|^2 \simeq \frac{1}{\sinh^2(\pi k)}$$  \hspace{1cm} (89)

Then, Eq. (87) becomes

$$\eta = \frac{1}{2\pi^2 c^3 R^3} \int_0^\infty k^2 \frac{1}{\sinh^2(\pi k)} \, dk$$  \hspace{1cm} (90)

which leads to (Gradshteyn and Ryzhik 1965)

$$\eta = \frac{1}{12\pi^3 c^3 t_s^4} e^{-3\tau} \left( \frac{\text{particles}}{\text{cm}^3} \right)$$  \hspace{1cm} (91)

where $t_s$ is measured in sec. Accordingly, the energy density of the massive particles created in ordinary space in the neighbourhood of the singular epoch $t = t_s$, is

$$\rho = \frac{\hbar}{2\pi^2 c^3 R^4} \int_0^\infty k^3 \frac{1}{\sinh^2(\pi k)} \, dk$$  \hspace{1cm} (92)

which results to (Gradshteyn and Ryzhik 1965)

$$\rho = \frac{3\hbar}{4\pi^6 c^3 t_s^4} \zeta(3) e^{-4\tau} \left( \frac{\text{erg}}{\text{cm}^3} \right)$$  \hspace{1cm} (93)

where $\zeta(3)$ is Riemann’s Zeta function of argument 3 ($\zeta(3) = 1.202$).

We see that, in the out-region (for $\tau \to \infty$) both $\eta$ and $\rho$ are exponentially suppressed. This is not an unexpected result. The production of high-mass particles should be exponentially small, because of the large amount of energy which must emerge from the changing gravitational field to supply the particles’ rest mass. Indeed, for modes with $k_5 \neq 0$, the physical frequency in the fifth dimension is strongly blueshifted during the contraction branch of the internal space (for $-\infty < \tau < 0$) and the particles produced at $\tau = 0$ acquire energies of the order of Planck mass. Consider, for instance, that the radius of the internal space is of the order of Planck length, $R_5 = l_{Pl} = 1.616 \times 10^{-33}$ cm, as indicated by modern Kaluza-Klein theories (Applequist et al 1987, Kolb and Turner 1990). Then, Eq. (70) results in an energy scale for the created particles, of the order

$$m_4c^2 = \ell \times 1.223 \times 10^{19} \text{ GeV}$$  \hspace{1cm} (94)
which corresponds to the typical energy scale at the Planck epoch (Planck mass). Eq. (94) raises a problem because (when \( R_5 = l_{Pl} \)), even if a single massive mode is excited, its contribution to the energy density becomes so large that backreaction should be taken into account and the cosmological model (6) that we have started with, breaks down.

Nevertheless, we expect that both the number and the energy density of the particles emerging in four dimensions, during the whole creation process (for \(-\infty < \tau < +\infty\)), are subsequently redshifted due to the simultaneous cosmological expansion of the visible space. In this context, the total number and energy density of the ”real” particles present at late times \( t \), are given by the sum of all the increments, \( \delta \eta(t) \) and \( \delta \rho(t) \) respectively, which were created at earlier times \( t' < t \) and being redshifted by a factor \( R(t')/R(t) \). Therefore, as regards the total number density of the ”real” particles present at late times, we have (Hu and Parker 1977, 1978, Nesteruk and Pritomanov 1990, Nesteruk 1991)

\[
\eta_R = \int_{t_{in}}^{t} \left[ \frac{R(t')}{R(t)} \right]^3 \left[ -\frac{d}{dt'} n(t') \right] dt' \tag{95}
\]

which gives

\[
\eta_R(t) = \frac{1}{4\pi^3 c^3} \frac{1}{t^3} \ln \left( \frac{t}{t_{in}} \right) \tag{96}
\]

while, as regards the corresponding energy density, we have

\[
\rho_R = \int_{t_{in}}^{t} \left[ \frac{R(t')}{R(t)} \right]^4 \left[ -\frac{d}{dt'} \rho(t') \right] dt' \tag{97}
\]

from which we obtain

\[
\rho_R(t) = \frac{3h}{\pi^6 c^6} \zeta(3) \frac{1}{t^4} \ln \left( \frac{t}{t_{in}} \right) \tag{98}
\]

Eqs. (96) and (98) represent the quantities which may be identified as the ”observable” number and energy density of the created particles in the (four-dimensional) out-region. Both \( \eta_R \) and \( \rho_R \) correspond to high-frequency modes, representing particles created in the immediate neighbourhood of a singular epoch (a narrow time-interval around \( t = t_s \)). Indeed, from Fig. 2 we see that those quantities are highly localized in time, around \( t = 1.4 t_{in} \), which may be considered as representing the singular epoch \( t = t_s \). For \( t \to t_{in} \), as well as for \( t \to \infty \), both \( \eta_R \) and \( \rho_R \) vanish and the process of particle production ceases, as expected.

Finally, we have to point out that, because of the low rate expansion experienced by the internal space for \( t > t_s \), the frequency component of the modes along the extra dimension \( \omega_5(k_5) \), is redshifted in the corresponding time-interval. Therefore, we expect that the actual values of the observable quantities \( \eta_R \) and \( \rho_R \) will be a little bit lower than those predicted by Eqs. (96) and (98).
5. Discussion and Conclusions

In the present article we have considered massive scalar particle production from vacuum, as a result of the anisotropic evolution of a five-dimensional cosmological model, arising from gravitational theories of quadratic Lagrangians.

The spacetime under consideration is an exact classical solution of the cosmological field equations, obtained from a gravity theory which includes terms quadratic in curvature tensor (Kleidis and Papadopoulos 1997b), when the matter that fills the Universe is in the form of a closed or heterotic superstring perfect gas (Matsuo 1987). The manifold consists of two anisotropically evolving subspaces (the external space and the internal one) and contains two singularities. The first one, at \( t = 0 \), is essential and is excluded from the analysis, together with the Planck epoch for which we have no information (Hu 1974, Grib et al 1980, Wald 1984). On the contrary, the second singularity, at \( t = t_s > 0 \), lies in the heart of the particle production process. At \( t = t_s \) the physical size of the internal space becomes zero, while the external Hubble parameter is finite and the corresponding scale function non-vanishing. The existence of such internal singularities may affect on compactification of the extra dimension, but it does not affect on its "invinsibility" at the present epoch (McInnes 1985).

In the search for renormalization conditions on the observable quantities arising from particle creation, i.e. \( \langle 0|T_{\mu\nu}|0 \rangle \), we have used the effective action method in the one-loop approximation. In terms of functional analysis we have found that, in a classical, \( C^\infty \), globally hyperbolic, \( n \)-dimensional spacetime, the total probability to produce a particle pair over the entire history of the Universe is finite, provided that the spacetime is odd-dimensional. Since the divergences in \( P \) are the same as those that afflict both the effective action \( W \) and the vacuum expectation value of the energy-momentum tensor for the scalar field, \( \langle 0|T_{\mu\nu}|0 \rangle \) (Birrell and Davies 1982), we may conclude that the physical, observable quantities relevant to any process of particle creation in a curved spacetime are not only one-loop renormalizable but actually one-loop finite, provided that this process is applied on a curved background of odd dimensions. The most simple extrapolation of the four-dimensional spacetime to an odd-dimensional one is a five-dimensional spacetime. Indeed, on supersymmetric considerations (Van Nieuwenhuizen 1977, Alvarez 1989), one-loop finiteness in five dimensions implies the existence of only \( N = 3 \) supersymmetric charges of spinorial character, while for \( n: \text{odd and } n \geq 7 \) we need at least four \( (N = 4) \) corresponding quantities. In this respect, we have studied particle production in a curved background of five dimensions.

According to Eq. (40), the total probability to produce a particle pair over the entire history of the Universe is a positive, covariant quantity. In the spacetime under consideration, the corresponding particle production probability density (i.e. probability per unit of three-dimensional coordinate external volume) \( P_{2^{ext}} \), may be calculated explicitly. Normalizing mass as

\[
m \to m_N = \frac{m}{M_{Pl}} \leq 1
\]

(99)
where $M_{Pl}$ is the Planck mass, we may exclude the influence of the inner dimension (see also Maeda 1986). Then, the major contribution to $P_2$ comes out solely from the quadratic term and Eq. (40), for $j = 2$, is reduced to

$$P_2 = \frac{1}{32\pi^2} \left[ \frac{1}{90} m_N \int d^4x \sqrt{-g(t)} \left( \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu}(t) - 3 \mathcal{R}_{\mu\nu\kappa\lambda} \mathcal{R}^{\mu\nu\kappa\lambda}(t) \right) + O(m_N^5) \right]$$

(100)

which is dimensionless, as required. The corresponding probability density $P_2^{\text{ext}}$, results to

$$P_2^{\text{ext}} \approx \frac{1}{16\pi^2} m_N \ln \frac{t_f}{t_i}$$

(101)

Accordingly, even at the present epoch ($t_f = t_0 = 10^{17}$ sec), $P_2^{\text{ext}}$ represents a well-defined quantity, since

$$P_2^{\text{ext}}|_{t_f=t_0} = 0.876 m_N < 1$$

(102)

Eq. (101) indicates that, the higher the masses of the created particles are, the greater the probability for them to be produced, will be. Nevertheless, as we have seen, the production of such “high-mass” particles is subsequently exponentially suppressed by the cosmological evolution.

From Eq. (40) [or Eq. (100)] we see that, $P_2 \neq 0$ (and therefore $|0_{\text{out}} \neq |0_{\text{in}}>$) only as long as $m \neq 0$. For the spacetime (6) this is not an unexpected result, since we have considered $p_{\text{int}} = 0$. This condition corresponds to an adiabaticity criterion as regards the thermodynamical treatment of two subspaces (Kleidis and Papadopoulos 1997a) and accordingly, we cannot have radiation exchange (i.e. particles with $m = 0$) between them.

If we consider a quantized massive scalar field in the curved background under consideration, the corresponding initial and final vacuum states are not identical, $|0_{\text{out}} \neq |0_{\text{in}}>$ and therefore an extra amount of particles is created in the visible space as a result of the semiclassical interaction between the quantum matter field and the classical gravitational one. Those particles correspond to field modes with non-vanishing momentum component in the fifth dimension ($k_5 \neq 0$) (Garriga and Verdaguer 1989). The underlying mechanism of massive particle production in four dimensions rests in the fact that, during a period of high-rate cosmological contraction of the internal space, at times $t_{Pl} < t < t_s$, the physical frequency of the scalar modes along the fifth dimension is blueshifted to bring about particle creation at high energies. According to Eq. (66), since for $m_4 = 0$ we have $k_5 = 0$, this mechanism is no longer applicable to massless modes. The corresponding analysis does not depend on the exact value of the coupling constant involved, since, for the spacetime chosen, we have $\mathcal{R} = 0$.

One of the major problems involved in the process of particle creation in a curved spacetime is how to define the initial vacuum state (Hu 1974, Isham 1981, Birrell and Davies 1982, Wald 1984). This problem is solved in the usual way (Hu 1974), by introducing an initial time parameter, $t_i$, prior to which the spacetime is matched to be static. This is justified by the fact that at the Planck epoch the semiclassical
approach, which was taken into account, must break down. Therefore, on mathematical grounds in order to discuss particle creation, we need to exclude the influence of the period $0 < t < t_{Pl} \sim 10^{-43}\text{sec}$ and to give an unambiguous definition for the positive frequency modes $u_{k,k_5}^{in}$. Hence, it seems reasonable to assume a static metric. Accordingly, we assume that the time evolution of both subspaces starts at $t = t_{in}$. Nevertheless, a more natural choice of the initial vacuum is necessary. On the other hand, as regards the out-vacuum state (as $t \to \infty$), there is no ambiguity, since the internal space becomes static (Fig. 1) and the external one exhibits adiabatic expansion, where no particles are produced (Birrell and Davies 1982, Garriga and Verdaguer 1989).

Once the two vacuum states ($|0_{in}>$ and $|0_{out}>$) are appropriately defined, the process of particle creation in four dimensions may be studied by means of the solutions of the equation governing the propagation of the massive quantum field in the curved spacetime (6). In this case, the wave equation can be solved analytically. Its exact solution has been obtained in terms of hypergeometric functions. Their linear transformation properties (Abramowitz and Stegun 1970) give the combination of positive and negative frequency “out-modes” which makes up a positive frequency “in-mode”, thus indicating particle production in the out-region with respect to $|0_{in}>$.

A corresponding mass-energy spectrum is attributed to the scalar modes as a consequence of the anisotropic evolution of the five-dimensional spacetime. This energy spectrum is discrete (and hence the energy is quantized) as a result of compactification of the extra dimension. In addition, the resulting energies of the field’s quanta in four dimensions are strongly related to the inverse ”radius” of the internal space, namely

$$m_4c^2 = \ell \times 1.977375 \times 10^{-11}\frac{1}{R_5} (\text{MeV})$$

(70)

where $\ell$ is an integer and $R_5$ is measured in cm. In the absence of real experimental data regarding the extra dimensions (Barrow 1987, Casas et al 1987), we can only make predictions, concerning some boundary values of $m_4$, with respect to the corresponding values of $R_5$.

The lower bound of $R_5$, which gives rise to the upper bound of $m_4$, is the Planck length (Chodos and Detweiller 1980, Kolb et al 1984, Kolb 1986, Accetta et al 1986, Applequist et al 1987, Barrow 1987, Garriga and Verdaguer 1989). In this case, the energies of the particles emerging in the visible space are $m_4c^2 \approx 10^{19}\text{GeV}$, i.e. of the order of the Planck mass and therefore undetectable at present.

On the other hand, an upper bound for $R_5$ could be any distance which "fits" the radius of the internal space, provided that the extra dimension would remain safely invisible inside the corresponding length. In this case, we may consider that $R_5 \approx 10^{-18}\text{cm}$. This distance is two orders of magnitude smaller than the distances at which accelerators can probe matter at present (Dominguez-Tenreiro and Quiros 1988, Collins et al 1989). Then, from Eq. (70), we obtain

$$m_4c^2 \approx \ell \times 1.977 \times 10^4 \text{Gev} \approx 20 \text{TeV}$$

(103)
thus providing a lower bound for the mass-energy of the field’s quanta. We see that it is of the order of the estimated energy for the Higgs boson (Eichten et al 1984) and therefore one would expect that the next generation of accelerators could be able to detect such particles, verifying or not the matter-creation mechanism under consideration and/or Kaluza-Klein theories in general.

The physical quantities corresponding to the number and the energy density of the created particles per unit proper volume have been explicitly demonstrated for the ultraviolet four-dimensional modes \( k^2 \gg 1 \), which correspond to particles created in the neighbourhood of the singular epoch \( t = t_s \). It is worth noting that, at \( t = t_s \), both \( \eta(t_s) \) and \( \rho(t_s) \) are well-defined and finite (see also Fig. 2)

\[
\eta(t_s) = \frac{1}{12\pi^3 c^3} \frac{1}{t_s^3} \left( \frac{\text{particles}}{\text{cm}^3} \right)
\]

\[
\rho(t_s) = \frac{3h}{4\pi^6 c^3} \zeta(3) \frac{1}{t_s^4} \left( \frac{\text{erg}}{\text{cm}^3} \right)
\]

From these formulas it becomes evident that the closer to the initial singularity the value \( t = t_s \) is, the larger the amount of the produced particles will be.

The local rate of particle production (\( LRP^2 \)), per unit of proper three-dimensional volume and per unit time (Zel’dovich and Starobinsky 1977, Koikawa and Yoshimura 1985) is

\[
LRP^2 = \frac{1}{4\pi^3 c^3} \frac{1}{t^4} \geq 0 \quad (106)
\]

Because of the spatial homogeneity of the spacetime under consideration, the total momentum of the quantum scalar field in the external space is conserved (Zel’dovich and Starobinsky 1972, Bernard and Duncan 1977, Birrell and Davies 1892). In the case of a complex scalar field, whose quantization may be carried out in analogous fashion, the corresponding total charge is also conserved (Zel’dovich and Starobinsky 1972). Therefore, the particles are created locally, in pairs, with opposite charges and momenta.

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Figure Captions

Fig. 1: The evolution of the internal scale factor $g_{55}(t) = S^2(t)$, as a function of time. Notice that, as $t \to \infty$ the physical size of the internal dimension, $L_5 = 2\pi R_5 \sqrt{g_{55}(t)}$, is stabilized, since $S^2(t)$ approaches to the static value $S^2(t = \infty) = S^2_0$.

Fig. 2: The time evolution (in dimensionless units) of the "observable" number (squares) and energy (asterisks) densities, for massive particles created in the external space. Notice the dramatic increase of both $\eta_R$ and $\rho_R$ at the beginning of the interaction procedure, to achieve their maximum values in the neighbourhood of the singular epoch $t_s \simeq 1.4t_m$. 