Interval-valued Fuzzy $SA$-ideals with Degree $(\lambda, \kappa)$ of $SA$-algebra

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Abstract. In this paper, the notion of interval-valued fuzzy $SA$-ideals (briefly i-v fuzzy $SA$-ideal) in $SA$-algebras is introduced. Several theorems are stated and proved. The image and inverse image of i-v fuzzy $SA$-ideals are defined and how the homomorphic images and inverse images of i-v fuzzy $SA$-ideals become i-v fuzzy $SA$-ideals in $SA$-algebras is studied as well.

Keywords. $SA$-algebras, fuzzy $SA$-ideals, interval-valued fuzzy $SA$-subalgebras, interval-valued fuzzy $SA$-ideals in $SA$-algebras.

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1. Introduction

Areej Tawfeeq Hameed and et al ([1]) introduced a new algebraic structure, called $SA$-algebra. They have studied a few properties of these algebras, the notion of $SA$-ideals on $SA$-algebras was formulated and some of its properties are investigated. The concept of a fuzzy set, was introduced by L.A. Zadeh [3]. In [5], S.M. Mostafa and A.T. Hameed made an extension of the concept of fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). This interval-valued fuzzy KUS-ideals on KUS-algebras is referred to as an i-v fuzzy KUS-ideals on KUS-algebras. They constructed a method of approximate inference using his i-v fuzzy KUS-ideals on KUS-algebras. In this paper, using the notion of interval-valued fuzzy set, we introduce the concept of an interval-valued fuzzy $SA$-ideals (briefly, i-v fuzzy $SA$-ideals) of a $SA$-algebra, and study some of their properties. Using an i-v level set of an i-v fuzzy set, we state a characterization of an i-v fuzzy $SA$-ideals. We prove that every $SA$-ideals of a $SA$-algebra $X$ can be realized as an i-v level $SA$-ideals of an i-v fuzzy $SA$-ideals of $X$. In connection with the notion of homomorphism, we study how the images and inverse images of i-v fuzzy $SA$-ideals become i-v fuzzy $SA$-ideals.

2. Preliminaries

Now, we give some definitions and preliminary results needed in the later sections.

Definition 2.1([1]). Let $(X;+,-,0)$ be an algebra with two binary operations $(+)$ and $(\cdot)$ and constant $(0)$. $X$ is called a $SA$-algebra if it satisfies the following identities: for any $x, y, z \in X$, 

$\lambda, \kappa, \mu, \nu \in [0,1]$ represent the degree of membership.
\((SA_1)\) \(x - x = 0\),  
\((SA_2)\) \(x - 0 = x\),  
\((SA_3)\) \((x - y) - z = x - (z + y)\),  
\((SA_4)\) \((x + y) - (x + z) = y - z\).

In X we can define a binary relation \((\leq)\) by: \(x \leq y\) if and only if \(x - y = 0\).

**Example 2.2((1)).** Let \(X = \{0, 1, 2, 3\}\) be a set with the following tables:

|     | 0 | 1 | 2 | 3 |
|-----|---|---|---|---|
| +   | 0 | 1 | 2 | 3 |
| 0   | 0 | 1 | 2 | 3 |
| 1   | 1 | 2 | 3 | 0 |
| 2   | 2 | 3 | 0 | 1 |
| 3   | 3 | 0 | 1 | 2 |

Then \((X;+,-,0)\) is a \(SA\)-algebra.

**Lemma 2.3((1)).** Let \((X;+,-,0)\) be a \(SA\)-algebra. Then for any \(x, y \in X\),

\[
\begin{align*}
(L_1) & \quad x + y = x - (-y), \\
(L_2) & \quad x - y = x + (-y), \\
(L_3) & \quad x - y = y + x.
\end{align*}
\]

**Proposition 2.4((1)).** Let \((X;+,-,0)\) be a \(SA\)-algebra. Then the following holds:

for any \(x, y, z \in X\),

\[
\begin{align*}
(a_1) & \quad (x - y) - z = (x - z) - y, \\
(a_2) & \quad 0 - (x - y) = (y - x), \\
(a_3) & \quad x - y \leq z \implies x - z \leq y, \\
(a_4) & \quad x \leq y \implies z + y \leq z + x, \\
(a_5) & \quad (x - y) - (y - z) \leq x - z \quad \text{and} \quad (x - y) - (x - z) \leq z - y, \\
(a_6) & \quad x \leq y \quad \text{and} \quad y \leq z \implies x \leq z.
\end{align*}
\]

**Definition 2.5((1)).** Let \((X;+,-,0)\) be a \(SA\)-algebra and let \(S\) be a nonempty set of \(X\). \(S\) is called a \(SA\)-subalgebra of \(X\) if

\[
\forall x, y, z \in X.\quad \begin{array}{l}
0 \in S, \\
x - y \in S, \\
x + y \in S, \\
x \leq y \implies x \leq y.
\end{array}
\]

**Definition 2.6 ((1)).** A nonempty subset \(I\) of a \(SA\)-algebra \(X\) is called a \(SA\)-ideal of \(X\) if it satisfies: for \(x, y, z \in X\),

\[
\begin{align*}
(1) & \quad 0 \in I, \\
(2) & \quad (x + y) \in I \quad \text{and} \quad (y - z) \in I \implies (x + y) \in I.
\end{align*}
\]

**Proposition 2.7((1)).** Every \(SA\)-ideal of \(SA\)-algebra \(X\) is a \(SA\)-subalgebra of \(X\) and the converse is not true.

**Lemma 2.8((1)).** A \(SA\)-ideal \(I\) of \(SA\)-algebra \(X\) has the following property:

1- for any \(x \in X\), for all \(y \in I\), \(x \leq y \implies x \in I\).

2- if for any \(x \in I \implies -x \in I\).

**Definition 2.9((3)).** Let be a nonempty set, a fuzzy subset \(\mu\) in \(X\) is a function \(\mu: X \rightarrow [0,1]\).
Definition 2.10 ([3]). Let $X$ be a nonempty set and $\mu$ be a fuzzy subset in $X$, for $t \in [0,1]$, the set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is called a level subset of $\mu$.

Remark 2.11 ([1]). Let $\lambda$ and $\kappa$ be members of $(0,1]$, and let $n$ and $r$ denote a natural number and a real number, respectively, such that $r < n$ unless otherwise specified.

Definition 2.12 ([1]). Let $(X;+,\cdot,0)$ be a $SA$-algebra, a fuzzy subset $\mu$ of $X$ is called a fuzzy $SA$-ideal with degree $(\lambda,\kappa)$ of $X$ if it satisfies: for all $x, y, z \in X$,

- $(FI1)$ $\mu(0) \geq \lambda \mu(x)$,
- $(FI2)$ $\mu(x + y) \geq \kappa \min\{\mu(x + z), \mu(y - z)\}$.

Example 2.13 ([1]). Let $X = \{0, 1, 2, 3\}$ be a set with the following tables:

|   | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

Then $(X;+,\cdot,0)$ is a $SA$-algebra. Define a fuzzy subset by:

$$\mu(x) = \begin{cases} 0.7 & \text{if } x \in \{0,1\} \\ 0.3 & \text{otherwise} \end{cases}$$

$I_1 = \{0, 1\}$ is a $SA$-ideal of $X$. Routine calculation gives that $\mu$ is a fuzzy $SA$-ideal with degree $(\frac{4}{7}, \frac{4}{7})$ of $SA$-algebras $X$.

Proposition 2.14 ([1]). Let $A$ be a $SA$-ideal of $SA$-algebra $X$. Then for any fixed number $t$ in an open interval $(0,1)$, there exists $\mu$ is a fuzzy $SA$-ideal with degree $(\lambda,\kappa)$ of $X$ such that $\mu_t = A$.

Theorem 2.15 ([1]). Let $A$ be a nonempty subset of a $SA$-algebra $X$ and $\mu$ be a fuzzy subset of $X$ such that $\mu$ is into $\{0, 1\}$, so that $\mu$ is the characteristic function of $A$. Then $\mu$ is a fuzzy $SA$-ideal with degree $(\lambda,\kappa)$ of $X$ if and only if $A$ is a $SA$-ideal of $X$.

Proposition 2.16 ([1]). Let $\mu$ be a fuzzy $SA$-ideal with degree $(\lambda,\kappa)$ of $X$, then the following hold: for all $x, y, z \in X$,

- $a)$ $x \leq y \Rightarrow \mu(x) \geq \lambda \kappa \mu(y)$,
- $b)$ $\mu(x + y) \geq \lambda \kappa \min\{\mu(x + z), \mu(y - z)\}$.

Proposition 2.17 ([1]). Let $\mu$ be a fuzzy $SA$-ideal with degree $(\lambda,\kappa)$ of a $SA$-algebra $X$ and let $\mu_{t_1}$, $\mu_{t_2}$ be level $SA$-ideals of $\mu$, where $t_1 < t_2$, then the following are equivalent.

- $E1)$ $\mu_{t_1} = \mu_{t_2}$.
- $E2)$ There is no $x \in X$ such that $t_1 \leq \mu(x) < t_2$.

Definition 2.18 ([1]). Let $(X;+,\cdot,0)$ be a $SA$-algebra, a fuzzy subset $\mu$ in $X$ is called a fuzzy $SA$-subalgebra of $X$ if for all $x, y \in X$, $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$.

Theorem 2.19 ([1]). Let $\mu$ be a fuzzy subset of $SA$-algebra $X$. If $\mu$ is a fuzzy $SA$-subalgebra of $X$ if and only if for every $t \in [0,1]$, $\mu_t$ is a $SA$-subalgebra of $X$, when $\mu_t \neq \emptyset$. 
Theorem 2.20([1]). Let $\mu$ be a fuzzy subset of $SA$-algebra $X$. $\mu$ is a fuzzy $SA$-ideal with degree $(\lambda, \kappa)$ of $X$ if and only if, for every $t \in [0,1]$ , $\mu_t$ is a $SA$-ideal of $X$, when $\mu_t \neq \emptyset$.

Proposition 2.21([1]). Every fuzzy $SA$-ideal with degree $(\lambda, \kappa)$ of $SA$-algebra $X$ is a fuzzy $SA$-subalgebra of $X$.

The converse of proposition (2.21) is not true as the following example:

Example 2.22([1]). Let $X = \{0, 1, 2, 3\}$ in which $(+)$ and $(\cdot)$ is defined by the following table:

|   | 0  | 1  | 2  | 3  |
|---|----|----|----|----|
| 0 | 0  | 1  | 2  | 3  |
| 1 | 1  | 2  | 3  | 0  |
| 2 | 2  | 3  | 0  | 1  |
| 3 | 3  | 0  | 1  | 2  |

It is easy to show that $(X;\cdot,-,0)$ is a $SA$-algebra. $I=\{0, 2\}$ is a $SA$-subalgebra of $X$ but $I$ is not a $SA$-ideal of $X$.

Define $\mu: X \rightarrow [0,1]$ by $\mu(x) = \begin{cases} 0.5 & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases}$. $\mu$ is a fuzzy $SA$-subalgebra of $X$, but $\mu$ is not fuzzy $SA$-ideal with degree $(\lambda, \kappa)$ of a $SA$-algebra $X$.

Definition 2.23([1]). Let $(X;\cdot,-,0)$ and $(Y;\cdot',-,0')$ be $SA$-algebras, the mapping $f : (X;\cdot,-,0) \rightarrow (Y;\cdot',-,0')$ is called a homomorphism if it satisfies:

$f(x+y) = f(x) + f(y)$, $f(x-y) = f(x) - f(y)$, for all $x, y \in X$.

Definition 2.24([2,4]). Let $f : (X;\cdot,-,0) \rightarrow (Y;\cdot',-,0')$ be a mapping nonempty sets $X$ and $Y$ respectively. If $\mu$ is a fuzzy subset of $X$, then the fuzzy subset $\beta$ of $Y$ defined by:

$$f(\mu)(y) = \begin{cases} \sup \{ \mu(x) : x \in f^{-1}(y) \} & \text{if } f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

is said to be the image of $\mu$ under $f$.

Similarly if $\beta$ is a fuzzy subset of $Y$, then the fuzzy subset $\mu = (\beta \circ f)$ in $X$ (i.e the fuzzy subset defined by $\mu(x) = \beta(f(x))$ for all $x \in X$) is called the pre-image of $\beta$ under $f$.

Theorem 2.25([1]). An into homomorphic pre-image of a fuzzy $SA$-ideal with degree $(\lambda, \kappa)$ is also a fuzzy $SA$-ideal with degree $(\lambda, \kappa)$.

Definition 2.26 ([2,4]). A fuzzy subset $\mu$ of a set $X$ has sup property if for any subset $T$ of $X$, there exist $t_0 \in T$ such that $\mu(t_0) = \sup \{\mu(t) | t \in T\}$.

Theorem 2.27([1]). Let $f : (X;\cdot,-,0) \rightarrow (Y;\cdot',-,0')$ be a homomorphism between $SA$-algebras $X$ and $Y$ respectively. For every fuzzy $SA$-ideal $\mu$ with degree $(\lambda, \kappa)$ of $X$ and with sup property, $f(\mu)$ is a fuzzy $SA$-ideal with degree $(\lambda, \kappa)$ of $Y$.

3. Interval-Valued fuzzy SA-subalgebra of $SA$-algebra

In the section, the notion of the interval-valued fuzzy $SA$-subalgebras.

Remark 3.1 [6]. An interval-valued fuzzy subset (briefly i-v fuzzy subset) $A$ of the set $X$ is defined by $A = \{[\mu_0^L(x), \mu_0^U(x)] | x \in X\}$. (briefly, it is denoted by $A = [\mu_0^L, \mu_0^U]$ where $\mu_0^L$ and $\mu_0^U$ are any two fuzzy subsets of $X$ such that
\[ \mu_A^l(x) \leq \mu_A^u(x) \text{ for all } x \in X. \]

Let \( \tilde{\mu}_A(x) = [\mu_A^l(x), \mu_A^u(x)] \), for all \( x \in X \) and let \( D[0,1] \) denotes the family of all closed sub-interval of \([0,1]\). It is clear that if \( \mu_A^l(x) = \mu_A^u(x) = c \), where \( 0 \leq c \leq 1 \), then \( \tilde{\mu}_A(x) = [c, c] \) in \( D[0,1] \), then \( \tilde{\mu}_A(x) \in [0,1] \), for all \( x \in X \). Therefore the i-v fuzzy subset \( A \) is given by: \( A = \{(x, \tilde{\mu}_A(x))\} \), for all \( x \in X \) where \( \tilde{\mu}_A : X \rightarrow D[0,1] \).

Now, we define the refined minimum (briefly r min) and order \( \leq \) on the elements \( D_1 = [a_1, b_1] \) and \( D_2 = [a_2, b_2] \) of \( D = [0,1] \) as follows:

\[
\text{r min}(D_1,D_2) = [\min(a_1,a_2), \min(b_1,b_2)], D_1 \leq D_2 \iff a_1 \leq a_2
\]

and \( b_2 \leq b_2 \). Similarly, we can define \( (\geq) \) and \( (=) \), [8].

**Definition 3.2.**
An i-v fuzzy subset \( A \) of \( S_A \)-algebra \( X \) is called an interval-valued fuzzy \( S_A \)-subalgebra of \( X \) (briefly i-v fuzzy \( S_A \)-subalgebra) if for all \( x, y \in X \),

1) \( \mu_A^l(x + y) \geq \text{r min}\{\mu_A^l(x), \mu_A^l(y)\} \)
2) \( \mu_A^l(x - y) \geq \text{r min}\{\mu_A^l(x), \mu_A^l(y)\} \).

**Remark 3.3.**
Since \( \mu_A^l(x + y) \geq \text{r min}\{\mu_A^l(x), \mu_A^l(y)\} \),

\[
= \text{r min}\{\min\{\mu_A^l(x), \mu_A^u(x)\}, \min\{\mu_A^l(y), \mu_A^u(y)\}\}
\]

\[
= \text{r min}\{\min\{\mu_A^l(x), \mu_A^l(y)\}, \min\{\mu_A^u(x), \mu_A^u(y)\}\}.
\]

**Example 3.4.**
Let \( X = \{0, 1, 2, 3\} \) in which \( (+) \) be defined by: Table 3.

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 0 |

Table 3.

Then \( (X, +, -, 0) \) is a \( S_A \)-algebra. Define \( \tilde{\mu}_A(x) \) as follows:

\[
\tilde{\mu}_A(x) = \begin{cases} 
(0.3, 0.9) & \text{if } x = \{0,1\} \\
(0.1, 0.6) & \text{otherwise} \end{cases}
\]

It is easy to check that \( A = [\mu_A^l, \mu_A^u] \) is an i-v fuzzy \( S_A \)-subalgebra.

**Proposition 3.5.**
If \( A = [\mu_A^l, \mu_A^u] \) is an i-v fuzzy \( S_A \)-subalgebra of \( S_A \)-algebra \( X \), then \( \tilde{\mu}_A(0) \geq \mu_A(x) \), for all \( x \in X \).

**Proof:**
For all \( x \in X \), we have

\[
\tilde{\mu}_A(0) = \tilde{\mu}_A(x - x) \geq \text{r min}\{\tilde{\mu}_A(x), \tilde{\mu}_A(x)\} \geq \mu_A(x).
\]

Hence \( \tilde{\mu}_A(0) \geq \mu_A(x) \), for all \( x \in X \). 

**Proposition 3.6.**
Let \( A = [\mu_A^l, \mu_A^u] \) be an i-v fuzzy \( S_A \)-subalgebra of \( S_A \)-algebra \( X \), if there exist a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} \tilde{\mu}_A(x_n) = [1,1] \), then

\[
\mu_A(0) = [1,1].
\]

**Proof:**
By Proposition (3.5), we have \( \tilde{\mu}_A(0) \geq \mu_A(x) \), for all \( x \in X \). Then \( \tilde{\mu}_A(0) \geq \mu_A(x_n) \), for every positive integer \( n \). Consider the inequality

\[
[1,1] \geq \mu_A(0) \geq \lim_{n \to \infty} \tilde{\mu}_A(x_n) = [1,1].
\]

Hence \( \mu_A(0) = [1,1] \).

**Theorem 3.7.**
An i-v fuzzy subset \( A = [\mu_A^l, \mu_A^u] \) of \( S_A \)-algebra \( X \) is an i-v fuzzy
If $\mu_A^I$ and $\mu_A^U$ are fuzzy $SA$-subalgebra of $SA$-algebra $X$, then

$$\bar{\mu}_A(x + y) = [\mu_A^I(x + y), \mu_A^U(x + y)]$$

$$\geq \min\{\mu_A^I(x), \mu_A^U(x), \mu_A^I(y), \mu_A^U(y)\}$$

$$= r \min\{\mu_A^I(x), \mu_A^U(x), \mu_A^I(y), \mu_A^U(y)\}$$

Thus we can conclude that $A$ is an i-v fuzzy $SA$-subalgebra of $X$.

Conversely, suppose that $\bar{\mu}_A(x - y) = [\mu_A^I(x - y), \mu_A^U(x - y)]$. And

$$\bar{\mu}_A(x - y) = [\mu_A^I(x - y), \mu_A^U(x - y)]$$

$$\geq \min\{\mu_A^I(x), \mu_A^U(x), \mu_A^I(y), \mu_A^U(y)\}$$

$$= r \min\{\mu_A^I(x), \mu_A^U(x), \mu_A^I(y), \mu_A^U(y)\}$$

Therefore,

$$\mu_A^I(x + y) \geq \min\{\mu_A^I(x), \mu_A^U(x), \mu_A^I(y), \mu_A^U(y)\}$$

$$\mu_A^U(x + y) \geq \min\{\mu_A^I(x), \mu_A^U(x), \mu_A^I(y), \mu_A^U(y)\}$$

Hence, we get that $\mu_A^I$ and $\mu_A^U$ are fuzzy $SA$-subalgebras of $X$.

**Theorem 3.8.**

Let $\{|i| \in \Lambda\}$ be a family of i-v fuzzy $SA$-subalgebras of $X$, then $\bigcap_{i \in \Lambda} \mu_i$ is also an i-v fuzzy $SA$-subalgebra of $X$.

**Proof.**

Let $\{|i| \in \Lambda\}$ be a family of i-v fuzzy $SA$-subalgebra of $X$, then $x, y \in \bigcap_{i \in \Lambda} \mu_i$, for all $i \in \Lambda$. Suppose, $x, y \in X$, such that

$$\bar{\mu}_{\bigcap_{i \in \Lambda} \mu_i}(x + y) = [\mu_{\bigcap_{i \in \Lambda} \mu_i}^I(x + y), \mu_{\bigcap_{i \in \Lambda} \mu_i}^U(x + y)]$$

$$\geq \min\{\mu_{\bigcap_{i \in \Lambda} \mu_i}^I(x), \mu_{\bigcap_{i \in \Lambda} \mu_i}^I(y), \mu_{\bigcap_{i \in \Lambda} \mu_i}^U(x), \mu_{\bigcap_{i \in \Lambda} \mu_i}^U(y)\}$$

$$= r \min\{\mu_{\bigcap_{i \in \Lambda} \mu_i}^I(x), \mu_{\bigcap_{i \in \Lambda} \mu_i}^I(y), \mu_{\bigcap_{i \in \Lambda} \mu_i}^U(x), \mu_{\bigcap_{i \in \Lambda} \mu_i}^U(y)\}$$

Hence $\bigcap_{i \in \Lambda} \mu_i$ is an i-v fuzzy $SA$-subalgebra of $X$.

**Remark 3.9.**

The union of i-v fuzzy $SA$-subalgebras of $SA$-algebra $X$ is not necessary in general, an i-v fuzzy $SA$-subalgebra of $X$ as seen in the following example.

**Example 3.10.**

In Example (3.4), we defined i-v fuzzy $SA$-subalgebra.

A$_1$: $X \rightarrow [0,1]$ by $\bar{\mu}_{A_1}(0) = [0.7, 0.8]$, $\bar{\mu}_{A_1}(1) = [0.3, 0.8]$, $\bar{\mu}_{A_1}(3) = [0.2, 0.8]$

A$_2$: $X \rightarrow [0,1]$ by $\bar{\mu}_{A_2}(0) = [0.7, 0.8]$, $\bar{\mu}_{A_2}(1) = [0.3, 0.8]$, $\bar{\mu}_{A_2}(3) = [0.2, 0.8]$. 

\[\text{doi:10.1088/1742-6596/1804/1/012068}\]
\[\tilde{\mu}_A(1) = \tilde{\mu}_A(2) = \tilde{\mu}_A(3) = [0.5, 0.6], \tilde{\mu}_A(4) = [0.5, 0.8]\]

By routine calculation union of \( \mu_{A_1 \cup A_2} \) not be i-v fuzzy \( SA \)-subalgebra, since \( \tilde{\mu}_{A_1 \cup A_2}(0) = [0.7, 0.8] \), \( \tilde{\mu}_{A_1 \cup A_2}(1) = [0.3, 0.8] \), \( \tilde{\mu}_{A_1 \cup A_2}(3) = \tilde{\mu}_{A_1 \cup A_2}(4) = [0.5, 0.8] \).

**Theorem 3.11.**

Let \( \{\mu_i : i \in \Lambda\} \) be a family of i-v fuzzy \( SA \)-subalgebras of \( SA \)-algebra \( X \), then \( \bigcup_{i \in \Lambda} \mu_i \) is an i-v fuzzy \( SA \)-subalgebra of \( X \), where \( \mu_i \subseteq \mu_{i+1} \), for all \( i \in \Lambda \).

**Proof:**

Since \( \{\mu_i : i \in \Lambda\} \) be a family of i-v fuzzy \( SA \)-subalgebra of \( X \) and \( \mu_i \subseteq \mu_{i+1} \), for all \( i \in \Lambda \), then for any \( x, y \in X \),

\[
\tilde{\mu}_{\bigcup_{i \in \Lambda} \mu_i}(x + y) = \left[ \min \left( \mu_{\bigcup_{i \in \Lambda} \mu_i}(x), \mu_{\bigcup_{i \in \Lambda} \mu_i}(y) \right) \right] \min \left( \mu_{\bigcup_{i \in \Lambda} \mu_i}(x), \mu_{\bigcup_{i \in \Lambda} \mu_i}(y) \right) = r \min \left( \mu_{\bigcup_{i \in \Lambda} \mu_i}(x), \mu_{\bigcup_{i \in \Lambda} \mu_i}(y) \right) = r \min \left( \mu_{\bigcup_{i \in \Lambda} \mu_i}(x), \mu_{\bigcup_{i \in \Lambda} \mu_i}(y) \right).
\]

Hence \( \bigcup_{i \in \Lambda} \mu_i \) is an i-v fuzzy \( SA \)-subalgebra of \( X \).

**Proposition 3.12.**

Let \( X \) be a \( SA \)-algebra, \( Y \) be a subset of \( X \) and let \( A \) be an i-v fuzzy subset on \( X \) defined by :

\[
\tilde{\mu}_A(x) = \begin{cases} [a_1, a_2] & \text{if } x \in Y \\ [0, 0] & \text{otherwise} \end{cases}
\]

Where \( a_1, a_2 \in (0, 1] \) with \( a_1 < a_2 \). If \( A \) is an i-v fuzzy \( SA \)-subalgebra of \( X \), then \( Y \) is a \( SA \)-subalgebra of \( X \).

**Proof:**

Since that \( A \) is an i-v fuzzy \( SA \)-subalgebra of \( X \). Let \( x, y \in Y \), then

\[
\tilde{\mu}_A(x + y) \geq r \min \{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} = r \min \{[a_1, a_2], [a_1, a_2] = [a_1, a_2] \}.
\]

And \( \tilde{\mu}_A(x - y) \geq r \min \{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} = [a_1, a_2] \),

this implies that \( x + y \in Y \) and \( x - y \in Y \).

Hence \( Y \) is a \( SA \)-subalgebra of \( X \).

**Definition 3.13.**

Let \( X \) be a \( SA \)-algebra and \( A \) be an i-v fuzzy subset of \( X \), the nonempty set \( \overline{U}(A; [\delta_1, \delta_2]) \) is called the i-v level set of \( A \), where

\[
\overline{U}(A; [\delta_1, \delta_2]) := \{ x \in X | \tilde{\mu}_A(x) \geq [\delta_1, \delta_2], \text{ for every } [\delta_1, \delta_2] \in D[0, 1] \}.
\]

**Theorem 3.14.**

Let \( X \) be a \( SA \)-algebra and \( A \) be an i-v fuzzy subset of \( X \). Then \( A \) is an i-v fuzzy \( SA \)-subalgebra of \( X \) if and only if, the nonempty set \( \overline{U}(A; [\delta_1, \delta_2]) \) is a \( SA \)-subalgebra of \( A \), for every \( [\delta_1, \delta_2] \in D[0, 1] \).

**Proof:**

Assume that \( A \) is an i-v fuzzy \( SA \)-subalgebra of \( X \) and let \( [\delta_1, \delta_2] \in D[0, 1] \) be such that \( x, y \in \overline{U}(A; [\delta_1, \delta_2]) \), then

\[
\tilde{\mu}_A(x + y) \geq r \min \{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} = r \min \{[\delta_1, \delta_2], [\delta_1, \delta_2] = [\delta_1, \delta_2] \}, \text{ so } (x + y) \in \overline{U}(A; [\delta_1, \delta_2]).
\]

And \( \tilde{\mu}_A(x - y) \geq r \min \{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} = r \min \{[\delta_1, \delta_2], [\delta_1, \delta_2] = [\delta_1, \delta_2] \}, \text{ so } (x - y) \in \overline{U}(A; [\delta_1, \delta_2]). \)

Therefore \( \overline{U}(A; [\delta_1, \delta_2]) \) is \( SA \)-subalgebra of \( A \).

Conversely, assume that \( \overline{U}(A; [\delta_1, \delta_2]) \neq \emptyset \) is a \( SA \)-subalgebra of \( X \), for every
\[ [\delta_1, \delta_2] \in D[0,1]. \quad \text{In the contrary, suppose that there exist } x_0, y_0 \in X \text{ such that } \]
\[ \mu_A(x_0 + y_0) < r \min \{ \mu_A(x_0), \mu_A(y_0) \}. \]
\[ \text{Let } \bar{\mu}_A(x_0) = [y_1, y_2], \bar{\mu}_A(y_0) = [y_3, y_4] \text{ and } \bar{\mu}_A(x_0 + y_0) = [\delta_1, \delta_2]. \]
\[ \text{If } [\delta_1, \delta_2] < r \min \{ \{ y_1, y_2 \}, \{ y_3, y_4 \} \} = \min \{ \min \{ y_1, y_2 \}, \min \{ y_3, y_4 \} \}. \]
\[ \text{So } \delta_1 < \min \{ y_1, y_2 \} \quad \text{and} \quad \delta_2 < \min \{ y_3, y_4 \}. \]
\[ \text{Now, consider } [\lambda_1, \lambda_2] = \frac{1}{2} \{ [\lambda_1, \lambda_2] + r \min \{ y_1, y_2 \}, [y_3, y_4] \} \]
\[ = \frac{1}{2} \{ \delta_1 + \min \{ y_1, y_3 \}, \delta_2 + \min \{ y_2, y_4 \} \}. \]
\[ \text{Therefore, } \min \{ y_1, y_3 \} > \lambda_1 = \frac{1}{2} (\delta_1 + \min \{ y_1, y_3 \}) > \delta_1, \quad \text{and } \min \{ y_2, y_4 \} > \lambda_2 = \frac{1}{2} (\delta_2 + \min \{ y_2, y_4 \}) > \delta_2. \]

\section{4. Interval-Valued fuzzy SA-ideal with degree \((\lambda, k)\)}

In the section, the interval-valued fuzzy \(SA\)-ideals (briefly i-v fuzzy \(SA\)-ideals) with degree \((\lambda, k)\) of \(SA\)-algebra is introduced. Some theorems and properties are stated and proved.

\textbf{Definition 4.1.}

An i-v fuzzy subset \( A = \{(x, \bar{\mu}_A(x))\} \), \( x \in X \) of \( SA\)-algebra \( X \) is called an interval-valued fuzzy \( SA\)-ideal with degree \((\lambda, k)\) (i-v fuzzy \(SA\)-ideal with degree \((\lambda, k)\), in short) if it satisfies the following conditions:

(A1) \( \bar{\mu}_A(0) \geq \lambda \bar{\mu}_A(x) \), for all \( x \in X \),

(A2) \( \bar{\mu}_A(x + y) \geq k r \min \{ \bar{\mu}_A(x + z), \bar{\mu}_A(y - z) \} \), for all \( x, y, z \in X \).

\textbf{Example 4.2.}

If \( X = \{0, 1, 2, 3\} \) and \((+)\) and \((-)\) is defined as in Example (3.1.4).

Define \( \bar{\mu}_A(x) \) as follows: \( \bar{\mu}_A(x) = \begin{cases} [0.3, 0.9] & \text{if } x = 0, 1, \\ [0.1, 0.6] & \text{otherwise}. \end{cases} \)

It is easy to check that \( A \) is an i-v fuzzy \(SA\)-ideal with degree \((\lambda, k)\) of \( X \).

\textbf{Theorem 4.3.}

An i-v fuzzy subset \( A = [\mu_A^L, \mu_A^U] \) of \( SA\)-algebra \( X \) is an i-v fuzzy \(SA\)-ideal with degree \((\lambda, k)\) of \( X \) if and only if, \( \mu_A^L \) and \( \mu_A^U \) are fuzzy \(SA\)-ideals with degree \((\lambda, k)\) of \( X \).

\textbf{Proof:}

Suppose that \( A \) is an i-v fuzzy \(SA\)-ideal with degree \((\lambda, k)\) of \( X \), the 
\[ \bar{\mu}_A(0) \geq \lambda \bar{\mu}_A(x), \quad \text{for all } x \in X, \]
then \( \bar{\mu}_A(0) = [\mu_A^L(0), \mu_A^U(0)] \geq \lambda [\mu_A^L(x), \mu_A^U(x)] = \lambda \bar{\mu}_A(x). \)

Hence, \( \mu_A^L(0) \geq \lambda \mu_A^L(x) \) and \( \mu_A^U(0) \geq \lambda \mu_A^U(x). \) For all \( x, y, z \in X \), we have 
\[ \mu_A^L(x + y), \mu_A^U(x + y) = \bar{\mu}_A(x + y) \]
\[ = k r \min \{ \bar{\mu}_A(x + z), \bar{\mu}_A(y - z) \} \]
\[ = k r \min \{ [\mu_A^L(x + z), \mu_A^U(x + z)], [\mu_A^L(y - z), \mu_A^U(y - z)] \} \]
\[ = k \min \{ \mu_A^L(x + z), \mu_A^U(x + z) \}, \min \{ \mu_A^L(y - z), \mu_A^U(y - z) \} \].

Therefore, \( \mu_A^L(x + y) \geq k \min \{ \mu_A^L(x + z), \mu_A^L(y - z) \} \) and \( \mu_A^U(x + y) \geq k \min \{ \mu_A^U(x + z), \mu_A^U(y - z) \} \).

Hence, we get that \( \mu_A^L \) and \( \mu_A^U \) are fuzzy \(SA\)-ideals with degree \((\lambda, k)\) of \( X \).
Conversely, if $\mu^+_A$ and $\mu^-_A$ are fuzzy $SA$-ideals with degree $(\lambda, k)$ of $X$, then $\mu^+_A(0) \geq \lambda \mu_A(x)$ and $\mu^-_A(0) \geq \lambda \mu_A(x)$ implies that $\bar{\mu}_A(0) \geq \bar{\lambda} \bar{\mu}_A(x)$, for all $x \in X$.

For all $x, y, z \in X$. Observe :

$$\bar{\mu}_A(x + y) = [\mu^+_A(x + y), \mu^-_A(x + y)]$$

$$\geq k \min \{ \mu^+_A(x + z), \mu^-_A(y - z) \}, \min \{ \mu^+_A(x + z), \mu^-_A(y - z) \}$$

$$= k r \min \{ [\mu^+_A(x + z), \mu^-_A(x + z)], [\mu^+_A(y - z), \mu^-_A(y - z)] \}$$

$$= k r \min \{ \bar{\mu}_A(x + z), \bar{\mu}_A(y - z) \}.$$ 

Thus we can conclude that $A$ is an i-v fuzzy $SA$-ideal with degree $(\lambda, k)$ of $X$.

**Theorem 4.4.**

The intersection of any set of i-v fuzzy $SA$-ideals with degree $(\lambda, k)$ of $SA$-algebra $X$ is also an i-v fuzzy $SA$-ideal with degree $(\lambda, k)$ of $X$.

**Proof:**

Let $\{\mu_i \mid i \in \Lambda\}$ be a family of i-v fuzzy $SA$-ideal of $X$, then $x, y \in \cap_{\mu_i}$ for all $i \in \Lambda$.

$$\bar{\mu}_{\cap \mu_i}(0) = [\mu^+_{\cap \mu_i}(0), \mu^-_{\cap \mu_i}(0)]$$

$$\geq \lambda \min \{ \mu^+_{\mu_i}(x), \mu^-_{\mu_i}(x) \}$$

$$= \lambda \bar{\mu}_{\cap \mu_i}(x).$$

Suppose $x, y, z \in X$ such that $x + z \in \cap \mu_i$ and $y - z \in \cap \mu_i$.

Since $\mu_i$ are i-v fuzzy $SA$-ideals with degree $(\lambda, k)$ of $X$, then we get

$$\bar{\mu}_{\cap \mu_i}(x + y) = [\mu^+_{\cap \mu_i}(x + y), \mu^-_{\cap \mu_i}(x + y)]$$

$$= k \min \{ \mu^+_{\mu_i}(x + z), \mu^-_{\mu_i}(y - z) \}, \min \{ \mu^+_{\mu_i}(x + z), \mu^-_{\mu_i}(y - z) \}$$

$$= k \min \{ \bar{\mu}_{\cap \mu_i}(x + z), \bar{\mu}_{\cap \mu_i}(y - z) \}.$$ 

Hence $\cap \mu_i$ is an i-v fuzzy $SA$-ideal with degree $(\lambda, k)$ of $X$.

**Remark 4.5.**

The union of an i-v fuzzy $SA$-ideal with degree $(\lambda, k)$ of $SA$-algebra $X$, is not necessarily an i-v fuzzy $SA$-ideal with degree $(\lambda, k)$ as seen in the following example.

**Example 4.6.**

Let $X = \{0, 1, 2, 3\}$ in which $(\ast)$ be defined by:

| 0 | 1 | 2 | 3 |
|---|---|---|---|
| 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 |
| 2 | 2 | 3 | 0 |
| 3 | 3 | 0 | 1 |
| 3 | 3 | 2 | 1 |

Then $(X, +, -)$ is a $SA$-algebra. Define $\bar{\mu}_A(x)$ as follows:

$$\bar{\mu}_A(x) = \begin{cases} [0.3, 0.9] & \text{if } x = \{0, 1\} \\ [0.1, 0.6] & \text{otherwise} \end{cases}$$

Let $I_1 = \{0, 1\}$ and $I_2 = \{0, 2\}$ it is easy to check that is $I_1$ and $I_2$ an i-v fuzzy $SA$-ideal but $I_1 \cup I_2$ not $SA$-ideal since if $x = 2$ & $y = 1$ & $z = 3$, then $\bar{\mu}_A(2 + 1) \geq k r \min \{ \bar{\mu}_A(2 + 3), \bar{\mu}_A(1 - 3) \}$ is not true.

**Theorem 4.7.**

Let $\{\mu_i \mid i \in \Lambda\}$ be a family of i-v fuzzy $SA$-ideals with degree $(\lambda, k)$ of $SA$-algebra $X$, then $\cup_{\mu_i}$ is an i-v fuzzy $SA$-ideal with degree $(\lambda, k)$ of $X$, where $\mu_i \subseteq \mu_{i+1}$, for all $i \in \Lambda$.

**Proof:**
Since \( \{ \mu_i : i \in \Lambda \} \) be a family of i-v fuzzy \( S_A \)-ideals with degree \((\lambda, k)\) of \( X \) and \( \mu_i \subseteq \mu_{i+1} \), for all \( i \in \Lambda \), then for any \( x, y, z \in X \),
\[
\mu_{\cup \lambda \in \Lambda} (0) = \left[ \mu_{\cup \lambda \in \Lambda} (0), \mu_{\cup \lambda \in \Lambda} (0) \right] \geq \lambda \left[ \mu_{\cup \lambda \in \Lambda} (x, \mu_{\cup \lambda \in \Lambda} (x) \right] = \lambda \mu_{\cup \lambda \in \Lambda} (x).
\]
Suppose \( x + z \in U_{\cup \lambda \in \Lambda} \) and \( y - z \in U_{\cup \lambda \in \Lambda} \).
\[
\mu_{\cup \lambda \in \Lambda} (x + y) = k \min \left[ \mu_{\cup \lambda \in \Lambda} (x + z), \mu_{\cup \lambda \in \Lambda} (y - z) \right] = k \min \left[ \mu_{\cup \lambda \in \Lambda} (x + z), \mu_{\cup \lambda \in \Lambda} (x + z), \mu_{\cup \lambda \in \Lambda} (y - z), \mu_{\cup \lambda \in \Lambda} (y - z) \right] = k \min \left[ \mu_{\cup \lambda \in \Lambda} (x + z), \mu_{\cup \lambda \in \Lambda} (y - z) \right].
\]
Hence \( U_{\cup \lambda \in \Lambda} \) is an i-v fuzzy \( S_A \)-ideal with degree \((\lambda, k)\) of \( X \).

**Theorem 4.8.**
Let \( X \) be a \( S_A \)-algebra and \( A \) be an i-v fuzzy subset of \( X \). Then \( A \) is an i-v fuzzy \( S_A \)-ideal of \( X \) if and only if the nonempty set \( \bar{U}(A; [\delta_1, \delta_2]) \) is a \( S_A \)-ideal of \( A \).

**Proof:**
Assume that \( A \) is an i-v fuzzy \( S_A \)-ideal of degree \((\lambda, k)\) of \( X \) and
\[
\bar{U}(A; [\delta_1, \delta_2]) = [\lambda \left[ \mu_{\cup \lambda \in \Lambda} (x), \mu_{\cup \lambda \in \Lambda} (x) \right], \lambda \mu_{\cup \lambda \in \Lambda} (x), \forall x \in X \),
\]
then \( \bar{U}(A; [\delta_1, \delta_2]) \neq \varnothing \) is a \( S_A \)-ideal of \( X \), for all \( x \in X \), then
\[
\bar{U}(A; [\delta_1, \delta_2]) = \lambda \left[ \mu_{\cup \lambda \in \Lambda} (x), \mu_{\cup \lambda \in \Lambda} (x) \right], \forall x \in X \).
\]
Let \( \delta_1, \delta_2 \in D \) such that \( x \in \bar{U}(A; [\delta_1, \delta_2]) \), then
\[
\bar{U}(A; [\delta_1, \delta_2]) = \lambda \left[ \mu_{\cup \lambda \in \Lambda} (x), \mu_{\cup \lambda \in \Lambda} (x) \right], \forall x \in X \).
\]
Therefore, \( \lambda \bar{U}(A; [\delta_1, \delta_2]) = [\lambda \left[ \mu_{\cup \lambda \in \Lambda} (x), \mu_{\cup \lambda \in \Lambda} (x) \right], \forall x \in X \).

**Theorem 4.9.**
Every \( S_A \)-ideal of a \( S_A \)-algebra \( X \) can be realized as a \( S_A \)-ideal of an i-v fuzzy \( S_A \)-ideal with degree \((\lambda, k)\) of \( X \).

**Proof:**
Let $Y$ be a $SA$-ideal of $X$ and let $A$ be an i-v fuzzy subset on $X$ defined by:
\[
\tilde{\mu}_A(x) = \begin{cases} 
[\alpha_1, \alpha_2] & \text{if } x \in Y \\
[0,0] & \text{otherwise}
\end{cases}
\]

Where $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 < \alpha_2$. It is clear that $\tilde{\mu}(A; [\alpha_1, \alpha_2]) = Y$.

We show that $A$ is an i-v fuzzy $SA$-ideal with degree $(\lambda, k)$ of $X$.

Let $x, y, z \in X$. If $(x + z), (y - z) \in Y$, then $(x + y) \in Y$, therefore
\[
\tilde{\mu}_A(x + y) = [\alpha_1, \alpha_2] = k r \min ([\alpha_1, \alpha_2], [\alpha_1, \alpha_2])
\]

If $(x + z), (y - z) \notin Y$, then $\tilde{\mu}_A(x + z) = \tilde{\mu}_A(y - z) = 0,0$.

Similarly, for the case $(x + z) \notin Y$ and $(y - z) \in Y$, we get
\[
\tilde{\mu}_A(x + y) \geq k r \min \{\tilde{\mu}_A(x + z), \tilde{\mu}_A(y - z)\},
\]

Therefore $A$ is an i-v fuzzy $SA$-ideal with degree $(\lambda, k)$ of $X$. ■

Proposition 4.10.

Let $X$ be a $SA$-algebra, $Y$ be a subset of $X$ and let $A$ be an i-v fuzzy subset on $X$ defined by:
\[
\tilde{\mu}_A(x) = \begin{cases} 
[\alpha_1, \alpha_2] & \text{if } x \in Y \\
[0,0] & \text{otherwise}
\end{cases}
\]

Where $\alpha_1, \alpha_2 \in (0,1]$ with $\alpha_1 < \alpha_2$ and $x \in X$. If $A$ is an i-v fuzzy $SA$-ideal with degree $(\lambda, k)$ of $X$, then $Y$ is a $SA$-ideal of $X$.

Proof:

Since that $A$ is an i-v fuzzy $SA$-ideal with degree $(\lambda, k)$ of $X$, it is clear that $0 \notin Y$.

Let $(x + z), (y - z) \notin Y$, then $\tilde{\mu}_A(x + z) = \tilde{\mu}_A(y - z) = 0,0$.

Therefore $A$ is an i-v fuzzy $SA$-ideal with degree $(\lambda, k)$ of $X$. ■

Theorem 4.11.

If $A$ is an i-v fuzzy $SA$-ideal with degree $(\lambda, k)$ of $X$, then the set
\[
X_{\tilde{\mu}_A} := \{x \in X : \tilde{\mu}_A(x) = \tilde{\mu}_A(0)\}
\]

is a $SA$-ideal of $X$.

Proof:

Since $X_{\tilde{\mu}_A} := \{x \in X : \tilde{\mu}_A(x) = \tilde{\mu}_A(0)\}$, then $0 \in X_{\tilde{\mu}_A}$.

Let $(x + z), (y - z) \in X_{\tilde{\mu}_A}$, implies that
\[
\tilde{\mu}_A(x + z) = \tilde{\mu}_A(0) = \tilde{\mu}_A(x) = \tilde{\mu}_A(y - z), \text{ so we have}
\]

Therefore $X_{\tilde{\mu}_A}$ is a $SA$-ideal of $X$. ■

Proposition 4.12.

Every i-v fuzzy $SA$-ideal with degree $(\lambda, k)$ of $SA$-algebra $X$ is an i-v fuzzy $SA$-subalgebra of $X$.

Proof:

Since $\mu$ is an i-v fuzzy $SA$-ideal with degree $(\lambda, k)$ of $SA$-algebra $X$, then by Theorem (4.8), $\tilde{\mu}(A; [\delta_1, \delta_2])$ is a $SA$-ideal of $X$. By Proposition (2.9) $\tilde{\mu}(A; [\delta_1, \delta_2])$ is a $SA$-subalgebra of $X$. Hence $\mu$ is an i-v fuzzy $SA$-subalgebra of $SA$-algebra $X$, by Theorem (3.14). ■

Remark 4.13.

The converse of Proposition (4.12) is not true as show in the following example.

Example 4.14.
In the Example (3.1.4), It is easy to show that \((X; +, - , 0)\) is a \(SA\)-algebra.

Define a fuzzy subset \(\mu: X \rightarrow [0, 1]\) by: 
\[
\mu(x) = \begin{cases} 
0.7 & \text{if } x \in [0, 2] \\
0.3 & \text{otherwise}
\end{cases}
\]

Routine calculations give that \(\mu\) is a fuzzy \(SA\)-subalgebra of \(X\).

Define \(\tilde{\mu}_A(x)\) as follows: 
\[
\tilde{\mu}_A(x) = \begin{cases} 
[0.3, 0.9] & \text{if } x = [0, 2] \\
[0.1, 0.6] & \text{otherwise}
\end{cases}
\]

It is easy to check that \(A\) is an i-v fuzzy \(SA\)-subalgebra, but not i-v fuzzy \(SA\)-ideal with degree \((\lambda, k)\).

5. Image (Pre-image) Interval-valued Fuzzy \(SA\)-ideals with degree \((\lambda, k)\) under Homomorphism of \(SA\)-algebras.

In this section, we study the homomorphic images and inverse images of i-v fuzzy \(SA\)-ideals with degree \((\lambda, k)\) become i-v fuzzy \(SA\)-ideals with degree \((\lambda, k)\) of \(SA\)-algebra is studied as well.

**Definition 5.1 [2,4]:**

Let \(f: (X; *, 0) \rightarrow (Y; *, 0)\) be a mapping from set \(X\) into a set \(Y\). Let \(B\) be an i-v fuzzy subset of \(Y\). Then the inverse image of \(B\), denoted by \(f^{-1}(B)\) is an i-v fuzzy subset of \(X\) with the membership function given by \(\mu_{f^{-1}(B)}(x) = \tilde{\mu}_B(f(x))\), for all \(x \in X\).

**Proposition 5.2 [6]:**

Let \(f: (X; *, 0) \rightarrow (Y; *, 0)\) be a mapping from set \(X\) into set \(Y\), let \(m = [m^L, m^U]\) and \(n = [n^L, n^U]\) be i-v fuzzy subsets of sets \(X\) and \(Y\) respectively. Then:

1. \(f^{-1}(n) = [f^{-1}(n^L), f^{-1}(n^U)]\),
2. \(f(m) = [f(m^L), f(m^U)]\).

**Theorem 5.3:**

Let \(f: (X; *, 0) \rightarrow (Y; *, 0)\) be homomorphism of \(SA\)-algebras. If \(B\) is an i-v fuzzy \(SA\)-subalgebra of \(Y\), then the inverse image \(f^{-1}(B)\) of \(B\) is an i-v fuzzy \(SA\)-subalgebra of \(X\).

**Proof:**

Since \(B = [\mu_B^L, \mu_B^U]\) is an i-v fuzzy \(SA\)-subalgebra of \(Y\), it follows from Theorem (3.7), that \(\mu_B^L\) and \(\mu_B^U\) are fuzzy \(SA\)-subalgebras of \(Y\). Using theorem (4.1.3.3) in [5], we know \(f^{-1}(\mu_B^L)\) and \(f^{-1}(\mu_B^U)\) are fuzzy \(SA\)-subalgebras of \(X\). Hence by Proposition (5.2(1)) and Theorem (3.7), we conclude that \(f^{-1}(B) = [f^{-1}(\mu_B^L), f^{-1}(\mu_B^U)]\) is an i-v fuzzy \(SA\)-subalgebra of \(X\).

**Theorem 5.4:**

Let \(f: (X; *, 0) \rightarrow (Y; *, 0)\) be a homomorphism of \(SA\)-algebras. If \(A\) is an i-v fuzzy \(SA\)-subalgebra of \(X\) with sup property, then \(f(A)\) is an i-v fuzzy \(SA\)-subalgebra of \(Y\).

**Proof:**

Assume that \(A = [\mu_A^L, \mu_A^U]\) is an i-v fuzzy \(SA\)-subalgebra of \(X\), it follows from Theorem (3.7), that \(\mu_A^L\) and \(\mu_A^U\) are fuzzy \(SA\)-subalgebras of \(X\). Using theorem (4.2.3.2) in [5], the images \(f(\mu_A^L)\) and \(f(\mu_A^U)\) are fuzzy \(SA\)-subalgebra of \(Y\). Hence by Proposition (5.2(2)) and Theorem (3.7), we conclude that \(f(A) = [f(\mu_A^L), f(\mu_A^U)]\) is an i-v fuzzy \(SA\)-subalgebra of \(Y\).

**Theorem 5.5:**

Let \(f: (X; *, 0) \rightarrow (Y; *, 0)\) be homomorphism of \(SA\)-algebras. If \(B\) is an i-v fuzzy \(SA\)-ideal with degree \((\lambda, k)\) of \(Y\), then the inverse image \(f^{-1}(B)\) of \(B\) is an i-v fuzzy \(SA\)-ideal with degree \((\lambda, k)\) of \(X\).

**Proof:**

Since \(B = [\mu_B^L, \mu_B^U]\) is an i-v fuzzy \(SA\)-ideal with degree \((\lambda, k)\) of \(Y\), it follows from Theorem (4.3), that \(\mu_B^L\) and \(\mu_B^U\) are fuzzy \(SA\)-ideals of \(Y\). Using theorem (4.2.3.2) in [5], we know \(f^{-1}(\mu_B^L)\) and \(f^{-1}(\mu_B^U)\) are fuzzy \(SA\)-ideals with degree \((\lambda, k)\) of \(X\). Hence by Proposition (5.2(1)) and Theorem (4.3), we conclude that \(f^{-1}(B) = [f^{-1}(\mu_B^L), f^{-1}(\mu_B^U)]\) is an i-v fuzzy \(SA\)-ideal with degree \((\lambda, k)\) of \(X\).

**Theorem 5.6:**

Let \(f: (X; *, 0) \rightarrow (Y; *, 0)\) be a homomorphism of \(SA\)-algebras. If \(A\) is an i-v fuzzy
$SA$-ideal with degree $(\lambda, k)$ of $X$ with sup property, then $f(A)$ is an i-v fuzzy $SA$-ideal with degree $(\lambda, k)$ of $Y$.

Proof:
Assume that $A = [\mu_A^1, \mu_A^U]$ is an i-v fuzzy $SA$-ideal with degree $(\lambda, k)$ of $X$, it follows from Theorem (4.3), that $\mu_A^1$ and $\mu_A^U$ are fuzzy $SA$-ideals with degree $(\lambda, k)$ of $X$. Using theorem (4.2.3.2) in [5], the images $f(\mu_A^1)$ and $f(\mu_A^U)$ are fuzzy $SA$-ideal with degree $(\lambda, k)$ of $Y$. Hence by Proposition (5.2(2)) and Theorem (4.3), we conclude that $f(A) = [f(\mu_A^1), f(\mu_A^U)]$ is an i-v fuzzy $SA$-ideal with degree $(\lambda, k)$ of $Y$.

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