PATTERN RECOGNITION ON ORIENTED MATROIDS:
SYMMETRIC CYCLES IN THE HYPERCUBE GRAPHS. V

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ABSTRACT. We consider decompositions of topes of the oriented matroid realizable as the arrangement of coordinate hyperplanes in $\mathbb{R}^t$, with respect to a distinguished symmetric $2 \cdot 2^t$-cycle in its hypercube graph of topes $H(2^t, 2)$. We seek interpretations of such decompositions in the context of subset families on the ground set $E_t := \{1, \ldots, t\}$ and of the families of their blocking sets, in the context of clutters on $E_t$ and of their blockers.

CONTENTS

1. Introduction 2

Decomposing 7
2. Topes, their relabeled opposites, and decompositions 7

Blocking 12
3. Increasing families of blocking sets, and blockers: Set covering problems 14
4. Families of subsets of the ground set $E_t$: Characteristic vectors and characteristic topes 17
5. Increasing families of blocking sets, and blockers: Characteristic vectors and characteristic topes 21
5.1. A clutter $\{\{a\}\}$ 22
5.1.1. The principal increasing family of blocking sets $\mathfrak{B}(\{\{a\}\}^\vee) = \{\{a\}\}^\vee$ 22
5.1.2. The blocker $\mathfrak{B}(\{\{a\}\}) = \{\{a\}\}$ 23
5.1.3. More on the principal increasing family $\mathfrak{B}(\{\{a\}\}^\vee) = \{\{a\}\}^\vee$ 23
5.2. A clutter $\{A\}$ 26
5.2.1. The increasing family of blocking sets $\mathfrak{B}(\{A\}^\vee) = \{\{a\}: a \in A\}^\vee$ 26
5.2.2. The blocker $\mathfrak{B}(\{A\}) = \{\{a\}: a \in A\}$ 27
5.2.3. More on the increasing families $\{A\}^\vee$ and $\mathfrak{B}(\{A\})^\vee$ 28
5.3. A clutter $\mathcal{A} := \{A_1, \ldots, A_n\}$ 28
5.3.1. The increasing family of blocking sets $\mathfrak{B}(\mathcal{A})^\vee$ 29
5.3.2. The blocker $\mathfrak{B}(\mathcal{A})$ 29
5.3.3. More on the increasing families $\mathcal{A}^\vee$ and $\mathfrak{B}(\mathcal{A})^\vee$ 30
5.3.4. The characteristic vector of the subfamily of inclusion-minimal
sets \( \text{min} \mathcal{F} \) in a family \( \mathcal{F} \)

5.3.5. More on the blocker \( \mathcal{B}(A) \)

**Blocking / Voting**

6. Decompositions of the characteristic topes and of the
characteristic vectors of families

6.1. A clutter \( \{ \{a\} \} \)

6.2. A clutter \( \{ A \} \)

6.3. A clutter \( \mathcal{A} := \{ A_1, \ldots, A_\alpha \} \)

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1. Introduction

Let \( \mathcal{H} := (E_t, \{1, -1\}^t) \) be the oriented matroid on its ground set \( E_t := [t] := [1, t] := \{1, \ldots, t\} \), where \( t \geq 3 \), and with its set of topes \( \{1, -1\}^t \). This oriented matroid is realizable as the arrangement of coordinate hyperplanes in the real Euclidean space \( \mathbb{R}^t \supset \{1, -1\}^t \) of row vectors, see [14, Example 4.1.4].

See, e.g., [8, 18, 19, 36, 62, 78, 84] on oriented matroids.

Each of the \( 2^t \) maximal covectors \( T := (T(1), \ldots, T(t)) \in \{1, -1\}^t \) of \( \mathcal{H} \) can be regarded as the characteristic tope of the negative part \( T^- := \{ e \in E_t : T(e) = -1 \} \). Conversely, given an arbitrary subset \( A \subseteq E_t \), we define the characteristic tope of \( A \) to be the reorientation \( -A T^+(+) \) of the positive tope \( T^+(+) := (1, \ldots, 1) \) on the subset \( A \); recall that \( (-A T^+(+))^- := A \).

Let \( \mathcal{H}(t, 2) \) denote the hypercube graph of topes of the oriented matroid \( \mathcal{H} \), that is, the vertex set of the graph \( \mathcal{H}(t, 2) \) is the set \( \{1, -1\}^t \), and the edges of \( \mathcal{H}(t, 2) \) are the pairs \( \{ T', T'' \} \subset \{1, -1\}^t \), such that \( |\{ e \in E_t : T'(e) \neq T''(e) \}| = 1 \).

Let \( R := (R^0, R^1, \ldots, R^{2t-1}, R^0) \) be a distinguished symmetric cycle in the graph \( \mathcal{H}(t, 2) \), where

\[
R^0 := T^+(+)
\]

\[
R^s := -[s] R^0, \quad 1 \leq s \leq t - 1,
\]

and

\[
R^{t+k} := -R^k, \quad 0 \leq k \leq t - 1.
\]

For any vertex \( T \in \{1, -1\}^t \) of the graph \( \mathcal{H}(t, 2) \), there exists a unique inclusion-minimal subset

\[
Q(T, R) \subset V(R) := (R^0, R^1, \ldots, R^{2t-1})
\]

of the vertex sequence \( V(R) \) of the cycle \( R \), such that

\[
T = \sum_{Q \in Q(T,R)} Q,
\]
and see [68, §11.1]. This subset \(Q(T, R) \subset \mathbb{R}^t\) is linearly independent, and it contains an odd number \(q(T) := q(T, R) := |Q(T, R)|\) of topes. In fact, the linear algebraic decomposition (1.4) is just a way to describe a particular mechanism of majority voting.

Let \(\sigma(e)\) denote the \(e\)th standard unit vector of the space \(\mathbb{R}^t\), \(e \in [t]\). The bijections
\[
\{1, -1\}^t \to \{0, 1\}^t: \quad T \mapsto \frac{1}{2}(T(+) - T), \quad (1.5)
\]
and
\[
\{0, 1\}^t \to \{1, -1\}^t: \quad \bar{T} \mapsto T(+) - 2\bar{T}, \quad (1.6)
\]
between the vertex set \(\{1, -1\}^t\) of the hypercube graph \(H(t, 2)\) and the vertex set \(\{0, 1\}^t\) of the hypercube graph \(\widetilde{H}(t, 2)\) allow us to associate with the symmetric cycle \(R\) in the graph \(H(t, 2)\) a symmetric cycle \(\bar{R} := (\bar{R}^0, \bar{R}^1, \ldots, \bar{R}^{2t-1}, \bar{R}^t)\) in the graph \(\widetilde{H}(t, 2)\), where
\[
\bar{R}^0 := (0, \ldots, 0), \quad \bar{R}^s := \sum_{e \in [s]} \sigma(e), \quad 1 \leq s \leq t - 1,
\]
and
\[
\bar{R}^{t+k} := T(+) - \bar{R}^k, \quad 0 \leq k \leq t - 1.
\]
For any vertex \(\bar{T}\) of the hypercube graph \(\widetilde{H}(t, 2)\), let us define a subset \(\bar{Q}(\bar{T}, \bar{R}) \subset V(\bar{R}) := (\bar{R}^0, \bar{R}^1, \ldots, \bar{R}^{2t-1})\) indirectly, via the mapping
\[
\bar{T} \overset{(1.6)}{\mapsto} T,
\]
and via the bijection
\[
Q(T, R) \overset{(1.5)}{\mapsto} \bar{Q}(\bar{T}, \bar{R}).
\]
Involving the quantity \(q(\bar{T}) := q(\bar{T}, \bar{R}) := |\bar{Q}(\bar{T}, \bar{R})| = q(T)\), we can write down the decomposition
\[
\bar{T} = -\frac{1}{2}(q(\bar{T}) - 1) \cdot T(+) + \sum_{\bar{Q} \in \bar{Q}(\bar{T}, \bar{R}) \setminus \bar{Q} \neq (0, \ldots, 0) =: \bar{R}^0} \bar{Q}, \quad (1.7)
\]
that describes yet another mechanism of majority voting,\(^1\) but this decomposition has no essential meaning from the linear algebraic viewpoint, since

\(^1\) Let \(2^{[t]}\) denote the power set (i.e., the family of all subsets) of \(E_t\). Recall that maps
\[
f: \{0, 1\}^t \to \mathbb{R}, \quad -\frac{1}{2}(q(\bar{T}) - 1) \cdot T(+) + \sum_{\bar{Q} \in \bar{Q}(\bar{T}, \bar{R}) \setminus \bar{Q} \neq (0, \ldots, 0) =: \bar{R}^0} \bar{Q} = \bar{T} \mapsto f(\bar{T}),
\]
and
\[
g: \{1, -1\}^t \to \mathbb{R}, \quad \sum_{\bar{Q} \in \bar{Q}(\bar{T}, \bar{R})} \bar{Q} = T \mapsto g(T),
\]
the set $\tilde{Q} \in \tilde{Q}(\tilde{T}, \tilde{R})$ can contain the origin $(0, \ldots, 0) =: \tilde{R}^0$ of the space $\mathbb{R}^t$, which should be omitted in calculations.

For the topes $T \in \{1, -1\}^t$ of the oriented matroid $\mathcal{H}$, we define topes $\text{ro}(T) \in \{1, -1\}^t$ by

$$\text{ro}(T) := -T \bar{U}(t),$$

where $\bar{U}(t)$ denotes the backward identity matrix (with the rows and columns indexed starting with 1) of order $t$ whose $(i, j)$th entry is the Kronecker delta $\delta_{i+j,t+1}$.

For vertices $\tilde{T}$ of the discrete hypercube $\{0, 1\}^t$, the counterparts of topes $\text{ro}(T)$ of the oriented matroid $\mathcal{H}$ are vertices $\text{rn}(\tilde{T}) \in \{0, 1\}^t$, defined by

$$\text{rn}(\tilde{T}) := T^{(+)} - \tilde{T} \bar{U}(t).$$

For example, suppose

$$T := (1, -1, 1, -1, -1) \in \{1, -1\}^5,$$

$$\tilde{T} := (0, 1, 0, 1) \in \{0, 1\}^5.$$

Then we have

$$\text{ro}(T) = (1, 1, -1, 1, -1),$$

$$\text{rn}(\tilde{T}) = (0, 0, 1, 0, 1).$$

- In the first part of the paper we compare the decompositions $Q(T, R)$ and $Q(\text{ro}(T), R)$ of topes $T$ and $\text{ro}(T)$ with respect to the symmetric cycle $R$ in the graph $H(t, 2)$, defined by (1.1)(1.2).

Our interest in considering relabeled opposites $\text{ro}(T)$ and relabeled negations $\text{rn}(\tilde{T})$ lies in their application to combined blocking/voting-models of increasing families of sets and of clutters. We study those impractical $2^t$-dimensional vector models in order to gain a better understanding of the structure of families.

Recall that a family $\mathcal{A} := \{A_1, \ldots, A_\alpha\} \subset 2^t$ of subsets$^5$ of the ground set $E_t$ is called a clutter$^6$ if no set $A_i$ from $\mathcal{A}$ contains another set $A_j$.

\[\text{footnotesize}
\begin{enumerate}
\item Maps $f: 2^t \to \mathbb{R}$ are set functions. Maps $f: \{0, 1\}^t \to \{0, 1\}$, and $g: \{1, -1\}^t \to \{1, -1\}$ are Boolean functions.
\item $\text{ro}(T)$ means the relabeled opposite of $T$.
\item In [68, Sect. 2.1] the similar notation $U(m)$ was used to denote the backward identity matrix of order $(m + 1)$ whose rows and columns were indexed starting with zero.
\item $\text{rn}(\tilde{T})$ means the relabeled negation of $\tilde{T}$.
\item We denote by $\emptyset$ the empty subset of the ground set $E_t$, and we let $\emptyset$ denote the empty family containing no sets.
\item Given a family $\mathcal{F} \subseteq 2^t$, such that $\emptyset \neq \mathcal{F} \neq \emptyset$, the set $E_t := [t]$ is the ground set of $\mathcal{F}$, while the set $V(\mathcal{F}) := \bigcup_{F \in \mathcal{F}} F \subseteq E_t$ is the vertex set of $\mathcal{F}$.
\item The families $\emptyset$ and $\{\emptyset\}$ are the two trivial clutters on the ground set $E_t$. The other clutters on $E_t$ are nontrivial.
\end{enumerate}
\[\text{footnotesize}^6\text{ Or Sperner family, antichain, simple hypergraph.}
Given a family $\mathcal{F} \subseteq 2^{[t]}$, we let $\min \mathcal{F}$ denote the clutter composed of the inclusion-minimal sets in $\mathcal{F}$.

We say that a family of subsets $\mathcal{F} \subseteq 2^{[t]}$ is an increasing family if the following implications hold:

$$A \in \mathcal{F}, \ 2^{[t]} \ni B \supseteq A \implies B \in \mathcal{F}.$$  

If $C \subseteq E_t$, then the family $\{C\}^\uparrow := \{D \subseteq E_t : D \supseteq C\}$ is called the principal increasing family generated by the one-member clutter $\{C\}$. Conversely, an increasing family $\mathcal{F} \subseteq 2^{[t]}$ is said to be principal if $\# \min \mathcal{F} = 1$.

Given an arbitrary nonempty family $C \subseteq 2^{[t]}$, we denote by $C^\uparrow$ the increasing family on $E_t$, generated by $C$:

$$C^\uparrow := \bigcup_{C \in C} \{C\}^\uparrow = \bigcup_{C \in \min C} \{C\}^\uparrow.$$  

“Decreasing” constructs are defined in the obvious similar way.

The duality philosophy behind clutters and increasing families is that any clutter is the blocker of a unique clutter, and any increasing family is the family of blocking sets of a unique clutter.

We often meet in the literature the free distributive lattice of antichains in the Boolean lattice of subsets of a finite nonempty set, ordered by containment of the corresponding generated order ideals, but an intrinsically related construct, the free distributive lattice of those antichains ordered by containment of the corresponding generated order filters has greater discrete

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7 Or up-set, upward-closed family of sets, filter of sets.

8 We denote by $| \cdot |$ the cardinality of a set, and we denote by $\# \cdot$ the number of sets in a family.

9 For a family $\mathcal{F} \subseteq 2^{[t]}$, we use the notation $\max \mathcal{F}$ to denote the clutter composed of the inclusion-maximal sets in $\mathcal{F}$.

A family $\mathcal{F} \subseteq 2^{[t]}$ is said to be a decreasing family (or down-set, downward-closed family of sets, ideal of sets) if the following implications hold:

$$B \in \mathcal{F}, \ A \subset B \implies A \in \mathcal{F}.$$  

If $\emptyset \neq \mathcal{F} \neq \{\hat{0}\}$, then this decreasing family is the abstract simplicial complex on its vertex set $\bigcup_{M \in \max \mathcal{F}} M$, with the facet family $\max \mathcal{F}$.

If $D \subseteq E_t$, then the family $\{D\}^\downarrow := \{C : C \subseteq D\}$ is called the principal decreasing family generated by the one-member clutter $\{D\}$. Conversely, a decreasing family $\mathcal{F} \subseteq 2^{[t]}$ is said to be principal if $\# \max \mathcal{F} = 1$.

Given an arbitrary nonempty family $D \subseteq 2^{[t]}$, we denote by $D^\downarrow$ the decreasing family on $E_t$, generated by $D$:

$$D^\downarrow := \bigcup_{D \in D} \{D\}^\downarrow = \bigcup_{D \in \max D} \{D\}^\downarrow.$$  

10 I enjoyed working with Ray [Fulkerson] and I coined the terms “clutter” and “blocker”.  

Jack Edmonds [37, p. 201]
mathematical expressiveness, because the latter lattice can be interpreted as the lattice of blockers, for which the blocker map is its anti-automorphism.\footnote{For this paper, we chose the language of power sets, clutters, and increasing and decreasing families. A parallel exposition could be presented in poset-theoretic terms of Boolean lattices, antichains, and order filters and ideals.}

Recall that a subset $B \subseteq E_t$ is called a blocking set\footnote{Or transversal, hitting set, vertex cover (or node cover), system of representatives.} of a subset family $F \subseteq 2^{[t]}$, where $\emptyset \neq F \neq 0$, if we have
\[ |B \cap F| > 0 , \]
for each set $F \in F$. The blocker\footnote{Or blocking hypergraph (or transversal hypergraph), blocking clutter, dual clutter, Alexander dual clutter.} $\mathcal{B}(F)$ of the family $F$ is the family of all inclusion-minimal blocking sets of $F$; note that we have $\mathcal{B}(F) = \mathcal{B}(\min_{min} F)$. The notation $\mathcal{B}(F)^\vee$ just means the increasing family of all blocking sets of the family $F$.

For a nonempty family of subsets $F \subseteq 2^{[t]}$, we define a family\footnote{Given a nonempty family of subsets $F \subseteq 2^{[t]}$, we define a family of complements $F^\perp$ by $F^\perp := \{ F^\perp : F \in F \}$, where $F^\perp := V(F) - F$.} of complements $F^\perp$ by $F^\perp := \{ F^\perp : F \in F \}$, where $F^\perp := E_t - F$.

Given a nontrivial clutter $A \subseteq 2^{[t]}$, one associates with $A$ the four extensively studied partitions of the power set of the ground set $E_t$:
\[ 2^{[t]} = A^\triangledown \cup (\mathcal{B}(A)^\triangledown)^\Delta , \]
\[ 2^{[t]} = A^\Delta \cup \mathcal{B}(A)^\triangledown , \]
\[ 2^{[t]} = \mathcal{B}(A)^\triangledown \cup (A^\Delta)^\Delta , \]
and
\[ 2^{[t]} = \mathcal{B}(A)^\Delta \cup \mathcal{B}(\mathcal{B}(A)^\Delta)^\triangledown . \]

• In the second part of the paper we arrange the subsets of the ground set $E_t$ in linear order. We then turn to the so-called characteristic vectors $\gamma(F) \in \{0, 1\}^{2^t}$ of subset families $F \subseteq 2^{[t]}$. If $A \subseteq 2^{[t]}$ is a nontrivial clutter on $E_t$, then relation (1.10) reformulated in the form (cf. (1.9))
\[ \gamma(\mathcal{B}(A)^\triangledown) = T^{(+)}_{2^t} - \gamma(A^\triangledown) \cdot \overline{U(2^t)} , \]
where $T^{(+)}_{2^t} := (1, \ldots, 1)$ is the $2^t$-dimensional row vector of all 1’s, provides us with the characteristic vector
\[ \gamma(\mathcal{B}(A)^\triangledown) = \gamma(\mathcal{B}(A)^\triangledown) \]
of the increasing family of blocking sets $\mathcal{B}(A)^\triangledown$ of the clutter $A$.

• In the third part of the paper we mention a blocking/voting-connection of the characteristic vectors $\gamma(A^\triangledown)$ and $\gamma(\mathcal{B}(A)^\triangledown)$ with the decompositions of the corresponding characteristic topes of the increasing families $A^\triangledown$.
and $\mathfrak{B}(\mathcal{A})^\top$ with respect to a distinguished symmetric cycle in the hypercube graph $H(2^t, 2)$, which is analogous to the cycle $(1.1)(1.2)$ in the graph $H(t, 2)$.

## Decomposing

### 2. Topes, their relabeled opposites, and decompositions

In this section we consider vertices $T$ of the discrete hypercube $\{1, -1\}^t$, their relabeled opposites $\text{ro}(T)$ defined by (1.8), and we discuss basic properties of the decompositions $Q(T, R)$ and $Q(\text{ro}(T), R)$ of the topes $T$ and $\text{ro}(T)$ with respect to the distinguished symmetric cycle $R := (R_0, R_1, \ldots, R_{2^t - 1}, R_0)$ in the graph $H(t, 2)$, defined by (1.1)(1.2).

- Definitions (1.8) and (1.9) determine the maps $\{1, -1\}^t \rightarrow \{1, -1\}^t$: $T \mapsto \text{ro}(T) := -T \overline{U}(t)$, \hspace{1cm} (2.1)

and since we deal with the standard one-to-one correspondences between the vertex sets of the discrete hypercubes $\{1, -1\}^t$ and $\{0, 1\}^t$, established by means of the maps (1.5) and (1.6), we mention the mappings $\{1, -1\}^t \ni \text{ro}(T) \overset{(1.5)}{\mapsto} \text{rn}(\overline{T}) = \frac{1}{2}(T^{(+) - 2T \overline{U}(t)}) \in \{0, 1\}^t$, \hspace{1cm} (2.2)

and since $t$ is odd, then we always have $\text{ro}(T) \neq T$, and $\text{rn}(\overline{T}) \neq \overline{T}$. 

Note that we have $\{1, -1\}^t \ni \text{ro}(T) = T \iff -T = T \overline{U}(t)$; \hspace{1cm} (2.3)
Let $\langle \cdot, \cdot \rangle$ denote the standard scalar product on the space $\mathbb{R}^t$.
For vertices $\vec{T} \in \{0,1\}^t$ and $T := -\supp(\vec{T}) \in \{1, -1\}^t$, we have

$$
\langle T, \text{ro}(T) \rangle = -T \text{U}(t)T^\top = - \sum_{e \in [t]} T(e)T(t - e + 1)
= \begin{cases}
-1 - 2 \sum_{e \in [(t-1)/2]} T(e)T(t - e + 1), & \text{if } t \text{ is odd}, \\
-2 \sum_{e \in [t/2]} T(e)T(t - e + 1), & \text{if } t \text{ is even};
\end{cases}
$$

$$
\langle \vec{T}, \text{rn}(\vec{T}) \rangle := \langle \vec{T}, \text{T}^{(+) \top} - \vec{T} \text{U}(t) \rangle = \text{hwt}(\vec{T}) - \sum_{e \in [t]} \tilde{T}(e)\tilde{T}(t - e + 1)
= \text{hwt}(\vec{T})\begin{cases}
\tilde{T}((t+1)/2) + 2 \sum_{e \in [(t-1)/2]} \tilde{T}(e)\tilde{T}(t - e + 1), & \text{if } t \text{ is odd}, \\
2 \sum_{e \in [t/2]} \tilde{T}(e)\tilde{T}(t - e + 1), & \text{if } t \text{ is even}.
\end{cases}
$$

Given two words $X, Y \in \{-1,0,1\}^t$, we let $d(X, Y) := |\{e \in E_t : X(e) \neq Y(e)\}|$ denote the Hamming distance between them.\(^{15}\)
Since the equal distances $d(T, \text{ro}(T)) = d(\vec{T}, \text{rn}(\vec{T}))$ can be calculated with the help of the formulas (see (1.6) and (2.3))

$$
d(T, \text{ro}(T)) = \frac{1}{2}(t - \langle T, \text{ro}(T) \rangle),
$$

$$
d(\vec{T}, \text{rn}(\vec{T})) = \frac{1}{2}(t - \langle \text{T}^{(+) \top} - 2\vec{T}, -\text{T}^{(+) \top} + 2\vec{T} \text{U}(t) \rangle),
$$

we see that

$$
d(T, \text{ro}(T)) = \frac{1}{2}(t + T \text{U}(t)T^\top) = \frac{1}{2}(t + \sum_{e \in [t]} T(e)T(t - e + 1))
= \frac{t}{2} + \left\{ \begin{array}{ll}
\frac{1}{2} + \sum_{e \in [(t-1)/2]} T(e)T(t - e + 1), & \text{if } t \text{ is odd}, \\
\sum_{e \in [t/2]} T(e)T(t - e + 1), & \text{if } t \text{ is even},
\end{array} \right.
$$

and

$$
d(\vec{T}, \text{rn}(\vec{T})) = t - 2 \cdot \text{hwt}(\vec{T}) + 2 \sum_{e \in [t]} \tilde{T}(e)\tilde{T}(t - e + 1)
= t - 2 \cdot \text{hwt}(\vec{T})
+ 2 \begin{cases} \tilde{T}((t+1)/2) + 2 \sum_{e \in [(t-1)/2]} \tilde{T}(e)\tilde{T}(t - e + 1), & \text{if } t \text{ is odd}, \\
2 \sum_{e \in [t/2]} \tilde{T}(e)\tilde{T}(t - e + 1), & \text{if } t \text{ is even}.
\end{cases}
$$

\(^{15}\) If $X$ and $Y$ are topes, then one says that $d(X, Y)$ is the graph distance.
• Suppose that \( 4 | t \) (i.e., \( t \) is divisible by 4). Note that

\[
\langle T, \text{ro}(T) \rangle = 0 \iff \sum_{e \in [t/2]} T(e)T(t - e + 1) = 0 ;
\]

\[
\langle T, \text{ro}(T) \rangle = 0 \iff \sum_{e \in [t/2]} \tilde{T}(e)\tilde{T}(t - e + 1) = \frac{4 \cdot \text{hwt}(\tilde{T}) - t}{8} .
\]

• Considering the restriction of the map (2.1) to the vertex set \( V(R) \) of the symmetric cycle \( R \) in the hypercube graph \( H(t, 2) \), defined by (1.1)(1.2), we have the mappings

\[
R^i \overset{(2.1)}{\mapsto} \text{ro}(R^i) = R^{3t-i} \mod 2t = \begin{cases} R^{t-i}, & \text{if } 0 \leq i \leq t , \\ R^{3t-i}, & \text{if } t + 1 \leq i \leq 2t - 1 . \end{cases}
\]

If \( t \) is even, then the following implication holds:

\[
R^i \in V(R), \; \text{ro}(R^i) = R^i \implies i \in \{ \frac{t}{2}, \frac{3t}{2} \} .
\]

**Remark 2.1.** Let \( R \) be the symmetric cycle in the hypercube graph \( H(t, 2) \), defined by (1.1)(1.2). Given a vertex \( T \in \{ -1, 1 \}^t \) of \( H(t, 2) \), suppose that

\[
(R^0, R^1, \ldots, R^{2t-1}) =: V(R) \supset Q(T, R) = (R^{i_0}, R^{i_1}, \ldots, R^{i_{q(T)-1}}) ,
\]

for some indices \( i_0 < i_1 < \cdots < q(T) - 1 \).

(i) We have

\[
Q(\text{ro}(T), R) = (R^{3t-i_0} \mod 2t, R^{3t-i_1} \mod 2t, \ldots, R^{3t-i_{q(T)-1}} \mod 2t) ,
\]

or, in other words,

\[
Q(\text{ro}(T), R) = \{ R^{3t-i} \mod 2t : R^i \in Q(T, R) \} .
\]

(ii) If \( t \) is even, then

\[
\text{ro}(T) = T \iff \left( Q \in Q(T, R) \implies \text{ro}(Q) \in Q(T, R) \right) .
\]

Note that the following implication holds:

\[
\text{ro}(T) = T \implies |\{ R^{t/2}, R^{3t/2} \} \cap Q(T, R)| = 1 .
\]

• Recall that for any vertex \( T \in \{ -1, 1 \}^t \) of the hypercube graph \( H(t, 2) \) with its distinguished symmetric cycle \( R \) defined by (1.1)(1.2), there exists a unique row vector \( \mathbf{x} := \mathbf{x}(T) := \mathbf{x}(T, R) := (x_1, \ldots, x_t) \in \{ -1, 0, 1 \}^t \) such that

\[
T = \sum_{i \in [t]} x_i \cdot R^{i-1} = \mathbf{x} \mathbf{M} ,
\]

where

\[
\mathbf{M} := \mathbf{M}(R) := \begin{pmatrix} R^0 \\ R^1 \\ \vdots \\ R^{t-1} \end{pmatrix} .
\]
In other words, the inclusion-minimal linearly independent set $Q(T,R)$ of odd cardinality, given in (1.3)(1.4), is described as

$$Q(T,R) = \{x_i \cdot R_i^{-1} : x_i \neq 0\}.$$ 

Recall that if $x_e \neq 0$ for some $e \in E_t$, then $x_e = T(e)$.

We will now give an explicit description of decompositions $Q(T,R)$ and $Q(\text{ro}(T),R)$ via the corresponding “$x$-vectors”.

Let $T(t)$ denote the forward shift matrix of order $t$ whose $(i,j)$th entry is $\delta_{i,j-1}$.

**Proposition 2.2.** [69, Prop. 2.4, extended] Let $R$ be the symmetric cycle in the hypercube graph $H(t,2)$, defined by (1.1)(1.2).

Let $A$ be a nonempty subset of the ground set $E_t$, regarded as a disjoint union

$$A = [i_1,j_1] \cup [i_2,j_2] \cup \cdots \cup [i_{\varrho-1},j_{\varrho-1}] \cup [i_{\varrho},j_{\varrho}]$$

of intervals such that

$$j_1 + 2 \leq i_2, \quad j_2 + 2 \leq i_3, \quad \ldots, \quad j_{\varrho-2} + 2 \leq i_{\varrho-1}, \quad j_{\varrho-1} + 2 \leq i_{\varrho},$$

for some $\varrho := \rho(A)$.

(i) (a) If $\{1,t\} \cap A = \{1\}$, then we have

$$|Q(\text{ro}(A),R)| = 2\varrho - 1,$$

$$x(\text{ro}(A),R) = \sum_{1 \leq k \leq \varrho} \sigma(j_k + 1) - \sum_{2 \leq \ell \leq \varrho} \sigma(i_\ell).$$

(b) Since

$$\{t-e+1: e \in E_t - A\} = [1,t-j_1] \cup [t-i_1 + 2, t-j_{\varrho-1}] \cup \cdots \cup [t-i_3 + 2, t-j_2] \cup [t-i_2 + 2, t-j_1],$$

and $\{1,t\} \cap \{t-e+1: e \in E_t - A\} = \{1\}$, we see that

$$|Q(\text{ro}(A),R)| = 2\varrho - 1,$$

$$x(\text{ro}(A),R) = \sum_{1 \leq k \leq \varrho} \sigma(t - j_k + 1) - \sum_{2 \leq \ell \leq \varrho} \sigma(t - i_\ell + 2).$$

(c) Note that

$$x(\text{ro}(A),R) = x(\text{ro}(A),R) \cdot U(t) \cdot T(t).$$

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In [68, Sect. 2.1] the similar notation $T(m)$ was used to denote the forward shift matrix of order $(m+1)$ whose rows and columns were indexed starting with zero.
(ii) (a) If \( \{1, t\} \cap A = \{1, t\} \), then
\[
|Q(\mathcal{T}^+, R)| = 2^q - 1,
\]
\[
x(\mathcal{T}^+, R) = -\sigma(1) + \sum_{1 \leq k \leq e-1} \sigma(j_k + 1) - \sum_{2 \leq \ell \leq e} \sigma(i_\ell).
\]

(b) Since
\[
\{t - e + 1: e \in E_t - A\} = [\nu, t - j] + 2, t - j_{e-1} \cup [t - i_{e-1} + 2, t - j_{e-2}]
\]
\[
\cup \cdots \cup [t - i_3 + 2, t - j_2] \cup [t - i_2 + 2, t - j_1],
\]
and \( \{1, t\} \cap \{t - e + 1: e \in E_t - A\} = 0 \), we have
\[
|Q(\mathcal{T}^+, R)| = 2^q - 1,
\]
\[
x(\mathcal{T}^+, R) = \sigma(1) + \sum_{1 \leq k \leq e-1} \sigma(t - j_k + 1) - \sum_{2 \leq \ell \leq e} \sigma(t - i_\ell + 2).
\]

(c) Note that
\[
x(\mathcal{T}^+, R) = \sigma(1) + x(\mathcal{T}^+, R) \cdot \overline{U}(t) \cdot \overline{T}(t).
\]

(iii) (a) If \( \{1, t\} \cap A = \emptyset \), then
\[
|Q(\mathcal{T}^+, R)| = 2^q + 1,
\]
\[
x(\mathcal{T}^+, R) = \sigma(1) + \sum_{1 \leq k \leq e} \sigma(j_k + 1) - \sum_{1 \leq \ell \leq e} \sigma(i_\ell).
\]

(b) Since
\[
\{t - e + 1: e \in E_t - A\} = [\nu, t - j] + 2, t - j_{e-1} \cup [t - i_{e-1} + 2, t - j_{e-2}]
\]
\[
\cup \cdots \cup [t - i_3 + 2, t - j_2] \cup [t - i_2 + 2, t - j_1],
\]
and \( \{1, t\} \cap \{t - e + 1: e \in E_t - A\} = \{1, t\} \), we have
\[
|Q(\mathcal{T}^+, R)| = 2^q + 1,
\]
\[
x(\mathcal{T}^+, R) = -\sigma(1) + \sum_{1 \leq k \leq e} \sigma(t - j_k + 1) - \sum_{1 \leq \ell \leq e} \sigma(t - i_\ell + 2).
\]

(c) Note that
\[
x(\mathcal{T}^+, R) \cdot \overline{U}(t), R) = -\sigma(1) + x(\mathcal{T}^+, R) \cdot \overline{U}(t) \cdot \overline{T}(t).
\]

(iv) (a) If \( \{1, t\} \cap A = \{t\} \), then
\[
|Q(\mathcal{T}^+, R)| = 2^q - 1,
\]
\[
x(\mathcal{T}^+, R) = \sum_{1 \leq k \leq e-1} \sigma(j_k + 1) - \sum_{1 \leq \ell \leq e} \sigma(i_\ell).
\]
(b) Since
\[ \{ t - e + 1 : e \in E_t - A \} = [t - i_\varrho + 2, t - j_{\varrho - 1}] \cup [t - i_{\varrho - 1} + 2, t - j_{\varrho - 2}] \]
\[ \cup \cdots \cup [t - i_2 + 2, t - j_1] \cup [t - i_1 + 2, t] , \]
and \( \{ 1, t \} \cap \{ t - e + 1 : e \in E_t - A \} = \{ t \} \), we see that
\[ |Q(\mathbf{ro}(-_A T^{(+)}), R)| = 2\varrho - 1 , \]
\[ x(\mathbf{ro}(-_A T^{(+)}), R) = \sum_{1 \leq k \leq \varrho - 1} \sigma(t - j_k + 1) - \sum_{1 \leq \ell \leq \varrho} \sigma(t - i_\ell + 2) . \]

(c) Note that
\[ x(\mathbf{ro}(-_A T^{(+)}), R) = x(-_A T^{(+)}, R) \cdot U(t) \cdot T(t) . \]

**Corollary 2.3.** Let \( R \) be the symmetric cycle in the hypercube graph \( H(t, 2) \), defined by (1.1)(1.2).

For any vertex \( T \in \{ 1, -1 \}^t \) of the graph \( H(t, 2) \) we have
\[ q(\mathbf{ro}(T)) := |\text{supp}(x(\mathbf{ro}(T), R))| = |\text{supp}(x(T, R))| =: q(T) . \]

(i) If \( |T^- \cap \{ 1, t \}| = 1 \), then
\[ x(\mathbf{ro}(T)) = x(T) \cdot U(t) \cdot T(t) . \]

(ii) If \( |T^- \cap \{ 1, t \}| = 2 \), then
\[ x(\mathbf{ro}(T)) = \sigma(1) + x(T) \cdot U(t) \cdot T(t) . \]

(iii) If \( |T^- \cap \{ 1, t \}| = 0 \), then
\[ x(\mathbf{ro}(T)) = -\sigma(1) + x(T) \cdot U(t) \cdot T(t) . \]

**Blocking**

Blocking sets and the blockers of set families (families are often regarded as the hyperedge families of hypergraphs) are discussed, e.g., in the monographs [11, 22, 29, 31, 32, 33, 44, 47, 49, 51, 52, 54, 55, 57, 68, 71, 73, 74, 79, 80, 82] and in the works [3, 4, 5, 6, 7, 9, 10, 12, 13, 16, 17, 20, 21, 23, 24, 25, 26, 27, 28, 30, 34, 35, 39, 40, 41, 42, 43, 45, 46, 48, 53, 58, 59, 60, 61, 66, 67, 72, 75, 76, 77, 83].

• Let
\[ \mathfrak{O}(t) := \{ A \subset 2^t : A = \min A = \max A \} \]
denote the family of clutters on the ground set \( E_t \). The map
\[ \mathfrak{O}(t) \rightarrow \mathfrak{O}(t) , \quad A \mapsto \mathfrak{B}(A) , \quad (2.4) \]
is called the blocker map on clutters [30].

• If the (abstract simplicial) complex \( \Delta := (\mathfrak{B}(A))^\Delta \) appearing in (2.5) and the complex \( \Delta' := \{ F \in \mathcal{A}^\Delta : F \in A_T \} \) both have the same vertex set \( E_t \), then the complex \( \Delta' \) is called the Alexander dual of the complex \( \Delta \); see, e.g., [82] and [15] on combinatorial Alexander duality.
Given a clutter $\mathcal{A}$, the quantity
$$\tau(\mathcal{A}) := \min\{|B| : B \in \mathcal{B}(\mathcal{A})\}$$
is called the \textit{transversal number}\textsuperscript{17} of $\mathcal{A}$.

- Recall a classical result in combinatorial optimization: For any clutter $\mathcal{A}$ we have
  $$\mathcal{B}(\mathcal{B}(\mathcal{A})) = \mathcal{A},$$
  see [38, 56, 64, 65].

- For a nontrivial clutter $\mathcal{A} \subset 2^{[t]}$ on the ground set $E_t$, we have
  $$\#(\mathcal{B}(\mathcal{A})^\uparrow \cap (E_s^t)) + \#(\mathcal{A}^\uparrow \cap (E_{t-s}^t)) = \binom{t}{s}, \quad (2.6)$$
  where
  $$\{E_s^t\} := \{F \subseteq E_t : |F| = s\}$$
is the \textit{complete s-uniform clutter} on the vertex set $E_t$.

The increasing families $\mathcal{A}^\uparrow$ and $\mathcal{B}(\mathcal{A})^\uparrow$ are \textit{comparable by containment}: either we have
$$\mathcal{B}(\mathcal{A})^\uparrow \subseteq \mathcal{A}^\uparrow,$$
or
$$\mathcal{B}(\mathcal{A})^\uparrow \supseteq \mathcal{A}^\uparrow.$$

The following implications hold:
$$\mathcal{B}(\mathcal{A})^\uparrow \subseteq \mathcal{A}^\uparrow \iff \#\mathcal{A}^\uparrow > 2^{t-1};$$
$$\mathcal{B}(\mathcal{A})^\uparrow \supseteq \mathcal{A}^\uparrow \iff \#\mathcal{A}^\uparrow < 2^{t-1}.$$

Note also that the following implications hold:
$$\#\mathcal{A}^\uparrow > 2^{t-1} \implies \min\{|A| : A \in \mathcal{A}\} \leq \min\{|B| : B \in \mathcal{B}(\mathcal{A})\};$$
$$\#\mathcal{A}^\uparrow < 2^{t-1} \implies \min\{|A| : A \in \mathcal{A}\} \geq \min\{|B| : B \in \mathcal{B}(\mathcal{A})\}.$$

A clutter $\mathcal{A}$ is called \textit{self-dual} [57, Ch. 9][68, §5.7] or \textit{identically self-blocking} [1, 2] if
$$\mathcal{B}(\mathcal{A}) = \mathcal{A};$$
see also the early reference [11, §2.1]. In other words, the self-dual clutters $\mathcal{A} \subset 2^{[t]}$ on the ground set $E_t$ are the \textit{fixed points} of the \textit{blocker map} (2.4); for each of them we also have
$$\mathcal{B}(\mathcal{A})^\uparrow = \mathcal{A}^\uparrow.$$

As noted in [68, Cor. 5.28(i)], one criterion for a clutter $\mathcal{A} \subset 2^{[t]}$ on the ground set $E_t$ to be \textit{self-dual} is as follows:
$$\mathcal{B}(\mathcal{A}) = \mathcal{A} \iff \#\mathcal{A}^\uparrow = 2^{t-1}.$$

- Let $X$ be a subset of the ground set $E_t$. Given a nontrivial clutter $\mathcal{A}$ on $E_t$, its \textit{deletion} $\mathcal{A} \setminus X$ is defined to be the clutter $\mathcal{A} \setminus X := \{A \in \mathcal{A} : |A \cap X| = 0\}$.

\textsuperscript{17} Or \textit{covering number}, \textit{vertex cover number}, \textit{blocking number}. 
The contraction \( \mathcal{A}/X \) is defined to be the clutter
\[
\mathcal{A}/X := \min \{ A - X : A \in \mathcal{A} \}.
\]

A classical result in combinatorial optimization is as follows:
\[
\mathcal{B}(\mathcal{A}) \setminus X = \mathcal{B}(\mathcal{A}/X), \quad \text{and} \quad \mathcal{B}(\mathcal{A})/X = \mathcal{B}(\mathcal{A} \setminus X),
\]
see [81].

We also have
\[
\mathcal{B}(\mathcal{A}) \setminus X \subseteq \mathcal{B}(\mathcal{A}/X) \subseteq (\mathcal{B}(\mathcal{A})/X) \subseteq \mathcal{B}(\mathcal{A} \setminus X),
\]
cf. [68, Eq. (5.4)]. Further,
\[
\#(\mathcal{A}/X) \cap (E_t^s) = \mathcal{B}(\mathcal{A}/X) \cap (E_t^s) \subseteq \mathcal{B}(\mathcal{A} \setminus X) \cap (E_t^s) = \#(\mathcal{A} \setminus X) \cap (E_t^s).
\]

- Let \( p \) be a rational number such that \( 0 \leq p < 1 \). Given a nontrivial clutter \( \mathcal{A} := \{ A_1, \ldots, A_\alpha \} \subset 2^{[t]} \) on the ground set \( E_t \), a subset \( B \subseteq E_t \) is called a \( p \)-committee of the clutter \( \mathcal{A} \), if we have
\[
|B \cap A_i| > p \cdot |B|,
\]
for each \( i \in [\alpha] \). The 0-committees of the clutter \( \mathcal{A} \) are its blocking sets.

- For a nontrivial clutter \( \mathcal{A} := \{ A_1, \ldots, A_\alpha \} \subset 2^{[t]} \) on the ground set \( E_t \), we have
\[
\#(\mathcal{B}(\mathcal{A}) \setminus X) \cap (E_t^s) = \binom{t}{k} + \sum_{S \subseteq [\alpha]: |S| > 0} (-1)^{|S|} \cdot \binom{t - \left| \bigcup_{i \in S} A_i \right|}{k} , \quad 1 \leq k \leq t.
\]

Several ways to count the blocking \( k \)-sets of clutters are mentioned in [68].

### 3. Increasing families of blocking sets, and blockers: Set covering problems

In this section we recall the set covering problem(s); see, e.g., [29, Sect. 2.4] and [31, Ch. 1].

Let \( \chi(A) := (\chi_1(A), \ldots, \chi_t(A)) \in \{0, 1\}^t \) denote the familiar row characteristic vector of a subset \( A \) of the ground set \( E_t \), defined for each element \( j \in E_t \) by
\[
\chi_j(A) := \begin{cases} 1, & \text{if } j \in A, \\ 0, & \text{if } j \not\in A. \end{cases}
\]

If \( \mathcal{A} := \{ A_1, \ldots, A_\alpha \} \subset 2^{[t]} \) is a nontrivial clutter on \( E_t \), then
\[
\mathcal{A} := \mathcal{A}(\mathcal{A}) := \left( \chi(A_1) : \cdots : \chi(A_\alpha) \right) \quad (3.1)
\]

By convention, a \( \frac{1}{2} \)-committee of a clutter \( \mathcal{A} \) is called its committee.
is its incidence matrix.

Consider the set covering collection

\[ \tilde{S} := \tilde{S}^c(A) := \{ \tilde{z} \in \{0,1\}^\ell : A\tilde{z}^\top \geq 1 \} , \]  

(3.2)

which is the collection of characteristic vectors of the blocking sets of the clutter \( A \), that is,

\[ \tilde{S} = \{ \chi(B) : B \in \mathcal{B}(A)^\uparrow \} , \quad \text{and} \quad \mathcal{B}(A)^\uparrow = \{ \text{supp}(\tilde{z}) : \tilde{z} \in \tilde{S} \} . \]

The latter expression just rephrases the convention according to which the supports of the vectors in the collection \( \tilde{S} \subset \{0,1\}^\ell \) are the blocking sets of the clutter \( A \).

Let us redefine the collection

\[ \tilde{S} := \{ \tilde{z} \in \{0,1\}^\ell : \left( \begin{array}{c} \chi(A_1) \\ \vdots \\ \chi(A_\alpha) \end{array} \right) \tilde{z}^\top \geq 1 \} \]

as

\[ \tilde{S} := \left\{ \frac{1}{2}(T^+ - z) \in \{0,1\}^\ell : \left( \begin{array}{c} \frac{1}{2}(T^+ - T^1) \\ \vdots \\ \frac{1}{2}(T^+ - T^\alpha) \end{array} \right) \cdot \frac{1}{2}(T^+ - z) \geq 1 \right\} , \]  

(3.3)

where the vertices \( T^i \) of the discrete hypercube \( \{1,-1\}^\ell \) and the vector of unknowns \( z \in \{1,-1\}^\ell \) are given by

\[ T^i := -A_i T^+ = T^+ - 2\chi(A_i) , \quad i \in [\alpha] , \]

and

\[ z := T^+ - 2\tilde{z} . \]

Let us now associate with the collection \( \tilde{S} \subset \{0,1\}^\ell \), described in (3.3), a collection \( S := S^c(A) \subset \{1,-1\}^\ell \), defined by

\[ S := \left\{ z \in \{1,-1\}^\ell : Az^\top \leq \left( \begin{array}{c} ||T^1|| \\ \vdots \\ ||T^\alpha|| \end{array} \right) - 2 \cdot 1 \right\} , \]

that is, the collection

\[ S := \left\{ z \in \{1,-1\}^\ell : Az^\top \leq \left( \begin{array}{c} |A_1| \\ \vdots \\ |A_\alpha| \end{array} \right) - 2 \right\} . \]  

(3.4)

We have defined the twin collections \( \tilde{S} \subset \{0,1\}^\ell \) and \( S \subset \{1,-1\}^\ell \), given in (3.2) and (3.4), respectively, that are equipped with the bijections \( \tilde{S} \rightarrow S : T \mapsto T^+ - 2T \), and \( S \rightarrow \tilde{S} : T \mapsto \frac{1}{2}(T^+ - T) \); see Example 3.1.

\[ ^{19} \] We will denote by \( 1 \) and \( 2 \) the \( \alpha \)-dimensional column vectors \( \left( \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right) \) and \( \left( \begin{array}{c} 2 \\ \vdots \\ 2 \end{array} \right) \), respectively.
Example 3.1. Consider the clutter $\mathcal{A} := \{A_1, A_{n-2}\} := \{\{1, 2\}, \{2, 3\}\}$, on the ground set $E_{l:3} := \{1, 2, 3\}$, with its incidence matrix

$$A := A(\mathcal{A}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$  

The set covering $\{0, 1\}$-collection

$$\vec{S} := \{\vec{z} \in \{0, 1\}^t : A\vec{z}^T \geq 1\}$$

$$= \{\vec{z} \in \{0, 1\}^3 : (1, 1, 0)\vec{z}^T \geq (1)\}$$

is the collection

$$\vec{S} = \{(0, 1, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}$$

$$= \{\chi(\{2\}), \chi(\{1, 2\}), \chi(\{1, 3\}), \chi(\{2, 3\}), \chi(\{1, 2, 3\})\}.$$  

The set covering $\{1, -1\}$-collection

$$S := \{\vec{z} \in \{1, -1\}^t : A\vec{z}^T \leq \left(\frac{|A_1|}{|A_2|}\right) - 2\}$$

$$= \{\vec{z} \in \{1, -1\}^3 : (1, 1, 0)\vec{z}^T \leq (2) - 2 = (0)\}$$

is the collection

$$S = \{(1, -1, 1), (-1, -1, 1), (1, 1, -1), (1, -1, -1), (1, -1, 1)\}$$

$$= \{-2, T^{(+)}, -\{1, 2\}T^{(+)}, -\{1, 3\}T^{(+)}, -\{2, 3\}T^{(+)}, -\{1, 2, 3\}T^{(+)}\}.$$  

- Let $\mathbf{w} \in \mathbb{R}^t$ be a row vector of nonnegative weights. The set covering problems are

$$\min\{\mathbf{w}\vec{z}^T : \vec{z} \in \vec{S}\} = \min\{\mathbf{w} : \frac{1}{2}(T^{(+)} - \vec{z})^T : \vec{z} \in S\}.$$  

- Suppose $\mathbf{w} := T^{(+)}$, and consider the (unweighted) set covering problems

$$\tau(\mathcal{A}) := \min\{\text{hwt}(\vec{z}) : \vec{z} \in \vec{S}\} = \min\{T^{(+)}\vec{z}^T : \vec{z} \in \vec{S}\}$$

$$= \min\{T^{(+)} : \frac{1}{2}(T^{(+)} - \vec{z})^T : \vec{z} \in S\} = \min\left\{\frac{1}{2}(t - \bigwedge_{t=2|z|}^{(+)\vec{z}}) : \vec{z} \in S\right\}$$

$$= \min\{|\vec{z}|^- : \vec{z} \in S\} =: \tau(\mathcal{A}),$$

that is, the problem

$$\underbrace{\min\{T^{(+)}\vec{z}^T : \vec{z} \in \vec{S}\}}_{\tau(\mathcal{A}) := \min\{\text{hwt}(\vec{z}) : \vec{z} \in \vec{S}\}} := \min\{T^{(+)}\vec{z}^T : \vec{z} \in \{0, 1\}^t, A\vec{z} \geq 1\}, \quad \quad (3.5)$$

$$\tau(\mathcal{A}) := \min\{\text{hwt}(\vec{z}) : \vec{z} \in \vec{S}\}$$
and the problem

\[ \tau(A) := \min \left\{ |z^+| : z \in \mathcal{S} \right\} \]

\[ = \frac{1}{2} \cdot \max \left\{ T^+(z^+) : z \in \{1, -1\}^t, Az^T \leq \left( \begin{array}{c} |A_1| \\ \vdots \\ |A_n| \end{array} \right) - 2 \right\} \]

(3.6)

For vectors \( \hat{z} \in \hat{\mathcal{S}} \) and \( z \in \mathcal{S} \), where \( \hat{z} := \frac{1}{2}(T^+(z) - z) \), we have the inclusions

\[ \hat{z} \in \operatorname{Arg} \min \left\{ T^+(z^+) : \hat{z} \in \hat{\mathcal{S}} \right\} \]

\[ z \in \operatorname{Arg} \max \left\{ T^+(z^+) : z \in \mathcal{S} \right\} \]

that is, \( \hat{z} \) and \( z \) provide the solution to the problems (3.5) and (3.6), respectively, if and only if the member

\[ B := \text{supp}(\hat{z}) = z^- \in \mathcal{B}(A) \]

of the blocker of the clutter \( A \) has the minimum cardinality

\[ |B| = \tau(A) \]

We conclude this section by noting that the rows of incidence matrices \( A \), as well as the vectors in the set covering collections \( \hat{\mathcal{S}} \subset \{0, 1\}^t \) and \( \mathcal{S} \subset \{1, -1\}^t \), admit their decompositions with respect to symmetric cycles in the corresponding hypercube graphs \( \overline{H}(t, 2) \) and \( H(t, 2) \).

4. Families of subsets of the ground set \( E_t \): Characteristic vectors and characteristic topes

The generation of fundamental combinatorial objects is extensively treated in [63].

Consider the family \( (E_t^s) \), for some \( s \), where \( 0 \leq s \leq t \). We denote this family of all \( s \)-subsets \( L_j^s \subseteq E_t \), ordered lexicographically, by \( (E_t^s) =: (L_1^s, \ldots, L_t^s) \).

For an \( s \)-uniform clutter \( \mathcal{G} := \{G_1, \ldots, G_k\} \subseteq (E_t^s) \), we define its row characteristic vector \( \gamma(s)(\mathcal{G}) := (\gamma_1(s)(\mathcal{G}), \ldots, \gamma_t(s)(\mathcal{G})) \in \{0, 1\}^{t(s)} \) in the familiar
way: for each $j$, where $1 \leq j \leq \binom{n}{1}$, we set
\[
\gamma_j^{(s)}(\mathbb{G}) := \begin{cases} 
1, & \text{if } \frac{E_j}{s} \ni L_j^s \in \mathbb{G}, \\
0, & \text{if } \frac{E_j}{s} \ni L_j^s \notin \mathbb{G};
\end{cases}
\]
see (4.2)–(4.8) in Example 4.1.

Now, given an arbitrary family $\mathcal{F} \subseteq 2^{[t]}$, we set
\[
\gamma^{(s)}(\mathcal{F}) := \gamma^{(s)}(\mathcal{F} \cap (\frac{E_i}{s})), \quad 0 \leq s \leq t,
\]
and in a natural way\(^{20}\) we define the characteristic vector $\gamma(\mathcal{F}) := (\gamma_1(\mathcal{F}), \ldots, \gamma_{2^t}(\mathcal{F})) \in \{0,1\}^{2^t}$ of the family $\mathcal{F}$ to be the concatenation
\[
\gamma(\mathcal{F}) := \gamma^{(0)}(\mathcal{F}) \cdot \gamma^{(1)}(\mathcal{F}) \cdot \ldots \cdot \gamma^{(t-1)}(\mathcal{F}) \cdot \gamma^{(t)}(\mathcal{F});
\]
see (4.9)–(4.22).

- The characteristic vector $\gamma(2^{[t]}) = T_{2^t}^{(+)}$, whose components are all 1’s, describes the linearly ordered power set $2^{[t]}$ of the ground set $E_t$; see (4.14).
- The Hamming weights $\text{hw}(\gamma^{(s)}(\mathcal{F}))$ of the vectors $\gamma^{(s)}(\mathcal{F})$, $0 \leq s \leq t$, are the components $f_s(\mathcal{F}; t)$ of the so-called long $f$-vectors $f(\mathcal{F}; t)$ associated with families $\mathcal{F} \subseteq 2^{[t]}$, see [68, Sect. 2.1].
- If $\mathcal{F}' \subseteq 2^{[t]}$ and $\mathcal{F}'' \subseteq 2^{[t]}$ are families of subsets of the ground set $E_t$, then we will use the componentwise product of their characteristic vectors
\[
\gamma(\mathcal{F}') \ast \gamma(\mathcal{F}'') := (\gamma_1(\mathcal{F}') \cdot \gamma_1(\mathcal{F}''), \ldots, \gamma_{2^t}(\mathcal{F}') \cdot \gamma_{2^t}(\mathcal{F}'')) \in \{0,1\}^{2^t}
\]
to describe\(^{21}\) the intersection of these families:
\[
\gamma(\mathcal{F}' \cap \mathcal{F}'') = \gamma(\mathcal{F}') \ast \gamma(\mathcal{F}'');
\]
- Let $\Gamma(k)$ denote the subset $A \subseteq E_t$, for which the characteristic vector of the corresponding one-member clutter $\{A\}$ on $E_t$ by convention is the $k$th standard unit vector $\sigma(k)$ of the space $\mathbb{R}^{2^t}$; we thus use the map
\[
\Gamma : [2^t] \to 2^{[t]}, \quad k \mapsto A: \gamma(\{A\}) = \sigma(k) \in \{0,1\}^{2^t};
\]
see (4.23)–(4.25). Conversely, we denote by $\Gamma^{-1}(A)$, where $A \subseteq E_t$, the position number $k$ such that the vector $\sigma(k)$ is the characteristic vector of the one-member clutter $\{A\}$ on $E_t$:
\[
\Gamma^{-1} : 2^{[t]} \to [2^t], \quad A \mapsto k: \sigma(k) = \gamma(\{A\}) \in \{0,1\}^{2^t};
\]
see (4.23)–(4.25).

\(^{20}\) Indeed, this is the most popular ordering of all subsets of a nonempty finite set, the lexicographic ordering subordinated to cardinality, that is, considering smaller subsets first, and in case of equal cardinality, the lexicographic ordering, see [50, p. 77].

\(^{21}\) The notation $\prod_{i}$ will be used to denote the componentwise product of several vectors.
By construction, we have the implications

\[ \ell', \ell'' \in [2^t], \quad \ell' < \ell'' \implies |\Gamma(\ell')| \leq |\Gamma(\ell'')|; \quad (4.1) \]

and, in particular,

\[ A, B \in 2^t, \quad |A| < |B| \implies \Gamma^{-1}(A) < \Gamma^{-1}(B), \]

Note also that for any index \( \ell \in [2^t] \), the disjoint union

\[ \Gamma(\ell) \cup \Gamma(2^t - \ell + 1) = E_t \]

is a partition of the ground set.

**Example 4.1.** Suppose \( t := 3 \), and \( E_t = \{1, 2, 3\} \). We have

\[ \gamma^{(0)}(\{E_0\}) := \gamma^{(0)}(\{\emptyset\}) = (1) \in \{0, 1\}^t, \quad (4.2) \]

\[ \gamma^{(1)}(\{E_1\}) := \gamma^{(1)}(\{\{1\}, \{\{2\}, \{3\}\}) = (1, 1, 1) \in \{0, 1\}^t, \quad (4.3) \]

\[ \gamma^{(2)}(\{E_2\}) := \gamma^{(2)}(\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}) = (1, 1, 1) \in \{0, 1\}^t, \quad (4.4) \]

\[ \gamma^{(3)}(\{E_3\}) := \gamma^{(3)}(\{\{1, 2, 3\}\}) = (1) \in \{0, 1\}^t, \quad (4.5) \]

\[ \gamma^{(1)}(\{\{2\}\}) = (0, 1, 0) \in \{0, 1\}^t, \quad (4.6) \]

\[ \gamma^{(2)}(\{\{1, 2\}, \{2, 3\}\}) = (1, 0, 1) \in \{0, 1\}^t, \quad (4.7) \]

\[ \gamma^{(2)}(\{\{1, 3\}\}) = (0, 1, 0) \in \{0, 1\}^t, \quad (4.8) \]

and

\[ \gamma(\emptyset) = (0, 0, 0, 0, 0, 0, 0) \in \{0, 1\}^{2^t}, \quad (4.9) \]

\[ \gamma(\{E_0\}) = (1, 0, 0, 0, 0, 0, 0), \quad (4.10) \]

\[ \gamma(\{E_1\}) = (0, 1, 1, 1, 0, 0, 0), \quad (4.11) \]

\[ \gamma(\{E_2\}) = (0, 0, 0, 0, 1, 1, 0), \quad (4.12) \]

\[ \gamma(\{E_3\}) = (0, 0, 0, 0, 0, 1), \quad (4.13) \]

\[ \gamma(2^t) = T_{2^t}^{(1)} := (1, 1, 1, 1, 1, 1, 1). \quad (4.14) \]
If \( A := \{A_1, A_2\} \) and \( B := \{B_1, B_2\} \) are clutters on \( E_t \), where \( A_1 := \{1, 2\}, A_2 := \{2, 3\}, B_1 := \{1, 3\}, B_2 := \{2\}, \) and \( B = \mathcal{B}(A) \), then we have

\[
\gamma(A) := \gamma(\{A_1, A_2\}) := \gamma(\{(1, 2), (2, 3)\}) = (0, 0, 0, 0, 1, 0, 1, 0) \in \{0, 1\}^{2t},
\]

\[
\gamma(B) := \gamma(\{B_1, B_2\}) := \gamma(\{(1, 3), (2)\}) = (0, 0, 1, 0, 0, 0, 0, 0),
\]

\[
\gamma(A^\uparrow) := \gamma(\{(1, 2), (2, 3)\}^\uparrow) = (0, 0, 0, 0, 1, 0, 1, 1),
\]

\[
\gamma(B^\uparrow) := \gamma(\{(1, 3), (2)\}^\uparrow) = (0, 0, 1, 0, 1, 1, 1, 1),
\]

\[
\gamma(\{A_1\}^\uparrow) := \gamma(\{(1, 2)\}^\uparrow) = (0, 0, 0, 0, 1, 1, 0, 0),
\]

\[
\gamma(\{A_2\}^\uparrow) := \gamma(\{(2, 3)\}^\uparrow) = (0, 0, 0, 0, 0, 0, 1, 1),
\]

\[
\gamma(\{B_1\}^\uparrow) := \gamma(\{(1, 3)\}^\uparrow) = (0, 0, 0, 0, 0, 1, 0, 1),
\]

\[
\gamma(\{B_2\}^\uparrow) := \gamma(\{(2)\}^\uparrow) = (0, 0, 1, 0, 1, 0, 0).
\]

We have

\[
\Gamma(3) = \{2\}, \quad \gamma(\{(2)\}) = \sigma(3) \in \{0, 1\}^{2t}, \quad \Gamma^{-1}(\{2\}) = 3,
\]

\[
\Gamma(6) = \{1, 3\}, \quad \gamma(\{(1, 3)\}) = \sigma(6) \in \{0, 1\}^{2t}, \quad \Gamma^{-1}(\{1, 3\}) = 6,
\]

\[
\Gamma(2^t) = E_t, \quad \gamma(\{E_t\}) = \sigma(2^t) \in \{0, 1\}^{2t}, \quad \Gamma^{-1}(E_t) = 2^t.
\]

- Given a family \( \mathcal{F} \subseteq 2^{[t]} \) of subsets of the ground set \( E_t \), we call the tope

\[
T_{\mathcal{F}} := -\supp(\gamma(\mathcal{F})) T_{2^t}^{(+) \downarrow} = T_{2^t}^{(+) \downarrow} - 2\gamma(\mathcal{F})
\]

of the oriented matroid \( \mathcal{H}_{2^t} := (E_{2^t}, \{1, -1\}^{2^t}) \) the characteristic tope of the family \( \mathcal{F} \); see Example 4.2.

If we let \( T_{s^t}^{(+)} := (1, \ldots, 1) \in \mathbb{R}^{(s^t)} \) denote the \( (s^t) \)-dimensional row vector of all 1’s, \( 0 \leq s \leq t \), then

\[
T_{\mathcal{F}}^{(s)} := -\supp(\gamma(s(\mathcal{F}))) T_{s^t}^{(+) \downarrow} = T_{s^t}^{(+) \downarrow} - 2\gamma(s(\mathcal{F})), \quad 0 \leq s \leq t.
\]

**Example 4.2.** Suppose \( t := 3 \) and \( E_t = \{1, 2, 3\} \). If \( A \) and \( B = \mathcal{B}(A) \) are clutters on the ground set \( E_t \), mentioned in Example 4.1 on page 20, then we have

\[
\gamma(A) = (0, 0, 0, 0, 1, 0, 1, 0) \in \{0, 1\}^{2t},
\]

\[
T_A := (1, 1, 1, 1, -1, -1, 1, -1) \in \{1, -1\}^{2t};
\]

\[
\gamma(A^\uparrow) = (0, 0, 0, 0, 1, 0, 1, 1),
\]

\[
T_A^\uparrow := (1, 1, 1, 1, -1, -1, 1, 1);
\]

\[
\gamma(B) = (0, 0, 1, 0, 0, 1, 0, 0),
\]

\[
T_B := (1, 1, -1, 1, 1, -1, 1, 1);
\]

\[
\gamma(B^\uparrow) = (0, 0, 1, 0, 1, 1, 1, 1),
\]

\[
T_B^\uparrow := (1, 1, -1, 1, -1, -1, -1, -1).
5. Increasing families of blocking sets, and blockers: Characteristic vectors and characteristic toposes

In this section, we begin with somewhat sophisticated restatements of the simple basic observations (1.10), (2.5) and (2.6) on set families in terms of their characteristic vectors.

- For a nontrivial clutter \( \mathcal{A} \subset 2^n \) on the ground set \( E_t \), we have

\[
\gamma(\mathcal{B}(\mathcal{A})^\uparrow) * \gamma(\mathcal{A}^\uparrow) = \text{rn}(\gamma(A^\uparrow)) * \gamma(\mathcal{A}^\uparrow) := (T_{2^t}^{(+)} - \gamma(A^\uparrow) \cdot \overline{U}(2^t)) * \gamma(\mathcal{A}^\uparrow)
\]

\[
= \gamma(\mathcal{B}(\mathcal{A})^\uparrow) * \gamma(\mathcal{A}^\uparrow) := \gamma(\mathcal{B}(\mathcal{A})^\uparrow) * (T_{2^t}^{(+)} - \gamma(\mathcal{B}(\mathcal{A})^\uparrow) \cdot \overline{U}(2^t))
\]

\[
= \begin{cases} 
\gamma(\mathcal{A}^\uparrow), & \text{if } \#A^\uparrow < 2^{t-1}, \\
\gamma(\mathcal{B}(\mathcal{A})^\uparrow), & \text{if } \#A^\uparrow > 2^{t-1}, \\
\gamma(\mathcal{A}^\uparrow) = \gamma(\mathcal{B}(\mathcal{A})^\uparrow), & \text{if } \#A^\uparrow = 2^{t-1}.
\end{cases}
\]

Remark 5.1. Let \( \mathcal{A} \subset 2^n \) be a nontrivial clutter on the ground set \( E_t \). We have

\[
\gamma_1(\mathcal{B}(\mathcal{A})^\uparrow) = 0, \quad \text{and} \quad \gamma_2(\mathcal{B}(\mathcal{A})^\uparrow) = 1;
\]

\[
\gamma(\mathcal{B}(\mathcal{A})^\uparrow) = \text{rn}(\gamma(A^\uparrow)) := T_{2^t}^{(+)} - \gamma(A^\uparrow) \cdot \overline{U}(2^t) \quad \text{if } \#A^\uparrow \leq 2^t - 1;
\]

\[
T_{\mathcal{B}(\mathcal{A})^\uparrow} = \text{ro}(T_{\mathcal{A}^\uparrow}) := -T_{2^t}^{(+)} \cdot \overline{U}(2^t) \quad \text{if } \#A^\uparrow \geq 2^t - 1;
\]

\[
\begin{aligned}
\text{hwt}(\gamma(\mathcal{B}(\mathcal{A})^\uparrow)) + \text{hwt}(\gamma(\mathcal{A}^\uparrow)) = 2^t; \\
\frac{|(T_{\mathcal{B}(\mathcal{A})^\uparrow})^s|}{\#(\mathcal{B}(\mathcal{A})^\uparrow)} + \frac{|(T_{\mathcal{A}^\uparrow})^s|}{\#(\mathcal{A}^\uparrow)} = 2^t;
\end{aligned}
\]

\[
\gamma^{(s)}(\mathcal{B}(\mathcal{A})^\uparrow) = \text{rn}(\gamma^{(t-s)}(A^\uparrow)) := T_{\binom{t}{s}}^{(+)} - \gamma^{(t-s)}(A^\uparrow) \cdot \overline{U}(\binom{t}{s}), \quad 0 \leq s \leq t;
\]

\[
T_{\mathcal{B}(\mathcal{A})^\uparrow}^{(s)} = \text{ro}(T_{\mathcal{A}^\uparrow}^{(t-s)}) := -T_{\binom{t}{s}}^{(+)} \cdot \overline{U}(\binom{t}{s}), \quad 0 \leq s \leq t;
\]

\[
\begin{aligned}
\text{hwt}(\gamma^{(s)}(\mathcal{B}(\mathcal{A})^\uparrow)) + \text{hwt}(\gamma^{(t-s)}(\mathcal{A}^\uparrow)) = \binom{t}{s}, \quad 0 \leq s \leq t;
\end{aligned}
\]

\[
\begin{aligned}
\frac{|(T_{\mathcal{B}(\mathcal{A})^\uparrow})^s|}{\#(\mathcal{B}(\mathcal{A})^\uparrow \cap (\binom{t}{s}))} + \frac{|(T_{\mathcal{A}^\uparrow})^s|}{\#(\mathcal{A}^\uparrow \cap (\binom{t}{s}))} = \binom{t}{s}, \quad 0 \leq s \leq t.
\end{aligned}
\]
In addition to (5.1) and (5.2), relations (5.3) and (5.4) imply that
\[
\gamma(\mathfrak{B}(A)^{\uparrow}) = \mathbf{rn}(\gamma(0)(A^{\uparrow})) \cdot \mathbf{rn}(\gamma(1)(A^{\uparrow})) \cdot \mathbf{rn}(\gamma(2)(A^{\uparrow})) \cdot \ldots \cdot \mathbf{rn}(\gamma(t-1)(A^{\uparrow})) \cdot \mathbf{rn}(\gamma(0)(A^{\uparrow})),
\]
\[
T_{\mathfrak{B}(A)}^{\uparrow} = \mathbf{ro}(T_{A^{\uparrow}}^{\downarrow}) \cdot \mathbf{ro}(T_{A^{\uparrow}}^{\downarrow}) \cdot \ldots \cdot \mathbf{ro}(T_{A^{\uparrow}}^{\downarrow}) \cdot \mathbf{ro}(T_{A^{\uparrow}}^{\downarrow}).
\]

- In view of (4.1), if
\[
\ell^* := \min \text{supp}(\gamma(\mathfrak{B}(A)^{\uparrow})) = \min (T_{\mathfrak{B}(A)^{\uparrow}})^{-},
\]
then the member \( \Gamma(\ell^*) \) of the blocker \( \mathfrak{B}(A) \) of a nontrivial clutter \( A \subset 2^E \) is a blocking set of minimum cardinality for \( A \), that is, the vectors \( \chi(\Gamma(\ell^*)) \) and \( -\Gamma(\ell^*) T^{(+)} \) provide the solution \( |\Gamma(\ell^*)| = \tau(A) \) to the set covering problems (3.5) and (3.6), respectively:
\[
\chi(\Gamma(\ell^*)) \in \text{Arg min}\{T^{(+)} \tilde{z}^\top: \tilde{z} \in \tilde{S}\},
\]
\[
-\Gamma(\ell^*) T^{(+)} \in \text{Arg max}\{T^{(+)} z^\top: z \in S\}.
\]

5.1. A clutter \( \{\{a\}\} \).

Let \( \{\{a\}\} \) be a (nontrivial) clutter on the ground set \( E_t \), whose only member is a one-element subset \( \{a\} \subset E_t \).

5.1.1. The principal increasing family of blocking sets \( \mathfrak{B}(\{\{a\}\})^{\uparrow} = \{\{a\}\}^{\uparrow} \).

- The increasing family of blocking sets \( \mathfrak{B}(\{\{a\}\})^{\uparrow} \) of the self-dual clutter \( \{\{a\}\} \) coincides with the principal increasing family \( \{\{a\}\}^{\uparrow} \).
- We will use the notation \( \tilde{a}(a) := \tilde{a}(a; 2^t) \) and \( a(a) := a(a; 2^t) \) to denote the characteristic vector and the characteristic tope, respectively, that are associated with the principal increasing family \( \{\{a\}\}^{\uparrow} = \mathfrak{B}(\{\{a\}\})^{\uparrow} : \)
\[
\tilde{a}(a) := \gamma(\{\{a\}\}^{\uparrow}) = \gamma(\mathfrak{B}(\{\{a\}\})^{\uparrow}) \in \{0, 1\}^{2^t},
\]
\[
a(a) := T_{\{\{a\}\}}^{\uparrow} = T_{\mathfrak{B}(\{\{a\}\})^{\uparrow}}^{\uparrow} \in \{1, -1\}^{2^t}.
\]

We have
\[
\tilde{a}(a) = \begin{pmatrix} 0 \end{pmatrix}, \quad \mathbf{x}(\{a\}) \cdot \gamma(2)(\{\{a\}\}^{\uparrow}) \cdot \ldots \cdot \gamma(t-1)(\{\{a\}\}^{\uparrow}) \cdot \begin{pmatrix} 1 \end{pmatrix},
\]
\[
a(a) = \begin{pmatrix} 1 \end{pmatrix} \cdot -a_1 T^{(+)_{\{\{a\}\}}} T^{(+)_{\{\{a\}\}}} \cdot \ldots \cdot T^{(+)_{\{\{a\}\}}} \cdot \begin{pmatrix} -1 \end{pmatrix},
\]

see (5.6), (5.11) and (5.16) in Example 5.5;
\[
a(a) = \begin{pmatrix} 1 \end{pmatrix} \cdot -a_1 T^{(+)_{\{\{a\}\}}} T^{(+)_{\{\{a\}\}}} \cdot \ldots \cdot T^{(+)_{\{\{a\}\}}} \cdot \begin{pmatrix} -1 \end{pmatrix},
\]

see (5.7), (5.12) and (5.17).
Remark 5.2 (see Remark 5.1, and cf. Remark 5.6). Note that

\[ \tilde{a}(a) = \text{rn}(a(a)) \quad \text{and} \quad a(a) = \text{ro}(a(a)) \quad \text{(5.5)} \]

\[ \text{hwt}(a(a)) = |a(a)| = |{a}| = |{a}|^\vee = |{a}|^\vee = 2^{t-1} ; \]

\[ \tilde{a}^{(s)}(a) = \text{rn}(a^{(t-s)}(a)) \quad \text{and} \quad a^{(s)}(a) = \text{ro}(a^{(t-s)}(a)) \quad 0 \leq s \leq t ; \]

\[ \text{hwt}(a^{(s)}(a)) = |a^{(s)}(a)| = \left(\frac{t-1}{s-1}\right) , \quad 0 \leq s \leq t . \]

5.1.2. The blocker \( \mathcal{B}({\{a\}}) = {\{a\}} \).

- The blocker \( \mathcal{B}({\{a\}}) \) coincides with the self-dual clutter \({\{a\}}\).
- We associate with the clutter \( \mathcal{B}({\{a\}}) = {\{a\}} \) its characteristic vector \( \gamma({\{a\}}) = \gamma(\mathcal{B}({\{a\}})) \in \{0,1\}^{|2^t|} \) and its characteristic tope \( T_{\{\{a\}\}} \)

\[ = T_{\mathcal{B}({\{a\}})} \in \{1,-1\}^{|2^t|} , \]

where

\[ \gamma({\{a\}}) = \gamma(\mathcal{B}({\{a\}})) = (0) \cdot \chi({\{a\}}) \cdot (0, \ldots, 0) \cdot \ldots \cdot (0) , \]

see (5.10), (5.15) and (5.20) in Example 5.5.

5.1.3. More on the principal increasing family \( \mathcal{B}({\{a\}})^\vee = {\{a\}}^\vee \).

In view of (5.5), we can make the following observation:

Remark 5.3 (cf. Remark 5.7). For any element \( a \in E_t \), we have

\[ \min \{ i \in E_{2t} : T_{\{\{a\}\}}(i) = -1 \} := \min \{ i \in E_{2t} : \gamma_i({\{a\}})^\vee = 1 \} \]

\[ = \Gamma^{-1}({\{a\}}) = 1 + a , \]

\[ \max \{ i \in E_{2t} : T_{\{\{a\}\}}(i) = 1 \} := \max \{ i \in E_{2t} : \gamma_i({\{a\}})^\vee = 0 \} \]

\[ = 2^t - \min \{ a \} = 2^t - a ; \]

\[ \min \{ j \in E_{2t} : T_{\mathcal{B}({\{a\}})}(j) = -1 \} := \min \{ j \in E_{2t} : \gamma_j(\mathcal{B}({\{a\}})^\vee) = 1 \} \]

\[ = 1 + \min \{ a \} = 1 + a , \]

\[ \max \{ j \in E_{2t} : T_{\mathcal{B}({\{a\}})}(j) = 1 \} := \max \{ j \in E_{2t} : \gamma_j(\mathcal{B}({\{a\}})^\vee) = 0 \} \]

\[ = 1 + 2^t - \Gamma^{-1}({\{a\}}) = 2^t - a . \]

- We have

\[ {\{a\}}^\vee \cup \{ E_t - \{a\} \}^\delta = 2^t . \]

Let us denote by \( \tilde{c}(a) := \tilde{c}(a;2^t) \) and \( c(a) := c(a;2^t) \) the characteristic vector and the characteristic tope, respectively, of the principal decreasing family \( \{ E_t - \{a\} \}^\delta \):

\[ \tilde{c}(a) := \gamma(\{ E_t - \{a\} \}^\delta) \in \{0,1\}^{2^t} , \]

see (5.8), (5.13) and (5.18);

\[ c(a) := T_{\{ E_t - \{a\} \}} \in \{1,-1\}^{2^t} , \]
see (5.9), (5.14) and (5.19). We have

\[ \tilde{c}(a) = T_2^{(+) - \tilde{a}(a) = \tilde{a}(a) \cdot \mathbf{U}(2^t)} , \]
\[ c(a) = -a(a) = a(a) \cdot \mathbf{U}(2^t) . \]

- For any two-element subset \( \{i, j\} \subset E_t \) of the ground set, we have

\[ \#(\{\{i\}\}^\vee \cap \{\{j\}\}^\vee) = \#\{\{i, j\}\}^\vee = 2^{t-2} , \]

and

\[ \#(\left(2^t\right) - \{\{i\}\}^\vee \cap \left(2^t\right) - \{\{j\}\}^\vee) = \#\{E_t - \{i, j\}\}^\Delta = 2^{t-2} . \]

Thus, if \( i \) and \( j \) are elements of the ground set \( E_t \), and \( i \neq j \), then we have

\[ d(\tilde{a}(i), \tilde{a}(j)) = d(a(i), a(j)) = d(\tilde{c}(i), \tilde{c}(j)) = d(c(i), c(j)) = 2^{t-1} . \]

**Remark 5.4.** For any two elements \( i \) and \( j \) of the ground set \( E_t \) we have

\[ \langle a(i), a(j) \rangle = \langle c(i), c(j) \rangle = \delta_{i,j} \cdot 2^t . \]

In other words, the sequences of \( t \) row vectors

\[ \left( \frac{1}{\sqrt{2^t}} \cdot a(1), \frac{1}{\sqrt{2^t}} \cdot a(2), \ldots, \frac{1}{\sqrt{2^t}} \cdot a(t) \right) \subset \mathbb{R}^{2^t} \]

and

\[ \left( \frac{1}{\sqrt{2^t}} \cdot c(t), \frac{1}{\sqrt{2^t}} \cdot c(t-1), \ldots, \frac{1}{\sqrt{2^t}} \cdot c(1) \right) \subset \mathbb{R}^{2^t} \]

are both orthonormal.
Example 5.5. Suppose \( t := 3 \), and \( E_t = \{1, 2, 3\} \). We have

\[
\begin{align*}
\tilde{a}(1) := \gamma(\{1\})^\gamma &= \gamma(\mathcal{B}(\{1\})^\gamma) = (0, 1, 0, 1, 0, 0, 0), \\
\tilde{a}(1) := T_{\{1\}}^\gamma &= T_{\mathcal{B}(\{1\})^\gamma} = (1, -1, 1, 1, -1, -1, 1), \\
\tilde{c}(1) := \gamma(\{E_t - \{1\}\})^\gamma &= (1, 0, 1, 0, 1, 0, 1), \\
c(1) := T_{\{E_t - \{1\}\}} = (-1, 1, -1, -1, 1, 1, 1), \\
\gamma(\{\{1\}\}) &= \gamma(\mathcal{B}(\{\{1\}\})) = (0, 1, 0, 0, 0, 0), \\
\tilde{a}(2) := \gamma(\{2\})^\gamma &= \gamma(\mathcal{B}(\{2\})^\gamma) = (0, 0, 1, 0, 1, 0, 1), \\
a(2) := T_{\{2\}}^\gamma &= T_{\mathcal{B}(\{2\})^\gamma} = (1, 1, -1, 1, -1, -1, 1), \\
\tilde{c}(2) := \gamma(\{E_t - \{2\}\})^\gamma &= (1, 1, 0, 1, 0, 1, 0, 1), \\
c(2) := T_{\{E_t - \{2\}\}} = (-1, -1, 1, -1, 1, 1, 1), \\
\gamma(\{\{2\}\}) &= \gamma(\mathcal{B}(\{\{2\}\})) = (0, 1, 0, 0, 0, 0), \\
\tilde{a}(t) := \gamma(\{\{t\}\})^\gamma &= \gamma(\mathcal{B}(\{\{t\}\})^\gamma) = (0, 0, 0, 1, 0, 1, 1, 1), \\
a(t) := T_{\{t\}}^\gamma &= T_{\mathcal{B}(\{t\})^\gamma} = (1, 1, 1, -1, 1, -1, -1, 1), \\
\tilde{c}(t) := \gamma(\{E_t - \{t\}\})^\gamma &= (1, 1, 0, 1, 0, 0, 0), \\
c(t) := T_{\{E_t - \{t\}\}} = (-1, -1, 1, -1, 1, 1, 1), \\
\gamma(\{\{t\}\}) &= \gamma(\mathcal{B}(\{\{t\}\})) = (0, 0, 1, 0, 0, 0, 0, 0).
\end{align*}
\]
Given a nontrivial clutter $A := \{A_1, \ldots, A_\alpha\} \subset 2^t$ on the ground set $E_t$, such that $A \neq \{E_t\}$, we have

$$
\gamma(A) := \sum_{i \in [\alpha]} \sigma(\Gamma^{-1}(A_i)) = \gamma(A^\vee) \ast \gamma(A^\Delta) = \sum_{i \in [\alpha]} (\gamma([A_i]^\vee) \ast \gamma([A_i]^\Delta))
$$

$$
= \sum_{i \in [\alpha]} \left( \prod_{a^i \in A_i} \mathcal{a}(a^i) \ast \prod_{c^i \in E_t-A_i} \mathcal{c}(c^i) \right)
$$

$$
= \sum_{i \in [\alpha]} \left( \prod_{a^i \in A_i} \mathcal{a}(a^i) \ast \left( \prod_{c^i \in E_t-A_i} \mathcal{c}(c^i) \ast \mathcal{U}(2^t) \right) \right)
$$

$$
= \sum_{i \in [\alpha]} \left( \prod_{a^i \in A_i} \mathcal{a}(a^i) \ast \mathcal{U}(2^t) \ast \prod_{c^i \in E_t-A_i} \mathcal{c}(c^i) \right).
$$

5.2. A clutter $\{A\}$.

Let $\{A\}$ be a (nontrivial) clutter on the ground set $E_t$, whose only member is a nonempty subset $A \subseteq E_t$.

5.2.1. The increasing family of blocking sets $\mathcal{B}(\{A\})^\vee = \{\{a\} : a \in A\}^\vee$.

- The family of blocking sets $\mathcal{B}(\{A\})^\vee$ of the clutter $\{A\}$ is the increasing family $\{\{a\} : a \in A\}^\vee$.

We have

$$
\{A\}^\vee = \bigcap_{a \in A} \{\{a\}\}^\vee, \quad \text{and} \quad \mathcal{B}(\{A\})^\vee = \bigcup_{a \in A} \{\{a\}\}^\vee.
$$

Let us associate with the increasing families $\{A\}^\vee$ and $\mathcal{B}(\{A\})^\vee$ their characteristic vectors $\gamma(\{A\}^\vee) \in \{0,1\}^{2^t}$ and $\gamma(\mathcal{B}(\{A\})^\vee) \in \{0,1\}^{2^t}$, and their characteristic topes $T_{\{A\}}^\vee \in \{1,-1\}^{2^t}$ and $T_{\mathcal{B}(\{A\})}^\vee \in \{1,-1\}^{2^t}$, where

$$
\gamma(\{A\}^\vee) = \gamma\left( \bigcap_{a \in A} \{\{a\}\}^\vee \right) = \left( \begin{array}{c}
0 \\
\gamma(0)(\{A\}^\vee) \\
\vdots \\
\gamma([A]-1)(\{A\}^\vee) \\
\gamma([A])(\{A\}^\vee) \\
\gamma([A]+1)(\bigcap_{a \in A} \{\{a\}\}^\vee) \\
\vdots \\
\gamma([A]-1)(\bigcap_{a \in A} \{\{a\}\}^\vee) \\
\gamma(\{A\})^\vee \\
\gamma([A]+1)(\{A\}^\vee) \\
\gamma([A]-1)(\{A\}^\vee) \\
\gamma(\{A\})^\vee
\end{array} \right),
$$
see (5.22), (5.26) and (5.30) in Example 5.5, and
\[
\gamma(\mathcal{B}(\{A\}^\circ)) = \gamma\left(\bigcup_{a \in A} \{a\}^\circ\right)
\]
\[
= \frac{\gamma(\mathcal{B}(\{A\}^\circ))}{\gamma(0)(\mathcal{B}(\{A\}^\circ))} \cdot \frac{\gamma(\mathcal{B}(\{A\}^\circ))}{\gamma(1)(\mathcal{B}(\{A\}^\circ))} \cdot \cdots \cdot \frac{\gamma(\mathcal{B}(\{A\}^\circ))}{\gamma(t-1)(\mathcal{B}(\{A\}^\circ))} \cdot \frac{\gamma(\mathcal{B}(\{A\}^\circ))}{\gamma(t)(\mathcal{B}(\{A\}^\circ))}.
\]

see (5.24), (5.28) and (5.32).

**Remark 5.6** (see Remark 5.1, and cf. Remark 5.2). **Note that**

\[
\gamma(\mathcal{B}(\{A\}^\circ)) = \mathbf{rn}(\gamma(\{A\}^\circ)) := T_{2t}^{(s)} - \gamma(\{A\}^\circ) \cdot \mathbf{U}(2t);
\]
\[
T_{\mathcal{B}(\{A\}^\circ)} = \mathbf{ro}(T_{\{A\}^\circ}) := -T_{\{A\}^\circ} \cdot \mathbf{U}(2t);
\]
\[
\mathbf{hwt}(\gamma(\mathcal{B}(\{A\}^\circ))) = \frac{|(T_{\mathcal{B}(\{A\}^\circ)})^{-}|}{\mathbf{hwt}(\mathcal{B}(\{A\}^\circ))} = 2t - 2^{t-|A|};
\]
\[
\gamma(s)(\mathcal{B}(\{A\}^\circ)) = \mathbf{rn}(\gamma(t-s)(\{A\}^\circ)) := T_{(t-s)}^{(s)} - \gamma(t-s)(\{A\}^\circ) \cdot \mathbf{U}(\binom{t}{s}), \quad 0 \leq s \leq t;
\]
\[
T_{\mathcal{B}(\{A\}^\circ)} = \mathbf{ro}(T_{(t-s)}^{(s)}) := -T_{(t-s)}^{(s)} \cdot \mathbf{U}(\binom{t}{s}), \quad 0 \leq s \leq t;
\]
\[
\mathbf{hwt}(\gamma(s)(\mathcal{B}(\{A\}^\circ))) = \frac{|(T_{\mathcal{B}(\{A\}^\circ)})^{-}|}{\mathbf{hwt}(\mathcal{B}(\{A\}^\circ))} = \binom{t}{s} - \binom{t-s}{s}, \quad 0 \leq s \leq t;
\]
\[
\mathbf{hwt}(\gamma(t-s)(\{A\}^\circ)) = \frac{|(T_{\{A\}^\circ})^{-}|}{\mathbf{hwt}(\mathcal{B}(\{A\}^\circ))} = \binom{t-s}{s}, \quad 0 \leq s \leq t.
\]

5.2.2. **The blocker** \(\mathcal{B}(\{A\}) = \{\{a\} : a \in A\}.

- **The blocker** of the clutter \(\{A\}\) is the clutter
\[
\mathcal{B}(\{A\}) = \{\{a\} : a \in A\}.
\]

Thus, \(\#\mathcal{B}(\{A\}) = |A|\), and the members of the blocker \(\mathcal{B}(\{A\})\) are the one-element subsets of the set \(A\).

- We associate with the clutters \(\{A\}\) and \(\mathcal{B}(\{A\})\) their characteristic vectors \(\gamma(\{A\}) \in \{0,1\}^{2t}\) and \(\gamma(\mathcal{B}(\{A\})) \in \{0,1\}^{2t}\), and their characteristic topes \(T_{\{A\}} \in \{1,-1\}^{2t}\) and \(T_{\mathcal{B}(\{A\})} \in \{1,-1\}^{2t}\), where
\[
\gamma(\{A\}) = \begin{pmatrix}
0
& \cdots
& 0
\end{pmatrix} \cdot \gamma(\mathcal{B}(\{A\})) = \begin{pmatrix}
0
& \cdots
& 0
\end{pmatrix}.
\]
see (5.21), (5.25) and (5.29) in Example 5.5, and
\[
\gamma(\mathcal{B}(\{A\})) = \left(\begin{array}{c} 0 \\ \gamma(\mathcal{B}(\{A\})) \\ \gamma(\{A\}) \\ \gamma(\mathcal{B}(\{A\})) \\ \gamma(\mathcal{B}(\{A\})) \end{array}\right),
\]
see (5.23), (5.27) and (5.31).

5.2.3. More on the increasing families \(\{A\}^\gamma\) and \(\mathcal{B}(\{A\})^\gamma\).

We can make the following observation:

**Remark 5.7** (cf. Remark 5.3). For a nonempty subset \(A \subseteq E_t\), we have
\[
\min\{i \in E_{2t} : T_{\{A\}}(i) = 1\} = \min\{i \in E_{2t} : \gamma_i(\{A\}^\gamma) = 1\} = T^{-1}(A),
\]
\[
\max\{i \in E_{2t} : T_{\{A\}}(i) = 1\} = \max\{i \in E_{2t} : \gamma_i(\{A\}^\gamma) = 0\} = 2^t - \min A,
\]
\[
\min\{j \in E_{2t} : T_{\mathcal{B}(\{A\})}(j) = 1\} = \min\{j \in E_{2t} : \gamma_j(\mathcal{B}(\{A\})^\gamma) = 1\} = 1 + \min A,
\]
\[
\max\{j \in E_{2t} : T_{\mathcal{B}(\{A\})}(j) = 1\} = \max\{j \in E_{2t} : \gamma_j(\mathcal{B}(\{A\})^\gamma) = 0\} = 1 + 2^t - T^{-1}(A).
\]

- Recall that the partition
  \[
  \{A\}^\gamma \cup (\mathcal{B}(\{A\})^\gamma)^\Delta = 2^{[t]}
  \]
implies that
  \[
  \mathcal{B}(\{A\})^\gamma = \{D^\mathcal{B} : D \in 2^{[t]} - \{A\}^\gamma\}.
  \]
- Note that
  \[
  \gamma(\{A\}^\gamma) = \prod_{a \in A}^* \gamma(\{a\}^\gamma) = \prod_{a \in A}^* \mathbf{a}(a)
  = \prod_{a \in A}^* (T^{(+)}_{2t} - \mathbf{c}(a)) = \left(\prod_{a \in A}^* \mathbf{c}(a)\right) : \mathbf{U}(2^t),
  \]
and recall that
  \[
  \gamma(\mathcal{B}(\{A\})^\gamma) = \mathbf{rn}(\gamma(\{A\}^\gamma)).
  \]

**Remark 5.8.** For a nonempty subset \(A \subseteq E_t\), we have:

(i)
\[
\gamma(\{A\}^\gamma) = \prod_{a \in A}^* \mathbf{a}(a).
\]

(ii)
\[
\gamma(\mathcal{B}(\{A\})^\gamma) = T^{(+)}_{2t} - \left(\prod_{a \in A}^* \mathbf{a}(a)\right) : \mathbf{U}(2^t).
\]

5.3. A clutter \(A := \{A_1, \ldots, A_n\}\).

Let \(A := \{A_1, \ldots, A_n\}\) be a nontrivial clutter on the ground set \(E_t\).
5.3.1. **The increasing family of blocking sets** $\mathcal{B}(A)^\triangleright$.

• See Remark 5.1, and note that

$$\mathcal{A}^\triangleright = \bigcup_{k \in [\alpha]} \{A_k\}^\triangleright = \bigcup_{k \in [\alpha]} \bigcap_{a^k \in A_k} \{a^k\}^\triangleright,$$

and

$$\mathcal{B}(A)^\triangleright = \bigcap_{k \in [\alpha]} \mathcal{B}\{A_k\}^\triangleright = \bigcap_{k \in [\alpha]} \bigcup_{a^k \in A_k} \{a^k\}^\triangleright.$$

• We associate with the increasing families $\mathcal{A}^\triangleright$ and $\mathcal{B}(A)^\triangleright$ their characteristic vectors $\gamma(\mathcal{A}^\triangleright) \in \{0, 1\}^{2^t}$ and $\gamma(\mathcal{B}(A)^\triangleright) \in \{0, 1\}^{2^t}$, and their characteristic topes $T_{\mathcal{A}^\triangleright} \in \{1, -1\}^{2^t}$ and $T_{\mathcal{B}(A)^\triangleright} \in \{1, -1\}^{2^t}$, where

$$\gamma(\mathcal{A}^\triangleright) = \gamma\left(\bigcup_{k \in [\alpha]} \bigcap_{a^k \in A_k} \{a^k\}^\triangleright\right) = \left(\mathcal{a}\right) \cdot \gamma^{(1)}(\bigcup_{k \in [\alpha]} \bigcap_{a^k \in A_k} \{a^k\}^\triangleright) \cdot \cdots \cdot \gamma^{(t-1)}(\bigcup_{k \in [\alpha]} \bigcap_{a^k \in A_k} \{a^k\}^\triangleright) \cdot \left(\mathcal{a}\right) \cdot \gamma^{(t)}(\mathcal{A}^\triangleright),$$

and

$$\gamma(\mathcal{B}(A)^\triangleright) = \gamma\left(\bigcap_{k \in [\alpha]} \bigcup_{a^k \in A_k} \{a^k\}^\triangleright\right) = \left(\mathcal{a}\right) \cdot \gamma^{(1)}(\bigcap_{k \in [\alpha]} \bigcup_{a^k \in A_k} \{a^k\}^\triangleright) \cdot \cdots \cdot \gamma^{(t-1)}(\bigcap_{k \in [\alpha]} \bigcup_{a^k \in A_k} \{a^k\}^\triangleright) \cdot \left(\mathcal{a}\right) \cdot \gamma^{(t)}(\mathcal{B}(A)^\triangleright).$$

5.3.2. **The blocker** $\mathcal{B}(A)$.

• The **blocker** of the clutter $\mathcal{A}$ is the clutter

$$\mathcal{B}(A) = \min \bigcap_{k \in [\alpha]} \mathcal{B}\{A_k\}^\triangleright = \min \bigcap_{k \in [\alpha]} \{a^k\} : a^k \in A_k \}^\triangleright = \min \bigcap_{k \in [\alpha]} \bigcup_{a^k \in A_k} \{a^k\}^\triangleright.$$

• We associate with the clutters $\mathcal{A}$ and $\mathcal{B}(A)$ their characteristic vectors $\gamma(\mathcal{A}) \in \{0, 1\}^{2^t}$ and $\gamma(\mathcal{B}(A)) \in \{0, 1\}^{2^t}$, and their characteristic topes $T_{\mathcal{A}} \in \{1, -1\}^{2^t}$ and $T_{\mathcal{B}(A)} \in \{1, -1\}^{2^t}$, where

$$\gamma(\mathcal{A}) := \gamma^{(0)}(\mathcal{A}) \cdot \gamma^{(1)}(\mathcal{A}) \cdot \cdots \cdot \gamma^{(t)}(\mathcal{A}),$$

$$\gamma^{(0)}(\mathcal{A})$$
and
\[ \gamma(\mathcal{B}(A)) = \gamma(0)_{(\mathcal{B}(A))} \cdot \gamma^{(1)}(\min\bigcap_{k \in [\alpha]} \bigcup_{a^k \in A_k} \{a^k\}) \bigcup_{k \in [\alpha]} \bigcup_{a^k \in A_k} \{a^k\}^\gamma. \]

\[ \cdots \cdot \gamma^{(t)}(\min\bigcap_{k \in [\alpha]} \bigcup_{a^k \in A_k} \{a^k\}) \bigcup_{k \in [\alpha]} \bigcup_{a^k \in A_k} \{a^k\}^\gamma. \]

5.3.3. More on the increasing families \( \mathcal{A}^\gamma \) and \( \mathcal{B}(A)^\gamma \).

- Recall that we have
\[ \mathcal{A}^\gamma \cup (\mathcal{B}(A)^\gamma)^\Delta = 2^{[\alpha]}, \]

that is,
\[ \mathcal{B}(A)^\gamma = \{D^\Delta : D \in 2^{[\alpha]} \setminus \mathcal{A}^\gamma\}. \]

- According to Remark 5.8(ii), we have
\[ \gamma(\mathcal{B}(A)^\gamma) = \prod_{i \in [\alpha]} \gamma(\mathcal{B}(\{A_i\})^\gamma) = \prod_{i \in [\alpha]} \left( T_{2t}^{(+)} - \left( \prod_{a^i \in A_i} a^i \right) \cdot U(2^t) \right) \]
\[ = \left( \prod_{i \in [\alpha]} T_{2t}^{(+)} - \prod_{a^i \in A_i} a^i \right) \cdot U(2^t). \]

Since
\[ \gamma(\mathcal{B}(A)^\gamma) = \text{rn}(\gamma(\mathcal{A}^\gamma)) := T_{2t}^{(+)} - \gamma(\mathcal{A}^\gamma) \cdot U(2^t), \]

by (5.1), we have
\[ \left( \prod_{i \in [\alpha]} T_{2t}^{(+)} - \prod_{a^i \in A_i} a^i \right) \cdot U(2^t) = T_{2t}^{(+)} - \gamma(\mathcal{A}^\gamma) \cdot U(2^t), \]

that is,
\[ \gamma(\mathcal{A}^\gamma) \cdot U(2^t) = T_{2t}^{(+)} - \left( \prod_{i \in [\alpha]} T_{2t}^{(+)} - \prod_{a^i \in A_i} a^i \right) \cdot U(2^t). \]

Theorem 5.9. If \( \mathcal{A} := \{A_1, \ldots, A_\alpha\} \) is a nontrivial clutter on the ground set \( E_t \), then we have:

(i)
\[ \gamma(\mathcal{A}^\gamma) = T_{2t}^{(+)} - \left( \prod_{i \in [\alpha]} T_{2t}^{(+)} - \prod_{a^i \in A_i} a^i \right) \cdot U(2^t). \]  

(ii)
\[ \gamma(\mathcal{B}(A)^\gamma) = \left( \prod_{i \in [\alpha]} T_{2t}^{(+)} - \prod_{a^i \in A_i} a^i \right) \cdot U(2^t). \]

Example 5.10. Suppose \( t := 3 \), and \( E_t = \{1, 2, 3\} \). We have in our hands the characteristic vectors
\[ \tilde{a}(1) := (0, 1, 0, 1, 0, 1) \in \{0, 1\}^{2^3}, \]
\[ \tilde{a}(2) := (0, 1, 0, 1, 0, 1) \in \{0, 1\}^{2^3}, \]
\[ \tilde{a}(3) := (0, 0, 0, 1, 1, 1) \in \{0, 1\}^{2^3}. \]
associated with the principal increasing families that are generated by the clutters \( \{\{a\}\} \), for the elements \( a \in E_t \) of the ground set.

We are given the clutter \( \mathcal{A} := \{A_1, A_2\} \) on the ground set \( E_t \), where \( A_1 := \{1, 2\} \) and \( A_2 := \{2, 3\} \), and we want to know the characteristic vector \( \gamma(\mathfrak{B}(\mathcal{A})^\vee) \) of the increasing family of blocking sets \( \mathfrak{B}(\mathcal{A})^\vee \) of the clutter \( \mathcal{A} \).

Turning to Theorem 5.9(ii), we see that
\[
\prod_{a^1 \in A_1}^* \tilde{a}(a^1) := \prod_{a^1 \in \{1,2\}}^* \tilde{a}(a^1) = (0, 0, 0, 0, 1, 0, 1, 1),
\]
\[
\prod_{a^2 \in A_2}^* \tilde{a}(a^2) := \prod_{a^2 \in \{2,3\}}^* \tilde{a}(a^2) = (0, 0, 0, 0, 1, 0, 1, 1),
\]
\[
T_{2^t}^{(+)} - \prod_{a^1 \in A_1}^* \tilde{a}(a^1) = (1, 1, 1, 1, 0, 1, 1, 0),
\]
\[
T_{2^t}^{(+)} - \prod_{a^2 \in A_2}^* \tilde{a}(a^2) = (1, 1, 1, 1, 1, 0, 0).
\]
\[
\prod_{i \in [2]} \left( T_{2^t}^{(+)} - \prod_{a^i \in A_i}^* \tilde{a}(a^i) \right) = (1, 1, 1, 1, 0, 1, 1, 0),
\]
and finally
\[
\gamma(\mathfrak{B}(\mathcal{A})^\vee) = \left( \prod_{i \in [2]} \left( T_{2^t}^{(+)} - \prod_{a^i \in A_i}^* \tilde{a}(a^i) \right) \right) \cdot \mathcal{U}(2^t) = (0, 0, 0, 0, 1, 1, 1, 1).
\]

In Example 5.12 on page 32, we will attempt to extract from the above vector \( \gamma(\mathfrak{B}(\mathcal{A})^\vee) \) the characteristic vector \( \gamma(\mathfrak{B}(\mathcal{A})) \) of the blocker \( \mathfrak{B}(\mathcal{A}) \).

5.3.4. The characteristic vector of the subfamily of inclusion-minimal sets \( \text{min} \mathcal{F} \) in a family \( \mathcal{F} \).

Suppose we are given the characteristic vector \( \gamma(\mathcal{F}) \) of a nonempty family \( \mathcal{F} \subset 2^{[t]} \) of subsets of the ground set \( E_t \), such that \( \mathcal{F} \neq \emptyset \). We can read off the position numbers of all the inclusion-minimal sets in the family \( \mathcal{F} \) in the following straightforward way (see Example 5.12 on page 32):

Algorithm 5.11.

Input: The char.-vector \( \gamma(\mathcal{F}) \) of a family \( \mathcal{F} \subset 2^{[t]} \), such that \( \emptyset \neq \mathcal{F} \neq \hat{0} \).

Output: A set \( M \) is the set \( \text{supp}(\gamma(\text{min} \mathcal{F})) \) of position numbers of the members of the clutter \( \text{min} \mathcal{F} \);

a vector \( \beta \) is the char.-vector \( \gamma(\text{min} \mathcal{F}) \) of the clutter \( \text{min} \mathcal{F} \) (this data is optional);
a family $\mathcal{B}$ is the clutter $\min \mathcal{F}$ (this data is optional).

(0). Define $\phi \in \{0, 1\}^{2t}$, and store $\phi \leftarrow \gamma(\mathcal{F})$; define $\beta \in \{0, 1\}^{2t}$, and store $\beta \leftarrow (0, \ldots, 0)$; % this action is optional define $\mathcal{B} \subset 2^{[t]}$, and store $\mathcal{B} \leftarrow \emptyset$; % this action is optional define $M \subset 2^{[t]}$, and store $M \leftarrow 0$; define $m \in \mathbb{N}$, and store $m \leftarrow 0$; define $B \in 2^{[t]}$, and store $B \leftarrow \emptyset$. % this action is optional

(1). If $|\text{supp}(\phi)| = 0$, then go to Step (3), else go to Step (2).

(2). Store $m \leftarrow \min \text{supp}(\phi)$, and store $M \leftarrow M \cup \{m\}$, and store $B \leftarrow \Gamma(m)$, and store $\mathcal{B} \leftarrow \mathcal{B} \cup \{B\}$; % this action is optional store $\beta \leftarrow \beta + \sigma(m)$; % this action is optional If $|\text{supp}(\phi)| = 1$, then go to Step (3), else store $\phi \leftarrow \phi - \phi \ast \prod_{e \in B} \tilde{a}(e)$. % this action is optional

Go to Step (1).

(3). Stop.

5.3.5. More on the blocker $\mathcal{B}(\mathcal{A})$.

If we know (see, e.g., Theorem 5.9(ii)) the characteristic vector $\gamma(\mathcal{F})$ of the increasing family of blocking sets $\mathcal{F} := \mathcal{B}(\mathcal{A})^\uparrow$ of a clutter $\mathcal{A} := \{A_1, \ldots, A_\alpha\}$ on the ground set $E_t$, then a description of the blocker $\min \mathcal{F} := \mathcal{B}(\mathcal{A})$ can be obtained by an application of Algorithm 5.11 to the vector $\gamma(\mathcal{F})$; see Example 5.12.

Example 5.12. Suppose $t := 3$, and $E_t = \{1, 2, 3\}$. Note that

$$\tilde{a}(2) := \tilde{a}(2; 2^t) := \gamma(\{\{2\}\}^\uparrow) = (0, 0, 1, 0, 1, 0, 1, 1) \in \{0, 1\}^{2t}.$$

We are given the characteristic vector

$$\gamma(\mathcal{F}) := (0, 0, 1, 0, 1, 1, 1, 1) \in \{0, 1\}^{2t}$$

of the increasing family of blocking sets $\mathcal{F} := \mathcal{B}(\mathcal{A})^\uparrow$ of the clutter $\mathcal{A} := \{\{1, 2\}, \{2, 3\}\}$ on the ground set $E_t$; see, e.g., Example 5.10 on page 30. In
order to find a description of the clutter \textbf{min} \mathcal{F} := \mathcal{B}(\mathcal{A}) \text{, let us apply Algorithm 5.11 to the vector } \gamma(\mathcal{F}):\n
\phi \leftarrow \gamma(\mathcal{F}) := (0, 0, 1, 0, 1, 1, 1) ; \n\|\text{supp}(\phi)\| > 0 ; \n\begin{align*}
m &\leftarrow \min \supp((0, 0, 0, 0, 0, 1, 1, 1)) , \n\end{align*}
\begin{align*}
M &\leftarrow \emptyset \cup \{3\} , \nB &\leftarrow \Gamma(3) , \n\mathcal{B} &\leftarrow \emptyset \cup \{2\} ; \n\beta &\leftarrow (0, 0, 0, 0, 0, 0, 0) + \sigma(3) ; \n\phi &\leftarrow (0, 0, 0, 1, 1, 1, 1) \quad \text{by } \phi = (0, 0, 0, 0, 1, 1, 1, 1) * \tilde{a}(2) ; \n\|\text{supp}(\phi)\| &> 0 ; \n\begin{align*}
m &\leftarrow \min \supp((0, 0, 0, 0, 1, 0, 0)) , \n\end{align*}
\begin{align*}
M &\leftarrow \{3\} \cup \{6\} , \nB &\leftarrow \Gamma(6) , \n\mathcal{B} &\leftarrow \{2\} \cup \{1, 3\} ; \n\beta &\leftarrow (0, 0, 0, 0, 0, 0, 0) + \sigma(6) ; \n\|\text{supp}(\phi)\| &> 1 ; \n\begin{align*}
\text{Stop.} \n\end{align*}
\end{align*}

We see that the set \text{supp}(\gamma(\text{min} \mathcal{F})) =: M \text{ of the position numbers of the members of the blocker } \mathcal{B}(\mathcal{A}) := \text{min} \mathcal{F} \text{ is the set } \{3, 6\} .

The characteristic vector \gamma(\text{min} \mathcal{F}) =: \beta \text{ of the blocker } \mathcal{B}(\mathcal{A}) := \text{min} \mathcal{F} \text{ is the vector } (0, 0, 1, 0, 0, 1, 0, 0) .

The blocker \mathcal{B}(\mathcal{A}) := \text{min} \mathcal{F} =: \mathcal{B} \text{ of the clutter } \mathcal{A} := \{1, 2\}, \{2, 3\} \text{ is the clutter } \{2\}, \{1, 3\} .

### Blocking / Voting

6. **Decompositions of the characteristic topes and of the characteristic vectors of families**

- The vertices \( R^i \in \{1, -1\}^t \) of the symmetric cycle \( R \) in the hypercube graph \( H(t, 2) \), given in (1.1)(1.2), are just simply defined and useful decomposition components of topes of the oriented matroid \( \mathcal{H} := (E_t, \{1, -1\}^t) \).

In the context of the combinatorics of finite sets, the vertices \( R^i \in \{1, -1\}^{2^t} \) of a distinguished symmetric cycle \( R := (R^0, R^1, \ldots, R^{2^t-1}, R^0) \):
in the hypercube graph of topes $H(2^t, 2)$ of the oriented matroid $\mathcal{H}_{2^t} := (E_{2^t}, \{1, -1\}^{2^t})$, where

$$R^0 := \mathbf{T}_{2^t}^{(+)},$$

$$R^s := -[s] R^0, \quad 1 \leq s \leq 2^t - 1,$$

and

$$R^{2^t+k} := -R^k, \quad 0 \leq k \leq 2^t - 1,$$

have an additional meaning:

**Remark 6.1.** Let $R$ be the symmetric cycle in the tope graph of the oriented matroid $\mathcal{H}_{2^t} := (E_{2^t}, \{1, -1\}^{2^t})$, defined by (6.1)(6.2).

(i) The vertex $R^0 := \mathbf{T}_{2^t}^{(+)}, V(R)$ is the characteristic tope $T_\emptyset$ of the empty family $\emptyset$ on the ground set $E_t$.

The vertex $R^2 := -\mathbf{T}_{2^t}^{(+)}, V(R)$ is the characteristic tope $T_{2^t[1]}$ of the power set $2^{[1]}$ of the set $E_t$.

(ii) If $1 \leq i \leq 2^t - 1$, then the vertex $R^i \in V(R)$ is the characteristic tope $T_\mathcal{F}$ of a decreasing family $\mathcal{F}$ of subsets of the ground set $E_t$. In other words, the family $\mathcal{F}$ is a particular abstract simplicial complex, when $1 \leq i \leq 2^t - 1$

Either the subfamily $\max \mathcal{F}$ is an $s$-uniform clutter, where $s := |\Gamma(\max(T_\mathcal{F})^−)|$, or we have $\{|F| : F \in \max \mathcal{F}\} = \{s, s - 1\}$. Indeed, we have

$$\max \mathcal{F} = \left(\mathcal{F} \cap \left(\binom{E_t}{s}\right)\right) \cup \left(\left(\binom{E_t}{s-1}\right) - \left(\mathcal{F} \cap \left(\binom{E_t}{s}\right)\right)\right).$$

(iii) If $2^t+1 \leq i \leq 2\cdot 2^t - 1$, then the vertex $R^i \in V(R)$ is the characteristic tope $T_\mathcal{F}$ of an increasing family $\mathcal{F}$ of subsets of the ground set $E_t$.

Either the subfamily $\min \mathcal{F}$ is an $s$-uniform clutter, where $s := |\Gamma(\min(T_\mathcal{F})^−)|$, or we have $\{|F| : F \in \min \mathcal{F}\} = \{s, s+1\}$. We have

$$\min \mathcal{F} = \left(\mathcal{F} \cap \left(\binom{E_t}{s}\right)\right) \cup \left(\left(\binom{E_t}{s}\right) - \left(\mathcal{F} \cap \left(\binom{E_t}{s}\right)\right)\right).$$

* A distinguished symmetric cycle $\tilde{R} := (\tilde{R}^0, \tilde{R}^1, \ldots, \tilde{R}^{2^t-1}, \tilde{R}^0)$ in the hypercube graph $H(2^t, 2)$ on the vertex set $\{0, 1\}^{2^t}$ is defined\(^{22}\) as follows:

$$\tilde{R}^0 := (0, \ldots, 0),$$

$$\tilde{R}^s := \sum_{e \in [s]} \sigma(e), \quad 1 \leq s \leq 2^t - 1,$$

---

\(^{22}\) Here $\sigma(e)$ is the $e$th standard unit vector of the space $\mathbb{R}^{2^t}$.
and
\[ \widetilde{R}_{2t+k} := T_{2t}^{(+) - \widetilde{R}_k}, \quad 0 \leq k \leq 2^t - 1. \]

We let \( V(\widetilde{R}) := (\widetilde{R}_0, \widetilde{R}_1, \ldots, \widetilde{R}_{2^t - 1}) \) denote the vertex sequence of the cycle \( \widetilde{R} \).

\( \boxed{} \)

\( \bullet \) Let \( \mathcal{F} \subset 2^{|E|} \) be a family of subsets of the ground set \( E_t, \emptyset \neq \mathcal{F} \neq \emptyset \). As earlier, we associate with the family \( \mathcal{F} \) its characteristic tope \( T_{\mathcal{F}} \in \{1, -1\}^{2^t} \), defined by (4.26).

Recall that there exists a unique inclusion-minimal subset \( Q(\mathcal{F}, R) \subset V(R) := (R_0, R_1, \ldots, R_{2^t - 1}) \) of the vertex sequence \( V(R) \) of the cycle \( R \), defined by (6.1)(6.2), such that
\[ T_{\mathcal{F}} = \sum_{Q \in Q(\mathcal{F}, R)} Q. \]

In other words, there exists a unique row vector \( x := x(T_{\mathcal{F}}) := x(T_{\mathcal{F}}, R) := (x_1, \ldots, x_{2^t}) \in \{-1, 0, 1\}^{2^t} \), such that
\[ T_{\mathcal{F}} = \sum_{i \in [2^t]} x_i \cdot R_{i-1} = xM, \quad (6.3) \]
where
\[ M := M(R) := \begin{pmatrix} R_0 & \cdots & R_{2^t - 1} \\ \end{pmatrix}. \quad (6.4) \]

One can see this matrix (in the case \( t := 3 \)) in Example 6.4 on page 40. Thus, we have
\[ x = T_{\mathcal{F}} \cdot M^{-1}, \]
and
\[ Q(T_{\mathcal{F}}, R) := \{x_i \cdot R_{i-1} \colon x_i \neq 0\}. \]
We use the notation \( q(T_{\mathcal{F}}) := q(T_{\mathcal{F}}, R) := |Q(T_{\mathcal{F}}, R)| \) to denote the cardinality of the set \( Q(T_{\mathcal{F}}, R) \).

\( \bullet \) Let us consider the subset
\[ \bar{Q}(\gamma(\mathcal{F}), \mathcal{R}) := \{\frac{1}{2}(T_{2^t}^{(+) - Q}) \colon Q \in Q(T_{\mathcal{F}}, R) \} \subset V(\widetilde{R}), \]
and let us use the notation \( q(\gamma(\mathcal{F})) := q(\gamma(\mathcal{F}), \widetilde{R}) := |\bar{Q}(\gamma(\mathcal{F}), \widetilde{R})| = q(T_{\mathcal{F}}) \) to denote its cardinality.

In analogy with (1.7), we have
\[ \gamma(\mathcal{F}) = -\frac{1}{2}(q(\gamma(\mathcal{F})) - 1) \cdot T_{2^t}^{(+) +} + \sum_{\bar{Q} \in \bar{Q}(\gamma(\mathcal{F}), \widetilde{R}) : \widetilde{Q} \neq (0, \ldots, 0) = \widetilde{R}_0} \bar{Q}. \quad (6.5) \]
• Let \( A \subset 2^T \) be a nontrivial clutter on the ground set \( E_t \), and let \( B := \mathcal{B}(A) \) be its blocker. We associate with the families \( A^\vee, B^\vee, A \) and \( B \) their characteristic topes \( T_A^\vee, T_B^\vee, T_A, T_B \in \{1, -1\}^{2^T} \), and their characteristic vectors \( \gamma(A^\vee), \gamma(B^\vee), \gamma(A), \gamma(B) \in \{0, 1\}^{2^T} \). See (6.6)–(6.13) in Example 6.2.

Example 6.2. Suppose \( t := 3 \), and \( E_t = \{1, 2, 3\} \). Let \( R \) by the symmetric cycle in the hypercube graph \( H(2^4, 2) \) on the vertex set \( \{1, -1\}^{2^T} \), defined by (6.1)(6.2).

We are given the blocking pair \( A := \{\{1, 2\}, \{2, 3\}\} \) and \( B := \mathcal{B}(A) = \{\{1, 3\}, \{2\}\} \) on the ground set \( E_t \).

The families \( A^\vee, B^\vee, A \) and \( B \) are described by their characteristic topes

\[
T_A^\vee := (1, 1, 1, 1, -1, 1, -1, -1) \in \{1, -1\}^{2^T}, \quad (6.6)
\]
\[
T_B^\vee := (1, 1, -1, 1, -1, -1, 1, -1), \quad (6.7)
\]
\[
T_A := (1, 1, 1, 1, -1, 1, -1, 1), \quad (6.8)
\]
\[
T_B := (1, 1, -1, 1, 1, -1, 1, 1), \quad (6.9)
\]

and by their characteristic vectors

\[
\gamma(A^\vee) := (0, 0, 0, 0, 1, 0, 1, 1) \in \{0, 1\}^{2^T}, \quad (6.10)
\]
\[
\gamma(B^\vee) := (0, 0, 1, 0, 1, 1, 1, 1), \quad (6.11)
\]
\[
\gamma(A) := (0, 0, 0, 0, 1, 0, 1, 0), \quad (6.12)
\]
\[
\gamma(B) := (0, 0, 1, 0, 0, 1, 0, 0). \quad (6.13)
\]

Turning to decompositions of the form (6.3), we see that

\[
\varphi(T_A^\vee) = (0, 0, 0, 0, -1, 1, -1, 0) \in \{-1, 0, 1\}^{2^T}, \quad (6.14)
\]
\[
\varphi(T_B^\vee) = (0, 0, -1, 1, -1, 0, 0, 0), \quad (6.15)
\]
\[
\varphi(T_A) = (1, 0, 0, 0, -1, 1, -1, 1), \quad (6.16)
\]
\[
\varphi(T_B) = (1, 0, -1, 1, 0, -1, 1, 0). \quad (6.17)
\]

Thus, we have the decompositions:

\[
T_A^\vee := (-1, -1, -1, -1, 1, 1, 1, 1) = -R_4 + R_5 - R_6
\]
\[
= -(1, -1, -1, -1, 1, 1, 1, 1) + (1, -1, -1, -1, 1, 1, 1, 1) + (-1, -1, -1, -1, 1, 1, 1, 1) = R_8 + R_12 + R_14
\]

\[
T_B^\vee := (1, 1, 1, 1, 1, 1, -1, -1) + (1, 1, 1, 1, 1, -1, -1, -1) + (1, 1, 1, 1, 1, -1, -1, -1),
\]

\[
T_A := (1, 1, 1, 1, 1, 1, -1, -1) = -R_{10} + R_{12} - R_{14}
\]
\[
= -(-1, -1, -1, -1, 1, 1, 1, 1) + (1, -1, -1, -1, 1, 1, 1, 1) + (-1, -1, -1, -1, 1, 1, 1, 1) = R_8 + R_12 + R_14
\]

\[
T_B := (1, 1, 1, 1, 1, 1, -1, -1) + (1, 1, 1, 1, 1, -1, -1, -1) + (1, 1, 1, 1, 1, -1, -1, -1),
\]
\[ T_{B^0} := (1, 1, -1, 1, -1, 1, -1, -1, 1) = - \left( R_2^2 + R_3^3 - R_4^{10} \right) - \frac{R_4^{12}}{-R_4^{10}} \]
\[ = (-1, -1, 1, 1, 1, 1, 1, 1) + (-1, -1, -1, 1, 1, 1, 1, 1) \]
\[ - (-1, -1, -1, 1, 1, 1, 1, 1) = R_3^4 + R_4^{10} + R_4^{12} \]
\[ = (-1, -1, 1, 1, 1, 1, 1, 1) + (1, 1, -1, -1, -1, 1, 1, 1) \]
\[ + (1, 1, 1, 1, 1, -1, -1, -1), \]

\[ T_A := (1, 1, 1, 1, -1, 1, -1, 1) = \frac{R_4^6}{R_4^{12}} - \frac{R_4^8}{R_4^{12}} + R_5^5 - \frac{R_6^6}{R_4^{14}} + \frac{R_7^7}{R_4^{14}} \]
\[ = (1, 1, 1, 1, 1, 1, 1, 1) \]
\[ - (-1, -1, -1, 1, 1, 1, 1, 1) \]
\[ + (-1, -1, -1, -1, 1, 1, 1, 1) \]
\[ - (-1, -1, 1, -1, -1, -1, 1, 1) \]
\[ + (-1, -1, 1, -1, -1, -1, 1, 1) \]
\[ = (1, 1, 1, 1, 1, 1, 1, -1, -1, -1) \]
\[ + (1, 1, 1, 1, 1, 1, -1, -1) \]

and

\[ T_B := (1, 1, 1, 1, -1, 1, 1, 1) = \frac{R_4^0}{R_4^{12}} - \frac{R_4^2}{R_4^{16}} + R_4^{3} - \frac{R_4^6}{R_4^{13}} + R_4^{6} \]
\[ = (1, 1, 1, 1, 1, 1, 1, 1) \]
\[ - (-1, -1, 1, 1, 1, 1, 1, 1) \]
\[ + (-1, -1, -1, 1, 1, 1, 1, 1) \]
\[ - (-1, -1, -1, -1, 1, 1, 1, 1) \]
\[ + (-1, -1, -1, -1, -1, 1, 1, 1) \]
\[ = (1, 1, 1, 1, 1, 1, 1, 1, 1) \]
\[ + (-1, -1, 1, 1, 1, 1, 1, 1) \]
\[ + (-1, -1, -1, -1, 1, 1, 1, 1) \]
\[ + (1, 1, 1, -1, -1, -1, -1, 1, 1) \]
\[ + (1, 1, 1, 1, 1, -1, -1, -1, 1, 1) \]
Relations of the form (6.5) imply that
\[ \gamma(A^\vee) := (0, 0, 0, 1, 0, 0, 1, 1) = -T_{2^t}^{(+)} + \vec{R}^5 + \vec{R}^{12} + \vec{R}^{14} \]
\[ = (-1, -1, -1, -1, -1, -1, -1, -1) + (1, 1, 1, 1, 1, 0, 0, 0) + (0, 0, 0, 0, 1, 1, 1, 1) + (0, 0, 0, 0, 0, 0, 1, 1), \]
\[ \gamma(B^\vee) := (0, 0, 1, 0, 1, 0, 1, 1) = -T_{2^t}^{(+)} + \vec{R}^3 + \vec{R}^{10} + \vec{R}^{12} \]
\[ = (-1, -1, -1, -1, -1, -1, -1, -1) + (1, 1, 1, 0, 0, 0, 0) + (0, 0, 1, 1, 1, 1, 1) + (0, 0, 0, 0, 0, 1, 1, 1), \]
\[ \gamma(A) := (0, 0, 0, 1, 0, 1, 0, 0) = -2T_{2^t}^{(+)} + \vec{R}^6 + \vec{R}^5 + \vec{R}^{12} + \vec{R}^{14} \]
\[ = -2T_{2^t}^{(+)} + \vec{R}^6 + \vec{R}^7 + \vec{R}^{12} + \vec{R}^{14} \]
\[ = (-2, -2, -2, -2, -2, -2, -2, -2) + (1, 1, 1, 1, 1, 0, 0, 0) + (0, 0, 0, 1, 1, 1, 1) + (0, 0, 0, 0, 0, 0, 1, 1), \]
\[ \gamma(B) := (0, 0, 1, 0, 1, 0, 0, 0) = -2T_{2^t}^{(+)} + \vec{R}^6 + \vec{R}^5 + \vec{R}^{10} + \vec{R}^{13} \]
\[ = -2T_{2^t}^{(+)} + \vec{R}^3 + \vec{R}^6 + \vec{R}^{10} + \vec{R}^{13} \]
\[ = (-2, -2, -2, -2, -2, -2, -2, -2) + (1, 1, 1, 0, 0, 0, 0, 0) + (0, 0, 1, 1, 1, 1, 1) + (0, 0, 0, 0, 0, 0, 1, 1), \]

and

Corollary 2.3(i) and Proposition 2.2(iv), restated in dimensionality \(2^t\), suggest the following:

**Theorem 6.3.** Let \( R \) be the symmetric cycle in the hypercube graph \( H(2^t, 2) \) on the vertex set \( \{1, -1\}^{2^t} \), defined by (6.1)(6.2).

Let \( A \subset 2^0 \) be a nontrivial clutter on the ground set \( E_t \), and let \( B := \mathcal{B}(A) \) be its blocker. Since the characteristic topes of the increasing families \( A^\vee \) and \( B^\vee \) obey the relation

\[ T_{B^\vee} = \text{ro}(T_{A^\vee}), \]

we have:
(i) 
\[ q(T_{S^V}) := |Q(T_{S^V}, R)| = |Q(T_{A^V}, R)| =: q(T_{A^V}) , \]
and 
\[ x(T_{S^V}) = x(T_{A^V}) \cdot \overline{U}(2^t) \cdot \overline{T}(2^t) . \]

(ii) Suppose that the subset \( (T_{A^V})^- = \text{supp}(\gamma(A^V)) \subset E_{2^t} \) is a disjoint union

\[ \{ i_1, j_1 \} \cup \{ i_2, j_2 \} \cup \ldots \cup \{ i_{\varrho - 1}, j_{\varrho - 1} \} \cup \{ i_{\varrho}, j_{\varrho} \} \]

of intervals such that

\[ j_1 + 2 \leq i_2, \ j_2 + 2 \leq i_3, \ldots, \ j_{\varrho - 2} + 2 \leq i_{\varrho - 1}, \ j_{\varrho - 1} + 2 \leq i_{\varrho} , \]

for some \( \varrho \). We have

\[ q(T_{S^V}) = q(T_{A^V}) = 2\varrho - 1 ; \]

\[ x(T_{A^V}) = \sum_{1 \leq k \leq \varrho - 1} \sigma(j_k + 1) - \sum_{1 \leq \ell \leq \varrho} \sigma(i_\ell) , \]

and

\[ x(T_{S^V}) = \sum_{1 \leq k \leq \varrho - 1} \sigma(2^t - j_k + 1) - \sum_{1 \leq \ell \leq \varrho} \sigma(2^t - i_\ell + 2) . \]

See expressions (6.6)(6.7) and (6.14)(6.15) in Example 6.2.

6.1. A clutter \( \{ \{a\} \} \).

As earlier (in Section 5.1), let \( \{ \{a\} \} \) be a clutter on the ground set \( E_t \), whose only member is a one-element subset \( \{a\} \subset E_t \).

- Let us associate with the characteristic tope \( a(a) := T(\{a\}) \) of the principal increasing family \( \{ \{a\} \} \) the row vector \( x(a(a)) := x(a(a), R) \in \{-1, 0, 1\}^{2^t} \), described in (6.3), where \( R \) is the symmetric cycle in the hypercube graph \( H(2^t, 2) \), defined by (6.1)(6.2). Recall that

\[ x(a(a)) = a(a) \cdot M^{-1} , \tag{6.18} \]

where the matrix \( M \) is defined by (6.4), and 

\[ Q(a(a), R) := \{ x_i \cdot R_i^{-1} : x_i \neq 0 \} , \quad \text{and} \quad a(a) = \sum_{Q \in Q(a(a), R)} Q , \]

see Example 6.4.

**Example 6.4.** Suppose \( t := 3 \), and \( E_t = \{1, 2, 3\} \).

The characteristic topes associated with the principal increasing families that are generated by the clutters \( \{ \{a\} \} \), for the elements \( a \in E_t \) of the ground set, are as follows:

\[ a(1) := a(1; 2^t) := T(\{1\})_{2^t} = (1, -1, 1, 1, -1, 1, -1, -1) \in \{1, -1\}^{2^t} , \]
\[ a(2) := a(2; 2^t) := T(\{2\})_{2^t} = (1, 1, -1, 1, -1, 1, -1, -1) , \]
\[ a(3) := a(3; 2^t) := T(\{3\})_{2^t} = (1, 1, -1, 1, -1, -1, -1, 1) \]
The corresponding “\(x\)-vectors”, given in (6.18) for the symmetric cycle \(R\) in the hypercube graph \(H(2^t, 2)\) defined by (6.1)(6.2), are:

\[
\begin{align*}
\mathbf{x}(\mathbf{a}(1)) &:= \mathbf{x}(\mathbf{a}(1), R) = (0, -1, 1, 0, -1, 0, -1, 1) \in \{-1, 0, 1\}^{2^t}, \\
\mathbf{x}(\mathbf{a}(2)) &:= \mathbf{x}(\mathbf{a}(2), R) = (0, 0, -1, 1, -1, 1, 0), \\
\mathbf{x}(\mathbf{a}(3)) &:= \mathbf{x}(\mathbf{a}(3), R) = (0, 0, 0, -1, 1, -1, 0, 0).
\end{align*}
\]

Indeed, we see that

\[
\mathbf{x}(\mathbf{a}(2)) \cdot \mathbf{M} = (0, 0, -1, 1, -1, 1, 1, 0) \cdot \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{pmatrix}
= (1, 1, -1, 1, -1, 1, -1, 1) =: \mathbf{a}(2).
\]

\[\text{• For the row vector } \mathbf{y}(1 + a) := \mathbf{y}(1 + a; 2^t) \in \{-1, 0, 1\}^{2^t}, \text{ defined by}
\]

\[
\mathbf{y}(1 + a) := \mathbf{x}(-\{1+a\} T^{(+)}_{2^t}) =: \mathbf{x}(T_{\{a\}}),
\]

we have (see [69, Sect. 2]):

\[
\mathbf{y}(1 + a) = \mathbf{\sigma}(1) - \mathbf{\sigma}(1 + a) + \mathbf{\sigma}(2 + a).
\]

In other words,

\[
\mathbf{Q}(T_{\{a\}}, R) = \{ \frac{\mathbf{R}^0}{T^{(+)}_{2^t}}, \mathbf{R}^{1+a}, \mathbf{R}^{2^t+a} \},
\]

and

\[
\mathbf{T}_\mathcal{B}(\{\{a\}\}) = T_{\{a\}} = T^{(+)}_{2^t} + \mathbf{R}^{1+a} + \mathbf{R}^{2^t+a}.
\]

Equivalently,

\[
\mathbf{Q}(\mathbf{\gamma}(\mathcal{B}(\{a\})), \mathbf{R}) = \{ \frac{\mathbf{R}^0}{(0,\ldots,0)}, \mathbf{R}^{1+a}, \mathbf{R}^{2^t+a} \},
\]

and

\[
\mathbf{\gamma}(\mathcal{B}(\{a\})) = \mathbf{\gamma}(\{a\}) = -\frac{1}{2}(3 - 1) \cdot T^{(+)}_{2^t} + \mathbf{R}^{1+a} + \mathbf{R}^{2^t+a} = -T^{(+)}_{2^t} + \mathbf{R}^{1+a} + \mathbf{R}^{2^t+a}.
\]

6.2. A clutter \(\{A\}\).

As in Section 5.2, let \(\{A\}\) be a clutter on the ground set \(E_t\), whose only member is a nonempty subset \(A \subseteq E_t\).

\[\text{• Dealing with the symmetric cycle } \mathbf{R} \text{ in the hypercube graph } H(2^t, 2), \text{ defined by (6.1)(6.2), with the matrix } \mathbf{M} \text{ given in (6.4), and with “}x\text{-vectors” described in (6.3), for the row vector}
\]

\[
\mathbf{y}(\Gamma^{-1}(A)) := \mathbf{x}(\{-\Gamma^{-1}(A)\} T^{(+)}_{2^t}) =: \mathbf{x}(T_{\{A\}}) = T_{\{A\}} \cdot \mathbf{M}^{-1} \in \{-1, 0, 1\}^{2^t},
\]

\[(6.19)\]
we have (see [69, Sect. 2]):

\[
y(\Gamma^{-1}(A)) = \begin{cases} 
\sigma(1) - \sigma(\Gamma^{-1}(A)) + \sigma(1 + \Gamma^{-1}(A)), & \text{if } A \neq E_t, \\
-\sigma(2^t), & \text{if } A = E_t.
\end{cases}
\]

In other words,

\[
Q(T_{\{A\}}, R) = \begin{cases} 
\{ \frac{R^0}{T_{2t}^{(+)}} \}, & \text{if } A \neq E_t, \\
\{ R^{2t-1} \}, & \text{if } A = E_t,
\end{cases}
\]

and

\[
T_{\{A\}} = \begin{cases} 
T_{2t}^{(+)}, & \text{if } A \neq E_t, \\
R^{2t-1}, & \text{if } A = E_t.
\end{cases}
\]

Equivalently,

\[
\bar{Q}(\gamma(\{A\}), \bar{R}) = \begin{cases} 
\{ \frac{R^0}{(0, \ldots, 0)} \}, & \text{if } A \neq E_t, \\
\{ R^{2t-1} \}, & \text{if } A = E_t,
\end{cases}
\]

and

\[
\gamma(\{A\}) = \begin{cases} 
-\frac{1}{2}(3 - 1) \cdot T_{2t}^{(+)}, & \text{if } A \neq E_t, \\
\bar{R}^{2t-1}, & \text{if } A = E_t.
\end{cases}
\]

• Recall that \(\gamma(\{A\}^\gamma) = \prod_{a \in A} \tilde{a}(a)\), and \(\gamma(\mathcal{B}(\{A\})^\gamma) = \text{rn}(\gamma(\{A\}^\gamma))\).

**Remark 6.5** (cf. Remark 5.8). For a nonempty subset \(A \subseteq E_t\), we have

(i)

\[
\gamma(\{A\}^\gamma) = \prod_{a \in A}^* \left( \frac{1}{2} \left( T_{2t}^{(+)}, x(a) \cdot \mathbf{M} \right) \right).
\]

(ii)

\[
\gamma(\mathcal{B}(\{A\})^\gamma) = T_{2t}^{(+)}, \left( \prod_{a \in A}^* \left( \frac{1}{2} \left( T_{2t}^{(+)}, x(a) \cdot \mathbf{M} \right) \right) \right) \cdot \mathbf{U}(2^t).
\]

• Since the *blocker* of the clutter \(\{A\}\) is the clutter \(\mathcal{B}(\{A\}) = \{\{a\} : a \in A\}\), and \(\gamma(\mathcal{B}(\{A\})) = \sum_{a \in A} \gamma(\{\{a\}\})\), we have

\[
\gamma(\mathcal{B}(\{A\})) = \sum_{a \in A} (-T_{2t}^{(+)}, \bar{R}^{1+a} + \bar{R}^{2t+a}),
\]

that is,

\[
\gamma(\mathcal{B}(\{A\})) = -|A| \cdot T_{2t}^{(+)}, \sum_{a \in A} (\bar{R}^{1+a} + \bar{R}^{2t+a}).
\]
6.3. A clutter $A := \{A_1, \ldots, A_\alpha\}$.

As in Section 5.3, let $A := \{A_1, \ldots, A_\alpha\}$ be a nontrivial clutter on the ground set $E_t$.

- In analogy with [69, Rem. 2.2], dealing with the symmetric cycle $R$ in the hypercube graph $H(2^t, 2)$, defined by (6.1)-(6.2), with the matrix $M$ given in (6.4), with “$x$-vectors” described in (6.3), and with “$y$-vectors” given in (6.19), we have

$$x(T_A) = (1 - \#A) \cdot \sigma(1) + \sum_{A \in A} y(\Gamma^{-1}(A)),$$

that is,

$$x(T_A) = \begin{cases} 
\sigma(1) + \sum_{A \in A} (\sigma(\Gamma^{-1}(A)) + \sigma(1 + \Gamma^{-1}(A))), & \text{if } A \neq \{E_t\}, \\
-\sigma(2^t), & \text{if } A = \{E_t\}, 
\end{cases}$$

or

$$T_A = \begin{cases} 
T_{2^t} + \sum_{A \in A} (R^{\Gamma^{-1}(A)} + R^{2^t + \Gamma^{-1}(A) - 1}), & \text{if } A \neq \{E_t\}, \\
2^t - 1, & \text{if } A = \{E_t\}. 
\end{cases}$$

We also have

$$\gamma(A) = \begin{cases} 
-(\#A) \cdot T_{2^t} + \sum_{A \in A} (\tilde{R}^{\Gamma^{-1}(A)} + \tilde{R}^{2^t + \Gamma^{-1}(A) - 1}), & \text{if } A \neq \{E_t\}, \\
2^t - 1, & \text{if } A = \{E_t\}. 
\end{cases}$$

- Theorem 5.9 can be accompanied with the following statement:

**Corollary 6.6.** If $A := \{A_1, \ldots, A_\alpha\}$ is a nontrivial clutter on the ground set $E_t$, then we have:

(i)

$$\gamma(A^\triangledown) = T_{2^t} - \left( \prod_{i \in [\alpha]} (T_{2^t} - \prod_{a_i \in A_i} (\frac{1}{2} (T_{2^t} - x(a_i)) \cdot M) \right).$$

(ii)

$$\gamma(\mathcal{B}(A)^\triangledown) = \left( \prod_{i \in [\alpha]} (T_{2^t} - \prod_{a_i \in A_i} (\frac{1}{2} (T_{2^t} - x(a_i)) \cdot M) \right) \cdot \mathcal{U}(2^t).$$

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