Velocity Dominance Near a Crushing Singularity *

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Abstract

The asymptotic behavior of geometry near the boundary of maximal Cauchy development is studied using a perturbative method, which at the zeroth order reduces Einstein’s equations to an exactly solvable set of equations—Einstein’s equations with all “space” derivatives dropped. The perturbative equations are solved to an arbitrarily-high order for the cosmological spacetimes admitting constant-mean-curvature foliation that ends in a crushing singularity, i.e., whose mean curvature blows up somewhere. Using a “new” set of dynamical variables (generalized Kasner variables) restrictions on the initial data are found that make the zeroth-order term in the expansion asymptotically dominant when approaching the crushing singularity. The results obtained are in agreement with the first order results of Belinskii, Lifshitz and Khalatnikov on general velocity-dominated cosmological singularities, and, in addition, provide clearer geometrical formulation.

1 Introduction

The set of all solutions of Einstein’s equations is too large from both the mathematical and physical standpoint. Many attempts have been made to formulate the conditions that would restrict the solutions of Einstein’s equations to a class that is:

- large enough to include all “reasonable” physical situations,
- small enough to exclude those deemed physically “pathological”,
- mathematically as simple as possible.

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The most promising candidate to date is (a generic subset of) the class of maximally-extended globally-hyperbolic spacetimes. The most serious problem—the problem that initiated the Strong Cosmic Censorship conjecture—for that class stems from the Hawking-Penrose-Geroch singularity theorems, which indicate that the globally-hyperbolic spacetimes, due to the possible incompleteness of non-curvature-singular timelike geodesics, might have to be extended beyond the boundary of their maximal Cauchy development. Possible incompleteness of curvature-singular geodesics is physically acceptable for two reasons. The first is that being crushed or torn apart by infinite tidal forces is believed to be a good reason for any observer to disappear, and the second is that there is some hope of rescue by a quantum modification of classical general relativity; as it is believed that quantum effects become dominant in the regions of strong curvature, and might eliminate the unpleasant classical infinite-curvature singularities.

It is, therefore, crucial to establish how often and under what conditions the non-curvature-singular (bad), as opposed to the curvature-singular (good), incomplete timelike geodesics occur near the boundary of the maximal Cauchy development; making the study of the asymptotic behavior of geometry near the Cauchy boundary crucial for the acceptability of the class of maximal globally-hyperbolic spacetimes in general relativity. Unfortunately, due to the complexity of Einstein’s equations, an exact or useful description of the asymptotic behavior seems to be unattainable, at least so in the foreseeable future. Some results of the work done by Belinski, Khalatnikov and Lifshitz, hereafter BKL, indicate that, generically, the approach to the boundary of the maximal Cauchy development (the singularity in their terminology) is extremely complex and can be thought of as a sequence of pointwise Mixmaster-like transitions between Kasner epochs. A satisfactory geometrical formulation of their method is still lacking, and it is still controversial whether their method can give any information about the global structure of the singularity. Nevertheless, an important discovery of BKL was that the asymptotic dynamics simplifies significantly for spacetimes for which, in a suitable foliation, the spatial derivative terms in the equations of motion can be neglected asymptotically. Since Einstein’s equations thus truncated are exactly solvable, we can extract all the asymptotic properties from the solution of the truncated equations, the Generalized Kasner Solution; a solution which evolves pointwise like the Kasner solution. Most importantly, the Generalized Kasner solution is generically curvature-singular, as there is only one, (1,0,0), non-curvature-singular Kasner solution.

The question I shall try to answer here, without going into technical details, is: Under what conditions is that approximation, usually called the Velocity-Dominated Approximation (VDA), valid in the class of globally-hyperbolic vacuum cosmological spacetimes, when approaching the boundary of the maximal Cauchy development?
2 Generalized Kasner Variables and Velocity-Dominated Approximation

Let \( K_j^i \) and \( g_{ij} \) be the extrinsic curvature and spatial metric tensors induced by an arbitrary foliation \( \Sigma_t \) on the spacetime, and let \( \tau = \text{Tr}(K) \) and \( \hat{K}_j^i = K_j^i / \tau \). There exists a unique triad \( E_i^q(p) \) of spacelike vector fields which puts \( K_j^i \) and \( g_{ij} \) into the following form:

\[
E g E^T = \begin{pmatrix}
\tau^{-2p_1} & 0 & 0 \\
0 & \tau^{-2p_2} & 0 \\
0 & 0 & \tau^{-2p_3}
\end{pmatrix},
\]

provided \( p_1 \neq p_2 \neq p_3 \) and \( \tau > 0 \). In that case there is a one to one mapping between the old dynamical variables \((K, g)\) and the new ones \((E, \tau, p)\) with \( \Sigma p_q = 1 \). If we restrict ourselves to a subclass of cosmological spacetimes that admit constant-mean-curvature (CMC) foliation, which is unique if it exists \([4]\), we obtain what I call generalized Kasner variables (GKV): a triad of spacelike vector fields \( E_i^q \) and three Kasner exponents \( p_q \) (two of them independent), invariantly defined throughout the spacetime. Since the globally-hyperbolic spacetimes with surfaces of infinite mean-curvature (crushing singularities) have Cauchy boundaries exactly at those surfaces, we cananalyse the Cauchy boundaries by taking limit \( \tau \to \pm \infty \). In spacetimes with no crushing singularities, however, there is no generic way to lock onto the Cauchy boundary, whose elusiveness prevents us from analysing the geometry near it. So we should restrict our attention to the crushing singularities.

In order to study the behavior of metric near the spacetime’s Cauchy boundary, I use a perturbative VDA method previously developed by Moncrief and myself. It was applied successfully to \( T^3 \times R \) Gowdy spacetimes, where it was found, by an inductive argument, that the VDA is valid to arbitrarily high perturbative order \([5]\). The method is applicable to spacetimes with no isometries as well, and, in a nutshell, consists of modifying Einstein’s equations by multiplying the shift vector in the ADM action by an \( \epsilon \),

\[
I_{\text{ADM}} \to I_{\text{ADM}\epsilon} = \int \left[ NH + \epsilon N^i H_i \right],
\]

and then expanding the solution of the modified Einstein’s equations with respect to that \( \epsilon \), about \( \epsilon = 0 \). The introduction of \( \epsilon \) is just a device to obtain a well defined series of recursively solvable perturbative equations which start with the generalized Kasner solution as the zeroth-order term. The case we are interested in is \( \epsilon = 1 \).
After rewriting Einstein’s equations in terms of the GKV, and choosing the time $t$ to be some monotonous function of the mean curvature $\tau$, we find that in the zeroth order the GKV are time independent, that the lapse function $N$ depends only on time $t$, and that the Hamiltonian constraint gives an additional restriction on (generalized) Kasner exponents, which together with the initial trace constraint gives just the standard Kasner relations for the zeroth order functions (only of “spacelike coordinates”):

$$\sum p^2_q = \sum p_q = 1$$

The zeroth order solution is just the generalized Kasner solution.

All higher order terms, in turn, are uniquely determined by the initial data for the zeroth order solution and, using an inductive argument, it can be showed that, as $\tau \to \infty$, they become negligible when compared with the zeroth order provided

$$\vec{E}_1 \cdot (\vec{\nabla} \times \vec{E}_1) = 0,$$

where $\vec{E}_1$ is the eigenvector with the only possible negative Kasner exponent. Roughly, the rate of decay is faster the higher the order.

### 3 Conclusions

The condition (5) has already been established by BKL [1] as a sufficient condition for the decay of the first-order correction. Using the generalized Kasner variables I showed that (5) is also sufficient condition for the decay of all higher orders in the expansion, asymptotically as $\tau \to \infty$, and, therefore, for the VDA to be valid perturbatively. Even though I did not use a synchronous foliation, the fact that the lapse function $N$ asymptotically approaches a spatially homogeneous function indicates that the CMC foliation asymptotically becomes a synchronous foliation when approaching $\tau = \infty$.

### References

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