New Results on Moment Generating Functions of Generalized Wireless Fading Distributions and Applications

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Abstract—In this letter, new exact and approximated moment generating functions (MGF) for generalized fading distributions are derived and presented. Specifically, we consider the \( \eta-\mu, \alpha-\mu, \alpha-\eta-\lambda, \alpha-\lambda-\mu, \alpha-\lambda-\eta-\mu \) and \( \alpha-\lambda-\mu-\eta \) generalized distributions. In contrast to earlier results in the literature, both the new exact and approximated MGFs avoids complicated and computationally expensive special functions such as the Meijer-G or the Fox H-functions. Hence, the new MGFs allow easier and more efficient analytical manipulation and code development. As an illustrative application, the average bit error rates (ABER) for each of the fading models are evaluated using the new MGFs, which are also compared with the numerically evaluated results. The presented analytical and numerical results agree with reported results from the literature for the special cases of the considered models.

Keywords—Moment Generating Function; \( \alpha-\eta-\mu, \alpha-\lambda-\mu; \alpha-\mu \); Generalized Distributions; Average Bit Error Rates.

I. INTRODUCTION

Wireless systems are inevitably subjected to additive white (generalized) Gaussian noise that predominantly degrades the system performance characterized by the bit error rates and ergodic capacity [1]. Propagation media is also characterized by various effects that worsen the performance even more, such as multipath fading and shadowing [2]. Recently reported generalized models which are used to characterize on these effects, e.g. the \( \alpha-\lambda-\mu \) and \( \alpha-\eta-\mu \) models, comprising many other distributions as special cases such as the \( \kappa-\mu \) and \( \eta-\mu \) (see e.g. [3] [4]), were proven to provide better fitting for experimental data as compared to classical models [5] [6]. In order to evaluate the performance limits of such transmission environments with the different possible system configurations, the moment generating function (MGF) based approaches were suggested to evaluate the error rates [7] [8], ergodic capacity [9] [8], and amount of dispersion [7]. However, the aforementioned MGF approaches are either dependent on the exact knowledge of the MGF closed-form expression and this is not always the case (and by itself is a challenging problem) for the generalized models, e.g. with cooperative diversity systems. Moreover, the MGFs reported in the literature for generalized fading models (e.g. \( \alpha-\mu \) [10] [11], \( \alpha-\lambda-\mu \) and \( \alpha-\eta-\mu \) [3]) are given in terms of the Fox \( H \)- or the Meijer-G functions, which are very computationally inefficient, or even in infinite series representation, and are not friendly for further manipulations. The MGF of the \( \alpha-\lambda-\eta-\mu \) was also elegantly reduced (and approximated) in [12] by reducing (estimating) this model as a generalized gamma summands and hence derived the MGF, but both of the exact and the approximate expressions are given in terms of the \( G \)- and \( H \)-functions.

In this paper, the aim is to develop unified MGF expressions for the \( \eta-\mu, \alpha-\mu, \alpha-\eta-\mu, \alpha-\lambda-\mu, \alpha-\lambda-\eta-\mu \) and \( \alpha-\lambda-\mu-\eta \) generalized models. The expressions are developed using simple and computationally efficient mathematical functions, mainly by re-exploiting the developed exponential approximation of the \( \exp[-x^r] \) function, with \( r < 1 \), from [3]. The resulting MGF expressions can then be easily used to derive the \( k \)th derivative of the MGF, given as \( \mathcal{M}_r^{(k)}(s) = (-1)^k \mathcal{E}_{k}^{[r]} \left[ y \cdot e^{-y\cdot s} \right] \) with \( y \) and \( \mathcal{E}_{k}^{[r]} \) denoting the instantaneous signal-to-noise ratio and the expectation operation, hence allowing simple and direct evaluation of essential performance metrics such as the error rates and ergodic capacity.

The rest of the letter is structured as follows. In section II, the considered generalized models are revisited along with the exponential approximation. In section III, the novel unified MGF expressions are derived for the six generalized fading distributions. Following to that, section IV is dedicated to illustrate some examples demonstrating the applicability of the derived results by comparing the average bit error rates obtained using the derived MGFs with those numerically simulated which are also available in the literature.

II. PRELIMINARIES

A. The Generalized Fading Models

This section is dedicated to present a brief overview of the six considered generalized fading distribution models, namely the \( \eta-\lambda-\mu, \alpha-\mu, \alpha-\eta-\mu, \alpha-\lambda-\mu, \alpha-\kappa-\mu \), and \( \alpha-\lambda-\eta-\mu \) models. In these models, the fading parameters \( \alpha, \lambda, \eta, \mu \) and \( \kappa \) account for the nonlinearity, the correlation between the in-phase and quadrature components, the unequal power of these two components, the number of the multipath clusters, and the ratio between the total power of the dominant components and the total power of the scattered waves, respectively [6] [5] [13]. In what follows, the symbols \( \gamma \) and \( \bar{\gamma} \) respectively denote the instantaneous signal-to-noise ratio and its average value.

i. The \( \eta-\lambda-\mu \) Fading Model

The power PDF of the \( \eta-\lambda-\mu \) generalized fading model, introduced in [14], is given by:

\[
f_\gamma(y) = \left[ \frac{\gamma}{(\gamma + \lambda)^{\alpha}} \right] \int_0^\infty \left[ \frac{y^{\alpha-1} e^{-\gamma y}}{\alpha \gamma^{\alpha-1} \lambda^{\alpha-1}} \right] e^{-\gamma y} I_{\alpha-1}(\alpha \gamma y) dy,
\]

where \( I_\alpha(.) \) is the modified Bessel function of the first type [15], \( \gamma \) and \( \bar{\gamma} \) denote the instantaneous and average signal-to-noise ratio, respectively. In addition, the parameters \( \lambda, \eta, \mu \) account for the correlation between the in-phase and quadrature components of the fading signal, the unequal power of these components, and the number of multipath clusters, respectively. The PDF in (1a) can be also rewritten, for compactness, as:

\[
f_\gamma(y) = \psi \gamma^{\alpha-1} e^{-\bar{\gamma} y} I_\alpha(dy),
\]

(1b)
with \( \psi, \beta, \upsilon, \) and \( m \) are as given in table I, and the internal parameter \( \hat{d} = \hat{b}\sqrt{(\eta - 1)^2 + 4\mu^2} \), and \( \hat{b} \) and \( \hat{c} \) are given in table II. This model includes the \( \lambda - \mu \), the \( \eta - \mu \), the Hoyt, the Rice, the Nakagami-\( m \), the Rayleigh, the Exponential, the Gamma, and the One-sided Gaussian models as special cases.

ii. The \( \alpha - \mu \) Fading Model

The PDF of the \( \alpha - \mu \) generalized fading is given by [10]:

\[
f_y(y) = \frac{\alpha \mu}{2!^m \mu^{m/2} \pi^{m/2}} \frac{y^{\alpha/2 - 1} e^{-\frac{\mu}{\sqrt{y}}}}{y^{m/2}} = \psi y^{m-1} e^{-\beta y^{\hat{d}}},
\]

(2)

with the parameters \( \psi, m, \) and \( \hat{d} \) given as shown in table I, \( y \), \( \beta \), and \( \alpha \) are as defined before, and \( \alpha \) accounts for the nonlinearities in the fading environment. This generalized model encompasses many other fading models as special cases [16] such as the Nakagami-\( m \) and the Weibull models. It is worth mentioning that the \( \alpha - \mu \) model itself is a special case of the generalized models discussed next.

iii. The \( \alpha - \eta, \lambda - \mu, \alpha - \kappa - \mu \) and \( \alpha - \lambda - \eta - \mu \) Fading Models

These generalized fading models were first introduced in [6] [5] [13]. The models enclose the \( \alpha - \mu \), the \( \lambda - \mu \), the \( \eta - \mu \), the \( \kappa - \mu \), the Weibull, the Hoyt, the Rice, the Nakagami-\( m \), the Rayleigh, the Lognormal, the Gamma, the Exponential, and the One-sided Gaussian as special cases, by setting the fading parameters to their appropriate values (see [16] [17] for more details). The four generalized models can be written in one general form, which is given by [1]:

\[
f_y(y) = \psi y^{m-1} e^{-\beta y^{\hat{d}}} I_0(dy^{\hat{d}}),
\]

(3)

where \( \psi, \beta, \alpha, \upsilon, \mu, \) and \( d \) are given in table I, with the internal parameters \( \hat{d}, \hat{c}, h, \) and \( H \) being defined in table II, where \( I_0(\cdot) \) is the modified Bessel function of the first type [15], and \( r = 0.5 \) for the \( \alpha - \kappa - \mu \) fading and otherwise is equal to 1.

B. The Utilized Approximation

In this section, we re-exploit our previously proposed and developed exponential approximation, given in [18] as:

\[
e^{-x/\alpha} \approx \sum_{i=1}^{4} a_i e^{-\beta_i x},
\]

(4)

where \( a_i \) and \( \beta_i \) are fitting parameters as discussed and given in [16], table III. Note that extending (4) to \( e^{-x^2/\alpha} \) function, where \( s \) is constant, is straightforward.

III. MOMENT GENERATING FUNCTIONS

The moment generating function (MGF) of the fading PDF can be evaluated as [10]:

\[
\mathcal{M}_y(s) = \mathcal{E}[e^{-s \psi}] = \int_0^\infty f_y(y)e^{-s y} dy ,
\]

(5)

where \( \mathcal{E}[\cdot] \) is the expectation (averaging) process. In what follows, we will derive the MGF of each of the generalized fading models described in section II.A.

TABLE I: GENERALIZED FADING MODELS PARAMETERS.

| Model | \( \alpha - \mu \) | \( \lambda - \mu \) | \( \eta - \mu \) | \( \alpha - \lambda - \eta - \mu \) |
|-------|------------------|------------------|-----------------|------------------|
| \( \alpha \) | \( \alpha \) | \( \alpha \) | \( \alpha \) | \( \alpha \) |
| \( \mu \) | \( \mu \) | \( \mu \) | \( \mu \) | \( \mu \) |
| \( d \) | \( d \) | \( d \) | \( d \) | \( 0 \) |
| \( m \) | \( m \) | \( m \) | \( m \) | \( m \) |

TABLE II: INTERNAL PARAMETERS OF THE GENERALIZED MODELS.

| Model | \( \alpha - \mu \) | \( \lambda - \mu \) | \( \eta - \mu \) | \( \alpha - \lambda - \eta - \mu \) |
|-------|------------------|------------------|----------------|------------------|
| \( \hat{c} \) | \( \hat{c} \) | \( \hat{c} \) | \( \hat{c} \) | \( \hat{c} \) |
| \( \hat{b} \) | \( \hat{b} \) | \( \hat{b} \) | \( \hat{b} \) | \( \hat{b} \) |
| \( h \) | \( h \) | \( h \) | \( h \) | \( h \) |
| \( H \) | \( H \) | \( H \) | \( H \) | \( H \) |

i. The MGF of \( \eta - \lambda - \mu \) Fading Model

Substituting (1) into (5) yields:

\[
\mathcal{M}_y(s) = \psi \int_0^\infty y^{m-1} e^{-[\beta + \upsilon]y} I_0(dy) dy ,
\]

(6)

which can straightforwardly be evaluated in the closed-form:

\[
\mathcal{M}_y(s) = \frac{\psi^{d}(m+s)}{2^{d}(m+\beta)(m+s+1)} \left[ \frac{m+s}{2} \right] \left[ \nu + 1 \right] + \frac{\mu^{d}}{\beta^{d} s \mu^{d}} \right),
\]

(7)

Alternatively, and following a similar approach to that in [19], one can see that \( \int_0^\infty (y^s I_0(dy)) e^{-\beta y} dy \) is the Laplace transform of \( y^n I_0(\nu y) \), which can be evaluated using Mathematica, and with some manipulations, (6) can be expressed without the need for any special function in the direct closed-form:

\[
\mathcal{M}_y(s) = \frac{4\pi(1 - \nu^2)}{(\nu^2 + \beta^2)}.
\]

(8)

Up to our knowledge, the expressions (7) and (8) are new.

ii. The MGF of \( \alpha - \mu \) Fading Model

Following from (5), and substituting (2) with some change of variable, the MGF of the \( \alpha - \mu \) fading model is given by:

\[
\mathcal{M}_y(s) = \psi \int_0^\infty \left[ e^{m/\alpha} - e^{-\beta z} \right] e^{-x^{2/\alpha}} dz ,
\]

(9)

and by utilizing (4), one can reach a simple closed-form approximation to the MGF of the \( \alpha - \mu \) model, and is given by:

\[
\mathcal{M}_y(s) \approx \sum_{i=1}^{4} \frac{\alpha \beta \mu^{m/\alpha} \beta_{i}^{m/\alpha}}{\left( \beta_{i}^{m/\alpha} \right)^{m/\alpha}}.
\]

(10)

Besides the novelty of the expression in (10), evaluating it is very computationally efficient, being a simple sum of scaled Gamma functions. The simplicity of (10) can also be observed by comparing it with the alternative solutions, e.g. eqn. (6) in [10], where the MGF is given by the Meijer-G function with convergence conditions as discussed therein.
iii. The MGF of $\alpha-\eta-\mu$, $\alpha-\lambda-\mu$, $\alpha-\alpha-\lambda-\eta-\mu$ and $\alpha-\kappa-\mu$ Models
The approximated moment generating functions of the three generalized fading models, namely the $\alpha-\eta-\mu$, $\alpha-\lambda-\mu$, $\alpha-\alpha-\lambda-\eta-\mu$ models, will be in exactly the same format since they have same compact form given in (3), with $\varkappa=1$. However, with the $\alpha-\kappa-\mu$ model, the value of $\varkappa=0.5$. By using (3) in (5) and using (4), the unified MGF is found in a simple closed-form as given in (11) for each of the two cases of $\varkappa$, at the top of this page. The new MGF expressions in (11) are novel.

### IV. AVERAGE ERROR RATES ANALYSIS
We accentuate and illustrate the usefulness of the derived MGF novel expressions (7) – (8) and (10) – (11) by evaluating the average symbol error rates (ASER) using the well-known MGF-approach. For various M-ary modulation schemes, such as M-ary pulse amplitude modulation (M-PAM), M-ary phase shift keying (M-PSK), M-ary differential phase shift keying (M-DPSK), as well as the M-ary square quadrature amplitude modulation (M-QAM) as [20]:

$$P_{SER} = \sum_{t=1}^{M} E_t f_{\theta_t} \left( \frac{\phi}{\sqrt{2a\sin^2(\theta)}} \right) d\theta,$$

where $N$, $E_t$, $\theta_t$, $\phi$, $V$ and $\Lambda$ are given in table III [20].

#### TABLE III: PARAMETERS SELECTION FOR $P_{SER}$ EVALUATION (12).

| Scheme | M-PAM | M-DPSK | M-QAM |
|--------|--------|--------|--------|
| N      | $2^m$  | $2^m$  | $4(1 - 1/\sqrt{M})$  |
| $E_t$  | $2/\pi$ | $\cos(\pi/M)$ | $1 + \Lambda$  |
| $\Lambda$ | $0$ | $\sin(\pi/M)$ | $\sin(t/\Lambda)$  |
| $\phi$ | $0$ | $1 + \Lambda$ | $0$  |
| $\theta_t$ | $\pi/2$ | $0$ | $0$  |
| $\theta_\phi$ | $\pi/2$ | $0$ | $0$  |

Due to space limitation, and for aim of comparing the results obtained with those existing in the literature, only the case of the binary PSK (i.e. M-PSK with $M=2$) is assumed here. Fig. 1 and Fig. 2 in [19] are regenerated using (11) and (12). The error rates for different $\eta-\lambda-\mu$ fading scenarios are also illustrated in Fig. 3 using (7) and (8). We finally compare the expression (10) for the $\alpha-\mu$ fading model with the numerically obtained results for different scenarios, as shown in Fig. 4. One can clearly see for the different testing conditions, the derived expressions provided very accurate results that closely match the numerical ones and proving their validity.

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Fig. 1: BPSK error rates for different $\alpha$-$\kappa$-$\mu$ fading scenarios ($\alpha=2$).

Fig. 2: BPSK error rates for different $\alpha$-$\lambda$-$\kappa$-$\mu$ fading scenarios ($\alpha=2$, $\lambda=0$).

Fig. 3: BPSK error rates for different $\eta$-$\lambda$-$\mu$ fading scenarios.

Fig. 4: BPSK error rates for different $\alpha$-$\mu$ fading scenarios.