LAN property for stochastic differential equations driven by fractional Brownian motion of Hurst parameter $H \in (1/4, 1/2)$

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Abstract: In this paper, we consider the problem of estimating the drift parameter of solution to the stochastic differential equation driven by a fractional Brownian motion with Hurst parameter less than $1/2$ under complete observation. We derive a formula for the likelihood ratio and prove local asymptotic normality when $H \in (1/4, 1/2)$. Our result shows that the convergence rate is $T^{-1/2}$ for the parameters satisfying a certain equation and $T^{-(1-H)}$ for the others.

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Contents

1 Introduction .......................................................... 2
2 Main results ........................................................... 3
3 On the stationary solution of the equation (1.1) ..................... 8
4 Likelihood ratio formula ................................................. 20
  4.1 Integral transformations involving fBM .......................... 20
  4.2 Proof of the first part of Theorem 2.6 ............................ 27
5 Local asymptotic structure of the likelihood ratio process .......... 29
  5.1 Proof of the second part of Theorem 2.6 ......................... 30
  5.2 Proof of lemmas in Section 5.1 ................................. 38
References ................................................................. 46

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1. Introduction

Let \((X_t^\theta)_{t \in [0,T]}\) be a solution of the (one-dimensional) stochastic differential equation

\[
dX_t = a(X_t, \theta) dt + \sigma dB_t, \quad X_0 = x_0, \quad t \in [0,T].
\]  

(1.1)

Here, \(X_0 = x_0 \in \mathbb{R}\) is an initial condition, \(a\) is a drift coefficient with parameter \(\theta \in \Theta\) (not necessarily be linear in \(x\) and \(\theta\)), \(\Theta\) is an open subset of \(\mathbb{R}^m\), \(\sigma \neq 0\) is a diffusion coefficient and \(B = (B_t)_{t \in [0,T]}\) is a fractional Brownian motion with Hurst parameter \(H \in (0,1)\) (later we restrict ourselves to the case where \(H \in (1/4,1/2)\)).

We are interested in estimating \(\theta \in \Theta\) from the completely observed data \((X_t^\theta)_{t \in [0,T]}\) when \(T \to \infty\). It is natural to consider the maximum likelihood estimation, which is successful in the case of ergodic diffusion processes (see Kutoyants (2004)). Kleptsyna and Le Breton (2002) proved strong consistency of the maximum likelihood estimator and derived explicit formulas for the asymptotic bias and mean square error when the observed process is the fractional Ornstein-Uhlenbeck process with Hurst parameter \(H \in (1/2,1)\). In the fractional Ornstein-Uhlenbeck case, Brouste and Kleptsyna (2010) and Bercu et al. (2011) proved asymptotic normality of the maximum likelihood estimator in a different way. In the case where the drift coefficient \(a\) is of the form \(a(x, \theta) = \theta a_0(x)\) for some function \(a_0\), Tudor and Viens (2007) proved strong consistency of the maximum likelihood estimator.

These results used an explicit expression for the maximum likelihood estimator. However, it is not available for a general drift coefficient \(a\). As is done in Kutoyants (2004), we rely on Ibragimov and Has’minskii’s framework when \(\theta \mapsto a(\cdot, \theta)\) is nonlinear. That is, we can derive asymptotic properties of maximum likelihood estimator (and Bayes estimator) from weak convergence of likelihood ratio fields. Therefore, it is important to specify the weak limit of likelihood ratio fields. In many cases, the limit of the likelihood ratio fields \((Z_{\epsilon,\theta})_{\epsilon>0}\) is of the form

\[
Z_{0,\theta}(u) = \exp \left( u^* \Delta(\theta) - \frac{1}{2} u^* I(\theta) u \right),
\]

where \(u \in \mathbb{R}^m\) (\(u^*\) denotes the transpose of \(u\)), \(I(\theta)\) is a \(m \times m\)-positive definite symmetric matrix, and \(\Delta(\theta)\) is a \(m\)-dimensional Gaussian random variable with mean zero and variance \(I(\theta)\). Local asymptotic normality ensures finite-dimensional convergence of \((Z_{\epsilon,\theta})_{\epsilon>0}\) to \(Z_{0,\theta}\), that is, \(Z_{\epsilon,\theta}(u) \to^d Z_{0,\theta}(u)\) as \(\epsilon \to 0\) for each \(u \in \mathbb{R}^m\). This is the reason to investigate local
asymptotic normality in this paper. Note that, to bridge local asymptotic normality and weak convergence of the random fields \((Z_{t,\theta})\), it is necessary to prove tightness of the family \((Z_{t,\theta})_{t>0}\). This will be investigated in a future work. For details of Ibragimov and Has’minskiı’s theory, we refer to Ibragimov and Has’minskiı (1981) and Yoshida (2011).

Although we focus on the maximum likelihood estimator in this paper, there are many papers investigating other estimators. We mention some literature. Properties of the least-square type estimators are investigated in, for example, Hu and Nualart (2010); Hu et al. (2017, 2018); Neuenkirch and Tindel (2014). For the problem of estimating the drift parameter of fractional diffusion processes, we also refer to monographs Kubilius et al. (2017); Mishura (2008); Rao (2011).

Recently Liu et al. (2015) proved local asymptotic normality in the case of \(H \in (1/2, 1)\). However, it is still unknown whether local asymptotic normality holds or not when \(H\) is less than 1/2. We partially solve this problem. More precisely, let \((\mu_T^\theta)_{\theta \in \Theta}\) be the probability measures on the space of continuous functions induced by the solution of the equation (1.1). The main aim of this paper is to prove local asymptotic normality of the probability measures \((\mu_T^\theta)_{\theta \in \Theta}\) when \(H \in (1/4, 1/2)\). In order to do this, it is necessary to derive a likelihood ratio formula for the probability measures \((\mu_T^\theta)_{\theta \in \Theta}\). We will derive a likelihood ratio formula in Section 4. Our proof of local asymptotic normality, which is given in Section 5, relies on the properties of the stationary solution of the equation (1.1). We will investigate the properties of the stationary solution of the equation (1.1) in Section 3. The results are summarized in Section 2.

2. Main results

As we explained in Section 1, we assume that the continuously observed data \((X^\theta_t)_{t \leq [0,T]}\) is available and is generated by the stochastic differential equation (1.1). In this paper, we impose the following assumptions on the coefficients in (1.1).

**Assumption 2.1.** Let the parameter space \(\Theta\) be an open subset of \(\mathbb{R}^m\) and \(\theta \in \Theta\). We assume that the initial value \(X_0\) is a deterministic constant \(x_0 \in \mathbb{R}\) and the diffusion coefficient \(\sigma\) is not equal to zero.

Furthermore, we impose the following conditions on the drift coefficient \(a: \mathbb{R} \times \Theta \to \mathbb{R}\).

(A1) All the partial derivatives of \(a\) appearing below exist.
(A2) There exist positive constant $\alpha = \alpha(\theta) > 0$ such that
\[ -\alpha^{-1} \leq \partial_x a(x, \theta) \leq -\alpha \]
for all $x \in \mathbb{R}$.

(A3) There is a positive constant $C = C(\theta) > 0$ and a nonnegative integer $p = p(\theta)$ such that
\[ |\partial_\theta a(x, \theta)| \leq C(1 + |x|^p). \]


\[ |\partial_\theta a(x, \theta) - \partial_\theta a(x, \theta')| \leq C(1 + |x|^p)|\theta - \theta'|. \]

(A4) The functions $\partial_x a(x, \theta)$ and $\partial_x \partial_\theta a(x, \theta)$ is uniformly bounded in $(x, \theta)$.

Remark 2.2. Since $x \mapsto a(x, \theta)$ is assumed to be differentiable for each $\theta \in \Theta$, the condition (A2) is equivalent to the one-sided dissipative Lipschitz condition plus the uniform boundedness of $\partial_x a(\cdot, \theta)$. Recall that a function $f: \mathbb{R}^n \to \mathbb{R}^n$ satisfies the one-sided dissipative Lipschitz condition if there exists a constant $L > 0$ such that
\[ \langle f(x) - f(y), x - y \rangle_{\mathbb{R}^n} \leq -L|x - y|^2 \]
for all $x$ and $y$ in $\mathbb{R}^n$. The one-sided dissipative Lipschitz condition is often imposed to ensure the ergodicity of the solution of the equation (1.1), see Cohen and Panloup (2011); Garrido-Atienza et al. (2009); Hairer et al. (2005) for example.

We can ensure the existence and uniqueness of the solution for the equation (1.1) under Assumption 2.1. More precisely, the following theorem holds. The proof of Theorem 2.3 is given in Section 3.

**Theorem 2.3.** Suppose that Assumption 2.1 is in force. Then there exists a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ such that

1. there exists a two-sided fractional Brownian motion $B = (B_t)_{t \in \mathbb{R}}$ on $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$, and
2. the SDE (1.1) has a unique pathwise solution $X_{t_0, \theta} = (X_{t_0, \theta}^t)_{t \geq 0}$ which is continuous and satisfy
\[ \sup_{t \geq 0} \mathbb{E}^*\{|X_{t_0, \theta}^t|^p\} < \infty \]
for all $p > 0$.  


Furthermore, on \((\Omega^*, \mathcal{F}^*, \mathbb{P}^*)\), there exists a unique stationary stochastic process \(\bar{X}^\theta = (\bar{X}^\theta_t)_{t \in \mathbb{R}}\) with the following properties.

(3) The process \(\bar{X}^\theta\) satisfies
\[
\bar{X}^\theta_t(\omega) - \bar{X}^\theta_s(\omega) = \int_s^t a(\bar{X}^\theta_r(\omega), \theta) \, dr + \sigma(B_t(\omega) - B_s(\omega)) \quad (2.2)
\]
for all \(s < t\) and \(\omega \in \Omega^*\).

(4) For any \(t \in \mathbb{R}\), the random variable \(\bar{X}^\theta_t\) is in \(\bigcap_{p > 0} D^{1,p}\), and its Malliavin derivative \(D\bar{X}^\theta_t\) is given by
\[
(D \bar{X}^\theta_t)(\omega) = \sigma \exp \left( \int_t^\infty \partial_\omega a(\bar{X}^\theta_r(\omega), \theta) \, dr \right) 1_{(-\infty, t]}(\cdot). \quad (2.3)
\]

Let \(C[0, T]\) be the space of \(\mathbb{R}\)-valued continuous functions on \([0, T]\) with the sup-norm \(\|\cdot\|_\infty\) and \(\pi = (\pi_t)_{t \in [0,T]}\) be the canonical process, i.e., \(\pi_t(x) = x(t)\). The \(\sigma\)-field generated by \((\pi_s)_{s \in [0,t]}\) is denoted as \(\mathcal{B}_t\). Note that \(\mathcal{B}_T\) coincides with the Borel \(\sigma\)-field generated by the sup-norm.

Thanks to Theorem 2.3, we can consider the probability distribution on \((C[0, T], \mathcal{B}(C[0, T]))\) induced by the solution of the equation (1.1). Let us consider the family of statistical experiments \((\mathcal{E}_T)_{T > 0}\), where
\[
\mathcal{E}_T = (C[0, T], \mathcal{B}_T, (\mu^T_\theta)_{\theta \in \Theta}).
\]
Here we consider the probability measures \((\mu^T_\theta)_{\theta \in \Theta}\) on \((C[0, T], \mathcal{B}_T)\) satisfying the following properties.

(1) The process \(B^\theta = (B^\theta_t)_{t \in [0, T]}\) defined by

\[
B^\theta_t(x) = \sigma^{-1} \left( \pi_t(x) - \pi_0(x) - \int_0^t a(\pi_s(x), \theta) \, ds \right)
\]

is a fractional Brownian motion on \([0, T]\) under the measure \(\mu^T_\theta\).

(2) There is a point \(x_0 \in \mathbb{R}\) such that \((\mu^T_\theta)^{\pi_0} = \delta_{x_0}\) for all \(\theta \in \Theta\), where

\[
(\mu^T_\theta)^{\pi_0}(A) = \mu^T_\theta(\{\pi_0 \in A\}) \quad \text{for} \quad A \in \mathcal{B}(\mathbb{R}).
\]

The aim of this paper is to prove local asymptotic normality (LAN) of the probability measures \((\mu^T_\theta)_{\theta \in \Theta}\).

**Notation.** Let \(m\) be a positive integer. For a vector \(u \in \mathbb{R}^m\), we denote the transpose of \(u\) by \(u^*\). Let \(\mu\) be in \(\mathbb{R}^m\) and \(\Sigma\) be a positive semi-definite \(m \times m\) matrix. The \(m\)-dimensional normal distribution with mean \(\mu\) and variance \(\Sigma\) is denoted by \(N_m(\mu, \Sigma)\).
Definition 2.4. A family \((\mu_T^\theta)_{\theta \in \Theta}\) is called \textit{locally asymptotically normal (LAN)} at a point \(\theta \in \Theta\) if there exists some nondegenerate \(m \times m\) matrix \(\varphi_T(\theta)\) such that for any \(u \in \mathbb{R}^m\) the likelihood ratio process

\[
Z_T^\theta(u) = \frac{d\mu_T^\theta + \varphi_T(\theta)u}{d\mu_T^\theta}
\]

can be represented as

\[
Z_T^\theta(u) = \exp \left( u^* \Delta_T^\theta - \frac{1}{2} u^* I(\theta) u + r_{T,u}^T \right),
\]

where the matrix \(I(\theta)\) is positive definite \(m \times m\) matrix (the Fisher information matrix), the random variable \(\Delta_T^\theta\) converges in distribution (with respect to \(\mu_T^\theta\)) to \(\mathcal{N}_m(0, I(\theta))\) as \(T \to \infty\), and \(r_{T,u}^T\) satisfies

\[
\lim_{T \to \infty} \mu_T^\theta \{ |r_{T,u}^T| > \epsilon \} = 0
\]

for any \(\epsilon > 0\).

Before stating our main theorem, we define a stochastic process which appears in the expression of the likelihood ratio process.

Definition 2.5. Let \(\beta_t(\theta)\) denote the process

\[
\beta_t(\theta) = c_H^{-1} \sigma^{-1} t^{H-1/2} \int_0^t (t-s)^{-1/2-H} s^{1/2-H} a(\pi_s, \theta) \, ds,
\]

where the constant \(d_H = c_H \Gamma(H + 1/2) = \sqrt{2H(3/2-H)\Gamma(H+1/2)} \Gamma(2-2H)\).

Notation. We denote the \(n \times n\) identity matrix by \(J_n\).

Here is our main result.

Theorem 2.6. Suppose that Assumption 2.1 is in force.

1. The family \((\mu_T^\theta)_{\theta \in \Theta}\) is equivalent (i.e., mutually absolutely continuous) and its Radon-Nikodym derivative is given by

\[
\frac{d\mu_{T,\theta_2}}{d\mu_{T,\theta_1}} = \exp \left( \int_0^T (\beta_t(\theta_2) - \beta_t(\theta_1)) \, dW_t^{\theta_1} - \frac{1}{2} \int_0^T (\beta_t(\theta_2) - \beta_t(\theta_1))^2 \, dt \right)
\]

for all \(\theta_1, \theta_2 \in \Theta\). Here the process \(W^{\theta_1}\) is a \((\mathcal{B}_t)\)-Brownian motion under the measure \(\mu_{T,\theta_1}\) (for the precise definition of \(W^\theta\), see Lemma 4.4 below).
(2) Let us fix $\theta \in \Theta$. Assume that $\mathbb{E}^\ast \{ \partial_{\theta} a(\bar{X}^\theta_0, \theta) \} = 0$ for $i = 1, \ldots, m_0(\theta)$ and $\mathbb{E}^\ast \{ \partial_{\theta} a(\bar{X}^\theta_0, \theta) \} \neq 0$ for $i = m_0(\theta) + 1, \ldots, m$. A family $(\mu_\theta^T)_{\theta \in \Theta}$ is LAN at a point $\theta \in \Theta$, with a normalizing matrix

$$
\varphi_T(\theta) = \begin{pmatrix}
T^{-1/2}J_{m_0(\theta)} & T^{-(1-H)}J_{m-m_0(\theta)}
\end{pmatrix}
$$

and the Fisher information matrix $I(\theta) = (I_{i,j}(\theta))_{i,j=1,\ldots,m}$, where

$$
I_{i,j}(\theta) = \sigma^{-2}d_H^{-2}\int_0^\infty dr \int_0^\infty du \ r^{-H-1/2}u^{-H-1/2}\mathbb{E}^\ast \{ \partial_{\theta} a(\bar{X}^\theta_r, \theta) \partial_{\theta_j} a(\bar{X}^\theta_u, \theta) \}
$$

for $i, j = 1, \ldots, m_0(\theta)$,

$$
I_{i,j}(\theta) = \sigma^{-2}d_H^{-2}\mathbb{E}^\ast \{ \partial_{\theta} a(\bar{X}^\theta_0, \theta) \} \mathbb{E}^\ast \{ \partial_{\theta_j} a(\bar{X}^\theta_0, \theta) \}
$$

for $i, j = m_0(\theta) + 1, \ldots, m$, and

$$
I_{i,j}(\theta) = 0
$$

else. Here the constant $d_H$ is $d_H = d_1^{-2}B(1/2-H, 3/2-H)^2(2-2H)^{-1}$.

The proof of the first part of Theorem 2.6 is given in Section 4, and the second part in Section 5.

**Remark 2.7.** Let us consider the case where the parameter space is one-dimensional, the diffusion coefficient $\sigma$ equals to 1, and the drift coefficient $a(\theta, x)$ is of the form $\theta b(x)$ for some function $b$. In this case, we can explicitly calculate the MLE $\hat{\theta}_T$ for the true parameter $\theta$ by the formula (2.4). An explicit calculation yields

$$
\varphi_T(\theta)^{-1}(\hat{\theta}_T - \theta) = \frac{\varphi_T(\theta) \int_0^T (\partial_{\theta} \beta)_t(\theta) dW^\theta_t}{\varphi_T(\theta)^2 \int_0^T (\partial_{\theta} \beta)_t(\theta)^2 dt}.
$$

As a consequence of (the proof of) Theorem 2.6, we have

$$
\varphi_T(\theta)^{-1}(\hat{\theta}_T - \theta) \to^d \mathcal{N}_1(0, I(\theta)^{-1})
$$

(2.5) as $T \to \infty$, where $\varphi_T(\theta) = T^{-1/2}$ and

$$
I(\theta) = \sigma^{-2}d_H^{-2}\int_0^\infty dr \int_0^\infty du \ r^{-H-1/2}u^{-H-1/2}\mathbb{E}^\ast \{ b(\bar{X}^\theta_r) b(\bar{X}^\theta_u) \}
$$

in the case of $\mathbb{E}^\ast \{ b(\bar{X}^\theta_0) \} = 0$, and $\varphi_T(\theta) = T^{H-1}$ and

$$
I(\theta) = \sigma^{-2}d_H^{-2}B(1/2 - H, 3/2 - H)^2(2-2H)^{-1}\mathbb{E}^\ast \{ b(\bar{X}^\theta_0) \}^2
$$

in the case of $\mathbb{E}^\ast \{ b(\bar{X}^\theta_0) \} \neq 0$. In particular, the MLE defined in Tudor and Viens (2007) is asymptotically normal.
3. On the stationary solution of the equation (1.1)

In this section, we investigate some properties of the stationary solution of the equation (1.1). In particular, we provide the proof of Theorem 2.3. First we specify the probability space \((Ω^*, F^*, P^*)\) in Theorem 2.3.

Let \(Ω^* = C_0(\mathbb{R})\) be the set of continuous function \(ω\) with \(ω(0) = 0\). We consider the topology of compact convergence and the corresponding Borel \(σ\)-algebra on \(Ω^*\). We denote this Borel \(σ\)-algebra as \(F^*_0\). Then there exists a probability measure \(P^*_0\) on \((Ω^*, F^*_0)\) such that the canonical process \(π = (π_t)_{t ∈ \mathbb{R}}\) is a two-sided fBM under \(P^*_0\). We define a \(σ\)-algebra \(F^*\) as the completion of \(F^*_0\) with respect to \(P^*_0\). The probability measure \(P^*_0\) can be naturally extended to the probability measure on \((Ω^*, F^*)\). This extension is denoted by \(P^*\).

It is known that there exists a set \(Ω^* ∈ F^*_0\) such that \(P^*_0\{Ω^*_0\} = 1\) and for each \(ω ∈ Ω^*\)

\[|B_t(ω)| ≤ K(ω)(1 + |t|^2)\]

holds for all \(t ∈ \mathbb{R}\), where \(K(ω) > 0\) is a random constant. For the proof of this fact, see Lemma 3.3 of Gess et al. (2011). We define \(B_t: Ω^* → \mathbb{R}\) by \(B_t(ω) = π_t(ω)1_{Ω^*_0}(ω)\) for each \(t ∈ \mathbb{R}\). We set \(B = (B_t)_{t ∈ \mathbb{R}}\).

Remark 3.1. Note that the process \(B\) is also a two-sided fBM under \(P^*_0\) and \(P^*\). We would rather regard \(B\) as the driving fBM than the canonical process \(π\).

We start with showing that the equation (1.1) has a unique continuous solution for a given initial condition \(X_0\). The next proposition gives the proof of Theorem 2.3 (1a).

Proposition 3.2. Suppose that Assumption 2.1 holds. Let \(s\) be a real number and \(ξ\) be a random variable on \((Ω^*, F^*)\). Then the equation

\[X_t = ξ + \int_s^t a(X_r, θ) dr + σ(B_t - B_s), \quad t ∈ [s, \infty)\]

has a unique continuous solution \(X^ξ,θ,s = (X^ξ,θ,s_t)_{t ∈ [s, \infty)}\) for each \(ω ∈ Ω^*\). Furthermore, if \(ξ\) is a constant, then \(X^ξ,θ,s\) satisfies

\[\sup_{t ∈ [s, \infty)} \mathbb{E}^s\{|X^ξ,θ,s_t|^p\} < ∞. \quad (3.1)\]

Proof. The existence and uniqueness of the solution is due to a standard Picard iteration argument. Therefore we omit the proof.

The inequality (3.1) can be proved by the same argument as in the proof of Proposition 2.2 of Neuenkirch and Tindel (2014).
In order to investigate the properties of the stationary solution of the equation (1.1), we use the theory of random dynamical systems and random attractors. In the sequel, we follow the terminologies of Garrido-Atienza et al. (2009) for random dynamical systems and random attractors.

**Definition 3.3.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space.

1. Suppose that a family of transformations \(\{\vartheta_t: \Omega \rightarrow \Omega; t \in \mathbb{R}\}\) satisfies that
   - \((t, \omega) \mapsto \vartheta_t \omega\) is \((\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}; \mathcal{F})\)-measurable,
   - \(\vartheta_0 \omega = \omega\) for all \(\omega \in \Omega\),
   - \(\vartheta_t \circ \vartheta_s = \vartheta_{t+s}\) for all \(s, t \in \mathbb{R}\) and
   - \(\mathbb{P}^{\vartheta_t} = \mathbb{P}\) for all \(t \in \mathbb{R}\).

   Then the quadruple \(\vartheta = (\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t; t \in \mathbb{R}\})\) is called a *(continuous)* metric dynamical system (MDS).

2. A map \(\phi: \mathbb{R}_\geq 0 \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d\) is called a *cocycle mapping* if
   - \(\phi\) is \((\mathcal{B}(\mathbb{R}_\geq 0) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d); \mathcal{B}(\mathbb{R}^d))\)-measurable,
   - \(\phi(0, \omega, x) = x\), for all \(\omega \in \Omega\) and \(x \in \mathbb{R}^d\), and
   - \(\phi(t + s, \omega, x) = \phi(s, \vartheta_t \omega, \phi(t, \omega, x))\) for all \(t, s \in \mathbb{R}_\geq 0, x \in \mathbb{R}^d\) and \(\omega \in \Omega\).

3. The pair \((\vartheta, \phi)\) of a (continuous) MDS and a cocycle mapping is called a *(continuous)* random dynamical system (RDS).

4. A *universe* \(\mathcal{D}\) is a collection of nonempty random sets \((D(\omega))_{\omega \in \Omega}\) of \(\mathbb{R}^d\) which is closed with respect to set inclusion: if \(D \in \mathcal{D}\) and \(D'(\omega) \subset D(\omega)\) for all \(\omega \in \Omega\), then \(D' \in \mathcal{D}\).

5. A random set \((A(\omega))_{\omega \in \Omega}\) is called a *random attractor* if it is
   - compact for all \(\omega \in \Omega\),
   - \(\phi\)-invariant: \(\phi(t, \omega, A(\omega)) = A(\vartheta_t \omega)\) for all \(t \in \mathbb{R}_\geq 0\), and
   - pathwise pullback attracting: for all \(D \in \mathcal{D}\)
     \[d^*(\phi(t, \vartheta_{-t} \omega, D(\vartheta_{-t} \omega)), A(\omega)) \rightarrow 0\]
     as \(t \rightarrow \infty\). Here \(d^*\) denotes the Hausdorff semi-distance on \(\mathbb{R}^d\).

We define the shift operator \(\vartheta_t: \Omega \rightarrow \Omega\) for each \(t \in \mathbb{R}\) by \(\vartheta_t(\omega)_s = \omega_{s+t} - \omega_t\). It is known that the set \(\Omega^*_0\) is shift-invariant: we have \(\vartheta_t(\Omega^*_0) = \Omega^*_0\) for all \(t \in \mathbb{R}\) (for the proof, see Gess et al. (2011)). Note that \(B(\vartheta_t \omega) = \vartheta_t B(\omega)\) holds for all \(t \in \mathbb{R}\) and \(\omega \in \Omega^*_0\).
We set $\phi(t, \omega, x) = X^{x, \theta}_t(\omega)$, where $X^{x, \theta}_t(\omega)$ denotes the solution of the stochastic differential equation

$$X_t(\omega) = x + \int_0^t a(X_s(\omega), \theta) \, ds + \sigma B_t(\omega), \quad t \geq 0$$
onumber

on $(\Omega^*, \mathcal{F}^*_0, \mathbb{P}^*_0)$.

In Garrido-Atienza et al. (2009), it is proved that

- the pair $(\theta, \phi)$ defines a continuous RDS, and
- this RDS has a random attractor consists of a random element $\{\bar{X}^0_\theta(\omega)\}$

assuming that the universe $\mathcal{D}$ is consists of the tempered random sets (see Garrido-Atienza et al. (2009) for detail). Note that $\omega \mapsto X^\theta_0(\omega)$ is $\mathcal{F}^*_0$-measurable. This is because $\bar{X}^\theta_0(\omega)$ can be written as

$$\bar{X}^\theta_0(\omega) = \lim_{t \to \infty} \phi(t, \theta - i \omega, 0).$$

We set $\bar{X}^\theta_t(\omega) = \bar{X}^\theta_0(\theta_t \omega)$ for all $t \in \mathbb{R}$. It is clear that $(t, \omega) \mapsto \bar{X}^\theta_t(\omega)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}^*_0$-measurable by Definition 3.3. Since $(\mathbb{P}^*_0)^{\theta_t} = \mathbb{P}^*_t$ holds for all $t \in \mathbb{R}$, the process $\bar{X}^\theta = (\bar{X}^\theta_t)_{t \in \mathbb{R}}$ is stationary. Let us check that the process $\bar{X}^\theta$ satisfies the equation (1.1). The following proposition gives the proof of Theorem 2.3 (2a).

**Proposition 3.4.** The stationary process $\bar{X}^\theta = (\bar{X}^\theta_t)_{t \in \mathbb{R}}$ satisfies

$$\bar{X}^\theta_t(\omega) - \bar{X}^\theta_s(\omega) = \int_s^t a(\bar{X}^\theta_r(\omega), \theta) \, dr + \sigma(B_t(\omega) - B_s(\omega)) \quad (3.2)$$

for all $s < t$ and $\omega \in \Omega^*$.

**Proof.** By $\phi$-invariance of a random attractor, we have

$$\bar{X}^\theta_t(\omega) = \phi(t, \omega, \bar{X}^\theta_0(\omega)) = \bar{X}^\theta_0(\omega) + \int_0^t a(\bar{X}^\theta_s(\omega), \theta) \, ds + \sigma B^\theta_t(\omega)$$

for all $t \geq 0$ and $\omega \in \Omega^*$. Since the (pathwise) solution of the equation (1.1) is unique, we have $X^{X^\theta_0(\omega), \theta}_t(\omega) = \bar{X}^\theta_t(\omega)$ for all $t \geq 0$ and $\omega \in \Omega^*$, i.e., we obtain

$$\bar{X}^\theta_t(\omega) = \bar{X}^\theta_0(\omega) + \int_0^t a(\bar{X}^\theta_s(\omega), \theta) \, ds + \sigma(B_t(\omega) - B_0(\omega)) \quad (3.3)$$

for all $t \geq 0$ and $\omega \in \Omega^*$. We can replace $\omega$ by $\theta - i \omega$. Then we have

$$\bar{X}^\theta_0(\omega) = \bar{X}^\theta_{-i}(\omega) + \int_{-i}^0 a(\bar{X}^\theta_s(\omega), \theta) \, ds + \sigma(B_0(\omega) - B_{-i}(\omega)) \quad (3.4)$$

for all $t \geq 0$ and $\omega \in \Omega^*$. Combining (3.3) and (3.4), we obtain (3.2).  \(\square\)
Let us consider applying Malliavin calculus to the stationary solution $\bar{X}^\theta$. First we introduce some fractional operators (for detail, see Samko et al. (1993)).

**Definition 3.5.** Let $\kappa$ be in $(0, 1)$ and $\varphi$ be a function on $\mathbb{R}$.

1. We define a fractional integral of order $\kappa$ of a function $\varphi$ by
   \[(I_\pm^\kappa \varphi)(x) = \frac{1}{\Gamma(\kappa)} \int_0^\infty t^{\kappa-1} \varphi(x \mp t) \, dt\]
   for $x \in \mathbb{R}$.

2. We define a (Marchaud) fractional derivative of order $\kappa$ of a function $\varphi$ by
   \[(D_\pm^\kappa \varphi)(x) = \frac{\kappa}{\Gamma(1 - \kappa)} \int_0^\infty \frac{\varphi(x) - \varphi(x \mp t)}{t^{1+\kappa}} \, dt\]
   for $x \in \mathbb{R}$.

Let $H$ denote the space $I_{1/2-H}^1(L^2(\mathbb{R}))$ with the inner product
\[
\langle f, g \rangle_H = e_H \langle D_{\pm}^{1/2-H} f, D_{\pm}^{1/2-H} g \rangle_{L^2(\mathbb{R})},
\]
where $e_H = \Gamma(2H + 1) \sin(\pi H)$. It is shown in Pipiras and Taqqu (2000) that the space $\mathcal{H}$ is a Hilbert space. Then the process $B$ defines an isonormal Gaussian process over $\mathcal{H}$ (see also Cheridito and Nualart (2005)).

Let $L^2(\Omega^*; \mathcal{H})$ be the set of $\mathcal{H}$-valued random variables that are square integrable: if $h \in L^2(\Omega^*; \mathcal{H})$, then $\mathbb{E}^*\{||h||^2_\mathcal{H}\} < \infty$. The subset of $L^2(\Omega^*; \mathcal{H})$, which consists of $\mathcal{H}$-valued random variables of the form
\[
\phi = \sum_{i=1}^n Z_i \phi_i,
\]
where $Z_i \in L^2(\mathbb{P}^*)$ and $\phi_i \in L^{1/2-H}_c(C_c^\infty(\mathbb{R}))$ for $i = 1, \ldots, n$, is denoted by $\mathcal{G}$. Here $C_c^\infty(\mathbb{R})$ denotes the set of smooth functions of compact support on $\mathbb{R}$. Note that the set $\mathcal{G}$ is dense in $L^2(\Omega^*; \mathcal{H})$.

Now we turn to show the Malliavin differentiability of the stationary solution $\bar{X}^\theta$. The following lemma reduces the Malliavin differentiability of $\bar{X}^\theta$ to that of $\bar{X}_t^0$.

**Lemma 3.6.** We set $\tau_s: \mathcal{H} \to \mathcal{H}$ by $(\tau_s f)(t) = f(t - s)$ for $s \in \mathbb{R}$ and $f \in \mathcal{H}$. Suppose that $F$ is in $\mathbb{D}^{1,2}$. Then we have $F \circ \tau_s \in \mathbb{D}^{1,2}$ and
\[
D(F \circ \tau_s) = \tau_s(DF) \circ \tau_s.
\]
Proof. Let $\tilde{1}_{(a,b]}$ denote the extended indicator function:

$$
\tilde{1}_{(a,b]} = \begin{cases} 
1_{(a,b]}, & \text{if } a \leq b \\
-1_{(b,a]}, & \text{if } a > b. 
\end{cases}
$$

Then we have

$$D((B_{b} - B_{a}) \circ \vartheta_{s}) = \tilde{1}_{(a+s,b+s]} = \tau_{s}\tilde{1}_{(a,b]}$$

for any real numbers $a$, $b$ and $s$. Therefore, by linearity of $\tau_{s}$ and $D$, we have $D(B(\phi) \circ \vartheta_{s}) = \tau_{s}B$ for a real number $s$ and a step function $\phi$. Since the set of step functions is dense in $(\mathcal{H}, \| \cdot \|_{\mathcal{H}})$ (see Theorem 3.3 of Pipiras and Taqqu (2000)), for each $\phi \in \mathcal{H}$ there exists a sequence of step functions $(\phi_{n})$ such that $\| \phi - \phi_{n} \|_{\mathcal{H}} \to 0$ as $n \to \infty$. It is clear that $E^{*}\{ (B(\phi_{n}) - B(\phi))^{2} \} = E^{*}\{ (B(\phi_{n}) - B(\phi))^{2} \} = \| \phi_{n} - \phi \|_{\mathcal{H}}^{2} \to 0$ as $n \to \infty$. Since $\tau_{s}$ and $D_{1/2-H}$ is commutative (see (5.61) in p. 111 of Samko et al. (1993)) and the Lebesgue measure is translation invariant, we have $E^{*}\{ \| D(B(\phi_{n}) \circ \vartheta_{s}) - D(B(\phi_{m}) \circ \vartheta_{s}) \|_{\mathcal{H}} \} = \| \tau_{s}(\phi_{n} - \phi_{m}) \|_{\mathcal{H}} \to 0$ as $n, m \to \infty$. Hence we have $D(B(\phi) \circ \vartheta_{s}) = \tau_{s}\phi$ for all $\phi \in \mathcal{H}$.

Let $\mathcal{S}$ denote the set of the random variables $F$ of the form

$$F = f(B(\phi_{1}), \ldots, B(\phi_{n}))$$

for some positive integer $n$ where $\phi_{i} \in \mathcal{H}$ ($i = 1, \ldots, n$) and $f$ is an infinitely continuously differentiable function such that all its partial derivatives are of polynomial growth. For $F \in \mathcal{S}$, we have

$$D(F \circ \vartheta_{s}) = \sum_{i=1}^{m} (\partial_{i}f)(B(\phi^{1}) \circ \vartheta_{s}, \ldots, B(\phi^{m}) \circ \vartheta_{s})\tau_{s}\phi^{i}$$

$$= \tau_{s}(DF) \circ \vartheta_{s}$$

and hence

$$E^{*}\{ \| D(F \circ \vartheta_{s}) \|_{\mathcal{H}}^{2} \} = E^{*}\{ \| D(F) \|_{\mathcal{H}}^{2} \}. \quad (3.6)$$

For each $F \in \mathbb{D}^{1,2}$, we can choose $(F_{n})_{n} \subset \mathcal{S}$ such that $\| F_{n} - F \|_{1,2} \to 0$ as $n \to \infty$. It is clear that $E^{*}\{ (F_{n} \circ \vartheta_{s} - F \circ \vartheta_{s})^{2} \} \to 0$ as $n \to \infty$. We also have $E^{*}\{ \| D(F_{n} \circ \vartheta_{s}) - D(F_{n} \circ \vartheta_{s}) \|_{\mathcal{H}}^{2} \} = E^{*}\{ \| DF_{n} - DF_{n} \|_{\mathcal{H}}^{2} \} \to 0$ as $n \to \infty$ by (3.6). On the other hand, we have $E^{*}\{ \| D(F_{n} \circ \vartheta_{s}) - \tau_{s}(DF) \circ \vartheta_{s} \|_{\mathcal{H}}^{2} \} = E^{*}\{ \| DF_{n} - DF \|_{\mathcal{H}}^{2} \} \to 0$ as $n \to \infty$. Hence we have $F \circ \vartheta_{s} \in \mathbb{D}^{1,2}$ and (3.5).
The next proposition gives the proof of Theorem 2.3 (2c).

**Proposition 3.7.** It holds that $X^θ_t \in D^{1,2}$ for all $t \in \mathbb{R}$ and its Malliavin derivative $D\overline{X}^θ_t$ is given by (2.3).

**Lemma 3.8.** Let $\xi \in D^{1,2} \cap \left( \bigcap_{p>0} L^p(\mathbb{P}^s) \right)$ and $Y^{θ,ξ,s}_t$ be a solution of the equation

$$Y_t = \xi + \int_s^t a(Y_r, θ) \, dr + σ(B_t - B_s), \quad t \in [s, \infty).$$

Then $Y^{θ,ξ,s}$ is in $D^{1,2}$ and

$$DY^{θ,ξ,s}_t = e^\int_s^t (∂_a a)(Y^{θ,ξ,s}_u, θ) \, du \, Dξ + 1_{[s,t]} e^\int_s^t (∂_a a)(Y^{θ,ξ,s}_u, θ) \, du. \quad (3.7)$$

**Proof.** Let us consider the Picard approximation $Y^0_t \equiv ξ$ for $t \in [s, \infty)$ and

$$Y^n_t = \xi + \int_s^t a(Y^{n-1}_r, θ) \, dr + σ(B_t - B_s)$$

for a positive integer $n$. For $ϕ \in G$, we have

$$DY^n_t = Dξ + \int_s^t (∂_a a)(Y^{n-1}_r, θ)DY^{n-1}_r \, dr + σ1_{[s,t]}$$

and

$$DY^{n+1}_t - DY^n_t = \int_s^t D(a(Y^n_r, θ) - a(Y^{n-1}_r, θ)) \, dr.$$

We can bound $\|DY^n_t\|_H$ independent of $n$. Indeed, we have

$$\|DY^n_t\|_H \leq \|Dξ\|_H + |σ|(t-s)^H + |α|^{-1} \int_s^t \|DY^{n-1}_{r1}\|_H \, dr_1$$

$$\leq \|Dξ\|_H + |σ|(t-s)^H + |α|^{-1} \int_s^t \, dr_1 \left( \|Dξ\|_H + |σ|(r1-s)^H \right)$$

$$+ |α|^{-2} \int_s^t \, dr_1 \int_s^{r1} \, dr_2 \|DY^{n-2}_{r2}\|_H$$

$$\leq \left( \|Dξ\|_H + |σ|(t-s)^H \right) (1 + |α|^{-1}(t-s))$$

$$+ |α|^{-2} \int_s^t \, dr_1 \int_s^{r1} \, dr_2 \|DY^{n-2}_{r2}\|_H$$

$$\leq \cdots$$

$$\leq \left( \|Dξ\|_H + |σ|(t-s)^H \right) e^{|α|^{-1}(t-s)}.$$
Next we bound $\|DY_t^{n+1} - DY_t^n\|_H$. Since $\|DY_t^n\|$ is bounded by a constant independent of $n$, we have

$$
\|D(a(Y_r^n, \theta) - a(Y_r^{n-1}, \theta))\|_H
= \left\| \int_0^1 (\partial_x \partial_a)(1 - \epsilon)Y_r^{n-1} + \epsilon Y_r^n, \theta)((1 - \epsilon)DY_r^{n-1} + \epsilon DY_r^n) \, de 
\times (Y_r^n - Y_r^{n-1}) + \int_0^1 (\partial_x \partial_a)(1 - \epsilon)Y_r^{n-1} + \epsilon Y_r^n, \theta) \, de(DY_r^n - DY_r^{n-1}) \right\|_H
\leq \|D(\partial_x \partial_a)(\cdot, \theta)\|_\infty \left( \|D\xi\|_H + |\sigma|(t-s)^H \right) e^{\|\alpha\|(t-s)} \int_s^t \, dr_1 \, |Y_r^n - Y_r^{n-1}|
+ |\alpha|^{-1} \|DY_r^n - DY_r^{n-1}\|_H.
$$

Therefore, we obtain

$$
\|DY_t^{n+1} - DY_t^n\|_H
\leq \int_s^t \|D(a(Y_r^n, \theta) - a(Y_r^{n-1}, \theta))\|_H \, dr
\leq \|D(\partial_x \partial_a)(\cdot, \theta)\|_\infty \left( \|D\xi\|_H + |\sigma|(t-s)^H \right) e^{\|\alpha\|(t-s)} \int_s^t \, dr_1 \, |Y_r^n - Y_r^{n-1}|
+ |\alpha|^{-1} \int_s^t \, dr_1 \, |DY_r^n - DY_r^{n-1}|\|_H
\leq \|D(\partial_x \partial_a)(\cdot, \theta)\|_\infty \left( \|D\xi\|_H + |\sigma|(t-s)^H \right) e^{\|\alpha\|(t-s)} \int_s^t \, dr_1 \, |Y_r^n - Y_r^{n-1}|
+ |\alpha|^{-1} \int_s^t \, dr_1 \int_s^{r_1} \, dr_2 \, |DY_{r_2}^{n-1} - DY_{r_2}^{n-2}|\|_H
\leq \cdots
\leq \|D(\partial_x \partial_a)(\cdot, \theta)\|_\infty \left( \|D\xi\|_H + |\sigma|(t-s)^H \right) e^{\|\alpha\|(t-s)} \int_s^t \, dr_1 \int_s^{r_1} \, dr_2 \cdots \int_s^{r_k} \, dr_{k+1} \, |Y_{r_{k+1}}^{n-k} - Y_{r_{k+1}}^{n-k-1}|
+ |\alpha|^{-n} \int_s^t \, dr_1 \int_s^{r_1} \, dr_2 \cdots \int_s^{r_{n-1}} \, dr_n \, |DY_r^1 - DY_r^0|\|_H
=: \mathcal{F}_1^n(t) + \mathcal{F}_2^n(t).
$$

Since

$$
|Y_t^{n+1} - Y_t^n| \leq (|a(\xi, \theta)|(t-s) + |\sigma||B_t - B_s|) \frac{(|\alpha|^{-1}(t-s))^{n}}{n!}
$$
holds, the term $\mathcal{J}_1^n(t)$ is bounded as

$$
\mathcal{J}_1^n(t) \leq \|\partial_x a(\cdot, \theta)\|_\infty (\|D\xi\|_\mathcal{H} + |\sigma|(t-s)^H) e^{\alpha^{-1}(t-s)}
\times (|a(\xi, \theta)|(t-s) + |\sigma| \sup_{s \leq r \leq t} |B_r - B_s|) 
\times \sum_{k=0}^{n-1} |\alpha|^{-k} \int_s^t dr_1 \int_s^{r_1} dr_2 \cdots \int_s^{r_k} dr_{k+1} \left( |\alpha|^{-1}(r_{k+1} - s) \right)^{n-k-1} (n-k-1)!
\leq \|\partial_x a(\cdot, \theta)\|_\infty (\|D\xi\|_\mathcal{H} + |\sigma|(t-s)^H) e^{\alpha^{-1}(t-s)}
\times (|a(\xi, \theta)|(t-s) + |\sigma| \sup_{s \leq r \leq t} |B_r - B_s|)(t-s) \left( |\alpha|^{-1}(t-s) \right)^{n-1} (n-1)!.
$$

The term $\mathcal{J}_2^n(t)$ can be bounded as

$$
\mathcal{J}_2^n(t) \leq (|\alpha|^{-1}\|D\xi\|_\mathcal{H}(t-s) + |\sigma|(t-s)^H) \frac{|\alpha|^{-1}(t-s)^n}{n!}.
$$

Therefore we have, for each $T > s$,

$$
\sup_{s \leq t \leq T} \|DY_t^n - DY_t^m\|_\mathcal{H} \to 0
$$

as $n, m \to \infty$ pathwisely and in $L^2(\mathbb{P}^*).$ Hence $Y_{\theta, \xi, s} \in \mathbb{D}^{1,2}.$

Let $\phi$ be in $L_{1/2-H}^\infty(C_c^\infty(\mathbb{R}))$. A continuous version of $(D^\phi Y_{t}^{\theta, \xi, s})_{t \in [s, \infty)}$ satisfies

$$
D^\phi Y_{t}^{\theta, \xi, s} = D^\phi \xi + \int_s^t (\partial_x a)(\cdot, \theta)(Y_{r}^{\theta, \xi, s}, \theta) D^\phi Y_{r}^{\theta, \xi, s} dr + \int_s^t b(r) dr, \quad (3.8)
$$

where $b(r) = \sigma(\mathbf{D}_{1/2-H}^{1/2-H} \mathbf{D}_{1/2-H}^{1/2-H} \phi)_r$. Here we used the integration by parts formula for fractional derivatives (see p.129 of Samko et al. (1993)):

$$
\sigma(1_{(s,t]}, \phi)_\mathcal{H} = \int_{\mathbb{R}} (\mathbf{D}_{1/2-H}^{1/2-H} 1_{(s,t]})(\mathbf{D}_{1/2-H}^{1/2-H} \phi)_s ds = \sigma \int_s^t (\mathbf{D}_{+1/2-H}^{1/2-H} \mathbf{D}_{-1/2-H}^{1/2-H} \phi)_s ds.
$$

Solving the equation (3.8), we have

$$
D^\phi Y_{t}^{\theta, \xi, s} = \langle e^\int_s^t (\partial_x a)(Y_{u}^{\theta, \xi, s}, \theta) du \rangle D^\phi \xi + 1_{(s,t]} \langle e^\int_s^t (\partial_x a)(Y_{u}^{\theta, \xi, s}, \theta) du \rangle \phi \rangle_\mathcal{H}.
$$

Since $L_{1/2-H}^\infty(C_c^\infty(\mathbb{R}))$ is dense in $\mathcal{H}$, we obtain the identity (3.7). \qed
Lemma 3.9. There is a positive constant $C > 0$ such that

$$\|DX_t^{0, \theta}\|^2_H \leq C$$  \hfill (3.9)

holds for all $t \geq 0$.

Proof. Thanks to Lemma 3.8, we have $X_t^{0, \theta} \in W^{1,2}$ and

$$DX_t^{0, \theta} = \sigma 1_{(0, t]}(\cdot) e^{\int_t^0 (\partial_x a)(X_u^{0, \theta}) \, du}.$$  

We set $\Phi(t, s) = \sigma 1_{(0, t]}(s) e^{\int_t^0 (\partial_x a)(X_u^{0, \theta}) \, du}$. Therefore $\|DX_t^{0, \theta}\|_H = \|\Phi(t, \cdot)\|_H$ holds, and so that it suffices to show that there is a positive constant $C > 0$ such that $\|\Phi(t, \cdot)\|_H \leq C$ holds for all $t \geq 0$.

First we consider the following decomposition:

$$\|\Phi(t, \cdot)\|^2_H = \int_{-\infty}^0 ds \left| \int_{-s}^{-s+t} d\xi \, \xi^{H-3/2} \Phi(t, s + \xi) \right|^2 + \int_0^t ds \left| \int_{-\infty}^\infty d\xi \, \xi^{H-3/2} (\Phi(t, s) - \Phi(t, s + \xi)) \right|^2$$

$$=: I_1(t) + I_2(t).$$

The term $I_1(t)$ can be bounded from above as follows:

$$I_1(t) = \int_{-\infty}^0 ds \left| \int_0^t d\xi \, (\xi - s)^{H-3/2} e^{\int_\xi^t \partial_x a(X_u^{0, \theta}) \, du} \right|^2$$

$$\leq \int_{-\infty}^0 ds \left| \int_0^t d\xi \, (\xi - s)^{H-3/2} e^{\alpha(t-\xi)} \right|^2$$

$$= \int_0^t d\xi \int_0^t d\eta \, e^{-\alpha(t-\xi)} e^{-\alpha(t-\eta)} \int_{-\infty}^0 ds \, (\xi - s)^{H-3/2} (\eta - s)^{H-3/2}$$

$$\leq (2 - 2H)^{-1} \left| \int_0^t d\xi \, \xi^{H-1} e^{-\alpha(t-\xi)} \right|.$$  

Here the last inequality is due to Hölder’s inequality. In order to obtain an upper bound for $I_2(t)$, we further decompose $I_2(t)$ as follows:

$$I_2(t) \leq 2 \int_0^t ds \left| \int_{-s}^{-s+t} d\xi \, \xi^{H-3/2} (\Phi(t, s) - \Phi(t, s + \xi)) \right|^2$$

$$+ 2 \int_0^t ds \left| \int_{-s+t}^\infty d\xi \, \xi^{H-3/2} \Phi(t, s) \right|^2$$

$$=: I_{2,1}(t) + I_{2,2}(t).$$
For the term $I_{2,1}$, we have

\[
I_{2,1}(t) \leq 4 \int_0^t ds \left| \int_s^{(s+1)\Lambda t} d\xi (\xi - s)^{H-3/2}(\phi(t,s) - \phi(t,\xi)) \right|^2 \\
+ 4 \int_0^t ds \left| \int_s^{(s+1)\Lambda t} d\xi (\xi - s)^{H-3/2}(\phi(t,s) - \phi(t,\xi)) \right|^2 \\
\leq 4\|\partial_x a(\cdot, \theta)\|_\infty^2 \int_0^t ds \left| \int_s^{(s+1)\Lambda t} d\xi (\xi - s)^{H-1/2}e^{-\alpha(t-\xi)} \right|^2 \\
+ 4 \int_0^t ds \left| \int_s^{(s+1)\Lambda t} d\xi (\xi - s)^{H-3/2}e^{-\alpha(t-\xi)} \right|^2 \\
= 4\|\partial_x a(\cdot, \theta)\|_\infty^2 \int_0^t ds \left| \int_s^{(s+1)\Lambda t} d\xi (\xi - s)^{H-1/2}e^{-\alpha(t-\xi)} \right|^2 \\
+ 4 \int_0^t ds \left| \int_s^{(s+1)\Lambda t} d\xi (\xi - s)^{H-3/2}e^{-\alpha(t-\xi)} \right|^2 \\
=: I_{2,1,1}(t) + I_{2,1,2}(t) + I_{2,1,3}(t)
\]

If $t \in [0,1]$, then only the term $I_{2,1,2}(t)$ appears and this is finite. If $t > 1$, we have

\[
I_{2,1,1}(t) \leq 4(H + 1/2)^{-2}\|\partial_x a(\cdot, \theta)\|_\infty^2 \int_0^{(t-1)-s} ds e^{-2\alpha(t-1)-s},
\]

\[
I_{2,1,2}(t) \leq 4\|\partial_x a(\cdot, \theta)\|_\infty^2 \int_0^t ds \left| \int_s^{(s+1)\Lambda t} d\xi (\xi - s)^{H-1/2} \right|^2 \\
= (H + 1/2)^{-2}(2H + 2)^{-1}
\]

and

\[
I_{2,1,3}(t) = 4 \int_0^{(t-1)-s} ds e^{-\alpha(t-s)} \int_1^{t-s} d\xi \xi^{H-3/2}e^{\alpha\xi} \\
\leq 4C \int_0^{(t-1)-s} ds (t - s)^{H-3/2}.
\]
Here we used Lemma 3.10 below. On the other hand, for the term $I_{2,2}(t)$, we have

$$I_{2,2}(t) = 2(1/2 - H)^{-2} \int_0^t ds \left| \Phi(t, s)(-s + t)^{H-1/2} \right|^2$$

$$\leq 2(1/2 - H)^{-2} \int_0^t ds \, e^{-2\alpha(t-s)}(t-s)^{2H-1}$$

$$= 2(1/2 - H)^{-2} \int_0^t ds \, s^{2H-1}e^{-2\alpha s}.$$  

Note that all these bounds are finite even if we take the supremum over $t \geq 0$.

**Lemma 3.10.** Let $\alpha > 0$ and $\beta > 0$ be positive constants. For $x \geq 1$, there exists a positive constant $C > 0$ which is independent of $x$ such that

$$e^{-\alpha x} \int_1^x d\xi \, \xi^{-\beta} e^{\alpha \xi} \leq C x^{-\beta}$$  \hspace{1cm} (3.10)

holds.

**Proof.** Let us set $I_{n,\beta}(x) = e^{\alpha x} \int_1^x d\xi \, \xi^{-\beta} e^{\alpha \xi}$. We have

$$I_{n,\beta}(x) = e^{-\alpha x} \left( \alpha^{-1} x^{-\beta} e^{\alpha x} - \alpha^{-1} e^{\alpha} + \beta \alpha^{-1} \int_1^x d\xi \, \xi^{-\beta-1} e^{\alpha \xi} \right) 1_{[2,\infty)}(x)$$

$$+ e^{-\alpha x} \int_1^x d\xi \, \xi^{-\beta} e^{\alpha \xi} 1_{[1,2]}(x)$$

$$\leq e^{-\alpha x} \left( \alpha^{-1} x^{-\beta} e^{\alpha x} + \beta^{-1} \alpha^{-1} (\beta(x/2)^{-\beta-1} e^{\alpha x/2} + \alpha^{-1} (x/2)^{-\beta+1} e^{\alpha}) \right) 1_{[2,\infty)}(x)$$

$$+ \alpha^\alpha (-\beta + 1)(2^{-\beta+1} - 1) 1_{[1,2]}(x).$$

This completes the proof. \hfill \square

**Proof of Proposition 3.7.** By Lemma 3.6, it suffices to show that $X_0^\theta \in \mathbb{D}^{1,2}$. Since a random attractor is pathwise pullback attracting, we have

$$|X_0^{0,\theta}(\vartheta t - t) - X_0^\theta(\vartheta)| \to 0$$

as $n \to \infty$ for all $\omega \in \Omega^*$. Note that for $p > 0$ it holds that $\mathbb{E}^*\{|X_t^{0,\theta}(\vartheta - t)|^p\} = \mathbb{E}^*\{|X_t^\theta|^p\} \leq c_p$ with a positive constant $c_p > 0$, which is independent of $t \geq 0$. Therefore the family $(|X_t^{0,\theta} \circ \vartheta - t|)^{t \geq 0}$ is uniformly integrable for all
\( p > 0 \) and in particular \( L^p(\mathbb{P}^*)\)-convergence \( X_t^{0,\theta} \circ \vartheta_{-t} \to \bar{X}_0^\theta \) as \( t \to \infty \) holds for all \( p > 0 \). Therefore, in order to prove that \( \bar{X}_0^\theta \in \mathbb{D}^{1,2} \), it suffices to show

\[
\sup_{t \geq 0} \mathbb{E}^* \{ \| D(X_t^{0,\theta} \circ \vartheta_{-t}) \|_H^2 \} < \infty
\]

(see Lemma 1.2.3 of Nualart (2006)). By Lemma 3.6, it suffices to prove

\[
\sup_{t \geq 0} \mathbb{E}^* \{ \| D X_t^{0,\theta} \|_H^2 \} < \infty,
\]

and this inequality follows from Lemma 3.9.

Let \( \phi \in \mathcal{G} \). By (3.7), we have

\[
D^\phi \bar{X}_t^\theta = e^\int_r^t (\partial_x a)(X_s^\theta, \vartheta) \, dr \, D^\phi \bar{X}_s^\theta + \sigma \int_s^t \sigma e^\int_r^s (\partial_x a)(X_u^\theta, \vartheta) \, du \, (D_1^{1/2-H} D_{-1}^{1/2-H} \phi) \, dr.
\]

(3.11)

By Assumption 2.1 and Lemma 3.6, the first term in (3.11) converges to zero in \( L^2(\mathbb{P}^*) \) as \( s \to -\infty \). Moreover, since \( (D_1^{1/2-H} D_{-1}^{1/2-H} \phi)_r = 0 \) for sufficiently small \( r \), the second term in (3.11) coincides with

\[
\sigma \int_{-\infty}^t e^\int_r^s (\partial_x a)(X_u^\theta, \vartheta) \, du \, (D_1^{1/2-H} D_{-1}^{1/2-H} \phi)_r \, dr,
\]

if \( s \) is sufficiently small. Hence we obtain

\[
D^\phi \bar{X}_t^\theta = \sigma \int_{\mathbb{R}} D_1^{1/2-H} (1_{(-\infty,t]}(r) e^\int_r^t (\partial_x a)(X_u^\theta, \vartheta) \, du) \, (D_{-1}^{1/2-H} \phi)_r \, dr.
\]

(3.12)

Let us denote the function \( r \mapsto \sigma 1_{(-\infty,t]}(r) e^\int_r^t (\partial_x a)(X_u^\theta, \vartheta) \, du \) by \( r \mapsto \Psi(t, r) \) for simplicity. Taking the expectation of the both sides in (3.12), we have

\[
\langle DX_t^\theta, \phi \rangle_{L^2(\Omega; \mathcal{H})} = \langle \Psi(t, \cdot), \phi \rangle_{L^2(\Omega; \mathcal{H})}.
\]

Since the set \( \mathcal{G} \) is dense in \( L^2(\Omega; \mathcal{H}) \), we obtain (2.3).

Finally we prove \( \mathbb{E}^* \{ \| D \bar{X}_t^\theta \|_H^p \} < \infty \) for all \( p > 0 \). It suffices to show that there is a constant \( C > 0 \) that is independent of \( \omega \in \Omega^* \) such that

\[
\| \Psi(0, \cdot)(\omega) \|_H^2 < C.
\]

(3.13)

A straightforward calculation yields

\[
(D_{-1}^{1/2-H} \Psi(0, \cdot))_t = (\sigma \int_0^{-t} d\xi \xi^{H-3/2} (e^\int_0^\xi (\partial_x a)(X_u^\theta, \vartheta) \, du - e^\int_0^\xi (\partial_x a)(\bar{X}_u^\theta, \vartheta) \, du) \\
+ \sigma \int_0^\infty d\xi \xi^{H-3/2} e^\int_\xi^\infty (\partial_x a)(X_u^\theta, \vartheta) \, du) 1_{(-\infty,0]}(t)
\]

\[=: J_1(t) + J_2(t).\]
By a simple calculation, we have
\[
|J_1(t)|1_{[-1,0]}(t) \leq |\alpha^{-1}\sigma| (H + 1/2)^{-1}(-t)^{H+1/2}1_{[-1,0]}(t),
\]
\[
|J_1(t)|1_{(-\infty,-1)}(t) \leq \left| \int_0^1 d\xi \xi^{H-3/2} \left( e^{\int_0^t (\partial_x a)(\bar{X}^a_\nu, \theta) \, dr} - e^{\int_0^{t+\xi} (\partial_x a)(\bar{X}^a_\nu, \theta) \, dr} \right) \right| 1_{(-\infty,-1)}(t)
\]
\[
+ \left| \int_0^{t^*} d\xi \xi^{H-3/2} \left( e^{\int_0^t (\partial_x a)(\bar{X}^a_\nu, \theta) \, dr} - e^{\int_0^{t+\xi} (\partial_x a)(\bar{X}^a_\nu, \theta) \, dr} \right) \right| 1_{(-\infty,-1)}(t)
\]
\leq \left( |\alpha^{-1}\sigma|e^{\alpha t}\int_0^1 d\xi \xi^{H-1/2}e^{\alpha \xi} + |\sigma|e^{\alpha t}\int_1^t d\xi \xi^{H-3/2}e^{\alpha \xi} \right) 1_{(-\infty,-1)}(t)
\]
and
\[
|J_2(t)| \leq |\sigma|(1/2 - H)e^{\alpha t}t^{H-1/2}.
\]
These upper bounds are square integrable and independent of \( \omega \in \Omega^* \).

4. Likelihood bounds

4.1. Integral ratio formula

First we introduce some transformations related to fBM.

**Definition 4.1.** (1) Let \( K_H^* : L^{1/2-H}_T(L^2[0,T]) \rightarrow L^2[0,T] \) be an isomorphism between the Cameron-Martin space associated with fBM on an interval and \( L^2[0,T] \), namely
\[
(K_H^* h)_s = d_H s^{1/2-H}D_{T-s}^{1/2-H}(H-1/2)h_s
= \frac{d_H s^{1/2-H}}{\Gamma(1 - (1/2 - H))} \left( s^{H-1/2}h_s \right) \left( \frac{T-s}{T-s} \right)^{1/2-H}
+ (1/2 - H) \int_s^T s^{H-1/2}h_s - r^{H-1/2}h_r \, dr.
\]

Note that the inverse \( K_H^{-1} : L^2[0,T] \rightarrow L^{1/2-H}_T(L^2[0,T]) \) of \( K_H^* \) is well-defined and given by
\[
(K_H^{-1} g)_s = d_H^{-1} s^{1/2-H}I_{T-s}^{1/2-H}(H-1/2)g_s.
\]

For properties of the operators \( K_H^* \) and \( K_H^{-1} \), we refer to Alós et al. (2001) and Nualart (2006).
We define a Volterra kernel $K_H(t,s)$ by
\[ K_H(t,s) = (K_H^*1_{[0,t]})(s). \]
Here the symbol $1_A(x)$ denotes an indicator function which is 1 if $x \in A$ and 0 otherwise. For explicit expressions of the kernel $K_H(t,s)$, see Decreusefond and Üstünel (1999), Alós et al. (2001) and Nualart (2006).

We set
\[ \eta_H(t,s) = \frac{1_{A_T}(t,s)}{d_H \Gamma(1/2-H)} s^{1/2-H} \int_s^t r^{H-1/2} (r-s)^{-1/2-H} dr, \]
where $A_T = \{(t,s) \in [0,T]^2 | 0 < t \leq T, 0 < s < t\}$. Then $\eta_H(t,\cdot)$ is a continuous version of $K^{*-1}1_{[0,t]}$.

**Proposition 4.2.** Let $t \in [0,T]$. The function $\eta_H(t,\cdot)$ is differentiable almost everywhere, and its derivative $s \mapsto \frac{\partial}{\partial s} \eta_H(t,s)$ is in $L^1[0,T]$.

**Proof.** If $t = 0$, then there is nothing to prove. Let $t \in (0,T]$. The almost everywhere differentiability is clear. The derivative $\frac{\partial}{\partial s} \eta_H(t,s)$ is given by
\begin{align*}
\frac{\partial}{\partial s} \eta_H(t,s) &= (1/2-H)s^{-1}\eta_H(t,s) \\
&\quad - (d_H \Gamma(1/2-H))^{-1}s^{-1/2-H}(t-s)^{-1/2-H}t^{1/2-H}.
\end{align*}
for $s \in (0, t)$, and $\frac{\partial}{\partial s} \eta(t, s) = 0$ for $s \in (t, T)$. It suffices to show that the first term of (4.1) is integrable around zero. We have, for $s \in (0, t/2)$,

$$|\eta_H(t, s)| \lesssim s^{1/2-H} \int_1^{t/s} r^{H-1/2}(r-1)^{-1/2-H} \, dr$$

$$\leq s^{1/2-H} \left( \int_1^2 r^{H-1/2}(r-1)^{-1/2-H} \, dr + \int_{2}^{t/s} (r-1)^{-1} \, dr \right)$$

$$= s^{1/2-H} \left( \int_1^2 r^{H-1/2}(r-1)^{-1/2-H} \, dr + \log(t) - \log(s) \right).$$

This completes the proof. \qed

Let $K_H: C[0, T] \to C[0, T]$ denotes a $\mathcal{B}_T$-measurable transform defined by

$$(K_H x)_t = -\int_0^t x_s \frac{\partial}{\partial s} \eta(t, s) \, ds.$$

Note that the transformation $K_H$ is well-defined thanks to Proposition 4.2.

**Remark 4.3.** Suppose that $x \in C^1[0, T]$. Then we have

$$(K_H x)_t = \int_0^t \eta_H(t, s) \dot{x}_s \, ds = \int_0^t \eta_H(t, s) \, dx_s. \tag{4.2}$$

Therefore, we can formally regard $(K_H B^\theta)_t$ as the Wiener integral

$$\int_0^T (K_H^{-1} x_{[0,t]}^\theta)_s \, dB_s^\theta,$$

which is a Brownian motion (see p.285 of Nualart (2006) for example).

By Remark 4.3, we expect that the process $K_H B^\theta$ is a Brownian motion under the measure $\mu_T^\theta$. This indeed holds.

**Lemma 4.4.** Let us set $W^\theta = K_H B^\theta$. Then the process $W^\theta$ is a continuous modification of the process $(\int_0^T (K_H^{-1} x_{[0,t]}^\theta)_s \, dB_s^\theta)_{t \in [0, T]}$. In particular, the process $W^\theta$ is a $(\mathcal{B}_t)$-Brownian motion.

**Proof.** We fix $t \in (0, T]$ and set $t^n_i = (i/n)t$ for $i = 0, 1, \ldots, n$. Let $x^n = \sum_{i=1}^n x(t^n_{i-1}) \mathbb{1}_{[t^n_{i-1}, t^n_i]} + x(t^n_n) \mathbb{1}_{[t^n_n, t]}$ and

$$(K_H^n x)_t = -\int_0^t x^n(s) \frac{\partial}{\partial s} \eta_H(t, s) \, ds.$$
Then clearly \( \lim_{n \to \infty} (K^n_H x)_t = (K_H x)_t \) holds for each \( x \in C[0, T] \). On the other hand, we have

\[
(K^n_H x)_t = -\sum_{i=1}^n (\eta_H(t, t^n_i) - \eta_H(t, t^n_{i-1})) x(t^n_{i-1})
\]

\[
= \sum_{i=1}^n \eta_H(t, t^n_i)(x(t^n_i) - x(t^n_{i-1})).
\]

Hence we have \( (K^n_H B^\theta)_t \to \int_0^T (K^n_H \xi s^{-1} 1_{[0,t]}_s) dB^\theta_s \) as \( n \to \infty \) in \( L^2(\mu^T_\theta) \). \( \square \)

**Notation.** Let \( \lambda \in (0, 1) \). We denote the space of \( \lambda \)-Hölder continuous functions on \([0, T]\) by \( C^\lambda[0, T] \). Furthermore, we set

\[
C^\lambda - [0, T] = \bigcap_{\epsilon \in (0, \lambda)} C^{\lambda-\epsilon}[0, T].
\]

For \( f \in C^\lambda[0, T] \), we also set \( |f|_\lambda = \sup_{0 \leq s < t \leq T} |f(t) - f(s)|/(t - s)^\lambda \) and \( \|f\|_\lambda = \|f\|_\infty + |f|_\lambda \). For \( \lambda \in (0, 1] \), we set \( C^\lambda_0[0, T] = C^\lambda[0, T] \cap \{ \pi_0 = 0 \} \). The set \( C^\lambda_0 - [0, T] \) is defined in the same way.

The operator \( K_H \) also transforms fBM into BM pathwisely. More precisely, the following proposition holds.

**Proposition 4.5.** Let \( x \) be in \( C^H_0[0, T] \). Then \( K_H x \) is in \( C^{(1/2)-}[0, T] \).

We begin with the following lemma.

**Lemma 4.6.** Let \( x \in C^1_0[0, T] \). Then we have

\[
\|K_H x\|_{(1/2)-} \leq C_{H, \lambda}\|x\|_{H-\lambda}
\]

for any \( \lambda \in (0, H) \). Here \( C_{H, \lambda} \) is a positive constant depends only on \( H \) and \( \lambda \).

**Proof.** By (4.2) and an integration by parts formula, one has

\[
(K_H x)_t = d_H^{-1} \int_0^t ds \ s^{1/2-H} \dot{x}_s \int_s^t dr \ r^{H-1/2}(r - s)^{-1/2-H}
\]

\[
= d_H^{-1} \int_0^t dr \ r^{H-1/2} \int_0^r ds \ (r - s)^{1/2-H} s^{-H-1/2} \dot{x}_{r-s}
\]

\[
= d_H^{-1} \int_0^t ds \ s^{-H-1/2} \int_s^t dr \ r^{H-1/2}(r - s)^{1/2-H} \dot{x}_{r-s}
\]
Note that we used the mean value theorem: for $0 < s < t$,

$$
= d_H^{-1} \int_0^t ds \ s^{-H-1/2}(t-s)^{1/2-H} \ t^{-H-1/2} x_{t-s}
+ (1/2 - H)d_H^{-1} \int_0^t ds \ s^{-H+1/2} \ \int_s^t dr \ (r-s)^{-1/2-H} \ t^{-H-3/2} x_{r-s}
= \mathcal{I}_1(t) + \mathcal{I}_2(t).
$$

**Calculation of $\mathcal{I}_1(t)$**. Since $x(0) = 0$, we have

$$
|\mathcal{I}_1(t) - \mathcal{I}_1(0)| \leq t^{1/2-\lambda/2} |x|_{H-\lambda} \int_0^1 ds \ s^{-H-1/2}(t-s)^{1/2-\lambda}. \quad (4.4)
$$

Next, suppose that $0 < t' < t < T$. Then $\mathcal{I}_1(t) - \mathcal{I}_1(t')$ can be written as

$$
\mathcal{I}_1(t) - \mathcal{I}_1(t') = \int_0^{t'} ds \ s^{-H-1/2} (1 - s/t)^{1/2-H} (x_{t-s} - x_{t'-s})
+ \int_0^{t'} ds \ s^{-H-1/2} (1 - s/t')^{1/2-H} (x_{t'-s} - x_{t-s})
= \mathcal{I}_{1,1}(t', t) + \mathcal{I}_{1,2}(t', t).
$$

The term $\mathcal{I}_{1,2}(t', t)$ can be bounded as follows:

$$
|\mathcal{I}_{1,2}(t', t)|
\leq (1/2 - H) \int_0^{t'} ds \ s^{-H+1/2} (1 - s/t')^{-1/2+H} (t - t') \int_0^1 ds \ s^{1/2-H} (t' - s)^{-1/2-\lambda}
\leq (1/2 - H)(1/2 - \lambda)^{-1} |x|_{H-\lambda} (t')^{-1/2-\lambda} (1 - t'),
$$

Note that we used the mean value theorem: for $0 < s < t'$,

$$
(1 - s/t)^{1/2-H} - (1 - s/t')^{1/2-H}
= (1/2 - H)(t - t') \int_0^1 d\epsilon \ (1 - s/t') + \epsilon(s/t' - s/t)^{-1/2-H}
\leq (1/2 - H)(t - t') \int_0^1 d\epsilon \ (1 - s/t')^{1/2-H}.
$$

Hence it holds that

$$
\frac{\mathcal{I}_{1,2}(t', t)}{(t - t')^{1/2-\lambda}}
\leq (1/2 - H)(1/2 - \lambda)^{-1} |x|_{H-\lambda} (t'/t)^{1/2-\lambda} (1 - t'/t)^{1/2+\lambda}. \quad (4.5)
$$
We can immediately bound the term $J_{1,1}(t', t)$:

$$|J_{1,1}(t', t)| \leq (1/2 - H)^{-1}|x|_{H-\lambda} (t - t')^{H-\lambda} (t^{1/2-H} - (t')^{1/2-H}).$$

Hence we have

$$\frac{|J_{1,1}(t', t)|}{(t - t')^{1/2-\lambda}} \leq (1/2 - H)^{-1}|x|_{H-\lambda} \frac{t^{1/2-H} - (t')^{1/2-H}}{(t - t')^{1/2-H}}. \quad (4.6)$$

Note that $[0, T] \ni t \mapsto t^{1/2-H}$ is $1/2-H$-Hölder continuous.

**Calculation of $J_2(t)$**. By changing the order of integration, we obtain

$$J_2(t) - J_2(t')$$

$$= (1/2 - H) d_H^{-1} \int_{t'}^t dr r^{H-3/2} \int_0^r ds s^{-H+1/2} (r - s)^{-1/2-H} x_{r-s}$$

$$\leq (1/2 - H)(1/2 - \lambda)^{-1} d_H^{-1} |x|_{H-\lambda} \int_{t'}^t dt r^{-1/2-\lambda}$$

$$\leq (1/2 - H)(1/2 - \lambda)^{-2} d_H^{-1} |x|_{H-\lambda} (t^{1/2-\lambda} - (t')^{1/2-\lambda}).$$

The proof is complete by the inequalities (4.4)-(4.7).

**Proof of Proposition 4.5**. Let $x$ be in $C_H^0[0, T]$. For each $\lambda \in (0, H)$, there exists a sequence $(x^{n,\lambda}) \subset C_H^1[0, T]$ such that $\lim_{n \to \infty} \|x - x_{n,\lambda}\|_{H-\lambda} = 0$. Since we have

$$\|K_{H}x^{n,\lambda} - K_{H}x^{m,\lambda}\|_{1/2-\lambda} \leq C_{H,\lambda}\|x^{n,\lambda} - x^{m,\lambda}\|_{H-\lambda} \to 0 \quad (n, m \to \infty)$$

by Lemma 4.6, there exists $y^\lambda \in C_H^{1/2-\lambda}[0, T]$ with $\lim_{n \to \infty} \|K_{H}x^{n,\lambda} - y^\lambda\|_{1/2-\lambda} = 0$. Since we have

$$\|K_{H}x - y^\lambda\|_{\infty} \leq \|K_{H}x - K_{H}x^{n,\lambda}\|_{\infty} + \|K_{H}x^{n,\lambda} - y^\lambda\|_{\infty}$$

$$\leq \|x - x^{n,\lambda}\|_{\infty} \sup_{t \in [0, T]} \int_0^t \left| \frac{\partial}{\partial s} \eta(t, s) \right| ds + \|K_{H}x^{n,\lambda} - y^\lambda\|_{\infty}$$

$$\to 0$$

as $n \to \infty$, it holds that $K_{H}x = y^\lambda$. Hence we obtain $K_{H}x \in C_0^{(1/2)-\lambda}[0, T]$ for any $\lambda \in (0, H)$ and therefore $K_{H}x \in C_0^{(1/2)-\lambda}[0, T].$ **□**

Let us define a “left inverse” operator $K_H^{-1}$ of $K_H$. As is shown in Proposition 5.3 of Nualart (2006), the kernel $K_H(t, s)$ can be represented as

$$K_H(t, s) = c_H 1_A(t, s) \left( (t/s)^{H-1/2} (t - s)^{H-1/2} \right).$$
\[-(H - 1/2)s^{1/2-H}s^{1/2-H} \int_s^t u^{H-3/2}(u - s)^{H-1/2} du)\].

A straightforward calculation yields
\[
\frac{\partial}{\partial s} \mathcal{K}_H(x) = c_H(1/2 - H) \left( t^{H-1/2}s^{1/2-H}(t - s)^{H-3/2} \right. \\
\left. - (1/2 - H) s^{H-3/2} \int_1^{t/s} u^{H-3/2}(u - 1)^{H-1/2} \, du \right)
\]
on the set \(A_T\).

We define a transformation \(\mathcal{K}_H^{-1} : C[0, T] \to C[0, T]\) as
\[
(\mathcal{K}_H^{-1}x)_t = \int_0^{t/2} x_s \frac{\partial}{\partial s} \mathcal{K}_H(t, s) \, ds - \int_{t/2}^t (x_s - x_t) \frac{\partial}{\partial s} \mathcal{K}_H(t, s) \, ds + \mathcal{K}_H(t, t/2)x_t
\]
for \(x \in C_0^{1/2-}[0, T]\), and \(\mathcal{K}_H^{-1}x = x\) otherwise.

**Remark 4.7.** The integrals in (4.8) are well-defined for \(x \in C_0^{1/2-}[0, T]\). Moreover, if \(x \in C_0^1[0, T]\), then \(\mathcal{K}_H^{-1}x\) becomes
\[
(\mathcal{K}_H^{-1}x)_t = \int_0^t \mathcal{K}_H(t, s)x_s \, ds
\]
by integration by parts.

**Proposition 4.8.** We have \(\mathcal{K}_H^{-1}\mathcal{K}_Hx = x\) for \(x \in C_0^{H-}[0, T]\).

**Proof.** Let \(x \in C_0^{H-}[0, T]\) and \(\lambda \in (0, H)\). Then there exists a sequence \((x^{n,\lambda})_n \subset C_0^{1}[0, T]\) such that \(\|x - x^{n,\lambda}\|_{H-\lambda} \to 0\) as \(n \to \infty\). Note that the limit of \(\|\cdot\|_{1/2-\lambda}\)-Cauchy sequence \((\mathcal{K}_H x^{n,\lambda})_n\) coincides with \(\mathcal{K}_Hx\) (see the proof of Proposition 4.5). The same holds for \(\mathcal{K}_H^{-1}\). In fact, for \(y \in C_0^{1/2-}[0, T]\) there is a sequence \((y^{n,\lambda}) \subset C_0^{1}[0, T]\) such that \(\|y - y^{n,\lambda}\|_{1/2-\lambda} \to 0\) as \(n \to \infty\). Then we have
\[
\|\mathcal{K}_H^{-1}y - \mathcal{K}_H^{-1}y^{n,\lambda}\|_\infty \\
\leq \sup_{t \in [0, T]} \left\{ \int_0^{t/2} \left| \frac{\partial}{\partial s} \mathcal{K}_H(t, s) \right| s^{1/2-\lambda} \, ds \\
+ \int_{t/2}^t \left| \frac{\partial}{\partial s} \mathcal{K}_H(t, s) \right| (t - s)^{1/2-\lambda} \, ds + |\mathcal{K}_H(t, t/2)| t^{1/2-\lambda} \right\}
\]
\[ \times \| y - y^{n,\lambda} \|_{1/2 - \lambda} \rightarrow 0 \]

as \( n \rightarrow \infty \). Hence we have \( \| K_H^+ K_H^{x^n,\lambda} - K_H^+ K_H^+ x \|_{\infty} \rightarrow 0 \) as \( n \rightarrow \infty \).

Therefore, it suffices to show \( K_H^{-} K_H x = x \) for \( x \in C_0^1[0,T] \). We have

\[
(K_H^{-} K_H x)_t = \int_0^t ds K_H(t, s) \frac{d}{ds} \int_0^s dr \eta_H(s, r) \dot{x}_r
= \int_0^t ds K_H(t, s) \int_0^s dr \frac{\partial}{\partial s} \eta_H(s, r) \dot{x}_r
= \int_0^t dr \dot{x}_r \int_r^t ds K_H(t, s) \frac{\partial}{\partial s} \eta_H(s, r)
\]

(note that \( \int_0^s dr \eta_H(s, r) \dot{x}_r = \int_0^T dr \eta_H(s, r) \dot{x}_r \)). Since

\[
\int_0^t ds K_H(t, s) \frac{\partial}{\partial s} \eta_H(s, r) = 1
\]

holds (for a proof, see Theorem 5.7 of Hu (2005)), we obtain \( K_H^{-} K_H x = x \).

\[ \square \]

4.2. Proof of the first part of Theorem 2.6

By the definition of \( B^\theta \), we have

\[
\sigma^{-1}(\pi_t(x) - \pi_0(x)) = \int_0^t \sigma^{-1}a(\pi_s(x), \theta) ds + B^\theta_t(x)
\]

for each \( t \in [0,T] \). Hence we obtain

\[
K_H \left[ \sigma^{-1}(\pi(x) - \pi_0(x)) \right]_t = K_H \left[ \int_0^t \sigma^{-1}a(\pi_s(x), \theta) ds \right]_t + (K_H B^\theta)_t. \quad (4.9)
\]

Since a straightforward calculation yields

\[
K_H \left[ \int_0^t \sigma^{-1}a(\pi_s(x), \theta) ds \right]_t = \int_0^t \beta_s(\theta) ds,
\]

the equation (4.9) can be written as

\[
K_H \left[ \sigma^{-1}(\pi(x) - \pi_0(x)) \right]_t = \int_0^t \beta_s(\theta) ds + W^\theta_t. \quad (4.10)
\]

Hence the process \( K_H \left[ \sigma^{-1}(\pi(x) - \pi_0(x)) \right] \) is a Brownian motion with drift under \( \mu_T^\theta \). Now we apply the Girsanov theorem. We check the Novikov condition is satisfied.
Lemma 4.9. Suppose that Assumption 2.1 holds. Then the Novikov condition holds:

\[ \mathbb{E}^{\mu^{(T)}_\theta} \left\{ \exp \left( \frac{1}{2} \int_0^T \beta^2_t(\theta) \, dt \right) \right\} < +\infty. \] (4.11)

**Proof.** See the proof of Proposition 1 of Tudor and Viens (2007). □

Therefore the process \( K_H[\sigma^{-1}(\pi.(x) - \pi_0(x))] \) is a Brownian motion under the probability measure \( \nu^{(T)}_\theta \) defined by \( d\nu^{(T)}_\theta = z^\theta_T d\mu^{(T)}_\theta \), where

\[ z^\theta_T = \exp \left( - \int_0^T \beta_t(\theta) \, dW^\theta_t - \frac{1}{2} \int_0^T \beta^2_t(\theta) \, dt \right) \, d\mu^{(T)}_\theta. \]

We can easily show that \( \mu^{(T)}_\theta \{ z^\theta_T = 0 \} = 0 \), and therefore it holds that \( \mu^{(T)}_\theta \ll \nu^{(T)}_\theta \) (absolutely continuous) and

\[ \frac{d\mu^{(T)}_\theta}{d\nu^{(T)}_\theta} = (z^\theta_T)^{-1}. \] (4.12)

We can prove that \( \nu^{(T)}_\theta \) actually is independent of \( \theta \in \Theta \).

**Notation.** Let \( n \) be a positive integer. The set of continuous and bounded functions \( f: \mathbb{R}^n \to \mathbb{R} \) is denoted by \( C_b(\mathbb{R}^n) \). Moreover, for each \( \xi \in \mathbb{R} \), let \( f_\xi \) denote the function \( (\eta_1, \ldots, \eta_n) \mapsto f(\eta_1 + \xi, \ldots, \eta_n + \xi) \).

Lemma 4.10. Let \( \theta_1 \) and \( \theta_2 \) be in \( \Theta \). Then we have \( \nu^{(T)}_{\theta_1} = \nu^{(T)}_{\theta_2} \).

**Proof.** The process

\[ K_H [\sigma^{-1}(\pi.(x) - \pi_0(x))]_t = \pi_t(\sigma^{-1}K_H x) \]

is a Brownian motion under the measures \( \nu^{(T)}_{\theta_1} \) and \( \nu^{(T)}_{\theta_2} \). Hence the image measures induced by the transformation \( \sigma^{-1}K_H \) coincide:

\[ (\nu^{(T)}_{\theta_1})^{\sigma^{-1}K_H} = (\nu^{(T)}_{\theta_2})^{\sigma^{-1}K_H}. \] (4.13)

To verify \( \nu^{(T)}_{\theta_1} = \nu^{(T)}_{\theta_2} \), it suffices to show that all finite dimensional marginal distributions coincide. Let \( n \) be a positive integer, \( 0 \leq t_1 < t_2 < \cdots < t_n \leq T \) and \( f \in C_b(\mathbb{R}^n) \). We show that

\[ \mathbb{E}^{\nu^{(T)}_{\theta_1}} \{ f(\pi_{t_1}, \ldots, \pi_{t_n}) \} = \mathbb{E}^{\nu^{(T)}_{\theta_2}} \{ f(\pi_{t_1}, \ldots, \pi_{t_n}) \}. \] (4.14)
Since $\nu_{\theta_1}^T(C^H - [0, T]^c) = \nu_{\theta_2}^T(C^H - [0, T]^c) = 0$ and $\nu_{\theta_1}^T(\pi_0 = x_0) = \nu_{\theta_2}^T(\pi_0 = x_0) = 1$, we have

$$E^{\nu_{\theta_1}^T} \{f(\pi_{t_1}, \ldots, \pi_{t_n})\}$$

$$= E^{\nu_{\theta_2}^T} \{1_{C^H - [0, T]^c}(\pi_0 = x_0)f_x(\pi_{t_1} - \pi_0, \ldots, \pi_{t_n} - \pi_0)\}$$

$$= E^{\nu_{\theta_2}^T} \{f_x(\pi_{t_1}, (\sigma K_H \sigma^{-1} K_H), \ldots, \pi_{t_n}, (\sigma K_H \sigma^{-1} K_H))\}$$

$$= E^{\nu_{\theta_1}^T} \{(\nu_{\theta_2}^T)^{-1}K_H f_x(\pi_{t_1}, (\sigma K_H), \ldots, \pi_{t_n}, (\sigma K_H))\}$$

$$= E^{\nu_{\theta_1}^T} \{(\nu_{\theta_2}^T)^{-1}K_H f_x(\pi_{t_1}, (\sigma K_H), \ldots, \pi_{t_n}, (\sigma K_H))\}$$

Therefore we obtain (4.14). \hfill \Box

Now let us turn to calculation of the likelihood ratio. By (4.12) and Lemma 4.10, we have

$$\frac{d\mu_{T_2}^T}{d\mu_{T_1}^T} = \exp \left( \int_0^T \beta_t(\theta_2) dW_t^{\theta_2} - \int_0^T \beta_t(\theta_1) dW_t^{\theta_1} \right) + \frac{1}{2} \int_0^T (\beta_t(\theta_2)^2 - \beta_t(\theta_1)^2) dt$$

under the measure $\mu_{T_1}^T$. By (4.10), we have

$$W_t^{\theta_2} = \int_0^t (\beta_s(\theta_1) - \beta_s(\theta_2)) ds + W_t^{\theta_1}. \quad (4.16)$$

Plugging (4.16) into (4.15), we obtain the formula (2.4). This completes the proof.

5. Local asymptotic structure of the likelihood ratio process

The aim of this section is to prove the second part of Theorem 2.6. Before proceeding to the proof, we recall the martingale central limit theorem.

Let ($\Omega, \mathcal{F}, P$) be a complete probability space and ($\mathcal{F}_t$)$_{t \in [0, T]}$ be a filtration. The class of ($\mathcal{F}_t$)-progressively measurable process $h$ satisfying

$$P \left\{ \int_0^T h_t^2 dt < \infty \right\} = 1$$
is denoted by $\mathcal{M}_T$. We assume that there is a $d_2$-dimensional Brownian motion $W = (W^1, \ldots, W^{d_2})$ on $(\Omega, \mathcal{F}, \mathbb{P})$, and the random processes

$$(h^{T,(i,j)}_t(\theta))_{i=1,\ldots,d_1,j=1,\ldots,d_2}$$

are in $\mathcal{M}_T$ for each $T > 0$ and $\theta \in \Theta$. We define

$$\mathcal{I}_T(\theta) = (\mathcal{I}_T^1(\theta), \ldots, \mathcal{I}_T^{d_1}(\theta))^\ast,$$

where

$$\mathcal{I}_T^i(\theta) = \sum_{j=1}^{d_2} \int_0^T h^{T,(i,j)}_t(\theta) dW^j_t.$$

The next result is taken from Kutoyants (2004).

**Theorem 5.1.** Suppose that there exists a (nonrandom) positive definite matrix $I(\theta) = (I_{i,j}(\theta))_{i,j=1,\ldots,d_1}$ such that the following convergence takes place:

$$\sum_{i=1}^{d_1} \int_0^T h^{T,(i,i)}_t(\theta)h^{T,(j,i)}_t(\theta) dt \to^p I_{i,j}(\theta).$$

Then it holds that

$$\mathcal{I}_T(\theta) \to^d \mathcal{N}_{d_1}(0, I(\theta)).$$

Here these limits are taken with respect to the measure $\mathbb{P}$.

5.1. **Proof of the second part of Theorem 2.6**

For each $u \in \mathbb{R}^m$, the likelihood ratio process $(d\mu^T_{\theta+\varphi_T(\theta)u}/d\mu^T_{\theta})$ is denoted by $Z^T_\theta(u)$. Let us define

$$R^T_t(\theta, u) = u^\ast \varphi_T(\theta) \int_0^1 (\partial_\theta \beta_t + \epsilon \varphi_T(\theta)u - \partial_\theta \beta_t(\theta)) d\epsilon,$$

then we have

$$\log Z^T_\theta(u) = u^\ast \varphi_T(\theta) \int_0^T \partial_\theta \beta_t(\theta) dW^\theta_t - \frac{1}{2} u^\ast I(\theta)u$$

$$- \frac{1}{2} u^\ast \left( \varphi_T(\theta) \int_0^T \partial_\theta \beta_t(\theta) (\partial_\theta \beta_t(\theta))^\ast dt \varphi_T(\theta) - I(\theta) \right) u$$
\[
\begin{align*}
+ & \int_0^T R_t^T(\theta, u) dW_t^\theta + \int_0^T u^* \varphi_T(\theta) \partial_\theta \beta_t(\theta) R_t^T(\theta, u) dt \\
- & \frac{1}{2} \int_0^T (R_t^T(\theta, u))^2 dt.
\end{align*}
\]

The first two terms are said to be the \textit{principal part}, and the last four terms the \textit{negligible part}.

First we identify the limit of the principal part. Let us rewrite the conditions of Theorem 5.1 in terms of our setting. We can choose the probability space \((\Omega^*, \mathcal{F}^*, \mathbb{P}^*)\) from Theorem 2.3 as underlying probability space. We consider the filtration generated by the fBM \(B\), which is denoted by \((\mathcal{F}_t^*)_{t \geq 0}\). A Brownian motion \(W\) in consideration is defined by \(W = W^\theta(X^\theta)\), and hence \(d_2 = 1\). The \(m\)-dimensional process

\[
(\partial_\theta \beta(\theta))(X^\theta) = ((\partial_{\theta_1} \beta(\theta))(X^\theta), \ldots, (\partial_{\theta_m} \beta(\theta))(X^\theta))^*
\]

corresponds to the random processes \((h^{T,(i,j)}(\theta))_{i=1, \ldots, d_1, j=1, \ldots, d_2}\) with \(d_1 = 1\) and \(d_2 = m\).

We set

\[
\kappa(i, \theta) = \begin{cases} 
-1/2 & \text{if } i = 1, \ldots, m_0(\theta) \\
-(1 - H) & \text{if } i = m_0(\theta) + 1, \ldots, m.
\end{cases}
\]

\textbf{Proposition 5.2.} Let \(\theta \in \Theta\). It holds that for any \(\epsilon > 0\)

\[
\mathbb{P}^* \left\{ \left| \int_0^T (\partial_\theta \beta_t(\theta))(X^\theta)(\partial_\theta \beta_t(\theta))(X^\theta) dt - I_{i,j}(\theta) \right| > \epsilon \right\} \rightarrow 0
\]

as \(T \rightarrow \infty\) for each \(i, j = 1, \ldots, m\). In particular, We have

\[
\varphi_T(\theta) \int_0^T \partial_\theta \beta_t(\theta) dW_t^\theta \rightarrow^d \mathcal{N}_m(0, I(\theta))
\]

as \(T \rightarrow \infty\). Here the limit is taken with respect to \(\mu_T^\theta\). Recall that the Fisher information matrix \(I(\theta) = (I_{i,j}(\theta))_{i,j=1,\ldots,m}\) is defined in Theorem 2.6.

\textbf{Proof.} We set

\[
I^T_{i,j}(\theta)(x) = T^{\kappa(i,\theta) + \kappa(j,\theta)} \int_0^T (\partial_\theta_i \beta_t(\theta))(x)(\partial_\theta_j \beta_t(\theta))(x) dt \quad (5.1)
\]

for \(x \in C[0, T]\).
Step 1. We approximate $I_{i,j}^T(\theta)(X^{\theta})$ by $I_{i,j}^T(\theta)(\bar{X}^{\theta})$ on $[0,T]$. For simplicity, we omit the restriction $|[0,T]$ in the following. Let $\Delta_t^{(i)}(\theta)$ denote $\partial_{\theta_i} \beta_t(\theta)(X^{\theta}) - \partial_{\theta_i} \beta_t(\theta)(\bar{X}^{\theta})$. Then we have

$$
I_{i,j}^T(\theta)(X^{\theta}) = I_{i,j}^T(\theta)(\bar{X}^{\theta}) + T^{\kappa(i,\theta) + \kappa(j,\theta)} \int_0^T \Delta_t^{(i)}(\theta)(\partial_{\theta_j} \beta_t(\theta))(\bar{X}^{\theta}) dt \\
+ T^{\kappa(i,\theta) + \kappa(j,\theta)} \int_0^T \Delta_t^{(j)}(\theta)(\partial_{\theta_i} \beta_t(\theta))(\bar{X}^{\theta}) dt \\
+ T^{\kappa(i,\theta) + \kappa(j,\theta)} \int_0^T \Delta_t^{(i)}(\theta) \Delta_t^{(j)}(\theta) dt.
$$

For $t > 0$, the quantity $\Delta_t^{(i)}(\theta)$ can be written as

$$
\Delta_t^{(i)}(\theta) = \int_0^t \sigma^{-1} H - 1/2 dt \int_0^1 ds \int_0^1 de \ (t - s)^{-1/2 - H} s^{1/2 - H} \\
\times (\partial_x \partial_{\theta_i} a) (X_s^{\theta} + \epsilon (X_s^{\theta} - \bar{X}_s^{\theta}))(X_s^{\theta} - \bar{X}_s^{\theta}).
$$

Since $|X_s^{\theta} - \bar{X}_s^{\theta}| \leq |X_0^{\theta} - \bar{X}_0^{\theta}| e^{-\alpha s}$ holds under Assumption 2.1 (for a proof, see Garrido-Atienza et al. (2009) or Neuenkirch and Tindel (2014)), we have

$$
|\Delta_t^{(i)}(\theta)| \leq ||\partial_x \partial_{\theta_i} a(\cdot, \theta)||_{\infty} |X_0^{\theta} - \bar{X}_0^{\theta}| t^{H - 1/2} \int_0^t ds \ (t - s)^{-1/2 - H} s^{1/2 - H} e^{-\alpha s} \\
= ||\partial_x \partial_{\theta_i} a(\cdot, \theta)||_{\infty} |X_0^{\theta} - \bar{X}_0^{\theta}| \int_0^t ds \ s^{-1/2 - H} \left(1 - \frac{s}{t}\right)^{1/2 - H} e^{-\alpha (t-s)} \\
\leq ||\partial_x \partial_{\theta_i} a(\cdot, \theta)||_{\infty} |X_0^{\theta} - \bar{X}_0^{\theta}| \int_0^t ds \ s^{-1/2 - H} e^{-\alpha (t-s)}.
$$

In particular, we obtain

$$
\mathbb{E}^* \int_0^T \Delta_t^{(i)}(\theta)^2 dt \leq C
$$

for some constant $C > 0$ that is independent of $T > 0$.

On the other hand, as we shall see in (5.3) below, it holds that

$$
\mathbb{E}^* \{I_{i,j}^T(\theta)(\bar{X}^{\theta})\} \rightarrow I_{i,j}(\theta)
$$

as $T \rightarrow \infty$ for all $i, j = 1, \ldots, m$. Hence it holds that

$$
\mathbb{P}^* \{|I_{i,j}^T(\theta)(X^{\theta}) - I_{i,j}^T(\theta)(\bar{X}^{\theta})|\} \rightarrow 0 \quad (5.2)
$$
as $T \to \infty$ for all $i, j = 1, \ldots, m$.

**Step 2.** Now we calculate the limit of $I_{i,j}^T(\theta)(\bar{X}^\theta)$ when $T \to \infty$. In the following, we show that the limit

$$\lim_{T \to \infty} E^* \left\{ \left| I_{i,j}^T(\theta)(\bar{X}^\theta) - I_{i,j}(\theta) \right| \right\} = 0$$

(5.3)

holds.

First we decompose $\partial_\theta \beta_i(\theta)(\bar{X}^\theta)$ into three terms:

$$\partial_\theta \beta_i(\theta) = \sigma^{-1} d_H^{-1} t^{H-1/2} \int_0^t (t - r)^{-H-1/2} t^{-H+1/2} (\partial_\theta a)(\bar{X}^\theta_r, \theta) \, dr$$

$$= \sigma^{-1} d_H^{-1} t^{H-1/2} \int_0^t (t - r)^{-H-1/2} t^{-H+1/2} H \{ (\partial_\theta a)(\bar{X}^\theta_r, \theta) - E^* \{ (\partial_\theta a)(\bar{X}^\theta_0, \theta) \} \} \, dr$$

$$+ \sigma^{-1} d_H^{-1} t^{H-1/2} \int_0^t (t - r)^{-H-1/2} t^{-H+1/2} H \{ (\partial_\theta a)(\bar{X}^\theta_r, \theta) - E^* \{ (\partial_\theta a)(\bar{X}^\theta_0, \theta) \} \} \, dr$$

$$= \sigma^{-1} d_H^{-1} B(-H + 1/2, -H + 3/2) t^{1/2-H} E^* \{ (\partial_\theta a)(\bar{X}^\theta_0, \theta) \}$$

$$+ \sigma^{-1} d_H^{-1} \int_0^t r^{-1/2-H} \{ (\partial_\theta a)(\bar{X}^\theta_{t-r}, \theta) - E^* \{ (\partial_\theta a)(\bar{X}^\theta_0, \theta) \} \} \, dr$$

$$+ \sigma^{-1} d_H^{-1} \int_0^t r^{-1/2-H} \{ (1 - r/t)^{1/2-H} - 1 \}$$

$$\times \{ (\partial_\theta a)(\bar{X}^\theta_{t-r}, \theta) - E^* \{ (\partial_\theta a)(\bar{X}^\theta_0, \theta) \} \} \, dr$$

$$=: \gamma_1^{1,i}(\theta) + \gamma_2^{2,i}(\theta) + \gamma_3^{3,i}(\theta).$$

The next lemma plays a crucial role to prove (5.3).

**Lemma 5.3.** We set

$$c_{i,j}(t) = E^* \{ (\partial_\theta a)(\bar{X}^\theta_t, \theta) - E^* \{ (\partial_\theta a)(\bar{X}^\theta_0, \theta) \} \}$$

for $t \geq 0$ and $i, j = 1, \ldots, d$.

1. We have

$$\int_0^\infty dr \int_0^\infty du \, r^{-H-1/2} u^{-H-1/2} |c_{i,j}(r-u)| < \infty$$

(5.4)

for all $i, j = 1, \ldots, m$. 
2. It holds that
\[
\lim_{T \to \infty} E^* \left\{ T^{-1} \int_0^T \gamma^{2i}_t(\theta) \gamma^{2j}_t(\theta) \, dt \right\} = \sigma^{-2} d_H^{-2} \int_0^\infty dr \int_0^\infty du \, r^{1-H/2} u^{-H/2} c_{i,j}(|r-u|)
\]
for all \( i, j = 1, \ldots, m \).

3. We obtain
\[
\lim_{T \to \infty} E^* \left\{ T^{-1} \int_0^T \gamma^{3i}_t(\theta)^2 \, dt \right\} = 0
\]
for all \( i = 1, \ldots, m \).

The proof of Lemma 5.3 requires a lengthy calculation, so that we postpone it until Section 4.2. In the following, we consider the following three cases:

(a) \( i = 1, \ldots, m_0(\theta) \) and \( j = 1, \ldots, m_0(\theta) \),

(b) \( i = 1, \ldots, m_0(\theta) \) and \( j = m_0(\theta) + 1, \ldots, m \) (or \( j = 1, \ldots, m_0(\theta) \) and \( i = m_0(\theta) + 1, \ldots, m \)), and

(c) \( i = m_0(\theta) + 1, \ldots, m \) and \( j = m_0(\theta) + 1, \ldots, m \).

**Case (a).** Let us assume that \( i = 1, \ldots, m_0(\theta) \) and \( j = 1, \ldots, m_0(\theta) \). Note that \( E^* \{(\partial_{\theta})_a(X^\theta_0, \theta) = 0 \) and \( \kappa(i, \theta) = \kappa(j, \theta) = -1/2 \) in this case, and so that \( I_{i,j}(\theta)(X^\theta) \) becomes

\[
I_{i,j}(\theta)(X^\theta) = T^{-1} \int_0^T (\gamma^{2i}_t(\theta) + \gamma^{3i}_t(\theta)) (\gamma^{2j}_t(\theta) + \gamma^{3j}_t(\theta)) \, dt.
\]

Hence

\[
E \left\{ \left| I_{i,j}(\theta)(X^\theta) - T^{-1} \int_0^T \gamma^{2i}_t(\theta) \gamma^{2j}_t(\theta) \, dt \right| \right\} \\
\leq \sum_{(p,q) \in \{2,3\}^2 \atop (p,q) \neq (2,2)} \left( T^{-1} E^* \int_0^T \gamma^{p,i}_t(\theta)^2 \, dt \right)^{1/2} \left( T^{-1} E^* \int_0^T \gamma^{q,j}_t(\theta)^2 \, dt \right)^{1/2}.
\]

Let us set \( J_{i,j}(\theta)^{(a),T} = T^{-1} \int_0^T \gamma^{2i}_t(\theta) \gamma^{2j}_t(\theta) \, dt \). By Lemma 5.3, the right hand side of (5.7) tends to 0 as \( T \to \infty \). Furthermore, we have

\[
E \left\{ \left| J_{i,j}(\theta)^{(a),T} - E^* \{ J_{i,j}(\theta)^{(a),T} \} \right|^2 \right\} \leq E^* \{ \| D J_{i,j}(\theta)^{(a),T} \|^2_H \}
\]
by the Poincaré inequality. The next lemma shows the law of large numbers for $J_{i,j}^{(a),T}(\theta)$.

**Lemma 5.4.** Suppose that $H \in (1/4,1/2)$ and $i,j = 1,\ldots,m_0(\theta)$. Then we have

$$
\mathbb{E}^*\{\|DJ_{i,j}^{(a),T}(\theta)\|_H^2\} \to 0
$$

as $n \to \infty$.

The proof of Lemma 5.4 is given in Section 4.2. On the other hand, $\lim_{T \to \infty} \mathbb{E}^*J_{i,j}^{(a),T}(\theta) = I_{i,j}(\theta)$ by (5.5). Therefore the limit (5.3) holds in this case.

**Case (b).** We assume $i = 1,\ldots,m_0(\theta)$ and $j = m_0(\theta) + 1,\ldots,m$. The case where $j = 1,\ldots,m_0(\theta)$ and $i = m_0(\theta) + 1,\ldots,m$ can be dealt with similarly. In this case, $\kappa(i,\theta) = -1/2$, $\kappa(j,\theta) = H - 1$, and

$$
I_{i,j}^{(b)}(\theta)(\bar{X}^\theta) = T^{H-3/2} \int_0^T (\gamma^2_{i,j}(\theta) + \gamma^3_{i,j}(\theta))(\gamma^1_{i,j}(\theta) + \gamma^2_{j}(\theta) + \gamma^3_{j}(\theta)) dt.
$$

Therefore we obtain

$$
\mathbb{E}\left\{\left|I_{i,j}^{(b)}(\theta)(\bar{X}^\theta) - T^{H-3/2} \int_0^T (\gamma^2_{i,j}(\theta) + \gamma^3_{i,j}(\theta))(\gamma^1_{i,j}(\theta) dt\right)\right\}
\leq T^{H-3/2} \sum_{(p,q) \in \{2,3\}^2} \left(\mathbb{E}^* \int_0^T \gamma^p_{i,j}(\theta)^2 dt\right)^{1/2} \left(\mathbb{E}^* \int_0^T \gamma^q_{i,j}(\theta)^2 dt\right)^{1/2}.
$$

(5.10)

Let us set $J_{i,j}^{(b),T}(\theta) = T^{H-3/2} \int_0^T (\gamma^2_{i,j}(\theta) + \gamma^3_{i,j}(\theta))(\gamma^1_{i,j}(\theta) dt$. As in the case (a), by Lemma 5.3, we have

$$
\mathbb{E}\left\{|I_{i,j}^{(b)}(\theta)(\bar{X}^\theta) - J_{i,j}^{(b),T}(\theta)|\right\} \to 0
$$

as $T \to \infty$. The Poincaré inequality gives

$$
\mathbb{E}\{|J_{i,j}^{(b),T}(\theta) - \mathbb{E}^*\{J_{i,j}^{(b),T}(\theta)\}|^2\} \leq \mathbb{E}^*\{|DJ_{i,j}^{(b),T}(\theta)\|_H^2\}.
$$

(5.11)

As in the case (a), the following lemma holds.

**Lemma 5.5.** Suppose that $i = 1,\ldots,m_0$ and $j = m_0(\theta) + 1,\ldots,m$. Then we have

$$
\mathbb{E}^*\{\|DJ_{i,j}^{(b),T}(\theta)\|_H^2\} \to 0
$$

as $n \to \infty$.
The proof of Lemma 5.5 is given in Section 4.2. The right hand side of (5.11) tends to 0 as \( T \to \infty \) thanks to Lemma 5.5. Since \( \mathbb{E}^* \{ J_{i,j}^{T(\theta)} \} = 0 \), we obtain (5.3).

**Case (c).** Let us consider the case where \( i = m_0(\theta) + 1, \ldots, m \) and \( j = m_0(\theta) + 1, \ldots, m \). In this case, we have \( \kappa(i, \theta) = \kappa(j, \theta) = H - 1 \) and

\[
I_{i,j}^{T(\theta)}(\bar{X}^\theta) = T^{2H-2} \int_0^T (\gamma_{i,t}^{1,i}(\theta) + \gamma_{i,t}^{2,i}(\theta) + \gamma_{i,t}^{3,i}(\theta))(\gamma_{t}^{1,j}(\theta) + \gamma_{t}^{2,j}(\theta) + \gamma_{t}^{3,j}(\theta)) \, dt.
\]

Note that \( I_{i,j}(\theta) = T^{2H-2} \int_0^T \gamma_{i,t}^{1,i}(\theta) \gamma_{t}^{1,j}(\theta) \, dt \). Therefore we can directly obtain

\[
\mathbb{E}^*\{|I_{i,j}^{T(\theta)}(\bar{X}^\theta) - I_{i,j}(\theta)|\} \leq T^{-2} t^{1-2H} \sum_{(p,q) \in \{1,2,3\}} \left( \mathbb{E}^* \int_0^T \gamma_{i,t}^{p,i}(\theta)^2 \, dt \right)^{1/2} \left( \mathbb{E}^* \int_0^T \gamma_{t}^{q,j}(\theta)^2 \, dt \right)^{1/2}
\]

in this case. Lemma 5.3 implies the right hand side of (5.13) converges to 0 as \( T \to \infty \). This completes the proof. ☐

Next we show that the negligible part is indeed negligible. We start with the following estimation for \( R_t^{T(\theta, u)} \).

**Lemma 5.6.** Let \( K \) be a compact subset of \( \mathbb{R}^m \). Then there exists a positive constant \( C = C(H, \sigma, \theta, K) > 0 \) depending only on \( \sigma, H, \theta \) and \( K \) such that

\[
\mathbb{E}^* \left\{ \sup_{u \in K \cap (\varphi_T(\theta)-1)\Theta} |R_t^{T(\theta, u)}(X^\theta)|^2 \right\} \leq CT^{-2} t^{1-2H}.
\]

**Proof.** Since

\[
\left| \partial_\theta \beta_t(\theta + \epsilon \varphi_T(\theta) u)(x) - \partial_\theta \beta_t(\theta)(x) \right|
\leq d_H^{-1} |\sigma|^{-1} t^{H-1/2} \int_0^t ds \ (t-s)^{-1/2-H} s^{1/2-H} \times |\partial_\theta a(x_s, \theta + \epsilon \varphi_T(\theta) u) - \partial_\theta a(x_s, \theta)|
\leq d_H^{-1} |\sigma|^{-1} C(\theta) \epsilon T^{-1/2} |u| \int_0^t ds \ (t-s)^{-1/2-H} s^{1/2-H} (1 + |x_s|^p),
\]

where \( H < 1 \).
holds by Assumption 2.1, we have

\[ |R^T_t(\theta, u)| \]
\[ \leq T^{-1/2}|u| \int_0^1 d\epsilon \, |\partial_\theta \beta_t(\theta + \epsilon \varphi_T(\theta) u) - \partial_\theta \beta_t(\theta)| \]
\[ \leq T^{-1}d_H^{-1}|\sigma|^{-1}C(\theta)|u|^2 t^{H-1/2} \int_0^t ds \, (t-s)^{-1/2-H} s^{1/2-H}(1 + |X^\theta_s|^p). \]

Therefore, we obtain

\[ E^* \left\{ \sup_{u \in K \cap (\varphi_T(\theta)^{-1}(\Theta - \theta))} |R^T_t(\theta, u)(X^\theta)|^2 \right\} \]
\[ \leq C(H, \sigma, \theta, K) T^{-2} t^{2H-1} \int_0^t ds \int_0^t dv \, (t-s)^{-1/2-H} \times (t-v)^{-1/2-H} s^{1/2-H} v^{1/2-H} E^* \{ (1 + |X^\theta_s|^p)(1 + |X^\theta_v|^p) \} \]
\[ \leq T^{-2} C(H, \sigma, \theta, K) t^{1-2H}. \]

Note that we used (2.1) in the last inequality.

Let us denote the four terms in the negligible part by \( N^T_i(\theta, u_T) \) (\( i = 1, 2, 3, 4 \)) in order.

**Proposition 5.7.** Let \( K \) be a compact subset of \( \mathbb{R}^m \). Then we have, for each \( i = 1, 2, 3, 4 \),

\[ |N^T_i(\theta, u_T)| \to^p 0 \quad (5.14) \]

as \( T \to \infty \) for any sequence \( u_T \in K \cap (\varphi_T(\theta)^{-1}(\Theta - \theta)) \).

**Proof.** For \( i = 1 \), the limit (5.14) follows from Proposition 5.2. The term \( N^T_2(\theta, u_T) \) can be bounded in \( L^2(\mathbb{P}^*) \):

\[ E^* \left\{ |N^T_2(\theta, u_T)|^2 \right\} = \int_0^T dt \, E^* \{ |R^T_t(\theta, u_T)|^2 \} \]
\[ \leq C(H, \sigma, \theta, K) T^{-2H}. \]

Therefore we have \( |N^T_2(\theta, u_T)| = o_p(1) \). For the term \( N^T_3(\theta, u_T) \), we have

\[ |N^T_3(\theta, u_T)| \leq (u^* I^T(\theta) u)^{1/2} \left( \int_0^T dt \, |R^T_t(\theta, u_T)|^2 \right)^{1/2} \quad (5.15) \]

by Hölder’s inequality. Here \( I^T(\theta) \) denotes the matrix \( (I^T_{i,j}(\theta))_{i,j=1,\ldots,m} \) (see (5.1)). We have \( |N^T_3(\theta, u_T)| = o_p(1) \) since \( u^* I^T(\theta) u = O_p(1) \) holds by Proposition 5.2. Finally, \( |N^T_4(\theta, u_T)| = o_p(1) \) is obvious. □
By Propositions 4.2 and 4.7, we complete the proof of the second part of Theorem 2.6.

5.2. Proof of lemmas in Section 5.1

First we prove Lemma 5.3. Finding a good upper bound of $c_{i,j}(t)$ is a key ingredient for the proof of Lemma 5.3.

**Lemma 5.8.** There exists a constant $C > 0$ that is independent of $t$ such that

$$|c_{i,j}(t)| \leq C t^{H-3/2}$$

for all $t \geq 1$ and $i, j = 1, \ldots, d$.

**Proof.** We have

$$c_{i,j}(t) = \mathbb{E}^* \{ (\partial_{\theta_i} a(X_t^\theta, \theta) - \mathbb{E}^* \{ \partial_{\theta_i} a(X_0^\theta, \theta) \}) (\partial_{\theta_j} a(X_0^\theta, \theta) \}$$

$$= \mathbb{E}^* \{ \partial_x \partial_{\theta_i} a(X_t^\theta, \theta \{D_{\bar{X}_t^\theta} - DL^{-1} \partial_{\theta_j} a(X_0^\theta, \theta) \} \}.$$

Let $(P_t)_{t \geq 0}$ be the Ornstein-Uhlenbeck semigroup. We can represent $-DL^{-1} \partial_{\theta_j} a(X_0^\theta, \theta)$ as

$$-DL^{-1} \partial_{\theta_j} a(X_0^\theta, \theta) = \int_0^\infty dt \ e^{-t} P_t(D\partial_{\theta_j} a(X_0^\theta, \theta))$$

$$= \int_0^\infty dt \ e^{-t} P_t(\partial_x \partial_{\theta_j} a(X_t^\theta, \theta)D_{\bar{X}_t^\theta})$$

$$= \int_0^\infty dt \ e^{-t} P_t((\partial_{\theta_j} a(X_0^\theta, \theta)\Psi(0, \cdot))$$

(recall that $\Psi(t, r) = \sigma 1_{(-\infty, t]}(r) e^{\int_t^r (\partial_x a)(X_u^\theta, \theta) du}$). We set

$$h_s(\omega) = \int_0^\infty dt \ e^{-t} P_t((\partial_{\theta_j} a(X_0^\theta, \theta)\Psi(0, \cdot))$$

$$= \int_0^\infty dt \ e^{-t} P_t((\partial_{\theta_j} a(X_0^\theta, \theta)\Psi(0, \cdot))$$

for each $s \in \mathbb{R}$ and $\omega \in \Omega^*$. Then we have

$$|h_s(\omega)| \leq \| (\partial_{\theta_j} a)(\cdot, \theta) \|_\infty e^{\alpha s} 1_{(-\infty, 0]}(s).$$

By (3.7), we obtain

$$c_{i,j}(t) = \mathbb{E}^* \{ (\partial_{\theta_j} a(X_t^\theta, \theta)\{D_{\bar{X}_t^\theta}, h\} \}$$
Hence

\[ \text{for the (Marchaud) fractional derivatives, we have} \]

Let us calculate an upper bound for \( \langle D \bar{X}^\theta_0, h \rangle_{\mathcal{H}} \). The term \( c_{i,j}^{(1)}(t) \) can be easily bounded:

\[ |c_{i,j}^{(1)}(t)| \leq \| (\partial_x \partial_{\theta_j} a)(\cdot, \theta) \|_{\infty} \| E^* \{ \langle D \bar{X}^\theta_0, h \rangle_{\mathcal{H}} \} |e^{-at}. \]

The term \( c_{i,j}^{(2)}(t) \) is bounded as

\[ |c_{i,j}^{(2)}(t)| \leq \| (\partial_x \partial_{\theta_j} a)(\cdot, \theta) \|_{\infty} E^* \{ |\langle \Phi(t, \cdot), h \rangle_{\mathcal{H}}| \}. \]

Let us calculate an upper bound for \( |\langle \Phi(t, \cdot), h \rangle_{\mathcal{H}}| \). By integration by parts for the (Marchaud) fractional derivatives, we have

\[ \langle \Phi(t, \cdot), h \rangle_{\mathcal{H}} = \int_{\mathbb{R}} D_+^{1/2-H} D_-^{1/2-H} \langle \Phi(t, \cdot) \rangle_s h_s \, ds. \]

The quantity \( D_+^{1/2-H} D_-^{1/2-H} \langle \Phi(t, \cdot) \rangle_s \) can be calculated as follows. Since \( h_s = 0 \) for \( s > 0 \), we can assume \( s < 0 \) without loss of generality. We have

\[
\begin{align*}
D_+^{1/2-H} D_-^{1/2-H} \langle \Phi(t, \cdot) \rangle_s &= \int_0^\infty d\xi \ \xi^{-3/2} \big( D_-^{1/2-H} \Phi(t, \cdot) \big)_s - D_-^{1/2-H} \Phi(t, \cdot)_{s-\xi} \\
&= \int_0^\infty d\xi \int_0^\infty d\eta \ \xi^{-3/2} \eta^{-3/2} (\Phi(t, s) - \Phi(t, s + \eta)) \\
&\quad - \Phi(t, s - \xi) + \Phi(t, s - \xi + \eta)) \\
&= \int_0^\infty d\xi \int_0^\infty d\eta \ \xi^{-3/2} \eta^{H-3/2} (\Phi(t, s - \xi) - \Phi(t, \eta)) \\
&= \int_s^\infty d\eta \int_{-\infty}^0 d\xi \ \xi^{-H-3/2} \eta^{-s + \eta} (\Phi(t, s - \xi + \eta)) - \Phi(t, s + \eta)) \\
&= \int_s^\infty d\eta \int_{-\infty}^0 d\xi \ (s - \eta)^{-H-3/2} \eta^{H-3/2} (\Phi(t, s - \xi) - \Phi(t, \eta)).
\end{align*}
\]

Hence \( |\langle \Phi(t, \cdot), h \rangle_{\mathcal{H}}| \) can be bounded as

\[
|\langle \Phi(t, \cdot), h \rangle_{\mathcal{H}}| \leq \| (\partial_x \partial_{\theta_j} a)(\cdot, \theta) \|_{\infty} \int_{-\infty}^0 ds \int_s^\infty d\eta \int_{-\infty}^0 d\xi \ (s - \eta)^{-H-3/2} \eta^{H-3/2} \]
A calculation of the term $I_1$ requires

An upper bound for $I_1(t)$. We decompose $I_1(t)$ as

Let us calculate an upper bound of each term: we have

A calculation of the term $I_{1,2}(t)$ is more involved than the former two terms. We further decompose $I_{1,2}(t)$ as
An upper bound for $I$ We have

$$I \leq \left( \int_{-\infty}^{t} ds \int_{0}^{\eta} d\xi \right) (\alpha) \left( \int_{-\infty}^{t} ds \right) e^{t\alpha s}$$

$$=: J_{1,2,1}(t) + J_{1,2,2}(t) + J_{1,2,3}(t).$$

We have

$$J_{1,2,1}(t) \leq |\alpha^{-1}\sigma| \int_{-\infty}^{-1} ds \int_{0}^{1} dt \int_{\eta-1}^{\eta} d\xi \left( \int_{-\infty}^{t} ds \right) e^{t\alpha s}$$

$$\leq (H + 1/2)^{-1} |\alpha^{-1}\sigma| \left( \int_{-\infty}^{t} ds \right) e^{t\alpha s} \left( \int_{0}^{1} dt \int_{\eta-1}^{\eta} \right) e^{-t\alpha s},$$

$$J_{1,2,2}(t) \leq |\alpha^{-1}\sigma| \int_{-\infty}^{-1} ds \int_{1}^{t} dt \int_{\eta-1}^{\eta} d\xi \left( \int_{-\infty}^{t} ds \right) e^{t\alpha s}$$

$$\leq (H + 1/2)^{-1} |\alpha^{-1}\sigma| \left( \int_{-\infty}^{t} ds \right) e^{t\alpha s} \left( \int_{0}^{1} dt \int_{\eta-1}^{\eta} \right) e^{-t\alpha s},$$

and

$$J_{1,2,3}(t) \leq \int_{-\infty}^{-1} ds \int_{1}^{t} dt \int_{\eta-1}^{\eta} d\xi \left( \int_{-\infty}^{t} ds \right) e^{t\alpha s}$$

$$\times \left( e^{-t\alpha s} + e^{-\alpha(t-\xi)} \right) e^{-t\alpha s}$$

$$\leq \int_{-\infty}^{-1} ds \int_{1}^{t} dt \int_{\eta-1}^{\eta} d\xi \left( \int_{-\infty}^{t} ds \right) e^{t\alpha s}$$

$$\times \left( e^{-t\alpha s} + e^{-\alpha(t-\xi)} + (1/2 - H)^{-1} \int_{-\infty}^{t} dt \int_{\eta-1}^{\eta} \right) e^{-t\alpha s},$$

$$J_{2}(t) \text{ as }$$

An upper bound for $J_{2}(t)$. We decompose $J_{2}(t)$ as

$$J_{2}(t)$$
\[
J_{2.1}(t) = \left( \int_{-1}^{0} ds \int_{0}^{t} dn \int_{-\infty}^{0} d\xi + \int_{-1}^{0} ds \int_{0}^{t} dn \int_{n}^{\eta} d\xi + \int_{-1}^{0} ds \int_{t}^{\infty} dn \int_{0}^{\eta} d\xi \right) \\
\times (\eta - \xi)^{H-3/2}(-s + \eta)^{H-3/2} |\Phi(t, \xi) - \Phi(t, \eta)| e^{\alpha s} \\
= J_{2.1}(t) + J_{2.2}(t) + J_{2.3}(t).
\]

We have

\[
J_{2.1}(t) \leq (1/2 - H)^{-1} e^{\alpha} \int_{-1}^{0} ds \int_{0}^{t} dn \int_{0}^{\eta} d\xi (\eta - \xi)^{H-1/2} (-s + \eta)^{H-3/2} e^{-\alpha(t-\eta)}
\]

\[
\leq (1/2 - H)^{-1} e^{\alpha} e^{-t} \left( \int_{1}^{t} dn \eta^{2H-2} e^{\alpha n} \right)
\]

\[
+ (1/2 - H)^{-2} e^{2\alpha} e^{-t} \left( \int_{0}^{1} dn \eta^{2H-1} \right),
\]

\[
J_{2.2}(t)
\]

\[
\leq e^{\alpha} \left( \int_{-1}^{0} ds \int_{0}^{t} dn \int_{0}^{\eta} d\xi + \int_{-1}^{0} ds \int_{0}^{t} dn \int_{n}^{\eta} d\xi + \int_{-1}^{0} ds \int_{t}^{\infty} dn \int_{0}^{\eta} d\xi \right) \\
\times (\eta - \xi)^{H-3/2} (-s + \eta)^{H-3/2} (\Phi(t, \eta) - \Phi(t, \xi))
\]

\[
\leq \|\partial_{x} a(\cdot, \theta)\| e^{\alpha} \int_{-1}^{0} ds \int_{0}^{t} dn \int_{0}^{\eta} d\xi (\eta - \xi)^{H-1/2} (-s + \eta)^{H-3/2} e^{-\alpha(t-\eta)}
\]

\[
+ e^{\alpha} \int_{-1}^{0} ds \int_{0}^{t} dn (-s + \eta)^{H-3/2} e^{-\alpha(t-\eta)}
\]

\[
+ ||\partial_{x} a(\cdot, \theta)\|| e^{\alpha} \int_{-1}^{0} ds \int_{0}^{t} dn \int_{n}^{\eta} d\xi (\eta - \xi)^{H-1/2} (-s + \eta)^{H-3/2} e^{-\alpha(t-\eta)}
\]

\[
\leq ||\partial_{x} a(\cdot, \theta)\|| e^{2\alpha} e^{-t} \int_{-1}^{0} ds \int_{0}^{t} dn \int_{0}^{\eta} d\xi (\eta - \xi)^{H-1/2} (-s + \eta)^{H-3/2}
\]

\[
+ e^{\alpha} e^{-\alpha t} \int_{1}^{t} dn \eta^{H-3/2} e^{\alpha n}
\]

\[
+ ||\partial_{x} a(\cdot, \theta)\|| e^{\alpha} (H + 1/2)^{-1} e^{\alpha} \int_{1}^{t} dn \eta^{H-3/2} e^{\eta}
\]

and

\[
J_{2.3}(t) \leq e^{\alpha} \int_{-1}^{0} ds \int_{t}^{\infty} dn \int_{0}^{\xi} d\xi (\eta - \xi)^{H-3/2} (-s + \eta)^{H-3/2} e^{-\alpha(t-\xi)}
\]

\[
\leq e^{\alpha} t^{H-3/2} \int_{0}^{t} d\xi e^{-\alpha(t-\xi)} \int_{t}^{\infty} dn (\eta - \xi)^{H-3/2}
\]
\[ \leq (1/2 - H)^{-1} e^{\alpha t H - 3/2} \int_0^t d\xi e^{-\alpha \xi H - 1/2}. \]

The proof is done. \( \square \)

**Proof of Lemma 5.3.** (1) We have

\[
\int_0^\infty dr \int_0^\infty du \ r^{-H-1/2} u^{-H-1/2} |c(|r - u|)|
= 2 \int_0^\infty du \int_0^u dr \ r^{-H-1/2} u^{-H-1/2} |c_{i,j}(u - r)|
= 2 \int_0^\infty du \int_0^u dr \ (u - r)^{-H-1/2} u^{-H-1/2} |c_{i,j}(r)|
= 2 \int_0^\infty dr \ |c_{i,j}(r)| \int_r^\infty du \ (u - r)^{-H-1/2} u^{-H-1/2}
= 2 \int_0^\infty dr \ |c_{i,j}(r)| r^{-2H} \int_1^\infty du \ (u - 1)^{-H-1/2} u^{-H-1/2}.
\]

The integral \( \int_0^\infty dr \ |c_{i,j}(r)| r^{-2H} \) converges by Lemma 5.8.

(2) By the change of variable, we obtain

\[
\mathbb{E}^* \left\{ T^{-1} \int_0^T \gamma_{t,1}^2(\theta) \gamma_{t,2}^2(\theta) dt \right\}
= \sigma^{-2} d_H^{-2T-1} \int_0^T dt \int_0^t dr \int_0^t du \ r^{-H-1/2} u^{-H-1/2} c_{i,j}(|r - u|)
= \sigma^{-2} d_H^{-2} \int_0^1 dt \int_0^t dr \int_0^t du \ r^{-H-1/2} u^{-H-1/2} c_{i,j}(|r - u|).
\]

By (5.4) and the dominated convergence theorem, we obtain (5.5).

(3) As in the proof of (5.5), we have

\[
\mathbb{E}^* \left\{ T^{-1} \int_0^T \gamma_{t,1}^3(\theta)^2 dt \right\}
= \sigma^{-2} d_H^{-2} \int_0^1 dt \int_0^t dr \int_0^t du \ r^{-H-1/2} u^{-H-1/2} \left\{ 1 - \left( 1 - \frac{r}{tT} \right)^{1/2-H} \right\}
\times \left\{ 1 - \left( 1 - \frac{r}{tT} \right)^{1/2-H} \right\} c_{i,j}(|r - u|).
\]

Again by (5.4) and the dominated convergence theorem, (5.6) follows. \( \square \)

Now we turn to prove Lemmas 5.4 and 5.5. The proofs are by straightforward calculation.
Proof of Lemma 5.4. By a straightforward calculation, we obtain

\begin{align*}
\|D_{1, j}^{(a), T}(\theta)\|_{\mathcal{H}}^2
&= \sigma^{-4} d_{\mathcal{H}^T}^{-4} T^{-2} \int_0^T dt_1 \int_0^T dt_2 \int_0^{t_1} dr_1 \int_0^{t_1} du_1 \int_0^{t_2} dr_2 \int_0^{t_2} du_2 \{ \\
&\quad (\partial_x \partial_t a)(\bar{X}_{r_1}^\theta, \theta)(\partial_t a)(\bar{X}_{u_1}^\theta, \theta)(\partial_x \partial_t a)(\bar{X}_{r_2}^\theta, \theta)(\partial_t a)(\bar{X}_{u_2}^\theta, \theta) \\
&\quad \times \langle D\bar{X}_{r_1}^\theta, D\bar{X}_{r_2}^\theta \rangle_{\mathcal{H}} \\
&\quad + (\partial_x \partial_t a)(\bar{X}_{r_1}^\theta, \theta)(\partial_t a)(\bar{X}_{u_1}^\theta, \theta)(\partial_x \partial_t a)(\bar{X}_{r_2}^\theta, \theta)(\partial_t a)(\bar{X}_{u_2}^\theta, \theta) \\
&\quad \times \langle D\bar{X}_{u_1}^\theta, D\bar{X}_{u_2}^\theta \rangle_{\mathcal{H}} \\
&\quad + (\partial_x \partial_t a)(\bar{X}_{u_1}^\theta, \theta)(\partial_t a)(\bar{X}_{r_1}^\theta, \theta)(\partial_x \partial_t a)(\bar{X}_{u_2}^\theta, \theta)(\partial_t a)(\bar{X}_{r_2}^\theta, \theta) \\
&\quad \times \langle D\bar{X}_{u_2}^\theta, D\bar{X}_{u_2}^\theta \rangle_{\mathcal{H}} \\
&\quad \times (t_1 - r_1)^{-H - 1/2}(t_1 - u_1)^{-H - 1/2}(t_2 - r_2)^{-H - 1/2}(t_2 - u_2)^{-H - 1/2}.
\end{align*}

Let us set \( \tilde{c}(t) = \mathbb{E}^\ast\{\|\langle D\bar{X}_t^\theta, D\bar{X}_0^\theta \rangle_{\mathcal{H}} \|_{\mathcal{H}}^2\}^{1/2} \). From (3.7), (3.13) and the proof of Lemma 5.8, there exists a positive constant \( C \) that is independent of \( t \) such that the inequality

\[ \tilde{c}(t) \leq Ct^{-3/2} \]

holds almost surely for each \( t \geq 1 \). Hence it holds that

\begin{align*}
\mathbb{E}^\ast\{\|D_{1, j}^{(a), T}(\theta)\|_{\mathcal{H}}^2\}
&\leq T^{-2} \int_0^T dt_1 \int_0^T dt_2 \int_0^{t_1} dr_1 \int_0^{t_1} du_1 \int_0^{t_2} dr_2 \int_0^{t_2} du_2 \{ \\
&\quad \tilde{c}(|r_1 - r_2|) + \tilde{c}(|r_1 - u_2|) + \tilde{c}(|u_1 - r_2|) + \tilde{c}(|u_1 - u_2|) \\
&\quad \times (t_1 - r_1)^{-H - 1/2}(t_1 - u_1)^{-H - 1/2}(t_2 - r_2)^{-H - 1/2}(t_2 - u_2)^{-H - 1/2} \\
&= 4T^{-2} \int_0^T dt_1 \int_0^T dt_2 \int_0^{t_1} dr_1 \int_0^{t_1} du_1 \int_0^{t_2} dr_2 \int_0^{t_2} du_2 \tilde{c}(|r_1 - r_2|) \\
&\quad \times (t_1 - r_1)^{-H - 1/2}(t_1 - u_1)^{-H - 1/2}(t_2 - r_2)^{-H - 1/2}(t_2 - u_2)^{-H - 1/2} \\
&= 4(1/2 - H)^{-2} T^{-2} \int_0^T dt_1 \int_0^T dt_2 \int_0^{t_1} dr_1 \int_0^{t_1} du_1 \int_0^{t_2} dr_2 \int_0^{t_2} du_2 \tilde{c}(|r_1 - r_2|) \\
&\quad \times (t_1 - r_1)^{-H - 1/2}(t_1 - u_1)^{-H - 1/2}(t_2 - r_2)^{-H - 1/2}(t_2 - u_2)^{-H - 1/2}.
\end{align*}
Since we assume

As in the proof of Lemma 5.4, we have

For notational simplicity, we set

Let us denote the last three terms by

It is clear that the term

Finally, the term

Let us denote the last three terms by 

It is clear that the term 

The term 

can be bounded as

Finally, the term 

is bounded as

Since we assume 

we obtain the limit (5.9)

Proof of Lemma 5.5. For notational simplicity, we set

As in the proof of Lemma 5.4, we have

\[ \|D_i^{(b),T}(\theta)\|_H^2 \]
Therefore we obtain

\[ E^*\{||D_j D_k (b)_{ij,j}D (\theta)\||_H^2 \} \]

\[ \leq \frac{C(\sigma, H, \theta)^2}{T^{3-2H}} \int_0^T dt_1 \int_0^T dt_2 \int_0^{t_1} dr_1 \int_0^{t_2} dr_2 (t_1 - r_1)^{-H-1/2} r_1^{1/2-H} \]

\[ \times (t_2 - r_2)^{-H-1/2} r_2^{1/2-H} c(|r_1 - r_2|) \]

\[ = \left( \frac{1/2 - H)^2 C(\sigma, H, \theta)^2}{T^{3-2H}} \right) \int_0^T dr_1 \int_0^T dr_2 (T - r_1)^{1/2-H} (T - r_2)^{1/2-H} \]

\[ \times r_1^{1/2-H} r_2^{1/2-H} c(|r_1 - r_2|) \]

\[ \leq \left( \frac{1/2 - H)^2 C(\sigma, H, \theta)^2}{T^2} \right) \int_0^T dr_1 \int_0^T dr_2 r_1^{1/2-H} r_2^{1/2-H} c(|r_1 - r_2|) \]

\[ \leq o(1) + \frac{2(1/2 - H)^2 C(\sigma, H, \theta)^2}{T^2} \int_1^T dr_2 \int_1^{r_2} dr_1 (r_2 - r_1)^{1/2-H} \]

\[ \times r_2^{1/2-H} r_1^{-3/2} \]

\[ \leq o(1) + \frac{2(1/2 - H)^2 C(\sigma, H, \theta)^2}{T^2} (2 - 2H)^{-1} T^{2-2H}. \]

This completes the proof.

References

E. Alós, O. Mazet, and D. Nualart. Stochastic calculus with respect to gaussian processes. *Annals of probability*, 29(2):766–801, 2001.

B. Bercu, L. Coutin, and N. Savy. Sharp large deviations for the fractional ornstein–uhlenbeck process. *Theory of Probability & Its Applications*, 55(4):575–610, 2011.

A. Brouste and M. Kleptsyna. Asymptotic properties of mle for partially observed fractional diffusion system. *Statistical inference for stochastic processes*, 13(1):1–13, 2010.

P. Cheridito and D. Nualart. Stochastic integral of divergence type with respect to fractional brownian motion with hurst parameter $H \in (0, \frac{1}{2})$. *Stochastic Processes and their Applications*, 41(6):1049–1081, 2005.

S. Cohen and F. Panloup. Approximation of stationary solutions of gaussian driven stochastic differential equations. *Stochastic Processes and their Applications*, 121(12):2776–2801, 2011.
L. Decreusefond and A. S. Üstünel. Stochastic analysis of the fractional brownian motion. *Potential analysis, 10*(2):177–214, 1999.
M. J. Garrido-Atienza, P. E. Kloeden, and A. Neuenkirch. Discretization of stationary solutions of stochastic systems driven by fractional brownian motion. *Applied Mathematics and Optimization, 60*(2):151–172, 2009.
B. Gess, W. Liu, and M. Röckner. Random attractors for a class of stochastic partial differential equations driven by general additive noise. *Journal of Differential Equations, 251*(4-5):1225–1253, 2011.
M. Hairer et al. Ergodicity of stochastic differential equations driven by fractional brownian motion. *The Annals of Probability, 33*(2):703–758, 2005.
Y. Hu. *Integral Transformations and Anticipative Calculus for Fractional Brownian Motions.* Number 825. American Mathematical Soc., 2005.
Y. Hu and D. Nualart. Parameter estimation for fractional ornstein–uhlenbeck processes. *Statistics & Probability Letters, 80*(11-12):1030–1038, 2010.
Y. Hu, D. Nualart, and H. Zhou. Parameter estimation for fractional ornstein–uhlenbeck processes of general hurst parameter. *Statistical Inference for Stochastic Processes, pages* 1–32, 2017.
Y. Hu, D. Nualart, and H. Zhou. Drift parameter estimation for nonlinear stochastic differential equations driven by fractional brownian motion. *arXiv preprint arXiv:1803.01032, 2018.*
I. A. Ibragimov and R. Z. Has’Minskiĭ. *Statistical Estimation: Asymptotic Theory, volume* 16. Springer Science & Business Media, 1981.
M. Kleptsyna and A. Le Breton. Statistical analysis of the fractional ornstein–uhlenbeck type process. *Statistical Inference for stochastic processes, 5*(3):229–248, 2002.
K. Kubilius, Y. Mishura, and K. Ralchenko. *Parameter Estimation in Fractional Diffusion Models, volume* 8 of *Bocconi & Springer Series.* Springer International Publishing, 2017.
Y. A. Kutoyants. *Statistical Inference for Ergodic Diffusion Processes.* Springer Science & Business Media, 2004.
Y. Liu, E. Nualart, and S. Tindel. Lan property for stochastic differential equations with additive fractional noise and continuous time observation. *arXiv preprint arXiv:1509.00003, 2015.*
Y. Mishura. *Stochastic Calculus for Fractional Brownian Motion and Related Processes, volume* 1929. Springer Science & Business Media, 2008.
A. Neuenkirch and S. Tindel. A least square-type procedure for parameter estimation in stochastic differential equations with additive fractional noise. *Statistical Inference for Stochastic Processes, 17*(1):99–120, 2014.
D. Nualart. *The Malliavin Calculus and Related Topics*, volume 1995. Springer, 2006.

D. Nualart and Y. Ouknine. Regularization of differential equations by fractional noise. *Stochastic Processes and their Applications*, 102(1):103–116, 2002.

V. Pipiras and M. S. Taqqu. Integration questions related to fractional brownian motion. *Probability theory and related fields*, 118(2):251–291, 2000.

B. P. Rao. *Statistical Inference for Fractional Diffusion Processes*. John Wiley & Sons, 2011.

S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional Integrals and Derivatives: Theory and Applications*. Switzerland: Gordon and Breach Science Publishers, 1993.

C. A. Tudor and F. G. Viens. Statistical aspects of the fractional stochastic calculus. *The Annals of Statistics*, 35(3):1183–1212, 2007.

N. Yoshida. Polynomial type large deviation inequalities and quasi-likelihood analysis for stochastic differential equations. *Annals of the Institute of Statistical Mathematics*, 63(3):431–479, 2011.