REPRODUCING KERNELS AND CHOICES OF ASSOCIATED FEATURE SPACES, IN THE FORM OF $L^2$-SPACES

PALLE JORGENSEN AND FENG TIAN

Abstract. Motivated by applications to the study of stochastic processes, we introduce a new analysis of positive definite kernels $K$, their reproducing kernel Hilbert spaces (RKHS), and an associated family of feature spaces that may be chosen in the form $L^2(\mu)$; and we study the question of which measures $\mu$ are right for a particular kernel $K$. The answer to this depends on the particular application at hand. Such applications are the focus of the separate sections in the paper.

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1. Introduction

The use of reproducing kernels and their reproducing kernel Hilbert spaces (RKHSs) was initially motivated by problems in classical analysis, and it was put into an especially attractive and useful form by Aronszajn in the 1950ties. Since then the applications of kernel theory has greatly expanded, both in pure and applied mathematics. An application of more recent vintage is machine learning. The number of applied areas include use of RKHSs in the study of stochastic processes, especially as a tool in Ito calculus; and in machine learning (ML). The last two are related, and they are the focus of our present paper. Dictated by a number of practical applications of the theory of ML, starting with a positive definite (p.d.)
kernel $K$, it has proved useful to study both the associated RKHS itself, as well as
a variety of choices of feature spaces (for details, see Remark 5.2 inside the paper);
and the interplay between them.

Now motivated by related applications to the study of stochastic processes, it is
of special significance to focus on the cases when the family of feature spaces may
be chosen in the form $L^2(\mu)$; but this then raises the question of which measures
$\mu$ are right for a particular kernel $K$, and its associated RKHS. The answer to this
depends on the particular application at hand. Such applications are the focus of
the separate sections below inside the paper.

In our study of RKHSs and choices of feature spaces, we have focused on those
of especial relevance to analysis of Gaussian calculus, but there are many others,
for example, functional and harmonic analysis, boundary value problems, PDE,
geometry and geometric analysis, operator algebras/theory, the theory of unitary
representations, mathematical physics, and the study of fractals and fractal measures.
Even this list is not exhaustive. Nonetheless, we have narrowed our scope,
and our choice of applications, for the present paper. The reader will be able to
follow up on the various other directions, not covered here, with the use of our cited
references, see especially our discussion of the literature below.

Discussion of the literature. The theory of RKHS and their applications
is vast, and below we only make a selection. Readers will be able to find more
cited there. As for the general theory of RKHS in the pointwise category, we find useful [ABDdS93, AD92, AD93, LMP09, PR16]. The applications include
fractals (see e.g., [AJSV13, Aro43]); probability theory [Sal16, MSF+16, HE15,
Jr68, EMESO17, PVPK17]; and learning theory [SZ05, CS07, CL10, XLT15,
Ste15, KDP+16, MPWZ16, BsdB16, GR17, CZ07].

2. Reproducing kernels

The present setting begins with a fixed positive definite (p.d.) kernel $K$, i.e., a
function $K : S \times S \rightarrow \mathbb{R}$ where $S$ is a set, and satisfying
\[
\sum_i \sum_j \alpha_i \alpha_j K(s_i, s_j) \geq 0
\]  
(2.1)
for all choices of $\{\alpha_i\}_{i=1}^n$, $\{s_i\}_{i=1}^n$, $\alpha_i \in \mathbb{R}$, $s_i \in S$, and $n \in \mathbb{N}$.

Remark 2.1. Even though we shall state our definitions and results in the special
case of real valued functions, the complex case will result from our present setting
with only minor modifications. But in order to minimize technical points, we have
restricted the present discussion to the real case.

The two more general settings are as follows: (i) complex; and (ii) operator
valued.

(i) There the definition is as in (2.1), but now $K : S \times S \rightarrow \mathbb{C}$, and the p.d.
assumption is instead:
\[
\sum_i \sum_j \alpha_i \bar{\alpha}_j K(s_i, s_j) \geq 0
\] for all choices of $\{\alpha_i\}$ and $\{s_i\}$, $\alpha_i \in \mathbb{C}$, $s_i \in S$, $1 \leq i \leq n$.

(ii) Let $H$ be a complex Hilbert space, and let $\mathcal{B}(H)$ = the algebra of all
bounded linear operators in $H$, i.e., $H \rightarrow H$. In this case, our setting for
the kernel \( K \) is: \( K : S \times S \rightarrow \mathcal{B}(H) \), and now we assume instead that, for all \( s_i, h_i \), with \( s_i \in S \), \( h_i \in H \), and \( 1 \leq i \leq n \); and all \( n \in \mathbb{N} \), we have:

\[
\sum_i \sum_j \langle K(s_i, s_j) h_i, h_j \rangle_H \geq 0.
\]

**Definition 2.2.** Given a positive definite (p.d.) kernel \( K \) on \( S \), we shall consider pairs \( (F, H) \) where \( H \) is a Hilbert space, and \( F : S \rightarrow H \) is a function satisfying

\[
(F(s), F(t))_H = K(s, t), \quad \forall s, t \in S.
\]

If \( (F, H) \) satisfies this, we say that \( H \) is a feature space, or a feature Hilbert space.

**Remark 2.3.** In a general setup, reproducing kernel Hilbert spaces (RKHSs) were pioneered by Aronszajn in the 1950s [Aro43, Aro50]; and subsequently they have been used in a host of applications. The key idea of Aronszajn is that a RKHS is a Hilbert space \( \mathcal{H}(K) \) of functions \( f \) on a set such that the values \( f(x) \) are “reproduced” from \( f \) and a vector \( K_x \) in \( \mathcal{H}(K) \), in such a way that the inner product \( \langle K_x, K_y \rangle =: K(x, y) \) is a positive definite kernel.

By a theorem of Kolmogorov, every Hilbert space may be realized as a (Gaussian) reproducing kernel Hilbert space (RKHS), see e.g., [PS75, IM65], and the details below.

Let \( (\Omega, \mathcal{C}, \mathbb{P}) \) be a probability space. We will be interested in centered Gaussian processes \( X_s \) \( s \in S \) (see e.g., [Kak16, KR60]), indexed by \( S \), satisfying

(i) \( X_s \) is Gaussian w.r.t. a probability space \( (\Omega, \mathcal{C}, \mathbb{P}) \),

(ii) \( X_s \in L^2(\Omega, \mathbb{P}) \), and

\[
\begin{align*}
\mathbb{E}(X_s) &= 0, \quad (2.2) \\
\mathbb{E}(X_sX_t) &= K(s, t), \quad \forall s, t \in S. \quad (2.3)
\end{align*}
\]

where \( \mathbb{E} \) denotes the expectation with respect to \( \mathbb{P} \).

Given a p.d. kernel \( K \), it is well known that a Gaussian realization as in (i)-(ii) always exists; in fact, we may choose \( \mathbb{P} \) such that \( \Omega = \mathbb{R}^S \) = all functions on \( S \), \( \mathcal{C} \) = the corresponding cylinder \( \sigma \)-algebra of subsets of \( \Omega \); and

\[
X_s(\omega) := \omega(s), \quad \forall \omega \in \Omega, \ s \in S. \quad (2.4)
\]

A p.d. kernel of particular interest in the present paper will be as follows:

Let \( (V, \mathcal{B}, \mu) \) be a measure space, where \( \mu \) is assumed positive and \( \sigma \)-finite. Set \( \mathcal{B}_{fin} := \{ A \in \mathcal{B} : \mu(< \infty) \} \). On \( \mathcal{B}_{fin} \times \mathcal{B}_{fin} \), then define \( K^{(\mu)} \) by

\[
K^{(\mu)}(A, B) := \mu(A \cap B), \quad \forall A, B \in \mathcal{B}_{fin}. \quad (2.5)
\]

It is immediate that \( K^{(\mu)} \) is p.d., and there is therefore a canonical associated centered Gaussian process \( X = X^{(\mu)} \), indexed by \( \mathcal{B}_{fin} \), satisfying

\[
\mathbb{E}(X^{(\mu)}_A X^{(\mu)}_B) = \mu(A \cap B), \quad \forall A, B \in \mathcal{B}_{fin}. \quad (2.6)
\]

We shall study this process in detail and show that it may be used to interpolate any Markov process built on \( (V, \mathcal{B}) \); see Theorem 4.2.

A tool in our analysis will be reproducing kernel Hilbert spaces (RKHSs). Recall that every p.d. kernel \( K \) has an associated and unique RKHS \( \mathcal{H}(K) \). The reproducing axiom is as follows: \( K(\cdot, s) \in \mathcal{H}(K) \), and

\[
F(s) = \langle F, K(\cdot, s) \rangle_{\mathcal{H}(K)} \quad \forall s \in S, \quad \text{for } F \in \mathcal{H}(K). \quad (2.7)
\]
White noise analysis serves as a versatile framework for stochastic and infinite-dimensional analysis, with a growing number of applications to neighboring areas, probability, mathematical statistics, and quantum physics. The setting is that of (Gaussian, continuous parameter) white noise — a generalized random process indexed by elements in a \( \sigma \)-algebra and with independent values at disjoint sets; informally, we may view it as an infinite system of coordinates on which to base an infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus. More precisely, the starting point is the \( L^2 \)-space of a white noise measure (Wiener measure). A common approach makes use of a certain infinite-dimensional calculus.
Theorem 3.2. Let $(V, \mathcal{B}, \mu)$ be a measure space, $\mu$ assumed $\sigma$-finite (positive). Let $K^{(\mu)} (A, B) := \mu (A \cap B)$, $A, B \in \mathcal{B}_{\text{fin}}$, be the corresponding p.d. kernel; and let $\mathcal{H}(K^{(\mu)})$ be the RKHS. Then

$$\mathcal{H}(K^{(\mu)}) = \{ F \text{ signed measures on } (V, \mathcal{B}) \text{ s.t.} \quad dF \ll d\mu \text{ (abs. cont) with } \frac{dF}{d\mu} \in L^2 (\mu) \}; \text{ and}$$

$$\| F \|_{\mathcal{H}(K^{(\mu)})} = \left\| \frac{dF}{d\mu} \right\|_{L^2(\mu)}. \quad (3.3)$$

Proof. We may use Lemma 2.4 to show that $F$, as in (3.2), is indeed in $\mathcal{H}(K^{(\mu)})$. Assume $F$ is as specified in (3.2); and set $\varphi = \frac{dF}{d\mu} \in L^2 (\mu)$, which is the condition from (3.2) on the Radon-Nikodym derivative. We will show that, if $n \in \mathbb{N}$, $(A_i)_{i=1}^n$, $A_i \in \mathcal{B}_{\text{fin}}$ (i.e., $\mu (A_i) < \infty$, $\alpha_i \in \mathbb{R}$, then

$$\left| \sum_i \alpha_i F (A_i) \right|^2 \leq \| \varphi \|_{L^2(\mu)}^2 \sum_i \sum_j \alpha_i \alpha_j K^{(\mu)} (A_i, A_j) =_{\mu (A_i \cap A_j)}$$

and so we conclude that $F \in \mathcal{H}(K^{(\mu)})$, with $\| F \|_{\mathcal{H}(K^{(\mu)})} \leq \| \varphi \|_{L^2(\mu)}$. It is in fact $\text{"=}$. See below.

We now give the verification of (3.4): Let $n$, $(A_i)_{i=1}^n$, $(\alpha_i)_{i=1}^n$, and $\varphi := \frac{dF}{d\mu} \in L^2 (\mu)$ be as stated in (3.2), and the discussion above; then

$$\text{LHS (3.4)} = \left| \sum_i \alpha_i F (A_i) \right|^2 = \left| \sum_i \alpha_i \int_{A_i} \varphi d\mu \right|^2$$

$$= \left| \int_V \varphi \cdot \sum_i \alpha_i 1_{A_i} d\mu \right|^2 \leq_{(\text{Schwarz})} \| \varphi \|_{L^2(\mu)}^2 \left| \sum_i \alpha_i 1_{A_i} \right|_{L^2(\mu)}^2$$

$$= \| \varphi \|_{L^2(\mu)}^2 \sum_i \sum_j \alpha_i \alpha_j \mu (A_i \cap A_j)$$

which is the desired conclusion (3.4).

Claim. Every $F \in \mathcal{H}(K^{(\mu)})$ is a $\sigma$-additive signed measure, i.e., if $A = \cup_{i=1}^\infty A_i$, $A_i \cap A_j = \emptyset$, $i \neq j$; (sets in $\mathcal{B}_{\text{fin}}$) then

$$F (A) = \sum_{i=1}^\infty F (A_i). \quad (3.5)$$

Proof of (3.5).

$$\text{LHS (3.5)} = F (A) = (F, \mu (\cdot \cap A))_{\mathcal{H}(K^{(\mu)})}, \quad \mu (\cdot \cap A) = K^{(\mu)} (\cdot, A),$$

$$= \left\langle F, \sum_{i=1}^\infty \mu (\cdot \cap A_i) \right\rangle_{\mathcal{H}(K^{(\mu)})}, \quad \mu (\cdot \cap A_i) = K^{(\mu)} (\cdot, A_i),$$

$$= \sum_{i=1}^\infty \left\langle F, K^{(\mu)} (\cdot, A_i) \right\rangle_{\mathcal{H}(K^{(\mu)})}$$

$$= \sum_{i=1}^\infty F (A_i).$$

We used $K^{(\mu)} (\cdot, A_i) \perp K^{(\mu)} (\cdot, A_j)$ for $i \neq j$. 


Claim. For \( A, B \in \mathcal{B}_{\text{fin}} \), we have
\[
\frac{dK^{(\mu)}(\cdot, A)}{d\mu} = 1_A = \text{the indicator function.} \tag{3.6}
\]

Proof. For \( A, B \in \mathcal{B}_{\text{fin}} \), we have
\[
K^{(\mu)}(A, B) = \int_B 1_A(x) d\mu(x) = \mu(B \cap A),
\]
and (3.6) follows.

Claim. If \( F \in \mathcal{H}(K^{(\mu)}) \), then \( dF \ll d\mu \) where \( dF \) is the signed measure in (3.5).

Proof. We show that \( \mu(A) = 0 \implies [F(A) = 0] \). From the reproducing property in \( \mathcal{H}(K^{(\mu)}) \), we have:
\[
F(A) = \left\langle F, K^{(\mu)}(\cdot, A) \right\rangle_{\mathcal{H}(K^{(\mu)})} = \left\langle F, \mu(\cdot \cap A) \right\rangle_{\mathcal{H}(K^{(\mu)})};
\]
hence, \( \mu(A) = 0 \implies F(A) = 0 \), since \( \mu(A) = 0 \implies K^{(\mu)}(\cdot, A) = 0 \), and so
\[
F(A) = \left\langle F, K^{(\mu)}(\cdot, A) \right\rangle_{\mathcal{H}(K^{(\mu)})} = 0.
\]

The proof of Theorem 3.2 is complete. \( \square \)

**Corollary 3.3.** Let \( (V, \mathcal{B}, \mu) \) be a fixed \( \sigma \)-finite measure space, and let \( \mathcal{H}(K^{(\mu)}) \) be the RKHS of \( K^{(\mu)}(A, B) := \mu(A \cap B), A, B \in \mathcal{B}_{\text{fin}} \) (see (3.1)). Define \( W^{(\mu)} \) as an isometry: \( W^{(\mu)} : L^2(\mu) \ni f \mapsto fd\mu \in \mathcal{H}(K^{(\mu)}) \), where \( W^{(\mu)}(f) = fd\mu \) is a signed measure on \( (V, \mathcal{B}) \); then
\[
W^{(\mu)} : L^2(\mu) \simeq \mathcal{H}(K^{(\mu)})
\]
is an isometric isomorphism onto \( \mathcal{H}(K^{(\mu)}) \).

**Theorem 3.4.** Let \( (V, \mathcal{B}) \) be a measure space, \( \mu \) a \( \sigma \)-finite measure on \( \mathcal{B} \), and set \( \mathcal{B}_{\text{fin}} = \{ A \in \mathcal{B} : \mu(A) < \infty \} \). Let \( K \) be a p.d. kernel on \( \mathcal{B}_{\text{fin}} \times \mathcal{B}_{\text{fin}} \), and let \( \mathcal{H}(K) \) be the corresponding RKHS. Suppose \( \mathcal{H}(K) \) consists of signed measures; and set
\[
\mathcal{H}(\mu) := \left\{ F \in \mathcal{H}(K) : dF \ll d\mu, \text{ and } \frac{dF}{d\mu} \in L^2(\mu) \right\}. \tag{3.7}
\]
Then
\[
\mathcal{H}(\mu) \subseteq \mathcal{H}(K^{(\mu)}), \tag{3.8}
\]
where
\[
K^{(\mu)}(A, B) := \mu(A \cap B), \forall A, B \in \mathcal{B}_{\text{fin}}; \tag{3.9}
\]
and therefore \( \exists c(\mu) < \infty \) such that
\[
c(\mu) K^{(\mu)} - K \tag{3.10}
\]
is positive definite.

Proof. Let \( F \in \mathcal{H}(\mu) \), see (3.7); and set \( \varphi^{(\mu)} := \frac{dF}{d\mu} \). Let \( n \in \mathbb{N} \), \( \{ A_i \}_1^n \), \( A_i \in \mathcal{B}_{\text{fin}} \), \( \{ \alpha_i \}_1^n \), \( \alpha_i \in \mathbb{R} \); then
\[
\left| \sum_i \alpha_i F(A_i) \right|^2 = \left| \sum_i \alpha_i \int_{A_i} \varphi^{(\mu)}(x) d\mu(x) \right|^2
\]
3.9 \sum_i \alpha_i \varphi_i \| \varphi_i \|_2^2 (\text{Schwarz})

Hence by Lemma 2.4, \( F \in \mathcal{H}(K^{(\mu)}) \), see (3.9); and \( \| F \|_{\mathcal{H}(K^{(\mu)})} \leq \| \varphi^{(F)} \|_{L^2(\mu)} \).

Conclusion (3.8) now follows. Finally conclusion (3.10) is immediate from Lemma 2.5. \( \square \)

**Example 3.5.** If \( V = [0, \infty), \mathscr{B} = \text{Borel} \sigma\text{-algebra, } \mu = dx = \lambda, \text{Lebesgue measure, } \) then \( X^{(\mu)} = \) standard Brownian motion.

**Proof.** Let \( A = [0, t], B = [0, s], s, t \in [0, \infty), \) then \( X_A^{(\mu)} = W_{[0,t]}, X_B^{(\mu)} = W_{[0,s]} \) satisfying

\[
E(X_A^{(\mu)} X_B^{(\mu)}) = E(W_{[0,t]} W_{[0,s]}) = \lambda([0, t] \cap [0, s]) = s \wedge t,
\]

so the standard p.d. kernel which determines Brownian motion; and \( d[W_{0,t}]_2 = dt, \) and \( [W_{0,1}]_2 = t, \) referring to the quadratic variation, see also Corollary 3.12 and Lemma 4.3. \( \square \)

Recall that in general, if \( K \) is a p.d. kernel, \( \mathcal{H}(K) \) the RKHS, then whenever \( F : S \rightarrow \mathcal{H} \) is a function from \( S \) into a Hilbert space \( \mathcal{H} \) s.t. \( K(t,s) = \langle F(t), F(s) \rangle_{\mathcal{H}} \), there is then a corresponding transform \( L = L_F : \mathcal{H} \rightarrow \mathcal{H}(K) \), given by

\[
(Lh)(t) = \langle h, F(t) \rangle_{\mathcal{H}}, \forall t \in S, \forall h \in \mathcal{H}.
\]

In Example 3.5, we may apply this to this to the kernel \( K = K^{(\mu)} \), \( S = \mathcal{B}_{fin}, \)

\[
K^{(\mu)}(A,B) := \mu(A \cap B), A,B \in \mathcal{B}_{fin}, \text{and let } F(A) := X_A^{(\mu)}, A \in \mathcal{B}_{fin}.
\]

**Corollary 3.6 (An explicit transform).** Let \( (V, \mathscr{B}, \mu) \) be as in Example 3.5. Let \( \Omega := \mathbb{R}^{\mathcal{B}_{fin}}, \) and set \( F : \mathcal{B}_{fin} \rightarrow L^2(\Omega, \mathcal{G}, \mathbb{P}), \)

\[
F(A) := X_A^{(\mu)}, A \in \mathcal{B}_{fin},
\]

where \( X_A^{(\mu)} \in L^2(\Omega, \mathbb{P}) \) is the centered Gaussian process with covariance kernel

\[
E(X_A^{(\mu)} X_B^{(\mu)}) = \mu(A \cap B).
\]

Then the transform \( L : L^2(\Omega, \mathbb{P}) \rightarrow \mathcal{H}(K^{(\mu)}) \) is

\[
(Lh)(A) = \mathbb{E}(h X_A^{(\mu)}), \forall A \in \mathcal{B}_{fin}, \forall h \in L^2(\Omega, \mathbb{P}).
\]

Let \( \mathcal{F}_V := \text{all measurable functions in } (V, \mathscr{B}), \) and \( f \in L^2(\mu) \subseteq \mathcal{F}_V \) (real valued), we get the Ito-integral

\[
\int_V f dX := \lim_{i} \sum_i f(s_i) X_{A_i}^{(\mu)}, \quad (3.12)
\]

where the limit is taken over all measurable partitions of \( V, \) mesh \( \rightarrow 0. \) Then

\[
\mathbb{E} \left( \left| \int_V f dX \right|^2 \right) = \int \| f \|^2 d\mu. \quad (3.13)
\]

**Proof.** (sketch) For all partitions \( \{A_i\} \) on \( V, s_i \in A_i, \) the Ito-isometry (3.13) follows from the approximation:

\[
\mathbb{E} \left( \left| \sum_i f(s_i) X_{A_i}^{(\mu)} \right|^2 \right) = \sum_i \left| f(s_i) \right|^2 \mu(A_i) \xrightarrow{\text{mesh} \rightarrow 0} \int_V \| f \|^2 d\mu.
\]
Remark 3.7. Using this version of Ito integral, we get the following conclusions.

Given a fixed $\sigma$-finite measure space $(V, \mathcal{B}, \mu)$, set $K = K^{(\mu)}$, 
\[ K^{(\mu)}(A, B) = \mu(A \cap B), \ A, B \in \mathcal{B}_{\text{fin}}, \]
and let $L^2(\Omega, \mathcal{F}, \mathbb{P})$ be the corresponding probability space s.t.
\[ X_\omega^{(\mu)}(\omega) = \omega(A), \ \omega \in \Omega = \mathbb{R}^\mathcal{B}. \]
Then
\[ \mathbb{E}(X_A^{(\mu)}X_B^{(\mu)}) = K^{(\mu)}(A, B) = \mu(A \cap B), \]
and the Ito integral $X_f^{(\mu)} = \int_V f dX$ is well defined with
\[ \mathbb{E}(|X_f^{(\mu)}|^2) = \int_V |f|^2 d\mu, \]
and
\[ \mu = QV(X) = [X, X] = [X]_2 \]
see also Corollary 3.12 and Lemma 4.3.

The correspondence $\{X_A^{(\mu)}\}_{A \in \mathcal{B}} \leftrightarrow \{X_f^{(\mu)}\}_{f \in L^2(\mu)}$ is bijective. Easy direction: given $X_f^{(\mu)}$ as above, $A \in \mathcal{B}$, set $f = 1_A$.

For details on Ito calculus and Brownian motion, see, e.g., [AJL11, AJ12, AJ15, Shr04, Hid71, Hid80].

Corollary 3.8. Given $(V, \mathcal{B}, \mu)$ fixed, $\sigma$-finite measure space, we introduce the kernel $K^{(\mu)}$, and the associated centered Gaussian process $X := X^{(\mu)}$. From our Ito-calculus, it follows that $X^{(\mu)}$ may be realized in two equivalent ways:

(i) $\Omega = \mathbb{R}^\mathcal{B} =$ all functions from $\mathcal{B}$ into $\mathbb{R}$, $X_A^{(\mu)}(\omega) = \omega(A), A \in \mathcal{B}$;

(ii) $\Omega = \mathbb{R}^V =$ all functions from $V$ into $\mathbb{R}$, $X_f^{(\mu)}(\omega) = \omega(f), f \in L^2(\mu)$.

From standard Kolmogorov consistency theory [Hid80, PS75], in (i) the probability measure $\mathbb{P}$ is defined on the cylinder $\sigma$-algebra of $\mathbb{R}^\mathcal{B}$, and in case (ii) it is defined on the $\sigma$-algebra for $\mathbb{R}^V$.

We also get two equivalent versions of the covariance function for $X$, which is indexed by $\mathcal{B}$ or by $L^2(\mu)$:

(iii) $\mathbb{E}(X_AX_B) = \mu(A \cap B), A, B \in \mathcal{B}$,

(iv) $\mathbb{E}(X_fX_g) = \int_V f(x) g(x) d\mu(x) = \mathbb{E} \left( (\int f dX) (\int g dX) \right), \forall f, g \in L^2(\mu)$, real valued, where $X_f = \int f dX$ is the Ito integral formula which made the link from (iii) to (iv).

Corollary 3.9. Let $X := X^{(\mu)}$ be the Gaussian process as above, and
\[ X_f := \int_V f dX, \quad (3.14) \]
then
\[ \mathbb{E}(e^{iX_f}) = e^{-\frac{1}{2} \int_V |f|^2 d\mu}, \ \forall f \in L^2(\mu). \quad (3.15) \]

Proof. Direct proof from the power series expansions. See, e.g., [JPT15, JT17c], and the papers cited there. \qed
Remark 3.10. Note that $(3.15)$ is analogous to the Gelfand triple construction, but more general. In the present setting, we do not need a Gelfand triple in order to make process $(3.15)$ above.

Corollary 3.11. If $\{f_n\}_{n \in \mathbb{N}_0}$ is an orthonormal basis (ONB) in $L^2(\mu)$, then $\{X_{f_n}\}_{n \in \mathbb{N}_0}$ is an i.i.d. $N(0,1)$ system, i.e., $X_{f_n} = Z_n \sim N(0,1)$, and the following Karhunen-Loeve decomposition holds:

$$X_A = \sum_{n=0}^{\infty} \left( \int_A f_n \, d\mu \right) Z_n, \forall A \in \mathcal{B}_{fin}. \quad (3.16)$$

Corollary 3.12. Assume $\mu$ is non-atomic. Then the quadratic variation of $X := X^{(\mu)}$ is $\mu$ itself, i.e., if $B \in \mathcal{B}$, $d[X,X] = d\mu$.

Proof. Let $B \in \mathcal{B}$ with a partition $\{A_i\}$ s.t. $B = \bigcup A_i$, $A_i \cap A_j = \emptyset$, $i \neq j$. If $\mu$ is non-atomic, then

$$\lim \sum_i (X_{A_i})^2 = \mu(B) \cdot 1$$

$$=: [X,X]_2$$

where $1$ denotes the constant function in $L^2(\Omega, \mathbb{P})$, and the limit is over the set of all partitions of $B$ with mesh tending to $0$. See Lemma 4.3 for additional details. $\square$

Corollary 3.13 (Generalized Ito lemma). Let $f : \mathbb{R} \to \mathbb{R}$, or $\mathbb{C}$, $f \in C^2$, then

$$df(X_s) = f'(X_s) \, dX_s + \frac{1}{2} f''(X_s) \, d\mu(s), \quad (3.18)$$

or equivalently,

$$f(X_B) = \int_B f'(X_s) \, dX_s + \frac{1}{2} \int_B f''(X_s) \, d\mu(s), \quad (3.19)$$

for $\forall B \in \mathcal{B}_{fin}$, where we used the Ito integral, and $d[X,X](s) = d\mu(s)$ for the quadratic variation.

Remark 3.14. We can do most of the white noise analysis in the more general setting, i.e., w.r.t $(3.19)$.

Corollary 3.15. Let the setting be as in Theorem 3.4, i.e., $(V, \mathcal{B}, \mu)$, and $K$ are specified as in $(3.7)$-$(3.8)$. In particular, in addition to $K$, we also have the $\mu$-kernel $K^{(\mu)}(A,B) = \mu(A \cap B)$ as in $(3.9)$. Let $X^{(K)}$ be the centered Gaussian process with kernel $K$, i.e.,

$$\mathbb{E}(X^{(K)}_A X^{(K)}_B) = K(A,B), \forall A,B \in \mathcal{B}_{fin}. \quad (3.20)$$

Let $X^{(\mu)}$ be the Gaussian process $(2.6)$ with Ito integral

$$X^{(\mu)}_f = \int_V f(x) \, dX^{(\mu)}_x, \quad f \in L^2(\mu), \quad (3.21)$$

and

$$\mathbb{E}\left(\left|X^{(\mu)}_f\right|^2\right) = \int_V |f|^2 \, d\mu; \quad (3.22)$$

see Corollary 3.6.

Then there is a function

$$G : \mathcal{B}_{fin} \times V \to \mathbb{R}, \quad (3.23)$$
measurable in the second variable, such that
\[ G(A, \cdot) \in L^2(\mu), \forall A \in \mathcal{B}_{\text{fin}}; \] (3.24)
\[ K(A, B) = \int_V G(A, x) G(B, x) d\mu(x) \] (3.25)
(compare with Definition 5.1 below), and
\[ X^{(K)}_A = \int_V G(A, x) dX^{(\mu)}_x. \] (3.26)

**Proof.** The existence of $G$ follows from Theorem 3.4, and the Hida-Cramer transform \cite{Gua15}. Hence, by (3.21), we may define a Gaussian process $X^{(K)}$ by (3.26); and for $A, B \in \mathcal{B}_{\text{fin}}$, we have
\[ E(X^{(K)}_A X^{(K)}_B) = K(A, B) \]
(by (3.22)) \[ \int_V G(A, x) G(B, x) d\mu(x), \]
which is the desired conclusion. \qed

4. **Gaussian interpolation of Markov processes**

Markov models, or hidden Markov models, are ubiquitous in model building, e.g., to models for speech and handwriting recognition, to software, and learning mechanisms in biological neural networks. Within the study of support vector machines, one use of Markov processes is to solve both the problem of classification, and that of clustering. The list of optimization tasks includes that of maximizing an “expected goodness of classification,” or a “goodness of clustering” criterion. This in turn leads to the study of specific kinds of probability distribution over sequences of vectors — for which we have good parameter estimation, and good marginal distribution algorithms.

Hidden Markov models tend to be robust in many uses, for example, in determining the nature of an input signal, given the corresponding an output. The model aims to determine the most probable set of parameters which dictate input states, when based on an observed sequence of output states.

The literature is quite large: Here we mention just \cite{Grz16, Yu16, MB17}, and the papers cited there.

4.1. **The Markov processes.** In our previous work \cite{JT17a}, we already discussed applications of the family of Gaussian processes from Section 3. Our present aim is to use them in an interpolation algorithm for non-atomic Markov processes.

Recall the Gaussian processes \( \{X^{(\mu)}_A : A \in \mathcal{B}_{\text{fin}}\} \), such that
\[ E(X^{(\mu)}_A X^{(\mu)}_B) = \mu(A \cup B), \forall A, B \in \mathcal{B}_{\text{fin}}; \] (4.1)
where \((V, \mathcal{B}, \mu)\) is a given measure space, and \(\mu\) assumed positive and \(\sigma\)-finite.
INTERPOLATION AND REPRODUCING KERNELS

Below we consider a family of Gaussian processes corresponding to a given Markov process \( P(x, \cdot) \), where \((V, \mathcal{B})\) is a measure space, \( x \in V \), \( P(x, \cdot) \) is a non-atomic probability measure, i.e., \( P(x, V) = 1 \). We shall denote \( P \) as the transition operator, defined for measurable functions \( f \) on \((V, \mathcal{B})\), by

\[
(Pf)(x) = \int_V f(y) P(x, dy), \quad \forall x \in V.
\] (4.2)

Thus \( P(1) = 1 \), and the constant function \( 1 \) is harmonic. (Also see [JT17b, JT17d], and the papers cited there.)

**Lemma 4.1.** Every generalized Markov process \( P(x, \cdot) \) induces a dual pairs of actions:

(i) action on measurable functions \( f \) on \((V, \mathcal{B})\),

\[
f \mapsto \int f(y) P(x, dy) = (Pf)(x), \quad x \in V;
\]

and

(ii) action on signed measures \( \nu \) on \((V, \mathcal{B})\),

\[
\nu \mapsto \int P(x, \cdot) d\nu(x) = P^*(\nu),
\]

where

\[
(P^*(\nu))(A) = \int P(x, A) d\nu(x), \quad \forall A \in \mathcal{B}.
\]

As in standard Markov theory,

\[
P_2(x, A) = \int P(x, dy) P(y, A) = P[P(\cdot, A)](x),
\]

and inductively

\[
P_{n+1}(x, A) = \int P_n(x, dy) P(y, A).
\] (4.3)

For each of the measures \( P(x, \cdot), P_2(x, \cdot) \cdots, P_n(x, \cdot) \), there is a corresponding white noise process \( X^{(x)} \), i.e., an indexed family of Gaussian processes \( X^{(x)}_A \sim P(x, A) \), where \( \mathbb{E}_x(X^{(x)}_A) = 0 \), and

\[
\mathbb{E}_x(X^{(x)}_A X^{(x)}_B) = P(x, A \cap B), \quad \forall A, B \in \mathcal{B}.
\] (4.4)

We now introduce a more general family of Ito integrals, and get a new process \( W^{(x)}_A \) which has \( P_2(x, A) \) as its covariance kernel. See Theorem 4.2 below.

**Theorem 4.2 (Interpolation).** Let \( P(x, \cdot) \) and \( X^{(x)}_A \) be as specified above, see (4.1)-(4.4). Set

\[
W^{(x)}_A := \int_V X^{(y)}_A dX^{(x)}(y),
\] (4.5)

defined as an Ito integral. Then

(i) \( \{W^{(x)}_A\} \) is a Gaussian process;

(ii) \( \mathbb{E}_x(W^{(x)}_A) = 0 \);

(iii) \( \mathbb{E}_x(W^{(x)}_A W^{(x)}_B) = P_2(x, A \cap B), \quad \forall A, B \in \mathcal{B}, \forall x \in V.\)
(iv) By induction, with an $n$-fold Ito integral from (4.5), we get $W^{(n)}_A(x)$ such that

$$E_x(W^{(n)}_A(x)W^{(n)}_B(x)) = P_n(x, A \cap B),$$

for $n \in \mathbb{N}$, $x \in V$, and $A, B \in \mathcal{B}$.

Proof. Let $P(x, A)$ and $X_A^{(x)}$ be as specified above. We then form the Ito integral $X_A^{(x)}$ with $P(x, \cdot)$ as covariance. Note that for every $y \in V$, $X^{(y)}$ is a centered Gaussian process with covariance kernel

$$E_y(X_A^{(y)}X_B^{(y)}) = P(y, A \cap B).$$

We shall show that $W^{(x)}$ is also a centered (i.e., mean zero) Gaussian process, now with $P_2(x, \cdot)$ as covariance measure, i.e., that $W^{(x)}$ from (4.5) will satisfy:

$$E_x(W^{(x)}_AW^{(x)}_B) = P_2(x, A \cap B), \forall A, B \in \mathcal{B};$$

The idea is that the white noise process interpolates the Markov process. Aside from the induction, the key step in the argument is an analysis of the Ito integral (4.5). By general Ito theory, we have

$$E\left(|W^{(x)}_A|^2\right) = \int E\left(|W^{(y)}_A|^2\right)dP^{(x)}_1, \quad (4.9)$$

In the last step, we used (4.7) on the term $E(|X_A^{(y)}|^2)$ in (4.9); and used the formula for the quadratic variation

$$dP^{(x)}_1, \quad (4.10)$$

See also Lemma 4.3 below. $\square$

Note that (4.10) is a special case of an analogous property of white noise, subject to a fixed measure $\mu$. Assume $\mu$ is non-atomic, then

$$dP^{(x)}_1, \quad (4.11)$$

where the $\mu$-martingale is a centered Gaussian process determined by

$$dP^{(x)}_1, \quad (4.12)$$

and so

$$dP^{(x)}_1, \quad (4.13)$$

Lemma 4.3. Give $(V, \mathcal{B}, \mu)$ as in above, and let $X := X^{(1)}$ be the corresponding Gaussian process, centered, with covariance given by (4.11). Let $d[X, X]$ be the quadratic variation measure, i.e., for $B \in \mathcal{B}$, $QV(B) = \lim \sum X_{A_i}^2$, where the limit is taken over all measurable partitions $\pi = \left\{A_i\right\}$ of $B$, as mesh $\pi \to 0$. Then $\mu(B) = QV(B)$, and $d\mu = d[X, X]$.

Proof. It is true in general that if $(V, \mathcal{B}, \mu)$ is a $\sigma$-finite non-atomic measure space, and $X = X^{(1)}$ is the white noise Gaussian process determined by

$$E(X_AX_B) = \mu(A \cap B), \forall A, B \in \mathcal{B}; \quad (4.11)$$

then for the quadratic variation measure $[X, X] = [X]_2$, we have:

$$[X]_2(B) = \mu(B),$$

and so

$$d[X]_2(s) = d\mu(s).$$
To see this, fix $B \in \mathcal{B}$, and take a limit on all measurable partitions $\pi = \{A_i\}$, where $A_i \cap A_j, i \neq j, \cup A_i = B$, and set $\alpha_i = \mu(A_i)$. A direct calculation gives

$$E(X_{A_i}^4) = 3(E(X_{A_i}^2))^2 = 3\alpha_i^2,$$

and

$$E \left( \left| \mu(B) - \sum_i X_{A_i}^2 \right|^2 \right) = E \left( \left| \sum_i \mu(A_i) - \sum_i X_{A_i}^2 \right|^2 \right) = \sum_i E \left( \left| \mu(A_i) - X_{A_i}^2 \right|^2 \right) + \sum_{i \neq j} E \left( (\mu(A_i) - X_{A_i}^2)(\mu(A_j) - X_{A_j}^2) \right)$$

$$= 2 \sum_i \mu(A_i)^2 = 2 \sum_i \alpha_i^2 \xrightarrow{n \to \infty} 0,$$

since $\sum_i \alpha_i = \mu(B) > 0$ is fixed.

In the proof of Lemma 4.3, we used the following fact:

**Lemma 4.4.** Let $n \in \mathbb{N}$ be fixed, and let $\alpha_i > 0$ satisfying $\sum_i \alpha_i = 1$, then

$$\inf_n \left\{ \sum_i \alpha_i^2 : \sum_1^n \alpha_i = 1 \right\} = 0.$$  

**Proof.** Fix $n \in \mathbb{N}$, and apply Lagrange multiplier to get $\alpha_i = 1/n$, for all $i$. Then

$$\sum_1^n \alpha_i^2 = \frac{1}{n} \xrightarrow{n \to \infty} 0.$$

\[\square\]

### 4.2. Fourier duality

**Example 4.5.** Let $(V, \mathcal{B}, \mu) = (\mathbb{R}, \mathcal{B}, \mu)$ where $\mu$ is a probability measure on $\mathbb{R}$. Let $X = X^{(\mu)}$ be the corresponding Gaussian process, and use the Ito integral to define

$$X_{et} = \int_{\mathbb{R}} e_t(x) dX_x, \text{ where } e_t(x) = \exp(i2\pi xt), \ x, t \in \mathbb{R}. \quad (4.12)$$

Then

$$E \left( X_{et} X_{es} \right) = \hat{\mu}(t - s), \ t, s \in \mathbb{R}, \quad (4.13)$$

where $\hat{\mu}$ denotes the standard Fourier transform.

**Proof.** Direct computation using (4.12):

$$E \left( X_{et} X_{es} \right) = \int e_t(x) e_s(x) d[X] = \int e_{t-s}(x) d\mu(x) \quad (4.14)$$

where we used that $d[X] = d\mu$, see Lemma 4.3. Then

$$\int e_t(x) e_s(x) E(dX^{(\mu)}_{x}dX^{(\mu)}_{y}) \xrightarrow{\text{use orthogonality}} \int e_t(x) e_s(x) d\mu(x). \quad (4.15)$$

Note that if $A \cap B = \emptyset$, then $E(X^{(\mu)}_B X^{(\mu)}_B) = \mu(A \cap B) = \mu(\emptyset) = 0.$
Theorem 4.6 (Duality). The Fourier transform $X_{e_t}^{(\mu)} = \int e_t(x) dX_x^{(\mu)}$ is well defined, and it is a stationary process with covariance kernel

$$K^F(s,t) = \hat{\mu}(t-s),$$

where $\hat{\mu}$ is the standard Fourier transform of the measure $\mu$.

Application. In one of our earlier papers [JT17a], and papers cited there, we studied tempered measures $\mu$ on $\mathbb{R}$, and processes $\{Y_\varphi\}$ indexed by $\varphi \in \mathcal{S}$ (= the Schwartz space), and we get

$$E(|Y_\varphi|^2) = \int_\mathbb{R} |\hat{\varphi}(x)|^2 d\mu(x),$$

or equivalently,

$$E(e^{iY_\varphi}) = e^{-\frac{1}{2} \int |\hat{\varphi}|^2 d\mu}. \hspace{1cm}(4.18)$$

But we can recover this setting from the case $(\mathbb{R}, \mathcal{B}, \mu)$ by setting

$$Y_\varphi = \int \hat{\varphi}(x) dX_x^{(\mu)} \hspace{1cm}(4.19)$$

as an Ito integral. (Note that the RHS in (4.18) is a continuous positive definite function in $\varphi \in \mathcal{S}$ (the Schwartz space), and so Minlos’ theorem applies; see [Hid80].) Then

$$Y_\varphi = \int \hat{\varphi}(x) dX_x^{(\mu)} = \int \varphi(t) \overline{e_t(x)} dt dX_x^{(\mu)}$$

and

$$E(|Y_\varphi|^2) = E\left(\left|\int \hat{\varphi}(x) dX_x^{(\mu)}\right|^2\right)$$

$$= E\left(\left|\int \varphi(t) dX_x^{(\mu)} dt\right|^2\right)$$

$$= \int \int \varphi(t) \overline{\varphi(s)} E(X_{e_t}X_{e_s}) dtds$$

$$= \int \int \varphi(t) \check{\mu}(t-s) \overline{\varphi(s)} dtds \hspace{1cm} \text{by (4.13)\&(4.16)}$$

$$= \int_\mathbb{R} |\hat{\varphi}(x)|^2 d\mu(x).$$

This is the desired conclusion (4.17). The idea is that we get all these conclusions without Gelfand triples.

Remark 4.7. The converse holds too. Suppose $\{Y_\varphi\}_{\varphi \in \mathcal{S}}$ is a Gaussian process (based on $\mathbb{R}$) computed from a tempered measure $\mu$,

$$\int_\mathbb{R} \frac{d\mu(x)}{1+x^2} < \infty,$$

with the Gelfand triple

$$\mathcal{S} \hookrightarrow L^2(\mathbb{R}) \hookrightarrow \mathcal{S'},$$

we get

$$E(|Y_\varphi|^2) = \int_\mathbb{R} |\hat{\varphi}(x)|^2 d\mu(x).$$
Proof. Here the process \(X^{(\mu)}\) is determined by measure \(\mu\),
\[
\mathbb{E}(X_{f}^{(\mu)}) = \mu(A \cap B), \quad \forall A, B \in \mathcal{B},
\]
and set \(X_{f}^{(\mu)} = \int f dX^{(\mu)} \quad \forall f \in L^{2}(\mathbb{R}, \mu)\), and
\[
Y_{\varphi} = \int \varphi(x) dX_{x}^{(\mu)} \quad \forall \varphi \in \mathcal{S}.
\]
Indeed, we already proved that \((4.24) \implies (4.23)\). Note that
\[
\mathbb{E}(\{Y_{\varphi}\}^2) = \int d\mu(x) \quad (4.22)
\]
holds by the generalized Ito isometry, and we define the transform \(L : H \rightarrow \mathcal{H}(K^{(\mu)})\),
\[
(Lh)(A) = \mathbb{E}(hX_{A}^{(\mu)}), \quad \forall A \in \mathcal{B},
\]
where \(H = L^{2}(\Omega, \mathcal{P})\), and \(\mathcal{H}(K^{(\mu)})\) is the RKHS of \(K^{(\mu)}(A, B) := \mu(A \cap B)\).
But using \((4.25)\) again, we get
\[
\mathbb{E}(hX_{A}^{(\mu)}) = \int_{V} hd\mu = \int_{A} h d\mu
\]
for \(h \in L^{2}(\Omega, \mathcal{P})\), and where \(1_{A}, \forall A \in \mathcal{B}\), is the indicator function on \(A\), i.e.,
\[
1_{A}(x) = \delta_{x}(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{else} \end{cases}
\]

We have realized \(\mathcal{H}(K^{(\mu)})\) as a Hilbert space of functions on \((V, \mathcal{B})\) viz \(L^{2}(\mu)\).
Note that since \(K^{(\mu)}(A, B) = \mu(A \cap B)\) is a p.d. kernel on \(\mathcal{B}_{\text{fin}} \times \mathcal{B}_{\text{fin}}\), initially \(\mathcal{H}(K^{(\mu)})\) is a Hilbert space of functions on \(\mathcal{B}_{\text{fin}}\), but not functions on \((V, \mathcal{B})\).
Corollary 4.8. Let \((V, \mathcal{B}, \mu)\) satisfy the axioms from above, let \(K^{(\mu)}\) be the p.d. kernel on \(\mathcal{B}_{fin} \times \mathcal{B}_{fin}\), and \(\mathcal{H}(K^{(\mu)})\) be the corresponding RKHS. Let \(X^{(\mu)}_f, f \in L^2(\mu)\), be the Gaussian process which extends \(\{X^{(\mu)}_A\}_{A \in \mathcal{B}_{fin}}\), and let \(H = L^2(\Omega, \mathcal{F})\) be the Gaussian Hilbert space with inner product \(\langle h_1, h_2 \rangle_H = \mathbb{E}(h_1 h_2), \forall h_i \in H\). Then the bilinear mapping
\[
L^2(\mu) \times H \ni (f, h) \mapsto \mathbb{E}(X^{(\mu)}_f h)
\]
defines two operators in duality:

\[
\begin{array}{c}
L^2(\mu) \\
\circlearrowleft
\end{array}
\begin{array}{c}
\xi^{(\mu)} \\
\circlearrowleft
\end{array}
\begin{array}{c}
H
\end{array}
\]

The Ito isometry \(I_\mu : L^2(\mu) \rightarrow H\),
\[
I_\mu(f) = X^{(\mu)}_f, \forall f \in L^2(\mu),
\]
and the co-isometry \(\xi^{(\mu)} : H \rightarrow L^2(\mu)\), determined by
\[
\langle I_\mu(f), h \rangle_H = \langle f, \xi^{(\mu)}(h) \rangle_{L^2(\mu)}, \forall f \in L^2(\mu), \forall h \in H.
\]
In particular, \(I_\mu^* = \xi^{(\mu)}\), \((\xi^{(\mu)})^* = I_\mu\).

Duality. Given \((V, \mathcal{B}, \mu)\) for \(\sigma\)-finite measure \(\mu\), let \(K^{(\mu)}(A, B) := \mu(A \cap B), A, B \in \mathcal{B}_{fin}\) where \(\mathcal{B}_{fin} := \{A \in \mathcal{B} : \mu(A) < \infty\}\). Set
\[
\mathcal{H}(K^{(\mu)}) := \text{the RKHS of functions on } \mathcal{B}_{fin},
\]
s.t. for all \(F \in \mathcal{H}(K^{(\mu)})\),
\[
F(A) = \langle F, K^{(\mu)}_A \rangle_{\mathcal{H}(K^{(\mu)})} = \langle F, \mu(A \cdot) \rangle_{\mathcal{H}(K^{(\mu)})}.
\]
Set
\[
L(h)(A) := \langle h, X^{(\mu)}_A \rangle = \mathbb{E}(h X^{(\mu)}_A), \quad L^*(K^{(\mu)}_A) = X^{(\mu)}_A,
\]
\(h \in H := L^2(\Omega, \mathcal{F}, \mathbb{P})\), \(X^{(\mu)}_A\) the Gaussian process of \(K^{(\mu)}\), where
\[
X^{(\mu)}_f = \int f(x) \, dX^{(\mu)}_x, f \in L^2(\mu).
\]
A new operator \(\xi := \xi^{(\mu)} : H \rightarrow L^2(\mu)\), where \(H := L^2(\Omega, \mathcal{F}, \mathbb{P})\), and \(L^2(\mu) := L^2(V, \mathcal{B}, \mu)\).

Lemma 4.9. With the setting as above, \(\xi^{(h)}(h) \in L^2(\mu), \forall h \in H\), is determined uniquely by
\[
\int \xi^{(\mu)}(h) f \, d\mu = \mathbb{E}(h X^{(\mu)}_f), \forall h \in H, \forall f \in L^2(\mu).
\]

Proof. \(\xi^{(h)}(h)\) is determined from (4.29) and Reisz since
\[
\left| \mathbb{E}(h X^{(\mu)}_f) \right|^2 \leq \|h\|_H^2 \int_V |f|^2 \, d\mu.
\]
So we only need to show the estimate (4.30), but it follows again from the Itô isometry, as follows: Let \( h \in H \), and \( f \in L^2(\mu) \), then
\[
\left| \mathbb{E}(h X_f^{(\mu)}) \right|^2 \leq \mathbb{E}(h^2) \mathbb{E}\left(|X_f^{(\mu)}|^2\right)
\]
\[
= \mathbb{E}(\|h\|^2) \int_V \|f\|^2 \, d\mu
\]
\[
= \|h\|^2 \mathbb{E}(\|f\|^2) = \|h\|^2 \mathbb{E}(\|f\|^2)
\]
where we used the Itô isometry in the last step. \( \square \)

**Corollary 4.10.** \( \xi^{(\mu)} \) is contractive,
\[
\|\xi^{(\mu)}(h)\|_{L^2(\mu)} \leq \|h\|_H, \quad \forall h \in H. \quad (4.31)
\]

**Corollary 4.11.** The two operators \( I^{(\mu)} \) and \( \xi^{(\mu)} \) are specified as follows:

\[
\begin{array}{ccc}
L^2(\mu) \ni f & \xrightarrow{I^{(\mu)}} & X_f^{(\mu)} \in H = L^2(\Omega, \mathscr{G}, \mathbb{P}) \\
L^2(\mu) \ni \xi^{(\mu)}(h) & \xleftarrow{\xi^{(\mu)}} & h \in H = L^2(\Omega, \mathscr{G}, \mathbb{P})
\end{array}
\]

We have
\[
\xi^{(\mu)} = \left( f \rightarrow X_f^{(\mu)} \right)^* = \text{the adjoint operator, and}
\]
\[
f \rightarrow X_f^{(\mu)} = (\xi^{(\mu)})^*, \quad \text{and}
\]
\[
f \rightarrow X_f^{(\mu)} = I^{(\mu)}(f) \text{ is isometric (Ito).} \quad (4.34)
\]

Moreover,
\[
(I^{(\mu)})^* I^{(\mu)} = I_{L^2(\mu)}, \quad \text{while}
\]
\[
I^{(\mu)}(I^{(\mu)})^* = Q_\mu = \text{projection on } H; \quad (4.35)
\]

\[
H = L^2(\Omega, \mathbb{P}), \quad Q = \text{proj on } R(I^{(\mu)}).
\]

Or, we may rewrite (4.35)-(4.36) as
\[
\xi^{(\xi)} I^{(\mu)} = \text{identity operator on } L^2(\mu) \quad (4.37)
\]
\[
I^{(\mu)} \xi^{(\mu)} = Q_\mu, \text{ the proj in } H \text{ onto the range of } I^{(\mu)}. \quad (4.38)
\]

5. Parseval frames in the measure category

Let \( U \) be a set, and let \( K : U \times U \rightarrow \mathbb{R} \) be a positive definite (p.d.) kernel. We assume that the corresponding RKHS \( \mathscr{H}(K) \) is separable. (The result below will apply \textit{mutatis mutandis} also to complex p.d. kernels \( K : U \times U \rightarrow \mathbb{C} \), but for simplicity, we shall state our theorem only in the real case.)

Let \((S, \mathscr{B}, \mu)\) be a measure space with \( \mu \) assumed positive and \( \sigma \)-finite.
Definition 5.1. We shall say that $L^2(\mu)$ is a feature space if there is a function $r : U \rightarrow L^2(\mu)$ such that

$$K(x, y) = \int_S r_x(s) r_y(s) \, d\mu(s), \forall x, y \in U.$$  \hfill (5.1)

(In the complex case, the RHS in 5.1 will instead be $\int_S r_x(s) \overline{r_y(s)} \, d\mu(s)$. See also Definition 2.2.)

Remark 5.2. The notion of feature space derives from the setting of machine learning [CS02, SZ05, ACM06, SZ07, CZ07, XLTLM15, LTLPD15, Ble15, Ste15, HKLW07], where learning optimization is made precise with the use of a choice of reproducing kernel Hilbert space (RKHS). In practical terms, a choice of feature space refers to a specified collections of features that used to characterize data. For example, feature space might be (Gender, Height, Weight) etc; and we will then arrive at mappings into other feature spaces. In finite dimension, we may have feature spaces referring to some fixed number $n$ of dimensions. Since data is “large” it is useful to consider feature spaces to be function spaces, especially a choice of $L^2$-spaces. The term feature space is used often in the machine learning (ML) literature because a task in ML is feature extraction. Hence we view all variables as features.

Lemma 5.3. We always have two distinguished feature spaces in the $L^2$-category: $l^2(\mathbb{N})$ vs Gaussian.

CASE 1. $S = \mathbb{N}$ (note the separability assumption.), and $\mu$ := the counting measure. If $\{h_n\}_{n \in \mathbb{N}}$ is such that

$$K(x, y) = \sum_{n=1}^\infty h_n(x) h_n(y), \forall x, y \in U;$$  \hfill (5.2)

then $h_n \in \mathcal{H}(K)$ for all $n \in \mathbb{N}$, and $\{h_n\}_{n \in \mathbb{N}}$ is a Parseval frame in $\mathcal{H}(K)$, i.e.,

$$\|F\|_{\mathcal{H}(K)}^2 = \sum_{n=1}^\infty \langle F, h_n \rangle_{\mathcal{H}(K)}^2;$$  \hfill (5.3)

and

$$F(x) = \sum_{n=1}^\infty \langle F, h_n \rangle_{\mathcal{H}(K)} h_n(x)$$  \hfill (5.4)

is also strongly convergent, for all $x \in U$, and $F \in \mathcal{H}(K)$.

Proof. The lemma follows from standard RKHS theory [Aro50, Aro43]. An important point is to note that if $K$ and $\{h_n\}_{n \in \mathbb{N}}$ satisfy the assumptions in CASE 1, then $h_n \in \mathcal{H}(K)$ for all $n \in \mathbb{N}$. We may get this as an application of Lemma 2.4. Indeed, let $n_0 \in \mathbb{N}$ be fixed. Let $k \in \mathbb{N}$, $\{\alpha_i\}_{i=1}^k$, $\{x_i\}_{i=1}^k$, $\alpha_i \in \mathbb{R}$, $x_i \in U$; then

$$\left| \sum_i \alpha_i h_{n_0}(x_i) \right|^2 \leq \sum_{n \in \mathbb{N}} \left| \sum_i \alpha_i h_n(x_i) \right|^2 \leq \sum_i \sum_j \alpha_i \alpha_j \sum_{n \in \mathbb{N}} h_n(x_i) h_n(x_j) \leq \sum_i \sum_j \alpha_i \alpha_j K(x_i, x_j).$$
and so the premise in Lemma 2.4 holds, and we conclude that \( h_{n_0} \in \mathcal{H}(K) \).

CASE 2. Set \( S = \mathbb{R}^U \), \( \mathcal{B} \) the cylinder \( \sigma \)-algebra, and \( \mu \) the Gaussian probability measure on \( S \) determined by its finite samples: \( k \in \mathbb{N} \), \( \{x_i\}_1^k \), \( x_i \in U \). On \( \mathbb{R}^k \), define the standard centered Gaussian, so with mean 0, and covariance matrix \( \{K(x_i, x_j)\}_{i,j=1}^k \). Then apply Kolmogorov consistency, and \( \mu = \mathbb{P}_{Kolm}(K) \) will be the corresponding measure, also called the Wiener measure. Setting, for \( x \in U \),

\[
  r_x(s) = s(x);
\]

and the desired conclusions follow:

(i) Each \( r_x \in L^2(\mu) \) is Gaussian,

(ii) \( \mathbb{E}(r_x) = \int r_x d\mu = 0 \),

(iii) \( \mathbb{E}(r_x r_y) = K(x, y), \forall x, y \in U \).

\[ \square \]

**Theorem 5.4.** Let \( K: U \times U \rightarrow \mathbb{R} \) be a positive definite \((p.d.) \) kernel, and let \( (S, \mathcal{B}, \mu, r) \) be a feature space as specified in Definition 5.1; in particular, we have \( r_x \in L^2(\mu) \), \( \forall x \in U \), and \( K(x, y) = \int_S r_x(s) r_y(s) d\mu(s) \), see (5.1).

(i) Set

\[
  R(x, A) = \int_A r_x(s) d\mu(s),
\]

for \( x \in U \), \( A \in \mathcal{B}_{fin} \), where

\[
  \mathcal{B}_{fin} = \{ A \in \mathcal{B} : \mu(A) < \infty \};
\]

then \( R \) in (5.6) is a measure in the second variable, and it is measurable in \( x \in U \).

(ii) For all \( F \in \mathcal{H}(K) \), we have

\[
  \|F\|^2_{\mathcal{H}(K)} = \int_S \langle F(\cdot), R(\cdot, ds) \rangle_{\mathcal{H}(K)}^2,
\]

and

\[
  F(x) = \int_S r_x(s) \langle F(\cdot), R(\cdot, ds) \rangle_{\mathcal{H}(K)}.
\]

**Remark 5.5.** We study the the parallel between the present conclusions (5.8)-(5.9), and the more familiar ones (5.3)-(5.4) from standard Parseval frame-theory, see, e.g., [Jor06, Pes13, FPWW14], and also see [Gab91, Cas93, Ky08] for direct integrals.

**Proof.** Note first that there is a natural isometry \( J \) defined by limits and closure as follows:

\[
  J \left( \sum_i \alpha_i K(\cdot, x_i) \right) = \sum_i \alpha_i r_{x_i}.
\]

Indeed for finite sample \( k \), \( \alpha_i \in \mathbb{R} \), \( x_i \in U \), \( 1 \leq i \leq k \), we have

\[
  \left\| \sum_i \alpha_i K(\cdot, x_i) \right\|^2_{\mathcal{H}(K)} = \left\| \sum_i \alpha_i r_{x_i} \right\|^2_{L^2(\mu)}
\]

since both sides in (5.11) reduce to \( \sum_i \alpha_i \alpha_j K(x_i, x_j) \). As a result, in order to verify (5.8)-(5.9), we need only consider the case \( F(\cdot) = K(\cdot, y) \) when \( y \in U \) is fixed. Then it is enough to show that (5.8) holds, and (5.9) will follow.
Let \( y \in U \) be fixed, and assume \( F(\cdot) = K(\cdot, y) \). Then
\[
\text{LHS}_{(5.8)} = K(y, y), \quad \text{and}
\]
\[
\text{RHS}_{(5.8)} = \int_S |r_y(s)|^2 \, d\mu = K(y, y),
\]
by (5.1). Similarly,
\[
\text{LHS}_{(5.9)} = K(x, y), \quad \text{and}
\]
\[
\text{RHS}_{(5.9)} = \int_S r_y(s) r_x(s) \, d\mu(s) = K(x, y),
\]
again from an application of assumption (5.1).

6. Transforms

Let \( K : U \times U \to \mathbb{R} \) be a positive kernel. (We shall state the result below for the real case but extensions to \( \mathbb{C} \) are straightforward; even to the case of operator valued kernels.) Now let \((S, \mathcal{B}, \mu)\) be a measure space with \( \mu \) fixed and assume \( \sigma \)-finite. We shall further assume that \( L^2(\mu) \) is a feature space; see Definition 5.1 and Remark 5.2, i.e., we assume that there is a function, \( U \xrightarrow{L} L^2(\mu), \ F(x) = r_x(\cdot) (\in L^2(\mu)) \) such that (5.1) holds.

**Proposition 6.1.**

(i) With the setting \((K, U, \mu, \{r_x\}_{x \in U})\) as above, there is then a unique isometry \( J : \mathcal{H}(K) \to L^2(\mu) \) specified by
\[
J(K(\cdot, x)) = r_x.
\]

(ii) The adjoint operator of \( J \) is \( L := J^* : L^2(\mu) \to \mathcal{H}(K) \), given by
\[
(Lh)(x) = \int_S h(s) r_x(s) \, d\mu(s),
\]
\( \forall h \in L^2(\mu), \text{ and } x \in U \).

(iii) \( Q := JJ^* = L^*L \) is the projection in \( L^2(\mu) \) onto the closed subspace spanned by \( \{r_x(\cdot) : x \in U\} \subseteq L^2(\mu) \).

**Proof.** (i) Let the setting be as specified in the Proposition. Define \( J \) as in (6.1) and extend by linearity, first on finite linear combinations
\[
J \left( \sum_{i} \alpha_i K(\cdot, x_i) \right) = \sum_{i} \alpha_i r_{x_i}(\cdot).
\]
It is isometric, since
\[
\left\| \sum_{i} \alpha_i K(\cdot, x_i) \right\|_{\mathcal{H}(K)}^2 = \sum_{i} \sum_{j} \alpha_i \alpha_j K(x_i, x_j),
\]
and
\[
\int_S \left| \sum_{i} \alpha_i r_{x_i}(\cdot) \right|^2 \, d\mu = \sum_{i} \sum_{j} \alpha_i \alpha_j \int_S r_{x_i} r_{x_j} \, d\mu
\]
\[
= \sum_{i} \sum_{j} \alpha_i \alpha_j K(x_i, x_j),
\]
where we used (5.1) in the last step. Since \( J \) is thus isometric on a dense subspace in \( \mathcal{H}(K) \), it extends uniquely by limits, to define an isometry \( J : \mathcal{H}(K) \to L^2(\mu) \) as required in (i).
(ii) We now turn to the operator $L : L^2(\mu) \to \mathcal{H}(K)$ as in (6.2). The important point is that $L$ maps into $\mathcal{H}(K)$. This follows from an application of Lemma 2.4 as follows. Let $n \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$, $x_i \in U$, $1 \leq i \leq n$; then
\[
\left| \sum_i \alpha_i (Lh)(x_i) \right|^2 = \left| \int S \left( \sum_i \alpha_i r_{x_i} \right) h d\mu \right|^2 \\
\leq \text{Schwarz} \int S \left| \sum_i \alpha_i r_{x_i} \right|^2 d\mu \int |h|^2 d\mu \\
= \sum_i \sum_j \alpha_i \alpha_j K(x_i, x_j) \|h\|^2_{L^2(\mu)},
\]
and the conclusion now follows from Lemma 2.4.

(iii) We have, since $Lh \in \mathcal{H}(K)$,
\[
\langle K(\cdot, x), Lh \rangle_{\mathcal{H}(K)} = (Lh)(x) \\
= \int S r_x h d\mu = \langle r_x, h \rangle_{L^2(\mu)},
\]
so $L^* (K(\cdot, x)) = r_x$, $L^* = J$, and $J^* = L$, now follow from (i). Since $J$ is isometric, $Q = JJ^*$ is the projection specified in (iii) in the Proposition. \hfill \square

7. Hilbert spaces of signed measures, and of distributions

Let $K : U \times U \to \mathbb{R}$ be a fixed positive definite (p.d.) kernel, and let $\mathcal{H}(K)$ be the corresponding RKHS. Suppose $U$ is a metric space, and that $K$ is continuous. (This is not a strict restriction since $K$ automatically induces a metric $d_K$ on $U$ given by
\[
d_K(x, y) = \|K(\cdot, x) - K(\cdot, y)\|_{\mathcal{H}(K)} \\
= (K(x, x) + K(y, y) - 2K(x, y))^{\frac{1}{2}},
\]
and
\[
|K(x_1, y) - K(x_2, y)| \leq d_K(x_1, x_2) K(y, y)^{\frac{1}{2}}, \forall x_1, x_2, y \in U.
\]

Introduce the Dirac delta measures $\{\delta_x\}, x \in U$, we get $\delta_x K \delta_y = K(x, y)$; or more precisely,
\[
\int_U \int_U K(s, t) d\delta_x(s) d\delta_y(t) = K(x, y).
\]
If $n \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$, $x_i \in U$, $1 \leq i \leq n$, set
\[
\xi := \sum_{i=1}^n \alpha_i \delta_{x_i},
\]
and we get
\[
\xi K \xi = \sum_i \sum_j \alpha_i \alpha_j K(x_i, x_j) = \left\| \sum_i \alpha_i K(\cdot, x_i) \right\|^2_{\mathcal{H}(K)}.
\]
Hence, if we complete the measures from (7.3) with respect to (7.4), we arrive at a Hilbert space $\mathcal{L}(K)$ consisting of signed measures, or of more general linear functionals, e.g., distributions.

**Proposition 7.1.** Let $K$, $U$, $\mathcal{H}(K)$ and $\mathcal{L}(K)$ be specified as above; then
\[
J \left( \sum \alpha_i \delta_i \right) := \sum \alpha_i K(\cdot, x_i)
\]
defines an isometry of $\mathcal{L}(K)$ onto $\mathcal{H}(K)$. 


Proof. This follows from the definitions. In particular, for \( \xi \in \mathcal{L}(K) \), we have, by (7.4)
\[
\xi K \xi = \| J \xi \|_{\mathcal{H}(K)}^2.
\] (7.6)

Corollary 7.2. Let \( K, U, \mathcal{H}(K) \) and \( \mathcal{L}(K) \) be as stated in Proposition 7.1, and let \((S, \mathcal{B}, \mu)\) be a \( \sigma \)-finite measure space such that \( L^2(\mu) \) is a feature space; see Definition 5.1. Then a signed measure \( \xi \) is in \( \mathcal{L}(K) \) if and only if
\[
\int_S \left| \int_U r_x(s) d\xi(x) \right|^2 d\mu(s) < \infty; \tag{7.7}
\]
and in this case \( \xi K \xi = \text{the RHS in (7.7)} \).

Proof. Since \( \mathcal{L}(K) \xrightarrow{J} \mathcal{H}(K) \) and \( K(\cdot, x) \mapsto r_x \) extends by limiting and closure to an isometry \( \mathcal{H}(K) \rightarrow L^2(\mu) \), by Proposition 6.1, the following computation is valid when \( \xi \in \mathcal{L}(K) \), and vice versa:
\[
\int_U \int_U K(x, y) d\xi(x) d\xi(y) = \int_U \int_U \int_S r_x(s) r_y(s) d\mu(s) d\xi(x) d\xi(y)
= \int_S \left( \int_U r_x(s) d\xi(x) \right)^2 d\mu(s) = \text{RHS (7.7)}.
\]

Remark 7.3. If \( U \times U \xrightarrow{K} \mathbb{R} \) (or \( \mathbb{C} \)) is \( C^\infty \), or analytic for suitable choices of \( U \) and \( K \), then we may have distribution solutions \( \xi \) to \( \xi K \xi < \infty \). A simple example illustrating this is \( U = (-1, 1) \) = the interval, and
\[
K(x, y) = \frac{1}{1 - xy}.
\]
For \( n \in \mathbb{N} \), let \( \xi = \delta_0^{(n)} \) = the \( n \)th derivative of the Dirac measure at \( x = 0 \). Then
\[
\int_U K(x, y) d\xi(y) = \frac{n! x^n}{(1 - xy)^{n+1}} \bigg|_{y=0} = n! x^n;
\]
and
\[
\int_U \int_U K(x, y) d\xi(x) d\xi(y) = (n!)^2.
\]
In fact,
\[
\delta_0^{(n)} K \delta_0^{(m)} = \delta_{n,m} (n!)^2, \forall n, m \in \mathbb{N};
\]
so the system \( \{\delta_0^{(n)} \}_{n \in \{0\} \cup \mathbb{N}} \) is orthogonal and total in \( \mathcal{L}(K) \). If \( x \in U \setminus \{0\} \), then
\[
\delta_x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \delta_0^{(n)},
\]
as an identity for compactly supported distributions.

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(Palle E.T. Jorgensen) Department of Mathematics, The University of Iowa, Iowa City, IA 52242-1419, U.S.A. E-mail address: palle-jorgensen@uiowa.edu URL: http://www.math.uiowa.edu/~jorgen/

(Feng Tian) Department of Mathematics, Hampton University, Hampton, VA 23668, U.S.A. E-mail address: feng.tian@hamptonu.edu