Attraction to and repulsion from a subset of the unit sphere for isotropic stable Lévy processes

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November 15, 2019

Abstract

Taking account of recent developments in the representation of $d$-dimensional isotropic stable Lévy processes as self-similar Markov processes, we consider a number of new ways to condition its path. Suppose that $\Omega$ is a region of the unit sphere $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$. We construct the aforesaid stable Lévy process conditioned to approach $\Omega$ continuously from either inside or outside of the sphere. Additionally, we show that these processes are in duality with the stable process conditioned to remain inside the sphere and absorb continuously at the origin and to remain outside of the sphere, respectively. Our results extend the recent contributions of [7], where similar conditioning is considered, albeit in one dimension. As in [7], we appeal to recent fluctuation identities related to the deep factorisation of stable processes, cf. [10, 12, 14].

Key words: Stable process, radial excursion, time reversal, duality.

Mathematics Subject Classification: 60J80, 60E10.

1 Introduction

Let $X = (X_t, t \geq 0)$ be a $d$-dimensional stable Lévy process with probabilities $(\mathbb{P}_x, x \in \mathbb{R}^d)$. This means that $X$ has càdlàg paths with stationary and independent increments as well as respecting a property of self-similarity: There is an $\alpha > 0$ such that, for $c > 0$, and $x \in \mathbb{R}^d \setminus \{0\}$, under $\mathbb{P}_x$, the law of $(cX_{c^{-\alpha}t}, t \geq 0)$ is equal to $\mathbb{P}_{cx}$. It turns out that stable Lévy processes necessarily have the scaling index $\alpha \in (0, 2]$. The case $\alpha = 2$ pertains to a standard $d$-dimensional Brownian motion, thus has a continuous path. The processes we
construct are arguably less interesting in the diffusive setting and thus we restrict ourselves to the pure jump setting of \( \alpha \in (0, 2) \).

Although Brownian motion is isotropic, this need not be the case in the stable case when \( \alpha \in (0, 2) \). Nonetheless, we will restrict our work to the isotropic setting. To be more precise, this means, for all orthogonal transformations \( U : \mathbb{R}^d \mapsto \mathbb{R}^d \) and \( x \in \mathbb{R}^d \),

\[
\text{the law of } (UX_t, t \geq 0) \text{ under } \mathbb{P}_x \text{ is equal to } (X_t, t \geq 0) \text{ under } \mathbb{P}_{UX}.
\]

For convenience, we will henceforth refer to \( X \) just as a stable process.

As a Lévy process, our stable Lévy process of index \((0,2)\) has a characteristic triplet \((0,0,\Pi)\), where the jump measure \( \Pi \) satisfies

\[
\Pi(B) = \frac{2^\alpha \Gamma((d + \alpha)/2)}{\pi^{d/2} |\Gamma(-\alpha/2)|} \int_B \frac{1}{|y|^\alpha d} dy, \quad B \subseteq B(\mathbb{R}^d).
\]

This is equivalent to identifying its characteristic exponent as

\[
\Psi(\theta) = -\frac{1}{t} \log \mathbb{E}(e^{i\theta \cdot X_t}) = |\theta|^\alpha, \quad \theta \in \mathbb{R}^d,
\]

where we write \( \mathbb{P} \) in preference to \( \mathbb{P}_0 \).

In this article, we characterise the law of a stable process conditioned to hit continuously a part of the surface, say \( \Omega \subseteq S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\} \), either from the inside or from the outside of the unit sphere. We develop an expression for the limiting point of contact on \( \Omega \). Moreover, we show that, when time reversed from the strike point on \( \Omega \), the resulting process can also be seen as a conditioned stable process. The extreme cases that \( \Omega = S^{d-1} \) (the whole unit sphere) and \( \Omega = \{\vartheta\} \in S^{d-1} \) (a single point on the unit sphere) are included in our analysis, however, we will otherwise insist that the Lebesgue surface measure of \( \Omega \) is strictly positive.

Our results relate to the recent work of [7], who considered a real valued Lévy process conditioned to continuously approach the boundary of the interval \([-1, 1]\) from the outside. In order to avoid repetition, we always remain in two or more dimensions. As in [7], we rely heavily on recent fluctuation identities that are connected to the deep factorisation of the stable process; cf. [10] [12] [14].

2 Attraction towards \( \Omega \)

For convenience, we will work with the definition \( \mathbb{B}_d = \{x \in \mathbb{R}^d : |x| < 1\} \). Let \( \mathcal{D}(\mathbb{R}^d) \) denote the space of càdlàg paths \( \omega : [0, \infty) \to \mathbb{R}^d \cup \varnothing \) with lifetime \( k(\omega) = \inf\{s > 0 : \omega(s) = \varnothing\} \), where \( \varnothing \) is a cemetery point. The space \( \mathcal{D}(\mathbb{R}^d) \) will be equipped with the Skorokhod topology, with its Borel \( \sigma \)-algebra \( \mathcal{F} \) and natural filtration \( (\mathcal{F}_t, t \geq 0) \). The reader will note that we will also use a similar notion for \( \mathbb{D}(\mathbb{R} \times S^{d-1}) \) later on in this text in the obvious way. We will always work with \( X = (X_t, t \geq 0) \) to mean the coordinate process defined on the space \( \mathbb{D}(\mathbb{R}^d) \). Hence, the notation of the introduction indicates that \( \mathbb{P} = (\mathbb{P}_x, x \in \mathbb{R}^d) \) is such that \( (X, \mathbb{P}) \) is our stable process.
We want to construct the law of $X$ conditioned to approach $\Omega$ continuously from within $\mathbb{B}_d^c := \mathbb{R}^d \setminus \mathbb{B}_d$. Similarly, we want the law of $X$ conditioned to approach $\Omega \subseteq S^{d-1}$ continuously from within $\mathbb{B}_d$. More precisely, via an appropriate limiting procedure, we want to build a new family of probabilities $P^\vee = (P^\vee_x, x \in \mathbb{B}_d^c)$ such that

$$P^\vee_x(X_s \in \mathbb{B}_d^c, s < k \text{ and } X_{s^-} \in \Omega) = 1, \quad x \in \mathbb{B}_d^c,$$

with a similar statement holding when the conditioning is undertaken from within $\mathbb{B}_d$.

As we are considering two or higher dimensions, the process $(X, P)$ is transient in the sense that $\lim_{t \to \infty} |X_t| = \infty$ almost surely. Defining

$$G(t) := \sup\{s \leq t : |X_s| = \inf_{u \leq s} |X_u|\}, \quad t \geq 0,$$

we thus have by monotonicity and the transience of $(X, P)$ that $G(\infty) := \lim_{t \to \infty} G(t)$ exists and, moreover, $X_{G(\infty)}$ describes the point of closest reach to the origin in the range of $X$.

We can similarly define $\overline{G}(t) = \sup\{s \leq t : |X_s| = \sup_{u \leq s} |X_u|\}$, $t \geq 0$, so that $\overline{G}(\tau_1^\ominus)$ is the point of furthest reach from the origin prior to exiting $\mathbb{B}_d$, where

$$\tau_1^\ominus = \inf\{t > 0 : |X_t| > 1\}.$$

Now define $A_\varepsilon = \{r\theta : r \in (1, 1+\varepsilon), \theta \in \Omega\}$ and $B_\varepsilon = \{r\theta : r \in (1-\varepsilon, 1), \theta \in \Omega\}$, for $0 < \varepsilon < 1$ and define the corresponding events $C_\varepsilon^\vee := \{X_{G(\infty)} \in A_\varepsilon\}$, and $C_\varepsilon^\wedge := \{X_{G(\tau_1^\ominus)} \in B_\varepsilon\}$.

Let

$$\tau_1^\ominus = \inf\{t > 0 : |X_t| < 1\}.$$

We are interested in the asymptotic conditioning

$$P^\vee_x(A, t < k) = \lim_{\varepsilon \to 0} P_x(A, t < \tau_1^\ominus | C_\varepsilon^\vee)$$

when $x \in \mathbb{B}_d^c$ and

$$P^\wedge_x(A, t < k) = \lim_{\varepsilon \to 0} P_x(A, t < \tau_1^\ominus | C_\varepsilon^\wedge)$$

when $x \in \mathbb{B}_d$, for all $A \in \mathcal{F}_t$.

In the setting that $\Omega = \{\emptyset\} \in \mathbb{S}^{d-1}$, we can adapt slightly the sets $A_\varepsilon$ and $B_\varepsilon$ so that $A_\varepsilon = \{r\phi : r \in (1, 1+\varepsilon), \phi \in \mathbb{S}^{d-1}, |\phi - \emptyset| < \varepsilon\}$ and $B_\varepsilon = \{r\phi : r \in (1-\varepsilon, 1), \phi \in \mathbb{S}^{d-1}, |\phi - \emptyset| < \varepsilon\}$.

We will go a little further in due course and give a fuller description of these two conditioned processes by including the cases that $X$ is issued from the unit sphere itself but not within $\Omega$, i.e. $\mathbb{S}^{d-1} \setminus \Omega$. For now, we have our first main result, given immediately below, for which we define the function

$$H_\Omega(x) = \begin{cases} \int_\Omega |\theta - x|^{-d}||x|^2 - 1|^{\alpha/2}\sigma_1(d\theta) & \text{if } \sigma_1(\Omega) > 0 \\ |\emptyset - x|^{-d}||x|^2 - 1|^{\alpha/2} & \text{if } \Omega = \{\emptyset\}, \end{cases}$$

for $|x| \neq 1$, where $\sigma_1(d\theta)$ is the Lebesgue surface measure on $S^{d-1}$ normalised to have unit mass.
Theorem 1 (Stable process conditioned to attract to Ω continuously from one side). Let \( \Omega \subseteq \mathbb{S}^{d-1} \) be an open set with \( \sigma_1(\Omega) > 0 \) or \( \Omega = \{ \vartheta \} \) for a fixed point \( \vartheta \in \mathbb{S}^{d-1} \). Then for all points of issue \( x \in \mathbb{R}^d \setminus \mathbb{S}^{d-1} \) we have

\[
\frac{d\mathbb{P}^\vee_x}{d\mathbb{P}^\wedge_x} \bigg|_{\mathcal{F}_t} = 1_{(t<\tau^\Omega)} \frac{H_\Omega(X_t)}{H_\Omega(x)}, \quad \text{if } x \in \mathbb{B}^c_d, \\
\frac{d\mathbb{P}^\wedge_x}{d\mathbb{P}^\vee_x} \bigg|_{\mathcal{F}_t} = 1_{(t<\tau^\Omega)} \frac{H_\Omega(X_t)}{H_\Omega(x)}, \quad \text{if } x \in \mathbb{B}_d.
\]

(3) and (4)

Remark 1. The choice of limiting conditioning procedure that we have used reflects a similar approach taken in [7] in one dimension. It is worth noting at this point that the choice of \( C^\vee_\epsilon \) and \( C^\wedge_\epsilon \) are by no means the only possibilities as far as performing a limiting conditioning that results in (3) and (4). For example, once the reader is familiar with the proof of Theorem 1, it will quickly become clear that, when \( \Omega \) is not a singleton, by defining e.g. \( C^\vee_\epsilon = \{ X^\tau_\vartheta \in B_\epsilon \} \), or indeed \( C^\vee_\epsilon = \{ X^\tau_\vartheta \in A_\epsilon \} \), the limit (1) will still produce the change of measure (3). Once the reader is familiar with the proof of Theorem 1, it is a worthwhile exercise to verify the two proposed alternative definitions of \( C^\vee_\epsilon \) for the limiting process by appealing to the fluctuation identities in e.g. [12]. Other definitions of \( C^\vee_\epsilon \) giving a consistent limit may indeed also be possible.

Whilst the above theorem deals with the construction of the conditioned process up to but not including its terminal position, we characterise the latter in the next result.

Proposition 1 (Distribution of the hitting location). Suppose that \( \Omega \subseteq \mathbb{S}^{d-1} \) be an open set with \( \sigma_1(\Omega) > 0 \). Let \( \Omega' \) be a measurable subset of \( \Omega \). Then for any \( x \in \mathbb{R}^d \setminus \mathbb{B}_d \), we have

\[
\mathbb{P}^\vee_x(X_{k^-} \in \Omega') = \frac{\int_{\Omega'} |\theta - x|^{-d}\sigma_1(d\theta)}{\int_{\Omega} |\theta - x|^{-d}\sigma_1(d\theta)},
\]

(5)

with an identical result holding for \( X_{k^-} \) under \( \mathbb{P}^\wedge_x \), with \( x \in \mathbb{B}_d \).

3 Lamperti–Kiu representation and radial excursions

The basic definition of the stable process conditioned to attract continuously to \( \Omega \) from one side is not quite complete. Strictly speaking, we could think about defining the process to include the points of issue in \( \mathbb{S}^{d-1} \setminus \Omega \). It turns out that this is possible. However, we first need to remind the reader of the recently described radial excursion theory, see [12, 13]. The starting point for the aforementioned is the Lamperti–Kiu transform which identifies the stable process as a self-similar Markov process.

To describe it, we need to introduce the notion of a Markov Additive Process, henceforth written MAP for short. Let \( \mathbb{S}^{d-1} = \{ x \in \mathbb{R}^d : |x| = 1 \} \). With an abuse of previous notation, we say that \((\Xi, \Upsilon) = ((\Xi_t, \Upsilon_t), t \geq 0)\) is a MAP if it is Feller process on \( \mathbb{R} \times \mathbb{S}^{d-1} \), with probabilities \( \mathbb{P}_{x, \theta}, x \in \mathbb{R}^d, \theta \in \mathbb{S}^{d-1} \), such that, for any \( t \geq 0 \), the conditional law of the
process \((\Xi_{s+t} - \Xi_t, \Upsilon_{s+t}) : s \geq 0\), given \((\Xi_u, \Upsilon_u), u \leq t\), is that of \((\Xi, \Upsilon)\) under \(P_{0,\theta}\), with \(\theta = \Upsilon_t\). For a MAP pair \(((\Xi_t, \Upsilon_t), t \geq 0)\), we call \(\Xi\) the ordinate and \(\Upsilon\) the modulator.

According to one of the main results in \([1]\), there exists a MAP, which we will henceforth write as \((\xi, \Theta)\), with probabilities \(P = (P_{x, \theta}, x \in \mathbb{R}^d, \theta \in \mathbb{S}^{d-1})\) such that the \(d\)-dimensional stable process can be written

\[
X_t = \exp\{\phi(t)\} \Theta \phi(t) \quad t \geq 0,
\]

where

\[
\phi(t) = \inf \left\{ s > 0 : \int_0^s e^{\alpha \xi_u} du > t \right\}.
\]

Whilst \(\Theta\) alone is a Feller process, it is not necessarily true that \(\xi\) alone is. However, it is a consequence of isotropy that this is the case here. Moreover, \(\xi\) alone is a Lévy process whose characteristic exponent is known (but not important in the current context); see for example \([5]\). What is important for our purposes is to note for now that it has paths of unbounded variation, and therefore is regular for the upper and lower half line (in the sense of Definition 6.4 of \([9]\)).

It is not difficult to show that the pair \(((\xi_t - \xi_\ell, \Theta_t), t \geq 0)\), forms a strong Markov process, where \(\xi_\ell := \inf_{s \leq t} \xi_s, t \geq 0\) is the running minimum of \(\xi\). On account of the fact that \(\xi\) alone, is a Lévy process, \((\xi_t - \xi_\ell, t \geq 0)\) is also a strong Markov process. Suppose we denote by \(\ell = (\ell_t, t \geq 0)\) the local time at zero of \(\xi - \xi_\ell\) then we can introduce the following processes

\[
H^- t = -\xi_\ell t, \quad \Theta^- t = \Theta_\ell t, \quad t \geq 0,
\]

and define \((H^- t, \Theta^- t) = (\partial, \Xi)^t\), a cemetery state, if \(\ell_t = \infty\). Then, the pair \((\ell_- t, H^-)\), without reference to the associated moduation \(\Theta^-\), are Markovian and play the role of the descending ladder time and height subordinators of \(\xi\). Moreover, the strong Markov property tells us that \((\ell^- t, H^- t, \Theta^- t), t \geq 0\), defines a Markov Additive Process, whose first two elements are ordinates that are non-decreasing. In this sense, \(\ell\) also serves as an adequate choice for the local time of the Markov process \((\xi - \xi_\ell, \Theta)\) on the set \(\{0\} \times \mathbb{S}^{d-1}\).

Suppose we define \(g_t = \sup\{s < t : \xi_s = \xi_\ell\}\), and recall that the regularity of \(\xi\) for \((-\infty, 0)\) and \((0, \infty)\) ensures that it is well defined, as is \(g_\infty = \lim_{t \to \infty} g_t\). Set

\[
d_t = \inf\{s > t : \xi_s = \xi_\ell\}.
\]

For all \(t > 0\) such that \(d_t > g_t\), the process

\[
(\epsilon_{g_t}(s), \Theta_{g_t} s) := (\xi_{g_t+s} - \xi_{g_t}, \Theta_{g_t+s}), \quad s \leq \xi_{g_t} := d_t - g_t,
\]

codes the excursion of \((\xi - \xi_\ell, \Theta)\) from the set \(\{0, \mathbb{S}^{d-1}\}\) which straddles time \(t\). Such excursions live in the space \(\mathbb{U}(\mathbb{R} \times \mathbb{S}^{d-1})\), the space of càdlàg paths in \(\mathbb{R} \times \mathbb{S}^{d-1}\), written in canonical form

\[
(\epsilon, \Theta^\epsilon) = ((\epsilon(t), \Theta^\epsilon(t)) : t \leq \zeta) \text{ with lifetime } \zeta = \inf\{s > 0 : \epsilon(s) < 0\},
\]
such that \((\epsilon(0), \Theta^\epsilon(0)) \in \{0\} \times \mathbb{S}^{d-1}, (\epsilon(s), \Theta^\epsilon(s)) \in (0, \infty) \times \mathbb{S}^{d-1}\), for \(0 < s < \zeta\), and \(\epsilon(\zeta) \in (-\infty, 0]\)
Taking account of the Lamperti–Kiu transform \([6]\) it is natural to consider how the excursion of \((\xi - \xi, \Theta)\) from \(\{0\} \times S^{d-1}\) translates into a radial excursion theory for the process

\[ Y_t := e^{\xi t} \Theta_t, \quad t \geq 0. \]

Ignoring the time change in \([6]\), we see that the radial minima of the process \(Y\) agree with the radial minima of the stable process \(X\). Indeed, each excursion of \((\xi - \xi, \Theta)\) from \(\{0\} \times S^{d-1}\) is uniquely associated to exactly one excursion of \((Y_t/\inf_{s \leq t}|Y_s|, t \geq 0)\), from \(S^{d-1}\), or equivalently an excursion of \(Y\) from its running radial infimum. Moreover, we see that, for all \(t > 0\) such that \(d_t > g_t\),

\[ Y_{g_t+s} = e^{\xi g_t} e^{\xi (s)} \Theta_{g+t}^\epsilon(s) = |Y_{g_t}| e^{\xi g_t} \Theta_{g+t}^\epsilon(s), \quad s \leq \xi_{g+t}. \]

This will be useful to keep in mind for the forthcoming excursion computations.

For \(t > 0\), let \(R_t = d_t - t\), and define the set \(G = \{ t > 0 : R_t = 0, R_t > 0 \} = \{ g_t : s \geq 0 \}\). The classical theory of exit systems in \([15]\) (see Theorem (4.1) therein) now implies that there exists an additive functional \((\Lambda_t, t \geq 0)\) and a family of excursion measures, \((N_\theta, \theta \in S^{d-1})\) such that:

(i) \(\Lambda\) is an additive functional of \((\xi, \Theta)\), has a bounded 1-potential and is carried by the set of times \(\{ t > 0 : (\xi_t - \xi, \Theta_t) \in \{0\} \times S^{d-1} \}\),

(ii) the map \(\theta \mapsto N_\theta\) is an \(S^{d-1}\)-indexed kernel on \(\mathbb{U}(\mathbb{R} \times S^{d-1})\) such that \(N_\theta(1 - e^{-\zeta}) < \infty\);

(iii) we have the exit formula

\[
\mathbb{E}_{x,\theta} \left[ \sum_{g \in G} F((\xi_s, \Theta_s) : s < g) H((\epsilon_g, \Theta_g^\epsilon)) \right] = \mathbb{E}_{x,\theta} \left[ \int_0^\infty F((\xi_s, \Theta_s) : s < t) N_{\Theta_t}(H(\epsilon, \Theta^\epsilon)) d\Lambda_t \right],
\]

for \(x \neq 0\), where \(F\) is continuous on the space of càdlàg paths \(\mathbb{D}(\mathbb{R} \times S^{d-1})\) and \(H\) is measurable on the space of càdlàg paths \(\mathbb{U}(\mathbb{R} \times S^{d-1})\);

(iv) under any measure \(N_\theta\), the process \(((\epsilon(s), \Theta^\epsilon(s)), s < \zeta)\) is Markovian with the same semigroup as \((\xi, \Theta)\) killed at its first hitting time of \((-\infty, 0] \times S^{d-1}\).

The couple \((\Lambda, N)\) is called an exit system. Note that in Maisonneuve’s original formulation, the pair \(\Lambda\) and the kernel \(N\) is not unique, but once \(\Lambda\) is chosen the measures \((N_\theta, \theta \in S^{d-1})\) are determined up to \(\Lambda\)-neglectable sets, i.e. sets \(A\) such that \(\mathbb{E}_{x,\theta}(\int_{t \geq 0} 1_{[(\xi_t - \xi, \Theta_t) \in A]} d\Lambda_t) = 0\).

Now referring back to the existence of \(\ell\), since it is an additive functional with a bounded 1-potential, there is an exit system which corresponds to \((\ell, N)\). Henceforth, this is the exit system we will work with and the system of excursion associated to it is what we call our radial excursion theory.

The reader will note that one may similarly construct a radial excursion theory based on the MAP \((\xi - \xi, \Theta)\), where \(\xi\) is the process \(\xi_t = \sup_{s \leq t} \xi_s, t \geq 0\). As such we can pair with
the local time of \((\xi - \zeta, \Theta)\) at the origin with a family of excursion measures \((\mathbb{N}_\theta, \theta \in \mathbb{S}^{d-1})\) on \(\mathbb{U}(\mathbb{R} \times \mathbb{S}^{d-1})\). Here, the set \(\mathbb{U}(\mathbb{R} \times \mathbb{S}^{d-1})\) consists of càdlàg paths on \(\mathbb{R} \times \mathbb{S}^{d-1}\) such that \((\varepsilon(0), \Theta(0)) \in \{0\} \times \mathbb{S}^{d-1}, (\varepsilon(s), \Theta(s)) \in (-\infty, 0) \times \mathbb{S}^{d-1}\), for \(0 < s < \zeta\), and \(\varepsilon(\zeta) \in [0, \infty)\).

With our excursion theory in hand, we can now proceed to identify the completion of Theorem 1.

**Theorem 2.** The processes \((X, P^\vee)\) and \((X, P^\wedge)\) can be extended in a consistent way to include points of issue \(x \in \mathbb{S}^{d-1} \setminus \Omega\) with pathwise continuous entry via

\[
P_x^\vee(A, t < \zeta) = \mathbb{N}_x \left(1_{(A, t<\zeta)} H_\Omega(X^\varepsilon(t))\right) \quad (9)
\]

and

\[
P_x^\wedge(A, t < \zeta) = \mathbb{N}_x \left(1_{(A, t<\zeta)} H_\Omega(X^\varepsilon(t))\right), \quad (10)
\]

where, for \((\varepsilon, \Theta^\varepsilon)\) selected from \(\mathbb{U}(\mathbb{R} \times \mathbb{S}^{d-1})\) or \(\mathbb{U}(\mathbb{R} \times \mathbb{S}^{d-1})\), respectively,

\[
X^\varepsilon(t) = e^{\varepsilon(t)} \Theta^\varepsilon(t) \quad \text{and} \quad \zeta = \varphi^{-1}(\zeta) = \int_0^\zeta |X^\varepsilon(u)|^{-\alpha} du. \quad (11)
\]

Here, pathwise continuous entry means that

\[
P_x^\vee(\lim_{t \to 0} X_t = x) = P_x^\wedge(\lim_{t \to 0} X_t = x) = 1
\]

for all \(x \in \mathbb{S}^{d-1} \setminus \Omega\).

### 4 Repulsion and duality

In this section, we want to introduce two new processes, which will turn out to be dual to \((X, P^\vee)\) and \((X, P^\wedge)\) in the sense of time reversal. The two processes we are interested give meaning to the stable process conditioned to remain in \(\mathbb{B}_d^c\) and \(\mathbb{B}_d\), respectively, in an appropriate sense.

An important tool that we will make use of in analysing the aforesaid time reversed processes comes through the so-called Riesz–Bogdan–Zak transform, which relates path behaviour of the stable process outside of the unit sphere to its behaviour inside the unit sphere. In order to state it, we need to introduce the process \((X, P^o)\), where the probabilities \(P^o = (P^o_x, x \neq 0)\) are given by

\[
\frac{dP^o_x}{dP_x} \bigg|_{X_t} = \frac{|X_t|^{-d}}{|x|^{-d}}, \quad t \geq 0.
\]

Since \(\alpha < 2 \leq d\), we note that the change of measure rewards paths that approach the origin and punishes paths that wander far from the origin. Intuitively, it is clear that \((X, P^o)\) describes the stable process conditioned to continuously approach the origin. Nonetheless, this heuristic can be made into a rigorous statement, see for example [11, 12, 13, 14]. The reader will also note from these references (and it is easy to prove that) that \((X, P^o)\) is also a self-similar Markov process with the same index of self-similarity as \((X, P)\).
Theorem 3 (Riesz–Bogdan–˙Zak transform). Suppose we write $Kx = x/|x|^2$, $x \in \mathbb{R}^d$ for the classical inversion of space through the sphere $\mathbb{S}^{d-1}$. Then, in dimension $d \geq 2$, for $x \neq 0$, $(KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}_K^0)$, where $\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} u > t\}$.

Let us return to our duality concerns. To this end, let us introduce the probabilities

$$H^\oplus(x) = \mathbb{P}_x(\tau^\oplus = \infty) = \frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)} \int_0^{\infty} (u+1)^{-d/2}u^{\alpha/2-1}du,$$

for $|x| > 1$, where the second inequality is lifted from [3], and,

$$H^\ominus(x) = |x|^d H^\ominus(Kx),$$

for $|x| < 1$.

These two functions can be used to define the two families of probabilities $\mathbb{P}^\ominus = (\mathbb{P}_x^\ominus, |x| > 1)$ and $\mathbb{P}^\oplus = (\mathbb{P}_x^\oplus, |x| < 1)$ via the changes of measure,

$$\frac{d\mathbb{P}_x^\ominus}{d\mathbb{P}_x} \bigg|_{\mathcal{F}_t} = \frac{H^\ominus(X_t)}{H^\ominus(x)} 1_{(t < \tau^\ominus)}, \quad t \geq 0, |x| > 1. \tag{13}$$

and,

$$\frac{d\mathbb{P}_x^\oplus}{d\mathbb{P}_x} \bigg|_{\mathcal{F}_t} = \frac{H^\ominus(X_t)}{H^\ominus(x)} 1_{(t < \tau^\oplus)}, \quad t \geq 0, |x| < 1. \tag{14}$$

The first of these two changes of measure corresponds to the stable process conditioned to avoid entering $\mathbb{B}_d$ by a simple restriction on the probability space (remembering that $\lim_{t \to \infty} |X_t| = \infty$). Noting from Theorem 3 that

$$H^\ominus(Kx) = \mathbb{P}_x^\ominus(\tau^\ominus = \infty) = \mathbb{P}_x^0(\tau^{(0)} < \tau^\ominus),$$

where $\tau^{(0)} = \inf\{t > 0 : |X_t| = 0\}$. The second change of measure, [14], is a composition of conditioning the stable process to be absorb continuously at the origin, followed by conditioning it not to exit $\mathbb{B}_d$ via a simple restriction on the probability space (noting that $\lim_{t \to \infty} |X_t| = 0$ under $\mathbb{P}_x^0$).

The reader will also note that the Riesz-Bogdan-˙Zak transform also implies a similar spatial inversion and time change must hold for the pair $(X, \mathbb{P}^\ominus)$ and $(X, \mathbb{P}^\oplus)$.

Corollary 1. For $|x| > 1$, $(KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}^\ominus_x$ is equal in law to $(X, \mathbb{P}^\ominus_K)$, where $\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} u > t\}$. Similarly, for $|x| < 1$, $(KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}^\oplus_x$ is equal in law to $(X, \mathbb{P}^\oplus_K)$.

Proof. Suppose that $F(X_s, s \leq t)$ is a bounded $\mathcal{F}_t$-measurable function for each $t \geq 0$. Then, for $|x| > 1$, appealing to Theorem 3 we have

$$\mathbb{E}^\ominus_x[F(KX_{\eta(s)}, s \leq t)] = \mathbb{E}_x \left[ F(KX_{\eta(s)}, s \leq t) \frac{H^\ominus(K(KX_{\eta(t)}))}{H^\ominus(x)} 1_{(\eta(t) < \tau^\ominus)} \right]$$

$$= \mathbb{E}^0_{Kx} \left[ F(X_s, s \leq t) \frac{H^\ominus(KX_t)}{H^\ominus(x)} 1_{(t < \tau^\ominus)} \right]$$

$$= \mathbb{E}_{Kx} \left[ F(X_s, s \leq t) \frac{|X_t|^{\alpha-d}}{Kx^{\alpha-d}} \frac{H^\ominus(KX_t)}{H^\ominus(K(Kx))} 1_{(t < \tau^\ominus)} \right]$$

$$= \mathbb{E}^\ominus_{Kx} [F(X_s, s \leq t)].$$
This shows the first half of the claim. The second part of the claim is proved using the same technique and the details are omitted for brevity given how straightforward they are. □

In the spirit of other cases of conditionings from an extreme boundary point (e.g. conditioning a Lévy process to avoid the origin, cf. [17], or to stay positive, cf. [6]), we can extend the definitions given in (13) and (14) by appealing to the Markov property of the excursion measures \( \mathbb{N}_x \) and \( \mathbb{N}_x, x \in S^{d-1} \).

**Theorem 4.** The processes \((X, P^\ominus)\) and \((X, P^\oplus)\) can be extended in a consistent way to include points of issue on \(S^{d-1}\). Specifically, for \(A \in \mathcal{F}_t\),

\[
P^\ominus(A) = \mathbb{N}_x \left( 1_{(A, t<\varsigma)} H^\ominus(X_t^\varsigma) \right)
\]

and similarly

\[
P^\oplus(A) = \mathbb{N}_x \left( 1_{(A, t<\varsigma)} H^\oplus(X_t^\varsigma) \right),
\]

where we have used the notation given in (11).

Our objective is to pair up \((X, P^\vee)\), \((X, P^\ominus)\) and \((X, P^\wedge)\), \((X, P^\oplus)\) via Nagasawa’s duality theorem for time reversal; cf [16]. To this end we need to introduce the notion of \(L\)-times.

Suppose that \(Y = (Y_t, t \leq \zeta)\) with probabilities \(P_x, x \in E\), is a regular Markov process on an open domain \(E \subseteq \mathbb{R}^d\) (or more generally, a locally compact Hausdorff space with countable base), with cemetery state \(\Delta\) and killing time \(\zeta = \inf \{t > 0 : Y_t = \Delta \}\). Let us additionally write \(P_\nu = \int_E \nu(da)P_a\), for any probability measure \(\nu\) on the state space of \(Y\).

Suppose that \(\mathcal{G}\) is the \(\sigma\)-algebra generated by \(Y\) and write \(\mathcal{G}(P_\nu)\) for its completion by the null sets of \(P_\nu\). Moreover, write \(\overline{\mathcal{G}} = \bigcap_\nu \mathcal{G}(P_\nu)\), where the intersection is taken over all probability measures on the state space of \(Y\), excluding the cemetery state. A finite random time \(k\) is called an \(L\)-time (generalized last exit time) if

(i) \(k \leq \zeta\) and \(k\) is measurable in \(\overline{\mathcal{G}}\),

(ii) \(\{s < k(\omega) - t\} = \{s < k(\omega_t)\}\) for all \(t, s \geq 0\),

where \(\omega_t\) is the Markov shift of \(\omega\) to time \(t\). The most important examples of \(L\)-times are killing times and last exit times.

**Theorem 5.** In what follows, we work with the probability distribution

\[
\nu(da) := \frac{\sigma_1(da)}{\sigma_1(\Omega)}, \quad a \in \mathbb{R}^d;
\]

if \(\Omega\) is open and \(\sigma_1(\Omega) > 0\) and, otherwise, if \(\Omega = \{\vartheta\}, \vartheta \in S^{d-1}\), we understand

\[
\nu(da) = \delta_{\{\vartheta\}}(da), \quad a \in \mathbb{R}^d.
\]

(i) For every \(L\)-time \(k\) of \((X, P^\ominus)\), the process \((X_{(k-t)-}, t < k)\) under \(P^\ominus_\nu\) has Markov increments which agree with those of \((X, P^\vee)\).

(ii) Similarly, for every \(L\)-time \(k\) of \((X, P^\ominus)\), the process \((X_{(k-t)-}, t < k)\) under \(P^\ominus_\nu\) has Markov increments which agree with those of \((X, P^\wedge)\).
5 Proof of Theorem 1

We start by recalling two useful identities. In Theorem 1.1 in [12], the law of $X_G(\infty)$ is given by

\[ \mathbb{P}_x(X_G(\infty) \in dz) = c_{\alpha,d} \frac{||x^2 - |z|^2||^{\alpha/2}}{|z|^\alpha} |x - z|^{-d} dz, \quad |x| > |z| > 0 \quad (19) \]

where

\[ c_{\alpha,d} = \pi^{-d/2} \frac{\Gamma(d/2)^2}{\Gamma(d - \alpha/2) \Gamma(\alpha/2)}. \]

Similarly, from Corollary 1.1 of [12], it was also shown that

\[ \mathbb{P}_x(X_{G(\tau_1^\beta)} \in dz, X_{\tau_1^\beta} \in dv) = C_{\alpha,d} \frac{(||z|^2 - |v|^2)^{\alpha/2}}{||z|^2 - |v|^2 |z - v|^d} |z - x|^d dz dv, \quad (20) \]

for $|x| < |z| < 1$ and $|v| > 1$, where

\[ C_{\alpha,d} = \frac{\Gamma(d/2)^2}{\pi^d \Gamma(-\alpha/2) \Gamma(\alpha/2)}. \]

First take $x \in B_0^\beta$. Let $\tau_\beta := \inf \{ t > 0 : |X_t| < \beta \}$ for any $\beta > 1$. For any $A \in \mathcal{F}_t$, define

\[ \mathbb{P}^\vee_x(A, t < \tau^\beta_\beta) = \lim_{\vee \to 0} \mathbb{P}_x(A, t < \tau^\beta_\beta | C^\vee_x) \quad (21) \]

The Markov property gives us

\[ \mathbb{P}_x(A, t < \tau^\beta_\beta | C^\vee_x) = \mathbb{E}_x \left[ 1_{\{A, t < \tau^\beta_\beta \}} \frac{\mathbb{P}_x(C^\vee_x)}{\mathbb{P}_x(C^\vee_x)} \right] \quad (22) \]

In order to prove the Theorem [11] it is enough to prove that, for all $\beta > 1$, (3) is true for sets of the form $A \cap \{ t < \tau^\beta_\beta \} \in \mathcal{F}_t$, in which case the full statement (3) follows by the Monotone Convergence Theorem as we take $\beta \downarrow 1$. Next note from (19) that

\[ \mathbb{P}_x(X_G(\infty) \in A_\varepsilon) = c_{\alpha,d} \int_{z \in A_\varepsilon} \frac{||x^2 - |z|^2||^{\alpha/2}}{|z|^\alpha} |x - z|^{-d} dz = c'_{\alpha,d} \int_1^{1+\varepsilon} \int_\Omega \frac{||x^2 - r^2||^{\alpha/2}}{r^\alpha} |x - r\theta|^{-d} r^{-d+1} dr d\sigma_1(d\theta), \]

where $c'_{\alpha,d}$ is an unimportant constant.

Since $||x^2 - r^2||^{\alpha/2} |x - r\theta|^{-d}$ is continuous at $r = 1$ with fixed $|x| > 1$, for any $\delta > 0$, there exists $\varepsilon > 0$ such that for all $1 < r < 1 + \varepsilon$,

\[ (1 - \delta)||x^2 - r^2||^{\alpha/2} |x - \theta|^{-d} < ||x^2 - r^2||^{\alpha/2} |x - r\theta|^{-d} < (1 + \delta)||x^2 - 1||^{\alpha/2} |x - \theta|^{-d} \]

and

\[ \int_1^{1+\varepsilon} r^{-d+\alpha+1} dr = c\varepsilon^{d-\alpha} + o(\varepsilon^{d-\alpha}). \]
Hence, we have
\[
\lim_{\varepsilon \to 0} \varepsilon^{\alpha-d} \mathbb{P}_x(X_{G(\infty)} \in \mathcal{A}_\varepsilon) = c'_{\alpha,d} \int_\Omega \frac{||x|^2 - 1|^\alpha|}{x - \theta}^{-d} \sigma_1(d\theta),
\]
where $c'_{\alpha,d}$ does not depend on $x$. Note, moreover, that
\[
\sup_{|x| > 1} \int_\Omega \frac{||x|^2 - 1|^\alpha|}{x - \theta}^{-d} \sigma_1(d\theta) < \infty \tag{23}
\]

We can both make use of the limit
\[
\lim_{\varepsilon \to 0} \frac{\mathbb{P}_x(X_{G(\infty)} \in \mathcal{A}_\varepsilon)}{\mathbb{P}_x(X_{G(\infty)} \in \mathcal{A}_\varepsilon)} = \frac{\int_\Omega \theta - X |x|^2 - 1|^{\alpha/2} \sigma_1(d\theta)}{\int_\Omega \theta - x |x|^2 - 1|^{\alpha/2} \sigma_1(d\theta)}. \tag{24}
\]
as well as [23] to ensure the limit may be passed through the expectation in [22] to give [3] on \{t < \tau^0\}, thus giving the desired result.

Next we look at the proof of [4]. From [20] recalling $C_\varepsilon := \{X_{G(\varepsilon)} \in \mathcal{B}_\varepsilon\}$, we have
\[
\mathbb{P}_x(C_\varepsilon) = \mathbb{P}_x(X_{G(\varepsilon \cdot r)} \in \mathcal{B}_\varepsilon)
\]
\[
= C_{\alpha,d} \int_{z \in B_\varepsilon} \int_{v \in \mathbb{B}^d} \frac{(|v|^2 - |x|^2)^{\alpha/2}}{|z - v|^d |z - x|^d} \, dz \, dv
\]
\[
= C'_{\alpha,d} \int_{z \in B_\varepsilon} \frac{(|v|^2 - |x|^2)^{\alpha/2}}{|z - x|^d} \, dz \int_1^\infty \frac{r^{-d-1} \, dr}{(r^2 - |z|^2)^{\alpha/2}} \int_{S^{d-1}(0,r)} \frac{1}{|z - \theta|^d} \, \sigma_r(d\theta), \tag{25}
\]
where $\sigma_r(d\theta)$ is the surface measure on $S^{d-1}(0,r)$, the sphere centred at 0 of radius $r$, normalised to have unit mass and $C'_{\alpha,d}$ is henceforth a constant whose value may change from line to line, which depends only on $\alpha$ and $d$. The Poisson formula (giving the probability that a $d$-dimensional Brownian motion issued from $z$ (with $|z| < 1$) will hit the sphere $S^{d-1}(0,r)$) tells us that
\[
\int_{S^{d-1}(0,r)} \frac{r^{-d-2} (r^2 - |z|^2)}{|z - \theta|^d} \sigma_r(d\theta) = 1, \quad |z| < 1 < r, \tag{26}
\]
see for example Remark III.2.5 in [11]. Putting [26] in [25] gives us
\[
\mathbb{P}_x(C_\varepsilon) = C'_{\alpha,d} \int_{z \in B_\varepsilon} \frac{(|v|^2 - |x|^2)^{\alpha/2}}{|z - x|^d} \, dz \int_1^\infty \frac{r^{-d-1} \, dr}{(r^2 - |z|^2)^{\alpha/2}} \int_{S^{d-1}(0,r)} \frac{1}{r^{d-2} (r^2 - |z|^2)} \, \sigma_r(d\theta)
\]
\[
= C'_{\alpha,d} \int_{z \in B_\varepsilon} \frac{(|v|^2 - |x|^2)^{\alpha/2}}{|z - x|^d} \frac{1}{(1 - |z|^2)^{\alpha/2}} \, dz
\]
\[
= C'_{\alpha,d} \int_{1 - \varepsilon}^1 \int_{\Omega} \frac{(u^2 - |x|^2)^{\alpha/2}}{(1 - u^2)^{\alpha/2}} \, du \, \sigma_1(d\theta)
\]
Since $|u^2 - |x|^2|^{\alpha/2} |x - u\theta|^{-d}$ is continuous at $u = 1$ with fixed $0 < |x| < 1$, for any $\delta > 0$, there exists $\varepsilon > 0$ such that for all $1 - \varepsilon < u < 1$,
\[
(1 - \delta) ||x|^2 - 1|^{\alpha/2} |x - \theta|^{-d} < ||x|^2 - u^2|^{\alpha/2} |x - u\theta|^{-d} < (1 + \delta) ||x|^2 - 1|^{\alpha/2} |x - \theta|^{-d}
\]
and
\[
\int_{1-\epsilon}^{1} \frac{u^{d-1}}{(1-u^2)^{\alpha/2}} \, du = \int_{0}^{\epsilon} \frac{(1-u)^{d-1}}{u^{\alpha/2}(2-u)^{\alpha/2}} \, du = c\epsilon^{-1/2} + o(\epsilon^{-1/2}),
\]
for an unimportant constant \( c > 0 \).

It is now clear that
\[
\lim_{\epsilon \to 0} \epsilon^{\alpha/2-1} \mathbb{P}_x(X_{\mathcal{G}(\tau_1^\varphi)} \in B_\epsilon) = \mathcal{C}_{\alpha,d} \int_{\Omega} ||x||^{2} - 1^{\alpha/2} |x - \theta|^{-d} \sigma_1(d\theta).
\]

Finally, we get again
\[
\lim_{\epsilon \to 0} \mathbb{P}_{x}(X_{\mathcal{G}(\tau_1^\varphi)} \in B_\epsilon) = \int_{\Omega} |\theta - X_t|^{-d}||X_t||^{2} - 1^{\alpha/2} \sigma_1(d\theta) = \int_{\Omega} |\theta - x|^{-d}||x||^{2} - 1^{\alpha/2} \sigma_1(d\theta)
\]
and we can proceed as in (21) noting again the use of (23) for the application of dominated convergence.

When \( \Omega = \{\vartheta\} \), we have \( A_\epsilon = \{r\varphi: r \in (1, 1 + \epsilon), \phi \in \mathbb{S}^{d-1}, |\phi - \vartheta| < \epsilon\} \) and \( B_\epsilon = \{r\varphi: r \in (1 - \epsilon, 1), \phi \in \mathbb{S}^{d-1}, |\phi - \vartheta| < \epsilon\} \), thus it is clear by similar analysis that
\[
\lim_{\epsilon \to 0} \mathbb{P}_{x}(X_{\mathcal{G}(\tau_1^\varphi)} \in A_\epsilon) = \lim_{\epsilon \to 0} \mathbb{P}_{x}(X_{\mathcal{G}(\tau_1^\varphi)} \in B_\epsilon) = \frac{\int_{\Omega} |\theta - X_t|^{-d}||X_t||^{2} - 1^{\alpha/2} \sigma_1(d\theta)}{\int_{\Omega} |\theta - x|^{-d}||x||^{2} - 1^{\alpha/2} \sigma_1(d\theta)}.
\]

The rest of the proof is otherwise a minor adjustment of what we have seen previously, now taking account of the continuity of \( (u, \theta) \mapsto |u^2 - |x||^{\alpha/2} |x - u\theta|^{-d} \) as well as the fact that \( \sup_{|x| > 1} (||x||^{2} - 1^{\alpha/2} |x - \theta|^{-d})/|x|^{\alpha-d} < \infty \).

\[\Box\]

5.1 Proof of Proposition 1

To calculate the hitting distribution, recall that \( A_\epsilon' = \{r\theta: r \in (1, 1 + \epsilon), \theta \in \Omega'\} \), that is the restriction of \( A_\epsilon \) from the set \( \Omega \) to its subset \( \Omega' \subset \Omega \). Then, due to Theorem 1.3 in [12], we have
\[
\lim_{\epsilon \to 0} \mathbb{P}_{x}(X_{\mathcal{G}(\infty)} \in A_\epsilon'|C_\epsilon') = \lim_{\epsilon \to 0} \mathbb{P}_{x}(X_{\mathcal{G}(\infty)} \in A_\epsilon'|X_{\mathcal{G}(\infty)} \in A_\epsilon)
\]
\[
= \lim_{\epsilon \to 0} \frac{\mathbb{P}_{x}(X_{\mathcal{G}(\infty)} \in A_\epsilon')}{\mathbb{P}_{x}(X_{\mathcal{G}(\infty)} \in A_\epsilon)} = \frac{\mathbb{P}_{x}(X_{\mathcal{G}(\infty)} \in A_\epsilon')}{\mathbb{P}_{x}(X_{\mathcal{G}(\infty)} \in A_\epsilon)} = \frac{\int_{\Omega'} |\theta - x|^{-d} \sigma_1(d\theta)}{\int_{\Omega} |\theta - x|^{-d} \sigma_1(d\theta)}
\]
which concludes the statement in the Proposition 1 for the case when \( X \) is issued from outside. Similar computations give the result when \( X \) is issued from inside \( \mathbb{B}_d \).

\[\Box\]

6 Proof of Theorems 2 and 4

Proof of Theorem 2: Let us restrict our attention to the extension of \( (X, \mathbb{P}^\varphi) \) to include \( \mathbb{S}^{d-1} \setminus \Omega \). We need to prove that the proposed definition of \( \mathbb{P}_{\varphi} \), for any \( \varphi \in \mathbb{S}^{d-1} \setminus \Omega \),
is consistent with the definition of \((X, \mathbb{P}^\vee)\) given in Theorem 1 on \(\mathbb{B}_d\), as well as offering continuous entry from the boundary \(S^{d-1} \setminus \Omega\).

From \([12]\), we know that the family of excursion measures \(N_\theta\) are consistent with the semigroup of the process \((\xi, \Theta)\) stopped at its first hitting time of \((-\infty, 0] \times S^{d-1}\). As a consequence, for \(\theta \in S^{d-1} \setminus \Omega\),

\[
\mathbb{E}_\theta^\vee(g(X_{t+s})) = N_\theta(H_\Omega(X_{t+s}^c)g(X_{t+s}^c)1_{(s+t<\zeta)})
\]

\[
= N_\theta \left(1_{(t<\zeta)}\mathbb{E}_{X_t^c} \left[ H_\Omega(X_s)g(X_s)1_{(s<\tau^\circ_0)} \right] \right)
\]

\[
= N_\theta \left( H_\Omega(X_t^c)1_{(t<\zeta)}\mathbb{E}_{X_t^c} \left[ \frac{H_\Omega(X_s)}{H_\Omega(X_t^c)}g(X_s)1_{(s<\tau^\circ_0)} \right] \right)
\]

\[
= N_\theta \left( H_\Omega(X_t^c)1_{(t<\zeta)}\mathbb{E}_{X_t^c}^\vee [g(X_s)] \right).
\]

Thus using the notation \(\mathcal{P}_t^\vee[g](x) := E_x^\vee[g(X_t)]\), we have \(\mathcal{P}_{t+s}^\vee[g](x) = \mathcal{P}_t^\vee[\mathcal{P}_s^\vee[g]](x)\) for any \(x \in \mathbb{R}^d \setminus (\mathbb{B}_d \cup \Omega)\), and the required consistency follows.

Now, we need to show that \(\mathbb{P}^\vee_\theta(X_{0+} = \theta) = 1\) for any \(\theta \in S^{d-1} \setminus \Omega\). Since \(\lim_{t \downarrow 0} \varphi(t) = 0\), it suffices to show that

\[
\mathbb{P}^\vee_\theta(X_0 \neq \theta) = N_\theta (\{\lim_{t \downarrow 0} \epsilon(t) = 0, \lim_{t \downarrow 0} \Theta^\circ(t) = \theta\}^c) = 0.
\]

Let us first observe \(\epsilon\) is an excursion of \(\xi\) from its running minimum and \(\xi\) is a hypergeometric Lévy process with unbounded variation, hence 0 is regular for \((0, \infty)\), that is

\[
P_{0,\theta}(\tau_0^+ = 0) = 1, \quad \theta \in S^{d-1},
\]

where \(\tau_0^+ = \inf\{t > 0 : \epsilon_t > 0\}\). Hence,

\[
N_\theta (\{\lim_{t \downarrow 0} \epsilon(t) = 0\}^c) = 0.
\]

Since the jump measure of \(X\) in radial form is

\[
\Pi(dr, d\theta) = \frac{1}{r^{1+\alpha}} \sigma_1(d\theta)d\theta, \quad r > 0, \theta \in S^{d-1},
\]

as a consequence, the process \((\xi, \Theta)\) has the property that both the modulator and the ordinate must jump simultaneously (the precise jump rate was explored in \([11]\)). If it were the case that \(N_\theta (\{\lim_{t \downarrow 0} \Theta^\circ(t) = \theta\}^c) > 0\), this would be tantamount to a discontinuity in \(\Theta\) but not in \(\xi\), which is a contradiction. The requirement \((31)\) now follows. This completes the proof of Theorem 2 as far as \(\mathbb{P}^\vee\) is concerned.

The proof of Theorem 2 for \((X, \mathbb{P}^\wedge)\) is exactly the same and we leave it as an exercise for the reader.

\[\square\]

**Proof of Theorem 3:** Given the proof of Theorem 2 above, we refrain from giving the proof of Theorem 3 noting only that it is a variant of the arguments given there. The details are, once again, left to the reader.  

\[\square\]
7 Proof of Theorem 5

Recall the notation for a general Markov process \((Y, \mathcal{P})\) on \(E\) preceding the statement of Theorem 5. We will additionally write \(\mathcal{P} := (\mathcal{P}_t, t \geq 0)\) for the semigroup associated to \((Y, \mathcal{P})\).

Theorem 3.5 of Nagasawa [16], shows that, under suitable assumptions on the Markov process, \(L\)-times form a natural family of random times at which the pathwise time-reversal

\[ \tilde{Y}_t := Y_{(k-t)^-}, \quad t \in [0, k], \]

is again a Markov process. Let us state Nagasawa’s principle assumptions.

(A) The potential measure \(U_Y(a, \cdot)\) associated to \(\mathcal{P}\), defined by the relation

\[
\int_E f(x)U_Y(a, dx) = \int_0^\infty \mathcal{P}_t[f](a)dt = E_a \left[ \int_0^\infty f(X_t) dt \right], \quad a \in E, \tag{33}
\]

for bounded and measurable \(f\) on \(E\), is \(\sigma\)-finite. Assume that there exists a probability measure, \(\nu\), such that, if we put

\[
\mu(A) = \int U_Y(a, A) \nu(da) \quad \text{for } A \in \mathcal{B}(\mathbb{R}), \tag{34}
\]

then there exists a Markov transition semigroup, say \(\hat{\mathcal{P}} := (\hat{\mathcal{P}}_t, t \geq 0)\) such that

\[
\int_E \mathcal{P}_t[f](x)g(x) \mu(dx) = \int_E f(x)\hat{\mathcal{P}}_t[g](x) \mu(dx), \quad t \geq 0, \tag{35}
\]

for bounded, measurable and compactly supported test-functions \(f, g\) on \(E\).

(B) For any continuous test-function \(f \in C_0(E)\), the space of continuous and compactly supported functions, and \(a \in E\), assume that \(\mathcal{P}_t[f](a)\) is right-continuous in \(t\) for all \(a \in E\) and, for \(q > 0\), \(U_Y^{(q)}[f](\tilde{Y}_t)\) is right-continuous in \(t\), where, for bounded and measurable \(f\) on \(E\),

\[
U_Y^{(q)}[f](a) = \int_0^\infty e^{-qt}\hat{\mathcal{P}}_t[f](a)dt, \quad a \in E
\]

is the \(q\)-potential associated to \(\hat{\mathcal{P}}\).

Nagasawa’s duality theorem, Theorem 3.5. of [16], now reads as follows.

Theorem 6 (Nagasawa’s duality theorem). Suppose that assumptions (A) and (B) hold. For the given starting probability distribution \(\nu\) in (A) and any \(L\)-time \(k\), the time-reversed process \(\tilde{Y}\) under \(\mathcal{P}_\nu\) is a time-homogeneous Markov process with transition probabilities

\[
\mathcal{P}_\nu(\tilde{Y}_t \in A | \tilde{Y}_r, 0 < r < s) = \mathcal{P}_\nu(\tilde{Y}_t \in A | \tilde{Y}_s) = p_Y(t - s, \tilde{Y}_s, A), \quad \mathcal{P}_\nu\text{-almost surely}, \tag{36}
\]

for all \(0 < s < t\) and Borel \(A\) in \(\mathbb{R}\), where \(p_Y(u, x, A)\), \(u \geq 0\), \(x \in \mathbb{R}\), is the transition measure associated to the semigroup \(\hat{\mathcal{P}}\).
Proof of Theorem 5. We give the proof of (i), the proof of (ii) is almost identical albeit requiring some straightforward adjustments. Once again, we leave the details to the reader.

We will make a direct application of Theorem 6 with $Y$ taken to be the process $(X, \mathbb{P}_\nu^\ominus)$ where $\nu$ satisfies (17) or (18) according to the nature of $\Omega$. Accordingly, we will write $U^\ominus_\nu$ in place of $U_Y$, $\mathcal{P}^\ominus_\nu$ in place of $\mathcal{P}$ etc. Moreover, the dual process, formerly $\hat{Y}$, is taken to be $(X, \mathbb{P}^\vee)$ and we will, in the obvious way, work with the notation $U^\vee$ in place of $U_Y$, $\mathcal{P}^\vee$ in place of $\mathcal{P}$ and so on. In essence we need only to verify the two assumptions (A) and (B). Let us momentarily take the former of these two cases.

In order to verify (A) we will make use of Proposition 5.2. in [12] which identifies, for $x \in \mathbb{R}^d \setminus \{0\}$, and continuous $g : \mathbb{R}^d \mapsto \mathbb{R}$ whose support is compactly embedded in the exterior of the ball of radius $|x|$,

$$\mathbb{N}_{\arg(x)} \left( \int_0^\varsigma g(|x| e^{\epsilon(t)} \Theta^\nu(u)) du \right) = K_{\alpha,d} \int_{|x| < |z|} g(z) \frac{|z|^2 - |x|^2/2}{|z|^\alpha |x - z|^d} dz,$$

where $K_{\alpha,d} \in (0, \infty)$ is a constant which only depends on $\alpha$ and $d$.

Next, noting that $e^{\alpha \varphi(t)} \phi(t) dt$, we have for $a \in S^d \setminus \Omega$ and bounded measurable $f : \mathbb{R}^d \setminus (\mathbb{B}_d \cup \Omega) \rightarrow [0, \infty)$,

$$U^\ominus_\nu f(a) = \mathbb{E}^\ominus_\nu \left[ \int_0^\infty f(X_t) dt \right]$$

$$= \mathbb{N}_a \left( \int_0^\infty H^\ominus(X_t) f(X_t) dt \right)$$

$$= \mathbb{N}_a \left( \int_0^\infty H^\ominus(e^{\alpha t} \Theta^\nu(u)) f(e^{\alpha t} \Theta^\nu(u)) e^{\alpha t} du \right)$$

$$= K_{\alpha,d} \int_{\mathbb{R}^d \setminus (\mathbb{B}_d \cup \Omega)} H^\ominus(y) f(y) (|y|^2 - 1)^{\alpha/2} |a - y|^{-d} dy,$$

where $U^\ominus_\nu f(a) = \int_{\mathbb{R}^d \setminus (\mathbb{B}_d \cup \Omega)} f(y) U^\ominus_\nu(a, dy)$ and we have used (15) in the second equality.

Next, we need to develop an expression for the reference measure $\eta$. This only needs to be identified up to a multiplicative constant. As such, irrespective of whether $\Omega$ is a singleton or not, we can take

$$\eta(dx) = H_\Omega(x) H^\ominus(x) dx, \quad x \in \mathbb{R}^d \setminus (\mathbb{B}_d \cup \Omega).$$

Next, we need to verify that (35) holds. Indeed, using Hunt’s switching identity (cf. Chapter II.1 of [2]) for the killed stable process $X^\ominus$, we have for $x, y \in \mathbb{R}^d \setminus \mathbb{B}_d$

$$\eta(dy) \mathcal{P}^\ominus_\nu(y, dx) = \mathcal{P}^\ominus_\nu(y, dx) H_\Omega(y) H^\ominus(y) dy$$

$$= \frac{H^\ominus(x)}{H^\ominus(y)} \mathcal{P}^\ominus_\nu(y, dx) H_\Omega(y) H^\ominus(y) dy$$

$$= \mathcal{P}^\ominus_\nu(x, dy) H_\Omega(y) H^\ominus(y) dx$$

$$= \mathcal{P}^\ominus_\nu(x, dy) \eta(dx)$$,
where \( \mathcal{P}_{t}^{B_d}(x, dy) = \mathbb{P}_{x}(X_t \in dy, t < \tau_1^{\infty}) \). Note, as the measure \( \eta \) is absolutely continuous with respect to Lebesgue measure, we do not need to deal with the case that \( x \) or \( y \) belong to \( S^{d-1} \setminus \Omega \).

Let us now turn to the verification of assumption (B). This assumption is immediately satisfied on account of the fact that both \( \mathcal{P}^{\ominus} \) and \( \mathcal{P}^{\vee} \) are right-continuous semigroups by virtue of their definition as a Doob \( h \)-transform with respect to the Feller semigroup \( \mathcal{P}^{B_d} \) of the stable process killed on entry to \( B_d \). With both (A) and (B) in hand, we can invoke Theorem 6 and the desired result follows. \( \square \)

Acknowledgements

TS acknowledges support from a Schlumberger Faculty of the Future award. SP acknowledges support from the Royal Society as a Newton International Fellow Alumnus (AL191032).

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