Lyapunov Analysis of Least Squares Based Direct Adaptive Control

Nursefa Zengin, Baris Fidan, Ladan Khoshnevisan

Abstract—Adaptive control strategies usually are designed based on gradient methods for the sake of simplicity in Lyapunov analysis. However, least squares (LS)-based parameter identifiers, with proper selection of design parameters, exhibit better transient performance than the gradient-based ones, from the aspects of convergence speed and robustness to measurement noise. On the other hand, most of the LS-based adaptive control procedures are designed via the indirect adaptive control approaches, due to the difficulty in integrating an LS-based adaptive law with the direct approaches that define a certain Lyapunov-like cost function and drive it to (a neighborhood of) zero. In this paper, a formal constructive analysis framework is proposed to integrate the recursive LS-based parameter identification with direct adaptive control. Application of the proposed procedure in adaptive cruise control design is studied through Matlab/Simulink and CarSim simulations, validating the analytical results.

I. INTRODUCTION

Stability and convergence analysis of adaptive controller schemes has traditionally been based on Lyapunov stability notions and techniques [1]–[5]; Lyapunov-like functions are selected to penalize the tracking or regulation error but at the same time to facilitate designing an adaptive law to generate the parameter estimates used by the control law. Adaptive control designs targeting to drive a Lyapunov-like function to zero mostly lead to gradient based adaptive laws with constant adaptive gain. On the other hand, it is well observed that least-squares (LS) algorithms provide faster convergence; hence, LS based adaptive control has potential to enhance convergence performance in direct adaptive control approaches as well [2], [6]–[9]. Despite wide use of gradient based online parameter identifiers, LS adaptive algorithms with forgetting factor are developed for faster settling and robustness to measurement noises [8]–[13]. Such properties have been justified by various simulation and experimental results [6], [7], [14]–[16].

In addition to the existing mathematical LS based adaptive control design studies, there are some publications in the recent literature on real-time applications, including those on robotic manipulators [15]–[17], unmanned aerial vehicles [14], [18], [19], and passenger vehicles [20]–[23]. Most of the existing studies on LS based adaptive control follow the indirect approach as opposed to the direct adaptive control. One reason for this is that constructive Lyapunov analysis of direct adaptive control is complicated for producing an LS based adaptive control scheme.

This paper proposes a constructive analysis framework for recursive LS (RLS) online parameter identifier based direct model reference adaptive control (MRAC). In the literature, [1], [2] considered the possible use of LS based online parameter identifiers in direct MRAC. However, the proof and the Lyapunov analysis were not provided in detail. Several techniques have been developed to robustify the LS based online parameter identifiers with respect to the loss of adaptation or parameter bursts related to the gain (covariance) matrix becoming arbitrarily small or arbitrarily large, including use of parameter projection, resetting, saturation, and forgetting factor [24]. The role of forgetting factor is extensively investigated in [25] and [6], where it is demonstrated that without forgetting factor the parameter estimates converge to the real values only asymptotically (and typically slower), whereas with forgetting factor the convergence becomes exponentially fast, which leads to specific design procedures for different applications. For instance, a composite LS method is provided in [16], where a Lyapunov-like function with time-varying gain matrix is utilized. However, the design in [16] is for a specific system model suitable for robot manipulators, which limits the applicability of the proposed procedure as is.

Constructive Lyapunov analysis of RLS parameter identifier based direct adaptive control, which is used to build the adaptive control laws, is studied in this paper. The main difference from the gradient based approaches is replacement of the constant adaptation gain with a time varying adaptive gain (covariance) matrix. For a systematic construction of the direct MRAC scheme with time-varying adaptive gain, a Lyapunov-like function is constructed through which an LS parameter identification based direct adaptive control scheme is established to guarantee asymptotic stability. The proposed procedure is utilized in an adaptive cruise control (ACC) application case study to demonstrate the transient performance, validate the analytical results, and compare the performance with the gradient based adaptive controllers through a set of Matlab/Simulink and CarSim simulations.
II. BACKGROUND

In model reference adaptive control (MRAC), desired plant behaviour is described by a reference model which is often formulated in the form of a transfer function driven by a reference signal. Then, a control law is developed via model matching so that the closed loop system has a transfer function equal to the reference model [1]–[4]. Consider the SISO LTI plant

\[ \dot{x}_p(t) = A_p x_p(t) + B_p u_p(t), \quad x(0) = x_0, \]

\[ y_p(t) = C_p x_p(t), \quad (1) \]

with state \( x_p \in \mathbb{R}^n \), input \( u_p \in \mathbb{R} \), output \( y_p \in \mathbb{R} \), and system matrices \( A_p, B_p, C_p \) of appropriate dimensions, and the reference model

\[ \dot{x}_m(t) = A_m x_m(t) + B_m r(t), \quad x_m(0) = x_m(0), \]

\[ y_m(t) = C_m x_m(t), \quad (2) \]

which is fed by the reference input signal \( r \in \mathbb{R} \). The transfer functions of the plant (1) and the reference model (2) are given by:

\[ G_p(s) = \frac{Z_p(s)}{R_p(s)} = C_p^T (sI - A_p)^{-1} B_p, \]

\[ W_m(s) = k_m \frac{Z_m(s)}{R_m(s)} = C_m^T (sI - A_m)^{-1} B_m, \quad (3), (4) \]

where \( k_p \) and \( k_m \) denote the high frequency gains, and \( Z_p(s), R_p(s), Z_m(s), R_m(s) \) are monic polynomials.

The MRAC task [1], [2] is to generate the control signal \( u_p \) so that all the closed-loop system signals are bounded and the plant output \( y_p \) tracks the reference model output \( y_m \), under the following assumptions:

**Assumption 1 (Plant Assumptions).** The plant is minimum phase, i.e., \( Z_p(s) \) is a monic Hurwitz polynomial. The upper bound \( n \) of the degree \( n_p \) of \( R_p(s) \), the relative degree \( n^* = n_p - m_p \) of \( G_p(s) \), where \( m_p \) denotes the degree of \( Z_p(s) \), and \( \text{sign}(k_p) \) are known.

**Assumption 2 (Reference Model Assumptions).** \( Z_m(s), R_m(s) \) are monic Hurwitz polynomials of degree \( q_m, p_m \), respectively. Relative degree \( n_m = p_m - q_m \) of \( W_m(s) \) is the same as that of \( G_p(s) \), i.e., \( n^* = n_m \).

Consider the fictitious feedback control law [1], [2]

\[ u_p = \theta^{T} \frac{\alpha(s)}{\Lambda(s)} u_p + \theta_2^T \frac{\alpha(s)}{\Lambda(s)} y_p + \theta_3^T y_p + c_0^T r, \quad (5) \]

where \( c_0 = k_p \frac{Z_p}{R_p} \), \( \alpha(s) \triangleq \alpha_{n-2}(s) = [s^{n-2}, s^{n-3}, \ldots, s, 1]^T \) for \( n \geq 2 \), \( \alpha(s) \equiv 0 \) for \( n = 1 \), and \( \Lambda(s) \) is an arbitrary monic Hurwitz polynomial of degree \( n - 1 \) containing \( Z_m(s) \) as a factor, i.e.,

\[ \Lambda(s) = \Lambda_0(s) Z_m(s) \]

implying that \( \Lambda_0(s) \) is monic and Hurwitz. The fictitious ideal model reference control (MRC) parameter vector \( \theta^* = [\theta_1^T \theta_2^T \theta_3^T \theta_0^T]^T \) is chosen so that the transfer function from \( r \) to \( y_p \) is equal to \( W_m(s) \). The closed-loop reference to output relation for the MRC scheme above is derived in [1], [2] as

\[ y_p = G_c(s) r, \quad (6) \]

\[ G_c = \frac{c_0^T k_p Z_p \Lambda}{R_p \Lambda - \theta_1^T \alpha R_p - k_p Z_p (\theta_2^T \alpha + \theta_3^T \Lambda)}. \]

The ideal MRC parameter vector \( \theta^* \) is chosen to match the coefficients of \( G_c(s) \) and \( W_m(s) \) in (4), (6). A state-space realization of the ideal MRC law (5) is given by [1], [2]

\[ \dot{\omega}_1(t) = F \omega_1(t) + g u_p(t), \quad \omega_1(0) = 0, \]

\[ \dot{\omega}_2(t) = F \omega_2(t) + g y_p(t), \quad \omega_2(0) = 0, \]

\[ u_p(t) = \theta^T \omega(t), \quad (7) \]

where \( \omega_1, \omega_2 \in \mathbb{R}^{n-1} \),

\[ \theta^* = [\theta_1^T \theta_2^T \theta_3^T \theta_0^T]^T, \quad \omega = [\alpha^{T} \omega^{T} y_p n]^{T}, \]

\[ F = \begin{bmatrix} -\lambda_{n-2} & -\lambda_{n-3} & -\lambda_{n-4} & \cdots & -\lambda_0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \]

\[ \Lambda(s) = s^{n-1} + \lambda_{n-2}s^{n-2} + \cdots + \lambda_1 s + \lambda_0 = \det(sI - F), \]

\[ g = [1 \ 0 \ \cdots \ 0]^T. \]

The MRAC scheme for the actual case where the plant parameters are unknown is derived by following the certainty equivalence approach and modifying (7) as

\[ \dot{\omega}_1(t) = F \omega_1(t) + g u_p(t), \quad \omega_1(0) = 0, \]

\[ \dot{\omega}_2(t) = F \omega_2(t) + g y_p(t), \quad \omega_2(0) = 0, \]

\[ u_p(t) = \theta^T(\omega(t), \theta(t) \text{ is the online estimate of the unknown ideal MRC parameter vector } \theta^*. \]

The adaptive law to generate \( \theta(t) \) can be formed considering the following composite state space representation of the closed-loop system [2]:

\[ \dot{Y}_c(t) = A_0 Y_c(t) + B_c u_p(t), \]

\[ y_p(t) = C_c^T Y_c(t), \quad (9) \]

where \( Y_c = [Y^T, \omega^T, Y_0^T]^T, \)

\[ A_0 = \begin{bmatrix} A_p & 0 & 0 \\ 0 & F & 0 \\ g C_p^T & 0 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} B_p \\ 0 \\ g \end{bmatrix}, \quad C_c = \begin{bmatrix} C_p^T & 0 & 0 \end{bmatrix}. \]

The system equation (9) can also be expressed as [2]

\[ \dot{Y}_c(t) = A_2 Y_c(t) + B_c u_p(t) - \theta^T \omega(t), \]

\[ y_p(t) = C_c^T Y_c(t), \quad (10) \]
\[ A_c = \begin{bmatrix} A_p + B_p \theta^* C_p^T & B_p \theta^* A_p^T & B_p \theta^* B_p^T \\ 0 & g C_p^T & g \theta^* T_2 \\ g \theta^* T_1 & g \theta^* T_2 & g C_p^T \end{bmatrix} . \]

Consider the fictitious system
\[ \dot{Y}_m(t) = A_c Y_m(t) + B_c \rho^*(t), \]
\[ \ddot{Y}_m(t) = C_c^T Y_m(t), \]
obtained by substituting (7) in (10). Noting that (7) guarantees matching of the closed-loop r-to-yp transfer function with \( W_m(s) \), the r-to-\( \ddot{y}_m \) transfer function of (11) satisfies
\[ W_m(s) = C_c^T (sI - A_c)^{-1} B_c \rho^*. \]

Since (2) and (11) are state-space representations of the same stable transfer function \( W_m(s) \), \( y_m(t) - \ddot{y}_m(t) \) converges to zero exponentially fast. Hence, defining \( e_1 = y_p - \ddot{y}_m \), exponential convergence of the tracking error \( y_p - y_m \) to zero is equivalent to exponential convergence of \( e_1 \) to zero. Subtracting (11) from (10), we obtain
\[ \dot{e}(t) = A_c e(t) + B_c (u_p(t) - \theta^* \omega(t)), \quad e(0) = 0, \]
\[ e_1 = C_c^T e \]
for the state mismatch vector \( e = Y_c - Y_m \in \mathbb{R}^{3n-2} \). Hence,
\[ e_1 = W_m(s) \rho^* (u_p - \theta^* \omega), \]
where \( \rho^* = 1/c_0 \). Substituting (8) into (13), we obtain
\[ \dot{e} = A_c e + B_c \tilde{\theta}^* \omega, \]
\[ e_1 = C_c^T e, \]
\[ \tilde{\theta}(t) = \theta(t) - \theta^*. \]

### III. LYAPUNOV-LIKE FUNCTION COMPOSITION AND ANALYSIS FOR LEAST-SQUARES BASED DIRECT MRAC

In the typical gradient based direct adaptive control designs of the literature, the Lyapunov-like function is chosen as
\[ V_1(\theta, e) = \frac{e^T P e}{2} + \frac{\tilde{\theta} \Gamma^{-1} \tilde{\theta}}{2} |\rho^*|, \]
where \( \tilde{\theta} = \theta - \theta^* \), \( \theta^* \) and \( \theta \), respectively, are the ideal MRC and MRAC parameter vectors defined in Section II, \( \Gamma = \Gamma^T \) is a constant positive definite adaptive gain matrix, and \( P_c = P_c^T \) is a positive definite matrix that satisfies the Meyer-Kalman-Yakubovich Lemma [2] equations
\[ P_c A_c + A_c^T P_c = -q \Gamma - V_c L_c, \]
\[ P_c B_c c_0 = C_c, \]
where \( q \) is a vector, \( L_c = L_c^T > 0 \), and \( V_c > 0 \) is small constant. The time derivative \( \dot{V}_1 \) of \( V_1 \) along (15), (17) is
\[ \dot{V}_1 = -\frac{e^T q \Gamma e}{2} - \frac{V_c}{2} e^T L_c e + e^T P_c B_c c_0 \rho^* \tilde{\theta}^* \omega + \tilde{\theta} \Gamma^{-1} \tilde{\theta} |\rho^*|. \]
Since \( e^T P_c B_c c_0 = e^T C_c = 1 \) and \( \rho^* = |\rho^*| \text{sgn}(\rho^*) \), defining the gradient based adaptive law
\[ \dot{\theta} = -\Gamma \omega e_1 \text{sgn}(\rho^*) \]
leads to
\[ \dot{V}_1 = -\frac{e^T q \Gamma e}{2} - \frac{V_c}{2} e^T L_c e \leq 0, \]
noting that \( \dot{\theta} = \tilde{\theta} \). (16) and (19) imply that \( V, \tilde{\theta}, \theta \in \mathcal{L}_m \) and \( e \in \mathcal{L}_2 \cap \mathcal{L}_\infty \). Furthermore, since the reference system (12) is stable, we have \( Y_m \in \mathcal{L}_m \), and hence \( Y_c = Y_m + e \), \( \ddot{y}_m = C_c^T Y_m \in \mathcal{L}_m \). By (1),(8),(10), this further implies that \( x_p, y_p, \omega_1, \omega_2, u_p = \theta^* \omega \in \mathcal{L}_m \), i.e., all the closed-loop signals are bounded. Moreover, since \( e \in \mathcal{L}_\infty \), based on Barbalat’s Lemma, \( \lim_{t \to \infty} e(t) = 0 \). Hence, the tracking error \( e_1 = y_p - \ddot{y}_m = C_c^T e \) converges to zero as \( t \to \infty \).

With the gradient adaptive law (18) with constant gain \( \Gamma \), adaptation speed can be increased only by setting \( \Gamma \) larger. However, this often leads to high-frequency oscillations, which adversely affects robustness of the adaptive control law. Unlike the gradient based adaptive law (18) with constant adaptive gain \( \Gamma \), initializing a time varying adaptive gain matrix \( P(t) \) at a large matrix norm value and then adjusting to lower values based on identification error during estimation process would allow robust tracking.

For generation of the time varying gain \( P(t) \), an efficient systematic approach is use of LS based adaptive laws, which are observed to have the advantage of faster convergence and robustness to measurement noise [2], [6]–[9]. Next, we propose a formal constructive analysis framework for integration of recursive LS (RLS) based parameter identification to direct adaptive control, following the steps above, but constructing a new Lyapunov-like function to replace (16), aiming to formally establish an adaptive control scheme law that involves the control structure (8) and an RLS based alternative of the adaptive law (18).

Aiming to replace the constant gain \( \Gamma \) with a time-varying matrix \( P(t) \), consider the Lyapunov-like function
\[ V_2(\tilde{\theta}, e, t) = \frac{e^T P e}{2} + \frac{\tilde{\theta} \Gamma^{-1} \tilde{\theta}}{2} |\rho^*|, \]
where \( P(t) \) is uniformly positive definite, in place of (16). The time derivative of \( V_2 \) along the solution of (20) is
\[ \dot{V}_2 = -\frac{e^T q \Gamma e}{2} - \frac{V_c}{2} e^T L_c e + e^T P_c B_c c_0 \rho^* \tilde{\theta}^* \omega + \frac{1}{2} \tilde{\theta} \frac{d(P^{-1})}{dt} \tilde{\theta} |\rho^*| + \tilde{\theta} \Gamma^{-1} \tilde{\theta} |\rho^*|, \]
where
\[ \frac{d(P^{-1})}{dt} = -P^{-1} \rho P^{-1}. \]
If \( P(t) \) is updated according to the RLS adaptive law
\[ \dot{P} = \beta P - P \omega \omega^T P \]
with forgetting factor $\beta > 0$, then (22) becomes
\[
\frac{d(P^{-1})}{dt} = -\beta P^{-1} + \omega \omega^T \tag{24}
\]
Substituting (24) into (21), we obtain
\[
\dot{V}_2 = -\frac{1}{2} e^T q q^T e - \frac{1}{2} e^T L e + e_1^T \rho^* \tilde{\theta}^T \omega - \frac{1}{2} \tilde{\theta}^T P^{-1} \tilde{\theta} |\rho^*|^2 + \tilde{\theta}^T P^{-1} \tilde{\theta} |\rho^*|^2 + \frac{1}{2} \tilde{\theta}^T \tilde{\theta} \omega^T |\rho^*|^2.
\tag{25}
\]
Defining the adaptive law
\[
\dot{\theta} = -P \omega e_1 \text{sgn}(\rho^*) - \frac{1}{2} P \omega \tilde{\theta} \omega,
\tag{26}
\]
where $P(t)$ is updated via (23), noting that $\dot{\theta} = \dot{\hat{\theta}}$, and substituting into (25), we obtain
\[
\dot{V}_2 = -\frac{1}{2} e^T q q^T e - \frac{1}{2} e^T L e - \frac{1}{2} \tilde{\theta}^T P^{-1} \beta |\rho^*|^2 \leq 0,
\tag{27}
\]
leading to the following theorem on the stability properties of the LS based direct MRAC scheme (8),(23),(26).

**Theorem III.1.** The RLS parameter estimation based MRAC scheme (8),(23),(26) has the following properties:

(i) All the closed-loop signals are bounded and tracking error $e_1$ converges to zero in time for any $r \in \mathcal{L}_\infty$.

(ii) If the reference input $r$ is sufficiently rich of order $2n$, i.e., $\dot{r} \in \mathcal{L}_\infty$ and $Z_p(s)R_p(s)$ are relatively coprime, then $\omega$ is persistently exciting (PE), viz.,
\[
\int_{t_{0}}^{t+T_0} \omega(\tau)\omega^T(\tau)d\tau \geq \alpha_0 T_0 I, \quad \alpha_0, T_0 > 0, \quad \forall t \geq 0,
\tag{28}
\]
which implies that $P, P^{-1} \in \mathcal{L}_\infty$ and $\theta(t) \rightarrow \theta^*$ as $t \rightarrow \infty$. In the case of $\beta = 0$ (pure-RLS), $\theta \rightarrow \theta^*$ and $e_1 \rightarrow 0$ as $t \rightarrow \infty$. When $\beta > 0$, which is RLS with forgetting factor, the parameter error $||\dot{\theta}|| = ||\dot{\theta} - \theta^*||$ and the tracking error $e_1$ converges to zero exponentially fast.

**Proof.** (i) $e \in \mathcal{L}_\infty$, $\omega \in \mathcal{L}_\infty$, and $\dot{e} \in \mathcal{L}_\infty$. Therefore, all signals in the closed loop plant are bounded. In order to complete the design, we need to show tracking error $e_1$ converges to the zero asymptotically with time. Using (20), (27), we know that $e, e_1 \in \mathcal{L}_2$. Using, $\theta, \omega, e \in \mathcal{L}_\infty$ in (15), we have $\dot{e}, e_1 \in \mathcal{L}_\infty$. Since $\dot{e}, e_1 \in \mathcal{L}_\infty$ and $e_1 \in \mathcal{L}_2$, the tracking error $e_1$ goes to zero as $t$ goes to infinity.

(ii) Considering pure-RLS, when $\beta = 0$, from (23) we have $\dot{P} = -P \omega \omega^T \leq 0$. So, $P(t)$ is non-increasing i.e., $P(t) \leq P_0$. As $P(t) = P(t)^T > 0$, it has a limit i.e., $\lim_{t \rightarrow \infty} \dot{P}(t) = \bar{P}$, where $\dot{P} = \bar{P}$ is a positive constant matrix. Furthermore, (23) results in
\[
P(t) = P_0 - \int_{0}^{t} \omega \omega^T P(\tau) d\tau 
\leq \lambda_{min}(P_0) - \lambda_{min}(P_0) \int_{t_{0}}^{t} \omega \omega^T d\tau,
\]
where $\lambda_{min}(P_0)$ is the minimum singular value of $P_0$. By Theorem 3.4.3 of [2], if $r$ is sufficiently rich of order $2n$ then the $2n$ dimensional regressor vector $\omega$ is PE. Hence,
\[
\dot{P} \leq P(t) \leq \lambda_{min}(P_0)(1 - \alpha T_0)I.
\]
So, $P(t) \in \mathcal{L}_\infty$ in pure-LS method. By (27), $V_2, \dot{\theta}, \bar{\theta} \in \mathcal{L}_\infty$ and $e \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

The exponential convergence in Theorem III.1 indicates fast adaptation and robustness against noise and external disturbances in practical applications. Comparing (18) and (26), we see the effect of time varying covariance matrix reflected to the second term of (26).
IV. A SIMULATION CASE STUDY

In this section, an ACC case study is performed, simulating and comparing gradient, pure-RLS (PRLS) ($\beta = 0$) and forgetting factor RLS ($\beta > 0$) adaptive laws. The ACC goal and parameters are illustrated in Fig. 1. An ACC scheme is to be designed to regulate the following vehicle’s speed $v$ towards the leading vehicle’s speed $v_l$ and to keep the distance $x_r$ between vehicles close to the desired spacing $s_d = s_0 + h v$, where $s_0$ is the fixed safety spacing and $h$ is constant time headway. Hence, the ACC objective can be expressed as having $v_r(t) = v_l(t) - v(t)$ and $\delta(t) = x_r(t) - s_0(t)$ converge to zero as $t \to \infty$.

It is also required to have $|\dot{v}_i|$ and $|\ddot{v}_i|$ bounded by some preset upper bounds. For ACC design, a simple longitudinal motion dynamics model is formed as

$$ \dot{v} = -a v + b u + d, $$

where $u$ is the throttle/brake command, $d$ is the modeling uncertainty, $a$ and $b$ are positive constant parameters. We assume that $d, \dot{v}_l, \dot{v}_l$ are all bounded. To generate $u$, an MRAC scheme is designed such that the vehicle speed $v$ tracks the output $v_m$ of the reference model

$$ v_m = \frac{a_m}{s + a_m} (v_l + k \delta), $$

where $a_m$ and $k$ are positive design parameters. The ideal MRAC law is formed, by treating $a, b,$ and $d$ as known, as

$$ u = \dot{k}_1 v_r + \dot{k}_2 \delta + \dot{k}_3, $$

$$ \dot{k}_1 = \frac{a_m - a}{b}, \quad \dot{k}_2 = \frac{a_m k}{b}, \quad \dot{k}_3 = \frac{a v_l - d}{b}. $$

The actual ACC is formed as the MRAC scheme

$$ u = k_1 v_r + k_2 \delta + k_3, $$

where $k_i$ is the estimate of $k^*_i$ to be generated by an adaptive law, aiming to minimize the tracking error

$$ e = v - v_m = \frac{b}{s + \dot{a}_m} (-k_1^* v_r - k_2^* \delta - k_3^* + u). $$

Substituting the control law in (36) into (37), we obtain

$$ e = \frac{b}{s + \dot{a}_m} (\tilde{k}_1 v_r + \tilde{k}_2 \delta + \tilde{k}_3), $$

where $\tilde{k}_i = k_i - k^*_i$ for $i = 1, 2, 3$. Considering [2]

$$ V = \frac{e^2}{2} + \sum_{i=1}^{3} \frac{b}{2\gamma_i} \tilde{k}_i^2, \quad \gamma_i > 0, b > 0 $$

and its time derivative

$$ \dot{V} = -a_m e^2 + b e (\tilde{k}_1 v_r + \tilde{k}_2 \delta + \tilde{k}_3) + \sum_{i=1}^{3} \frac{b}{h_i} \tilde{k}_i^2, $$

the following gradient based adaptive laws are obtained:

$$ \dot{k}_1 = Pr\{-\gamma_1 e v_r\}, \quad \dot{k}_2 = Pr\{-\gamma_2 e \delta\}, \quad \dot{k}_3 = Pr\{-\gamma_3 e\}, $$

where the projection operator keeps $k_i$ within the lower and upper intervals and $\gamma_i$ are the positive constant adaptive gains. These adaptive laws lead to $\dot{V} = -a_m e^2$, implying that $e \in L_\infty$ and in turn all other signals in the closed loop are bounded. We apply RLS based adaptive law to (38) and obtain following equations to be used in simulations

$$ \dot{\theta} = \text{Pr}\{P \phi e\}, \quad \dot{\phi} = \beta P - \phi \phi^T P, $$

where $e = v - v_m$, $\theta = [k_1, k_2, k_3]^T$, $\phi = \left[ \frac{v_r}{\sigma^2 + \dot{a}_m}, \frac{\delta}{\sigma^2 + \dot{a}_m}, \frac{1}{\sigma^2 + \dot{a}_m} \right]^T$. For gradient based adaptive law, the gains are set as $\gamma_1 = 50, \gamma_2 = 30, \gamma_3 = 40$. The constants of the RLS based adaptive law are set as $\beta = 0.95$ and $P(0) = 100I_3$. For both RLS and gradient based adaptive laws, a Gaussian noise is applied ($\sigma = 0.05$). Matlab/Simulink based simulation results are shown in Figures 2 and 3. Fig. 2 indicates better velocity tracking performance and exponential tracking convergence with the RLS based adaptive law.

We also implemented RLS based adaptive control algorithm in (39), with the same adaptive law constants above, in more realistic CarSim simulations, with vehicle parameters $m = 567.75$ kg, $R = 0.3$ m, $I = 1.7$ kgm$^2$, $B = 0.01$ kg/s. The results are shown in Fig. 4.
V. CONCLUSIONS

In this paper, a constructive Lyapunov analysis of RLS based parameter estimation direct adaptive control has been proposed. A systematic design of a time varying adaptive gain (covariance) matrix is proposed via Lyapunov analysis, where it is analytically demonstrated that the adaptive parameters converge to the actual ones exponentially fast. The simulation results on an ACC case study, and the quantitative comparison of gradient and pure-LS based designs validate the analysis results, demonstrating performance superiority of the LS-based design. Detailed simulations via a realistic vehicle software, CarSim, are used to scrutinize the real-life applicability of the proposed method.

REFERENCES
[1] P. Ioannou and J. Sun, Robust Adaptive Control. Pren.-Hall, 1996.
[2] P. Ioannou and B. Fidan, Adaptive Control Tutorial. SIAM, 2006.
[3] K. Narendra and A. Annaswamy, Stable Adaptive Systems. Prentice-Hall, 1989.
[4] G. Goodwin and K. Sin, Adaptive Filtering Prediction and Control. Prentice-Hall, 1989.
[5] M. Krstic, P. Kokotovic, and I. Kanellakopoulos, Nonlinear and Adaptive Control Design. Wiley, 1995.
[6] B. Fidan, A. Camlica, and S. Gulér, “Least-squares-based adaptive target localization by mobile distance measurement sensors,” Int J of Adaptive Con. and Signal Proc., vol. 29, no. 2, pp. 259–271, 2015.
[7] S. Gulér, B. Fidan, S. Dasgupta, B. Anderson, and I. Shames, “Adaptive source localization based station keeping of autonomous vehicles,” IEEE Tr. Auto. Con., vol. 62, no. 7, pp. 3122–3135, 2016.
[8] M. Krstic, “On using least-squares updates without regressor filtering in identification and adaptive control of nonlinear systems,” Automatica, vol. 45, p. 731–736, March 2009.
[9] P. R. Kumar, “Convergence of adaptive control schemes using least-squares parameter estimates,” IEEE Tr. Automatic Control, vol. 35, no. 4, pp. 416–424, 1990.
[10] C. Hu, B. Yao, and Q. Wang, “Integrated direct/indirect adaptive robust contouring control of a biaxial gantry with accurate parameter estimations,” Automatica, vol. 46, no. 4, pp. 701–707, 2010.
[11] T. Tay, J. Moore, and R. Horowitz, “Indirect adaptive techniques for fixed controller performance enhancement,” International Journal of Control, vol. 50, no. 5, pp. 1941–1959, 1989.
[12] I. Karafyllis and M. Krstic, “Adaptive certainty-equivalence control with regulation-triggered finite-time least-squares identification,” IEEE Tr. on Automatic Control, vol. 63, no. 10, pp. 3261–3275, 2018.
[13] K. Jiang, A. Victorino, and A. Charara, “Adaptive estimation of vehicle dynamics through RLS and Kalman Filter approaches,” in Proc. IEEE Int. Conf. Intelligent Transp. Sys., 2015, pp. 1741–1746.
[14] N. Koksal, H. An, and B. Fidan, “Backstepping-based adaptive control of a quadrotor UAV with guaranteed tracking performance,” ISA Transactions, vol. 105, pp. 98–110, 2020.
[15] A. Mohanty and B. Yao, “Integrated direct/indirect adaptive robust control of hydraulic manipulators with valve deadband,” IEEE/ASME Transactions on Mechatronics, vol. 16, no. 4, pp. 707–715, 2011.
[16] J.-J. Slotine and W. Li, “Composite adaptive control of robot manipulators,” Automatica, vol. 25, no. 4, pp. 509–519, 1989.
[17] A. Mohanty and B. Yao, “Indirect adaptive robust control of hydraulic manipulators with accurate parameter estimates,” IEEE Tr. Control Systems Technology, vol. 19, no. 3, pp. 567–575, 2011.
[18] Y. Ameho, F. Niel, F. Defay, J. Biannic, and C. Berard, “Adaptive control for quadrotors,” in Proc. Int. Conf. Robotics and Automation, 2013, pp. 5396–5401.
[19] R. Leishman, J. Macdonald, R. Beard, and T. McLain, “Quadrotors and accelerometers: State estimation with an improved dynamic model,” IEEE Control Systems, vol. 34, no. 1, pp. 28–41, 2014.
[20] A. Vahidi, A. Stefanopoulou, and H. Peng, “Recursive least squares with forgetting for online estimation of vehicle mass and road grade,” Vehicle System Dynamics, vol. 43, no. 1, pp. 31–55, 2005.
[21] H. Bae, J. Ryu, and J. C. Gerdes, “Road grade and vehicle parameter estimation for longitudinal control using gps,” in Proc. IEEE Int. Conf. on Intelligent Transportation Systems, 2001, pp. 25–29.
[22] D. Pavkovic, J. Deur, G. Burgio, and D. Hrovat, “Estimation of tire static curve gradient and related model-based traction control application,” in Proc. IEEE Conf. Control App., 2009, pp. 594–599.
[23] W. Chen, D. Tan, and L. Zhao, “Vehicle sideslip angle and road fricition estimation using online gradient descent algorithm,” IEEE Tr. Vehicle Technology, vol. 67, no. 12, pp. 11475–11485, 2018.
[24] R. Ortega, V. Nikiforov, and D. Gerasimov, “On modified parameter estimators for identification and adaptive control,” Annual Reviews in Control, vol. 50, pp. 278–293, 2020.
[25] A. Goel, A. L. Bruce, and D. S. Bernstein, “Recursive least squares with variable-direction forgetting: Compensating for the loss of persistency,” IEEE Control Systems, vol. 40, no. 4, pp. 80–102, 2020.