A primal discontinuous Galerkin method with static condensation on very general meshes

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Abstract

We propose an efficient variant of a primal Discontinuous Galerkin method with interior penalty for
the second order elliptic equations on very general meshes (polytopes with eventually curved bound-
daries). Efficiency, especially when higher order polynomials are used, is achieved by static condensation,
i.e. a local elimination of certain degrees of freedom element by element. This alters the original method
in a way that preserves the optimal error estimates. Numerical experiments confirm that the solutions
produced by the new method are indeed very close to that produced by the classical one.

Keywords: Discontinuous Galerkin, static condensation, polyhedral (polygonal) meshes.

1 Introduction

The recent years have seen the emergence (or the revival) of several numerical methods capable to solve
approximately elliptic partial differential equations using general polygonal/polyhedral meshes. This is
witnessed for example by the book [4]. The methods reviewed in this book (if we restrict our attention only
to finite element type methods using piecewise polynomial approximation spaces in one form or another)
include interior penalty discontinuous Galerkin (DG) methods [2], hybridizable discontinuous Galerkin
(HDG) methods ([6], introduced in [8]), the Virtual Element (VE) method ([18], introduced in [17, 5]),
the Hybrid High-Order (HHO) method ([12], introduced in [10, 11]). One can add to this list the weak
Galerkin finite element [19], which is similar to HDG. The relations between HHO and HDG methods
were exhibited in [7].

Among the above, the primal interior penalty DG methods are the most classical. In the symmetric
form, also referred to as SIP – symmetric interior penalty, this method dates back to [20, 3] and is now
presented and thoroughly studied in several monographs, for example [16, 9]. It is well suited to the
discretization on very general meshes because its approximation space is populated by polynomials of
degree, say \( \leq k \), on each mesh element without any constraints linking the polynomials on two adjacent
elements. It leaves thus a lot of freedom on the choice of the mesh elements which can be not only
polytopes but also virtually any geometrical shapes. It is generally admitted however that the SIP method is
too expensive especially when higher order polynomials are employed. Indeed, its cost, i.e. the dimension
of the approximation space, is the product of the number of mesh elements and the dimension of the space
of polynomials of degree \( \leq k \). The cost on a given mesh is thus proportional to \( k^2 \) in 2D (resp. \( k^3 \) in 3D).
This should be contrasted with the cost of HDG or HHO methods which is proportional to \( k \) in 2D (resp.
\( k^2 \) in 3D).

The goal of the present article is to modify the SIP method so that its cost is reduced to that of HDG or
HHO methods. In doing so, we inspire ourselves from the static condensation procedure for the standard
continuous Galerkin (CG) finite element methods. It is indeed well known that the dimension of the
approximation space in CG is proportional to \( k^2 \) in 2D on a given mesh, but the degrees of freedom interior
to each mesh element can be locally eliminated which leaves a global problem of the size proportional to \( k \) (these numbers are changed to, respectively, \( k^3 \) and \( k^2 \) in 3D). Although the notion of interior degrees of freedom does not make sense in the DG context, we shall be able to select, on each mesh element, a subspace of the approximating polynomials that can be used to construct a local approximation through the solution of a local problem. The remaining degrees of freedom will then be used in a global problem. We shall thus achieve a significant reduction of the global problem size in the DG SIP-like method, similarly to that achieved in CG by the static condensation. The resulting DG method, which can be refereed to as scSIP (static condensation SIP), will not produce exactly the same approximation as the original SIP method. We shall prove however that these two solutions satisfy the same optimal a priori error bounds in \( H^1 \) and \( L^2 \) norms. Moreover, they turn out to be very close to each other in our numerical experiments.

We treat here only the diffusion equation with variable, but sufficiently smooth, coefficients. The extension to other problems, such as convection-reaction-diffusion, linear elasticity, Stokes, as well as to other DG variants (IIP, NIP) seems relatively straight-forward. Our assumptions on the mesh allow for elements of general shape, not necessarily the polytopes.

The article is organized as follows: in the next section, we present the idea of our method starting by the description of the governing equations. We recall the static condensation for the classical CG FEM. Our variants of DG FEM (SIP and scSIP) are first introduced in Subsection 2.2. The convergence proofs are in Section 3. They are done assuming some properties of the discontinuous FE spaces and the underlying mesh. In Section 4, we give an example of the hypotheses on the mesh under which the necessary properties of the FE spaces can be established. Finally, a numerical illustration is presented in Section 5.

## 2 Description of the problem and static condensation for FEM (CG and DG cases)

We consider the second-order elliptic problem

\[
\mathcal{L}u = f \text{ in } \Omega, \quad u = g \text{ on } \partial \Omega
\]  

(1)

where \( \Omega \subset \mathbb{R}^d, d = 2 \text{ or } 3 \), is a bounded Lipschitz domain, \( f \in L^2(\Omega) \) and \( g \in H^{1/2}(\partial \Omega) \) are given functions. The differential operator \( \mathcal{L} \) is defined by

\[
\mathcal{L}u = -\partial_i (A_{ij}(x) \partial_j u)
\]

with \( \partial_i \) denoting the partial derivative in the direction \( x_i, i = 1, \ldots, d \). The coefficients \( A_{ij} \) are supposed to form a positive definite matrix \( A = A(x) \) for any \( x \in \Omega \) which is sufficiently smooth with respect to \( x \) so that

\[
\alpha|\xi|^2 \leq \xi^T A(x) \xi \leq \beta|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, x \in \Omega
\]  

(2)

and

\[
|\nabla A_{ij}(x)| \leq M, \quad \forall x \in \Omega, i, j = 1, \ldots, d
\]  

(3)

with some constants \( \beta \geq \alpha > 0, M > 0 \).

### 2.1 Static condensation for CG FEM

To present our idea, we start by recalling the idea of static condensation, going back to [14]. as applied to the usual CG finite element method for problem (1). Let us assume for the moment (in this subsection only) that \( \Omega \) is a polygon (polyhedron) and introduce a conforming mesh \( \mathcal{T}_h \) on \( \Omega \) consisting of triangles (tetrahedrons). Assuming for simplicity \( g = 0 \), the usual continuous \( P_k \) finite element discretization of (1) is then written: find \( u_h \in V_h \) such that

\[
a(u_h, v_h) := \int_{\Omega} A \nabla u_h \cdot \nabla v_h = \int_{\Omega} f v_h, \quad \forall v_h \in V_h
\]  

(4)

\[
\tag{4}
\]
where $V_h$ is the space of continuous piecewise polynomial functions (polynomials of degree $\leq k$ on each mesh element $T \in \mathcal{T}_h$ for some $k \geq 1$) vanishing on $\partial \Omega$. The size of this problem, i.e. the dimension of $V_h$, is of order $k^3$ on a given mesh in 2D (resp. $k^3$ in 3D). To reduce this cost, one can decompose the space $V_h$ as follows

$$V_h = V_h^{loc} \oplus V_h'$$

where the subspace $V_h^{loc}$ consists of functions of $V_h$ that vanish on the boundaries of all the mesh elements $T \in \mathcal{T}_h$, and $V_h'$ is the complement of $V_h^{loc}$, orthogonal with respect to the bilinear form $a$. Decomposing $u_h = u_h^{loc} + u_h'$ with $u_h^{loc} \in V_h^{loc}$ and $u_h' \in V_h'$ we see that $\Omega^+$ is split into two independent problems

$$u_h^{loc} \in V_h^{loc} : \quad a(u_h^{loc}, v_h^{loc}) = \int_{\Omega} f v_h^{loc}, \quad \forall v_h^{loc} \in V_h^{loc}$$

(5)

$$u_h' \in V_h' : \quad a(u_h', v_h') = \int_{\Omega} f v_h', \quad \forall v_h' \in V_h'$$

(6)

The first problem above is further split into a collection of mutually independent problems on every mesh element $T \in \mathcal{T}_h$:

Find $u_h^{loc,T}$ such that

$$\int_T A \nabla u_h^{loc,T} \cdot \nabla v_h^{loc,T} = \int_T f v_h^{loc,T}, \quad \forall v_h^{loc,T} \in V_h^{loc,T}$$

(7)

where $V_h^{loc,T}$ is the restriction of $V_h^{loc}$ on $T$, i.e. the set of all polynomials of degree $\leq k$ vanishing on $\partial T$. The cost of solution of these local problems is negligible and we thus get very cheaply $u_h^{loc} |_T = u_h^{loc,T}$. Note also that Problem (7) can be recast as

$$\pi_T \mathcal{L}(u_h^{loc,T}) = \pi_T f, \quad \forall T \in \mathcal{T}_h$$

(8)

where $\pi_T$ is the projection to $V_h^{loc,T}$, orthogonal in $L^2(T)$.

On the other hand, Problem (6) remains global but its size is only proportional to $k$ in 2D (resp. $k^2$ in 3D) which is much smaller than that of the original problem (4). Indeed, the degrees of freedom are associated to the standard interpolation points of $P_k$ finite elements on the edges of the mesh. Note also that a basis for $V_h'$ can be constructed solving cheap local problems of the type

$$\pi_T \mathcal{L}(v_h' |_T) = 0, \quad \forall T \in \mathcal{T}_h$$

(9)

with appropriate boundary conditions on $\partial T$ insuring the continuity of functions in $V_h'$.

### 2.2 DG FEM: SIP and scSIP methods

We turn now to the main subject of this paper: the DG methods. We now let $\Omega$ be a bounded domain of general shape, and $\mathcal{E}_h$ be a splitting of $\Omega$ into a collection of non-overlapping subdomains (again of general shape, the precise definitions and assumptions on the mesh will be given Sections 3 and 4). Let $V_h$ denote the space of discontinuous piecewise polynomial functions (polynomials of degree $\leq k$ on each mesh element $T \in \mathcal{T}_h$ for some $k \geq 1$). The SIP method is then written as: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = L_h(v_h), \quad \forall v_h \in V_h$$

(10)

with the bilinear form $a_h$ and the linear form $L_h$ defined by

$$a_h(u, v) = \sum_{T \in \mathcal{T}_h} \int_T A \nabla u \cdot \nabla v - \sum_{E \in \mathcal{E}_h} \int_E ([A \nabla u \cdot n] | v) + ([A \nabla v \cdot n] | u)$$

$$+ \sum_{E \in \mathcal{E}_h} h_E \int_E [u][v]$$

(11)

and

$$L_h(v) = \sum_{T \in \mathcal{T}_h} \int_T f v + \sum_{E \in \mathcal{E}_h} \int_E g \left( \frac{h_E}{n} v - A \nabla v \cdot n \right)$$

(12)
where $\partial_h$ is the set of all the edges/faces of the mesh, $\partial_h^b \subset \partial_h$ regroups the edges/faces on the boundary $\partial \Omega$, $n$, $\cdot$ and $\{\cdot\}$ denote the unit normal, the jump and the mean over $E \in \partial_h$. More precisely, for any internal edge $E$ shared by two mesh elements $T_i^E$ and $T_j^E$, we choose $n$ as the unit vector, normal to $E$ and looking from $T_i^E$ to $T_j^E$. We then define for any function $v$ which is $H^1$ on both $T_i^E$ and $T_j^E$ but discontinuous across $E$

$$[v]|_E := v|_{T_i^E} - v|_{T_j^E}, \quad \{A\nabla v \cdot n\}|_E := \frac{1}{2} A \left( \nabla v|_{T_i^E} + \nabla v|_{T_j^E} \right) \cdot n$$

On a boundary edge $E \in \partial_h^b$, $n$ is the unit normal looking outward $\Omega$ and $[v] = v$, $\{A\nabla v \cdot n\} = A\nabla v \cdot n$.

Unlike the case of continuous finite elements, Problem (10) does not allow directly for a static condensation. However, we can construct a modification of (10) that mimics the characterization of local and global components of the solution by the projectors on local polynomial spaces (8)–(9). These spaces are now defined simply as

$$V_{h}^{\text{loc},T} = \mathbb{P}^{k-2}$$

We also let $\pi_{T,k-2}$ to be the projection to $V_{h}^{\text{loc},T}$, orthogonal in $L^2(T)$, and propose the following DG scheme:

- Compute $v_{h}^{\text{loc}} \in V_{h}$ by solving

  $$\pi_{T,k-2} \mathcal{L}(u_{h}^{\text{loc}}|_T) = \pi_{T,k-2} f, \quad \forall T \in \mathcal{T}_h$$

  We shall show that this problem admits an infinity of solutions. We choose arbitrarily a solution on each mesh element.

- Define the subspace of $V_{h}$

  $$V_{h}' = \{ v_{h}' \in V_{h} : \pi_{T,k-2} \mathcal{L}(v_{h}'|_T) = 0, \quad \forall T \in \mathcal{T}_h \}$$

  and compute $u_{h}' \in V_{h}'$ such that

  $$a_h(u_{h}', v_{h}') = L_h(v_{h}') - a_h(u_{h}^{\text{loc}}, v_{h}'), \quad \forall v_{h}' \in V_{h}'$$

- Set

  $$u_{h} = u_{h}^{\text{loc}} + u_{h}'.$$  

The local projection step (13) is not necessarily consistent with the original formulation (10) so that the solution $u_{h}$ given by (13)–(16) is different from that of (10). We shall prove however that SIP and scSIP approximations satisfy the same a priori error bounds. Moreover, they turn out to be very close to each other in numerical experiments. Note that the dimension of the “global space $V_{h}'$ on a given mesh is of order $k$ in 2D ($k^2$ in 3D) so that global problem (15) is much cheaper than (10) for large $k$.

### 3 Well posedness of SIP and scSIP methods and a priori error estimates

Let us now be more precise about the hypotheses on the mesh. Recall that $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, is a Lipschitz bounded domain and $\mathcal{T}_h$ is a general (not necessarily polygonal or polyhedral) mesh on $\Omega$. We mean by this that $\mathcal{T}_h$ is a decomposition of $\Omega$ into mutually disjoint cells $\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} \bar{T}$ so that each cell $T$ is a Lipschitz subdomain of $\Omega$ and for every $T_1, T_2 \in \mathcal{T}_h$ we have either $T_1 = T_2$ or $T_1 \cap T_2 = \emptyset$ (the cells $T_i \in \mathcal{T}_h$ are
and choosing with some $c$ treated here as open sets). We also introduce the sets of internal and boundary edges/faces as respectively

$$\mathcal{E}_h^i = \{ E = \bar{T}_1 \cap \bar{T}_2 \text{ for some } T_1, T_2 \in \mathcal{T}_h \}$$

$$\mathcal{E}_h^b = \{ E = T \cap \partial \Omega \text{ for some } T \in \mathcal{T}_h \}$$

and denote by $\mathcal{E}_h := \mathcal{E}_h^i \cup \mathcal{E}_h^b$ the union of all the edges/faces.

Let $B_T$, for any $T \in \mathcal{T}_h$, denote the smallest ball containing $T$, and $B_T^{\text{in}}$ denote the largest ball inscribed in $T$. Set $h_T = \text{diam}(T)$ and $h = \max_{T \in \mathcal{T}_h} h_T$. From now on, we assume that mesh $\mathcal{T}_h$ is

- **Shape regular**: there is a mesh-independent parameter $\rho_1 > 1$ such that, for all $T \in \mathcal{T}_h$,

  $$R_T \leq \rho_1 r_T$$

  where $r_T$ is the radius of $B_T^{\text{in}}$ and $R_T$ is the radius of $B_T$. This also implies $h_T \leq 2 \rho_1 r_T$ and $R_T \leq \rho_1 h_T$.

- **Locally quasi-uniform**: there is a mesh-independent parameter $\rho_2 > 1$ such that for all $T_1, T_2 \in \mathcal{T}_h$ with $\partial T \cap \partial T' \neq \emptyset$ one has

  $$h_{T_1} \leq \rho_2 h_{T_2}$$

Choose an integer $k \geq 1$ and introduce the FE space

$$V_h = \{ v \in L^2(\Omega) : v|_T \in \mathcal{P}_k(T), \forall T \in \mathcal{T}_h \}$$

In the sequel, we assume that $V_h$ has two following properties (and we shall prove in Section 3 that these properties hold under some additional assumptions on the mesh):

- **Optimal interpolation**: there exists an operator $I_h : H^{k+1}(\Omega) \rightarrow V_h$ such that for any $v \in H^{k+1}(\Omega)$

  $$\left( \sum_{T \in \mathcal{T}_h} \left( \| v - I_h v \|_{H^1(T)}^2 + \frac{1}{h_T^2} \| \nabla v - \nabla I_h v \|_{L^2(T)}^2 + h_T^{-1} \| v - I_h v \|_{H^2(T)}^2 \right) + h_T \| \nabla v - \nabla I_h v \|_{L^2(\partial T)}^2 + \frac{1}{h_T} \| v - I_h v \|_{L^2(\partial T)}^2 \right)^{\frac{1}{2}} \leq C \left( \sum_{T \in \mathcal{T}_h} h_T^2 \| v \|_{H^{k+1}(T)}^2 \right)^{\frac{1}{2}}$$

- **Inverse inequalities**: for any $v_h \in V_h$ and any $T \in \mathcal{T}_h$

  $$\| v_h \|_{L^2(\partial T)} \leq \frac{C}{\sqrt{h_T}} \| v_h \|_{L^2(T)} \quad \| \nabla v_h \|_{L^2(\partial T)} \leq \frac{C}{\sqrt{h_T}} \| \nabla v_h \|_{L^2(T)}$$

  and

  $$|v_h|_{H^1(T)} \leq \frac{C}{h_T} \| v_h \|_{H^1(T)}$$

We can now study the well posedness and establish optimal *a priori* error estimates for the classical DG method (10). These results are well known but we shall sketch their proofs for the sake of completeness.

**Lemma 1.** Under the above assumptions on the mesh and on $V_h$, setting

$$h_E = 2 \left( \frac{1}{h_{T_1}} + \frac{1}{h_{T_2}} \right)^{-1}$$

for any $E \in \mathcal{E}_h^i$ with $E = \partial T_1 \cap \partial T_2$,

$$h_E = h_T$$

for any $E \in \mathcal{E}_h^b$ with $E = \partial T \cap \partial \Omega$,

and choosing $\gamma$ large enough, $\gamma \geq \gamma_0$, the bilinear form $a_h$ defined by (11) is coercive, i.e.

$$a_h(v_h, v_h) \geq c \| v_h \|^2, \quad \forall v_h \in V_h$$

with some $c > 0$ and the triple norm defined by

$$\| u \|^2 = \sum_{T \in \mathcal{T}_h} |u|_{H^1(T)}^2 + \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} \| u \|_{L^2(E)}^2$$

The constants $c, \gamma_0$ depend only on the parameters in the assumptions on the mesh and on $V_h$, as well as on $\alpha, \beta$ in (3).
Consider the linear map $Q$ defined by

$$Q = \sum_{\ell \in \partial\Omega} A\nabla v_h \cdot \nabla v_h - 2 \sum_{E \in \partial\Omega} \int_E \{A\nabla v_h \cdot n\} [v_h] + \sum_{E \in \partial\Omega} \frac{\gamma}{h_T} \int_E |v_h|^2$$

$$\geq \alpha \sum_{T \in \mathcal{T}_h} |v_h|_{H^1(T)}^2 - \sum_{T \in \mathcal{T}_h} \left( \int_{\partial T \cap \Omega} A\nabla v_h \cdot n [v_h] + 2 \int_{\partial T \cap \partial\Omega} (A\nabla v_h \cdot n) v_h \right)$$

$$+ \sum_{T \in \mathcal{T}_h} \left( \frac{\gamma}{2h_T} \int_{\partial T \cap \Omega} |v_h|^2 + \frac{\gamma}{h_T} \int_{\partial T \cap \partial\Omega} v_h^2 \right)$$

$$\geq \frac{\alpha}{2} \sum_{T \in \mathcal{T}_h} |v_h|_{H^1(T)}^2 + \sum_{T \in \mathcal{T}_h} \frac{\gamma - C^2\beta^2/(2\alpha)}{h_T} \|v_h\|^2_{L^2(\partial T)}$$

with $C$ the constant from the inverse inequality (21). Rewriting the sum over the boundaries of mesh cells $T \in \mathcal{T}_h$ back to the sum over the edges and recalling the local quasi-uniformity assumption we get (24). \qed

Lemma 1 implies that problem (10) of the SIP method is well posed. Moreover, we have the error estimate:

**Theorem 1.** Assume that the solution $u$ to (1) is in $H^{k+1}(\Omega)$. Under the assumptions of Lemma 1, there exists the unique solution $u_h$ to (10) and it satisfies

$$|u - u_h|_{H^1(\mathcal{T}_h)} \leq C h^k |u|_{H^{k+1}(\Omega)}$$

where $H^1(\mathcal{T}_h)$ is the broken $H^1$ space on the mesh $\mathcal{T}_h$ and $| \cdot |_{H^1(\mathcal{T}_h)} := \left( \sum_{T \in \mathcal{T}_h} | \cdot |_{H^1(T)}^2 \right)^{\frac{1}{2}}$. If, moreover, the elliptic regularity property holds for (1), then

$$\|u - u_h\|_{L^2(\Omega)} \leq C |u|_{H^{k+1}(\Omega)} h^{k+1}$$

We skip the proof of this well known result. Actually, it goes along the same lines as that of our forthcoming Theorem 2.

We turn now to the study of the scSIP method (13)–(15) and start by the following technical lemma.

**Lemma 2.** There exists $h_0 > 0$ such that for all $T \in \mathcal{T}_h$ with $h_T \leq h_0$ and for all $q_T \in P_{k-2}(T)$ one can find $q_T \in P_k(T)$ such that

$$\int_T q_T (\mathcal{L} u_T) \geq \frac{1}{2} \|q_T\|^2_{L^2(T)}$$

(26) and

$$|q_T|^2_{H^1(T)} + \frac{1}{h_T} \|q_T\|^2_{L^2(\partial T)} \leq C h_T^2 \|q_T\|^2_{L^2(T)}$$

(27)

The constants $h_0$ and $C$ depend only on the regularity of the mesh and on $\alpha$, $\beta$ and $M$ in (2) and (3).

**Proof.** Let $\chi_T$ be the polynomial of degree 2 vanishing on $\partial B_T'$, i.e.

$$\chi_T(x) = \left( \sum_{i=1}^d (x_i - x_i^0)^2 - r_T^2 \right)$$

where $x^0 = (x_1^0, \ldots, x_d^0)$ is the center of $B_T'$ and $r_T$ is its radius. Set $A^0_{ij} = A_{ij}(x^0)$ and $\mathcal{L}^0 = -\partial A^0_{ij} \partial j$. Consider the linear map

$$Q : P_{k-2}(T) \to P_{k-2}(T)$$

defined by

$$Q(v) = \mathcal{L}^0 (\chi_T v)$$

The kernel of $Q$ is $\{0\}$. Indeed, if $Q(v) = 0$ then $w := \chi_T v$ is the solution to

$$\mathcal{L}^0 w = 0 \text{ in } B_T', \quad w = 0 \text{ on } \partial B_T'$$
so that \( w = 0 \) as a solution to an elliptic problem with vanishing right-hand side and boundary conditions. Since \( Q \) is a linear map on the finite dimensional space \( \mathbb{P}_{k-2}(T) \), this means that \( Q \) is one-to-one.

Take any \( q_T \in \mathbb{P}_{k-2}(T) \) and let \( u_T = \chi_T v_T \) with \( v_T \in \mathbb{P}_{k-2}(T) \) such that \( Q v_T = q_T \). We have thus constructed \( u_T \in \mathbb{P}_k(T) \) such that \( L^0 u_T = q_T \). This immediately proves (26) in the case of an operator \( \mathcal{L} = \mathcal{L}^0 \) with constant coefficients. Moreover, by scaling,

\[
|u_T|_{W^{2,r}(T)} + \frac{1}{h_T} |u_T|_{W^{1,r}(T)} + \frac{1}{h_T^2} ||u_T||_{L^2(B_T)} \leq C \frac{h_T}{h_T^2} ||q_T||_{L^2(B_T)}
\]

with a constant \( C \) depending only on \( \alpha, \beta \) and the ratio \( R_T/r_T \). Thus,

\[
|u_T|_{H^{1}(T)} \leq |T|^{1/2} |u_T|_{W^{1,r}(T)} \leq C h_T ||q_T||_{L^2(T)}
\]

which proves the estimate in \( H^1(T) \) norm in (27). Similarly, \( ||u_T||_{L^2(T)} \leq C h_T ||q_T||_{L^2(T)} \) and the estimate in \( L^2(\partial T) \) norm in (27) follows by the trace inverse inequality.

It remains to prove (26) in the case of operator \( \mathcal{L} \) with variable coefficients. To this end, we use the estimates in (28) as follows

\[
\int_T q_T \mathcal{L} u_T = \sum_{T \in \mathcal{T}_h} \int_T q_T \mathcal{L}^0 u_T + \int_T q_T \partial_i \left( (A_{ij} - A^0_{ij}) \partial_j u_T \right)
\]

\[
\geq ||q_T||_{L^2(T)}^2 \left( \sum_{T \in \mathcal{T}_h} \int_T \left( \partial_i \left( (A_{ij} - A^0_{ij}) \partial_j u_T \right) \right) \right)
\]

\[
\geq ||q_T||_{L^2(T)}^2 \left( \sum_{T \in \mathcal{T}_h} \int_T \left( \partial_i \left( (A_{ij} - A^0_{ij}) \partial_j u_T \right) \right) \right)
\]

\[
\geq ||q_T||_{L^2(T)}^2 - C h_T ||q_T||_{L^2(T)}^2 \right) \geq \frac{1}{2} ||q_T||_{L^2(T)}
\]

for sufficiently small \( h_T \).

\[
\text{Corollary 1. Introduce the bilinear form}
\]

\[
b_h(q, v) = \sum_{T \in \mathcal{T}_h} h_T^2 \int_T q \mathcal{L} u
\]

and the space

\[
M_h = \{ v \in L^2(\Omega) : v|_T \in \mathbb{P}_{k-2}(T), \forall T \in \mathcal{T}_h \}
\]

Equip the space \( V_h \) with the triple norm (25) and the space \( M_h \) with

\[
||q||_h = \left( \sum_{T \in \mathcal{T}_h} h_T^2 ||q||_{L^2(T)}^2 \right)^{1/2}
\]

The bilinear form \( b \) satisfies the inf-sup condition

\[
\inf_{q_h \in M_h} \sup_{v_h \in V_h} \frac{b(q_h, v_h)}{||v_h||_h} \geq \delta
\]

with a mesh-independent constant \( \delta > 0 \). Moreover, \( b \) is continuous on \( M_h \times V_h \) with a mesh-independent continuity bound.

\[
\text{Proof. Take any } q_h \in M_h, \text{ denote } q_T = q_h|_T, \text{ construct } u_T \text{ as in Lemma 2 and introduce } u_h \in V_h \text{ by } u_h|_T = u_T \text{ on all } T \in \mathcal{T}_h. \text{ This yields using (26) and (27),}
\]

\[
\frac{b_h(q_h, v_h)}{||v_h||_h} \geq \frac{\sum_{T \in \mathcal{T}_h} h_T^2 ||q_h||_{L^2(T)}^2}{\left( \sum_{T \in \mathcal{T}_h} C h_T^2 ||q_h||_{L^2(T)}^2 \right)^{1/2}} = \frac{2}{\sqrt{C}} ||q_h||_h
\]

which is equivalent to (29) with \( \delta = 2/\sqrt{C} \). Finally, the continuity of \( b \) is easily seen from inverse inequality (22).
Theorem 2. Assume that the solution $u$ to (1) is in $H^a$ the coercivity of indeed a solution at least on sufficiently refined meshes. The existence of a solution to (15) follows from the coercivity of $a_h$. Thus, scheme (13)–(16) produced indeed a solution $u_h \in V_h$. In order to establish the error estimates for this solution, we reinterpret its definition as a saddle point problem.

Lemma 3. The problem of finding $u_h \in V_h$ and $p_h \in M_h$ such that

$$a_h(u_h,v_h) + b_h(p_h,v_h) = L_h(v_h), \quad \forall v_h \in V_h \quad (30)$$

$$b_h(q_h,u_h) = \sum_{T \in \mathcal{T}_h} h_T^2 \int_T f q_h, \quad \forall q_h \in M_h \quad (31)$$

has the unique solution. Moreover, $u_h$ given by (30)–(31) coincides with $u_h$ given by (13)–(16). This implies that $u_h$ produced by the scheme (13)–(16) is unique.

Proof. The existence and uniqueness of the solution to (30)–(31) follows from the standard theory of saddle point problems, cf. for example Proposition 2.36 from [13], thanks to the coercivity of $a_h$ (Lemma 1) and to the inf-sup property on $b_h$ (Corollary 1).

In order to explore its relation with $u_h$ from (13)–(16), we note

$$b_h(q_h,u^{1_{\text{loc}}} + u'_h, v'_h) = \sum_{T \in \mathcal{T}_h} h_T^2 \int_T f q_h \quad \forall q_h \in M_h$$

We obtain thus

$$b_h(q_h,u^{1_{\text{loc}}} + u'_h, v'_h) = \sum_{T \in \mathcal{T}_h} h_T^2 \int_T f q_h, \quad \forall q_h \in M_h$$

which is equivalent to (31). Eq. (15) can be rewritten as

$$a_h(u_h^{1_{\text{loc}}} + u'_h, v'_h) = L_h(v'_h), \quad \forall v'_h \in V'_h$$

This is equivalent to (30) after eliminating $p_h$ since $V'_h$ is precisely the kernel of the bilinear form $b_h$. We conclude that $u_h = u^{1_{\text{loc}}} + u'_h$ coincides with the solution to (30)–(31).

Theorem 2. Assume that the solution $u$ to (1) is in $H^{k+1}(\Omega)$. Under the assumptions of Lemma 1 and $h$ sufficiently small, the scSIP method (13)–(16) produces the unique solution $u_h \in V_h$, which satisfies

$$|u - u_h|_{H^1(\mathcal{T}_h)} \leq C h^k |u|_{H^{k+1}(\Omega)}$$

If, moreover, the elliptic regularity property holds for (1), then

$$|u - u_h|_{L^2(\Omega)} \leq C |u|_{H^{k+1}(\Omega)} h^{k+1}$$

Proof. We shall use the saddle point reformulation (30)–(31). This discretization is consistent. Indeed setting $p = 0$ we have

$$a_h(u,v_h) + b_h(p,v_h) = L_h(v_h), \quad \forall v_h \in V_h$$

$$b_h(q_h,u) = \sum_{T \in \mathcal{T}_h} h_T^2 \int_T f q_h, \quad \forall q_h \in M_h$$

Thus, by the standard theory of saddle point problems, cf. for example Proposition 2.36 from [13], recalling the coercivity of $a_h$ (Lemma 1) and the inf-sup property on $b_h$ (Corollary 1), we get

$$\|u_h - I_h u\| + \|p_h\| \leq C \sup_{(v_h, q_h) \in V_h \times M_h} \frac{a_h(u_h - I_h u, v_h, q_h) + b_h(p_h, v_h, q_h, u_h - I_h u)}{\|v_h\| + \|q_h\|}$$

$$= C \sup_{(v_h, q_h) \in V_h \times M_h} \frac{a_h(u - I_h u, v_h) + b_h(q_h, u - I_h u)}{\|v_h\| + \|q_h\|}$$

$$\leq C \left( |u - I_h u|_a^2 + \sum_{T \in \mathcal{T}_h} h_T^2 \|\mathcal{L}(u - I_h u)|_{L^2(T)}^2 \right)^{\frac{1}{2}}$$
with the augmented triple norm $\|\cdot\|_a$ defined by

$$\|v\|_a^2 := \|v\|^2 + \sum_{E \in \mathcal{E}_h} h_E \|A \nabla v \cdot n\|_{L^2(E)}^2$$

Applying the interpolation estimates (20) and the triangle inequality gives

$$\|u - I_h u\|_a + \|p_h\|_h \leq Ch^k |u|_{H^{k+1}}$$

(34)

This implies in particular (32).

To prove the $L^2$ error estimate, we consider the auxiliary problem for $z \in H^2(\Omega)$

$$\mathcal{L}z = u - u_h \text{ in } \Omega, \quad z = 0 \text{ on } \partial \Omega$$

Then, for all $v \in H^1(\mathcal{T}_h)$ and $q \in L^2(\Omega),$

$$a_h(v, z) + b_h(q, z) = \int_{\Omega} (u - u_h)v + \sum_{T \in \mathcal{T}_h} h_T^2 \int_T (u - u_h)q$$

Setting $v = u - u_h$ and $q = p - p_h$ (with $p = 0$) and using Galerkin orthogonality yields

$$\|u - u_h\|_{L^2(\Omega)}^2 + \sum_{T \in \mathcal{T}_h} h_T^2 \int_T (u - u_h)(p - p_h) = a_h(u - u_h, z) + b_h(p - p_h, z)$$

$$= a_h(u - u_h, z - z_h) + b_h(p - p_h, z - z_h)$$

for any $z_h \in V_h$. Thus, taking $z_h = I_h z$ and applying the interpolation estimates,

$$\|u - u_h\|_{L^2(\Omega)}^2 \leq |a_h(u - u_h, z - z_h)| + h\|p - p_h\|_h \left( \sum_{T \in \mathcal{T}_h} \|\mathcal{L}z - \mathcal{L}z_h\|_{L^2(T)}^2 \right)^{1/2}$$

$$+ h\|p - p_h\|_h \|u - u_h\|_{L^2(\Omega)}$$

$$\leq Ch \|u - I_h u\|_a \|z\|_{H^2(\Omega)} + Ch\|p_h\|_h (\|z\|_{H^2(\Omega)} + \|u - u_h\|_{L^2(\Omega)})$$

Recalling $\|z\|_{H^2(\Omega)} \leq C\|u - u_h\|_{L^2(\Omega)}$ and (34) yields (33).

4 An example of assumptions on the mesh that guarantee the interpolation and inverse estimates

In this section, we adopt the following assumptions on the mesh.

M1: $\mathcal{T}_h$ is shape regular in the sense (17) with a parameter $\rho_1 > 1$.

M2: $\mathcal{T}_h$ is locally quasi-uniform in a sense stronger than (18): one can assign a local meshsize $h_x > 0$ to any point $x \in \Omega$ so that for all $T \in \mathcal{T}_h$ and any $x \in B_T$

$$\frac{1}{\rho_2} h_x \leq h_T \leq \rho_2 h_x$$

with a parameter $\rho_2 > 1$.

M3: The cell boundaries are not too wiggly: for all $T \in \mathcal{T}_h$

$$|\partial T| \leq \rho_3 h_T^{d-1}$$

with a parameter $\rho_3 > 0$. 

Under Assumptions M1 and M3, we have for any \( v \in \mathbb{R}^d \) at least for \( k \geq 2 \) and to prove the optimal error (20) the inverse estimates (21)–(22).

First of all, the assumptions that the mesh is shape regular and locally quasi-uniform entail the following

**Lemma 4.** Let \( N_{ball}(x) \) be the number of balls \( B_T \), \( T \in \mathcal{T}_h \) covering a point \( x \), \( x \in \mathbb{R}^d \). Under assumptions M1 and M2, we have \( N_{ball}(x) \leq N_{int} \) for all \( x \in \mathbb{R}^d \) with a constant \( N_{int} \) depending only on \( \rho_1 \) and \( \rho_2 \).

**Proof.** Take any \( x \in \mathbb{R}^d \). If \( x \in B_T \) for some \( T \in \mathcal{T}_h \) then \( h_T \leq \rho_2 h \). Hence, \( T \) is inside the ball \( B_x \) of radius \( \rho_2 h \), centered at \( x \). Recall that \( T \) contains an inscribed ball of radius \( r_T \geq \frac{h_T}{\rho_1} \geq \frac{h_T}{\rho_2} \). If there are several such mesh cells covering \( x \), then their inscribed balls do not intersect each other and they are all inside \( B_x \). Thus, their number is bounded by

\[
\frac{|B_x|}{\min_{v \in B_T} |B_T^m|} \leq \left( \frac{\rho_2 h_x}{\rho_1 \rho_2 h} \right)^d = \left( \frac{\rho_1 \rho_2^2}{2} \right)^d
\]

Recall that \( V_h \) is the discontinuous FE space on \( \mathcal{T}_h \) of degree \( k \geq 1 \), cf. (19). Take any \( v \in H^{k+1}(\Omega) \).

**Lemma 5. (Local interpolation estimate)** Take any \( T \in \mathcal{T}_h \). Let \( \pi_h \) denote the \( L^2(B_T) \)-orthogonal projection to the space of polynomials, i.e. given \( v \in L^2(B_T) \), \( v_h = \pi_h v \) is a polynomial from \( \mathbb{P}^k \) such that

\[
\int_{B_T} v_h \phi_h = \int_{B_T} v \phi_h \quad \forall \phi_h \in \mathbb{P}^k
\]

Under Assumptions M1 and M3, we have then for any \( v \in H^{k+1}(B_T) \)

\[
|v - v_h|_{H^1(B_T)} + \frac{1}{h_T} \|v - v_h\|_{L^2(B_T)} + h_T |v - v_h|_{H^2(B_T)} + \frac{1}{\sqrt{h_T}} \|\nabla(v - v_h)\|_{L^2(\partial T)} + \frac{1}{\sqrt{h_T}} \|v - v_h\|_{L^2(\partial T)} \leq C h_T^k |v|_{H^{k+1}(B_T)}
\]

with a constant \( C > 0 \) depending only on \( \rho_3 \).

**Proof.** Since the embedding of \( H^{k+1}(B_T) \) into \( L^\infty(B_T) \) is compact, Petree-Tartar lemma together with a scaling argument entail

\[
\|v - v_h\|_{L^\infty(B_T)} \leq C h_T^{k+1-\frac{d}{2}} |v|_{H^{k+1}(B_T)}
\]

Hence,

\[
\|v - v_h\|_{L^2(B_T)} \leq |T|^\frac{d}{2} \|v - v_h\|_{L^\infty(B_T)} \leq C h_T^{k+1} |v|_{H^{k+1}(B_T)}
\]

and, in view of the hypothesis \( |\partial T| \leq \rho_3 h_T^{d-1} \),

\[
\|v - v_h\|_{L^2(\partial T)} \leq (\rho_3 h_T^{d-1})^\frac{1}{2} \|v - v_h\|_{L^\infty(B_T)} \leq C h_T^{k+1/2} |v|_{H^{k+1}(B_T)}
\]

The estimates for \( |v - v_h|_{H^1(B_T)} \) and \( \|\nabla v - \nabla v_h\|_{L^2(\partial T)} \) are proven in the same way starting from

\[
\|\nabla v - \nabla v_h\|_{L^\infty(B_T)} \leq C h_T^{k-\frac{d}{2}} |v|_{H^{k+1}(B_T)}
\]

This is valid since the embedding of \( H^{k+1}(B_T) \) into \( W^{1,\infty}(B_T) \) is compact for \( k \geq 2 \).

Finally, the estimate for \( |v - v_h|_{H^2(B_T)} \) holds by the compactness of the embedding of \( H^{k+1}(B_T) \) into \( H^2(B_T) \) (\( k \geq 2 \)).
Lemma 6. (Global interpolation estimate) Let \( I_h : H^{k+1}(\Omega) \rightarrow V_h \) denote the operator obtained by first extending a function \( v \in H^{k+1}(\Omega) \) by a function \( \tilde{v} \in H^{k+1}(\mathbb{R}^d) \) and then applying the local operator \( \pi_h \) from Lemma 5 to \( \tilde{v} \) on every \( T \in \mathcal{T}_h \), i.e. \( I_h v|_T := (\pi_h \tilde{v})|_T \) on any \( T \in \mathcal{T}_h \). Then, under Assumptions M1–M3, (20) holds for any \( v \in H^{k+1}(\Omega) \).

**Proof.** First, extension theorem for Sobolev spaces [1] insure that there exists \( \tilde{v} \in H^{k+1}(\mathbb{R}^d) \) such that
\[
\tilde{v} = v \quad \text{on} \quad \Omega \quad \text{and} \quad \| \tilde{v} \|_{H^{k+1}(\mathbb{R}^d)} \leq C \| \tilde{v} \|_{H^{k+1}(\Omega)}
\]

To prove (20), we sum the local interpolation estimates of Lemma 5 over all the mesh elements and then use the estimates of Lemma 4:
\[
\sum_{T \in \mathcal{T}_h} \left( |v - v_h|^2_{H^2(T)} + \frac{1}{h_T^2} \| v - v_h \|^2_{L^2(T)} + h_T \| \nabla v - \nabla v_h \|^2_{L^2(\partial T)} + \frac{1}{h_T^2} \| v - v_h \|^2_{L^2(\partial T)} \right) \leq C \sum_{T \in \mathcal{T}_h} h_T^{2k} \int_{T} |v|^{k+1} |v|^2 dx \leq C N_m \rho_2 \sum_{T \in \mathcal{T}_h} h_T^{2k} \| v \|^2_{H^{k+1}(T)}
\]

Lemma 7. (Inverse inequalities) Under assumptions one M1 and M3, (21) and (22) hold for any \( v_h \in V_h \) and any \( T \in \mathcal{T}_h \).

**Proof.** Both bounds in (21) follow immediately from the following one: for any polynomial \( v_h \) of degree \( \leq l \) (with \( l = k \) or \( l = k - 1 \)) one has
\[
\| v_h \|_{L^\infty(T)} \leq \frac{C}{h_T^{2l/2}} \| v_h \|_{L^2(T)}
\]
which in turn follows from
\[
\| v_h \|_{L^\infty(B_T)} \leq \frac{C}{h_T^{2l/2}} \| v_h \|_{L^2(B_T^{\rho_0})}
\]
where \( B_T^{\rho_0} \) is the largest ball inscribed in \( T \). Scaling the ball \( B_T \) to a ball of radius 1 \( B_1 \) and considering all the possible positions of the inscribed ball, the last inequality can be rewritten as
\[
\| v_h \|_{L^\infty(B_1)} \leq \min \left\{ C \rho_0 \leq B_1, B_{\rho_0} \text{ a ball of radius } \geq \rho_1 \right\} \| v_h \|_{L^2(B_T^{\rho_0})}
\]
This is valid for any polynomial of degree \( l \) by equivalence of norms.

The remaining inverse inequality (22) can be proven similarly:
\[
|v_h|_{H^2(T)} \leq C h_T^{2l/2} |v_h|_{L^\infty(B_T^{\rho_0})} \leq C h_T^{d/2-1} \| v_h \|_{H^1(B_T)} \leq \frac{C}{h_T^2} \| v_h \|_{H^1(B_T^{\rho_0})} \leq \frac{C}{h_T^2} \| v_h \|_{H^1(T)}
\]

5 Numerical results

We shall illustrate the convergence of SIP and scSIP methods on polygonal meshes obtained by agglomerating the elements of a background triangular mesh. Both the mesh construction and the following calculations are done in FreeFEM++ [15]. An example of such a mesh is given in Fig. 1. To construct it, we take a positive integer \( n \) (\( n = 4 \) in the Figure), let FreeFEM++ to construct a Delaunay triangulation of \( \Omega = (0, 1)^2 \) with \( 4n \) boundary nodes on each side of the square, and finally agglomerate the triangles of this mesh into \( n \times n \) cells as follows. We start by attributing the triangle containing the point
\[
\left( \frac{i-1/2}{n}, \frac{j-1/2}{n} \right), \quad i, j = 1, \ldots, n
\]
Figure 1: On the left, a polygonal mesh consisting of $4 \times 4$ cells, which are obtained by the agglomeration of the triangles of a finer mesh seen on the right.

Figure 2: The error in $L^2$ norm and $H^1$ semi-norm vs. mesh-size $h$. The solid lines with squares represent the SIP method. The dashed lines with circles represent the scSIP method.
to the cell number $i + jn$. Then, iteratively, we run over all the cells and attach yet unattributed triangles neighboring a triangle from a cell to the same cell, until all the triangles are attributed.

We have considered Poisson equation, i.e. \((1)\) with $A = I$, on $\Omega = (0, 1)^2$ with homogeneous Dirichlet boundary conditions $g = 0$ and the exact solution $u = \sin(\pi x) \sin(\pi y)$. We have applied SIP method \((10)\) and scSIP method \((13)-(15)\) to this problem on the agglomerated meshes as described above. The results are presented in Fig. 2. In a slight deviation from the general notations, we set here the mesh-size as $h = 1/n$ on the $n \times n$ mesh, and $h_E = h$ on all the edges in \((11)\). Three choices for the polynomial space degree $k$ were investigated, namely $k = 2, 3, 4$ and the penalty parameter $\gamma$ in \((11)\) was set, respectively, to 10, 30, 50. The numerical results confirm the theoretically expected order of convergence in both $L^2$ norm and $H^1$ semi-norm. They also demonstrate that the approximation produced by SIP and scSIP methods are very close to each other.

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