Casimir entropy for magnetodielectrics

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Abstract
We find the analytic expressions for the Casimir free energy, entropy and pressure at low temperature in the configuration of two parallel plates made of magnetodielectric material. The cases of constant and frequency-dependent dielectric permittivity and magnetic permeability of the plates are considered. Special attention is paid to the account of dc conductivity. It is shown that in the case of finite static dielectric permittivity and magnetic permeability the Nernst heat theorem for the Casimir entropy is satisfied. If the dc conductivity is taken into account, the Casimir entropy goes to a positive nonzero limit depending on the parameters of a system when the temperature vanishes, i.e. the Nernst theorem is violated. The experimental situation is also discussed.

Keywords: Casimir effect, Nernst heat theorem, magnetodielectrics

1. Introduction

The Casimir force [1] acts between closely spaced material bodies. It is caused by the zero-point and thermal fluctuations of the electromagnetic field. The Casimir force is of the same physical nature as the van der Waals force, but it acts at larger separation distances (typically starting from a few nanometers to a few micrometers), where the effects of relativistic retardation become essential. The Casimir effect finds numerous and diverse applications ranging from condensed matter physics and nanotechnology to quantum field theory, gravitation and cosmology [2–4]. Specifically, in nanotechnology the Casimir force can lead to a stiction and a loss of functionality in microdevices [5], on the one hand and plays a useful role by actuating microdevices [6], on the other hand.

Advances in microfabrication techniques allowed to perform precise measurements of the Casimir force and its gradient at separations up to a few hundred nanometers at room and lower temperature (see reviews [7–9] and some of later experiments [10–15]). The measurement results have been compared with theoretical predictions of the Lifshitz theory [16] of the van der Waals and Casimir forces. This comparison resulted in a big surprise. The Lifshitz theory uses the frequency-dependent dielectric permittivity $\epsilon(\omega)$ of interacting bodies as an input data for calculation. It turned out that if the low-frequency behavior of $\epsilon(\omega)$ is described by the most precise Drude model, taking into account the relaxation properties of conduction electrons in metals, the theoretical results are excluded by the data at a high confidence level [4, 7, 12]. If, alternatively, the low-frequency behavior of $\epsilon(\omega)$ is described by the nondissipative plasma model, the theoretical results are consistent with the measurement data. (There is only one experiment [17], which is claimed to be in agreement with the Drude model; it measures, however, not the Casimir force, but up to an order of magnitude larger force of unclear origin and subtracts the major part of it by means of a fitting procedure which was critically discussed in the literature [18, 19].)

In parallel with the experimental progress, the analytic behavior of the Casimir free energy and entropy at low temperature has been found in the framework of the Lifshitz theory for the case of two metallic test bodies [20–22]. Here also a big surprise has been found supposedly connected with the above puzzle in theory-experiment comparison. On the one hand, for metal plates with perfect crystal lattices, the Casimir entropy calculated by using the Lifshitz theory combined with the Drude model was shown to go to a nonzero negative limit depending on the parameters of the system when the temperature vanishes [20–22]. This means the violation of the third law of thermodynamics (the Nernst heat theorem). If the dielectric permittivity of the plasma model is used, the Casimir
entropy was shown to go to the zero limit when temperature goes to zero, i.e. the Nernst heat theorem is satisfied. On the other hand, at large separations, where the Casimir force is classical, the plasma model was shown\textsuperscript{[23]} to violate the Bohr–van Leeuwen theorem, whereas the Drude model is consistent with it. Later on, the thermodynamic inconsistency was also revealed for dielectric plates with taken into account conductivity at a constant current (the dc conductivity). It was shown\textsuperscript{[24–26]} that in this case the Casimir entropy goes to a nonzero positive limit depending on the parameters of a system when the temperature vanishes. The experimental data of several experiments using dielectric materials were found in agreement with theoretical predictions of the Lifshitz theory only if the contribution of free charge carriers into the dielectric permittivity is omitted\textsuperscript{[4, 7, 9, 11, 27–29]}.

Many attempts were undertaken in order to solve the problem of thermodynamic inconsistency of the Lifshitz theory. Specifically, it was argued\textsuperscript{[30]} that for metallic crystal lattices with some concentration of impurities there is some small but nonzero residual relaxation at zero temperature. As a result, at very low temperature the Casimir entropy undergoes an abrupt increase to zero and the Nernst heat theorem is formally satisfied. The problem remains, however, unsolved for metals with perfect crystal lattices because the perfect lattice is a truly equilibrium system with the nondegenerate dynamical state of lowest energy. Thus, the Nernst heat theorem should be satisfied in this case. Another attempt to avoid thermodynamic inconsistency for dielectrics\textsuperscript{[31]} and metals\textsuperscript{[32]} goes beyond the Lifshitz theory and modifies it by taking into account screening effects and diffusion currents of free charge carriers. It was shown, however, that for metals with perfect crystal lattices, as well as for those dielectrics whose concentration of free charge carriers does not go to zero with vanishing temperature (for such dielectric materials conductivity goes to zero due to vanishing mobility), the Nernst heat theorem remains to be violated in the modified theory\textsuperscript{[33–35]}. Furthermore, the modified theory turned out to be excluded by the measurement data of experiments with metallic test bodies at the same high level of confidence as the standard Drude model\textsuperscript{[34, 35]}. For dielectric test bodies the modified theory was found to be consistent with the data of experiment\textsuperscript{[28]}, but still excluded by the experiment\textsuperscript{[27]}.

In this paper, we investigate the low-temperature behavior of the Casimir free energy, entropy and pressure for two parallel plates made of a magnetodielectric material. The Lifshitz theory was generalized for the case of magnetic materials long ago\textsuperscript{[36]} and this subject recently received considerable attention in the literature\textsuperscript{[37–40]}. It became especially topical after successful measurements of the gradient of the Casimir force between magnetic materials\textsuperscript{[41–43]}, which again excluded the Drude model and turned out to be consistent with the plasma model. Here, we show that for magnetodielectric plates the low-temperature behavior of all physical quantities is typically the same as for dielectric ones and only the respective coefficients are generalized due to the presence of an additional parameter—the magnetic permeability $\mu(\omega)$. As a result, we have proven that for magnetodielectric plates with omitted dc conductivity the Nernst heat theorem is satisfied and, if the dc conductivity is included, the Nernst theorem is violated. This result is obtained for both constant and frequency-dependent magnetic permeability.

The paper is organized as follows. Section 2 contains brief presentation of the formalism adapted for use in perturbation expansions. In section 3 we derive the asymptotic expressions for the Casimir free energy, entropy and pressure at low temperature for magnetodielectrics with constant dielectric permittivity and magnetic permeability. Section 4 is devoted to the account of frequency dependence. In section 5 we formulate our conclusions.

## 2. Basic formalism

We consider two parallel thick magnetodielectric plates (semispaces) characterized by the frequency-dependent dielectric permittivity $\epsilon$ and magnetic permeability $\mu$ at temperature $T$ in thermal equilibrium with an environment. In terms of dimensionless variables, the Lifshitz formula for the free energy of the Casimir interaction per unit area of the plates is given by\textsuperscript{[4, 16]}

$$F(a, T) = \frac{\hbar c T}{32\pi^2 a^3} \sum_{l=0}^{\infty} \int_0^\infty dy \left[ \ln \left( 1 - r_{TM}^2 e^{-y} \right) + \ln \left( 1 - r_{TE}^2 e^{-y} \right) \right].$$

(1)

Here, $a$ is the separation distance between the plates and the dimensionless quantity $\tau = 4\pi k_B a T/(\hbar c)$ where $k_B$ is the Boltzmann constant. The prime near the summation sign means that the term with $l = 0$ is divided by two. The reflection coefficients for the two independent polarizations of the electromagnetic field, transverse magnetic (TM) and transverse electric (TE), are given by

$$r_{TM} = r_{TM}(i\xi_l, y) = \frac{\epsilon_l y + \sqrt{y^2 + \xi_l^2} (\epsilon_l \mu_l - 1)}{\epsilon_l y + \sqrt{y^2 + \xi_l^2} (\epsilon_l \mu_l - 1)}$$

$$r_{TE} = r_{TE}(i\xi_l, y) = \frac{\mu_l y + \sqrt{y^2 + \xi_l^2} (\epsilon_l \mu_l - 1)}{\mu_l y + \sqrt{y^2 + \xi_l^2} (\epsilon_l \mu_l - 1)}$$

(2)

In these expressions, we use the notations $\epsilon_l \equiv \epsilon(i\xi_l)$, $\mu_l \equiv \mu(i\xi_l)$, $\xi_l = 2\pi k_B T l / \hbar$ and $\xi_l' = \xi_l / \xi_c = l \tau$, where $\xi_c \equiv c/(2a)$, are the dimensional and dimensionless Matsubara frequencies, respectively. Note also that the dimensionless variable $y = 2a[k_B^2 + \xi_l^2/c^2]^{1/2}$ where $k_B$ is the projection of the wave vector on the plane of plates.

By applying the Abel-Plana formula\textsuperscript{[4]}

$$\sum_{l=0}^{\infty} F(l) = \int_0^\infty F(t) \, dt + \int_0^\infty dt \frac{F(it) - F(-it)}{e^{2\pi it} - 1},$$

(3)

one can rearrange (1) to the form\textsuperscript{[4]}

$$F(a, T) = E(a) + \Delta_T F(a, T),$$

(4)
where the first term on the right-hand side is the Casimir energy per unit area at zero temperature

$$E(a) = \frac{\hbar c}{32\pi^2a^3} \int_0^\infty d\xi F(\xi),$$

$$F(\xi) \equiv \int_\xi^\infty dy f(\xi, y).$$  \hspace{1cm} (5)

The second term on the right-hand side of (4) has the meaning of the thermal correction

$$\Delta_T F(a, T) = \frac{\hbar c T}{32\pi^2a^3} \int_0^\infty d\xi \Phi(\xi),$$

$$\Phi(\xi) \equiv \int_\xi^\infty dy g(\xi, y).$$ \hspace{1cm} (6)

In a similar way, the Casimir pressure can be represented in the form

$$P(a, T) = P_0(a) + \Delta_T P(a, T),$$ \hspace{1cm} (7)

where the Casimir pressure at $T = 0$ is [4]

$$P_0(a) = -\frac{\hbar c}{32\pi^2a^3} \int_0^\infty d\xi \Phi(\xi),$$

$$g(\xi, y) \equiv y^2 \left[ \frac{r_{TM}^2(\xi, y)}{\epsilon^2} - \frac{r_{TE}^2(\xi, y)}{\epsilon^2} \right],$$

and the thermal correction to it is given by [4]

$$\Delta_T P(a, T) = -\frac{\hbar c T}{32\pi^2a^3} \int_0^\infty d\xi \Phi(i\tau) - \Phi(-i\tau) \frac{\epsilon^2}{e^{2\pi\tau} - 1}. \hspace{1cm} (9)$$

3. Casimir free energy, entropy and pressure at low temperature for constant $\epsilon$ and $\mu$

In this section we describe the magnetodielectric plates by their static dielectric permittivity $\epsilon_0 \equiv \epsilon(0)$ and magnetic permeability $\mu_0 \equiv \mu(0)$. In fact at sufficiently low $T$ the Matsubara frequencies, giving the leading contribution to the thermal corrections (6) and (9) for insulating material with no free charge carriers, belong to the region where $\epsilon$ is almost constant and equal to $\epsilon_0$. Because of this, the qualitative behavior of the Casimir free energy at low temperature for dielectric materials described by constant and frequency-dependent $\epsilon$ is similar (only a power in the power-type law can be different [24, 26]).

The account of dc conductivity in the dielectric permittivity and frequency-dependence of the magnetic permeability are discussed in section 4.

3.1. Thermal correction to the Casimir energy in the lowest order

Now we assume that $\xi \ll 1$ and expand the function $f(\xi, y)$ defined in (5) in powers of this small parameter

$$f(\xi, y) = y \sum_{k=1,2} \left[ \ln(1 - r_{0,k}^2 e^{-y}) + \frac{2\alpha_k \beta}{y (1 + \alpha_k)^2} \frac{r_{0,k} e^{-y}}{1 - r_{0,k}^2 e^{-y}} \right] + O(\xi^4).$$ \hspace{1cm} (10)

Here, the following notations are introduced:

$$\alpha_1 = \epsilon_0, \quad \alpha_2 = \mu_0, \quad \beta = \epsilon_0 \mu_0 - 1, \quad r_{0,k} = \frac{\alpha_k - 1}{\alpha_k + 1}. \hspace{1cm} (11)$$

Integrating equation (10) in accordance to (5), one obtains

$$F(\xi) = -\sum_{k=1,2} \left[ Li_3 \left( r_{0,k}^2 \right) - \frac{\xi^2}{2} \ln(1 - r_{0,k}^2) \right] - \frac{\xi^2}{2} \sum_{n=1}^\infty \frac{r_{0,k}^{2n}}{n} Ei(-n\xi) + O(\xi^3). \hspace{1cm} (12)$$

where $Li_n(z)$ is the polylogarithm function and $Ei(z)$ is the exponential integral function.

Now we take into account the identity [44]

$$Ei(ix) - Ei(-ix) = -i\pi + O(x) \hspace{1cm} (13)$$

and calculate the quantity $F(i\tau) - F(-i\tau)$ using equation (12), where we put $\xi = \tau t \ll 1$:

$$F(i\tau) - F(-i\tau) = i\pi t t \sum_{k=1,2} \sum_{n=1}^\infty \frac{r_{0,k}^{2n}}{2n} \left[ \left( r_{0,k}^2 \right)^n + 1 \right] + O(\tau^3). \hspace{1cm} (14)$$

Substituting this expression in (6) and performing the integration with respect to $t$, one finds

$$\Delta_T F(a, T) = -\frac{\hbar c}{32\pi^2a^3} \int_0^\infty d\xi \Phi(i\tau) - \Phi(-i\tau) \frac{\epsilon^2}{e^{2\pi\tau} - 1}. \hspace{1cm} (15)$$

where $\zeta(z)$ is the Riemann zeta function. This is the lowest order thermal correction to the Casimir energy for magnetodielectric plates with constant $\epsilon$ and $\mu$. For $\mu_0 = 1$ it coincides with the previously obtained result [25].

From (15) it is straightforward to get the analytic behavior of the Casimir entropy per unit area at low temperature

$$S(a, T) = -\frac{\partial \Delta_T F(a, T)}{\partial T} = \frac{3\zeta(3) k_B^2 (\epsilon_0 \mu_0 - 1)^2}{2\pi (\hbar c)^2 (\epsilon_0 + 1)(\mu_0 + 1)} T^2 + O(T^3). \hspace{1cm} (16)$$

Here, we returned to the dimensional temperature $T$. As can be seen from this expression, the Casimir entropy goes to zero when the temperature vanishes, i.e. the Nernst heat theorem is satisfied for the magnetodielectric plates.

Note that the right-hand side of (15) does not depend on $a$ because $\tau \sim a$. This does not allow to obtain from (15) the asymptotic behavior of the thermal correction to the Casimir pressure at low temperature. It is also difficult to find the next, fourth order, term of the perturbation expression in (15) because all powers of the expansion of $f(\xi, y)$ in equation (10) contribute to its coefficient. In the next subsection the perturbation asymptotic expansion for the Casimir pressure is obtained in a direct way starting from equation (9). This makes possible to find the next terms in the asymptotic expressions (15) and (16) for the thermal correction to the Casimir energy and for the Casimir entropy, respectively.
3.2. Thermal correction to the Casimir pressure in the lowest order

We begin with the function \( g(\zeta, y) \) defined in (8) and expand it in powers of a small parameter \( \zeta \)

\[
g(\zeta, y) = \sum_{k=1,2} \left[ \frac{\pi^2}{3} \frac{\beta}{2} \frac{\zeta^2}{r_{0,k}} - \frac{2\alpha_k}{\beta} \frac{\zeta^2}{(1+\alpha_k)^2} \left( 1 - \frac{r_{0,k}^2 e^{-2y}}{1 - r_{0,k}^2 e^{-2y}} \right)^2 \right] + O(\zeta^4). \tag{17}\]

Then we integrate (17) in accordance with (8) and obtain

\[
\Phi(\zeta) = \sum_{k=1,2} \left[ 2\mathrm{Li}_3\left( \frac{\zeta^2}{r_{0,k}} \right) - \frac{\beta}{2} \frac{\zeta^2}{r_{0,k}} \right] + \frac{\alpha_k}{6} \left( \beta - 1 - \sqrt{\beta + 1 + 1} \right)^2 \zeta^3 + O(\zeta^4). \tag{18}\]

Using this expression, the difference of functions \( \Phi(\zeta, y) \) entering equation (9) takes the form

\[
\Phi(\zeta) = \Phi(-\zeta) = \Phi(\zeta) + \Phi(0) \tag{19}\]

Substituting this in (9) and integrating with respect to \( t \), one finds

\[
\Phi(\zeta, y) = \frac{\mu_0 (\epsilon_0 + \mu_0) (\epsilon_0 \mu_0 - 2) \sqrt{\epsilon_0 \mu_0 + 2}}{720} \tau^4 + O(\tau^5). \tag{20}\]

This result presents the low-temperature behavior of the thermal correction to the Casimir pressure for magnetodielectric plates. In the absence of magnetic properties (\( \mu_0 = 1 \)) it coincides with the known result [25].

Equation (20) allows to find the more exact than (15) and (16) asymptotic expansions for the thermal correction to the Casimir energy and Casimir entropy. For this purpose we use the relationship

\[
\Delta_T P(a, T) = -\frac{\partial \Delta_T F(a, T)}{\partial a} \tag{21}\]

and by integrating with respect to \( a \) find

\[
\Delta_T F(a, T) = \frac{\mu_0 (\epsilon_0 + \mu_0) (\epsilon_0 \mu_0 - 2) \sqrt{\epsilon_0 \mu_0 + 2}}{720} \tau^4 + \text{const} + O(\tau^5). \tag{22}\]

An independent on \( \alpha \) constant was already determined in (15). Thus, combining (15) and (22), we obtain a more exact asymptotic expansion for the thermal correction to the Casimir energy of magnetodielectric plates

\[
\Delta_T F(a, T) = \frac{\mu_0 (\epsilon_0 + \mu_0) (\epsilon_0 \mu_0 - 2) \sqrt{\epsilon_0 \mu_0 + 2}}{720} \tau^4 + O(\tau^5) \tag{23}\]

The respective more exact expression for the Casimir entropy per unit area at low temperature is given by

\[
S(a, T) = \frac{k_B^2 T^2}{(\hbar c)^2} \left\{ \frac{3\pi^2}{3} \frac{(\epsilon_0 \mu_0 - 1)^2}{2\pi} \frac{(\epsilon_0 + 1)(\mu_0 + 1)}{2\pi} \right\} - 2\pi^2 k_B a \frac{\mu_0}{45\hbar c} \left[ (\epsilon_0 + \mu_0)(\epsilon_0 \mu_0 - 2) \sqrt{\epsilon_0 \mu_0 + 2} \right] + T + O(T^2) \tag{24}\]

The second term in this asymptotic expression depends both on the temperature and on separation distance.

4. Account for frequency dependence

As was mentioned in the beginning of section 3, the frequency dependence of the dielectric permittivity of dielectric materials plays only a minor role in the behavior of the Casimir free energy and entropy at low temperature [24, 26]. In this section we investigate the impact of the frequency dependence of the magnetic permeability. Then we consider what happens when the magnetodielectric material contains some small fraction of free charge carriers which is the case for any real dielectric at nonzero temperature.

4.1. Frequency-dependent magnetic permeability

The dependence of the magnetic permeability on the frequency along the imaginary frequency axis is described by the Debye formula [45]

\[
\mu(i\xi) = 1 + \frac{\mu_0 - 1}{1 + \omega_m / \omega}, \tag{25}\]

where \( \omega_m \) is some characteristic frequency which is different for different magnetic materials. Using the dimensionless variables introduced in section 2, one has

\[
\mu(i\xi) = 1 + \frac{\mu_0 - 1}{1 + \kappa_m \xi}, \tag{26}\]

where \( \kappa_m = c/(2\pi a \omega_m) \).

Now we expand in powers of \( \xi \) the function \( f(\zeta, y) \) defined in (5), where \( \epsilon(i\xi) = \epsilon_0 \) remains constant but \( \mu(i\xi) \) is given by (26). As a result, we again arrive to the asymptotic expansion (10) with the following additional contribution to the right-hand side:

\[
f(\zeta, y) = f(\zeta) + \frac{4k_m r_{0,2}^2}{(\mu_0 + 1)^2} \left( \sqrt{\epsilon_0 - \epsilon_{0,2}^2} \right) y + O(\xi^3). \tag{27}\]

It is seen that now the asymptotic expansion of the function \( f(\zeta, y) \) contains the first order term in \( \xi \). In the case of constant magnetic permeability \( \kappa_m = 0 \) and \( f(\zeta; y) = 0 \).

We substitute the sum \( f(\zeta, y) + f(\zeta, y) \) to the definition of the function \( F(\zeta) \) in (5) in place of \( f(\zeta, y) \) and in the first perturbation order arrive at

\[
F(\zeta) = -\sum_{k=1,2} \mathrm{Li}_3\left( r_{0,k}^2 \right) + \frac{4k_m r_{0,2}^2}{(\mu_0 + 1)^2} \mathrm{Li}_2\left( r_{0,2}^2 \right) \zeta + O(\zeta^2). \tag{28}\]

Note that the term of the order of \( \zeta^2 \) in (28) does not contribute to the difference \( F(it\tau) - F(-it\tau) \). Then with the help of (28) we obtain

\[
F(it\tau) - F(-it\tau) = 8i\kappa_m \mathrm{Li}_2\left( r_{0,2}^2 \right) \left( \mu_0 + 1 \right) - i\tau \frac{\beta}{2} \sum_{k=1,2} r_{0,k}^2 \zeta^2 + O(\tau^3). \tag{29}\]
As a result, instead of (15) in the lowest order in $\tau$ one finds
\[
\Delta_T F(a, T) = -\frac{\hbar c}{32\pi a^2} \frac{k_m L_i r_0^2}{\omega_m (\mu_0 + 1)} \tau^2 + O(\tau^3).
\] (30)

It is seen that now the perturbation expansion starts from the second-order term and, thus, the thermal correction (30) depends on separation distance. This gives the possibility to obtain the dominant term of the thermal correction to the Casimir pressure directly from (30) using equation (21)
\[
\Delta_T P(a, T) = -\frac{\hbar c^2}{96\pi^2 a^2} \left[ \frac{L_i r_0^2}{\omega_m (\mu_0 + 1)} \tau^2 + O(\tau^3) \right].
\] (31)

The Casimir entropy per unit area is found from (30) using equation (16). Returning to the dimensional temperature, one has in the dominant order
\[
S(a, T) = -\frac{k_b L_i r_0^2}{6\hbar \omega_m (\mu_0 + 1)} a^2 T + O(T^2).
\] (32)

This goes to zero when the temperature vanishes, i.e., the Nernst heat theorem is satisfied. If to compare with (24), it is seen that now the leading order in $T$ is the first instead of the second. The reason is the account of the frequency dependence of the magnetic permeability. Thus, an account of the frequency dependence of $\mu$ results in the change of power in the power-type law like this holds for $\epsilon$ [24, 26].

4.2. Influence of dc conductivity

It is common knowledge that all dielectric materials at any nonzero temperature contain some small concentration of free charge carriers. As a result, the dielectric permittivity $\epsilon_i$ calculated at the imaginary Matsubara frequencies should be replaced with
\[
\tilde{\epsilon}_i = \epsilon_i + \frac{4\pi \sigma_0(T)}{\epsilon_i} \equiv \epsilon_i + \frac{\beta(T)}{\epsilon_i},
\] (33)

where $\sigma_0(T)$ is the static (dc) conductivity of the dielectric and $\beta = 2\hbar \sigma_0 / (k_b T)$. In previous sections we have neglected by the effect of dc conductivity in magnetodielectric materials and considered only $\epsilon_i$ with a finite value of $\epsilon_0$. It is justified by the fact that for all $l \geq 1$ it holds
\[
\epsilon_i \gg \frac{4\pi \sigma_0(T)}{\epsilon_i}.
\] (34)

For example, for quartz glass (SiO$_2$) at $T = 300$ K one has $\beta \sim 10^{-12}$.

Furthermore, $\sigma_0(T) \sim \exp(-b/T)$, where $b$ is some constant different for different materials and, thus, $\sigma_0(T) \to 0$ when $T \to 0$ [46]. For $l = 0$, however, $\tilde{\epsilon}_i$ diverges. Because of this, the impact of dc conductivity on the Casimir free energy of magnetodielectric materials is determined by the zero-frequency term of the Lifshitz formula (1).

To find this impact, we note as $F_i(a, T)$ the Casimir free energy per unit area (1) calculated using the dielectric permittivity $\tilde{\epsilon}_i$ and magnetic permeability $\mu_i$. From equation (1) it is evident that
\[
F_i(a, T) = F_{i0}(a, T) + \tilde{F}_{i\geq1}(a, T),
\]
\[
F(a, T) = F_{i0}(a, T) + F_{i\geq1}(a, T),
\] (35)

where we have separated the contributions of the zero Matsubara frequency in the free energies calculated using $\tilde{\epsilon}_i$ and $\tilde{\epsilon}_i^2$. It can be shown [24] that
\[
\tilde{F}_{i\geq1}(a, T) - F_{i\geq1}(a, T) = Q(a, T) \sim e^{-b/T}
\] (36)

independently of the value of magnetic permeability $\mu$. Calculating the zero-frequency contributions to (1) with account of (2) and (33), we obtain
\[
F_{i=0}(a, T) = -\frac{k_b T}{16\pi^2 a^2} \left[ \xi(3) + L_i r_0^2 \right],
\]
\[
\tilde{F}_{i=0}(a, T) = -\frac{k_b T}{16\pi^2 a^2} \left[ \xi(3) + L_i r_0^2 \right].
\] (37)

Subtracting the two equations in (35) and using (36) and (37), one finds
\[
\tilde{F}(a, T) = F(a, T) - \frac{k_b T}{16\pi^2 a^2} \left[ \xi(3) - L_i r_0^2 \right] + Q(a, T).
\] (38)

Note that contributions of the magnetic permeability in the zero frequency terms (37) have been canceled.

Differentiating the Casimir free energy (38) with respect to $T$ and taking the negative sign, we arrive to the Casimir entropy per unit area with account of the effect of dc conductivity
\[
\tilde{S}(a, T) = S(a, T) + \frac{k_b T}{16\pi^2 a^2} \left[ \xi(3) - L_i r_0^2 \right] - \frac{\partial Q(a, T)}{\partial T},
\] (39)

where $S(a, T)$ is defined in (32). Taking into consideration that
\[
S(a, T) \to 0, \quad \frac{\partial Q(a, T)}{\partial T} \to 0 \quad \text{when} \quad T \to 0,
\] (40)

one obtains at $T = 0$
\[
\tilde{S}(a, 0) = \frac{k_b T}{16\pi^2 a^2} \left[ \xi(3) - L_i r_0^2 \right] > 0.
\] (41)

This means the violation of the Nernst heat theorem because the right-hand side of (41) depends on the parameters of a system under consideration (the separation distance between the plates and their dielectric permittivity). Thus, if the dc conductivity of magnetodielectric materials is taken into consideration, the Lifshitz theory violates the third law of thermodynamics in the same way as for ordinary dielectrics which do not possess magnetic properties [4, 7, 24–26].

5. Conclusions and discussion

In this paper, we have investigated the low-temperature behavior of the Casimir entropy for two parallel plates made of magnetodielectric materials. For this purpose, the Casimir free energy at low temperature was found analytically for both constant and frequency-dependent dielectric permittivity and magnetic permeability in the form of perturbation expansions. The effect of dc conductivity was also taken into account. It was shown that for magnetodielectric materials characterized by the finite static dielectric permittivity and magnetic permeability the Casimir entropy goes to zero when the...
temperature vanishes, i.e. the third law of thermodynamics (the Nernst heat theorem) is satisfied. By contrast, if the dc conductivity is taken into account, in the limiting case of zero temperature the Casimir entropy goes to the positive constant depending on the parameters of a system, i.e. the Nernst heat theorem is violated. This is the same effect, as was discovered previously for dielectrics which do not possess magnetic properties [4, 7, 24–26]. In so doing, the value of the entropy at zero temperature does not depend on the magnetic permeability of the plates.

As is mentioned in section 1, the measurement data of several experiments using dielectric plates are consistent with theoretical predictions of the Lifshitz theory only if the contribution of dc conductivity is omitted in computations. This means that experimentally consistent is the theoretical approach, which is also in agreement with thermodynamics. This would be not surprising if the dc conductivity of dielectrics at room temperature were not an observable physical effect contributing in many experiments with an exception of the Casimir effect. To conclude we can say that the violation of the Nernst heat theorem for magnetodielectric materials in the Lifshitz theory is a challenging problem which awaits for its resolution.

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