Local Fields without Restrictions on the Spectrum of 4-Momentum Operator and Relativistic Lindblad Equation

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Abstract

Quantum theory of Lorentz invariant local scalar fields without restrictions on 4-momentum spectrum is considered. The mass spectrum may be both discrete and continuous and the square of mass as well as the energy may be positive or negative. Such fields can exist as part of a hidden matter in the Universe if they interact with ordinary fields very weakly. Generalization of Kallen-Lehmann representation for propagators of these fields is found. The considered generalized fields may violate CPT-invariance. Restrictions on mass-spectrum of CPT-violating fields are found. Local fields that annihilate vacuum state and violate CPT-invariance are constructed in this scope. Correct local relativistic generalization of Lindblad equation for density matrix is written for such fields. This generalization is particularly needed to describe the evolution of quantum system and measurement process in a unique way. Difficulties arising when the field annihilating the vacuum interacts with ordinary fields are discussed.

keywords: tachyons, CPT-violation, collapse of the state vector, Lindblad equation, renormalizability

1. Introduction.

It is known that there exists a lot of hidden mass in the Universe, and its interaction with usual matter is very weak. Such weakness of interaction allows us to suppose without contradiction with experiment, that hidden mass contains fields, which do not satisfy usual restrictions imposed on the spectrum of 4-momentum. In this work we consider the properties of scalar nonhermitian fields of such type and discuss applications of these fields. Although the mentioned fields are of special interest in cosmology, we consider as the first

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step the case of flat space-time. With the exception of the requirements for the spectrum of 4 momentum, which we refuse, all other postulates of Quantum Field Theory and in particular the property of local commutativity are kept in the following consideration. We consider scalar local fields both with discrete and continues mass spectrum. These fields may have positive or negative square of mass and positive or negative energy in the case of positive square of mass. Let us remark, that some time ago a theory of tachyonic fields (i.e. fields with negative square of mass) was proposed based on the assumption that only particles with positive energy can be created [1]. Such tachyonic scalar fields fulfill anticommutation relations. No field, annihilating the vacuum exists in this model. On the contrary we suppose that states with arbitrary sign of energy can be created, assuming for consistency with experiment that the interaction with conventional matter is extremely weak. This permits us to use commutation relations between scalar fields only. In such a way we get the local field annihilating the vacuum and apply it in subsequent consideration.

In the second section the generalization of Kallen-Lehmann representation for the propagator of nonhermitian scalar field is deduced under assumptions described above. It is clarified that CPT-violation in local Lorentz invariant theory may take place, if we abandon ordinary requirements for the spectrum of 4-momentum. At the cost of such violation it is possible to construct nonzero local field annihilating the vacuum state. We shall use free field of such type in the following consideration for particular applications.

In the third chapter it is shown, how generalized free field with arbitrary spectrum of 4-momentum can be constructed, if its propagator coincides with the one, given by general expression found in section 2.

In the forth section the local relativistic generalization of Lindblad equation for density matrix is introduced as an example of usage of the fields mentioned above. In nonrelativistic Quantum Mechanics such equation allows, in particular, to describe in a unique way the evolution of a quantum system over time as well as the measurement process. Generalization of quantum theory by the passage from Schrödinger equation to Lindblad equation is especially advisable in cosmology to describe the early Universe. In this case it is meaninglessly to speak about any external devices making measurements over the Universe. That’s why one needs an equation which provides spontaneous transformation of superposition of macroscopically different states of the Universe into one of them. In order to apply Lindblad equation for this purpose, one has to write its relativistic generalization for flat space-time, and then pass to Riemann-space of gravitational theory. Here we consider only the first part of this problem, and assume the space-time to be flat. Trying to construct a local relativistic generalization of Lindblad equation, using only fields with conventional spectrum of 4-momentum operator, one meets irresistible ultraviolet (u.v. below) divergencies [2]. However, in section 4 it is shown that such generalization exists for the free field annihilating the vacuum. It should be noted, that the authors of the work [3] with an eye to describe the evolution of nonrelativistic quantum system and measurement process in a unique way, used stochastic Schrödinger equation, which corresponds to specified Lindblad equation. These authors em-
phasize, that the usage of corresponding Lindblad equation does not mean yet the passage to realistic description which provides all macroscopic quantities with definite values. It is connected with the fact, that the density matrix \( \rho \) can be presented as a sum of projectors onto pure states in different ways and the Lindblad equation does not give an instruction which of them should be chosen. Nevertheless, if the density matrix \( \rho(t) \) satisfies the Lindblad equation and can be presented as a sum of projectors onto macroscopically definite states at every moment of time \( t \) at least in one way, then apparently one can construct stochastic Schrödinger equation which generates this Lindblad equation. So we describe the Lindblad equation only.

In the fifth section we consider u.v. divergencies arising when one introduces the interaction between the field annihilating the vacuum and ordinary fields. In the investigated examples these divergencies have unusual nature and can not be removed by renormalization. In particular, it inhibits to use the considered local Lindblad equation for description of collapse of the state vector of usual fields.

2. Kallen-Lehmann representation at arbitrary spectrum of 4-momentum operator.

For the sake of simplicity we shall consider the case of nonhermitian scalar field \( \varphi(x) \) \( (\varphi^\dagger(x) \neq \varphi(x)) \) only. It is more interesting than a hermitian one, that could be easily described in a similar way. We assume that the space-time is flat with the metric \( g_{\mu\nu} = diag(+1,-1,-1,-1) \) and the theory is Lorentz and translation invariant. In the whole section 2 the Heisenberg representation is used, and the symbol \( |0\rangle \) means physical (Heisenberg) vacuum state, normalized by the condition \( \langle 0|0 \rangle = 1 \). We assume that the \( |0\rangle \) is Lorentz- and translation invariant. The field \( \varphi(x) \) is required to satisfy the locality conditions:

\[
[\varphi(x), \varphi(x')] = 0 \quad \text{at} \quad (x - x')^2 < 0, \quad (2.1a)
\]

\[
[\varphi(x), \varphi^\dagger(x')] = 0 \quad \text{at} \quad (x - x')^2 < 0, \quad (2.1b)
\]

where \( (x)^2 = g_{\mu\nu} x^\mu x^\nu; \mu, \nu, ... = 0, 1, 2, 3 \). For simplicity the theory is considered to be invariant under the global phase transformations

\[
\varphi \rightarrow e^{i\alpha} \varphi, \quad (2.2)
\]

where \( \alpha \) is an arbitrary real constant. The vacuum \( |0\rangle \) is assumed to be invariant under (2.2). Therefore

\[
\langle 0|\varphi(x)\varphi(x')|0 \rangle = 0. \quad (2.3)
\]

Due to (2.3) the casual and the retarded Green functions of \( \varphi(x) \) and \( \varphi(x') \) are also equal to zero. The following Wightman functions and casual Green function are nonzero:

\[
u(x) \equiv \langle 0|\varphi(x)\varphi^\dagger(0)|0 \rangle, \quad w(x) \equiv \langle 0|\varphi^\dagger(x)\varphi(0)|0 \rangle, \quad (2.4)
\]
\[ G(x) \equiv \langle 0 | T \{ \varphi(x) \varphi^\dagger(0) \} | 0 \rangle, \quad (2.5) \]

where \( T \{ \varphi(x) \varphi^\dagger(y) \} = \theta(x^0 - y^0) \varphi(x) \varphi^\dagger(y) + \theta(y^0 - x^0) \varphi^\dagger(x) \varphi(y) \) and \( \theta(a) = 1 \) at \( a > 0 \), \( \theta(a) = 0 \) at \( a < 0 \). Because of translation invariance \( \langle 0 | \varphi(x) \varphi^\dagger(y) | 0 \rangle = u(x - y) \), and likewise for other two-point functions. The problem under consideration is to find general representation for \( G(x) \) without any limitations for the spectrum of 4-momentum operator, i.e. to generalize Kallen-Lehmann representation.

For every function \( \xi(x) \) we introduce the Fourier transform \( \tilde{\xi}(k) \), assuming that
\[
\tilde{\xi}(k) = \int d^4 x e^{ikx} \xi(x), \quad \xi(x) = \frac{1}{(2\pi)^4} \int d^4 k e^{-ikx} \tilde{\xi}(k), \quad (2.6)
\]
where \( kx \equiv k^0 x^0 - k^i x^i \equiv k^0 \vec{x} - \vec{k} \vec{x}, \ i = 1, 2, 3 \). Because of Lorentz invariance, the Fourier-transforms \( \tilde{u}(k) \), \( \tilde{w}(k) \) of the quantities \( u(x) \), \( w(x) \), defined by the formulas (2.4), can depend only on \( k^2 \) when \( k^2 < 0 \), and on \( k^2 \) as well as on the sign of \( k^0 \) when \( k^2 \geq 0 \), so
\[
\tilde{u}(k) = \alpha(k^2, \theta(k^2) sgn(k_0)), \quad \tilde{w}(k) = \beta(k^2, \theta(k^2) sgn(k_0)). \quad (2.7)
\]

Now we introduce complete set of eigenvectors \( |k\rangle \) of 4-momentum operator \( P^\mu \):
\[
P^\mu |k\rangle = k^\mu |k\rangle. \quad (2.8)
\]
The states \( |k\rangle \) are normalized in such a manner, that the unity operator \( I \) has the following form \( I = \int d^4 k |k\rangle \langle k| \). Due to translation invariance
\[
\varphi(x) = e^{iP_x} \varphi(0) e^{-iP_x}, \quad \varphi^\dagger(x) = e^{iP_x} \varphi^\dagger(0) e^{-iP_x}. \quad (2.9)
\]
Based on the formulas (2.4), (2.6), (2.7), (2.9) and the relations
\[
P^\mu |0\rangle = 0, \quad \langle 0 | P^\mu = 0 \quad (2.10)
\]
one finds, that
\[
\tilde{u}(k) \equiv \alpha(k^2, \theta(k^2) sgn(k_0)) = (2\pi)^4 \langle 0 | \varphi(0) |k\rangle \langle k| \varphi^\dagger(0) | 0 \rangle, \quad (2.11)
\]
\[
\tilde{w}(k) \equiv \beta(k^2, \theta(k^2) sgn(k_0)) = (2\pi)^4 \langle 0 | \varphi^\dagger(0) |k\rangle \langle k| \varphi(0) | 0 \rangle. \quad (2.12)
\]
This implies, that
\[
\alpha(k^2, \theta(k^2) sgn(k_0)) \geq 0, \quad \beta(k^2, \theta(k^2) sgn(k_0)) \geq 0. \quad (2.13)
\]
One sees, that nonnegative functions \( \alpha \) and \( \beta \) are defined by 4-momentum spectrum of physical states of the theory. Further we show, that casual Green function \( G(x) \) can be expressed through \( \alpha \) and \( \beta \), and it turns out, that this quantities are subordinated to some conditions due to (2.1b).

\[1\] Further in those places where it does not cause misunderstandings we shall sometimes write \( \alpha(k) \) and \( \beta(k) \) instead of \( \alpha(k^2, \theta(k^2) sgn(k_0)) \) and \( \beta(k^2, \theta(k^2) sgn(k_0)) \) for short.
Let us define the function
\[ F(x) = i\langle 0| [\varphi(x), \varphi^\dagger(0)]|0\rangle, \] (2.14)
and also retarded and advanced Green functions:
\[ F_r(x) = \theta(x^0)F(x), \quad F_a(x) = -\theta(-x^0)F(x), \] (2.15)
so that
\[ F(x) = F_r(x) - F_a(x). \] (2.16)

Owing to locality condition (1) the functions \( F_r(x), F_a(x) \) are Lorentz invariant, and
\[ F(x) = 0 \quad \text{at} \quad (x^2 < 0), \] (2.17)
\[ F_r(x) \quad \text{is nonzero only if} \quad (x^2 \geq 0, \ x^0 \geq 0), \] (2.18)
\[ F_a(x) \quad \text{is nonzero only if} \quad (x^2 \geq 0, \ x^0 \leq 0). \] (2.19)

Then we form Fourier transforms \( \widetilde{F}(k), \widetilde{F}_r(k), \widetilde{F}_a(k) \) of functions \( F(x), F_r(x), F_a(x) \) according to (2.6). Because of (2.4), (2.7), (2.14)
\[ \widetilde{F}(k) = i(u(k) - \bar{w}(-k)) = i(\alpha(k^2, \theta(k^2)sgn(k^0)) - \beta(k^2, -\theta(k^2)sgn(k^0))). \] (2.20)

So far, it was assumed that \( k^\mu \) is a real vector. Now we shall investigate the analytical continuation into the region of complex \( k^\mu \), i.e. we allow the vector \( k^\mu \) in the formulas
\[ \widetilde{F}_r(k) = \int d^4xe^{ik\cdot x}F_r(x), \quad \widetilde{F}_a(k) = \int d^4xe^{ik\cdot x}F_a(x) \] (2.21)
to take complex values. We shall assume, that \( F(x), F_r(x), F_a(x) \) are distributions of slow growth (from \( S' \)). Then \( \widetilde{F}_r(k), \widetilde{F}_a(k) \) are analytical functions of the argument \( k^\mu \) at those its values for which the first integral (2.21), or correspondingly the second, exists.\footnote{Here are applicable all the considerations described in the book [4] with changing \( x^\mu \) to \( k^\mu \) and vice versa.}

According to (2.18), (2.19)
\[ \widetilde{F}_r(k) \quad \text{is analytical at} \quad (Im \ k)^2 > 0, \ Im \ k^0 > 0, \] (2.22)
\[ \widetilde{F}_a(k) \quad \text{is analytical at} \quad (Im \ k)^2 > 0, \ Im \ k^0 < 0. \] (2.23)

But due to Lorentz invariance, the functions \( \widetilde{F}_r \) and \( \widetilde{F}_a \) can depend only on \( k^2 \) in those region, where they are analytical. The region of analyticity contains those \( k^2 \), which can be expressed through \( k^\mu \) under the condition (2.22) (correspondingly (2.23))at least in a one way. It is easy to check, that conditions (2.22), (2.23) can be satisfied at every \( k^2 \), expect those ones, for which
\[ Im(k^2) = 0, \quad Re(k^2) \geq 0. \] (2.24)
So, in the region of analyticity we have
\[ \tilde{F}_r(k) = f_r(k^2), \quad \tilde{F}_a(k) = f_a(k^2), \]  
(2.25)
and the functions \( f_r(k^2) \), \( f_a(k^2) \) are analytical in a whole complex \( k^2 \)-plane, probably with the exception of the positive real axis. The functions \( \tilde{F}_r(k) \), \( \tilde{F}_a(k) \) at \( \Im k^\mu = 0 \), \( \Re(k^2) \geq 0 \) can be expressed through \( f_r(k^2) \), \( f_a(k^2) \), going to the limit from those \( k^\mu \), for which \( \tilde{F}_r(k) \), (correspondingly \( \tilde{F}_a(k) \)) is analytical. Taking into account (2.22),(2.23), it is easy to conclude, that at real \( k^\mu \)
\[ \tilde{F}_r(k) = \lim_{\epsilon \to 0} f_r(k^2 + i\epsilon \operatorname{sgn}(k_0)), \quad \tilde{F}_a(k) = \lim_{\epsilon \to 0} f_a(k^2 - i\epsilon \operatorname{sgn}(k_0)). \]  
(2.26)
In the following we consider \( k^\mu \) to be real and write the formulas (2.26) without the sign of limit. Due to the definition (2.14)
\[ F(-x) = -F^*(x) \]  
(2.27)
(*-is a sign of complex conjugation). Thus, using (2.15), we find, that
\[ \tilde{F}_a(k) = \tilde{F}_r^*(k), \quad \tilde{F}(k) = -\tilde{F}^*(k), \]  
(2.28)
and because of (2.26)
\[ f_a(k^2 - i\epsilon \operatorname{sgn} k_0) = f_r^*(k^2 + i\epsilon \operatorname{sgn} k_0) = f_r^*((k^2 - i\epsilon \operatorname{sgn} k_0)^*). \]  
(2.29)
This means, that analytical functions \( f_a(s) \), \( f_r(s) \) satisfy the condition
\[ f_a(s) = f_r^*(s^*) \]  
(2.30)
at every complex \( s \). In the following we shall write \( f(s) \) instead of \( f_r(s) \). So according to (2.26), (2.29)
\[ \tilde{F}_r(k) = f(k^2 + i\epsilon \operatorname{sgn}(k_0)), \quad \tilde{F}_a(k) = f^*(k^2 + i\epsilon \operatorname{sgn}(k_0)), \]  
(2.31)
\[ \tilde{F}(k) = f(k^2+i\epsilon \operatorname{sgn}(k_0)) - f^*(k^2+i\epsilon \operatorname{sgn}(k_0)) = 2\Im f(k^2+i\epsilon \operatorname{sgn}(k_0)). \]  
(2.32)
Thus due to (2.20)
\[ \Im f(k^2+i\epsilon \operatorname{sgn}(k_0)) = \frac{1}{2} \left( \alpha(k^2, \theta(k^2)\operatorname{sgn}(k^0)) - \beta(k^2, \theta(k^2)\operatorname{sgn}(k^0)) \right). \]  
(2.33)

The analytical function \( f(s) \) has a cut along the positive real axis, where the relation (2.33) is true. So \( f(s) \) can be reconstructed\textsuperscript{4} using the functions \( \alpha \) and \( \beta \), if they are defined at \( k^2 \geq 0 \). We shall do it at first imposing restrictions on the function \( f(s) \) strong enough, to provide its uniqueness, and later we shall discuss the arbitrariness, caused by weakening these restrictions. Let us

\textsuperscript{4}Long time ago Yuriy Petrovich Scherbin, who is no more with us, taught one of the authors (V. A. Franke) the following mathematical procedure.
introduce into consideration the function $\sqrt{s}$ of complex variable $s$ and let us define the branch of the square root as follows: if $s = |s|e^{i\mu}$, $-\pi < \mu < \pi$, then
\[
\sqrt{s} = -i \sqrt{|s|}e^{\frac{i\mu}{2}} \text{ at } \mu \geq 0,
\sqrt{s} = i \sqrt{|s|}e^{\frac{i\mu}{2}} \text{ at } \mu \leq 0.
\] (2.34)

Thus at real $k^2$ in the limit $\epsilon \to 0$ with $\epsilon > 0$
\[
\sqrt{-(k^2 \pm i\epsilon)} = \mp i\theta(k^2)\sqrt{k^2} + \theta(-k^2)\sqrt{|k^2|}.
\] (2.35)

Let us define the functions
\[
\xi_+(s) \equiv \frac{1}{2} (f(s) + f^*(s^*)) , \quad \xi_-(s) \equiv -\frac{i}{2\sqrt{-s}} (f(s) - f^*(s^*)),
\] (2.36).

Then because of (2.34)
\[
\xi_\pm^*(s^*) = \xi_\pm(s).
\] (2.37)

Furthermore the functions $\xi_\pm(s)$ have the same analytical properties, as the function $f(s)$. Let us assume, that, when $|s| \to \infty$, the functions $\xi_\pm(s)$ decrease not slower than $|s|^{-\alpha}$, but when $s \to 0$ they increase not faster than $|s|^{-1+\alpha}$, where $\alpha > 0$. Then due to (37) the following dispersion relations are true:
\[
\xi_\pm(s) = \frac{1}{\pi} \int_0^\infty ds' \frac{1}{s' - s} \text{Im} \xi_\pm(s' + i\epsilon),
\] (2.38)

so according to (2.35), (2.36) at $s \geq 0$
\[
\text{Im} \xi_+(s + i\epsilon) = \frac{1}{2} (\text{Im} f(s + i\epsilon) - \text{Im} f(s - i\epsilon)),
\] (2.39)
\[
\text{Im} \xi_-(s + i\epsilon) = \frac{1}{2\sqrt{s}} (\text{Im} f(s + i\epsilon) + \text{Im} f(s - i\epsilon)).
\] (2.40)

Taking into account the relations (2.33),(2.38),(2.39),(2.40), we conclude that
\[
\xi_+(s) = \frac{1}{4\pi} \int_0^\infty ds' \frac{1}{s' - s} \left( \alpha(s', 1) + \beta(s', 1) - \alpha(s', -1) - \beta(s', -1) \right),
\] (2.41)
\[
\xi_-(s) = \frac{1}{4\pi} \int_0^\infty ds' \frac{1}{(s' - s)\sqrt{s'}} \left( \alpha(s', 1) - \beta(s', 1) + \alpha(s', -1) - \beta(s', -1) \right).
\] (2.42)

Finally, because of (2.36)
\[
f(s) = \frac{1}{4\pi} \int_0^\infty ds' \frac{1}{s' - s} \left( \alpha(s', 1) + \beta(s', 1) - \alpha(s', -1) - \beta(s', -1) \right) +
\] (2.43)
\[
+ \frac{i\sqrt{-s}}{4\pi} \int_0^\infty ds' \frac{1}{(s' - s)\sqrt{s'}} \left( \alpha(s', 1) - \beta(s', 1) \right) + \left( \alpha(s', -1) - \beta(s', -1) \right).
\]
Here \( \alpha(s',1) = \alpha(k^2, \theta(k^2) \text{sgn}(k^0)) \) at \( k^2 = s' \geq 0 \), \( \theta(k^2) \text{sgn}(k^0) = 1 \) and similarly for \( \beta(s',1), \alpha(s',-1), \beta(s',-1) \).

Let us consider the equality (2.43) from the point of view of CPT-transformation. At such transformation the matrix element \( \langle 0|\phi(x)\varphi^\dagger(0)|0 \rangle \) goes into \( \langle 0|\varphi^\dagger(0)\varphi(-x)|0 \rangle = \langle 0|\varphi^\dagger(x)\varphi(0)|0 \rangle \) and vice versa, or according to (2.4), (2.6), (2.7)

\[
\alpha(k^2, \theta(k^2) \text{sgn}(k^0)) \quad \text{and} \quad \beta(k^2, \theta(k^2) \text{sgn}(k^0)) \quad \text{pass into each other} . \quad (2.44)
\]

So, the first term in the right hand side of the equality (2.43) is CPT-invariant, while the second one is not CPT-invariant. Assuming that \( s = k^2 < 0 \) in the formula (2.43), taking the imaginary part of the function \( f(k^2) \) and using (2.33) at \( k^2 < 0 \), we obtain the relation

\[
\theta(-k^2) \left( \alpha(k^2,0) - \beta(k^2,0) \right) = \theta(-k^2) \frac{-\sqrt{-k^2}}{2\pi} \times \\
\int_0^\infty ds' \frac{1}{(s' - k^2) \sqrt{s'}} \left( \left( \alpha(s',1) - \beta(s',1) \right) + \left( \alpha(s',-1) - \beta(s',-1) \right) \right). \quad (2.45)
\]

One sees, that CPT-noninvariant parts of the spectrum are not arbitrary, but they are coupled by the relation (2.45). This is a result of the locality of the theory. In the equality (2.45) the expression \( (\alpha(s',1) - \beta(s',1)) \) describes the CPT-violation in the spectrum of states with \( s' = k^2 \geq 0 \) and positive energy, the difference \( (\alpha(s',-1) - \beta(s',-1)) \) in the spectrum of states with nonnegative \( s' = k^2 \), but with negative energy and, finally, \( \theta(-k^2)(\alpha(k^2,0) - \beta(k^2,0)) \) in the spectrum of tachyonic states \( k^2 < 0 \).

Let us notice, that CPT-invariant parts of the spectrum \( \theta(k^2)(\alpha(k^2,1) + \beta(k^2,1)), \theta(k^2)(\alpha(k^2,-1) + \beta(k^2,-1)), \theta(-k^2)(\alpha(k^2,0) + \beta(k^2,0)) \) can be defined arbitrary.

Now let us consider the propagator \( G(x) \), defined by the equality (2.5). Using the relations (2.4), (2.5), (2.14), (2.15) one sees, that \( G(x) \) might be written, in particular, in two ways:

\[
G(x) = F_r(x) + iu(-x) = F_u(x) + iu(x) . \quad (2.46)
\]

Using the equalities (2.31), (2.7) we obtain for the Fourier transform

\[
\tilde{G}(k) = f(k^2 + i\epsilon \text{sgn}(k^0)) + i\beta(k^2, -\theta(k^2) \text{sgn}(k^0)) = \\
f^*(k^2 + i\epsilon \text{sgn}(k^0)) + i\alpha(k^2, \theta(k^2) \text{sgn}(k^0)), \quad (2.47)
\]

i.e. in accordance with (2.43)

\[
\tilde{G}(k) = \frac{1}{4\pi} \int_0^\infty ds' \frac{1}{s' - k^2 - i\epsilon \text{sgn}(k^0)} \left( (\alpha(s',1) + \beta(s',1)) - (\alpha(s',-1) + \beta(s',-1)) \right) + \\
+ i\frac{\sqrt{-k^2 - i\epsilon \text{sgn}(k^0)}}{4\pi} \int_0^\infty \frac{1}{(s' - k^2 - i\epsilon \text{sgn}(k^0)) \sqrt{s'}} \times 
\]
\[ \times \left( (\alpha(s', 1) - \beta(s', 1)) + (\alpha(s', -1) - \beta(s', -1)) \right) + 2i\beta(k^2, -\theta(k^2)sgn(k^0)), \]
or

\[ \tilde{G}(k) = \frac{1}{4\pi} \int_0^\infty ds' \frac{1}{s' - k^2 + i\epsilon sgn(k^0)} \left( (\alpha(s', 1) + \beta(s', 1)) - (\alpha(s', -1) + \beta(s', -1)) \right) + \]

\[ -i\sqrt{-k^2 + i\epsilon sgn(k^0)} \times \]

\[ \times \left( (\alpha(s', 1) - \beta(s', 1)) + (\alpha(s', -1) - \beta(s', -1)) \right) + i\alpha(k^2, \theta(k^2)sgn(k^0)). \]

This is the required generalization of Kallen-Lehmann representation. Let us rewrite it in another form, which allows us to understand more clearly the meaning of each term of the sum. Using well known formulas like

\[ \frac{1}{x + i\epsilon} + 2\pi i\delta(x) = \frac{1}{x - i\epsilon}, \]

one can show, that (we omit simple, but lengthy calculations)

\[ \tilde{G}(k) = \frac{1}{4\pi} \int_0^\infty ds' \left( \frac{\alpha(s', 1) + \beta(s', 1)}{s' - k^2 - i\epsilon} - \frac{\alpha(s', -1) + \beta(s', -1)}{s' - k^2 + i\epsilon} \right) + \]

\[ + \theta(-k^2) \frac{i}{2} \left( \alpha(k^2, 0) + \beta(k^2, 0) \right) + \]

\[ + \theta(k^2) \sqrt{k^2} sgn(k^0) \int_0^\infty ds' \frac{1}{\sqrt{s'}} \left( \frac{\alpha(s', 1) - \beta(s', 1)}{s' - k^2 - i\epsilon} + \frac{\alpha(s', -1) - \beta(s', -1)}{s' - k^2 + i\epsilon} \right). \]

Ordinary Kallen-Lehmann representation [5][6] could be obtained from (2.50), by putting \( \alpha(k^2, -1) = \beta(k^2, -1) = \alpha(k^2, 0) = \beta(k^2, 0) = 0, \) \( \alpha(k^2, 1) = \beta(k^2, 1). \) So only the first term under the sign of the first integral remains.

In the general case there is the similar term in the first integral, which contains \( \alpha(s', -1) + \beta(s', -1). \) It describes the contribution of the states with \( k^2 \geq 0, \) \( k^0 < 0. \) Both of these terms are CPT- invariant. Further, outside the integral there is a CPT-invariant tachyonic contribution, which contains \( \alpha(k^2, 0) + \beta(k^2, 0). \) Finally there is CPT- noninvariant contribution, given by the last integral in the formula (2.50). This contribution is nonzero only if \( k^2 \geq 0. \) Values of function \( \tilde{G}(k) \) at \( k^2 < 0 \) are CPT- invariant. We emphasize that the quantities \( \alpha(k^2, 1), \beta(k^2, 1), \alpha(k^2, -1), \beta(k^2, -1), \alpha(k^2, 0), \beta(k^2, 0) \) are coupled with each other by the relation (2.45), although the difference \( (\alpha(k^2, 0) - \beta(k^2, 0)) \) is absent at \( k^2 < 0 \) in the formula (2.50).
If there are no restrictions on the spectrum of 4-momentum operator and the CPT- invariance is violated, then it is possible to introduce nonzero local field $\varphi(x)$, annihilating the vacuum state i.e. fulfilling the condition

$$\varphi(x)|0\rangle = 0 \quad \text{at arbitrary} \quad x. \quad (2.51)$$

In this case due to (2.4) and (2.7)

$$\beta(k^2, \theta(k^2) \text{sgn}(k^0)) = 0, \quad (2.52)$$

and according to the expression (2.48a)

$$\tilde{G}(k) = \frac{1}{4\pi} \int_{0}^{\infty} ds' \frac{1}{s' - k^2 - i\epsilon \text{sgn}(k^0)} \left( \alpha(s', 1) - \alpha(s', -1) \right) +$$

$$+ \frac{i \sqrt{-k^2 - i\epsilon \text{sgn}(k^0)}}{4\pi} \int_{0}^{\infty} ds' \frac{1}{(s' - k^2 - i\epsilon \text{sgn}(k^0)) \sqrt{s'}} \times$$

$$\times \left( \alpha(s', 1) + \alpha(s', -1) \right). \quad (2.53)$$

Furthermore because of (2.45)

$$\theta(-k^2) \alpha(k^2, 0) = \theta(-k^2) \frac{\sqrt{-k^2}}{2\pi} \times$$

$$\times \int_{0}^{\infty} ds' \frac{1}{(s' - k^2 - i\epsilon \text{sgn}(k^0)) \sqrt{s'}} \left( \alpha(s', 1) + \alpha(s', -1) \right). \quad (2.54)$$

It is easy to see from (2.54), that in the case (2.51) the spectrum of tachyons is continuous for every $\alpha(s', 1)$ and $\alpha(s', -1)$ and covers all negative real axis $-\infty < k^2 < 0$.

Let us do several remarks. Writing the formula (2.38), we assumed, that the function $\xi^-(s)$ increases at $s \to 0$ not faster, then $|s|^{-1+\alpha}$ where $\alpha > 0$. Due to the equality (2.36) it means, that the difference $f(s) - f^*(s^*)$ increases in the limit $s \to 0$ not faster then $|s|^{-\frac{1}{2}+\alpha}$. One could weaken this condition and consider the function

$$\xi'_-(s) = \frac{i \sqrt{-s}}{2} (f(s) - f^*(s^*)) \quad (2.55)$$

instead of $\xi_-(s)$, putting on the functions $\xi_+(s)$ and $\xi'_+(s)$ the requirement to decrease in the limit $|s| \to \infty$ not slower then $|s|^{-\alpha}$ and increased in the limit $|s| \to 0$ not faster then $|s|^{-1+\alpha}$. After repeating the calculations performed earlier with the function $\xi'_-(s)$ instead of $\xi_-(s)$, we get the following formula which replaces (2.43),

$$f'(s) = \frac{1}{4\pi} \int_{0}^{\infty} ds' \frac{1}{s' - s} \left( \alpha(s', 1) + \beta(s', 1) - \alpha(s', -1) - \beta(s', -1) \right) -$$

10
\[-\frac{i}{4\pi \sqrt{-s}} \int_0^\infty ds' \frac{\sqrt{s'}}{(s' - s)} \left( \left( \alpha(s', 1) - \beta(s', 1) \right) + \left( \alpha(s', -1) - \beta(s', -1) \right) \right), \quad (2.56)\]

where \( f'(s) \) is an analogue of \( f(s) \). One sees, that in the cases, when properties of the functions \( \alpha \) and \( \beta \) provide the existence of integrals in both formulas (2.43) and (2.56), the following relation is true

\[ f(s) - f'(s) = \frac{iC}{\sqrt{-s}}, \quad (2.57) \]

where \( C \) is a real constant. Using the function \( f'(s) \) instead of the function \( f(s) \) we get the equality

\[ \theta(-k^2)(\alpha(k^2, 0) - \beta(k^2, 0)) = -\theta(-k^2) \frac{1}{2\pi \sqrt{-k^2}} \times \]

\[ \times \int_0^\infty ds' \frac{\sqrt{s'}}{s' - k^2} \left( \left( \alpha(s', 1) - \beta(s', 1) \right) + \left( \alpha(s', -1) - \beta(s', -1) \right) \right) \quad (2.58) \]

instead of (2.45). So, for the most natural restrictions imposed on the function \( f(s) \), which have under absence of massless particles the form

\[ |f(s)| < |s|^{-1 + \alpha_1} \quad \text{at} \quad s \to 0, \quad \alpha_1 > 0, \quad (2.59a) \]

\[ |f(s)| < |s|^{-\alpha_2} \quad \text{at} \quad s \to \infty, \quad \alpha_2 > 0, \quad (2.59b) \]

this function can be reconstructed from its imaginary part above and below the cut along the positive real axis with some arbitrariness. This arbitrariness affects the relations like (2.45) and (2.58). One can easily figure out the most general arbitrariness. Let us assume that the functions \( f_1(s) \) and \( f_2(s) \) are analytical on the whole complex \( s \) - plane, with the exception of the cut at \( s \geq 0 \), and that they meet the condition

\[ \text{Im} \ f_1(s \pm i \epsilon) = \text{Im} \ f_2(s \pm i \epsilon) \quad \text{at} \quad s > 0. \quad (2.60) \]

We impose the condition (2.60) only at \( s > 0 \), but not at \( s = 0 \) in order to consider specially a possible singularity at \( s = 0 \). The function

\[ \nu(s) \equiv f_1(s) - f_2(s) \quad (2.61) \]

satisfies the equality

\[ \text{Im} \ \nu(s \pm i \epsilon) = 0, \quad \text{at} \quad s > 0. \quad (2.62) \]

Let us construct the functions

\[ \chi_+(s) = \frac{1}{2} \left( \nu(s) + \nu^*(s^*) \right), \quad \chi_-(s) = \frac{i\sqrt{-s}}{2} \left( \nu(s) - \nu^*(s^*) \right). \quad (2.63) \]
Because of $(\sqrt{-s^*})^* = \sqrt{-s}$, one gets
\[ \chi_\pm(s^*) = \pm \chi_\pm(s) \] (2.64)
and
\[ Re \chi_\pm(s + i \epsilon) = Re \chi_\pm(s - i \epsilon) \quad \text{at} \quad s > 0. \] (2.65)
Further more due to (2.62)
\[ Im \chi_\pm(s + i \epsilon) = 0 = Im \chi_\pm(s - i \epsilon), \quad \text{at} \quad s > 0. \] (2.66)
Consequently, at $s > 0$
\[ \chi_\pm(s + i \epsilon) = \chi_\pm(s - i \epsilon). \] (2.67)
Due to this fact the functions $\chi_\pm(s)$ are analytical in a whole complex $s$-plane probably with the exception of singularity at $s = 0$. According to (2.61), (2.63)
\[ \nu(s) \equiv f_1(s) - f_2(s) = \chi_+(s) - \frac{1}{\sqrt{s}} \chi_-(s). \] (2.68)
Due to analytical properties of functions $\chi_\pm$ under the conditions (2.59), imposed on both quantities $f_1$ and $f_2$, there is only one possibility 4
\[ f_1(s) - f_2(s) = \frac{iC}{\sqrt{-s}}, \quad \text{where} \quad C \text{ is a real constant}. \] (2.69)
We have already met this arbitrariness earlier (formula (2.57)). So, under conditions (2.59) the formula (2.69) describes all the arbitrariness of restored function $f(s)$.

Further we shall be primarily interested in the case of the field $\varphi$, annihilating the vacuum $(\varphi(x)|0\rangle = 0)$. As already established, this corresponds to the equality $\beta \equiv 0$. Under such condition the relation (2.58) cannot take place for nonzero functions $\alpha$, as far as all of this functions are nonnegative. That’s why we previously used the formula (2.43) but not (2.56), and we shall follow this assumption below. Let us notice, in connection with this, that, if we included the additional term (2.69) in the right hand side of the formula (2.43), we would not improve the u.v. behavior of the function $f$, at positive $\alpha$ without disturbing a relation like (2.45).

Let us furthermore notice, that, weakening the condition (2.59a), one could take into account massless particles, by including into one of the functions $\alpha(k^2, +1), \beta(k^2, +1), \alpha(k^2, -1), \beta(k^2, -1)$ an additional term $const \cdot \theta(k_0) \delta(k^2)$ or correspondingly $const \cdot \theta(-k_0) \delta(k^2)$ and by using the formula (2.56). The relation (2.43) in doing so is unapplicable, because of infinity at $s \to 0$. As it was just figured out, the formula (2.56) is incompatible with the condition $\beta \equiv 0$. So we see, that the Wightman functions of the field $\varphi$ annihilating the vacuum are not allowed to contain terms, which correspond to massless particles. Further we assume, that there are no such particles, and then we use the formula (2.43) and its consequences.

4We abandon simple proof of this fact for space saving.
3. Constructing of free local fields with given propagators.

In section 2 the general expression was deduced for the propagator of local scalar field not forced to any restrictions imposed on the spectrum of 4-momentum operator. Generally speaking the field was not supposed to be free. In order to use it further in perturbation theory we shall construct now a free quantum field such that its propagator coincides with the expression described in section 2. Under the phrase "construct quantum field" we understand introducing the Hilbert space of physical states and corresponding operators acting on it.

Obviously it suffices to construct free field, whose Wightman functions in the momentum representation \( \tilde{u}(k) \) and \( \tilde{w}(k) \) coincide with preassigned ones, since the propagator and the commutator in this case, can be restored from Wightman functions (see section 2). All assumptions and notations in this section are the same as in section 2, but we assume moreover that only two-point connected Green functions of fields under consideration are nonzero. We understand here the term "free fields" in this meaning only. We choose a scheme of quantization which permits to describe fields with different types of 4-momentum spectrum in a unique way. Let us assume in the sake of simplicity that \( \alpha \) and \( \beta \) as functions of \( k^2 \) with fixed second argument \( \theta(k^2) \text{ sgn}(k_0) \) are continuous everywhere probably with the exception of no more than countable set of points and also can have no more than countable set of singularities like \( c_1 \delta(k^2 - m^2), \ c_1 > 0 \). Unless otherwise stated the sign of \( m^2 \) is though here and below to be arbitrary. The discrete \( \delta \)-like singularities of \( \alpha \) (\( \beta \)) correspond to particles (antiparticles) with fixed square of mass. Regions of continuity of \( \alpha \) and \( \beta \) correspond to continues mass spectrum i.e. "unparticle matter".

Further for arbitrary function \( y(x) \) we shall designate through \( \text{Supp} y \) the variety of its arguments \( x \) for which \( y(x) \neq 0 \).

At first let us discuss the case when \( \delta \)-like singularities are absent. We assume below that \( \alpha \) and \( \beta \) are defined and satisfy all conditions of section 2. Furthermore we postulate that translation and Lorentz invariant state \( |0\rangle \) exists and shall call it "vacuum". Let us build Hilbert space as a Fock space upon the vacuum \( |0\rangle \), fixing the Lorentz frame of reference. For this purpose let us introduce operators \( a(k) \) and \( a^\dagger(k) \) (correspondingly \( b(k) \) and \( b^\dagger(k) \)) for every \( k \in \text{Supp} \alpha \ (k \in \text{Supp} \beta) \), which we call "annihilation and creation operators of particles with 4-momentum \( k \)" (correspondingly antiparticles). Let us postulate that under Lorentz transformation \( a'(k') = a(k), \ b'(k') = b(k) \). Further let us introduce the conditions

\[
\begin{align*}
a(k)|0\rangle = b(k)|0\rangle = 0, \\
[a(k), a^\dagger(k')] = \frac{\delta^4(k - k')D_\alpha(k)}{\alpha(k^2, \theta(k^2) \text{ sgn}(k_0))},
\end{align*}
\]

\[\text{(3.1)}\]

and commutation relations

\[\text{(3.2)}\]

\[\text{We exclude from consideration derivatives of } \delta \text{- function because of its sign indeterminateness.}\]
where \( D_\alpha(k) = 1 \), if \( k \in \text{Supp} \, \alpha \), and 0 otherwise. In a similar manner for \( b \) and \( b^\dagger \)
\[
[b(k), b^\dagger(k')] = \frac{\delta^4(k-k')D_\beta(k)}{\beta(k^2, \theta(k^2) \, \text{sgn}(k_0))}.
\]
(3.3)

All other commutators between creation and annihilation operators are equal to zero.

Now we build Fock space of ket vectors, acting on the vacuum by creation operators. We assume that the operators \( \alpha^\dagger(k) \) and \( \alpha(k) \) (correspondingly \( b^\dagger(k) \) and \( b(k) \)) are Hermitian conjugated and consequently \( \langle 0 | \alpha^\dagger(k) = \langle 0 | b^\dagger(k) = 0 \). This is a Hilbert space of states in our theory.

Further let us introduce local field \( \varphi(x) \) in terms of creation and annihilation operators by the formulas:
\[
\varphi(x) \equiv \frac{1}{(2\pi)^2} \int d^4k \{ \alpha(k)e^{-ikx}a(k) + \beta(k)e^{ikx}b^\dagger(k) \},
\]
(3.4)
\[
\varphi^\dagger(x) \equiv \frac{1}{(2\pi)^2} \int d^4k \{ \alpha(k)e^{ikx}a^\dagger(k) + \beta(k)e^{-ikx}b(k) \}.
\]
(3.5)

Such field is obviously a scalar and satisfies the locality conditions from section 2, if \( \alpha \) and \( \beta \) fulfill the relation (2.45). Now let us build 4-momentum operator \( P_\mu \) for this field by means of creation and annihilation operators and define it as follows
\[
P_\mu \equiv \int d^4k \, k_\mu \{ \alpha(k)a^\dagger(k)a(k) + \beta(k)b^\dagger(k)b(k) \}.
\]
(3.6)

It is easy to see that standard relations take place:
\[
\varphi(x) = e^{iPx}\varphi(0)e^{-iPx}, \quad P_\mu a^\dagger(k)\langle 0 \rangle = k_\mu a^\dagger(k)\langle 0 \rangle.
\]
(3.7)

Let us further describe a case of discrete mass spectrum. For short all relations are written for the particles creation and annihilation operators. All formulas for antiparticles are similar. Let us assume without loss of generality that \( \alpha(k^2, \theta(k^2) \, \text{sgn}(k_0)) \) is nonzero only at fixed value of its second argument and has only one \( \delta \)-like singularity by its first argument. Trying to write the relation (3.2) we have a difficulty due to presence of \( \delta(k^2 - m^2) \) in the denominator in right hand side. Therefore let us postulate the following relation that generalizes (2) in the case of discrete spectrum
\[
\alpha(k^2, \theta(k^2) \, \text{sgn}(k_0))|a(k), a^\dagger(k')\rangle = \delta(k - k')D_\alpha(k).
\]
(3.8)

One can rewrite the equality (3.3) in a similar manner. Relations (3.4), (3.5), (3.6), (3.7) remain the same. Obviously, in the case of discrete mass spectrum, the argument \( k \) of creation and annihilation operators has only 3 independent components and we choose its spatial components as independent ones. So let us rewrite (3.8) in terms of operators that depend on \( \vec{k} \) only. Arbitrary function \( \alpha(k^2, \theta(k^2) \, \text{sgn}(k_0)) \) can be presented in the form
\[
\alpha(k) = \theta(k^2)\theta(k^0)\alpha(k^2, 1) + \theta(k^2)\theta(-k^0)\alpha(k^2, -1) + \theta(-k^2)\alpha(k^2, 0).
\]
(3.9)
Therefore, without loss of generality, it is enough to describe 3 cases only:

\[ \alpha(k^2, 1) \equiv \delta(k^2 - m^2), \quad \alpha(k^2, -1) \equiv 0, \quad \alpha(k^2, 0) \equiv 0, \quad m^2 > 0, \quad (3.10) \]

\[ \alpha(k^2, 1) \equiv 0, \quad \alpha(k^2, -1) \equiv \delta(k^2 - m^2), \quad \alpha(k^2, 0) \equiv 0, \quad m^2 > 0, \quad (3.11) \]

\[ \alpha(k^2, 1) \equiv 0, \quad \alpha(k^2, -1) \equiv 0, \quad \alpha(k^2, 0) \equiv \delta(k^2 - m^2), \quad m^2 < 0. \quad (3.12) \]

Let us notice that the case (3.10) corresponds to the ordinary scalar field with mass \( m \). The case (3.11) corresponds to the scalar field with positive square of mass but negative energy, and we call this field a "phantom". The case (3.12) corresponds to a tachyon with fixed negative square of mass. Of course, all spectrum should satisfy (2.45). So in the case of tachyons, discrete parts of the functions \( \alpha \) and \( \beta \) must coincide.

Let us notice that \( \delta(k^2 - m^2) \) with arbitrary sign of \( m^2 \) can be written in the form:

\[ \delta(k^2 - m^2) = \frac{\delta(k_0 - \sqrt{k^2 + m^2}) - \delta(k_0 + \sqrt{k^2 + m^2})}{2k_0} \]

At first we consider the case (3.10). Let us integrate both parts of the relation (3.8) by \( k^0 \), with the following result

\[ [a(\vec{k}), a^\dagger(\vec{k}')] = 2k^0 \delta^3(\vec{k} - \vec{k}'), \quad k^0 = \sqrt{k^2 + m^2}, \quad (3.13) \]

where \( a(\vec{k}) \) obviously is \( a(k) \) at \( k_0 = \sqrt{k^2 + m^2} \). In the case (3.12) we get in the same way

\[ [a(\vec{k}), a^\dagger(\vec{k}')] = -2k^0 \delta^3(\vec{k} - \vec{k}'), \quad k^0 = -\sqrt{k^2 + m^2}. \quad (3.14) \]

The case (3.12) is less trivial. In the fixed Lorentz frame of reference let us introduce two sorts of annihilation operators \( a_+(\vec{k}) \equiv a(k) \) at \( k_0 = +\sqrt{k^2 + m^2} \) and \( a_-(\vec{k}) \equiv a(k) \) at \( k_0 = -\sqrt{k^2 + m^2} \) (correspondingly creation operators). Let us notice that \( a_+ \) and \( a_- \) transfer into each other under Lorentz transformation for some \( k_\mu \). This scheme is Lorentz invariant because equalities (3.4), (3.5), (3.6), (3.7), (3.8) are true. Then the relation (3.9) leads to the following commutation relations for \( a_+(\vec{k}) \) and \( a^\dagger_+(\vec{k}) \):

\[ [a_+(\vec{k}), a^\dagger_+(\vec{k}')] = 2k^0 \delta^3(\vec{k} - \vec{k}') \theta(k^2 + m^2) \quad \text{at} \quad k^0 = \sqrt{k^2 + m^2}, \quad (3.15a) \]

\[ [a_-(\vec{k}), a^\dagger_-(\vec{k}')] = -2k^0 \delta^3(\vec{k} - \vec{k}') \theta(k^2 + m^2) \quad \text{at} \quad k^0 = -\sqrt{k^2 + m^2}. \quad (3.15b) \]

For example let us rewrite the formula (3.4) in the case (3.12) as follows:

\[ \varphi(x) = \int \frac{d^3\vec{k}}{2\sqrt{k^2 + m^2}} \left( e^{-i\sqrt{k^2 + m^2} x_0 + i\vec{k} \vec{x}} a_+(\vec{k}) + e^{i\sqrt{k^2 + m^2} x_0 + i\vec{k} \vec{x}} a_-(\vec{k}) \right) \]

\[ \text{This case corresponds to tachyons with fixed square of mass, and these tachyons must be CPT-invariant i.e. } \alpha \equiv \beta \text{ (see (2.45)).} \]
Each term including $a_+$ or $a_-$ (correspondingly $b_{+}$ or $b_{-}$) is not Lorentz invariant but their sum is, because it can be presented as the first term (correspondingly the second) of the expression (3.4). Let us remark, that in the example (3.16) the commutator $[\varphi(x), \varphi^\dagger(y)]$ is identically equal to zero, but the Wightman functions and the propagator are nonzero and the latter has the following unconventional form: $G(k) = i\delta(k^2 - m^2)$ (see (2.50)).

Finally let us consider a question about classical action and how it should be quantized in order to get the local free quantum field $\varphi(x)$ described above. We shall built this field $\varphi$ as a sum with certain coefficients of nonlocal fields with fixed square of mass, and shall quantize the last ones Lorentz invariantly in a way which leads to the formulas (3.4) and (3.5).

In the following let the index $j$ run through values $0, \pm 1$. For every $m^2 \in \text{Supp } \alpha(m^2, j)$ let us introduce classical scalar nonhermitian field with fixed square of a mass $\varphi_{m^2,\alpha,j}(x)$. In the same manner we define $\varphi_{m^2,\beta,j}(x)$ for every $m^2 \in \text{Supp } \beta(m^2, j)$. At this stage there are no principal differences in a properties of this fields but they will be quantized in a different ways. Let us construct an action of a usual type for each of the fields introduced above

$$S_{m^2, n, j} \equiv \int d^4x \varphi_{m^2, n, j}(x) (\partial \mu \partial^n + m^2) \varphi_{m^2, n, j}(x), \quad (3.17)$$

where index $n$ ranges over $\alpha, \beta$.

As far as each of the fields $\varphi_{m^2, n, j}(x)$ satisfies the corresponding Klein-Fock-Gordon equation, the following equalities are true

$$\varphi_{m^2, n, j}(x) = \frac{1}{(2\pi)^2} \int d^4k \delta(k^2 - m^2) \tilde{\varphi}_{m^2, n, j}(x) e^{-ikx}, \quad (3.18)$$

$$\varphi_{m^2, n, j}(x) = \frac{1}{(2\pi)^2} \int d^4k \delta(k^2 - m^2) \tilde{\varphi}^*_{m^2, n, j}(x) e^{ikx}. \quad (3.19)$$

In the expressions (3.18) and (3.19) the functions $\tilde{\varphi}(k)$ and $\tilde{\varphi}^*(k)$ are arbitrary scalar functions defined on $\text{Supp } \alpha$ and $\text{Supp } \beta$ (depending on the value of the index "n"). Now let us postulate the following nonstandard recipe of quantization, i.e. rules by which one transforms classical fields, written in the form (3.18) and (3.19) into operators acting on Hilbert space introduced above. These operators satisfy commutation relations (3.2),(3.3) (or (3.13),(3.14),(3.15) for discrete mass spectrum. The rules look as follows:

$$\tilde{\varphi}_{m^2, \alpha, +1}(k) \rightarrow a(k)\theta(\theta k^2)\theta(\theta k-\theta k_0), \quad \tilde{\varphi}^*_{m^2, \alpha, +1}(k) \rightarrow a^\dagger(k)\theta(\theta k^2)\theta(\theta k+k_0),$$

$$\tilde{\varphi}_{m^2, \alpha, -1}(k) \rightarrow a(k)\theta(\theta k^2)\theta(-\theta k_0), \quad \tilde{\varphi}^*_{m^2, \alpha, -1}(k) \rightarrow a^\dagger(k)\theta(\theta k^2)\theta(-\theta k),$$

$$\tilde{\varphi}_{m^2, \alpha, 0}(k) \rightarrow a(k)\theta(-\theta k^2), \quad \tilde{\varphi}^*_{m^2, \alpha, 0}(k) \rightarrow a^\dagger(k)\theta(-\theta k^2).$$

\footnote{We do not write sub-indexes for short.}
There are similar equalities for the \( b \) and \( b^\dagger \). Let us notice that quantum fields \( \varphi_{m^2, n, j}(x) \) and ones conjugated with them, do not satisfy locality conditions. Now, using the operators just defined, we construct local field \( \varphi(x) \) according to the formula:

\[
\varphi(x) \equiv \int_{-\infty}^{+\infty} dm^2 \sum_{j=-1, 0, +1} \{ \alpha(m^2, j)\varphi_{m^2, \alpha, j}(x) + \beta(m^2, j)\varphi_{m^2, \beta, j}(x) \}. \tag{3.20}
\]

One can built the field \( \varphi^\dagger(x) \) in a similar manner. It is easy to see that our last constructions coincide with formally defined ones by equalities (3.4) and (3.5).

Let us emphasize that in contrast to usual theory where commutators between creation and annihilation operators follow from canonical commutation relations between generalized coordinates and momenta, in this scheme we deduced them from the requirement of getting the given Wightman functions. The Fourier transforms of these functions satisfy all limitations from section 2 and in particular the formula (2.45). Finally one gets free local scalar fields defined by (3.4) and (3.5) and the free Hamiltonian equal to \( P_0 \) from the equality (3.6). In the sectors of particles with positive square of mass and positive or negative energy, this two approaches give the same results.

Having such free fields one can introduce an interaction of them with other fields and investigate it in the interaction picture, drawing Feynman diagrams.

4. Local relativistic generalization of Lindblad equation.

One of the most interesting applications of local field annihilating the vacuum consists in using it for relativistic generalization of Lindblad equation for density matrix \( \rho \). It is known that in nonrelativistic Quantum Mechanics this equation has the form \[7\]

\[
\frac{d\rho}{dt} = -i[H, \rho] + \sum_n \lambda_n \left( 2A_n\rho A_n^\dagger - A_n^\dagger A_n \rho - \rho A_n^\dagger A_n \right), \tag{4.1}
\]

where \( H \) is usual Hamiltonian, \( A_n \) are arbitrary operators, \( \lambda_n \) are positive constants.

Let us consider the case of one nonhermitian scalar field in a flat space-time and require Lorentz invariance. To establish the latter in the simplest way one may use the Tomonaga-Schwinger formalism \[9\][10] in the interaction picture. We assume that the operators \( \varphi(x) \) form in this picture a free local field and that the density matrix \( \rho(\sigma) \) describing the state depends on space-like hypersurface \( \sigma \). This density matrix is defined for each such surface. Along with the field \( \varphi(x) \) there may exist other fields and their state is described by \( \rho(\sigma) \) too. Then

\[8\] See also [8].
one can write the following relativistic generalization of Lindblad equation

$$\frac{\delta \rho(\sigma)}{\delta \sigma(x)} = -i[H_1(x), \rho(\sigma)] + \lambda \left(2\varphi(x)\rho\varphi^\dagger(x) - \varphi^\dagger(x)\varphi(x)\rho - \rho\varphi^\dagger(x)\varphi(x)\right),$$

(4.2)

where $\frac{\delta \rho(\sigma)}{\delta \sigma(x)}$ is variational derivative of $\rho(\sigma)$ induced by infinitely small change $\delta \sigma(x)$ of the surface $\sigma$ in the vicinity of the point $x$, $H_1(x)$ is the part of hamiltonian density describing the interaction of the field $\varphi(x)$ with itself and with other fields as well as of other fields with themselves in the interaction picture. The $\lambda$ is positive constant, $\delta \sigma(x)$ is the 4-space volume enclosed between space-like surface $\sigma$ and the varied surface $\sigma + \delta \sigma(x)$. Resolvability condition of the equation (4.2) is

$$\frac{\delta^2 \rho(\sigma)}{\delta \sigma(x_1)\delta \sigma(x_2)} = \frac{\delta^2 \rho(\sigma)}{\delta \sigma(x_2)\delta \sigma(x_1)},$$

(4.3)

and it may be fulfilled under local commutativity of the operators $\varphi(x)$, $\varphi^\dagger(x)$ and $H_1(x)$. In other words, if $(x_1 - x_2)^2 < 0$, the following conditions must hold:

$$[\varphi(x_1), \varphi(x_2)] = 0, \quad [\varphi(x_1), \varphi^\dagger(x_2)] = 0, \quad [\varphi(x_1), H_1(x)] = 0.$$  

(4.4)

All other fields presented in our system are thought to be local. Let us show that for local scalar field $\varphi(x)$ nonannihilating the vacuum the equation (4.2) leads to irremovable u.v. divergences. For the sake of simplicity consider the case when there are no other fields with the exception of the $\varphi$ and when $H_1 = 0$. Let us assume, in particular, that on the surface $\sigma$ the state $\rho(\sigma)$ is the vacuum of the interaction picture: $\rho(\sigma) = |0\rangle\langle 0|$. Under the variation $\delta \sigma(x)$ of the surface $\sigma$ in vicinity of the point $x$ it appears that $\rho(\sigma + \delta \sigma) = |0\rangle\langle 0| + \left(\frac{\delta \rho(\sigma)}{\delta \sigma(x)}\right) \delta \sigma(x)$, where $\frac{\delta \rho(\sigma)}{\delta \sigma(x)}$ is defined by the equation (4.2). Let us ask the question: what is the probability of the vacuum state $|0\rangle\langle 0|$ to remain unchanged? To answer this question one should calculate $Sp \{ |0\rangle\langle 0| \rho(\sigma + \delta \sigma(x)) \} = |0\rangle\rho(\sigma + \delta \sigma(x))\langle 0|$. As much as $\varphi$ is in the interaction picture with the vacuum $|0\rangle$, we assume that $\langle 0|\varphi(x)|0\rangle = \langle 0|\varphi^\dagger(x)|0\rangle = 0$ at every $x$. So we have:

$$\langle 0|\rho(\sigma + \delta \sigma(x))|0\rangle = 1 - 2\lambda\langle 0|\varphi^\dagger(x)\varphi(x)|0\rangle \delta \sigma(x) + O((\delta \sigma(x))^2).$$

(4.5)

The $\langle 0|\varphi^\dagger(x)\varphi(x)|0\rangle$ is the Wightman function of the fields $\varphi(x)$ and $\varphi^\dagger(x)$ at coinciding arguments. Even in the general case considered above in section 2 this is equal to $+\infty$, if it is not identically zero. The last possibility corresponds to the field annihilating the vacuum. Indeed, according with (2.6), (2.8), (2.16)

$$\langle 0|\varphi^\dagger(x)\varphi(x)|0\rangle = \frac{1}{(2\pi)^2} \int dk^4 \beta(k^2, \theta(k^2) \text{ sgn}(k_0)).$$

(4.6)

Because $\beta \geq 0$ one sees by going to the hyperbolic coordinates that this expression is proportional to the infinite volume of the hyperboloid $\theta(\pm k_0)(k^2 - 1) = 0$, or correspondingly $k^2 + 1 = 0$, unless $\beta \equiv 0$. Let us also notice that the
mentioned divergence is the divergence of probability but not of probability amplitude. So due to positivity condition we are not allowed to subtract any counterterms to renormalize it.

The considered problem was discussed by Pearl [2], who suggested to refuse local commutativity of the field $\varphi$ and consequently to renounce the proof of Lorentz invariance using an equation like (4.2). Under this conditions the Lorentz invariance must be proved directly by passage from one Lorentz frame to another. But even in this case it appears to be necessary to introduce tachyonic field.

The described above problem of u.v. divergence connected with an equation like (4.2), does not arise, if the field $\varphi(x)$ annihilates the vacuum state, i.e. if $\beta \equiv 0$. If simultaneously $H_1 = 0$, then due to the equation (4.2) the vacuum state $\rho = |0\rangle\langle 0|$ will not change with the course of time at all. Consequently this is the only possibility, accordant with the local field theory. This possibility should be investigated.

As found out above the field $\varphi(x)$ annihilating the vacuum can exist only in a theory with continuous tachyonic spectrum. If under absence of Lindblad-terms (i.e at $\lambda = 0$) this field is free, then it belongs to the class of generalized free fields and its pure tachyonic part describes "unparticle matter". The existence of such field might be assumed without contradiction with experimental data supposing only, that it interacts with other matter very weakly. Just the possibility to apply such a field in relativistic Lindblad equation impelled the authors to undertake the present investigation. To make a theory with the field $\varphi$ practically useful one has to introduce very weak interaction of this field with usual matter. As it will be figured out below we meet very hard difficulties on this way and so far we did not remove them. Now let us describe this problem.

5. Difficulties caused by interaction between the field $\varphi$ annihilating the vacuum and ordinary fields.

Consider now a practically important case when the system described by Lindblad equation (4.2) contains ordinary fields interacting with the field $\varphi$. We assume for simplicity that there is only one ordinary hermitian scalar field $\xi(x)$ with the mass $m, m^2 > 0$. The Hamiltonian density $H_{int}(x)$ in the Lindblad equation (4.2) describes the interaction between the fields $\varphi(x), \varphi^\dagger(x)$ and $\xi(x)$ only. All the fields are thought in the interaction picture. Before the interaction is taken into account the following formula holds

$$\varphi(x)|0\rangle = 0, \quad (5.1)$$

Let us require that the interaction does not destroy the property (1) of the vacuum state. This means that the vacuum in the Schrödinger picture coincides with the one in the interaction picture. It leads to the condition

$$H_{int}(x)|0\rangle = 0. \quad (5.2)$$
Without this requirement it is impossible to introduce a pure state which is stable in time and fulfills the Lindblad equation, i.e. it is impossible to define the vacuum state correctly for this equation.

The simplest density of interaction Hamiltonian, fulfilling the condition (5.2) looks like this \( H_{int}(x) = g\xi(x)\varphi\dagger(x)\varphi(x) \). All operators written here and below are in the interaction picture. One can develop invariant perturbation theory and draw Feynman diagrams for this theory. Free propagator \( D_\varphi(k) \) of the field \( \varphi(x) \) in the momentum representation is defined by the formula (2.53). We shall represent it on diagrams by an arrowed line (because \( \varphi \) is nonhermitian). The propagator of the field \( \xi(x) \) equals to \( D_\xi(k) = \frac{1}{m^2 - k^2 - i\epsilon} \). We shall represent it on diagrams using wavy line. Let us show that this theory is nonrenormalizable. Consider the diagram with 5 tales shown in fig 1.

\[
\begin{align*}
\text{Fig 1:} & \quad \varphi & \quad \xi & \quad g & \quad g & \quad g & \quad g & \quad g \\
& \quad P_1 & \quad g & \quad g & \quad g & \quad g & \quad g & \quad P_2 \quad \text{P}_2 \quad \text{Fig 1}
\end{align*}
\]

We shall show in a moment that although the difference between momentum degrees in the numerator and in the denominator is negative, this diagram has u.v. divergence. The last fact is closely connected with special analytical structure of the \( \varphi \) field propagator. let us beforehand notice that the divergence of such diagram leads to nonrenormalizability of considered theory. Really, the presence of this divergence forces us to introduce a vertex of 5-th degree in fields during renormalization: \( g_5\xi^3(x)\varphi\dagger(x)\varphi(x) \). It is easy to see that canonical dimensions of the fields \( \varphi \) and \( \xi \) are equal to +1 (see the expressions for the propagators), and hence the canonical dimension of the \( g_5 \) is \(-1\). Therefore the theory is nonrenormalizable. Indeed, in this case there is a new divergence connected with 7- tale diagram shown in fig 2. Consequently one needs to introduce an additional term \( g_7\xi^5(x)\varphi\dagger(x)\varphi(x) \) in the \( H_{int} \). But this leads to a new divergence (see fig 3) and so on. Finally we conclude that our theory is nonrenormalizable.

So let us now return to the diagram in fig 1. At first, we shall consider a special case of the (2.53) for which

\[
\alpha(k^2, 1) = \delta(k^2 - m_1^2) \quad \alpha(k^2, -1) = 0 \quad \text{where} \quad m_1^2 > 0. \quad (5.3)
\]
So one should calculate the following integral

\[
\int d^4k D_\xi(k) D_\varphi(p_1 - k) D_\varphi(p_1 + p_2 - k) D_\varphi(p_1 + p_2 + p_3 - k) D_\varphi(p_1 + \ldots + p_4 - k).
\]

From the formula (2.53) one sees, that the propagator \(D_\varphi\) as a function of \(k\) at fixed \(\vec{k}\), has two branch points at \(k_0 = \pm|\vec{k}|\) and two poles in the points \(k_0 = \pm\sqrt{(\vec{k})^2 + m^2}\). Due to presence of "\(i\epsilon \, sgn(k^0)\)" in the denominator and under the sign of the square root, the contour of integration by \(k^0\) passes above this singularities in complex plane of the variable \(k^0\). In turn, the singularities by \(k^0\) of the \(\xi\) field propagator are passed over by the contour of integration in the following manner: \(k^0 = -\sqrt{k^2 + m^2}\) - from below, and \(k^0 = +\sqrt{k^2 + m^2}\) - from above. This implies that in contrast with the ordinary theory we are not allowed to make "Euclidian rotation" in the complex plane of the variable \(k^0\).

On the other hand, we can perform the integration over \(k^0\) using the theorem of residues, by closing path of integration around the upper complex semiplane. So, the quantity (4) turns into

\[
\pi i \int \frac{d^3k}{\sqrt{k^2 + m^2}} (D_\varphi(p_1 - k) \cdot \ldots \cdot D_\varphi(p_1 + \ldots + p_4 - k)) \bigg|_{k^0 = -\sqrt{k^2 + m^2}}. \tag{5.5}
\]

Let us consider the behavior of the expression under integral sign in the (5.5) at \(|\vec{k}| \to \infty\). There are 4 propagators \(D_\varphi\) in the fig 1; let us number them in the following way: the momentum \(p_1 + \ldots + p_j - k\) \((j = 1, 2, 3, 4)\) flows through the propagator of number \(j\). From the equality (2.53) and our additional assumption (5.3) one can see that anyway at \(p_1^2 + \ldots + p_j^2 > |p_1 + \ldots + p_j|\), the \(j\)-th propagator \(D_\varphi\) decreases not faster than \(const(j, p_1, \ldots, p_j)\sqrt{|\vec{k}|}\), when \(|\vec{k}| \to \infty\). So, their product decreases not faster than \(\frac{1}{|\vec{k}|^j}\) and the expression under integral sign as a whole - not faster than \(\frac{1}{|\vec{k}|}\), hence the integral diverges. We would get the same result, if we consider a case when \(\alpha(k^2, -1) = \delta(k^2 - m_2^2)\), \(m_2^2 > 0\). Presenting nonnegative continuous functions \(\alpha(k^2, \pm1)\) in the form \(\alpha(k^2, \pm1) = \int_0^\infty dm^2 (k^2 - m^2) \rho(m^2, \pm1)\), we make sure that the most general expression (2.53) does not make the situation better.

Let us notice that similar result takes place if we use arbitrary scalar field considered in the section 2 and nonannihilating vacuum as the field \(\xi\). Really, the propagator of just considered ordinary field \(\xi\) can be written in the form (compare with (2.48a)):

\[
\frac{1}{m^2 - k^2 - i\epsilon} = \frac{1}{m^2 - k^2 - i\epsilon \, sgn(k^0)} + 2\pi i\theta(-k_0)\delta(k^2 - m^2) \tag{5.6}
\]

In a discourse made above only the second term of the last expression contributed to our diagram (as integration contour passed above both singularities
of the first term). If we consider the similar diagram taking the expression $2\pi i \theta(-k_0) \delta(k^2 - m^2)$ as a wavy line, then the result will be obviously the same. Meanwhile, it is easy to see that the sign of $k^0$ and the sign of $m^2$ are not important for our discourse i.e. if we calculate diagram shown in fig 1 and use the expression $\theta(\pm \text{sgn}(k_0)) \delta(k^2 - m^2)$ at arbitrary sign of $m^2$ as wavy line, it still diverges. Finally let us look at the formula (2.48 a). Only the second term $"i\beta(k^2, -\theta(k^2) \text{sgn}(k^0))"$ (compare with (5.6)) gives contribution to the diagram similar to the one, presented in fig 1. Let us present $\beta$ in the form:

$$\beta(k) = \theta(k^0)\beta_1(k^2) + \theta(-k^0)\beta_2(k^2) =$$

$$= \int_{-\infty}^{+\infty} dm^2 \beta_1(m^2)\theta(k^0) \delta(k^2 - m^2) + \int_{-\infty}^{+\infty} dm^2 \beta_2(m^2)\theta(-k^0) \delta(k^2 - m^2),$$

where $\beta_{1,2} \geq 0$. One can see that u.v. divergence, similar to that one described above, takes places every time with the exception of the case $\beta \equiv 0$ that corresponds to the field $\xi$ annihilating the vacuum.

So we conclude that under condition (5.2) the field $\varphi(x)$ is not able to interact with the ordinary fields in renormalizable way. If we took vector or spinor field as a field of matter and made discourse like one written above, we would have the same result. Moreover, if we try to introduce the interaction between matter fields and the field $\varphi$ not directly but through other local scalar fields of a general form described in the section 2, we have nonrenormalizability again.

Everywhere above we required that the interaction of the field $\varphi$, subordinated to the condition $\varphi|0\rangle = 0$, with other fields does not disturb the stability of the vacuum $|0\rangle$. As a matter of principle it is possible that this stability is only approximate but the rebuilding of the vacuum flows extremely slowly. Then one may assume that at the birth of the Universe, the vacuum fulfilled the condition $\varphi|0\rangle = 0$ and it did not change significantly till nowadays. We did not investigate this possibility in full scale, but after consideration of several variants we found out that the rebuilding of the vacuum in these examples is infinitely rapid.

**Conclusion.**

As a result of our investigation the generalization of Kallen-Lehmann representation for the propagator of local scalar field with arbitrary spectrum of 4-momentum operator is established. Under this conditions the $CPT$ violation becomes possible. The 4-momentum spectrum of $CPT$-violating field is not arbitrary but obeys the relation like (2.45) which follows from the locality condition. Furthermore the tachyons with discrete mass spectrum are not allowed to violate $CPT$-invariance. It is found out that at this assumption and under $CPT$- violation nonzero local field annihilating the vacuum state can exist. Such field is composite: it may have terms which correspond to positive square of mass and positive or negative sign of energy, but it must contain continuous tachyonic spectrum of mass.
Using such field we succeeded to write a correct (in particular without u.v. divergences) local relativistic generalization of Lindblad equation for statistical operator in the case of no interaction with other fields.

It turned out that for the simplest interaction of described field with ordinary ones, the theory becomes nonrenormalizable. We investigated other simple examples of interaction of local tachyonic and phantomic fields (in particular, nonannihilating the vacuum) with ordinary ones. These examples, not described in this work, also did not lead to renormalizable theories. Such results are closely connected with the general form of propagator (2.50) and with impossibility to make "Euclidian rotation".

In the light of the consideration made above, the question arises, whether the local tachyonic and phantomic fields can interact in a renormalizable way with the ordinary fields at all. In the case of positive answer only, the CPT noninvariant local fields can interact with ordinary fields in a renormalizable theory. At the present time we do not know the answer.

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\footnote{We remind that we call the field a ”phantom” if its square of mass is positive, but the energy is negative.}