Journal of Physics Communications

PAPER

Gauge invariant resolution of the $\mathbf{\tilde{A}} \cdot \tilde{p}$ versus $\tilde{r} \cdot \mathbf{E}$ controversy

Hsiang Shun Chou

Institute of Optoelectronic Sciences, National Taiwan Ocean University, Keelung, Taiwan 202, Republic of China
E-mail: hschou@mail.ntou.edu.tw

Abstract

We delve into the problem of the choice of the interaction Hamiltonian in quantum transitions by using the gauge invariant formalism. We demonstrate that the gauge invariant transition amplitudes for the bare-bare transitions, which is the case in the majority of experiments, are identical to the conventional results with the velocity form interaction Hamiltonian. Under the electric dipole approximation, the gauge invariant formalism derives the modified length form interaction Hamiltonian in any gauge. We propose a simple experiment that one can perform to verify our prediction. On the other hand, the gauge invariant transition amplitudes for the dressed-dressed transitions, which is the case in the Lamb shift experiment, are identical to the conventional results with the length form interaction Hamiltonian. There is no preferred gauge in the calculations of transition amplitudes. The present investigation resolves the long-standing $\mathbf{\tilde{A}} \cdot \tilde{p}$ versus $\tilde{r} \cdot \mathbf{E}$ controversy and clarifies the prevailing misconception in the literature.

1. Introduction

The $\mathbf{\tilde{A}} \cdot \tilde{p}$ versus $\tilde{r} \cdot \mathbf{E}$ controversy is a long-standing problem in quantum physics. In the conventional formalism of nonrelativistic quantum mechanics, the time-independent Hamiltonian $\hat{H}_0$ is identified as the energy operator. The interaction Hamiltonian in the Coulomb gauge takes the velocity form $\hat{H}_{\text{int}}^{(v)} = -\frac{e}{m} \mathbf{\tilde{A}} \cdot \tilde{p} + \frac{e}{2m} \mathbf{\tilde{A}}^2$, while that in the electric field gauge takes the length form $\hat{H}_{\text{int}}^{(l)} = -e\mathbf{\tilde{r}} \cdot \mathbf{E}$. The wave function is expanded in terms of the eigenstates of $\hat{H}_0$. The expansion coefficients, which are interpreted as the transition amplitudes, depend on the gauge. The transition amplitude in the Coulomb gauge and that in the electric field gauge differ by the ratio of the transition energy over the photon energy [1]. These considerations have a parallel in the relativistic theory. It has been shown [2] that in the nonrelativistic limit the transition amplitudes in the Coulomb gauge reduce to the velocity form amplitudes, while those in the Babushkin gauge reduce to the length form amplitudes. Relativistic transition amplitudes for one-electron system are gauge invariant only for resonant transitions.

A typical example of the length-velocity discrepancy was found in the Lamb shift measurement [3]. In the Lamb shift experiment, the hydrogen atoms in the metastable $2S_1$ state were exposed to a rf field. This field couples the $2S_1$ and $2P_2$ states, thereby causes the metastable atoms to decay to the $1S_0$ state. The decay curve depends on the matrix elements for the transition from $2S_1$ to $2P_2$. Lamb found that it was the length form interaction Hamiltonian that yielded the correct decay curve. The use of the velocity form interaction Hamiltonian gave rise to a significant distortion of the decay curve. Lamb then asserted that the length form interaction Hamiltonian should be used under the electric dipole approximation.

In the mid-seventies of the twenty century, the problem was investigated in detail from the point of view of gauge invariance. Gauge invariance in quantum mechanics is elucidated in the textbook by Cohen-Tannoudji et al [4]. Yang [5] developed the gauge invariant interpretation of quantum mechanics. A summary of the publications about the gauge invariant interpretation can be found in [6]. Yang emphasized the importance of formulating the interpretation using only physical operators and physical states. Physical operators are defined as those operators whose expectation values are gauge invariant [1, 4]. The eigenstates of physical operators are physical states. It is well known [1, 4] that the time-independent Hamiltonian $\hat{H}_0$ is not a physical operator.
Consequently, the eigenstates of $\hat{H}_0$ are not physical states. This causes the gauge dependence of the transition amplitudes in the conventional formalism. Yang discriminated between the time-independent Hamiltonian and the energy operator. The energy operator for one-electron system $\hat{\mathcal{E}} = \frac{1}{2m} \hat{\mathbf{p}}^2 - eA^\mathcal{E}(\mathbf{r}, t) + eV(\mathbf{r})$, which is just the sum of the electron’s kinetic energy and potential energy, is a physical operator. Yang expanded the wave function in terms of the eigenstates of the energy operator. The expansion coefficients are interpreted by Yang as the gauge invariant transition amplitudes. Yang’s transition amplitudes, with the exception of the electric dipole transition, differ from their counterparts in the conventional formalism. Under the electric dipole approximation wherein the magnetic field is neglected and the electric field varies slowly over atomic dimensions, Yang’s gauge invariant transition amplitudes are identical to the conventional results with the length form interaction Hamiltonian. Yang’s finding completely agree with the conclusion of Lamb [3]. Since then there has been a common assertion [1, 5–8] that the length form interaction Hamiltonian should be used in the conventional formalism of quantum mechanics.

The subject of gauge invariance in many-body system is more subtle and controversial because approximate wave functions must be used. Lin [9] claimed, in the conventional formalism of relativistic quantum mechanics, that the Hartree–Fock theory violates the gauge invariance even for resonant transitions because of the nonlocal potential. One of the elegant methods to treat the correlation effects in many-body system is the many-body perturbation theory. Chi and Chou [10] showed, also in the conventional formalism, that the relativistic many-body perturbation theory calculations are gauge invariant order by order for resonant transitions, provided that the calculations start from a local potential and include the contributions from the negative-energy states. Kobe [11] extended Yang’s formulation in the second-quantization formalism, to many-body system. The formulation was then applied to demonstrate that Hartree–Fock theory is gauge invariant in spite of being nonlocal. The only proviso is that the Hamiltonian for the many-body system is replaced by Yang’s energy operator.

The gauge invariant interpretation of Yang has raised critical comments in the literature [12–14]. Feuchtwang et al [14] pointed out that the main difficulties of Yang’s interpretation are as follows. First, Yang’s energy operator has time dependent eigenvalues which cannot be measured precisely. Second, Yang’s transition amplitudes, while gauge invariant, are neither measurable nor do they relate to probabilities of physically significant observations in the study of electromagnetic interactions with matter. The aim of this paper is to illustrate the confusion inherent in Yang’s interpretation and to make a gauge invariant resolution of the long-standing $\hat{\mathbf{A}} \cdot \hat{\mathbf{p}}$ versus $\hat{\mathbf{r}} \cdot \hat{\mathbf{E}}$ controversy. We point out that the careful distinction between the bare-bare transitions and the dressed-dressed transitions is necessary for the choice of the interaction Hamiltonian. We obtain the rule over when to use $\hat{\mathbf{A}} \cdot \hat{\mathbf{p}}$ and when to use $\hat{\mathbf{r}} \cdot \hat{\mathbf{E}}$.

### 2. Gauge transformation, gauge invariance, and gauge covariance

We consider an electron subject to a static potential $V(\mathbf{r})$ and a time-varying electromagnetic field $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$. The fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ can be written in the form

$$
\mathbf{E}(\mathbf{r}, t) = -\nabla \Phi^\mathbf{E}(\mathbf{r}, t) - \frac{\partial}{\partial t} \mathbf{A}^\mathbf{E}(\mathbf{r}, t), \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}^\mathbf{E}(\mathbf{r}, t),
$$

(1)

where $\Phi^\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{A}^\mathbf{E}(\mathbf{r}, t)$ are the electromagnetic potentials in an arbitrary gauge. In this gauge, the Hamiltonian and the Schrödinger equation are given by

$$
\hat{H}^\mathbf{E}(\mathbf{r}, t) = \frac{1}{2m} \hat{\mathbf{p}}^2 - e\mathbf{A}^\mathbf{E}(\mathbf{r}, t) + eV(\mathbf{r}),
$$

$$
\hat{\psi}(\mathbf{r}, t) = \frac{\partial}{\partial t} \psi(\mathbf{r}, t).
$$

(2)

(3)

Here $e = -|e|$ is the electron charge. In classical mechanics, the mechanical momentum and the conanical momentum are given by $\hat{\mathbf{p}} = m\hat{\mathbf{r}}$ and $\hat{\mathbf{p}} = \nabla \times \mathbf{A}^\mathbf{E}(\mathbf{r}, t)$, respectively. In quantum mechanics, the conanical momentum operator is $\hat{\mathbf{p}} = -i\hbar \nabla$ in any gauge.

A common gauge is the Coulomb (or transverse) gauge ($\mathbf{A}^\mathbf{E}$, $\Phi^\mathbf{E}$) which satisfies the condition

$\nabla \cdot \mathbf{A}^\mathbf{E}(\mathbf{r}, t) = 0, \quad \Phi^\mathbf{E}(\mathbf{r}, t) = 0.$

(4)

The wave function of the electron in the Coulomb gauge is denoted by $\psi(\mathbf{r}, t)$. The electromagnetic potentials and the electron’s wave function in other gauges can be obtained by the gauge transformation [1, 4, 7]:

$$
\mathbf{A}^\mathbf{E}(\mathbf{r}, t) = \mathbf{A}^\mathbf{C}(\mathbf{r}, t) + \nabla \chi^\mathbf{E}(\mathbf{r}, t),
$$

$$
\Phi^\mathbf{E}(\mathbf{r}, t) = -\frac{\partial \chi^\mathbf{E}(\mathbf{r}, t)}{\partial t},
$$

(5)

(6)
\[ \Psi^\xi(\vec{r}, t) = T^\xi(\vec{r}, t) \Psi^\xi(\vec{r}, t), \]

where \( T^\xi(\vec{r}, t) = e^{i\chi^\xi(\vec{r}, t)/\hbar} \). In equation (5), \( \vec{A}^\xi(\vec{r}, t) \) and \( \nabla \chi^\xi(\vec{r}, t) \) are the transverse and the longitudinal components of \( \vec{A}^\xi(\vec{r}, t) \), respectively. The electromagnetic field \( \vec{E}(\vec{r}, t) \) and \( \vec{B}(\vec{r}, t) \) are invariant under the gauge transformation (5)–(6). The Schrödinger equation in (3) is covariant, i.e. form invariant under the gauge transformation (5)–(7).

3. Schrödinger equation in a manifestly gauge covariant form

The Schrödinger equation in (3) can be written in a manifestly gauge covariant form:

\[ \hat{E}^\xi(\vec{r}, t) \Psi^\xi(\vec{r}, t) = \hat{\mathcal{E}}^\xi(\vec{r}, t) \Psi^\xi(\vec{r}, t), \]

where

\[ \hat{E}^\xi(\vec{r}, t) = i\hbar \frac{\partial}{\partial t} - e\Phi^\xi(\vec{r}, t) = i\hbar \frac{\partial}{\partial t} + e\partial \chi^\xi(\vec{r}, t). \]

The energy operator \( \hat{\mathcal{E}}^\xi(\vec{r}, t) \) can be expressed as

\[ \hat{\mathcal{E}}^\xi(\vec{r}, t) = \frac{[\hat{p}^\xi - e\chi^\xi(\vec{r}, t)]^2}{2m} + eV(\vec{r}) \]

\[ = \frac{[\hat{p} - e\chi^\xi(\vec{r}, t)]^2}{2m} - \frac{e}{m} \vec{A}^\xi(\vec{r}, t) : [\hat{p} - e\chi^\xi(\vec{r}, t)] \]

\[ + \frac{e^2}{2m} [\vec{A}^\xi(\vec{r}, t)]^2 + eV(\vec{r}). \]

In arriving at equation (10), we have used the identity \([\hat{p}^\xi, \vec{A}^\xi] = 0\).

We can show that

\[ \hat{\mathcal{E}}^\xi(\vec{r}, t) = T^\xi t^\xi(\vec{r}, t) \hat{\mathcal{E}}^\xi(\vec{r}, t) T^\xi t^\xi(\vec{r}, t), \]

and

\[ \hat{\mathcal{E}}^\xi(\vec{r}, t) = T^\xi t^\xi(\vec{r}, t) \hat{\mathcal{E}}^\xi(\vec{r}, t) T^\xi t^\xi(\vec{r}, t) \]

Equations (11) and (12) indicate that \( \hat{E}^\xi(\vec{r}, t) \) and \( \hat{\mathcal{E}}^\xi(\vec{r}, t) \) are physical operators [1, 4]. For a physical operator \( \hat{G}^\xi, \hat{G}^\xi \) and \( \hat{G}^\xi \) represent the same operator in different gauges. For example, \( \hat{E}^\xi(\vec{r}, t) \) and \( \hat{\mathcal{E}}^\xi = i\hbar \frac{\partial}{\partial t} \) represent the same operator in different gauges.

It is straightforward to show that \( \hat{E}^\xi \) and \( \hat{\mathcal{E}}^\xi \) form a 4-vector \( i\hbar D^\mu = (\hat{E}^\xi, \hat{\pi}^\xi) \), where \( D^\mu = \partial^\mu + \frac{e}{\hbar} \vec{A}^\xi \) is the gauge covariant derivative [15]. Therefore the physical significance of \( \hat{E}^\xi(\vec{r}, t) \) is \( i\hbar \) times the time component of the gauge covariant derivative. Note that \( e \Phi^\xi \) is an integral part of \( \hat{E}^\xi \). It cannot be interpreted as the interaction energy because it is not a physical operator. Another reason is that only static fields can be used to define a potential energy [5]. We shall derive the expression for the interaction energy operator in subsequent paragraphs.

A heuristic procedure to obtain the Schrödinger equation in (8) involves starting from the energy-momentum relation \( E = \frac{\hat{p}^2}{2m} + e V \) and making the substitution

\[ E \rightarrow \hat{E}^\xi = i\hbar \frac{\partial}{\partial t} - e\Phi^\xi, \quad \pi \rightarrow \hat{\pi}^\xi = -i\hbar \nabla - e\vec{A}^\xi. \]

The substitution (13) is Lorentz covariant, since it is a correspondence between two 4-vectors \( p^\mu = (E, \vec{p}) \) and \( i\hbar D^\mu \).

4. Bare states versus dressed states

4.1. Bare states

The electromagnetic potentials in the absence of the electromagnetic field can in general be chosen as

\[ \vec{A}^\xi(\vec{r}, t) = \nabla \chi^\xi(\vec{r}, t), \quad \Phi^\xi(\vec{r}, t) = - \frac{\partial \chi^\xi(\vec{r}, t)}{\partial t}. \]

Equation (14) indicates that the vector potential is longitudinal in the absence of the electromagnetic field. The energy operator in the absence of the electromagnetic field is called the bare energy operator. From equations (10) and (14) it follows that the bare energy operator is given by
Thus the interaction energy operator in the Coulomb gauge coincides with the velocity form interaction Hamiltonian.

The eigenvalue equation of \( \hat{\mathcal{E}}_{\text{bare}}(\vec{r}, t) \) reads

\[
\hat{\mathcal{E}}_{\text{bare}}(\vec{r}, t) \phi_i(\vec{r}) = E_i \phi_i(\vec{r}),
\]

where \( \phi_i(\vec{r}) \) are the eigenfunctions of \( \hat{\mathcal{E}}_0 \) with eigenvalues \( E_i \). It follows from (16), (17), (19) that

\[
E_i^f = E_i
\]

and

\[
\phi_i^f(\vec{r}, t) = T^{\mathcal{E}}(\vec{r}, t) \phi_i(\vec{r}).
\]

It is noticed that the eigenvalues of \( \hat{\mathcal{E}}_{\text{bare}}(\vec{r}, t) \) are time independent despite of the time dependence of \( \hat{\mathcal{E}}_{\text{bare}}(\vec{r}, t) \).

### 4.2. Dressed states

In the presence of the electromagnetic field, the vector potential contains the transverse component \( \vec{A}(\vec{r}, t) \). The energy operator in the presence of the electromagnetic field is called the dressed energy operator. It follows from (10) and (15) that the dressed energy operator \( \hat{\mathcal{E}}_{\text{dressed}}(\vec{r}, t) \) can be decomposed into the bare energy operator \( \hat{\mathcal{E}}_{\text{bare}}(\vec{r}, t) \) and the interaction energy operator \( \hat{\mathcal{E}}_{\text{int}}(\vec{r}, t) \):

\[
\hat{\mathcal{E}}_{\text{dressed}}(\vec{r}, t) = \hat{\mathcal{E}}_{\text{bare}}(\vec{r}, t) + \hat{\mathcal{E}}_{\text{int}}(\vec{r}, t),
\]

where

\[
\hat{\mathcal{E}}_{\text{int}}(\vec{r}, t) = -\frac{e}{m} \vec{A}(\vec{r}, t) \cdot \vec{p} - e \nabla \chi^f(\vec{r}, t) + \frac{e^2}{2m} [\nabla \chi^f(\vec{r}, t)]^2.
\]

It is straightforward to show that \( \hat{\mathcal{E}}_{\text{int}}(\vec{r}, t) \) satisfies

\[
\hat{\mathcal{E}}_{\text{int}}(\vec{r}, t) = T^{\mathcal{E}}(\vec{r}, t) \hat{\mathcal{E}}_{\text{int}}(\vec{r}, t) T^{\mathcal{E}^*}(\vec{r}, t),
\]

which indicates that the interaction energy operator is a physical operator. Therefore, the decomposition in (22) is gauge invariant. The interaction energy operator in the Coulomb gauge is given by

\[
\hat{\mathcal{E}}_{\text{int}}(\vec{r}, t) = -\frac{e}{m} \vec{A}(\vec{r}, t) \cdot \vec{p} + \frac{e^2}{2m} [\vec{A}(\vec{r}, t)]^2 = \hat{\mathcal{E}}_{\text{int}}(t).
\]

Thus the interaction energy operator in the Coulomb gauge coincides with the velocity form interaction Hamiltonian.

The eigenvalue equation of \( \hat{\mathcal{E}}_{\text{dressed}}(\vec{r}, t) \) reads

\[
\hat{\mathcal{E}}_{\text{dressed}}(\vec{r}, t) \psi^f_\epsilon(t) = \mathcal{E}^f(t) \psi^f_\epsilon(t).
\]

The eigenvalues \( \mathcal{E}^f(t) \) and the eigenfunctions \( \psi^f_\epsilon(\vec{r}, t) \) of \( \hat{\mathcal{E}}_{\text{dressed}}(\vec{r}, t) \) are the dressed energies and the dressed states [16], which represent the energies and the eigenstates of the electron in the presence of the electromagnetic field. The eigenvalues \( \mathcal{E}^f(t) \) is gauge invariant because of (12). Therefore we simply denote it by \( \mathcal{E}(\vec{r}) \). Note that \( \mathcal{E}(\vec{r}) \) is in general time dependent. Thus we resolve the first difficulty of Yang’s interpretation [5] by correctly identifying Yang’s energy operator as the dressed energy operator.

Under the electric dipole approximation, the vector potential is approximated by

\[
\vec{A}(\vec{r}, t) = \vec{A}(0, t).
\]

In the study of the dressed state, the electric field gauge in which

\[
\chi^\mu(\vec{r}, t) = -\vec{r} \cdot \vec{A}(0, t)
\]
is of particular interest. The vector and the scalar potential in this gauge are given by

\[ \mathbf{A}^g(\mathbf{r}, t) = \mathbf{A}^g(0, t) + \nabla \chi(\mathbf{r}, t) = 0, \]

\[ \phi^e(\mathbf{r}, t) = -\frac{\partial \chi^e(\mathbf{r}, t)}{\partial t} = \mathbf{r} \cdot \frac{\partial \mathbf{A}^g(0, t)}{\partial t} = -\mathbf{r} \cdot \mathbf{E}(0, t). \]

Therefore the dressed energy operator takes the form

\[ \hat{\mathcal{E}}^{\text{dressed}}(\mathbf{r}) = \frac{\hat{p}^2}{2m} + eV(\mathbf{r}) = \hat{H}_0. \]

Thus the dressed energy operator in the electric field gauge coincides with the time-independent Hamiltonian. Consequently, the eigenvalues and the eigenfunctions of \( \hat{\mathcal{E}}^{\text{dressed}}(\mathbf{r}) \) are identical to those of \( \hat{H}_0 \):

\[ \hat{\mathcal{E}}^{\text{dressed}}(\mathbf{r}) \phi_i(\mathbf{r}) = E_i \phi_i(\mathbf{r}), \]

It follows from (12), (26), (32) that

\[ E_i(t) = E_i, \]

\[ \psi^g_i(\mathbf{r}, t) = T^g(\mathbf{r}, t) \phi_i(\mathbf{r}). \]

It is noticed that only in the electric dipole approximation the eigenvalues of \( \hat{\mathcal{E}}^{\text{dressed}}(\mathbf{r}, t) \) are time independent.

5. Quantum transitions

5.1. Quantum transitions between the bare states

We are now in a position to investigate the quantum transitions. In an experiment the electromagnetic field is turned on at an instant \( t_1 \) and then turned off at another instant \( t_2 \). In the prior-interaction sector \( (t < t_1) \), the electron is in a bare state with a time-independent energy. In the interaction sector \( (t_1 \leq t \leq t_2) \), the electron switches to a dressed state with a time-dependent energy. In the post-interaction sector \( (t > t_2) \), the electron is again in a bare state.

In the majority of experiments, the initial and the final states belong to the prior- and the post-interaction sectors, respectively. Therefore, one has to consider transitions between the bare states which are eigenstates of \( \hat{\mathcal{E}}^{\text{bare}}(\mathbf{r}, t) \). It is thus useful to expand the wave function in terms of the eigenfunctions of \( \hat{\mathcal{E}}^{\text{bare}}(\mathbf{r}, t) \):

\[ \Psi^g(\mathbf{r}, t) = \sum_i C_i^g(t) \phi_i^g(\mathbf{r}, t). \]

The expansion coefficients \( C_i^g(t) \) then coincide with the probability amplitudes for transitions to an eigenstate of \( \hat{\mathcal{E}}^{\text{bare}}(\mathbf{r}, t) \) with energy \( E_i \). Inserting the wave function (35) into (8) and projecting with \( \langle \phi_i^g(t) \rangle \) on this equation yields

\[ \frac{d}{dt} C_i^g(t) = -\frac{i}{\hbar} \left\{ E_i C_i^g(t) + \sum_j C_j^g(t) \langle \phi_j^g(t) | \hat{\mathcal{E}}^g(t) | \phi_i^g(t) \rangle \right\}. \]

In arriving at (36), we have used the orthogonality of \( \phi_i^g(\mathbf{r}, t) \) and the relation

\[ \langle \phi_i^g(t) | \hat{\mathcal{E}}^g(t) | \phi_j^g(t) \rangle = \langle \phi_i | \hat{\mathcal{E}}^g | \phi_j \rangle = 0, \]

which is a consequence of (11) and (21). The transition matrix elements in g gauge are given by

\[ T_{ij}^g = \langle \phi_j^g(t) | \hat{\mathcal{E}}_{\text{int}}^g(t) | \phi_i^g(t) \rangle = \langle \phi_j | \hat{\mathcal{E}}_{\text{int}}^g(t) | \phi_i \rangle = \langle \phi_j | \hat{H}_{\text{int}}^g(t) | \phi_i \rangle, \]

where we have used (19), (21), (24), and (25). Equation (38) indicates that it is \( \hat{\mathcal{E}}^{\text{bare}}(\mathbf{r}, t) \) alone which causes transitions between the bare states. Within the conventional formalism, the velocity form interaction Hamiltonian should be used in any gauge for the transitions between the bare states. There is no preferred gauge in the calculations of transition amplitudes. This result is contrary to the common assertion [1, 5–8] that the length form interaction Hamiltonian should be used in the conventional formalism. Note that the transition matrix elements in equation (38) are identical to their counterparts in the conventional formalism for all multipole transitions.

In the electric dipole approximation, the \( (\mathbf{A})^2 \) term in equation (25) cannot give rise to transitions between different states. It follows from equations (25) and (38) that

\[ T_{ij}^g = \langle \phi_j | -\frac{e}{m} \mathbf{A}(0, t) \cdot \hat{\mathbf{p}} | \phi_i \rangle. \]
For a monochromatically linearly polarized plane wave field, the electric field takes the form
\[ \vec{E}(0, t) = \vec{E}_0 \sin \omega t = \vec{E}(0, \omega) e^{-i\omega t} + \vec{E}^\ast(0, \omega) e^{i\omega t}, \]  
and the vector potential in the Coulomb gauge reads
\[ \vec{A}(0, t) = \vec{A}_0 \cos \omega t = \vec{A}(0, \omega) e^{-i\omega t} + \vec{A}^\ast(0, \omega) e^{i\omega t}, \]  
with \( \vec{A}_0 = \frac{\vec{E}}{\omega} \) and \( \vec{A}(0, \omega) = \frac{-i\vec{E}(0, \omega)}{\omega} \). By using the commutator relation
\[ [\vec{r}, \hat{H}_0] = i\hbar \frac{\vec{e}}{m}, \]  
we obtain
\[ \langle \phi_i | - \frac{\vec{e}}{m} \vec{A}(0, \omega) \cdot \vec{p} | \phi_j \rangle = \frac{E_i - E_j}{\hbar \omega} \langle \phi_i | e^{-i\vec{r} \cdot \vec{E}(0, \omega)} | \phi_j \rangle. \]  
Under the electric dipole approximation, the gauge invariant formalism derives the modified length form interaction Hamiltonian \( \hat{H}^{(l)} = -\frac{e}{\hbar \omega} \vec{r} \cdot \hat{E}(0, \omega) |g\rangle \langle e| \) in any gauge for the transitions between the bare states. Note that the modified length form interaction Hamiltonian depends on the transition energy.

One can perform a simple experiment to verify our prediction. Consider the interaction of a radiation field of angular frequency \( \omega \) with a two-level atom with ground state \( |g\rangle \) and excited state \( |e\rangle \). The population of the two-level atom undergoes a Rabi oscillation between the ground and excited states at the angular frequency \( \Omega = \sqrt{\Omega_0^2 + \Delta^2} \), where \( \Omega_0 \) is the Rabi frequency and \( \Delta \) is the detuning of the radiation field from the atomic resonance. The Rabi frequency \( \Omega_0 \) in the conventional formalism is given by \( \frac{\hbar \Omega}{2} = |\langle e | - e \vec{r} \cdot \vec{E}(0, \omega) | g \rangle| \), where \( |g\rangle \) and \( |e\rangle \) are the eigenstates of \( \hat{H}_0 \). The gauge invariant Rabi frequency \( \Omega_0 \) is, however, given by \( \frac{\hbar \Omega}{2} = \frac{E_i - E_j}{\hbar \omega} \langle \phi_i | e^{-i\vec{r} \cdot \vec{E}(0, \omega)} | \phi_j \rangle \). It is possible to determine the frequency of the Rabi oscillation in a very high precision. As long as both \( \Delta \) and \( \Omega \) are precise, \( \Omega_0 \) can be determined precisely. One can measure the values of \( \Omega_0 \) as a function of \( \Delta \). The result will be able to answer the question which form of the interaction Hamiltonian should be used.

5.2. Quantum transitions between the dressed states

In the Lamb shift experiment, the induced transition from \( 2S_1/2 \) to \( 2P_1/2 \) takes place inside the interaction sector. Therefore one has to consider transitions between the dressed states which are eigenstates of \( \hat{E}^{d}_{\text{dressed}}(\vec{r}, t) \). It is thus useful to expand the wave function in terms of the eigenfunctions of \( \hat{E}^{d}_{\text{dressed}}(\vec{r}, t) \):
\[ \Psi(\vec{r}, t) = \sum_i C_i^g(t) \psi_i^g(\vec{r}, t). \]  
The expansion coefficients \( C_i^g(t) \) then coincide with the probability amplitudes for transitions to an eigenstate of \( \hat{E}^{d}_{\text{dressed}}(\vec{r}, t) \) with energy \( E_i(t) \). Inserting the wave function (44) into (8) and projecting with \( \psi_i^g(t) \) on this equation yields
\[ \frac{d}{dt} C_i^g(t) = \frac{i}{\hbar} \{ E_i(t) C_i^g(t) - \sum_j C_j^g(t) \langle \psi_i^g(t) | \hat{E}^{d}_{\text{dressed}}(\vec{r}, t) | \psi_j^g(t) \rangle \}. \]  
The transition matrix elements in g gauge are given by
\[ T_i^g = -\langle \psi_i^g(t) | \hat{E}^{d}_{\text{dressed}}(\vec{r}, t) | \psi_j^g(t) \rangle = -\langle \phi_i | \hat{E}(0, t) | \phi_j \rangle = -\langle \phi_i | e^{-i\vec{r} \cdot \vec{E}(0, t)} | \phi_j \rangle, \]  
where we have used (9), (11), (30), and (34).

The transition amplitude in (46) was first obtained by Yang [5]. However, it was mistakenly interpreted as the transition amplitude for the transitions between the bare states. The confusion in the literature arises from this misinterpretation. We resolve the second difficulty of Yang’s interpretation [5] by correctly identifying Yang’s transition amplitudes as the transition amplitudes for the transitions between the dressed states. We conclude that it is \( E^{d}_{\text{dressed}}(\vec{r}, t) \) alone which causes transitions between the dressed states. Within the conventional formalism, the length form interaction Hamiltonian should be used in any gauge for the transitions between the dressed states.

6. Summary

In summary, we delve into the problem of the choice of the interaction Hamiltonian in quantum transitions by using the gauge invariant formalism. We point out that there exists so much confusion in the common assertion. The confusion arises from the wrong identification of the dressed energy operator as the bare energy operator and the failure to distinguish between the bare-bare transitions and the dressed-dressed transitions. We
demonstrate that within the conventional formalism the velocity form interaction Hamiltonian should be used in any gauge for the transitions between the bare states. In the electric dipole approximation, the gauge invariant formalism derives the modified length form interaction Hamiltonian in any gauge. We propose a simple experiment that one can perform to verify our prediction. On the other hand, within the conventional formalism the length form interaction Hamiltonian should be used in any gauge for the transitions between the dressed states. There is no preferred gauge in the calculations of transition amplitudes. The present investigation leads to a gauge invariant resolution of the long-standing $\mathbf{A} \cdot \mathbf{p}$ versus $\mathbf{r} \cdot \mathbf{E}$ controversy and a clarification of the prevailing misconception in the literature. Gauge invariant formulation of many-body theory is in progress.

Acknowledgments

The author thanks Professor I A Yu and Professor R L Chang for useful discussions. This work was supported by the Ministry of Science and Technology of the Republic of China under Grant No. NSC104-2112-M-019-003-.

ORCID iDs

Hsiang Shun Chou https://orcid.org/0000-0001-6916-778X

References

[1] Scully M O and Zubairy M S 1997 Quantum Optics (London: Cambridge) pp 178–82
[2] Grant I P 1974 J. Phys. B 7 1458
[3] Lamb W E Jr. 1952 Phys. Rev. 85 239
Lamb W E Jr., Schlüter R R and Scully M O 1987 Phys. Rev. 36 2763
[4] Cohen-Tannoudji C, Diu B and Laloe F 1977 Quantum Mechanics (Paris: Hermann/Wiley) sect. Complement HIII.
[5] Yang K H 1976 Ann. Phys. (N.Y.) 101 62
[6] Schlüter R R, Becker W, Bergou J and Scully M O 1984 Quantum Electrodynamics and Quantum Optics ed A O Barut (N.Y.: Plenum) p 405
[7] Cohen-Tannoudji C, Dupont-Roc J and Grynberg G 1977 Photons and Atoms (N.Y.: J. Wiley) p 271
[8] Rzazewski K and Boyd R W 2004 J. Mod. Optic. 51 1137
[9] Lin D L 1977 Phys. Rev. A 16 600
[10] Chi H C and Chou H S 2010 Phys. Rev. A 82 032518
[11] Kobe D H 1979 Phys. Rev. A 19 1876
[12] Aharonov Y and Au C K 1981 Phys. Lett. A 86 269
Aharonov Y and Au C K 1983 Phys. Lett. A 95 412
[13] Au C K 1984 J. Phys. B 17 L59
[14] Kazes E, Feuchtwang T E, Groth H and Cutler P H 1983 Phys. Rev. D 27 1388
Feuchtwang T E, Kazes E, Groth H and Cutler P H 1982 Phys. Lett. A 93 4
Feuchtwang T E, Kazes E and Cutler P H 1984 J. Phys. A 17 1157
Feuchtwang T E, Kazes E, Cutler P H and Groth H 1984 J. Phys. A 17 151
[15] Ref. [7], p. 411
[16] Courtens E and Szöke A 1977 Phys. Rev. A 15 1588
[17] Weissbluth M 1989 Photon-Atom Interactions (London: Academic Press) pp 214–15