Triaxially-deformed Freely-precessing Neutron Stars: Continuous electromagnetic and gravitational radiation

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ABSTRACT

The shape of a neutron star (NS) is closely linked to its internal structure and the equation of state of supranuclear matters. A rapidly rotating, asymmetric NS in the Milky Way undergoes free precession, making it a potential source for multimessenger observation. The free precession could manifest in (i) the spectra of continuous gravitational waves (GWs) in the kilohertz band for ground-based GW detectors, and (ii) the timing behavior and pulse-profile characteristics if the NS is monitored as a pulsar with radio and/or X-ray telescopes. We extend previous work and investigate in great detail the free precession of a triaxially deformed NS with analytical and numerical approaches. In particular, its associated continuous GWs and pulse signals are derived. Explicit examples are illustrated for the continuous GWs, as well as timing residuals in both time and frequency domains. These results are ready to be used for future multimessenger observation of triaxially-deformed freely-precessing NSs, in order to extract scientific implication as much as possible.

Key words: gravitational waves – pulsars: general – methods: analytical

1 INTRODUCTION

Pulsars are magnetized rotating neutron stars (NSs). Using the so-called pulsar timing technique, the Hulse-Taylor pulsar provided the first validation for the existence of gravitational waves (GWs; Taylor et al. 1979). In the new era after the direct observation of GWs with ground-based laser interferometric detectors (Aasi et al. 2015; Acernese et al. 2015; Abbott et al. 2016, 2017b, 2019a), pulsars continue to play an important role in the context of GW astrophysics. They can be perceived as GW sources radiating continuous GWs, as well as GW detectors in the form of pulsar timing arrays (Janssen et al. 2015; Perera et al. 2019). In this work, we are interested in freely precessing, asymmetric NSs, which can produce both modulated pulse signals and continuous GW radiations with characteristic features (Zimmermann & Szedenits 1979; Zimmermann 1980; Bisnovatyi-Kogan et al. 1990; Jones & Andersson 2001, 2002), and thus become potential multimessenger sources of great scientific interest.

The most compelling evidence for NS free precession comes from PSR B1828−11. Timing observation over 13 years for this isolated pulsar showed strong Fourier power at periods of about 250, 500 and 1000 days (Stairs et al. 2000), which could be an indication for free precession. Link & Epstein (2001) suggested the period at 500 days as the precession period of a biaxial NS with a precessing angle of ∼3° and a dipole moment nearly orthogonal to the symmetric axis. They also interpreted the period at 250 days as a result of the electromagnetic dipole torque. Timing data of another pulsar, PSR B1642−03, provided additional support for the idea of NS free precession (Cordes 1993; Shanabova et al. 2001). Recently, the CHIME/FRB Collaboration reported the detection of a 16.35 ± 0.15 day periodicity in the radio burst from FRB 180916.J0158+65 (Amiri et al. 2020). This periodicity may arise from the free precession of a magnetar (Zanazzi & Lai 2020; Levin et al. 2020). Another possible evidence of precession comes from oscillations in gamma-ray burst afterglows. The afterglow may be powered by a long-lived remnant (Dai & Lu 1998; Zhang et al. 2006), which might be a millisecond magnetar (Suvorov & Kokkotas 2020). In the early stages of its life, the magnetar is likely to precess and result in modulation of X-ray luminosity. Suvorov & Kokkotas (2020) found that the data of two bursts are highly consistent with precessing oblique rotators. While more evidence is needed to solidify the phenomenon

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of NS free precession, it is nowadays certainly interesting to study it in the context of GW astrophysics.

The conventional model for NS structure consists of a liquid core and a thin solid crust. Jones & Andersson (2001) conjectured that the core of a NS does not participate in the free precession. Assuming the crust-only precession, they constructed a simple model with a thin radio beam fixed on the body of a biaxially deformed NS, and they assumed that the beam is aligned with the dipole moment. The model was applied to some potential candidates which may be undergoing free precession (Jones & Andersson 2001). Although our understanding related to NSs has advanced remarkably (Baym et al. 2018), NS structure is still unclear and alternative models have already been proposed. NSs could actually be strange stars if Witten’s conjecture is correct (Witten 1984), and they could globally be in a solid state if quarks are condensed in position space (Xu 2003) or momentum space (Mammarelli et al. 2007), resulting in highly elastic quadrupole deformations (Owen 2005) and thus large free precession amplitudes. Therefore, a multimeasurer study of freely-precessing NSs would help in understanding the equation of state of cold matter at supranuclear density and distinguish between different models.

Previous studies dominantly focused on biaxial NSs. In the most generic case, the deformation of a NS does not need to be biaxial. A triaxially deformed NS can demonstrate new features in its free precession. We extend the simple model in Jones & Andersson (2001) and study the timing residual of a freely precessing triaxial NS in this work. The internal dissipation from the frictional-type coupling between the crust and the core may damp the wobble angle in a relatively short timescale (Jones & Andersson 2002). As an illustrative work, we do not consider the damping here, but use different wobble angles in our calculation, from large ones to small ones, to display the modulations of spin period and spin period derivative in both time and frequency domains. The precession modulates the pulse width as well, which provides a good way to probe the beam shape of the pulsar radiation (Link & Epstein 2001; Desvignes et al. 2019). We use a simple cone model (Gil et al. 1984; Lorimer & Kramer 2005) to study the pulse-width modulation of triaxially-deformed freely-precessing NSs, and investigate the change of pulse width with different choices of wobble angles.

From the GW perspectives, precessing NSs have been recognized as potential sources of continuous GWs for decades (Zimmermann & Szedlnitz 1979; Zimmermann 1980; Alpar & Pines 1985). In the new era of GW astronomy, the detection of GWs from precessing NSs with ground-based detectors is imminent. Using the LIGO data from its first and second observing runs, the search of continuous GWs at once and twice rotation frequencies from 222 pulsars has been performed (Abbott et al. 2019b). Stringent upper limits are set on the GW amplitude, the fiducial ellipticity, and the mass quadrupole moment via the search at the twice of the rotation frequency. These results can be used for testing various alternatives to the General Relativity (e.g., Xu et al. 2020).

Zimmermann (1980) treated precessing triaxial NSs as rigid bodies and derived the quadrupole waveform for them. In addition, he simplified the waveform assuming a small wobble angle, and showed that the spectral lines of the continuous GWs are located at angular frequencies of $\Omega_r + \Omega_p$ and $2\Omega_r$, where $\Omega_r$ is the rotation angular frequency, and $\Omega_p$ is the free precession angular frequency of the NS. The first-order spectral lines yield little information about other physical properties of NSs beyond the rotation and precession frequencies. Based on Zimmermann (1980), Van Den Broeck (2005) obtained a third angular frequency at $2(\Omega_r + \Omega_p)$ by expanding the waveform to the second order of the wobble angle. The feasibility to detect continuous GWs from precessing triaxial NSs was re-examined, and it is found that the deviation from axisymmetry, the oblateness, and the wobble angle can be determined if the second-order line is observed (Van Den Broeck 2005).

Following Zimmermann (1980) and Van Den Broeck (2005), in this work we use a Newtonian treatment for the precession, augmented with the GW radiation formalism in GR (Misner et al. 1973). In this problem, the Newtonian treatment is indeed also valid for strong-field objects like NSs, if the GR expressions for the integrals of various moments are used (Thorne 1980). Similarly to Van Den Broeck (2005), we expand the GW waveform to the second order of the precessing angle. However, unlike Van Den Broeck (2005), where a hierarchy for small parameters is assumed, more generically we treat the deviation from axisymmetry as a small parameter independent of the wobble angle in the expansion. Consequently, we obtain three more frequencies in the continuous GW spectra that are useful for a more complete extraction of physical information.

The structure of this paper is as follows. In Section 2, we provide both analytical and numerical solutions for freely precessing triaxial rigid bodies. Estimations of the oblateness, the nonaxisymmetry, and the wobble angle for elastically deformed NSs are given based on existing literature. In Section 3, we show the timing residuals and pulse-width modulations of precessing triaxial NSs. These features could be identified if the NS is observed as a radio and/or X-ray pulsar. In Section 4, after briefly reviewing the quadrupole formula in Zimmermann (1980), we expand the waveform to the second order assuming a small wobble angle, a small nonaxisymmetry and a small oblateness. Because of the relaxation in the assumption about the small quantities, three new spectral lines are obtained with respect to previous studies. In Section 5, we discuss the extraction of physical information of NSs from radio signals and continuous GWs. We briefly summarize our work in Section 6.

2 FREE PRECESSION OF TRIAXIAL RIGID BODIES

In general, the rotation axis of a rigid body does not coincide with its principal axes. As a consequence, a freely rotating rigid body precesses around the direction of the total angular momentum (Landau & Lifshitz 1960). The motion of the body can be described by three Euler angles, $\theta$, $\phi$, and $\psi$, and their time derivatives. In Fig. 1 we denote the coordinates of the inertial reference frame by upper-case letters, $X$, $Y$, and $Z$, with unit basis vectors $\hat{e}_X$, $\hat{e}_Y$, and $\hat{e}_Z$. The vector $\hat{e}_Z$ is chosen to be in the direction of the angular momentum of the rigid body, $\mathbf{J}$. We use lowercase letters, $x_1$, $x_2$, and $x_3$, to denote the coordinates in the body frame, which is attached to the rigid body. Their unit basis vectors are $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$, chosen to be parallel to the three individual eigenvectors.
of the moment of inertia tensor. We use \( I_1, I_2, \) and \( I_3 \) as the diagonal components of the moment of inertia tensor in the body frame.

For freely precessing rigid bodies, the dynamical equations of motion in the body frame are (Landau & Lifshitz 1960),

\[
\begin{align*}
I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 &= 0, \\
I_2 \dot{\omega}_2 - (I_1 - I_3) \omega_3 \omega_1 &= 0, \\
I_3 \dot{\omega}_3 - (I_1 - I_2) \omega_1 \omega_2 &= 0,
\end{align*}
\]

where \( \omega_1, \omega_2, \) and \( \omega_3 \) represent the angular velocities along \( \hat{e}_1, \hat{e}_2, \) and \( \hat{e}_3. \) The dots denote the derivatives with respect to time \( t. \) Considering the kinematics of the rigid body, the evolution of the three Euler angles can be described by (Landau & Lifshitz 1960),

\[
\begin{align*}
\omega_1 &= \dot{\theta} \sin \phi \sin \psi + \dot{\phi} \cos \psi, \\
\omega_2 &= \dot{\phi} \sin \theta \cos \psi - \dot{\psi} \sin \phi, \\
\omega_3 &= \dot{\phi} \cos \theta + \dot{\psi},
\end{align*}
\]

where the Euler angles are defined in Fig. 1.

As the rigid body is torque free, both the kinetic energy

\[
E = \frac{1}{2} \left( I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 \right),
\]

and the angular momentum

\[
J = \left( I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 \right)^{1/2},
\]

are conserved. In the following, we assume that the principal moments of inertia satisfy \( I_1 < I_2 < I_3. \) We also assume \( J^2 > 2EI_2, \) which is equivalent to the condition that the tail of the angular momentum \( J \) moves around \( \hat{e}_3 \) along a closed curve in the body frame (Landau & Lifshitz 1960). Results for other choices can be obtained by properly relabeling the indices.

The motion of a rigid body described by Eqs. (1–6) is an initial value problem. In principle, one can obtain the evolution of the orientation of the triaxial rigid body at any time once the initial values of the three Euler angles and the angular velocities are specified. In subsequent calculations, we choose the initial values such that at \( t = 0, \) one has \( \phi = 0, \) \( \psi = \pi/2, \) and \( \theta \) is at its minimum value \( \theta_{\text{min}}. \) The initial values of the angular velocities in the body frame are denoted as \( \omega_1 = a, \omega_2 = 0, \) and \( \omega_3 = b \) at \( t = 0. \) These assumptions can easily be extended to generic cases. Worth to stress that, throughout the calculation, we have assumed that the moments \( I_i \) \( (i = 1, 2, 3) \) are constant and the body undergoes free precession. We have ignored the damping of the precession due to some physical processes for NSs (e.g., the fluid dynamics of the NS interior). Now, we discuss the analytical solution and the numerical method to solve the equations of motion in Eqs. (1–6).

2.1 Analytical solution

The exact analytical solution to Eqs. (1–6) for a precessing triaxial rigid body has been obtained in terms of the elliptic functions (Landau & Lifshitz 1960; Zimmermann 1980; Whittaker 1988; Shakura et al. 1998; Van Den Broeck 2005; Akgun et al. 2006; Pina 2015; Lasky & Melatos 2013). Here we briefly review the solution according to Landau & Lifshitz (1960) for readers’ convenience.

The angular velocities in the body frame are,

\[
\begin{align*}
\omega_1(t) &= a \, \text{cn}(\tau, m), \\
\omega_2(t) &= a \left( \frac{I_1(I_3 - I_2)}{I_2(I_1 - I_2)} \right)^{1/2} \, \text{sn}(\tau, m), \\
\omega_3(t) &= b \, \text{dn}(\tau, m),
\end{align*}
\]

where \( \tau \) is the dimensionless time variable,

\[
\tau = t \sqrt{\frac{(I_2 - I_3)(J^2 - 2EI_2)}{I_1I_2I_3}},
\]

and \( \text{sn}, \text{cn}, \) and \( \text{dn} \) are the elliptic functions (see e.g., Olver et al. 2010). The parameter \( m \) can be expressed as,

\[
m = \frac{(I_2 - I_1)I_1a^2}{(I_2 - I_3)I_3b^2}.
\]

The angular velocities in the body frame are periodic with a period,

\[
T = \frac{4K(m)}{b} \left( \frac{I_1I_2}{(I_1 - I_2)(I_3 - I_2)} \right)^{1/2},
\]

where \( K(m) \) is the complete elliptic integral of the first kind (Olver et al. 2010). The period \( T \) is the free precession period. If \( I_1 \) is nearly equal to \( I_2, \) the parameter \( m \) is close to zero. In this case, the period \( T \) is approximately \( 2\pi I_1/[\omega_3(I_3 - I_2)] \), which is the well-known free precession period for a biaxial body.

The Euler angles \( \theta \) and \( \psi \) are also periodic, and can be expressed as,

\[
\begin{align*}
\cos \theta &= \frac{I_3b}{J} \, \text{dn}(\tau, m), \\
\tan \psi &= \left( \frac{I_1(I_3 - I_2)}{I_2(I_3 - I_1)} \right)^{1/2} \frac{\text{sn}(\tau, m)}{\text{cn}(\tau, m)}.
\end{align*}
\]

From the above two equations, one finds that the angle \( \theta \)
has a period of $T/2$, while the angle $\psi$ has a period of $T$. In contrast, the angle $\phi$ is not periodic. It can be represented as a sum of two parts, $\phi = \phi_1 + \phi_2$. The “periodic” part $\phi_1$ has a period of $T/2$, and is defined via,

$$\exp[2i\phi(t)] = \frac{\theta_a(\frac{2\pi t + \iota \alpha a}{T})}{\theta_a(\frac{2\pi t - \iota \alpha a}{T})},$$  

(17)

where $\theta_a$ is the fourth Jacobi theta function with nome $q = \exp[-\pi \kappa(1 - m) / K(m)]$. In Eq. (17), $a$ is determined via

$$\sin[2\alpha K(m)] = \frac{U_1 b}{I_1 a}.$$  

(18)

The “linear-in-time” part $\phi_2$ is given by,

$$\phi_2 = \frac{2\pi t}{T} = \left(\frac{J}{I_1} + \frac{2\pi \theta_3^\prime(\iota \alpha a, q)}{\theta_3(\iota \alpha a, q)}\right) t,$$

(19)

where $\theta_3^\prime(u, q)$ is the derivative of $\theta_3(u, q)$ with respect to $u$.\footnote{Note that the solution of the Euler angle $\phi$ in Landau & Lifshitz (1960) and Zimmermann (1980) has sign typos when the following theta function and its derivative (Whittaker 1988) is used as they claimed. The fourth Jacobi theta function is defined as $\theta_4(u, q) = 4 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2\pi mu)$, and the derivative of $\theta_4$ with respect to $u$ is $\theta_4^\prime(u, q) = 4 \pi \sum_{n=1}^{\infty} (-1)^n q^{n^2} \sin(2\pi nu)$.}

As $I_1$ approaches $I_2$, the period $T_1$ in Eq. (19) approaches $2\pi I_1 / J$. Generally, $T_1$ and $T_2$ are not commensurate with each other, so the motion of the body is not periodic in the inertia frame. For simplicity, we define (Zimmermann 1980; Van Den Broeck 2005)

$$\Omega_0 = \frac{2\pi}{T} = \frac{\pi \theta_2}{2K(m)} \left[(I_3 - I_2)(I_1 - I_1)\right]^{1/2},$$

(20)

$$\Omega_1 = \frac{2\pi}{T} = \frac{J}{I_1} + \frac{2\pi \theta_3^\prime(\iota \alpha a, q)}{\theta_3(\iota \alpha a, q)} - \Omega_0,$$

(21)

for later use.

2.2 Numerical approach using quaternions

Although the analytical solution given in the above subsection is exact, the use of it is not intuitive. Here we discuss a numerical method to integrate the equations of motion. There are two reasons for the need of a numerical method. First, numerical methods can avoid the use of the elliptic functions. Second, they can be easily applied to general differential equations that govern the time evolution of the quaternion $q$ with the Euler angles through

$$q_0 = \cos \frac{\theta}{2} \cos \left(\frac{1}{2}(\phi + \psi)\right),$$

(25)

$$q_1 = \sin \frac{\theta}{2} \cos \left(\frac{1}{2}(\phi - \psi)\right),$$

(26)

$$q_2 = \sin \frac{\theta}{2} \sin \left(\frac{1}{2}(\phi - \psi)\right),$$

(27)

$$q_3 = \cos \frac{\theta}{2} \sin \left(\frac{1}{2}(\phi + \psi)\right).$$

(28)

In addition, the differential equation that governs the time evolution of the quaternion $q$ can be expressed as (Coutsias & Romero 2004; Betsch & Siebert 2009)

$$\frac{dq}{dt} = -\frac{1}{2} q(0, \omega) \begin{pmatrix} 0 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & -\omega_3 \\ -\omega_2 & \omega_3 & 0 \end{pmatrix} q(t).$$

(29)

Once the initial orientation of the rigid body is given, one can translate it into the initial value of the quaternion $q$ using Eqs. (25–28). Together with the initial values of $\omega_1$, $\omega_2$, and $\omega_3$, Eqs. (1–3) and Eq. (29) can be integrated to obtain the angular velocities in the body frame and the time evolution of the quaternion $q(t)$, hence the elements of the rotation matrix $\mathcal{R}$ at any given time. The Euler angles can be recovered by the elements of the matrix $\mathcal{R}$ via

$$\phi = \tan^{-1}\left(-\frac{\mathcal{R}_{13}}{\mathcal{R}_{33}}\right),$$

(30)

$$\theta = \tan^{-1}\left(-\frac{\mathcal{R}_{13}}{\mathcal{R}_{33}}\right),$$

(31)

$$\psi = \tan^{-1}\left(-\frac{\mathcal{R}_{31}}{\mathcal{R}_{32}}\right),$$

(32)

where $\mathcal{R}_{ij}$ ($i, j = 1, 2, 3$) is the component of the matrix $\mathcal{R}$. 

A quaternion, $q = q_0 + q_1 i + q_2 j + q_3 k$, in the quaternion basis, $\{1, i, j, k\}$, is usually denoted as $q = (q_0, q)$. with $q = (q_1, q_2, q_3)$ being a 3-vector when used in calculating 3-dimensional rotations. The rotation transformation from $r$ to $r'$ is performed via (Coutsias & Romero 2004)

$$\mathbf{R}(\mathbf{q}) = \mathbf{R}(0, \mathbf{r}) \mathbf{q} \mathbf{R}(0, \mathbf{r})^{-1} = \mathbf{R}(0, \mathbf{r}) \\ = \left(0, (q_0^2 - q \cdot q) \mathbf{r} + 2 q (q \cdot \mathbf{r}) + 2 q_0 q \times \mathbf{r}\right).$$

(33)

where $\mathbf{q} = (q_0, -q)$ is the conjugate quaternion of $q$. Note that from the first to the second line we have used the multiplication rule of quaternions, and from the second to the third line the usual dot product and cross product of 3-vectors are applied. Explicitly, the rotation matrix $\mathcal{R}$ in Eq. (23) is

$$\mathcal{R} = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 + 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix},$$

(24)

which equals the normal Euler rotation matrix. Comparing the elements of them, we can relate the quaternion $q$ with the Euler angles through

$$\mathcal{R} = \begin{pmatrix} 0 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & -\omega_3 \\ -\omega_2 & \omega_3 & 0 \end{pmatrix}.$$
parameter $\eta$ is unity for a perfectly rigid star and zero for a liquid star. By assuming the shear modulus in the crust to be constant, the rigidity parameter for a NS with a liquid core can be approximated as (Baym & Pines 1971)

$$\eta \approx \frac{57 \mu V_c}{10 |E_G|} \approx 2.3 \times 10^{-5} \left(\frac{\mu}{10^{30} \text{erg cm}^{-3}}\right) R_6^3 M_{1.4}^{-2},$$

(37)

where $\mu$ is the shear modulus of the crust, $V_c$ is the volume of the crust, and $E_G = -3GM^2/5R$ is the gravitational binding energy of the NS. The notations $M_{1.4}$ and $R_6$ represent the dimensionless NS mass $M_{1.4} \equiv M/(1.4M_{\odot})$ and the dimensionless NS radius $R_6 \equiv R/(10^6 \text{cm})$. Cutler et al. (2003) adopted a relativistic NS structure and solved for the strain in the crust which evolves as the NS spins down. They found that $\eta$ is smaller than the estimation in Baym & Pines (1971) by a factor of $\sim 40$. The estimation of the zero-strain oblateness $\epsilon_0$ is (Cutler et al. 2003; Van Den Broeck 2005),

$$\epsilon_0 \approx \frac{\Omega^2 R^3}{GM} = 2.1 \times 10^{-3} \left(\frac{f}{100 \text{ Hz}}\right)^2 R_6^3 M_{1.4}^{-1} \cdot$$

(38)

where $f = \Omega / 2\pi$ is the spin frequency of the NS, and $\Omega$ is defined in Eq. (21). Combining Eqs. (36–38), we obtain the oblateness due to the elastic deformation

$$\epsilon_{\text{elas}} \approx 4.9 \times 10^{-8} \left(\frac{f}{100 \text{ Hz}}\right)^2 \left(\frac{\mu}{10^{30} \text{erg cm}^{-3}}\right) R_6^3 M_{1.4}^{-3} \cdot$$

(39)

The oblateness for a NS due to elastic deformation is also limited by the breaking strain $\sigma_{\text{break}}$. According to Owen (2005), the largest oblateness is

$$\epsilon_{\text{max}} = 3.4 \times 10^{-7} \left(\frac{\sigma_{\text{break}}}{10^2}\right)^{2} \left(M_{1.4}^{-2} R_6^{-1} + 0.7M_{1.4} R_6^{-1}\right)^{2} \cdot$$

(40)

The value of $\sigma_{\text{break}}$ is uncertain. Early estimations are in the range from $10^{-4}$ to $10^{-2}$ (Ruderman et al. 1992). Horowitz & Kadu (2009) found that $\sigma_{\text{break}}$ is around 0.1 by simulating the crust as Coulomb solids. Recent semi-analytical lattice studies of Baiko & Chugunov (2018) showed that $\sigma_{\text{break}}$ is more like to be 0.04. Note that with this value of $\sigma_{\text{break}}$, the largest oblateness is about $\epsilon_{\text{max}} = 8 \times 10^{-7}$.

Now we turn to the nonaxisymmetry. NSs are biaxial when $\delta$ is zero or infinity, and triaxial when $\delta$ has a finite value. Due to the complex evolution and relaxation of the crust after the star’s birth and during the accretion in the late lifetime (Link et al. 1998; Shakura et al. 1998; Link 2003; Akgun et al. 2006), deformed NSs are possible to be triaxial, characterized by finite values of $\delta$. The magnetic stresses might contribute to the triaxility as well (Wasserman 2003). The nonaxisymmetry depends on the evolution and relaxation of the crust and the magnetic stresses, which are complex, especially during dynamical or explosive processes. Because of our lack of knowledge about $\delta$, a measurement of it would be particularly exciting.

As for the wobble angle $\theta$, there is no physical limitation for slowly rotating NSs. However, for a fast rotating one, as the rotational bulge of the NS turns larger, more matter needs to be displaced during the precession, leading to a larger crust strain (Jones & Andersson 2001; Van Den Broeck 2005). In order to keep the strain below the limit of $\sigma_{\text{break}}$, the star can only possess a small wobble angle. Jones & Andersson (2001) estimated the maximum allowed
wobble angle,
\[ \theta_{\text{max}} \approx 0.45 \left( \frac{100 \text{ Hz}}{f_e} \right)^2 \left( \sigma_{\text{break}} \right) 10^{-3} M_{1.4} R_{\text{g}}^{-3}. \] (41)

The constraint on the wobble angle depends on the rotation frequency and the breaking strain \( \sigma_{\text{break}} \). For a NS with \( f_e = 100 \text{ Hz} \) and \( \sigma_{\text{break}} = 10^{-3} \), the wobble angle is smaller than 0.45 radians. If we take the breaking strain in the extreme case where \( \sigma_{\text{break}} = 0.1 \), the wobble angle is basically unlimited even for a fast rotating NS at a spin frequency of \( f_e = 500 \text{ Hz} \).

For the theoretical analysis in subsequent sections, we apply series expansion for the trigonometric functions of the three Euler angles in Section 2.1, assuming a small oblateness, a small nonaxisymmetry, and a small wobble angle. The benefit of the perturbative treatment is the great simplification it brings and the explicit harmonics appearing in the spectra. As for generic cases when one or more of these parameters are large, one can always restore back to the exact solution (or the numerical scheme) for a careful check.

Following Zimmermann (1980) and Van Den Broeck (2005), we find that practically, it is more convenient to use\( \kappa \) than \( \delta \) in the expansion. Up to the leading order, \( \kappa \) and \( \delta \) are related by \( \kappa \approx \delta/16 \). A constant of \( 1/16 \) is included for the convenience of later computation (Van Den Broeck 2005). The parameter \( m \) in Eq. (13) is then simplified to \( m = 16x\gamma^2 \). In the series expansion, different from Van Den Broeck (2005), we treat \( \gamma \) and \( \kappa \) independently and do not assume any hierarchy between them. The series expansions of trigonometric functions of the three Euler angles up to the second order of \( \gamma \) and \( \kappa \) are,

\[ \cos \phi = \cos \left( \Omega_p \tau \right), \]
\[ \sin \phi = \sin \left( \Omega_p \tau \right), \]
\[ \cos \theta = 1 - \frac{y^2}{2}, \]
\[ \sin \theta = y + 8xy \sin^2 \left( \Omega_p \tau \right), \]
\[ \cos \psi = \sin \left( \Omega_p \tau \right) \left[ 1 + (8x + 32x^2) \cos^2 \left( \Omega_p \tau \right) \right] \]
\[ \quad + \sin \left( \Omega_p \tau \right) \left[ 16x^2 \cos^2 \left( \Omega_p \tau \right) (3 \cos (2\Omega_p \tau) + 1) \right], \]
\[ \sin \psi = \cos \left( \Omega_p \tau \right) \left[ 1 - (8x + 32x^2) \sin^2 \left( \Omega_p \tau \right) \right] \]
\[ \quad - \cos \left( \Omega_p \tau \right) \left[ 96x^2 \sin^2 \left( \Omega_p \tau \right) \cos^2 \left( \Omega_p \tau \right) \right]. \] (46)

In subsequent sections, we apply the analytical solution in Eqs. (9–19) and above estimation for the oblateness, the nonaxisymmetry, and the wobble angle to investigate the timing behavior and the GW radiation of precessing triaxial NSs. Note that the internal, fluid-dynamical mechanisms that might reorient the NS over time are not included in the calculation. They should exist in general, but depend on the equation of state of NSs (see e.g. Xu 2003, for solid strange stars). In addition, the magnetic field evolution or pulsation will lead to time-dependent changes in the oblateness and the nonaxisymmetry as well.

The braking torques (Goldreich 1970) and the gravitational radiation reaction (Cutler & Jones 2001) are also omitted for simplicity, which will make the spin frequency decrease and change the wobble angle. For very young and/or highly-magnetized NSs, this might be quite important. Magnetic pressures can significantly adjust the precession period and lead to a nonzero time derivative for the free precession angular frequency, \( \dot{\omega}_p \), because the oblateness might evolve rapidly and the NS spins down much faster due to large spin-down torques (Zanazzi & Lai 2020; Levin et al. 2020; Suvorov & Kokkotas 2020). For some systems, the braking torques and the gravitational radiation reaction may become important over long timescales. The details are beyond the scope of this paper and deserve further investigation.

3 MODULATED TIMING AND PULSE SIGNALS

If a triaxially-deformed freely-precessing NS is observed as a pulsar, the free precession will introduce characteristic modulations on the timing and pulse signals. These modulations might be revealed by radio and/or X-ray observations. In Section 3.1, we discuss the phase modulations of precessing triaxial NSs and show the residuals of spin period and spin period derivative for different initial values of the wobble angle. In Section 3.2, the pulse-width modulations for different initial values of the wobble angle are displayed.

3.1 Phase modulation

Following Jones & Andersson (2001), we assume for simplicity that the pulsar beam is in the same direction as the magnetic dipole moment \( \hat{m} \). Once the dipole moment sweeps through the plane defined by the line of sight and the spin angular momentum, a pulse can be observed. In Fig. 3, we show the geometry of a freely precessing triaxial NS. We denote the polar angle between \( \hat{e}_Z \) and \( \hat{m} \) as \( \Theta \). We denote the

\[
\theta_{\text{max}} \approx 0.45 \left( \frac{100 \text{ Hz}}{f_e} \right)^2 \left( \sigma_{\text{break}} \right) 10^{-3} M_{1.4} R_{\text{g}}^{-3}. \]

azimuthal angle between \( \tilde{r}_\chi \) and the projection of \( \tilde{m} \) on the X-Y plane as \( \Phi \). It is related to the Euler angles via (Jones & Andersson 2001),

\[
\Phi = \phi - \frac{\pi}{2} + \arctan \left( \frac{\cos \psi \sin \chi}{\sin \theta \cos \chi - \sin \psi \sin \chi \cos \theta} \right),
\]

where \( \chi \) is the magnetic inclination angle between \( \tilde{r}_\chi \) and \( \tilde{m} \).

The time derivative of \( \Phi \) is the instantaneous spin angular frequency of the NS. Its time-averaged value corresponds to the mean spin angular frequency obtained in the observation. Note that for a precessing triaxial NS, all of the three Euler angles change with time. Especially, the wobble angle varies in a range from its minimum value \( \theta_{\text{min}} \) to the maximum, \( \theta_{\text{max}} \). We discuss separately the two situations for \( \theta_{\text{min}} > \chi \) and \( \theta_{\text{max}} < \chi \) to obtain the timing residual of the precessing NS.

- When \( \theta_{\text{min}} > \chi \), the time-averaged spin frequency of the NS is \( \phi \). Therefore, the precession-induced phase residual is (Jones & Andersson 2001)

\[
\Delta \Phi = \Phi - \left( \phi - \frac{\pi}{2} \right) = \arctan \left( \frac{\cos \psi \sin \chi}{\sin \theta \cos \chi - \sin \psi \sin \chi \cos \theta} \right).
\]

- When \( \theta_{\text{max}} < \chi \), the time-averaged spin frequency is \( \phi + \psi \), and the precession-induced phase residual is (Jones & Andersson 2001)

\[
\Delta \Phi = \Phi - (\phi + \psi) = \arctan \left( \frac{(\cos \theta - 1) \sin \psi \sin \chi - \sin \theta \cos \chi}{\cos \psi \sin \chi + (\cos \theta \sin \psi \sin \chi - \sin \theta \cos \chi) \tan \psi} \right).
\]

The precession-induced residuals of the spin period, \( P \), and the spin period derivative, \( \dot{P} \), can be calculated using the time derivatives of the precession-induced phase residual,

\[
\Delta P = -\frac{P_0^2}{2\pi} \Delta \Phi, \quad \dot{\Delta P} = -\frac{P_0^2}{2\pi} \Delta \dot{\Phi},
\]

where \( P_0 \) is the mean spin period of the NS. Substituting the time-evolving Euler angles into Eqs. (50–53), one can obtain the modulations of the spin period and the spin period derivative for different choices of parameters.

In Fig. 4 and Fig. 5 we respectively present examples for cases of a large wobble angle where \( \theta_{\text{min}} > \chi \), and a small wobble angle where \( \theta_{\text{max}} < \chi \). In the calculation, we take the magnetic inclination angle \( \chi = \pi/6 \), the oblateness \( \epsilon = 4.9 \times 10^{-8} \), the nonaxisymmetry parameter \( \delta = 0.1 \), and a wobble angle in the range \( \theta \in (0.79, 0.84) \). With these parameters, we have \( T_1 = 0.010 \) s, and a free precession period \( T = 3.1 \times 10^5 \) s.

![Figure 4.](image)

Figure 4. (Upper) Precession-induced residuals of the spin period and the spin period derivative for a large wobble angle where \( \theta_{\text{min}} > \chi \). (Lower) The Fourier amplitude for the spin period residual. In this figure, we have chosen the magnetic inclination angle \( \chi = \pi/6 \), the oblateness \( \epsilon = 4.9 \times 10^{-8} \), the nonaxisymmetry parameter \( \delta = 0.1 \), and a wobble angle in the range \( \theta \in (0.79, 0.84) \). With these parameters, we have \( T_1 = 0.010 \) s, and a free precession period \( T = 3.1 \times 10^5 \) s.

amplitude is shown in the lower panel of Fig. 5. Notice that a logarithmic scale is used for the Fourier amplitude. Compared to the case of a large wobble angle, the harmonics at \( n\Omega_p/2\pi \) (\( n \geq 2 \)) are much weaker than the line at \( \Omega_p/2\pi \) (now, \( \Omega_p = 3.0 \times 10^{-5} \) s\(^{-1} \)) in the case of a small wobble angle. In the small wobble angle limit, the precession-induced spin phase residual can be approximated as (Jones & Andersson 2001; Link & Epstein 2001)

\[
\Delta \Phi = -\sin \theta \cot \chi \cos \psi - \frac{1}{4} \sin^2 \theta (1 + 2 \cot^2 \chi) \sin 2\psi.
\]

Applying the series expansion of the Euler angles in Eqs. (43–48) and using Eq. (51), the spin period residual is given by

\[
\Delta P = \frac{P_0^2}{2\pi} \Omega_p \gamma (8\kappa + 1) \cot \chi \cos (\Omega_p t)
\]

\[
+ \frac{P_0^2}{4\pi} \Omega_p \gamma^2 \left( 1 + 2 \cot^2 \chi \right) \cos (2\Omega_p t).
\]

It shows that at the second order of the wobble angle, the modulation of the spin period includes the first and the second harmonics of the free precession angular frequency \( \Omega_p \), corresponding to the first two peaks in the lower panel of Fig. 5. The third peak comes from higher-order terms that
are not included in the approximation. The residual of the spin period derivative $\Delta P$ can be obtained by taking the time derivative of $\Delta P$, which gives

$$
\frac{\Delta P}{dP} = \frac{P^2}{\Omega^2} \frac{\sin(2\kappa + 1) \cot \chi \sin (\Omega_T)}{2} 
- \frac{P^2}{\Omega^2} \frac{\sin^2(\beta/2) \sin(2\Omega_T)}{\sin(\Theta + \beta) \sin \Theta} .
$$

When $\kappa = 0$, Eqs. (55–56) reduce to the corresponding results for a precessing biaxial NS (Link & Epstein 2001). Note that we have ignored the intrinsic spindown of NSs for simplicity in deriving Eqs. (55–56).

### 3.2 Pulse-width modulation

In order to analyse the pulse-width modulation, we adopt a simple cone model to describe the radiation of a pulsar. For a more complicated radiation geometry, our method can be extended as well. In the cone model, from the geometry in Fig. 6 we have (Gil et al. 1984; Lorimer & Kramer 2005,

$$
\sin^2 \left( \frac{W}{4} \right) = \frac{\sin^2(\rho/2) - \sin^2(\beta/2)}{\sin(\Theta + \beta) \sin \Theta} ,
$$

where $\Theta$ is defined in Fig. 3, and $W, \rho, \beta$ are defined in Fig. 6.

Equation (57) is not exact for the pulse width because in our case the angle $\Theta$ changes with time. However, as the spin frequency is much higher than the free precession frequency, the change in $\Theta$ during a spin period is negligible, thus this approximation is good enough for our calculation. In this cone model, the observer can observe the pulse signal once the line of sight enters into the emission cone. The variation of pulse width can be determined by the angle $\Theta$ once the inclination angle (denoted as $\iota$) between the angular momentum and the line of sight to the NS is determined. The angle $\Theta$ can be expressed through the Euler angles and the magnetic inclination angle via (Jones & Andersson 2001; Link & Epstein 2001),

$$
\cos \Theta = \sin \theta \sin \psi \sin \chi + \cos \theta \cos \chi .
$$

In Fig. 7 we present examples for the pulse-width modulation with a large wobble angle where $\theta_{\text{min}} > \chi$ and a small wobble angle where $\theta_{\text{max}} < \chi$. The choices of the oblateness, the nonaxisymmetry, and the magnetic inclination angle are the same as in Fig. 4 and Fig. 5 for the large and small wobble angles, respectively. The example for the large wobble angle is displayed in the upper panel of Fig. 7. For our (extreme) choice of the parameters, the angle between the angular momentum and the dipole moment changes significantly, $\Theta \in (0.26, 1.31)$. As a consequence, the pulse width changes in a wide range with $W \in (0, 2.68)$. The line of sight...
leaves out of the emission cone due to the free precession during certain time ranges, and then the pulse width diminishes to zero accordingly. The modulation of pulse width in the case of a small wobble angle is shown in the lower panel of Fig. 7. In this case, the angle $\Theta$ is in the range of $\Theta \in (0.51, 0.54)$, and the change of pulse width is much milder with $W \in (2.14, 2.21)$.

In the case of a small wobble angle, applying the series expansions of $\theta$ and $\psi$ in Eqs. (43–48), the angle $\Theta$ can be approximated as

$$\Theta \approx \chi - \sin \theta \sin \psi \approx \chi - \gamma \cos (\Omega pt),$$

which reduces to the corresponding result for a precessing biaxial NS (Link & Epstein 2001).

4 CONTINUOUS GRAVITATIONAL WAVES

We discuss generic continuous GWs from a triaxially-deformed freely-precessing NS in Section 4.1, and the approximation to the waveform with small oblateness, a small wobble angle, and small nonaxisymmetry in Section 4.2. The results mainly follow Zimmermann (1980) and Van Den Broeck (2005), but we make further extensions by assuming no hierarchy in the three small parameters.

4.1 Generic waveform

We use the quadrupole approximation for the continuous GWs from freely precessing triaxial NSs. In the transverse-traceless (TT) gauge, the metric perturbation is (Misner et al. 1973)

$$h_{ij}^{TT} = \frac{2G}{r c^4} \frac{d^2 I_{ij}}{dt^2},$$

where $G$ is the gravitational constant, $c$ is the speed of light, $I_{ij}$ is the trace-free part of the moment of inertia tensor, and $r$ is the luminosity distance from the source to the observer.

Alternatively, the quadrupole formula in Eq. (60) can be expressed as (Zimmermann 1980)

$$h_{ij}^{TT} = -\frac{2G}{rc^4} R_{ik} R_{jl} A_{kl},$$

where $R$ is the rotation matrix (24), and $A_{kl}$ is determined by the body-frame angular velocities and the body-frame moments of inertia. For example, we have

$$A_{11} = 2 \left( \Delta_2 \omega_2^2 - \Delta_1 \omega_1^2 \right),$$

$$A_{12} = \left( \Delta_1 - \Delta_2 + \frac{\Delta_3^2}{I_3} \right) \omega_1 \omega_2,$$

where

$$\Delta_1 \equiv I_2 - I_3, \quad \Delta_2 \equiv I_3 - I_1, \quad \Delta_3 \equiv I_1 - I_2.$$

The other components of $A_{kl}$ can be obtained from Eqs. (62–63) by cyclic permutation of the indices.

The waveform in Eq. (61) is usually decomposed into $h_+$ and $h_\times$,

$$h_{ij}^{TT} = h_+ (\vec{e}_+)_{ij} + h_\times (\vec{e}_\times)_{ij},$$

where $h_+$ and $h_\times$ represent the radiation of the two independent polarizations. The polarization tensors, $\vec{e}_+$ and $\vec{e}_\times$, are

$$\vec{e}_+ = \vec{p} \otimes \vec{p} - \vec{q} \otimes \vec{q},$$

$$\vec{e}_\times = \vec{p} \otimes \vec{q} + \vec{q} \otimes \vec{p},$$

where $\vec{p}$ and $\vec{q}$ are two unit vectors with $\vec{p} \times \vec{q}$ in the propagation direction of the GWs. We assume that the observer lies in the Y-Z plane and define the inclination angle $\iota$ as the angle between the direction of the angular momentum $\vec{e}_Z$ and the line of sight to the NS. In the inertial frame, the unit vectors $\hat{p}$ and $\hat{q}$ are

$$\hat{p} = -\hat{e}_Y \cos \iota - \hat{e}_Z \sin \iota,$$

$$\hat{q} = -\hat{e}_X.$$

Combining Eq. (61) and Eqs. (65–69), the waveforms of the two polarizations are (Zimmermann 1980; Van Den Broeck 2005)

$$h_+ = -\frac{G}{rc^4} \left( R_{2k} \cos \iota + R_{3k} \sin \iota \right) (R_{2l} \cos \iota + R_{3l} \sin \iota) - R_{1k} R_{1l} A_{kl},$$

$$h_\times = -\frac{2G}{rc^4} \left( R_{2k} \cos \iota + R_{3k} \sin \iota \right) R_{1l} A_{kl}.$$

In Section 2, we have discussed the time evolution of the angular velocities in the body frame and the three Euler angles, so we can obtain $h_+$ and $h_\times$ at any given time $t$. In Fig. 8, we plot waveforms of $h_+$ and $h_\times$ in the time domain at different inclination angles. Note that the parameters that we have chosen for the plot are exaggerated for NSs for illustrative purposes. Physical parameters consistent with the estimates in Section 2.3 can be easily implemented, but the
effects will be too small for visual inspection. We also show
the Fourier transformation of the waveforms at the inclina-
tion angle of $\iota = \pi/6$ in Fig. 9. The peaks of the spectra are
dominantly at angular frequencies
\[ \Omega_r + n \Omega_p, \quad 2\Omega_r + n \Omega_p, \]
where $n = 0, \pm 1, \pm 2, \ldots$ is an integer number.

4.2 Waveform for small oblateness, small wobble
angle and small nonaxisymmetry
Following Zimmermann (1980) and Van Den Broeck (2005),
we investigate the waveforms in the limit of small oblateness
$\epsilon$, small wobble angle $\theta$, and small nonaxisymmetry $\delta$. The
difference between our work and the previous work is that,
instead of assuming a hierarchy between $\kappa$ and $\gamma$, namely
$\kappa \sim O(\gamma^2)$ in Van Den Broeck (2005), we treat $\gamma$ and $\kappa$
as small quantities independent to each other. As discussed
in Section 2.3, it is more plausible to assume no intrinsic
hierarchy between $\kappa$ and $\gamma$, in particular when the internal
structure of NSs is still rather uncertain.

The procedure to derive the expansion of the waveform
is as follows. First we expand $R_{ij}$ and the angular velocities
$\omega_1$, $\omega_2$, and $\omega_3$ to the second order of $\gamma$ and $\kappa$ using the
expansion of Euler angles in Eqs. (43–48). Second, we sub-
stitute the expansion of $R_{ij}$ and $A_{ij}$ into Eqs. (70–71). Third,
we retain the GW waveform to the second order of $\gamma$ and $\kappa$
and combine the trigonometric functions using trigonometric
identities. Such an extension with independent $\kappa$ and $\gamma$
parameters gives us more features than what was discovered
before.

With the above procedure, we obtain six components of
emitted at angular frequencies of NS (Zimmermann & Szedenits 1979). The continuous GW radiation is detected by the kilohertz laser-interferometric GW detectors, including LIGO (Aasi et al. 2015), Virgo (Acernese et al. 2015), and KAGRA (Akutsu et al. 2019). As we will see below, this should surely provide invaluable constraints on the NS structure, complementary to traditional observables, including masses, radii and tidal deformabilities of NSs.

Radio/X-ray signals and continuous GWs from precessing triaxial NSs will provide valuable information about the wobble angle, the nonaxisymmetry, and the oblateness of the source. These measurements are ultimately related to the long-standing question on the equation of state for supranuclear matters inside NSs (Lattimer & Prakash 2001). Below we take the case of a small wobble angle as an example to discuss the extraction of physical properties from such measurements (Van Den Broeck 2005).

For pulsar signals, the amplitude of the spin period residual in Eq. (55) at the frequencies of $\Omega_p$ and $2\Omega_p$ can be expressed as
\[
\Delta P_1 = 1.6 \times 10^{-10} \cot \Omega_p \left( \frac{P_0}{0.01 \text{ s}} \right)^2 \left( \frac{\Omega_p}{10^{-5} \text{ rad s}^{-1}} \right) (\gamma + \chi \gamma) \text{ s,}
\]
\[
\Delta P_2 = 8.0 \times 10^{-11} \left( 1 + 2 \cot^2 \chi \right) \left( \frac{P_0}{0.01 \text{ s}} \right)^2 \left( \frac{\Omega_p}{10^{-5} \text{ rad s}^{-1}} \right) \gamma^2 \text{ s,}
\]
where $\Delta P_1$ is the amplitude of the spin period residual at the frequency $\Omega_p$, and $\Delta P_2$ is the amplitude of the spin period residual at the frequency $2\Omega_p$.

The elliptic integral of the first kind $K(m)$ approaches $\pi/2$ in the small-wobble-angle and small-nonaxisymmetry limit, which leads to $\Omega_p \to 0\Omega_p$. Therefore, in such a limiting case, the precession angular frequency $\Omega_p$ can be approximated as (Van Den Broeck 2005)
\[
\Omega_p \approx \frac{\gamma}{2K(m)} t \Omega_c.
\]

5 DISCUSSIONS

The first detection of the coalescence of a binary NS system opened the avenue for multimessenger astrophysics (Abbott et al. 2017b,c,d). In this paper, we discuss another possibility to achieve multimessenger observation with electromagnetic and GW detectors, namely the observation of precessing NSs. Multimessenger astrophysics can be greatly advanced if a precessing NS is observed as a pulsar via radio and/or X-ray telescopes, and in the mean time, its continuous GW radiation is detected by the kilohertz laser-interferometric GW detectors, including LIGO (Aasi et al. 2015), Virgo (Acernese et al. 2015), and KAGRA (Akutsu et al. 2019). As we will see below, this should surely provide invaluable constraints on the NS structure, complementary to traditional observables, including masses, radii and tidal deformabilities of NSs.

Radio/X-ray signals and continuous GWs from precessing triaxial NSs will provide valuable information about the wobble angle, the nonaxisymmetry, and the oblateness of the source. These measurements are ultimately related to the long-standing question on the equation of state for supranuclear matters inside NSs (Lattimer & Prakash 2001). Below we take the case of a small wobble angle as an example to discuss the extraction of physical properties from such measurements (Van Den Broeck 2005).

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\]
\[
\Delta P_2 = 8.0 \times 10^{-11} \left( 1 + 2 \cot^2 \chi \right) \left( \frac{P_0}{0.01 \text{ s}} \right)^2 \left( \frac{\Omega_p}{10^{-5} \text{ rad s}^{-1}} \right) \gamma^2 \text{ s,}
\]
where $\Delta P_1$ is the amplitude of the spin period residual at the frequency $\Omega_p$, and $\Delta P_2$ is the amplitude of the spin period residual at the frequency $2\Omega_p$.

The elliptic integral of the first kind $K(m)$ approaches $\pi/2$ in the small-wobble-angle and small-nonaxisymmetry limit, which leads to $\Omega_p \to 0\Omega_p$. Therefore, in such a limiting case, the precession angular frequency $\Omega_p$ can be approximated as (Van Den Broeck 2005)
\[
\Omega_p \approx \frac{\gamma}{2K(m)} t \Omega_c.
\]

The free precession period $T$ can be directly obtained from the positions of spectral lines in the frequency domain of timing residuals. The wobble angle $\gamma$, the nonaxisymmetry $\kappa$, and the magnetic inclination angle $\chi$ cannot be fully determined with two spectral lines. But if the nonaxisymmetry is small enough, the second-order contribution to the amplitude of the first line can be ignored. Then the wobble angle $\gamma$ and the magnetic inclination angle $\chi$ can be determined.

The pulse-width modulations will provide important information on the beam shape of pulsars. In our work, we used a simple cone model (Gil et al. 1984) to describe the modulations of pulse width. We find that up to the second order, the pulse-width modulation is the same as that in the biaxial case. From the perspective of observation, if the pulse-width variations from precessing NSs are observed, the

\[ h_+, \text{ which are distinguished by different GW frequencies,}
\]
\[ h_1^+ = \frac{-GeyI_b^2 \sin 2t}{rc^4} \cos \left[ \Omega_t + \Omega_p t \right], \]
\[ h_2^+ = \frac{-\gamma GeyI_b^2 (1 + \cos^2 \ell)}{rc^4} \cos \left[ 2\Omega_t t \right], \]
\[ h_3^+ = \frac{2GeyI_b^2 (1 + \cos^2 \ell)}{rc^4} \cos \left[ 2\Omega_t + \Omega_p t \right], \]
\[ h_4^+ = \frac{-14GeyI_b^2 \sin 2t}{rc^4} \cos \left[ \Omega_t - \Omega_p t \right], \]
\[ h_5^+ = \frac{2GeyI_b^2 \sin 2t}{rc^4} \cos \left[ \Omega_t + 3\Omega_p t \right], \]
\[ h_6^+ = \frac{-128GeyI_b^2 (1 + \cos^2 \ell)}{rc^4} \cos \left[ 2\Omega_t - \Omega_p t \right]. \]

Similarly, we obtain six components of $h_\times$,
\[ h_1^\times = \frac{-6GeyI_b^2 \sin^2 t}{rc^4} \sin \left[ \Omega_t + \Omega_p t \right], \]
\[ h_2^\times = \frac{-4GeyI_b^2 \cos^2 t}{rc^4} \sin \left[ 2\Omega_t t \right], \]
\[ h_3^\times = \frac{28GeyI_b^2 \sin^2 t}{rc^4} \sin \left[ 2\Omega_t + \Omega_p t \right], \]
\[ h_4^\times = \frac{-4GeyI_b^2 \sin^2 t}{rc^4} \sin \left[ \Omega_t - \Omega_p t \right], \]
\[ h_5^\times = \frac{-256GeyI_b^2 \cos^2 t}{rc^4} \sin \left[ 2\Omega_t - \Omega_p t \right]. \]

Note that the components with frequencies $\Omega_t + \Omega_p$ and $2\Omega_t$ are the leading order contributions found in Zimmermann (1980), and that the one with frequency $2(\Omega_t + \Omega_p)$ is the third spectral line in Van Den Broeck (2005) once $k$ is treated as $O(\gamma^2)$.

Below we briefly discuss the waveform for different choices of $\gamma$ and $k$, and for presentation reasons, we leave their observational aspects to the next section, together with the possible radio/X-ray counterparts.

- When $\gamma = 0$ and $k \neq 0$, the NS does not precess and GWs are radiated at twice of the rotation frequency. The radiation is caused by the asymmetry between $I_1$ and $I_2$.
- When $k = 0$ and $\gamma \neq 0$, GWs are radiated at $\Omega_t + \Omega_p$ and $2(\Omega_t + \Omega_p)$. This is the classical result of a precessing biaxial NS (Zimmermann & Szedenits 1979).
- When $k \neq 0$ and $\gamma \neq 0$, the NS is a precessing triaxial body. At the first order of $\gamma$ and $k$, continuous GWs are emitted at angular frequencies of $\Omega_t + \Omega_p$ and $2\Omega_t$ (Zimmermann 1980). Previously, Van Den Broeck (2005) treated $k$ as small as $\gamma^2$ and got a new line. It is at the second order of $\gamma$, but still at the first order of $k$. The continuous GWs corresponding to this spectral line have an angular frequency of $2(\Omega_t + \Omega_p)$. Here in our work, we treat $k$ and $\gamma$ independently and expand the waveform to the second order that includes $\gamma^2$, $k^2$, and $\gamma k$. We find three new spectral lines at angular frequencies of $\Omega_t - \Omega_p$, $\Omega_t + 3\Omega_p$, and $2(\Omega_t - \Omega_p)$. We consider it a natural extension of the results in Van Den Broeck (2005).
beams shape can be inferred via different cuts by the line of
sight (Link & Epstein 2001).

From above discussions, we find that the inclusion of the
nonaxisymmetry of NS only slightly changes the timing
residuals and the pulse width compared with the biaxial
results. The reason is that the parameter $m = 16\epsilon\gamma^2$ plays
an important role in determining the behavior of free pre-
cession. As $m$ approaches zero, the biaxial approximation is
robust. Even for a large nonaxisymmetry, if the wobble an-
gle $\gamma < 1$, the dynamics of the NS still only deviates from
the biaxial one slightly. In the case of large wobble angles
and large nonaxisymmetries, the parameter $m$ can be of
order unity. Then the amplitudes of the harmonics are cor-
respondingly large in timing residuals. In this case, if the angle
between the beam and the line of sight changes during the
free precession, the observer might lose the radiating beam
when the line of sight does not cut the radiating region (see
the upper panel in Fig. 7). Some pulsars display episodes of
interpulsing (i.e. a pulse occurring midway between success-
seive main pulses). The existence of interpulses in a specific
system in principle leads to a constraint on the orientation
between the magnetic and rotation axes (Akgun et al. 2006),
which may give us more information on the pulsar geom-
eties and pulse profiles. More exploration along this line is
worthwhile.

For active pulsars, magnetospheric processes may af-
fect the pulse signals from precessing NSs and make the
interpretation of free precession complicated. For example,
the precession may itself introduce changes on the emission
geometry and the activities of the magnetosphere (Link &
Epstein 2001). Besides, the changes of the emission height
can contribute to pulse arrival time (Link & Epstein 2001).
Depending on the properties of the observed pulsars, these
complications need to be considered. Compared to the radio
signals, the X-ray signals are hardly affected by dispersion
and scattering during the propagation. The NICER mission
can give phase-resolved X-ray spectroscopy for pulsars (Ri-
ley et al. 2019; Bilous et al. 2019). High-precision X-ray tim-
ing for millisecond pulsars have also been conducted (Den-
eneva et al. 2019). In the near future, NICER X-ray timing of pul-
sars may help to track the precession of NSs better.

As a new observation window, GWs from precessing
NSs can give complementary physical information on these
trialaxial NSs. Following Van Den Broeck (2005), we present
the procedures to extract physical parameters from contin-
uous GWs. We take the “$x$” mode as an example, and the
discussion for the “$z$” mode is similar. For the “$x$”-polarized
GW, the amplitudes of the first-order lines at $\Omega_p + \Omega_3$ and
$
\Omega_p + 2\Omega_3$ are

$$A_x^1 = 1.0 \times 10^{-28} \gamma \sin \left( \frac{\epsilon}{4.9 \times 10^{-8}} \right) \left( \frac{f_c}{100 \text{ Hz}} \right)^2 \left( \frac{10 \text{ kpc}}{r} \right),$$

$$A_x^2 = 3.3 \times 10^{-27} \kappa \cos \left( \frac{\epsilon}{4.9 \times 10^{-8}} \right) \left( \frac{f_c}{100 \text{ Hz}} \right)^2 \left( \frac{10 \text{ kpc}}{r} \right),$$

where we have assumed that the moment of inertia of the NS is
$10^{45}$ g cm$^2$ and applied Eq. (39) to estimate the oblateness
at specific rotation frequency for a NS with $M = 1.4 M_\odot$
and $R = 10$ km.

If the first-order lines are observed, the rotation fre-
cency and the free precession frequency can be determined.

The inclination angle $\iota$ can be obtained by comparing
the amplitudes of different polarizations for the two first-order
lines. Note that the determination of the inclination angle
$\iota$ is model dependent with the radio signals (Jones 2007).
The derived $\iota$ from continuous GWs is less model dependent
and can help to probe the pulsar geometry (Jones 2007). In
our work, the inclination angle $\iota$ is needed to determine the
pulse-width modulations.

The oblateness, nonaxisymmetry, and wobble angle are
degenerate in the first-order waveform. For the “$x$”-polarized
GW, the amplitudes of the second-order lines at
$2(\Omega_3 + \Omega_p)$, $\Omega_3 - \Omega_p$, $\Omega_3 + 3\Omega_p$, and $(2\Omega_3 - \Omega_p)$ are respectively

$$A_x^3 = -2.1 \times 10^{-28} \left( 64\epsilon^2 + \gamma^2 \right) \times \cos \iota \left( \frac{\epsilon}{4.9 \times 10^{-8}} \right) \left( \frac{f_c}{100 \text{ Hz}} \right)^2 \left( \frac{10 \text{ kpc}}{r} \right),$$

$$A_x^4 = 1.5 \times 10^{-27} \gamma \sin \iota \left( \frac{\epsilon}{4.9 \times 10^{-8}} \right) \left( \frac{f_c}{100 \text{ Hz}} \right)^2 \left( \frac{10 \text{ kpc}}{r} \right),$$

$$A_x^5 = -2.1 \times 10^{-28} \gamma \sin \iota \left( \frac{\epsilon}{4.9 \times 10^{-8}} \right) \left( \frac{f_c}{100 \text{ Hz}} \right)^2 \left( \frac{10 \text{ kpc}}{r} \right),$$

$$A_x^6 = 1.3 \times 10^{-26} \kappa \cos \iota \left( \frac{\epsilon}{4.9 \times 10^{-8}} \right) \left( \frac{f_c}{100 \text{ Hz}} \right)^2 \left( \frac{10 \text{ kpc}}{r} \right).$$

Theoretically, by comparing the amplitudes of the two first-
order lines and one of the second-order lines, the wobble angle
$\gamma$ and the nonaxisymmetry $\kappa$ can be determined. Then,
once can obtain the parameter $m = 16\epsilon\gamma^2$ so that the oblate-
ness can be determined using Eq. (87).

From the observational perspective, however, these am-
plitudes at the second order are very small, and unlikely
to be detectable with the Advanced LIGO/Virgo detectors.
Besides, if the coherent time of the observation is shorter
than the free precession period, the free precession angular
frequency $\Omega_3$ cannot be resolved in frequency domain. How-
ever, in the optimistic situation when they are observed with
the next-generation GW detectors (e.g., the Einstein Tele-
scope and Cosmic Explorer (Hild et al. 2011; Sathyaprakash
et al. 2012; Punturo et al. 2010; Abbott et al. 2017a)), they
can be used to infer the oblateness, nonaxisymmetry, and
wobble angle of the star. The distance to the NS and the
moment of inertia always enter the waveform through the
combination $I_3/r$. Therefore, we cannot obtain them inde-
dependently. By inserting an educated guess of $I_3$, the dis-
tance to the NS can be roughly determined (Van Den Broeck
2005). Or conversely, if the distance can be determined via
parallax or dispersion measure in pulsar timing data, one
can get a measurement of $I_3$, thus putting new constraints
on the equation of state. Detailed analysis along this line is
beyond the scope of this paper, and we leave it to future
study.

6 SUMMARY

To summarize, in this paper we describe both the analytical
and numerical methods to calculate the dynamical evolu-

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tion of precessing triaxial rigid bodies. We discuss the timing residuals and the pulse-width modulations for precessing triaxial NSs. We also present concrete examples of the timing residuals and the pulse-width modulations for large and small wobble angles. For the GWs from triaxial precessing NSs, after reviewing the general solution of the quadrupole waveform (Zimmermann 1980) and showing examples of the waveform in both time and frequency domains, we extend the work by Van Den Broeck (2005) at the second order by relaxing the assumption on the small parameters γ and κ. We obtain three new lines in the continuous GWs spectra, which might be useful for future continuous GW analysis using the third-generation ground-based detectors (Abbott et al. 2017a), depending on the distance of the sources. If the prospects of the multimessenger astrophysics discussed in this work become reality, numerous information on the shape of NSs and the equation of state of supranuclear matters will be obtained, enabling a new frontier for fundamental physics.

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DATA AVAILABILITY

The data underlying this article will be shared on reasonable request to the corresponding author.

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