Repeated randomized algorithm for the Multicovering Problem

Abbass Gorgi\textsuperscript{a,}\textsuperscript{*}, Mourad El Ouali\textsuperscript{b}, Anand Srivastav\textsuperscript{b}, Mohamed Hachimi\textsuperscript{a}

\textsuperscript{a}Engineering Science Laboratory, University Ibn Zohr, Agadir, Morocco

\textsuperscript{b}Department of Computer Science, Christian Albrechts University, Kiel, Germany

Abstract

Let \( \mathcal{H} = (V, \mathcal{E}) \) be a hypergraph with maximum edge size \( \ell \) and maximum degree \( \Delta \). For given numbers \( b_v \in \mathbb{N}_{\geq 2}, v \in V \), a set multicover in \( \mathcal{H} \) is a set of edges \( C \subseteq \mathcal{E} \) such that every vertex \( v \) in \( V \) belongs to at least \( b_v \) edges in \( C \). Set Multicover is the problem of finding a minimum-cardinality set multicover. Peleg, Schechtman and Wool conjectured that unless \( P = NP \), for any fixed \( \Delta \) and \( b := \min_{v \in V} b_v \), no polynomial-time approximation algorithm for the SET MULTICOVER problem has an approximation ratio less than \( \delta := \Delta - b + 1 \). Hence, it’s a challenge to know whether the problem of SET MULTICOVER is not approximable within a ratio of \( \beta \delta \) with a constant \( \beta < 1 \).

This paper proposes a repeated randomized algorithm for the SET MULTICOVER problem combined with an initial deterministic threshold step. Boosting success by repeated trials, our algorithm yields an approximation ratio of \( \max \left\{ \frac{15}{16} \delta, \left( 1 - \frac{(b-1) \exp \left( \frac{3b+1}{72} \right)}{72} \right) \delta \right\} \). The crucial fact is not only that our result improves over the approximation ratio presented by Srivastav et al (Algorithmica 2016) for any \( \delta \geq 13 \), but it’s more general since we set no restriction on the parameter \( \ell \).

Furthermore, we prove that it is NP-hard to approximate the SET MULTICOVER problem on \( \Delta \)-regular hypergraphs within a factor of \( (\delta - 1 - \epsilon) \).

\textsuperscript{*}Corresponding author

Email addresses: abbass.gorgi@gmail.com (Abbass Gorgi), Elouali.math.uni-kiel.de (Mourad El Ouali), srivastavi.math.uni-kiel.de (Anand Srivastav), m.hachimi@uiz.ac.ma (Mohamed Hachimi)
Moreover we show that the integrality gap for the SET MULTICOVER problem is at least \(\frac{\ln(2(n+1))}{2b}\), which for constant \(b\) is \(\Omega(\ln n)\).

**Keywords:** Integer linear programs, hypergraphs, approximation algorithms, randomized rounding, set cover and set multicover.

1. Introduction

This work was intended as an attempt to solve approximately the SET MULTICOVER problem. A nice formulation of this problem may be given by the notion of hypergraphs.

A hypergraph is a pair \(\mathcal{H} = (V, \mathcal{E})\), where \(V\) is a finite set and \(\mathcal{E} \subseteq 2^V\) is a family of some subsets of \(V\). We call the elements of \(V\) vertices and the elements of \(\mathcal{E}\) (hyper-)edges. Further, let \(n := |V|, m := |\mathcal{E}|\). W.l.o.g. let the vertices be enumerated as \(v_1, v_2, \ldots, v_n\) and the edges as \(E_1, E_2, \ldots, E_m\). As usually the degree of a vertex \(v\) (notation \(d(v)\)) is the number of hyperedges it appears in.

Let \(\Delta := \max_{v \in V} d(v)\) be the maximum degree. Furthermore, if the degree of every vertex is exactly \(\Delta\), then \(\mathcal{H}\) is called \(\Delta\)-regular. We define the number of vertices of a hyperedge as its size. If the size of all hyperedges is exactly \(\ell\), i.e., \(\forall E \in \mathcal{E}, |E| = \ell\), then \(\mathcal{H}\) is \(\ell\)-uniform. Let \(b := (b_1, b_2, \ldots, b_n) \in \mathbb{N}^n_{\geq 2}\) be given. If a vertex \(v_i, i \in [n]\), is contained in at least \(b_i\) edges of some subset \(C \subseteq \mathcal{E}\), we say that the vertex \(v_i\) is fully covered by \(b_i\) edges in \(C\). A set multicover in \(\mathcal{H}\) is a set of edges \(C \subseteq \mathcal{E}\) such that every vertex \(v_i\) in \(V\) is fully covered by \(b_i\) edges in \(C\). The SET MULTICOVER problem is the task of finding a set multicover of minimum cardinality.

**Related Work.** The set cover problem \((b = 1)\) is known to be NP-hard \([14]\) and has been intensively explored for decades. Several deterministic approximation algorithms are exhibited for this problem \([1, 10, 12, 16]\), all with approximation ratios \(\Delta\). Furthermore, Johnson \([13]\) and Lovász \([17]\) gave a greedy algorithm with performance ratio \(H(\ell)\), where \(H(\ell) = \sum_{i=1}^{\ell} \frac{1}{i}\) is the harmonic number. Notice that \(H(\ell) \leq 1 + \ln(\ell)\). For hypergraphs with bounded \(\ell\), Duh and Fürer \([4]\) used the technique called semi-local optimization, improving \(H(\ell)\) to
Unlike the set cover problem, the case \( b \geq 2 \) of the set multicover problem is less known. Let us give a summary of the known approximability results. In paper \cite{21}, Vazirani using primal-dual schema extended the result of Lovász \cite{17} for \( b \geq 1 \). Later Fujito et al. \cite{9} improved the algorithm of Vazirani and achieved an approximation ratio of \( H(\ell) - \frac{1}{6} \) for \( \ell \) bounded. Hall and Hochbaum \cite{11} achieved by a greedy algorithm based on LP duality an approximation ratio of \( \Delta \). By a deterministic threshold algorithm Peleg, Schechtman, and Wool in 1997 \cite{19, 20} improved this result and gave an approximation ratio of \( \delta \). They were also the first to propose an approximation algorithm for the set multicover problem with approximation ratio below \( \delta \), namely a randomized rounding algorithm with performance ratio \( (1 - \frac{\gamma}{n}) \cdot \delta \) for a small constant \( c > 0 \). However, their ratio is depending on \( n \), and asymptotically tends to \( \delta \). Furthermore Peleg, Schechtman and Wool conjectured that for any fixed \( \Delta \) and \( b := \min_{i \in [n]} b_i \) the problem cannot be approximated by a ratio smaller than \( \delta := \Delta - b + 1 \) unless \( P = NP \). Hence it remained an open problem whether an approximation ratio of \( \beta \delta \) with \( \beta < 1 \) constant can be proved. A randomized algorithm of hybrid type was later given by Srivastav et al. \cite{7}. Their algorithm achieves for hypergraphs with \( l \in O\left(\max\{nb^{1/2}, n^{1/2}\}\right) \) an approximation ratio of \( \left(1 - \frac{11(\Delta - b)}{72}\right) \cdot \delta \) with constant probability.

Concerning the algorithmic complexity, the set multicover problem has still not been investigated. In contrast to the set cover problem, it is known that the problem is hard to approximate to within \( \Delta - 1 - \epsilon \), unless \( P = NP \) \cite{2}, and to within \( \Delta - \epsilon \) under the UGC \cite{15} for any fixed \( \epsilon > 0 \). Unless \( P = NP \) there is no \( (1 - \epsilon) \ln n \) approximation \cite{8}. This motivated us to study this aspect of the problem.

**Our Results.** The main contribution of our paper is the combination of a deterministic threshold-based algorithm with repeated randomized rounding steps. The idea is to algorithmically discard instances that can be handled deterministically in favor of instances for which we obtain a constant-factor approximation less than \( \delta \) using a repeated randomized strategy.
Our hybrid randomized algorithm is designed as a cascade of a deterministic and a repeated randomized rounding step followed by greedy repair if the randomized solution is not feasible. First, the relaxed problem of the set multicover problem is solved. The successive actions depend on the cardinality of a set of hyperedges that will be defined according to the relaxed problem output. Our algorithm is an extension of an example given in [5, 6, 7, 10, 11, 20] for the vertex cover, partial vertex cover and set multicover problem in graphs and hypergraphs.

The methods used in this paper rely on an application of an extension of the Chernoff-Hoeffding bound theorem for sums of independent random variables and are based on estimating the variance of the summed random variables for invoking the Chebyshev-Cantelli inequality. Our algorithm yields a performance ratio of \[ \max \left\{ \frac{15}{16} \delta, \left(1 - \frac{(b-1) \exp(\frac{4\delta}{b^2})}{2^7}\right) \delta \right\}. \] This ratio means a constant factor of less than \( \delta \) for many settings of the parameters \( \delta, b, \) and \( \ell \). It is asymptotically better than the former approximation ratios due to Peleg et al. and Srivastav et al. Furthermore, using a reduction of the set cover problem on \( \Delta \)-regular hypergraphs to the set multicover problem on \( \Delta + b - 1 \)-regular hypergraphs, we show that it is NP-hard to approximate the set multicover problem on \( \Delta \)-regular hypergraphs within a factor of \( (\delta - 1 - \epsilon) \). Moreover, we show that the integrality gap for the natural LP formulation of the set multicover problem is at least \( \ln \frac{n+1}{2b} \), which for constant \( b \) is \( \Omega(\ln n) \).

**Outline of the paper.** In Section 2, we give all the definitions and the tools needed for our analysis. In Section 3, we present a randomized algorithm of hybrid type and its analysis. In Section 4, we give a lower bound for the problem. In Section 5, we discuss the integrality gap of the LP formulation of the problem.

2. Definitions and preliminaries

For the later analysis we will use the following extension of Chernoff-Hoeffding Bound inequality for a sum of independent random variables. It is often used if one only has a bound on the expectation:
Fundamental results and approximations for set multicover problem

| Hypergraph | Approximation ratio |
|------------|---------------------|
| -          | $H(\ell)[21]$       |
| bounded $\ell$ | $H(\ell) - \frac{1}{\ell} [9]$ |
| -          | $\delta [11, 20]$   |
| $l \in \mathcal{O} \left( \max\{(nb)^{\frac{1}{5}}, n^{\frac{1}{4}}\} \right)$ | $\left(1 - \frac{11(\Delta-b)}{72\ell}\right) \cdot \delta [7]$ |
| -          | $\max \left\{ \frac{15}{16} \delta, \left(1 - \frac{(b-1)\exp\left(\frac{41\ell}{2}\right)}{72\ell}\right) \delta \right\}$ (this paper) |

**Theorem 1** (see [3]). Let $X_1, \ldots, X_n$ be independent $\{0, 1\}$-random variables. Let $X = \sum_{i=1}^{n} X_i$ and suppose $\mathbb{E}(X) < \mu$. For every $0 < \beta \leq 1$ we have

$$\Pr[X \geq (1 + \beta)\mu] \leq \exp\left(-\frac{\beta^2\mu}{3}\right).$$

A further useful concentration theorem we will use is the Chebychev-Cantelli inequality:

**Theorem 2** (see [18], page 64). Let $X$ be a non-negative random variable with finite mean $\mathbb{E}(X)$ and variance $\text{Var}(X)$. Then for any $a > 0$ it holds that

$$\Pr(X \leq \mathbb{E}(X) - a) \leq \frac{\text{Var}(X)}{\text{Var}(X) + a^2}.$$

Our lower bound proof for the problem relies on extending the following theorem from the case of $b = 1$ to the case of $b \geq 2$.

**Theorem 3** (I. Dinur et al, 2005 [2]). For every integer $\ell \geq 3$ and every $\epsilon > 0$, it is NP-hard to approximate the minimum vertex cover problem on $\ell$-uniform hypergraphs within a factor of $(\ell - 1 - \epsilon)$.

A key notion of linear programming relaxations is the concept of Integrality Gap.

**Definition 1.** Let $\mathcal{I}$ be a set of instances, the Integrality Gap for minimization problems is defined as

$$\sup_{i \in \mathcal{I}} \frac{\text{Opt}(I)}{\text{Opt}^*(I)}.$$
3. The multi-randomized rounding algorithm

Let \( \mathcal{H} = (V, \mathcal{E}) \) be a hypergraph with maximum vertex degree \( \Delta \) and maximum edge size \( \ell \). An integer linear programming formulation of the \textsc{set multicover} problem is the following:

\[
\min \sum_{j=1}^{m} x_j,
\]

\[
\text{ILP}(\Delta, b) : \quad \sum_{j=1}^{m} a_{ij} x_j \geq b_i \quad \text{for all } i \in [n],
\]

\[
x_j \in \{0, 1\} \quad \text{for all } j \in [m],
\]

where \( A = (a_{ij})_{i \in [n], j \in [m]} \in \{0, 1\}^{n \times m} \) is the vertex-edge incidence matrix of \( \mathcal{H} \) and \( b = (b_1, b_2, \ldots, b_n) \in \mathbb{N}_2^n \) is the given integer vector. For every vertex \( v \), we define \( \Gamma(v) := \{ E \in \mathcal{E} \mid v \in E \} \) the set of edges incident to \( v \).

The linear programming relaxation \( \text{LP}(\Delta, b) \) of \( \text{ILP}(\Delta, b) \) is given by relaxing the integrality constraints to \( x_j \in [0, 1] \) for all \( j \in [m] \). Let \( \text{Opt} \) resp. \( \text{Opt}^* \) be the value of an optimal solution to \( \text{ILP}(\Delta, b) \) resp. \( \text{LP}(\Delta, b) \). Let \((x_1^*, \ldots, x_m^*)\) be the optimal solution of the \( \text{LP}(\Delta, b) \). So \( \text{Opt}^* = \sum_{j=1}^{m} x_j^* \) and \( \text{Opt}^* \leq \text{Opt} \).

The next lemma shows that the \( b_i \) greatest values of the LP variables corresponding to the incident edges for any vertex \( v_i \) are all greater than or equal to \( \frac{1}{\delta} \).

**Lemma 1 (see [20]).** Let \( b_i, d, \Delta, n \in \mathbb{N} \) with \( 2 \leq b_i \leq d - 1 \leq \Delta - 1, i \in [n] \). Let \( x_j \in [0, 1], j \in [d] \), such that \( \sum_{j=1}^{d} x_j \geq b_i \). Then at least \( b_i \) of the \( x_j \) fulfill the inequality \( x_j \geq \frac{1}{\delta} \).

Our second lemma shows that the \( b_i - 1 \) greatest values of the LP variables corresponding to the incident edges for any vertex \( v_i \) are all greater than or equal to \( \frac{2}{\delta + 1} \) and with Lemma 1, we take the sum over the \( b_i \) greatest values of the LP variables corresponding to the incident edges for any vertex \( v_i \).

**Lemma 2.** Let \( b_i, d, \Delta, n \in \mathbb{N} \) with \( 2 \leq b_i \leq d - 1 \leq \Delta - 1, i \in [n] \). Let \( x_j \in [0, 1], j \in [d] \), such that \( \sum_{j=1}^{d} x_j \geq b_i \). Then at least \( b_i - 1 \) of the \( x_j \) fulfill the
inequality \( x_j \geq \frac{2}{\delta + 1} \) and there exists an element \( x_j \), distinct to all of them, that fulfills the inequality \( x_j \geq \frac{1}{\delta} \).

**Proof.** W.l.o.g. we suppose \( x_1 \geq x_2 \geq \cdots \geq x_{b_i} \geq \cdots \geq x_d \).

Hence \( b_i - 2 \geq \sum_{j=1}^{b_i-1} x_j \) and \( (d - b_i + 2)x_{b_i-1} \geq \sum_{j=b_i-1}^{d} x_j \).

Then

\[
\begin{align*}
    b_i - 2 + (\Delta - b + 2)x_{b_i-1} &\geq b_i - 2 + (\Delta - b_i + 2)x_{b_i-1} \\
    &\geq \sum_{j=1}^{b_i-2} x_j + \sum_{j=b_i-1}^{d} x_j = \sum_{j=0}^{d} x_j \\
    &\geq b_i
\end{align*}
\]

So we have \( x_{b_i-1} \geq \frac{2}{\delta + 1} \).

Since for all \( j \in [b_i - 1] \), \( x_j \geq x_{b_i-1} \) then for all \( j \in [b_i - 1] \), \( x_j \geq \frac{2}{\delta + 1} \).

Furthermore, by Lemma 1 and the assumption on the orders of the variables \( x_j \), for all \( j \in [b_i] \) we have \( x_j \geq \frac{1}{\delta} \) and particularly \( x_{b_i} \geq \frac{1}{\delta} \).

### 3.1. The algorithm

In this section we present an algorithm with conditioned randomized rounding based on the properties satisfied by two generated sets, \( C_1 \) and \( C_2 \).

In step 2 we solve the linear programming relaxation \( \text{LP}(\Delta, b) \) in polynomial time, using some known polynomial-time procedure, e.g. the interior point method. Next we take into the cover all edges of the sets \( C_1 \) resp. \( C_2 \). Since the LP variable value \( x_j^* \) that corresponds to an edge \( E_j \) from the set \( C_1 \) is greater than or equal to \( \frac{2}{\delta+1} \) and the value \( x_j^* \) that corresponds to an edge \( E_j \) from the set \( C_2 \) is less than \( \frac{2}{\delta+1} \), we have

\[
    |C_1| + |C_2| = |C| \quad \text{and} \quad C_1 \cap C_2 = \emptyset \tag{1}
\]
Algorithm 1: SET MULTICOVER

Input: A hypergraph $\mathcal{H} = (V, E)$ with maximum degree $\Delta$ and maximum hyperedge size $\ell$, numbers $b_i \in \mathbb{N}_{\geq 2}$ for $i \in [n]$, $\delta := \min_{i \in [n]} b_i$, $\epsilon \in (0, 1)$, a constant $k \in \mathbb{N}_{\geq 2}$ and $\delta = \Delta - b + 1$.

Output: A set multicover $C$

1. Initialize $C := \emptyset$. Set $\lambda = \frac{\delta + 1}{2}$, $\alpha = \frac{(b-1)\delta^k}{6\ell}$ and $\lambda_0 = (1-\epsilon)\delta$.
2. Obtain an optimal solution $x^* \in [0, 1]^m$ by solving the LP($\Delta$, $b$) relaxation.
3. Set $C_1 := \{E_j \in E \mid x^*_j \geq \frac{1}{\lambda}\}$, $C_2 := \{E_j \in E \mid \frac{1}{\lambda} > x^*_j \geq \frac{1}{\delta}\}$ and $C_3 := \{E_j \in E \mid 0 < x^*_j < \frac{1}{\delta}\}$.
4. Take all edges of the set $C_1$ in the cover $C$.
5. if $|C_1| \geq \alpha \cdot \text{Opt}^*$ then return $C = C_1 \cup C_2$.
   Else (Multi-randomized Rounding)
   (a) For all edges $E_j \in C_2$ include the edge $E_j$ in the cover $C$, independently for all such $E_j$, with probability $\lambda_0 x^*_j$, $k$ times.
      (If, in any of these $k$ biased coin flips shows head, include the edge $E_j$ in the cover.)
   (b) For all edges $E_j \in C_3$ include the edge $E_j$ in the cover $C$, independently for all such $E_j$, with probability $(1-\epsilon^k)\delta x^*_j$.
   (c) (Repairing) Repair the cover $C$ (if necessary) as follows: Include arbitrary edges from $C_2$, incident to the vertices $v_i$ not fully covered, to $C$ until all vertices are fully covered.
   (d) Return the cover $C$. 

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3.2. Analysis of the algorithm

Case $|C_1| \geq \alpha \cdot \text{Opt}^*$.

**Theorem 4.** Let $H$ be a hypergraph with maximum vertex degree $\Delta$ and maximum edge size $\ell$. Let $\alpha = \frac{(b-1)\delta k}{6\ell} \times \exp(a_{k,\epsilon})$ with $a_{k,\epsilon} = \frac{k(1-\epsilon)+(k-1)(1-\epsilon^*)}{2}$ as defined in Algorithm 1. If $|C_1| \geq \alpha \cdot \text{Opt}^*$ then Algorithm 1 returns a set multicover $C$ such that

$$|C| \leq \left(1 - \frac{(b-1)e^k}{18\ell} \times \exp(a_{k,\epsilon}) \right) \delta \cdot \text{Opt}^*$$

**Proof.** The proof is straightforward, using the definitions of the sets $C_1$ and $C_2$.

$$\delta \text{Opt}^* = \sum_{j=1}^{m} \delta x_j^* \geq \sum_{E_j \in C_1} \delta x_j^* + \sum_{E_j \in C_2} \delta x_j^*$$

$$\geq \frac{2\delta}{\delta + 1} |C_1| + |C_2|$$

$$\geq \frac{2\delta}{\delta + 1} |C_1| + (|C| - |C_1|)$$

$$\geq \frac{\delta - 1}{\delta + 1} |C_1| + |C|$$

$$\delta \geq \frac{1}{3} |C_1| + |C|$$

$$\geq \frac{1}{3} \alpha \cdot \text{Opt}^* + |C|.$$ 

Hence

$$|C| \leq \left(1 - \frac{(b-1)e^k}{18\ell} \times \exp(a_{k,\epsilon}) \right) \delta \cdot \text{Opt}^*$$

Case $|C_1| < \alpha \cdot \text{Opt}^*$.

Let $X_1, \ldots, X_m$ be $\{0, 1\}$-random variables defined as follows:

$$X_j = \begin{cases} 1 & \text{if the edge } E_j \text{ was picked into the cover before repairing} \\ 0 & \text{otherwise.} \end{cases}$$
Note that the $X_1, \ldots, X_m$ are independent for a given $x^* \in [0,1]^m$. For all $i \in [n]$ we define the \{0,1\}-random variables $Y_i$ as follows:

$$Y_i = \begin{cases} 1 & \text{if the vertex } v_i \text{ is fully covered before repairing} \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $X := \sum_{j=1}^m X_j$ and $Y := \sum_{i=1}^n Y_i$ the cardinality of the cover and the cardinality of the set of fully covered vertices before the step of repairing, respectively. At this step by Lemma 2 one more edge for each vertex is at most needed to be fully covered. The cover $C$ obtained by Algorithm 1 is bounded by

$$|C| \leq X + n - Y. \quad (2)$$

Our next lemma provides upper bounds on the expectation of the random variable $X$ and the expectation and variance of the random variable $Y$, which we will use to proof Theorem 5. This is a restriction of Lemma 4 in [7] to the last case in Algorithm 1.

**Lemma 3.** Let $l$ and $\Delta$ be the maximum size of an edge and the maximum vertex degree, respectively. Let $\epsilon \in \left[ \frac{\Delta - 1}{2\Delta}, \left( \frac{4 - 1}{2\Delta} \right)^\frac{1}{2} \right]$, $a_{k,\epsilon} = \frac{k(1-\epsilon)+(\Delta-1)(1-\epsilon^*)}{2}$, $\lambda_0 = (1-\epsilon)\delta$ and $\lambda = \frac{1+\epsilon}{1-\epsilon}$ as in Algorithm 1. We have

(i) $E(Y) \geq (1 - \exp (-2a_{k,\epsilon}))n$.

(ii) $\text{Var}(Y) \leq 2n^2 \exp (-2a_{k,\epsilon})$.

(iii) $E(X) \leq (1 - \epsilon^k)\delta \text{Opt}^*$.

(iv) $\frac{(b-1)n}{\alpha \ell} < \text{Opt}^*$.

**Proof.** (i) Let $i \in [n]$, $r = d(i) - b_i + 1$. If $|C_1 \cap \Gamma(v_i)| \geq b_i$, then the vertex $v_i$ is fully covered and $\Pr(Y_i = 0) = 0$. Otherwise we get by Lemma 2 that $|C_1 \cap \Gamma(v_i)| = b_i - 1$ and there exists at least one more edge from $C_2$ with $x_j \geq \frac{1}{\delta}$, so we have $\sum_{E_j \in \Gamma(v_i) \cap C_2} x_j \geq \frac{1}{\delta}$ and by the inequality constraints it
holds that \( \sum_{E_j \in \Gamma(v_i) \cap (C_2 \cup C_3)} x_j^* \geq 1 \). Therefore

\[
\Pr(Y_i = 0) = \left( \prod_{E_j \in \Gamma(v_i) \cap C_2} (1 - \lambda_0 x_j^*) \right)^k \prod_{E_j \in \Gamma(v_i) \cap C_3} (1 - (1 - \epsilon^k) \delta x_j^*)
\]

\[
\leq \prod_{E_j \in \Gamma(v_i) \cap C_2} \exp(-k \lambda_0 x_j^*) \prod_{E_j \in \Gamma(v_i) \cap C_3} \exp(-(1 - \epsilon^k) \delta x_j^*)
\]

\[
= \exp \left( -k(1 - \epsilon) \delta \sum_{E_j \in \Gamma(v_i) \cap C_2} x_j^* \right) \cdot \exp \left( -(1 - \epsilon^k) \delta \sum_{E_j \in \Gamma(v_i) \cap C_3} x_j^* \right)
\]

\[
= \exp \left( -(k - 1 + \epsilon^k) \delta \sum_{E_j \in \Gamma(v_i) \cap C_2} x_j^* \right) \cdot \exp \left( -(1 - \epsilon^k) \delta \sum_{E_j \in \Gamma(v_i) \cap (C_2 \cup C_3)} x_j^* \right).
\]

Since \( 1 - \epsilon^k = (1 - \epsilon) \sum_{i=0}^{k-1} \epsilon^i \leq k(1 - \epsilon) \), we have \(-k(1 - \epsilon) + 1 - \epsilon^k \leq 0\).

It follows that

\[
\Pr(Y_i = 0) \leq \exp \left( -(k - 1 + \epsilon^k) \delta \right) \cdot \exp \left( -(1 - \epsilon^k) \delta \right)
\]

\[
= \exp \left( -2a_{k, \epsilon} \right).
\]

Therefore

\[
\mathbb{E}(Y) = \sum_{i=1}^{n} \Pr(Y_i = 1) = \sum_{i=1}^{n} \left( 1 - \Pr(Y_i = 0) \right)
\]

\[
\geq \sum_{i=1}^{n} \left( 1 - \exp \left( -2a_{k, \epsilon} \right) \right)
\]

\[
\geq (1 - \exp \left( -2a_{k, \epsilon} \right))n.
\]

(ii) Since

\[
Y = \sum_{i=1}^{n} Y_i \leq n,
\]

we have

\[
\mathbb{E}(Y^2) \leq n^2.
\]
Thus,

\[
\text{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 \leq n^2 - (1 - \exp(-2a_{k, \epsilon}))^2 n^2
\]

\[
\leq n^2 \left(1 - (1 - \exp(-2a_{k, \epsilon}))^2\right)
\]

\[
\leq 2n^2 \exp(-2a_{k, \epsilon}).
\]

(iii) Let \( E_j \) be an edge from \( C_2 \). By Lemma 2 we have \( \frac{1}{\delta} \leq x^*_j < \frac{2}{\delta + 1} \).

Recall that we include independently the edge \( E_j \) in the cover \( C \), with probability \( \lambda_0 x^*_j \), \( k \) times. Since \( \frac{\delta - 1}{\delta + 1} \leq \epsilon \), we have \( 1 - \epsilon \leq \lambda_0 x^*_j < \frac{2}{\delta + 1} (1 - \epsilon) \delta \leq \frac{2}{\delta + 1} (1 - \frac{\delta - 1}{2\delta}) \delta = 1 \).

Furthermore with \( \epsilon \leq \left(\frac{\delta - 1}{2\delta}\right)^k \) we have \( (1 - \epsilon^k) \delta \geq (1 - \frac{\delta - 1}{2\delta}) \delta = \frac{\delta + 1}{2} = \lambda \).

Then

\[
\lambda \leq (1 - \epsilon^k) \delta. \tag{3}
\]

Clearly \( \Pr(X_j = 1) = 1 - (1 - \lambda_0 x^*_j)^k \).

Define the function \( f \) by \( f(x) = \frac{1 - (1 - x)^k}{x} \).

\( f \) is strictly decreasing on \((0, 1]\). Therefore,

\[
\frac{1 - (1 - \lambda_0 x^*_j)^k}{\lambda_0 x^*_j} \leq \frac{1 - (1 - \epsilon)^k}{1 - \epsilon} = \frac{1 - \epsilon^k}{1 - \epsilon}.
\]

It follows that \( \Pr(X_j = 1) \leq \frac{1 - \epsilon^k}{1 - \epsilon} \cdot \lambda_0 x^*_j \).

Then

\[
\Pr(X_j = 1) \leq (1 - \epsilon^k) \delta x^*_j. \tag{4}
\]

By using the LP relaxation and the definition of the sets \( C_1 \) and \( C_2 \), and since \( \lambda x^*_j \geq 1 \) for all \( E_j \in C_1 \), we get

\[
\mathbb{E}(X) = |C_1| + \sum_{E_j \in C_2} \Pr(X_j = 1) + \sum_{E_j \in C_3} \Pr(X_j = 1)
\]

\[
\leq \sum_{E_j \in C_1} \lambda x^*_j + \sum_{E_j \in C_2} (1 - \epsilon^k) \delta x^*_j + \sum_{E_j \in C_3} (1 - \epsilon^k) \delta x^*_j
\]

\[
\leq (1 - \epsilon^k) \delta \sum_{E_j \in \mathcal{E}} x^*_j
\]

\[
\leq (1 - \epsilon^k) \delta \text{Opt}^*.
\]

(iv) Let us consider \( \mathcal{H} \) the subhypergraph induced by \( C_1 \) in which degree equality gives
\[
\sum_{i \in V} d(i) = \sum_{E_j \in C_1} |E_j|.
\]

As the minimum vertex degree in the subhypergraph \( \tilde{H} \) is \( b - 1 \) with \( b := \min_{i \in [n]} b_i \), we have

\[
(b - 1)n \leq \sum_{i \in V} d(i) = \sum_{E_j \in C_1} |E_j| \leq \ell |C_1|.
\]

Therefore

\[
\frac{(b - 1)n}{\ell} \leq |C_1|.
\]

Since \( |C_1| < \alpha \cdot \text{Opt}^* \) we obtain

\[
\frac{(b - 1)n}{\alpha \ell} < \text{Opt}^*.
\]

**Theorem 5.** Let \( H \) be a hypergraph with fixed maximum vertex degree \( \Delta \) and maximum edge size \( \ell \). Let \( \alpha = \frac{1}{2} \left( 1 - \frac{1}{2} + \epsilon \right) \delta \) with \( a_{k,\epsilon} = \frac{k(1-\epsilon) + (4-1)(1-\epsilon^2)}{2} \) and \( \epsilon \in \left[ \frac{\delta - 1}{25}, \frac{\delta - 1}{25} \right] \) as in Algorithm 4. The Algorithm 4 returns a set multicolor cover \( C \) such that

\[
|C| < \max \left\{ \left( 1 - \frac{1}{2} (1 - \epsilon)^k \right) \delta, \left( 1 - \frac{(b - 1)\epsilon^k}{18\ell} \times \exp(a_{k,\epsilon}) \right) \delta \right\} \cdot \text{Opt}^*
\]

with probability greater than 0.65.

**Proof.** Let \( C \) be the event that the inequality \( |C| < \left( 1 - \frac{1}{2} (1 - \epsilon)^k \right) \delta \cdot \text{Opt}^* \) is satisfied. It suffices to prove that event \( C \) holds with the given probability in the case \( |C_1| < \alpha \cdot \text{Opt}^* \) since the opposite case is discussed in Theorem 4. For this purpose, we estimate both the concentration of \( X \) and \( Y \) around their expectation. Choose \( t = 2n \exp(-a_{k,\epsilon}) \) and consider \( A \) the event \( Y \leq n(1 - \exp(-2a_{k,\epsilon})) - t \).
This involves

\[ n \exp (-2a_{k,\epsilon}) + t = n \exp (-2a_{k,\epsilon}) + 2n \exp (-a_{k,\epsilon}) \]

\[ \leq 3n \exp (-a_{k,\epsilon}) \]

\[ = \frac{n(b - 1)}{\ell} \cdot \frac{6\ell}{(b - 1)\delta e^k \exp (a_{k,\epsilon}) \cdot \frac{1}{2}e^k} \cdot \frac{1}{2}\delta \]

\[ = \frac{n(b - 1)}{\alpha \ell} \cdot \frac{1}{2}e^k \delta \]

\[ \leq \frac{1}{2}e^k \delta \cdot \text{Opt}^*. \]

And by Lemma 3(ii) we have \( \frac{t^2}{\text{Var}(Y)} \geq \frac{4n^2 \exp(-2a_{k,\epsilon})}{2n^2 \exp(-2a_{k,\epsilon})} = 2. \)

Therefore

\[ \Pr (A) \leq \Pr (Y \leq E(Y) - t) \]

\[ \leq \frac{\text{Var}(Y)}{\text{Var}(Y) + t^2} \]

\[ = \frac{1}{1 + \frac{t^2}{\text{Var}(Y)}} \]

\[ \leq \frac{1}{3}. \]

Consider now B the event \( X \geq (1 - (1 - \frac{1}{2}\epsilon) e^k) \delta \text{Opt}^*. \) Our basic assumption is to consider \( \delta, k \) and \( \epsilon \) constants, and we can certainly assume that \( n \geq \frac{16 \exp(a_{k,\epsilon})}{e^{k+1}} \), since otherwise we obtain an optimal solution for the SET MULTICOVER problem in polynomial time.

Choosing \( \beta = \frac{1}{2}e^{k+1} \) we have

\[ (1 + \beta)(1 - \epsilon^k) = 1 - \epsilon^k + \frac{1}{2}e^{k+1} - \frac{1}{2}e^{2k+1} \]

\[ = 1 - \epsilon^k \left( 1 - \frac{1}{2}e + \frac{1}{2}e^{k+1} \right) \]

\[ \leq 1 - \left( 1 - \frac{1}{2} \right) \epsilon^k. \]

Note that \( \epsilon \in \left[ \frac{4-1}{2k}, \left( \frac{4-1}{2k} \right)^{\frac{1}{k}} \right] \) therewith \( 1 - \epsilon^k \geq 1 - \frac{4-1}{2k} = \frac{4-1}{2k} > \frac{1}{2}. \)
We thus get

\[
\Pr(B) \leq \Pr(X \geq (1 + \beta) \cdot (1 - \epsilon^k)\delta \text{Opt}^*)
\]

\[
\text{Th} \leq \exp\left(-\frac{\beta^2(1 - \epsilon^k)\delta \text{Opt}^*}{3}\right)
\]

\[
\text{Lem} \leq \exp\left(-\frac{\epsilon^k(1 - \epsilon^k)n}{12\ell} \cdot \frac{6\ell}{(b - 1)\delta \epsilon^k \times \exp(a_{k,\epsilon})}\right)
\]

\[
\leq \exp\left(-\frac{\epsilon^k(1 - \epsilon^k)n}{12\ell} \cdot \frac{6\ell}{(b - 1)\delta \epsilon^k \times \exp(a_{k,\epsilon})}\right)
\]

\[
\leq \exp\left(-\frac{\epsilon^k(1 - \epsilon^k)n}{2\exp(a_{k,\epsilon})}\right)
\]

\[
\leq \exp\left(-\frac{\epsilon^k(1 - \epsilon^k)n}{4\exp(a_{k,\epsilon})}\right)
\]

Therefore it holds that

\[
\Pr(\overline{A} \cap \overline{B}) \geq 1 - \left(\frac{1}{3} + \exp(-4)\right)
\]

where $\overline{A}$ and $\overline{B}$ denote the complement events of $A$ and $B$ respectively. We conclude that

\[
\Pr(C) = \Pr\left(|C| \leq \left(1 - \frac{1}{2}(1 - \epsilon)\delta \text{Opt}^*\right)\right)
\]

\[
= \Pr\left(|C| \leq \left(1 - \left(1 - \frac{1}{2}\epsilon\right)\epsilon^{k} + \frac{1}{2}\epsilon^{k}\right)\delta \text{Opt}^*\right)
\]

\[
\geq \Pr\left(X + n - Y \leq \left(1 - \left(1 - \frac{1}{2}\epsilon\right)\epsilon^{k} + \frac{1}{2}\epsilon^{k}\right)\delta \text{Opt}^*\right)
\]

\[
\geq \Pr\left(X \leq \left(1 - \left(1 - \frac{1}{2}\epsilon\right)\epsilon^{k}\right)\delta \text{Opt}^* \text{ and } n - Y \leq \frac{1}{2}\epsilon^{k}\delta \text{Opt}^*\right)
\]

\[
\geq \Pr\left(X \leq \left(1 - \left(1 - \frac{1}{2}\epsilon\right)\epsilon^{k}\right)\delta \text{Opt}^* \text{ and } Y \geq n - \frac{1}{2}\epsilon^{k}\delta \text{Opt}^*\right)
\]

\[
\geq \Pr\left(X \leq \left(1 - \left(1 - \frac{1}{2}\epsilon\right)\epsilon^{k}\right)\delta \text{Opt}^* \text{ and } Y \geq n - n \exp(-2a_{k,\epsilon}) - t\right)
\]

\[
\geq \Pr\left(X \leq \left(1 - \left(1 - \frac{1}{2}\epsilon\right)\epsilon^{k}\right)\delta \text{Opt}^* \text{ and } Y \geq n(1 - \exp(-2a_{k,\epsilon})) - t\right)
\]

\[
\geq 1 - \left(\frac{1}{3} + \exp(-4)\right)
\]

\[
\geq 0.65.
\]

\[\square\]

**Remark 2.** The proof above gives for $k = 2$ and $\epsilon = \frac{1}{2}$ an approximation ratio
of max \( \left\{ \frac{45}{16} \delta, \left( 1 - \frac{(b-1)\exp\left(\frac{3\delta+1}{8}\right)}{72\ell} \right) \right\} \). Note that \( \frac{b-1}{25} < \frac{1}{2} < \left( \frac{b-1}{25} \right)^{\frac{3}{2}} \) therewith the condition of Theorem 5 on \( \epsilon \) is satisfied.

As mentioned above our performance guaranty improves over the ratio presented by Srivastav et al.\([7]\), and this without restriction on the parameter \( \ell \).

Namely, for \( \delta \geq 13 \) we have

\[
11(\delta - 1) < \exp\left(\frac{3\delta + 1}{8}\right) \quad \Rightarrow \quad \frac{11(\delta - 1)}{72\ell} < \frac{\exp\left(\frac{3\delta + 1}{8}\right)}{72\ell}
\]

\[
b - 1 \geq 1 \quad \Rightarrow \quad \frac{11(\Delta - b)}{72\ell} < \frac{(b - 1)\exp\left(\frac{3\delta + 1}{8}\right)}{72\ell}
\]

\[
\Rightarrow \quad \left( 1 - \frac{(b-1)\exp\left(\frac{3\delta + 1}{8}\right)}{72\ell} \right) \delta < \left( 1 - \frac{11(\Delta - b)}{72\ell} \right) \delta.
\]

4. Lower Bound

One of the features of the proof is the duality of hypergraphs. In dual hypergraphs, vertices and edges just swap the roles. So the SET MULTICOVER problem in dual hypergraphs becomes as follows: find a minimum cardinality set \( C \subseteq V \) such that for every \( E \in \mathcal{E} \) it holds \( |E \cap C| \geq b \). This problem is known as the \( b \)-vertex cover problem and we have that the SET MULTICOVER problem in \( \Delta \)-regular hypergraphs is equivalent to the \( b \)-vertex cover problem in \( \Delta \)-uniform hypergraphs.

**Theorem 6.** Let \( \epsilon > 0, \Delta \) and \( b \in \mathbb{N}_0 \) be given and \( b = \min_i b_i \). Then, it is NP-hard to approximate the SET MULTICOVER problem on \( \Delta \)-regular hypergraphs within a factor of \( \Delta - b - \epsilon \).

**Proof.** Assume, for a contradiction, that the theorem is false. Then there exists an algorithm \( A \) that returns a \( (\Delta - b - \epsilon) \)-approximation in polynomial time for the \( b \)-vertex cover problem on \( \Delta \)-uniform hypergraphs.

We give a reduction of the minimum vertex cover problem on \( \Delta \)-uniform hypergraphs to the \( b \)-vertex cover problem on \( \Delta + b - 1 \)-uniform hypergraphs.

Let \( \hat{\mathcal{H}} = (V, \mathcal{E}) \) be a \( \Delta \)-uniform hypergraph and let \( \alpha = \frac{\Delta - 1 - \epsilon}{2}(b - 1) \).

Now we consider the following algorithm:
1. Consider all subsets $T \subseteq V$ with $|T| \leq \alpha$. Check if any of these subsets is a vertex cover in $\tilde{H}$. If it’s the case then return the smallest one of them, else go to step 2.

2. Add $b - 1$ vertices $v_1, \ldots, v_{b-1}$ to $V$. Define for every hyper-edge $E$ a new edge $E^* := E \cup \{v_1, \ldots, v_{b-1}\}$ and the set $\mathcal{E}^* := \{E^*|E \in \mathcal{E}\}$. Finally set $H = (V \cup \{v_1, \ldots, v_{b-1}\}, \mathcal{E}^*)$. We execute $\mathcal{A}$ on $H$. Return $T := \mathcal{A}(H) \cap V$.

Claim. The algorithm given above returns a vertex cover in $\tilde{H}$ in polynomial-time with an approximation ratio of $\Delta - 1 - \frac{\epsilon}{2}$.

Proof. Correctness and approximation ratio. If $T$ is selected by the algorithm in step 1 then $T$ is an optimal vertex cover in $\tilde{H}$.

If $T$ is selected by the algorithm in step 2 then $\mathcal{A}(H) = T \cup K$ for some $K \subset \{v_1, \ldots, v_{b-1}\}$. Note that $T$ and $K$ are disjoint sets. Consider an edge $E \in \mathcal{E}$. Because $\mathcal{A}(H)$ is a $b$-vertex cover in $H$, we have $|\mathcal{A}(H) \cap E^*| = |T \cap E^*| + |K \cap E^*| \geq b$. Since $T \cap E^* = T \cap E$ and $|K \cap E^*| \leq b - 1$, it follows that $|T \cap E| \geq 1$. Hence $T$ is a vertex cover in $\tilde{H}$.

Now, let $C$ and $C'$ denote a minimum vertex cover in $\tilde{H}$ and a minimum $b$-vertex cover in $H$, respectively. Since $D' := C \cup \{v_1, \ldots, v_{b-1}\}$ is a feasible $b$-vertex cover in $H$, it holds that $|C'| \leq |C| + b - 1$. On the other hand, it is clear that $H$ is a $(\Delta + b - 1)$-uniform hypergraph, and by the assumption we get

$$|\mathcal{A}(H)| \leq (\Delta + b - 1) - b - \epsilon |C'|$$

$$\leq (\Delta - 1 - \epsilon) |C| + (\Delta - 1 - \epsilon)(b - 1) = (\Delta - 1 - \epsilon) |C| + \frac{\epsilon}{2} \alpha$$

$$\leq (\Delta - 1 - \frac{\epsilon}{2}) |C|.$$  

Since $|T| = |\mathcal{A}(H) \cap V| \leq |\mathcal{A}(H)|$, it follows that $|T| \leq (\Delta - 1 - \frac{\epsilon}{2}) |C|$.

Running time. In step 1 we test at most $n^\alpha$ sets of vertices to be a vertex cover in $\tilde{H}$. Since $\alpha = \frac{\epsilon}{2}(\Delta - 1 - \epsilon)(b - 1)$ is a constant, the running time in this step is polynomial. In step 2 we add a constant number of vertices to $V$ and execute the algorithm $\mathcal{A}$. Hence the algorithm runs in polynomial time in both steps.
With Claim 1 there is a factor \( \Delta - 1 - \frac{\varepsilon}{2} \) approximation algorithm for the minimum vertex cover problem on \( \Delta \)-uniform hypergraphs, which contradicts the statement of Theorem 3.

5. The \( \frac{\ln_2(n+1)}{2b} \)-Integrality Gap

The integrality gap for set multicover problem is defined as the supremum of the ratio \( \frac{\text{Opt}_b(H)}{\text{Opt}_\ast_b(H)} \) over all instances \( H \) of the problem. In this section we give a slight modification of the proof presented in [22] for the integrality gap. We present in the following a specific class of instances of the set multicover problem, where \( b := (b, \ldots, b) \in \mathbb{N}^n \) for which the integrality gap is at least \( \frac{\ln_2(n+1)}{2b} \).

**Theorem 7.** Let \( b := (b, \ldots, b) \in \mathbb{N}^n \). The integrality gap of the set multicover problem is at least \( \frac{\ln_2(n+1)}{2b} \).

Define \( V = F_2^k \setminus \{0\} \) as the set of all \( k \)-dimensional non-zero vectors with component values of \( \mathbb{Z}_2 = \{0, 1\} \) for a fixed integer \( k \) and we define \( \mathcal{E} \) as a collection of the sets \( E_v = \{u \in V : < v, u > \equiv 1[2]\} \) for each \( v \in V \), where \( < \ldots > \) is the usual dot product in \( V \).

We remark that each element \( v \in V \) is contained in exactly half of the sets of \( \mathcal{E} \) therewith the hypergraph \( H = (V, \mathcal{E}) \) is regular and \( n = |V| = 2^k - 1 \).

**Lemma 4.** Let \( H = (V, \mathcal{E}) \) the hypergraph defined and \( b \in \mathbb{N}_{\geq 1} \). It holds that the vector \( x = (\frac{2b}{|\mathcal{E}|}, \ldots, \frac{2b}{|\mathcal{E}|}) \) is a feasible solution for LP(\( \Delta, b \)).

**Proof.** It is clear that \( x = (\frac{2b}{|\mathcal{E}|}, \ldots, \frac{2b}{|\mathcal{E}|}) \) is a feasible solution for LP(\( \Delta, b \)), namely since \( H \) is regular with \( \Delta = \frac{|\mathcal{E}|}{2} \) we have for every \( i \in \{1, \ldots, n\} \)

\[
\sum_{E \in \Gamma(v_i)} \frac{2b}{|E|} = 2b \cdot \frac{\Delta}{|\mathcal{E}|} = 2b \cdot \frac{|\mathcal{E}|}{2} \geq b
\]
therewith \( \text{Opt}^* \leq 2b \).

**Lemma 5.** The optimal integral solution to the previous LP formulation of the set multicover problem requires at least \( k \) sets.
Proof. Let \( \{ E_{v_1}, E_{v_2} \ldots E_{v_t} \} \) a collection of sets such that \( \bigcup_{i \in [t]} E_{v_i} = F_2^k \setminus \{0\} \).
This implies that the intersection of their complements contains exactly the zero vector, i.e., \( \bigcap_{i \in [t]} E_{v_i}^C = \{0\} \). It follows that 0 is the only solution in \( F_2^k \) of the system
\[
< x, v_i > = 0 \quad \forall i \in [t]
\]
Then it holds that \( t \geq k \), since the dimension of \( F_2^k \) is \( k \) while the number of the equations in the system is \( t \). From this we conclude Opt \( \geq \ln_2(n+1) \).

Proof of Theorem 7. Theorem 7 follows from Lemma 4 and Lemma 5.

6. Future Work

We believe now that the conjecture of Peleg et al. holds in the general setting.
Hence proving the trueness of the conjecture remains a big challenge for our future works.

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