Quantum mechanics of a particle on a torus knot: Curvature and torsion effects

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Abstract – In this paper, we study the subtle effect of constraints on the quantum dynamics of a point particle moving on a non-trivial torus knot. The particle is kept on the knot by the constraints, generated by curvature and torsion. In the Geometry-Induced Potential (GIP) approach, the Schrödinger equation for the system yields new results in particle energy eigenvalues and eigenfunctions, in contrast with existing results that ignored curvature and torsion effects. Our results depend on $\Gamma$, parameter that characterizes the global features of both the embedding torus and, more interestingly, the knottedness of the path.

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Introduction. – The improved expertise in synthesizing low-dimensional nanostructures with curved geometries \cite{1-4} has generated interest in quantum physics on curved low-dimensional spaces. De Witt \cite{5} directly considered quantum particle motion in a curved $n$-dimensional space. On the other hand, Jensen and Koppe \cite{6} and da Costa \cite{7} proposed to treat the quantum dynamics in an unconstrained ($n+1$)-dimensional Euclidean space in the presence of appropriate confining (or constraint) forces that restrict the particle motion on the designated surface. Subsequently, Ortix \cite{8} took into account the quantum mechanical properties of particles constrained to space curves through electron spin-orbit coupling and revealed a torsion-induced term along with the curvature-induced term in the geometric potential. Following \cite{7,8} Wang \emph{et al.} \cite{9} also included torsion and curvature terms in the effective potential. Classically the two procedures are complimentary and equivalent, but in quantum physics, the curved space quantization programme of de Witt \cite{5} is plagued with operator ordering ambiguities whereas the constrained dynamics framework of \cite{6-8,10} is well defined and rigorous.

Quantum mechanics of particles constrained to move on specified space curves is directly applicable to complex three-dimensional nanostructures such as helical nanowires \cite{11} or multiple helices, toroids, and conical spirals \cite{12}. Similar to a particle moving on a circle and a (single) rotor mapping, a particle on a torus can be identified with a double rotor, acting as a non-planar extension of the planar rotor \cite{13,14}. However, the particle on a torus knot is qualitatively distinct since the knots on the torus (a surface of genus 1) form a class of non-contractible loops. The quantum Hamiltonian for particles constrained to move on surfaces or curves in Euclidean space $\mathbb{R}^n$ contains a unique \cite{7} Geometry-Induced Potential (GIP) \cite{7,8,10}, which depends on the local property of curvature \cite{10}. Furthermore, in \cite{7,8,10} the confining potential approach is free from operator ordering ambiguities. We will investigate the quantum dynamics of a particle constrained to move on a torus knot, starting from the GIP approach.

The paper is organized as follows. In the second section, we discuss the curvature and torsion terms in a generic GIP. The third section discusses the torus knot along with our particular parameterization, explicit expressions for curvature and torsion terms. In the sect. “Numerical comparison of torsion and curvature terms”, we show comparative strengths of the curvature and torsion terms in a numerical study. In the fourth section, we write down the Schrödinger equation and reveal the significance of a constant parameter in our study. In the fifth section, we solve the Schrödinger equation in a thin-torus limit and show the implications of torus knot parameters. In the sixth
section, we consider the full solution of the Schrödinger equation. We conclude in the seventh section, with a summary of the present work along with open problems.

**Generic form of effective potential from GIP.** – We consider a $d$-dimensional surface, $N^d$, embedded in a manifold, $M^{d+k}$, of $d + k$ dimensions [10]. If we use the intrinsic coordinates of $N^d$, it does not invoke any property of $M^{d+k}$. However, if we consider a constraining potential approach, in which the motion of the particle is constrained in $M^{d+k}$, on $N^d$, due to some associated potential, it may also depend on properties of the ambient space $M^{d+k}$ [10]. We mention the results of da Costa [7] together with the modification by Wang et al. [9]. Consider the constrained motion of a particle along a space curve $C$ given by $\vec{r}(q_1)$ with $q_1$ given by the arc-length. The close proximity of $C$ in 3-space is denoted by

$$ \vec{n} = \cos(\theta(q_1))\vec{n}(q_1) - \sin(\theta(q_1))\vec{b}(q_1), $$

$$ \vec{b} = \cos(\theta(q_1))\vec{b}(q_1) + \sin(\theta(q_1))\vec{n}(q_1). $$

To restrict the particle motion along $C$, one may choose a binding potential $V_\omega(q_2, q_3)$ (independent of $q_1$), where $\omega$ is a “squeezing parameter” [7], which decides the strength of the potential, such that

$$ \lim_{\omega \to \infty} V_\omega(q_2, q_3) = \begin{cases} 0, & q_2 = q_3 = 0, \\ \infty, & q_2 \neq 0, q_3 \neq 0. \end{cases} $$

Following the calculations of Wang et al. [9], we directly write the Schrödinger equation for a curve as [7,10]

$$ i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \left( \Delta_s + \frac{\kappa^2}{4} \right) \Psi, $$

where $\Delta_s = \frac{\partial^2}{\partial s^2}$ is the Laplace-Beltrami operator, or in this case, just the Laplacian on the curve, with respect to the arc-length coordinate $s$, and $\kappa$ is the local curvature of the path $C$. However Wang et al. [9] noted that for curves with non-zero torsion, instead of (4), the correct way is to relax the condition (3) to

$$ \lim_{\omega \to \infty} V_\omega(q_2, q_3) = \begin{cases} 0, & |q_2| \leq \epsilon_2, |q_3| \leq \epsilon_3, \\ \infty, & |q_2| > \epsilon_2, \quad |q_3| > \epsilon_3, \end{cases} $$

where $\epsilon_2, \epsilon_3$ are regularization (length) scales in the plane normal to $C$ that do not appear in the final result. This induces harmonic confining potentials along $q_2, q_3$ that in infinite limit freezes motion along $q_2, q_3$, leaving the system in the ground state in the normal plane: [9] took

$$ \lim_{\omega \to \infty} V_\omega(q_2, q_3) = \lim_{\omega \to \infty}(m/2)\omega^2(q_2^2, q_3^2) \quad \text{with ground states} \quad |\chi_{q_2}^0 \rangle \sim \exp(-\frac{m\omega^2 q_2^2}{2}) \quad \text{and} \quad |\chi_{q_3}^0 \rangle \sim \exp(-\frac{m\omega^2 q_3^2}{2}). $$

The effective $\hat{H}_{\text{eff}}$ is recovered as [9]

$$ \langle (|\chi_{q_2}^0 \rangle \otimes |\chi_{q_3}^0 \rangle) \hat{H}(|\chi_{q_2}^0 \rangle \otimes |\chi_{q_3}^0 \rangle) \rangle = \hat{H}_{\text{eff}} $$

$$ = -\frac{\hbar^2}{2m} \left( \nabla_s^2 + \frac{\kappa^2}{4} + \frac{\tau^2}{2} \right), $$

where $\hat{H}$ is the full three-dimensional Hamiltonian and $\frac{\tau^2}{2}$ is the additional torsion-induced potential. In the next section we use (6) (differing from [13]) for a torus knot, using toroidal coordinates. We will exploit the conditions where torsion effects are small compared to curvature effects.

**Torus knots — definitions and properties.** – Torus knots are a special class of non-self-intersecting, closed curves, wound on the surface of a geometric torus in $\mathbb{R}^3$. The family of knots appears because the first homotopy group for the torus is $\pi_1(\text{torus}) \approx \mathbb{Z}_1 \times \mathbb{Z}_1$. For our purposes, each knot is defined using 2 co-prime integers, $(p, q) \in \mathbb{N}$. A $(p, q)$-torus knot winds $p$ times in the toroidal direction, about the axis of rotational symmetry of the torus $k$ times in the poloidal direction, about the cross-sectional axis of symmetry (see fig. 1). The quantity $\alpha = \frac{q}{p}$ is called the “winding number” of the $(p, q)$-torus knot, and is a simple measure of the complexity of the knot. One may note immediately that two different torus knots (differing in at least one of $p$ or $q$) will have different values of $\alpha$. In this sense, $\alpha$ may be regarded as the “unique” identity of a $(p, q)$-torus knot.

**Parametrisation of torus knots in toroidal coordinates.** Henceforth, we shall use toroidal coordinates, to exploit the inherent symmetry of the problem. We consider a torus of major radius $R$ and minor radius $d$ (see fig. 1). Then, we can define the parameter $a^2 = R^2 - d^2$, and the aspect ratio of the torus, $\cosh(q_0) = \frac{a^2}{d^2}$. For a thin torus, $a \gg 0$, the aspect ratio can be quite large.

The parametrisation for torus knots in toroidal coordinates is then given by [13]

$$ x = \frac{a \sinh(q_0) \cos(\phi)}{\cosh(q_0) - \cos(\alpha \phi)}, \quad y = \frac{a \sin(q_0) \sin(\phi)}{\cosh(q_0) - \cos(\alpha \phi)}, $$

$$ z = \frac{-a \sin(\alpha \phi)}{\cosh(q_0) - \cos(\alpha \phi)}. $$

Fig. 1: Left: the major and minor radii of a torus (picture courtesy — P. Das et al. [14]). Right: (2, 3) knot or trefoil knot.
where \(x, y, z\) are Cartesian coordinates, \(\eta_0\) fixes the toroidal surface on which the knot is wound, \(\alpha\) is the winding number of the knot as defined above, and \(0 < \phi < 2\pi\).
Note that the winding number condition has been imposed so that the only variable is \(\phi\). Since the relations in the second section involved the arc-length \(s\) we need to express \(s\) in terms of \(\phi\). Computing the differential arc-length \(ds\), from (7), we get
\[
\frac{ds}{d\phi} = \frac{1}{a\sigma + \lambda^2 - 1},
\]
where
\[
\sigma = \frac{1}{b - \cos(\alpha \phi)}, \quad \lambda^2 = a^2 + c^2, \quad c = \sinh(\eta_0), \quad b = \cosh(\eta_0),
\]
and \(\beta^2 = \lambda^2 - 1\). As we will see in the next section, these calculations will be required to rewrite the Schrödinger equation (6).

Curvature of a torus-knot. Let us start by rewriting the space curve \(C\) of the previous section as \(\vec{r}(\phi) = x(\phi)\hat{i} + y(\phi)\hat{j} + z(\phi)\hat{k}\). The curvature of a general curve \(\kappa(s)\), in Euclidean space, is given by (s is the arc-length), \(\kappa(s) = \frac{d^2 \vec{r}(s)}{ds^2}\). For a general parametrisation using \(\phi\), given by \(\vec{r}(\phi)\), \(\kappa\) is rewritten as
\[
\kappa = \frac{\sqrt{(x''y' - y''x)^2 + (y''z - z''y)^2 + (z''x - x''z)^2}}{(x'^2 + y'^2 + z'^2)^{3/2}}.
\]
In our case, we get
\[
\kappa^2 = \frac{c^2}{4} [2 - 6a^2 + 4a^4 + 2\cos(2\alpha \phi) - 2a^2 \cos(2\alpha \phi)] + 8(a^2 - 1)b \cos(\alpha \phi) + 4b^2 [4a^2(2b^2 + 2a^2 - 2)]^{-1}.
\]
(8)

Torsion of a torus knot. For a curve \(\vec{r}(s)\), the unit normal \(\hat{n}\) and binormal \(\hat{b}\) are given by \(\hat{n} = \frac{\hat{T}}{\kappa}\) and \(\hat{b} = \hat{T} \times \hat{n}\), respectively. Then, the torsion is simply given by \(\tau(s) = \kappa(s) \frac{d\phi}{ds}\). Similar to the curvature, the torsion \(\tau(\phi)\), of a curve parametrized by \(\phi\), is given by
\[
\tau = \frac{x''(y''z' - y'z'') + y''(x''z' - x'z'') + z''(x'y'' - x'y')}{{(y''z - y'z')}^2 + (x''z - x'z'')^2 + (x'y'' - x'y')^2}.
\]
(10)

Substituting in (10), \(x(\phi), y(\phi), z(\phi)\) from (7), and calculating \(\frac{\tau^2}{2}\) we get
\[
\frac{\tau^2}{2} = \frac{32\alpha^2(a^2 - 1)^2(\cos(\alpha \phi) - b)^4(\cos(\alpha \phi) + a^2 - 1)^2}{a^2(2b^2 + 2a^2 - 2)^2(3 - 6a^2 + 4a^4 - 2(a^2 - 1)\cos(2\alpha \phi) + 8b(a^2 - 1)\cos(\alpha \phi) + 4b^2 - 1)^2}.
\]
(11)

In the following sub-section, we show numerically, using Wolfram Mathematica, the regions in the parameter space of the torus knot where the torsion effect dominates over curvature effects and where the reverse occurs. Quite interestingly, through \(\alpha\) and \(\beta\) parameters, the properties of both the knot and the host torus get intertwined. We perform this exercise because later on we will study analytic solutions for energy eigenvalues and eigenfunctions that are difficult to obtain if the torsion term is present in the effective Hamiltonian. The numerical study will help us to concentrate on those sectors where the torsion contribution can be neglected, thereby reducing (6) into (4).

Numerical comparison of torsion and curvature terms. We use Mathematica to draw surface plots of \(\tau^2(\alpha, \phi)\) and \(\kappa^2(\alpha, \phi)^2\), by varying both \(\alpha\) and \(\phi\), for \(0 \leq \phi \leq 2\pi\), \(0 \leq \alpha \leq 100\). We consider essentially a thin torus with \(R = 4\) and \(d = 1\) (\(\eta_0 = 4\)) to be the torus parameters for this numerical analysis. The plots obtained are depicted in fig. 2. The vertical axis, denoted by \(T(\alpha, \phi)\), is the common axis for both \(\tau^2(\alpha, \phi)^2\) and \(\kappa^2(\alpha, \phi)^2\). It can be seen from fig. 2 that \(\frac{\tau^2}{\kappa^2}\) begins to dominate \(\frac{\tau^2}{\kappa^2}\), for \(\alpha \approx 25 \approx \cosh(\eta_0)\). Therefore, for \(\alpha \geq 10\), the torsion term in (6), is much smaller compared to the curvature term. Figure 3 shows two projections of the surface plots for fixed \(\phi\), for better clarity, clearly showing how \(\frac{\tau^2}{\kappa^2}\) begins to dominate \(\frac{\tau^2}{\kappa^2}\), for large \(\alpha\). Hence we will be considering from now on large \(\alpha\) torus knots on a thin torus so that the torsion term can be safely neglected. It is important to note that we keep the curvature contribution which will yield new features in the expression of particle wave function. We will compare and contrast with the result...
We now proceed to solve the time-independent Schrödinger equation for (4) (with torus term neglected).

The Laplacian \( \frac{d^2}{ds^2} \) can be reparametrised in terms of \( \phi \), whereby we obtain the time-independent Schrödinger equation, in toroidal coordinates, to be

\[
\left[ \left( \frac{d}{ds} \right)^2 + \left( \frac{d^2 \phi}{d\phi^2} \right) \frac{d}{d\phi} + \frac{\kappa^2}{4} \right] \psi = -\epsilon \psi, \tag{12}
\]

where \( \epsilon = \frac{2mE}{\hbar^2} \), and \( E \) is the energy. We now consider \( \psi = f(\phi)G(\phi) \), and substitute it back in (12). Putting the coefficient of \( G' \) to be zero, in this case, we obtain

\[
f(\phi) = \sigma(\phi) = \frac{1}{\sqrt{b - \cos(\alpha \phi)}}. \tag{13}
\]

For \( \kappa(\phi) = 0 \), (13) reduces to the one considered in [13]. However, a priori there is no reason to drop the curvature term. Clearly it will affect the results in a non-trivial way since \( \kappa(\phi) \) is not a numerical constant. Note the distinction between canonical quantization and confining potential approaches to the same problem, as mentioned in [7]. Now comes the unexpected result. Substituting the expression for \( \sigma \) in (13), we obtain

\[
\left( \frac{\sigma'' - 2\sigma'}{\sigma} \right) + \frac{\alpha^2 \left[ 3 - 4\beta \cos(\alpha \phi) + \cos(2\alpha \phi) \right]}{8\left[ b - \cos(\alpha \phi) \right]^2} = 0 \tag{14}
\]

Notice that the \( \phi \)-dependent terms, other than the energy, in (13), add up to a numerical constant, consisting of the winding number of the torus loop and aspect ratio of the torus. We can see this, by adding \( (a^2 \beta^2 \sigma^4 \frac{\kappa^2}{4}) \) and (14) to get

\[
\left( \frac{\sigma'' - 2\sigma'}{\sigma} \right) + a^2 \beta^2 \sigma^4 \frac{\kappa^2}{4} = \frac{b^2 + \alpha^4 - 1}{4(b^2 + \alpha^2 - 1)}. \tag{15}
\]

The numerical parameter \( \Gamma = \frac{b^2 + \alpha^4 - 1}{4(b^2 + \alpha^2 - 1)} \) incorporates global features of the knot (through \( \alpha \)) and that of the embedding torus (through \( b \)).

Substituting back in (13), we get

\[
\left[ \frac{d^2}{d\phi^2} + \frac{b^2 + \alpha^4 - 1}{4(b^2 + \alpha^2 - 1)} \right] G = 0. \tag{16}
\]

Thereby, we shall solve (16) for \( G(\phi) \). The complete solutions to (4) may be obtained by multiplying \( G \) with \( \sigma(\phi) \). We shall also obtain the energy eigenvalues for the thin-torus approximation, by imposing the condition of periodicity on the wave functions.

**Solving for the energy eigenvalues.** – To have a better understanding of (16), we expand the third term in the equation, as a binomial series, to obtain

\[
\left[ \frac{d^2}{d\phi^2} + \left( \Gamma + \frac{a^2 \beta^2 \epsilon}{b^2} \sum_{k=0}^{\infty} (-1)^k \left( k + 1 \right) \frac{1}{k} \right) \frac{\cos(\alpha \phi)}{b} \right] G = 0. \tag{17}
\]

Note that this expansion is meaningful, since \( b = \cosh(\eta_0) > 1 \geq \cos(\alpha \phi), \forall \eta_0 \in (0, \infty) \).

For thin-torus approximation, we consider large \( b \), for which the higher-order terms in (17) may be neglected.

On substituting \( 2z = \alpha \phi \), (17) can immediately be seen to be of the form

\[
\left[ \frac{d^2}{dz^2} + \Theta_0 + \sum_{r=1}^{\infty} \Theta_{2r} \cos(2rz) \right] G = 0, \tag{18}
\]

where the \( \Theta_i \) are constants depending on the energy and torus knot parameters. Equation (18) is the well-known Hill equation.

Although the Hill equation was also obtained in [13], the parameters \( \Theta_i \) we have obtained differ from their results, since we have introduced the curvature term in the “confining potential approach”, unlike the canonical quantization used by them. More importantly our result contains the knot characteristic parameter \( \alpha \) of the path which is absent in [13].
Since there are no closed-form, analytic solutions to (18), we look for solutions to simpler cases, arising in the thin-torus limit. 

**Thin-torus approximation.** For a thin torus, we may assume \( b^2 \approx c^2 \), and neglect terms in (17), having overall order \( O(\frac{1}{b}) \) and above, in \( \frac{1}{b} \). With these approximations, we obtain from (17),

\[
\begin{align*}
\left[ \frac{d^2}{dz^2} + \left( \frac{4\ell}{\alpha^2} + \frac{4a^2\beta^2 c}{b^2\alpha^2} \right) - \frac{2a^2\beta^2 c}{b^3} \cos(2z) \right] G &= 0. \tag{19}
\end{align*}
\]

This is the Mathieu differential equation, also obtained for the thin-torus limit, in [13], albeit with different coefficients. The applicable solutions to (19), with the correction boundary conditions, are given by Mathieu functions \( \nu \) coefficients. The general solutions to (19), with the correct boundary conditions, are given by Mathieu functions of fractional order \( \nu \) which yields

\[
G = A se_{\nu}(z,b) + B ce_{\nu}(z,b), \tag{22}
\]

where \( A \) and \( B \) may be determined from normalization conditions. The functions, \( se_{\nu}(z,b) \) and \( ce_{\nu}(z,b) \), are of the form \([15]\)

\[
\begin{align*}
se_{\nu}(z,b) &= \sin(\nu z) + \frac{\hbar^2}{16mb^2} \sin(3\nu z) + O\left( \frac{1}{b^2} \right), \tag{23}
ce_{\nu}(z,b) &= \cos(\nu z) + \frac{\hbar^2}{16mb^2} \cos(3\nu z) + O\left( \frac{1}{b^2} \right). \tag{24}
\end{align*}
\]

The final eigenfunction, given by \( \psi = \sigma(\phi)G(\phi) \), with \( B = 0 \) in \( G \), is plotted in fig. 4 (ground state).

**Brief discussion on the results obtained.** Our result in (20), for a thin torus, describes how the energy eigenvalues depend both on the geometric properties of the torus, through \( b \) and \( c \), as well as on the property of the torus knot, via the “unique” quantity \( \alpha \) (third section). We notice that, for the case \( \mathcal{O}(\alpha^2) \approx \mathcal{O}(c^2) \approx \mathcal{O}(b^2) \), i.e., for large winding numbers \( \alpha \), the obtained result deviates from that obtained in [13], even in the thin-torus limit.

This is a crucial difference from the existing literature (to the best of our knowledge), as the topological complexity of knots, manifested through the parameter \( \alpha \) in (20) does indeed play a role in modifying the energy eigenvalues, even in the thin-torus limit, as opposed to the results obtained in Sreedhar [13], which remains unchanged for arbitrary \( \alpha \) (with fixed values of \( p \)). Note that the energy eigenvalues in (20) degenerate into the well-known energy expression for a particle on a ring (with \( \tau = 0 \)), if we consider \( \mathcal{O}(b^2) \approx \mathcal{O}(c^2) \) and \( \alpha^2 \ll c^2 \approx b^2 \), i.e., for small winding numbers. We also observe from (23) and (24), that for large values of \( \mathcal{O}(b^2) \), the solutions degenerate into sine and cosine functions, which is expected for a particle on a ring (thin-torus knot) since a thin-torus knot, is essentially a putative circle. The complete eigenfunctions of the original problem, may be obtained from (22), by multiplying by \( \sigma(\phi) \).

**Surface plot of the correction factor.** In this subsection, we construct a graphical representation (fig. 5) of the regions in the \((\alpha, \eta)\) parameter space, where we indicate how our results deviate from [13] resulting from the curvature effect. From the surface plot for \( F(\eta, \alpha) \), in fig. 5, we see that the factor \( F(\eta, \alpha) \approx 1 \), for low values of \( \alpha \) and high values of \( \eta \). Hence, for the parameters in the “red” region, our result for the energy eigenvalue expression (20) reduces to the energy expression \( E_{0,n} \), obtained by Sreedhar.

The “bluer” regions depict subspaces (of the parameter space), with \( \mathcal{O}(b^2) \approx \mathcal{O}(\alpha^2) \), while the “reddish” region

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Fig. 4: Left: a surface plot of $\sigma(\eta, \phi)G(\eta, \phi)$ showing the un-normalised ground-state eigenfunction $\psi(\alpha = 1)$ in (12). Right: two slices of the surface plot for $\eta = 1.5$ and $\eta = 6$. The sinusoidal nature is more pronounced for $\eta = 6$, as expected for torus knots with large aspect ratio $\eta$.

Fig. 5: Left: the surface plot with $\alpha \in (0, 1000)$ and $\eta \in (3, 10)$. Right: top view of the same plot. In the red zone $F \approx 1$ where our results roughly agree with [13] whereas in the blue region the results greatly disagree.

Fig. 6: Left: the plot for $\alpha = 1.5$. Right: the plot for $\alpha = 3.5$. In the left panel both results, ours (red curve) and that of [13] (blue curve) agree with the simulated result (green curve). In the right panel, the result of [13] deviates from the green line but our result is closer to the simulated result.

is for $O(\alpha^2) \ll O(b^2)$. This is in accordance with the predictions from the theoretical results, as discussed in the sect. “Brief discussion on the results obtained”. However, our result deviates significantly from [13] in the “blue” regions of the surface plot, since we have correctly taken into account the curvature term.

**Numerical simulation.** – Finally we briefly demonstrate the validity of the analytic expression of energy eigenvalues computed from the thin-torus approximated expression, with a numerical result obtained by considering the exact equation (16). We use Wolfram Mathematica to numerically solve (16), by moving $\epsilon$ to the RHS and...
cal calculations, using 6 eigenvalues of this equation, for the result with the numerical values to confirm the theoretical predictions. The idea is to demonstrate and compare our thin-torus model and new results are the following. i) We have computed energy eigenvalues (with torsion neglected) analytically and numerically. The new feature in our result is that the energy eigenvalues change with \( \alpha \) for large \( \alpha \), even in the thin-torus limit, and more so than that of [13] even though both have the same \( n^2 \)-dependence. Indeed, in other geometries, such as in the Archimedean spiral path (see, e.g., [16]), the geometric potential changes the energy eigenvalues and eigenfunctions in a qualitative way. However, note that if we kept the torsion term in our final analysis, the Schrödinger equation would differ qualitatively from the Hill equation (18) leading to non-trivial effects. iv) In previous works, one of us [17,18] has considered the Hannay angle and Berry phase for a similar problem with the torus itself revolving around its vertical axis, omitting torsion and curvature effects. We hope that inclusion of those effects can account for the topologically non-trivial nature of the torus knot to induce new results.

**Conclusion and open problems.** – The present work deals with the quantum mechanics of a point particle moving along a torus knot on a regular torus. Novelty of our model and new results are the following. i) We have correctly taken into account the torsion and curvature terms in the Hamiltonian. ii) Explicit dependence of the torsion and curvature terms on the nature of torus and torus knot has led us to conditions where the latter dominates such that the torsion term is ignored. iii) We have computed energy eigenvalues and eigenfunctions in the thin torus approximation (with torsion neglected) analytically and numerically. The new feature in our result is that the energy depends on both \( q,p \) that depicts the knottedness of the path as well as on the embedding torus parameter. Because of this difference our result is richer in structure and more natural than that of [13] even though both have the same \( n^2 \)-dependence.

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