On the Continuity of Bounded Weak Solutions to Parabolic Equations and Systems with Quadratic Growth in Gradients.

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Abstract

We establish the pointwise continuity of bounded weak solutions to of a class of scalar parabolic equations and strongly coupled parabolic systems. Our approach to the regularity theory of parabolic scalar equations is quite elementary and its applications to strongly coupled systems does not require higher $L^p$ integrability of derivatives.

1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial \Omega$ and $T$ be a positive number. Denote $Q = \Omega \times (0,T)$ and $z = (x,t)$ a generic point in $Q$.

In the first part of this paper, we consider the following scalar parabolic equation

$$v_t = \text{div}(aDv + b(v)) + e(Dv) + g(v) \quad \text{in } Q.$$  \hfill (1.1)

Here, $a, e, g$ are scalar functions and $b \in \mathbb{R}^N$. The equation is regular parabolic in the sense that $a$ is bounded and $a \geq \lambda_0 > 0$ for some constant $\lambda_0$.

Under suitable integrability conditions, much weaker than those in literature (e.g. [5, 6]), on the data of this equation we will show that any bounded weak solution $v$ of (1.1) is pointwise (or Hölder) continuous. Also important, we allow $e(Dv)$ to have a quadratic growth in $Dv$. That is, $e(Dv) \leq C(|Dv|^2 + 1)$.

In the second part, we will apply the theory for scalar equations to systems of $m$ equations, $m \geq 2$, with linear or quadratic growth in gradients

$$u_t = \text{div}(A(u)Du) + e(Du) + f(u) \quad \text{in } Q.$$ \hfill (1.2)

Here, $u = [u_i]_{i=1}^m$ and $A(u)$ is a $m \times m$ matrix and $e, f$ are vectors in $\mathbb{R}^m$. We will always assume that there is $\lambda_0 > 0$ such that for all $u \in \mathbb{R}^m$, $\zeta \in \mathbb{R}^{mN}$ and $i = 1, \ldots, m$

$$\langle A(u)\zeta, \zeta \rangle \geq \lambda_0 |\zeta|^2.$$ \hfill (1.3)

Interestingly, we are able to establish the regularity of bounded weak solutions to parabolic systems on planar domains, where the 'hole filling' technique of Widman [14] for elliptic systems (e.g. [1]) does not seem to be extendable. We also consider triangular systems on any dimension domains and assert that bounded weak solutions are pointwise continuity continuous. Our methods does not require higher $L^p$ integrability of derivatives as in classical works (e.g. [4]).

In Section 2 we recall a simple parabolic version of the usual Sobolev inequality. We discuss scalar equations in Section 3. We conclude the paper with applications to systems in Section 4.

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Mathematics Subject Classifications: 35K40, 35B65, 42B37.
Key words: Cross diffusion systems, weak solutions regularity.
2 Some technical lemmas

In this section, we will present some technical results which will be used throughout this paper. We recall the following simple parabolic version of the usual Sobolev inequality

**Lemma 2.1** Let \( r = 2/N \) if \( N > 2 \) and \( r \in (0, 1) \) if \( N \leq 2 \). If \( g, G \) are sufficiently smooth then

\[
\iint_{\Omega \times I} |g|^{2r} |G|^2 \, dz \leq C \sup_I \left( \int_{\Omega} |g|^2 \, dx \right)^r \left( \iint_{\Omega \times I} (|DG|^2 + |G|^2) \, dz \right).
\]

If \( G = 0 \) on \( \partial \Omega \) then we can drop the integrand \( |G|^2 \) on the right hand side.

In particular, if \( g = G \) we have

\[
\iint_{\Omega \times I} |g|^{2(1+r)} \, dz \leq C(N, |\Omega|) \sup_I \left( \int_{\Omega} |g|^2 \, dx \right)^r \left( \iint_{\Omega \times I} (|Dg|^2 + |g|^2) \, dz \right).
\]

Putting \( g = G = |u|^p \) (with \( p > 1 \)) and using Young’s inequality, we can see that for some constant \( c_0 > 0 \)

\[
\left( \iint_Q |u|^{2p(1+\frac{2}{N})} \, dz \right)^{\frac{N}{N+2}} \leq c_0 \left( \sup_{(0,T)} \int_{\Omega} |u|^{2p} \, dx + \iint_Q |u|^{2p-2} |Du|^2 \, dz \right). \tag{2.1}
\]

In addition, if \( \gamma \in (1, 1 + 2/N) \), then an use of Young and Sobolev’s inequalities also gives that for any \( \epsilon > 0 \) there is a constant \( C(\epsilon) \) such that

\[
\|u\|_{L^{2p\gamma}(Q)} \leq \epsilon \left( \sup_{(0,T)} \int_{\Omega} |u|^{2p} \, dx + \iint_Q |u|^{2p-2} |Du|^2 \, dz \right)^{\frac{1}{2p}} + C(\epsilon)\|u\|_{L^{2p}(Q)}. \tag{2.2}
\]

3 On scalar equations

We now revisit the regularity theory of scalar equations with integrable coefficients, in the class \( \mathcal{M}(\Omega, T) \) defined below. These new improvements serve well our purposes in the next section.

For any \( x_0 \in \Omega, R > 0 \) and \( t_0 \geq 4R^2 \), we define \( \Omega_R(x_0) = \Omega \cap B_R(x_0) \) and \( Q_R(x_0) = \Omega_R(x_0) \times (t_0 - R^2, t_0) \). If \( x_0, t_0 \) are understood from the context, we simply drop them from the notations.

**Definition of the class \( \mathcal{M} \):** We say that a function \( f : Q \to \mathbb{R} \) (or \( \mathbb{R}^m \)) is of class \( \mathcal{M}(\Omega, T) \) if for any \( \epsilon > 0 \) there is \( R(\epsilon) > 0 \) such that \( \forall R \in (0, R(\epsilon)) \) either

\[ i) \quad \sup_{(0,T)} \|f\|_{L^\infty(\Omega_R)} < \epsilon, \]

or

\[ ii) \quad \|f\|_{L^{\frac{N+2}{N}}(Q_R)} < \epsilon. \]

Alternatively, we also define the class \( \mathcal{M}(\Omega, T) \)
Definition of the class $\mathcal{M}$: We say that a function $f : Q \rightarrow \mathbb{R}$ (or $\mathbb{R}^n$) is of class $\mathcal{M}(\Omega, T)$ if for some $p_0 > N/2$ such that either one of the quantities i) $\sup_{(0,T)} \|f\|_{L^{p_0}}(\Omega)$ or ii) $\|f\|_{L^{p_0+1}}(Q)$ is finite.

By Hölder inequality, it is easy to see that $\mathcal{M}(\Omega, T) \subset \mathcal{M}(\Omega, T)$.

3.1 Global boundedness and a local estimate

We consider scalar equation

$$
\begin{cases}
    v_t = \text{div}(ADv + B) + G & \text{in } Q, \\
    v = v_0 & \text{in } \Omega.
\end{cases}
$$

(3.1)

Here $A, G$ are scalar functions. As usual, we will assume that there is a positive number $\lambda_0$ such that

$$A \in \mathcal{M}(\Omega, T) \text{ and } A \geq \lambda_0. \tag{3.2}$$

We also assume that there is a function $\Phi \in \mathcal{M}(\Omega, T)$ such that $\Phi \geq \lambda_0$ on $Q$ and

$$|B|^2, |G| \leq \Phi. \tag{3.3}$$

By using Steklov average, a weak solution of (3.1) satisfies for all $\eta \in C^1(Q)$

$$\iint_Q u_t \eta + ADv D\eta \, dz = \iint_Q G \eta \, dz. \tag{3.4}$$

Applying the usual Moser iteration argument, we derive

**Lemma 3.1** Assume (3.2) and the growth condition (3.3) and $v$ is a solution of (3.4). Then there is a constant $C$ such that

$$\sup_{Q} |v| \leq C \left( \iint_Q v^2 \, dz \right)^{\frac{1}{2}}. \tag{3.5}$$

The proof of this lemma bases on a Moser iteration technique by testing the equation with $|v|^{2p-2}v$ similar to the local version below.

We discuss the local estimates. This type of estimates will be useful for later investigations on the regularity of weak solutions. We will assume that the function $A$ is bounded. Note that $A$ may depend on $v$ in general. But $|v|$ is globally bounded by the above lemma.

**Lemma 3.2** Assume (3.2), the growth condition (3.3) and that $A$ is bounded. Let $v$ be a solution of (3.4) and $B_R$ be a ball in $\mathbb{R}^N$. Then there is a constant $C$ such that

$$\sup_{Q_R} |v| \leq C \left( \frac{1}{R^{N+2}} \iint_{Q_{2R}} v^2 \, dz \right)^{\frac{1}{2}}. \tag{3.6}$$

We will make use of the Sobolev inequality for any $q \in (1, 2N/(N-2))$ and $\varepsilon > 0$ there is $C(\varepsilon)$ such that

$$\left( \int_{\Omega} |v|^q \, dx \right)^{\frac{2}{q}} \leq \varepsilon \int_{\Omega} |Dv|^2 \, dx + C(\varepsilon) \int_{\Omega_R} |v|^2 \, dx. \tag{3.7}$$
If $q = 2N/(N-2)$ then we can only assert that

$$\left( \int_{\Omega} |v|^{\frac{2N}{N-2}} \, dx \right)^{\frac{N-2}{N}} \leq C \int_{\Omega} |Dv|^2 \, dx + C \int_{\Omega_R} |v|^2 \, dx. \quad (3.8)$$

The proof is the standard Moser iteration argument by testing (3.4) with $|v|^{2p-2}v\phi^2\eta$ with some $p \geq 1$ and $\phi, \eta$ are respectively cutoff functions for concentric balls $B_R, B_{2R}$ and intervals $[-2R^2, -R^2], [-R^2, 0]$ with $|D\phi| \leq C/R, |D\eta| \leq C/R^2$. Let $V = |v|^{2p}$. Using (3.3) and Young’s inequality it is standard to derive (see [6, 9])

$$\sup_{(0,T)} \int_{\Omega} |V|^2 \phi^2\eta^2 \, dx + \iint_{Q} |Dv|^2 \phi^2\eta^2 \, dz \leq C \iint_{Q} \Phi |V|^2 \phi^2\eta^2 \, dz + C \iint_{Q} |V|^2 \phi^2\eta^2 \, dz.

By H"older inequality with $q = 2N/(N-2)$ so that $q/(q-2) = N/2$ and because $\Phi \in M(\Omega, T)$ we see that, assuming i) in the definition of $M$, for any $\varepsilon > 0$ if $R$ is sufficiently small then

$$\iint_{Q_{2R}} \Phi |V|^2 \phi^2\eta^2 \, dz \leq \varepsilon \int_{-2R^2}^{0} \left( \int_{\Omega_R} \Phi \frac{|V|^2}{\phi^2} \, dx \right)^{\frac{1}{q}} \left( \int_{\Omega} |V\phi\eta|^q \, dx \right)^{\frac{2}{q}} \, dt \leq \varepsilon \int_{-2R^2}^{0} \left( \int_{\Omega} |V\phi\eta|^q \, dx \right)^{\frac{2}{q}} \, dt.$$ 

Because $|D\phi| \leq C/R$, we derive

$$\varepsilon \left( \int_{\Omega} |V\phi\eta|^q \, dx \right)^{\frac{2}{q}} \leq \varepsilon \int_{\Omega} |Dv|^2 \phi^2\eta^2 \, dx + \frac{C(\varepsilon)}{R^2} \int_{Q_{2R}} |V|^2 \, dx.$$

If $\varepsilon$ is sufficiently small in terms of $\lambda_0$ then it follows that

$$\sup_{(-R^2,0)} \int_{\Omega_R} |V|^2 \, dx + \iint_{Q_R} |Dv|^2 \, dz \leq C \frac{R^2}{R^2} \iint_{Q_{2R}} |V|^2 \, dz. \quad (3.9)$$

By the parabolic Sobolev inequality for $\gamma = 1 + 2/N > 1$, we obtain for any $p \geq 1$

$$\left( \iint_{Q_{2R}} |v|^{2p\gamma} \, dz \right)^{\frac{1}{\gamma}} \leq C(\varepsilon) \frac{R^2}{R^2} \iint_{Q_{2R}} |v|^{2p} \, dz.$$

A standard Moser iteration argument (e.g. [8]) implies the local estimate of the lemma. If ii) in the definition of $M$ holds then for $\gamma = 1 + 2/N$ we can also use H"older inequality ($\gamma' = (N+2)/2$) to have for $R$ small

$$\iint_{Q_{2R}} \Phi V^2 \, dz \leq \left( \iint_{Q_{2R}} \Phi' \, dz \right)^{\frac{1}{\gamma'}} \left( \iint_{Q_{2R}} V^{2\gamma} \, dz \right)^{\frac{1}{\gamma}} \leq \varepsilon \left( \iint_{Q_{2R}} V^{2\gamma} \, dz \right)^{\frac{1}{\gamma}}. \quad (3.10)$$

Therefore, we can use the above estimate and the parabolic Sobolev inequality to treat the last integral to obtain the local estimate (3.9) again and the proof can go on.
If $\Phi \in M(\Omega, T)$, depending whether i) or ii) in its definition holds, we define a positive function $\nu$ in $R \geq 0$ by

$$
\nu(R) := \sup_{(-R^2,0)} \left( \int_{B_R} \Phi^{\frac{N}{2}} \; dx \right)^{\frac{2}{N}} \text{ or } \left( \int_{Q_R} \Phi^{\frac{N+2}{2}} \; dz \right)^{\frac{2}{N+2}}.
$$

(3.11)

It is clear that $\nu$ is increasing and continuous at 0 ($\nu(0) = 0$). Also, because either

$$
\int_{Q_R} \Phi \; dz \leq \int_{-R^2}^0 \left( \int_{B_R} \Phi^{\frac{N}{2}} \; dx \right)^{\frac{2}{N}} (R^N)^{\frac{N-2}{N}} \; dt \leq \nu(R) R^N
$$

or

$$
\int_{Q_R} \Phi \; dz \leq \left( \int_{Q_R} \Phi^{\frac{N+2}{2}} \; dz \right)^{\frac{2}{N+2}} (R^{N+2})^{\frac{N}{N+2}} = \nu(R) R^N,
$$

we observe that in both cases

$$
\int_{Q_R} \Phi \; dz \leq \nu(R) R^N.
$$

(3.12)

### 3.2 Hölder continuity

We will study the Hölder regularity in this subsection. Note that the bounds for the Hölder norm and exponents will depend only on the generic constants the parameters in their definition $M(\Omega, T)$. This fact will play a crucial role when we estimate the derivatives which appear in cross diffusion systems.

**Theorem 3.3** Assume that $\mathbf{a} \geq \lambda_0$ and $\mathbf{a}$ is bounded and $|\mathbf{b}|^2, |\mathbf{g}| \leq \Phi \in M(\Omega, T)$. Let $v$ be a bounded weak solution of

$$
v_t = \text{div}(\mathbf{a} Dv + \mathbf{b}) + \mathbf{e}(Dv) + \mathbf{g}.
$$

(3.13)

If $\mathbf{e}(Dv) \leq \varepsilon_0 |Dv|^2 + \Phi$ for some $\varepsilon_0 > 0$ with $\varepsilon_0 \sup_Q |v|$ is sufficiently small compared to $\lambda_0$ then $v$ is pointwise continuous. The continuity of $v$ depends on those of the integrals in (3.12) on the measure of their domains.

For simplicity we assume first that $\mathbf{b} \equiv 0$. The case $\mathbf{b} \neq 0$ is similar will be discussed in Remark 3.4 after this proof.

The idea based on that of [7]. We present the details and nontrivial modification.

Fixing any $x_0 \in \Omega$, $t_0 > 0$ and $4R^2 < t_0$, we denote $Q_{iR} = \Omega_{iR} \times (t_0 - iR^2, t_0)$ where $\Omega_{R} = \Omega \cap B_R(x_0)$.

Let $M_i = \sup_{Q_{iR}} v$, $m_i = \inf_{Q_{iR}} v$ and $\omega_i = M_i - m_i$. For some number $\nu_\ast > 0$ will be determined later and the function $\nu$ as in (3.12) we define $\delta(R) = \nu_\ast \sqrt{\nu(R)}$ and

$$
N_1(v) = 2(M_4 - v) + \delta(R), \quad N_2(v) = 2(v - m_4) + \delta(R),
$$

$$
w_1(v) = \log \left( \frac{\omega_4 + \delta(R)}{N_1(v)} \right), \quad w_2(v) = \log \left( \frac{\omega_4 + \delta(R)}{N_2(v)} \right).
$$
We will prove that either \( w_1 \) or \( w_2 \) is bounded from above on \( Q_{2R} \) by a constant \( C \) independent of \( R \). This implies a decay estimate for some \( \varepsilon \in (0,1) \) and all \( R > 0 \)

\[
\omega_2 \leq \varepsilon \omega_1 + C \delta(R). \tag{3.14}
\]

It is standard to iterate (3.14) to obtain (see [4, Lemma 8.23])

\[
\omega(R) \leq C \left[ \left( \frac{R}{R_0} \right)^{\alpha} \omega(R_0) + \delta(R) \right] \quad \forall R \in (0, R_0)
\]

for any \( \mu \in (0,1) \) and some \( R_0, \alpha > 0 \). This gives the continuity of \( v \) as \( \lim_{R \to 0} \delta(R) = 0 \). To see (3.14), if either \( w_1 \) or \( w_2 \) is bounded from above by \( C > 0 \) in \( Q_{2R} \) then either

\[
\omega_4 + \delta(R) \leq 2C(\omega_4 + m_4 - v) + C \delta(R) \quad \text{or} \quad \omega_4 + \delta(R) \leq 2C(\omega_4 + v - M_4) + C \delta(R).
\]

Taking the supremum (respectively infimum) over \( Q_{2R} \) and replacing \( m_4 \) by \( M_4 \) respectively \( M_4 \) by \( M_2 \), we obtain \( \omega_2 \leq \varepsilon \omega_4 + C(1-\delta(R)) \) for \( \varepsilon = \frac{2C-1}{2\alpha} < 1 \). This yields (3.14).

Thus, we just need to show that either \( w_1 \) or \( w_2 \) is bounded from above on \( Q_{2R} \). Before proving this, we note the following crucial property of the functions \( w_1, w_2 \). We will see that \( w_1 \leq 0 \Leftrightarrow w_2 \geq 0 \) and vice versa. Indeed,

\[
w_1 \leq 0 \Leftrightarrow \omega_4 + \delta(R) \leq 2(M_4 - v) + \delta(R) \Leftrightarrow 2(v - M_4) + \delta(R) \leq \omega_4 + \delta(R) \Leftrightarrow w_2 \geq 0. \tag{3.15}
\]

**Proof:** For any \( \eta \in C^1(Q) \) and \( \eta \geq 0 \), observe that

\[
Dw_1 = \frac{2Dw}{N_1(v), (w_1)_t = \frac{2v}{N_1(v)} \quad \text{and} \quad Dw_2 = -\frac{2Dv}{N_2(v)} \quad (w_2)_t = -\frac{2v}{N_2(v)}.
\]

So, by multiplying the equation of \( v \) by \( \eta/N_1(v) \) and \( -\eta/N_2(v) \) respectively by \( w, N(v) \) we obtain

\[
\int_\Omega \frac{\partial w}{\partial t} \eta \, dx + \int_\Omega \langle aDw, d_\eta \rangle \, dx + \int_\Omega \langle aDv, \eta Dv/N_2(v) \rangle \, dx = 2 \int_\Omega \frac{e(Dv) + g}{N(v)} \eta \, dx. \tag{3.16}
\]

Because \( \langle aDv, Dv \rangle \geq \lambda_0|Dv|^2 \) and the assumption \( e(Dv) \leq \varepsilon_*|Dv|^2 + \Phi \)

\[
\left| \frac{e(Dv)}{N(v)} \right| \leq N(v) \frac{\varepsilon_*|Dv|^2 + \Phi}{N^2(v)} \leq \varepsilon_* \left( 4 \sup_Q |v| + 1 \right) \frac{|Dv|^2}{N^2(v)} + \frac{\Phi}{N(v)},
\]

assuming \( \nu(R_0) < 1 \), we can absorb (and discard) the integral of \( \frac{e(Dv)\eta}{N(v)} \) into that of \( \langle aDv, \eta Dv/N_2(v) \rangle \) if \( \varepsilon_* \sup_Q |v| \) is sufficiently small (compared to \( \lambda_0 \)) to get

\[
\int_\Omega \frac{\partial w}{\partial t} \eta \, dx + \int_\Omega \langle aDv, d_\eta \rangle \, dx \leq 4 \int_\Omega \frac{\Phi}{N(v)} \eta \, dx. \tag{3.17}
\]

Testing (3.17) with \( (w^+)^{2^{p-1} - 2} \) as in Lemma 3.2 with \( G = \Phi/N(v) \). Because \( \lambda_0 \leq a \) and \( a \) is bounded and since \( N(v) \geq \nu_* \nu(R) \) and by the definition (3.11) of \( \nu \) then either

\[
\|G\|_{L^\infty(\Omega_R)} \leq \frac{1}{\nu_* \nu(R)} \|\Phi\|_{L^\infty(\Omega_R)} = \frac{1}{\nu_*}
\]

or \( \|G\|_{L^{\frac{N+2}{N-2}}(Q_R)} \leq \frac{1}{\nu_* \nu(R)} \|\Phi\|_{L^{\frac{N+2}{N-2}}(Q_R)} = \frac{1}{\nu_*} \). Thus, we see that \( G \in M(\Omega, T) \) if \( \nu_* \) large.

By Lemma 3.2 which applies if \( 1/\nu_* \) is sufficiently small, we find a constant \( C \) such that

\[
\sup_{\Omega_{2R} \times (t_0 - 2R^2, t_0)} w^+ \leq C \left( \frac{1}{R^{N+2}} \int_{\Omega_{14R} \times (t_0 - 4R^2, t_0)} w^2 \, dz \right)^{\frac{1}{2}}. \tag{3.18}
\]
If we can show that for any $R > 0$ there is a constant $C$ such that
\[
\frac{1}{R^{N+2}} \int_{Q_{2R} \times (t_0 - 4R^2, t_0)} w^2 \, dz \leq C
\] (3.19)
then this implies $w^+$ is bounded on $Q_{2R}$. So, the decay estimate (3.14) holds.

Let $\eta$ be a cut-off function for $B_{2R}, B_{4R}$. Replacing $\eta$ in (3.16) by $\eta^2$ (keeping the third term on the left hand side and using the assumption that $\varepsilon_* \sup_Q |v|$ is small again), we get
\[
\frac{d}{dt} \int_{Q_{2R}} w \eta^2 \, dx + \int_{Q_{2R}} a |Dw|^2 \eta^2 \, dx \leq \frac{1}{N} \int_{Q_{2R}} a |Dw| |D\eta| \, dx + 4 \int_{Q_{2R}} \frac{\Phi}{N(v)} \eta^2 \, dx
\] (3.20)
Applying Young’s inequality we derive (as $|D\eta| \leq C/R$ and $a$ is bounded)
\[
\frac{d}{dt} \int_{Q_{2R}} w \eta^2 \, dx + \int_{Q_{2R}} a |Dw|^2 \eta^2 \, dx \leq \frac{1}{N} \int_{Q_{2R}} a \, dx + 4 \int_{Q_{2R}} \frac{\Phi}{N(v)} \eta^2 \, dx
\] (3.21)
\[
\leq CR^{N-2} + 4 \int_{Q_{2R}} \frac{\Phi}{N(v)} \eta^2 \, dx.
\]

Set $I_* = [t_0 - 4R^2, t_0 - 2R^2]$, $Q_* = B_{2R} \times I_*$ and $Q_v = \{(x, t) \in Q_* : w_1 \leq 0\}$. It is easy to see that $w_2 \leq 0$ on $Q_* \setminus Q_v$ (see (3.15)). Therefore one of $w_2^+, w_2^-$ must vanish on a subset $Q_0^*$ of $Q_*$ with $|Q_0^*| \geq \frac{1}{2} |Q_*|$. We denote by $w$ such function. Let $Q_t^0$ be the slice $Q_0 \cap (B_{2R} \times \{t\})$ then $Q_t^0 = \cup_{\Omega \in L} \Omega_t$. For $t \in I_*$ let
\[
\Omega_t^0 = \{x : \omega^+(x, t) = 0\}, \quad m(t) = |Q_t^0|.
\]
The fact that $|Q^0| \geq \frac{1}{2} |Q_*|$ implies $\int_{I_*} m(t) \, dt \geq \frac{1}{2} R^{N+2}$.

We now set
\[
V(t) = \frac{\int_{Q_{2R}} w \eta^2 \, dx}{\int_{Q_{2R}} \eta^2 \, dx}.
\]
By the weighted Poincaré’ inequality ([10] Lemma 3])
\[
\int_{Q_{2R}} (w - V)^2 \eta^2 \, dx \leq CR^2 \int_{Q_{2R}} |Dw|^2 \eta^2 \, dx.
\]
Reducing the integral on the left to the set $Q_t^0$ where $w \leq 0$ (so that $V^2 \leq (w - V)^2$), we have
\[
V^2(t) m(t) \leq CR^2 \int_{Q_t^0} |Dw|^2 \eta^2 \, dx.
\]
Since $N(v) \geq \nu_* \nu(R)$ on $Q_{4R}$, the above estimate and (3.21) implies that ($V'$ denotes the $t$ derivative)
\[
R^N V'(t) + \frac{1}{R^2} V^2(t) m(t) \leq CR^{N-2} + \frac{4}{\nu_* \nu(R)} \int_{Q_{2R}} |\Phi| \eta^2 \, dx.
\] (3.22)
Because $\nu$ is given by (3.11), we also get from (3.12)
\[
\|G\|_{L^1(Q_R)} \leq \frac{1}{\nu_* \nu(R)} \int_{Q_R} \Phi \, dz \leq CR^N.
\] (3.23)
We show that \(V(t_1)\) is bounded on for some \(t_1 \in I_\ast\). Indeed, suppose \(V(t) \geq A > 0\) in \(I_\ast\). We have from (3.22)

\[
R^{N+2} \frac{V'(t)}{V^2(t)} + m(t) \leq \frac{C}{A^2} \left( R^N + \frac{R^2}{\nu_\ast \nu(R)} \int_\Omega |\Phi| \eta^2 \, dx \right).
\]

Because \(\int_{I_\ast} m(t) dt \geq \frac{1}{2} R^{N+2}\) and \(|I_\ast| \sim R^2\), we integrate this over \(I_\ast\) and use (3.23) to see that

\[
R^{N+2} \leq \int_{I_\ast} m(t) dt \leq R^{N+2} \left( \frac{2}{A} + \frac{C}{\nu_\ast A^2} \right).
\]

By choosing \(A\) large we get a contradiction. So, we must have \(V(t_1) \leq A\) for some \(t_1 \in I_\ast\).

Integrating (3.21) over \([t_1, t_2]\) for any \(t_2 \in I_0 = [t_0 - 2R^2, t_0]\), we have

\[
V(t_2) \int_{\Omega \times \{t_2\}} \eta^2 \, dx + \int_{\Omega \times I_0} a |Dw|^2 \eta^2 \, dz \leq CR^N + V(t_1) \int_{\Omega \times \{t_1\}} \eta^2 \, dx.
\]

This implies that \(V(t_2) \leq C\) for all \(t_2 \in I_0\) and \(\int_{\Omega \times I_0} a |Dw|^2 \eta^2 \, dz \leq CR^N\). This implies that \(V(t_2) \leq C\) for all \(t_2 \in I_0\) and \(\int_{\Omega \times I_0} a |Dw|^2 \eta^2 \, dz \leq CR^N\). Since we can always assume that \(\omega_1 \geq \nu(R)\) (otherwise there is nothing to prove) so that \(V(t)\) is bounded from below by \(-\log(\nu_\ast)\). Hence, \(|V(t)|\) is bounded.

By Poincaré’s inequality again we have

\[
\int_{\Omega \times I_0} (w - V)^2 \eta^2 \, dz \leq CR^2 \int_{\Omega \times I_0} |Dw|^2 \eta^2 \, dz \leq CR^{N+2}.
\]

Since \(|V(t)|\) is bounded on \(I_0\), replacing \(R\) by \(2R\), the above implies the desired (3.19) and concludes our proof.

**Remark 3.4** The assertion still holds if \(b \neq 0\) and \(|b|^2 \in \mathcal{M}(\Omega, T)\). Indeed, there will be an extra term in our argument. Namely, replacing \(\eta\) by \(\eta/N(v)\) as before we have the following extra terms in (3.16)

\[
\int_\Omega \left( b, \frac{D\eta}{N(v)} + \frac{\eta Dv}{N^2(v)} \right) \, dx = \int_\Omega \left( \frac{b}{N(v)}, D\eta + \frac{1}{2} D\eta \right) \, dx.
\]

We take \(\eta\) to be \(|w|^{2p-2} w \eta^2\) and think of \(B_1 := b/N(v), B_2 := 1/2 b/N(v)\) in

\[
w_t = \text{div}(ADw + B_1) + B_2 Dw + G
\]

(compared with (3.11)). Again noting that \(Dw = 2Dv/N(v)\). Because \(|b|^2, |B_1|^2, |B_2|^2 \leq \Phi\) so that the Moser’s technique is applied as before. We get the local estimate (3.18) and the proof can go on (we can redefine \(N(v)\) such that \(N^2(v) \geq \nu_\ast \nu(R)\) because certainly \(\nu(R)^{\frac{1}{2}} \geq \nu(R)\)).

**Remark 3.5** It is worth to noting that if \(\Phi\) verifies (3.10) then (3.18) holds. Once this is true we need only (3.23) which is valid if the \(L^1(Q)\) norm of \(\Phi\) satisfies \(\|\Phi\|_{L^1(Q,R)} \leq \nu(R) R^N\) (see (3.12)).
Since $\mathcal{M}(\Omega, T) \subset \mathcal{M}(\Omega, T)$ a similar argument shows that we can take $\nu_\ast = 1$ and $\nu(R) = R^\gamma$ for some appropriate $\gamma > 0$ to have a stronger version of Lemma 3.3.

**Lemma 3.6** Assume that $a \geq \lambda_0$ and $a$ is bounded and $|b|^2, |g| \leq \Phi \in \mathcal{M}(\Omega, T)$. Let $v$ be a bounded weak solution of

$$v_t = \text{div}(aDv + b) + e(Dv) + g.$$  

If $e(Dv) \leq \varepsilon_\ast |Dv|^2 + \Phi$ for some $\varepsilon_\ast > 0 \sup_Q |v|$ small compared to $\lambda_0$ then $v$ is Hölder continuous. Its Hölder norm is bounded in terms of the $L^p_0(\Omega)$ (or $L^{p_0+1}(_Q)$) norms of $|b|^2, g$.

It is also important to mention the following

**Corollary 3.7** Assume that $a = a(v)$ is Hölder continuous in $v$ and $a \geq \lambda_0$ and that $|b|^2, e$ are as in Theorem 3.3. Let $v$ be a bounded weak solution of

$$v_t = \text{div}(a(v)Dv + b) + e(Dv) + g.$$  

If $g \in L^\infty_{\text{loc}}(_Q)$ then $Dv$ is Hölder continuous and locally bounded.

**Proof:** Knowing that $v$ is continuous and so is $a(v)$, we can use [3, Theorem 3.2] applying to scalar equations to see that $Dv$ is Hölder continuous and thus locally bounded. Note that we don’t have to use the $L^p$ estimate of $Dv$ here once we know that $v$ is continuous. This continuity of $v$ suffices to obtain [3] (3.4) in the proof of Proposition 3.1 to obtain a decay estimate in proving that $Dv$ is Hölder continuous in the proof of [3, Theorem 3.2].

4 Applications to systems

In this section, we apply the theory to the system

$$u_t = \text{div}(A(u)Du) + e(Du) + f(u)$$

in $Q$. Here, $u = [u_i]_{i=1}^m$ and $A(u)$ is a $m \times m$ matrix and $e, f$ are vectors in $\mathbb{R}^m$. We will always assume that there is $\lambda_0 > 0$ such that for all $u \in \mathbb{R}^m$, $\zeta \in \mathbb{R}^{mN}$ and $i = 1, \ldots, m$

$$\langle A(u)\zeta, \zeta \rangle \geq \lambda_0 |\zeta|^2 \quad \text{and} \quad \alpha_{ii}(u) \geq \lambda_0. \quad (4.2)$$

In addition, there is $\varepsilon_\ast > 0$ such that

$$|e(\zeta)| \leq \varepsilon_\ast |\zeta|^2 \quad \forall \zeta \in \mathbb{R}^{mN}. \quad (4.3)$$

The well known ‘hole-filling’ trick of Widman (e.g. [1]) has been applied in the regularity of strongly coupled *elliptic systems* on planar domains. Roughly speaking, the idea is that one tests the elliptic system with $u\phi^2$ where $\phi$ is a cutoff function for $B_R, B_{2R}$ (that is $\phi \equiv 1$ in $B_R$ and $\phi \equiv 0$ outside $B_{2R}$ and $|D\phi| \leq C/R$) to obtain a decay estimate for
This and imbedding theorems of Campanato spaces (e.g. [6]) implies that \( u \) is Hölder continuous if \( N = 2 \).

However, this trick does not seem to apply to the corresponding parabolic systems like (4.1) even when \( \varepsilon^* = 0 \) in (4.3). Following the same idea to (4.1) with \( \phi \) is a cutoff function for \( Q_R, Q_{2R} \), one can not obtain a decay estimate for \( \| Du \|_{L^2(Q_R)} \). Even so, the extra time dimension does not imply any continuity of \( u \).

We will apply the theory for scalar equations in previous section to each equation in (4.1) for appropriate conditions on \( a, b \) and \( g \) to establish the pointwise continuity of bounded weak solutions to systems like (4.1). Namely, under appropriate settings and assumptions, we will show that if a bounded weak solution \( u \) of (4.1) is averagely continuous

\[
\liminf_{R \to 0} \iint_{Q_R} |u - u_R|^2 \, dz = 0 \tag{4.4}
\]
then it is pointwise continuous.

### 4.1 Full systems (SKT) on planar domains

The checking of the average continuity assumption (4.4) for a bounded weak solution to (4.1) is a hard problem in general. Here, we consider the case \( N = 2 \) and present examples when this can be done.

Let \( A = P_u \) for some \( P : \mathbb{R}^m \to \mathbb{R}^m \). We consider a special case of (4.1). The following model was introduced in [12] and studied widely in the context of mathematical biology (e.g. [13])

\[
u_t = \Delta(P(u)) + e(Du) + f(u) \tag{4.5}
\]

and assume that the nonlinearity is sublinear. That is, for some constant \( C \) (4.3) is now

\[
|e(\zeta)| \leq C|\zeta| \quad \forall \zeta \in \mathbb{R}^{mN}. \tag{4.6}
\]

We will prove that \( \| Du \|_{L^2(\Omega)} \leq C \) for some constant \( C \) for all \( t \in (0, T) \). The following calculation is formal and it can be justified by replacing the operator \( \frac{\partial}{\partial t} \) in the proof of [9, Lemma 2.2] with the partial difference operator \( \frac{u_i(t)}{h} \) (or \( u, P(u) \) by their Steklov average as in [5]). Multiplying the \( i \)th equation with \( \frac{1}{h} P_i(u) \eta \) where \( \eta \) is function in \( t \) and summing the results, we obtain for \( Q^t = \Omega \times (t_1, t) \)

\[
\iint_{Q^t} \langle u_t, P_u u_t \rangle \eta \, dz + \iint_{Q^t} \langle D(P(u)), D(P(u))_t \rangle \eta \, dz = \iint_{Q^t} \langle f, P_u u_t \rangle \eta \, dz.
\]

We now choose \( \eta \) such that \( \eta(t) = 1, \eta(t_1) = 0 \) and \( |\eta'| \leq C \). Since \( \langle u_t, P_u u_t \rangle \geq \lambda u|u_t|^2 \), \( |u| \) is bounded, by a simple use of Young’s inequality to the last integral (assuming \( f(u) \in L^2(Q) \) for any given bounded solution \( u \)) and rearranging, we easily derive (\( C \) denotes a constant depending on \( \sup_Q |u| \))

\[
\int_{\Omega} |Du(x, t)|^2 \, dx \leq C \int_{t_1}^t \int_{\Omega} |Du(x, t)|^2 \, dx \, dt + C.
\]
This is an integral Grönwall inequality for \( y(t) = \| Du \|_{L^2(\Omega \times \{t\})} \) and implies for all \( t \in (0, T) \) that \( \| Du \|_{L^2(\Omega \times \{t\})} \leq C \).

For each \( i = 1, \ldots, m \) we apply Theorem 3.3 by simply set \( a = P_{u_i}, b = \sum_{j \neq i} P_{u_j} D u_j \) and \( g = e(D u) + f \) (we see that \( |b|^2, |g| \) belong to \( \mathcal{M}(\Omega, T) \) as \( N = 2 \) as either i) or ii) of the definition of \( \mathcal{M} \) is satisfied). Hence, \( u \) is pointwise continuous.

**Corollary 4.1** Consider the system (4.5). Assume that \( f \in L^2(Q) \). Then any bounded weak solution of (4.5) is pointwise continuous.

**Remark 4.2** If \( f \in L^\infty(Q) \) then the derivatives are bounded and Hölder continuous (see the discussion leading to Corollary 4.3 below).

### 4.2 Triangular systems on \( N \)-dimensional domains

The result of Corollary 4.1 holds for full systems with nonlinearities grow at most linear in gradients (see (4.6)). If the system (4.1) is of the special triangular form then we can resume the quadratic growth in gradients (4.3) (and some what more general) for general \( N \).

We will present now an example of a class of triangular systems whose nonlinearities having quadratic growth in gradients.

We start with a system of two equations

\[
\begin{cases}
  u_t = \text{div}(\alpha(u, v) Du + \beta(u, v) D v) + \varepsilon_1 |Du|^2 + c|Dv|^2 + g_1 & \text{in } Q, \\
  v_t = \text{div}(\delta(v) Dv) + \varepsilon_2 |Dv|^2 + g_2 & \text{in } Q, \\
  \text{Homogeneous Dirichlet or Neumann boundary conditions} & \text{on } \partial \Omega \times (0, T), \\
  (u, v) = (u_0, v_0) & \text{on } \Omega.
\end{cases}
\]

That is we will consider (4.1) with \( A = \begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix} \), \( e(D u, D v) = \begin{bmatrix} \varepsilon_1 |Du|^2 + c|Dv|^2 \\ \varepsilon_2 |Dv|^2 \end{bmatrix} \).

We assume, instead of (4.2) which implies, that \( \alpha(u, v), \delta(v) \geq \lambda_0 \). We also assume that \( g_i \in \mathcal{M}(\Omega, T) \) and \( \alpha, \beta \) are continuous and \( \delta \) is Hölder continuous in \( u, v \).

Assume that \( u, v \) are locally bounded. If \( \varepsilon_2 \sup_Q |v| \) is small then we see that \( v \) is continuous by Theorem 3.3 with \( b = 0 \) and \( g = g_2 \).

Knowing that \( \delta(v) \) is continuous and assuming that \( g_1 \in L^\infty_{loc}(Q) \), we can use 3 Theorem 3.2] applying to scalar equations to see that \( Dv \) is Hölder continuous and thus bounded (see Corollary 3.7]). This can be used in the equation of \( u \) with \( b = \beta(u, v) D v \) and \( g = c|Dv|^2 + g_2 \) and we can apply Theorem 3.3 again to prove that \( u \) is continuous if \( \varepsilon_1 \sup_Q |u| \) is small (and \( c \) can be large). Again, note that we don’t have to use the higher integrability \( L^p \) estimate of \( Dv \) here once we know that \( v \) is continuous. This continuity of \( v \) suffices to obtain [3] (3.4) in the proof of Proposition 3.1] to obtain a decay estimate in proving that \( Dv \) is Hölder continuous in the proof of [3 Theorem 3.2].

Again, we remark that the continuity of \( u, v \) also shows that \( Dv, Du \) are Hölder continuous by [3]. But to apply the theory in [3], we need the elliptic condition (4.2) for the whole system when \( m > 2 \).

By induction, the above argument can be extended to systems for \( m > 2 \) unknowns \( u_i \ (i = 1, \ldots, m) \) satisfying homogeneous Dirichlet or Neumann boundary conditions on
The system consists of $m$ equations of the form
\begin{equation}
(u_i)_t = \text{div}(\alpha_i(\hat{u}_i) Du_i) + \sum_{j<i} \beta_{ij}(\hat{u}_i) Du_j + \sum_{j<i} \epsilon_{ij} |Du_j|^2 + g_i, \tag{4.8}
\end{equation}
where we denote $\hat{u}_i = (u_1, \ldots, u_i)$ and assume that $\epsilon_i \sup_{Q} |u_i|$ is small for all $i = 1, \ldots, m$.

**Corollary 4.3** Consider the system (4.1) of $m$ equations of the form (4.8). If $g_i \in L^\infty_{\text{loc}}(Q)$ then any bounded weak solution has bounded derivatives.

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