The mechanics of shuffle products and their siblings

Gérard H. E. Duchamp*, Jean-Yves Enjalbert*, Vincel Hoang Ngoc Minh†‡, Christophe Tollu*

* LIPN, Institut Galilée - UMR CNRS 7030, 93430 Villetaneuse, France.
† University Lille 2, 1 Place Délion, 59024 Lille, France.

Abstract
Nous poursuivons ici le travail commencé dans [15] en décrivant des produits de mélange d’algèbres de fonctions spéciales (issues d’équations différentielles à pôles simples) de plus en plus grandes. Les étudier nous conduit à définir une classe de produits de mélange, que nous nommons ϕ-shuffles. Nous étudions cette classe d’un point de vue combinatoire, en commençant par étendre (sous conditions) le théorème de Radford à celle-ci, puis en construisant (toujours sous conditions) sa bigèbre. Nous analysons les conditions des résultats précités pour les simplifier en les rendant visible dès la définition du produit de mélange. Nous testons enfin ces conditions sur les produits introduits en début d’article.

We carry on the investigation initiated in [15]: we describe new shuffle products coming from some special functions and group them, along with other products encountered in the literature, in a class of products, which we name ϕ-shuffle products. Our paper is dedicated to a study of the latter class, from a combinatorial standpoint. We consider first how to extend Radford’s theorem to the products in that class, then how to construct their bi-algebras. As some conditions are necessary do carry that out, we study them closely and simplify them so that they can be seen directly from the definition of the product. We eventually test these conditions on the products mentioned above.

Keywords: polyzêtas functions, combinatorics of ϕ-shuffle products, comultiplication, Hopf algebra.

Version du: 08-12-2015 02:18

1. Introduction

As a matter of fact, mathematics (in particular number theory), physics and other sciences provide, for their theories, algebras of special functions indexed by parameters¹, with a product, defined at first as a function $X^* \times X^*$ to $A(X)$ and satisfying a simple

¹The combinatorial supports of these parameters will finally resolve themselves into words.
recurrence of the type

$$\forall (a, b) \in X^2, \forall (u, v) \in (X^*)^2, \quad au \uplus \varphi vb = a(u \uplus \varphi bv) + b(au \uplus \varphi v) + \varphi(a, b)(u \uplus \varphi v), \quad (1)$$

the initialization being provided by the fact that $1_{X^*}$ should be a unit. Of course, we will address the question of the existence of such a product, and will extend it by linearity to $A(X)$.

However, recall that these special functions are indexed by parameters but, unfortunately, sometimes do not exist for some of their values: the prototype of this case is the Riemann zeta function $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$ for $s = 1$. Nevertheless, if these “functions” are seen formally, one can in many cases\(^2\), define a product on the indices which governs the effective product on the functions\(^3\).

Once the formal identity is obtained, there are many ways to write the divergent quantities as limits of terms which fulfil the same identities (truncated or power series)\(^4\).

Returning to this family of products, we will use a typology based on examples frequently encountered in the literature as well as new ones that we supply in Section 2.

1. Type I : factor $\varphi$ comes from a product (possibly with zero) between letters (i.e. $X \cup \{0\}$ is a semigroup).
2. Type II : factor $\varphi$ comes from the deformation of a semigroup product by a bicharacter.
3. Type III : factor $\varphi$ comes from the deformation of a semigroup product by a colour factor.
4. Type IV : factor $\varphi$ is the commutative law of an associative algebra (CAA) on $A.X$
5. Type V : factor $\varphi$ is the law of an associative algebra (AA) on $A.X$

These classes are ordered by the following (strict) inclusion diagram:

$$\begin{array}{cccc}
I & \hookrightarrow & II & \hookrightarrow \quad III & \hookrightarrow \quad V \\
& & & \quad \quad IV \\
\end{array}$$

Figure 1: Hasse diagram of the inclusions between classes.

\(^2\)That includes in particular all the cases under consideration in our paper
\(^3\)That is the domain of symbolic computation in the vein of Euler and Arbourast\(^2\),\(^1\).
\(^4\)That is the domain of renormalisation and asymptotic analysis initiated by Du Bois-Reymond and Hardy\(^1\),\(^2\).
We have collected examples from the literature, with the corresponding formulas, in the following table.

| Name            | Formula (recursion)                                                                 | \( \varphi \) | Type |
|-----------------|------------------------------------------------------------------------------------|----------------|------|
| Shuffle [21]    | \( au \shuffle bv = a(uy bv) + b(au \shuffle v) \)                              | \( \varphi \equiv 0 \) | I    |
| Shuffle [19]    | \( x_i u \shuffle x_j v = x_i(u \shuffle x_j)v + x_j(x_i u \shuffle v) \) + \( x_{i+j}(u \shuffle v) \) | \( \varphi(x_i, x_j) = x_{i+j} \) | I    |
| Min-stuffle [7] | \( x_i u \shuffle x_j v = x_j(u \shuffle x_i)v + x_j(x_i u \shuffle v) \) - \( x_{i+j}(u \shuffle v) \) | \( \varphi(x_i, x_j) = -x_{i+j} \) | III  |
| Muffle [14]     | \( x_i u \shuffle x_j v = x_j(u \shuffle x_i)v + x_j(x_i u \shuffle v) \) + \( x_{i\times j}(u \shuffle v) \) | \( \varphi(x_i, x_j) = x_{i\times j} \) | I    |
| q-shuffle [3]   | \( x_i u \shuffle q x_j v = x_i(u \shuffle q x_j)v + x_j(x_i u \shuffle q v) \) + \( x_{i+j}(u \shuffle q v) \) | \( \varphi(x_i, x_j) = q x_{i+j} \) | III  |
| q-shuffle \_2   | \( x_i u \shuffle q x_j v = x_i(u \shuffle q x_j)v + x_j(x_i u \shuffle q v) \) + \( q^i j x_{i+j}(u \shuffle q v) \) | \( \varphi(x_i, x_j) = q^i j x_{i+j} \) | II   |
| LDIAG(1, q) [10]| \( au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v) \) + \( q_s[|a|][|b|] a.b(u \shuffle v) \) | \( \varphi(a, b) = q_s[|a|][|b|] (a, b) \) | II   |
| q-Infiltration [12]| \( au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v) \) + \( q_s a.b(u \uparrow v) \) | \( \varphi(a, b) = q \delta_{a,b} \) | III  |
| AC-stuffle      | \( au \shuffle \varphi bv = a(u \shuffle \varphi bv) + b(au \shuffle \varphi v) \) + \( q_s a.b(u \shuffle \varphi v) \) + \( \varphi(b, a) = q \delta_{a,b} \) | \( \varphi(a, b, c) = \varphi(a, \varphi(b, c)) \) | IV   |
| Semigroup-       | \( x_i u \shuffle \perp x_s v = x_i(u \shuffle \perp x_s)v + x_s(x_i u \shuffle \perp v) \) + \( x_{i\perp s}(u \shuffle \perp v) \) | \( \varphi(x_t, x_s) = x_{t\perp s} \) | I    |
| -shuffle        | \( au \shuffle \varphi bv = a(u \shuffle \varphi bv) + b(au \shuffle \varphi v) \) + \( \varphi(a, b)(u \shuffle \varphi v) \) | \( \varphi(a, b) \) law of AAU | V    |

Of course, the q-shuffle is equal to the (classical) shuffle when \( q = 0 \). As for the \( q \)-infiltration, when \( q = 1 \), one recovers the infiltration product defined in [6].

Many shuffle products arise in number theory when one studies polylogarithms, harmonic sums and polyzêtas: it was in order to study all these products that two of us introduced Type IV (see above) [13].

On the other hand, in combinatorial physics, one has coproducts with bi-multiplicative (and noncommutative) perturbation factors (see [11]).

The structure of the paper is the following: in part 2, we complete the first products of [13] with the description of products which come from Hurwitz polyzêta functions (the product given in [13] was not valid in all cases) and from generalized Polyléerch functions. We are able to give the complete recursive relation which allows to define all kinds of products; we verify that it implies the existence and uniqueness of this product, which can be extended to \( A(X) \). We examine the “known” and the “new” products in order to determine their classes. In part 3, we consider how to extend Radford’s theorem and we prove that it can be carried over to the whole class of AC-products (class IV): the Lyndon words constitute a pure transcendence basis of the corresponding commutative algebra, which can moreover be endowed, under additional growth conditions, with a Hopf algebra structure.

The basis of Lyndon words is the key to effective computations on the algebra of
special functions ruled by such products. In part 4, we determine the necessary and sufficient conditions on \( \varphi \) so that \( \varphi \) belong to the class of AC-products; we give also necessary and sufficient conditions for such a product to be dualizable (i.e. to be the adjoint of a comultiplication).

**Preliminary remark.** It is worth emphasizing at the outset that, although some of the objects/results under review in the present paper have already been defined/proved elsewhere, we include them in our study to lay out as complete a picture as possible and to exemplify the rather ‘pedestrian’ approach we have adopted. In particular, we have refrained throughout the paper from using more sophisticated algebraic techniques.

**Notation.** In the sequel, \( X \) will denote an alphabet, \( k \) a \( \mathbb{Q} \)-algebra, and \( A \) a \( k \)-commutative and associative algebra with unit (a \( k \)-CAAU).

2. Hurwitz Polyzêtas and Generalized Polylerch Functions

We remind the reader of some special functions introduced in [14] and complete their study: we prove that they follow a product law which we describe.

2.1. Some special functions and their products

The Riemann Polyzêta is the function which maps every composition \( s = (s_1, \ldots, s_r) \in (\mathbb{N}_{\geq 1})^r \), tc[5]

\[
\zeta(s) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}
\]  

We now make an observation which, however simple, will appear in different disguises as a building block of many a construction of the paper: There is a (linear) bijection between the module freely generated by (all) compositions and \( \mathbb{Q}[Y] \) (where \( Y = \{y_k \}_{k \geq 1} \)) defined by

\[
\beta_s : (s_1, \ldots, s_r) \mapsto y_{s_1} \cdots y_{s_r}
\]  

So, if \( s = (s_1, \ldots, s_r) \in (\mathbb{N}_{\geq 1})^r \), \( s_1 > 1 \) and \( s' = (s'_1, \ldots, s'_r) \), \( s'_1 > 1 \) are compositions, one knows [14] that[6]

\[
\zeta(s \sqcup s') = \zeta(s) \zeta(s')
\]  

That function \( \zeta \) is well-known and is a special case of the following special functions.

---

5 The decomposition algorithm (which we shall not describe in detail) is based on formula (36) of lemma (4).
6 The following series converges for \( s_1 > 1 \). Under that condition, the definition can be extended by linearity to the module generated by the set of so-called admissible composition.
7 With a slight abuse of language. Strictly speaking, equation (4) actually reads

\[
\zeta\left(\beta_s^{-1}(\beta_s(s \sqcup \beta_s(s'))\right) = \zeta(s)\zeta(s').
\]
2.1.1. Coloured Polyzéta

The coloured polyzéta is the function which, to a composition \( s = (s_1, \ldots, s_r) \) and a tuple of complex numbers of the same length \( \xi = (\xi_1, \ldots, \xi_r) \), associates

\[
\zeta(s, \xi) = \sum_{n_1 > \ldots > n_r > 0} \frac{\xi_1^{n_1} \cdots \xi_r^{n_r}}{n_1^{s_1} \cdots n_r^{s_r}}.
\]  

(5)

It should be noted that \( \zeta(s, \xi) \) appears – with the notation \( \text{Li}_s(\xi) \) – in particule physics \[25\].

To describe the product here, we will use two alphabets \( Y = \{y_i\}_{i \in \mathbb{N}^*}, X = \{x_i\}_{i \in \mathbb{C}^*} \) and \( M \) be the (free) submonoid generated by \( Y \times X \). One easily checks that \[8\]

\[
M = \{(u, v) \in Y^* \times X^* \mid |u| = |v|\}
\]

As above, to make things rigorous (but slightly more difficult to read), one considers the (linear) bijection defined, on \( M \), by

\[
\beta_c : ((s_1, \ldots, s_r), (\xi_1, \ldots, \xi_r)) \mapsto (y_{s_1} \ldots y_{s_r}, x_{\xi_1} \ldots x_{\xi_r}).
\]

The duffle product is defined as follows.

**Definition 1 ([15]).** (Product of coloured polyzéta) Let \( Y = \{y_i\}_{i \in \mathbb{N}^*}, X = \{x_i\}_{i \in \mathbb{C}^*} \) and \( M \) be as above.

The duffle is defined as a bilinear product over \( k[M] = k \langle Y \times X \rangle \) such that

\[
\forall w \in M^*, \quad w \texttt{[1]}_{M^*} = 1_{M^*} \texttt{[w]} = w,
\]

\[
\forall y_i, y_j \in Y^2, \forall x_k, x_l \in X^2, \forall u, v \in M^{*^2}, \quad (y_i, x_k).u \texttt{[u]} (y_j, x_l).v = (y_i, x_k)(u \texttt{[u]} (y_j, x_l)v) + (y_j, x_l)(u \texttt{[u]} (y_i, x_k)v) + (y_{i+j}, x_{k+l})(u \texttt{[u]} v).
\]

Again, we will show that, under suitable conditions \[9\]

\[
\zeta( (s, \xi) \texttt{[u]} (s', \xi') ) = \zeta(s, \xi)\zeta(s', \xi').
\]

(6)

2.1.2. Hurwitz Polyzéta

The Hurwitz polyzéta is the function which, to a composition \( s = (s_1, \ldots, s_r) \) and a tuple of parameters \[10\] of the same length \( t = (t_1, \ldots, t_r) \), associates

\[
\zeta(s, t) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{(n_1 - t_1)^{s_1} \cdots (n_r - t_r)^{s_r}}.
\]

(7)

---

8 Throughout the paper \( |w| \) stands for the length of the word \( w \).

9 Again, rigorously speaking, the left-hand side of the following equation should read

\[
\zeta \left( \beta_c^{-1} \left( \beta_c(s, \xi) \texttt{[u]} \beta_c(s', \xi') \right) \right).
\]

10 All parameters in the tuple are taken in some subring of \( \mathbb{C} \) and none of them is a strictly positive integer.
This series converges if and only if \( s_1 > 1 \) (for a “global” way to expand (7) as a meromorphic function of \( s \in \mathbb{C}^* \), see [13]). To be able to cope with the case \( s_1 = 1 \), we have to use the truncated Hurwitz poly\( \zeta \)etas function given by:

\[
\forall N \in \mathbb{N}_{>0}, \quad \zeta_N(s, t) = \sum_{N > n_r > \ldots > n_1 > 0} \frac{1}{(n_1 - t_1)^{s_1} \ldots (n_r - t_r)^{s_r}} \quad (8)
\]

In order to obtain the product law, we will use here two alphabets \( Y = \{ y_i \}_{i \in \mathbb{N}_{>0}}, Z = \{ z_t \}_{t \in \mathbb{N} \setminus \mathbb{N}_{>0}} \), the (free) submonoid \( N \) generated by \( Y \times Z \) and, as usual, the bijection

\[
\beta_h : ((s_1, \ldots, s_r), (t_1, \ldots, t_r)) \mapsto (y_{s_1} \ldots y_{s_r}, z_{t_1} \ldots z_{t_r})
\]

suitably extended by linearity. We have now the following product

**Definition 2. (Product of Formal Hurwitz Poly\( \zeta \)etas)** Let \( Y = \{ y_i \}_{i \in \mathbb{N}^*}, Z = \{ z_t \}_{t \in k} \) and \( N \) be as above.

The huffle is defined as a bilinear product over \( k[N] = k(Y \times Z) \) such that

\[
\forall w \in N^*, \quad w \boxplus 1_{N^*} = 1_{N^*}, \quad \boxplus w = w,
\]

\[
\forall y_i, y_j \in Y^2, \forall z_t, z_{t'} \in Z^2, \forall u, v \in N^*2,
\]

\[
t = t' \Rightarrow (y_i, z_t) u \boxplus (y_j, z_{t'}) v = (y_i, z_t)(u \boxplus (y_j, z_{t'}) v) + (y_j, z_{t'})((y_i, z_t) u \boxplus v)
\]

\[
t \neq t' \Rightarrow (y_i, z_t). u \boxplus (y_j, z_{t'}) v = (y_i, z_t). (u \boxplus (y_j, z_{t'}) v) + (y_j, z_{t'}). ((y_i, z_t) u \boxplus v)
\]

\[
+ \sum_{n=0}^{i-1} \binom{j-1+n}{i-1} \frac{(-1)^n}{(t-t')^{i+n}} (y_{i-n}, z_t) (u \boxplus v)
\]

\[
+ \sum_{n=0}^{j-1} \binom{i-1+n}{j-1} \frac{(-1)^n}{(t-t')^{i+n}} (y_{j-n}, z_{t'}) (u \boxplus v) .
\]

We also will show that\(^\dagger\) for all integer \( N \)

\[
\zeta_N((s, t) \boxplus (s', t')) = \zeta_N(s, t)\zeta_N(s', t').
\]

\[(10)\]

**Remark 1.** The functions we call 'Hurwitz poly\( \zeta \)etas', a term coined in the last century (see for example [18]), must not be confused with the moncenter poly\( \zeta \)etas, defined only for a composition \( s \) and a parameter \( t \) by

\[
\zeta(s, t) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{(n_1 - t)^{s_1} \ldots (n_r - t)^{s_r}},
\]

which follow a much simpler rule, namely the stuffle product on the compositions.

\[^\dagger\]Again, rigorously speaking, the left-hand side of the following equation should read

\[
\zeta_N(\beta_h^{-1}(\beta_h(s, t) \boxplus \beta_h(s', t'))).
\]
2.1.3. Generalized Polylérch functions

The generalized Polylérch function is the function which maps a composition \( s = (s_1, \ldots, s_r) \), a tuple \( \xi = (\xi_1, \ldots, \xi_r) \) of complex numbers, and a tuple \( t = (t_1, \ldots, t_r) \) of parameters\(^{10}\), all three of the same length, to

\[
\zeta(s, t, \xi) = \sum_{n_1, \ldots, n_r > 0} \frac{\xi_1^{n_1} \cdots \xi_r^{n_r}}{(n_1 - t_1)^{s_1} \cdots (n_r - t_r)^{s_r}}. \tag{12}
\]

Here, we will need three alphabets \( Y = \{ y_i \}_{i \in \mathbb{N}^r} \), \( X = \{ x_i \}_{i \in \mathbb{C}^r} \), \( Z = \{ z_t \}_{t \in k} \) and the (free) submonoid \( T \) generated by \( Y \times Z \times X \). The bijection

\[
\beta_i : ((s_1, \ldots, s_r), (t_1, \ldots, t_r), (\xi_1, \ldots, \xi_r)) \mapsto (y_{s_1} \cdots y_{s_r}, z_{t_1} \cdots z_{t_r}, x_{\xi_1} \cdots x_{\xi_r}) \tag{13}
\]

still extended by linearity. The product \( \# \) is given by the following definition:

**Definition 3. Product of Generalized Lerch functions**

Let \( Y = \{ y_i \}_{i \in \mathbb{N}^r} \), \( X = \{ x_i \}_{i \in \mathbb{C}^r} \), \( Z = \{ z_t \}_{t \in k} \) and \( T \) be the (free) submonoid generated by \( Y \times Z \times X \).

The **luffle** is defined as the bilinear product over \( k[T] = k(Y \times Z \times X) \) satisfying the following recursive relation:

\[
\forall w \in A^*, \quad w \# 1_{A^*} = 1_{A^*} \# w = w, \\
\forall (y_i, y_j) \in Y^2, \forall (z_t, z_{t'}) \in Z^2, \forall (x_k, x_l) \in X^2, \forall (u, v) \in A^{*2}, \\
t = t' \Rightarrow (y_i, z_t, x_k).u \# (y_j, z_t, x_l).v = (y_i, z_t, x_k).((y_j, z_t).u \# v) + (y_{i+j}, z_t, x_{k+l}).(u \# v) \\
t \neq t' \Rightarrow (y_i, z_t, x_k).u \# (y_j, z_{t'}, x_l).v = (y_i, z_t, x_k).((y_j, z_{t'}).u \# v) + \sum_{n=0}^{i-1} \binom{j-1+n}{j-1} (-1)^n (y_{i-n}, z_t, x_{k+l}).(u \# v) + \sum_{n=0}^{j-1} \binom{i-1+n}{i-1} (-1)^n (y_{j-n}, z_{t'}, x_{k+l}).(u \# v).
\]

We also show\(^{12}\)

\[
\zeta((s, t, \xi) \# (s', t', \xi')) = \zeta(s, t, \xi)\zeta(s', t', \xi'). \tag{15}
\]

2.2. General framework of study

Other products from table\(^{11}\) belong to the same family as the examples examined so far, and so pertain to the same kind of approach. As we aim to offer as comprehensive a framework as possible, we now concentrate on the most general class of \( \varphi \)-products, i.e. class \( V \), which emerges from definition \(^{11}\) below. We will use a unitary ring as the ground set of scalars (and not a field as it would be expected in combinatorics) because some applications require to work with rings of (analytic or arithmetic) functions.

\(^{10}\)Again, rigorously speaking, the left-hand side of equation\(^{15}\) should read

\[
\zeta \left( \beta_t^{-1} \left( \beta_t(s, t, \xi) \# \beta_t(s', t', \xi') \right) \right). \tag{14}
\]
Proposition 1. Let $A$ be a unitary commutative ring, $X$ be an alphabet and $\varphi : X \times X \to A(X)$ is an arbitrary mapping. Then there exists a unique mapping $\ast : X^* \times X^* \to A(X)$ satisfying the conditions :

$$(R) \begin{cases} 
\text{for any } w \in X^*, \ 1_{X^*} \ast w = w \ast 1_{X^*} = w, \\
\text{for any } a, b \in X \text{ and } u, v \in X^*, \\
au \ast bv = a(u \ast bv) + b(au \ast v) + \varphi(a, b)(u \ast v).
\end{cases} \quad (16)$$

Proof — By recurrence over $n = |u| + |v|$. □

Definition 4. With the notations of Proposition 1, the unique mapping from $X \times X$ to $A(X)$ satisfying conditions $(R)$ will be noted $\ast_{\varphi}$ and will be called $\varphi$-shuffle product.

From now on, we suppose that $\varphi$ takes its values in $AX$ the space of homogeneous polynomials of degree 1. We still denote by $\varphi$ its linear extension to $AX \otimes AX$ given by

$$\varphi(P, Q) = \sum_{x, y \in X} \langle P|x\rangle \langle Q|y\rangle \varphi(x, y) \quad (17)$$

and $\ast_{\varphi}$ the extension of the mapping of Definition 4 by linearity to $A(X) \otimes A(X)$. Then $\ast_{\varphi}$ becomes a law of algebra (with $1_{X^*}$ as unit) on $A(X)$.

2.3. Extending quasi-stuffle relations

The following elementary result can be found in any complex analysis textbook. It is freely used throughout this section.

Lemma 1. For any integers $s, r \geq 1$, for any complex numbers $a, b \neq a$ :

$$\forall x \in \mathbb{C} \setminus \{a, b\}, \quad \frac{1}{(x-a)^s(x-b)^r} = \sum_{k=1}^{s} \frac{a_k}{(x-a)^k} + \sum_{k=1}^{r} \frac{b_k}{(x-b)^k} \quad (18)$$

where, for all $k \in \{1, \ldots, s\}$, $a_k = \binom{s + r - k - 1}{s - 1} \frac{(-1)^{s-k}}{(a-b)^{s+r-k}}$

and, for all $k \in \{1, \ldots, r\}$, $b_k = \binom{s + r - k - 1}{s - 1} \frac{(-1)^{r-k}}{(b-a)^{s+r-k}}$.

Let $t = (t_1, \ldots, t_r)$ be a set of parameters, $s = (s_1, \ldots, s_r)$ a composition, $\xi = (\xi_1, \ldots, \xi_r) \in \mathbb{C}^r$. We define, for $N \in \mathbb{N}_{>0}$,

$$M^N_{s, \xi, t} = \sum_{N \geq n_1 \ldots n_r > 0} \prod_{i=1}^{r} \frac{\xi_i^{n_i}}{(n_i - t_i)^{n_i}} \quad (19)$$

and $M^N_{0,0,0} = 1$.

Of course, it is a truncated series of $\zeta(s; t; \xi)$.

\^We recall that $AX$ (resp. $A(X)$) admits $X$ (resp. $X^*$) as linear basis, therefore $AX \otimes AX$ (resp. $A(X) \otimes A(X)$) is free with basis $X \times X$ (resp. $X^* \times X^*$) or more precisely, the image family $(x \otimes y)_{x, y \in X}$ (resp. $(u \otimes v)_{u, v \in X^*}$).
Proposition 2. For every composition \( s \), tuple \( \xi \) of complex numbers, tuple \( t \) of parameters all of the same length \( l \in \mathbb{N} \), and for every composition \( r \), tuple \( \rho \) of complex numbers, tuple \( t' \) of parameters also of the same length \( k \in \mathbb{N} \), one has

\[
\forall N \in \mathbb{N}, \quad M_{s,\xi,t}^N M_{r,\rho,t'}^N = M_{(s,\xi,t) \boxplus (r,\rho,t')}. \tag{20}
\]

Proof — If \( l = 0 \) or \( k = 0 \), that is immediate.
Let \( l \in \mathbb{N}^* \), \( k \in \mathbb{N}^* \) and \( s = (s_1, \ldots, s_l) \) and \( r = (r_1, \ldots, r_k) \) two compositions, \( \xi = (\xi_1, \ldots, \xi_l) \subset \mathbb{C}^l \), \( \rho = (\rho_1, \ldots, \rho_k) \subset \mathbb{C}^k \), and \( t = (t_1, \ldots, t_l) \), \( t' = (t'_1, \ldots, t'_k) \) two sets of parameters and put \( s_2 = (s_2, \ldots, s_l) \), \( r_2 = (r_2, \ldots, r_k) \), \( \xi_2 = (\xi_2, \ldots, \xi_l) \), \( \rho_2 = (\rho_2, \ldots, \rho_k) \), \( t_2 = (t_2, \ldots, t_l) \) and \( t'_2 = (t'_2, \ldots, t'_k) \),

- If \( t'_1 = t_1 \),

\[
M_{s,\xi,t}^N M_{r,\rho,t}^N = \sum_{N \geq n_1, N \geq n'_1} \frac{\xi_{n_1}^m}{(n_1 - t_1)^{s_1}} M_{s_2,\xi_2,t_2}^m M_{r,\rho,t}^m \frac{\rho_{n'_1}}{(n'_1 - t'_1)^{r_1}} M_{r,\rho,t'}^m \tag{21}
\]

Classically, we decompose the sum \( \sum_{N \geq n_1, N \geq n'_1} \) into three sums corresponding to the simplices \( n_1 > n'_1 \), \( n'_1 > n_1 \) and \( n_1 = n'_1 \) and get

\[
M_{s,\xi,t}^N M_{r,\rho,t}^N = \sum_{N \geq n_1} \frac{\xi_{n_1}^m}{(n_1 - t_1)^{s_1}} M_{s_2,\xi_2,t_2}^m M_{r,\rho,t}^m + \sum_{N \geq n'_1} \frac{\rho_{n'_1}}{(n'_1 - t'_1)^{r_1}} M_{s,\xi,t}^m M_{r_2,\rho_2,t_2}^m + \sum_{N \geq m} \frac{1}{(m - t_1)^{s_1}} \frac{1}{(m - t'_1)^{r_1}} M_{s_2,\xi_2,t_2}^m M_{r_2,\rho_2,t_2}^m \tag{22}
\]

so that,

\[
\forall N \in \mathbb{N}, \quad M_{s,\xi,t}^N M_{r,\rho,t}^N = M_{(s,\xi,t) \boxplus (r,\rho,t')}^N. \tag{23}
\]

- In the same way, when \( t_1 \neq t'_1 \)

\[
M_{s,\xi,t}^N M_{r,\rho,t}^N = \sum_{N \geq n_1} \frac{\xi_{n_1}^m}{(n_1 - t_1)^{s_1}} M_{s_2,\xi_2,t_2}^m M_{r,\rho,t}^m + \sum_{N \geq n'_1} \frac{\rho_{n'_1}}{(n'_1 - t'_1)^{r_1}} M_{s,\xi,t}^m M_{r_2,\rho_2,t_2}^m + \sum_{N \geq m} \frac{1}{(m - t_1)^{s_1}} \frac{1}{(m - t'_1)^{r_1}} M_{s_2,\xi_2,t_2}^m M_{r_2,\rho_2,t_2}^m
\]

\[
+ \sum_{k=1}^{s_1} \binom{s_1 + r_1 - k - 1}{r_1 - 1} \frac{(-1)^{s_1-k}(\xi_1 \rho_1)^m}{(t_1 - t'_1)^{s_1+r_1-k}(m - t'_1)^k} M_{s_2,\xi_2,t_2}^m M_{r_2,\rho_2,t_2}^m \tag{24}
\]

so

\[
\forall N \in \mathbb{N}, \quad M_{s,\xi,t}^N M_{r,\rho,t}^N = M_{(s,\xi,t) \boxplus (r,\rho,t')}^N. \tag{25}
\]
Remark 2. Let $r$ a integer, $\chi = (\chi_1, \ldots, \chi_r)$ a tuple of multiplicative characters\footnote{Endomorphisms of the semigroup $(\mathbb{C}, \times)$.} and $(s, \xi, t$ being as above) let us define

$$M_{s, \xi, t}^N(\chi) = \sum_{N > n_1 > \ldots > n_r > 0} \prod_{i=1}^r \frac{\chi_i^{n_i}(\xi_i)}{(n_i - t_i)^{s_i}}. \quad (26)$$

The same proof shows that, for any $(s, \xi, t)$ $\in \mathbb{Z}_{>0}^l \times \mathbb{C}^l$ and $(r, \rho) \in \mathbb{Z}_{>0}^k \times \mathbb{C}^k$, for any $l$-tuple $t$ and $k$-tuple $t'$ of parameters,$^{10}$

$$\forall N \in \mathbb{N}, \quad M_{s, \xi, t}^N(\chi) M_{r, \rho, t'}^N(\chi) = M_{(s, \xi, t) \# (r, \rho, t')}(\chi). \quad (27)$$

This result allows to deduce some product relations on the different multi-zèta functions:

**Theorem 2.1.** Let $s = (s_1, \ldots, s_l)$ and $r = (r_1, \ldots, r_k)$ two compositions, $\xi = (\xi_1, \ldots, \xi_l)$ a $l$-tuple, $\rho = (\rho_1, \ldots, \rho_k)$ a $k$-tuple of complex numbers of which the first compositant has a modulus strictly less than $1$, $t = (t_1, \ldots, t_s)$ and $t' = (t'_1, \ldots, t'_k)$ two tuples of parameters not in $\mathbb{N}_{>0}$ and $N \in \mathbb{N}$

(i) For the coloured polyzèta function:

$$\zeta(s, \xi) \zeta(s', \xi') = \zeta((s, \xi) \# (s', \xi')) \quad (28)$$

(ii) For the truncated Hurwitz polyzèta function:

$$\zeta_N(s, t) \zeta_N(s', t') = \zeta_N((s, t) \# (s', t')) \quad (29)$$

(iii) In particular, for the monocentered polyzèta function:

$$\zeta(s, (t, \ldots, t)) \zeta(s', (t, \ldots, t)) = \zeta((s, (t, \ldots, t)) \# (s', (t, \ldots, t))) \quad (30)$$

where $t$ is a parameter s.t. $t \notin \mathbb{N}_{>0}$.

(iv) For the Polymerch generalized function:

$$\zeta(s, t, \xi) \zeta(s', t', \xi') = \zeta((s, t, \xi) \# (s', t', \xi')) \quad (31)$$

**Proof —** (ii) comes directly from Proposition\footnote{Endomorphisms of the semigroup $(\mathbb{C}, \times)$.} because $\zeta_N(s, t) = M_{s, (1, \ldots, 1), t}^N$; for (i), (iii) and (iv), apply Proposition\footnote{Endomorphisms of the semigroup $(\mathbb{C}, \times)$.} with, respectively, the functions

$$M_{s, \xi, (0, \ldots, 0)}^N, \quad M_{s, (1, \ldots, 1), (t, \ldots, t)}^N \quad \text{and} \quad M_{s, \xi, t}^N$$

and take both sides of the equality to the limit as $N$ grows to infinity. \qed
Remark 3. We cannot use this method for the Hurwitz polyzetas because in the decomposition, some divergent terms (which have $s_1 = 1$) appear: for example, for $t \neq t'$,

\[
(y_2, z_t) \shuffle (y_3, z_{t'}) = (y_2y_3, z_tz_{t'}) + (y_3y_2, z vz_t) + \sum_{n=0}^{1} \binom{2 + n}{2} \frac{(-1)^n}{(t - t')^{3+n}}(y_2 - n, z_t)
\]

\[
+ \sum_{n=0}^{2} \frac{(1 + n)}{1} \frac{(-1)^n}{(t - t')^{2+n}}(y_3 - n, z_{t'})
\]

\[
= (y_2y_3, z_tz_{t'}) + (y_3y_2, z vz_t) + \frac{1}{(t - t')^3}(y_2, z_{t'}) - \frac{3}{(t - t')^4}(y_1, z_t)
\]

\[
+ \frac{1}{(t - t')^2}(y_3, z_{t'}) - \frac{2}{(t - t')^3}(y_1, z_{t'}) + \frac{3}{(t - t')^4}(y_1, z_{t'})
\]

(32)

Separately, the terms $-\frac{3}{(t - t')^4}(y_1, z_t)$ and $\frac{3}{(t - t')^4}(y_1, z_{t'})$, corresponding respectively to $-\frac{3}{(t - t')^4} \frac{1}{n-t}$ and $\frac{3}{(t - t')^4} \frac{1}{n-t'}$ give a divergent series although all other terms correspond to convergent series. Of course, the sum of the two

\[
\frac{3}{(t - t')^4} \left(-\frac{1}{n-t} + \frac{1}{n-t'}\right) = \frac{3}{(t - t')^4} \left(\frac{t'-t}{(n-t)(n-t')}\right)
\]

(33)

is a term of a convergence series, but the series is not a Hurwitz Polyzetas.

3. Radford’s theorem for the AC-stuffle.

In this subsection, $A$ is supposed to be a ring with unit; when we need it to be commutative or to contain the set of rational numbers, we will state it explicitly.

Let $<$ be a total ordering on the alphabet $X$, and $\mathcal{Lyn}(X)$ denote the family of Lyndon words [24] constructed from $X$ w.r.t. this ordering. We will prove that the largest framework in which Radford’s theorem holds true [23] is when $\varphi$ is commutative (and associative).

3.1. Computing $\varphi$-shuffle expressions using shuffles

In this subsection $A$ is a ring with unit and $\varphi : AX \otimes AX \to AX$ an associative law. We can express the result of the $\varphi$-shuffle product thanks to the shuffle product (and some terms of lower degree). First we observe what happens with the product of two words :

Lemma 2. For $u, v \in X^*$, there exists $(C^w_{u,v})_{|w| < |u| + |v|} \in A^{(N)}$ such that :

\[
u \shuffle^\varphi v = u \shuffle v + \sum_{|w| < |u| + |v|} C^w_{u,v,w}.
\]

Proof — Omitted.

Now, because the Lyndon words are candidates to be a transcendental basis, we see what happens when they are $\varphi$-shuffled.
Definition 5. Let $*: A(X) \times A(X) \mapsto A(X)$ be an associative law with unit and $X = \text{Lyn}(X)$. For any $\alpha \in N^{(X)}$ and $\{l_1, \cdots, l_r\} \supset \text{supp}(\alpha)$ in strict decreasing order (i.e. $l_1 > \cdots > l_r$), we set

$$X^{*\alpha} = l_1^{*\alpha_1} \cdots l_r^{*\alpha_r} \quad (34)$$

One easily checks easily that the product (34) does not depend on the choice of $\{l_1, \cdots, l_r\} \supset \text{supp}(\alpha)$. We will also need the following parameter (which will turn out to be the length of the dominant terms in the product)

$$||\alpha|| = \sum_{l \in \text{Lyn}(X)} \alpha(l)|l| \quad (35)$$

Lemma 3. If $\shuffle$ is associative,

$$\forall \alpha \in N^{(\text{Lyn}(X))}, \exists (C^{\alpha}_\beta)_{\beta} \in A^{(\text{Lyn}(X))}/X^{\shuffle}\shuffle^{\alpha} = X^{\shuffle} \alpha + \sum_{\beta \in N^{(\text{Lyn}(X))}, ||\beta|| < ||\alpha||} C^{\alpha}_\beta X^{\shuffle} \beta. \quad (36)$$

Proof — Omitted

3.2. Radford’s theorem in $\varphi$-shuffle algebras

Lemma 4. If $\shuffle$ is associative,

$$\forall p \in N^*, \text{span} \left( \left( X^{\shuffle}\varphi^{\alpha} \right)_{\alpha \in N^{(\text{Lyn}(X))}, ||\alpha|| < p} \right) = \text{span} \left( \left( X^{\shuffle}\varphi^{\alpha} \right)_{\alpha \in N^{(\text{Lyn}(X))}, ||\alpha|| < p} \right) \quad (37)$$

Proof — Lemma\[3\] give $\forall p \in N^*$, span $\left( \left( X^{\shuffle}\varphi^{\alpha} \right)_{\alpha \in N^{(\text{Lyn}(X))}, ||\alpha|| < p} \right) \subset \text{span} \left( \left( X^{\shuffle}\varphi^{\alpha} \right)_{\alpha \in N^{(\text{Lyn}(X))}, ||\alpha|| < p} \right)$.

We just have to prove, for any $p \in N^*$, the property $\mathcal{P}(p)$:

$$\text{span} \left( \left( X^{\shuffle}\varphi^{\alpha} \right)_{\alpha \in N^{(\text{Lyn}(X))}, ||\alpha|| < p} \right) \subset \text{span} \left( \left( X^{\shuffle}\varphi^{\alpha} \right)_{\alpha \in N^{(\text{Lyn}(X))}, ||\alpha|| < p} \right)$$

- It is true for $p = 1$.
- Assume $\mathcal{P}(p)$ true for an integer $p$.

Let $\alpha \in N^{(\text{Lyn}(X))}$ such that $||\alpha|| < p + 1$.

We can find $(C^{\alpha}_\beta)_{\beta} \in A^{(\text{Lyn}(X))}$ such that $X^{\shuffle}\varphi^{\alpha} = X^{\shuffle} \alpha + \sum_{\beta \in N^{(\text{Lyn}(X))}, ||\beta|| < ||\alpha||} C^{\alpha}_\beta X^{\shuffle} \beta$, so

$$X^{\shuffle} \alpha = X^{\shuffle}\varphi^{\alpha} - \sum_{\beta \in N^{(\text{Lyn}(X))}, ||\beta|| < ||\alpha||} C^{\alpha}_\beta X^{\shuffle} \beta.$$

But every term of the sum is of the form $C^{\alpha}_\beta X^{\shuffle} \beta$ with $\beta \in N^{(\text{Lyn}(X))}$ and $||\beta|| < ||\alpha|| < p + 1$ so $||\beta|| < p$.

Consequently, they are in $\text{span} \left( \left( X^{\shuffle}\varphi^{\alpha} \right)_{\alpha \in N^{(\text{Lyn}(X))}, ||\alpha|| < p} \right)$, and so is the sum. By the induction hypothesis, the sum is in $\text{span} \left( \left( X^{\shuffle}\varphi^{\alpha} \right)_{\alpha \in N^{(\text{Lyn}(X))}, ||\alpha|| < p} \right)$, therefore $X^{\shuffle} \alpha \in \text{span} \left( \left( X^{\shuffle}\varphi^{\alpha} \right)_{\alpha \in N^{(\text{Lyn}(X))}, ||\alpha|| < p+1} \right)$.
Theorem 3.1. Let $A$ be a commutative ring (with unit) such that $Q \subset A^{15}$ and $\omega_\varphi : A(X) \otimes A(X) \to A(X)$ is associative.
If $X$ is totally ordered by $<$, then $(X^{\omega_\varphi \alpha})_{\alpha \in \mathbb{N}(\text{Lyn}(X))}$ is a linear basis of $A(X)$.

Proof — Since this family is a generating family by lemma 4, only freeness remains to be proven.
Let $\sum_{\alpha \in J} \beta_\alpha X^{\omega_\varphi \alpha} = 0$ be a null linear combination of $(X^{\omega_\varphi \alpha})_{\alpha \in \mathbb{N}(\text{Lyn}(X))}$, with $J$ a nonempty finite subset of $\mathbb{N}(\text{Lyn}(X))$. Thanks to lemma 3, for any $\alpha \in J$, we can find a finite family $B_\alpha \subset \mathbb{N}(\text{Lyn}(X))$ and $(C^\alpha_\beta)_{\beta \in B_\alpha} \in A^{B_\alpha}$ such that

\[ X^{\omega_\varphi \alpha} = X^{\omega \alpha} + \sum_{\beta \in B_\alpha : ||\beta|| < ||\alpha||} C^\alpha_\beta X^{\omega \beta}. \]

Set $B = J \cup \left( \bigcup_{\alpha \in J} B_\alpha \right)$; $B$ is a finite set. Then $(X^{\omega_\varphi \alpha})_{\alpha \in J}$ is a triangular family for $|.|$ with respect to the family $\mathcal{F} = (X^{\omega_\varphi \alpha})_{\alpha \in B}$ in the vector space $\text{span}(\mathcal{F})$, which is of finite dimension. But $\mathcal{F}$ is a basis, so $(X^{\omega_\varphi \alpha})_{\alpha \in J}$ is free and $\forall \alpha \in J, \beta_\alpha = 0$. □

Corollary 1. Under the same hypotheses, if in addition $\omega_\varphi$ is commutative in $A$ then

i) The algebra $\mathcal{A} = (A(X), \omega_\varphi, 1_X^*)$ is a polynomial algebra.

ii) $\text{Lyn}(X)$ is a transcendence basis of $\mathcal{A}$.

Remark 4. It is necessary to suppose $Q \subset A$ as, in case $\varphi \equiv 0$, one has

\[ \forall n \in \mathbb{N}_{>0}, a^n = \frac{1}{n!}(a^{\omega \alpha n}) \quad (38) \]

Proof —

i) Immediate result.

ii) Comes from proposition 3.1 and theorem 4.1, which proves in an elementary (so independent) way that the commutativity of $\varphi$ is equivalent to the commutativity of $\omega_\varphi$. □

\[ ^{15} \text{Precisely, } N^+ \cdot 1_A \subset A^x \]
3.3. Bialgebra structure

**Definition 6.** A law $\ast$ defined over $A(X)$ is a dual law (or dualizable) if there exists a linear mapping $\Delta_\ast : A(X) \to A(X) \otimes A(X)$ such

$$\forall(u, v, w) \in X^* \times X^* \times X^*, \quad (u \ast v|w) = (u \otimes v|\Delta_\ast(w))^{\otimes 2}. \quad (39)$$

In this case, $\Delta_\ast$ will be called the comultiplication dual to $\ast$.

**Theorem 3.2.** If $A$ is a commutative ring (with unit), if $Q \subset A$, and if in addition the product $\cup : A(X) \otimes A(X) \to A(X)$ is an associative and commutative law on $A(X)$, then the algebra $(A(X), \cup, 1_{X^*})$ can be endowed with the comultiplication $\Delta_{\text{conc}}$ dual to the concatenation

$$\Delta_{\text{conc}}(w) = \sum_{uv=w} u \otimes v \quad (40)$$

and the “constant term” character $\epsilon(P) = \langle P|1_{X^*}\rangle$.

With this setting

$$B_\varphi = (A(X), \cup, 1_{X^*}, \Delta_{\text{conc}}, \epsilon) \quad (41)$$

is a bialgebra.

**Remark 5.** Let, classically, $\Delta_{\text{conc}}^+$ be defined by

$$\forall w \in X^*, \Delta_{\text{conc}}^+(w) = \sum_{u \neq v \neq w} u \otimes v$$

We remark that $\Delta_{\text{conc}}^+$ is coassociative and locally nilpotent, i.e.

$$\forall w \in X^*, \exists n \in \mathbb{N}^*: (\Delta_{\text{conc}}^+)^n(w) = 0.$$

Thus the bialgebra $(41)$ is, in fact, a Hopf Algebra.

**Proof —** It is a classical combinatorial verification, done in [15]. The following identity remains to be proven:

$$\forall(w_1, w_2) \in X^*, \Delta_{\text{conc}}(w_1 \cup \varphi w_2) = \Delta_{\text{conc}}(w_1)\Delta_{\text{conc}}(w_2) \quad (42)$$

which can be done by a (lengthy) induction or by duality. \qed

4. Conditions for AC-shuffle and dualizability

4.1. Commutative and associative conditions

We have obtained an extended version of Radford’s theorem and other properties with conditions stated w.r.t. $\cup$, we will see in this subsection that these conditions can be set uniquely in terms of properties of $\varphi$ itself.

\footnote{Commutative and, when $|X| \geq 2$, noncocommutative.}
Definition 7. For $P \in A(X)$, we note $\text{supp}(P)$ the support of $P$ and $\deg(P) = \max \{|l|, l \in \text{supp}(P)\}$

Lemma 5. Let $A$ be a unitary commutative ring, $X$ be an alphabet and $\varphi : X \times X \to A(X)$ is an arbitrary mapping. Then,

$$\forall (u, v) \in (X^*)^2, \deg(u \circ \varphi v) \leq |u| + |v|$$

(43)

Proof — If $|u| = 0$ or $|v| = 0$, then $u \circ \varphi v$ is one of $\{u, v\}$ so its length is $|u| + |v|$.

Let $X^+$ be the set of nonempty words. We prove $\forall (u, v) \in (X^*)^2, \deg(u \circ \varphi v) = |u| + |v|$ by induction on $|u| + |v|$.

For any letters $a$ and $b$, $a \circ \varphi b = ab + ba + \varphi(a, b)1_A$, so $\deg(a \circ \varphi b) = 2 = |a| + |b|$.

One assumes the property true for all words $u, v \in X^+$ such that $|u| + |v| = n$, where $n$ is an integer. Let $u$ and $v$ be now two words of $X^+$ such that $|u| + |v| = n + 1$.

There exist $x, y$ in $X$, $u', v'$ in $X^*$ such that $u = xu'$, $v = yv'$ (because $(u, v) \in (X^*)^2$). Then $|u| + |v'| = |u| + |v| - 1 \leq n$, so $\deg(y(u \circ \varphi v')) \leq n + 1$. Also $|u'| + |v'| = |u| - 1 + |v| \leq n$ so $\deg(x(u \circ \varphi v)) \leq n + 1$, and $|u'| + |v'| = |u| - 1 + |v| - 1 \leq n$ so $\deg(\varphi(x, y)u' \circ \varphi v') \leq n + 1$.

Hence, $\deg(u \circ \varphi v) = n + 1 :$ the induction is proved. □

Theorem 4.1. In the context of definition 4,

(i) The law $\circ \varphi$ is commutative if and only if the extension $\varphi : AX \otimes AX \to AX$ is commutative.

(ii) The law $\circ \varphi$ is associative if and only if the extension $\varphi : AX \otimes AX \to AX$ is associative.

Proof — We give an elementary proof.

(i) $[\circ \varphi \text{ commutative} \implies \varphi \text{ commutative}]$

Let us suppose $\forall (u, v) \in (X^*)^2, u \circ \varphi v = v \circ \varphi u$.

In particular, $\forall (x, y) \in (X^*)^2, x \circ \varphi y = x \circ \varphi y$. But, for any $(x, y) \in X^2$,

$$x \circ \varphi y = xy + yx + \varphi(x, y) \quad \text{and} \quad y \circ \varphi x = yx + xy + \varphi(y, x)$$

(44)

and so $(\forall x, y \in X)(\varphi(x, y) = \varphi(y, x))$.

$[\varphi \text{ commutative} \implies \circ \varphi \text{ commutative}]$

Now let us suppose $\varphi$ is commutative then let us prove by recurrence on $|uv|$ that $\circ \varphi$ is commutative:

$-$ The previous equivalence proves that the recurrence holds for $|u| = |v| = 1$.

$-$ Suppose the recurrence holds for any $u, v \in X^*$ such that $2 \leq |uv| \leq n$ and $|u|, |v| \neq 1$.

Let $u = xu'$ and $v = yv'$ with $x, y \in X$ and $u', v' \in X^*$. Then,

$$u \circ \varphi v = x(u' \circ \varphi yv) + y(xu' \circ \varphi v) + \varphi(x, y)(u' \circ \varphi v')$$

$$= x(yv \circ \varphi u) + y(v' \circ \varphi xu') + \varphi(y, x)(v' \circ \varphi u') \quad \text{(by the induction hypothesis)}$$

(45)
Let us suppose
\[ \forall u, v, w \in X^* , \quad (u \varphi v) \varphi w = u \varphi (v \varphi w). \] (46)

Then, for any \( x, y, z \in X \), one has
\[ (x \varphi y) \varphi z = x \varphi (y \varphi z). \] (47)

But
\[
(x \varphi y) \varphi z = (x y + yx + \varphi(x,y)) \varphi z
= xy \varphi z + yx \varphi z + \varphi(x,y) \varphi z
= x(y \varphi z) + z(yx \varphi y) + \varphi(x,z)y + xy \varphi z + \varphi(y,z)x
+ \varphi(x,y)z + z \varphi(x,y) + \varphi(x,y), z
= x(yz + yz + \varphi(y,z)) + zxy + \varphi(x,z)y + y(xz + zx + \varphi(x,z)) + zyx
+ \varphi(y,z)x + \varphi(x,y)z + z \varphi(x,y) + \varphi(x,y), z.
\] (48)

One can then deduce that
\[ (\forall x, y, z \in X)(x \varphi y) \varphi z = (x \varphi y) \varphi z \quad \iff \quad (\forall x, y, z \in X)(\varphi(x, \varphi(y,z)) = \varphi(\varphi(x,y), z)). \] (49)

[\varphi associative \implies \varphi associative] Now if \( \varphi \) is associative then let us prove by induction on \(|u| + |v| + |w|\) that \( \varphi \varphi \) is associative:

- The previous equivalence proves that the induction holds for \(|u| = |v| = |w| = 1\).
- Suppose the recurrence holds for any \( u, v \in X^* \) such that \( 3 \leq |u| + |v| + |w| \leq n \) and \(|u|, |v|, |w| \neq 1\).
- Let \( u = xu', v = yv' \) and \( w = zw' \) with \( x, y, z \in X \) and \( u', v', w' \in X^* \). Then,
\[
u \varphi (v \varphi w) = u \varphi (y \varphi (v' \varphi w')) + z(v \varphi v' \varphi w') + \varphi(y,z)(v' \varphi w')
= x(u \varphi y \varphi (v' \varphi w')) + y(u \varphi v' \varphi w') + \varphi(x,y)(u \varphi (v' \varphi w'))
+ x(u \varphi z \varphi (v' \varphi w')) + (u \varphi (v' \varphi w')) + \varphi(x,z)(u \varphi (v' \varphi w'))
+ (u \varphi (v' \varphi w')) + \varphi(x,z)(u \varphi (v' \varphi w'))
+ u \varphi (v' \varphi w') + \varphi(x,y)(u \varphi (v' \varphi w'))
+ z(u \varphi (v' \varphi w')) + \varphi(x,z)(u \varphi (v' \varphi w'))
+ \varphi(y,z)(u \varphi (v' \varphi w')) + (u \varphi (v' \varphi w'))(u \varphi (v' \varphi w'))
\] (50)
\[(u \cdot \varphi^z v) \cdot \varphi w \]
\[= (x (u' \cdot \varphi^z v) + y (u \cdot \varphi^z v') + \varphi(x, y) (u' \cdot \varphi^z v')) \cdot \varphi w \]
\[= x ((u' \cdot \varphi^z v) \cdot \varphi w + z (x (u' \cdot \varphi^z v) \cdot \varphi w')) + \varphi(x, z) ((u' \cdot \varphi^z v) \cdot \varphi w') \]
\[+ y ((u \cdot \varphi^z v') \cdot \varphi w) + z (y (u \cdot \varphi^z v') \cdot \varphi w') + \varphi(y, z) ((u \cdot \varphi^z v') \cdot \varphi w') \]
\[+ \varphi(x, y) ((u' \cdot \varphi^z v') \cdot \varphi w) + \varphi(x, y) ((u \cdot \varphi^z v') \cdot \varphi w') + \varphi(\varphi(x, y), z) ((u' \cdot \varphi^z v') \cdot \varphi w') \]
\[+ z (u \cdot \varphi^z v) \cdot \varphi w') \]
\[(53)\]

Indeed, thanks to the induction hypothesis and the commutativity of \(\varphi\), \(u \cdot \varphi^z (v \cdot \varphi^z w)\) and \((u \cdot \varphi^z v) \cdot \varphi^z w\) are equal.

\[\square\]

4.2. Dualizability conditions

**Proposition 3.** We call \(\gamma^z_{x,y} := \langle \varphi(x, y) | z \rangle\) the structure constants of \(\varphi\) (w.r.t. the basis \(X\)).
The product \(\varphi \cdot \psi\) is a dual law if and only if \((\gamma^z_{x,y})_{x,y,z \in X}\) is dualizable in the following sense
\[(\forall z \in X)(\# \{ (x, y) \in X^2 | \gamma^z_{x,y} \neq 0 \} < +\infty) \ . \quad (54)\]

**Proof —** [\(\varphi \cdot \psi\) dual law \(\Rightarrow\) \(\gamma^z_{x,y}\) dualizable]. Let \(\Delta\) be the dual of \(\varphi \cdot \psi\), that is, for all \(u, v, w \in X^*\)
\[\langle u \cdot \varphi^z v | w \rangle = \langle u \otimes v | \Delta(w) \rangle^{\otimes 2} \ . \quad (55)\]
For all \(z \in X\), one must have \(\Delta(z) = \sum_{i=1}^n \alpha_i u_i \otimes v_i\). On the other hand, for all \(x, y \in X\), one has \((x \cdot \varphi^z y) = (x y + y x) = \varphi(x, y)\). Hence
\[\gamma^z_{x,y} = \langle \varphi(x, y) | z \rangle = \langle (x \cdot \varphi^z y) - (x y + y x) \rangle | z \rangle = \langle (x \cdot \varphi^z y) | z \rangle - \langle (x y + y x) | z \rangle \]
\[= \langle (x \otimes y) | \Delta(z) \rangle = \langle (x \otimes y) | \sum_{i=1}^n \alpha_i u_i \otimes v_i \rangle \ . \quad (56)\]

We can deduce from the preceding argument that
\[\gamma^z_{x,y} \neq 0 \quad (x \in \cup_{i=1}^n Alph(u_i) \quad \text{and} \quad y \in \cup_{i=1}^n Alph(v_i) )\]
which proves the point.

\([\gamma^z_{x,y}\ \text{dualizable} \iff \varphi \cdot \psi\ \text{dual law}])\. This is, combinatorially speaking, the most interesting point. We first define a comultiplication \(\Delta\) on \(A(X)\) by transposing the structure constants of \(\varphi \cdot \psi\) by
\[\Delta(z) := z \otimes 1 + 1 \otimes z + \sum_{x,y \in X} \gamma^z_{x,y} x \otimes y \quad (57)\]
and, as the sum is finite (see however the comment after this theorem), this quantity belongs to \(A(X) \otimes A(X)\). One then has a linear mapping \(\Delta : AX \rightarrow A(X) \otimes A(X)\) which is extended, by universal property, into a morphism of algebras \(\Delta : A(X) \rightarrow A(X) \otimes A(X)\). Explicitly, for all \(w = z_1 z_2 \cdots z_n\), one has
\[\Delta(z_1 z_2 \cdots z_n) = \Delta(z_1) \Delta(z_2) \cdots \Delta(z_n) \ . \quad (58)\]
Now, we prove that the dual law of the latter coproduct is exactly \( \omega_\varphi \).

First remark: by (57) and (58), one has

\[
\Delta(w) = w \otimes 1 + 1 \otimes w + \sum_{u,v \in X^+} \beta(u,v) u \otimes v
\]

the last sum being finitely supported. This shows by duality that

\[
\omega_{\Delta} 1 = 1_{\omega_{\Delta}} u = u
\]

(60)

(here, \( \omega_{\Delta} \) stands for the dual law of \( \Delta \)). Moreover

\[
\begin{align*}
au_{\omega_{\Delta}} bv &= \sum_{w \in X^*} \langle au_{\omega_{\Delta}} bv|w\rangle w \\
&= \sum_{w \in X^*} \langle au \otimes bv|\Delta(w)\rangle w \\
&= \langle au \otimes bv|1 \otimes 1\rangle 1 + \sum_{w \in X^+} \langle au \otimes bv|\Delta(w)\rangle w \\
&= \sum_{x \in X; m \in X^*} \langle au \otimes bv|\Delta(x)\Delta(m)\rangle xm \\
&= \sum_{x \in X; m \in X^*} \langle au \otimes bv|(x \otimes 1 + 1 \otimes x + \sum_{y,z \in X} \langle \Delta(x)|y \otimes z\rangle y \otimes z\rangle \Delta(m)\rangle xm \\
&= \sum_{x \in X; m \in X^*} \langle au \otimes bv|(x \otimes 1)\Delta(m)\rangle xm + \sum_{x \in X; m \in X^*} \langle au \otimes bv|(1 \otimes x)\Delta(m)\rangle xm + \sum_{x \in X; m \in X^*} \langle au \otimes bv|\Delta(x) \Delta(m)\rangle xm \\
&= \sum_{m \in X^*} \langle au \otimes bv|(a \otimes 1)\Delta(m)\rangle am + \sum_{m \in X^*} \langle au \otimes bv|(1 \otimes b)\Delta(m)\rangle bm + \sum_{x \in X; m \in X^*} \langle au \otimes bv|\Delta(x) \Delta(m)\rangle xm \\
&= a \sum_{m \in X^*} \langle au \otimes bv|\Delta(m)\rangle m + b \sum_{m \in X^*} \langle au \otimes v|\Delta(m)\rangle m + \sum_{x \in X} \langle \Delta(x)|a \otimes b\rangle \langle u \otimes v|\Delta(m)\rangle m \\
&= a(u_{\omega_{\Delta}} bv) + b(au_{\omega_{\Delta}} v) + \varphi(a,b)(u_{\omega_{\Delta}} v)
\end{align*}
\]

(61)

This proves that the dual law \( \omega_{\Delta} \) equals \( \omega_\varphi \) and we are done.

\[\square\]

4.3. The Hopf-Hurwitz algebra

In section 4.1 we provided the law on indices followed by the product of Formal Hurwitz polyzetæs, we now prove that the law \( \varphi \) associated with it is associative. The “centres” will be taken from a subfield \( k \) of \( \mathbb{C} \) and the set of coefficients \( A \) is a \( k \)-CAAU.
Proposition 4. i) The law $\varphi : AN \otimes AN \to AN$ associated to $\circ$ is defined, on the basis $N$, by the multiplication table $T_{\text{Formal Hurwitz}}$

\[
\begin{align*}
\text{if } t = t'; & \quad \varphi((y_i, z_t), (y_j, z_{t'})) = (y_{i+j}, z_t) \\
\text{if } t \neq t'; & \quad \varphi((y_i, z_t), (y_j, z_{t'})) = \\
& \quad \sum_{n=0}^{i-1} \left( j - 1 + n \right) \left( -1 \right)^n \frac{1}{(t-t')^{j+n}} (y_{i-n}, z_t) + \\
& \quad \sum_{n=0}^{j-1} \left( i - 1 + n \right) \left( -1 \right)^n \frac{1}{(t'-t)^{i+n}} (y_{j-n}, z_{t'})
\end{align*}
\]

(62)

ii) The product $\shuffle$ is associative, commutative and unital, making $(A\langle N \rangle, \circ, 1_N)$ into a $A$-CAAU.

Proof — i) Let first $j : kN \to k(X)$ be the linear mapping defined by $j((y_i, z_t)) = \frac{1}{(X-y)^n}$. In fact, as the $\left\{ \frac{1}{(X-y)^n} \right\}$ are linearly independent, $j$ is into. On the other hand, $j$ is a morphism of $k$-AAU due to the fact that the multiplication table is identical. Hence $\varphi$ is a law of $\mathbb{C}$-AA on $AN \simeq A \otimes kN$. ii) Is a consequence of the general theorems.

Now, we have the following bialgebra

\[
\mathcal{H}_{\text{Formal Hurwitz}} = (A\langle N \rangle, \shuffle, 1_N, \Delta_{\text{conc}}, \varepsilon)
\]

which is a Hopf algebra. Note that $\shuffle$ is not dualisable which means that the adjoint

\[
\Delta_{\shuffle} : N^* \to A\langle N^* \otimes N^* \rangle
\]

does not have its image in $A\langle N \rangle \otimes A\langle N \rangle$. See next paragraph for tools and proofs.

Corollary 2. The product $\shuffle$ is associative, commutative and unital, making $(A\langle N \rangle, \shuffle, 1_N)$ into a $A$-CAAU.

Proof — It comes that the product $\shuffle$ is a direct product of the products $\circ$ and $\cdot$.

5. Conclusion

We have been able to give a useful extended version of Radford’s theorem.

Let us observe that:

- For the shuffle product, $\varphi_{\shuffle} \equiv 0$, so the shuffle $\shuffle$ is associative, commutative and dualizable.
- The shuffle product over an alphabet indexed by $\mathbb{N}$ is associative and commutative because $\varphi(x_i, x_j) = x_{i+j}$ is so; moreover it is dualizable.
- The muffle product over an alphabet indexed by $\mathbb{C}$ is associative and commutative because $\varphi_{\shuffle}(x_i, x_j) = x_{i\times j}$ is so; it is not dualizable because for all $n \in \mathbb{N}_{>0}$, $x_1 = \varphi_{\shuffle}(x\frac{1}{n}, x_n)$. However, there are multiplicative subsemigroups $S$ of $\mathbb{C}$ such that $\varphi$ restricted to the alphabet $(x_i)_{i \in S}$ is dualizable.
• The duffle product over an alphabet indexed by $\mathbb{N}^* \times \mathbb{C}^*$ is associative and commutative because $\varphi \psi \left( (y_i, x_k), (y_j, x_l) \right) = (y_{i+j}, x_{k+l})$ is associative and commutative; it is not dualizable either (for the same reason).

But we can do the same remark as the muffle about the possibility to restrict the alphabet so that $\varphi$ becomes dualizable.

• The Formal Polyzèta product is associative, commutative but, for all $j \in \mathbb{N}_0$, with $t \neq t'$,

$$
\varphi_\circ \left( (y_2, z_t), (y_j, z_{t'}) \right) = \sum_{n=0}^{j-1} \binom{j-1+n}{j} \frac{(-1)^n}{(t-t')^{j+n}} (y_{2-n}, z_t) + \sum_{n=0}^{j-1} \binom{1+n}{1} \frac{(-1)^n}{(t'-t)^{2+n}} (y_{j-n}, z_{t'})
$$

(65)

and then $\forall j \in \mathbb{N}_0$, $\langle \varphi_\circ \left( (y_2, z_t), (y_j, z_{t'}) \right) | (y_1, z_t) \rangle = - \binom{j}{j-1} \frac{1}{(t-t')^{j+1}} \neq 0$

which implies that $\circ$ is not dualizable.

• The Lerch product is associative, commutative and not dualizable (for the same reason as $\psi$).

So, if we work in the Riemann polyzèta algebra, in the coloured polyzèta algebra, or in the Generalized Lerch polyzèta algebra, we can use a representation with the Lyndon set as a transcendence basis. Moreover, in the Riemann polyzèta algebra and the truncated Hurwitz polyzèta algebra can both be completed into Hopf algebras.

References

[1] P. Du Bois-Reymond, Théorie générale des fonctions, traduit de l’allemand par G. Milhaud et A. Girot, Imprimerie niçoise, Nice, 1887.

[2] N. Bourbaki, Theory of sets, Springer, Berlin, 2004.

[3] C. Bui, Hopf algebras of shuffles and quasi-shuffles. Constructions of dual bases, Master dissertation, Villetaneuse, 2012.

[4] C. Bui, G. H. E. Duchamp, V. Hoang Ngoc Minh, Schützenberger’s factorization on the (completed) Hopf algebra of $q$-stuffle product, JP Journal of Algebra, Number Theory and Applications, 30(2) (September 2013), 191–215.

[5] K.T. Chen, Iterated path integrals Bull. Amer. Math. Soc., 83 (1977), 831–879.

[6] K.T. Chen, R.H. Fox, R.C. Lyndon, Free differential calculus, IV. The quotient groups of the lower central series, Ann. Math., 68(1) (July 1958), 81–95.

[7] C. Costermans, Calcul symbolique non commutatif : analyse des constantes d’arbres de fouille, Thèse de doctorat, Lille, 2008.

[8] J. Désarménien, G. Duchamp, D. Krob, G. Mélançon, Quelques remarques sur les superalgèbres de Lie libres, C. R. Acad. Sci. Paris, 318 (1994), série I, 419–424.
[9] G. Duchamp, D. Krob, B. Leclerc, J.-Y. Thibon, Noncommutative symmetric functions III: Deformations of Cauchy and convolution structures, *Discrete Mathematics and Theoretical Computer Science*, 1 (1997), 159–216.

[10] G. H. E. Duchamp, C. Tollu, K. A. Penson, G. A. Koshevoy, Deformations of Algebras: Twisting and Perturbations, *Séminaire Lotharingien de Combinatoire*, B62e (2010), 14 pp.

[11] G. H. E. Duchamp, P. Blasiak, A. Horzela, K. A. Penson, A. I. Solomon, A three-parameter Hopf deformation of the algebra of Feynman-like diagrams, *Journal of Russian Laser Research*, 31(1) (March 2010), 162–181.

[12] G. Duchamp, M. Floreut, É. Laugerotte, J.-G. Luque, Direct and dual laws for automata with multiplicities, *Theoretical Computer Science*, 267(1–2) (September 2001) 105–120.

[13] J.-Y. Enjalbert, Hoang Ngoc Minh, Propriétés combinatoires et prolongement analytique effectif de polyzètas de Hurwitz et de leurs homologues, *J. Théor. Nombres Bordeaux*, 19(3) (2007), 595–640.

[14] J.-Y. Enjalbert, Hoang Ngoc Minh, Analytic and combinatoric aspects of Hurwitz polyzètas, *J. Théor. Nombres Bordeaux*, 23(2) (2011) 353–386.

[15] J.-Y. Enjalbert, Hoang Ngoc Minh, Combinatorial study of colored Hurwitz polyzètas, *Discrete Math.*, 312(24) (December 2012) 3489–3497.

[16] M. Fréchet, Biographie du mathématicien alsacien Arbogast, *Thalès*, 4 (1940) 43–55.

[17] G. H. Hardy, *Orders of Infinity: The “Infinitärcalcül” of Paul Du Bois-Reymond*, Cambridge University Press (Cambridge Tracts in Mathematics and Mathematical Physics, vol. 12), viii + 62 pages, 1910.

[18] Hoang Ngoc Minh, G. Jacob, M. Petitot, N. E. Oussous, De l’algèbre des ζ de Riemann multivariées à l’algèbre des ζ de Hurwitz multivariées, *Séminaire Lotharingien de Combinatoire*, B44i (2001), 21 pp.

[19] M. E. Hoffman, Quasi-symmetric functions, multiple zeta values, and rooted trees, arxiv: math/0609413 [math.QA] (14 Sep 2006) 4 pp.

[20] L. A. Lusternik, S. S. Petrova, Les premières étapes du calcul symbolique, *Revue d’Histoire des Sciences*, 25(3) (1972) 201–206.

[21] R. Ree, Lie elements and an algebra associated with shuffles, *Ann. Math.*, 68(2) (September 1958) 210–220.

[22] R. Ree, Generalized Lie elements, *Canad. J. Math.*, 12 (1960) 493–502.

[23] D. E. Radford, A natural ring basis for shuffle algebra and an application to group schemes, *J. Algebra*, 58(2) (June 1979) 432–454.
[24] C. Reutenauer, *Free Lie Algebras*, Clarendon Press (London Mathematical Society Monographs, vol. 7), Oxford, 286 pages, 1993.

[25] Stefan Weinzierl. *Hopf algebras and Dyson-Schwinger equations* lectures given at the workshop ”Dyson-Schwinger Equations in Modern Physics and Mathematics”, Trento, 30 pages, September 2014. [http://arxiv.org/abs/1506.09119](http://arxiv.org/abs/1506.09119)