Cluster Probability in Bootstrap Percolation

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We develop a recursive formula for the probability of a $k$-cluster in bootstrap percolation.

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I. INTRODUCTION

The model of bootstrap percolation (or $k$-core percolation, as we will refer to it) has a connection to jamming transitions, including the glass transition. Many features of dynamical arrest in glassforming liquids are captured by kinetically-constrained spin models which map onto $k$-core percolation and its variants. In addition, Schwarz, et al. have argued that there is an analogy (but not a strict mapping) between $k$-core percolation and the jamming transition of frictionless granular packings, similar to the analogy between $k$-core percolation and rigidity percolation. In the mean-field, or infinite-dimensional limit, the $k$-core percolation transition was shown to be peculiar in Ref. and subsequently in Ref. The transition is discontinuous but also exhibits power-law scaling and a diverging susceptibility and length scale. This unusual type of transition has been called a random first-order phase transition. What makes the mean-field $k$-core percolation transition particularly interesting is that its exponents are identical to the mean-field exponents of several models that have been proposed for jamming transitions, raising the possibility that $k$-core percolation may be in the same universality class as these models.

For jamming transitions, the random first-order nature of the transition does not appear to be an artifact of the mean-field approximation. Numerical simulations and scaling arguments suggest that the jamming transition of frictionless granular packings (Point J) is discontinuous with a diverging length scale in two and three dimensions. It has also been argued that the finite-dimensional glass transition, if it exists as a true phase transition, has a mixed character, with a discontinuity in the infinite-time limit of the dynamical structure factor and a diverging dynamical length scale. This raises the question of the nature of the $k$-core percolation transition in finite dimensions $d$. Expansions in $1/d$ indicate that the mixed nature of the transition should persist for a range of large but finite $d$. In two dimensions, specific variants of $k$-core percolation have been identified that are rigorously known to exhibit discontinuous transitions with diverging length scales, both at $p = 1$ (full occupancy) and $p < 1$ (partial occupancy). Numerical studies of other variants also point to the possibility of mixed transitions in finite dimensions at $p < 1$. However, it is also known that some realizations of the model, such as $k = 3$ on the two-dimensional triangular lattice, exhibit continuous transitions. Thus, in finite dimensions, it appears that different versions of $k$-core models can exhibit at least three different types of transitions: continuous transitions, mixed transitions at $p = 1$ and mixed transitions at $p < 1$.

In order to sort through this confusing state of affairs, it would be useful to map $k$-core percolation onto a spin model, which could then be studied by a wide variety of methods. Such an approach proved extremely powerful for ordinary percolation. A necessary first step towards this goal is to be able to count $k$-core clusters with the correct probabilities. In this paper, we show how this can be done by deriving a relation for the probability distribution of clusters.

The $k$-core ensemble is defined as follows. We start with the ensemble of $2^N$ percolation configurations on a finite lattice of $N$ bonds. Each percolation configuration $C_p$ has probability

$$P_p(C_p) = p^{n_{occ}(C_p)} (1-p)^{n_{vac}(C_p)},$$

where $n_{occ}(C_p)$ and $n_{vac}(C_p)$ are, respectively, the number of occupied and vacant bonds in the percolation configuration $C_p$. A $k$-core configuration $C_k$ is obtained from $C_p$ by removing all bonds which intersect a site to which there are less than $k$ remaining bonds. This process is recursively continued because as bonds are so removed, they may cause more sites to have less than $k$ neighbors. After this culling, the remaining $k$-cluster is one in which all sites have at least $k$ occupied bonds. The weight of the configuration $C_k$ is the sum of the percolation probabilities over all the percolation configurations which gave rise to $C_k$.

In the percolation problem one can easily establish that the probability of finding an isolated cluster $\Gamma_p$ is given by

$$P_p(\Gamma_p) = p^{n(\Gamma_p)} (1-p)^{t(\Gamma_p)},$$

where $n(\Gamma_p)$ is the number of occupied bonds in the cluster $\Gamma_p$ and $t(\Gamma_p)$ is the number of perimeter bonds which must be unoccupied in order to define the limit of the percolation cluster $\Gamma_p$. One would like to write a similar formula for $P_k(\Gamma)$, the probability of a $k$-cluster $\Gamma_k$, but it is obvious that this is not quite so trivial because of the complications of the culling process. It is the purpose of this paper to develop such a formula.

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II. FORMULATION

To proceed we introduce the function $e_T(C_p)$ which is defined as unity if the percolation configuration $C_p$, when culled, gives rise to the rooted $k$-cluster $\Gamma$. A rooted cluster includes the site at the origin and two rooted clusters are distinct unless their list of occupied bonds coincides exactly. The final culled configuration must contain the rooted $k$-cluster $\Gamma$, but can also contain an arbitrary number of $k$-clusters disconnected from $\Gamma$. If culling of $C_p$ does not produce a configuration containing the rooted cluster $\Gamma$, then $e_T(C_p)$ is zero. With this definition it seems obvious that

$$P_k(\Gamma) = \sum_{C_p} e_T(C_p) P_p(C_p).$$

This relation expresses the fact that the cluster $\Gamma$ inherits the total percolation probability of all configurations which contain the cluster $\Gamma$ after culling. Note that a fixed $\Gamma$ inherits probability from not only single clusters which, when culled, yield $\Gamma$, but also from the vast array of configurations which, when culled, yield $\Gamma$ together with arbitrary disconnected $k$-clusters. Now it is clear that the sum over $C_p$ can be restricted to configurations which either are equal to $\Gamma$ or include it as a proper subset. So we write

$$P_k(\Gamma) = \sum_{C_p: C_p \geq \Gamma} e_T(C_p) P_p(C_p)$$

$$= \sum_{C_p: C_p \geq \Gamma} \left(1 - [1 - e_T(C_p)]\right) P_p(C_p).$$

But

$$\sum_{C_p: C_p \geq \Gamma} P_p(C_p) = p^{n(\Gamma)}$$

because after we occupy the bonds of $\Gamma$ (with probability $p^{n(\Gamma)}$) we sum over all states of all the bonds not in $\Gamma$ (which are occupied with probability $p$ and vacant with probability $1-p$). Thus

$$P_k(\Gamma) = p^{n(\Gamma)} - \sum_{C_p: C_p \geq \Gamma} [1 - e_T(C_p)] P_p(C_p).$$

The result of Eq. (2) for the probability of a percolation cluster can be calculated in this same style. Note the meaning of the factor $[1 - e_T(C_p)]$: it is an indicator function such that culling does not lead to the $k$-cluster $\Gamma$. But since $C_p$ contains $\Gamma$ but does not cull to $\Gamma$, it must be that the configuration $C_p$ leads to a cluster $\Gamma'$ which includes $\Gamma$ as a proper subset. So we have

$$P_k(\Gamma) = p^{n(\Gamma)} - \sum_{\Gamma': \Gamma' \geq \Gamma} \sum_{C_p: C_p \geq \Gamma'} e_T(C_p) P_p(C_p)$$

$$= p^{n(\Gamma)} - \sum_{\Gamma': \Gamma' \geq \Gamma} P_k(\Gamma').$$

Note that the sum over $k$-clusters $\Gamma$ does not include $\Gamma' = \Gamma$. Equation (7) is the principal result of this paper. On a qualitative level we can see the following: for percolation Eq. (2) gives what could be called a perimeter renormalization $(1 - p)^{n(\Gamma)}$, whereas the subtraction terms in Eq. (4) give a much weaker “culling renormalization.” In the extreme case of lattice animals, there is no perimeter renormalization at all. In that case $P(C) = p^n(\Gamma)$ (where $p$ is interpreted to be the bond fugacity, usually denoted $K$). The continuous aspect of $k$-core percolation has the same anomalous correlation length exponent $\nu = 1/4$ as does lattice animals in infinite dimensions. A heuristic explanation of this is that the culling renormalization is probably closer to the absence of a perimeter renormalization in lattice animals than to the perimeter renormalization of percolation.

Now we briefly explore some consequences of the above result. We iterate Eq. (4) to get

$$P_k(\Gamma_k) = p^{n(\Gamma_k)} - \sum_{\Gamma'_k: \Gamma'_k > \Gamma_k} p^{n(\Gamma'_k)} + \sum_{\Gamma'_k: \Gamma'_k > \Gamma_k} p^{n(\Gamma'_k)} + \ldots + (-1)^m \sum_{\Gamma'_k: \Gamma'_k > \Gamma_k} p^{n(\Gamma'_k)} + \ldots$$

$$= p^{n(\Gamma_k)} + \sum_{m=1}^{m_{\text{max}}} (-1)^m \sum_{\{\Gamma_{k}(m)\}} p^{n(\Gamma_{k}(m))},$$

where, for simplicity, we do not write out (in the last line) the inclusions. Because we are dealing with a finite lattice, the index $m$ is bounded by a maximum value, denoted $m_{\text{max}}$.

Next we consider the calculation of the connectedness susceptibility, $\chi$, which can be written as

$$\chi = \sum_{\Gamma_k} P_k(\Gamma_k) n(\Gamma_k)^2.$$
This result no longer requires that \( N \) be finite because for any finite cluster \( \Gamma_k \), the number of terms coming from included subclusters in the sum inside the square brackets is independent of system size. Thus this formula presumably can be used as long as \( p < p_k \) and it may yield a possible route to an analytic approach to the calculation of \( \chi \) for finite dimensional lattices.

The result Eq. (11) can be generalized to any observable \( G(\Gamma_k) \) that can be defined for a \( k \)-cluster \( \Gamma_k \). In that case, the ensemble-averaged value of \( G \) is

\[
\langle G \rangle = \sum_{\Gamma_k} p^{n(\Gamma_k)} G_c(\Gamma_k),
\]

where the cumulant function \( G_c \) is

\[
G_c(\Gamma_k) = G(\Gamma_k) - \sum_{\Gamma_k^{(1)} < \Gamma_k} G(\Gamma_k^{(1)}),
\]

or more simply,

\[
G_c(\Gamma_k) = G(\Gamma_k) - \sum_{\Gamma_k^{(1)} < \Gamma_k} G_c(\Gamma_k^{(1)}).
\]

This cumulant subtraction for the \( k \)-core percolation problem is much more difficult to handle than it is for ordinary percolation.

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