BLOW-UP CRITERIA OF STRONG SOLUTIONS TO THE ERICKSEN-LESLIE SYSTEM IN $\mathbb{R}^3$

MIN-CHUN HONG, JINKAI LI AND ZHOUPING XIN

Abstract. In this paper, we establish the local well-posedness and blow-up criteria of strong solutions to the Ericksen-Leslie system in $\mathbb{R}^3$ for the well-known Oseen-Frank model. The local existence of strong solutions to liquid crystal flows is obtained by using the Ginzburg-Landau approximation approach to guarantee the constraint that the direction vector of the fluid is of length one. We establish four kinds of blow-up criteria, including (i) the Serrin type; (ii) the Beal-Kato-Majda type; (iii) the mixed type, i.e., Serrin type condition for one field and Beal-Kato-Majda type condition on the other one; (iv) a new one, which characterizes the maximal existence time of the strong solutions to the Ericksen-Leslie system in terms of Serrin type norms of the strong solutions to the Ginzburg-Landau approximate system. Furthermore, we also prove that the strong solutions of the Ginzburg-Landau approximate system converge to the strong solution of the Ericksen-Leslie system up to the maximal existence time.

1. Introduction

The Ericksen-Leslie theory is successful in describing dynamic flows of liquid crystals in physics, which is based on the fundamental Oseen-Frank model. Mathematically, the static theory of nematic liquid crystals involves a unit vector field $u$ in a region $\Omega \subset \mathbb{R}^3$. The Oseen-Frank density $W(u, \nabla u)$ is given by

$$W(u, \nabla u) = k_1 (\text{div } u)^2 + k_2 (u \cdot \text{curl } u)^2 + k_3 |u \times \text{curl } u|^2 + k_4 [\text{tr}(\nabla u)^2 - (\text{div } u)^2],$$

where $k_1$, $k_2$, $k_3$ and $k_4$ are positive constants. The free energy for a configuration $u \in H^1(\Omega; S^2)$ is

$$E(u; \Omega) = \int_{\Omega} W(u, \nabla u) \, dx.$$ 

The Euler-Lagrange system for the Oseen-Frank energy $E(u, \Omega)$ is:

$$\nabla \alpha \left[ W_{p\alpha} (u, \nabla u) - u^i u^i W_{p\alpha} (u, \nabla u) \right] - W_{a\alpha} (u, \nabla u) + W_{a\alpha} (u, \nabla u) u^i u^i + W_{p\alpha} (u, \nabla u) \nabla_\alpha u^i + W_{p\alpha} (u, \nabla u) u^i \nabla_\alpha u^i = 0 \quad \text{in } \Omega$$

for $i = 1, 2, 3$ (see [13]), where the standard summation convention is adopted.

Since the divergence of $\text{tr}(\nabla u)^2 - (\text{div } u)^2$ is free ([3]), one can rewrite the density $W(u, \nabla u)$ as

$$W(u, \nabla u) = a |\nabla u|^2 + V(u, \nabla u), \quad a = \min\{k_1, k_2, k_3\} > 0,$$

where

$$V(u, \nabla u) = (k_1 - a) (\text{div } u)^2 + (k_2 - a) (u \cdot \text{curl } u)^2 + (k_3 - a) |u \times \text{curl } u|^2.$$

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Hardt, Kinderlehrer and Lin in [10] proved that a minimizer $u$ of the energy $E$ is smooth away from a closed set $\Sigma$ of $\Omega$. Moreover, $\Sigma$ has Hausdorff dimension strictly less than one. See further contributions in [5] and [11] about the static theory of liquid crystals.

Dynamic motion of liquid crystals are described by the Ericksen-Leslie system, including the velocity vector $v$ and the direction vector $u$ of the fluid (see [9] and [18]). More precisely, let

$$v = (v^1, v^2, v^3)$$

be the velocity vector of the fluid and

$$u = (u^1, u^2, u^3)$$

the unit direction vector. The Ericksen-Leslie system is given by (e.g. [21] and [22])

$$v_t^i + (v \cdot \nabla)v^i - \Delta v^i + \nabla_x (\nabla_x u^k W_{\rho_k}^i (u, \nabla u)),$nabla \cdot v = 0,$nabla_x (\nabla_x u^k W_{\rho_k}^i (u, \nabla u)) + W_{\rho_k} (u, \nabla u) W_{\rho_k} (u, \nabla u)$$

for $i = 1, 2, 3$. Here $\nu$, $\lambda$ are given positive constants, and $p$ is the pressure.

The system (1.1)-(1.3) is a system of the Navier-Stokes equations coupled with the gradient flow for the Oseen-Frank model, which is an extension of the harmonic map flow ([6]). Caffarelli, Kohn and Nirenberg [3] established the fundamental result on the existence and partial regularity of the global modified weak solutions of the Navier-Stokes equations (See also [19], [26]). On the other hand, Struwe [25] and Chen-Struwe [4] established the existence and partial regularity of global weak solutions of the harmonic map flow between manifolds. There is an interesting question to establish the global existence of weak solutions of (1.1)-(1.3) supplemented with initial or initial-boundary conditions. The question for the case of $k_1 = k_2 = k_3$ was answered by the first author in [12] in $\mathbb{R}^2$ and Lin-Lin-Wang [20] in a bounded domain of $\mathbb{R}^2$ independently. Recently, the first and third authors [13] proved the global existence of weak solutions of the general Ericksen-Leslie system (1.1)-(1.3) in $\mathbb{R}^2$. However, the question on the global weak solution on the system in 3D is still unknown. In the study of the Navier-Stokes equations, there are two well-known blow-up criteria for the strong (smooth) solutions: the Serrin (also called Ladyzhenskaya-Prodi-Serrin type) criterion [23] and the Beal-Kato-Majda type criteria [2]. Recently, for the simplified model, i.e. $k_1 = k_2 = k_3$, the local strong solutions was obtained by Wen and Ding [14], and the blow up criterions were obtained by Huang and Wang [15], and there have been many new results developed in this direction [16].

In this paper, we consider the Cauchy problem to the Ericksen-Leslie system (1.1)-(1.3) for the general Oseen-Frank model in $\mathbb{R}^3$. Suppose that the initial data is given by

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x).$$

Throughout this paper, we always assume that $(u_0, v_0)$ satisfies

$$v_0 \in H^1(\mathbb{R}^3), \quad \text{div} v_0 = 0, \quad u_0 - b \in H^2(\mathbb{R}^3), \quad |u_0| = 1$$

for some constant unit vector $b$.

In order to state our results, we give the definition of strong solutions and introduce some notations.
Definition 1.1. For any $T > 0$, a couple $(u, v)$ is called a strong solution to the system (1.1)–(1.3) in $\mathbb{R}^3 \times (0, T)$ if and only if

$$u \in L^2(0, T; H^3(\mathbb{R}^3)), \quad \partial_t u \in L^2(0, T; H^1(\mathbb{R}^3)), \quad |u| = 1,$$

$$v \in L^2(0, T; H^3(\mathbb{R}^3)), \quad \partial_t v \in L^2(0, T; L^2(\mathbb{R}^3))$$

and it satisfies the equation (1.1)–(1.3) a.e.

Definition 1.2. A finite time $T^* > 0$ is called the maximal existence time of a strong solution $(u, v)$ to the system (1.1)–(1.3) if and only if $(u, v)$ is a strong solution in $\mathbb{R}^3 \times (0, T)$ for all $T < T^*$ and

$$\lim_{T \nearrow T^*} \|\nabla u, v\|_{L^2(0, T; H^2(\mathbb{R}^3))} = \infty.$$

The maximal existence time of the strong solution to the approximate system (1.5)–(1.7) can be defined similarly. For $T > 0$, we denote

$$J_1(T) = \inf_{(q, r) \in \mathcal{O}} \|\nabla u\|_{L^q(\mathbb{R}^3; L^r(\mathbb{R}^3))} + \inf_{(q, r) \in \mathcal{O}} \|v\|_{L^q(\mathbb{R}^3; L^r(\mathbb{R}^3))},$$

$$J_2(T) = \|\omega\|_{L^1(\mathbb{R}^3; BMO(\mathbb{R}^3))} + \|\Delta u\|_{L^1(\mathbb{R}^3; L^\infty(\mathbb{R}^3))},$$

$$J_3(T) = \inf_{(q, r) \in \mathcal{O}} \|v\|_{L^q(\mathbb{R}^3; L^r(\mathbb{R}^3))} + \|\Delta u\|_{L^1(\mathbb{R}^3; L^\infty(\mathbb{R}^3))},$$

$$J_4(T) = \|\omega\|_{L^1(\mathbb{R}^3; BMO(\mathbb{R}^3))} + \inf_{(q, r) \in \mathcal{O}} \|\nabla u\|_{L^q(\mathbb{R}^3; L^r(\mathbb{R}^3))},$$

where $\omega = \nabla \times v$ and

$$\mathcal{O} = \left\{(q, r) \in \mathbb{R}^3 \mid \frac{2}{q} + \frac{3}{r} = 1, q \in [2, \infty), r \in (3, \infty)\right\}.$$

Then, we have the following results on the local existence and blow up criteria of strong solutions to the system (1.1)–(1.3).

Theorem 1. The system (1.1)–(1.3) has a unique strong solution $(u, v)$ in $\mathbb{R}^3 \times (0, T^*)$ for some positive number $T^*$ depending only on the initial data. The maximal existence time $T^* < \infty$ can be described as

$$J_1(T^*) = J_2(T^*) = J_3(T^*) = J_4(T^*) = \infty.$$

Moreover, for any $T > 0$, $J_1(T), J_2(T), J_3(T)$ and $J_4(T)$ are equivalent in the following sense:

$$J_1(T) = \infty \iff J_2(T) = \infty \iff J_3(T) = \infty \iff J_4(T) = \infty.$$

The proof of Theorem 1 is divided into two parts: local existence and blow up criterion of the strong solution. For the proof of the local existence of the Ericksen-Leslie system, the main difficulty is that the system (1.4)–(1.6) is not a standard parabolic system in the sense described in [17] or [8]. As a result, the constraint $|u| = 1$ cannot be derived directly from the system by using the maximum principle. To overcome this difficulty, we follow the same idea in [13] to consider the approximating Ericksen-Leslie system in the following:

(1.5) \[ v^i_t + (v \cdot \nabla)v^i - \Delta v^i + \nabla_x, D = -\nabla_x, (\nabla_x, u^k W^i_{\rho}(u, \nabla u)), \]

(1.6) \[ \nabla \cdot v = 0, \]

(1.7) \[ u^i_t + (v \cdot \nabla) u^i = \nabla_x \left[ W^i_{\rho}(u, \nabla u) \right] - W^i_{\rho}(u, \nabla u) + \frac{1}{\varepsilon^2} u^i(1 - |u|^2) \]
for $i = 1, 2, 3$, prescribing the initial condition \((1.4)\). However, it should be noted that the condition that $u_0 \in H^1_0$ and $v_0 \in L^2$ is insufficient to establish the local existence of the Ericksen-Leslie system in 3D. Instead we must assume that $u_0 \in H^2_0$ and $v_0 \in H^1$. Under this condition, we can establish uniform estimates in $\varepsilon$ on higher derivatives of solutions $(u_\varepsilon, v_\varepsilon)$ to the approximation system \((1.5)-(1.7)\) in a short time and prove the local existence. In order to obtain such uniform estimates of $u_\varepsilon$ and its higher derivatives, we impose a Beal-Kato-Majda condition on the velocity field $v$, the first key idea is to prove that $|u_\varepsilon|$ is close to zero and the second key idea is to control a difficult term $\int \frac{1-|u_\varepsilon|^2}{|\varepsilon|^2} |\partial_t u_\varepsilon|^2$ by using the decomposition

$$\partial_t u_\varepsilon = \frac{1}{|u_\varepsilon|^2} (\partial_t u_\varepsilon \cdot u_\varepsilon) u_\varepsilon - \frac{1}{|u_\varepsilon|^2} (\partial_t u_\varepsilon \times u_\varepsilon) \times u_\varepsilon.$$  

We note that $\partial_t u_\varepsilon \times u_\varepsilon$ is independent of $\varepsilon$ by equation \((1.7)\).

To establish the blow up criteria of the Ericksen-Leslie system, we need a prior estimates on high derivatives of the solution before the maximal existence time $T^*$. Two kinds of estimates are established, which roughly speaking involve the $L^\infty(H^1)$ and $L^\infty(H^2)$ bounds of $(v, \nabla u)$, respectively. One of the key ideas in establishing such estimates is using the constraint $|u| = 1$ to handle the terms like $u \cdot \Delta^2 u$ by reducing the order of the derivatives. In Theorem 1, we impose a Serrin type condition or Beal-Kato-Majda condition on $u$ or $v$. If we impose a Serrin type condition on the velocity field $v$, the $L^\infty(H^1)$-bounds on $(v, \nabla u)$ is sufficient for the proof, no matter what kind of condition is imposed on the direction field $u$. If imposing a Beal-Kato-Majda condition on the velocity field $v$, we have to analyze the second kind estimate $L^\infty(H^2)$. In this case, a new logarithmic Sobolev type inequality is needed to control the $L^1([-T, T]; L^\infty(\mathbb{R}^3))$ norm of $\nabla v$ in terms of its $L^1([-T, T]; BMO(\mathbb{R}^3))$ and the norms of higher order derivatives.

**Remark 1.1.** (i) $J_4(T^*) = \infty$ is a Serrin type condition for both fields $u$ and $v$; $J_2(T^*) = \infty$ is a Beal-Kato-Majda type condition for both fields; $J_3(T^*) = \infty$ and $J_4(T^*) = \infty$ are a Serin type condition for one field and a Beal-Kato-Majda type for the other one.

(ii) Recently, Huang-Wang \[15\] established the blow up criterion of the form

$$\|\omega\|_{L^1_t(L^\infty_x)} + \|\nabla u\|_{L^2_t(L^\infty_x)} = \infty,$$

for the simplified model, which is a special case of $J_4$ in Theorem 1.

(ii) Theorem 4 shows that the Serrin type condition is equivalent to the Beal-Kato-Majda type in our case.

By comparing with the well-known result of Chen-Struwe \[4\] on the harmonic map flow, it is of interests to investigate the convergence problem of solutions of the approximating system \((1.5)-(1.7)\). In fact, the approximating Ericksen-Leslie system \((1.5)-(1.7)\) was first introduced by Lin-Liu in \[21\] through the Ginzburg-Landau approximation. They proved global existence of the classical solution of the approximate system \((1.5)-(1.7)\) with \((1.4)\) in dimension two and the weak solution of the same system in dimension three for the case of $k_1 = k_2 = k_3$. Since their estimates depends on the parameter $\varepsilon$ (also see \[22\]), it is unknown whether as $\varepsilon \to 0$ the solutions $(u_\varepsilon, v_\varepsilon)$ of \((1.5)-(1.7)\) converge to the solution of the original Ericksen-Leslie system \((1.1)-(1.3)\). In this paper, we can answered this problem...
and prove that these strong solutions \((u_\varepsilon, v_\varepsilon)\) of the approximate system \((1.5) - (1.7)\) converge to the strong solution \((u, v)\) of the original Ericksen-Leslie system up to the maximal existence time of \((u, v)\). More precisely, we have:

**Theorem 2.** Let \((u, v)\) be a strong solution to the system \((1.1) - (1.4)\) in \(\mathbb{R}^3\times (0, T^*)\). Let \((u_\varepsilon, v_\varepsilon)\) be the unique strong solution to the system \((1.5) - (1.7)\) in \(\mathbb{R}^3 \times (0, T_\varepsilon^*)\) with \((1.4)\), where \(T_\varepsilon^*\) is the maximal existence time of \((1.5) - (1.7)\). Then for sufficiently small \(\varepsilon\), \(T_\varepsilon^* \geq T^*\) and for any \(T \in (0, T^*)\), it holds that

\[
(\nabla u_\varepsilon, v_\varepsilon) \to (\nabla u, v), \quad \text{in } L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))
\]

and

\[
\lim_{\varepsilon \to 0} \| (\nabla u_\varepsilon, v_\varepsilon) \|_{L^\infty(0, T; H^1(\mathbb{R}^3))} + \| (\nabla u_\varepsilon, v_\varepsilon) \|_{W^{2,1}(\mathbb{R}^3 \times (0, T))} < \infty,
\]

where \(\| f \|_{W^{2,1}(\mathbb{R}^3 \times (0, T))} = \| f \|_{L^2(0, T; H^2(\mathbb{R}^3))} + \| \partial_t f \|_{L^2(\mathbb{R}^3 \times (0, T))}\).

Furthermore, \(T^* < \infty\) is the maximal existence time if and only if

\[
\lim_{\varepsilon \to 0} \| (\nabla u_\varepsilon, v_\varepsilon) \|_{L^4(0, T^*; L^r(\mathbb{R}^3))} = \infty
\]

for any \((q, r) \in \mathcal{O}\), with \(\mathcal{O}\) being the same set stated as before.

The key in the proof of Theorem 2 is to establish the strong convergence and uniform estimates, which is divided in three steps: in step 1, we prove the strong convergence and uniform estimate up to a time \(T_M\), where \(M\) is a constant depending only on the initial data and \(T\); in step 2, we show that if the strong convergence and uniform estimate hold true up to \(T_1\) with \(T_1 < T\), then they hold true up to another time \(T_2 := \min\{T, T_1 + T_M\}\); in step 3, we prove the strong convergence and uniform estimate up to time \(T\). To prove the strong convergence up to \(T_M\), we need to derive high order estimates up to time \(T_M\) and prove that the energy of \((u_\varepsilon, v_\varepsilon)\) is small outside a big ball uniformly for \(\varepsilon\). High order estimates of these strong solutions are guaranteed by Proposition 2.1, which, roughly speaking, states that the existence time and the uniform estimates of these strong solutions depend only on the \(H^1\) bounds of the initial data \((\nabla u_\varepsilon(0), v_\varepsilon(0))\) and the \(L^2\) bounds of \(\frac{1 - |u_\varepsilon(0)|^2}{\varepsilon} u_\varepsilon(0)\), while the uniform smallness outsider a big ball can be guaranteed by our Lemma 2.3, which is a local type of energy inequality.

Using these two tools, we can prove the strong convergence of these solutions up to the time \(T_M\). If the strong convergence and uniform estimate hold true up to time \(T_1\) for some \(T_1 < T\), by the aid of the strong convergence and the uniform estimates up to time \(T_1\), we show that the \(H^1\) bounds of \((\nabla u_\varepsilon(T_1), v_\varepsilon(T_1))\) and the \(L^2\) bounds of \(\frac{1 - |u_\varepsilon(T_1)|^2}{\varepsilon} u_\varepsilon(T_1)\) is controlled by \(M\). As a result, starting from \(T_1\) and taking \((u_\varepsilon(T_1), v_\varepsilon(T_1))\) as initial data, we obtain high order estimates up to time \(T_2 = \min\{T, T_1 + T_M\}\). With this estimate in hand, using the same argument as in step 1, we can show the strong convergence up to \(T_2\). Continuing this procedure, we prove the strong convergence up to \(T\), and thus complete the proof of Theorem 2.

By the aid of the strong convergence and uniform estimate, we can characterize the maximal existence time of the strong solutions to the Ginzburg-Landau system.

**Remark 1.2.** Theorem 2 can be viewed as a blow up criterion for the strong solutions to the Ericksen-Leslie system \((1.1) - (1.4)\) in term of the Serrin type norms of the strong solutions to the Ginzburg-Landau approximation system \((1.5) - (1.7)\). It is a new kind of blow up criterion for the Ericksen-Leslie system even for the simplified case that \(k_1 = k_2 = k_3\).
The rest of the paper is organized as follows. In Section 2, we prove the local existence part of Theorem 1; the blow-up criteria part of Theorem 2 is proved in Section 3. Finally, we give the proof of Theorem 2 in Section 4.

2. LOCAL EXISTENCE

In this section, we prove the local existence of strong solutions to the Ericksen-Leslie system by using the Ginzburg-Landau approximation mentioned in Introduction. One can easily check that the following hold

\[ W(z, p) \geq a|p|^2, \quad W_{p_1, p_2}(z, p) \xi_1 \xi_2 \geq a|\xi|^2, \quad \forall z \in \mathbb{R}^3, p, \xi \in M^{3 \times 3}, \]

\[ |W(u, \nabla u)| \leq C|u|^2|\nabla u|^2, \quad |W_{u, u}(u, \nabla u)| \leq C|\nabla u|^2, \]

\[ |W_{u, \nabla}(u, \nabla u)| \leq C|\nabla u|^2, \quad |W_{p, \nabla}(u, \nabla u)| \leq C|\nabla u|^2, \quad |W_{u, \nabla}(u, \nabla u)| \leq C|\nabla u|^2. \]

These inequalities will be used in the following text without any further mentions. For the Ginzburg-Landau approximate system (1.5)–(1.7), the following local existence result holds.

**Lemma 2.1.** Suppose that the initial data \((u_{0\varepsilon}, v_{0\varepsilon})\) satisfies

\[ u_{0\varepsilon} - b \in H^2(\mathbb{R}^3), \quad v_{0\varepsilon} \in H^1(\mathbb{R}^3), \quad \text{div} \ v_{0\varepsilon} = 0, \]

where \(b\) is a constant unit vector. Then there is a positive number \(T^0\), such that the system (1.5)–(1.7) with initial data \((u_{0\varepsilon}, v_{0\varepsilon})\) admits a unique solution \((u_{\varepsilon}, v_{\varepsilon})\) on \(\mathbb{R}^3 \times (0, T^0)\), satisfying

\[ u_{\varepsilon} \in L^2(0, T^0; H^3(\mathbb{R}^3)), \quad \partial_t u_{\varepsilon} \in L^2(0, T^0; H^1(\mathbb{R}^3)), \]

\[ v_{\varepsilon} \in L^2(0, T^0; H^2(\mathbb{R}^3)), \quad \partial_t v_{\varepsilon} \in L^2(\mathbb{R}^3 \times (0, T^0)). \]

**Proof.** We can apply the standard contraction mapping principle based on the following linearized problem

\[ \tilde{u}_i^t - \Delta \tilde{u}_i + \nabla_i P = -\nabla_j (\nabla_j u^k W_{p_1} (u, \nabla u)) - v \nabla u^i, \]

\[ \nabla \cdot \tilde{v} = 0, \]

\[ \tilde{u}_i^t - \nabla_\alpha [W_{p_1} (u, \nabla u)] = -W_{u, u} (u, \nabla u) - v \nabla u^i + \frac{1}{\varepsilon^2} u^i (1 - |u|^2) \]

for \(i = 1, 2, 3\). The argument is standard, and thus omitted. \( \square \)

For strong solutions to the system (1.5)–(1.7), it holds the following basic energy balance.

**Lemma 2.2.** Let \((u_{\varepsilon}, v_{\varepsilon})\) be a strong solution to the system (1.5)–(1.7) in \(\mathbb{R}^3 \times (0, T)\). Then

\[ \frac{d}{dt} \int_{\mathbb{R}^3} \left[ \frac{|v_{\varepsilon}|^2}{2} + W(u_{\varepsilon}, \nabla u_{\varepsilon}) + \frac{(1 - |u_{\varepsilon}|^2)^2}{4\varepsilon^2} \right] dx \]

\[ + \int_{\mathbb{R}^3} \left[ |\nabla v_{\varepsilon}|^2 + |\partial_t v_{\varepsilon} + (v_{\varepsilon} \cdot \nabla) u_{\varepsilon}|^2 \right] dt = 0 \]

for any \(t \in (0, T)\).
Proof. Multiplying \( \frac{d}{dt} \) by \( v^i_t \) and \( \nabla u^i_t \) by \( \partial_t u^i_t + v^i \cdot \nabla u^i_t \) respectively and then summing the resulting equations up and integrating over \( \mathbb{R}^3 \), we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \left[ \frac{|v^i_t|^2}{2} + W(u^i_e, \nabla u^i_e) + \frac{1}{4\varepsilon^2} \right] dx
+ \int_{\mathbb{R}^3} \left[ (\nabla v^i_e)^2 + |\partial_t u^i_e + (v^i \cdot \nabla)u^i_e|^2 \right] dx
= \int_{\mathbb{R}^3} \left( \nabla u^i_e \nabla (W_{p^i_e}) (u^i_e, \nabla u^i_e) \right)
- \int_{\mathbb{R}^3} \left( v^i \cdot \nabla u^i_e \right) dx
= - \int_{\mathbb{R}^3} \left[ -v^i \nabla \cdot (\nabla (W_{p^i_e}) (u^i_e, \nabla u^i_e)) \right] dx = 0,
\]
which proves the claim.

The following high order estimate is one of key lemmas in this paper.

**Lemma 2.3.** Let \( (u^i_e, v^i_e) \) be a strong solution to the system (1.5)–(1.7) in \( \mathbb{R}^3 \times (0, T) \). Assume that \( 4 \leq |u^i_e| \leq 5 \) on \( \mathbb{R}^3 \times (0, T) \). Then for any \( t \in (0, T) \), it holds that
\[
\frac{d}{dt} \int_{\mathbb{R}^3} |\Delta u^i|^2 + |\partial_t u^i|^2 + |\nabla v^i_e|^2 dx
+ \int_{\mathbb{R}^3} \left( a|\nabla^3 u^i_e|^2 + a|\nabla \partial_t u^i_e|^2 + \frac{1}{\varepsilon^2} |\Delta u^i_e|^2 \right) dx
\leq C \int_{\mathbb{R}^3} (|\nabla u^i_e|^2 + |v^i_e|^2)(|\nabla^2 u^i_e|^2 + |\partial_t u^i_e|^2 + |\nabla v^i_e|^2) dx,
\]
where \( C \) is a positive constant independent of \( \varepsilon \).

**Proof.** Since \( 4 \leq |u^i_e| \leq 5 \) on \( \mathbb{R}^3 \times (0, T) \), it follows from (1.7) that
\[
\left| \frac{1}{\varepsilon^2} (1 - |u^i_e|^2) \right| \leq \frac{4}{3} \left| \frac{1}{\varepsilon^2} (1 - |u^i_e|^2) u^i_e \right|
\leq C |\partial_t u^i_e + (v^i \cdot \nabla) u^i_e| + C |W_{p^i_e p^j_e} (u^i_e, \nabla u^i_e) \nabla \alpha \beta u^j_e|
+ C |W_{p^i_e p^j_e} (u^i_e, \nabla u^i_e) \nabla \alpha u^j_e| + C |W_{p^i_e} (u^i_e, \nabla u^i_e)|
\leq C |\partial_t u^i_e + (v^i \cdot \nabla) u^i_e| + C |\nabla^2 u^i_e| + C |\nabla v^i_e|^2.
\]

Differentiating (1.7) in \( x_\beta \), multiplying the resulting equation by \( \nabla \beta \Delta u^i_e \) and integrating by parts, one obtains
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\Delta u^i_e|^2 dx
- \int_{\mathbb{R}^3} \nabla \beta (v^i \nabla u^i_e) \nabla \beta \Delta u^i_e dx
= - \int_{\mathbb{R}^3} \left[ \nabla \alpha \beta (W_{p^i_e} (u^i_e, \nabla u^i_e)) - \nabla \beta (W_{p^i_e} (u^i_e, \nabla u^i_e)) \nabla \beta \Delta u^i_e \right] dx
- \int_{\mathbb{R}^3} \nabla \beta \left[ \frac{1}{\varepsilon^2} (1 - |u^i_e|^2) u^i_e \right] \nabla \beta \Delta u^i_e dx.
\]
We will estimate the terms on the right hand side of (2.2) term by term. Estimates on the first term can be found in [13]. For completeness, we outline it here. Recalling that \( W(u, \nabla u) \) is quadratic in \( u \) and \( \nabla u \), one has

\[
\nabla_\gamma \beta W_{p_3} (u_\varepsilon, \nabla u_\varepsilon) = \nabla_\gamma \left[ W_{u^j p_0^j} (u_\varepsilon, \nabla u_\varepsilon) \nabla_\beta u_\varepsilon^j + W_{p_3^j} (u_\varepsilon, \nabla \beta u_\varepsilon^j) \right] \\
= W_{u^j p_0^j} (u_\varepsilon, \nabla u_\varepsilon) \nabla_\gamma \beta u_\varepsilon^j + W_{u^j p_0^j} (\nabla_\gamma u_\varepsilon, \nabla u_\varepsilon) \nabla_\beta u_\varepsilon^j \\
+ W_{u^j p_0^j} (u_\varepsilon, \nabla_\gamma \nabla u_\varepsilon) \nabla_\beta u_\varepsilon^j + W_{p_3^j} (u_\varepsilon, \nabla \beta u_\varepsilon^j) \nabla_\gamma \beta u_\varepsilon^j \\
+ W_{u^j p_0^j} (u_\varepsilon, \nabla \beta u_\varepsilon^j) \nabla_\gamma u_\varepsilon^j.
\]

Since \( W_{p_3^j} (u_\varepsilon, \nabla u_\varepsilon) \xi^j \xi^j \geq a |\xi|^2 \) and

\[
|W_{u^j p_0^j} (u_\varepsilon, \nabla u_\varepsilon) \nabla_\gamma \beta u_\varepsilon^j + W_{u^j p_0^j} (\nabla_\gamma u_\varepsilon, \nabla u_\varepsilon) \nabla_\beta u_\varepsilon^j \\
+ W_{u^j p_0^j} (u_\varepsilon, \nabla_\gamma \nabla u_\varepsilon) \nabla_\beta u_\varepsilon^j + W_{p_3^j} (u_\varepsilon, \nabla \beta u_\varepsilon^j) \nabla_\gamma \beta u_\varepsilon^j| \leq C |\nabla u_\varepsilon| \nabla^2 u_\varepsilon \bigg) \epsilon dx 
\]

it follows that

\[
- \int_{\mathbb{R}^3} W_{p_3^j} (u_\varepsilon, \nabla \beta \nabla u_\varepsilon) \nabla_\beta \gamma u_\varepsilon^j \nabla_\gamma \alpha u_\varepsilon^j dx \leq -a \int_{\mathbb{R}^3} |\nabla^3 u_\varepsilon|^2 dx
\]

and

\[
\int_{\mathbb{R}^3} \left( W_{u^j p_0^j} (u_\varepsilon, \nabla u_\varepsilon) \nabla_\gamma \beta u_\varepsilon^j + W_{u^j p_0^j} (\nabla_\gamma u_\varepsilon, \nabla u_\varepsilon) \nabla_\beta u_\varepsilon^j \\
+ W_{u^j p_0^j} (u_\varepsilon, \nabla_\gamma \nabla u_\varepsilon) \nabla_\beta u_\varepsilon^j + W_{p_3^j} (u_\varepsilon, \nabla \beta u_\varepsilon^j) \nabla_\gamma \beta u_\varepsilon^j \nabla_\gamma \alpha u_\varepsilon^j \epsilon dx 
\]

\[
\leq \eta \int_{\mathbb{R}^3} |\nabla^3 u_\varepsilon|^2 dx + C \int_{\mathbb{R}^3} (|\nabla u_\varepsilon|^2 |\nabla^2 u_\varepsilon|^2 + |\nabla u_\varepsilon|^6 \epsilon dx.
\]

Combining the above two inequalities yields

\[
- \int_{\mathbb{R}^3} \nabla_\beta (W_{p_3} (u_\varepsilon, \nabla u_\varepsilon)) \nabla_\beta \Delta u_\varepsilon^j dx = - \int_{\mathbb{R}^3} \nabla_\beta W_{p_3} (u_\varepsilon, \nabla u_\varepsilon) \nabla_\gamma \beta \alpha u_\varepsilon^j dx
\]

(2.3) \((-a - \eta) \int_{\mathbb{R}^3} |\nabla^3 u_\varepsilon|^2 dx + C \int_{\mathbb{R}^3} (|\nabla u_\varepsilon|^2 |\nabla^2 u_\varepsilon|^2 + |\nabla u_\varepsilon|^6 \epsilon dx
\]

for a sufficient small \( \eta > 0 \). Hence

\[
\int_{\mathbb{R}^3} \nabla_\beta [W_{u^j} (u_\varepsilon, \nabla u_\varepsilon)] \nabla_\beta \Delta u_\varepsilon^j dx
\]

\[
= \int_{\mathbb{R}^3} \left( W_{u^j} (u_\varepsilon, \nabla u_\varepsilon) \nabla_\beta u_\varepsilon^j + W_{u^j} (u_\varepsilon, \nabla u_\varepsilon) \nabla_\alpha \beta u_\varepsilon^j \nabla_\beta \Delta u_\varepsilon^j dx
\]

(2.4) \( \leq \eta \int_{\mathbb{R}^3} |\nabla^3 u_\varepsilon|^2 dx + C \int_{\mathbb{R}^3} (|\nabla u_\varepsilon|^2 |\nabla^2 u_\varepsilon|^2 + |\nabla u_\varepsilon|^6 \epsilon dx.
\]

Now we estimate the second term on the right hand side of (2.2). Direct calculations give

\[
u_\varepsilon^j \Delta^2 u_\varepsilon^j = \Delta (u_\varepsilon^j \Delta u_\varepsilon^j) - 2 \nabla u_\varepsilon^j \Delta \nabla u_\varepsilon^j - |\Delta u_\varepsilon^j|^2 = \Delta \left( \frac{1}{2} |\nabla u_\varepsilon^j|^2 - |\nabla u_\varepsilon^j|^2 \right) - 2 \nabla u_\varepsilon^j \Delta \nabla u_\varepsilon^j - |\Delta u_\varepsilon^j|^2
\]

\[
= \Delta \left( \frac{1}{2} |\nabla u_\varepsilon^j|^2 - |\nabla u_\varepsilon^j|^2 \right) - 2 \nabla u_\varepsilon^j \Delta \nabla u_\varepsilon^j - |\Delta u_\varepsilon^j|^2
\]

\[
= \frac{1}{2} \Delta^2 |\nabla u_\varepsilon^j|^2 - 2 \nabla u_\varepsilon^j \Delta \nabla u_\varepsilon^j - 2 |\nabla^2 u_\varepsilon^j|^2 - 2 \nabla u_\varepsilon^j \Delta \nabla u_\varepsilon^j - |\Delta u_\varepsilon^j|^2
\]

\[
= \frac{1}{2} \Delta^2 |\nabla u_\varepsilon^j|^2 - 4 \nabla u_\varepsilon^j \Delta \nabla u_\varepsilon^j - 2 |\nabla^2 u_\varepsilon^j|^2 - |\Delta u_\varepsilon^j|^2
\]

\]
Due to (2.1), one has

\[- \int_{\mathbb{R}^3} \nabla_\beta \left( \frac{1}{\epsilon^2} (1 - |u_\epsilon|^2) u_\epsilon \right) \nabla_\beta \Delta u_\epsilon dx = \int_{\mathbb{R}^3} \frac{|u_\epsilon|^2 - 1}{\epsilon^2} u_\epsilon^2 \Delta^2 u_\epsilon dx\]

\[= \int_{\mathbb{R}^3} \frac{|u_\epsilon|^2 - 1}{\epsilon^2} \left( \frac{1}{2} \Delta^2 |u_\epsilon|^2 - 4 \nabla u_\epsilon \nabla \Delta u_\epsilon - 2 |\nabla^2 u_\epsilon|^2 - |\Delta u_\epsilon|^2 \right) dx\]

\[= - \frac{1}{2\epsilon^2} \int_{\mathbb{R}^3} \Delta |u_\epsilon|^2 dx + \int_{\mathbb{R}^3} \frac{|u_\epsilon|^2 - 1}{\epsilon^2} (4 \nabla u_\epsilon \nabla \Delta u_\epsilon + 2 |\nabla^2 u_\epsilon|^2 + |\Delta u_\epsilon|^2) dx\]

\[\leq \eta \int_{\mathbb{R}^3} |\nabla \Delta u_\epsilon|^2 dx - \frac{1}{2\epsilon^2} \int_{\mathbb{R}^3} \Delta |u_\epsilon|^2 dx + C \int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 (|\partial_t u + v_\epsilon \nabla u_\epsilon|^2 + |\nabla^2 u_\epsilon|^2) dx\]

(2.5)

Substituting (2.3)–(2.5) into (2.2) leads to

\[\frac{d}{dt} \int_{\mathbb{R}^3} |\Delta u_\epsilon|^2 dx + \int_{\mathbb{R}^3} \left( \frac{3a}{2} |\nabla^3 u_\epsilon|^2 + \frac{1}{\epsilon^2} |\Delta |u_\epsilon|^2| \right) dx\]

\[\leq 2 \int_{\mathbb{R}^3} \nabla_\beta [(v_\epsilon \cdot \nabla) u_\epsilon] \nabla_\beta \Delta u_\epsilon dx + C \int_{\mathbb{R}^3} \left( |(\partial_t u + v_\epsilon \nabla u_\epsilon) + |\nabla^2 u_\epsilon| \right) |\nabla^2 u_\epsilon|^2 dx\]

(2.6)

Then it follows from Young inequality that

\[\frac{d}{dt} \int_{\mathbb{R}^3} |\Delta u_\epsilon|^2 dx + \int_{\mathbb{R}^3} \left( a |\nabla^3 u_\epsilon|^2 + \frac{1}{\epsilon^2} |\Delta |u_\epsilon|^2| \right) dx\]

\[\leq C \int_{\mathbb{R}^3} \left( |\nabla u_\epsilon|^2 + |v_\epsilon|^2 \right) \left( |\nabla^2 u_\epsilon|^2 + |\partial_t u_\epsilon|^2 + |\nabla v_\epsilon|^2 \right)\]

(2.7)

+ |\nabla^2 u_\epsilon|^2 (|\nabla^2 u_\epsilon| + |\partial_t u_\epsilon|) + |v_\epsilon|^2 |\nabla u_\epsilon|^4 + |\nabla u_\epsilon|^6) dx.

Differentiating equation (1.7) with respect to \(t\), multiplying the resulting equation by \(\partial_t u_\epsilon\) and integrating over \(\mathbb{R}^3\), recalling (2.1), we have

\[\frac{d}{dt} \int_{\mathbb{R}^3} |\partial_t u_\epsilon|^2 dx + \int_{\mathbb{R}^3} W_{p_\alpha, p_\delta} (u_\epsilon, \nabla u_\epsilon) \partial_t \nabla_\beta u_\epsilon \partial_t \nabla_\alpha u_\epsilon dx\]

\[= - \int_{\mathbb{R}^3} \left[ W_{u, \alpha} u_\epsilon \partial_t \nabla u_\epsilon \partial_t u_\epsilon + (v_\epsilon \nabla \partial_t u_\epsilon + \partial_t v_\epsilon \nabla u_\epsilon) \partial_t u_\epsilon\right.

\[\left. + W_{u, \alpha} u_\epsilon \partial_t \nabla u_\epsilon \partial_t u_\epsilon + W_{u, \alpha} u_\epsilon \nabla \partial_t u_\epsilon \partial_t u_\epsilon \right]

\[+ \frac{1}{2\epsilon^2} |\partial_t u_\epsilon|^2 + \frac{1}{\epsilon^2} (|u_\epsilon|^2 - 1)|\partial_t u_\epsilon|^2) dx.\]

This, together with \(W_{p_\alpha, p_\delta} (z, p) |\xi|^2 \geq a |\xi|^2\) and \(\int_{\mathbb{R}^3} v_\epsilon \nabla \partial_t u_\epsilon \partial_t v_\epsilon dx = 0\), shows that

\[\frac{d}{dt} \int_{\mathbb{R}^3} |\partial_t u_\epsilon|^2 dx + \int_{\mathbb{R}^3} \left( \frac{3a}{2} |\nabla \partial_t u_\epsilon|^2 + \frac{|\partial_t u_\epsilon|^2}{\epsilon^2} + \frac{2(|u_\epsilon|^2 - 1)}{\epsilon^2} |\partial_t u_\epsilon|^2 \right) dx\]

(2.8)

\[\leq \eta \int_{\mathbb{R}^3} |\partial_t u_\epsilon|^2 dx + C \int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 |\partial_t u_\epsilon|^2 dx.\]

Due to the identity

\[\partial_t u_\epsilon = |u_\epsilon|^{-2} (\partial_t u_\epsilon \cdot u_\epsilon) u_\epsilon - |u_\epsilon|^{-2} (\partial_t u_\epsilon \times u_\epsilon) \times u_\epsilon,\]
it holds
\[
\int_{\mathbb{R}^3} \left[ \frac{|\partial_t u_\varepsilon|^2}{\varepsilon^2} + \frac{2(|u_\varepsilon|^2 - 1)}{\varepsilon^2} |\partial_t u_\varepsilon|^2 \right] dx
\]
\[
= \int_{\mathbb{R}^3} \left[ \frac{|\partial_t u_\varepsilon|^2}{\varepsilon^2} + \frac{(|u_\varepsilon|^2 - 1)}{2\varepsilon^2 |u_\varepsilon|^2} |\partial_t u_\varepsilon|^2 \right.
\]
\[
- \frac{2(|u_\varepsilon|^2 - 1)}{2\varepsilon^2 |u_\varepsilon|^2} ((\partial_t u_\varepsilon \times u_\varepsilon) \times u_\varepsilon) \cdot \partial_t u_\varepsilon \bigg] \int_{\mathbb{R}^3} dx,
\]
(2.9)
where in the last step, the assumption $|u_\varepsilon| \geq \frac{3}{4} \geq \frac{1}{\sqrt{2}}$ has been used.

Now, we estimate the term $\int_{\mathbb{R}^3} \frac{2(|u_\varepsilon|^2 - 1)}{\varepsilon^2 |u_\varepsilon|^2} ((\partial_t u_\varepsilon \times u_\varepsilon) \times u_\varepsilon) \cdot \partial_t u_\varepsilon dx$ in (2.9). It follows (1.7) that
\[
\frac{(|u_\varepsilon|^2 - 1)}{\varepsilon^2 |u_\varepsilon|^2} = \frac{1}{\varepsilon^2 |u_\varepsilon|^2} \left( \frac{1}{2} |\partial_t u_\varepsilon|^2 - \nabla_\alpha (W_{p_\alpha}(u_\varepsilon, \nabla u_\varepsilon)) \cdot u_\varepsilon \right.
\]
\[
+ (W_u(u_\varepsilon, \nabla u_\varepsilon) + (v_\varepsilon \cdot \nabla) u_\varepsilon) \cdot u_\varepsilon \left.ight),
\]
and thus
\[
\int_{\mathbb{R}^3} \frac{2(|u_\varepsilon|^2 - 1)}{\varepsilon^2 |u_\varepsilon|^2} ((\partial_t u_\varepsilon \times u_\varepsilon) \times u_\varepsilon) \cdot \partial_t u_\varepsilon dx
\]
\[
= \int_{\mathbb{R}^3} \frac{1}{\varepsilon^2 |u_\varepsilon|^2} \left( \frac{1}{2} |\partial_t u_\varepsilon|^2 - \nabla_\alpha (W_{p_\alpha}(u_\varepsilon, \nabla u_\varepsilon)) \cdot u_\varepsilon \right.
\]
\[
+ (W_u(u_\varepsilon, \nabla u_\varepsilon) + (v_\varepsilon \cdot \nabla) u_\varepsilon) \cdot u_\varepsilon \left((\partial_t u_\varepsilon \times u_\varepsilon) \times u_\varepsilon \right) \cdot \partial_t u_\varepsilon dx
\]
\[
= \int_{\mathbb{R}^3} \frac{2}{\varepsilon^2 |u_\varepsilon|^2} (W_u(u_\varepsilon, \nabla u_\varepsilon) + (v_\varepsilon \cdot \nabla) u_\varepsilon) \cdot u_\varepsilon ((\partial_t u_\varepsilon \times u_\varepsilon) \times u_\varepsilon) \cdot \partial_t u_\varepsilon dx
\]
\[
+ \int_{\mathbb{R}^3} \frac{1}{\varepsilon^2 |u_\varepsilon|^2} \nabla_\alpha (W_{p_\alpha}(u_\varepsilon, \nabla u_\varepsilon)) \cdot u_\varepsilon ((\partial_t u_\varepsilon \times u_\varepsilon) \times u_\varepsilon) \cdot \partial_t u_\varepsilon dx
\]
\[
+ \int_{\mathbb{R}^3} \frac{1}{\varepsilon^2 |u_\varepsilon|^2} |\partial_t u_\varepsilon|^2 ((\partial_t u_\varepsilon \times u_\varepsilon) \times u_\varepsilon) \cdot \partial_t u_\varepsilon dx
\]
\[
= I_1 + I_2 + I_3.
\]

To estimate $I_1$ and $I_2$, we have
\[
I_1 \leq C \int_{\mathbb{R}^3} (|v_\varepsilon|^2 + |\nabla u_\varepsilon|^2) |\partial_t u_\varepsilon|^2 dx.
\]
and
\[
I_2 = - \int_{\mathbb{R}^3} W_{p_\alpha}(u_\varepsilon, \nabla u_\varepsilon) \nabla_\alpha \left( \frac{2}{|u_\varepsilon|^4} u_\varepsilon ((\partial_t u_\varepsilon \times u_\varepsilon) \times u_\varepsilon) \cdot \partial_t u_\varepsilon \right) dx
\]
\[
\leq C \int_{\mathbb{R}^3} |\nabla u_\varepsilon| (|\partial_t u_\varepsilon||\nabla \partial_t u_\varepsilon| + |\nabla u_\varepsilon||\partial_t u_\varepsilon|^2) dx
\]
\[
\leq \eta \int_{\mathbb{R}^3} |\nabla \partial_t u_\varepsilon|^2 dx + C \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 |\partial_t u_\varepsilon|^2 dx.
\]
To estimate $I_3$, we use (1.7) to obtain
\[
\partial_t u_\varepsilon \times u_\varepsilon = [\nabla_\alpha (W_{p_\alpha}(u_\varepsilon, \nabla u_\varepsilon)) - W_u(u_\varepsilon, \nabla u_\varepsilon) - (v_\varepsilon \cdot \nabla u_\varepsilon)] \times u_\varepsilon.
\]
Thus

\[
I_3 = \int_{\mathbb{R}^3} \frac{-1}{|u_\varepsilon|^4} \partial_t |u_\varepsilon|^2 [((\nabla \alpha(W_{p_\varepsilon}(u_\varepsilon, \nabla u_\varepsilon)) - W_u(u_\varepsilon, \nabla u_\varepsilon) - (v_\varepsilon \cdot \nabla)u_\varepsilon) \times u_\varepsilon \times u_\varepsilon] \cdot \partial_t u_\varepsilon dx
\]

\[
\leq C \int_{\mathbb{R}^3} \left[ \left[ W_{p_\varepsilon}(u_\varepsilon, \nabla u_\varepsilon) \right] \left[ |\nabla u_\varepsilon| |\partial_t u_\varepsilon|^2 + |\partial_t |u_\varepsilon|^2 |\nabla \partial_t u_\varepsilon| + |\nabla \partial_t |u_\varepsilon|^2 |\partial_t u_\varepsilon| + |\partial_t |u_\varepsilon|^2 |\nabla u_\varepsilon| + |v_\varepsilon|^2 \right] dx
\]

\[
\leq C \int_{\mathbb{R}^3} \left[ |\nabla u_\varepsilon| |\partial_t u_\varepsilon|^2 + |\partial_t u_\varepsilon| |\nabla u_\varepsilon| + |\partial_t |u_\varepsilon|^2 |\nabla u_\varepsilon| + |v_\varepsilon|^2 \right] dx
\]

\[
\leq \eta \int_{\mathbb{R}^3} |\nabla \partial_t u_\varepsilon|^2 dx + C \int_{\mathbb{R}^3} ((|\nabla u_\varepsilon|^2 + |v_\varepsilon|^2) |\partial_t u_\varepsilon|^2 dx.
\]

Combining above estimates of $I_1, I_2, I_3$ shows

\[
\int_{\mathbb{R}^3} \frac{2(|u_\varepsilon|^2 - 1)}{\varepsilon^2 |u_\varepsilon|^2} ((|\partial_t u_\varepsilon| \times u_\varepsilon) \times u_\varepsilon) \cdot \partial_t u_\varepsilon dx
\]

\[
\leq \eta \int_{\mathbb{R}^3} |\nabla \partial_t u_\varepsilon|^2 dx + C \int_{\mathbb{R}^3} ((|\nabla u_\varepsilon|^2 + |v_\varepsilon|^2) |\partial_t u_\varepsilon|^2 dx,
\]

which, together with (2.8)–(2.9), shows

\[
\frac{d}{dt} \int_{\mathbb{R}^3} |\partial_t v_\varepsilon|^2 dx + \int_{\mathbb{R}^3} \left( a |\nabla \partial_t u_\varepsilon|^2 + \frac{1}{\varepsilon^2} |\partial_t |u_\varepsilon|^2 |\nabla \partial_t u_\varepsilon| + |\partial_t |u_\varepsilon|^2 |\nabla u_\varepsilon| + |v_\varepsilon|^2 \right) dx
\]

\[
\leq \eta \int_{\mathbb{R}^3} |\partial_t v_\varepsilon|^2 dx + C \int_{\mathbb{R}^3} ((|\nabla u_\varepsilon|^2 + |v_\varepsilon|^2) |\partial_t u_\varepsilon|^2 dx.
\]

Multiplying equation (1.5) by $\partial_t v_\varepsilon - \Delta v_\varepsilon$ and integrating over $\mathbb{R}^3$ yields

\[
\frac{d}{dt} \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx + \int_{\mathbb{R}^3} \left( |\partial_t v_\varepsilon|^2 + |\Delta v_\varepsilon|^2 \right) dx
\]

\[
= - \int_{\mathbb{R}^3} (v_\varepsilon \cdot \nabla v_\varepsilon + \nabla_j (\nabla_i \kappa_{ij} W_{p_\varepsilon} W_{p_\varepsilon} (u_\varepsilon, \nabla u_\varepsilon))) (|\partial_t v_\varepsilon|^2 - \Delta v_\varepsilon) dx
\]

\[
\leq \eta \int_{\mathbb{R}^3} ((|\partial_t v_\varepsilon|^2 + |\Delta v_\varepsilon|^2) dx + C \int_{\mathbb{R}^3} ((|v_\varepsilon|^2 |\nabla v_\varepsilon|^2 + |\nabla u_\varepsilon| |\nabla^2 u_\varepsilon| + |v_\varepsilon|^2 |\nabla u_\varepsilon|)^2) dx
\]

for a sufficient small $\eta > 0$. Therefore

\[
\frac{d}{dt} \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx + \frac{3}{4} \int_{\mathbb{R}^3} ((|\partial_t v_\varepsilon|^2 + |\Delta v_\varepsilon|^2) dx
\]

\[
\leq C \int_{\mathbb{R}^3} ((|\Delta u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2 dx + \int_{\mathbb{R}^3} (a |\nabla^3 u_\varepsilon|^2 + a |\nabla \partial_t u_\varepsilon|^2 + |\partial_t v_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + \frac{1}{\varepsilon^2} |\Delta |u_\varepsilon|^2| + \frac{1}{\varepsilon^2} |\partial_t |u_\varepsilon|^2 |\nabla u_\varepsilon| + |v_\varepsilon|^2 |\nabla u_\varepsilon| + |v_\varepsilon|^2 |\nabla v_\varepsilon|)^2) dx
\]

\[
\leq C \int_{\mathbb{R}^3} ((|\nabla u_\varepsilon|^2 + |v_\varepsilon|^2)|\nabla^2 u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2)
\]

\[
\text{for a sufficient small } \eta > 0. \text{ Therefore}
\]
\[(2.12) \quad + |∇^2 u_ε|^2(|∇^2 u_ε| + |∂_t u_ε|) + |v_ε|^2|∇u_ε|^4 + |∇u_ε|^6| dx.
\]

Note that
\[\int_{\mathbb{R}^3} |∇^2 u_ε|^3 dx = \int_{\mathbb{R}^3} |∇_i u_ε| |∇_i u_ε| \cdot |∇_i u_ε| dx\]
\[= - \int_{\mathbb{R}^3} ∇_i(|∇_i u_ε| |∇_j u_ε|) |∇_j u_ε| dx \leq C \int_{\mathbb{R}^3} |∇^2 u_ε||∇^3 u_ε||∇ u_ε| dx\]
\[(2.13) \quad \leq η \int_{\mathbb{R}^3} |∇^3 u_ε|^2 dx + C \int_{\mathbb{R}^3} |∇ u_ε|^2 |∇^2 u_ε|^2 dx\]

and
\[\int_{\mathbb{R}^3} |∇^2 u_ε|^2 |∂_t u_ε| dx\]
\[= \int_{\mathbb{R}^3} |∇_i u_ε| |∇_i u_ε| |∂_t u_ε| dx = - \int_{\mathbb{R}^3} ∇_i(|∇_i u_ε| |∂_t u_ε|) \cdot |∇_j u_ε| dx\]
\[\leq C \int_{\mathbb{R}^3} |∇ u_ε|(|∇^3 u_ε||∂_t u_ε| + |∇^2 u_ε||∇_i u_ε|) dx\]
\[(2.14) \quad \leq η \int_{\mathbb{R}^3} (|∇^3 u_ε|^2 + |∇∂_t u_ε|^2) dx + C \int_{\mathbb{R}^3} |∇ u_ε|^2 (|∂_t u_ε|^2 + |∇^2 u_ε|^2) dx\]

for sufficient small \(η > 0\). On the other hand, integrating by parts gives
\[\int_{\mathbb{R}^3} |v_ε|^2 |∇ u_ε|^4 dx = - \int_{\mathbb{R}^3} \text{div}(|v_ε|^2 |∇ u_ε|^2 |∇ u_ε|)(u_ε - b) dx\]
\[= C \int_{\mathbb{R}^3} (|v_ε||∇ v_ε||∇ u_ε|^3 + |v_ε|^2 |∇ u_ε|^2 |∇^2 u_ε|) dx\]
\[\leq \frac{1}{2} \int_{\mathbb{R}^3} |v_ε|^2 |∇ u_ε|^4 dx + C \int_{\mathbb{R}^3} (|∇ u_ε|^2 |∇ v_ε|^2 + |v_ε|^2 |∇^2 u_ε|^2) dx\]

and
\[\int_{\mathbb{R}^3} |∇ u_ε|^6 dx = - \int_{\mathbb{R}^3} \text{div}(|∇ u_ε|^4 |∇ u_ε|)(u_ε - b) dx\]
\[\leq C \int_{\mathbb{R}^3} |∇ u_ε|^4 |∇^2 u_ε| dx \leq \frac{1}{2} \int_{\mathbb{R}^3} |∇ u_ε|^6 dx + C \int_{\mathbb{R}^3} |∇ u_ε|^2 |∇^2 u_ε|^2 dx.\]

These imply
\[(2.15) \quad \int_{\mathbb{R}^3} |v_ε|^2 |∇ u_ε|^4 dx \leq C \int_{\mathbb{R}^3} (|∇ u_ε|^2 |∇ v_ε|^2 + |v_ε|^2 |∇^2 u_ε|^2) dx\]

and
\[(2.16) \quad \int_{\mathbb{R}^3} |∇ u_ε|^6 dx \leq C \int_{\mathbb{R}^3} |∇ u_ε|^2 |∇^2 u_ε|^2 dx.\]

Substituting (2.13)–(2.16) into (2.12) leads to
\[\frac{d}{dt} \int_{\mathbb{R}^3} (|Δ u_ε|^2 + |∂_t u_ε|^2 + |∇ v_ε|^2) dx + \int_{\mathbb{R}^3} (a |∇^3 u_ε|^2 + a |∇∂_t u_ε|^2\]
\[+ |Δ v_ε|^2 + |∂_t v_ε|^2 + \frac{1}{ε^2} |Δ |u_ε|^2|^2 + \frac{1}{ε^2} |∂_t |u_ε|^2|^2) dx\]
\[\leq C \int_{\mathbb{R}^3} (|∇ u_ε|^2 + |v_ε|^2)(|∇^2 u_ε|^2 + |∂_t u_ε|^2 + |∇ v_ε|^2) dx,\]

which completes the proof. \(\Box\)
Due to the above lemma, we can prove the following uniform estimates (independent of $\varepsilon$) on the strong solutions to the system (1.5)–(1.7).

**Proposition 2.1.** Suppose that the initial data $(u_{0\varepsilon}, v_{0\varepsilon})$ satisfies

\[
\frac{7}{8} \leq |u_{0\varepsilon}| \leq \frac{9}{8}, \quad u_{0\varepsilon} - b \in H^2(\mathbb{R}^3), \quad v_{0\varepsilon} \in H^1(\mathbb{R}^3), \quad \text{div} v_{0\varepsilon} = 0 \text{ in } \mathbb{R}^3
\]

\[
\|\nabla u_{0\varepsilon}, v_{0\varepsilon}\|_{H^2(\mathbb{R}^3)}^2 + \|Q_{\varepsilon}(u_{0\varepsilon}, v_{0\varepsilon})\|_{L^2(\mathbb{R}^3)}^2 \leq M^2
\]

for some positive constant $M$ and constant unit vector $b$, where

\[
Q_{\varepsilon}(u, v) = \nabla_\alpha(W_p(u, \nabla u)) - W_u(u, \nabla u) + \frac{1 - |u|^2}{\varepsilon^2} u - (v \cdot \nabla)u.
\]

Then there is an absolute constant $C^* > 0$ such that the system (1.5)–(1.7) with initial data $(u_{0\varepsilon}, v_{0\varepsilon})$ has a unique strong solutions $(u_\varepsilon, v_\varepsilon)$ in $\mathbb{R}^3 \times (0, T_M^\varepsilon)$ with $T_M = C^* M^{-4}$, satisfying

\[
\frac{7}{8} \leq |u_\varepsilon| \leq \frac{9}{8} \quad \text{on } \mathbb{R}^3 \times [0, C^* M^{-4}]
\]

and

\[
\sup_{0 \leq t \leq T_M^\varepsilon} \int_{\mathbb{R}^3} (|\Delta u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx
\]

\[
+ \int_0^{T_M^\varepsilon} \int_{\mathbb{R}^3} (|\nabla \Delta u_\varepsilon|^2 + |\nabla \partial_t u_\varepsilon|^2 + |\Delta v_\varepsilon|^2 + |\partial_t v_\varepsilon|^2) dx dt \leq C^* M^{-4},
\]

provided $\varepsilon \leq \varepsilon_M$, where $\varepsilon_M$ is a positive constant depending only on $M$.

**Proof.** By Lemma 2.1, there is a unique local solution to the system (1.5)–(1.7) with initial data $(u_{0\varepsilon}, v_{0\varepsilon})$, which can be extended to the maximum time $T_{\varepsilon}^{\max}$. Note that the properties of $u$ stated in Lemma 2.1 imply that $u$ is Hölder continuous on $\mathbb{R}^3 \times [0, T_{\varepsilon}^{\max}]$ due to the well-known Gagliardo-Nirenberg-Sobolev inequality. Since $\frac{7}{8} \leq |u_{0\varepsilon}| \leq \frac{9}{8}$, there is a maximal time $T_{\varepsilon}^0 \in (0, T_{\varepsilon}^{\max}]$, such that $\frac{3}{4} \leq |u_\varepsilon| \leq \frac{5}{4}$ on $\mathbb{R}^3 \times [0, T_{\varepsilon}^0]$. It follows from Lemma 2.3 that

\[
\frac{d}{dt} \int_{\mathbb{R}^3} (|\Delta u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx
\]

\[
+ \int_{\mathbb{R}^3} (a|\nabla^3 u_\varepsilon|^2 + a|\nabla \partial_t u_\varepsilon|^2 + |\Delta v_\varepsilon|^2 + |\partial_t v_\varepsilon|^2) dx
\]

\[
\leq C \int_{\mathbb{R}^3} (|\nabla u_\varepsilon|^2 + |v_\varepsilon|^2)(|\nabla^2 u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx
\]

for any $t \in (0, T_{\varepsilon}^0)$. Using the Gagliardo-Nirenberg-Sobolev inequality in the above inequality yields

\[
\frac{d}{dt} \int_{\mathbb{R}^3} (|\Delta u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx
\]

\[
+ \int_{\mathbb{R}^3} (a|\nabla^3 u_\varepsilon|^2 + a|\nabla \partial_t u_\varepsilon|^2 + |\Delta v_\varepsilon|^2 + |\partial_t v_\varepsilon|^2) dx
\]

\[
\leq C \left[ \int_{\mathbb{R}^3} (|\nabla u_\varepsilon|^6 + |v_\varepsilon|^6) dx \right]^{1/3} \left[ \int_{\mathbb{R}^3} (|\nabla^2 u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx \right]^{1/2}
\]

\[
\times \left[ \int_{\mathbb{R}^3} (|\nabla^2 u_\varepsilon|^6 + |\partial_t u_\varepsilon|^6 + |\nabla v_\varepsilon|^6) dx \right]^{1/6}
\]
\[
\begin{align*}
\leq & C \left[ \int_{\mathbb{R}^3} (|\Delta u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx \right] \left[ \int_{\mathbb{R}^3} (|\Delta u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx \right]^{1/2} \\
\times & \left[ \int_{\mathbb{R}^3} (|\nabla \Delta u_\varepsilon|^2 + |\nabla \partial_t u_\varepsilon|^2 + |\Delta v_\varepsilon|^2) dx \right]^{1/2} \\
\leq & \eta \int_{\mathbb{R}^3} (|\nabla \Delta u_\varepsilon|^2 + |\nabla \partial_t u_\varepsilon|^2 + |\Delta v_\varepsilon|^2) dx \\
+ & C \left[ \int_{\mathbb{R}^3} (|\Delta u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx \right]^3 
\end{align*}
\]
and thus
\[
\frac{d}{dt} \int_{\mathbb{R}^3} (|\Delta u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx \\
+ \int_{\mathbb{R}^3} \left( \frac{\alpha}{2} |\nabla^3 u_\varepsilon|^2 + \frac{\alpha}{2} |\nabla \partial_t u_\varepsilon|^2 + |\Delta v_\varepsilon|^2 + |\partial_t v_\varepsilon|^2 \right) dx 
\leq C \left[ \int_{\mathbb{R}^3} (|\Delta u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx \right]^3
\]
for any \( t \in (0, T^1_\varepsilon) \).

Define
\[
f(t) = \int_{\mathbb{R}^3} (|\Delta u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx \\
+ \int_0^t \int_{\mathbb{R}^3} \left( \frac{\alpha}{2} |\nabla^3 u_\varepsilon|^2 + \frac{\alpha}{2} |\nabla \partial_t u_\varepsilon|^2 + |\Delta v_\varepsilon|^2 + |\partial_t v_\varepsilon|^2 \right) dx d\tau.
\]
It follows from (2.17) that
\[
f'(t) \leq C_1 f(t)^3,
\]
which implies
\[
f(t) \leq \left( \frac{1}{1 - 2C_1 f(0)^2 t} \right)^{1/2} f(0) \leq 2 f(0)
\]
for any \( t \leq \min \left\{ T^1_\varepsilon, \frac{3}{8C_1 f(0)^2} \right\} \). Due to equation (1.7), it holds that
\[
f(0) = \int_{\mathbb{R}^3} (|\Delta u_{0\varepsilon}|^2 + |\nabla v_{0\varepsilon}|^2 + |Q_\varepsilon(u_{0\varepsilon}, v_{0\varepsilon})|^2) dx \leq C_2 M^2
\]
and thus
\[
f(t) \leq 2C_2 M^2, \quad \forall t \leq \min \{ T^1_\varepsilon, C_3 M^{-4} \},
\]
where \( C_3 = \frac{3}{8C_1 f(0)^2} \).

By Lemma 2.2 one has
\[
\sup_{0 \leq t \leq T^2_{\varepsilon, max}} (\| \nabla u_{\varepsilon} \|_{L^2} \varepsilon^{-3/2} + \varepsilon^{-3/2} \| 1 - |u_{\varepsilon}|^2 \|_{L^2}^2) \leq C M^2.
\]
Combining the above inequality with (2.18) and using Gagliardo-Nirenberg-Sobolev inequality, we deduce
\[
\| 1 - |u_{\varepsilon}|^2 \|_{L^\infty(\mathbb{R}^3)} \leq C \| 1 - |u_{\varepsilon}|^2 \|_{L^2(\mathbb{R}^3)}^{1/4} \| \nabla^2 (1 - |u_{\varepsilon}|^2) \|_{L^2(\mathbb{R}^3)}^{3/4} \\
\leq C (\varepsilon M)^{1/4} (\| \nabla^2 u_{\varepsilon} \|_{L^2(\mathbb{R}^3)}^{3/4} + \| \nabla u_{\varepsilon} \|_{L^4(\mathbb{R}^3)}^{3/4}) \\
\leq C (\varepsilon M)^{1/4} (1 + \| \nabla u_{\varepsilon} \|_{H^1(\mathbb{R}^3)}^{3/2})
\]
for any \( t \leq \min\{T_{\varepsilon}^1, C_3M^{-4}\} \), and thus
\[
\frac{7}{8} \leq |u_{\varepsilon}| \leq \frac{9}{8}
\]
on \( \mathbb{R}^3 \times [0, \min\{T_{\varepsilon}^1, C_3M^{-4}\}] \),
provided \( \varepsilon \leq \varepsilon_M := \left[ \frac{(9/8)^2 - 1}{C_4(1 + M^{-7/4})} \right]^{1/4} \). Note that (2.18) implies \( \min\{T_{\varepsilon}^1, C_3M^{-4}\} < T_{\varepsilon}^{\text{max}} \), otherwise we can extend \((u_{\varepsilon}, v_{\varepsilon})\) beyond \( T_{\varepsilon}^{\text{max}} \), contradicting the definition of \( T_{\varepsilon}^{\text{max}} \). Due to this fact and the above inequality, there is another time \( T_{\varepsilon}' \) with
\[
\min\{T_{\varepsilon}^1, C_3M^{-4}\} < T_{\varepsilon}' \leq T_{\varepsilon}^{\text{max}},
\]
such that
\[
\frac{3}{4} \leq |u_{\varepsilon}| \leq \frac{5}{4}
\]
on \( \mathbb{R}^3 \times [0, T_{\varepsilon}' \] .

The definition of \( T_{\varepsilon}^1 \) gives \( T_{\varepsilon}' \geq \mathfrak{T}_{2}^1 \), and thus
\[
\min\{T_{\varepsilon}^1, C_3M^{-4}\} < \mathfrak{T}_{2}^1 \leq T_{\varepsilon}^1,
\]
which forces \( T_{\varepsilon}^1 > C_3M^{-4} \). As a result, it follows from (2.18) that
\[
f(t) \leq 2C_2M^2, \quad \forall t \leq C_3M^{-4},
\]
which implies the conclusion with \( C^* = C_3 \).

We will use the following version of the Aubin-Lions lemma.

**Lemma 2.4. (Aubin-Lions Lemma, See Simon [24 Corollary 4])** Assume that \( X, B \) and \( Y \) are three Banach spaces, with \( X \hookrightarrow B \hookrightarrow Y \). Then it holds that

(i) If \( F \) is a bounded subset of \( L^p(0,T;X) \) where \( 1 \leq p < \infty \), and \( \frac{\partial F}{\partial t} = \left\{ \frac{\partial f}{\partial t} | f \in F \right\} \) is bounded in \( L^1(0,T;Y) \), then \( F \) is relatively compact in \( L^p(0,T;B) \);

(ii) If \( F \) is bounded in \( L^\infty(0,T;X) \) and \( \frac{\partial F}{\partial t} \) is bounded in \( L^r(0,T;Y) \) where \( r > 1 \), then \( F \) is relatively compact in \( C([0,T];B) \).

Now we can prove the local existence and uniqueness of strong solutions to the Ericksen-Leslie system.

**Proof of the local existence and uniqueness part of Theorem 1.** For any \( \varepsilon > 0 \), by Proposition 2.1, there is a positive number \( T \) independent of \( \varepsilon \), such that the system (1.5)-(1.7) with the initial condition (1.4) has a unique solution \((u_{\varepsilon}, v_{\varepsilon})\), with the properties
\[
\begin{align*}
u_{\varepsilon} & \in L^2(0,T;H^3_0(\mathbb{R}^3)), \quad \partial_t u_{\varepsilon} \in L^2(0,T;H^1(\mathbb{R}^3)), \\
v_{\varepsilon} & \in L^2(0,T;H^2_0(\mathbb{R}^3)), \quad \partial_t v_{\varepsilon} \in L^2(\mathbb{R}^3 \times (0,T)), \\
\frac{7}{8} \leq |u_{\varepsilon}| \leq \frac{9}{8} & \quad \text{on } \mathbb{R}^3 \times (0,T)
\end{align*}
\]
and
\[
\begin{align*}
\sup_{0 \leq \tau \leq T} \int_{\mathbb{R}^3} \left( \frac{(1 - |u_{\varepsilon}|^2)}{\varepsilon^2} + |\nabla u_{\varepsilon}|^2 + |\nabla^2 u_{\varepsilon}|^2 + |\partial_t u_{\varepsilon}|^2 + |v_{\varepsilon}|^2 + |\nabla v_{\varepsilon}|^2 \right) & \leq C, \\
\int_0^T \int_{\mathbb{R}^3} (|\nabla^2 u_{\varepsilon}|^2 + |\partial_t u_{\varepsilon}|^2 + |\nabla^3 u_{\varepsilon}|^2 + |\nabla \partial_t u_{\varepsilon}|^2 + |\nabla v_{\varepsilon}|^2 + |\nabla^2 v_{\varepsilon}|^2) & \leq C, \\
+ |\partial_t v_{\varepsilon}|^2 & \leq C
\end{align*}
\]
Due to (1.5) and (1.6), the pressure \( p_\varepsilon \) satisfies
\[
\Delta p_\varepsilon = -\nabla^2 (v^i_\varepsilon v^j_\varepsilon + \nabla_i u^k_\varepsilon W_{\rho^j_\rho^k}(u_\varepsilon, \nabla u_\varepsilon))
\]
and
\[
\Delta p_\varepsilon = -\nabla \cdot ((v_\varepsilon \cdot \nabla)v_\varepsilon) - \nabla_i \nabla^2 u^k_\varepsilon W_{\rho^j_\rho^k}(u_\varepsilon, \nabla u_\varepsilon) + \nabla_i u^k_\varepsilon (W_{\rho^j_\rho^k}(u_\varepsilon, \nabla u_\varepsilon) \nabla_j \alpha u^j_\varepsilon + W_{u^j}(u_\varepsilon, \nabla u_\varepsilon) \nabla_j u^j_\varepsilon)]
\]
from which, using elliptic estimates, we obtain
\[
\int_0^T \int_{\mathbb{R}^3} |p_\varepsilon|^2 \, dx \, dt \leq C \int_0^T \int_{\mathbb{R}^3} (|v_\varepsilon|^4 + |\nabla u_\varepsilon|^4) \, dx \, dt \leq C
\]
and
\[
\int_0^T \int_{\mathbb{R}^3} |\nabla p_\varepsilon|^2 \, dx \, dt
\]
\[
\leq C \int_0^T \int_{\mathbb{R}^3} (|v_\varepsilon|^2 |\nabla v_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 |\nabla u_\varepsilon|^2 + |\nabla u_\varepsilon|^2 (|\nabla^2 u_\varepsilon|^2 + |\nabla u_\varepsilon|^4)) \, dx \, dt
\]
\[
\leq C \int_0^T \int_{\mathbb{R}^3} (|v_\varepsilon|^6 + |\nabla u_\varepsilon|^6 + |\nabla v_\varepsilon|^3 + |\nabla^2 u_\varepsilon|^3) \, dx \, dt \leq C
\]
for some positive constant \( C \) independent of \( \varepsilon \). In the above, we have used the Gagliado-Nirenberg-Sobolev inequality and the estimates stated in the previous.

On account of all the estimates obtained in the above, there is a subsequence, still denoted by \((u_\varepsilon, v_\varepsilon, p_\varepsilon)\), and \((u, v, p)\), such that
\[
\begin{align*}
&u \in L^2(0, T; H^3(\mathbb{R}^3; S^2)), \quad \partial_t u \in L^2(0, T; H^1(\mathbb{R}^3)), \\
v &\in L^2(0, T; H^2(\mathbb{R}^3)), \quad \partial_t v \in L^2(\mathbb{R}^3 \times (0, T)), \\
p &\in L^2(0, T; H^1(\mathbb{R}^3)), \quad (u, v) \text{ satisfies the initial condition}
\end{align*}
\]
and for any \( R \in (0, \infty) \)
\[
\begin{align*}
&u_\varepsilon \to u \text{ in } L^2(0, T; H^2(B_R)) \cap C([0, T]; H^1(B_R)), \\
&u_\varepsilon \to u \text{ in } L^2(0, T; H^3(\mathbb{R}^3)), \quad \partial_t u_\varepsilon \to \partial_t u \text{ in } L^2(0, T; H^1(\mathbb{R}^3)), \\
v_\varepsilon \to v \text{ in } L^2(0, T; H^1(B_R)) \cap L^2([0, T]; L^2(B_R)), \\
v_\varepsilon \to v \text{ in } L^2(0, T; H^2(\mathbb{R}^3)), \quad \partial_t v_\varepsilon \to \partial_t u \text{ in } L^2(\mathbb{R}^3 \times (0, T)), \\
p_\varepsilon \to p \text{ in } L^2(0, T; L^2(\mathbb{R}^3)),
\end{align*}
\]
where \(|u| = 1\) follows from the estimate that \( \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \frac{(1-|u|^2)^2}{\varepsilon^2} \, dx \leq C \), while the strong convergence stated above follows from the Aubin-Lions lemma.

By (1.7), we have
\[
\partial_t v^i + v \nabla v^i - \Delta v^i + v \nabla p = -\nabla_j [\nabla_i u^k W_{\rho^j_\rho^k}(u, \nabla u)]
\]
and
\[
\nabla \cdot v = 0
\]
and
\[
\partial_t u \times u + (v \cdot \nabla) u \times u
\]
Lemma 3.2. Let \( u \) be the Ericksen-Leslie system, which is one of the key lemmas of this paper.

Recalling that \(|u| = 1\), one can calculate to get
\[
-\partial_t u \times u = (u \cdot u) \partial_t u - (u \cdot u)u = \partial_t u,
\]
\[
-(v_k \nabla_k u \times u) \times u = v_k(u \cdot u)\nabla_k u - v_k(\nabla_k u \cdot u)u = (v \cdot \nabla)u,
\]
\[
-\nabla_\alpha [W_{pa}(u, \nabla u)] \times u \times u = (u \cdot u)\nabla_\alpha [W_{pa}(u, \nabla u)] - [\nabla_\alpha (W_{pa}(u, \nabla u)) \cdot u]u
\]
\[
= \nabla_\alpha [W_{pa}(u, \nabla u)] - \nabla_\alpha [(V_{pa}(u, \nabla u) \cdot u)u]
\]
\[
+ [V_{pa}(u, \nabla u) \cdot u]\nabla_\alpha u + [W_{pa}(u, \nabla u) \cdot \nabla_\alpha u]u,
\]
and
\[
-W_u(u, \nabla u) \times u \times u = (u \cdot u)W_u(u, \nabla u) - (W_u(u, \nabla u) \cdot u)u
\]
\[
= W_u(u, \nabla u) - (W_u(u, \nabla u) \cdot u)u.
\]

By the aid of the above identities, we obtain
\[
\partial_t u + (v \cdot \nabla)u = \nabla_\alpha [W_{pa}(u, \nabla u)] - (V_{pa}(u, \nabla u) \cdot u)u - W_u(u, \nabla u)
\]
\[
+(W_u(u, \nabla u) \cdot u)u + (V_{pa}(u, \nabla u) \cdot \nabla_\alpha u)u + (V_{pa}(u, \nabla u) \cdot u)\nabla_\alpha u,
\]
which is exactly \ref{1.3}.

The uniqueness of strong solutions follows from the regularities stated in \ref{2.19} by using the standard argument. The proof is completed. \(\square\)

3. Blow up criteria

In this section, we establish Serrin type or Beal-Kato-Majda type or mixed type (Serrin condition on one field and Beal-Kato-Majda condition on the other one) blow up criteria to strong solutions to the Ericksen-Leslie system, in other words, which will complete the proof of Theorem \ref{1} on the blow up criteria.

Strong solutions to the Ericksen-Leslie system satisfy the following basic energy balance law.

**Lemma 3.1.** Let \((u, v)\) be a strong solution to \ref{1.1}--\ref{1.3} in \(\mathbb{R}^3 \times (0, T)\). Then we have
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{|v|^2}{2} + W(u, \nabla u) \right) dx + \int_{\mathbb{R}^3} (|\nabla v|^2 + |\partial_t u + (v \cdot \nabla u)|^2) dx = 0
\]
for any \(t \in (0, T)\).

**Proof.** Equality \ref{3.1} follows from by multiplying \ref{1.1} by \(v^i\) and \ref{1.3} by \(\partial_t u^i + v^i \nabla u^i\), summing the resulting equations up and integrating over \(\mathbb{R}^3\). Details can be found in the proof of Lemma 3.1 in \ref{[15]}.

The following lemma states high order energy inequalities on the strong solutions to the Ericksen-Leslie system, which is one of the key lemmas of this paper.

**Lemma 3.2.** Let \((u, v)\) be a strong solution to \ref{1.1}--\ref{1.3} in \(\mathbb{R}^3 \times (0, T)\). Then
\[
\frac{d}{dt} \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta u|^2) dx + \int_{\mathbb{R}^3} (|\Delta v|^2 + \frac{3a}{2} |\nabla^3 u|^2) dx
\]
\[
\leq C \min \left\{ \int_{\mathbb{R}^3} |v|^2 (|\nabla v|^2 + \nabla^2 u^2) dx, \int_{\mathbb{R}^3} |\nabla v| (|\nabla v|^2 + \nabla^2 u^2) dx \right\}
\]
\begin{equation}
C \int_{\mathbb{R}^3} |\nabla u|^2 (|\nabla^2 u|^2 + |\nabla v|^2) dx
\end{equation}
and
\begin{align*}
\frac{d}{dt} \int_{\mathbb{R}^3} (|\nabla^3 u|^2 + |\Delta v|^2) dx + \int_{\mathbb{R}^3} \frac{3\alpha}{2} |\nabla^4 u|^2 + |\nabla^3 v|^2) dx \\
\leq C \int_{\mathbb{R}^3} |\nabla u|^2 (|\Delta v|^2 + |\nabla^3 u|^2) + |\nabla^2 u|^4 + |\nabla v|^4) dx
\end{align*}
\begin{equation}
+C \min \left\{ \int_{\mathbb{R}^3} |v|^2 |\nabla^2 v|^2 dx, \int_{\mathbb{R}^3} |\nabla v||\nabla^2 v|^2 \right\}
\end{equation}
for any \( t \in (0, T) \).

**Proof.** Differentiating (3.3) with respect to \( x_\beta \), multiplying the resulting equations by \( -\nabla_\beta \Delta u^i \) and integrating over \( \mathbb{R}^3 \), we then obtain
\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}^3} \frac{\Delta u^i}{2} dx = - \int_{\mathbb{R}^3} \nabla_\alpha \beta (W_{\alpha \beta}^k (u, \nabla u)) \nabla_\beta \Delta u^i dx + \int_{\mathbb{R}^3} \nabla_\alpha \beta (u^k u^i V_{\alpha \beta}^k (u, \nabla u)) \\
+ W_{\alpha \beta}^k (u, \nabla u) \nabla_\beta \Delta u^i dx - \int_{\mathbb{R}^3} \nabla_\beta (u^k u^i) W_{\alpha \beta}^k (u, \nabla u) + \nabla_\alpha u^k W_{\alpha \beta}^k (u, \nabla u) w^i
\end{equation}
\begin{equation}
V_{\alpha \beta}^k (u, \nabla u) u^k \nabla_\alpha u^i |\nabla_\beta \Delta u^i dx + \int_{\mathbb{R}^3} \nabla_\beta (v \cdot \nabla u^i) \nabla_\beta \Delta u^i dx.
\end{equation}
The terms on the right hand side of the above identity can be estimated term by term as follows. Estimates on all terms, except the last one, can be found in [13].
For reader’s convenience, we rewrite them here. Similar to (2.3), there holds
\begin{equation}
- \int_{\mathbb{R}^3} \nabla_\alpha \beta (W_{\alpha \beta}^k (u, \nabla u)) \nabla_\beta \Delta u^i dx
\end{equation}
\begin{equation}
\leq - (a - \eta) \int_{\mathbb{R}^3} |\nabla^3 u|^2 dx + C \int_{\mathbb{R}^3} (|\nabla u|^2 |\nabla^2 u|^2 + |\nabla u|^6) dx.
\end{equation}
One checks that
\begin{align*}
\nabla_\alpha \beta (u^k u^i V_{\alpha \beta}^k (u, \nabla u)) &= \nabla_\alpha \beta (u^k u^i) V_{\alpha \beta}^k (u, \nabla u) + \nabla_\alpha (u^k u^i) \nabla_\beta (V_{\alpha \beta}^k (u, \nabla u)) \\
&+ \nabla_\beta (u^k u^i) \nabla_\alpha (V_{\alpha \beta}^k (u, \nabla u)) + u^k u^i \nabla_\alpha \beta (V_{\alpha \beta}^k (u, \nabla u)) \\
&= \nabla_\alpha \beta (u^k u^i) V_{\alpha \beta}^k (u, \nabla u) + \nabla_\alpha (u^k u^i) \nabla_\beta (V_{\alpha \beta}^k (u, \nabla u)) \\
&+ \nabla_\beta (u^k u^i) \nabla_\alpha (V_{\alpha \beta}^k (u, \nabla u)) + u^k u^i \nabla_\alpha \nabla_\beta u^j + u^k u^i V_{\alpha \beta}^k (u, \nabla u) \nabla_\alpha \beta u^i,
\end{align*}
which implies
\begin{equation}
\int_{\mathbb{R}^3} \nabla_\alpha \beta (u^k u^i V_{\alpha \beta}^k (u, \nabla u)) \nabla_\beta \Delta u^i dx
\end{equation}
\begin{align*}
\leq & \int_{\mathbb{R}^3} u^k u^i V_{\alpha \beta}^k (u, \nabla u) \nabla_\alpha \beta u^i \nabla_\beta \Delta u^i dx \\
&+ C \int_{\mathbb{R}^3} (|\nabla u| |\nabla^2 u| + |\nabla u|^3) |\nabla^3 u| dx \\
\leq & C \int_{\mathbb{R}^3} (|\nabla u| |\nabla^2 u| + |\nabla u|^3) |\nabla^3 u| + |u^i \nabla_\beta \Delta u^i| |\nabla^3 u| dx
\end{align*}
\[ C \int_{\mathbb{R}^3} [(|\nabla u||\nabla^2 u| + |\nabla u|^3)|\nabla^3 u| + |\nabla_\beta (u^i \Delta u^i) - \nabla_\beta u^i \Delta u^i||\nabla^3 u|] \, dx \]
\[ \leq C \int_{\mathbb{R}^3} [(|\nabla u||\nabla^2 u| + |\nabla u|^3)|\nabla^3 u| + |\nabla_\beta (|\nabla u|^2 + \nabla_\beta u^i \Delta u^i)|\nabla^3 u|] \, dx \]
(3.6) \[ \leq \eta \int_{\mathbb{R}^3} |\nabla^3 u|^2 \, dx + C \int_{\mathbb{R}^3} (|\nabla u|^2|\nabla^2 u|^2 + |\nabla u|^6) \, dx. \]

Here, we have used the fact that \( \Delta u \cdot u = -|\nabla u|^2 \) guaranteed by \( |u| = 1 \). One can check easily that

\[ |\nabla_\beta [W_{\alpha'}(u, \nabla u) - u^k u^i W_{\alpha^k}(u, \nabla u)| - \nabla_\alpha u^k W_{\rho^k}(u, \nabla u)u^i - V_{\rho^k}(u, \nabla u)u^k \nabla_\alpha u^i| |\nabla_\beta \Delta u^i| \]
\[ \leq C (|\nabla u||\nabla^2 u| + |\nabla u|^3). \]

Thus

\[ \int_{\mathbb{R}^3} \nabla_\beta [W_{\alpha'}(u, \nabla u) - u^k u^i W_{\alpha^k}(u, \nabla u) - \nabla_\alpha u^k W_{\rho^k}(u, \nabla u)u^i - V_{\rho^k}(u, \nabla u)u^k \nabla_\alpha u^i] |\nabla_\beta \Delta u^i| \, dx \]
(3.7) \[ \leq \eta \int_{\mathbb{R}^3} |\nabla^3 u|^2 \, dx + C \int_{\mathbb{R}^3} (|\nabla u|^6 + |\nabla u|^2|\nabla^2 u|^2) \, dx. \]

It follows that

\[ \int_{\mathbb{R}^3} \nabla_\beta (v \cdot \nabla u^i) \nabla_\beta \Delta u^i \, dx \]
\[ = \int_{\mathbb{R}^3} \nabla_\beta v \cdot \nabla u^i \nabla_\beta \Delta u^i \, dx + \int_{\mathbb{R}^3} v \cdot \nabla \nabla_\beta v \nabla_\beta \Delta u^i \, dx \]
\[ = \int_{\mathbb{R}^3} \nabla_\beta v \cdot \nabla u^i \nabla_\beta \Delta u^i \, dx - \int_{\mathbb{R}^3} (\nabla_\alpha v \cdot \nabla \nabla_\beta u^i \nabla_\alpha u^i + v \cdot \nabla \nabla_\beta u^i \nabla_\alpha u^i) \, dx \]
\[ = \int_{\mathbb{R}^3} (\nabla_\beta v \cdot \nabla u^i \nabla_\beta \Delta u^i - \nabla_\alpha v \cdot \nabla \nabla_\beta u^i \nabla_\alpha u^i) \, dx \]
\[ \leq C \int_{\mathbb{R}^3} (|\nabla v||\nabla^3 u| + |\nabla v||\nabla^2 u|^2) \, dx \]
\[ \leq \eta \int_{\mathbb{R}^3} |\nabla^3 u|^2 \, dx + C \int_{\mathbb{R}^3} (|\nabla v|^2|\nabla u|^2 + |\nabla v||\nabla^2 u|^2) \, dx, \]
and

\[ \int_{\mathbb{R}^3} \nabla_\beta (v \cdot \nabla u^i) \nabla_\beta \Delta u^i \, dx = \int_{\mathbb{R}^3} (\nabla_\beta v \cdot \nabla u^i + v \cdot \nabla \nabla_\beta u^i) \nabla_\beta \Delta u^i \, dx \]
\[ \leq \eta \int_{\mathbb{R}^3} |\nabla^3 u|^2 \, dx + C \int_{\mathbb{R}^3} (|\nabla v|^2|\nabla u|^2 + |v|^2|\nabla^2 u|^2) \, dx. \]

Hence, it holds that

\[ \int_{\mathbb{R}^3} \nabla_\beta (v \cdot \nabla u^i) \nabla_\beta \Delta u^i \, dx \leq C \int_{\mathbb{R}^3} |\nabla u|^2|\nabla v|^2 \, dx + \eta \int_{\mathbb{R}^3} |\nabla^3 u|^2 \, dx \]
(3.8) \[ + C \min \left\{ \int_{\mathbb{R}^3} |v|^2|\nabla^2 u|^2 \, dx, \int_{\mathbb{R}^3} |\nabla v||\nabla^2 u|^2 \, dx \right\}. \]
Substituting (3.5)–(3.8) into (3.4) yields
\[
\frac{d}{dt} \int_{\mathbb{R}^3} |\Delta u|^2 dx + a \int_{\mathbb{R}^3} |\nabla^3 u|^2 dx \\
\leq C \int_{\mathbb{R}^3} |\nabla u|^2 (|\nabla u|^4 + |\nabla^2 u|^2 + |\nabla v|^2) dx \\
(3.9) + C \min \left\{ \int_{\mathbb{R}^3} |v|^2 |\nabla^2 u|^2 dx, \int_{\mathbb{R}^3} |\nabla v||\nabla^2 u|^2 dx \right\}.
\]

Multiplying (3.11) by \(-\Delta v^i\) and integrating the resulting equation over \(\mathbb{R}^3\) yields
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \frac{1}{2} |\nabla v|^2 dx + \int_{\mathbb{R}^3} |\Delta v|^2 dx = \int_{\mathbb{R}^3} (v \cdot \nabla v^i + \nabla j (\nabla_i u^k W_{pj}^k (u, \nabla u)) \Delta v^i) dx.
\]
It follows that
\[
\int_{\mathbb{R}^3} v \cdot \nabla v^i \Delta v^i dx \leq \eta \int_{\mathbb{R}^3} |\Delta v|^2 dx + C \int_{\mathbb{R}^3} |v|^2 |\nabla v|^2 dx,
\]
and
\[
\int_{\mathbb{R}^3} v \cdot \nabla v^i \Delta v^i dx = - \int_{\mathbb{R}^3} (\nabla_\beta v \cdot \nabla v^i \nabla_\beta v^i + v \cdot \nabla v_\beta v^i \nabla_\beta v^i) dx \leq \int_{\mathbb{R}^3} |\nabla v|^3 dx.
\]
Since \(|\nabla_j (\nabla_i u^k W_{pj}^k (u, \nabla u)| \leq C (|\nabla u| |\nabla^2 u| + |\nabla u|^3)\), it follows that
\[
\int_{\mathbb{R}^3} \nabla_j (\nabla_i u^k W_{pj}^k (u, \nabla u)) \Delta v^i dx \leq \eta \int_{\mathbb{R}^3} |\Delta v|^2 dx + C \int_{\mathbb{R}^3} |\nabla u|^2 (|\nabla u|^4 + |\nabla^2 u|^2) dx.
\]
Hence
\[
\frac{d}{dt} \int_{\mathbb{R}^3} |\nabla v|^2 dx + \int_{\mathbb{R}^3} |\Delta v|^2 dx \\
\leq C \int_{\mathbb{R}^3} |\nabla u|^2 (|\nabla u|^4 + |\nabla^2 u|^2) dx + C \min \left\{ \int_{\mathbb{R}^3} |v|^2 |\nabla v|^2 dx, \int_{\mathbb{R}^3} |\nabla v|^3 dx \right\}.
\]
Combining this with (3.10) gives
\[
\frac{d}{dt} \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta u|^2) dx + \int_{\mathbb{R}^3} (|\Delta v|^2 + |\nabla^3 u|^2) dx \\
\leq C \min \left\{ \int_{\mathbb{R}^3} |v|^2 (|\nabla v|^2 + |\nabla^2 u|^2) dx, \int_{\mathbb{R}^3} |\nabla v||\nabla v|^2 + |\nabla^2 u|^2) dx \right\} \\
+ C \int_{\mathbb{R}^3} |\nabla u|^2 (|\nabla u|^4 + |\nabla^2 u|^2 + |\nabla v|^2) dx \\
\leq C \min \left\{ \int_{\mathbb{R}^3} |v|^2 (|\nabla v|^2 + |\nabla^2 u|^2) dx, \int_{\mathbb{R}^3} |\nabla v||\nabla v|^2 + |\nabla^2 u|^2) dx \right\} \\
(3.10) + C \int_{\mathbb{R}^3} |\nabla u|^2 (|\nabla^2 u|^2 + |\nabla v|^2) dx.
\]
In the last step of the above inequality, we have used the fact that \(|\nabla u|^2 = -\Delta u \cdot u|.

Now we prove (3.3). Multiplying \((1.1)\) by \(\Delta^2 v^i\) and integrating the resulting equation over \(\mathbb{R}^3\) yield
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \frac{1}{2} |\Delta v|^2 dx + \int_{\mathbb{R}^3} |\nabla^3 v|^2 dx = \int_{\mathbb{R}^3} \nabla_j (\nabla_i u^k W_{pj}^k (u, \nabla u)) - v \cdot \nabla v^i |\Delta^2 v^i| dx.
\]
It follows that
\[-\int_{\mathbb{R}^3} v \cdot \nabla^i \Delta v^i dx = - \int_{\mathbb{R}^3} (\Delta v \cdot \nabla v^i + 2 \nabla v \cdot \nabla^2 v^i + v \nabla \Delta v^i) \Delta v^i dx\]
(3.12)
\[\leq C \int_{\mathbb{R}^3} |\nabla v||\nabla^2 v|^2,\]
and
\[-\int_{\mathbb{R}^3} v \cdot \nabla^i \Delta^2 v^i dx = \int_{\mathbb{R}^3} (\nabla_j v \cdot \nabla v^i + v \cdot \nabla \nabla_j v^i) \nabla_j \Delta v^i dx\]
(3.13)
\[\leq \eta \int_{\mathbb{R}^3} |\nabla^3 v|^2 dx + C \int_{\mathbb{R}^3} (|\nabla v|^4 + |v|^2 |\nabla^2 v|^2) dx.\]

Notice that
\[\int_{\mathbb{R}^3} |\nabla v|^4 dx = - \int_{\mathbb{R}^3} \text{div}(|\nabla v|^2 \nabla v) dx \leq C \int_{\mathbb{R}^3} |v||\nabla v|^2 |\nabla^2 v| dx\]
\[\leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla v|^4 dx + C \int_{\mathbb{R}^3} |v|^2 |\nabla^2 v|^2 dx,\]
which implies
\[\int_{\mathbb{R}^3} |\nabla v|^4 dx \leq C \int_{\mathbb{R}^3} |v|^2 |\nabla^2 v|^2 dx.\]

This, together with (3.12)–(3.13), shows
(3.14)
\[-\int_{\mathbb{R}^3} v \cdot \nabla^i \Delta^2 v^i dx \leq \eta \int_{\mathbb{R}^3} |\nabla^3 v|^2 dx + C \min \left\{ \int_{\mathbb{R}^3} |v|^2 |\nabla^2 v|^2 dx, \int_{\mathbb{R}^3} |\nabla v||\nabla^2 v|^2 \right\}.\]

Note that
\[|\nabla_j [\nabla_i u^k W_{p_j}^k (u, \nabla u)]| \leq C(|\nabla u||\nabla^3 u| + |\nabla u|^2 |\nabla^2 u|^2 + |\nabla u|^4)\]
\[\leq C(|\nabla u||\nabla^3 u| + |\nabla^2 u|^2),\]
where we have used |\nabla u|^2 \leq |\Delta u|. Then
\[\int_{\mathbb{R}^3} \nabla_j (\nabla_i u^k W_{p_j}^k (u, \nabla u)) \Delta^2 v^i dx\]
\[= - \int_{\mathbb{R}^3} \nabla_j (\nabla_i u^k W_{p_j}^k (u, \nabla u)) \nabla_i \Delta v^i dx\]
\[\leq C \int_{\mathbb{R}^3} (|\nabla u||\nabla^3 u| + |\nabla^2 u|^2) |\nabla v| dx\]
(3.15)
\[\leq \eta \int_{\mathbb{R}^3} |\nabla^3 v|^2 dx + C \int_{\mathbb{R}^3} (|\nabla u|^2 |\nabla^3 u|^2 + |\nabla^2 u|^4) dx.\]

It follows that
\[\int_{\mathbb{R}^3} |\nabla^2 u|^4 dx = - \int_{\mathbb{R}^3} \nabla_j (|\nabla^2 u|^2 \nabla_i u) \nabla_i u dx \leq \int_{\mathbb{R}^3} |\nabla u||\nabla^2 u|^2 |\nabla^3 u| dx\]
\[\leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla^2 u|^4 dx + C \int_{\mathbb{R}^3} |\nabla u|^2 |\nabla^3 u|^2 dx,\]
which, together with (3.13), gives
\[\int_{\mathbb{R}^3} \nabla_j (\nabla_i u^k W_{p_j}^k (u, \nabla u)) \Delta^2 v^i dx \leq \eta \int_{\mathbb{R}^3} |\nabla^3 v|^2 dx + C \int_{\mathbb{R}^3} |\nabla u|^2 |\nabla^3 u|^2 dx.\]
Substitute the above inequality and (3.14) into (3.11) to get
\[
\frac{d}{dt} \int_{\mathbb{R}^3} |\Delta v|^2 dx + \int_{\mathbb{R}^3} |\nabla^3 v|^2 dx \\
\leq C \int_{\mathbb{R}^3} |\nabla u|^2 |\nabla^3 u|^2 dx + C \min \left\{ \int_{\mathbb{R}^3} |v|^2 |\nabla^2 v|^2 dx, \int_{\mathbb{R}^3} |\nabla v||\nabla^2 v|^2 \right\}.
\]

Multiplying (3.13) by \(-\Delta u^i\) and integrating over \(\mathbb{R}^3\) lead to
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \frac{|\nabla^3 u|^2}{2} dx = - \int_{\mathbb{R}^3} \nabla_\alpha (W_{p_i}^k (u, \nabla u)) \Delta^3 u^i dx + \int_{\mathbb{R}^3} [\nabla_\alpha (u^k u^i V_{p_i}^k (u, \nabla u)) \\
+ W_{u^i} (u, \nabla u)] \Delta^3 u^i dx - \int_{\mathbb{R}^3} W_{p_i}^k (u, \nabla u) u^k u^i + W_{p_i}^k (u, \nabla u) \nabla_\alpha u^k u^i \\

+ V_{p_i}^k (u, \nabla u) u^k \nabla_\alpha u^i] \Delta^3 u^i dx + \int_{\mathbb{R}^3} v \nabla u^i \Delta^3 u^i dx.
\]

Direct calculations give
\[
\nabla_{\beta \gamma \sigma} (W_{p_i}^k (u, \nabla u)) = \nabla_{\beta} [W_{u^i} (u, \nabla u)] \nabla_\sigma u^i + W_{p_i}^k (u, \nabla u) \nabla_\gamma u^i \\
= \nabla_{\beta} [W_{u^i} (u, \nabla u)] \nabla_\sigma u^i + \nabla_{\beta} [W_{p_i}^k (u, \nabla u)] \nabla_\sigma u^i + \nabla_\beta [W_{p_i}^k (u, \nabla u)] \nabla_\gamma u^i \\
+ \nabla_\beta [W_{p_i}^k (u, \nabla u)] \nabla_\gamma u^i + W_{p_i}^k (u, \nabla u) \nabla_\beta \nabla_\gamma u^i.
\]

Due to \(W_{p_i}^k (z, p) \xi_{\alpha} \xi_{\beta} \geq a|\xi|^2\) and
\[
|\nabla_{\beta} [W_{u^i} (u, \nabla u)] | \nabla_\sigma u^i | + \nabla_{\beta} [W_{p_i}^k (u, \nabla u)] \nabla_\gamma u^i \\
+ \nabla_\beta [W_{p_i}^k (u, \nabla u)] \nabla_\gamma u^i + \nabla_\beta [W_{p_i}^k (u, \nabla u)] \nabla_\beta \nabla_\gamma u^i \\
\leq C(|\nabla u| |\nabla^3 u| + |\nabla u|^2 |\nabla^2 u| + |\nabla^2 u|^2) \leq C(|\nabla u| |\nabla^3 u| + |\nabla^2 u|^2),
\]

one can get
\[
- \int_{\mathbb{R}^3} \nabla_\alpha (W_{p_i}^k (u, \nabla u)) \Delta^3 u^i dx = - \int_{\mathbb{R}^3} \nabla_{\beta \gamma \sigma} (W_{p_i}^k (u, \nabla u)) \nabla_\alpha \nabla_\beta \nabla_\gamma \nabla_\sigma u^i dx \\
\leq - a \int_{\mathbb{R}^3} |\nabla^4 u|^2 dx + C \int_{\mathbb{R}^3} (|\nabla u| |\nabla^3 u| + |\nabla^2 u|^2) |\nabla^4 u| dx \\
(3.18) \leq - (a - \eta) \int_{\mathbb{R}^3} |\nabla u|^2 |\nabla^3 u|^2 + |\nabla^2 u|^2 dx.
\]

Next
\[
\nabla_\alpha \Delta [u^k u^i V_{p_i}^k (u, \nabla u)] = \nabla_\alpha [\Delta (u^k u^i) V_{p_i}^k (u, \nabla u)] \\
+ 2 \nabla (u^k u^i) \nabla (V_{p_i}^k (u, \nabla u)) + u^k u^i \Delta (V_{p_i}^k (u, \nabla u)) \\
= \nabla_\alpha [\Delta (u^k u^i) V_{p_i}^k (u, \nabla u)] + \Delta (u^k u^i) \nabla_\alpha [V_{p_i}^k (u, \nabla u)] \\
+ 2 \nabla_\alpha (u^k u^i) \nabla (V_{p_i}^k (u, \nabla u)) + 2 \nabla (u^k u^i) \nabla_\alpha (V_{p_i}^k (u, \nabla u)) \\
+ \nabla_\alpha (u^k u^i) \Delta (V_{p_i}^k (u, \nabla u)) + u^k u^i \Delta \nabla_\alpha (V_{p_i}^k (u, \nabla u)).
\]

Note that
\[
|\nabla_\alpha [\Delta (u^k u^i) V_{p_i}^k (u, \nabla u)] + \Delta (u^k u^i) \nabla_\alpha [V_{p_i}^k (u, \nabla u)] \\
+ 2 \nabla_\alpha (u^k u^i) \nabla (V_{p_i}^k (u, \nabla u)) + 2 \nabla (u^k u^i) \nabla_\alpha (V_{p_i}^k (u, \nabla u)) \\
+ \nabla_\alpha (u^k u^i) \Delta (V_{p_i}^k (u, \nabla u))|
\]
and
\[ |\nabla \alpha(V_{\rho_k}^i (u, \nabla u))| \leq C(|\nabla^4 u| + |\nabla^3 u| |\nabla u| + |\nabla^2 u| |\nabla u|^2) \]
\[ \leq C(|\nabla^4 u| + |\nabla^3 u| |\nabla u| + |\nabla^2 u|^2), \]
where we have used again |\nabla u|^2 = -\Delta u \cdot u$. Hence
\[ \int_{\mathbb{R}^3} \nabla_\alpha [u^k u^i V_{\rho_k}^i (u, \nabla u)] \Delta^3 u^i dx = \int_{\mathbb{R}^3} \nabla_\alpha \Delta [u^k u^i V_{\rho_k}^i (u, \nabla u)] \Delta^2 u^i dx \]
\[ \leq C \int_{\mathbb{R}^3} (|\nabla^3 u|^2 + |\nabla^2 u|^2) |\Delta^2 u|^2 dx + \int_{\mathbb{R}^3} u^k u^i \Delta^2 u^i \Delta \nabla_\alpha (V_{\rho_k}^i (u, \nabla u)) dx. \]
Since
\[ |u^i \Delta^2 u^i| = |\Delta (u^i \Delta u^i) - 2 \nabla u^i \nabla \Delta u^i - |\Delta u^i|^2| \]
\[ = |\Delta (|\nabla u|^2) + 2 \nabla u^i \nabla \Delta u^i + |\Delta u^i|^2| \leq C(|\nabla u|^3 u + |\nabla^2 u|^2), \]
we arrive at
\[ \int_{\mathbb{R}^3} \nabla_\alpha [u^k u^i V_{\rho_k}^i (u, \nabla u)] \Delta^3 u^i dx \leq C \int_{\mathbb{R}^3} (|\nabla^3 u|^2 + |\nabla^2 u|^2) |\Delta^2 u|^2 dx + \int_{\mathbb{R}^3} \nabla u \Delta \nabla_\alpha (V_{\rho_k}^i (u, \nabla u)) dx. \]
(3.19)
\[ \leq \eta \int_{\mathbb{R}^3} |\nabla u|^4 dx + C \int_{\mathbb{R}^3} (|\nabla u|^3 u|^2 + |\nabla^2 u|^4) dx. \]
One can check that
\[ |\Delta (W_{\nu^i} (u, \nabla u) - W_{\nu^i}^k (u, \nabla u) u^k u^i - W_{\rho_k}^i (u, \nabla u) \nabla_\alpha u^k u^i - V_{\rho_k}^i (u, \nabla u) u^k \nabla_\alpha u^i)| \]
\[ \leq C(|\nabla u|^3 u + |\nabla u|^2 |\nabla^2 u| + |\nabla^2 u|^2 + |\nabla u|^4) \leq C(|\nabla u|^3 u + |\nabla^2 u|^2), \]
which implies
\[ \int_{\mathbb{R}^3} W_{\nu^i} (u, \nabla u) - W_{\nu^i}^k (u, \nabla u) u^k u^i \]
\[ - W_{\rho_k}^i (u, \nabla u) \nabla_\alpha u^k u^i - V_{\rho_k}^i (u, \nabla u) u^k \nabla_\alpha u^i |\Delta^3 u^i dx \]
\[ = \int_{\mathbb{R}^3} \Delta [W_{\nu^i} (u, \nabla u) - W_{\nu^i}^k (u, \nabla u) u^k u^i \]
\[ - W_{\rho_k}^i (u, \nabla u) \nabla_\alpha u^k u^i - V_{\rho_k}^i (u, \nabla u) u^k \nabla_\alpha u^i |\Delta^2 u^i dx \]
\[ \leq C \int_{\mathbb{R}^3} (|\nabla u|^3 u + |\nabla^2 u|^2) |\Delta u|^2 dx \]
(3.20)
\[ \leq \eta \int_{\mathbb{R}^3} |\nabla^4 u|^2 dx + C \int_{\mathbb{R}^3} (|\nabla u|^3 u|^2 + |\nabla^2 u|^4) dx. \]
Integrating by parts gives
\[ \int_{\mathbb{R}^3} v \cdot \nabla u |\Delta^3 u^i dx = \int_{\mathbb{R}^3} (\Delta v \cdot \nabla u^i + 2 \nabla v \cdot \nabla^2 u^i + v \cdot \nabla \Delta u^i) |\Delta^2 u^i dx \]
Lemma 3.3. For any \( \nabla \) norm of \((0, T; BMO(\mathbb{R}^3)) \cap L^1(0, T; L^q(\mathbb{R}^3)) \) and \( \nabla f \in L^1(0, T; L^p(\mathbb{R}^3)) \) with \( p \in (3, \infty) \) and \( q \in [1, \infty) \), it holds that

\[
\int_s^t \| f(\tau) \|_{L^\infty} \, d\tau \leq C \left[ \int_s^t [f(\tau)]_{BMO} \, d\tau \ln \left( 1 + \int_s^t \| \nabla f(\tau) \|_{L^p} \, d\tau \right) \right. \\
+ \left. \int_s^t \| f \|_{L^q} \, d\tau + 1 \right],
\]

with \( C \) being a positive constant depending only on \( p \) and \( q \).

Proof. Set

\[
f_r(x) = \frac{1}{|B_r|} \int_{B_r(x)} f(y) \, dy.
\]

For any \( r \geq 1 \), we apply Hölder’s inequality to obtain

\[
|f_r(x)| \leq \frac{C}{r^q} \int_{B_r(x)} |f(y)| \, dy \leq C \left( \frac{1}{r^q} \int_{B_r(x)} |f(y)|^q \, dy \right)^{1/q} \leq C \| f \|_{L^q(\mathbb{R}^3)}.
\]
For any $r < 1$, there is a unique $k \in \mathbb{N}$, such that
\[
\frac{1}{2^k} \leq r < \frac{1}{2^{k-1}}, \quad k \leq C(1 + |\ln r|),
\]
and thus
\[
|f_r(x)| \leq \sum_{j=1}^{k} |f_{2^{j-1}r}(x) - f_{2^jr}(x)| + |f_{2^kr}(x)|
\]
\[
\leq \sum_{j=1}^{k} \frac{1}{|B_{2^j-1r}|} \int_{B_{2^j-1r}(x)} |f(y) - f_{2^jr}(x)| dy + C\|f\|_{L^s(\mathbb{R}^3)}
\]
\[
\leq \sum_{j=1}^{k} \frac{C}{|B_{2^j r}|} \int_{B_{2^j r}(x)} |f(y) - f_{2^jr(x)}| dy + C\|f\|_{L^s(\mathbb{R}^3)}
\]
\[
\leq Ck[f]_{BMO(\mathbb{R}^3)} + C\|f\|_{L^s(\mathbb{R}^3)}
\]
\[
\leq C(1 + |\ln r|)[f]_{BMO(\mathbb{R}^3)} + C\|f\|_{L^s(\mathbb{R}^3)}.
\]
By a variant of the Sobolev embedding theorem (see e.g. page 268 of [7]), the above inequalities give
\[
|f(x)| \leq |f(x) - f_r(x)| + |f_r(x)|
\]
\[
\leq Cr^{1-3/p}\|\nabla f\|_{L^p(\mathbb{R}^3)} + C(1 + |\ln r|)[f]_{BMO(\mathbb{R}^3)} + C\|f\|_{L^s(\mathbb{R}^3)}
\]
for any $r < 1$. Integrating the above inequality over $(s, t)$ yields
\[
\int_s^t \|f\|_{L^s(\mathbb{R}^3)} d\tau \leq Cr^{1-3/p} \int_s^t \|\nabla f\|_{L^p(\mathbb{R}^3)} d\tau + C(1 + |\ln r|) \int_s^t [f]_{BMO(\mathbb{R}^3)} d\tau
\]
\[
+ \int_s^t \|f\|_{L^s(\mathbb{R}^3)} d\tau
\]
for any $r < 1$. Taking $r = \left(1 + \int_s^t \|\nabla f\|_{L^p(\mathbb{R}^3)} d\tau \right)^{-p/(p-3)}$ in the above inequality proves the lemma. \hfill \Box

Now we finish the proof of the blow up criteria in Theorem [1]

Proof of the blow up criteria in Theorem [7] Let $T^*$ be the maximum existence time for the strong solution $(u, v)$ to the system \((1.1) - (1.3)\). Suppose, by contradiction, that the conclusion fails. Then both the following two hold true

- $\|\Delta u\|_{L^1(0,T^*;L^\infty(\mathbb{R}^3))} < \infty$ or $\|\nabla u\|_{L^{q_1}(0,T^*;L^{r_1}(\mathbb{R}^3))} < \infty$ for some $q_1 \in [2, \infty)$, $r_1 \in (3, \infty)$ with $\frac{2}{q_1} + \frac{3}{r_1} = 1$;
- $\|\omega\|_{L^1(0,T^*;BMO(\mathbb{R}^3))} < \infty$ or $\|\omega\|_{L^q(0,T^*;L^r(\mathbb{R}^3))} < \infty$ for some $q_2 \in [2, \infty)$, $r_2 \in (3, \infty)$ with $\frac{q}{q_2} + \frac{r}{r_2} = 1$.

By the Sobolev embedding inequality and the Hölder inequality, there holds
\[
\int_{\mathbb{R}^3} |w|^2 |f|^2 dx \leq \|w\|_{L^q(\mathbb{R}^3)}^2 \|f\|_{L^r(\mathbb{R}^3)}^2 \leq \eta \|\nabla f\|_{L^2}^2 + C\|w\|_{L^q}^2 \|f\|_{L^r}^2
\]
for any $r \in (3, \infty)$, $q \in [2, \infty)$ with $\frac{q}{q_2} + \frac{r}{r_2} = 1$. Note that $[\nabla v]_{BMO(\mathbb{R}^3)} \leq [\omega]_{BMO(\mathbb{R}^3)}$ by elliptic estimates. By the aid of this inequality, \((3.22)\), and $|\nabla u|^2 \leq |\Delta u|$, one
can get from Lemma 3.2 and Lemma 3.3 that
\[
\left[ \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta u|^2) dx \right] (t) + \int_s^t \int_{\mathbb{R}^3} (|\Delta v|^2 + a|\nabla^3 u|^2) dxdr \\
\leq \left[ \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta u|^2) dx \right] (s) \exp \left\{ C \min \left\{ \int_s^t \|v\|_{L^q}^q dr, \int_s^t \|\nabla v\|_{L^\infty} dr \right\} \right\} \\
\times \exp \left\{ C \min \left\{ \int_s^t \|\nabla u\|_{L^r}^r dr, \int_s^t \|\Delta u\|_{L^\infty} dr \right\} \right\} \\
\leq C \exp \left\{ C \min \left\{ \int_s^t \|v\|_{L^q}^q dr, \int_s^t \|\nabla u\|_{L^r}^r dr \right\} \right\} \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta u|^2) dx (s) \\
\leq C \exp \left\{ C \min \left\{ \int_s^t \|v\|_{L^q}^q dr, \int_s^t \|\nabla u\|_{L^r}^r dr \right\} \right\} \left[ \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta u|^2) dx \right] (s)
\]
(3.23)
for any 0 < s < t < T^*.
If v \in L^{q_2}(T^*_2, T^*; L^{r_2}) for some q_2 \in [2, \infty), r_2 \in (3, \infty) with \( \frac{2}{q_2} + \frac{3}{r_2} = 1 \), then the above inequality shows that
\[
\left[ \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta u|^2) dx \right] (t) + \int_s^t \int_{\mathbb{R}^3} (|\Delta v|^2 + a|\nabla^3 u|^2) dxdr \\
\leq C \left[ \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta u|^2) dx \right] (s)
\]
for any \( \frac{T^*_2}{2} \leq s \leq t < T^* \). In particular, it holds that
\[
\sup_{\frac{T^*_2}{2} \leq t < T^*} \left[ \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta u|^2) dx \right] (t) + \int_{\frac{T^*_2}{2}}^{T^*} \int_{\mathbb{R}^3} (|\Delta v|^2 + a|\nabla^3 u|^2) dxdt < \infty.
\]
Consequently, one can apply the local existence to extend the strong solution (u, v) beyond \( T^* \), which contradicts to the definition of \( T^* \).
If \( \omega \in L^1(\frac{T^*_2}{2}, T^*; BMO) \), then it follows from (3.23) that
\[
\left[ \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta u|^2) dx \right] (t) + \int_s^t \int_{\mathbb{R}^3} (|\Delta v|^2 + a|\nabla^3 u|^2) dxdr \\
\leq C \exp \left\{ \int_s^t [\omega]_{BMO} dr \ln \left( 1 + \int_s^t \|\nabla^3 v\|_{L^2} dr \right) \right\} \left[ \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta u|^2) dx \right] (s)
\]
for any \( \frac{T^*_2}{2} \leq s \leq t < T^* \). For any \( \delta > 0 \), we can choose \( s \in (\frac{T^*_2}{2}, T^*) \) such that \( \int_s^t [\omega]_{BMO} dr < \delta \) for any \( s \leq t < T^* \), and thus
\[
\left[ \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta u|^2) dx \right] (t) + \int_s^t \int_{\mathbb{R}^3} (|\Delta v|^2 + a|\nabla^3 u|^2) dxdr \\
\leq C \left[ 1 + \left( \int_s^t \|\nabla^3 v\|_{L^2} dr \right)^C \right] \left[ \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta u|^2) dx \right] (s)
\]
By the Gagliardo-Nirenberg-Sobolev inequality, it follows from the above two inequalities (3.24) and Lemma 3.3 that
\[
\left\| \nabla^3 v \right\|_{L^2}^2 + |\Delta v|^2 dx \right\|^2 \rightarrow \left( \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta u|^2) dx \right)(s)
\]
for any \( s \leq t < T^* \). Due to (3.22) and (3.23), it follows from Lemma 3.2 and Lemma 3.3 that
\[
\left\| \nabla^3 u \right\|_{L^2}^2 + |\Delta u|^2 dx \right\|^2 \rightarrow \left( \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta u|^2) dx \right)(s)
\]
and
\[
\left\| |\nabla v|^2 \right\|_{L^2}^2 \rightarrow \left( \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta u|^2) dx \right)(s)
\]
for any \( s \leq t < T^* \). By (3.24) and (3.25), it holds
\[
\sup_{s \leq t \leq T} \int_{\mathbb{R}^3} (|\Delta u|^2 + |\nabla v|^2) dx \leq C(1 + f^{C_\delta}(t)) \left[ \int_{\mathbb{R}^3} (|\Delta u|^2 + |\nabla v|^2) dx \right](s),
\]
(3.26)
\[
f(t) \leq C(1 + f^{C_\delta}(t)) \left[ f(s) + \int_s^t \int_{\mathbb{R}^3} (|\nabla v|^4 + |\nabla^2 u|^4) dx d\tau \right] .
\]
(3.27)
\[
\leq C \int_s^t \| \nabla v \|_{L^2} \| \nabla^3 v \|_{L^2}^{3/2} dB^t_t d\tau \\
\leq C \left( \int_s^t \| \nabla^3 v \|_{L^2}^2 d\tau \right)^{3/4} \left( \int_s^t \| \nabla v(\tau) \|_{L^2}^{10} d\tau \right)^{1/4} \\
\leq C \left( \int_s^t \| \nabla^3 v \|_{L^2}^2 d\tau \right)^{3/4} \left( \sup_{s \leq \tau \leq t} \| \nabla v \|_{L^2}^2 \right)^{5/4} \\
\leq C(1 + f^{3/4+C\delta}(t)) \left[ \int_{\mathbb{R}^3} (|\Delta u|^2 + |\nabla v|^2) \right] (s)
\]

and similarly

\[
\int_s^t \int_{\mathbb{R}^3} |\nabla^2 u|^4 |dx| d\tau \leq C(1 + f^{3/4+C\delta}(t)) \left[ \int_{\mathbb{R}^3} (|\Delta u|^2 + |\nabla v|^2) \right] (s).
\]

Combining the above two inequalities with (3.27) yields

\[
f(t) \leq C(1 + f^{3/4+C\delta}(t)) \left[ f(s) + \left( \int_{\mathbb{R}^3} (|\Delta u|^2 + |\nabla v|^2) \right) (s) \right]
\]

for any \( s \leq t < T^* \). This, together with (3.24), gives

\[
\sup_{s \leq t < T^*} \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta u|^2) dx + \int_{s}^{T^*} (|\Delta v|^2 + |\nabla^3 u|^2) dx < \infty.
\]

As a consequence, one can apply Theorem 1 to extend \((u, v)\) to be a strong solution beyond \(T^*\), which contradicts to the definition of \(T^*\) again.

Now we prove the equivalency of the quantities \(J_i, i = 1, 2, 3, 4\). Suppose that \(J_1(T)\) is finite, then the statements proved in the above implies that \(T\) is not the maximal time existence; as a result, \((u, v)\) can be extended to be a strong solution beyond \(T\), and thus

\[
u \in L^2(0,T;H^3_0(\mathbb{R}^3)) \cap L^\infty(0,T;H^2_0(\mathbb{R}^3)),
\]

\[
u \in L^2(0,T;H^3(\mathbb{R}^3)) \cap L^\infty(0,T;H^1(\mathbb{R}^3)).
\]

Due to these facts, by Lemma 3.2 one can easily prove that

\[
u \in L^2(\frac{T}{2},T;H^1_0(\mathbb{R}^3)), \quad \nu \in L^2(\frac{T}{2},T;H^3(\mathbb{R}^3)).
\]

Thus, one can check easily that \(J_2(T), J_3(T)\) and \(J_4(T)\) are all finite. Other cases can be proved in the same way. The proof of Theorem 1 is complete. \(\square\)

Finally, it should be noted that Theorem 1 has an equivalent version:

**Theorem 3.** Let \((u_0, v_0) \in H^2_6(\mathbb{R}^3) \times H^1(\mathbb{R}^3)\) be given initial data with \(\text{div} \ v_0 = 0\). Then, there exists a unique strong solution \((u, v) : \mathbb{R}^3 \times [0,T^*) \to S^2 \times \mathbb{R}^3\) of (1.1)-(1.3) with initial values \((u_0, v_0)\). Moreover, the maximal time \(T^*\) can be characterized by the condition that there are two constants \(\epsilon_0 > 0\) and \(R_0 > 0\) such that at a singular point \(x_i\),

\[
\limsup_{t \nearrow T^*} \int_{B_R(x_i)} |\nabla u (\cdot, t)|^3 + |v (\cdot, t)|^3 \, dx \geq \epsilon_0
\]

for any \(R > 0\) with \(R \leq R_0\).
Lemma 4.2. Let $u$ be a bounded subset of $\mathbb{R}^3$, every $u \in \mathbb{R}^3$. By a standard covering argument of Adams and Fournier, we have proved that

$$\frac{d}{dt} \int_{\mathbb{R}^3} (|\nabla^3 u|^2 + |\Delta v|^2) dx + \int_{\mathbb{R}^3} (a|\nabla^4 u|^2 + |\nabla^3 v|^2) dx \leq C \int_{\mathbb{R}^3} (|\nabla u|^2 + |v|^2)(|\Delta u|^2 + |\nabla^3 u|^2) dx.$$ 

By a standard covering argument of $\mathbb{R}^3$, one can obtain

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + |v|^2)(|\Delta v|^2 + |\nabla^3 u|^2) dx \leq C \sum_i \int_{B_{R_0}(x_i)} (|\nabla u|^2 + |v|^2)(|\Delta u|^2 + |\nabla^3 u|^2) dx \leq C \sum_i \left[ \int_{B_{R_0}(x_i)} (|\nabla u|^3 + |v|^3) \right]^{2/3} \left[ \int_{B_{R_0}(x_i)} (|\Delta v|^6 + |\nabla^3 u|)^6 dx \right]^{1/3} \leq C \varepsilon_0^{2/3} \int_{\mathbb{R}^3} (|\nabla^4 u|^2 + |\nabla^3 v|^2 + \frac{1}{R_0^3}(|\nabla^3 u|^2 + |\Delta v|^2) dx.$$ 

We now can prove Theorem 3 by similar arguments as before, details are omitted.

4. Convergence of Ginzburg-Landau to Ericksen-Leslie

In this section, we prove that the strong solutions to the Ginzburg-Landau approximate system converge to the strong solution of the Ericksen-Leslie system and give a new blow up criterion of the strong solutions to the Ericksen-Leslie system in term of Serrin type norms of the strong solutions to the Ginzburg-Landau approximate system.

The following lemma is a characterization of precompact subset of $L^p(\mathbb{R}^N)$.

**Lemma 4.1.** (see Adams and Fournier [1] Theorem 2.32) Let $1 \leq p < \infty$. A bounded subset $K \subseteq L^p(\mathbb{R}^N)$ is precompact in $L^p(\mathbb{R}^N)$ if and only if for every number $\varepsilon > 0$ there exists a number $\delta > 0$ and a compact subset $G$ such that for every $u \in K$ and $h \in \mathbb{R}^N$ with $|h| < \delta$ both of the following inequalities hold:

$$\int_{\mathbb{R}^N} |u(x + h) - u(x)|^p dx \leq \varepsilon^p, \quad \int_{\mathbb{R}^N \setminus G} |u(x)|^p dx \leq \varepsilon^p. $$

We need the following local type energy inequality.

**Lemma 4.2.** Let $(u_\varepsilon, v_\varepsilon)$ be a strong solution to the system (1.3)–(1.7) in $\mathbb{R}^3 \times (0, T)$, satisfying $\frac{3}{2} \leq \frac{|u_\varepsilon|}{\varepsilon} \leq \frac{9}{8}$ on $\mathbb{R}^3 \times (0, T)$. Then for any $\varphi \in C^\infty(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$, there holds

$$\frac{d}{dt} \int_{\mathbb{R}^3} \left[ |v_\varepsilon|^2 + \frac{(1 - |u_\varepsilon|^2)^2}{2\varepsilon^2} + 2W(u_\varepsilon, \nabla u_\varepsilon) \right] \varphi^2 dx + \int_{\mathbb{R}^3} (|\nabla v_\varepsilon|^2 + |\partial_t u_\varepsilon + v_\varepsilon \nabla u_\varepsilon|^2) \varphi^2 dx \leq C \int_{\mathbb{R}^3} \left[ (|v_\varepsilon|^2 + |\nabla u_\varepsilon|^2 + |\partial_t u_\varepsilon| + |\nabla v_\varepsilon| + |\nabla^2 u_\varepsilon|)|\varphi| \nabla \varphi| \right. $$

$$\left. + (|u_\varepsilon|^2 + |\nabla u_\varepsilon|^2)|\nabla \varphi|^2 \right] dx$$

for any $t \in (0, T)$, where $C$ is an absolute constant.
Proof. Multiplying (1.4) by $v^i \varphi^2$ and integrating over $\mathbb{R}^3$ yield
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \frac{|v_\varepsilon|^2}{2} \varphi^2 dx + \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 \varphi^2 dx
= \int_{\mathbb{R}^3} \left( \frac{|v_\varepsilon|^2}{2} + p_\varepsilon \right) \text{div}(v_\varepsilon \varphi^2) - \frac{1}{2} \frac{|v_\varepsilon|^2}{\varepsilon^2} \nabla |\nabla \varphi^2|^2 \right) dx
+ \int_{\mathbb{R}^3} W_p^j(u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon^j \nabla_j (v_\varepsilon \varphi^2) dx.
\]
(4.1)

Multiplying (1.6) by $(\partial_t u_\varepsilon^i + v_\varepsilon \nabla u_\varepsilon^i) \varphi^2$ and integrating over $\mathbb{R}^3$, one can get
\[
\left( \frac{\partial_t u_\varepsilon^i + v_\varepsilon \cdot \nabla u_\varepsilon^i}{\varepsilon} \right)^2 \varphi^2 dx
\]
(4.2)

It follows from integrating by parts that
\[
\int_{\mathbb{R}^3} [\nabla_\alpha (W_{p_\varepsilon}^j(u_\varepsilon, \nabla u_\varepsilon)) - W_{u_\varepsilon}^i(u_\varepsilon, \nabla u_\varepsilon)] \partial_t u_\varepsilon^i \varphi^2 dx
= - \int_{\mathbb{R}^3} [W_{p_\varepsilon}^j(u_\varepsilon, \nabla u_\varepsilon) \partial_t \nabla_\alpha u_\varepsilon^i + W_{u_\varepsilon}^i(u_\varepsilon, \nabla u_\varepsilon) \partial_t u_\varepsilon^i] \varphi^2 dx
- \int_{\mathbb{R}^3} W_p^j(u_\varepsilon, \nabla u_\varepsilon) \partial_t u_\varepsilon^i \nabla_\alpha \varphi^2 dx
- \int_{\mathbb{R}^3} [\partial_t W(u_\varepsilon, \nabla u_\varepsilon) \varphi^2 + W_{p_\varepsilon}^j(u_\varepsilon, \nabla u_\varepsilon) \partial_t u_\varepsilon^i \nabla_\alpha \varphi^2] dx
= - \frac{d}{dt} \int_{\mathbb{R}^3} W(u_\varepsilon, \nabla u_\varepsilon) \varphi^2 dx - \int_{\mathbb{R}^3} W_{p_\varepsilon}^j(u_\varepsilon, \nabla u_\varepsilon) \partial_t u_\varepsilon^i \nabla_\alpha \varphi^2 dx
\]

and
\[
\int_{\mathbb{R}^3} [\nabla_\alpha (W_{p_\varepsilon}^j(u_\varepsilon, \nabla u_\varepsilon)) - W_{u_\varepsilon}^i(u_\varepsilon, \nabla u_\varepsilon)] v \nabla u_\varepsilon^i \varphi^2 dx
= - \int_{\mathbb{R}^3} [W_{p_\varepsilon}^j(u_\varepsilon, \nabla u_\varepsilon) v \nabla_\alpha u_\varepsilon^i + W_{u_\varepsilon}^i(u_\varepsilon, \nabla u_\varepsilon) v \nabla u_\varepsilon^i] \varphi^2 dx
- \int_{\mathbb{R}^3} W_p^j(u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon^i \nabla_\alpha (v_\varepsilon \varphi^2) dx
- \int_{\mathbb{R}^3} [v \nabla W(u_\varepsilon, \nabla u_\varepsilon) \varphi^2 + W_{p_\varepsilon}^j(u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon^i \nabla_\alpha (v_\varepsilon \varphi^2)] dx
= \int_{\mathbb{R}^3} [W(u_\varepsilon, \nabla u_\varepsilon) \text{div}(v_\varepsilon \varphi^2) - W_{p_\varepsilon}^j(u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon^i \nabla_\alpha (v_\varepsilon \varphi^2)] dx.
\]

Moreover, direct calculations give
\[
\int_{\mathbb{R}^3} \frac{1 - |u_\varepsilon|^2}{\varepsilon^2} u_\varepsilon^i (\partial_t u_\varepsilon^i + v_\varepsilon \cdot \nabla u_\varepsilon^i) \varphi^2 dx
= - \frac{d}{dt} \int_{\mathbb{R}^3} \frac{(1 - |u_\varepsilon|^2)^2}{4 \varepsilon^2} \varphi^2 dx + \int_{\mathbb{R}^3} \frac{(1 - |u_\varepsilon|^2)^2}{4 \varepsilon^2} \text{div}(v_\varepsilon \varphi^2) dx.
\]
Substituting the above three equalities into (4.2) gives
\[ \frac{d}{dt} \int_{\mathbb{R}^3} \left[ \left( \frac{1}{4\varepsilon^2} - |u_\varepsilon|^2 \right)^2 + W(u_\varepsilon, \nabla u_\varepsilon) \right] \varphi^2 \, dx + \int_{\mathbb{R}^3} |\partial_t u_\varepsilon + v_\varepsilon \cdot \nabla u_\varepsilon|^2 \varphi^2 \, dx \]
\[ = \int_{\mathbb{R}^3} \left[ \left( \frac{1}{4\varepsilon^2} - |u_\varepsilon|^2 \right)^2 + W(u_\varepsilon, \nabla u_\varepsilon) \right] \text{div}(v_\varepsilon \varepsilon^2) - W_{p_\alpha} (u_\varepsilon, \nabla u_\varepsilon) \partial_t u_\varepsilon \nabla \alpha \varphi^2 \right] \, dx \]
(4.3)
\[ - \int_{\mathbb{R}^3} W_{p_\alpha} (u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon \nabla \alpha (v_\varepsilon \varepsilon^2) \, dx. \]

Combining (4.1) with (4.3), we obtain
\[ \frac{d}{dt} \int_{\mathbb{R}^3} \left[ \left( \frac{1}{4\varepsilon^2} - |u_\varepsilon|^2 \right)^2 + W(u_\varepsilon, \nabla u_\varepsilon) \right] \varphi^2 \, dx \]
\[ + \int_{\mathbb{R}^3} \left( |\nabla v_\varepsilon|^2 + |\partial_t u_\varepsilon + v_\varepsilon \cdot \nabla u_\varepsilon|^2 \right) \varphi^2 \, dx \]
\[ = \int_{\mathbb{R}^3} \left[ \left( \frac{1}{4\varepsilon^2} - |u_\varepsilon|^2 \right)^2 + W(u_\varepsilon, \nabla u_\varepsilon) \right] \text{div}(v_\varepsilon \varepsilon^2) \]
\[ - \frac{1}{2} \nabla v_\varepsilon \nabla \varphi^2 - W_{p_\alpha} (u_\varepsilon, \nabla u_\varepsilon) \partial_t u_\varepsilon \nabla \alpha \varphi^2 \right] \, dx \]
\[ \leq \eta \int_{\mathbb{R}^3} |\partial_t u_\varepsilon + v_\varepsilon \cdot \nabla u_\varepsilon|^2 \varphi^2 \, dx + C \int_{\mathbb{R}^3} \left( |v_\varepsilon|^2 + |\nabla u_\varepsilon|^2 \right) \varphi^2 \, dx \]
\[ + |p_\varepsilon| + |\nabla v_\varepsilon| |v_\varepsilon| \varphi ||\nabla \varphi| + |\nabla u_\varepsilon|^2 |\nabla \varphi|^2 \, dx \]
\[ + C \int_{\mathbb{R}^3} \left( \frac{1}{\varepsilon^2} \right) |v_\varepsilon| \varphi ||\nabla \varphi| \, dx. \]

This, together with the facts that \( \frac{7}{9} \leq |u_\varepsilon| \leq \frac{9}{7} \) and
\[ \left| \frac{1 - |u_\varepsilon|^2}{\varepsilon^2} \right| \leq C \left( |\partial_t u_\varepsilon + v_\varepsilon \cdot \nabla u_\varepsilon| + |\nabla^2 u_\varepsilon| + |\nabla u_\varepsilon|^2 \right), \]
gives
\[ \frac{d}{dt} \int_{\mathbb{R}^3} \left[ \left( \frac{1}{4\varepsilon^2} - |u_\varepsilon|^2 \right)^2 + W(u_\varepsilon, \nabla u_\varepsilon) \right] \varphi^2 \, dx \]
\[ + \int_{\mathbb{R}^3} \left( |\nabla v_\varepsilon|^2 + \frac{3}{4} |\partial_t u_\varepsilon + v_\varepsilon \cdot \nabla u_\varepsilon|^2 \right) \varphi^2 \, dx \]
\[ \leq C \int_{\mathbb{R}^3} \left( |v_\varepsilon|^2 + |\nabla u_\varepsilon|^2 \right) \varphi^2 \, dx + C \int_{\mathbb{R}^3} \left( |v_\varepsilon|^2 + |\nabla u_\varepsilon|^2 \right) \varphi^2 \, dx \]
\[ + |p_\varepsilon| + |\nabla v_\varepsilon| |v_\varepsilon| \varphi ||\nabla \varphi| + |\nabla u_\varepsilon|^2 |\nabla \varphi|^2 \, dx \]
\[ + C \int_{\mathbb{R}^3} \left( |v_\varepsilon|^2 + |\nabla u_\varepsilon|^2 \right) \varphi^2 \, dx \]
\[ \leq \eta \int_{\mathbb{R}^3} |\partial_t u_\varepsilon + v_\varepsilon \nabla u_\varepsilon|^2 \varphi^2 \, dx + C \int_{\mathbb{R}^3} \left( |v_\varepsilon|^2 + |\nabla u_\varepsilon|^2 \right) \varphi^2 \, dx \]
\[ + |p_\varepsilon| + |\nabla v_\varepsilon| + |\nabla^2 u_\varepsilon| |v_\varepsilon| \varphi ||\nabla \varphi| + (|v_\varepsilon|^2 + |\nabla u_\varepsilon|^2) |\nabla \varphi|^2 \, dx, \]
which implies the conclusion. This completes a proof. \( \square \)

The following lemma will be used in the proof of strong convergence and uniform estimates.
Lemma 4.3. Let \((u_\varepsilon, v_\varepsilon)\) and \((u, v)\) be strong solutions to the systems \((1.2) - (1.7)\) and \((1.4) - (1.5)\) in \(\mathbb{R}^3\times (0, T)\) with the same initial data \((u_0, v_0)\), respectively. Suppose that

\[
(\nabla u_\varepsilon, v_\varepsilon) \to (\nabla u, v) \quad \text{in} \quad L^2(0, T; H^1(\mathbb{R}^3))
\]

and

\[
\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \| (\nabla u_\varepsilon, v_\varepsilon) \|_{H^1(\mathbb{R}^3)} < \infty.
\]

Let \(K > 0\) be a constant such that

\[
\sup_{0 \leq t \leq T} \| (\nabla u, v) \|_{H^1(\mathbb{R}^3)}^2 + \int_0^T (\| \nabla^2 u \|_2^4 + \| \nabla v \|_4^4) dt \leq K.
\]

Then there are two positive constants \(\varepsilon_0\) and \(S_0\), with \(S_0\) depending only on the initial data \((u_0, v_0)\) and \(K\), such that

\[
\sup_{0 \leq t \leq \varepsilon_0} \sup_{0 \leq t \leq T} \left( \| (\nabla u_\varepsilon, v_\varepsilon) \|_{H^1(\mathbb{R}^3)}^2 + \| \partial_t u_\varepsilon \|_{L^2(\mathbb{R}^3)}^2 \right) \leq S_0.
\]

Proof. Set \(M_1 = \| \nabla u_0 \|_{H^1(\mathbb{R}^3)} + \| v_0 \|_{H^1(\mathbb{R}^3)}\). Using Lemma 2.3 and the Gagliardo-Nirenberg-Sobolev inequality, we have

\[
\frac{d}{dt} (\| u_\varepsilon \|_3^2 + \| \partial_t u_\varepsilon \|_3^2 + \| \nabla v_\varepsilon \|_3^2)
\]

\[
+ a \left( \| \nabla^3 u_\varepsilon \|_2^2 + \| \nabla \partial_t u_\varepsilon \|_2^2 + \| \Delta u_\varepsilon \|_2^2 + \| \partial \varepsilon \|_2^2 \right)
\]

\[
\leq C \int_{\mathbb{R}^3} (\| \nabla u_\varepsilon \|_2^2 + \| v_\varepsilon \|_2^2) (\| \nabla^2 u_\varepsilon \|_2 + \| \nabla \partial_t u_\varepsilon \|_2 + \| \nabla^2 v_\varepsilon \|_2)
\]

\[
	imes (\| \Delta u_\varepsilon \|_2 + \| \partial_t u_\varepsilon \|_2 + \| \nabla v_\varepsilon \|_2)
\]

\[
\leq \eta (\| \nabla^3 u_\varepsilon \|_2^2 + \| \nabla \partial_t u_\varepsilon \|_2^2 + \| \nabla^2 v_\varepsilon \|_2^2) + C (\| \nabla^2 u_\varepsilon \|_2^2 + \| \nabla v_\varepsilon \|_2^2)
\]

\[
	imes (\| \Delta u_\varepsilon \|_2^2 + \| \partial_t u_\varepsilon \|_2^2 + \| \nabla v_\varepsilon \|_2^2).
\]

Thus

\[
\frac{d}{dt} (\| u_\varepsilon \|_3^2 + \| \partial_t u_\varepsilon \|_3^2 + \| \nabla v_\varepsilon \|_3^2)
\]

\[
\leq C (\| \nabla u_\varepsilon \|_3^2 + \| \nabla v_\varepsilon \|_3^2) (\| \Delta u_\varepsilon \|_2^2 + \| \partial_t u_\varepsilon \|_2^2 + \| \nabla v_\varepsilon \|_2^2).
\]

Using equation \((1.7)\) and \(|u_0| = 1\), one has

\[
\| \partial_t u_\varepsilon(0) \|_2^2 \leq C (\| \nabla^2 u_0 \|_2^2 + \| \nabla u_0 \|_4^4 + \| v_0 \|_4^4) \leq C(M_1^4 + 1).
\]

Due to the assumptions in the lemma, there is a constant \(\varepsilon_0\), such that for any \(\varepsilon \in (0, \varepsilon_0)\), it holds that

\[
\int_0^T (\| \nabla^2 u_\varepsilon \|_2^4 + \| \nabla v_\varepsilon \|_2^4) dt
\]

\[
\leq 8 \int_0^T (\| \nabla u \|_2^4 + \| \nabla v \|_2^4) dt + 8 \int_0^T (\| \nabla^2 (u_\varepsilon - u) \|_2^4 + \| \nabla (v_\varepsilon - v) \|_2^4) dt
\]

\[
\leq 8K + \sup_{0 \leq t \leq T} \left( \| (\nabla^2 u_\varepsilon, \nabla v_\varepsilon) \|_2^4 + \| (\nabla^2 u_\varepsilon, \nabla v) \|_2^4 \right)
\]

\[
\times \int_0^T (\| \nabla^2 (u_\varepsilon - u) \|_2^4 + \| \nabla (v_\varepsilon - v) \|_2^4) dt.
\]
\[ \leq 8K + C(K + 2 \lim_{\varepsilon \to 0} \| (\nabla u_\varepsilon, v_\varepsilon) \|_{H^1(\mathbb{R}^3)}^2) \int_0^T (\| \nabla^2 (u_\varepsilon - u) \|_2 + \| \nabla (v_\varepsilon - v) \|_2^2) dt \]

\[ \leq 8K + 1. \]

It follows from these two inequalities and (4.4) that

\[ \sup_{0 \leq t \leq T} (\| \Delta u_\varepsilon \|_2^2 + \| \partial_t u_\varepsilon \|_2^2 + \| \nabla v_\varepsilon \|_2^2) \]

\[ \leq e^{C \int_0^T (\| \nabla^2 u_\varepsilon \|_2^4 + \| \nabla v_\varepsilon \|_2^4) dt} (\| \Delta u_0 \|_2^2 + \| \partial_t u_\varepsilon (0) \|_2^2 + \| \nabla v_0 \|_2^2) \]

(4.5)

\[ \leq C(M_1^4 + 1)e^{CK} =: M_2^2 \]

for any \( \varepsilon \in (0, \varepsilon_0] \). By Lemma 2.2

\[ \sup_{0 \leq t \leq T} \| v_\varepsilon \|_2^2 + \| \nabla u_\varepsilon \|_2^2 \leq CM_1^2. \]

Combining this with (4.5), we have

\[ \sup_{0 \leq t \leq T} \sup_{0 < \varepsilon \leq \varepsilon_0} (\| (\nabla u_\varepsilon, v_\varepsilon) \|_{H^1(\mathbb{R}^3)}^2 + \| \partial_t u_\varepsilon \|_{L^2(\mathbb{R}^3)}^2) \]

\[ \leq C(M_1^2 + M_2^2) := S_0. \]

This completes the proof. \( \square \)

The following lemma will be used to prove the new blow up criterion.

**Lemma 4.4.** Let \((u_\varepsilon, v_\varepsilon)\) be a strong solution in \(\mathbb{R}^3 \times (0, T_\varepsilon)\) to the system (1.4) – (1.7) with (1.2). Suppose that

\[ \| (\nabla u_\varepsilon, v_\varepsilon) \|_{L^q(0, T_\varepsilon; L^r(\mathbb{R}^3))} \leq L \]

for some positive constant \(L\) and \(\frac{2}{q} + \frac{3}{r} = 1\) with \(q \in [2, \infty)\) and \(r \in (3, \infty)\). Then there is a constant \(N\) depending only on \(L\) and the initial data such that

\[ \sup_{0 \leq t \leq T_\varepsilon} \| (\nabla u_\varepsilon, v_\varepsilon) \|_{H^1}^2 + \int_0^{T_\varepsilon} (\| \nabla^3 u_\varepsilon, \nabla^2 v_\varepsilon \|_{L^2}^2 + \| (\nabla \partial_t u_\varepsilon, \partial_t v_\varepsilon) \|_{L^2}^2) dt \leq N \]

for small \(\varepsilon\).

**Proof.** Let \(T_\varepsilon^1 \in (0, T_\varepsilon)\) be the maximal time such that \(\frac{3}{4} \leq |u_\varepsilon| \leq \frac{5}{4}\) on \(\mathbb{R}^3 \times (0, T_\varepsilon^1)\). By Lemma 2.3

\[ \frac{d}{dt} \int_{\mathbb{R}^3} (|\Delta u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx \]

\[ + \int_{\mathbb{R}^3} (a|\nabla^3 u_\varepsilon|^2 + a|\nabla \partial_t u_\varepsilon|^2 + |\Delta v_\varepsilon|^2 + |\partial_t v_\varepsilon|^2) dx \]

\[ \leq C \int_{\mathbb{R}^3} (|\nabla u_\varepsilon|^2 + |v_\varepsilon|^2) (|\nabla^2 u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx \]

for any \(t \in (0, T_\varepsilon^1)\). By the Sobolev embedding inequality and the Hölder inequality, it holds that

\[ \int_{\mathbb{R}^3} |w|^2 |f|^2 dx \leq \| w \|_{L^r}^2 \| f \|_{L^q}^2 \leq \eta \| \nabla f \|_{L^2}^2 + C \| w \|_{L^r}^2 \| f \|_{L^2}^2. \]

Combining the above two inequalities shows that

\[ \frac{d}{dt} \int_{\mathbb{R}^3} (|\Delta u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx \]

\[ + \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla^3 u_\varepsilon|^2 + a|\nabla \partial_t u_\varepsilon|^2 + |\Delta v_\varepsilon|^2 + |\partial_t v_\varepsilon|^2) dx \]
\[ \leq C \| (\nabla u_\varepsilon, v_\varepsilon) \|_{L^2(0,T; L^2(\mathbb{R}^d))} (\| \nabla^2 u_\varepsilon \|^2_{L^2(\mathbb{R}^d)} + \| \partial_t u_\varepsilon \|^2_{L^2(\mathbb{R}^d)} + \| \nabla v_\varepsilon \|^2_{L^2(\mathbb{R}^d)}). \]

Equation (1.7) implies that
\[ \| \partial_t u_\varepsilon(0) \|^2_{L^2(\mathbb{R}^d)} \leq C (\| \nabla^2 u_0 \|^2_{L^2(\mathbb{R}^d)} + \| v_0 \|^2_{L^2(\mathbb{R}^d)} + \| \nabla u_0 \|^2_{L^2(\mathbb{R}^d)}) \leq C. \]

It follows from above two inequalities that
\[ \sup_{0 \leq t \leq T_\varepsilon} (\| \nabla^2 u_\varepsilon \|^2_{L^2(\mathbb{R}^d)} + \| \partial_t u_\varepsilon \|^2_{L^2(\mathbb{R}^d)} + \| \nabla v_\varepsilon \|^2_{L^2(\mathbb{R}^d)}) + \int_0^{T_\varepsilon} (\| \nabla^3 u_\varepsilon \|^2_{L^2(\mathbb{R}^d)} + \| \nabla \partial_t u_\varepsilon \|^2_{L^2(\mathbb{R}^d)} + \| \nabla^2 v_\varepsilon \|^2_{L^2(\mathbb{R}^d)} + \| \partial_t v_\varepsilon \|^2_{L^2(\mathbb{R}^d)}) dt \]
\[ \leq C \varepsilon^2 \int_0^{T_\varepsilon} (\| \nabla u_\varepsilon \|^2_{L^2(\mathbb{R}^d)} + \| v_\varepsilon \|^2_{L^2(\mathbb{R}^d)}) dt \leq C \varepsilon^2. \]

Apply Lemma 2.2 to obtain
\[ \sup_{0 \leq t \leq T_\varepsilon} (\| \nabla^2 u_\varepsilon, v_\varepsilon \|^2_{L^2(\mathbb{R}^d)} + \varepsilon^{-2} \| u_\varepsilon \|^2_{L^2(\mathbb{R}^d)}) \leq C \| (\nabla u_0, v_0) \|^2_{L^2(\mathbb{R}^d)}, \]
which, together with (4.6), gives
\[ \sup_{0 \leq t \leq T_\varepsilon} (\| (\nabla^2 u_\varepsilon, v_\varepsilon) \|^2_{H^1} + \int_0^{T_\varepsilon} (\| \nabla^3 u_\varepsilon, \nabla \partial_t u_\varepsilon \|^2_{L^2} + \| (\nabla^2 v_\varepsilon, \partial_t v_\varepsilon) \|^2_{L^2}) dt \leq N \]
for some constant \( N \) depending only on \( L \) and the initial data. By the aid of the above two inequalities, one can use a similar argument as in the proof of (2.1) to conclude by using the Gagliardo-Nirenberg-Sobolev inequality that
\[ \frac{7}{8} \leq |u_\varepsilon| \leq \frac{9}{8} \quad \text{on} \quad \mathbb{R}^3 \times [0, T_\varepsilon) \]
for small \( \varepsilon \). Recalling the definition of \( T_\varepsilon \), the above inequality implies \( T_\varepsilon = T_\varepsilon \), and thus the conclusion holds true. \( \square \)

Now, let us give the proof of Theorem 2.

Proof of Theorem 2. We first prove the strong convergence and the uniform estimates, which are given in three steps as follows.

Given arbitrary \( T \in (0, T^*) \), set
\[ K = \sup_{0 \leq t \leq T} \| (\nabla u, v) \|^2_{H^1(\mathbb{R}^d)} + \int_0^T (\| \nabla^2 u \|^2_{L^2} + \| \nabla v \|^2_{L^2}) dt. \]

Let \( S_0 \) be the constant stated in Lemma 4.3 and put
\[ M = \| (\nabla u_0, v_0) \|^2_{H^1(\mathbb{R}^d)} + S_0. \]

Step 1. In this step, we prove that the strong convergence and estimate hold true up to a time \( T_M \). By Proposition 2.1 and Lemma 2.2 \((u_\varepsilon, v_\varepsilon)\) can be defined on \( \mathbb{R}^3 \times [0, T_M] \) such that \( \frac{7}{8} \leq |u_\varepsilon| \leq \frac{9}{8} \) on \( \mathbb{R}^3 \times [0, T_M] \) and
\[ \sup_{0 \leq t \leq T_M} \| (\nabla u_\varepsilon, v_\varepsilon) \|^2_{H^1} + \int_0^{T_M} (\| (\nabla^3 u_\varepsilon, \nabla^2 v_\varepsilon, \partial_t v_\varepsilon, \partial_t v_\varepsilon) \|^2_{L^2}) dt \leq C(M) \]
for small \( \varepsilon \).

Using the same argument as the proof of Theorem 1, we can prove that
\[ u_\varepsilon \rightarrow u \quad \text{in} \quad L^2(0, T_M; H^2(B_R(0))), \]
\[ v_\varepsilon \rightarrow v \quad \text{in} \quad L^2(0, T_M; H^1(B_R(0))) \]
for any $R \in (0, \infty)$. In fact, to prove these convergence, by the aid of the uniqueness of the strong solutions to system $\text{(1.1)}$, it suffices to show that any sequence $\{(u_{\varepsilon}, v_{\varepsilon})\}_{i=1}^{\infty}$ has convergent subsequence. While such sequentially convergence has already been justified in the proof of Theorem $\text{[1]}$.

The aim is to show that

$$\text{(4.10)} \quad \nabla u_{\varepsilon} \to \nabla u \quad \text{and} \quad v_{\varepsilon} \to v \quad \text{in} \quad L^2(0, T_M; H^1(\mathbb{R}^3)).$$

By the aid of $\text{(4.7)}$–$\text{(4.9)}$, using Lemma $\text{[4.1]}$ and the Gagliado-Nirenberg-Sobolev inequality, one needs to show that for any $\eta > 0$, there is $R > 0$, such that

$$\text{(4.11)} \quad \int_0^{T_M} \int_{\mathbb{R}^3 \setminus B_R(0)} (|\nabla u_{\varepsilon}|^2 + |v_{\varepsilon}|^2) dx dt \leq \eta.$$

Take function $\varphi_0 \in C^\infty(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, such that $\varphi_0 \equiv 0$ on $(-\infty, 1)$, $\varphi_0 \equiv 1$ on $(2, \infty)$ and $|\varphi'| \leq 2$ on $\mathbb{R}$. For $R \geq 1$, set $\varphi_R(x) = \varphi_0(\frac{|x|}{R})$, then $\varphi_R(x) \equiv 0$ on $B_R(0)$, $\varphi_R \equiv 1$ on $\mathbb{R}^3 \setminus B_{2R}(0)$ and $|\nabla \varphi_R| \leq \frac{2}{R}$ on $\mathbb{R}^3$. For $\varphi = \varphi_R$ in Lemma $\text{[4.2]}$, it holds that

$$\text{sup} \int_{0 \leq t \leq T M} \int_{\mathbb{R}^3 \setminus B_{2R}(0)} (|v_{\varepsilon}|^2 + |\nabla u_{\varepsilon}|^2) dx$$

$$\leq \int_{\mathbb{R}^3 \setminus B_{2R}(0)} (|v_0|^2 + |\nabla u_0|^2) dx + \frac{C}{R} \int_0^{T M} \int_{\mathbb{R}^3} (|v_{\varepsilon}|^4 + |\nabla u_{\varepsilon}|^4$$

$$+ |p_{\varepsilon}|^2 + |\nabla v_{\varepsilon}|^2 + |\nabla^2 u_{\varepsilon}|^2 + |v_{\varepsilon}|^2 + |\nabla u_{\varepsilon}|^2) dx dt.$$

Applying elliptic estimates for the Stokes equations, it follows from equation $\text{(1.5)}$ that

$$\int_0^{T M} \int_{\mathbb{R}^3} |p_{\varepsilon}|^2 dx dt \leq C \int_0^{T M} \int_{\mathbb{R}^3} (|v_{\varepsilon}|^4 + |\nabla u_{\varepsilon}|^4) dx dt.$$

Combining the above two inequalities, using the Gagliado-Nirenber-Sobolev inequality and the absolute continuity of integrals, one obtains from $\text{(4.4)}$ that

$$\text{sup} \int_{0 \leq t \leq T M} \int_{\mathbb{R}^3 \setminus B_{2R}(0)} (|v_{\varepsilon}|^2 + |\nabla u_{\varepsilon}|^2) dx$$

$$\leq \int_{\mathbb{R}^3 \setminus B_{2R}(0)} (|v_0|^2 + |\nabla u_0|^2) dx + \frac{C}{R} \int_0^{T M} \left( \|v_{\varepsilon}\|_{H^1}^4 + \|\nabla u_{\varepsilon}\|_{H^1}^4 + 1 \right) dt$$

$$\leq \frac{\eta}{2} + \frac{C}{R} T_M C(M) \leq \eta$$

for large $R$, which shows $\text{(4.11)}$ and thus $\text{(4.10)}$.

Next we prove

$$\text{(4.12)} \quad (\nabla u_{\varepsilon}, v_{\varepsilon}) \to (\nabla u, v) \quad \text{in} \quad L^\infty(0, T_M; L^2(\mathbb{R}^3)).$$

Due to $\text{(4.7)}$, it suffices to show that each sequence $(u_{\varepsilon}, v_{\varepsilon})$ has an convergent subsequence in $L^\infty(0, T; L^2(\mathbb{R}^3))$. Let $(u_{\varepsilon}, v_{\varepsilon})$ be a sequence. By $\text{(4.7)}$, there is a subsequence, still denoted by $(u_{\varepsilon}, v_{\varepsilon})$, such that

$$\text{(4.13)} \quad \lim_{i \to \infty} \| \nabla u_{\varepsilon}(\cdot, t) - \nabla u(\cdot, t), v_{\varepsilon}(\cdot, t) - v(\cdot, t) \|_{H^1(\mathbb{R}^3)} = 0, \quad \text{for a.e.} \quad t \in [0, T_M].$$

By $\text{(4.7)}$, it holds that

$$\| \nabla u_{\varepsilon}(\cdot, t) - \nabla u_{\varepsilon}(\cdot, s) \|_{L^2(\mathbb{R}^3)} = \left\| \int_s^t \partial_t \nabla u_{\varepsilon}(\tau) d\tau \right\|_{L^2(\mathbb{R}^3)}$$
to the system (1.1)–(1.4). We prove that

\[ \sup_{T < T} v_\varepsilon (\cdot, t) - v_\varepsilon (\cdot, s) \leq C(M)(t - s)^{1/2}, \]

and similarly

\[ \sup_{T < T} \|u_\varepsilon(\cdot, t) - u_\varepsilon(\cdot, s)\|_{L^2(\mathbb{R}^3)} \leq C(M)(t - s)^{1/2}. \]

By the aid of the two inequalities above and (4.13), one can prove easily (4.12) by a density argument. This completes the proof of Step 1.

**Step 2.** In this step, we prove that if the strong convergence and uniform estimate hold true up to time \( T_1 < T \), then they hold true up to time \( T_2 := \min\{T, T_1 + T_M\} \). Suppose that \( \frac{7}{8} \leq |u_\varepsilon| \leq \frac{9}{8} \) on \( \mathbb{R}^3 \times [0, T_1] \),

\[
\sup_{0 \leq t \leq T_1} \|(\nabla u_\varepsilon, v_\varepsilon)\|^2_{H^1(\mathbb{R}^3)} + \int_0^{T_1} \|(\nabla^3 u_\varepsilon, \nabla^2 v_\varepsilon, \partial_\varepsilon \nabla u_\varepsilon, \partial_\tau \nabla v_\varepsilon)\|^2_{L^2(\mathbb{R}^3)} dt \leq C(M)
\]

and

\[
(\nabla u_\varepsilon, v_\varepsilon) \to (\nabla u, v) \quad \text{in} \quad L^\infty(0, T_1; L^2(\mathbb{R}^3)) \cap L^2(0, T_1; H^1(\mathbb{R}^3))
\]

for some \( T_1 < T \). Due to the above two inequality, we apply Lemma 4.3 to conclude that

\[
\sup_{0 \leq t \leq T_1} \left( \|(\nabla u_\varepsilon, v_\varepsilon)\|^2_{H^1(\mathbb{R}^3)} + \|\partial_\varepsilon u_\varepsilon\|^2_{L^2(\mathbb{R}^3)} \right) \leq S_0,
\]

which, using equation (1.7), gives

\[
\sup_{0 \leq t \leq T_1} \left( \|(\nabla u_\varepsilon, v_\varepsilon)\|^2_{H^1(\mathbb{R}^3)} + \|Q_\varepsilon u_\varepsilon, v_\varepsilon\|^2_{L^2(\mathbb{R}^3)} \right) \leq S_0 \leq M
\]

for small \( \varepsilon \). Recall that \( \frac{7}{8} \leq |u_\varepsilon| \leq \frac{9}{8} \) on \( \mathbb{R}^3 \times [0, T_1] \), starting from time \( T_1 \) and taking \((u_\varepsilon(T_1), v_\varepsilon(T_1))\) as the initial data, we can apply Proposition 2.1 again to extend \((u_\varepsilon, v_\varepsilon)\) to time \( T_2 := \min\{T_1 + T_M, T\} \), such that \( \frac{7}{8} \leq |u_\varepsilon| \leq \frac{9}{8} \) on \( \mathbb{R}^3 \times [0, T_2] \) and

\[
\sup_{0 \leq t \leq T_2} \|(\nabla u_\varepsilon, v_\varepsilon)\|^2_{H^1(\mathbb{R}^3)} + \int_0^{T_2} \|(\nabla^3 u_\varepsilon, \nabla^2 v_\varepsilon, \partial_\varepsilon \nabla u_\varepsilon, \partial_\tau \nabla v_\varepsilon)\|^2_{L^2(\mathbb{R}^3)} dt \leq C(M)
\]

for small \( \varepsilon \). Using the same argument as Step 1, we can prove the strong convergence up to time \( T_2 \), that is

\[
\nabla u_\varepsilon \to \nabla u \quad \text{and} \quad v_\varepsilon \to v \quad \text{in} \quad L^2(0, T_2; H^1(\mathbb{R}^3)) \cap L^\infty(0, T_2; L^2(\mathbb{R}^3)).
\]

This completes the proof of Step 2.

**Step 3.** Combining Step 1 and Step 2, one can easily prove that the strong convergence and uniform estimate hold true for any time \( T < T^* \).

Now, we turn to the proof of the characterization of the maximal existence time. Suppose that \( T^* < \infty \) is the maximal existence time of the strong solution \((u, v)\) to the system (1.1)–(1.4). We prove that

\[
\lim_{\varepsilon \to 0} \|(\nabla u_\varepsilon, v_\varepsilon)\|_{L^q(0, T^*; L^r(\mathbb{R}^3))} = \infty
\]

for any \((q, r) \in \mathcal{O}\). Suppose, by contradiction, that the above is not true. Then there is some \((q, r) \in \mathcal{O}\) and a sequence \( \varepsilon_i \to 0 \), such that

\[
\|(\nabla u_{\varepsilon_i}, v_{\varepsilon_i})\|_{L^q(0, T^*; L^r(\mathbb{R}^3))} \leq L
\]
for a positive number $L$. By Lemma 4.4 there is a positive constant $N$ depending only on $L$ and the initial data, such that
\[
\sup_{0 \leq t \leq T^*} \| (\nabla^2 u_{e_\varepsilon}, v_{e_\varepsilon}) \|_{L^q}^2 + \int_0^{T^*} \left( \| (\nabla^3 u_{e_\varepsilon}, \nabla \partial_t u_{e_\varepsilon}) \|_{L^2}^2 + \| (\nabla^2 v_{e_\varepsilon}, \partial_t v_{e_\varepsilon}) \|_{L^2}^2 \right) dt \leq N.
\]
Due to this estimates, using the same argument to the proof of Theorem 1 in Section 2 a subsequence of $(u_{e_\varepsilon}, v_{e_\varepsilon})$ converges to $(u, v)$ and
\[
\sup_{0 \leq t \leq T^*} \| (\nabla^2 u, v) \|_{H^1}^2 + \int_0^{T^*} \left( \| (\nabla^3 u, \nabla \partial_t u) \|_{L^2}^2 + \| (\nabla^2 v, \partial_t v) \|_{L^2}^2 \right) dt \leq N.
\]
As a result, by Theorem 1 we can extend the strong solution $(u, v)$ beyond $T^*$, contradicting to the definition of $T^*$. This contradiction implies that (4.14) holds true.

Now we prove that (4.14) implies that $T^*$ is the maximal existence time. Suppose, by contradiction, that $T^*$ is not the maximal existence time. By what we proved in the strong convergence and uniform estimates, we have
\[
\lim_{\varepsilon \to 0} \| (\nabla u_{e_\varepsilon}, v_{e_\varepsilon}) \|_{L^q(0,T^*;H^1(\mathbb{R}^3) \cap W^{1,1}_2(\mathbb{R}^3 \times [0,T^*]))} \leq M,
\]
for a positive constant $M$. It follows from the Gagliado-Nirenberg-Sobolev inequality and the above inequality that
\[
\| (\nabla u_{e_\varepsilon}, v_{e_\varepsilon}) \|_{L^q(0,T^*;L^r(\mathbb{R}^3))} = \left( \int_0^{T^*} \| (\nabla u_{e_\varepsilon}, v_{e_\varepsilon}) \|_{L^r(\mathbb{R}^3)}^q \right)^{1/q} \leq C \sup_{0 \leq t \leq T^*} \| (\nabla u_{e_\varepsilon}, v_{e_\varepsilon}) \|_{L^r(\mathbb{R}^3)} \leq CMT^*1/q
\]
for $r \in (3, q)$, and
\[
\| (\nabla u_{e_\varepsilon}, v_{e_\varepsilon}) \|_{L^q(0,T^*;L^r(\mathbb{R}^3))} = \left( \int_0^{T^*} \| (\nabla u_{e_\varepsilon}, v_{e_\varepsilon}) \|_{L^r(\mathbb{R}^3)}^q \right)^{1/q} \leq C \left( \int_0^{T^*} \| (\nabla^2 u_{e_\varepsilon}, \nabla v_{e_\varepsilon}) \|_{L^2(\mathbb{R}^3)}^{\frac{q+2}{q}} \| (\nabla^3 u_{e_\varepsilon}, \nabla^2 v_{e_\varepsilon}) \|_{L^2(\mathbb{R}^3)}^{\frac{2}{q}} \right)^{1/q} \leq C \left( \int_0^{T^*} \| (\nabla^2 u_{e_\varepsilon}, \nabla v_{e_\varepsilon}) \|_{L^2(\mathbb{R}^3)}^{\frac{2}{q}} \| (\nabla^3 u_{e_\varepsilon}, \nabla^2 v_{e_\varepsilon}) \|_{L^2(\mathbb{R}^3)}^{\frac{2}{q}} \right)^{1/q} \leq C \sup_{0 \leq t \leq T^*} \| (\nabla^2 u_{e_\varepsilon}, \nabla v_{e_\varepsilon}) \|_{L^2(\mathbb{R}^3)}^{\frac{2}{q}} \left( \int_0^{T^*} \| (\nabla^3 u_{e_\varepsilon}, \nabla^2 v_{e_\varepsilon}) \|_{L^2(\mathbb{R}^3)}^{\frac{2}{q}} \right)^{1/q} \leq CMT^*\frac{2}{q} \left( \int_0^{T^*} \| (\nabla^3 u_{e_\varepsilon}, \nabla^2 v_{e_\varepsilon}) \|_{L^2(\mathbb{R}^3)}^{\frac{2}{q}} \right)^{1/q} \leq CMT^*\frac{2}{q} M^{\frac{1}{2} - \frac{1}{q}} = CMT^*\frac{1}{2}.
\]
for $r \in (6, \infty)$. Due to the above two inequalities, we have
\[
\lim_{\varepsilon \to 0} \| (\nabla u_{e_\varepsilon}, v_{e_\varepsilon}) \|_{L^q(0,T^*;L^r(\mathbb{R}^3))} \to \infty,
\]
contradicting to (4.14). This contradiction implies that $T^*$ must be the maximal existence time, completing the proof.

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Min-Chun Hong, Department of Mathematics, The University of Queensland, Brisbane, QLD 4072, Australia
Jinkai Li, The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong
Zhiping Xin, The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong