Wavelet Characterizations of the Atomic Hardy Space $H^1$ on Spaces of Homogeneous Type

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Abstract  Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type in the sense of R. R. Coifman and G. Weiss and $H^1_{at}(\mathcal{X})$ be the atomic Hardy space. Via orthonormal bases of regular wavelets and spline functions recently constructed by P. Auscher and T. Hytönen, together with obtaining some crucial lower bounds for regular wavelets, the authors give an unconditional basis of $H^1_{at}(\mathcal{X})$ and several equivalent characterizations of $H^1_{at}(\mathcal{X})$ in terms of wavelets, which are proved useful.

1 Introduction

The real variable theory of Hardy spaces $H^p(\mathbb{R}^D)$ on the $D$-dimensional Euclidean space $\mathbb{R}^D$ plays essential roles in various fields of analysis such as harmonic analysis and partial differential equations; see, for example, [35, 33, 7, 34]. Meyer [30] established the equivalent characterizations of $H^1(\mathbb{R}^D)$ via wavelets. Liu [27] obtained several equivalent characterizations of the weak Hardy space $H^{1,\infty}(\mathbb{R}^D)$ via wavelets. Wu [37] further gave a wavelet area integral characterization of the weighted Hardy space $H^p_\omega(\mathbb{R}^D)$ for $p \in (0,1]$. Later, via the vector-valued Calderón-Zygmund theory, García-Cuerva and Martell [9] obtained a characterization of $H^p_\omega(\mathbb{R}^D)$ for $p \in (0,1]$ in terms of wavelets without compact supports.

It is well known that many classical results of harmonic analysis over Euclidean spaces can be extended to spaces of homogeneous type in the sense of Coifman and Weiss [4, 5], or to RD-spaces introduced by Han, Müller and Yang [16] (see also [15, 39]).

Recall that a quasi-metric space $(\mathcal{X}, d)$ equipped with a nonnegative measure $\mu$ is called a space of homogeneous type in the sense of Coifman and Weiss [4, 5] if $(\mathcal{X}, d, \mu)$ satisfies the following measure doubling condition: there exists a positive constant $C_{(\mathcal{X})} \in [1, \infty)$ such that, for all balls $B(x, r) := \{y \in \mathcal{X} : d(x, y) < r\}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, 2r)) \leq C_{(\mathcal{X})} \mu(B(x, r)),$$

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which further implies that there exists a positive constant \( \tilde{C}(\mathcal{X}) \) such that, for all \( \lambda \in [1, \infty) \),

\[
\mu(B(x, \lambda r)) \leq \tilde{C}(\mathcal{X}) \lambda^n \mu(B(x, r)),
\]

where \( n := \log_2 C(\mathcal{X}) \). Let

\[
n_0 := \inf \{ n \in (0, \infty) : \text{n satisfies (1.1)} \}.
\]

It is obvious that \( n_0 \) measures the dimension of \( \mathcal{X} \) in some sense and \( n_0 \leq n \). Observe that (1.1) with \( n \) replaced by \( n_0 \) may not hold true.

A space of homogeneous type, \( (\mathcal{X}, d, \mu) \), is called a metric measure space of homogeneous type in the sense of Coifman and Weiss if \( d \) is a metric.

Recall that an RD-space \( (\mathcal{X}, d, \mu) \) is defined to be a space of homogeneous type satisfying the following additional reverse doubling condition (see [16]): there exist positive constants \( a_0, \hat{C}(\mathcal{X}) \in (1, \infty) \) such that, for all balls \( B(x, r) \) with \( x \in \mathcal{X} \) and \( r \in (0, \text{diam}(\mathcal{X})/a_0) \),

\[
\mu(B(x, a_0 r)) \geq \hat{C}(\mathcal{X}) \mu(B(x, r))
\]

(see [39] for more equivalent characterizations of RD-spaces). Here and hereafter,

\[
\text{diam}(\mathcal{X}) := \sup \{ d(x, y) : x, y \in \mathcal{X} \}.
\]

Let \( (\mathcal{X}, d, \mu) \) be a space of homogeneous type. In [5], Coifman and Weiss introduced the atomic Hardy space \( H^{p,q}_{\text{at}}(\mathcal{X}, d, \mu) \) for all \( p \in (0, 1] \) and \( q \in [1, \infty) \cap (p, \infty] \) and showed that \( H^{p,q}_{\text{at}}(\mathcal{X}, d, \mu) \) is independent of the choice of \( q \), which is hereafter simply denoted by \( H^p_{\text{at}}(\mathcal{X}, d, \mu) \), and that its dual space is the Lipshitz space \( \text{Lip}_{1/p-1}(\mathcal{X}, d, \mu) \) when \( p \in (0, 1) \), or the space BMO(\( \mathcal{X}, d, \mu \)) of functions with bounded mean oscillations when \( p = 1 \).

Recall that Coifman and Weiss [5] introduced the following measure distance \( \rho \) which is defined by setting, for all \( x, y \in \mathcal{X} \),

\[
\rho(x, y) := \inf \{ \mu(B_d) : B_d \text{ is a ball containing } x \text{ and } y \},
\]

where the infimum is taken over all balls in \( (\mathcal{X}, d, \mu) \) containing \( x \) and \( y \); see also [28]. It is well known that, although all balls defined by \( d \) satisfy the axioms of the complete system of neighborhoods in \( \mathcal{X} \) [and hence induce a (separated) topology in \( \mathcal{X} \)], the balls \( B_d \) are not necessarily open with respect to the topology induced by the quasi-metric \( d \). However, by [28, Theorem 2], we see that there exists a quasi-metric \( \tilde{d} \) such that \( \tilde{d} \) is equivalent to \( d \), namely, there exists a positive constant \( C \) such that, for all \( x, y \in \mathcal{X} \),

\[
C^{-1} d(x, y) \leq \tilde{d}(x, y) \leq Cd(x, y),
\]

and the balls in \( (\mathcal{X}, \tilde{d}, \mu) \) are open.

Recall also that a quasi-metric measure space \( (\mathcal{X}, \rho, \mu) \) is said to be normal in [28] if there exists a fixed positive constant \( C(\rho) \) such that, for all \( x \in \mathcal{X} \) and \( r \in (0, \infty) \),

\[
C(\rho)^{-1} r \leq \mu(B_\rho(x, r)) \leq C(\rho) r.
\]
Assuming that all balls in \((X, d, \mu)\) are open, Coifman and Weiss [5, p.594] claimed that the topologies of \(X\) induced, respectively, by \(d\) and \(\rho\) coincide and \((X, \rho, \mu)\) is a normal space, which were rigorously proved by Macías and Segovia in [28, Theorem 3], and also that the atomic Hardy space \(H^p_{at}(X, d, \mu)\) associated with \(d\) and the atomic Hardy space \(H^p_{at}(X, \rho, \mu)\) associated with \(\rho\) coincide with equivalent quasi-norms for all \(p \in (0, 1]\).

Macías and Segovia [28, Theorem 2] further showed that there exists a normal quasi-metric \(\tilde{\rho}\) such that \(\tilde{\rho}\) is equivalent to \(\rho\) and \(\tilde{\rho}\) is \(\theta\)-Hölder continuous with \(\theta \in (0, 1)\), namely, there exists a positive constant \(C\) such that, for all \(x, \bar{x}, y \in X\),

\[
|\tilde{\rho}(x, y) - \tilde{\rho}(\bar{x}, y)| \leq C \left[ \tilde{\rho}(x, \bar{x}) \right]^{\theta} \left[ \tilde{\rho}(x, y) + \tilde{\rho}(\bar{x}, y) \right]^{1-\theta}.
\]

Via establishing certain geometric measure relations between \((X, d, \mu)\) and \((X, \rho, \mu)\), Hu, Yang and Zhou [18, Theorem 2.1] rigorously verified the claim of Coifman and Weiss [5, p.594] on the coincidence of both atomic Hardy spaces \(H^p_{at}(X, d, \mu)\) and \(H^p_{at}(X, \rho, \mu)\), which was also used by Macías and Segovia [29, pp.271-272].

When \((X, \rho, \mu)\) is a normal quasi-metric measure space, Coifman and Weiss [5] further established the molecular characterization for \(H^1_{at}(X, \rho, \mu)\). When \((X, \tilde{\rho}, \mu)\) is a normal quasi-metric measure space and \(\tilde{\rho}\) is \(\theta\)-Hölder continuous, Macías and Segovia [29] obtained the grand maximal function characterization for \(H^p_{at}(X, \tilde{\rho}, \mu)\) with \(p \in (\frac{1}{1+\theta}, 1]\) via distributions acting on certain spaces of Lipschitz functions; Han [14] obtained their Lusin-area function characterization; Duong and Yan [6] then characterized these atomic Hardy spaces via Lusin-area functions associated with some Poisson semigroups; Li [25] also obtained a characterization of \(H^p_{at}(X, \tilde{\rho}, \mu)\) in terms of the grand maximal function defined via test functions introduced in [17].

Over RD-spaces \((X, d, \mu)\) with \(d\) being a metric, for \(p \in (\frac{n_0}{n_0+1}, 1]\) with \(n_0\) as in (1.2), Han, Müller and Yang [15] developed a Littlewood-Paley theory for atomic Hardy spaces \(H^p_{at}(X, d, \mu)\); Grafakos, Liu and Yang [12] established their characterizations via various maximal functions. Moreover, it was shown in [16] that these Hardy spaces coincide with Triebel-Lizorkin spaces on \((X, d, \mu)\). Some basic tools, including spaces of test functions, approximations of the identity and various Calderón reproducing formulas on RD-spaces, were well developed in [15, 16], in order to develop a real-variable theory of Hardy spaces or, more generally, Besov spaces and Triebel-Lizorkin spaces on RD-spaces. From then on, these basic tools play important roles in harmonic analysis on RD-spaces (see, for example, [11, 13, 15, 16, 21, 22, 38, 39]).

Recently, Auscher and Hytönen [2] built an orthonormal basis of Hölder continuous wavelets with exponential decay via developing randomized dyadic structures and properties of spline functions over general spaces of homogeneous type. Motivated by [2], in this article, we obtain an unconditional basis of \(H^1_{at}(X, d, \mu)\) and several equivalent characterizations of \(H^1_{at}(X, d, \mu)\) in terms of wavelets.

We point out that the main result (Theorem 4.4 below) of this article was applied in [8] to confirm the conjecture suggested by A. Bonami and F. Bernicot affirmatively (This conjecture was presented by L. D. Ky in [23]). More applications are also expectable.

Throughout this article, for the presentation simplicity, we always assume that \((X, d, \mu)\) is a metric measure space of homogeneous type, \(\text{diam}(X) = \infty\) and \((X, d, \mu)\) is non-atomic, namely, \(\mu(\{x\}) = 0\) for any \(x \in X\). It is known that, if \(\text{diam}(X) = \infty\), then \(\mu(X) = \infty\).
Wavelet Characterizations of the Atomic Hardy Space $H^1$ (see, for example, [2, Lemma 8.1]). Also, from now on, for the notational simplicity, on function spaces over $(\mathcal{X}, d, \mu)$ such as $H^1_{at}(\mathcal{X}, d, \mu)$, we will simply write it as $H^1_{at}(\mathcal{X})$ by omitting $d$ and $\mu$.

The organization of this paper is as follows.

In Section 2, we first recall some preliminary notions on wavelets and discover some crucial lower bounds for regular wavelets via the continuous functional calculus (see Theorem 2.8 below).

In Section 3, we give an unconditional basis of $H^1_{at}(\mathcal{X})$. To this end, we first establish two useful lemmas which are generalizations of [36, Proposition 8.8 and Corollary 7.10], respectively. Via these, we show that the orthonormal basis of regular wavelets is just an unconditional basis of $H^1_{at}(\mathcal{X})$, where the molecular characterization of $H^1_{at}(\mathcal{X})$ from [18] and the boundedness of Calderón-Zygmund operators from [38] play important roles.

Section 4 is devoted to the equivalent wavelet characterizations of $H^1_{at}(\mathcal{X})$. Via the unconditional basis of $H^1_{at}(\mathcal{X})$ in Section 3, combined with the aforementioned obtained lower bounds for regular wavelets, the Lebesgue differential theorem associated to the dyadic cubes (see Lemma 4.7 below), and the technical Lemma 4.8, we then finish the proof of Theorem 4.4, the equivalent characterizations of $H^1_{at}(\mathcal{X})$ via wavelets.

Finally, we make some conventions on notation. Throughout the whole paper, $C$ stands for a positive constant which is independent of the main parameters, but it may vary from line to line. Moreover, we use $C(\rho, \alpha, \ldots)$ to denote a positive constant depending on the parameters $\rho, \alpha, \ldots$. Usually, for a ball $B$, we use $c_B$ and $r_B$, respectively, to denote its center and radius. Moreover, for any $x, y \in \mathcal{X}$, $r, \rho \in (0, \infty)$ and ball $B := B(x, r)$, $\rho B := B(x, \rho r)$, $V(x, r) := \mu(B(x, r)) =: V_r(x)$, $V(x, y) := \mu(B(x, d(x, y)))$.

If, for two real functions $f$ and $g$, $f \leq C g$, we then write $f \lesssim g$; if $f \lesssim g \lesssim f$, we then write $f \sim g$. For any subset $E$ of $\mathcal{X}$, we use $\chi_E$ to denote its characteristic function. Furthermore, $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ represent the duality relation and the $L^2(\mathcal{X})$ inner product, respectively.

2 Preliminaries on Wavelets over $(\mathcal{X}, d, \mu)$

In this section, we first recall some preliminary notions and then obtain some crucial lower bounds for regular wavelets from [2].

The following notion of the geometrically doubling is well known in analysis on metric spaces, for example, it can be found in Coifman and Weiss [4, pp. 66-67].

Definition 2.1. A metric space $(\mathcal{X}, d)$ is said to be geometrically doubling if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0$, where, for all $i$, $x_i \in \mathcal{X}$.

Remark 2.2. Let $(\mathcal{X}, d)$ be a geometrically doubling metric space. In [19], Hytönen showed that the following statements are mutually equivalent:

(i) $(\mathcal{X}, d)$ is geometrically doubling.
(ii) For any $\epsilon \in (0, 1)$ and any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$, there exists a finite ball covering $\{B(x_i, \epsilon r)\}_i$, with $x_i \in \mathcal{X}$ for all $i$, of $B(x, r)$ such that the cardinality of this covering is at most $N_0 \epsilon^{-G_0}$, here and hereafter, $N_0$ is as in Definition 2.1 and $G_0 := \log_2 N_0$.

(iii) For every $\epsilon \in (0, 1)$, any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$ contains at most $N_0 \epsilon^{-G_0}$ centers of disjoint balls $\{B(x_i, \epsilon r)\}_i$ with $x_i \in \mathcal{X}$ for all $i$.

(iv) There exists $M \in \mathbb{N}$ such that any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$ contains at most $M$ centers $\{x_i\}_i \subset \mathcal{X}$ of disjoint balls $\{B(x_i, r/4)\}_{i=1}^M$.

Recall that metric measure spaces of homogeneous type are geometrically doubling, which was proved by Coifman and Weiss in [4, pp. 66-68].

Before we introduce the orthonormal basis of regular wavelets from [2], we first recall some notions and notation from [2]. For every $k \in \mathbb{Z}$, a set of reference dyadic points, $\{x_{\alpha}^k\}_{\alpha \in \mathcal{A}_k}$, here and hereafter,

$$\mathcal{A}_k \text{ denotes some countable index set for each } k \in \mathbb{Z},$$

is chosen as follows [the Zorn lemma (see [31, Theorem I.2]) is needed since we consider the maximality]. For $k = 0$, let $\mathcal{X}^0 := \{x_0^0\}_{\alpha \in \mathcal{A}_0}$ be a maximal collection of 1-separated points. Inductively, for any $k \in \mathbb{N}$, let

$$\mathcal{X}^k := \{x_{\alpha}^k\}_{\alpha \in \mathcal{A}_k} \supset \mathcal{X}^{k-1} \quad \text{and} \quad \mathcal{X}^{-k} := \{x_{\alpha}^{-k}\}_{\alpha \in \mathcal{A}_k} \subset \mathcal{X}^{-(k-1)}$$

be maximal $\delta^k$-separated and $\delta^{-k}$-separated collections in $\mathcal{X}$ and in $\mathcal{X}^{-(k-1)}$, respectively. Fix $\delta$ a small positive parameter, for example, it suffices to take $\delta \leq \frac{1}{1000}$. From [2, Lemma 2.1], it follows that

$$d\left(x_{\alpha}^k, x_{\beta}^k\right) \geq \delta^k \text{ for all } \alpha, \beta \in \mathcal{A}_k \text{ and } \alpha \neq \beta, \quad d(x, \mathcal{X}) := \inf_{\alpha \in \mathcal{A}_k} d\left(x, x_{\alpha}^k\right) < 2\delta^k.$$

It is obvious that the dyadic reference points $\{x_{\alpha}^k\}_{k \in \mathbb{Z}, \alpha \in \mathcal{A}_k}$ satisfy [20, (2.3) and (2.4)] with $A_0 = 1$, $c_0 = 1$ and $C_0 = 2$, which further induces a dyadic system of dyadic cubes over geometrically doubling metric spaces as in [20, Theorem 2.2]. We re-state it in the following theorem, which is applied to the construction of the orthonormal basis of regular wavelets as in [2].

**Theorem 2.3.** Let $(\mathcal{X}, d)$ be a geometrically doubling metric space. Then there exist families of sets, $\mathcal{Q}_k^\alpha \subset \mathcal{Q}_k^\alpha \subset \mathcal{Q}_k^\alpha$ (called, respectively, open, half-open and closed dyadic cubes) such that:

(i) $\mathcal{Q}_k^\alpha$ and $\overline{\mathcal{Q}}_k^\alpha$ denote, respectively, the interior and the closure of $\mathcal{Q}_k^\alpha$;

(ii) if $\ell \in \mathbb{Z} \cap [k, \infty)$ and $\alpha, \beta \in \mathcal{A}_k$, then either $\mathcal{Q}_\beta^\ell \subset \mathcal{Q}_\alpha^k$ or $\mathcal{Q}_\alpha^k \cap \mathcal{Q}_\beta^\ell = \emptyset$;

(iii) for any $k \in \mathbb{Z}$,

$$\mathcal{X} = \bigcup_{\alpha \in \mathcal{A}_k} \mathcal{Q}_k^\alpha \quad \text{(disjoint union);}$$
(iv) for any \( k \in \mathbb{Z} \) and \( \alpha \in \mathcal{A}_k \) with \( \mathcal{A}_k \) as in (2.1),
\[
B \left( x^k_{\alpha}, \frac{1}{3} \delta^k \right) \subset Q^k_{\alpha} \subset B \left( x^k_{\alpha}, 4 \delta^k \right) =: B \left( Q^k_{\alpha} \right);
\]

(v) if \( k \in \mathbb{Z}, \ell \in \mathbb{Z} \cap [k, \infty), \alpha, \beta \in \mathcal{A}_k \) and \( Q^k_{\beta} \subset Q^k_{\alpha} \), then \( B(Q^k_{\beta}) \subset B(Q^k_{\alpha}) \).

The open and closed cubes \( \tilde{Q}^k_{\alpha} \) and \( \overline{Q}^k_{\alpha} \), with \( (k, \alpha) \in \mathcal{A} \), here and hereafter,
\[
(2.3) \quad \mathcal{A} := \{(k, \alpha): k \in \mathbb{Z}, \alpha \in \mathcal{A}_k\},
\]
depend only on the points \( x^k_{\beta} \) for \( \beta \in \mathcal{A}_\ell \) and \( \ell \in \mathbb{Z} \cap [k, \infty) \). The half-open cubes \( Q^k_{\alpha} \),
with \( (k, \alpha) \in \mathcal{A} \), depend on \( x^k_{\beta} \) for \( \beta \in \mathcal{A}_\ell \) and \( \ell \in \mathbb{Z} \cap [\min\{k, k_0\}, \infty) \), where \( k_0 \in \mathbb{Z} \) is a
preassigned number entering the construction.

Remark 2.4. (i) In what follows, let \( \leq \) be the partial order for dyadic points as in [20, Lemma 2.10]. It was shown in [20, Lemma 2.10] with \( C_0 = 2 \) that, if \( k \in \mathbb{Z}, \alpha \in \mathcal{A}_k \) with \( \mathcal{A}_k \) as in (2.1), \( \beta \in \mathcal{A}_{k+1} \) and \( (k+1, \beta) \leq (k, \alpha) \), then \( d(x^{k+1}_{\beta}, x^k_{\alpha}) < 2 \delta^k \).

(ii) For any \( (k, \alpha) \in \mathcal{A} \), let
\[
(2.4) \quad L(k, \alpha) := \{\beta \in \mathcal{A}_{k+1}: (k+1, \beta) \leq (k, \alpha)\}.
\]

By the proof of [20, Theorem 2.2] and the geometrically doubling property, we have the
following conclusions: \( 1 \leq \# L(k, \alpha) \leq \tilde{N}_0 \) and
\[
(2.5) \quad Q^k_{\alpha} = \bigcup_{\beta \in L(k, \alpha)} Q^{k+1}_{\beta},
\]
where \( \tilde{N}_0 \in \mathbb{N} \) is independent of \( k \) and \( \alpha \). Here and hereafter, for any finite set \( \mathcal{C} \), \( \# \mathcal{C} \)
denotes its cardinality.

The following useful estimate about the 1-separated set is from [2, Lemma 6.4].

Lemma 2.5. Let \( \Xi \) be a 1-separated set in a geometrically doubling metric space \((\mathcal{X}, d)\)
with positive constant \( N_0 \). Then, for all \( \epsilon \in (0, \infty) \), there exists a positive constant \( C(\epsilon, N_0) \),
depending on \( \epsilon \) and \( N_0 \), such that
\[
\sup_{a \in \mathcal{X}} e^{\epsilon d(a, \Xi)/2} \sum_{b \in \Xi} e^{-\epsilon d(a, b)} \leq C(\epsilon, N_0),
\]
here and hereafter, for any set \( \Xi \subset \mathcal{X} \) and \( x \in \mathcal{X}, d(x, \Xi) := \inf_{a \in \Xi} d(x, a) \).

Now we recall more notions and notation from [2]. Let \((\Omega, \mathcal{F}, \mathbb{P}_\omega)\) be the natural probability measure space with the same notation as in [2], where \( \mathcal{F} \) is defined as the
smallest \( \sigma \)-algebra containing the set
\[
\left\{ \prod_{k \in \mathbb{Z}} A_k : A_k \subset \Omega_k := \{0, 1, \ldots, L\} \times \{1, \ldots, M\} \text{ and only finite many } A_k \neq \Omega_k \right\},
\]
where $L$ and $M$ are defined as in [2]. For every $(k, \alpha) \in \mathcal{A}$ with $\mathcal{A}$ as in (2.3), the spline function is defined by setting
\[
s_k^\alpha(x) := \mathbb{P}_\omega \left( \left\{ \omega \in \Omega : x \in \mathcal{Q}_\alpha^k(\omega) \right\} \right), \quad x \in \mathcal{X}.
\]
Then the splines have the following properties:

(i) for all $(k, \alpha) \in \mathcal{A}$ and $x \in \mathcal{X}$, $\chi_B(x_k^\alpha, \delta_k^\alpha)(x) \leq s_k^\alpha(x) \leq \chi_B(x_k^\alpha, s_k^\alpha)(x)$;

(ii) for all $k \in \mathbb{Z}$, $\alpha, \beta \in \mathcal{A}_k$, with $\mathcal{A}_k$ as in (2.1), and $x \in \mathcal{X}$,
\[
s_k^\alpha(x_k^\beta) = \delta_{\alpha\beta}, \quad \sum_{\alpha \in \mathcal{A}_k} s_k^\alpha(x) = 1 \quad \text{and} \quad s_k^\alpha(x) = \sum_{\beta \in \mathcal{A}_{k+1}} p_{\alpha\beta}^k s_{\beta}^{k+1}(x),
\]
where, for each $k \in \mathbb{Z}$, $\mathcal{A}_{k+1} \subset \mathcal{A}_k$ denotes some countable index set
\[
\delta_{\alpha\beta} := \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{if } \alpha \neq \beta,
\end{cases}
\]
and $\{p_{\alpha\beta}^k\}_{\beta \in \mathcal{A}_{k+1}}$ is a finite nonzero set of nonnegative numbers with $p_{\alpha\beta}^k \leq 1$ for all $\beta \in \mathcal{A}_{k+1}$;

(iii) there exist positive constants $\eta \in (0, 1]$ and $C$, independent of $k$ and $\alpha$, such that, for all $(k, \alpha) \in \mathcal{A}$ and $x, y \in \mathcal{X}$,
\[
\left| s_k^\alpha(x) - s_k^\alpha(y) \right| \leq C \left[ \frac{d(x, y)}{\delta_k} \right]^\eta.
\]

By [2, Theorem 5.1], we know that there exists a linear, bounded uniformly on $k \in \mathbb{Z}$, and injective map $U_k : \ell^2(\mathcal{A}_k) \to L^2(\mathcal{X})$ with closed range, defined by
\[
U_k \lambda := \sum_{\alpha \in \mathcal{A}_k} \frac{\lambda_\alpha}{\sqrt{\mu_\alpha^k}} s_k^\alpha, \quad \lambda := \left\{ \chi^k_\alpha \right\}_{\alpha \in \mathcal{A}_k} \in \ell^2(\mathcal{A}_k),
\]
here and hereafter, $\mu_\alpha^k := \mu(B(x_k^\alpha, \delta_k^\alpha)) =: V(x_k^\alpha, \delta_k)$ for all $(k, \alpha) \in \mathcal{A}$, $\ell^2(\mathcal{A}_k)$ denotes the space of all sequences $\lambda := \{\chi^k_\alpha\}_{\alpha \in \mathcal{A}_k} \subset \mathbb{C}$ such that
\[
\|\lambda\|_{\ell^2(\mathcal{A}_k)} := \left\{ \sum_{\alpha \in \mathcal{A}_k} |\chi^k_\alpha|^2 \right\}^{1/2} < \infty.
\]
Observe that, if $k \in \mathbb{Z}$, $\lambda, \tilde{\lambda} \in \ell^2(\mathcal{A}_k)$, $f = U_k \lambda$ and $\tilde{f} = U_k \tilde{\lambda}$, then
\[
\left( f, \tilde{f} \right)_{L^2(\mathcal{X})} = \left( M_k \lambda, \tilde{\lambda} \right)_{\ell^2(\mathcal{A}_k)}
\]
with $M_k$ being the infinite matrix which has entries $M_k(\alpha, \beta) = \frac{(s_k^\alpha, s_{\beta}^k)}{\sqrt{\mu_\alpha^k \mu_\beta^k}}$ for $\alpha, \beta \in \mathcal{A}_k$.

Let $U_k^*$ be the adjoint operator of $U_k$ for all $k \in \mathbb{Z}$. Thus, for each $k \in \mathbb{Z}$, $M_k = U_k^* U_k$ is bounded, invertible, positive and self-adjoint on $\ell^2(\mathcal{A}_k)$. Let $V_k := U_k(\ell^2(\mathcal{A}_k))$ for all $k \in \mathbb{Z}$. The following result from [2] (with $\mu_\alpha^k$ replaced by $\nu_\alpha^k := \int_\mathcal{X} s_k^\alpha d\mu$) shows that $\{V_k\}_{k \in \mathbb{Z}}$ is a multiresolution analysis (for short, MRA) of $L^2(\mathcal{X})$. 

Theorem 2.6. Suppose that \((X, d, \mu)\) is a metric measure space of homogeneous type. Let \(k \in \mathbb{Z}\) and \(V_k\) be the closed linear span of \(\{s^k_\alpha\}_{\alpha \in \mathcal{A}_k}\). Then \(V_k \subset V_{k+1}, \bigcup_{k \in \mathbb{Z}} V_k = L^2(X)\) and \(\bigcap_{k \in \mathbb{Z}} V_k = \{0\}\).

Moreover, the functions \(\{s^k_\alpha/\sqrt{\nu^k_\alpha}\}_{\alpha \in \mathcal{A}_k}\) form a Riesz basis of \(V_k\): for all sequences of complex numbers \(\{\lambda^k_\alpha\}_{\alpha \in \mathcal{A}_k}\),

\[
\left\| \sum_{\alpha \in \mathcal{A}_k} \lambda^k_\alpha s^k_\alpha \right\|_{L^2(X)} \sim \left[ \sum_{\alpha \in \mathcal{A}_k} \left| \lambda^k_\alpha \right|^2 \nu^k_\alpha \right]^{1/2}
\]

with equivalent positive constants independent of \(k\) and \(\{\lambda^k_\alpha\}_{\alpha \in \mathcal{A}_k}\).

Recall that [2, Theorem 6.1] gives the system \(\{s^k_\alpha\}_{\alpha \in \mathcal{A}_k}\) of biorthogonal splines in \(V_k\) satisfying with \(\left(s^k_\alpha, \overline{s}^k_\beta\right)_{L^2(X)} = \delta_{\alpha\beta}\). Now we sketch the construction of the wavelet basis \(\{\psi^k_\beta\}_{k \in \mathbb{Z}, \beta \in \mathcal{A}_k}\), here and hereafter, for all \(k \in \mathbb{Z}\),

\[
\mathcal{A}_k := \mathcal{A}_{k+1} \setminus \mathcal{A}_k
\]

with \(\mathcal{A}_k\) as in (2.1). Let \(k \in \mathbb{Z}\). The inverse of \(U_{k+1}, U_{k+1}^{-1} : f \mapsto \{f(x^k_{\beta})\sqrt{\mu^k_{\beta}}\}_{\beta \in \mathcal{A}_{k+1}}\),

is an isomorphism from \(V_{k+1}\) onto \(\ell^2(\mathcal{A}_{k+1})\). Let

\[
Y_k := U_{k+1} \left( \left\{ \lambda : = \left( \lambda^k_{\beta+1} \right)_{\beta \in \mathcal{A}_{k+1}} \in \ell^2(\mathcal{A}_{k+1}) : \lambda^k_{\beta+1} = 0 \text{ for all } \beta \in \mathcal{A}_k \right\} \right),
\]

which is identified with \(U_{k+1}(\ell^2(\mathcal{A}_k))\). Obviously, \(V_{k+1} = V_k \oplus Y_k\). Let \(W_k\) be the orthogonal complement (in \(L^2(X)\)) of \(V_k\) in \(V_{k+1}\) and \(Q_k\) the orthogonal projection onto \(W_k\). Then the restriction of \(Q_k\) to \(Y_k\) is an isomorphism from \(Y_k\) onto \(W_k\). Then \(\{s^{k+1}_\beta\}_{\beta \in \mathcal{A}_k}\) is an unconditional basis of \(Y_k\) and its image under \(Q_k\) is an unconditional basis of \(W_k\). Thus, for all \(f \in V_{k+1}\),

\[
Q_k f = f - \sum_{\alpha \in \mathcal{A}_k} (f, \overline{s}^k_\alpha)_{L^2(X)} s^k_\alpha.
\]

Moreover, the matrix \(\tilde{M}(\alpha, \beta)\) with

\[
\tilde{M}(\alpha, \beta) := \frac{(Q_k s^k_\alpha, Q_k s^{k+1}_\beta)_{L^2(X)}}{\sqrt{V(y^k_\alpha, \delta^k)} V(y^k_\beta, \delta^k)} = \frac{(s^{k+1}_\alpha, s^{k+1}_\beta)_{L^2(X)}}{\sqrt{V(y^k_\alpha, \delta^k)} V(y^k_\beta, \delta^k)}
\]

for all \((\alpha, \beta) \in \mathcal{A}_k \times \mathcal{A}_k\), is bounded uniformly, invertible, positive and self-adjoint on \(\ell^2(\mathcal{A}_k)\), where \(y^k_\beta := x^k_{\beta+1}\) for all \(k \in \mathbb{Z}\) and \(\beta \in \mathcal{A}_k\). Indeed, observe that \(\tilde{M} = (U_k|_{\ell^2(\mathcal{A}_k)})^*(U_k|_{\ell^2(\mathcal{A}_k)}), \) where \(U_k|_{\ell^2(\mathcal{A}_k)}\) denotes the restriction of \(U_k\) to \(\ell^2(\mathcal{A}_k)\) whose adjoint operator is denoted by \((U_k|_{\ell^2(\mathcal{A}_k)})^*\), is the restriction of \(M_{k+1}\) to \(\ell^2(\mathcal{A}_k)\), which implies the desired result.
From [32, Theorem 12.33], it follows that $\widetilde{M}^{-1/2}$ exists and is bounded, invertible, positive and self-adjoint on $\ell^2(\mathcal{G}_k)$. Then the wavelet functions are defined by setting, for all $k \in \mathbb{Z}$, $\alpha \in \mathcal{G}_k$ and $x \in \mathcal{X}$,

$$
\psi^k_\alpha(x) := (U_k|_{\ell^2(\mathcal{G}_k)}) \widetilde{M}^{-1/2} \delta^{k+1}_\alpha(x) = \sum_{\beta \in \mathcal{G}_k} \widetilde{M}^{-1/2}(\alpha, \beta) \frac{\delta^{k+1}_\beta}{\mu^{k+1}_\beta} \delta^{k+1}_\beta(\alpha)(x) = \sum_{\beta \in \mathcal{G}_k} \widetilde{M}^{-1/2}(\alpha, \beta) \frac{\delta^{k+1}_\beta}{\mu^{k+1}_\beta} \psi^k_\beta(x),
$$

where $\{\delta^{k+1}_\beta\}_{\beta \in \mathcal{G}_k}$ is the canonical orthonormal basis of $\ell^2(\mathcal{G}_k)$.

Now we are ready to introduce the following notable orthonormal basis of regular wavelets constructed by Auscher and Hytönen ([2, Theorem 7.1]) with a slight difference on the notation

$$
\{\psi^k_{\alpha, \beta}\}_{(k, \alpha) \in \widetilde{\mathcal{A}}, \beta \in \widetilde{L}(k, \alpha)} := \{\psi^k_\beta\}_{k \in \mathbb{Z}, \beta \in \mathcal{G}_k},
$$

where

$$
\widetilde{\mathcal{A}} := \{(k, \alpha) \in \mathcal{A} : \#L(k, \alpha) > 1\}
$$

and, for all $(k, \alpha) \in \widetilde{\mathcal{A}}$,

$$
\widetilde{L}(k, \alpha) := L(k, \alpha) \setminus \{\alpha\},
$$

via the fact that, for any $k \in \mathbb{Z}$,

$$
\mathcal{A}_{k+1} \setminus \mathcal{A}_k = \bigcup_{\{\alpha \in \mathcal{A}_k : \#L(k, \alpha) > 1\}} \widetilde{L}(k, \alpha).
$$

**Theorem 2.7.** Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type. Then there exists an orthonormal basis $\{\psi^k_{\alpha, \beta}\}_{(k, \alpha) \in \widetilde{\mathcal{A}}, \beta \in \widetilde{L}(k, \alpha)}$ of $L^2(\mathcal{X})$ and positive constants $\eta \in (0, 1]$ as in (2.7), $\nu$ and $C(\eta)$, independent of $k$, $\alpha$ and $\beta$, such that

$$
|\psi^k_{\alpha, \beta}(x)| \leq \frac{C(\eta)}{V(x^{k+1}_\beta, \delta^k)} e^{-\nu \delta^k d(x^{k+1}_\beta, x)}
$$

for all $x \in \mathcal{X}$, and

$$
|\psi^k_{\alpha, \beta}(x) - \psi^k_{\alpha, \beta}(y)| \leq \frac{C(\eta)}{V(x^{k+1}_\beta, \delta^k)} \left[\frac{d(x, y)}{\delta^k}\right]^\eta e^{-\nu \delta^k d(x^{k+1}_\beta, x)}
$$

for all $x, y \in \mathcal{X}$ satisfying $d(x, y) \leq \delta^k$, and

$$
\int_{\mathcal{X}} \psi^k_{\alpha, \beta}(x) \, d\mu(x) = 0.
$$

Now we give out an important property of $\psi^k_{\alpha}$ which is crucial to the succeeding context.
Theorem 2.8. Let \((X, d, \mu)\) be a metric measure space of homogeneous type. Then there exist positive constants \(c_0\) and \(C\), independent of \(k, \alpha\) and \(\beta\), such that, for all \((k, \alpha) \in A\) with \(A\) as in (2.11), \(\beta \in L(k, \alpha)\) with \(L(k, \alpha)\) as in (2.12), and \(x \in B(y^k_\beta, c_0 \delta^k) \subset Q^k_\alpha\),
\[
\left| \psi^k_{\alpha, \beta}(x) \right| \geq C \frac{1}{\sqrt{\mu(Q^k_\alpha)}}.
\]

Proof. Let \((k, \alpha) \in A\) and \(\beta \in L(k, \alpha)\). We first show that
\[
(2.16) \quad \left| \tilde{M}^{1/2}(\beta, \beta) \right| \geq c_3,
\]
where \(\tilde{M} := \{\tilde{M}(\alpha, \beta)\}_{(\alpha, \beta) \in \mathcal{G}_k \times \mathcal{G}_k}\) is as in (2.9) and \(c_3\) is a positive constant independent of \(\alpha, \beta\) and \(k\). To this end, we adopt an idea from the proof of [24, Theorem 5]; see also the proof of [2, Lemma 6.5].

Indeed, denote the spectrum and the resolvent set of \(\tilde{M}\) by \(\sigma(\tilde{M})\) and \(\rho(\tilde{M})\), respectively. The spectral radius of \(\tilde{M}\) is defined by setting \(r(\tilde{M}) := \sup\{\lambda : \lambda \in \sigma(\tilde{M})\}\). Then, since \(\tilde{M}\) is positive and self-adjoint, it follows that \(\{\lambda \in (0, \infty) : \lambda > r(\tilde{M})\} \subset \rho(\tilde{M})\) and hence
\[
\sigma(\tilde{M}) \subset \left\{ \lambda \in (0, \infty) : \lambda \leq r(\tilde{M}) \right\}.
\]

Furthermore, by the facts that \(L^2(\mathcal{G}_k)\), with \(\mathcal{G}_k\) as in (2.8), is a Hilbert space and that \(\tilde{M}\) is self-adjoint, and [31, Theorem VI.6], we see that \(r(\tilde{M}) = \|\tilde{M}\|_{L^2(\mathcal{G}_k)}\), which, combined with the fact that \(\tilde{M}\) is positive, invertible and bounded, implies that \(\sigma(\tilde{M}) \subset [a, b]\) for some \(a, b \in (0, \infty)\) satisfying \(0 < a < b \leq \|\tilde{M}\|_{L^2(\mathcal{G}_k)}\), since \(\sigma(\tilde{M})\) is closed (see [31, Theorem VI.5]). Here and hereafter, for a normed linear space \(E\) and a bounded linear operator \(T\) from \(E\) to \(E\), we use \(\|T\|_{L(E)}\) to denote the operator norm of \(T\).

Now we claim that there exists a positive, bounded and self-adjoint operator \(A\) such that \(\tilde{M} = 2\|\tilde{M}\|_{L^2(\mathcal{G}_k)}(Id - A)\), where \(Id\) denotes the identity operator on \(L^2(\mathcal{G}_k)\), and
\[
\|A\|_{L^2(\mathcal{G}_k)} \leq 1 - \frac{a}{2\|\tilde{M}\|_{L^2(\mathcal{G}_k)}} < 1.
\]

Indeed, let \(g(t) := 1 - \frac{t}{2\|\tilde{M}\|_{L^2(\mathcal{G}_k)}}\) for all \(t \in \sigma(\tilde{M})\) and \(A := g(\tilde{M})\). From [31, Theorem VII.1(e), (g)], we deduce that
\[
\sigma(A) = \sigma \left( g \left( \tilde{M} \right) \right) = \left\{ g(t) : t \in \sigma(\tilde{M}) \right\} \subset \left\{ g(t) : t \in [a, b] \right\}
\]
and
\[
\|A\|_{L^2(\mathcal{G}_k)} = \|g \left( \tilde{M} \right) \|_{L^2(\mathcal{G}_k)} = \|g\|_{L^\infty(\sigma(\tilde{M}))} \leq 1 - \frac{a}{2\|\tilde{M}\|_{L^2(\mathcal{G}_k)}} < 1.
\]
which show the above claim.

Thus, \((I_0 - A)^{-1/2} = \sum_{n=0}^\infty p_n A^n\), where \(0 < p_n \lesssim n^{1/2}\) for all \(n \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}\).

Observe that, for all \(\beta \in \mathcal{G}_k\),

\[
A(\beta, \beta) = 1 - \frac{\tilde{M}(\beta, \beta)}{2\|M\|_{\mathcal{L}(l^2(\mathcal{G}_k))}} = 1 - \frac{(\tilde{M}\delta_\beta^{k+1}, \delta_\beta^{k+1})_{l^2(\mathcal{G}_k)}}{2\|M\|_{\mathcal{L}(l^2(\mathcal{G}_k))}} \geq 1 - 1/2 = 1/2
\]

with \(\{\delta_\beta^{k+1}\}_{\beta \in \mathcal{G}_k}\) being the canonical orthonormal basis of \(l^2(\mathcal{G}_k)\).

By this, [31, Theorem VII.1(e), (g)], and the fact that \(\tilde{M}\) is bounded uniformly on \(k\), we conclude that, for all \((k, \alpha) \in \mathcal{A}\) and \(\beta \in \tilde{L}(k, \alpha)\),

\[
\tilde{M}^{-1/2}(\beta, \beta) = \left(2\|\tilde{M}\|_{\mathcal{L}(l^2(\mathcal{G}_k))}\right)^{-1/2} (I_0 - A)^{-1/2}(\beta, \beta)
\]

\[
= \left(2\|\tilde{M}\|_{\mathcal{L}(l^2(\mathcal{G}_k))}\right)^{-1/2} \sum_{n=0}^\infty p_n A^n(\beta, \beta)
\]

\[
\geq \left(2\|\tilde{M}\|_{\mathcal{L}(l^2(\mathcal{G}_k))}\right)^{-1/2} p_1 A(\beta, \beta) \gtrsim 1,
\]

where we used the fact that, if the infinite matrix \(A^n\) \((n \in \mathbb{Z}_+)\) is positive, then the diagonal entries \(A^n(\beta, \beta) = (A^n\delta_\beta^{k+1}, \delta_\beta^{k+1})_{l^2(\mathcal{G}_k)} \geq 0\). This finishes the proof of (2.16).

Then we turn to estimate \(\psi_k^\beta\) for all \(k \in \mathbb{Z}\) and \(\beta \in \mathcal{G}_k\), where \(\psi_k^\beta\) is as in (2.10) with \(\alpha\) replaced by \(\beta\). From the definition of \(\psi_k^\beta\) in (2.10), (2.6) and (2.16), it follows that

\[
(2.17) \quad \left|\psi_k^\beta \left(y_k^\beta\right)\right| = \left|\sum_{\gamma \in \mathcal{G}_k} \tilde{M}^{-1/2}(\beta, \gamma) s_{\gamma}^{k+1}(y_k^\beta) \right| \geq \frac{c_3}{\sqrt{\mu_\beta^{k+1}}},
\]

where \(\mu_\beta^{k+1} := V(x_\beta^{k+1}, \delta_\beta^{k+1})\).

Moreover, let \(\epsilon_0 \in (0, 1)\) be a constant which will be determined later. Recall \(y_k^\beta := x_\beta^{k+1}\) for all \(k \in \mathbb{Z}\) and \(\beta \in \mathcal{G}_k\). By (2.14), we know that, if \(x \in B(y_k^\beta, \epsilon_0 \delta_k) \subset Q_\alpha^k\) (provided that \(\epsilon_0\) is small enough), then there exists a positive constant \(\tilde{C}\) such that, for all \(x \in B(y_k^\beta, \epsilon_0 \delta_k)\),

\[
\left|\psi_k^\beta(x) - \psi_k^\beta \left(y_k^\beta\right)\right| \leq \frac{1}{\sqrt{\mu_\beta^{k+1}}} \epsilon_0 e^{-\mu_\beta^{k+1} d(y_k^\beta, x) \leq \frac{\tilde{C} \epsilon_0}{\sqrt{\mu_\beta^{k+1}}},
\]

which, combined with (2.17), further implies that, if we choose \(\epsilon_0\) small enough, then, for all \(x \in B(y_k^\beta, \epsilon_0 \delta_k)\),

\[
(2.18) \quad \left|\psi_k^\beta(x)\right| \geq \left|\psi_k^\beta \left(y_k^\beta\right)\right| - \left|\psi_k^\beta(x) - \psi_k^\beta \left(y_k^\beta\right)\right| \geq \frac{c_3 - \tilde{C} \epsilon_0}{\sqrt{\mu_\beta^{k+1}}} \geq \frac{c_3}{2\sqrt{\mu_\beta^{k+1}}} \geq \frac{1}{\sqrt{\mu_\beta^{k+1}}}.
\]
Now we are ready to estimate $\psi_{k,\beta}$ for all $(k, \alpha) \in \tilde{\mathcal{A}}$ and $\beta \in \tilde{L}(k, \alpha)$. By $y^k_\beta := x^{k+1}_\beta$, $d(x^{k+1}_\beta, x^k_\alpha) < 2\delta^k$ [see Remark 2.4(i)] and $B(x^k_\alpha, \frac{1}{3} \delta^k) \subset Q^k_\alpha$ [see Theorem 2.3(iv)], we have

$$\mu^{k+1}_\beta \leq V(x^{k+1}_\beta, \delta^k) \leq V(x^k_\alpha, 3\delta^k) \lesssim V(x^k_\alpha, \frac{1}{3} \delta^k) \lesssim \mu(Q^k_\alpha).$$

This, together with (2.18), then finishes the proof of Theorem 2.8. \hfill \Box

**Remark 2.9.** Let $(k, \alpha) \in \tilde{\mathcal{A}}$ with $\tilde{\mathcal{A}}$ as in (2.11), $\beta \in L(k, \alpha)$ with $L(k, \alpha)$ as in (2.4), and $B(y^k_\beta, \epsilon_0 \delta^k)$ be as in Theorem 2.8. Now we claim that

$$V(y^k_\beta, \epsilon_0 \delta^k) \sim \mu(Q^k_\alpha).$$

Indeed, from Remark 2.4(i), $(k+1, \beta) \leq (k, \alpha)$, (1.1) and $B(y^k_\beta, \epsilon_0 \delta^k) \subset Q^k_\alpha$, it follows that

$$\mu(Q^k_\alpha) \leq V(x^k_\alpha, 4\delta^k) \leq V(y^k_\beta, 6\delta^k) \lesssim V(y^k_\beta, \epsilon_0 \delta^{k+1}) \lesssim V(y^k_\beta, \epsilon_0 \delta^k) \lesssim \mu(Q^k_\alpha),$$

which shows the above claim.

### 3 An Unconditional Basis of $H^1_{at}(\mathcal{X})$

In this section, we obtain an unconditional basis of $H^1_{at}(\mathcal{X})$. Now we first recall the following notion of Hardy spaces $H^1_{at}(\mathcal{X})$, which was introduced in [5].

**Definition 3.1.** Let $q \in (1, \infty]$. A function $a$ on $\mathcal{X}$ is called a $(1, q)$-atom if

(i) $\text{supp}(a) \subset B$ for some ball $B \subset \mathcal{X};$

(ii) $\|a\|_{L^q(\mathcal{X})} \leq [\mu(B)]^{1/q-1};$

(iii) $\int_{\mathcal{X}} a(x) d\mu(x) = 0.$

A function $f \in L^1(\mathcal{X})$ is said to be in the Hardy space $H^1_{at}(\mathcal{X})$ if there exist $(1, q)$-atoms $\{a_j\}_{j=1}^\infty$ and numbers $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that

$$f = \sum_{j=1}^\infty \lambda_j a_j,$$

which converges in $L^1(\mathcal{X})$, and

$$\sum_{j=1}^\infty |\lambda_j| < \infty.$$

Moreover, the norm of $f$ in $H^1_{at}(\mathcal{X})$ is defined by setting

$$\|f\|_{H^1_{at}(\mathcal{X})} := \inf \left\{ \sum_{j \in \mathbb{N}} |\lambda_j| \right\},$$

where the infimum is taken over all possible decompositions of $f$ as in (3.1).
Coifman and Weiss [5] proved that \( H^1_{a_t}(\mathcal{X}) \) and \( H^1_{a_t}(\mathcal{X}) \) coincide with equivalent norms for all different \( q \in (1, \infty) \). Thus, from now on, we denote \( H^1_{a_t}(\mathcal{X}) \) simply by \( H^1_{a_t}(\mathcal{X}) \).

**Remark 3.2.** It was shown in [5] that \( H^1_{a_t}(\mathcal{X}) \) is a Banach space which is the predual of BMO(\( \mathcal{X} \)).

We then recall the molecular characterization of \( H^1_{a_t}(\mathcal{X}) \) from [18], which plays important roles in establishing equivalent characterizations of \( H^1_{a_t}(\mathcal{X}) \) via wavelets, since it partially compensates the defect of the regular wavelets without bounded supports.

The following notions of \((1,q,\eta)\)-molecules are from [18].

**Definition 3.3.** Let \( q \in (1, \infty] \) and \( \{\eta_k\}_{k \in \mathbb{N}} \subset [0, \infty) \) satisfy

\[
\sum_{k \in \mathbb{N}} k\eta_k < \infty. \tag{3.2}
\]

A function \( m \in L^q(\mathcal{X}) \) is called a \((1,q,\eta)\)-molecule centered at a ball \( B := B(x_0, r) \), for some \( x_0 \in \mathcal{X} \) and \( r \in (0, \infty) \), if

- (M1) \( \|m\|_{L^q(\mathcal{X})} \leq [\mu(B)]^{1/q-1} \);
- (M2) for all \( k \in \mathbb{N} \),
  \[
  \left\| m\chi_{B(x_0, 2^k r) \setminus B(x_0, 2^{k-1} r)} \right\|_{L^q(\mathcal{X})} \leq \eta_k 2^{k(1/q-1)} [\mu(B)]^{1/q-1} ;
  \]
- (M3) \( \int_{\mathcal{X}} m(x) \, d\mu(x) = 0 \).

Then the following molecular characterization of the space \( H^1_{a_t}(\mathcal{X}) \) is a slight variant of [18, Theorem 2.2] which is originally related to the quasi-metric \( \rho \) as in (1.3) and is obviously true with \( \rho \) replaced by \( d \).

**Theorem 3.4.** Suppose that \( (\mathcal{X}, d, \mu) \) is a metric measure space of homogeneous type. Let \( q \in (1, \infty] \) and \( \eta = \{\eta_k\}_{k \in \mathbb{N}} \subset [0, \infty) \) satisfy (3.2). Then there exists a positive constant \( C \) such that, for any \((1,q,\eta)\)-molecule \( m, m \in H^1_{a_t}(\mathcal{X}) \) and

\[
\|m\|_{H^1_{a_t}(\mathcal{X})} \leq C.
\]

Moreover, \( f \in H^1_{a_t}(\mathcal{X}) \) if and only if there exist \((1,q,\eta)\)-molecules \( \{m_j\}_{j \in \mathbb{N}} \) and numbers \( \{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C} \) such that

\[
f = \sum_{j \in \mathbb{N}} \lambda_j m_j,
\]

which converges in \( L^1(\mathcal{X}) \). Furthermore,

\[
\|f\|_{H^1_{a_t}(\mathcal{X})} \sim \inf \left\{ \sum_{j \in \mathbb{N}} |\lambda_j| \right\},
\]

where the infimum is taken over all the decompositions of \( f \) as above and the equivalent positive constants are independent of \( f \).
In order to show that \( \{ \psi^k_{\alpha, \beta} \}_{(k, \alpha) \in \mathcal{I}, \beta \in \mathcal{L}(k, \alpha)} \) is an unconditional basis of \( H^1_{at}(X) \), we need some notions and basic properties of the unconditional convergence and the unconditional basis from [26, 36].

**Definition 3.5.** (i) Let \( A \) be some countable index set and \( \{ x_n \}_{n \in A} \) a countable family of vectors in a Banach space \( B \). The series \( \sum_{n \in A} x_n \) is said to be *unconditionally convergent* if, for each permutation \( \sigma : \mathbb{N} \to A \), namely, a bijection, the series \( \sum_{k=0}^{\infty} x_{\sigma(k)} \) still converges in \( B \).

(ii) A countable family \( \{ x_n \}_{n \in A} \) of vectors in a Banach space \( B \) is called an *unconditional basis* if, for any \( x \in B \), there exists a unique sequence of scalars, \( \{ \lambda_n \}_{n \in A} \subset \mathbb{C} \), such that

\[
x = \sum_{n \in A} \lambda_n x_n \quad \text{in} \quad B
\]

and the expansion \( \sum_{n \in A} \lambda_n x_n \) of \( x \) converges unconditionally.

**Remark 3.6.** It was shown in [26, Proposition 1.c.1] that \( \{ x_n \}_{n \in A} \) is an unconditional basis of a Banach space \( B \) if and only if, for any sequence \( \{ \epsilon_n \}_{n \in A} \subset \{-1, 1\}, \sum_{n \in A} \epsilon_n x_n \) converges in \( B \).

The following useful lemma is a variant of [36, Proposition 8.8] on Euclidean spaces.

**Lemma 3.7.** Suppose that \( (X, d, \mu) \) is a metric measure space of homogeneous type. Let \( a \) be a \((1, \infty)\)-atom. Then \( \sum_{(k, \alpha) \in \mathcal{I}, \beta \in \mathcal{L}(k, \alpha)} (a, \psi^k_{\alpha, \beta}) \psi^k_{\alpha, \beta} \) converges unconditionally in \( H^1_{at}(X) \). Moreover, there exists a positive constant \( C \), independent of \( a \), such that, for all subsets \( \mathcal{S} \subset \{(k, \alpha, \beta) : (k, \alpha) \in \mathcal{I}, \beta \in \mathcal{L}(k, \alpha)\} =: \mathcal{I} \) with \( \mathcal{I} \) and \( \mathcal{L}(k, \alpha) \) being, respectively, as in (2.11) and (2.12),

\[
\left\| \sum_{(k, \alpha, \beta) \in \mathcal{S}} (a, \psi^k_{\alpha, \beta}) \psi^k_{\alpha, \beta} \right\|_{H^1_{at}(X)} \leq C.
\]

**Proof.** Let \( a \) be a \((1, \infty)\)-atom supported in the ball \( B := B(c_B, r_B) \) with \( c_B \in X \) and \( r_B \in (0, \infty) \), and \( N \in \mathbb{Z} \) satisfy \( \delta^{N+1} < r_B \leq \delta^N \). We first show that \( \sum_{(k, \alpha, \beta) \in \mathcal{I}} (a, \psi^k_{\alpha, \beta}) \psi^k_{\alpha, \beta} \) converges unconditionally in \( H^1_{at}(X) \) and (3.3) holds true for \( \mathcal{S} = \mathcal{I} \).

Let \( \mathcal{A} := \{(k, \alpha, \beta) \in \mathcal{I} : k \leq N\}, \mathcal{B} := \{(k, \alpha, \beta) \in \mathcal{I} : k > N, x_{\beta}^{k+1} \notin 2B\} \) and \( \mathcal{C} := \{(k, \alpha, \beta) \in \mathcal{I} : k > N, x_{\beta}^{k+1} \in 2B\} \). Then we write

\[
\sum_{(k, \alpha, \beta) \in \mathcal{I}} (a, \psi^k_{\alpha, \beta}) \psi^k_{\alpha, \beta} = \sum_{(k, \alpha, \beta) \in \mathcal{A}} (a, \psi^k_{\alpha, \beta}) \psi^k_{\alpha, \beta} + \sum_{(k, \alpha, \beta) \in \mathcal{B}} \cdots + \sum_{(k, \alpha, \beta) \in \mathcal{C}} \cdots
\]

\[
=: \Sigma_A + \Sigma_B + \Sigma_C.
\]

Let \( (k, \alpha, \beta) \in \mathcal{I} \). We first claim that \( \frac{\psi^k_{\alpha, \beta}}{\sqrt{V(x_{\beta}^{k+1}, \delta^k)}} \) is a \((1, 2, \eta)\)-molecule multiplied by a positive constant independent of \( k, \alpha \) and \( \beta \), where \( \eta := \{\eta_{\ell}\}_{\ell=1}^{\infty} \) and

\[
\eta_{\ell} := |2C_{(x_{\ell})}|^{\ell/2} \exp\{-\gamma 2^{\ell-1}\} \quad \text{for any} \quad \ell \in \mathbb{N}
\]
with $\tilde{C}(\mathcal{X})$ as in (1.1). Indeed, by $\|\psi_{k,\alpha,\beta}^k\|_{L^2(\mathcal{X})} = 1$ [since $\{\psi_{k,\alpha,\beta}^k\}_{(k,\alpha,\beta)\in\mathcal{F}}$ is an orthonormal basis of $L^2(\mathcal{X})$ (see Theorem 2.7)], we find that

\begin{equation}
(3.4) \quad \left\| \frac{\psi_{k,\alpha,\beta}^k}{\sqrt{V(x_{\beta}^{k+1}, \delta^k)}} \right\|_{L^2(\mathcal{X})} = \left[ V\left(x_{\beta}^{k+1}, \delta^k\right) \right]^{-1/2}.
\end{equation}

On the other hand, by (2.13) and (1.1), we know that, for any $\ell \in \mathbb{N}$,

\[
\left\| \frac{\psi_{k,\alpha,\beta}^k}{\sqrt{V(x_{\beta}^{k+1}, \delta^k)}} \chi_{B(x_{\beta}^{k+1}, 2^\ell \delta^k) \setminus B(x_{\beta}^{k+1}, 2^{\ell-1} \delta^k)} \right\|_{L^2(\mathcal{X})} \leq \frac{1}{V(x_{\beta}^{k+1}, \delta^k)} \left\{ \int_{B(x_{\beta}^{k+1}, 2^\ell \delta^k) \setminus B(x_{\beta}^{k+1}, 2^{\ell-1} \delta^k)} e^{-2\nu \delta^{-k} d(x_{\beta}^{k+1}, x)} \, d\mu(x) \right\}^{1/2} \leq \frac{1}{V(x_{\beta}^{k+1}, \delta^k)} e^{-\nu 2^{\ell-1}} \left[ V\left(x_{\beta}^{k+1}, 2^\ell \delta^k\right) \right]^{1/2} \leq e^{-\nu 2^{\ell-1}} \left[ \tilde{C}(\mathcal{X}) \right]^{\ell/2} \left[ V\left(x_{\beta}^{k+1}, \delta^k\right) \right]^{-1/2} \sim \eta 2^{-\ell/2} \left[ V\left(x_{\beta}^{k+1}, \delta^k\right) \right]^{-1/2}.
\]

This, combined with (3.4) and $\sum_{\ell=1}^{\infty} \ell \eta \ell < \infty$, implies the above claim. Moreover, by this claim and Theorem 3.4, we conclude that, for all $(k,\alpha,\beta) \in \mathcal{F}$,

\[
\|\psi_{k,\alpha,\beta}^k\|_{H^1_\alpha(\mathcal{X})} \lesssim \sqrt{V(x_{\beta}^{k+1}, \delta^k)},
\]

where the implicit positive constant is independent of $k, \alpha$ and $\beta$.

In order to estimate $\Sigma_A$, we first control $|(a, \psi_{k,\alpha,\beta}^k)|$ for all $(k,\alpha,\beta) \in \mathcal{A}$. From the vanishing moment of $a$, (2.14) and $r_B \leq \delta^N \leq \delta^k$, we deduce that

\[
(a, \psi_{k,\alpha,\beta}^k) = \int_B a(x) (\psi_{k,\alpha,\beta}^k(x) - \psi_{k,\alpha,\beta}^k(c_B)) \, d\mu(x) \leq \|a\|_{L^\infty(\mathcal{X})} \int_B \left| \psi_{k,\alpha,\beta}^k(x) - \psi_{k,\alpha,\beta}^k(c_B) \right| \, d\mu(x) \leq \frac{1}{\mu(B)} \int_B \left[ \frac{r_B}{\delta^k} \right]^\eta e^{-\nu \delta^{-k} d(x_{\beta}^{k+1}, x)} \, d\mu(x),
\]

which, combined with the above claim, Theorem 3.4, Lemma 2.5, (1.1) and $\eta \in (0,1]$, implies that

\begin{equation}
(3.5) \quad \sum_{(k,\alpha,\beta) \in \mathcal{A}} |(a, \psi_{k,\alpha,\beta}^k)| \left\| \psi_{k,\alpha,\beta}^k \right\|_{H^1_\alpha(\mathcal{X})} \lesssim \sum_{(k,\alpha,\beta) \in \mathcal{A}} \sqrt{V\left(x_{\beta}^{k+1}, \delta^k\right)} \left| (a, \psi_{k,\alpha,\beta}^k) \right|.
\end{equation}
converges unconditionally in $\{a, \beta \in \mathcal{B}(k, \alpha) : \#L(k, \alpha) > 1\}$.

Then we estimate $\Sigma_B$. By the above claim, Theorem 3.4, the size condition of $a$, (2.13), $d(x, x_{\beta}^{k+1}) \geq \frac{1}{\delta}d(x_{\beta}^{k+1}, c_B)$ for $x \in B$ and $x_{\beta}^{k+1} \notin 2B$, Lemma 2.5 and $r_B > \delta^{N+1}$, we conclude that

$$\sum_{(k, \alpha, \beta) \in B} |(a, \psi_{\alpha, \beta}^k)| \left\|\psi_{\alpha, \beta}^k\right\|_{H^1_{at}(\mathcal{X})} \leq \frac{1}{\mu(B)} \int_B \sum_{k=N+1}^{\infty} \sum_{\alpha, \beta \in \mathcal{L}(k, \alpha) : x_{\beta}^{k+1} \notin 2B, \#L(k, \alpha) > 1} e^{-\nu\delta^{-k}d(x_{\beta}^{k+1}, x)} d\mu(x) \leq \sum_{k=N+1}^{\infty} e^{-2-2\nu\delta^{-k}d(c_B, \mathcal{Y}^{k+1})B} \leq \sum_{k=N+1}^{\infty} e^{-\nu\delta^{-k}r_B} \leq \sum_{k=N+1}^{\infty} e^{-\nu\delta^{-k-N+1}} \leq 1,$$

where the implicit positive constant is independent of $a$. Thus, similar to $\Sigma_A$, we know that $\Sigma_B$ converges unconditionally in $H^1_{at}(\mathcal{X})$ and $\|\Sigma_B\|_{H^1_{at}(\mathcal{X})} \leq 1$.

Finally, we prove that $\Sigma_C$ unconditionally converges in $H^1_{at}(\mathcal{X})$. For any $M \in \mathbb{Z} \cap [N, \infty)$, let

$$\Sigma_C^M := \sum_{k>M, \alpha, \beta \in \mathcal{L}(k, \alpha) : x_{\beta}^{k+1} \in 2B, \#L(k, \alpha) > 1} \epsilon_{\alpha, \beta}^k (a, \psi_{\alpha, \beta}^k) \psi_{\alpha, \beta}^k,$$

where $\epsilon_{\alpha, \beta}^k \in \{-1, 1\}$ for any $k > M, \alpha, \beta \in \mathcal{L}(k, \alpha)$ with $x_{\beta}^{k+1} \in 2B$ and $\#L(k, \alpha) > 1$. By Remark 3.6, it suffices to show that $\|\Sigma_C^M\|_{H^1_{at}(\mathcal{X})} \leq 1$ for all $M \geq N$ and all choices of $\epsilon_{\alpha, \beta}^k \in \{-1, 1\}$, and $\|\Sigma_C^M\|_{H^1_{at}(\mathcal{X})} \to 0$ as $M \to \infty$.

Without loss of generality, we may assume that $\|\Sigma_C^M\|_{L^2(\mathcal{X})} > 0$ for all $M \in \mathbb{Z} \cap [N, \infty)$. Otherwise, we only need to consider all those $M \in \mathbb{Z} \cap [N, \infty)$ such that $\|\Sigma_C^M\|_{L^2(\mathcal{X})} > 0$. 
From Theorem 2.7 and Definition 3.1(ii), it follows that, for any $M \in \mathbb{Z}$,

\[(3.7) \quad \|\Sigma^M_c\|_{L^2(\mathcal{X})} \leq \|\Sigma_c\|_{L^2(\mathcal{X})} \leq \|a\|_{L^2(\mathcal{X})} \leq [\mu(B)]^{-1/2}\]

and $\|\Sigma^M_c\|_{L^2(\mathcal{X})} \to 0$ as $M \to \infty$.

Let $\mu_M := \|\Sigma^M_c\|_{L^2(\mathcal{X})}/[\mu(4B)]^{1/2}$ for all $M \in \mathbb{Z} \cap [N, \infty)$. Now we claim that

\[(3.8) \quad \Sigma^M_c := \Sigma^M_c / \mu_M \quad \text{is a} \quad (1, 2, \eta) - \text{molecule,} \]

centered at ball $4B$, multiplied by some positive constant, where $\eta := \{\eta_\ell\}_{\ell=0}^\infty \subset [0, \infty)$ and $\eta_\ell := [2\widetilde{C}(\mathcal{X})]^{\ell/2}2^{-(\ell+1)K_0}/\mu_M$ for some large positive integer $K_0$ such that $K_0 \geq G_0 + n + 1$, with $G_0$ and $n$ respectively as in Remark 2.2(ii) and (1.1), and

\[
\sum_{\ell=1}^\infty 2^{n\ell/2} \left[2\widetilde{C}(\mathcal{X})\right]^{\ell/2} 2^{-(\ell+1)K_0}/\mu_M < \infty.
\]

Obviously,

\[(3.9) \quad \|\Sigma^M_c\|_{L^2(\mathcal{X})} = \|\Sigma^M_c\|_{L^2(\mathcal{X})}/\mu_M = [\mu(4B)]^{-1/2}.
\]

On the other hand, by (1.1), we observe that, for any $r_0, \nu_0 \in (0, \infty)$ and $x_0 \in \mathcal{X}$,

\[(3.10) \quad \int_{\mathcal{X}} e^{-i\nu_0 d(x, x_0)/r_0} \, d\mu(x) \leq \int_{B(x_0, r_0)} e^{-i\nu_0 d(x, x_0)/r_0} \, d\mu(x) + \sum_{\ell=1}^\infty \int_{B(x_0, (\ell+1)r_0) \setminus B(x_0, \ell r_0)} \cdots \leq V(x_0, r_0) + \sum_{\ell=1}^\infty e^{-\nu_0 \ell} V(x_0, [\ell + 1]r_0) \leq V(x_0, r_0) + \sum_{\ell=1}^\infty e^{-\nu_0 \ell} (\ell + 1)^n V(x_0, r_0) \leq V(x_0, r_0).
\]

From (3.10) and (2.13), we further deduce that, for all $(k, \alpha, \beta) \in \mathcal{I}$,

\[(3.11) \quad \left\|\psi^k_{\alpha, \beta}\right\|_{L^1(\mathcal{X})} \leq \sqrt{V(x^{k+1}_\beta, \delta^k)}.
\]

Moreover, for any $\ell \in \mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ and $x \in 2^{\ell+3}B \setminus 2^{\ell+2}B$, by (3.11), (2.13), $x^{k+1}_\beta \in 2B$ [and hence $d(x, x^{k+1}_\beta) \geq \frac{1}{2}d(x, c_B)$ and $B(x^{k+1}_\beta, \delta^k) \subset 3B$], the geometrically doubling condition, (i) and (iii) of Remark 2.2, $K_0 \geq G_0 + n + 1$ and $\delta^{M+1} < r_B$, we conclude that

\[
|\Sigma^M_c(x)| \leq \sum_{k=M+1}^\infty \sum_{\{\alpha \in \delta^k, \beta \in \tilde{L}(k, \alpha) : x^{k+1}_\beta \in 2B, \#L(k, a) > 1\}} \left|\left(a, \psi^k_{\alpha, \beta}\right)\right| \left|\psi^k_{\alpha, \beta}(x)\right|
\]
Thus, by this and (1.1), we further have
\[
\leq \|a\|_{L^\infty(X)} \sum_{k=M+1}^{\infty} \sum_{\alpha \in \mathcal{A}_k, \beta \in \mathcal{L}(k, \alpha)} e^{-\nu \delta^{-k}d(x, x_{\beta}^{k+1})} \| \psi_{\alpha, \beta}^k \|_{L^1(X)} \| \psi_{\alpha, \beta}^k \|_{L^1(X)}.
\]
\[
\leq \|a\|_{L^\infty(X)} \sum_{k=M+1}^{\infty} \sum_{\alpha \in \mathcal{A}_k, \beta \in \mathcal{L}(k, \alpha)} e^{-\xi \delta^{-k}d(x, \epsilon B)}
\]
\[
\leq [\mu(B)]^{-1} \sum_{k=M+1}^{\infty} \sum_{\alpha \in \mathcal{A}_k, \beta \in \mathcal{L}(k, \alpha)} e^{-\nu 2^k \epsilon B \delta^{-k}k}
\]
\[
\leq [\mu(B)]^{-1} \sum_{k=M+1}^{\infty} 2^{-(\ell+1)K_0} \left[ \frac{\delta^k}{T_B} \right]^{K_0} \left[ \frac{T_B}{\delta^k} \right]^{G_0} \sum_{k=M+1}^{\infty} \frac{\delta^k}{T_B} \lesssim [\mu(B)]^{-1} 2^{-(\ell+1)K_0}.
\]

Thus, by this and (1.1), we further have
\[
(3.12) \quad \left\| \tilde{\Sigma}_C^M \chi_{2^\ell+3B \setminus 2^{\ell+2}B} \right\|_{L^2(X)} \lesssim \frac{1}{\mu(B)} [\mu(B)]^{-1} 2^{-(\ell+1)K_0} \left[ \frac{\delta^k}{T_B} \right]^{K_0} [\mu(B)]^{1/2} \lesssim 2^{-\ell/2} \eta \mu(4B)^{-1/2}.
\]

To prove the claim in (3.8), we need to further show that
\[
(3.13) \quad \int_X \tilde{\Sigma}_C^M \, d\mu = 0.
\]

By the Hölder inequality, (3.9), (3.12), (1.1) and $K_0 \geq G_0 + n + 1$, we know that
\[
\left\| \tilde{\Sigma}_C^M \right\|_{L^1(X)} \leq \left\| \tilde{\Sigma}_C^M \chi_{4B} \right\|_{L^1(X)} + \sum_{\ell=0}^{\infty} \left\| \tilde{\Sigma}_C^M \chi_{2^\ell+3B \setminus 2^{\ell+2}B} \right\|_{L^1(X)}
\leq \left\| \tilde{\Sigma}_C^M \right\|_{L^2(X)} [\mu(4B)]^{1/2} + \sum_{\ell=0}^{\infty} \left\| \tilde{\Sigma}_C^M \chi_{2^\ell+3B \setminus 2^{\ell+2}B} \right\|_{L^2(X)} \left[ \mu \left( 2^{\ell+3}B \right) \right]^{1/2}
\lesssim 1 + \sum_{\ell=0}^{\infty} 2^{2\ell} 2^{-\ell} \eta \lesssim 1.
\]

Moreover, let $U_\ell(B) := 2^{\ell+3}B \setminus 2^{\ell+2}B$ for any $\ell \in \mathbb{Z}_+$. By $\Sigma_C^M \in L^1(X)$, Theorem 2.7
and (2.15), we conclude that

\[
\int_{\mathcal{X}} \sum_{\ell=0}^{M} d\mu = \int_{\mathcal{X}} \chi_{\mathcal{B}} \sum_{\ell=0}^{\infty} \chi_{U_{\ell}(B)} \sum_{\ell=0}^{M} d\mu \\
= (\sum_{\ell=0}^{M}, \chi_{\mathcal{B}}) + \sum_{\ell=0}^{\infty} (\sum_{\ell=0}^{M}, \chi_{U_{\ell}(B)}) \\
= \sum_{k=M+1}^{\infty} \sum_{\alpha \in \mathcal{A}_{k}} \sum_{\beta \in \mathcal{L}(k, \alpha)} e_{k}^{A} (a, \psi_{k}^{A}, \chi_{\mathcal{B}}) (a, \psi_{k}^{A}, \chi_{\mathcal{B}}) \\
+ \sum_{\ell=0}^{\infty} \sum_{k=M+1}^{\infty} \sum_{\alpha \in \mathcal{A}_{k}} \sum_{\beta \in \mathcal{L}(k, \alpha)} e_{k}^{A} (a, \psi_{k}^{A}, \chi_{\mathcal{B}}) (a, \psi_{k}^{A}, \chi_{\mathcal{B}}) \\
\times \left( \psi_{k}^{A}, \chi_{U_{\ell}(B)} \right).
\]

Now we show that

\[
\sum_{k=M+1}^{\infty} \sum_{\alpha \in \mathcal{A}_{k}} \sum_{\beta \in \mathcal{L}(k, \alpha)} \sum_{x_{\beta}^{k+1} \in 2B} \#L(k, \alpha) > 1 \left| (a, \psi_{k}^{A}, \chi_{U_{\ell}(B)}) \right| < \infty.
\]

Indeed, from the Hölder inequality and Theorem 2.7, it follows that

\[
\sum_{k=M+1}^{\infty} \sum_{\alpha \in \mathcal{A}_{k}} \sum_{\beta \in \mathcal{L}(k, \alpha)} \sum_{x_{\beta}^{k+1} \in 2B} \#L(k, \alpha) > 1 \left| (a, \psi_{k}^{A}, \chi_{U_{\ell}(B)}) \right| \\
\leq \sum_{k=M+1}^{\infty} \sum_{\alpha \in \mathcal{A}_{k}} \sum_{\beta \in \mathcal{L}(k, \alpha)} \sum_{x_{\beta}^{k+1} \in 2B} \#L(k, \alpha) > 1 \left| (a, \psi_{k}^{A}, \chi_{U_{\ell}(B)}) \right|^2 \\
\leq \left\| a \right\|_{L^{2}(\mathcal{X})} \sum_{\ell=0}^{\infty} \sum_{k=M+1}^{\infty} \sum_{\alpha \in \mathcal{A}_{k}} \sum_{\beta \in \mathcal{L}(k, \alpha)} \sum_{x_{\beta}^{k+1} \in 2B} \#L(k, \alpha) > 1 \left| (a, \psi_{k}^{A}, \chi_{U_{\ell}(B)}) \right|^2 \leq \infty.
\]

Now we estimate \( (\psi_{k}^{A}, \chi_{U_{\ell}(B)}) \) for any \( \ell \in \mathbb{Z}, \ k \in \mathbb{Z} \cap [M + 1, \infty), \alpha \in \mathcal{A}_{k} \) and \( \beta \in \mathcal{L}(k, \alpha) \) with \( x_{\beta}^{k+1} \in 2B \) and \( \#L(k, \alpha) > 1 \). Indeed, we choose \( M_{0} \) to be a large enough positive constant such that \( M_{0} \geq G_{0} + 1 \) with \( G_{0} \) as in Remark 2.2(ii). From (2.13) [together with (3.10)], (1.1), \( \delta^{k} \leq \delta^{M+1} \leq \delta^{N+1} < r_{B} \) for all \( k \in \mathbb{Z} \cap [M + 1, \infty) \) and \( B(x_{\beta}^{k+1}, \delta^{k}) \subset B(x_{\beta}^{k}, r_{B}) \subset 3B \), we deduce that

\[
\left| \left( \psi_{k}^{A}, \chi_{U_{\ell}(B)} \right) \right| \lesssim \frac{1}{V(x_{\beta}^{k+1}, \delta^{k})} \int_{U_{\ell}(B)} e^{-\nu \delta^{-k} d(x, x_{\beta}^{k+1})} d\mu(x)
\]
\[
\lesssim \frac{1}{\sqrt{V(x^{k+1}_\beta, \delta^k)}} e^{\frac{\nu}{2} \delta^{-k} d(x, CB)} \int_{U_\ell(B)} e^{-\frac{\nu}{2} \delta^{-k} d(x, CB)} e^{-\frac{\nu}{2} \delta^{-k} d(x, x^{k+1}_\beta)} d\mu(x)
\]
\[
\lesssim e^{\nu \delta^{-k} r_B e^{-\nu \ell r_B} \sqrt{V(x^{k+1}_\beta, \delta^k)}}
\]
\[
\lesssim e^{-\nu \ell M_0} \frac{\delta^k}{r_B} \sqrt{\mu(B)} \lesssim e^{-\nu \ell M_0} \frac{\delta^k}{r_B} \mu(B),
\]
which, combined with the elementary inequality
\[
\sum_{j=0}^{\infty} |a_j|^p \leq \sum_{j=0}^{\infty} |a_j|^p \quad \text{for all } \{a_j\}_{j=0}^{\infty} \subset \mathbb{C} \text{ and } p \in (0, 1],
\]
and the fact that
\[
\# \left\{ \alpha \in \mathcal{A}_k, \beta \in \tilde{L}(k, \alpha) : x^{k+1}_\beta \in 2B, \# L(k, \alpha) > 1 \right\} = \# \left\{ \beta \in \mathcal{G}_k : x^{k+1}_\beta \in 2B \right\} \lesssim \left( \frac{r_B}{\delta^k} \right)^{G_0}
\]
[see Remark 2.2(ii)], further implies that
\[
\sum_{\ell=0}^{\infty} \sum_{k=M+1}^{\infty} \sum_{\left\{ \alpha \in \mathcal{A}_k, \beta \in \tilde{L}(k, \alpha) : x^{k+1}_\beta \in 2B, \# L(k, \alpha) > 1 \right\}} \left| \left( a, \psi^k_{\alpha, \beta} \right) \left( \psi^k_{\alpha, \beta}, \chi_{U_\ell(B)} \right) \right| 
\]
\[
\lesssim \left[ \mu(B) \right]^{-1/2} \sum_{\ell=0}^{\infty} 2^{-\ell M_0} \sum_{k=M+1}^{\infty} \left( \frac{r_B}{\delta^k} \right)^{G_0} \left( \frac{\delta^k}{r_B} \right)^{M_0} \left[ \mu(B) \right]^{1/2}
\]
\[
\lesssim \sum_{\ell=0}^{\infty} 2^{-\ell M_0} \sum_{k=M+1}^{\infty} \frac{\delta^k}{r_B} \lesssim 1.
\]
This shows (3.15).

From (3.14), (3.15) and (2.15), we further deduce that
\[
\int_X \sum_{C}^{M} d\mu = \sum_{k=M+1}^{\infty} \sum_{\left\{ \alpha \in \mathcal{A}_k, \beta \in \tilde{L}(k, \alpha) : x^{k+1}_\beta \in 2B, \# L(k, \alpha) > 1 \right\}} \epsilon^k_{\alpha, \beta} \left( a, \psi^k_{\alpha, \beta} \right)
\]
\[
\times \left[ \left( \psi^k_{\alpha, \beta}, \chi_{4B} \right) + \sum_{\ell=0}^{\infty} \left( \psi^k_{\alpha, \beta}, \chi_{U_\ell(B)} \right) \right]
\]
\[
= \sum_{k=M+1}^{\infty} \sum_{\left\{ \alpha \in \mathcal{A}_k, \beta \in \tilde{L}(k, \alpha) : x^{k+1}_\beta \in 2B, \# L(k, \alpha) > 1 \right\}} \epsilon^k_{\alpha, \beta} \left( a, \psi^k_{\alpha, \beta} \right) \left( \psi^k_{\alpha, \beta}, 1 \right) = 0,
\]
which shows (3.13) and hence completes the proof of the above claim in (3.8).
From the above claim, Theorem 3.4, (3.7) and the fact that \( \|\Sigma^M_c\|_{L^2(X)} \to 0 \) as \( M \to \infty \), we further deduce that, for all integer \( M \geq N \),

\[
\|\Sigma^M_c\|_{H^1_{at}(X)} \lesssim \mu_M \lesssim 1 \quad \text{and} \quad \|\Sigma^M_c\|_{H^1_{at}(X)} \to 0 \quad \text{as} \quad M \to \infty.
\]

This, combined with (3.5) and (3.6), shows that \( H \) conditionally in \( C \).

To obtain an unconditional basis of \( H^1_{at}(X) \), we need the boundedness of Calderón-Zygmund operators from \( H^1_{at}(X) \) to \( L^1(X) \) and from \( H^1_{at}(X) \) to itself. We first recall some notions and notation from [4]; see also [2]. Let \( s \in (0, \eta] \) with \( \eta \) as in (2.7) and \( C^s_b(X) \) be the set of all \( s \)-Hölder continuous functions \( f \) [namely, \( \sup_{x, y \in X: x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^s} < \infty \)] with bounded supports, whose dual space is denoted by \( (C^s_b(X))^* \). We point out that \( C^s_b(X) \) is dense in \( L^2(X) \) (see, for example, [2, Proposition 4.5]).

Now we introduce the notion of Calderón-Zygmund operators from [4]; see also [2].

**Definition 3.8.** A function \( K \in L^1_{loc}(\{X \times X\} \setminus \{(x, x) : x \in X\}) \) is called a Calderón-Zygmund kernel if there exists a positive constant \( C(K) \), depending on \( K \), such that

(i) for all \( x, y \in X \) with \( x \neq y \),

\[
(3.17) \quad |K(x, y)| \leq C(K) \frac{1}{V(x, y)};
\]

(ii) there exist positive constants \( s \in (0, 1] \) and \( c(K) \in (0, 1) \), depending on \( K \), such that

\[
(3.18) \quad |K(x, y) - K(\bar{x}, y)| \leq C(K) \left[ \frac{d(x, \bar{x})}{d(x, y)} \right]^s \frac{1}{V(x, y)};
\]

(iii) for all \( x, \bar{x}, y \in X \) with \( d(x, y) \geq c(K)d(x, \bar{x}) > 0 \),

\[
(3.19) \quad |K(x, y) - K(x, \bar{y})| \leq C(K) \left[ \frac{d(y, \bar{y})}{d(x, y)} \right]^s \frac{1}{V(x, y)}.
\]

Let \( T : C^s_{b}(X) \to (C^s_{b}(X))^* \) be a linear continuous operator. The operator \( T \) is called a Calderón-Zygmund operator with kernel \( K \) satisfying (3.17), (3.18) and (3.19) if, for all \( f \in C^s_{b}(X) \),

\[
(3.20) \quad Tf(x) := \int_{X} K(x, y) f(y) \, d\mu(y), \quad x \notin \text{supp}(f).
\]
Wavelet Characterizations of the Atomic Hardy Space $H^1$

Then we recall some results from [38, Proposition 3.1] (see also [18, Theorem 4.2]) about the boundedness of Calderón-Zygmund operators. In what follows $T^*1 = 0$ means that, for all $(1, 2)$-atom $a$, $\int_X Ta(x) d\mu(x) = 0$. By some careful examinations, we see that this result remains valid over the metric measure space of homogeneous type without resorting to the reverse doubling condition, the details being omitted.

**Theorem 3.9.** Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type. Suppose that $T$ is a Calderón-Zygmund operator as in (3.20) which is bounded on $L^2(\mathcal{X})$.

(i) Then there exists a positive constant $C$, depending only on $\|T\|_{L^2(\mathcal{X})}$, $s$, $C_K$, $c_K$ and $\bar{C}(\mathcal{X})$, such that, for all $f \in H^1_{at}(\mathcal{X})$, $Tf \in L^1(\mathcal{X})$ and $\|Tf\|_{L^1(\mathcal{X})} \leq C\|f\|_{H^1_{at}(\mathcal{X})}$.

(ii) If further assuming that $T^*1 = 0$, then there exists a positive constant $\tilde{C}$, depending only on $\|T\|_{L^2(\mathcal{X})}$, $s$, $C_K$, $c_K$ and $\bar{C}(\mathcal{X})$, such that, for all $f \in H^1_{at}(\mathcal{X})$, $Tf \in H^1_{at}(\mathcal{X})$ and $\|Tf\|_{H^1_{at}(\mathcal{X})} \leq C\|f\|_{H^1_{at}(\mathcal{X})}$.

Now we show the following conclusion on an unconditional basis of $H^1_{at}(\mathcal{X})$.

**Theorem 3.10.** Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type. Then

$$\{\psi^k_{\alpha, \beta}\}_{(k, \alpha, \beta) \in \mathcal{I}}$$

with $\mathcal{I}$ as in Lemma 3.7, is an unconditional basis of $H^1_{at}(\mathcal{X})$.

**Proof.** We first show that, for any $(1, \infty)$-atom $a$

$$a = \sum_{(k, \alpha, \beta) \in \mathcal{I}} (a, \psi^k_{\alpha, \beta}) \psi^k_{\alpha, \beta} \text{ in } H^1_{at}(\mathcal{X}).$$

(3.21)

Observe that, by Lemma 3.7, $\sum_{(k, \alpha, \beta) \in \mathcal{I}} (a, \psi^k_{\alpha, \beta}) \psi^k_{\alpha, \beta}$ converges unconditionally in $H^1_{at}(\mathcal{X})$.

Let

$$\{\mathcal{I}_N : N \in \mathbb{N}, \mathcal{I}_N \subset \mathcal{I} \text{ and } \mathcal{I}_N \text{ is finite}\}$$

be any collection satisfy $\mathcal{I}_N \uparrow \mathcal{I}$ (namely, for any $N \in \mathbb{N}$, $\mathcal{I}_N \subset \mathcal{I}_{N+1}$ and $\mathcal{I} = \bigcup_{N \in \mathbb{N}} \mathcal{I}_N$) and

$$S_N(a) := \sum_{(k, \alpha, \beta) \in \mathcal{I}_N} (a, \psi^k_{\alpha, \beta}) \psi^k_{\alpha, \beta}.$$

By Lemma 3.7, we conclude that there exists $\bar{a} \in H^1_{at}(\mathcal{X})$ such that

$$\bar{a} = \lim_{N \to \infty} \sum_{(k, \alpha, \beta) \in \mathcal{I}_N} (a, \psi^k_{\alpha, \beta}) \psi^k_{\alpha, \beta} \text{ in } H^1_{at}(\mathcal{X}),$$

(3.23)
which, together with $H^1_{at}(X) \subset L^1(X)$ and the Riesz lemma, further implies that there exists a subsequence $\{\sum_{(k, \alpha, \beta) \in I_N} (a, \psi^k_{\alpha, \beta}) \psi^k_{\alpha, \beta}\}_{m \in \mathbb{N}}$ of $\{\sum_{(k, \alpha, \beta) \in I_N} (a, \psi^k_{\alpha, \beta}) \psi^k_{\alpha, \beta}\}_{N \in \mathbb{N}}$ such that

$$\tilde{a} = \lim_{m \to \infty} \sum_{(k, \alpha, \beta) \in I_{N_m}} (a, \psi^k_{\alpha, \beta}) \psi^k_{\alpha, \beta} \mu \text{ almost everywhere on } X. \tag{3.24}$$

On the other hand, from Theorem 2.7, together with $a \in L^2(X)$, it follows that

$$a = \lim_{m \to \infty} \sum_{(k, \alpha, \beta) \in I_{N_m}} (a, \psi^k_{\alpha, \beta}) \psi^k_{\alpha, \beta} \text{ in } L^2(X),$$

which, combined with the Riesz lemma and (3.24), further implies that

$$\tilde{a} = a \mu \text{ almost everywhere on } X.$$

This, together with (3.23), then finishes the proof of (3.21).

For all $(k, \alpha, \beta) \in \mathscr{I}$ with $\mathscr{I}$ as in Lemma 3.7, from $\psi^k_{\alpha, \beta} \in L^\infty(X) \subset \text{BMO}(X)$, it follows that

$$\langle f, \psi^k_{\alpha, \beta} \rangle := \int_X f \psi^k_{\alpha, \beta} d\mu$$

is well defined in the sense of duality between $H^1_{at}(X)$ and $\text{BMO}(X)$.

Then we claim that, for any $f \in H^1_{at}(X)$,

$$f = \sum_{(k, \alpha, \beta) \in \mathscr{I}} \langle f, \psi^k_{\alpha, \beta} \rangle \psi^k_{\alpha, \beta} \text{ in } H^1_{at}(X). \tag{3.25}$$

By Definition 3.1, we see that there exist sequences $\{a_j\}_{j \in \mathbb{N}}$ of $(1, \infty)$-atoms and numbers $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$ satisfying $f = \sum_{j=1}^{\infty} \lambda_j a_j$ in $L^1(X)$ and $\sum_{j=1}^{\infty} |\lambda_j| \lesssim \|f\|_{H^1_{at}(X)}$.

From (3.21), it follows that, for any $M \in \mathbb{N}$, $f_M := \sum_{j=1}^{M} \lambda_j a_j$ satisfies

$$f_M = \sum_{(k, \alpha, \beta) \in \mathscr{I}} \langle f_M, \psi^k_{\alpha, \beta} \rangle \psi^k_{\alpha, \beta} \text{ in } H^1_{at}(X). \tag{3.26}$$

Let $N \in \mathbb{N}$ and, for any suitable function $f$,

$$S_N(f) := \sum_{(k, \alpha, \beta) \in \mathscr{I}_N} \langle f, \psi^k_{\alpha, \beta} \rangle \psi^k_{\alpha, \beta} \text{ with } \mathscr{I}_N \text{ as in } (3.22).$$

Then, by (3.26), we see that, for any fixed $M \in \mathbb{N}$,

$$\lim_{N \to \infty} \|S_N(f_M) - f_M\|_{H^1_{at}(X)} = 0. \tag{3.27}$$

Observe that, for any $N \in \mathbb{N}$, $S_N$ and $S_N^*$, where $S_N^*$ denotes the adjoint operator of $S_N$, are integral operators with kernels

$$K_N(x, y) := \sum_{(k, \alpha, \beta) \in \mathscr{I}_N} \psi^k_{\alpha, \beta}(x) \psi^k_{\alpha, \beta}(y).$$
and \( K_N^*(x, y) := K_N(y, x) \) for all \( x, y \in \mathcal{X} \) with \( x \neq y \), respectively. By [2, Proposition 10.3], we know that, for each \( N \in \mathbb{N} \), \( K_N \) satisfies (3.17), (3.18) and (3.19). From Theorem 2.7, we deduce that \( S_N^*(1) = 0 \) and \( \| S_N(f) \|_{L^2(\mathcal{X})} \leq \| f \|_{L^2(\mathcal{X})} \) for all \( f \in L^2(\mathcal{X}) \).

By this and Theorem 3.9(ii), we conclude that \( \{ S_N \}_{N \in \mathbb{N}} \) are bounded on \( H^1_{at}(\mathcal{X}) \) uniformly in \( N \in \mathbb{N} \), which further implies that, for each \( N \in \mathbb{N} \),

\[
(3.28) \quad \| S_N(f_M) - S_N(f) \|_{H^1_{at}(\mathcal{X})} = \| S_N(f_M - f) \|_{H^1_{at}(\mathcal{X})} \lesssim \| f_M - f \|_{H^1_{at}(\mathcal{X})}.
\]

This, combined with (3.27), further implies that

\[
\limsup_{N \to \infty} \| S_N(f) - f \|_{H^1_{at}(\mathcal{X})} \\
\leq \limsup_{N \to \infty} \left[ \| S_N(f) - S_N(f_M) \|_{H^1_{at}(\mathcal{X})} + \| S_N(f_M) - f_M \|_{H^1_{at}(\mathcal{X})} + \| f_M - f \|_{H^1_{at}(\mathcal{X})} \right] \\
\lesssim \| f_M - f \|_{H^1_{at}(\mathcal{X})} + \lim_{N \to \infty} \| S_N(f_M) - f_M \|_{H^1_{at}(\mathcal{X})} \\
\sim \| f_M - f \|_{H^1_{at}(\mathcal{X})} \to 0, \quad \text{as } M \to \infty,
\]

which completes the proof of the claim (3.25).

Now we show the uniqueness of the representations

\[
f = \sum_{(k, \alpha, \beta) \in \mathcal{I}} \lambda^k_{\alpha, \beta} \psi^k_{\alpha, \beta} \quad \text{in} \quad H^1_{at}(\mathcal{X})
\]

for all numbers \( \{ \lambda^k_{\alpha, \beta} \}_{(k, \alpha, \beta) \in \mathcal{I}} \subset \mathbb{C} \). Indeed, by the fact that, for all \( (k, \alpha, \beta) \in \mathcal{I}, \psi^k_{\alpha, \beta} \in L^\infty(\mathcal{X}) \subset \text{BMO}(\mathcal{X}) \) and the orthogonality of \( \{ \psi^k_{\alpha, \beta} \}_{(k, \alpha, \beta) \in \mathcal{I}} \) (see Theorem 2.7), we conclude that, for any \( (\ell, \gamma, \theta) \in \mathcal{I} \),

\[
\left\langle f, \psi^\ell_{\gamma, \theta} \right\rangle = \sum_{(k, \alpha, \beta) \in \mathcal{I}} \lambda^k_{\alpha, \beta} \left( \langle f, \psi^k_{\alpha, \beta} \rangle, \psi^\ell_{\gamma, \theta} \right) = \lambda^\ell_{\gamma, \theta},
\]

which is the desired result.

Finally, we prove that \( \sum_{(k, \alpha, \beta) \in \mathcal{I}} (f, \psi^k_{\alpha, \beta}) \psi^k_{\alpha, \beta} \) converges unconditionally. By Remark 3.6, we know that it suffices to show that, for any sequence \( \{ \epsilon^k_{\alpha, \beta} \}_{(k, \alpha, \beta) \in \mathcal{I}} \subset \{-1, 1\}, \)

\[
(3.29) \quad \sum_{(k, \alpha, \beta) \in \mathcal{I}} \epsilon^k_{\alpha, \beta} \langle f, \psi^k_{\alpha, \beta} \rangle \psi^k_{\alpha, \beta} \quad \text{converges in} \quad H^1_{at}(\mathcal{X}).
\]

Let \( N \in \mathbb{N} \) and

\[
\tilde{S}_N(f) := \sum_{(k, \alpha, \beta) \in \mathcal{I}_N} \epsilon^k_{\alpha, \beta} \langle f, \psi^k_{\alpha, \beta} \rangle \psi^k_{\alpha, \beta} \quad \text{with } \mathcal{I}_N \text{ as in (3.22)}.
\]

By some arguments similar to those used in (3.28), we conclude that \( \tilde{S}_N \) is bounded on \( H^1_{at}(\mathcal{X}) \) uniformly in \( N \in \mathbb{N} \) and hence, for any \( N, M \in \mathbb{N} \), if \( f_M \) is as above, then

\[
\left\| \tilde{S}_N(f) - \tilde{S}_N(f_M) \right\|_{H^1_{at}(\mathcal{X})} \lesssim \| f - f_M \|_{H^1_{at}(\mathcal{X})}.
\]
Observe also that, by Lemma 3.7 and Remark 3.6, we know that \( \{ \tilde{S}_N(a_j) \}_{N \in \mathbb{N}} \) for \( j \in \{1, \ldots, M\} \) is a Cauchy sequence in \( H^1_{at}(\chi) \). By these facts, we further conclude that, for all \( M \in \mathbb{N} \),

\[
\limsup_{N, K \to \infty} \left\| \tilde{S}_N(f) - \tilde{S}_K(f) \right\|_{H^1_{at}(\chi)} \\
\leq \limsup_{N \to \infty} \left\| \tilde{S}_N(f) - \tilde{S}_N(f_M) \right\|_{H^1_{at}(\chi)} + \limsup_{N, K \to \infty} \left\| \tilde{S}_N(f_M) - \tilde{S}_K(f_M) \right\|_{H^1_{at}(\chi)} \\
+ \limsup_{K \to \infty} \left\| \tilde{S}_K(f_M) - \tilde{S}_K(f) \right\|_{H^1_{at}(\chi)} \\
\lesssim \| f - f_M \|_{H^1_{at}(\chi)} + \sum_{j=1}^{M} |\lambda_j| \lim_{N, K \to \infty} \left\| \tilde{S}_N(a_j) - \tilde{S}_K(a_j) \right\|_{H^1_{at}(\chi)} \\
\lesssim \| f - f_M \|_{H^1_{at}(\chi)} \to 0 \quad \text{as} \quad M \to \infty,
\]

which, together with the completeness of \( H^1_{at}(\chi) \), implies that (3.29) holds true. This finishes the proof of Theorem 3.10. \( \square \)

4 Equivalent Wavelet Characterizations of \( H^1_{at}(\chi) \)

In this section, we establish several equivalent wavelet characterizations of \( H^1_{at}(\chi) \). To this end, we first recall a version of the Khintchine inequality; see, for example, [10, Theorem 12.5.1].

**Lemma 4.1.** Let \( A \) be a countable index set and \( \Omega \) be the product set \( \{1,-1\}^A \), associated with the Bernoulli probability measure \( d\mathbb{P}(\omega) \), namely, the product \( \prod_{a \in A} d\mathbb{P}_a(\omega) \) of measures \( d\mathbb{P}_a(\omega) \) (\( a \in A \)) such that \( \mathbb{P}_a(\{-1\}) = 1/2 = \mathbb{P}_a(\{1\}) \), where \( \omega \) is an element of \( \Omega \) in the form of \( \{\omega(a)\}_{a \in A} \subset \{-1,1\} \). Suppose that \( q \in (0, \infty) \). Then there exists a positive constant \( C \) such that, for all \( \{\lambda(a)\}_{a \in A} \subset \mathbb{C} \) and functions of the form, \( S(\omega) := \sum_{a \in A} \lambda(a) w(a) \), it holds true that

\[
C^{-1} \left[ \sum_{a \in A} |\lambda(a)|^2 \right]^{\frac{1}{2}} \leq \left[ \int_\Omega |S(\omega)|^q d\mathbb{P}(\omega) \right]^{\frac{1}{q}} \leq C \left[ \sum_{a \in A} |\lambda(a)|^2 \right]^{\frac{1}{2}}.
\]

The following lemma is a slight variant of [36, Corollary 7.10].

**Lemma 4.2.** Suppose that \( (\chi, d, \mu) \) is a metric measure space of homogeneous type, \( A \) is a countable index set and the series \( \sum_{a \in A} f_a \) converges unconditionally in \( L^q(\chi) \) with \( q \in (0, \infty) \). Then

\[
\left\| \left( \sum_{a \in A} |f_a|^2 \right)^{1/2} \right\|_{L^q(\chi)} \leq \sup \left\{ \left\| \sum_{a \in A} \epsilon_a f_a \right\|_{L^q(\chi)} : \{\epsilon_a\}_{a \in A} \subset \{-1,1\} \right\} < \infty,
\]

where the supremum is taken over all choices of \( \{\epsilon_a\}_{a \in A} \subset \{-1,1\} \).
Proof. Let $q \in (0, \infty)$. From the Khintchine inequality (Lemma 4.1), the Fubini-Tonelli theorem, the unconditional convergence of $\sum_{a \in A} f_a$, [36, Corollary 7.4] and $\mathbb{P}(\Omega) = 1$, it follows that

$$\left\| \left( \sum_{a \in A} |f_a|^2 \right)^{1/2} \right\|_{L^q(\mathcal{X})}^q \overset{\text{at}}{=} \int_{\mathcal{X}} \int_{\Omega} \left| \sum_{a \in A} \omega(a) f_a(x) \right|^q d\mathbb{P}(\omega) d\mu(x)$$

$$\sim \int_{\Omega} \int_{\mathcal{X}} \left| \sum_{a \in A} \omega(a) f_a(x) \right|^q d\mu(x) d\mathbb{P}(\omega)$$

$$\overset{\text{at}}{=} \sup \left\{ \left\| \sum_{a \in A} \epsilon_a f_a \right\|_{L^q(\mathcal{X})}^q : \{\epsilon_a\}_{a \in A} \subset \{-1, 1\} \right\} < \infty,$$

which completes the proof of Lemma 4.2. ☐

Corollary 4.3. Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type. Then there exists a positive constant $C$ such that, for all $f \in H^1_{at}(\mathcal{X})$,

$$\int_{\mathcal{X}} \left[ \sum_{(k, \alpha, \beta) \in \mathcal{J}} \left| \left\langle f, \psi^k_{\alpha, \beta} \right\rangle \right|^2 \left\| \psi^k_{\alpha, \beta}(x) \right\|^2 \right]^{1/2} d\mu(x) \leq C \|f\|_{H^1_{at}(\mathcal{X})},$$

with $\mathcal{J}$ as in Lemma 3.7.

Proof. Let $f \in H^1_{at}(\mathcal{X})$. From Theorem 3.10 and $H^1_{at}(\mathcal{X}) \subset L^1(\mathcal{X})$, we deduce that

$$\sum_{(k, \alpha, \beta) \in \mathcal{J}} \left\langle f, \psi^k_{\alpha, \beta} \right\rangle \psi^k_{\alpha, \beta}$$

converges unconditionally in $L^1(\mathcal{X})$. For any sequence $\mathcal{E} := \{\epsilon_{\alpha, \beta}^k\}_{(k, \alpha, \beta) \in \mathcal{J}} \subset \{-1, 1\}$, the operator $T_{\mathcal{E}} : L^2(\mathcal{X}) \to L^2(\mathcal{X})$ is defined by setting, for any $(k, \alpha, \beta) \in \mathcal{J}$,

$$T_{\mathcal{E}} \left( \psi^k_{\alpha, \beta} \right) := \epsilon_{\alpha, \beta}^k \psi^k_{\alpha, \beta},$$

which can be extended to an isometric isomorphism on $L^2(\mathcal{X})$.

Let $\{\mathcal{J}_N\}_{N \in \mathbb{N}}$ be any sequence of finite subsets of $\mathcal{J}$ as in (3.22), $g \in L^2(\mathcal{X})$ and, for all $N \in \mathbb{N}$, $g_N := \sum_{(k, \alpha, \beta) \in \mathcal{J}_N} \left\langle g, \psi^k_{\alpha, \beta} \right\rangle \psi^k_{\alpha, \beta}$,

$$K_{\mathcal{E}, N}(x, y) := \sum_{(k, \alpha, \beta) \in \mathcal{J}_N} \epsilon_{\alpha, \beta}^k \psi^k_{\alpha, \beta}(x) \overline{\psi^k_{\alpha, \beta}(y)}$$

for all $x, y \in \mathcal{X}$,

and

$$K_{\mathcal{E}}(x, y) := \sum_{(k, \alpha, \beta) \in \mathcal{J}} \epsilon_{\alpha, \beta}^k \psi^k_{\alpha, \beta}(x) \overline{\psi^k_{\alpha, \beta}(y)}$$

for all $x, y \in \mathcal{X}$ with $x \neq y$. 
Now we claim that $K_{\tilde{c}}$ is the Calderón-Zygmund kernel of $T_{\tilde{c}}$. Indeed, by [2, Proposition 10.3], we conclude that $K_{\tilde{c}, N}, K_{\tilde{c}} \in L^1_{\text{loc}}(\mathcal{X} \times \mathcal{X})$ are Calderón-Zygmund kernels satisfying (3.17), (3.18) and (3.19) with $s := \eta$ and $C(K_{\tilde{c}, N})$ and $C(K_{\tilde{c}})$ independent of $N \in \mathbb{N}$, which, together with the boundedness of $T_{\tilde{c}}$ on $L^2(\mathcal{X})$, the Lebesgue dominated convergence theorem and the Fubini theorem, further implies that, for all $g, h \in C^0_b(\mathcal{X})$ with $\text{supp} (g) \cap \text{supp} (h) = \emptyset$,

$$
\langle K_{\tilde{c}}, g \otimes h \rangle = \lim_{N \to \infty} \langle K_{\tilde{c}, N}, g \otimes h \rangle
= \lim_{N \to \infty} \int_{\mathcal{X}} \int_{\mathcal{X}} K_{\tilde{c}, N}(x, y) g(y) h(x) \, d\mu(y) \, d\mu(x)
= \lim_{N \to \infty} (T_{\tilde{c}}(g_N), h) = (T_{\tilde{c}}(g), h).
$$

Therefore, the above claim holds true, which, combined with Theorem 3.9(i), further implies that, for all $f \in H^1_{\text{at}}(\mathcal{X})$ and sequences $\tilde{c} \subset \{-1, 1\}$,

$$
\|T_{\tilde{c}}(f)\|_{L^1(\mathcal{X})} \lesssim \|f\|_{H^1_{\text{at}}(\mathcal{X})}.
$$

From this and Lemma 4.2 with $q = 1$, we further deduce that

$$
\int_{\mathcal{X}} \left[ \sum_{(k, \alpha, \beta) \in \mathcal{I}} \left| \left\langle f, \psi^k_{\alpha, \beta} \right\rangle \right|^2 \left| \psi^k_{\alpha, \beta}(x) \right|^2 \right]^{1/2} \, d\mu(x)
\lesssim \sup \left\{ \left\| \sum_{(k, \alpha, \beta) \in \mathcal{I}} \epsilon^k_{\alpha, \beta} \left\langle f, \psi^k_{\alpha, \beta} \right\rangle \psi^k_{\alpha, \beta} \right\|_{L^1(\mathcal{X})} : \left\{ \epsilon^k_{\alpha, \beta} \right\}_{(k, \alpha, \beta) \in \mathcal{I}} \subset \{-1, 1\} \right\}
\sim \sup \left\{ \|T_{\tilde{c}}(f)\|_{L^1(\mathcal{X})} : \tilde{c} \subset \{-1, 1\} \right\} \lesssim \|f\|_{H^1_{\text{at}}(\mathcal{X})},
$$

which completes the proof of Corollary 4.3. \qed

Now we establish several equivalent characterizations for $H^1_{\text{at}}(\mathcal{X})$ in terms of wavelets. To this end, we need more notation. We point out that, for any $(k, \alpha, \beta) \in \mathcal{I}$ with $\mathcal{I}$ as in Lemma 3.7, we have $\psi^k_{\alpha, \beta} \in L^\infty(\mathcal{X})$ and hence $\langle f, \psi^k_{\alpha, \beta} \rangle$ is well defined for any $f \in L^1(\mathcal{X})$ in the sense of duality between $L^1(\mathcal{X})$ and $L^\infty(\mathcal{X})$.

**Theorem 4.4.** Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type. Suppose that $f \in L^1(\mathcal{X})$ and

$$
\sum_{(k, \alpha, \beta) \in \mathcal{I}} \left\langle f, \psi^k_{\alpha, \beta} \right\rangle \psi^k_{\alpha, \beta} = \sum_{(k, \alpha, \beta) \in \mathcal{I}} \left\langle f, \psi^k_{\alpha, \beta} \psi^k_{\alpha, \beta} \right\rangle \psi^k_{\alpha, \beta} \in L^1(\mathcal{X}).
$$

Then the following statements are mutually equivalent:

(i) $f \in H^1_{\text{at}}(\mathcal{X})$;

(ii) $\sum_{(k, \alpha, \beta) \in \mathcal{I}} \left\langle f, \psi^k_{\alpha, \beta} \right\rangle \psi^k_{\alpha, \beta}$ converges unconditionally in $L^1(\mathcal{X})$;
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(iii) $\|f\|_{(iii)} := \left\{ \sum_{(k, \alpha, \beta) \in \mathcal{I}} \left| \langle f, \psi_{\alpha, \beta}^k \rangle \right|^2 \left| \psi_{\alpha, \beta}^k \right|^2 \right\}^{1/2} < \infty$;

(iv) $\|f\|_{(iv)} := \left\{ \sum_{(k, \alpha, \beta) \in \mathcal{I}} \left| \langle f, \psi_{\alpha, \beta}^k \rangle \right|^2 \frac{\lambda_{Q_k}^\alpha}{\mu(Q_k^\alpha)} \right\}^{1/2} < \infty$;

(v) $\|f\|_{(v)} := \left\{ \sum_{(k, \alpha, \beta) \in \mathcal{I}} \left| \langle f, \psi_{\alpha, \beta}^k \rangle \right|^2 \left[ R_{\alpha, \beta}^k \right]^2 \right\}^{1/2} < \infty$,

here and hereafter,

$$R_{\alpha, \beta}^k := \frac{\chi_{W_{\alpha, \beta}^k}}{\sqrt{\mu(Q_k^\alpha)}}$$

and

$$W_{\alpha, \beta}^k := B \left( y_{\beta}^k, \varepsilon_0 \delta^k \right) \subset Q_k^\alpha$$

as in Theorem 2.8.

Moreover, $\| \cdot \|_{(iii)}$, $\| \cdot \|_{(iv)}$ and $\| \cdot \|_{(v)}$ give norms on $H^1_{at}(\mathcal{X})$, which are equivalent to $\| \cdot \|_{H^1_{at}(\mathcal{X})}$, respectively.

Before we prove Theorem 4.4, we first establish several useful lemmas which are of independent interest.

In what follows, let

$$\mathcal{D} := \left\{ Q_k^\alpha \right\}_{(k, \alpha) \in \mathcal{J}}$$

be the dyadic system as in Theorem 2.3. The following notion of the dyadic maximal function is taken from [1]. Namely, for any $f \in L^1_{loc}(\mathcal{X})$, the dyadic maximal function is defined by setting

$$M^{dy}(f)(x) := \sup_{x \in Q \in \mathcal{D}} \frac{1}{\mu(Q)} \int_Q \left| f(y) \right| d\mu(y), \quad x \in \mathcal{X}.$$ 

The following lemma is on the boundedness of $M^{dy}(f)$, whose proof is completely analogous to that of [1, Theorem 3.1], the details being omitted.

**Lemma 4.5.** Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type. Then the following conclusions hold true:

(a) For any $\lambda \in (0, \infty)$ and $f \in L^1(\mathcal{X})$, there exists a disjoint family $\mathcal{F} \subset \mathcal{D}$ such that

$$\left\{ x \in \mathcal{X} : M^{dy}(f)(x) > \lambda \right\} = \bigcup_{Q \in \mathcal{F}} Q;$$

(b) the weak type $(1,1)$ inequality

$$\mu \left( \left\{ x \in \mathcal{X} : M^{dy}(f)(x) > \lambda \right\} \right) \leq \frac{1}{\lambda} \int_{\mathcal{X}} \left| f(y) \right| d\mu(y)$$

holds true for all $f \in L^1(\mathcal{X})$ and $\lambda \in (0, \infty)$;

(c) for any $p \in (1, \infty]$, there exists a positive constant $C(p)$, depending on $p$, such that, for all $f \in L^p(\mathcal{X})$,

$$\left\| M^{dy}(f) \right\|_{L^p(\mathcal{X})} \leq C(p) \| f \|_{L^p(\mathcal{X})}.$$
Remark 4.6. Let $M$ be the Hardy-Littlewood maximal function defined by setting

$$M(f)(x) := \sup_{B \ni x: B \text{ ball}} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y)$$

for all $f \in L^1_{\text{loc}}(\mathcal{X})$ and $x \in \mathcal{X}$. It follows easily from Theorem 2.3(iv) and (1.1) that there exists a positive constant $C$ such that, for all $f \in L^1_{\text{loc}}(\mathcal{X})$,

$$M^dy(f) \leq CM(f).$$

It is still unclear whether the inverse of the above inequality holds true or not; see [1] for some comparisons between the level sets of $M^dy$ and $M$.

By Lemma 4.5, the classical Lebesgue differentiation theorem associated to the dyadic cubes on $\mathbb{R}^D$ can be easily generalized to metric measure spaces of homogeneous type as follows (see, for example, the proof of [36, Theorem 6.4] on $\mathbb{R}^D$), the details being omitted.

Lemma 4.7. Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type and $f \in L^1(\mathcal{X})$. Then, for $\mu$-almost every $x \in \mathcal{X}$ and for every decreasing sequence of dyadic cubes $\{Q_j\}_{j=1}^\infty \subset \mathcal{D}$ such that $\bigcap_{j=1}^\infty Q_j = \{x\}$, it holds true that

$$\lim_{j \to \infty} \frac{1}{\mu(Q_j)} \int_{Q_j} f(y) \, d\mu(y) = f(x).$$

Now we introduce a key lemma, which is an extension of [36, Proposition 8.15] on $\mathbb{R}^D$.

Lemma 4.8. Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type. For any family of numbers, $\{a(j, \alpha, \beta)\}_{(j, \alpha, \beta) \in \mathcal{I}} \subset \mathbb{C}$ with $\mathcal{I}$ as in Lemma 3.7, let $\mathcal{S}$ be any finite subset of $\mathcal{I}$ and

$$\varphi_{\mathcal{S}}(x) := \left\{ \sum_{(j, \alpha, \beta) \in \mathcal{S}} |a(j, \alpha, \beta)|^2 \left[ R_{\alpha, \beta}^j(x) \right]^2 \right\}^{1/2}, \quad x \in \mathcal{X},$$

where $R_{\alpha, \beta}^j$ is as in (4.1). Suppose that and $\varphi_{\mathcal{S}} \in L^1(\mathcal{X})$. Then the function

$$\sum_{(j, \alpha, \beta) \in \mathcal{S}} a(j, \alpha, \beta) \psi_{\alpha, \beta}^j \in H^1_{\text{at}}(\mathcal{X})$$

and there exists a positive constant $C$, independent of $\mathcal{S}$, such that

$$\left\| \sum_{(j, \alpha, \beta) \in \mathcal{S}} a(j, \alpha, \beta) \psi_{\alpha, \beta}^j \right\|_{H^1_{\text{at}}(\mathcal{X})} \leq C \| \varphi_{\mathcal{S}} \|_{L^1(\mathcal{X})}.$$
into a sum of molecules. This will be done by partitioning the index set $S$ into sets of \( D(k, \theta) \) for any \( k \in \mathbb{Z} \), where \( \mathcal{B}_k \) denotes some index set which will be determined later, in a way such that

\[
A^k_\theta := \sum_{(j, \alpha, \beta) \in D(k, \theta)} a(j, \alpha, \beta) \psi^j_{\alpha, \beta}
\]

is an appropriate multiple of a \((1, 2, \eta)\)-molecule centered at some ball \( B \), where \( \eta \) and \( B \) will also be determined later.

For any \( k \in \mathbb{Z} \), let \( \Omega_k := \{ x \in X : \varphi_S(x) > 2^k \} \). Obviously, \( \Omega_{k+1} \subset \Omega_k \) for all \( k \in \mathbb{Z} \). Thus, by this and the facts that \( \mu(\Omega_{k+1}) \leq \| \varphi_S \|_{L^1(X)} / 2^{(k+1)} \to 0 \), as \( k \to \infty \), and \( \bigcup_{k \in \mathbb{Z}} \Omega_k = X \), we know that

\[
\sum_{k=-\infty}^{\infty} 2^k \mu(\Omega_k) = \sum_{k=-\infty}^{\infty} 2^k \sum_{j=k}^{\infty} \mu(\Omega_j \setminus \Omega_{j+1})
\]

\[
\leq \sum_{k=-\infty}^{\infty} 2^k \sum_{j=k}^{\infty} 2^{-j} \int_{\Omega_j \setminus \Omega_{j+1}} \varphi_S(x) \, d\mu(x)
\]

\[
= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{j} 2^{(k-j)} \int_{\Omega_j \setminus \Omega_{j+1}} \varphi_S(x) \, d\mu(x)
\]

\[
\sim \sum_{j=-\infty}^{\infty} \int_{\Omega_j \setminus \Omega_{j+1}} \varphi_S(x) \, d\mu(x) \sim \int_X \varphi_S(x) \, d\mu(x).
\]

For any \( k \in \mathbb{Z} \), let

\[
C_k := \left\{ (j, \alpha, \beta) \in S : \mu(\Omega_k \cap Q^j_\alpha) > \frac{1}{2C_2} \mu(Q^j_\alpha) \right\},
\]

where \( C_2 \in [1, \infty) \) is a constant, independent of \( j, \alpha \) and \( \beta \), satisfying

\[
\mu(Q^j_\alpha) \leq C_2 \mu(W^j_{\alpha, \beta})
\]

with \( W^j_{\alpha, \beta} \) defined as in (4.2) (see Remark 2.9). From the decreasing property of \( k \mapsto \Omega_k \), we deduce that \( C_k \supset C_{k+1} \) for all \( k \in \mathbb{Z} \). Define

\[
\Omega^*_k := \bigcup_{(j, \alpha, \beta) \in C_k} Q^j_\alpha.
\]

Now we choose a sequence of decreasing dyadic cubes, \( \{Q^j_{\alpha(j)}\}_{j \in \mathbb{N}} \subset D \), where \( D \) is as in (4.3) and \( \alpha(j) \in \mathcal{A}_j \) with \( \mathcal{A}_j \) as in (2.1) for all \( j \in \mathbb{N} \) such that \( \bigcap_{j=1}^{\infty} Q^j_{\alpha(j)} = \{ x \} \). Indeed, by Theorem 2.3(iii), we see that \( x \in X = \bigcup_{n \in \mathcal{A}_1} Q_\alpha \). Thus, there exists \( \alpha(1) \in \mathcal{A}_1 \) such that \( x \in Q^1_{\alpha(1)} \). Moreover, from Remark 2.4(ii), we deduce that \( x \in Q^1_{\alpha(1)} = \bigcup_{\alpha \in L(1, \alpha(1))} Q^a_\alpha \), which further implies that there exists \( \alpha(2) \in L(1, \alpha(1)) \).
such that \( x \in Q^2_{\alpha(2)} \subset Q^1_{\alpha(1)} \). Repeating this procedure, we obtain a decreasing sequence of dyadic cubes, \( \{Q^j_{\alpha(j)}\}_{j \in \mathbb{N}} \subset \mathcal{D} \), satisfying \( \bigcap_{j=1}^{\infty} Q^j_{\alpha(j)} \supset \{x\} \). On the other hand, by Theorem 2.3(iv) with \( C_1 := 4 \), we see that \( \bigcap_{j=1}^{\infty} Q^j_{\alpha(j)} \subset \bigcap_{j=1}^{\infty} B(x^j_{\alpha(j)},4\delta^j) \).

Now we claim that \( \bigcup_{j=1}^{\infty} B(x^j_{\alpha(j)},4\delta^j) = \{x\} \). Obviously, by Theorem 2.3(v), we have \( B(x^j_{\alpha(j)+1},4\delta^j+1) \subset B(x^j_{\alpha(j)},4\delta^j) \) for any \( j \in \mathbb{N} \), and \( \bigcap_{j=1}^{\infty} B(x^j_{\alpha(j)},4\delta^j) \supset \{x\} \). Conversely, if \( y \in \bigcap_{j=1}^{\infty} B(x^j_{\alpha(j)},4\delta^j) \), then
\[
d(x, y) \leq d(x, x^j_{\alpha(j)}) + d(x^j_{\alpha(j)}, y) < 8\delta^j \to 0
\]
as \( j \to \infty \). This shows that \( y = x \) and hence the above claim, which further implies that \( \bigcap_{j=1}^{\infty} Q^j_{\alpha(j)} = \{x\} \).

By Lemma 4.7, we know that, for \( \mu \)-almost every \( x \in \mathcal{X} \),
\[
\lim_{j \to \infty} \frac{1}{\mu(Q^j_{\alpha(j)})} \int_{Q^j_{\alpha(j)}} \chi_{\Omega_k}(y) \, d\mu(y) = \chi_{\Omega_k}(x).
\]
Thus, for \( \mu \)-almost every \( x \in \Omega_k \), there exists \( j_0 \in \mathbb{N} \) such that \( x \in Q^{j_0}_{\alpha(j_0)} \) and
\[
\mu \left( \Omega_k \cap Q^{j_0}_{\alpha(j_0)} \right) > \frac{1}{2C_2^2} \mu \left( Q^j_{\alpha} \right),
\]
which further implies that \( Q^{j_0}_{\alpha(j_0)} \in \mathcal{C}_k \) and \( x \in \Omega^*_k \). That is, there exists a set \( Z \) of measure zero such that
\[
\Omega_k \subset \Omega^*_k \cup Z.
\]

For a fixed \( k \in \mathbb{Z} \), let \( \{Q(k, \theta)\}_{\theta \in \tilde{\mathcal{B}}_k} := \{Q^k(\theta)\}_{\theta \in \tilde{\mathcal{B}}_k} \subset \mathcal{D} \), where \( \mathcal{D} \) is as in (4.3) and \( \tilde{\mathcal{B}}_k \) denotes some unique index set such that \( \{Q(k, \theta)\}_{\theta \in \tilde{\mathcal{B}}_k} \) is the class of all maximal dyadic cubes in \( \{Q^k_{\alpha} : (j, \alpha, \beta) \in \mathcal{C}_k \} \) and, for any \( \theta \in \tilde{\mathcal{B}}_k \), \( k(\theta) \) denotes some integer depending on \( \theta \). It is easy to see that \( \{Q(k, \theta)\}_{\theta \in \tilde{\mathcal{B}}_k} \subset \mathcal{D} \) is pairwise disjoint and
\[
\Omega^*_k = \bigcup_{\theta \in \tilde{\mathcal{B}}_k} Q(k, \theta).
\]
By this, (4.7) and (4.8), we conclude that
\[
\mu \left( \Omega^*_k \right) = \sum_{\theta \in \tilde{\mathcal{B}}_k} \mu(Q(k, \theta)) \lesssim \sum_{\theta \in \tilde{\mathcal{B}}_k} \mu(\Omega_k \cap Q(k, \theta))
\sim \mu \left( \Omega_k \cap \left( \bigcup_{\theta \in \tilde{\mathcal{B}}_k} Q(k, \theta) \right) \right) \sim \mu \left( \Omega_k \cap \Omega^*_k \right) \sim \mu(\Omega_k).
\]

Observe that, for any \( (j, \alpha, \beta) \in \mathcal{S} \), if \( a(j, \alpha, \beta) \neq 0 \), then there exists \( \tilde{k} \in \mathbb{Z} \), depending on \( j, \alpha \) and \( \beta \), such that \( |a(j, \alpha, \beta)| > 2^{\tilde{k}} |\mu(Q^j_{\alpha})|^{1/2} \). Thus, for all \( x \in W^j_{\alpha, \beta}, \varphi_S(x) > 2^{\tilde{k}}, \)}
which shows that \( W_{k, \beta}^j \subset \Omega_k \). From this, (4.2) and (4.5), it follows that
\[
\mu (Q^j_{k, \alpha} \cap \Omega_k) \geq \mu (W_{k, \beta}^j) \geq \frac{1}{C_2} \mu (Q^j_{k, \alpha}) > \frac{1}{2C_2} \mu (Q^j_{k, \alpha}) ,
\]
which shows that \( (j, \alpha) \in C^k_k \) and hence there exists \( k \in \mathbb{Z} \) such that \( (j, \alpha, \beta) \in C_k \setminus C_{k+1} \).

Let \( k \in \mathbb{Z} \), \( \theta \in \mathcal{B}_k \), \( \mathcal{E}_k := C_k \setminus C_{k+1} \) and
\[
D(k, \theta) := \{ (j, \alpha, \beta) \in \mathcal{E}_k : Q^j_{k, \alpha} \subset Q(k, \theta) \} .
\]

Now we claim that this is the desired splitting. Indeed, for any \( (j, \alpha, \beta) \in \mathcal{S} \) such that \( a(j, \alpha, \beta) \neq 0 \), by the above proof, we know that there exists \( k \in \mathbb{Z} \) such that \( (j, \alpha, \beta) \in C_k \setminus C_{k+1} =: \mathcal{E}_k \), which, together with (4.6) and (4.9), further implies that there exists \( \theta \in \mathcal{B}_k \) such that \( Q^j_{k, \alpha} \subset Q(k, \theta) \) and hence \( (j, \alpha, \beta) \in D(k, \theta) \). On the other hand, it is obvious that \( \bigcup_{k \in \mathbb{Z}, \theta \in \mathcal{B}_k} D(k, \theta) \subset \mathcal{S} \). Thus, to show the above claim, it suffices to prove that \( \{D(k, \theta)\}_{k \in \mathbb{Z}, \theta \in \mathcal{B}_k} \) are mutually disjoint. To this end, for \( k, \tilde{k} \in \mathbb{Z} \) and \( \theta, \tilde{\theta} \in \mathcal{B}_k \), if there exists \( (j, \alpha, \beta) \in D(k, \theta) \cap D(\tilde{k}, \tilde{\theta}) \), then, by the pairwise disjointness of \( \{\mathcal{E}_k\}_{k \in \mathbb{Z}} \), we know that \( k = \tilde{k} \). Moreover, from \( Q^j_{k, \alpha} \subset Q(k, \theta) \cap Q(k, \tilde{\theta}) \neq \emptyset \) and the maximality of \( Q(k, \theta) \) and \( Q(k, \tilde{\theta}) \), we deduce that \( Q(k, \theta) = Q(k, \tilde{\theta}) \) and hence \( \theta = \tilde{\theta} \), which, combined with \( k = k \), further implies that \( D(k, \theta) = D(\tilde{k}, \tilde{\theta}) \). This finishes the proof of the above claim.

As a consequence of the above claim, we have
\[
(4.11) \quad \sum_{(j, \alpha, \beta) \in \mathcal{S}} a(j, \alpha, \beta) \psi^j_{k, \alpha, \beta} = \sum_{k \in \mathbb{Z}} \sum_{\theta \in \mathcal{B}_k} A^j_{k, \theta}
\]
and, for all \( k \in \mathbb{Z} \) and \( \theta \in \mathcal{B}_k \),
\[
(4.12) \quad \| A^j_{k, \theta} \|^2_{L^2(\mathcal{X})} = \sum_{(j, \alpha, \beta) \in D(k, \theta)} |a(j, \alpha, \beta)|^2 .
\]

Let \( k \in \mathbb{Z} \) and \( \theta \in \mathcal{B}_k \). Observe that \( (j, \alpha, \beta) \in D(k, \theta) \) implies that \( Q^j_{k, \alpha} \subset Q(k, \theta) \) and \( Q^j_{k, \alpha} \notin C_{k+1} \). Thus,
\[
\mu (Q^j_{k, \alpha} \setminus \Omega_{k+1}) = \mu (Q^j_{k, \alpha}) - \mu (Q^j_{k, \alpha} \cap \Omega_{k+1}) \geq \left( 1 - \frac{1}{2C_2} \right) \mu (Q^j_{k, \alpha}) .
\]
By the finiteness of \( \mathcal{S} \) and the above claim, we easily conclude that there are only finitely many \( D(k, \theta) \neq \emptyset \). Thus, assuming that, if \( D(k, \theta) = \emptyset \), then \( A^j_{k, \theta} := 0 \), there are only finitely many \( A^j_{k, \theta} \) in (4.11) are non-zero.

Since \( [\varphi_{\mathcal{S}}(x)]^2 \geq \sum_{(j, \alpha, \beta) \in D(k, \theta)} |a(j, \alpha, \beta)|^2 |R^j_{\alpha, \beta}(x)|^2 \) for all \( x \in \mathcal{X} \), where \( R^j_{\alpha, \beta} \) is as in (4.1), we have
\[
(4.13) \quad \int_{Q(k, \theta) \setminus \Omega_{k+1}} [\varphi_{\mathcal{S}}(x)]^2 \, d\mu(x)
\]
By \( W_{\alpha,\beta}^j \subset Q^j \subset Q(k, \theta) \) [see (4.2)], we find that \( W_{\alpha,\beta}^j \cap [Q(k, \theta) \setminus \Omega_{k+1}] = W_{\alpha,\beta}^j \setminus \Omega_{k+1} \). From this, \( Q_{\alpha}^j \notin C_{k+1} \) and (4.5), it follows that

\[
\begin{align*}
\mu \left( W_{\alpha,\beta}^j \cap [Q(k, \theta) \setminus \Omega_{k+1}] \right) \\
= \mu \left( W_{\alpha,\beta}^j \setminus \Omega_{k+1} \right) = \mu \left( W_{\alpha,\beta}^j \right) - \mu \left( W_{\alpha,\beta}^j \cap \Omega_{k+1} \right) \\
\geq \mu \left( W_{\alpha,\beta}^j \right) - \mu \left( Q_{\alpha}^j \cap \Omega_{k+1} \right) \geq \frac{1}{2C_2} \mu \left( Q_{\alpha}^j \right). 
\end{align*}
\]

Moreover, combining (4.12), (4.13) and (4.14), we conclude that

\[
\begin{align*}
\left\| A_{\alpha}^k \right\|^2_{L^2(\mathcal{X})} & \lesssim \int_{Q(k, \theta) \setminus \Omega_{k+1}} \left| \varphi_S(x) \right|^2 d\mu(x) \\
& \lesssim 2^{2(k+1)} \mu \left( Q(k, \theta) \setminus \Omega_{k+1} \right) \lesssim 4^k \mu(Q(k, \theta)).
\end{align*}
\]

Thus, by (4.15) and \( Q(k, \theta) \subset B(x_{\theta}^{k(\theta)}, 4\delta k(\theta)) \) [see Theorem 2.3(iv)], we obtain

\[
\left\| A_{\alpha}^k \right\|^2_{L^2(\mathcal{X})} \lesssim \mu(Q(k, \theta)) \lesssim V \left( x_{\theta}^{k(\theta)}, 8\delta k(\theta) \right).
\]

Let \( \widetilde{A}_{\alpha}^k := A_{\alpha}^k / \lambda(k, \theta) \), where \( \lambda(k, \theta) := [V(x_{\theta}^{k(\theta)}, 8\delta k(\theta))]^{1/p - 1/2} \| A_{\alpha}^k \|^2_{L^2(\mathcal{X})} \in (0, \infty) \). Now we claim that \( \widetilde{A}_{\alpha}^k \) is a \((p, 2, \eta)\)-molecule centered at \( B(x_{\theta}^{k(\theta)}, 8\delta k(\theta)) \) multiplied by a positive constant, where \( \eta := \{ \eta_\ell \}_{\ell=1}^\infty \) and \( \eta_\ell := 2^{-\frac{\ell}{2}(M_0-1)2^\frac{n}{p}} \) for any \( \ell \in \mathbb{N} \) and a fixed large enough constant \( M_0 \) satisfying \( M_0 > 1 + n + G_0 \) with \( n \) and \( G_0 \), respectively, as in (1.1) and Remark 2.2(ii), \( \sum_{\ell=1}^\infty \ell \eta_\ell < \infty \). Indeed, obviously, we have

\[
\left\| \widetilde{A}_{\alpha}^k \right\|_{L^2(\mathcal{X})} = \left( V(x_{\theta}^{k(\theta)}, 8\delta k(\theta)) \right)^{1/2-1/p}.
\]

For any \( \ell \in \mathbb{N} \), by the Minkowski inequality and the Hölder inequality, we see that

\[
\begin{align*}
J := \left\| \widetilde{A}_{\alpha}^k \chi_{B(x_{\theta}^{k(\theta)}, 2^\ell 8\delta k(\theta)) \setminus B(x_{\theta}^{k(\theta)}, 2^\ell-1 8\delta k(\theta))} \right\|_{L^2(\mathcal{X})} \\
\leq \frac{1}{\lambda(k, \theta)} \sum_{(j, \alpha, \beta) \in D(k, \theta)} \left| a(j, \alpha, \beta) \right| \left\| \varepsilon_{j, \alpha, \beta} \chi_{B(x_{\theta}^{k(\theta)}, 2^\ell 8\delta k(\theta)) \setminus B(x_{\theta}^{k(\theta)}, 2^\ell-1 8\delta k(\theta))} \right\|_{L^2(\mathcal{X})} \\
\lesssim \frac{1}{\lambda(k, \theta)} \left[ \sum_{(j, \alpha, \beta) \in D(k, \theta)} \left| a(j, \alpha, \beta) \right|^2 \right]^{1/2}.
\end{align*}
\]
\[
\times \left[ \sum_{(j, \alpha, \beta) \in D(k, \theta)} \left\| \psi_j^\alpha \beta^j \chi_{B(x_\theta^k, 2^j \delta^k)} \backslash B(x_\theta^k, 2^{j-1} \delta^k) \right\|_{L^2(\mathcal{X})}^2 \right]^{1/2}.
\]

Moreover, for any \((j, \alpha, \beta) \in D(k, \theta)\), by Theorem 2.3(iv) and (2.5), we have
\[
x_\beta^{j+1} \in Q_{\beta}^{j+1} \subset Q_{\beta}^j \subset Q(k, \theta) \subset B\left(x_\theta^k, 4\delta^k\right)
\]
and hence \(d(x_\beta^{j+1}, x_\theta^k) < 4\delta^k\). From this, we deduce that, for any \((j, \alpha, \beta) \in D(k, \theta)\) and \(x \in B(x_\theta^k, 2^\ell \delta^k) \backslash B(x_\theta^k, 2^{\ell-1} \delta^k)\),
\[
d\left(x_\beta^{j+1}, x\right) \geq d\left(x, x_\theta^k\right) - d\left(x_\theta^k, x_\beta^{j+1}\right) > 2^{\ell+2} \delta^k - 4\delta^k \geq 2^{\ell+1} \delta^k,
\]
which, together with (2.13), (1.1) and \(k(\theta) \geq j\), further implies that
\[
\left\| \psi_j^\alpha \beta^j \chi_{B(x_\theta^k, 2^j \delta^k)} \backslash B(x_\theta^k, 2^{j-1} \delta^k) \right\|_{L^2(\mathcal{X})}^2 \leq \int_{B(x_\theta^k, 2^\ell \delta^k) \backslash B(x_\theta^k, 2^{\ell-1} \delta^k)} \frac{1}{V(x_\beta^{j+1}, \delta^j)} e^{-2^\ell \nu \delta^k(\theta) - j} V(x_\beta^{j+1}, 8\delta^k(\theta)) \, d\mu(x)
\]
\[
\leq e^{-2^\ell \nu \delta^k(\theta) - j} V(x_\beta^{j+1}, 8\delta^k(\theta)) \leq e^{-2^\ell \nu \delta^k(\theta) - j} 2^{n_\ell} V(x_\beta^{j+1}, 8\delta^k(\theta))
\]
\[
\leq e^{-2^\ell \nu \delta^k(\theta) - j} 2^{n_\ell} \delta^k(\theta) - j \}
\]

By this, \(D(k, \theta) \subset \{(j, \alpha, \beta) \in S : j \geq k(\theta), d(x_\alpha^j, x_\theta^k) < 4\delta^k\}\), (i) and (iii) of Remark 2.2 and \(M_0 > G_0 + n + 1\), we conclude that
\[
\sum_{(j, \alpha, \beta) \in D(k, \theta)} \left\| \psi_j^\alpha \beta^j \chi_{B(x_\theta^k, 2^j \delta^k)} \backslash B(x_\theta^k, 2^{j-1} \delta^k) \right\|_{L^2(\mathcal{X})}^2 \leq \sum_{j=k(\theta)}^\infty \sum_{\alpha \in \xi_j : d(x_\alpha^j, x_\theta^k) < 4\delta^k}\{j \geq k(\theta)\}
\]
\[
\leq \sum_{j=k(\theta)}^\infty \sum_{\alpha \in \xi_j : d(x_\alpha^j, x_\theta^k) < 4\delta^k}\{j \geq k(\theta)\} 2^{-M_0 \ell} \delta^k(\theta) - j \}
\]
\[
\leq 2^{-M_0 \ell} 2^{n_\ell} \sum_{j=k(\theta)}^\infty \delta^k(\theta) - j \}
\]
which further implies that
\[
(4.17) \quad J \lesssim \eta 2^{-\ell/2} \left\| A_{k, \theta}^+ \right\|_{L^2(\mathcal{X})} \sim \eta 2^{-\ell/2} \left[ V\left(x_\theta^k, 8\delta^k(\theta)\right) \right]^{1/2-1/p}.
\]
By (2.15) and the finiteness of $\mathcal{S}$, we obtain

$$\int_X \tilde{A}_0^k(x) \, d\mu(x) = 0.$$ 

From this, (4.16) and (4.17), together with $\sum_{\ell=1}^{\infty} \ell \eta_{\ell} < \infty$, we deduce that the above claim holds true.

By the above claim, (4.11),

$$\sum_{(j, \alpha, \beta) \in \mathcal{S}} a(j, \alpha, \beta) \psi_{\alpha, \beta}^j = \sum_{k \in \mathbb{Z}} \sum_{\theta \in \mathcal{B}_k} \lambda(k, \theta) \tilde{A}_0^k$$

with only finitely many $\lambda(k, \theta) \tilde{A}_0^k \neq 0$, and Theorem 3.4, we conclude that

$$\sum_{(j, \alpha, \beta) \in \mathcal{S}} a(j, \alpha, \beta) \psi_{\alpha, \beta}^j \in H^1_{\text{at}}(X).$$

Moreover, by (4.15), $Q(k, \theta) = Q^k_{\theta}(\theta)$, together with Theorem 2.3(iv), (1.1), disjoint property of $\{Q(k, \theta)\}_{\theta \in \mathcal{B}_k}$, (4.10) and (4.4), we conclude that

$$\sum_{k \in \mathbb{Z}} \sum_{\theta \in \mathcal{B}_k} \lambda(k, \theta) \tilde{A}_0^k \lesssim \mu(Q(k, \theta)) \lesssim \sum_{k \in \mathbb{Z}} 2^k \mu(\Omega_k^*),$$

$$\lesssim \sum_{k \in \mathbb{Z}} 2^k \mu(\Omega_k) \sim \int_X \varphi \psi_1 \, d\mu(x) < \infty.$$

Thus, from this, (4.18) and Theorem 3.4, it follows that

$$\left\| \sum_{(j, \alpha, \beta) \in \mathcal{S}} a(j, \alpha, \beta) \psi_{\alpha, \beta}^j \right\|_{H^1_{\text{at}}(X)} \lesssim \sum_{k \in \mathbb{Z}} \sum_{\theta \in \mathcal{B}_k} \lambda(k, \theta) \lesssim \|\varphi\psi\|_{L^1(\mathcal{X})},$$

which completes the proof of Lemma 4.8.

Now we are ready to prove Theorem 4.4.

**Proof of Theorem 4.4.** Let $f \in L^1(\mathcal{X})$ and

$$f = \sum_{(k, \alpha, \beta) \in \mathcal{F}} \langle f, \psi_{\alpha, \beta}^k \rangle \psi_{\alpha, \beta}^k \quad \text{in} \quad L^1(\mathcal{X}).$$

From $H^1_{\text{at}}(\mathcal{X}) \subset L^1(\mathcal{X})$ and Theorem 3.10, we deduce that (i) implies (ii).
By Lemma 4.2, we know that (ii) implies (iii).

Now we prove “(iii) ⇒ (i)”. Indeed, let \( \mathcal{N} \) be any sequence of finite subsets of \( \mathcal{I} \) as in (3.22) and

\[
S_N(f) := \sum_{(k, \alpha, \beta) \in \mathcal{N}} \langle f, \psi_{\alpha, \beta}^k \rangle \psi_{\alpha, \beta}^k, \quad N \in \mathbb{N}.
\]

For any \( N, M \in \mathbb{N} \) with \( M < N \), by Theorem 2.8, we have

\[
\sum_{(k, \alpha, \beta) \in \mathcal{N} \setminus \mathcal{I}_M} \left| \langle f, \psi_{\alpha, \beta}^k \rangle \right|^2 \left[ R_{\alpha, \beta}^k \right]^2 \leq \sum_{(k, \alpha, \beta) \in \mathcal{N} \setminus \mathcal{I}_M} \left| \langle f, \psi_{\alpha, \beta}^k \rangle \right|^2 \left| R_{\alpha, \beta}^k \right|^2 \leq \sum_{(k, \alpha, \beta) \in \mathcal{I}} \left| \langle f, \psi_{\alpha, \beta}^k \rangle \right|^2 \left[ R_{\alpha, \beta}^k \right]^2 \in L^1(\mathcal{X}),
\]

which, together with Lemma 4.8, further implies that

\[
\| S_N(f) - S_M(f) \|_{H^1_{at}(\mathcal{X})} \lesssim \sum_{(k, \alpha, \beta) \in \mathcal{N} \setminus \mathcal{I}_M} \left| \langle f, \psi_{\alpha, \beta}^k \rangle \right|^2 \left[ R_{\alpha, \beta}^k \right]^2 \to 0,
\]
as \( N, M \to \infty \).

Thus, \( \{S_N(f)\}_{N \in \mathbb{N}} \) is a Cauchy sequence in \( H^1_{at}(\mathcal{X}) \) and hence, by Remark 3.2, there exists \( g \in H^1_{at}(\mathcal{X}) \) such that

\[
g = \lim_{N \to \infty} S_N(f) \quad \text{in} \quad H^1_{at}(\mathcal{X}).
\]

From this, the fact that \( H^1_{at}(\mathcal{X}) \subset L^1(\mathcal{X}) \) and (4.19), we deduce that

\[
g = \lim_{N \to \infty} S_N(f) = f \quad \text{in} \quad L^1(\mathcal{X}),
\]

which, combined with \( g \in H^1_{at}(\mathcal{X}) \), further implies that \( f \in H^1_{at}(\mathcal{X}) \). This finishes the proof of “(iii) ⇒ (i)” and hence (i), (ii) and (iii) are mutually equivalent.

“(iii) ⇒ (v)” follows from Theorem 2.8.

“(v) ⇒ (i)” is an implicit consequence of the proof of “(iii) ⇒ (i)”. Thus, (i), (ii), (iii) and (v) are mutually equivalent.

“(iv) ⇒ (v)” is obvious by (4.2).

To show “(v) ⇒ (iv)”, we first claim that, for all \( s \in (0, \infty) \) and \( (k, \alpha, \beta) \in \mathcal{I} \),

\[
\chi_{Q^k_{\alpha, \beta}} \lesssim \left[ M \left( \chi_{W^k_{\alpha, \beta}} \right) \right]^{1/s}.
\]
Indeed, by Remark 2.9, Theorem 2.3(iv) and (1.1), we know that, for all \( x \in Q^k_a \subset B(x^k_{\alpha}, 4\delta^k) \),

\[
1 \sim \left[ \frac{\mu(W^k_{\alpha,\beta})}{\mu(Q^k_a)} \right]^{1/s} \sim \left\{ \frac{1}{\mu(Q^k_a)} \int_{Q^k_a} \left[ \chi W^k_{\alpha,\beta}(y) \right]^s d\mu(y) \right\}^{1/s} 
\]

\[
\lesssim \left\{ \frac{1}{\mu(B(x^k_{\alpha}, (1/3)\delta^k))} \int_{B(x^k_{\alpha}, 4\delta^k)} \left[ \chi W^k_{\alpha,\beta}(y) \right]^s d\mu(y) \right\}^{1/s} \lesssim \left[ M \left( \chi W^k_{\alpha,\beta} \right)^s (x) \right]^{1/s},
\]

which shows the above claim.

Moreover, by (4.21), with \( s := 2/r \) and \( r \in (0, 1) \), and the Fefferman-Stein vector-valued maximal function inequality (see, for example, [13, Theorem 1.2]), we obtain

\[
\left\| \left\{ \sum_{(k,\alpha,\beta)\in J} \left| \left\langle f, \psi^k_{\alpha,\beta} \right\rangle \right|^2 \left[ \mu(Q^k_a) \right]^{-1} \chi Q^k_a \right\} \right\|_{L^1(X)}^{1/2} 
\]

\[
\lesssim \int_X \left\{ \sum_{(k,\alpha,\beta)\in J} \frac{\left| \left\langle f, \psi^k_{\alpha,\beta} \right\rangle \right|^2}{\mu(Q^k_a)} \left[ M \left( \chi W^k_{\alpha,\beta} \right)^{r/2} (x) \right]^{2/r} \right\}^{1/2} d\mu(x) 
\]

\[
\lesssim \left\{ \sum_{(k,\alpha,\beta)\in J} \left[ \left| \left\langle f, \psi^k_{\alpha,\beta} \right\rangle \right|^2 \mu(Q^k_a)^{1/2} \chi W^k_{\alpha,\beta} \right] \right\}^{r/2} \left[ R^{k,\alpha,\beta}_1 \right]^2 \left\| \left\{ \sum_{(k,\alpha,\beta)\in J} \left| \left\langle f, \psi^k_{\alpha,\beta} \right\rangle \right|^2 \chi Q^k_a \right\} \right\|_{L^1(X)}^{1/2},
\]

which shows “(v) \( \iff \) (iv)”.

Thus, (i) through (v) are mutually equivalent.

Finally, we show that \( \| \cdot \|_{(iii)} \), \( \| \cdot \|_{(iv)} \) and \( \| \cdot \|_{(v)} \) give norms on \( H^1_{at}(X) \), which are equivalent to \( \| \cdot \|_{H^1_{at}(X)} \), respectively. Indeed, by (4.20), Corollary 4.3, Theorem 3.10 and Lemma 4.8 we conclude that, for all \( f \in H^1_{at}(X) \),

\[
\| f \|_{(v)} \lesssim \| f \|_{(iii)} \sim \left\{ \sum_{(k,\alpha,\beta)\in J} \left| \left\langle f, \psi^k_{\alpha,\beta} \right\rangle \right|^2 \left| \psi^k_{\alpha,\beta}(x) \right|^2 \right\} \left\| \left\{ \sum_{(k,\alpha,\beta)\in J} \left| \left\langle f, \psi^k_{\alpha,\beta} \right\rangle \right|^2 \right\} \right\|_{L^1(X)}^{1/2} 
\]

\[
\lesssim \| f \|_{H^1_{at}(X)} \sim \lim_{N \to \infty} \left\| \sum_{(k,\alpha,\beta)\in J_N} \left\langle f, \psi^k_{\alpha,\beta} \right\rangle \psi^k_{\alpha,\beta}(x) \right\|_{H^1_{at}(X)} 
\]

\[
\lesssim \lim_{N \to \infty} \| \varphi_{f_N} \|_{L^1(X)} \lesssim \| f \|_{(v)}.
\]

Thus, \( \| \cdot \|_{(iv)} \sim \| \cdot \|_{H^1_{at}(X)} \sim \| \cdot \|_{(iii)} \). Moreover, by the proof of “(iv) \( \iff \) (v)”, we see that \( \| \cdot \|_{(iv)} \sim \| \cdot \|_{(v)} \), which implies the desired conclusion and hence completes the proof of Theorem 4.4.
Remark 4.9. By arguments essentially the same as those used in the case of $d$, we conclude that all the results obtained in this article remain valid with the metric $d$ replaced by a quasi-metric $\rho$, since most of the tools we need are from [2, 3], which were established in the context of spaces of homogeneous type. Some minor modifications are needed when dealing with the inclusion relations between two balls, where the quasi-triangle constant is involved, which only alter the corresponding results by additive positive constants via (1.1).

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