Finite Volume Method with Explicit Scheme Technique for Solving Heat Equation

F Y Saptaningtyas¹, A D Setyarsi²

¹Mathematics Education Departement, Faculty of Mathematic and Natural Sciences, Yogyakarta State University
²Graduate Program of Mathematics Education Departement, Faculty of Mathematic and Natural Sciences, Yogyakarta State University

E-mail: fitrianatya@uny.ac.id

Abstract. The Finite Volume Method is one of the numerical procedures that uses integrals on each volume control. It can be used to solve complex problems. This research aims to implement the Finite Volume method with explicit scheme for solving the heat equations. The research methods are modelling of heat equations, integration on each control volume, discretization using explicit scheme, solving linear algebra system, simulation, and comparing with analytical solutions. The separation variable method is used to compare the results. The results showed that the finite volume method provides a good solution with an average error of 0.0016. This showed that both solutions are similar.

1. Introduction

The heat equation is one of the diffusion problems associated with particle propagation. The heat equation is one of the clumps of parabolic partial differential Equations which have many applications. [1] Investigates the flow field and turbulent flow heat transfer around an array of plain and perforated fin using Fluent software. While [2] analyze heat transfer of an incompressible viscous water-based nano fluid over a stretching/shrinking sheet both analytically and numerically. To get numerical solutions of diffusion equation, there are many numerical methods available in the literatures [3,4,5,6, 13], for examples finite element, finite fourier, finite difference, and finite volume methods. Finite difference methods are based on the differential form of the equation. Some drawbacks in the finite different methods are low accuracy solution and non-continuous issues. In comparison, finite volume methods are based on the integral form. Integral form does not assume continuous of their solutions, and hence finite volume methods are able to resolve continuous and non-continuous solutions [7,8,9,10,11].

The finite volume method is one of the approximation methods that can produce a good solution to the diffusion problem [12]. However, the accuracy of numerical methods will be debased on the integration with respect to both time and space. In this paper we investigate the application of finite volume methods when solving the heat Equation. We compare the performance of the solutions with analytical solution which are solved by separation method.

This paper is organized as follows. The modelling of heat Equation is recalled in Section 2. We present the finite volume method that we use to solve the heat Equation in Section 3. The application on
heat propagation in metal rods are presented in Section 4. We obtain some concluding remarks in Section 5.

2. Modelling of Heat Equation

Given a metal rod with length $l$ lies along the $x$-axis as shown in Figure 1. Metal rods are partitioned into small sections and selected one small part which will represent as volume control. Assume some things as follows. The cross section of the metal bar $(A)$ is constant, the amount of heat in all parts of $(A)$ is constant, metal rods are made of a homogeneous material, The metal rod is perfectly insulated throughout its surface, The heat flow travels from high temperatures to lower temperatures, Heat density and thermal conduction are constant.

![Figure 1. Metal Rod with Heat Energy Flowing in the x-axis](image)

Next, we will review the metal bar partition by $\Delta x$. Given $Q(t)$ is the total heat energy and $e(x,t)$ is the amount of heat energy per unit volume, here in after referred to as the heat density mass. When the heat density mass is constant throughout the volume of the metal rod, the amount of heat energy in $\Delta x$ is the result of the heat density mass and volume. It can be formulated as follows.

$$e(x,t) = \frac{Q(t)}{V}$$

With $V = A\Delta x$, so we get

$$Q(t) = e(x,t)A\Delta x$$

The heat changes occur at the interval $[x, x+\Delta x]$ when there are heat flow along the point $x$ to $[x+\Delta x]$ . Under the Heat Conservation Law, the basis of the heat flow process is the rate of change of heat equal to the heat energy that flows per unit of time plus the heat energy produced from within the metal rods per unit time. Since the metal rods are homogeneous and isolated throughout the surface there is no heat generated from inside the metal rod. obtained the formulation of the rate of heat change as follows.

$$\frac{\partial}{\partial t}(e(x,t)A\Delta x)$$

In Figure 1 there is a temperature difference between the two ends of the metal rod, i.e. $W(x,t)$ and $W(x+\Delta x,t)$. The heat energy that travels on the piece of metal per unit time is as follows.

$$w = W(x,t)A - W(x+\Delta x,t)A$$

According to Fick’s Law about the rate of diffusion, which states that the flux is directly proportional to the rate of heat change. Obtained the following formula:

$$\frac{\partial}{\partial t}(e(x,t)) = \frac{w}{A\Delta x}$$

Since $\Delta x$ is very small, the limit value is close to zero. So, Eq. (5) becomes.
\[
\frac{\partial}{\partial t}(e(x,t)) = \lim_{\Delta x \to 0} \frac{w}{A}\Delta x \\
\frac{\partial}{\partial t}(\rho e)(x,t) = \frac{1}{A} \frac{\partial w}{\partial x}
\]  
\hspace{1cm} (6)

Given \( c \) is heat density mass that must be supplied to a unit of mass of a substance to raise the temperature of one unit. Since it has been assumed that the metal rods are made of a homogeneous material then \( c \) is constant, so that the heat energy per unit mass is given by \( cW(x,y) \). Then given \( \rho \) which is the mass density of mass per unit volume, since the metal rod is homogeneous then the total mass on the metal piece is \( \frac{m}{V} \). Total heat energy on a piece of metal can be written as.

\[
Q = mc\Delta W
\]  
\hspace{1cm} (7)

Since \( \rho = \frac{m}{V} \) and \( V = A\Delta x \), so Eq. (7) can be written to.

\[
Q = \rho A\Delta xc
\]  
\hspace{1cm} (8)

If Eq. (7) and Eq. (8) are simplified, the following results are obtained.

\[
e(x,t) = \rho cW(x,t)
\]  
\hspace{1cm} (9)

If Eq. (9) is substituted in Eq. (8) the result is obtained.

\[
\frac{\partial}{\partial t}(\rho cW(x,t)) = \frac{1}{A} \frac{\partial w}{\partial x}
\]  
\hspace{1cm} (10)

According to Fourier’s Law, the rate of heat propagation passing through the surface of the plane is directly proportional to the temperature change passing through the metal strip and the thickness of the wall. It can be written as follows.

\[
w = -KA\Delta W(x,t)
\]  
\hspace{1cm} (11)

In Eq. (11), \( K \) is the thermal conductivity. With the approach \( \Delta x \to 0 \), then Eq. (11) changes to.

\[
W = -KA\lim_{\Delta x \to 0} \frac{\Delta W(x,t)}{\Delta x}
\]  
\hspace{1cm} (12)

If Eq. (10) is substituted in Eq. (12) it is obtained.

\[
\rho c \frac{\partial W(x,t)}{\partial t} = \frac{1}{A} \frac{\partial}{\partial x} \left( -KA \frac{\partial W(x,t)}{\partial x} \right)
\]  
\hspace{1cm} (13)

Let \( k^2 = \frac{K}{\rho c} \), so Eq. (13) can be written to

\[
\frac{\partial W(x,t)}{\partial t} = k^2 \frac{\partial^2 W(x,t)}{\partial x^2}
\]  
\hspace{1cm} (14)

Eq. (14) is called the one-dimension Heat Equation.
3. Methods in Solving Using Finite Volume Method

The finite volume method is one of the numerical methods for solving partial differential equation on physical problems.

![Finite Volume Flow Chart](image)

According to Figure 2. We build mathematical equations from real problems and physics problems. Next We discretized the mathematical equations and also the volume control objects. In this study explicit methods were used. From these results, an algebraic equation system is obtained. Determined the solutions of the system using the appropriate method. The sistem solutions give numerical solutions.

### 4. Results and Discussion

In this section we present the numerical solutions of heat Equation in a homogeneous rod. Given a wax and a homogeneous metal rod with a length of 0.1 m. The wax is placed under the metal rod in the left position, after which the candle is turned on some time and then turned off. In this case, the temperature changes at both $x = 0$ and $x = 0.1$ are maintained at zero degrees. heat only flows from high temperature to lower temperature. A solution of a one dimensional heat Equation in a metal rod shall be determined using the finite volume method. Given the one-dimensional heat Equation as follows.

$$\frac{\partial W(x,t)}{\partial t} = k \left( \frac{\partial^2 W(x,t)}{\partial x^2} \right), 0 \leq x \leq 0.1, t > 0$$ (15)

The initial values are
\[ W(x,0) = 50 ; 0 \leq x \leq 0.1 \]  

The boundary values are,  
\[ W_i(0,t) = 0, \quad t > 0 \]  
\[ W(0,1,t) = 0, \quad t > 0 \]  

Metal rods stretched out along the length \( x \), partitioned for \( \Delta x \) and will be selected partitions at intervals \([x_i , x_i + \Delta x]\), \(i = 0,1,2...n\). Here in after referred to as volume control. Assume \( \Delta t \) is the heat propagation time from \( x_i \) to \( x_i + \Delta x \). If Eq. (15) is integral to \( x \) with the interval \([x_i , x_i + \Delta x]\), so Eq. (15) becomes.

\[ \int_{x_i}^{x_i + \Delta x} \rho c \frac{\partial W}{\partial t} \, dx = \int_{x_i}^{x_i + \Delta x} \frac{\partial^2 W}{\partial x^2} \, dx \]  

If Eq. (19) is integral to \( t \) by the interval \([t,t+\Delta t]\), thus Eq. (19) becomes

\[ \int_{t}^{t+\Delta t} \left( \int_{x_i}^{x_i + \Delta x} \rho c \frac{\partial W(x,t)}{\partial t} \, dx \right) dt = \int_{t}^{t+\Delta t} \left( \int_{x_i}^{x_i + \Delta x} K \frac{\partial^2 W(x,t)}{\partial x^2} \, dx \right) dt \]  

If it is assumed that the temperature at point \( i \) is the temperature at all volume control \( i \). Then the left side of Eq. (20) can be accomplished as follows.

\[ \int_{x_i}^{x_i + \Delta x} \rho c \frac{\partial W}{\partial t} \, dx = \int_{x_i}^{x_i + \Delta x} \rho c W(x,t) \, dx = \int_{x_i}^{x_i + \Delta x} \rho c W(x,t+\Delta t) - \rho c W(x,t) \, dx \]  

Eq. (21) is integrated over time, the following results are obtained.

\[ \int_{t}^{t+\Delta t} \int_{x_i}^{x_i + \Delta x} \rho c \frac{\partial W}{\partial t} \, dx \, dt = \int_{t}^{t+\Delta t} \int_{x_i}^{x_i + \Delta x} \rho c W(x,t) \, dx \, dt = \int_{t}^{t+\Delta t} \rho c W(x,t+\Delta t) - \rho c W(x,t) \, dx \, dt \]  

The integration process continues by integrating Eq. (22) over the volume control and the following results are obtained.

\[ \int_{x_i}^{x_i + \Delta x} \rho c (W_{i+\Delta t} - W_i) \, dx = \rho c (W_{i+\Delta t} - W_i)(x_i + \Delta x - x_i) = \rho c (W_{i+\Delta t} - W_i) \Delta x = \rho c (W_{i+\Delta t} - W_i^0) \Delta x \]  

Eq. (23), \( W_i \) is the temperature at \( i \) at time \( t+\Delta t \) and \( W_i^0 \) is the temperature at \( i \) at time \( t \). After obtaining the integral result from the left side of Eq. (23), then the integral result of the right-hand side of Eq. (20) as follows.

\[ \int_{t}^{t+\Delta t} \int_{x_i}^{x_i + \Delta x} K \frac{\partial^2 W}{\partial x^2} \, dx \, dt = \int_{t}^{t+\Delta t} \int_{x_i}^{x_i + \Delta x} \frac{\partial^2 W}{\partial x^2} \, dx \, dt = \int_{t}^{t+\Delta t} K \left( \frac{\partial W}{\partial x} \right)_{x_i}^{x_i + \Delta x} \, dt \]

\[ = \int_{t}^{t+\Delta t} K \left( \frac{\partial W}{\partial x} \right)_{x_i}^{x_i + \Delta x} \, dt = \int_{t}^{t+\Delta t} K \left( \frac{\partial W}{\partial x} \right)_{x_i}^{x_i + \Delta x} + \frac{\partial W}{\partial x} \right)_{x_i}^{x_i + \Delta x} \, dt \]  

The average integral theorem is used to derive results from \( \frac{\partial W}{\partial x} \) and \( \frac{\partial W}{\partial x} \), so obtained the following results. \( \frac{\partial W}{\partial x} \) and \( \frac{\partial W}{\partial x} \), with \( W_{i-1} \) being the temperature at \( i+1 \) at time \( t \), \( W_i \) represents the temperature at \( i \) at time \( t \) and \( W_{i-1} \) represents the temperature at \( i-1 \) at time \( t \). So the integral result as follows.
\[ \int_{i}^{i+\Delta t} \int_{j}^{j+\Delta x} K \frac{\partial^2 W}{\partial x^2} \partial x \partial t = \int_{i}^{i+\Delta t} K \left( \frac{W_{i+1} - W_{i}}{\Delta x} - \frac{W_{i} - W_{i-1}}{\Delta x} \right) \partial t \]  

(25)

If Eq. (24) and Eq. (25) are substituted in Eq. (19) then the following results are obtained.

\[ \rho c (W_i - W_i^0) \Delta x = \int_{i}^{i+\Delta t} K \left( \frac{W_{i+1} - W_{i}}{\Delta x} - \frac{W_{i} - W_{i-1}}{\Delta x} \right) \partial t \]  

(26)

The two sections of Eq. (26) when divided by \( \Delta t \) will be obtained as follows.

\[ \frac{\rho c (W_i - W_i^0) \Delta x}{\Delta t} = \int_{i}^{i+\Delta t} K \left( \frac{W_{i+1} - W_{i}}{\Delta x} - \frac{W_{i} - W_{i-1}}{\Delta x} \right) \partial x \]  

(27)

To obtain an integral result over the time in the right-hand side of Eq. (27), an assumption is given for \( W_i, W_i^0 \). To calculate the integrals of time on the right side of Eq. (27), we may use temperature at time \( t \) or temperature at \( t + \Delta t \), or it may be using a combination of temperatures at time \( t \) and \( t + \Delta t \). Next is approximation using parameter \( \theta \) which \( 0 \leq \theta \leq 1 \). So we get the assumption of temperature integral with time as follows.

\[ I_T = \int_{i}^{i+\Delta t} [\partial W_i + (1 - \theta) W_i^0] \Delta t \]  

(28)

with,

\[
\begin{array}{c|c|c|c}
\theta & 0 & \frac{1}{2} & 1 \\
\hline
I_T & W_i^0 \Delta t & \frac{1}{2} (W_i + W_i^0) \Delta t & W_i \Delta t \\
\end{array}
\]

By applying Eq. (28) the following results are obtained.

\[ \frac{\rho c (W_i - W_i^0) \Delta x}{\Delta t} = \left( \theta \left( \frac{K W_{i+1} - W_i}{\Delta x} - \frac{K W_i - W_{i-1}}{\Delta x} \right) \right) + (1 - \theta) \left( \frac{K W_i^0 - W_i^0}{\Delta x} - \frac{K W_i^0 - W_{i-1}^0}{\Delta x} \right) \]  

(29)

In this case the heat Eq. of one dimension corresponds to Eq. (15), with the initial value \( W(x, 0) = 50 \). The boundary conditions of this case are \( W_x(0, t) = 0 \) and \( W(0.1, t) = 0 \). It is also known that \( \rho c = 10 \times 10^6 \) and \( K = 10 \). Using the Explicit Method for discretization techniques, the value for \( \theta \) is \( \theta = 0 \) to obtain the results of the general Equation of the solution of the case as follows.

\[ \frac{\rho c (W_i - W_i^0) \Delta x}{\Delta t} = K \frac{W_{i+1}^0 - W_i^0}{\Delta x} - K \frac{W_i^0 - W_{i-1}^0}{\Delta x} \]  

(30)

To determine the time step on an explicit method must satisfy

\[ \Delta t < \frac{\rho c (\Delta x)^2}{2k} \]  

(31)

From Inequality (31) we get the limit for time step as follows.

\[ \Delta t < 50s \]  

(32)

Having obtained the limit for time step, then for this case will be taken time step by \( \Delta t = 2s \). Next, we will calculate the value of the constant in Eq. (32) to facilitate further calculations as follows

\[ \frac{\rho c \Delta x}{\Delta t} = 10 \times 10^6 \times \frac{0.01}{2} = 50000 \]

\[ \frac{K \Delta x}{\Delta t} = \frac{10}{0.01} = 1000 \]  

(33)
The following will be shown numerical solutions of metal rod heating using finite volume method with explicit discretization methods technique at $t = 0$ to $t = 120$.

Figure 3. Finite Volume Solution of Heat Eq.

From Figure 3, it can be seen that the temperature at any $x$ when $t = 0$ is 50, it corresponds to the initial value applied to the case. At time $t > 0$ the temperature starts to decrease gradually until it reaches 0 at the point $x = 0.1$. It also conforms to the boundary conditions of $W(0.1, t) = 0$ with $t > 0$.

Next, an analytical solution will be determined by using the variable separation method. Taken substitution $W(x, t) = X(x)T(t)$ on heat Equation, is obtained.

$$\frac{\partial W(x, t)}{\partial t} = X(x)T'(t)$$ (34)

$$k^2 \left( \frac{\partial^2 W(x, t)}{\partial x^2} \right) = k^2 X''(x)T(t)$$ (35)

If Eq. (34) and Eq. (35) are substituted in Eq. (15) then obtained.

$$X(x)T'(t) = k^2 \left( X''(x)T(t) \right)$$ (36)

Separation variable will be done, where the Equation containing the variable $x$ is grouped on the right and the Equation containing the variable $t$ will be grouped on the left side.

$$T'(t) = \frac{X''(x)}{k^2 T(t)}$$ (37)

determined the real separation constant is negative $\lambda$, so Eq. (31) becomes

$$\frac{T'(t)}{k^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$ (38)

from Eq. (32) obtained the following Sturm-Liouville problem

$$\frac{T'(t)}{k^2 T(t)} = -\lambda$$ (39)
\[
\frac{X\prime(x)}{X(x)} = -\lambda
\] (40)

Each Equation is solved so that the analytical solution is obtained.

\[
W(x,t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \cos\left(\frac{2n-1}{0.2} \pi x\right) e^{-\left(\frac{2n-1}{0.2} \pi \xi\right)^2}
\] (41)

For \(W(x,t)\) is the temperature at \(x\) at time \(t\). From Eq. (41), the following sample of heat propagation will be taken at \(t=0, 40, 80, 120, 160, 200\). The result of temperature at \(t\) which has been determined can be seen in Table 1.

**Table 1. Analytical Solution of Heat Equation**

| Point | Time       |
|-------|------------|
|       | \(t = 0\) | \(t = 40\) | \(t = 80\) | \(t = 120\) | \(t = 160\) | \(t = 200\) |
| 0     | 0          | 48.4142    | 49.9719    | 49.9996    | 50         | 50         | 49.9999    |
| 1     | 0.005      | 50.0062    | 50.0010    | 50.0000    | 50.0000    | 49.9999    |
| 2     | 0.015      | 49.9806    | 49.9970    | 49.9999    | 50.0000    | 49.9999    |
| 3     | 0.025      | 50.0352    | 50.0053    | 50.0001    | 49.9999    | 49.9986    |
| 4     | 0.035      | 49.9438    | 49.9918    | 49.9998    | 49.9986    |
| 5     | 0.045      | 50.0874    | 50.0122    | 49.9996    | 49.9808    |
| 6     | 0.055      | 49.8609    | 49.9821    | 49.9810    |
| 7     | 0.065      | 50.2372    | 50.0216    | 49.7175    | 48.8066    |
| 8     | 0.075      | 49.5376    | 49.7030    | 47.5942    | 44.6708    |
| 9     | 0.085      | 51.1826    | 45.3741    | 38.2166    |
| 10    | 0.095      | 43.646     | 21.1321    |
| 11    | 0.1        | 50         | 0          | 0          |

Next, it will be shown analytical solutions in the graph. If the analytical solutions is plotted in graphical form, then the result is as follows.
Figure 4. Analytical Solution of Heat Equations

From Figure 4 it can be seen that the temperature at any $x$ when $t = 0$ to around $t = 50$, it corresponds to the initial value applied to the case. At time $> 0$ the temperature starts to decrease gradually until it reaches 0 at the point $x = 0.1$. It also conforms to the boundary conditions of $W(0.1,t) = 0$ with $t > 0$. Next, the error will be analyzed by comparing with analytical solutions. Relative error was associated with how close the solution with the real solution. Relative error ($e_R$) can be obtained with the following formula:

$$e_R = \frac{|a - \hat{a}|}{a}$$

(42)

It shows the relative error of numerical resolution in the Table 2 below:

| Points | Error relative $t = 0$ | Error relative $t = 80$ | Error relative $t = 160$ | Error relative $t = 200$ |
|--------|------------------------|------------------------|------------------------|------------------------|
| 0      | 0.0328                 | 0.0000                 | 0.0000                 | 0.0000                 |
| 0.005  | 0.0001                 | 0.0000                 | 0.0000                 | 0.0000                 |
| 0.015  | 0.0004                 | 0.0000                 | 0.0000                 | 0.0001                 |
From Table 2, it has been shown that the solutions has small errors. Both of methods give relatively the same results. It can be seen that numerical methods using finite volume method can be used to approach analytical solutions. In addition, it also fulfilled the initial value and boundary conditions.

5. Conclusion
The modelling of heat equation has been investigated. The heat Equation is integrated to the volume control. Having obtained the integral result, the Equation is discretized using explicit scheme. To get a convergent solution, time steps at $\Delta t < 0.5\, \text{s}$ is allowed. Error has been calculated from the difference of numerical solutions with analytical solution which is using separation variable. Numerical solution with finite volume quite well in solving heat Equation on metal heating. The result showed that the finite volume method provides a good solution with an average error of 0.0016.

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