On nonlinear equations associated with developable, ruled and minimal surfaces

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Abstract

An examples of solutions of nonlinear differential equations associated with developable, ruled and minimal surfaces are constructed.

1 Developable surfaces

The equation of developable surface is defined by the condition [1]

\[
\begin{bmatrix}
\frac{\partial^2}{\partial x^2} F(x, y, z) & \frac{\partial^2}{\partial x \partial y} F(x, y, z) & \frac{\partial^2}{\partial x \partial z} F(x, y, z) & \frac{\partial}{\partial x} F(x, y, z) \\
\frac{\partial^2}{\partial x \partial y} F(x, y, z) & \frac{\partial^2}{\partial y^2} F(x, y, z) & \frac{\partial^2}{\partial y \partial z} F(x, y, z) & \frac{\partial}{\partial y} F(x, y, z) \\
\frac{\partial^2}{\partial x \partial z} F(x, y, z) & \frac{\partial^2}{\partial y \partial z} F(x, y, z) & \frac{\partial^2}{\partial z^2} F(x, y, z) & \frac{\partial}{\partial z} F(x, y, z) \\
\frac{\partial}{\partial x} F(x, y, z) & \frac{\partial}{\partial y} F(x, y, z) & \frac{\partial}{\partial z} F(x, y, z) & 0
\end{bmatrix} = 0. \tag{1}
\]

It is equivalent to the second order partial differential equation

\[
- \left( \frac{\partial^2}{\partial x^2} F(x, y, z) \right) \left( \frac{\partial^2}{\partial y^2} F(x, y, z) \right) \left( \frac{\partial}{\partial z} F(x, y, z) \right)^2 + \\
+ 2 \left( \frac{\partial^2}{\partial x^2} F(x, y, z) \right) \left( \frac{\partial^2}{\partial y \partial z} F(x, y, z) \right) \left( \frac{\partial}{\partial y} F(x, y, z) \right) \frac{\partial}{\partial z} F(x, y, z) - \\
- \left( \frac{\partial^2}{\partial x^2} F(x, y, z) \right) \left( \frac{\partial}{\partial y} F(x, y, z) \right)^2 \frac{\partial^2}{\partial z^2} F(x, y, z) + \\
+ \left( \frac{\partial^2}{\partial x \partial y} F(x, y, z) \right)^2 \left( \frac{\partial}{\partial z} F(x, y, z) \right)^2 - \\
\]

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\[-2 \left( \frac{\partial^2}{\partial x \partial y} F(x,y,z) \right) \left( \frac{\partial^2}{\partial y \partial z} F(x,y,z) \right) \left( \frac{\partial}{\partial x} F(x,y,z) \right) \left( \frac{\partial}{\partial z} F(x,y,z) \right) - \]
\[-2 \left( \frac{\partial^2}{\partial x \partial y} F(x,y,z) \right) \left( \frac{\partial}{\partial y} F(x,y,z) \right) \left( \frac{\partial^2}{\partial x \partial z} F(x,y,z) \right) \left( \frac{\partial}{\partial z} F(x,y,z) \right) + \]
\[+2 \left( \frac{\partial^2}{\partial x \partial y} F(x,y,z) \right) \left( \frac{\partial^2}{\partial y^2} F(x,y,z) \right) \left( \frac{\partial}{\partial x} F(x,y,z) \right) \frac{\partial^2}{\partial z^2} F(x,y,z) + \]
\[+2 \left( \frac{\partial^2}{\partial x^2} F(x,y,z) \right) \left( \frac{\partial^2}{\partial y^2} F(x,y,z) \right) \left( \frac{\partial}{\partial x} F(x,y,z) \right) \frac{\partial}{\partial z} F(x,y,z) + \]
\[+ \left( \frac{\partial^2}{\partial x \partial z} F(x,y,z) \right)^2 \left( \frac{\partial}{\partial y} F(x,y,z) \right)^2 \]
\[-2 \left( \frac{\partial^2}{\partial x \partial z} F(x,y,z) \right) \left( \frac{\partial}{\partial y} F(x,y,z) \right) \frac{\partial^2}{\partial z^2} F(x,y,z) - \]
\[-2 \left( \frac{\partial^2}{\partial y^2} F(x,y,z) \right) \left( \frac{\partial}{\partial x} F(x,y,z) \right)^2 \frac{\partial^2}{\partial z^2} F(x,y,z) + \]
\[+ \left( \frac{\partial}{\partial x} F(x,y,z) \right)^2 \left( \frac{\partial^2}{\partial y \partial z} F(x,y,z) \right)^2 = 0. \quad (2) \]

2 Method of solutions

To obtain particular solutions of nonlinear partial differential equations

\[F(x, y, f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{xxx}, f_{xxy}, f_{xyy}, ...) = 0 \quad (3)\]

we use the parametric presentation of the functions and variables [2],[3],[4]

\[f(x,y) \rightarrow u(x,t), \quad y \rightarrow v(x,t), \quad f_x \rightarrow u_x = \frac{u_t}{v_t}, \quad f_y \rightarrow \frac{u_t}{v_t}, \quad f_{xy} \rightarrow \frac{(u_x - \frac{u_t}{v_t} v_x)}{v_t}, \quad f_{yy} \rightarrow \frac{(u_x - \frac{u_t}{v_t} v_x)}{v_t}, \quad f_{xy} \rightarrow \frac{(u_x - \frac{u_t}{v_t} v_x)}{v_t}, \quad \ldots \quad (4)\]

where variable \( t \) is considered as parameter.

Remark that conditions of equality of mixed derivatives

\[f_{xy} = f_{yx}\]

are fulfilled at the such type of presentation.

In result instead of equation (3) one gets the relation between a new variables \( u(x,t) \) , \( v(x,t) \) and their partial derivatives

\[\Psi(u, v, u_x, u_t, v_x, v_t, ...) = 0. \quad (5)\]

At the condition \( v(x,t) = t \) the relation (5) coincides with the equation (3) and takes a more general form after the reduction \( u(x,t) = F(\omega(x,t), \omega_t, \ldots) \) and \( v(x,t) = \Phi(\omega(x,t), \omega_t, \ldots) \).

The substitution \( u(x,t) \) into the relation (5) leads to the p.d.e. with respect the function \( v(x,t) \) and it can be considered as the partner equation to the equation (3).
Classification of possible reductions of the relation (5) connected with a given equation (3) has an important interest for development of the \((u, v)\)-transformation method.

The most popular reductions of the relation (5) are in the form

\[
\begin{align*}
    u(x, t) &= \frac{\partial}{\partial t} \omega(x, t, z) t, \\
    v(x, t) &= t \frac{\partial}{\partial t} \omega(x, t, z) - \omega(x, t, z),
\end{align*}
\]

or

\[
\begin{align*}
    v(x, t) &= \frac{\partial}{\partial t} \omega(x, t, z) t, \\
    u(x, t) &= t \frac{\partial}{\partial t} \omega(x, t, z) - \omega(x, t, z).
\end{align*}
\]

3 An examples of solutions

The equation (2) after the applying transformation (4) with the conditions

\[
\begin{align*}
    v(x, t, z) &= t \frac{\partial}{\partial t} \omega(x, t, z) - \omega(x, t, z), \\
    u(x, t, z) &= \frac{\partial}{\partial t} \omega(x, t, z)
\end{align*}
\]

is reduced to the form

\[
\begin{align*}
    &\left( \frac{\partial^2}{\partial t^2} \omega(x, t, z) \right) \left( \frac{\partial^2}{\partial x \partial z} \omega(x, t, z) \right)^2 - 2 \left( \frac{\partial}{\partial t} \omega(x, t, z) \right) \left( \frac{\partial^2}{\partial t \partial x} \omega(x, t, z) \right) \frac{\partial}{\partial x} \omega(x, t, z) + \\
    &\quad + \left( \frac{\partial}{\partial t} \omega(x, t, z) \right)^2 \frac{\partial^2}{\partial x^2} \omega(x, t, z) - \left( \frac{\partial^2}{\partial t^2} \omega(x, t, z) \right) \left( \frac{\partial}{\partial x} \omega(x, t, z) \right) \frac{\partial}{\partial x} \omega(x, t, z) + \\
    &\quad + \left( \frac{\partial}{\partial t} \omega(x, t, z) \right)^2 \frac{\partial^2}{\partial z^2} \omega(x, t, z) = 0,
\end{align*}
\]

which is equivalent the condition

\[
\begin{bmatrix}
    \frac{\partial^2}{\partial x \partial t} \omega(x, t, z) & \frac{\partial^2}{\partial x^2} \omega(x, t, z) & \frac{\partial^2}{\partial x \partial z} \omega(x, t, z) \\
    \frac{\partial^2}{\partial x^2} \omega(x, t, z) & \frac{\partial^2}{\partial x \partial t} \omega(x, t, z) & \frac{\partial^2}{\partial x \partial z} \omega(x, t, z) \\
    \frac{\partial^2}{\partial x \partial z} \omega(x, t, z) & \frac{\partial^2}{\partial x \partial z} \omega(x, t, z) & \frac{\partial^2}{\partial z^2} \omega(x, t, z)
\end{bmatrix} = 0.
\]

From solutions of the equation (7) can be derived solutions of the equation (2) with the help of elimination of the parameter \(t\) from the relations

\[
y - t \frac{\partial}{\partial t} \omega(x, t, z) + \omega(x, t, z) = 0
\]

and

\[
F(x, y, z) = \frac{\partial}{\partial t} \omega(x, t, z) = 0.
\]
To integrate the equation (7) we rewrite it in the form

\[
\left( \frac{\partial^2}{\partial y^2} h(x, y, z) \right) \left( \frac{\partial^2}{\partial x \partial z} h(x, y, z) \right)^2 - 2 \left( \frac{\partial^2}{\partial x \partial y} h(x, y, z) \right) \left( \frac{\partial^2}{\partial y \partial z} h(x, y, z) \right) \left( \frac{\partial^2}{\partial x \partial z} h(x, y, z) \right) + \\
+ \left( \frac{\partial^2}{\partial y \partial z} h(x, y, z) \right)^2 \frac{\partial^2}{\partial x^2} h(x, y, z) - \left( \frac{\partial^2}{\partial y^2} h(x, y, z) \right) \left( \frac{\partial^2}{\partial x^2} h(x, y, z) \right) \frac{\partial^2}{\partial z^2} h(x, y, z) + \\
+ \left( \frac{\partial^2}{\partial y \partial z} h(x, y, z) \right)^2 \frac{\partial^2}{\partial z^2} h(x, y, z) = 0
\]  

(8)

where we change the parameter \( t \) on the variable \( y \) and the function \( \omega(x, t, z) \) on the function \( h(x, y, z) \).

After the \((u, v)\)-transformation with conditions

\[
v(x, t, z) = t \frac{\partial}{\partial t} \theta(x, t, z) - \theta(x, t, z), \\
u(x, t, z) = \frac{\partial}{\partial t} \theta(x, t, z)
\]

this equation is reduced at the equation on the function \( \theta(x, t, z) \)

\[
\left( \frac{\partial^2}{\partial x^2} \theta(x, t, z) \right) \frac{\partial^2}{\partial t} \theta(x, t, z) - \left( \frac{\partial^2}{\partial x \partial z} \theta(x, t, z) \right)^2 = 0.
\]  

(9)

From solutions of the equation (9) we find the function \( h(x, y, z) \) by the way of elimination of the parameter \( t \) from the relations

\[
y - t \frac{\partial}{\partial t} \theta(x, t, z) + \theta(x, t, z) = 0,
\]

and

\[
h(x, y, z) - \frac{\partial}{\partial t} \theta(x, t, z) = 0.
\]

Using the function \( h(x, y, z) \) we get the function \( \omega(x, t, z) = h(x, t, z) \) and then the solutions of the equation (2).

Solutions of the equation (9) can be derived by the \((u, v)\) transformation and are determined with the help of elimination of the parameter \( \tau \) from the relations

\[
\tau x + \phi(\tau, t) z + \psi(\tau, t) \theta(x, t, z) - 1 = 0,
\]

\[
x + \phi(\tau, t) z + \psi(\tau, t) \theta(x, t, z) = 0
\]

where \( \phi \) and \( \psi \) are arbitrary functions.
4 An example

After the substitution

$$\varphi(\tau, t) = -A(t) \tau^2,$$
$$\psi(\tau, t) = B(t) \tau$$

from the system of equations

$$x\tau - A(t) \tau^2 z + B(t) \tau \theta(x, t, z) - 1 = 0,$$
$$x - 2A(t) \tau z + B(t) \theta(x, t, z) = 0$$

we find

$$\theta(x, t, z) = \frac{-x + 2\sqrt{A(t)z}}{B(t)}.$$  

From the equations

$$h(x, y, z) - \frac{\partial}{\partial t}\theta(x, t, z) = 0,$$
$$y - \frac{\partial}{\partial t}\theta(x, t, z) + \theta(x, t, z) = 0$$

at the conditions

$$A(t) = (B(t))^2,$$
$$B(t) = \frac{t + 1}{t - 1}$$

we get the function

$$h(x, y, z) = -\left(\frac{1/2 \sqrt{2}x + 1/2 \sqrt{6 \cdot x^2 - 4yx + 8 \sqrt{zx}}}{x}\right)^2$$

and corresponding function

$$\omega(x, t, z) = -1/2 \left(\frac{x + \sqrt{x(3x - 2t + 4\sqrt{z})}}{x}\right)^2.$$  

Now after the elimination of the parameter $t$ from the system of equations

$$F(x, y, z) - \frac{\partial}{\partial t}\omega(x, t, z) = 0$$
$$y - t\frac{\partial}{\partial t}\omega(x, t, z) + \omega(x, t, z) = 0$$

we find the function

$$F(x, y, z) = \frac{x + y + 2\sqrt{z} + \sqrt{y^2 - 4yx - 4y\sqrt{z} + x^2 + 4\sqrt{zx} + 4z}}{4\sqrt{z} + 3x}$$

which is solution of the equation (2).
5 Partner equation

After application of the \((u,v)\)-transformation with the condition

\[ u(x,t,z) = t \]

the equation (8) is transformed to the partner equation

\[
- \left( \frac{\partial^2}{\partial t^2} v(x,t,z) \right) \left( \frac{\partial^2}{\partial x^2} v(x,t,z) \right)^2 - \left( \frac{\partial^2}{\partial t \partial z} v(x,t,z) \right)^2 \frac{\partial^2}{\partial x^2} v(x,t,z) - \\
\left( \frac{\partial^2}{\partial t \partial x} v(x,t,z) \right)^2 \frac{\partial^2}{\partial z^2} v(x,t,z) + 2 \left( \frac{\partial^2}{\partial t^2} v(x,t,z) \right) \left( \frac{\partial^2}{\partial t \partial z} v(x,t,z) \right) \frac{\partial^2}{\partial x \partial z} v(x,t,z) + \\
+ \left( \frac{\partial^2}{\partial t^2} v(x,t,z) \right) \left( \frac{\partial^2}{\partial x^2} v(x,t,z) \right) \frac{\partial^2}{\partial z^2} v(x,t,z) = 0. \tag{10}
\]

After the substitution

\[ v(x,t,z) = A(x,t) \]

we get the equation with respect the function \(A(x,t)\)

\[
\left( \frac{\partial^2}{\partial t^2} A(x,t) \right) \left( \frac{\partial}{\partial x} A(x,t) \right)^2 + \left( \frac{\partial}{\partial t} A(x,t) \right)^2 \frac{\partial^2}{\partial x^2} A(x,t) - \\
-2 \left( \frac{\partial^2}{\partial t \partial x} A(x,t) \right) \left( \frac{\partial}{\partial t} A(x,t) \right) \frac{\partial}{\partial x} A(x,t) = 0. \tag{11}
\]

A simplest solution of this equation has the form

\[ A(x,t) = -F_1(x) \cdot F_2(t), \]

where the functions \(-F_1(x)\) and \(-F_2(x)\) are defined from the equations

\[
\frac{d^2}{dx^2} F_1(x) = \left( \frac{d}{dx} F_1(x) \right)^2 \frac{c_1}{-c_1} = 0,
\]

\[
\frac{d^2}{dt^2} F_2(t) = 2 \frac{d}{dt} F_2(t)^2 - \frac{d}{dt} F_2(t) \frac{c_1}{-c_1} = 0
\]

and has the form

\[ F_1(x) = \left( \frac{-c_1}{C_1 x + C_1 x c_1 - C_2 + C_2 c_1} \right)^{-1}, \]

\[ F_2(t) = \left( \frac{-C_3 t + C_3 t c_1 + C_4 - C_4 c_1}{-c_1} \right)^{-1}. \]

In result we obtain the function \(\omega(x,t,z)\)

\[ \omega(x,t,z) = z \left( \frac{-C_3 y + C_3 y c_1 - C_4 + C_4 c_1}{-c_1} \right)^{-1} \times \]
\begin{align*}
&\times \left( \frac{-c_1}{C1 x + C1 x_c1 - C2 + C2 c_1} \right)^{-1}.
\end{align*}

With the help of the $\omega(x, t, z)$ the function $F(x, y, z)$ can be found from the relations

$$F(x, y, z) - \frac{\partial}{\partial t} \omega(x, t, z) = 0$$

and

$$y - t \frac{\partial}{\partial t} \omega(x, t, z) + \omega(x, t, z) = 0.$$ 

after elimination of the parameter $t$.

As example in the case $c_1 = 2$, 

the function $F(x, y, z)$ which is solution of the equation (2) is defined by the equation

$$4 (F(x, y, z))^3 C_4^3 + 27 (F(x, y, z))^2 C_3 z C_1^2 x^2 + 54 (F(x, y, z))^2 C_3 z C_1 x C_2 +$$

$$+ 27 (F(x, y, z))^2 C_3 z C_2^2 + 12 (F(x, y, z))^2 C_3 C_4^2 y + 12 F(x, y, z) C_3^2 C_4 y^2 +$$

$$+ 4 C_3^3 y^3 = 0.$$ 

The equation (11) can be integrated by the $(u, v)$- or the Legendre -transformation and its solutions may be used to construction of solutions of the equation (2).

### 6 Ruled surfaces

The equation of any ruled $f = f(x, y)$ surface is derived by elimination of the parameter $\tau$ from the relations

$$f - \alpha(\tau)x - \alpha(\tau) = 0,$$

$$y - \beta(\tau)x - b(\tau) = 0.$$ 

It can be presented as [5]

$$\left( \frac{\partial^2}{\partial x^2} f(x, y) \right) \left( \frac{\partial}{\partial y} Q(x, y) \right)^2 - 2 \left( \frac{\partial^2}{\partial x \partial y} f(x, y) \right) \left( \frac{\partial}{\partial x} Q(x, y) \right) \frac{\partial}{\partial y} Q(x, y) +$$

$$+ \left( \frac{\partial^2}{\partial y^2} f(x, y) \right) \left( \frac{\partial}{\partial x} Q(x, y) \right)^2 - 8 QT = 0$$

where

$$T = \det \begin{bmatrix}
\frac{\partial^2}{\partial x^2} f(x, y) & \frac{\partial^3}{\partial x^2 \partial y} f(x, y) & \frac{\partial^2}{\partial y} f(x, y) \\
\frac{\partial^3}{\partial x^2 \partial y} f(x, y) & \frac{\partial^3}{\partial x^3} f(x, y) & \frac{\partial^2}{\partial y} f(x, y) \\
\frac{\partial^3}{\partial x \partial y^2} f(x, y) & \frac{\partial^3}{\partial y^3} f(x, y) & \frac{\partial^2}{\partial y^2} f(x, y)
\end{bmatrix},$$

and

$$Q = \left( \frac{\partial^2}{\partial x^2} f(x, y) \right) \frac{\partial^2}{\partial y^2} f(x, y) - \left( \frac{\partial}{\partial x} f(x, y) \right)^2,$$

7
In explicit form it looks as

\[-18 \left( \frac{\partial^2}{\partial x^2} f(x, y) \right) \left( \frac{\partial^2}{\partial y^2} f(x, y) \right) \left( \frac{\partial^3}{\partial x^2 \partial y} f(x, y) \right) \left( \frac{\partial^2}{\partial x \partial y} f(x, y) \right) \frac{\partial^3}{\partial y \partial x \partial y} f(x, y) +\]

\[+6 \left( \frac{\partial^2}{\partial x \partial y} f(x, y) \right) \left( \frac{\partial^3}{\partial x^2} f(x, y) \right) \left( \frac{\partial^2}{\partial x^2} f(x, y) \right) \left( \frac{\partial^2}{\partial x^2} f(x, y) \right) \frac{\partial^3}{\partial y^3} f(x, y) -\]

\[-6 \left( \frac{\partial^2}{\partial y^2} f(x, y) \right) \left( \frac{\partial^3}{\partial x^2 \partial y} f(x, y) \right) \left( \frac{\partial^2}{\partial x^2} f(x, y) \right) \frac{\partial}{\partial y} f(x, y) +\]

\[-6 \left( \frac{\partial^2}{\partial x^2} f(x, y) \right)^2 \left( \frac{\partial^3}{\partial y^3} f(x, y) \right) \left( \frac{\partial^2}{\partial x \partial y} f(x, y) \right) \frac{\partial^3}{\partial y^3} f(x, y) +\]

\[+12 \left( \frac{\partial^3}{\partial x^3} f(x, y) \right) \left( \frac{\partial^2}{\partial y^2} f(x, y) \right) \left( \frac{\partial^2}{\partial x^2} f(x, y) \right) \frac{\partial^3}{\partial y^3} f(x, y) -\]

\[-6 \left( \frac{\partial^3}{\partial x^3} f(x, y) \right) \left( \frac{\partial^2}{\partial y^2} f(x, y) \right)^2 \left( \frac{\partial^2}{\partial x^2} f(x, y) \right) \frac{\partial^3}{\partial y^3} f(x, y) -\]

\[-6 \left( \frac{\partial^3}{\partial x^3} f(x, y) \right) \left( \frac{\partial^2}{\partial y^2} f(x, y) \right)^2 \left( \frac{\partial^2}{\partial x^2} f(x, y) \right) \frac{\partial^3}{\partial y^3} f(x, y) +\]

\[+ \left( \frac{\partial^2}{\partial x^2} f(x, y) \right)^3 \left( \frac{\partial^3}{\partial y^3} f(x, y) \right)^2 + \left( \frac{\partial^2}{\partial x^2} f(x, y) \right)^3 \left( \frac{\partial^3}{\partial y^3} f(x, y) \right)^3 +\]

\[+9 \left( \frac{\partial^2}{\partial x^2} f(x, y) \right) \left( \frac{\partial^2}{\partial y^2} f(x, y) \right)^2 \left( \frac{\partial^3}{\partial x \partial y} f(x, y) \right)^2 +\]

\[+9 \left( \frac{\partial^2}{\partial y^2} f(x, y) \right) \left( \frac{\partial^2}{\partial x^2} f(x, y) \right)^2 \left( \frac{\partial^3}{\partial y \partial x} f(x, y) \right)^2 -\]

\[-8 \left( \frac{\partial^2}{\partial x \partial y} f(x, y) \right)^3 \left( \frac{\partial^3}{\partial x^3} f(x, y) \right) \frac{\partial^3}{\partial y^3} f(x, y) = 0 \] (12)

After the \((u, v)\)-transformation

\[u(x, t) = t \frac{\partial}{\partial t} \omega(x, t) - \omega(x, t) ,\]

\[v(x, t) = \frac{\partial}{\partial t} \omega(x, t)\]

we get the equation with respect the function \(\omega(x, t)\)

\[-9 \left( \frac{\partial^3}{\partial x \partial t \partial x} \omega(x, t) \right)^2 \left( \frac{\partial^2}{\partial x^2} \omega(x, t) \right) \left( \frac{\partial^2}{\partial t^2} \omega(x, t) \right)^2 -\]

\[-6 \left( \frac{\partial^2}{\partial t^2} \omega(x, t) \right) \left( \frac{\partial^3}{\partial x \partial t \partial x} \omega(x, t) \right) \left( \frac{\partial^2}{\partial x^2} \omega(x, t) \right)^2 \frac{\partial^3}{\partial t^3} \omega(x, t) +\]
\[
+6 \left( \frac{\partial^3 \omega(x,t)}{\partial x^3} \right) \left( \frac{\partial^2 \omega(x,t)}{\partial t^2} \right)^2 \left( \frac{\partial^2 \omega(x,t)}{\partial x^2} \right) \frac{\partial^3 \omega(x,t)}{\partial t^2 \partial x} + \\
+ \left( \frac{\partial^3 \omega(x,t)}{\partial x^3} \right)^2 \left( \frac{\partial^2 \omega(x,t)}{\partial t^2} \right)^3 + 9 \left( \frac{\partial^2 \omega(x,t)}{\partial t^2} \right) \left( \frac{\partial^2 \omega(x,t)}{\partial x^2} \right)^2 \left( \frac{\partial^3 \omega(x,t)}{\partial t^2 \partial x} \right)^2 - \\
- \left( \frac{\partial^2 \omega(x,t)}{\partial x^2} \right)^3 \left( \frac{\partial^3 \omega(x,t)}{\partial t^3} \right)^2 = 0
\] (13)

It has the solution of the form

\[
\omega(x,t) = A(t) + B(x) t.
\]

where the functions \(A(t)\) and \(B(t)\) satisfy the equations

\[
\left( \frac{d^3}{dx^3} B(x) \right)^2 - \mu \left( \frac{d^2}{dx^2} B(x) \right)^3 = 0
\]

and

\[
9 \left( \frac{d^2}{dt^2} A(t) \right)^2 + 6 \left( \frac{d^2}{dt^2} A(t) \right) t \frac{d^3}{dt^3} A(t) - \mu t \left( \frac{d^2}{dt^2} A(t) \right)^3 + t^2 \left( \frac{d^3}{dt^3} A(t) \right)^2 = 0.
\]

From here we find

\[
A(t) = \left( -4 \frac{\ln(-C1 t - 1)}{\mu} + 4 \frac{\ln(t)}{\mu} - 4 \frac{1}{\mu C1 t} - \frac{-C2}{t} + \mu C3 \right) t
\]

and

\[
B(t) = -4 \frac{\ln(x + -C4)}{\mu} + \mu C5 x + \mu C6.
\]

Using these expressions we find the function \(f(x,y)\) satisfying the equation (12).

In particular case

\[
\mu C5 = 0, \quad \mu C2 = 0, \quad \mu C3 = 0, \quad \mu C4 = 0, \quad \mu C6 = 0
\]

it is determined from the equation

\[
y\mu + 8 \ln(2) + 4 \ln \left( -\frac{1}{f(x,y) \mu C1} \right) - 4 \ln \left( \frac{-4 + f(x,y) \mu C1}{f(x,y) \mu C1^2} \right) - \\
-f(x,y) \mu C1 + 4 \ln(x) = 0
\]

### 6.1 Partner equation

After the \((u,v)\)-transformation with condition

\[
u(x,t) = t
\]

the equation (12) takes the form

\[
-6 \left( \frac{\partial^2}{\partial x \partial t} v(x,t) \right) \left( \frac{\partial^3}{\partial x^3} v(x,t) \right) \left( \frac{\partial^2}{\partial t^2} v(x,t) \right) \left( \frac{\partial^2}{\partial x^2} v(x,t) \right) \frac{\partial^3}{\partial t^3} v(x,t) + 
\]
\[
+18 \left( \frac{\partial^2}{\partial x^2} v(x, t) \right) \left( \frac{\partial^2}{\partial t^2} v(x, t) \right) \left( \frac{\partial^3}{\partial x^3} v(x, t) \right) \left( \frac{\partial^2}{\partial x^2 \partial t} v(x, t) \right) \left( \frac{\partial^3}{\partial x \partial t^2} v(x, t) \right) - v(x, t) + \\
+6 \left( \frac{\partial^2}{\partial x \partial t} v(x, t) \right) \left( \frac{\partial^3}{\partial x^3} v(x, t) \right) \left( \frac{\partial^2}{\partial t^2} v(x, t) \right) \left( \frac{\partial^2}{\partial x^2} v(x, t) \right) ^2 \frac{\partial^3}{\partial x^2 \partial t} v(x, t) + \\
-12 \left( \frac{\partial^2}{\partial x \partial t} v(x, t) \right) ^2 \left( \frac{\partial^3}{\partial x^3} v(x, t) \right) \left( \frac{\partial^2}{\partial x^2} v(x, t) \right) \frac{\partial^3}{\partial x \partial t^2} v(x, t) - \\
-12 \left( \frac{\partial^3}{\partial x^3} v(x, t) \right) \left( \frac{\partial^2}{\partial t^2} v(x, t) \right) \left( \frac{\partial^2}{\partial x^2} v(x, t) \right) ^2 \frac{\partial^3}{\partial x \partial t^2} v(x, t) + \\
+8 \left( \frac{\partial^2}{\partial x \partial t} v(x, t) \right) ^3 \left( \frac{\partial^3}{\partial x^3} v(x, t) \right) \frac{\partial^3}{\partial t^3} v(x, t) - 9 \left( \frac{\partial^2}{\partial t^2} v(x, t) \right) \left( \frac{\partial^2}{\partial x^2} v(x, t) \right) ^2 \left( \frac{\partial^3}{\partial x \partial t^2} v(x, t) \right) ^2 - \\
-9 \left( \frac{\partial^2}{\partial x^2} v(x, t) \right) \left( \frac{\partial^2}{\partial t^2} v(x, t) \right) ^2 \left( \frac{\partial^3}{\partial x^3} v(x, t) \right) ^2 - \left( \frac{\partial^3}{\partial x^3} v(x, t) \right) \left( \frac{\partial^2}{\partial t^2} v(x, t) \right) ^3 = 0
\] (14)

It has solution of the form

\[ v(x, t) = A(t) + B(x) \]

which looks as

\[ v(x, t) = -4 \frac{\ln(-C1 + x) + C2 + C34 \ln(-C4 + t + C5 t + C6)}{\mu} \]

Using these expressions and the conditions

\[ y - v(x, t) = 0, \quad t = f(x, y) \]

we can find the function \( f(x, y) \) which is solution of the equation (12).

In particular case

\[ C6 = 0, \quad C3 = 0, \quad C1 = 0, \quad C4 = 0, \quad C5 = 1 \]

one gets

\[ f(x, y) = 4 \frac{\text{LambertW}\left( 1/4 \mu e^{1/4 y \mu - 1/4 - C2 \mu x} \right)}{\mu} \]

where the function \( \text{LambertW}(x) \) is defined by the relation

\[ \text{LambertW}(x) e^{\text{LambertW}(x)} = x. \]
7 Minimal surfaces

Minimal surfaces are defined by solutions of the equation [6]

\[
\left(1 + \left(\frac{\partial}{\partial y} f(x,y)\right)^2\right) \frac{\partial^2}{\partial x^2} f(x,y) - 2 \left(\frac{\partial}{\partial y} f(x,y)\right) \left(\frac{\partial}{\partial x} f(x,y)\right) \frac{\partial^2}{\partial y \partial x} f(x,y) + \\
+ \left(1 + \left(\frac{\partial}{\partial x} f(x,y)\right)^2\right) \frac{\partial^2}{\partial y^2} f(x,y) = 0
\]  

(15)

After the \((u,v)\)-transformation

\[v(x,t) = t \frac{\partial}{\partial t} \omega(x,t) - \omega(x,t),\]

\[u(x,t) = \frac{\partial}{\partial t} \omega(x,t),\]

one gets the equation

\[-\left(\frac{\partial^2}{\partial x \partial t} \omega(x,t)\right)^2 + t^2 \left(\frac{\partial^2}{\partial t^2} \omega(x,t)\right) \frac{\partial^2}{\partial x^2} \omega(x,t) - t^2 \left(\frac{\partial^2}{\partial x \partial t} \omega(x,t)\right)^2 + \\
+ 2t \left(\frac{\partial^2}{\partial x \partial t} \omega(x,t)\right) \frac{\partial}{\partial x} \omega(x,t) - \left(\frac{\partial}{\partial x} \omega(x,t)\right)^2 - 1 + \left(\frac{\partial^2}{\partial t^2} \omega(x,t)\right) \frac{\partial^2}{\partial x^2} \omega(x,t) = 0.
\]

(16)

It has the solution

\[\omega(x,t) = 1/4t \arctan(t) + (1 + t^2)x^2.\]

Corresponding solution of the equation (16) can be presented in a parametric form

\[f(x,y) = \frac{1}{4} \arctan(t) + \arctan(t) \frac{t^2 + t + 8tx^2 + 8t^3x^2}{1 + t^2},\]

\[t = \frac{\sqrt{2} \sqrt{4y - 1 + \sqrt{16y^2 - 8y + 1 + 64x^2y + 64x^4}}}{16x^2y + 64x^4}.
\]

8 Partner equation

After the \((u,v)\)-transformation with the condition

\[u(x,t) = t\]

the equation (15) is transformed into the partner equation

\[-\left(\frac{\partial}{\partial x} v(x,t)\right)^2 \frac{\partial^2}{\partial t^2} v(x,t) + 2 \left(\frac{\partial}{\partial x} v(x,t)\right) \left(\frac{\partial^2}{\partial x \partial t} v(x,t)\right) \frac{\partial}{\partial t} v(x,t) - \frac{\partial^2}{\partial x^2} v(x,t) - \left(\frac{\partial^2}{\partial x^2} v(x,t)\right) \left(\frac{\partial}{\partial t} v(x,t)\right)^2 = 0.
\]

(17)
It has the same form with the initial equation (15).
This property can be used to construction a new solutions of the equation (15) by the way of elimination of the parameter $t$ from the relations

$$y - v(x, t) = 0, \quad t - f(x, y) = 0. \quad (18)$$

Let us consider an examples.
The function

$$f(x, y) = \ln \left( \sqrt{x^2 + y^2 + \sqrt{x^2 + y^2 - 1}} \right)$$

is solution of the equation (15).

Then the function

$$v(x, t) = \ln \left( \sqrt{x^2 + t^2 + \sqrt{x^2 + t^2 - 1}} \right)$$

is the solution of the equation (17).

Elimination of the parameter $t$ from the conditions

$$y - \ln \left( \sqrt{x^2 + t^2 + \sqrt{x^2 + t^2 - 1}} \right) = 0,$$

$$t = f(x, y)$$

leads to the relation

$$y - \ln \left( \sqrt{x^2 + (f(x, y))^2 + \sqrt{x^2 + (f(x, y))^2 - 1}} \right) = 0$$

from which we get the function

$$f(x, y) = 1/2 \sqrt{2 - 4x^2 + e^{-2y} + e^{2y}}$$

which is solution of the equation (15).

If the

$$f(x, y) = \ln \left( \frac{\cos(y)}{\cos(x)} \right)$$

is solution of the equation (15), then the function

$$v(x, t) = \ln \left( \frac{\cos(t)}{\cos(x)} \right)$$

satisfies the equation (17).

Now from the conditions (18) we find the equation

$$y - \ln \left( \frac{\cos(f(x, y))}{\cos(x)} \right) = 0$$

from which we get a new solution of the equation (15)

$$f(x, y) = \arccos (e^y \cos(x)).$$

In the case of the substitution

$$v(x, t) = H \left( x^2 + t^2 + 1 \right)$$
we find the solution of the equation (17)

\[ v(x,t) = \ln \left( -2C1^2 + x^2 + t^2 + \sqrt{\left( -t^2 + 4C1^2 - x^2 \right)} C1 \right) C1^{-1} C1. \]

Now from the conditions (18) we obtain the equation

\[ y = \ln \left( -2C1^2 + x^2 + f(x,y)^2 + \sqrt{\left( \frac{x^2+f(x,y)^2}{C1^2} \right) \left( f(x,y)^2 - 4C1^2 + x^2 \right)} C1 \right) C1^{-1} C1 \]

to determination of the function \( f(x,y) \).

Corresponding solution has the form

\[ f(x,y) = \frac{1}{2} \sqrt{2} e^{\frac{-x}{\sqrt{2}}} \left( -C1^2 e^{\frac{x}{\sqrt{2}}} + 4C1^2 e^{\frac{2x}{\sqrt{2}}} - 2 e^{\frac{2x}{\sqrt{2}}} x^2 + 4C1^3 \right) e^{-\frac{x}{\sqrt{2}}}. \]

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