SPECTRAL PAIRS AND POSITIVE DEFINITE TEMPERED DISTRIBUTIONS

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Abstract. The present paper presents two new approaches to Fourier series and spectral analysis of singular measures.

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“There cannot be a language more universal and more simple, more free from errors and obscurities...more worthy to express the invariable relations of all natural things [than mathematics]. [It interprets] all phenomena by the same language, as if to attest the unity and simplicity of the plan of the universe, and to make still more evident that unchangeable order which presides over all natural causes” — Joseph Fourier, The Analytical Theory of Heat

A spectral pair in $\mathbb{R}^k$, is a pair $(\nu, \Lambda)$ where $\nu$ is a finite positive Borel measure on $\mathbb{R}^k$, and where $\Lambda$ is a strictly discrete subset of $\mathbb{R}^k$ (see [JP93, JP98, HJW17, Str98a, Str98b, Str00, Str12]). From a given set $\Lambda$, there is then an associated positive definite tempered distribution $F_{\Lambda}$, and a corresponding generalized reproducing kernel Hilbert space $\mathcal{H}_{\Lambda}$. We say that $(\nu, \Lambda)$ is a spectral pair iff the set of Fourier exponentials from $\Lambda$ forms an orthogonal basis in $L^2(\nu)$; intuitively, $L^2(\nu)$ admits an orthogonal $\Lambda$-Fourier series representation. In this paper, we study the occurrence of spectral pairs within the framework of positive definite tempered distributions, as defined by Laurent Schwartz [Sch64b]. Given $\Lambda$, we present a necessary and sufficient condition for the existence of a finite positive Borel measure $\nu$ such that

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$(\nu, \Lambda)$ is a spectral pair. Our method relies on use of the generalized reproducing kernel Hilbert space $\mathcal{H}_\Lambda$.

1. Preliminaries

In our theorems and proofs, we shall make use the particular reproducing kernel Hilbert spaces (RKHSs) which allow us to give explicit formulas for our solutions. The general framework of RKHSs were pioneered by Aronszajn in the 1950s [Aro50]; and subsequently they have been used in a host of applications; e.g., [SZ07, SZ09].

Positive definite functions on groups $G$ yield a special class of positive definite kernels. In general, the covariance kernel of a Gaussian process, indexed by $G$, is a positive definite kernel. If the $G$-process is $G$-stationary, then its covariance kernel will be defined from a positive definite function. While these notions make sense in general, we shall restrict attention here to the case of locally compact abelian groups, with an emphasis on the case $G = \mathbb{R}^k$. But to simplify notation, it is convenient to begin the discussion with the case $k = 1$. The extension to the cases $k > 1$ is fairly straightforward, and we shall concentrate on the general case $G = \mathbb{R}^k$ in the main part of the paper. Our analysis will go beyond the case of positive definite functions. Indeed, our main theorem is stated in the context of positive definite tempered distributions. A comparison of the analysis in the two cases will be given in section 2 below.

The RKHS $\mathcal{H}_F$. For simplicity we focus on the case $G = \mathbb{R}$.

**Definition 1.1.** Let $\Omega$ be an open domain in $\mathbb{R}$. A function $F : \Omega \to \mathbb{C}$ is positive definite if
\[
\sum_i \sum_j c_i \overline{c}_j F(x_i - x_j) \geq 0
\]
for all finite sums with $c_i \in \mathbb{C}$, and all $x_i \in \Omega$. We assume that all the p.d. functions are continuous and bounded.

**Lemma 1.2** (Two equivalent conditions for p.d.). If $F$ is given continuous on $\mathbb{R}$, we have the following two equivalent conditions for the positive definite property:

(i) $\forall n \in \mathbb{N}, \forall \{x_i\}_{i=1}^n, \forall \{c_i\}_{i=1}^n, x_i \in \mathbb{R}, c_i \in \mathbb{C}$,
\[
\sum_i \sum_j c_i \overline{c}_j F(x_i - x_j) \geq 0;
\]

(ii) $\forall \varphi \in C_c(\mathbb{R})$, we have:
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x) \overline{\varphi(y)} F(x - y) \, dx \, dy \geq 0.
\]

**Proof.** Use Riemann integral approximation, and note that $F(\cdot - x) \in \mathcal{H}_F$, and $\varphi \ast F \in \mathcal{H}_F$. (See details below.)

Consider a continuous positive definite function so $F$ is defined on $\Omega$. Set
\[
F_y(x) := F(x - y), \quad \forall x, y \in \Omega.
\]
Let $\mathcal{H}_F$ be the reproducing kernel Hilbert space (RKHS), which is the completion of
\[
\left\{ \sum_{\text{finite}} c_j F_{x_j} \mid x_j \in \Omega, c_j \in \mathbb{C} \right\}
\]
with respect to the inner product
\[ \left\langle \sum_i c_i F_{x_i}, \sum_j d_j F_{y_j} \right\rangle_{\mathcal{H}_F} := \sum_i \sum_j c_i d_j F(x_i - y_j); \tag{1.4} \]
modulo the subspace of functions of \(\|\cdot\|_{\mathcal{H}_F}\)-norm zero.

Below, we introduce an equivalent characterization of the RKHS \(\mathcal{H}_F\), which we will be working with in the rest of the paper.

**Lemma 1.3.** Fix \(\Omega = (0, \alpha)\). Let \(\varphi_{n,x}(t) = n\varphi(n(t - x))\), for all \(t \in \Omega\); where \(\varphi\) satisfies

(i) \(\text{supp}(\varphi) \subset (-\alpha, \alpha)\);
(ii) \(\varphi \in C_c^\infty, \varphi \geq 0\);
(iii) \(\int \varphi(t) \, dt = 1\). Note that \(\lim_{n \to \infty} \varphi_{n,x} = \delta_x\), the Dirac measure at \(x\).

**Lemma 1.4.** The RKHS, \(\mathcal{H}_F\), is the Hilbert completion of the functions
\[ F_{\varphi}(x) = \int_\Omega \varphi(y) F(x - y) \, dy, \forall \varphi \in C_c^\infty(\Omega), \, x \in \Omega \tag{1.5} \]
with respect to the inner product
\[ \langle F_{\varphi}, F_{\psi} \rangle_{\mathcal{H}_F} = \int_\Omega \int_\Omega \varphi(y) \overline{\psi(y)} F(x - y) \, dx \, dy, \forall \varphi, \psi \in C_c^\infty(\Omega). \tag{1.6} \]
In particular,
\[ \|F_{\varphi}\|_{\mathcal{H}_F}^2 = \int_\Omega \int_\Omega \varphi(x) \overline{\varphi(y)} F(x - y) \, dx \, dy, \forall \varphi \in C_c^\infty(\Omega) \tag{1.7} \]
and
\[ \langle F_{\varphi}, F_{\psi} \rangle_{\mathcal{H}_F} = \int_\Omega F_{\varphi}(x) \overline{\psi(x)} \, dx, \forall \varphi, \psi \in C_c^\infty(\Omega). \tag{1.8} \]

**Proof.** Indeed, by Lemma 1.4, we have
\[ \|F_{\varphi_{n,x}} - F(\cdot - x)\|_{\mathcal{H}_F} \to 0, \text{ as } n \to \infty. \tag{1.9} \]
Hence \(\{F_{\varphi}\}_{\varphi \in C_c^\infty(\Omega)}\) spans a dense subspace in \(\mathcal{H}_F\).

For more details, see [Jor86, Jor87].

The following two conditions (1.10)/(⇔(1.11)) below will be used to characterize elements in the Hilbert space \(\mathcal{H}_F\).

**Theorem 1.5.** A continuous function \(\xi : \Omega \to \mathbb{C}\) is in \(\mathcal{H}_F\) if and only if there exists \(A_0 > 0\), such that
\[ \sum_i \sum_j c_i \overline{c_j} \xi(x_i) \overline{\xi(x_j)} \leq A_0 \sum_i \sum_j c_i \overline{c_j} F(x_i - x_j) \tag{1.10} \]
for all finite system \(\{c_i\} \subset \mathbb{C}\) and \(\{x_i\} \subset \Omega\).

Equivalently, for all \(\psi \in C_c^\infty(\Omega)\),
\[ \left| \int_\Omega \psi(y) \overline{\xi(y)} \, dy \right|^2 \leq A_0 \int_\Omega \int_\Omega \psi(x) \overline{\psi(y)} F(x - y) \, dx \, dy \tag{1.11} \]
Note that, if \(\xi \in \mathcal{H}_F\), then the LHS of (1.11) is \(\|F_{\psi}, \xi \|_{\mathcal{H}_F}^2\). Indeed,
\[ \left| \langle \xi, F_{\psi} \rangle_{\mathcal{H}_F} \right|^2 = \left| \langle \xi, \int_\Omega \psi(y) F_{\psi} \, dy \rangle_{\mathcal{H}_F} \right|^2 \]
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\[ = \left| \int_\Omega \psi(y) \langle \xi, F_y \rangle_{\mathcal{H}_F} \, dy \right|^2 \]

\[ = \left| \int_\Omega \psi(y) \xi(y) \, dy \right|^2 \text{ (by the reproducing property)}. \]

2. THE PARALLELS OF P.D. FUNCTIONS VS DISTRIBUTIONS

The study of positive definite (p.d.) functions, and p.d. kernels, is motivated by diverse themes in analysis and operator theory, in white noise analysis, applications of reproducing kernel (RKHS) theory, extensions by Laurent Schwartz, and in reflection positivity from quantum physics (see the cited references.) The parallels between Bochner’s theorem (for continuous p.d. functions), and the generalization to Bochner/Schwartz representations for positive definite tempered distributions will be made clear. In the first case, we have the Bochner representation via finite positive measures \( \mu \); and in the second case, instead via tempered positive measures. This parallel also helps make precise the respective reproducing kernel Hilbert spaces (RKHSs). This further leads to a more unified approach to the treatment of the stationary-increment Gaussian processes \([AJL11, AJ12, AJ15]\). A key argument will rely on the existence of a unitary representation \( U \) of \((\mathbb{R}, +)\), acting on the particular RKHS under discussion. In fact, the same idea (with suitable modifications) will also work in the wider context of locally compact groups. In the abelian case, we shall make use of the Stone representation for \( U \) in the form of orthogonal projection valued measures; and in more general settings, the Stone-Naimark-Ambrose-Godement (SNAG) representation \([Sto32]\).

**Theorem 2.1.**

(a) Let \( F \) be a continuous positive definite (p.d.) function on \( \mathbb{R} \) (a p.d. tempered distribution \([Sch64a, Sch64b]\)); then there is a unique finite positive Borel measure \( \mu \) on \( \mathbb{R} \) (resp., a unique tempered measure on \( \mathbb{R} \)) such that \( F = \hat{\mu} \).

(b) Given \( F \) as above, let \( \mathcal{H}_F \) denote the corresponding kernel Hilbert space, i.e., the Hilbert completion of \( \{ \varphi * F \} \varphi \in C_c(\Omega) \) (resp. \( \varphi \in \mathcal{S} \)) w.r.t

\[ \| \varphi * F \|^2_{\mathcal{H}_F} = \int_\mathbb{R} \int_\mathbb{R} \varphi(x) \varphi(y) F(x-y) \, dx \, dy \]

resp., \( \langle F(x-y), \varphi * \varphi \rangle \); action in the sense of distributions. Then there is a unique isometric transform

\[ \mathcal{H}_F \xrightarrow{T_F} L^2(\mathbb{R}, \mathcal{B}, \mu), \quad T_F(\varphi * F) = \hat{\varphi}, \text{ i.e.,} \]

\[ \| \varphi * F \|^2_{\mathcal{H}_F} = \int_\mathbb{R} |\hat{\varphi}|^2 \, d\mu = \| T_F \varphi \|^2_{L^2(\mu)}. \]

(c) If \( \mu \) is tempered, e.g., i.e., \( \int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} < \infty \), then

\[ \| \varphi * F \|^2_{\mathcal{H}_F} = \int \left( |\hat{\varphi}|^2 + \left| \left( D_x \varphi \right) \right|^2 \right) \frac{d\mu(\lambda)}{1 + \lambda^2}; \]

where \( D_x \varphi = \frac{d\varphi}{dx} \), and where “\( \cdot \)” denotes the standard Fourier transform on \( \mathbb{R} \).

**Proof.** See \([JT17]\). \( \square \)
OVERVIEW.

Continuous p.d. functions on $\mathbb{R}$

**Lemma.** Let $F$ be a continuous function on $\mathbb{R}$. Then the following are equivalent:

(i) $F$ is p.d., i.e., $\forall \varphi \in C_c(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x) \overline{\varphi(y)} F(x - y) \, dx \, dy \geq 0.$$  

(ii) $\forall \{x_j\}_{j=1}^n \subset \mathbb{R}$, $\forall \{c_j\}_{j=1}^n \subset \mathbb{C}$, and $\forall n \in \mathbb{N}$, we have

$$\sum_{j=1}^n \sum_{k=1}^n c_j \overline{c_k} F(x_j - x_k) \geq 0.$$  

Note. In both cases, we have the following representation for vectors in the RKHS $\mathcal{H}^*_F$:

$$\langle \varphi \ast F, \psi \ast F \rangle_{\mathcal{H}^*_F} = \langle \varphi \ast \overline{\psi}, F \rangle, \quad \forall \varphi, \psi \in \mathcal{S};$$  

where $\varphi \ast F$ := the standard convolution w.r.t. Lebesgue measure.

p.d. tempered distributions on $\mathbb{R}$

**Lemma.** Let $F$ be a tempered distribution on $\mathbb{R}$. Then $F$ is p.d. if and only if

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x) \overline{\varphi(y)} F(x - y) \, dx \, dy \geq 0$$  

hold, for all $\varphi \in \mathcal{S}$, where $\mathcal{S}$ is the Schwartz space.

Equivalently,

$$\langle F(x - y), \varphi \otimes \overline{\varphi} \rangle \geq 0, \quad \forall \varphi \in \mathcal{S}. \quad (2.4)$$

Here $(\cdot, \cdot)$ denotes distribution action.

RKHS

**Bochner’s theorem.**

$\exists$ positive finite measure $\mu$ on $\mathbb{R}$ such that

$$F(x) = \int_{\mathbb{R}} e^{ix\lambda} d\mu(\lambda).$$

Let $\mathcal{H}_F$ be the RKHS of $F$.

- Then

$$\| \varphi \ast F \|_{\mathcal{H}_F}^2 = \int_{\mathbb{R}} |\hat{\varphi}(\lambda)|^2 d\mu(\lambda) \quad (2.5)$$

where $\hat{\varphi}$ is the Fourier transform.

- $F$ admits the factorization

$$F(x_1 - x_2) = \langle F(\cdot - x_1), F(\cdot - x_2) \rangle_{\mathcal{H}_F}$$

$\forall x_1, x_2 \in \mathbb{R}$, with $\mathbb{R} \ni x \mapsto F(\cdot - x) \in \mathcal{H}_F.$

**Applications**

Now applied to Bochner’s theorem.

Set $\mathcal{H}_F = \text{RKHS of } F$, and $w_0 = F(\cdot - 0)$. Then

$$U_t w_0 = w_t = F(\cdot - t), \quad t \in \mathbb{R}$$

defines a strongly continuous unitary representation of $\mathbb{R}$.

Bochner/Schwartz

$\exists$ positive tempered measure $\mu$ on $\mathbb{R}$ such that

$$F = \hat{\mu}$$

where $\hat{\mu}$ is in the sense of distribution.

Let $\mathcal{H}_F$ denote the corresponding RKHS.

- For all $\varphi \in \mathcal{S}$, we have

$$\| \varphi \ast F \|_{\mathcal{H}_F}^2 = \langle F(x - y), \varphi \otimes \overline{\varphi} \rangle,$$  

distribution action.

- $\mathcal{S} \ni \varphi \mapsto \varphi \ast F \in \mathcal{H}_F$, where

$$(\varphi \ast F)(\cdot) = \int \varphi(y) F(\cdot - y) \, dy.$$

On white noise space:

$$\mathbb{E} \left( e^{i\langle \cdot, \cdot \rangle} \right) = e^{-\frac{1}{2} \int |\varphi|^2 d\mu}$$

where $\mathbb{E}(\cdots)$ = expectation w.r.t the Gaussian path-space measure.

(The proof for the special case when $F$ is assumed p.d. and continuous carries over with some changes to the case when $F$ is a p.d. tempered distribution.)
3. Spectral pairs and p.d. distributions

**Definition 3.1** (L. Schwartz). Introduce the following locally convex topological (LCT) spaces, such that we have continuous inclusions

\[ C_c^\infty (\mathbb{R}^k) \hookrightarrow \mathcal{S}_k \hookrightarrow C^\infty (\mathbb{R}^k), \]

with corresponding duals for the distribution spaces,

\[ (C_c^\infty (\mathbb{R}^k))' \hookrightarrow \mathcal{S}_k' \hookrightarrow (C^\infty (\mathbb{R}^k))'. \]

where \( \mathcal{D} := C_c^\infty (\mathbb{R}^k) \) = all distributions, and \( \mathcal{E} := (C^\infty)' = \) all distributions of compact support.

**Definition 3.2.** A subset \( \Lambda \subset \mathbb{R}^k, k \in \mathbb{N}, \) is said to be uniformly discrete iff (Def.) there exists \( r > 0 \) such that \( |\lambda - \lambda'| \geq r, \) for all \( \lambda, \lambda' \in \Lambda, \lambda \neq \lambda'. \) In this case, we say \( \Lambda \in \mathcal{D}_k. \)

**Definition 3.3** ([JP93, JP98, HJW17]).

(a) Let \( \nu \) be a finite Borel measure on \( \mathbb{R}^k, \) assumed normalized for simplicity, then a subset \( \Lambda \subset \mathbb{R}^k \) is said to be a spectrum iff \( \{ e^{i\lambda x} \}_{\lambda \in \Lambda} \) is an orthonormal basis (ONB) in \( L^2 (\nu). \) In this case, we say that \( (\nu, \Lambda) \) is a spectral pair.

(b) If instead

\[ \int \Omega |f(x)|^2 \, d\nu(x) = \sum_{\lambda \in \Lambda} \left| \int \mathbb{R} f(x) e^{-i\lambda x} \, d\nu(x) \right|^2 \]

for all \( f \in L^2 (\nu), \) then we say that \( (\nu, \Lambda) \) is a Parseval spectral pair. Clearly a spectral pair is also a Parseval spectral pair.

(c) Given \( \nu, \) set

\[ \mathcal{L}(\nu) := \{ \Lambda \subset \mathbb{R}^k ; (\nu, \Lambda) \text{ is a spectral pair} \}; \quad (3.1) \]

and given \( \Lambda \in \mathcal{D}_k, \) set

\[ \mathcal{M}(\Lambda) := \{ \nu \in \text{Prob}_{\mathcal{B}} (\mathbb{R}^k) ; (\nu, \Lambda) \text{ is a spectral pair} \}. \quad (3.2) \]

(d) If \( (\nu, \Lambda) \) is a spectral pair, we say that \( \nu \in \mathcal{M}(\Lambda), \) and \( \Lambda \in \mathcal{L}(\nu). \) If \( (\nu, \Lambda) \) is a Parseval spectral pair, we say that \( \Lambda \in \mathcal{L}_{Pa}(\nu), \) and \( \nu \in \mathcal{M}_{Pa}(\Lambda). \)

**Remark 3.4.** For details on Parseval frames and their recent applications, we refer to [DJ13, DJ15, BH15b, LH17, LW17, DHSW11, aSW14, PW17].

The case when \( \nu \in \text{Prob}_{\mathcal{B}} (\mathbb{R}^k) \) is an IFS-Cantor measure has been studied extensively in the literature. See, e.g, [JP93, JP98, DHS09, DHL13, Mal89, Dau88, Law91, DJ09, DJ12].

**Definition 3.5** (L. Schwartz [Sch64b]). A tempered distribution \( F \in \mathcal{S}' \) is positive definite (p.d.) on \( \mathbb{R}^k \) if

\[ \langle F(x - y) \cdot \varphi \otimes \varphi' \rangle \geq 0, \quad \forall \varphi \in \mathcal{S}_k. \quad (3.3) \]

where \( \mathcal{S}_k := \) the Schwartz space on \( \mathbb{R}^k. \)

**Definition 3.6.** A positive measure \( \mu \) on \( \mathbb{R}^k \) is said to be tempered iff (Def.) \( \exists M \in \mathbb{N} \) such that

\[ \int_{\mathbb{R}^k} \frac{d\mu(\lambda)}{1 + \|\lambda\|^{2M}} < \infty. \quad (3.4) \]
Theorem 3.7 (Schwartz [Sch64b]). A tempered distribution $F$ is positive definite if and only if there exists a positive tempered Borel measure $\mu$ on $\mathbb{R}^k$ such that

$$\int_{\mathbb{R}^k} F(x) \varphi(x) \, dx = \int_{\mathbb{R}^k} \hat{\varphi}(\lambda) \, d\mu(\lambda), \quad \forall \varphi \in S_k. \quad (3.5)$$

In particular (see (3.3)) the measure $\mu$ will satisfy:

$$\int_{\mathbb{R}^k} \int_{\mathbb{R}^k} F(x-y) \varphi(x) \psi(y) \, dx \, dy = \int_{\mathbb{R}^k} |\hat{\varphi}(\lambda)|^2 \, d\mu(\lambda) \quad (3.6)$$

with a slight abuse of notation. Here, “$\hat{\cdot}$” denotes the standard Fourier transform.

Definition 3.8. Let $F$ be a positive definite tempered distribution, then on the functions

$$(\varphi * F)(x) = \int_{\mathbb{R}^k} \varphi(y) F(x-y) \, dy, \quad \varphi \in S_k, \quad (3.7)$$

set

$$(\varphi * F, \psi * F)_{\mathcal{H}_F} := \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \varphi(x) \overline{\psi(y)} F(x-y) \, dx \, dy, \quad \forall \varphi, \psi \in S_k. \quad (3.8)$$

Then $\{\varphi * F; \varphi \in S_k\}$ forms a pre-Hilbert space with respect to the norm specified in (3.8) and (3.10). Set

$$\mathcal{H}_F := \text{the Hilbert completion from (3.7) & (3.8)}. \quad (3.9)$$

It follows that

Lemma 3.9. A function $\varphi * F$ is in $\mathcal{H}_F$ if and only if $\hat{\varphi} \in L^2(\mu)$, and then

$$\|\varphi * F\|_{\mathcal{H}_F}^2 = \int_{\mathbb{R}^k} |\hat{\varphi}(\lambda)|^2 \, d\mu(\lambda) \quad (3.10)$$

where $\mu$ is the tempered measure from (3.5).

Lemma 3.10. If $\Lambda \in \mathcal{D}_k$, then

$$F_{\Lambda}(x) := \sum_{\lambda \in \Lambda} e^{i\lambda x} \quad (3.11)$$

is a positive definite (p.d.) tempered distribution.

Proof. Let $\varphi \in S_k$, and let $r > 0$ be as in Definition 3.2. Then

$$\langle F_{\Lambda}, \varphi \rangle = \int_{\mathbb{R}^k} \sum_{\lambda \in \Lambda} e^{i\lambda x} \varphi(x) \, dx$$

where $dx = dx_1 \cdots dx_k$ is the usual $k$-Lebesgue measure on $\mathbb{R}^k$; and so

$$\langle F_{\Lambda}, \varphi \rangle = \sum_{\lambda \in \Lambda} \hat{\varphi}(\lambda). \quad (3.12)$$

Set $|\lambda| = (\lambda_1^2 + \cdots + \lambda_k^2)^{\frac{1}{2}}$. From the assumption in the lemma; see Definition 3.2, it follows that there is an $M \in \mathbb{N}$, depending on $k$ and $r$ such that

$$C_M(\Lambda) := \sum_{\lambda \in \Lambda} \frac{1}{1 + |\lambda|^{2M}} < \infty. \quad (3.13)$$
Let $\Delta_k$ be the usual Laplacian on $\mathbb{R}^k$, then for $\varphi \in \mathcal{S}_k$ (the space of Schwartz test functions), we get

$$\hat{\varphi} (\lambda) = \left( (I - \Delta_k^M) \varphi \right)^\wedge (\lambda) / (1 + |\lambda|^{2M}), \forall \lambda \in \mathbb{R}^k. \tag{3.14}$$

Again, using $\varphi \in \mathcal{S}_k$,

$$A (\varphi) := \sup_{\lambda \in \mathbb{R}^k} \left| \left( (I - \Delta_k^M) \varphi \right)^\wedge (\lambda) \right| < \infty. \tag{3.15}$$

Combining the estimates (3.13) & (3.15), we get

$$|\langle F_\Lambda, \varphi \rangle | \leq A (\varphi) C_M (\Lambda).$$

Since $A (\varphi)$ is one of the seminorms given in the definition of the topology on Schwartz' space of tempered test functions $\mathcal{S}_k$, the desired conclusion follows. □

**Example 3.11.** Let $\nu = \nu_4$ be the $\frac{1}{4}$-Cantor measure, i.e., the unique solution to the IFS-equation in $\text{Prob}_{\mathcal{S}} (\mathbb{R})$

$$\frac{1}{2} \int_\mathbb{R} \left( f \left( \frac{x}{4} \right) + f \left( \frac{x + 2}{4} \right) \right) d\nu_4 (x) = \int_\mathbb{R} f (x) d\nu_4 (x), \tag{3.16}$$

and let

$$\Lambda_4 = \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, \ldots \} \tag{3.17}$$

$$= \left\{ \sum_{0}^{\text{finite}} b_i 4^i ; b_i \in \{0, 1\} \right\}.$$

Then $(\nu_4, \Lambda_4)$ is a spectral pair; and so $\Lambda_4 \in \mathcal{L} (\nu_4)$, and $\nu_4 \in \mathcal{M} (\Lambda_4)$. See Figure 3.1.

It is known that $(\nu_3, \Lambda_3)$ is not a spectral pair, where $\nu_3$ is the unique solution in $\text{Prob}_{\mathcal{S}} (\mathbb{R})$ to

$$\frac{1}{2} \int_\mathbb{R} \left( f \left( \frac{x}{3} \right) + f \left( \frac{x + 2}{3} \right) \right) d\nu_3 (x) = \int_\mathbb{R} f (x) d\nu_3 (x), \tag{3.18}$$

and

$$\Lambda_3 := \{0, 1, 3, 4, 9, 10, 12, 13, 27, 28, \ldots \} \tag{3.19}$$

$$= \left\{ \sum_{0}^{\text{finite}} b_i 3^i ; b_i \in \{0, 1\} \right\}.$$

In fact, Jorgensen & Pedersen [JP98] showed that

$$\mathcal{L} (\nu_3) = \emptyset. \tag{3.20}$$

However, $\mathcal{M} (\Lambda_3)$ is not known. See also [Str98a, Str98b, Str00, Str12].
Support of $\nu_4$; note $\nu_4$ makes a spectral pair $(\nu_4, \Lambda_4)$.

The Sierpinski gasket; the corresponding IFS measure is not part of a spectral pair.

The Sierpinski Eiffle Tower; the corresponding IFS measure $\nu$ is part of a spectral pair $(\nu, \Lambda)$ with some $\Lambda \in \mathcal{D}_3$; see [JP98], and also [Str98a, Str98b, Str00, Str12].

Figure 3.1. Examples of support sets in $\mathbb{R}^k$, $k = 1, 2, 3$, for IFS Cantor measures. See [JP93].
Theorem 3.12. Let $k \in \mathbb{N}$, and $\Lambda \in \mathcal{D}_k$ (see Definition 3.2), and set $F_\Lambda(x) = \sum_{\lambda \in \Lambda} e^{i\lambda x}$, $x \in \mathbb{R}^k$ as a tempered p.d. distribution, and let $\mathcal{H}_F$ be the generalized RKHS of Schwartz [Sch64b]. Then a function $h$ on $\mathbb{R}^k$ is in $\mathcal{H}_F$ if and only if it has a convolution-factorization

$$h = \varphi \ast F_\Lambda$$

(3.21)

where $\varphi$ is a measurable function such that $\hat{\varphi}(\lambda)$ exists for all $\lambda \in \Lambda$, and $\{\hat{\varphi}(\lambda)\}_{\lambda \in \Lambda}$ is in $l^2(\Lambda)$. In this case

$$\|h\|_{\mathcal{H}_F}^2 = \sum_{\lambda \in \Lambda} |\hat{\varphi}(\lambda)|^2;$$

(3.22)

see (3.21).

Proof. We shall include below only a sketch; additional details will follow inside the proof of Theorem 3.14 below. The key formula for $\|\cdot\|_{\mathcal{H}_F}$ is

$$\|\varphi \ast F_\Lambda\|_{\mathcal{H}_F}^2 = \sum_{\lambda \in \Lambda} |\hat{\varphi}(\lambda)|^2,$$

(3.23)

which in turn follows from Theorem 3.7 above. \qed

Remark 3.13. The tempered distributions $F_\Lambda$ considered here play an important role in the literature on aperiodic phenomena, especially the study of diffraction patterns; see e.g., [LO17, BH15a, BG16, BL17].

Theorem 3.14. Let $\Lambda \in \mathcal{D}_k$, and set

$$F_\Lambda(x) := \sum_{\lambda \in \Lambda} e^{i\lambda x},$$

(3.24)

see (3.11), then $\mathcal{H}_F$ (eqns. (3.7)-(3.8)) has the form $\mathcal{H}_F = L^2(\nu)$ for a finite positive Borel measure $\nu$ on $\mathbb{R}^k$ if and only if $\nu \in \mathcal{M}_{Pa}(\Lambda)$.

Proof. For $\varphi \in \mathcal{S}_k$, let $\hat{\varphi}$ denote the standard Fourier transform. It is known [Sch64a] that $\hat{S}_k = S_k$, and so $\hat{S}'_k = S'_k$. One checks that

$$(\varphi \ast F_\Lambda)(x) = \sum_{\lambda \in \Lambda} \hat{\varphi}(\lambda) e^{i\lambda x}, \quad x \in \mathbb{R}^k,$$

(3.25)

and

$$\|\varphi \ast F_\Lambda\|_{\mathcal{H}_F}^2 = \sum_{\lambda \in \Lambda} |\hat{\varphi}(\lambda)|^2;$$

(3.26)

see (3.7)-(3.8) for definitions. Note that the transforms are computed in the sense of Schwartz distributions.

We now turn to the proof of the theorem. Note that by Definition 3.2, given $\Lambda$, then $\nu \in \mathcal{M}(\Lambda)$ iff $L^2(\nu)$ admits the representation: $f \in L^2(\nu) \iff \exists \{c_\lambda\}_{\lambda \in \Lambda} \in l^2(\Lambda)$ such that

$$f(x) = \sum_{\lambda \in \Lambda} c_\lambda e^{i\lambda x},$$

(3.27)

and

$$\int_{\mathbb{R}^k} |f(x)|^2 \, d\nu(x) = \sum_{\lambda \in \Lambda} |c_\lambda|^2,$$

(3.28)

where, for simplicity, we assume that $\nu$ is normalized. The interpretation of (3.27) is that of $L^2(\nu)$, and so when $f \in L^2(\nu)$ has the representation (3.27), then we
have convergence
\[
\int \left| f(x) - \sum_{\lambda \in \Lambda} c_{\lambda} e^{i\lambda x} \right|^2 d\nu(x) \to 0
\]
where the limit in (3.29) is over the filter of all finite subsets of the given set \( \Lambda \in \mathcal{D}_k \).

The conclusion of the theorem now follows from this, combined with (3.26)-(3.27), and the previous lemmas.

Conversely, suppose \( \mathcal{H}_{F_\Lambda} = L^2(\nu) \) for some \( \nu \), a finite positive Borel measure; then \( e_\lambda(x) := e^{i\lambda x} \) is in \( \mathcal{H}_{F_\Lambda} \) for all \( \lambda \in \Lambda \). Now, by the above, we know that every \( f \in \mathcal{H}_{F_\Lambda} (= L^2(\nu)) \) has the representation
\[
f(x) = \sum_{\lambda \in \Lambda} \hat{\varphi}(\lambda) e_\lambda(x)
\]
(3.30)
where \( \hat{\varphi}(\lambda) \) denotes the standard Fourier transform, i.e., \( \hat{\varphi}(\lambda) = \int_{\mathbb{R}^k} \varphi(x) e^{-i\lambda x} dx \), and \( dx \) = the \( k \)-dimensional Lebesgue measure; see (3.27). We now verify that \( \nu \in \mathcal{M}_{Pa}(\Lambda) \), i.e.,
\[
\sum_{\lambda \in \Lambda} \left| \langle f, e_\lambda \rangle_{L^2(\nu)} \right|^2 = \|f\|^2_{L^2(\nu)}.
\]
(3.32)
We have
\[
\text{RHS}_{(3.32)} = \left\| \sum_{\lambda \in \Lambda} \hat{\varphi}(\lambda) e_\lambda(\cdot) \right\|^2_{\mathcal{H}_{F_\Lambda}} \quad \text{(by (3.30))}
\]
\[
= \sum_{\lambda \in \Lambda} |\hat{\varphi}(\lambda)|^2 \quad \text{(by (3.28) and (3.31))}
\]
\[
= \sum_{\lambda \in \Lambda} \left| \langle f, e_\lambda \rangle_{\mathcal{H}_{F_\Lambda}} \right|^2
\]
\[
= \sum_{\lambda \in \Lambda} \left| \langle f, e_\lambda \rangle_{L^2(\nu)} \right|^2 \quad \text{(by (3.30))}
\]
\[
= \text{LHS}_{(3.32)}
\]
which is the desired conclusion, i.e., \( \nu \in \mathcal{M}_{Pa}(\Lambda) \).

\[\square\]

**Corollary 3.15.** Let the setting be as in Theorem 3.14, i.e., \( \Lambda \in \mathcal{D}_k \) is given, and a positive finite measure \( \nu \) on \( \mathbb{R}^k \) is assumed to satisfy \( \mathcal{H}_{F_\Lambda} = L^2(\nu) \), see Theorem 3.7. Then \( \nu(\mathbb{R}^k) \leq 1 \), and \( \nu(\mathbb{R}^k) = 1 \) holds if and only if \( (\nu, \Lambda) \) is a spectral pair, i.e., \( \nu \in \mathcal{M}(\Lambda) \).

**Proof.** The result follows from Theorem 3.14, combined with the following:

**Lemma 3.16.** Let \( \mathcal{H} \) be a Hilbert space, and let \( \{\varphi_j\}_{j \in J} \), \( J \) countable, be a Parseval frame, i.e.,
\[
\|h\|^2_{\mathcal{H}} = \sum_{j \in J} |\langle \varphi_j, h \rangle_{\mathcal{H}}|^2
\]
(3.33)
holds for all \( h \in \mathcal{H} \).

Then \( \{\varphi_j\}_{j \in J} \) is an ONB if and only if \( \|\varphi_j\|_{\mathcal{H}} = 1 \), for all \( j \in J \). In general, \( \|\varphi_j\|_{\mathcal{H}} \leq 1 \), \( \forall j \in J \).

**Proof.** Pick \( j \in J \), and apply (3.33) to \( h = \varphi_j \), we get
\[
\|\varphi_j\|^2_{\mathcal{H}} = \|\varphi_j\|^4_{\mathcal{H}} + \sum_{i \in J \setminus \{j\}} |\langle \varphi_i, \varphi_j \rangle_{\mathcal{H}}|^2 \geq \|\varphi_j\|^4_{\mathcal{H}}.
\]
(3.34)
The conclusion in the lemma is immediate from this: In particular, \( \| \varphi_j \|_{\mathcal{M}} = 1 \) holds iff the terms on the RHS in (3.34) \( \langle \varphi_i, \varphi_j \rangle_{\mathcal{M}}, i \neq j, i, j \in J \), vanish. \( \square \)

The corollary follows since when \( \nu, \Lambda \) are as specified as stated, then

\[
\| e_\lambda \|^2_{L^2(\nu)} = \int_{\mathbb{R}^k} |e_\lambda|^2 \, d\nu = \nu(\mathbb{R}^k).
\]

\( \square \)

4. A Result from [HJW17]

The conclusions in [HJW17] introduced a new and different approach to the study of spectral theory for \( L^2(\nu) \) in the case where \( \nu \) is a given compactly supported singular measure on \( \mathbb{R} \). Below we include a brief sketch.

Let \( \nu \) be a singular measure supported on \([0,1] = \mathbb{R}/\mathbb{Z} = \) the boundary of the disk \( D \), \( D = \{ z \in \mathbb{C} : |z| < 1 \} \). For \( n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), set \( e_n(x) = e^{i 2\pi n x} \), and pass to the following Kaczmarz algorithm (see, e.g., [Szw07, CT13]):

\[
f_0 = \langle f, e_0 \rangle_{\nu} e_0
\]

\[
\cdots
\]

\[
f_n(x) = f_{n-1}(x) + \langle f - f_{n-1}, e_n \rangle_{\nu} e_n(x).
\]

Then the associated sequence

\[
g_0 = e_0
\]

\[
\cdots
\]

\[
g_n(x) = e_n(x) - \sum_{j=0}^{n-1} \langle e_n, e_j \rangle_{\nu} g_j(x)
\]

has the form:

\[
\sum_{j=0}^{n} \alpha_{n-j} e_j(x) = g_n(x),
\]

and

**Theorem 4.1** ([HJW17]). Let \( \nu \) and \( \{ g_n \}_{n \in \mathbb{N}_0} \) be as sketched; then every \( f \in L^2(\nu) \) has the following frame expansion:

\[
f(x) = \sum_{n \in \mathbb{N}_0} \langle f, g_n \rangle_{\nu} e_n(x)
\]

\[
= \sum_{n \in \mathbb{N}_0} \left( \sum_{j=0}^{n} \hat{f}(j) \alpha_{n-j} \right) e_n(x),
\]

where \( (\alpha_{n-j}) \) is the \((\infty \times \infty)\)-matrix inverse to the Gramian \( (\hat{\nu}(n-j))_{n \leq n} \).

Moreover, the system \( \{ g_n \}_{n \in \mathbb{N}_0} \) in (4.3) is a Parseval frame in \( L^2(\nu) \), i.e.,

\[
\| f \|_{L^2(\nu)}^2 = \sum_{n \in \mathbb{N}_0} \| \langle f, g_n \rangle_{L^2(\nu)} \|^2
\]

holds for all \( f \in L^2(\nu) \).

**Proof.** Readers are referred to [HJW17] for proof details. \( \square \)
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