Gauge-ready formulation of cosmological perturbations in scalar-vector-tensor theories

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In scalar-vector-tensor (SVT) theories with parity invariance, we perform a gauge-ready formulation of cosmological perturbations on the flat Friedmann-Lemaître-Robertson-Walker (FLRW) background by taking into account a matter perfect fluid. We derive the second-order action of scalar perturbations and resulting linear perturbation equations of motion without fixing any gauge conditions. Depending on physical problems at hand, most convenient gauges can be chosen to study the development of inhomogeneities in the presence of scalar and vector fields coupled to gravity. This versatile framework, which encompasses Horndeski and generalized Proca theories as special cases, is applicable to a wide variety of cosmological phenomena including nonsingular cosmology, inflation, and dark energy. By deriving conditions for the absence of ghost and Laplacian instabilities in several different gauges, we show that, unlike Horndeski theories, it is possible to evade no-go arguments for the absence of stable nonsingular bouncing/genesis solutions in both generalized Proca and SVT theories. We also apply our framework to the case in which scalar and vector fields are responsible for dark energy and find that the separation of observables relevant to the evolution of matter perturbations into tensor, vector, and scalar sectors is transparent in the unitary gauge. Unlike the flat gauge chosen in the literature, this result is convenient to confront SVT theories with observations associated with the cosmic growth history.

I. INTRODUCTION

The cosmological perturbation theory is a fundamental framework for understanding the growth of cosmic structures \cite{1–4}. The perturbations of metric and matter can be generally decomposed into scalar, vector, and tensor sectors arising from irreducible representations of the \textit{SO}(3) background field configuration. Among them, scalar perturbations are the main source for the development of inhomogeneities in the Universe. For example, it is believed that the energy density of a scalar degree of freedom (DOF) drives inflation \cite{5}, during which the field perturbation \( \delta \phi \) is stretched over the Hubble radius \( \delta \phi \). After inflation, the primordial curvature perturbation is converted to the radiation perturbation, which is observed as temperature anisotropies in the Cosmic Microwave Background (CMB) \cite{7}. The CMB temperature fluctuation works as a source for the growth of matter density perturbations due to the gravitational instability \cite{8}.

General Relativity (GR) with standard matter (baryons and radiation) is not sufficient to account for the observed evidence of inflation, dark energy, dark matter etc. It is possible to explain such phenomena by taking into account new DOFs like scalar or vector fields. As in the case of string dilaton \cite{9}, these new DOFs can have direct couplings to the gravity sector with two tensor polarizations. For a single scalar field \( \phi \) coupled to gravity, most general scalar-tensor theories with second-order equations of motion are known as Horndeski theories \cite{10–13}. Indeed, the application of Horndeski theories to inflation and dark energy has been extensively performed in the literature \cite{14–19}. Since different models in Horndeski theories predict different cosmic growth histories, one can distinguish them from the observations of CMB, redshift space distortions, weak lensing etc \cite{20–28}.

For a massive vector field \( A_\mu \) with broken \( U(1) \) gauge symmetry, one can also construct self-interactions and nonminimal couplings to gravity similar to those appearing in Horndeski theories \cite{29–32}. The vector-tensor theories with second-order equations of motion are dubbed generalized Proca (GP) theories (see Refs. \cite{33–38} for further extensions). If we apply GP theories to cosmology, the temporal vector component \( A_0 \) plays a role of the auxiliary field directly related to the Hubble expansion rate \( H \) \cite{39}. Then, there exists a de Sitter fixed point responsible for the late-time cosmic acceleration.

The important difference of GP theories from scalar-tensor theories is the presence of intrinsic vector modes in the former, which work as dynamical vector perturbations on the FLRW background. By choosing the flat gauge, the authors of Ref. \cite{39} obtained the second-order actions of scalar, vector, and tensor perturbations for the purpose of deriving stability conditions and observables relevant to the cosmic growth history. Existence of intrinsic vector modes affects the effective gravitational coupling with matter through a quantity \( q_v \) associated with the no-ghost condition of vector perturbations \cite{40}. Around local massive objects, nonlinear vector-field self-interactions can suppress the propagation of fifth forces through the operation of the Vainshtein mechanism \cite{41, 42}.

In the presence of both scalar and vector fields coupled to gravity, it is possible to construct a unified version of Horndeski and GP theories with second-order equations of motion \cite{43} (dubbed SVT theories). There are two
versions of SVT theories, depending on whether the $U(1)$ gauge symmetry is respected or not. The $U(1)$-invariant SVT theories have been already applied to the static and spherically symmetric configuration, in which case hairy black hole solutions endowed with scalar and vector hairs are present [44–46]. The $U(1)$-broken SVT theories can be applied to the cosmological setup, in which the temporal vector component $A_0$ affects the background dynamics [47]. In this case, the longitudinal vector component works as a dynamical scalar perturbation. In Ref. [48], the second-order actions of tensor, vector, and scalar perturbations were derived in $U(1)$-broken SVT theories with parity invariance by choosing the flat gauge [48]. These results can be used for the studies of linear perturbations during inflation and late-time cosmic acceleration. See Ref. [49] for a recent review on the systematic approach to generalizations of GR, where the novel progress in constructing consistent field theories of gravity based on additional scalar, vector and tensor fields together with their cosmological implications is reviewed.

In this paper, without fixing any gauge conditions from the beginning, we derive the second-order actions of scalar perturbations and resulting linear perturbation equations of motion in $U(1)$-broken SVT theories with parity invariance by taking into account a matter perfect fluid. The motivation of such analysis is that, depending on the problems at hand, the gauge should be appropriately chosen. If we choose the flat gauge and apply GP/SVT theories to the bouncing cosmology, for example, the quantity $q_s$ relevant to no-ghost conditions of scalar perturbations (given by Eq. (5.32) of Ref. [48]) vanishes at the bounce ($H = 0$). Apparently, this signals the appearance of a strong coupling problem, but in the flat gauge the curvature perturbation $\mathcal{R}$ vanishes at $H = 0$ (as we will see in Sec. V). Hence the flat gauge is not suitable for describing the evolution of scalar perturbations across the bounce. If we choose other gauges like the Newtonian gauge, neither $q_v$ nor $\mathcal{R}$ vanishes at $H = 0$. Then, the strong coupling problem is not actually present at the bounce by choosing suitable gauges.

If the flat gauge is chosen for the computation of observables relevant to the growth of matter perturbations in the dark energy cosmology, the coefficients of second-order scalar action do not explicitly contain terms associated with the stability of tensor perturbations. This reflects the fact that, unlike tensor perturbations, there are no scalar perturbations arising from spatial metric components in the flat gauge. In other gauges like the unitary gauge, we show that the second-order scalar action contains quantities related to stability conditions of tensor, vector, and scalar perturbations. Then, unlike the flat gauge, the separation between tensor, vector, and scalar modes in the effective gravitational coupling $G_{\text{eff}}$ of linear perturbations becomes transparent. This is convenient for testing dark energy models in SVT theories with observational data of the cosmic growth history.

Our gauge-ready formulation of cosmological perturbations is versatile in that any convenient gauge can be chosen depending on the problem under consideration (see Sec. 12.1 in Ref. [49] for further discussion on the gauge choice). We would like to stress that, provided the gauge is suitably chosen, physical results are equivalent to each other among different gauges. For example, the effective gravitational coupling mentioned above can be expressed in several different ways by choosing different gauges, but they are actually identical to each other. If the flat gauge is chosen from the beginning, expressing $G_{\text{eff}}$ in terms of quantities associated with the stability conditions of tensor perturbations is a nontrivial and complicated procedure. This is attributed to the mixture of those quantities among coefficients of the second-order action of scalar perturbations. This problem can be avoided in our gauge-ready formalism in which the gauge choice can be performed at the level of scalar perturbation equations of motion. Apart from a subclass of Horndeski theories [50], this gauge-ready formulation was not performed yet even for full Horndeski theories. Our results are sufficiently general to accommodate both Horndeski and GP theories as specific cases.

Our paper is organized as follows. In Sec. II, we revisit the background equations of motion in SVT theories as well as the second-order actions of tensor and vector perturbations. In Sec. III, we derive the second-order action of scalar perturbations and resulting perturbation equations of motion without fixing any gauges. We also discuss the issues of gauge transformations, gauge-invariant variables, and gauge choices. In Sec. IV, we obtain conditions for the absence of ghost and Laplacian instabilities of scalar perturbations in the small-scale limit by choosing several different gauges. In Sec. V, our general results are applied to the discussion for the realization of stable nonsingular bouncing/genesis cosmologies. In Sec. VI, we compute observables relevant to the evolution of Newtonian and weak lensing gravitational potentials by choosing the unitary gauge in scalar perturbation equations of motion. Sec. VII is devoted to conclusions.

**II. SVT THEORIES ON THE COSMOLOGICAL BACKGROUND**

In SVT theories with broken $U(1)$ gauge symmetry [43], there exist a scalar field $\phi$ and a vector field $A_\mu$ coupled to gravity. For the vector field, we define the antisymmetric field strength tensor $F_{\mu\nu}$, its dual $\tilde{F}_{\mu\nu}$, and the symmetric tensor $S_{\mu\nu}$, as

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}, \quad S_{\mu\nu} = \nabla_\mu A_\nu + \nabla_\nu A_\mu, \quad (2.1)$$
where $\nabla_\mu$ represents the covariant derivative operator and $\mathcal{E}^{\mu\nu\alpha\beta}$ is the antisymmetric Levi-Civita tensor. The SVT theories contain the following Lorentz-invariant combinations:

$$X_1 = \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi, \quad X_2 = -\frac{1}{2} A_\mu \nabla_\mu \phi, \quad X_3 = -\frac{1}{2} A_\mu A^\mu,$$

and

$$F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad Y_1 = \nabla_\mu \phi \nabla_\nu \phi F^{\mu\alpha\nu\beta} F_{\alpha\beta}, \quad Y_2 = \nabla_\mu \phi A_\nu F^{\mu\alpha\nu\beta} F_{\alpha\beta}, \quad Y_3 = A_\mu A_\nu F^{\mu\alpha\nu\beta} F_{\alpha\beta}. \quad (2.3)$$

The quantities $X_1$ and $X_3$ correspond to the kinetic term of $\phi$ and the mass term of $A_\mu$, respectively, while $X_2$ characterizes their mixings. The quantities $F, Y_1, Y_2, Y_3$ arise from intrinsic vector modes.

The Ricci scalar $R$ and Einstein tensor $G_{\mu\nu}$ are generally coupled to scalar and vector fields. To keep the equations of motion up to second order, we need to take into account additional derivative interactions of those fields. In SVT theories, there are also nonminimal couplings with the double dual Riemann tensor defined by

$$L^{\mu\nu\alpha\beta} = \frac{1}{4} \mathcal{E}^{\mu\nu\rho\sigma} \mathcal{E}_{\alpha\beta\gamma\delta} R_{\rho\sigma\gamma\delta}, \quad (2.4)$$

where $R_{\rho\sigma\gamma\delta}$ is the Riemann tensor.

### A. Action of SVT theories with broken $U(1)$ gauge invariance

The full action of parity-invariant SVT theories with broken $U(1)$ gauge invariance is given by [43]

$$S = \int d^4x \sqrt{-g} \left( \sum_{n=3}^{5} L^{(n)}_{\text{ST}} + \sum_{n=2}^{6} L^{(n)}_{\text{SVT}} \right) + S_m, \quad (2.5)$$

where $g$ is the determinant of metric tensor $g_{\mu\nu}$. The Lagrangians $L^{(n)}_{\text{ST}}$ and $L^{(n)}_{\text{SVT}}$ are those arising in scalar-tensor (Horndeski) theories and SVT theories, respectively, whose explicit forms are

$$L^{(3)}_{\text{ST}} = G_3(\phi, X_1) \Box \phi, \quad (2.6)$$

$$L^{(4)}_{\text{ST}} = G_4(\phi, X_1) R + G_{4, X_2}(\phi, X_3) \left[ (\Box \phi)^2 - (\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi) \right] + \frac{1}{6} G_{5, X_1}(\phi, X_1) \left[ (\Box \phi)^3 - 3(\Box \phi)(\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi) + 2(\nabla_\mu \nabla_\alpha \phi)(\nabla^\alpha \nabla_\beta \phi)(\nabla^\beta \nabla_\mu \phi) \right], \quad (2.8)$$

and

$$L^{(2)}_{\text{SVT}} = f_2(\phi, X_1, X_2, X_3, F, Y_1, Y_2, Y_3), \quad (2.9)$$

$$L^{(3)}_{\text{SVT}} = f_3(\phi, X_3) g^{\mu\nu} S_{\mu\nu} + \tilde{f}_3(\phi, X_3) A^\mu A^\nu S_{\mu\nu}, \quad (2.10)$$

$$L^{(4)}_{\text{SVT}} = f_4(\phi, X_3) R + f_{4, X_2}(\phi, X_3) \left[ (\nabla_\mu A^\mu)^2 - \nabla_\mu A_\nu A^\nu A^\mu \right], \quad (2.11)$$

$$L^{(5)}_{\text{SVT}} = f_5(\phi, X_3) G^{\mu\nu} \nabla_\mu A_\nu = \frac{1}{6} f_{5, X_1}(\phi, X_3) \left[ (\nabla_\mu A^\mu)^3 - 3 \nabla_\mu A^{\mu\nu} \nabla_\nu A^\sigma \nabla_\sigma A^\rho + 2 \nabla_\rho A_\sigma \nabla_\gamma A^\rho A^\sigma A_\gamma \right]$$

$$+ M_5^{\mu\nu} \nabla_\mu \nabla_\nu \phi + N_5^{\mu\nu} S_{\mu\nu} \equiv \tilde{f}_5(\phi, X_3) G^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} \tilde{f}_6(\phi, X_3) \tilde{G}^{\mu\nu} \tilde{F}_{\mu\nu}, \quad (2.12)$$

$$L^{(6)}_{\text{SVT}} = f_6(\phi, X_1) L^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} + 2 f_{6, X_1}(\phi, X_1) \tilde{F}^{\mu\nu} \tilde{F}^{\alpha\beta} \nabla_\mu \phi \nabla_\nu \phi \nabla_\alpha \phi \nabla_\beta \phi + \tilde{f}_6(\phi, X_3) L^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = \frac{1}{2} \tilde{f}_6(\phi, X_3) \tilde{G}^{\mu\nu} \tilde{F}_{\mu\nu} \tilde{F}^{\alpha\beta} S_{\alpha\beta}, \quad (2.13)$$

with the notations $\Box \phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi$ and $G_{i, X_1} = \partial G_i / \partial X_1, f_j, X_3 = \partial f_j / \partial X_3$ etc. The functions $G_3, G_4, G_5$ depend on $\phi$ and its kinetic energy $X_1$. The quadratic Horndeski Lagrangian $G_2(\phi, X_1)$ is accommodated in the SVT Lagrangian $f_2$, which is a function of $\phi, X_1, F, Y_i$ (where $i = 1, 2, 3$). The functions $f_3, f_4, f_5, f_6$ depend on $\phi$ and $X_3$, while $f_6$ is a function of $\phi$ and $X_1$.

The 2-rank tensors $M_5^{\mu\nu}$ and $N_5^{\mu\nu}$ in $L^{(5)}_{\text{SVT}}$ are defined, respectively, by

$$M_5^{\mu\nu} = \mathcal{G}^{\mu\nu}_{\rho\sigma} \tilde{F}^{\rho\sigma}, \quad N_5^{\mu\nu} = \mathcal{G}^{\rho_{\rho\sigma}}_{\rho\sigma} \tilde{F}^{\rho\sigma} \tilde{F}^{\nu\sigma},$$

(2.14)
where
\[ \mathcal{G}^{h_5}_{\mu \nu} = h_{51}(\phi, X_i)g_{\mu \sigma} + h_{52}(\phi, X_i) \nabla_\mu \phi \nabla_\sigma \phi + h_{53}(\phi, X_i)A_\mu A_\sigma + h_{54}(\phi, X_i)A_\mu \nabla_\sigma \phi, \]
with the functions \( h_{5j} \) and \( \tilde{h}_{5j} \) \((j = 1, 2, 3, 4)\) depending on \( \phi \) and \( X_i \). The Lagrangians \( \mathcal{M}^{\mu \nu}_{\phi} \nabla_\mu \nabla_\nu \phi \), \( \mathcal{N}^{\mu \nu}_{S_{\mu \nu}} \), and \( \mathcal{L}^{(6)}_{\text{SVT}} \) correspond to intrinsic vector modes. The last two terms in Eq. (2.13) appear in GP theories with the \( X_3 \) dependence alone in \( f_6 \). The first two terms in Eq. (2.13) and the \( \phi \) dependence in \( \tilde{f}_6 \) arise in the context of SVT theories. As pointed out in Ref. [43], the full dependence of tensors \( \mathcal{M}^{\mu \nu}_{\phi} \) and \( \mathcal{N}^{\mu \nu}_{S_{\mu \nu}} \) on all the functions \( h_{5j} \) and \( \tilde{h}_{5j} \) in the effective metric would introduce dynamics for the temporal component of the vector field on a general background and hence an additional restriction is needed. To guarantee the absence of ghosts on arbitrary backgrounds, the dependence of \( \mathcal{M}^{\mu \nu}_{\phi} \) has to be restricted to \( X_1 \) only and similarly the dependence of \( \mathcal{N}^{\mu \nu}_{S_{\mu \nu}} \) to \( X_3 \), but for the purpose of cosmological applications, we keep the analysis general here.

In Eq. (2.5), we have taken into account the matter action \( S_m \) to include additional DOFs like radiation, dark matter, and baryons. For this matter sector, we consider a perfect fluid minimally coupled to gravity.

**B. Background equations of motion**

We consider the flat FLRW background given by the line element
\[ ds^2 = -dt^2 + a(t)^2 dx^i dx^j, \]
where \( a(t) \) is the time-dependent scale factor. The Hubble expansion rate is defined by \( H(t) = \dot{a}(t)/a(t) \), where a dot represents a derivative with respect to \( t \). The scalar and vector fields compatible with the background (2.17) are of the forms \( \phi = \phi(t) \) and \( A_\mu(t) = (A_0(t), 0, 0, 0) \), where the temporal component \( A_0(t) \) corresponds to a time-dependent auxiliary field. The matter sector is described by a perfect fluid with energy density \( \rho_m \) and pressure \( P_m \).

The background equations of motion on the flat FLRW spacetime (2.17) were already derived in Ref. [48] (see also Ref. [47]). By using coefficients of the second-order action of scalar perturbations, they can be expressed in compact forms, as
\[ 6(f_4 + G_4)H^2 + f_2 - \dot{\phi}^2 f_{2,X_1} - \frac{1}{2}\dot{\phi} A_0 f_{2,X_2} + \dot{\phi}^2 \left(3H \phi G_{3,X_1} - G_{3,\phi}\right) + 6H \left(\dot{\phi} f_{4,\phi} - H A_{0}^2 f_{4,X_3}\right) \]
\[ + 6H \phi \left(G_{4,\phi} + \dot{\phi}^2 G_{4,X_1,\phi} - 2H \dot{\phi} G_{4,X_1} - H \dot{\phi} G_{4,X_1,X_1}\right) + 2A_0 H^2 \left(3\ddot{\phi} f_{5,\phi} - H A_{0}^2 f_{5,X_3}\right) \]
\[ + H^2 \dot{\phi}^2 \left(9G_{5,\phi} + 3\dot{\phi}^2 G_{5,X_1,\phi} - 5H \dot{\phi} G_{5,X_1} - H \ddot{\phi} G_{5,X_1,X_1}\right) = \rho_m, \]
\[ 2q_\dot{H} - D_6 \ddot{\phi} + \frac{w_5}{A_0} \dot{A}_0 + D_7 \dot{\phi} = -\rho_m - P_m, \]
\[ 3D_6 \ddot{H} + 2D_1 \ddot{\phi} - D_8 \dot{A}_0 + 3D_7 H - D_9 A_0 - D_5 = 0, \]
\[ 2 \left(f_{2,X_3} + 6H^2 f_{4,X_3} - 6H \phi f_{4,X_3,\phi}\right) A_0 - 2 \left(6H f_{3,X_3} + 6H f_3 + 2\dot{\phi} f_{3,\phi} - 3H^2 f_{3,X_5} + 3H \dot{\phi} f_{5,X_5,\phi}\right) A_0^2 \]
\[ + 12H^2 f_{4,X_3,X_3} A_0^3 + 2H^2 f_{5,X_3,X_3} A_0^3 + \left(f_{2,X_3} + 4f_{3,\phi} - 6H^2 f_{5,\phi}\right) \dot{\phi} = 0, \]
\[ \dot{\rho}_m + 3H (\rho_m + P_m) = 0, \]
where \( D_1, D_5, D_6, D_7, D_8, D_9 \) and \( w_2 \) are given in Appendix A. The quantity \( q_t \) in Eq. (2.19) is defined by
\[ q_t = 2f_4 + 2G_4 - 2A_0^2 f_{4,X_3} - 2\dot{\phi}^2 G_{4,X_1} + A_0 \dot{\phi} f_{5,\phi} - HA_{0}^2 f_{5,X_3} + \dot{\phi}^2 G_{5,\phi} - H \dot{\phi} G_{5,X_1}, \]
whose positivity is required for the absence of ghosts in the tensor sector (see Sec. II C). We note that Eqs. (2.18)-(2.19) follow from Hamiltonian and momentum constraints, whereas Eqs. (2.20), (2.21), and (2.22) correspond to the equations of motion for \( \phi, A_0 \), and the perfect fluid, respectively. Differentiating Eq. (2.21) with respect to \( t \), it follows that
\[ \frac{2w_5}{A_0} \dot{A}_0 - D_6 \ddot{\phi} - \frac{3w_2}{A_0} \dot{H} - D_9 \dot{\phi} = 0, \]
where \( w_5 \) is given in Appendix A. Then, we can solve Eqs. (2.19), (2.20), and (2.24) for \( \dot{A}_0, \ddot{\phi}, \) and \( \dot{H} \) under the condition
\[ \mathcal{D} = 2 \left(4D_1 q_t w_3 + 3D_1 w_2^2 + 3D_6^2 w_5 - A_0^2 D_8^2 q_t - 3A_0 D_6 D_8 w_2\right) \neq 0. \]
The determinant $\mathcal{D}$ cannot change its sign to avoid divergences of the quantities $\dot{A}_0, \ddot{\phi}, \dot{H}$. Indeed, $\mathcal{D}$ is proportional to a quantity $q_s$ associated with the no-ghost condition of scalar perturbations [48], so that the positivity of $q_s$ corresponds to $\mathcal{D} > 0$.

C. Stability conditions of tensor and vector perturbations

The conditions for the absence of ghost and Laplacian instabilities of tensor perturbations $h_{ij}$ were derived in Ref. [48]. The perturbed line element in the tensor sector is given by

$$ ds^2_t = -dt^2 + a^2(t) (\delta_{ij} + h_{ij}) dx^i dx^j, $$

(2.26)

where the nonvanishing components of $h_{ij}$ can be chosen as $h_{11} = a^2(t) h_1(t, z)$, $h_{22} = -a^2(t) h_1(t, z)$, and $h_{12} = h_{21} = a^2(t) h_2(t, z)$ to satisfy the transverse and traceless conditions $\partial^i h_{ij} = 0$ and $h_{ii} = 0$. Expanding the action (2.5) in terms of $h_{ij}$ up to quadratic order, the resulting second-order action of tensor perturbations yields

$$ S^{(2)}_t = \int dt d^3x \sum_{i=1}^2 \frac{a^3}{q_t} \left[ h_{ij}^2 - \frac{c_i^2}{a^2} (\partial h_{ij})^2 \right], $$

(2.27)

where $q_t$ is given by Eq. (2.23), and

$$ c_i^2 = \frac{1}{q_t} \left[ 2 f_4 + 2 G_4 - A_0 \dot{\phi} f_5,\phi - \dot{A}_0 A_0^2 f_5, X_3 - \ddot{\phi}^2 G_5,\phi - \ddot{\phi}^2 \ddot{G}_{5, X_1} \right]. $$

(2.28)

Since we are considering the theories with a massless graviton, the term proportional to $h_i^2$ in the second-order action vanishes after the integration by parts. We require the two conditions $q_t > 0$ and $c_i^2 > 0$ to avoid ghost and Laplacian instabilities.

For vector perturbations, the perturbed line element in the flat gauge is given by

$$ ds^2_v = -dt^2 + 2V_i dt dx^i + a^2(t) \delta_{ij} dx^i dx^j, $$

(2.29)

where $V_i$ satisfies the transverse condition $\partial^i V_i = 0$. The spatial components of $A_\mu$ can be expressed as $A_i = Z_i + \partial_i \psi$, where $Z_i$ is the vector perturbation obeying $\partial^i Z_i = 0$ and $\psi$ is the longitudinal scalar perturbation discussed later in Sec. III. For the components of $Z_i$, we choose $Z_i = (Z_1(t, z), Z_2(t, z), 0)$ without loss of generality. The matter perfect fluid can be described by a Schutz-Sorkin action [51, 52], see Eq. (2.16) of Ref. [48]. However, it gives rise to only nondynamical perturbations like the velocity perturbation $v_i$. After integrating out all the nondynamical perturbations and taking the small-scale limit, we are left with two dynamical DOFs $Z_1$ and $Z_2$ with the quadratic action [48]

$$ S^{(2)}_v = \int dt d^3x \sum_{i=1}^2 \frac{q_v}{2} \left[ Z_i^2 - \frac{c_v^2}{a^2} (\partial Z_i)^2 - \alpha_2 Z_i^2 \right], $$

(2.30)

where

$$ q_v = f_2 F + 2 \dot{\phi}^2 f_2, V_1 + 2 \phi A_0 f_2, V_2 + 2 A_0^2 f_2, V_3 - 4 H \left( \dot{\phi} h_{51} + 2 A_0 \ddot{h}_{51} \right) + 8 H^2 \left( f_6 + \ddot{f}_6 + \phi^2 f_6, X_1 + A_0^2 \ddot{f}_6, X_3 \right), $$

(2.31)

$$ c_v^2 = \frac{2 \alpha_1 q_t + \alpha_2^2}{2 q_v q_v}, $$

(2.32)

with

$$ \alpha_1 = f_2 F - 4 A_0 \ddot{h}_{51} + 8 \left( H^2 + \dot{H} \right) \left( f_6 + \ddot{f}_6 \right) - 2 \ddot{\phi} h_{51} + H \left[ 2 \dot{\phi} \left( \phi^2 h_{52} - h_{51} + 4 \phi f_6, X_1 \right) - 4 A_0 \ddot{h}_{51} \right], $$

(2.33)

$$ \alpha_2 = -w_7 $$

$$ = f_2, X_3 + 4 H f_4, X_3 - 2 \left( A_0 + 3 H A_0 \right) \left( f_4, X_3 + \dot{f}_3 \right) - 2 \ddot{\phi} A_0 \ddot{f}_3, \phi + 2 H (3 H f_4, X_3 + 3 H A_0^2 f_4, X_3 X_3 + 2 A_0 \ddot{A}_0 f_4, X_3 X_3 $$

$$ - \ddot{\phi} f_{4, X_3} \phi + H \left( H A_0 + 2 H A_0 + 3 H^2 A_0 \right) f_5, X_3 + H^2 A_0 \left( H A_0^2 f_5, X_3 X_3 + A_0 \ddot{A}_0 f_5, X_3 X_3 - 2 \ddot{\phi} f_5, X_3 \phi \right), $$

(2.34)

$$ \alpha_3 = -2 A_0 f_4, X_3 - H A_0^2 f_5, X_3 + \phi f_5, \phi. $$

(2.35)
The quantity $w_\gamma$, which has the opposite sign to $\alpha_2$, appears in the second-order action of scalar perturbations, see Appendix A. The term $\alpha_2$ is associated with the mass squared of vector perturbations. Provided that $\alpha_2 > 0$, there is no tachyonic instability of vector perturbations. Even for $\alpha_2 < 0$, as long as the mass $\sqrt{-\alpha_2}$ is as light as today’s Hubble constant $H_0$, the tachyonic instability does not arise for perturbations inside the Hubble radius. In the small-scale limit, there are neither ghost nor Laplacian instabilities for $q_v > 0$ and $c_v^2 > 0$, whose conditions are independent of the choice of gauges.

III. GAUGE-READY FORMULATION OF SCALAR PERTURBATIONS

In this section, we derive the second-order action of scalar perturbations without fixing gauge conditions. The resulting linear perturbation equations of motion are written in the gauge-ready form, so that one can choose convenient gauges depending on the problems at hand. Let us consider the perturbed line element containing four scalar metric perturbations $\alpha, \chi, \zeta$, and $E$ [1]:

$$ds^2 = -(1 + 2\alpha)dt^2 + 2\partial_i\chi dx^i + a^2(t)\left[(1 + 2\zeta)\delta_{ij} + 2\partial_i\partial_j E\right]dx^i dx^j ,$$

where $\partial_i \equiv \partial/\partial x^i$ and $\partial_i \partial_j E \equiv \partial^2 E / \partial x^i \partial x^j$. The scalar and vector fields are expressed in the forms

$$\phi = \bar{\phi}(t) + \delta \phi, \quad A^0 = -\bar{A}_0(t) + \delta A, \quad A_i = \partial_i \psi,$$

where $\bar{\phi}(t), \bar{A}_0(t)$ are background quantities, $A_i$ is the spatial component of $A_\mu$, and $\delta \phi, \delta A, \psi$ are scalar perturbations. In the following, we omit the over-bar from background quantities.

A. Second-order matter action

To describe scalar perturbations in the matter sector, we consider the matter perfect fluid described by the Schutz-Sorkin action [51, 52]:

$$S_m = -\int d^4x \left[\sqrt{-g}\rho_m(n) + J^\mu \partial_\mu \ell \right].$$

The quantity $J^\mu$ is related to the number density $n$, as

$$n = \sqrt{\frac{J^\mu J^\nu g_{\mu\nu}}{g}} .$$

The temporal and spatial components of $J^\mu$ can be decomposed into background and perturbed parts, as

$$J^0 = N_0 + \delta J, \quad J^k = \frac{1}{a^2(t)} \delta k^i \partial_i \delta j ,$$

where $N_0$ is a constant, and $\delta J, \delta j$ are scalar perturbations. The background number density $n_0$ is given by $n_0 = N_0 / a^3$. The scalar quantity $\ell$ has a relation to the velocity potential $v$, as

$$\ell = -\int^t \rho_{m,n}(\bar{t}) d\bar{t} - \rho_{m,n} v ,$$

where $\rho_{m,n} \equiv \partial \rho_m / \partial n$. We introduce the matter density perturbation $\delta \rho_m$ in the form

$$\delta \rho_m = \frac{\rho_{m,n}}{a^3} \left[\delta J - N_0(3\zeta + \partial^2 E)\right],$$

where we use the notation $\partial^2 E \equiv (\partial_i E)(\partial_i E)$ with the same latin subscripts summed over. The perturbation of fluid number density $n$ is given by

$$\delta n = \frac{\delta \rho_m}{\rho_{m,n}} - \frac{(N_0 \partial_X + \partial \delta j)^2}{2N_0 a^3} - \frac{(3\zeta + \partial^2 E)\delta \rho_m}{\rho_{m,n}} - \frac{N_0(\zeta + \partial^2 E)(3\zeta - \partial^2 E)}{2a^3} .$$
At linear order, this reduces to $\delta n = \delta \rho_m/\rho_{m,n}$.

Expanding the Schutz-Sorkin action (3.4) up to quadratic order in scalar perturbations, it follows that

$$
(S_{m}^{(2)})_s = \int dt d^3 x \, a^3 \left[ \frac{\rho_{m,n}}{2\alpha^3 n_0} (\delta \phi)^2 + \frac{\rho_{m,n}}{\alpha^3} (\delta \chi + \delta v)(\partial \delta j) + (\dot{v} - 3H c_m^2 v - \alpha) \delta \rho_m - \frac{c_m^2}{2n_0 \rho_{m,n}} \delta \rho_m^2 + \frac{\rho_m}{2} \alpha^2 \\
+ \frac{n_0 \rho_{m,n} - \rho_m}{2} \left\{ \left( \frac{\partial^2 \delta j}{a^2} + (\zeta + \delta^2 E)(3\zeta - \delta^2 E) \right) + (3\zeta + \delta^2 E) \left\{ n_0 \rho_{m,n}(\dot{v} - 3H c_m^2 v) - \rho_m \alpha \right\} \right\} \right],
$$

(3.10)

where $c_m^2$ is the matter sound speed squared defined by

$$
c_m^2 = \frac{P_{m,n}}{\rho_{m,n}} = \frac{n_0 \rho_{m,n}}{\rho_{m,n}}.
$$

(3.11)

Varying Eq. (3.10) with respect to $\delta j$, we obtain

$$
\partial \delta j = -a^3 n_0 (\partial \dot{v} + \partial \chi).
$$

(3.12)

Substituting this relation into Eq. (3.10), the second-order matter action reduces to

$$
(S_{m}^{(2)})_s = \int dt d^3 x \, a^3 \left[ (\dot{v} - 3H c_m^2 v - \alpha) \delta \rho_m - \frac{c_m^2}{2n_0 \rho_{m,n}} \delta \rho_m^2 - \frac{n_0 \rho_{m,n}}{2\alpha^2} \left\{ (\partial \dot{v})^2 + 2\partial v \partial \chi \right\} - \frac{\rho_m}{2\alpha^2} (\partial \chi)^2 + \frac{\rho_m}{2} \alpha^2 \\
+ \frac{P_m}{2} \left( \zeta + \delta^2 E \right)(3\zeta - \delta^2 E) + (3\zeta + \delta^2 E) \left\{ n_0 \rho_{m,n}(\dot{v} - 3H c_m^2 v) - \rho_m \alpha \right\} \right],
$$

(3.13)

where we used the property that the background pressure is given by

$$
P_m = n_0 \rho_{m,n} - \rho_m.
$$

(3.14)

The second-order matter action (3.13) is written in a gauge-ready form.

**B. Full second-order action and perturbation equations of motion in gauge-ready form**

Now, we expand the total action (2.5) up to quadratic order in scalar perturbations. On using Eq. (2.18), the term $\rho_m \alpha^2 / 2$ in Eq. (3.13) is cancelled by a part of contributions proportional to $\alpha^2$ arising from $L_{ST}^{(n)} + L_{SVT}^{(n)}$. After integrations by parts, the full second-order action is expressed in the form

$$
S_s^{(2)} = \int dt d^3 x \left( L_1^{\text{flat}} + L_2^{\text{flat}} + L_3^{\text{flat}} + L_\zeta + L_\chi \right),
$$

(3.15)

where

$$
L_1^{\text{flat}} = a^3 \left[ D_1 \delta \phi^2 + D_2 \left( \frac{\partial \delta \phi}{a^2} \right)^2 + D_3 \delta \phi^2 + \left( D_4 \delta \phi + D_5 \delta \phi + D_6 \frac{\partial^2 \delta \phi}{a^2} \right) \alpha - \left( D_6 \delta \phi - D_7 \delta \phi \right) \frac{\partial^2 \chi}{a^2} \\
+ \left( D_8 \delta \phi + D_9 \delta \phi \right) \delta \chi + D_{10} \frac{\partial^2 \psi}{a^2} \right],
$$

(3.16)

$$
L_2^{\text{flat}} = a^3 \left[ \left( w_1 \alpha - w_2 \frac{\delta A}{A_0} \right) \frac{\partial^2 \chi}{a^2} - w_3 \left( \frac{\partial \alpha}{a^2} \right)^2 + w_4 \alpha^2 - \left( w_5 \frac{\partial^2 \delta A}{a^2 A_0} - w_6 \delta A + w_7 \frac{\partial^2 \psi}{a^2 A_0} \right) \alpha \\
- w_8 \frac{(\partial \delta A)^2}{4a^2 A_0^2} + \frac{w_9 \delta A^2}{A_0^2} + \left( w_{10} \delta \psi - (w_2 - A_0 w_6) \psi \right) \frac{\partial^2 \delta A}{2a^2 A_0^2} - w_3 \frac{(\partial \psi)^2}{2a^2 A_0^2} + w_{11} \frac{(\partial \psi)^2}{a^2} \right],
$$

(3.17)

$$
L_3^{\text{flat}} = a^3 \left[ (\rho_m + P_m) \frac{\partial^2 \chi}{a^2} - v_0 \delta \rho_m - 3H(1 + c_m^2) v \delta \rho_m - \frac{1}{2} \left( \rho_m + P_m \right) \frac{(\partial \dot{v})^2}{a^2} - \frac{c_m^2}{2\left( \rho_m + P_m \right)} \delta \rho_m^2 - \alpha \delta \rho_m \right],
$$

(3.18)

$$
L_\zeta = a^3 \left[ 3D_9 \delta \phi - 3D_7 \delta \phi - 3w_1 \alpha + 3w_2 \frac{\delta A}{A_0} - 3(\rho_m + P_m) v + \frac{2}{a^2} (q_1 \partial^2 \chi + \alpha_3 \partial^2 \psi) \right] \zeta - 3q_4 \zeta^2
$$
where \( q, c_s^2, \alpha_3 \) are given by Eqs. (2.23), (2.28), (2.35), respectively, and the explicit forms of coefficients \( D_{1, \ldots, 10}, w_{1, \ldots, 8} \) are shown in Appendix A. The effect of intrinsic vector modes on scalar perturbations appears through the quantity

\[
\omega_3 = -2A_0^2 q_c .
\]

The coefficients \( B_{1, \ldots, 5} \) in Eqs. (3.19)-(3.20) can be expressed by using other coefficients, as

\[
B_1 = \frac{2}{3} \left( \dot{q}_t + (1 - c_s^2)Hq_t - A_0(\alpha_3 + H\alpha_3) \right) , \quad B_2 = \dot{q}_t + 3Hq_t , \quad B_3 = \dot{D}_6 + 3HD_6 - D_7, \\
B_4 = \dot{D}_7 + 3HD_D , \quad B_5 = \frac{1}{A_0} \left[ \dot{w}_2 + 3Hw_2 + \dot{A}_0(w_6 - 4H\alpha_3) \right] .
\]

The first three Lagrangians \( \mathcal{L}_0^\text{flat}, \mathcal{L}_1^\text{flat}, \mathcal{L}_2^\text{flat} \) in Eq. (3.15) are equivalent to those derived for the flat gauge in Ref. [48]. The other two Lagrangians \( \mathcal{L}_c, \mathcal{L}_E \) arise from metric perturbations \( \zeta, E \), respectively.

Since the perturbations \( \alpha, \chi, \delta A, v, E \) do not possess their kinetic terms in the second-order action (3.15), they correspond to nondynamical variables. Varying the action (3.15) with respect to \( \alpha, \chi, \delta A, v, E \), we obtain their equations of motion in Fourier space, as

\[
\begin{align*}
\mathcal{E}_\alpha & \equiv D_4 \dot{\delta} \phi - 3w_1 \zeta + D_5 \delta \phi + 2w_4 \alpha + w_8 \frac{\delta A}{A_0} + \frac{k^2}{a^2} \left( 2(q_t - 2A_0\alpha_3)\zeta + w_6 \psi - w_1 \chi - D_6 \delta \phi - \chi + a^2 w_1 E \right) - \delta \rho_m = 0 , \\
\mathcal{E}_\chi & \equiv D_6 \delta \phi - 2q_1 \zeta - 7D_7 \delta \phi - w_1 \alpha - (\rho_m + P_m) v + w_2 \delta \frac{A}{A_0} = 0 , \\
\mathcal{E}_{\delta A} & \equiv D_8 \dot{\delta} \phi + 3w_2 \frac{\dot{\zeta}}{A_0} - D_9 \delta \phi + w_8 \frac{\alpha}{A_0} + 2w_5 \delta \frac{A}{A_0} + \frac{k^2}{a^2} \frac{1}{A_0} \left( 2A_0w_6 - w_2 \frac{A_0}{2A_0} \psi + \chi - a^2 w_1 E \right) = 0 , \\
\mathcal{E}_v & \equiv \delta \rho_m + 3H(1 + c_m^2) \delta \rho_m + 3(\rho_m + P_m) \zeta + \frac{k^2}{a^2} (\rho_m + P_m) \left( v + \chi - a^2 E \right) = 0 , \\
\mathcal{E}_E & \equiv 2q_1 \zeta + 2B_2 \dot{\zeta} - D_2 \delta \phi - B_3 \delta \phi + B_4 \delta \phi + w_1 \alpha + w_1 \chi + 3Hw_1 \alpha - \frac{w_2}{A_0} \delta \frac{A}{A_0} + B_5 \delta A + (\rho_m + P_m)(\dot{v} - 3Hc_m^2 v) = 0 ,
\end{align*}
\]

where \( k \) is a comoving wavenumber, and

\[
\chi = -\frac{w_3}{A_0} \left( \dot{\psi} + \delta A - 2A_0\alpha \right) .
\]

To simplify Eq. (3.25), we used Eq. (3.22) and the following relation

\[
w_2 + A_0w_6 = 4HA_0\alpha_3 .
\]

Variations of the action (3.15) with respect to the remaining perturbations \( \psi, \delta \phi, \delta \rho_m, \zeta \) lead to

\[
\begin{align*}
\mathcal{E}_\psi & \equiv \dot{\chi} + \left( H - \frac{\dot{A}_0}{A_0} \right) \chi + 4A_0\alpha_3 \chi - \frac{1}{A_0} \left[ (2w_6 \alpha + 2w_7 \psi - 2D_{10} \delta \phi) A_0^2 + (w_2 - w_6 A_0) \delta A \right] = 0 , \\
\mathcal{E}_{\delta \phi} & \equiv \dot{Z} + 3HZ + 3D_7 \dot{\zeta} - 2D_9 \delta \phi - D_5 \alpha - D_8 \delta A - \frac{k^2}{a^2} \left( 2D_2 \delta \phi - D_6 \dot{\alpha} - D_7 \chi - D_{10} \psi + B_1 \zeta - a^2 B_4 E \right) = 0 , \\
\mathcal{E}_{\delta \rho_m} & \equiv \dot{v} - 3Hc_m^2 v - \frac{c_m^2}{\rho_m + P_m} \delta \rho_m = 0 , \\
\mathcal{E}_\zeta & \equiv \dot{W} + 3HW + (\rho_m + P_m)(\dot{v} - 3Hc_m^2 v) + \frac{k^2}{3a^2} \left[ 2(q_t - 2A_0\alpha_3)\alpha + 2q_1c_s^2 \zeta + B_1 \delta \phi + 2\alpha_3 \delta A \right] = 0 ,
\end{align*}
\]
where

\[ Z = 2D_1 \dot{\phi} + 3D_6 \dot{\zeta} + D_4 \alpha + D_8 \delta A + \frac{k^2}{a^2} \left[ D_6 \chi - a^2 (D_6 \dot{E} + D_7 E) \right], \]  
(3.34)

\[ W = 2q_l \dot{\zeta} - D_9 \delta \phi + D_7 \delta \phi + w_1 \alpha - \frac{w_2}{A_0} \delta A + \frac{2k^2}{3a^2} (q_l \chi + \alpha_3 \psi - q_l a^2 \dot{E}). \]  
(3.35)

The second-order time derivatives \( \ddot{\zeta} \) and \( \ddot{\phi} \) can be eliminated by combining Eq. (3.27) with (3.33). On using Eqs. (3.22) and (3.29) as well, we obtain

\[ q_l \left( \alpha + \dot{\chi} + e^2 \zeta + H \chi - a^2 \dot{E} - 3a^2 H \dot{E} \right) + \dot{q}_l \left( \chi - a^2 \dot{E} \right) + \frac{B_1}{2} \left( \delta \phi - \frac{\dot{\phi}}{\dot{A_0} \psi} \right) - \left[ H(e^2 - 1) \dot{q}_l - \dot{q}_l \right] \frac{\dot{\psi}}{\dot{A_0}} = \frac{A_0 \alpha_3 Y}{w_3} = 0. \]  
(3.36)

The second-order action (3.15) and the linear perturbation Eqs. (3.23)-(3.27), (3.30)-(3.33), and (3.36) are valid for arbitrary gauges and hence they are written in gauge-ready forms.

Let us confirm the consistency of scalar perturbation equations of motion derived above. In doing so, we employ the following relations:

\[ D_1 \ddot{\phi} = -3H^2 q_l - 3H (w_1 - w_2) + w_4 + w_5 + w_8, \]  
(3.37)

\[ D_4 \ddot{\phi} = 3H w_1 - 2w_4 - w_8, \]  
(3.38)

\[ D_8 \ddot{\phi} A_0 = -3H w_2 - 2w_5 - w_8, \]  
(3.39)

\[ D_6 \ddot{\phi} = w_1 - w_2 + 2H q_1, \]  
(3.40)

\[ A_0 \ddot{\phi} D_{10} = A_0^2 w_7 + A_0 w_6 - 2(A_0 \dot{H} + \dot{A_0} H) \alpha_3, \]  
(3.41)

\[ 2A_0 \ddot{\phi} (\phi D_2 + D_7) = A_0^2 w_7 + 2A_0^2 (H \alpha_3 - \dot{H} \alpha_3 + H^2 \alpha_3) - 2A_0 w_2 \]  
\[ + A_0 \left[ 2H^2 q_1 (e^2_1 - 2) + H(2\dot{A_0} \alpha_3 + w_2 - w_1) - 2\phi D_6 + w_1 - w_2 - \rho_m - P_m \right], \]  
(3.42)

\[ 2\phi D_3 = \frac{1}{a^3} \frac{d}{dt} (a^3 D_5) - \frac{3H}{a^3} \frac{d}{dt} (a^3 D_7) + \frac{1}{a^3 A_0} \frac{d}{dt} (a^3 A_0^2 D_0), \]  
(3.43)

as well as their time derivatives. Using these properties and the background Eqs. (2.19), (2.20), (2.22), (2.24) and (3.29), it follows that there are two particular relations among the perturbation equations:

\[ \frac{1}{a^3} \frac{d}{dt} (a^3 E_\alpha) - 3HE_\zeta + \frac{1}{a^3 A_0} \frac{d}{dt} (a^3 A_0^2 E_\delta A) - \phi E_{\delta \phi} - \frac{k^2}{a^2} (E_\chi + A_0 E_\psi) + 3H (\rho_m + P_m) E_{\delta \rho_m} + E_v = 0, \]  
(3.44)

\[ E_E - \frac{1}{a^3} \frac{d}{dt} (a^3 E_\chi) = 0, \]  
(3.45)

which correspond to the temporal and spatial components of the Bianchi identity, respectively. Thus, we have confirmed the consistency of Eqs. (3.23)-(3.27) and (3.30)-(3.33) with the Bianchi identity.

C. Gauge transformations and the choice of gauges

Now, we discuss the issue of gauge transformations, gauge-invariant variables, and gauge fixings. We consider the scalar gauge transformation from the coordinate \( x^\mu = (t, x^i) \) to another coordinate \( \tilde{x}^\mu = (\tilde{t}, \tilde{x}^i) \), as

\[ \tilde{t} = t + \xi, \]  
\[ \tilde{x}^i = x^i + \delta^i_j \partial_j \xi, \]  
(3.46)

where \( \xi \) determines the time slicing and spatial threading, respectively. The four scalar metric perturbations \( \alpha, \chi, \zeta, E \) transform as

\[ \tilde{\alpha} = \alpha - \xi, \]  
\[ \tilde{\chi} = \chi + \xi - a^2 \zeta, \]  
\[ \tilde{\zeta} = \zeta - H \xi, \]  
\[ \tilde{E} = E - \xi. \]  
(3.47)

The transformations of scalar-field perturbation \( \delta \phi \) and matter density perturbation \( \delta \rho_m \) are given by

\[ \tilde{\delta} \phi = \delta \phi - \dot{\phi} \xi, \]  
\[ \tilde{\delta} \rho_m = \delta \rho_m - \rho_m \xi. \]  
(3.48)

For the vector field \( A_\mu \), we use the property that the scalar product \( A_\mu dx^\mu \) is invariant under the gauge transformation. This leads to the following relations

\[ \tilde{\delta} A = \delta A - A_0 \xi, \]  
\[ \tilde{\psi} = \psi - A_0 \xi. \]  
(3.49)
The velocity potential $v$ transforms as

$$\tilde{v} = v - \xi^0. \quad (3.50)$$

We can construct several perturbed quantities invariant under the transformation (3.46). The gauge-invariant Bardeen gravitational potentials are given by [1]

$$\Psi = \alpha + \dot{\chi} - \frac{d}{dt} \left( a^2 \dot{E} \right), \quad \Phi = \zeta + H\chi - a^2 \dot{H}, \quad (3.51)$$

which are commonly used for the study of cosmic growth history in the presence of dark energy. There are also the following gauge-invariant quantities:

$$\delta\phi_f = \delta\phi - \frac{\dot{\phi}}{H} \zeta, \quad \delta\phi_e = \delta\phi - \frac{\dot{\phi}}{A_0} \psi, \quad \delta\phi_N = \delta\phi + \dot{\phi} \chi - a^2 \dot{\phi} E, \quad (3.52)$$

$$\psi_f = \psi - \frac{A_0}{H} \zeta, \quad \psi_u = \psi - \frac{A_0}{\phi} \delta\phi, \quad \psi_N = \psi + A_0 \chi - a^2 A_0 \dot{E}, \quad (3.53)$$

$$\delta\rho_f = \delta\rho_m - \frac{\dot{\rho}_m}{H} \zeta, \quad \delta\rho_u = \delta\rho_m - \frac{\dot{\rho}_m}{\phi} \delta\phi, \quad \delta\rho_N = \delta\rho_m + \dot{\rho}_m \chi - a^2 \dot{\rho}_m E, \quad (3.54)$$

$$\delta m = \frac{\delta\rho_m}{\rho_m} + 3H \left( 1 + \frac{P_m}{\rho_m} \right) v, \quad (3.55)$$

where $\delta\phi_f$ is called the Mukhanov-Sasaki variable [53, 54].

In the context of inflationary cosmology, it is convenient to introduce the following gauge-invariant curvature perturbations [55, 56]:

$$R_\phi = \zeta - \frac{H}{\phi} \delta\phi, \quad R_\psi = \zeta - \frac{H}{A_0} \psi. \quad (3.56)$$

We define the time derivative of an adiabatic field $\sigma$ representing the velocity along the background trajectory, as [57]

$$\dot{\sigma} = \left( \cos \theta \right) \dot{\phi} + \left( \sin \theta \right) A_0, \quad (3.57)$$

where

$$\cos \theta = \frac{\dot{\phi}}{\sqrt{\dot{\phi}^2 + A_0^2}}, \quad \sin \theta = \frac{A_0}{\sqrt{\dot{\phi}^2 + A_0^2}}. \quad (3.58)$$

The adiabatic field perturbation $\delta\sigma$ and the entropy perturbation $\delta s$ orthogonal to the background trajectory are defined, respectively, by

$$\delta\sigma = \left( \cos \theta \right) \delta\phi + \left( \sin \theta \right) \psi, \quad (3.59)$$

$$\delta s = \left( \cos \theta \right) \psi - \left( \sin \theta \right) \delta\phi, \quad (3.60)$$

where $\delta s$ is gauge-invariant by construction. We also introduce the total gauge-invariant curvature perturbation incorporating both $\delta\phi$ and $\psi$, as

$$R = \left( \cos^2 \theta \right) R_\phi + \left( \sin^2 \theta \right) R_\psi$$

$$= \zeta - \frac{H\left( \dot{\phi} \delta\phi + A_0 \psi \right)}{\dot{\phi}^2 + A_0^2}. \quad (3.61)$$

In terms of the adiabatic field $\sigma$ and its perturbation $\delta\sigma$, Eq. (3.61) can be expressed as

$$R = \zeta - \frac{H\delta\sigma}{\dot{\sigma}}. \quad (3.62)$$

For the background field trajectory satisfying $\dot{\theta} \neq 0$, the entropy perturbation $\delta s$ generally works as a source term for the adiabatic perturbation $\delta\sigma$ [57]. Hence the evolution of $R$ is known by studying how $\delta s$ and $\delta\sigma$ evolve in time [58, 59].
On using the gauge-invariant variables (3.51) and (3.56), we can write Eq. (3.36) in the following simple form

\[ \Psi + (1 + \alpha_M) \Phi + (c_2^2 - 1 - \alpha_M) R_\phi + \frac{A_0}{H q_t} (\dot{\alpha}_3 + H \alpha_3) (R_\phi - R_\psi) - \frac{A_0 \alpha_3}{q_t w_3} Y = 0, \]  

(3.63)

where

\[ \alpha_M \equiv \frac{\dot{q}_t}{H q_t}. \]  

(3.64)

We note that the perturbation \( Y \) defined by Eq. (3.28) is also gauge-invariant.

Let us consider theories satisfying the condition

\[ \alpha_3 = -2A_0 f_4 X_3 - HA_0^2 f_5 X_3 + \dot{\phi} f_5, \phi = 0. \]  

(3.65)

Then, Eq. (3.63) reduces to

\[ \Psi + (1 + \alpha_M) \Phi + (c_2^2 - 1 - \alpha_M) R_\phi = 0 \quad \text{for} \quad \alpha_3 = 0. \]  

(3.66)

The condition (3.65) is satisfied not only for Horndeski theories but also for SVT theories with the couplings:

\[ f_4 = f_4(\phi), \quad f_5 = \text{constant}. \]  

(3.67)

In such cases, the time variation of \( q_t \) (i.e., \( \alpha_M \neq 0 \)) and the deviation of \( c_2^2 \) from 1 give rise to the gravitational slip \((-\Psi \neq \Phi\)) for SVT theories with the couplings \( f_4 = f_4(X_3) \) and \( f_5 = f_5(\phi, X_3) \). The last two terms in Eq. (3.63) also work as additional anisotropic stresses.

Under the transformation (3.46), there are residual gauge DOFs for fixing \( \xi^0 \) and \( \xi \). Several gauge conditions commonly used in the literature are

\[ \zeta = 0, \quad E = 0, \quad \text{(Flat gauge)}, \]  

(3.68)

\[ \delta \phi = 0, \quad E = 0, \quad \text{(Unitary gauge)}, \]  

(3.69)

\[ \psi = 0, \quad E = 0, \quad \text{(Uniform vector gauge)}, \]  

(3.70)

\[ \chi = 0, \quad E = 0, \quad \text{(Newtonian gauge)}, \]  

(3.71)

\[ \alpha = 0, \quad \chi = 0, \quad \text{(Synchronous gauge)}. \]  

(3.72)

Apart from the synchronous gauge in which \( \xi^0 \) is not unambiguously fixed, the other gauges (3.68)-(3.71) completely fix \( \xi^0 \) and \( \xi \).

For the flat gauge, the dynamical DOFs correspond to the perturbations \( \delta \phi_t, \psi_t, \) and \( \delta \rho_t \). In Refs. [47, 48], the second-order action of these dynamical fields was derived by choosing the flat gauge. This gauge choice is valid in the expanding Universe \((H > 0)\), but as we see in Sec. V, it is not suitable for describing the evolution of perturbations in bouncing cosmologies. This is not generally the case for gauges in which the perturbation \( \zeta \) does not vanish, e.g., (3.69)-(3.71).

If we apply SVT theories to dark energy and choose the flat gauge, the contributions of tensor, vector, and scalar perturbations in observables associated with the cosmic growth are not transparent [48]. In Sec. VI, we show that the separation between tensor, vector, and scalar modes becomes clear by choosing gauges in which \( \zeta \) does not vanish, e.g., the unitary gauge.

Thus, in our gauge-ready formulation, we can choose most suitable and convenient gauges depending on the problem under consideration. While the underlying physics is not affected by the choice of different gauges, it makes sense to choose most appropriate gauges in which the physical meaning and interpretation of results are transparent.

IV. STABILITY CONDITIONS IN UNITARY AND NEWTONIAN GAUGES

In this section, we derive conditions for the absence of ghost and Laplacian instabilities of scalar perturbations in the small-scale limit by choosing the unitary and Newtonian gauges. In Ref. [48], the similar analysis was performed in the flat gauge, but this gauge choice is not necessarily suitable for studying the evolution of curvature perturbations in the bouncing cosmology (as we will discuss in Sec. V). This problem can be circumvented by choosing other suitable gauges like unitary and Newtonian gauges. In Appendix B, we also obtain stability conditions in the uniform vector gauge.
A. Unitary gauge

Let us first consider the unitary gauge characterized by Eq. (3.69). In this case, the dynamical perturbations correspond to \( \psi_\mu = \psi, \phi, \zeta \), and \( \delta \rho_a = \delta \rho_m \), which are represented by the matrix
\[
\chi^a = (\psi_\mu, \phi, \zeta) / k .
\] (4.1)

From Eqs. (3.23)-(3.26), the nondynamical perturbations \( \alpha, \chi, \delta A, v \) can be expressed in terms of \( \psi_\mu, \phi, \delta \rho_m \) and their time derivatives. Substituting them into Eq. (3.15) and integrating it by parts, the second-order scalar action in Fourier space reduces to
\[
S_s^{(2)} = \int dt d^3x a^3 \left( \hat{\chi}^a K \hat{\chi}^a - \frac{k^2}{a^2} \hat{\chi}^a G \hat{\chi}^a - \hat{\chi}^a M \hat{\chi}^a - \hat{\chi}^a B \hat{\chi}^b \right),
\] (4.2)
where \( K, G, M, B \) are \( 3 \times 3 \) matrices. The leading-order contributions to \( M \) and \( B \) correspond to the order of \( k^0 \). In the small-scale limit, the nonvanishing matrix components of \( K \) and \( G \) are given by
\[
K_{11} = \frac{w^2 w_5 + w_2 w_4 + w_1 w_2 w_8}{A_0^2 (w_1 - 2w_2)^2}, \quad K_{22} = q t \left[ 3 + \frac{4q(4w_4 + 4w_5 + 2w_8)}{(w_1 - 2w_2)^2} \right],
\]
\[
K_{12} = K_{21} = \frac{q t [4w_4w_5 + 2w_2(4w_4 + w_8)]}{A_0(w_1 - 2w_2)^2}, \quad K_{33} = \frac{a^2}{2(\rho_m + P_m)},
\] (4.3)

and
\[
G_{11} = \frac{\alpha_2^2}{2} + \frac{2E_1}{q_v} - \frac{w^2 (\rho_m + P_m)}{2A_0^2 (w_1 - 2w_2)^2} + \frac{1}{a} \frac{d}{dt} (a E_1), \quad G_{22} = -q t c^4_1 + \frac{2E_2}{q_v} - \frac{2q^2 (\rho_m + P_m)}{(w_1 - 2w_2)^2} + \frac{1}{a} \frac{d}{dt} (a E_2),
\]
\[
G_{12} = G_{21} = \frac{2E_1 E_3}{q_v} - \frac{w_2 q (\rho_m + P_m)}{A_0 (w_1 - 2w_2)^2} + \frac{1}{a} \frac{d}{dt} (a E_3), \quad G_{33} = \frac{c^4_2 a^2}{2(\rho_m + P_m)},
\] (4.4)

where we used the relation \( \alpha_2 = -w_7 \), and
\[
E_1 = \frac{w_6}{4A_0} - \frac{w_1 w_2}{4A_0^2 (w_1 - 2w_2)^2}, \quad E_2 = -\frac{2q^2}{w_1 - 2w_2}, \quad E_3 = -\frac{w_2 + A_0 w_6}{4HA_0} - \frac{q t w_2}{A_0 (w_1 - 2w_2)}.
\] (4.5)

The last time derivatives in \( G_{11}, G_{22}, G_{12} \) arise from partial integrations of the terms containing \( k^2/a^2 \) in \( \hat{\chi}^a B \hat{\chi}^a \). The matter perfect fluid is decoupled from other fields \( \psi_\mu \) and \( \phi \), so that the ghost and Laplacian instabilities are absent for \( \rho_m + P_m > 0 \) and \( c^4_2 > 0 \). The quantities \( K_{11} \) and \( G_{11} \) are identical to those derived for the flat gauge in Ref. [48], but \( K_{22}, K_{12}, G_{22}, G_{12} \) are different by reflecting the fact that \( R_\phi \) corresponds to the dynamical DOF in the unitary gauge (unlike \( \delta \phi_t \) in the flat gauge).

The conditions for the absence of scalar ghosts are given by \( K_{11} > 0 \) or \( K_{22} > 0 \), and
\[
q_s \equiv K_{11} K_{22} - K_{12}^2 > 0 .
\] (4.6)

In the unitary gauge, the quantity \( q_s \) reduces to
\[
q_s(\mu) = \frac{q_t [4w_4 w_5 - w^2_2] + 3(w^2 w_5 - w_1 w_2 w_8 + w^2_2 w_4)]}{A_0^2 (w_1 - 2w_2)^2}.
\] (4.7)

On using the properties (3.37)-(3.40), the determinant \( D \) defined by Eq. (2.25) is expressed in the form
\[
D = \frac{2}{\rho_m^2} \left[ q_t (4w_4 w_5 - w^2_2) + 3(w^2 w_5 - w_1 w_2 w_8 + w^2_2 w_4) \right] .
\] (4.8)

Then, \( q_s(\mu) \) is proportional to \( D \), as
\[
q_s(\mu) = \frac{\rho_m^2 q_t}{2A^2_0 (w_1 - 2w_2)^2} D.
\] (4.9)

Since \( q_s(\mu) > 0 \) and \( q_t > 0 \) for the absence of scalar and tensor ghosts, the determinant associated with the closed-form background equations of motion needs to be in the range
\[
D > 0 .
\] (4.10)
In the flat gauge chosen in Refs. [43, 48], the quantity (4.6) is given by
\[ q_s^{(f)} = \frac{H^2 q_t}{2A_0^2(w_1 - 2w_2)^2} D = \frac{H^2}{\phi^2} q_s^{(u)}, \] (4.11)
which is different from \( q_s^{(u)} \) only by an overall factor \( H^2/\phi^2 \).

Taking the small-scale limit in Eq. (4.2), the dispersion relation yields \( \det(c_s^2 K - G) = 0 \), where \( c_s \) is the propagation speed of scalar perturbations. One of the solutions is the matter propagation speed squared \( c_m^2 \), while the other two solutions are
\[ c_{s1}^2 = \frac{F_s}{2q_s} \left[ 1 + \sqrt{1 - \frac{4q_s G_s}{F_s^2}} \right], \quad c_{s2}^2 = \frac{F_s}{2q_s} \left[ 1 - \sqrt{1 - \frac{4q_s G_s}{F_s^2}} \right], \] (4.12)
where
\[ F_s = K_{11}G_{22} + K_{22}G_{11} - 2K_{12}G_{12}, \quad G_s = G_{11}G_{22} - G_{12}^2. \] (4.13)

To avoid small-scale Laplacian instabilities, we require the two conditions \( c_{s1}^2 > 0 \) and \( c_{s2}^2 > 0 \).

In the flat gauge, the matrix components of \( K \) and \( G \) contain the terms \( D_1, D_2, D_4, D_6, D_7, D_8, D_{10} \) besides \( w_i \), see Eqs. (5.22) and (5.23) of Ref. [48]. On using Eqs. (3.23)-(3.26) and Eq. (3.36), the nondynamical perturbations \( \delta\phi, \dot{\delta}\phi, \delta\dot{\phi} \) can be expressed in terms of \( \psi_N, \Phi, \delta\rho_N \) and their first-order time derivatives. Substituting those relations into Eq. (3.15) and integrating it by parts, the second-order scalar action reduces to the form (4.2). In the small-scale limit, the nonvanishing matrix components of \( K \) and \( G \) are
\[ K_{11} = \frac{D_1 w_2^2 + D_2 w_5 - A_0 D_6 D_8 w_2}{A_0^2 D_6^2}, \quad K_{22} = 3q_t + \frac{4D_1 q_t^2}{D_6^2}, \quad K_{12} = K_{21} = \frac{q_t (2D_1 w_2 - A_0 D_6 D_8)}{A_0 D_6^2}, \quad K_{33} = \frac{a^2}{2(\rho_m + P_m)}, \] (4.16)
and
\[ G_{11} = \frac{\alpha_2}{2} + \frac{(w_2 - A_0 w_6)^2}{8A_0^2 q_t} + \frac{w_2 (A_0 D_6 D_{10} - D_2 w_2)}{A_0^2 D_6^2} - \frac{1}{a} \frac{d}{dt} \left[ \frac{a(w_2 - A_0 w_6)}{4A_0^2} \right], \]
\[ G_{22} = -q_t c_t^2 + \frac{2\alpha_3}{q_t} - \frac{q_t (2\dot{q}_t D_2 + D_6 B_1)}{D_6^2}, \]
\[ G_{12} = G_{21} = \frac{q_t (2D_2 w_2 - A_0 D_6 D_{10})}{A_0 D_6^2} - \frac{w_2 B_1}{2A_0 D_6} + \frac{\alpha_3 (w_2 - A_0 w_6)}{2A_0^2 q_t} - \frac{1}{a} \frac{d}{dt} (a\alpha_3), \quad G_{33} = \frac{c_m^2 a^2}{2(\rho_m + P_m)}. \] (4.17)

Again, the matter perturbation \( \delta\rho_N \) is decoupled from other fields \( \psi_N \) and \( \Phi \).

The matrix components \( K_{11}, K_{22}, K_{12} \) in the Newtonian gauge are not the same as those in the unitary gauge, but the combination \( q_s = K_{11}K_{22} - K_{12}^2 \) is related to each other among different gauges up to positive overall factors. In the Newtonian gauge, the quantity \( q_s \) is given by
\[ q_s^{(N)} = \frac{q_t}{2A_0^2 D_6} D. \] (4.18)
On using Eqs. (3.40) and (4.11), we find that $q_s^{(N)}$ is related to $q_s^{(u)}$ and $q_s^{(f)}$, as

$$ q_s^{(N)} = \frac{(w_1 - 2w_2)^2}{(w_1 - w_2 + 2Hq_t)^2} q_s^{(u)} = \frac{(w_1 - 2w_2)^2}{(w_1 - w_2 + 2Hq_t)^2} \frac{\dot{\phi}^2}{H^2} q_s^{(f)}. \tag{4.19} $$

If $q_s > 0$ in one of the gauges, then this property holds in other gauges as well.

Employing Eqs. (2.19), (3.29), and (3.37)-(3.42) for computing the quantities defined by Eq. (4.13), it follows that $\mathcal{F}_s$ and $\mathcal{G}_s$ in the Newtonian gauge are related to those in the unitary gauge, as

$$ \mathcal{F}_s^{(N)} = \frac{(w_1 - 2w_2)^2}{(w_1 - w_2 + 2Hq_t)^2} \mathcal{F}_s^{(u)}, \quad \mathcal{G}_s^{(N)} = \frac{(w_1 - 2w_2)^2}{(w_1 - w_2 + 2Hq_t)^2} \mathcal{G}_s^{(u)}. \tag{4.20} $$

Then, the scalar sound speed squares (4.12) are the same in both Newtonian and unitary gauges. As we see in Appendix B, this property also holds in the uniform vector gauge. Thus, we explicitly showed that $c_{s1}^2$ and $c_{s2}^2$ are gauge-invariant quantities.

V. APPLICATION TO NONSINGULAR COSMOLOGY

In this section, we apply the stability conditions derived in Sec. IV to the nonsingular cosmology in which the scale factor is always in the region $a > 0$. Our main interest is to discuss the possibility for realizing nonsingular bouncing/genesis solutions free from ghost and Laplacian instabilities as well as a strong coupling problem.

A. No-ghost condition at the bounce

Let us first consider the bouncing cosmology in which the Universe transits from the collapse to the expansion. Then, the Hubble parameter $H$ vanishes at the point of bounce. At $H = 0$, the gauge-invariant perturbations $\delta \phi, \psi, \delta \rho_T$ in Eqs. (3.52)-(3.54) are not well defined because their denominators vanish. For the flat gauge ($\zeta = 0$), it looks as if such divergences can be circumvented, but the problem manifests in curvature perturbations defined by Eq. (3.56). Since $\mathcal{R}_T = -H \delta \phi / \dot{\phi}$ and $\mathcal{R}_\psi = -H \psi / \dot{\phi}$ in the flat gauge, both $\mathcal{R}_\phi$ and $\mathcal{R}_\psi$ vanish at $H = 0$.

Provided that $A_0(w_1 - 2w_2) \neq 0$, the quantity $q_s^{(f)}$ given by Eq. (4.11) is 0 at $H = 0$. Then, it looks as if there is the strong coupling problem at the bounce, but this is an artifact of choosing the flat gauge in which $\mathcal{R}_\phi$ and $\mathcal{R}_\psi$ vanish at $H = 0$. The gauge-invariant variables $\psi_u, \mathcal{R}_\theta,$ and $\delta \rho_{\psi}$, which reduce to $\psi, \zeta,$ and $\delta \rho_m$ respectively in the unitary gauge ($\delta \phi = 0$), are well defined except for $\dot{\phi} = 0$. At the bounce, both $\mathcal{R}_\phi$ and $\mathcal{R}_\psi$ reduce to $\zeta$. The right hand side of Eq. (4.9) does not contain terms proportional to $H^2$, so $q_s^{(u)}$ does not vanish at $H = 0$. This means that there is no strong coupling problem at the bounce.

The gauge-invariant perturbations $\psi_N, \Phi$, and $\delta \rho_{\Phi}$, which reduce to $\psi, \zeta$, and $\delta \rho_m$ respectively in the Newtonian gauge ($\chi = E = 0$), are also well defined during the transition across the bounce. From Eq. (4.18), the quantity $q_s^{(N)}$ does not generally vanish at $H = 0$, so there is no strong coupling problem at the bounce under this gauge choice. The same property also holds for the uniform vector gauge ($\psi = 0$) with the dynamical perturbations $\delta \phi_N, \mathcal{R}_\psi,$ and $\delta \rho_{\psi}$.

The above discussion shows that the strong coupling problem arises only when the combination $q_t \mathcal{D}$, which appears in $q_s$, for any gauge choice, crosses 0. Under the no-ghost condition of tensor perturbations ($q_t > 0$), the strong coupling only occurs when $\mathcal{D}$ approaches 0. In the limit that $\mathcal{D} \to 0$, however, the background equations of motion exhibit the divergence. Thus, the strong coupling problem of scalar perturbations can be avoided for the nonsingular background cosmology in which the determinant is always in the range $\mathcal{D} > 0$ and does not approach 0.

B. Possibility for realizing nonsingular cosmology

There have been attempts for constructing bouncing/genesis cosmological solutions without the initial singularity. This requires the violation of null energy condition, which is not realized by conventional matter satisfying $\rho_m + P_m \geq 0$. Galileons and its generalizations [60–62] can be the candidates for violating the null energy condition. Indeed, generalized Galileons allow the existence of nonsingular bouncing solutions with neither ghost nor Laplacian instabilities around the bounce [63–66]. In the original genesis scenario and its variants [67–73], it is possible to realize an initial super-accelerating stage without ghost and Laplacian instabilities.
Although generalized Galileons can give rise to nonsingular solutions stable near the bounce or during the super-accelerating stage, the stability of cosmological solutions is not necessarily guaranteed during the whole cosmological history. Indeed, for the cubic-order generalized Galileon and its extensions, the Laplacian instabilities arise during the transition from the bouncinggenesis period to the subsequent stage [74–78]. In Ref. [79], it was shown that this conclusion also holds for full Horndeski theories. In what follows, we first revisit the no-go argument in Horndeski theories for the absence of stable nonsingular solutions throughout the cosmic history and then discuss what happens in GP and SVT theories.

1. Horndeski theories

In Horndeski theories with the matter perfect fluid, there are two dynamical scalar DOFs. In the unitary gauge (3.69), these DOFs are characterized by the perturbations

$$X^t = (R_\phi, \delta \rho_a / k).$$

(5.1)

After integrating out nondynamical DOFs, the second-order action of scalar perturbations is of the form (4.2) with $2 \times 2$ matrices $K$, $G$, $M$, $B$. In the small-scale limit, the stability conditions of $\delta \rho_a$ are the same as those in SVT theories, i.e., $\rho_m + P_m > 0$ and $c_s^2 > 0$. For the perturbation $R_\phi$, the no-ghost condition corresponds to

$$q_s^{(u)} \equiv q_t \left( 3 + \frac{4q_t w_3}{w_1^2} \right) > 0,$$

(5.2)

where $q_s^{(u)}$ is equivalent to the matrix component $K_{22}$ in Eq. (4.3) with $w_2 = w_5 = w_8 = 0$. In Horndeski theories, the product of $q_s^{(u)}$ and the scalar propagation speed squared $c_s^2$ is equivalent to $G_{22}$ in Eq. (4.4) without the term $2E_2^2 / q_t$. Then, we obtain the following relation

$$\frac{1}{a} \frac{d}{dt} (a E_2) = q_s^{(u)} c_s^2 + q_t c_t^2 + \frac{2q_t^2 (\rho_m + P_m)}{w_1^2},$$

(5.3)

where

$$E_2 = -\frac{2q_t^2}{w_1}.$$  

(5.4)

In the bouncing Universe, the scale factor $a(t)$ reaches a positive minimum at the bounce and it approaches a positive constant or diverges in the asymptotic past ($t \to -\infty$) and future ($t \to \infty$). The genesis model corresponds to the case in which the scale factor and its time derivative are finite for all $-\infty < t < \infty$. Since we require that the perturbations are prone to neither ghost nor Laplacian instabilities, the three terms on the right hand side of Eq. (5.3) are positive. Then, the following inequality holds

$$\frac{1}{a} \frac{d}{dt} > q_t c_t^2 > 0,$$

(5.5)

where

$$\xi \equiv a E_2 = -\frac{2aq_t^2}{w_1}.$$  

(5.6)

Integrating Eq. (5.5) from $t = t_i$ to $t = t_f$ (> $t_i$), we obtain

$$\xi_f - \xi_i > \int_{t_i}^{t_f} a q_t c_t^2 \, dt > 0.$$  

(5.7)

In the following, we consider the case in which the quantity $q_t c_t^2$ does not approach 0 for $t_i \to -\infty$ and $t_f \to \infty$. The limit $q_t c_t^2 \to 0$ corresponds to either $q_t \to 0$ or $c_t^2 \to 0$. For $q_t \to 0$, the strong coupling problem of tensor perturbations arises. In another limit $c_t^2 \to 0$ the gradient term in Eq. (2.27) vanishes, so nonlinear contributions to the tensor action are out of control. From the view point of quantum field theory, the leading-order solution to $h_i$ corresponding to the Bunch-Davies vacuum is proportional to $1/(c_t k)^{3/2}$ [80], which diverges for $c_t \to 0$.

Since $q_t c_t^2$ does not decrease toward 0 in the asymptotic future, the integral in Eq. (5.7) is a positive growing function of $t_f$. Hence the consistency of Eq. (5.7) demands that $\xi_f > 0$ for sufficiently large $t_f$. The integral also
increases toward the asymptotic past \((t_i \to -\infty)\), so we require the condition \(\xi_i < 0\). Then, the function \(\xi\) crosses 0 at some time between \(-\infty < t < \infty\), which correspond to \(a = 0\) from Eq. (5.6). This behavior is at odds with the nonsingular bouncing/genesis cosmology in which \(a > 0\) throughout the cosmological evolution.

The no-go argument given above has been proven in the unitary gauge. The same argument also holds for other gauges in which the perturbations are well defined at the bounce. In the Newtonian gauge, for example, the structure of \(q_s^{(N)} c_s^2\) apparently looks different from that in the unitary gauge in that there are neither \((1/a)d(aE_2)/dt\) nor \(2q^2_t(\rho_m + P_m)/w^2_t\) terms in \(G_{22}\) of Eq. (4.17). However, we can exploit the facts that \(q_s^{(N)}\) is related to \(q_s^{(u)}\) according to \(q_s^{(N)} = w^2_t q_s^{(u)}/(w_1 + 2H q_t)^2\) and that \(c_s^2\) is a gauge-invariant quantity. Then, we obtain the following relation

\[
\frac{1}{a} \frac{d}{dt} (aE_2) = \left( \frac{w_1 + 2H q_t}{w_1^2} \right) q_s^{(N)} c_s^2 + q_t c_t^2 + \frac{2q^2 q_t (\rho_m + P_m)}{w_1^2} > qv c_t^2, \tag{5.8}
\]

where the last inequality holds due to the stability conditions \(q_s^{(N)} c_s^2 > 0\) and \(\rho_m + P_m > 0\). This means that the same no-go statement also holds in the Newtonian gauge. Hence, the absence of consistent bouncing solutions in Horndeski theories is not an artifact of the gauge choice, but it is a real unavoidable physical problem. We will see in the following how this can be naturally avoided in GP and SVT theories due to the presence of intrinsic vector modes.

2. GP theories

The crucial point of the no-go argument in Horndeski theories is that, besides the term \(q_t c_t^2\), all the other terms on the right hand side of Eq. (5.3) are positive for the absence of ghost and Laplacian instabilities. Let us consider GP theories in the presence of a matter perfect fluid. Choosing the uniform vector gauge (3.70), the dynamical scalar DOFs are given by

\[ \mathcal{X}^t = (R_{\psi}, \delta \rho_c, /k) . \tag{5.9} \]

Since the scalar-field perturbation \(\delta \phi\) is absent in GP theories, the computation of \(q_s^{(v)}\) and \(c_s^2\) in the uniform vector gauge \((\psi = 0)\) is analogous to that of \(q_s^{(u)}\) and \(c_s^2\) in Horndeski theories with the choice of unitary gauge \((\delta \phi = 0)\). For the matter perturbation \(\delta \rho_c\), the conditions for the absence of ghost and Laplacian instabilities are given by \(\rho_m + P_m > 0\) and \(c_m^2 > 0\). In GP theories, there are the following relations

\[
w_1 = w_2 - 2H q_t, \quad w_4 = w_5 + \frac{3}{2} H (w_1 + w_2), \quad w_8 = 3H w_1 - 2w_4. \tag{5.10}
\]

For the perturbation \(R_{\psi}\), the ghost is absent for

\[
q_s^{(v)} = \frac{q_t (3w_2^2 + 4w_5 q_t)}{(2H q_t + w_2)^2} > 0, \tag{5.11}
\]

which is equivalent to \(K_{22}\) in Eq. (4.3) after the substitution of Eq. (5.10). Since the Hubble parameter \(H\) does not appear in the numerator of \(q_s^{(v)}\), the strong coupling problem does not arise at \(H = 0\). The product of \(q_s^{(v)}\) and \(c_s^2\) reduces to the same form as \(G_{22}\) in Eq. (4.4) with the particular relations (5.10). Then, it follows that

\[
\frac{1}{a} \frac{d}{dt} (aE_2) = q_s^{(v)} c_s^2 + q_t c_t^2 + \frac{2q^2 q_t (\rho_m + P_m)}{(2H q_t + w_2)^2} - \frac{2E_2^2}{q_v}, \tag{5.12}
\]

where

\[
E_2 = \frac{2q^2}{2H q_t + w_2}, \quad E_3 = \frac{1}{4H} \left[ \frac{w_2 (2H q_t - w_2)}{A_0 (2H q_t + w_2)} - w_6 \right]. \tag{5.13}
\]

Compared to the relation (5.3) in Horndeski theories, there is the additional term \(-2E_3^2/q_v\) in Eq. (5.12). This new term arises from the existence of intrinsic vector modes. Since \(q_v > 0\) for the absence of vector ghosts, the term \(-2E_3^2/q_v\) needs to be negative. Then, unlike Horndeski theories, the right hand side of Eq. (5.12) is no longer bounded from below with the minimum value \(q_t c_t^2\).
Integration of Eq. (5.12) from \( t = t_i \) to \( t = t_f \) leads to

\[
\xi_f - \xi_i = \int_{t_i}^{t_f} a \left[ q_0^2 c_s^2 + q_0 c_t^2 + \frac{2E_1^2(\rho_m + P_m)}{(2Hq_t + w_2)^2} - \frac{2E_2}{q_v} \right] dt,
\]

where \( \xi = aE_2 = 2aq_0^2/(2Hq_t + w_2) \). If the contribution \(-2E_2^2/q_v\) to the square bracket of Eq. (5.14) dominates over the other terms in the asymptotic past \( (t_i \to -\infty) \), then the integral goes to \(-\infty\) and hence \( \xi_i > 0 \). If the term \(-2E_2^2/q_v\) is subdominant to \( q_0 c_t^2 > 0 \) in the asymptotic future \( (t_f \to \infty) \), the integral grows toward \( \infty \) and hence \( \xi_f > 0 \). In this case, it is possible to have \( \xi > 0 \) throughout the cosmological evolution. This means that, in GP theories, there is a possibility for realizing nonsingular bouncing/genesis solutions where the scale factor is always in the region \( a > 0 \). This is a very promising property of GP theories for bouncing solutions compared to Horndeski theories.

3. SVT theories

In SVT theories, there are two scalar propagation speed squares given by Eq. (4.12) and hence

\[
q_s c_{s1}^2 + q_s c_{s2}^2 = F_s ,
\]

\[
q_s c_{s1}^2 c_{s2} = G_s .
\]

The positivities of \( F_s \) and \( G_s \) are required to avoid ghost and Laplacian instabilities of scalar perturbations. From Eq. (5.16), it follows that

\[
G_{11} G_{22} = q_s c_{s1}^2 c_{s2} + G_{12}^2 > 0 ,
\]

which means that either (i) \( G_{11} > 0 \) and \( G_{22} > 0 \), or (ii) \( G_{11} < 0 \) and \( G_{22} < 0 \). In the unitary gauge, the expressions of \( G_{11} \) and \( G_{22} \) have been derived in Eq. (4.4), so that

\[
\frac{1}{a} \frac{d}{dt} (aE_1) = G_{11} - \frac{\alpha_2}{2} - 2E_1^2q_v + \frac{w_2^2(q_0 + P_m)}{2A_s^2(w_1 - 2w_2)^2},
\]

\[
\frac{1}{a} \frac{d}{dt} (aE_2) = G_{22} + q_s c_t^2 - \frac{2E_2^2}{q_v} + \frac{2q_0^2(\rho_m + P_m)}{(w_1 - 2w_2)^2},
\]

where \( E_1, E_2, E_3 \) are defined by Eq. (4.5). The tachyonic instability of vector perturbations can be avoided for \( \alpha_2 > 0 \), but this condition is not obligatory compared to conditions for the absence of ghost and Laplacian instabilities. For \( G_{11} > 0 \) and \( G_{22} > 0 \), the situation is analogous to what we discussed in GP theories. The intrinsic vector-mode contributions \(-2E_1^2/q_v\) and \(-2E_2^2/q_v\) to Eqs. (5.18) and (5.19), which are required to be negative, allow the possibility for evading the no-go argument in Horndeski theories, in such a way that the quantities \( aE_1 \) and \( aE_2 \) can remain positive throughout the cosmological evolution. When \( G_{11} < 0 \) and \( G_{22} < 0 \), the no-go statement does not hold either. Thus, in SVT theories, it would be possible to realize nonsingular bouncing/genesis solutions without theoretical pathologies. We note that such nonsingular solutions should be constructed to satisfy the conditions \( F_s > 0 \) and \( G_s > 0 \) besides \( q_s > 0 \), without having the behavior \( q_s c_t^2 \to 0 \) in the asymptotic past and future.

VI. APPLICATION TO DARK ENERGY

In this section, we apply the gauge-ready formulation of Sec. III to the case in which the scalar field \( \phi \) and the vector field \( A_\mu \) are the source for the late-time cosmic acceleration. For the matter action \( S_m \), we consider a nonrelativistic perfect fluid satisfying \( P_m \simeq 0 \) and \( c_m^2 \simeq 0 \). We are interested in observables relevant to the evolution of matter perturbations and gravitational potentials to test dark energy models in SVT theories with the measurements of redshift-space distortions, weak lensing, and CMB.

From Eqs. (3.26) and (3.32), the matter perturbation \( \delta \rho_m \) and the velocity potential \( v \) obey

\[
\delta \rho_m + 3H \delta \rho_m + \dot{\rho}_m \left[ 3\dot{c}^2 + \frac{k^2}{a^2} \left( v + \chi - a^2 \dot{E} \right) \right] = 0 ,
\]

\[
\dot{v} = \alpha .
\]
Taking the time derivative of Eq. (6.1) and using Eq. (6.2), the gauge-invariant density contrast \( \delta_m = \delta \rho_m / \rho_m + 3 H v \) satisfies

\[
\dot{\delta}_m + 2H \delta_m + \frac{k^2}{a^2} \Psi = 3 \left( \dot{\Phi} + 2HB \right),
\]

(6.3)

where \( B = H v - \zeta \), and \( \Psi \) is the gauge-invariant gravitational potential defined in Eq. (3.51). We relate the Newtonian gravitational potential \( \Psi \) and the weak lensing potential \( \psi_{\text{eff}} = \Phi - \Psi \) with \( \delta_m \), as

\[
\frac{k^2}{a^2} \Psi = -4\pi G \mu \rho_m \delta_m , \quad \frac{k^2}{a^2} \psi_{\text{eff}} = 8\pi G \mu \rho_m \delta_m ,
\]

(6.4)

where \( \mu \) and \( \Sigma \) are dimensionless quantities, and \( G \) is the Newton gravitational constant. The quantity \( \Sigma \) can be expressed as

\[
\Sigma = \frac{1 + \eta}{2} \mu , \quad \eta \equiv \frac{\delta \phi}{\Psi} ,
\]

(6.5)

where \( \eta \) is dubbed the gravitational slip parameter. The deviations of \( \mu \) and \( \Sigma \) from 1 lead to the modified evolution of \( \Psi \), \( \psi_{\text{eff}} \), and \( \delta_m \) compared to the case of GR.

In Ref. [48], the calculations of \( \mu \) and \( \Sigma \) were performed by choosing the flat gauge (3.68), but the separation of those quantities between tensor, vector, and scalar contributions is not transparent. Since \( \zeta = 0 \) in the flat gauge, the quantities \( q_t \) and \( c_t^2 \) do not explicitly appear as coefficients of the flat-gauge Lagrangians (3.16)-(3.18). As we see in Eqs. (3.19)-(3.20), this situation is different in other gauges where \( \zeta \) does not vanish. In the following, we choose the unitary gauge given by

\[
\delta \phi = 0 , \quad E = 0 .
\]

(6.6)

We employ the quasi-static approximation on sub-horizon scales [20, 81, 82], under which the dominant contributions to the perturbation equations of motion are those containing \( k^2 / a^2 \) and \( \delta \rho_m \). In doing so, we introduce the dimensionless quantities:

\[
\epsilon_H \equiv \frac{H}{H^2} , \quad \Omega_m \equiv \frac{\rho_m}{3H^2 q_t} , \quad \epsilon_A \equiv \frac{\dot{A}_0}{H A_0} , \quad \alpha_B \equiv -\frac{w_1 - 2w_2 + 2H q_t}{2H q_t} , \quad x_2 \equiv \frac{w_2}{H q_t} , \quad x_6 \equiv \frac{A_0 w_6}{H q_t} ,
\]

\[
y_B \equiv \frac{\dot{\alpha}_B}{H \alpha_B} , \quad y_2 \equiv \frac{\dot{x}_2}{H x_2} , \quad y_6 \equiv \frac{\dot{x}_6}{H x_6} , \quad \varphi_u \equiv \frac{H}{A_0} \psi_u , \quad \beta_2 \equiv \frac{A_0^2 \alpha_2}{H^2 q_t} , \quad q_r \equiv \frac{q_t}{A_0^2 q_0} ,
\]

(6.7)

and \( \alpha_M \) defined by Eq. (3.64). If we switch off the vector field, the parameters \( \alpha_M \) and \( \alpha_B \) reduce to those introduced in Horndeski theories in Ref. [24], which represent the running of gravitational constant and the kinetic mixing between the scalar field and gravity, respectively [24]. In Appendix C, we also show the correspondence with other dimensionless parameters introduced in Ref. [24] (such as \( \alpha_Y \) and \( \alpha_K \)).

In the unitary gauge, there are three dynamical perturbations \( \psi_u = \psi , \mathcal{R}_\phi = \zeta \), and \( \delta \rho_u = \delta \rho_m \) with the gravitational potentials \( \Psi = \alpha + \chi \) and \( \Phi = \zeta + H \chi \). Applying the quasi-static approximation to Eqs. (3.25) and (3.23), respectively, it follows that

\[
\mathcal{Y} = \frac{A_0 w_6 - w_2}{A_0} \psi_u - 2w_2 \chi - 4A_0 \alpha_3 \mathcal{R}_\phi
\]

\[
= 2 (q_t - 2A_0 \alpha_3) \mathcal{R}_\phi + w_6 \psi_u - w_1 \chi - \frac{a^2}{k^2} \delta \rho_u .
\]

(6.8)

Then, the term \( \mathcal{Y} \) can be eliminated to give

\[
\delta \rho_u = \frac{k^2}{a^2} \left[ (w_1 - 2w_2) \chi - 2q_t \mathcal{R}_\phi - \frac{w_2}{A_0} \psi_u \right]
\]

\[
= \frac{k^2}{a^2} q_t \left[ 2 \left( 1 + \alpha_B \right) \Phi - 2 \alpha_B \mathcal{R}_\phi + x_2 \varphi_u \right] .
\]

(6.10)

We take the time derivative of Eq. (6.10) and substitute \( \delta \rho_u \) and \( \delta \rho_m \) into Eq. (3.26). In doing so, we exploit Eq. (3.24) to remove the perturbation \( v \) from Eq. (3.26) and eliminate the time derivative \( \dot{\psi}_u \) in \( \delta \rho_u \) by using Eqs. (3.28) and (6.8). This process finally leads to the disappearance of \( \mathcal{R}_\phi \). After replacing the combination \( \alpha + \chi \) with \( \Psi \), we obtain

\[
b_1 \Phi + 4 \left( 1 + \alpha_B \right) \psi + b_2 \mathcal{R}_\phi + b_3 \varphi_u = 0 ,
\]

(6.12)
where
\[
\begin{align*}
b_1 &= 4(1 + \alpha B)(1 + \alpha M + \epsilon H) + 4\alpha B y B + 6\Omega_m - 2x_2^2 q_r, \\
b_2 &= 4(1 + \alpha M) - x_2 (x_2 + x_6) q_r - b_1, \\
b_3 &= x_2 [2(1 + \alpha M + \epsilon H - \epsilon_A + y_2) - (x_2 - x_6) q_r].
\end{align*}
\]  
(6.13)  
(6.14)  
(6.15)

We also differentiate Eq. (6.8) with respect to $t$ and eliminate the terms $\dot{Y}$ and $Y$ from Eq. (3.30). This gives
\[
-2b_3 \Phi - 4x_2 \Psi + b_4 R_\phi + b_5 \varphi_u = 0,
\]  
(6.16)

where
\[
\begin{align*}
b_4 &= 2x_2 (1 + \alpha M + 2\epsilon H - \epsilon_A + y_2) - 2x_6 (1 + \alpha M - \epsilon_A + y_6) - (x_2 - x_6)^2 q_r, \\
b_5 &= -2x_2 (1 + \alpha M + \epsilon H - 2\epsilon_A + y_2) + 2x_6 (1 + \alpha M + \epsilon H - 2\epsilon_A + y_6) + (x_2 - x_6)^2 q_r + 4\beta_2.
\end{align*}
\]  
(6.17)  
(6.18)

Substituting Eq. (6.8) into Eq. (3.36), it follows that
\[
2(b_1 + b_2) \Phi + 8\Psi + b_6 R_\phi + (2b_3 - b_4) \varphi_u = 0,
\]  
(6.19)

where
\[
b_6 = 8(c_1^2 - 1 - \alpha M) + (x_2^2 - x_6^2) q_r.
\]  
(6.20)

Solving Eqs. (6.11), (6.12), (6.16), and (6.19) for $\Phi, \Psi, R_\phi,$ and $\varphi_u$, we obtain
\[
\begin{align*}
\Phi &= \frac{4|b_2(2x_2b_3 - x_6b_4 + 2b_5) + b_3(2\alpha B b_4 - x_2b_6) - (1 + \alpha B)(b_1^2 + b_5b_6)| a^2}{k^2 \delta \rho_u}, \\
\Psi &= -\frac{b_1(b_2b_5 - b_1^2 - b_5b_6) + 2b_2(b_2b_5 + 2b_3^2 - 2b_3b_4) - 2b_5b_6 a^2}{k^2 \delta \rho_u}, \\
R_\phi &= \frac{4[x_2(b_1b_4 + 2b_2b_3) + 2\alpha B(b_2b_4 + b_3b_5) + 2b_2b_5 - 3b_4b_3] a^2}{k^2 \delta \rho_u}, \\
\varphi_u &= \frac{4[x_2(2b_1b_2 - b_1b_6 + 2b_3^2) + 2\alpha B(b_2b_4 + b_3b_5) + 4b_2b_5 - 4b_3b_4] a^2}{k^2 \delta \rho_u},
\end{align*}
\]  
(6.21)  
(6.22)  
(6.23)  
(6.24)

where the determinant $\Delta$ can be expressed in terms of the quantity $q_s^{(u)} c_{s,1}^2 c_{s,2}^2$, as
\[
\Delta = \frac{512A^2 G(1 + \alpha B)^2}{H^2 q_t} q_s^{(u)} c_{s,1}^2 c_{s,2}^2.
\]  
(6.25)

For the derivation of the relation (6.25), we used the fact that $q_s^{(u)} c_{s,1}^2 c_{s,2}^2 = G_{11}G_{22} - G_{12}^2$ with $G_{11}, G_{22}, G_{12}$ given by Eq. (4.4). Since the approximation $\delta_m \approx \delta \rho_u/\rho_m$ holds for the perturbations deep inside the Hubble radius, the quantities $\mu$ and $\eta$ defined in Eqs. (6.4) and (6.5) yield
\[
\mu = H^2 q_t [b_1(2b_2b_3 - b_1^2 - b_5b_6) + 2b_2(b_2b_5 + 2b_3^2 - 2b_3b_4) - 2b_5b_6], \\
\eta = \frac{2048\pi G A^2 G(1 + \alpha B)^2 q_s^{(u)} c_{s,1}^2 c_{s,2}^2}{b_1(2b_2b_5 - b_1^2 - b_5b_6) + 2b_2(b_2b_5 + 2b_3^2 - 2b_3b_4) - 2b_5b_6}.
\]  
(6.26)  
(6.27)

We note that $\mu$ contains the matter density parameter $\Omega_m$ through $b_1$ and $b_2$. From Eq. (4.4), the product $q_s^{(u)} c_{s,1}^2 c_{s,2}^2 = G_{11}G_{22} - G_{12}^2$ also contains the term linear in $\Omega_m$. After using this relation to eliminate $\Omega_m$ from Eq. (6.26), we find that $\mu$ is expressed in the form
\[
\mu = \mu_0 \left[1 + \frac{\mu_1}{\mu_2 q_s^{(u)} c_{s,1}^2 c_{s,2}^2}\right],
\]  
(6.28)

where
\[
\begin{align*}
\mu_0 &= \frac{[2(b_1 + b_2) + b_6] \xi_0}{8\pi G q_t} \left[8\xi_0 - q_r A^2 G \left\{ \left( b_1 + b_2 + \frac{b_6}{2} \right) x_2 - 2 \left( b_3 - \frac{b_4}{2} \right) \right\} \right]^{-1}, \\
\mu_1 &= H q_t q_s \left[ x_2 \left\{ (b_1 + b_2) b_4 + b_3b_6 \right\} + (2b_3 - b_4) \{2\alpha B b_4 - (1 + \alpha B) b_4 \} + b_5 \{2\alpha B (b_1 + b_2) + (1 + \alpha B) b_6 \} \right], \\
\mu_2 &= q_s \xi_0.
\end{align*}
\]  
(6.29)  
(6.30)  
(6.31)
with
\[ \xi_0 \equiv \frac{A_0^2 q_e}{8} \left( b_1 + b_2 + \frac{b_6}{2} \right) b_5 + 2 \left( b_3 - \frac{b_4}{2} \right)^2 \right]. \] (6.32)

In Eqs. (6.29)-(6.32), the quantities \( b_1 \) and \( b_2 \) appear only through the combination \( b_1 + b_2 \), which does not contain \( \Omega_m \).

In GR we have \( \mu_0 = 1 \) and \( \mu_1 = 0 \), but in SVT theories the modifications arising from tensor, vector, scalar sectors generally lead to \( \mu_0 \neq 1 \) and \( \mu_1 \neq 0 \). Since \( b_6 \) contains \( c_i^2 \), the term \( \mu_0 \) depends on \( q_e, c_i^2, q_v \), i.e., the quantities associated with the stabilities of tensor and vector perturbations. The second term in the square bracket of Eq. (6.28) is dependent on \( q_e, c_i^2, q_v, c_s^2, c_2^2 \), so that this characterizes the matter interaction with tensor, vector, and scalar sectors. Thus, the separation of \( \mu \) between tensor, vector, and scalar contributions is clear in the unitary gauge, but this is not the case for the flat gauge chosen in Ref. [48]. Even though \( \mu \) and \( \eta \) are gauge-invariant quantities, the unitary gauge is more convenient than the flat gauge for this problem in that the physical interpretation of gravitational interactions becomes transparent. Provided the ghost and Laplacian instabilities are absent in the scalar sector, the condition
\[ - q_s < \beta_4 < \beta_5 \] (6.33)

If we do not admit any tuning among functions in Eq. (2.28), the couplings are constrained to be
\[ G_4 = G_4(\phi), \quad G_5 = 0, \quad f_4 = 0, \quad f_5 = 0, \] (6.34)

with all the other functions like \( f_6(\phi, X_1) \) allowed. Note that the \( \phi \) dependence in \( f_4 \) has been absorbed into \( G_4(\phi) \).

For the couplings (6.34) the quantity \( \alpha_3 \) defined by Eq. (2.35) vanishes, so there is the particular relation \( w_2 = -A_0 w_6 \) from Eq. (3.29). Then, the following relations hold
\[ x_2 = -x_6, \quad y_2 = y_6, \] (6.35)

under which we have
\[ b_1 + b_2 = 4(1 + \alpha_M), \quad b_1 + b_2 + \frac{b_6}{2} = 4, \quad b_3 - \frac{b_4}{2} = 0. \] (6.36)

Substituting these relations into Eqs. (6.29), (6.30) and (6.31), we obtain
\[ \mu_0 = \frac{b_5}{8\pi G q_e(b_5 - 2x_2^2)}, \quad \mu_1 = \frac{H q_e q_v[b_2 x_2 + b_5(\alpha_B - \alpha_M)]}{4(1 + \alpha_B)}, \quad \mu_2 = \frac{A_0^2 q_e^2 b_5}{2}, \] (6.37)

with \( \xi_0 = A_0^2 q_e b_5/2 \), and
\[ b_5 = 4\beta_2 + 4x_2^2 q_r - 4x_2(1 + \alpha_M + \epsilon_H - 2\epsilon_A + y_2). \] (6.38)

When \( b_5 > 0 \), the positivity of \( \mu_0 \) requires that
\[ b_5 > 2x_2^2. \] (6.39)

In this case we have \( \mu > \mu_0 > 1/(8\pi G q_e) \), so the gravitational interaction is stronger than that in GR for linear cosmological perturbations.

If \( b_5 < 0 \), it follows that \( \mu < \mu_0 < 1/(8\pi G q_e) \). Then, the gravitational interaction is weaker than that in GR. If the vector mass squared is positive (\( \beta_2 > 0 \)), the first two terms on the right hand side of Eq. (6.38) are positive under the absence of tensor and vector ghosts. Then, the only possibility for realizing \( b_5 < 0 \) is that the contribution \(-4x_2(1 + \alpha_M + \epsilon_H - 2\epsilon_A + y_2)\) in Eq. (6.38) is negative and it overwhelms other positive terms. It remains to be seen whether this behavior is possible for concrete dark energy models in the framework of SVT theories.

Finally, we further specify cubic couplings in the form
\[ f_3 = f_3(\phi), \quad \tilde{f}_3 = 0, \] (6.40)
in addition to the functions (6.34). Since \( w_2 = 0 \) in this case, we have \( x_2 = 0 \) and \( b_5 = 4\beta_2 = 4A_0^2\alpha_2/(H^2q_t) \). Substituting these relations into Eq. (6.37), the quantity (6.28) reduces to

\[
\mu = \frac{1}{8\pi G q_t} \frac{1 + \alpha_2}{2(1 + \alpha_2)q/(\alpha_B - \alpha_M)^2} q_t. 
\] (6.41)

From Eq. (6.27), the gravitational slip parameter yields

\[
\eta = \frac{2(1 + \alpha_B)q/(\alpha_B - \alpha_M)^2 + \alpha_2q_0\alpha_B(\alpha_B - \alpha_M)}{2(1 + \alpha_B)q/(\alpha_B - \alpha_M)^2 + \alpha_2q_t(\alpha_B - \alpha_M)^2}. 
\] (6.42)

Now, the explicit dependence on \( q_t \) disappears from \( \mu \) and \( \eta \). The intrinsic vector modes implicitly affect \( \mu \) and \( \eta \) through the dependence of \( c_1^2 \) and \( c_2^2 \) on \( q_t \). Provided that \( \alpha_B \) and \( \alpha_M \) do not vanish with \( \alpha_B \neq \alpha_M \), \( \mu \) differs from the value \( 1/(8\pi G q_t) \). This property is analogous to what happens in Horndeski theories with \( c_i^2 = 1 \), in which case the braiding parameter \( \alpha_B \) and the running parameter \( \alpha_M \) of \( q_t \) lead to the gravitational interaction different from that in GR (see, e.g., Eqs. (3.36) and (3.37) of Ref. [85]).

Compared to the values of \( \mu \) and \( \eta \) in Horndeski theories, the vector mass squared \( \alpha_2 \) appears in Eqs. (6.41)-(6.42), in addition to the presence of the product \( c_1^2 c_2^2 \) instead of a single sound speed squared \( c_1^2 \). If the condition

\[
\alpha_2 = f_{2,X_i} > 0 
\] (6.43)
is satisfied, the gravitational interaction is enhanced (i.e., \( \mu > 1/(8\pi G q_t) \)) compared to that in GR under the stability conditions \( q_t > 0 \) and \( q_t/(\alpha_B - \alpha_M)^2 > 0 \). Since \( w_2 = 0 \) in the present theory, the matrix component \( K_{11} \) in the unitary gauge reduces to \( w_5/A_0^2 \). On using the background Eq. (2.21), i.e., \( (f_{2,X_2} + 4f_{3,\phi})\phi = -2f_{2,X_2}A_0 \) to simplify \( w_5 \), it follows that

\[
K_{11} = \frac{1}{5} \left( 4f_{2,X_2} + 4f_{2,X_2}(\phi^2 + 4f_{2,X_2} \phi A_0 + 4f_{2,X_2} A_0^2) \right). 
\] (6.44)

For the theories in which \( f_2 \) contains only linear functions of \( X_2 \) and \( X_3 \), we have \( K_{11} = f_{2,X_2}/2 = 3/2 \) and hence \( K_{11} \) and \( \alpha_2 \) have the same sign. For the tachyonic vector mass squared \( \alpha_2 < 0 \), the negative value of \( K_{11} \) implies that \( K_{22} \) needs to be negative to satisfy the condition \( q_t = K_{11}K_{22} - K_{12}^2 > 0 \). In this case the scalar ghost appears, so we require the condition \( \alpha_2 > 0 \). Then the gravitational interaction is stronger than that in GR. The only possibility for realizing \( \mu < 1/(8\pi G q_t) \) is to introduce nonlinear terms in \( X_2 \) and \( X_3 \) which overwhelm the negative term \( f_{2,X_3} A_0 \) in \( K_{11} \). Since the last three terms in the bracket of Eq. (6.44) contain the time-dependent fields \( \phi \) and \( A_0 \), we generally require the tuning of functions to keep the condition \( K_{11} > 0 \) throughout the cosmological evolution for \( f_{2,X_2} < 0 \).

The gravitational slip parameter (6.42) is generally different from 1, but there are specific theories in which \( \eta \) is equivalent to 1. They are characterized by three cases: (i) \( \alpha_2 = 0 \), (ii) \( \alpha_B = \alpha_M \), and (iii) \( \alpha_M = 0 \). In cases (i) and (ii) the quantity (6.41) simply reduces to \( \mu = 1/(8\pi G q_t) \), but in case (iii) the second term in the square bracket of Eq. (6.41) does not vanish for \( \alpha_B \neq 0 \). For example, the cubic coupling \( G_3(X_1) \) gives rise to a nonvanishing contribution to \( \alpha_B \). Apart from the specific cases (i), (ii), (iii), the quantity \( \Sigma = (1 + \eta)\mu/2 \) differs from \( \mu \). We note that the quartic nonminimal coupling \( G_4(\phi) \) affects \( \mu \) and \( \Sigma \) through the nonvanishing contributions to \( \alpha_M \) as well as to \( \alpha_B \).

VII. CONCLUSIONS

In parity-invariant SVT theories with broken \( U(1) \) gauge symmetry, we developed the gauge-ready formulation of scalar cosmological perturbations by taking into account a matter perfect fluid. In such theories, there are three scalar DOFs arising from a scalar field \( \phi \), the longitudinal component of a vector field \( A_\mu \), and the matter field, besides two tensor polarizations and two transverse vector components. So far the computation of the second-order action of scalar perturbations in SVT theories was performed in the flat gauge, but the gauge choice from the beginning shows some limitations depending on the problems under consideration. This motivates us to derive the second-order action of scalar perturbations and linear perturbation equations of motion without fixing any gauge conditions. Our gauge-ready formulation of SVT theories is sufficiently general to accommodate Horndeski and GP theories as specific cases.

The second-order scalar action (3.15) consists of the Lagrangians \( \mathcal{L}_1^\text{flat}, \mathcal{L}_2^\text{flat}, \mathcal{L}_3^\text{flat} \) derived for the flat gauge in Ref. [48] and the new Lagrangians \( \mathcal{L}_\zeta, \mathcal{L}_E \) arising from the perturbations \( \zeta \) and \( E \). The coefficients of terms in \( \mathcal{L}_\zeta, \mathcal{L}_E \)
can be expressed by using those appearing in \(\mathcal{L}^{\text{flat}}_1, \mathcal{L}^{\text{flat}}_2, \mathcal{L}^{\text{flat}}_3\) as well as the coefficients present in the second-order actions of tensor and vector perturbations. This means that the choice of flat gauge does not lose any physical content for the purpose of studying the evolution of scalar perturbations. As we observe in Eqs. (3.16)-(3.18), however, the quantities \(q_0\) and \(c_1^2\) relevant to the stability conditions of tensor perturbations do not explicitly appear in \(\mathcal{L}^{\text{flat}}_1, \mathcal{L}^{\text{flat}}_2, \mathcal{L}^{\text{flat}}_3\), while this is not the case for \(\mathcal{L}_c, \mathcal{L}_v\). If we choose gauges in which the perturbation \(\zeta\) does not vanish, this allows one to identify contributions to scalar perturbations arising from the tensor sector much easier.

In Sec. III C, we studied the issue of gauge transformations and constructed a number of gauge-invariant variables associated with scalar perturbations. In SVT theories, the time-dependent temporal vector component \(A_0\) contributes to the background evolution besides the scalar field \(\phi\), so the dynamics is effectively described by a multi-scalar system with an adiabatic velocity (3.57). In Eq. (3.56), we introduced gauge-invariant curvature perturbations \(\mathcal{R}_\phi\) and \(\mathcal{R}_\psi\) associated with the scalar perturbation \(\delta\phi\) and the longitudinal scalar curvature \(\psi\). The total curvature perturbation (3.61), which incorporates both the perturbations \(\delta\phi\) and \(\psi\), can be used for the computation of primordial scalar power spectrum generated during inflation. We also derived the general relation between two gauge-invariant gravitational potentials \(\Psi\) and \(\Phi\) in the form (3.63). In Horndeski theories and SVT theories with the couplings (3.67), this relation reduces to the even simpler form (3.66).

In Sec. IV, we derived conditions for avoiding scalar ghost and Laplacian instabilities by choosing several different gauges introduced in Eqs. (3.68)-(3.71). The quantity \(q_c\) defined by Eq. (4.6), whose positivity is required for the absence of scalar ghosts, contains the common factor \(q_0D\) irrespective of the gauge choices. Provided that the tensor ghost is absent \((q_0 > 0)\) and that the determinant \(D\) appearing in the closed-form background equations of motion remains positive, the scalar ghost does not appear. By computing the scalar propagation speed squares \(c_{s1}^2\) and \(c_{s2}^2\) in several different gauges, we explicitly showed that they are gauge-independent quantities.

In Sec. V, we applied our general results of Sec. III to nonsingular bouncing and genesis cosmologies. If we choose the flat gauge, the quantity \(q_\alpha\) is proportional to \(H^2q_0D\) and hence it vanishes at the bounce \((H = 0)\). Apparently, this implies the strong coupling problem, but for this gauge the curvature perturbations \(\mathcal{R}_\phi\) and \(\mathcal{R}_\psi\) are not well defined at \(H = 0\) in that they vanish identically. If we choose other gauges like unitary or Newtonian gauges, the strong coupling problem does not appear at the bounce (i.e., \(q_\alpha \neq 0\)) with nonvanishing curvature perturbations. We also studied the possibility for realizing nonsingular bouncing/genesis cosmologies under the condition that the product \(q_0c_1^2\) does not asymptotically approach 0 and showed that, in GP and SVT theories, the existence of intrinsic vector modes (with \(q_0 > 0\)) can evade the no-go statement for the absence of stable nonsingular cosmologies made in Horndeski theories.

In Sec. VI, we computed observables associated with the growth of nonrelativistic matter perturbations for SVT theories in which the scalar and vector fields are responsible for the late-time cosmic acceleration. By choosing the unitary gauge and using the quasi-static approximation on sub-horizon scales, we obtained the effective gravitational coupling \(\mu\) and the gravitational slip parameter \(\eta\) in the forms (6.26) and (6.27), respectively. The quantity \(\mu\) can be also expressed as Eq. (6.28), where \(\mu_0\) depends on \(q_1, c_1^2, q_\alpha\). The second term in Eq. (6.28), which depends on \(q_1, c_1^2, q_v, c_{s1}, c_{s2}\), corresponds to the interaction of matter with tensor, vector, scalar sectors. Unlike the choice of flat gauge [48], this separation into tensor, vector, scalar contributions is convenient to study the cases in which the gravitational interaction is stronger or weaker than that in GR.

In SVT theories satisfying the condition \(c_1^2 = 1\), the quantities \(\mu_0, \mu_1, \mu_2\) in Eq. (6.28) simply reduce to Eq. (6.37). In cubic functions of the forms (6.40), \(\mu\) and \(\eta\) can be expressed as Eqs. (6.41) and (6.42), respectively. These expressions are analogous to those in Horndeski theories with \(c_1^2 = 1\) but the important difference is that the vector mass squared \(\alpha_2\) appears in SVT theories. For \(\alpha_2 < 0\) the gravitational interaction can be weaker than that in GR, but in this case it is nontrivial to construct consistent dark energy models in which the scalar ghost never appears. It will be of interest to study such a possibility further to distinguish SVT theories from Horndeski theories.

Our gauge-ready formulation of scalar perturbations can be directly applicable to the construction of concrete bouncing/genesis models in the framework of GP and SVT theories. In such cases, the intrinsic vector modes should play crucial roles for realizing stable solutions. In the context of inflationary cosmology, it will be interesting to study the effect of the vector field on the primordial power spectrum of total curvature perturbations \(\mathcal{R}\). These issues are left for future works.

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Appendix A: Coefficients in the second-order action of scalar perturbations

The coefficients $D_1, \ldots, D_4$ appearing in the background Eqs. (2.19), (2.20), (2.24) and the second-order action of scalar perturbations are given by

$$D_1 = H^2 \ddot{\phi} \left( 3G_{4,X_1} + \frac{7}{2} \dot{\phi}^2 G_{5,X_1,X_1} + \frac{1}{2} \dot{\phi}^4 G_{5,X_1,X_1,X_1} \right) + 3H^2 \left[ G_{4,X_1} - G_{5,\phi} + \dot{\phi}^2 \left( 4G_{4,X_1,X_1} - \frac{5}{2} G_{5,X_1,\phi} \right) \right]$$

$$+ \frac{1}{2} \ddot{f}_{2,x_1} + 2G_{3,\phi} + \dot{\phi}^2 \left( f_{2,x_1} + G_{3,\phi} \right) + \phi A_0 f_{2,x_1} + \frac{A_0^2}{4} f_{2,x_2} \right),$$

$$D_2 = - 2G_{4,X_1} - 5G_{5,\phi} + \phi^2 \left( 2G_{4,X_1} - G_{5,\phi} \right) + H \phi \left( 2G_{4,X_1} + \phi^2 G_{5,X_1,X_1} \right) \dot{H}$$

$$+ \left[ G_{3,\phi} + 3G_{4,\phi,\phi} + \phi^2 \left( \frac{G_{5,X_1,X_1}}{2} + G_{4,X_1,X_1} \right) - 2H \phi \left( 3G_{4,X_1} - 2G_{5,X_1,\phi} \right) \right] - H \phi \left( 2G_{4,X_1} - G_{5,\phi} \right) - H^2 \left( G_{5,\phi} + \frac{5}{2} \ddot{\phi}^2 G_{5,X_1,X_1} \right) \right] \ddot{\phi}$$

$$+ H^3 \dot{\phi} \left( G_{4,X_1} - G_{5,\phi} + \frac{5}{2} \ddot{\phi}^2 G_{5,X_1,X_1} \right) + \frac{3}{2} H^2 \phi \left( 2G_{4,X_1} - G_{5,\phi} \right) + \phi^2 \left( \frac{G_{5,X_1,X_1}}{2} + G_{4,X_1,X_1} \right) + H^3 \phi \left( 3G_{5,\phi} + \frac{7}{2} \ddot{\phi}^2 G_{5,X_1,X_1} \right)$$

$$- \frac{3}{2} H^2 \phi \left( 2G_{5,X_1} + \phi^2 G_{5,X_1,X_1} \right) + H^3 \left[ \frac{1}{2} A_0 \left( 9f_{5,\phi} + A_0^2 f_{5,\phi} \right) - 9 \phi \left( G_{4,X_1} - G_{5,\phi} \right) \right]$$

$$\left[ \frac{3}{2} A_0 \left( f_{4,\phi} + 2G_{4,\phi} + A_0^2 f_{4,\phi} \right) + \frac{1}{2} \phi A_0 + \phi A_0 f_{2,x_1,\phi} + \phi \left( \frac{1}{2} f_{2,x_1,\phi} + G_{3,\phi} \right) - \phi^3 \left( \frac{1}{2} G_{3,\phi,\phi} + G_{4,\phi,\phi} \right) \right]$$

$$- \frac{1}{4} \phi \left( f_{2,x_1,\phi} + A_0 f_{2,x_1,\phi} + 2G_{3,\phi,\phi} \right) - \frac{1}{2} A_0 \left( 4f_{2,x_1,\phi} + f_{2,x_2,\phi} \right)$$

$$- A_0 \left( f_{3,\phi} - A_0^2 f_{3,\phi} + \frac{f_{2,x_2,\phi} + A_0^2 f_{2,x_2,\phi}}{4} \right) + \frac{f_{2,x_2,\phi}}{2},$$

$$D_3 = \left[ \frac{A_0^2 f_{5,\phi} + A_0^2 f_{5,\phi}}{2} + \phi^2 \left( \frac{3}{2} G_{3,\phi} + 3G_{4,\phi,\phi} + \frac{1}{2} G_{5,\phi,\phi} \right) - \phi^4 \left( G_{4,X_1,\phi} - \frac{1}{2} G_{5,\phi,\phi} \right) \right]$$

$$- 3H \left[ \frac{A_0 \left( f_{2,x_2,\phi} + 4f_{3,\phi,\phi} \right)}{4} - A_0 \dot{A}_0 f_{4,X_3,\phi} + \phi \left( \frac{1}{2} f_{2,x_1,\phi} + G_{3,\phi} \right) - \phi^3 \left( \frac{1}{2} G_{3,\phi,\phi} + G_{4,\phi,\phi} \right) \right]$$

$$- \frac{1}{4} \phi \left( f_{2,x_1,\phi} + A_0 f_{2,x_1,\phi} + 2G_{3,\phi,\phi} \right) - \frac{1}{2} A_0 \left( 4f_{2,x_1,\phi} + f_{2,x_2,\phi} \right)$$

$$- A_0 \left( f_{3,\phi} - A_0^2 f_{3,\phi} + \frac{f_{2,x_2,\phi} + A_0^2 f_{2,x_2,\phi}}{4} \right) + \frac{f_{2,x_2,\phi}}{2},$$

$$D_4 = - H^3 \phi^2 \left( 15G_{5,X_1} + 10 \phi^2 G_{5,X_1,X_1} + \phi^4 G_{5,X_1,X_1,X_1} \right) + 3H^2 \left[ A_0 \left( f_{5,\phi} - A_0^2 f_{5,\phi} \right) - 6 \phi \left( G_{4,X_1} - G_{5,\phi} \right) \right]$$

$$- \phi^3 \left( 12G_{4,X_1} - 7G_{5,X_1} \right) - \phi^5 \left( 2G_{4,X_1} - G_{5,X_1,X_1} \right) + 3H \left( f_{4,\phi} + G_{4,\phi} - 2A_0^2 f_{4,\phi} \right)$$

$$+ \phi^2 \left( 3G_{5,X_1} + 8G_{4,X_1} \right) + \phi^4 \left( G_{3,X_1} + 2G_{4,X_1,X_1} \right) - \phi^3 \left( f_{2,x_1,\phi} + G_{3,\phi,\phi} \right) - \frac{1}{2} \phi^2 A_0 f_{2,x_1,\phi}$$
\[ -\dot{\phi}(f_{2,x_1} - A_0^3 f_{2,x_1} - 2G_{3,x_1} + 2G_{3,x_1}) + \frac{1}{2} A_0(f_{2,x_2} + A_0^2 f_{2,x_2} + 4f_{4,\phi} - 4A_0^2 \tilde{f}_{3,\phi}) , \]

\[ D_6 = H^3 \left[ A_0^3 f_{5,x_1} + A_0^2 f_{5,x_1} - \dot{\phi}^3(5G_{5,x_1} + \dot{\phi}^2 G_{5,x_1}) \right] + 3H^2 \left[ 2(f_{4,\phi} + G_{4,\phi} + A_0^4 f_{4,\phi} X_1) \right] \\
+ \phi A_0(f_{5,\phi} - A_0^2 f_{5,\phi} - \dot{\phi}^2(4G_{4,x_1} - 3G_{5,x_1}) - \dot{\phi}^4(2G_{4,x_1} X_1 + G_{5,x_1} - G_{5,x_1}) \right] \\
- 3H \left[ 2A_0^3 (f_{3,x_1} + \tilde{f}_{3,\phi}) - 2\phi(f_{4,\phi} - A_0^2 f_{5,\phi} + G_{4,\phi}) - \dot{\phi}^3(3G_{3,x_1} + 2G_{4,x_1}) \right] \\
- \dot{\phi}^2(f_{2,x_1} + G_{3,\phi}) + 2\phi A_0(f_{3,\phi} - A_0^2 \tilde{f}_{3,\phi} + f_{2,\phi} + A_0^2 f_{2,\phi}) , \]

\[ D_7 = H^3 \left[ A_0^3 f_{5,x_1} + \dot{\phi}^2 G_{5,x_1}, \right] - H^2 \left[ A_0(f_{3,\phi} + A_0^2 f_{5,\phi}) - 6\phi(G_{4,x_1} - G_{5,\phi}) - 2\dot{\phi}^3(3G_{4,x_1} - 2G_{5,x_1}) \right] \\
- H \left[ 2(f_{4,\phi} + A_0^2 f_{4,\phi} + G_{4,\phi}) - 2A_0 \phi f_{5,\phi} + \dot{\phi}^2(3G_{3,x_1} + 10G_{4,x_1}) - 2G_{5,\phi} \right] \\
+ \phi(f_{2,x_1} + f_{2,\phi} + 2G_{3,\phi} + 2G_{4,\phi}) + \frac{1}{2} A_0(f_{2,x_2} + 4f_{4,\phi}) , \]

\[ D_8 = -\frac{2\phi D_1 + D_4 + 3H D_6}{A_0} , \]

\[ D_9 = -H^3 A_0^3 f_{5,x_1} + A_0^2 f_{5,x_1} X_1 - 3H^2 \left[ 2A_0(f_{4,x_1} + A_0^2 f_{4,x_1} X_1 - \dot{\phi}(f_{5,\phi} + A_0^2 f_{5,\phi}) \right] \\
+ 6HA_0 \left[ A_0(f_{3,x_1} + \tilde{f}_{3,\phi}) + f_{4,x_1,\phi} \right] - \phi \left( \frac{1}{2} f_{2,x_2} + 2f_{3,\phi} - 2A_0^2 \tilde{f}_{3,\phi} \right) - A_0 f_{2,x_3} , \]

\[ D_{10} = -2H f_{5,\phi} - H^2 \left( 3f_{3,\phi} + A_0^2 f_{5,\phi} \right) = 2H A_0 \left( f_{4,x_1,\phi} + \tilde{A}_0 f_{5,\phi} \right) - 2A_0 f_{4,x_1,\phi} + 2f_{3,\phi} + \frac{1}{2} f_{2,x_2} , \] (A1)

and

\[ w_1 = -H^2 \left[ A_0^4 (f_{5,x_1} + A_0^2 f_{5,x_1} X_1) - \dot{\phi}^3(5G_{5,x_1} + \dot{\phi}^2 G_{5,x_1}) \right] - 2H \left[ 2f_{4} + A_0^4 f_{4,x_1} + G_{4} \right] \\
+ A_0 \phi(f_{5,\phi} - A_0^2 f_{5,\phi}) - \dot{\phi}^2(4G_{4,x_1} - 3G_{5,\phi}) - \dot{\phi}^4(2G_{4,x_1} X_1 + G_{5,\phi}) \]

\[ - \phi^3(2G_{3,x_1} + 2G_{4,x_1}) - 2\phi(f_{4,\phi} - A_0^2 f_{5,\phi} + G_{4,\phi}) + 2A_0^3 (f_{3,x_1} + \tilde{f}_{3}) \]

\[ w_2 = w_1 + 2H q_0 - \phi D_6 , \]

\[ = A_0 \left[ -H^2 A_0^2 (3f_{5,x_1} + A_0^2 f_{5,x_1} X_1) + 2H \left[ 2A_0(f_{4,x_1} + A_0^2 f_{4,x_1} X_1) - \dot{\phi}(f_{5,\phi} + A_0^2 f_{5,\phi}) \right] + 2A_0 \phi f_{4,x_1} \phi \right. \]

\[ + A_0^3 (f_{3,x_1} + \tilde{f}_{3}) \],

\[ w_3 = -2A_0^2 q_0 , \]

\[ w_4 = w_5 - H^3 \left[ 3A_0^3 (2f_{5,x_1} + A_0^2 f_{5,x_1} X_1) - \dot{\phi}^3 \left( 15G_{5,x_1} + \frac{13}{2} \dot{\phi}^2 G_{5,x_1} + \frac{1}{2} \dot{\phi}^4 G_{5,x_1} X_1 \right) \right] - 3H^2 \left[ 2f_{4} + G_{4} \right] \\
+ A_0^2 \left[ 2f_{4,x_1} + 4A_0^2 f_{4,x_1} X_1 + 3A_0 \phi f_{5,\phi} \right] - \dot{\phi}^2(7G_{4,x_1} - 6G_{5,\phi}) - \dot{\phi}^4 \left( 8G_{4,x_1} X_1 - 9 \phi G_{5,x_1} \phi \right) \]

\[ - \phi^3(2G_{3,x_1} + 5G_{4,x_1}) - \phi \left( \frac{1}{2} G_{3,x_1} X_1 + G_{4,x_1} X_1 \phi \right) \right] + \frac{1}{2} \phi^4(f_{2,x_1,\phi} + G_{3,x_1}) \]

\[ + \phi^2 \left( \frac{1}{2} f_{2,x_1} - A_0^2 f_{2,x_1,\phi} - \frac{1}{2} A_0^2 f_{2,x_1,\phi} + G_{3,\phi} \right) - \frac{1}{2} A_0 \phi^2 \left( f_{2,x_2} + A_0^2 f_{2,x_2,\phi} + 4f_{3,\phi} - 4A_0^2 \tilde{f}_{3,\phi} \right) , \]

\[ w_5 = \frac{1}{2} H^3 A_0^3 (3f_{5,x_1} + 6A_0^2 f_{5,x_1} + A_0^4 f_{5,x_1} X_1) + 3H^2 A_0 \left[ A_0^3 (3f_{4,x_1} X_1 + A_0^2 f_{4,x_1} X_1) \right] \\
+ \frac{1}{2} \phi(f_{5,\phi} - A_0^2 f_{5,\phi} + A_0^4 f_{5,\phi}) - 3H A_0^3 \left[ f_{3,x_1} + \tilde{f}_{3} + A_0^2 (f_{5,x_1} + \tilde{f}_{3}) X_1 + A_0 \phi f_{4,x_1} \phi \right] \]

\[ + \frac{1}{8} A_0^2 \phi^2 \left( f_{2,x_2} - \frac{1}{4} A_0^2 \phi \left( f_{2,x_2} + 4f_{3,\phi} - 2A_0^2 (f_{2,x_2} - 2\tilde{f}_{3,\phi} + 2f_{3,\phi}) + 4A_0^2 \tilde{f}_{3,\phi} \right) + \frac{1}{2} A_0^2 f_{2,x_1} . \]
in the action (3.15) and taking the small-scale limit, the resulting action is of the form (4.2) with the same values of $\phi$. We note that the background Eqs. (2.18) and (2.21) are used for the derivation of these coefficients.

Let us consider the uniform vector gauge characterized by Eq. (3.70). In this case, the dynamical DOFs are given by $w$ and $f$. Compared to $w$ and $F$, the following relations hold in flat, unitary and Newtonian gauges, the following relations hold

$$ w_6 = \frac{-w_1 - \dot{D}_6 + 2Hq}{A_0} - 4H \left(2A_0 f_{4,X_3} - \dot{\phi}f_{5,\phi} + H A_0^2 f_{5,X_3}\right), $$

$$ w_7 = -2H^2 A_0^2 (f_{5,X_3} - A_0^2 f_{5,X_3}, X_3) - 2H \left[2A_0 (f_{4,X_3} - A_0^2 f_{4,X_3}) - \dot{\phi} (f_{5,\phi} - A_0^2 f_{5,X_3})\right] - 2A_0 \dot{\phi} f_{4,X_3}, $$

$$ w_7 = -2H (2f_{4,X_3} + HA_0 f_{5,X_3}) - H^2 \left[\frac{\dot{\phi}}{A_0} \left(3f_{5,\phi} + 2A_0^2 f_{5,X_3}\right) + \dot{A}_0 \left(f_{5,X_3} + A_0^2 f_{5,X_3}\right)\right] $$

$$ -4H \left(\dot{\phi} f_{4,X_3} + A_0 \dot{A}_0 f_{4,X_3}\right) + 2 \dot{A}_0 \left(f_{3,X_3} + \dot{f}_3\right) + \frac{\dot{\phi}}{2A_0} (f_{2,X_2} + 4f_{4,\phi}), $$

$$ w_8 = 3Hw_1 - 2w_4 - \dot{\phi}D_4. $$

We note that the background Eqs. (2.18) and (2.21) are used for the derivation of these coefficients.

Appendix B: Stability conditions in the uniform vector gauge

Let us consider the uniform vector gauge characterized by Eq. (3.70). In this case, the dynamical DOFs are given by the matrix $X^t = (\delta \phi \nu, R_{\rho e}, \delta \rho_e / k)$. Solving Eqs. (3.23)-(3.26) for $\alpha, \chi, \delta A, v$ to eliminate these nondynamical variables in the action (3.15) and taking the small-scale limit, the resulting action is of the form (4.2) with the same values of $K_{22}, K_{33}$ and $G_{22}, G_{33}$ as those in Eqs. (4.3) and (4.4). The other nonvanishing matrix components of $K$ and $G$ are

$$ K_{11} = D_1 + \frac{D_6}{w_1 - 2w_2} \left[D_4 + 2A_0 D_8 + \frac{D_6 (w_4 + 4w_5 + 2w_8)}{w_1 - 2w_2}\right], $$

$$ K_{12} = K_{21} = -\frac{q_1}{w_1 - 2w_2} \left[D_4 + 2A_0 D_8 + \frac{2D_6 (w_4 + 4w_5 + 2w_8)}{w_1 - 2w_2}\right], $$

$$ G_{11} = -D_2 - \frac{1}{w_1 - 2w_2} \left[D_6 D_7 + \frac{w_2^2 F_1}{2A_0 q_e} - (\rho_m + P_m) F_1\right] + \frac{1}{a t} \left(\dot{a} F_1\right), $$

$$ G_{12} = G_{21} = -\frac{B_1}{2} \frac{\alpha_3 w_2 F_2}{A_0 q_e} + \frac{1}{w_1 - 2w_2} \left[q_1 D_7 - \frac{w_2^2 F_2}{2A_0 q_e} + (\rho_m + P_m) F_2\right] + \frac{1}{a t} \left(\dot{a} F_2\right), $$

where

$$ F_1 = -\frac{D_6^2}{2(w_1 - 2w_2)} , \quad F_2 = \frac{q_1 D_6}{w_1 - 2w_2}. $$

On using Eqs. (3.37)-(3.40), the quantity $q_s = K_{11} K_{22} - K_{12}^2$ in the uniform vector gauge is expressed as

$$ q_s^{(v)} = \frac{q_1 D}{2(w_1 - 2w_2)^2}, $$

where $D$ is given by Eq. (2.25). Compared to $q_s$ in flat, unitary and Newtonian gauges, the following relations hold

$$ q_s^{(v)} = A_0^2 \frac{H^2 q_s^{(f)}}{\phi^2} = A_0^2 \frac{H^2 q_s^{(u)}}{\phi^2} = \frac{A_0^2 D_6^2}{(w_1 - 2w_2)^2} q_s^{(N)}. $$

Similarly, the quantities $\mathcal{F}_s$ and $\mathcal{G}_s$ in the uniform vector gauge are related to those in flat, unitary and Newtonian gauges, as

$$ \mathcal{F}_s^{(v)} = A_0^2 \frac{H^2 \mathcal{F}_s^{(f)}}{\phi^2} = A_0^2 \frac{H^2 \mathcal{F}_s^{(u)}}{\phi^2} = \frac{A_0^2 D_6^2}{(w_1 - 2w_2)^2} \mathcal{F}_s^{(N)}, \quad \mathcal{G}_s^{(v)} = A_0^2 \frac{H^2 \mathcal{G}_s^{(f)}}{\phi^2} = A_0^2 \frac{H^2 \mathcal{G}_s^{(u)}}{\phi^2} = \frac{A_0^2 D_6^2}{(w_1 - 2w_2)^2} \mathcal{G}_s^{(N)}. $$

Then, it follows that the scalar propagation speed squares (4.12) are gauge-independent.
Appendix C: $\alpha_T$ and $\alpha_K$

Besides the quantities $\alpha_M$ and $\alpha_B$ given in Eqs. (3.64) and (6.7), we define the following dimensionless quantities:

\[
\alpha_T = \frac{1}{q_t} \left[ 2A_0^2f_{4,X_3} + 2\dot{\phi}^2G_{4,X_3} - 2\dot{\phi}A_0f_{5,\phi} - 2\dot{\phi}^2G_{5,\phi} + A_0^2(HA_0 - \dot{A}_0)f_{5,X_3} + \dot{\phi}^2(H\dot{\phi} - \ddot{\dot{\phi}})G_{5,X_3} \right], \quad (C1)
\]

\[
\alpha_K = 6 + 12\alpha_B + \frac{2(w_4 + 4w_5 + 2w_8)}{H^2q_t}. \quad (C2)
\]

After switching off the vector field, Eqs. (C1) and (C2) reduce to those in Horndeski theories introduced in Ref. [24]. The quantity $\alpha_T$ represents the deviation of $c_T^2$ from that of light, i.e., $c_T^2 = 1 + \alpha_T$, while $\alpha_K$ corresponds to the kinetic term for scalar perturbations. The matrix component $K_{22}$ given in Eq. (4.3), which is computed in the unitary gauge, can be simply expressed in terms of $q_t$, $\alpha_B$, and $\alpha_K$, as

\[
K_{22} = \frac{q_t(\alpha_K + 6\alpha_B^2)}{2(1 + \alpha_B)^2}. \quad (C3)
\]
