CONFORMAL AMBIENT METRIC CONSTRUCTION AND WEINGARTEN ENDOMORPHISMS OF SPACELIKE SUBMANIFOLDS

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ABSTRACT. In this paper we propose an approach to the Feffermann-Graham conformal ambient metric construction for Riemannian conformal structures from the Lorentzian geometry perspective. We adopt the notions introduced by A. Čap and R. Gover which satisfy a weaker condition compared to the original ones. We provide a method to construct a family of such ambient manifolds and metrics for Riemannian conformal structures on \((n \geq 2)\)-dimensional manifolds. The construction relies on the choice of a metric in the conformal class and a smooth \(1\)-parameter family of self-adjoint tensor fields. Now, every metric in the conformal class is the induced metric on a codimension two spacelike submanifold into the Lorentzian ambient manifold. Under suitable choices of the \(1\)-parameter family of self-adjoint tensor fields, there exists a lightlike normal vector field along such spacelike submanifolds which Weingarten endomorphisms provide a Möbius structure on the Riemannian conformal structure. Conversely, every Möbius structure on a Riemannian conformal structure arises in this way. The Ricci-flatness condition along the scale bundle as a lightlike hypersurface into the ambient manifold is studied by means of the initial velocity of the \(1\)-parameter family of self-adjoint tensor fields. Moreover, flat Möbius structures are characterized in terms of the extrinsic geometry of spacelike surfaces into the ambient manifold.

1. Introduction

A Riemannian conformal structure on a manifold \(M\) is an equivalence class of Riemannian metrics on \(M\) where two metrics are equivalent if they differ by a factor that is a smooth positive function on the manifold \(M\). Conformal structures (in Lorentzian signature) was introduced by Hermann Weyl in order to formulate a unified fields theory. Weyl wrote “To derive the values of the quantities \(g_{ik}\) from directly observed phenomena, we use light-signals .... By observing the arrival of light at the points neighbouring to \(O\) we can thus determine the ratios of the values of the \(g_{ik}\)'s ..... It is impossible, however, to derive any further results from the phenomenon of the propagation of light...” [19, Chap. 4, Sec. 27].

From a mathematical perspective, the problem of the equivalence for conformal structures on \((n \geq 3)\)-dimensional manifolds was solved by E. Cartan by means of the now called canonical normal Cartan connection [5]. For dimension \(n \geq 3\), Riemannian conformal structures \((M, c)\) correspond bijectively (up to isomorphism) with normal Cartan geometries of type \((G, P)\) where \(G = O(1, n + 1)/\{\pm \text{Id}\}\) is the Möbius group and \(P\) is the Poincaré conformal group defined to be the isotropy group of the line through an isotropic (lightlike) vector (see details in [8, Theor. 1.6.7]). That is, conformal structures on an \((n \geq 3)\)-dimensional manifold \(M\) gives rise to a principal \(P\)-bundle \(\mathcal{P} \to M\) and a unique Cartan connection \(\omega \in \Omega^1(\mathcal{P}, g)\) where \(g\) is the Lie algebra of the Möbius group \(G\) such that \(\omega\)

The second author is partially supported by the Regional Government of Andalusia and ERDEF project PY20-01391 and both of them by Spanish MICINN project PID2020-118452GB-100.

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2020 Mathematics Subject Classification. Primary 53C18, 53C50, 53C40, 53C42. Secondary 53C05, 53B15.

Key words and phrases. Ambient manifold, Ambient metric, Tractor conformal bundle, Möbius structure, Lorentzian geometry, Spacelike submanifold, Weingarten endomorphism.
satisfies certain normalization conditions and conversely. In an equivalent way, the canonical normal Cartan connection can be given by means of Tractor conformal bundles and Tractor connections as was introduced by Thomas in the 1920s [17]. This description of the conformal structures relies on vector bundles and linear connections. These two tools have permitted to get invariants for conformal structures. In a more general setting, the relation between Tractor connections on Tractor bundles and Cartan geometries was clarified in [6]. A detailed exposition of these facts, further details and background can be found in [8].

In the 1980s, Fefferman and Graham introduced a new technique to obtain invariants for conformal structures: the ambient metric in [9] (see also [10]). Roughly speaking, starting with a Riemannian conformal structure $(M, c)$, the space of scales $Q$ consists of the rays of metrics $y := t^2 g_x$ on $T_x M$ where $x \in M$, $t \in \mathbb{R}^+$ and $g \in c$, [10]. Then, an ambient metric $\tilde{g}$ can be defined so that $(M, \tilde{g})$ is a Lorentzian manifold that admits $Q$ as an embedded lightlike hypersurface, Section 3. The original Fefferman-Graham metric requires certain Ricci-flatness properties and depends only on the conformal class of Riemannian metrics, thus immediately leading to a construction for conformal invariants, [10]. An in-depth study and several generalisations of this kind of ideas can be cast in the context of Cartan geometries, [8]. In this paper, we will adopt the treatment given in [7], where Čap and Gover provide an ambient metric, which is a weakening of the Fefferman-Graham ambient metric introduced in [10], Definitions 2.1 and 2.2.

For conformal structures on $(n \geq 3)$-dimensional manifolds, there is an equivalence (up to isomorphism) between canonical normal Cartan connections and Tractor conformal bundles endowed with Tractor connections with certain normalization conditions, [7]. This result has been extended to dimensions one and two by means of the notion of Möbius structure [3]. Section 2. A Möbius structure on a manifold $M$ is essentially equivalent to defining a conformal class of metrics $c$ on $M$ and a “Schouten type-tensor” for $c$, Definition 2.3. The notion of Möbius structure provides a uniform description of Cartan geometries of type $(G, P)$ for all dimensions in the following terms: a Cartan geometry of type $(G, P)$ on a manifold $M$ is essentially equivalent to defining a conformal class of metrics $c$ on $M$ and a “Schouten type-tensor” for $c$, [3].

The aim of this paper is to bring together the extrinsic geometry of spacelike submanifolds of Lorentzian manifolds, Section 2 and aforementioned notions of ambient metric and Möbius structure. Spacelike submanifolds in Lorentzian geometry has been studied for a long time, both from the physical and mathematical points of view (see for instance [16] and references therein). In this paper, we show first several basic properties from the Lorentzian geometry perspective of the ambient metric construction, Section 3. Then, we provide an explicit family of ambient metrics in Proposition 4.1. The main results are in Section 5 where it is essentially shown that Möbius structures can be recovered from certain Weingarten endomorphisms of codimension two spacelike submanifolds in these ambient manifolds. We hope that our viewpoint sheds some light on the interplay between the theory of spacelike submanifolds in Lorentzian geometry and the ambient metric for conformal structures.

The content of this paper is distributed into six sections. Section 2 introduces the basic notions on Tractor conformal bundles and Tractor connections, Definitions 2.1 and 2.2. Then, taking into account ideas from [3] and [4], we recall the notion of Möbius structure as was introduced in [15], extended to arbitrary dimension in an obvious way. This section also includes basic facts about spacelike submanifolds of Lorentzian manifolds and the notions of ambient manifold and ambient metric from [7]. After establishing the requisite language, Section 3 shows basic properties of these ambient manifolds and ambient metrics from the Lorentzian geometry perspective. As was mentioned, the scale bundle $Q$ of a Riemannian conformal structure $(M, c)$ is a lightlike hypersurface, Definition 3.1.
in any ambient manifold endowed with an ambient metric \((\tilde{M}, \tilde{g})\) and every choice of the metric \(g \in c\) provides an isometric spacelike immersion from \((M, g)\) into \((\tilde{M}, \tilde{g})\), Lemma 5.2.

Section 4 exhibits a wide family of ambient manifolds and timelike orientable ambient metrics \((\tilde{M}, \tilde{g})\) for a Riemannian conformal structure \((M, c)\) on an \((n \geq 2)\)-dimensional manifold \(M\), Proposition 4.1. This construction starts from a choice of a metric \(g \in c\) and a smooth 1-parameter family \(\alpha(\rho)\) of self-adjoint tensor fields with respect to any representative \(g \in c\) such that \(\alpha(0) = \text{Id}\) and the symmetric tensor \(g(\alpha(\rho)(\cdot), \cdot)\) is positive definite for \(|\rho| < \varepsilon\). Although these ambient manifolds are product manifolds \(\tilde{M} = B \times M\), where \(B\) is a suitable open set of \(\mathbb{R}^2\), the metrics \(\tilde{g}\) are not warped product metrics, in general. Hence, the classical formulas for the Levi-Civita connection in [12, Prop. 7.36] do not work for \(\tilde{g}\). Proposition 4.5 partially provides the Levi-Civita connection of \(\tilde{g}\) and then, as a consequence of Remark 4.3 and Corollary 4.7, we obtain the components of the Ricci tensor of \(\tilde{g}\) along \(Q \subset \tilde{M}\). We would like to point out that the spacelike submanifolds \(\mathcal{F} = \{(t, \rho)\} \times M \subset \tilde{M}\) at fixed \((t, \rho) \in B\) are not totally umbilical, in general, Remark 4.6.

The main results of this paper are introduced in Section 5. Suppose that \(g \in c\) and \(\alpha(\rho)\) are the starting pieces to build the aforementioned ambient manifold and ambient metric \((\tilde{M}, \tilde{g})\). For every \(u \in C^\infty(M)\), we define the codimension two isometric immersion \(\Psi^u: (M, e^{2u}g) \to (\tilde{M}, \tilde{g})\), (20). Then, there are two normal vector fields \(\xi^u\) and \(\eta^u\) along \(\Psi^u\) (22) and the pair \(\{\xi^u, \eta^u\}\) provides a global lightlike normal frame (see Section 2). Let \(\nabla^\tilde{g}\) be the Levi-Civita connection of the chosen metric \(g\). A key fact for this paper is encoded in Proposition 5.3 as follows. The Weingarten endomorphisms associated to \(\xi^u\) and \(\eta^u\) are given, respectively, by \(A_{\xi^u} = -\text{Id}\) and \(A_{\eta^u} = e^{-2u} \left[ \frac{\dot{\alpha}(0)}{2} - \frac{||\nabla^\tilde{g} u||^2}{2} \text{Id} + g(\nabla^\tilde{g} u, \text{Id})\nabla^\tilde{g} u - \nabla^\tilde{g} \nabla^g u \right]\)

where \(\nabla^g u\) is the gradient of the function \(u \in C^\infty(M)\) and \(\nabla^g \nabla^g(V) = \nabla^g_v \nabla^g u\) for \(V \in \mathcal{X}(M)\). As consequence of the above formula for \(A_{\eta^u}\), we observe that he assignment

\[
P: c \to \mathcal{T}_{(0,2)} M, \quad e^{2u} g \mapsto e^{2u} g (A_{\eta^u}(-), -),
\]

satisfies the same conformal transformation law of the Schouten tensor (28) with \(P(g)(V, W) = \frac{1}{2} g(\dot{\alpha}(0)(V), W)\) for \(V, W \in \mathcal{X}(M)\). Therefore, when \(\text{trace}(\dot{\alpha}(0)) = \frac{\scal^g}{(n-1)}\), where \(\scal^g\) is the scalar curvature of \(g\), \(P\) is a Möbius structure as in Definition 2.3 for the Riemannian conformal structure \((M, c)\) on the \((n \geq 2)\)-dimensional manifold \(M\).

On the other hand, for \(n = 2\), Corollary 5.4 shows that the Ricci tensor of \(\tilde{g}\) along \(Q \subset \tilde{M}\) vanishes if and only if \(\text{trace}(\dot{\alpha}(0)) = 2\tilde{K}^g\), where \(\tilde{K}^g\) is the Gauss curvature of \(g\). For \(n \geq 3\), the same vanishing property for the Ricci tensor along \(Q \subset \tilde{M}\) is equivalent to the condition \(\dot{\alpha}(0) = 2\tilde{P}^g\), where \(\tilde{P}^g\) is the metrically equivalent endomorphism tensor field to the Schouten tensor \(P^g\).

Theorems 5.6 and 5.9 delve into this set of ideas as follows. For every Riemannian conformal structure \((M, c)\) on an \((n \geq 2)\)-dimensional manifold \(M\), \(g \in c\) and a smooth 1-parameter family \(\alpha(\rho)\) of self-adjoint tensor fields with respect to any representative \(g \in c\) such that \(\alpha(0) = \text{Id}\) and the symmetric tensor \(g(\alpha(\rho)(\cdot), \cdot)\) is positive definite for \(|\rho| < \varepsilon\), let us consider the ambient manifold and ambient metric \((\tilde{M}, \tilde{g})\) as in Proposition 4.1 and then:

- Theorem 5.6 states that, for \(n \geq 3\) and \(\dot{\alpha}(0) = 2\tilde{P}^g\), the Schouten tensor of the metric \(e^{2u} g\) can be recovered from the immersion \(\Psi^u\) by means of the following formula

\[
P^{e^{2u} g}(V, W) = e^{2u} g(A_{\eta^u}(V), W), \quad V, W \in \mathcal{X}(M).
\]
Theorem 5.9 states that, for \( n = 2 \) and \( \text{trace}(\dot{\alpha}(0)) = 2Kg \), the assignment
\[
e^{2u}g \in c \mapsto P(e^{2u}g)(V, W) = e^{2u}g(A_{\eta u}(V), W), \quad V, W \in \mathcal{X}(M),
\]
defines a Möbius structure on \((M, c)\).

In both cases, the Ricci tensor of \( \tilde{g} \) along \( Q \subset \tilde{M} \) vanishes.

Moreover, as is shown in Theorem 5.11, any Möbius structure \( P \) on \((M, c)\), for \( n = 2 \), arises essentially by means of Theorem 5.9.

Section 5 also includes several properties on the family of spacelike immersions \( \Psi^u \). For example, Corollary 5.4 shows that the normal curvature tensor of \( \Psi^u \) vanishes and Remark 5.5 provides explicit formulas for the second fundamental form and the mean curvature vector field of each \( \Psi^u \). Under the assumptions of Theorems 5.6 and 5.9, the mean curvature vector field is given by
\[
H^u = -\frac{1}{2n(n-1)}\text{scal}e^{2u}g \xi^u + \eta^u,
\]
and therefore, \( \|H^u\|^2 = \frac{\text{scal}e^{2u}g}{n(n-1)} \), Remark 5.12.

For a Möbius structure \( P \) on \((M, c)\) with \( n = 2 \), Section 6 delves into the extrinsic geometry of the isometric immersions \( \Psi^u : (M, e^{2u}g) \to (\tilde{M}, \tilde{g}) \). In this section, we consider the ambient manifolds and ambient metrics obtained from \((M, c, P)\) by means of Theorem 5.11. We write down the Codazzi equation (4) in terms of the Cotton-York tensor (30), Lemma 6.1. Then, Proposition 6.3 shows that tangent spaces of \( M \) along \( \Psi^u \) are invariant under the curvature tensor of \((\tilde{M}, \tilde{g})\) if and only if the Cotton-York tensor of \( c \) vanishes. In this case, the Möbius structure \( P \) is said to be flat in the terminology of [4], [15]. Finally, from the Gauss equation of \( \Psi^u \), we deduce that the sectional curvature with respect to \( \tilde{g} \) of every spacelike tangent plane to \( Q \subset M \) vanishes.

2. Preliminaries

All the manifolds are assumed to be smooth, Hausdorff, satisfying the second axiom of countability and without boundary. Let \( M \) be a manifold with \( \dim M = n \geq 2 \). A Riemannian conformal structure on \( M \) is an equivalence class \( c = [g] \) of Riemannian metrics where two metrics \( g \) and \( g' \) are said to be equivalent when \( g' = e^{2u}g \) for a smooth function \( u \) on \( M \).

There are several equivalent ways of describing a conformal structure. We will use here the linear description. This point of view goes back to the works of Thomas in the 1920s, [17]. We follow closely the treatment given in [7].

**Definition 2.1.** ([3], [7]) A Riemannian Tractor conformal bundle on an \((n \geq 2)\)-dimensional manifold \( M \) is a rank \( n + 2 \) real vector bundle \( T \to M \) endowed with a bundle metric \( h \) of Lorentzian signature with a distinguished lightlike line subbundle \( T^1 \subset T \).

**Definition 2.2.** ([3], [7]) A Tractor connection \( \nabla^T \) on a Tractor conformal bundle \( T \to M \) is a linear connection such that \( \nabla^T h = 0 \) and the following map \( \beta \) is an isomorphism of vector bundles on \( M \)
\[
\begin{array}{cccccc}
T M & \xrightarrow{\beta} & \text{Hom}(T^1, (T^1)^{\perp}/T^1) \\
& & \\
& & M
\end{array}
\]
where \( \beta(v)(\xi_x) = \nabla^T_v \sigma + T^1_x \) for \( x \in M, v \in T_xM, \xi_x \in T^1_x \) and \( \sigma \in \Gamma(T^1) \) is any section with \( \sigma(x) = \xi_x \).
Starting from the aforementioned \((T, T^1, h, \nabla^T)\), it is possible to define a conformal structure on \(M\). Nevertheless, for a given conformal structure, the Tractor connection is not unique. This situation is similar to the case of the Levi-Civita connection for semi-Riemannian metrics. To achieve a result of uniqueness, we require \(n \geq 3\) as follows. For \((n \geq 3)\)-dimensional manifolds \(M\), there is an unique normal Tractor connection \(\nabla^T\) for a conformal structure \((M, c)\) that satisfies some normalization conditions, (see details in [7]).

The following result will be a key fact in this paper. For any \((n \geq 2)\)-dimensional manifold \(M\), there is a one-to-one correspondence (up to isomorphism) between Möbius structures and Tractor connections on Tractor conformal bundles \((T, T^1, h, \nabla^T)\) on \(M\). A Möbius structure on a manifold \(M\) is essentially equivalent to defining a conformal class of metrics on \(M\) and a “Schouten type-tensor” for \(c\). This problem was addressed in [4] and [3, Sec. 5]. For our purposes, we adopt the following definition.

**Definition 2.3.** ([4], [15]) A Möbius structure on an \((n \geq 2)\)-dimensional manifold \(M\) is a triple \((M, c, P)\) where \(c\) is a Riemannian conformal structure on \(M\) and

1. \(P\) is a map \(P : c \to \mathcal{T}_{(0,2)}M\) such that for every \(g \in c\), the tensor \(P(g)\) is symmetric with
   \[
   \text{trace}_g P(g) = \frac{\text{scal}^g}{2(n-1)},
   \]
   where \(\text{scal}^g\) is the scalar curvature of the metric \(g \in c\) and \(\text{trace}_g P(g)\) denotes the \(g\)-metric trace of the corresponding tensor \(P(g)\).

2. \(P\) satisfies the following conformal transformation law
   \[
   P(e^{2u}g) = P(g) - \frac{\|\nabla^g u\|^2_g}{2} g - \text{Hess}^g(u) + du \otimes du,
   \]
   where \(\nabla^g u\) and \(\text{Hess}^g(u)\) are the gradient and the Hessian of the function \(u \in C^\infty(M)\) for the metric \(g\), respectively.

We mean the map \(P\) as a Möbius structure for the conformal structure \(c\). The conformal transformation law (1) implies that a Möbius structure \(P\) for a conformal class \(c\) is completely determined by the value at a single \(g \in c\). In fact, the relationship between the scalar curvatures of two conformally related metrics and the conformal transformation law (1) imply that \(\text{trace}_{e^{2u}g} P(e^{2u}g) = \frac{\text{scal}^{e^{2u}g}}{2(n-1)}\).

For \((n \geq 3)\)-dimensional Riemannian conformal structures \((M, c)\), there is a preferred Möbius structure. In fact, recall that Schouten tensor is defined by

\[
P^g(X, Y) = \frac{1}{n-2} \left( \text{Ric}^g(X, Y) - \frac{\text{scal}^g}{2(n-1)} g \right),
\]
where \(\text{Ric}^g\) denotes the Ricci tensor of the Riemannian metric \(g \in c\). The well-known conformal transformation law for the Schouten tensor implies that \(P(g) = P^g\) provides a Möbius structure for the conformal class \(c\). Therefore, for conformal structures on \((n \geq 3)\)-dimensional manifolds, the Schouten tensor gives a canonical Möbius structure. But for the two dimensional case there is something new. A quadruple \((T, T^1, h, \nabla^T)\) on a 2-dimensional manifold \(M\) is equivalent to specifying a conformal class of Riemannian metrics and a “Schouten type-tensor”, ([3], [4]).

**Remark 2.4.** For \(n \geq 3\) and taking into account \(2 \div \text{Ric}^g = d\text{scal}^g\) (see for instance [12, Cor. 3.54]), one gets that \(\text{div} P^g = \frac{1}{2(n-1)} d\text{scal}^g\). This property is not satisfied for Möbius structures, in general.
Remark 2.5. The Uniformization Theorem states that a 2-dimensional Riemannian manifold \((M, g)\) admits a metric \(g'\) of constant Gauss curvature \(k\) conformal to \(g\). This fact leads to a preferred choice of the Möbius structure determined by \(P(g') = (k/2)g'\) and the conformal transformation law. On the other hand, recall that for a connected oriented 2-dimensional manifold \(M\), there is a well-known one-to-one correspondence between conformal classes and complex structures. A Riemannian manifold endowed with a metric tensor \(\tilde{g}\) of signature \((1, m-1)\). A smooth immersion \(\Psi : M \to (\tilde{M}, \tilde{g})\) of a (connected) \(n\)-dimensional manifold \(M\) is said to be spacelike when the induced metric \(\tilde{g} := \Psi^*\cdot (\tilde{g})\) is Riemannian.

In this section we also fix some terminology and notations for spacelike immersions in Lorentzian manifolds. Let \((\tilde{M}, \tilde{g})\) be an \((m \geq 2)\)-dimensional Lorentzian manifold. That is, \((\tilde{M}, \tilde{g})\) is a semi-Riemannian manifold endowed with a metric tensor \(\tilde{g}\) of signature \((1, m-1)\). A smooth immersion \(\Psi : M \to (\tilde{M}, \tilde{g})\) of a (connected) \(n\)-dimensional manifold \(M\) is said to be spacelike when the induced metric \(g := \Psi^*\cdot (\tilde{g})\) is Riemannian.

Let \(\mathfrak{X}(M)\) be the \(C^\infty(M)\)–module of vector fields along the spacelike immersion \(\Psi\). Every vector field \(X \in \mathfrak{X}(\tilde{M})\) provides, in a natural way, the vector field \(X|_\Psi \in \mathfrak{X}(M)\). As usual, for \(V \in \mathfrak{X}(M)\), we have the decomposition \(V = V^\perp + V^\perp\), where \(V_x^\perp \in T_x\tilde{g}(T_x\tilde{M})\) and \(V_x^\perp \in (T_x\tilde{g}(T_x\tilde{M}))^\perp\) for all \(x \in M\). We have agreed to denote by \(T\Psi\) the differential map of \(\Psi\). We call \(V^\perp\) the tangent part of \(V\) and \(V^\perp\) the normal part of \(V\). The \(C^\infty(M)\)–submodule of \(\mathfrak{X}(M)\) of all normal vector fields along \(\Psi\) is denoted by \(\mathfrak{X}^\perp(M)\), that is, \(\mathfrak{X}^\perp(M) = \{V \in \mathfrak{X}(M) : V^\perp = 0\}\). The set of vector fields \(\mathfrak{X}(M)\) may be seen as a \(C^\infty(M)\)–submodule of \(\mathfrak{X}(M)\) by meaning of

\[
\mathfrak{X}(M) \to \mathfrak{X}(M), \quad V \mapsto T\Psi \cdot V,
\]

where \((T\Psi \cdot V)(x) := T_x\Psi \cdot V_x\) for all \(x \in M\). In order to avoid ambiguities, we explicitly write the differential map \(T\Psi\) when necessary.

We write \(\nabla^g\) and \(\nabla\) for the Levi-Civita connections of \((M, g)\) and \((\tilde{M}, \tilde{g})\), respectively. As usual, we also denote by \(\nabla\) the induced connection and by \(\nabla^\perp\) the normal connection on \(M\). The decomposition of the induced connection \(\nabla\), into tangent and normal parts, leads to the Gauss and Weingarten formulas of \(\Psi\) as follows

\[
(2) \quad \nabla_W V = T\Psi(\nabla^\perp_W V) + \Pi(V, W) \quad \text{and} \quad \nabla_V \xi = -T\Psi(A_\xi V) + \nabla^\perp_V \xi,
\]

for every tangent vector fields \(V, W \in \mathfrak{X}(M)\) and \(\xi \in \mathfrak{X}^\perp(M)\). Here \(\Pi\) denotes the second fundamental form and \(A_\xi\) the Weingarten endomorphism (or shape operator) associated to \(\xi\). For vector fields \(U, V, W \in \mathfrak{X}(M)\), we let

\[
(3) \quad (\nabla_U \Pi)(V, W) = \nabla^\perp_U (\Pi(V, W)) - \Pi(\nabla_U V, W) - \Pi(V, \nabla_U W).
\]

Then the Codazzi equation reads as follows (see for instance \cite{12} Prop. 4.33), taking into account that our convention on the sign of the Riemannian curvature tensor is the opposite to \cite{12}.

\[
(4) \quad (\nabla_U \Pi)(V, W) - (\nabla_V \Pi)(U, W) = \left(\tilde{R}(T\Psi \cdot U, T\Psi \cdot V)T\Psi \cdot W\right)^\perp,
\]

where \(\tilde{R}\) is the curvature tensor of \(\nabla\). Every Weingarten endomorphism \(A_\xi\) is self-adjoint and the second fundamental form is symmetric. They are also related by the following formula

\[
(5) \quad g(A_\xi V, W) = \tilde{g}(\Pi(V, W), \xi).
\]

The normal curvature tensor \(R^\perp\) is given by

\[
R^\perp(V, W)\xi = \nabla^\perp_W \nabla^\perp_V \xi - \nabla^\perp_W \nabla^\perp_V \xi - \nabla^\perp_{[V,W]} \xi.
\]
and the mean curvature vector field by \( H = \frac{1}{n} \text{trace}_g \Pi \).

A particular case occurs when, working with a codimension two immersion \( \Psi \), we are able to find a global lightlike normal frame \( \{ \xi, \eta \} \) along \( \Psi \). That is, \( \xi \) and \( \eta \) are two globally defined normal vector fields along \( \Psi \) which are lightlike (i.e., \( g(\tilde{g}(\xi, \xi)) = g(\eta, \eta) = 0 \)) with the normalization condition \( \tilde{g}(\xi, \eta) = -1 \). Let \( A_\xi \) and \( A_\eta \) be the associated Weingarten endomorphisms. Then, for every \( V, W \in \mathcal{X}(M) \), the second fundamental form can be written as

\[
\Pi(V, W) = -g(A_\eta V, W)\xi - g(A_\xi V, W)\eta.
\]

Taking traces in this expression, we obtain for the mean curvature vector field

\[
H = -\frac{1}{n} \left( \text{trace}(A_\xi)\xi + \text{trace}(A_\eta)\eta \right).
\]

Now, we end this section recalling the notion of ambient manifold and ambient metric as appears in [7]. Let \((M, c)\) be a Riemannian conformal structure on an \((n \geq 2)\)-dimensional manifold \(M\). Let us consider the \(\mathbb{R}^+\)-principal fiber bundle \(\pi : Q \to M\) defined as the ray fiber subbundle in the fiber bundle of Riemannian metrics given by metrics in the conformal class \(c\). Thus, the fiber over \(x \in M\) is formed by the values of \(g_x\) for all metrics \(g \in c\). Every section of \(\pi\) provides a Riemannian metric in the conformal class \(c\) and the principal \(\mathbb{R}^+\)-action on \(Q\) is given by \(\varphi(\tau, g_x) = \tau^2 g_x, x \in M\). Let us denote by \(Z_Q\) the fundamental vector field for the action \(\varphi\), that is,

\[
Z_Q(g_x) = \frac{d}{dt} \bigg|_{t=0} \varphi(e^t g_x) = \frac{d}{dt} \bigg|_{t=0} (e^{2t} g_x).
\]

The principal bundle \(\pi : Q \to M\) is called the scale bundle of \((M, c)\).

**Definition 2.6.** ([7]) An \((n + 2)\)-dimensional manifold \(\tilde{M}\) is called an ambient manifold for \((M, c)\) when

1. There is a free \(\mathbb{R}^+\)-action \(\tilde{\varphi}\) on \(\tilde{M}\).
2. There is an embedding \(\iota : Q \to \tilde{M}\) such that the following diagram commutes

\[
\begin{array}{ccc}
\mathbb{R}^+ \times Q & \xrightarrow{id_{\mathbb{R}^+} \times \iota} & \mathbb{R}^+ \times \tilde{M} \\
\varphi \downarrow & & \tilde{\varphi} \\
Q & \xrightarrow{\iota} & \tilde{M}
\end{array}
\]

Hence, the fundamental vector field \(Z \in \mathcal{X}(\tilde{M})\) for the action \(\tilde{\varphi}\) and the vector field \(Z_Q \in \mathcal{X}(Q)\) are \(\iota\)-related, i.e., \(T_{g_x} \iota \cdot Z_Q(g_x) = Z(\iota(g_x))\) for all \(g_x \in Q\).

**Definition 2.7.** ([7]) A Lorentzian metric \(\tilde{g}\) on an ambient manifold \(\tilde{M}\) for \((M, c)\) is called an ambient metric when

1. For \(Z \in \mathcal{X}(\tilde{M})\), the fundamental vector field for the action \(\tilde{\varphi}\), we have \(L_Z \tilde{g} = 2\tilde{g}\), where \(L\) is the Lie derivative.
2. For any \(g_x \in Q\) and \(\xi, \eta \in T_{g_x} Q\), the following equality holds

\[
e^\iota (\tilde{g})_{g_x}(\xi, \eta) = g_x(T_{g_x} \pi \cdot \xi, T_{g_x} \pi \cdot \eta).
\]

In particular, we have \(e^\iota (\tilde{g})(Z_Q, -) = 0\).

The condition \(L_Z \tilde{g} = 2\tilde{g}\) tells us that the vector field \(Z\) is homothetic with respect to the ambient metric \(\tilde{g}\). We would like to recall that the original definition of ambient metric in [10] imposes conditions on the Ricci tensor of the metric \(\tilde{g}\).
3. Basic properties from the Lorentzian geometry

**Definition 3.1.** A lightlike manifold is a pair \((N, h)\) where \(N\) is an \((n + 1)\)-dimensional smooth manifold with \(n \geq 2\) and furnished with a lightlike metric \(h\). That is, \(h\) is a symmetric \((0, 2)\)-tensor field on \(N\) such that

1. \(h(\xi, \xi) \geq 0\) for all \(\xi \in \mathfrak{X}(N)\).
2. The radical \(\text{Rad}(h)(\xi) = \{\xi \in T_y N : h(\xi, -) = 0\}\) for every \(y \in N\) defines a 1-dimensional distribution on \(N\).

A smooth immersion \(\Psi: N^{n+1} \to (\tilde{M}^{n+2}, \tilde{g})\) in an arbitrary Lorentzian manifold is said to be a lightlike hypersurface when the induced tensor \(\Psi^*\tilde{g}\) is a lightlike metric.

Now, let \((M, c)\) be a Riemannian conformal structure on an \((n \geq 2)\)-dimensional manifold \(M\) and \((\tilde{M}, \tilde{g})\) an ambient manifold \(\tilde{M}\) endowed with an ambient metric \(\tilde{g}\) for \((M, c)\). Then, condition (2) in Definition [2,7] implies that the hypersurface \(\iota: Q \to \tilde{M}\) is a lightlike hypersurface and the induced lightlike metric \(h := \iota^*(\tilde{g})\) does not depend on the particular ambient metric \(\tilde{g}\). In the terminology of [10], the lightlike metric \(h\) is called the tautological tensor. The radical distribution \(\text{Rad}(h)\) is globally generated by the vector field \(Z_Q\).

Recall that every choice of a metric \(g \in c\) provides a section of \(\pi: Q \to M\) and conversely.

**Lemma 3.2.** Let \((M, c)\) be a Riemannian conformal structure and \((\tilde{M}, \tilde{g})\) an ambient manifold for \((M, c)\). Then, for every \(g \in c\), the map \(\Psi^g := \iota \circ g: M \to (\tilde{M}, \tilde{g})\) is a codimension two spacelike immersion with induced metric \((\Psi^g)^*(\tilde{g}) = g\). Moreover, the vector field \(\xi := Z_{\Psi^g}\) is normal and lightlike along \(\Psi^g\) with \(A_\xi = -\text{Id}\).

**Proof.** A direct computation gives

\[
(\Psi^g)^*(\tilde{g})_x = g^*(\iota^*(\tilde{g})_{g_x}) = g^*(\pi^*(g)_{g_x}) = (\pi \circ g)^*(g)_x = g_x,
\]

for every \(x \in M\). Taking into account that \(\xi_x = Z(\Psi^g(x)) = T_{g_x}(Z_Q(g_x))\), for every \(x \in M\), we get \(\xi \in \mathfrak{X}^\perp(M)\) (for the immersion \(\Psi^g\)) and \(\tilde{g}(\xi, \xi) = 0\). In order to see that \(A_\xi = -\text{Id}\), recall that the condition \(\mathcal{L}_Z\tilde{g} = 2\tilde{g}\) is equivalent to

\[
\tilde{g}(\tilde{\nabla}_X Z, Y) + \tilde{g}(X, \tilde{\nabla}_Y Z) = 2\tilde{g}(X, Y), \quad X, Y \in \mathfrak{X}(\tilde{M}).
\]

In particular, for vector fields \(V, W \in \mathfrak{X}(M)\) we get

\[
\tilde{g}(\tilde{\nabla}_T \Psi^g \cdot V, T \Psi^g \cdot W) + \tilde{g}(T \Psi^g \cdot V, \tilde{\nabla}_T \Psi^g \cdot W) = 2\tilde{g}(T \Psi^g \cdot V, T \Psi^g \cdot W),
\]

and from the polarization identity we arrive to

\[
\tilde{g}(\tilde{\nabla}_T \Psi^g \cdot V, T \Psi^g \cdot W) = \tilde{g}(T \Psi^g \cdot V, T \Psi^g \cdot W).
\]

We are in position to compute \(\tilde{\nabla}_V \xi\) as follows

\[
\tilde{\nabla}_V \xi = \tilde{\nabla}_{T \Psi^g \cdot V} Z = (\tilde{\nabla}_{T \Psi^g \cdot V} Z) \uparrow + (\tilde{\nabla}_{T \Psi^g \cdot V} Z) \downarrow = T \Psi^g \cdot V + \nabla^\perp \xi
\]

and now the assertion \(A_\xi = -\text{Id}\) is clear.
4. A FAMILY OF AMBIENT METRICS

In this section, we introduce a family of ambient metrics in the sense of Definition 2.7. Let \((M, c)\) be a Riemannian conformal structure on an \((n \geq 2)\)-dimensional manifold \(M\) and \(\alpha : \mathbb{R} \to \mathcal{T}_{(1,1)} M\) be a smooth 1-parameter family of self-adjoint tensor fields with respect to any representative \(g \in c\) with \(\alpha(0) = \text{Id}\). Here, the smoothness of \(\alpha\) means that for every \(V \in \mathfrak{X}(M)\) and \(x \in M\), there exists

\[ \dot{\alpha}(\rho)(V_x) = \lim_{h \to 0} \frac{\alpha(\rho + h)(V_x) - \alpha(\rho)(V_x)}{h} \in T_x M. \]

In particular, we have \(\dot{\alpha}(0) \in \mathcal{T}_{(1,1)} M\). Let us fix a metric \(g \in c\), for every \(\rho \in \mathbb{R}\), we define the following symmetric tensor on \(M\),

\[ \langle V, W \rangle_{\rho} = g(\alpha(\rho)(V), W). \]

Clearly, \(\langle , \rangle^0 = g\) and \(\langle , \rangle^\rho\) can be seen as a 1-parameter deformation of the metric \(g\). Assume there is \(\varepsilon > 0\) such that \(\langle , \rangle^\rho\) is positive definite on \(M\) for \(|\rho| < \varepsilon\). A standard argument shows that there is such an \(\varepsilon\) when \(M\) is assumed to be compact. At least locally, such an \(\varepsilon\) always exists.

Henceforth, let us consider the manifold \(\tilde{M} := B \times M\), where \(B := \mathbb{R}^+ \times (-\varepsilon, +\varepsilon)\) with coordinates \((t, \rho)\). This manifold \(\tilde{M}\) can be endowed with the Lorentzian metric

\[ \tilde{g} = d(t \rho) \otimes dt + dt \otimes d(t \rho) + t^2 \langle \cdot, \cdot \rangle^0 \]

and with the free \(\mathbb{R}^+\)-action \(\tilde{\varphi}(t, (\rho, x)) = (\tau t, \rho, x)\). The choice of the metric \(g \in c\) provides the global trivialization of \(\pi : Q \to M\) given by \(t^2g_x \in Q \mapsto (t, x) \in \mathbb{R}^+ \times M\) and then also gives rise to the following embedding of \(Q\) in \(M\) at \(\rho = 0\),

\[ t_g : Q \to \tilde{M}, \quad t^2g_x \mapsto (t, 0, x). \]

A direct computation shows that \(t_g \circ \varphi(\tau, t^2g_x) = \tilde{\varphi}(id_{\mathbb{R}^+} \times t_g)(\tau, t^2g_x) = (\tau t, 0, x)\) and then, \(\tilde{M}\) is an ambient manifold for \((M, c)\). On the other hand, the fundamental vector field \(Z \in \mathfrak{X}(\tilde{M})\) corresponding to the action \(\tilde{\varphi}\) is \(Z = t \frac{\partial}{\partial t}\). One directly checks that \(\mathcal{L}_Z \tilde{g} = 2\tilde{g}\). Finally, for \(t^2g_x \in Q\) and \(\xi, \eta \in T_{t^2g_x} Q\), we have

\[ (t_g^* \tilde{g})_z(t^2g_x(\xi, \eta)) = g_{(t,0,x)}(T_{t^2g_x} t_g : \xi, T_{t^2g_x} t_g : \eta) = t^2g_x (T_{t^2g_x} \pi : \xi, T_{t^2g_x} \pi : \eta), \]

hence, \(\tilde{g}\) is an ambient metric on the ambient manifold \(\tilde{M}\) for \((M, c)\).

We have thus led to the following result.

**Proposition 4.1.** Let \((M, c)\) be a Riemannian conformal structure on an \((n \geq 2)\)-dimensional manifold \(M\). For every choice of a metric \(g \in c\) and \(\alpha : (-\varepsilon, +\varepsilon) \to \mathcal{T}_{(1,1)} M\) be a smooth 1-parameter family of self-adjoint tensor fields with respect to \(c\) with \(\alpha(0) = \text{Id}\) such that \(\langle , \rangle^\rho\) is positive definite on \(M\) for \(|\rho| < \varepsilon\), the manifold \(\tilde{M} = B \times M\) is an ambient manifold for \((M, c)\) and the Lorentzian metric \(\tilde{g}\) in \((\tilde{M}, \tilde{g})\) is an ambient metric.

**Remark 4.2.** In the particular case that \(\alpha(\rho) = f^2(\rho) \text{Id}\) with \(f : (-\varepsilon, +\varepsilon) \to \mathbb{R}, f(0) = 1\) and \(f > 0\), the ambient manifold \((\tilde{M}, \tilde{g})\) with metric \(\tilde{g} = d(t \rho) \otimes dt + dt \otimes d(t \rho) + (t f(\rho))^2 g\) is a warped product in the terminology of [12] Chap. 7.

**Remark 4.3.** The one-form \(\omega\) metrically equivalent to the vector field \(Z\) is

\[ \omega = t^2 d\rho + 2t dt. \]

In particular, we have \(d\omega = 0\).
Example 4.4. Let \( \mathbb{L}^{n+2} \) be the \((n+2)\)-dimensional Lorentz-Minkowski spacetime. That is, \( \mathbb{R}^{n+2} \) endowed with the Lorentzian metric \( g_L = -dx_0^2 + dx_1^2 + \cdots + dx_{n+1}^2 \), where \((x_0, \ldots, x_{n+1})\) are the canonical coordinates of \( \mathbb{R}^{n+2} \). The Lorentz-Minkowski spacetime \( \mathbb{L}^{n+2} \) essentially is the Fefferman-Graham ambient space for the conformal sphere \((S^n, c)\), where \( g \in c \) is the usual round metric of constant sectional curvature 1. As an application of Proposition 4.1, we consider \( \alpha(\rho) = (1+\rho/2)^2 \text{Id} \) for \( |\rho| < 2 \), then the metric in (\ref{eq:metric}) reduces to

\[
\tilde{g} = d(\rho t) \otimes dt + dt \otimes d(\rho t) + t^2 \left( 1 + \frac{\rho}{2} \right)^2 g
\]
on \( \mathbb{R}^+ \times (-2, 2) \times S^n \). In this case, the metric \( \tilde{g} \) agrees with the induced metric from the immersion

\[
F : \mathbb{R}^+ \times (-2, 2) \times S^n \to \mathbb{L}^{n+2}, \quad (t, \rho, x) \mapsto \left( \left( 1 - \frac{\rho}{2} \right) t, \left( 1 + \frac{\rho}{2} \right) t x \right).
\]
The choice of the metric \( g \in c \) gives the following embedding

\[
i_g : Q \to \mathbb{R}^+ \times (-2, 2) \times S^n, \quad t^2 g_x \mapsto (t, 0, x).
\]
By means of the map \( F \), the scale bundle \( Q \) of \((S^n, c)\) is isometric to the future lightlike cone

\[
\mathcal{N}^{n+1} = \{ v = (v_0, \cdots, v_{n+1}) \in \mathbb{L}^{n+2} : g_L(v, v) = 0, \ v_0 > 0 \}.
\]

The ambient manifold \((\tilde{M}, \tilde{g})\) is timelike orientable, that is, there exists a globally defined timelike vector field, namely,

\[
T := \frac{1}{t} \frac{\partial}{\partial t} - \left( 1 + \frac{\rho}{t^2} \right) \frac{\partial}{\partial \rho} \in \mathfrak{X}(\tilde{M}),
\]
which satisfies \( \tilde{g}(T, T) = -2 \). To be used later, we also introduce the spacelike vector field

\[
E := \frac{1}{t} \frac{\partial}{\partial t} + \left( 1 - \frac{\rho}{t^2} \right) \frac{\partial}{\partial \rho} \in \mathfrak{X}(\tilde{M}),
\]
with \( \tilde{g}(E, E) = 2 \) and \( \tilde{g}(T, E) = 0 \).

For simplicity of notation, from now on, we write \( \partial_t \) instead of \( \frac{\partial}{\partial t} \) and similarly for \( \partial_\rho \). The set of all natural lifts of vector fields \( V \in \mathfrak{X}(M) \) to \( \mathfrak{X}(\tilde{M}) \) is denoted by \( \Sigma(M) \). For a vector field \( V \in \mathfrak{X}(M) \), its lift to \( \Sigma(M) \subset \mathfrak{X}(\tilde{M}) \) is also denoted by \( V \).

As was mentioned in Remark 4.2, the ambient metrics (\ref{eq:metric}) are not warped product metrics, in general. Hence, the formulas for the Levi-Civita connection of warped products metrics in [12] Prop. 7.36] do not work.

Proposition 4.5. The Levi-Civita connection \( \tilde{\nabla} \) of \((\tilde{M}, \tilde{g})\) satisfies

\[
\tilde{\nabla}_{\partial_t} \partial_t = \tilde{\nabla}_{\partial_\rho} \partial_\rho = 0, \quad \tilde{\nabla}_{\partial_\rho} \partial_\rho = \tilde{\nabla}_{\partial_t} \partial_t = \frac{1}{t} \partial_\rho,
\]

\[
\tilde{\nabla}_{\partial_t} V = \frac{1}{t} V, \quad \tilde{\nabla}_{\partial_\rho} V = \frac{1}{2} \alpha(\rho)^{-1}(\dot{\alpha}(\rho)(V)),
\]

\[
\tilde{\nabla}_V W |_{\iota_g(\mathbb{Q})} = -\frac{1}{2t} \tilde{g}(\dot{\alpha}(0)(V), W) \partial_t - \frac{1}{t^2} \tilde{g}(V, W) \partial_\rho + \nabla^\mathbb{Q}_V W,
\]
where \( V, W \in \Sigma(M) \).
The vector fields $\nabla_{\partial_t} = \tilde{\nabla}_{\partial_t} \partial_t = 0$ and $\nabla_{\partial_t} \partial_{\rho} = \frac{1}{t} \partial_{\rho}$. On the other hand, the Koszul formula also implies $\tilde{g}(\tilde{\nabla}_{\partial_t} V, \partial_t) = \tilde{g}(\tilde{\nabla}_{\partial_t} V, \partial_{\rho}) = 0$ and $2\tilde{g}(\tilde{\nabla}_{\partial_t} V, W) = \partial_{\rho} \tilde{g}(V, W)$. By definition of the metric $\tilde{g}$,

$$\partial_{\rho} \tilde{g}(V, W) = 2tg(\alpha(\rho)(V), W) = \frac{2}{t} \tilde{g}(V, W),$$

and then we get $\tilde{\nabla}_{\partial_t} V = \frac{1}{t} V$. In the same manner, we compute

$$2\tilde{g}(\tilde{\nabla}_{\partial_t} V, W) = \partial_{\rho} \left(t^2 \tilde{g}(\alpha(\rho)(V), W)\right) = t^2 \tilde{g}(\alpha^{-1}(\dot{\alpha}(\rho)(V)), W).$$

From (14), it follows that

$$\tilde{g}(\tilde{\nabla}_V W, \partial_t) = -\tilde{g}(\tilde{\nabla}_V \partial_t, W) = -\frac{1}{t} \tilde{g}(V, W), \quad \tilde{g}(\tilde{\nabla}_V W, \partial_{\rho}) = -\frac{1}{2} \tilde{g} \left(\alpha^{-1}(\dot{\alpha}(\rho)(V)), W\right).$$

In order to compute $\tilde{g}(\tilde{\nabla}_V W, U_{\rho})$ for $U \in \mathcal{L}(M)$ and $p = (t, 0, x) \in \iota_{\rho}(Q)$, we can assume $U, V, W \in \mathcal{L}(M)$ so that all their brackets are zero at the point $p$. Then, the Koszul formula yields

$$2\tilde{g}(\tilde{\nabla}_V W, U_{\rho}) = V_p \tilde{g}(W, U) + W_p \tilde{g}(V, U) - U_p \tilde{g}(V, W)$$

$$= t^2 \left( V_x g(W, U) + W_x g(V, U) - U_x g(V, W) \right)$$

$$= 2t^2 g(\nabla_{U_{\rho}} W, U_x) = 2\tilde{g} \left( \nabla_{U_{\rho}} W, U_{\rho} \right).$$

Therefore, we conclude that

$$\tilde{\nabla}_V W \big|_{\iota_{\rho}(Q)} = -\frac{1}{2} \tilde{g}(\tilde{\nabla}_V W, T) T + \frac{1}{2} \tilde{g}(\tilde{\nabla}_V W, E) E + \nabla^\rho V W$$

$$= -\frac{1}{2t} \tilde{g}(\alpha^{-1}(\dot{\alpha}(\rho)(V)), W) \partial_t - \frac{1}{t^2} \tilde{g}(V, W) \partial_{\rho} + \nabla^\rho V W.$$

\[\square\]

**Remark 4.6.** Let us fix $(t, \rho) \in B$ and consider the spacelike submanifold $\mathcal{F} := \{(t, \rho)\} \times M \subset \tilde{M}$. The vector fields $T|_{\mathcal{F}}$ and $E|_{\mathcal{F}}$ span the normal bundle of $\mathcal{F}$ and Proposition 4.5 implies

$$\tilde{\nabla}_V T|_{\mathcal{F}} = \frac{1}{t^2} V - \frac{1}{2} \left(1 + \frac{\rho}{t^2}\right) \alpha^{-1}(\dot{\alpha}(\rho)(V))$$

and $\tilde{\nabla}_V E|_{\mathcal{F}} = \frac{1}{t^2} V + \frac{1}{2} \left(1 - \frac{\rho}{t^2}\right) \alpha^{-1}(\dot{\alpha}(\rho)(V))$.

for every $V \in \mathcal{L}(M)$. Therefore, the second fundamental form $\Pi_{\mathcal{F}}$ is given by

$$\Pi_{\mathcal{F}}(V, W) = -\frac{1}{2t} \tilde{g}(\alpha^{-1}(\dot{\alpha}(\rho)(V)), W) \partial_t - \frac{1}{t^2} \tilde{g}(V, W) \partial_{\rho} + \nabla^\rho V W,$$

where $V, W \in \mathcal{X}(M)$. Thus, the fibers $\mathcal{F}$ are not totally umbilical, in general. It is not difficult to show that for a fixed $(t, \rho)$, the corresponding fiber $\mathcal{F}$ is totally umbilical if and only if the endomorphism field $\alpha^{-1} \circ \dot{\alpha}(\rho) = f \text{Id}$ for some $f \in \mathcal{C}^\infty(M)$.

As was mentioned in Remark 4.2, when $\alpha(\rho) = f^2(\rho) \text{Id}$ with $f > 0$, the metric $\tilde{g}$ is a warped metric with warping function $h(t, \rho) = tf(\rho)$. In this case (16) reduces to

$$\Pi_{\mathcal{F}}(V, W) = -\frac{\tilde{g}(V, W)}{tf(\rho)} \left( f'(\rho) \partial_t + \frac{f(\rho) - 2\rho f'(\rho)}{t} \partial_{\rho} \right).$$

A direct computation shows that the above formula agrees with [12, Prop. 7.35 (3)].
As was mentioned, the original definition of ambient metric in [10] imposes conditions on the Ricci tensor of the metric $\tilde{g}$. Therefore, we end this section with several properties of the Ricci tensor $\overline{\text{Ric}}$ of $(M, \tilde{g})$ on the hypersurface $\rho = 0$, that is, on $\iota_\rho(Q)$. From [7], the Ricci tensor $\overline{\text{Ric}}$ of $(M, \tilde{g})$ restricted to $\iota_\rho(Q)$ satisfies, for $V \in \mathcal{L}(M)$,

\begin{equation}
\overline{\text{Ric}}|_{\iota_\rho(Q)}(\partial_t, \partial_t) = \overline{\text{Ric}}|_{\iota_\rho(Q)}(\partial_t, V) = 0
\end{equation}

if and only if $d\omega|_{\iota_\rho(Q)} = 0$. As a consequence of Remark 4.3, formula (17) holds for the metric $\tilde{g}$. The following result provides the other components of $\overline{\text{Ric}}$ on $\iota_\rho(Q)$.

**Corollary 4.7.** The Ricci tensor $\overline{\text{Ric}}$ of $(M, \tilde{g})$ satisfies

\begin{equation}
\overline{\text{Ric}}|_{\iota_\rho(Q)}(V, W) = \text{Ric}^\rho(V, W) - \frac{\text{trace}(\dot{\alpha}(0))}{2}g(V, W) - \left(\frac{n-2}{2}\right)g(\dot{\alpha}(0)(V), W),
\end{equation}

where $V, W \in \mathcal{L}(M)$. Also, for all $\xi, \eta \in \mathfrak{X}(Q)$, it follows that:

- For $n = 2$, we have $\overline{\text{Ric}}|_{\iota_\rho(Q)}(T_{\iota_\rho}, \xi, T_{\iota_\rho}, \eta) = 0$ if and only if $\text{trace}(\dot{\alpha}(0)) = 2K^\rho$, where $K^\rho$ is the Gauss curvature of $g$.
- For $n \geq 3$, we have $\overline{\text{Ric}}|_{\iota_\rho(Q)}(T_{\iota_\rho}, \xi, T_{\iota_\rho}, \eta) = 0$ if and only if $g(\dot{\alpha}(0)(-), -) = 2P^\rho$, where $P^\rho$ is the Schouten tensor of $g$.

**Proof.** Let $(e_1, \ldots, e_n)$ be an orthonormal local frame on $(M, g)$ and consider the orthonormal local frame for $(M, \tilde{g})$ on $\rho = 0$ given by

$$
\left(\frac{1}{\sqrt{2}}T, \frac{1}{\sqrt{2}}E_i, \ldots, E_n\right),
$$

where $E_i = \frac{1}{\sqrt{2}}e_i$ and the vector fields $T, E$ are given in (11) and (12), respectively. Then, we get

$$
\overline{\text{Ric}}|_{\iota_\rho(Q)}(V, W) = \sum_{i=1}^{n} \tilde{g}\left(\overline{\text{R}}(E_i, V)W, E_i\right) + \frac{1}{2}\tilde{g}\left(\overline{\text{R}}(E, V)W, E\right) - \frac{1}{2}\tilde{g}\left(\overline{\text{R}}(T, V)W, T\right) = \sum_{i=1}^{n} \tilde{g}\left(\overline{\text{R}}(E_i, V)W, E_i\right) + \frac{1}{t}\tilde{g}\left(\overline{\text{R}}(\partial_t, V)W, \partial_t\right).
$$

For every vector field $X \in \mathfrak{X}(M)$, we have the following decomposition

$$
X = \sum_{i=1}^{n} f_i E_i + \frac{1}{2}\tilde{g}(X, E)E - \frac{1}{2}\tilde{g}(X, T)T = \sum_{i=1}^{n} f_i E_i + \frac{1}{t}\tilde{g}(X, \partial_t)\partial_t + \tilde{g}(X, \partial_\rho)\partial_\rho - \frac{2\rho}{t^2}\tilde{g}(X, \partial_\rho)\partial_\rho.
$$

where $f_i \in C^\infty(M)$. Let us note that $f_i|_{\rho=0} = \tilde{g}(X, E_i)$. Now, it is a straightforward computation, from Proposition 4.5, to obtain that

\begin{equation}
\tilde{g}\left(\overline{\text{R}}(\partial_t, V)W, \partial_t\right) + \tilde{g}\left(\overline{\text{R}}(\partial_\rho, V)W, \partial_t\right) = 0.
\end{equation}
Lemma 5.2. Let the lightlike normal vector field $\xi$ in (9).

From (21) and (22), it is easy to check that

$$\text{Ric}_{\xi}(V, W) = \sum_{i=1}^{n} g\left(\nabla_{\xi_i} W, e_i\right) - \sum_{i=1}^{n} g\left(\nabla_{e_i} W, e_i\right) - \sum_{i=1}^{n} g\left(\nabla_{[e_i, V]} W, e_i\right)$$

The vanishing properties of the Ricci tensor on $\iota_g(\mathcal{Q})$ are direct consequences of (18).

5. Main results

Let $(M, c)$ be a Riemannian conformal structure on an $(n \geq 2)$-dimensional manifold $M$ and let us fix a metric $g \in c$. Assume $\alpha: (-\varepsilon, +\varepsilon) \to T_{\{1\}}M$ is a smooth 1-parameter family of self-adjoint tensor fields with respect to $c$ as in Proposition 4.1. From the same proposition, we know that $\tilde{M} = B \times M$ is an ambient manifold for $(M, c)$ with embedding given in (10) and ambient metric $\tilde{g}$ in (9).

For every $u \in C^\infty(M)$, the spacelike immersion $\Psi^{e^{2u}g}$ given in Lemma 3.1 satisfies

$$\Psi^{e^{2u}g} : M \to (\tilde{M}, \tilde{g}), \quad x \mapsto (e^{u(x)}, 0, x)$$

and $(\Psi^{e^{2u}g})^* (\tilde{g}) = e^{2u} g$. For simplicity of notation, from now on, we write $\Psi^u$ instead of $\Psi^{e^{2u}g}$. The differential map of $\Psi^u$ satisfies

$$T\Psi^u : V = V(u)e^u \partial_t|_{\Psi^u} + V|_{\Psi^u},$$

where $V \in \mathfrak{X}(M)$.

A direct computation from (21) shows that the vector fields

$$\xi^u = e^u \partial_t|_{\Psi^u} \quad \text{and} \quad \eta^u = e^{-u} \frac{\|\nabla^g u\|^2}{2} e^{-u} \partial_u|_{\Psi^u} + e^{-2u} \partial_u|_{\Psi^u} + e^{-2u} \nabla^g u|_{\Psi^u}$$

span the normal bundle of $\Psi^u$ and one easy checks that $\{\xi^u, \eta^u\}$ is a global lightlike normal frame.

Remark 5.1. The lightlike normal vector field $\xi^u$ agrees with $Z|_{\Psi^u}$ where $Z \in \mathfrak{X}(\tilde{M})$ is the fundamental vector field corresponding to the action $\tilde{\xi}$.

Lemma 5.2. Let $\Psi^u : M \to (\tilde{M}, \tilde{g})$ be the immersion given in (20). Then, for every $V \in \mathfrak{X}(M) \subset \mathfrak{X}(\tilde{M})$, the following formulas hold

$$\left(V|_{\Psi^u}\right)^\top = T\Psi^u \cdot V, \quad \left(\partial_{t|_{\Psi^u}}\right)^\top = T\Psi^u \cdot \nabla^g u.$$
Proposition 5.3. Let \( A_{\xi^u}, A_{\eta^u} \) be the Weingarten endomorphisms associated to the lightlike normal vector fields \( \xi^u, \eta^u \) given in (22), then \( A_{\xi^u} = -\text{Id} \) and

\[
A_{\eta^u} = e^{-2u} \left[ \frac{\dot{\alpha}(0) - \|\nabla g u\|_2^2 \text{Id}}{2} + g(\nabla g u, \text{Id}) \nabla g u - \nabla g \nabla g u \right],
\]

where \( \nabla g \nabla g u(V) := \nabla_V \nabla g u \) for all \( V \in \mathcal{X}(M) \).

**Proof.** The first assertion is a direct consequence of Lemma 5.2. On the other hand, according again to (21) and Proposition 4.5 we have for \( V \in \mathcal{L}(M) \),

\[
\tilde{\nabla}_V \left( e^{-u} \frac{\|\nabla g u\|_2^2}{2} \partial\right) = V \left( e^{-u} \frac{\|\nabla g u\|_2^2}{2} \right) \partial |_{\psi^u} + e^{-2u} \frac{\|\nabla g u\|_2^2}{2} V |_{\psi^u},
\]

(24)

\[
\tilde{\nabla}_V (e^{-2u} \partial_k) = -2e^{-2u} V(u) \partial_k |_{\psi^u} + e^{-2u} \left( \dot{\alpha}(0) (V) \right) |_{\psi^u},
\]

(25)

and

\[
\tilde{\nabla}_V (e^{-2u} \nabla g u) = -2e^{-2u} V(u) \nabla g u |_{\psi^u} + e^{-2u} \left( \tilde{\nabla}_V \nabla g u \right) |_{\psi^u}.
\]

Taking into account that \( (\partial_k |_{\psi^u})^\top = 0 \) and, from Lemma 5.2 we also get

\[
\left( \partial_k |_{\psi^u} - \nabla g u |_{\psi^u} \right)^\top = 0.
\]

Then, from (24), (25) and (26), we arrive to

\[
\left( \tilde{\nabla}_V \eta^u \right)^\top = e^{-2u} \left[ \frac{\|\nabla g u\|_2^2}{2} (V |_{\psi^u})^\top \right]^\top - \frac{1}{2} \left( \left( \dot{\alpha}(0) (V) \right) |_{\psi^u} \right)^\top + \left( \left( \tilde{\nabla}_V \nabla g u \right) |_{\psi^u} \right)^\top.
\]

The proof ends by means of a straightforward computation from (15) and Lemma 5.2. \( \square \)

**Corollary 5.4.** Let \( \Psi^u : M \to (\tilde{M}, \tilde{g}) \) be the immersion given in (20). The normal vector fields \( \xi^u \) and \( \eta^u \) are parallel with respect to the normal connection. In particular, the normal curvature tensor vanishes, that is, \( R^\perp(V, W) = 0 \) for every \( V, W \in \mathcal{X}(M) \).

**Proof.** From Proposition 5.3 we know that \( A_{\xi^u} = -\text{Id} \). Then, the Weingarten formula reads as follows

\[
\tilde{\nabla}_V \xi^u = T \Psi^u \cdot V + \tilde{\nabla}^\perp \xi^u = V(u) e^u \partial_k |_{\psi^u} + V |_{\psi^u} + \tilde{\nabla}_V \xi^u.
\]

On the other hand, from (14), we get

\[
\tilde{\nabla}_V \xi^u = \tilde{\nabla}_V (e^u \partial_k) = V(u) e^u \partial_k |_{\psi^u} + e^u e^{-u} V |_{\psi^u} = V(u) e^u \partial_k |_{\psi^u} + V |_{\psi^u},
\]

and therefore \( \tilde{\nabla}^\perp \xi^u = 0 \). Now, taking into account that \( \{\xi^u, \eta^u\} \) is a global lightlike normal frame, we have \( V \tilde{g}(\xi^u, \eta^u) = \tilde{g}(\xi^u, \tilde{\nabla}_V \eta^u) = 0 \) for every \( V \in \mathcal{X}(M) \). Thus, since \( \Psi^u \) is a codimension two spacelike submanifold, there is a smooth function \( f \in C^\infty(M) \) such that \( \tilde{\nabla}_V \eta^u = f \xi^u \) and then \( 0 = \tilde{g}(\eta^u, \tilde{\nabla}_V \eta^u) = -f \) and so \( \tilde{\nabla}^\perp \eta^u = 0 \). \( \square \)

**Remark 5.5.** From Proposition 5.3 and formula (6), one obtains the second fundamental form \( \Pi^u \) of \( \Psi^u \) as follows

\[
\Pi^u(V, W) = -g \left( \frac{\dot{\alpha}(0) (V)}{2} \right) + V(u) \nabla g u - \tilde{\nabla}_V \nabla g u, W \right) \xi^u + e^{2u} g(V, W) \eta^u,
\]

where \( g = \tilde{g} + \nabla g = \tilde{g} + \nabla \xi^u \) and \( \tilde{g} = \frac{\dot{\alpha}(0)}{2} \).
for every \( V, W \in \mathfrak{X}(M) \). In particular, the corresponding mean curvature vector field is

\[
H^u = \frac{e^{-2u}}{n} \left( \triangle^g u - \frac{\text{trace}(\hat{\alpha}(0))}{2} - (n-2)\frac{\|\nabla^g u\|_g^2}{2} \right) \xi^u + \eta^u,
\]

where \( \triangle^g \) denotes the Laplace operator of the metric \( g \).

Let \( \Psi^u : M \to (\tilde{M}, \tilde{g}) \) be the immersions given in (20) and consider the assignment

\[
P : c \to T_{(0,2)}M, \quad e^{2u}g \mapsto e^{2u}g(A_{\eta^u}(-), -),
\]

where \( A_{\eta^u} \) is the Weingarten endomorphism corresponding to \( \eta^u \). From Proposition 5.3 this assignment \( P \) is explicitly expressed as follows

\[
P(e^{2u}g)(V, W) = \frac{1}{2}g(\hat{\alpha}(0)(V), W) - \frac{\|\nabla^g u\|_g^2}{2}g(V, W) + V(u)W(\mu) - \text{Hess}\tilde{g}(u)(V, W),
\]

for \( V, W \in \mathfrak{X}(M) \). In particular, for \( u = 0 \), we have \( P(g)(V, W) = \frac{1}{2}g(\hat{\alpha}(0)(V), W) \) and then,

\[
P(e^{2u}g) = P(g) - \frac{\|\nabla^g u\|_g^2}{2}g - \text{Hess}\tilde{g}(u) + du \otimes du.
\]

Therefore, the assignment \( P \) satisfies the conformal transformation law of the Schouten tensor for an \((n \geq 3)\)-dimensional Riemannian manifold \( M \) and then, \( P \) also obeys the conformal transformation law of a Möbius structure for a Riemannian conformal structure \((M, c)\) given in Definition 2.3.

As usual, for any tensor field \( P \in T_{(0,2)}M \) and any Riemannian metric \( g \) on \( M \), we consider the endomorphism tensor field \( \hat{P}^g \) by means of the relation \( P(V, W) = g(\hat{P}^g(V), W) \) for \( V, W \in \mathfrak{X}(M) \). With this notation, we have stated the following results.

**Theorem 5.6.** Let \((M, c)\) be a Riemannian conformal structure on an \((n \geq 3)\)-dimensional manifold \( M \) and let us fix a metric \( g \in c \). Assume \( \alpha : (-\varepsilon, +\varepsilon) \to T_{(1,1)}M \) is a smooth 1-parameter family of self-adjoint tensor fields as in Proposition 4.1 with \( \hat{\alpha}(0) = 2\hat{P}^g \), where \( \hat{P}^g \) is the Schouten tensor of \( g \). Then, for every \( u \in C^\infty(M) \), the Schouten tensor of the metric \( e^{2u}g \) can be recovered from the immersion \( \Psi^u \) given in (20) by means of the following formula

\[
P(e^{2u}g)(V, W) = e^{2u}g(A_{\eta^u}(V), W), \quad V, W \in \mathfrak{X}(M),
\]

where \( A_{\eta^u} \) is the Weingarten endomorphism of \( \eta^u \). Moreover, from Corollary 4.7, the Ricci tensor of \( \tilde{g} \) satisfies \( \text{Ric}_{\hat{P}^g}(T_{\eta^u}g \cdot \xi, T_{\eta^u}g \cdot \eta) = 0 \), for all \( \xi, \eta \in \mathfrak{X}(\mathbb{Q}) \).

**Remark 5.7.** This result could be compared with the classical Brinkmann result [2] in the 1920s which stated that an \((n \geq 3)\)-dimensional simply connected Riemannian manifold is (locally) conformally flat if and only if it can be isometrically immersed in the future lightlike cone \( \Lambda^{n+1} \subset \mathbb{L}^{n+2} \). This classical result is presented in a modern form in [1].

**Remark 5.8.** A slight modification of Theorem 5.6 permits to obtain non-canonical Möbius structures on a conformal structure \((M, c)\) for an \((n \geq 3)\)-dimensional manifold \( M \).

With the same notation as above and in a similar way, we get.

**Theorem 5.9.** Let \((M, c)\) be a Riemannian conformal structure on a 2-dimensional manifold \( M \) and let us fix a metric \( g \in c \). Assume \( \alpha : (-\varepsilon, +\varepsilon) \to T_{(1,1)}M \) is a smooth 1-parameter family of self-adjoint tensor fields as in Proposition 4.1 with \( \text{trace}(\hat{\alpha}(0)) = 2K^g \), where \( K^g \) is the Gauss curvature of \( g \). Then, for every \( u \in C^\infty(M) \), the assignment

\[
e^{2u}g \in c \mapsto P(e^{2u}g)(V, W) = e^{2u}g(A_{\eta^u}(V), W), \quad V, W \in \mathfrak{X}(M),
\]
defines a Möbius structure on \((M, c)\), where \(A_\eta\) is the Weingarten endomorphism of \(\eta^u\). Moreover, from Corollary \ref{cor:mobius} the Ricci tensor of \(\widetilde{\gamma}\) satisfies \(\text{Ric}_{\text{c}(Q)}(T\eta, \xi, T\eta, \eta) = 0\), for all \(\xi, \eta \in \mathfrak{X}(Q)\).

**Example 5.10.** For example, when we take the metric \(g' \in c\) with constant Gauss curvature \(K_{g'} = k\), the curve \(\alpha(\rho) = \frac{1 + \frac{ak}{2}}{2}\text{Id}\) satisfies the assumptions in Theorem \ref{thm:5.9} for \(|\rho| < \frac{2}{a}\) when \(k \neq 0\) and for \(\rho \in \mathbb{R}\) when \(k = 0\). The Möbius structure deduced from \(\alpha\) agrees with \(P(g') = (k/2)g'\) in Remark \ref{rem:2.5}. In this case, the metric \(\widetilde{g}\) obtained from this \(\alpha\) is a warped metric, Remark \ref{rem:4.2}.

At least locally, any Möbius structure \(P\) on \((M, c)\), with \(\dim M = 2\), arises by means of Theorem \ref{thm:5.9} as follows.

**Theorem 5.11.** Let \((M, c, P)\) be a Möbius structure on a 2-dimensional manifold \(M\) and let us fix a metric \(g \in c\). Assume \(\alpha: (-\varepsilon, +\varepsilon) \to T(1,1)M\) is a smooth 1-parameter family of self-adjoint tensor fields as in Proposition \ref{prop:4.1} with \(\alpha(0) = 2\hat{P}(g)\). Then, the Möbius structure on \((M, c)\) obtained from \(\alpha\) by means of Theorem \ref{thm:5.9} agrees with \(P\).

**Proof.** As was mentioned, it suffices to check that the assignment given in \(\alpha\) satisfies \(g \mapsto P(g)\). This fact is a direct consequence of \(\alpha\).

**Remark 5.12.** Under the assumptions of Theorems \ref{thm:5.9} and \ref{thm:5.9} we have \(\text{trace}(\hat{\alpha}(0)) = \frac{\text{scal}_{2ug}}{n-1}\) with \(n = \dim M\). In such a case, by means of the relationship between the scalar curvature of conformally related metrics, formula \(\mathcal{I}(\sigma, \tau)\) reduces to

\[
\mathcal{H}^u = -\frac{1}{2n(n-1)}\text{scal}_{2ug} \xi^u + \eta^u,
\]

and therefore, \(\Vert \mathcal{H}^u \Vert^2 = \frac{\text{scal}_{2ug}}{n(n-1)}\). Taking into account Example \ref{ex:4.4} this formula widely generalizes \[\text{Cor. 4.5}\] and \[\text{Cor. 3.7}\]. For \(n = 2\), we have \(\Vert \mathcal{H}^u \Vert^2 = K^{2ug}\) and then, for \(M\) compact, making use of the Gauss–Bonnet theorem, we obtain

\[
\int_M \Vert \mathcal{H}^u \Vert^2 d\mu_{2ug} = 2\pi\chi(M),
\]

where \(d\mu_{2ug}\) is the metric volume element of \(e^{2ug}\) and \(\chi(M)\) is the Euler characteristic of \(M\). Also, from Corollary \ref{cor:5.4} the condition \(\nabla^u \mathcal{H}^u = 0\) is equivalent to \(\text{scal}_{2ug}\) being constant (compare with \[\text{Cor. 3.10}\]).

6. **An Application**

For a Möbius structure \((M, c, P)\) on a 2-dimensional manifold \(M\), the Cotton-York tensor for \(g \in c\) has been introduced in \[\text{Cor. 3.7}\] and \[\text{Cor. 4.5}\] as follows

\[
C(g)(U, V, W) = g\left(\left(\nabla_{U}^{g} \hat{P}(g)\right)(V) - \left(\nabla_{V}^{g} \hat{P}(g)\right)(U), W\right), \quad U, V, W \in \mathfrak{X}(M).
\]

This definition formally agrees with the usual Cotton-York tensor defined from the Schouten tensor of an \((n \geq 3)\)-dimensional Riemannian manifold \((M, g)\). The Cotton-York tensor given in \(\ref{eqn:cotton-york}\) for \(n = 2\) satisfies \(C(g) = C(e^{2ug})\), e.g., \[\text{Cor. 3.7}\]. In this section, we consider the ambient manifolds and ambient metrics obtained from \((M, c, P)\) by means of Theorem \ref{thm:5.11}.

**Lemma 6.1.** Let \((M, c, P)\) be a Möbius structure on a 2-dimensional manifold \(M\). Then, the Cotton-York tensor satisfies

\[
\left(\nabla_{U}^{g} \Pi^{u}\right)(V, W) - \left(\nabla_{V}^{g} \Pi^{u}\right)(U, W) = C(g)(V, U, W)\xi^u,
\]
for the fixed metric $g$ and every immersion $\Psi^u : M \to (\tilde{M}, \tilde{g})$ as in (20). Hence, the Codazzi equation (4) reduces to

$$\left( \tilde{R}(T\Psi^u \cdot U, T\Psi^u \cdot V)T\Psi^u \cdot W \right)_{\xi^u} = C(g)(V, U, W)\xi^u.$$  

Proof. According to Remark 5.5, the second fundamental form of $\Psi^u$ reduces to

$$(\ref{curvature})$$  

$$\Pi^u(V, W) = -P(e^{2u}g)(V, W)\xi^u + e^{2u}g(V, W)\eta^u.$$  

From Corollary 5.4 we have $\nabla^h_U \xi^u = \nabla^h_U \eta^u = 0$ and then, a direct computation gives

$$\nabla^h_U(\Pi^u(V, W)) = -e^{2u}g\left( \left( \nabla^h_U e^{2u}g \tilde{P}(e^{2u}g) \right)(V, W) \right)\xi^u.$$  

Now, the covariant derivative of the second fundamental form in (5) is easily computed. The proof ends by means of (30) and $C(g) = C(e^{2u}g)$ for $n = 2$. \qed

**Definition 6.2.** (4), (15) A M"obius structure $(M, c, P)$ on a 2-dimensional manifold $M$ is called flat when $C(g) = 0$ for every $g \in c$.

As a direct consequence of Lemma 6.1 we have.

**Proposition 6.3.** A M"obius structure $(M, c, P)$ on a 2-dimensional manifold $M$ is flat if and only if for every immersion $\Psi^u : M \to (\tilde{M}, \tilde{g})$ as in (20), the curvature tensor $\tilde{R}$ of the ambient manifold $(\tilde{M}, \tilde{g})$ satisfies

$$\tilde{R}(T\Psi^u \cdot U, T\Psi^u \cdot V)T\Psi^u \cdot W \in \mathcal{X}(M) \subset \mathcal{X}(M),$$  

for all $U, V, W \in \mathcal{X}(M)$.

**Remark 6.4.** For a flat M"obius structure $(M, c, P)$, Proposition 6.3 states that tangent spaces of $M$ along $\Psi^u$ are invariant under the curvature tensor of $(\tilde{M}, \tilde{g})$. As far as we know, the theory of immersions satisfying this condition appeared for the first time in [11]. K. Ogieu called these immersions as invariant immersions. This condition generalizes properties of the immersions into manifolds of constant sectional curvature. The existence of curvature invariant tangent subspaces in a general Riemannian manifold is related with the existence of totally geodesic submanifolds (see [18] for more details).

**Remark 6.5.** The well-known relationship between the Gauss curvature of conformally related metrics and (31) give, from the Gauss equation [12] Cor. 4.6] of $\Psi^u$,

$$\frac{K^g - \Delta g_U}{e^{2u}} = \tilde{K} + \text{trace } \tilde{P}(e^{2u}g),$$  

where $\tilde{K}(x)$ is the sectional curvature of the plane $T_x\Psi^u \cdot T_xM$ with respect to $\tilde{g}$ for $x \in M$. In particular, from Definition 2.3 we have $\tilde{K} = 0$ for every $u \in C^\infty(M)$. That is, the sectional curvature with respect to $\tilde{g}$ of every spacelike tangent plane to $\iota_g(Q)$ vanishes.

**REFERENCES**

[1] A. Asperti and M. Dajczer, Conformally flat Riemannian manifolds as hypersurfaces of the light cone, Can. Math. Bull., 32 (1989), 281–285.

[2] W.H. Brinkmann, On Riemannian spaces conformal to Euclidean space, Proc. Nat. Acad. Sci. USA, 9 (1923), 1–3.

[3] F.E. Burstall and D.M.J. Calderbank, Conformal Submanifold Geometry I–III, [arXiv:1006.5700v1](http://arxiv.org/abs/1006.5700v1) (2010).

[4] D.M.J. Calderbank, Möbius structures and two-dimensional Einstein-Weyl geometry, J. Reine Angew. Math., 504 (1998), 37–53.

[5] E. Cartan, Les Espaces à Connexion Conform, Les Annales de la Société Polonaise de Mathématiques, 2 (1923), 171–202.
[6] A. Čap and R. Gover, Tractor calculi for parabolic geometries, *Trans. Amer. Math. Soc.*, 354 (2002), 1511–1548.
[7] A. Čap and R. Gover, Standard tractors and the conformal ambient metric construction, *Ann. Global Anal. Geom.*, 24 (3)(2003), 231–259.
[8] A. Čap and J. Slovák, *Parabolic Geometries I. Background and General Theory*, Mathematical Surveys and Monographs 154, AMS 2009.
[9] C. Fefferman and C. Graham, Conformal invariants, *The mathematical heritage of Élie Cartan* (Lyon, 1984), *Astérisque, hors serie* (1985), 95–116.
[10] C. Fefferman and C. Graham, The ambient metric, *arXiv:0710.0919v2* (2008).
[11] K. Ogiue, On invariant immersions, *Ann. Mat. Pura Appl.*(4), 80 (1968), 387–397.
[12] B. O’Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.
[13] O. Palmas, F.J. Palomo and A. Romero, On the total mean curvature of a compact space-like submanifold in Lorentz-Minkowski spacetime, *Proc. Roy. Soc. Edinburgh Sect. A*, 148 (2018), no. 1, 199–210.
[14] F.J. Palomo and A. Romero, On spacelike surfaces in four-dimensional Lorentz-Minkowski spacetime through a light cone, *Proc. Roy. Soc. Edinburgh Sect. A*, 143 (2013), no. 4, 881–892.
[15] M. Randall, Local obstructions to a conformally invariant equation on Möbius surfaces, *Differ. Geom. Appl.*, 33 Suppl. (2014), 112–122.
[16] A. Romero, Constant mean curvature spacelike hypersurfaces in spacetimes with certain causal symmetries, *Hermitian-Grassmannian submanifolds, Springer Proc. Math. Stat.*, 203 (2017), 1–15.
[17] T.Y. Thomas, On conformal geometry, *Proc. Nat. Acad. Sci. USA*, 12 (1926), 352–359.
[18] K. Tsukada, Totally geodesic submanifolds of Riemannian manifolds and curvature-invariant subspaces, *Kodai Math. J.*, 19 (1996), 395–437.
[19] H. Weyl, *Space, Time, Matter*, Dover Publications, INC, 2016.

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