The Potts-$q$ random matrix model: loop equations, critical exponents, and rational case

B. Eynard and G. Bonnet

† Department of Physics and Astronomy, University of British Columbia 6224 Agricultural Road, Vancouver, British Columbia, V6T 1Z1
‡ CEA/Saclay, Service de Physique Théorique, F-91191, Gif-sur-Yvette Cedex, France

Abstract

In this article, we study the $q$-state Potts random matrix models extended to branched polymers, by the equations of motion method. We obtain a set of loop equations valid for any arbitrary value of $q$. We show that, for $q = 2 - 2\cos\frac{l}{r}\pi$ ($l$, $r$ mutually prime integers with $l < r$), the resolvent satisfies an algebraic equation of degree $2r - 1$ if $l + r$ is odd and $r - 1$ if $l + r$ is even. This generalizes the presently-known cases of $q = 1, 2, 3$. We then derive for any $0 \leq q \leq 4$ the Potts-$q$ critical exponents and string susceptibility.
1 Introduction

Random matrix models [1] have proven to be a powerful mathematical tool for the study of statistical physics systems on a fluctuating two dimensional lattice [2]. In particular, the Potts model [3] on a random lattice [4], which was first partially solved by Daul [5], has received a recent renewed interest [6, 7] due to new approaches to the problem. It is a $q$-matrices model where all the matrices are coupled to each other, which prevents one from using the formula [8, 9] to integrate out the relative angles between the matrices (they no longer are independent variables) and deal with the eigenvalues only. In this paper we use the equations of motion method which does not involve integration over angular variables. We obtain non-trivial loop equations relating even moments to odd moments of a single matrix $M_i$, and we show how to extend these relations to the case of Potts-$q$ plus branched polymers (gluing of surfaces). Such relations (which could not be obtained by previous methods [5, 6]) are needed to apply the renormalization group method [10] to Potts-$q$ models with added branching interactions, which is hoped to provide an understanding of the $c = 1$ ($q = 4$) barrier.

We obtain an $O(n)$-like equation, the solution of which is known [11] and involves elliptical functions [12]. What was the resolvent in the $O(n)$ model, however, is now (up to some transformations) the functional inverse of the Potts-$q$ resolvent, with $n$ replaced by $2 - q$. When $q = 2 - 2 \cos(\nu \pi)$ with $\nu$ rational, the general elliptic solution degenerates into an algebraic function. It has already been observed [3, 7] that for the particular cases of $q = 1, 2$ or 3, the resolvent obeys an algebraic equation of degree 2, 3, 5 respectively. In this article, we will derive from the value of $\nu$, for any “rational $q$”, the degree of the algebraic equation obeyed by the Potts-$q$ resolvent. We will also derive the Potts-$q$ critical exponent for general values of $q$, which agrees with Daul’s [5] expression.

2 The Potts-$q$ matrix model

The Potts-$q$ model with branching interactions (which appear if one wants to apply the renormalization group method) is defined by the partition function:

$$Z = \int dM_1 \ldots dM_q \, e^{-N^2 \sum \frac{1}{N} \text{tr} M_i^3 + \psi(\frac{1}{N} \text{tr} M_i^2, \frac{1}{N} \text{tr} \sum_{j \neq i} M_i M_j)}$$

(2.1)

with $M_i$ hermitian matrices $N \times N$.

The partial derivatives of $\psi$ with respect to $\text{tr} M_i^2/(2N)$ and $\text{tr} \sum_{j \neq i} M_i M_j/N$ are $\bar{U}$ and $\bar{c}/2$ respectively, and their expectation values (which are numbers) $U$ and $c$. When $\bar{U} = 1$ and $\bar{c}$ is a constant, this reduces to the ordinary Potts-$q$ model.

We will also define the following functions:

$$W(z) = \frac{1}{N} \langle \text{tr} \frac{1}{z - M_i} \rangle$$

$$\bar{W}(z) = \frac{1}{N} \langle \text{tr} \frac{1}{z - M_i} M_j \rangle$$

$$F(z, z') = \frac{1}{N} \langle \text{tr} \frac{1}{z - M_i} \frac{1}{z' - M_j} \rangle$$

$$\bar{F}(z, z') = \frac{1}{N} \langle \text{tr} \frac{1}{z - M_i} \frac{1}{z' - M_j} M_k \rangle + \frac{1}{N} \langle \text{tr} \frac{1}{z - M_i} \frac{1}{z' - M_j} M_k \rangle$$

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which do not depend on the indices provided \( i \neq j \neq k \). The function \( F(z, z') \) and \( \tilde{F}(z, z') \) are thus symmetric:

\[
F(z, z') = F(z', z) , \quad \tilde{F}(z, z') = \tilde{F}(z', z)
\]  

We will also define:

\[
f(z) = W(z) - gz^2 + (c - U)z
\]  

Recall that \( U \) and \( c \) can be general functions of the numbers \( \langle \text{tr} M_i^2 \rangle \) and \( \langle \text{tr} M_i M_j \rangle \).

The moments \( t_k \) of the resolvent \( W(z) \) are defined by the large \( z \) expansion:

\[
W(z) \sim \frac{1}{z} + \frac{t_1}{z^2} + \ldots + \frac{t_k}{z^{k+1}} + \ldots \quad \text{when} \quad z \to \infty
\]

and we define:

\[
u = \frac{c - U}{g}
\]

### 3 Equations of motion

We are going to work in the large \( N \) (planar) limit, where we have the factorization property:

\[
\langle \text{tr} A \text{tr} B \rangle = \langle \text{tr} A \rangle \langle \text{tr} B \rangle
\]

The following changes of variables in Eq. (2.1) then give the following equations:

- \( \delta M_1 = \frac{1}{z - M_1} \):

\[
g(z^2W(z) - z - t_1) + U(zW(z) - 1) + c(q - 1)\tilde{W}(z) = W^2(z)
\]

- \( \delta M_2 = \frac{1}{2} \left[ \frac{1}{z - M_1 - z' - M_2} + \frac{1}{z - M_2 - z - M_1} \right] \):

\[
g \left( z^2 F(z, z') - z' W(z) - \tilde{W}(z) \right) + U \left( z' F(z, z') - W(z) \right) + c \left( zF(z, z') - W(z') \right) + c(q - 2)\tilde{F}(z, z') = W(z')F(z, z')
\]

Subtracting Eq. (3.2) with \( z \leftrightarrow z' \), and using Eq. (2.3) we can get rid of \( \tilde{F} \) in Eq. (3.2):

\[
(f(z) - f(z'))F(z, z') = g \left( z'W(z) - zW(z') + \tilde{W}(z') - \tilde{W}(z) - uW(z) + uW(z') \right)
\]

In particular, if we choose \( z' \) such that \( f(z') = f(z) \) and \( z' \neq z \), then we have:

\[
(z' - u)W(z) - (z - u)W(z') + \tilde{W}(z) - \tilde{W}(z') = 0
\]
Eq. (3.1) allows us to eliminate $\tilde{W}$, and we then have an equation involving only $W$ or equivalently $f$:

$$\begin{cases} f(z') = f(z) \\ \frac{1}{c}(z + z' - u - q^2) f(z) = (z + z')^2 - qzz' - (2 - q)u(z + z') + (1 - q)u^2 - \frac{1}{c} \end{cases}$$

(3.5)

This seemingly difficult non-local equation is sufficient to compute $f(z)$.

Let us first study it perturbatively. When $z \to \infty$, we have $f(z) \sim -gz^2$, thus $z' \sim -z$. If we expand $z'(z)$ in powers of $z$ by solving $f(z) = f(z')$ perturbatively, then insert this expansion into the second equation of Eq. (3.5), we obtain a set of equations of motion, the first of which are:

$$gt_2 + (cq - gu)t_1 = 0$$
$$g^2 t_4 + g(cq - 2gu)t_3 - gu(cq - 2gu)t_2 - g(gu^3 + 2)t_1 + c(1 - q) + gu = 0$$

$$\ldots$$

(3.6)

with $t_k = \langle \text{tr} M^k \rangle$. Let us recall that, in the general case, $u$ and $c$ are functions of $t_2$ and $t_{1,1} = \text{tr} M_i M_j / N$. However, we can compute $t_{1,1}$ thanks to Eq. (3.1) in function of the $t_k$'s:

$$c(q - 1)t_{1,1} + gt_3 + Ut_2 - 1 = 0$$

Finally, our equations, contrary to ordinary equations of motion, allow us to relate even traces of a given matrix $M_i$ to odd traces of the same matrix.

Only even traces appear as the higher order traces in these equations. Let us explain why. If we write the expansion of $f(z)$ as: $f(z) = -gz(z - u) + 1/z + \sum_{i=1}^{\infty} t_{i-1} / z^i$, the $z^{-i}$ coefficient of the $f(z') - f(z) = 0$ equation reads, as $z' = -z + u + \ldots$:

$$((-1)^i - 1) t_{i-1} + (-1)^{i-1} (i - 1) t_{i-2} u + \ldots = 0$$

Thus, our loop equations give us a relation between the even and odd parts of $f(z)$, and $f(z)$ can be completely determined by the requirement that it has only one cut in the physical sheet (i.e. $f(-z)$ is regular along this cut).

Finally, let us stress that these equations of motion, which are valid for branched polymers, are a precious tool whenever one wants to study Potts-$q$ models by the renormalization group method [10].

4 Correspondence with the O(n) model

Let us now see how to deal with a non-local equation such as Eq. (3.5).

The function $z'(z)$ defined above maps one solution of the equation $f(z) = y$ on another. It is involutive in the sense of multivaluated functions. We have:

$$z'(z'(z)) = z$$

Let $z_0$ be a fixed point:

$$z'(z_0) = z_0$$
and \( f_0 = f(z_0) \).

Then let us set

\[ \zeta = \sqrt{f_0 - f} \]

and consider \( z \) as a function of \( \zeta \). Then we have \( z' = z(-\zeta) \) and \( z(\zeta) \) is regular at \( \zeta = 0 \). Let us set

\[ \omega(\zeta) = z(\zeta) + \frac{1}{c} \frac{1}{4 - q} (\zeta^2 - f_0 - (2 - q)cu) \]

Eq. (3.5) rewritten in term of \( \omega(\zeta) \) is now an \( O(n) \)-like quadratic equation:

\[
\omega^2(\zeta) + \omega^2(-\zeta) + (2 - q)\omega(\zeta)\omega(-\zeta) = R(\zeta) 
\] (4.1)

where the right-hand side of the equation is an even polynomial of degree 4. The similarity between Potts-\( q \) and the \( O(n) \) model had already been noted \[5, 6\], but it had always been said to be unphysical. We shall show here how one can relate the results for the \( O(n) \) model to those for Potts-\( q \).

Eq. (4.1) can be solved exactly, as in the case of the \( O(n) \) model \[11\]. Here, we will assume for simplicity that \( \omega(\zeta) \) has only one physical cut \([a, b] \) with \( ab > 0 \). Let us denote \( q = 2 - 2 \cos(\nu \pi) \), with \( 0 \leq \nu \leq 1 \). Then we have, by writing

\[
R(\zeta + i0) - R(\zeta - i0) = 0
\]

\[
(\omega(\zeta + i0) - \omega(\zeta - i0))(\omega(\zeta + i0) + \omega(\zeta - i0) + 2 \cos(\nu \pi)\omega(-\zeta)) = 0 \quad \text{for} \quad a \leq \zeta \leq b
\]

Thus we have the linear equation:

\[
\omega(\zeta + i0) + \omega(\zeta - i0) + 2 \cos(\nu \pi)\omega(-\zeta) = 0
\] (4.2)

The general solution for \( \omega(\zeta) \) is known and can be expressed with elliptical functions. It degenerates, in the rational case (i.e. when \( \nu \) is rational), into the solution of an algebraic equation. However, \( \omega(\zeta) \) is not the resolvent of the model as it is in the \( O(n) \) model. Indeed, it is rather the functional inverse of the resolvent for the Potts-\( q \) model, up to some transformations. Thus, the phase diagrams and critical exponents of Potts-\( q \), as expected, are not the same as for the \( O(n) \) model.

5 Rational case

Here we will assume that \( \nu \) is rational: \( \nu = l/r \) where \( l \) and \( r \) are relatively prime integers. We first recall how to obtain Eq. (5.4), which is an algebraic equation for \( \omega(\zeta) \) \[11\].

If we denote \( \omega_+(\zeta) = e^{i\frac{\nu \pi}{2}} \omega(\zeta) + e^{-i\frac{\nu \pi}{2}} \omega(-\zeta) \) and \( \omega_-(\zeta) = \omega_+(\zeta) \), then Eq. (4.1) reads

\[
\omega_+(\zeta)\omega_-(\zeta) = R(\zeta)
\] (5.1)

and Eq. (4.2) becomes:

\[
\omega_+(\zeta + i0) = -e^{i\nu \pi} \omega_-(\zeta - i0) \quad \omega_-(\zeta + i0) = -e^{-i\nu \pi} \omega_+(\zeta - i0)
\] (5.2)
If $\phi(\zeta)$ is defined by:

$$\omega_+ = \sqrt{R} \ e^{i\phi - (\nu + 1)\pi/2}$$

then we have

$$\omega_- = \sqrt{R} \ e^{-i\phi - (\nu + 1)\pi}$$

and

$$\omega(\zeta) = -\frac{\sqrt{R} \cos(\phi)}{\sin \nu \pi}$$

Eq. (5.2) shows that

$$S(\zeta) = \frac{1}{2}(\omega_+ + (-1)^{r+l}\omega_-)$$

has no cut, and behaves as a polynomial at infinity (as does $\omega(\zeta)$), i.e. it is a polynomial. We thus have the algebraic equation for $\omega(z)$:

$$S(\zeta) = R(\zeta)^{2r} \ e^{-it_{r+1}\pi} \ T_r(-\frac{\omega(\zeta) \sin(\nu \pi)}{\sqrt{R(\zeta)}})$$

(5.4)

where $T_r$ is the order $r$ Chebychev polynomial: $T_r(\cos \phi) = \cos r \phi$.

Let us now examine the degree of this equation. It is polynomial in $\omega$ and $\zeta$, but we have to keep in mind that the resolvent of our problem is not $\omega(\zeta)$, but $W_0 = f(z) + g z(z-u)$, with $\zeta = \sqrt{f_0 - f}$ and $\omega = z - \frac{1}{c(4-q)}(f + (2-q)cu)$.

The right-hand side of Eq. (5.4), seen as a polynomial in $\zeta$ and $z$, is even and of order $2r$ in $\zeta$, thus it is also a polynomial in $f$.

As for the left-hand side of the equation, $S(\zeta)$ verifies $S(-\zeta) = (-1)^{r+l} S(\zeta)$, thus, when $r+l$ is odd, we have to take the square of Eq. (5.4) to have our final algebraic equation for $f$ which should be of degree $2r$. When $r+l$ is even, however, Eq. (5.4) is already polynomial in $f$, and the degree of the equation should be $r$.

Moreover, it is easy to check that the order $2r$ terms on both sides of Eq. (5.4) are the same. Finally, the degree $d$ in $f(z)$ of the final equation is:

$$d = 2r - 1 \text{ if } r + l \text{ is odd}$$

$$d = r - 1 \text{ if } r + l \text{ is even}$$

(5.5)

while it is of degree $d + 1$ in $z$.

This formula generalizes the presently known results [6, 7]:

for $\nu = 1/3$, $q = 1$, and $d = 2$

for $\nu = 1/2$, $q = 2$, and $d = 3$

for $\nu = 2/3$, $q = 3$, and $d = 5$

(5.6)

In [6], the author investigated the dilute Potts-1,2,3, and 4 cases by the saddle point method. These models coincide, when there is no dilution, with the particular case of our models with no branching interactions. This gave rise to algebraic equations when $q \neq 4$ which correspond to our results. However, no general result was given. In [7], the
author used the equations of motion to solve Potts-3 with branching interactions, as well as for the resolution of Potts-∞. But, there again, there was no general expression.

We are now going to derive the general Potts-$q$ exponent from Eq. (4.1). If, when $\omega(\zeta)$ is singular (i.e. $\zeta$ is close to one of the bounds of the physical cut of $\omega$), $\omega(-\zeta)$ is not, then $\omega$ cannot have more complex singularities than half-integer exponents. Thus, whereas in the generic case the physical cut $[a, b]$ verifies $ab > 0$, when the model is at the Potts critical point, $a$ (or $b$) is equal to zero. This means that the bound of the unphysical semi-infinite cut we have suppressed by changing variables from $f$ to $\zeta = \sqrt{f_0 - f}$ coincides in that case with the physical cut.

Let us express: $\omega(\zeta) = C(-\zeta)^{\alpha} + $ regular part Eq. (4.1) shows immediately that the regular part is equal to zero, and that

$$e^{2i\pi\alpha} + 1 + 2\cos(\nu\pi)e^{i\pi\alpha} = 0$$

i.e. $2 - 2\cos(\nu\pi) = q = 2 + 2\cos(\alpha\pi)$

thus

$$\alpha = \pm\nu + 1 + 2p \quad \omega(\zeta) \sim (f_0 - f)^{\frac{\pm\nu + 1 + 2p}{2}} \quad p \in \mathbb{Z}$$

and

$$f \sim (z - \text{const})^{\frac{2}{\pm\nu + 1 + 2p}}$$

As we expect the exponent for $f$ to be greater than one, we have the exponent $\frac{2}{\pm\nu + 1}$ for $f$ and the string exponent, using [14]'s formula, is:

$$\gamma_s = 1 - \frac{2}{\pm\nu + 1} = \frac{(1 \pm \nu)}{(1 - \pm\nu)}$$

6 Conclusion

In this article, we have obtained general loop equations for the Potts-$q$ model extended to branched polymers, which allow us to relate the even and odd parts of the resolvent. This relation is then equivalent to an $O(n)$-like equation, from which we derived the Potts critical exponents and the degree of the algebraic equation which appears in the rational case. This last result generalizes the known results for $q = 1, 2$ and 3 [3, 7] to any $q = 2 - 2\cos(l/r\pi)$ with $l < r$ integers. Moreover, such loop equations as we have obtained are necessary when one wishes to apply the renormalization group techniques to Potts-$q$. The study of the renormalization group flows near $q = 4$ may then provide us with useful information about the $C = 1$ transition. Since $q = \infty$ is a real $C = \infty$ model [13] and the orders of the phase transitions differ between the flat and random surface Potts models, we may find for large $q$ a set of real $C > 1$ models.

Let us also stress that the equations of motion method may be used when $N$ is finite i.e. on non-planar surfaces, in contrast to the saddle point method. Furthermore, they are also less dependent on the analytic structure of the resolvent than the saddle point method of [3, 3]. Finally, as we know how to solve the $O(n)$ model exactly for general values of $n$, we hope to be soon able to obtain the general expression of the Potts-$q$ free energy and operators for any value of $q$. 
7 Acknowledgments

We are grateful to R. MacKenzie and I. Kostov for careful reading of the manuscript and F. David for useful discussions.

References

[1] Theory of random matrices in mesoscopic quantum physics, P. Mello, Les Houches 1994, E. akkermans, G. Montambaux, J.-L. Pichard, J. Zinn-Justin Ed (North-Holland, 1995)

[2] E. Brezin, C. Itzykson, G. Parisi, J.-B. Zuber, Comm. Math. Phys. 59 (1978) 35

[3] R. B. Potts, Proc. Cambridge Philos. Soc. 48 (1952) 106

[4] V.A. Kazakov, Nucl. Phys. B (Proc. Suppl.) 4 (1998) 93

[5] J.M. Daul hep-th/9502014

[6] P. Zinn-Justin, cond-mat/9903385

[7] G. Bonnet, hep-th/9904058

[8] C. Itzykson, J.-B. Zuber, J. Math. Phys. 21(1980) 411.

[9] Harish Chandra, Amer. J. Math. 79 (1957), 87

[10] G. Bonnet, F. David, hep-th/9811210, S. Higuchi, C. Itoi, S. Nishigaki, N. Sakai, Phys. Lett. B318 (1993) 63, Nucl. Phys. B434 (1995) 283-318, Phys. Lett. B398 (97) 123, F. David, Nucl. Phys. B487 [FS] (1997) 633-649, E. Brezin, J. Zinn-Justin, Phys. Lett. B288 (1992) 54-58

[11] B. Eynard, C. Kristjansen, Nucl. Phys. B466 (1996) 463-487, Nucl. Phys. B455 (1995) 577, B. Eynard, J. Zinn-Justin, Nucl. Phys. B386 (1992) 558-591

[12] Handbook of mathematical functions with formulas, graphs and mathematical tables, M. Abramovitz, I.A. Stegun, US Department of Commerce, National Bureau of Standards, Applied Mathematics Series 55 (1972)

[13] Yu. M. Makeenko, A. Migdal, Phys. Lett. B88 (1979) 135, A. Migdal, Ann. Phys. 109 (1997) 365, Veneziano, Nucl. Phys. B117 (1976) 519, ’t Hooft, Nucl. Phys. B72 (1974) 461

[14] I. Kostov, Nucl. Phys. B376 (1992) 539

[15] M. Wexler, Nucl. Phys. B410 (93) 377, J. Ambjorn, G. Thorleifsson, M. Wexler, Nucl. Phys. B439 (1995) 187-204