Motivated by the recent proposition by Buniy, Hsu and Zee with respect to discrete space-time and finite spatial degrees of freedom of our physical world with a short- and a long-distance scales, $l_P$ and $L$, we reconsider the Lorentz-covariant Yang’s quantized space-time algebra (YSTA), which is intrinsically equipped with such two kinds of scale parameters, $\lambda$ and $R$. In accordance with their proposition, we find the so-called contracted representation of YSTA with finite spatial degrees of freedom associated with the ratio $R/\lambda$, which gives a possibility of the divergence-free noncommutative field theory on YSTA. The canonical commutation relations familiar in the ordinary quantum mechanics appear as the cooperative Inonu-Wigner’s contraction limit of YSTA, $\lambda \to 0$ and $R \to \infty$.

Key words: Yang’s quantized space-time algebra; Inonu-Wigner contraction; finite spatial degrees of freedom; ultraviolet divergence in quantum field theory.
1 Introduction

As was pointed out recently by Jackiw\cite{1}, the idea of current noncommutative quantized space-time was first suggested by Heisenberg in the late 1930’s so as to regulate the short-distance singularities in local quantum-field theories by virtue of the noncommutative-coordinate uncertainty. In 1947, Snyder\cite{2} worked out it successfully over a lot of challenges by prominent theoretical physicists in those days. Immediately after the Snyder’s pioneering work, Yang\cite{3} proposed the so-called Yang’s space-time algebra by introducing quantized space-time and momentum in parallel, in order to improve the Snyder’s theory so as to satisfy the translation invariance in addition to the Lorentz-invariance, which nicely holds in both theories. Unfortunately, however, in those approaches\cite{4} we have never succeeded so far to find a clear-cut conclusion on the original Heisenberg’s intention to get rid of short-distance singularities, in spite of the fact that especially in the Yang’s theory the so-called contraction parameters, \( \lambda \) and \( R \) are explicitly introduced which might be related to short-distance (ultraviolet) and long-distance (infrared) regularization, respectively, as will be shown later in the present paper.

In this connection, the recent idea proposed by Buniy, Hsu and Zee\cite{5}, let us refer it hereafter by the B-H-Z’s proposition, is noticeable. Let us summarize its essence according to our present viewpoint:

1. There are many indications that space-time may be discrete rather than continuous. A consequence of spatial discreteness with spacing \( l_P \) is that in any finite region of size \( L \) there are only a finite number of degrees of freedom \( N \sim (L/l_P)^3 \).

2. Although our universe might be infinite in extent, any experiment performed by scientists must take place over a finite period of time. By causality, this implies that the experiment takes place in a region of finite size, which we take to be \( L \). We therefore assume the existence of a long-distance (infrared) regulator \( L \) in addition to a short-distance (ultraviolet) regulator \( l_P \).

Indeed, it is well known that a reasonable derivation of a finite number of spatial degrees of freedom is crucial for arriving at divergence-free quantum field theory within an ordinary Hilbert space of countable degrees of freedom as in quantum mechanics. The problem will be our main subject in this paper.

The present author has long studied the Snyder-Yang’s quantized space-time, through a series of the preliminary studies\cite{6,7}, since the first proposal\cite{8} of Snyder-Yang’s quantized space-time applied to the matrix model\cite{9}. Especially, the so-called Yang’s quantized space-time algebra has been there noticed, because of the fact that it is intrinsically equipped with two scale parameters, \( \lambda \) and \( R \), deeply related to \( l_P \) and \( L \) in the above B-H-Z’s
proposition and has the background symmetry equivalent to a conformal symmetry\cite{6}. As will be shown in the present paper, it has a strong possibility of leading us to a noncommutative field theory free from the ultraviolet divergences with aid of the B-H-Z’s proposition.

The present paper is organized as follows. In Sec. 2, we shortly recapitulate Yang’s space-time algebra (YSTA) and its spatial discrete structure in connection with the short-distance parameter in the B-H-Z’s proposition. In Sec. 3, based on the consideration of the quantum-mechanical limit of YSTA, we present a postulate of a certain contraction of representation space of YSTA in connection with the long-distance scale in the B-H-Z’s proposition. Sec. 4 is devoted to the group-theoretical consideration of our postulate of contracted representation of YSTA with finite spatial degrees of freedom. In Sec. 5, we show a possibility of divergence-free noncommutative field theory on YSTA. In the final section, Sec. 6, the physical and theoretical implication of our postulate and the B-H-Z’s proposition is conclusively discussed.

2 Yang’s Space-Time Algebra (YSTA) and B-H-Z’s Proposition

Let us first review the essence of Yang’s space-time algebra, focusing our attention on the B-H-Z’s proposition\cite{5}.

As was explicitly shown in Ref. \cite{7}, D-dimensional Yang’s quantized space-time algebra (YSTA) is derived as the result of the so-called Inonu-Wigner\cite{10} contraction procedure with two contraction parameters, \( R \) and \( \lambda \), from \( SO(D + 1, 1) \) algebra with generators \( \hat{\Sigma}_{MN} \):

\[
\hat{\Sigma}_{MN} \equiv i(q_M \partial / \partial q_N - q_N \partial / \partial q_M),
\]

(2.1)

which work on \( (D + 2) \)-dimensional parameter space \( q_M (M = \mu, a, b) \) satisfying

\[
-q_0^2 + q_1^2 + ... + q_{D-1}^2 + q_a^2 + q_b^2 = R^2.
\]

(2.2)

Here, \( q_0 = -iq_D \) and \( M = a, b \) denote two extra dimensions with space-like metric signature.

D-dimensional space-time and momentum operators, \( \hat{X}_\mu \) and \( \hat{P}_\mu \), with \( \mu = 1, 2, ..., D \), are defined in parallel by

\[
\hat{X}_\mu \equiv \lambda \hat{\Sigma}_{\mu a}
\]

(2.3)

\[
\hat{P}_\mu \equiv \hbar / R \hat{\Sigma}_{\mu b},
\]

(2.4)
together with $D$-dimensional angular momentum operator $\hat{M}_{\mu\nu}$

$$\hat{M}_{\mu\nu} \equiv \hbar \hat{\Sigma}_{\mu\nu} \quad (2.5)$$

and the so-called reciprocity operator

$$\hat{N} \equiv \lambda/R \hat{\Sigma}_{ab}. \quad (2.6)$$

Operators $(\hat{X}_\mu, \hat{P}_\mu, \hat{M}_{\mu\nu}, \hat{N})$ defined above satisfy the so-called contracted algebra of the original $SO(D + 1, 1)$, or Yang’s space-time algebra (YSTA):

$$[\hat{X}_\mu, \hat{X}_\nu] = -i\lambda^2/\hbar \hat{M}_{\mu\nu} \quad (2.7)$$

$$[\hat{P}_\mu, \hat{P}_\nu] = -i\hbar/R^2 \hat{M}_{\mu\nu} \quad (2.8)$$

$$[\hat{X}_\mu, \hat{P}_\nu] = -i\hbar \hat{N} \delta_{\mu\nu} \quad (2.9)$$

$$[\hat{N}, \hat{X}_\mu] = -i\lambda^2/\hbar \hat{P}_\mu \quad (2.10)$$

$$[\hat{N}, \hat{P}_\mu] = i\hbar/R^2 \hat{X}_\mu, \quad (2.11)$$

with familiar relations among $\hat{M}_{\mu\nu}$’s omitted.

Here, one finds that the so-called contraction-parameters $\lambda$ and $R$ with dimension of length introduced in the above expression are fundamental constants of YSTA, which are to be compared with two fundamental scales of short-distance $l_P$ and long-distance $L$, respectively, in the B-H-Z’s proposition mentioned in the preceding section.

Before entering into the central problem in the present paper, it is important to notice the following basic fact that $\hat{\Sigma}_{MN}$ defined in Eq. (2.1) with $M, N$ being the same metric signature have discrete eigenvalues, i.e., $0, \pm 1, \pm 2, \ldots$, and those with $M, N$ being opposite metric signature have continuous eigenvalues, as was shown in Ref. [7] explicitly. This fact leads to the remarkable result of YSTA: The spatial components of position and momentum defined in Eqs. (2.3) and (2.4), respectively, have discrete eigenvalues in units of $\lambda$ and $\hbar/R$, and temporal components of them continuous eigenvalues, consistently with Lorentz invariance. As was emphasized by Yang and explicitly shown in Ref. [7], this conspicuous fact is entirely due to non-commutativity between individual components of space-time and momentum operators. This aspect of YSTA is well understood by means of the familiar example of the three-dimensional angular momentum in quantum mechanics, where individual components, which are noncommutative among themselves, are able to have discrete eigenvalues, consistently with the three-dimensional rotation-invariance.

It should be noted that in YSTA the requirement of spatial discreteness in the B-H-Z’s proposition is well satisfied in a Lorentz-covariant way and
the fundamental scale parameter \( \lambda \) actually has the physical meaning as the minimal length of spatial distance in Eq. (2.3). With respect to another fundamental scale parameter \( R \), however, the preceding argument indicates only the fact that it gives the minimal scale of momenta, \( \hbar/R \). As a matter of fact, Eq. (2.3), as it stands, tells us that the maximal eigenvalue of spatial components of position operators \( \hat{X}_i \) becomes infinite corresponding to the large limit of integer eigenvalue of \( \hat{\Sigma}_{ia} \). In consequence, one finds that this is the central problem of YSTA in arriving at the B-H-Z’s proposition.

3 Quantum-Mechanical Limit of YSTA and Contraction of Representation Space

It is important here to notice that a clue to this problem is found in our previous consideration\[^7\] on the quantum-mechanical limit of YSTA, in which the canonical commutation relations familiar in the ordinary quantum mechanics appear. This consideration was given in the course of investigating the translation operation, which was introduced by Yang beyond the original Snyder’s space-time algebra, as mentioned above. Let us here review it shortly.

In fact, D-dimensional translation operator \( \hat{T} \) with infinitesimal parameters \( \alpha_\mu \) is defined by

\[
\hat{T}(\alpha_\mu) = \exp i (\alpha_\mu \hat{P}_\mu) = \exp \left( \frac{i\hbar\alpha_\mu}{R} \hat{\Sigma}_{b\mu} \right) \tag{3.1}
\]

One finds that this operator induces infinitesimal transformation on \( \hat{X}_\mu \)

\[
\hat{X}_\mu \rightarrow \hat{X}_\mu + \alpha_\mu \hat{N} \tag{3.2}
\]

together with

\[
\hat{N} \rightarrow \hat{N} - \alpha_\mu \hat{X}_\mu/R^2. \tag{3.3}
\]

This result is well understood, if one notices that the momentum operator \( \hat{P}_\mu \) in YSTA is nothing but generator of infinitesimal rotation on \( b-\mu \) plane, \( \Sigma_{\mu b} \) given in Eq. (2.4).

However, let us here notice that the reciprocity operator \( \hat{N} (= \lambda R^{-1} \hat{\Sigma}_{ab}) \) defined in Eq. (2.6) is an operator with discrete eigenvalues \( n (\lambda R^{-1}) \), \( n \) being \( \pm \) integer and the displacement \( \alpha_\mu \hat{N} \) in Eq. (3.2) is noncommutative with \( \hat{X}_\mu \). Therefore, it is important to see in what limit the ordinary translations
familiar in quantum mechanics may appear. Indeed, one finds them in a following cooperative limit of contraction-parameters, $\lambda$ and $R$

$$\lambda \to 0$$

$$R \to \infty,$$  \hspace{1cm} (3.4)

in conformity with a condition

$$\hat{N} (= \lambda R^{-1} \hat{\Sigma}_{ab}) \to 1.$$  \hspace{1cm} (3.5)

In fact, the above condition (3.5) in the limit (3.4) can never be a simple operator relation of YSTA and turns out below to be of the so-called Inonu-Wigner “contraction of groups and their representations”\[10\]. One finds immediately that it necessitates a large limit of discrete eigenvalues of $\hat{\Sigma}_{ab}$ in order for $\hat{N}$ to survive with nonvanishing value 1.

Furthermore, it should be noted here that the canonical commutation relations in the ordinary quantum mechanics just appear as the above cooperative Inonu-Wigner contraction limit of YSTA with respect to $\lambda$ and $R$, as seen from (2.7) to (2.11). This fact reminds us the Bohr’s correspondence principle at the birth of quantum mechanics, that is, quantum mechanics tends to classical mechanics in a large limit of quantum numbers. Hereafter, let us call the limit of contraction parameters (3.4) under the constraint (3.5), the quantum-mechanical limit of YSTA.

At this point, let us rewrite the last Eq. (3.5) in the following form

$$\hat{\Sigma}_{ab} \to R/\lambda \ (infinite).$$  \hspace{1cm} (3.6)

As was remarked after Eq. (3.5), this limiting relation as well as (3.5) does not imply a simple operator relation of $\hat{\Sigma}_{ab}$ in YSTA, but seems strongly to suggest that in YSTA equipped with contraction parameters $R$ and $\lambda$ in accordance with the B-H-Z’s proposition, eigenvalues of $\hat{\Sigma}_{ab}$ or more covariantly eigenvalues $n_{MN}$ of generators $\hat{\Sigma}_{MN}$ of $SO(D+1)$, with $M, N = a, b, 1, 2, \ldots, D - 1$, are to be limited to integers between $+R/\lambda$ and $-R/\lambda$ with the largest integer $[R/\lambda]$, so that Eq. (3.6) is replaced by the more comprehensive expression

$$|n_{MN}| \leq N \ (\equiv [R/\lambda]) \to infinite, \hspace{0.5cm} (M, N = a, b, 1, 2, \ldots, D - 1)$$  \hspace{1cm} (3.7)

with

$$N \equiv [R/\lambda].$$  \hspace{1cm} (3.8)

In fact, this postulate applied to Eqs. (2.3) and (2.4), respectively, leads to the result that the contraction parameter $R$ indicates the maximal spatial
scale in YSTA, as was expected in the B-H-Z's proposition, and the maximal
eigenvalue of spatial components of momenta becomes $\hbar/\lambda$.

The mathematical and physical implication of the postulate (3.7) will be
clarified in the next Sec. 4.

4 Irreducible Decomposition of $SO(D + 1)$- Representation in
Quasi-Regular Representation of YSTA ($SO(D + 1, 1)$)

In this section, let us first give a mathematical consideration of our postu-
late given by Eq. (3.7) in the preceding section. In Ref. [7], we studied
the unitary infinite-dimensional representation of YSTA, let us call it the
quasi-regular representation of $SO(D + 1, 1)$ [11], in which the (noncompact)
time operator $\hat{X}_0 \sim \hat{\Sigma}_0^a$ is diagonal with continuous eigenvalue $t$ and the repre-
sentation bases at a fixed time $t$ are given by simultaneous eigenstates of
operators, which are commutative with time operator $\hat{X}_0 (= \lambda \hat{\Sigma}_0^a)$ and com-
mutative among themselves, that is, the maximal commuting set of $SO(D)$,
for instance, in $D (= 11)$-dimensional YSTA, $\hat{\Sigma}_{12}, \hat{\Sigma}_{34}, \ldots, \hat{\Sigma}_{910}$. The whole
of the representation bases are then given by the following infinite set of
eigenstates, neglecting their multiplicity:

$$ | t; n_{12}, \ldots, n_{910} > \equiv | \hat{\Sigma}_{0a} = t/\lambda; \hat{\Sigma}_{12} = n_{12}, \ldots, \hat{\Sigma}_{910} = n_{910} > \, . $$

(4.1)

Our postulate stated in the preceding section, implies that the absolute
values of eigenvalues $n_{ij}$ in the above expression do not exceed $\mathcal{N} (= [R/\lambda])$.
In consequence, the spatial degrees of freedom of the representation space of
YSTA, we called it Hilbert Space I in Refs. [6,7], at a fixed time turn out to
be finite and of the order of

$$ D \equiv \mathcal{N}^{[D/2]} (= [R/\lambda]^{[D/2]}) $$

(4.2)

with $[D/2]$ denoting the integer either of $D/2$ and $(D - 1)/2$.

Our preceding postulate, i.e., the idea of setting a certain maximum
$\mathcal{N} (= [R/\lambda])$ to the eigenvalues of $\hat{\Sigma}_{ab}$, more covariantly those of genera-
tors of $SO(D + 1)$, $\hat{\Sigma}_{MN}$ with $M, N = a, b, 1, 2, \ldots, D - 1$ may be defined
in a more refined mathematical expression. Indeed, the quasi-regular repre-
sentation of $SO(D + 1, 1)$ mentioned above, whose basis vectors are given

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1Our postulate stated above suggests a certain possibility of a theoretical refinement of
the Inonu-Wigner contraction scheme [10] applied in physics, in which the representation
space of the contracted algebra with finite values of contraction parameters, might be so
constrained that contraction parameters, not merely as mathematical ones, but acquire
their own physical meanings, as was shown above in the case of YSTA.
by Eq. (4.1), must provide a certain unitary infinite-dimensional representation of its maximal compact subgroup $SO(D+1)$ under consideration, which can be completely reducible into an infinite series of unboundedly increasing-dimensional unitary irreducible representation of $SO(D+1)$. Therefore, our idea of setting a certain upper limit to eigenvalues of generators of $SO(D+1)$ means now to contract the above infinite series of irreducible decomposition into the finite series.

Let us explain the situation more in detail\cite{12}. The above unitary representation of $S(D + 1)$ appearing in the quasi-regular representation of $SO(D + 1, 1)$ is now explicitly decomposed into a series of the irreducible representations expressed in terms of the so-called spherical functions defined on $S^D = SO(D + 1)/SO(D)$, by taking $SO(D)$ with generators $\hat{\Sigma}_{MN}$ with $M, N = a, b, 1, 2, \ldots, D - 1$.

It is well-known that the irreducible representation $\rho_l$ of $SO(D + 1)$ on the representation space $S^D$, is uniquely designated by the maximal integer $l$ of eigenvalues $n_{ab}$ of $\hat{\Sigma}_{ab}$ in the representation, which is known to be a possible Cartan subalgebra of the so-called compact symmetric pair $(SO(D + 1), SO(D))$ of rank 1\cite{12}. The whole set of basis spherical functions belonging to the irreducible representation $\rho_l$ is given by

$$\Phi_{ln_{12}n_{34}n_{56}}(\theta_1, \theta_2, \ldots, \theta_D) \quad (l \geq n_{12} \geq n_{34} \geq n_{56} \geq \ldots). \quad (4.3)$$

They are labeled by $n_{12}, n_{34}, n_{56}, \ldots$, i.e., eigenvalues of generators, $\hat{\Sigma}_{12}, \hat{\Sigma}_{34}, \hat{\Sigma}_{56}, \ldots$, respectively, which constitute together with $\hat{\Sigma}_{ab}$ a Cartan subalgebra of $SO(D + 1)$, and expressed in terms of the products of associated Gegenbauer functions of polar coordinates of $S^D$, $\theta_1, \theta_2, \ldots, \theta_D$.

Eigenvalues of Casimir operator of $SO(D + 1)$ and of Laplace-Beltrami operator $\Delta$ operating on the spherical functions (4.3), are equally given by

$$-l(l + D - 1). \quad (4.4)$$

The dimension of the representation $\rho_l$ is given by

$$\dim(\rho_l) = \frac{(l + \nu)(l + 2\nu - 1)!}{\nu \cdot l!(2\nu - 1)!}, \quad (4.5)$$

where $\nu \equiv (D - 1)/2$ and $D \geq 2$.

From the above argument, one finds out that our postulate to impose a certain maximum $N(= [R/\lambda])$ upon the eigenvalues of generators of $S(D+1)$, that is, $\hat{\Sigma}_{MN}$ with $M, N = a, b, 1, 2, \ldots, D - 1$, is nothing but to impose the
upper limit upon the series of irreducible representations \( \rho_l \)'s of \( SO(D + 1) \) to be \( l \leq \mathcal{N} (= [R/\lambda]) \).

At this point, it is important to reconsider the physical implication of our postulate to limit or contract the representation space of YSTA in accordance with the B-H-Z’s proposition. In fact, it is clear that the Lorentz-covariant quasi-regular representation of YSTA or \( SO(D + 1, 1) \), whose representation bases are given in Eq. (4.1), formally gives rise to the infinite series of unitary irreducible representations \( \rho_l \)'s of \( SO(D + 1) \) which clearly involves \( l \)'s beyond \( \mathcal{N} (= [R/\lambda]) \). However, according to the B-H-Z’s proposition, as stated in Introduction, our physical experiments performed by scientists take place over a finite period of time, at most, of the order of \( R/c \), \( R \) being of a cosmological scale, so the experiments must take place, by causality, in a region of finite spatial size, at most, of the order of \( R \). In other words, any processes, which might evolve beyond the contracted representation space of YSTA stated above, are to be regarded as unphysical and excluded in our present scheme tightly based on the B-H-Z’s proposition, even though they are conceivable metaphysically.

This means that although, as was mentioned above, the quasi-regular representation of \( SO(D + 1, 1) \) formally provides the infinite series of irreducible decomposition of the unitary representation of its subgroup \( SO(D + 1) \), physical processes in our physical world necessitate and actually utilize for their own descriptions, the contracted part of representation space, which corresponds to the following order of finite spatial degrees of freedom

\[
\mathcal{D}(\mathcal{N}) = \sum_{l \leq \mathcal{N}} \text{dim} (\rho_l), \quad (4.6)
\]

with \( \mathcal{N} \) given in Eq. (3.8).

As seen in the next section, this consideration will play an important role in the arguments of ultraviolet divergence problem in the quantum field theory developed on YSTA.

5 Quantum Field Theory on YSTA with Finite Spatial Degrees of Freedom and B-H-Z’s Proposition

In a series of our studies of YSTA referred in the preceding argument, we have constantly tried to reformulate the so-called matrix model\[^{[9]}\] as quantum

\[^{[9]}\]It is interesting to find a certain similar procedure in Ref. [10], where a contraction limit of \( SO(3) \) into the inhomogeneous two-dimensional rotation group is shown through the infinite sequence of irreducible representations of \( SO(3) \).
mechanics of many-$D_0$ branes in terms of noncommutative quantum field theory on Yang’s quantized space-time. One encounters there the concept of creation-annihilation operators of $D_0$ brane, which construct the Fock space of $D_0$ branes, which we call Hilbert space II, distinguished from Hilbert space I so far discussed.

At this point, it is very important to note that the number of different creation-annihilation operators at fixed time $t$ is now not only countable but also finite, being of the order of $\mathcal{D}(N)^2$, because of the fact that creation-annihilation operators similarly to noncommutative $D_0$ brane field on YSTA are to be described by $\mathcal{D}(N) \times \mathcal{D}(N)$ matrices operating on $\mathcal{D}(N)$-dimensional Hilbert space I, according to our preceding postulate. This fact leads us to the important result that the Fock space or Hilbert space II constructed by a finite number of different creation-annihilation operators remains to be the ordinary Hilbert space with countable degrees of freedom.

In contrast, it should be noted that in the ordinary local quantum field theory, there appear a countable, but infinite number of different creation-annihilation operators. They are introduced, for instance, by Fourier analysis applied to a local field which is defined in the range $(R, -R)$ for each spatial coordinate and formally extended to an infinite spatial region by assuming an infinite repetition of the functional behavior in $(R, -R)$ or by taking the limit, $R \to \infty$. It is well-known that an infinite number of different creation-annihilation operators lead us to the continuous degrees of freedom of Fock space beyond the framework of the ordinary Hilbert space which must be a fundamental origin of ultraviolet-divergence.

In addition, it should be noticed that the important concept of asymptotic field defined in the limit of time $t \to \pm \infty$ in the well-known Lehmann-Symanzik-Zimmermann’s formalism of conventional local quantum field theory becomes inapplicable under the B-H-Z’s proposition.

\footnote{In this connection, it is interesting to note that in Ref. [7], we have derived the following $D_0$ brane field equation by applying the method of covariant Moyal star product to the $D_0$ brane field $D(\Sigma_{aK})(= D(X_\mu, N))$ defined on YSTA,

$$[\Sigma_{aK}^2 (\partial/\partial \Sigma_{aL})^2 - (\Sigma_{aK} \partial/\partial \Sigma_{aK})^2 - (D-1)\Sigma_{aK} \partial/\partial \Sigma_{aK}] D(X_\mu, N) = 0,$$

with $K, L = b, \mu$. It satisfies the translation invariance (see Eq. (3.1)) and tends to the massless Klein-Gordon equation in the quantum-mechanical limit, but is not fitted for a simple Fourier analysis, as was noticed there.}
6 Concluding Remarks

In the present paper, we have found that our postulate of a certain contraction of representation space of YSTA equipped with two scale parameters, $\lambda$ and $R$, in accordance with the B-H-Z’s proposition leads us to the important result that the noncommutative field theory on YSTA applied to the actual physical world may be free from the ultraviolet divergence. It should be noticed here that, as seen from the argument given at the end in Sec. 4, this postulate becomes rather nominal and might be removable with fixed finite values of $\lambda$ and $R$ whenever our physical processes under consideration take place in accordance with the B-H-Z’s proposition. In this case, however, one sees that the theoretical aspect of the finite spatial degrees of freedom of YSTA ceases to be explicit and instead its Lorentz covariance becomes explicit.

At this point, it is very interesting to ask how two scale parameters, $\lambda$ and $R$ in YSTA are to be fixed. According to the present view, it must be an experimental problem largely related to the applicability limit of the present quantum mechanics or local quantum field theory, because the canonical commutation relations in the ordinary quantum mechanics just appear as the quantum-mechanical limit of YSTA, that is, the cooperative Inonu-Wigner’s contraction limit, $N(\equiv R/\lambda) \to \infty$, as was discussed in Sec. 3. It is doubtless that $\lambda$ might be identified with something like the Planck length $l_P$ and $R$ with one of cosmological scales like $L$ in the B-H-Z’s proposition.

In this case, however, the relation of Yang’s quantized space-time to the curved space-time or gravitation may be asked. As was discussed in Ref. [6], it is important to note again that YSTA which begins with the flat $(D+1, 1)$-dimensional parameter space $q_M$’s subject to Eq. (2.2), might be regarded as a kind of flat local reference frame taken at any point in the $(D+1, 1)$-dimensional curved space-time, on the analogy of the familiar local Lorentz frame in general theory of relativity, while the curved space-time must be self-consistent with the gravitation induced by the Matrix Model or $D_0$-brane field theory constructed on YSTA.

Several important subjects must be postponed to be discussed to the forthcoming paper, which should clarify the physical meaning of the reciprocity operator$^{[13]}$ $\hat{N}$ possibly underlying the relation between the present quantum mechanics and YSTA from the more profound level and study the Bekenstein-Hawking area-entropy relation in the light of our present scheme.
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