WRINKLED EMBEDDINGS

To Paul Schweitzer on his 70th birthday

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Abstract

A wrinkled embedding $f : V^n \to W^m$ is a topological embedding which is a smooth embedding everywhere on $V$ except a set of $(n-1)$-dimensional spheres, where $f$ has cuspidal corners. In this paper we prove that any rotation of the tangent plane field $T_V \subset T_W$ of a smoothly embedded submanifold $V \subset W$ can be approximated by a homotopy of wrinkled embeddings $V \to W$.

Contents

1 Introduction 2
1.1 Wrinkled embeddings 2
1.2 Main result and the idea of the proof 3
1.3 Applications 4
1.4 History of the problem 4
1.5 Remark 5

2 Integrable approximations of tangential homotopies 5
2.1 Tangential homotopies of embeddings 5
2.2 Local integrable approximations 6
2.3 Wrinkled embeddings 7
2.4 Fibered wrinkled embeddings 10

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1 Introduction

1.1 Wrinkled embeddings

A wrinkled embedding $f : V^n \to W^m$, $n < m$, is a topological embedding which is a smooth embedding everywhere on $V$ except a finite set of $(n-1)$-dimensional spheres $S_i \subset V$, where $f$ has cuspidal corners: threefold corners along an equator $S'_i \subset S_i$ and twofold corners along the complement $S_i \setminus S'_i$. For $n = 2$ and $q = 3$ see Fig. 1 and Fig. 2. The spheres $S_i$ are called wrinkles.

A formal definition of wrinkled embeddings is given in Section 2.3 below.

Note that for $n = 1$ each wrinkle consists of two twofold cuspidal points. Families of wrinkled embeddings may have, in addition to wrinkles, embryos of wrinkles, and therefore wrinkles may appear and disappear in a homotopy of wrinkled embeddings.

Let us point out that a wrinkled embedding is not a wrinkled map in the sense of [EM97]. The relation between the two notions is discussed in Section 2.3 below.
1.2 Main result and the idea of the proof

In this paper we prove (see Theorem 2.5.1 below) that any homotopy of the tangent plane field $TV \subset TW$ (tangential homotopy) of a smoothly embedded submanifold $V \subset W$ can be approximated by a wrinkled isotopy, i.e., an isotopy through wrinkled embeddings $V \to W$. For $n = 1$ the idea of the proof is presented on Fig. 3. Here we consider the counterclockwise tangential rotation of an interval in $\mathbb{R}^2$. 
The implementation of this general idea for \( n > 1 \) is far from being straightforward. There are a lot of similarities here with the Nash-Kuiper theorem about isometric \( C^1 \)-embeddings \( V^n \to W^{n+1} \), where the proof in the case \( n = 1 \) is more or less trivial, while already contains the general idea (goffering). However, its realization for \( n > 1 \) is highly non-trivial.

1.3 Applications

Our main theorem can be reformulated as an *h-principle for A-directed wrinkled embeddings*, see [3.1.1]. As an application of the main theorem we prove an *h-principle for embeddings* \( f : V^n \to (W, \xi) \), \( n \geq q = \text{codim} \xi \), whose tangency singularities with respect to a distribution \( \xi \) (integrable or non-integrable) are simple, e.g. folds, or alternatively *generalized wrinkles* (see Theorem [3.2.1] and Section [3.3]). This *h-principle* allows us, in particular, to simplify the singularities of an individual embedding or a family of embeddings \( V^n \to \mathbb{R}^m \), \( n < m \), with respect to the projection \( \mathbb{R}^m = \mathbb{R}^q \times \mathbb{R}^{m-q} \to \mathbb{R}^q \), \( n \geq q \), i.e. with respect to the standard foliation of \( \mathbb{R}^m \) by \((m-q)\)-dimensional affine subspaces, parallel to \( 0 \times \mathbb{R}^{m-q} \).

1.4 History of the problem

**A. Directed embeddings.** An embedding \( V^n \to W^m \) is called *A-directed*, if its tangential (or Gaussian) image belongs to a given subset \( A \) of the Grassmannian bundle \( \text{Gr}_n W \). Using his convex integration method, Gromov proved in [Gr86] a general *h-principle for A-directed embedding* in the case when \( V \) is an *open* manifold and \( A \subset \text{Gr}_n W \) is an open subset. C. Rourke and B. Sanderson gave two independent proofs of this theorem in [RS01]. A different proof based on our holonomic approximation theorem was given in our book [EM02]. For some special \( A \subset \text{Gr}_n W \) Gromov also proved in [Gr73] and [Gr86] the *h-principle* for *closed* manifolds. However, for closed \( V \) the *h-principle* for *A-directed embeddings* fails for a *general* open \( A \). For example, for any closed \( V^n \) there is no *A-directed embeddings* \( V^n \to \mathbb{R}^{n+1} \) unless \( \pi(A) = S^n \), where

\[
\pi : \text{Gr}_n \mathbb{R}^{n+1} = S^n \times \mathbb{R}^{n+1} \to S^n
\]

is the projection. The main theorem of the current paper states that *this h-principle can be saved by relaxing the notion of embedding*. D. Spring in
proved, using Gromov’s convex integration method and the geometry of spiral curves from [Sp02], an existence theorem for directed embeddings with twofold spherical corners, which is equivalent to the non-parametric version of our main theorem, see the discussion in Sections 2.10 and 3.1 below.

B. Simplification of singularities. First result allowing to simplify singularities of an individual embedding $V^n \to \mathbb{R}^m$, $n < m$, with respect to the projection $\mathbb{R}^m = \mathbb{R}^q \times \mathbb{R}^{m-q} \to \mathbb{R}^q$, $n \geq q$, was proven in [El72]. In fact, in [El72] this result was formulated for $n = m - 1$; the general case $n < m$ can be derived from this basic one by induction which was done in [EM00]. A different proof based on convex integration method was given by D. Spring in [Sp02]. In [EM00] we also proved the parametric version, but only for $q = 1$, and only the epimorphism part of the corresponding parametric $h$-principle. However, the approach in [EM00] does not seem to be suitable to recover the main results of the current paper.

The reader may find additional interesting information related to the subject of this paper in [Sp02] and [Sp05]. A different approach to the problem of simplification of singularities can be found in [RS03].

1.5 Remark

We assume that the reader is familiar with the general philosophy of the $h$-principle, see [Gr86] and [EM02]. It is useful, though not necessary for the reader to be also familiar with the Holonomic Approximation Theorem from [EM02] and the wrinkling philosophy (see [EM97] and [EM98]). We recall in Section 4 for a convenience of the reader, some definitions and results from [EM97] and [EM98] and introduce there the notions of generalized wrinkles and generalized wrinkled maps.

2 Integrable approximations of tangential homotopies

2.1 Tangential homotopies of embeddings

In what follows we assume that $V \subset W$ is an embedded compact submanifold and denote by $f_0$ the inclusion $i_V : V \hookrightarrow W$. We also assume that the manifolds $W$ and $\text{Gr}_n W$ are endowed with Riemannian metrics.
Let $\pi : \text{Gr}_n W \to W$ be the Grassmannian bundle of tangent $n$-planes to a $m$-dimensional manifold $W$, $m > n$, and $V$ a $n$-dimensional manifold. Given a monomorphism (fiberwise injective homomorphism) $F : TV \to TW$, we will denote by $GF$ the corresponding map $V \to \text{Gr}_n W$. Thus the tangential (Gaussian) map associated with an immersion $f : V \to W$ can be written as $Gdf$.

A tangential homotopy of an embedding $f_0$ is a homotopy $G_t : V \to \text{Gr}_n W$, such that $G_0 = Gdf_0$ which covers an isotopy $g_t = \pi \circ G_t : V \to W$, $g_0 = f_0$. A tangential homotopy $G_t : V \to \text{Gr}_n W$ is called integrable, if $G_t = Gdf_t$. A tangential homotopy $G_t : V \to \text{Gr}_n W$ is called a tangential rotation, if $G_0 = Gdf_0$ and $\pi \circ G_t = f_0$, i.e. $G_t$ covers the constant isotopy $g_t = f_0 : V \to W$.

**Problem.** Let $G_t : V \to \text{Gr}_n W$ be a tangential homotopy. We want to construct an arbitrarily close integrable approximation of $G_t$, i.e an isotopy of embeddings $f_t : V \to W$, such that $Gdf_t$ is arbitrarily close to $G_t$. One can reduce the problem to the case when $G_t$ is a tangential rotation. Indeed, we can consider, instead of $G_t$, the rotation $(d\hat{g}_t)^{-1} \circ G_t$, where $\hat{g}_t : W \to W$ is a diffeotopy which extends the isotopy $g_t = \pi \circ G_t : V \to W$. Note that if $G_t$ is an integrable tangential homotopy, then $(d\hat{g}_t)^{-1} \circ G_t$ is the constant homotopy $G_t = G_0$.

Of course, an integrable approximation of a tangential rotation does not exist in general. However, as we shall see, one can always construct an integrable approximation of $G_t$ by a family of wrinkled embeddings.

### 2.2 Local integrable approximations

Let $X \subset W$ be a tubular neighborhood of $V \subset W$, and $\pi : X \to V$ the normal projection. Denote by $\mathcal{N}$ the normal foliation of $X$ by the fibers of $\pi$. An isotopy $f_t : V \to W$, $f_0 = i_V$, is called graphical, if all the images $f_t(V)$ are transversal to $\mathcal{N}$. In other words, the graphical isotopy $V \to W$ is an isotopy of sections $V \to X$, up to reparameterizations of $V$. A tangential rotation $G_t : V \to \text{Gr}_n W$ is called small, if $G_t(v)$ is transversal to $\mathcal{N}$ for all $t$ and $v$.

Following Gromov’s book [Gr86] we will use the notation $O p A$ as a replacement of the expression an open neighborhood of $A \subset V$. In other words, $O p A$ is an arbitrarily small but not specified open neighborhood of a subset $A \subset V$. 

6
2.2.1. (Local integrable approximation of a small tangential rotation)

Let \( G_t : V \to \text{Gr}_n W \) be a small tangential rotation of \( V \subset W \) and \( K \subset V \) a stratified subset of positive codimension. Then there exists an arbitrarily \( C^0 \)-small graphical isotopy \( f_t : V \to W \) such that the homotopy \( Gdf_t|_{O_p K} : O_p K \to \text{Gr}_n W \) is arbitrarily \( C^0 \)-close to the tangential rotation \( G_t|_{O_p K} \).

**Proof.** The space \( X^{(1)} \) of 1-jets of sections \( V \to X \) can be interpreted as a space of tangent to \( X \)-planes which are non-vertical, i.e. transversal to \( N \). Hence the inclusion \( f_0 : V \hookrightarrow X \) together with the tangential homotopy \( G_t : V \to \text{Gr}_n W \) can be viewed as a homotopy of sections \( F_t : V \to X^{(1)} \). For arbitrarily small \( \varepsilon \) and \( \delta \) we can construct, using Holonomic Approximation Theorem 3.1.2 in [EM02], a holonomic \( \varepsilon \)-approximation \( \tilde{F}_t \) of \( F_t \) over \( O_p \tilde{h}_t(K) \), where \( \tilde{h}_t : V \to V \) is a \( \delta \)-small diffeotopy. The 0-jet part \( \tilde{f}_t \) of the section \( \tilde{F}_t \) is automatically an embedding, because \( \tilde{f}_t \) is a section of the normal bundle. Thus, we have a family \( \tilde{f}_t \) of integrable approximations of \( G_t \) over \( O_p \tilde{h}_t(K) \), and hence one can define the required isotopy \( f_t : V \to W \) as the composition \( \tilde{f}_t \circ \tilde{h}_t \), where \( \tilde{f}_t : V \to X \) is an extension of the isotopy \( \tilde{f}_t : O_p \tilde{h}_t(K) \to X \) to an isotopy of sections \( V \to X \). \( \square \)

**Remarks.**

1. One can also apply the above construction to any section \( \tilde{V} \subset X \) instead of the zero-section \( V \subset X \), provided that the tangential rotation of \( T\tilde{V} \) is transversal to \( N \) (here \( N \) is the original foliation on \( X \) by the fibers of the projection \( \pi : X \to V \)).

2. Without the “graphical” restriction on \( f_t \) Theorem [2.2.1] remains true for any tangential rotation, see 4.4.1 in [EM02].

3. The relative and the parametric versions of the holonomic approximation theorem similarly prove the relative and the parametric versions of Theorem [2.2.1]. In the relative version the homotopy \( G_t \) and the diffeotopy \( h_t \) are constant near a compact subset \( L \subset K \). In the parametric version we deal with a family of embedded submanifolds \( V_p \hookrightarrow W, p \in B \). The parameters space \( B \) usually is a manifold, possibly with non-empty boundary \( \partial B \), and the homotopies \( G_{t,p}, p \in B \), are constant for \( p \in O_p \partial B \). In this case the diffeotopies \( h_{t,p}, p \in B \), are constant for \( p \in O_p \partial B \). \( \triangleright \)

### 2.3 Wrinkled embeddings

**A. Definition.** A (smooth) map \( f : V^n \to W^m, \ n < m \), is called a wrinkled
embedding, if

- $f$ is a topological embedding;
- any connected component $S_i$ of the singularity $S = \Sigma(f)$ is diffeomorphic to the standard $(n-1)$-dimensional sphere $S^{n-1}$ and bounds an $n$-dimensional disk $D_i \subset V$;
- the map $f$ near each sphere $S_i$ is equivalent to the map

$$Z(n,m) : \mathcal{O}_p \mathbb{R}^n S^{n-1} \rightarrow \mathbb{R}^m$$

given by the formula

$$(y_1, \ldots, y_{n-1}, z) \mapsto (y_1, \ldots, y_{n-1}, z^3 + 3(|y|^2 - 1)z, \int_0^z (z^2 + |y|^2 - 1)^2 dz, 0, \ldots, 0). \quad (1)$$

The compositions of the canonical form (1) with the projection to the space of first $n$ local coordinates in $\mathbb{R}^m$ is the standard equidimensional wrinkled map $\mathbb{R}^n \rightarrow \mathbb{R}^n$, see Section 4.2 or [EM97]. Thus, the canonical form for the wrinkled embeddings contains, in comparison to the canonical form for the wrinkled mappings, the unfolding function $\int_0^z (z^2 + |y|^2 - 1)^2 dz$ and zero functions as additional coordinates in the image. The spheres $S_i$ and its images $f(S_i)$ are called wrinkles of the wrinkled embedding $f$. According to the formula (1), each wrinkle $S_i$ has a marked $(n-2)$-dimensional equator $S'_i \subset S_i$ such that

- the local model for $f$ near each point of $S_i \setminus S'_i$ is given by the formula

$$(y_1, \ldots, y_{n-1}, z) \mapsto (y_1, \ldots, y_{n-1}, z^2, z^3, 0, \ldots, 0), \quad (2)$$

see Fig. 1;

- the local model for $f$ near each point of $S'_i$ is given by the formula

$$(y_1, \ldots, y_{n-1}, z) \mapsto (y_1, \ldots, y_{n-1}, z^3 - 3y_1 z, \int_0^z (z^2 - y_1)^2 dz, 0, \ldots, 0), \quad (3)$$

see Fig. 2
The compositions of the canonical forms (2) and (3) with the projection to the space of first \( n \) local coordinates in \( \mathbb{R}^m \) are the standard fold and cusp of the equidimensional map \( \mathbb{R}^n \rightarrow \mathbb{R}^n \).

We will denote the union \( \bigcup_i S'_i \) of all the equators \( S'_i \) by \( S' \).

The restriction of a wrinkled embedding \( f \) to \( \text{Int} \ D_i \) may have singularities, which are, of course, again wrinkles. We will say, that a wrinkle \( S_i \) has a depth \( d + 1 \), if \( S_i \) is contained in \( d \) “exterior” wrinkles. The depth of a wrinkled embedding is the maximal depth of its wrinkles; according to this definition the smooth embeddings are the wrinkled embeddings of depth 0.

\[ \blacktriangleright \text{Remark.} \text{ Note that a wrinkled embedding } f \text{ near } S \setminus S' \text{ is equivalent to the restriction of a generic } \Sigma^{11} \text{-map of an } (n+1) \text{-dimensional manifold to its fold } \Sigma^1 \text{ near the cusp } \Sigma^{11}. \text{ However, near } S' \text{ a wrinkled embedding } f \text{ is not equivalent to the restriction of a generic } \Sigma^{111} \text{-map to } \Sigma^1 \text{ near the swallow tail singularity } \Sigma^{111}. \ \blacktriangleright \]

B. Regularization. Any wrinkled embedding can be canonically regularized by changing the unfolding function \( u(y, z) = \int (z^2 + |y|^2 - 1)^2 \, dz \) in the canonical form to a \( C^1 \)-close function \( \tilde{u}(y, z) \) such that \( \partial_z \tilde{u}(y, z) > 0 \), see Fig.4 and Fig.5. One can chose \( \tilde{u} \) such that \( \tilde{u}(y, z) = u(y, z) \) for all \( (y, z) \in S \setminus Op S' \). Then the respective regularization does not moves the twofold corner points \( f(S \setminus S') \) everywhere except an arbitrarily small neighborhood of the threefold corner points \( f(S') \).

\[ \text{Figure 4: Regularization of a wrinkled embedding near } \Sigma_i \setminus \Sigma'_i \]

C. Gaussian map.

For any wrinkled embedding \( f : V \rightarrow W \) and any \( v_0 \in \Sigma_f \) there exists a limit

\[
\lim_{v \to v_0, v \in V \setminus \Sigma_f} f_*(T_v V).
\]

Hence, we can associate with a wrinkled embedding its wrinkled tangent bundle \( T(f) \). If \( V \) is oriented then the bundle \( T(f) \) is oriented as well. The
D. Normal foliation. Though the image $f(V) \subset W$ of a wrinkled embedding $f : V \to W$ is not a smooth submanifold, one can still define an analog of normal foliation.

There exists an $n$-dimensional submanifold $\hat{V} \subset W$, such that $f(\Sigma) \subset \hat{V}$ and $\hat{V}$ is tangent to $f(V)$ along $f(\Sigma)$. Let $X \subset W$ be a small neighborhood of $f(V)$. We supply $X$ with an “almost normal” (to $f(V)$) foliation $\mathcal{N}$, which coincides with the normal foliations to $\hat{V}$ near $f(\Sigma)$ and with the normal foliation to $f(V \setminus \mathcal{O}p \Sigma)$ near $f(V \setminus \mathcal{O}p \Sigma)$.

2.4 Fibered wrinkled embeddings

The notion of a wrinkled embedding can be extended to the parametric case. Considering $k$-parametric families $f_p, p \in B$, of the wrinkled embeddings we allow, in addition to the wrinkles, their embryos. Near each embryo $v_i \in V$ the map $f_p$ is equivalent to the map

$$Z_0(n, m) : \mathcal{O}_p \mathbb{R}, 0 \to \mathbb{R}^m$$

given by the formula

$$(y_1, ..., y_{n-1}, z) \mapsto \left( y_1, ..., y_{n-1}, z^3 + 3|y|^2 z, \int (z^2 + |y|^2)^2 dz, 0, ..., 0 \right). \quad (4)$$

Figure 5: Regularization of a wrinkled embedding near $\Sigma_i'$
Thus, wrinkles may appear and disappear when we consider a homotopy of wrinkled embeddings \( f_t : V \to W \).

In a more formal mode one can use, quite similar to the case of wrinkled mappings, the “fibered” terminology (see [EM97]). A fibered (over \( B \)) map is a commutative diagram

\[
\begin{array}{ccc}
V^{k+n} & \xrightarrow{f} & W^{k+m} \\
\downarrow p & & \downarrow q \\
B^k & \xleftarrow{\phi} & \\
\end{array}
\]

where \( f \) is a smooth map and \( p, q \) are submersions. For the fibered map \( f \) we denote by \( T_BV \) and \( T_BW \) the subbundles \( \text{Ker} \ p \subset TV \) and \( \text{Ker} \ q \subset TW \). They are tangent to foliations of \( V \) and \( W \) formed by preimages \( p^{-1}(b) \subset V \), \( q^{-1}(b) \subset W \), \( b \in B \). We will often denote a fibered map simply by \( f : V \to W \) when \( B \), \( p \) and \( q \) are implied from the context. Fibered homotopies, fibered differentials, fibered submersions, and so on can be naturally defined in the category of fibered maps. For example, the fibered differential \( d_Bf \) of a fibered map \( f : V \to W \) is the restriction

\[
d_Bf = df|_{T_BV} : T_BV \to T_BW.
\]

Notice that \( d_Bf \) itself has the structure of a map fibered over \( B \).

Two fibered maps, \( f : V \to W \) over \( B \) and \( f' : V' \to W' \) over \( B' \), are called equivalent if there exist open subsets \( A \subset B \), \( A' \subset B' \), \( Y \subset W \), \( Y' \subset W' \) with \( f(V) \subset Y \), \( p(V) \subset A \), \( f'(V') \subset Y' \), \( p'(V') \subset A' \) and diffeomorphisms \( \varphi : U \to U' \), \( \psi : Y \to Y' \), \( s : A \to A' \) such that they form the following commutative diagram
The canonical form $Z(k+n,k+m)$, being considered as a fibered map over the space of first $k$ coordinates, gives us the canonical form for the fibered wrinkle of the fibered wrinkled embedding. Thus, in a $k$-parametric family of the wrinkled embeddings each fibered wrinkle bounds a $(k+n)$-dimensional fibered disk, fiberwise equivalent to the standard $(k+n)$-disk in the space $\mathbb{R}^{k+n}$, fibered over $\mathbb{R}^k$. In addition, the canonical form $Z(k+n,k+m)$ over the half space $\mathbb{R}^k(y_1 \leq 0)$ gives us the model for the fibered “half-wrinkles” near the boundary of $B$.

2.5 Main theorem

The following Theorem 2.5.1 and its fibered analog 2.9.1 in Section 2.9 below are the main results of the paper.

2.5.1. (Integrable approximation of a tangential rotation) Let $G_t : V \to \text{Gr}_n W$ be a tangential rotation of an embedding $i_V : V \hookrightarrow W$. Then there exists a homotopy of wrinkled embeddings $f_t : V \to W$, $f_0 = i_V$, such that the homotopy $Gdf_t : V \to \text{Gr}_n W$ is arbitrarily $C^0$-close to $G_t$. If the rotation $G_t$ is fixed on a closed subset $C \subset V$, then the homotopy $f_t$ can be chosen also fixed on $C$.

Remark. The parametric version of Theorem 2.5.1 is also true, see Section 2.9 below.

A small rotation $G_t : V \to \text{Gr}_n W$ is called simple, if $G_t(v)$ is a rotation in a $(n+1)$-dimensional subspace $L_v \subset T_v W$ for every $v \in V$ and the angle of the rotation $G_t(v)$ is less than $\pi/4$. In particular, for $q = n+1$ any small rotation with the maximum angle $< \pi/4$ is simple. Any tangential rotation can be approximated by a finite sequence of simple rotations and hence Theorem 2.5.1 follows from

2.5.2. (Integrable approximation of a simple tangential rotation) Let $G_t : V \to \text{Gr}_n W$ be a simple tangential rotation of a wrinkled embedding $f_0 : V \to W$. Then there exists a homotopy of wrinkled embeddings $f_t : V \to W$, such that the homotopy $Gdf_t : V \to \text{Gr}_n W$ is arbitrarily $C^0$-close to $G_t$. If the rotation $G_t$ is fixed on a closed subset $C \subset V$, then the homotopy $f_t$ can be also chosen fixed on $C$.

Remark. The proof of Theorem 2.5.2 will give us the following additional information: the homotopy $f_t$ increases the depth of $f_0$ at most by 1. In particular,
• if $f_0$ is a smooth embedding, then $f_t$ consists of wrinkled embeddings of depth $\leq 1$;

• the depth of the final map $f_1$ in Theorem 2.5.1 is equal to the number of simple rotations in the decomposition of the rotation $G_t$. ▶

2.6 Local integrable approximations near wrinkles

We will distinguish the notions of homotopy of wrinkled embeddings and their isotopy. A homotopy $f_t$, $t \in [0, 1]$, of wrinkled embeddings is called an isotopy, if for all $t \in [0, 1]$ the wrinkled embedding $f_t$ has no embryos, i.e. its wrinkles do not die, and no new wrinkles are born. Equivalently, $f_t$ is an isotopy if $f_t = h_t \circ f_0$ where $h_t : W \to W$ is a diffeotopy.

A $C^0$-small isotopy $f_t : V \to W$, $f_0 = i_V$, of wrinkled embeddings is called graphical, if all the images $f_t(V)$ are transversal to the normal foliation $\mathcal{N}$ of $V$. As in the smooth case, the tangential rotation $G_t : V \to \Gr_n W$ of the wrinkled embedding $f_0 : V \to W$ is called small, if $G_t(v)$ is transversal to $\mathcal{N}$ for all $t$ and $v$. We will reformulate now 2.2.1 for the situation, when $f_0 : V \to W$ is a wrinkled embedding and $K = \Sigma = \Sigma(f_0)$.

2.6.1. (Local integrable approximation of small tangential rotation near wrinkles) Let $G_t : V \to \Gr_n W$ be a small tangential rotation of a wrinkled embedding $f_0 : V \to W$. Then there exists an arbitrarily $C^0$-small graphical isotopy of wrinkled embeddings $f_t : V \to W$ such that the homotopy

$$Gdf_t|_{O_p \Sigma} : O_p \Sigma \to \Gr_n W$$

is arbitrarily $C^0$-close to the tangential rotation $G_t|_{O_p \Sigma}$.

Proof. The image $f_0(O_p \Sigma) \subset W$ is not a submanifold and hence the proof of 2.2.1 formally does not work. Let $\hat{V} \subset W$ be an $n$-dimensional submanifold, such that $f_0(\Sigma) \subset \hat{V}$ and $\hat{V}$ is tangent to $f_0(\Sigma)$ along $f_0(\Sigma)$. Let $\hat{G}_t$ be an extension of the rotation $G_t|_{\Sigma}$ to a tangential rotation $\hat{V} \to \Gr_n W$. Apply 2.2.1 to the pair $(\hat{V}, f_0(\Sigma))$, the normal foliation $\mathcal{N}$ and the tangential rotation $\hat{G}_t$; denote the produced isotopy $\hat{V} \to W$ by $\hat{f}_t$. Let $g_t : W \to W$ be an ambient diffeotopy for $\hat{f}_t$, such that $g_t^* \mathcal{N} = \mathcal{N}$. Then $f_t = g_t \circ f_0$ is the desired isotopy of wrinkled embeddings on $O_p \Sigma$, which can be then extended to the whole $V$ as a graphical isotopy. □

13
2.7 Main lemma

Lemma 2.7.1, which we prove in this section, is the main ingredient in the proof of Theorem 2.5.2.

Let us denote by $\tilde{G} : S \to S^n$ the oriented Gaussian map of an oriented hypersurface $S \subset \mathbb{R}^{n+1}$. For the angle metric on $S^n$ denote by $U_\varepsilon(s)$ the open metric $\varepsilon$-neighborhood of the north pole $s = (0, \ldots, 1) \in S^n$.

An oriented hypersurface $S \subset \mathbb{R}^{n+1}$ is called:

- $\varepsilon$-horizontal, if $\tilde{G}(S) \subset U_\varepsilon(s) \subset S^n$;
- graphical, if $\tilde{G}(S) \subset U_{\pi/2}(s) \subset S^n$;
- $\varepsilon$-graphical, if $\tilde{G}(S) \subset U_{(\pi/2)+\varepsilon}(s) \subset S^n$;
- quasi-graphical, if $\tilde{G}(S) \subset U_{\pi}(s) \subset S^n$.

Similarly a wrinkled embedding $f : S \to \mathbb{R}^{n+1}$, is called $\varepsilon$-horizontal, if its Gaussian image $\tilde{G}(S) \subset S^n$ is contained in $U_\varepsilon(s)$.

In what follows we will often say almost horizontal and almost graphical instead of $\varepsilon$-horizontal and $\varepsilon$-graphical, assuming that $\varepsilon$ is appropriately small.

2.7.1. (Approximation of an embedded hypersurface by almost horizontal wrinkled embeddings) Let $S \subset \mathbb{R}^{n+1}$ be an oriented quasi-graphical hypersurface, such that $S$ is almost horizontal near the boundary $\partial S$. Then there exists a $C^0$-approximation of the embedding $i_S : S \to \mathbb{R}^{n+1}$ by an almost horizontal wrinkled embedding $f : S \to \mathbb{R}^{n+1}$ of depth 1, such that $f$ coincides with $i_S$ near $\partial S$.

Proof. Cutting from $S$ a small neighborhood of the critical set of the function $h = x_{n+1} : S \to \mathbb{R}$, where $S$ is already almost horizontal, we may consider this new $S$, and thus assume that the function $h$ has no critical points on $S$. Let $\tilde{S} \subset \text{Int}S$ be a slightly smaller compact hypersurface such that $S \setminus \tilde{S}$ is still almost horizontal. On a neighborhood $O\partial S$ consider a non-vanishing vector field $v$ which is transversal to $S$, defines its given co-orientation, and horizontal, i.e. tangent to the level sets of the function $h$. Let $v^t$ denotes the flow of $v$. We may assume that for $z \in S$ the flow-line $v^t(z)$ is defined for all $|t| \leq 1$. 

14
Consider two foliations on $S$: the 1-dimensional foliation $G$, formed by the gradient trajectories of the function $h$, and the $(n-1)$-dimensional foliation $L$, formed by level surfaces of the function $h$.

A subset $C \subset S$ is called cylindrical, if $C$ equipped with $L$ and $G$ is diffeomorphic to $D^{n-1} \times D^1$ equipped with the standard horizontal and vertical foliations. We call $C$ special if, in addition, $C$ is almost horizontal near its top and bottom.

Fix a covering of $\tilde{S}$ by special cylindrical sets $C_\alpha \subset S$, $\alpha = 1, \ldots, K$, and fix the respective parameterizations $\varphi_\alpha : D^{n-1} \times D^1 \rightarrow C_\alpha$, which send the positive direction on $D^1$ to the directions on $G$ defined by the gradient vector field $\nabla h$. Given a function $\tau : S \rightarrow \mathbb{R}$, $|\tau| \leq 1$, we will denote by $I_\tau^S$ a perturbation of the inclusion $i_S$ defined by the formula

$$I_\tau^S(z) = v^{\tau(z)}(z), \quad z \in S.$$  

To clarify the geometric meaning of the construction of the necessary approximation we will give first a slightly imprecise description without technical details. Let $x_i \in D^1$, $i = 1, \ldots, N$, be a finite set of points such that $x_1 = -1+d/2$, $x_N = 1-d/2$ and $x_{i+1}-x_i = d$. Set $w(x) = \sum_i \delta(x-x_i) - 1/d$, where $\delta(x)$ is the $\delta$-function, and define $W(x) := \int_0^x w(x)dx$ (see Fig.6). Take

![Figure 6: Functions w and W](image_url)

a cut-off function $\lambda : D^{n-1} \times D^1 \rightarrow \mathbb{R}_+$, equal to 1 on a slightly lesser than $D^{n-1} \times D^1$ subset $A \subset D^{n-1} \times D^1$ and equal to 0 on $\partial(D^{n-1} \times D^1)$, and define a function $\tau : D^{n-1} \times D^1 \rightarrow \mathbb{R}$ by the formula $\tau(r,x) = \lambda(r,x)W(x)$ and a function $\tau_\alpha : C_\alpha \rightarrow \mathbb{R}^1$ by the formula $\tau_\alpha(c) = \gamma(\tau \circ \varphi_\alpha^{-1}(c))$, $\gamma \in \mathbb{R}_+$. For sufficiently small $d$ and $\gamma$ the map $I_\tau^S$ (see Fig.[7]) is a good candidate for the wrinkled embedding. Take this candidate for a moment; then assuming that all the levels $h = \varphi_\alpha(x_i)$ for all $C_\alpha$ are distinct from each other, we can apply the described local goffering simultaneously for all $C_\alpha$ and thus
get the desired global approximation of $S$. Unfortunately, our map $I_{S}^{\tau_{n}}$ does not match the canonical form (1) and requires an additional perturbation near the singularity. Rather than doing this, we describe below the whole construction again in a slightly modified form with all the details.

First, choose a family of curves $A_{t} \subset \mathbb{R}^{2}$, $t \in \mathbb{R}$, which is given by parametric equations

$$x(u, t) = \frac{15}{8} \int_{0}^{u} (u^{2} - t)^{2} du, \quad y(u, t) = -\frac{1}{2}(u^{3} - 3tu).$$

The curve $A_{t}$ is a graph of a continuous function $a_{t} : \mathbb{R} \to \mathbb{R}$ which is smooth for $t < 0$ and smooth on $\mathbb{R} \setminus \{\sqrt{t}, -\sqrt{t}\}$ for $t \geq 0$, see Fig. 8,9. Note that

(a) $a_{1}(\pm 1) = \pm 1$;

(b) the composition $a_{t}(x(u, t))$ is a smooth function.

Choose $\sigma \in (0, \frac{1}{8})$ and consider a family of odd 1-periodic functions $\mathcal{W}_{\sigma, t} : \mathbb{R} \to \mathbb{R}$ with the following properties:

$$\mathcal{W}_{\sigma, t}(x) = \begin{cases} a_{t}(\frac{x}{\sigma}) & \text{for } x \in \mathcal{O} \mathbb{P} [-\sigma, \sigma] \\ 0 & \text{for } x = \frac{1}{2} \end{cases}$$

$$\frac{d\mathcal{W}_{\sigma, t}}{dx}(x) \begin{cases} \geq 3 & \text{for } x \in (-\sigma, \sigma) \\ \leq -2 & \text{for } x \in [\sigma, 2\sigma] \end{cases}$$

$\sigma, 2\sigma$.
Note that the composition $W_{\sigma,t}(a_t(x(u,t)))$ is a smooth function. Pick an $\varepsilon > 0$ and integer $N > 0$ such that the cylinders

$$\tilde{C}_\alpha = \varphi_\alpha(D_{n-1}^{n-1} \times D_1^{1-1/N})$$

are still special and cover the surface $\tilde{S}$. We will keep $\varepsilon$ fixed while $N$ will need to be increased to achieve the required approximation. Choose a non-strictly decreasing smooth function $\beta : [0, 1] \to \mathbb{R}$ such that

$$\beta(r) = \begin{cases} 1 & r \in [0, 1 - 3\varepsilon]; \\ 1 - r - \varepsilon & r \in [1 - 2\varepsilon, 1]. \end{cases}$$

Given an integer $N > 0$, pick also a cut-off function $\lambda : [0, 1] \to [0, 1]$ which is equal to 1 on $[1 - \rho]$ and to 0 near 1, where $\rho = \min(\frac{1}{2N}, \frac{\varepsilon}{2})$, and define a function

$$\tau = \tau_{N,\sigma} : D^{n-1} \times D^1 \to \mathbb{R}$$

by the formula

$$\tau(r, x) = \gamma \lambda(|x|)\lambda(|r|)W_{\sigma,\beta(|r|)}\left(\frac{2N + 1}{2} x\right), \ (r, x) \in D^{n-1} \times D^1, \ \gamma \in \mathbb{R}_+. $$

Let us push-forward the function $\tau$ to $S$ by $\varphi_\alpha$,

$$\tau_\alpha := \tau \circ \varphi_\alpha^{-1}.$$

Figure 8: Function $a_t, t = 1$
2.7.2. (Local wrinkling) For sufficiently big $N$ and sufficiently small $\gamma$ the map $I_S^o \circ g : S \to \mathbb{R}^{n+1}$ is a wrinkled embedding. Here $g : S \to S$ is an appropriate smooth homeomorphism, which makes the composition smooth.

Remark. Of course, $I_S^o(S) = (I_S^o \circ g)(S)$. However, the map $I_S^o$ itself is not smooth. Using the property (b) of the function $a_t$ one can write down the reparameterization $g$ explicitly.

Let $D^1_Q$ be the set of rational points in $D^1$. We can choose parameterizations $\varphi_\alpha$ is such a way that the images $\varphi_\alpha(D^{n-1} \times D^1_Q) \subset S$ are pairwise disjoint. Then for a sufficiently large $N$ the images

$$\varphi_\alpha(\bigcup_{-N}^{N}[x_k - \bar{\sigma}, x_k + \bar{\sigma}]) \subset S,$$

where $x_k = \frac{2k}{2N+1}$, $\bar{\sigma} = \frac{4\sigma}{2N+1}$, are pairwise disjoint as well.

Choose a partition of unity $\sum_{1}^{K} \eta_\alpha = 1$ on $\tilde{S}$ realized by functions $\eta_\alpha$ supported in Int $C_\alpha$ and such that $\eta_\alpha|_{\tilde{C}_\alpha} > 0$. Finally, for a sufficiently large $N$ and sufficiently small $\gamma > 0$, the map $I_S^\Psi \circ g$, where

$$\Psi := \sum_{\alpha=1}^{K} \eta_\alpha \tau_\alpha,$$
Figure 10: Function \( W_{\sigma,1}, \ x \in [-\frac{1}{2}, \frac{1}{2}] \), \( \sigma \simeq 0.02 \)

is the required approximation of \( i_S \) by an almost horizontal wrinkled embedding. Here, like in \ref{2.7.2}, \( g : S \to S \) is an appropriate reparameterization. \( \square \)

We will need the following \textit{parametric} version of Lemma \ref{2.7.1}:

\textbf{2.7.3. (Approximation of a family of embedded hypersurfaces by a family of almost horizontal wrinkled embeddings)} \( \) Let \( S_t \subset \mathbb{R}^{n+1}, \ t \in I, \) be a family of oriented quasi-graphical hypersurfaces, such that \( S_t \) is almost horizontal for \( t = 0 \) and \( S_t \) is almost horizontal near the boundary \( \partial S_t \) for all \( t \in I. \) Then there exists a \( C^0 \)-approximation of the family of embeddings \( i_{S_t} : S_t \hookrightarrow \mathbb{R}^{n+1} \) by a family of almost horizontal wrinkled embeddings \( f_t : S_t \to \mathbb{R}^{n+1} \) of depth \( \leq 1, \) such that \( f_t \) coincide with \( i_{S_t} \) for \( t = 0 \) and \( f_t \) coincide with \( i_{S_t} \) near \( \partial S \) for all \( t \in I. \)

In order to prove this version, we need to do the following modification in the previous proof. The family \( S_t, \ t \in I, \) can be considered as a fibered over \( I \) quasi-graphical hypersurface \( S \) in \( I \times \mathbb{R}^{n+1}. \) Then, all above constructions should be done in the fibered category. In particular, instead of the cylinder \( D^n \times D^1 \) we need to use the fibered cylinder \( I \times D^{n-1} \times D^1 \) for the parameterization of the fibered cylindrical sets.

The hypersurface \( S \) is almost horizontal everywhere near the boundary \( \partial S \) except the right side, \( S_1 \subset \partial S. \) Hence, in order to use the above scheme,
we will work with the doubled, fibered over \([0, 2]\) family \(S_t, t \in [0, 2]\), where \(S_t = S_{2-t}\) for \(t \in [1, 2]\), while making all the above constructions equivariant with respect to the involution \(t \mapsto 2 - t, t \in [0, 2]\).

### 2.8 Proof of Theorem \[2.5.2\]

In order to get rid of non-essential details, we will consider only the case \((W, g_W) = (\mathbb{R}^q, dx_1^2 + \ldots + dx_m^2)\). The proof can be easily adjusted to the general case.

Working with a triangulation \(\Delta\) of the manifold \(V\), we always assume that \(S' \subset S = \Sigma(f_0)\) is contained in the \((n-2)\)-skeleton of \(\Delta\) and \(S\) is contained in its \((n-1)\)-skeleton. Given a triangulation \(\Delta\) of \(V\) and a map \(G: V \to Gr_n \mathbb{R}^m\), we will denote by \(G^\Delta\) the piecewise constant map \(G^\Delta: V \to Gr_n \mathbb{R}^m\), defined on each \(n\)-simplex \(\Delta^m\) by the condition that \(G^\Delta(v)\) is parallel to \(G(v_i)\), where \(v_i\) is the barycenter of the simplex \(\Delta^m\). The map \(G^\Delta\) is multivalued over the \((n-1)\)-skeleton of \(\Delta\).

**The case** \(m = n+1\). We can choose a triangulation \(\Delta\) of \(V\) with sufficiently small simplices (fine triangulation), such that for any \(i\) the image \(f_0(\Delta_i^m)\) is arbitrarily \(C^1\)-close to \(G_0(v_i)\) and for any \(t\) the map \(G^\Delta_t\) is arbitrarily \(C^0\)-close to \(G_t\). In what follows we will approximate the homotopy \(G^\Delta_t\) instead of \(G_t\), and therefore the approximation \(G_t \approx G^\Delta_t\) must have at least the same order as the desired approximation \(G_t \approx Gdf_t\).

First, we apply Theorems \[2.2.1\] and \[2.6.1\] to the \((n-1)\)-skeleton \(\Delta^{n-1}\) and construct a graphical isotopy \(\tilde{f}_t: V \to \mathbb{R}^{n+1}\), \(\tilde{f}_0 = f_0\), such that \(Gd\tilde{f}_t|_{\partial \Delta^{n-1}}\) is arbitrarily \(C^0\)-close to \(G_t|_{\partial \Delta^{n-1}}\). For every \(i\) and \(t\) the hypersurface \(\tilde{f}_t(\Delta_i^m)\), i.e. the image of \(f_0(\Delta_i^m)\) under the graphical isotopy, is almost graphical with respect to \(G_0(v_i)\), and hence it is quasi-graphical with respect to the hyperplane \(G_t(v_i)\) because the angle between \(G_0(v_i)\) and \(G_t(v_i)\) is less then \(\pi/4\).

Then, \(\tilde{f}_t(\Delta_i^m)\) is almost horizontal with respect to \(G_t(v_i)\) near the boundary \(\partial \tilde{f}_t(\Delta_i^m)\). Therefore, over each \(n\)-simplex \(\Delta_i^m\) we can apply Lemma \[2.7.3\] with \(S_t = \tilde{f}_t(\Delta_i^m)\) and \(G_t(v_i)\) as the horizontal hyperplane, i.e. for every \(t\) the hyperplane \(G_t(v_i)\) plays the role of the horizontal hyperplane \(\mathbb{R}^n \subset \mathbb{R}^{n+1}\).

Thus we can deform the isotopy of embeddings \(\tilde{f}_t\) to the desired homotopy \(f_t\) of the wrinkled embeddings.

**The case** \(m > n+1\). Let \(\Delta\) be a fine triangulation of \(V\). For every \(i\) the image \(f_0(\Delta_i^m)\) is arbitrarily \(C^1\)-close to \(G_0(v_i)\) and, therefore, we can work...
over each simplex $\Delta^i$ with the projection of the isotopy $\tilde{f}_t$ to $L_{v_i} \simeq \mathbb{R}^{n+1}$ (the $(n+1)$-dimensional subspace where the rotation $G_t$ goes on) exactly as in the previous case, keeping the coordinates in $L_{v_i}^\perp$ unchanged.

2.9 Fibered case

Let us formulate and discuss the parametric version of Theorem 2.5.1. For a fibration $q : W \to B$ denote by $\text{Gr}_n W_B$ the Grassmannian of $n$-planes tangent to the fibers of the fibration $q$.

2.9.1. (Global integrable approximation of fibered tangential rotation) Let $G_t : V \to \text{Gr}_n W$ be a fibered tangential rotation of a fibered (over $B$) embedding $i_V : V \hookrightarrow W$. Then there exists a homotopy of fibered wrinkled embeddings $f_t : V \to W$, $f_0 = i_V$, such that the (fibered) homotopy $Gd_B f_t : V \to \text{Gr}_n W_B$ is arbitrarily $C^0$-close to $G_t$. If the rotation $G_t$ is fixed on a closed subset $C \subset V$, then the homotopy $f_t$ can be chosen also fixed on $C$. In particular, if $G_t$ is fixed over a closed subset $B' \subset B$, then the homotopy $f_t$ can be chosen also fixed over $B'$.

This theorem follows from

2.9.2. (Global integrable approximation of simple fibered tangential rotation) Let $G_t : V \to \text{Gr}_n W$ be a simple fibered tangential rotation of a fibered wrinkled embedding $f_0 : V \to W$. Then there exists a homotopy of fibered wrinkled embeddings $f_t : V \to W$, such that the (fibered) homotopy $Gd_B f_t : V \to \text{Gr}_n W_B$ is arbitrarily $C^0$-close to $G_t$.

Now we need a version of the main lemma 2.7.1 which is parametric with respect to both, time and space:

2.9.3. (Approximation of family of fibered embedded hypersurfaces by family of almost horizontal fibered wrinkled embeddings) Let $S_t \subset B^k \times \mathbb{R}^{n+1}$, $t \in I$, be a family of fibered over $B^k$ oriented quasi-graphical hypersurfaces, such that $S_t$ is almost horizontal for $t = 0$ and $S_t$ is almost horizontal near the boundary $\partial S_t$ and over $\partial B$ for all $t \in I$. Then there exists a $C^0$-approximation of the family of fibered embeddings $i_{S_t} : S_t \hookrightarrow B \times \mathbb{R}^{n+1}$ by a family of almost horizontal fibered wrinkled embeddings $f_t : S_t \to \mathbb{R}^{n+1}$ of depth $\leq 1$, such that $f_t$ coincides with $i_{S_t}$ for $t = 0$ and $f_t$ coincides with $i_{S_t}$ near $\partial S$ and over $\partial B$ for all $t \in I$. 21
The proof of the time-parametric version 2.7.3 of the main lemma can be rewritten for this fibered case. In the proof of Theorem 2.9.2, in order to apply the fibered version of the main lemma, the fine triangulation $\Delta$ ought to be transversal to the fibers on $\text{Int} \Delta_i$ for all its simplices $\Delta_i \subset S \setminus \mathcal{O} \partial S$. Let us point that the existence of such a triangulation is non-obvious and was proven in a more general case (for foliation) by W. Thurston in [Th74].

2.10 Double folds

A (smooth) map $f : V^n \to W^m$, $n < m$, is called folded embedding, if

- $f$ is a topological embedding;
- the singularity $\Sigma = \Sigma_f$ of the map $f$ is an $(n-1)$-dimensional submanifold in $V$;
- near each connected components $S$ of $\Sigma$ the map $f$ is equivalent to the map
  \[ \mathcal{O} p_{S \times \mathbb{R}^1} S \times 0 \to S \times \mathbb{R}^{m-n+1} \]
  given by the formula $(y, z) \mapsto (y, z^2, z^3, 0, ..., 0)$, where $y \in S$.

Thus, folded embeddings have only two-fold corners, i.e. on Fig. The submanifolds $S$ will be called the folds of the folded embedding $f$.

We say that a folded embedding $f$ has spherical double-folds if all its folds diffeomorphic to the the $(n-1)$-sphere, and organized in pairs $(S_0, S_1)$ which bounds annuli diffeomorphic to $S^{n-1} \times I$. These annuli are allowed to be nested, i.e. the annulus bounded by one double fold may contain the annulus associated with another double fold.

Considering families of folded embeddings with spherical double folds we will also allow, similar to the case of wrinkled embeddings, embryo double folds, so that a double fold could die or be born during the deformation. The local model for a map $\mathcal{O} p_{S \times \mathbb{R}^1} S \times 0 \to S \times \mathbb{R}^{m-n+1}$ near an embryo double fold is given by the formula

\[ (y, z) \mapsto (y, z^3, z^5, 0, ..., 0). \]

Theorem 2.5.1 has an equivalent reformulation for folded embeddings with spherical double folds.
2.10.1. (Global integrable approximation of tangential rotation by folded embeddings with spherical double folds) Let \( G_t : V \to \text{Gr}_n W \) be a tangential rotation of an embedding \( i_V : V \hookrightarrow W \). Then there exists a homotopy of folded embeddings with spherical double folds \( f_t : V \to W \), \( f_0 = i_V \), such that the homotopy \( G df_t : V \to \text{Gr}_n W \) is arbitrarily \( C^0 \)-close to \( G_t \). If the rotation \( G_t \) is fixed on a closed subset \( C \subset V \), then the homotopy \( f_t \) can be chosen also fixed on \( C \).

A fibered version of this theorem also holds.

The following surgery construction allows one to deduce 2.10.1 from 2.5.1. Let \( f : V^n \to W^m \) be a wrinkled embedding, \( n > 1 \). One can modify each wrinkle of \( f \) by a connected sum construction for cusps (Whitney surgery) such that the resulting map \( \tilde{f} : V^n \to W^m \) will be a folded embedding with spherical double folds \( S^{n-1} \times S^0 \). For maps \( V^2 \to W^m \) the construction (in the pre-image) is shown on Fig.11. Next two propositions contain a formal description of the construction, see [El72] or [EM98] for more details.

2.10.2. (Preparation for the cusp surgery) Let

\[
Z(n,q) : \mathcal{O}_{\mathbb{R}^n} S^{n-1} \to \mathbb{R}^m
\]

be the standard wrinkled embedding with the wrinkle \( S^{n-1} \subset \mathbb{R}^n \). If \( n > 1 \) then there exists an embedding \( h : D^{n-1} \to \mathcal{O}_{\mathbb{R}^n} S^{n-1} \) such that

- \( h(\partial D^{n-1}) = S^{n-2} \), the cusp (three-fold points) of the wrinkle;
- \( h(D^{n-1} \setminus \text{Int } D^{n-1}) = (D^{n-1} \setminus \text{Int } D^{n-1}) \times 0 \subset \mathbb{R}^{n-1} \times \mathbb{R}^1 \);
- \( h(\text{Int } D^{n-1}) \) does not intersect the wrinkle \( S^{n-1} \).

Figure 11: Whitney surgery (in the pre-image); \( n = 2 \)
2.10.3. (Surgery of cusps) Let \( h : D^{n-1} \to \mathcal{O}_p \mathbb{R}^n S^{n-1} \) be an embedding, as in \ref{2.10.2}. There exists a \( C^0 \)-small perturbation of the map \( Z(n, m) \) in an arbitrarily small neighborhood of the embedded disk \( h(D^{n-1}) \) such that the resulting map \( \tilde{Z}(n, m) \) is a folded embedding with two spherical folds.

The construction also implies that each double fold \( S^{n-1} \times S^0 \) of the folded embeddings \( \tilde{f} \) bounds an annulus \( S^{n-1} \times D^1 \) in \( V \).

3 Applications

3.1 Homotopy principle for directed wrinkled embeddings

Using the wrinkled embeddings one can reformulate the \( h \)-principle for \( A \)-directed embeddings of open manifolds (see 4.5.1 in \cite{EM02}) to the case of \( A \)-directed wrinkled embeddings of closed manifolds.

3.1.1. (\( A \)-directed wrinkled embeddings of closed manifolds) If \( A \subset \text{Gr}_n W \) is an open subset and \( f_0 : V \hookrightarrow W \) is an embedding whose tangential lift

\[
G_0 = Gd f_0 : V \to \text{Gr}_n W
\]

is homotopic to a map

\[
G_1 : V \to A \subset \text{Gr}_n W,
\]

then there exists a homotopy of wrinkled embeddings \( f_i : V \to W \) such that \( f_1 : V \to W \) is an \( A \)-directed wrinkled embedding. Such a homotopy can be chosen arbitrarily \( C^0 \)-close to \( f_0 \).

One can formulate 3.1.1 also in the case when \( A \) is an open subset in the Grassmannian of oriented \( n \)-planes in \( W \). In both cases the \( h \)-principle holds also in the relative and parametric versions. Then, one can reformulate this \( h \)-principle for the folded embeddings with spherical double folds.

3.2 Embeddings into foliations

A. Wrinkled mappings and generalized wrinkled mappings. In Section 4 below we provide for a convenience of the reader the basic definitions
and results of the wrinkling theory from [EM97] and [EM98]. However, in this section we talk about generalized wrinkled maps. Let us explain here the difference between the two notions.

A wrinkled mapping $f : V^n \to W^q$, $n \geq q$, by definition, is a map whose singularity set consists of $(n,q)$-wrinkles (= wrinkles), where each wrinkle is a $(q-1)$-dimensional sphere $S_i \subset V$ of fold points divided by an equator of cusp points, see Fig. 12. In addition, each wrinkle $S_i \subset V$ is required to bound an embedded $q$-dimensional disk $D_i \subset V$, such that the restriction of the map $f$ to $D_i \setminus S_i$ is an equidimensional embedding.

![Figure 12: Wrinkle and its image; $n = q = 2$](image)

In the case of a generalized wrinkled mapping each $(n,q)$-wrinkle $S_i$ also bounds a disk $D_i$ in $V$. However, the restriction $f|_{D_i}$ is not required to be a smooth embedding on $D_i \setminus S_i$. Instead, this restriction is allowed itself to be wrinkled, see Fig. 13. See Section 4.6 below for more details.

![Figure 13: Nested wrinkles in the pre-image (i.e. in $V$); $n = 2$](image)

**B. Main theorem (statement).** Let $(W^m, \mathcal{F})$ be a foliated manifold, codim $\mathcal{F} = q \leq n$. We say that the singularities of an embedding $f : V^n \to (W, \mathcal{F})$ with respect to foliation $\mathcal{F}$ are generalized wrinkles, if $f$ is a generalized wrinkled map with respect to $\mathcal{F}$, see Section 4.6.
3.2.1. (Embeddings into foliations) Let $F_p, p \in B$, be a family of foliations on $W^m$, codim $F_p = q \leq n$, and $f_p : V^n \rightarrow (W^m, F_p)$ be a family of embeddings such that $f_p$ is transversal to $F_p$ for $p \in O \cup \partial B$. Suppose, in addition, that there exists a family of tangential rotations $G^t_p : V \rightarrow Gr_n W$, $p \in B, t \in I$, such that

- $C^t_p$ is constant for $p \in O \cup \partial B$;
- $C^1_p$ is transversal to the foliation $F_p$ for all $p$.

Then there exists a family of embeddings $f^t_p, p \in B, t \in I$, such that

- $f^0_p = f_p$ for all $p$ and $f^1_p = f_p$ for $p \in O \cup \partial B$;
- for any $p$ the singularities of the $f^1_p$ with respect to $F_p$ are generalized $(n, q)$-wrinkles and embryos.

◧ Remarks. 1. The family $f^t_p$ is $C^0$-close to the constant family $\bar{f}^t_p = f^0_p$.
2. The theorem is true also relative to a closed subset $A \subset V$, i.e. in the situation when the embeddings $f_p$ are already transversal to $F$ on $O \cup V \cup A$.
3. In a more formal way, one can say that $f^1_p$ is a fibered generalized wrinkled map with respect to $F_p$. ◦

◧ Examples. 1. Let $f_p : S^{n-1} \times I \rightarrow \mathbb{R}^{n+1}, p \in D^k$, be a family of embeddings, such that $f_p$ is the standard inclusion $i_{S^{n-1} \times I} : S^{n-1} \times I \hookrightarrow \mathbb{R}^{n+1}$ for $p \in O \cup \partial D^k$ and $f_p = i_{S^{n-1} \times I}$ near $\partial(S^{n-1} \times I)$ for all $p$. Then there exists an isotopy of the family $f_p$ which is fixed near $\partial S^{n-1} \times I$ and for $p \in \partial D^k$ and such that the projections of the resulting cylinders to the axis $x_{n+1}$ have only Morse and birth-death type singularities.

Indeed, the corresponding homotopical condition for the family of normal vector fields is automatically holds here, as it was observed by A. Douady and F. Laudenbach, see [La76] and [EM00]. Similarly, we have

2. Let $f_p : D^n \rightarrow \mathbb{R}^{n+1}, p \in D^k$ be a family of embeddings, such that $f_p$ is the standard inclusion $i_{D^n} : D^n \hookrightarrow \mathbb{R}^{n+1}$ for $p \in O \cup \partial D^k$, and $f_p = i_{D^n}$ near $\partial D^n$ for all $p$. Then there exists an isotopy of the family $f_p$, which is fixed near $\partial D^n$ and for $p \in \partial D^k$, such that all the singularities of the projections of the resulting disks on $\mathbb{R}^n$ are generalized wrinkles and embryos. ◦

Theorem 3.2.1 is proved below in D.

C. $F$-regularization (lemmas). For the canonical form

$Z(n, n+1) : O \cup \mathbb{R} \rightarrow \mathbb{R}^{n+1}$

26
of the wrinkled embedding (see 2.3 A) the regularizing foliation is, by definition, the one-dimensional affine foliation in \( \mathbb{R}^{n+1} \), parallel to the axis \( x_{n+1} \).

In order to make the situation more transparent and simplify the notation, we formulate our lemmas only in the case \( m = n + 1 \). All statements remain true also for an arbitrary \( m \geq n + 1 \).

**3.2.2.** Let \( f : V^n \to (W^{n+1}, \mathcal{F}) \) be a wrinkled embedding, transversal to the foliation \( \mathcal{F} \), \( \dim \mathcal{F} = 1 \) (\( \text{codim} \mathcal{F} = n \)). Then there exists a regularization \( \tilde{f} \) of the wrinkled embedding \( f \) such that the singularities of the smooth embedding \( \tilde{f} \) with respect to \( \mathcal{F} \) are generalized wrinkles.

**Proof.** We can choose the canonical coordinates near each wrinkle \( f(S_i) \) such that \( \mathcal{F} \) will be the regularizing foliation and then apply the standard regularization (see 2.3 B).

Let \( f : V^n \to (W^{n+1}, \mathcal{F}) \) be a wrinkled embedding. Denote by \( \tilde{V}_i \) an \( n \)-dimensional submanifold in \( W^m \), such that \( f(S_i) \subset \tilde{V}_i \) and \( \tilde{V}_i \) is tangent to \( f(Op S_i) \) along \( f(S_i) \) (see 2.3 D). Notice that if \( f \) is transversal to \( \mathcal{F} \) then \( \tilde{V}_i \) is transversal to \( \mathcal{F} \) near \( f(S_i) \).

**3.2.3.** Let \( f : V^n \to (W^{n+1}, \mathcal{F}) \) be a wrinkled embedding, transversal to the foliation \( \mathcal{F} \), \( \dim \mathcal{F} \geq 2 \) (\( \text{codim} \mathcal{F} \leq n - 1 \)). Suppose, in addition, that the restriction \( f|_{S_i} : S_i \to \tilde{V}_i \) is transversal to the foliation \( \mathcal{F} \cap \tilde{V}_i \) for all \( i \). Then any regularization \( \tilde{f} \) of the wrinkled embedding \( f \) (i.e. for any choice of the canonical coordinates) gives us a smooth embedding transversal to \( \mathcal{F} \).

**3.2.4.** Let \( f : V^n \to (W^{n+1}, \mathcal{F}) \) be a wrinkled embedding, transversal to the foliation \( \mathcal{F} \), \( \dim \mathcal{F} \geq 2 \) (\( \text{codim} \mathcal{F} \leq n - 1 \)). Suppose, in addition, that the singularities of the restriction \( f|_{S_i} : S_i \to \tilde{V}_i \) with respect to \( \mathcal{F} \cap \tilde{V}_i \) are generalized \( (n - 1, q) \)-wrinkles for all \( i \). Then there exists a regularization \( \tilde{f} \) of the map \( f \) such that the singularities of the smooth embedding \( \tilde{f} \) with respect to \( \mathcal{F} \) are generalized \( (n, q) \)-wrinkles.

**Proof.** We can choose the canonical coordinates near each wrinkle \( f(S_i) \) in such a way that the one-dimensional regularizing foliation is inscribed into \( \mathcal{F} \), and then apply the standard regularization. Such a regularization adds \( \pm t^2 \) to the canonical form of any generalized \( (n - 1, q) \)-wrinkle and thus transform it to a \( (n, q) \)-wrinkle. See Fig.11.

**Remark.** Lemmas 3.2.2, 3.2.4 remain true in the parametric form.
D. Main theorem (proof). We consider only a non-parametric situation, i.e. when $B$ is just a point. The proof in the parametric case is similar with a systematic use of the fibered terminology. Thus, we will drop “$p$” from the notation. Moreover, we will consider only the case $m = n + 1$; the proof can be easily rewritten for any $m > n$.

First of all, let us apply Theorem 2.5.1 to the family $G^t : V^n \to \text{Gr}_n W^{n+1}$ and construct a family of wrinkled embeddings $\hat{f}^t : V^n \to W^{n+1}$, $t \in [0, 1]$, transversal to $F$.

a) The base of the induction: $\dim F = 1$ (the equidimensional case). We regularize the family $\hat{f}^1$ in such a way that for $\hat{f}^1$ our regularization is the $F$-regularization as in Lemma 3.2.2 and get the required family $f^t$ of embeddings where $f^1$ is a generalized wrinkled map with respect to $F$.

b) Wrinkling of wrinkles: $\dim F > 1$. Let $\hat{V}_i \subset W$ be an $n$-dimensional submanifold, such that $\Sigma_i = \hat{f}^1(S_i) \subset \hat{V}_i$ and $\hat{V}_i$ is tangent to $\hat{f}^1(V)$ along $\Sigma_i$. Here $S_i$ is a wrinkle of the wrinkled embedding $\hat{f}^1$. The foliation $F$ is transversal to $\hat{V}_i$ near $\Sigma_i$. Each wrinkle originates from embryo (i.e. from a point) and hence one can rotate the tangent $(n - 1)$-planes to $\Sigma_i \subset \hat{V}_i$ in $\hat{V}_i$ to a position transversal to the foliation $F \cap \hat{V}_i$. Such a rotation can be approximated near $\Sigma'_i = \hat{f}^1(S'_i)$ by an isotopy of $\Sigma_i$ in $\hat{V}_i$. Then, by our inductive hypothesis there exists an isotopy of $\Sigma_i$ in $\hat{V}_i$, fixed near $\Sigma'_i$ and such that the singularities of the final embedding with respect to $F \cap \hat{V}_i$ are generalized $(n - 1, q)$-wrinkles. The resulting isotopies $g^i_t$, $t \in [1, 2]$ (in $\hat{V}_i$, for all $i$) can be extended to an isotopy $\hat{f}^2$, $t \in [1, 2]$, of the wrinkled embedding $\hat{f}^1$. Finally, we regularize the family $\hat{f}^2$, $t \in [0, 2]$, in such a way that for $\hat{f}^2$ our regularization is the $F$-regularization as in Lemma 3.2.4, and get the required family $f^\tau$, $\tau \in [0, 1], \tau = 2t$, of embeddings such that $f^1$ is a
generalized wrinkled map with respect to $F$. □

◆ Remark. Using in the above proof Theorem 2.10.1 instead of 2.5.1 we get a version of Theorem 3.2.1 about embeddings with the double fold type tangency singularities to a foliation.◆

3.3 Embeddings into distributions

In this section we sketch a generalization of Theorem 3.2.1 to the case of distributions.

Let $\xi$ be a distribution on a manifold $W^m$, $\text{codim} \xi = q$. An embedding $f : V^n \to (W^m, \xi)$, $n \geq \text{codim} \xi$, is called transversal to $\xi$, if the reduced differential

$$d\xi f : TV \xrightarrow{df} TW \xrightarrow{\pi_\xi} TW/\xi$$

is surjective. The non-transversality defines a variety $\Sigma_\xi$ of the 1-jet space $J^1(V, W)$. For a general non-integrable $\xi$ one cannot define fold and cusp type tangency through normal forms. However, the original Whitney-Thom definition is applicable in this situation as well.

We say that $f$ has at a point $p \in V$ a tangency to $\xi$ of fold type if

- Corank $d_p^\xi = 1$.
- $J^1(f) : V \to J^1(V, W)$ is transverse to $\Sigma$;
- $d_p^\xi f|_{T_p\Sigma(f)} : T_p\Sigma(f) \to TW_{f(p)}/\xi$ is injective.

For a codimension 1 distribution $\xi$ defined by a Pfaffian equation $\alpha = 0$, the fold tangency point of an embedding $f : V \to W$ to $\xi$ are isolated non-degenerate zeroes of the induced 1-form $f^*\alpha$ on $V$.

Similarly, we define a tangency of cusp type by the same first two conditions and, in addition by requiring that $d_p^\xi f|_{T_p\Sigma(f)} : T_p\Sigma(f) \to TW_{f(p)}/\xi$ is not injective and the 2-jet section $J^2(f) : V \to J^2(V, W)$ to be transversal to the singularity in $J^2(V, W)$ defined by the non-injectivity condition.

Combining the following observations, one can associate with a co-oriented fold an index, as in the case of a foliation.

3.3.1. (Index for fold-type tangency to distribution)
A. Let $J^2(\xi) \to W^m$ be the bundle of 2-jets of codimension $q$ submanifolds of $W$ tangent to $\xi$. There exists a section $W \to J^2(\xi)$ which can be uniquely characterized by the following property: for every point $p \in W$ there exists an embedding $\varphi_p : \mathbb{R}^{m-q} \to W$ such that $d_p\varphi_p(\mathbb{R}^{m-q}) = \xi_p$, and for all $t \in \mathbb{R}$, $x \in \mathbb{R}^{m-q}$ we have $\frac{d\varphi_p(tx)}{dt} \in \xi$.

B. Let $(W, \xi)$ be as above, and $\Sigma$ a submanifold of dimension $< \text{codim} \xi$ such that $\pi^\xi : T\Sigma \to TW/\xi$ is injective. Then, given any section $\sigma : \Sigma \to J^2(\xi)$ there exists codimension $q$ foliation $F$ on $O_p \Sigma$ whose 2-jet along $\Sigma$ is equal to $\sigma$.

C. Suppose that an embedding $f : V \to W$ has a fold type tangency of index $k$ to a foliation $F$ along a submanifold $\Sigma \subset W$. Let $\tilde{F}$ be another foliation of the same codimension and which have the same 2-jet as $F$ along $\Sigma$. Then $f : V \to W$ has at $\Sigma$ a fold type tangency to a foliation $\tilde{F}$ of the same index $k$.

This allows us define generalized $(n, q)$-wrinkles and embryo type singularities of tangency of an embedding $f : V^n \to (W, \xi)$ to a distribution $\xi$.

With these definitions Theorem 3.2.1 can be generalized without any changes to the case of an arbitrary distribution $\xi$ instead of a foliation. The inductive proof for the foliation case presented in Section 3.2 works in this more general with a few additional remarks. Namely, one note that the base of the induction, i.e. the case when $\dim \xi = 1$ is the same in this case, as 1-dimensional distributions are integrable. When applying in the inductive step the standard regularization from 2.3B, one need to choose the regularizing function $\tilde{u}(y, z)$ sufficiently $C^1$-close to $u(y, z)$. With this modification the proof goes through as is.

4 Appendix: Wrinkling

We recall here, for a convenience of the reader, some definitions from [EM97] and [EM98] and introduce the notions of generalized wrinkles and generalized wrinkled maps. We also formulate here some results from [EM97] and [EM98] though we do not use these theorems in the paper.
4.1 Folds and cusps

Let $V$ and $W$ be smooth manifolds of dimensions $n$ and $q$, respectively, and $n \geq q$. For a smooth map $f : V \to W$ we will denote by $\Sigma(f)$ the set of its singular points, i.e.

$$\Sigma(f) = \{ p \in V, \ \text{rank} \ d_pf < q \} .$$

A point $p \in \Sigma(f)$ is called a fold type singularity or a fold of index $s$ if near the point $p$ the map $f$ is equivalent to the map

$$\mathbb{R}^{q-1} \times \mathbb{R}^{n-q+1} \to \mathbb{R}^{q-1} \times \mathbb{R}^1$$
given by the formula

$$(y, x) \mapsto \left( y, -\sum_{i=1}^{s} x_i^2 + \sum_{s+1}^{n-q+1} x_j^2 \right)$$

where $x = (x_1, \ldots, x_{n-q+1}) \in \mathbb{R}^{n-q+1}$ and $y = (y_1, \ldots, y_{q-1}) \in \mathbb{R}^{q-1}$. For $W = \mathbb{R}^1$ this is just a nondegenerate index $s$ critical point of the function $f : V \to \mathbb{R}^1$.

Let $q > 1$. A point $p \in \Sigma(f)$ is called a cusp type singularity or a cusp of index $s + \frac{1}{2}$ if near the point $p$ the map $f$ is equivalent to the map

$$\mathbb{R}^{q-1} \times \mathbb{R}^1 \times \mathbb{R}^{n-q} \to \mathbb{R}^{q-1} \times \mathbb{R}^1$$
given by the formula

$$(y, z, x) \mapsto \left( y, z^3 + 3y_1z - \sum_{i=1}^{s} x_i^2 + \sum_{s+1}^{n-q} x_j^2 \right)$$

where $x = (x_1, \ldots, x_{n-q}) \in \mathbb{R}^{n-q}$, $z \in \mathbb{R}^1$, $y = (y_1, \ldots, y_{q-1}) \in \mathbb{R}^{q-1}$.

For $q \geq 1$ a point $p \in \Sigma(f)$ is called an embryo type singularity or an embryo of index $s + \frac{1}{2}$ if $f$ is equivalent near $p$ to the map

$$\mathbb{R}^{q-1} \times \mathbb{R}^1 \times \mathbb{R}^{n-q} \to \mathbb{R}^{q-1} \times \mathbb{R}^1$$
given by the formula

$$(y, z, x) \mapsto \left( y, z^3 + 3|y|^2z - \sum_{i=1}^{s} x_i^2 + \sum_{s+1}^{n-q} x_j^2 \right)$$

(7)
where \( x \in \mathbb{R}^{n-q}, \ y \in \mathbb{R}^{q-1}, \ z \in \mathbb{R}^1, \ |y|^2 = \sum_{i=1}^{q-1} y_i^2. \)

Notice that folds and cusps are stable singularities for individual maps, while embryos are stable singularities only for 1-parametric families of mappings. For a generic perturbation of an individual map embryos either disappear or give birth to wrinkles which we consider in the next section.

### 4.2 Wrinkles and wrinkled mappings

Consider the map

\[
 w(n, q, s) : \mathbb{R}^{q-1} \times \mathbb{R}^1 \times \mathbb{R}^{n-q} \rightarrow \mathbb{R}^{q-1} \times \mathbb{R}^1
\]

given by the formula

\[
 (y, z, x) \mapsto \left( y, z^3 + 3(|y|^2 - 1)z - \sum_{i=1}^{s} x_i^2 + \sum_{s+1}^{n-q} x_j^2 \right), \tag{8}
\]

where \( y \in \mathbb{R}^{q-1}, \ z \in \mathbb{R}^1, \ x \in \mathbb{R}^{n-q} \) and \( |y|^2 = \sum_{i=1}^{q-1} y_i^2. \)

Notice that the singularity \( \Sigma(w(n, q, s)) \) is the \((q - 1)\)-dimensional sphere

\[
 S^{q-1} = S^{q-1} \times 0 \subset \mathbb{R}^q \times \mathbb{R}^{n-q}.
\]

Its equator \( \{|y| = 1, z = 0, x = 0\} \subset \Sigma(w(n, q, s)) \) consists of cusp points of index \( s + \frac{1}{2} \). The upper hemisphere \( \Sigma(w) \cap \{z > 0\} \) consists of folds of index \( s \) and the lower one \( \Sigma(w) \cap \{z < 0\} \) consists of folds of index \( s + 1 \). Also it is useful to notice that the restrictions of the map \( w(n, q, s) \) to subspaces \( y_1 = t \), viewed as maps \( \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{q-1} \), are non-singular maps for \(|t| > 1\), equivalent to \( w(n - 1, q - 1, s) \) for \(|t| < 1\) and to embryos for \( t = \pm 1 \).

Although the differential \( dw(n, q, s) : T(\mathbb{R}^n) \rightarrow T(\mathbb{R}^q) \) degenerates at points of \( \Sigma(w) \), it can be canonically regularized over \( \mathcal{O}_p \mathbb{R}^q \mathcal{D}^q \), an open neighborhood of the disk \( D^q = D^q \times 0 \subset \mathbb{R}^q \times \mathbb{R}^{n-q} \). Namely, we can change the element \( 3(z^2 + |y|^2 - 1) \) in the Jacobi matrix of \( w(n, q, s) \) by a function \( \gamma \) which coincides with \( 3(z^2 + |y|^2 - 1) \) on \( \mathbb{R}^n \setminus \mathcal{O}_p \mathbb{R}^q \mathcal{D}^q \) and does not vanish along the \( q \)-dimensional subspace \( \{x = 0\} = \mathbb{R}^q \times 0 \subset \mathbb{R}^n \). The new bundle map \( \mathcal{R}(dw) : T(\mathbb{R}^n) \rightarrow T(\mathbb{R}^q) \) provides a homotopically canonical extension of the
map $dw : T(R^n \setminus \mathcal{O}_{p \in R^q} D^q) \to T(R^q)$ to an epimorphism (fiberwise surjective bundle map) $T(R^n) \to T(R^q)$. We call $\mathcal{R}(dw)$ the regularized differential of the map $w(n,q,s)$.

A smooth map $f : V^n \to W^q$, $n \geq q$, is called wrinkled, if any connected component $S_i$ of the singularity $\Sigma(f)$ is diffeomorphic to the standard $(q-1)$-dimensional sphere $S^{q-1}$ and bounds in $V$ a $q$-dimensional disk $D_i$, such that the map $f|_{\mathcal{O}_p \cdot D_i}$ is equivalent to the map $w(n,q,s)|_{\mathcal{O}_p \cdot R^n \cdot D^q}$. The spheres $S_i$ and its images $f(S_i)$ are called wrinkles of the wrinkled mapping $f$. The differential $df : T(V) \to T(W)$ of the wrinkled map $f$ can be regularized (near each wrinkle and hence globally) to obtain an epimorphism $\mathcal{R}(df) : T(V) \to T(W)$.

### 4.3 Fibered wrinkles and fibered wrinkled mappings

For any integer $k > 0$ the map $w(k+n,q,s)$ can be considered as a fibered map over $R^k \times 0 \subset R^{k+n}$. We shall refer to this fibered map as $w_k(k+n,q,s)$. The regularized differential $\mathcal{R}(dw_k(k+n,q,s))$ is a fibered (over $R^k$) epimorphism

$$R^k \times T(R^{q-1} \times R^1 \times R^{n-q}) \xrightarrow{\mathcal{R}(dw_{k+n,q,s})} R^k \times T(R^{q-1} \times R^1)$$

A fibered (over $B$) map $f : V^{k+n} \to W^{k+q}$, $n \geq q$, is called fibered wrinkled, if any connected component $S_i$ of the singularity $\Sigma(f)$ is diffeomorphic to the standard $(k + q - 1)$-dimensional sphere $S^{k+q-1}$ and bounds in $V$ a $(k + q)$-dimensional disk $D_i$, such that the fibered map $f|_{\mathcal{O}_p \cdot D_i}$ is equivalent to the fibered map $w_k(k+n,q,s)|_{\mathcal{O}_p \cdot R^{k+n} \cdot D^q}$. The spheres $S_i$ and its images $f(S_i)$ are called fibered wrinkles of the fibered wrinkled mapping $f$. The restrictions of a fibered wrinkled map to a fiber may have, in addition to wrinkles, endrophy singularities. For a fibered wrinkled map $f : V \to W$ one can define its regularized differential which is a fibered (over $B$) epimorphism $\mathcal{R}(d_B f) : T_B V \to T_B W$.

### 4.4 Main theorems about wrinkled mappings

The following Theorem 4.4.1 and its parametric version 4.4.2 are the main results of our paper [EM97]:
4.4.1 (Wrinkled mappings). Let \( F : T(V) \to T(W) \) be an epimorphism which covers a map \( f : V \to W \). Suppose that \( f \) is a submersion on a neighborhood of a closed subset \( K \subset V \), and \( F \) coincides with \( df \) over that neighborhood. Then there exists a wrinkled map \( g : V \to W \) which coincides with \( f \) near \( K \) and such that \( R(dg) \) and \( F \) are homotopic rel. \( T(V)|_K \). Moreover, the map \( g \) can be chosen arbitrarily \( C^0 \)-close to \( f \), and his wrinkles can be made arbitrarily small.

4.4.2 (Fibered wrinkled mappings). Let \( f : V \to W \) be a fibered over \( B \) map covered by a fibered epimorphism \( F : T_B(V) \to T_B(W) \). Suppose that \( F \) coincides with \( df \) near a closed subset \( K \subset V \) (in particular, \( f \) is a fibered submersion near \( K \)), then there exists a fibered wrinkled map \( g : V \to W \) which extends \( f \) from a neighborhood of \( K \), and such that the fibered epimorphisms \( R(dg) \) and \( F \) are homotopic rel. \( T_B(M)|_K \). Moreover, the map \( g \) can be chosen arbitrarily \( C^0 \)-close to \( f \), and his wrinkles can be made arbitrarily small.

4.5 Wrinkled mappings into foliations

Here we formulate a slightly strengthened version of Theorems 4.4.1 and 4.4.2. Let us start with some definitions.

Let \( F \) be a foliation on a manifold \( W \), \( \text{codim} \ F = q \). A map \( f : V \to W \) is called \emph{transversal to} \( F \), if the reduced differential

\[
TV \xrightarrow{df} TW \xrightarrow{\pi_F} \nu(F)
\]

is an epimorphism. Here \( \nu(F) = TW/\tau(F) \) is the normal bundle of the foliation \( F \).

An open subset \( U \subset W \) is called \emph{elementary} (with respect to \( F \)), if \( F|_U \) is generated by a submersion \( p_U : U \to \mathbb{R}^q \). An open subset \( U \subset V \) is called \emph{small} (with respect to \( f \) and \( F \)), if \( f(U) \) is contained in an elementary subset \( U' \) of \( W \). A map \( f : V \to W \) is called \emph{\( F \)-wrinkled}, or \emph{wrinkled with respect to} \( F \), if there exist disjoint small subsets \( U_1, ...U_l \subset V \) such that \( f|_{V\setminus(U_1\cup...\cup U_l)} \) is transversal to \( F \) and for each \( i = 1, ..., l \) the composition

\[
U_i \xrightarrow{f|_{U_i}} U_i' \xrightarrow{p_{U_i}} \mathbb{R}^q
\]
(where $U'_i \supset f(U_i)$ is an elementary subset of $W$), is a wrinkled map. In order to get the regularized reduced differential

$$\mathcal{R}(\pi_F \circ df) : TM \to \nu(F)$$

of the $F^\perp$-wrinkled map $f$, we regularize the differential of each wrinkled map $w_i = p_{U'_i} \circ f|_{U_i}$ as in Sections 4.2 and then set

$$\mathcal{R}(\pi_F \circ df|_{U_i}) = [dp_{U'_i}|_{\nu(F)}]^{-1} \circ \mathcal{R}(dw_i).$$

Similarly to Section 4.3 we can define a fibered $F^\perp$-wrinkled map $f : V \to Q$, where the foliation $F$ on $W$ is fibered over the same base $B$. Finally we define, in a usual way, the regularization

$$\mathcal{R}(\pi_F \circ d_B f) : T_B V \to \nu_B(F)$$

of the fibered reduced differential

$$T_B V \xrightarrow{d_B f} TW \xrightarrow{\pi_F} \nu_B(F).$$

4.5.1. (Wrinkled mappings of manifolds into foliations) Let $F$ be a foliation on a manifold $W$ and let $F : TV \to \nu(F)$ be an epimorphism which covers a map $f : V \to W$. Suppose that $f$ is transversal to $F$ in a neighborhood of closed subset $K$ of $V$, and $F$ coincides with the reduced differential $\pi_F \circ df$ over that neighborhood. Then there exists a $F^\perp$-wrinkled map $g : V \to W$ such that $g$ coincides with $f$ near $K$, and $\mathcal{R}(\pi_F \circ dg)$ is homotopic to $F$ relative to $TV|_K$.

**Proof.** Take a triangulation of the manifold $V$ by small simplices. First we use Gromov-Phillips' theorem (see [Ph67], [Gr86] or [EM02]) to approximate $f$ near the $(n-1)$-skeleton of the triangulation by a map transversal to $F$. Then, using Theorem 4.4.1, for a neighborhood $U_i$ of every $n$-simplex $\sigma_i$ and an elementary set $U'_i \supset f(U_i)$ we can approximate the map $p_{U'_i} \circ f|_{U_i}$ by a wrinkled map. This approximation can be realized by a deformation of the map $f$, keeping it fixed on a closed subset of $U_i$, where $f$ was already previously defined. This process produces the desired $F^\perp$-wrinkled map. □

Similarly, Theorem 4.4.2 can be generalized to the following fibered version of Theorem 4.5.1.
4.5.2. (Fibered wrinkled mappings of manifolds into foliations) Let \( f : V \to W \) be a fibered over \( B \) map, \( \mathcal{F} \) be a fibered over \( B \) foliation on \( W \) and let \( F : T_B(V) \to \nu_B(\mathcal{F}) \) be a fibered epimorphism which covers \( f \). Suppose that \( f \) is fiberwise transversal to \( \mathcal{F} \) near a closed subset \( K \subset V \), and \( \mathcal{F} \) coincides with fibered reduced differential \( \nu_B(f) \) near \( K \). Then there exists a fibered \( \mathcal{F}^\perp \)-wrinkled map \( g : V \to W \) which extends \( f \) from a neighborhood of \( K \), and such that the fibered epimorphisms \( R(\pi_F \circ d_Bg) \) and \( \mathcal{F} \) are homotopic rel. \( T_B(V)|_K \).

4.6 Generalized wrinkled mappings

A smooth map \( f : V^n \to W^q, n \geq q \), is called generalized wrinkled, if any connected components \( S_i \) of the singularity \( \Sigma = \Sigma_f \) is diffeomorphic to the standard \((q-1)\)-dimensional sphere \( S^{q-1} \), which bounds in \( V \) a \( q \)-dimensional disk \( D_i \), and for each such sphere the map \( f|_{\mathcal{O}_p S_i} \) is equivalent to the map \( w(n, q, s)|_{\mathcal{O}_p S^{q-1}} \). The spheres \( S_i \) and its images \( f(S_i) \) are called generalized wrinkles of the generalized wrinkled mapping \( f \).

A fibered map \( f : V^{k+n} \to W^{k+q}, n \geq q \), is called generalized fibered wrinkled map, if any connected components \( S_i \) of the singularity \( \Sigma = \Sigma_f \) is diffeomorphic to the standard \((k+q-1)\)-dimensional sphere \( S^{k+q-1} \), which bounds in \( V \) a \((k+q)\)-dimensional disk \( D_i \), and for each such sphere the fibered map \( f|_{\mathcal{O}_p S_i} \) is equivalent to the fibered map \( w_k(k + n, q, s)|_{\mathcal{O}_p S^{k+q-1}} \). The spheres \( S_i \) and its images \( f(S_i) \) are called generalized fibered wrinkles of the generalized wrinkled mapping \( f \).

Let \( \mathcal{F} \) be a foliation on a manifold \( W \), \( \text{codim} \mathcal{F} = q \). A map \( f : V \to W \) is called generalized wrinkled with respect to \( \mathcal{F} \), if there exist disjoint small subsets \( U_1, \ldots U_l \subset V \) such that \( f|_{V \setminus (U_1 \cup \ldots \cup U_l)} \) is transversal to \( \mathcal{F} \) and for each \( i = 1, \ldots, l \) the composition

\[
U_i \xrightarrow{f|_{U_i}} U'_i \xrightarrow{p'_i} \mathbb{R}^q
\]

(where \( U'_i \supset f(U_i) \) is an elementary subset of \( W \)), is a generalized wrinkled map.
References

[Ce68] J. Cerf, *Sur les difféomorphismes de la sphere de dimension trois $(\Gamma_4 = 0)$*, Lecture Notes in Mathematics, No. 53. Springer-Verlag, Berlin-New York 1968.

[El72] Y. Eliashberg, *Surgery of singularities of smooth maps*, Izv. Akad. Nauk SSSR Ser. Mat., **36**(1972), 1321–1347.

[EM97] Y. Eliashberg and N. Mishachev, *Wrinkling of smooth mappings and its applications - I*, Invent. Math., **130**(1997), 345–369.

[EM00] Y. Eliashberg and N. Mishachev, *Wrinkling of smooth mappings - II. Wrinkling of embeddings and K.Igusa’s theorem*, Topology, **39**(2000), 711-732.

[EM98] Y. Eliashberg and N. Mishachev, *Wrinkling of smooth mappings - III. Foliation of codimension greater than one*, Topol. Methods in Nonlinear Analysis, **11**(1998), 321-350.

[EM02] Y. Eliashberg and N. Mishachev, *Introduction to the h-principle*, AMS, Graduate Studies in Mathematics, v.48, 2002.

[Gr73] M. Gromov, *Convex integration of partial differential relations*, Izv. Akad. Nauk SSSR Ser. Mat., **37**(1973), 329–343.

[Gr86] M. Gromov, *Partial differential relations*, Springer-Verlag, 1986.

[Ha83] A. Hatcher, *A proof of a Smale conjecture, Diff$(S^3) \simeq O(4)$*, Ann. of Math., (2) 117 (1983), no. 3, 553–607.

[Ig84] K. Igusa, *Higher singularities are unnecessary*, Annals of Math., **119**(1984), 1–58.

[La76] F. Laudenbach, *Formes différentielles de degré 1 non singulières: classes d’homotopie de leurs noyaux*, Comment. Math. Helvet., **51**(1976), 447–464.

[Ph67] A. Phillips, *Submersions of open manifolds*, Topology **6**(1967), 171–206.
[RS01] C. Rourke and B. Sanderson, *The compression theorem - I,II*, Geom. and Topology, 5(2001), 399–429, 431–440.

[RS03] C. Rourke and B. Sanderson, *The compression theorem III: applications*, Alg. and Geom. Topology, 3, (2003), 857–872.

[Sp02] D. Spring, *Directed embeddings and the simplification of singularities*, Comm. in Cont. Math., 4(2002), 107–144.

[Sp05] D. Spring, *Directed embeddings of closed manifolds*, Comm. in Cont. Math., 7(2005), 707–725.

[Th74] W. Thurston, *The theory of foliations of codimension greater than one*, Comm. Math. Helvet., 49(1974), 214–231.