THE CALABI–YAU EQUATION FOR $T^2$-BUNDLES OVER $T^2$: THE NON-LAGRANGIAN CASE

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ABSTRACT. In the spirit of [9, 2], we study the Calabi-Yau equation on $T^2$-bundles over $T^2$ endowed with an invariant non-Lagrangian almost-Kähler structure showing that for $T^2$-invariant initial data it reduces to a Monge-Ampère equation having a unique solution. In this way we prove that for every total space $M^4$ of an orientable $T^2$-bundle over $T^2$ endowed with an invariant almost-Kähler structure the Calabi-Yau problem has a solution for every normalized $T^2$-invariant volume form.

1. Introduction

Let $(M^{2n}, J, \Omega)$ be a $2n$-dimensional compact Kähler manifold with associated complex structure $J$ and symplectic form $\Omega$. In view of a celebrated Yau’s theorem [12] for every volume form $\sigma$ on $M^{2n}$ satisfying

\begin{equation}
\int_{M^{2n}} \Omega^n = \int_{M^{2n}} \sigma
\end{equation}

there exists a unique Kähler form $\tilde{\Omega}$ in the same de Rham cohomology class of $\Omega$ and such that

\begin{equation}
\tilde{\Omega}^n = \sigma.
\end{equation}

Equation (1.2) still makes sense in the almost-Kähler context when $J$ is merely an almost-complex structure and $\Omega$ remains closed. The almost complex structure $J$ is still orthogonal relative to a Riemannian metric $g$ for which $\Omega(X, Y) = g(JX, Y)$, and

\begin{equation}
\tilde{\Omega} = \Omega + d\alpha
\end{equation}

is again assumed to be a positive-definite $(1,1)$-form relative to $J$. In this context the equations (1.1), (1.2) and (1.3) constitute the Calabi-Yau problem, which in the last years has been intensively studied in four dimensions (see [1, 8, 9, 2] and the references therein).

In [1] Donaldson introduced the Calabi-Yau problem for almost-Kähler manifolds showing that equation (1.2) has unique solution in dimensions four and that it is related to some other central problems in symplectic geometry. In [8] Tosatti, Weinkove and Yau gave a sufficient condition for the existence of solution to the Calabi-Yau equation in terms of the Chern connection. This condition fails in case of the Kodaria-Thurston surface, which is a 4-dimensional nilmanifold, i.e. a compact quotient of the nilpotent Lie group $\text{Nil}^3 \times \mathbb{R}$ by a lattice, where $\text{Nil}^3$ denotes the 3-dimensional real Heisenberg group.
The Kodaira-Thurston surface is a typical example of a compact almost-Kähler 4-dimensional manifold which does not admit any Kähler structure. More precisely, it is the total space of a principal $T^2$-bundle over a torus $T^2$ (in our notation $T^2$ denotes the torus on the fibres, while $T^2$ is the torus at the basis) and it has an invariant almost-Kähler structure whose symplectic form vanishes along the fibres of the $T^2$-fibration, where by invariant structure we mean a structure induced by a left-invariant one on $\text{Nil}^3 \times \mathbb{R}$. The almost-Kähler structures on a total space of a fibration whose symplectic form vanishes along the fibres are usually called Lagrangian, since the fibers are Lagrangian submanifolds.

In [9] Tosatti and Weinkove studied the Calabi-Yau equation on the Kodaira-Thurston surface endowed with an invariant Lagrangian almost-Kähler structure, showing the existence of a solution for every $T^2$-invariant normalized volume form $\sigma$. In [2] the previous result obtained by Tosatti and Weinkove was simplified and extended to other $T^2$-bundles over a $T^2$ endowed with an invariant Lagrangian almost-Kähler structure.

We recall that in view of [10] every orientable $T^2$-bundle over a $T^2$ is a infra-solvmanifold, i.e. a smooth quotient $\Gamma \backslash G$ covered by a solvmanifold $\tilde{\Gamma} \backslash \tilde{G}$, compact quotient by a co-compact discrete subgroup of one of the following four Lie groups

\[ \mathbb{R}^4, \text{Nil}^3 \times \mathbb{R}, \text{Nil}^4, \text{Sol}^3 \times \mathbb{R}. \]

These Lie groups are all diffeomorphic to $\mathbb{R}^4$. The Lie groups $\text{Nil}^3, \text{Nil}^4$ are nilpotent and $\text{Sol}^3$ is a particular solvable (non-nilpotent) Lie group.

In particular, if the total space $M^4$ of an orientable $T^2$-bundle over a $T^2$ is a solvmanifold, then it must be the compact quotient of one of the above Lie groups $G$. It is well known that all the orientable $T^2$-bundles over $T^2$ admit symplectic structures (see [4]). The notion of invariant almost-Kähler structure makes sense for orientable $T^2$-bundles over $T^2$, meaning one induced from a left-invariant structure on $G$ which is invariant by the discrete subgroup $\Gamma$.

As a main result of [2] it was shown that if $M^4 = \Gamma \backslash G$ is an orientable $T^2$-bundle over a $T^2$ with $G = \text{Nil}^3 \times \mathbb{R}$ or $\text{Nil}^4$, and if $M^4$ admits an invariant Lagrangian almost-Kähler structure $(\Omega, J)$, then for every normalized volume form $\sigma = e^F \Omega^2$ with $F \in C^\infty(T^2)$, the corresponding Calabi–Yau problem has a unique solution.

The Lagrangian condition may or may not apply in the case of $G = \text{Nil}^3 \times \mathbb{R}$, but is automatic when $M^4$ is modelled on the 3-step nilpotent Lie group $\text{Nil}^4$. In the case of $G = \text{Sol}^3$ every invariant almost-Kähler on $\Gamma \backslash G$ is non-Lagrangian.

The aim of this paper is to extend the main result in [2] to the non-Lagrangian cases, i.e. to some $T^2$-fibrations modelled on $\text{Nil}^3 \times \mathbb{R}$ and to all the $T^2$-fibrations modelled on $\text{Sol}^3 \times \mathbb{R}$.

Our main result is the following

**Theorem 1.1.** Let $M^4 = \Gamma \backslash G$ be an orientable $T^2$-bundle over a $T^2$ with $G = \text{Nil}^3 \times \mathbb{R}$ or $\text{Sol}^3 \times \mathbb{R}$, and suppose that $M^4$ admits an invariant non-Lagrangian almost-Kähler
structure \((\Omega, J)\). Then for every normalized volume form \(\sigma = e^{F}\Omega^2\) with \(F \in C^\infty(\mathbb{T}^2)\), the corresponding Calabi–Yau problem has a unique solution.

The proof of this theorem consists in showing that the Calabi-Yau problem can be reduced to a single elliptic Monge-Ampère equation which has solution.

The trick of reducing the problem to a Monge-Ampère equation was the core of [2], but the class of equations which appear in the present paper differs from the ones considered in [2].

As a consequence we show that for every total space \(M^4\) of an orientable \(T^2\)-bundle over a \(T^3\) endowed with an invariant almost-Kähler structure \((\Omega, J)\) the Calabi-Yau problem has a solution for every normalized \(T^2\)-invariant volume form.

The paper is organized as follows: In Section 2 we recall the classification of \(T^2\)-bundles over \(T^2\) and we briefly describe the main result in [2]. Sections 3 and 4 contain the proof of Theorem 1.1 where the case of \(G = \text{Nil}^3 \times \mathbb{R}\) and \(G = \text{Sol}^3 \times \mathbb{R}\) are treated separately. In each of the two cases we can reduce the problem to a Monge-Ampère equation for which we show the existence of a solution.

2. The Calabi-Yau equation on \(T^2\)-bundles over \(T^2\)

Orientable \(T^2\)-bundles over \(T^2\) were classified by Fukuhara and Sakamoto in [3] and it was shown by Ue in [10, 11] that all these manifolds are infra-solvmanifolds. A compact manifold \(M\) is called an \textit{infra-solvmanifold} if it admits a finite cover \(\tilde{\pi}: \tilde{M} \to M\), where \(\tilde{M} = \tilde{\Gamma}\backslash G\) is the compact quotient of a solvable Lie group \(G\) by a lattice \(\tilde{\Gamma}\). Alternatively, \(M\) can be written as a quotient \(M = \Gamma\backslash G\), where \(\Gamma\) is a discrete group containing a lattice \(\tilde{\Gamma}\) of \(G\) such that \(\tilde{\Gamma}\backslash \Gamma\) is finite. In the case that \(\tilde{\Gamma}\) is a lattice, \(M\) is simply called a \textit{solvmanifold}.

It turns out that in the classification of \(T^2\)-bundles over \(T^2\), the solvable Lie group \(G\) must be one of the following four patterns

\[
(2.1) \quad \mathbb{R}^4, \quad \text{Nil}^3 \times \mathbb{R}, \quad \text{Nil}^4, \quad \text{Sol}^3 \times \mathbb{R},
\]

while the classification of the possible \(\Gamma\)'s determines eight families. For the Lie groups \(\text{Nil}^3 \times \mathbb{R}, \text{Nil}^4, \text{Sol}^3 \times \mathbb{R}\) we have the following description:

(1) \(\text{Nil}^3\) is the 3-dimensional Heisenberg group of matrices

\[
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}
\]

and \(\text{Nil}^3 \times \mathbb{R}\) is a 2-step nilpotent Lie group which can be regarded as \(\mathbb{R}^4\) with the product

\[
(x_0, y_0, z_0, t_0)(x, y, z, t) = (x_0 + x, \ y_0 + y, \ z_0 + z + x_0y, \ t_0 + t).
\]
(2) $\text{Nil}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is the 3-step nilpotent Lie group of real matrices

$$
\begin{pmatrix}
1 & t & \frac{1}{2}t^2 & x \\
0 & 1 & t & y \\
0 & 0 & 1 & z \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

(3) $\text{Sol}^3 = \mathbb{R} \ltimes \mathbb{R}^2$ is a unimodular 2-step solvable Lie group with $\varphi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$

and $\text{Sol}^3 \times \mathbb{R}$ can be regarded as $\mathbb{R}^4$ with the product

$$(x_0, y_0, z_0, t_0)(x, y, z, t) = (x_0 + e^{t_0}x, y_0 + e^{-t_0}y, z_0 + z, t_0 + t).$$

The diffeomorphism classes of (the total space of) $T^2$-bundles over $\mathbb{T}^2$ can be summarized in Geiges’ eight families [4, Table 1], which can be explicitly described in terms of the generators of the discrete groups $\Gamma$, the monodromy matrices along the two curves generating $\pi_1(\mathbb{T}^2)$, as well as the Euler class for the corresponding $T^2$-bundle.

In the case of $G = \text{Nil}^3 \times \mathbb{R}$ one has two inequivalent fibrations

$$
\pi_{xy} : M^4 \to \mathbb{T}^2_{xy},
$$

$$
\pi_{yt} : M^4 \to \mathbb{T}^2_{yt},
$$

induced from the following coordinate mappings:

$$(x, y, z, t) \mapsto (x, y),$$

$$(x, y, z, t) \mapsto (y, t).$$

If $\Gamma$ is not a lattice of $G$, we have that $\Gamma$ contains a lattice $\tilde{\Gamma}$ of $G$ such that the quotient $\tilde{\Gamma}/\Gamma$ is a finite group. Therefore there exists a covering map $p: \tilde{\Gamma}/G \to \Gamma/G$ which preserves the $T^2$-bundle structure over $\mathbb{T}^2$.

We recall that an almost-Kähler structure on a manifold $M$ is a pair $(\Omega, J)$, where $\Omega$ is a symplectic form and $J$ is an endomorphism of the tangent bundle to $M$ satisfying $J^2 = -I$ and

$$
\Omega(JX, JY) = \Omega(X, Y), \quad \Omega(Z, JZ) > 0
$$

for every tangent vector fields $X, Y, Z$ on $M$ with $Z$ nowhere vanishing. Every almost-Kähler structure induces the Riemannian metric

$$
g(X, Y) = \Omega(X, JY).$$

In this paper (as in [2]) we consider on the total space $M^4 = \Gamma/G$ of $T^2$-bundles over $\mathbb{T}^2$ invariant almost-Kähler structures, i.e. ones induced from left-invariant structures on $G$ which are invariant by the discrete group $\Gamma$ and we study for these almost-Kähler manifolds the Calabi-Yau problem. In particular every invariant almost-Kähler structure on $M^4 = \Gamma/G$ induces an invariant almost-Kähler structure on the solvmanifold $\tilde{\Gamma}/G$. 
The case \( G = \mathbb{R}^4 \) (which corresponds to two of the Geiges’ families) is not interesting from our point of view, since in this case every invariant almost-Kähler structure is Kähler and Yau’s theorem can be applied. For the other cases we have to distinguish the Lagrangian case from the non-Lagrangian one. If \( M^4 \) is modelled on \( G = \text{Nil}^4 \) or on \( G = \text{Nil}^3 \times \mathbb{R} \) with bundle structure given by the projection \( \pi_{xy} \), then every invariant almost-Kähler structure is Lagrangian. If \( M^4 \) is modelled on \( \text{Nil}^3 \times \mathbb{R} \) with bundle structure given by the projection \( \pi_{yt} \) then it admits Lagrangian and non-Lagrangian almost-Kähler structures as well. In the case \( G = \text{Sol}^3 \times \mathbb{R} \), every invariant almost-Kähler structure is non-Lagrangian.

Let now \( M^4 = \Gamma/G \) be an orientable \( T^2 \)-bundle over \( T^2 \) and denote by \( \mathfrak{g} \) the Lie algebra of \( G \). Then every basis \((e^i)\) of the dual space \( \mathfrak{g}^* \) induces a global frame of 1-forms on \( M^4 \). Furthermore we fix an invariant almost-Kähler structure \((\Omega, J)\) on \( M^4 \). Let \( \sigma = e^F \Omega^2 \) be a volume form and let \( F \) be a smooth map on the base \( T^2 \) of \( M^4 \) satisfying

\[
\int_{T^2} (e^F - 1) = 0.
\]

Then in this case the Calabi-Yau problem reads as

\[
\begin{cases}
(\Omega + da)^2 = e^F \Omega^2, \\
J(da) = da,
\end{cases}
\]

on \( M^4 \) whose components with respect to the coframe \((e^i)\) are defined on the torus \( T^2 \). Thus the Calabi-Yau problem reduces to a system of partial differential equations on the base \( T^2 \).

Although the system (2.2) depends on the choice of \( G \), \((\Omega, J)\) and the structure of \( T^2 \)-fibration, for all the cases we can proceed in the following way: first we parametrize \((\Omega, J)\) using a suitable invariant coframe on \( M^4 \) in order to simplify the formulation of (2.2) as far as possible and then we perform a suitable change of variables transforming the system (2.2) in a Monge-Ampère equation on the base \( T^2 \).

The Lagrangian cases have been considered in [2], where it has been proved that has a unique solution. In the next two sections we will consider the non-Lagrangian cases for the manifolds modelled on \( \text{Nil}^3 \times \mathbb{R} \) and \( \text{Sol}^3 \times \mathbb{R} \).

3. MANIFOLDS MODELLLED ON \( \text{Nil}^3 \times \mathbb{R} \): THE NON-LAGRANGIAN CASE

In this section we study the Calabi-Yau problem for \( T^2 \)-bundles over \( T^2 \) modelled on \( \text{Nil}^3 \times \mathbb{R} \) and equipped with an invariant non-Lagrangian almost-Kähler structure.

The structure of \( T^2 \)-bundle over a \( T^2 \) is then induced by the projection \( \pi_{yt} \) onto the torus \( T^2_{yt} \). The total space \( M^4 \) of the \( T^2 \)-fibration has the global invariant coframe

\[
e^1 = dy, \quad e^2 = dx, \quad e^3 = dt, \quad e^4 = dz - xdy
\]
which satisfies the structure equations
\[
(3.1) \quad de^1 = de^2 = de^3 = 0, \quad de^4 = e^{12}.
\]

**Lemma 3.1.** Let $M^4$ be the total space of an oriented $T^2$-bundle over $\mathbb{T}^2$ modelled on $\text{Nil}^3 \times \mathbb{R}$ and induced by the projection $\pi_{yt}$ onto the torus $\mathbb{T}^2_{yt}$.

Let $(\Omega, J)$ be an invariant almost-Kähler structure on $M^4$ with induced Riemannian metric $g$. Then there exists an orthonormal invariant coframe $(f^i)$ for which
\[
(3.2) \quad \Omega = f^{14} + f^{23},
\]
and
\[
(3.3) \quad f^1 \in \langle e^1 \rangle, \quad g(e^3, f^2) = 0, \quad g(e^3, f^3)g(e^4, f^4) \geq 0.
\]

**Proof.** We can certainly find an orthonormal invariant coframe $(f^i)$ for which (3.2) is valid and $f^1 \in \langle e^1 \rangle$. Since $f^4 = J(f^1)$, we still have the freedom of rotate $f^{23}$ in the plane orthogonal to $\langle f^1, f^4 \rangle$. After a suitable rotation we obtain $g(e^3, f^2) = 0$. Eventually, we may invert the direction of $f^2$ and $f^3$ to meet the condition $g(e^3, f^3)g(e^4, f^4) \geq 0$, without reversing $f^{23}$. \qed

3.1. **The Calabi-Yau equation on** $M^4$. Consider on $M^4$ an invariant non-Lagrangian almost-Kähler structure $(\Omega, J)$ with induced Riemannian metric $g$. Let $\sigma = e^F \Omega^2$ be a volume form where $F = F(y, t)$ is a smooth map on the base satisfying
\[
(3.4) \quad \int_{\mathbb{T}^2} (e^F - 1) = 0.
\]
Consider the Calabi-Yau equation
\[
\begin{cases}
(\Omega + da)^2 = \sigma, \\
J(da) = da,
\end{cases}
\]
where $a$ is a 1-form on $M^4$ whose components with respect to the basis $(e^i)$ depend on $(y, t)$ only. Let $(f^i)$ be a coframe as in Lemma 3.1 and set
\[
G_i^j = g(e^i, f^j);
\]
then we have
\[
e^i = G_i^j f^j.
\]
In particular we have
\[
(3.5) \quad e^1 = G^1_1 f^1
\]
and
\[
(3.6) \quad G^1_1 \neq 0, \quad G^1_2 = G^1_3 = G^1_4 = 0.
\]
Let $H = G^{-1}$ be the inverse matrix of $G = (G_j^i)$. Then
\[
f^i = H^i_j e^j.
From (3.5) and (3.6), we have

\[ H_1^1 = (G_1^1)^{-1} \neq 0, \]

and

\[ H_2^1 = H_3^1 = H_4^1 = 0. \]  

(3.7)

Thanks to structure equations (3.1), we have

\[ df_i = H_{ij}^j de_j = H_{ij}^4 de_4 = H_{ij}^4 e_{12} = H_{ij}^4 G_1^1(G_2^3 f_{12} + G_3^3 f_{13} + G_4^3 f_{14}). \]

The condition that \((\Omega, J)\) is non-Lagrangian implies that

\[ G_4^3 \neq 0. \]

Moreover, since \(G_2^3 = g(e^3, f^2) = 0\), we have

\[ e^3 = G_1^3 f^1 + G_3^3 f^3 + G_4^3 f^4, \]  

(3.8)

where

\[ G_3^3 G_4^3 \geq 0, \]  

(3.9)

thanks to (3.3).

Differentiating (3.8) we get

\[ G_3^3 df^3 + G_4^3 df^4 = (G_3^3 H_4^3 + G_4^3 H_4^4) e^{12} = 0, \]

i.e.

\[ G_3^3 H_4^3 + G_4^3 H_4^4 = 0. \]  

(3.10)

Furthermore, the symplectic condition \(d\Omega = 0\) gives

\[ df^{23} = 0 \]

that is

\[ 0 = H_4^2 G_1^1(G_2^3 f_{12} + G_3^3 f_{13} + G_4^3 f_{14}) \wedge f^3 - H_4^3 G_1^1 f^2 \wedge (G_2^3 f_{12} + G_3^3 f_{13} + G_4^3 f_{14}) \]

\[ = G_1^1(H_4^2 G_2^2 + G_1^1 H_4^3 G_3^2) f^{123} - G_1^1 H_4^2 G_4^2 f^{134} + G_1^1 H_4^3 G_4^2 f^{124}. \]

It follows that

\[ H_4^2 G_2^2 + H_4^3 G_3^2 = 0, \quad H_4^2 G_4^2 = H_4^3 G_4^3 = 0. \]  

(3.11)

From (3.10) and (3.11) we have

\[ G_4^2 G_1^3 H_4^1 = G_4^2 (G_3^3 H_4^3 + G_4^3 H_4^4) = 0, \]

hence, since \(H_1^1 = 0\), and \(H_2^3, H_3^3\) and \(H_4^4\) cannot vanish all together, from (3.7), (3.11) and (3.12) we obtain that

\[ G_4^3 = 0. \]

Write

\[ a = a_k f^k \]
and compute
\[ da = (G_1 a_{2,y} + G_1 G_2 H_4^k a_k + G_1^3 a_{2,t}) f^{12} + (G_1 a_{3,y} + G_1 G_2^2 H_4^k a_k + G_1^3 a_{3,t} - G_3^3 a_{1,t}) f^{13} - G_3^3 a_{2,t} f^{24} + (G_1 a_{4,y} + G_1^3 a_{4,t} - G_3^3 a_{1,t}) f^{14} - G_3^3 a_{2,t} f^{23} + (G_3^3 a_{4,t} - G_4^3 a_{3,t}) f^{34}. \]

Hence \( da \) is of type \((1, 1)\) with respect to \( J \) if and only if
\[
\begin{cases}
G_1 a_{2,y} + G_1 G_2^2 (H_4^2 a_2 + H_4^4 a_3 + H_4^4 a_4) + G_1^3 a_{2,t} = -G_1^3 a_{4,t} + G_4^3 a_{3,t}, \\
G_1 a_{3,y} + G_1 G_2^3 (H_4^2 a_2 + H_4^4 a_3 + H_4^4 a_4) + G_1^3 a_{3,t} - G_3^3 a_{1,t} = -G_3^3 a_{2,t}
\end{cases}
\]
and in this case \( da \) reduces to
\[
da = (-G_1^3 a_{4,t} + G_4^3 a_{3,t}) f^{12} - G_3^3 a_{2,t} f^{13} - G_3^3 a_{2,t} f^{24} + (G_1 a_{4,y} + G_1^3 a_{4,t} - G_3^3 a_{1,t}) f^{14} - G_3^3 a_{2,t} f^{23} + (G_3^3 a_{4,t} - G_4^3 a_{3,t}) f^{34}.
\]

The Calabi-Yau equation now reads as
\[
e^F = (1 + G_1 a_{4,y} + G_3^3 a_{4,t} - G_3^3 a_{1,t})(1 - G_3^3 a_{2,t}) - G_3^3 G_4^3 (a_{2,t})^2 - (-G_3^3 a_{4,t} + G_4^3 a_{3,t})^2
\]
and the Calabi-Yau problem is equivalent to the following system of partial differential equations:
\[
(3.13) \quad \begin{cases}
G_1 a_{2,y} + G_1 G_2^2 (H_4^2 a_2 + H_4^4 a_3 + H_4^4 a_4) + G_1^3 a_{2,t} + G_4^3 a_{4,t} - G_3^3 a_{3,t} = 0, \\
G_1 a_{3,y} + G_1 G_2^3 (H_4^2 a_2 + H_4^4 a_3 + H_4^4 a_4) + G_1^3 a_{3,t} - G_3^3 a_{1,t} + G_4^3 a_{2,t} = 0, \\
(1 + G_1 a_{4,y} + G_1^3 a_{4,t} - G_4^3 a_{1,t})(1 - G_3^3 a_{2,t}) - G_3^3 G_4^3 (a_{2,t})^2 - (-G_3^3 a_{4,t} + G_4^3 a_{3,t})^2 = e^F.
\end{cases}
\]

In the system \((3.13)\) the parameter \(G_3^3\) has a special role. We will study separately the cases \(G_3^3 = 0\) and \(G_3^3 \neq 0\).

3.2. **The case \(G_3^3=0\)**. This case is quite trivial since condition \(G_3^3 = 0\) implies \( dt \in \langle f^3, f^4 \rangle \) and \( f^{14} = dy \wedge dt \). Therefore if \(G_3^3 = 0\) the corresponding Calabi-Yau equation has the explicit solution
\[
\tilde{\Omega} = (e^F - 1) f^{14} + f^{23}.
\]

3.3. **The Case \(G_3^3 \neq 0\)**. Under this assumption we consider the transformation
\[
\begin{align*}
a_1 &= -G_3^3 u_t - G_1 (H_4^2 G_3^2 + H_4^4 G_2^2) u, \\
a_2 &= -G_3^3 u_t - H_4^4 G_1^2 G_2^2 u, \\
a_3 &= -H_4^4 G_1 G_3^2 u, \\
a_4 &= G_1^3 u_y + G_4^3 u_t.
\end{align*}
\]

A long but straightforward computation shows that the first two equations of system \((3.13)\) are identically satisfied, while the third one becomes
\[
(3.14) \quad (u_{yy} + B_{11} u_t + C_{11})(u_{tt} + B_{22} u_t + C_{22}) - (u_{yt} + B_{12} u_t + C_{12})^2 = E_1 + E_2 e^F,
\]
where
\[
B_{11} = \frac{G_2^2(G_3^1)^2H_4^4}{G_3^1G_3^3} - 2\frac{G_3^2G_3^1G_3^4H_4^4}{G_3^1G_3^3} + \frac{G_3^2G_3^3H_4^2}{G_3^1},
\]
\[
B_{12} = -\frac{G_2^2G_3^1H_4^4}{G_3^1} + \frac{G_3^2G_3^4H_4^1}{G_3^1}, \quad B_{22} = \frac{G_3^1G_2^2H_4^4}{G_3^1},
\]
\[
C_{11} = \frac{1}{(G_3^1)^2} + \frac{(G_3^1)^2}{(G_3^1)^2(G_3^3)^2} + \frac{G_3^3}{(G_3^1)^2G_3^3},
\]
\[
C_{12} = -\frac{G_1^3}{G_1^1(G_3^3)^2}, \quad C_{22} = \frac{1}{(G_3^3)^2},
\]
\[
E_1 = \frac{G_3^3G_4^1}{(G_1^1)^2(G_3^3)^4}, \quad E_2 = (G_1^1G_3^3)^2.
\]

In particular we have
\[
B_{11}B_{22} - (B_{12})^2 = 0
\]
and
\[
C_{11}C_{22} - (C_{12})^2 = E_1 + E_2.
\]

4. Manifolds modelled on $\text{Sol}^3 \times \mathbb{R}$

In this section we study the Calabi-Yau equation for the total spaces $M^4$ of $T^2$-bundles over torus $T^2$ modelled on $\text{Sol}^3 \times \mathbb{R}$.

Since the Lie group $\text{Sol}^3 \times \mathbb{R}$ can be seen as $\mathbb{R}^4$ equipped with the product
\[
(x_0,y_0,z_0,t_0)(x,y,z,t) = (x_0 + x, y_0 + y, z_0 + e^x z, t_0 + e^{-x} t)
\]
$M^4$ inherits the global invariant coframe
\[
e^1 = dx, \quad e^2 = dy, \quad e^3 = e^x dz, \quad e^4 = e^{-x} dt
\]
satisfying the following structure equations
\[
de^1 = de^2 = 0, \quad de^3 = e^{13}, \quad de^4 = -e^{14}.
\]

Moreover, invariant almost-Kähler structure on $M^4$ should be parametrized as claimed in the following lemma proved in [2]

**Lemma 4.1.** Let $(\Omega, J)$ be an invariant almost-Kähler structure on the total space $M^4$ of a $T^2$-bundle over a torus $T^2$ modelled on $\text{Sol}^3 \times \mathbb{R}$. Let $g$ be the induced Riemannian metric. Then there exists an orthonormal global coframe $(f^i)$ for which
\[
\Omega = f^{12} + f^{34},
\]
and
\[
f^1 \in \langle e^1 \rangle, \quad f^3 \in \langle e^3 \rangle, \quad f^4 \in \langle e^3, e^4 \rangle,
\]
with
\begin{equation}
(4.4) \quad g(e^1, f^1) > 0.
\end{equation}

Notice that in this case every invariant almost-Kähler structure is non-Lagrangian.

### 4.1. The Calabi-Yau equation on $M^4$

Let $(\Omega, J)$ be an invariant almost-Kähler structure on $M^4$, $(f^k)$ be a coframe as in the previous lemma and $\sigma = e^F \Omega^2$ be a volume form where $F = F(x, y) \in C^\infty(T^2)$ satisfies the condition
\begin{equation}
\int_{T^2} (e^F - 1) = 0.
\end{equation}

Then we consider the Calabi-Yau equation
\begin{equation}
(4.5) \quad (\Omega + da)^2 = \sigma,
\end{equation}
where
\begin{equation}
a = \sum_{k=1}^4 a_k f^k,
\end{equation}
is a 1-form whose components $a_k$ are functions on the base $T^2$.

Let $g$ be the Riemannian metric induced by $(\Omega, J)$ and set
\begin{equation}
G^i_j = g(e^i, f^j).
\end{equation}

Then
\begin{equation}
e^i = G^i_j f^j
\end{equation}
and
\begin{equation}
f^i = H^i_j e^j,
\end{equation}
where $H = G^{-1}$ is the inverse matrix of $G = (G^i_j)$. In particular this implies that
\begin{equation}
G^3_4 = H^2_4 G^2_2 + H^4_4 G^4_4 = H^3_3 G^3_3 + H^3_3 G^4_4 = 0.
\end{equation}

From (4.3) we have that
\begin{equation}
f^1 = H^1_1 e^1, \quad f^2 = H^2_2 e^2 + H^3_2 e^3 + H^4_2 e^4, \quad f^3 = H^3_3 e^3, \quad f^4 = H^4_4 e^3 + H^4_4 e^4.
\end{equation}

Then, making use of (4.2) and (4.6), we obtain
\begin{align*}
df^1 &= 0, \quad df^2 = G^1_1(H^2_3 G^3_3 - H^2_4 G^4_4) f^{13} - G^1_1 H^2_4 G^4_4 f^{14}, \\
df^3 &= G^1_1 f^{13}, \quad df^4 = G^1_1(H^3_3 G^3_3 - H^4_4 G^4_4) f^{13} - G^1_1 f^{14}.
\end{align*}

Let
\begin{equation}
a = a_1 f^1 + a_2 f^2 + a_3 f^3 + a_4 f^4
\end{equation}
be a $T^2$-invariant form on $M^4$. 
Hence $da$ is $J$-invariant if and only if its components satisfy

\[
\begin{cases}
G^1_{a,3,x} - G^2_{3,a,1,y} + G^1_1(H^2_3 G^3_3 - H^2_4 G^4_3) a_2 + G^1_{a,3} + G^1_1(H^3_3 G^3_3 - H^4_4 G^4_3) a_4 \\
= G^2_{a,4,y} - G^2_{4,a,2,y},
\end{cases}
\]

and equation (4.5) becomes

\[
(1 + G^1_{a,2,x} - G^2_{a,2,y})(1 + G^2_{a,4,y} - G^2_{3,a,3,y})
= (G^2_{a,4,y} - G^2_{4,a,2,y})^2 - (G^2_{a,3,y} - G^2_{3,a,2,y})^2 = e^F.
\]

Therefore the Calabi-Yau problem reduces to the following system of partial differential equations

\[
\begin{cases}
G^1_{a,3,x} - G^2_{3,a,1,y} + G^1_1(H^2_3 G^3_3 - H^2_4 G^4_3) a_2 + G^1_{a,3} + G^1_1(H^3_3 G^3_3 - H^4_4 G^4_3) a_4 \\
= G^2_{a,4,y} - G^2_{4,a,2,y},
\end{cases}
\]

(4.7)

\[
(1 + G^1_{a,2,x} - G^2_{a,2,y})(1 + G^2_{a,4,y} - G^2_{3,a,3,y})
= (G^2_{a,4,y} - G^2_{4,a,2,y})^2 - (G^2_{a,3,y} - G^2_{3,a,2,y})^2 = e^F.
\]

4.2. Reduction of (4.7) to a single equation. Consider $u \in C^2(\mathbb{T}^2)$ such that

\[
\int_{\mathbb{T}^2} u = 0,
\]

(4.8)
and let

\[
\begin{align*}
    a_1(x, y) &= -\frac{H_1^1 G_2^2 (u_y(x, y) - u_y(x, 0)) + 2H_1^1 G_2^4 (u(x, y) - u(x, 0))}{(G_3^2)^2 + (G_4^2)^2} \\
    &\quad - \frac{G_1^1 H_2^2}{(G_3^2)^2 + (G_4^2)^2} \left( \int_0^x (u_{xx}(x, t) - u(x, t)) \, dt - y \int_0^1 (u_{xx}(x, t) - u(x, t)) \, dt \right), \\
    a_2(x, y) &= -\frac{1}{(G_3^2)^2 + (G_4^2)^2} \left( \int_0^x \left( \int_0^1 u(s, t) \, dt \right) \, ds - \int_0^1 (u_x(x, t) - u_x(0, t)) \, dt \right), \\
    a_3(x, y) &= -\frac{H_2^3 G_2^2 u_x(x, y) + H_1^1 G_4^2 u_y(x, y) - H_2^3 (G_3^2 - 2G_4^2 H_4^4 G_4^4) u(x, y)}{(G_3^2)^2 + (G_4^2)^2} \\
    &\quad - \frac{H_2^2 G_2^3}{(G_3^2)^2 + (G_4^2)^2} \left( \int_0^x \left( \int_0^1 u(s, t) \, dt \right) \, ds - \int_0^1 (u_x(x, t) - u_x(0, t)) \, dt \right), \\
    a_4(x, y) &= -\frac{H_2^3 G_4^2 u_x(x, y) - H_1^1 G_3^2 u_y(x, y) + H_2^2 G_2^4 u(x, y)}{(G_3^2)^2 + (G_4^2)^2} \\
    &\quad - \frac{H_2^2 G_4^3}{(G_3^2)^2 + (G_4^2)^2} \left( \int_0^x \left( \int_0^1 u(s, t) \, dt \right) \, ds - \int_0^1 (u_x(x, t) - u_x(0, t)) \, dt \right).
\end{align*}
\]

Thanks to condition (1.8) we have that the functions \(a_1\) to \(a_4\) are periodic. A long computation shows that the first two equations of system (1.7) are identically satisfied, while the third one becomes:

\[
(4.9) \quad (u_{xx} + B_{11} u_y + C_{11} + D u)(u_{yy} + B_{22} u_y + C_{22}) - (u_{xy} + B_{12} u_y)^2 = E_1 + E_2 e^F
\]

where

\[
B_{11} = \frac{2H_1^1 G_2^2 G_3^3 (G_4^2 + G_3^2 H_4^4 G_3^4)}{(G_3^2)^2 + (G_4^2)^2},
\]

\[
B_{12} = \frac{(G_3^2)^2 - (G_4^2)^2 + 2G_3^2 G_4^2 H_4^4 G_3^4}{(G_3^2)^2 + (G_4^2)^2},
\]

\[
B_{22} = -\frac{2G_1^1 H_2^2 G_3^4 (G_3^2 - G_3^2 H_4^4 G_3^4)}{(G_3^2)^2 + (G_4^2)^2},
\]

\[
C_{11} = H_1^1 \left( (G_3^2)^2 + (G_4^2)^2 + (G_4^2)^2 \right), \quad C_{22} = G_1^1,
\]

\[
D = -1, \quad E_1 = (G_2^2)^2, \quad E_2 = (G_3^2)^2 + (G_4^2)^2.
\]

In particular we have

\[
B_{11} B_{22} - (B_{12})^2 = -1
\]

and

\[
C_{11} C_{22} - (C_{12})^2 = E_1 + E_2.
\]
5. The Monge-Ampère equation

Both equations (3.14) and (4.9) are generalized Monge-Ampère equations of the following type:

\[
A_{11}[u]A_{22}[u] - (A_{12}[u])^2 = E_1 + E_2 e^F,
\]

where

\[
A_{11}[u] = u_{xx} + B_{11}u_y + C_{11} + Du,
A_{12}[u] = u_{xy} + B_{12}u_y + C_{12},
A_{22}[u] = u_{yy} + B_{22}u_y + C_{22},
\]

with \(B_{ij}, C_{ij}, D, E_i\) real numbers such that

\[
C_{11} + C_{22} > 0, \quad D \leq 0,
\]
\[
E_1 > 0, \quad E_2 > 0,
\]
\[
B_{11}B_{22} - (B_{12})^2 = D
\]

and

\[
C_{11}C_{22} - (C_{12})^2 = E_1 + E_2.
\]

Moreover

\[
F \in C^\infty(T^2)
\]

and satisfies the condition

\[
\int_{T^2} (e^F - 1) = 0.
\]

In Theorem 5.7 we shall prove that equation (5.1) has a solution belonging to \(C^\infty(T^2)\) and satisfying the condition

\[
\int_{T^2} u = 0.
\]

For all \(n \in \mathbb{N}, 0 < \epsilon < 1\), consider the semi-norms

\[
|u|_{C^n} = \max_{0 \leq j \leq n} \sup_{(x,y) \in \mathbb{R}^2} |\partial_x^j \partial_y^{n-j} u(x,y)|
\]
\[
|u|_{C^n,\epsilon} = \max_{0 \leq j \leq n} \sup_{(x,y) \in \mathbb{R}^2} \sup_{(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}} \frac{|\partial_x^j \partial_y^{n-j} u(x + h, y + k) - \partial_x^j \partial_y^{n-j} u(x, y)|}{(h^2 + k^2)\epsilon/2}
\]

and the norms

\[
\|u\|_{C^n} = \max_{0 \leq k \leq n} |u|_{C^k}, \quad \|u\|_{C^n,\epsilon} = \max\{\|u\|_{C^n}, |u|_{C^n,\epsilon}\}.
\]
Lemma 5.1. Under the hypotheses (5.3), for all $u \in C^2(\mathbb{T}^2)$ satisfying (5.1) we have that
\begin{equation}
\begin{aligned}
A_{11}[u] > 0, \\
A_{22}[u] > 0.
\end{aligned}
\end{equation}

Proof. Equation (5.1) implies that $A_{11}[u] A_{22}[u] > E_1$. Then $A_{11}[u]$ and $A_{22}[u]$ never vanish and have the same sign. At a point where $u$ reaches its minimum value, we have $u_y = 0$ and $u_{yy} \leq 0$. Then
$$A_{22}[u] = u_{yy} + C_{22} > 0$$
and both $A_{11}[u]$ and $A_{22}[u]$ must be positive everywhere. \hfill \Box

Lemma 5.2. Consider a function $u \in C^2(\mathbb{T}^2)$ satisfying equation (5.1). Under the hypotheses (5.2) and (5.3) we have that
\begin{equation}
Du(x, y) \geq C_{11}, \quad \forall (x, y) \in \mathbb{R}^2.
\end{equation}

Proof. Consider a point where $Du$ attains its minimum value. Since $D \leq 0$, this corresponds to a point where $u$ reaches its maximum value. Then we have $u_y = 0$ and $u_{xx} \leq 0$ and from (5.9) we have
$$C_{11} + Du \geq u_{xx} + C_{11} + Du > 0,$$
which implies
$$Du \geq C_{11},$$
at the maximum and therefore everywhere. \hfill \Box

We need Lemma 6.3 of [2]:

Lemma 5.3. Consider $w \in C^2(\mathbb{T})$ and two real numbers $\alpha$ and $\beta$ such that
\begin{equation}
\begin{aligned}
w''(t) + \alpha w'(t) \geq \beta, \\
\forall t \in \mathbb{R}.
\end{aligned}
\end{equation}

Then we have
\begin{equation}
|w'(t)| \leq 2|\beta| e^{2|\alpha|}, \quad \forall t \in \mathbb{R}.
\end{equation}

Theorem 5.4. Assume hypotheses (5.2) and (5.5) are satisfied. Then all solutions of (5.1) satisfy the following estimate:
$$\|u\|_{C^2} \leq 2(|B_{11}| + 1) |B_{22}| e^{2C_{22}} + C_{11} + C_{22}.$$

Proof. From (5.9) we obtain that
$$u_{yy} + B_{22}u_y \geq -C_{22},$$
hence from Lemma 5.3 we obtain that
\begin{equation}
|u_y| \leq 2|B_{22}| e^{2C_{22}}.
\end{equation}
From (5.9), (5.10) and (5.13), we obtain
$$u_{xx} \geq -2|B_{11}B_{22}| e^{2C_{22}} - C_{11} - C_{22},$$
hence from Lemma 5.3 we obtain

\begin{equation}
|u_x| \leq 2|B_{11}B_{22}|e^{2C_{22}} + C_{11} + C_{22}.
\end{equation}

Now consider a point \((x_0, y_0) \in [0, 1] \times [0, 1]\) where \(u\) vanishes. Then we have

\[
\begin{align*}
    u(x, y) &= \int_0^1 u_x((1-t)x + tx_0, (1-t)y + ty_0) \, dt \, (x - x_0) \\
    &\quad + \int_0^1 u_y((1-t)x + tx_0, (1-t)y + ty_0) \, dt \, (y - y_0),
\end{align*}
\]

which, together periodicity, implies

\[
|u|_{C^0} \leq 2|u|_{C^1}.
\]

This estimate, together \((5.13)\) and \((5.14)\), implies

\[
|u|_{C^0} \leq 2(|B_{11}| + 1)|B_{22}|e^{2C_{22}} + C_{11} + C_{22}.
\]

Let \(\tau \in [0, 1]\) and set

\begin{equation}
\mathcal{S}_\tau = \left\{ u \in C^2(\mathbb{T}^2) \mid A_{11}[u]A_{22}[u] - (A_{12}[u])^2 = E_1 + (1-\tau)E_2 + \tau E_2e^F, \int_{\mathbb{T}^2} u = 0 \right\}.
\end{equation}

**Theorem 5.5.** Assume hypotheses \((5.2)\) to \((5.6)\) are satisfied. Then

\[\mathcal{S}_\tau \subset C^{2,1/2}(\mathbb{T}^2), \quad \forall \tau \in [0, 1],\]

and

\[
\sup_{0 \leq \tau \leq 1} \sup_{u \in \mathcal{S}_\tau} \|u\|_{C^{2,1/2}} < \infty.
\]

**Proof.** Thanks to lemma 5.1 and hypothesis \((5.3)\) the equation

\[
A_{11}[u]A_{22}[u] - (A_{12}[u])^2 = E_1 + (1-\tau)E_2 + \tau E_2e^F
\]

is uniformly elliptic and we can apply Theorem 2 of \([6]\). \(\square\)

**Corollary 5.6.** Under the same hypotheses of Theorem 5.5 we have that

\[\mathcal{S}_\tau \subset C^\infty(\mathbb{T}^2)\]

for all \(0 \leq \tau \leq 1\).

**Proof.** It follows from Theorems 1 and 3 of \([7]\). \(\square\)

**Theorem 5.7.** Under the hypotheses \((5.2)\) to \((5.7)\), we have that for all \(\tau \in [0, 1]\) the equation

\[
A_{11}[u]A_{22}[u] - (A_{12}[u])^2 = E_1 + (1-\tau)E_2 + \tau E_2e^F
\]

has a solution in \(C^\infty(\mathbb{T}^2)\) satisfying condition \((5.8)\).

In particular, for \(\tau = 1\) we obtain that equation \((5.1)\) is solvable.
Proof. In view of Corollary [5.6] it is sufficient to prove the existence of a $C^2$-solution. Hence we have to prove that $\mathcal{G}_\tau(\mathbb{T}^2) \neq \emptyset$ for all $\tau \in [0, 1]$. If $\tau = 0$, thanks to (5.5) we have that $0 \in \mathcal{G}_0$. Then we may set

$$\rho = \sup \{ \sigma \in [0, 1] \mid \mathcal{G}_\tau \neq \emptyset, \forall \tau \in [0, \sigma] \}$$

We must show that $\mathcal{G}_\rho \neq \emptyset$ and that $\rho = 1$.

Consider a sequence $\tau_n \in [0, \rho]$ converging to $\rho$ and such that $\mathcal{G}_{\tau_n} \neq \emptyset$. Let $u_n \in \mathcal{G}_{\tau_n}$ for all $n$. By Theorem [5.5] the sequence $(u_n)$ is bounded in $C^{2,1/2}(\mathbb{T}^2)$, hence, by Ascoli-Arzelà theorem, it contains a subsequence $(v_n)$ which converges in $C^2(\mathbb{T}^2)$ to a function $v$, which is a solution belonging to $\mathcal{G}_\rho$.

Now we show that $\rho = 1$. Assume by contradiction $\rho < 1$ and let $C^{k,1/2}_*(\mathbb{T}^2)$ be the space of functions $u \in C^{k,1/2}(\mathbb{T}^2)$ satisfying $\int_{\mathbb{T}^2} u = 0$. Consider the map

$$T : C^{2,1/2}_*(\mathbb{T}^2) \times [0, 1] \to C^{0,1/2}_*(\mathbb{T}^2),$$

defined as

$$T(u, \tau) = A_{11}[u]A_{22}[u] - (A_{12}[u])^2 - E_1 - (1 - \tau)E_2 - \tau E_2 e^\rho.$$

Observe that

$$\int_{\mathbb{T}^2} T(u, \tau) = 0,$$

thanks to (5.4), (5.5) and (5.7).

We know that there exists $v \in \mathcal{G}_\rho \subset C^{2,1/2}_*(\mathbb{T}^2)$ such that $T(v, \rho) = 0$. We have that

$$T'[v, \rho](w, 0) = Lw,$$

with

$$L : C^{2,1/2}_*(\mathbb{T}^2) \to C^{0,1/2}_*(\mathbb{T}^2)$$

given by

$$Lw = (A_{22}[v] + C_{22})w_{xx} - 2(A_{12}[v] + C_{12})w_{xy} + (A_{11}[v] + C_{11})w_{yy}$$

$$+ \left( B_{11}(A_{22}[v] + C_{22}) - 2B_{12}(A_{12}[v] + C_{12}) + B_{22}(A_{11}[v] + C_{11}) \right)w_y$$

$$+ D(A_{22}[v] + C_{22})w.$$ (5.16)

Now from Lemma [5.1] and hypotheses (5.3) the matrices

$$\begin{bmatrix} A_{11}[v] & A_{12}[v] \\ A_{12}[v] & A_{22}[v] \end{bmatrix}, \quad \begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix}$$

are positive, so their sum is positive too, and the operator $L$ is uniformly elliptic. Since $D(A_{22}[u] + A_{22}[v]) \leq 0$, we may apply the strong maximum principle ([5], Theorem 3.5) and obtain that $Lw = 0$ implies that $w$ is constant, that is $w = 0$, by condition (5.3). Ellipticity and classical Schauder estimates ([5], Theorem 6.2) show that $L$ is onto. Since $L$ is one-to-one, it must be an isomorphism. Then by the implicit function theorem there
exists $\epsilon > 0$ such that $\mathcal{S}_\rho(\mathbb{T}^2) \neq \emptyset$ for $\rho < \tau < \rho + \epsilon$, in contradiction with the definition of $\rho$. \hfill \Box

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