THE HOMOTOPY SEQUENCE OF NORI’S FUNDAMENTAL GROUP

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Abstract. In this paper, we investigate the necessary sufficient conditions for the exactness of the homotopy sequence of Nori’s fundamental group and apply these to various special situations to regain some classical theorems and give a counter example to show the conditions are not always satisfied. This work is partially based on the earlier work of H.Esnault, P.H.Hai, E.Viehweg.

1. Introduction

If \( f : X \to S \) is a separable proper morphism with geometrically connected fibres between locally noetherian connected schemes, \( x \to X \) is a geometric point with image \( s \to S \), Grothendieck shows in [Gr, Exposé X, Corollaire 1.4] that one has a homotopy exact sequence for the étale fundamental group:

\[
\pi^\text{ét}_1(\bar{X}_s, x) \to \pi^\text{ét}_1(X, x) \to \pi^\text{ét}_1(S, s) \to 1.
\]

A similar case is that one can take \( X, Y \) to be two locally noetherian connected \( k \)-schemes with \( k = \bar{k} \) and suppose \( Y \) is proper over \( k \), so if \( K \) is an algebraically closed field containing \( k \) and if we take a \( K \)-point \( z = (x, y) : \text{Spec}(K) \to X \times_k Y \), then we get a canonical morphism of topological groups

\[
\pi^\text{ét}_1(X \times_k Y, z) \to \pi^\text{ét}_1(X, x) \times \pi^\text{ét}_1(Y, y).
\]

Again Grothendieck shows in [Gr, Exposé X, Corollaire 1.7] that the canonical homomorphism is an isomorphism. This is called the Küneth formula for the étale fundamental group. If \( X \times_k Y \) admits a \( k \)-rational point then the Küneth formula is a direct consequence of the homotopy exact sequence.

Let \( X \) be a reduced connected locally noetherian scheme over a field \( k \), \( x \in X(k) \) be a rational point. Let \( N(X, x) \) be the category whose objects consist of triples \( (P, G, p) \) (where \( P \) is an FPQC \( G \)-torsor over \( X \), \( G \) is a finite group scheme, \( p \in P(k) \) is a \( k \)-rational point lying over \( x \)), whose morphisms are morphisms of \( X \)-schemes which intertwine the group actions and preserve the points. M.Nori proved in [Nori, Part I, Chapter II, Proposition 2] that the projective limit \( \varprojlim_{N(X, x)} G \) exists in the category of \( k \)-group schemes (in the projective system we associate to each index \( (P, G, p) \) the group \( G \)). Then he defined the fundamental group \( \pi^N(X, x) \) to be the projective limit \( \varprojlim_{N(X, x)} G \). If \( X \) is in addition proper over \( k \) and if \( k \) is perfect, Nori gave in [Nori, Part I, Chapter I] a Tannakian

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description of his fundamental group: he defined $\pi^N(X, x)$ to be the Tannakian group of the neutral Tannakian category of $\text{Ess}(X)$ (the essentially finite vector bundles on $X$) with the fibre functor $x^* : V \mapsto V|_x$, and he showed that this definition is the same as the one defined by the projective limit.

The main purpose of this article is to study the analogues of the homotopy sequence and Künneth formula for Nori’s fundamental group. Since in Nori’s fundamental group we only deal with rational points, if the homotopy sequence is exact then Künneth formula always automatically follows.

In [EHV][Section 2] H.Esnault, P.H.Hai, E.Viehweg, give a counterexample which shows that homotopy sequence of Nori’s fundamental group is not always exact even for $X \to S$ projective smooth and $S$ projective smooth as well. And then they give a necessary and sufficient condition for the exactness of the homotopy sequence of Nori’s fundamental group under the assumption that $S$ is a proper $k$-scheme. But unfortunately there is a gap in the argument for the necessary and sufficient condition. In this article, our first goal is to reformulate some similar conditions to make everything work. These works are contained in Theorem 2.1 and Theorem 3.1, where we correct the mistake, improve the arguments and make the wonderful ideas hidden in that article right and clean. The upshot is that in Theorem 2.1 we don’t have to assume $S$ to be proper, so the result applies to the general definition of Nori’s fundamental group.

Then we make two applications of Theorem 2.1 and Theorem 3.1. We first apply the criterion to show that the homotopy sequence for the étale quotient of Nori’s fundamental group is exact. The argument is independent of Grothendieck’s theory of the étale fundamental group which was developed in [Gr, Exposé X], so it can be seen as a new proof of the homotopy exact sequence for the étale fundamental group (in the language of Nori’s fundamental group).

In [MS][Theorem 2.3] V.B.Mehta and S.Subramanian proved that Künneth formula holds for Nori’s fundamental group if both $X$ and $Y$ are proper $k$-schemes. In §3, we apply Theorem 3.1 to give a neat proof for the Künneth formula of the local quotient of Nori’s fundamental group. This can be thought of as a new proof of [MS][Proposition 2.1] which is the key point for the proof of [MS][Theorem 2.3].

In the end of this paper, we give a counterexample to show that [MS][Theorem 2.3] does not work if $X$ or $Y$ is not proper. We prove that if $X = A_k^1$ and $Y = E$ is a supersingular elliptic curve and $k$ is an algebraically closed field in positive characteristic, then the Künneth formula is always false. But, in contrast, the Künneth formula always holds in this case for the étale fundamental group. This also provides another counter example to show the failure of the exactness of the homotopy sequence for Nori’s fundamental group.

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2. The General Criterion

Definition 1. Let $X$ be a reduced connected locally noetherian scheme over a field $k$, $x \in X(k)$ be a rational point. We call a triple $(P, G, p) \in N(X, x)$ a $G$-saturated torsor if the canonical map $\pi^N(X, x) \to G$ is surjective.

Definition 2. Let $f : X \to S$ be a map of schemes, $\mathcal{F}$ be a sheaf of $O_X$-modules, $s : \text{Spec}(\kappa(s)) \hookrightarrow S$ a point, then we get a Cartesian diagram:

$$
\begin{array}{ccc}
X_s & \xrightarrow{t} & X \\
\downarrow{g} & & \downarrow{f} \\
\text{Spec}(\kappa(s)) & \xrightarrow{s} & S
\end{array}
$$

We say $\mathcal{F}$ satisfies base change at $s$ if the canonical map

$$s^*f_*\mathcal{F} \to g_*t^*\mathcal{F}$$

is surjective. Note that if $f$ is proper, $S$ is locally noetherian, $\mathcal{F}$ is coherent and flat over $S$ then $\mathcal{F}$ satisfies base change at $s$ if and only if the above canonical map is an isomorphism (see [Hart][Chapter III, Theorem 12.11]).

Definition 3. We call a morphism of schemes $f : X \to S$ separable if it is flat and if for $\forall s \in S$ the fibre $X_s$ is geometrically reduced over $\kappa(s)$. [Gr][Exposé X, Définition 1.1]

Theorem 2.1. [EHV] (H.Esnault, P.H.Hai, E.Viehweg) Let $f : X \to S$ be a separable proper morphism with geometrically connected fibres between two reduced connected locally noetherian schemes over a perfect field $k$. We suppose further that $S$ is irreducible. Let $x \in X(k)$, $s \in S(k)$ and assume $f(x) = s$. Then the following conditions are equivalent:

1. the sequence

$$\pi^N(X_s, x) \to \pi^N(X, x) \to \pi^N(S, s) \to 1$$

is exact;

2. for any $G$-saturated torsor $(P, G, p)$ with structure map $\pi : P \to X$, $\pi_*O_P$ satisfies base change at $s$ and the image of the composition $\pi^N(X_s, x) \to \pi^N(X, x) \to G$ is a normal subgroup of $G$;

3. for any $G$-saturated torsor $(P, G, p)$ with structure map $\pi : P \to X$, $\pi_*O_P$ satisfies base change at $s$ and there is a $G'$-saturated torsor $\pi' : P' \to S$ together with a morphism $(P, G) \xrightarrow{\theta} (P', G')$ satisfy that the $\theta$-induced map $(\pi'_*O_{P'})_s \to (f_*\pi_*O_P)_s$ is an isomorphism.

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In the original article [EHV] the statement was that the exactness is equivalent to base change. But it seems that only base change is not enough to deduce the exactness, so we add extra conditions to make the argument. In addition, we removed the properness assumption on $S$ in the original statement.
Proof. "(1) \implies (2)" If the homotopy sequence is exact then clearly the image of \( \pi^N(X_s, x) \to \pi^N(X, x) \to G \) (which is denoted by \( H \)) is normal in \( G \). The exactness also gives us a commutative diagram

\[
\begin{array}{ccc}
\pi^N(X, x) & \to & \pi^N(S, s) \\
\downarrow & & \downarrow \\
G & \to & G/H
\end{array}
\]

This commutative diagram gives us a \( G/H \)-saturated torsor \( (P', G/H, p') \) over \( S \) and a morphism in \( N(X, x) \):

\[
\lambda : (P, G, p) \to (P' \times_S X, G/H, p' \times_S X) \cong (P/H, G/H, p).
\]

Let \( W' \) be the push forward of the structure sheaf of \( P' \) to \( S \), \( V := \pi_*O_P, W := f^*W' \). Let \( \lambda^* : W \to V \) be the map induced by \( \lambda \). If we pull-back \( \lambda^* \) to \( X_s \) then we get a morphism in the category of essentially finite vector bundles because \( V|_{X_s} \) (resp. \( W|_{X_s} \)) is the 0-th direct image of the structure sheaf of the torsor \( P \times_X X \) (resp. \( P' \times_S X \)). From [Nori][Part I, Chapter I, Proposition 2.9], this \( \lambda^* \) corresponds, via Tannakian duality, to the morphism

\[
k[G]^{\pi^N(X_s, x)} = k[G]^H = k[G/H] \to k[G]
\]

in the category of \( \text{Rep}_k(\pi^N(X_s, x)) \). Hence \( W|_{X_s} \) is the maximal trivial subbundle of \( V|_{X_s} \). But \( H^0(X_s, V|_{X_s}) \otimes_k O_{X_s} \subseteq V|_{X_s} \) is the maximal trivial subbundle (see lemma 2.2 below), thus the canonical map

\[
W|_{X_s} = H^0(X_s, W|_{X_s}) \otimes_k O_{X_s} \to H^0(X_s, V|_{X_s}) \otimes_k O_{X_s}
\]

is an isomorphism. But note that the above map factors \( W|_{X_s} \to f_*f^*W|_{X_s} \). This implies \( f_*f^*W|_{X_s} \to H^0(X_s, W|_{X_s}) \otimes_k O_{X_s} \) is an isomorphism, so base change is satisfied.

"(2) \implies (3)" Let \( H \subseteq G \) be the image of the composition \( \pi^N(X_s, x) \to \pi^N(X, x) \to G \). Since it is normal we get a \( G/H \)-torsor \( P/H \) on \( X \). If \( W \) is the push-forward of the structure sheaf of \( P/H \) to \( X \) and \( V := \pi_*O_P \), then we know from our assumption that \( W \) and \( V \) satisfy base change at \( s \). Let \( \lambda : W \to V \) be the imbedding induced from \( P \to P/H \), then we have the following commutative diagram of sheaves on \( X_s \):

\[
\begin{array}{ccc}
f_*f^*W|_{X_s} & \xrightarrow{a_1} & H^0(X_s, W|_{X_s}) \otimes_k O_{X_s} \xrightarrow{a_2} W|_{X_s} \\
\downarrow f_*f_*\lambda & & \downarrow H^0(X_s, \lambda|_{X_s}) \\
f_*f^*V|_{X_s} & \xrightarrow{a_3} & H^0(X_s, V|_{X_s}) \otimes_k O_{X_s} \xrightarrow{a_4} V|_{X_s}
\end{array}
\]

By base change \( a_1, a_2 \) are isomorphisms. Since \( \lambda|_{X_s} \) corresponds via Tannakian duality to \( k[G]^H \to k[G] \) (in the category \( \text{Rep}_k(\pi^N(X_s, x)) \)), \( W|_{X_s} \) is imbedded as the maximal trivial subbundle of \( V|_{X_s} \). Hence \( a_2 \) and \( H^0(X_s, \lambda|_{X_s}) \) are isomorphisms. So \( f_*f^*\lambda \) is also an isomorphism. In particular

\[
(f_*\lambda)_s : (f_*W)_s \to (f_*V)_s
\]
is an isomorphism. Let \( r \in \mathbb{N} \) be the rank of \( W \). For any point \( t \in S \), by applying lemma 2.2 to \( X_t \times_{\kappa(t)} \kappa(t)/\kappa(t) \) we know that the following map is an imbedding:

\[
H^0(X_t, W|_{X_t}) \otimes_{\kappa(t)} O_{X_t} \to W|_{X_t}.
\]

So \( \dim_{\kappa(t)}(H^0(X_t, W|_{X_t})) \leq r \). But on the other hand, since \( W \) satisfies base change at \( s \), \( r = \dim_k(H^0(X_s, W|_{X_s})) \) reaches the minimal dimension (the dimension at the generic point), so by semi-continuity theorem we have

\[
\dim_{\kappa(t)}(H^0(X_t, W|_{X_t})) \geq \dim_k(H^0(X_s, W|_{X_s})) = r.
\]

This implies \( H^0(X_t, W|_{X_t}) \) has constant dimension \( r \), and hence \( W \) satisfies base change all over \( S \). So \( f_sW \) a vector bundle. Since \( f^*f_sW \to W \) is injective after restricting to all the points of \( X \), we have it is an embedding as a subbundle (i.e. injectively and locally split). But since \( a_1, a_2 \) are isomorphisms, we have \( f^*f_sW \to W \) is an isomorphism. Now we can check easily that \( \text{Spec}(f_sW) \to S \) with the canonical \( G/H \)-action induced from \( P/H \) is an FPQC-torsor which satisfies all our conditions in (3).

"(3) \implies (1)" Let \( (P, G, p) \) be any \( G \)-saturated torsor over \( X \), by the assumption \( \pi_*O_P \) satisfies base change and there is a \( G' \)-torsor \( P' \) over \( S \) with a morphism \( \theta : (P, G) \to (P', G') \) which satisfies the conditions in (3). Let \( N \) be the image of \( \text{Ker}(\pi^N(f)) \) in \( G \) (where \( \pi^N(f) \) is the map \( \pi^N(X, x) \to \pi^N(S, s) \)), \( N' \) be the kernel of \( G \to G' \), and \( H \subseteq G \) be the image of the composition \( \pi^N(X_s, x) \to \pi^N(X, x) \to G \). We also write \( W := \pi_*O_{P'} \) and \( V := \pi_*O_P \). We first note that the \( \theta \)-induced map \( f^*W|_{X_s} \to V|_{X_s} \) corresponds to \( k[G/N'] \to k[G] \) in \( \text{Rep}_k(\pi^N(X_s, x)) \). But from base change of \( V \) and the fact that the \( \theta \)-induced map \( W_s \to (f_*V)_s \) is an isomorphism we know that \( f^*W|_{X_s} \to V|_{X_s} \) should be the same as \( H^0(X_s, V|_{X_s}) \otimes_k O_{X_s} \to V|_{X_s} \) as subobjects. Thus the canonical imbedding \( k[G/N'] \to k[G/H] \) should be an isomorphism. Hence \( N' = H \) as subgroups. But since we have \( H \subseteq N \subseteq N' \), so \( H = N \) as well. Because the equality holds for all \( G \)-saturated torsor \( (P, G, p) \), we have \( \pi^N(X_s, x) \to \text{Ker}(\pi^N(f)) \) is surjective. This completes the proof. \( \square \)

**Lemma 2.2.** If \( X \) is a reduced connected proper scheme over a perfect field \( k \) with a point \( x \in X(k) \), then for any essentially finite vector bundle \( V \) on \( X \) the canonical morphism \( \Gamma(X, V) \otimes_k O_X \to V \) imbeds \( \Gamma(X, V) \otimes_k O_X \) as the maximal trivial subbundle of \( V \).

**Proof.** Let \( \text{Ess}(X) \) be the category of essentially finite vector bundles, \( \omega_x : \text{Ess}(X) \to \text{Vec}_k \) be the fibre functor. Then applying \( \omega_x \) to the canonical morphism \( \Gamma(X, V) \otimes_k O_X \to V \) we get \( \text{Hom}_{O_X}(O_X, V) \cong \Gamma(X, V) \to V_x \otimes_{O_{X,x}} k = \omega_x(V) \). But note that we have \( \text{Hom}_{O_X}(O_X, V) \cong \text{Hom}_{\pi^N(X_x)}(k, \omega_x(V)) \) where \( k \) stands for the dim 1 vector space with trivial \( \pi^N(X, x) \) action. One checks readily that under these isomorphisms we get exactly the canonical injection \( \text{Hom}_{\pi^N(X_x)}(k, \omega_x(V)) \to \omega_x(V) \) sending any morphism \( k \to \omega_x(V) \) to the image of 1 \( \in k \). Since this map imbeds \( \text{Hom}_{\pi^N(X_x)}(k, \omega_x(V)) \) as the maximal trivial sub of \( \omega_x(V) \). Using Tannakian duality we get our result. \( \square \)

**2.1. Application to the étale quotient.**

**Definition 4.** Let \( X \) be a connected reduced locally noetherian scheme over a perfect field \( k \) which admits a rational point \( x \in X(k) \). Let \( N^\text{ét}(X, x) \) be the full subcategory of
$N(X, x)$ whose objects consist of those $(P, G, p)$ with $G$ finite étale. This sub category is filtered so we can define the étale quotient of $\pi^N(X, x)$ to be $\pi^\text{ét}(X, x) := \varprojlim_{N \in \text{ét}(X, x)} G$.

We have an obvious surjection: $\pi^N(X, x) \twoheadrightarrow \pi^\text{ét}(X, x)$.

**Lemma 2.3.** Let $X$ be a geometrically connected reduced locally noetherian scheme over a perfect field $k$ which admits a rational point $x \in X(k)$. Let $(P, G, p)$ be an étale torsor over $(X, x)$. This torsor is $G$-saturated if and only if $P$ is geometrically connected.

**Proof.** Since the formation of Nori’s fundamental group is compatible with separable field extensions, we can assume $k = \bar{k}$. [Nori][Part I, Chapter II, Proposition 5]

“⇒” Let’s take $Q \subseteq P$ to be the connected component of $P$ containing $p$. Now $G$ is an abstract group we can write the action $\rho : P \times_k G \to P$ as $\coprod G P \to P$ where each component in the direct union is mapped to $P$ via a unique element in $G$. Since $P$ is an $G$-torsor we have the following cartesian diagram:

$$
\begin{array}{ccc}
\coprod G P & \overset{\rho}{\longrightarrow} & P \\
\downarrow{id^G} & & \downarrow{id} \\
P & \longrightarrow & X
\end{array}
$$

If we let $H \subseteq G$ be the maximal subgroup of $G$ which fix $Q$, then we can see by definition that $Q \times_k H \subseteq P \times_k G$ is the intersection of $\rho^{-1}(Q)$ and $(id^G)^{-1}(Q)$. Thus the square

$$
\begin{array}{ccc}
\coprod H Q & \overset{\rho}{\longrightarrow} & Q \\
\downarrow{id^H} & & \downarrow{id} \\
Q & \longrightarrow & X
\end{array}
$$

is cartesian. Hence $Q$ is an $H$-torsor. But from the assumption the imbedding $H \to G$ should be surjective. This tells us $H = G$. But then the map of $G$-torsors $Q \subseteq P$ should also be an isomorphism. So $P$ is connected.

“⇐” Let $(P', G', p') \to (P, G, p)$ be any morphism in $N(X, x)$. Since $P \to X$ is étale, we know $P' \to P$ is finite flat. Thus the image must be both open and closed, and hence it must be the whole of $P$. But if we pull-back the surjective map $P' \to P$ via $x \in X(k)$, we will get the group homorphism $G' \to G$. Thus this homomorphism must be surjective. Since $(P', G', p')$ is taken arbitrarily, it actually shows that $(P, G, p)$ is $G$-saturated. □

**Corollary 2.4.** Let $f : X \to S$ be a separable proper morphism with geometrically connected fibres between two reduced geometrically connected locally noetherian schemes over a perfect field $k$. Let $x \in X(k), s \in S(k)$ and assume $f(x) = s$. Then the homotopy sequence:

$$
\pi^\text{ét}(X, x) \to \pi^\text{ét}(X, s) \to 1
$$

is exact.

**Proof.** Without loss of generality one may assume $k = \bar{k}$. [Nori][Part I, Chapter II, Proposition 5]. Now let $(P, G, p)$ be a $G$-saturated étale torsor over $X$, $\pi : P \to X$ be the
structure map $V := \pi_*O_P$. Let $P \xrightarrow{\phi} Q \xrightarrow{\varpi} S$ be the Stein factorization of the proper map $P \xrightarrow{f} X \xrightarrow{f} S$. Since $f \circ \pi$ is proper separable $\varpi$ is finite étale \cite{EGA}[7.8.10 (i)]. Thus $\phi$ is proper separable surjective with geometrically connected fibres. But then the pull back $\phi_s : P_s \rightarrow Q_s$ along the rational point $s \rightarrow S$ is also proper separable surjective with geometrically connected fibres. Hence $O_{Q_s} \rightarrow (\phi_s)_*O_{P_s}$ is an isomorphism \cite{EGA}[7.8.6]. But since $\varpi : Q \rightarrow S$ is affine, $H^0(Q_s, O_{Q_s}) \cong \varpi_*O_{Q}|_s \cong f_*\pi_*O_P|_s = f_*V|_s$. Thus base change is satisfied for $V$ at $s$.

The action $P \times_k G \rightarrow P$ induces a map $V \rightarrow V \otimes_k k[G]$. Push it to $S$ we get $f_*V \rightarrow f_*V \otimes_k k[G]$. Thus there is an action of $G$ on $Q = \text{Spec}_{O_S}(f_*V)$ which makes $\phi : P \rightarrow Q$ $G$-equivariant. If we pull back the map $P \rightarrow Q \times_S X$ along the rational point $x \in X(k)$, we get a $G$-equivariant map $t : G \rightarrow G'$, where we identify $G$ with $P \times_X x$ via the rational point $p \in P$ and $G' := Q \times_S X \times_X x = Q_s$ is a $G$-set with a distinguished point $q := t(e)$. Since $t$ is $G$-equivariant, the subgroup $H := t^{-1}(q)$ of $G$ is the stabilizer of $q$. Now let $h \in H$ be an element. Consider the $S$-isomorphism $Q \rightarrow Q$ induced by $h$. Evidently $h$ sends $q$ to $q$, and since $Q$ is a connected finite étale cover of $S$, the $S$-isomorphism induced by $h$ must be the identity \cite{Gr}[Exposé I, Corollaire 5.4]. Hence $H$ acts trivially on $Q$ and in particular it also acts trivially on $G'$. But since $t : G \rightarrow G'$ is surjective, $G'$ is the quotient of $G$ by $H$, and $t$ is the quotient map. And also for any $x \in G$, we have $qxhx^{-1} = t(x)hx^{-1} = t(x)x^{-1} = t(x)x^{-1} = q$, so $H$ is a normal subgroup of $G$. Thus $G'$ is the quotient group of $G$ by $H$. The following commutative diagram:

$$
\begin{array}{c}
P \times_k G \xrightarrow{\sim} P \times_X P \\
\downarrow \\
Q \times_k G' \xrightarrow{\rho} Q \times_S Q
\end{array}
$$

tells us that $\rho$ is a finite étale surjective $Q$-morphism. Let $r$ be the degree of the connected finite étale cover $\varpi : Q \rightarrow S$. Then one sees easily that both $Q \times_k G'$ and $Q \times_S Q$ are finite étale of degree $r$ over $Q$. Thus $\rho$ must be an isomorphism for it is an isomorphism on all the geometric fibres of $Q$. Because the Now $\varpi : Q \rightarrow S$ has a structure of a $G'$-torsor which satisfies all the conditions in (3) of our main theorem. So we can use the same argument we have used in ”(3) \implies (1)” to conclude our proof. \hfill \square

3. The Proper Case

Theorem 3.1. (H.Esnault, P.H.Hai , E.Viehweg) Let $f : X \rightarrow S$ be a proper separable morphism with geometrically connected fibres between two reduced connected proper schemes over a perfect field $k$, $x \in X(k)$, $s \in S(k)$, $f(x) = s$. Assume further that $S$ is irreducible. Then the homotopy sequence

$$
\pi^N(X_s, x) \rightarrow \pi^N(X, x) \rightarrow \pi^N(S, s) \rightarrow 1
$$

\footnote{In the original article \cite{EHV} the statement was that the exactness is equivalent to base change. We add the condition that $f_*V$ is essentially finite to complete the argument.}
is exact if and only if for any $G$-saturated torsor $(P, G, p) \in N(X, x)$ with structure map $\pi : P \to X$, $V := \pi_*O_P$ satisfies base change at $s$ and $f_*V$ is essentially finite.

**Proof.** " $\iff$ " Since $f_*V$ satisfies base change, the canonical map $f^*f_*V \to V$ is of the form

$$\Gamma(X_s, V|_{X_s}) \otimes_k O_{X_s} \to V|_{X_s}$$

after restricting to the fibre $X_s$. Because $f^*f_*V \to V$ is a map of essentially finite vector bundles, the kernel of it is also a vector bundle. But the kernel is trivial on $X_s$, so the kernel itself is trivial. Thus $f^*f_*V \subseteq V$ is a subobject in the category of essentially finite vector bundles on $X$ and it becomes the maximal trivial subobject after restricting to $X_s$. Now let $G'$ be the Tannakian group of the sub Tannakian category of $\text{Ess}(S)$ generated by $f_*V$. The imbedding $f^*f_*V \to V$ gives us a surjection $\lambda : G \to G'$. Let $H$ be the kernel of $\lambda$. Then $f^*f_*V \to V$ corresponds via Tannakian duality to an inclusion $M \subseteq k[G]$ in $\text{Rep}_k(G)$. Note that since $M$ comes from an object in $\text{Rep}_k(G')$ via $\lambda : G \to G'$, so $M \subseteq k[G]$ factors through the inclusion $k[G]^H \subseteq k[G]$. On the other hand, since we have a surjection $\pi^N(S, s) \to G'$, by [Nori] [Chapter I, Proposition 3.11] we have a $G'$-saturated torsor $(P', G', p') \in N(S, s)$ with a map

$$\theta : (P, G, p) \to f^*(P', G', p')$$

in $N(X, x)$ extending $\lambda$. Let $V' := \pi'_*O_{P'}$, $\pi' : P' \to S$. Then since $P \to f^*P' \cong P/H$ is faithfully flat, $f^*V' \subseteq V$ is a subbundle, and this subbundle corresponds via Tannakian duality to the inclusion $k[G]^H \subseteq k[G]$. But clearly $f^*V' \subseteq V$ factors $f^*f_*V \to V$, so $k[G]^H \subseteq k[G]$ factors $M \subseteq k[G]$, which means $k[G]^H = M$. So we have $V' \cong f_*V$. Now the triple $(P', G', p')$ satisfies all our conditions in Theorem 2.1 (3), so we get the exact sequence.

" $\implies$ " By Theorem 2.1 we have a $G'$-saturated torsor $(P', G', p') \in N(S, s)$ and a morphism

$$\theta : (P, G, p) \to f^*(P', G', p') \in N(X, x)$$

such that the induced map $V'_s \to (f_*V)_s$ is an isomorphism, where $V' := \pi'_*O_{P'}$ and $\pi' : P' \to S$ is the structure map. Because $V$ satisfies base change at $s$, there is a neighborhood $s \in U$ such that $f_*V$ is a vector bundle on $U$ and the adjunction map $f^*f_*V \to V$ is an imbedding of subbundles (locally split) on $f^{-1}(U)$. Since $P \to f^*P'$ is finite faithfully flat, the induced map $f^*V' \to V$ is an imbedding of subbundles over $X$ (i.e. $V/f^*V'$ is a vector bundle on $X$). Because $f^*V' \to V$ factors through the adjunction map, we get a map $f^*V' \to f^*f_*V$ which is an imbedding of subbundles on $f^{-1}(U)$. Since $V'_s \cong (f_*V)_s$, $f^*V' \to f^*f_*V$ is an isomorphism on $f^{-1}(U)$. Hence the injective map $V' \to f_*V$ is also an isomorphism on $U$. Now by [EGA] [Théorème 7.7.6] there is a coherent sheaf $\mathcal{Q}$ on $S$ such that

$$f_*V/f^*V' \cong \mathbb{H}\text{om}_{O_S}(\mathcal{Q}, O_S).$$

Since locally $\mathbb{H}\text{om}_{O_S}(\mathcal{Q}, O_S)$ is contained in a vector bundle and by applying the left exact functor $f_*$ to the sequence $0 \to f^*V' \to V \to V/f^*V' \to 0$ we have

$$f_*V/V' \subseteq f_*(V/f^*V') = \mathbb{H}\text{om}_{O_S}(\mathcal{Q}, O_S),$$
so if there is \( t \in S \setminus U \) such that \((f_*V/V')_t \neq 0\), then we can choose an open affine \( t \in \text{Spec}(A) \subseteq S \) such that

\[
(f_*V/V')_{|_{\text{Spec}(A)}} \subseteq \bigoplus_{i=0}^n A_i,
\]

where \( A_i \) is a rank 1 free \( A \)-module for all \( 0 \leq i \leq n \). Notice that since \( S \) is integral and \( \text{Spec}(A) \) is non-empty, so \( A \) is an integral ring. This implies \( f_*V/V' \) is non-zero at the generic point which contradicts to the fact that \( f_*V/V' \) has support in \( S \setminus U \). So \( V' \to f_*V \) is an isomorphism on \( S \). But \( V' \) is certainly essentially finite. This completes the proof. \( \square \)

3.1. Application to the Künneth formula.

**Definition 5.** Let \( X \) be a reduced connected locally noetherian scheme over a field \( k \) with a rational point \( x \in X(k) \). Let \( N^F(X, x) \) be the full subcategory of \( N(X, x) \) whose objects consist of pointed torsors with finite local groups. This category is also filtered so we can write \( \pi^F(X, x) := \lim_{\leftarrow} N^F(X, x) \). If \( X \) is also proper and \( k \) is perfect, then \( \pi^F(X, x) \) is the Tannakian group of the full subcategory of the category of essentially finite vector bundles \( \text{Ess}(X) \) consisting of \( F \)-trivial bundles, i.e. vector bundles which are trivial after pull back along some relative Frobenius \( \phi_{(-t)} : X^{(-t)} \to X \) with \( t \in \mathbb{N} \).

**Corollary 3.2.** Let \( X \) and \( Y \) be two reduced connected proper schemes over a perfect field \( k \). Let \( x \in X(k), y \in Y(k) \). Then the canonical map

\[
\pi^F(X \times_k Y, (x, y)) \to \pi^F(X, x) \times_k \pi^F(Y, y)
\]

is an isomorphism of \( k \)-group schemes.

**Proof.** As usual we may assume \( k = \bar{k} \). We will use the obvious analogues of Theorem 3.1 to prove this theorem. Note that after replacing \( \pi^N(X, x) \) by \( \pi^F(X, x) \), "torsor" by "local torsor" (torsors whose groups are local), essentially finite vector bundle by \( F \)-trivial vector bundle, Theorem 2.1 and Theorem 3.1 are still true.

To prove this corollary we only need to show that the sequence

\[
1 \to \pi^F(Y, y) \to \pi^F(X \times_k Y, (x, y)) \to \pi^F(X, x) \to 1
\]

is exact. So we have to check that for any \( G \)-saturated local torsor \( (P, G, p) \in N^F(X, x) \), \( V := \pi_*O_P \) (\( \pi : P \to X \) is the structure map) satisfies base change at \( x \) and \( f_*V \) is an \( F \)-trivial vector bundle.

Now suppose that \( V \) is trivialized by \( X^{(-t)} \times_k Y^{(-t)} = (X \times_k Y)^{(-t)} \to X \times_k Y \).

Consider the following commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{x} & X^{(-t)} \times_k Y \\
\downarrow & & \downarrow_{\phi_{(-t)} \times \text{id}} \\
Y & \xrightarrow{x} & X \times_k Y \\
\end{array}
\]

where the vertical arrows are base change and the horizontal arrows are pull-backs.
Let $W$ be the pull back of $V$ via $X^{(-t)} \times_k Y \to X \times_k Y$. Let $W$ be the pull back of $V$ via $X^{(-t)} \times_k Y \to X \times_k Y$. Since $W$ has trivial fibres along the projection $p_2 : X^{(-t)} \times_k Y \to Y$ and $X^{(-t)}$ is proper separable and geometrically connected scheme, if we set $E := p_{2*}W$ then by [Mum] [Chapter 2, §5, Corollary 2] we see that $E$ is a vector bundle and that the canonical map $p_2^*E \to W$ is an isomorphism. Hence for all closed point of $X$ (or equivalently $X^{(-t)}), V|_Y \cong W|_Y \cong E$ i.e. $V$ has constant fibres along $f : X \times_k Y \to X$. Consequently base change is satisfied for $V$ along $f$ (at any point of $X$). On the other hand we have the following trivial cartesian diagram

$$
\begin{array}{ccc}
X^{(-t)} \times_k Y & \xrightarrow{p_2} & Y \\
\downarrow p_1 & & \downarrow b \\
X^{(-t)} & \xrightarrow{a} & \text{Spec } (k)
\end{array}
$$

where $a$ and $b$ are structure maps. Because of base change we have $a^*b_*E \cong p_{1*}p_2^*E$. This implies $p_{1*}W = p_{1*}p_2^*E$ is a trivial vector bundle. But since base change is satisfied for $V$ along $f$, so we have a canonical isomorphism

$$p_{1*}W = p_{1*}(\phi_{(-t)} \times id)^*V \cong \phi_{(-t)}^*f_*V.
$$

Thus $\phi_{(-t)}^*f_*V$ is a trivial vector bundle. By definition $f_*V$ is $F$-trivial. \qed

Remarks 3.3. (1) Here we didn’t assume $X$ or $Y$ is irreducible, this is because we have only used the sufficiency part of Theorem 3.1 in which only the citation of Theorem 2.1 used the irreducibility. But we only used (3) $\Rightarrow$ (1) part of Theorem 2.1 where the irreducibility plays no role.

(2) This corollary gives another way to see [MS] [Proposition 2.1] which is the key point in the proof of the Künneth formula for Nori’s fundamental group. But unfortunately, for the full proof of Künneth formula we have to use the same trick employed in [MS] to reduce the problem for $\pi^N$ to the problem for $\pi^F$. At the moment, I can not find any easy way to reduce the problem to $\pi^F$ using our language here.

4. A COUNTEREXAMPLE

Lemma 4.1. Let $X$ and $Y$ be two reduced connected schemes locally of finite type over an algebraically closed field $k$. Let $x \in X(k), y \in Y(k)$. If the canonical map

$$
\pi^N(X \times_k Y, (x, y)) \to \pi^N(X, x) \times_k \pi^N(Y, y)
$$

is an isomorphism of $k$-group schemes, then for any $(P, G, p) \in N(X \times_k Y, (x, y))$ and any other point $x' \in X(k)$, the restriction of $(P, G, p)$ to $Y$ along $x$ and $x'$ are isomorphic.

Proof. By the assumption we have a $(P_1, G_1, p_1) \in N(X, x)$ and $(P_2, G_2, p_2) \in N(Y, y)$ and a morphism

$$(P_1 \times_k P_2, G_1 \times_k G_2, p_1 \times p_2) \to (P, G, p)$$
in $N(X \times_k Y, (x, y))$. But the restrictions of $(P_1 \times_k P_2, G_1 \times_k G_2, p_1 \times p_2)$ to $Y$ along $x$ and $x'$ are all isomorphic to $(G_1 \times_k X \times_k P_2, G_1 \times_k G_2, e \times p_2)$. This implies the restrictions of $(P, G, p)$ are all isomorphic to

$$((G_1 \times_k X \times_k P_2) \times (G_1 \times_k G_2), G, p)$$

the contracted product. This finishes the proof. □

Now consider $k$ an algebraically closed field of characteristic 2, $X = \mathbb{A}^1_k$, $Y$ is a supersingular elliptic curve over $k$. In the following we will construct an $\alpha_2$-torsor $\pi': Q \to X \times_k Y$ in FPQC-topology, and we will show that the restrictions of $Q$ to $Y$ along the two rational points $x = 0$ and $x = 1$ in $X$ can not be isomorphic. This shows by our lemma that the canonical map

$$\pi^N(X \times_k Y, (x, y)) \to \pi^N(X, x) \times_k \pi^N(Y, y)$$

could not be an isomorphism for any rational point $y \in Y$ and $x = 0$ or $x = 1$.

**Construction.** Suppose $\pi: P \to Y$ be a non-trivial $\alpha_2$-torsor in FPQC-topology. Note that we can choose $\pi: P \to Y$ to be the Frobenius endomorphism $F: Y \to Y$, it naturally carries a translation by $Y[F] \cong \alpha_2$ which makes it into a non-trivial $\alpha_2$-torsor.

Let $V := \pi_*O_P$. Let $\mathcal{L}$ be the cokernel of the structure map $O_Y \to V$, then $\mathcal{L}$ is an essentially finite line bundle, so it has degree 0. Since $H^1(Y, \mathcal{L}^{-1}) = \text{Ext}^1(O_Y, \mathcal{L}^{-1}) \neq 0$, so by Riemann-Roch $h^0(Y, \mathcal{L}^{-1}) = h^1(Y, \mathcal{L}^{-1}) \neq 0$, but this implies $\mathcal{L}^{-1}$ is $O_Y$, hence we have $\mathcal{L} \cong O_Y$. This gives us an exact sequence

$$0 \to O_Y \to V \to O_Y \to 0. $$

We know that $P \to Y$ already becomes a trivial torsor after pulling back along the relative Frobenius $\phi_{(-1)}: Y^{(-1)} \to Y$. Thus after choosing a section $Y^{(-1)} \to P \times_Y Y^{(-1)}$ we get an $Y^{(-1)}$-scheme isomorphism $\alpha_2 \times_k Y^{(-1)} \cong P \times_Y Y^{(-1)}$ which gives us an isomorphism of $O_{Y^{(-1)}}$-algebras

$$\delta: \phi_{(-1)}^*V \xrightarrow{\cong} O_{Y^{(-1)}}[T]/T^2$$

making the diagram (*):

$$\begin{array}{ccc}
0 & \longrightarrow & O_{Y^{(-1)}} \\
\downarrow & & \downarrow \delta \\
0 & \longrightarrow & O_{Y^{(-1)}}[T]/T^2 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & O_{Y^{(-1)}}[T]/T^2 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & O_{Y^{(-1)}} \\
\end{array}$$

commutative. By Grothendieck’s FPQC descent theory there is an essentially unique isomorphism of $O_{Y^{(-1)} \times_Y Y^{(-1)}}$-algebras $\varepsilon$ corresponding to $V$

$$\begin{array}{ccc}
p_1^*\phi_{(-1)}^*V & \cong & p_2^*\phi_{(-1)}^*V \\
\varepsilon^{-1} & & \varepsilon \\
O_{Y^{(-1)} \times_Y Y^{(-1)}}[T]/T^2 & \longrightarrow & O_{Y^{(-1)} \times_Y Y^{(-1)}}[T]/T^2 \\
\end{array}$$
where $p_1$ and $p_2$ are the two projections of $Y^{(-1)} \times_Y Y^{(-1)}$. From the commutative diagram (*) we know that this $\varepsilon$ is expressible by a matrix

$$
\begin{pmatrix}
1 & a \\
0 & 1
\end{pmatrix}
$$

in $GL_2(\Gamma(Y^{(-1)} \times_Y Y^{(-1)}, O_{Y^{(-1)} \times_Y Y^{(-1)}}))$ regarding $\{1, T\}$ as a basis for $O_{Y^{(-1)} \times_Y Y^{(-1)}}[T]/T^2$. Since $P$ is not a trivial torsor $a \neq 0$. Moreover since $T \mapsto a + T$ by $\varepsilon$ we have $(T + a)^2 = 0$, thus $a^2 = T^2 + a^2 = (T + a)^2 = 0$.

Let $x$ be the indeterminate in $X = \mathbb{A}^1_k = \text{Spec}(k[x])$. Let $A := X \times_k Y^{(-1)} \times_Y Y^{(-1)}$, $\mathcal{A} := O_{X \times_k Y^{(-1)} \times_Y Y^{(-1)}}$ and $T$ be an indeterminate. Then the $2 \times 2$-matrix:

$$
\begin{pmatrix}
1 & ax \\
0 & 1
\end{pmatrix}
$$

in $GL_2(A, O_A)$ determines an isomorphism

$$
\varepsilon' : A[T]/T^2 \xrightarrow{\cong} A[T]/T^2
$$

One checks readily that the pair

$$(A[T]/T^2, \varepsilon')$$

gives us a descent data of affine schemes, i.e. the cocycle condition is satisfied and the isomorphism $\varepsilon'$ is an automorphism of the $A$-algebra $A[T]/T^2$. So it gives us an affine scheme $\pi' : Q \to X \times_k Y$, which is obviously finite and faithfully flat.

Next we will check that $Q$ is an $\alpha_2$-torsor in FPQC topology. Let $S := T$. Now we consider the commutative diagram of $A$-algebras

$$
\begin{array}{ccc}
A[T]/T^2 \otimes_A A[S]/S^2 & \xrightarrow{\varepsilon' \otimes id} & A[T]/T^2 \otimes_A A[S]/S^2 \\
\downarrow \lambda & & \downarrow \lambda \\
A[T]/T^2 \otimes_A A[S]/S^2 & \xrightarrow{\varepsilon' \otimes id} & A[T]/T^2 \otimes_A A[S]/S^2
\end{array}
$$

where $\lambda$ denotes the sheaf version of the canonical map

$$
\alpha_{2,A} \times_A \alpha_{2,A} \xrightarrow{\cong} \alpha_{2,A} \times_A \alpha_{2,A}
$$

sending $(x, y) \mapsto (x, xy)$. This commutative diagram means that $\lambda$ defines an isomorphism between the descent data of $Q \times_k \alpha_{2,k}$ and $Q \times_{X \times_k Y} Q$. Let

$$
\rho : Q \times_k \alpha_{2,k} \to Q \times_{X \times_k Y} Q
$$

be the corresponding isomorphism. It is clear that the composition of $\rho$ with the second projection of $Q \times_{X \times_k Y} Q$ defines an action

$$
\rho : Q \times_k \alpha_{2,k} \to Q
$$

$$(q, g) \mapsto q \cdot g$$
of $\alpha_2$ on $Q$, so $\rho$ is the canonical map $(q, g) \mapsto (q, q \cdot g)$. This shows that $\pi' : Q \to X \times_k Y$ is an $\alpha_2$-torsor in FPQC-topology.

It is clear that the fiber of $Q$ on $Y$ along $x = 0$ is the trivial torsor and the fiber along $x = 1$ is just $\pi : P \to X$ which is non-trivial. This violates the necessary condition in our Lemma 4.1.

**Remark 4.2.** Let $Y_i (i = 0, 1)$ be the fibre of $x = i$ along the map $X \times_k Y \to X$ and let $W := \pi'_* Q$. In the above construction we see that $W|_{Y_0} = O_{Y_0} \oplus O_{Y_0}$ and $W|_{Y_1} = V$. If the chosen non-trivial torsor $\pi : P \to Y$ is the relative Frobenius endomorphism $F : Y \to Y$, then we have $\dim_k \Gamma(Y, W|_{Y_1}) = 1$. But $\dim_k \Gamma(Y, W|_{Y_0}) = 2$. This tells us that $W$ does not satisfy base change at $x = 0$. So it provides an example for which the base change condition in Theorem 2.1 is not satisfied. But note that the normality condition is still OK, since in this example the group is commutative.

**Remark 4.3.** The above construction actually shows that for any smooth proper connected scheme $Y$ over an algebraically closed field $k$ of characteristic 2 which admits a non-trivial $\alpha_2$-torsor, and for any rational points $(x, y) \in \mathbb{A}_k^1 \times_k Y$ the Künneth formula does not hold for $\mathbb{A}_k^1 \times_k Y$. In fact the only place where we used the assumption that $E$ is an elliptic curve is in the argument to show that $\mathcal{L}$ is $O_Y$, but everything still works without knowing that $\mathcal{L} \cong O_Y$, i.e. just that the computation is a little more complicated. Furthermore one can show easily that in this situation if Künneth formula holds for any rational point $(x, y) \in \mathbb{A}_k^1 \times_k Y(k)$ then it holds for any other rational point.

**Remark 4.4.** Hélène Esnault and André Chatzistamatiou pointed to us the following non-constructive improvement of the above example. Thanks to their suggestion our counterexample may work for any algebraically closed field $k$ of characteristic $p > 0$. Now we consider the exact sequence of abelian sheaves in the FPPF-topology

$$0 \to \alpha_p \to \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \to 0,$$

we then get a long exact sequence of abelian groups:

$$\cdots \to H^0_{fl}(Y, \mathbb{G}_a) \to H^1_{fl}(Y, \alpha_p) \to H^1_{fl}(Y, \mathbb{G}_a) \xrightarrow{H^1(F)} H^1_{fl}(Y, \mathbb{G}_a) \to \cdots$$

where $Y$ is a smooth proper connected $k$-variety so that the map $H^1(F)$ has none trivial kernel (e.g. a supersingular elliptic curve). Then we can choose some $a \neq 0$ in the that kernel. We can also put $\mathbb{A}_k^1 \times_k Y$ into the above long exact sequence instead of $Y$, then since

$$a \otimes x \in H^1_{fl}(\mathbb{A}_k^1 \times_k Y, \mathbb{G}_a) = H^1_{fl}(Y, \mathbb{G}_a) \otimes_k k[x]$$

is still in the kernel of $H^1(F)$, so we can choose an element $b \in H^1_{fl}(\mathbb{A}_k^1 \times_k E, \alpha_p)$ such that $b \mapsto a \otimes x$. But then $b$ is an $\alpha_p$-torsor with non-constant fibres along the projection $\mathbb{A}_k^1 \times_k Y \to \mathbb{A}_k^1$ because $b$ has trivial image at $x = 0$ and non-trivial image at $x = 1$. This can not happen if Künneth formula was true.
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