The complete low-energy collective-excitation spectrum of vortex lattices is discussed for rotating Bose-Einstein condensates (BEC) by solving the Bogoliubov-de Gennes (BdG) equation, yielding, e.g., the Tkachenko mode recently observed at JILA. The totally symmetric subset of these modes includes the transverse shear, common longitudinal, and differential longitudinal modes. We also solve the time-dependent Gross-Pitaevskii (TDGP) equation to simulate the actual JILA experiment, obtaining the Tkachenko mode and identifying a pair of breathing modes. Combining both the BdG and TDGP approaches allows one to unambiguously identify every observed mode.

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Collective Oscillations of Vortex Lattices in Rotating Bose-Einstein Condensates

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Owing to their fundamental significance for superfluidity, quantized vortices have attracted widespread interest in many different physical systems ranging from superconductors, superfluid 3He and 4He, to neutron stars and extending to cosmology. Recently, vortices have been created in dilute alkali-atom gases using three different methods: phase imprinting, topological phase engineering and optical spoon stirring. Several groups are now able to routinely prepare a vortex array with hundreds of vortices in a BEC.

In the mid 1960’s, Tkachenko predicted that a vortex lattice would sustain a collective vortex-oscillation mode in which the vortex cores move elliptically around the equilibrium positions. Subsequent theoretical advances were made in 1982. In superfluid 4He, the Tkachenko modes were first observed in 1982.

Recently, Coddington et al. succeeded in observing the Tkachenko mode in (1982) a rotating BEC of 87Rb. The TK wave was excited by removing condensate from the central region of the rotating cloud. They created the lowest and second-lowest TK modes and measured their energies ω1,0 and ω2,0 as functions of the rotation frequency Ω. They also discovered various new phenomena, some of which were explained by two groups: Anglin and Crescimanno extended the previous hydrodynamic description for infinite systems to a finite harmonically trapped system, and Baym discovered subtle effects due to the finite compressibility.

Here we choose a different approach to this problem: We wish to treat the whole low-lying collective-excitation spectrum of trapped BECs; not only the TK mode but also the other important modes and their intrinsic relationships. We construct a first-principles theory, namely the Bogoliubov-de Gennes equation (BdG) coupled with the Gross-Pitaevskii equation (GP). The set of equations within the Bogoliubov framework is regarded as the fully microscopic theory of dilute Bose gases, in the sense that there remain no adjustable parameters once we have fixed the atomic species and the atomic number. Thus it is quite reasonable to expect that this formalism must well apply to analyzing the TK mode and also to provide the complete spectral features of the low-energy excitations, beyond any limitations of hydrodynamics which has thus far been the only way to describe the TK mode. We can then better characterize the various modes, such as the three classes of compressional modes: the transverse, common longitudinal and the differential longitudinal waves. These are characteristic to the two-component system consisting of a vortex lattice and the superfluid. Some of the selected common longitudinal modes in a vortex lattice (the breathing and quadrupole modes) have been recently examined within hydrodynamics. This theory may help one to gain improved understanding of related problems, such as vortex pinning and vortex melting in superconductors which have thus far only been analyzed phenomenologically.

In a frame rotating with the frequency Ω, the time-dependent Gross-Pitaevskii equation (TDGP) may be expressed for the condensate wavefunction ψ as

\[
\frac{i\hbar}{\partial t} \psi = \left[ \frac{\vec{p}^2(\Omega)}{2m} + V_{\text{eff}}(\vec{r},\Omega) - \mu + g|\psi|^2 \right] \psi, \quad (1)
\]

where \(\vec{p}(\Omega) = -i\hbar \nabla - m\vec{\Omega} \times \vec{r}\) and the effective confining potential is \(V_{\text{eff}}(\vec{r},\Omega) = \frac{1}{2}m(\omega_r^2 - \Omega^2)r^2\) with the radial trap frequency \(\omega_r\).

In order to study the collective oscillations of vortex lattices microscopically, we first consider the equation of motion for a small perturbation around the stationary state \(\phi_0\), i.e., \(\psi(\vec{r},t) = \phi_0(\vec{r}) + u_q(\vec{r})e^{-i\omega_q t} - v_q(\vec{r})e^{i\omega_q t}\), where the equilibrium state \(\phi_0\) is determined by the stationary GP equation. By retaining terms up to first order in \(u\) and \(v\), we derive the BdG equation:

\[
\begin{pmatrix}
L(\vec{r},\Omega) - g\phi_0^2(r)
& g\phi_0 v_q(r)
& u_q(r)
\end{pmatrix}
\begin{pmatrix}
L(\vec{r},-\Omega) - g\phi_0^2(r)
& -g\phi_0 v_q(r)
& -v_q(r)
\end{pmatrix} = \hbar \omega_q \begin{pmatrix}
u_q(r)
u_q(r)
\end{pmatrix}, \quad (2)
\]
where $L(r, \Omega) = \hat{p}^2(\Omega)/2m + V_{ext}(r, \Omega) - \mu + 2g|\phi_g(r)|^2$.
We consider the JILA experiment with 2.0 \times 10^6 atoms of $^{87}$Rb confined in a trap with the radial frequency $\omega_r/2\pi = 8.3$ Hz and an axial one $\omega_z/2\pi = 5.2$ Hz\cite{13}. Employing Thomas-Fermi (TF) theory and the assumption of solid-body rotation, the condensate aspect ratio is given as $\lambda_{TF} \equiv R_{TF}/Z_{TF} \propto (\omega^2_r - \Omega^2)^{-1/2}$ where $R_{TF}$ and $Z_{TF}$ are the condensate lengths along the $r$- and $z$-axes. This relation allows one to assume the system to constitute a two-dimensional geometry at high rotation frequencies. Under this assumption, we introduce the linear density $n_z(\Omega)$ along the $z$ axis. Therefore, the equilibrium state $\phi_g$ must fulfill the normalization condition $n_z(\Omega) = \int \int \phi_g^2(r) dx dy$, where the linear density is obtained as $n_z(\Omega) = \frac{R_{TF}^2}{16a_d^2}$, with $a$ an s-wave scattering length, and $d_2^2 \equiv \hbar/(m\sqrt{\omega_r^2 - \Omega^2})$. We discretize the two-dimensional space typically into a 300$^2 \sim 1000^2$ mesh to solve the TDGP and BdG equations.

Here, since we consider a vortex array with sixfold symmetry, the wavefunction of the stationary state obeys the condition, $\phi_g(R^n r) = \phi_g(r) e^{in\pi/3}$, where $R^n r$ describes a rotation $n\pi/3$ (n integer) around the center of the trap, i.e., $R^n r = (x \cos (n\pi/3) - y \sin (n\pi/3), x \sin (n\pi/3) + y \cos (n\pi/3))$. We then obtain the following relation from Eq.\textsuperscript{2}:

\[ u_{q,m}(R^n r) = u_q(r) \exp \left( \frac{i\pi}{3} (m + 1) \right) \]

and $u_{q,m}(R^n r) = u_q(r) \exp \left( \frac{i\pi}{3} (m - 1) \right)$, where $m = 0, \pm 1, \pm 2, 3$. In order to classify the collective excitations, we introduce the following classifying function:

\[ F_q^{(n)}(m) = \frac{1}{\int d^2 r \int u_q^2(r)} \left[ \sum_{n=0}^{n_{\text{Max}}} u_{q,m}(R^n r) \right] \int d^2 r |u_q(r)|^2, \]

which tends to 1 for a suitable $m$ and 0 for the others. Furthermore, we define the average angular momentum as $q_\theta \equiv (\langle L_z \rangle_u + (L_z)_v - \langle L_z \rangle_g)/\int d^2 r |u(r)|^2 + |v(r)|^2$, where $\langle L_z \rangle_u = \int d^2 r \phi_g^* \hat{L}_z \phi_g/\int d^2 r |\phi_g(r)|^2$, and $\langle L_z \rangle_v = \int d^2 r \chi_u^* \hat{L}_z \chi_u$. In an axisymmetric situation, $m$ and $q_\theta$ merge into the same integer quantum number.

The low-energy excitations are illustrated in Fig.\textsuperscript{1}. Here, the equilibrium states are taken as a regular array with sixfold symmetry. The resulting configuration at $\Omega = 0.7\omega_r$ is formed by 37 vortices, displayed in Fig.\textsuperscript{1}(a). In Fig.\textsuperscript{1}(b), the excitation energies up to $\omega = 3.5\omega_r$ are shown as functions of the average angular momentum $q_\theta$. Each collective mode is classified by a symmetry index, $m$, obtained from the above function, $F_q^{(n)}(m)$, which characterizes the oscillation pattern. For example, the breathing (BR), dipole (DP), and quadrupole (QP) modes have $m = 0, \pm 1, \pm 2$, respectively. The branch extending from the origin towards higher $q_\theta$-values consists of surface modes which are spaced periodically with $m$ (mod 6), where the outer condensate surface oscillates, the vortex cores being almost stationary at their equilibrium positions. The other branch situated at higher energies and starting at $\omega = 2\omega_r$ represents another class of surface modes characterized by a node along the radial direction. The energy spectra for the low-lying modes are depicted in Fig.\textsuperscript{1}(c) as functions of the symmetry index $m$. These low-lying eigenstates are the lattice-oscillation modes coupled with the condensate motion and lead to the distortion of both the lattice and the condensate surface towards an $m$-fold symmetric shape. In addition to the lowest and second-lowest TK with $m = 0$, embedded among the other modes, we found a parallel-precession mode with $m = -1$ and having the lowest energy where all the vortices precess in phase. For increasing energy, the modes for each $m$ feature nodes along $r$ and/or $\theta$. The TK mode (with $m = 0$) is selectively excited by the Gaussian laser beam with 0-fold symmetry in the experiment\cite{14}. Likewise, a mode with $m = \pm \ell$ may be excited by a disturbance (e.g., magnetic) with $\ell$-fold symmetry\cite{14}.\[\text{FIG. 1: (a) Density profile of the condensate with 37 vortices at } \Omega = 0.7\omega_r.\text{ (b) Collective excitations up to } \omega = 3.5\omega_r.\text{ (c) Lowest excitations marked by a dotted line as functions of } q_\theta \text{ are displayed as functions of the symmetry index } m. \text{ Here, TK, BR, DP, and QP denote the Tlachenko, breathing, dipole, and quadrupole modes, respectively.}\]
served that the modulation displays a node at amplitude is magnified by a factor of 30). It is observed that the equilibrium data of $\delta \psi$ of the nominally longitudinal BR mode all the vortex motion for each circle, $\theta_j(t)$ in terms of the averaged angle $\theta_j(t) = \frac{\omega}{\bar{\omega}_e} \sum_{n=0}^{\infty} [\arctan (y_{j,n}(t)/x_{j,n}(t)) - \arctan (y_{j,n}(0)/x_{j,n}(0))]$ where $n$ denotes the vortices aligned along the line and extending the angle $\pi n/3$ from the vertical, see Fig. 1(a). It is apparent in Fig. 3(a) where we plot the time dependence of the vortex motion for each circle, $j = 1, 2, 3$, that (i) there exist several superposed oscillations. (ii) The outermost vortices ($j = 3$) are in opposite phase with the inner vortices ($j = 1, 2$). This result coincides with the above calculations based on the BdG equations. In fact, the nodal positions obtained from both calculations agree quite well.

In Fig. 3(b), the Fourier analyses of these transverse oscillations, $\theta_j(t)$, and the longitudinal motions, $r_j(t) = \overline{V(r_j)}$, are depicted. It is seen that the sharp peak at $\omega = 0.16 \omega_r$ precisely coincides with the first TK mode $\omega_{1,0}$ identified above, see Fig. 1(c). The second peak at $\omega = 2.0 \omega_r$ corresponds to the 1BR mode, belonging to the common longitudinal modes. The third peak at $\omega \sim 3.1 \omega_r$ may be assigned as the 2BR mode, see Figs. 1(b) and 2(d), and as a differential longitudinal mode. These three collective modes closely match the observed characteristics. In particular, the third mode which was tentatively assigned by Cozzini et al., independently, as a higher-order hydrodynamic mode, is now identified above. It should be noted
that the resonances of these three modes for a Gaussian potential have been numerically reproduced over a wide range of rotation rates $\Omega = 0.7 \sim 0.92$, corresponding to $37 \sim 121$ vortices.

In Fig. 4, we plot the first TK energy $\omega_{1,0}$ as a function of $\Omega/\omega_r$, together with the experimental data [13] and a hydrodynamic prediction by Anglin and Crescimanno [15]. There prevails close quantitative overall agreement between our results and the experimental data. We emphasize that our calculations contain no adjustable parameters and also that our computations within the BdG and TDGP approaches are compared with the JILA data (filled squares, diamonds, and triangles) as functions of $\Omega$. The solid line represents the dispersion relation in Ref. [15] and TDGP approaches agree within numerical accuracy $\sim 3\%, below \Omega \leq 0.8\omega_r$. Calculations for larger rotation rates, where BdG cannot be feasible from a numerical point of view, are done with TDGP, which enables us to extrapolate the BdG results to larger rotation rates.

In summary, we have discussed the complete low-energy excitation spectrum in a vortex lattice by solving the BdG equations. The $m = 0$ subset of these solutions includes the transverse shear, common longitudinal, and differential longitudinal modes. We have also succeeded in simulating the actual experimental results, identifying a new pair of modes.

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During the preparation of this manuscript, we learned about two closely related preprints [21] and [22]. The former (latter) presents BdG (TDGP) treatments of the TK mode.

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