Toll Caps in Privatized Road Networks

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Abstract

We consider a nonatomic routing game on a parallel link network in which link owners set tolls for travel so as to maximize profit. A central authority is able to regulate this competition by means of a (uniform) price cap. The first question we want to answer is how such a cap should be designed in order to minimize the total congestion. We provide an algorithm that finds an optimal price cap for networks with affine latency functions and a full support Wardrop equilibrium. Second, we consider the induced network performance at an optimal price cap. We show that for two link networks with affine latency functions, the congestion costs at the optimal price cap are at most 8/7 times the optimal congestion costs. For more general latency functions, this bound goes up to 2 under the assumption that an uncapped subgame perfect Nash equilibrium exists. However, in general such an equilibrium need not exist and this can be used to show that optimal price caps can induce arbitrarily inefficient flows.

Keywords: Competition regulation, toll caps, congestion, Wardrop equilibrium, subgame perfect Nash equilibrium, efficiency.

1 Introduction

With the ongoing privatization of public road infrastructure, toll charging on highways and roads is common practice in many cities around the world (see e.g. Bergen [44], London [22], Santiago de Chile [24], Singapore [25], and Stockholm [3]). The toll market in the United States (starting already in the 18th century) is built on the key idea that private firms obtain the right to construct the infrastructure (usually via an auction, see Porter and Zona [41]) and as compensation are allowed to charge tolls for road usage. Firms are further obligated to reinvest parts of the revenues to maintain the infrastructure. To date about 35 States use this mechanism and while it has worked well in some cases, several problems arose in the past and still prevail. We report here one incident. In 2012, large toll rate increases have been implemented by the Port Authority of New York and New Jersey (justified in part to finance its World Trade Center project). In response, a bill was introduced in Congress that would allow the Secretary of Transportation to regulate tolls on every bridge on the country’s interstates and other federally aided highways.

If a regulation authority introduces toll caps, how should this cap be designed in order to induce a socially beneficial outcome and how does this outcome perform? This question is at

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the core of our paper. We model the situation using a three-level optimization model, where in
the lowest (third) level, there are commuters that want to travel from an origin to a destination
and each commuter minimizes a combination of latency costs plus toll costs. We model the
route choice of a fixed volume of commuters using the concept of Wardrop equilibrium [46].
We further assume that the underlying network consists of \( n \geq 2 \) parallel edges connecting
a common origin with a common destination. While this assumption is indeed restrictive, it
still models relevant situations, for instance, when there are parallel access roads (e.g. tollable
bridges) connecting the central district of a city with a suburb, and it is used in recent related
work (e.g. Acemoglu and Ozdaglar [1], and Wan [45]).

In the middle (second) layer, there are firms owning the edges and charging tolls on them.
Each firm maximizes revenue, but faces competition from other firms. We assume that each
firm owns only one edge. A subgame perfect Nash equilibrium is a vector of tolls, one toll
value per firm, so that no firm can improve by unilaterally deviating to another toll value. A
subgame perfect Nash equilibrium takes into account that commuters will (potentially) change
their route choices once a firm deviates to a different toll.

The authority in the first and highest level can decide on some toll cap. We assume that
the authority is not allowed to discriminate between the different firms and puts one (uniform)
price cap applicable to all firms. We consider the situation in which the authority is purely
interested in minimizing the resulting total latency costs (clearly other objectives may be
feasible as well).

1.1 Our Results

We study a three-level optimization problem in which a central authority can impose a (uni-
form) price cap so as to minimize the total induced equilibrium congestion costs. We motivate
this form of competition regulation by showing that without any price regulation, the induced
equilibrium congestion costs can be arbitrarily higher than the optimal congestion costs be-
cause of too high prices of one of the firms.

For solving the three-level optimization problem, we first derive a set of structural results
for instances with affine latency functions. In particular, we show that a subgame perfect Nash
equilibrium exists for all price caps \( c \), and is in fact unique whenever the Wardrop equilibrium
has full support. Moreover, under the full support assumption, we derive a complete char-
acterization of subgame perfect Nash equilibria using the KKT conditions of the underlying
optimization problems of the involved firms. Based on this characterization, we devise an
algorithm that finds the optimal price cap in polynomial time. The algorithm computes break-
points at which the set of firms putting their price equal to the cap changes. We show that
these sets only increase (with decreasing cap) so that there are at most \( n \) breakpoints, where
\( n \) denotes the number of firms. The breakpoints essentially divide the real line into at most \( n \)
intervals and for each of these intervals we show that the cost function is quadratic. Thus, the
algorithm needs to solve at most \( n \) quadratic one-dimensional minimization problems, which
can be done in polynomial time.

We then consider the performance of an optimal price cap. Given that a subgame perfect
Nash equilibrium always exists for networks with affine latency functions, we show a tight
bound of \( 8/7 \) between the congestion costs at an optimal price cap and the optimal congestion
costs for duopoly instances. For more general latency functions, this bound goes up to 2 if we
assume that an uncapped subgame perfect Nash equilibrium exists.

In general, however, the objective function of a firm in the second level is not concave, the
set of best replies is not convex and therefore standard techniques for proving existence of a 
subgame perfect duopoly equilibrium fail. In fact, we give an example where an equilibrium 
need not exist. Given that a subgame perfect duopoly equilibrium need not exist, we are able 
to provide a sequence of instances for which no uncapped subgame perfect Nash equilibrium 
exists such that the ratio between the congestion costs at the optimal price cap and the optimal 
congestion costs goes to infinity.

1.2 Related Work

The inefficiency of selfish behavior in congestion games has been well recognized since the 
work of Pigou [40] in economics and Wardrop [46] in transportation networks. A natural way 
to quantify this inefficiency is the price of anarchy as defined by Koutsoupias and Papadimitriou [35]. For routing games this is first done by Roughgarden and Tardos [43] and later by Correa et al. [16].

One way to restore efficiency is by means of centralized pricing. Beckmann et al. [5] and 
Dafermos and Sparrow [17] showed that marginal cost pricing induces an equilibrium flow 
that is optimal. Despite the effectiveness of marginal cost tolls, there are two drawbacks. First, potentially each edge in the network is tolled; an issue that is considered by Hoefer et al. [30] and Harks et al. [27]. Second, the imposed tolls can be arbitrary large; an issue that is considered by Bonifaci et al. [7], Fotakis et al. [23], Jelink et al. [31] and Kleer and Schäfer [34]. Cole et al. [13] show that optimal tolls also exist when users are heterogeneous with respect to the tradeoff between time and money. Later this result was extended to general networks by Fleischer et al. [21], Karakostas and Kolliopoulos [33] and Yang and Huang [48].

Instead of improving efficiency, tolls can also be used to minimize or maximize the profit 
of one or multiple leaders. The model that computes tolls that induce the optimal flow at minimal profit was analyzed by Dial [19, 20]. The model with one profit maximizing leader and no congestion effects, was first analyzed by Labbé et al. [36] and later by Briest et al. [8].

Acemoglu and Ozdaglar [1] introduce a model of price competition between link owners in 
a parallel network that is very similar to ours. The main difference is that in their model users 
have a reservation value for travel. This implies that if links become too expensive users choose 
not to travel, whereas in our model users always travel through the network. Their main result 
is a (tight) bound on the inefficiency of equilibria. Later several generalizations of the model 
were introduced. The follow-up work of Acemoglu and Ozdaglar [2] allow for a slightly more 
general topology, namely parallel paths with multiple links, and show that equilibria can be 
arbitrarily inefficient. Hayrapetyan et al. [29] consider the model with elastic traffic demand. 
Their bounds on inefficiency, however, are not tight, which is improved upon by Ozdaglar [38].

Recently, following our question on competition regulation of price competition between 
edge owners, Correa et al. [15] consider the setting in which a central authority is allowed 
to put different toll caps on different edges of the network. Their main result is that for all 
network topologies there are toll caps so that firms are willing to put their toll equal to the cap, 
and the induced equilibrium flow is the optimal flow. An example of such caps are marginal 
tolls as introduced by Beckmann et al. [5]. Notice that the assumption of individual toll caps is 
important. In practice, however, toll cap discrimination is often not allowed, which motivates 
our work on uniform toll caps.

The model of Xiao et al. [47] studies competition in both tolls and capacities and finds that 
tolls are higher, but capacities are lower than socially desired. Other recent models of Bertrand 
competition in a network setting that use different ways of modelling congestion effects are
Anshelevich and Sekar [4], Chawla and Niu [11], Chawla and Roughgarden [12], Papadimitriou and Valiant [39] and Caragiannis et al. [9].

2 The Model

An instance of the three-level optimization problem is given by the tuple \(I = (N, (\ell_i)_{i \in N})\), where \(N = \{1, \ldots, n\}\), with \(n \geq 2\), is a set of parallel links connecting a common source with a common destination. There is one unit of flow to be sent over the links. Let \(x_i \in \mathbb{R}_+\) be the total flow on link \(i\) and \(x = (x_1, \ldots, x_n)\) be the vector of flows. Each link has a flow-dependent latency function \(\ell_i(x_i)\) that we assume to be strictly increasing, convex and continuously differentiable. We denote this class of functions by \(\mathcal{L}_c\). By \(\mathcal{L}_d\) we denote the class of strictly increasing polynomial latency functions with nonnegative coefficients and degree at most \(d\). In particular, \(\mathcal{L}_1\) denotes the class of strictly increasing affine latency functions.

Given a flow \(x\), define the total latency costs by

\[
C(x) = \sum_{i \in N} \ell_i(x_i) \cdot x_i.
\]

A flow \(x^*\) is optimal, if \(C(x^*) \leq C(x)\) for all flows \(x\).

Let \(t_i \in \mathbb{R}_+\) be the toll on link \(i\) and \(t = (t_1, \ldots, t_n)\) be the vector of tolls. The effective cost of a user of link \(i\) is \(\ell_i(x_i) + t_i\). Given a toll vector \(t\), a flow \(x\) is a Wardrop equilibrium for \(t\), if for all \(i, j \in N\) with \(x_i > 0\),

\[
\ell_i(x_i) + t_i \leq \ell_j(x_j) + t_j.
\]

In particular, if \(x\) is an equilibrium for \(t\), then all links with positive flow have equal effective costs, i.e., there is \(K \in \mathbb{R}_+\) with \(\ell_i(x_i) + t_i = K\) for all \(i \in N\) with \(x_i > 0\). It is well-known that given our assumptions on the latency functions, an equilibrium for \(t\) exists, is unique and can be described by means of the following inequality (see Beckmann et al. [3] and Dafermos and Sparrow [17]).

**Lemma 2.1.** A flow \(x\) is a Wardrop equilibrium for \(t\) if and only if for all feasible flows \(x'\),

\[
\sum_{i \in N} (\ell_i(x_i) + t_i) \cdot (x_i - x'_i) \leq 0
\]

For each toll vector \(t\), we denote by \(x(t)\) the unique equilibrium for \(t\). The equilibrium for \(t = 0\) is called the Wardrop equilibrium.

2.1 Subgame Perfect Nash Equilibrium

We assume that each link \(i\) is owned by a firm (also denoted by \(i\)) and that the objective of each firm is to maximize profit. Given a toll vector \(t \in \mathbb{R}_+^N\), define \(\Pi_i(t_i, t_{-i}) = t_i \cdot x_i(t)\) for all \(i \in N\). A toll vector \(t\) is a \(c\)-capped subgame perfect Nash equilibrium, if for all \(i \in N\), \(t_i \leq c\) and for all \(0 \leq t'_i \leq c\),

\[
\Pi_i(t_i, t_{-i}) \geq \Pi_i(t'_i, t_{-i}).
\]

Observe that the flow adapts to the toll vector \((t'_i', t_{-i})\) in the calculation of \(\Pi_i(t'_i, t_{-i})\). Given \(t_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}\) and \(c \in \mathbb{R}_+\), define \(B^c_i(t_{-i}) = \arg \max_{0 \leq t_i \leq c} \Pi_i(t_i, t_{-i})\) and \(B^\infty_i(t_{-i}) = \arg \max_{t_i \geq 0} \Pi_i(t_i, t_{-i})\) for all \(i \in N\).
Given \( c \in \mathbb{R}_+ \), we denote by
\[
\mathcal{T}(c) = \{ t \in \mathbb{R}_+^n \mid t \text{ is a } c\text{-capped subgame perfect Nash equilibrium} \}
\]
the set of \( c \)-capped subgame perfect Nash equilibria. By \( \mathcal{T}(\infty) \) we denote the set of uncapped subgame perfect Nash equilibria. Note that there might be instances for which \( \mathcal{T}(c) \) is not a singleton for some \( c \in \mathbb{R}_+ \), and for which \( \mathcal{T}(c) = \emptyset \) for some \( c \in \mathbb{R}_+ \).

### 2.2 Designing Price Caps

The following example provides the motivation for our research: in terms of latency costs, a subgame perfect Nash equilibrium can be arbitrarily worse than the optimal flow.

**Example 2.1.** Consider the network of Figure 1.

![Figure 1: An inefficient subgame perfect Nash equilibrium as \( a_2 \to \infty \).](image)

Then
\[
B_1^\infty(t_2) = \begin{cases} 
\frac{a_2 + t_2}{2}, & \text{if } t_2 \leq a_2 + 2, \\
 t_2 - 1, & \text{if } t_2 > a_2 + 2,
\end{cases}
\]
and
\[
B_2^\infty(t_1) = \begin{cases} 
\frac{1 + t_1}{2}, & \text{if } t_1 \leq 2a_2 + 1, \\
 t_1 - a_2, & \text{if } t_1 > 2a_2 + 1.
\end{cases}
\]
Combining \( B_1^\infty(t_2) \) and \( B_2^\infty(t_1) \), if \( t \in \mathcal{T}(\infty) \), then \( t = (\frac{2a_2 + 1}{3}, \frac{a_2 + 2}{3}) \). Since \( x(t) = \left( \frac{2a_2 + 1}{3a_2 + 3}, \frac{a_2 + 2}{3a_2 + 3} \right) \) and \( x^* = \left( \frac{a_2}{a_2 + 1}, \frac{1}{a_2 + 1} \right) \), we have \( C(x(t)) = \frac{a_2^2 + 7a_2 + 1}{9a_2 + 9} \) and \( C(x^*) = \frac{a_2}{a_2 + 1} \). Hence \( \frac{C(x(t))}{C(x^*)} = \frac{a_2^2 + 7a_2 + 1}{9a_2} \to \infty \) as \( a_2 \to \infty \). As \( a_2 \to \infty \), the price of firm 1 grows faster than the price of firm 2, which pushes too much flow onto link 2, creating inefficiency.

The bad performance of the subgame perfect Nash equilibrium calls for competition regulation. The regulation policy we focus on is that of a price cap \( c \in \mathbb{R}_+ \): no firm is allowed to set its toll above the price cap, i.e., \( t_i \leq c \) for all \( i \in N \). Formally, we seek to choose \( c \) so as to solve the following three-level optimization problem:

\[
\inf_{c \in \mathbb{R}_+} \sup_{t \in \mathcal{T}(c)} C(x(t)). \quad (3L-P)
\]

If \( \mathcal{T}(c) = \emptyset \) for some \( c \in \mathbb{R}_+ \), we assume that \( C(x(t)) = \infty \). Observe that \( \mathcal{T}(0) \neq \emptyset \).
Note that this is a quite appealing and robust formulation: we minimize total costs for the worst possible realization of a subgame perfect Nash equilibrium. It is known, however, that multi-level optimization problems are notoriously hard in terms of proving existence and computability of optimal solution.

In the next section, we derive structural results of subgame perfect equilibria and their corresponding Wardrop equilibria. These structural results contain existence and uniqueness of equilibria for networks with affine latencies and a comparative statics result with respect to different price caps. These insights form the basis of our polynomial time algorithm computing an optimal price cap for the case of affine latency functions.

3 Structural Properties of Subgame Perfect Equilibria

Assume that \( \ell_i(x_i) = a_i \cdot x_i + b_i \) with \( a_i > 0 \) and \( b_i \geq 0 \). The first result shows that for affine latency functions, a subgame perfect Nash equilibrium exists for all price caps. A similar result is obtained for the model in which users have a reservation value for travel and there are no latency functions, a subgame perfect Nash equilibrium exists for all price caps. A similar result is obtained for the model in which users have a reservation value for travel and there are no latency functions, a subgame perfect Nash equilibrium exists for all price caps. A similar result is obtained for the model in which users have a reservation value for travel and there are no latency functions, a subgame perfect Nash equilibrium exists for all price caps. A similar result is obtained for the model in which users have a reservation value for travel and there are no latency functions, a subgame perfect Nash equilibrium exists for all price caps. A similar result is obtained for the model in which users have a reservation value for travel and there are no latency functions, a subgame perfect Nash equilibrium exists for all price caps. A similar result is obtained for the model in which users have a reservation value for travel and there are no latency functions, a subgame perfect Nash equilibrium exists for all price caps. A similar result is obtained for the model in which users have a reservation value for travel and there are no latency functions, a subgame perfect Nash equilibrium exists for all price caps. A similar result is obtained for the model in which users have a reservation value for travel and there are no latency functions, a subgame perfect Nash equilibrium exists for all price caps. A similar result is obtained for the model in which users have a reservation value for travel and there are no latency functions, a subgame perfect Nash equilibrium exists for all price caps. A similar result is obtained for the model in which users have a reservation value for travel and there are no latency functions, a subgame perfect Nash equilibrium exists for all price caps.

**Proposition 3.1.** Let \( \ell_i \in \mathcal{L}_i \) for all \( i \in N \). Then, there exists a c-capped subgame perfect Nash equilibrium for all \( c \in \mathbb{R}_+ \).

*Proof.* Assume that \( \ell_i(x_i) = a_i \cdot x_i + b_i \) with \( a_i > 0 \) and \( b_i \geq 0 \). For all toll vectors \( t \in \mathbb{R}_+^N \), define \( N(t) = \{ i \in N \mid x_i(t) > 0 \} \).

Since \( x(t) \) is an equilibrium for \( t \), we have \( a_i \cdot x_i(t) + b_i + t_i = K \) for all \( i \in N(t) \). With \( \sum_{i \in N(t)} x_i(t) = 1 \), we get

\[
K = 1 + \frac{\sum_{j \in N(t)} b_i + t_j}{\sum_{j \in N(t)} a_j} \quad \text{and thus}
\]

\[
x_i(t) = \frac{1 + \sum_{j \in N(t)} \frac{b_i + t_j - b_j - t_i}{a_j}}{\sum_{j \in N(t)} \frac{a_n}{a_j}} \quad \text{for all } i \in N(t).
\]

By the Theorem of the Maximum [6], it follows that \( B_i^c(t) = (B_i^c(t_{-i}))_{t_{-i} \in N} \) is upper semi-continuous and hence has a closed graph. We show that \( B_i^c(t_{-i}) \) is a convex set for all \( t_{-i} \).

Assume \( t_{-i} \in \mathbb{R}_+^N \setminus \{0\} \) and consider the following two cases: (a) \( \Pi_i(t_i, t_{-i}) = 0 \) for all \( t_i \in \mathbb{R}_+ \) and (b) \( \Pi_i(t_i, t_{-i}) > 0 \) for some \( t_i \in \mathbb{R}_+ \).

Case (a): Suppose \( \Pi_i(t_i, t_{-i}) = 0 \) for all \( t_i \in \mathbb{R}_+ \). Obviously \( B_i^c(t_{-i}) \) is convex.

Case (b): Suppose there is some \( t_i' \in \mathbb{R}_+ \) with \( \Pi_i(t_i', t_{-i}) > 0 \). We will prove that \( \Pi_i(t_i, t_{-i}) \) is a concave function in \( t_i \) until there is some \( t_i' \) such that \( \Pi_i(t_i', t_{-i}) = 0 \) for all \( t_i \geq t_i' \). Observe the following from [1].

(i) If \( j \in N(t'_i, t_{-i}) \), with \( j \neq i \), for some \( t'_i \in \mathbb{R}_+ \), then \( j \in N(t_i, t_{-i}) \) for all \( t_i \geq t'_i \). That is, the set \( N(t) \) increases as \( t_i \) increases.

(ii) For a fixed \( N(t) \), \( \Pi_i(t_i, t_{-i}) \) is either linear or quadratic in \( t_i \). \( \Pi_i(t) \) is linear if and only if \( N(t) = \{ i \} \), which can only happen for \( t_i \in [0, \hat{t}_i] \) for some \( \hat{t}_i \in \mathbb{R}_+ \). In all other cases, \( \Pi_i(t) \) is quadratic with a decreasing slope.
Now in order to prove that $\Pi_i(t_i, t_{-i})$ is a concave function in $t_i$, it is sufficient to show that the slope decreases at every toll $t_i$ for which new links start to receive flow, i.e., a new set of players joins $N(t_i, t_{-i})$. To this end, let $t_i$ denote a price at which a new set of firms starts to receive flow and let us denote this set by $\bar{N}$. Notice that

$$
\frac{\partial_- \Pi_i(t_i, t_{-i})}{\partial t_i} = 1 + \sum_{j \in N(t); j \neq i} \frac{b_j + t_j - b_i - 2t_i}{a_j} - \sum_{j \in N(t)} \frac{a_i}{a_j},
$$

$$
\frac{\partial_+ \Pi_i(t_i, t_{-i})}{\partial t_i} = 1 + \sum_{j \in N(t) \cup N; j \neq i} \frac{b_j + t_j - b_i - 2t_i}{a_j} - \sum_{j \in N(t) \cup N} \frac{a_i}{a_j},
$$

where $\frac{\partial_- \Pi_i(t_i, t_{-i})}{\partial t_i}$ and $\frac{\partial_+ \Pi_i(t_i, t_{-i})}{\partial t_i}$ denote the left and right partial derivative with respect to $t_i$, respectively. So in order to prove that the slope decreases it is sufficient to show that

$$
\left(1 + \sum_{j \in N(t); j \neq i} \frac{b_j + t_j - b_i - 2t_i}{a_j}\right) \cdot \sum_{k \in \bar{N}} \frac{a_i}{a_k} \geq \sum_{k \in N(t)} \frac{b_k + t_k - b_i - 2t_i}{a_k} \cdot \sum_{j \in N(t)} \frac{a_i}{a_j}.
$$

We get

$$
\sum_{k \in \bar{N}} \frac{b_k + t_k - b_i - 2t_i}{a_k} \cdot \sum_{j \in N(t)} \frac{a_i}{a_j} = \sum_{k \in N(t)} \frac{a_i \cdot x_i(t) - t_i}{a_k} \cdot \sum_{j \in N(t)} \frac{a_i}{a_j}
$$

$$
= \sum_{k \in N(t)} \frac{1 + \sum_{j \in N(t); j \neq i} \frac{b_j + t_j - b_i - 2t_i}{a_j} - \sum_{j \in N(t)} \frac{a_i}{a_j}}{a_k} \cdot \sum_{j \in N(t)} \frac{a_i}{a_j}
$$

$$
= \sum_{k \in \bar{N}} \left(1 + \sum_{j \in N(t); j \neq i} \frac{b_j + t_j - b_i - 2t_i}{a_j}\right) \cdot \frac{a_i}{a_k} - \frac{t_i}{a_i}
$$

$$
= \left(1 + \sum_{j \in N(t); j \neq i} \frac{b_j + t_j - b_i - 2t_i}{a_j} - \frac{t_i}{a_i}\right) \cdot \sum_{k \in \bar{N}} \frac{a_i}{a_k},
$$

where the first equality follows from $b_k + t_k = a_i \cdot x_i(t) + b_i + t_i$ for all $k \in \bar{N}$, the second equality from the definition of $x_i(t)$, and the last two inequalities from rewriting. Since $\frac{1}{a_i} > 0$, equation (2) holds true. Hence $B_i(t_i, t_{-i})$ is convex-valued.

Kakutani’s fixed point theorem [32] now implies that there exists an equilibrium for all price caps $c \in \mathbb{R}_+$.

The following lemma states a very natural comparative statics result, showing that a unilateral increase in toll by one firm only decreases the flow on the corresponding link while the flow on other links only increases.

**Lemma 3.1.** Let $i \in N$ and $t = (t_i, t_{-i}), t' = (t'_i, t_{-i}) \in \mathbb{R}_+^n$ with $t_i \leq t'_i$. Then $x_i(t) \geq x_i(t')$ and $x_j(t) \leq x_j(t')$ for all $j \neq i$.  

7
Proof. It is well known (see Beckmann et al. [5] and Dafermos and Sparrow [17]) that $x$ is an equilibrium for $t$ if $x$ solves the following minimization problem
\[
\min_x \sum_{i \in N} \int_0^{x_i} (\ell_i(y) + t_i) \, dy.
\]

Hence
\[
\sum_{i \in N} \int_0^{x_i(t)} (\ell_i(y) + t_i) \, dy \leq \sum_{i \in N} \int_0^{x_i(t')} (\ell_i(y) + t_i) \, dy
\]
\[
\sum_{i \in N} \int_{x_i(t')}^{x_i(t)} (\ell_i(y) + t_i) \, dy \leq \sum_{i \in N} \int_0^{x_i(t)} (\ell_i(y) + t_i') \, dy
\]

Combining these inequalities we get
\[
\sum_{j \in N} (t_j - t_j') \cdot (x_j(t) - x_j(t')) \leq 0. \tag{3}
\]

Using $t' = (t_i', t_{-i})$, we get
\[
(t_i - t_i') \cdot (x_i(t) - x_i(t')) \leq 0,
\]
and thus $x_i(t) \geq x_i(t')$.

Before we prove that $x_j(t) \leq x_j(t')$ for all $j \neq i$, we first show that $\ell_i(x_i(t)) + t_i \leq \ell_i(x_i(t')) + t_i'$. Since $x(t) + (x(t') - x(t)) = x(t')$ is a feasible flow, $x(t') - x(t)$ is a feasible direction for $x(t)$. By the first-order optimality conditions,
\[
\sum_{i \in N} (\ell_i(x(t)) + t_i) \cdot (x_i(t) - x_i(t')) \leq 0.
\]

Analogously, $x(t) - x(t')$ is a feasible direction for $x(t')$, and thus
\[
\sum_{i \in N} (\ell_i(x(t')) + t_i') \cdot (x_i(t') - x_i(t)) \leq 0.
\]

Adding up these inequalities, we obtain
\[
\sum_{j \neq i} (x_j(t) - x_j(t')) \cdot (\ell_j(x_j(t)) - \ell_j(x_j(t'))) + (x_i(t) - x_i(t')) \cdot (\ell_i(x_i(t)) + t_i - \ell_i(x_i(t')) - t_i') \leq 0.
\]

Notice that the first summation term is nonnegative, as $\ell_i$ is increasing for all $i \in N$. Thus,
\[
\ell_i(x_i(t)) + t_i \leq \ell_i(x_i(t')) + t_i'. \tag{4}
\]

Now we prove that $x_j(t) \leq x_j(t')$ for all $j \neq i$. Notice that we can assume that $x_i(t) > 0$, as otherwise the conclusion follows trivially. Also notice that if the result is true for all $t_i'$ with $x_i(t') > 0$, then it is true for all $t_i'$. We use a proof by contradiction. Suppose that $x_i(t) \geq x_i(t') > 0$ and $x_j(t) > x_j(t')$ for some $j \neq i$. Then
\[
\ell_j(x_j(t')) + t_j < \ell_j(x_j(t)) + t_j \leq \ell_i(x_i(t)) + t_i \leq \ell_i(x_i(t')) + t_i' \leq \ell_j(x_j(t')) + t_j,
\]
where the first inequality follows from $x_j(t) > x_j(t')$, the second inequality from $x_j(t) > 0$, the third inequality from (4), and the fourth inequality from $x_i(t') > 0$. This is a contradiction and finishes the proof. \hfill \Box
From now on, we assume that \( x_i(0) > 0 \) for all \( i \in N \). In terms of practical applications, this assumption seems not overly restrictive since a road segment is usually only priced if there is traffic without any tolls. However, the assumption is important for proving uniqueness of subgame perfect Nash equilibria. Example 4.2 shows that multiple subgame perfect Nash equilibria may exist in case the full Wardrop support assumption is not satisfied.

The full support assumption \( x_i(0) > 0 \) for all \( i \in N \) does not imply a priori that any subgame perfect equilibrium \( x_i(t), i \in N \) for some \( t \in T(c), c \in \mathbb{R}_+ \) also has the full support property. In the following, however, we show that given the full support property for \( t = 0 \), it continues to hold for all \( t \in T(c), c \in \mathbb{R}_+ \).

**Lemma 3.2.** Let \( c > 0 \) and \( t \in T(c) \). If \( x_i(0) > 0 \) for all \( i \in N \), then \( x_i(t) > 0 \) and \( t_i > 0 \) for all \( i \in N \).

**Proof.** By Lemma 3.1 if \( x_i(0) > 0 \) for all \( i \in N \), then \( x_i(0, t_{-i}) > 0 \) for all \( i \in N \) and all \( t_{-i} \). As the profit function is continuous in the toll vector (Hayrapetyan et al. [29]), this implies that for all \( t_{-i} \), there is some \( t_i \) so that \( \Pi_i(t_i, t_{-i}) > 0 \). Hence if \( t \in T(c) \), then for all \( i \in N \), \( \Pi_i(t_i, t_{-i}) > 0 \) and thus \( x_i(t) > 0 \) and \( t_i > 0 \). \( \square \)

In the following lemma we derive an explicit price representation of subgame perfect equilibria.

**Lemma 3.3.** Let \( c \in \mathbb{R}_+ \cup \{\infty\} \) and \( t \in T(c) \). If \( x_i(t) > 0 \) for all \( i \in N \), then for all \( i \in N \),

\[
t_i = \min \left\{ \left( \ell'_i(x_i(t)) + \frac{1}{\sum_{j \neq i} \ell'_i(x_j(t))} \right) \cdot x_i(t), c \right\}.
\]

**Proof.** Assume that \( x_i(t) > 0 \) for all \( i \in N \). Given \( t_{-i} \in \mathbb{R}_+^{N \setminus \{i\}} \), each firm \( i \in N \) solves the following maximization problem:

\[
\begin{align*}
\max & \quad t_i \cdot x_i \\
\text{s.t.} & \quad \ell_j(x_j) + t_j = K \text{ for all } j \in N, \\
& \quad \sum_{j \in N} x_j = 1, \\
& \quad t_i \leq c.
\end{align*}
\]

The corresponding Lagrangian is

\[
L(t_i, x, K, \lambda, \mu, \nu) = t_i \cdot x_i - \sum_{j \in N} \lambda_j \cdot (\ell_j(x_j) + t_j - K) - \nu \cdot (\sum_{j \in N} x_j - 1) - \mu \cdot (t_i - c).
\]

So the Kuhn-Tucker conditions reduce to

\[
\begin{align}
\frac{\partial L(t_i, x, K, \lambda, \mu, \nu)}{\partial t_i} &= x_i - \lambda_i - \mu = 0, \\
\frac{\partial L(t_i, x, K, \lambda, \mu, \nu)}{\partial x_i} &= t_i - \lambda_i \cdot \ell'_i(x_i) - \nu = 0, \\
\frac{\partial L(t_i, x, K, \lambda, \mu, \nu)}{\partial x_j} &= -\lambda_j \cdot \ell'_j(x_j) - \nu = 0, \\
\frac{\partial L(t_i, x, K, \lambda, \mu, \nu)}{\partial K} &= \sum_{j \in N} \lambda_j = 0.
\end{align}
\]
By (5),
\[ \lambda_i = x_i - \mu. \quad (9) \]

By (6) and plugging in (9),
\[ t_i = \ell'_i(x_i) \cdot \lambda_i + \nu = \ell'_i(x_i) \cdot (x_i - \mu) + \nu. \quad (10) \]

By (7), for all \( j \neq i \),
\[ \lambda_j = -\nu \cdot \frac{1}{\ell'_j(x_j)}. \quad (11) \]

By (11),
\[ \sum_{j \neq i} \lambda_j = -\nu \cdot \sum_{j \neq i} \frac{1}{\ell'_j(x_j)} \quad (12) \]

By (8), (9) and (12),
\[ \nu = \frac{\lambda_i - \sum_{j \in N} \lambda_j}{\sum_{j \neq i} \frac{1}{\ell'_j(x_j)}} = \frac{x_i - \mu}{\sum_{j \neq i} \frac{1}{\ell'_j(x_j)}}. \quad (13) \]

Combining (10) and (13) yields
\[ t_i = \left( \ell'_i(x_i) + \frac{1}{\sum_{j \neq i} \ell'_j(x_j)} \right) \cdot (x_i - \mu). \]

If \( t_i < c \), then we know \( \mu = 0 \), and thus,
\[ t_i = \left( \ell'_i(x_i) + \frac{1}{\sum_{j \neq i} \ell'_j(x_i)} \right) \cdot x_i. \]

Now we derive a complete characterization of equilibria.

**Theorem 3.1.** Let \( x_i(0) > 0 \) for all \( i \in N \) and let \( c \in \mathbb{R}_+ \). The tuple \((t, x)\) is a c-capped subgame perfect Nash equilibrium if and only if the following conditions hold:

\[ a_i x_i + b_i + t_i = K \text{ for all } i \in N, \quad (14) \]
\[ \sum_{i \in N} x_i = 1, \quad (15) \]
\[ t_i = \min \left\{ \left( a_i + \frac{1}{\sum_{j \neq i} a_j} \right) \cdot x_i, c \right\} \text{ for all } i \in N, \quad (16) \]
\[ x_i > 0 \text{ for all } i \in N. \quad (17) \]
Proof. We first show $\Rightarrow$: Conditions (14) and (17) follow from Lemma 3.2 and the Wardrop condition. Condition (15) is trivial. Condition (16) follows from Lemma 3.3.

Now we prove $\Leftarrow$: let $(t', x')$ be a tuple that satisfies (14)-(17). We want to show that $(t', x') = (t, x)$, where $(t, x)$ is a subgame perfect equilibrium, which by Proposition 3.1 exists. By (16), we get that tolls $t'$ are feasible w.r.t. $c$. Condition (14) implies that $x'$ is a Wardrop equilibrium with full support with respect to $t'$. Hence, similarly as in equation (3), we get

$$\sum_{j \in \mathbb{N}} (t_j - t'_j) \cdot (x_j - x'_j) \leq 0. \quad (18)$$

Assume by contradiction that there is $i \in \mathbb{N}$ with $t'_i < t_i$ (the case $t'_i > t_i$ follows similarly). By (16), we get

$$t'_i = \left( a_i + \frac{1}{\sum_{j \neq i} 1/a_j} \right) \cdot x'_i < t_i \leq \left( a_i + \frac{1}{\sum_{j \neq i} 1/a_j} \right) \cdot x_i.$$

From this $x'_i < x_i$ follows. Putting things together, we get

$$\sum_{j \in \mathbb{N}} (t_j - t'_j)(x_j - x'_j) > 0$$

a contradiction to (18). \qed

Corollary 3.1. There is at most one tuple $(t, x)$ that satisfies (14)-(17), thus the subgame perfect Nash equilibrium is unique.

If $\mathcal{T}(c)$ is singular, we let $t(c)$ denote $t \in \mathcal{T}(c)$, and $x(c)$ denote $x(t(c))$.

4 Optimal Price Caps for Affine Latencies

Assume that $\ell_i(x_i) = a_i \cdot x_i + b_i$ with $a_i > 0$ and $b_i \geq 0$, and $x_i(0) > 0$ for all $i \in \mathbb{N}$. Now we have everything together to derive an optimal polynomial time algorithm for networks with affine latencies, which is the main result of this section.

Theorem 4.1. Let $\ell_i \in \mathcal{L}_1$ and $x_i(0) > 0$ for all $i \in \mathbb{N}$. Algorithm 1 computes in polynomial time an optimal price cap.

Proof. We show by induction on the iterations of the while loop of Algorithm 1 (indexed by $k \in \mathbb{N}$) that the algorithm computes breakpoints $c_1 > \cdots > c_k > \cdots > c_j$ so that in the intervals $\mathcal{I}_1 = [c_1, \infty)$ and $\mathcal{I}_k = [c_k, c_{k-1}]$ for $k = 2, \ldots, j + 1$ with $c_{j+1} = 0$ the following invariant holds: for all $k = 1, \ldots, j + 1$, the flows $x_i(c), i \in \mathbb{N}$ as defined in Lines 7 and 8 together with prices $t_i(c), i \in \mathbb{N}$ as defined in Lines 9 and 10 constitute the unique subgame perfect Nash equilibrium for all $c \in \mathcal{I}_k$.

Consider the base case $k = 1$. First, observe that Line 12 is well defined since the maximum in Line 12 obviously exists (it is attained at $c_1 = \max_{i \in \mathbb{N}} t_i(\infty)$). The parameterized flow $x(c)$ as defined in lines Line 7 and 8 is a solution of the following system of linear equations:

$$a_i \cdot x_i(c) + b_i + \left( a_i + \frac{1}{\sum_{j \neq i} 1/a_j} \right) \cdot x_i(c) = K(c) \text{ for all } i \in \mathbb{N},$$

$$\sum_{i \in \mathbb{N}} x_i(c) = 1.$$
Algorithm 1: An optimal algorithm for affine latencies.

input: $I = (N, (\ell_i)_{i \in N})$
output: An optimal price cap $c^*$

1. initialize
2. $c_0 \leftarrow \infty$;
3. $A(c_0) \leftarrow \emptyset$;
4. $j \leftarrow 0$;
5. while $A(c_j) \neq N$ do
6. \[ K(c) \leftarrow \frac{1}{2} + \sum_{k \in A(c_j)} \frac{b_k}{a_k} + \sum_{k \in N \setminus A(c_j) \setminus i} \frac{b_i}{2a_i + \sum_{l \in N \setminus \{k\}} \frac{1}{a_l}}; \]
7. $x_i(c) \leftarrow K(c) - \frac{b_i - c}{a_i}$ for all $i \in A(c_j)$;
8. $x_i(c) \leftarrow \frac{K(c) - b_i}{2a_i + \sum_{j \neq i} \frac{1}{a_j}}$ for all $i \in N \setminus A(c_j)$;
9. $t_i(c) \leftarrow c$ for all $i \in A(c_j)$;
10. $t_i(c) \leftarrow \left( a_i + \frac{1}{\sum_{j \neq i} \frac{1}{a_j}} \right) \cdot x_i(c)$ for all $i \in N \setminus A(c_j)$;
11. $c_{j+1} \leftarrow \max \left\{ c \left| \left( a_i + \frac{1}{\sum_{j \neq i} \frac{1}{a_j}} \right) \cdot x_i(c) = c \text{ for some } i \in N \setminus A(c_j) \right\} ;$
12. $c_{j+1} \leftarrow \max \left\{ c_{j+1} \left| i \in N \setminus A(c_j) \right\} ;$
13. $A(c_{j+1}) \leftarrow A(c_j) \cup \arg \max \left\{ c_{j+1} \mid i \in N \setminus A(c_j) \right\} ;$
14. $j \leftarrow j + 1$;
15. $c_{j} \leftarrow \arg \min_{c \in [c_{j-1}, c_{j-1}]} \sum_{i \in N} \ell_i(x_i(c)) \cdot x_i(c)$
16. end
17. output $c^* \in \arg \min \left\{ C(x(c_k^*)), k = 1, \ldots, j \right\}$
By Theorem 3.1 and Corollary 3.1, $x(c), t(c)$ is the unique subgame perfect Nash equilibrium for all $c \in \mathcal{I}_1$.

For the inductive step $k \to k + 1$, assume $x(c), t(c)$ as defined in Lines 7[10] is the unique subgame perfect Nash equilibrium for all $c \in \mathcal{I}_k$, $\ell = 1, \ldots, k$.

First, we show again that the maximum $c_{k+1}$, $i \in N \setminus A(c_k)$ in Line 11 exists so that Lines 11 and 12 are well defined, and, that $c_{k+1} = \max \{c_{k+1}[i] \in N \setminus A(c_k) \} < c_k$ as claimed. To see this, observe that for $i \in N \setminus A(c_k)$, we have $\left( a_i + \frac{1}{\sum_{j \neq i} 1/a_j} \right) x_i(c_k) < c_k$. On the other hand, by the assumption that $x_i(0) > 0$ for all $i \in N$ we have $\left( a_i + \frac{1}{\sum_{j \neq i} 1/a_j} \right) x_i(0) > 0$. As the function $\left( a_i + \frac{1}{\sum_{j \neq i} 1/a_j} \right) x_i(c)$ is continuous in $c$, by the intermediate value theorem, there exists $c_{k+1}'$ with

$$\left( a_i + \frac{1}{\sum_{j \neq i} 1/a_j} \right) x_i(c_{k+1}') = c_{k+1},$$

implying $c_{k+1}' < c_k$ and thus $c_{k+1} < c_k$.

We next prove that $x(c), t(c)$ as defined in Lines 7[10] is the unique subgame perfect Nash equilibrium for all $c \in \mathcal{I}_{k+1}$. We prove this by showing that $x(c), t(c)$ satisfies the conditions of Theorem 3.1. Observe that $x(c)$ is a solution of the following system of linear equations:

$$a_i \cdot x_i(c) + b_i + t_i = K(c)$$

for all $i \in N$, $t_i = c$ for all $i \in A(c_k)$,

$$t_i = \left( a_i + \frac{1}{\sum_{j \neq i} 1/a_j} \right) x_i(c)$$

for all $i \in N \setminus A(c_k)$,

$$\sum_{i \in N} x_i(c) = 1.$$

By Theorem 3.1 it is suffices to show that $t_i(c)$ satisfies condition (110) for all $i \in N$, $c \in \mathcal{I}_{k+1}$. From Lines 11 and 12 we get $t_i(c) \leq c$ for all $i \in N \setminus A(c_k)$ and all $c \in \mathcal{I}_{k+1}$. It remains to show that for all $i \in A(c_k)$ and all $c \in \mathcal{I}_{k+1}$,

$$\left( a_i + \frac{1}{\sum_{j \neq i} 1/a_j} \right) x_i(c) \geq c.$$

We know by the induction hypothesis that $x(c_k)$ is the unique subgame perfect Nash equilibrium, and thus satisfies $\left( a_i + \frac{1}{\sum_{j \neq i} 1/a_j} \right) x_i(c_k) \geq c_k$ for all $i \in A(c_k)$. For all $c \in (c_{k+1}, c_k)$, we have

$$\frac{\partial K(c)}{\partial c} = \sum_{i \in A(c_j)} \frac{1}{a_i} \sum_{i \in A(c_j)} \frac{1}{\sum_{j \neq i} 1/a_j} + \sum_{i \in N \setminus A(c_k)} 2a_i \left( a_i + \frac{1}{\sum_{j \neq i} 1/a_j} \right) a_i \leq 1,$$

and thus by Line 7 we get for all $i \in A(c_k)$ and all $c \in (c_{k+1}, c_k)$,

$$\frac{\partial x_i(c)}{\partial c} \leq 0.$$
So when decreasing $c$ from $c_k$ to $c_{k+1}$, it follows that \( \left(a_i + \frac{1}{\sum_{j \neq i} 1/a_i}\right) \cdot x_i(c) \) increases, and thus \( \left(a_i + \frac{1}{\sum_{j \neq i} 1/a_i}\right) \cdot x_i(c) \geq c \) for all $i \in A(c_k)$ and all $c \in \mathcal{I}_{k+1}$, which completes the induction proof.

For each $\mathcal{I}_k$, with $k = 1, \ldots, j + 1$, the objective function
\[
C(x(c)) = \sum_{i \in N} \ell_i(x_i(c)) \cdot x_i(c)
\]
is quadratic in $c$, thus, we can find a local minimum by comparing the two endpoints and a possible interior minimum point. For the interior minimum, we just need to check first-order optimality conditions, thus, solving a linear equation in $c$. This way, we have effectively partitioned the search space $\mathbb{R}_+$ into at most $n$ intervals. As we solved each segment optimally, taking the best solution leads to the optimal $c^*$.

\[\square\]

**Remark 4.1.** The algorithm is still polynomial for arbitrary demands $d \geq 0$ since solving the linear equation systems appearing in the algorithm can be done in polynomial time in the encoding length of $d$.

The following example demonstrates the calculation process of the algorithm.

**Example 4.1.** Consider the network of Figure 2.

![Diagram](image)

\[\ell_1(x_1) = x_1\]

\[\ell_2(x_2) = x_2 + \frac{1}{2}\]

Figure 2: A demonstration of the algorithm.

By Lemma 3.3, the uncapped subgame perfect Nash equilibrium prices can be found by solving the following system of linear equations
\[
x_1(c) + 2x_1(c) = K(c),
\]
\[
x_2(c) + \frac{1}{2} + 2x_2(c) = K(c),
\]
\[
x_1(c) + x_2(c) = 1,
\]
and thus are given by $t(\infty) = (\frac{7}{5}, \frac{5}{8})$. Initialize $c_1 = \frac{7}{6}$ and $A(c_1) = \{1\}$. Solve the following system of linear equations
\[
x_1(c) + c = K(c),
\]
\[
x_2(c) + \frac{1}{2} + 2x_2(c) = K(c),
\]
\[
x_1(c) + x_2(c) = 1,
\]
yields $x_2(c) = \frac{2c + 1}{8}$. Solving $2 \cdot \frac{2c + 1}{8} = c$ yields $c = \frac{1}{2}$. Hence, $c_2 = \frac{1}{2}$ and $A(c_2) = \{1, 2\}$.  

14
Therefore,

\[
C(x(c)) = \begin{cases} 
\frac{3}{4}, & \text{if } c \leq \frac{1}{2}, \\
\frac{4c^2 - 9c + 27}{32}, & \text{if } \frac{1}{2} < c \leq \frac{7}{6}, \\
\frac{13}{18}, & \text{if } c > \frac{171}{224},
\end{cases}
\]

which is minimized for \( c = 1 \). Notice that we induce the optimal flow with a uniform price cap of \( c = 1 \).

The last example of this section shows that the full Wardrop support assumption is important. Without this assumption, a subgame perfect Nash equilibrium need not be unique.

**Example 4.2.** Consider the network of Figure 3.

\[
\ell_1(x_1) = x_1 \\
\ell_2(x_2) = x_2 \\
\ell_3(x_3) = \frac{2}{5} + \frac{x_3}{5}
\]

Figure 3: Multiple subgame perfect Nash equilibria.

Assume that \( a_3 = \frac{1}{2} \) and \( t_3 = 0 \). Then

\[
B_1^\infty(t_2, t_3) = \begin{cases} 
1 = \frac{t_2 + 1}{2}, & \text{if } 0 \leq t_2 \leq \frac{3}{5}, \\
t_1 = \frac{7 - 5t_2}{5}, & \text{if } \frac{3}{5} < t_2 \leq \frac{5}{7}, \\
t_1 = \frac{5t_2 + 17}{30}, & \text{if } \frac{5}{7} < t_2 \leq \frac{17}{20}, \\
t_1 = \frac{17}{20}, & \text{if } t_2 > \frac{17}{20}.
\end{cases}
\]

and

\[
B_2^\infty(t_1, t_3) = \begin{cases} 
t_2 = \frac{t_1 + 1}{2}, & \text{if } 0 \leq t_1 \leq \frac{3}{5}, \\
t_2 = \frac{7 - 5t_1}{5}, & \text{if } \frac{3}{5} < t_1 \leq \frac{5}{7}, \\
t_2 = \frac{5t_1 + 17}{30}, & \text{if } \frac{5}{7} < t_1 \leq \frac{17}{20}, \\
t_2 = \frac{17}{20}, & \text{if } t_1 > \frac{17}{20}.
\end{cases}
\]

Combining \( B_1^\infty(t_2, t_3) \) and \( B_2^\infty(t_1, t_3) \) (see Figure 4) implies that the set \( \{(t_1, \frac{2}{5} - t_1, 0) \mid \frac{24}{35} \leq t_1 \leq \frac{5}{7}\} \) are subgame perfect Nash equilibria.

**Remark 4.2.** The network pricing game of Example 4.2 exhibits another interesting phenomenon: if \( \ell_3(x_3) = a_3 \cdot x_3 + \frac{6}{5} \) with \( a_3 \leq 2/3 \), the equilibrium flow and profit of firm 3 is 0, whereas if \( \ell_3(x_3) = a_3 \cdot x_3 + \frac{6}{5} \) with \( a_3 > 2/3 \), the equilibrium flow and profit is strictly positive. This implies that firm 3 is worse off by being congestion free than by being sufficiently congestion dependent. The reason is that in the former case firm 3 is too competitive, and thus its two competitors set prices in such a way that prevents the third firm from entering the market.
Figure 4: Best reply correspondence: in blue for player 1, and in (dashed) red for player 2.

5 Optimal Price Caps for Duopolies

In this section, we compare the latency costs of optimal price caps with those of optimal solutions minimizing total congestion (ignoring price competition). The aim of this comparison is to reveal the possible strength of introducing price caps as a mechanism. Our results, however, only hold for duopolies, that is, we assume $n = 2$ for the following analysis. We conjecture though that our results carry over to general $n$ (in the spirit of Pigou instances).

Let $c^*$ denote an optimal price cap (a solution to problem $3L-P$), let $x^*$ denote the optimal flow (ignoring price caps), and let $L \subseteq L_c$ be a class of latency functions. We are interested in the ratio $\rho(L)$ between the latency costs at the optimal price cap and the latency costs of the optimal flow, defined by, for all $\ell \in L$,

$$C(x(c^*)) \leq \rho(L) \cdot C(x^*).$$

Our main results are as follows.

**Theorem 5.1.** Let $T(c) \neq \emptyset$ for all $c \in \mathbb{R}_+$. Then

(i) $\rho(L_d) \leq (1 - \frac{d}{2(2+1)(d+1/2)})^{-1},$

(ii) $\rho(L_c) \leq 2.$

Suppose that we have a subgame perfect duopoly equilibrium in which one firm receives all the flow, say firm 1. The proof of Lemma 5.3 implies that in that case $\ell_1(1) + \ell'_1(1) \leq \ell_2(0)$. This again implies that the duopoly and optimal flow coincide. So in the remainder of this section, we assume that $x_i(t) > 0$ for all $i = 1, 2$ for $t \in T(\infty)$.

Before we prove our the main result, we introduce some helpful lemmas.

**Lemma 5.1.** Let $t \in T(\infty)$. Suppose that $x_i(t) > 0$ for $i = 1, 2$. Then $x_1(0) \geq x_1(t)$ if and only if $x_1(t) \geq \frac{1}{2}$.

**Proof.** Suppose that $x_1(0) \geq x_1(t)$. Then

$$\ell_1(x_1(t)) \leq \ell_1(x_1(0)) \leq \ell_2(x_2(0)) \leq \ell_2(x_2(t)),$$
where the first inequality follows from $x_1(t) \leq x_1(0)$ and $\ell_1$ increasing, the second inequality from $x_1(0) \geq x_1(t) > 0$, and the third inequality from $x_2(0) \leq x_2(t)$ and $\ell_2$ increasing. Since $x_i(t) > 0$ for $i = 1, 2$, we have that $\ell_1(x_1(t)) + t_1 = \ell_2(x_2(t)) + t_2$ and thus $t_1 \geq t_2$. Hence Lemma 3.3 implies that $x_1(t) \geq x_2(t)$ and thus $x_1(0) \geq x_1(t) \geq \frac{1}{2}$.

Suppose that $x_1(0) < x_1(t)$. Then
\[
\ell_2(x_2(t)) < \ell_2(x_2(0)) \leq \ell_1(x_1(0)) < \ell_1(x_1(t)),
\]
where the first inequality follows from $x_2(t) < x_2(0)$ and $\ell_2$ increasing, the second inequality from $x_2(0) > 0$, and the third inequality from $x_1(0) < x_1(t)$ and $\ell_1$ increasing. Since $x_i(t) > 0$ for $i = 1, 2$, we have that $\ell_1(x_1(t)) + t_1 = \ell_2(x_2(t)) + t_2$ and thus $t_1 < t_2$. Hence Lemma 3.3 implies that $x_1(t) < x_2(t)$ and thus $x_1(t) < \frac{1}{2}$.

Define
\[
\mu_1(\ell_i) = \sup_{x_i, x_i^* \geq 0} \frac{(\ell_i(x_i) - \ell_i(x_i^*)) \cdot x_i^*}{\ell_i(x_i) \cdot x_i}
\]
for each $\ell_i \in \mathcal{L}$ and
\[
\mu_1(\mathcal{L}) = \sup_{\ell_i \in \mathcal{L}} \mu_1(\ell_i).
\]
The parameter $\mu_1(\mathcal{L})$ is a smoothness parameter that is well studied in the context of bounding the price of anarchy in routing games. It is well known that this parameter is the important for determining an upper bound on the price of anarchy, and not other characteristics of the instance like the topology of the network. See, for example, Correa et al. [16] and Roughgarden [12] for more details. Observe that $\mu_1(\mathcal{L}) \leq 1$.

**Lemma 5.2.** If $x_i(0) > x_i^*$ and $x_i(0) \leq \frac{1}{2}$ for some $i = 1, 2$, then $C(x(0)) \leq \frac{1}{1 - \mu_1(\mathcal{L})/2} \cdot C(x^*)$.

**Proof.** W.l.o.g. suppose that $x_1(0) > x_1^*$ and $x_1(0) \leq \frac{1}{2}$. By Lemma 2.1 we have
\[
C(x(0)) \leq C(x^*) + \sum_{i=1}^{2} (\ell_i(x_i(0)) - \ell_i(x_i^*)) \cdot x_i^*,
\]
where the second inequality follows from $x_2(0) < x_2^*$ and $\ell_2$ increasing. By definition of $\mu_1(\mathcal{L})$,
\[
\frac{(\ell_1(x_1(0)) - \ell_1(x_1^*)) \cdot x_1^*}{\ell_1(x_1(0)) \cdot x_1(0)} \leq \mu_1(\mathcal{L}).
\]
The lemma then follows because $\ell_1(x_1(0)) \leq C(x(0))$ and $x_1(0) \leq \frac{1}{2}$.

Define
\[
\mu_2(\ell_i) = \sup_{x_i \geq \frac{1}{2}, 0 \leq x_i^* \leq x_i} \frac{(\ell_i(x_i) - \ell_i(x_i^*)) \cdot (x_i^* + 1 - 2x_i)}{\ell_i(x_i)}
\]
for each $\ell_i \in \mathcal{L}$ and
\[
\mu_2(\mathcal{L}) = \sup_{\ell_i \in \mathcal{L}} \mu_2(\ell_i).
\]
The parameter $\mu_2(\mathcal{L})$ is a new smoothness parameter that takes into account the pricing behavior of the firms. It plays an important role in providing an upper bound on the loss in efficiency due to competition. Observe that $\mu_2(\mathcal{L}) \in [0, 1]$.
Lemma 5.3. Let \( t \in T(\infty) \) and \( x_i(t) > 0 \) for all \( i = 1, 2 \). If \( x_i(t) > x_i^* \) and \( x_i(t) \geq \frac{1}{2} \) for some \( i = 1, 2 \), then \( C(x(t)) \leq \frac{1}{1 - \mu_2(L_c)} \cdot C(x^*) \).

Proof. W.l.o.g. suppose that \( x_1(t) > x_1^* \) and \( x_1(t) \geq \frac{1}{2} \). By Lemma 2.1 and 3.3 we have

\[
C(x(t)) \leq C(x^*) + \sum_{i=1}^{2} (\ell_i(x_i(t)) - \ell_i(x_i^*)) \cdot x_i^* + \sum_{i=1}^{2} t_i(x_i^* - x_i(t)),
\]

where the second inequality follows from Lemma 3.3, \( x^* \) tight.

Proof of Theorem 5.1. Let \( x_i(t) \geq 0 \) for all \( i = 1, 2 \). Suppose that either \( x_1(t) < x_1(0) \), or \( x_1(t) < x_1^* < x_1(t) \). By the Theorem of the Maximum (Berge [6]), the profit function of each firm is continuous in \( c \). Given that the profit function of each firm is continuous in the toll vector (Hayrapetyan et al. [29]), and the toll vector continuously changes the induced flow (Beckmann et al. [5]), there exist a price cap \( c \) such that \( x_1(c) = x_1^* \). So, we can assume that \( x_1^* < x_1(0) \) and \( x_1^* < x_1(t) \).

Suppose that \( x_1(t) \leq \frac{1}{2} \). Then by Lemma 5.1 \( x_1(0) \leq x_1(t) \leq \frac{1}{2} \). Since \( \mu_1(L_d) \leq \frac{d}{(d+1)^{(d+1)/d}} \) (Correa et al. [16]) and \( \mu_1(L_c) \leq 1 \), the result follows by Lemma 5.2.

Suppose that \( x_1(t) \geq \frac{1}{2} \). Then

\[
\mu_2(L_d) = \sup_{x_i(t) \geq \frac{1}{2}, 0 \leq x_i^* \leq x_i(t)} \frac{(\ell_1(x_1(t)) - \ell_1(x_1^*)) \cdot (x_1^* + 1 - 2x_1(t))}{\ell_1(x_1(t))} \leq \frac{d}{2(d+1)^{(d+1)/d}},
\]

and

\[
\mu_2(L_c) = \sup_{x_i(t) \geq \frac{1}{2}, 0 \leq x_i^* \leq x_i(t)} \frac{(\ell_1(x_1(t)) - \ell_1(x_1^*)) \cdot (x_1^* + 1 - 2x_1(t))}{\ell_1(x_1(t))} \leq \sup_{\frac{1}{2} \leq x_1(t)} 1 - x_1(t) \leq \frac{1}{2},
\]

and the result follows by Lemma 5.3.

The following example shows that the bound of 8/7 in Theorem 5.1 for affine latencies is tight.

18
Figure 5: The bound for affine latencies is tight.

Example 5.1. Consider the network of Figure 5 with $n \geq 2$.

Let $t \in T(\infty)$. By Lemma 3.3, $t = x(t) = x(0) = \left( \frac{1}{2(n-1)}, \ldots, \frac{1}{2(n-1)}, \frac{1}{2(n-1)} \right)$ and $x^* = \left( \frac{1}{4(n-1)}, \ldots, \frac{1}{4(n-1)}, \frac{3}{4} \right)$.

If $0 \leq c \leq \frac{1}{2(n-1)}$, then $t_1 = \ldots = t_n = c$ and thus $x(c) = \left( \frac{1}{2(n-1)}, \ldots, \frac{1}{2(n-1)}, \frac{1}{2(n-1)} \right)$. If $c > \frac{1}{2(n-1)}$, then $t_1 = \ldots = t_2 = \frac{1}{2(n-1)}$ and thus $x(c) = \left( \frac{1}{2(n-1)}, \ldots, \frac{1}{2(n-1)}, \frac{1}{2} \right)$. Hence any price cap is optimal. So

$$\min_{c \in \mathbb{R}_+} \frac{C(x(c))}{C(x^*)} = \frac{8}{7}.$$

The main result in Theorem 5.1 assumes that a $c$-capped subgame perfect Nash equilibrium exists for all $c \in \mathbb{R}_+$. The following example shows that an equilibrium need not exist for quadratic latency functions.

Example 5.2. Consider the network of Figure 6.

$$\ell_1(x_1) = x_1$$

$$\ell_2(x_2) = 0$$

Figure 6: No subgame perfect Nash equilibrium.

Then

$$B_1^\infty(t_2) = \begin{cases} t_1 = \frac{2t_2}{3}, & \text{if } 0 \leq t_2 \leq 3, \\ t_1 = t_2 - 1, & \text{if } t_2 > 3, \end{cases}$$

and

$$B_2^\infty(t_1) = \begin{cases} t_2 = \arg \max \left( 1 - (t_2 - t_1)^{1/2} \right) \cdot t_2, & \text{if } 0 \leq t_1 \leq \frac{1}{4}, \\ t_2 = t_1, & \text{if } t_1 \geq \frac{1}{4}. \end{cases}$$

Observe that $B_2^\infty(t_1)$ is not convex at $t_1 = \frac{1}{4}$. Combining $B_1^\infty(t_2)$ and $B_2^\infty(t_1)$ (see Figure 7) implies that there is no uncapped subgame perfect Nash equilibrium. In fact, the result is even stronger: there is no $c$-capped subgame perfect Nash equilibrium whenever $c \geq \frac{1}{2}$. 

19
Remark 5.1. The nonexistence result in Example 5.2 violates the assumption that latency functions are strictly increasing. Nevertheless, we can change the latency function of link 2 into \( \ell_2(x_2) = a_2 \cdot x_2 \) with \( a_2 < \sqrt{17} - 4 \) and obtain the same result: an uncapped subgame perfect duopoly equilibrium need not exist.

The next and final example shows that the latency costs at the optimal price cap can be arbitrarily worse than the optimal latency costs due to nonexistence of an uncapped subgame perfect duopoly equilibrium.

Example 5.3. Consider the parallel network of Figure 8, where \( d \geq 3 \).

Then

\[
B_1^\infty(t_2) = \begin{cases} 
    t_1 = \frac{d(t_2 + (\frac{d}{d+1})^d)}{d+1}, & \text{if } 0 \leq t_2 \leq d + 1 - \left(\frac{d-1}{d}\right)^d, \\
    t_1 = t_2 + \left(\frac{d}{d+1}\right)^d - 1, & \text{if } t_2 > d + 1 - \left(\frac{d-1}{d}\right)^d,
\end{cases}
\]

and

\[
B_2^\infty(t_1) = \begin{cases} 
    t_2 = \arg \max_{t_2} \left(1 - (t_2 - t_1 + b_2)^{1/d}\right) \cdot t_2, & \text{if } 0 \leq t_1 \leq \frac{d(\frac{d-1}{d})^d - 1}{d^d}, \\
    t_2 = t_1 - \left(\frac{d}{d+1}\right)^d, & \text{if } t_1 \geq \frac{d(\frac{d-1}{d})^d - 1}{d^d}.
\end{cases}
\]

Combining \( B_1^\infty(t_2) \) and \( B_2^\infty(t_1) \) implies that there is only an equilibrium for all price caps \( c \leq \left(\frac{d}{d+1}\right)^{d-1} \). If \( c \leq \left(\frac{d}{d+1}\right)^{d-1} \), then \( t_1(c) = t_2(c) = c \). Similarly as in Example 5.2 an
important reason for the nonexistence of equilibria seems to be the nonconvexity of the set of best replies for player 2 at $t_1 = (\frac{d-1}{d})^{d-1}$. Since $x(c) = x(0) = (\frac{d-1}{d}, \frac{1}{d})$ for all $c \leq (\frac{d-1}{d})^{d-1}$, and $x^* = \left(\frac{d-1}{d(d+1)^{1/d}}, 1 - \frac{d-1}{d(d+1)^{1/d}}\right)$, we have

$$\rho(\mathcal{L}_d) \geq \min_{0 \leq c \leq (\frac{d-1}{d})^{d-1}} \frac{C(x(c))}{C(x^*)} = \frac{(d + 1)^{(d+1)/d}}{(d + 1)^{(d+1)/d} - (d - 1)} \to \infty \text{ as } d \to \infty.$$

6 Discussion

We consider a network pricing game in which, in the first stage, edge owners set prices so as to maximize profit, and, in the second stage, users choose paths that minimize their total costs. The problem with these games is that subgame perfect Nash equilibria might not exist, and if they exist, they can induce arbitrarily inefficient flows. We therefore allow for competition regulation and consider a (uniform) price cap regulation policy. Our main goal is, firstly, to find a price cap that minimizes the inefficiency of the induced flow, and, secondly, to quantify the loss in efficiency due to competition even in the presence of competition regulation. Our main results are the following. For parallel networks with affine latency functions and a full support Wardrop equilibrium, we provide an algorithm that finds the optimal price cap in polynomial time. Due to multiplicity of subgame perfect Nash equilibria, the algorithm is not valid for instances without a full support Wardrop equilibrium. Then we show that the ratio between the congestion costs at an optimal price cap and the optimal congestion costs are at most 2 for duopoly instances with an uncapped subgame perfect Nash equilibrium. This bound lowers down to 8/7 for affine latency functions. However, due to the nonexistence of subgame perfect Nash equilibria, we can give a sequence of instances such that the performance of the induced flow at an optimal price cap is arbitrarily bad.

The following questions remain open. First, is there a (polynomial) algorithm that finds the optimal price cap for more general instances? In particular, for parallel networks with affine latency functions that have no full support Wardrop equilibrium. Secondly, we have provided an instance that shows that the flow at an optimal price cap can be 8/7 times as costly as the optimal flow. Is this bound tight for parallel networks with affine latencies? Thirdly, for two-link networks with affine latency functions, a worst-case guarantee of 8/7 also holds for the simple algorithm that selects the best flow from the Wardrop flow and the uncapped subgame perfect Nash equilibrium flow. How does this simple algorithm perform for arbitrary parallel networks? Fourthly, how do results change if we assume a different user behavior, like elastic users (Chau and Sim [10], Hayrapetyan et al. [29] or Ozdaglar [38]), atomic splittable users (Haurie and Marcotte [28], Orda et al. [37] or Cominetti et al. [14]), or stochastic users (Daganzo and Sheffi [18] or Guo et al. [26]?).

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