Correlation Functions of the XXZ model for $\Delta < -1$

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Dedicated to Professor Chen Ning Yang on the occasion of his seventieth birthday

Abstract. A new approach to the correlation functions is presented for the XXZ model in the anti-ferroelectric regime. The method is based on the recent realization of the quantum affine symmetry using vertex operators. With the aid of a boson representation for the latter, an integral formula is found for correlation functions of arbitrary local operators. As a special case it reproduces the spontaneous staggered polarization obtained earlier by Baxter.
\section{Introduction}

In this article we consider the one-dimensional infinite spin-chain

\begin{equation}
H_{XXZ} = -\frac{1}{2} \sum_{k=-\infty}^{\infty} (\sigma_{k+1}^{x} \sigma_{k}^{x} + \sigma_{k+1}^{y} \sigma_{k}^{y} + \Delta \sigma_{k+1}^{z} \sigma_{k}^{z}) \tag{0.1}
\end{equation}

known classically as the XXZ model. We shall limit ourselves strictly to the anti-ferroelectric regime $\Delta < -1$. Our aim is to find an exact expression for the spin correlation functions $\langle \sigma_{1}^{i} \cdots \sigma_{n}^{i} \rangle$, where $\sigma_{i}^{x} = \sigma_{i}^{x}, \sigma_{i}^{y}, \sigma_{i}^{z}$ and $\langle \cdot \rangle = \langle \text{vac} \mid \cdot \mid \text{vac} \rangle$ denotes the ground state average. Here the ground state $|\text{vac}\rangle$ means one of the two ground states in the anti-ferroelectric regime. In Baxter’s paper \cite{Baxter1971} they are denoted by $|\pm\rangle$. The present paper is based entirely on the framework of the recent work \cite{Jimbo1990}, which describes the realization of the quantum affine symmetry for (0.1) using the $q$-deformed vertex operators. Let us recall below the contents of \cite{Jimbo1990} which are relevant to the subsequent discussions.

The Hamiltonian (0.1) is an operator acting on the infinite tensor product $V^{\otimes \infty} = \cdots V \otimes V \otimes V \otimes \cdots$ of the two dimensional space $V = \mathbb{C}^{2}$. The space $V^{\otimes \infty}$ admits also an action of the quantum affine algebra $U = U_{q}(\hat{sl}_{2})$ via the iterated coproduct. A naive computation shows that the algebra $U$ provides an exact symmetry for (0.1). Namely if $\Delta = (q + q^{-1})/2$, then $[H_{XXZ}, U'] = 0$ where $U' = U_{p}(\hat{sl}_{2})$ denotes the subalgebra ‘without the grading operator $d$’ \cite{Jimbo1990}, while $d$ plays the role of the boost operator. However the actions of (0.1) and $U$ are both defined only formally, and the issue is how to extract the theory free from the difficulties of divergence.

The basic idea in \cite{Jimbo1990} is to replace the formal object $V^{\otimes \infty}$ by the level 0 $U$-module

\begin{equation}
\mathcal{F}_{\lambda, \mu} = V(\lambda) \hat{\otimes} V(\mu)^{\ast} \simeq \text{Hom}(V(\mu), V(\lambda)), \tag{0.2}
\end{equation}

where $V(\lambda)$ $(\lambda = \Lambda_{0}, \Lambda_{1})$ denotes the level 1 highest weight $U$-module, and $V(\lambda)^{\ast}$ signifies its dual (the suffix indicates that the $U$-module structure is given via the antipode $a$). The choice $\lambda = \mu$ (resp. $\lambda \neq \mu$) is responsible for the even (resp. odd) particle sector. $V(\Lambda_{0})$ (resp. $V(\Lambda_{1})$) means that we are working in the boundary condition $\sigma_{2k}^{z} = 1, \sigma_{2k+1}^{z} = -1$ (resp. $\sigma_{2k}^{z} = -1, \sigma_{2k+1}^{z} = 1$) for $k \gg 0$, and $V(\Lambda_{0})^{\ast}$ (resp. $V(\Lambda_{1})^{\ast}$) means the same thing for $k \ll 0$. The tensor product should be completed in the $q$-adic sense to allow for infinite sums (see \cite{Jimbo1990} for a precise treatment). To make contact with the naive picture of $V^{\otimes \infty}$, one utilizes the embedding of $V(\lambda)$ into the half infinite tensor product $\cdots V \otimes V \otimes V^{\otimes \infty}$. This is supplied by iterating the vertex operators

\begin{equation}
\Phi_{\lambda}^{\mu V} : V(\lambda) \longrightarrow V(\mu) \otimes V. \tag{0.3}
\end{equation}

(It is conjectured \cite{Jimbo1990} that there is a unique normalization of (0.3) which makes the infinite iteration convergent.) Similarly $V(\mu)^{\ast a}$ embeds to the other half infinite tensor product $V \otimes V \otimes V \otimes \cdots$, giving altogether the embedding

\begin{equation}
\mathcal{F}_{\lambda, \mu} \longrightarrow \cdots V \otimes V \otimes V \otimes \cdots. \tag{0.4}
\end{equation}

Eq. (0.4) provides the principle of interpreting the notions defined in the picture $V^{\otimes \infty}$ by pulling them back to $\mathcal{F}_{\lambda, \mu}$. The translation operator $T$ is the first such example. (The Hamiltonian (0.1) itself is defined in terms of $T$ and the grading operator $d$; see \cite{Jimbo1990}.) In this paper we follow the same principle to formulate the local operators $\sigma_{i}^{\lambda}$ as acting on $\mathcal{F}_{\lambda, \mu}$. 


In the language of (0.2) the ground state (vacuum) vector is given by the identity element of $F_{\lambda,\lambda} = \text{Hom}(V(\lambda), V(\lambda))$. The inner product of two vectors $f, g \in F_{\lambda,\lambda}$ is given as the trace

$$\langle f|g \rangle = \frac{\text{tr}_{V(\lambda)}(q^{-\rho}fg)}{\text{tr}_{V(\lambda)}(q^{-\rho})} \quad (0.5)$$

where $\rho = \Lambda_0 + \Lambda_1$ (we have changed the normalization from [2], see Sect. 1 below). Thus the correlation functions $\langle \text{vac}|\sigma_{i_1}^{z_1}\cdots\sigma_{i_n}^{z_n}|\text{vac} \rangle$ can be expressed as the trace of products of the vertex operators (0.3).

To perform the evaluation of these traces, we invoke the bosonization method. The realization of the $q$-deformed currents on level one modules was done in [4] using (ordinary) bosons. In the same spirit we derive the formulas for the vertex operators (0.3) in terms of bosons. This leads to an explicit formula for the correlators in terms of certain integrals of meromorphic functions. We verify that in the simplest case of the one-point function $\langle \sigma_z^k \rangle$ this formula reproduces the known result for the spontaneous staggered polarization due to Baxter [1]. Jacques Perk noted a strong similarity between our formula and the formula for the Ising model spin-spin correlation functions given in [5,6].

The bosonization of the vertex operators also enables us to compute the $n$-point functions discussed in Sect. 6.8 of [2]. They are the matrix elements (not the trace) of the product of the vertex operators with respect to the highest weight vectors. The conjectural formula (6.39) of [2] is thus proved. In this paper, we do not go into details on this matter.

The plan of the paper is as follows. In Sect. 1 we formulate the local operators and their correlators using the scheme mentioned above. In our algebraic formulation the algebra $U$, intertwiners, etc. are a priori defined over the base field $F = Q(q)$ ($q$ an indeterminate, see the remark at the end of Sect. 2); we shall see however that the resulting formulas are meaningful for complex values of $q$ with $|q| < 1$. In Sect. 2 we outline the bosonization of the vertex operators. The basic ingredients are the Drinfeld realization of the algebra $U'$ and the result of [4], which we recall briefly. Unlike the classical case ($q = 1$) only one of the components $\Phi_-(z)$ of (0.3) has the exponential form, while the other one does not but is given as the $q$-commutator of $\Phi_-(z)$ with a generator $f_1$ of $U$. Sect. 3 is devoted to the formula for the correlators.
§1. Vacuum expectation values of local operators

Let us recall some notations of [2]. We denote by $V = F_{\uparrow} \oplus F_{\downarrow}$ the 2-dimensional vector space on which the Pauli matrices $\sigma^x, \sigma^y, \sigma^z$ act. We consider $V$ as the 2-dimensional $U_q(\mathfrak{sl}(2))$-module. The XXZ-Hamiltonian formally acts on the infinite tensor product $V^\otimes \infty$ of $V$. We label the components of the tensor product by integers $k \in \mathbb{Z}$ from right ($k \to -\infty$) to left ($+\infty \leftarrow k$). In our mathematical scheme, we replace the semi-infinite tensor product of the components $k \geq 1$ by the irreducible highest weight $U_q(\mathfrak{sl}(2))$-module $V(L_i)$ ($i = 0,1$). Let us define the action of the local operators on $V(L_i)$. Let $L \in \text{End}(V \otimes \cdots \otimes V)$. The operator $L$ naturally acts on the tensor product of the components $n \geq k \geq 1$ of $V^\otimes \infty$. We want to interpret this action as one in $\text{End}(V(L_i))$.

For this purpose, we use the vertex operators as the generating functions of their Fourier components in terms of the spectral parameter $\sigma$. They are normalized as

\[
\hat{\Phi}_L^\mu \ (z) : V(\sigma) \rightarrow V(\sigma) \otimes V_z,
\hat{\Phi}_{L+}^\mu (z)(v) = \hat{\Phi}_L^\mu (z)(v) \otimes v_+ + \hat{\Phi}_L^\mu (z)(v) \otimes v_- \quad (1.1)
\]

where $\lambda = \Lambda_0, \mu = \Lambda_1$ or $\lambda = \Lambda_1, \mu = \Lambda_0$, and $V_z = V \otimes F[z, z^{-1}]$ signifies the $U_q(\mathfrak{sl}(2))$-module associated to $V$ with the spectral parameter $z$ (see [2], Sect. 6). Precisely speaking, we consider the vertex operators as the generating functions of their Fourier components in terms of the spectral parameter $z$. They are normalized as

\[
\hat{\Phi}_{\Lambda_0} \ (u_{\Lambda_0}) = |u_{\Lambda_0}\rangle \otimes v_- + \cdots, \quad \hat{\Phi}_{\Lambda_1} \ (|u_{\Lambda_1}\rangle) = |u_{\Lambda_1}\rangle \otimes v_+ + \cdots, \quad (1.2)
\]

\[
\hat{\Phi}_{\Lambda_0}^\Lambda (z) = \hat{\Phi}_{\Lambda_0}^\Lambda (z/q^2), \quad \hat{\Phi}_{\Lambda_1}^\Lambda (z) = -q^{-1} \hat{\Phi}_{\Lambda_0}^\Lambda (z/q^2),
\hat{\Phi}_{\Lambda_0}^\Lambda (z) = -q \hat{\Phi}_{\Lambda_0}^\Lambda (z/q^2), \quad \hat{\Phi}_{\Lambda_1}^\Lambda (z) = \hat{\Phi}_{\Lambda_0}^\Lambda (z/q^2),
\]

\[
\frac{(q^2; q^4)_\infty}{(q^4; q^4)_\infty} \hat{\Phi}_{\Lambda_0} (z) \hat{\Phi}_{\Lambda_0} (z) = \text{id}_{V(L)}.
\]

Here $|u_{\Lambda_0}\rangle$ denotes the highest weight vector of $V(L_i)$. We have used the standard notation $(z;p)_\infty = \prod_{j=0}^{\infty} (1 - zp^j)$.

Set $\Lambda = \Lambda_{n-2}$ for simplicity of notation. Given an $L$ as above, we define the operator $L = \rho(l_{z_1}, ..., z_l) \in \text{End}(V(L_i))$ by

\[
L = \frac{(q^2; q^4)_\infty}{(q^4; q^4)_\infty} \hat{\Phi}_{\Lambda_0}^\Lambda (z_1) \circ \cdots \circ \hat{\Phi}_{\Lambda_0}^\Lambda (z_n)
\circ (\text{id}_{V(L_{i+n})} \otimes L) \circ \hat{\Phi}_{\Lambda_0}^\Lambda (z_{n+1}) \circ \cdots \circ \hat{\Phi}_{\Lambda_0}^\Lambda (z_l).
\]

In this paper, we use the following convention (different from that of [2]) for the invariant bilinear form on $P = \mathbb{Z} \Lambda_0 \oplus \mathbb{Z} \Lambda_1 \oplus \mathbb{Z} \delta$:

\[
(\Lambda_0, \Lambda_0) = 0, (\Lambda_0, \alpha_1) = 0, (\Lambda_0, \delta) = 1,
(\alpha_1, \alpha_1) = 2, (\alpha_1, \delta) = 0, (\delta, \delta) = 0.
\]

Note that $\Lambda_1 = \Lambda_0 + \alpha_1/2, \delta = \alpha_0 + \alpha_1$. We set $\rho = \Lambda_0 + \Lambda_1$ as usual, and also $\alpha = \alpha_1$ for simplicity. We identify $P^* = \mathbb{Z} \Lambda_0 \oplus \mathbb{Z} \Lambda_1 \oplus \mathbb{Z} \delta$ as a subset of $P$ via $(\, , \, )$. We have $\rho = 2d + \alpha_1/2$. 

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The vacuum expectation value \( \langle L \rangle_{z_n, \ldots, z_1}^{(i)} \) of the local operator \( L \) is given by

\[
\langle L \rangle_{z_n, \ldots, z_1}^{(i)} = \frac{\text{tr}_{V(A_i)}(q^{-2\rho} \rho_{z_n, \ldots, z_1}(L))}{\text{tr}_{V(A_i)}(q^{-2\rho})}.
\]

This is a consequence of the formula for the invariant inner product in the space \( V(A_i) \otimes V(A_i)^{sa} \) [2]. For the XXZ model correlator we specialize the spectral parameters to \( z_1 = \cdots = z_n \). We expect that the formula unspecialized would give the equal-row vertical arrow correlator for the inhomogeneous six-vertex model with \( z_k \) being the trigonometric spectral parameter of the \( k \)-th vertical line. Here we use the usual language of vertex models on the 2-dimensional square lattice, in which the fluctuation variables are described as arrows sitting on vertical or horizontal edges. Note the following selection rule, which is specific to the six vertex model: \( \langle L \rangle_{z_n, \ldots, z_1}^{(i)} = 0 \) for \( L = E_{\varepsilon_n, \varepsilon_n} \otimes \cdots \otimes E_{\varepsilon_1, \varepsilon_1} \) \( (E_{ij} \) is a matrix unit) such that \( \varepsilon_1 + \cdots + \varepsilon_n \neq \varepsilon_1' + \cdots + \varepsilon_n' \).

§2. Bosonization

Let us first recall Drinfeld’s realization of the quantum affine algebra \( U' = U_q'(\widehat{sl}(2)) \) [7]. It is an associative algebra generated by the letters \( \{ x_k^\pm \mid k \in \mathbb{Z} \}, \{ a_l \mid l \in \mathbb{Z}_{#0} \}, \gamma^{\pm 1/2} \) and \( K \), satisfying the following defining relations.

\[
\gamma^{\pm 1/2} \in \text{the center of the algebra},
\]

\[
[a_k, a_l] = \delta_{k+l,0} \frac{1}{k} [2k] \frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}, \quad [a_k, K] = 0,
\]

\[
K x_k^+ K^{-1} = q^{+2} x_k^+,
\]

\[
[a_k, x_l^+] = \pm \frac{1}{k} [2k] \gamma^{(k-l)/2} x_{k+l}^+, \quad x_{k+1}^+ x_l^+ - q^{+2} x_l^+ x_{k+1}^+ = q^{+2} x_k^+ x_{l+1}^+ - x_{l+1}^+ x_k^+,
\]

\[
[x_k^+, x_l^-] = \frac{1}{q - q^{-1}} (\gamma^{(k-l)/2} \psi_{k+l} - \gamma^{(l-k)/2} \varphi_{k+l}),
\]

where \( [n] = (q^n - q^{-n})/(q - q^{-1}) \) and \( \{ \psi_r, \varphi_s \mid r \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}_{\leq 0} \} \) are related to \( \{ a_l \mid l \in \mathbb{Z}_{#0} \} \) by

\[
\sum_{k=0}^{\infty} \psi_k z^{-k} = K \exp \left( (q - q^{-1}) \sum_{k=1}^{\infty} a_k z^{-k} \right), \quad \sum_{k=0}^{\infty} \varphi_k z^k = K^{-1} \exp \left( -q^{-1} \sum_{k=1}^{\infty} a_{-k} z^k \right).
\]

The standard Chevalley generators \( \{ e_i, f_i, t_i \} \) are given by the identification

\[
t_0 = \gamma K^{-1}, \quad t_1 = K, \quad e_1 = x_0^+, \quad f_1 = x_{-1}^-, \quad e_0 t_1 = x_1^-, \quad t_1^{-1} f_0 = x_{-1}^+.
\]

We use the coproduct

\[
\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta(t_i) = t_i \otimes t_i.
\]
are isomorphic to the irreducible highest weight modules $V$ vectors $u$ describe the vertex operators in terms of the representation constructed above, and then With these actions $W$ representations of $M.\text{Jimbo et al.}$ \textit{sl}$F$ $U$ determine $\tilde{\Phi}$ from them only. We better use the relations with Drinfeld’s generators information which will suffice for our purpose [9].

formulas for the coproduct of them, in general. However we have the following partial information which will suffice for our purpose [9].

$$a_k = \text{the left multiplication by } a_k \otimes 1 \text{ for } k < 0,$$

$$e^{\beta_i}(f \otimes e^{\beta_j}) = f \otimes e^{\beta_i + \beta_j},$$

$$\partial_\alpha(f \otimes e^\beta) = (\alpha, \beta)f \otimes e^\beta.$$

We let also

$$K = 1 \otimes q^{\partial_\alpha}, \quad \gamma = q \otimes \text{id}.$$

The actions of the generators $\{x_n^\pm\}$ are given through the generating functions $X^\pm(z) = \sum_{n \in \mathbb{Z}} x_n^\pm z^{-n-1}$ as follows.

$$X^+(z) = \exp \left( \sum_{n=1}^\infty a_n z^{-n/2} \right) \exp \left( - \sum_{n=1}^\infty a_n z^{-n/2} \right) e^{\alpha z} \partial_\alpha,$$

$$X^-(z) = \exp \left( - \sum_{n=1}^\infty a_n z^{-n/2} \right) \exp \left( \sum_{n=1}^\infty a_n z^{-n/2} \right) e^{-\alpha z} \partial_\alpha.$$

With these actions $W$ becomes a $U'$-module. The submodules $F[a_{-1}, a_{-2}, \cdots] \otimes F[P]_i$ are isomorphic to the irreducible highest weight modules $V(\Lambda_i)$ with the highest weight vectors $u_{\Lambda_0} = 1 \otimes 1$ and $u_{\Lambda_1} = 1 \otimes e^{\alpha/2}$. The grading operator $d$ is introduced by

$$-d(a_{-1_{i_1}} \cdots a_{-r_{i_r}} \otimes e^\beta) = \left( \sum_{j=1}^r n_{ij} + \frac{\langle \beta, \beta \rangle}{2} - \frac{\langle \Lambda_i, \Lambda_i \rangle}{2} \right) (a_{-1_{i_1}} \cdots a_{-r_{i_r}} \otimes e^\beta).$$

We remark that the trace of $p^{-d} X^+(z) X^-(z)$ on $V(\Lambda_0)$ is calculated in [8]. We shall describe the vertex operators in terms of the representation constructed above, and then compute a similar trace for them (see Sect. 3).

Let the components $\tilde{\Phi}_\pm(z) = \tilde{\Phi}_{\Lambda_0}^\nu(z)$ be defined as in (1.1). From the condition that it intertwines the action of $x_0^-$ we find

$$\tilde{\Phi}_+(z) = [\tilde{\Phi}_-(z), x_0^-]_q,$$  \hspace{1cm} (2.1)

where $[X, Y]_q = XY - qYX$. The intertwining relations with the Chevalley generators determine $\Phi_{\pm}(z)$ uniquely, but it is not easy to get the expression of the vertex operators from them only. We better use the relations with Drinfeld’s generators $a_{\alpha}$. We lack the formulas for the coproduct of them, in general. However we have the following partial information which will suffice for our purpose [9].
Proposition 2.1. For \( k \geq 0 \) and \( l > 0 \) we have

\[
\Delta(x^+_k) = x^+_k \otimes \gamma^k + \gamma^{2k} K \otimes x^+_k + \sum_{i=0}^{k-1} \gamma^{(k+3)i/2} \psi_{k-i} \otimes \gamma^{k-i} x^+_i \mod N_- \otimes N^2_+, \\
\Delta(x^-_l) = x^-_l \otimes \gamma^{-l} + K^{-1} \otimes x^-_l + \sum_{i=1}^{l-1} \gamma^{-(l-i)/2} \varphi_{-l+i} \otimes \gamma^{-l+i} x^+_i \mod N_- \otimes N^2_+, \\
\Delta(a_l) = a_l \otimes \gamma^{l/2} + \gamma^{3l/2} a_l \mod N_- \otimes N_+, \\
\Delta(a_{-l}) = a_{-l} \otimes \gamma^{-3l/2} + \gamma^{-l/2} a_{-l} \mod N_- \otimes N_+.
\]

Here \( N_\pm \) and \( N^2_\pm \) are left \( F[\gamma^\pm, \psi^r, \varphi^s | r, -s \in \mathbb{Z}_{\geq 0}] \)-modules generated by \( \{x^\pm_k | k \in \mathbb{Z}\} \) and \( \{x^\pm_m x^\pm_n | m, n \in \mathbb{Z}\} \) respectively.

By using Proposition 2.1 and noting that \( N_\pm v_\pm = 0, N_+ v_- \subset F[z, z^{-1}] v_+ \), we get the exact relations

\[
[a_k, \tilde{\Phi}_-(z)] = q^{7k/2} \frac{[k]}{k} z^k \tilde{\Phi}_-(z) \quad k > 0, \\
[a_{-k}, \tilde{\Phi}_-(z)] = q^{-5k/2} \frac{[k]}{k} z^{-k} \tilde{\Phi}_-(z) \quad k > 0, \\
[\tilde{\Phi}_-(z), X^+(w)] = 0, \quad t_1 \tilde{\Phi}_-(z) t_1^{-1} = q \tilde{\Phi}_-(z).
\]

These conditions along with the normalization (1.2) determine the form of \( \tilde{\Phi}_-(z) \) completely. Explicitly we have, on \( V(\Lambda_i) \) \((i = 0, 1)\),

\[
\tilde{\Phi}_-(z) = \exp \left( \sum_{n=1}^{\infty} \frac{a_n}{[2n]} q^{7n/2} z^n \right) \exp \left( - \sum_{n=1}^{\infty} \frac{a_n}{[2n]} q^{-5n/2} z^{-n} \right) e^{a/2} (-q^3 z)^{(\partial_\alpha + i)/2}, \quad (2.2)
\]

It can be checked directly that the operators given by (2.1, 2.2) enjoy the correct intertwining properties.

Let us make a comment on the base field \( F = \mathbb{Q}(q) \). In the above expression of the bosonization of the vertex operators, the square root of \( q \) appears. But, it can be absorbed in the boson oscillators if we change the definition of their commutation relations. Note also that the combination \((\partial_\alpha + i)/2\) in the power of \(-q^3 z\) produces only an integer power acting on \( V(\Lambda_i) \). In any event, the appearance of \( q^{1/2} \) is only superficial, and our theory is free from the ambiguity in the choice of the square root.
§3. Calculation of correlators

As explained in Sect. 1, for the evaluation of the correlators it is necessary to calculate the trace of the product of the vertex operators. Using the boson realization described in Sect. 2, we shall treat the trace of the form \( \text{tr}(x^{-d}y^a) \) where \( x, y \) are complex parameters with \( |x| < 1 \). Since \( q^{-2\rho} = q^{-4d-\alpha} \), the choice \( x = q^{\alpha}, y = q^{-1} \) will be relevant to the correlators. The calculation is simplified by the technique of Clavelli and Shapiro ([10], Appendix C). Their prescription is as follows. Introduce a copy of bosons \( \{b_n\} \) satisfying \([a_m, b_n] = 0\) and the same commutation relations as the \(a_n\). Let

\[
\tilde{a}_n = \frac{a_n}{1 - x^n} + b_{-n} \quad (n > 0), \quad a_n + \frac{b_{-n}}{x^n - 1} \quad (n < 0).
\]

For a linear operator \( O = O(\{a_n\}) \) on the Fock space \( \mathcal{F}_a = F[a_{-1}, a_{-2}, \cdots] \), let \( \hat{O} = O(\{\tilde{a}_n\}) \) be the operator on \( \mathcal{F}_a \otimes \mathcal{F}_b \) (\( \mathcal{F}_b = F[b_{-1}, b_{-2}, \cdots] \)) obtained by substituting \( \tilde{a}_n \) for \( a_n \). We have then

\[
\text{tr}_{\mathcal{F}_a}(x^{-d} O) = \frac{\langle 0|\hat{O}|0\rangle}{\prod_{n=1}^{\infty}(1 - x^n)},
\]

where \( d = -\sum_{n>0} \frac{n^2}{|n|} a_{-n} a_n \) and \( \langle 0|\hat{O}|0\rangle \) denotes the usual expectation value with respect to the Fock vacuum \( |0\rangle = 1 \otimes 1 \in \mathcal{F}_a \otimes \mathcal{F}_b \), \( \langle 0|0\rangle = 1 \).

In order to apply their method to our case, we need the following. Set

\[
f(z) = (xz; x)_\infty (q^2x/z; x)_\infty, \quad g(z) = \prod_{n=1}^{\infty} \frac{(zq^2x^n; q^4)_\infty}{(zq^4x^n; q^4)_\infty}.
\]

Let us denote the operators \( X^- (z) \) and \( \phi_\pm (z) \) with \( \tilde{a}_n \) substituted for \( a_n \) by \( J(z) \) and \( \phi_\pm (z) \) (after the normal ordering) are

\[
J(z) = f(1) \exp\left(-\sum_{n=1}^{\infty} \frac{q^{n/2}}{|n|} (z^n a_{-n} - z^{-n} b_{-n})\right) \times \exp\left(\sum_{n=1}^{\infty} \frac{q^{n/2} z^{-n} a_{-n} - (xz)^n b_{-n}}{1 - x^n}\right) e^{-\alpha z^{-d}},
\]

\[
\phi_-(z) = g(1) \exp\left(\sum_{n=1}^{\infty} \frac{q^{n/2} z^n a_{-n} - q^{5n/2} z^{-n} b_{-n}}{|2n|} \right) \times \exp\left(-\sum_{n=1}^{\infty} \frac{q^{5n/2} z^{-2n} a_{-n} - q^{2n/2} (xz)^n b_{-n}}{|2n|(1 - x^n)}\right) e^{\alpha/2 (-q^3 z)^{d_{+1}/2}} \text{ on } V(\Lambda_i).
\]

They satisfy the following relations,

\[
J(\xi_1)J(\xi_2) =: J(\xi_1)J(\xi_2) : = \left(1 - \frac{\xi_2}{\xi_1}\right) \left(1 - \frac{q^2 \xi_2}{\xi_1}\right) f\left(\frac{\xi_2}{\xi_1}\right) f\left(\frac{\xi_1}{\xi_2}\right),
\]

\[
\phi_-(z_1) \phi_-(z_2) =: \phi_-(z_1) \phi_-(z_2) : = g\left(\frac{z_1}{z_2}\right) g\left(\frac{z_2}{z_1 x}\right),
\]

\[
J(\xi) \phi_-(z) =: J(\xi) \phi_-(z) : = \frac{1}{(1 - q^2 w^{-1}) f(w)},
\]

\[
\phi_-(z) J(\xi) =: J(\xi) \phi_-(z) : = -\frac{q^2 w^{-1}}{1 - w} f(w).
\]

\[
\phi_+(z) = (q - q^{-1}) \int_{q^2 |w| < 1} \frac{d\xi}{2\pi i} : J(\xi) \phi_-(z) : \left(\frac{w}{1 - w(1 - q^2 w^{-1}) f(w)}\right).
\]
Here $w = \xi/q^2 z$ and $\cdots$ denotes the normal ordering with respect to the bosons $a_n$ and $b_n$ (We do not normal order the operators $e^{i\alpha}$ and $\theta_n$). In the above equations, all factors of the form $1/(1 - z)$ should be understood as $\sum_{n=0}^\infty z^n$. This fact specifies the contour for the expression $\phi_\pm$. Using the formulas above, (1.3) is evaluated as follows.

Set

$$P_{\bar{z}_1, \cdots, \bar{z}_n}(z_1, \cdots, z_n \mid x, y \mid i) = \frac{(q^2; q^4)_n}{(q^4; q^4)_n} \text{tr}_{V(\Lambda)}(x^{-d} y^\alpha \hat{\Phi}_{A+1}^{(1)}(z_1) \cdots \hat{\Phi}_{A+n}^{(1)}(z_n) \hat{\Phi}_{A+1}^{(1)}(z_n) \cdots \hat{\Phi}_{A+n}^{(1)}(z_1)) \text{tr}_{V(\Lambda)}(x^{-d} y^\alpha).}$$

(3.1)

We have

$$(L|_{\bar{z}_1, \cdots, \bar{z}_n} = P_{\bar{z}_1, \cdots, \bar{z}_n}(z_1, \cdots, z_1 \mid q^4, q^{-1} | i).$$

for $L = E_{\bar{z}_1, \cdots, \bar{z}_n} \otimes \cdots \otimes E_{\bar{z}_1, \bar{z}_1}$ (see the remark at the end of Sect. 1).

Introduce the following notations

$$A = \{a_1, \cdots, a_s\} = \{j \mid \varepsilon_j = -1\}, \quad B = \{b_1, \cdots, b_t\} = \{j \mid \varepsilon_j = +1\},$$

$$(s + t = n, \quad a_i < a_j, \quad b_i < b_j \quad \text{for} \quad i < j),$$

$$h(z) = (q^2 z; x)_\infty (q^2 z^{-1}; x)_\infty (q^2 z^{-1}; x)_\infty (q^2 z^{-1}; x)_\infty.$$

We prepare the integration variables $\xi_a (a \in A), \zeta_b (b \in B)$ and set $\eta_j = \xi_{a_j} (1 \leq j \leq s), \zeta_b = \zeta_{b_n+1...}$ ($s < j \leq n), \eta = \prod_j \eta_j$ and $\bar{z} = \prod_j z_j$. Then we have

$$P_{\bar{z}_1, \cdots, \bar{z}_n}(z_1, \cdots, z_1 \mid x, y \mid i)$$

$$= (-1)^t q \sum_{a \in A} q^{a+n} (q^2 z_a)_\infty \prod_{a \in A, j \leq n} \frac{d\xi_a}{2\pi i (\xi_a - z_a)} \prod_{b \in B, j < n} \frac{d\zeta_b}{2\pi i (\zeta_b - z_j)}$$

$$\times \prod_{a \in A, a < j \leq n} \frac{z_j - q^2 \xi_a}{z_j - \xi_a} \prod_{b \in B, b < j \leq n} \frac{\zeta_b - q^2 z_j}{\zeta_b - z_j} \prod_{j < k} \frac{\eta_k - \eta_j}{\eta_k - q^2 \eta_j}$$

$$\times \frac{h(1)^n \prod_{j < k} h(z_j/z_k) h(\eta_j/\eta_k) \sum_{m \in \mathbb{Z}_{+1/2}} (z/\eta)^m y^{2m} x^{m^2-j/4}}{(x; x)_\infty \text{tr}_{V(\Lambda)}(x^{-d} y^\alpha)}.$$  

(3.2)

Note that the last factor of the above equation can be rewritten into

$$\left(\frac{z}{\eta}\right)^i (-y \bar{z}/\bar{\eta})^2 x^{1-i}; x^2 \infty (-\eta/y \bar{z})^2 x^{1-i}; x^2 \infty (-y/\eta \bar{z})^2 x^{1-i}; x^2 \infty (-\eta/y \bar{z})^2 x^{1-i}; x^2 \infty.$$
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