Boundary K-matrices for the six vertex and the $n(2n - 1) \ A_{n-1}$ vertex models

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Abstract

Boundary conditions compatible with integrability are obtained for two dimensional models by solving the factorizability equations for the reflection matrices $K^\pm(\theta)$. For the six vertex model the general solution depending on four arbitrary parameters is found. For the $A_{n-1}$ models all diagonal solutions are found. The associated integrable magnetic Hamiltonians are explicitly derived.
1 Introduction

As is by now well known integrability is a consequence of the Yang-Baxter equation in two-dimensional lattice models and quantum field theory [1]. The Yang-Baxter equation takes the form:

\[
[1 \otimes R(\theta - \theta')][R(\theta) \otimes 1][1 \otimes R(\theta')] = [R(\theta') \otimes 1][1 \otimes R(\theta)][R(\theta - \theta') \otimes 1] \tag{1}
\]

where the R matrix elements \( R_{cd}^{ab}(\theta) \) with \( 1 \leq a, b, c, d \leq n, n \geq 2 \) define the statistical weights for a vertex model in two dimensions.

Not all boundary conditions (bc) are compatible with integrability in the bulk. Eq. (1) guarantees the integrability on the bulk. Integrability holds for bc defined by matrices \( K^- (\theta) \) and \( K^+ (\theta) \) (associated to the left and right boundaries respectively) provided the R matrix has P,T and crossing symmetry and the following equations proposed by Cherednik and Sklyanin [2] are fulfilled:

\[
R(\theta - \theta')[K^-(\theta) \otimes 1]R(\theta + \theta')[K^-(\theta') \otimes 1] = [K^-(\theta') \otimes 1]R(\theta + \theta')[K^-(\theta) \otimes 1]R(\theta - \theta') \tag{2}
\]

\[
R(\theta - \theta')[1 \otimes K^+(\theta)]R(\theta + \theta')[1 \otimes K^+(\theta')] = [1 \otimes K^+(\theta')]R(\theta + \theta')[1 \otimes K^+(\theta)]R(\theta - \theta') \tag{3}
\]

In addition the Yang-Baxter equation (1) guarantees the factorizability for the S matrix where:

\[
S_{cd}^{ab}(\theta) = R_{dc}^{ab}(\theta) \tag{4}
\]

is a two-particle S-matrix in two spacetime dimensions. In this context, \( K_{ab}^-(\theta) \) and \( K_{ab}^+(\theta) \) describe the scattering of the particles by the left and right boundaries respectively. Therefore for each integrable model (a given solution to the Yang-Baxter equation \( R(\theta) \)) in order to find the integrable boundary conditions , one must find the solutions \( K_{ab}^-(\theta) \) and \( K_{ab}^+(\theta) \) of eqs. (2) and (3) for the given \( R(\theta) \).

We present in this note the general solution of eqs.(2)-(3) for the six vertex model and a family of solutions to the corresponding modifications of these equations for the \( A_{n-1} \) vertex model [1]. Also the associated magnetic hamiltonians are derived and their properties investigated.

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2
2 General $K^\pm$ matrices for the six vertex model and its associated hamiltonians

Let us first consider the six vertex model [1]. The R matrix has the form:

$$R(\theta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{\sinh \gamma}{\sinh (\theta + \gamma)} & \frac{\sinh \theta}{\sinh (\theta + \gamma)} & 0 \\
0 & \frac{\sinh \theta}{\sinh (\theta + \gamma)} & \frac{\sinh \gamma}{\sinh (\theta + \gamma)} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$  \hspace{1cm} (5)

Since this R matrix enjoys P symmetry:

$$PR(\theta)P = R(\theta)$$  \hspace{1cm} (6)

eq(2) \text{ and } \eq(3) \text{ are equivalent for the six vertex model.}

We seek for the general solution of these equations:

$$K(\theta) = \begin{pmatrix}
x(\theta) \\
y(\theta) \\
z(\theta) \\
t(\theta)
\end{pmatrix}$$  \hspace{1cm} (7)

where $x(\theta)$, $y(\theta)$, $z(\theta)$ and $t(\theta)$ are unknown functions. Inserting eqs. (5) and (7) in eqs. (2) or (3) yields ten functional equations for these four functions.

The relevant ones are:

$$z(\theta)y(\theta') = z(\theta')y(\theta)$$  \hspace{1cm} (8)

$$\sinh(\theta - \theta') [x(\theta)x(\theta') - t(\theta)t(\theta')] + \sinh(\theta + \theta')[x(\theta')t(\theta) - x(\theta)t(\theta')] = 0$$  \hspace{1cm} (9)

$$y(\theta)x(\theta') \sinh 2\theta' = [\sinh (\theta + \theta')x(\theta) + \sinh (\theta - \theta')t(\theta)]y(\theta')$$  \hspace{1cm} (10)

Eq.(8) imply:

$$y(\theta') = k_1 z(\theta)$$  \hspace{1cm} (11)

where $k_1$ is an arbitrary constant.

Eq.(8) can be rewritten as:
\[
[tanh(\theta) - tanh(\theta')] [1 - a(\theta) a(\theta')] + [tanh(\theta) + tanh(\theta')] [a(\theta) - a(\theta')] = 0 \tag{12}
\]

where \(a(\theta) = t(\theta)/x(\theta)\). Differentiating eq. (12) with respect to \(\theta'\) and setting \(\theta' = 0\) yields an algebraic equation with solution:

\[
\frac{t(\theta)}{x(\theta)} = \frac{\sinh(\xi - \theta)}{\sinh(\xi + \theta)} \tag{13}
\]

where \(\xi\) is another arbitrary constant. Then, eq. (11) tells us that:

\[
y(\theta) = \mu \sinh 2\theta \tag{14}
\]

The remaining equations are identically satisfied. In summary, the general solution \(K(\theta)\) for the six-vertex model can be written as:

\[
K(\theta, k, \lambda, \mu, \xi) = \begin{pmatrix} k \sinh(\xi + \theta) & \mu \sinh 2\theta \\ \lambda \sinh 2\theta & k \sinh(\xi - \theta) \end{pmatrix} \tag{15}
\]

where \(k, \lambda, \mu\) and \(\xi\) are arbitrary parameters.

The special case \(\lambda = \mu = 0\) reproduces the solution given in ref. [2].

Let us now discuss the integrable hamiltonians associated to the K-matrix (15). They follow by the procedure discussed in [2] from the equation:

\[
H = C \left\{ \sum_{n=1}^{N-1} h_{n,n+1} + \frac{1}{2} \dot{K}^{-}_1(0) + \frac{tr_0[K^+_0(-\eta) h_{N0}]}{tr[K^+(-\eta)]} \right\} \tag{16}
\]

Here \(C\) is an arbitrary constant and:

\[
h_{n,n+1}^{n,n+1} = \dot{R}_{n,n+1}(0) \tag{17}
\]

gives the two sites hamiltonian in the bulk and the indices \((n, n+1)\) label the sites in which the matrix acts. In the present, case we can choose:

\[
K^\pm(\theta) = K(\theta, k_\pm, \lambda_\pm, \mu_\pm, \xi_\pm) \tag{18}
\]

If we want to maintain the bulk part of the XXZ hamiltonian with the first derivative of the transfer matrix we are led to take \(K^-\) matrices with value in \(\theta = 0\) different from zero, this leads us to \(k_- \neq 0\) and without loss of
generality we can take $K^-(0) = 1$ that is $k_- = 1/\sinh \xi$. For the same reason we choose $k_+ \neq 0$.

Inserting eqs. (5), (15) and (18) in eq. (16) yields:

$$H = \sum_{n=1}^{N-1} \left( \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cosh \gamma \sigma_n^z \sigma_{n+1}^z \right) + \sinh \gamma \left( b_- \sigma_1^- - b_+ \sigma_N^- + c_- \sigma_1^+ - c_+ \sigma_N^+ + d_- \sigma_1^+ - d_+ \sigma_N^+ \right)$$  \hspace{1cm} (19)

we have chosen $C = 2 \sinh \gamma$ and omitted the terms proportional to the identity. The parameters $b_\pm, c_\pm$ and $d_\pm$ follow from $\lambda_\pm, \mu_\pm, \xi_\pm$ and $k_+$ as shown:

$$b_- = \coth \xi_-$$
$$b_+ = \coth \xi_+$$
$$c_- = 2\lambda_-$$
$$c_+ = \frac{2\lambda_+}{k_+ \sinh \xi_+}$$
$$d_- = 2\mu_-$$
$$d_+ = \frac{2\mu_+}{k_+ \sinh \xi_+}$$  \hspace{1cm} (20)

The bulk part in eq.(19) is just the well-known XXZ Heisenberg hamiltonian. When $c_\pm = d_\pm = 0$ we recover the hamiltonian discussed in ref [4]. Equation (19) provides the more general choice of boundary terms compatible with integrability for the XXZ chain besides periodic and twisted bc. It contains three parameters in each boundary ($b_\pm, c_\pm$ and $d_\pm$). In particular when $b_\pm = \pm 1$ and $c_\pm = d_\pm = 0$, one recovers the $SU_q(2)$ invariant hamiltonian ($q = e^{\mp \gamma}$) [3], [4].

3 Diagonal $K^\pm(\theta)$ matrices for the $A_{n-1}$ vertex models

Let us now consider the $n(2n-1)$ integrable vertex model associated to the $A_{n-1}$ Lie algebra. The $R$-matrix takes the form (see ref.[4])
\[ R^{ab}_{ij}(\theta) = \frac{\sinh \gamma}{\sinh(\gamma + \theta)} \delta_{ia}\delta_{jb} e^{\theta \text{sign}(a-b)} + \frac{\sinh \theta}{\sinh(\gamma + \theta)} \delta_{ib}\delta_{ja}, \quad a \neq b \]

\[ R^{aa}_{ii} = \delta_{ia} \quad 1 \leq a, b \leq n \] (21)

This reduces to eq. (4) when \( n = 2 \) upon a gauge transformation \([1]\).

Contrary to the six vertex \( R \)-matrix, the \( R \)-matrix (21) does not enjoy \( P \) and \( T \) symmetry but just \( PT \) invariance. It is not crossing invariant either but it obeys the weaker property \([5]\).

\[ \left\{ [S_{12}(\theta)]^{t_2} \right\}^{t_2} = L(\theta, \gamma)M_{21} \left[ S_{12}(\theta + 2\eta) \right]^{-1} \] (22)

where \( L(\theta, \gamma) \) is a c-number function, \( \eta \) a constant and \( M \) a symmetry of the \( R \)-matrix (21). That is

\[ [M_1 \otimes M_2, R_{12}(\theta)] = 0 \] (23)

We find by direct calculation from eqs. (21) and (22):

\[ \eta = \frac{n}{2} \gamma \]

\[ M_{ab} = \delta_{ab} e^{(n-2a+1)\gamma} \quad 1 \leq a, b \leq n \]

\[ L(\theta, \gamma) = \frac{\sinh(\theta + \gamma) \sinh[\theta + (n - 1)\gamma]}{\sinh(\theta) \sinh(\theta + n\gamma)} \] (24)

In this case, when the weak condition (22) holds, the integrability requirements on the boundary matrices \( K^-(\theta) \) and \( K^+(\theta) \) need some modifications \([3]\). \( K^-(\theta) \) must still obey eq (2) and \( K^+(\theta) \) obeys:

\[ R(\theta - \theta')K_1^+(\theta')^t_1 M_1^{-1} R(-\theta - \theta' - 2\eta)K_1^+(\theta')^t_1 M_2 = K_1^+(\theta')^t_1 M_2 R(-\theta - \theta' - 2\eta)M_1^{-1}K_1^+(\theta')^t_1 R(-\theta - \theta') \] (25)

There is an automorphism between \( K^- \) and \( K^+ \)

\[ K^+(\theta) = K^-(-\theta - \eta)^t M \] (26)

For simplicity, we start for searching diagonal matrices \( K^- \):

6
Inserting eqs. (21) and (27) in (2) yields:

\[
\sinh(\theta + \theta') \left[ K_a^-(\theta) K_b^-(\theta') e^{\text{sign}(a-b)(\theta-\theta')} - K_a^-(\theta') K_b^-(\theta) e^{-\text{sign}(a-b)(\theta-\theta')} \right] + \\
\sinh(\theta - \theta') \left[ K_a^-(\theta) K_b^-(\theta') e^{\text{sign}(a-b)(\theta+\theta')} - K_a^-(\theta') K_b^-(\theta) e^{\text{sign}(a-b)(\theta+\theta')} \right] = 0
\]

These equations are generalizations of eq. (9). By a similar procedure we find their general solution:

\[
K_a^-(\theta) = k \sinh(\xi + \theta) e^\theta \quad 1 \leq a \leq l_-
\]
\[
K_a^-(\theta) = k \sinh(\xi - \theta) e^{-\theta} \quad l_- + 1 \leq a \leq n
\]  

(29)

Here \(k\), \(\xi\) and \(l_-\) are arbitrary parameters. For \(n = 2\) and \(l_- = 1\) we recover the diagonal case of eq. (15) after a gauge transformation. In general, for \(n > 2\), we have the extra discrete parameter \(l_-\) that tells where the diagonal element change from one to another.

The integrable Hamiltonian associated to the \(R\)-matrix (21) with b.c. (29) follows using equation (16). We find after some straightforward calculations:

\[
H = \sum_{j=1}^{N-1} \left\{ \cosh \gamma \sum_{r=1}^{n} e_{rr}^{(j)} e_{rr}^{(j+1)} + \sum_{r, s = 1, r \neq s}^{n} e_{rs}^{(j)} e_{sr}^{(j+1)} + \sinh \gamma \sum_{r, s = 1}^{n} \text{sign}(r - s) e_{rr}^{(j)} e_{ss}^{(j+1)} \right\} \\
+ \frac{\sinh \gamma}{2} \left[ (1 + \coth \xi_-) \sum_{r=1}^{l_-} e_{rr}^{(1)} - \sum_{r=1}^{L_+} e_{rr}^{(1)} \right] + \cosh \gamma \left\{ \frac{\sinh(\xi_+ - \frac{n \gamma}{2})}{\sinh \xi_+} e^\gamma \sum_{r=1}^{l_+} e_{rr}^{(N)} e^{-2r \gamma} \right\} \\
+ \frac{\sinh(\xi_+ + \frac{n \gamma}{2})}{\sinh \xi_+} e^{(n+1)\gamma} \sum_{r=1}^{l_-} \sum_{s=1}^{l_-} e_{rr}^{(N)} e^{-2r \gamma} + \frac{1}{\text{tr}K^+(0)} \sum_{r,s=1}^{n} \text{sign}(r - s) e_{rr}^{(N)} K_s^+(0)
\]

(30)

Where \(N\) is the number of sites of the chain and \(l_+, l_-\) arbitrary integers running from 1 to \(n\). We have extracted a global factor of \(1/\sinh \gamma\) and
omitted the terms proportional to the unit operator.
In particular with \( \xi_{\pm} = -\infty \) we get a \( SU_q(n) \) invariant hamiltonian, with \( q = e^{-\gamma} \), that can be written as:

\[
H = \sum_{j=1}^{N-1} \sum_{r, s = 1}^{n-1} \left( \prod_{l=s}^{r-1} (J^+_l)^{(j)} \sum_{l=r-1}^{s} (J^-_l)^{(j+1)} + \prod_{l=s}^{r-1} (J^-_l)^{(j)} \sum_{l=r-1}^{s} (J^+_l)^{(j+1)} \right) + \]

\[
\cosh \gamma \sum_{r, s = 1}^{n-1} s(n - r)(h_r^{(j)}h_r^{(j+1)} + h_s^{(j)}h_s^{(j+1)}) + \frac{s(n - r)h_r^{(j)}h_r^{(j+1)}}{n} + \frac{\sinh \gamma \sum_{r = 1}^{n-1} (n - r)(h_r^{(N)} - h_r^{(1)})}{n}
\]

(31)

Here \( N \) is the number of sites, \( J^+_l \equiv e_{ll+1}, J^-_l \equiv e_{l+1l} \) and \( h_l \equiv e_{ll} - e_{l+1,l+1} \) are the \( su(n) \) generators in the fundamental representation with \( (e_{lm})_{ij} \equiv \delta_{li}\delta_{mj} \). It is easily seen that this hamiltonian coincides for \( n = 2 \) with the \( SU_q(2) \) invariant one, discussed in [3],[4].

4 Conclusions

We have obtained the general solution to the surface factorization equations for the six-vertex \( R \) matrix providing in this way the more general boundary terms compatible with integrability. The Bethe ansatz in these systems must change drastically as the hamiltonian does not commute with \( J_z \). For the \( A_n \) chain, a generalization of the nested Bethe Ansatz [3] will be needed.
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Nota Added: After completion of this paper, we hear from A. B. Zamolodchikov that he has independently obtained eq.(15) for the six-vertex model.

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