On the quest for generalized Hamiltonian descriptions of $3D$-flows generated by curl of a vector potential

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Abstract

We study Hamiltonian analysis of three-dimensional advection flow $\dot{x} = v(x)$ of incompressible nature $\nabla \cdot v = 0$ assuming that dynamics is generated by the curl of a vector potential $v = \nabla \times A$. More concretely, we elaborate Nambu-Hamiltonian and bi-Hamiltonian characters of such systems under the light of vanishing or non-vanishing of the quantity $A \cdot \nabla \times A$. We present an example (satisfying $A \cdot \nabla \times A \neq 0$) which can be written as in the form of Nambu-Hamiltonian and bi-Hamiltonian formulations. We present another example (satisfying $A \cdot \nabla \times A = 0$) which we cannot able to write it in the form of a Nambu-Hamiltonian or bi-Hamiltonian system. On the hand, this second example can be manifested in terms of Hamiltonian one-form and yields generalized or vector Hamiltonian equations $\dot{x}_i = -\epsilon_{ijk} \partial \eta_j / \partial x_k$.

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1 Introduction

Hamiltonian analysis of finite dimensional systems has a huge literature and still attracts deep attention of many autors. There are several types of 3D flows which you can interpret in terms of Nambu-Poisson Hamiltonian dynamics. As it is well known, the generalized force field can be decomposed into the sum of two vector fields. One which is equal to minus of the gradient of some potential function whereas the other does not come from a potential. This decomposition resembles the decomposition of a vector field into a curl-free (irrotational) component and a solenoidal (divergence-free) component arising from the celebrated Helmholtz theorem. A particular instance of this Helmholtz-like decomposition is the one where the irrotational and the divergence-free parts are orthogonal [33].

We start with a three dimensional system in form

\[ \dot{x} = v(x, t), \]  

(1.1)

where we denote a three-tuple by \( x = (x, y, z) \), describing the evolution with time \( t \) of the spatial position \( x \) of a fluid particle under the action of the fluid velocity field \( v(x, t) \), known as the advection equation, which can be described briefly as follows [33]. The transport of a passive scalar \( \psi \) embedded in a fluid flow governed by the ideal advection equation

\[ \frac{\partial \psi}{\partial t} + v(x, t) = 0 \quad \Leftrightarrow \quad \frac{d\psi}{dt} = 0. \]

Since the passive scalar is frozen into the fluid element, the distribution function \( \psi \) at arbitrary time can be found to be

\[ \frac{dx(\xi, t)}{dt} = v(x, t) \]

with the initial condition \( x(\xi, t = 0) = \xi \). In general, (1.1) is a nonlinear dynamical system capable of exhibiting chaotic dynamics, then this flow is a mixing flow over some region of Lagrangian topology. The flow is mixing means whether the flow trajectory is globally ergodic, i.e., trajectory visits every point in a closed domain.

If the flow is incompressible \( \nabla \cdot v = 0 \), then the dynamical system (1.1) is conservative. The representation of divergence-free vector fields as curls in two and three dimensions has been studied in [2]. In such a situation \( v \) can be represented as \( v(x) = \nabla \gamma_1 \times \nabla \gamma_2 \), which can be manifested in terms of Nambu-Hamiltonian form [7, 25]. In this paper we show an example involving an additional condition on the dynamics of the flow \( v(x) = \nabla \times A \), i.e., null helicity condition etc, then Hamiltonian formalism can not be given in terms of Nambu-Hamiltonian. This can be restored if we break the symmetry of the null helicity. Accordingly, we notice one can grossly divide these into two categories.

In the first category, there are 3D flows satisfying both the divergence free condition and the integrability condition, that is

\[ \nabla \cdot v = 0, \quad v \cdot \nabla \times v = 0. \]  

(1.2)
Here, the integrability condition $\mathbf{v} \cdot \nabla \times \mathbf{v} = 0$ of the vector field $\mathbf{v}$ is defined up to a multiplier $\mu$. This can be shown as follows. Let us consider a system of equation

$$
\dot{x} = P(x, y, z), \quad \dot{y} = Q(x, y, z), \quad \dot{z} = R(x, y, z).
$$

Then a direct calculation follows

$$
\mu \mathbf{v} \cdot (\nabla \times \mu \mathbf{v}) \\
= \mu P \left( \frac{\partial}{\partial y}(\mu R) - \frac{\partial}{\partial z}(\mu Q) \right) + \mu Q \left( \frac{\partial}{\partial z}(\mu P) - \frac{\partial}{\partial x}(\mu R) \right) + \mu R \left( \frac{\partial}{\partial x}(\mu Q) - \frac{\partial}{\partial y}(\mu P) \right) \\
= \mu^2 \left( P \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + Q \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right) \\
= \mu^2 (\mathbf{v} \cdot \nabla \times \mathbf{v}) = 0.
$$

Note that, in the second line the other terms are identically vanishing. It is immediate to see that, if $\nabla \cdot \mathbf{v} = 0$ we have that

$$
\mathbf{v} = K \nabla H, \quad \text{or} \quad \nabla \times \mathbf{v} = \nabla K \times \nabla H, \quad (1.3)
$$

for two real valued functions $K$ and $H$. As an example for this case, consider the Euler top equation

$$
\dot{x} = Ayz, \quad \dot{y} = Bxz, \quad \dot{z} = Cxy, \quad (1.4)
$$

where $A$, $B$ and $C$ being constants. The divergence free condition generates Liouville's theorem of Nambu mechanics, therefore the state space can be regarded as an incompressible fluid. It is interesting to notice that if the coefficients $A$, $B$ and $C$ in the system (1.4) are all equal to each other, then $\nabla \times \mathbf{v}$ vanishes identically. This motivates the following special case of such flows

$$
\nabla \cdot \mathbf{v} = 0, \quad \nabla \times \mathbf{v} = \mathbf{0}. \quad (1.5)
$$

As an example, consider the Lagrange system, the case $A = B = C = 1$,

$$
\dot{x} = yz, \quad \dot{y} = xz, \quad \dot{z} = xy, \quad (1.5)
$$

and 3D circle map equation $\mathbf{x} = x^2$. These flows are the usual (conservative) potential flows $\mathbf{v} = -\nabla V$ generating by a potential function $V$.

As an another category, let the generator vector field $\mathbf{v}$ in a 3D system is curl of a potential field $\mathbf{A}$. In general, the integrability condition is not fulfilled. In this case, we have

$$
\mathbf{v} = \nabla \times \mathbf{A}, \quad \nabla \times \mathbf{v} \neq \mathbf{0}, \quad (1.6)
$$

where $\mathbf{A}$ is the potential vector field. Motivating from the Hamiltonian curl forces in two dimensions discussed in [3,4,5], our focus in this study is to elaborate Hamiltonian analysis of three dimensional differential system satisfying (1.6). In this present work, we distinguish two subcategories according to the satisfaction of the potential vector field $\mathbf{A}$ of the integrability condition.
In the first subcategory, we study
\[ \mathbf{A} \cdot \nabla \times \mathbf{A} \neq 0. \]

It is known from the standard text book that for a solenoidal and continuously differentiable vector function such as a vector field \( \mathbf{v} \) can be expressed as a cross-product of two gradients,
\[ \mathbf{v} = \nabla H \times \nabla K, \]
where \( H \) and \( K \) are assumed to smooth functions \([2]\). The standard proof assumes that there exist two families of surfaces, \( H(x, y, z) = c_1 \) and \( K(x, y, z) = c_2 \), for which all the flow lines are confined to the surface of intersections. This is the essential ingredient of the Nambu mechanics too. It is worth noting that if we start with \( \mathbf{A} = H \nabla K \) or \( \mathbf{v} = \nabla H \times \nabla K \), the scalar product \( \mathbf{A} \cdot \mathbf{v} = 0 \). It is known that the vector potential corresponding to vector field \( \mathbf{v} \) can be presented in the Clebsch representation \([6]\) form
\[ \mathbf{A} = -K \nabla H + \nabla M, \quad (1.7) \]
where the function \( M \) must be multi-valued. This implies that the function \( M \) has a surface \( \Sigma \) inside the volume \( \Omega \) where it has a jump, then the contribution from the jump surface \( \Sigma \) is added to the integral over \( \partial \Omega \), which results in the nonzero helicity. It is a challenging question to find the function \( M \) and to understand why it has to be multi-valued \([6, 30]\). In three dimensions, the geometric aspects can be discussed in terms of bi-Hamiltonian and Nambu-Hamiltonian formulations apart from the two dimensional ones. Accordingly, we shall discuss the Hamiltonian characters of the systems \([1, 6]\) in terms of the bi-Hamiltonian and the Nambu-Hamiltonian frameworks. We give two different examples for this subcategory, one possesses the Nambu structure whereas the other one has the almost Nambu structure involving a multiplier.

The second subcategory covers a divergence free flow satisfying integrability condition
\[ \mathbf{A} \cdot \nabla \times \mathbf{A} = 0. \]

We should expect if these conditions are satisfied then \( \mathbf{A} = H \nabla K \), then it would yield Nambu Hamiltonian system, unfortunately we fail to get it. The closed two form \( \omega \) associated to dynamics is the product of two one-forms \( \omega = J_1 \wedge J_2 \), unlike the previous cases in this \( J_1 \) and \( J_2 \) are not exact. The Euler potentials could be discontinuous, although the vector potential and \( \mathbf{v} = \nabla \times \mathbf{A} \) might not have discontinuities on the intersection of surfaces. In this paper, we demonstrate if we deform the system to another which satisfies divergence free condition and \( \mathbf{A} \cdot \nabla \times \mathbf{A} \neq 0 \), then it becomes Nambu-Hamiltonian.

Our motivation to study such flows coming from the work of Berry and Shukla on curl forces we have studied in this paper curl velocities on 3D spaces. We have studied 3D vector fields generated by a curl of a vector potential. In general, the theory for 3D systems is less well-developed \([36]\), there are many unanswered questions regarding the nature of dynamics in 3D flows. The flows which have been investigated in this paper are also divergence free vector fields like Berry-Shukla, and these could be either Hamiltonian or non-Hamiltonian.
In this point, we want to make a final remark that 3D flows possess extremely diverse characters. Consider, for example, the following 3D flow, called as SIR equation, given by
\[ \dot{S} = -rSI, \quad \dot{I} = rSI - aI, \quad \dot{R} = aI, \]
which neither satisfies \( \nabla \cdot v = 0 \) nor \( v = \nabla \times A \), but still one can express it in terms of Nambu Hamiltonian form.

**Organization of the paper.** In Section 2, we present basics of Hamiltonian realizations of 3D systems. In Section 3 we exhibit some illustrations that posses Nambu-Hamiltonian and bi-Hamiltonian character. In Section 4, we examine a counter example which cannot in the Nambu-Hamiltonian and bi-Hamiltonian formalism.

## 2 Hamiltonian Analysis of Three Dimensional Velocities

### 2.1 Three Dimensional Hamiltonian Systems

Let \((P, \{\cdot, \cdot\})\) be a 3-dimensional Poisson manifold equipped with Poisson bracket \(\{\cdot, \cdot\}\) satisfying the Jacobi identity. The Hamilton’s equation generated by a Hamiltonian function is
\[ \dot{x} = \{x, H\}, \]
for local coordinates \((x)\) on \(P\). In 3-dimensions, we can replace the role of a Poisson bracket with a Poisson vector \(J\), [10, 17, 18]. In this case, the Jacobi identity turns out to be the following equation
\[ J \cdot (\nabla \times J) = 0, \]
whereas the Hamilton’s equation takes the particular form
\[ \dot{x} = J \times \nabla H. \]

Here, \(H\) is a Hamiltonian function defined on \(P\), and \(\nabla H\) is the gradient of \(H\). The following theorem is exhibiting all possible solutions of Jacobi identity given in (2.2) so that characterizes all Poisson Poisson structures in 3-dimensions, [1, 19, 20, 21].

**Theorem 2.1** The general solution of the vector equation (2.2) is \(J = (1/M) \nabla F\) for arbitrary functions \(M\) and \(F\).

The existence of scalar multiple \(1/M\) in the solution is a manifestation of conformal invariance of the identity (2.2). In the literature, \(M\) is called as the Jacobi’s last multiplier [22, 23]. In this picture, a Hamiltonian system has the following generic form
\[ \dot{x} = \frac{1}{M} \nabla F \times \nabla H. \]
A dynamical system is bi-Hamiltonian if it admits two different Hamiltonian structures

\[ \dot{x} = \{x, H_2\}_1 = \{x, H_1\}_2, \quad (2.5) \]

with the requirement that the Poisson brackets \( \{\bullet, \bullet\}_1 \) and \( \{\bullet, \bullet\}_2 \) be compatible \[13, 28\].

Recalling the system (2.4), we arrive at that a Hamiltonian system in the form (2.4) is bi-Hamiltonian

\[ \dot{x} = \frac{1}{M} \nabla F \times \nabla H = J_1 \times \nabla H = J_2 \times \nabla F, \quad (2.6) \]

where, the first Poisson vector \( J_1 \) is given by \((1/M)\nabla F\) whereas the second Poisson vector \( J_2 \) is given by \(- (1/M)\nabla H\). The following theorem determine the Hamiltonian picture of three dimensional dynamical systems admitting an integral invariant. For the proof, we refer \[10, 11\].

**Theorem 2.2** A three dimensional dynamical system \( \dot{x} = v(x) \) having a time independent first integral is bi-Hamiltonian if and only if there exist a Jacobi’s last multiplier \( M \) which makes \( Mv \) divergence free.

### 2.2 3D Nambu-Poisson manifolds

Let \( \mathcal{P} \) be a three dimensional manifold. A Nambu-Poisson bracket of order 3 is a ternary operation, denoted by \( \{\bullet, \bullet, \bullet\} \), on the space of smooth functions, satisfying both the generalized Leibniz identity

\[ \{F_1, F_2, FH\} = \{F_1, F_2, F\} H + F \{F_1, F_2, H\} \quad (2.7) \]

and the fundamental (or Takhtajan) identity

\[ \{F_1, F_2, \{H_1, H_2, H_3\}\} = \sum_{k=1}^{3} \{H_1, ..., H_{k-1}, \{F_1, F_2, H_k\}, H_{k+1}, ..., H_3\}, \quad (2.8) \]

for arbitrary functions \( F, F_1, F_2, H, H_1, H_2, \) see \[26, 32\].

Assume that \( (\mathcal{P}, \{\bullet, \bullet, \bullet\}) \) be a Nambu-Poisson manifold. For a pair \( (H_1, H_2) \) of Hamiltonian functions, the associated Nambu-Hamiltonian vector field \( X_{H_1, H_2} \) is defined through

\[ X_{H_1, H_2}(F) = \{F, H_1, H_2\}. \quad (2.9) \]

The distribution of the Nambu-Hamiltonian vector fields are in involution and defines a foliation of the manifold \( \mathcal{P} \). A dynamical system is called Nambu-Hamiltonian with a pair \( (H_1, H_2) \) of Hamiltonian functions if it can be recasted as

\[ \dot{x} = \{x, H_1, H_2\}. \quad (2.10) \]

If a system is in the Nambu-Hamiltonian form (2.10) then by fixing one of the Hamiltonian functions in the pair \( (H_1, H_2) \), we can write it in the bi-Hamiltonian form as well

\[ \dot{x} = \{x, H_1\}^{H_2} = \{x, H_2\}^{H_1} \quad (2.11) \]
where the brackets $\{\cdot,\cdot\}^{H_2}$ and $\{\cdot,\cdot\}^{H_1}$ are compatible Poisson structures defined by

\[
\{F,H\}^{H_2} = \{F,H,H_2\}, \quad \{F,H\}^{H_1} = \{F,H_1,H\},
\]

respectively.

Let $\mathcal{P}$ be a 3 dimensional manifold equipped with a non-vanishing volume manifold $\mu$. Then the following identity

\[
\{F_1,F_2,F_3\}\mu = dF_1 \wedge dF_2 \wedge dF_3
\]

defines a Nambu-Poisson bracket on $\mathcal{P}$ \[15\] \[16\]. In this case, the equation (2.9) relating a Hamiltonian pair $(H_1,H_2)$ and a Nambu-Hamiltonian vector field $X_{H_1,H_2}$ can be written, in a covariant formulation, as

\[
\iota_{X_{H_1,H_2}}\mu = dH_1 \wedge dH_2,
\]

where $\iota$ is the interior derivative. We call (2.14) as the Nambu-Hamilton’s equations \[12\]. Note that, by taking the exterior derivative of both hand side of (2.14), we arrive at the preservation of the volume form by the Nambu-Hamiltonian vector field, that is

\[
\mathcal{L}_{X_{H_1,H_2}}\mu = 0.
\]

Integration of this conservation law gives that the flow of a Nambu-Hamiltonian vector field is a volume preserving diffeomorphism.

Consider a local frame (called as the standard basis) given by a three-tuple $(u,v,w)$ such that the volume form is

\[
\mu = du \wedge dv \wedge dw.
\]

In this picture the Nambu-Poisson three-vector takes the particular form

\[
N = \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial w}.
\]

Locally, the Nambu-Hamiltonian vector field $X_{H_1,H_2}$ defined in (2.14) for a pair $(H_1,H_2)$ of Hamiltonian functions can be computed as

\[
X_{H_1,H_2} = \{H_1,H_2\}_u \frac{\partial}{\partial w} + \{H_1,H_2\}_v \frac{\partial}{\partial u} + \{H_1,H_2\}_w \frac{\partial}{\partial v},
\]

where the coefficients are, for example,

\[
\{H_1,H_2\}_{a,b} = \frac{\partial H_1}{\partial a} \frac{\partial H_2}{\partial b} - \frac{\partial H_1}{\partial b} \frac{\partial H_2}{\partial a}.
\]

For the particular case of a three dimensional Euclidean space, the present discussion reduces to the following form. Let $F_1$, $F_2$ and $F_3$ be three real valued functions, and consider the triple product

\[
\{F_1,F_2,F_3\} = \nabla F_1 \cdot \nabla F_2 \times \nabla F_3
\]
of the gradients of these functions. It is evident that the bracket (2.20) is a Nambu-Poisson bracket with corresponding Nambu–Poisson three–vector field in the standard form (2.17). The Nambu-Hamiltonian vector field presented in (2.18) takes the particular form

\[ X_{H_1, H_2} = \nabla H_1 \times \nabla H_2. \]

It follows that, the Nambu-Hamilton’s equations (2.10) turn out to be

\[ \dot{x} = \{x, H_1, H_2\} = \nabla H_1 \times \nabla H_2. \] (2.21)

The bi-Hamiltonian character of this system can easily be observed by employing (2.11). The divergence of (2.21) generates Liouvilles theorem of Nambu mechanics:

\[ \nabla \cdot \dot{x} = \nabla \cdot (\nabla H_1 \times \nabla H_2) = 0. \]

The main advantage of using the two conserved quantities (Hamiltonians) in Nambu formulation, is the representation of the phase space trajectory as intersection line of two surfaces based on the conserved quantities. This geometric application illustrates the kind of motion without explicitly solving the equations of motion, this is the key feature of the (maximal) superintegrability.

### 2.3 Three Dimensional Systems

Our motivation stems from the equation of the chaotic advection of dye

\[ \dot{x} = v, \quad \nabla \times v \neq 0 \quad (2.22) \]

where the velocity field is assumed to have been determined a priori and satisfies the incompressibility condition \( \nabla \cdot v = 0 \). All the flows we consider in this section satisfy the Frobenius integrability condition. An handy way to describe Frobenius integrability condition has been prescribed by Ollagnier and Strelcyn [27]. Consider a smooth one form \( J \) corresponding to a first integral \( I \) of a smooth dynamical vector field \( v \) defined on \( \mathbb{R}^3 \). It is tautological that the volume form \( \Omega \) and \( J \) satisfy \( \Omega \wedge J = 0 \). If we take interior product with respect to the vector field \( v \), we get \( i_v \Omega \wedge J = 0 \). We now state a general remark that a smooth function \( I \) is a first integral of a smooth vector field \( v \) defined on \( \mathbb{R}^3 \) if and only if \( dI \wedge i_v \Omega = 0 \). The advection equations are non-integrable in general. A large class of conservative systems, which exhibit chaotic behavior, has a Hamiltonian representation. Two of the well-known examples are the magnetic field \( B \) and the velocity field \( v \) of a divergence free fluid.

Let us start by considering a vector potential \( A = (A_x, A_y, A_z) \), and the following system of equations

\[ \dot{x} = v(x) = \nabla \times A(x). \] (2.23)

It is easy to observe now that the vector potential \( A \) is far from being unique since

\[ \nabla \times (A + \nabla \phi) = \nabla \times A + \nabla \times \nabla \phi = \nabla \times A \] (2.24)
for an arbitrary real valued function $\phi$. This is the gauge invariance of the dynamics. It is immediate to observe that $\nabla \times \mathbf{v}$ of the velocity field does not necessarily zero for arbitrary vector potential $\mathbf{A}$.

In terms of the differential forms, the picture is as follows. Consider a differential one-form
\[ \alpha = \mathbf{A}(x) \cdot dx, \]
so that we have
\[ * d\alpha = \nabla \times \mathbf{A} \cdot dx = \mathbf{v}(x) \cdot dx, \]
where $*$ is the Hodge star operator with respect to the Euclidean norm. Alternatively, start with a volume form $\Omega = dx \wedge dy \wedge dz$. If we contract with the dynamical vector field $\mathbf{\hat{v}} = v_i \partial / \partial x_i$ we obtain the two form $d\alpha$. Therefore, the equations of motion presented in (2.23) can also be expressed in the following form
\[ \dot{x} = *(d\alpha \wedge dx). \]

In 3-dimensions, any two-form is decomposable. Accordingly, we write the exact two-form $d\alpha$ as the wedge product of two one-forms $J_1$ and $J_2$ that is
\[ d\alpha = dv_x \wedge dx + dv_y \wedge dy + dv_z \wedge dz = J_1 \wedge J_2. \]

Here, $J_1$ and $J_2$ are two integral invariants of the system. If both $J_1$ and $J_2$ are closed then by Poincaré lemma, we arrive at two first integrals $I_1$ and $I_2$ satisfying $J_1 = dI_1$ and $J_2 = dI_2$ respectively. If this is the case, then 3-dimensional phase flow can be described by means of the first integrals. From geometric point of view, this gives that a solution to the system, that is an integral curve, can be realized as the intersection of two level surfaces defined by the first integrals $I_1$ and $I_2$. So that we can write the system (2.22) as follows
\[ \dot{x} = \nabla I_1 \times \nabla I_2. \]

In this case, we can write $\mathbf{A} = I_1 \nabla I_2$. Note that this representation of the dynamics coincide with the standard form of the Nambu-Hamilton equations exhibited in (2.21). Further, by employing (2.11), we see that this description is bi-Hamiltonian as well.

We will illustrate two types of flows; one kind of 3D flows yield Nambu-Hamiltonian mechanics and they satisfy $\mathbf{A} \cdot \nabla \times \mathbf{A} \neq 0$, where $\mathbf{v} = \nabla \times \mathbf{A}$. Other type flow does not possess Hamiltonian framework and it satisfies $\mathbf{A} \cdot \nabla \times \mathbf{A} = 0$.

### 3 Illustrations

In this section we give two examples, one is a relatively simple one and the other one is more complicated.
### 3.1 A superintegrable system

We now consider an example of 3D system \([9]\) generated by the vector potential

\[
A = \frac{1}{4} \left( z^2 - xy^2, x^2y - 2yz, y^2 - 2xz \right). \tag{3.1}
\]

The equations of motion (2.23) can be written as

\[
\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = xy. \tag{3.2}
\]

One can check trivially \(A \cdot (\nabla \times A) \neq 0\). It is immediate to see that \(\nabla \times v\) does not vanish for \(v = (y, x, xy)\). The two-forms in (2.28) turns out to be

\[
zdz \wedge dx + ydy \wedge dz + xydx \wedge dy = J_1 \wedge J_2
\]

where the integral one-forms \(J_1\) and \(J_2\) can be computed to be

\[
J_1 = zdx - y^2/2 - x^3/3, \quad J_2 = x^2 - z \tag{3.3}
\]

respectively. It is immediate to check the invariance of the one-forms by \(\iota_v J_i = 0\). Note that, \(J_1 = dI_1\) and \(J_2 = dI_2\) are exact with the potential functions

\[
I_1 = xz - y^2/2 - x^3/3, \quad I_2 = x^2 - z \tag{3.4}
\]

See that \(I_1\) and \(I_1\) are smooth first integrals of the dynamics (3.2). Thus (3.2) is a maximal superintegrable 3D dynamics generated by the curl of the vector potential \(A\).

Using these two first integrals, we can express equation (3.2) in the Nambu-Hamiltonian form (2.21) as follows

\[
\dot{x} = \{x, I_1, I_2\} = \nabla I_1 \times \nabla I_2 \tag{3.5}
\]

where \(I_1\) and \(I_1\) are the ones in (3.4). According to (2.11), we can represent the maximal superintegrable system (3.2) in the bi-Hamiltonian formulation

\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{pmatrix} =
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & x \\
0 & -x & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial I_1}{\partial x} \\
\frac{\partial I_1}{\partial y} \\
\frac{\partial I_1}{\partial z}
\end{pmatrix} =
\begin{pmatrix}
0 & x & -y \\
-x & 0 & -z + x^2 \\
y & -x & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial I_2}{\partial x} \\
\frac{\partial I_2}{\partial y} \\
\frac{\partial I_2}{\partial z}
\end{pmatrix}. \tag{3.6}
\]

### 3.2 Lotka-Volterra equation

The generalized Lotka-Volterra equation is given by

\[
\begin{align*}
\dot{x}_1 &= x_1(a_3x_2 + x_3 + l_1), \\
\dot{x}_2 &= x_2(a_1x_3 + x_1 + l_2), \\
\dot{x}_3 &= x_3(a_2x_1 + x_2 + l_3). \tag{3.7}
\end{align*}
\]

The divergence free condition \(\partial v_i/\partial x_i = 0\) imposes conditions on \(a_i\) and \(l_i\), such that \(a_i = -1\) and \(l_1 + l_2 + l_3 = 0\). Then the reduced set of equations becomes

\[
\begin{align*}
\dot{x}_1 &= x_1(-x_2 + x_3 + l_1), \\
\dot{x}_2 &= x_2(-x_3 + x_1 + l_2), \\
\dot{x}_3 &= x_3(-x_1 + x_2 + l_3). \tag{3.8}
\end{align*}
\]
One can check directly that \( \nabla \times \mathbf{v} \neq 0 \). Recast this in the vector potential form \( \mathbf{A} \) with

\[
A_i = x_1 x_2 x_3 + \tilde{l}_i,
\]

where the constants are satisfying

\[
l_1 = \tilde{l}_3 - \tilde{l}_2, \quad l_2 = \tilde{l}_1 - \tilde{l}_3, \quad l_3 = \tilde{l}_2 - \tilde{l}_1.
\]

One can also check that \( \mathbf{A} \cdot (\nabla \times \mathbf{A}) \neq 0 \). Just like the previous example we can express \( d^* \alpha \) in terms of \( J_1 \) and \( J_2 \) using a multiplier \( m = x_1 x_2 x_3 \):

\[
x_1(-x_2 + x_3 + l_1)dx_2 \wedge dx_3 + x_2(-x_3 + x_1 + l_2)dx_3 \wedge dx_1 + x_3(a_2 x_1 + x_2 + l_3)dx_1 \wedge dx_2 = m(J_1 \wedge J_2),
\]

(3.9)

where

\[
J_1 = \frac{dx_1}{x_1} + \frac{dx_2}{x_2} + \frac{dx_3}{x_3}, \quad J_2 = dx_1 + dx_2 + dx_3 + \frac{l_3 dx_2}{x_2} - \frac{l_2 dx_3}{x_3}.
\]

This immediately shows \( dJ_1 = dJ_2 = 0 \) and the two Hamiltonians are

\[
H_1 = \ln x_1 + \ln x_2 + \ln x_3, \quad H_2 = x_1 + x_2 + x_3 + l_3 \ln x_2 - l_2 \ln x_3.
\]

(3.10)

Thus equations can be obtained from the standard Nambu-Hamiltonian formalism.

### 4 Generalized Hamiltonian case and vector equations

Consider the following vector potential, see for example [8],

\[
\mathbf{A} = \frac{1}{2} (y^3 - x^2 z, z^3 - xy^2, x^3 - yz^2).
\]

(4.1)

The dynamics generated by the curl of \( \mathbf{A} \) is computed to be

\[
\dot{x} = z^2, \quad \dot{y} = x^2, \quad \dot{z} = y^2.
\]

(4.2)

See that \( \nabla \times \mathbf{v} \) is not vanishing for \( \mathbf{v} = (z^2, x^2, y^2) \). The vector potential satisfies

\[
\mathbf{A} \cdot (\nabla \times \mathbf{A}) = z^2(y^3 - x^2 z) + x^2(z^3 - xy^2) + y^2(x^3 - yz^2) = 0.
\]

Thus the helicity of the local flow is identically zero, due to this identity Sposito’s result shows that the flow streamlines are confined to flat 2D manifolds. Hence we can not get the Nambu-Hamiltonian structure here.

The identity (2.28) becomes

\[
y^2 dx \wedge dy + x^2 dz \wedge dx + z^2 dy \wedge dz = J_1 \wedge J_2,
\]

(4.3)

where

\[
J_1 = (z^2 dy - x^2 dx), \quad J_2 = (dz - \frac{y^2}{z^2} dx).
\]

(4.4)
It is easy to check that $J_1$ and $J_2$ are invariant one-forms that is $\iota_\omega J_i = 0$. The integrals one-forms $J_1$ and $J_2$ presented in (4.4) are not closed, i.e.,

$$dJ_1 = 2zdz \wedge dy \neq 0, \quad dJ_2 = 2\frac{y^2}{z^3}dz \wedge dx - 2\frac{y}{z^2}dy \wedge dx \neq 0.$$  

Hence we can not express the equation (4.1) neither in the Nambu-Hamiltonian nor the bi-Hamiltonian forms.

We can find two more such pairs like $J_1$ and $J_2$ which also satisfy

$$y^2dx \wedge dy + x^2dz \wedge dx + z^2dy \wedge dz = K_1 \wedge K_2 = L_1 \wedge L_2,$$

where $(K_1, K_2) = (y^2dx - z^2dz, dy - \frac{x^2}{z^2}dx)$ and $(L_1, L_2) = (z^2dy - x^2dx, dz - \frac{y^2}{z^2}dy)$ but none of them are closed.

### 4.1 Homotopy operator and closed form

Let $M$ be a manifold and let $I = [0, 1]$ Suppose $\omega \in \Omega^k(M \times I)$, every $\omega$ can be uniquely decomposed to $\omega = \alpha_1 + \alpha_2 \wedge dt$ with $\alpha_1(0, t) \in \Omega^k(M)$ and $\alpha_2(0, t) \in \Omega^{k-1}(M)$. We define the following mapping

$$D_k : \Omega^k(M \times I) \to \Omega^{k-1}(M), \quad (D_k \omega)(m) := (-1)^{k-1} \int_0^1 \alpha_2(m, t)dt,$$  

(4.5)

where the integral is to be understood as an integral of a function on the interval $I$ with values in the vector space $\wedge^{k-1}T^*_m M$, [29]. Notice that, this satisfies

$$d(D_k \omega) + D_{k+1}(d\omega) = \omega|_{t=1} - \omega|_{t=0}. \quad (4.6)$$

Let $\alpha \in \Omega^k(N)$, let $\phi_0, \phi_1 : M \to N$ be smooth mapping, Then we set $\Omega = F^*\alpha$ and obtain

$$d(D_k F^*\alpha) + D_{k+1}(dF^*\alpha) = \phi_1^*\alpha - \phi_0^*\alpha. \quad (4.7)$$

Here, the operator $H_k = D_k \circ F^*$ is called the homotopy operator. If $\alpha$ belongs to a cohomology class in $H^k(N)$, then $dD_k F^*\alpha = \phi_1^*\alpha - \phi_0^*\alpha$. So $(\phi_1^*\alpha - \phi_0^*\alpha)$ differ by an exact form, hence they define the same cohomology class.

The operator $D_k$ defined in (4.7) yields an explicit potential form of a closed form on a contractible manifold. We recall that if $M$ is contractible then by Poincaré lemma $H_k(M) = \mathbb{R}$ for $k = 0$ otherwise it is zero for all other $k > 0$. Let $F : M \times I \to N$ be a homotopy fulfilling $F(m, 0) = \phi_0(m)$ and $F(m, 1) = \phi_1(m)$, where $I = [0, 1]$. From (4.7), we obtain

$$dD_k F^*\alpha + D_{k+1}dF^*\alpha = -\alpha,$$

for all $\alpha, d\alpha = 0$, then

$$\beta = -D_k F^*\alpha. \quad (4.8)$$
Claim 4.1 Let $\alpha = y^2 dx \wedge dy + x^2 dz \wedge dx + z^2 dy \wedge dz$ be a two form and $F((x,y,z),t) = (tx, ty, tz)$ be a homotopy mapping. Then the exact one form $\eta = -D_2 F^* \alpha$ is given by

$$\eta = -\frac{1}{4} ((y^3 - x^2 z)dx + (z^3 - xy^2)dy + (x^3 - z^2 y)dz),$$

(4.9)

such that $\alpha = d\eta$.

Notice that $\eta = \eta_1 dx + \eta_2 dy + \eta_3 dz$ plays the role of Hamiltonian (one) form.

Let us consider first order Hamiltonian equations in 2D, we can express it

$$\dot{x} \Omega = dx \wedge dH, \quad \dot{p} \Omega = dp \wedge dH, \text{ where } \Omega = dx \wedge dp,$$

which yields Hamiltonian equations in the standard form $\dot{x} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial x$. Similarly, we can express 3D equations of motion as

$$\dot{x}_i \tilde{\Omega} = dx_i \wedge d\eta, \text{ where } \tilde{\Omega} = dx \wedge dy \wedge dz,$$

(4.10)

where $x_i = x, y, z$. Expanding in components we obtain

$$\dot{x} = -\frac{\partial \eta_2}{\partial z} + \frac{\partial \eta_3}{\partial y}, \quad \dot{y} = -\frac{\partial \eta_3}{\partial x} + \frac{\partial \eta_1}{\partial z}, \quad \dot{z} = -\frac{\partial \eta_1}{\partial y} + \frac{\partial \eta_2}{\partial x}.$$  

(4.11)

This set of equations are also obtained by Dumachev [8, 9] and he called as vector Hamiltonian equation.

### 4.2 Deformation and Hamiltonization

In this section we deform the two form (4.3) in such a way that the dynamics involved in the modified system also yields velocity as a curl of potential vector field and during this process we will get rid of null condition, i.e. $A \cdot \nabla \times A = 0$. Let us assume one form $\tilde{J}_1 = xdx + ydy + zdz$ and demand another form $\tilde{J}_2 = A_1 dx + A_2 dy + A_3 dz$ in such a way that it would yield the original form (4.3) and at the same time it should be exact. We arrive at the following set of (deformed) equations

$$\dot{x} = z^2 - y^2 + xz - xy, \quad \dot{y} = x^2 - z^2 + xy - yz, \quad \dot{z} = y^2 - x^2 + yz - xz.$$  

(4.12)

It is easy to check that (4.12) yields a divergence free vector field $v$ and it satisfies $v = \nabla \times A$, where the vector potential $A$ is given by

$$A = ((y^2 + z^2)x + xyz, (x^2 + z^2)y + xyz, (x^2 + y^2)z + xyz).$$  

(4.13)

It is easy to check that the vector potential $A$ for the deformed equation satisfies $A \cdot \nabla \times A \neq 0$. The identity (2.28) becomes

$$(y^2 - x^2 + yz - xz) dx \wedge dy + (x^2 - z^2 + xy - yz) dz \wedge dx + (z^2 - y^2 + xz - xy) dy \wedge dz = \tilde{J}_1 \wedge \tilde{J}_2.$$  

(4.14)
where $\hat{J}_1$ and $\hat{J}_2$ are exact one forms, $\hat{J}_1 = dI_1$ and $\hat{J}_2 = dI_2$ with the potential functions

$$I_1 = xy + yz + zx, \quad I_2 = \frac{1}{2}(x^2 + y^2 + z^2). \quad (4.15)$$

Using these two first integrals, we can express equation (4.12) in the Nambu-Hamiltonian form (2.21) as follows

$$\dot{x} = \{x, I_1, I_2\} = \nabla I_1 \times \nabla I_2 \quad (4.16)$$

where $I_1$ and $I_1$ are the ones in (4.15).

## 5 Outlook

We studied Hamiltonian aspects of divergence-free vector fields in dimension 3, chaotic aspects of these kind of equations have been studied in [33][34], in general handful of papers are known in the literature for three-dimensional divergence-free vector fields. In particular, we have studied $\dot{x} = \nabla \times A$ type flows, and all these flows satisfy Frobenius integrability condition in a sense that $i_\nu \Omega \wedge J = 0$, or in other words, $i_\nu \Omega = J \wedge K$, where $J$ and $K$ are 1-forms. We explored that not all the flows yield (Nambu) Hamiltonian framework, it depends on the nature of $J$ and $K$, whether they are closed or not. We have demonstrated that when we deform the second class of system to get rid of null condition $A \cdot \nabla \times A = 0$, the system possesses the Hamiltonian realization. In this paper we could not prove the existence theorem for Hamiltonian framework but instead of that we have demonstrated the existence of both Hamiltonian and non-Hamiltonian type flows for $v = \nabla \times A$ type 3D flows. Also, very little is known about divergence-free vector fields in dimension $n \geq 4$, so we will focus on this problem in our next project. This piece of work also raised several questions regarding the applicability of the Euler theorem of potential.

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## References

[1] Ay, A., Gürses, M., & Zheltukhin, K. (2003). Hamiltonian equations in $\mathbb{R}^3$, J.Math. Phys. 44(12) 5688-5705.
[2] Barbarosie, C. (2011). Representation of divergence-free vector fields, Quart. Appl. Math. 69, no. 2, 309-316.

[3] Berry, M. V., & Shukla, P. (2012). Classical dynamics with curl forces, and motion driven by time-dependent flux. Journal of Physics A: Mathematical and Theoretical, 45(30), 305201.

[4] Berry, M. V., & Shukla, P. (2013). Physical curl forces: dipole dynamics near optical vortices. Journal of Physics A: Mathematical and Theoretical, 46(42), 422001.

[5] Berry, M. V., & Shukla, P. (2015). Hamiltonian curl forces. Proc. R. Soc. A, 471(2176), 20150002.

[6] Biskamp, D. (1993). Nonlinear magnetohydrodynamics, Cambridge University Press.

[7] Cariñena, J. F., Guha, P. and Rañada, M. F. (2008). Hamiltonian and quasi-Hamiltonian systems, NambuPoisson structures and symmetries, J. Phys. A: Math. Theor. 41, 335209.

[8] Dumachev, V. N. (2018). Vector hamiltonians in Nambu mechanics. Russian Mathematics, 62(2), 28-33.

[9] Dumachev, V. N. (2011). Phase flows and vector Hamiltonians. Russian Mathematics, 55(3), 1-7.

[10] Esen, O., Ghose Choudhury, A., & Guha, P. (2016). Bi-Hamiltonian Structures of 3D Chaotic Dynamical Systems. International Journal of Bifurcation and Chaos, 26(13), 1650215.

[11] Gao, P. (2000). Hamiltonian structure and first integrals for the LotkaVolterra systems. Physics Letters A, 273(1-2), 85-96.

[12] Fecko, M. (1992). On a variational principle for the Nambu dynamics. Journal of mathematical physics, 33(3), 930-933.

[13] Fernandes, R. L. (1994). Completely integrable bi-Hamiltonian systems. Journal of Dynamics and Differential Equations, 6(1), 53-69.

[14] Ghose-Choudhury A, Guha P, Paliathanasis A, Leach P G L (2017), Noetherian symmetries of noncentral forces with drag term, Int. J. Geom. Methods Mod. Phys. 14, 1750018 (2017) [14 pages]

[15] Gautheron, P. (1996). Some remarks concerning Nambu mechanics. Letters in Mathematical Physics, 37(1), 103-116.

[16] Guha, P. (2001). Volume preserving multidimensional integrable systems and NambuPoisson geometry. Journal of Nonlinear Mathematical Physics, 8(3), 325-341.
[17] Gümṛal, H. (2010). Existence of Hamiltonian Structure in 3D. Advances in Dynamical Systems and Applications, 5(2), 159-171.

[18] Gümṛal, H., & Nutku, Y. (1993). Poisson structure of dynamical systems with three degrees of freedom. Journal of Mathematical Physics, 34(12), 5691-5723.

[19] Hernandez-Bermejo, B. (2001). New solutions of the Jacobi equations for three-dimensional Poisson structures. Journal of Mathematical Physics, 42(10), 4984-4996.

[20] Hernandez-Bermejo, B. (2001). One solution of the 3D Jacobi identities allows determining an infinity of them. Physics Letters A, 287(5), 371-378.

[21] Hernández-Bermejo, B. (2007). New solution family of the Jacobi equations: Characterization, invariants, and global Darboux analysis. Journal of mathematical physics, 48(2), 022903.

[22] Jacobi, C.G.J. (1844). Sul principio dell’ultimo moltiplicatore, e suo uso come nuovo principio generale di meccanica, Giornale Arcadico di Scienze, Lettere ed Arti 99, 129-146.

[23] Jacobi, C.G.J. Theoria novi multiplicatoris systemati aequationum differentialium vulgarium applicandi. J. Reine Angew. Math 27 (1844), 199-268, Ibid 29(1845), 213-279 and 333-376. Astrophys. Journal, 342, 635-638, (1989).

[24] Moffat, H. K. (1978), Magnetic field generation in electrically conducting fluids, Cambridge University Press.

[25] Morando, P. (1996), Liouville condition, Nambu mechanics, and differential forms. J. Phys. A: Math. Gen. 29 L329–31.

[26] Nambu, Y. (1973). Generalized hamiltonian dynamics. Physical Review D, 7(8), 2405.

[27] Moulin Ollagnier J., & Strelcyn, J.-M, On first integrals of linear systems, Frobenius integrability theorem and linear representations of Lie algebras, in Bifurcations of Planar Vector Fields, Proceedings, Luminy 1989, J.-P. Francoise and R. Roussarie (Eds.), Lecture Notes in Mathematics 1455, Springer-Verlag, Berlin, Heidelberg, New-York (1991).

[28] Olver, P. J. (2000). Applications of Lie groups to differential equations (Vol. 107). Springer Science & Business Media.

[29] Rudolph, G., Schmidt, M., Differential geometry and mathematical physics. Part I. Manifolds, Lie groups and Hamiltonian systems. Theoretical and Mathematical Physics. Springer, Dordrecht, 2013. xiv+759 pp

[30] Semenov, V.S., Korovinski, D.B., & Biernat, H.K.(2001). Euler potentials for the MHD Kamchatnov-Hopf soliton solution, arXiv:physics/0111212 [physics.plasm-ph].
[31] Sposito, G., (2001), Topological groundwater hydrodynamics, Adv. Wat. Res., 24, 793-801.

[32] Takhtajan, L. (1994). On foundation of the generalized Nambu mechanics. Communications in Mathematical Physics, 160(2), 295-315.

[33] Tang, X.Z. & Boozer, A.H. (1999). A Lagrangian analysis of advection-diffusion equation for a three dimensional chaotic flow. Physics of Fluids 11, 1418.

[34] Thiffeault, J.-L. & Boozer, A.H. (2001). Geometrical constraints on finite-time Lyapunov exponents in two and three dimensions. Chaos 11, 16.

[35] Wedemann, R. S., Plastino, A. R., & Tsallis, C. (2016). Curl forces and the nonlinear Fokker-Planck equation. Physical Review E, 94(6), 062105.

[36] Wiggins, S., (2010), Coherent structures and chaotic advection in three dimensions, J. Fluid. Mech, 654, 1-4.