In [Shi86], Shi proved Lusztig’s conjecture that the number of two-sided cells for the affine Weyl group of type $A_{n-1}$ is the number of partitions of $n$. As a byproduct, he introduced the Shi arrangement of hyperplanes and found a few of its remarkable properties. The Shi arrangement has since become a central object in algebraic combinatorics. This article is intended to be a fairly gentle introduction to the Shi arrangement, intended for readers with a modest background in combinatorics, algebra, and Euclidean geometry. After background material in Section 1, this introduction to the arrangement will be by way of a discussion in Section 2 of how it arose, some of its marvelous enumerative properties in Section 3, and some of its surprising connections to algebra in Section 4. For some brief comments on recent extensions, see Section 5 and for an incomplete list of topics we left out, see Section 6.

1. Background

In this section, we will give very brief introductions to some of the ingredients needed to define the Shi arrangement.

1.1. Root systems and Coxeter group notation. Let $V$ be a finite dimensional real vector space with fixed inner product $(\langle | \rangle)$. We’ll use $\Delta$ to denote a root system: a finite set of vectors in $V$ which satisfies

(1) $\Delta \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Delta$ and
(2) $s_\alpha \Delta = \Delta$ for all $\alpha \in \Delta$,

where $s_\alpha$ is the reflection about the hyperplane with normal $\alpha$. We use $\Delta^+$ to denote a choice of positive roots of $\Delta$, so that $\Delta = \Delta^+ \cup -\Delta^+$, and $\Pi$ to denote the simple roots, which are a basis for the $\mathbb{R}$-span of $\Delta$. The reflections $S = \{s_\alpha\}_{\alpha \in \Pi}$ generate a finite reflection group $W$. The rank of the system and of $W$ is the dimension of the space spanned by $\Delta$.

Coxeter groups generalize finite reflection groups. Let $W$ be a group with a set of generators $S \subset W$. Let $m_{st}$ be the order of the element $st$, with $s, t \in S$. If there is no relation between $s$ and $t$, we set $m_{st} = \infty$. If $W$ has a presentation such that

(1) $m_{ss} = 1$
(2) $m_{st} < \infty$ for $s, t \in S, s \neq t$,

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The roots and reflecting hyperplanes of affine type $A_2$. The reflection $s_1$ (resp. $s_2$) flips the plane over the hyperplane $H_{\alpha_{1,0}} (H_{\alpha_{2,0}}).$ The reflection $s_0$ reflects over $H_{\theta,1}$.}
\end{figure}
then $W$ is a Coxeter group. We refer to $(W, S)$ as a Coxeter system. If $m_{st} \in \{2, 3, 4, 6\}$ when $s \neq t$, then the Coxeter group is called crystallographic and, if finite, is a Weyl group. It is also a reflection group. The product in any order of all the elements in $S$ is called a Coxeter element; all Coxeter elements for a given $W$ are conjugate and their order is the Coxeter number of $W$.

The expression for the reflection $s_\alpha$, $\alpha \in \Delta$, is

$$s_\alpha(v) = v - 2\frac{\langle v | \alpha \rangle}{\langle \alpha | \alpha \rangle} \alpha$$

for $v \in V$. For any $k \in \mathbb{R}$, we can define an affine reflection $s_{\alpha,k}$ by

$$s_{\alpha,k}(v) = v - 2\frac{\langle v | \alpha \rangle - k}{\langle \alpha | \alpha \rangle} \alpha.$$  

We define the affine Weyl group to be the group generated by all affine reflections $s_{\alpha,k}$ for $\alpha \in \Delta$ and $k \in \mathbb{Z}$. It is also a Coxeter group. Its simple reflections are the simple reflections of the finite Weyl group. Our proofs will not be detailed enough to need the full set of affine roots, and we will not define them. Given a root system $\Delta$, we write $W_\Delta$ for the corresponding finite group. Please see [Hum90] or [Kac90] for more information.

We put a partial order on any root system. The root poset of $\Delta$ is the set of positive roots $\Delta^+$, partially ordered by setting $\alpha \leq \beta$ if $\beta - \alpha$ is a nonnegative linear combination of simple roots. If the root system is irreducible, then there is a unique highest root relative to this ordering. We will denote this root by $\theta$. See Figure 1.2 for a picture of the root poset for type $A_4$.

Let $W$ be a Coxeter group. Every $w \in W$ has an expression as a product of elements of $S$: $w = s_{i_1} \cdots s_{i_k}$. If $k$ is minimal among all expressions for $w$, then $k$ is the length $\ell(w)$. Any expression for $w$ of length $\ell(w)$ is a reduced expression.

We will often refer to the type of a group or root system, particularly “type $A$” and “affine type $A$,” which are the symmetric group or affine symmetric group if we are referring to groups. Please see [Hum90] for more information on the classification of finite reflection groups and Coxeter groups. Humphreys and [BB05] are good sources for the definitions of irreducible, Bruhat order, and other material omitted here.

1.1.1. Type $A$. We will be seeing type $A$ often, so we’ll be a little more concrete. For $A_{n-1}$, the vector space $V$ is $\{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid a_1 + \cdots + a_n = 0\}$. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$ and $\langle | \rangle$ be the bilinear form for which this is an orthonormal basis. The set of roots is $\Delta = \{e_i - e_j \mid i \neq j\}$ and a root $\alpha \in \Delta$ is positive, written $\alpha > 0$, if $\alpha \in \Delta^+ = \{e_i - e_j \mid i < j\}$. Set $\alpha_i = e_i - e_{i+1}$; the simple roots are $\{\alpha_1, \ldots, \alpha_{n-1}\}$. The set $\Pi$ is a basis of $V$.

The Coxeter group that Shi studied was the affine symmetric group $\hat{S}_n$, and we review that here. There are several possible descriptions, here we give one due to Lusztig [Lus83]. It is the set of permutations $w$ of $\mathbb{Z}$ such that

1. $w(i + n) = w(i) + n$ for all $i \in \mathbb{Z}$
2. $\sum_{i=1}^n w(i) = \binom{n}{2}$

It’s a Coxeter group: for any $i$, $0 \leq i < n$, $s_i$ corresponds to the permutation

$$t \mapsto \begin{cases} 
    t & \text{if } t \mod n \neq i \text{ and } t \mod n \neq i + 1 \\
    t - 1 & \text{if } t \mod n = i + 1 \\
    t + 1 & \text{if } t \mod n = i
\end{cases}$$

The set of reflections $S$ is $\{s_1, \ldots, s_{n-1}, s_0\}$ and

$$\hat{S}_n = \langle s_1, \ldots, s_{n-1}, s_0 \rangle.$$  

The affine symmetric group contains the symmetric group $S_n$ as a subgroup. $S_n$ is the subgroup generated by the $s_i$, $0 < i < n$. We identify $S_n$ as permutations of $\{1, \ldots, n\}$ by identifying $s_i$ with the simple transposition $(i, i+1)$. We act on the right, as did Shi.
1.2. A taste of Coxeter combinatorics, type $A$. The number of parking functions and the Catalan numbers appear in every discussion of the Shi arrangement. We’ll define the parking functions when we first see them, in Section 3.5, but we’ll collect some facts on the Catalan objects here, mostly type $A$. There are an awful lot of Catalan objects, but only a few of them will appear in this survey.

A partition is a finite sequence $\lambda = (\lambda_1, \ldots, \lambda_r)$ of positive integers in decreasing order: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0$. We identify a partition with its Young diagram, the left-justified array of boxes where the $i^{\text{th}}$ row from the top has $\lambda_i$ boxes. A box is called removable (respectively addable) if we can remove (respectively add) it and still have a diagram of a partition. We use $|\lambda| = \lambda_1 + \cdots + \lambda_r$ and $\ell(\lambda) = r$.

1.2.1. Set partitions. We denote the set $\{1, 2, \ldots, n\}$ by $[n]$. The nonempty subsets $B_1, \ldots, B_k$ of $[n]$ are a set partition of $[n]$ if they are pairwise disjoint and their union is $[n]$. We denote the set partition $\{B_1, \ldots, B_k\}$ by $B_1 \cdots B_k$. For example, $13|25|64$ is a partition of $[6]$. The arc diagram of a set partition $\pi$ is defined as follows: place the numbers $1, 2, \ldots, n$ in order on a line and draw an arc between each pair $i < j$ such that

- $i$ and $j$ are in the same block of $\pi$, and
- there is no $k$ such that $i < k < j$ and $i, k, j$ are in the same block.

See Figure 1.2.

The partition $\pi$ has $k$ blocks if and only if it has $n - k$ arcs. This is easy to see if the partition has no arcs. Consider a partition with $k$ blocks and $n - k$ arcs, where $i$ and $j$ are in different blocks. Suppose we add an arc from $i$ to $j$. We have joined $i$’s and $j$’s blocks, and we now have $k - 1$ blocks and $n - k + 1$ arcs.

A set partition is called noncrossing if there does not exist $i < j < k < l$ such that there is an arc from $i$ to $k$ and an arc from $j$ to $l$. There are $C_n$ noncrossing set partitions, where

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

is the Catalan number (type $A$). It is called nonnesting if there does not exist $i < j < k < l$ such that there is an arc from $i$ to $l$ and an arc from $j$ to $k$.

![Figure 1.2](image)

**Figure 1.2.** On the left is the nonnesting set partition $\pi = \{\{1, 3\}, \{2, 4, 5\}\}$. Next to it is the root poset of type $A_4$ with the filter corresponding to $\pi$ circled. The third figure is the partition inside the staircase which corresponds to $\pi$. On the right is the corresponding Dyck path.

1.2.2. Dyck paths. A Dyck path of length $n$ is a lattice path which starts at $(0,0)$, takes only north or east steps of length 1, never goes below the line $y = x$, and ends at $(n,n)$. A north step followed by an east step is called a valley of the path.

1.2.3. Root poset. An ideal of a poset $P$ is a subset $I$ of the elements of $P$ such that if $x \in I$ and $y \leq x$ then $y \in I$. A filter is like an ideal, except that the condition becomes that $x \in I$ and $x \leq y$ implies $y \in I$. A subset $X$ of the elements of $P$ is an antichain if no two elements in $X$ are comparable. An ideal is determined by its maximal elements, which form an antichain, just as a filter is by its minimal elements. These antichains are also called nonnesting partitions; in type $A$ the bijection to the nonnesting partition described above is simply sending the root $\varepsilon_i - \varepsilon_j$ in an antichain to the arc from $i$ to $j$. There are Catalan number of ideals, filters, and antichains in the root poset defined in Section 1.1. We can map a filter $F$ in the root poset for
type $A_{n-1}$ to a partition $\lambda$ whose diagram fits inside the staircase partition $(n-1, n-2, \ldots, 1)$ by the rule that $\varepsilon_i - \varepsilon_j \in F$ if and only if the box in row $i$ and column $n+1-j$ is in the diagram of $\lambda$. The minimal elements of $F$ correspond to the removable boxes of $\lambda$.

There are well-known bijections among all these objects; please see Stanley’s book on the subject [Sta15].

1.3. **Deformation of Coxeter arrangements.** A (real) hyperplane arrangement $\mathcal{H}$ is a set of hyperplanes, possibly affine hyperplanes, in a real vector space. For us, the vector space will be $V$, the span of some root system $\Delta$, with a fixed inner product $\langle | \rangle$ which is $W_\Delta$ invariant. We’ll be looking at connected components of a hyperplane arrangement’s complement $V \setminus \bigcup_{H \in \mathcal{H}} H$. We will refer to these as the regions of the arrangement. The closure $\bar{R}$ of the region $R$ is a convex polyhedron. A face of $\mathcal{H}$ is a nonempty set of the form $\bar{R} \cap x$, where $x$ is an intersection of hyperplanes in $\mathcal{H}$. The dimension of a face is the dimension of its affine span. See Stanley [Sta96] for more details. A wall $H$ of $R$ is a hyperplane $H \in \mathcal{H}$ such that $\dim(H \cap R) = \dim(H)$. The word “bounded” applied to a region has its usual meaning: a region is bounded if there is a real number $M$ such that all points in the region are within distance $M$ of the origin. Let $\tau(A)$ and $b(A)$ be the number of regions and number of bounded regions, respectively, of the arrangement $A$.

Let $\Delta$ be a root system. The roots (plus the integers) define a system of affine hyperplanes

$$H_{\alpha,k} = \{ v \in V \mid \langle v \mid \alpha \rangle = k \}.$$ 

Note $H_{-\alpha,-k} = H_{\alpha,k}$. In type $A$, we will sometimes write $x_i - x_j = k$ instead of $H_{\alpha_0, \ldots, \alpha_j, k}$.

The Coxeter arrangement, also called the braid arrangement, is defined

$$Cox_\Delta = \{ H_{\alpha,0} : \alpha \in \Delta^+ \}.$$ 

We give the regions of this arrangement the special name chambers. Each chamber corresponds to an element of $W = W(\Delta)$. The dominant chamber of $V$ is $\bigcap_{i=1}^{n-1} H_{\alpha_i, 0^+}$, where $H_{\alpha_i, k^+}$ is the half-space $\{ v \in V \mid \langle v \mid \alpha \rangle \geq k \}$. The dominant chamber corresponds to the identity of $W$. It is also referred to as the fundamental chamber in the literature.

The affine Coxeter arrangement is all integer translates of the hyperplanes in $Cox_\Delta$; that is, it is the whole system of hyperplanes $\{ H_{\alpha,k} \}_{\alpha \in \Delta^+, k \in \mathbb{Z}}$. In this arrangement, each region is called an alcove and the fundamental alcove is $A_0$, the interior of $H_{\theta,1^-} \cap \bigcap_{\alpha \in \Pi} H_{\alpha,0^+}$, where $\theta$ is the highest root. A dominant alcove is one contained in the dominant chamber.

We also have the $m$-Catalan arrangement:

$$Cat_\Delta^m = \{ H_{\alpha,r} : \alpha \in \Delta, 0 \leq r \leq m \}.$$ 

Let $W$ be an affine Weyl group and $V$ the vector space spanned by its roots. $W$ acts on $V$ via affine linear transformations, and acts freely and transitively on the set of alcoves. In affine type $A_{n-1}$, $s_i$ reflects over $H_{\alpha_i,0}$ for $1 \leq i \leq n-1$ and $s_0$ reflects over $H_{\theta,1}$, where $\theta$ is the highest root. We identify each alcove $A$ with the unique $w \in W$ such that $A = A_0w$. For example, if $w$ is the element of affine $A_2$ whose reduce decomposition is $s_0s_1$, then $A_0w$ is the image of $A_0$ after reflecting first across $H_{\theta,1}$ and then across $H_{\alpha_1,0}$. See Figure 3.1.
We can be even more specific for type $A_{n-1}$. The action on $V$ is given by

$$s_i(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_n) \quad \text{for } i \neq 0, \text{ and}$$

$$s_0(a_1, \ldots, a_n) = (a_n + 1, a_2, \ldots, a_{n-1}, a_1 - 1).$$

Note $S_n$ preserves $\langle | \rangle$, but $\hat{S}_n$ does not.

An alcove $A$ can be described by the hyperplanes it is between. For example, in Figure 3.1, the alcove labeled by 20 is between $H_{\alpha_1,1}$ and $H_{\alpha_1,2}$, between $H_{\alpha_2,0}$ and $H_{\alpha_2,1}$ and between $H_{\theta,1}$ and $H_{\theta,2}$. Given a positive root $\alpha$, there is a unique integer $k = k_\alpha(A)$ such that $k < \langle \alpha | x \rangle < k + 1$ for all $x \in A$. Let $K(A) = \{k_\alpha(A)\}_{\alpha \in \Delta^+}$ denote the set of coordinates for $A$, indexed by the positive roots. Returning to Figure 3.1, the alcove labeled by 20 has coordinates $k_{\alpha_1} = k_\theta = 1$ and $k_{\alpha_2} = 0$.

Shi characterized the sets of integers which can arise as $K(A)$ for some alcove $A$; for type $A$ in [Shi86, Chapter 6] and for general affine Weyl groups in [Shi87a]. The situation in general is rather messy, but if we assume our root system is an irreducible crystallographic one, then Shi found (see also [Ath04]) that a collection of integers indexed by the positive roots $\Delta^+$ corresponds to an alcove if and only if

$$k_\alpha + k_\beta \leq k_{\alpha+\beta} \leq k_\alpha + k_\beta + 1$$

for all $\alpha, \beta \in \Delta^+$ such that $\alpha + \beta \in \Delta^+$. We call the set $K(A)$ the coordinates of $A$.

2. Origin

2.1. Kazhdan-Lusztig Cells. We provide a bare-bones introduction to Kazhdan-Lusztig theory. For more information, see [Shi86], [KL79], [BB05]. If you are willing to believe that Kazhdan and Lusztig defined an equivalence relation on the elements of a Coxeter group, then skip this section. We include this collection of definitions for completeness. We’ll need the Hecke algebra $H$ and the Kazhdan-Lusztig polynomials in order to define the $W$-graph and then the cells. We will prove none of our claims.

Let $W$ be a Coxeter group and let $S$ be the corresponding set of simple reflections. We first follow [KL79], who follow [Bou68], for the definition of the Hecke algebra. Let $A$ be the ring of Laurent polynomials in the indeterminate $q^{1/2}$ with integral coefficients. The Hecke algebra $H = H(W, S)$ is a free module over $A$ with basis $T_w$, one for each $w \in W$. The multiplication is defined by the rules

1. $T_w T_w' = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$;
2. $(T_s + 1)(T_s - q) = 0$ if $s \in S$;

here $\ell(w)$ is the length of $w$.

Now for the polynomials. The involution on $A a \mapsto \bar{a}$ defined by $\overline{q^{1/2}} = q^{-1/2}$ extends to an involution of the ring $H$:

$$\overline{a_w T_w} = \sum a_w T_{w^{-1}}.$$  

Kazhdan’s and Lusztig’s theorem, Theorem 2.1 in this survey, asserts the existence of elements $C_w \in H$, one for each $w \in W$, and simultaneously defines the Kazhdan-Lusztig polynomials $P_{y,w}$, where $y, x \in W$. The order is the Bruhat order on $W$.

**Theorem 2.1 ([KL79]).** For any $w \in W$, there is a unique element $C_w \in H$ such that

* $\overline{C_w} = C_w$
* $C_w = \sum_{y \leq w} (-1)^{\ell(y) + \ell(w)} q^{\ell(w)/2 - \ell(y)} T_{y,w} T_y$

where $P_{y,x} \in A$ is a polynomial in $q$ of degree at most $\frac{1}{2}(\ell(w) - \ell(y)) - 1$ for $y < w$, and $P_{w,w} = 1$.

Kazhdan and Lusztig used the polynomials to define a graph and from there the cells. Now we follow the exposition given in [BB05], simplified just a bit because we will not prove anything. For $u, w \in W$, define $\mu(u, w)$ to be the coefficient of $q^{\frac{1}{2}(\ell(w) - \ell(u)) - 1}$ in $P_{u,w}(q)$ if $u < w$ and $\frac{1}{2}(\ell(w) - \ell(y)) - 1$ is an integer; otherwise, $\mu(u, w)$ is 0. Let $\mathcal{L}(w)$ be the set of left
descents of $w$: $L(w) = \{ s \in S \mid sw < w \}$. The directed, labeled graph $\tilde{\Gamma}_{(W,S)}^L$ is the graph with vertices $x \in W$ and edges $x \xrightarrow{(s,\mu)} y \in E$. There are two types of edges in $E$:

1. $x, y \in W, x \neq y$, either $\mu(x, y) \neq 0$ or $\mu(y, x) \neq 0$, and $s \in L(x) \setminus L(y)$. Let $\mu$ be either $\mu(x, y)$ or $\mu(y, x)$, whichever is not 0.
2. Loops at $x$: labeled by $s \in S$ and

$$\mu = \begin{cases} 
1 & \text{if } s \not\in L(x), \\
-1 & \text{if } s \in L(x).
\end{cases}$$

The graph $\tilde{\Gamma}_{(W,S)}^R$ has an analogous definition, using right descents of $w$: $R(w) = \{ s \in S \mid ws < w \}$. The graph $\tilde{\Gamma}_{(W,S)}^{LR}$ is the superposition of the $\tilde{\Gamma}_{(W,S)}^L$ and $\tilde{\Gamma}_{(W,S)}^R$. We describe the cells in graph theoretic terms. A directed graph is strongly connected if there is a directed path between all pairs of vertices. A strongly connected component of a directed graph is a maximal strongly connected subgraph. Finally, the left cells are the strongly connected components of $\tilde{\Gamma}_{(W,S)}^L$, the right cells the strongly connected components of $\tilde{\Gamma}_{(W,S)}^R$, and the two-sided cells the strongly connected components of $\tilde{\Gamma}_{(W,S)}^{LR}$.

The list of areas in math where cells appear is mind-boggling. Please see the short survey by Gunnells [Gun06], for example, for references, as well as for insight into the geometry of the cells. The book by Björner and Brenti [BB05] explains much of the combinatorial connection.

2.2. Shi regions and Kazhdan-Lusztig cells. Shi was studying cells in [Shi86]. He concentrated on the affine Weyl groups of type $A$, because of the following conjectures of Lusztig. In [Lus83], Lusztig defined a map $\sigma$ from $\tilde{E}_n$ to partitions of $n$. He conjectured that for any partition $\lambda$ of $n$, $\sigma^{-1}(\lambda)$, a set of affine permutations, is in fact a two-sided cell. Lusztig also conjectured a formula for the number of left (or right) cells which make up the two-sided cell $\sigma^{-1}(\lambda)$. The description of the map is simple enough and we define here. Let $w \in \tilde{E}_n$ and define $d_k = d_k(w)$ to be the maximum size of a subset of $\mathbb{Z}$ whose elements are noncongruent to each other modulo $n$ and which is a disjoint union of $k$ subsets each of which has its natural order reversed by $w$. The partition $\lambda$ is given by $(d_1, d_2 - d_1, \ldots, d_n - d_{n-1})$

**Example 2.2.** Let $n = 3$. The permutation $s_0$ is

$$\begin{pmatrix}
\cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
\cdots & -2 & -3 & -1 & 1 & 0 & 2 & 4 & 3 & 5 & 3 & \cdots
\end{pmatrix}.$$

The set $\{3, 4\}$ has its order reversed by $s_0$ and there is no larger set, so $d_1 = 2$. The sets $\{3, 4\}$ and $\{2\}$ show that $d_2 = 3$. Therefore $\sigma(s_0) = (2, 1)$. In our notation, $s_0 s_2$ is

$$\begin{pmatrix}
\cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
\cdots & -2 & -4 & 0 & 1 & -1 & 3 & 4 & 2 & 6 & 7 & \cdots
\end{pmatrix}$$

and $\sigma(s_0 s_2) = (2, 1)$ also. The identity maps to $(1, 1, 1)$ under $\sigma$ and $s_1 s_2 s_1$, for example, maps to $(3)$.

Shi proved both of Lusztig’s conjectures, and more. Shi used the identification of $\tilde{E}_n$ with alcoves to describe the cells of affine type $A$. He showed that the two-sided cells correspond to connected sets of alcoves, one set of alcoves for each partition $\lambda$ of $n$. A two-sided cell is a disjoint union of left-cells. Inside the two-sided cell corresponding to the partition $\lambda$, there is one left-cell for each tabloid of shape $\lambda$. See Figure 2.1.

What came to be known as the Shi arrangement was not initially defined in terms of hyperplanes. Shi began by defining rank $n$ sign types as triangular arrays $X = (x_{ij})_{1 \leq i < j \leq n}$ with entries from $\{+, -, \bigcirc\}$. The admissible sign types correspond to the regions of his arrangement. He defined them as the sign types which satisfy the following condition: for all $1 \leq i < t < j \leq n$, the triple

$$\begin{pmatrix}
x_{ij} \\
x_{it} \\
x_{tj}
\end{pmatrix}$$
Figure 2.1. The cells for affine $A_2$. The yellow region is the two-sided cell $\sigma^{-1}(1, 1, 1)$, which is also a single left-cell. The six pink regions are left-cells, whose union is the two-sided cell $\sigma^{-1}(3))$. The three gray regions are also left cells, and their union is the two-sided cell $\sigma^{-1}(2, 1))$. See [Shi86, Page 98].

is a member of the set $G_A$ of admissible sign types of rank 3 ($A_2$). $G_A$ is the set

$$\{+++, + + \cdot, ++ \cdot, + \cdot+, + \cdot-, - + \cdot, \cdot + +, \cdot + \cdot, + \cdot \cdot, \circ \cdot \cdot, + \cdot+, + \cdot- \}$$

(2.1)

Two comments on $G_A$. If we order the symbols $\{\circ, +, -\}$ as $- < 0 < +$, then $G_A$ can be seen as the rank 3 sign types where either $x_{12} \leq x_{13} \leq x_{23}$ or $x_{23} \leq x_{13} \leq x_{12}$, together with $x_{13} = +, x_{12} = x_{23} = 0$. The set $G_A$ has cardinality 16, which is $(n + 1)^{n-1}$ for $n = 3$.

Shi connected the admissible sign types to geometry using (1.1) and the map $\zeta$. If $K$ is the set of coordinates for an alcove $A$, then define the sign type $X = \zeta(A)$ by

$$x_{ij} = \begin{cases} + & \text{if } k_{ij} > 0 \\ \circ & \text{if } k_{ij} = 0 \\ - & \text{if } k_{ij} < 0. \end{cases}$$

He then calculated the hyperplanes so that the regions defined by them were made up of alcoves with the same image under the map $\zeta$. We use admissible sign type, region in the Shi arrangement, and Shi region interchangeably.

Shi showed in [Shi86] that the left-cells for affine type $A$ are themselves disjoint unions of admissible sign types. Admissible sign types were not used directly in the proofs of the Lusztig conjectures in Shi’s monograph, but describe the structure of the cells. They have taken on a life of their own.

Later, in [Shi87b], Shi extended the definition of admissible sign types, thereby generalizing the Shi arrangement. This is the definition we give below.

We give the definition for any irreducible, crystallographic root system $\Delta$. When the root system is type $A_{n-1}$, we will sometimes write $Shi_n$ instead of $Shi_\Delta$.

**Definition 2.3.** The Shi arrangement $Shi_\Delta$ is the collection of hyperplanes

$$\{H_{\alpha,k} \mid \alpha \in \Delta^+, 0 \leq k \leq 1\}.$$ 

**Example 2.4.** All alcoves in the region labeled by

$$+ \cdot-$$
2.2. The Shi arrangement for type $A_2$. See [Shi86, Page 102]. Each region has been labeled with its sign type. See Example 2.4

in Figure 2.2 have positive coordinate $k_{12} = k_{a_1}$, have negative coordinate $k_{a_2}$, and have the coordinate $k_\theta$ equal to zero. Likewise, all three coordinates of all alcoves in the region labeled by

$$++$$

are positive. See Section 1.3 for the definition of coordinates of an alcove.

We mention here that in the case where the Coxeter graph of the system contains an edge with a label greater than 3, it is not true that all the left-cells of the affine Weyl group are unions of admissible sign types. It may be conjectured that it holds for any affine Weyl group of simply-laced type. The cells in affine $D_4$ have been explicitly described by Shi in [Shi94], so the conjecture may not be difficult to verify. It is known that any left-cell in the lowest or highest two-sided cell of any irreducible affine Weyl group forms a single admissible sign type; see Shi [Shi87c, Shi88]. We thank Jian-Yi Shi for this information.

By a dominant region of the Shi arrangement, we mean a connected component of the hyperplane arrangement complement $V \setminus \bigcup_{H \in S_{\text{sh}}} H$ that is contained in the dominant chamber. Both the formula for the number of regions in the whole arrangement and for the number of dominant regions are intriguing and will be discussed in the next section.

3. Enumeration

The Shi regions have been counted multiple times. We discuss four different approaches to enumerating them.

3.1. The number of Shi regions, part 1. Shi concentrated on the admissible sign types in Chapter 7 of his book [Shi86], where he introduced the arrangement for type $A$.

He enumerates them for type $A$ by considering the alcove closest to the origin in each region. We’ll call this the minimal alcove of the region and denote it $A_R$ if the region is $R$. Shi called such an alcove the shortest alcove [Shi86, Section 7.3]. He characterized $A_R$ using left descents. Left descents are key to the definition of cells, so it is not surprising that they appear in the description of minimal alcoves. Basically, an alcove is minimal if any reflection which brings it closer to the origin flips it out of the region.
**Example 3.1.** See Figure 3.1. Let \( w = s_1 s_2 s_1 s_0 \). This \( w \) has two left descents, \( s_1 \) and \( s_2 \), and both \( s_1 w = s_2 s_1 s_0 \) and \( s_2 w = s_1 s_2 s_0 \) are in different regions the minimal permutation \( w \). On the other hand, \( w = s_1 s_2 s_0 \) is not minimal and indeed \( \ell(s_1 w) < \ell(w) \) and \( s_1 w = s_2 s_0 \) is in the same region as \( w \).

Every alcove corresponds to an affine permutation. We’ll call the affine permutations whose alcoves are minimal *minimal permutations*. Shi showed that the collection of alcoves corresponding to the inverses of minimal permutations is exactly a scaled version of the fundamental alcove. See Figure 3.1. Thus to calculate the number of regions in his newfound arrangement, he calculated the number of alcoves in this scaled fundamental alcove. He calculated something a bit more general: if the fundamental alcove is expanded by the positive integer \( m \), then it is made up of \( m \dim(V) \) alcoves. The alcoves corresponding to the inverses of minimal permutation land in the fundamental alcove scaled by \( n + 1 \), which showed that there are \((n + 1)^{n-1}\) regions in the Shi arrangement of type \( A \). The expression \((n + 1)^{n-1}\) pops up frequently in combinatorics and algebra; see [Hag08] for their connection to \(q,t\)-Catalan numbers, for example.

Shi’s enumeration of the regions is perhaps more complicated than the others described here. However, his discovery that the inverses of the minimal permutations correspond to a simplex is worth the price of admission. The minimal alcoves have been useful in other enumeration; see [Ath05, FV10], for example. For another example, Hohlweg, Nadeau, and Williams, in [HNW16], generalize the Shi arrangement to any Coxeter group (and beyond!) and conjecture that the inverses of the analogues of minimal permutations form a convex body. See also Sommers [Som05] where the simplex was generalized to what is now called the Sommers region. Thomas and Williams [TW14] show that the set of alcoves in this region, and by extension the Shi regions, exhibit the cyclic sieving phenomenon.

In [Shi87b], Shi generalized sign types to other affine Weyl groups. He defined sets analogous to (2.1) for other types. The hyperplane arrangements were still given by Definition 2.3, but now he considered root systems other than type \( A \). He used the map \( \zeta \) on alcoves and described the sign types which arose and again characterized the element in each region with the minimal number of hyperplanes separating it from the origin. As above, we identify elements \( w \in W \) with \( A_0 w \) and refer to \( w \) as minimal if its alcove is minimal. The fact that

\[
\bigcup_{w \text{ minimal}} A_0 w^{-1}
\]

is a simplex is not just a type \( A \) phenomenon. Shi proved it for other affine Weyl groups and used it to prove that there are \((h + 1)^n\) regions, where \( h \) is the Coxeter number of the system.

Shi counted the number of regions in the dominant chamber for affine Weyl groups in [Shi97]. He calls the admissible sign types corresponding to regions in the dominant chamber \( \oplus \)-sign types. For types \( A, B, C, \) and \( D \), he finds a bijection from \( \oplus \)-sign types to filters in the root poset for \( \Delta^+ \). Here is a technical detail: Shi finds the bijection to the positive coroots \((\Delta^+)^\vee\), which we won’t define, then mentions that it has the same type as \( \Delta^+ \) except when \( \Delta^+ \) has type \( B \) and \( C \). He deals with...
types $B$ and $C$ separately. We will continue using $\Delta^+$. He further maps the filters to subdiagrams of certain Young diagrams; see Section 1.2.3. For example, for type $A$, the subdiagrams are those of partitions whose diagrams fit inside the staircase shape. In the exceptional cases, he enumerates increasing subsets directly. He shows the $\oplus$-sign types for affice Weyl groups are enumerated by the Catalan numbers, although Shi does not mention them.

A few more words are in order on this important bijection to filters in the root poset. The key proposition, from Section 1.2 of [Shi97], follows (using roots instead of coroots).

**Proposition 3.2.** Assume that $X = (X_\alpha)_{\alpha \in \Delta^+}$ is a $\Delta^+$-tuple with $X_\alpha \in \{+, \emptyset\}$. Then $X$ is an $\oplus$-sign type if and only if the following condition on $X$ holds: if $\alpha, \beta \in \Delta^+$ satisfy $\beta > \alpha$ and $X_\alpha = +$, then $X_\beta = +$.

We’ll use type $A$ as an example. In the set $G_A$ in (2.1), there are five triples which contain only $\emptyset$ and $+$, but there are eight which are possible. The condition in Proposition 3.2 rules the other three out, proving sufficiency. For necessity, Shi uses induction to reduce to the rank two case and shows that the five $\oplus$-sign types in $G$ satisfy the condition.

3.2. **Interlude.** We’ll need these standard definitions for Sections 3.3 and 3.4. See [Sta12] for the definitions of the rank function $\rho$ and Möbius function $\mu$ of a poset.

**Definition 3.3.** [Sta12] Let $P$ be a finite graded poset with $\hat{0}$. Let $\rho$ be its rank function and $n$ the rank of $P$. Define the characteristic polynomial $\chi_P(x)$ of $P$ by

$$\chi_P(x) = \sum_{t \in P} \mu(\hat{0}, t) x^{n-\rho(t)}.$$  

**Definition 3.4.** [Sta12] Let $A$ be a hyperplane arrangement in a vector space $V$ and let $L(A)$ be the set of all nonempty intersections of hyperplanes in $A$. Include $V$ itself, by considering it as the intersection over the empty set. Order $L(A)$ by reverse inclusion, so that $\hat{0}$ is $V$.

See Figure 3.2 for an example of the poset of intersections.

If the intersection of all the hyperplanes in $A$ is nonempty, then $L(A)$ is a lattice. The intersection of all hyperplanes in $\text{Shi}_\Delta$ is empty and $L(\text{Shi}_\Delta)$ will only be a meet semi-lattice. It is finite and graded by $\rho(t) = n - \dim(t)$, where $n = \dim(V)$. The characteristic polynomial of an arrangement is

$$\chi_A(x) = \sum_{t \in L(A)} \mu(\hat{0}, t) x^{n-\rho(t)}$$

and the Poincaré polynomial is

$$P_A(x) = \sum_{t \in L(A)} \mu(\hat{0}, t) (-x)^{\rho(t)}.$$  

The Poincaré polynomial is a rescaled version of the characteristic polynomial.

The characteristic polynomial is invaluable for studying hyperplane arrangements, thanks to a theorem of Zaslavsky [Zas75b, Zas75a]. See also Stanley’s notes on hyperplanes [Sta07]. In Section 1.3, we defined $\tau(A)$ and $b(A)$ be the number of regions and number of bounded regions of the arrangement $A$.

**Theorem 3.5 ([Zas75b]).** Let $A$ be an arrangement in an $n$-dimensional real vector space. Then

$$\tau(A) = (-1)^n \chi_A(-1) = P_A(1)$$  

$$b(A) = (-1)^{\text{rank} A} \chi_A(1) = P_A(-1).$$  

3.3. **The number of Shi regions, part 2.** Headley [Hea97, Hea94] calculated the Poincaré polynomial $P_{\text{Shi}_\Delta}$ of the Shi arrangement for an irreducible root system. He found a recursion for its coefficients, which we will now present.

Let $\overline{\text{Shi}_\Delta}$ be the subarrangement of $\text{Shi}_\Delta$ consisting of all hyperplanes which contain the origin. For $Y \in L(\overline{\text{Shi}_\Delta})$, let $W_Y$ be the group generated by the reflection through all the hyperplanes containing $Y$. For a polynomial $p(t)$, let $[t^k]p(t)$ be the coefficient of $t^k$ in $p(t)$.
In his thesis [Hea94], he used Lagrange inversion to calculate

\[ P_{Shi}(t) = (1 + (n + 1)t)^n \]

Later, in [Hea97], he recognized the sum in (3.1) to be the number of labeled forests on \( n + 1 \) vertices of \( n + 1 - k \) trees and used [Moo70]. In both his thesis and later paper, he showed that the coefficient of \( t^k \) in \( P_{Shi}(t) \) and \((1 + (n + 1))^n \) are the same for \( 1 \leq k \leq n - 1 \). Then since the degree of \( P_{Shi}(t) \) is \( n \) and since \( P_{Shi}(1) = (n + 2)^n \) by Shi’s result, he showed

\[ P_{Shi}(t) = (1 + (n + 1)t)^n = (1 + ht)^n \]

in \( A_n \).
3.4. The number of Shi regions, part 3. Crapo and Rota [CR70, Chapter 16] described the critical problem: let $S$ be a set of points in an $n$-dimensional vector space $V_n$ over the field $\mathbb{F}$, with $q$ elements. The set $S$ must not contain the origin. Find the minimum number $c$ of projective hyperplanes $H_1, \ldots, H_c$ with the property that the intersection $H_1 \cap \ldots \cap H_c \cap S$ is null. They were able to solve the problem using the poset of intersections and characteristic polynomial.

Athanasiadis [Ath96] turned Crapo and Rota’s theorem around and used it to calculate the characteristic polynomial of subspace arrangements. Blass and Sagan [BS98] had previously used a similar idea, but not for all subspaces and not for the Shi arrangement. We present first the the Crapo and Rota theorem, then describe how Athanasiadis used it to get his hands on the characteristic polynomial for irreducible crystallographic root systems.

**Theorem 3.8** ([CR70]). The number of linearly ordered sequences $(L_1, L_2, \ldots, L_k)$ of $k$ linear functionals in $V_n$, which distinguish the set $S$ is given by $p(q^{k})$ where $p(v)$ is the characteristic polynomial of the geometric lattice spanned by the set $S$.

Athanasiadis needed to count the number of $n$-tuples $(x_1, \ldots, x_n) \in \mathbb{P}^n_q$ which satisfy $x_i \neq x_j$ and $x_i \neq x_j + 1$ for $i < j$. The argument is simple (and lovely) enough in type $A_{n-1}$ for the full Shi arrangement that we reproduce it here. See also [Sta12].

We first solve a related problem. Find the number of ways there are to place $n$ labeled balls in $q$ unlabeled boxes, where

1. the boxes are in a circle,
2. there is never more than one ball in a box, and
3. if $i < j$, then ball $i$ is not placed in the box immediately following, in the clockwise direction, the box holding ball $j$.

There will be $q - n$ empty boxes, so first place them in a circle. There is one way to do that. There are now $q - n$ spaces between the empty boxes, where the boxes holding the balls will go. By cyclic symmetry, there is one way to place the box holding the 1-ball. Then there are $(q-n)^{n-1}$ ways to place the rest of the boxes holding balls in the empty spaces. It is enough to pick the space between empty boxes; to avoid violating condition (3), the boxes between a consecutive pair of empty boxes must placed in increasing order of the labels on the balls inside. That is, our final answer to the related problem is $(q-n)^{n-1}$.

Now back to counting $n$-tuples. We are essentially done, if we think of each $n$-tuple representing a distribution of $n$ labeled balls into a circle of $q$ labeled boxes, where the distribution satisfies conditions (2) and (3). We place the ball labeled $i$ in box $x_i$. We need only label the boxes, and there are $q$ ways to do this. Thus there are $q(q-n)^{n-1}$ $n$-tuples which satisfy $x_i \neq x_j$ and $x_i \neq x_j + 1$ for $i < j$ and we have that the characteristic polynomial is $\chi_{L_n(\text{Sh}_n)} = q(q-n)^{n-1}$.

Crapo and Rota’s finite field method has since been used to calculate other characteristic polynomials. See Armstrong [Arm13], Armstrong and Rhoades [AR12], and Ardila [Ard07], for example. See Athanasiadis [Ath10] for reciprocity results for the characteristic polynomial for the Shi arrangement.

3.5. The number of Shi regions, part 4. Pak and Stanley, in [Sta96], give a bijection from Shi regions (type $A$) to parking functions, which refines (3.2). It is proved to be a bijection in [Sta07, Lecture 6]. A parking function of length $n$ is a tuple of nonnegative integers $(p_1, \ldots, p_n)$ such that when rearranged in nondecreasing order and relabeled as $b_1 \leq b_2 \leq \cdots \leq b_n$, then $b_i \leq i - 1$. Parking functions generalize inversion vectors of permutations. Pak and Stanley recursively defined the label $\lambda(R)$ of a region $R$. We use the description given in [Sta07, Lecture 6].

Let $R_0$ be the fundamental alcove $A_0$. Set $\lambda(R_0) = (0, \cdots, 0)$. Suppose we have labeled the region $R$ and its label $\lambda(R)$ is $(a_1, \ldots, a_n)$.

- If the regions $R$ and $R'$ are separated by the single hyperplane $H$ with the equation $x_i - x_j = 0, i < j$, and if $R$ and $R_0$ lie on the same side of $H$, then $\lambda(R') = (a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n)$.
- If the regions $R$ and $R'$ are separated by the single hyperplane $H$ with the equation $x_i - x_j = 1, i < j$, and if $R$ and $R_0$ lie on the same side of $H$, then $\lambda(R') = (a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{j-1}, a_j + 1, a_{j+1}, \ldots, a_n)$.

The bijection generalizes the well-known bijection from permutations to inversion vectors [Sta12, Chapter 1]. Although the map $\lambda$ is simply stated, the proof that it is a bijection is not simple. To show that the labeling is a bijection, Stanley encodes each region as a permutation and antichain pair. He builds the inverse map step-by-step from the parking functions to the pairs. The summary by Armstrong [Arm13, Theorem 3] of the proof that the Pak-Stanley map is bijective is particularly
good. Recall that the filters/antichains/ideals in the root poset for type \(A\) correspond to partitions in a staircase, and define the non-inversions of a permutation \(w\) to be the pairs \((i, j)\) such that \(i < j\) and \(w(i) < w(j)\). Then the proof can be summarized as showing that the Shi regions are in bijection with pairs \((w, I)\) where \(w\) is a permutation in \(S_n\) and \(I\) is an ideal in the root poset \(\Delta^+\) such that the minimal elements of \(I\), which are labels in the valleys of the Dyck path corresponding to \(I\), are non-inversions of \(w\).

The Pak and Stanley bijection from regions to parking functions \((m = 1)\) case can be composed with a bijection from trees to parking functions. The number of regions \(R\) for which \(i\) hyperplanes separate \(R\) from the region \(R_0\) is equal to the number of trees on the vertices \(0, \ldots, n\) with \(\binom{n}{2} - i\) inversions. The pair \((i, j)\), where \(1 \leq i < j\), is an inversion for \(T\) if the vertex \(j\) lies on the unique path in \(T\) from \(0\) to \(i\). See [Sta96, Theorem 5.1].

![Shi regions labeled by parking functions](image)

**Figure 3.3.** The Shi regions labeled by parking functions, using the Pak-Stanley labeling. A label is nondecreasing if and only if the region is in the dominant region.

We mention a few papers which build on the Pak-Stanley bijection. Duertes and Guedes de Oliveira, for example, further analyze this bijection in [DGdO15]. Rincón [Rin07] extends the Pak-Stanley labeling to the poset of faces of the Shi arrangement. See also Section 5.

### 3.6. **More.** Believe it or not, there are still other wonderful proofs concerning the number of regions.

For example, Athanasiadis and Linusson [AL99] defined a bijection, different from Pak and Stanley’s, from parking functions to the Shi regions (type \(A\)). Theirs gives a simple proof of the number of regions. Their bijection was generalized to type \(C\) by Mészáros in [Mész13]. In [AR12, Section 5.2] there is another proof of the formula for the number of regions, using Armstrong’s and Rhoades’ ceiling diagrams, which we define in Section 3.7. The ceiling diagrams are related to the diagrams Athanasiadis and Linusson used. Armstrong, Reiner, and Rhoades [ARR15] define nonnesting parking functions using the root poset and permutations from the finite Weyl group and label the Shi regions with these. Their definition can also be used for types which are not crystallographic. Other, more recent and more general bijections include [HP16, BBD+15] for example.

### 3.7. **The Ish and the Shi.** We’ll start off by writing the \(q, t\)-Catalan polynomial combinatorially:

\[
C_n(q, t) = \sum_{\pi} q^{\text{area}(\pi)} t^{\text{bounce}(\pi)},
\]

where the sum is over all Dyck paths of length \(n\). The \((q, t)\)-Catalan polynomials are remarkable generating functions coming from representation theory. They have been intensely studied since their introduction by Garsia and Haiman in [GH96]. See Haglund’s monograph [Hag08, Chapter 3] for more information. There are the same number of dominant Shi regions as there are Dyck paths, and one of Armstrong’s results in [Arm13] (and the one we’ll describe) was to transfer the statistics \(\text{area}\) and
bounce to dominant regions. His statistics are actually for all regions. The statistic $shi$ will correspond to the statistic area and $shi(R)$ is defined as the number of hyperplanes which must be crossed on a trip to the region $R$ from $R_0 = A_0$. The statistic $ish$ is defined using a second hyperplane arrangement, the Ish arrangement $Ish_{n}$. It is defined for type $A$ and is a deformation of the Coxeter arrangement. Let $\Delta$ be the set of roots for type $A$, so $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ as in Section 1.1.1. Denote $\alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-1}$ by $\tilde{\alpha}_i$. Then the definition of the Ish arrangement is

$$
Ish_{n} = Cox_n \cup \{ H_{\tilde{\alpha}_j,k} | 1 \leq j \leq n-1, k \in \{ 1, 2, \ldots, n-j \} \} = Cox_n \cup \{ x_j - x_n = k | k \in \{ 1, \ldots, n-j \}, 1 \leq j \leq n-1 \}.
$$

The $ish$ statistic is defined on Shi regions using the hyperplanes in $Ish_{n}$. Let $R$ be region of the arrangement $Shi_n$ with minimal alcove $A_R$. There is a unique $w \in S_n$ such that $A_R = A_0w$. The affine permutation $w$ has a unique factorization $w^f \cdot w_I$ [BB05], where $w_I \in S_n$ and $w^f$ is a minimal length coset representative, which we won’t define. What is important for us is that $A_0w^f$ is an alcove in the dominant chamber since $w^f$ is a minimal length coset representative. Then $ish(R) = ish(A_R)$ is the number of hyperplanes in $Ish_{n}$ which must be crossed in traveling from $A_0$ to $Aw^f$.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure3.4.png}
\caption{The Ish arrangement for $n = 3$ is on the left. The Shi arrangement is on the right, where region $R$ is labeled by the pair $(shi(R), ish(R)).$ The dominant regions are also labeled by $q^{\binom{n}{2} - shi(R)}ish(R)$.}
\end{figure}

The sum of the monomials is $C_3(q, t) = q^3 + q^2t + qt + qt^2 + t^3$.

Each dominant Shi region $R$ corresponds to a Dyck path $\pi_R$. Armstrong showed that $(\binom{n}{2} - shi(R)) = area(\pi_R)$ and $ish(R) = bounce(\pi_R)$ Notice that the Ish and Shi statistics are defined on all regions, not just the dominant ones. Armstrong was able to show they agree with bounce and area on all diagonally labeled Dyck paths. See [Hag08, Chapter 5] and [Arm13, Section 3].

Armstrong and Rhoades concentrated on properties of the Ish arrangement, especially its uncanny similarities to the Shi arrangement, in [AR12]. Their definition of the arrangement changes just a bit: replace $x_j - x_n = i$ by $x_1 - x_{n-j} = n-i+1$. Their main theorem is for deleted versions (more general) of the arrangements (see Section 6), but we’ll stick with the full arrangements. That is, $G$ is the complete graph in this survey. We need to define a few terms before we can state the main theorem. The wall $H$ of a region $R$ is called a ceiling if it does not contain the origin and if the origin and $R$ are not separated by $H$ (they lie in the same half-space of $H$). The regions of both $Ish_{n}$ and $Shi_{n}$ are convex, so every region has a recession cone:

$$
\mathcal{R}(R) = \{ v \in V : v + R \subseteq R \}.
$$

The cone is closed under nonnegative linear combinations and has a dimension. The dimension of $\mathcal{R}(R)$ is called the degrees of freedom of $R$. It’s worth mentioning that the region $R$ is bounded if and only if $\mathcal{R}(R) = \{0\}$.

A simplified version of their main theorem can now be stated:

**Theorem 3.9** ([AR12]). Let $c$ and $d$ be nonnegative integers. The $Ish_{n}$ and $Shi_{n}$ have the same
We are not presenting their theorem in its full generality, and as written here, (1) was proved in [Arm13].

We want to define the ceiling diagrams for the Shi arrangement because they show the properties of the corresponding region so clearly. We will also need variations on the root poset, which they use to prove Theorem 3.9. First the definition of the Shi ceiling diagram of a region $R$. Suppose the region is in the chamber $Dw$, where $D$ is the dominant chamber and $w \in \mathfrak{S}_n$. Then we define the set partition $\Delta$: there is an arc from $i$ to $j$, $i < j$, in the diagram of $\sigma$ if and only if the hyperplane $x^w_{w(i)} - x^w_{w(j)} = 1$ is a ceiling of $R$. We draw the Shi ceiling diagram $(w, \sigma)$ by placing the arc diagram for $\sigma$ above $w(1), w(2), \ldots, w(n)$. See Example 3.10. Armstrong and Rhoades show that $\sigma$ is a nonnesting set partition. The number of arcs is $c$. What about $d'$? Let $d'$ be the number of $k$, $1 \leq k \leq n - 1$ where there is no arc covering the space between $k$ and $k + 1$; that is, the number of $k$ where for which there is no $i < j$ such that $i \leq k < j$ and there is an arc from $i$ to $j$. For example, $d' = 0$ for the set partition in Figure 1.2 and $d' = 1$ for the set partition $12\{35\}4$. Then set $d = d' + 1$. Additionally, the recession cone $\mathcal{R}(R)$ can be read from the diagram.

There is still a key point: for a fixed $w \in \mathfrak{S}_n$, both the regions and the ceiling diagrams are in bijection with antichains in $\Delta^+(w)$. The poset $\Delta^+(w)$ is the first variation on the root poset:

$$\Delta^+(w) = \{x^w_{w(i)} - x^w_{w(j)} = 1 : w(i) < w(j)\}.$$  

The elements of $\Delta^+(w)$ are the affine hyperplanes in the Shi arrangement which intersect $Dw$. The partial order on the hyperplanes is given by

$$x^w_{w(i')} - x^w_{w(j')} = 1 \leq x^w_{w(i)} - x^w_{w(j)} = 1$$

if $w(i) \leq w(i') < w(j') \leq w(j)$. The partial order is defined so that ceilings of any Shi region $R$, $R \in Dw$, are the maximal elements of an order ideal. The number $c$ shows up as the number of these maximal elements.

We’ll need the second variation on $\Delta^+$, $\Psi^+(w)$, for discussing the Ish arrangement:

$$\Psi^+(w) = \{x^w_1 - x^w_j = i : w^{-1}(i) < w^{-1}(j)\}.$$ 

Its elements are the Ish hyperplanes that intersect $wD$ and its partial order is chosen so that the ceilings of any Ish region $R$, $R \in wD$, are minimal elements of a filter, and $c$ for the region is the number of these minimal elements.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{shi_arrangement.png}
\caption{The Shi arrangement for $A_2$. The chamber $(D)w$ is labeled by $w$.}
\end{figure}

**Example 3.10.** This example refers to Figures 3.5 and 3.6. In 3.5, the chamber $Dw$ is labeled by $w \in \mathfrak{S}_3$. We’ve picked two Shi regions to consider in this example—the ones we have labeled $R_1$ and $R_2$. 

(1) characteristic polynomial,  
(2) number of dominant regions with $c$ ceilings, and  
(3) number of regions with $c$ ceilings and $d$ degrees of freedom.
$x_1 - x_3 = 1$

$\begin{align*}
  x_1 - x_2 &= 1 \\
  x_2 - x_3 &= 1
\end{align*}$

$\begin{align*}
  x_1 - x_3 &= 1 \\
  x_1 - x_2 &= 1
\end{align*}$

\textbf{Figure 3.6.} On the left, the poset $\Delta^+(e)$. The poset $\Delta^+(s_2)$ is on the right.

$R_1$ is in the chamber $Dw$ for $w = e$, the identity. There are three hyperplanes of the form $x_i - x_j = 1$ which intersect $Dw$ and the poset $\Delta^+(w)$ is on the left in Figure 3.6. The hyperplane $x_2 - x_3 = 1$ is a ceiling for $R_1$ and the region is also labeled with its ceiling diagram in Figure 3.5.

The region $R_2$ is in the chamber $Dw$ for $w = s_2$ and poset $\Delta^+(s_2)$ is on the right in Figure 3.6. The poset $\Delta^+(s_2)$ has three ideals, corresponding to the three Shi regions in $Dw$. Our region $R_2$ has ceiling $x_1 - x_3 = x_{w(1)} - x_{w(2)} = 1$ and we have placed the arc diagram for the partition 123 above $w(1)w(2)w(3)$ to build the ceiling diagram.

The relationship between the filters and the regions is bijective in the Ish case, just as between ideals and regions in the Shi case. The posets $\Delta^+(w)$ and $\Psi^+(w)$ are dual to each other when $w$ is the identity permutation $e$. The final step in the proof of Theorem 3.9, part (3), is simply to send an order ideal in $\Delta^+(e)$ to the corresponding filter in $\Psi^+(e)$. Since the maximal elements in the ideal become the minimal elements in the filter, $c$ is preserved. There are also ceiling diagrams for the Ish arrangement, but we won’t define them.

To prove Theorem 3.9, part (3), Armstrong and Rhoades used ceiling partitions, which are set partitions of $[n]$. We now define a simplified version of them. First suppose $R$ is a Shi region. The ceiling partition $\pi_H$ has an arc from $i$ to $j$, $i < j$, if and only if the hyperplane $x_i - x_j = 1$ is a ceiling of $R$. Next suppose we have an Ish region $R$. Its ceiling partition has an arc from $i$ to $j$, $i < j$, if and only if $x_1 - x_j = i$ is a ceiling of $R$. The definition of the ceiling partition does not depend on the chamber of $R$ for either arrangement. Surprisingly, the distribution of the set partitions is the same for the Ish and Shi arrangements.

\textbf{Theorem 3.11 ([AR12])}. Let $A$ be either the Ish or the Shi arrangement. Let $\pi$ be a partition of $[n]$ with $k$ blocks and let $1 \leq d \leq k$.

1. The number of regions of $A$ with ceiling partition $\pi$ is

\[
\frac{n!}{(n-k+1)!}.
\]

2. The number of regions of $A$ with ceiling partition $\pi$ and $d$ degrees of freedom is

\[
\frac{d(n-d-1)!(k-1)!}{(n-k-1)!(k-d)!}.
\]

To obtain the number of regions with $c$ ceilings and $d$ degrees of freedom, thereby proving Theorem 3.9, part (3), sum the expression in Theorem 3.11, part (2), over all partitions $\pi$ with $k = n - c$ blocks.

For space reasons, we cannot include the arguments here for Theorem 3.9 and Theorem 3.11. This is a shame, because we thereby don’t present evidence for their observation [AR12]:

The Ish arrangement is something of a “toy model” for the Shi arrangement (and other Catalan objects). That is, for any property $P$ that $\text{Shi}_n$ and $\text{Ish}_n$ share, the proof that $\text{Ish}_n$ satisfies $P$ is easier than the proof that $\text{Shi}_n$ satisfies $P$.

Many of the theorems of [AR12] are proved bijectively by Leven, Rhoades, and Wilson in [LRW14].

3.8. \textbf{Extended Shi arrangement.} In [PS00], Postnikov and Stanley introduced the extended Shi arrangement of type $A_{n-1}$:

\[
\text{Shi}_{m} = \{ H_{\alpha,k} \mid \alpha \in \Delta^+, -m + 1 \leq k \leq m \}.
\]
This kind of extension is sometimes denoted by Fuss, as in Fuss-Catalan [Arm09]. Up until now, we have been discussing $m = 1$. Postnikov and Stanley show that $\text{Shi}_n^m$ has $(mn + 1)^{n-1}$ regions. They fix $m$, set $f_n = \tau(\text{Shi}_n^m)$ to be the number of regions, and show that the exponential generating function

$$f = \sum_{n \geq 0} f_n \frac{x^n}{n!}$$

satisfies

$$f = e^{xf^m}.$$ 
The extended Shi arrangement is a special case ($a = m$, $b = m + 1$) of what they named truncated affine arrangements; see [PS00, Section 9] for more details. The dominant regions of the $m$-Shi are the same as the dominant regions of the $m$-Catalan.

Many of the other enumerative treats of the Shi arrangement generalize well to the extended Shi arrangement. For example, Stanley [Sta98] labeled the $m$-Shi regions with $m$-parking functions of length $n$ using an extended version of the bijection described here in Section 3.5. Stanley defined $m$-parking functions of length $n$. He replaced the condition that “$b_i \leq i - 1$” in the definition of parking function (see Section 3.5) by “$b_i \leq m(i - 1)$.” If we set $d(R) = \text{area}(R)$ (see Section 3.7), then Stanley’s bijection showed [Sta98, Corollary 2.2] that

$$\sum_R q^{d(R)} = \sum_{(p_1, \ldots, p_n)} q^{p_1 + \cdots + p_n},$$

where the sum on the left is over all regions in $\text{Shi}_n^m$ and the sum on the right is over all $m$-parking functions of length $n$.

In 2004, Athanasiadis wrote two papers on the extended Catalan arrangement for crystallographic $\Delta$, concentrating on the dominant regions. The Catalan arrangement has more hyperplanes than the Shi arrangement, but it has the same dominant regions, so we record his results here in terms of the $m$-Shi arrangement. We’ll need a definition. The Narayana numbers (type $A$) are given by [Pet15]

$$N_{n,k} = \frac{1}{k+1} \binom{n}{k} \binom{n-1}{k}.$$ 

They refine the Catalan numbers by counting the Dyck paths of length $n$ with $k$ peaks. In other words, $C_n = \sum_{k=0}^{n-1} N_{n,k}$. Athanasiadis

(1) generalized and extended the Narayana numbers, finding what they enumerate in terms of dominant $m$-Shi regions;
(2) counted the number of $m$-Shi regions in the dominant chamber, generalizing Shi’s result described in Section 3.1; and
(3) used co-filtered chains of ideals in the root poset to describe the dominant $m$-Shi regions.

We will describe (3) in a bit more depth. Let $\Delta^+ = I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_m$ be a a decreasing chain $\mathcal{I}$ of ideals in $\Delta^+$, set $I_i = I_m$ for all $i > m$, and set $J_i = \Delta^+ \setminus I_i$. The chain $\mathcal{I}$ is a co-filtered chain of ideals of length $m$ if

(1) $(I_i + J_j) \cap \Delta^+ \subseteq I_{i+j}$ and
(2) $(J_i + J_j) \cap \Delta^+ \subseteq J_{i+j}$.
is true for all indices \(i, j \geq 1\) with \(i + j \leq m\). The coordinates of the chain are
\[
k_\alpha(\mathcal{I}) = \max\{k_1 + k_2 + \cdots + k - r : \alpha = \beta_1 + \cdots + \beta_r \text{ with } \beta_i \in I_k \text{ for all } i\}.
\]

Athanasiadis showed that
\[
k_\alpha(\mathcal{I}) + k_\beta(\mathcal{I}) \leq k_{\alpha+\beta}(\mathcal{I}) \leq k_\alpha(\mathcal{I}) + k_\beta(\mathcal{I}) + 1
\]
whenever \(\alpha, \beta, \alpha + \beta \in \Delta^+\). Equation (3.5) generalizes Shi’s bijection between filters in the root poset and dominant Shi regions (see Section 3.1). Finally, to define the fundamental \(m\)-Shi region associated to \(\mathcal{I}\), he sets \(R^\mathcal{I}\) to be the set of points \(x \in V\) which satisfy
\[
(1) \langle \alpha \mid x \rangle > k, \text{ if } \alpha \in I_k \text{ and } \\
(2) 0 < \langle \alpha \mid x \rangle < k, \text{ if } \alpha \in J_k, \text{ for } 0 \leq k \leq m. 
\]
The coordinates of the ideal are then the coordinates of a region.

Certain elements in an ideal are called indecomposable. These elements correspond to the walls of \(R^\mathcal{I}\) which separate \(R^\mathcal{I}\) from \(R_0\), and take the place of peaks in Dyck paths when defining the Narayana numbers in terms of Shi regions.

Here we mention a few other enumerative results concerning the regions of the extended Shi arrangement. Any fixed hyperplane in the \(m\)-Shi arrangement is dissected into regions by the other hyperplanes in the arrangements. Fishel, Tzanaki, and Vazirani enumerate the number of regions for certain fixed hyperplanes in type \(A\) in [FTV13]. Fishel, Kalipoliti, and Tzanaki [FKT13] defined a bijection between dominant regions of the \(m\)-Shi arrangement in type \(A_n\) and dissections of an \(m(n+1)+2\)-gon. These dissections represent facets of the \(m\)-generalized cluster complex. In 2008, Sivasubramanian [Siv08] gave combinatorial interpretations for the coefficients of a two-variable version of Stanley’s distance enumerator (3.4) in type \(A\) for \(m = 1\). Forge and Zaslavsky study the integral points in \([m]^m\) that do not lie in any hyperplane of the arrangement [FZ07]. Thiel resolves a conjecture of Armstrong [Arm09, Conjecture 5.1.24] on the distribution of floors and ceilings in the dominant regions of the \(m\)-Shi arrangements for all types. See Section 3.7 for the definition of ceiling.

4. Connections

4.1. Decompositions numbers and the Shi arrangement. The dominant regions make an appearance in the study of decomposition numbers for certain Hecke algebras. To describe this appearance, we’ll first need a host of combinatorial definitions, then we’ll indicate briefly how these arose from algebra, and finally we’ll relate this back to the Shi arrangement. We thank Matthew Fayers for not only pointing out this connection, but carefully explaining it.

4.1.1. Combinatorics. Here we define \(n\)-cores, review some well-known facts about them, and review the abacus construction, which will be useful for us. Details can be found in [JK81].

The \((k, l)\)-hook of an integer partition \(\lambda\) consists of the box in row \(k\) and column \(l\) of \(\lambda\), all the boxes to the right of it in row \(k\) together with all the nodes below it and in column \(l\). The hook length \(h^\lambda_{(k, l)}\) of this box is the number of boxes in the \((k, l)\)-hook. Let \(n\) be a positive integer. An \(n\)-core is a partition \(\lambda\) such that \(n \nmid h^\lambda_{(k, l)}\) for all boxes \(\lambda\). An \(n\)-regular partition has no (nonzero) \(n\) parts which equal each other. For example, \((7, 6, 6, 6)\) is not 3-regular. We’ll sometimes use \(p\) or \(e\) instead of \(n\), depending on the context. The definition is the same.

![Figure 4.1](attached-image)

**Figure 4.1.** The Young diagram of the partition \(\lambda = (5, 2, 1, 1, 1)\). The hooklengths are the entries in the boxes of its Young diagram. The partition \(\lambda\) is a 4-core but not a 5-core.

The \(\beta\)-numbers of the partition \(\lambda\) are the hook lengths from its first column. The \(\beta\)-numbers can be displayed on an abacus: a \(p\)-abacus is a diagram with \(p\) runners, labeled \(0, 1, \ldots, p - 1\). Runner \(i\) has positions labeled by integers \(pj + i\), for all \(j \in \mathbb{Z}\).
We make a $p$-abacus for $\lambda$ by placing a bead at position $\beta_k$, for each $\beta$-number $\beta_k$ of $\lambda$ and at all negative positions. We say two $p$-abaci are equivalent if we can change one to the other by moving the bead at position $i$ to position $i + C$ for some $C \in \mathbb{Z}$ and for all positions $i$ where there is a bead. The positive integer $p$ is arbitrary for now, but will be related to the characteristic of a field when we see abaci in their algebraic context. See the 4-abacus in Figure 4.2.

![Figure 4.2](image1)

**Figure 4.2.** On the left, the beads are placed on the positions labeled by the first column hook-lengths of $\lambda = (5, 2, 1, 1, 1)$. On the right is an equivalent abacus, where all bead positions have been shifted by $C = -1$.

![Figure 4.3](image2)

**Figure 4.3.** The beads are placed on a 4-abacus on the positions labeled by the first column hook-lengths of $\lambda = (5, 2, 1, 1, 1)$. The partition $\lambda$ is not a 5-core, since there are gaps on the 5-abacus. If we push the beads in positions 5 and 9 up to fill in the gaps, we obtain the 5-abacus of the empty partition.

We can give an equivalent description of $p$-core partitions: a partition $\lambda$ is an $p$-core if and only if whenever there is a bead at position $j$ of its $p$-abacus, there is also a bead at position $j - p$ [JK81].

Suppose we have a partition which is not an $p$-core. Then there is at least one bead at a position $j$ of its $p$-abacus which can be pushed up into the vacant position $j - p$. This gives us the $p$-abacus of another partition. We can repeat this until no beads can be pushed up, at which point we have the $p$-abacus of an $p$-core. The final partition $\gamma$ is called the $p$-core of $\lambda$ and the number of beads we moved in called the $p$-weight of $\lambda$. With a little work, which we won’t do, it is possible to show that $|\lambda| = |\gamma| + wp$.

The last combinatorial ingredient we need are residues for the boxes of a partition. Let $n$ be a positive integer. We call the box in row $i$, column $j$ a $k$-box if $(j - i) \mod n$ is equal to $k$.

![Figure 4.4](image3)

**Figure 4.4.** The Young diagram of the partition $\lambda = (5, 2, 1, 1, 1)$. The entries are the residues modulo 4.
4.1.2. Algebra. It is time to say a few words about how the combinatorics from the last section relate to algebra.

Roughly speaking, the $p$-regular partitions of $n$ index the irreducible modules $D^\lambda$ of $FS_n$, where $p$ is the characteristic of the field $F$. The $p$-blocks are the equivalence classes of a certain equivalence relation on the irreducible modules. By Nakayama’s celebrated conjecture, and Brauer and Robinson’s theorem, two irreducibles $D^\lambda$ and $D^\mu$ belong to the same $p$-block if and only if $\lambda$ and $\mu$ have the same $p$-core. Thus each $p$-block is labeled by a $p$-core and $p$-weight $w$. The $p$-weight keeps track of the difference between the $p$-core labeling a block and the partitions in the block: the $p$-core is a partition of $n - pw$. The weights are nonnegative integers. We will assume $w$ is at least one, because the blocks where $w = 0$ are singletons consisting of the $p$-core.

Scopes [Sco91] investigated the classes of $p$-blocks under Morita equivalence. She characterized families of Morita equivalent $p$-blocks using the $p$-core and $p$-weight which label $p$-blocks. Suppose $B$ is $p$-block for $F\Sigma_n$ labeled by weight $w$ and $p$-core $\gamma$. Let $k$ be an integer at least as large as $w$, and suppose that in a $p$-abacus for $\gamma$, there is a runner $i$ which has $k$ more beads in positive positions than runner $(i - 1)$ has. Now move $k$ beads from runner $i$ to runner $i - 1$ in such a $p$-abacus for $\gamma$. This new abacus determines another $p$-core, say $\hat{\gamma}$. See Figure 4.5. The operation changes the size of the partition: $|\gamma| - k = |\hat{\gamma}|$. We are glossing over details here involving the $\beta$-numbers. Let $\hat{B}$ be the block of $F\Sigma_{n-k}$ labeled by $w$ and $\hat{\gamma}$. Scopes took the transitive closure of the relation $B \sim \hat{B}$ and showed that within an equivalence class, the $p$-blocks have the same decomposition matrix, among other results.

![Figure 4.5](image)

**Figure 4.5.** On the left, the abacus for $\gamma = (6, 3, 1, 1, 1)$, on the right, the abacus for $\hat{\gamma} = (5, 2, 1, 1, 1)$. We have moved $k = 2$ beads from runner 2 to runner 1.

Richards was studying the decomposition numbers for the Hecke algebra; see Section 2.1 for at least the definition. He used the classes from the Scopes equivalence on $e$-cores, where $e$ depends on the characteristic $p$ of the field and the element $q$ used in the definition of the Hecke algebra. He called the classes families. Richards was interested in these families because the blocks of the Hecke algebras for $\Sigma_n$ and for $\Sigma_{n-k}$ corresponding to $\gamma$ and $\hat{\gamma}$ respectively have essentially the same decomposition numbers [Ric96].

Richards wanted to count such families. He built the following *pyramid* $\{u a_v\}_{0 \leq u < v \leq e - 1}$ for an $e$-core $\gamma$ based on $\gamma$’s $e$-abacus. Note the similarity in shape to admissible sign types and to the arrangement of roots in a staircase shape diagram in Figure 1.2. For $i = 0, 1, \ldots, e - 1$, let $p_i$ be the position of the first free space on runner $i$. Arrange these $e$ numbers in ascending order and relabel as $p_0 < p_1 < \cdots < p_{e-1}$. If $0 \leq u < v \leq e - 1$, then $p_u - p_v$ is a positive integer not divisible by $e$. We may use any $e$-abacus for $\gamma$; it doesn’t affect the set of differences. Richards defined the pyramid of numbers by

$$a_v = \begin{cases} 
w - 1 & \text{if } 0 < p_v - p_u < e \\
2 & \text{if } e < p_u - p_v < 2e \\
\vdots & \\
1 & \text{if } (w - 2)e < p_v - p_u < (w - 1)e \\
0 & \text{if } (w - 1)e < p_v - p_u
\end{cases}$$

where $w$ is the weight. Richards proved that two $e$-cores are in the same family if and only if they have the same pyramid and that there are exactly

$$\frac{1}{e} \left( \frac{ew}{e - 1} \right)$$

families. What’s more, he characterized the triangles of numbers which form a pyramid.
To show the connection to the Shi arrangement, we transform $u \hat{a}_v$ into $\hat{a}_u$ by $u \hat{a}_v + \hat{a}_v = w - 1$. Then Richard’s Proposition 3.4 becomes

**Proposition 4.1 ([Ric96]).** Let $e \geq 2$ and $w > 0$, and for $0 \leq u < v < e - 1$ let $0 \leq \hat{a}_v \leq w - 1$. Then the $\hat{a}_u$ form a pyramid if and only if for all $0 \leq u < t < v < e - 1$,

$$\hat{a}_u = \begin{cases} \hat{a}_t + \hat{a}_v & \text{if both of these have all entries no bigger than } w - 1 \\ w - 1 & \text{otherwise.} \end{cases}$$

Please see Example 4.2 for the calculation of a few pyramids from cores.

Let’s examine the case $e = 3$ and $w = 2$. There are five families. We choose five 3-cores $\{\emptyset, (1), (2), (1, 1), (3, 1, 1)\}$ and calculate their pyramids:

$$\emptyset \quad 0^00 \quad 0^10 \quad 0^11 \quad 1^10 \quad 1^11$$

The pyramids are all different, so we have found all the families. If we look back at the set $G$ in Section 2.2 and consider the subset where all entries are either + or ⨿, we see a similarity to the pyramids (replace + with 1). This is true in general. Richard’s proposition is the type $A$ version of (3.5) from Section 3.8. The pyramid is also an admissible sign type for a dominant $m$-Shi region of type $A$, where $m = w - 1$.

4.1.3. **Geometry.** We’ll just say a few more words about the geometry here. Richard’s pyramids have connected the core partitions to regions. We’ll describe Lascoux’s [Las01] well-known bijection between $n$-cores and certain elements of $\hat{S}_n$, and by extension, between $n$-cores and alcoves in the dominant chamber. Please see Lapointe’s and Morse’s paper [LM05] for details. We describe the bijection, as another way of seeing why core partitions pop up here. An $n$-core partition may have several removable boxes of a given residue or it may have several addable boxes of a given residue, but it will never have both addable and removable boxes of the same residue. Given an $n$-core partition $\lambda$ and the generators $s_0, s_1, \ldots, s_{n-1}$ of $\hat{S}_n$, let $s_i(\lambda)$ be the partition where all boxes of residue $i$ have been removed (added) if there are removable (addable) boxes. Any $n$-core partition can be expressed as $w(\emptyset)$. See Figure 4.6 and Example 4.2. We associate the $n$-core $w(\emptyset)$ with the alcove $A_0w$.

We mention that Fishel and Vazirani mapped partitions which are both $n$ and $nm + 1$ cores to dominant regions in the $m$-Shi arrangement of type $A_{n-1}$ in [FV10] using abacus diagrams and the root lattice.

**Example 4.2.** First, we construct the two pyramids for the partition $(4, 2)$, one each for $w = 2$ and $w = 3$. From Figure 4.7, we see that $(p_0', p_1', p_2') = (p_0, p_1, p_2) = (0, 1, 8)$. When $w = 2$, the pyramid is $(a_2, a_1, a_2) = (0, 1, 0)$ and when $w = 3$, it
A finite state automaton is a finite directed graph, with one vertex designated as the initial state $S_0$ and a subset of vertices as final states, and with every edge labeled by an element of $B$. We call the vertices states. A word $w \in L$ is accepted by the automaton if the sequence of edge labels along some directed path starting at $S_0$ and ending at a final state is equal to $w$.

Headley was not the first nor the last to construct an automaton to accept reduced words; see Björner and Brenti [BB05], Hohlweg, Nadeau, and Williams [HNW16], and Gunnells [Gun10], for instance. However, Headley realized that if $s_{i_2} s_{i_3} \cdots s_{i_k}$ is reduced, then the fundamental alcove $A_0$ and $A_0 s_{i_2} s_{i_3} \cdots s_{i_k}$ lie on the same side of the hyperplane fixed by $s_{i_2}$ if and only if $s_{i_1} s_{i_2} s_{i_3} \cdots s_{i_k}$ is reduced. He built his automaton on this observation. The key lemma is

**Lemma 4.3.** [Hea94] Let $W$ be an irreducible affine Weyl group with root system $\Delta$. Let $R$ be a region of the Shi arrangement. If $R$ and $A_0$ lie on the same side of the hyperplane fixed by $s \in S$, then $Rs$ lies in a single region.
on the same side of the hyperplane, then \( R \) has no outgoing arrow labeled by \( s \). Headley not only showed that the language accepted by this automaton is the set of all reduced words, he showed that if \( W \) is the affine symmetric group, the automaton has the minimal number of states. There were actually two automata in Headley’s thesis. The first produced a nice generating function, but it has more states.

![Diagram of Headley’s automaton](image)

**Figure 4.8.** Headley’s automaton based on the Shi arrangement. It accepts reduced words in \( \tilde{S}_3 \). Each state is labeled with the affine permutation corresponding to the minimal alcove of the region. We use \( i \) instead of \( s_i \). Solid arrows represent an edge labeled by \( s_0 \) and dashed (respectively dotted) edges represent edges labeled \( s_1 \) (respectively \( s_2 \)). See Example 4.4.

**Example 4.4.** This example refers to Figure 4.8. The path
\[
\begin{align*}
e & \xrightarrow{0} 0 \xrightarrow{2} 02 \xrightarrow{1} 21 \xrightarrow{0} 10 \xrightarrow{1} 101 \xrightarrow{2} 1012
\end{align*}
\]
represents the expression \( s_0s_2s_1s_0s_1s_2 \), which is reduced. Since \( s_0s_1s_0s_1 \) is not reduced, there is no path which starts at \( e \) and follows solid, dashed, solid, dashed arrows.

4.3. **More connections.** This connection is to the filters in \( \Delta^+ \), not to the Shi arrangement directly. In [CP00, CP02, CP04], Cellini and Papi investigate ad-nilpotent ideals in a Borel subalgebra. They associate each ideal to a filter in \( \Delta^+ \) and also to an element of the affine Weyl group. Very roughly speaking, they use the Cartan decomposition \( L = H \oplus N \) and \( N = \bigoplus_{\alpha \in \Delta^+} L_{\alpha} \) and the definition of an ad-nilpotent ideal as an ideal contained in \( N \) to define the antichain
\[
\Delta_I = \{ \alpha \in \Delta^+ : L_\alpha \subseteq I \}
which defines a filter. See also Suter [Sut04]. Dong extends Cellini and Papi’s work from Borel subalgebras to parabolic subalgebras in [Don13]. He uses deleted Shi arrangements, which we don’t address in this survey. Panyushev [Pan04] develops combinatorial aspects of the theory of $ad$-nilpotent ideals, giving a geometric interpretation for the number of generators of an ideal, for example.

Gunnells and Sommers study Dynkin elements, which we won’t define, in [GS03]. They define $N$-regions, which turn out to be unions of Shi regions. A simplified version of their main theorem is that if $x$ is the point of minimal Euclidean length in the closure of an $N$-region, then $2x$ is a Dynkin element.

5. Further

We briefly mention a few recent results. In [GMV16], Gorsky, Mazin, and Vazirani developed “rational slope” versions of much of what has been discussed here. A tuple $(b_1, \ldots, b_n)$ of nonnegative integers is called an $M/n$-parking function if the Young diagram with row lengths equal to $b_1, \ldots, b_n$ arranged in decreasing order fits above the diagonal in an $n \times M$ rectangle. We’ve stated it a bit differently than in Sections 3.5 and 3.8, but if we let $M = n + 1$ and $M = mn + 1$ respectively and reverse the order of the tuple, we obtain the same functions. Gorsky, Mazin, and Vazirani defined $M$-stable permutations to take the place of minimal permutations of Shi regions and generalized the Pak-Stanley bijection, as well as another map defined by Anderson [And02] in her study of core partitions. They conjectured their generalization of the Pak-Stanley map is injective for all relatively prime $M$ and $n$. In 2017, McCammond, Thomas, and Williams [MTW17] proved the conjectures in [GMV16]. Additionally, Gorsky, Mazin, and Vazirani connect their maps to the combinatorics of $q, t$-Catalan polynomials. Sulzgruber [Sul15] built on [GMV16] by finding the coordinates of the $M$-stable permutations, generalizing (3.5). Thiel [Thi14] extended their work to other types, among other results.

As mentioned in Section 3 Hohlweg, Nadeau, and Williams generalized the Shi arrangement to any Coxeter group, using $n$-small roots, and then to indefinite Coxeter systems. They also investigated automata.

6. Themes we haven’t included

We give a short and incomplete list of topics we have not discussed.

(1) The Shi arrangement is free. Either see original article by Athanasiadis [Ath98] or his excellent summary [Ath00]. Abe, Suyama, and Tsujie [AST17] show that the Ish arrangement is free.

(2) In graphical arrangements or deleted arrangements, some of the hyperplanes have been removed. We survey only the complete Shi arrangement.

(3) We have no discussion of the connections to the torus $\hat{Q}/(1 + mh)\hat{Q}$, where $\hat{Q}$ is the coroot lattice of a root system, $(mh + 1)\hat{Q}$ is its dilate, and $h$ is the Coxeter number of the root system. See Athanasiadis [Ath05] or Haiman [Hai94] for more information.

(4) The enumeration of bounded regions has nice results, which we have not discussed. See Athanasiadis and Tzanaki [AT06], for example and Sommers [Som05].

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References

[AST17] Takuro Abe, Daisuke Suyama, and Shuhei Tsujie, The freeness of Ish arrangements, J. Combin. Theory Ser. A 146 (2017), 169–183. MR 3574228

[And02] Jaclyn Anderson, Partitions which are simultaneously $t_1$- and $t_2$-core, Discrete Math. 248 (2002), no. 1-3, 237–243. MR 1892698

[Ard07] Federico Ardila, Computing the Tutte polynomial of a hyperplane arrangement, Pacific J. Math. 230 (2007), no. 1, 1–26. MR 2318445
A SURVEY OF THE SHI ARRANGEMENT
