On Ashbaugh-Benguria’s Conjecture about Lower Order Dirichlet Eigenvalues of the Laplacian

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Abstract
We prove an isoperimetric inequality for lower order eigenvalues of the Dirichlet Laplacian on bounded domains of a Euclidean space which strengthens the celebrated Ashbaugh-Benguria inequality conjectured by Payne-Pólya-Weinberger on the ratio of the first two Dirichlet eigenvalues and makes an important step toward the proof of a conjecture by Ashbaugh-Benguria.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. Let us denote by $\Delta$ the Laplace operator on $\mathbb{R}^n$ and consider the homogeneous membrane problem

$$\begin{cases}
\Delta u = -\lambda u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases} \quad (1.1)$$

It is well known that the spectrum of (1.1) is real and discrete consisting in a sequence

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots \rightarrow +\infty,$$

where each eigenvalue is repeated with its multiplicity. An important issue in analysis and geometry is to give good estimates to these and other eigenvalues, especially to obtain isoperimetric bounds for them. When $\Omega = \mathbb{B}^n$ is the $n$-dimensional unit ball in $\mathbb{R}^n$, it is well known that $\lambda_1(\mathbb{B}^n) = j_{n/2-1,1}^2$ and $\lambda_2(\mathbb{B}^n) = \cdots = \lambda_{n+1}(\mathbb{B}^n) = j_{n/2,1}^2$, where $j_{p,k}$ denotes the $k$th positive zero of the Bessel function $J_p(x)$ of the first kind of order $p$. One of the earliest isoperimetric inequalities for an eigenvalue is the Faber-Krahn inequality \cite{15, 18, 19} conjectured by Rayleigh \cite{23} in 1877:

$$\lambda_1(\Omega) \geq \left(\frac{\mathbb{B}^n}{|\Omega|}\right)^{\frac{2}{n}} j_{\frac{n}{2}-1,1}^2, \quad (1.2)$$

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with equality if and only if $\Omega$ is an $n$-ball. Here, $|\Omega|$ denotes the volume of $\Omega$. In 1956, Payne-Pólya-Weinberger proposed the following well-known conjecture [21]:

**Payne-Pólya-Weinberger Conjecture.** The eigenvalues of (1.1) satisfy

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(B^n)}{\lambda_1(B^n)},$$

(1.3)

$$\frac{\lambda_2(\Omega) + \cdots + \lambda_{n+1}(\Omega)}{\lambda_1(\Omega)} \leq n \frac{\lambda_2(B^n)}{\lambda_1(B^n)},$$

(1.4)

The conjecture (1.3) was studied by many mathematicians, for examples, Payne, Pólya and Weinberger [21, 22], Brands [9], Chiti [12, 13], de Vries [14], Hile and Protter [17]. Finally, Ashbaugh and Benguria proved this conjecture [2, 3, 4]. Ashbaugh-Benguria [7] and Benguria-Linde [8] also proved similar inequalities for the first $(n+1)$ Dirichlet eigenvalues of the Laplacian on bounded domains in a hemisphere or a hyperbolic space.

The conjecture (1.4) is stronger than (1.3) and was also studied by many authors. In 1956, Payne, Pólya and Weinberger [22] proved that for $\Omega \subset \mathbb{R}^2$,

$$\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq 6,$$

(1.5)

which was improved by Brands [9] to

$$\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq 3 + \sqrt{7}.$$  

(1.6)

Furthermore, Hile-Protter [17] obtained

$$\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq 5.622.$$  

(1.7)

In [20], Marcellini obtained the bound

$$\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq \frac{15 + \sqrt{345}}{6}.$$  

(1.8)

Chen-Zheng proved in [11]

$$\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq 5.3507^{-}.$$  

(1.9)

For general dimensions $n \geq 2$, Thompson [26] obtained the bound (see also [6])

$$\frac{\lambda_1 + \lambda_2 \cdots + \lambda_{n+1}}{\lambda_1} \leq (n + 4).$$  

(1.10)
In \cite{6}, Ashbaugh-Benguria proved
\begin{equation}
\frac{1}{\lambda_2 - \lambda_1} + \cdots + \frac{1}{\lambda_{n+1} - \lambda_1} \geq \frac{2j_{\frac{2}{n}-1,1}^2 + n(n - 4)}{6\lambda_1}.
\end{equation}

They observed that \cite{6}
\begin{equation}
\frac{2j_{\frac{2}{n}-1,1}^2 + n(n - 4)}{6} \approx \frac{n^2}{4} \left[ 1 + \frac{2}{3}(1.8557571) \frac{2^{\frac{2}{n}}}{n^{\frac{2}{n}}} - \frac{4}{n} + O(n^{-\frac{4}{n}}) \right],
\end{equation}
whereas
\begin{equation}
\frac{n}{\left( \frac{j_{\frac{2}{n}}}{j_{\frac{2}{n}-1}} \right)^2 - 1} \approx \frac{n^2}{4} \left[ 1 + \frac{2}{3}(1.8557571) \frac{2^{\frac{2}{n}}}{n^{\frac{2}{n}}} - \frac{2}{n} + O(n^{-\frac{4}{n}}) \right]
\end{equation}
and also conjectured that \cite{5, 6}
\begin{equation}
\frac{\lambda_1}{\lambda_2 - \lambda_1} + \cdots + \frac{\lambda_1}{\lambda_{n+1} - \lambda_1} \geq \frac{n}{\left( \frac{j_{\frac{2}{n}}}{j_{\frac{2}{n}-1}} \right)^2 - 1}
\end{equation}
with equality if and only if \( \Omega \) is an \( n \)-ball.

Ashbaugh \cite{1} and Henrot \cite{16} mentioned this conjecture again. One can also formulate a similar conjecture for the first \( (n + 1) \) eigenvalues of the Dirichlet Laplacian on bounded domains in a hemisphere or a hyperbolic space.

In this paper, we prove the following isoperimetric inequality which supports strongly the conjecture (1.14).

**Theorem 1.1** Let \( \Omega \) be a bounded domain with smooth boundary in \( \mathbb{R}^n \). Then the first \( n \) Dirichlet eigenvalues of \( \Omega \) satisfy
\begin{equation}
\frac{\lambda_1}{\lambda_2 - \lambda_1} + \cdots + \frac{\lambda_1}{\lambda_{n+1} - \lambda_1} \geq \frac{n - 1}{\left( \frac{j_{\frac{2}{n}}}{j_{\frac{2}{n}-1}} \right)^2 - 1},
\end{equation}
with equality holding if and only if \( \Omega \) is an \( n \)-ball.

For eigenvalues \( 0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \to +\infty \) of the Neumann problem
\begin{equation}
\begin{cases}
\Delta u = \mu u & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}
where \( \frac{\partial}{\partial \nu} \) is the outer normal derivative, the well-known Szegö-Weinberger inequality states that \cite{25, 28}
\begin{equation}
\mu_1(\Omega)|\Omega|^{2/n} \leq \mu_1(\mathbb{B}^n)|\mathbb{B}^n|^{2/n},
\end{equation}
where \( \mathbb{B}^n \) denotes the \( n \)-dimensional unit ball.
with equality holding if and only if $\Omega$ is a ball in $\mathbb{R}^n$. Ashbaugh and Benguria conjectured in [5] that
\[
\sum_{i=1}^{n} \frac{1}{\mu_i(\Omega)} \geq \frac{n}{\mu_1(B_\Omega)}, \quad \text{with equality if and only if } \Omega \text{ is a ball,} \tag{1.18}
\]
where $B_\Omega \subset \mathbb{R}^n$ is a ball of same volume as $\Omega$. In [27], the authors proved the following inequality
\[
\sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)} \geq \frac{n-1}{\mu_1(B_\Omega)}, \quad \text{with equality if and only if } \Omega \text{ is a ball,} \tag{1.19}
\]
which supports this conjecture of Ashbaugh and Benguria.

2 A proof of Theorem 1.1.

Before proving Theorem 1.1, let us recall some known facts (Cf. [2, 3, 4, 10, 16, 24, 29]). Let \( \{u_j\}_{j=1}^\infty \) be an orthonormal set of eigenfunctions of the problem (1.1), that is,
\[
\begin{aligned}
\Delta u_i &= -\lambda_i u_i \quad \text{in } \Omega, \\
\left. u_i \right|_{\partial \Omega} &= 0, \\
\int_{\Omega} u_i u_j dx &= \delta_{ij},
\end{aligned}
\tag{2.1}
\]
where $dx$ denotes the volume element of $\Omega$. For each $k = 1, 2, \cdots$, the variational characterization of $\lambda_{k+1}(\Omega)$ is given by
\[
\lambda_{k+1}(\Omega) = \inf_{\phi \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla \phi|^2 dx}{\int_{\Omega} \phi^2 dx}. \tag{2.2}
\]

Let $B_r$ be a ball of radius $r$ centered at the origin in $\mathbb{R}^n$. It is known that
\[
\lambda_1(B_r) = \left( \frac{j_{n-2,1}}{r} \right)^2 \tag{2.3}
\]
with its corresponding eigenfunction given by the radial function
\[
u(x) := c |x|^{1-\frac{n}{2}} J_{\frac{n}{2}-1} \left( \frac{j_{n-2,1}}{r} |x| \right), \tag{2.4}
\]
where $c$ is a nonzero constant. The second Dirichlet eigenvalue of $B_r$ has multiplicity $n$, that is,
\[
\lambda_2(B_r) = \cdots = \lambda_{n+1}(B_r) = \frac{j_{n/2,1}^2}{r^2} \tag{2.5}
\]
and a basis for the eigenspace corresponding to $\lambda_2(B_r)$ consists of

$$\xi_i(x) = |x|^{1-\frac{n}{2}} J_{n/2} \left(\frac{j_{n/2,1}|x|}{r} \right) \frac{x_i}{|x|}, \quad i = 1, \ldots, n. \quad (2.6)$$

Define a function $w : [0, +\infty) \to \mathbb{R}$ by

$$w(t) \equiv \begin{cases} J_{n/2}^{(\beta t)} & \text{for } 0 \leq t < 1, \\ J_{n/2}^{(\alpha t)} & \text{for } t \geq 1, \end{cases} \quad (2.7)$$

where $\alpha = j_{n/2-1,1}$, $\beta = j_{n/2,1}$. We have $w(0) = 0$, $w(t) > 0$, $\forall t \in (0, +\infty)$ and for any $t \geq 0$, one concludes from Theorem 3.3 in [3] that

$$(w'(t))^2 \leq \left( \frac{w(t)}{t} \right)^2. \quad (2.8)$$

Let $\gamma = \sqrt{\lambda_1/\alpha}$ and set

$$B(t) \equiv w'(t)^2 + (n-1) \frac{w(t)^2}{t^2}; \quad (2.9)$$

then (Cf. (2.14), (2.15) and (2.22) in [3])

$$\int_{\Omega} B(\gamma|x|) u_1^2 dx \int_{\Omega} w(\gamma|x|)^2 u_1^2 dx \leq \beta^2 - \alpha^2. \quad (2.10)$$

**Proof of Theorem 1.1.** Observe that if

$$Q \neq 0 \quad \text{and} \quad \int_{\Omega} Qu_1^2 dx = \int_{\Omega} Qu_1 u_2 dx = \cdots = \int_{\Omega} Qu_1 u_k dx = 0, \quad (2.11)$$

then (2.2) gives

$$\lambda_{k+1} \leq \frac{\int_{\Omega} \nabla (Qu_1)^2 dx}{\int_{\Omega} Q^2 u_1^2 dx}, \quad (2.12)$$

which, yields by integration by parts that

$$\lambda_{k+1} - \lambda_1 \leq \frac{\int_{\Omega} |\nabla Q|^2 u_1^2 dx}{\int_{\Omega} Q^2 u_1^2 dx}. \quad (2.13)$$

We define $g : [0, +\infty) \to \mathbb{R}$ by

$$g(t) = w(\gamma t) \quad (2.14)$$

and fix an orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathbb{R}^n$. By using the Brouwer fixed-point theorem, we can choose the origin of $\mathbb{R}^n$ so that (Cf. [3])

$$\int_{\Omega} \langle x, e_i \rangle \frac{g(|x|)}{|x|} u_1^2 dx = 0, \quad i = 1, \cdots, n. \quad (2.15)$$
Next we show that there exists a new orthonormal basis \( \{ e'_i \}_{i=1}^n \) of \( \mathbb{R}^n \) such that

\[
\int_{\Omega} \langle x, e'_i \rangle \frac{g(|x|)}{|x|} u_{j+1} \, dx = 0,
\]

for \( j = 1, \cdots, i - 1 \) and \( i = 2, \cdots, n \). To see this, we define an \( n \times n \) matrix \( P = (p_{ij}) \) by

\[
p_{ij} = \int_{\Omega} \langle x, e_i \rangle \frac{g(|x|)}{|x|} u_{j+1} \, dx, \quad i, j = 1, 2, \cdots, n.
\]

Using the orthogonalization of Gram and Schmidt (QR-factorization theorem), one can find an upper triangle matrix \( T \) such that \( T = (T_{ij}) \) and an orthogonal matrix \( U = (u_{ij}) \) such that \( T = UP \). Hence,

\[
T_{ij} = \sum_{k=1}^{n} a_{ik} p_{kj} = \int_{\Omega} \sum_{k=1}^{n} a_{ik} (x, e_k) \frac{g(|x|)}{|x|} u_{j+1} \, dx = 0, \quad 1 \leq j < i \leq n.
\]

Letting \( e'_i = \sum_{k=1}^{n} a_{ik} e_k, \quad i = 1, \cdots, n \), one gets (2.16). Let us denote by \( x_1, x_2, \cdots, x_n \) the coordinate functions of \( \mathbb{R}^n \) with respect to the base \( \{ e'_i \}_{i=1}^n \), that is, \( x_i = \langle x, e'_i \rangle, \ x \in \mathbb{R}^n \). From (2.15) and (2.16), we have

\[
\int_{\Omega} g(|x|) \frac{x_i}{|x|} u_{j+1} \, dx = 0, \quad i = 1, \cdots, n, \quad j = 0, \cdots, i - 1.
\]

Let

\[
\phi_k = \frac{x_k}{|x|}, \quad k = 1, \cdots, n;
\]

then

\[
\phi_k \neq 0 \quad \text{and} \quad \int_{\Omega} \phi_k u_i^2 \, dx = \cdots = \int_{\Omega} \phi_k u_1 u_k \, dx = 0.
\]

It then follows from (2.13) that

\[
(\lambda_{k+1} - \lambda_1) \int_{\Omega} \phi_k^2 u_i^2 \, dx \leq \int_{\Omega} |\nabla \phi_k|^2 u_i^2 \, dx, \quad k = 1, \cdots, n.
\]

Substituting

\[
|\nabla \phi_k|^2 = g'(|x|)^2 \frac{x_k^2}{|x|^2} + \frac{g(|x|)^2}{|x|^2} \left(1 - \frac{x_k^2}{|x|^2}\right)
\]

into (2.20) and dividing by \( (\lambda_{k+1} - \lambda_1) \), we have for \( k = 1, \cdots, n \), that

\[
\int_{\Omega} \phi_k^2 u_i^2 \, dx \leq \frac{1}{\lambda_{k+1} - \lambda_1} \int_{\Omega} \left(g'(|x|)^2 - \frac{g(|x|)^2}{|x|^2}\right) \frac{x_k^2}{|x|^2} u_i^2 \, dx + \frac{1}{\lambda_{k+1} - \lambda_1} \int_{\Omega} \frac{g(|x|)^2}{|x|^2} u_i^2 \, dx
\]
Summing on \( k \) from 1 to \( n \), one gets

\[
\int_{\Omega} g(|x|)^2 u_1^2 \, dx \leq \sum_{k=1}^{n} \frac{1}{\lambda_{k+1} - \lambda_1} \int_{\Omega} \frac{g(|x|)^2}{|x|^2} u_1^2 \, dx \quad (2.23)
\]

\[
+ \sum_{k=1}^{n} \frac{1}{\lambda_{k+1} - \lambda_1} \int_{\Omega} \left( g'(|x|)^2 - \frac{g(|x|)^2}{|x|^2} \right) \frac{x_k^2}{|x|^2} u_1^2 \, dx.
\]

Observe that

\[
\sum_{k=1}^{n} \frac{1}{\lambda_{k+1} - \lambda_1} x_k^2 = \sum_{k=1}^{n-1} \frac{1}{\lambda_{k+1} - \lambda_1} x_k^2 + \frac{1}{\lambda_{n+1} - \lambda_1} x_n^2
\]

\[
= \sum_{k=1}^{n-1} \frac{1}{\lambda_{k+1} - \lambda_1} x_k^2 + \frac{1}{\lambda_{n+1} - \lambda_1} \left( 1 - \sum_{k=1}^{n-1} \frac{x_k^2}{|x|^2} \right).
\]

Therefore,

\[
\sum_{k=1}^{n} \frac{1}{\lambda_{k+1} - \lambda_1} \int_{\Omega} \left( g'(|x|)^2 - \frac{g(|x|)^2}{|x|^2} \right) \frac{x_k^2}{|x|^2} u_1^2 \, dx \quad (2.25)
\]

\[
= \sum_{k=1}^{n-1} \frac{1}{\lambda_{k+1} - \lambda_1} \int_{\Omega} \left( g'(|x|)^2 - \frac{g(|x|)^2}{|x|^2} \right) \frac{x_k^2}{|x|^2} u_1^2 \, dx
\]

\[
+ \frac{1}{\lambda_{n+1} - \lambda_1} \int_{\Omega} \left( g'(|x|)^2 - \frac{g(|x|)^2}{|x|^2} \right) u_1^2 \, dx - \frac{1}{\lambda_{n+1} - \lambda_1} \int_{\Omega} \left( g'(|x|)^2 - \frac{g(|x|)^2}{|x|^2} \right) \sum_{k=1}^{n-1} \frac{x_k^2}{|x|^2} u_1^2 \, dx.
\]

\[
= \sum_{k=1}^{n-1} \int_{\Omega} \left( \frac{1}{\lambda_{k+1} - \lambda_1} - \frac{1}{\lambda_{n+1} - \lambda_1} \right) \left( g'(|x|)^2 - \frac{g(|x|)^2}{|x|^2} \right) \frac{x_k^2}{|x|^2} u_1^2 \, dx
\]

\[
+ \frac{1}{\lambda_{n+1} - \lambda_1} \int_{\Omega} \left( g'(|x|)^2 - \frac{g(|x|)^2}{|x|^2} \right) u_1^2 \, dx.
\]

We have

\[
\frac{1}{\lambda_{k+1} - \lambda_1} - \frac{1}{\lambda_{n+1} - \lambda_1} \geq 0, \quad k = 1, \ldots, n - 1.
\]

It follows from (2.8) and (2.14) that

\[
g'(|x|)^2 - \frac{g(|x|)^2}{|x|^2} \leq 0 \quad \text{on } \Omega.
\]
Thus,

\[
\sum_{k=1}^{n-1} \int_\Omega \left( \frac{1}{\lambda_{k+1} - \lambda_1} - \frac{1}{\lambda_{n+1} - \lambda_1} \right) \left( g'(\|x\|)^2 - \frac{g(\|x\|)^2}{\|x\|^2} \right) \frac{x_k^2}{\|x\|^2} u_1^2 dx \leq 0,
\]

which, combining with (2.23) and (2.25), gives

\[
\int_\Omega g(\|x\|)^2 dx \leq \frac{1}{\lambda_{n+1} - \lambda_1} \int_\Omega g'(\|x\|)^2 u_1^2 dx + \sum_{k=1}^{n-1} \frac{1}{\lambda_{k+1} - \lambda_1} \int_\Omega g(\|x\|)^2 u_1^2 dx
\]

\[
= \frac{1}{\lambda_{n+1} - \lambda_1} \int_\Omega g'(\|x\|)^2 u_1^2 dx + \sum_{k=1}^{n-1} \frac{1}{\lambda_{k+1} - \lambda_1} \int_\Omega g(\|x\|)^2 u_1^2 dx
\]

\[
\leq \frac{1}{n-1} \sum_{k=1}^{n-1} \frac{1}{\lambda_{k+1} - \lambda_1} \int_\Omega \left( g'(\|x\|)^2 + (n-1) \frac{g(\|x\|)^2}{\|x\|^2} \right) u_1^2 dx.
\]

Consequently, we have from (2.9), (2.10), (2.14) and (2.26) that

\[
\frac{1}{n-1} \sum_{k=1}^{n-1} \frac{1}{\lambda_{k+1} - \lambda_1} \geq \frac{\int_\Omega g(\|x\|)^2 u_1^2 dx}{\int_\Omega \left( g'(\|x\|)^2 + (n-1) \frac{g(\|x\|)^2}{\|x\|^2} \right) u_1^2 dx}
\]

\[
= \frac{\alpha^2}{\lambda_1} \int_\Omega w(\gamma|x|)^2 u_1^2 dx
\]

\[
\geq \frac{\alpha^2}{\lambda_1 (\beta^2 - \alpha^2)},
\]

which proves (1.15). Also, one can see that the equality holds in (1.15) if and only if \(\Omega\) is an \(n\)-ball. This completes the proof of Theorem 1.1.

### 3 Lower Order Dirichlet eigenvalues of general elliptic equations

By using the arguments in the proof of Theorem 1.1 and the work of Ashbaugh-Benguria [3] one can generalize the inequality (1.15) to the first \(n\) eigenvalues of the following general problem

\[
\begin{cases}
- \sum_{i,j} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + q(x)u = \lambda r(x)u \quad \text{in } \Omega, \\
u|_{\partial \Omega} = 0,
\end{cases}
\]

where \(\Omega\) is bounded domain with smooth boundary in \(\mathbb{R}^n\) and \([a_{ij}(x)]\) is symmetric positive definite for any \(x \in \Omega\). Namely, we have
Theorem 3.1 For equation (3.1), assume that \( q \geq 0 \) on \( \Omega \) and that there are positive numbers \( a, A, c \) and \( C \) such that the matrix \( [a_{ij}] \) satisfies
\[
a \leq [a_{ij}] \leq A
\]
in the sense of quadratic forms throughout \( \Omega \) and
\[
c \leq r(x) \leq C \text{ on } \Omega.
\]
Then the first \( n \) eigenvalues of the problem (3.1) satisfy
\[
\frac{\lambda_1}{\lambda_2 - \lambda_1} + \cdots + \frac{\lambda_1}{\lambda_n - \lambda_1} \geq \frac{(n-1)ac}{AC \left( \left( \frac{\lambda_1}{\lambda_2 - \lambda_1} \right)^2 - 1 \right)}.
\]
Furthermore, equality holds if and only if \( c = C, a = A, q \equiv 0, \) and \( \Omega \) is a ball in \( \mathbb{R}^n \).

Proof. Let \( \{v_k\}_{k=1}^\infty \) be an orthonormal set of eigenfunctions of the problem (3.1), that is,
\[
\begin{cases}
- \sum_{i,j} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial v_k}{\partial x_j} \right) + q(x)v_k = \lambda_k r(x)v_k \text{ in } \Omega, \\
v_k|_{\partial \Omega} = 0, \\
\int_\Omega r v_k v_l dx = \delta_{kl}.
\end{cases}
\]
For each \( k = 1, 2, \ldots \), the variational characterization of \( \lambda_{k+1} \) of the problem (3.1) is given by
\[
\lambda_{k+1}(\Omega) = \inf_{\psi \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega \left( \sum_{i,j} a_{ij}(x) \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} + q\psi^2 \right) dx}{\int_\Omega r \psi^2 dx}.
\]
Thus if \( Q \) is such that \( Q \neq 0 \) and
\[
\int_\Omega r Q v_1^2 dx = \cdots = \int_\Omega r Q v_k dx = 0,
\]
then
\[
\lambda_{k+1} \leq \frac{\int_\Omega \left( \sum_{i,j} a_{ij}(x) \frac{\partial (Q v_1)}{\partial x_i} \frac{\partial (Q v_1)}{\partial x_j} + q Q^2 v_1^2 \right) dx}{\int_\Omega r(x) Q^2 v_1^2 dx}.
\]
It then follows from integration by parts, (3.2), (3.3) and the fact that \( v_1 \) is an eigenfunction corresponding to the eigenvalue \( \lambda_1 \) that
\[
\lambda_{k+1} - \lambda_1 \leq \frac{\int_\Omega \left( \sum_{i,j} a_{ij}(x) \frac{\partial Q}{\partial x_i} \frac{\partial Q}{\partial x_j} v_1^2 \right) dx}{\int_\Omega r(x) Q^2 v_1^2 dx} \leq \frac{A \int_\Omega |\nabla Q|^2 v_1^2 dx}{c \int_\Omega Q^2 v_1^2 dx},
\]
Let $\omega$ and $B$ be as in Section 2 and set
\[ \gamma = \frac{1}{j_{n/2-1,1}} \sqrt{\frac{C \lambda_1}{a}}, \quad g(t) = \omega(\gamma t), \quad t \geq 0. \] (3.9)

From the proof of Theorem 1.1, we know that one can choose the origin and the coordinate system of $\mathbb{R}^n$ properly so that
\[ \int_{\Omega} rg(|x|) \frac{x_i}{|x|} v_{j+1} dx = 0, \quad i = 1, \ldots, n, \quad j = 0, \ldots, i - 1 \] (3.10)

and so for $k = 1, \ldots, n$,
\[ (\lambda_{k+1} - \lambda_1) \int_{\Omega} \left( g(|x|) \frac{x_i}{|x|} \right)^2 v_1^2 dx \leq \frac{A}{a} \int_{\Omega} \left| \nabla \left( g(|x|) \frac{x_i}{|x|} \right) \right|^2 v_1^2 dx. \] (3.11)

Dividing (3.11) by $(\lambda_{k+1} - \lambda_1)$ and summing on $k$ from 1 to $n$, one gets as in Section 2 that
\[ \int_{\Omega} g(|x|)^2 v_1^2 dx \leq \frac{1}{n-1} \sum_{k=1}^{n-1} \frac{1}{\lambda_{k+1} - \lambda_1} A \int_{\Omega} \left( g'(|x|)^2 + (n-1) \frac{g(|x|)^2}{|x|^2} \right) v_1^2 dx. \] Hence,
\[ \frac{1}{n-1} \sum_{k=1}^{n-1} \frac{1}{\lambda_{k+1} - \lambda_1} \geq \frac{c}{A} \int_{\Omega} \frac{g(|x|)^2 v_1^2 dx}{\left( g'(|x|)^2 + (n-1) \frac{g(|x|)^2}{|x|^2} \right) v_1^2 dx} \] (3.12)
\[ = \frac{c}{A} \frac{\int_{\Omega} w(\gamma |x|)^2 v_1^2 dx}{\int_{\Omega} \frac{w(\gamma |x|)^2 v_1^2 dx}{B(\gamma |x|) v_1^2 dx}} \]
\[ = \frac{c}{A} \frac{a j_{n/2-1,1}}{C \lambda_1} \frac{\int_{\Omega} w(\gamma |x|)^2 v_1^2 dx}{\int_{\Omega} B(\gamma |x|) v_1^2 dx}. \]

The proof of Theorem 4.1 in [3] shows that
\[ \frac{\int_{\Omega} w(\gamma |x|)^2 v_1^2 dx}{\int_{\Omega} B(\gamma |x|) v_1^2 dx} \geq \frac{1}{j_{n/2-1,1}^2 - j_{n/2-1,1}^2}. \] (3.13)

Combining (3.11) and (3.12), we get (3.2). Also, we can see that the equality holds in (3.2) if and only if $c = C$, $a = A$, $q \equiv 0$, and $\Omega$ is a ball in $\mathbb{R}^n$. This completes the proof of Theorem 3.1.

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