PROP profile of deformation quantization and
graph complexes with loops and wheels

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§1. Introduction

The first instances of graph complexes have been introduced in the theory of operads
and props which have found recently lots of applications in algebra, topology and ge-
ometry. Another set of examples has been introduced by Kontsevich [Ko1] as a way to
expose highly non-trivial interrelations between certain infinite dimensional Lie algebras
and topological objects, including moduli spaces of curves, invariants of odd dimensional
manifolds, and the group of outer automorphisms of a free group.

Motivated by the problem of deformation quantization we introduce and study di-
rected graph complexes with oriented loops and wheels. We show that universal quanti-
zations of Poisson structures can be understood as morphisms of dg props,

\[ Q : \text{DefQ} \rightarrow \text{Lie}^\circlearrowleft_{1} \]

where

- \( \text{DefQ} \) is the dg free prop whose representations in a graded vector space \( V \) describe
  Maurer-Cartan elements in the Hochschild dg Lie algebra, \( \mathcal{D}_V \), of polydifferential op-
  erators on the ring, \( \mathcal{O}_V := \hat{\otimes} V^* \), of smooth formal functions on \( V \) (see §2.7 for a precise
  definition); we call such Maurer-Cartan elements star products; if \( V \) is \( \mathbb{R}^n \) concentrated in
  degree zero, then this notion coincides with the ordinary notion of star product on smooth
  formal functions on \( \mathbb{R}^n \).

- \( \text{Lie}^\circlearrowleft_{1} \) is the wheeled completion of the minimal resolution, \( \text{Lie}^\circlearrowleft_{1} \), of the prop,
  \( \text{Lie}^\circlearrowleft_{1} \), of Lie 1-bialgebras; it is defined explicitly in §2.6 and is proven to have the prop-
  erty that its representations in a finite-dimensional graded vector space \( V \) correspond to
  Maurer-Cartan elements in the Schouten Lie algebra, \( \wedge^\bullet \mathcal{T}_V \), of polievector fields, where
  \( \mathcal{T}_V := \text{Der}\mathcal{O}_V \); such Maurer-Cartan elements are called Poisson structures on the formal
  graded manifold \( V \); if \( V \) is \( \mathbb{R}^n \) concentrated in degree zero, then this notion coincides with
  the ordinary notion of Poisson structure on \( \mathbb{R}^n \).

In the theory of props one is most interested in those directed graph complexes which
contain no loops and wheels. A major advance in understanding the cohomology groups of
such complexes was recently accomplished in [Ko3, MaVo, Va] using key ideas of \( \frac{1}{2} \) prop and

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Koszul duality. In particular, these authors were able to compute cohomologies of directed versions (without loops and wheels though) of Kontsevich’s ribbon graph complex and “commutative” graph complex, and show that they both are acyclic almost everywhere. One of our purposes in this paper is to extend some of the results of [Ko3, MaVo, Va] to a more difficult situation when the directed graphs are allowed to contain loops and wheels (i.e. directed paths which begin and end at the same vertex). In this case the answer differs markedly from the unwheeled case: we prove, for example, that while the cohomology of the wheeled extension, $\mathfrak{Lie}_\infty^\rightarrow$, of the operad of $\mathfrak{Lie}_\infty$-algebras remains acyclic almost everywhere (see Theorem 4.1.1 for a precise formula for $H^\ast(\mathfrak{Lie}_\infty^\rightarrow)$), the cohomology of the wheeled extension of the operad $\mathfrak{Ass}_\infty$ gets more complicated. Both these complexes describe irreducible summands of directed “commutative” and, respectively, ribbon graph complexes with the restriction on absence of wheels dropped.

The wheeled complex $\mathfrak{Lie}_\infty^\rightarrow$ is a subcomplex of the above mentioned graph complex $\mathfrak{Lie}_\infty^\leftrightarrow$ which describes Poisson structures\(^1\). Using Theorem 4.1.1 on $H^\ast(\mathfrak{Lie}_\infty^\rightarrow)$ we show in §4.2 that a subcomplex of $\mathfrak{Lie}_\infty^\leftrightarrow$ which is spanned by graphs with at most genus 1 wheels is also acyclic almost everywhere. However this acyclicity breaks for graphs with higher genus wheels: we find an explicit cohomology class with 3 wheels in §4.2.4 which proves that the natural epimorphism, $\mathfrak{Lie}_\infty^\rightarrow \to \mathfrak{Lie}_\infty^\leftrightarrow$, fails to stay quasi-isomorphism when extended to the wheeled completions, $\mathfrak{Lie}_\infty^\rightarrow \to \mathfrak{Lie}_\infty^\leftrightarrow$.

Kontsevich’s famous universal deformation quantization formulae involve graphs with wheels which encode traces of tensor powers of polyvector fields and their partial derivatives and hence make, in general, sense only in finite-dimensions. We show in this paper that every universal deformation quantization of Poisson structures on graded manifolds must involve such traces, i.e. graphs with wheels are unavoidable. This fact is one of the motivation behind our study of wheeled extensions of props, especially the wheeled extensions, $\mathfrak{Lie}_\infty^\rightarrow$ and $\mathfrak{Lie}_\infty^\leftrightarrow$, of the props related to Poisson geometry.

We show that there exists a natural dg free prop $[\mathfrak{Lie}_\infty^\rightarrow]_\infty$ which extends the above prop, $\mathfrak{Lie}_\infty^\rightarrow$, of polyvector fields and fits into a commutative diagram

$$
\begin{array}{ccc}
[\mathfrak{Lie}_\infty^\rightarrow]_\infty & \xrightarrow{\alpha} & \mathfrak{Lie}_\infty^\rightarrow \\
\downarrow \scriptscriptstyle{qis} & & \downarrow \\
\mathfrak{Lie}_\infty^\leftrightarrow & \xrightarrow{} & \mathfrak{Lie}_\infty^\rightarrow
\end{array}
$$

where $\alpha$ is an epimorphism of nondifferential props, and $qis$ a quasi-isomorphism of differential props. It is called a quasi-minimal prop resolution of $\mathfrak{Lie}_\infty^\rightarrow$.

As we mentioned above, representations of $\mathfrak{Lie}_\infty^\rightarrow$ in a finite-dimensional dg space $V$ are precisely Poisson structures on $V$. This prompts us to call representations of $[\mathfrak{Lie}_\infty^\rightarrow]_\infty$ in a finite dimensional dg space $V$ wheeled Poisson structures on the formal graded manifold $V$. Geometric meaning of such wheeled Poisson structures is not clear at

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\(^1\)Strictly speaking, a representation of $\mathfrak{Lie}_\infty^\rightarrow$ in $V$ gives a degree 1 element $\gamma \in \wedge^\ast T_V$ which satisfies not only the Maurer-Cartan condition, $[\gamma, \gamma] = 0$, but also the vanishing condition, $\gamma|_{x=0} = 0$, at $0 \in V$ (see §2.6); given, however, $\gamma \in \wedge^\ast T_V$ with $\gamma|_{x=0} \neq 0$, then, for a formal parameter $h$ viewed as a coordinate on $\mathbb{R}$, the element $\tilde{\gamma} := h\gamma \in \wedge^\ast T_{\tilde{V}}$, $\tilde{V} := V \times \mathbb{R}$, satisfies $\tilde{\gamma}|_{x=0} = 0$ and hence comes from a representation of the prop $\mathfrak{Lie}_\infty^\rightarrow$; thus such representations encompass arbitrary formal Poisson structures.
present — the differential equations behind these new structures involve not only Schouten brackets but also traces of derivatives of tensor powers of polyvector fields\(^2\) (and hence makes sense only in finite dimensions). Moreover, polyvector fields are only part of the data — there are other tensors (including new ones in degree 0) in the content list of a wheeled Poisson structure. Remarkably, one can construct “star products” out of wheeled Poisson structures, i.e. they can be deformation quantized:

**Main Theorem.** There exists a morphism of dg props,

\[ \hat{Q} : \text{DefQ} \to [\text{LieB}^\omega]_\infty. \]

Moreover, this morphism exists in the category of props over the field, \(\mathbb{Q}\), of rational numbers.

**Corollary.** Every wheeled Poisson structure on a finite dimensional formal manifold can be deformation quantized, i.e. there exists an associated Maurer-Cartan element — “star product” — in the Hochschild dg Lie algebra \(\mathcal{D}_V[[\hbar]]\).

The quantization morphism \(\hat{Q}\) is very non-trivial: the proof of Theorem §5.2 below implies that \(\hat{Q}\) involves, e.g., infinite jets of the polyvector fields constituent of a wheeled Poisson structure. There is a canonical monomorphism of dg props, \(i : \text{LieB}^\omega_\infty \to [\text{LieB}^\omega]_\infty\), so that those quantization morphisms \(\hat{Q}\) which factor though \(i\) provide us with quantizations of ordinary Poisson structures; we shall discuss a purely propic construction of such quantization morphisms elsewhere.

One can get some intuition into the geometric meaning of wheeled Poisson structures from their simpler analogues — representations of quasi-minimal prop resolutions, \([\text{Ass}^\omega]_\infty\) and \([\text{Com}^\omega]_\infty\), of the operads of associative and, respectively, commutative algebras which have been computed in [MMS]. We show some details in §4.5.2: if an \([\text{Ass}^\omega]_\infty\)-structure in a finite-dimensional dg space \(V\) is just a homological vector field, \(\partial\), on \(V[1]\) viewed as a non-commutative formal manifold, then an \([\text{Ass}^\omega]_\infty\)-structure in \(V\) is a pair \((\partial, f)\), involving a new piece of data — a cyclically invariant function \(f\) on \(V[1]\) — all satisfying a system of differential equations involving trace of \(\partial\). Unfortunately, such a detailed picture of wheeled Poisson structures is out of reach at present: this problem appears to have complexity level comparable with that of the problem of computing homologies of the directed versions of famous Kontsevich’s graph complexes [Ko1] (in fact, the graph complex behind wheeled Poisson structures is closely related to the directed version of Kontsevich’s “commutative” graph complex).

A few words about our notations. The cardinality of a finite set \(I\) is denoted by \(|I|\). The degree of a homogeneous element, \(a\), of a graded vector space is denoted by \(|a|\) (this should never lead to a confusion). \(S_n\) stands for the group of all bijections, \([n] \to [n]\), where \([n]\) denotes (here and everywhere) the set \(\{1, 2, \ldots, n\}\). The set of positive integers is denoted by \(\mathbb{N}^*\). If \(V = \bigoplus_{i \in \mathbb{Z}} V^i\) is a graded vector space, then \(V[k]\) is a graded vector space with \(V[k]^i := V^{i+k}\). All our free props are assumed to be completed with respect

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\(^2\)It is proven in §4.2.2 that this system of differential equations can not contain equations involving just a single trace of some tensorial expression built from the Poisson tensor and its derivatives (which correspond to graphs with genus one wheels).
to the number of vertices (so that extra care should be in place when composing prop
morphism).

We work throughout over the field $\mathbb{K}$ of characteristic $0$.

The paper is organized as follows. In §2 we remind some basic facts about props
and graph complexes and describe a universal construction which associates dg props to a
class of sheaves of dg Lie algebras on smooth formal manifolds; we then illustrate it with
examples which are relevant to Poisson geometry and deformation quantization. In §3 we
develop new methods for computing cohomology of directed graph complexes with wheels,
and prove several theorems on cohomology of wheeled completions of minimal resolutions
of dioperads. In §4 we apply these methods and results to compute cohomology of sev
eral concrete graph complexes. In §5 we prove the main theorem formulated above. In the
appendix we use ideas of cyclic homology to construct a cyclic bicomplex computing
cohomology of wheeled completions of dg operads.

§2. Dg props versus sheaves of dg Lie algebras

2.1. Props. An $S$-bimodule, $E$, is, by definition, a collection of graded vector spaces,
$\{E(m,n)\}_{m,n \geq 0}$, equipped with a left action of the group $S_m$ and with a right action of $S_n$
which commute with each other. For any graded vector space $M$ the collection, $\text{End}(M) = \{\text{End}(M)(m,n) := \text{Hom}(M^{\otimes m}, M^{\otimes n})\}_{m,n \geq 0}$, is naturally an $S$-bimodule. A morphism
of $S$-bimodules, $\phi : E_1 \to E_2$, is a collection of equivariant linear maps, $\{\phi(m,n) : E_1(m,n) \to E_2(m,n)\}_{m,n \geq 0}$. A morphism $\phi : E \to \text{End}(M)$ is called a representation of
an $S$-bimodule $E$ in a graded vector space $M$.

There are two natural associative binary operations on the $S$-bimodule $\text{End}(M)$,

$$\otimes : \text{End}(M)(m_1, n_1) \otimes \text{End}(M)(m_2, n_2) \to \text{End}(M)(m_1 + m_2, n_1 + n_2),$$

$$\circ : \text{End}(M)(p, m) \otimes \text{End}(M)(m, n) \to \text{End}(M)(p, n),$$

and a distinguished element, the identity map $1 \in \text{End}(M)(1,1)$.

Axioms of prop ("products and permutations") are modelled on the properties of $(\otimes, \circ, 1)$
in $\text{End}(M)$ (see [Mc]).

2.1.1. Definition. A prop, $E$, is an $S$-bimodule, $E = \{E(m,n)\}_{m,n \geq 0}$, equipped with the following data,

- a linear map called horizontal composition,

$$\otimes : E(m_1, n_1) \otimes E(m_2, n_2) \to E(m_1 + m_2, n_1 + n_2)$$

such that

$$e_1 \otimes e_2 = (-1)^{|e_1||e_2|} \sigma_{m_1,m_2} e_2 \otimes e_1 \otimes e_3$$

where $\sigma_{m_1,m_2}$ is the following permutation

in $S_{m_1+m_2}$,
A morphism of props, $\phi : E_1 \rightarrow E_2$, is a morphism of the associated $\mathbb{S}$-bimodules which respects, in the obvious sense, all the prop data.

A differential in a prop $E$ is a collection of degree 1 linear maps, $\{\delta : E(m,n) \rightarrow E(m,n)\}_{m,n \geq 0}$, such that $\delta^2 = 0$ and

$$\delta(e_1 \boxtimes e_2) = (\delta e_1) \boxtimes e_2 + (-1)^{|e_1|} e_1 \boxtimes \delta e_2,$$

$$\delta(e_3 \circ e_4) = (\delta e_3) \circ e_4 + (-1)^{|e_3|} e_3 \circ \delta e_4,$$

for any $e_1, e_2 \in E$ and any $e_3, e_4 \in E$ such that $e_3 \circ e_4$ makes sense. Note that $d1 = 0$.

For any dg vector space $(M, d)$ the associated prop $\text{End}\langle M \rangle$ has a canonically induced differential which we always denote by the same symbol $d$.

A representation of a dg prop $(E, \delta)$ in a dg vector space $(M, d)$ is, by definition, a morphism of props, $\phi : E \rightarrow \text{End}\langle M \rangle$, which commutes with differentials, $\phi \circ \delta = d \circ \phi$. (Here and elsewhere $\circ$ stands for the composition of maps; it will always be clear from the context whether $\circ$ stands for the composition of maps or for the vertical composition in props.)

2.1.2. Remark. If $\psi : (E_1, \delta) \rightarrow (E_2, \delta)$ is a morphism of dg props, and $\phi : (E_2, \delta) \rightarrow (\text{End}\langle M \rangle, d)$ is a representation of $E_2$, then the composition, $\phi \circ \psi$, is a representation of $E_1$. Thus representations can be “pulled back”.

2.1.3. Free props. Let $\mathcal{G}^{1}(m, n), m, n \geq 0$, be the space of directed $(m, n)$-graphs, $G$, that is, connected 1-dimensional CW complexes satisfying the following conditions:

(i) each edge (that is, 1-dimensional cell) is equipped with a direction;

(ii) if we split the set of all vertices (that is, 0-dimensional cells) which have exactly one adjacent edge into a disjoint union, $V_{in} \sqcup V_{out}$,

with $V_{in}$ being the subset of vertices with the adjacent edge directed from the vertex,
and \( V_{\text{out}} \) the subset of vertices with the adjacent edge directed towards the vertex,

then \(|V_{\text{in}}| \geq n\) and \(|V_{\text{out}}| \geq m\);

(iii) precisely \( n \) of vertices from \( V_{\text{in}} \) are labelled by \( \{1, \ldots, n\} \) and are called inputs;

(iv) precisely \( m \) of vertices from \( V_{\text{out}} \) are labelled by \( \{1, \ldots, m\} \) and are called outputs;

(v) there are no oriented wheels, i.e. directed paths which begin and end at the same vertex; in particular, there are no loops (oriented wheels consisting of one internal edge). Put another way, directed edges generate a continuous flow on the graph which we always assume in our pictures to go from bottom to the top.

Note that \( G \in \mathcal{G}^\uparrow(m,n) \) may not be connected. Vertices in the complement,

\[ v(G) := \overline{\text{inputs} \cup \text{outputs}}, \]

are called internal vertices. For each internal vertex \( v \) we denote by \( In(v) \) (resp., by \( Out(v) \)) the set of those adjacent half-edges whose orientation is directed towards (resp., from) the vertex. Input (resp., output) vertices together with adjacent edges are called input (resp., output) legs. The graph with one internal vertex, \( n \) input legs and \( m \) output legs is called the \((m,n)\)-corolla.

We set \( \mathcal{G}^\uparrow := \sqcup_{m,n} \mathcal{G}^\uparrow(m,n) \).

The free prop, \( P\langle E \rangle \), generated by an \( S \)-module, \( E = \{E(m,n)\}_{m,n\geq0} \), is defined by (see, e.g., [MaVo, Va])

\[
P\langle E \rangle(m,n) := \bigoplus_{G \in \mathcal{G}^\uparrow(m,n)} \left( \bigotimes_{v \in v(G)} E(Out(v),In(v)) \right)_{\text{Aut}G}
\]

where

- \( E(Out(v),In(v)) := \text{Bij}([m],Out(v)) \times_{S_m} E(m,n) \times_{S_n} \text{Bij}(In(v),[n]) \) with \( \text{Bij} \) standing for the set of bijections,

- \( \text{Aut}(G) \) stands for the automorphism group of the graph \( G \).

An element of the summand above, \( G\langle E \rangle := \left( \bigotimes_{v \in v(G)} E(Out(v),In(v)) \right)_{\text{Aut}G} \), is often called a graph \( G \) with internal vertices decorated by elements of \( E \), or just a decorated graph.

A differential, \( \delta \), in a free prop \( P\langle E \rangle \) is completely determined by its values,

\[
\delta : E(Out(v),In(v)) \longrightarrow P\langle E \rangle(|Out(v)|,|In(v)|),
\]

on decorated corollas (whose unique internal vertex is denoted by \( v \)).
Prop structure on an $S$-bimodule $E = \{E(m, n)\}_{m,n \geq 0}$ provides us, for any graph $G \in \mathfrak{S}^+(m, n)$, with a well-defined evaluation morphism of $S$-bimodules,

$$ev : G\langle E \rangle \rightarrow E(m, n).$$

In particular, if a decorated graph $C \in P\langle E \rangle$ is built from two corollas, $C_1 \in \mathfrak{S}(m_1, n_1)$ and $C_2 \in \mathfrak{S}(m_2, n_2)$ by gluing $j$th output leg of $C_2$ with $i$th input leg of $C_1$, and if the vertices of these corollas are decorated, respectively, by elements $a \in E(m_1, n_1)$ and $b \in E(m_2, n_2)$, then we reserve a special notation,

$$a \circ_j b := ev(C) \in E(m_1 + m_2 - 1, n_1 + n_2 - 1),$$

for the resulting evaluation map.

2.1.4. Completions. Any free prop $P\langle E \rangle$ is naturally a direct sum, $P\langle E \rangle = \oplus_{n \geq 0} P_n\langle E \rangle$, of subspaces spanned by decorated graphs with $n$ vertices. Each summand $P_n\langle E \rangle$ has a natural filtration by the genus, $g$, of the underlying graphs (which is, by definition, equal to the first Betti number of the associated CW complex). Hence each $P_n\langle E \rangle$ can be completed with respect to this filtration. Similarly, there is a filtration by the number of vertices. We shall always work in this paper with completed with respect to these filtrations free props and hence use the same notation, $P\langle E \rangle$, and the same name, free prop, for the completed version. Note that not every pair of morphisms of props, $f : P \rightarrow Q$ and $g : Q \rightarrow R$, can be composed into $g \circ f : P \rightarrow R$. In concrete examples one must be careful to check that no divergences occur.

2.2. Dioperads and $\frac{1}{2}$props. A dioperad is an $S$-bimodule, $E = \{E(m, n)\}_{m,n \geq 1}$, equipped with a set of compositions,

$$\{i \circ_j : E(m_1, n_1) \otimes E(m_2, n_2) \rightarrow E(m_1 + m_2 - 1, n_1 + n_2 - 1)\}_{1 \leq i \leq n_1, 1 \leq j \leq n_2},$$

which satisfy the axioms imitating the properties of the compositions $\circ_j$ in a generic prop. We refer to [Ga], where this notion was introduced, for a detailed list of these axioms. The free dioperad generated by an $S$-bimodule $E$ is given by,

$$D\langle E \rangle(m, n) := \bigoplus_{G \in \mathfrak{T}(m, n)} G\langle E \rangle$$

where $\mathfrak{T}(m, n)$ is a subset of $\mathfrak{S}(m, n)$ consisting of connected trees (i.e., connected graphs of genus 0).

Another and less obvious reduction of the notion of prop was introduced by Kontsevich in [Ko3] and studied in detail in [MaVo]: a $\frac{1}{2}$prop is an $S$-bimodule, $E = \{E(m, n)\}_{m,n \geq 1}$, equipped with two sets of compositions,

$$\{i_1 \circ_j : E(m_1, 1) \otimes E(m_2, n_2) \rightarrow E(m_1 + m_2 - 1, n_2)\}_{1 \leq j \leq m_2}$$

and

$$\{i_0 : E(m_1, 1) \otimes E(1, n_2) \rightarrow E(m_1 + m_2 - 1, n_2)\}_{1 \leq i \leq n_1}.$$
satisfying the axioms which imitate the properties of the compositions $1 \circ j$ and $i \circ 1$ in a generic dioperad. The free $\frac{1}{2}$prop generated by an $\mathcal{S}$-bimodule $E$ is given by,

$$\frac{1}{2}P\langle E \rangle(m,n) := \bigoplus_{G \in \frac{1}{2}\mathcal{T}(m,n)} G\langle E \rangle,$$

where $\frac{1}{2}\mathcal{T}(m,n)$ is a subset of $\mathcal{T}(m,n)$ consisting of those directed trees which, for each pair of internal vertices, $(v_1, v_2)$, connected by an edge directed from $v_1$ to $v_2$ have either $|\text{Out}(v_1)| = 1$ or/and $|\text{In}(v_2)| = 1$. Such trees have at most one vertex $v$ with $|\text{Out}(v)| \geq 2$ and $|\text{In}(v)| \geq 2$.

Axioms of dioperad (resp., $\frac{1}{2}$prop) structure on an $\mathcal{S}$-bimodule $E$ ensure that there is a well-defined evaluation map,

$$\text{ev} : G\langle E \rangle \rightarrow E(m,n),$$

for each $G \in \mathcal{T}(m,n)$ (resp., $G \in \frac{1}{2}\mathcal{T}(m,n)$).

2.2.1. Free resolutions. A free resolution of a dg prop $P$ is, by definition, a dg free prop, $(P\langle E \rangle, \delta)$, generated by some $\mathcal{S}$-bimodule $E$ together with a morphism of dg props, $\alpha : (P\langle E \rangle, \delta) \rightarrow P$, which is a homology isomorphism.

If the differential $\delta$ in $P\langle E \rangle$ is decomposable (with respect to prop’s vertical and /or horizontal compositions), then $\alpha : (P\langle E \rangle, \delta) \rightarrow P$ is called a minimal model of $P$.

Similarly one defines free resolutions and minimal models, $(D\langle E \rangle, \delta) \rightarrow P$ and $(\frac{1}{2}P\langle E \rangle, \delta) \rightarrow P$, of dioperads and $\frac{1}{2}$props.

2.2.2. Forgetful functors and their adjoints. There is an obvious chain of forgetful functors, $\text{Prop} \rightarrow \text{Diop} \rightarrow \frac{1}{2}\text{Prop}$. Let

$$\Omega_{\frac{1}{2}\text{Prop} \rightarrow \text{Diop}} : \frac{1}{2}\text{Prop} \rightarrow \text{Diop}, \quad \Omega_{\text{Diop} \rightarrow \text{Prop}} : \text{Diop} \rightarrow \text{Prop}, \quad \Omega_{\frac{1}{2}\text{Prop} \rightarrow \text{Prop}} : \frac{1}{2}\text{Prop} \rightarrow \text{Prop},$$

be the associated adjoints. The main motivation behind introducing the notion of $\frac{1}{2}$prop is a very useful fact that the functor $\Omega_{\frac{1}{2}\text{Prop} \rightarrow \text{Prop}}$ is exact [Ko3, MaVo], i.e., it commutes with the cohomology functor. Which in turn is due to the fact that, for any $\frac{1}{2}$prop $P$, there exists a kind of PBW lemma which represents $\Omega_{\frac{1}{2}\text{Prop} \rightarrow \text{Prop}}(P)$ as a vector space freely generated by a family decorated graphs,

$$\Omega_{\frac{1}{2}\text{Prop} \rightarrow \text{Prop}}(P)(m,n) := \bigoplus_{G \in \overline{\mathcal{G}}(m,n)} G\langle E \rangle,$$

where $\overline{\mathcal{G}}(m,n)$ is a subset of $\mathcal{G}(m,n)$ consisting of so called reduced graphs, $G$, which satisfy the following defining property [MaVo]: for each pair of internal vertices, $(v_1, v_2)$, of $G$ which are connected by a single edge directed from $v_1$ to $v_2$ one has $|\text{Out}(v_1)| \geq 2$ and $|\text{In}(v_2)| \geq 2$. The prop structure on $\Omega_{\frac{1}{2}\text{Prop} \rightarrow \text{Prop}}(P)$ is given by
(i) horizontal compositions := disjoint unions of decorated graphs,

(ii) vertical compositions := graftings followed by $\frac{1}{2}$ prop compositions of all those pairs of vertices $(v_1, v_2)$ which are connected by a single edge directed from $v_1$ to $v_2$ and have either $|Out(v_1)| = 1$ or/and $|In(v_2)| = 1$ (if there are any).

2.3. Graph complexes with wheels. Let $\mathcal{G}^\circ(m, n)$ be the set of all directed $(m, n)$-graphs which satisfy conditions 2.1.3(i)-(iv), and set $\mathcal{G}^\circ := \sqcup_{m,n} \mathcal{G}(m, n)$. A vertex (resp., edge or half-edge) of a graph $G \in \mathcal{G}^\circ$ which belongs to an oriented wheel is called a cyclic vertex (resp., edge or half-edge).

Note that for each internal vertex of $G \in \mathcal{G}^\circ(m, n)$ there is still a well defined separation of adjacent half-edges into input and output ones, as well as a well defined separation of legs into input and output ones.

For any $\mathcal{S}$-bimodule $E = \{E(m, n)\}_{m,n \geq 0}$, we define an $\mathcal{S}$-bimodule,

$$P^\circ(E)(m, n) := \bigoplus_{G \in \mathcal{G}(m, n)} \left( \bigotimes_{v \in v(G)} E(Out(v), In(v)) \right)_{\text{Aut} G},$$

and notice that $P^\circ(E)$ has a natural prop structure with respect to disjoint union and grafting of graphs. Clearly, this prop contains the free prop $P(E)$ as a natural sub-prop.

A derivation in $P^\circ(E)$ is, by definition, a collection of linear maps, $\delta : P^\circ(E)(m, n) \to P^\circ(E)(m, n)$ such that, for any $G \in \mathcal{G}$ and any element of $P^\circ(E)(m, n)$ of the form,

$$\epsilon = \text{coequalizer} \left( \epsilon_1 \otimes \epsilon_2 \otimes \ldots \otimes \epsilon_{|v(G)|} \right), \quad \epsilon_k \in E(Out(v_k), In(v_k)) \text{ for } 1 \leq k \leq |v(G)|,$

one has

$$\delta \epsilon = \text{coequalizer} \left( \sum_{k=1}^{|v(G)|} (-1)^{|\epsilon_1| + \ldots + |\epsilon_{k-1}|} \epsilon_1 \otimes \ldots \otimes \delta \epsilon_k \otimes \ldots \otimes \epsilon_{|v(G)|} \right).$$

Put another way, a graph derivation is completely determined by its values on decorated corollas,

$$\begin{array}{c}
\begin{array}{c}
1 \quad 2 \quad \ldots \quad m \\
\downarrow \quad \downarrow \\
1 \quad 2 \quad \ldots \quad n
\end{array}
\end{array}$$

that is, by linear maps,

$$\delta : E(m, n) \to P^\circ(E)(m, n).$$

A differential in $P^\circ(E)$ is, by definition, a degree 1 derivation $\delta$ satisfying the condition $\delta^2 = 0$. 

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2.3.1. Remark. If \((\mathcal{P}(E), \delta)\) is a dg free prop generated by an \(\mathcal{S}\)-bimodule \(E\), then \(\delta\) extends naturally to a differential on \(\mathcal{P}^\circ(E)\) which we denote by the same symbol \(\delta\). It is worth pointing out that such an induced differential may not preserve the number of oriented wheels. For example, if \(\delta\) applied to an element \(a \in E(m, n)\) (which we identify with the \(a\)-decorated \((m, n)\)-corolla) contains a summand of the form,

\[
\delta \left( \begin{array}{c}
\alpha \\
(j_1 \ j_2) \\
(i_1 \ i_2)
\end{array} \right) = \ldots + \delta \left( \begin{array}{c}
\alpha \\
(j_1 \ j_2) \\
(i_1 \ i_2)
\end{array} \right) + \ldots
\]

then the value of \(\delta\) on the graph obtained from this corolla by gluing output \(i_1\) with input \(j_1\) into a loop,

\[
\delta \left( \begin{array}{c}
\alpha \\
(j_1 \ j_2) \\
(i_1 \ i_2)
\end{array} \right) = \ldots + \delta \left( \begin{array}{c}
\alpha \\
(j_1 \ j_2) \\
(i_1 \ i_2)
\end{array} \right) + \ldots .
\]

contains a term with no oriented wheels at all. Thus propic differential can, in general, decrease the number of wheels. Notice in this connection that if \(\delta\) is induced on \(\mathcal{P}(E)\) from the minimal model of a \(\frac{1}{2}\)prop, then such summands are impossible and hence the differential preserves the number of wheels.

Vector spaces \(\mathcal{P}(E)\) and \(\mathcal{P}^\circ(E)\) have a natural positive gradation,

\[
\mathcal{P}(E) = \bigoplus_{k \geq 1} \mathcal{P}_k(E), \quad \mathcal{P}^\circ(E) = \bigoplus_{k \geq 1} \mathcal{P}^\circ_k(E),
\]

by the number, \(k\), of internal vertices of underlying graphs. In particular, \(\mathcal{P}_1(E)(m, n)\) is spanned by decorated \((m, n)\)-corollas and can be identified with \(E(m, n)\).

2.3.2. Representations of \(\mathcal{P}^\circ(E)\). Any representation, \(\phi : E \rightarrow \text{End}(M)\), of an \(\mathcal{S}\)-bimodule \(E\) in a finite dimensional vector space \(M\) can be naturally extended to representations of props, \(\mathcal{P}(E) \rightarrow \text{End}(M)\) and \(\mathcal{P}^\circ(E) \rightarrow \text{End}(M)\). In the latter case decorated graphs with oriented wheels are mapped into appropriate traces.

2.3.3. Remark. Prop structure on an \(\mathcal{S}\)-bimodule \(E = \{E(m, n)\}\) can be defined as a family of evaluation linear maps,

\[
\mu_G : G(E) \rightarrow E(m, n), \quad \forall \ G \in \mathcal{G}^+, \quad \mu_G : G(E) \rightarrow E(m, n), \quad \forall \ G \in \mathcal{G}^\circ,
\]

satisfying certain associativity axiom (cf. §2.1.3). Analogously, one can define a wheeled prop structure on \(E\) as a family of linear maps,

\[
\mu_G : G(E) \rightarrow E(m, n), \quad \forall \ G \in \mathcal{G}^\circ,
\]

such that
\( \mu_{(m,n)\mathrm{-corolla}} = \text{Id} \),

(ii) \( \mu_G = \mu_{G/H} \circ \mu_H \) for every subgraph \( H \in \mathcal{G}^\circ \) of \( G \), where \( G/H \) is obtained from \( G \) by collapsing to the single vertex every connected component of \( H \), and \( \mu_H : G\langle E \rangle \to G/H\langle E \rangle \) is the evaluation map on the subgraph \( H \) and identity on its complement.

**Claim.** For every finite-dimensional vector space \( M \) the associated endomorphism prop \( \text{End}\langle M \rangle \) has a natural structure of wheeled prop.

The notion of representation of \( \mathcal{P}^\circ\langle E \rangle \) in a finite dimensional vector space \( M \) introduced above is just a morphism of wheeled props, \( \mathcal{P}^\circ\langle E \rangle \to \text{End}\langle M \rangle \). We shall discuss these issues in detail elsewhere as in the present paper we are most interested in computing cohomology of dg free wheeled props \( (\mathcal{P}^\circ\langle E \rangle, \delta) \), where the composition maps \( \mu_G \) are tautological.

2.4. Formal graded manifolds. For a finite-dimensional vector space \( M \) we denote by \( \mathcal{M} \) the associated formal graded manifold. The distinguished point of the latter is always denoted by \( * \). The structure sheaf, \( \mathcal{O}_{\mathcal{M}} \), is (non-canonically) isomorphic to the completed graded symmetric tensor algebra, \( \hat{\otimes} M^* \). A choice of a particular isomorphism, \( \phi : \mathcal{O}_{\mathcal{M}} \to \hat{\otimes} M^* \), is called a choice of a local coordinate system on \( \mathcal{M} \). If \( \{ e_\alpha \}_{\alpha \in I} \) is a basis in \( M \) and \( \{ t^\alpha \}_{\alpha \in I} \) the associated dual basis in \( M^* \), then one may identify \( \mathcal{O}_{\mathcal{M}} \) with the graded commutative formal power series ring \( \mathbb{R}[[(t^\alpha)]] \).

Free modules over the ring \( \mathcal{O}_{\mathcal{M}} \) are called locally free sheaves (=vector bundles) on \( \mathcal{M} \). The \( \mathcal{O}_{\mathcal{M}}^- \)-module, \( \mathcal{T}_{\mathcal{M}} \), of derivations of \( \mathcal{O}_{\mathcal{M}} \) is called the tangent sheaf on \( \mathcal{M} \). Its dual, \( \Omega_{\mathcal{M}} \), is called the cotangent sheaf. One can form their (graded skewsymmetric) tensor products such as the sheaf of polyvector fields, \( \wedge^\bullet \mathcal{T}_{\mathcal{M}} \), and the sheaf of differential forms, \( \wedge^\bullet \Omega_{\mathcal{M}} \). The first sheaf is naturally a sheaf of Lie algebras on \( \mathcal{M} \) with respect to the Schouten bracket.

One can also define a sheaf of polydifferential operators, \( \mathcal{D}_{\mathcal{M}} \subset \bigoplus_{i \geq 0} \text{Hom}_\mathbb{R}(\mathcal{O}_{\mathcal{M}}^\otimes i, \mathcal{O}_{\mathcal{M}}) \). The latter is naturally a sheaf of dg Lie algebras on \( \mathcal{M} \) with respect to the Hochschild differential, \( d_H \), and brackets, \([ \cdot, \cdot ]_H \).

2.5. Geometry \( \Rightarrow \) graph complexes. We shall sketch here a simple trick which associates a dg free prop, \( \mathcal{P}^\circ\langle E_G \rangle \), to a sheaf of dg Lie algebras, \( \mathcal{G}_\mathcal{M} \), over a smooth graded formal manifold \( \mathcal{M} \).

We assume that

(i) \( \mathcal{G}_\mathcal{M} \) is built from direct sums and tensor products of (any order) jets of the sheaves \( \mathcal{T}_{\mathcal{M}}^\otimes \otimes \Omega_{\mathcal{M}}^\bullet \) and their duals (thus \( \mathcal{G}_\mathcal{M} \) can be defined for any formal smooth manifold \( \mathcal{M} \), i.e., its definition does not depend on the dimension of \( \mathcal{M} \)),

(ii) the differential and the Lie bracket in \( \mathcal{G}_\mathcal{M} \) can be represented, in a local coordinate system, by polydifferential operators and natural contractions between the duals.
The motivating examples are $\Lambda^*\mathcal{T}_\mathcal{M}$, $\mathcal{D}_\mathcal{M}$ and the sheaf of Nijenhuis dg Lie algebras on $\mathcal{M}$ (see [Me2]).

By assumption (i), a choice of a local coordinate system on $\mathcal{M}$, identifies $\mathcal{G}_\mathcal{M}$ with a subspace in

$$\bigoplus_{q,m \geq 0} \mathcal{O}_\mathcal{M} \otimes \text{Hom}(M^{\otimes q}, M^{\otimes m}) = \prod_{p,q,m \geq 0} \text{Hom}(\otimes^p M^{\otimes q}, M^{\otimes m}) \subset \prod_{m,n \geq 0} \text{Hom}(M^{\otimes n}, M^{\otimes m}).$$

Let $\Gamma$ be a degree 1 element in $\mathcal{G}_\mathcal{M}$. Denote by $\Gamma^m_{\alpha_1\ldots\alpha_n}$ the bit of $\Gamma$ which lies in $\text{Hom}(\otimes^\beta M^{\otimes \gamma}, M^{\otimes m})$ and set $\Gamma_n^m := \oplus_{p+q=n} \Gamma^m_{p,q} \in \text{Hom}(M^{\otimes n}, M^{\otimes m})$.

There exists a uniquely defined finite-dimensional $\mathbb{S}$-bimodule, $E_\mathcal{G} = \{E_\mathcal{G}(m,n)\}_{m,n \geq 0}$, whose representations in the vector space $M$ are in one-to-one correspondence with Taylor components, $\Gamma^m_{\alpha_1\ldots\alpha_n}$, of a degree 1 element $\Gamma$ in $\mathcal{G}_\mathcal{M}$. Set $\mathcal{P}^\mathcal{O}(\mathcal{G}) := \mathcal{P}^\mathcal{O}(E_\mathcal{G})$ (see Sect. 2.3).

Next we employ the dg Lie algebra structure in $\mathcal{G}_\mathcal{M}$ to introduce a differential, $\delta$, in $\mathcal{P}^\mathcal{O}(\mathcal{G})$. The latter is completely determined by its restriction to the subspace of $\mathcal{P}^\mathcal{O}_{1}(\mathcal{G})$ spanned by decorated corollas (without attached loops).

First we replace the Taylor coefficients, $\Gamma^m_{n}$, of the section $\Gamma$ by the decorated $(m,n)$-corollas

- with the unique internal vertex decorated by a basis element, $\{e_r\}_{r \in J}$, of $E_\mathcal{G}(m,n)$,
- with input legs labeled by basis elements, $\{e_\alpha\}$, of the vector space $M$ and output legs labeled by the elements of the dual basis, $\{t^\beta\}$.

Next we consider a formal linear combination,

$$\bar{\Gamma}_n^m = \sum_r \sum_{\beta_1\ldots\beta_m} t^\beta_1 t^\beta_2 \ldots t^\beta_m e_\alpha_1 e_\alpha_2 \ldots e_\alpha_n \in \mathcal{P}^\mathcal{O}_{1}(\mathcal{G}) \otimes \text{Hom}(M^{\otimes n}, M^{\otimes m}).$$

This expression is essentially a component of the Taylor decomposition of $\Gamma$,

$$\Gamma_n^m = \sum_{\alpha_1\ldots\alpha_n} \Gamma^\beta_1\ldots\beta_m t^{\alpha_1} \otimes \ldots \otimes t^{\alpha_n} \otimes e_\beta_1 \otimes \ldots \otimes e_\beta_m,$$

in which the numerical coefficient $\Gamma^\beta_1\ldots\beta_n_{\alpha_1\ldots\alpha_n}$ is substituted by a decorated labeled graph. More precisely, the interrelation between $\Gamma = \bigoplus_{m,n \geq 0} \Gamma_n^m$ and $\Gamma = \bigoplus_{m,n \geq 0} \Gamma^m_{n} \in \mathcal{G}_\mathcal{M}$ can be described as follows: a choice of any particular representation of the $\mathbb{S}$-bimodule $E_\mathcal{G}$,

$$\phi : \{E_\mathcal{G}(m,n) \to \text{Hom}(M^{\otimes n}, M^{\otimes m})\}_{m,n \geq 0},$$

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defines an element \( \Gamma = \phi(\Gamma) \in G_M \) which is obtained from \( \Gamma \) by replacing each graph,

\[
\begin{array}{c}
ed_\alpha_1 e_\alpha_2 \ldots e_\alpha_n \\
\end{array}
\]

by the value, \( \Gamma_{\alpha_1 \ldots \alpha_n} \in \mathbb{R} \), of \( \phi(\epsilon_r) \in \text{Hom}(M^{\otimes n}, M^{\otimes m}) \) on the basis vector \( e_\alpha_1 \otimes \ldots \otimes e_\alpha_n \otimes t^\beta_1 \otimes \ldots \otimes t^\beta_m \) (so that \( \Gamma_{\alpha_1 \ldots \alpha_n} = \sum_r \Gamma_{\alpha_1 \ldots \alpha_n} \)).

In a similar way one can define an element,

\[
\left[ \cdots [d \Gamma, \Gamma], \Gamma \right] \cdots \in P_n^\wedge(G) \otimes \text{Hom}(M^{\otimes \bullet}, M^{\otimes \bullet})
\]

for any Lie word,

\[
\left[ \cdots [d \Gamma, \Gamma], \Gamma \right] \cdots ,
\]

built from \( \Gamma, d \Gamma \) and \( n-1 \) Lie brackets. In particular, there are uniquely defined elements,

\[
\overline{d \Gamma} \in P_1^\wedge(G) \otimes \text{Hom}(M^{\otimes \bullet}, M^{\otimes \bullet}), \quad \frac{1}{2} [\Gamma, \Gamma] \in P_2^\wedge(G) \otimes \text{Hom}(M^{\otimes \bullet}, M^{\otimes \bullet}),
\]

whose values, \( \phi(\overline{d \Gamma}) \) and \( \phi(\frac{1}{2} [\Gamma, \Gamma]) \), for any particular choice of representation \( \phi \) of the \( \mathbb{S} \)-bimodule \( E_G \), coincide respectively with \( d \Gamma \) and \( \frac{1}{2} [\Gamma, \Gamma] \).

Finally one defines a differential \( \delta \) in the graded space \( P^\wedge(G) \) by setting

\[
\delta \Gamma = \overline{d \Gamma} + \frac{1}{2} [\Gamma, \Gamma],
\]

i.e. by equating the graph coefficients of both sides. That \( \delta^2 = 0 \) is clear from the following calculation,

\[
\delta^2 \Gamma = \delta \left( \overline{d \Gamma} + \frac{1}{2} [\Gamma, \Gamma] \right) = \delta \overline{d \Gamma} + \left[ \overline{d \Gamma}, \frac{1}{2} [\Gamma, \Gamma] \right] = - \overline{d (d \Gamma + \frac{1}{2} [\Gamma, \Gamma])} + \left[ \overline{d \Gamma}, \frac{1}{2} [\Gamma, \Gamma] \right] = 0,
\]

where we used both the axioms of dg Lie algebra in \( G_M \) and the axioms of the differential in \( P^\wedge(G) \). This completes the construction of \( (P^\wedge(G), \delta)^3 \).

\(^3\)As a first approximation to the propic translation of non-flat geometries (Yang-Mills, Riemann, etc.) one might consider the following version of the “trick”: in addition to generic element \( \Gamma \in G_M \) of degree 1 take into consideration (probably, non-generic) element of degree 2, \( F \in G_M \), extend appropriately the \( \mathbb{S} \)-bimodule \( E_G \) to accommodate the associated “curvature” \( F \)-corollas, and then (attempt to) define the differential \( \delta \) in \( P^\wedge(E_G) \) by equating graph coefficients in the expressions, \( \delta F = \overline{d F} + \frac{1}{2} [\Gamma, \Gamma] \) and \( \delta \Gamma = \overline{d \Gamma} + \frac{1}{2} [\Gamma, \Gamma] \).
2.5.1. Remarks. (i) If the differential and Lie brackets in \( G \) contain no traces, then the expression \( \frac{d}{2} \Gamma + \frac{1}{2} [\Gamma, \Gamma] \) does not contain graphs with oriented wheels. Hence formula (**) can be used to introduce a differential in the free prop, \( P(G) \), generated by the \( S \)-bimodule \( E_G \).

(ii) If the differential and Lie brackets in \( G \) contain no traces and are given by first order differential operators, then the expression \( d\Gamma + \frac{1}{2} [\Gamma, \Gamma] \) is a tree. Therefore formula (**) can be used to introduce a differential in the free dioperad, \( D(G) \).

2.5.2. Remark. The above trick works also for sheaves, 
\[
(G_M, \mu_n : \wedge^n G_M \to G_M[2-n], n = 1, 2, \ldots ),
\]
of \( L_\infty \) algebras over \( M \). The differential in \( P_{\wedge}(G) \) (or in \( P(G) \), if appropriate) is defined by,
\[
\delta \Gamma = \sum_{n=1}^{\infty} \frac{1}{n!} \mu_n(\Gamma, \ldots, \Gamma).
\]
The term \( \mu_n(\Gamma, \ldots, \Gamma) \) corresponds to decorated graphs with \( n \) internal vertices.

2.5.3. Remark. Any sheaf of dg Lie subalgebras, \( G'_M \subset G_M \), defines a dg prop, \( (P_{\wedge}(G'), \delta) \), which is a quotient of \( (P_{\wedge}(G), \delta) \) by the ideal generated by decorated graphs lying in the complement, \( P_{\wedge}(G) \setminus P_{\wedge}(G') \). Similar observation holds true for \( P(G) \) and \( P(G') \) (if they are defined).

2.6. Prop profile of Poisson structures. Let us consider the sheaf of polyvector fields, 
\[
\wedge^* T_M := \sum_{i \geq 0} \wedge^i T_M[1-i],
\]
equipped with the Schouten Lie bracket, \([, [ \cdot, \cdot ]_S \), and vanishing differential. A degree one section, \( \Gamma \), of \( \wedge^* T_M \) decomposes into a direct sum, \( \oplus_{i \geq 0} \Gamma_i \), with \( \Gamma_i \in \wedge^i T_M \) having degree \( 2 - i \) with respect to the grading of the underlying manifold. In a local coordinate system \( \Gamma \) can be represented as a Taylor series,
\[
\Gamma = \sum_{m,n \geq 0} \sum_{\alpha_1 \ldots \alpha_n} \Gamma^{\beta_1 \ldots \beta_m}_{\alpha_1 \ldots \alpha_n} (e_{\beta_1} \wedge \ldots \wedge e_{\beta_m}) \otimes (t^{a_1} \circ \ldots \circ t^{a_n}).
\]
As \( \Gamma^{\beta_1 \ldots \beta_m}_{\alpha_1 \ldots \alpha_n} = \Gamma^{[\beta_1 \ldots \beta_m]}_{(\alpha_1 \ldots \alpha_n)} \) has degree \( 2 - m \), we conclude that the associated \( S \)-bimodule \( E_{\wedge^* T} \) is given by
\[
E_{\wedge^* T}(m, n) = \text{sgn}_m \otimes 1_n[m - 2], \quad m, n \geq 0,
\]
where \( \text{sgn}_m \) stands for the one dimensional sign representation of \( \Sigma_m \) and \( 1_n \) stands for the trivial one-dimensional representation of \( \Sigma_n \). Then a generator of \( P(\wedge^* T) \) can be represented by the directed planar \((m, n)\)-corolla,
with skew-symmetric outgoing legs and symmetric ingoing legs. The formula \((\star\star)\) in Sect. 2.5 gives the following explicit expression for the induced differential, \(\delta\), in \(P(\wedge^T)\),

\[
\delta = \sum_{I_1 \sqcup I_2 = (1, \ldots, m), \begin{array}{c} j_1 \geq 0, j_2 \geq 1 \\ j_1 \geq 1, j_2 \geq 0 \end{array}} (-1)^{\sigma(I_1 \sqcup I_2) + |I_1||I_2|} \cdot \delta_{J_1 \sqcup J_2}^{\uparrow \uparrow \downarrow \downarrow \nonumber}
\]

where \(\sigma(I_1 \sqcup I_2)\) is the sign of the shuffle \(I_1 \sqcup I_2 = (1, \ldots, m)\).

2.6.1. Proposition. There is a one-to-one correspondence between representations,

\[
\phi : (P(\wedge^T), \delta) \rightarrow (\operatorname{End}(M), d),
\]

of \((P(\wedge^T), \delta)\) in a dg vector space \((M, d)\) and Maurer-Cartan elements, \(\gamma\), in \(\wedge^T_M\), that is, degree one elements satisfying the equation, \([\gamma, \gamma]_S = 0\).

Proof. Let \(\phi\) be a representation. Images of the above \((m, n)\)-corollas under \(\phi\) provide us with a collection of linear maps, \(\Gamma^m_n : \odot^n M \rightarrow \wedge^m M[2 - m]\) which we assemble, as in Sect. 2.5, into a section, \(\Gamma = \sum_{m,n} \Gamma^m_n\), of \(\wedge^T_M\).

The differential \(d\) in \(M\) can be interpreted as a linear (in the coordinates \(\{t^\alpha\}\)) degree one section of \(T_M\) which we denote by the same symbol.

Finally, the commutativity of \(\phi\) with the differentials implies

\[
[-d + \Gamma, -d + \Gamma]_S = 0.
\]

Thus setting \(\gamma = -d + \Gamma\) one gets a Maurer-Cartan element in \(\wedge^T_M\).

Reversely, if \(\gamma\) is a Maurer-Cartan element in \(\wedge^T_M\), then decomposing the sum \(d + \gamma\) into a collection of its Taylor series components as in Sect. 2.5, one gets a representation \(\phi\). \(\square\)

Let \(\wedge^T_M = \sum_{i \geq 1} \wedge^i_i T_M[1 - i]\) be a sheaf of Lie subalgebras of \(\wedge^T_M\) consisting of those elements which vanish at the distinguished point \(\star \in M\), have no \(\wedge^0 T_M[2]\)-component, and whose \(\wedge^1 T_M[1]\)-component is at least quadratic in the coordinates \(\{t^\alpha\}\). The associated dg free prop, \(P(\wedge^T_M)\), is generated by \((m, n)\)-corollas with \(m, n \geq 1\), \(m + n \geq 1\), and has a surprisingly small cohomology, a fact which is of key importance for our proof of the deformation quantization theorem.

2.6.2. Theorem. The cohomology of \((P(\wedge^T_M), \delta)\) is equal to a quadratic prop, \(\text{Lie}^e \mathcal{B}\), which is a quotient,

\[
\text{Lie}^e \mathcal{B} = \frac{P(A)}{\text{Ideal} < R >},
\]

of the free prop generated by the following \(S\)-bimodule \(A\).
all $A(m, n)$ vanish except $A(2, 1)$ and $A(1, 2)$,

- $A(2, 1) := \text{sgn}_2 \otimes 1_1 = \text{span} \left( \begin{array}{c} 1 \\ 2 \\ \end{array} \right) = - \left( \begin{array}{c} 2 \\ 1 \\ \end{array} \right)$
- $A(1, 2) := 1_1 \otimes 1_2[-1] = \text{span} \left( \begin{array}{c} 1 \\ 2 \\ 1 \\ \end{array} \right)$

modulo the ideal generated by the following relations, $R$,

\begin{align*}
R_1 : & \quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\end{array}
\begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow \\
\end{array}
\begin{array}{c}
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\end{array}
\end{array}

\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\end{array}
\begin{array}{c}
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\end{array}
\end{array}

\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\end{array}
\begin{array}{c}
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\end{array}
\end{array}
\end{align*}

$\in \mathcal{P}(A)(3, 1)$

\begin{align*}
R_2 : & \quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\begin{array}{c}
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\end{array}
\end{array}

\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\begin{array}{c}
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\end{array}
\end{array}

\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\begin{array}{c}
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\end{array}
\end{array}
\end{align*}

$\in \mathcal{P}(A)(1, 3)$

\begin{align*}
R_3 : & \quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\end{array}
\begin{array}{c}
\Rightarrow \\
\Rightarrow \\
\end{array}
\end{array}

\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\end{array}
\begin{array}{c}
\Rightarrow \\
\Rightarrow \\
\end{array}
\end{array}

\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\end{array}
\begin{array}{c}
\Rightarrow \\
\Rightarrow \\
\end{array}
\end{array}
\end{align*}

$\in \mathcal{P}(A)(2, 2)$.

**Proof.** The cohomology of $(\mathcal{P}(T), \delta)$ can not be computed directly. At the dioperadic level the theorem was established in [Me1]. That this result extends to the level of props can be easily shown using either ideas of perturbations of 1/2props and path filtrations developed in [Ko3, MaVo] or the idea of Koszul duality for props developed in [Va]. One can argue, for example, as follows: for any $f \in \mathcal{P}(T)$, define the natural number,

$$|f| := \text{number of directed paths in the graph } f$$

which connect input legs with output ones.

and notice that the differential $\delta$ preserves the filtration,

$$F_p \mathcal{P}(T) := \{\text{span } f \in \mathcal{P}(T) : |f| \leq p\}.$$

The associated spectral sequence, $\{E_r \mathcal{P}(T), \delta_r \}_{r \geq 0}$, is exhaustive and bounded below so that it converges to the cohomology of $(\mathcal{P}(T), \delta)$.

By Koszulness of the operad $\text{Lie}$ and exactness of the functor $\Omega^1_{\mathcal{P} \to \mathcal{P}}$, the zeroth term of the spectral sequence, $(E_0 \mathcal{P}(T), \delta_0)$, is precisely the minimal resolution of a quadratic prop, $\text{Lie} B'$, generated by the $S$-bimodule $A$ modulo the ideal generated by relations $R_1$, $R_2$ and the following one,

$$R'_3 : \quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\end{array}
\begin{array}{c}
\Rightarrow \\
\Rightarrow \\
\end{array}
\end{array}
\end{align*}

$= 0$.

As the differential $\delta$ vanishes on the generators of $A$, this spectral sequence degenerates at the first term, $(E_1 \mathcal{P}(T), d_1 = 0)$, implying the isomorphism

$$\bigoplus_{p \geq 1} F_p H(\mathcal{P}(T), \delta) \cong \text{Lie} B'.$$
There is a natural surjective morphism of dg props, \( p : (P⟨∧₀T⟩, δ) \to \text{Lie}B \). Define the dg prop \((X, δ)\) via an exact sequence,

\[
0 \to (X, δ) \to (P⟨∧₀T⟩, δ) \to (\text{Lie}B, 0) \to 0.
\]

The filtration on \((P⟨∧₀T⟩, δ)\) induces filtrations on sub- and quotient complexes,

\[
0 \to (F_pX, δ) \to (F_pP⟨∧₀T⟩, δ) \to (F_p\text{Lie}B, 0) \to 0,
\]

and hence an exact sequence of 0th terms of the associated spectral sequences,

\[
0 \to (E_0X, δ) \to (E_0P⟨∧₀T⟩, δ) \to (\bigoplus_{p \geq 1} F_{p+1}\text{Lie}B, 0) \to 0.
\]

By the above observation,

\[
E_1P⟨∧₀T⟩ = \bigoplus_{p \geq 1} \frac{F_{p+1}H(P⟨∧₀T⟩, δ)}{F_pH(P⟨∧₀T⟩, δ)} = \text{Lie}B'.
\]

On the other hand, it is not hard to check that

\[
\bigoplus_{p \geq 1} \frac{F_{p+1}\text{Lie}B}{F_p\text{Lie}B} = \text{Lie}B'.
\]

Thus the map \( p_0 \) is a quasi-isomorphism implying vanishing of \( E_1X \) and hence acyclicity of \((X, δ)\). Thus the projection map \( p \) is a quasi-isomorphism.

### 2.6.3. Corollary

The dg prop \((P⟨∧₀T⟩, δ)\) is a minimal model of the prop \(\text{Lie}B\): the natural morphism of dg props,

\[
p : (P⟨∧₀T⟩, δ) \to (\text{Lie}B, \text{vanishing differential}).
\]

which sends to zero all generators of \(P⟨∧₀T⟩\) except those in \(A(2,1)\) and \(A(1,2)\), is a quasi-isomorphism. Hence we can and shall re-denote \(P⟨∧₀T⟩\) as \(\text{Lie}B∞\).

### 2.6.4. Definition

A representation of the dg prop \(\text{Lie}B∞\) in a dg space \(V\) is called a Poisson structure on the formal graded manifold \(V\).

Thus Poisson structure on \(V\) is the same as a Maurer-Cartan element, \(γ ∈ ∧^*TV\), in the Lie algebra of formal polyvector fields on \(V\) satisfying the conditions. \([γ, γ] = 0\), and \(γ|_x=0 = 0\).

### 2.6.5. Remark

The condition \(γ|_x=0 = 0\) above is no serious restriction: given an arbitrary Poisson structure \(π\) on \(V\) (not necessary vanishing at \(0 ∈ V\)), then, for any parameter \(h\) viewed as a coordinate on \(K\), the product \(γ := hπ\) is a Poisson structure on \(V = V × K\) vanishing at zero \(0 ∈ V\) and hence is a representation of the prop \(\text{Lie}B∞\).
2.7. Prop profile of “star products”. Let us consider a sheaf of dg Lie algebras,

\[ D_M \subset \bigoplus_{k \geq 0} \text{Hom}(O_M^k, O_M)[1-k], \]

consisting of polydifferential operators on \( O_M \) which, for \( k \geq 1 \), vanish on every element \( f_1 \otimes \cdots \otimes f_k \in O_M^k \) with at least one function, \( f_i, i = 1, \ldots, k \), constant. A degree one section, \( \Gamma \in D_M \), decomposes into a sum, \( \sum_{k \geq 0} \Gamma_k \), with \( \Gamma_k \in \text{Hom}_{2-k}(O_M^k, O_M) \). In a local coordinate system, \( (t^\alpha, \partial/\partial t^\alpha \simeq e_\alpha) \) on \( M \), \( \Gamma \) can be represented as a Taylor series,

\[ \Gamma = \sum_{k \geq 0} \sum_{I_1, \ldots, I_k} \Gamma_{I_1, \ldots, I_k} e_{I_1} \otimes \cdots \otimes e_{I_k} \otimes t^J, \]

where for each fixed \( k \) and \( |J| \) only a finite number of coefficients \( \Gamma_{I_1, \ldots, I_k} \) is non-zero. The summation runs over multi-indices, \( I := \alpha_1 \alpha_2 \ldots \alpha_{|I|} \), and \( e_I := e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_{|I|}} \), \( t^I := t^{\alpha_1} \otimes \cdots \otimes t^{\alpha_{|I|}} \). Hence the associated \( S \)-bimodule \( E_D \) is given by

\[ E_D(m, n) = E(m) \otimes 1_n, \quad m, n \geq 0, \]

where

\[ E(0) := \mathbb{R}[-2], \quad E(m \geq 1) := \bigoplus_{k \geq 1} \bigoplus_{[m] = I_1 \sqcup \cdots \sqcup I_k} \text{Ind}_{S_{|I_1|} \times \cdots \times S_{|I_k|}}^{S_m} 1_{|I_1|} \otimes \cdots \otimes 1_{|I_k|}[k-2] \]

Let \( \text{Def}Q \) stand for the free prop, \( P\langle E_D \rangle \), generated by the above bimodule. The generators of \( \text{Def}Q \) can be identified with directed planar corollas of the form,

\[ \Gamma_{I_1, \ldots, I_k} \]

where

- the input legs are labeled by the set \( [n] := \{1, 2, \ldots, n\} \) and are symmetric (so that it does not matter how labels from \( [n] \) are distributed over them),

- the output legs (if there are any) are labeled by the set \( [m] \) partitioned into \( k \) disjoint non-empty subsets,

\[ [m] = I_1 \sqcup \cdots \sqcup I_i \sqcup I_{i+1} \sqcup \cdots \sqcup I_k, \]

and legs in each \( I_i \)-bunch are symmetric (so that it does not matter how labels from the set \( I_i \) are distributed over legs in \( I_i \)-th bunch).

The \( \mathbb{Z} \)-grading in \( \text{Def}Q \) is defined by associating degree \( 2-k \) to such a corolla. The formula \( (**) \) in Sect. 2.5 provides us with the following explicit expression for the differential, \( \delta \).
where the first sum comes from the Hochschild differential $d_H$ and the second sum comes from the Hochschild brackets $[,]_H$. The $s$-summation in the latter runs over the number, $s$, of edges connecting the two internal vertices. As $s$ can be zero, the r.h.s. above contains disconnected graphs (more precisely, disjoint unions of two corollas).

2.7.1. Proposition. There is a one-to-one correspondence between representations,

$$
\phi : (\text{DefQ}, \delta) \longrightarrow (\text{End}(M), d),
$$

of $(\text{DefQ}, \delta)$ in a dg vector space $(M, d)$ and Maurer-Cartan elements, $\gamma$, in $D_M$, that is, degree one elements satisfying the equation, $d_H \gamma + \frac{1}{2} [\gamma, \gamma]_H = 0$.

Proof is similar to the proof of Proposition 2.6.1.

2.8. Remark. Kontsevich’s formality map [Ko2] can be interpreted as a morphism of dg props,

$$
F_\infty : (\text{DefQ}, \delta) \longrightarrow (\mathcal{P}(\wedge^\bullet T)\hat{\otimes}, \delta).
$$

Vice versa, any morphism of the above dg props gives rise to a universal formality map in the sense of [Ko2]. Note that the dg prop of polyvector fields, $\mathcal{P}(\wedge^\bullet T)$, appears above in the wheel extended form, $\mathcal{P}(\wedge^\bullet T)\hat{\otimes}$. This is not accidental: we shall show below in §5 (by quantizing a pair consisting of a linear Poisson structure and quadratic homological vector field) that there does not exist a morphism between ordinary (i.e. unwheeled) dg props, $(\text{DefQ}, \delta) \longrightarrow (\mathcal{P}(\wedge^\bullet T), \delta)$, satisfying the quasi-classical limit condition.

2.9. Prop profile of perturbative “star products”. Let $\mathbb{K}[[\hbar]]$ be the formal power series in a formal parameter $\hbar$, and let $h\mathbb{K}[[\hbar]]$ be its (maximal) ideal spanned by series which vanish at $\hbar = 0$. Then $D^h_V := D_V \otimes h\mathbb{K}[[\hbar]]$ is a dg Lie algebra of polydifferential operators on $\mathcal{O}_V[[\hbar]]$ which vanish at $\hbar = 0$. A solution, $\Gamma$, of Maurer-Cartan equations in
$\mathcal{D}_V^h$ gives a (generalized) $A_\infty$-structure on $\mathcal{O}_V[[\hbar]]$ which at $\hbar = 0$ reduces to the ordinary graded commutative multiplication in $\mathcal{O}_V$. Thus Maurer-Cartan elements in this Lie algebra describe perturbative deformations of the ordinary product in $\mathcal{O}_V$. Again, the set of all possible Maurer-Cartan elements in $\mathcal{D}_V^h$ can be understood as the set of all possible representations of a certain dg free prop, $\text{DefQ}^h := \mathcal{P}(\mathcal{D}^h)$, defined as follows:

- the $\mathcal{S}$-bimodule of generators, $E_{\mathcal{P}^h} = \{E_{\mathcal{P}^h}(m, n)\}$, is a direct sum, $E_{\mathcal{P}^h}(m, n) = \bigoplus_{a=1}^\infty E^a_D(m, n)$, $m, n \geq 0$, of labeled by $a \in \mathbb{N}^*$ copies, $E^a_D(m, n) := E_D(m, n)$, of the $\mathcal{S}$-module corresponding to the sheaf $\mathcal{D}_V$; each copy corresponds to the $a^{\text{th}}$ coefficient in the formal power series;

- generators of $\text{DefQ}^h$ can be identified with planar corollas, which are exactly the same as in §2.7 except that now the vertex gets a numerical label $a \in \mathbb{N}^*$;

- the differential $\delta$ is given on generators by

$$
\delta \left( \begin{array}{c}
I_1 & I_2 & \ldots & I_n \\
1 & 2 & \ldots & n
\end{array} \right) = \sum_{i=1}^k (-1)^{i+1} \begin{array}{c}
I_1 & I_2 & \ldots & I_n \\
1 & 2 & \ldots & n
\end{array} + \sum_{1 < i < m} \sum_{p \geq 1, q \geq 0} \sum_{i=0}^{p-1} \sum_{j=0}^{i+q-1} \sum_{a=0}^{[n]=J_1\cup J_2} (-1)^{(p+1)q+i(q-1)} \begin{array}{c}
I_1 I_2 \ldots I_n \\
1 \ldots n
\end{array}
$$

Note that $\text{DefQ}^h$ is spanned by graphs over $\mathbb{K}$, not over $\mathbb{K}[[\hbar]]$. At the prop level the only remnant of the presence of the Plank constant $\hbar$ in the input geometry is in the decoration of vertices by a natural number $a \in \mathbb{N}^*$. In this respect the notation $\text{DefQ}^h$ could be misleading.

2.9.1. Proposition. There is a one-to-one correspondence between representations,

$$
\phi : (\text{DefQ}^h, \delta) \longrightarrow (\text{End}_{\mathbb{K}[[\hbar]]}(V[[\hbar]]), d),
$$

of $\text{DefQ}^h$ in an $\mathbb{K}[[\hbar]]$-extension of a dg vector space $(V, d)$, and star products on $\mathcal{O}_V[[\hbar]]$, i.e. Maurer-Cartan elements, $\Gamma$, in $\mathcal{D}_V[[\hbar]]$ satisfying the equation, $d_H\Gamma + \frac{1}{2}[\Gamma, \Gamma]_{H} = 0$ and the condition $\Gamma|_{\hbar=0} = d$. 

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Proof is similar to the proof of Proposition 2.6.1.

2.9.2. Remark. By constructions of \((\text{Def}Q, \delta)\) and \((\text{Def}Q^h, \delta)\), there is a canonical morphism of dg props,

\[
\chi_h : \text{Def}Q \rightarrow \text{Def}Q^h[[h]]
\]

As we assume that the prop \(\text{Def}Q^h\) is completed with respect to the number of vertices, we can set \(h = 1\) above and get a well-defined morphism of dg props, \(\chi_{h=1} : \text{Def}Q \rightarrow \text{Def}Q^h\). However, in general a morphism of dg props \(\phi : \text{Def}Q^h \rightarrow P\) can not be composed with \(\chi_{h=1}\) while \(\phi \circ \chi_h : \text{Def}Q \rightarrow P[[h]]\) is always well defined.

We shall next investigate how wheeled completion of directed graph complexes affects their cohomology.

§3. Directed graph complexes with loops and wheels

3.1. \(\mathcal{G}^\bigcirc\) versus \(\mathcal{G}^\uparrow\). One of the most effective methods for computing cohomology of dg free props (that is, decorated \(\mathcal{G}^\uparrow\)-graph complexes) is based on the idea of interpreting the differential as a perturbation of its \(\frac{1}{2}\)propic part which, in this \(\mathcal{G}^\uparrow\)-case, can often be singled out by the path filtration [Ko3, MaVo]. However, one can \(\textit{not}\) apply this idea directly to graphs with wheels — it is shown below that a filtration which singles out the \(\frac{1}{2}\)propic part of the differential does \(\textit{not}\) exist in general even for dioperadic differentials. Put another way, if one takes a \(\mathcal{G}^\uparrow\)-graph complex, \((P(E), \delta)\), enlarges it by adding decorated graphs with wheels while keeping the original differential \(\delta\) unchanged, then one ends up in a very different situation in which the idea of \(\frac{1}{2}\)props is no longer directly applicable.

3.2. Graphs with back-in-time edges. Here we suggest the following trick: we further enlarge our set of graphs with wheels, \(\mathcal{G}^\bigcirc \rightarrow \mathcal{G}^+\), by putting a mark on one (and only one) of the edges in each wheel, and then study the natural “forgetful” surjection, \(\mathcal{G}^+ \rightarrow \mathcal{G}^\bigcirc\). The point is that \(\mathcal{G}^+\)-graph complexes again admit a filtration which singles out the \(\frac{1}{2}\)prop part of the differential and hence their cohomology are often easily computable.

More precisely, let \(\mathcal{G}^+(m, n)\) be the set of all directed \((m, n)\)-graphs \(G\) which satisfy conditions 2.1.3(i)-(iv), and the following one:

(v) every oriented wheel in \(G\) (if any) has one and only one of its internal edges marked (say, dashed) and called \textit{back-in-time} edge.
For example,

\[
\begin{array}{c}
\bullet - \bullet - \bullet \\
\bigcirc - \bigcirc - \bigcirc \\
\end{array}
\]

are four different graphs in \(\mathfrak{G}^+(1, 1)\).

Clearly, we have a natural surjection,

\[
u : \mathfrak{G}^+(m, n) \to \mathfrak{G}^\bigcirc(m, n),
\]

which forgets the markings. For example, the four graphs above are mapped under \(u\) into the same graph,

\[
\begin{array}{c}
\bullet - \bullet - \bullet \\
\end{array}
\in \mathfrak{G}^\bigcirc(1, 1),
\]

and, in fact, span its pre-image under \(u\).

3.3. Graph complexes. Let \(E = \{E(m, n)\}_{m,n \geq 1, m+n \geq 3}\) be an \(\mathcal{S}\)-bimodule and let \((P\langle E \rangle, \delta)\) be a dg free prop on \(E\) with the differential \(\delta\) which preserves connectedness and genus, that is, \(\delta\) applied to any decorated \((m, n)\)-corolla creates a connected \((m, n)\)-tree. Such a differential can be called dioperadic, and from now on we restrict ourselves to dioperadic differentials only. This restriction is not that dramatic: every dg free prop with a non-dioperadic but connected differential always admits a filtration which singles out its dioperadic part [MaVo]. Thus the technique we develop here in §3 can, in principle, be applied to a wheeled extension of any dg free prop with a connected differential.

We enlarge the \(\mathfrak{G}^\bigcirc\)-graph complex \((P\langle E \rangle, \delta)\) in two ways,

\[
P^\bigcirc\langle E \rangle(m, n) := \bigoplus_{G \in \mathfrak{G}^\bigcirc(m, n)} \left( \bigotimes_{v \in v(G)} E(\text{Out}(v), \text{In}(v)) \right)_{\text{Aut}G},
\]

\[
P^+\langle E \rangle(m, n) := \bigoplus_{G \in \mathfrak{G}^+(m, n)} \left( \bigotimes_{v \in v(G)} E(\text{Out}(v), \text{In}(v)) \right)_{\text{Aut}G},
\]

and notice that both \(P^\bigcirc\langle E \rangle := \{P^\bigcirc\langle E \rangle(m, n)\}\) and \(P^+\langle E \rangle := \{P^+\langle E \rangle(m, n)\}\) have a natural prop structure with respect to disjoint unions and grafting of decorated graphs. The original differential \(\delta\) in \(P\langle E \rangle\) extends naturally to \(P^\bigcirc\langle E \rangle\) and \(P^+\langle E \rangle\) making the

\footnote{A differential \(\delta\) in \(P\langle E \rangle\) is called connected if it preserves the filtration of \(P\langle E \rangle\) by the number of connected (in the topological sense) components.}
latter into $dg$ props. However that differential preserves, in general, neither the number of wheels in $P^\otimes\langle E \rangle$ nor the number of marked edges in $P^+\langle E \rangle$. Clearly, they both contain $(P\langle E \rangle, \delta)$ as a $dg$ subprop. There is a natural morphism of $dg$ props,

$$u : (P^+\langle E \rangle, \delta) \longrightarrow (P^\otimes\langle E \rangle, \delta),$$

which forgets the markings. Let $(L^+\langle E \rangle, \delta) := \text{Ker} \ u$ and denote the natural inclusion $L^+\langle E \rangle \subset P^+\langle E \rangle$ by $i$.

### 3.4. Fact.

There is a short exact sequence of graph complexes,

$$0 \longrightarrow (L^+\langle E \rangle, \delta) \xrightarrow{i} (P^+\langle E \rangle, \delta) \xrightarrow{u} (P^\otimes\langle E \rangle, \delta) \longrightarrow 0,$$

Thus, if the natural inclusion of complexes, $i : (L^+\langle E \rangle, \delta) \rightarrow (P^+\langle E \rangle, \delta)$, induces a monomorphism in cohomology, $[i] : H(L^+\langle E \rangle, \delta) \rightarrow H(P^+\langle E \rangle, \delta)$, then

$$H(P^\otimes\langle E \rangle, \delta) = \frac{H(P^+\langle E \rangle, \delta)}{H(L^+\langle E \rangle, \delta)}.$$ 

Put another way, if $[i]$ is a monomorphism, then $H(P^\otimes\langle E \rangle, \delta)$ is obtained from $H(P^+\langle E \rangle, \delta)$ simply by forgetting the markings.

### 3.5. Functors which adjoin wheels.

We are interested in this paper in dioperads, $D$, which are either free, $D\langle E \rangle$, on an $S$-bimodule $E = \{E(m, n)\}_{m,n \geq 1, m+n \geq 3}$, or are naturally represented as quotients of free dioperads, $D = \frac{D\langle E \rangle}{<I>}$, modulo the ideals generated by some relations $I \subset D\langle E \rangle$. Then the free prop, $\Omega_{D\rightarrow P}\langle E \rangle$, generated by $D$ is simply the quotient of the free prop, $P\langle E \rangle$,

$$\Omega_{D\rightarrow P}\langle D \rangle := \frac{P\langle E \rangle}{<I>},$$

by the ideal generated by the same relations $I$. Now we define two other props\textsuperscript{5},

$$\Omega_{D\rightarrow P^\otimes}\langle D \rangle := \frac{P^\otimes\langle E \rangle}{<I>^\otimes}, \quad \Omega_{D\rightarrow P^+}\langle D \rangle := \frac{P^+\langle E \rangle}{<I>^+},$$

where $<I>^\otimes$ (resp., $<I>^+$) is the subspace of those graphs $G$ in $P^\otimes\langle E \rangle$ (resp., in $P^+\langle E \rangle$) which satisfy the following condition: there exists a (possibly empty) set of cyclic edges whose breaking up into two legs produces a graph lying in the ideal $<I>$ which defines the prop $\Omega_{D\rightarrow P}\langle D \rangle$.

Analogously one defines functors $\Omega_{P\rightarrow P^\otimes}$ and $\Omega_{P\rightarrow P^+}$.

\textsuperscript{5}The prop $\Omega_{D\rightarrow P^\otimes}\langle D \rangle$ is a particular example of a \textit{wheeled prop} which will be discussed in detail elsewhere.
From now on we abbreviate notations as follows,

\[ D^\uparrow := \Omega_{D \to \mathcal{P}}(D), \quad D^+ := \Omega_{D \to \mathcal{P}^+}(D), \quad D^\circ := \Omega_{D \to \mathcal{P}_0}(D), \]

for values of the above defined functors on dioperads, and, respectively

\[ D_0^\uparrow := \Omega_{\frac{1}{2}P \to \mathcal{P}}(D_0), \quad D_0^+ := \Omega_{\frac{1}{2}P \to \mathcal{P}^+}(D_0), \quad D_0^\circ := \Omega_{\frac{1}{2}P \to \mathcal{P}_0}(D_0). \]

for their values on \( \frac{1}{2} \) props.

3.5.1. Facts. (i) If \( D \) is a dg dioperad, then both \( D^\circ \) and \( D^+ \) are naturally dg props. (ii) If \( D \) is an operad, then both \( D^\circ \) and \( D^+ \) may contain at most one wheel.

3.5.2. Proposition. Any finite-dimensional representation of the dioperad \( D \) lifts to a representation of its wheeled prop extension \( D^\circ \).

Proof. If \( \phi : D \to \text{End}(M) \) is a representation, then we first extend it to a representation, \( \phi^\circ \), of \( \mathcal{P}^\circ(E) \) as in Sect. 2.3.2 and then notice that \( \phi^\circ(f) = 0 \) for any \( f \in <I>^\circ \). \( \square \)

3.5.3. Definition. Let \( D \) be a Koszul dioperad with \( (D_\infty, \delta) \to (D, 0) \) being its minimal resolution. The dioperad \( D \) is called stably Koszul if the associated morphism of the wheeled completions,

\[ (D_\infty^\circ, \delta) \longrightarrow (D^\circ, 0) \]

remains a quasi-isomorphism.

3.5.4. Example. The notion of stable Koszulness is non-trivial. Just adding oriented wheels to a minimal resolution of a Koszul operad while keeping the differential unchanged may alter the cohomology group of the resulting graph complex as the following example shows.

Claim. The operad \( \text{Ass} \), of associative algebras is not stably Koszul.

Proof. The operad \( \text{Ass} \) can be represented a quotient,

\[ \text{Ass} = \frac{\text{Oper}(E)}{\text{Ideal} < R >}, \]

of the free operad, \( \text{Oper}(E) \), generated by the following \( S \)-module \( E \),

\[ E(n) := \begin{cases} k[S_2] = \text{span} \left( \begin{array}{c} 1 \\ 2 \\ 2 \\ 1 \end{array} \right) & \text{for } n = 2 \\ 0 & \text{otherwise,} \end{cases} \]

modulo the ideal generated by the following relations,

\[ s^{(1)} s^{(2)} - s^{(3)} s^{(1)} = 0, \quad \forall \sigma \in S_3. \]
Hence the minimal resolution, \((\text{Ass}_\infty, \delta)\) of \text{Ass} contains a degree -1 corolla \(\frac{1}{3} \frac{2}{3}\) such that
\[
\delta \frac{1}{3} \frac{2}{3} = \frac{1}{3} \frac{2}{3} - \frac{1}{3} \frac{2}{3}.
\]
Therefore, in its wheeled extension, \((\text{Ass}^\wedge_\infty, \delta)\), one has
\[
\delta \frac{1}{2} = \frac{1}{2} - \frac{1}{2} = 0,
\]
implying existence of a non-trivial cohomology class \(\frac{1}{2}\) in \(H(\text{Ass}^\wedge_\infty, \delta)\) which does not belong to \(\text{Ass}^\wedge_\infty\). Thus \text{Ass} is Koszul, but not stably Koszul.

It is instructive to see explicitly how the map \([i] : H(L^+(\text{Ass}_\infty), \delta) \to H(P^+(\text{Ass}_\infty), \delta)\) fails to be a monomorphism. As \(L^+(\text{Ass}_\infty)\) does not contain loops, the element
\[
a := \frac{1}{2} - \frac{1}{2}
\]
defines a non-trivial cohomology class, \([a]\), in \(H(L^+(\text{Ass}_\infty), \delta)\), whose image, \([i][a]\), in \(H(P^+(\text{Ass}_\infty), \delta)\) vanishes.

3.6. Koszul substitution laws. Let \(P = \{P(n)\}_{n \geq 1}\) and \(Q = \{Q(n)\}_{n \geq 1}\) be two quadratic Koszul operads generated,
\[
P := \frac{\mathcal{P}\langle E_P(2) \rangle}{< I_P >}, \quad Q := \frac{\mathcal{P}\langle E_Q(2) \rangle}{< I_Q >},
\]
by \(\mathbb{S}_2\)-modules \(E_P(2)\), and, respectively, \(E_Q(2)\).

One can canonically associate \([\text{MaVo}]\) to such a pair the \(\frac{1}{2}\)prop, \(P \circ Q^\dagger\), with
\[
P \circ Q^\dagger(m, n) = \begin{cases} P(n) & \text{for } m = 1, n \geq 2, \\ Q(m) & \text{for } n = 1, m \geq 2, \\ 0 & \text{otherwise}, \end{cases}
\]

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and the $\frac{1}{2}$-prop compositions,

\[
\{ 1 \circ_j : P \circ Q^\dagger(m_1, 1) \otimes P \circ Q^\dagger(m_2, n_2) \to P \circ Q^\dagger(m_1 + m_2 - 1, n_2) \}_{1 \leq j \leq m_2}
\]

being zero for $n_2 \geq 2$ and coinciding with the operadic composition in $Q$ for $n_2 = 1$, and

\[
\{ i \circ_1 : P \circ Q^\dagger(m_1, n_1) \otimes P \circ Q^\dagger(1, n_2) \to P \circ Q^\dagger(m_1 + m_2 - 1, n_2) \}_{1 \leq i \leq n_1}
\]

being zero for $m_1 \geq 2$ and otherwise coinciding with the operadic composition in $P$ for $m_1 = 1$.

Let $D_0 = \Omega_{P \to P} (P \circ Q^\dagger)$ be the associated free dioperad, $D_0^!$ its Koszul dual dioperad, and $(D_{0\infty} := D D_0^0, \delta_0)$ the associated cobar construction [Ga]. As $D_0$ is Koszul [Ga, MaVo], the latter provides us with the dioperadic minimal model of $D_0$. By exactness of $\Omega_{P \to P}$, the dg free prop, $(D_{0\infty}^\dagger := \Omega_{D \to P} (D_{0\infty}), \delta_0)$, is the minimal model of the prop $D_0^\dagger := \Omega_{D \to P} (D_0) \simeq \Omega_{P \to P} (P \circ Q^\dagger)$.

3.6.1. Remark. The prop $D_0^\dagger$ can be equivalently defined as the quotient,

\[
\frac{P \ast Q^\dagger}{I_0}
\]

where $P \ast Q^\dagger$ is the free product of props associated to operads $P$ and $Q^\dagger$, and the ideal $I_0$ is generated by graphs of the form,

\[
I_0 = \text{span} \left\langle \begin{array}{c}
\uparrow \\
\downarrow
\end{array} \right\rangle \simeq D_0(2, 1) \otimes D_0(1, 2) = E_Q(2) \otimes E_P(2)
\]

with white vertex decorated by elements of $E_Q(2)$ and black vertex decorated by elements of $E_P(2)$.

Let us consider a morphism of $S_2$-bimodules,

\[
\lambda : \begin{array}{c}
D_0(2, 1) \otimes D_0(1, 2) \\
\text{span} \left\langle \begin{array}{c}
\uparrow \\
\downarrow
\end{array} \right\rangle
\end{array} \to \begin{array}{c}
D_0(2, 2) \\
\text{span} \left\langle \begin{array}{c}
\uparrow \\
\downarrow
\end{array}, \begin{array}{c}
\uparrow \\
\downarrow
\end{array} \right\rangle
\end{array}
\]

and define [MaVo] the dioperad, $D_\lambda$, as the quotient of the free dioperad generated by the two spaces of binary operations, $D_0(2, 1) = E_Q(2)$ and $D_0(1, 2) = E_P(2)$, modulo the ideal generated by relations in $P$, relations in $Q$ as well as the followings ones,

\[
I_\lambda = \text{span} \left\{ f - \lambda f : \forall f \in D_0(2, 1) \otimes D_0(1, 2) \right\}.
\]

Note that in notations of § 3.6.1 the associated prop, $D_\lambda^\dagger := \Omega_{D \to P} (D_\lambda)$, is just the quotient, $P \ast Q^\dagger / I_\lambda$.  

---

6the symbol $^\dagger$ stands for the functor on props, $P = \{P(m, n)\} \to P^\dagger = \{P^\dagger(m, n)\}$ which reverses “time flow”, i.e. $P^\dagger(m, n) := P(n, m)$.  

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The substitution law $\lambda$ is called Koszul, if $D_\lambda$ is isomorphic to $D_0$ as an $S$-bimodule. Which implies that $D_\lambda$ is Koszul [Ga]. Koszul duality technique provides the space $DD_0^\lambda \simeq DD_\lambda^0$ with a perturbed differential $\delta_\lambda$ such that $(DD_0^\lambda, \delta_\lambda)$ is the minimal model, $(D_{\lambda\infty}, \delta_\lambda)$, of the dioperad $D_\lambda$.

3.7. Theorem [MaVo, Va]. The dg free prop $D_{\lambda\infty}^\dagger := \Omega_{D\to P}(D_{\lambda\infty})$ is the minimal model of the prop $D_{\lambda\infty}^\dagger$, i.e. the natural morphism

$$(D_{\lambda\infty}^\dagger, \delta_\lambda) \longrightarrow (D_{\lambda\dagger}^\dagger, 0),$$

which sends to zero all vertices of $D_{\lambda\infty}^\dagger$ except binary ones decorated by elements of $E_P(2)$ and $E_Q(2)$, is a quasi-isomorphism.

Proof. The main point is that

$F_p := \left\{ \text{span}(f \in D_{\lambda\infty}^\dagger) : \begin{array}{l} \text{number of directed paths in the graph } f \\ \text{which connect input legs with output ones} \end{array} \leq p \right\}.$

defines a filtration of the complex $D_{\lambda\infty}^\dagger$. The associated spectral sequence, $\{E_r, d_r\}_{r \geq 0}$, is exhaustive and bounded below so that it converges to the cohomology of $(D_{\lambda\infty}^\dagger, \delta_\lambda)$.

The zeroth term of this spectral sequence is isomorphic to $(D_{0\infty}^\dagger, \delta_0)$ and hence, by Koszulness of the dioperad $D_0$ and exactness of the functor $\Omega_{\frac{1}{2}P\to P}$, has the cohomology, $E_1$, isomorphic to $D_0^\dagger$. Which, by Koszulness of $D_\lambda$, is isomorphic to $D_{\lambda\dagger}$ as an $S$-bimodule. Hence $\{d_r = 0\}_{r \geq 1}$ and the result follows along the same lines as in the second part of the proof of Theorem 2.6.2. 

3.8. Cohomology of graph complexes with marked wheels. In this section we analyze the functor $\Omega_{\frac{1}{2}P\to P}$. The following statement is one of the motivations for its introduction (it does not hold true for the “unmarked” version $\Omega_{\frac{1}{2}P\to P^0}$).

3.8.1. Theorem. The functor $\Omega_{\frac{1}{2}P\to P}$ is exact.

Proof. Let $T$ be an arbitrary dg $\frac{1}{2}$prop. The main point is that we can use $\frac{1}{2}$prop compositions and presence of marks on cyclic edges to represent $\Omega_{\frac{1}{2}P\to P}(T)$ as a vector space freely generated by a family of decorated graphs,

$$\Omega_{\frac{1}{2}P\to P}(T)(m, n) = \bigoplus_{G \in \mathcal{G}(m, n)} G\langle P \rangle,$$

where $\mathcal{G}(m, n)$ is a subset of $\mathcal{G}(m, n)$ consisting of so called reduced graphs, $G$, which satisfy the following defining property: for each pair of internal vertices, $(v_1, v_2)$, of $G$ which are connected by an unmarked edge directed from $v_1$ to $v_2$ one has $|Out(v_1)| \geq 2$ and $|In(v_2)| \geq 2$. Put another way, given an arbitrary $T$-decorated graph with wheels, one can perform $\frac{1}{2}$prop compositions (“contractions”) along all unmarked internal edges $(v_1, v_2)$ which do not satisfy the above conditions. The result is a reduced decorated graph.
(with wheels). Which is uniquely defined by the original one. Notice that marks are vital for this contraction procedure, e.g.

\[
\begin{array}{ccc}
\begin{array}{c}
\text{[Diagram]}
\end{array}
& \rightarrow &
\begin{array}{c}
\text{[Diagram]}
\end{array}
\end{array}
\]

to be well-defined.

Then we have

\[
H^* \left( \Omega_{\frac{1}{2}p \to p+} (T) (m, n) \right) = H^* \left( \bigoplus_{G \in \mathcal{G}^+(m, n)} \left( \bigotimes_{v \in v(G)} T(Out(v), In(v)) \right) \right)
\]

\[
= \bigoplus_{G \in \mathcal{G}^+(m, n)} H^* \left( \bigotimes_{v \in v(G)} T(Out(v), In(v)) \right) \text{ by Maschke's theorem}
\]

\[
= \bigoplus_{G \in \mathcal{G}^+(m, n)} \left( \bigotimes_{v \in v(G)} H^* (T) (Out(v), In(v)) \right) \text{ by K"unneth formula}
\]

\[
= \Omega_{\frac{1}{2}p \to p+} \langle H^* (T) \rangle (m, n).
\]

In the second line we used the fact that the group $AutG$ is finite. \qed

Another motivation for introducing graph complexes with marked wheels is that they admit a filtration which singles out the $\frac{1}{2}$propic part of the differential. A fact which we heavily use in the proof of the following

3.8.2. Theorem. Let $D_\lambda$ be a Koszul dioperad with Koszul substitution law and let $(D_{\lambda \infty}, \delta)$ be its minimal resolution. The natural morphism of graph complexes,

\[
(D_{\lambda \infty}^+, \delta_\lambda) \to (D_\lambda^+, 0)
\]

is a quasi-isomorphism.

Proof. Consider first a filtration of the complex $(D_{\lambda \infty}^+, \delta_\lambda)$ by the number of marked edges, and let $(D_{\lambda \infty}^+, b)$ denote 0th term of the associated spectral sequence (which, as we shall show below, degenerates at the 1st term).

To any decorated graph $f \in D_{\lambda \infty}^+$ one can associate a graph without wheels, $\overline{f} \in D_{\lambda \infty}^+$, by breaking every marked cyclic edge into two legs (one of which is input and another one is output). Let $|\overline{f}|$ be the number of directed paths in the graph $\overline{f}$ which connect input legs with output ones. Then

\[
F_p := \{ f \in D_{\lambda \infty}^+ : |\overline{f}| \leq p \}.
\]
defines a filtration of the complex \((D^+_{\lambda\infty}, b)\).

The zeroth term of the spectral sequence, \(\{E_r, d_r\}_{r \geq 0}\), associated to this filtration is isomorphic to \((D^+_{0\infty}, \delta_0)\) and hence, by Theorem 3.8.1, has the cohomology, \(E_1\), equal to \(D^+_{0}\). Which, by Koszulness of \(D_{\lambda}\), is isomorphic as a vector space to \(D^+_{\lambda}\). Hence the differentials of all higher terms of both our spectral sequences vanish, and the result follows.

\[\square\]

3.8.3. Remark. In the proof of Theorem 3.8.2 the \(\frac{1}{2}\)propic part, \(\delta_0\), of the differential \(\delta_{\lambda}\) was singled out in two steps: first we introduced a filtration by the number of marked edges, and then a filtration by the number of paths, \(|\tilde{f}|\), in the unwheeled graphs \(\tilde{f}\). As the following lemma shows, one can do it in one step. Let \(w(f)\) stand for the number of marked edges in a decorated graph \(f \in D^+_{\lambda\infty}\).

3.8.4. Lemma The sequence of vector spaces \(F_p := \{\text{span}\langle f \in D^+_{\lambda\infty} : ||f|| := 3w(f)|\tilde{f}| \leq p \}\}\), defines a filtration of the complex \((D^+_{\lambda\infty}, \delta_{\lambda})\) whose spectral sequence has 0-th term isomorphic to \((D^+_{0\infty}, \delta_0)\).

Proof. It is enough to show that for any graph \(f\) in \(D^+_{\lambda\infty}\) with \(w(f) \neq 0\) one has, \(||\delta_{\lambda} f|| \leq ||f||\).

We can, in general, split \(\delta_{\lambda} f\) into two groups of summands,

\[\delta_{\lambda} f = \sum_{a \in I_1} g_a + \sum_{b \in I_2} g_b\]

where \(w(g_a) = w(f), \forall a \in I_1\), and \(w(g_b) = w(f) - p_b\) for some \(p_b \geq 1\) and all \(b \in I_2\).

For any \(a \in I_1\),

\[||g_a|| = 3w(f)|g_a| \leq 3w(f)|\tilde{f}| = ||f||.\]

So it remains to check the inequality \(||g_b|| \leq ||f||, \forall b \in I_2\).

There is an associated splitting of \(\delta_{\lambda} \tilde{f}\) into two groups of summands,

\[\delta_{\lambda} \tilde{f} = \sum_{a \in I_1} h_a + \sum_{b \in I_2} h_b\]

where \(\{h_b\}_{b \in I_2}\) is the set of all those summands which contain two-vertex subgraphs of the form,

having half-edges of the type \(x\) and \(y\) corresponding to broken wheeled paths in \(f\). Every graph \(g_b\) is obtained from the corresponding \(h_b\) by gluing some number of path connected
to y output legs with the same number of path connected to x input legs into new internal non-cyclic edges. This gluing operation creates \( p_b \) new paths connecting some internal vertices in \( h_b \), and hence may increase the total number of paths in \( h_b \) but no more than by the factor of \( p_b + 1 \), i.e. 

\[ |\mathcal{G}_b| \leq (p_b + 1)|h_b|, \quad \forall b \in I_2. \]

Finally, we have

\[ ||g_b|| = 3^{w(f) - p_b}|\mathcal{G}_b| \leq 3^{w(f) - p_b}(p_b + 1)|h_b| < 3^{w(f)}|\mathcal{F}| = ||f||, \quad \forall b \in I_2. \]

The part of the differential \( \delta_\lambda \) which preserves the filtration must in fact preserve both the number of marked edges, \( w(f) \), and the number of paths, \( |\mathcal{F}| \), for any decorated graph \( f \). Hence this is precisely \( \delta_0 \). \( \square \)

3.9. Graph complexes with unmarked wheels built on \( \frac{1}{2} \)props.

Let \( (T = \frac{1}{2}\mathcal{P}(E)/<I>, \delta) \) be a dg \( \frac{1}{2} \)prop. In §3.5 we defined its wheeled extension, 

\[ (T^\otimes := \mathcal{P}^\otimes(E)_{<I>^\otimes}, \delta) \]

Now we specify a dg subprop, \( \Omega_{\text{no-oper}}(T) \subset T^\otimes \), whose cohomology is easy to compute.

3.9.1. Definition. Let \( E = \{E(m, n)\}_{m,n \geq 1, m+n \geq 3} \) be an S-bimodule, and \( \mathcal{P}^\otimes(E) \) the associated prop of decorated graphs with wheels. We say that a wheel \( W \) in a graph \( G \in \mathcal{P}^\otimes(E) \) is operadic if all its cyclic vertices \( v \in W \) are decorated either by elements of \( \{E(1, n_v)\}_{n_v \geq 2} \) only, or by elements \( E(n_v, 1)_{n_v \geq 2} \) only. Vertices of operadic wheels are called operadic cyclic vertices. Notice that operadic wheels can be of geometric genus 1 only.

Let \( \mathcal{P}^\otimes_{\text{no-oper}}(E) \) be the subspace of \( \mathcal{P}^\otimes(E) \) consisting of graphs with no operadic wheels, and let 

\[ \Omega_{\text{no-oper}}(T) = \frac{\mathcal{P}^\otimes_{\text{no-oper}}(E)}{<I>^\otimes}, \]

be the associated dg sub-prop of \( (T^\otimes, \delta) \).

Clearly, \( \Omega_{\text{no-oper}} \) is a functor from the category of dg \( \frac{1}{2} \)props to the category of dg props. It is worth pointing out that this functor can not be extended to dg dioperads as differential can, in general, create operadic wheels from non-operadic ones.

3.9.2. Theorem. The functor \( \Omega_{\text{no-oper}} \) is exact.

Proof. Let \( (T, \delta) \) be an arbitrary dg \( \frac{1}{2} \)prop. Every wheel in \( \Omega_{\text{no-oper}}(T) \) contains at least one cyclic edge along which \( \frac{1}{2} \)prop composition in \( T \) is not possible. This fact allows one to non-ambiguously perform such compositions along all those cyclic and non-cyclic edges at which such a composition makes sense, and hence represent \( \Omega_{\text{no-oper}}(T) \) as a vector space freely generated by a family of decorated graphs,

\[ \Omega_{\text{no-oper}}(T)(m, n) = \bigoplus_{G \in \mathcal{G}^\otimes(m,n)} G(T) \]

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where $\mathbf{G}^\subset(m,n)$ is a subset of $\mathbf{G}^\subset(m,n)$ consisting of reduced graphs, $G$, which satisfy the following defining properties: (i) for each pair of internal vertices, $(v_1, v_2)$, of $G$ which are connected by an edge directed from $v_1$ to $v_2$ one has $|\text{Out}(v_1)| \geq 2$ and $|\text{In}(v_2)| \geq 2$; (ii) there are no operadic wheels in $G$. The rest of the proof is exactly the same as in §3.8.1. □

Let $P$ and $Q$ be Koszul operads and let $D_0$ be the associated Koszul dioperad (defined in §3.6) whose minimal resolution is denoted by $(D_0, \delta_0)$.

3.9.3. Corollary. $H(\Omega_{\text{no-oper}}\langle D_0, \delta_0 \rangle) = \Omega_{\frac{2}{2}P \rightarrow P} \langle D_0, \delta_0 \rangle$.

Proof. By Theorem 3.9.2,

$$H(\Omega_{\text{no-oper}}\langle D_0, \delta_0 \rangle) = \Omega_{\text{no-oper}}\langle H(D_0, \delta_0) \rangle = \Omega_{\text{no-oper}}\langle D_0 \rangle.$$ 

But the latter space cannot have graphs with wheels as any such a wheel would contain at least one “non-reduced” internal cyclic edge corresponding to composition,

$$\circ_{1,1} : D_0(m,1) \otimes D_0(1,n) \longrightarrow D_0(m,n),$$

which is zero by the definition of $D_0$ (see §3.6). □

3.10. Theorem. For any Koszul operads $P$ and $Q$,

(i) the natural morphism of graph complexes,

$$(D_0^\subset, \delta_0) \longrightarrow (D_0^\subset, 0)$$

is a quasi-isomorphism if and only if the operads $P$ and $Q$ are stably Koszul;

(ii) there is, in general, an isomorphism of $S$-bimodules,

$$H(D_0^\subset, \delta_0) = \frac{H(P_\infty^\subset) \ast H(Q_\infty^\subset)}{I_0}$$

where $H(P_\infty^\subset)$ and $H(Q_\infty^\subset)$ are cohomologies of the wheeled completions of the minimal resolutions of the operads $P$ and $Q$, $\ast$ stands for the free product of PROPs, and the ideal $I_0$ is defined in §3.6.1.

Proof. (i) The necessity of the condition is obvious. Let us prove its sufficiency.

Let $P$ and $Q$ be stably Koszul operads so that the natural morphisms,

$$(P_\infty^\subset, \delta_P) \rightarrow P^\subset \quad \text{and} \quad (Q_\infty^\subset, \delta_Q) \rightarrow Q^\subset,$$

are quasi-isomorphisms, where $(P_\infty, \delta_P)$ and $(Q_\infty, \delta_Q)$ are minimal resolutions of $P$ and $Q$ respectively.

Consider a filtration of the complex $(D_0^\subset, \delta_0)$,

$$F_p := \langle \text{span}(f \in D_0^\subset) : |f|_2 - |f|_1 \leq p \rangle,$$

where
- $|f|_1$ is the number of cyclic vertices in $f$ which belong to operadic wheels;
- $|f|_2$ is the number of non-cyclic half-edges attached to cyclic vertices in $f$ which belong to operadic wheels.

Note that $|f|_2 - |f|_1 \geq 0$. Let $\{E_r, d_r\}_{r \geq 0}$ be the associated spectral sequence. The differential $d_0$ in $E_0$ is given by its values on the vertices as follows:

(a) on every non-cyclic vertex and on every cyclic vertex which does not belong to an operadic wheel one has $d_0 = \delta_0$;

(b) on every cyclic vertex which belongs to an operadic wheel one has $d_0 = 0$.

Hence modulo the action of finite groups (which we can ignore by Maschke theorem) the complex $(E_0, d_0)$ is isomorphic to the complex $(\Omega_{\text{no-oper}}(D_{0\infty}), \delta_0)$, tensored with a trivial complex (i.e. one with vanishing differential). By Corollary 3.9.3 and Künneth formula we obtain,

$$E_1 = H(E_0, d_0) = W_1/h(W_2)$$

where

- $W_1$ is the subspace of $P^\circ (E_P \oplus E_Q^\dagger)$ consisting of graphs whose wheels (if any) are operadic; here the $\mathbb{S}$-bimodule $E_P \oplus E_Q^\dagger$ is given by

$$E_P(2), \text{the space of generators of } P, \quad \text{if } m = 1, n = 2$$

$$E_Q(2), \text{the space of generators of } Q, \quad \text{if } m = 2, n = 1$$

$$0, \quad \text{otherwise};$$

- $W_2$ is the subspace of $P^\circ (E_P \oplus E_Q^\dagger \oplus I_P \oplus I_Q^\dagger)$ consisting of graphs, $G$, whose wheels (if any) are operadic and satisfy the following condition: the elements of $I_P$ and $I_Q^\dagger$ are used to decorate at least one non-cyclic vertex in $G$. Here $I_P$ and $I_Q^\dagger$ are $\mathbb{S}$-bimodules of relations of the quadratic operads $P$ and $Q^\dagger$ respectively.

- the map $h : W_2 \rightarrow W_1$ is defined to be the identity on vertices decorated by elements of $E_P \oplus E_Q^\dagger$, and the tautological (in the obvious sense) morphism on vertices decorated by elements of $I_P$ and $I_Q^\dagger$.

To understand all the remaining terms $\{E_r, d_r\}_{r \geq 1}$ of the spectral sequence we step aside and contemplate for a moment a purely operadic graph complex with wheels, say, $(P_\infty^\circ, \delta_P)$. The complex $(P_\infty^\circ, \delta_P)$ is naturally a subcomplex of $(D_{0\infty}^\circ, \delta_0)$. Let

$$F_p := \{\text{span}(f \in P_\infty^\circ) : |f|_2 - |f|_1 \leq p\},$$

be the induced filtration, and let $\{E^P_r, d^P_r\}_{r \geq 0}$ be the associated spectral sequence. Then $E^P_1 = H(E^P_0, d^P_0)$ is a subcomplex of $E_1$.

The main point is that, modulo the action of finite groups, the spectral sequence $\{E_r, d_r\}_{r \geq 1}$ is isomorphic to the tensor product of spectral sequences of the form
\( \{E^P_r, d^P_r\}_{r \geq 1} \) and \( \{E^Q_r, d^Q_r\}_{r \geq 1} \). By assumption, the latter converge to \( P^\circ \) and \( Q^\circ \) respectively. Which implies the result.

(ii) The argument is exactly the same as in (i) except the very last paragraph: the spectral sequences of the form \( \{E^P_r, d^P_r\}_{r \geq 1} \) and \( \{E^Q_r, d^Q_r\}_{r \geq 1} \) converge, respectively, to \( H(P^\circ) \) and to \( H(Q^\circ) \) (rather than to \( P^\circ \) and \( Q^\circ \)). □

3.11. Operadic wheeled extension. Let \( D_\lambda \) be a dioperad and \( D^\lambda_{\infty} \) its minimal resolution. Let \( D^\lambda_{\infty} \) be a dg subprop of \( D^\lambda_{\infty} \) spanned by graphs with at most operadic wheels (see §3.9.1). Similarly one defines a subprop, \( D^\lambda_{\infty} \), of \( D^\lambda_{\infty} \).

3.11.1. Theorem. For any Koszul operads \( P \) and \( Q \) and any Koszul substitution law \( \lambda \),

(i) the natural morphism of graph complexes,

\[ (D^\lambda_{\infty}, \delta_\lambda) \rightarrow (D^\lambda_{\infty}, \delta_\lambda), \]

is a quasi-isomorphism if and only if the operads \( P \) and \( Q \) are both stably Koszul.

(ii) there is, in general, an isomorphism of \( S \)-bimodules,

\[ H(D^\lambda_{\infty}, \delta_\lambda) = H(D^\lambda_{0\infty}, \delta_0) = \frac{H(P^\circ) \ast H(Q^\circ)}{I_0}, \]

where \( H(P^\circ) \) and \( H(Q^\circ) \) are cohomologies of the wheeled completions of the minimal resolutions of the operads \( P \) and \( Q \).

Proof. Use spectral sequence of a filtration, \( \{F_p\} \), defined similar to the one introduced in the proof of Theorem 3.10. We omit full details as they are analogous to §3.10. □

In the next section we apply some of the above results to compute cohomology of several concrete graph complexes with wheels.

§4. Wheeled Poisson structures and other examples

4.1. Wheeled operad of strongly homotopy Lie algebras. Let \( (\operatorname{Lie}_\infty, \delta) \) be the minimal resolution of the operad, \( \operatorname{Lie} \), of Lie algebras. It can be identified with the subcomplex of \( (\operatorname{LieB}_\infty, \delta) \) spanned by connected trees built on degree one \((1, n)\)-corollas, \( n \geq 2 \),

with the differential given by

\[ \delta \]
Let $\text{Lie}_\infty^\odot$ and $\text{Lie}^\odot$ and wheeled extensions of $\text{Lie}_\infty$, and, respectively, $\text{Lie}$ (see §3.5 for precise definitions).

4.1.1. Theorem. The operad, $\text{Lie}$, of Lie algebras is stably Koszul, i.e. $H(\text{Lie}_\infty^\odot) = \text{Lie}^\odot$.

Proof. We shall show that the natural morphism of dg props,

$$\begin{align*}
(\text{Lie}_\infty^\odot, \delta) & \longrightarrow (\text{Lie}^\odot, 0)
\end{align*}$$

is a quasi-isomorphism. Consider a surjection of graph complexes (cf. Sect. 3.4),

$$u : (\text{Lie}^\infty_+^\odot, \delta) \longrightarrow (\text{Lie}_\infty^\odot, \delta)$$

where $\text{Lie}_\infty^\odot$ is the marked extension of $\text{Lie}_\infty^\odot$, i.e. the one in which one cyclic edge in every wheel is marked. This surjection respects the filtrations,

$$F_p \text{Lie}_\infty^\infty := \{ \text{span}\{f \in \text{Lie}_\infty^\infty\} : \text{total number of cyclic vertices in } f \geq p \},$$

$$F_p \text{Lie}_\infty^\odot := \{ \text{span}\{f \in \text{Lie}_\infty^\odot\} : \text{total number of cyclic vertices in } f \geq p \},$$

and hence induces a morphism of the associated 0-th terms of the spectral sequences,

$$u_0 : (E^+_0, \partial_0) \longrightarrow (E^\odot_0, \partial_0).$$

The point is that the (pro-)cyclic group acting on $(E^+_0, \partial_0)$ by shifting the marked edge one step further along orientation commutes with the differential $\partial_0$ so that $u_0$ is nothing but the projection to the coinvariants with respect to this action. As we work over a field of characteristic 0 coinvariants can be identified with invariants in $(E^+_0, \partial_0)$. Hence we get, by Maschke theorem,

$$H(E^\odot_0, \partial_0) = \text{cyclic invariants in } H(E^+_0, \partial_0).$$

The next step is to compute the cohomology of the complex $(E^+_0, \partial_0)$. Consider its filtration,

$$F_p := \{ \text{span}\{f \in E^+_0\} : \text{total number of non-cyclic input edges at cyclic vertices in } f \leq p \},$$

and let $\{E_r, \delta_r\}_{r \geq 0}$ be the associated spectral sequence. We shall show below that the latter degenerates at the second term (so that $E_2 \simeq H(E^+_0, \partial_0)$). The differential $\delta_0$ in $E_0$ is given by its values on the vertices as follows:

(i) on every non-cyclic vertex one has $\delta_0 = \delta$, the differential in $\text{Lie}_\infty$;

(ii) on every cyclic vertex $\delta_0 = 0$.

Hence the complex $(E_0, \delta_0)$ is isomorphic to the direct sum of tensor products of complexes $(\text{Lie}_\infty, \delta)$. By Künneth theorem, we get,

$$E_1 = V_1/h(V_2),$$

where

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- $V_1$ is the subspace of $\text{Lie}_\infty^+$ consisting of all those graphs whose every non-cyclic vertex is $\bullet$;

- $V_2$ is the subspace of $\text{Lie}_\infty^+$ whose every non-cyclic vertex is either $\bullet$ or $\circ$ with the number of vertices of the latter type $\geq 1$;

- the map $h : V_2 \to V_1$ is given on non-cyclic vertices by

$$h \left( \begin{array}{c} \circ \\ \circ \\ \circ \\ \circ \\ 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \\ 1 \end{array} \right) = \begin{array}{c} \bullet \\ \bullet \\ \circ \\ \circ \\ 1 \\ 2 \\ 3 \\ 1 \\ 3 \\ 2 \\ 1 \end{array} + \begin{array}{c} \bullet \\ \bullet \\ \circ \\ \circ \\ 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \end{array} + \begin{array}{c} \bullet \\ \bullet \\ \circ \\ \circ \\ 1 \\ 2 \\ 3 \\ 1 \\ 3 \end{array}$$

and on all cyclic vertices $h$ is set to be the identity.

The differential $\delta_1$ in $\mathcal{E}_1$ is given by its values on vertices as follows:

(i) on every non-cyclic vertex one has $\delta_1 = 0$;

(ii) on every cyclic $(1, n + 1)$-vertex with cyclic half-edges denoted by $x$ and $y$, one has

$$\delta_1 \begin{array}{c} \circ \\ \circ \\ \cdots \\ \circ \\ x \\ 1 \\ 2 \\ \cdots \\ n \end{array} = \sum_{[n] = J_1 \cup J_2, |J_1| = 2, |J_2| \geq 0} \begin{array}{c} \circ \\ \circ \\ \cdots \\ \circ \\ y \\ J_1 \\ J_2 \\ x \end{array}.$$ 

To compute the cohomology of $(\mathcal{E}_1, \delta_1)$ let us step aside and compute the cohomology of the minimal resolution, $(\text{Lie}_\infty, \delta)$ (which we, of course, already know to be equal to $\text{Lie}$), in a slightly unusual way:

$$F_p^{\text{Lie}} := \{\text{span}(f \in \text{Lie}_\infty) : \text{number of edges attached to the root vertex of } f \leq p\}$$

is clearly a filtration of the complex $(\text{Lie}_\infty, \delta)$. Let $\{E_r^{\text{Lie}}, d_r^{\text{Lie}}\}_{r \geq 0}$ be the associated spectral sequence. The cohomology classes of $E_1^{\text{Lie}} = H(E_0^{\text{Lie}}, d_0^{\text{Lie}})$ resemble elements of $\mathcal{E}_1$: they are trees whose root vertex may have any number of edges while all other vertices are binary, $\bullet$. The differential $d_1^{\text{Lie}}$ is non-trivial only on the root vertex on which it is given by,

$$d_1^{\text{Lie}} \begin{array}{c} \circ \\ \circ \\ \cdots \\ \circ \\ 1 \\ 2 \\ \cdots \\ n-1 \end{array} = \sum_{[n] = J_1 \cup J_2, |J_1| = 2, |J_2| \geq 1} \begin{array}{c} \circ \\ \circ \\ \cdots \\ \circ \\ y \\ J_1 \\ J_2 \end{array}.$$ 

The cohomology of $(E_1^{\text{Lie}}, d_1^{\text{Lie}})$ is equal to the operad of Lie algebras. The differential $d_1^{\text{Lie}}$ is identical to the differential $\delta_1$ above except for the term corresponding to $|J_2| = 0$. Thus let us define another complex, $(E_1^{\text{Lie}+}, d_1^{\text{Lie}+})$, by adding to $E_1^{\text{Lie}}$ trees whose root vertex is a degree $-1$ corolla $\downarrow$ while all other vertices are binary $\bullet$. The differential $d_1^{\text{Lie}+}$ is defined
Claim. The cohomology of the complex \((E_{1}^{\text{Lie}+}, d_{1}^{\text{Lie}+})\) is a one dimensional vector space spanned by \(\hat{1}\).

Proof of the claim. Consider the 2-step filtration, \(F_0 \subset F_1\) of the complex \((E_{1}^{\text{Lie}+}, d_{1}^{\text{Lie}+})\) by the number of \(\hat{1}\). The zero-th term of the associated spectral sequence is isomorphic to the direct sum of the complexes,

\[(E_{1}^{\text{Lie}}, d_{1}^{\text{Lie}}) \oplus (E_{1}^{\text{Lie}}[1], d_{1}^{\text{Lie}}) \oplus (\text{span}(\hat{1}), 0)\]

so that the next term of the spectral sequence is

\[\text{Lie} \oplus \text{Lie}[1] \oplus \langle \hat{1} \rangle\]

with the differential being zero on \(\text{Lie}[1] \oplus \langle \hat{1} \rangle\) and the the natural isomorphism,

\[\text{Lie} \rightarrow \text{Lie}[1]\]

on the remaining summand. Hence the claim follows.

The point of the above Claim is that the graph complex \((\mathcal{E}_1, \delta_1)\) is isomorphic to the tensor product of a trivial complex with complexes of the form \((E_{1}^{\text{Lie}+}, d_{1}^{\text{Lie}+})\). Which immediately implies that \(\mathcal{E}_2 = \mathcal{E}_\infty \simeq H(E_0^+, \partial_0)\) is the direct sum of \(\text{Lie}\) and the vector space spanned by marked wheels of the type,

\[
\begin{array}{c}
\bullet\ \\
/\ \\
\bullet\ \\
/\ \\
\bullet\ \\
\end{array}
\]

whose every vertex is cyclic. Hence the cohomology group \(H(E_0^+, \partial_0) = E_1^\circ\) we started with is equal to the direct sum of \(\text{Lie}\) and the space, \(Z\), spanned by unmarked wheels of the type,

\[
\begin{array}{c}
\bullet\ \\
/\ \\
\bullet\ \\
/\ \\
\bullet\ \\
\end{array}
\]

whose every vertex is cyclic. As every vertex is binary, the induced differential \(\partial_1\) on \(E_1^\circ\) vanishes, the spectral sequence by the number of cyclic vertices we began with degenerates, and we conclude that this direct sum, \(\text{Lie} \oplus Z\), is isomorphic to the required cohomology group \(H(\text{Lie}_\infty^\circ, d)\).

Finally one checks using Jacobi identities that every element of \(\text{Lie}^\circ\) containing a wheel can be uniquely represented as a linear combination of graphs from \(Z\) implying

\[\text{Lie}^\circ \simeq \text{Lie} \oplus Z \simeq H(\text{Lie}_\infty^\circ, d)\]

and completing the proof. \(\square\)
4.2. Wheeled prop of polyvector fields. Let \( \text{Lie} B \) be the prop of Lie 1-bialgebras and \( (\text{Lie} B^\otimes, \delta) \) its minimal resolution (see §2.6.3). We denote their wheeled extensions by \( \text{Lie} B^\otimes \) and \( (\text{Lie} B^\otimes, \delta) \) respectively (see §3.5), and their operadic wheeled extensions by \( \text{Lie} B^\otimes \) and \( (\text{Lie} B^\otimes, \delta) \) (see §3.11). By Theorems 3.11.1 and 4.1.1, we have

4.2.1. Proposition. The natural epimorphism of dg props,

\[
(\text{Lie} B^\otimes, \delta) \rightarrow (\text{Lie} B^\otimes, 0)
\]

is a quasi-isomorphism.

We shall study next a subcomplex (not a subprop!) of the complex \( (\text{Lie} B^\otimes, \delta) \) which is spanned by directed graphs with at most one wheel, i.e. with at most one closed path which begins and ends at the same vertex. We denote this subcomplex by \( \text{Lie} B^\circ \). Similarly we define a subspace \( \text{Lie} B^\circ \subset \text{Lie} B^\otimes \) spanned by equivalence classes of graphs with at most one wheel.

4.2.2. Theorem. \( H(\text{Lie} B^\circ, \delta) = \text{Lie} B^\circ \).

Proof. (a) Consider a two step filtration, \( F_0 \subset F_1 := \text{Lie} B^\circ \) of the complex \( (\text{Lie} B^\circ, \delta) \), with \( F_0 := \text{Lie} B^\circ \) being the subspace spanned by graphs with no wheels. We shall show below that the cohomology of the associated direct sum complex,

\[
F_0 \bigoplus F_1 / F_0
\]

is equal to \( \text{Lie} B \bigoplus \text{Lie} B^\circ / \text{Lie} B \). In fact, the equality \( H(F_0) = \text{Lie} B \) is obvious so that it is enough to show below that the cohomology of the complex, \( C := F_1 / F_0 \), is equal to \( \text{Lie} B^\circ / \text{Lie} B \).

(b) Consider a filtration of the complex \( (C, \delta) \),

\[
F_p C := \text{span} \{ f \in C : \text{number of cyclic vertices in } f \geq p \},
\]

and a similar filtration,

\[
F_p C^+ := \text{span} \{ f \in C^+ : \text{number of cyclic vertices in } f \geq p \},
\]

of the marked version of \( C \). Let \( \{ E_r^\circ, \partial_r \}_{r \geq 0} \) and \( \{ E_r^+, \partial_r \}_{r \geq 0} \) be the associated spectral sequences. There is a natural surjection of complexes,

\[
u_0 : (E_0^+, \partial_0) \rightarrow (E_0^\circ, \partial_0).
\]

It is easy to see that the differential \( \partial_0 \) in \( E_0^+ \) commutes with the action of the (pro-)cyclic group on \( (E_0^+, \partial_0) \) by shifting the marked edge one step further along orientation so that \( u_0 \) is nothing but the projection to the coinvariants with respect to this action. As we work over a field of characteristic 0, we get by Maschke theorem,

\[
H(E_0^\circ, \partial_0) = \text{cyclic invariants in } H(E_0^+, \partial_0).
\]
so that we can work from now on with the complex \((E_0^+, \partial_0)\). Consider a filtration of the latter,\[ \mathcal{F}_p := \left\{ \text{span}(f \in E_0^+) : \text{total number of non-cyclic input edges at cyclic vertices in } f \leq p \right\}, \]
and let \(\{\mathcal{E}_r, d_r\}_{r \geq 0}\) be the associated spectral sequence. The differential \(\partial_0\) in \(\mathcal{E}_0\) is given by its values on the vertices as follows:

(i) on every non-cyclic vertex one has \(d_0 = \delta\), the differential in \(\text{LieB}_\infty\);

(ii) on every cyclic vertex \(d_0 = 0\).

Hence the complex \((\mathcal{E}_0, d_0)\) is isomorphic to the direct sum of tensor products of complexes \((\text{Lie}_\infty, \delta)\) with trivial complexes. By Künneth theorem, we conclude that \(\mathcal{E}_1 = H(\mathcal{E}_0, d_0)\) can be identified with the quotient of the subspace in \(\mathbb{C}\) spanned by graphs whose every non-cyclic vertex is ternary, e.g. either \(\cup\) or \(\rhd\), with respect to the equivalence relation generated by the following equations among non-cyclic vertices,

\[
(x) \quad \begin{array}{c}
\begin{array}{ccc}
1 & 2 & 3 \\
3 & 1 & 2 \\
2 & 3 & 1 \\
1 & 3 & 2 \\
1 & 2 & 3 \\
\end{array}
\end{array} = 0 , \quad \begin{array}{c}
\begin{array}{ccc}
1 & 2 & 3 \\
3 & 1 & 2 \\
2 & 3 & 1 \\
1 & 3 & 2 \\
1 & 2 & 3 \\
\end{array}
\end{array} = 0 , \quad \begin{array}{c}
\begin{array}{ccc}
1 & 2 & 3 \\
3 & 1 & 2 \\
2 & 3 & 1 \\
1 & 3 & 2 \\
1 & 2 & 3 \\
\end{array}
\end{array} = 0 , \quad \begin{array}{c}
\begin{array}{ccc}
1 & 2 & 3 \\
3 & 1 & 2 \\
2 & 3 & 1 \\
1 & 3 & 2 \\
1 & 2 & 3 \\
\end{array}
\end{array} = 0 ,
\]

The differential \(d_1\) in \(\mathcal{E}_1\) is non-zero only on cyclic vertices,

\[
d_1 = \sum_{|m|=I_1 \cup I_2} (-1)^{\sigma(I_1 \cup I_2)+1} + \sum_{|n|=I_1 \cup I_2} (-1)^{\sigma(I_1 \cup I_2)} + \sum_{|m|=I_1 \cup I_2} (-1)^{\sigma(I_1 \cup I_2)+m}
\]

where cyclic half-edges (here and below) are dashed. Then \(\mathcal{E}_1\) can be interpreted as a bicomplex, \((\mathcal{E}_1 = \bigoplus_{m,n} \mathcal{E}_1^{m,n}, d_1 = \partial + \bar{\partial})\), with, say, \(m\) counting the number of vertices attached to cyclic vertices in “operadic” way (as in the first two summands above), and \(n\) counting the number of vertices attached to cyclic vertices in “non-operadic” way (as
in the last two summands in the above formula). Note that the assumption that there is only one wheel in $C$ is vital for this splitting of the differential $d_1$ to have sense. The differential $\partial$ (respectively, $\bar{\partial}$) is equal to the first (respectively, last) two summands in $d_1$.

Using Claim in the proof of Theorem 4.1.1 it is not hard check that $H(E_1, \partial)$ is isomorphic to the quotient of the subspace of $C$ spanned by graphs whose

— every non-cyclic vertex is ternary, e.g. either $\Uparrow$ or $\Uparrow$;

— every cyclic vertex is either $\Uparrow$, or $\Uparrow$, or $\Uparrow$,

with respect to the equivalence relation generated by equations $(\ast)$ and the following ones,

$$(\ast\ast) \quad \Uparrow = 0, \quad \Uparrow = 0, \quad \Uparrow = 0, \quad \Uparrow = 0. $$

The differential $\bar{\partial}$ is non-zero only on cyclic vertices of the type $\Uparrow$, on which it is given by

$$\bar{\partial} \Uparrow = - \Uparrow - \Uparrow.$$

Hence $A := H(H(E_1, \partial), \bar{\partial})$ can be identified with the quotient of the subspace of $C$ spanned by graphs whose

— every non-cyclic vertex is ternary, e.g. either $\Uparrow$ or $\Uparrow$;

— every cyclic vertex is also ternary, e.g. either $\Uparrow$ or $\Uparrow$,

with respect to the equivalence relation generated by equations $(\ast)$, $(\ast\ast)$ and, say, the following one,

$$\Uparrow = 0.$$

As all vertices are ternary, all higher differentials in our spectral sequences vanish, and we conclude that

$$H(C, \delta) \simeq A,$$

which proves the Theorem. \[\square\]

4.2.3. Remark. As an independent check of the above arguments one can show using relations $R_1 - R_3$ in §2.6.2 that every element of $\text{Lie}B^\circ_{\text{Lie}B}$ can indeed be uniquely represented as a linear combinations of graphs from the space $A$.

4.2.4. Remark. Proposition 4.2.1 and Theorem 4.2.2 can not be extended to the full wheeled prop $\text{Lie}B^\circ_{\infty}$, i.e. the natural surjection,

$$\pi : (\text{Lie}B^\circ_{\infty}, \delta) \longrightarrow (\text{Lie}B^\circ, 0),$$

is not a quasi-isomorphism. For example, the graph

\[\text{Graphs}\]

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represents a non-trivial cohomology class in $H^1(\text{LieB}_{\infty}^\odot, \delta)$.

4.3. Wheeled prop of Lie bialgebras. Let $\text{LieB}$ be the prop of Lie bialgebras which is generated by the dioperad very similar to $\text{LieB}$ except that both generating Lie and coLie operations are in degree zero. This dioperad is again Koszul with Koszul substitution law so that the analogue of Proposition 4.2.1 holds true for the operadic wheelification, $\text{LieB}_{\infty}^\odot$. In fact, the analogue of Theorem 4.2.2 holds true for $\text{LieB}_{\infty}^\circ$.

4.4. Prop of infinitesimal bialgebras. Let $\text{IB}$ be the dioperad of infinitesimal bialgebras $\text{[Ag]}$ which can be represented as a quotient, 

$$\text{IB} = \frac{D\langle E \rangle}{\text{Ideal } < R >},$$

of the free prop generated by the following $S$-bimodule $E$,

- all $E(m, n)$ vanish except $E(2, 1)$ and $E(1, 2)$,

$$E(2, 1) := k[S_2] \otimes 1_1 = \text{span} \left( \begin{array}{c} 1 \ \ \ 2 \\ 2 \ \ \ 1 \end{array} \right),$$

$$E(1, 2) := 1_1 \otimes k[S_2] = \text{span} \left( \begin{array}{c} 1 \ \ \ 1 \\ 2 \ \ \ 2 \ \ \ 1 \\ 2 \end{array} \right),$$

modulo the ideal generated by the associativity conditions for $\uparrow\downarrow\uparrow\downarrow$, co-associativity conditions for $\uparrow\downarrow\uparrow\downarrow$,

$$\uparrow\downarrow\uparrow\downarrow - \uparrow\downarrow\uparrow\downarrow - \uparrow\downarrow\uparrow\downarrow = 0.$$ 

This is a Koszul dioperad with a Koszul substitution law. Its minimal prop resolution, $(\text{IB}_\infty, \delta)$ is a dg prop freely generated by the $S$-bimodule $E = \{E(m, n)\}_{m,n \geq 1, m+n \geq 3}$, with

$$E(m, n) := k[S_m] \otimes k[S_n][3 - m - n] = \text{span} \left( \begin{array}{c} 1 \ \ \ 2 \ldots \ m-1 \\ 1 \ \ \ 2 \ldots \ n-1 \ n \end{array} \right).$$

By Claim 3.5.4, the analogue of Proposition 4.2.1 does not hold true for $\text{IB}_\infty^\circ$. Moreover, it is not hard to check that the graph

$$\uparrow\downarrow\uparrow\downarrow - \uparrow\downarrow\uparrow\downarrow$$

represents a non-trivial cohomology class in $H^{-1}(\text{IB}_\infty^\odot)$. Thus neither the analogue of Theorem 4.2.2 holds true for $\text{IB}_\infty^\circ$ nor the natural surjection, $\text{IB}_\infty^\odot \rightarrow \text{IB}_\infty^\circ$, is a quasi-isomorphism. This example is of interest because the wheeled dg prop $\text{IB}_\infty^\odot$ controls the cohomology of a directed version of Kontsevich’s ribbon graph complex.
4.5. **Wheeled quasi-minimal resolutions.** Let $P$ be a graded prop with zero differential admitting a minimal resolution,

$$\pi : (P_\infty = \langle E \rangle, \delta) \to (P, 0).$$

We shall use in the following discussion of this pair of props, $P_\infty$ and $P$, a so called *Tate-Josefak* grading\(^7\) which, by definition, assigns degree zero to all generators of $P$ and hence make $P_\infty$ into a non-positively graded differential prop, $P_\infty = \bigoplus_{i \leq 0} P_i$, with cohomology concentrated in degree zero, $H^0(P_\infty, \delta) = P$. Both props $P_\infty$ and $P$ admit canonically wheeled extensions,

$$P^\otimes_\infty := \bigoplus_{G \in \Theta^\otimes} G(\langle E \rangle),$$

$$P^\otimes := H^0(P^\otimes_\infty, \delta).$$

However, the natural extension of the epimorphism $\pi$,

$$\pi^\otimes : (P^\otimes_\infty, \delta) \to (P, 0).$$

fails in general to stay a quasi-isomorphism.

Note that the dg prop, $(P^\otimes_\infty, \delta)$, defined above is a *free* prop

$$P^\otimes_\infty := \bigoplus_{G \in \Theta^\otimes} G(\langle E \rangle)$$

on the $\mathbb{S}$-bimodule, $E^\otimes = \{E(m, n)\}_{m,n \geq 0}$,

$$E^\otimes(m, n) := \bigoplus_{G \in \Theta^\otimes_{indecomposable}(m, n)} G(\langle E \rangle)$$

generated by indecomposable (with respect to graftings and disjoint unions) decorated wheeled graphs. Note that the induced differential is *not* quadratic with respect to the generating set $E^\otimes$.

4.5.1. **Theorem-definition.** There exists a dg free prop, $(P^\otimes_\infty, \delta)$, which fits into a commutative diagram of morphisms of props,

$$[P^\otimes_\infty] \xrightarrow{\alpha} P^\otimes_\infty \xrightarrow{\pi^\otimes} P^\otimes,$$

where $\alpha$ is an epimorphism of (nondifferential) props $\text{qis}$ a quasi-isomorphism of dg props. The prop $[P^\otimes_\infty]$ is called a quasi-minimal resolution of $P^\otimes$.

\(^7\)The Tate-Josefak grading of props $\text{Lie}_\infty$ and $\text{Lie}_B$, for example, assigns to generating $(m, n)$ corollas degree $3 - m - n$. 

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Proof. Let \( s_1 : H^{-1}(P^\infty_\infty) \to P^\infty_\infty \) be any representation of degree \(-1\) cohomology classes (if there are any) as cycles. Set \( E_1 := H^{-1}(P^\infty_\infty)[1] \) and define a differential graded prop,

\[
Q_1 := P\langle E^\infty \oplus E_1 \rangle
\]

with the differential \( \delta \) extended to new generators as \( s_1[1] \). By construction, \( H^0(Q_1) = P^\infty \), and \( H^{-1}(Q_1) = 0 \).

Let \( s_2 : H^{-2}(Q_1) \to Q_1 \) be any representation of degree \(-2\) cohomology classes (if there are any) as cycles. Set \( E_2 := H^{-2}(Q_1)[1] \) and define a differential graded prop,

\[
Q_2 := P\langle E^\infty \oplus E_1 \oplus E_2 \rangle
\]

with the differential \( \delta \) extended to new generators as \( s_2[1] \). By construction, \( H^0(Q_2) = P^\infty \), and \( H^{-1}(Q_2) = H^{-2}(Q_2) = 0 \).

Continuing by induction we construct a dg free prop, \([P^\infty]_\infty := \lim_{n \to \infty} Q_n = P\langle E^\infty \oplus E_1 \oplus E_2 \oplus E_3 \oplus \ldots \rangle\) with all the cohomology concentrated in Tate-Jozefak degree 0 and equal to \( P^\infty \). \( \square \)

4.5.2. Example. The prop \([Ass^\infty]_\infty\) has been explicitly described in [MMS]: this is a dg free prop \([Ass^\infty]_\infty\) generated by planar \((1,n)\)-corollas in degree \(2 - n\),

\[
\begin{array}{c}
1 \\
\vdots \\
n
\end{array}
\]

\( n \geq 2 \),

and planar \((0,m + n)\)-corollas in degree \(-m - n\),

\[
\begin{array}{c}
1 \\
\vdots \\
m \\
m+1 \\
m+n
\end{array}
\]

\( m, n \geq 1 \),

having the cyclic skew-symmetry

\[
\begin{array}{c}
1 \\
\vdots \\
m \\
m+1 \\
m+n
\end{array} \quad = (-1)^{\text{sgn}(\zeta)} \quad \begin{array}{c}
\zeta(1) \\
\zeta(2) \\
\vdots \\
\zeta(m) \\
\zeta(m+1) \\
\vdots \\
m+n \\
1 \\
\vdots \\
m \\
m+1 \\
m+n
\end{array} \quad = (-1)^{\text{sgn}(\xi)} \quad \begin{array}{c}
\xi(m+1) \\
\xi(m+2) \\
\vdots \\
\xi(m+n) \\
1 \\
\vdots \\
m \\
m+1 \\
m+n
\end{array}
\]

with respect to the cyclic permutations \( \zeta = (12 \ldots m) \) and \( \xi = ((m+1)(m+2) \ldots (m+n)) \).
The differential is given on generators as

\[
\delta = \sum_{k=0}^{n-2} \sum_{l=2}^{n-k} (-1)^{k+l+1(n-k-l)+1} \delta^{1 \ldots k \ldots 1}_{1 \ldots k \ldots 1},
\]

\[
\delta = \sum_{i=0}^{m-1} (-1)^{m+1} \zeta^i \sum_{j=1}^{n-1} (-1)^{n+1} \xi^j \left( \begin{array}{c}
\delta^{1 \ldots m+1}_{1 \ldots m+1} \\
\delta^{1 \ldots m+1}_{1 \ldots m+1}
\end{array} \right),
\]

\[
+ \sum_{k=2}^{m} (-1)^{k(m+n)} \delta^{1 \ldots k \ldots 1}_{1 \ldots k \ldots 1}, + \sum_{k=2}^{n-2} (-1)^{m+k+nk+1} \delta^{1 \ldots m+k+1}_{1 \ldots m+k+1}.
\]

4.6. Wheeled Poisson structures. By Theorem 4.5.1, there exists a natural extension, \([\text{Lie}\mathcal{B}_C]_\infty\), of the dg prop \(\text{Lie}\mathcal{B}_C\) which provides us with a quasi-minimal prop resolution of \(\text{Lie}\mathcal{B}_C\).

It follows from Proposition 3.4.1 that representations of the dg prop \(\text{Lie}\mathcal{B}_C\) in a finite-dimensional dg space \(V\) are the same as Poisson structures on the formal manifold \(V\). This fact prompts us to make the following

**Definition.** Representations of the dg prop \([\text{Lie}\mathcal{B}_C]_\infty\) in a finite-dimensional dg space \(V\) are called **wheeled Poisson structures** on the formal graded manifold \(M\).

As generators of \([\text{Lie}\mathcal{B}_C]_\infty\) contain the generators of \(\text{Lie}\mathcal{B}_C\), a wheeled Poisson structure includes a degree one polyvector field \(\gamma \in \wedge^* \mathcal{T}_V\) satisfying the Poisson equations, \([\gamma, \gamma] = 0\). However, there are other generators of \([\text{Lie}\mathcal{B}_C]_\infty\], hence other new tensor fields on \(V\) enter, in general, the content list of a wheeled Poisson structure. One can get some insight into that content list from the above example of the prop \([\text{Ass}_C]_\infty\) where new fields are cyclically invariant “functions” on the noncommutative manifold \(V\) — representations of the type \((0, m + n)\) corollas for all \(n \geq 1, m \geq 1\).

The new fields of a wheeled Poisson structure satisfy systems of differential equations which involve *traces* of tensors formed from polyvector fields and their partial derivatives. Again the above example §4.6.5 (more precisely the value of the differential on \((0, m + n)\)-corollas) gives some intuition into the possible structure of the equations but not that much: it follows from Theorem 4.2.2 that the terms in that equations which involve traces of polyvector fields \(\gamma\) can be neither linear in \(\gamma\) nor contain only wheels of genus one (i.e. “wheeled” genus must be at least 2).

Geometric meaning of a wheeled Poisson structure is not clear to the author at present. One can only be sure that this notion is a **canonical** generalization of ordinary Poisson...
structure in finite dimensions. Clearer picture might emerge from computation of the cohomology of the complex \((\text{Lie}^\mathbb{B}_\infty^\mathcal{O}, \delta)\) which is a highly non-trivial problem comparable in complexity with the problem of computing the homology of the directed versions of famous Kontsevich’s graph complexes [Ko1].

§5. Deformation quantization via dg props

5.1. Reminder. Recall that in §2.6 we introduced the dg prop of polyvector fields, \(\text{Lie}^\mathbb{B}_\infty\), and then in §4 we studied its wheeled completion \(\text{Lie}^\mathbb{B}_\infty^\mathcal{O}\) and proved that the latter can be further extended into a dg free prop, \([\text{Lie}^\mathbb{B}_\infty^\mathcal{O}]_\infty\), which fits the commutative diagram,

\[
\begin{array}{c}
\text{Lie}^\mathbb{B}_\infty^\mathcal{O} \\
\downarrow \pi^\mathcal{O} \\
\text{Lie}^\mathbb{B}_\infty
\end{array}
\stackrel{\alpha}{\longrightarrow}
\begin{array}{c}
[\text{Lie}^\mathbb{B}_\infty^\mathcal{O}]_\infty \\
\downarrow \text{qis} \\
[\text{Lie}^\mathbb{B}_\infty^\mathcal{O}]_\infty
\end{array}
\]

with \(\alpha\) being an epimorphism of non-differential props and \(\text{qis}\) a quasi-isomorphism of differential props. Representations of \([\text{Lie}^\mathbb{B}_\infty^\mathcal{O}]_\infty\) in a dg vector space \(V\) are called wheeled Poisson structures (see §4.6).

In §2.7 we introduced the dg prop, \(\text{DefQ}\), of “star products”.

The main purpose of this section is to prove Theorem 5.2 which says that wheeled Poisson structures can be deformation quantized.

5.2. Theorem. There exists a morphism of dg props,

\[
\hat{Q}: \text{DefQ} \longrightarrow [\text{Lie}^\mathbb{B}_\infty^\mathcal{O}]_\infty,
\]

such that

\[
\pi_1 \circ \alpha \circ \hat{Q}
\begin{pmatrix}
1 & 2 & \cdots & k-1 & k \\
1 & 2 & \cdots & n-1 & n
\end{pmatrix}
= \begin{cases}
1, 2, \ldots, k-1, k \\
1, 2, \ldots, n-1, n
\end{cases}
\]

for \(|I_1| = \ldots = |I_k| = 1\) otherwise.

where \(\pi_1\) is the projection to the subspace of \(\text{Lie}^\mathbb{B}_\infty^\mathcal{O}\) spanned by one-vertex graphs.

Proof. We prove it below in three movements:

Step I: we construct a morphism of dg props, \(q: (\text{DefQ}^\hbar, \delta) \longrightarrow (\text{Lie}^\mathbb{B}_\infty^\mathcal{O}, 0)\).

Step II: we prove cofibrancy of \((\text{DefQ}^\hbar, \delta)\) and then use this property to show that \(q\) can be lifted to a morphism \(Q\) making the diagram

\[
\begin{array}{c}
[\text{Lie}^\mathbb{B}_\infty^\mathcal{O}]_\infty \\
\downarrow \text{qis} \\
[\text{Lie}^\mathbb{B}_\infty^\mathcal{O}]
\end{array}
\stackrel{\hat{Q}}{\longrightarrow}
\begin{array}{c}
\text{DefQ}^\hbar \\
\downarrow q
\end{array}
\]

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commutative;

**Step III** : we set $\hat{Q}$ to be the composition,

$$\operatorname{Def}_Q \xrightarrow{x_h} \operatorname{Def}_Q[[\hbar]] \xrightarrow{Q} [\operatorname{Lie}^B_\infty][[\hbar]]$$

at $\hbar = 1$ (which makes sense as a morphism between completed props).

**Step I.** To construct a morphism $q : (\operatorname{Def}_Q^\hbar, \delta) \longrightarrow (\operatorname{Lie}^B_\infty, 0)$ is the same as to deformation quantize an arbitrary finite-dimensional Lie 1-bialgebra, that is, a pair $(\nu, \xi)$, consisting of a linear Poisson structure $\nu$ and a quadratic homological vector field $\xi$ such that $[\xi, \nu] = 0$.

As a first approximation to $q$ we discuss first a well-known Poincare-Birkhoff-Witt quantization of linear Poisson structures, which, in the language of props, translates into existence of a morphism,

$$\mathcal{PBW} : (\operatorname{Def}_Q^\hbar, \delta) \longrightarrow (\operatorname{CoLie}, 0),$$

where $\operatorname{CoLie}$ is the prop of coLie algebras. The morphism $q$ we construct below in II.2 will make the diagram

$$\begin{array}{ccc}
\operatorname{Def}_Q^\hbar & \xrightarrow{q} & \operatorname{Lie}^B_\infty \\
\downarrow{s} & & \downarrow{s} \\
\mathcal{PBW} & \longrightarrow & \operatorname{CoLie}
\end{array}$$

commutative, where $s$ is a natural surjection,

$$s : \operatorname{Lie}^B \longrightarrow \operatorname{CoLie}$$

$(\nu, \xi) \longrightarrow \nu$,

“forgetting” the quadratic homological vector field $\xi$.

**I.1. Poincare-Birkhoff-Witt quantization.** A representation of $\operatorname{CoLie}$ in a vector space $V$ is the same as a linear Poisson structure on $V$ viewed as a graded manifold. This Poisson structure makes the graded commutative algebra $\mathcal{O}_V[[\hbar]] := \bigotimes^* V^* \otimes \mathbb{K}[[\hbar]]$ into a Lie algebra with respect to the Poisson brackets, $\{,\}$. Clearly, $V^*[[\hbar]] \subset \mathcal{O}_V[[\hbar]]$ is a Lie subalgebra consisting of linear functions. Let $\mathcal{U}_h$ be the associated universal enveloping algebra defined as a quotient,

$$\mathcal{U}_h := \bigotimes^* V^*[[\hbar]]/\mathcal{I},$$

where the ideal $\mathcal{I}$ is generated by all expressions of the form

$$X_1 \otimes X_2 - (-1)^{|X_1||X_2|} X_2 \otimes X_1 - \hbar \{X_1, X_2\}, \quad X_i \in V^*.$$

To construct a morphism of props $\mathcal{PBW}$ is the same as to deformation quantize an arbitrary linear Poisson structure $\nu$. Which is a well-known trick: the Poincare-Birkhoff-Witt theorem says that the natural morphism,

$$s : \mathcal{O}_V[[\hbar]] \longrightarrow \mathcal{U}_h,$$
is an isomorphism of vector spaces so that one can quantize \( \nu \) by the formula,
\[
  f \star \hbar g := s^{-1}(s(f) \circ s(g)), \quad \forall f, g \in \mathcal{O}_V,
\]
where \( \circ \) is the product in \( \mathcal{U}(V_\hbar) \). This proves the existence of the required prop morphism \( \mathcal{P}BW \).

### I.2. Deformation quantization of Lie\(^1\)B algebras

A representation of \( \text{Lie}^1\mathcal{B} \) in a graded vector space \( V \) is the same as a degree 0 linear Poisson structure,
\[
  \nu = \hbar \sum_{\alpha,\beta,\gamma} \Phi_{\gamma}^{\alpha \beta} x_\gamma \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta},
\]

together with a degree 1 quadratic vector field,
\[
  \xi = \hbar \sum_{\alpha,\beta,\gamma} C_{\alpha \beta}^{\gamma} x_\alpha x_\beta \frac{\partial}{\partial x^\gamma},
\]
on \( V \) satisfying \([\xi, \xi]_S = 0\) and \( \text{Lie}_\xi \nu = 0 \). Here \( \{x^a\} \) are arbitrary linear coordinates on \( V \) and \( \text{Lie}_\xi \) stands for the Lie derivative along the vector field \( \xi \). The association,
\[
  \Phi_{\gamma}^{\alpha \beta} \simeq \left\downarrow{}_{\gamma}^{\alpha} \right\uparrow{}_{\beta}, \quad C_{\alpha \beta}^{\gamma} \simeq \left\downarrow{}_{\alpha}^{\gamma} \right\uparrow{}_{\beta},
\]

translates the equations,
\[
  [\nu, \nu]_S = 0, \quad [\xi, \xi]_S = 0, \quad \text{Lie}_\xi \nu = 0,
\]

precisely into the graph relations \( R_1-R_3 \) in \( \S2.6.2 \).

To prove existence of a morphism \( q : \text{DefQ}_h \to \text{LieB}_h^0 \) is the same as to deformation quantize such a pair \( (\nu, \xi) \), that is, to construct from \( (\nu, \xi) \) a degree 2 function \( \Gamma_0 \in \text{Hom}_2(\mathbb{K}, \mathcal{O}_V)[[\hbar]] = \mathcal{O}_V[2][[\hbar]] \), a differential operator \( \Gamma_1 \in \text{Hom}_1(\mathcal{O}_V, \mathcal{O}_V)[[\hbar]] \), and a bi-differential operator \( \Gamma_2 \in \text{Hom}_0(\mathcal{O}_V^\otimes 2, \mathcal{O}_V)[[\hbar]] \) such that the following equations are satisfied
\[
  \Gamma_1 \Gamma_0 = 0, \quad \Gamma_1^2 + [\Gamma_0, \Gamma_2]_H = 0,
\]
\[
  d_H \Gamma_1 + [\Gamma_1, \Gamma_2]_H = 0, \quad d_H \Gamma_2 + \frac{1}{2} [\Gamma_2, \Gamma_2]_H = 0.
\]

We can solve the last equation by setting \( \Gamma_2 \) to be related to \( \Phi_{\gamma}^{\alpha \beta} \) via the \( \mathcal{P}BW \) quantization, i.e. we choose the star product,
\[
  \star_h = \text{usual product of functions} + \Gamma_2
\]
to be given as before, \( f \star_h g := \sigma^{-1}(\sigma(f) \cdot \sigma(g)), \forall f, g \in \mathcal{O}_V. \)

\[\text{There can not be non-vanishing terms } \Gamma_k \in \text{Hom}_{2-k}(\mathcal{O}_V^\otimes k, \mathcal{O}_V)[[\hbar]] \text{ with } k \geq 3 \text{ for degree reasons.}\]
Our next task is to find a degree 1 differential operator \( \Gamma_1 \) such that \( d_H \Gamma_1 + [\Gamma_1, \Gamma_2]_H = 0 \) which is equivalent to saying that \( \Gamma_1 \) is a derivation of the star product,

\[
\Gamma_1(f \ast_h g) = (\Gamma_1 f) \ast_h g + (-1)^{|f|} f \ast_h (\Gamma_1 g), \quad \forall f, g \in \mathcal{O}_V.
\]

Consider first a derivation of the tensor algebra \( \otimes \mathbb{T} V[[\hbar]] \) given on the generators, \( \{t^\gamma\} \), by

\[
\hat{\xi}(t^\gamma) := \hbar \sum_{\alpha, \beta} C^\gamma_{\alpha\beta} x^\alpha \otimes x^\beta.
\]

It is straightforward to check using equations \([\xi, \xi]_S = 0\) and \( \text{Lie}_\xi \nu = 0\) that

\[
\hat{\xi} \left( x^\alpha \otimes x^\beta - (-1)^{|\alpha||\beta|} x^\beta \otimes x^\alpha - \hbar \sum_{\gamma} \Phi^\alpha_{\beta\gamma} x^\gamma \right) = 0 \mod I
\]

so that \( \hat{\xi} \) descends to a derivation of the star product. Hence setting \( \Gamma_1 = \hat{\xi} \) we solve the equation \( d_H \Gamma_1 + [\Gamma_1, \Gamma_2]_H = 0 \). However, \( \hat{\xi}^2 \neq 0 \), which is enough to check on a generator \( t^\alpha \),

\[
\hat{\xi}^2(x^\alpha) = -\sum a^\alpha_{\beta\gamma} C^\beta_{\mu\nu} \Phi^\mu_{\sigma\tau} \Phi^\nu_{\sigma\tau} x^\gamma,
\]

or, in terms of graphs,

\[
-\frac{1}{3} \begin{array}{l} \rightarrow \end{array} \gamma \begin{array}{c} \rightarrow \end{array} \begin{array}{c} \rightarrow \end{array} \begin{array}{c} \rightarrow \end{array} \begin{array}{c} \rightarrow \end{array} .
\]

The data, \( (\Gamma_2, \Gamma_1 = \hat{\xi}) \), are given by directed graphs without cycles (in particular, these data make sense for \( \text{infinite} \) dimensional representations of the prop \( \text{LieB} \)). However, to solve the next equation, \( \Gamma_1^2 + [\Gamma_0, \Gamma_2]_H = 0 \), one has to construct from the generators of \( \text{LieB} \) a non-vanishing graph, \( \Gamma_0 \), with no output legs which is impossible to do without using graphs with oriented wheels. This is the reason why in Theorem 5.2 we attempt to construct a morphism into a \( \text{wheeled extension} \) of the prop \( \text{LieB}_{\infty} \): there already does not exist a morphism, \( \text{DefQ}^\hbar \rightarrow \text{LieB} \), into the original unwheeled version of the prop of \( \text{Lie} \) 1-bialgebras which satisfies quasi-classical limit condition that terms linear in \( \hbar \) are \( \nu + \xi \).

In fact, one has to modify the naive choice, \( \Gamma_1 = \hat{\xi} \), to get a solution of \( \Gamma_1^2 + [\Gamma_0, \Gamma_2]_H = 0 \). Consider first a derivation of the tensor algebra \( \otimes \mathbb{T} V[[\hbar]] \) given on the generators, \( \{x^\gamma\} \), by

\[
\hat{\xi}(t^\gamma) := a \hbar^3 \Theta^\gamma + \hbar \sum_{\alpha, \beta} C^\gamma_{\alpha\beta} x^\alpha \otimes x^\beta,
\]

where

\[
\Theta^\gamma := \sum a^\gamma_{\alpha\beta\mu\nu} \Phi^\mu_{\nu\beta} \Phi^\nu_{\mu\beta} = \begin{array}{c} \rightarrow \end{array} \begin{array}{c} \rightarrow \end{array} \begin{array}{c} \rightarrow \end{array} \begin{array}{c} \rightarrow \end{array} .
\]

and \( a \) is a constant.

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I.2.1. Lemma. $\Phi^\alpha_\gamma \Theta^\gamma = 0$, i.e., \[ \begin{array}{c}
\end{array} = 0. \]

Proof. Using relations

\[
\begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array},
\end{array}
\]

we first obtain,

\[
\begin{array}{c}
\end{array} = 2 \begin{array}{c}
\end{array} - 2 \begin{array}{c}
\end{array}.
\end{array}
\]

Next, using relations,

\[
\begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} = 0,
\end{array}
\]

we finally obtain the desired result,

\[
\begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} = 0. \quad \square
\end{array}
\]

I.2.2. Corollary-definition. For any constant $a$ the operator $\xi$ descends to a derivation of the star product $\star_h$ which we denote from now on by $\Gamma_1$.

Proof. \[
\begin{array}{c}
\end{array} \left( x^\alpha \otimes x^\beta - (-1)^{|\alpha||\beta|} x^\beta \otimes x^\alpha - h \sum_\gamma \Phi^\alpha_\gamma x^\gamma \right) = -ah \sum_\gamma \Phi^\alpha_\gamma \Theta^\gamma \mod \mathcal{I}
\end{array}
\]

\[
\begin{array}{c}
\end{array} = 0 \mod \mathcal{I}. \quad \square
\end{array}
\]

Let us next define a linear function, $\Gamma_0 := bh^3 \sum_\gamma \Xi_\gamma x^\gamma$, where $b$ is a constant and

\[
\Xi_\gamma := \sum_{\alpha, \beta, \mu, \nu} C^\mu_\alpha C^\beta_\mu \Phi^{\nu\alpha}_\gamma = \begin{array}{c}
\end{array}
\end{array}
\]

I.2.3. Lemma. $C^\gamma_\alpha \Xi_\gamma = 0$, i.e., \[ \begin{array}{c}
\end{array} = 0. \]

Proof is very similar to the proof of Lemma I.2.1. We omit details.
I.2.4. Corollary. $\Gamma_1 \Gamma_0 = 0$.

Therefore, for any Lie 1-bialgebra, $(C_{\alpha\beta}, \Phi_{\mu\nu})$, we constructed differential operators, $\Gamma_0 \in \text{Hom}_2(\mathbb{R}, \mathcal{O}_V)[[\hbar]]$, $\Gamma_1 \in \text{Hom}_1(\mathcal{O}_V, \mathcal{O}_V)[[\hbar]]$ and $\Gamma_2 \in \text{Hom}_0(\mathcal{O}_V^2, \mathcal{O}_V)[[\hbar]]$ such that the equations, $\Gamma_1 \Gamma_0 = 0$, $d_H \Gamma_1 + [\Gamma_1, \Gamma_2]_H = 0$ and $d_H \Gamma_2 + \frac{1}{2} [\Gamma_2, \Gamma_2]_H = 0$, are satisfied. It remains to check whether or not we can adjust the free parameters $a$ and $b$ in such a way that the last equation,

$$\Gamma_1^2 + [\Gamma_0, \Gamma_2]_H = 0,$$

holds. As the l.h.s. of this equation is obviously a derivation of the star product, it is enough to check the latter only on generators $x^\alpha$,

$$\Gamma_1^2(x^\alpha) + \Gamma_0 \star \hbar x^\alpha - x^\alpha \star \hbar \Gamma_0 = 0.$$

We have

$$\hbar^{-4}(\Gamma_0 \star \hbar x^\alpha - x^\alpha \star \hbar \Gamma_0) = b \sum \Xi_\gamma \Phi_{\alpha\beta}^\gamma x^\beta \simeq b \begin{array}{c} \alpha \end{array} = 2b \begin{array}{c} \alpha \end{array}.$$

We also have,

$$\hbar^{-4} \Gamma_1^2(x^\alpha) = -\frac{1}{3} \begin{array}{c} \alpha \end{array} - 2a \begin{array}{c} \alpha \end{array} = -\frac{1}{3} \begin{array}{c} \alpha \end{array} + 4a \begin{array}{c} \alpha \end{array}.$$

Consider a graph, $\begin{array}{c} \alpha \end{array}$. Replacing its lower two vertices together with their half-edges by the following linear combination of four graphs,

$$\begin{array}{c} 1 \begin{array}{c} 2 \end{array} \end{array} = -\begin{array}{c} 2 \end{array} + 2 \begin{array}{c} 1 \end{array} + 2 \begin{array}{c} 1 \end{array} - 1 \begin{array}{c} 2 \end{array}$$

one gets, after a cancelation of two terms, the identity,

$$\begin{array}{c} \alpha \end{array} = \begin{array}{c} \alpha \end{array} - \begin{array}{c} \alpha \end{array}.$$
Playing a similar trick with upper two vertices one gets another identity,

\[
\begin{align*}
 \begin{array}{c}
 \begin{array}{c}
 \includegraphics[width=0.2\textwidth]{identity1} \\
 \includegraphics[width=0.2\textwidth]{identity2}
 \end{array}
 \end{array}
 \end{align*}
\]

These both identities imply

\[
\begin{align*}
 \begin{array}{c}
 \begin{array}{c}
 \includegraphics[width=0.2\textwidth]{identity1} \\
 \includegraphics[width=0.2\textwidth]{identity2}
 \end{array}
 \end{array}
 \end{align*}
\]

which in turn implies,

\[
\begin{align*}
 h^{-4} \left( \Gamma_1^2(x^\alpha) + \Gamma_0 \ast_h x^\alpha - x^\alpha \ast_h \Gamma_0 \right) &= \frac{1}{2} \begin{array}{c}
 \begin{array}{c}
 \includegraphics[width=0.2\textwidth]{identity1} \\
 \includegraphics[width=0.2\textwidth]{identity2}
 \end{array}
 \end{array} \end{align*}
\]

Hence setting \( a = 1/24 \) and \( b = -1/12 \), i.e. adding to the PBW star product, \( \Gamma_2 \), the operators,

\[
\begin{align*}
 \Gamma_0 &= -\hbar^3 \frac{1}{12} \sum_{\alpha,\beta,\gamma,\mu,\nu} C_{\alpha\beta}^{\mu} C_{\mu\nu}^{\beta} \Phi_{\gamma}^{\alpha \nu} x^\gamma = -\hbar^3 \frac{1}{12} \sum_{\gamma} \begin{array}{c}
 \begin{array}{c}
 \includegraphics[width=0.2\textwidth]{identity1} \\
 \includegraphics[width=0.2\textwidth]{identity2}
 \end{array}
 \end{array} x^\gamma \\
 \Gamma_1(x^\gamma) &= \hbar \sum_{\alpha,\beta} C_{\alpha\beta}^{\gamma} x^\alpha x^\beta + \frac{1}{24} \hbar^3 \sum_{\alpha,\beta,\mu,\nu} C_{\alpha\beta}^{\gamma} \Phi_{\gamma}^{\alpha \mu} \Phi_{\beta}^{\gamma \nu} = \hbar \sum_{\alpha,\beta} \begin{array}{c}
 \begin{array}{c}
 \includegraphics[width=0.2\textwidth]{identity1} \\
 \includegraphics[width=0.2\textwidth]{identity2}
 \end{array}
 \end{array} x^\alpha x^\beta + \frac{1}{24} \hbar^3 \begin{array}{c}
 \begin{array}{c}
 \includegraphics[width=0.2\textwidth]{identity1} \\
 \includegraphics[width=0.2\textwidth]{identity2}
 \end{array}
 \end{array}.
\end{align*}
\]

we complete deformation quantization of Lie 1-bialgebras. Thus we proved the following,

**I.2.5. Proposition.** There exists a morphism of dg props, \( q : \text{DefQ}^h \to \text{LieB}^\odot \), making the diagram

\[
\begin{array}{c}
 \begin{array}{c}
 \includegraphics[width=0.2\textwidth]{diagram1} \\
 \includegraphics[width=0.2\textwidth]{diagram2}
 \end{array}
 \end{array}
\]

commutative.

Hence Step 1 is done.
Step II. First we explain iterative procedure behind our construction of a morphism $Q$ fitting the commutative diagram

$$\begin{array}{ccc}
\text{Def}Q^h & \xrightarrow{\text{qis}} & \text{Lie}^\delta \\
\downarrow Q & & \downarrow \text{qis} \\
\text{[Lie}^\delta\text{B]}_\infty & & \\
\end{array}$$

and then illustrate it with concrete examples. Define $E_s$ to be zero for negative $s$ and, for $s \geq 0$,

$$E_s := \text{span}\left\{ \begin{array}{c} a \\
1 \\
2 \\
3 \\
\vdots \\
n \\
\end{array} \in \text{Def}Q^h \right\}$$

where $a$ satisfies

$$2a + k - 2 = s, \quad k \geq 0, a \geq 1, n \geq 0$$

For example,

$$E_0 = \text{span}\left\{ \begin{array}{c} 1 \\
2 \\
\vdots \\
n \\
\end{array} \right\}, \quad E_1 = \text{span}\left\{ \begin{array}{c} 1 \\
2 \\
\vdots \\
n \\
\end{array} \right\}, \quad E_2 = \text{span}\left\{ \begin{array}{c} 1 \\
2 \\
\vdots \\
n \\
\end{array} \right\}.$$ 

Let $\text{Def}Q^h_s \subset \text{Def}Q^h$ be the free prop generated by $\oplus_{i=0}^s E_i$. Thereby we get an increasing filtration, $0 \subset \text{Def}Q^h_0 \subset \ldots \subset \text{Def}Q^h_s \subset \text{Def}Q^h_{s+1} \ldots$ with

$$\lim_{s \to \infty} \text{Def}Q^h_s = \text{Def}Q^h.$$

A straightforward inspection of the formula for differential $\delta$ in §2.9 implies

$$\delta E_{s+1} \subset \text{Def}Q^h_s,$$

i.e. that the dg prop $(\text{Def}Q^h, \delta)$ has a “cell” structure analogous to that of CW complex. This simple but crucial for our purposes observation permits us to apply the well-known in algebraic topology Whitehead lifting trick and construct a morphism $Q : \text{Def}Q^h \to [\text{Lie}^\delta\text{B}]_\infty$ by defining its values on the generators $E_s$ via an induction on $s$. We begin the induction by specifying its values on $E_0 \oplus E_1 \oplus E_2$ as follows

$$Q\left( \begin{array}{c} 1 \\
2 \\
\vdots \\
n \\
\end{array} \right) = 0, \quad Q\left( \begin{array}{c} I_1 \\
I_2 \\
\vdots \\
n \\
\end{array} \right) = \left\{ \begin{array}{ll} 0 & \text{for } |I_1| = 1, n = 2 \\
\bigcup & \text{otherwise.} \\
\end{array} \right.$$

$$Q\left( \begin{array}{c} 1 \\
2 \\
\vdots \\
n \\
\end{array} \right) = 0, \quad Q\left( \begin{array}{c} I_1 \\
I_2 \\
\vdots \\
n \\
\end{array} \right) = \left\{ \begin{array}{ll} 0 & \text{for } |I_1| = 1, |I_2| = 1, n = 1 \\
\bigcup & \text{otherwise.} \\
\end{array} \right.$$
Note that this choice respects both commutativity of the diagram and the condition on \( \pi_1 \circ Q \) in the Proposition. Thus we constructed a morphism of dg props,

\[
Q_3 : \text{Def}Q^h_3 \rightarrow [\text{Lie}B^\odot]\_\infty
\]
satisfying the required conditions.

Assume now that a morphism

\[
Q_s : \text{Def}Q^h_s \rightarrow [\text{Lie}B^\odot]\_\infty
\]
satisfying the required conditions is already constructed for some \( s \geq 3 \). We want to extend \( Q_s \) to a morphism of dg props,

\[
Q_{s+1} : \text{Def}Q^h_{s+1} \rightarrow [\text{Lie}B^\odot]\_\infty
\]
such that \( q_{is} \circ Q_{s+1} = q \) and the condition on \( \pi_1 \circ Q_{s+1} \) is fulfilled. Let \( e' \) be a lift of \( q(e) \) along the surjection \( q_{is} \). Then

\[
Q_s(\delta e) - \delta e' = \delta e''
\]
is a cycle in \( [\text{Lie}B^\odot]\_\infty \) which projects under \( q_{is} \) to zero. As \( q_{is} \) is a homology isomorphism, this element is exact,

\[
Q_s(\delta e) - \delta e' = \delta e''
\]
for some \( e'' \in [\text{Lie}B^\odot]\_\infty \). We set \( Q_{s+1}(e) := e' + e'' \) completing thereby the inductive construction of \( Q \) as a morphism of dg props. The condition \( q_{is} \circ Q = q \) is automatically satisfied. Another condition on \( \pi_1 \circ Q \circ Q \) is also satisfied because the Hochschild brackets, \([\gamma, \gamma]_H\) of a degree 1 polyvector field viewed as a polydifferential operator contain the Schouten brackets, \([\gamma, \gamma]_S\), as one of the irreducible (in the \( S \)-bimodule sense) summands. Using this fact it is easy to check by induction on \( 2a + k - 2 \) that

\[
\pi_1 \circ Q = \left\{ \begin{array}{ll}
1 & \text{for } a = n + k - 2 \text{ and } |I_1| = \ldots = |I_k| = 1 \\
0 & \text{otherwise}
\end{array} \right.
\]

where \( \pi_1 : \text{Lie}B^\odot_\infty \rightarrow (\text{Lie}B^\odot_\infty)_1 \) is the projection to the subspace, \((\text{Lie}B^\odot_\infty)_1\), consisting of graphs with precisely 1 internal vertex.

\[\text{II.2. Corollary.}\]  (i) If \( a_s \in E_s \) is such that \( Q_{s-1}(\delta a_s) = 0 \) then we can set \( Q_s(a_s) := q_{is}^{-1} \circ q(a_s) \), an arbitrary lifting of \( q(a_s) \in \text{Lie}B^\odot \) to \( \text{Lie}B^\odot_\infty \subset [\text{Lie}B^\odot]\_\infty \).

(ii) If \( a_s \in E_s \) is such that \( Q_{s-1}(\delta a_s) = 0 \) and \( q(a_s) = 0 \), then we can set \( Q(a_s) = 0 \).

\[\text{II.3. Iteration.}\]  Iteration (with respect to the "weight" parameter \( s = 2a + k - 2 \)) formula has the form,

\[
Q = \left\{ \begin{array}{ll}
1 & \text{for } a = n + k - 2 \text{ and } |I_1| = \ldots = |I_k| = 1 \\
0 & \text{otherwise}
\end{array} \right.
\]

\[= e' + e''\]
where \( e' \) is an arbitrary lift of \( q \) to a cycle in \( \text{Lie}^\infty B \supset [\text{Lie}^\infty B]_\infty \), and \( e'' \) is a solution of the following equation in \( [\text{Lie}^\infty B]_\infty \),

\[
\delta e'' = Q \left( \begin{array}{c}
I_1 \\
\vdots \\
I_k \\
\end{array} \right) = \sum_{i=1}^k (-1)^{i+1} Q \left( \begin{array}{c}
I_1 \\
\vdots \\
I_k \\
\end{array} \right) + \sum_{b+c=a} \sum_{p+q=k+1} \sum_{i=0}^{p-1} \sum_{i'q=p+1,a\geq 0} \sum_{[n]=J_1\cup J_2} (-1)^{(p+1)q+i(q-1)}
\]

Note that the r.h.s. of the above equation contains values of \( Q \) on \([a', k', n']\)-corollas with the weight \( 2a' + k' - 2 < 2a + k - 2 \) which are, by induction assumption, are already known.

**II.4. How it works.** We shall illustrate the above construction of \( Q \) in a few examples.

**Iteration level \( s=3 \):** We have

\[
E_3 = \text{span} \left\{ \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array} \end{array} , \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array} \end{array} \right\}
\]

The equation (\( \star \)) for the first generator takes the form,

\[
\delta e'' = Q \left( \begin{array}{c}
I_1 \\
\vdots \\
I_k \\
\end{array} \right) = \sum_{l_1=l_1'\cup l_1''} \sum_{[n]=J_1\cup J_2} \frac{1}{s!} Q \left( \begin{array}{c}
I_1' \\
\vdots \\
I_k' \\
\end{array} \right) \quad \text{for } |I_1| = 1, n = 3
\]

\[
= \left\{ \begin{array}{c}
\begin{array}{ccc}
1 & 2 & 3 \\
\cdots & \cdots & \cdots \\
1 & 2 & 3 \\
\end{array} \\
0 & \text{otherwise.}
\end{array} \right\}
\]
implying
\[ e'' = \begin{cases} 1 & \text{for } |I_1| = 1, n = 3 \\ 0 & \text{otherwise.} \end{cases} \]

As
\[ q \left( \begin{array}{c} \halftree \halftree \\ 1 & 2 & \ldots & n \end{array} \right) = \begin{cases} -1 & -2 & -1 & -2 \\ -2 & -1 & -2 & -1 \\ 1 & 2 & 3 \end{cases} & \text{for } |I_1| = 2, n = 2 \\ 0 & \text{otherwise.} \]
and identifying the r.h.s. with its natural lift, \( e' \), into \( \text{Lie}B_\infty \), we finally obtain
\[ Q \left( \begin{array}{c} \halftree \halftree \\ 1 & 2 & \ldots & n \end{array} \right) = \begin{cases} -1 & -2 & -1 & -2 \\ -2 & -1 & -2 & -1 \\ 1 & 2 & 3 \end{cases} & \text{for } |I_1| = 1, n = 3 \\ -1 & -2 & -1 & -2 \\ -2 & -1 & -2 & -1 \\ 1 & 2 & 3 \end{cases} & \text{for } |I_1| = 2, n = 2 \\ 0 & \text{otherwise.} \]

By Corollary II.2(ii), we can set
\[ Q \left( \begin{array}{c} \halftree \halftree \\ 1 & 2 & \ldots & n \end{array} \right) = 0 \]
completing thereby the \( s = 3 \) iteration step.

**Iteration level \( s=4 \):** We have
\[ E_4 = \text{span} \left\{ \begin{array}{c} \halftree \\ 1 & 2 & \ldots & n \\ \halftree \halftree \\ 1 & 2 & \ldots & n \end{array} \right\} \]

By Corollary II.1(i) the values of the morphism \( Q \) on corollas of the first type are completely determined by lifts of the values of the morphism \( q \), i.e.
\[ Q \left( \begin{array}{c} \halftree \halftree \\ 1 & 2 & \ldots & n \end{array} \right) = qis^{-1} \circ q \left( \begin{array}{c} \halftree \halftree \\ 1 & 2 & \ldots & n \end{array} \right) = \begin{cases} -\frac{1}{12} & \text{for } n = 1 \\ 0 & \text{otherwise.} \end{cases} \]
It is also not hard to check that

\[
Q \left( \begin{array}{c}
\begin{array}{c}
\text{diagram}
\end{array}
\end{array} \right) = \begin{cases}
\frac{1}{2} & \text{for } |I_1| = 2, |I_2| = 2, n = 2 \\
\frac{1}{6} & \text{for } |I_1| = 2, |I_2| = 1, n = 1 \\
\frac{1}{6} & \text{for } |I_1| = 1, |I_2| = 2, n = 1 \\
\ldots & \text{for } |I_1| = 1, |I_2| = 1, n = 2 \\
0 & \text{otherwise.}
\end{cases}
\]

Here round brackets stand for symmetrization of the labels.

By Corollary II.2(ii), we can set \(Q \left( \begin{array}{c}
\begin{array}{c}
\text{diagram}
\end{array} \right) = 0\) completing thereby the \(s = 4\) iteration step.

**Iteration level \(s=6\):** as a last but least trivial example we compute the value of the morphism \(Q\) on the generator

\[
\begin{array}{c}
\begin{array}{c}
\text{diagram}
\end{array}
\end{array} \in E_6.
\]

As the value of the morphism \(q\) on such a generator is zero, we have

\[
Q \left( \begin{array}{c}
\begin{array}{c}
\text{diagram}
\end{array} \right) = e'',
\]

where \(e''\) is a solution of the equation in \([\text{Lie}\mathbf{B}^{\mathcal{C}}]\)\(\infty\),

\[
\delta e'' = Q \left( \begin{array}{c}
\begin{array}{c}
\text{diagram}
\end{array} \right) = \sum_{b+c=4} \sum_{s \geq 0} \frac{1}{s!} Q \left( \begin{array}{c}
\begin{array}{c}
\text{diagram}
\end{array} \right)
\right)
\]

\[
= \begin{cases}
-\frac{1}{12} & \text{for } n = 2 \\
0 & \text{otherwise.}
\end{cases}
\]
It is not hard to solve the latter for $e''$ and finally get

$$Q \begin{pmatrix}
\begin{array}{c}
4 \\
1 \ 2 \ \ldots \ n
\end{array}
\end{pmatrix} = \begin{cases}
-\frac{1}{12} & \text{for } n = 2 \\
-\frac{1}{6} & \\
0 & \text{otherwise}.
\end{cases}$$

**Step III.** One can show by induction on the parameter $s = 2a + k - 2$ (we omit these details) that the map $Q$ can be chosen so that the composition $Q \circ \chi_{h=1}$ (see §2.9.2) makes sense as a morphism into a completion of $[\text{LieB}^\odot]_\infty$ with respect to the number of vertices. We finally set $\hat{Q} := Q \circ \chi_{h=1}$ completing the construction. □

5.3. **Remark.** There exists a canonical monomorphism of dg props, $i : \text{LieB}^\odot_\infty \to [\text{LieB}^\odot]_\infty$. Hence any morphism $\hat{Q}$ which factors through $i$ gives rise to a universal quantization of ordinary Poisson structures.

5.4. **Remark.** The condition on the projection $\pi_1 \circ \alpha \circ \hat{Q}$ in Theorem 5.2 implies high non-triviality of the quantization morphism $\hat{Q}$ in the sense that $\hat{Q}$ involves all possible jets of the polyvector part of the input wheeled Poisson structure.

**Appendix**

A.1. **Genus 1 wheels.** Let $(P\langle E \rangle, \delta)$ be a dg free prop, and let $(P^\odot \langle E \rangle, \delta)$ be its wheeled extension. We assume in this section that the differential $\delta$ preserves the number of wheels$^9$. Then it makes sense to define a subcomplex, $(T^\odot \langle E \rangle \subset P^\odot \langle E \rangle$, spanned by graphs with precisely one wheel. In this section we use the ideas of cyclic homology to define a new cyclic bicomplex which computes cohomology of $(T^\odot \langle E \rangle, \delta)$.

All the above assumptions are satisfied automatically if $(P\langle E \rangle, \delta)$ is the free dg prop associated with a free dg operad.

We denote by $T^+ \langle E \rangle$ the obvious “marked wheel” extension of $T^\odot \langle E \rangle$ (see §3.2).

A.2. **Abbreviated notations for graphs in $T^+ \langle E \rangle$.** The half-edges attached to any internal vertex of split into, say $m$, ingoing and, say $n$, outgoing ones. The differential $\delta$ is uniquely determined by its values on such $(m,n)$-vertices for all possible $m, n \geq 1$. If the vertex is cyclic, then one of its input half-edges is cyclic and one of its output half-edges is also cyclic. In this section we show in pictures only those (half-)edges attached to vertices which are cyclic (unless otherwise is explicitly stated), so that

- \[ \begin{array}{c}
- \end{array} \] stands for a non-cyclic $(m,n)$-vertex decorated by an element $e \in E(m,n)$,

$^9$This is not that dramatic loss of generality in the sense there always exits a filtration of $(P^\odot \langle E \rangle, \delta)$ by the number of wheels whose spectral sequence has zero-th term satisfying our condition on the differential.
- \( \blacktriangleleft \) is a decorated cyclic \((m,n)\)-vertex with no input or output cyclic half-edges marked,

- \( \blacktriangleright \) is a decorated cyclic \((m,n)\)-vertex with the output cyclic half-edge marked,

- \( \blacktriangleleft \) is a decorated cyclic \((m,n)\)-vertex with the input cyclic half-edge marked.

The differential \( \delta \) applied to any vertex of the last three types can be uniquely decomposed into the sum of the following three groups of terms,

\[
\delta \blacktriangleleft = \sum_{a \in I_1} e_{a}^{a'} + \sum_{a \in I_2} e_{a}^{a'} + \sum_{b \in I_3} e_{b}^{b'},
\]

where we have shown also non-cyclic internal edges in the last two groups of terms. The differential \( \delta \) applied to \( \blacktriangleleft \) and \( \blacktriangleright \) is given by exactly the same formula except for the presence/position of dashed markings.

**A.3. New differential in** \( T^+(E) \). Let us define a new derivation, \( b \), in \( T^+(E) \), as follows:

- \( b \blacktriangleleft := \delta \blacktriangleleft \),

- \( b \blacktriangleright := \delta \blacktriangleright \),

- \( b \blacktriangleleft = \delta \blacktriangleleft \),

- \( b \blacktriangleright := \delta \blacktriangleright + \sum_{a \in I_1} e_{a}^{a'} e_{a}^{a'} \).

**A.3.1. Lemma.** The derivation \( b \) satisfies \( b^2 = 0 \), i.e. \( (T^+(E),b) \) is a complex.

Proof is a straightforward but tedious calculation based solely on the relation \( \delta^2 = 0 \).

**A.4. Action of cyclic groups.** The vector space \( T^+(E) \) is naturally bigraded,

\[
T^+(E) = \sum_{m \geq 0, n \geq 1} T^+(E)_{m,n},
\]

where the summand \( T^+(E)_{m,n} \) consists of all graphs with \( m \) non-cyclic and \( n \) cyclic vertices. Note that \( T^+(E)_{m,n} \) is naturally a representation space of the cyclic group \( \mathbb{Z}_n \) whose generator, \( t \), moves the mark to the next cyclic edge along the orientation. Define also the operator, \( N := 1 + t + \ldots + t^n : T^+(E)_{m,n} \to T^+(E)_{m,n} \), which symmetrizes the marked graphs.
A.4.1. Lemma. $\delta(1-t) = (1-t)b$ and $N\delta = bN$.

Proof is a straightforward calculation based on the definition of $b$.

Following the ideas of the theory of cyclic homology (see, e.g., [Lo]) we introduce a 4th quadrant bicomplex,

$$C_{p,q} := C_q, \quad C_q := \sum_{m+n=q} T^+\langle E\rangle_{m,n}, \quad p \leq 0, q \geq 1,$$

with the differentials given by the following diagram,

\[
\begin{array}{cccccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

A.4.2. Theorem. The cohomology group of the unmarked graph complex, $H(T^\langle E\rangle, \delta)$, is equal to the cohomology of the total complex associated with the cyclic bicomplex $C_{p,q}$.

Proof. The complex $(T^\langle E\rangle, \delta)$ can be identified with the cokernel, $C_q/(1-t)$, of the endomorphism $(1-t)$ of the total complex, $C_q$, associated with the bicomplex $C_{p,q}$. As the rows of $C_{p,q}$ are exact [Lo], the claim follows.  

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References

[Ag] M. Aguiar, *Infinitesimal Hopf Algebras*, In: New trends in Hopf algebra theory, vol. 267, Contemporary Math. (2000), 1-29.

[BFFLS] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowics, and D. Sternheimer, *Deformation theory and quantization. I. Deformations of symplectic structures*. Ann. Phys. 111 (1978), 61-110.

[Ga] W.L. Gan, *Koszul duality for dioperads*, Math. Res. Lett. 10 (2003), 109-124.

[Ko1] M. Kontsevich. *Formal (non)commutative symplectic geometry*. In: The Gel’fand mathematics seminars 1990–1992, Birkhäuser, 1993.

[Ko2] M. Kontsevich, *Deformation quantization of Poisson manifolds I*, math/9709040.

[Ko3] M. Kontsevich, letter to Martin Markl, November 2002.

[Lo] J.L. Loday, *Cyclic homology*, Springer 1998.

[MMS] M. Markl, S. Merkulov and S. Shadrin, *Wheeled PROPs and the master equation*, math.AG/0610683.

[MaVo] M. Markl and A.A. Voronov, *PROPped up graph cohomology*, math.QA/0307081.

[Mc] S. McLane, *Categorical algebra*, Bull. Amer. Math. Soc. 71 (1965), 40-106.

[Me1] S.A. Merkulov, *PROP profile of Poisson geometry*, preprint math.DG/0401034, Commun.Math.Phys. 262 (2006), 117-135.

[Me2] S.A. Merkulov, *Nijenhuis infinity and contractible dg manifolds*, math.AG/0403244, Compositio Mathematica 141 (2005), 1238-1254.

[Ta] D.E. Tamarkin, *Another proof of M. Kontsevich formality theorem*, math.QA/9803025.

[Va] B. Vallette, *Dualité de Koszul des props*, PhD thesis, Université Louis Pasteur, 2003, math.QA/0405057. English translation: *A Koszul duality for props*, math.AT/0411542.

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