On The Equivalence Of Four Dimensional
And Two dimensional Field Theories

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Abstract
We investigate the dimensional, the dynamical and the topological structures
of four dimensional Einstein and Yang-Mills theories. It is shown that these
theories are constructed from two dimensional quantities, so that they pos-
sess always a distinguished two dimensional substructure. In this sense the
four dimensional field theories are equivalent to related two dimensional field
theories.
The theories of fundamental interactions of physics are given by the four dimensional field theories, from which the three Yang-Mills theories can be considered only as partly renormalizable quantum field theories \[1\], whereas the Einstein theory of gravitation is a non-renormalizable and hence non-quantizable theory. The mathematical framework of these theories is based on the one hand on the class of "non-trivial" and "simple" forms up to the two form, i.e. up to the curvature two forms \[2\]. On the other hand it is based on the class of differential operators up to the second order operators, i.e. up to the Laplace/d’Alembert operators. In other words, beyond the trivial volume forms, all higher forms in field theories are restricted to be constructed by exterior products of simple one and two forms. A fact which is related, in view of the relations between invariant dimensions of cohomology elements and invariant indicies of differential operators, with the mentioned restriction of physical differential operators to the differential operators of first and second order. Such relations are due to the duality between the homology and cohomology elements in accord with de Rham theorem and the equality of their dimensions: The Betti numbers. In other words it is due to the relation between the cohomology dimensions and the indicies of differential operators which determine the dynamics of field theories \[3\].

In view of these restrictions, the physical or dynamical content of four dimensional theories, which can be derived from an action principle, should be equivalent to the physical or dynamical content of related two dimensional theories where the two forms are the natural geometrical limit \[4\]. Thus in both cases the dynamical content is restricted to be given by differential operators up to the second order, which act on quantities up to the two forms.

Now differential operators on the physically interesting compact manifolds have invariant indices which do not depend on the dimension of the underlying compact manifolds \[5\]. Thus the topological dimensions of forms, as the dynamical quantities of field theories, which are given by the dimensions of the harmonic forms or the cohomology elements, i.e. Betti numbers, also do not depend on the dimension of the underlying manifold \(\sim S^4\). Furthermore such numbers are equal for Poincare dual elements on a manifold in view of \(H^n \cong H^{n-a}\), which reduces such invariant numbers, in the case of four dimensional theory, to the invariants of up to two forms, i.e. to those of the related two dimensional theory. Therefore, in view of such a reducibility and related circumstances one can speak from the reduction of four dimensional field theories on compact manifolds \(\sim S^4\), to related two dimensional theories on
compact manifolds ($\sim S^2$), with respect to such topological characteristics \[5\]. We show after dimensional, dynamical and topological discussions of structure of four dimensional field theories that, in view of the fact that the topological invariants of the four dimensional field theories, i.e. the dimensions of cohomology/homology elements of the theory or the indicies of differential operators which determine the dynamics of equations of motion of four dimensional field theory, are given by the invariants of the two dimensional field theory. Therefore the four dimensional theories can be considered to be equivalent to related two dimensional field theories. Note that the same differential operators and forms determine the dynamical behaviour of the physical quantities of the field theories for example by the number of solutions of the related differential equations of motion, which is given by the index of the involved differential operator. In other words considering the topological structure of field theories disclose also the invariant local or dynamical structure of these theories.

From physical point of view, the above restriction of physical theories to quantities which are represented by forms up to the second order, i.e. up to the curvature form, can be understood as follows: Any regular physical effect is due to some force or potential which is related with a curvature. Thus in view of the fact that a true force have dimension $\frac{1}{L \cdot t} \sim L^{-2}$ in geometric units \[6\]. Therefore such effects, i.e. all regular physical effects, should be of such a two dimensional origin ($\sim L^{-2}$) in the mentioned sense: Since the two dimensional structure is characterised by its highest and lowest dimensions: The dimension of its area: $L^2$ and its curvature tensor: $L^{-2}$ (see also below).

Thus we will show that from dynamical and topological point of views the four dimensional theories are equivalent to the related two dimensional theories where the action function are given by the topological invariant surface integrals of related curvature two forms. Thus one can describe the same physical fact by a four dimensional theory with \textit{four dependent variables} from which only two are independent, or by a related two dimensional theory with the \textit{two independent variables}. Note that the two dimensional theories have the advantage to be integrable and hence quantizable, without constraints and renormalization problems of four dimensional theories which could not be solved yet in a satisfactory manner. Thus the two dimensional theory of gravity can be considered as the only quantizable theory of gravity, since as it is mentioned above the four dimensional theory of gravity is non-quantizable.

There are various hints for a two dimensional foundation of physical theories of fundamental interactions,
from the mathematical- and physical frame works of such theoris: Among them, as already mentioned, the very fundamental fact that we are given, not only from physical side but even from the mathematical side, only simple forms up to the second order. In other words the mathematical quantities to describe mathematical and physical processes are restricted to differential forms or antisymmetric tensors up to two forms or second rank tensors, respectively: \(\Omega^0, \Omega^1 = \Omega_i dx^i\) and \(\Omega^2 = \Omega_{ij} dx^i \wedge dx^j\) which correspond to the scalar functions, the connections or momentums or potentials and the curvatures or field strengths or forces on manifolds or in field theories, respectively. Note that all forms are of invariant dimension \(L^0\), since the invariant dimensions of their tensor components and their coframe basis are reciprocal, in accord with the dimensional convention: \((dx \sim x \sim L)\), \(((dx \wedge dy) \sim L^2)\) etc.. Thus such dimensions are determined in accord with the coframe structure of the volumes invariants on orientable manifolds: \(((dx^1 \wedge ..., \wedge dx^n) \sim L^n)\). 

In this sense the symmetric tensors are considered with respect to the exterior algebra as zero forms or scalar functions, i.e. of dimension \(L^0\): Since symmetric tensors possess no antisymmetric components and the volumes which define the fundamental dimensions of manifolds and theories, are defined entirely by the antisymmetric coframe basis. This consideration is also in accord with the definition of the fundamental symmetric metric tensor by: 

\[
ds^2 := g_{ij} dx^i \cdot dx^j
\]

where \(ds^2 \sim (dx^i \cdot dx^j)\) and also the metric \(g_{ij}\), all can be considered of dimension \(L^0\), in view of their symmetric structure.

Note that in the same sense, the tensor components of differential operators, such as \((\partial_i \sim \frac{\partial}{\partial x^i} \sim \frac{1}{x^i})\) and \((\epsilon_{ij} \partial_i \partial_j \sim \frac{1}{x \wedge y})\) can be considered of dimensions \(L^{-1}\) and \(L^{-2}\), respectively. Whereas the symmetric derivatives like Laplace operator: \(\Delta = (dd^i + d^i d)\), can be considered of dimension \(L^0\) in accord with their symmetric structures, thus their application on forms does not change their order. In the same sense, the antisymmetric tensor components of vector- and tensor fields on a n-dimensional manifold has dimensions \(L, ..., L^n\) and their antisymmetric frame basis are of dimensions \(L^{-1}, ..., L^{-n}\). Note further that such invariant dimensions of tensor components are independent of the dimension of the underlying manifold, thus they are in accord with the well known physical dimensions of related physical quantities such as \(((\text{force or curvature or field strength}) \sim L^{-2})\) and \(((\text{potential or momentum or mass}) \sim L^{-1})\), etc. in geometric units. Thus a force is the gradient of a regular potential which is a component of a connection one form.
Recall also that in accord with the phase space philosophy the time is not a true phase space variable, i. e. it is not an independent variable beside the canonical position and momentum variables in phase space, but it can be introduced only as a *parameter* for this variables. Thus the ”canonical conjugate variable” of time parameter , i. e. the Hamiltonian function, is a *function* of momentum and position, i. e. a *dependent* variable. This circumstance reduces the number of true variables of phase space of four dimensional theories to three. Thus any theory like the four dimensional field theories which consider, in spite of this, the time as a true variable; will possess constraints and the time component of its field variable should be eliminated by a gauge fixing condition.

There is a further hint for the fundamental role played by the two dimensional theories in physics, which is described by the fact that in view of KAM and Morse-Smale stability theorems on the integrability of physical sytems, only systems with two *independent* degrees of freedom are stable against small perturbations [7]. In other words only two dimensional theories, i. e. those with only *two independent* degrees of freedom, are integrable and stable with respect to small perturbations of related systems.

Note that in this sense the *small perturbation* or deviation from the two dimensional structure of the system can not be considered as an independent variable or degree of freedom [7]. Therefore it is plausible to consider that any physical theory which modelizes a *stable* physical process, e. g. the above mentioned fundamental theories, should be itself an stable model and hence should possess such a two dimensional foundation, in accord with the mentioned relation between the two dimensionality and the stability of sytems. This will means that integrable physical theories should possess two dimensional substructures.

Note further that in view of the fundamentality of such a *possible* invariant equivalence between the four dimensional and two dimensional field theories, the essential features of this equivalence should be embodied in the internal invariant structure of four dimensional field theories. Nevertheless the dominance of *local* point of view in theoretical physics diverted the interest from such features of theories.

Thus the main interest in the *local approach* in physics was to obtain the equations of motion and their solutions, where the conditions or assumptions to obtain the equations of motion, and the topological or global character of these conditions and equations were not of much interest.

Therefore the simplest way to consider the invariant or topological character of boundary conditions
and assumptions on physical quantities under which one obtains equations of motion, is to consider the invariant dimensions of physical quantities and their mathematical representants, i. e. the tensor components of differential forms, in accord with the above invariant determinations.

To investigate the aspired equivalence, recall that any reasonable field theory is defined on a differentiable manifold of some dimension \( n \). Thus in the same sense that such a \( n \)-dimensional manifold implies invariant or constant volume of dimension: \( L^n \), i. e. \( L^n = (\text{constant}) \) and constant tensor components of highest form of dimension \( L^{-n} \), i. e. \( L^{-n} = (\text{constant}) \). The \( n \)-dimensional theory implies also an invariant or constant volume of dimension: \( L^n \) and an invariant Lagrangian density of dimension \( L^{-n} \), i. e. again \( L^n = (\text{constant}) \) and \( L^{-n} = (\text{constant}) \). Note also that the product \( L^{-n} \cdot L^n = L^0 = (\text{constant}) \) represents the well known action invariant of the \( n \)-dimensional theory. Therefore it is possible to write an \( n \) dimensional action function in two equivalent ways: By \( \int \sum_{(nD)} \Omega^0 \cdot dx^1 \wedge \cdots \wedge dx^n \) where the Lagrangian density is a zero form, or by \( \int \sum_{(nD)} \Omega^n_1 \wedge \cdots \wedge dx^n \) where the Lagrangian density is an \( n \) form which is the Hodge dual of the mentioned zero form in \( n \) dimensions. This is in accord with the well known Hodge duality of zero forms and the highest forms on a manifold, from which the \( L^{-n} \) dimensional tensor components are proportional to the \( L^{-n} \) dimensional Lagrangian density of the related theory. In other words any \( n \)-dimensional manifold and theory implies two equivalent characteristic invariants of dimensions \( L^n \) and \( L^{-n} \) and one invariant product of them. Thus any of such characteristic invariants implies the other one by definition. Hence a theory which possess one of these characteristic invariants can be considered as \( n \)-dimensional theory or equivalent to this, thus it possess the other invariant also. Furthermore a ”higher” dimensional theory which implies lower dimensional characteristic invariants, as well as higher dimensional invariants, should be considered as equivalent to the lower dimensional theory which is defined by the related lower dimensional invariant Lagrangian and volume: Since the higher dimensional invariants in this case can be constructed from the lower dimensional invariants, e. g. by their product or by multiplication with dimensional constants, as in the case of Einstein theory (see below).

Therefore a theory with an \( L^2 \) invariant, or equivalently an \( L^{-2} \) invariant, can be considered as a two dimensional theory which is formulated on a two dimensional manifold, or it is equivalent to such a two dimensional theory, in view of the fact that both \( L^2 \) dimensional volume element and the ”dual” \( L^{-2} \)
dimensional Lagrangian density in a two dimensional theory should be invariants. In other words from the invariant dimensional point of view, a higher dimensional theory with $L^2$ and/or $L^{-2}$ invariants or constants, should be equivalent to a two dimensional theory which possess these invariants by definition. Thus one can formulate a theory either with higher number of dependent variables on a higher dimensional manifold with "conditions" which reduce the number of independent variables to the independent ones; or one can formulate an equivalent theory with these independent variables on the lower dimensional manifold.

Another direct fact about the $n$ dimensionality of a field theory is that, if its solutions, i.e. the actual field variable of theory possess $n$ independent components, then the theory should be reducible to an $n$-dimensional theory: Since a field theory with $n$ independent variables, or with a field variable with only $n$ independent components, can be defined as a $n$-dimensional theory on a $n$-dimensional manifold. In other words a theory with only $n$ independent solutions is actually a $n$ dimensional theory. This statement is in accord with the dynamical, invariant dimensional and topological statements on the $n$ dimensionality of a theory.

Further note that the above introduced invariant dimensions of tensor- and coframe basis components of forms which are independent of the dimension of the underlying manifold and therefore called invariant dimensions, are also independent of the local transformations of coordinates. The reason is that we consider these invariant dimensions in accord with the measures of invariant integrals of forms: $\int_\mathcal{C}_a \omega^a \wedge \eta$; $a \leq n$ on the $n$-dimensional manifold. In other words the invariant dimension of the coframe components of forms is considered to be equal to the dimension of the Cartesian coframe basis of forms which is equal to the invariant dimension of the invariant integral measure. This means that we consider the invariant dimensions of tensor- and coframe components of equivalence classes of forms with respect to their local transformations. It is in this sense that such a dimension is independent of local transformations and can be considered as a globally or topologically invariant dimension. In other words the introduced invariant dimensional consideration of components of forms, as the main ingredients of field theories, are in accord with their integrals, where $\Omega^0 = \int_\mathcal{C}_a \omega^a$ and $dim(\Omega^0) = L^0 = dim(\int_\mathcal{C}_a \omega^a dx^i \wedge dx^j \wedge ... \wedge dx^a) = L^{-a} \cdot L^a$.

We will show that in accord with these statements the classical four dimensional theories of electrodynamics and gravitation are indeed equivalent to two dimensional theories. Note however that the phase
space structure of four dimensional and two dimensional theories are different and hence their quantum structures are also different. Thus the two dimensional theories have the advantage to be renormalizable, whereas the four dimensional theories have essential problems with constraints and renormalization [1].

To begin with the discussion of topological and dynamical equivalence of Einstein theory of gravity with a two dimensional theory: Note that the four dimensional theory possess the Einstein constant $\kappa$, which is an invariant of dimension $L^2$, that is introduced in the theory as the reciprocal $L^{-2}$ dimensional constant. Note further that it is this invariant which makes possible to formulate the Einstein-Hilbert action invariant in four dimensions, i.e. on a four dimensional manifold with the $L^4$ dimensional volume, since the dimension of the Lagrangian density of the theory: $(\sqrt{-g}R)$ is only $L^{-2}$: In view of the $L^0$ dimensionality of metric and its determinant, and the $L^{-2}$ dimensionality of curvature: $(R \sim \frac{1}{r^2})$.

Recall that the construction of Einstein-Hilbert action bears already a distinguished two dimensional structure within the four dimensional envelope of this theory, in view of the necessity of $L^{-2}$ dimensionality of the Ricci scalar $R$ to define this theory in four dimension: Since as a scalar function or zero form $R$ should be considered of dimension $L^0 = (\text{invariant})$ with respect to the exterior calculus. Therefore in view of the neccessary equivalence between these two dimensions of $R$, i.e. $(\text{invariant}) = L^0 \cong L^{-2}$, there is a neccessary distinguished two dimensional structure in this theory, which is just defined by the equivalence relation: $(\text{invariant}) = L^0 \cong L^{-2})$. Thus the Ricci tensor and Ricci scalar can be considered to be of dimension $L^{-2}$, also in view of their extraction from the $L^{-2}$ dimensional tensor components of the curvature form.

Note further that the theory possess the cosmological constant $\Lambda$, which is an invariant of dimension $L^{-2}$.

Therefore in accord with the above discussion of characteristic invariants of a n-dimensional theory, the four dimensional Einstein theory of gravity should be equivalent to a two dimensional theory of gravity which is formulated on a two dimensional manifolds, in view of the fact that Einstein theory possess with $\kappa$ and $\Lambda$, both $L^2$- and $L^{-2}$-dimensional characteristic invariants of a two dimensional theory.

There is a further fundamental fact that implies the two dimensionality of Einstein theory from algebraic point of view: If the equations of motion of a theory with a non-Abelian curvature, e.g. Einstein theory with Riemann curvature, are obtained in local geodesic coordinate system, then this theory can be
considered to be equivalent to a two dimensional theory. The reason is that in local geodesic coordinate system the non-Abelian algebra and curvature transit to the Abelian algebra and curvature, respectively. On the other hand the Abelian algebra is equivalent to the $SO(2)$ algebra which defines the symmetry group on a two dimensional manifold. Thus an Abelian curvature is equivalent to rotation on the space of connections or to a curvature on the related two dimensional base manifold; and a theory with such a curvature should be equivalent to a two dimensional theory.

In retrospect, the fact that the Einstein equations of motion are obtainable and hence they are obtained in the local geodesic coordinates manifests the two dimensional nature of Einstein theory, although there is an equivalent method to obtain them.

The main differential geometric reason that the Einstein equations and so the whole dynamical content of the four dimensional theory of gravity is of two dimensional nature is the existence of the second Bianchi identity: $d\bar{R} = 0$ where $\bar{R}$ is the curvatur form $\bar{R} \in \Omega^2$. Thus, a trace of this identity results in the "contracted" Bianchi identity with respect to the Einstein tensor: $\partial_i(R_{ij} - \frac{1}{2}g_{ij}R) = 0$ : $i, j = 1, ..., 4$. However in view of the fact that the symmetric Einstein tensor can be considered as a scalar function with respect to the exterior calculus, i. e. in view of $(R_{ij} - \frac{1}{2}g_{ij}R) \in \Omega^0$: Therefore in accord with relation: $(d\bar{R} = 0 \leftrightarrow \partial_i(R_{ij} - \frac{1}{2}g_{ij}R) = 0)$, i. e. in accord with $(d\Omega^2 = 0 \leftrightarrow d\dual\Omega^0 = 0)$, the Einstein theory which contains such an identity is a two dimensional theory or equivalent to that: Since only a two dimensional theory implies a relation: $(d\Omega^2 = 0 \leftrightarrow d\dual\Omega^0 = 0)$, in accord with the Hodge duality: $\Omega^2 \cong \dual\Omega^0$ (see also below). Thus in a two dimensional theory on the two dimensional manifold the action function of theory: $S_{2D} = \int_{2D} \bar{R}, S_{2D} \in \Omega^0$ and the curvature two form of theory $\bar{R} \in \Omega^2$ can be considered as the dual zero- and two forms $\Omega^2 \cong \dual\Omega^0$ of the two dimensional- theory and manifold. Recall that this fact is consistent with the fact that, on the one hand the Ricci tensor is of dimension $L^{-2}$, in view of its extraction from the tensor components of the $L^{-2}$ dimensional curvature form. On the other hand it is of dimension $L^0$, in view of its symmetric tensor character which is a zero form or scalar function. Thus the invariant $L^{-2}$ dimension, is not missed here in view of the trace operation over the symmetric Riemann tensor components of curvature two form; but it is just rewritten in its equivalent dimension: $L^0$, in accord with the invariant relation $L^{-2} = (constant) = L^0$ on the two dimensional manifold.
Nevertheless we prove further arguments in favour of this statement, which concern the conditions to obtain the Einstein equations, and its structure: In other words we show in the following that even the invariant dimensional character of conditions to obtain the Einstein equations, shows a distinguished two dimensional substructure within the assumed four dimensional structure, in accord with the discussed invariant structures of the two and four dimensional theories:

Recall that the main condition under which one obtains the Einstein equations from the Einstein-Hilbert action function, is the vanishing of the term which contains the variation of Ricci tensor: $\delta R_{ij}$, in the variation of action function. Such a condition is fulfilled, without lose of generality, by the assumption of $\delta R_{ij} = 0$, and hence it is equivalent to such an assumption. In other words one can obtain from the same action function the same Einstein equations by the assumption of $\delta R_{ij} = 0$. Considering the equivalence of variation with the exterior derivative, i. e. $\delta \Omega^a \cong d \Omega^a [8]$, the condition to obtain the Einstein equations is given by: $d R_{ij} = 0$. Further note that the Bianchi identity is equivalent to the divergencelessness of Ricci tensor: $d R_{ij} = 0$. These results together mean that the Ricci tensor which obeys the Einstein equation, is a harmonic or constant function, since both its derivatives are zero. Thus symmetric tensors are considered as functions with respect to the exterior calculus and the Hodge-de Rham theory. Hence in view of $L^{-2}$ dimensionality of such a Ricci tensor and its constancy, i. e. in view of $L^{-2} \sim (constant) = L^0$, the underlying Einstein theory possess by such a $R_{ij}$ a constant of dimension $L^{-2}$. Therefore in accord with the dimensional analyses, the Einstein theory is, in principle, a two dimensional theory in agreement with the above analysis of the Einstein constant and the cosmological constant. In other words the two dimensionality of Einstein theory or its equivalence to a two dimensional theory is embodied in the structure of this theory and is incorporated in several fundamental properties and quantities of the theory.

Recall also that the wave equation for gravitational field which results from Einstein equations by linearization, i. e. $\Box g_{ij} = 0$, is no more than the statement of harmonicity of metric field. Further recall that such a metric which obeys the Einstein equations, possess only two components or two degrees of freedom, which are manifested by the two directions of polarizations of gravitational waves. Accordingly also the Riemann- and Ricci tensors which obeys the Einstein equations in form of wave equation, should possess only two independent components, since they are obtained from such a two component metric
Moreover note that in view of the fact that the gravitational field is represented, equivalently, by the metric $g_{ij}$ or by the Ricci field $R_{ij}$, therefore the dynamical equations of gravitational field can be represented also by the harmonicity of Ricci field: $\bigtriangleup R_{ij} = 0$, in accord with the above statements. Therefore also the actual or dynamical Ricci field which obeys the Einstein equation, possess only two degrees of freedom, which are represented by the two non-vanishing components of Ricci tensor for gravitational wave, that are given by the second time derivative of the two components of the metric $\mathcal{M}$. Nevertheless one can prove the two componentness of the curvature- or Ricci tensor also directly:

Thus the ten symmetric components of Ricci tensor should obey in addition to the four conditions to fulfil the Bianchi identities: $d^\dagger R_{ij} = 0$, also the four variation conditions: $\delta R_{ij} = dR_{ij} = 0$ to obtain the Einstein equations, as it is discussed above. The reason that the number of last conditions is four, is that the symmetric Ricci tensor should be considered as a scalar function with respect to the exterior differentiation $d$. Then in accord with this condition, the one form $dR_{ij} \in d\Omega^0$ on the space-time four manifold, should possess *four* vanishing components in order to fulfil the variation conditions. Therefore only two components of Ricci tensor which obeys the Einstein equations and Bianchi identities, remain independent. Hence, in accord with the statement on the independent number of dimensions of a theory and the number of its independent solutions, the Einstein theory of gravity is a two dimensional theory; or at least, it is dynamically equivalent to such a two dimensional theory.

Note with respect to the four last conditions that, as it was asserted above, the symmetric metric- and Ricci tensors are indeed zero forms, i.e. scalar functions, with respect to the exterior differential calculus:

Thus the above mentioned relation: $d^\dagger R_{ij} = 0$ which are equivalent to Bianchi identity, can be considered indeed as an identity, since in view of the absence of less than zero forms one has: $d^\dagger \Omega^0 \equiv 0$ or $d^\dagger R_{ij} \equiv 0$. Such an accordance between the identity: $d^\dagger R_{ij} \equiv 0$ and the identity of divergencelessness of zero forms or scalar functions, shows the consistency of our considerations, in accord with our consideration of symmetric tensors of second rank as zero forms with respect to the exterior differential calculus.

Note further, the fact of harmonicity of gravitational field and that it possess only two degrees of freedom, can be considered as the main invariant statement of Einstein equations. Nevertheless such a harmonicity and two dimensionality of gravitational field can be described also by the two dimensional gravitational curvature field with its two dimensional action function: $S(2D) = \int_{(2D)} \bar{R}(2D)$ over a
suitable two dimensional submanifold of the original four manifold, which should belong to the second homology class, in view of invariant theoretical requirements (see below). Here \( \bar{R}(2D) = d\Gamma(2D) = R_{mn}dx^m \wedge dx^n s \), \( m, n = 1, 2 \) is the curvature form in two dimensions: Since the equations of motion which follow from this action by its variation with respect to the gravitational connection form \( \Gamma(2D) \) are: \( d\bar{R} = 0 \). This equation, together with the trivial closedness of two forms on a two dimensional manifold: \( d\bar{R}(2D) \cong 0 \), result in the harmonicity of the \textit{two dimensional} gravitational curvature field \( \bar{R}(2D) \). Thus such a two dimensional result is equivalent to the above discussed two dimensional result for the four dimensional gravitational field or the Ricci tensor which obeys the Einstein equations. Thus in view of the fact that the Ricci tensor is given by the mixed trace of tangential- or algebraic components of curvature two form: The invariant result of Einstein equations, that their solutions possess only two degrees of freedom, should be obtainable from the equations of motion which are obtained from the two dimensional action function of Einstein-Hilbert type in two dimensions: \( S_{2D} = \int_{(2D)} \sqrt{-g(2D)} R(2D). \) Therefore the dynamical content of the four dimensional Einstein theory of gravity is equivalent to the dynamical content of the two dimensional theory of gravity, in accord with the above discussions.

The topological equivalence between the four dimensional Einstein theory of gravity and the discussed two dimensional theory is partly obvious from the above invariant dimensional considerations. To see this directly, note that on the one hand, in view of the principle of least action, the vanishing of variation of Einstein-Hilbert action function results in Einstein equations of motion which are equivalent to the harmonicity of Ricci tensor: \( (\delta S_{(EH)} = 0 \leftrightarrow R_{ij} \in Harm^0) \), in accord with the above discussions of variation condition and Bianchi identities. On the other hand, since the Einstein-Hilbert action function is a zero form, therefore the vanishing of its variation or its gradient: \( \delta S_{(EH)} = dS_{(EH)} = 0, S \in \Omega^0 \), means that it is a harmonic zero form, i. e. \( (\delta S_{(EH)} = 0 \leftrightarrow S_{(EH)} \in Harm^0) \). Therefore this theory possess the relation: \( (S_{(EH)} \in Harm^0 \leftrightarrow R_{ij} \in Harm^0) \), thus both \( S_{(EH)} \) and \( R_{ij} \) are zero forms with respect to the exterior calculus. Further, in view of the fact that the Ricci tensor is a mixed traced tensor component of the four dimensional curvature two form: \( \bar{R}(4D) \in \Omega^2 \): Therefore its harmonicity can be obtained from, and hence it can be considered to be equivalent to, the harmonicity of curvature two form: \( (R_{ij} \in Harm^0 \cong \bar{R} \in Harm^2) \). Therefore, in this theory, the harmonicity of the action zero form \( S_{(EH)} \) results in the harmonicity of the curvature two form \( \bar{R}(4D) \) and vice versa, i. e.: \( (S_{(EH)} \in \Omega^0 \cong \bar{R} \in \Omega^2) \).
Harm^0 \leftrightarrow \mathcal{R}(4D) \in \text{Harm}^2) or they are equivalent to each other: \text{Harm}^0_{(G)}(4D) \cong \text{Harm}^2_{(G)}(4D). Here the subscript \((G)\) means the gravitational field on a gravitational manifold, where other interactions are neglected.

Nevertheless such a relation is a typical relation of a two dimensional theory, i.e. on a two dimensional manifold, since only in a two dimensional theory or on a two dimensional manifold one has the very general invariant relation: \text{Harm}^0(2D) \cong \text{Harm}^2(2D) in accord with Poincare duality and Hodge’s theorem.

Therefore in view of the fact that the actual gravitational field \(R_{ij} \text{ or } g_{ij}\) of the four dimensional theory of gravity which fulfil the equations of motion, express the above invariant relation and possess only two independent components, so the theory can be considered to be equivalent to a two dimensional theory of gravity, in accord with the above discussions also from the invariant theoretical point of view.

Now we investigate the dynamical, the invariant dimensional and the topological structures of Yang-Mills theories by the example of four dimensional electrodynamics which is the only stablished Yang-Mills theory \[1\]:

Here first recall that only two of six components of the antisymmetric field strength tensor \(F_{ij}\), which obey the identities \(dF(A) = ddA \equiv 0\), are independent components: Since these identities can be written in the component form by \(\omega^i = \epsilon^{ijkl} \partial_j F_{kl} = 0\) which are 4-relations between the 6-components of \(F\).

Thus the actual number of independent components of the electromagnetic field is \((6 - 4 = 2)\) which is in accord with the well known fact that the photon posses only two polarization directions or two components. Therfore in view of our statement about the relation between the independent number of field variables of a field theory and the actual dimension of the theory, the four dimensional theory of electrodynamics is equivalent to a two dimensional theory of electrodynamics.

Further recall that, as it is mentioned above, from the two groups of Maxwell equations: \(d^4 F(4D) = 0\) and \(dF(4D) \equiv 0\) which follows from the action function: \(S_{(4D)} = \int_{(4D)} F(A(4D)) \wedge *F(A(4D))\), the second group are identities, in view of \(F := dA\). Thus the non-trivial part of Maxwell equations, i.e. the first group, with its two independent components does not need to be derived from a four dimensional theory. In other words the first group of these equations for a two component field strength form: \(F(2D)\), can be derived in accord with the usual variation with respect to \(A(2D)\), from a two dimensional
theory with the action function: \( S_{(2D)} = \int_{(2D)} F(A(2D)) \) over a suitable two dimensional submanifold of the original four manifold which should belong to the second homology class in view of invariant theoretical requirements (see below). Note also that, as it is known from Maxwell theory, even in this four dimensional theory the actual electromagnetic field \( F(A(4D)) \) or \( A(4D) \) possess only two degrees of freedom or two independent actual components. Thus one reduces the assumed four degrees of freedom to these two by the application of gauge conditions. Hence, in view of the fact that the two dimensional field \( F(A(2D)) \) also possess two degrees of freedom, by definition, therefore the two dimensional field \( F(A(2D)) \) with its two dimensional action invariant: \( S = \int_{(2D)} F(A(2D)) \) can also represent the usual electromagnetic field with its two degrees of freedom. Thus the non-trivial equations of motion in both cases are the same.

From invariant dimensional point of view the harmonicity of \( F \) in view of Maxwell equations, i.e. the constancy of its tensorial components \( F_{ij}(4D) \), means that \( L^{-2} = \text{(constant)} = L^0 \), since dimension of \( (F_{ij}) \) is \( L^{-2} \). This is but, as we discussed for the general case of \( n \)-dimensional theories, a two dimensional relation, since only on a two dimensional manifold and in a two dimensional theory, the \( L^2 \)-dimensional area and so the \( L^{-2} \)-dimensional Lagrangian density are invariants or constants. Thus this is exactly what one has in the two dimensional theory of electrodynamics with the action function \( S = \int_{(2D)} F(A(2D)) \) and the equations of motion: \( (d^+ F(2D) = 0 \text{ and } d F(2D) \equiv 0) \sim (F(2D) \in Harm^2) \).

Hence the dynamical structure of electromagnetic field and its true number of degrees of freedom are well represented and described by the two dimensional theory of electrodynamics. Therefore from the dynamical and invariant dimensional point of views the four dimensional theory of electrodynamics can be considered as equivalent to its two dimensional theory.

The topological equivalence between the four dimensional Yang-Mills theory and its two dimensional theory is given beyond of the discussed invariant dimensional considerations, by the following aspects of topological contents of these two theories:

First note in this relation that invariants of a four dimensional manifold and hence of a four dimensional theory, are given by the dimensions of cohomology elements or of harmonic forms, i.e. by Betti numbers: \( b^4(4D) = dimH^4(4D), b^3(4D) = dimH^3(4D), b^2(4D) = dimH^2(4D), b^1(4D) = dimH^1(4D) \).
and $b^0(4D) = \dim H^0(4D)$. Nevertheless in view of Poincare dualities on this manifold, i.e.: $H^4(4D) \cong H^0(4D)$ and $H^3(4D) \cong H^1(4D)$ one has $b^4 = b^0$ and $b^3 = b^1$. Therefore all invariants of a four dimensional field theory can be given by only dimensions of cohomologies up to the second order: $H^2, H^1$ and $H^0$, i.e. by: $b^2(4D), b^1(4D)$ and $b^0(4D)$; which are also the cohomologies of two manifolds or of two dimensional field theories: $b^2(2D) = \dim H^2(2D), b^1(2D) = \dim H^1(2D)$ and $b^0(2D) = \dim H^0(2D)$. Thus in view of de Rham theorem one can choose a homology basis where: $b^2(4D) = b^2(2D), b^1(4D) = b^1(2D)$ and $b^0(4D) = b^0(2D)$ [3]: Since also in the four dimensional case the Betti numbers are given by the integrals of related cohomology elements over the related homology elements, which are two and one dimensional submanifolds of the four dimensional manifolds. In other words the invariants of four dimensional case are given by the invariants of the two dimensional substructures. This fact shows on the one hand the fundamental relation between the four dimensional field theories and the two dimensional field theories as intended in this work, in view of their common differential structure basis up to the second order and the resulting common invariants. On the other hand it underlines so the topological background of the restriction of four dimensional field theories to differential forms up to the second order.

Note that in a four dimensional or two dimensional manifold without boundary $b^1 = b^3 = 0$ or $b^1 = 0$. Thus the desired two dimensional field theory can be formulated on the mentioned two dimensional submanifold of the four manifold of the original four dimensional field theory, i.e. on the mentioned second homology manifold, which enables also the definition of Betti number $b^2(2D)$ in the four dimensional case. Thus in the four and two dimensional cases without boundary, all invariants can be obtained from these two: $b^2$ and $b^0$, in view of the above discussion.

Therefore in view of the above discussed and also following consideration, one can establish a closed topological equivalence between the four- and two dimensional field theories:

1. In view of the mentioned restriction of mathematical-physical simple forms to zero-, one- and two forms, the invariants of a four dimensional Yang-Mills theory result from the invariants of these forms, i.e. from the invariants of cohomology elements $H^0(4D) \in \Omega^0(4D), H^1(4D) \in \Omega^1(4D)$ and $H^2(4D) \in \Omega^2(4D)$.

2. The invariants of $H^0(4D), H^1(4D)$ and $H^2(4D)$ are given by their Betti numbers which are defined by:
\[ b^0 = \dim H^0 = \dim \text{Harm}^0, \quad b^1 = \oint_{\mathcal{C}_1} \Omega^1(4D) \quad \text{and} \quad b^2 = \int_{\mathcal{C}_2} \Omega^2(4D), \] respectively; where the integrations of cohomology elements \( \Omega^1(4D) \) and \( \Omega^2(4D) \) are considered over the related homology elements \( C_1 \) and \( C_2 \) which are, respectively, one- and two dimensional submanifolds of the four dimensional basis manifold.

Thus in this sense in accord with de Rham duality \( b^a = \dim H_a \), the classes of homological submanifolds of the four manifold, \textit{up to the two dimensional ones}, play the same essential role like the cohomology elements up to the second order in the topology and invariant aspects of four dimensional theories.

3. From (1.) and (2.) it follows that the invariants of a four dimensional Yang-Mills theory result from the mentioned Betti numbers as the dimensions of the harmonic- or cohomology- or homology elements on the four dimensional manifold.

4. The invariants of the above discussed related two dimensional theory, result from the invariants of \( H^0(2D) \in \Omega^0(2D), H^1(2D) \in \Omega^1(2D) \) and \( H^2(2D) \in \Omega^2(2D) \) on the underlying two dimensional manifold of the theory which is a submanifold of the four dimensional manifold. These are given, as in (2.) by \( b^0 = \dim H^0 = \dim \text{Harm}^0, \quad b^1 = \oint_{\mathcal{C}_1} \Omega^1(2D) \quad \text{and} \quad b^2 = \int_{\mathcal{C}_2} \Omega^2(2D), \] respectively; where the integrations are considered over the homology elements \( C_1 \) and \( C_2 \) which are, respectively, one- and two dimensional submanifolds of the two dimensional basis manifold.

5. From (3.) and (4.), i.e. from (1.) - (4.), it follows that the invariants of the four dimensional Yang-Mills theory should be the same as the invariants of the related two dimensional theory.

6. The result in (5.) is in accord with the invariants of the four dimensional theory and of the related two dimensional theory which are deduced from their equations of motion, in view of the above discussed invariant dimensional considerations.

Furthermore note that considering the action function as a zero form which is a harmonic form in view of the principle of least action, if we identify as above the variation of action zero form, by its exterior derivative \( \delta \). Then the resulting equations of motion in the four dimensional Yang-Mills theory, i.e. the harmonicity of the \( F \) form: or \( F \in \text{Harm}^2 \), manifests the following invariant relation: That the harmonicity of the action zero form of the four dimensional Yang-Mills theory results in the harmonicity of the four dimensional Yang-Mills two form and vice versa. In other words in the four dimensional Yang-Mills theory these two harmonicities are isomorphic to each other:
Nevertheless such a relation manifests a two dimensional theory with the above discussed action function

\[ S_{(2D)} = \int F, \]

since only in a two dimensional theory or on a two dimensional manifold one has the very general invariant relation: \( \text{Harm}^0(2D) \cong \text{Harm}^2(2D) \), in accord with Poincare duality and Hodge’s theorem. Thus this result is in accord with the above result for the Betti numbers of both theories, in view of the fact that \( b^a = \text{dim}H^a = \text{dim}\text{Harm}^a \).

This proves again the topological equivalence of the four dimensional Yang-Mills theory and its two dimensional counterpart.

Therefore one can understand why one was forced to reduce the number of components of electromagnetic field in the four dimensional electrodynamics from four to the two independent ones by gauge conditions.

The reason is that the four dimensional abelian Yang-Mills theory describes, as we showed, from dynamical and invariant theoretical point of views, nothing else than the two dimensional dynamical and invariant facts, in view of the only two independent dynamical degrees of freedom of electromagnetic field and in accord with the two polarization of photons.

Furthermore with respect to both type of two dimensional theories on compact manifolds note that, since on a two dimensional manifold the general element of the second cohomology group \( \Omega^2 \in H^2 \) is given by: \( \Omega^2 = d\Omega^1 \oplus \text{Harm}^2 \). Therefore Gauss’s law: \( \int_{S^2} F = \int_{S^2} dA = 0 \) does not apply here, since: \( \int_{S^2} \Omega^2 \neq \int_{S^2} d\Omega^1 \). Thus the integral \( \int_{S^2} \Omega^2 \) is an invariant of \( S^2 \) which is a multiple of the area of \( S^2 \), and is proportional to the value of the applied constant field strenght or the constant curvature: \( \text{Harm}^2 \) on \( S^2 \), since not only the area is an invariant of \( S^2 \), but also the \( \text{Harm}^2 \).

Thus we proved by various consistent methodes the equivalence of both types of four- and the related two dimensional theories with respect to their dynamical, invariant dimensional and topological contents.

In conclusion let us recall that the mentioned topological invariants, such as Betti numbers \( b^2 \) or \( b^1 \), are related with the main quantum numbers in Bohr-Sommerfeld quantization and in flux quantization, which are integrals of closed one or two forms over two or one dimensional manifolds, respectively. Thus there is no known main quantum number which is related with higher Betti numbers than the second one.
Footnotes and references

References

[1] For example in view of the non-convergence of renormalized perturbation series in QED (see F. Dyson, Phys. Rev. 85, (1952) 631).

[2] We mean by ”non-trivial” forms on a \( n \)-dimensional manifold, those beyond the \( n \)-dimensional volume forms: \( dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \) which is dual to the zero form \( \Omega^0 \). Thus we mean by ”simple” forms, those which are not given by the exterior product of other forms. In this sense the ”non-trivial” and ”simple” forms are connection one forms \( \Omega^1 \) and curvature two forms \( \Omega^2 \). The Chern-Simons form: \( \Omega^1 \wedge \Omega^2 \) and the Yang-Mills form: \( \Omega^2 \wedge *\Omega^2 \) are examples of non-simple or product forms.

[3] For topological notations see among others: M. Nakahara: ”Geometry, Topology and Physics” (Adam Hilger, 1990); C. Nash: ”Differential Topology and Quantum Field Theory”, (Academic Press 1991).

Note that on a two dimensional manifold the general element of the second cohomology group \( \Omega^2 \in H^2 \) is given in accord with the Hodge decomposition: \( \Omega^n = d\Omega^{n-1} \oplus d^*\Omega^{n+1} \oplus \text{Harm}^n \), by:
\[
\Omega^2 = d\Omega^1 \oplus \text{Harm}^2,
\]
since there is no three form \( \Omega^3 \) on a two dimensional manifold. In this sense the Gauss’s law:
\[
\int_{S^2} \tilde{\Omega}^2 = \int_{S^2} d\Omega^1 = 0
\]
does not apply here, since:
\[
\int_{S^2} \Omega^2 \neq \int_{S^2} d\Omega^1.
\]
Note further that in this paper we consider: \( (F, \bar{R}) \in \Omega^2 \).

[4] On a two dimensional manifold the highest form is the two form. Note further that we use the concept of degrees of freedom of a field, equivalent to the independent dynamical or actual components of the field. Thus the assumed number of coordinates or field components that fulfil dynamical equations (of motion) is reduced always to such independent components or degrees of freedom which fulfil such equations.

[5] Since the Euler characteristic and thereby the index of differential operator on \( S^2 \) and \( S^4 \) manifolds are equal. In other words, since these indices can be given by the Euler characteristics on compact
manifolds, which is equal to two for the even dimensional spheres as the important class of compact manifolds; whereas it is zero for the spheres of odd dimensions. Therefore not the number of dimensions of compact manifolds, but only its odd- or evenness plays a role for this index. Note that we consider, as usual, the $S^4$ as the mathematical model of the underlying four dimensional manifold of four dimensional field theories, specially with respect to the compactness requirement of the underlying manifold of quantum field theories.

[6] As it will be clarified in the text: In geometric units a scalar function is dimensionless $L^0$, the components of connection one forms, i.e. potentials and momentums, are of dimension $L^{-1}$ and the components of two forms, i.e. curvatures or field strengths or forces, are of dimension $L^{-2}$. Note that in geometric units the dimension of time is the same as length, i.e. $L$.

[7] R. Abraham, J. E. Marsden: "Foundattions Of Mechanics", (Benjamin/Cummings Publishing Co., Inc. 1978); V. I. Arnol’d: "Mathematical methodes in classical mechanics", (graduate text in mathematics, Springer-Verlag 1978).

Note also the so called restricted three body problem, which is a two body problem with a small third body on the same plane as a small perturbation.

[8] This is the only possibility of a non-trivial definition of variation for functions, since a function is, as a zero form, divergenceless by definition: $d^0 \Omega^0 \equiv 0$, and $R_{ij} \in \Omega^0$. Note that also in the phase space, the variation of variable functions are considered to be equal to their differentials, i.e. $\delta P_i = dP_i$ and $\delta Q_i = dQ_i$.

[9] H. Stephani: "General Relativity", (Cambridge University Press 1982).