Optimal estimation and discrimination of excess noise in thermal and amplifier channels

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We determine a fundamental upper bound on the performance of any adaptive protocol for discrimination or estimation of a channel which has an unknown parameter encoded in the state of its environment. Since our approach relies on the principle of data processing, the bound applies to a variety of discrimination measures, including quantum relative entropy, hypothesis testing relative entropy, Rényi relative entropy, fidelity, and quantum Fisher information. We apply the upper bound to thermal (amplifier) channels with a known transmissivity (gain) but unknown excess noise. In these cases, we find that the upper bounds are achievable for several discrimination measures of interest, and the method for doing so is non-adaptive, employing a highly squeezed two-mode vacuum state at the input of each channel use. Estimating the excess noise of a thermal channel is of principal interest for the security of quantum key distribution, in the setting where a fiber-optic cable has a known transmissivity but a tampering eavesdropper alters the excess noise on the channel, so that estimating the excess noise as precisely as possible is desirable. Finally, we outline a practical strategy which can be used to achieve these limits.

Introduction—One of the primary goals of quantum information theory is to identify limitations on how well one can process information or estimate an unknown parameter, when allowing for quantum effects \[1\]–\[4\]. Along with this goal, there is great interest in determining whether it is possible to approach these limits in principle, and furthermore, if this can be done in practice with realistic constraints taken into account, such as time, energy, scalability, etc.

In this paper, we are interested in the fundamental limitations on channel discrimination and estimation for a particular class of quantum channels. Suppose that an unknown parameter \(x\) is encoded in an environmental state, which subsequently interacts with an input quantum system \(A\) via a fixed unitary quantum interaction. Suppose further that the unitary interaction has two output quantum systems, one of which is available and denoted as \(B\) and the other is lost or discarded to the environment. The transformation of the input system \(A\) to the output system \(B\) is called a quantum channel. Let us call such channels environment-parametrized channels, given that the unknown parameter \(x\) is encoded exclusively in the environment and not in the unitary interaction \([5]\). Important environment-parametrized channels of practical interest are thermal channels with a fixed, known transmissivity and unknown excess noise. Other examples are amplifier channels with a fixed, known gain but unknown excess noise.

We consider two tasks: first, we suppose that the parameter \(x\) takes one of two values and the goal is to figure out which value it takes. Second, we suppose that the parameter \(x\) takes a value from a continuum and the goal is to estimate the unknown parameter. The former task is called channel discrimination \([6]\)–\([11]\) and the latter channel estimation \([12]\)–\([16]\), both topics having an extensive literature already. Also, there are strong connections between the two tasks \([17]\), as one might suspect. In these tasks, we would like for the error probability or the mean-square error, respectively, to be as small as possible when determining the unknown parameter.

For both tasks, the most general strategy one could allow for is an adaptive strategy, when trying to determine an unknown parameter \(x\) encoded in a quantum channel \(\mathcal{N}^{A_{1}\rightarrow B}\) (see Figure 1). An adaptive strategy that makes \(M\) calls to the channel is specified in terms of an input quantum state \(\rho_{R_{1}A_{1}}\), a set of adaptive, interleaved channels \(\mathcal{A}^{i}_{R_{i}B_{i}\rightarrow R_{i+1}A_{i+1}}\) \(i=1,\ldots,M-1\), and a final quantum measurement \(\mathcal{A}^{\ast}_{R_{M}B_{M}}\) that outputs an estimate \(\hat{x}\) of the unknown parameter. The strategy begins with the discriminator preparing the input quantum state \(\rho_{R_{1}A_{1}}\) and sending the \(A_{1}\) system into the channel \(\mathcal{N}^{A_{1}\rightarrow B_{1}}\). The channel \(\mathcal{N}^{A_{1}\rightarrow B_{1}}\) outputs the system \(B_{1}\), which is then available to the discriminator. The discriminator adjoins the system \(B_{1}\) to system \(R_{1}\) and applies the channel \(\mathcal{A}^{i}_{R_{i}B_{i}\rightarrow R_{i+1}A_{i+1}}\). We say that the channel \(\mathcal{A}^{i}_{R_{i}B_{i}\rightarrow R_{i+1}A_{i+1}}\) is adaptive because it can take an action conditioned on information in the system \(B_{i}\), which itself might contain some partial information about the unknown parameter \(x\). The discriminator then inputs the system \(A_{2}\) into the second use of the channel \(\mathcal{N}^{A_{2}\rightarrow B_{2}}\), which outputs a system \(B_{2}\). This process repeats \(M-2\) more times, and at the end, the discriminator has systems \(R_{M}\) and \(B_{M}\). The discriminator finally performs a measurement \(\mathcal{A}^{\ast}_{R_{M}B_{M}}\) that outputs an estimate \(\hat{x}\) of the unknown parameter \(x\). The conditional probability for the estimate \(\hat{x}\) given the unknown parameter \(x\) is given by the...
Born rule:

\[
P_{\hat{x}|x}(\hat{x}|x) = \frac{\text{Tr}\{A_{M}^{\hat{x}}B_{M}(N_{A_{M}B_{M}}^{n}A_{M}^{M-1}B_{M-1}A_{M}\cdots A_{R_{1}B_{1}}A_{2})\rho_{R_{1}A_{1}}\}}{\text{Tr}\{A_{M}^{\hat{x}}B_{M}\}}
\]

Note that such an adaptive strategy contains a non-adaptive strategy as a special case: the system \(R_1\) can be arbitrarily large and divided into subsystems, with the only role of the interleaved channels \(A_{R_{1}B_{1}}\rightarrow R_{2}A_{2}\) being that they redirect these subsystems to be the inputs of future calls to the channel (as would be the case in any non-adaptive strategy for estimation or discrimination).

Our first main result is a general upper bound on the performance of adaptive discrimination and estimation of environment-parametrized channels. We establish this upper bound for any discrimination measure that satisfies a data-processing inequality (that is, it is monotone non-increasing with respect to the action of a quantum channel). Our result thus holds for all known and useful discrimination measures, given that the data-processing inequality is the most basic requirement needed for any discrimination measure. This includes well known discrimination measures such as quantum relative entropy [18], Rényi relative entropy [19, 20], quantum fidelity [22], trace distance, Chernoff information [23, 24], hypothesis testing relative entropy [25, 26], etc., each of which have operational interpretations for certain information-processing tasks. The essential statement of the upper bound is that one’s ability to discriminate or estimate environment-parametrized channels is limited by how well one can discriminate or estimate the environmental states that encode the unknown parameter.

In our second main result, we show that it is possible to attain this upper bound in principle for a number of the discrimination measures listed above, when estimating excess noise in thermal channels or excess noise in amplifier channels. For these particular channels, the unknown parameter is the mean photon number of an environmental thermal state, while the transmissivity or gain is known in our scenario. We find that the optimal strategy does not involve any adaptation whatsoever and consists solely in sending one share of a highly squeezed two-mode squeezed vacuum state into each use of the channel, followed by a measurement on the output systems. What we find remarkable about this result is that, in the limit of large squeezing, several of the discrimination measures mentioned above depend only on the mean photon number of the environmental thermal state and have no dependence on the transmissivity or gain of the channel. Thus, such a strategy with a highly squeezed two-mode squeezed vacuum state allows for removing the effect of loss or gain in the channel, and we provide a physical interpretation for this phenomenon in what follows.

Our results for estimating excess noise in thermal channels should be useful for the security of quantum key distribution [28]. There, the transmissivity is typically known when the communication medium is a fiber-optic cable, but the excess noise in the channel can be attributed to a tampering eavesdropper. Thus, estimating excess noise in the channel is of primary interest and plays a critical role in security analyses.

Environment-parametrized channels—We begin by defining an environment-parametrized quantum channel [15, 20]. Let \(x\) be an unknown parameter, and let \(\theta_{E}^{x}\) be a quantum state that depends on \(x\). Let \(U_{AE_{1}B_{1}E_{1}}\) be a unitary operator that takes vectors in a tensor-product input Hilbert space \(\mathcal{H}_{A} \otimes \mathcal{H}_{E_{1}}\) to vectors in a tensor-product output Hilbert space \(\mathcal{H}_{B} \otimes \mathcal{H}_{E_{1}}\). Then we define an environment-parametrized channel \(N_{A_{1}B_{1}}^{x}\) as follows:

\[
N_{A_{1}B_{1}}^{x}(L_{A}) \equiv \text{Tr}_{E_{1}}\{U_{AE_{1}B_{1}E_{1}}(L_{A} \otimes \theta_{E}^{x})(U_{AE_{1}B_{1}E_{1}})^{\dagger}\},
\]

where \(L_{A}\) is an operator acting on \(\mathcal{H}_{A}\) and \(\text{Tr}_{E_{1}}\) denotes the partial trace. By inspecting the above definition, we see that it is only the environment state \(\theta_{E}^{x}\) that depends on the unknown parameter \(x\) and the unitary interaction \(U_{AE_{1}B_{1}E_{1}}\) is fixed and independent of \(x\). Thus, all of the information that distinguishes one channel \(N_{A_{1}B_{1}}^{x}\) from another channel \(N_{A_{1}B_{1}}^{x}\) is encoded in the environment of these channels.

Particular examples of environment-parametrized channels are thermal channels, noisy amplifier channels, Pauli channels, and erasure channels. We review the first two here and sketch later why the latter two are environment-parametrized. The unitary \(U_{AE_{1}B_{1}E_{1}}\) for a thermal channel is defined from the following Heisenberg input-output relations:

\[
\hat{b} = \sqrt{\eta} \hat{a} + \sqrt{1 - \eta} \hat{e},
\]

\[
\hat{e}' = -\sqrt{1 - \eta} \hat{a} + \sqrt{\eta} \hat{e},
\]

where \(\hat{a}, \hat{b}, \hat{e}, \text{and} \hat{e}'\) are the field-mode annihilation operators for the sender’s input, the receiver’s output, the environment’s input, and the environment’s output of these channels, respectively. The environmental mode \(\hat{e}\) is prepared in a thermal state \(\theta(N_{B})\) of mean photon number \(N_{B} \geq 0\), defined as

\[
\theta(N_{B}) \equiv \frac{1}{N_{B} + 1} \sum_{n=0}^{\infty} \left(\frac{N_{B}}{N_{B} + 1}\right)^{n} |n\rangle\langle n|.
\]
where \( \{ |n\} \) is the orthonormal, photonic number-state basis. The parameter \( N_B \) is the excess noise of the thermal channel. When \( N_B = 0 \), \( \theta(N_B) \) reduces to the vacuum state, in which case the resulting channel in (3) is called the pure-loss channel—it is said to be quantum-limited in this case because the environment is injecting the minimum amount of noise allowed by quantum mechanics. The parameter \( \eta \in [0, 1] \) is the transmissivity of the amplifier channel, representing the average fraction of photons making it from the input to the output of the channel. Let \( L_{\eta,N_B} \) denote this channel. In our application, we set the unknown parameter \( x = N_B \), and we suppose that the transmissivity \( \eta \) is known.

The unitary \( U_{AE ightarrow B} \) for an amplifier channel is defined from the following Heisenberg input-output relations:

\[
\hat{b} = \sqrt{G} \hat{a} + \sqrt{G - 1} \hat{e}^\dagger, \\
\hat{e}^\dagger = \sqrt{G - 1} \hat{a} + \sqrt{G} \hat{e}^\dagger.
\]

The parameter \( G \geq 1 \) is the gain of the amplifier channel. For this channel, the environment is prepared in the thermal state \( \theta(N_B) \). The parameter \( N_B \) is the excess noise of the amplifier channel. If \( N_B = 0 \), the amplifier channel is said to be quantum-limited for a similar reason as stated above. Let \( A_{G,N_B} \) denote this channel. The class of amplifier channels we consider are those with a fixed gain \( G \) and the unknown parameter \( x = N_B \).

**General bound from quantum data processing**—We now establish our first main result. Let \( D(\rho||\sigma) \) denote a generalized divergence \([21,30]\), which is a function accepting two quantum states as input and producing a non-negative real number as its output. The only property that we demand to hold for a generalized divergence is that the following data-processing inequality hold:

\[
D(\rho||\sigma) \geq D(N(\rho)||N(\sigma)),
\]

where \( N \) is a quantum channel. The inequality in (8) asserts that a generalized divergence \( D \), interpreted as a measure of distinguishability of the states \( \rho \) and \( \sigma \), does not increase under the action of a quantum channel \( N \). Particular examples of generalized divergences include quantum relative entropy \([18]\), hypothesis testing relative entropy \([26,27]\), quantum fidelity \([22]\), trace distance, Rényi relative entropy \([19,21]\), etc. Note that any generalized divergence is unitarily invariant \([21]\); i.e., the following equality holds for any unitary operator \( U \):

\[
D(\rho||\sigma) = D(U\rho U^\dagger||U\sigma U^\dagger),
\]

because \( (\cdot) \rightarrow (\cdot)U^\dagger \) and \( (\cdot) \rightarrow U(\cdot)U^\dagger \) are quantum channels, \( D(\rho||\sigma) \geq D(U\rho U^\dagger||U\sigma U^\dagger) \), and \( D(U\rho U^\dagger||U\sigma U^\dagger) \geq D(U^\dagger U\rho U^\dagger||U^\dagger U\sigma U^\dagger) = D(\rho||\sigma) \). Furthermore, it is invariant with respect to tensoring in the same state \( \tau \) \([21]\):

\[
D(\rho||\tau) = D(\rho \otimes \tau||\sigma \otimes \tau),
\]

because \( (\cdot) \rightarrow (\cdot) \otimes \tau \) is a quantum channel and partial trace is a quantum channel, so that \( D(\rho||\sigma) \geq D(\rho \otimes \tau||\sigma \otimes \tau) \) and \( D(\rho \otimes \tau||\sigma \otimes \tau) \geq D(\rho||\sigma) \).

Suppose that the discriminator is attempting to distinguish two environment-parametrized channels of the form in (2), where the environmental state is either \( \theta_E^{x_1} \) or \( \theta_E^{x_2} \). In such a case, the conditional probability for outputting \( \hat{x} \) is \( p_{X} | X(\hat{x}|x_i) \) for \( i \in \{1, 2\} \) as given in (1), whenever the discrimination strategy is the most general adaptive strategy as outlined before. Then our first main result is the following inequality

\[
D([\theta_{E}^{x_1}]^{\otimes M}||[\theta_{E}^{x_2}]^{\otimes M}) \geq D(p_{X} | X(\hat{x}|x_1)||p_{X} | X(\hat{x}|x_2)).
\]

(11)

Manifest in the above inequality is the following intuitive statement: the discriminator’s ability to distinguish the two channels, if given \( M \) calls to the channel, cannot be any better than if the discriminator were presented with \( M \) copies of the environmental state \( \theta_E^{x_i} \) and then asked to decide with which one he was presented. If the generalized divergence is also additive with respect to tensor-product states, which holds for many examples of divergences as we discuss below, then \([11]\) reduces to

\[
M D(\theta_{E}^{x_1}||\theta_{E}^{x_2}) \geq D(p_{X} | X(\hat{x}|x_1)||p_{X} | X(\hat{x}|x_2)).
\]

(12)

We note that results bearing some similarities to (11) have appeared in previous papers \([15,16]\), but the previous statements are not given in such generality (i.e., for all generalized divergences) nor were the previous statements argued to apply to the most general adaptive strategy one could consider and instead only argued for non-adaptive strategies.

We now prove the inequality in (11). For simplicity, let us suppose that the adaptive discrimination strategy consists of two calls to the unknown channel, and then it will be easy to see how to generalize the result to get (11). Then, in this case,

\[
p_{X} | X(\hat{x}|x_i) = Tr(A_{R_2B_2}(N_{A_1 \rightarrow B_2} \circ A_{R_1B_1 \rightarrow R_2A_2} \circ N_{A_1 \rightarrow B_1}(\rho_{R_1A_1})),
\]

(13)

and let us abbreviate the expression on the right as \( Tr(A_{X}^{X_1} A_{1}^{A_1} N_{X_1}(\rho)) \). Then

\[
D(p_{X} | X(\hat{x}|x_1)||p_{X} | X(\hat{x}|x_2)) \\
\leq D(N_{X_1}^{X_1} A_{1}^{A_1} N_{X_2}(\rho)||N_{X_2}^{X_1} A_{1}^{A_1} N_{X_2}(\rho)) \\
\leq D(U(A_{1}^{A_1} N_{X_1}(\rho) \otimes \theta_{E}^{x_2}) U^\dagger||U(A_{1}^{A_1} N_{X_2}(\rho) \otimes \theta_{E}^{x_2}) U^\dagger) \\
= D(A_{1}^{A_1} N_{X_1}(\rho) \otimes \theta_{E}^{x_2}||A_{1}^{A_1} N_{X_2}(\rho) \otimes \theta_{E}^{x_2}) \\
\leq D(N_{X_1}(\rho) \otimes \theta_{E}^{x_2}||N_{X_2}(\rho) \otimes \theta_{E}^{x_2}) \\
\leq D(U(\rho \otimes \theta_{E}^{x_2}) U^\dagger \otimes \theta_{E}^{x_2}||U(\rho \otimes \theta_{E}^{x_2}) U^\dagger \otimes \theta_{E}^{x_2}) \\
= D(\rho \otimes \theta_{E}^{x_1} \otimes \theta_{E}^{x_2}||\rho \otimes \theta_{E}^{x_1} \otimes \theta_{E}^{x_2} \\
= D(\theta_{E}^{x_1} \otimes \theta_{E}^{x_2}||\theta_{E}^{x_1} \otimes \theta_{E}^{x_2}).
\]

(14)

All of the steps given above are a consequence of the data-processing inequality in \([3]\). The first inequality follows because the final measurement can be considered as
a quantum channel acting on the states $N^x_2A^1N^x_2(\rho)$ and $N^x_2A^2N^x_2(\rho)$ that produces the respective output probability distributions $p_{\tilde{X}|X}(\tilde{x}|x_1)$ and $p_{\tilde{X}|X}(\tilde{x}|x_2)$. The second inequality follows from the definition of environment-parametrized channels in [2] and because a partial trace is a quantum channel. The first equality follows because any generalized divergence is unitarily invariant, as recalled in [3]. The third inequality follows by discarding the adaptive channel $A^1$. The next few steps follow the same reasoning and the last equality follows from (10). Thus we establish the inequality in (11) for $M = 2$, but it is easy to see that repeating the above steps establishes (11) for arbitrary $M$.

Examples of generalized divergences—One notable generalized divergence is the quantum hypothesis testing relative entropy $D_H^\varepsilon(\rho||\sigma)$, defined for $\varepsilon \in [0, 1]$ as follows:

$$D_H^\varepsilon(\rho||\sigma) \equiv -\log \inf_{\Lambda} \text{Tr}[\Lambda \sigma], \quad (15)$$

where the infimum is with respect to all operators $\Lambda$ satisfying $0 \leq \Lambda \leq I$ and $\text{Tr}[\Lambda \rho] \geq 1 - \varepsilon$. The physical interpretation of this quantity is in asymmetric hypothesis testing: if it is desired that the error probability in identifying the state $\rho$ by a measurement $\{\Lambda, I - \Lambda\}$ be less than $\varepsilon$, then $\inf_{\Lambda} \text{Tr}[\Lambda \rho]$ is the minimum error that one could have in identifying the state $\sigma$ using the same binary-outcome quantum measurement. The hypothesis testing relative entropy is a quantity of deep interest in quantum information theory because various relevant information measures can be built from it, which are useful in assessing the performance of a variety of information-processing tasks [31][35]. It obeys the data processing inequality in [8] by its very definition, for the simple reason that applying the same quantum channel to the states $\rho$ and $\sigma$ never decreases the two different error probabilities discussed above [27].

Applying the result in [11] leads to the following bound:

$$D_H^\varepsilon(p_{\tilde{X}|X}(\tilde{x}|x_1)||p_{\tilde{X}|X}(\tilde{x}|x_2)) \leq D_H^\varepsilon(\theta_E^{x,1}||\theta_E^{x,2}) + \sqrt{MV(\theta_E^{x,1}||\theta_E^{x,2})\Phi^{-1}(\varepsilon) + O(\log M)}, \quad (16)$$

where, in the last equality, we have used the quantum relative entropy $D(\rho||\sigma) \equiv \text{Tr}[\rho \log \rho - \log \sigma]$ [18], the quantum relative entropy variance $V(\rho||\sigma) \equiv \text{Tr}[\rho \log \rho - \log \sigma - D(\rho||\sigma)^2]$ [32][35], the inverse of the cumulative Gaussian distribution function $\Phi$, and an expansion of the hypothesis testing relative entropy that holds for tensor-power states [32][35][37]. The bound in (16) thus places a fundamental limitation on the performance of any adaptive channel discrimination strategy in the context of asymmetric hypothesis testing.

Notable additive generalized divergences are given by the Rényi relative entropies [19][21], defined for $\alpha \in (0, 1) \cup (1, \infty)$ as

$$D_\alpha(\rho||\sigma) \equiv \frac{1}{\alpha - 1} \log \text{Tr}[\rho^{\alpha} \sigma^{1-\alpha}], \quad (17)$$

$$\tilde{D}_\alpha(\rho||\sigma) \equiv \frac{2\alpha}{\alpha - 1} \log \|\rho^{1/2}(1-\alpha)^{1/2}\sigma^{1/2}\|_2, \quad (18)$$

where $\|A\|_p \equiv \text{Tr}[\{\|A\|^p\}]^{1/p}$ and $|A| \equiv \sqrt{A^*A}$. The first one $D_\alpha(\rho||\sigma)$ satisfies [8] for $\alpha \in (0, 1) \cup [1, 2]$ [19], and the second one satisfies [8] for $\alpha \in (1/2, 1) \cup (1, \infty)$ [20][21]. Both are additive with respect to tensor-product states, converge to the quantum relative entropy in the limit as $\alpha \to 1$, and thus satisfy [8] in this limit. Applying [11] we find that

$$D_\alpha(p_{\tilde{X}|X}(\tilde{x}|x_1)||p_{\tilde{X}|X}(\tilde{x}|x_2)) \leq M D_\alpha(\theta_E^{x,1}||\theta_E^{x,2}),$$

$$\tilde{D}_\alpha(p_{\tilde{X}|X}(\tilde{x}|x_1)||p_{\tilde{X}|X}(\tilde{x}|x_2)) \leq M \tilde{D}_\alpha(\theta_E^{x,1}||\theta_E^{x,2}), \quad (19)$$

for the ranges of $\alpha$ for which data processing holds. As these quantities have operational meaning in the context of asymmetric hypothesis testing as error exponents and strong converse exponents in the quantum Hoefting bound [40], the above inequalities place fundamental limitations on the exponential convergence rate of error probabilities of adaptive channel discrimination strategies in this setting (see also [41] for results on adaptive channel discrimination and Rényi relative entropies).

A special case of the Rényi relative entropy in (18) when $\alpha = 1/2$ is $-\log F(\rho, \sigma)$, the negative logarithm of the quantum fidelity, the latter defined as $F(\rho, \sigma) \equiv \|\sqrt{\rho} \sqrt{\sigma}\|^2_1$. Applying (12), we find that

$$-M \log F(\theta_E^{x,1}||\theta_E^{x,2}) \geq -\log F(p_{\tilde{X}|X}(\tilde{x}|x_1), p_{\tilde{X}|X}(\tilde{x}|x_2)). \quad (20)$$

An important measure in quantum estimation theory is the quantum Fisher information [42][45], related to quantum fidelity and defined for a continuously parametrized set $\{\sigma^x\}$ of states as [22][Theorem 6.3]

$$I_F(x; \{\sigma^x\}) \equiv \lim_{\delta \to 0} \frac{8}{9} \left[1 - \sqrt{F(\sigma^x, \sigma^{x+\delta})}\right]/\delta^2 \quad (21)$$

(See the appendix for a derivation of the second equality.) The importance of the quantum Fisher information is that it is a lower bound on the variance of an unbiased estimator $\hat{x}$ of $x$ [42][45].

$$\Var(\hat{x} - x) \geq \left[I_F(x; \{\sigma^x\})\right]^{-1}. \quad (22)$$

One can apply the same reasoning to adaptive protocols for estimating an unknown parameter $x$ encoded in a family $\{N^x\}^x$ of channels, and we find that

$$\Var(\hat{x} - x) \geq \left[I_F^M(x; \{N^x\})\right]^{-1}, \quad (23)$$

where $I_F^M(x; \{N^x\})$ is the Fisher information with respect to the conditional probability defined in [1]. Applying the bound in (20) and the relation between fidelity...
and Fisher information in [15, 40, 46], given that those works did not consider adaptive protocols for estimating \( x \).

**Application to thermal channels**—We now show that several of the above upper bounds are in fact achievable, whenever the goal is to determine the excess noise in a thermal channel with known transmissivity. We begin with channel discrimination. Suppose that we are given two thermal channels \( \mathcal{L}_{\eta, N_B^1} \) and \( \mathcal{L}_{\eta, N_B^2} \), each having a known transmissivity \( \eta \in [0, 1) \) with excess noise equal to \( N_B^1 \geq 0 \) or \( N_B^2 \geq 0 \). (If \( \eta = 1 \) or \( N_B^1 = N_B^2 \), then it is impossible to distinguish the channels and so we do not consider these cases.)

In all cases for discrimination or estimation, we find that a non-adaptive strategy involving \( M \) copies of a highly squeezed, two-mode squeezed vacuum state suffices to attain the upper bounds given above, proving that this non-adaptive strategy suffices for achieving the best possible performance. The two-mode squeezed vacuum state is equivalent to a purification of the thermal state in [15] and is defined as

\[
|\psi_{\text{TMS}}(N_S)\rangle_{RA} \equiv \frac{1}{\sqrt{N_S + 1}} \sum_{n=0}^{\infty} \sqrt{\binom{N_S}{n} \frac{1}{n} \frac{1}{n} A} \langle n| \rangle_R|n\rangle_A. \tag{25}
\]

The strategy we are employing in all cases leads to the following, final pre-measurement state for \( i \in \{1, 2\} \):

\[
\sigma_{N_B^i} \equiv \left[ (\text{id}_R \otimes \mathcal{L}_{\eta, N_B^i})(|\psi_{\text{TMS}}(N_S)\rangle \langle \psi_{\text{TMS}}(N_S)|_{RA}) \right] \otimes M. \tag{26}
\]

Starting with quantum relative entropy, we find the following expansion for large \( N_S \) and for \( \eta \in [0, 1) \), by employing a formula for the quantum relative entropy of Gaussian states [47, 48]:

\[
D(\sigma_{N_B^1} || \sigma_{N_B^2}) = -g(N_B^1, N_B^1) + g(N_B^1, N_B^2) + O(1/N_S) \tag{27}
\]

\[
= D(\theta(N_B^1) || \theta(N_B^2)) + O(1/N_S). \tag{28}
\]

where \( g(x, y) \) is a relative entropic generalization of the well known formula for the entropy of a bosonic thermal state (see, e.g., [49]) and is defined for \( x, y \geq 0 \) as

\[
g(x, y) \equiv (x + 1) \log(y + 1) - x \log y. \tag{29}
\]

In fact, as indicated in [28], we find for all \( \eta \in [0, 1) \) that

\[
\lim_{N_S \to \infty} D(\sigma_{N_B^1} || \sigma_{N_B^2}) = D(\theta(N_B^1) || \theta(N_B^2)), \tag{30}
\]

so that the relative entropy in the limit of high squeezing converges to the classical relative entropy between the two thermal states that distinguish the channels (here we say classical relative entropy because the states \( \theta(N_B^i) \) and \( \theta(N_B^i) \) commute).

Similarly, we find the following expansion for the quantum relative entropy variance for large \( N_S \) and for \( \eta \in [0, 1) \), by employing a formula for the quantum relative entropy variance of Gaussian states [50]:

\[
D(\sigma_{N_B^1} || \sigma_{N_B^2}) = N_B^1(N_B^1 + 1) \log(\frac{1 + 1/N_B^1}{1 + 1/N_B^2}) + O(1/N_S) \tag{31}
\]

\[
= V(\theta(N_B^1)||\theta(N_B^2)) + O(1/N_S). \tag{32}
\]

As indicated in [41], we also find for all \( \eta \in [0, 1) \) that

\[
\lim_{N_S \to \infty} D(\sigma_{N_B^1} || \sigma_{N_B^2}) = D(\theta(N_B^1) || \theta(N_B^2)). \tag{33}
\]

The formula in [30] is an expression for the relative entropy variance of two thermal states, which generalizes the entropy variance formula from [51] for a thermal state. See the appendix for a derivation.

By the statement in [15], we find the following upper bound on the performance of any adaptive strategy when discriminating the channels

\[
D_H^*(p_{X|\theta}(\hat{x}|N_B^1)p_{\bar{X}|\theta}(\hat{x}|N_B^2)) \leq MD(\theta(N_B^1)||\theta(N_B^2))
\]

\[
+ \sqrt{MV(\theta(N_B^1)||\theta(N_B^2))} \Phi^{-1} (\varepsilon) + O(\log M). \tag{34}
\]

Since we know from prior work [32, 30, 37] the following lower bound on the hypothesis testing relative entropy

\[
D_H^*(\sigma_{N_B^1} || \sigma_{N_B^2}) \geq MD(\sigma_{N_B^1} || \sigma_{N_B^2})
\]

\[
+ \sqrt{MV(\sigma_{N_B^1} || \sigma_{N_B^2})} \Phi^{-1} (\varepsilon) + O(\log M), \tag{35}
\]

the expansions for large \( N_S \) in [28] and [41] establish that the upper bound in [32] is achievable in the limit as \( N_S \to \infty \). As a consequence, by using a highly squeezed state as a probe and in the limit of high squeezing, it is as if the loss in the channel has no effect on the transmitted state and one’s ability to distinguish the channels is as good as one’s ability to distinguish the environmental states \( \theta(N_B^1) \) and \( \theta(N_B^2) \), which correspond to the excess noise in the channels. We offer an explanation for this phenomenon later on.

Turning to the fidelity, we find similar results. Applying a formula for the fidelity of two-mode Gaussian states [52], we find for \( \eta \in [0, 1) \) that

\[
F(\sigma_{N_B^1}, \sigma_{N_B^2}) = \left[ \sqrt{(N_B^1 + 1)(N_B^2 + 1)} - \sqrt{N_B^1 N_B^2} \right]^{-2} + O(1/N_S) \tag{36}
\]

\[
= F(\theta(N_B^1), \theta(N_B^2)) + O(1/N_S). \tag{37}
\]

Consistent with our previous observations and as indicated in [15], we also find for \( \eta \in [0, 1) \) that

\[
\lim_{N_S \to \infty} F(\sigma_{N_B^1}, \sigma_{N_B^2}) = F(\theta(N_B^1), \theta(N_B^2)). \tag{38}
\]
We finally consider the quantum Fisher information \( I_F(N_B; \{\sigma_{N_B}\}_B) \) as defined in \([21]\). Applying a formula for the fidelity of two-mode Gaussian states \([22]\) and expanding about small \(\delta > 0\) and large \(N_S\), we find for \(N^1_B = N_B\) and \(N^2_B = N_B + \delta\) that

\[
\sqrt{F}(\sigma_{N_B^1}, \sigma_{N_B^2}) = 1 - \frac{1 - \eta \left[(N_S(1 - \eta)(2N_B + 1) + 1)\right]^{-1}\delta^2}{8N_B(N_B + 1)} + O(\delta^3/N_S^2). \tag{36}
\]

Thus, by applying \([21]\), we find that the quantum Fisher information in the large \(N_S\) limit is equal to

\[
\lim_{N_S \to \infty} I_F(N_B; \{\sigma_{N_B}\}_B) = \frac{1}{N_B(N_B + 1)}, \tag{37}
\]

in agreement with \([23]\) Eq. (63)]. By applying the bound from \([24]\), the fact that the quantum Fisher information of an ensemble \(\{\theta(N_B)\}_{N_B}\) of thermal states is equal to \([N_B(N_B + 1)]^{-1}\), and the fact that the quantum Fisher information is achievable in principle by a measurement \([42-45]\), we can conclude that there exists a non-adaptive strategy that achieves the ultimate precision possible in the limit of high squeezing. Furthermore, the form of the quantum Fisher information in \([27]\) has an intuitive form: the noisier the state, the lower the Fisher information, and vice versa.

**Concrete Discrimination Strategy**—All of the convergences of the quantum discrimination measures to the discrimination of two thermal states begs for an intuitive explanation. Here we give some explanation for this phenomenon, by establishing a physical relation between a thermal state with mean photon number \(N_B\) and the state \(\sigma_{N_B}\) defined in \([26]\), in the limit as \(N_S \to \infty\). At the same time, this explanation leads to a concrete discrimination strategy consisting of applying the unitary transformation given below followed by photodetection.

The Wigner characteristic function covariance matrix for \(\sigma_{N_B}\) in \([26]\) is as follows (see, e.g., \([48]\):

\[
V = \begin{pmatrix}
a & c & 0 & 0 \\
c & b & 0 & 0 \\
0 & 0 & a & -c \\
0 & 0 & -c & b
\end{pmatrix}, \tag{38}
\]

where

\[
a = \eta N_S + (1 - \eta)N_B + 1/2, \tag{39}
\]

\[
b = N_S + 1/2, \tag{40}
\]

\[
c = \sqrt{\eta N_S(N_S + 1)}. \tag{41}
\]

Consider the following symplectic transformation:

\[
S = \begin{pmatrix}
\omega_+ & -\omega_- & 0 & 0 \\
-\omega_- & \omega_+ & 0 & 0 \\
0 & 0 & \omega_+ & \omega_- \\
0 & 0 & \omega_- & \omega_+
\end{pmatrix}, \tag{42}
\]

where

\[
\omega_+ = \sqrt{\frac{1 + N_S}{1 + (1 - \eta)N_S}}, \tag{43}
\]

\[
\omega_- = \frac{\eta N_S}{1 + (1 - \eta)N_S}. \tag{44}
\]

The symplectic transformation \(S\) is independent of \(N_B\) and diagonalizes \(V\) when \(N_B = 0\). Also, \(S\) can be realized by a two-mode squeezer, which corresponds to a unitary transformation acting on the tensor-product Hilbert space of the two modes. Applying \(S\) to \(V\) with finite \(N_B\), we get

\[
SVS^T = \begin{pmatrix}
a_s & -c_s & 0 & 0 \\
-c_s & b_s & 0 & 0 \\
0 & 0 & a_s & c_s \\
0 & 0 & c_s & b_s
\end{pmatrix}, \tag{45}
\]

where

\[
a_s = N_B + 1/2 + O(N_S^{-1}), \tag{46}
\]

\[
b_s = (1 - \eta)N_S + \eta N_B + 1/2 + O(N_S^{-1}), \tag{47}
\]

\[
c_s = \sqrt{\eta N_B + O(N_S^{-1})}. \tag{48}
\]

One can physically eliminate the off-diagonal terms by randomizing the two modes (or just by simply treating them separately). Then in the limit as \(N_S \to \infty\), we find that the above state is equivalent to a product of two thermal states with photon numbers \(N_B\) and \((1 - \eta)N_S + \eta N_B\). So a concrete discrimination strategy consists in applying the above unitary transformation to the output of each channel, tracing over the second mode, and performing photodetection on the first mode, which is the optimal measurement for distinguishing two thermal states.

**Application to amplifier channels**—For quantum amplifier channels with a fixed known gain but unknown excess noise, we find results similar to the ones given above for thermal channels. The upper bound from \([11]\) results in a generalized divergence between two thermal states. Also, the quantum relative entropy, the quantum relative entropy variance, the fidelity, and the quantum Fisher information evaluated for the state \(\langle \psi_{\text{TMS}}(N_S) \rangle |\langle \psi_{\text{TMS}}(N_S) \rangle_{RA}\rangle^\otimes M\) converge to the same expressions given above in the limit of high squeezing, having no dependence on the gain of the amplifier channel. There is a similar explanation for the convergences as given above and a resulting concrete discrimination strategy in the limit of high squeezing.

**Teleportation method**—One can also arrive at our results for thermal and amplifier channels in terms of a technique called teleportation simulation \([41]\) Section V]. In \([44]\) Section V], the authors showed how any protocol consisting of adaptive operations interleaved between many independent uses of the same channel can be reduced to a non-adaptive protocol if the channel is simulable by teleportation. This method was reviewed recently in \([48]\) and therein extended to continuous-variable
bosonic channels and others as well. Recently, the technique was also applied in the context of channel discrimination and estimation of particular channels [55].

Briefly, the main idea of the teleportation method is to 1) replace every channel in the protocol by its simulation with teleportation and 2) rearrange all of the uses of the channel to the start of the protocol, such that all of the adaptive operations act at the end of the protocol and the resulting protocol no longer has the adaptive form. For the channels considered in [55] (limited to Pauli channels or erasure channels), the resulting protocol is such that one feeds in $M$ shares of a maximally entangled state to each channel use. Then a final measurement is performed on this state to discriminate two channels in a given class.

In the examples that we consider here, including thermal channels of a fixed transmissivity or amplifier channels of a fixed gain, we can instead use the two-mode squeezed vacuum state and continuous-variable teleportation [56] to effect the teleportation reduction discussed above. One critical aspect of the problem setup is that the channels being discriminated or estimated have the same transmissivity or gain, so that the teleportation correction operations are independent of the particular channel being discriminated or estimated. In order for the teleportation simulation to be perfect, it is necessary to consider the limit of high squeezing, as we have done above, and the result is to recover all of the convergences of quantum discrimination measures discussed previously.

The teleportation simulation approach to understanding our results is interesting, but we think that the data-processing method outlined in this paper is simpler and more powerful when applicable. The data-processing method applies independently of whether a channel is teleportation simulable, and furthermore, we only need a generalized divergence for the argument in [14] to hold, whereas one further requires continuity (albeit a natural property) in order for the teleportation argument to go through in the continuous-variable case. Finally, the data-processing method outlined here recovers all of the results established in [55] because all of the channels considered there are in fact environment-parametrized. To see this, for Pauli channels, we can take the environment state $\theta^E_\rho$ in [2] to be $\sum_{i=0}^{d^2-1} p_i |i\rangle\langle i|_E$ and the unitary interaction to be $\sum_{i=0}^{d^2-1} U_A^i \otimes |i\rangle\langle i|_E$, where the parameter $x$ is the probability vector $\{p_i\}$, and $U_A^i$ is a Pauli operator. For erasure channels, we can take the environment state in [2] to be $\sum_{i=0}^{d^2-1} p_i |i\rangle\langle i|_E \otimes |e\rangle\langle e|_{E_2}$ and the unitary interaction to be $|0\rangle\langle 0|_{E_1} \otimes I_{AE_2} + |1\rangle\langle 1| \otimes \text{SWAP}_{AE_2}$.

It would be interesting to determine if there are teleportation-simulable channels that are not environment-parametrized. If it were the case, then the teleportation simulation method could be used to analyze adaptive discrimination and estimation protocols, whereas the data-processing method would not necessarily apply.

**Conclusion**—We have outlined a general method for bounding the performance of adaptive channel discrimination or estimation of environment-parametrized channels, in which an unknown parameter is encoded in the environment of the channel. The method applies to any generalized divergence, a function whose sole property is data processing (monotonicity under the action of a quantum channel). We applied the approach to several discrimination measures that have operational meaning in a variety of contexts. As a concrete example, we considered thermal (amplifier) channels with known transmissivity (gain) and unknown excess noise. We derived limitations on the performance of the most general adaptive discrimination or estimation strategies for these channels, and we also showed that these limits are achievable in principle if highly squeezed states are available.

Going forward from here, it would be interesting to generalize the approach to channels encoding multiple unknown parameters that need to be estimated or discriminated—the results from [53, 57] should be helpful here, at least in the case of quantum Gaussian channels. We also wonder whether there are other approaches, besides the data-processing method or the teleportation simulation approach, that could be used to simplify adaptive protocols for channel discrimination or estimation.

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**Appendix**—Here we establish the formula in [30] for the relative entropy variance of two thermal states and the formula in [21] for the quantum Fisher information.

We begin by establishing [30]. Let

$$\rho = \frac{1}{N_B + 1} \sum_{n=0}^{\infty} \left( \frac{N_B^1}{N_B^1 + 1} \right)^n |n\rangle\langle n|, \tag{49}$$

$$\sigma = \frac{1}{N_B^2 + 1} \sum_{n=0}^{\infty} \left( \frac{N_B^2}{N_B^2 + 1} \right)^n |n\rangle\langle n|. \tag{50}$$

The relative entropy variance is defined as

$$V(\rho||\sigma) = \text{Tr} \{ \rho \log \rho - \log \sigma - D(\rho||\sigma)^2 \}. \tag{51}$$

Consider that

$$D(\rho||\sigma) = -g(N_B^1, N_B^1) + g(N_B^1, N_B^2) \tag{52}$$

$$= -(N_B^1 + 1) \log(N_B^1 + 1) + N_B^1 \log N_B^1$$

$$+ (N_B^1 + 1) \log(N_B^2 + 1) - N_B^1 \log N_B^2. \tag{53}$$
Also,
\[
\log \rho - \log \sigma = -\log(N_B^1 + 1) + \log(N_B^2 + 1)
+ \sum_{n=0}^{\infty} n \log \left( \frac{N_B^1}{N_B^1 + 1} \left[ \frac{N_B^2}{N_B^2 + 1} \right]^{-1} \right) |n\rangle\langle n| \tag{54}
\]
where \(n\) is the photon-number operator. So then
\[
\log \rho - \log \sigma - D(\rho||\sigma)
= -N_B^1 \log(N_B^1 + 1) + N_B^1 \log N_B^1
+ N_B^1 \log(N_B^2 + 1) - N_B^1 \log N_B^2
+ \hat{n} \log \left( \frac{N_B^1}{N_B^1 + 1} \left[ \frac{N_B^2}{N_B^2 + 1} \right]^{-1} \right) \tag{55}
\]
and
\[
\left[ \log \rho - \log \sigma - D(\rho||\sigma) \right]^2
= (\hat{n} - N_B^1)^2 \log^2 \left( \frac{N_B^1}{N_B^1 + 1} \left[ \frac{N_B^2}{N_B^2 + 1} \right]^{-1} \right) \tag{56}
\]
Finally, we find that
\[
V(\rho||\sigma)
= \langle (\hat{n} - N_B^1)^2 \rangle_{\rho(N_B^1)} \log^2 \left( \frac{N_B^1}{N_B^1 + 1} \left[ \frac{N_B^2}{N_B^2 + 1} \right]^{-1} \right) \tag{59}
\]
\[
= N_B^1 (N_B^1 + 1) \log^2 \left( \frac{N_B^1}{N_B^1 + 1} \left[ \frac{N_B^2}{N_B^2 + 1} \right]^{-1} \right) \tag{60}
\]
Now we derive the formula in [21] for the quantum Fisher information:
\[
I_F(x; \rho_x) = \lim_{\delta \to 0} -4 \log F(\rho_x, \rho_{x+\delta}) / \delta^2 \tag{61}
\]
To begin with, consider the known formula for quantum Fisher information [2, Theorem 6.3]:
\[
\lim_{\delta \to 0} \frac{8}{\delta^2} \left[ 1 - \sqrt{F(\rho_x, \rho_{x+\delta})} \right] \tag{62}
\]
To evaluate this, we apply L'Hospital’s rule, and find that
\[
\lim_{\delta \to 0} -8 \frac{\delta}{\delta^2} \sqrt{F(\rho_x, \rho_{x+\delta})} / 2\delta
= \lim_{\delta \to 0} \left[ -4 \frac{\delta^2}{\delta^2} \sqrt{F(\rho_x, \rho_{x+\delta})} \right] \tag{63}
= -4 \frac{\delta^2}{\delta^2} \sqrt{F(\rho_x, \rho_{x+\delta})} \bigg|_{\delta=0} \tag{64}
\]
so that
\[
I_F(x; \rho_x) = -4 \frac{\delta^2}{\delta^2} \sqrt{F(\rho_x, \rho_{x+\delta})} \bigg|_{\delta=0} \tag{65}
\]
Now we move on to showing (61). Consider that
\[
-2 \log F(\rho_x, \rho_{x+\delta}) = -4 \log \sqrt{F(\rho_x, \rho_{x+\delta})} \tag{66}
\]
Furthermore,
\[
\frac{d^2}{d\delta^2} \left[ -\log \sqrt{F(\rho_x, \rho_{x+\delta})} \right]
= \frac{d}{d\delta} \left[ \frac{d}{d\delta} \left[ -\log \sqrt{F(\rho_x, \rho_{x+\delta})} \right] \right] \tag{67}
\]
\[
= \frac{d}{d\delta} \left[ -\left( \sqrt{F(\rho_x, \rho_{x+\delta})} \right)^{-1} \left( \frac{d}{d\delta} \sqrt{F(\rho_x, \rho_{x+\delta})} \right) \right] \tag{68}
\]
\[
= \left( \sqrt{F(\rho_x, \rho_{x+\delta})} \right)^{-2} \left( \frac{d}{d\delta} \sqrt{F(\rho_x, \rho_{x+\delta})} \right)^2 \tag{69}
\]
Then we find that
\[
\frac{d^2}{d\delta^2} \left[ -\log \sqrt{F(\rho_x, \rho_{x+\delta})} \right] \bigg|_{\delta=0}
= \left( \sqrt{F(\rho_x, \rho_{x+\delta})} \right)^{-2} \left( \frac{d}{d\delta} \sqrt{F(\rho_x, \rho_{x+\delta})} \right)^2 \bigg|_{\delta=0}
\]
\[
- \left( \sqrt{F(\rho_x, \rho_{x+\delta})} \right)^{-1} \left( \frac{d^2}{d\delta^2} \sqrt{F(\rho_x, \rho_{x+\delta})} \right) \bigg|_{\delta=0} \tag{70}
\]
The quantity
\[
\left( \frac{d}{d\delta} \sqrt{F(\rho_x, \rho_{x+\delta})} \right)^2 \bigg|_{\delta=0} = 0 \tag{72}
\]
because
\[
\frac{d}{d\delta} \sqrt{F(\rho_x, \rho_{x+\delta})}
= \frac{d}{d\delta} \text{Tr} \left\{ \sqrt{\rho_x \rho_{x+\delta}} \sqrt{\rho_x} \right\} \tag{73}
\]
\[
= \frac{1}{2} \text{Tr} \left\{ \left( \sqrt{\rho_x \rho_{x+\delta}} \sqrt{\rho_x} \right)^{-1/2} \sqrt{\rho_x} \frac{d}{d\delta} (\rho_{x+\delta}) \sqrt{\rho_x} \right\} \tag{74}
\]
and so

\[
\frac{d}{d\delta} \sqrt{F(\rho_x, \rho_{x+\delta})} \bigg|_{\delta=0} = \frac{1}{2} \text{Tr} \left\{ \left( \sqrt{\rho_x} \rho_x \rho_x \sqrt{\rho_x} \right)^{-1/2} \sqrt{\rho_x} \left( \frac{d}{d\delta} \rho_{x+\delta} \right) \bigg|_{\delta=0} \right\} \sqrt{\rho_x} \]

(75)

\[
= \frac{1}{2} \text{Tr} \left\{ \rho_x^{-1} \sqrt{\rho_x} \left( \frac{d}{d\delta} \rho_{x+\delta} \bigg|_{\delta=0} \right) \sqrt{\rho_x} \right\} \]

(76)

\[
= \frac{1}{2} \text{Tr} \left\{ \frac{d}{d\delta} \rho_{x+\delta} \bigg|_{\delta=0} \right\} \]

(77)

\[
= 0,
\]

(78)

where the last line follows from the definition of the derivative and because the difference of two density operators is equal to zero. (In the above we assumed that the density operators \(\rho_x\) are full rank but one can arrive at the same conclusion when they are not necessarily full rank \([58]\).) So we find that

\[
\frac{d^2}{d\delta^2} [-2 \log F(\rho_x, \rho_{x+\delta})] \bigg|_{\delta=0} = -4 \frac{d^2}{d\delta^2} \sqrt{F(\rho_x, \rho_{x+\delta})} \bigg|_{\delta=0}, \quad (79)
\]

which is consistent with \([63]\).

Thus we can conclude \([61]\) because after applying L’Hospital’s rule, we find that

\[
\lim_{\delta \to 0} -\frac{4 \log F(\rho_x, \rho_{x+\delta})}{\delta^2} = \lim_{\delta \to 0} -\frac{d}{d\delta} \frac{1}{2} \log F(\rho_x, \rho_{x+\delta}) \]

(80)

\[
= \lim_{\delta \to 0} \left[ -\frac{d^2}{d\delta^2} 2 \log F(\rho_x, \rho_{x+\delta}) \right]. \quad (81)
\]
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