AN OPERATOR APPROACH TO THE RATIONAL SOLUTIONS OF THE CLASSICAL YANG-BAXTER EQUATION

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Abstract. Motivated by the study of the operator forms of the constant classical Yang-Baxter equation given by Semonov-Tian-Shansky, Kupershmidt and the others, we try to construct the rational solutions of the classical Yang-Baxter equation with parameters by certain linear operators. The fact that the rational solutions of the CYBE for the simple complex Lie algebras can be interpreted in terms of certain linear operators motivates us to give the notion of \( \mathcal{O} \)-operators such that these linear operators are the \( \mathcal{O} \)-operators associated to the adjoint representations. Such a study can be generalized to the Lie algebras with nondegenerate symmetric invariant bilinear forms. Furthermore, we give a construction of a rational solution of the CYBE from an \( \mathcal{O} \)-operator associated to the coadjoint representation and an arbitrary representation with a trivial product in the representation space respectively.

1. Introduction

The classical Yang-Baxter equation (CYBE) first arose in the study of the inverse scattering theory (see [1], [2]) and has played an important role in the study of the classical integrable systems ([3], [4], [5], [6], [7], [8], [9], etc.). There are some close relations between it and many branches of mathematical physics and pure mathematics, like symplectic geometry, quantum groups, quantum field theory and so on (see [10] and the references therein).

The classical Yang-Baxter equation with spectral parameters is given as
\[
[r, r] = [r_{12}(u_1, u_2), r_{13}(u_1, u_3)] + [r_{12}(u_1, u_2), r_{23}(u_2, u_3)] + [r_{13}(u_1, u_3), r_{23}(u_2, u_3)] = 0,
\]
where \( r \) is a function \( r : \mathbb{F} \otimes \mathbb{F} \to \mathfrak{g} \otimes \mathfrak{g} \) with \( \mathfrak{g} \) being a Lie algebra over a field \( \mathbb{F} \) and the notations \( r_{ij} \) are given as follows. For any \( r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g} \), set
\[
\begin{align*}
    r_{12} &= \sum_i a_i \otimes b_i \otimes 1, & r_{13} &= \sum_i a_i \otimes 1 \otimes b_i, & r_{23} &= \sum_i 1 \otimes a_i \otimes b_i,
\end{align*}
\]
and the commutation relations in (1.1) are given in the universal enveloping algebra \( U(\mathfrak{g}) \) of the Lie algebra \( \mathfrak{g} \).

Most of the study on the classical Yang-Baxter equation (1.1) is concentrated on the following cases ([11], [12], [13], [14], etc.): \( \mathfrak{g} \) is taken as a finite-dimensional simple Lie algebra over the
complex number field $\mathbb{C}$ and $r$ is nondegenerate which depends on a single parameter. That is, $r$ satisfies
\[
r(u_1, u_2) = r(u_1 - u_2),
\]
and there is no proper subalgebra $h$ of $g$ such that $r(u) \in h \otimes h$.

According to Belavin and Drinfeld ([11], [12]), the nondegenerate solutions of the classical Yang-Baxter equation (1.1) depending on a single parameter for the simple complex Lie algebras are divided into three cases: trigonometric, elliptic and rational. In this paper, we pay our main attention to the rational solutions $r$ with exactly one pole. In fact, a general form of a rational solution $r$ of the CYBE can be written as ([11], [12], [13], [14], [15], [16])
\[
r(u_1, u_2) = \frac{t}{u_1 - u_2} + r_0(u_1, u_2),
\]
where $t$ is the Casimir element of $g$ and $r_0$ is a polynomial in $g[u_1] \otimes g[u_2]$. However, it is not easy to get an explicit expression of $r_0$ from the equation (1.4). Moreover, it is also difficult to extend the study from the simple complex Lie algebras to the other Lie algebras.

On the other hand, for any $r \in g \otimes g$, $r$ can be expressed by a matrix under a basis. So it is natural to consider the conditions satisfied by the linear maps corresponding to the matrices (classical $r$-matrices) satisfying the CYBE. For the constant solutions of the CYBE, Semonov-Tian-Shansky ([5]) first gave an operator form of the CYBE as a linear map $R : g \rightarrow g$ satisfying
\[
[R(x), R(y)] = R([R(x), y] + [x, R(y)]), \quad \forall x, y \in g.
\]
It is equivalent to the tensor form of the CYBE when the following two conditions are satisfied: (a) there exists a nondegenerate symmetric invariant bilinear form on $g$ and (b) $r$ is skew-symmetric. Note that equation (1.5) is exactly the Rota-Baxter relation of weight-zero in the version of Lie algebras ([17], [18], [19]), whereas the Rota-Baxter relations were introduced to generalizes the integration-by-parts formula ([20], [21], [22]) and then (the versions of associative algebras) play important roles in many fields in mathematics and mathematical physics (cf. [23] and the references therein).

Furthermore, Kupershmidt ([24]) replaced the above condition (a) by letting $r$ be a linear map from $g^*$ to $g$ and when $r$ is skew-symmetric, the tensor form of the CYBE is equivalent to such a linear map $r$ satisfying
\[
[r(a^*), r(b^*)] = r(ad^*r(a^*)(b^*) - ad^*r(b^*)(a^*)), \quad \forall a^*, b^* \in g^*,
\]
where $g^*$ is the dual space of $g$ and $ad^*$ is the dual representation of the adjoint representation (coadjoint representation) of the Lie algebra $g$. Moreover, Kupershmidt generalized the above
ad to be an arbitrary representation $\rho : g \to gl(V)$ of $g$, that is, a linear map $T : V \to g$ satisfying

$$[T(u), T(v)] = T(\rho(T(u))v - \rho(T(v))u), \forall u, v \in V, \quad (1.7)$$

which was regarded as a natural generalization of the CYBE. Such an operator $T$ is called an $O$-operator associated to $\rho$ by Kupershmidt ([24]). It was also mentioned in [25]. Moreover, such an $O$-operator indeed gives a constant solution of the CYBE in a larger Lie algebra ([26]).

Then it is natural to consider how to extend such an idea to study the rational solutions of the CYBE (1.1), which is the main aim in our paper. We would like to point out that this study is not a simple generalization since it is quite different with the study of the constant solutions (see the discussion in Section 5), although the idea is quite similar to the study in [26]. On the other hand, Xu also considered to use the operator form to study the CYBE (1.1) in [27] (even he extended his study to any nonassociative algebra). We would like to point out that although the ideas are quite similar (which both are in fact motivated by the study of Semonov-Tian-Shansky ([5])), they are two different approaches. One of the main differences is that Xu’s approach is the direct generalization of equation (1.5) with a similar form (thus the existence of nondegenerate associative symmetric bilinear form and the skew-symmetry is necessary for his study on the general nonassociative algebras including Lie algebras) and he focused on the trigonometric solutions with a similar form on certain more general algebras, whereas our approach are essentially the generalizations of equations (1.5)-(1.7) with certain “modified” forms for a general Lie algebra without many additional constraints and we paid our main attention to the rational solutions with the form (1.4). More comparisons between the two approaches are given in the following sections.

The paper is organized as follows. In Section 2, we interpret the rational solutions of the CYBE for the simple complex Lie algebras in term of certain linear operators which motivates us to give the notion of $O$-operators such that these linear operators are the $O$-operators associated to the adjoint representations. Such a study can be generalized to the Lie algebras with nondegenerate symmetric invariant bilinear forms. In Section 3, we generalize the Casimir element appearing in the rational solutions of the CYBE in Section 2 to a symmetric invariant tensor under the action of the adjoint representation, which gives a construction of a rational solution of the CYBE from an $O$-operator associated to the coadjoint representation. In Section 4, we give a construction of a rational solution of the CYBE from an $O$-operator associated to an arbitrary
representation with a trivial product in the representation space. In Section 5, we give some conclusions and discussion.

2. AN $O$-OPERATOR ASSOCIATED TO A RATIONAL SOLUTION OF THE CYBE FOR A LIE ALGEBRA WITH A NONDEGENERATE SYMMETRIC INVARIANT BILINEAR FORM

Let $g$ be a Lie algebra. Let $\sigma: g \otimes g \rightarrow g \otimes g$ be the exchanging operator satisfying $\sigma(x \otimes y) = y \otimes x$ for any $x, y \in g$. For any $r = \sum a_i \otimes b_i$, we set

$$r_{21} = \sigma(r) = \sum b_i \otimes a_i.$$  \hfill (2.1)

A bilinear form $B$ on $g$ is invariant if $B$ satisfies

$$B([x, y], z) = B(x, [y, z]), \quad \forall x, y, z \in g.$$  \hfill (2.2)

We begin our study from the case of $g$ being a simple complex Lie algebra. Let $k(\ , \ )$ be the Killing form on $g$ which is the unique nondegenerate symmetric invariant bilinear form on $g$ up to a scalar multiplication. Let $r$ be a nondegenerate rational solution of the CYBE (1.1). In addition, $r$ usually satisfies the unitary condition:

$$r(u_1, u_2) + r_{21}(u_2, u_1) = 0.$$  \hfill (2.3)

As in the Introduction, a general form of $r$ is given as

$$r(u_1, u_2) = \frac{t}{u_1 - u_2} + r_0(u_1, u_2),$$  \hfill (2.4)

where $t = \sum e_i \otimes e_i$ is the Casimir element of $g$, $\{e_i\}$ is an orthonormal basis of $g$ associated to the Killing form $k(\ , \ )$ and $r_0(u_1, u_2) \in g[u_1] \otimes g[u_2]$. According to Stolin’s study in [14], [15] and [16], we can set

$$r_0(u_1, u_2) = \sum_{i=1}^{K} \sum_{p=0}^{M} \mu(e_i u_1^{-p-1}) \otimes e_i u_2^p,$$  \hfill (2.5)

where $\mu$ is a linear operator from $g[u^{-1}]u^{-1}$ to $g[u]$, $M, K \in \mathbb{N}$, and dim$g = K$. Note that Stolin has proved that deg$_{u_1} r_0 \leq 1$ when $g$ is the simple Lie algebra $sl(n)$. But it is not necessary to consider this conclusion because the following study can be generalized to some more general Lie algebras. On the other hand, in [27], the operator form $r'(z)$ related to a solution $r(z)$ of the CYBE (1.1) satisfying equation (1.3) is given by

$$r(z) = \sum_{i \in \Omega} r'(z)(e_i) \otimes e_i,$$  \hfill (2.6)

where $\{e_i| i \in \Omega\}$ is a basis of $g$, $r(z)$ is a function with domain $D \subset \mathbb{C}$ and range $g \otimes g$, $r'(z) \in \text{End}(g)$. Comparing equations (2.5) and (2.6), we know that the domain of the linear operator $\mu$
in equation (2.5) is $g[u^{-1}]u^{-1}$ (later we will extend it to be the whole algebra $g[u, u^{-1}]$), whereas the linear operator $r'(z)$ appearing in equation (2.6) can be regarded as a family of the linear transformations on $g$ with the parameter $z$. In fact, the latter $r'(z)$ gives a kind of trigonometric solutions from an identity on $e^z$ (27).

Substituting the form (2.5) into the CYBE (1.1), we have

\[
[r, r] = \sum_{i,j=1}^{K} \sum_{p,q=0}^{M} \left[ \mu(e_i u_1^{-p-1}), \mu(e_j u_1^{-q-1}) \right] \otimes e_i u_2^p \otimes e_j u_3^q + \sum_{i,j=1}^{K} \sum_{p,q=0}^{M} \left[ \mu(e_i u_1^{-p-1}), \mu(e_j u_2^{-q-1}) \right] \otimes e_i u_2^p \otimes e_j u_3^q - \sum_{i,j=1}^{K} \sum_{p,q=0}^{M} \left[ \mu(e_i u_1^{-p-1}), e_j u_2^p \otimes \mu(e_j u_2^{-q-1}) \right] \otimes e_i u_3^q + \sum_{i,s=1}^{K} \sum_{p=0}^{M} \left[ \mu(e_i u_1^{-p-1}) - \mu(e_i u_2^{-p-1}) \right] \otimes e_i u_2^p \otimes e_s + \sum_{i,s=1}^{K} \sum_{p=0}^{M} \left[ \mu(e_i u_1^{-p-1}) - \mu(e_i u_3^{-p-1}) \right] \otimes e_i u_2^p \otimes e_s + \sum_{i,s=1}^{K} \sum_{p=0}^{M} \mu(e_i u_1^{-p-1}) \otimes [e_i, e_s] \otimes e_s \frac{u_2^p - u_3^p}{u_2 - u_3}. \tag{2.7}
\]

Since $r$ satisfies the unitary condition (2.3), we know that

\[
\sum_{j,q} \mu(e_j u_1^{-q-1}) \otimes e_j u_2^q + \sum_{j} e_j u_1^{-q} \otimes \mu(e_j u_2^{-q-1}) = 0; \tag{2.8}
\]

that is,

\[
\sum_{j,q} \mu(e_j u_1^{-q-1}) \otimes e_j u_2^q \text{ can be replaced by } - \sum_{j,q} e_j u_1^{-q} \otimes \mu(e_j u_2^{-q-1}). \tag{2.9}
\]

Furthermore, due to the unitary condition (2.3) again, we know that $\deg(\mu(e_i u^{-p-1})) \leq M$.

Hence we can let

\[
\mu(e_i u^{-p-1}) = \sum_{l=0}^{M} \alpha_l (e_i u^{-p-1})^l, \tag{2.10}
\]

where $\alpha_l$ is a linear operator from $g[u^{-1}]u^{-1}$ to $g$, $l = 0, 1, \ldots, M$. Since $r$ is a rational function and $r_0$ is a polynomial, $\mu$ can be defined on the whole $g[u^{-1}]u^{-1}$ by the zero-extension. Set

\[
\alpha_l \equiv 0, \text{ when } l > M. \tag{2.11}
\]

We divide the right hand side of the equation (2.7) into four parts

\[
[r, r] = \sum_{i,j=1}^{K} \sum_{p,q=0}^{M} \left[ \mu(e_i u_1^{-p-1}), \mu(e_j u_1^{-q-1}) \right] \otimes e_i u_2^p \otimes e_j u_3^q + (A) + (B) + (C), \tag{2.12}
\]

\[ \begin{align*}
(A) & = \sum_{i,j=1}^{K} \sum_{p,q=0}^{M} \left[ \mu(e_i u_1^{-p-1}), \mu(e_j u_2^{-q-1}) \right] \otimes e_i u_2^p \otimes e_j u_3^q, \\
(B) & = \sum_{i,s=1}^{K} \sum_{p=0}^{M} \left[ \mu(e_i u_1^{-p-1}) - \mu(e_i u_2^{-p-1}) \right] \otimes e_i u_2^p \otimes e_s, \\
(C) & = \sum_{i,s=1}^{K} \sum_{p=0}^{M} \left[ \mu(e_i u_1^{-p-1}) - \mu(e_i u_3^{-p-1}) \right] \otimes e_i u_2^p \otimes e_s, \\
(D) & = \sum_{i,s=1}^{K} \sum_{p=0}^{M} \mu(e_i u_1^{-p-1}) \otimes [e_i, e_s] \otimes e_s \frac{u_2^p - u_3^p}{u_2 - u_3}. 
\end{align*} \]
where

\[(A) = \sum_{i,j=1}^{K} \sum_{p,q=0}^{M} \mu(e_i u_1^{-p-1}) \otimes [e_i u_2^p, \mu(e_j u_2^{-q-1})] \otimes e_j u_3^q \]

\[+ \sum_{i,s=1}^{K} \sum_{p=0}^{M} \left[ e_s, \frac{\mu(e_i u_1^{-p-1}) - \mu(e_i u_2^{-p-1})}{u_1 - u_2} \right] \otimes e_s \otimes e_i u_3^p; \quad (2.13)\]

\[(B) = -\sum_{i,j=1}^{K} \sum_{p,q=0}^{M} \mu(e_i u_1^{-p-1}) \otimes e_j u_2^q \otimes [e_i u_3^p, \mu(e_j u_3^{-q-1})] \]

\[+ \sum_{i,s=1}^{K} \sum_{p=0}^{M} \left[ \frac{\mu(e_i u_1^{-p-1}) - \mu(e_i u_3^{-p-1})}{u_1 - u_3}, e_s \right] \otimes e_i u_2^p \otimes e_s; \quad (2.14)\]

\[(C) = \sum_{i,s=1}^{K} \sum_{p=0}^{M} \mu(e_i u_1^{-p-1}) \otimes [e_i, e_s] \otimes e_i u_3^p \frac{u_1^p - u_2^p}{u_2 - u_3} \frac{u_2^p - u_3^p}{u_2 - u_3}. \quad (2.15)\]

It is easy to know that

\[\mathcal{B}(f, g) = -\text{Res}_{u=0} k(f, g), \quad \forall \ f, g \in \mathfrak{g}[u, u^{-1}] \quad (2.16)\]

is an invariant bilinear form on the Lie algebra \(\mathfrak{g}[u, u^{-1}]\). Hence for any \(f \in \mathfrak{g}[u]\), we know that

\[f = \sum_{i=1}^{K} \sum_{p=0}^{\infty} \mathcal{B}(f, e_i u^{-p-1}) e_i u^p, \quad (2.17)\]

where there are always finite terms not zero in the above equation. Extend the linear operator \(\mu\) from \(\mathfrak{g}[u^{-1}] u^{-1}\) to \(\mathfrak{g}[u, u^{-1}]\) by \((id \text{ is the identity operator})\)

\[\mu|_{\mathfrak{g}[u]} = -id. \quad (2.18)\]

Therefore

\[(A) = \sum_{i,j,k=1}^{K} \sum_{q,n=0}^{M} 2M \mu(e_i u_1^{-n-1}) \otimes \mathcal{B}([e_i u^n, \mu(e_j u^{-q-1})], e_k u^{-p-1}) e_k u_2^p \otimes e_j u_3^q \]

\[+ \sum_{i,s=1}^{K} \sum_{p=0}^{M} \left[ e_s, \sum_{l=0}^{M} \alpha_{l+1} (e_i u_1^{-l-1}) u_1^{-q} \right] \otimes e_s u_2^p \otimes e_i u_3^p \]

\[= \sum_{i,j,k=1}^{K} \sum_{q,n=0}^{M} 2M \mathcal{B}([e_i u^n, \mu(e_k u^{-q-1})], e_j u^{-p-1}) \mu(e_i u_1^{-n-1}) \otimes e_j u_2^p \otimes e_k u_3^q \]

\[+ \sum_{i,j=1}^{K} \sum_{p=0}^{M} \sum_{l=0}^{M} \sum_{q=0}^{M} \left[ e_i u_1^{-l-q}, \alpha_{l+1} (e_j u_1^{-p-1}) \right] \otimes e_i u_2^q \otimes e_j u_3^p \]
\[
\begin{align*}
&= \sum_{i,j,k=1}^{K} \sum_{q,n=0}^{M} \sum_{p=0}^{2M} \mathcal{B}(e_i u^n, [\mu(e_k u^{q-1})], e_j u^{-p-1}) \mu(e_i u_1^{n-1}) \otimes e_j u_2^p \otimes e_k u_3^q \\
&\quad + \sum_{i,j=1}^{K} \sum_{q,p=0}^{M} \sum_{l=p}^{M-1} [e_i u_1^{l-p}, \alpha_{l+1}(e_j u_1^{q-1})] \otimes e_i u_2^p \otimes e_j u_3^q \\
&= \sum_{i,j=1}^{K} \sum_{q=0}^{M} \sum_{p=0}^{2M} \mu[\sum_{l=0}^{p} \alpha_l(e_j u_1^{q-1}) u_1^l, e_i u_1^{p-1}] \otimes e_i u_2^p \otimes e_j u_3^q \\
&\quad - \sum_{i,j=1}^{K} \sum_{p,q=0}^{M} \sum_{l=p+1}^{M} [\alpha_l(e_j u_1^{q-1}) u_1^l, e_i u_1^{p-1}] \otimes e_i u_2^p \otimes e_j u_3^q \\
&= \sum_{i,j=1}^{K} \sum_{q=0}^{M} \sum_{p=0}^{2M} \mu[\mu(e_j u_1^{q-1}), e_i u_1^{p-1}] \otimes e_i u_2^p \otimes e_j u_3^q,
\end{align*}
\]

Note that for convenience, all the degree parameters can be taken from 0 to \(2M\) (here and in the following sections), that is,

\[
(A) = \sum_{i,j=1}^{K} \sum_{p,q=0}^{2M} \mu[\mu(e_j u_1^{q-1}), e_i u_1^{p-1}] \otimes e_i u_2^p \otimes e_j u_3^q.
\]

Similarly,

\[
(B) = - \sum_{i,j=1}^{K} \sum_{p,q=0}^{2M} \mu[\mu(e_i u_1^{p-1}), e_j u_1^{q-1}] \otimes e_i u_2^p \otimes e_j u_3^q.
\]

Set \([e_i, e_s] = C^k_{is} e_k\), where \((\text{and in the following})\) the repeated (up and down) indices mean summation. Then \(C_{ik}^k = C_{ik}^i\) and we have

\[
\begin{align*}
(C) &= \sum_{i,s=1}^{K} \sum_{p,q=0}^{2M} \mu(e_i u_1^{p-q-2}) \otimes [e_i, e_s] u_2^p \otimes e_s u_3^q \\
&= \sum_{i,s,k=1}^{K} \sum_{p,q=0}^{2M} \mu(C_{ia}^k e_i u_1^{p-q-2}) \otimes e_k u_2^p \otimes e_s u_3^q \\
&= \sum_{s,k=1}^{K} \sum_{p,q=0}^{2M} \mu[e_s u_1^{q-1}, e_k u_1^{p-1}] \otimes e_k u_2^p \otimes e_s u_3^q, \quad (2.22)
\end{align*}
\]

Therefore, we have the following conclusion.
Theorem 1 Let $g$ be a complex simple Lie algebra. Let $r$ be given by the equation (2.4) satisfying the unitary condition (2.3). Then $r$ is a nondegenerate solution of the CYBE for $g$ if and only if the linear operator $\mu : g[u, u^{-1}] \to g[u] \subset g[u, u^{-1}]$ defining the polynomial $r_0$ by the equation (2.5) satisfies $\mu|_{g[u]} = -id$ and

$$[\mu(f), \mu(g)] = \mu[\mu(f), g] + \mu[f, \mu(g)] + \mu[f, g], \quad \forall f, g \in g[u, u^{-1}],$$  

(2.23)

$$\sum_{i=1}^{K} \sum_{p=0}^{M} (\mu(e_i u_1^{-p-1}) \otimes e_i u_2^p + e_i u_1^p \otimes \mu(e_i u_2^{-p-1})) = 0.$$  

(2.24)

In fact, the above conclusion can be implied by an (equivalent) result in [15] given by Stolin with a different approach (see Theorem 1.1 and its proof in [15]) as follows. When $g$ is a simple Lie algebra, as a key point of his study, Stolin proved that there is a natural one-to-one correspondence between the rational solutions of CYBE in $g$ and the subspaces $W \subset g((u^{-1}))$ (where $g((u^{-1})) = \{ \sum_{i=-\infty}^{m} x_i u^i | x_i \in g, \text{ certain } m \in \mathbb{N} \}$) such that

(a) $W$ is a subalgebra in $g((u^{-1}))$ such that $W \supset u^{-N}g[[u^{-1}]]$ for some $N > 0$;

(b) $W \otimes g[u] = g((u^{-1}))$;

(c) $W$ is a Lagrangian subspace with respect to the bilinear form $B'$ of $g((u^{-1}))$ given by

$$B'(f, g) = -\text{Res}_{u=0} B(f, g), \quad \forall f, g \in g((u^{-1})).$$  

(2.25)

It is straightforward to prove that the linear operator $\mu$ given by equation (2.23) (the domain can be extended to $u^{-1}g[[u^{-1}]]$ by zero-extension) is exactly the operator satisfying

$$B'([f + \mu(f), g + \mu(g)], h + \mu(h)) = 0, \quad \forall f, g, h \in u^{-1}g[[u^{-1}]],$$  

(2.26)

which was given by Stolin to decide the corresponding subspace $W \subset g((u^{-1}))$ satisfying the above three conditions by

$$W = (1 + \mu)u^{-1}g[[u^{-1}]].$$  

(2.27)

Note that the study of Stolin on the correspondence between the rational solutions of CYBE and the subspaces $W \subset g((u^{-1}))$ is valid only for $g$ being a complex simple Lie algebra.

We call a linear operator $\mu : g[u, u^{-1}] \to g[u, u^{-1}]$ satisfying the equation (2.23) an $O$-operator. In fact, in the next section, we will give an exact definition of an $O$-operator associated to any representation which the notion is due to its similarity with the notion $O$-operator (1.7) for the constant CYBE given by Kupershmidt ([24]). In this sense, equation (2.23) just gives an $O$-operator associated to the adjoint representation. On the other hand, note that equation (2.23) is exactly the Rota-Baxter relation of weight $-1$ in the version of Lie algebras ([18, 19]). Obviously, such a notion of an $O$-operator (the equation (2.23)) can be defined for any
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Furthermore, note that in the above study, the nondegenerate symmetric invariant bilinear form (the Killing form) on the Lie algebra $g$ plays an essential role. So by a similar study, we can extend Theorem 1 as follows (which is a new conclusion to our knowledge).

**Theorem 2** Let $g$ be a Lie algebra with a nondegenerate symmetric invariant bilinear form. Let $\{e_i\}$ be an orthonormal basis of $g$ associated to the bilinear form and $t = \sum e_i \otimes e_i$. Let $r$ be given by the equation (2.4) satisfying the unitary condition (2.3). Then $r$ is a nondegenerate solution of the CYBE for $g$ if and only if the linear operator $\mu : g[u,u^{-1}] \to g[u] \subset g[u,u^{-1}]$ defining the polynomial $r_0$ by the equation (2.5) is an $O$-operator (that is, the equation (2.23) holds) satisfying $\mu|_{g[u]} = -id$ and the equation (2.24).

Note that when the Lie algebra $g$ is simple, we can get all the nondegenerate rational solutions satisfying unitary condition (2.3) from the $O$-operators as we have interpreted after Theorem 1 ([11], [12], [13], [14], [15], [16]). But it may fail for the general case. In fact, the $O$-operators for the Lie algebras with nondegenerate symmetric invariant bilinear forms are only “sufficient”, that is, they can only give a kind of the rational solutions of the CYBE (maybe not all!).

Furthermore, for a Lie algebra with a nondegenerate symmetric invariant bilinear form, Xu in [27] gave another kind of operator form (see equation (2.6) for the notations)

$$[r'(z_1 + z_2)(x), r'(z_2)(y)] = r'(z_1 + z_2)[x, r'(-z_1)(y)] + r'(z_2)[r'(z_1)(x), y], \ \forall x, y \in g,$$

which is equivalent to the CYBE (1.1) under certain more conditions. Obviously, it is quite different with Theorem 2 (also see the comparison the differences between equations (2.5) and (2.6) given at the beginning of this section).

On the other hand, one may think that the above study on the rational solutions of the CYBE by introducing the notion of an $O$-operator is not very effective since merely the terms of the polynomial part $r_0$ of $r$ have been concerned. This weakness is rather evident when $g$ is taken as a general Lie algebra with a nondegenerate symmetric invariant bilinear form because there probably exist other forms of the rational solutions of the CYBE. In fact, it would not be difficult to give a definition which covers the whole $r$ by considering how to extend the terms with certain poles (see [25]). However, the corresponding operator product expansion would be very complicated and it would not be easy to give a further study explicitly since one might be entangled with paying more attention to the parameter $u$.

At the end of this section, we give a special example of constructing a rational solution of the CYBE from an $O$-operator for the classical double of a Lie bialgebra. Recall that a Lie bialgebra structure on a Lie algebra $g$ is a skew-symmetric linear map $\delta_g : g \to g \otimes g$ such that
\( \delta^*_g : g^* \otimes g^* \to g^* \) is a Lie bracket on \( g^* \) and
\[
\delta([x,y]) = [x \otimes 1 + 1 \otimes x, \delta(y)] - [y \otimes 1 + 1 \otimes y, \delta(x)], \quad \forall x, y \in g. \tag{2.29}
\]
It is equivalent to a Manin triple \((g,g^*, B)\), that is, there is a Lie algebra structure on a direct sum \( g \oplus g^* \) of the underlying vector spaces of \( g \) and \( g^* \) such that \( g \) and \( g^* \) are subalgebras and the natural symmetric bilinear form on \( g \oplus g^* \):
\[
B(x + a^*, y + b^*) = \langle a^*, y \rangle + \langle x, b^* \rangle, \quad \forall x, y \in g, a^*, b^* \in g^*, \tag{2.30}
\]
is invariant, where \( \langle , \rangle \) is the ordinary pair between \( g \) and \( g^* \). The Lie algebra \( g \oplus g^* \) with the bilinear form (2.30) is still a Lie bialgebra which is called a classical double of the Lie bialgebra \( (g, \delta_g) \) ([10]). Let \( \{e_1, \cdots, e_K\} \) be a basis of \( g \) and \( \{e^*_1, \cdots, e^*_K\} \) be its dual basis. Set
\[
[e_i, e_j] = C^k_{ij}e_k, \quad [e^*_i, e^*_j] = \Gamma^k_{ij}e^*_k. \tag{2.31}
\]
Then the Lie algebraic structure on the classical double \( g \oplus g^* \) satisfies
\[
[e^*_i, e_j] = \sum_k (C^k_{jk}e^*_k - \Gamma^k_{ij}e^*_k). \tag{2.32}
\]
and the bilinear form (2.30) is invariant. It is obvious that the bilinear form on \((g^* \oplus g)[u, u^{-1}]\) given by
\[
B'(f, g) = -\text{Res}_{u=0} B(f, g), \quad \forall f, g \in (g \oplus g^*)[u, u^{-1}] \tag{2.33}
\]
is invariant, too. Furthermore, let \( \mu : (g \oplus g^*)[u, u^{-1}] \to (g \oplus g^*)[u] \) be a linear operator satisfying the following conditions:

1. \( \mu \) is an \( \mathcal{O} \)-operator \( \mu \) on the Lie algebra \((g \oplus g)[u, u^{-1}]\), that is, \( \mu \) satisfies
\[
[\mu(f), \mu(g)] = \mu[\mu(f), g] + \mu[f, \mu(g)] + \mu[f, g], \quad \forall f, g \in (g \oplus g^*)[u^{-1}, u]; \tag{2.34}
\]

2. \( \mu|_{(g \oplus g^*)[u]} = -id; \)

3. There exists an \( L \in \mathbb{N} \) such that \( \mu(xu^{-n-1}) = 0 \) for any \( n > L \) and \( x \in g \oplus g^* \). Moreover,
\[
\sum_{i=1}^K \sum_{n=0}^L \{(\mu(e^*_i u^{-n-1}) \otimes e_i u^n_2 + e_i u^n_1 \otimes \mu(e^*_i u^{-n-1})) + [\mu(e_i u^{-n-1}) \otimes e^*_i u^n_2 + e^*_i u^n_1 \otimes \mu(e_i u^{-n-1})]\} = 0. \tag{2.35}
\]

It is easy to know that \( \{\frac{e_i + e^*_i}{\sqrt{2}}, \frac{e_i - e^*_i}{\sqrt{2}}\}_{i \leq K} \) is an orthonormal basis of \( g \oplus g^* \) associated to the bilinear form (2.33). Therefore by Theorem 2 with a direct computation, we know that
\[
r = \sum_{i=1}^K \frac{e_i \otimes e^*_i + e^*_i \otimes e_i}{u_1 - u_2} + \sum_{n=0}^L \mu(e_i u^{-n-1} \otimes e_i u^n_1) + \sum_{n=0}^L \mu(e_i u^{-n-1} \otimes e^*_i u^n_2) \tag{2.36}
\]
is a rational solution of the CYBE satisfying the unitary condition for the classical double \( g \oplus g^* \).
3. Constructing a rational solution of the CYBE from an $O$-operator: 
coadjoint representations

We have known that the construction of the rational solutions from the $O$-operators satisfying the equation (2.23) in Theorem 2 partly depends on the existence of the Casimir element $t$ given by the nondegenerate symmetric invariant bilinear form, where we use the key fact that $t \in g \otimes g$ is invariant under the adjoint representation of a Lie algebra $g$, that is,

$$[x \otimes 1 + 1 \otimes x, t] = 0, \quad \forall x \in g.$$  \hspace{1cm} (3.1)

Actually, for any symmetric invariant tensor $t \in g \otimes g$, it is easy to know that (10)  

$$r(u_1, u_2) = \frac{t}{u_1 - u_2} \hspace{1cm} \text{(3.2)}$$

satisfies the CYBE (1.1). In fact, it follows from

$$\sum_{i,j} \left( \frac{[a_i, a_j] \otimes b_i \otimes b_j}{(u_1 - u_2)(u_1 - u_3)} + \frac{a_i \otimes [b_i, a_j] \otimes b_j}{(u_1 - u_2)(u_2 - u_3)} + \frac{a_i \otimes a_j \otimes [b_i, b_j]}{(u_1 - u_3)(u_2 - u_3)} \right),$$

$$\sum_{i,j} \left( \frac{-a_i \otimes [b_i, a_j] \otimes b_j}{(u_1 - u_2)(u_1 - u_3)} + \frac{a_i \otimes [b_i, a_j] \otimes b_j}{(u_1 - u_2)(u_2 - u_3)} + \frac{-a_i \otimes [b^i, a_j] \otimes b^j}{(u_1 - u_3)(u_2 - u_3)} \right) = 0, \hspace{1cm} \text{(3.3)}$$

where $t = \sum a_i \otimes b_i = \sum b_i \otimes a_i$. Note that here there are not any constraint conditions for the Lie algebra $g$ itself any more. Therefore it is natural to consider how to construct a rational solution of the CYBE with a form (2.4) from certain operators, where $t \in g \otimes g$ is symmetric invariant under the adjoint representation, as a generalization of the study in Section 2.

First we give some notations. Let $g$ be a (finite-dimensional) Lie algebra. Any $t \in g \otimes g$ can be regarded as a linear operator from $g^* \to g$ by the following way

$$\langle t, a^* \otimes b^* \rangle = \langle t(a^*), b^* \rangle, \quad \forall a^*, b^* \in g^*. \hspace{1cm} \text{(3.4)}$$

It can be defined from $g^*[u, u^{-1}]$ to $g[u, u^{-1}]$ by (it is still denoted by $t$)

$$t(a^* \otimes u^m) = t(a^*) \otimes u^m, \quad \forall a^* \in g^*, m \in \mathbb{Z}. \hspace{1cm} \text{(3.5)}$$

On the other hand, let $\rho : g \to gl(V)$ be a representation. The $V[u, u^{-1}]$ is still a representation of $g[u, u^{-1}]$ by (we still denote it by $\rho$)

$$\rho(x \otimes u^m)(v \otimes u^n) = \rho(x)(v) \otimes u^{m+n}, \quad \forall x \in g, v \in V, m, n \in \mathbb{Z}. \hspace{1cm} \text{(3.6)}$$

Let $ad$ be the adjoint representation of $g$ and $ad^*$ be the coadjoint representation (the dual representation of the adjoint representation), that is,

$$ad(x)y = [x, y], \quad \langle ad^*(x)a^*, y \rangle = -\langle a^*, [x, y] \rangle, \quad \forall x, y \in g, a^* \in g^*. \hspace{1cm} \text{(3.7)}$$
In particular, if \( t \in \mathfrak{g} \otimes \mathfrak{g} \) is symmetric invariant under the adjoint representation, then
\[
t(\text{ad}^*(x)a^*) = [x, t(a^*)], \quad \forall x \in \mathfrak{g}, a^* \in \mathfrak{g}^*.
\] (3.8)

In fact, let \( t = \sum_i a_i \otimes b_i \). Then for any \( x \in \mathfrak{g}, a^*, b^* \in \mathfrak{g}^* \), we know that
\[
\langle t(\text{ad}^*(x)a^*), b^* \rangle = \langle \text{ad}^*(x)a^* \otimes b^*, t \rangle = \sum_i \langle -[x, a_i], a^* \rangle \langle b_i, b^* \rangle
\]
\[
= \langle -(\text{ad}(x) \otimes 1)t, a^* \otimes b^* \rangle = \langle (1 \otimes \text{ad}(x))t, a^* \otimes b^* \rangle
\]
\[
= -\langle a^* \otimes \text{ad}^*(x)b^*, t \rangle = -\langle t(a^*), \text{ad}^*(x)b^* \rangle = \langle [x, t(a^*)], b^* \rangle.
\] (3.9)

Moreover, since \( t \) is symmetric, we have (the left hand side of the equation (3.9))
\[
\langle t(\text{ad}^*(x)a^*), b^* \rangle = \langle \text{ad}^*(x)a^*, t(b^*) \rangle = \langle -[x, t(b^*)], a^* \rangle = -\langle \text{ad}^*(t(b^*))a^*, x \rangle, \quad \forall x \in \mathfrak{g}, a^*, b^* \in \mathfrak{g}^*.
\] (3.10)

By the equation (3.9), we know that
\[
\text{ad}^*(t(a^*))b^* + \text{ad}^*(t(b^*))a^* = 0, \quad \forall a^*, b^* \in \mathfrak{g}^*.
\] (3.11)

**Theorem 3**  Let \( \mathfrak{g} \) be a Lie algebra and \( t \in \mathfrak{g} \otimes \mathfrak{g} \) be symmetric invariant under the action of the adjoint representation. Let \( \{e_1, \ldots, e_K\} \) be a basis of \( \mathfrak{g} \) and \( \{e_1^*, \ldots, e_K^*\} \) be its dual basis. Then
\[
r = \frac{t}{u_1 - u_2} + \sum_{i=1}^K \sum_{n=0}^L T(e_i^* u_1^{n-1}) \otimes e_i u_2^n
\] (3.12)
is a rational solution of the CYBE satisfying the unitary condition (2.3) for \( \mathfrak{g} \) if the linear operator \( T : \mathfrak{g}^*[u, u^{-1}] \rightarrow \mathfrak{g}[u] \subset \mathfrak{g}[u, u^{-1}] \) satisfies the following conditions:
\[
[T(f), T(g)] = T(\text{ad}^*(T(f))g - \text{ad}^*(T(g))f - \text{ad}^*(t(g))f), \quad \forall f, g \in \mathfrak{g}^*[u, u^{-1}];
\] (3.13)
\[
\sum_{i=1}^K \sum_{n=0}^L T(e_i^* u_1^{n-1}) \otimes e_i u_2^n + \sum_{i=1}^K \sum_{n=0}^L e_i u_1^n \otimes T(e_i^* u_2^{n-1}) = 0;
\] (3.14)
\[
T(a^* u^p) = -t(a^*)u^p, \quad \forall a^* \in \mathfrak{g}^*; p \geq 0.
\] (3.15)

In fact, let \( t = t^{ij} e_i \otimes e_j \in \mathfrak{g} \otimes \mathfrak{g} \). Then \( t(e_i^*) = t^{ij} e_j \). Obviously, by the equations (3.8), (3.11) and (3.15), for any \( f \in \mathfrak{g}[u] \) or \( g \in \mathfrak{g}[u] \), the equation (3.13) holds automatically. With a similar study as in Section 2, after substituting the equation (3.12) into the CYBE (1.1), we can divide \([r, r]\) into four parts
\[
[r, r] = \sum_{i,j=1}^K \sum_{p,q=0}^{2L} \left[ T(e_i^* u_1^{p-1}), T(e_j^* u_1^{q-1}) \right] \otimes e_i u_2^p \otimes e_j u_2^q + (A) + (B) + (C),
\] (3.16)
(A) \quad = \quad \sum_{i,j=1}^{K} \sum_{p,q=0}^{2L} T(e_i^* u_1^{-p-1}) \otimes [e_i u_2^p, T(e_j^* u_2^{-q-1})] \otimes e_j u_3^q \\
- \sum_{i,j=1}^{K} \sum_{p=0}^{2L} \left[ T(e_i^* u_1^{-p-1}) - T(e_i^* u_2^{-p-1}) \right] \otimes e_j \otimes e_i u_3^p; \quad (3.17) \\
(B) \quad = \quad \sum_{i,j=1}^{K} \sum_{p,q=0}^{2L} T(e_i^* u_1^{-p-1}) \otimes T(e_j^* u_2^{-q-1}) \otimes [e_i u_3^p, e_j u_3^q] \\
+ \sum_{i,j=1}^{K} \sum_{p=0}^{2L} \left[ T(e_i^* u_1^{-p-1}) - T(e_i^* u_3^{-p-1}) \right] \otimes e_i u_2^p \otimes e_j; \quad (3.18) \\
(C) \quad = \quad \sum_{i,j=1}^{K} \sum_{p=0}^{2L} T(e_i^* u_1^{-p-1}) \otimes [e_i, t(e_j^*)] \otimes e_j \frac{u_2^p - u_3^p}{u_2 - u_3}. \quad (3.19) \\

Let B' be the bilinear form on the vector space \((g^* \oplus g)[u, u^{-1}]\) given by the equation (2.28). By the equation (3.14), we know that \(\text{deg} \ \text{Im}T \leq L\), where \(\text{Im}T\) is the image of the linear operator \(T\). So we can set \\

\[ T(e_i^* u_1^{-n-1}) = \sum_{l=0}^{L} \alpha_l(e_i^* u_1^{-n-1}) u_l, \quad (3.20) \]

where \(\alpha_l\) is a linear operator from \(g^*[u^{-1}]u^{-1}\) to \(g\), \(l = 0, 1, \cdots, L\). Let \(\alpha_l \equiv 0\) when \(l > L\). So

(A) \quad = \quad \sum_{i,j,k=1}^{K} \sum_{p,q,n=0}^{2L} T(e_i^* u_1^{-p-1}) \otimes B'(e_i u_2^p, \text{ad}^*(T(e_j^* u_2^{-q-1}))(e_k^* u_2^{-n-1})) e_k u_2^n \otimes e_j u_3^q \\
+ \sum_{i,j=1}^{K} \sum_{n,q=0}^{2L} \left[ t(e_i^*) u_1^{-n-1}, \sum_{l=0}^{2L-1} \alpha_{l+1}(e_j^* u_1^{-q-1}) u_1^{l+1} \right] \otimes e_i u_2^n \otimes e_j u_3^q \\
= \quad \sum_{i,j,k=1}^{K} \sum_{p,q,n=0}^{2L} T(e_i^* u_1^{-p-1}) B'(e_i u_2^p, \text{ad}^*(\sum_{l=0}^{n} \alpha_l(e_j^* u_1^{-q-1}) u_1^l))(e_k^* u_1^{-n-1})) \otimes e_k u_2^n \otimes e_j u_3^q \\
+ \sum_{i,j=1}^{K} \sum_{n,q=0}^{2L} \left[ t(e_i^*) u_1^{-n-1}, \sum_{l=0}^{2L-1} \alpha_{l+1}(e_j^* u_1^{-q-1}) u_1^{l+1} \right] \otimes e_i u_2^n \otimes e_j u_3^q \\
= \quad \sum_{i,j=1}^{K} \sum_{n,q=0}^{2L} T(\text{ad}^*(\sum_{l=0}^{n} \alpha_l(e_j^* u_1^{-q-1}) u_1^l))(e_i^* u_1^{-n-1})) \otimes e_i u_2^n \otimes e_j u_3^q \\
+ \sum_{i,j=1}^{K} \sum_{n,q=0}^{2L} \left[ t(e_i^*) u_1^{-n-1}, \sum_{l=0}^{2L-1} \alpha_{l+1}(e_j^* u_1^{-q-1}) u_1^{l+1} \right] \otimes e_i u_2^n \otimes e_j u_3^q \\
= \quad \sum_{i,j=1}^{K} \sum_{n,q=0}^{2L} T(\text{ad}^*T(e_j^* u_1^{-q-1}))(e_i^* u_1^{-n-1})) \otimes e_i u_2^n \otimes e_j u_3^q. \quad (3.21)
Note that in the last equation, we use the equations (3.8) and (3.15). By the equation (3.14) and a similar study as above, we know that

\[
(B) = - \sum_{i,j=1}^{K} \sum_{n,q=0}^{2L} T(\text{ad}^*T(e_i^*u_1^{-q-1})(e_j^*u_1^{-n-1})) \otimes e_i u_2^q \otimes e_j u_3^n. \tag{3.22}
\]

Set \([e_i, e_j] = C_{ij}^k e_k\). Then

\[
(C) = \sum_{i=1}^{K} \sum_{p,q=0}^{2L} T(e_i^*u_1^{-p-q-2})t^{kj}C_{ik}^l \otimes e_l u_2^p \otimes e_j u_3^q
\]

\[
= - \sum_{i,j=1}^{K} \sum_{p,q=0}^{2L} T(\text{ad}^*(t(e_i^*u_1^{-p-1})e_j^*u_1^{-q-1})) \otimes e_i u_2^p \otimes e_j u_3^q. \tag{3.23}
\]

Therefore \(r\) given by the equation (3.12) is a solution of the CYBE if the equations (3.13)-(3.15) hold. \(\square\)

**Example** We give a concrete example of Theorem 3 as follows. Let \(\mathfrak{h}\) be the 3-dimensional Heisenberg Lie algebra. That is, there exists a basis \(\{e_1, e_2, e_3\}\) of \(\mathfrak{h}\) satisfying

\[
[e_1, e_2] = e_3, \quad [e_1, e_3] = [e_2, e_3] = 0. \tag{3.24}
\]

Let \(\{e_1^*, e_2^*, e_3^*\}\) be the dual basis. The coadjoint representation \(\text{ad}^*\) is given as (only the non-zero actions are given)

\[
\text{ad}^*(e_1)e_3^* = -e_2^*, \quad \text{ad}^*(e_2)e_3^* = e_1^*. \tag{3.25}
\]

Since \(e_3\) is in the center of \(\mathfrak{h}\), \(t = e_3 \otimes e_3\) is invariant under the action of the adjoint representation of \(\mathfrak{h}\). Then

\[
t(e_1^*) = 0, \quad t(e_2^*) = 0, \quad t(e_3^*) = e_3. \tag{3.26}
\]

So \(\text{ad}^*(t(a^*))b^* = 0\) for any \(a^*, b^* \in \mathfrak{h}^*\). Moreover, let \(T_{\lambda_1, \lambda_2} : \mathfrak{h}^*[u, u^{-1}] \to \mathfrak{h}[u]\) be a linear operator satisfying (only the non-zero actions are given)

\[
T(e_3^*u^{-2}) = -(\lambda_1 e_1 + \lambda_2 e_2), T(e_1^*u^{-1}) = \lambda_1 e_3 u, T(e_2^*u^{-1}) = \lambda_2 e_3 u; \quad T(e_3^*u^p) = -e_3 u^p, \quad \forall p \geq 0, \tag{3.27}
\]

where \(\lambda_1, \lambda_2 \in \mathbb{C}\). It is easy to know that \(T_{\lambda_1, \lambda_2}\) satisfies the equations (3.13)-(3.15). So

\[
r(u_1, u_2) = \frac{e_3 \otimes e_3}{u_1 - u_2} + e_3 \otimes (\lambda_1 e_1 + \lambda_2 e_2)u_1 - (\lambda_1 e_1 + \lambda_2 e_2) \otimes e_3 u_2 \tag{3.28}
\]

is a rational solution of the CYBE satisfying the unitary condition (2.3) for the Lie algebra \(\mathfrak{h}\). Although the solution (3.28) seems a little trivial (all the commutators of \(r\) are zero), it is enough to illustrate the essential roles of the \(\mathcal{O}\)-operators here. Moreover, it is easy to know that the above construction can be generalized to any Lie algebra with a nonzero center. \(\square\)
Furthermore, in fact, the above construction can be regarded as a natural generalization of Theorem 2 in the following sense. Let \( \mathfrak{g} \) be a Lie algebra with a nondegenerate symmetric invariant bilinear form \( \mathcal{B} \). Let \( \{ e_1, \cdots, e_K \} \) be an orthonormal basis of \( \mathfrak{g} \) and \( \{ e^*_1, \cdots, e^*_K \} \) be its dual basis. Let \( t = \sum e_i \otimes e_i \) be the Casimir element of \( \mathfrak{g} \). Then as a linear operator from \( \mathfrak{g}^* \) to \( \mathfrak{g} \), \( t \) satisfies \( t(e^*_i) = e_i, i = 1, \cdots, K \). Then as the representations of \( \mathfrak{g} \), \( \mathfrak{g}^* \) can be identified with \( \mathfrak{g} \) by the linear isomorphism \( t \) in the following sense

\[
\langle a^*, x \rangle = \mathcal{B}(t(a^*), x), \quad t(ad^*(x)a^*) = [x, t(a^*)], \quad \forall x \in \mathfrak{g}, a^* \in \mathfrak{g}^*.
\]

(3.29)

Therefore, we can get Theorem 2 from Theorem 3 from the following correspondence:

- equation (3.14) \( \iff \) equation (2.24);
- equation (3.13) \( \iff \) equation (2.23);
- equation (3.15) \( \iff \) \( \mu|_{\mathfrak{g}[u]} = -id \).

In particular, the equation (3.13) which is in fact well-defined on \( \mathfrak{g}^*[u^{-1}]u^{-1} \) with a consistent extension (3.15) to \( \mathfrak{g}^*[u] \) can be regarded as a generalization of the \( \mathcal{O} \)-operator given by the equation (2.23) which is also well-defined on \( \mathfrak{g}[u^{-1}]u^{-1} \) with a consistent extension \( \mu|_{\mathfrak{g}[u]} = -id \) to \( \mathfrak{g}[u] \). Note that by the equation (3.11) or by skew-symmetry of the Lie bracket (3.13),

\[
ad^*(t(f))g + ad^*(t(g))f = 0, \quad \forall f, g \in \mathfrak{g}^*[u, u^{-1}].
\]

(3.30)

Therefore, the above facts motivate us to give a more general definition of an \( \mathcal{O} \)-operator which is related to the rational solutions of the CYBE with the form (2.4) ([24]):

**Definition** Let \( \mathfrak{g} \) be a Lie algebra and \( \rho : \mathfrak{g} \to gl(V) \) be a representation of \( \mathfrak{g} \). Suppose that there exists a skew-symmetric (bilinear) product \( * \) on the vector space \( V \) which gives a skew-symmetric bilinear product on \( V[u, u^{-1}] \) naturally by

\[
(x \otimes u^m) \ast (y \otimes u^n) = (x \ast y) \otimes u^{m+n}, \quad \forall x, y \in V, m, n \in \mathbb{Z}.
\]

(3.31)

A linear map \( T : V[u^{-1}]u^{-1} \to \mathfrak{g}[u] \subset \mathfrak{g}[u, u^{-1}] \) with a suitable extension to \( V[u] \) is called an \( \mathcal{O} \)-operator associated to \( (\rho, V, *) \) if \( T \) satisfies

\[
[T(f), T(g)] = T(\rho(T(f))g - \rho(T(g))f + f \ast g), \forall f, g \in V[u, u^{-1}].
\]

(3.32)

Obviously, in the above sense, the equation (2.23) (with \( \mu|_{\mathfrak{g}[u]} = -id \)) in Theorems 1 and 2 gives an \( \mathcal{O} \)-operator associated to the adjoint representation while the equation (3.13) (with the equation (3.15)) in Theorem 3 gives an \( \mathcal{O} \)-operator associated to the coadjoint representation.
4. Constructing a rational solution of the CYBE from an \( \mathcal{O} \)-operator: general cases

Not like the study on the construction of the constant solutions of the CYBE from \( \mathcal{O} \)-operators in [26], it is not easy to given an explicit construction of a rational solution of the CYBE from an \( \mathcal{O} \)-operator for a general representation \((\rho, V)\) and an arbitrary (skew-symmetric bilinear) product \(*\) on \(V\). In this section, we consider the case that \((\rho, V)\) is still arbitrary but the product \(*\) on \(V\) is trivial. Similar to the study given in [26], the rational solutions (from the following construction) of the CYBE from such \( \mathcal{O} \)-operators are not for the Lie algebra \( \mathfrak{g} \) itself but for a larger Lie algebra.

Let \( \mathfrak{g} \) still be a Lie algebra and \( \rho : \mathfrak{g} \to gl(V) \) be a representation of \( \mathfrak{g} \). It is known that there is a Lie algebra structure on a direct sum \( \mathfrak{g} \oplus V \) of the underlying vector spaces \( \mathfrak{g} \) and \( V \) given by

\[
[e_1 + x_1, e_2 + x_2] = [e_1, e_2] + \rho(e_1)(x_2) - \rho(e_2)(x_1), \quad \forall e_1, e_2 \in \mathfrak{g}, x_1, x_2 \in V.
\]

It is denoted by \( \mathfrak{g} \ltimes \rho V \). On the other hand, let \( \rho^* : \mathfrak{g} \to gl(V^*) \) be the dual representation of \((\rho, V)\) of the Lie algebra \( \mathfrak{g} \), that is,

\[
\langle \rho^*(e)x^*, y \rangle = -\langle x^*, \rho(e)y \rangle, \quad \forall e \in \mathfrak{g}, x^* \in V^*, y \in V.
\]

Then both \( V \) and \( V^* \) can be the representations of the Lie algebra \( \mathfrak{g} \ltimes \rho \ast V^* \) by the zero-extension, that is (we still denote them by \( \rho \) and \( \rho^* \) respectively),

\[
\rho(e + x^*)y = \rho(e)y; \quad \rho^*(e + x^*)z^* = \rho^*(e)z^*, \quad \forall e \in \mathfrak{g}, x^*, z^* \in V^*, y \in V.
\]

Moreover, by the equation (3.6), both \( V[u, u^{-1}] \) and \( V^*[u, u^{-1}] \) are the representations of the Lie algebra \( \mathfrak{g} \ltimes \rho^* V^*[u, u^{-1}] \).

**Theorem 4** Let \( \mathfrak{g} \) be a Lie algebra and \( \rho : \mathfrak{g} \to gl(V) \) be a representation of \( \mathfrak{g} \). Let \( t \in V^* \otimes V^* \) be symmetric invariant under the action of the dual representation \( \rho^* \). Let \( \{w_1, \cdots, w_N\} \) be a basis of \( V \) and \( \{w_1^*, \cdots, w_N^*\} \) be its dual basis. Then

\[
r = \frac{2t}{u_1 - u_2} + \sum_{i=1}^{N} \sum_{k=0}^{L} T(w_i u_1^{-k-1}) \otimes w_i^* u_2^k - \sum_{i=1}^{N} \sum_{k=0}^{L} w_i^* u_1^k \otimes T(w_i u_2^{-k-1})
\]

is a rational solution of the CYBE satisfying the unitary condition (2.3) for the Lie algebra \( \mathfrak{g} \ltimes \rho^* V^* \) if the linear operator \( T : V[u, u^{-1}] \to (\mathfrak{g} \ltimes \rho^* V^*)[u] \) with \( \deg \text{Im}T \leq L \) satisfies the following conditions:

\[
[T(f), T(g)] = T(\rho(T(f))g - \rho(T(g))f), \forall f, g \in V[u, u^{-1}];
\]

\[
[T(f), T(g)] = T(\rho(T(f))g - \rho(T(g))f), \forall f, g \in V[u, u^{-1}];
\]

\[
[T(f), T(g)] = T(\rho(T(f))g - \rho(T(g))f), \forall f, g \in V[u, u^{-1}];
\]

\[
[T(f), T(g)] = T(\rho(T(f))g - \rho(T(g))f), \forall f, g \in V[u, u^{-1}];
\]
In fact, let \( t = t^{ij} v_i^* \otimes v_j^* \in V^* \otimes V^* \). Then \( t(w) = t^{ij} w_i^* w_j^* \). Moreover, since \( t \) is symmetric invariant under the action of the dual representation \( \rho^* \), we know that

\[
t(\rho(e)w) = \rho^*(e)t(w), \quad \forall e \in g, w \in V,
\]

by replacing \( \text{ad}^* \) by \( \rho \) in the equation (3.9). Obviously, by the equations (4.6) and (4.7), for any \( f \in V[u] \) or \( g \in V[u] \), the equation (4.5) holds automatically. Let \( B' \) be the bilinear form on the vector space \((V^* \oplus V)[u, u^{-1}]\) given by the equation (2.28). On the other hand, as in the above section, from \( \text{deg} \text{Im}T \leq L \), we can set

\[
T(w_i u^{-k-1}) = \sum_{l=0}^{L} \alpha_l(u_i u^{-k-1}) u_l,
\]

where \( \alpha_l \) is a linear operator from \( V[u^{-1}]u^{-1} \) to \( g \), \( l = 0, 1, \ldots, L \). Let \( \alpha_l \equiv 0 \) when \( l > L \).

Substituting the equation (4.4) into the CYBE (1.1), we know that

\[
[[r, r]] = (A) + (B) + (C),
\]

where

\[
(A) = \sum_{i,j=1}^{N} \sum_{n,q=0}^{2L} \frac{T(w_i u_1^{-n-1})T(w_j u_2^{-q-1})}{u_1 - u_2} w_i^* u_2^* \otimes w_j^* u_3^q
\]

\[
+ \sum_{i,j=1}^{N} \sum_{p,q=0}^{2L} T(w_i u_1^{-p-1})T(w_j u_2^{-q-1}) w_j^* u_3^q
\]

\[
- \sum_{i,j=1}^{N} \sum_{p=0}^{2L} \frac{T(w_i u_1^{-p-1}) - T(w_i u_2^{-p-1})}{u_1 - u_2} t(w_j) \otimes w_j^* \otimes w_i^* u_3^p
\]

\[
- \sum_{i,j=1}^{N} \sum_{p,q=0}^{2L} T(w_i u_1^{-p-1}) w_j^* u_3^q \otimes w_i^* u_3^p
\]

\[
+ \sum_{i,j=1}^{N} \sum_{p=0}^{2L} \frac{T(w_i u_1^{-p-1}) - T(w_i u_3^{-p-1})}{u_1 - u_3} t(w_j) \otimes w_i^* u_2^p \otimes w_j^*;
\]

\[
(B) = \sum_{i,j=1}^{N} \sum_{p,q=0}^{2L} \rho^*(T(w_j u_1^{-p-1})w_i^* u_3^q) \otimes T(w_i u_2^{-q-1}) \otimes w_j^* u_3^p
\]

\[
- \sum_{i,j=1}^{N} \sum_{p,q=0}^{2L} w_i^* u_3^p \otimes T(w_i u_2^{-q-1}) \otimes w_j^* u_3^q
\]

\[
- \sum_{i,j=1}^{N} \sum_{p,q=0}^{2L} w_i^* u_3^p \otimes T(w_j u_2^{-q-1}) \otimes \rho^*(T(w_i u_3^{-q-1})) w_j^* u_3^p
\]

\[
(4.6)
\]
\[-\sum_{i,j=1}^{N} \sum_{p=0}^{2L} \frac{T(w_i u_1^{-p-1}) - T(w_i u_2^{-p-1})}{u_1 - u_2} \cdot t(w_j) \otimes w_j^* \otimes w_i^* u_3^p \]
\[-\sum_{i,j=1}^{N} \sum_{p=0}^{2L} w_i^* u_1^p \otimes [T(w_i u_2^{-p-1}) - T(w_i u_3^{-p-1})], t(w_j) \otimes w_j^* \]
\[= \sum_{i,j=1}^{N} \sum_{p,q=0}^{2L} \rho^*(T(w_i u_1^{-p-1}))(w_j^* u_1^p) \otimes w_i^* u_2^p \otimes T(w_j u_3^{-q-1}) \]
\[+ \sum_{i,j=1}^{N} \sum_{p,q=0}^{2L} w_i^* u_1^p \otimes \rho^*(T(w_i u_2^{-q-1}))(w_j^* u_2^q) \otimes T(w_j u_3^{-q-1}) \]
\[+ \sum_{i,j=1}^{N} \sum_{p,q=0}^{2L} w_i^* u_1^p \otimes w_j^* u_2^q \otimes [T(w_i u_3^{-p-1}), T(w_j u_3^{-q-1})] \]
\[+ \sum_{i,j=1}^{N} \sum_{p,q=0}^{2L} \frac{T(w_i u_1^{-p-1}) - T(w_i u_3^{-p-1})}{u_1 - u_3} \cdot t(w_j) \otimes w_i^* u_2^p \otimes w_j^* \]
\[-\sum_{i,j=1}^{N} \sum_{p=0}^{2L} w_i^* u_1^p \otimes [T(w_i u_2^{-p-1}) - T(w_i u_3^{-p-1}), t(w_j) \otimes w_j^*] \]
\[= \sum_{i,j=1}^{N} \sum_{n,q=0}^{2L} [T(w_i u_1^{-n-1}), T(w_j u_1^{-q-1})] \otimes w_i^* u_2^n \otimes w_j^* u_3^q + (A_1) + (A_2), \] (4.11)

We can divide \((A)\) given by the equation (4.10) into three parts:
\[= \sum_{i,j=1}^{N} \sum_{n,q=0}^{2L} [T(w_i u_1^{-n-1}), T(w_j u_1^{-q-1})] \otimes w_i^* u_2^n \otimes w_j^* u_3^q + (A_1) + (A_2), \] (4.13)

where
\[\begin{align*}
(A_1) & = \sum_{i,j=1}^{N} \sum_{p,q=0}^{2L} T(w_i u_1^{-p-1}) \otimes [w_i^* u_2^p, T(w_j u_2^{-q-1})] \otimes w_j^* u_3^q \\
& - \sum_{i,j=1}^{N} \sum_{p=0}^{2L} \frac{T(w_i u_1^{-p-1}) - T(w_i u_2^{-p-1})}{u_1 - u_2} \cdot t(w_j) \otimes w_j^* \otimes w_i^* u_3^p; \quad (4.14)
\end{align*}\]
\[\begin{align*}
(A_2) & = -\sum_{i,j=1}^{N} \sum_{p,q=0}^{2L} T(w_i u_1^{-p-1}) \otimes w_j^* u_2^q \otimes [w_i^* u_3^p, T(w_j u_3^{-q-1})] \\
& + \sum_{i,j=1}^{N} \sum_{p=0}^{2L} \frac{T(w_i u_1^{-p-1}) - T(w_i u_3^{-p-1})}{u_1 - u_3} \cdot t(w_j) \otimes w_i^* u_2^p \otimes w_j^*. \quad (4.15)
\end{align*}\]

Moreover,
\[\begin{align*}
(A_1) & = \sum_{i,j,k=1}^{N} \sum_{p,q=n}^{2L} T(w_i u_1^{-p-1})B'(w_i^* u_1^p) \cdot \rho^* \left( \sum_{l=0}^{n} \alpha_l(w_j u_1^{-q-1}) u_1^l (w_k u_1^{-n-1}) \right) \otimes w_k^* u_2^n \otimes w_j^* u_3^q \\
& + \sum_{i,j=1}^{N} \sum_{n,q=0}^{2L} [t(w_i) u_1^{-n-1}, \sum_{l=n}^{2L-1} \alpha_{l+1}(w_j u_1^{-q-1}) u_1^{l+1}] \otimes w_i^* u_2^n \otimes w_j^* u_3^q
\end{align*}\]
\[
\begin{align*}
&= \sum_{i,j=1}^{N} \sum_{n,q=0}^{2L} T(\rho \sum_{l=0}^{n} \alpha_l (w_j u_1^{-q-1}) w_1^l ) (w_i u_1^{-n-1}) \otimes w_i^* u_2^n \otimes w_j^* u_3^q \\
&\quad + \sum_{i,j=1}^{N} \sum_{n,q=0}^{2L} [t(w_i) u_1^{-n-1}, \sum_{l=0}^{2L-1} \alpha_{l+1} (w_j u_1^{-q-1}) u_1^{l+1}] \otimes w_i^* u_2^n \otimes w_j^* u_3^q \\
&= \sum_{i,j=1}^{N} \sum_{n,q=0}^{2L} T(\rho (T(w_j u_1^{-q-1}))(w_i u_1^{-n-1})) \otimes w_i^* u_2^n \otimes w_j^* u_3^q. 
\end{align*}
\]

Note that in the last equation, we use the equations (4.6) and (4.7). Similarly, we have

\[
(A_2) = -\sum_{i,j=1}^{N} \sum_{n,q=0}^{2L} T(\rho (T(w_i u_1^{-q-1}))(w_j u_1^{-n-1})) \otimes w_i^* u_2^n \otimes w_j^* u_3^q. 
\]

Therefore

\[
(A) = \sum_{i,j=1}^{N} \sum_{n,q=0}^{2L} \{ [T(w_i u_1^{-n-1}), T(w_j u_1^{-q-1})] + T(\rho(T(w_j u_1^{-q-1}))(w_i u_1^{-n-1})) \\
- T(\rho(T(w_i u_1^{-n-1}))(w_j u_1^{-q-1})) \} \otimes w_i^* u_2^n \otimes w_j^* u_3^q. 
\]

With a similar discussion as above, we know that

\[
[r,r] = \sum_{i,j=1}^{N} \sum_{n,q=0}^{2L} \{ [T(w_i u_1^{-n-1}), T(w_j u_1^{-q-1})] + T(\rho(T(w_j u_1^{-q-1}))(w_i u_1^{-n-1})) \\
- T(\rho(T(w_i u_1^{-n-1}))(w_j u_1^{-q-1})) \} \otimes w_i^* u_2^n \otimes w_j^* u_3^q \\
- \sum_{i,j=1}^{N} \sum_{n,q=0}^{2L} w_i^* u_1^n \otimes \{ [T(w_i u_2^{-n-1}), T(w_j u_2^{-q-1})] + T(\rho(T(w_j u_2^{-q-1}))(w_i u_2^{-n-1})) \\
- T(\rho(T(w_i u_2^{-n-1}))(w_j u_2^{-q-1})) \} \otimes w_i^* u_3^q \\
+ \sum_{i,j=1}^{N} \sum_{n,q=0}^{2L} w_i^* u_1^n \otimes w_j^* u_2^q \otimes \{ [T(w_i u_3^{-n-1}), T(w_j u_3^{-q-1})] \\
+ T(\rho(T(w_j u_3^{-q-1}))(w_i u_3^{-n-1})) - T(\rho(T(w_i u_3^{-n-1}))(w_j u_3^{-q-1})) \}. 
\]

Therefore \( r \) given by the equation (4.4) is a solution of the CYBE if the equations (4.5)-(4.6) hold. Obviously, \( r \) satisfies the unitary condition (2.3). \( \square \)

Note that in the equation (4.4), \( t \) is taken in the vector space \( V^* \otimes V^* \subset (g \ltimes \rho^* V^*) \otimes (g \ltimes \rho^* V^*) \) which there is not any part in \( g \). Otherwise, it would involve the actions between the Lie algebra \( g \) itself which might be very complicated and a little far away from the equation (4.5) defining the \( \mathcal{O} \)-operator associated to any arbitrary representation (in fact, it might involve the coadjoint representation as given in Section 3).

At the end of this section, we consider two special cases and then compare them with the relative study in Section 2 and Section 3 respectively.
(Case I) The representation $\rho$ is taken as the adjoint representation $\text{ad}$. Then by Theorem 4, we can get a rational solution of the CYBE with the form (4.4) for the Lie algebra $\mathfrak{g} \ltimes_{\text{ad}} \mathfrak{g}^*$, where $t \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ and the linear operator $T$ is from $\mathfrak{g}[u, u^{-1}]$ to $\mathfrak{g} \ltimes_{\text{ad}} \mathfrak{g}^*[u]$. On the other hand, it is known [10] that $\mathfrak{g} \ltimes_{\text{ad}} \mathfrak{g}^*$ is the classical double of the trivial Lie bialgebra structure (that is, $\delta_{\mathfrak{g}} = 0$) on the Lie algebra $\mathfrak{g}$. Therefore by the study at the end of section 2, there is another (completely different) rational solution of the CYBE with the form (2.31) for the same Lie algebra $\mathfrak{g} \ltimes_{\text{ad}} \mathfrak{g}^*$, where $t \in (\mathfrak{g} \ltimes_{\text{ad}} \mathfrak{g}^*) \otimes (\mathfrak{g} \ltimes_{\text{ad}} \mathfrak{g}^*)$ and $T = \mu$ is a linear operator from $\mathfrak{g} \ltimes_{\text{ad}} \mathfrak{g}^*[u, u^{-1}]$ to $\mathfrak{g} \ltimes_{\text{ad}} \mathfrak{g}^*[u]$.

(Case II) The representation $\rho$ is taken as the coadjoint representation $\text{ad}^*$. Then by Theorem 4, we can get a rational solution of the CYBE with the form (4.4) for the Lie algebra $\mathfrak{g} \ltimes_{\text{ad}} \mathfrak{g}$, where $t \in \mathfrak{g} \otimes \mathfrak{g}$ and the linear operator $T$ is from $\mathfrak{g}^*[u, u^{-1}]$ to $\mathfrak{g} \ltimes_{\text{ad}} \mathfrak{g}[u]$. In particular, $r(u_1, u_2)$ given by the equation (4.4) in this case is in the vector space $\mathfrak{g}[u_1] \otimes \mathfrak{g}[u_2]$. Despite it, this $r(u_1, u_2)$ is not a rational solution of the CYBE for the Lie algebra $\mathfrak{g}$ itself which only involves the adjoint action in general. In fact, although the forms of $t$ and $T(w_i u_1^{-k-1}) \otimes w_i^* u_2^k$ involve the vector space $\mathfrak{g} \otimes \mathfrak{g}$, they actually involve the vector space $(0, \mathfrak{g}) \otimes (0, \mathfrak{g})$ and $(0, \mathfrak{g}) \otimes (0, \mathfrak{g})$ respectively. However, the commutators (zero) between $(0, \mathfrak{g})$ and $(0, \mathfrak{g})$ are different with the commutators (adjoint action) between $(\mathfrak{g}, 0)$ and $(\mathfrak{g}, 0)$ or $(\mathfrak{g}, 0)$ and $(0, \mathfrak{g})$. Nevertheless, by the proof of Theorems 3 and 4, we can prove that if $T : \mathfrak{g}[u, u^{-1}] \to \mathfrak{g}[u]$ satisfies equation (3.15), the unitary condition (3.14) and

\[ [T(f), T(g)] = T(\text{ad}^*(T(f))g - \text{ad}^*(T(g))f), \quad \forall f, g \in \mathfrak{g}^*[u^{-1}, u], \tag{4.20} \]

and

\[ \text{ad}^*(t(g))f = 0, \quad \forall f, g \in \mathfrak{g}^*[u^{-1}, u], \tag{4.21} \]

then

\[ r = \frac{2t}{u_1 - u_2} + 2 \sum_{i=1}^{K} \sum_{k=0}^{L} T(e_i u_1^{-k-1}) \otimes e_i u_2^k, \tag{4.22} \]

is a rational solution of the CYBE for the Lie algebra $\mathfrak{g}$, which coincides with the construction from Theorem 3 under the condition (4.21) since $r$ is a solution of the CYBE if and only if $2r$ is a solution of the CYBE.

5. Conclusions and discussion

From the study in the previous sections, we give the following conclusions and discussion.
(1) The rational solutions of the CYBE for the complex simple Lie algebras are interpreted in terms of the $\mathcal{O}$-operators (associated to the adjoint representations). Furthermore, the $\mathcal{O}$-operators (associated to the suitable representations) can be used to construct explicitly the rational solutions for a Lie algebras with a nondegenerate symmetric invariant bilinear form (adjoint representation), a Lie algebra with a symmetric invariant tensor under the action of the adjoint representation (coadjoint representation), a semidirect sum of a Lie algebra and the dual representation of its representation with a symmetric invariant tensor under the action of the dual representation (arbitrary representation). All of the solutions have a uniform form as

$$r(u_1, u_2) = \frac{t}{u_1 - u_2} + r_0(u_1, u_2), \quad (5.1)$$

where $t$ is the symmetric invariant tensor and $r_0$ is a polynomial defined by an $\mathcal{O}$-operator. We call the equation (5.1) a Drinfeld form ([13], [29]). Note that in the above construction the existence of a symmetric invariant tensor $t$ is necessary and it plays an essential role in the concrete definition of an $\mathcal{O}$-operator defining the polynomial $r_0$.

(2) Comparing with the study of the operator approach to the constant CYBE ([26]), we find that, roughly speaking, there are the following “correspondence”:

- adjoint representation : equation (2.23) $\leftrightarrow$ equation (1.5) (Semonov-Tian-Shansky)
- coadjoint representation : equation (3.13) $\leftrightarrow$ equation (1.6) (Kupershmidt)
- an arbitrary representation : equation (4.5) $\leftrightarrow$ equation (1.7) (Kupershmidt and [14])

However, the “correspondence” is in a “rough” level, since not like in the study of the constant CYBE that the construction from the $\mathcal{O}$-operators by the equations (1.5) and (1.6) can be obtained through the uniform construction from $\mathcal{O}$-operators by the equation (1.7) as the special cases ([26]), the construction from $\mathcal{O}$-operators by the equations (2.23) and (3.13) are usually quite different with the construction from $\mathcal{O}$-operators by the equation (4.5) in the corresponding cases (see the discussion at the end of section 4). In fact, it is due to the inhomogeneous term $T(f \ast g)$ appearing in the definition (3.32) which is closely related to the symmetric invariant tensor $t$.

(3) The form of the equation (2.23) with the inhomogeneous term $\mu[f, g]$ which gives an $\mathcal{O}$-operator associated to the adjoint representation corresponds precisely to the linear operator $R' : \mathfrak{g} \to \mathfrak{g}$ in the constant case satisfying

$$[R'(x), R'(y)] = R'([R'(x), y] + [x, R'(y)]) + R'[x, y], \quad \forall x, y \in \mathfrak{g}. \quad (5.2)$$
The above equation is exactly the Rota-Baxter relation of weight \(-1\) in the version of Lie algebras (\[18\], \[19\]). By letting \( R' = 1 - 2R \), the equation (5.2) is equivalent to the operator form (\[5\])

\[
[R(x), R(y)] = R([R(x), y] + [x, R(y)]) - [x, y], \quad \forall x, y \in \mathfrak{g}
\]

(5.3)
of the (constant) modified classical Yang-Baxter equation (MCYBE, \[10\]) satisfying

\[[[r, r]] \text{ is invariant under the adjoint action.} \tag{5.4}\]

Moreover, the product \([ , , 1\] given by

\[ [x, y]_1 = [R'(x), y] + [x, R'(y)] + [x, y], \quad \forall x, y \in \mathfrak{g} \tag{5.5} \]
defines a Lie algebra and \( R' \) is a homomorphism between two Lie algebras. On the other hand, for the equation (2.23), if we define

\[ f \circ g = [\mu(f), g] + 1/2[f, g], \quad \forall f, g \in \mathfrak{g}[u, u^{-1}], \tag{5.6}\]

then \((\mathfrak{g}[u, u^{-1}], \circ)\) is a Lie-admissible algebra satisfying that

\[ [f, g]_1 = f \circ g - g \circ f = [\mu(f), g] + \mu([f, \mu(g)]) + [f, g], \quad \forall f, g \in \mathfrak{g}[u, u^{-1}] \tag{5.7} \]
defines a Lie algebra and \( \mu \) is a homomorphism of Lie algebras.

(4) As we have already pointed out in Section 2, besides the complex simple Lie algebras, the construction from \( \mathcal{O} \)-operators in this paper may not give all the rational solutions of the CYBE. What we have done in this paper is just an effort to provide certain helpful ideas to construct the solutions of the CYBE with parameters for the general Lie algebras which is still an open question.

(5) It is natural to consider the corresponding Lie bialgebra structures from the rational solutions (5.1) of the CYBE constructing from the \( \mathcal{O} \)-operators through

\[ \delta(f)(u, v) = [f(u) \otimes 1 + 1 \otimes f(v), r(u, v)], \quad \forall f \in \mathfrak{g}[u, u^{-1}]. \tag{5.8} \]

It is also natural and important to consider the quantization of these Lie bialgebra structures.

ACKNOWLEDGMENTS

This work was supported in part by the National Natural Science Foundation of China (10571091, 10621101), NKBRCPC (2006CB805905), SRFD (200800550015), Program for New Century Excellent Talents in University. Part of this work was done while the second author was visiting Korea Institute for Advanced Study (KIAS). He would like to thank KIAS and his host Dr. Dafeng Zuo for the hospitality and the valuable discussions.
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