Incorporation of Generalized Uncertainty Principle into Lifshitz Field Theories

Mir Faizal\(^1\) and Barun Majumder\(^2\)
\(^1\)Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada
\(^2\) Indian Institute of Technology Gandhinagar, Ahmedabad, 382424, India

Abstract

In this paper, we will incorporation the generalized uncertainty principle into field theories with Lifshitz scaling. We will first construct both bosonic and fermionic theories with Lifshitz scaling based on generalized uncertainty principle. After that we will incorporation the generalized uncertainty principle into an non-abelian gauge theory with Lifshitz scaling. We will observe that even though the action for this theory is non-local, it is invariant under local gauge transformations.

1 Introduction

The classical picture of spacetime breaks down in most approaches to quantum gravity. This is due to the fluctuations in the geometry being of order one at Planck scale. Thus, the picture of spacetime as a continuous differential manifold cannot be valid below Planck length. Furthermore, the existence of a minimum length scale is also a feature of string theory [1]-[5]. In fact, in loop quantum gravity the existence of minimum length turns big bang into a big bounce [6]. However, the existence of minimum length is not consistent with conventional uncertainty principle, which states that one can measure length with arbitrary accuracy, if one takes no measurement of momentum [7]-[21]. Thus, the uncertainty principle has to be modified if one wants to incorporate the existence of minimum length scale. These considerations have led to a modification of the Heisenberg uncertainty principle, which in turn has led to a modification of the Heisenberg algebra. It may be noted that the implications of this modified uncertainty principle for quantum field theory have also been studied [22]-[24]. In this paper, we analyse a quantum field theory based on generalized uncertainty with Lifshitz scaling. Lifshitz field theories are quantum field theories based on an anisotropic scaling between space and time.

Lifshitz theories were first introduced in condensed matter physics to model quantum criticality [25]-[28]. In fact, a Fermi-surface-changing Lifshitz transition occurs for some heavy fermion compounds [29]. The location of this Fermi-surface-changing Lifshitz transition is influenced by carrier doping. Due to strong correlations, a heavy band does not shift rigidly with the chemical
potential and the actual shift is determined by the interplay of heavy and additional light bands crossing the Fermi level. Furthermore, meta-magnetic transitions in models for heavy fermions has also been analysed using doped Kondo lattice model in two dimensions [30]. Some heavy fermion metals displays a field-driven quantum phase transition due to a breakdown of the Kondo effect [31]-[32]. Many of the properties have been described by a Zeeman-driven Lifshitz transition of narrow heavy fermion bands [33]. Materials that cannot be described with the local dielectric response have been described by a generalization of the usual Lifshitz theory [34]. In fact, the temperature correction to the Casimir-Lifshitz free energy between two parallel plates made of dielectric material, possessing a constant conductivity at low temperatures, has been calculated [35]. Lifshitz theory have also been used for calculating the van der Waals and Casimir interaction between graphene and a material plate, graphene and an atom or a molecule, and between a single-wall carbon nanotube and a plate [36]. In this model the reflection properties of electromagnetic oscillations on graphene are governed by the specific boundary conditions imposed on the infinitely thin positively charged plasma sheet, carrying a continuous fluid with some mass and charge density.

Fermionic retarded Green’s function with $z = 2$ has been studied at finite temperature and finite chemical potential [37]. Here the usual Lifshitz geometry was replaced by a Lifshitz black hole. Hawking radiation for Lifshitz fermions has also been studied [38]. Fermionic theories with $z = 2$ Lifshitz scaling have also been constructed using a non-local differential operator [39]. This non-local differential operator is defined using harmonic extension of a compactly supported function [40]-[44]. It appears as a map from the Dirichlet-type problem to the Neumann type problem. It may be noted that fermionic theories with $z = 3$ have also been studied [45]-[46]. It has been demonstrated that Nambu-Jona-Lasinio type four-fermion coupling at the $z = 3$ Lifshitz fixed point in four dimensions is asymptotically free and generates a mass scale [47]. In this paper, we will study both bosonic and fermionic Lifshitz field theory, consistent with generalized uncertainty principle. We will also study the gauge symmetry for these theories.

2 Generalized Uncertainty Principle

In the Lifshitz field theories the scaling is usually taken as $x \rightarrow bx$ and $t \rightarrow b^{z}t$, where $b$ is called the scaling factor and $z$ is called the degree of anisotropy. For $z = 1$, this reduces to the usual conformal transformation. In this paper, we will analyze the Lifshitz theories with $z = 2$. The Lifshitz action for a bosonic field with $z = 2$, can be written as [39]

$$S_b = \frac{1}{2} \int d^{d+1}x \ (\phi \partial^\mu \partial_\mu \phi - \kappa^2 (\partial^i \partial_i)^2 \phi).$$

(1)

The Lifshitz theories are unitarity because they contain no higher order temporal derivatives. So, we will leave the temporal part of the Lifshitz action for a bosonic field undeformed. However, we will deform its spatial part, to make it consistent with the existence of a minimum measurable length [21]-[22]. The Heisenberg uncertainty principle is not consistent with the existence of a minimum measurable length, as according to it, we can measure length up to
arbitrary accuracy, if we do not measure the momentum. So, to accommodate the existence of a minimum measurable length scale, the Heisenberg uncertainty principle has to be modified to the generalized uncertainty principle. The generalized uncertainty principle can derived from a deformed Heisenberg algebra. The deformation of the Heisenberg algebra in turn deforms the coordinate representation of the momentum operator, and this deforms the Laplacian to \( \partial^i \partial_i \rightarrow \partial^i \partial_i (1 - \beta \partial^i \partial_i) \) \[22\]. Now using this definition of the deforms the Laplacian, the deformed Lifshitz action can be written as

\[
S_b = \frac{1}{2} \int d^{d+1}x \left( \phi \partial^0 \partial_0 \phi - \kappa^2 \phi \partial^i \partial_i (1 - \beta \partial^i \partial_i)^2 \phi \right). \tag{2}
\]

Here we have to promote that parameter \( \beta \) to a background field, such that it scales as \( \beta \rightarrow b^2 \beta \). This ensures that theory still has Lifshitz scaling after it has been deformed by the generalized uncertainty principle. It may be noted that it is common to promote parameters in conformal field theories to background field in this way \[48\]-\[49\]. These background fields have scaling properties that ensures the conformal invariance of the deformed theory. Now we can write this action as

\[
S_b = \frac{1}{2} \int d^{d+1}x \left( \phi \partial^0 \partial_0 \phi - \kappa^2 \partial^i \phi \mathcal{T}_0^i (1 - \beta \partial^i \partial_i)^2 \partial_i \phi \right), \tag{3}
\]

where \( \mathcal{T}_0 = \sqrt{-\partial \partial_i} \). It may be noted that the non-local differential operator \( \mathcal{T}_0 \) is crucial in constructing the fermionic action with Lifshitz scaling.

Even though this operator is non-local it can be effectively viewed as a local operator, by using the theory of harmonic extension of functions from \( \mathbb{R}^d \) to \( \mathbb{R}^d \times (0, \infty) \) \[39\]-\[44\]. Thus, we can define \( \mathcal{T}_0 \) by its action on functions \( f : \mathbb{R}^d \rightarrow \mathbb{R} \), such that its harmonic extension \( u : \mathbb{R}^d \times (0, \infty) \rightarrow \mathbb{R} \) satisfies, \( \mathcal{T}_0 f(x) = -\partial_y u(x, y)|_{y=0} \). This is because if we start with a function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \), and find a harmonic function \( u : \mathbb{R}^d \times (0, \infty) \rightarrow \mathbb{R} \), such that its restriction to \( \mathbb{R}^d \) coincides with the original function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \), then it is possible to find \( u \) by solving a Dirichlet problem. This Dirichlet problem can be expressed in terms of the Laplacian in \( \mathbb{R}^{d+1} \), which is denoted by \( \partial^2_{d+1} \). So, for \( x \in \mathbb{R}^d \) and \( y \in \mathbb{R} \), we have, \( u(x, 0) = \phi(x) \) and \( \partial^2_{d+1} u(x, y) = 0 \). In fact, for a smooth function \( C^\infty(\mathbb{R}^d) \), there is a unique harmonic extension \( u \in C^\infty(\mathbb{R}^d \times (0, \infty)) \).

Now as \( \mathcal{T}_0 \phi(x) \) also has a harmonic extension to \( \mathbb{R}^d \times (0, \infty) \), we can obtain the following result, \( \mathcal{T}_0^2 \phi(x) = \partial^2_{d+1} u(x, y)|_{y=0} = -\partial^2 \partial_y u(x, y)|_{y=0} = -\partial^2 \partial_y \phi(x) \). Thus, it is possible to define \( \mathcal{T}_0 = \sqrt{-\partial \partial_i} \), because \( \mathcal{T}_0^2 \phi(x) = -\partial^2 \partial_y \phi(x) \). So, we can write \( \mathcal{T}_0 \exp ikx = |k|\exp ikx \), because \( \mathcal{T}_0^2 \exp ikx = |k|^2 \exp ikx \), Furthermore, if we start with two fields \( \phi_1(x) \) and \( \phi_2(x) \), such that \( u_1(x, y) \) and \( u_2(x, y) \) are their harmonic extensions to \( C = \mathbb{R}^d \times (0, \infty) \), and both these harmonic extensions vanish for \( |x| \rightarrow \infty \) and \( |y| \rightarrow \infty \), then we can write \[50\]

\[
\int_C d^dxdy\ u_1(x, y)\partial^2_{n+1}u_2(x, y) - \int_C d^dxdy\ u_2(x, y)\partial^2_{n+1}u_1(x, y) = 0. \tag{4}
\]

Thus, we get the following expression

\[
\int_{\mathbb{R}^d} d^d x \ (u_1(x, y)\partial_y u_2(x, y) - u_2(x, y)\partial_y u_1(x, y))|_{y=0} = 0. \tag{5}
\]
This can now be written in terms of $\phi_1(x)$ and $\phi_2(x)$ as

$$\int_{\mathbb{R}^d} d^d x \ (\phi_1(x) \partial_y \phi_2(x) - \phi_2(x) \partial_x \phi_1(x)) = 0.$$  \hspace{1cm} (6)

So, the operator $T_\partial$ can be moved from $\phi_2(x)$ to $\phi_1(x)$,

$$\int_{\mathbb{R}^d} d^d x \ \phi_1(x) T_\partial f \phi_2(x) = \int_{\mathbb{R}^d} d^d x \ \phi_2(x) T_\partial \phi_1(x).$$  \hspace{1cm} (7)

Now the Lifshitz bosonic action, consistent with generalized uncertainty principle, can also be written as

$$S_b = \frac{1}{2} \int d^{d+1}x \ \partial^\mu \phi \ G_{\mu \nu} \partial^\nu \phi,$$  \hspace{1cm} (8)

where $G_{\mu \nu}$ can be written as

$$G_{\mu \nu} = \left( \begin{array}{cc} I_{1 \times 1} & 0_{1 \times d} \\ 0_{d \times 1} & -\kappa^2 T_0^2 (1 - \beta \partial^i \partial_j) I_{d \times d} \end{array} \right).$$

This equation can now be regarded as defining a scalar product for vector fields, such that for any two vectors $V$ and $W$, we have

$$(V(x), W(x)) = \int d^{d+1}x \ \left( V_0 W_0 - \kappa^2 V_i T_0^2 (1 - \beta \partial^i \partial_j) W_i \right).$$  \hspace{1cm} (9)

The under group of isometries this inner product remains invariant. So, we can write, $(\Lambda(V), \Lambda(W)) = (V, W)$. Thus, we can write $\Lambda_{0\mu} \Lambda_{\mu \nu} - \kappa^2 \Lambda_{ai} \Lambda_{\mu \nu} T_\partial (1 - \beta \partial^i \partial_j) = G_{\mu \nu}$. From this we can infer that $\Lambda_{00} = \Lambda_{i0} = \Lambda_{0i} = 0$ and $\Lambda_{ij} = \delta_{ij}$. Now a set of local gamma matrices can be defined, such that $\{\Gamma_\mu, \Gamma_\nu\} = 2G_{\mu \nu}$. Furthermore, an appropriate choice for these local gamma matrices is $\Gamma_0 = \gamma_0$ and $\Gamma_1 = \kappa T_0 (1 - \beta \partial^i \partial_j) \gamma_i$, where $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$. We can thus define a fermionic Lifshitz operator as $\Gamma^\mu \partial_\mu = \gamma^0 \partial_0 + \kappa \gamma^i T_0 (1 - \beta \partial^i \partial_j) \partial_i$. We observe that $\Gamma^\mu \partial_\mu \Gamma^\nu \partial_\nu = \partial^\nu \partial_\nu - \kappa^2 [\partial^i \partial_j, (1 - 2\beta \partial^i \partial_j)]^2$. The Lifshitz action for a massless fermionic field can be written as

$$S_f = \frac{1}{2} \int d^{d+1}x \ \bar{\psi} \ (\Gamma^\mu \partial_\mu) \psi = \frac{1}{2} \int d^{d+1}x \ \bar{\psi} \ (\gamma^0 \partial_0 + \gamma^i \kappa T_0 (1 - \beta \partial^i \partial_j) \partial_i) \psi.$$  \hspace{1cm} (10)

3 Gauge Symmetry

In this section, we will analyse gauge theories with Lifshitz corresponding to generalized uncertainty principle. We note that if the covariant derivative is gauge covariant, then so, is any function of the covariant derivative. We will construct a covariant derivative from using a non-abelian gauge field $A_\mu = A_\mu^a T_a$, where $[T_A, T_B] = ij f_{AB}^C T_C$. Now if $\psi \rightarrow U \psi$, then we should have $D_\mu \psi \rightarrow UD_\mu \psi$.

We can construct a covariant derivative with this transformation property if, we assume that the gauge field transforms as $A_\mu \rightarrow iU D_\mu U^{-1}$ and define the gauge covariant derivative as, $D_\mu = \partial_\mu + i A_\mu$. This is because now the covariant derivative will transform as

$$D_\mu \rightarrow UD_\mu U^{-1},$$  \hspace{1cm} (11)
and so, $D_{\mu}\psi \rightarrow UD_{\mu}\psi$, if $\psi \rightarrow U\psi$. Now any function of the covariant derivative is also gauge covariant. So, if we take a general function of $D_{\mu}$, $f(D' D_{\nu}) D_{\mu}$, then it transforms as

$$f(D' D_{\nu}) D_{\mu} \rightarrow U f(D' D_{\nu}) D_{\mu} U^{-1},$$

(12)

such that, $f(D' D_{\nu}) D_{\mu} \psi \rightarrow U f(D' D_{\nu}) D_{\mu} \psi$.

We can now use different $f(D' D_{\nu})$, for the spatial and temporal part of the covariant derivative. Now we define $f_1(D' D_{\nu})$ to be the function for the temporal part of the derivative and $f_2(D' D_{\nu})$ to be the function for the spatial part of the derivative. The theory has Lifshitz scaling, if we choose

$$f_1(D' D_{\nu}) D_0 = D_0,$$

$$f_2(D' D_{\nu}) D_i = \kappa \mathcal{T}_D D_i,$$

(13)

where $\mathcal{T}_D = \sqrt{-D' D}$. The covariant derivative will still transform as

$$D_0 \rightarrow U D_0 U^{-1},$$

$$\kappa \mathcal{T}_D D_i \rightarrow U \kappa \mathcal{T}_D D_i U^{-1}.$$  

(14)

Here we have to again assumed that $\beta$ is background field which scales like $\beta \rightarrow \beta' \beta$. [18-19]. This ensures that the theory constructed also has Lifshitz scaling. However, we also want a theory that will correspond to generalized uncertainty principle. In particular the matter part of the Lagrangian should reduce to the Lagrangian derived in the previous section, if we set all the gauge field to zero. Thus, we re-define $f_1(D' D_{\nu})$ and $f_2(D' D_{\nu})$ as

$$f_1(D' D_{\nu}) D_0 = D_0,$$

$$f_2(D' D_{\nu}) D_i = \kappa \mathcal{T}_D (1 - \beta D^j D_j) D_i.$$  

(15)

It may be noted that the covariant derivative will still transforms as

$$D_0 \rightarrow U D_0 U^{-1},$$

$$\kappa \mathcal{T}_D (1 - \beta D^j D_j) D_i \rightarrow U \kappa \mathcal{T}_D (1 - \beta D^j D_j) D_i U^{-1}.$$  

(16)

So, we can now write the final action as

$$S_{fg} = \frac{1}{2} \int d^{4+1}x Tr[\bar{\psi}(\gamma^0 D_0 + \gamma^i \kappa \mathcal{T}_D (1 - \beta D^j D_j) D_i)\psi].$$  

(17)

Now the temporal part of this action is invariant under local gauge transformations because, $Tr[\bar{\psi}(\gamma^0 D_0) \psi] \rightarrow Tr[\bar{\psi} U^{-1} U(\gamma^0 D_0) U^{-1} U\psi] = Tr[\bar{\psi}(\gamma^0 D_0) \psi]$, and the spatial part of this action is also invariant under local gauge transformations because, $Tr[\bar{\psi}(\gamma^i \kappa \mathcal{T}_D (1 - \beta D^j D_j) D_i) \psi] \rightarrow Tr[\bar{\psi} U^{-1} U(\gamma^i \kappa \mathcal{T}_D (1 - \beta D^j D_j) D_i) U^{-1} U\psi] = Tr[\bar{\psi}(\gamma^i \kappa \mathcal{T}_D (1 - \beta D^j D_j) D_i) \psi]$. So, even though this action is non-local, it is invariant under local gauge transformations, $A_{\mu} \rightarrow i U D_{\mu} U^{-1}$.

It may be noted that we can now define a gauge field tensor for this theory as

$$F_{\phi} = -i[D_0, \kappa \mathcal{T}_D (1 - \beta D^j D_j) D_i],$$

$$F_{ij} = -i[\kappa \mathcal{T}_D (1 - \beta D^k D_k) D_i, \kappa \mathcal{T}_D (1 - \beta D^j D_j) D_j].$$  

(18)
It transforms as
\[
F_{i0} \rightarrow -i[U D_0 U^{-1}, U \kappa T_D(1 - \beta D^j D_j)D_i U^{-1}],
\]
\[
= U F_{i0} U^{-1},
\]
\[
F_{ij} \rightarrow -i[U \kappa T_D U^{-1}(1 - \beta U D^k U^{-1} UD_k U^{-1})U D_i U^{-1},
\]
\[
U \kappa T_D U^{-1}(1 - \beta U D_i U^{-1} UD_j U^{-1})U D_j U^{-1}]
\]
\[
= U F_{ij} U^{-1}.
\]

Now we can write the action for the gauge part of the action as follows,
\[
S_g = -\frac{1}{4} \int d^{d+1}x \ Tr[F^{\mu\nu} F_{\mu\nu}].
\]

It may be noted that even thought this action is non-local, it is invariant under local gauge transformations, \( A_\mu \rightarrow i U D_\mu U^{-1} \), because,
\[
Tr[F^{\mu\nu} F_{\mu\nu}] \rightarrow Tr[U F^{\mu\nu} U^{-1} U F_{\mu\nu} U^{-1}] = Tr[F^{\mu\nu} F_{\mu\nu}].
\]
Now we can write the gauge fixing term for this theory,
\[
S_{gh} = \int d^{d+1}x \ Tr[b \partial_0 A_0 - b \kappa \partial^i T_D(1 - \beta \partial^j \partial_j)A_i].
\]

The ghost term for corresponding to this gauge fixing term, can be written as
\[
S_{gf} = \int d^{d+1}x \ Tr[\bar{c} \partial_0 D_0 c - \kappa^2 \bar{c} \partial^i T_D(1 - \beta \partial^j \partial_j)D_i (1 - \beta D^k D_k)T_D c].
\]

4 Conclusion

In this paper we deformed the Lifshitz field theories to make them consistent with the existence of a minimum measurable length. This was done by incorporating generalized uncertainty principle in them. We had to promote a parameter used in the theory to a background field with interesting scaling properties, to preserve the Lifshitz scaling of the deformed theory. We also analysed a deformed Lifshitz theory gauge theory based on the generalized uncertainty principle. We observed that even though this theory is non-local, it is invariant under local gauge transformations. We are expect to obtain similar results, if we generalize this work by incorporating terms linear in the momentum, in the deformed Heisenberg algebra [51]-[55]. It would also be interesting to analyse the BRST symmetry for this theory.

The holographic dual to the Lifshitz field theory has also been studied [57]-[60]. The dual of the field theory vacuum has a bulk metric,
\[
ds^2 = -r^2 dt^2 + r^2 dx^2 + L^2 r^{-2} dr^2,
\]
where \( L^2 \) represents the overall curvature scale. It is obvious that for \( z = 1 \), this metric reduces to the usual AdS metric. In these Lifshitz theories, the renormalization group flow at finite temperature is used for evaluating the dependence of physical quantities such as the energy density on the momentum scale [61]. Furthermore, the holographic renormalization of gravity in asymptotically Lifshitz spacetimes naturally reproduces the structure of gravity with anisotropic scaling [62]. The holographic counter-terms induced near anisotropic infinity
take the form of the action for gravity at a Lifshitz point, with the appropriate value of the dynamical critical exponent. The holographic renormalization of Horava-Lifshitz gravity reproduces the full structure of the $z = 2$ anisotropic Weyl anomaly in dual field theories in three dimensions [63]. In fact, Lifshitz theories have also become important because of the development of Horava-Lifshitz gravity [64]-[68]. Horava-Lifshitz gravity is a renormalizable theory of gravity, in which unitarity is not spoiled. Even though gravity is not renormalizable, it can be made renormalizable by adding higher order curvature terms to it. However, the addition of higher order temporal derivatives spoils the unitarity of the theory. A way out of this problem is to add higher order spatial derivatives without adding any higher order temporal derivatives. Even though this break Lorentz symmetry, the Horava-Lifshitz theory of gravity reproduces General Relativity in the infrared limit. It may be noted that the string theory comes naturally equipped with a minimum measurable length scale, which is the string length scale. This is because the spacetime cannot be probed below this scale [69]. Furthermore, the existence of a minimum measurable length scale in a theory produces higher derivative corrects terms, due to the existence of generalized uncertainty principle [70]. So, it will be interesting to analyse the Lifshitz deformation of $AdS/CFT$ correspondence, consistent with generalized uncertainty principle.

As we have both the fermionic and bosonic actions, it will be interesting to analyse supersymmetric theories based on such deformations. In fact, according to $AdS/CFT$ correspondence type $IIB$ superstring on $AdS_5 \times S^5$ is dual to the maximally non-local supersymmetric $\mathcal{N} = 4$ super-Yang-Mills theory in four dimensions [50]-[74], so, this result will can be used to study the gravity dual to such a theory. Furthermore, as $AdS_5 \times S^5 \sim SO(2,4)/SO(1,4) \times SO(6)/SO(5) \subset SU(2,2|4)/SO(1,4) \times SO(5)$, so, the superisometries of this background are generated by the supergroup $SU(2,2|4)$, which also generates the superconformal invariance of $\mathcal{N} = 4$ super-Yang-Mills theory in four dimensions. Here the four dimensional superconformal transformations are generated by $SO(2,4)$ and the $R$-symmetry is generated by $SO(6) \sim SU(4)$. Furthermore, $\mathcal{N} = 4$ super-Yang-Mills theory, with $U(N)$ as the gauge group, is the low-energy limit for a stack of multiple coincident D3-branes on $AdS_5 \times S^5$. Here the transverse D3-brane coordinates give rise to six scalar fields in the $\mathcal{N} = 4$ super-Yang-Mills theory. Apart from these six bosons, there are also sixteen fermions. Thus, a Lifshitz deformation of bulk theory, consistent with generalized uncertainty, may produce interesting deformation of the $\mathcal{N} = 4$ super-Yang-Mills theory. It will be interesting to analyse such a deformed super-Yang-Mills theories.

Acknowledgement

We would like to thank Ali Nassar for pointing out to us an interesting technique used in conformal field theories, i.e., the parameters in a conformal field theory can be promoted to background fields. These background fields can have interesting scaling properties.
References

[1] D. Amati, M. Ciafaloni and G. Veneziano, Phys. Lett. B 216, 41 (1989)
[2] A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. D 52, 1108 (1995)
[3] L. N. Chang, D. Minic, N. Okamura, and T. Takeuchi, Phys. Rev. D 65, 125027 (2002)
[4] L. N. Chang, D. Minic, N. Okamura, and T. Takeuchi, Phys. Rev. D 65, 125028 (2002)
[5] S. Benczik, L. N. Chang, D. Minic, N. Okamura, S. Rayyan, and T. Takeuchi, Phys. Rev. D 66, 026003 (2002)
[6] P. Dzierzak, J. Jezierski, P. Malkiewicz, and W. Piechocki, Acta Phys. Polon. B41, 717 (2010)
[7] D. Amati, M. Ciafaloni, and G. Veneziano, Phys. Lett. B 216, 41 (1989)
[8] M. Magnie, Phys. Lett. B 394, 5 (1993)
[9] M. Magnie, Phys. Rev. D 49, 2182 (1994)
[10] M. Magnie, Phys. Lett. B 319, 83 (1993)
[11] L. J. Garay, Int. J. Mod. Phys. A 10, 145 (1995)
[12] F. Scardigli, Phys. Lett. B 452, 39 (1999)
[13] C. Bambi, F. R. Urban, Class. Quantum Grav. 25, 095006 (2008)
[14] K. Nozari, Phys. Lett. B. 629, 41 (2005)
[15] K. Nozari, T. Azizi, Gen. Relativ. Gravit. 38, 735 (2006)
[16] P. Pedram, Int. J. Mod. Phys. D 19, 2003 (2010)
[17] A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. D 52, 1108 (1995)
[18] A. Kempf, J. Phys. A 30, 2093 (1997)
[19] F. Brau, J. Phys. A 32, 7691 (1999)
[20] K. Nozari, and B. Fazlpour, Chaos, Solitons and Fractals, 34, 224 (2007)
[21] S. Das, and E. C. Vagenas, Phys. Rev. Lett. 101, 021301 (2008)
[22] M. Kober, Phys. Rev. D 82, 085017 (2010)
[23] V. Husain, D. Kothawala and S. S. Seahra, Phys. Rev. D 87, 025014 (2013)
[24] M. Kober, Int. J. Mod. Phys. A 26, 4251 (2011)
[25] R. M. Hornreich, M. Luban and S. Shtrikman, , Phys. Rev. Lett. 35, 1678 (1975)
[26] G. Grinstein, Phys. Rev. B 23, 4615 (1981)
[27] P. M. Chaikin and T. C. Lubensky, Principles of Condensed Matter Physics, Cambridge University Press, Cambridge, UK (1995)

[28] S. Sachdev, Quantum Phase Transitions, Cambridge University Press, Cambridge, UK (2001)

[29] A. Benlagra and M. Vojta, Phys. Rev. B 87, 165143 (2013)

[30] M. Bercx and F. F. Assaad, Phys. Rev. B 86, 075108 (2012)

[31] P. Gegenwart, Nature Phys. 4, 186 (2008)

[32] P. Coleman, C. Pepin, Q. Si and R. Ramazashvili, J. Phys. Condens. Matt. 13, R723 (2001)

[33] A. Hackl and M. Vojta, Phys. Rev. Lett. 106, 137002 (2011)

[34] V. B. Svetovoy, Phys. Rev. Lett. 101, 163603 (2008)

[35] S. A. Ellingsen, I. Brevik, J. S. Hoye and K. A. Milton, Phys. Rev. E 78, 021117 (2008)

[36] M. Bordag, B. Geyer, G. L. Klimchitskaya and V. M. Mostepanenko, Phys. Rev. B 74, 205431 (2006)

[37] M. Alishahiha, M. R. M. Mozaffar and A. Mollabashi, Phys. Rev. D 86, 026002 (2012)

[38] M. Liu, J. Lu and J. Lu, Class. Quant. Grav. 28, 125024 (2011)

[39] H. Montani and F. A. Schaposnik, Phys. Rev. D 86, 065024 (2012)

[40] L. Caffarelli and L. Silvestre, Comm. Part. Diff. Eqs. 32, 1245 (2007)

[41] R. T. Seeley, Proc. Symp. Pure Math. 10, 288 (1967)

[42] C. Laemmerzahl, J. Math. Phys. 34, 3918 (1993)

[43] J. J. Giambiagi, Nuovo Cim. A 104, 1841 (1991)

[44] C. G. Bollini and J. J. Giambiagi, J. Math. Phys. 34, 610 (1993)

[45] D. Anselmi and M. Halat, Phys. Rev. D 76, 125011 (2007)

[46] D. Anselmi, Eur. Phys. J. C 65, 523 (2010)

[47] A. Dhar, G. Mandal and S. R. Wadia, Phys. Rev. D 80, 105018 (2009)

[48] Z. Komargodski, JHEP. 1207, 069 (2012)

[49] Z. Komargodski, JHEP 1112, 099 (2011)

[50] J. Tan, Calc. Var 42, 21 (2011)

[51] A. F. Ali, S. Das and E. C. Vagenas, Phys. Rev. D84, 044013 (2011)

[52] P. Pedram, K. Nozari and S. H. Taheri, JHEP 1103, 093 (2011)

[53] M. Asghari, P. Pedram and K. Nozari, Phys. Lett. B 725, 451 (2013)
[54] S. Das, E. C. Vagenas and A. F. Ali, Phys. Lett. B 690, 407 (2010)
[55] W. Chemissany, S. Das, A. F. Ali and E. C. Vagenas, JCAP. 1112, 017 (2011)
[56] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998)
[57] S. Kachru, X. Liu and M. Mulligan, Phys. Rev. D78, 106005 (2008)
[58] K. Balasubramanian and K. Narayan, JHEP. 1008, 014 (2010)
[59] R. Gregory, S. L. Parameswaran, G. Tasinato and I. Zavala, JHEP. 1012, 047 (2010)
[60] A. Donos and J. P. Gauntlett, JHEP. 1012, 002 (2010)
[61] M. Park and R. B. Mann, JHEP. 1207, 173 (2012)
[62] T. Griffin, P. Horava and C. M. M. Thompson, JHEP. 1205, 010 (2012)
[63] T. Griffin, P. Horava and C. M. M. Thompson, Phys. Rev. Lett. 110, 081602 (2013)
[64] P. Horava, Phys. Lett. B 694,172 (2010)
[65] P. Horava, Phys. Rev. D 79, 084008 (2009)
[66] P. Horava, JHEP. 03, 020 (2009)
[67] O. Obregon and J. A. Preciado, Phys. Rev. D 86, 063502 (2012)
[68] A. Sheykhi, Phys. Rev. D 87, 024022 (2013)
[69] D. Amati, M. Ciafaloni and G. Veneziano, Phys. Lett. B 216, 41 (1989)
[70] S. Das and E. C. Vagenas, Phys. Rev. Lett. 101, 221301 (2008)
[71] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. 323, 183 (2000)
[72] E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998)
[73] J. M. Maldacena and C. Nunez, Phys. Rev. Lett. 86, 588 (2001)
[74] A. Karch and E. Katz, JHEP. 0206, 043 (2002)