CARLESON MEASURE PROBLEMS FOR PARABOLIC BERGMAN SPACES AND HOMOGENEOUS SOBOLEV SPACES

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Abstract. Let $b^p_0(\mathbb{R}^{1+n}_+)$ be the space of solutions to the parabolic equation
$$\partial_t u + (-\Delta)^\alpha u = 0 \quad (\alpha \in (0,1])$$
having finite $L^p(\mathbb{R}^{1+n}_+)$ norm. We characterize nonnegative Radon measures $\mu$ on $\mathbb{R}^{1+n}_+$ having the property
$$\|u\|_{L^q(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|\dot{W}^{\alpha,p}(\mathbb{R}^{1+n}) \cap \mathbb{R}^{1+n}_+\|_{L^q(\mathbb{R}^{1+n}_+,\mu)},$$
whenever $u(t,x) \in b^p_0(\mathbb{R}^{1+n}_+) \cap \dot{W}^{1,p}(\mathbb{R}^{1+n})$. Meanwhile, denoting by $v(t,x)$ the solution of the above equation with Cauchy data $v_0(x)$, we characterize nonnegative Radon measures $\mu$ on $\mathbb{R}^{1+n}_+$ satisfying
$$\|v(t^{2\alpha},x)\|_{L^q(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|v_0\|_{W^{\beta,p}(\mathbb{R}^n)}, \quad \beta \in (0,n), \quad p \in [1, n/\beta], \quad q \in (0,\infty).$$
Moreover, we obtain the decay of $v(t,x)$, an iso-capacitary inequality and a trace inequality.

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1. Introduction and Statement of Results

Carleson measure was first introduced in classical Hardy space (see Carleson [7]) and have been extensively studied, for example, see Dafni-Karadzhov-Xiao [11], Dafni-Xiao [12], Hastings [13], Johnson [15], and Xiao [38]-[40] and the references therein. This paper considers Carleson measure problems via the parabolic equation

$$\partial_t u(t,x) + (-\Delta)^\alpha u(t,x) = 0, \quad (t,x) \in \mathbb{R}^{1+n}_+ = (0,\infty) \times \mathbb{R}^n$$

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with $\alpha \in (0, 1]$, and its Cauchy problem
\begin{equation}
\begin{cases}
\partial_t v + (-\Delta)^\alpha v = 0, & (t, x) \in \mathbb{R}^{1+n}_+; \\
v(0, x) = v_0(x), & x \in \mathbb{R}^n ,
\end{cases}
\end{equation}
where $\Delta$ is the Laplacian with respect to $x$ and
\[(-\Delta)^\alpha u(t, x) = F^{-1}(|\xi|^{2\alpha} F(u(t, \xi)))(x)\]
with $F$ and $F^{-1}$ being the Fourier transform and the inverse Fourier transform. More specifically, we characterize nonnegative Radon measures $\mu$ on $\mathbb{R}^{1+n}_+$ having the property
\begin{equation}
\|u(t, x)\|_{L^p(\mathbb{R}^{1+n}_+, \mu)} \lesssim \|\nabla (t, x) u(t, x)\|_{L^p(\mathbb{R}^{1+n}_+)} , \forall u \in \mathcal{B}_0^\alpha(\mathbb{R}^{1+n}_+) \cap \dot{W}^{1, p}(\mathbb{R}^{1+n}_+) ,
\end{equation}
for $1 \leq p \leq q < \infty$, or
\begin{equation}
\|u(t_2, x)\|_{L^p(\mathbb{R}^{1+n}_+, \mu)} \lesssim \|u_0(x)\|_{\dot{W}^{\beta, p}(\mathbb{R}^n)} , \forall u_0 \in \dot{W}^{\beta, p}(\mathbb{R}^n) ,
\end{equation}
for $\beta \in (0, n)$, $p \in [1, n/\beta]$ and $q \in (0, \infty)$. Here
\[v(t, x) = S_\alpha(t)v_0(x) := K^\alpha_t(x) * v_0(x)\]
solves (1.2),
\[K^\alpha_t(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{-t|\xi|^{2\alpha}} d\xi \geq 0 , \forall (t, x) \in \mathbb{R}^{1+n}_+\]
and $g(x) * h(x)$ means the convolution between $g(x)$ and $f(x)$ on the space variable.

The main motivation of considering embeddings (1.3) and (1.4) comes from the so called trace inequalities problem. Particularly, for a nonnegative Borel measure $\mu$ on $\mathbb{R}^n$, when working on spectral problems for Schrodinger operators, Maz'ya first discovered in 1962 (see Maz'ya 17-18 and 22) that if $1 < p \leq q$ and $pl < n$ then
\begin{equation}
\|u\|_{L^q(\mathbb{R}^n, \mu)} \lesssim \|u\|_{h^l(\mathbb{R}^n)} , \forall u \in h^l_p(\mathbb{R}^n)\]
holds if and only if
\begin{equation}
\sup \left\{ \frac{\mu(E)^{p/q}}{cap(E, h^l_p)} : E \subset \mathbb{R}^n , cap(E, h^l_p) > 0 \right\} < \infty .
\end{equation}
Here $U \lesssim V$ denotes $U \leq \theta V$ for some positive $\theta$ which is independent of the sets or functions under consideration in both $U$ and $V$, $h^l_p(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to
\[\|f\|_{h^l_p(\mathbb{R}^n)} = \|(-\Delta)^{l/2} f\|_{L^p}\]
and $cap(E, h^l_p(\mathbb{R}^n))$ is the capacity of $E$ associated with $h^l_p(\mathbb{R}^n)$. Such embeddings like (1.5) are referred to as trace inequalities, see Adams-Hedberg 5. Meanwhile, (1.0) is called isocapacitary inequality, see Maz'ya 23. Since Maz'ya established the pioneer work in 17-18, other equivalent conditions of trace inequalities were established by Maz'ya 19-20, Maz'ya-Preobraženskii 25, Maz'ya-Verbitsky 29, Adams 11 and new advances of such problems were made by Casacante-Ortega-Verbitsky 8 in which they established similar trace inequality for a general class of radially decreasing convolution kernels. When $0 < q < p$ and $1 < p < \infty$, (1.5) holds if and only if
\begin{equation}
\int_0^\infty \left( \frac{t^p}{\theta(t)^q} \right)^{1/(p-q)} \frac{dt}{t} < \infty
\end{equation}
with \( \vartheta(t) = \inf \{ \text{cap}(E, S^t_x) : E \subset \mathbb{R}^n, \mu(E) \geq t \} \), see Maz’ya-Netrusov \[27\] and Verbitsky \[36\]. When \( 1 < p < q < \infty \), \( (1.5) \) holds if and only if

\[
(1.8) \sup_{x \in \mathbb{R}^n, r > 0} \frac{\mu(B(x, r)))^{p/q}}{r(n-1p)} < \infty,
\]

where \( B(x, r) \) is a ball of radius \( r \) centered at \( x \in \mathbb{R}^n \), see Adams-Hedberg \[3\] Theorem 7.2.2. When \( 0 < q < p \) and the references therein. These equivalent conditions were widely applied to harmonic analysis, operator theory, function spaces, linear and nonlinear partial differential equations, etc., see, Adams-Hedberg \[4\], Maz’ya \[22\] and Maz’ya-Shaposhnikova \[29\] and the references therein.

This paper characterizes \( (1.3) \) or \( (1.4) \) by conditions like \( (1.6), (1.7) \) and \( (1.8) \). To do this, we need the following preliminary materials.

We always assume that \( \beta \in (0, n) \cap \mathbb{N} \) when \( p = 1 \) or \( n/\beta \). \( \dot{W}^{1,p} (\mathbb{R}^{1+n}) \) is the completion of \( C_0^0 (\mathbb{R}^{1+n}) \) with respect to the norm

\[
\|f\|_{\dot{W}^{1,p} (\mathbb{R}^{1+n})} = \left( \int_{\mathbb{R}^{1+n}} |\nabla (t,x)f|^p \, dt \, dx \right)^{1/p}.
\]

\( b^p_\alpha (\mathbb{R}^{1+n})(\alpha \in (0, 1]) \) introduced by Nishio-Shimomura-Suzuki \[32\] is the parabolic Bergman space on \( \mathbb{R}^{1+n} \), which is the set of all solutions of the parabolic equation \( (1.1) \) having finite \( L^p (\mathbb{R}^{1+n}) \) norm. \( \dot{W}^{\beta,p} (\mathbb{R}^n) \) is the homogeneous Sobolev space which is the completion of \( C_0^0 (\mathbb{R}^n) \) with respect to the norm

\[
\|f\|_{\dot{W}^{\beta,p} (\mathbb{R}^n)} = \begin{cases} \|(-\Delta)^{\beta/2} f\|_{L^p}, & p \in (1, n/\beta), \\ \left( \int_{\mathbb{R}^n} \frac{\|\nabla_x f\|^p}{|\nabla_x f|^{n-\beta p}} \, dx \right)^{1/p}, & p = 1 \text{ or } p = n/\beta, \beta \in (0, n) \cap \mathbb{N}, \end{cases}
\]

where

\[
\Delta^k f(x) = \begin{cases} \Delta^k_1 \Delta^k_2 \cdots \Delta^k_{k-1} f(x), & k > 1, \\ f(x+h) - f(x), & k = 1, \end{cases}
\]

\( k = 1 + \lfloor \beta \rfloor, \beta = \lfloor \beta \rfloor + \{\beta\} \) with \( \{\beta\} \in (0, 1) \).

If \( X = \mathbb{R}^{1+n}, \beta = 1 \) and \( p \geq 1 \), or \( X = \mathbb{R}^n, \beta \in (0, n) \) and \( p \in [1, n/\beta] \), \( \text{cap}_{\dot{W}^{\beta,p}(X)} (S) \) (see Maz’ya \[22\]) is the variational capacity of an arbitrary set \( S \subset X \):

\[
\text{cap}_{\dot{W}^{\beta,p}(X)} (S) = \inf \left\{ \|f\|_{\dot{W}^{\beta,p}(X)} : f \in V_X (S) \right\}.
\]

Here

\[
V_{\mathbb{R}^{1+n}} (S) = \{ f \in \dot{W}^{1,p} (\mathbb{R}^{1+n}) : S \subset \text{Int}(\{ x \in \mathbb{R}^{1+n} : f \geq 1 \}) \}
\]

and

\[
V_{\mathbb{R}^n} (S) = \{ f \in \dot{W}^{\beta,p} (\mathbb{R}^n) : f \geq 0, S \subset \text{Int}(\{ x \in \mathbb{R}^n : f \geq 1 \}) \}.
\]
with \( \text{Int}(E) \) be the interior of a set \( E \subseteq X \). For \( t \in (0, \infty) \), \( c_p^\alpha(\mu; t) \) is the \((p, \beta)\)-variational capacity minimizing function associated with both \( W^{\beta, p}(\mathbb{R}^n) \) and a nonnegative measure \( \mu \) on \( \mathbb{R}^{1+n}_+ \) defined by

\[
c_p^\alpha(\mu; t) = \inf \{ \text{cap}_{W^{\beta, p}(\mathbb{R}^n)}(O) : \text{bounded open } O \subseteq \mathbb{R}^n, \mu(T(O)) > t \},
\]

where \( T(O) \) is the tent based on an open subset \( O \) of \( \mathbb{R}^n \):

\[
T(O) = \{(r, x) \in \mathbb{R}^{1+n}_+ : B(x, r) \subseteq O \},
\]

with \( B(x, r) \) be the open ball centered at \( x \in \mathbb{R}^n \) with radius \( r > 0 \).

For handling the endpoint case \( p = n/\beta \) we also need the definition of the Riesz potentials (see Adams-Xiao [6] and Adams [2]) on \( \mathbb{R}^{2n}_+ \) as follows. The Riesz potential of order \( \gamma \in (0, 2n) \) is defined by

\[
I_{\gamma}^{(2n)} \ast f(z) = \int_{\mathbb{R}^{2n}_+} |z - y|^{\gamma - 2n} f(y) dy, \quad z \in \mathbb{R}^{2n}_+.
\]

From Adams [2] Theorem 5.1, we have that if \( u(x) \) and \( I_{\gamma}^{(2n)} \ast |f|(x, 0) \) are both in \( L^1_{loc}(\mathbb{R}^n) \) with

\[
(1.9) \quad f(x, h) = |h|^{-\gamma} \Delta_h^k u(x),
\]

then \( u(x) = C I_{\gamma}^{(2n)} \ast f(x, 0) \), for a.e. \( x \in \mathbb{R}^n \) and some \( C > 0 \). Note that if \( u \in W^{\beta, n/\beta}(\mathbb{R}^n) \) and \( \gamma = 2\beta \in (0, 2n) \) then the function \( f(\cdot, \cdot) \) in (1.9) belongs to the space \( L^{n/\beta}(\mathbb{R}^{2n}) \). For any \( \gamma \in (0, 2n), L^{\gamma}_p(\mathbb{R}^{2n}) = I_{\gamma}^{2n} \ast L^p(\mathbb{R}^{2n}) \) defined by

\[
\|I_{\gamma}^{2n} \ast f\|_{L^p(\mathbb{R}^{2n})} = \|f\|_{L^p(\mathbb{R}^{2n})}.
\]

To state our main results, let us agree to more conventions. \( U \approx V \) if \( U \subseteq V \) and \( V \subseteq U \); for \( 0 < p, q < \infty \) and a nonnegative Radon measure \( \mu \) on \( X = \mathbb{R}^{1+n}_+ \) or \( \mathbb{R}^n \), \( L^{q,p}(X, \mu) \) and \( L^q(X, \mu) \) denote the Lorentz space and the Lebesgue space of all functions \( f \) on \( X \) for which

\[
\|f\|_{L^{q,p}(X, \mu)} = \left( \int_0^\infty \mu(\{x \in X : |f(x)| > \lambda\})^{p/q} d\lambda \right)^{1/p} < \infty
\]

and

\[
\|f\|_{L^q(X, \mu)} = \left( \int_X |f(x)|^q d\mu \right)^{1/q} < \infty,
\]

respectively. Moreover, we use \( L^{q,\infty}(X, \mu) \) as the set of all \( \mu \)-measurable functions \( f \) on \( X \) with

\[
\|f\|_{L^{q,\infty}(X, \mu)} = \sup_{\lambda > 0} \lambda \mu(\{x \in X : |f(x)| > \lambda\})^{1/q} < \infty.
\]

**Theorem 1.1.** Let \( 1 \leq p < q < \infty \) and \( \mu \) be a nonnegative Radon measure on \( \mathbb{R}^{1+n}_+ \). Then the following statements are equivalent:

(a) \[
\|u\|_{L^{q,p}(\mathbb{R}^{1+n}_+, \mu)} \lesssim \|u\|_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_+)} \quad u \in \dot{W}^{1,p}(\mathbb{R}^{1+n}_+), \quad b_p(\mathbb{R}^{1+n}_+),
\]

(b) \[
\|u\|_{L^{q,p}(\mathbb{R}^{1+n}_+, \mu)} \lesssim \|u\|_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_+)} \quad u \in \dot{W}^{1,p}(\mathbb{R}^{1+n}_+), \quad b_p(\mathbb{R}^{1+n}_+),
\]

(c) \[
\|u\|_{L^{q,\infty}(\mathbb{R}^{1+n}_+, \mu)} \lesssim \|u\|_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_+)} \quad u \in \dot{W}^{1,p}(\mathbb{R}^{1+n}_+), \quad b_p(\mathbb{R}^{1+n}_+),
\]
Theorem 1.2. Let $R$ measure on $1 < p < \infty$. If $0 < q < p$, then (b) \( \Rightarrow \) (c) \( \Rightarrow \) (d) \( \Rightarrow \) (a).

In the following, $v(t, x)$ is the solution of equation (1.2) with Cauchy data $v_0(x)$.

Theorem 1.3. Let $\beta \in (0, n)$, $1 \leq p \leq n/\beta$ and $\mu$ a nonnegative Radon measure on $\mathbb{R}^{1+n}_+$. Then the following five conditions are equivalent:

(a) \[ \|v(t^{2\alpha}, x)\|_{L^p(R^{1+n}_+, \mu)} \lesssim \|v_0\|_{\dot{W}^{\beta,p}(\mathbb{R}^n)}, \forall v_0 \in \dot{W}^{\beta,p}(\mathbb{R}^n). \]

(b) \[ \int_0^\infty \left( \frac{t^{p/q}}{c_p^\beta(\mu; t)} \right)^{(q-p)/q} dt < \infty. \]

If we change $1 < p \leq n/\beta$ and $0 < q < p$ into $1 \leq p \leq n/\beta$ and $p \leq q < \infty$, then the conditions (a) and (b) of Theorem 1.2. can be replaced by a weak-type one and two simpler ones, respectively.

Theorem 1.4. Let $\beta \in (0, n)$ and $\mu$ a nonnegative Radon measure on $\mathbb{R}^{1+n}_+$. If $1 < p < \min\{q, n/\beta\}$ or $1 = p \leq q < \infty$, then the following two conditions are equivalent:

(a) \[ \|v(t^{2\alpha}, x)\|_{L^p(R^{1+n}_+, \mu)} \lesssim \|v_0\|_{\dot{W}^{\beta,p}(\mathbb{R}^n)}, \forall v_0 \in \dot{W}^{\beta,p}(\mathbb{R}^n). \]

(b) \[ \sup_{x \in \mathbb{R}^n, t > 0} \frac{\mu(T(B(x, r)))^{p/q}}{\text{cap}_{\dot{W}^{\beta,p}(\mathbb{R}^n)}(B(x, r))} < \infty. \]

But, this equivalence fails to hold when $1 < p = q < n/\beta$. 

Furthermore, the family of all bounded open sets in the inequality (c) of Theorem 1.3 in some situation can be replaced by the family of all open balls.
Remark 1.5. We plan to check in our future work that whether or not the operator
\[ v_0 \rightarrow S_{t^{2\alpha}} v_0(x) \]
being compact from \( \dot{W}^{\beta,p}(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^{1+n}, \mu) \) is equivalent to
\[ \lim_{t \to 0} \frac{t^{p/q}}{c_p^\beta(\mu; t)} = 0, \quad \text{if } 1 \leq p \leq n/\beta, p \leq q < \infty; \]
(1.10)
\[ \int_0^\infty \left( \frac{t^{p/q}}{c_p^\beta(\mu; t)} \right) t^{q/(p-q)} \frac{dt}{t} < \infty \text{ if } 0 < q < p, 1 < p \leq n/\beta; \]
or when \( 1 < p < \min\{q, n/\beta\} \) or \( 1 = p \leq q < \infty \),
(1.11)
\[ \lim_{\delta \to 0} \sup_{x \in \mathbb{R}^n, r \in (0, \delta)} \frac{(\mu(T(B(x,r))))^{p/q}}{cap_{W^{\beta,p}}(B(x,r))} = 0 \]
and
(1.12)
\[ \lim_{|x| \to 0} \sup_{r \in (0,1)} \frac{(\mu(T(B(x,r))))^{p/q}}{cap_{W^{\beta,p}}(B(x,r))} = 0. \]
Similar results hold for the embedding (1.5), see Maz’ya [22] section 8.5, 8.6 and [23] or Adams-Hedberg [5] section 7.3.

Remark 1.6. Since the inequality (b) in Theorem 1.4 corresponds to the classical Carleson criterion for \( L^p(\mathbb{R}^n) \) to be embedded in \( L^p(\mathbb{R}^{1+n}, \mu) \) via Poisson’s kernel (see for example Grafakos [13], p. 539, Theorem 7.37), we can refer to the embeddings in Theorems 1.2, 1.4 as the Carleson embeddings for the homogeneous Sobolev spaces per the Cauchy problem for the \( \alpha \)-parabolic equation.

When \( 0 < q < p = 1 \) we obtain necessary conditions for such embeddings.

Theorem 1.7. Let \( \beta \in (0, n) \), \( 0 < q < p = 1 \) and \( \mu \) a nonnegative Radon measure on \( \mathbb{R}^{1+n} \). Then (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c) \( \Rightarrow \) (d):
(a) \[ \|v(t^{2\alpha}, x)\|_{L^q(\mathbb{R}^{1+n}, \mu)} \lesssim \|v_0\|_{W^{\beta,1}(\mathbb{R}^n)}, \forall v_0 \in \dot{W}^{\beta,1}(\mathbb{R}^n). \]
(b) \[ \|v(t^{2\alpha}, x)\|_{L^q(\mathbb{R}^{1+n}, \mu)} \lesssim \|v_0\|_{W^{\beta,1}(\mathbb{R}^n)}, \forall v_0 \in \dot{W}^{\beta,1}(\mathbb{R}^n). \]
(c) \[ \sup \{ (\mu(T(O)))^{1/q}/cap_{W^{\beta,1}}(O) : \text{open } O \subseteq \mathbb{R}^n \} < \infty. \]
(d) \[ \|v(t^{2\alpha}, x)\|_{L^q(\mathbb{R}^{1+n}, \mu)} \lesssim \|v_0\|_{W^{\beta,1}(\mathbb{R}^n)}, \forall v_0 \in \dot{W}^{\beta,1}(\mathbb{R}^n). \]

We can establish the following decay of the solutions of equation (1.2).

Theorem 1.8. If \( v_0 \in \dot{W}^{\beta,p}(\mathbb{R}^n) \) for \( 1 \leq p < n/\beta \) and \( \beta \in (0, n) \), then
\[ \|v(t^{2\alpha}, x_0)\| \lesssim t_0^{p\beta-n} \|v_0\|_{W^{\beta,p}(\mathbb{R}^n)}, \forall (t_0, x_0) \in \mathbb{R}^{1+n}. \]
Equivalently
\[ \|v(t_0, x_0)\| \lesssim t_0^{p\beta-n} \|v_0\|_{W^{\beta,p}(\mathbb{R}^n)}, \forall (t_0, x_0) \in \mathbb{R}^{1+n}. \]
The special case $\alpha = p = 1$ of Theorem 1.8 was proved by Xiao in [39]. Working from $\mathbb{R}^{1+n}$ to $\mathbb{R}^n$, a trace inequality can be derived from $W_{\beta,p}(\mathbb{R}^n)$.

**Theorem 1.9.** Let $\beta \in (0, n)$, $1 < p \leq q < \infty$, $p < n/\beta$ and $\mu$ be a nonnegative Radon measure on $\mathbb{R}^n$. Then

$$
\|f\|_{L^p(\mathbb{R}^n, \mu)} \lesssim \|f\|_{W_{\beta,p}(\mathbb{R}^n)}, \quad f \in W_{\beta,p}(\mathbb{R}^n) \iff \sup_{O \subseteq \mathbb{R}^n} \frac{(\mu(O))^{p/q}}{\text{cap}_{W_{\beta,p}(\mathbb{R}^n)}(O)} < \infty.
$$

If $1 = p \leq q < \infty$, or $1 < p < \min\{q, n/\beta\}$ then

$$
\|f\|_{L^p(\mathbb{R}^n, \mu)} \lesssim \|f\|_{W_{\beta,p}(\mathbb{R}^n)}, \quad f \in W_{\beta,p}(\mathbb{R}^n) \iff \sup_{x \in \mathbb{R}^n, r > 0} \frac{(\mu(B(x, r)))^{p/q}}{\text{cap}_{W_{\beta,p}(\mathbb{R}^n)}(B(x, r))} < \infty.
$$

Similarly to Theorem 1.9 we obtain the following result which covers Theorem 1.10.

**Theorem 1.10.** Let $\beta \in (0, n)$, $1 < p \leq q < \infty$, $p < n/\beta$ and $\mu$ be a nonnegative Radon measure on $\mathbb{R}^n$. Then the following statements are equivalent:

(a) $$
\|f\|_{L^p(\mathbb{R}^n, \mu)} \lesssim \|f\|_{W_{\beta,p}(\mathbb{R}^n)}, \quad f \in W_{\beta,p}(\mathbb{R}^n),
$$

(b) $$
\|f\|_{L^q(\mathbb{R}^n, \mu)} \lesssim \|f\|_{W_{\beta,p}(\mathbb{R}^n)}, \quad f \in W_{\beta,p}(\mathbb{R}^n),
$$

(c) $$
\|f\|_{L^{q,\infty}(\mathbb{R}^n, \mu)} \lesssim \|f\|_{W_{\beta,p}(\mathbb{R}^n)}, \quad f \in W_{\beta,p}(\mathbb{R}^n),
$$

(d) $$
(\mu(O))^{p/q} \lesssim \text{cap}_{W_{\beta,p}(\mathbb{R}^n)}(O), \quad \text{open } O \subseteq \mathbb{R}^n.
$$

If $1 = p \leq q < \infty$, or $1 < p < \min\{q, n/\beta\}$ then all of them are equivalent to

(e) $$
\sup_{r > 0, x \in \mathbb{R}^n} \frac{(\mu(B(x, r)))^{p/q}}{\text{cap}_{W_{\beta,p}(\mathbb{R}^n)}(B(x, r))} < \infty.
$$

If $0 < q < p = 1$, then (b) $\iff$ (c) $\iff$ (d) $\iff$ (a).

**Remark 1.11.** The equivalences (b) $\iff$ (d) and (b) $\iff$ (e) in Theorem 1.10 can be verified directly from Casacan-Ortega-Verbiksky [9, Theorem 3.1 & 3.2], Maz’ya [17-18] and [22], and Adams-Hedberg [5, Theorem 7.2.2]. Moreover, the case $1 = p < q < \infty$ of Theorem 1.10 was shown by Xiao in [39].

Nishio-Yamada [33] gave a characterization for a nonnegative Radon measure $\mu$ on $\mathbb{R}^{1+n}$ to be a Carleson type measure on $b_0^p(\mathbb{R}^{1+n})$, which is called $\beta$-type Carleson measure and means that $|\nabla (t,x)u(t,x)| \in L^p(\mathbb{R}^{1+n}, \mu)$, that is,

$$
\|\nabla (t,x)u(t,x)\|_{L^p(\mathbb{R}^{1+n}, \mu)} \lesssim \|u(t,x)\|_{L^p(\mathbb{R}^{1+n})}, \quad \forall u \in b_0^p(\mathbb{R}^{1+n}).
$$

We find a sufficient condition for a nonnegative Radon measure $\mu$ on $\mathbb{R}^{1+n}$ to be a Carleson type measure on $b_0^p(\mathbb{R}^{1+n})$.

**Theorem 1.12.** If $\mu$ is a nonnegative Radon measure on $\mathbb{R}^{1+n}$ satisfying the property

$$
\sup_{x \in \mathbb{R}^n, r > 0} \frac{(\mu(T(B(x, r))))^{p/q}}{\text{cap}_{W_{\beta,1/2,p}(\mathbb{R}^{1+n})}(B(x, r))} < \infty
$$

...
for $1 \leq p < 2n$ and $\frac{2pn+2n}{2n-p} \leq q < \infty$, then $\mu$ is a $(0,1)$-type Carleson measure on $b^p_1(R_n^{1+n})$ for $p = \frac{2(2n-p)}{2(p+1)} - 1$.

**Corollary 1.13.** Let $\beta \in (0, n)$, $1 < p < \min\{q, n/\beta\}$ or $1 = p \leq q < \infty$, $\gamma \in (0, 1]$, $\zeta > 0$ and $\zeta + n\gamma > n-p\beta$. If $d\mu_{\gamma, \zeta}(t, x) = t^{\omega-1}|x|^{n(\beta-1)}dt$, then

$$\left(\mu_{\gamma, \zeta}(T(O))\right)^{\frac{n-\gamma}{n-\omega}} \lesssim \text{cap}_{W^{\beta,p}}(O), \quad O \subseteq \mathbb{R}^n.$$  

Equivalently

$$\|v(t^{2\alpha}, x)\|_{L^{\frac{q}{q-n/\beta}}(\mathbb{R}_+^{1+n}, \mu_{\alpha, \zeta})} \lesssim \|v_0\|_{W^{\beta,p}}, \quad \forall v_0 \in \dot{W}^{\beta,p}.$$

**Proof.** This assertion follows from the case $q = p(\zeta + n\gamma)/(n-p\beta)$ and $\mu = \mu_{\gamma, \zeta}$ of Theorem 1.13. \qed

The first inequality of Corollary 1.13 is the iso-capacitary inequality (see Maz'ya [22] for more) and the second is its analytic form attached to the kernel $K^{\omega}_{t^{2\alpha}}(x)$. Both of them were firstly stated by Xiao in [39] for $\alpha = p = 1$.

**Corollary 1.14.** Let $\alpha \in (0, 1], \beta \in (0, n)$, $1 \leq p < n/\beta$ and $\gamma \in (-1, \infty)$. Then the following two conditions hold:

(a) \[ \left(\int_{\mathbb{R}_+^{1+n}} |v(t^{2\alpha}, x)|^\frac{q}{n-\omega-\beta} t^{\omega}dt\right)^\frac{n-\gamma}{n-\omega} \lesssim \|v_0\|_{\dot{W}^{\beta,p}}, \quad \forall v_0 \in \dot{W}^{\beta,p}(\mathbb{R}^n). \]

(b) \[ \sup_{t>0} \left(\int_{\mathbb{R}^n} |v(t^{2\alpha}, x)|^\frac{qm}{n-p\beta} dx\right)^\frac{n-p\beta}{qm} \lesssim \|v_0\|_{\dot{W}^{\beta,p}}, \quad \forall v_0 \in \dot{W}^{\beta,p}(\mathbb{R}^n). \]

**Proof.** In Theorem 1.14 we take $\mu(t, x) = (1+\gamma)^{-1}t^{\omega}dt$, $q = \frac{p(1+n+\gamma)}{n-p\beta}$, $\gamma > -1$, respective

$$d\mu(t, x) = \delta_{t_0}(t) \otimes dx, \quad q = \frac{pm}{n-p\beta}, \quad \gamma \to -1,$$

where $\delta_{t_0}(t)$ is the Dirac measure at $t_0 > 0$, then an application of the capacitary estimate of ball (see Maz'ya [22] p. 356) for $p \in (1, n/\beta)$, Xiao [39] p. 833 for $p = 1$:

$$\text{cap}_{W^{\beta,p}}(B(x, r)) \approx r^{n-p\beta}, \quad x \in \mathbb{R}^n, r > 0,$$

we can finish the proof. \qed

According to Miao-Yuan-Zhang [31], Proposition 2.1, the condition (a) of Corollary 1.14 amounts to that $W^{\beta,p}(\mathbb{R}^n)$ is embedded in the homogeneous Besov or Triebel-Lizorkin space (see Triebel [35] for more details about these spaces)

$$B^{\frac{n}{n-\beta}}_{q, \frac{q}{\beta}}(\mathbb{R}^n) = F^{\frac{n}{n-\beta}}_{q, \frac{q}{\beta}}(\mathbb{R}^n), \quad q = \frac{p(1+n+\gamma)}{n-p\beta}.$$

At the same time, the condition (b) of Corollary 1.14 can be treated as extreme case of the condition (a) in Corollary 1.14.

The rest of this paper is organized as follows. In the next section, we give six preliminary results: Lemma 2.2—a strong-type inequality for the Hardy-littlewood
maximal operator with respect to the variational capacity (whose new generalizations were made by Maz’ya \cite{mazya21} and Costea-Maz’ya \cite{costea-mazya10}). Lemma 2.2—an elementary Riesz integral upper estimate of the kernel $K_{t}^{\alpha}(x)$, Lemma 2.3—a lower estimate for the kernel $K_{t}^{\alpha}(x)$, Lemma 2.4—four standard estimates involving capacity, measure and nontangential maximal functions, Lemma 2.5—an integral representation of fractional order homogeneous Sobolev functions, and Lemma 2.6—a homogeneous version of the extension/restriction theorem. In the third section, we prove our theorems and corollaries: Theorem 1.1 is proved by using Lemma 2.1. Theorem 1.2 is showed from applying Lemmas 2.1, 2.4, 2.6 and the dyadically discrete forms of the left-hand integrals in (a) – (b) of Theorem 1.2. Theorem 1.3 is demonstrated by using Lemmas 2.1 & 2.3. Theorem 1.4 is derived from the equivalence established in Theorem 1.3. Lemmas 2.2, 2.5 and more delicate estimates for measures, functions and integrals under consideration. Theorem 1.7 is verified from Lemmas 2.1 & 2.4. Theorem 1.8 is obtained by applying Theorems 1.3 & 1.7 and estimating the norm of Dirac measure on $\mathbb{R}^{1+n}_{+}$. Theorem 1.2 is established through comparing $1/2-$parabolic rectangles in $\mathbb{R}^{1+n}_{+}$ with the tents of $n-$dimensional balls.

2. Preliminary Lemmas

This section contains six technical results needed for proving the main results of this paper. The first is the capacity strong-type inequalities for $f \in \dot{W}^{1,p}(\mathbb{R}^{n})$ and its Hardy-Littlewood maximal operator

$$\mathcal{M}f(x) = \sup_{r > 0} r^{-n} \int_{B(x,r)} |f(y)|dy, \ x \in \mathbb{R}^{n}.$$  

Lemma 2.1. The following three inequalities hold:

(a) If $\beta \in (0, n)$ and $p \in [1, n/\beta]$, then, $\forall f \in \dot{W}^{1,p}(\mathbb{R}^{n})$,

$$\int_{0}^{\infty} \text{cap}_{\dot{W}^{1,p}(\mathbb{R}^{n})}(\{x \in \mathbb{R}^{n} : |f(x)| \geq \lambda\}) d\lambda^{p} \lesssim \|f\|_{\dot{W}^{1,p}(\mathbb{R}^{n})}^{p}.$$  

If $1 \leq p < \infty$, then, $\forall f \in \dot{W}^{1,p}(\mathbb{R}^{1+n}_{+})$,

$$\int_{0}^{\infty} \text{cap}_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_{+})}(\{x \in \mathbb{R}^{1+n}_{+} : |f(t,x)| \geq \lambda\}) d\lambda^{n} \lesssim \|f\|_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_{+})}^{p}.$$  

(b) If $\beta \in (0, n)$ and $p \in [1, n/\beta]$, then, $\forall f \in \dot{W}^{1,p}(\mathbb{R}^{n})$,

$$\int_{0}^{\infty} \text{cap}_{\dot{W}^{1,p}(\mathbb{R}^{n})}(\{x \in \mathbb{R}^{n} : |\mathcal{M}f(x)| \geq \lambda\}) d\lambda^{p} \lesssim \|f\|_{\dot{W}^{1,p}(\mathbb{R}^{n})}^{p}.$$  

Proof. (a) Part 1, $f \in \dot{W}^{1,p}(\mathbb{R}^{1+n}_{+})$: This assertion is due to Maz’ya \cite{mazya22} Section 2.3.1 or his another work \cite{mazya17}. Part 2, $f \in \dot{W}^{1,p}(\mathbb{R}^{n})$: Case 1, $p \in (1, n/\beta)$: This case is due to Maz’ya \cite{mazya21} Proposition 4.1 or Maz’ya \cite{mazya22} p. 368 Theorem. Case 2, $p = 1$: This case is essentially proved by Wu \cite{wu34} when $\beta \in (0,1)$ and Xiao \cite{xiao39} when $\beta \in (0, n)$. Case 3, $p = n/\beta$: It can be found in Maz’ya \cite{mazya20} or Adams-Xiao \cite{adams-xiao06}.

(b) If $f \in \dot{W}^{1,p}(\mathbb{R}^{n})$: We divide the proof into three cases. Case 1, $p = 1$: It is due to Xiao \cite{xiao39}. Case 2, $p = n/\beta$: This is proved by Adams-Xiao \cite{adams-xiao06}. Case 3, $p \in (1, n/\beta)$: It follows from Maz’ya \cite{mazya22} p. 347, Theorem 2) or his earlier work \cite{mazya20} that for $1 < p < n/\beta$, $f \in \dot{W}^{1,p}(\mathbb{R}^{n})$ if and only if

$$f = (-\Delta)^{-\beta/2} g = I_{\beta} \ast g(x) \quad \text{and} \quad \|f\|_{\dot{W}^{1,p}(\mathbb{R}^{n})} = \|g\|_{L^{p}},$$  

where

$$I_{\beta}(g)(x) = \int_{\mathbb{R}^{n}} g(y) (|x - y|^{\beta}/\beta) dy.$$  

This completes the proof of the lemma.
for some \( g \in L^p(\mathbb{R}^n) \), where
\[
I_\beta * g(x) = \frac{1}{\gamma_\beta} \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-\beta}} \, dy
\]
with \( \gamma_\beta = \pi^{n/2} 2^\beta \Gamma(\beta/2)/\Gamma(n+\beta) \). Then for fixed \( f \in \dot{W}^{\beta,p}(\mathbb{R}^n) \), and \( g \in L^p(\mathbb{R}^n) \) with \( f(x) = I_\beta * g(x) \), according to R. Johnson [15, p. 33, Proof of Theorem 1.9], we have
\[
\mathcal{M}(I_\beta * g) \leq I_\beta * (\mathcal{M}g)
\]
and
\[
M_\lambda(\mathcal{M}f(x)) \leq M_\lambda(I_\beta * (\mathcal{M}g)).
\]
It follows from Maximal Theorem Stein [34, p. 13, Theorem 1] that
\[
\mathcal{M}(g) \in L^p(\mathbb{R}^n) \quad \text{and} \quad \|\mathcal{M}(g)\|_p \lesssim \|g\|_p.
\]
Thus (a) implies (b). \( \square \)

**Lemma 2.2.** If \( \alpha \in (0,1] \), \( \beta \in (0,n) \) and \((t,x) \in \mathbb{R}^{1+n}_+\), then
\[
\int_{\mathbb{R}^n} K_\alpha^n(t) |y-x|^{\beta-n} \, dy \lesssim (t^2 + |x|^2)^{\frac{\beta-n}{2}}.
\]

**Proof.** By Miao-Yuan-Zhang [31], Nishio-Shimomura-Suzuki [32] or Nishio-Yamada [33], we have the following point-wise estimate
\[
|K_\alpha^n(t)| \leq C \frac{t}{(t^{1/2\alpha} + |x|)^{n+2\alpha}}, \quad \forall (t,x) \in \mathbb{R}^{1+n}_+.
\]
So, it suffices to verify
\[
J(t,x) := \int_{\mathbb{R}^n} t^{2\alpha}(t+|y|)^{-n-2\alpha} |y-x|^{\beta-n} \, dy \lesssim (t^2 + |x|^2)^{\frac{\beta-n}{2}}.
\]
Changing variables: \( x \to tx, \ y \to ty \), we see the previous estimate is equivalent to the following one:
\[
J(1,x) \lesssim (1 + |x|^2)^{\frac{\beta-n}{2}}.
\]
Since \( J(1,0) \lesssim 1 \) we may assume that \( |x| > 0 \). Then
\[
J(1,x) \lesssim \left( \int_{B(x,|x|/2)} + \int_{\mathbb{R}^n \setminus B(x,|x|/2)} \right) \frac{1}{(1+|y|)^{n+2\alpha}} \frac{1}{|y-x|^{n-\beta}} \, dy = I_1(x) + I_2(x).
\]
Since \( |x-y| \leq |x|/2 \) implies that \(|y| \approx |x|\), we have
\[
I_1(x) = \int_{B(x,|x|/2)} \frac{1}{(1+|y|)^{n+2\alpha}} \frac{1}{|y-x|^{n-\beta}} \, dy
\]
\[
\lesssim (1+|x|)^{-n-2\alpha} \int_{B(x,|x|/2)} \frac{1}{|y-x|^{n-\beta}} \, dy
\]
\[
\lesssim (1+|x|)^{-n-2\alpha} \int_{|x|/2}^{\infty} s^{\beta-1} \, ds
\]
\[
\lesssim |x|^\beta (1+|x|)^{-n-2\alpha}
\]
\[
\lesssim (1+|x|)^{\beta-n}.
\]
Lemma 2.3. If \( |x - y| > |x|/2 \), then

\[
I_2(x) = \int_{\mathbb{R}^n \setminus B(x, |x|/2)} \frac{1}{(1 + |y|)^{n+2\alpha} |x - y|^{\beta}} dy
\]

\[
\lesssim |x|^{\beta - n} \int_{\mathbb{R}^n \setminus B(x, |x|/2)} \frac{1}{(1 + |y|)^{n+2\alpha} |y|^{\beta}} dy
\]

with the last inequalities using the fact \( \frac{1}{(1 + |y|)^{n+2\alpha} |y|^{\beta}} \in L^1(\mathbb{R}^n) \). Since \( |x - y| > |x|/2 \) implies \( |y| < 3|x - y| \),

\[
I_2(x) \lesssim \int_{\mathbb{R}^n \setminus B(x, |x|/2)} \frac{1}{(1 + |y|)^{n+2\alpha} |y|^{\beta}} dy \lesssim 1.
\]

Thus \( I_2 \lesssim (1 + |x|)^{\beta - n} \) and \( J(1, x) \lesssim (1 + |x|^2)^{\frac{\beta - n}{2}} \). \( \square \)

Lemma 2.3. For \( \alpha \in (0, 1) \), there are positive constants \( \sigma \) and \( C \) such that

\[
\inf \{|K_\alpha^n(x) : |x| \leq \sigma t^{\frac{n}{m}}\} \geq Ct^{-\frac{\alpha}{m}}.
\]

where \( \sigma \) and \( C \) depend only on \( n, \alpha \).

Lemma 2.4. Let \( \alpha \in (0, 1) \) and \( \beta \in (0, n) \). Given \( f \in \tilde{W}^{\beta,p}(\mathbb{R}^n), \lambda > 0 \), and a nonnegative measure \( \mu \) on \( \mathbb{R}^{1+n} \), let

\[
E_\lambda^{\alpha,\beta}(f) = \{ (t, x) \in \mathbb{R}^{1+n} : |S_\alpha(t^{2\alpha})f(x)| > \lambda \}
\]

and

\[
O_\lambda^{\alpha,\beta}(f) = \{ y \in \mathbb{R}^n : \sup_{|y-x|<t} |S_\alpha(t^{2\alpha})f(x)| > \lambda \}.
\]

Then the following four statements are true:

(a) For any natural number \( k \),

\[
\mu \left( E_\lambda^{\alpha,\beta}(f) \cap T(B(0,k)) \right) \leq \mu \left( T(O_\lambda^{\alpha,\beta}(f) \cap B(0,k)) \right).
\]

(b) For any natural number \( k \),

\[
cap_{W^{\beta,p}(\mathbb{R}^n)} \left( O_\lambda^{\alpha,\beta}(f) \cap B(0,k) \right) \geq c_\beta \left( \mu \left( T(O_\lambda^{\alpha,\beta}(f) \cap B(0,k)) \right) \right).
\]

(c) There exists a dimensional constant \( \theta_1 > 0 \) such that

\[
\sup_{|y-x|<t} |S_\alpha(t^{2\alpha})f(y)| \leq \theta_1 Mf(x), x \in \mathbb{R}^n.
\]

(d) There exists a dimensional constant \( \theta_2 > 0 \) such that

\[
(t, x) \in T(O) \implies S_\alpha(t^{2\alpha})f(x) \geq \theta_2,
\]

where \( O \) is a bounded open set contained in \( \text{Int}(\{x \in \mathbb{R}^n : f(x) \geq 1\}) \).

Proof. (a) Since \( \sup_{|y-x|<t} |S_\alpha(t^{2\alpha})f(x)| \) is lower semicontinuous on \( \mathbb{R}^n \), \( O_\lambda^{\alpha,\beta}(f) \)

is an open subset of \( \mathbb{R}^n \). By the definition of \( E_\lambda^{\alpha,\beta}(f) \) and \( O_\lambda^{\alpha,\beta}(f) \), we have

\[
E_\lambda^{\alpha,\beta}(f) \subseteq T(O_\lambda^{\alpha,\beta}(f)) \quad \text{and} \quad \mu(E_\lambda^{\alpha,\beta}(f)) \leq T(\mu(O_\lambda^{\alpha,\beta}(f))).
\]

Then

\[
\mu \left( E_\lambda^{\alpha,\beta}(f) \cap T(B(0,k)) \right) \leq \mu(T(O_\lambda^{\alpha,\beta}(f) \cap T(B(0,k)))) = \mu \left( T(O_\lambda^{\alpha,\beta}(f) \cap B(0,k)) \right).
\]
(b) It follows from the definition of $c^2_p(\mu; t)$.
(c) By (2.1), we have
\[ |S_\alpha(t^{2\alpha}) f(x)| = |K_{t^{2\alpha}}^\alpha(x) * f(x)| \leq \int_{\mathbb{R}^n} \frac{C t^{2\alpha}}{(t + |x - y|)^{n+2\alpha}} |f(y)| dy =: H_t(x) * |f(x)|. \]
Thus
\[ \sup_{|y-x|<t} |S_\alpha(t^{2\alpha}) f(y)| \leq \sup_{|y-x|<t} H_t(y) * |f(y)| \leq \theta_1 M f(x). \]
The last inequality follows from Stein [34, p. 57, Proposition].
(d) For any $(t, x) \in T(\mathcal{O})$, we have
\[ B(x, t) \subset O \subset \text{Int}(|x : f(x) \geq 1|). \]
It follows from Lemma 2.3 that there exist $\sigma$ and $C$ which are only depending on $n$ and $\alpha$ such that
\[ \inf \{K^\alpha_1(x) : |x| \leq \sigma t^{\frac{n}{\alpha}} \} \geq C t^{-\frac{n}{\alpha}}. \]
Then
\[ S_\alpha(t^{2\alpha}) |f|(x) = \int_{\mathbb{R}^n} K_{t^{2\alpha}}^\alpha(x - y) |f(y)| dy \leq C t^{-n} \int_{\{x : f(x) \geq 1\}} |f(y)| dy. \]
If $\sigma > 1$, then
\[ B(x, \sigma t) \cap \text{Int}(|x : f(x) \geq 1|) \supseteq B(x, t) \cap \text{Int}(|x : f(x) \geq 1|) = B(x, t); \]
if $\sigma \leq 1$ then
\[ B(x, \sigma t) \cap \text{Int}(|x : f(x) \geq 1|) = B(x, \sigma t). \]
Thus $S_\alpha(t^{2\alpha}) |f|(x) \geq \theta_2$ for some dimensional constant $\theta_2 > 0$. □

Using $f(x) = (-\Delta)^{-\beta/2}((-\Delta)^{\beta/2} f(x))$ and the definition of Riesz potentials we can easily derive an integral representation of homogeneous Sobolev functions.

**Lemma 2.5.** [3] Let $p \in (1, n/\beta)$, $0 < \beta < n$ and $f \in \dot{W}^{\beta, p}(\mathbb{R}^n)$. Then
\[ f(x) = \frac{1}{\gamma_\beta} \int_{\mathbb{R}^n} \frac{(-\Delta)^{\beta/2} f(y)}{|y - x|^{n-\beta}} dy, \]
where $\gamma_\beta = \pi^{n/2} 2^{\beta} \Gamma(\beta/2) / \Gamma(n-\beta)$.

The following result is a special case of Adamas [2] Theorem 5.2] or Adams-Xiao [6] Theorem A).

**Lemma 2.6.** Let $\beta \in (0, n)$. Then there are a linear extension operator
\[ \mathcal{E} : \dot{W}^{\beta, n/\beta}(\mathbb{R}^n) \rightarrow \dot{L}^{n/\beta}_{2\beta}(\mathbb{R}^{2n}) \]
and a linear restriction operator
\[ \mathcal{R} : \dot{L}^{n/\beta}_{2\beta}(\mathbb{R}^{2n}) \rightarrow \dot{W}^{\beta, n/\beta}(\mathbb{R}^n) \]
such that $\mathcal{R} \mathcal{E}$ is the identity, and
(a) $\|\mathcal{E} f\|_{\dot{L}^{n/\beta}_{2\beta}(\mathbb{R}^{2n})} \lesssim \|f\|_{\dot{W}^{\beta, n/\beta}(\mathbb{R}^n)}$, $\forall f \in \dot{W}^{\beta, n/\beta}(\mathbb{R}^n)$;
(b) $\|\mathcal{R} g\|_{\dot{W}^{\beta, n/\beta}(\mathbb{R}^n)} \lesssim \|g\|_{\dot{L}^{n/\beta}_{2\beta}(\mathbb{R}^{2n})}$, $\forall g \in \dot{L}^{n/\beta}_{2\beta}(\mathbb{R}^{2n})$. 
3. Proofs of Main Results

3.1. Proof of Theorem 1.1 Assume that $1 \leq p \leq q < \infty$. In what follows, for $\lambda > 0$ and $u \in W^{1,p}(\mathbb{R}_+^{1+n}) \cap b_0^p(\mathbb{R}_+^{1+n})$, let

$$M_\lambda(u) = \{(t,x) \in \mathbb{R}_+^{1+n} : |u(t,x)| \geq \lambda\}.$$ 

(a) $\implies$ (b) $\implies$ (c). Since $0 < \lambda_1 < \lambda_2$ implies $\mu(M_{\lambda_2}(u)) \leq \mu(M_{\lambda_1}(u))$, we can conclude

$$q\mu(M_\lambda(u))\lambda^{q-1} \leq \frac{d}{d\lambda} \left( \int_0^\lambda (\mu(M_s(u)))^{p/q} ds \right)^{q/p}.$$ 

This implies

$$(s^{p/q}\mu(M_s(u)))^{p/q} \leq \left( q \int_0^\infty \mu(M_\lambda(u))\lambda^{q-1} d\lambda \right)^{p/q} \leq \int_0^\infty (\mu(M_\lambda(u)))^{p/q} d\lambda,$$

and obtains the desired implications.

(c) $\implies$ (d). Let (c) be true. For an given open set $O \subseteq \mathbb{R}_+^{1+n}$, and any function $u \in \dot{W}^{1,p}(\mathbb{R}_+^{1+n}) \cap b_0^p(\mathbb{R}_+^{1+n})$ with

$$O \subseteq \text{Int} \{(t,x) \in \mathbb{R}_+^{1+n} : u(t,x) \geq 1\},$$

we have $\mu(O) \leq \mu(M_1(u)) \lesssim \|u\|_{\dot{W}^{1,p}}^p$. This derives (d).

(d) $\implies$ (a). If (d) is true, then for $u \in \dot{W}^{1,p}(\mathbb{R}_+^{1+n})$, $k \in \mathbb{N}$ and $B(0,k) \subseteq \mathbb{R}_n$, Lemma 2.1 (a) implies

$$\int_0^\infty (\mu(M_\lambda(u) \cap ((0,k) \times B(0,k))))^{p/q} d\lambda \leq \int_0^\infty (\mu(M_\lambda(u)))^{p/q} d\lambda \leq \int_0^\infty (\mu(M_\lambda(u)))^{p/q} d\lambda \\ \lesssim \int_0^\infty \text{cap}_{\dot{W}^{1,p}(\mathbb{R}_+^{1+n})}(M_\lambda(u)) d\lambda \lesssim \|u\|_{\dot{W}^{1,p}(\mathbb{R}_+^{1+n})}.$$ 

Letting $k \rightarrow \infty$ we see (a) hold. When $0 < q < p = 1$, the implications are obviously.

\[\] 

3.2. Proof of Theorem 1.2 Let $0 < q < p$. Then we finish the proof in two steps.

Part 3.2.1: (b) $\implies$ (a). If

$$I_{p,q}(\mu) = \int_0^\infty \left( \frac{t^{p/q}}{c_p^q(\mu;t)} \right)^{\frac{q}{p-q}} dt < \infty,$$

then for each $v_0 \in \dot{W}^{\beta,p}(\mathbb{R}_n)$, each $j = 0, \pm 1, \pm 2, \cdots$ and each natural number $k$, Lemma 2.4 (c) implies

$$\text{cap}_{\dot{W}^{\beta,p}(\mathbb{R}_n)}(v_0 \cap B(0,k)) \leq \text{cap}_{\dot{W}^{\beta,p}(\mathbb{R}_n)}(\{x \in \mathbb{R}_n : \mathcal{M}v_0(x) > 2^j \}) \cap B(0,k).$$

\[\]
Let $\mu_{j,k}(v_0) = \mu(T(O_2(v_0) \cap B(0, k)))$, and

$$S_{p,q,k}(\mu; v_0) = \sum_{j=-\infty}^{\infty} \left( \frac{(\mu_{j,k}(v_0) - \mu_{j+1,k}(v_0))^{p-q}}{\text{cap}_{W^{p,q}(\mathbb{R}^n)}(O_{2^j}(v_0) \cap B(0, k))} \right)^{\frac{q}{p}}.$$

Lemma 2.4 (b) implies that

$$\left( S_{p,q,k}(\mu; v_0) \right)^{\frac{p}{p-q}} = \left( \sum_{j=-\infty}^{\infty} \left( \frac{(\mu_{j,k}(v_0) - \mu_{j+1,k}(v_0))^{p-q}}{\text{cap}_{W^{p,q}(\mathbb{R}^n)}(O_{2^j}(v_0) \cap B(0, k))} \right)^{\frac{q}{p}} \right)^{\frac{p}{p-q}}.$$

On the other hand, using Hölder’s inequality and Lemmas 2.4 (b) and 2.4 (b)–(c), we have

$$\int_{T(B(0,k))} |v(2^n, x)|^q d\mu(t, x)$$

$$= \int_{0}^{\infty} \mu \left( E^{\alpha,\beta}_x(v_0) \cap T(B(0, k)) \right) d\lambda^q$$

$$\lesssim \sum_{j=-\infty}^{\infty} (\mu_{j,k}(v_0) - \mu_{j+1,k}(v_0)) 2^{jq}$$

$$\lesssim \left( S_{p,q,k}(\mu; v_0) \right)^{\frac{p}{p-q}} \left( \sum_{j=-\infty}^{\infty} 2^{jq} \text{cap}_{W^{p,q}(\mathbb{R}^n)}(O_{2^j}(v_0) \cap B(0, k)) \right)^{\frac{q}{p}}.$$

$$\lesssim \left( S_{p,q,k}(\mu; v_0) \right)^{\frac{p}{p-q}} \left( \sum_{j=-\infty}^{\infty} 2^{jq} \text{cap}_{W^{p,q}(\mathbb{R}^n)}(\{x \in \mathbb{R}^n : \theta_1 M v_0(x) > 2^j \} \cap B(0, k)) \right)^{\frac{q}{p}}.$$

$$\lesssim \left( S_{p,q,k}(\mu; v_0) \right)^{\frac{p}{p-q}} \left( \int_{O_{2^j}(v_0)} \text{cap}_{W^{p,q}(\mathbb{R}^n)}(\{x \in \mathbb{R}^n : \theta_1 M v_0(x) > \lambda \}) d\lambda^p \right)^{\frac{q}{p}}.$$

$$\lesssim \left( S_{p,q,k}(\mu; v_0) \right)^{\frac{p}{p-q}} \|v_0\|_{W^{p,q}(\mathbb{R}^n)}^q.$$
Given integers $O$ open set $c$. The definition of $W$ we divide the following proof into two cases.

By this equivalent definition we can find thus for each $\mu$ (3.2) this along with the definition of $c$ for fixed positive $0$ $(g)$.

If (a) is true, then $\{\frac{\mu}{\langle t^2, x \rangle} \}^q d\mu(t, x)$.

Thus for each $v_0 \in \tilde{W}^{0, p} (\mathbb{R}^n)$, with $\|v_0\|_{\tilde{W}^{0, p} (\mathbb{R}^n)} > 0$, we have

$$\left( \int_{\mathbb{R}^{1+n}_+} |v(t^2, x)|^q d\mu(t, x) \right)^{1/q} \leq J_{p, q}(\mu) \|v_0\|_{\tilde{W}^{0, p} (\mathbb{R}^n)}.$$ 

Since $\mu(E_v^{\alpha, \beta}(v_0))$ is nonincreasing in $\lambda$, we have

$$(3.1) \sup_{\lambda > 0} \lambda \left( \mu(E_v^{\alpha, \beta}(v_0)) \right)^{1/q} \lesssim J_{p, q}(\mu) \|v_0\|_{\tilde{W}^{0, p} (\mathbb{R}^n)}.$$ 

This along with the definition of $cap_{\tilde{W}^{0, p} (\mathbb{R}^n)}(\cdot)$ give

$$(3.2) \mu(T(O)) \lesssim (J_{p, q}(\mu))^q \left( cap_{\tilde{W}^{0, p} (\mathbb{R}^n)}(O) \right)^{q/p}.$$ 

It follows from (3.2) and the definition of $c_p^{\beta}(\mu; t)$ that for $0 < t < \infty$, $c_p^{\beta}(\mu; t) > 0$. The definition of $c_p^\alpha(\mu; t)$ implies that for every integer $j$ there exists a bounded open set $O_j \subseteq \mathbb{R}^n$ such that

$\cap_{\tilde{W}^{0, p} (\mathbb{R}^n)}(O_j) \leq 2 c_p^{\beta}(\mu; 2^j)$ and $\mu(T(O_j)) > 2^j$.

We divide the following proof into two cases.

Case 1, $p \in (1, n/\beta)$:

It follows from Maz’ya [22] that $\cap_{\tilde{W}^{0, p} (\mathbb{R}^n)}(S) \lesssim \inf \{ \|g\|^p : g \in L^p(\mathbb{R}^n), g \geq 0, S \subseteq \text{Int}(\{ x \in \mathbb{R}^n : I_\beta \ast g(x) \geq 1 \}) \}$.

By this equivalent definition we can find $g_j(x) \in L^p(\mathbb{R}^n)$ such that

$g_j \geq 0, I_\beta \ast g_j(x) \geq 1, \forall x \in O_j$ and $\|g_j\|^p_{L^p} \leq 2 \cap_{\tilde{W}^{0, p} (\mathbb{R}^n)}(O_j) \leq 4 c_p^{\beta}(\mu; 2^j)$.

Given integers $i, k$ with $i < k$ define

$$g_{i,k} = \sup_{i \leq j \leq k} \left( \frac{2^j}{c_p^{\beta}(\mu; 2^j)} \right)^{\frac{1}{p-q}} g_j.$$
Since $L^p(\mathbb{R}^n)$ is a lattice, we can conclude that $g_{i,k} \in L^p(\mathbb{R}^n)$ and

$$
\|g_{i,k}\|_{L^p}^p \lesssim \sum_{j=i}^k \left( \frac{2^j}{c_p^{\beta}(\mu; 2^j)} \right)^{\frac{p}{p-q}} \|g_j\|_{L^p}^p \lesssim \sum_{j=i}^k \left( \frac{2^j}{c_p^{\beta}(\mu; 2^j)} \right)^{\frac{p}{p-q}} c_p^{\beta}(\mu; 2^j).
$$

Note that for $i \leq j \leq k$,

$$
x \in O_j \implies I_\beta \ast g_{i,k}(x) \geq \left( \frac{2^j}{c_p^{\beta}(\mu; 2^j)} \right)^{\frac{1}{p-q}}.
$$

It follows from Lemma 2.4(d) that there exists a dimensional constant $\theta_2$ such that

$$(t, x) \in T(O_j) \implies S_\alpha(t^{2\alpha})|I_\beta \ast g_{i,k}(x)|(x) \geq \left( \frac{2^j}{c_p^{\beta}(\mu; 2^j)} \right)^{\frac{1}{p-q}} \theta_2.
$$

This gives

$$2^j < \mu(T(O_j)) \leq \mu \left( E^{\alpha,\beta} \left( \frac{2^j}{c_p^{\beta}(\mu; 2^j)} \right) \right) \left( I_\beta \ast g_{i,k}(x) \right).
$$

Thus

$$
(J_{p,q}(\mu)\|g_{i,k}\|_{L^p})^q \gtrsim \int_{\mathbb{R}^n} |S_\alpha(t^{2\alpha})(I_\beta \ast g_{i,k}(x))|^q d\mu(t, x)
\approx \int_0^\infty \left( \inf \{ \lambda : \mu \left( E^{\alpha,\beta}_{\lambda}(I_\beta \ast g_{i,k}(x)) \right) \leq s \} \right)^q ds
\gtrsim \sum_{j=i}^k \left( \inf \{ \lambda : \mu \left( E^{\alpha,\beta}_{\lambda}(I_\beta \ast g_{i,k}(x)) \right) \leq 2^{j} \right)^q 2^{j}
\gtrsim \sum_{j=i}^k \left( \frac{2^j}{c_p^{\beta}(\mu; 2^j)} \right)^{\frac{q}{p-q}} 2^j
\gtrsim \left( \sum_{j=i}^k \left( \frac{2^j}{c_p^{\beta}(\mu; 2^j)} \right)^{\frac{q}{p-q}} 2^j \right)^{\frac{p}{p-q}} \|g_{i,k}\|_{L^p}^q
\simeq \left( \sum_{j=i}^k \left( c_p^{\beta}(\mu; 2^j) \right)^{\frac{p}{p-q}} \right)^{\frac{p}{p-q}} \|g_{i,k}\|_{L^p}^q.
$$

This tells us

$$
\sum_{j=i}^k \left( \frac{2^j}{c_p^{\beta}(\mu; 2^j)} \right)^{\frac{p}{p-q}} \lesssim (J_{p,q}(\mu))^{\frac{p}{p-q}}.
$$

Case 2, $p = \frac{n}{\beta}$: By the definition of $\text{cap}_{\dot{W}^{\beta,p}(\mathbb{R}^n)}(O_j)$, there is $f_j \in \dot{W}^{\beta,p}(\mathbb{R}^n)$ such that

$$f_j \geq 0, f_j(x) \geq 1, \forall x \in O_j \text{ and } \|f_j\|_{\dot{W}^{\beta,p}(\mathbb{R}^n)}^p \leq 2 \text{ cap}_{\dot{W}^{\beta,p}(\mathbb{R}^n)}(O_j) \leq 4 c_p^{\beta}(\mu; 2^j).
$$
Lemma 2.6 implies that for each $j$ there is $g_j(\cdot, \cdot) \in L^p(\mathbb{R}^n)$ such that

$$f_j(x) = I_{2^j}^{(2n)} * g_j(x, 0) = \mathcal{R} f_j(x)$$

and

$$\|I_{2^j}^{(2n)} * g_j\|_{L^p_{\mathcal{E}}(\mathbb{R}^n)} = \|\mathcal{E} f_j\|_{L^p_{\mathcal{E}}(\mathbb{R}^n)} \leq \|f_j\|_{W^p, p(\mathbb{R}^n)}.$$ 

(3.3)

Given integers $i, k$ with $i < k$, define

$$g_{i,k} = \sup_{i \leq j \leq k} \left( \frac{2^j}{c_p^\beta(\mu; 2^j)} \right)^{\frac{1}{p-\theta}} g_j.$$

Since $L^p(\mathbb{R}^n)$ is a lattice, we can conclude that $g_{i,k} \in L^p(\mathbb{R}^n)$ and $I_{2^j}^{(2n)} * g_{i,k} \in L^p_{\mathcal{E}}(\mathbb{R}^n)$. Then (3.3) and Lemma 2.6 imply that

$$\|\mathcal{R}(I_{2^j}^{(2n)} * g_{i,k})\|_{L^p_{\mathcal{E}}(\mathbb{R}^n)} \leq \sum_{j=1}^{k} \left( \frac{2^j}{c_p^\beta(\mu; 2^j)} \right)^{\frac{1}{p-\theta}} \|f_j\|_{W^p, p(\mathbb{R}^n)}^p.$$

Note that for $i \leq j \leq k$,

$$x \in O_j \implies \mathcal{R}(I_{2^j}^{(2n)} * g_{i,k})(x) \geq \left( \frac{2^j}{c_p^\beta(\mu; 2^j)} \right)^{\frac{1}{p-\theta}}.$$

Then Lemma 2.4 (d) implies that

$$(t, x) \in T(O_j) \implies S_\alpha(t^{2\alpha}) I_{2^j}^{(2n)} * g_{i,k} \|_{\mathcal{E}}(x) \geq \left( \frac{2^j}{c_p^\beta(\mu; 2^j)} \right)^{\frac{1}{p-\theta}} \theta_2.$$

This gives

$$2^j < \mu(T(O_j)) \leq \mu \left( \frac{E^{\alpha, \beta}}{\left( \frac{2^j}{c_p^\beta(\mu; 2^j)} \right)^{\frac{1}{p-\theta}}} \right).$$
3.3. Proof of Theorem 1.3. Let $p < q$. The proof consists two parts.

Part 3.3.1: We prove $(a) \implies (b) \implies (c) \implies (e) \implies (a)$.

$(a) \implies (b) \implies (c)$. Since $\mu(E_\lambda(v_0))$ is nonincreasing in $\lambda$,

$$q\mu(E_\lambda^{\alpha,\beta}(v_0))\lambda^{p-1} \leq \frac{d}{d\lambda} \left( \int_0^\lambda (\mu(E_\nu^{\alpha,\beta}(v_0)))^{p/q} d\nu\right)^{q/p}.$$ 

This gives, for $s > 0$

$$(s^q \mu(E_\lambda^{\alpha,\beta}(v_0)))^{\frac{p}{q}} \leq q \int_0^\infty \mu(E_\nu^{\alpha,\beta}(v_0))\lambda^{p-1} d\lambda \leq \int_0^\infty \left( \mu(E_\nu^{\alpha,\beta}(v_0)) \right)^{p/q} d\nu \leq \int_0^\infty \left( \mu(E_\nu^{\alpha,\beta}(v_0)) \right)^{p/q} d\lambda.$$
and establishes the desired implications.

If (c) is true, then

\[ K_{p,q}(\mu) = \sup_{v_0 \in \hat{W}^{\beta,p}(\mathbb{R}^n), \|v_0\|_{\hat{W}^{\beta,p}} > 0} \sup_{\lambda > 0} \lambda \left( \mu \left( \{ (t, x) \in \mathbb{R}_{+}^{1+n} : |v(t^{2\alpha}, x)| > \lambda \} \right) \right)^{\frac{1}{p}} \]

For a given \( v_0 \in \hat{W}^{\beta,p}(\mathbb{R}^n) \) and a bounded set \( O \subseteq \text{Int} \left( \{ x \in \mathbb{R}^n : v_0(x) \geq 1 \} \right) \), then Lemma 2.4 (d) implies

\[ (\mu(T(O)))^{\frac{1}{q}} \lesssim K_{p,q}(\mu) \| v_0 \|_{\hat{W}^{\beta,p}} \]

and hence (e) follows from the definition of \( \text{cap}_{W^{\beta,p}}(O) \). To prove (e) \( \Rightarrow \) (a), we assume (e). Then

\[ Q_{p,q}(\mu) = \sup \left\{ \frac{(\mu(T(O)))^{p/q}}{\text{cap}_{W^{\beta,p}}(O)} : \text{bounded open } O \subseteq \mathbb{R}^n \right\} < \infty. \]

If \( v_0 \in \hat{W}^{\beta,p}(\mathbb{R}^n) \) and \( k = 1, 2, 3, \ldots \), then Lemmas 2.4 (a)-(c) and 2.1 (b) imply

\[
\int_0^\infty \left( \mu \left( E^{\alpha,\beta}_0(v_0) \cap T(B(0,k)) \right) \right)^{p/q} \, d\lambda^p \leq \int_0^\infty \left( \mu \left( T(O^{\alpha,\beta}_0(v_0) \cap B(0,k)) \right) \right)^{p/q} \, d\lambda^p \leq \int_0^\infty (\mu_k(T(\{ x \in \mathbb{R}^n : \theta_1 M v_0(x) > \lambda \} \cap B(0,k))))^{p/q} \, d\lambda^p \leq \int_0^\infty (\mu(T(\{ x \in \mathbb{R}^n : \theta_1 M v_0(x) > \lambda \} \cap B(0,k))))^{p/q} \, d\lambda^p \leq Q_{p,q}(\mu) \int_0^\infty \text{cap}_{W^{\beta,p}}(\{ x \in \mathbb{R}^n : \theta_1 M v_0(x) > \lambda \} \cap B(0,k)) \, d\lambda^p \leq Q_{p,q}(\mu) \int_0^\infty \text{cap}_{W^{\beta,p}}(\{ x \in \mathbb{R}^n : \theta_1 M v_0(x) > \lambda \}) \, d\lambda^p \leq Q_{p,q}(\mu) \| v_0 \|^p_{\text{cap}_{W^{\beta,p}}}. \]

Letting \( k \to \infty \) in the above inequality we have

\[ \int_0^\infty (\mu \left( E^{\alpha,\beta}_0(v_0) \right) \right)^{p/q} \, d\lambda^p \leq Q_{p,q}(\mu) \| v_0 \|^p_{W^{\beta,p}}. \]

This derives (a).

Part 3.3.2. We verify (c) \( \Rightarrow \) (d) \( \Rightarrow \) (a).

If (c) holds, then for any bounded open set \( O \subseteq \text{Int} \left( \{ x \in \mathbb{R}^n : v_0(x) \geq 1 \} \right) \), we have

\[ (\mu(T(O)))^{1/q} \lesssim K_{p,q}(\mu) \| v_0 \|_{W^{\beta,p}}. \]

Note that

\[ t^{p/q} \lesssim (K_{p,q}(\mu))^p \text{cap}_{W^{\beta,p}}(O) \text{ wthenever } 0 < t < \mu(T(O)). \]

Hence

\[ t^{p/q} \lesssim (K_{p,q}(\mu))^p c^\beta_p(\mu,t). \]

Therefore (d) holds.
If (d) holds, then Lemmas 2.3 (b)-(c) and 2.1 (b) imply that for each $k = 1, 2, 3, \ldots$,
\[
\int_0^\infty \left( \mu \left( E_{\lambda}^{\alpha, \beta} (v_0) \cap T(B(0, k)) \right) \right)^{p/q} \, d\lambda^p 
\lesssim \int_0^\infty \left( \mu \left( E_{\lambda}^{\alpha, \beta} (v_0) \cap T(B(0, k)) \right) \right)^{p/q} \, d\lambda^p 
\lesssim \left( \sup_{t>0} \frac{t^{p/q}}{c_p^2 (\mu; t)} \right) \int_0^\infty \cap \lambda^p \left( \left\{ x \in \mathbb{R}^n : \theta_1 \cdot \mu (v_0) (x) > \lambda \right\} \cap B(0, k) \right) \, d\lambda^p 
\lesssim \left( \sup_{t>0} \frac{t^{p/q}}{c_p^2 (\mu; t)} \right) \left\| v_0 \right\|^p_{W^{\alpha, \beta}, p}.
\]

Letting $k \to \infty$ in the previous inequality we have
\[
\int_0^\infty \left( \mu \left( E_{\lambda}^{\alpha, \beta} (v_0) \right) \right)^{p/q} \, d\lambda^p \lesssim \left( \sup_{t>0} \frac{t^{p/q}}{c_p^2 (\mu; t)} \right) \left\| v_0 \right\|^p_{W^{\alpha, \beta}, p}.
\]

This implies that (a) holds. \(\square\)

3.4. Proof of Theorem 1.4

Part 3.4.1: We prove (a) \iff (b).

It follows from Theorem 1.3 that it is enough to prove that (b) implies (c) or (e) in Theorem 1.3. We consider the following three cases.

Case 1, $1 = p < q < \infty$: If (b) holds, then $\| u \|_{1,q} < \infty$. Suppose that $O \subseteq \mathbb{R}^n$ is a bounded open set and is covered by a sequence of dyadic cubes $\{I_j\}$ in $\mathbb{R}^n$ with $\sum_j |I_j|^n < \infty$. According to Dafni-Xiao [8, Lemma 4.1] there exists another sequence of dyadic cubes $\{J_j\}$ in $\mathbb{R}^n$ such that
\[
\text{Int}(I_j) \cap \text{Int}(I_k) = \emptyset \text{ for } j \neq k, \quad \bigcup_j I_j = \bigcup_k I_k,
\]
\[
\sum_j |J_j|^\frac{n-\beta}{n} \leq \sum_k |I_k|^\frac{n-\beta}{n}, \quad T(O) \subseteq \bigcup_j T(\text{Int}(\sqrt{n}J_j)).
\]
Then
\[
\mu(T(O)) \lesssim \| u \|_{1,q} \sum_j |J_j|^n \frac{\| u \|_q}{|J_j|^n} \lesssim \| u \|_{1,q} \left( \sum_j |J_j|^\frac{\| u \|_q}{|J_j|^n} \right)^q
\]
\[
\lesssim \| u \|_{1,q} \left( \sum_j |J_j|^\frac{\| u \|_q}{|J_j|^n} \right)^q.
\]

By Xiao [39] (see also Adams [3] or [4]) we have $\cap_1^\beta (\cdot) \approx H_{\infty}^{\alpha-\beta} (\cdot)$, where the $H_{\infty}^{\alpha-\beta} (\cdot)$ is the $d-$dimensional Hausdorff capacity. Thus, these along with the definition of $H_{\infty}^{\alpha-\beta} (O)$ imply
\[
\mu(T(O)) \lesssim \| u \|_{1,q} (\cap_1^{W^{\alpha, \beta}, 1} (O))^q;
\]
that is, the inequality (e) in Theorem 1.3 holds.
Case 2: 1 < p < min\{q, n/\beta\} : Let \( v_0 \in \tilde{W}^{\beta,p}(\mathbb{R}^n) \) and \( \mu_\lambda \) be the restriction of \( \mu \) to \( E_\lambda^{\alpha,\beta}(\tilde{W}^{\beta,p}(\mathbb{R}^n)) \). If (b) holds, then

\[
\|\mu\|_{p,q} := \sup_{x \in \mathbb{R}^n, r > 0} \frac{\mu(B(x, r))}{r^{\frac{n-\beta}{p}}} < \infty.
\]

It follows from Lemma 2.5 that

\[
|f(x)| \lesssim \int_{\mathbb{R}^n} \frac{(\triangle)^{\beta/2}f(y)}{|y-x|^{n-\beta}} \, dy, \quad f \in \tilde{W}^{\beta,p}(\mathbb{R}^n), x \in \mathbb{R}^n.
\]

This inequality along with Lemma 2.2 and Fubini’s theorem tell us

\[
\lambda \mu \left( E_\lambda^{\alpha,\beta}(v_0) \right) \lesssim \int_{E_\lambda^{\alpha,\beta}(v_0)} |S_\alpha(t^{2\alpha})(v_0(x))| \, d\mu(t, x)
\]

\[
\lesssim \int_{E_\lambda^{\alpha,\beta}(v_0)} \left( \int_{\mathbb{R}^n} K_{2\alpha}^\lambda(y)(v_0(x-y)) \, dy \right) d\mu(t, x)
\]

\[
\lesssim \int_{\mathbb{R}^1} \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \frac{K_{2\alpha}^\lambda(y)}{|z-(x-y)|^{n-\beta}} \, dy \right) \right) \left( (-\triangle)^{\beta/2}v_0(z) \right) \, dz \, d\mu(t, x)
\]

\[
\lesssim \int_{\mathbb{R}^1} \left( \int_{\mathbb{R}^n} \left( t^2 + |z-x|^2 \right)^{\frac{\beta-n}{2}} \right) \left( (-\triangle)^{\beta/2}v_0(z) \right) \, dz \, d\mu(t, x)
\]

\[
\lesssim \int_{\mathbb{R}^n} (-\triangle)^{\beta/2}v_0(z) \left( \int_{\mathbb{R}^1} \mu_\lambda(T(B(z, r))) r^{\alpha-n-1} \, dr \right) \, dz
\]

\[
\lesssim (I_1(s) + I_2(s)),
\]

where

\[
I_1(s) = \int_0^s \left( \int_{\mathbb{R}^n} (-\triangle)^{\beta/2}v_0(z) \mu_\lambda(T(B(z, r))) \, dz \right) r^{\alpha-n-1} \, dr
\]

and

\[
I_2(s) = \int_s^\infty \left( \int_{\mathbb{R}^n} (-\triangle)^{\beta/2}v_0(z) \mu_\lambda(T(B(z, r))) \, dz \right) r^{\alpha-n-1} \, dr.
\]

By the definition of \( \|\mu\|_{p,q} \), we have

\[
\mu_\lambda(T(B(z, r))) \leq (\mu_\lambda(T(B(z, r))))^{1/p} \|\mu\|_{p,q}^{1/p} r^{\frac{\alpha-n-\beta}{p'}}
\]

for \( \frac{1}{p} + \frac{1}{p'} = 1 \). So, using Hölder’s inequality and the estimate

\[
\int_{\mathbb{R}^n} \mu_\lambda(T(B(x, r))) \, dx \lesssim r^n \mu_\lambda \left( E_\lambda^{\alpha,\beta}(v_0) \right),
\]
we obtain
\[
I_1(s) \lesssim \int_0^s \left( \int_{\mathbb{R}^n} |(-\Delta)^{\beta/2} v_0(z)| \mu_\lambda(T(B(z,r))) \mu_\mu^{-\frac{1}{p'}} \frac{\lambda^{(n-p)\beta}}{r^{n-p\beta}} \, dz \right) r^{\beta-n-1} \, dr
\]
\[
\lesssim \|v_0\|_{\dot{W}^{\beta,p}} \|\mu\|_{p,q}^{1/p} \int_0^s \left( \int_{\mathbb{R}^n} \mu_\lambda(T(B(z,r))) \mu_\mu^{-\frac{1}{p'}} \frac{\lambda^{(n-p)\beta}}{r^{n-p\beta}} \, dz \right) r^{\beta-n-1} \, dr
\]
\[
\lesssim \|v_0\|_{\dot{W}^{\beta,p}} \|\mu\|_{p,q}^{1/p} \int_0^s \left( \int_{\mathbb{R}^n} \mu \left( E^{\alpha,\beta}_\lambda(v_0) \right) \frac{\lambda^{(n-p)\beta}}{r^{n-p\beta}} \, dz \right) r^{\beta-n-1} \, dr
\]
\[
\lesssim \|v_0\|_{\dot{W}^{\beta,p}} \|\mu\|_{p,q}^{1/p} \left( \mu \left( E^{\alpha,\beta}_\lambda(v_0) \right) \right)^{1/p'} \frac{\lambda^{(n-p)\beta}}{s^{\frac{(n-p)\beta}{p^2}}}
\]
Similarly, we have
\[
I_2(s) \lesssim \int_s^\infty \left( \int_{\mathbb{R}^n} |(-\Delta)^{\beta/2} v_0(z)| \mu_\lambda(T(B(z,r))) \mu_\mu^{-\frac{1}{p'}} \frac{\lambda^{(n-p)\beta}}{r^{n-p\beta}} \, dz \right) r^{\beta-n-1} \, dr
\]
\[
\lesssim \int_s^\infty \|v_0\|_{\dot{W}^{\beta,p}} \left( \mu_\lambda(T(B(z,r))) \right)^{1/p} \left( \int_{\mathbb{R}^n} \mu_\lambda(T(B(z,r))) \mu_\mu^{-\frac{1}{p'}} \frac{\lambda^{(n-p)\beta}}{r^{n-p\beta}} \, dz \right) r^{\beta-n-1} \, dr
\]
\[
\lesssim \|v_0\|_{\dot{W}^{\beta,p}} \left( \mu \left( E^{\alpha,\beta}_\lambda(v_0) \right) \right)^{1/p} \int_s^\infty \frac{\lambda^{(n-p)\beta}}{r^{n-p\beta}} \mu(\mu_\lambda(T(B(z,r)))) \, dr
\]
\[
\lesssim \|v_0\|_{\dot{W}^{\beta,p}} \left( \mu \left( E^{\alpha,\beta}_\lambda(v_0) \right) \right)^{1/p} \frac{\lambda^{(n-p)\beta}}{s^{\frac{(n-p)\beta}{p^2}}}
\]
Combining the above estimates on $I_1(s)$ and $I_2(s)$ together, we have
\[
\lambda \mu(E^{\alpha,\beta}_\lambda(v_0)) \lesssim \|v_0\|_{\dot{W}^{\beta,p}} \mu_\lambda \left( E^{\alpha,\beta}_\lambda(v_0) \right)
\]
\[
\times \left( \frac{\lambda^{(n-p)\beta}}{s^{\frac{(n-p)\beta}{p^2}}} \right)^{1/p} \frac{\lambda^{(n-p)\beta}}{s^{\frac{(n-p)\beta}{p^2}}}
\]
Taking
\[
s = \left( \|\mu\|_{p,q}^{-1} \mu \left( E^{\alpha,\beta}_\lambda(v_0) \right) \right)^{\frac{n-p}{(n-p)\beta}}
\]
in the above inequality, we have
\[
\lambda \left( \mu(E^{\alpha,\beta}_\lambda(v_0)) \right)^{1/q} \lesssim \|\mu\|_{p,q}^{1/q} \|v_0\|_{\dot{W}^{\beta,p}}
\]
This implies the condition (c) of Theorem 1.3

**Part 3.4.2** We find a nonnegative Radon measure to show that if $1 < p = q < n/\beta$ then (b) does not imply (a) in general.

In fact, suppose $K \subset \mathbb{R}^n$ is a compact set with the $(n-p)$-dimensional Hausdorff measure $H^{(n-p)\beta}(K) > 0$, then by Maz'ya [22] p. 358, Proposition 3] we have $\text{cap}_{\dot{W}^{\beta,p}}(K) = 0$, on the other hand by Adams-Hedberg [5] p. 132, Proposition 5.1.5 & p. 136, Theorem 5.1.12 we can find a nonnegative Radon measure $\nu$ on $\mathbb{R}^n$ such that
\[
\sup_{x \in \mathbb{R}^n, r > 0} \frac{\nu(B(x,r))}{r^{n-p\beta}} < \infty \quad \text{and} \quad 0 < H^{(n-p)\beta}_\infty(K) \lesssim \nu(K).
\]
Define $\mu(t, x) = \delta_1(t) \otimes \nu(x)$. Then (b) hold for this nonnegative Radon measure on $\mathbb{R}^{1+n}_+$. However, (a) is not true, otherwise, we would have $0 < \nu(K) \lesssim \text{cap}_{\mathcal{W}, \beta, r}(K) = 0$. Contradiction. □

3.5. **Proof of Theorem 1.4** Suppose $0 < q < 1$. Since the proof of $(a) \implies (b) \implies (c)$ is similar to that of $(b) \implies (c) \implies (e)$ of Theorem 1.3 we only need to verify $(c) \implies (d)$. Let $(c)$ be true. Then Lemma 2.1 (a)-(c) imply

$$
\mu \left( E^{\alpha, \beta}_\lambda(v_0) \right) \leq \left( \mu \left( T(O^{\alpha, \beta}_\lambda(v_0)) \right) \right) ^{\frac{p}{q}} \\
\leq (\mu (T(\{x \in \mathbb{R}^n : \theta_1 M v_0(x) > \lambda\}))) \\
\leq (\text{cap}_{\mathcal{W}, \beta, r}(\{x \in \mathbb{R}^n : \theta_1 M v_0(x) > \lambda\}))^{\frac{q}{p}}.
$$

This and Lemma 2.1 (b) imply that $(d)$ holds. □

3.6. **Proof of Theorem 1.8** From the proof of Theorems 1.3 & 1.4 for $1 \leq p < n/\beta$ and $q > p$, we have

$$
\|\mu\|_{p,q} = \sup_{x \in \mathbb{R}^{1+n}} \frac{\mu(T(B(x, r)))^{\frac{p}{q}}}{\text{cap}_{\mathcal{W}, \beta, r}(B(x, r))} < \infty
$$

$$
\Rightarrow \left( \int_{\mathbb{R}^{1+n}} |v(t^{2\alpha}, x)|^q \text{d} \mu(t, x) \right) ^{\frac{1}{q}} \lesssim \|\mu\|_{p,q} \|v_0\|_{\mathcal{W}, \beta, r}, \ \forall v_0 \in \mathcal{W}^{\beta, p}(\mathbb{R}^n).
$$

Given $(t_0, x_0) \in \mathbb{R}^{1+n}$. Let $q = \frac{np}{n-p\beta}$ and $\mu(t, x) = \delta_{(t_0, x_0)}$ be the Dirac measure at $(t_0, x_0)$. It suffices to prove $\|\delta_{(t_0, x_0)}\|_{p,q} \leq t_0^{\beta-n}$. In fact, if $(t_0, x_0)$ is not in $T(B(x, r))$, then $\delta_{(t_0, x_0)}(T(B(x, r))) = 0$. If $(t_0, x_0) \in T(B(x, r))$, then $B(x_0, t_0) \subseteq B(x, r)$ and $r^n \geq t_0^n$. This gives the estimate

$$
\delta_{(t_0, x_0)}(T(B(x, r))) \leq \frac{r^n}{t_0^n} = t_0^{-n} r^{\frac{(n-p\beta)q}{p}}.
$$

The above estimate and $\text{cap}_{\mathcal{W}, \beta, r}(B(x, r)) \approx r^{n-p\beta}$ verify

$$
\frac{(\delta_{(t_0, x_0)}(T(B(x, r))))^{p/q}}{\text{cap}_{\mathcal{W}, \beta, r}(B(x, r))} \leq t_0^{-n} r^{\frac{(n-p\beta)q}{p}}.
$$

Therefore, $\|\delta_{(t_0, x_0)}\|_{p,q} \leq t_0^{\beta-n}$. □

3.7. **Proof of Theorem 1.12** Assume that $\mu$ is a nonnegative Radon measure such that

$$
\sup_{x \in \mathbb{R}^{1+n}, r > 0} \frac{\mu(T(B(x, r)))^{p/q}}{\text{cap}_{\mathcal{W}, \beta, r}(B(x, r))} < \infty
$$

for $1 \leq p < 2n$ and $\frac{4n-4p}{2n-p} \leq q < \infty$. According to the definition of 1/2−parabolic rectangle (see Nishio-Yamada [33])

$$
Q^{1/2}(s, y) = \{ (s, y) \in \mathbb{R}^{1+n} : |x_j - y_j| < s/2, 1 \leq j \leq n, s \leq t \leq 2s \}
$$

with center $(s, y)$, we have

$$
Q^{1/2}(s, y) = [s, 2s] \times B(y, \sqrt{n}s/2).
$$
The definition of $T(B(y, r))$ implies that there is a dimensional constant $c(n)$ such that

$$Q^{1/2}(s, y) \subseteq T(B(y, c(n)s)),$$

for each $(s, y) \in \mathbb{R}_+^{1+n}$, so

$$\mu(Q^{1/2}(s, y)) \leq \mu(T(B(y, c(n)s))) \lesssim s^{q(n-p/2)/p}.$$

If $p_1 = \frac{q(2n-p)}{2p(n+1)} - 1$, then for each $(s, y) \in \mathbb{R}_+^{1+n}$,

$$\mu(Q^{1/2}(s, y)) \lesssim s^{(n+1)(1+p_1)}.$$

Note that $p_1 \geq 1$ since $q \geq \frac{4p(n+1)}{2n-p}$ and $p < 2n$. It follows from Nishio-Yamada [33, p. 91 Theorem 2] that $\nu$ is a $(0,1)$-type Carleson measure on $b_{q/2}$ if and only if $\nu(Q^{1/2}(s, y)) \lesssim s^{(n+1)(1+q)}$, for each $(s, y) \in \mathbb{R}_+^{1+n}$. Thus $\mu$ is a $(0,1)$-type Carleson measure on $b_{1/2}^{p_1}$. □

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