**Revision of Using Eigenvalues of Covariance Matrices in Boundary-Based Corner Detection**

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**SUMMARY** In this paper, we present a revision of using eigenvalues of covariance matrices proposed by Tsai et al. as a measure of significance (i.e., curvature) for boundary-based corner detection. We first show the pitfall of Tsai et al.’s approach. We then further investigate the properties of eigenvalues of covariance matrices of three different types of curves and point out a mistake made by Tsai et al.’s method. Finally, we propose a modification of using eigenvalues as a measure of significance for corner detection to remedy their defect. The experiment results show that under the same conditions of the test patterns, in addition to correctly detecting all true corners, the spurious corners detected by Tsai et al.’s method disappear in our modified measure of significance.

**key words:** corner detection, covariance matrix, curvature estimation, eigenvalue, measure of significance

1. Introduction

Corners have been one of the most important features in computer vision since they are invariant to geometric transformations, such as translation, rotation and scaling. Some corner detection methods are based on gray-level differences of pixels [1]–[4]. Others use morphological operators to extract corners [5]. Boundary-based corner detectors, segmenting objects from an image first and then locating the discontinuities on the object boundaries [6]–[13], have been widely applied to spline curve fitting [14], [15], automated visual inspection [16]–[19], image segmentation [20]–[22], object recognition [23], etc.

The accuracy of corner detection on boundaries is primarily influenced by quantization and noises. In continuous space, arcs are smooth. However, in digitized domain, smooth arcs become zigzag after quantization which may mix them up with discontinuous corners. Besides, sensitive detection algorithms may be disturbed by noises and give false alarms on smooth arcs, while insensitive detectors may resist noises well but also miss the true corners.

In 1999, Tsai et al. [12] first proposed the usage of eigenvalues of covariance matrices over boundaries to distinguish corners from smooth arcs. Based on extensive tests, they found that small eigenvalues have good performance on detection and localization of corners for curved objects under different rotation and scale changes [12]. Since then, many studies have applied the small eigenvalues to detect corners directly [15], [16], [18], [21]. In addition, some others studies were also inspired by Tsai et al.’s small eigenvalues approach [6], [7], [13], [14], [17], [19], [22], [23].

Although Sossa Azuela et al. [20] commented that Tsai et al.’s method was the best one among the methods they had tested, Guru et al. [8] later discovered that Tsai et al.’s method may also detect unwanted spurious corners. Besides, some other researchers [24], [25] suggested that the small eigenvalues are more suitable for line segment detection, rather than corners.

In order to continue using eigenvalues of covariance matrices as a robust measure of significance (i.e., curvature) for boundary-based corner detection, in this paper, we first illustrate the pitfall of spurious corners detected in Tsai et al.’s [12] method. We then explore the full range of parameters of three different types of curves to further investigate the properties of eigenvalues of covariance matrices. After the thorough study, we shall point out a mistake made by Tsai et al.’s original proposal on the small eigenvalues for corner detection. Finally, we present a revision to Tsai et al.’s method to get rid of their flaw to become a more robust measure of significance for corner detection.

The rest of the paper is organized as follows. In Sect. 2, we briefly review Tsai et al.’s method and show their drawback. In Sect. 3, we further investigate the properties of eigenvalues and reveal the mistake made by Tsai et al.’s method. In Sect. 4, we propose a revision to Tsai et al.’s method as a new measure of significance for corner detection to correct their flaw. In Sect. 5, we perform some experiments to compare our proposed revision with Tsai et al.’s original method. Finally, we conclude the paper in the last section.

2. Review of Tsai et al.’s Method

2.1 Eigenvalues of Covariance Matrices

Let the closed boundary \( P \) of an object be represented by a sequence of \( n \) digital points,

\[
P = \{ p_i = (x_i, y_i) | 0 \leq i \leq n - 1 \}
\]

where \((x_i, y_i)\) are the Cartesian coordinates of point \( p_i \), and \( p_{(i+1) \mod n} \) is a neighbor of \( p_i \). For notational simplicity, \( p_j \) is also used to represent \( p_{j \mod n} \) for \( j \geq n \). Let the region of support \( S_j(p_i) \) denote a small curve segment of \( P \) between \( p_{i-k} \) and \( p_{i+k} \) for some integer \( k \), i.e.,
\(S_k(p_i) = \{p_j | i-k \leq j \leq i+k\}\) \hspace{1cm} (2)

Note that \(p_i\) is the middle point of \(S_k(p_i)\) whose length (the number of data points) is \(2k + 1\). The curve segments from \(p_{i-1}\) to \(p_{i-k}\) and from \(p_{i+1}\) to \(p_{i+k}\) constitute the two arms of point \(p_i\) with arm length \(k\).

Let \((c_x, c_y)\) be the geometric center of \(S_k(p_i)\), i.e.,
\[
c_x = \frac{1}{2k+1} \sum_{j=-k}^{i+k} x_j
\]
\[
c_y = \frac{1}{2k+1} \sum_{j=-k}^{i+k} y_j
\]

Then, the covariance matrix \(C\) of \(S_k(p_i)\) is given by
\[
C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}
\]
where
\[
c_{11} = \left( \frac{1}{2k+1} \sum_{j=-k}^{i+k} x_j^2 \right) - c_x^2
\]
\[
c_{12} = c_{21} = \left( \frac{1}{2k+1} \sum_{j=-k}^{i+k} x_j y_j \right) - c_x c_y
\]
\[
c_{22} = \left( \frac{1}{2k+1} \sum_{j=-k}^{i+k} y_j^2 \right) - c_y^2
\]

The two eigenvalues (the small eigenvalue \(\lambda_S\) and the large eigenvalue \(\lambda_L\)) of the covariance matrix \(C\) are computed as follows:
\[
\lambda_S = \frac{1}{2} \left[ c_{11} + c_{22} - \sqrt{(c_{11} - c_{22})^2 + 4c_{12}^2} \right]
\]
\[
\lambda_L = \frac{1}{2} \left[ c_{11} + c_{22} + \sqrt{(c_{11} - c_{22})^2 + 4c_{12}^2} \right]
\]

2.2 Tsai et al.’s Observations on Eigenvalues

Tsai et al. [12] considered the following three different types of digitized curves, as shown in Fig. 1:
- Straight line: \(y = (\tan \theta)x\), where \(\theta\) is the slope angle of the line.
- Circle: \(x^2 + y^2 = r^2\), where \(r\) is the radius of the circle.
- Angle (or jointed line): defined by two intersecting lines \(y = [\tan(\frac{\pi}{2} - \frac{\varphi}{2})]x\), where \(\varphi\) is the included angle of the two intersecting lines.

They analyzed the eigenvalues of the above three different types of curves with various parameters, such as slope angles \(\theta\) for straight lines, radii \(r\) for circles, and included angles \(\varphi\) for jointed lines. They then obtained the following observations for a smaller region of support with \(k = 10\) (21 data points) and a larger one with \(k = 15\) (31 data points); this is because \(k\) is selected typically in the range between 5 and 15.

- For straight-line segments, the values of \(\lambda_S\) approximate to zero regardless of their orientations.
- The values of \(\lambda_S\) of circular arcs are much smaller than those of jointed lines, as shown in Table 1.
- All values of \(\lambda_S\) of circular arcs are relatively larger than those of straight-line segments.

From the above analyses, they inferred that smaller included angles have larger \(\lambda_S\) values. Finally, they concluded that the eigenvalue \(\lambda_S\) is a robust measure of significance for corner detection [12].

2.3 General Corner Detection Procedure

For a given boundary \(P\), a general corner detection procedure can be divided into three steps as follows [26].

1. Select a suitable \(k\) value to determine the length of region of support (i.e., \(2k + 1\)). For each point \(p_i\) in \(P\), estimate its curvature over \(S_k(p_i)\) according to some chosen measure of significance.
2. Define a threshold \(T\), and eliminate those points whose estimated curvatures are less than \(T\).
3. Perform the non-maxima suppression to eliminate the remaining points in \(P\) whose estimated curvatures are not local maxima in \(S_k(p_i)\). The remaining boundary points are the detected corners.

2.4 Tsai et al.’s Performance Evaluation Procedure

Using the small eigenvalues \(\lambda_S\) as a new measure of significance, Tsai et al. first defined a point \(p_i\) to be a corner if its \(\lambda_S\) value is greater than some predetermined threshold and
each corner must be separated at least \( k \) points apart [12]. In order to show that \( \lambda_5 \) can be used to detect corners effectively, based on the general corner detection procedure, Tsai et al. suggested the following parameter-selection method for performance evaluation so that corner detectors to be compared are on the same basis:

1. Select a suitable value of \( k \) (\( k = 10 \) in their experiments).
2. Specify a set \( \{d_1, d_2, \ldots, d_m\} \) of \( m \) desired corners from a boundary \( P \). Then, define the threshold \( T = \min(\lambda_5(k, d_j)) \) where \( j = 1, 2, \ldots, m \) so that the desired corners will all be detected, where \( \lambda_5(k, d_j) \) is the \( \lambda_5 \) value over \( S_k(d_j) \).

Finally, count the number of spurious corners for performance evaluation.

2.5 Pitfall of Tsai et al.’s Method

Some researchers, such as Guru et al. [8], have discovered that Tsai et al.’s corner detection method may suffer from the problem of spurious corners on a smooth curve segment with a smaller radius of curvature. A cone shape illustrated in Fig. 2 (a) is one of such case. If point \( A \) is selected as a reference corner and let the threshold \( T = \lambda_5(k, A) \) for \( k = 15 \), then points \( B \) and \( C \) will also be detected by Tsai et al.’s performance evaluation procedure as corner points. This is because \( \lambda_5(k, B) \) and \( \lambda_5(k, C) \) are larger than \( T \), as shown in Fig. 2 (b).

3. Exploring Properties of Eigenvalues

In this section, we further investigate the properties of eigenvalues of covariance matrices to reveal the reasons of the pitfall in using the small eigenvalues as a measure of significance proposed by Tsai et al. for corner detection.

3.1 Properties of Eigenvalues

Let \( x \) and \( y \) be the two axes in the two-dimensional continuous space. Suppose that \( x' \) and \( y' \) are the two new rotated transformation axes after performing the principal components transformation [27], where \( x' \) and \( y' \) are the major and minor component axes, respectively. Let \( D \) be a set of sample data in the \( x-y \) space, \( C \) the covariance matrix of \( D \), and \( \lambda_L \) and \( \lambda_S \) the two eigenvalues of \( C \). Let \( \text{Var}(x') \) and \( \text{Var}(y') \) denote the variances of \( D \) projected onto the \( x' \) and \( y' \) axes, respectively. Härđle and Simar have proved that \( \lambda_L \) and \( \lambda_S \) are equal to the variances of data projected onto the rotated transformation major and minor axes, respectively, as follows [27].

**Theorem 1:** For a given set \( D \) of sample data in the \( x-y \) continuous space, let \( x' \)-\( y' \) space be the principal component transformation. Then, \( \text{Var}(x') = \lambda_L \) and \( \text{Var}(y') = \lambda_S \).

In order to further investigate the properties of eigenvalues used for corner detection in digital images (i.e., in the discrete domain), we show the distributions of \( \lambda_L \) and \( \lambda_S \) of the three different types of curves in Fig. 1. Note that for the case of angles, the symmetric axis of an included angle is allowed to be of any orientation, as shown in Fig. 3, rather than restricted to the \( y \)-axis only as in Tsai et al.’s original proposal [12] (refer to Fig. 1 (c)). In the following discussions, the length of the region of support \( S_k(p_i) \) is set to 31 data points (i.e., \( k = 15 \)).

Figure 4 shows the eigenvalues \( \lambda_L \) and \( \lambda_S \) of various straight lines of different slope angles \( \theta \). It can be seen that \( \lambda_S \) values are very close to zero, while \( \lambda_L \) values are almost equal to the total variance of region of supports. Note that
the \( \lambda_L \) of the line segment with \( \theta = \pi/4 \) has a maximum value. This is due to the quantization effect of digital images; the length of the diagonal of a square is longer than the lengths of the horizontal and the vertical line segment of the square (the ratio of their lengths is \( \sqrt{2} : 1 \)).

Figure 5 illustrates the eigenvalues \( \lambda_L \) and \( \lambda_S \) of various circular arcs of different radii \( r \). It is shown that the \( \lambda_S \) values are quite small if radii \( r \) are greater than the length of the region of support. However, when radii \( r \) are smaller than the length of the region of support, the \( \lambda_S \) values increase more significantly as radii \( r \) decrease.

Figure 6 depicts the eigenvalues \( \lambda_L \) and \( \lambda_S \) of angles of different \( \varphi \) with the symmetric axis \( y = x \) (refer to Fig. 3). Figure 7 shows another case of angles symmetric at \( y \)-axis (refer to Fig. 1 (c)). In both cases, \( \lambda_S \) and \( \lambda_L \) meet at \( \varphi \approx 53^\circ \); this means that \( \lambda_S \) has a maximum value when the included angle approximates \( 53^\circ \). Note that, in both of these two cases of digital jointed lines, the length of the line segment from the corner point \( p_i \) to one end point \( p_{i-k} \) is equal to the length of the line segment from \( p_i \) to the other end point \( p_{i+k} \). These are called the Euclidean arm lengths to differentiate with the digital arm lengths on the digital boundaries. Note that two arms with the same digital arm length do not necessarily have the same Euclidean arm length.

In the following, we are going to show that in the continuous space, when the two eigenvalues of a jointed line are equal, then its included angle will be approximating \( 2 \tan^{-1}(1/2) \approx 53^\circ \).

**Lemma 1:** Given a set \( D \) of sample data in the \( x-y \) continuous space, let \( D_T \) be \( D \) after a translation. Then, \( \lambda_L(D) = \lambda_L(D_T) \) and \( \lambda_S(D) = \lambda_S(D_T) \).

**Lemma 2:** Given a set \( D \) of sample data in the \( x-y \) continuous space, let \( D_R \) be \( D \) after a rotation. Then, \( \lambda_L(D) = \lambda_L(D_R) \) and \( \lambda_S(D) = \lambda_S(D_R) \).

By the definitions of translation and rotation, it is easy to prove the above two lemmas, where \( \lambda_L(D) \) and \( \lambda_S(D) \) denote the \( \lambda_L \) and \( \lambda_S \) values of the sample data \( D \), respectively. These two lemmas state that the eigenvalues of a data set are invariant to translations and rotations. Now, for the case of joint lines in the continuous space, we are going to show the following theorem.

**Theorem 2:** In the \( x-y \) continuous space, when the two eigenvalues of a jointed line are equal, its included angle will approximate \( 2 \tan^{-1}(1/2) \).

**Proof:** Consider a continuous jointed line in the \( x-y \) continuous space, in which the included angle is \( \varphi \), and the length of region of support is \( 2k + 1 \) (i.e., each Euclidean arm length is \( k \)). Suppose that each of the \( k \) data points on each arm (a straight-line segment) is one unit apart. Since the eigenvalues of a data set are invariant to translations and rotations, we can first translate the jointed line such that the angle point is on the origin, and then rotate it such that it is symmetric at \( y \)-axis (like Fig. 1 (c)). Let \( p_i \) denote the angle point, \( p_{i-j} \) (for \( 1 \leq j \leq k \)) denote points on one arm of \( p_i \), and \( p_{i+j} \) (for \( 1 \leq j \leq k \)) denote points on another arm, where

\[
p_i = (x_i, y_i) = (0, 0) \quad (11)
\]

\[
p_{i-j} = (x_{i-j}, y_{i-j}) = (j \sin(\varphi/2), j \cos(\varphi/2)) \quad (12)
\]

\[
p_{i+j} = (x_{i+j}, y_{i+j}) = (-j \sin(\varphi/2), j \cos(\varphi/2)) \quad (13)
\]

Let \( \lambda_S \) and \( \lambda_L \) be the two eigenvalues of the transformed jointed line. By Lemmas 1 and 2, \( \lambda_L \) and \( \lambda_S \) are equal to the large and the small eigenvalues of the original, untransformed jointed line, respectively.

Let \( C \) be the covariance matrix of the transformed
jointed line (as defined in Eq. (5)). Then, \( \lambda_L = \lambda_S \) implies \( c_{11} = c_{22} \) by Eqs. (9) and (10), where \( c_{11} \) is the variance of the transformed jointed line projected onto the \( x \)-axis, while \( c_{22} \) is the variance of the transformed jointed line projected onto the \( y \)-axis.

By Eq. (6), \( c_{11} \) is computed as

\[
c_{11} = \left( \frac{1}{2k+1} \sum_{j=-k}^{k} ((-j) \sin(\varphi)/2)^2 \right) - c_x^2
\]

where

\[
c_x = \frac{1}{2k+1} \sum_{j=-k}^{k} (-j) \sin(\varphi)/2 = 0
\]

is the mean of the transformed jointed line projected onto the \( x \)-axis. Substituting Eq. (15) into Eq. (14), it is easy to obtain

\[
c_{11} = \frac{k(k+1)}{3} \sin^2(\varphi/2)
\]

Similarly, by Eq. (8), \( c_{22} \) is computed as

\[
c_{22} = \left( \frac{1}{2k+1} \sum_{j=-k}^{k} ([j] \cos(\varphi)/2)^2 \right) - c_y^2
\]

where

\[
c_y = \frac{1}{2k+1} \sum_{j=-k}^{k} [j] \cos(\varphi)/2
\]

\[
= \frac{k(k+1)}{2k+1} \cos(\varphi/2)
\]

is the mean of the transformed jointed line projected onto the \( y \)-axis. Substituting Eq. (18) into Eq. (17), we can obtain

\[
c_{22} = \left( \frac{k(k+1)}{3} - \frac{k(k+1)^2}{2k+1} \right) \cos^2(\varphi/2)
\]

Then, by equating \( c_{11} \) with \( c_{22} \), we can derive

\[
\tan^2(\varphi/2) = \frac{k^2 + k + 1}{4k^2 + 4k + 1}
\]

If \( k \) is large enough, then \( \tan^2(\varphi/2) \approx 1/4 \), which means \( \tan(\varphi/2) \approx 1/2 \). Thus, \( \varphi \approx 2 \tan^{-1}(1/2) \).

**Corollary 1:** For a digital jointed line, if its included angle is symmetric at \( x \)-axis, \( y \)-axis, \( x = y \), or \( x = -y \), then its two Euclidean arm lengths are equal. Thus, \( \varphi \approx 2 \tan^{-1}(1/2) \) if \( \lambda_L = \lambda_S \).

However, if a digital jointed line is not symmetric at the above four cases, its two Euclidean arm lengths are not equal; and thus, its two eigenvalues will not meet. Figure 8 show an example of an angle with symmetric axis being \( y = (\tan 26.5^\circ)x \).

3.2 Revealing Tsai et al.’s Mistake

Recall that in Table 1, Tsai et al. showed that the value of \( \lambda_S \) decreases as the included angle \( \varphi \) increases from \( 30^\circ \) to \( 90^\circ \) for the region of support with \( k = 10 \) and \( 15 \) (i.e., 21 and 31 data points). However, as illustrated in Fig. 7 (under the same condition as Tsai et al. did), \( \lambda_S \) has a maximum value when \( \varphi \) is around \( 53^\circ \). It is obvious from this figure that the sharper the included angle, the smaller the \( \lambda_S \) value. Table 2 lists the values of \( \lambda_S \) for the case of included angle with symmetric axis being \( x = 0 \) for comparison. Note that Fig. 7 only shows the case of region of support of 31 data points, in which there is a plateau for the values of \( \lambda_S \) ranging from about \( 54^\circ \) to \( 92^\circ \), and the values decrease gradually toward two ends (\( 0^\circ \) and \( 180^\circ \)). This observation contradicts the conclusion made by Tsai et al.: “a smaller included angle results in a larger \( \lambda_S \) value” [12]. Up to this point, it is clear that a smaller \( \lambda_S \) value may be resulted from

- a straight-line segment,
- a smooth circular arc with a larger radius,
- a jointed line with a very small included angle or a very large included angle.

In Sect. 2.5, Fig. 2(b) shows the distribution of the small eigenvalues of the boundary of Fig. 2(a). It is clear that the \( \lambda_S(k, A) \) is smaller than \( \lambda_S(k, B) \) and \( \lambda_S(k, C) \) for \( k = 15 \). If point \( A \) on the lower sharp angle is selected as a reference corner, then according to Tsai et al.’s performance evaluation procedure using the small eigenvalues as the measure of significance for corner detection, points \( B \) and \( C \) on the top circular arc will also be detected as corners, which result in the unwanted spurious corner points.
4. Improved Corner Detection

4.1 Estimation of Included Angles

In Sect. 3, it has been shown that, theoretically, if $\lambda_S$ and $\lambda_L$ are equal for a continuous jointed line, then $\varphi \approx 2\tan^{-1}(1/2)$, in which $\lambda_S$ has a maximum value. To exclude the dilemma of smaller included angles of jointed lines having smaller $\lambda_S$ values, which are also shared by straight lines (Fig. 4), circular arcs with larger radii (Fig. 5), and larger included angles of jointed lines (Figs. 6 and 7), the large eigenvalues $\lambda_L$ can be considered to be used for corner detection when the included angle $\varphi < 2\tan^{-1}(1/2)$.

Let $\lambda(k, p_i)$ denote the included angle of the region of support $S_k(p_i)$. It can be estimated by the arccosine of the two arm vectors of $S_k(p_i)$ [23], [28], i.e.

$$\varphi(k, p_i) = \cos^{-1}\left(\frac{A \cdot B}{|A||B|}\right)$$

(21)

where

$$A = (x_A, y_A) = (x_{i-k} - x_i, y_{i-k} - y_i)$$

(22)

$$B = (x_B, y_B) = (x_{i+k} - x_i, y_{i+k} - y_i)$$

(23)

$$A \cdot B = x_A \cdot x_B + y_A \cdot y_B$$

(24)

$$|A| = \sqrt{x_A^2 + y_A^2}$$

(25)

$$|B| = \sqrt{x_B^2 + y_B^2}$$

(26)

4.2 Naive Approach

A naive approach to correct the usage of eigenvalues of covariance matrices as a measure of significance for corner detection in Tsai et al.’s method [12] can be revised as follows. Let $\lambda_m(k, p_i)$ be the naive corner index of point $p_i$ in $S_k(p_i)$, which is defined by

$$\lambda_m(k, p_i) = \begin{cases} 
\lambda_L(k, p_i) & \text{if } \varphi(p_i) < 2\tan^{-1}(1/2) \\
\lambda_S(k, p_i) & \text{otherwise}
\end{cases}$$

(27)

Figure 9 (b) shows the naive corner indices $\lambda_m$ of Fig. 2 (a). Comparing with Fig. 2 (b), the naive corner indices $\lambda_m$ are enhanced by the large eigenvalues $\lambda_L$ around the sharp corner $A$ significantly in Fig. 9 (b). However, the estimated included angles $\varphi(k, p_i)$ of the neighbors around the sharp corner $A$ in Fig. 2 (a) are also smaller than $2\tan^{-1}(1/2)$; these neighbors obtain even larger $\lambda_L$ as $\lambda_m$ than the corner $A$ itself. Therefore, if the local maxima of $\lambda_m$ above some threshold are selected as corners, the loci of corners will be deflected. As shown in Fig. 9 (a), point $D$ is detected as the corner by using $\lambda_m$, which is obvious not the correct corner point, though it is near the exact corner.

4.3 Modified Approach

In fact, a corner is usually located at the local minimum of $\lambda_L$ and the local maximum of $\lambda_S$ along a boundary. To locate true corners more accurately, the modified corner index $\lambda_M(k, p_i)$ for point $p_i$ can be defined as

$$\lambda_M(k, p_i) = \begin{cases} 
\lambda_L(k, p_i) & \text{if } \varphi(p_i) < 2\tan^{-1}(1/2) \\
\lambda_L(k, p_i) & \text{and } \lambda_L(k, p_i) = \min_L \\
\lambda_S(k, p_i) & \text{otherwise}
\end{cases}$$

(28)

where $\min_L = \min(\lambda_L(k(p), p_i) | p_i \in S_k(p_i))$. That is, if the included angle $\varphi(k, p_i)$ is smaller than $2\tan^{-1}(1/2)$ and $\lambda_L(k, p_i)$ is the local minimum in $S_k(p_i)$, then $\lambda_L(k, p_i)$ is used as $\lambda_M(k, p_i)$; otherwise, $\lambda_S(k, p_i)$ is used instead. The modified corner index $\lambda_M(k, p_i)$ can be used as a measure of significance for corner detection to locate corners more precisely.

Figure 10 compares $\lambda_M$ with $\lambda_S$ of angles with different $\varphi$ symmetric at $y = x$. It shows that the modified corner indices $\lambda_M$ decrease when the included angles $\varphi$ increase, while there is a local maximum at $\varphi \approx 53^\circ$ for $\lambda_S$. Note that for included angles greater than $53^\circ$, $\lambda_M$ overlaps $\lambda_S$.

Figure 11 (b) shows the modified corner indices $\lambda_M$ of Fig. 2 (a). Comparing with Fig. 9 (b), $\lambda_M$ of the sharp corner $A$ in Fig. 2 (a) is enhanced significantly under the smaller in-
termine the threshold we adopt Tsai et al.’s parameter-selection suggestion to de-
ner detection. As to comparing with other corner detectors, 
consistent. Tsai et al. claimed that using the small eigenval-
both Tsai et al. [12] and Guru et al. [8], but the results are in-
5. Experimental Results

4.4 Corner Detection Procedure

Our corner detection procedure follows the general corner
detection procedure in Sect. 2.3, in which the modified cor-
index λₘ is used as the measure of significance for cor-
detection. As to comparing with other corner detectors,
we adopt Tsai et al.’s parameter-selection suggestion to de-
determine the threshold T in Sect. 2.4 for performance evalua-

5. Experimental Results

An oxalis-like object, as shown in Fig. 12 (a), was tested by
both Tsai et al. [12] and Guru et al. [8], but the results are in-
consistent. Tsai et al. claimed that using the small eigenval-
ues λ₅ could identify all desired corners without any false
alarm, while Guru et al. reported that Tsai et al.’s method
would detect some spurious points as corners on smooth
curve segments of small radii of curvatures. In our exper-
iment, we also create an oxalis-like object (Fig. 12 (a)) of
size 240 × 240 pixels as our root test pattern to compare our
method with Tsai et al.’s method. Figure 12 (b) shows the
preselected eight reference corner points for subsequent per-
formance evaluation. To verify the reliability of both meth-
ods, we scale the root oxalis-like object to 75% (180 × 180)
and 150% (360 × 360). Besides, we also rotate the above
two different sizes of objects 22° and 45° clockwise to ob-
tain nine test patterns for comparison.

Table 3 lists some of the experimental results of our
modified method (using λₘ) and Tsai et al.’s method (us-
ing λ₅) for corner detection on the above nine test patterns.
Several different lengths of region of supports (ROS) rags-
ing from 7 to 33 data points (i.e., k = 3, ..., 16) are used for
comparisons; only the cases of k = 9, 12, and 15 are

![Figure 11](image1.png)

Fig. 11 Corner detection by modified corner indices λₘ.

![Figure 12](image2.png)

(a) the root test pattern (b) preselected corners

Fig. 12 An oxalis-like object of size 240 × 240 pixels.

| Scaling | ROS Length | Rotation Degrees | Spurious (use λ₅) | Spurious (use λₘ) |
|---------|------------|------------------|------------------|------------------|
| 75%     | 19 (k = 9) | 0°               | 0                | 0                |
| 75%     | 25 (k = 12)| 0°               | 18               | 0                |
| 75%     | 31 (k = 15)| 0°               | 12               | 0                |
| 75%     | 19 (k = 9) | 22°              | 0                | 0                |
| 75%     | 25 (k = 12)| 22°              | 17               | 0                |
| 75%     | 31 (k = 15)| 22°              | 13               | 0                |
| 75%     | 19 (k = 9) | 45°              | 0                | 0                |
| 75%     | 25 (k = 12)| 45°              | 16               | 0                |
| 75%     | 31 (k = 15)| 45°              | 16               | 0                |
| 100%    | 19 (k = 9) | 0°               | 0                | 0                |
| 100%    | 25 (k = 12)| 0°               | 0                | 0                |
| 100%    | 31 (k = 15)| 0°               | 16               | 0                |
| 100%    | 19 (k = 9) | 22°              | 0                | 0                |
| 100%    | 25 (k = 12)| 22°              | 0                | 0                |
| 100%    | 31 (k = 15)| 22°              | 16               | 0                |
| 100%    | 19 (k = 9) | 45°              | 0                | 0                |
| 100%    | 25 (k = 12)| 45°              | 0                | 0                |
| 100%    | 31 (k = 15)| 45°              | 16               | 0                |
| 150%    | 19 (k = 9) | 0°               | 0                | 0                |
| 150%    | 25 (k = 12)| 0°               | 0                | 0                |
| 150%    | 31 (k = 15)| 0°               | 0                | 0                |
| 150%    | 19 (k = 9) | 22°              | 0                | 0                |
| 150%    | 25 (k = 12)| 22°              | 0                | 0                |
| 150%    | 31 (k = 15)| 22°              | 0                | 0                |
| 150%    | 19 (k = 9) | 45°              | 0                | 0                |
| 150%    | 25 (k = 12)| 45°              | 0                | 0                |
| 150%    | 31 (k = 15)| 45°              | 0                | 0                |
corner index $\lambda_M$ as the measure of significance for corner detection.

Figures 13 (a) to 13 (i) show the results of Tsai et al.'s method with spurious corners, where the label $(12, 75\%, 0^\circ)$ in Fig. 13 (a) stands for $k = 12$ with scaling factor $75\%$ and $0^\circ$ rotation, and the detected true corners are marked with ‘•’ and the spurious corners with ‘◦’. Figure 14 illustrates the small eigenvalues $\lambda_S$ of Fig. 12 (a), the root test pattern, with $k = 15$, in which all of the detected corners are marked with ‘+’. In Tsai et al.'s method, the threshold $T = 2.4483$ is decided by selecting the minimum value of $\lambda_S$ of the eight preselected corners. Note that these 16 spurious corners on smooth curve segments of small radii of curvatures is the case of weakness of Tsai et al.'s method pointed out in [8]. Note also that these spurious corners also appear after rotations.

The reason of these spurious corners detected in Tsai et al.'s method is as follows. In Tsai et al.'s paper, their oxalis-like object (512×480) is almost four times of our root test pattern (Fig. 12 (a)) in size. If we reduce their image to one quarter (approximating to the size of our root test pattern), then its length of boundary is also reduced to one half of its original length. If the length of region of support is also reduced to one half of its original length for corner detection, then the detected corners will be the same as those in the original image. As shown in Table 3, for our root test pattern with $k \leq 12$, Tsai et al.'s method using $\lambda_S$ does not detect spurious corners. However, when $k = 15$, the 16 spurious corners are detected.

Figure 15 shows the modified corner indices $\lambda_M$ of the root test pattern with $k = 15$, in which the detected corner points in our method are marked with ‘+’, which are exactly the same as the preselected corners, as shown in Fig. 12 (b). In our approach, the threshold $T = 18.9677$ is decided by selecting the minimal value of the modified corner indices $\lambda_M$ of the eight preselected corners. Note that the 16 spurious corner points detected by Tsai et al.'s method disappear in our method.

In addition, we also perform further experiments using the four real objects tested in Tsai et al.'s paper. During the reproduction from their paper, additional noises, such as quantization errors, are also included to these objects, which are shown in Figs. 16 (a) to 16 (d). The sizes of these reproduced objects are 256×256, 256×272, 520×280, and 300×160, respectively. Figures 17 (a) to 17 (d) are the detected corners using $\lambda_M$ as the measure of significance for corner detection with $k = 10$. The results are exactly the same as Tsai et al. did.

From the above experimental results, we show that us-
ing the modified corner index $\lambda_M$ as the measure of significance for corner detection in our method indeed eliminates the spurious corners in Tsai et al.'s method, which simply uses the small eigenvalue $\lambda_S$ for corner detection. In addition, our method is also immune to rotation and scaling for smooth curve segments with smaller radii of curvature.

6. Conclusion

In this paper, we have presented a revision to Tsai et al.'s corner detection method using small eigenvalues of covariance matrices. We first showed the defect of spurious corners in Tsai et al.'s original proposal. By exploring the properties of eigenvalues, we revealed a mistake made by Tsai et al.'s method. We then proposed the modified corner index $\lambda_M$ as a measure of significance for corner detection instead of simply using $\lambda_S$ as did in Tsai et al.'s method. In addition to excluding spurious corners, the corner index $\lambda_M$ in our method is also immune to rotation and scaling for smooth curve segments with smaller radii of curvature. The experimental results showed that the modified corner index $\lambda_M$ is a robust measure for boundary-based corner detection.

To obtain better performance, Teh and Chin [26] have pointed out that in addition to the accurate measure of significance, a good corner detector primarily depends on the precise determination of region of support. Besides, for a general corner detection procedure, the threshold depends on the measure of significance as well as the length of region of support, which in turn depends on the target pattern. How to determine an optimal region of support and how to estimate the threshold to obtain better results are the future research topics for corner detection.

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