Compressive Imaging and Characterization of Sparse Light
Deflection Maps

P. Sudhakar, L Jacques, X. Dubois, P. Antoine, and L. Joannes

1ELEN Department, ICTEAM, Université catholique de Louvain, Belgium.
2Lambda-X, Nivelles, Belgium.

June 26, 2014

Abstract

Light rays incident on a transparent object of uniform refractive index undergo deflections, which uniquely characterize the surface geometry of the object. Associated with each point on the surface is a deflection map which describes the pattern of deflections in various directions and it tends to be sparse when the object surface is smooth. This article presents a novel method to efficiently acquire and reconstruct sparse deflection maps using the framework of Compressed Sensing (CS). To this end, we use a particular implementation of schlieren deflectometer, which provides linear measurements of the underlying maps via optical comparison with programmable spatial light modulation patterns. To optimize the number of measurements needed to recover the map, we base the design of modulation patterns on the principle of spread spectrum CS. We formulate the map reconstruction task as a linear inverse problem and provide a complete characterization of the proposed method, both on simulated data and experimental deflectometric data. The reconstruction techniques are designed to incorporate various types of prior knowledge about the deflection spectrum. Our results show the capability and advantages of using a CS based approach for deflectometric imaging.

Further, we present a method to characterize deflection spectra that captures its essence in a few parameters. We demonstrate that these parameters can be extracted directly from a few compressive measurements, without needing any costly reconstruction procedures, thereby saving a lot of computations. Then, a connection between the evolution of these parameters as a function of spatial locations and the optical characteristics of the objects under study is made. The experimental results with simple plano-convex lenses and multifocal intra-ocular lenses show how a quick characterization of the objects can be obtained using compressed sensing.

Keywords  deflectometry, compressive imaging, compressed sensing, optical metrology, proximal methods, Chambolle-Pock

1 Introduction

When light travels through a medium of varying refractive index, the refractive index gradient in the direction of light propagation causes light to deflect from its original path. By carefully studying the patterns of deflection, the composition of the medium under consideration can be discovered. A set of imaging techniques, known as schlieren imaging, allows us to optically visualise the extent of
light deflection [45]. These schlieren deflectometers operate by converting deflections into grayscale values and can be employed for any medium (solid, liquid and gases). Applications of schlieren techniques include flow modeling and computer graphics [15, 46, 26]. In this article, we consider an instance of schlieren deflectometer, which is used for characterizing solid transparent objects such as optical lenses.

Most of the alternative optical modalities for characterizing transparent objects are based on interferometry [33]. However, techniques based on optical interferometry are very sensitive to vibrations and also need precise calibrations for the methods to work successfully. On the other hand, thanks to the nature of deflectometry, problems due to vibrations and sensitive calibrations are avoided and hence they make an excellent alternative for industrial deployment in applications such as optical metrology and quality control.

Consider a (thin) transparent object with a beam of parallel light rays incident on one of its sides, as shown in Fig. 1(left). At each surface location $p$ (a non-zero area defined by the spatial resolution of the instrument), light deviates in multiple directions and they can be characterized in a local coordinate system $(e_1, e_2, e_3)$, with $e_1$ being parallel to the incident light beam. Using the spherical coordinates $(\theta, \varphi)$ in this system (see Fig. 1(left)), the resulting deflection spectrum $\tilde{s}_p(\theta, \varphi)$, which is non-negative in nature, represents the flux of light deviated in each direction $(\theta, \varphi)$.

As we consider a different location $p'$ on the surface of the object, the local information about its shape is given by the corresponding deflection spectrum $\tilde{s}_{p'}(\theta, \varphi)$. Therefore, by studying deflection spectra across all the locations on the object, we can understand the overall shape of the object, thereby characterizing the same.

In this article, the deflection spectrum $\tilde{s}_p$ is conveniently represented by its projection on to the plane $\Pi$ that is normal to the $e_1$, i.e., according to the projected function $s_p(r(\theta), \varphi) = \tilde{s}_p(\theta, \varphi)$ with $r(\theta) = \tan \theta$. Moreover, the object surface is assumed sufficiently smooth to be parameterized by a projection of the location $p$ on to the same plane (with an arbitrarily fixed origin), so that $p$ is basically parameterized as a 2-D vector in the coordinate system $\{e_2, e_3\}$.

For most objects (e.g., with smooth surfaces), deflections at any location $p$ are well behaved and
occur in a limited range of angles. The deflection spectra therefore tend to be naturally sparse in plane $\Pi_p$ or even in some appropriate basis of this domain, such as redundant wavelets. Fig. 1(right) shows an example of a discretized deflection spectrum $s_p$ for one location of a plano-convex lens, obtained using the setup that will be described in Sec. 2. The bright spot in the illustrated deflection spectrum signifies that light deflects only in a few directions around a dominant deflection (as governed by classical optics) and deflections elsewhere are negligible.

Measuring deflection angles in a straightforward way using goniophotometer is a cumbersome and elaborate process, and it is only suited for large dynamic of deflection angles and high contrast [44]. However, in the present context, the deflection information is observed using an indirect method. The particular optical setup at our disposal, which will be described in Sec. 2, measures the spectrum indirectly by optical comparison with a certain number of programmable modulation patterns. These optical comparisons are actually modeled as inner products between the modulation patterns and the underlying deflection spectrum. The discretization of the modulation pattern dictates the discretization of the deflection spectrum, and hence the inner products can also be envisaged as being discrete. Furthermore, the optical setup is capable of simultaneously performing the optical comparisons of a single modulation pattern with deflection spectra corresponding to a regular sampling of the test object’s surface. These parallel optical comparisons, which are inner-products, are in turn collected in a Charged Coupled Device (CCD) array. By this arrangement, each CCD pixel probes one location on the object, and to reconstruct each spectrum one needs to solve a linear inverse problem from the inner products collected at the corresponding CCD pixel.

Indirectly probing a signal through its inner products, with known patterns, is a generalization of the classical sampling procedure where the modulation patterns are simply shifted delta functions [3]. Therefore, inner products with known modulation patterns are generalized samples of the signal. From now on, we will simply refer to these inner product samples as measurements. In the context of sampling band limited signals using shifted delta functions, the number of samples per unit time needed to reconstruct the signal is dictated by the overall bandwidth of the considered signal space. This is the classical Shannon’s sampling theorem. Alternatively, if the underlying signal can be expressed or approximated by a linear combination of a few basis vectors, then results in sparse signal recovery and in compressed sensing [19, 20] show that with appropriate (random) patterns, one can successfully recover such a signal from a fewer number of measurements compared to the dimension of its ambient domain. This signal recovery procedure is non-linear in general and the performance greatly depends on the nature of the modulation patterns.

In the context of our schlieren deflectometry, we have a situation where the unknown signals (deflection spectra) are sparse and can be observed only through their inner products with programmable modulation patterns. This is a tailor-made situation for adopting compressed sensing for recovering deflection spectra.

Compressed sensing, despite some difficulty in implementation for optical systems [55], has found several applications in imaging, beginning with the famous single pixel camera [17] to recent applications in, for example, magnetic resonance imaging [31, 32, 38], astronomical imaging [6], radio interferometry [54], hyper spectral imaging [22] and biological imaging [48].

Contributions and organization of the article:
The contributions of this article are twofold. Firstly, we demonstrate a novel way to compressively acquire sparse deflection spectra in a schlieren deflectometer and present a numerical method for their reconstruction along with experimental results. To this end, we present a design of optical
modulation patterns based on *spread spectrum*\(^1\) compressive sensing [39]. This framework not only enables us to leverage the power of random measurements, as advocated by compressed sensing theory, but also makes the numerical methods exploit the advantage of fast algorithms for matrix-vector multiplications. Sec. 2 describes the schlieren deflectometric system and Sec. 4 contains the design of optical modulation patterns based on spread spectrum compressed sensing.

The performance of compressive sensing reconstruction methods depend on how well the underlying signals are sparsely represented. As we have commented earlier, for smooth objects the spectrum tend to be sparse in their natural domain itself, but much can be gained by transforming them into other domain. In Sec. 5, we formulate the compressive recovery problem using three different strategies of sparse representations of deflection spectra. These formulations are convex optimization problems. In Sec. 6, we detail the numerical method, based on a primal-dual approach, used to solve the three problems.

Sec. 7 contains the experimental results, both on simulated data and actual data obtained using the deflectometer. Optical systems are prone to noise and the deflectometric system under consideration is not an exception. Fortunately, the system can be calibrated relative to its intrinsic noise and the noise calibration procedure is discussed in Sec. 7.2.1. Our experimental results show the potential of compressed sensing based deflectometric imaging in reducing the number of measurements.

A description of the optical setup and a basic reconstruction method, along with the results, were briefly presented in our conference communications [50, 49]. However, the present article contains several new material including the details of modulation pattern design and also new strategies for reconstruction, along with new contributions on how to use deflection information for object characterization.

The second part of the contribution deals with the local characterization of lenses using compressive measurements, such as mapping local dioptic power (related to the local focal length) across the surface of the lens. The deflection spectrum provides detailed information about the deflection pattern at a particular location on the object. However, the price to be paid to obtain such rich information is the computation effort spent on its numerical reconstruction. When computational resource is scarce, we can still extract meaningful information about the deflection spectra, sufficient to characterize the shapes of objects under consideration. This work is on the lines of compressed domain signal processing and parameter estimation [14, 13, 21].

We develop a simplified description of deflection spectrum, which is characterized by a pair of translation parameters. This is inspired by the fact that when the surface of an object is smooth, the deflection spectra at consecutive locations tend to have similar shape and size but translated on the spectral plane. Therefore, we propose a *compressed domain matched filtering*, in Sec. 8, to extract these parameters directly from the measurements, without involving any expensive reconstruction method.

A method for estimating parameters in compressed domain, up to sub-pixel accuracy, is presented and comparison is made with the parameters estimated from reconstructed deflection spectra. We then describe the relationship between the optical power of the object under consideration and the evolution of the parameters, using the schematic of the optical system. Our experiments show the capability of compressive method to provide relevant characterization of simple test objects such as plano-convex lenses and also complicated objects such as diffractive multifocal intra-ocular

\(^1\)“Spread Spectrum” is not related to the studied deflection “spectrum” but it refers to the signal frequency spectrum.
lenses [29], i.e., lenses displaying multiple foci due to their special design of surfaces.

2 Optical setup and Notations

The schematic of the schlieren deflectometer used in our work to measure deflection spectra is shown in Fig. 2(a), which is a simplified 2D representation associated to one 2D slice. Fig. 2(b) shows a picture of a commercially available schlieren deflectometer sold under the name of NIMO™ by Lambda-X. Its key components are (i) a Spatial Light Modulator (SLM), (ii) a schlieren lens with focal length $f$, (iii) a Telecentric System (TS) and (iv) a Charged Coupled Device (CCD) camera to collect light.

The object to be analyzed is placed in between the schlieren lens and the telecentric system. On its left side, light rays from a coherent source are incident. Due to the telecentric system, only those light rays emerging out of the object that are parallel to the optical axis pass through and get collected by the CCD. Up to a flipping around the optical axis $O$, each location $p$ on the object, at a distance $d$ from the optical axis $O$ (dashed line), is probed by a corresponding CCD pixel also at a distance of $d$ from $O$. Each location $p$ is thus in one-to-one correspondence with a CCD pixel and therefore, the spatial resolution of the system is dictated by the resolution of the CCD array and the size of the pinhole of the telecentric system.

From classical optics, a light ray that is incident on location $p$ at an angle $\theta_p$ originates from the light source at a distance of $r = f \tan \theta_p$ from the optical axis. Likewise, the light rays originating from different locations of the source have different incident angles at $p$. Since we can always invert the direction of light propagation virtually in an optical system, we can view the system as though parallel rays of light were incident from the right side of the object, and the light undergoes deflections at $p$ on the object, and they form an image on the SLM. Therefore, up to a global scaling by $f$, the SLM plane is exactly the plane $\Pi$, that was discussed in Sec. 1, on which the deflection spectra corresponding to every location $p$ is observed. With this arrangement, modulating the SLM amounts to modulating the deflection spectrum $s_p$, while the light collected in CCD pixel $p$ is just an inner product of $s_p$ with the modulation pattern.

Even though deflection spectrum is a continuous domain object, the use of a discrete SLM, and hence discrete modulation patterns, renders the inner products to be discrete in nature, i.e., we
observe the continuous spectrum through a discrete representation where each sample is associated to the integration of light in a single SLM pixel. Hence, the recoverable deflection spectra is also discrete and has the same resolution as the SLM. Similarly, the discreteness of the CCD also limits the spatial resolution of the locations that can be probed. Henceforth, we shall use the notation \( k \) to indicate the discrete locations on the object and the corresponding CCD pixel locations.

Even though the deflection spectrum and the modulation patterns are 2D quantities, we shall represent them as 1D vectors for brevity of notation so that the action of modulation patterns on a deflection spectrum can be written as a matrix vector product. The 1D vector can be obtained by simply stacking all the columns of the 2D representation. If we generate \( M \) modulation patterns \( \phi_i \in \mathbb{R}^N \) with \( 1 \leq i \leq M \) in the SLM of \( N \) pixels, considering the discrete nature of the CCD camera (having \( N_C \) pixels), the discretized deflection spectra are observed through

\[
y_k = \Phi s_k + n, \quad 1 \leq k \leq N_C,
\]

where \( \Phi^T = (\phi_1, \ldots, \phi_M) \in \mathbb{R}^{N \times M} \) is the sensing matrix, \( k \) is the CCD pixel index, \( s_k \in \mathbb{R}^N \) is the discretized spectrum at the \( k^{th} \) pixel/object location, and \( n \) models the additive measurement noise of finite power.

For each modulation pattern, the CCD collects one measurement of the underlying corresponding deflection spectrum. The system is designed in such a way that each CCD pixel is independent of others and collects only the samples of its corresponding deflection spectrum. The quest now is to optimize the design of the sensing matrix \( \Phi \) in order to maximize the information captured in each sample. To this end, we design the modulation patterns relying on the theory of spread spectrum compressed sensing [35, 39]. As the forward measurement model in Eq. (1) and the recovery problem formulation (developed in future sections) are the same for all the locations of the CCD, we suspend the use of the subscript \( k \) to simplify the notations, and resume the usage when the situation demands.

### 3 Phase shifting schlieren

The schlieren deflectometer which we consider in our work has been already used, outside the context of compressive sensing, to measure light deflections in transparent objects. For the purpose of emphasizing the advantage of the compressive sensing method, we shall briefly explain the existing method. For a detailed description of the same, the readers are encouraged to refer to [28, 27]. The system is configured to work using several phase shifted sinusoidal patterns (for modulation) in order to measure deflections and this configuration is named Phase Shifting Schlieren (PSS). In PSS, the deflection information is directly encoded into the intensity of the CCD through sinusoidal modulation.

The key assumption in PSS is that the deflection spectrum for each location consists of a single peak, instead of a distribution. This means, the object is hypothesized to contain only one deflection at each of its location. the measurements are made using multiple phase shifted sinusoidal modulations in both horizontal and vertical directions, for making the measurements. For example, a vertical sinusoidal pattern (modulation in \( x \) direction) with phase \( \xi \) is given by

\[
h(x, y) = 1 + \cos \left( \frac{2\pi}{\Lambda} x + \xi \right),
\]

where \( x \) represents the horizontal axis, \( \Lambda \) is the period of the sinusoid and \( \phi \) is its phase. With such a pattern and suitable approximations of the optical system, the intensity recorded on a CCD
pixel at location $k$ can be modelled as

$$y_k(r_x) = a + b \cos \left( \frac{2\pi}{K} r_x + \xi \right),$$

(3)

where $r_x$ is the horizontal component of the displacement $r$ on the SLM plane as shown in Fig. 2(a). Likewise, by using horizontal sinusoidal patterns, the vertical component $r_y$ of the displacement is encoded in the CCD pixel intensity.

The rest of the PSS method is in extracting the values $r_x$ and $r_y$ from Eq. (3). Moreover, as the measurements are made with sinusoidal patterns, the deviation information are wrapped over in $2\pi$ phases. Hence, several measurements using phase shifted sinusoids are made to obtain extra information to solve for the unknowns in Eq. (3). Then deviation angles are numerically decoded and $2\pi$ phase unwrapped from the measurements $y_k$ using $n$-step algorithms [24, 2].

PSS is a simple and effective method for measuring deflection angles and has been successfully used in tomographic applications such as refractive index map reconstruction [5, 23]. However, this method is not stable for deflection spectra that are spread out, unlike a point or even of estimating several main deflection angles for each object location. Therefore, the richness of deflection spectra to reveal interesting information is lost. Moreover, the use of non-binary modulation patterns in Eq. (2) brings in the problem of non-linearity in the SLM response and without a careful calibration, it is prone to computational errors. Our proposed method is capable of recovering full deflection spectra and also it uses binary modulation patterns to avoid SLM non-linearities.

4 Spread Spectrum Optical Compressive Sensing

4.1 Compressive sensing

Compressive Sensing (CS) [16, 10, 3, 30, 20] is a new paradigm in signal sampling which envisages that any signal $x = \Psi \alpha \in \mathbb{C}^N$, where $\alpha$ is either sparse or compressible, can be tractably recovered from a few corrupted linear measurements of the form

$$y = \Theta x + n,$$

where $\Theta$ is a $M \times N$ random Gaussian measurement matrix\(^2\). If the number of measurements satisfies $M = O(K \log(N/K))$, then the solution $\tilde{\alpha}$ of the following convex optimization problem

$$\tilde{\alpha} := \underset{\alpha \in \mathbb{C}^N}{\text{arg min}} \|\alpha\|_1 \text{ subject to } \|y - \Phi \alpha\|_2 \leq \epsilon,$$

(5)

where $\Phi = \Theta \Psi$ and $\epsilon$ is a bound on the noise power, i.e., $\|n\|_2 \leq \epsilon$, satisfies

$$\|\alpha - \tilde{\alpha}\|_2 = O \left( \frac{\|\alpha - \alpha_K\|_1}{\sqrt{K}} + \epsilon \right)$$

(6)

with a probability at least $1 - N^{-\gamma \log^3(N)}$, where $\alpha_K$ is the best $K$-term approximation to the vector $\alpha$, $\|\alpha - \alpha_K\|_2 \leq \|\alpha - \alpha'\|_2$ for all $\alpha'$ such that $\|\alpha'\|_0 \leq K$ [9, 40].

Similar type of result also holds when the measurement matrix is derived from an orthonormal basis $\Gamma \in \mathbb{C}^{N \times N}$. In such a case, a $M \times N$ measurement matrix is of the form $\Gamma \Omega$, the conjugate

\(^2\)A matrix where each entry is a Gaussian random variable.
transpose of the submatrix matrix $\Gamma_{\Omega}$ formed by restricting the columns of $\Gamma$ to those index by subset $\Omega \subset [N] := \{1, \ldots, N\}$ randomly chosen at uniform.

If the coherence $\mu := \sqrt{N} \max_{1 \leq i,j \leq N} |\langle \Gamma_j, \psi_i \rangle|$ between $\Gamma$ and $\Psi$ is very close to 1, then

$$M = O(\mu^2 K \log^4(N))$$

measurements are enough for the optimization problem (with $\Phi = \Gamma_{\Omega}^* \Psi$) to provide an estimate $\hat{\alpha}$ that satisfies Eq. (6) [9, 40]. The number of the measurements for successful recovery scales quadratically with respect to the coherence $\mu$ and hence it is desirable to have the two bases as incoherent as possible to make $\mu$ close to 1.

While fully random matrices are optimal in terms of sampling efficiency, measurement matrices derived from orthonormal bases are equipped with fast matrix-vector multiplication making them attractive for practically solving Eq. (5). In the next section, we describe a method that achieves a tradeoff between optimal random measurement strategy and structured computations.

4.2 Spread spectrum sensing

The key issue in using orthonormal bases as measurement matrices is that of its coherence with the sparsity basis. In order to statistically minimize coherence between the two bases, we employ spread spectrum modulation of the data vector $x$ [35, 39].

Spread spectrum randomly scrambles the phases of signal by point-wise multiplication of the signal samples with another modulator signal whose samples have random phases but unit magnitude. If the sensing basis is a Fourier or Fourier-like, then this point-wise modulation amounts to performing a convolution of the signal and the modulator in the spectral domain. As a result of this, the energy of the signal is spread over its entire spectrum, while preserving its norm, and hence the name “spread spectrum”.

Further, whenever all the entries of the sensing basis $\Gamma$ have the same amplitude, then with high probability, spread spectrum technique ensures that the number of measurements required for successful reconstruction of the signal is comparable to that of fully random measurements, the optimal measurement strategy according to compressive sensing. Such bases are called universal sensing basis and Fourier and Hadamard bases are some examples of them.

Mathematically, the spread spectrum vector is a random vector $\sigma$ with the amplitudes of each of its entry $|\sigma_i| = 1$, e.g., a Steinhaus or Rademacher sequence. The sensing matrix that incorporates modulation by $\sigma$ is $\Phi = \Gamma_{\Omega}^* \Sigma \Psi$, where $\Sigma = \text{diag}(\sigma)$ is a diagonal matrix. In this case, we need

$$M \geq C_\rho K \log^5(N)$$

measurements in order to recover a solution $\alpha^* \text{ of } (5)$ satisfying (6) with a probability at least $1 - O(N^{-\rho})$, for some $0 < \rho < \log^3(N)$. Noticeably, the coherence $\mu$ has disappeared from the condition implying that with spread spectrum and universal sensing basis, the recovery guarantee is universal, irrespective of the sparsity basis. From the perspective of computations, spread spectrum involves only point-wise multiplication and hence does not break down the structure of the fast algorithms used for the transforms.

On a related note, some researchers have investigated other ways of designing structured random sensing matrices. Notable of them are the ones based on convolution with random sequences [25, 51, 42, 56], where the signal is convolved with a random sequence before subsampling. Even this scheme is shown to be universal and works well with any sparsity basis [42]. Convolutions can be
implemented as multiplication in the Fourier domain and hence they are computationally efficient too. However, as our application is for an optical system, we would like to have a sensing matrix that is binary valued in nature to avoid non-linearities of the system and hence we stick to spread spectrum.

4.3 Optical sensing basis

As the sensing operation has to be finally implemented in a physical deflectometric system, it is essential to have the sensing basis and the spread spectrum vector $\sigma$ to be non-negative real valued, and also preferably binary in nature to avoid optical non-linearities.

A natural choice for a Fourier-like basis having binary valued entries is the Hadamard basis $H$, which is also a universal basis as remarked earlier [52]. As the modulation vector $\sigma$ should be real valued and having unit magnitude in each of its entry, they are chosen randomly from $\{\pm 1\}$ with equal probability. The sensing basis is then composed as

$$\Gamma = H\Sigma,$$  \hspace{1cm} (7)

where $\Sigma = \text{diag}(\sigma)$ is a diagonal matrix. Randomly selected subset of columns of $\Gamma$, indexed by a set $\Omega$, are then used as the sensing vectors.

As the matrix $H\Sigma \in \{-1,1\}^{N\times N}$, it has to be properly biased to make it non-negative so that it can be implemented as a on-off pattern in the SLM. This is achieved by setting

$$\Gamma = \frac{1}{2} (H\Sigma + 11^T),$$  \hspace{1cm} (8)

where $1$ is an $N$ dimensional vector of all ones. Then, the $M$ dimensional measurements from the optical system are given by $z_k = \Gamma^*_{\Omega} s_k$.

Before using the measurements for reconstruction, the bias introduced in the sensing basis, Eq. (8), has to be undone, so that the canonical form of the Hadamard transform can be used during the reconstruction procedure. The bias corrected measurements are then given by

$$y_k = 2z_k - 1_M 1_N^T s_k,$$  \hspace{1cm} (9)

where $(1_M 1_N^T s_k)$ is a $M$ dimensional vector filled with the value $(1^T s_k)$. This value is simply the sum of all the entries in the vector $s_k$, which has to be obtained separately by an extra measurement using a SLM pattern fully filled with the value 1, meaning, all the SLM pixels are made transparent. Once the required number of measurements are made, then they can be used to reconstruct the spectrum by solving a recovery optimization problem.

5 Recovery problem formulations

As noted previously, each CCD location $k$ collects the measurements corresponding to its deflection spectrum $s_k$ and it is reconstructed by solving the optimization problem (6). The choice of the sparsity basis $\Psi$ plays an important role in the quality of reconstruction. Note that, the knowledge of $\Psi$ is used only during reconstruction and has no bearing on either the measurement basis or the measurement process itself. Hence, with the same set of measurements, we can apply the reconstruction method for different choices of $\Psi$. 


5.1 Sparsity basis and overcomplete dictionaries

Even though it is remarked earlier that the spectra we wish to reconstruct are sparse in the canonical domain itself, due to the nature of the objects we investigate, it is advantageous nevertheless to use some other basis for sparse representation of deflection spectra. One choice is to use wavelets as they are suitable for signals which are piecewise regular. In our work, we use Daubechies 16 tap orthogonal wavelet basis [34], due to the expected smooth and non-dispersive shapes of deflection spectra.

Representation in a wavelet basis is translation variant and hence the sparsity of the representation might be affected by even a small translation of the signal, which may result in reconstruction artefacts. One way to overcome this is to use a translation invariant frame of wavelet associated with the UnDecimated Wavelet Transform (UDWT), i.e., the usual wavelet transform without the decimation [34]. The non-uniqueness of the decomposition of a signal in such a redundant frame allows us to represent signals in their sparsest possible way [19, 8]. Our work also study the advantage of using a UDWT built on the Daubechies 16 tap filter in all our reconstruction experiments.

5.2 Synthesis and analysis formulations

The solution of the optimization problem (6) actually provides the wavelet coefficients of the deflection spectrum, from which the spectrum itself is reconstructed. The notion of sparsity in such a formulation is that the deflection spectrum can be approximately synthesised using a few wavelet coefficients, and hence the formulation is traditionally called as synthesis approach to sparse recovery.

On the other hand, we can set out to work in the deflection spectrum domain itself and find a spectrum that agrees with the measurements $y_k$. At the same time, we wish to have a spectrum which has sparse wavelet coefficients when analyzed with a wavelet dictionary/basis. That is, we can solve the following optimization problem [19, 38]:

$$\hat{s} := \arg \min_s \| \Psi^* s \|_1 \quad \text{subject to} \quad \| y - \Gamma \Omega^* s \|_2 \leq \epsilon,$$

where $\epsilon$ is a bound on $\| n \|_2 \leq \epsilon$. In the case where $\Psi$ is a tight frame, reconstruction guarantees similar to that of Eq. (6) exists for analysis formulation as well, and they are based on another property of sensing matrices known as Dictionary-Restricted Isometry Property (D-RIP), quantified by a constant $\delta_K$ [8]. Specifically, if the analysis dictionary $\Psi$ is a tight frame and $\Gamma$ satisfies D-RIP with $\delta_{2K} < 0.08$, then the solution of Eq. (10) satisfies [8, Thm. 1.4]

$$\| s - \hat{s} \|_2 = O \left( \frac{\| \Psi^* s - (\Psi^* \alpha)_K \|_1}{\sqrt{K}} + \epsilon \right).$$

In our work, we make use of tight frames based sparse representation for analysis based recovery. Recovery guarantees when $\Psi$ is not a tight frame and measurements are Gaussian can be found in [41].

At the outset, the two problems, Eqs. (5) and (10), look identical, except for the placement of the matrix $\Psi$. In fact, when $\Psi$ is an orthonormal basis the location of the sparsity basis has no bearing on the solution and hence the two problems are exactly the same. When $\Psi$ is a frame or an overcomplete dictionary, the implication of the changed position of the matrix $\Psi$ is not trivial and the synthesis and analysis problems are completely different. In (10), one looks for a solution directly in the signal space, which, when analyzed with $\Psi$, has a sparse representation and also
agrees with the measurements. Hence, this approach is called as the analysis approach [18, 36]. In our work, we solve both the synthesis and analysis problems with a UDWT dictionary.

To summarize, our work will analyze the benefit of using one of the following schemes for reconstructing sparse deflection spectra:

1. Recovery by synthesis using orthonormal wavelet basis,
2. Recovery by synthesis using UDWT and
3. Recovery by analysis using UDWT.

As the deflection spectrum represents the deflection of light in different directions, it is natural that it is non-negative. This is an important prior information which constrains the space of candidate solutions of the optimization problems we would like to solve.

Letting \( \mathbb{R}_{+}^{N} = \{ z = [z_1, \ldots, z_N]^T \in \mathbb{R}^N | z_i \geq 0, 1 \leq i \leq N \} \) denote the \( N \)-dimensional non-negative quadrant, and denoting \( \Phi_S := \Gamma_\Omega^* \Psi \) and \( \Phi_A := \Gamma_\Psi \), the two optimization problems we would like to solve, with non-negativity constraint, can be written as

\[
\hat{\alpha} := \arg \min_{\alpha \in \mathbb{C}^N} \|\alpha\|_1 \text{ subject to } \|y - \Phi_S \alpha\|_2 \leq \epsilon \text{ and } \Psi \alpha \in \mathbb{R}_{+}^{N}; \tag{12}
\]

and

\[
\hat{s} := \arg \min_{s \in \mathbb{R}^N} \|\Psi^* s\|_1 \text{ subject to } \|y - \Phi_A s\|_2 \leq \epsilon \text{ and } s \in \mathbb{R}_{+}^{N}. \tag{13}
\]

### 6 Numerical method

To solve the problems Eqs. (12) and (13), we make use of a primal-dual method called the Chambolle-Pock (CP) algorithm [11]. Chambolle-Pock algorithm solves primal-dual forms of unconstrained convex problems and it relies on proximal operators of the functions involved in the objective. It has a flexible structure, which allows easy inclusion of additional terms in the objective function. Furthermore, it has guarantees of convergence under under broad conditions on the objective function.

#### 6.1 Chambolle-Pock algorithm

Let \( K : \mathbb{R}^N \rightarrow \mathbb{R}^M \) be a continuous linear operator with a bounded norm. Let \( F : \mathbb{R}^M \rightarrow [0, +\infty] \) and \( G : \mathbb{R}^N \rightarrow [0, +\infty] \) be two proper, convex, lower-semicontinuous functions [4]. The Chambolle-Pock algorithm is used to solve saddle-point problems of the form

\[
\min_{u \in \mathbb{R}^N} \max_{v \in \mathbb{R}^M} \langle Ku, v \rangle + G(u) - F^*(v). \tag{14}
\]

It can be seen that Eq. (14) is the primal-dual formulation of the primal minimization problem

\[
\min_{u \in \mathbb{R}^N} F(Ku) + G(u). \tag{15}
\]

By denoting \( F^* \) and \( G^* \) to be the convex conjugates of \( F \) and \( G \) respectively, the Chambolle-Pock algorithm to solve Eq. (14) can be summarized as:
Algorithm 1: Chambolle-Pock

1. **Input:** Choose $\tau, \sigma > 0$, $(u^0, v^0) \in \mathbb{R}^N \times \mathbb{R}^M$, $\theta \in [0, 1]$ and $\bar{u}^0 = u^0$

2. **Iterate:** For $n \geq 0$, until stopping condition

   \[
   \begin{align*}
   v^{n+1} &= \text{prox}_{\sigma F^*}(v^n + \sigma K \bar{u}^n) \\
   u^{n+1} &= \text{prox}_{\tau G}(u^n - \tau K^* v^{n+1}) \\
   \bar{u}^{n+1} &= u^{n+1} + \theta(u^{n+1} - u^n)
   \end{align*}
   \] (16)

3. **Output:** $(u^n, v^n)$

The proximal operator $\text{prox}_{\gamma F}$ of a proper, convex, lower semicontinuous function $F$, with a parameter $\gamma$, is defined as

\[
\text{prox}_{\gamma F}(u) := \arg \min_{z \in \mathbb{R}^N} \frac{1}{2\gamma} \|u - z\|^2_2 + F(z).
\] (17)

It can be shown that the proximal operator generalizes a simple gradient descent of $F$, in the form an implicit subgradient descent, even if $F$ is not differentiable [37]. Moreover, proximal operators of several functions commonly used in signal processing have closed forms which are easy to evaluate, e.g., the proximal operator of $\ell_1$ norm is soft thresholding [12, 37]. Further, the proximal operator of the conjugate function $F^*$ is easily derived using Moreau’s identity [11]. The proximal operators for the functions appearing in the minimization problems Eq. (5) and Eq. (10) are discussed in Sec. 6.2 and Appendix A.

To ensure the convergence of the algorithm, the parameters $\tau$ and $\sigma$ have to be chosen such that the product $\tau \sigma \|K\|^2 < 1$, where $\|K\|$ is the operator norm of $K$. Details on the conditions and type of convergence can be found in [11].

6.2 Adaptation of Chambolle-Pock to our setting

In order to express the two recovery problems Eq. (5) and Eq. (10) in the form of Eq. (15), we need to convert the constraints into some functions which can be included in the unconstrained objective. This is done by considering convex indicator functions onto the appropriate convex sets defined by the constraints. The convex indicator function of a convex set $C$ is defined by

\[
\iota_C(z) = \begin{cases} 
0 & \text{if } z \in C, \\
\infty & \text{otherwise.}
\end{cases}
\] (18)

The convex indicator function is a proper, convex and lower-semicontinuous function and hence it satisfies the requirements of the Chambolle-Pock algorithm [7]. Let $B = \{z \in \mathbb{R}^N \mid \|y - z\|_2 \leq \epsilon\}$ be the $\ell_2$ ball of radius $\epsilon$, centred on the measurement vector $y$ which is a convex set. Then the constraints $\|y - \Phi_S \alpha\|_2 \leq \epsilon$ and $\|y - \Phi_A s\|_2 \leq \epsilon$ can be inserted into unconstrained problems by including the convex indicator function of the set $B$ into the objective functions.

The additional non-negative constraint of Eq. (12) and Eq. (13) can be included in the synthesis and analysis problems through the convex indicator function of the non-negative orthant $\mathbb{R}^N_+$. With
these considerations, the unconstrained formulations of the reconstruction problems Eq. (5) and Eq. (10) respectively take the form:

\[
\hat{\alpha} := \arg \min_{\alpha \in \mathbb{R}^N} \left( \|\alpha\|_1 + \mathcal{I}_B(\Phi_S\alpha) + \mathcal{I}_{\mathbb{R}_+^N}(\Psi\alpha) \right), \tag{19}
\]

\[
\hat{s} := \arg \min_{s \in \mathbb{R}^N} \left( \|\Psi^*s\|_1 + \mathcal{I}_B(\Phi_As) + \mathcal{I}_{\mathbb{R}_+^N}(s) \right). \tag{20}
\]

The closed forms of the proximal operators for the three functions involved in the objective are listed in Appendix A. What remains is the generalization of the CP algorithm to handle more than two functions in the objective function.

### 6.3 Generalization of CP algorithm to several functions

For completeness, we briefly recall the procedure described in [23] for generalizing the CP algorithm to handle more than two functions in the objective. This is done by expanding the domain of the variable to a product space of the domains of individual objective functions [12].

Let us consider a general situation where we have to minimize a sum \( P + 1 \) proper, convex and lower-semicontinuous functions of \( s \)

\[
\min_{s \in \mathbb{R}^N} \sum_{p=1}^P F_p(K_ps) + H(s), \tag{21}
\]

where \( K_p : \mathbb{R}^N \to \mathbb{R}^{W_p}, 1 \leq p \leq P \) are bounded linear operators.

To bring Eq. (21) into the standard CP formulation in Eq. (15) we consider the \( P \) fold product space \( \mathbb{R}^{PN} = \mathbb{R}^N \times \mathbb{R}^N \ldots \mathbb{R}^N \) (\( P \) times) and a expanded vector \( \bar{s} = [s_1^T, s_2^T, \ldots, s_P^T]^T \in \mathbb{R}^{PN} \), with each \( s_p \in \mathbb{R}^N, 1 \leq p \leq P \). The optimization is carried out in this larger space by enforcing the condition that \( s_1 = s_2 = \ldots = s_P \). This is easily done by considering the consensus sets

\[
\Pi_{1,j} = \{ \bar{s} \in \mathbb{R}^{PN} : s_p = s_1 \}, \quad 2 \leq p \leq P \tag{22}
\]

and their convex indicator functions \( \mathcal{I}_{\Pi_{1,j}} \).

Let the dimension \( W = \sum_p W_p \) and define a new operator \( K : \mathbb{R}^{PN} \to \mathbb{R}^W \), such that \( K := \text{diag}(K_1, K_2, \ldots, K_P) \), a block diagonal operator which acts on individual pieces of a vector \( \bar{s} \). Any vector \( \bar{u} = K\bar{s} \in \mathbb{R}^W \) thus can be broken down into its own pieces \( \bar{u} = [u_1^T, u_2^T, \ldots, u_P^T]^T \), such that \( u_p = K_ps_p \). With this, a new function \( F \) with the domain \( \mathbb{R}^W \) can be defined such that

\[
F(\bar{u}) := \sum_{p=1}^P F_p(u_p). \tag{23}
\]

With these definitions, we can rewrite the problem in Eq. (21) as

\[
\min_{\bar{s} \in \mathbb{R}^{PN}} F(K\bar{s}) + \sum_{p=2}^P \mathcal{I}_{\Pi_{1,p}}(\bar{s}) + H(s_1). \tag{24}
\]

The first term encapsulates all the actions of \( P \) operators and and function evaluations in the form of a single operator and function, the second term enforces that individual \( N \) dimensional
pieces of the solution should be the same and the last term is simply retained from the original form. Finally, by setting \( G(\bar{s}) := \sum_{p=2}^P \eta_{1,p}(\bar{s}) + H(s_1) \), we can recover the original form of the CP algorithm in Eq. (15).

Moreover, the proximal operators of \( F^* \) and \( G \) are easily computable from the proximal operators of \( F_p^* \) and \( H \) as [23]

\[
\text{prox}_{\gamma F^*}(\bar{u}) = \begin{pmatrix} \text{prox}_{\gamma F_1^*}(u_1) \\ \vdots \\ \text{prox}_{\gamma F_P^*}(u_P) \end{pmatrix}
\]

and

\[
\text{prox}_{\gamma G}(\bar{s}) = \mathbb{I}_N^P \text{prox}_{\gamma H} \left( \frac{1}{P} \sum_p s_p \right),
\]

where \( \mathbb{I}_N^P \) is a vertical concatenation of \( P \) identity matrices \( I_N \) of size \( N \times N \)

\[
\mathbb{I}_N^P = [I_N, I_N, \ldots, I_N]^T.
\]

Essentially, the result of the proximal operator of \( G \) is simply the \( P \) fold repetition of the result obtained by the result of the proximal operator of \( H \), operating on the average of the \( P \) individual pieces of \( \bar{s} \).

7 Experimental results

The spread spectrum Hadamard sensing approach to measure and reconstruct deflection spectra was first evaluated with synthetic experiments in an ideal noiseless condition. Subsequently, it was implemented and evaluated on an actual deflectometric system.

7.1 Spectrum reconstruction from synthetic measurements

In Sec. 1, we discussed that objects having smooth surfaces tend to have sparse deflection spectra, i.e., light deflections are concentrated around a primary deflection angle in a bright spectral spot on the spectral plane. If the object has locations which are not smooth, light may deflect in multiple directions and hence corresponding spectra may contain several spectral spots. In this section, we report the results of compressive sensing and reconstruction of synthetically generated sparse deflection spectra, under noiseless conditions. To mimic the content of actual deflection spectra of transparent objects, we generate and use synthetic spectra which contains one or more bright spots for all our experiments.

7.1.1 Data generation

The size of each spectrum was chosen to be \( 64 \times 64 \), so that \( N = 4096 \) with each pixel indexed by an ordered pair of integers \( (x, y) \in [64] \times [64] \), where \( [C] := \{1, 2, \ldots, C\} \). A synthetic spectrum with one spectral spot was synthesized by picking \( \mu = (\mu_x, \mu_y)^T \) uniformly at random in \([1, 64] \times [1, 64] \) and evaluating the function \( s(x) = \exp(-\frac{1}{2\alpha^2}||x - \mu||^2) \) for \( x \in [64] \times [64] \subseteq \mathbb{Z}^2 \), with parameter \( \alpha = 3 \). Deflection spectra with multiple spots were obtained by simply repeating the described procedure as many times as needed.

Typically, light deflections on smooth surfaces tend to be strong around a mean deflection angle and they gradually decay away from the mean angle. Moreover, when the surfaces are symmetric,
the deflection spectral spot is also symmetric and hence Gaussian spots make good candidates for synthetic verification of compressed sensing scheme for deflection spectra.

For a given set $\Omega \subseteq [N]$ of $M$ random indices, a typical vector of compressive measurements was generated using the sensing matrix $\Gamma_\Omega^*$, defined in Eq. (8). No noise was added either to the synthetic spectra or the measurement vector to maintain ideal sensing conditions. The spectra were reconstructed by solving all the three problems listed in Sec. 5 using the Chambolle-Pock algorithm presented in Sec. 6. In all the problems, the corresponding operators were normalized to have unit norm $\|K\| = 1$ and the iterations were chosen to be stopped when the relative error of successive iterates crossed below a threshold $10^{-4}$.

### 7.1.2 Performance measure

In case of synthetic experiments, as the ground truth $s$ was available to us, the reconstruction performance for a given number of measurements $M$ was evaluated by the output SNR defined by

$$\text{SNR}_{out} = 20 \log_{10} \left( \frac{\|s\|_2}{\|s - \hat{s}\|_2} \right), \quad (25)$$

where $\hat{s}$ is the reconstructed spectrum obtained by solving either Eq. (5) or Eq. (10) and $s$ is the true spectrum.

In the case of experiments conducted using actual deflectometric measurements, due to the lack of ground truth. Instead, the reference for comparison will be the spectrum reconstructed using 100% of the measurements, i.e., with $M = N$.

### 7.1.3 Results

Fig. 3 shows a typical synthetic deflection spectrum with (a) one and (b) two deflection spots, without any noise. The first row of the figure array in Fig. 4 contains the results of reconstruction using $M/N = 3.6\%$ of the measurements. The three columns correspond to reconstructions using (i) DWT in synthesis, (ii) UDWT in synthesis and (iii) UDWT in analysis modes.

Fig. 5(a) shows the SNR curve for the three reconstruction modes on synthetically generated data as a function of the ratio $M/N$ in percentage, for deflection spectra containing one and two spots (as depicted in Fig. 3). Each point was obtained as an average over 20 independent trials.\(^3\)

---

\(^3\)Within each trial, the same vector of measurements were input to all the three reconstruction methods.
The synthesis DWT method performs inferior to the methods based on UDWT while the number of measurements are small, and it starts to get comparable to the synthesis UDWT method as the ratio $M/N$ increases. All the three methods can reconstruct deflection spectrum with one spot marginally better than spectrum with two spots, most of the times, with the same number of measurements.

Fig. 5(b) shows the reconstruction performance of all the three methods as a function of the number of spots in the deflection spectrum, for a fixed $M/N = 10.99\%$. As expected, the performance decreases as the number of spots increases. This is in concurrence with the fact that as the number of spots increase in the spectrum, its sparsity (number of non-zero coefficients) increases and hence would actually require larger number of measurements to have a comparable reconstruction quality. In other words, the error terms $\|\alpha - \alpha_K\|_1/\sqrt{K}$ and $\|\Psi^* s - (\Psi^* s)_K\|_1/\sqrt{K}$ in the cases of synthesis and analysis methods (Eq. (6) and Eq. (11)) decay more slowly if $K$ is high.

The synthetic experiments in the ideal conditions show that the method of spread spectrum compressed sensing is capable of reconstructing deflection spectra with good SNR, using a small number of measurements. It also demonstrates the fact that there is a significant gain in reconstruction performance while using UDWT dictionary over ordinary orthogonal wavelet sparsity basis. With this positive note, we shall now present the reconstruction results obtained using actual deflectometric measurements.
7.2 Preliminary deflectometric experiments

An important input information to perform the reconstruction using actual deflectometric data is noise, both intrinsic signal noise and measurement noise. It is important to calibrate the noise so that an appropriate value of $\epsilon$ can be used during the reconstruction process. To this end, we conducted preliminary controlled experiments before conducting the experiments of actual spectrum reconstruction. All the deflectometric experiments were carried out using Lambda-X’s NIMO™[1] (see Fig. 2(b)) deflectometric system, which has a programmable interface to load optical modulation patterns into its SLM.

For all the experiments, we chose to use a region of $64 \times 64$ pixels on the SLM, positioned at its physical centre, for making the measurements. The SLM pixels outside this area were always set to be opaque. Hence, the size of the modulation patterns and the underlying deflection spectra are $64 \times 64$ pixels in size, which in the vectorized version have the size $N = 4096$. The total number of possible modulation patterns is therefore also 4096, because they are drawn from a Hadamard basis for the space of $N$ dimensional vectors.

7.2.1 Noise calibration

The deflectometric system is under the influence of several sources of noise such as the inherent signal noise itself, and many measurement noises such as (a) leakage in SLM, (b) noise in the light source, (c) quantization noise due to limited word length of the CCD and (d) CCD readout noise. Fortunately, the light source of the system can be operated at high intensity and for each compressive measurement, one half of the SLM pixels are turned on (transparent) due to the nature of Hadamard basis. Hence sufficient light is collected at the CCD for every measurement and we can safely ignore the photon counting noise.

To calibrate the noise in the system, deflectometric measurement were made in the absence of any test object. Ideally, in the absence of any test object, all the light rays go undeflected and hence the spectrum should consist of a single peak at the origin. However, the pinhole in the schematic Fig. 2(a) has a finite size and hence the deflection spectrum consists of the image of the pinhole at its origin, which is nothing but a disk of a constant amplitude and 12 pixels in diameter. Further, as there is no test object, the deflection spectrum is agnostic to CCD locations. Let us denote such
Figure 6: (a) Deflection spectrum obtained by inverting the measurements made in the absence of test object, (b) spectrum found by least squares fitting and (c) the difference of the two spectra.

a deflection spectrum by $s^{\text{no}}$.

With the full (i.e., with $M = N$) measurement matrix $\Phi = \Gamma^*$, the measurements on an arbitrary CCD pixel $k$ in the absence of test object is given by:

$$y^{\text{no}}_k = \Phi(s^{\text{no}} + n_s) + n_y,$$

(26)

where $n_s \in \mathbb{R}^N$ is the signal noise, which is independent of the signal, and $n_y \in \mathbb{R}^M$ is the measurement noise. The quantity of interest to us is the norm of the sum of two noises: $\epsilon_N = \|\Phi n_s + n_y\|_2$.

Since in the case of spread spectrum compressed sensing the matrix $\Phi$ is orthonormal when $M = N$ (see Eq. (7)), we simply invert the measurement operation to obtain a noisy version of $s^{\text{no}}$,

$$\tilde{s}^{\text{no}} = s^{\text{no}} + n_s + \Phi^* n_y,$$

(27)

which is shown in Fig. 6(a). While the measurement noise can be modelled as zero mean Gaussian noise, the signal noise however itself is non-negative and hence cannot be treated as zero-mean Gaussian. Instead, we model it as

$$n_s = \bar{n}_s + (\mathbb{E}n_s) \mathbf{1},$$

(28)

a sum of zero mean Gaussian signal noise $\bar{n}_s$ and a constant unknown bias.

Although the ground truth $s^{\text{no}}$ is unknown, thanks to the knowledge of its shape (a disk of certain constant amplitude), we can find a least squares estimate $\hat{s}^{\text{no}}$ by optimizing over the height and the location of the disk. Then, the estimate is subtracted from $\tilde{s}^{\text{no}}$ and the operator $\Phi$ is re-applied to obtain an estimate of the norm of noise

$$\epsilon_N = \|\Phi \hat{n}_s + \Phi (\mathbb{E}n_s) \mathbf{1} + n_y\|_2 \approx \|\Phi (s^{\text{no}} - \hat{s}^{\text{no}})\|_2 = \|y^{\text{no}}_k - \Phi \hat{s}^{\text{no}}\|_2.$$

(29)

This least squares fit is shown in Fig. 6(b) and Fig. 6(c) shows the difference which is purely noise. The estimate thus obtained is actually for the number of measurements $M = N$. In order to scale it to an arbitrary number of measurements $M$, we assume that the noise is Gaussian\(^4\) and use the $\chi^2$ scaling to obtain, for $M < N$

$$\epsilon^2(M) = \frac{M + 2c\sqrt{M}}{N} \epsilon_N, \quad c = O(1),$$

(30)

where the term $\sqrt{M}$ guarantees that the bound holds with a high probability [23]. We just take $c = 1$ in the rest of this paper. This estimate is plugged in the optimization problems for spectrum reconstruction. However, this discrepancy in scaling only manifests in the form of a systematic bias in the reconstructed spectrum, which is tolerable for our purposes.

\(^4\)A similar argument could be given if the noise is assumed to be sub-Gaussian, e.g., for bounded noise [53].
7.2.2 Input SNR

We also define a notion of input SNR using the measurements made without any test object. In the absence of any signal noise and measurement noise, the norm of the measurement vector is simply \( \| \Phi s^{\text{no}} \|_2 \). However, the actual measurement vector \( y \) contains contributions due to signal noise and measurement noise. Therefore, we define a notion of input SNR as

\[
\text{SNR}_{\text{in}} := 20 \log_{10} \left( \frac{\| \Phi s^{\text{no}} \|_2}{\| y - \Phi s^{\text{no}} \|_2} \right).
\]

However, as the ideal deflection spectrum \( s^{\text{no}} \) is not known exactly, an estimate of \( \text{SNR}_{\text{in}} \) is obtained by plugging in the least squares fit \( s = \hat{s}^{\text{no}} \). From our preliminary experiments, the value of \( \text{SNR}_{\text{in}} \) was computed to be 4.8 dB.

7.3 Spectrum reconstruction from deflectometric measurements

In this section, we will present the spectrum reconstruction results for actual test objects. All the deflectometric measurements are made using Lambda-X’s NIMO system™.

7.3.1 Data generation

For experiments with the deflectometric data, we chose two plano-convex lenses of optical powers 9.99\( D \) and 60\( D \). The optical power of a lens is defined as \( D = 1/f \), the reciprocal of its focal length \( f \) and it is measured in the unit \( m^{-1} \) or dioptries (\( D \)). The size of the deflection spectrum to be reconstructed was set to 64 \( \times \) 64 pixels, so that \( N = 4096 \). All the 4096 possible modulation patterns were generated according to Eq. (8) and the corresponding number of measurements for all locations \( k \) were collected on the CCD by loading the modulation patterns one by one into the SLM. For a given value of \( M \), a set of indices \( \Omega \subseteq [N], |\Omega| = M \), was drawn uniformly and the measurements corresponding the indices in \( \Omega \) were selected to form the vector \( y_k \), for a given CCD location \( k \). The spectrum was reconstructed by solving either Eq. (5) or Eq. (10) with either \( \Phi = \Gamma^* \Omega \Sigma \Psi \) or \( \Phi = \Gamma^* \Omega \Sigma \) respectively, for the two choices of sparsity basis \( \Psi \) described in Sec. 5.1. The value of \( \epsilon \) was appropriately set in all the experiments according to the procedure described in Sec. 7.2.1.

7.3.2 Results

Fig. 7 shows typical reconstructions of a deflection spectrum for the 9.99\( D \) lens, at some arbitrary location \( k \). The first row of the array in Fig. 7 are the results of reconstruction from 100\% of the measurements, middle one from 3.6\% of measurements and on the bottom from 6.1\% measurements. The three columns correspond to reconstruction using (i) DWT in synthesis, (ii) UDWT in synthesis and (iii) UDWT in analysis modes. It can be seen from these example reconstructions that the analysis UDWT method gives the best reconstruction for a given number of measurements, and the quality of reconstruction improves as the number of measurements increases for all the three methods.

Fig. 8(a) and Fig. 8(b) show the SNR curves for the three reconstruction methods, for the spectra of the two specified lens (9.99\( D \) and 60\( D \)), as a function of the ratio \( M/N \) in \%. Each point on the plot was obtained by considering 4 different locations on the CCD and performing 20 independent
Figure 7: Reconstruction examples of a spectrum of a 9.99\(D\) plano-convex lens: (a)-(c) correspond to 100\% of measurements, (d)-(f) correspond to 3.6\% of measurements and (g)-(i) correspond to 6.1\% of measurements. The three columns correspond to synthesis with DWT, synthesis with UDWT and analysis with UDWT reconstruction modes.

Figure 8: Output SNR versus number of measurements for all the three reconstruction modes, for a plano-convex lens of optical powers (a) 9.99\(D\) and (b) 60\(D\).
reconstruction trials on each of them for every value of $M/N$ considered and subsequently averaging over all locations and trials. For each new trial, a new index set $\Omega$ was generated and used.

The output SNR steadily increases as the ratio $M/N$ increases for both the lenses and the UDWT based methods consistently outperform the DWT based method by at least 5 dB. However, unlike the synthetic experiments reported in Sec. 7.1, the difference between the performance of analysis UDWT methods is marginal compared to synthesis UDWT method. In case of the 60$D$ lens, the analysis UDWT method tends to perform marginally inferior to the synthesis UDWT method. However, the analysis UDWT method is faster than the synthesis UDWT method as the signal dimensions are significantly smaller for analysis based reconstruction than their synthesis counterpart.

The horizontal dotted line near 5 dB in both the plots is the input SNR, computed in Sec. 7.2.2. To recollect, the input SNR tells us the relative strengths of signal and noise in the input which is used for reconstruction. From the plots, we can infer that the compressive sensing method produces a reconstruction that has better SNR compared to the input SNR. This performance is owed to the fact that the reconstruction procedure has the capability to take advantage of the sparse structure of the signal to carefully discriminate it from noise and suppress the latter.

It is also clear from the performance curves that for a good quality (about 20 dB output SNR) reconstruction, the minimum number of measurements needed is dependent on the reconstruction method used. The performance is always better with UDWT based methods compared to orthonormal wavelet basis. With UDWT methods, the number of measurements needed for a 20 dB reconstruction is about $M/N = 11\%$, which is far lesser than the number of measurements needed according to classical Shannon sampling.

The time required to reconstruct a spectrum is mainly dependent on the kind of reconstruction scheme used. On a Macbook Pro with a 2GHz processor and 4GB RAM, the synthesis DWT method takes about 2 minutes for reconstruction, whereas the synthesis UDWT takes about 13 minutes. This is due to the larger size of the unknown vector. The analysis UDWT takes about 3 minutes, slightly more than the DWT scheme. However, when the ratio $M/N$ increases, the time required for reconstruction marginally decreases. Reconstruction of spectra for all the locations on the object surface is still a bottleneck due to computational requirement, but quicker characterization of objects can be achieved without reconstruction as described in the next section.

\section{Compressive characterization of light deflection spectra}

The final goal of measuring and reconstructing deflection spectra is to characterize and understand the object being studied. Though deflection spectra at individual locations themselves contain rich information about the local geometry of the object, it is essential to see how the spectrum evolves as a function of the spatial location $k$. This evolution pattern is related to the geometry of the object we would like to summarize each deflection spectrum by a few parameters. In this section, we elaborate on how to characterize a spectrum by defining the notion of centrum of a deflection spectrum, \textit{by assuming that each deflection spectrum contains only one deflection spot}, and present a method to compute it using compressive measurements.

The definition of centrum involves a fully reconstructed spectrum, and naturally it is expected that in order to find the centrum of a spectrum it is necessary to reconstruct it. In the case when the centrum is to be found for deflection spectra at all locations of an object, it becomes computationally challenging. In order to overcome this situation, we present a novel method to
estimate the centrum of a deflection spectrum directly from the compressive measurements, without involving reconstruction.

The PSS method in Sec. 3 is also aimed at obtaining local deflection information across the whole surface. However, it is effective and accurate only when each deflection spectra contains only one Dirac spike. Compressed sensing method, on the other hand, considers a generic deflection pattern and hence it is expected to be more effective.

8.1 Parametric characterization of a deflection spectrum

We have seen from the experimental results in Sec. 7.3 that, for sufficiently smooth objects such as plano-convex lenses, the deflection spectrum usually consists of a single bright spot and the deflections are negligible elsewhere. This means, for a given location \( k \) on the object, i.e., for a given CCD pixel location, most of the deflections are concentrated around a mean deflection angle. This mean deflection angle is necessarily dependent on the location for which the spectrum is considered. Furthermore, the shape of the bright spectral spot depends on the local curvature\(^5\) of the object at the considered location \( k \).

Due to the spherical symmetry of a plano-convex lens, the shape of the bright spectral spot tends to remain identical for all the spectra but the location of the bright spot inside the spectral coordinates changes as a function of the location \( k \). Therefore, all the deflection spectra are simply planar translates of each other. The evolution of the deflection spectra across the surface of the object can be obtained by simply tracking the position of the spectral spot inside each spectrum, with respect to an origin. The position encodes information about the mean deflection angle and hence it is a good summarization of the deflection spectrum itself.

Concretely, we designate the geometric centroid of the spectral spot, with respect to a spectral origin \( O \), as its position in a deflection spectrum. The spectral origin \( O \) corresponds to the coordinates in the spectrum for zero deflection and it can be conveniently calibrated from the system. If the reconstructed deflection spectrum is noiseless, then the position of the spectral spot is simply found by computing the geometric centroid of the whole spectrum. In this case, the centroid of the spectral spot coincides with the centroid of the whole deflection spectrum. However, as this is rarely the case, especially when we have few compressive measurements, we need a method which computes the centroid of the spectral spot in isolation by rejecting the values elsewhere in the whole spectrum.

A simple method to locate the spectral spot in noisy conditions is to find the peak of the autocorrelation function of the spectrum. However, this is insufficient because the position of the peak thus computed always lies on the pixel grid, whereas the actual centroid of the spectral spot might lie at a sub-pixel position. Hence, the method should be capable of working at a sub-pixel accuracy.

In an ideal situation when the exact continuous analytical expression for the deflection spectrum is known, we can locally fit the spectrum and find the centroid of the spectral spot most accurately. This fit provides a sub-pixel precision because of the continuous nature of the fitting function. However the exact analytical expression for the spectrum is unknown in reality. Alternatively, we design a template function that has an analytical form and has the same overall appearance (shape and size) as the spectral spot. Then, we use the template to perform a matched filtering with

---

\(^5\)The shape of the spectral spot is a function of both the object geometry and also the deflectometric system parameters. However, as the system parameters remain fixed throughout, we can safely ignore this factor.
the full deflection spectrum to locate the spectral spot. As the template function has analytical expression, then sub-pixel level localization can be achieved with a sufficient accuracy.

In our work, we chose to use a 2-D Gaussian shape $G_\rho$ of variance $\rho$, located on the spectral origin $O$, as the template function. The radius $\rho$ was set by analyzing the width of the spectral spot in a sample reconstruction of deflection spectrum. By letting $g_\rho^\rho$ to denote the vectorized form of $G_\rho$, translated in 2 dimensions by \( \mathbf{r} = (r_x, r_y)^T \in \mathbb{R}^2 \), we propose to find the position of the spectral spot within the deflection spectrum by solving

\[
\mathbf{r}^*_k = \arg \max_{\mathbf{r}} |\langle \hat{s}_k, g_\rho^\rho, \mathbf{r} \rangle|.
\] (32)

The solution $\mathbf{r}^*_k$ tells us by how much the 2-D Gaussian $G_\rho$ has to be translated in 2 dimensions with respect to the origin $O$, so that it matches best with the spectral spot in the estimated deflection spectrum $\hat{s}_k$, and this is precisely the notion of the position of a spectral spot.

As it is evident from the discussion so far, the actual shape of the spectral spot may not be exactly having a Gaussian shape, but we are proposing to localize it using such a shape. This may not be totally disadvantageous because we are only interested in finding the best argument $\mathbf{r}$ that maximizes the inner-product in Eq. (32), and not the value of the matching criterion itself.

Finding $\mathbf{r}^*_k$ in this manner still involves a fully reconstructed spectrum $\hat{s}_k$, which as remarked earlier, can be a computational challenge. However, thanks to the embedding properties of compressed sensing matrices, characterized by the restricted isometry property [20], a similar matched filtering operation can be instead performed on the vector of measurements $\mathbf{y}_k$ itself, without performing any reconstruction.

8.2 Matched filtering using compressive samples

Many common signal processing tasks such as detection, classification and parameter estimation can be performed using the compressive samples, without fully reconstructing the signals [14, 13]. On similar lines, we propose to localize the spectral spot of a deflection spectrum directly from its measurements $\mathbf{y}_k$ by posing the matched filtering problem (32) in the measurement domain, appropriately named smashed filtering [14]

\[
\mathbf{r}^*_k = \arg \max_{\mathbf{r}} |\langle \mathbf{y}_k, \Phi g_\rho^\rho, \mathbf{r} \rangle| = \arg \max_{\mathbf{r}} |\langle \Phi^T \mathbf{y}_k, g_\rho^\rho, \mathbf{r} \rangle|.
\] (33)

Notice that solving (33) is similar to solving (32), but instead of using the fully reconstructed $\hat{s}_k$, we simply apply the adjoint of $\Phi$ to $\mathbf{y}_k$ and perform the matched filtering. This saves a great deal of computational effort. The key reason for this to be successful is that the matrix $\Phi$ has to satisfy the Restricted Isometry Property (RIP) [20]. The Restricted Isometry Constant (RIC) associated with $\Phi$ controls the discrepancy between the solutions of the two problems Eq. (32) and Eq. (33). Smaller the RIC, closer the solutions.

To solve Eq. (33), one needs to evaluate the inner product for every possible translation $\mathbf{r}$ of the template function $G$. However, in practice, one can first efficiently solve Eq. (33) at a pixel level precision through a convolution and peak detection. Subsequently, the finer precision can be achieved either by using a gradient descent or an interpolation technique, around the already found coarse position.

Let us denote the coarse estimate of the centrum, for a spectrum $s_k$, by $\hat{r}_k$. Then, a gradient descent method can use $\mathbf{r} = \hat{r}_k$ as the initialization in Eq. (33) to obtain the best sub-pixel location $\mathbf{r}^*_k$. 

23
Figure 9: Average error in centroid estimation (in pixels) using compressive measurements, as a function of number of measurements, in %.

8.3 Results

For the experimental evaluation of spectral spot estimation using compressive measurements, we retained the configuration described in Sec. 7.3. For each of the 5 CCD locations $k$, the centroid $r_k$ was computed by solving Eq. (33). A “ground truth” centroid $r^*_k$ was also found by solving Eq. (32), using the fully reconstructed spectrum $s_k$ (solving (5)) using $M = 4096$ (100%) measurements and $\Psi$ being the UDWT dictionary.

Fig. 9 shows the centroid computation error $\|r - r^*\|_2$ as a function of the number of measurements $M/N$. Each data point is obtained by averaging 50 independent trials for each value of $M$ over all the five locations. The horizontal dotted line indicates a unit pixel error and it can be seen that the compressive centroid estimation achieves sub-pixel accuracy, even with the number of measurements as low as 2.4% (50 measurements) for the 60D lens, and about 3.7% for the 9.99D lens. This demonstrates the ability of the compressive samples to extract certain parametric information about the underlying signal, without even fully reconstructing it.

The time required to compressively compute the centroid of a single deflection spectra is less than a second on the same machine that was used for spectrum reconstruction. Now that we have a faster method to obtain parameters that localize the spectral spot, we can study how it evolves as a function of spatial dimensions and how it is related to the global geometry of the object.

9 Characterization of objects from deflection spectra

In order to interpret the evolution of deflection spectra along the spatial dimensions of the object, it is necessary to understand the physics of spectrum formation in the schlieren deflectometric system.

The relationship between the object locations $k$ and the position $r_k$ of the spectral spot in the spectrum $s_k$ can be understood with the 1-D schematic in Fig. 10, which is the arrangement of the schlieren lens, a plano-convex lens and the SLM. Note that this is exactly the same as described in Fig. 2(a), without the tele-centric arrangement, which is not necessary for the following discussion.

A ray of light parallel to the optical axis $OO'$, entering the plano-convex lens $L$ at location $k$,
 exits the lens making an angle \( \theta \) with the optical axis. If there is nothing in front of the lens, then light ray meets \( OO' \) at \( Q \), which is at a distance of \( f_o \), the focal length of the plano-convex lens. By assuming that this lens is thin (i.e., its focal length \( f_o \) is much larger compared to its thickness), we have the relation

\[
\tan \theta \approx \frac{d}{f_o}.
\]

(34)

However in the schlieren deflectometer, the light ray enters the schlieren lens making an angle \( \theta \) with \( OO' \), and as described in Sec. 2, the schlieren lens converts this angular deflection into a linear displacement \( r \) from \( OO' \) on the SLM plane. This relationship is given by

\[
r = f \tan \theta.
\]

(35)

Combining Eq. (34) and Eq. (35) we have

\[
r \approx \frac{f}{f_o}d.
\]

(36)

For a fixed focal length \( f \) of the schlieren lens, the position of the spectral spot \( r \) on SLM is a linear function of the location on the object \( d \) and the slope is given by the reciprocal of the focal length \( f_o \) of the plano-convex lens, which is nothing but its dioptric power.

Inverting the relationship, we have

\[
\frac{1}{f_o} \approx \frac{1}{f} \frac{r}{d}.
\]

(37)

By computing the distances \( r \) and \( d \) in physical units (with the knowledge of the pitch of the CCD and the SLM pixels) and knowing the focal length \( f \) of the schlieren lens, we can compute the local optical power for each location \( k \) on the object. For a plano-convex lens, the local optical power is the same across all the object locations.

### 9.1 Characterization of plano-convex lenses

We shall now demonstrate the validity of the relationship Eq. (36) from experimental data. For the two plano-convex lenses of optical powers \( 9.99D \) and \( 60D \), already used in Sec. 7.3, we considered an array of \( 65 \times 65 \) CCD pixels (around the centre of the full CCD array of size \( 1392 \times 1040 \)). The pitch of CCD pixel is \( 7.8989 \text{\mu m} \) and hence a region of \( 65 \times 65 \) pixels corresponds to a small region of \( 513.42 \times 513.42 \text{\mu m}^2 \) on the object’s surface. Then, the positions \( r_k \) of all the spectral spots \( s_k \)
corresponding to the $65 \times 65$ CCD locations were found using the compressive matched filtering method, described in Sec. 8.2, for different values of the number of measurements $M$. The total time required to find the centroid of all the $65 \times 65$ spectra was about one hour and ten minutes on the same machine that was used for the experiments in Sec. 7.

Fig. 11(a) and Fig. 11(b) show the surface plots of $\|r_k\|_2$ as the function of location $k \in \mathbb{R}^2$, computed using $M/N = 100\%$ measurements for the 9.99$D$ and 60$D$ plano-convex lenses respectively. Let us refer to these centroid evolution plots as deflection maps. Note that the axes are in the physical units ($\mu m$) of the object. Firstly, the deflection map corresponding to a lens of higher power is much smoother than the one for lower power. This is because, in case of a high power lens, the spectral spot for consecutive CCD pixels are well separated and hence it is easier for the compressive sensing matched filtering to find it accurately. However, when the lens power is low, the distinction between two consecutive spectral spots is weak and hence the compressive matched filter might err.

As a plano-convex lens has symmetric surface geometry, the deflection map is also radially symmetric from the spectral origin and it has a constant rate in every direction. To robustly estimate the constant rate of centroid evolution, a two dimensional inverted cone is fit to each deflection map, in the least squares sense. The inverted cone is fit by optimizing its slope to minimize its squared error with the deflection map. Table 1 shows the values of the slopes computed for the centroid evolution plots corresponding to the two lenses.

| Optical power | $m_o$ $M/N = 100\%$ | $M/N = 8.5\%$ | Estimated power $M/N = 100\%$ | $M/N = 8.5\%$ |
|---------------|----------------------|---------------|--------------------------------|---------------|
| 9.99$D$ | 0.1128 | 0.1065 | 10.2785$D$ | 9.7115$D$ |
| 60$D$ | 0.6731 | 0.6743 | 61.3544$D$ | 61.4612$D$ |

Table 1: Table of computed slopes and dioptric powers using compressive measurements with $M/N = 100\%$ and $M/N = 8.5\%$ measurements.

Further, by Eq. (37), the dioptric power of the lens is given by

$$\frac{1}{f_o} = \frac{m_o}{f}.$$  \hspace{1cm} (38)

The schlieren deflectometer that was used in our experiments had the schlieren lens of focal length 50$mm$ and a SLM pixel pitch of 36$\mu m$. The third column of Table 1 lists the estimated dioptric powers of the two lenses.

Till now, all the experimental demonstration of the relationship between the slope of the deflection maps and the focal length of the plano-convex lens were done using $100\%(M = N)$ of the measurements. However, we can show that the computations are valid, up to a margin of error, even when we have far lesser number of measurements.

Fig. 11(c)-(d) show the deflection maps for the 9.99$D$ and 60$D$ lenses with $M/N = 8.5\%$ (250) measurements. Note that these maps are generated from one particular random selection of $M$ measurements. Table 1 lists the slopes $m_o$ deduced from the fitted cones as described earlier and the corresponding estimated lens powers.

To assess the accuracy of the dioptric powers computed using the compressive measurements ($M/N < 100\%$), we plot the absolute percentage error (with respect to the true power) in Fig. 12. As expected, in general the accuracy of the estimated optical power of the plano-convex lens, using
Figure 11: Evolution of deflection spectrum along the spatial locations of the object, computed using (a)-(b) $M/N = 100\%$ and (c)-(d) $M/N = 8.5\%$ of measurements.
compressive samples improves as the number of measurements increases. The systematic error in
the power estimation of the 60D lens is unexplained and it may be arising from the bad calibration
of the system parameters.

We see that the error incurred in power computation for the high powered lens is smaller than
that of low power lens. As reasoned out earlier, the accuracy of spectral spot estimation is good
for a high powered lens as compared to a low powered one and hence it translates to better power
estimation. However, the estimation error is well below 6% of the true power even for the lower
power lens. It is also important to remember that there is no deflection spectrum reconstruction
involved whatsoever in all the calculations and hence the technique is actually performing under
the input SNR conditions (4.8 dB), which is quite poor.

9.2 Characterization of diffractive multi-focal intra-ocular lenses

In this section, we shall demonstrate the ability of the compressive centroid method to characterize
a complicated object such as a diffractive Multifocal Intra-Ocular Lens (MIOL). Multifocal lenses,
as the name suggests, have more than one focal point and have non-smooth surfaces by design.
In diffractive MIOLs, this feature is achieved by carving gratings having sawtooth profile on the
surface of a normal uni-focal refractive lens. The gratings make the light diffract at multiple orders
thereby resulting in multiple foci. Fig. 13(a) shows an example profile of a diffractive MIOL. The
blue dotted curve represents the original refractive lens on which grooves are carved.

The diffraction orders, and hence the foci, are defined by the groove widths. The overall
curvature of the lens, which is defined by the refractive aspect of the diffractive lens, controls the
location of zero-order diffraction, which is usually utilized for far distance vision.

When analyzing such a lens using compressive centroid method, we expect to observe the
 evolution of the deflection spectra that traces the underlying shape of the lens. For the study
reported here, we considered a multifocal diffractive IOL having two dioptric powers (hence, two
foci): 28D and 30.25D. In order to cover a larger range of the deflections, we considered a region of
256 × 256 pixels around the centre of the SLM. On the CCD array, the deflections were computed
using compressive centroid method on an array of 500 × 500 pixels, which corresponds to a physical
area of 3.95mm × 3.95mm. As the total estimated time (from previous experiments) to compute
the centroids at all the locations was very high, a computational grid with several engines was made
use of and the total time required to do all the estimations was about five hours.

Fig. 13(c) shows the 3D surface plot of centroid evolution of the considered diffractive MIOL, where the centroids were computed using $M/N = 7.63\%$ measurements, which corresponds to 5000 measurements out of the possible 65536. Similarly, Fig. 13(d) shows the centroid evolution computed using $M/N = 0.76\%$ measurements. The unusual peaks in the corners are due to the CCD pixels outside the boundary of the object. The prominent feature of a multifocal IOL, the grating pattern is clearly reproduced in both the surface plots. The central region which has the same structure as a refractive lens reproduces the cone like structure observed in Sec. 9.1.

The same lens was also analyzed with the phase shifting (PSS) method, described in Sec. 3, without disturbing the experimental setup. The PSS method provides horizontal and vertical deflection values $r_k^{\text{PSS}} \in \mathbb{R}^2$ for each CCD pixel $k$, from which a 3D deflection map is constructed by first computing the norm $\|r_k^{\text{PSS}}\|_2$, and appropriately rescaling them (accounting for unknown PSS algorithmic parameters) to make them comparable with the deflection map provided by the compressive sensing method. Fig. 13(b) shows the 3D deflection map of the diffractive MIOL obtained by the PSS method.

At the outset, the deflection map provided by the PSS method also contains the feature of the
diffractive MIOL. Therefore, for comparison, Fig. 14(a) and Fig. 14(b) display cross sections\(^6\) of the deflection maps obtained from CS method using \(M/N = 0.76\%\) and \(M/N = 7.63\%\) measurements. For comparison, the corresponding cross-section of the deflection map obtained using PSS method is also overlaid. It can be seen from the two plots that even though both the methods agree in the refractive region of the lens, the CS method provides a smoother deflection map in the diffractive region compared to the PSS method.

The PSS method relies on a phase-unwrapping stage in its implementation and hence neighbouring pixels influence the deflection information at a given pixel. However, the CS method independently estimates the deflections at each pixel and hence they are more robust to errors in neighbouring pixels. Therefore, even though both methods recover similar deflection maps, the one obtained using the CS method is more reliable.

Additionally, as discussed earlier in this section, diffractive MIOLs have multiple foci and hence deflection spectrum is expected to have multiple spectral spots. This information is impossible to extract from the PSS method, whereas with the CS method, we can reconstruct the underlying spectrum from the measurements to actually visualize the light deflection pattern.

Fig. 15(a) shows a zoom of the reconstructed spectrum from \(M/N = 7.63\%\) measurements for the location indicated by a circle (\(\bigcirc\)) mark in Fig. 14(a). The blue cross mark (X) indicates the deflection as estimated by the PSS method and the red cross mark indicates the deflection estimated by the compressive centroid estimation method. Interestingly, the deflection spectrum clearly has an elongated profile and seems to be made up of two spots. However, the PSS method has no way of considering this fact in its computations.

Fig. 15(a) shows a zoom of the reconstructed spectrum from \(M/N = 7.63\%\) measurements for another location indicated by a diamond (\(\bigdiamond\)) mark in Fig. 14(a). For this location, the deflection spectrum resembles a deflection spectrum of a plano-convex lens. However, the estimate due to the CS method seems more accurate at the centre of the spectral spot than the one provided by the PSS method. This is possibly a reason for the smoothness of the overall deflection map computed using the CS method.

In order to measure the surface smoothness, we compute the total-variation (TV) norm of

\(^6\)The cross-sections are selected in the horizontal direction such that the corresponding minima are included.
the deflection maps. The total-variation norm of a 2-dimensional discrete signal $I(x, y)$ is defined as \[47, 43\]

$$\|I\|_{TV} = \sum_{x,y} \|\nabla I(x, y)\|_2, \quad (39)$$

where $\nabla I(x, y) \in \mathbb{R}^2$ is the gradient (horizontal and vertical) vector of the image at coordinates $(x, y)$. The TV norm measures the amount of gradient in a signal, and hence signals with smooth transitions tend to have smaller TV norm and vice-versa.

We restrict the computation of the TV norm of deflection maps to a region of $400 \times 400$ pixels at the centre to avoid the CCD pixels outside the boundary of the object. Fig. 16 shows a plot of TV norm of the deflection map obtained by the CS method as a function of the number of measurements $M/N$. For comparison, the TV norm of the deflection map obtained by the PSS method is also plotted, which is independent of the number of measurements. We see from the plot that the smoothness of the deflection map obtained by the CS method improves as the number of measurements increase. When the number of measurements is at least 3%, the CS method produces deflection maps smoother than the one provided by the PSS method.

While the CS method needs more number of measurements to provide a smoother deflection map than the PSS method, CS method has the unique ability to provide individual spectrum reconstruction, from which richer information about deflection can be deduced. This is very important in the case where deflection spectrum contains multiple spots, the case which PSS is totally incapable to handle. Also, there is no spatial phase unwrapping involved which makes the CS method robust.

In addition to bringing richer information about deflections, CS method relies only on binary modulation patterns. This fully avoids the SLM non-linearities which is inherent to PSS method and makes itself amenable to be implemented with faster digital micro-mirror arrays, which are much faster than the SLM technology.

### 10 Conclusions and perspectives

This paper presents a compressed sensing approach to acquire and reconstruct deflection spectra of transparent objects, using schlieren deflectometry. The design of the sensing matrix considers
the practical aspects of optical implementation and also algorithmic implementation. A noise calibration procedure is also described which provides a reasonable bound on the inherent system noise, which is then used to tune the reconstruction algorithm. The experimental results show a great reduction in the required number of measurements for a good reconstruction SNR, thereby demonstrating the power of compressed sensing. Moreover, by decoupling the sensing and the reconstruction stages, compressed sensing framework provides a flexibility to tailor the reconstruction method by using appropriate sparsity prior.

As a second major contribution, the paper contains a method to extract relevant low-dimensional parameters of the deflection spectra can be directly extracted from the compressive measurements, thereby saving computations. This helps in quick characterization of the shape of the object under study and also provides estimates of optical parameters such as dioptic power.

A fully reconstructed deflection spectrum is certainly rich in information and contains a lot more than simply a mean deflection angle. Though reconstruction is a computationally intensive task, it can be selectively used to pinpoint features at particular locations on the object. These locations could be guided by a first analysis provided by the compressive characterization which is computationally economical. The accuracy of the compressive characterization method is certainly dependent on the dioptic power of the object as well as the number of measurements one can afford. However, accuracy versus complexity tradeoff is always a design choice.

While the applicability of compressed sensing for deflectometric imaging is very promising, this is at best a good starting point for further exploration. Firstly, the reconstruction method can possibly be improved for speed so that it is viable to reconstruct deflection spectra at all CCD locations, thereby enabling the user to exploit the rich information in deflection spectra. Simultaneous spectra reconstruction involving low-rank models and other techniques could be investigated. A thorough analysis of the optical system noise also helps in better spectrum reconstruction quality.

While the compressive characterization of deflection spectra presented in this paper is capable of providing reasonable information about the objects, it can be further made more robust by having more parameters in addition to the translation parameters. Modelling the underlying deflection
spectra with complex functions that are closer to reality will result in much better estimation of parameters. This requires a thorough understanding of the physics of deflection spectrum formation.

Acknowledgements

Prasad Sudhakar is funded by the DETROIT project (WIST3), convention no. 1017073, Walloon Region, Belgium. Laurent Jacques is supported by the Belgian FRS-FNRS fund. Computational resources were provided by the supercomputing facilities of the Université catholique de Louvain (CISM/UCL) and the Consortium des Équipements de Calcul Intensif en Fédération Wallonie Bruxelles (CÉCI) funded by the F.R.S.-FNRS under convention 2.5020.11.

A Proximal operators of $\|u\|_1$, $\iota_B(u)$ and $\iota_{\mathbb{R}_+^N}(u)$

The proximal operator for the $\ell_1$-norm function is the simple component wise soft thresholding operator, defined here for a scalar $u$:

$$\text{prox}_{\gamma \|\cdot\|_1}(u) := \begin{cases} 0 & \text{if } |u| \leq \gamma, \\ (|u| - \gamma) \text{sgn}(u) & \text{otherwise}. \end{cases}$$

(40)

The proximal operator of the function $\iota_B(u)$ is simply the following function that implements a projection onto the convex set $\mathcal{B}$:

$$\text{prox}_{\iota_B}(u) := y + (u - y) \min \left(1, \frac{\epsilon}{\|u - y\|_2}\right).$$

(41)

The proximal operator of the function $\iota_{\mathbb{R}_+^N}(u)$ is simply the projection operator onto the non-negative orthant:

$$\text{prox}_{\iota_{\mathbb{R}_+^N}}(u) := \begin{cases} u_i & \text{if } u_i \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$

(42)

References

[1] Lambda-X. http://www.lambda-x.com.

[2] S. Almazán-Cuéllar and D. Malacara-Hernández. Two-step phase-shifting algorithm. *Optical Engineering*, 42(12):3524–3531, 2003.

[3] RG Baraniuk. Compressive sensing. *IEEE Signal Processing Magazine*, 24(4):118, 2007.

[4] H. H. Bauschke and P. L. Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. Springer, 2011.

[5] D. Beghuin, J. L. Dewandel, L. Joannes, E. Fournou, and P. Antoine. Optical deflection tomography with the phase-shifting schlieren. *Optics letters*, 35(22):3745–3747, 2010.

[6] J. Bobin, J-L Starck, and R. Ottensamer. Compressed sensing in astronomy. *Selected Topics in Signal Processing, IEEE Journal of*, 2(5):718–726, Oct 2008.
[7] S P Boyd and L Vandenberghe. Convex optimization, 2004.

[8] E Candes, Y Eldar, and D Needell. Compressed sensing with coherent and redundant dictionaries. *Arxiv preprint arXiv:1005.2613*, Jan 2010.

[9] E. Candes and J. Romberg. Sparsity and incoherence in compressive sampling. *Inverse problems*, 23(3):969, 2007.

[10] E.J. Candes and T. Tao. Near-optimal signal recovery from random projections: Universal encoding strategies? *Information Theory, IEEE Transactions on*, 52(12):5406–5425, 2006.

[11] A. Chambolle and T. Pock. A first-order primal-dual algorithm for convex problems with applications to imaging. *Journal of Mathematical Imaging and Vision*, 40(1):120–145, May 2011.

[12] Patrick L Combettes and Jean-Christophe Pesquet. Proximal splitting methods in signal processing. *Fixed-Point Algorithms for Inverse Problems in Science 185 and Engineering*, Jan 2011.

[13] M. A. Davenport, P. T. Boufounos, M. B. Wakin, and R. G. Baraniuk. Signal processing with compressive measurements. *IEEE Journal on Selected Topics in Signal Processing*, 4(2):445–460, 2010.

[14] M. A. Davenport, M. F. Duarte, M. B. Wakin, J. N. Laska, D. Takhar, K. F. Kelly, and R. G. Baraniuk. The smashed filter for compressive classification and target recognition. In *Proceedings of Computational Imaging V at SPIE Electronic Imaging*, San Jose, CA, Jan. 2007.

[15] T.P. Davies. Schlieren photography a short bibliography and review. *Optics and Laser Technology*, 13(1):37 – 42, 1981.

[16] DL Donoho. Compressed sensing. *IEEE Transactions on Information Theory*, 52(4):1289–1306, 2006.

[17] M. F. Duarte, M. A. Davenport, D. Takhar, J. N. Laska, T. Sun, K. F. Kelly, and R. G. Baraniuk. Single-pixel imaging via compressive sampling. *IEEE Signal Processing Magazine*, 25(2):83–91, Mar. 2008.

[18] M. Elad, P Milanfar, and R Rubinstein. Analysis versus synthesis in signal priors. *Inverse Problems*, 23:947, 2007.

[19] Michael Elad. *Sparse and Redundant Representations: From Theory to Applications in Signal and Image Processing*. Springer, 1st edition, 2010.

[20] Simon Foucart and Holger Rauhut. *A Mathematical Introduction to Compressive Sensing (Birkhäuser series in applied and numerical harmonic analysis)*. Birkhäuser, 2013.

[21] Karsten Fyhn, Marco F Duarte, and Søren Holdt Jensen. Compressive Parameter Estimation for Sparse Translation-Invariant Signals Using Polar Interpolation. *arXiv.org*, May 2013.

[22] M. Golbabaee, S. Arberet, and P. Vandergheynst. Compressive source separation: Theory and methods for hyperspectral imaging. *Image Processing, IEEE Transactions on*, 22(12):5096–5110, Dec 2013.
[23] A. Gonzalez, L. Jacques, C. De Vleeschouwer, and P. Antoine. Compressive optical deflectometric tomography: A constrained total-variation minimization approach. *Inverse Problems and Imaging Journal*, 8(2):421–457, 2014.

[24] P. Hariharan, B. F. Oreb, and T. Eiju. Digital phase-shifting interferometry: a simple error-compensating phase calculation algorithm. *Appl. Opt.*, 26(13):2504–2506, Jul 1987.

[25] Jarvis Haupt, Waheed U. Bajwa, Gil Raz, and Robert Nowak. Toeplitz compressed sensing matrices with applications to sparse channel estimation. *IEEE Transactions on Information Theory*, 56(11):5862–5875, November 2010.

[26] Ivo Ihrke and Marcus Magnor. Image-based tomographic reconstruction of flames. In *Proceedings of the 2004 ACM SIGGRAPH/Eurographics Symposium on Computer Animation, SCA ’04*, pages 365–373, Aire-la-Ville, Switzerland, Switzerland, 2004. Eurographics Association.

[27] L. Joannes, D. Beghuin, R. Ligot, S. Farinotti, and O. Dupont. High-resolution shape measurements with phase-shifting schlieren (PSS), 2004.

[28] L. Joannes, F. Dubois, and J. C. Legros. Phase-shifting schlieren: high-resolution quantitative schlieren that uses the phase-shifting technique principle. *Applied optics*, 42(25):5046–5053, 2003.

[29] S. A. Klein. Understanding the diffractive bifocal contact lens. *Optometry and Vision Science*, 70(6):439–60, 1993.

[30] G Kutyniok. Compressed sensing: Theory and applications. *Arxiv preprint arXiv:1203.3815*, Jan 2012.

[31] M Lustig, D Donoho, and J M Pauly. Sparse MRI: The application of compressed sensing for rapid MR imaging. *Magnetic Resonance in Medicine*, 58(6), 2007.

[32] M Lustig, D.L Donoho, J M Santos, and J M Pauly. Compressed Sensing MRI. *IEEE Signal Processing Magazine*, 25(2):72–82.

[33] Daniel Malacara. *Optical Shop Testing (Wiley Series in Pure and Applied Optics)*. Wiley-Interscience, 2007.

[34] S. Mallat. *A Wavelet Tour of Signal Processing: The Sparse Way*. Academic Press, 3rd edition, 2008.

[35] F.M. Naini, R. Gribonval, L. Jacques, and P. Vandergheynst. Compressive sampling of pulse trains: Spread the spectrum! In *Acoustics, Speech and Signal Processing, 2009. ICASSP 2009. IEEE International Conference on*, pages 2877–2880, April 2009.

[36] Sangnam Nam, Mike E Davies, Michael Elad, and Rémi Gribonval. The cosparse analysis model and algorithms. *Applied and Computational Harmonic Analysis*, 34(1):30–56, 2013.

[37] Neal Parikh and Stephen Boyd. Proximal algorithms. *Foundations and Trends in optimization*, 1(3):123–231, 2013.
[38] G. Puy, J.P. Marques, R. Gruetter, J. Thiran, D. Van De Ville, P. Vandergheynst, and Y. Wiaux. Spread spectrum magnetic resonance imaging. *Medical Imaging, IEEE Transactions on*, 31(3):586–598, March 2012.

[39] G. Puy, P. Vandergheynst, R. Gribonval, and Y. Wiaux. Universal and efficient compressed sensing by spread spectrum and application to realistic fourier imaging techniques. *EURASIP Journal on Advances in Signal Processing*, 2012:1–13, 2012.

[40] H. Rauhut. Compressive sensing and structured random matrices. *Theoretical Foundations and Numerical Methods for Sparse Recovery*, 2010.

[41] H. Rauhut and M. Kabanza. Analysis $\ell_1$-recovery with frames and gaussian measurements. *CoRR*, abs/1306.1356, 2013.

[42] J. Romberg. Compressive sensing by random convolution. *SIAM Journal on Imaging Sciences*, 2(4):1098–1128, 2009.

[43] Leonid I Rudin, Stanley Osher, and Emad Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D: Nonlinear Phenomena*, 60(1):259–268, 1992.

[44] G. Sauter. Goniophotometry: new calibration method and instrument design. *Metrologia*, 32(6):685, 1995.

[45] G. S. Settles. *Schlieren and Shadowgraph Techniques: Visualizing Phenomena in Transparent Media*. Springer, New York, NY, USA, 2001.

[46] G.S. Settles. Colour-coding schlieren techniques for the optical study of heat and fluid flow. *International Journal of Heat and Fluid Flow*, 6(1):3–15, 1985.

[47] Jean-Luc Starck, Fionn Murtagh, and Mohamed-Jalal Fadili. *Sparse Image and Signal Processing - Wavelets, Curvelets, Morphological Diversity*. Cambridge University Press, 2010.

[48] Vincent Studer, Jérôme Bobin, Makhlad Chahid, Hamed Shams Mousavi, Emmanuel Candès, and Maxime Dahan. Compressive fluorescence microscopy for biological and hyperspectral imaging. *Proceedings of the National Academy of Sciences*, 109(26):E1679–E1687, 2012.

[49] P. Sudhakar, L. Jacques, X. Dubois, P. Antoine, and L. Joannes. Compressive acquisition of sparse deflectometric maps. In *Sampling Theory and Applications (SampTA)*, 2013.

[50] P. Sudhakar, L. Jacques, X. Dubois, P. Antoine, and L. Joannes. Compressive schlieren deflectometry. In *Acoustics, Speech and Signal Processing (ICASSP), 2013 IEEE International Conference on*, pages 5999–6003, 2013.

[51] J A Tropp, M B Wakin, M F Duarte, D Baron, and R G Baraniuk. Random filters for compressive sampling and reconstruction. *Proc. Int. Conf. Acoustics, Speech, Signal Processing (ICASSP)*, 2006.

[52] Y. Tsaig and D. L. Donoho. Extensions of compressed sensing. *Signal Processing*, 86(3):549–571, March 2006.
[53] Roman Vershynin. Introduction to the non-asymptotic analysis of random matrices. *Compressed Sensing and Applications*, 2010.

[54] Y. Wiaux, L. Jacques, G. Puy, A. M. M. Scaife, and P. Vandergheynst. Compressed sensing imaging techniques for radio interferometry. *Monthly Notices of the Royal Astronomical Society*, 395(3):1733–1742, 2009.

[55] Rebecca M Willett, Roummel F Marcia, and Jonathan M Nichols. Compressed sensing for practical optical imaging systems: a tutorial. *Optical Engineering*, 50(7):1–14, July 2011.

[56] Wotao Yin, Simon Morgan, Junfeng Yang, and Yin Zhang. Practical compressive sensing with Toeplitz and circulant matrices, 2010.