APPLICATIONS OF FRACTIONAL CALCULUS TO
k-UNIFORMLY STARLIKE AND k-UNIFORMLY
CONVEX FUNCTIONS OF ORDER α

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Abstract. A new subclass of analytic functions $k-S^{\lambda}_{P}(\alpha)$ is introduced by applying certain
operators of fractional calculus to $k$-uniformly starlike and $k$-uniformly convex functions of order
$\alpha$. Some theorems on coefficient bounds and growth and distortion theorems for this subclass
are found. The radii of close to convexity, starlikeness and convexity for this subclass is also
derived.

1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n\quad (1.1)$$

analytic and normalized in the open unit disk $U = \{ z : |z| < 1 \}$ and $S$ denote the subclass of $A$ that are univalent in $U$. Let $T$ denote the subclass of $S$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0\quad (1.2)$$

We denote by $ST$ and $CV$ the subclasses of $S$ that are respectively starlike and convex. Goodman [2,3] introduced and defined the following subclasses of $CV$ and $ST$.

A function $f(z)$ is uniformly convex (uniformly starlike) in $U$ if $f(z)$ is in $CV(ST)$ and has the property that for every circular arc $\gamma$ contained in $U$, with center $\xi$ also in $U$, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\xi)$. The class of uniformly convex functions is denoted by $UCV$ and the class of uniformly starlike functions by $S_{p}$. From Ronning [7] and Ma and Minda [6], it is well known that

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \left| \frac{zf'(z)}{f(z)} - 1 \right|.$$ 

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and
\[ f \in UCV \iff \text{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq \left| \frac{zf''(z)}{f'(z)} \right|. \]

Note that \( f(z) \in UCV \iff zf'(z) \in S_p. \)

Kanas and Wisniowska [4][5] defined the functions \( f \in S_p \) to be \( k \)-uniformly convex \((k\text{-uniformly starlike}) \) if for \( 0 \leq k < \infty \) the image of every circular arc \( \gamma \) contained in \( U \) with center \( \xi \) where \( \xi \leq k \) is convex (starlike). In the same period, Kanas and Srivastava [11] studied extensively on linear operators associated with \( k \)-uniformly convex functions.

Bharati, Parvatham and Swaminathan [1] defined \( k^{-SP}(\alpha) \) to be the class of functions \( f(z) \) of the form (1.1) with \( 0 \leq k < \infty \) and \( 0 \leq \alpha < 1 \) that satisfies the condition
\[ \text{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq k \left| \frac{zf''(z)}{f'(z)} \right| + \alpha. \] (1.3)

In [1], \( k^{-UCV}(\alpha) \) is defined to be the class of functions \( f(z) \) of the form (1.1) with \( 0 \leq k < \infty \) and \( 0 \leq \alpha < 1 \) that satisfies the condition
\[ \text{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq k \left| \frac{zf''(z)}{f'(z)} \right| + \alpha. \] (1.4)

Owa and Srivastava [7] introduced the operator \( \Omega : A \to A \) defined by
\[ \Omega^\lambda f(z) := \Gamma(2 - \lambda)z^\lambda D^\lambda_z f(z), \quad (\lambda \neq 2, 3, 4...) \] (1.5)
where \( D^\lambda_z f(z) \) is the fractional derivative of \( f \) of order \( \lambda \) defined by Owa [8] to be
\[ D^\lambda_z f(z) = \frac{1}{\Gamma(1 - \lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z - \xi)^\lambda} d\xi, \quad 0 \leq \lambda < 1. \] (1.6)

Following the work of Srivastava and Mishra [10] we introduce a class of analytic functions related to \( k^{-SP}(\alpha) \) and \( k^{-UCV}(\alpha) \) using the operator \( \Omega^\lambda \) defined by (1.5).

**Definition 1.1.** Let \( f \) be of the form (1.1), \( 0 \leq k < \infty \), and \( 0 \leq \alpha < 1 \). Then \( f \in k^{-SP}\lambda(\alpha) \) if and only if \( f \) satisfies the condition
\[ \text{Re}\left\{ \frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)} \right\} \geq k \left| \frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)} \right| - 1 \] + \alpha. \] (1.7)

It follows that \( k^{-SP}_0(\alpha) \equiv k^{-SP}(\alpha) \) and \( k^{-SP}_1(\alpha) \equiv k^{-UCV}(\alpha) \).

Define \( k^{-TSP}_\lambda(\alpha) \) to be \( k^{-TSP}_\lambda(\alpha) = k^{-SP}_\lambda(\alpha) \cap T. \)

The aim of this paper is to investigate several basic properties of the class \( k^{-TSP}_\lambda(\alpha). \)
2. Coefficient estimates

**Theorem 2.1** Let $f$ be of the form (1.1). Then for $0 \leq k < \infty$ and $0 \leq \alpha < 1$, $f \in k - SP_\alpha (\alpha)$ if

$$
\sum_{n=2}^{\infty} [n(1 + k) - (\alpha + k)] \frac{\Gamma(2 - \lambda) \Gamma(n + 1)}{\Gamma(n + \lambda + 1)} |a_n| \leq 1 - \alpha.
$$

(2.1)

**Proof.** It suffices to show that

$$
k \left| \frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)} - 1 \right| - \text{Re} \left\{ \frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)} - 1 \right\} \leq 1 - \alpha.
$$

First of all, it is easily seen from (1.6) that

$$
D_z \{z^n\} = \frac{\Gamma(n + 1)}{\Gamma(n - \lambda + 1)} z^{n-\lambda}.
$$

We have

$$
k \left| \frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)} - 1 \right| - \text{Re} \left\{ \frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)} - 1 \right\} \leq (1 + k) \left| \frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)} - 1 \right| \leq \frac{(1 + k) \sum_{n=2}^{\infty} (n - 1) \frac{\Gamma(2 - \lambda) \Gamma(n + 1)}{\Gamma(n - \lambda + 1)} |a_n|}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(2 - \lambda) \Gamma(n + 1)}{\Gamma(n - \lambda + 1)} |a_n|}
$$

The last expression is bounded above by $(1 - \alpha)$ if

$$
\sum_{n=2}^{\infty} [n(1 + k) - (\alpha + k)] \frac{\Gamma(2 - \lambda) \Gamma(n + 1)}{\Gamma(n + \lambda + 1)} |a_n| \leq 1 - \alpha
$$

and hence the proof is complete.

**Theorem 2.2.** A necessary and sufficient condition for $f$ of the form (1.2) to be in the class $k - TSP_\alpha (\alpha)$ for $0 \leq k < \infty$ and $0 \leq \alpha < 1$ is that

$$
\sum_{n=2}^{\infty} [n(1 + k) - (\alpha + k)] \frac{\Gamma(2 - \lambda) \Gamma(n + 1)}{\Gamma(n - \lambda + 1)} |a_n| \leq 1 - \alpha.
$$

(2.2)

**Proof.** In view of Theorem 2.1, we need only to prove that if (2.1) is true then $f(z) \in k - TSP_\alpha (\alpha)$. Now suppose $f(z) \in k - TSP_\alpha (\alpha)$ and $z$ is real, then

$$
\left| \frac{1 - \sum_{n=2}^{\infty} \frac{\Gamma(2 - \lambda) \Gamma(n + 1)}{\Gamma(n - \lambda + 1)} a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(2 - \lambda) \Gamma(n + 1)}{\Gamma(n - \lambda + 1)} a_n z^{n-1}} \right| \geq k \left| \sum_{n=2}^{\infty} \frac{(1 - n) \frac{\Gamma(2 - \lambda) \Gamma(n + 1)}{\Gamma(n - \lambda + 1)} a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(2 - \lambda) \Gamma(n + 1)}{\Gamma(n - \lambda + 1)} a_n z^{n-1}} \right|.
$$
Letting $z \to 1$ along the real axis we obtain
\[
(1 - \alpha) - \sum_{n=2}^{\infty} \frac{n(1 + k) - (\alpha + k) \Gamma(2 - \lambda) \Gamma(n + 1)}{\Gamma(n - \lambda + 1)} |a_n| 
\]
\[
1 - \sum_{n=2}^{\infty} \frac{\Gamma(2 - \lambda) \Gamma(n + 1)}{\Gamma(n - \lambda + 1)} |a_n| 
\]
This is only possible if (2.1) holds. Therefore we obtain the desired results.

3. Growth and distortion theorems

In this section we prove some growth and distortion theorems for the subclass $k - TSP_{\lambda}(\alpha)$.

**Theorem 3.1.** If $f(z) \in k - TSP_{\lambda}(\alpha)$ then
\[
r - \frac{(1 - \alpha)(2 - \lambda)}{2(2 - \alpha + k)} r^2 \leq |f(z)| \leq r + \frac{(1 - \alpha)(2 - \lambda)}{2(2 - \alpha + k)} r^2 \quad (|z| = r)
\]
with equality for
\[
f(z) = z - \frac{(1 - \alpha)(2 - \lambda)}{2(2 - \alpha + k)} z^2 \quad (z = \mp r).
\]

**Proof.** Since $f(z) \in k - TSP_{\lambda}(\alpha)$ by applying assertion (2.1) of Theorem 2.1, we obtain
\[
\frac{[2(1 + k) - (\alpha + k)] \Gamma(2 - \lambda) \Gamma(3)}{\Gamma(3 - \lambda)} \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \frac{n(1 + k) - (\alpha + k) \Gamma(2 - \lambda) \Gamma(n + 1)}{\Gamma(n - \lambda + 1)} |a_n| \leq 1 - \alpha
\]
which immediately yields
\[
\sum_{n=2}^{\infty} |a_n| \leq \frac{(1 - \alpha)(2 - \lambda)}{2(2 - \alpha + k)}. \quad (3.1)
\]
From (1.2) and (3.1) we have
\[
|f(z)| \leq |z| + \sum_{n=2}^{\infty} |a_n| |z|^n \leq |z| + \frac{(1 - \alpha)(2 - \lambda)}{2(2 - \alpha + k)} |z|^2.
\]
Also from (1.2) and (3.1)
\[
|f(z)| \geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^n \geq |z| - \frac{(1 - \alpha)(2 - \lambda)}{2(2 - \alpha + k)} |z|^2.
\]
Thus, the proof of Theorem 3.1 is complete.
**Theorem 3.2.** If \( f(z) \in k - TSP_\lambda(\alpha) \) then
\[
1 - \frac{(1 - \alpha)(2 - \lambda)}{2 - \alpha + k} r \leq |f'(z)| \leq 1 + \frac{(1 - \alpha)(2 - \lambda)}{2 - \alpha + k} r \quad (|z| = r)
\]
with equality for
\[
f(z) = z - \frac{(1 - \alpha)(2 - \lambda)}{2(2 - \alpha + k)} z^2 \quad (z = \mp r).
\]

**Proof.** From the proof of Theorem 3.1
\[
|f(z)| \leq |z| + \frac{(1 - \alpha)(2 - \lambda)}{2(2 - \alpha + k)} |z|^2.
\]
Therefore
\[
|f'(z)| \leq 1 + \frac{2(1 - \alpha)(2 - \lambda)}{2(2 - \alpha + k)} |z|.
\]
Also, from the proof of Theorem 3.1,
\[
|f(z)| \geq |z| - \frac{(1 - \alpha)(2 - \lambda)}{2(2 - \alpha + k)} |z|^2.
\]
Therefore
\[
|f'(z)| \geq 1 - \frac{2(1 - \alpha)(2 - \lambda)}{2(2 - \alpha + k)} |z|.
\]

4. **Close-to-convexity, starlikeness and convexity**

For some \( 0 \leq \delta < 1 \) and all \( z \in U \):
A function \( f \in T \) is said to be close-to-convex of order \( \delta \) if it satisfies
\[
\Re \{ f'(z) \} > \delta.
\] (4.1)
A function \( f \in T \) is said to be starlike of order \( \delta \) if it satisfies
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \delta.
\] (4.2)
A function \( f \in T \) is said to be convex of order \( \delta \) if and only if \( zf'(z) \) is starlike of order \( \delta \) that is, if
\[
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \delta.
\] (4.3)

**Theorem 4.1.** If \( f(z) \in k - TSP_\lambda(\alpha) \) then \( f \) is close-to-convex of order \( \delta \) in \( |z| < r_1(k, \alpha, \lambda, \delta) \) where
\[
r_1(k, \alpha, \lambda, \delta) = \inf_n \left[ \frac{(1 - \delta)n(1 + k) - (\alpha + k)\Gamma(2 - \lambda)\Gamma(n + 1)}{n(1 - \alpha)\Gamma(n - \lambda + 1)} \right]^{1/n}.
\]
Proof. It is sufficient to show that
\[ |f'(z) - 1| < \sum_{n=2}^{\infty} na_n|z|^{n-1} \leq 1 - \delta. \] (4.4)
But in view of (2.1) the inequality (4.4) holds true if
\[ \frac{n|z|^{n-1}}{1 - \delta} \leq \frac{[n(1 + k) - (\alpha + k)]\Gamma(2 - \lambda)\Gamma(n + 1)}{(1 - \alpha)\Gamma(n - \lambda + 1)}. \] (4.5)
Solving (4.5) for \(|z|\) we obtain
\[ |z| \leq \left[ \frac{(1 - \delta)[n(1 + k) - (\alpha + k)]\Gamma(2 - \lambda)\Gamma(n + 1)}{(n - \delta)(1 - \alpha)\Gamma(n - \lambda + 1)} \right]^{\frac{1}{n-1}} \quad n = 2, 3, 4, \ldots. \]

Theorem 4.2. If \( f(z) \in k - TSP_\lambda(\alpha) \) then \( f \) is starlike of order \( \delta \) in \(|z| < r_2(k, \alpha, \lambda, \delta)\) where
\[ r_2(k, \alpha, \lambda, \delta) = \inf_n \left[ \frac{(1 - \delta)[n(1 + k) - (\alpha + k)]\Gamma(2 - \lambda)\Gamma(n + 1)}{(n - \delta)(1 - \alpha)\Gamma(n - \lambda + 1)} \right]^{\frac{1}{n-1}}. \]
Proof. We must show that
\[ \left| \frac{zf''(z)}{f'(z)} - 1 \right| \leq \sum_{n=2}^{\infty} \frac{na_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n|z|^{n-1}} < 1 - \delta. \] (4.6)
But in view of (2.1) the inequality (4.6) holds true if
\[ \frac{(n - \delta)|z|^{n-1}}{1 - \delta} \leq \frac{[n(1 + k) - (\alpha + k)]\Gamma(2 - \lambda)\Gamma(n + 1)}{(1 - \alpha)\Gamma(n - \lambda + 1)}. \] (4.7)
Solving (4.7) for \(|z|\) we obtain
\[ |z| \leq \left[ \frac{(1 - \delta)[n(1 + k) - (\alpha + k)]\Gamma(2 - \lambda)\Gamma(n + 1)}{(n - \delta)(1 - \alpha)\Gamma(n - \lambda + 1)} \right]^{\frac{1}{n-1}} \quad n = 2, 3, 4, \ldots. \]

Theorem 4.3. If \( f(z) \in k - TSP_\lambda(\alpha) \) then \( f \) is convex of order \( \delta \) in \(|z| < r_3(k, \alpha, \lambda, \delta)\) where
\[ r_3(k, \alpha, \lambda, \delta) = \inf_n \left[ \frac{(1 - \delta)[n(1 + k) - (\alpha + k)]\Gamma(2 - \lambda)\Gamma(n + 1)}{n(n - \delta)(1 - \alpha)\Gamma(n - \lambda + 1)} \right]^{\frac{1}{n-1}}. \]
Proof. We must show that
\[ \left| 1 + \frac{zf'''(z)}{f''(z)} \right| \leq \sum_{n=2}^{\infty} \frac{n(n - 1)a_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n|z|^{n-1}} < 1 - \delta. \] (4.8)
But in view of (2.1) the inequality (4.8) holds true if

\[
\frac{n(n - \delta)|z|^{n-1}}{1 - \delta} \leq \frac{n(1 + k) - (\alpha + k)\Gamma(2 - \lambda)\Gamma(n + 1)}{(1 - \alpha)\Gamma(n - \lambda + 1)}.
\]  

(4.9)

Solving (4.9) for $|z|$ we obtain

\[
|z| \leq \left[ \frac{(1 - \delta)[n(1 + k) - (\alpha + k)\Gamma(2 - \lambda)\Gamma(n + 1)]}{n(n - \delta)(1 - \alpha)\Gamma(n - \lambda + 1)} \right]^{\frac{1}{n-1}} \quad n = 2, 3, 4, \ldots
\]

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