Large deviation analysis for quantum security via smoothing of Rényi entropy of order 2

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Abstract

It is known that the security evaluation can be done by smoothing of Rényi entropy of order 2 in the classical and quantum settings when we apply universal₂ hash functions. Using the smoothing of Rényi entropy of order 2, we derive security bounds for $L_1$ distinguishability and modified mutual information criterion under the classical and quantum setting, and have derived these exponential decreasing rates. These results are extended to the case when we apply $\varepsilon$-almost dual universal₂ hash functions. Further, we apply this analysis to the secret key generation with error correction.

Index Terms

exponential rate, non-asymptotic setting, secret key generation, universal hash function, almost dual universal₂ hash function

I. INTRODUCTION

A. Overview

When a random number is correlated to the third party, the random number is not secure. In this case, in order to amplify the privacy, one can apply a hash function to the original random number. This process is called privacy amplification or secret key extraction. Bennett et al. [4] and Håstad et al. [21] proposed to use universal₂ hash functions for privacy amplification and derived two universal hashing lemma, which provides an upper bound for leaked information based on Rényi entropy of order 2. In the quantum setting, Renner and König [38] showed that the trace norm of the difference between the real state and the ideal state is universally composable. Hence, we use the trace norm and call it the $L_1$ distinguishability criterion. Renner [22] extended two universal hashing lemma to the quantum case and evaluated the $L_1$ distinguishability criterion with universal₂ hash functions based on a quantum version of conditional Rényi entropy of order 2. In order to apply Renner’s two universal hashing lemma to a realistic setting, Renner [22] attached the smoothing to min entropy, which is a lower bound on the above quantum version of conditional Rényi entropy of order 2. That is, he proposed to maximize the min-entropy among the sub-states whose trace norm distance to the true state is less than a given threshold. However, it is not easy to find the maximizing sub-state. Instead of the rigorous maximization of min entropy under this condition, we can consider a lower bound of the maximum of min entropy. In the following, we say that this type lower bound or the method based on this type lower bound is an approximate smoothing of min entropy. In contrast with an approximate smoothing, we say that the tight value of min entropy under the given condition or the method based on the tight value is the rigorous smoothing of min entropy.

Indeed, the same difficulty still holds even for the maximum of Rényi entropy of order 2 under the same condition. Hence, we can consider an approximate smoothing of Rényi entropy of order 2. Considering an approximate smoothing of Rényi entropy of order 2, the previous paper [17] derived an upper bound of the $L_1$ distinguishability criterion after an application of universal₂ hash functions in the classical setting. In the $n$-fold independent and identical case, the upper bound yields a lower bound of the exponential decreasing rate of the $L_1$ distinguishability criterion. The obtained lower bound is tight with no side information [17]. The same fact is also shown with classical side information by combination of [71] and the forthcoming paper [53]. This fact shows that the approximate smoothing gives a sub-distribution that is sufficiently close to the sub-distribution maximizing the Rényi entropy of order 2 in the classical setting. However, no study treats the approximate smoothing of Rényi entropy of order 2 in the quantum case. One of the purposes of this paper is to attach the approximate smoothing of Rényi entropy of order 2 and to evaluate the $L_1$ distinguishability criterion in the quantum case.

Further, when we employ the rigorous smoothing of min entropy instead of approximate smoothing of Rényi entropy of order 2, we can derive another lower bound of the exponential decreasing rate of the $L_1$ distinguishability criterion. When there is no side information, it has been shown in [17] that the lower bound based on the rigorous smoothing of min entropy is not tight, i.e., strictly weaker than the bound based on the approximate smoothing of Rényi entropy of order 2 given in [17]. Further, the paper [71] showed the same fact when the side information classical. Due to this superiority of approximate
smoothing of Rényi entropy of order 2 over the rigorous smoothing of min entropy, it is natural to extend the bound given by

\[ \text{to the quantum case.} \]

The security of secret key generation by universal hash function has been discussed mainly in the cryptography community and has not been studied in the information theory community while the problem can be described by information theoretic quantity. However, the mutual information has not been discussed in this topic while the mutual information has been widely accepted as the criterion of information security by so many papers \cite{67, 68, 7, 6, 69, 70}. In fact, the security of the wiretap channel model has been mainly discussed with the mutual information among information theory community \cite{40, 41, 42, 35}. Watanabe \cite{39} gave an interesting example in the classical setting, in which, the mutual information is not close to zero while the \( L_1 \) distinguishability criterion is close to zero. His example suggests the demand of the convergence of the mutual information. Therefore, it is needed to evaluate the security based on the mutual information as well as the security based on the \( L_1 \) distinguishability criterion because so many recent literatures \cite{34, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65} still accept the mutual information.

However, the mutual information does not reflect the uniformity while it reflects the independence. In order to address the uniformity as well as leaked information, we need the modification of mutual information, which is called the modified mutual information criterion and is explained in Subsection \( \text{II-C} \). As is shown in Appendix \( \text{A} \) if we suppose several natural conditions for the security criterion, it is limited to the modified mutual information criterion. Hence, it is needed to evaluate the modified mutual information criterion as well as the \( L_1 \) distinguishability criterion.

In fact, when one of two security criteria goes to zero exponentially, the other also goes to zero exponentially due to the relations given in Subsection \( \text{II-C} \). Hence, the asymptotic key generation rate does not depend on the choice of the security criterion. However, the relations given in Subsection \( \text{II-C} \) cannot decide one of their exponential decreasing rates from the other exponent. Hence, we need to consider both exponents separately.

### B. Main results

As our result, first, we obtain upper bounds of the above two kinds of secrecy criteria when Alice and Bob share the same random number and Eve has a correlated quantum state by using approximate smoothing of Rényi entropy of order 2 (Theorems \text{24} and \text{25}). This problem is called the secret key generation without errors. Then, in the independent and identical distributed (i.i.d.) case, we obtain lower bounds on the exponential decreasing rate of the above two kinds of secrecy criteria (Theorems \text{26}). We also show that the obtained lower bound for universal composable criterion is tight in a typical example, in which, the leaked information is given as a pure state and can be regarded as the environment of Pauli channel. This fact suggests the superiority of our method even in the quantum setting.

Further, we apply this result to the case when there exist errors between Alice’s and Bob’s random variables and Eve has a correlated quantum state (Theorems \text{29} and \text{31}). This problem is called the secret key generation with error correction. The classical case has been treated by Ahlswede & Csiszár \cite{7}, Maurer \cite{6}, and Muramatsu \cite{10} et al. Renner \cite{22} treated the quantum case while he did not discuss the exponential decreasing rate. Our analysis derives the exponential decreasing rate even for the secret key generation with error correction (Theorems \text{32} and \text{34}). For derivation of these results, we need to invent several information quantities and several original technical lemmas, which are given in Section \( \text{II} \).

Further, we should note that the presentation style of this paper has is different from that of existing researches \cite{11, 22, 37}, with respect to security evaluation in the single-shot form. These papers \cite{11, 22, 37} bound the length of generated keys when the amount of leaked information is fixed. In contrast, this paper bounds the amount of leaked information when the length of generated keys is fixed. The latter style is useful for evaluation of the exponential decreasing rate.

### C. Generalization of main results

Recently, Tomamichel et al. \cite{37} extended two universal hashing lemma, i.e., they showed the security with a larger class of hash functions, which is the class of \( \varepsilon \)-almost universal hash functions in the sense of \cite{11, 2} when \( \varepsilon \) is close to 1 while they \cite{37} used a different terminology. Tsurumaru et al \cite{19} proposed the concept “\( \varepsilon \)-almost dual universal hash functions” for linear universal hash functions, which are defined as the dual functions of \( \varepsilon \)-almost universal hash functions. They also showed that the \( \varepsilon \)-almost dual universal hash functions contain the original universal hash functions when \( \varepsilon = 2 \). Tsurumaru et al \cite{19} showed the security of \( \varepsilon \)-almost dual universal hash functions when \( \varepsilon \) increases polynomially with respect to the coding length while Tomamichel et al. \cite{37} showed the security of \( \varepsilon \)-almost universal hash functions when \( \varepsilon \) is close to 1. Tsurumaru et al \cite{19} also gave an insecure example for 2-almost universal hash functions over the finite field \( \mathbb{F}_2 \). This example suggests that \( \varepsilon \)-almost dual universal hash functions have a larger expandability than \( \varepsilon \)-almost universal hash functions. Further, the forthcoming paper \cite{40} gives concrete examples of \( \varepsilon \)-almost universal hash functions that have a smaller calculation amount and a smaller number of random variables than the concatenation of Toeplitz matrix and the identity matrix, which is a typical example of universal hash functions. Hence, it is useful from a applied viewpoint to evaluate the security with \( \varepsilon \)-almost universal hash functions.

On the other hand, Dodis and Smith \cite{12} proposed the concept “\( \delta \)-biased family” for a family of random variables. The concept “\( \varepsilon \)-almost dual universal hash functions” can be converted to a part of “\( \delta \)-biased family” \cite{12, 19}. Indeed, Dodis
D. Relation with second order analysis

In the i.i.d. case, when the rate of generated random numbers is smaller than the entropy rate (or conditional entropy rate) of the original information source, it is possible to generate the random variable, in which, the $L_1$ distinguishability criterion approaches zero asymptotically. In the realistic setting, we can manipulate only a finite size operation. In order to treat the performance in the finite length setting, we have two kinds of formalism for the i.i.d. setting.

The first one is the second order formalism, in which, we focus on the asymptotic expansion up to the second order $\sqrt{n}$ of the length of generated keys $l_n$ as $l_n = Hn + C \sqrt{n} + o(\sqrt{n})$ with the constant constraint for the $L_1$ distinguishability criterion. The second one is the exponent formalism, in which, we fixed the generation rate $R := l_n/n$ and evaluate the exponential decreasing rate of convergence of the $L_1$ distinguishability criterion. In the exponent formalism, it is not sufficient to show that the security parameter goes zero exponentially, and it is required to explicitly give lower and/or upper bounds for the exponential decreasing rate. The exponent formalism has been studied by various information theoretical problems, e.g., channel coding\[9, 33\], source coding\[44, 43, 46\], and wire-tap channel\[35, 18\]. In the quantum case, the same topic has been studied also in channel coding\[28\], source coding\[43\], wire-tap channel\[50\], and entanglement concentration\[49\], \[51\]. As the second order formalism, the optimal coding length with the fixed error probability has been derived up to the second order $\sqrt{n}$ in the various setting\[28\], \[29\], \[30\] in the case of classical channel coding. The previous paper\[29\] treats the secret key generation with the second order formalism based on the information spectrum approach\[52\], which is closely related to $\epsilon$-smooth min-entropy. Then, another previous paper\[27\] discusses the randomness extraction with quantum side information with the second order formalism by using the relation with $\epsilon$-smooth min-entropy\[22\] and quantum versions of the information spectrum\[15\], \[26\]. The classical case of the result\[27\] can be regarded as a finite-length bound based on smoothing of min-entropy. Note that, as is mentioned by Han\[52\], the information spectrum approach can not yield the optimal exponent of error probability in the channel coding. This fact suggests that we have to treat the exponent formalism with a method different from the second order formalism.

Since the secret key generation by universal $L_2$ hash functions has been studied mainly in the cryptography community, it has not been studied with the exponent formalism sufficiently while the exponential decreasing rate is a standard topic in the information theory community. Since the exponential decreasing rate of the decoding error probability in the source coding is characterized by Rényi entropy in the classical\[44\] and the quantum\[43\] case, many information theoretical people might be interested in whether a similar characterization holds in the secret key generation.

Recently, the previous paper\[17\] derived an exponential decreasing rate of leaked information in the $L_1$ distinguishability criterion in the classical setting. The tightness of the rate is shown in the forthcoming paper\[53\]. Based on the results\[17\], \[27\], another recent paper\[52\] numerically dealt with the $L_1$ distinguishability criterion in the independently and identically distribution of the binary distribution with the finite-length setting. It compared the bound based on the second order formalism and the bound based on the exponent formalism in this setting. It numerically showed that the comparative merits between both depend on the length of the data and the required amount of the $L_1$ distinguishability criterion. That is, when the length of the data is not so many and the required amount of the $L_1$ distinguishability criterion is too small, the bound based on the exponent formalism is better than the bound based on the second order formalism. Indeed, when the required amount of the $L_1$ distinguishability criterion is too small, the convergence of the second order rate is not uniform. Hence, the second order formalism does not necessarily work properly for an approximation of the finite-length case. In such a case, from a mathematical viewpoint, we often take the limit of the length of generated keys under the condition that the required amount of the $L_1$ distinguishability criterion depends on the length of the data because such a limit often gives a better approximation of the finite-length case. The exponent formalism is a particular case of this type of limit. The numerical analysis in\[52\] shows the importance of the exponent formalism when the required amount of the $L_1$ distinguishability criterion is too small at least in the classical case.

While the paper\[27\] derives a finite-length bound achieving the optimal second order rate by using smoothing of min entropy, the bound in the classical case requires the evaluation of the tail probability, which causes the following drawback. In the case of binary distribution, the tail probability can be numerically calculated. Otherwise, its calculation is not easy when the data has a huge size. Hence, we often apply the Berry-Esseen theorem (the central limit theorem). However, the convergence of Berry-Esseen theorem is not so good when the tail probability is too small. Instead of Berry-Esseen theorem, we often apply Chernoff bound, which essentially gives the exponential decreasing rate. This is because Chernoff bound gives
a smaller upper bound of the tail probability than Berry-Esseen theorem in this case. When the tail probability is bounded by Chernoff bound, this type bound essentially gives an exponential decreasing upper bound based on an approximate smoothing of min entropy. This fact suggests the importance of the exponent formalism when the data has a huge size. We have the similar importance of the exponent formalism in the quantum case because the numerical calculation based on the bound given in [27] is more difficult in the quantum case except for the special example given in [27]. Hence, we need to discuss the finite-length bound given in [27] from the exponent formalism. As is shown in the paper [71], the upper bound by the rigorous smoothing of min entropy does not give the optimal exponential decreasing rate when the side information is classical. That is, the finite-length bound given in [27] cannot attain the optimal exponent, and the smoothing of Rényi entropy of order 2 is required for the optimal exponent. Therefore, this paper addresses only the smoothing of Rényi entropy of order 2 under the exponent formalism.

E. Organization

Now, we give the outline of the preliminary parts. In Section II we introduce the information quantities for evaluating the security and derive several useful inequalities for the quantum case. We also give a clear definition for security criteria. In section III, we introduce several class of hash functions (universal2 hash functions and ε-almost dual universal2 hash functions). We clarify the relation between ε-almost dual universal2 hash functions and δ-biased family. We also derive an ε-almost dual universal2 version of Renner’s two universal hashing lemma [23] Lemma 5.4.3](Lemma 17) based on Lemma for δ-biased family given by Dodis et al. [12] and Fehr et al. [13] in the classical and quantum setting. These parts give the definitions for concepts and quantities describing the main results. The latter preliminary parts are more technical and used for proofs of the main results. In section IV under the universal2 condition or the ε-almost dual universal2 condition, we evaluate the L1 distinguishability criterion and the modified mutual information based on Rényi entropy of order 2 for the quantum setting.

Next, we outline the main results. In Section V, we obtain a suitable bound for the quantum setting in the single-shot setting by attaching an approximate smoothing of Rényi entropy of order 2 to the evaluation obtained in the previous section. In Section VI, we derive an exponential decreasing rate for both criteria for the quantum setting when we simply apply hash functions and there is no error between Alice’s and Bob’s information.

In Section VII, we proceed to the secret key generation with error correction for the quantum setting. In this case, we need error correction as well as the privacy amplification. We derive Gallager bound for the error probability in this setting. We also derived upper bounds for the L1 distinguishability criterion and the modified mutual information for a given sacrifice rate. Based on these upper bounds, we derive the exponential decreasing rates for both criteria.

In Section VIII, we apply our result to the QKD case. That is, the state is given by the quantum communication via Pauli channel, which is a typical case in quantum key distribution. For this example, we showed that our approximate smoothing is tight in the sense of exponents. This evaluation is shown in Appendix E.

| TABLE I | SUMMARY OF OBTAINED RESULTS. |
|------------------|-----------------------------|
| **task**         | **setting** | **hash functions** | **L1** | **MMI** |
| PV               | single-shot | universal2         |   69   | Corollary 20 | Corollary 20 |
|                  | exponent    | ε-almost dual universal2 |   69   | Corollary 20 | Corollary 20 |
| PV & fixed EC    | single-shot | universal2         |   73   | Corollary 24 | Corollary 24 |
|                  | exponent    | ε-almost dual universal2 |   73   | Corollary 24 | Corollary 24 |
| PV & randomized EC | single-shot | universal2         |   76   | Corollary 27 | Corollary 27 |
|                  | exponent    | ε-almost dual universal2 |   76   | Corollary 27 | Corollary 27 |

Roman letters express obtained results. Italic letters express existing results or results with the same performance as existing results. PV is privacy amplification. EC is error correction. L1 is the L1 distinguishability criterion. MMI is the modified mutual information criterion. P(n) is a polynomial.
II. Preparation

A. Information quantities for single system

1) Case of sub-states: In order to discuss the security problem in the quantum systems, we prepare several information quantities in the single quantum system. In the following, a non-negative Hermitian matrix $\rho$ is called a sub-state when $\text{Tr} \rho \leq 1$. First, we define the following quantities:

\[
D(\rho || \sigma) := \text{Tr} \rho (\log \rho - \log \sigma) \tag{1}
\]

\[
\psi(s | \rho || \sigma) := \log \text{Tr} \rho^{1+s} - s \tag{2}
\]

\[
\psi(-s | \rho || \sigma) := \log \text{Tr} \rho^{-1-s} - s \tag{3}
\]

Then, we obtain the following lemma:

Lemma 1: The functions $s \mapsto \psi(s | \rho || \sigma)$, $\psi(-s | \rho || \sigma)$ are convex. In particular, they are strictly convex when $\rho$ and $\sigma$ are not completely mixed.

The proof of Lemma 1 is given in Appendix C.

Lemma 1 yields the following lemma.

Lemma 2: $\psi(s | \rho || \sigma)$ and $\psi(-s | \rho || \sigma)$ are monotonically increasing with respect to $s$ in $(0, \infty)$ and $(-\infty)$. In particular, they are strictly monotonically increasing with respect to $s$ when $\rho$ and $\sigma$ are not completely mixed.

For any quantum operation $\Lambda$, the following information processing inequalities

\[
D(\Lambda(\rho) || \Lambda(\sigma)) \leq D(\rho || \sigma), \quad \psi(s | \Lambda(\rho) || \Lambda(\sigma)) \leq \psi(s | \rho || \sigma) \tag{4}
\]

hold for $s \in (0, 1]$. However, this kind of inequality does not hold for $\psi(s | \rho || \sigma)$ in general.

Lemma 3: The relation

\[
\psi(s | \rho || \sigma) \leq \psi(s | \rho || \sigma) \tag{5}
\]

holds for $s \in (0, 1]$.

Lemma 3 is shown in Appendix B. For the latter discussion, we define the pinching map, which is used for our proof of another lemma. For a given Hermitian matrix $X$, we focus on its spectral decomposition $X = \sum_{i=1}^{v} x_{i} E_{i}$, where $v$ is the number of the eigenvalues of $X$. Then, the pinching map $\mathcal{E}_{X}$ is defined as

\[
\mathcal{E}_{X}(\rho) := \sum_{i} E_{i} \rho E_{i} \tag{6}
\]

Then, the inequality

\[
\rho \leq v \mathcal{E}_{\sigma}(\rho) \tag{7}
\]

holds. Inequality (7) is used in the proof of Lemma 3.

2) Case of normalized states: When $\rho$ and $\sigma$ are normalized states, we can show several additional useful properties as follows. In this case, the inequality $D(\rho || \sigma) \geq 0$ holds. The equality holds if and only if $\rho = \sigma$.

Since $\psi(0 | \rho || \sigma) = 0$ and $\psi(0 | \rho || \sigma) = 0$, we have $\lim_{s \to 0} \frac{1}{s} \psi(s | \rho || \sigma) = D(\rho || \sigma)$ and $\lim_{s \to 0} \frac{1}{s} \psi(-s | \rho || \sigma) = D(\rho || \sigma)$. Hence, Lemma 2 yields the following lemma.

Lemma 4: When $\rho$ and $\sigma$ are normalized states, we have

\[
-\psi(-s | \rho || \sigma) \leq s D(\rho || \sigma) \leq \psi(s | \rho || \sigma) \tag{8}
\]

\[
-\psi(-s | \rho || \sigma) \leq s D(\rho || \sigma) \leq \psi(s | \rho || \sigma) \tag{9}
\]

for $s > 0$.

B. Information quantities in composite system

1) Case of joint sub-state: Next, we prepare several information quantities in the composite system $\mathcal{H}_{A} \otimes \mathcal{H}_{E}$, in which $\mathcal{H}_{A}$ is a classical system spanned by the basis $\{|a\rangle\}$. A composite sub-state $\rho$ is called a c-q sub-state when it has a form $\rho_{A,E} = \sum_{a} \rho(a) |a\rangle \langle a| \otimes \rho_{E|a}$, in which the conditional state $\rho_{E|a}$ is normalized. For a given c-q state $\rho_{A,E}$, we define the sub-states $\rho_{E} := \text{Tr}_{A} \rho_{A,E}$ and $\rho_{A} := \text{Tr}_{E} \rho_{A,E}$. Then, we define the normalized states $\rho_{E,\text{normal}} := \rho_{E} / \text{Tr} \rho_{E}$ and $\rho_{A,\text{normal}} := \rho_{A} / \text{Tr} \rho_{A}$. Then, the von Neumann entropies and Rényi entropies of order $1 + s$ are given as

\[
H(A, E | \rho_{A,E}) := -\text{Tr} \rho_{A,E} \log \rho_{A,E} 
\]

\[
H_{1+s}(A, E | \rho_{A,E}) := -\frac{1}{s} \log \text{Tr} \rho_{A,E}^{1+s}
\]
with \( s \in \mathbb{R} \setminus \{0\} \).

Quantum versions of the conditional entropy and the min entropy, and two kinds of quantum versions of conditional Rényi entropy of order \( 1 + s \) are given as

\[
H(A|E|\rho_{A,E}) := H(A,E|\rho_{A,E}) - H(E|\rho_{E,\text{normal}})
\]

and

\[
H_{\text{min}}(A|E|\rho_{A,E}) := -\log ||(I_A \otimes \rho_{E,\text{normal}}^{-1/2})\rho_{A,E}(I_A \otimes \rho_{E,\text{normal}}^{-1/2})||,
\]

\[
H_{1+s}(A|E|\rho_{A,E}) := \frac{-1}{s} \log \text{Tr} \rho_{A,E}^{1+s}(I_A \otimes \rho_{E,\text{normal}}^{-s}),
\]

\[
\overline{H}_{1+s}(A|E|\rho_{A,E}) := \frac{-1}{s} \log \text{Tr} \rho_{A,E}^{1+s}(I_A \otimes \rho_{E,\text{normal}}^{-s/2})\rho_{A,E}^{1+s}(I_A \otimes \rho_{E,\text{normal}}^{-s/2})
\]

with \( s \in \mathbb{R} \setminus \{0\} \). These quantities can be written in the following way:

\[
H(A|E|\rho_{A,E}) = \log |A| - D(\rho_{A,E}||\rho_{\text{mix},A} \otimes \rho_{E,\text{normal}}) \tag{10}
\]

\[
H_{1+s}(A|E|\rho_{A,E}) = \log |A| - \frac{1}{s} \psi(s)\rho_{A,E}||\rho_{\text{mix},A} \otimes \rho_{E,\text{normal}}) \tag{11}
\]

\[
\overline{H}_{1+s}(A|E|\rho_{A,E}) = \log |A| - \frac{1}{s} \psi(s)\rho_{A,E}||\rho_{\text{mix},A} \otimes \rho_{E,\text{normal}}), \tag{12}
\]

where \( \rho_{\text{mix},A} \) is the completely mixed state on \( \mathcal{H}_A \). When we replace \( \rho_{E,\text{normal}} \) by another normalized state \( \sigma_E \) on \( \mathcal{H}_E \), we obtain the following generalizations:

\[
H(A|E|\rho_{A,E}\|\sigma_E) := \log |A| - D(\rho_{A,E}\|\rho_{\text{mix},A} \otimes \sigma_E)
\]

\[
H_{1+s}(A|E|\rho_{A,E}\|\sigma_E) := \log |A| - \frac{1}{s} \psi(s)\rho_{A,E}\|\rho_{\text{mix},A} \otimes \sigma_E)
\]

\[
\overline{H}_{1+s}(A|E|\rho_{A,E}\|\sigma_E) := \log |A| - \frac{1}{s} \psi(s)\rho_{A,E}\|\rho_{\text{mix},A} \otimes \sigma_E),
\]

\[
H_{\text{min}}(A|E|\rho_{A,E}\|\sigma_E) := -\log ||(I_A \otimes \sigma_{E,\text{normal}}^{-1/2})\rho_{A,E}(I_A \otimes \sigma_{E,\text{normal}}^{-1/2})||
\]

Lemma 3 implies that

\[
\overline{H}_{1+s}(A|E|\rho_{A,E}\|\sigma_E) \geq H_{1+s}(A|E|\rho_{A,E}\|\sigma_E) \tag{13}
\]

for \( s \in (0,1] \). Using Lemma 2 we obtain the following lemma.

**Lemma 5:** \( H_{1+s}(A|E|\rho_{A,E}\|\sigma_E) \) and \( \overline{H}_{1+s}(A|E|\rho_{A,E}\|\sigma_E) \) are monotonically decreasing with respect to \( s \) in \( (0,\infty) \) and \( (-\infty,0) \). In particular, they are strictly monotonically decreasing with respect to \( s \) in \( (0,\infty) \) and \( (-\infty,0) \) when \( \rho_{A,E} \) and \( \sigma_E \) are not completely mixed.

Further, since

\[
e^{-\overline{H}_2(A|E|\rho_{A,E}\|\sigma_E)} = \text{Tr} \rho_{A,E}(I_A \otimes \sigma_{E,normal}^{-1/2})\rho_{A,E}^{-1/2}(I_A \otimes \sigma_{E,normal}^{-1/2}) \\
\leq ||(I_A \otimes \sigma_{E,normal}^{-1/2})\rho_{A,E}(I_A \otimes \sigma_{E,normal}^{-1/2})|| = e^{-H_{\text{min}}(A|E|\rho_{A,E}\|\sigma_E)},
\]

Lemma 5 implies the relation \( \overline{H}_{1+s}(A|E|\rho_{A,E}\|\sigma_E) \geq H_{\text{min}}(A|E|\rho_{A,E}\|\sigma_E) \) for \( s \in (0,1) \). A similar relation \( H_{1+s}(A|E|\rho_{A,E}\|\sigma_E) \geq H_{\text{min}}(A|E|\rho_{A,E}\|\sigma_E) \) has been shown for \( s \in (0,1) \) in [22].

When we apply a quantum operation \( \Lambda \) on \( \mathcal{H}_E \), since it does not act on the classical system \( A \), (4) implies that

\[
H(A|E|\Lambda(\rho_{A,E})\|\Lambda(\sigma_E)) \geq H(A|E|\rho_{A,E}\|\sigma_E) \tag{14}
\]

\[
H_{1+s}(A|E|\Lambda(\rho_{A,E})\|\Lambda(\sigma_E)) \geq H_{1+s}(A|E|\rho_{A,E}\|\sigma_E)). \tag{15}
\]

When we apply the function \( f \) to the classical random number \( a \in A \), \( H(f(A), E|\rho_{A,E}) \leq H(A,E|\rho_{A,E}) \), i.e.,

\[
H(f(A)|E|\rho_{A,E}) \leq H(A|E|\rho_{A,E}). \tag{16}
\]

2) **Case of joint normalized state:** When the joint state \( \rho_{A,E} \) is normalized, we can show several additional useful properties as follows. In this case, since \( D(\rho_E||\sigma_E) \geq 0 \), we obtain

\[
H(A|E|\rho_{A,E}\|\sigma_E) = H(A|E|\rho_{A,E}) + D(\rho_E||\sigma_E) \geq H(A|E|\rho_{A,E}). \tag{17}
\]

Further, using Lemma 4 we obtain the following lemma.

**Lemma 6:** In particular,

\[
H_{1-s}(A|E|\rho_{A,E}\|\sigma_E) \geq H(A|E|\rho_{A,E}\|\sigma_E) \geq H_{1+s}(A|E|\rho_{A,E}\|\sigma_E), \tag{18}
\]

\[
\overline{H}_{1-s}(A|E|\rho_{A,E}\|\sigma_E) \geq H(A|E|\rho_{A,E}\|\sigma_E) \geq \overline{H}_{1+s}(A|E|\rho_{A,E}\|\sigma_E). \tag{19}
\]
for $s > 0$.

Now, we introduce another kind of conditional Rényi entropy for a joint normalized state as

$$H_{1+s}^G(A|E|\rho_{A,E}) := \frac{1+s}{s} \log \text{Tr}_E(\text{Tr}_AP_{1+s}A|E|\rho_{A,E})^\frac{1}{1+s}.$$  

This quantity can be expressed as

$$H_{1+s}^G(A|E|P_{A,E}) = \frac{1+s}{s} \phi(s|A|E|\rho_{A,E})$$  

by using the Gallager type function \cite{17}:

$$\phi(s|A|E|\rho_{A,E}) := \log \text{Tr}_E(\text{Tr}_AP_{1+s}A|E|\rho_{A,E})^{1-s} = \log \text{Tr}_E(\sum_a P_a(a)\rho_{E|a}^{1-s})^{1-s}.$$  

Taking the limit $s \to 0$, we obtain

$$\lim_{s \to 0} H_{1+s}^G(A|E|P_{A,E}) = \lim_{s \to 0} \frac{\phi(s|A|E|\rho_{A,E})}{s} = \frac{d\phi(s|A|E|\rho_{A,E})}{ds}|_{s=0}$$

Then, we obtain the following lemma:

\textbf{Lemma 7}: The relation

$$\max_{\sigma} H_{1+s}(A|E|\rho_{A,E}||\sigma_E) = H_{1+s}^G(A|E|P_{A,E})$$  

holds for $s \in (-1, \infty)$. The maximum can be realized when $\sigma_E = (\text{Tr}_AP_{1+s}A|E|\rho_{A,E})^{1/(1+s)}/\text{Tr}_E(\text{Tr}_AP_{1+s}A|E|\rho_{A,E})^{1/(1+s)}$.

The proof of Lemma 7 is given in Appendix D.

As a corollary of Lemma 7, we have the following.

\textbf{Corollary 8}: The map $s \mapsto H_{1+s}^G(A|E|\rho_{A,E})$ is monotonically decreasing for $s \in (-1, \infty)$. In particular, it is strictly decreasing for $s \in (-1, \infty)$ when $\rho_{A,E}$ is not completely mixed.

\textbf{Proof}: For $-1 < s < t$, we choose $\sigma_E$ such that $H_{1+t}(A|E|\rho_{A,E}||\sigma_E) = H_{1+s}^G(A|E|\rho_{A,E})$. Since $s \mapsto H_{1+s}^G(A|E|\rho_{A,E})$ is monotonically decreasing (Lemma 5),

$$H_{1+t}^G(A|E|\rho_{A,E}) = H_{1+s}^G(A|E|\rho_{A,E}||\sigma_E) \leq H_{1+s}^G(A|E|\rho_{A,E}||\sigma_E)$$  

for $s < t$. In particular, when $\rho_{A,E}$ is not completely mixed, Inequality (22) is strict. Hence, the function is strictly decreasing for $s \in (-1, \infty)$.  

\section*{5. TP-CP maps on the c-q sub state $\rho_{A,E} = \sum_a P_a(a)|a\rangle\langle a| \otimes \rho_{E|a}$}

Given a c-q sub state $\rho_{A,E} = \sum_a P_a(a)|a\rangle\langle a| \otimes \rho_{E|a}$, any TP-CP map $\Lambda$ on $\mathcal{H}_E$ satisfies that

$$H_{1+s}^G(A|E|\rho_{A,E}) \leq H_{1+s}^G(A|E|\Lambda(\rho_{A,E}))$$  

for $1 > s \geq 0$.

\textbf{Proof}: Due to (15) and Lemma 7, we obtain

$$sH_{1+s}^G(A|E|\rho_{A,E}) = \max_{\sigma_E} sH_{1+s}(A|E|\rho_{A,E}||\sigma_E)$$

$$\leq \max_{\sigma_E} sH_{1+s}(A|E|\Lambda(\rho_{A,E})||\sigma_E) = \max_{\sigma_E} sH_{1+s}(A|E|\Lambda(\rho_{A,E})||\sigma_E) = sH_{1+s}^G(A|E|\Lambda(\rho_{A,E})).$$  

\section*{Appendix D}

\textbf{Appendix D}: The proof of Lemma 7 is given in Appendix D.
C. Criteria for secret random numbers

1) Case of joint sub-state: Next, we introduce criteria for the amount of the information leaked from the secret random number $A$ to $E$ for joint sub-state $\rho_{A,E}$. Using the trace norm, we can evaluate the secrecy for the state $\rho_{A,E}$ as follows:

$$d_1(A|E|\rho_{A,E}) := \|\rho_{A,E} - \rho_A \otimes \rho_E\|_1. \quad (25)$$

Taking into account the randomness, Renner [22] defined the following criteria for security of a secret random number:

$$d'_1(A|E|\rho_{A,E}) := \|\rho_{A,E} - \rho_{\text{mix},A} \otimes \rho_E\|_1. \quad (26)$$

It is known that the quantity is universally composable [38]. We call it the $L_1$ distinguishability criterion.

Renner [22] defined the conditional $L_2$-distance from uniform of $\rho_{A,E}$ relative to a normalized state $\sigma_E$ on $\mathcal{H}_E$:

$$d_2(A|E|\rho_{A,E}\|\sigma_E) := \text{Tr}\left(\left((I \otimes \sigma_E^{-1/4})(\rho_{A,E} - \rho_{\text{mix},A} \otimes \rho_E)(I \otimes \sigma_E^{-1/4})\right)^2\right)$$

$$= \text{Tr}\left((I \otimes \sigma_E^{-1/4})\rho_{A,E}(I \otimes \sigma_E^{-1/4})\right) - \frac{1}{|A|}\text{Tr}\left(\sigma_E^{-1/4}(\rho_E - \sigma_E^{-1})^2\right) = e^{-\sqrt{2}d_2(A|E|\rho_{A,E}\|\sigma_E)} - \frac{1}{|A|}\left(|\sqrt{2}d_2(A|E|\rho_{A,E}\|\sigma_E)|\right). \quad (27)$$

Using this value, we can evaluate $d'_1(A|E|\rho_{A,E})$ as follows [22, Lemma 5.2.3]

$$d_1'(A|E|\rho_{A,E}) \leq \sqrt{|A|} \sqrt{d_2(A|E|\rho_{A,E}\|\sigma_E).} \quad (28)$$

2) Case of joint normalized state: In the remaining part of this subsection, we assume that the state $\rho_{A,E}$ is a normalized state. The correlation between the classical system $\mathcal{A}$ and the quantum system $\mathcal{H}_E$ can be evaluated by the mutual information

$$I(A : E|\rho) := D(\rho||\rho_A \otimes \rho_E). \quad (29)$$

This quantity has been adopted by many literatures [34], [54], [55], [56], [57], [58], [59], [60], [61], [62], [63], [64], [65], [67], [68], [69], [70] as a criteria of independence. In order to take account into uniformity as well as independence, we modify the mutual information by using the completely mixed state $\rho_{\text{mix},A}$ on $\mathcal{A}$:

$$I'(A|E|\rho_{A,E}) := D(\rho_{A,E}||\rho_{\text{mix},A} \otimes \rho_E), \quad (30)$$

which is called the modified mutual information and satisfies

$$I'(A|E|\rho_{A,E}) = I(A : E|\rho_{A,E}) + D(\rho_A||\rho_{\text{mix},A}) \quad (31)$$

and

$$H(A|E|\rho_{A,E}) = -I'(A|E|\rho_{A,E}) + \log |A|. \quad (32)$$

This quantity $I(A : E|\rho_{A,E})$ represents the amount of information leaked by $E$, and the remaining quantity $D(\rho_A||\rho_{\text{mix},A})$ describes the difference of the random number $A$ from the uniform random number. So, if the quantity $I'(A|E|\rho_{A,E})$ is small, we can conclude that the random number $A$ has less correlation with $E$ and is close to the uniform random number. In particular, if the quantity $I'(A|E|\rho_{A,E})$ goes to zero, the mutual information $I(A : E|\rho_{A,E})$ goes to zero, and the marginal distribution $\rho_A$ goes to the uniform distribution. In this paper, we can adopt the quantity $I'(A|E|\rho_{A,E})$ as a criterion for qualifying the secret random number. The detail validity of the quantity $I'(A|E|\rho_{A,E})$ is given in Appendix A.

Using the quantum version of Pinsker inequality, we obtain

$$d_1(A|E|\rho_{A,E})^2 \leq 2I(A|E|\rho_{A,E}) \quad (33)$$

$$d_1'(A|E|\rho_{A,E})^2 \leq 2I'(A|E|\rho_{A,E}). \quad (34)$$

Conversely, we can evaluate $I(A : E|\rho_{A,E})$ and $I'(A|E|\rho_{A,E})$ by using $d_1(A|E|\rho_{A,E})$ and $d_1'(A|E|\rho_{A,E})$ in the following way. When $\rho_{A,E}$ is a normalized c-q state, applying the Fannes inequality, we obtain

$$0 \leq I(A : E|\rho_{A,E}) = H(A|\rho_{A,E}) + H(E|\rho_{A,E}) - H(A,E|\rho_{A,E}) = H(A, E|\rho_A \otimes \rho_E) - H(A, E|\rho_{A,E})$$

$$= \sum_a P_A(a)H(E|\rho_E) - H(E|P_{E|a})$$

$$\leq \sum_a P_A(a)\eta(\|\rho_{E|a} - \rho_E\|_1, \log d_E) = \eta(\|\rho_{E|a} - \rho_E\|_1, \log d_E) = \eta(d_1(A|E|\rho_{A,E}), \log d_E) \quad (35)$$

where $d_E$ is the dimension of $\mathcal{H}_E$. Similarly, we obtain

$$0 \leq I'(A|E|\rho_{A,E}) = H(A|\rho_{\text{mix},A}) + H(E|\rho_{A,E}) - H(A, E|\rho_{A,E}) = H(A, E|\rho_{\text{mix},A} \otimes \rho_E) - H(A, E|\rho_{A,E})$$

$$\leq \eta(\|\rho_{\text{mix},A} \otimes \rho_E - \rho_{A,E}\|_1, \log |A| d_E) = \eta(d_1'(A|E|\rho_{A,E}), \log |A| d_E). \quad (36)$$
III. Ensemble of Hash functions

A. Ensemble of general hash functions

In this section, we focus on an ensemble \{f_X\} of hash functions \(f_X\) from \(A\) to \(B\), where \(X\) is a random variable identifying the function \(f_X\). In this case, the total information of Eve’s system is written as the composite system of \(\mathcal{H}_E\) and \(X\). By using the state \(\rho_{f_X(A),E,X} := \sum_{a \in f_X^{-1}(b),x} P_X(x)P_{A}(a)|b\rangle\langle b| \otimes \rho_{E|a} \otimes |x\rangle\langle x|\), the \(L_1\) distinguishability criterion is written as

\[
d'_1(f_X(A)|E,X|\rho_{f_X(A),E,X}) = \|\rho_{f_X(A),E,X} - \rho_{\text{mix}, B} \otimes \rho_{E,X}\|_1
\]

\[
= \sum_x P_X(x)\|\rho_{f_X=x(A),E} - \rho_{\text{mix}, B} \otimes \rho_{E}\|_1 = E_X\|P_{f_X(A),E} - \rho_{\text{mix}, B} \otimes \rho_{E}\|_1.
\]  

(37)

Then, the modified mutual information is written as

\[
I'(f_X(A)|E,X|\rho_{f_X(A),E,X}) = D(\rho_{f_X(A),E,X}\|\rho_{\text{mix}, B} \otimes \rho_{E,X})
\]

\[
= \sum_x P_X(x)D(\rho_{f_X=x(A),E}\|\rho_{\text{mix}, B} \otimes \rho_{E}) = E_XD(\rho_{f_X(A),E}\|\rho_{\text{mix}, B} \otimes \rho_{E}).
\]  

(38)

We say that a function ensemble \(\{f_X\}\) is \(\varepsilon\)-almost universal \(2\) if, for any pair of different inputs \(a_1,a_2\), the collision probability of their outputs is upper bounded as

\[
\Pr [f_X(a_1) = f_X(a_2)] \leq \frac{\varepsilon}{|B|}.
\]  

(39)

The parameter \(\varepsilon\) appearing in (39) is shown to be confined in the region

\[
\varepsilon \geq \frac{|A| - |B|}{|A| - 1},
\]  

(40)

and in particular, an ensemble \(\{f_X\}\) with \(\varepsilon = 1\) is simply called a universal \(2\) function ensemble.

Two important examples of universal \(2\) hash function ensembles are the Toeplitz matrices (see, e.g., [3]), and multiplications over a finite field (see, e.g., [11, 4]). A modified form of the Toeplitz matrices is also shown to be universal \(2\), which is given by a concatenation \((X, I)\) of the Toeplitz matrix \(X\) and the identity matrix \(I\) \([11]\). The (modified) Toeplitz matrices are particularly useful in practice, because there exists an efficient multiplication algorithm using the fast Fourier transform algorithm with complexity \(O(n \log n)\) (see, e.g., [5]). The following proposition holds for any universal \(2\) function ensemble.

Proposition 10 (Renner \[22\] Lemma 5.4.3): Given any composite c-q sub-state \(\rho_{A,E}\) on \(\mathcal{H}_A \otimes \mathcal{H}_E\) and any normalized state \(\sigma_E\) on \(\mathcal{H}_E\), any universal \(2\) ensemble of hash functions \(f_X\) from \(A\) to \(\{1, \ldots, M\}\) satisfies

\[
E_Xd_2(f_X(A)|E|\rho_{A,E}\|\sigma_E) \leq e^{-\mathcal{H}_2(A|E|\rho_{A,E}\|\sigma_E)}.
\]  

(41)

More precisely, the inequality

\[
E_Xe^{-\mathcal{H}_2(f_X(A)|E|\rho_{A,E}\|\sigma_E)} \leq (1 - \frac{1}{M})e^{-\mathcal{H}_2(A|E|\rho_{E}\|\sigma_E)} + \frac{1}{M}e^{\psi(1|\rho_{A,E}\|\sigma_E)}
\]  

(42)

holds.

B. Ensemble of linear hash functions

Tsurumaru and Hayashi\[19\] focus on linear functions over the finite field \(\mathbb{F}_2\). Now, we treat the case of linear functions over a finite field \(\mathbb{F}_q\), where \(q\) is a power of a prime number \(p\). That is, the following contents are generalization of the arguments given in \[19\]. Further, the contents with respect to the modified mutual information are not given in \[19\] even with \(q = 2\). We assume that sets \(A, B\) are \(\mathbb{F}_q^n, \mathbb{F}_q^m\) respectively with \(n \geq m\), and \(f\) are linear functions over \(\mathbb{F}_q\). Note that, in this case, there is a kernel \(C\) corresponding to a given linear function \(f\), which is a vector space of the dimension \(n - m\) or more. Conversely, when given a vector subspace \(C \subset \mathbb{F}_q^n\) of the dimension \(n - m\) or more, we can always construct a linear function

\[
f_C : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n/C \cong \mathbb{F}_q^l,\quad l \leq m.
\]  

(43)

That is, we can always identify a linear hash function \(f_C\) and a code \(C\).

When \(C_X = \text{Ker} f_X\), the definition of \(\varepsilon\)-universal \(2\) function ensemble of \(39\) takes the form

\[
\forall x \in \mathbb{F}_q^n \setminus \{0\},\quad \Pr [f_X(x) = 0] \leq q^{-m}\varepsilon,
\]  

(44)

which is equivalent with

\[
\forall x \in \mathbb{F}_q^n \setminus \{0\},\quad \Pr [x \in C_X] \leq q^{-m}\varepsilon.
\]  

(45)

This shows that the ensemble of kernel \(\{C_X\}\) contains sufficient information for determining if a function ensemble \(\{f_X\}\) is \(\varepsilon\)-almost universal \(2\) or not.
For a given ensemble of codes \( \{C_X\} \), we define its minimum (respectively, maximum) dimension as \( \tau_{\min} := \min_X \dim C_X \) (respectively, \( \tau_{\max} := \max_X \dim C_X \)). Then, we say that a linear code ensemble \( \{C_X\} \) of minimum (or maximum) dimension \( \tau \) is an \( \varepsilon \)-almost universal\( _2 \) code ensemble, if the following condition is satisfied

\[
\forall x \in \mathbb{F}_q^n \setminus \{0\}, \quad \Pr[ x \in C \geq q^{t-n} \varepsilon. \tag{46}
\]

In particular, if \( \varepsilon = 1 \), we call \( \{C_X\} \) a universal\( _2 \) code ensemble.

C. Dual universality of a code ensemble

Based on Tsurumaru and Hayashi\cite{19}, we define several variations of the universality of an ensemble of error-correcting codes and the linear functions as follows. First, we define the dual code ensemble \( \{C_X\}^\perp \) of a given linear code ensemble \( \{C_X\} \) as the set of all dual codes of \( C_X \). That is, \( \{C_X\}^\perp := \{C_X^\perp\} \). We also introduce the notion of dual universality as follows. We say that a code ensemble \( \{C_X\} \) in \( \mathbb{F}_q^n \) is \( \varepsilon \)-almost dual universal\( _2 \) with minimum dimension \( \tau \) (with maximum dimension \( t \)), if the dual ensemble \( \{C_X\}^\perp \) is \( \varepsilon \)-almost universal\( _2 \) with maximum dimension \( n - \tau \) (with minimum dimension \( n - t \)).

Hence, we say that a linear function ensemble \( \{f_X\} \) from \( \mathbb{F}_q^n \) to \( \mathbb{F}_q^m \) is \( \varepsilon \)-almost dual universal\( _2 \), if the kernels \( C_X^\perp \) of \( f_X \) forms an \( \varepsilon \)-almost dual universal\( _2 \) code ensemble with minimum dimension \( n - m \). This condition is equivalent with the condition that the ensemble of the linear spaces spanned by the generating matrix of \( f_X \) forms an \( \varepsilon \)-almost universal\( _2 \) code ensemble with maximum dimension \( m \).

An explicit example of a dual universal\( _2 \) function ensemble (with \( \varepsilon = 1 \)) can be given by the modified Toeplitz matrices mentioned earlier\cite{16} i.e., a concatenation \( (X, I) \) of the Toeplitz matrix \( X \) and the identity matrix \( I \). This example is particularly useful in practice because it is both universal\( _2 \) and dual universal\( _2 \), and also because there exists an efficient algorithm with complexity \( O(n \log n) \).

With these preliminaries, we can present the following theorem as \( \mathbb{F}_q \) extension of \cite{19} Corollary 2:

**Proposition 11:** An \( \varepsilon \)-almost universal\( _2 \) surjective linear hash function ensemble \( \{f_X\} \) from \( \mathbb{F}_q^n \) to \( \mathbb{F}_q^m \) is \( q(1 - q^{-m}) + (\varepsilon - 1)q^{n-m} \)-almost dual universal\( _2 \) linear hash function ensemble.

As a special case, we obtain the following.

**Corollary 12:** Any universal\( _2 \) linear function ensemble \( \{f_X\} \) over a finite filed \( \mathbb{F}_q \) is \( q \)-almost dual universal\( _2 \) function ensemble.

D. Permutated code ensemble

In order to treat an example of \( \varepsilon \)-almost universal\( _2 \) functions, we consider the case when the distribution is invariant under permutations of the order in \( \mathbb{F}_q^n = \mathbb{A}^n \). Now, \( S_n \) denotes the symmetric group of degree \( n \), and \( \sigma(i) = j \) means that \( \sigma \in S_n \) maps \( i \) to \( j \), where \( i, j \in \{1, \ldots, n\} \). The code \( \sigma(C) \) is defined by \( \{x^{\sigma} := (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \mid x = (x_1, \ldots, x_n) \in C\} \). Then, we introduce the permuted code ensemble \( \{\sigma(C)\}_{\sigma \in S_n} \) of a code \( C \). In this ensemble, \( \sigma \) obeys the uniform distribution on \( S_n \).

For an element \( x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n \), we can define the empirical distribution \( p_x \) on \( \mathbb{F}_q \) as \( p_x(a) := \#\{i \mid x_i = a\} / n \). So, we denote the set of the empirical distributions on \( \mathbb{F}_q^n = \mathbb{A}^n \) by \( \mathbb{P}_{\mathbb{A}^n} \). The cardinality \( |\mathbb{P}_{\mathbb{A}^n}| \) is bounded by \( (n + 1)^{q-1} \).

Similarly, we define \( T_{n, A} := T_{n, A} \setminus \{1\} \), where \( 1 \) is the deterministic distribution on \( 0 \in \mathbb{F}_q \). For given a code \( C \subset \mathbb{F}_q^n \), we define \( p(C) := \frac{\#\{x \in C \mid p_x = p\}}{\#\{x \in \mathbb{P}_{\mathbb{A}^n} \mid p_x = p\} \} \) and \( \varepsilon(C) := \max_{p \in T_{n, A}^+} p(C) \).

Then, we obtain the following lemma.

**Lemma 13:** The permuted code ensemble \( \{\sigma(C)\}_{\sigma \in S_n} \) of a code \( C \) is \( \varepsilon(C) \)-almost universal\( _2 \).

**Proof:** For any non-zero element \( x' \in \mathbb{F}_q^n \), we fix an empirical distribution \( p := p_x \). Then, \( x' \) belongs to \( \sigma(C) \) with the probability \( \frac{\#\{x \in C \mid p_x = p\}}{\#\{x \in \mathbb{P}_{\mathbb{A}^n} \mid p_x = p\} \} \). That is, the probability that \( x' \) belongs to \( \sigma(C) \) is less than \( \frac{\varepsilon(C) |C|}{q^n} \). \( \blacksquare \)

**Lemma 14:** For any \( t \leq n \), there exists a \( t \)-dimensional code \( C \in \mathbb{F}_q^n \) such that

\[
\varepsilon(C) < (n + 1)^{q-1}. \tag{47}
\]

**Proof:** Let \( \{C_X\}_X \) be a universal\( _2 \) code ensemble. Then, any \( p \in T_{n, A}^+ \) satisfies \( E_X \varepsilon(C_X) \leq 1 \). The Markov inequality yields

\[
\Pr\{\varepsilon(C) \geq |T_{n, A}|\} \leq \frac{1}{|T_{n, A}|} \tag{48}
\]

and thus

\[
\Pr\{\exists p \in T_{n, A}^+, \varepsilon(C_X) \geq |T_{n, A}|\} \leq \frac{|T_{n, A}| - 1}{|T_{n, A}|}. \tag{49}
\]

Hence,

\[
\Pr\{\forall p \in T_{n, A}^+, \varepsilon(C_X) < |T_{n, A}|\} \geq \frac{1}{|T_{n, A}|}. \tag{50}
\]

Therefore, there exists a code \( C \) satisfying the desired condition \( (47) \). \( \blacksquare \)
E. δ-biased ensemble

Next, according to Dodis and Smith[12], we introduce δ-biased ensemble of random variables {W_X}. For a given δ > 0, an ensemble of random variables {W_X} on F_q^n is called δ-biased when the inequality

\[ E_X(E_{W_X}(-1)^{x \cdot W_X}) \leq \delta^2 \]  

(51)

holds for any x ∈ F_q^n.

We denote the random variable subject to the uniform distribution on a code C ∈ F_q^n by W_C. Then,

\[ E_{W_C}(-1)^{x \cdot W_C} = \begin{cases} 0 & \text{if } x \notin C^⊥ \\ 1 & \text{if } x \in C^⊥. \end{cases} \]  

(52)

Using the above relation, as is suggested in [12, Case 2], we obtain the following lemma.

**Lemma 15:** When a code ensemble \( \{C_X\} \) in F_q^n is \( \varepsilon \)-almost dual universal with minimum dimension t, the ensemble of random variables \( \{W_{C_X}\} \) in F_q^n is \( \varepsilon \sqrt{q}^{-t} \)-biased.

**Proof:** \( \{C_X\} \) is \( \varepsilon \)-almost universal with maximum dimension \( n - t \) in F_q^n. Hence, for any \( x \in F_q^n \), the probability \( \Pr\{x \in C^⊥\} \) is less than \( \varepsilon q^{-t} \). Thus, \( \{W_{C_X}\} \) guarantees that the ensemble of random variables \( \{W_{C_X}\} \) in F_q^n is \( \varepsilon \sqrt{q}^{-t} \)-biased.

In the following, we treat the case of \( A = F_q^n \). Given a composite state \( \rho_{A,E} \) on \( \mathcal{H}_A \otimes \mathcal{H}_E \) and a distribution \( P_W \) on \( A \), as a quantum generalization of \( P_{A,E} \) * \( P_W \), we define another composite state \( \tilde{\rho}_{A,E} = \sum_w P_W(w) \sum_a P_A(a) |a + w\rangle \langle a + w| \otimes \rho_{E}^w \). Then, using this concept, Fehr and Schaffner [13] obtain the following proposition as a quantum extension of Lemma 4 of Dodis and Smith[12]. Their proof is based on discrete Fourier transform and is easy to understand.

**Proposition 16** ([13 Theorem 3.2]): For any c-q sub-state \( \rho_{A,E} \) on \( \mathcal{H}_A \otimes \mathcal{H}_E \) and any normalized state \( \sigma_E \) on \( \mathcal{H}_E \), a δ-biased ensemble of random variables \( \{W_X\} \) on \( A = F_q^n \) satisfies

\[ E_X d_2(A|E|\rho_{A,E} \ast P_{W_X}||\sigma_E) \leq \delta^2 e^{-T_2(A|E|\rho_{A,E}||\sigma_E)}. \]  

(53)

More precisely,

\[ E_X d_2(A|E|\rho_{A,E} \ast P_{W_X}||\sigma_E) \leq \delta^2 (1 - \frac{1}{q^n}) e^{-T_2(A|E|\rho_{A,E}||\sigma_E)}. \]  

(54)

Using the above proposition, we can show the following lemma.

**Lemma 17:** Given a c-q sub-state \( \rho_{A,E} \) on \( \mathcal{H}_A \otimes \mathcal{H}_E \) and a normalized state \( \sigma_E \) on \( \mathcal{H}_E \). When \( \{C_X\} \) is an \( \varepsilon \)-almost dual universal code ensemble with minimum dimension t, the ensemble of hash functions \( \{f_{C_X}\} \) satisfies

\[ E_X d_2(f_{C_X}(A)|E|\rho_{A,E}||\sigma_E) \leq \varepsilon e^{-T_2(A|E|\rho_{A,E}||\sigma_E)}. \]  

(55)

More precisely,

\[ E_X e^{-T_2(f_{C_X}(A)|E|\rho_{A,E}||\sigma_E)} \leq \varepsilon (1 - \frac{1}{q^n}) e^{-T_2(A|E|\rho_{A,E}||\sigma_E)} + \frac{1}{q^n - 1} e^{\psi(1|\rho_{A,E}||\sigma_E)}. \]  

(56)

In other words, an \( \varepsilon \)-almost dual universal function family \( \{f_X\} \) from \( F_q^n \) to \( F_q^{n-t} \) satisfies (55) and (56).

**Proof:** Due to Lemma 15 and (53), we obtain

\[ E_X d_2(A|E|\rho_{A,E} \ast P_{W_X}||\sigma_E) \leq \varepsilon q^{-t} e^{-T_2(A|E|\rho_{A,E}||\sigma_E)}. \]  

(57)

Now, we focus on the relation \( A \cong A/C \times C \cong f_C \times C \) for any code C. Then, we obtain

\[ \tilde{\rho}(W_C) = \sum_{w \in C} q^{-t} \sum_a P_A(a) |a + w\rangle \langle a + w| \otimes \rho_{E}^w = \sum_{w \in C} q^{-t} |w\rangle \langle w| \otimes \sum_{[a] \in A/C} P_A([a]) |[a]\rangle \langle [a]| \otimes \rho_{[a]}^E. \]

Thus, (57) implies

\[ d_2(A|E|\rho_{A,E} \ast P_{W_C}||\sigma_E) = q^{-t} d_2(f_{C}(A)|E|\rho_{f_{C}(A),E}||\sigma_E) = q^{-t} d_2(f_{C}(A)|E|\rho_{A,E}||\sigma_E). \]  

(58)

Therefore,

\[ E_X q^{-t} d_2(f_{C_X}(A)|E|\rho_{A,E}||\sigma_E) \leq \varepsilon q^{-t} e^{-T_2(A|E|\rho_{A,E}||\sigma_E)}, \]

which implies (53).

Similarly, Lemma 15, (54), and (58) imply that

\[ E_X q^{-t} d_2(f_{C_X}(A)|E|\rho_{A,E}||\sigma_E) \leq \varepsilon q^{-t} (1 - \frac{1}{q^n}) e^{-T_2(A|E|\rho_{A,E}||\sigma_E)}. \]

Since \( E_X d_2(f_{C_X}(A)|E|\rho_{A,E}||\sigma_E) = E_X e^{-T_2(f_{C_X}(A)|E|\rho_{A,E}||\sigma_E)} - \frac{1}{q^n} e^{\psi(1|\rho_{E}||\sigma_E)}, \) we have (56).
IV. Security bounds with Rényi entropy of order 2

Next, we consider the quantum case for the security bound based on the Rényi entropy of order 2. Renner[22] Lemma 5.2.3] essentially evaluated $\text{Ex}_n d'_1(f_x(A)|E|\rho_{A,E})$ by using $\text{Ex}_n d'_2(f_x(A)|E|\rho_{A,E}||\sigma_E)$ as follows.

**Lemma 18:** Given a composite c-q sub-state $\rho_{A,E}$ on $\mathcal{H}_A \otimes \mathcal{H}_E$ and a normalized state $\sigma_E$ on $\mathcal{H}_E$, any ensemble of hash functions $f_X$ from $A$ to $\{1, \ldots, M\}$ satisfies

$$\text{Ex}_n d'_1(f_x(A)|E|\rho_{A,E}) \leq M\frac{2}{2}\sqrt{\text{Ex}_n d'_2(f_x(A)|E|\rho_{A,E}||\sigma_E)}$$

Further, the inequalities used in proof of Renner[22 Corollary 5.6.1] imply that

$$\text{Ex}_n d'_1(f_x(A)|E|\rho_{A,E}) \leq 2\|\rho_{A,E} - \rho_{A,E}'\|_1 + \text{Ex}_n d'_1(f_x(A)|E|\rho_{A,E}')$$

$$\leq 2\|\rho_{A,E} - \rho_{A,E}'\|_1 + M\frac{2}{2}\sqrt{\text{Ex}_n d'_2(f_x(A)|E|\rho_{A,E}||\sigma_E)}.$$ 

Applying the same discussion to the von Neumann entropy, we can evaluate the average of the modified mutual information criterion by using $\text{Ex}_n d'_2(f_x(A)|E|\rho_{A,E}||\sigma_E)$ as follows.

**Lemma 19:** Assume that $\rho_{A,E}$ is a normalized composite c-q state $\rho_{A,E}$ on $\mathcal{H}_A \otimes \mathcal{H}_E$. Any ensemble of hash functions $f_X$ from $A$ to $\{1, \ldots, M\}$ satisfies

$$\text{Ex}_n I'(f_x(A)|E|P_{A,E}) \leq \log(1 + M \text{Ex}_n d'_2(f_x(A)|E|\rho_{A,E}))$$

$$\leq \text{ME}_n d'_2(f_x(A)|E|\rho_{A,E}).$$

Further, when a composite c-q sub-state $\rho_{A,E}'$ satisfies $\rho_{E}' \leq \rho_{E}$ and $\rho_{A}' \leq \rho_{A}$,

$$\text{Ex}_n I'(f_x(A)|E|\rho_{A,E}) \leq 2\eta(\|\rho_{A,E} - \rho_{A,E}'\|_1, \log \tilde{M}) + \log(1 + M \text{Ex}_n d'_2(f_x(A)|E|\rho_{A,E}'||\rho_{E}'))$$

$$\leq 2\eta(\|\rho_{A,E} - \rho_{A,E}'\|_1, \log \tilde{M}) + M \text{Ex}_n d'_2(f_x(A)|E|\rho_{A,E}'||\rho_{E}').$$

where $\tilde{M} := \max\{M, d_E\}$.

**Proof:** The inequality $\psi(1|\rho_{E}'||\rho_{E}) \leq 0$ holds because $\rho_{E}' \leq \rho_{E}$. Since

$$d'_2(f_x(A)|E|\rho_{A,E}'||\rho_{E}) = e^{-T'_2(f_x(A)|E|\rho_{A,E}'||\rho_{E})} - \frac{1}{M} e^{-T'_2(f_x(A)|E|\rho_{E}'||\rho_{E})} - \frac{1}{M},$$

we have

$$e^{-T'_2(f_x(A)|E|\rho_{A,E}'||\rho_{E})} \leq d'_2(f_x(A)|E|\rho_{A,E}'||\rho_{E}) + \frac{1}{M}.$$

Taking the logarithm, we obtain

$$-\log M + \log(1 + M d'_2(f_x(A)|E|\rho_{A,E}'||\rho_{E})) \geq -T'_2(f_x(A)|E|\rho_{A,E}'||\rho_{E}) \geq H(f_x(A)|E|\rho_{A,E}'||\rho_{E}).$$

Substituting $\rho_{A,E}$ to $\rho_{A,E}'$, we obtain $H(f_x(A)|E|\rho_{A,E}'||\rho_{E}) = H(f_x(A)|E|\rho_{A,E})$ and

$$I'(f_x(A)|E|\rho_{A,E}) = \log M - H(f_x(A)|E|\rho_{A,E}) \leq \log(1 + M d'_2(f_x(A)|E|\rho_{A,E})).$$

Since the function $x \mapsto \log(1 + x)$ is concave, we obtain

$$\text{Ex}_n I'(f_x(A)|E|\rho_{A,E}) \leq \log(1 + M \text{Ex}_n d'_2(f_x(A)|E|\rho_{A,E})),$$

which implies (59). The inequality $\log(1 + x) \leq x$ yields (60).

Fannes inequality guarantees that

$$|\text{Tr}(\rho_{E} - \rho_{E}')\log\rho_{E}| \leq \eta(\|\rho_{E} - \rho_{E}'\|_1, \log d_E) \leq \eta(\|\rho_{A,E} - \rho_{A,E}'\|_1, \log \tilde{M}),$$

and

$$|H(E|f_x(A)|\rho_{A,E}||\rho_{A}) - H(E|f_x(A)|\rho_{A,E}'||\rho_{A})| = \sum b P_{f_x(A)}(b)H(E|f_x(A)|\rho_{E}b) - H(E|\rho_{E}b|f_x(A)) = \sum b P_{f_x(A)}(b)\log d_E||\rho_{E}b|f_x(A) = \rho_{A,E}'b||_1 - \log d_E||\rho_{A,E} - \rho_{A,E}'||_1,$$

$$\leq \eta(\|\rho_{A,E} - \rho_{A,E}'\|_1, \log d_E) \leq \eta(\|\rho_{A,E} - \rho_{A,E}'\|_1, \log \tilde{M}).$$

Since the condition $\rho_{A}' \leq \rho_{A}$ implies $-\text{Tr}(\rho_{f_x(A)} - \rho_{f_x(A)})\log\rho_{f_x(A)} \geq 0$, we have

$$H(f_x(A)|E|\rho_{A,E}||\rho_{E}) - H(f_x(A)|E|\rho_{A,E}'||\rho_{E}) = H(f_x(A)|E|\rho_{A,E}||\rho_{E}) + \text{Tr}\rho_{E}\log\rho_{E} - H(f_x(A)|E|\rho_{A,E}'||\rho_{E}) - \text{Tr}\rho_{E}'\log\rho_{E}$$

$$= H(f_x(A)|E|\rho_{A,E}||\rho_{E}) - H(f_x(A)|\rho_{A,E}||\rho_{E}) - \text{Tr}\rho_{f_x(A)} - \rho_{f_x(A)}\log\rho_{f_x(A)} + \text{Tr}(\rho_{E} - \rho_{E}')\log\rho_{E}$$

$$\geq H(f_x(A)|\rho_{A,E}||\rho_{E}) - H(f_x(A)|\rho_{A,E}'||\rho_{E}) + \text{Tr}(\rho_{E} - \rho_{E}')\log\rho_{E} \geq -2\eta(\|\rho_{A,E} - \rho_{A,E}'\|_1, \log \tilde{M}).$$
Therefore, (65) and (64) imply that

\[ I'(f_X(A)|E|\rho_{A,E}) = \log M - H(f_X(A)|E|\rho_{A,E}) \leq 2\eta(\|\rho_{A,E} - \rho'_{A,E}\|_1, \log \hat{M}) + \log M - H(f_X(A)|E|\rho'_{A,E}|\rho_E) \leq 2\eta(\|\rho_{A,E} - \rho'_{A,E}\|_1, \log \hat{M}) + \log(1 + M_{\delta_2}(f_X(A)|E|\rho'_{A,E}|\rho_E)). \]

Therefore, taking the expectation of \(X\), we obtain (61), which implies (62).

In this proof, the condition \(\rho'_E \leq \rho_E\) is crucial because Inequality (63) cannot be shown without this condition.

Now, we evaluate the security by combining Proposition 10 and Lemmas 18 and 19. For this purpose, we introduce the quantities

\[ \Delta_{d,2}(M,\varepsilon|\rho_{A,E}) := \min_{\sigma_E} 2\|\rho_{A,E} - \rho'_{A,E}\|_1 + \sqrt{M} e^{-2\|\rho_{A,E}|\sigma_E\|} = \min_{\sigma_E,\varepsilon_1>0} 2\varepsilon_1 + \sqrt{M} e^{-2\|\rho_{A,E}|\sigma_E\|}, \]

\[ \Delta_{I,2}(M,\varepsilon|\rho_{A,E}) := \min_{\sigma_E} \min_{\rho'_{A,E},\varepsilon_1>0} \eta(\|\rho_{A,E} - \rho'_{A,E}\|_1, \log \hat{M}) + \varepsilon M e^{-2\|\rho_{A,E}|\sigma_E\|} \]

where \(\hat{M} := \max\{M, d_E\}\) and

\[ \mathcal{H}_{2}^{-1}(A|E|\rho_{A,E}|\sigma_E) := \max_{\rho'_{A,E},\varepsilon_1>0,\varepsilon_1\leq \sigma_E} \mathcal{H}_{2}(A|E|\rho'_{A,E}|\sigma_E) \]

\[ \mathcal{H}_{2}^{+}(A|E|\rho_{A,E}|\sigma_E) := \max_{\rho_{A,E},\varepsilon_1>0,\varepsilon_1\leq \sigma_E} \mathcal{H}_{2}(A|E|\rho'_{A,E}|\sigma_E) \]

Note that \(\mathcal{H}_{2}^{+}(A|E|\rho_{A,E}|\sigma_E)\) is different from \(\mathcal{H}_{2}^{-1}(A|E|\rho_{A,E}|\sigma_E)\) because the definition of \(\mathcal{H}_{2}^{+}(A|E|\rho_{A,E}|\sigma_E)\) has additional constraints for \(\rho'_{A,E}\). Then, we obtain the following lemma under the universal2 condition.

**Lemma 20:** Given a normalized state \(\sigma_E\) on \(H_E\) and a q sub-states \(\rho_{A,E}\), any universal2 ensemble of hash functions \(f_X\) from \(A\) to \(\{1, \ldots, M\}\) satisfies

\[ E_X d'(f_X(A)|E|\rho_{A,E}) \leq M e^{-2\|\rho_{A,E}|\sigma_E\|} \]

\[ E_X d'_1(f_X(A)|E|\rho_{A,E}) \leq \Delta_{d,2}(M, 1|\rho_{A,E}) \]

(68)

When \(\rho_{A,E}\) is a normalized c-q state, it satisfies

\[ E_X I'(f_X(A)|E|\rho_{A,E}) \leq M e^{-2\|\rho_{A,E}|\sigma_E\|} \]

\[ E_X I'(f_X(A)|E|\rho_{A,E}) \leq \Delta_{I,2}(M, 1|\rho_{A,E}), \]

(69)

While the above evaluations of the \(L_1\) distinguishability criterion has been shown in Renner[22, Corollary 5.6.1], those of the modified mutual information criterion have not been shown until now.

Further, since \(\mathcal{H}_{2}(A|E|\rho'_{A,E}|\sigma_E) \geq H_{\min}(A|E|\rho'_{A,E}|\sigma_E)\), Renner[22] introduced the idea to replace \(\mathcal{H}_{2}(A|E|\rho'_{A,E}|\sigma_E)\) by the min entropy \(H_{\min}(A|E|\rho'_{A,E}|\sigma_E)\) in (68). For this purpose, based on \(H_{\min}(A|E|\rho_{A,E}|\sigma_E)\), Renner[22] introduced \(\varepsilon_1\)-smooth min entropy as

\[ H_{\min}^{\varepsilon_1}(A|E|\rho_{A,E}|\sigma_E) := \max_{\|\rho_{A,E} - \rho'_{A,E}\|_1 \leq \varepsilon_1} H_{\min}(A|E|\rho'_{A,E}|\sigma_E). \]

(70)

Then, Renner[22, Corollary 5.6.1] obtained another upper bound:

\[ E_X d'_1(f_X(A)|E|\rho_{A,E}) \leq \Delta_{d,\min}(M, \varepsilon|\rho_{A,E}), \]

(71)

where

\[ \Delta_{d,\min}(M, \varepsilon|\rho_{A,E}) := \min_{\sigma_E} 2\varepsilon_1 + \sqrt{M} e^{-2\|\rho_{A,E}|\sigma_E\|}. \]

That is, he proposed to evaluate \(\Delta_{d,\min}(M, 1|\rho_{A,E})\) instead of \(\Delta_{d,2}(M, 1|\rho_{A,E})\). However, the bound \(\Delta_{d,2}(M, 1|\rho_{A,E})\) gives a strictly better bound in the following sense.

When there is no side information, i.e., the state is given as a distribution \(P_A\) on \(A\), the previous paper [17] showed that

\[ \frac{-1}{n} \log \Delta_{d,\min}(e^n R, 1|P_A^n) < \frac{-1}{n} \log \Delta_{d,2}(e^n R, 1|P_A^n). \]

Further, when the side information is classical, i.e., the state is given as a joint distribution \(P_{A,E}\) on the joint system, the paper [71] showed that

\[ \frac{-1}{n} \log \Delta_{d,\min}(e^n R, 1|P_{A,E}^n) < \frac{-1}{n} \log \Delta_{d,2}(e^n R, 1|P_{A,E}^n). \]
That is, in these cases, \( \Delta_{d,2}(e^{nR}, 1|\rho_{A,E}^{\otimes n}) \) gives a strictly better exponential decreasing rate. Hence, we focus on the bounds based on Rényi entropy of order 2 rather than those based on min entropy.

Since the function \( x \mapsto \eta(x,y) \) is concave, combining Inequality (36), we obtain the following corollary.

**Corollary 21:** Any universal 2 ensemble of hash functions \( f_X \) from \( A \) to \( \{1, \ldots, M\} \) and any normalized c-q state \( \rho_{A,E} \) on \( \mathcal{H}_A \otimes \mathcal{H}_E \) satisfy

\[
\mathbb{E}_X I'(f_X(A)|E|\rho_{A,E}) \leq \eta(\Delta_{d,2}(M,1|\rho_{A,E}), \log |A|d_E).
\]

(72)

for \( s \in (0,1] \).

Since the function \( x \mapsto \sqrt{x} \) is concave, combining Inequality (33), we obtain the following corollary.

**Corollary 22:** Any universal 2 ensemble of hash functions \( f_X \) from \( A \) to \( \{1, \ldots, M\} \) and any normalized c-q state \( \rho_{A,E} \) on \( \mathcal{H}_A \otimes \mathcal{H}_E \) satisfy

\[
\mathbb{E}_X d'_1(f_X(A)|E|\rho_{A,E}) \leq 2\sqrt{\Delta_{l,2}(M,1|\rho_{A,E})}
\]

(73)

for \( s \in (0,1] \).

Similarly, combining Lemmas 17, 18 and 19 under the \( \varepsilon \)-almost dual universal 2 condition and employing the same discussion as Corollaries 21 and 22, we can evaluate the average of both security criteria as follows.

**Lemma 23:** Given a normalized state \( \sigma_E \) on \( \mathcal{H}_E \) and c-q sub-states \( \rho_{A,E} \) on \( \mathcal{H}_A \otimes \mathcal{H}_E \). When an ensemble of linear hash functions \( \{f_X\}_X \) from \( A \) to \( \{1, \ldots, M\} \) is \( \varepsilon \)-almost universal 2, we obtain

\[
\mathbb{E}_X d'_1(f_X(A)|E|\rho_{A,E}) \leq \sqrt{\varepsilon M + e^{-2H_1(A|E|\rho_{A,E})}},
\]

\[
\mathbb{E}_X I'(f_X(A)|E|\rho_{A,E}) \leq \Delta_{d,2}(M,|\rho_{A,E}|, \log |A|d_E).
\]

(74)

When \( \rho_{A,E} \) is a normalized c-q state, we have

\[
\mathbb{E}_X d'_1(f_X(A)|E|\rho_{A,E}) \leq 2\sqrt{\Delta_{l,2}(M,|\rho_{A,E}|)}
\]

(75)

\[
\mathbb{E}_X I'(f_X(A)|E|\rho_{A,E}) \leq e^{-2H_1(A|E|\rho_{A,E})}.
\]

(76)

\[
\mathbb{E}_X I'(f_X(A)|E|\rho_{A,E}) \leq \eta(\Delta_{d,2}(M,|\rho_{A,E}|, \log |A|d_E)).
\]

(77)

Hence, the quantities \( \mathbb{E}_X d'_1(f_X(A)|E|\rho_{A,E}) \) and \( \mathbb{E}_X I'(f_X(A)|E|\rho_{A,E}) \) can be evaluated by bounding the quantities \( \Delta_{d,2}(M,\varepsilon|\rho_{A,E}|) \) and \( \Delta_{l,2}(M,\varepsilon|\rho_{A,E}|) \). In the next section, we derive upper bounds of these quantities.

V. SECRET KEY GENERATION WITH NO ERROR: SINGLE-SHOT CASE

In this section, in order to evaluate the security of secret key generation with no error in the single-shot case, we evaluate the upper bounds \( \Delta_{d,2}(M,\varepsilon|\rho_{A,E}|) \) and \( \Delta_{l,2}(M,\varepsilon|\rho_{A,E}|) \).

A. \( L_1 \) distinguishability criterion

In order to describe our upper bound of \( \Delta_{d,2}(M,\varepsilon|\rho_{A,E}|) \), we introduce two notations. We denote the number of eigenvalues by \( \nu(\sigma_E) \), and define the real number \( \lambda(\sigma_E) := \log a_1 - \log a_0 \) by using the maximum eigenvalue \( a_1 \) and the minimum eigenvalue \( a_0 \) of \( \sigma_E \). Then, we obtain the following theorem:

**Theorem 24:** Given any c-q sub-state \( \rho_{A,E} \) on \( \mathcal{H}_A \otimes \mathcal{H}_E \) and any normalized state \( \sigma_E \) on \( \mathcal{H}_E \), we have

\[
\Delta_{d,2}(M,\varepsilon|\rho_{A,E}|) \leq (4 + \sqrt{\varepsilon \nu(\sigma_E) M e^{-2H_{1+}(A|E|\rho_{A,E})}})
\]

(78)

\[
\Delta_{d,2}(M,\varepsilon|\rho_{A,E}|) \leq (4 + \sqrt{\varepsilon \lambda(\sigma_E) M e^{-2H_{1+}(A|E|\rho_{A,E})}})
\]

(79)

for \( s \in (0,1] \). Further, when \( \rho_{A,E} \) is normalized,

\[
\Delta_{d,2}(M,\varepsilon|\rho_{A,E}|) \leq (4 + \sqrt{\varepsilon \nu_s M e^{-2H_{1+}(A|\rho_{A,E})}})
\]

(80)

\[
\Delta_{d,2}(M,\varepsilon|\rho_{A,E}|) \leq (4 + \sqrt{\varepsilon \lambda_s M e^{-2H_{1+}(A|\rho_{A,E})}})
\]

(81)

for \( s \in (0,1] \), where \( \nu_s := \nu(\text{Tr}_A \rho_{A,E}^{1+s}/\text{Tr} \rho_{A,E}^{1+s}) \) and \( \lambda_s := \lambda(\text{Tr}_A \rho_{A,E}^{1+s}/\text{Tr} \rho_{A,E}^{1+s}) \).

Indeed, the number \( \nu(\sigma_E) \) in crease at most polynomially when \( \sigma_E \) is i.i.d. However, otherwise, it does not generally behaves polynomially with respect to the system size when the system size increases. On the other hand, the number \( \lambda(\sigma_E) \) is decided
only by the ratio between the maximum and the minimum eigenvalues. In many cases, we can expect that the number $\lambda(\sigma_E)$ behaves linearly with respect to the system size when the system size increases.

**Proof of Theorem 24.** When $\rho'_{A,E} = P\rho_{A,E}P$ with a projection $P$, we have $\|\rho'_{A,E} - \rho_{A,E}\|_1 \leq 2\sqrt{\text{Tr}\rho_{A,E}(I - P)}$. Any projection $P$ satisfies

$$\Delta_{d,2}(M,\varepsilon|\rho_{A,E}) \leq 4\sqrt{\text{Tr}\rho_{A,E}(I - P)} + M^{1/2}e^{-\frac{1}{2}H_{1+s}(A|E|\rho_{A,E})},$$

(82)

We choose $P = \{\mathcal{E}_{\sigma_E}(\rho_{A,E}) - \frac{1}{M}I \otimes \sigma_E \leq 0\}$, where we simplify $\mathcal{E}_{I_A \otimes \sigma_E}$ to $\mathcal{E}_{\sigma_E}$. Since $P$ is commutative with $I \otimes \sigma_E$,

$$\text{Tr}\rho_{A,E}(I - P) = \text{Tr}\rho_{A,E}\mathcal{E}_{\sigma_E}(I - P) = \text{Tr}\mathcal{E}_{\sigma_E}(\rho_{A,E})(I - P) \leq \text{Tr}\mathcal{E}_{\sigma_E}(\rho_{A,E})^{1+s}M^s(I \otimes \sigma_E^s)(I - P) \leq \text{Tr}\mathcal{E}_{\sigma_E}(\rho_{A,E})^{1+s}M^s(I \otimes \sigma_E^s) = M^s e^{-sH_{1+s}(A|E|\mathcal{E}_{\sigma_E}(\rho_{A,E})||\sigma_E}).$$

(83)

Further, using (7), we have

$$e^{-\frac{1}{2}H_{1+s}(A|E|\rho_{A,E})^2} = \text{Tr}P\rho_{A,E}^{s/2}P\rho_{A,E}^{s/2} = \text{Tr}P\rho_{A,E}^{s/2}P\rho_{A,E}^{s/2} = v\epsilon^{-2H_{1+s}(A|E|\rho_{A,E})^2}.$$  

Thus,

$$\frac{M}{2}e^{-\frac{1}{2}H_{1+s}(A|E|\rho_{A,E})^2} \leq \epsilon^{-2H_{1+s}(A|E|\rho_{A,E})^2} = \text{Tr}\mathcal{E}_{\sigma_E}(\rho_{A,E})^{2}M(I \otimes \sigma_E^{-1})^{P} \leq \text{Tr}\mathcal{E}_{\sigma_E}(\rho_{A,E})^{1+s}M^s(I \otimes \sigma_E^s) = M^s e^{-sH_{1+s}(A|E|\mathcal{E}_{\sigma_E}(\rho_{A,E})||\sigma_E)}.$$  

(84)

Substituting (83) and (84) into RHS of (82), we obtain

$$\Delta_{d,2}(M,\varepsilon|\rho_{A,E}) \leq (4 + \sqrt{\epsilon V})M^{s/2}e^{-\frac{1}{2}H_{1+s}(A|E|\rho_{A,E})^2} \leq (4 + \epsilon V)M^{s/2}e^{-\frac{1}{2}H_{1+s}(A|E|\rho_{A,E})^2}.$$  

Hence, we obtain (78).

Next, we show (79). For this purpose, we choose a positive integer $l$. For the given $\lambda = \lambda(\mathcal{E}_E)$, we define $\sigma'_E$ by the following procedure. First, we diagonalize $\mathcal{E}_E$ as $\sigma_E = \sum_y s_y |y\rangle\langle y|$. We define $s_y^l := a_0e^{\lambda y}$ when $s_y \geq a_0$. Hence, $\sigma_E \leq \sigma'_E \leq e^\frac{\lambda}{2} \sigma_E$ and $1 \leq \text{Tr}\sigma'_E \leq \epsilon$. Then, $e^{-\frac{1}{2}H_{1+s}(A|E|\rho_{A,E})^2} \leq \epsilon^{-\frac{1}{2}H_{1+s}(A|E|\rho_{A,E})^2}$, Inequality (78) implies that

$$\Delta_{d,2}(M,\varepsilon|\rho_{A,E}) \leq (4 + \sqrt{\epsilon V})M^{s/2}e^{-\frac{1}{2}H_{1+s}(A|E|\rho_{A,E})^2} \leq (4 + \epsilon V)M^{s/2}e^{-\frac{1}{2}H_{1+s}(A|E|\rho_{A,E})^2}.$$  

Substituting $[\lambda]$ into $l$, we obtain (79).

Applying Lemma 7, we obtain (80) from (78) with $\sigma_E = \frac{\text{Tr}A^{s/2}e^{-\frac{1}{2}H_{1+s}(A|E|\rho_{A,E})^2}}{\text{Tr}A^{s/2}e^{-\frac{1}{2}H_{1+s}(A|E|\rho_{A,E})^2}}$. Similarly, (78) yields (81). Therefore, we obtain Theorem 24.

**Remark 1:** In our proof of the above theorems, the state $\rho'_{A,E}$ is chosen by the information-spectrum-smoothing of the pinched state $\mathcal{E}_{\sigma_E}(\rho_{A,E})$. Since the choice in 27 is also characterized by the information-spectrum-smoothing of the pinched state, our choice is the same as the choice in 27.

**B. Modified mutual information**

The bound $\Delta_{d,2}(M,\varepsilon|\rho_{A,E})$ can be evaluated by using the conditional Rényi entropy $H_{1+s}(A|E|\rho_{A,E})$ as follows.

**Theorem 25:**

$$\Delta_{d,2}(M,\varepsilon|\rho_{A,E}) \leq \frac{1}{2\eta}(2M^{s/2}e^{-\frac{1}{2}H_{1+s}(A|E|\mathcal{E}_{\sigma_E}(\rho_{A,E})^2)}, v \varepsilon / 4 + \log \tilde{M}) \leq \frac{1}{2\eta}(2M^{s/2}e^{-\frac{1}{2}H_{1+s}(A|E|\rho_{A,E})^2)}, v \varepsilon / 4 + \log \tilde{M})$$

(85)

for $s \in (0, 1]$, where $\tilde{M} := \max\{M, d_E\}$ and $v$ is the number of eigenvalues of $\rho_E$.

**Proof of Theorem 25:** When $\rho'_{A,E} = P\rho_{A,E}P$ with a projection $P$, $\|\rho'_{A,E} - \rho_{A,E}\|_1 \leq 2\sqrt{\text{Tr}\rho_{A,E}(I - P)}$. Any projection $P$ satisfies

$$\Delta_{d,2}(M,\varepsilon|\rho_{A,E}) \leq \frac{1}{2\eta}(2\sqrt{\text{Tr}\rho_{A,E}(I - P)}, \log \tilde{M}) + \epsilon M^{s/2}e^{-\frac{1}{2}H_{1+s}(A|E|\rho_{A,E})^2)}.$$  

(87)

We choose $P = \{\mathcal{E}_{\sigma_E}(\rho_{A,E}) - \frac{1}{M}I \otimes \sigma_E \leq 0\}$ with arbitrary real number $\tilde{M}$. Since $P$ is commutative with $I_A \otimes \rho_E$ and $\rho_A \otimes I_E$, the sub-state $\rho'_{A,E} = P\rho_{A,E}P$ satisfies $\rho_E \leq \rho_P$ and $\rho'_A \leq \rho_A$. Hence, we can apply Lemmas 19 and 23.
Further, since $P$ is commutative with $I_A \otimes \rho_E$, similar to (83) and (84), we obtain

$$\text{Tr} \rho_{A,E}(I - P) = \text{Tr} \mathcal{E}_{\rho_E}(\rho_{A,E})(I - P) \leq M^{\ast} e^{-sH_{1+s}(A|E|\rho_{A,E})}$$

(88)

and

$$M e^{-\frac{7}{2}H_{1+s}(A|E|\rho_{A,E})} \leq \text{MMM} e^{-\frac{7}{2}H_{1+s}(A|E|\rho_{A,E})}.$$  

(89)

We choose $M := e^{-\frac{7}{2}H_{1+s}(A|E|\rho_{A,E})}$. Then, we obtain

$$\text{Tr} \rho_{A,E}(I - P) \leq M e^{-\frac{7}{2}H_{1+s}(A|E|\rho_{A,E})}$$

(90)

and

$$M e^{-\frac{7}{2}H_{1+s}(A|E|\rho_{A,E})} \leq eM e^{-\frac{7}{2}H_{1+s}(A|E|\rho_{A,E})}.$$  

(91)

Substituting (90) and (91) to (87), we obtain (85). Then, (86) follows from (15). Therefore, we obtain Theorem 25. 

VI. SECRET KEY GENERATION WITH NO ERROR: ASYMPTOTIC CASE

A. Approximate smoothing of Rényi entropy of order 2

Next, we consider the quantum case when our state is given by the $n$-fold independent and identical state $\rho_{A,E}$, i.e., $\rho_{\otimes n}^{\otimes n}$. In this case, we focus on the optimal generation rate

$$G(\rho_{A,E}) := \sup_{\{f_n, n\}} \left\{ \lim_{n \to \infty} \frac{\log M_n}{n} \mid d'_1(f_n(A_n)|E_n|\rho_{A,E}^{\otimes n}) \to 0 \right\}.$$  

Due to Theorem 24, when the generation rate $R = \lim_{n \to \infty} \frac{\log M_n}{n}$ is smaller than $H(A|E)$, there exists a sequence of functions $f_n : A \to \{1, \ldots, e^{nR}\}$ such that

$$d'_1(f_n(A)|E|\rho_{A,E}^{\otimes n}) \leq 4(\sqrt{\Delta})e^{\frac{7}{2}H_{1+s}(A|E|\rho_{A,E}^{\otimes n})} + \frac{2\Delta}{\sqrt{n}},$$

(92)

where $\Delta$ is the number of eigenvalues of $(\text{Tr} \rho_{A,E}^{\otimes n})$. Thus, we have a polynomial increasing for $n$. Since $\text{lim}_{n \to \infty} \frac{1}{2}H_{1+s}(A|E|\rho_{A,E}) = H(A|E|\rho_{A,E})$, there exists a sequence $s \in (0, 1)$ such that $\frac{1}{2}H_{1+s}(A|E|\rho_{A,E}) - \frac{nR}{2} > 0$. Thus, the right hand side of (92) goes to zero exponentially. Conversely, due to (16), any sequence of functions $f_n : A \to \{1, \ldots, e^{nR}\}$ satisfies that

$$\lim_{n \to \infty} \frac{H(f_n(A)|E|\rho_{A,E}^{\otimes n})}{n} \leq \frac{H(A|E|\rho_{A,E})}{n} = H(A|E|\rho_{A,E}).$$

(93)

Therefore,

$$\lim_{n \to \infty} \frac{H(f_n(A)|E|\rho_{A,E}^{\otimes n})}{n} = R - \lim_{n \to \infty} \frac{H(f_n(A)|E|\rho_{A,E}^{\otimes n})}{n} \geq R - H(A|E|\rho_{A,E}).$$

(94)

That is, when $R > H(A|E|\rho_{A,E})$, $\frac{H(f_n(A)|E|\rho_{A,E}^{\otimes n})}{n}$ does not go to zero. Due to (34), $d'_1(f_n(A)|E|\rho_{A,E}^{\otimes n})$ does not go to zero. Hence, we can recover the result by (45) as

$$G(\rho_{A,E}) = H(A|E|\rho_{A,E}).$$

(95)

In order to treat the speed of this convergence, we focus on the exponentially decreasing rate (exponent) of $d'_1(f_n(A)|E|\rho_{A,E}^{\otimes n})$ for a given $R$. As another criterion, we also focus on a variant $I'(f_n(A_n)|E_n|\rho_{A,E}^{\otimes n}) = I(f_n(A_n) : E_n|\rho_{A,E}^{\otimes n}) + D(\rho_{f_n(A_n)}||\rho_{\text{mix}, f_n(A_n)})$ of the mutual information.

For this purpose, we evaluate the exponential decreasing rates of upper bounds. For a given polynomial $P(n)$, Theorems 24 and 25 yield that

$$\lim_{n \to \infty} \frac{-1}{n} \log \Delta_2(d_2(e^{nR}, P(n)|\rho_{A,E}^{\otimes n}) \geq \varepsilon_{G,n}(\rho_{A,E}|R),$$

(96)

$$\lim_{n \to \infty} \frac{-1}{n} \log \Delta_1(d_1(e^{nR}, P(n)|\rho_{A,E}^{\otimes n}) \geq \varepsilon_{H,n}(\rho_{A,E}|R),$$

(97)

where

$$\varepsilon_{G,n}(\rho_{A,E}|R) := \max_{0 \leq s \leq 1} \frac{s}{2}H_{1+s}(A|E|\rho_{A,E}) - \frac{s}{2}R,$$

(98)

$$\varepsilon_{H,n}(\rho_{A,E}|R) := \max_{0 \leq s \leq 1} \frac{s}{2}H_{1+s}(A|E|\rho_{A,E}) - \frac{3}{2}R.$$
When the side information is classical, i.e., the state is given as a joint distribution $P_{A,E}$ on the joint system, the equation

$$\lim_{n \to \infty} -\frac{1}{n} \log \Delta_{A,2}(e^{nR}, \varepsilon|P_{A,E}) = \max_{0 \leq s \leq 1/2} t(H_{1+s}(A|E|P_{A,E}) - R)$$

(98)

is shown by combination of [71] and the forthcoming paper [53]. Hence, our evaluation is useful in the quantum setting. Since the example given in Subsection VIII-B is very natural in the quantum case, our evaluation is useful in the quantum setting.

Applying Lemma 20 we obtain the following theorem.

**Theorem 26:** When a function ensemble $f_{X^n}$ from $A^n$ to $\{1, \ldots, |e^{nR}|\}$ is universal, 2,

$$\lim_{n \to \infty} -\frac{1}{n} \log EX_n d_1'(f_{X^n}(A_n)|E_n|\rho_{A,E}^{\otimes n}) \geq e_{G,q}(\rho_{A,E}|R).$$

(99)

$$\lim_{n \to \infty} -\frac{1}{n} \log EX_n I'(f_{X^n}(A_n)|E_n|\rho_{A,E}^{\otimes n}) \geq e_{H,q}(\rho_{A,E}|R).$$

(100)

Similarly, using Lemma 23 we obtain the following theorem.

**Theorem 27:** When an ensemble of linear functions $f_{X^n}$ from $A^n$ to $\{1, \ldots, |e^{nR}|\}$ is $P(n)$-almost dual universal, we have

$$\lim_{n \to \infty} -\frac{1}{n} \log EX_n d_1'(f_{C^n}(A_n)|E_n|\rho_{A,E}^{\otimes n}) \geq e_{G,q}(\rho_{A,E}|R).$$

(101)

$$\lim_{n \to \infty} -\frac{1}{n} \log I'(f_{C^n}(A_n)|E_n|\rho_{A,E}^{\otimes n}) \geq e_{H,q}(\rho_{A,E}|R).$$

(102)

In particular, when codes $C_n$ satisfies condition [47], we have

$$\lim_{n \to \infty} -\frac{1}{n} \log d_1'(f_{C_n}(A_n)|E_n|\rho_{A,E}^{\otimes n}) \geq e_{G,q}(\rho_{A,E}|R).$$

(103)

$$\lim_{n \to \infty} -\frac{1}{n} \log I'(f_{C_n}(A_n)|E_n|\rho_{A,E}^{\otimes n}) \geq e_{H,q}(\rho_{A,E}|R).$$

(104)

**B. Comparison for exponents**

Now, we compare exponents given in Theorem 26 with exponents derived by Corollaries 21 and 22. When a function ensemble $f_{X^n}$ from $A^n$ to $\{1, \ldots, |e^{nR}|\}$ is universal, Corollary 22 yields the inequality

$$\lim_{n \to \infty} -\frac{1}{n} \log EX_n d_1'(f_{X^n}(A_n)|E_n|\rho_{A,E}^{\otimes n}) \geq \frac{1}{2} e_{H,q}(\rho_{A,E}|R).$$

(105)

Similarly, Corollary 21 yields the inequality

$$\lim_{n \to \infty} -\frac{1}{n} \log I'(f_{X^n}(A_n)|E_n|\rho_{A,E}^{\otimes n}) \geq e_{G,q}(\rho_{A,E}|R).$$

(106)

under the same condition for a function ensemble $f_{X^n}$.

In order to compare (105) and (106) with (99) and (100), respectively, we prepare the following lemma for two exponents $e_{H,q}(\rho_{A,E}|R)$ and $e_{G,q}(\rho_{A,E}|R)$.

**Lemma 28:** We obtain

$$\frac{1}{2} e_{H,q}(\rho_{A,E}|R) \leq e_{G,q}(\rho_{A,E}|R).$$

(107)

Further, when the relations

$$H_{1+s}(A|E|\rho_{A,E}) = H_{1-s}(A|E|\rho_{A,E})$$

(108)

and

$$R \geq R(2/3) := \left( \frac{2-s}{2} d \right) \left( \frac{s}{2-s} H_{1+s}(A|E|\rho_{A,E}) \right)_{s=\frac{2}{3}}$$

(109)

hold, we obtain a stronger inequality

$$e_{H,q}(\rho_{A,E}|R) \leq e_{G,q}(\rho_{A,E}|R).$$

(110)

Hence, we can conclude that (99) is better than (105). Similarly, under the condition in Lemma 28 (106) is better than (100). However, the relation between (106) and (100) is not clear in general, now. The condition (108) seems too restrictive. However a typical example given in Section VIII satisfies the condition. Hence, Lemma 28 is often useful.

1The part $\geq$ is shown in [71]. The part $\leq$ with $\varepsilon = 1$ is shown in [53]. The LHS is monotonically decreasing for $\varepsilon$. Hence, we have (99) with $\varepsilon \geq 1$. 
Therefore, when the number $n$ is sufficiently large, Inequalities 80 and 81 are better evaluations for the average $\mathbb{E}_n d'_1(\mathcal{F}_{\mathcal{A}}(A_n)|E_n|ho_{A,E}^{\otimes n})$ of the $L_1$ distinguishability criterion than Corollary 22. In this case, if (110) holds, Corollary 21 gives a better evaluation for the average of the modified mutual information criterion $\mathbb{E}_n I'(\mathcal{F}_{\mathcal{A}}(A_n)|E_n|ho_{A,E}^{\otimes n})$ than Inequality 86.

Proof of Lemma 28: Lemma 7 yields that

$$\frac{1}{2} e_{H,q}(\rho_{A,E}|R) = \max_{0 \leq s \leq 1} \frac{1}{2 - s} (H_{1+s}(A|E|\rho_{A,E}) - s R) \leq \max_{0 \leq s \leq 1} \frac{1}{2 - s} (H_{1+s}^G(A|E|\rho_{A,E}) - s R),$$

where Inequality (111) follows from the non-negativity of the RHS of (111) and the inequality $\frac{1}{2 - s} \leq 1$.

Next, we show (109). Assume that the relations (109) and (108) hold. We choose $\mu(s) \triangleq s H_{1+s}(A|E|\rho_{A,E})$. Then, $\mu'(s) \geq 0$ and $\mu''(s) \leq 0$. Defining $R(s) \triangleq \frac{(2-s)^2}{2} \mu(s) \pm \frac{\mu(s)}{2} + \frac{2-s}{2} \mu'(s)$, we have

$$\frac{1}{2} \frac{d}{ds} (\mu(s) - s R) = \frac{d}{ds} \frac{\mu(s)}{2 (2-s)} R = \frac{d}{ds} \frac{\mu(s)}{2 (2-s)} - \frac{2}{(2-s)^2} R(s) + \frac{(R(s) - R)}{(2-s)^2} = \frac{2 (R(s) - R)}{(2-s)^2}.$$  

Since $\frac{d}{ds} R(s) = \frac{2-s}{2} \mu''(s) \leq 0$, the maximum $\max_{0 \leq s \leq 1} (\mu(s) - s R)$ can be attained only when $R = R(s)$. Hence, when (109) holds,

$$e_{H,q}(\rho_{A,E}|R) = \max_{0 \leq s \leq 1} \frac{1}{2 - s} (H_{1+s}(A|E|\rho_{A,E}) - R) = \max_{0 \leq s \leq 1} \frac{1}{2 - s} (H_{1+s}(A|E|\rho_{A,E}) - R).$$

Now, we choose $t$ by $\frac{t}{2 - 2t} = \frac{s}{2 - s}$. Then, $0 \leq t \leq 1/2$ and $t \leq s$ when $0 \leq s \leq 2/3$. Hence, $H_{1+s}(A|E|\rho_{A,E}) - R \leq H_{1+t}(A|E|\rho_{A,E}) - R$, which implies that

$$\max_{0 \leq s \leq 2/3} \frac{s}{2} (H_{1+s}(A|E|\rho_{A,E}) - R) \leq \max_{0 \leq t \leq 1/2} \frac{t}{2 - 2t} (H_{1+t}(A|E|\rho_{A,E}) - R).$$

Since the relation (108) holds,

$$\max_{0 \leq t \leq 1/2} \frac{t}{2 - 2t} (H_{1+t}(A|E|\rho_{A,E}) - R) \geq \max_{0 \leq t \leq 1/2} \frac{t}{2 - 2t} (H_{1+t}(A|E|\rho_{A,E}) - R) = e_{G,q}(\rho_{A,E}|R).$$

C. Non i.i.d. case

Finally, we consider our bounds when the state $\rho_{A,E}^{(n)}$ is given as non i.i.d. state on the system $(\mathcal{H}_A \otimes \mathcal{H}_E)^{\otimes n}$.

In this case, the speeds of increase of $v$ and $v_s$ are not polynomial with respect to the size $n$ of the system, in general. Hence, when $n$ is sufficiently large, the factor $v$ and $v_s$ are not negligible.

However, when the minimum eigenvalue of $\rho_{A,E}^{(n)}$ is greater than $c^n$ with a constant $c > 0$, the minimum eigenvalue of $T(A_n, 1+1/s) / T(A_n, 1+s)$ is greater than $c^{1+s} n$. Hence, $\lambda_n$ increases linearly with respect to $n$. Thus, when the key generation rate is $R$, the upper bound (61) for the $L_1$ distinguishability criterion has the factor of the order $O(\sqrt{n})$ with the term $e^{2\mu_0 - \frac{R}{2} H_{1+s}^G(A|E|\rho_{A,E}) + \frac{R}{2}}$. For the modified mutual information criterion, Theorem 25 cannot derive a good bound because the factor $v$ does not behave polynomially. Instead of Theorem 25, Corollary 21 gives a better upper bound, which has the factor of the order $O(n^{3/2})$ with the term $e^{2\mu_0 - \frac{R}{2} H_{1+s}^G(A|E|\rho_{A,E}) + \frac{R}{2}}$. Hence, Theorem 24 and Corollary 21 have a larger applicability beyond the i.i.d. case.

VII. SECRET KEY GENERATION WITH ERROR CORRECTION

A. Protocol

Next, we apply the above discussions to secret key generation with public communication. Alice is assumed to have an initial random variable $a \in \mathcal{A}$, which generates with the probability $p_a$, and Bob and Eve are assumed to have their initial quantum states $\rho_{B|a}$ and $\rho_{E|a}$ on their quantum systems $\mathcal{H}_B$ and $\mathcal{H}_E$, respectively. The task for Alice and Bob is to share a common random variable almost independent of Eve’s quantum state by using a public communication. The quality is evaluated by three quantities: the size of the final common random variable, the probability of the disagreement of their final variables (error probability), and the information leaked to Eve, which can be quantified by the $L_1$ distinguishability criterion or the modified mutual information criterion between Alice’s final variables and Eve’s state.

In order to construct a protocol for this task, we assume that the set $\mathcal{A}$ is a vector space on a finite field $\mathbb{F}_q$. Indeed, even if the cardinality $|\mathcal{A}|$ is not a prime power, it become a prime power by adding elements with zero probability. Hence, we can assume that the cardinality $|\mathcal{A}|$ is a prime power $q$ without loss of generality. Then, the secret key agreement can be realized...
by the following two steps: The first is the error correction, and the second is the privacy amplification. In the error correction, Alice and Bob prepare a linear subspace $C_1 \subset \mathcal{A}$ and the representatives $a(x)$ of all cosets $x \in \mathcal{A}/C_1$. Alice sends the coset information $[A] \in \mathcal{A}/C_1$ to Bob in stead of her random variable $A \in \mathcal{A}$, and Bob obtain his estimate $\hat{A}$ of $A \in \mathcal{A}$ from his quantum state on $\mathcal{H}_B$ and $[A] \in \mathcal{A}/C_1$. Alice obtains her random variable $A_1 := A - a([A]) \in C_1$, and Bob obtains his random variable $\hat{A}_1 := \hat{A} - a([B]) \in C_1$. In the privacy amplification, Alice and Bob prepare a common hash function $f$ on $C_1$. Then, applying the hash function $f$ to the their variables $A_1$ and $\hat{A}_1$, they obtain their final random variables $f(A_1)$ and $f(\hat{A}_1)$.

Indeed, the above protocol depends on the choice of estimator that gives the estimate $\hat{A}$ from $[A] \in \mathcal{A}/C_1$ and his random variable $B \in B$ (or his quantum state on $\mathcal{H}_B$). In the remaining part of this section, we give the estimator depending on the setting and discuss the performance of this protocol.

### B. Error probability

In the following, we give the concrete form of the estimator and evaluate the error probability when Bob’s information is quantum. In this case, we construct an estimator for $\hat{A}$ in the following way. For a given code $C_1 \subset \mathcal{A}$ and a normalized c-q state $\rho_{A,B} = \sum_a P_A(a)\langle a \rangle \otimes \rho_{B|a}$, our decoder is given as follows: First, we define the projection:

$$P_a := \{ P_A(a)\rho_{B|a} - \frac{q}{|A|} \rho_B \geq 0 \},$$

where $t$ is the dimension of $C_1$. When Bob receives the coset $[A]$, he applies the POVM $\{ P'_a \}$:

$$P'_a := Q_{[A]}^{-1/2} P_a Q_{[A]}^{-1/2}, \quad Q_{[A]} := \sum_{a \in [A]} P_a.$$  

Then, Bob chooses the outcome $a$ as the estimate $\hat{A}$.

Next, we evaluate the performance of the error probability. Using the operator inequality ([14] Lemma 4.5), we obtain

$$I - P'_a \leq 2(I - P_a) + 4 \sum_{a' \in C_1 + a \setminus \{a\}} P_{a'}.$$  

Thus, the error probability $P_e[\rho_{A,B}, C_1]$ is evaluated as follows.

$$P_e[\rho_{A,B}, C_1] = \sum_a P_A(a)\text{Tr} \rho_{B|a}(I - P'_a) \leq 2 \sum_a P_A(a)\text{Tr} \rho_{B|a}(I - P_a) + 4 \sum_a P_A(a)\text{Tr} \rho_{B|a} \sum_{a' \in C_1 + a \setminus \{a\}} P_{a'}.$$  

Now, we choose the code $C_1$ from $\varepsilon$-almost universal2 code ensemble $\{ C_X \}$ with the dimension $t$. Then, the average of the error probability can be evaluated as

$$\mathbb{E}_{P}[\rho_{A,B}, C_X] \leq 2 \sum_a P_A(a)\text{Tr} \rho_{B|a}(I - P_a) + 4\mathbb{E} \sum_a P_A(a)\text{Tr} \rho_{B|a} \sum_{a' \in C_1 + a \setminus \{a\}} P_{a'}$$

$$\leq 2 \sum_a P_A(a)\text{Tr} \rho_{B|a}(I - P_a) + 4\varepsilon \frac{q^t}{|A|} \sum_{a' \neq a} P_{a'}$$

$$\leq 2 \sum_a \text{Tr} (P_A(a)\rho_{B|a})^{1-s} \rho_B^{s} (\frac{q^t}{|A|})^s + 4\varepsilon \sum_{a'} \text{Tr} (P_A(a')\rho_{B|a'})^{1-s} \rho_B^{s} (\frac{q^t}{|A|})^s$$

$$= (2 + 4\varepsilon) \sum_{a'} \text{Tr} (P_A(a')\rho_{B|a'})^{1-s} \rho_B^{s} (\frac{q^t}{|A|})^s = (2 + 4\varepsilon) (\frac{q^t}{|A|})^s e^{sH_{1-s}[A|B]\rho_{A,n}}.$$  

### C. Leaked information with fixed error correction

As is mentioned in the previous sections, we have two criteria for quality of secret random variables. Given a code $C_1 \subset \mathcal{A}$ and a hash function $f$, the first criterion is $d'_f(f(A_1)||[A], E[|A,E])$, and the second criterion is $I'(f(A_1)||[A], E[|A,E])$. Note that the random variable $A$ can be written by the pair of $A_1$ and $[A]$ given in Subsection VII-A.

**Theorem 29**: When $\{ f_X \}$ is a universal2 ensemble of hash functions from $\mathcal{A}/C_1$ to $\{1, \ldots, M\}$, the relations

$$\mathbb{E}d'_f(f(A_1)||[A], E[|A,E]) \leq (4 + \sqrt{\nu})(|A|/L)^{s/2}e^{sH_{1-s}[A|B]\rho_{A,n}},$$

$$\mathbb{E}I'(f(A_1)||[A], E[|A,E]) \leq \eta((4 + \sqrt{\nu})(|A|/L)^{s/2}e^{sH_{1-s}[A|B]\rho_{A,n}} \log |A|d_E)$$
hold for $s \in (0,1]$, where $v'$ is the number of eigenvalues of $\operatorname{Tr} A \rho'_{A,E}$, $v$ is the number of eigenvalues of $\rho_E$, $L$ is the amount of sacrifice information $|C_1|/|M|$, and $\tilde{M} := \max\{M,d_E\}$.

**Proof:** The relations (80) and (24) guarantee that

$$E_X d'_1(E_X(\rho_{A,E}) || [A], E | \rho_{A,E}) \leq (4 + \sqrt{v'})M^{1/2}e^{\frac{\lambda}{2}} H_{1+s}^{\leq}(A_1[A], E | \rho_{A,E}) \leq (4 + \sqrt{v'})M^{1/2}(|A|/|C_1|)^{1/2}e^{\frac{\lambda}{2}} H_{1+s}^{\leq}(A_1[A], E | \rho_{A,E})$$

$$= (4 + \sqrt{v'})(|A|/|C_1|)^{1/2}e^{\frac{\lambda}{2}} H_{1+s}^{\leq}(A_1[E | \rho_{A,E})}$$

for $s \in (0,1]$, which implies (115). A simple combination of (86) and (115) yields (116).

Similarly, Lemma (23), (80), and (24) yield the following theorem.

**Theorem 30:** When $\{f_X\}$ is an $\varepsilon_1$-almost dual universal $2$ ensemble of hash functions from $A/C_1$ to $\{1, \ldots, M\}$, the relation

$$E_X d'_1(f_X(A_1) || [A], E | \rho_{A,E}) \leq (4 + \sqrt{v'})M^{1/2}e^{\frac{\lambda}{2}} H_{1+s}^{\leq}(A_1[A], E | \rho_{A,E})$$

$$E_X I'(f_X(A_1) || [A], E | \rho_{A,E}) \leq \eta((4 + \sqrt{v'})(|A|/L)^{1/2}e^{\frac{\lambda}{2}} H_{1+s}^{\leq}(A_1[E | \rho_{A,E}), \log |A|d_E)$$

holds for $s \in (0,1]$, where $v'$ is the number of eigenvalues of $\operatorname{Tr} A \rho_{A,E}$, $v$ is the number of eigenvalues of $\rho_E$, $L$ is the amount of sacrifice information $|C_1|/|M|$, and $\tilde{M} := \max\{M,d_E\}$. Similarly, when $\{f_Y\}$ is universal $2$ ensemble of hash functions from $A/C_1$ to $\{1, \ldots, M\}$,

$$E_X I'(f_Y(A_1) || [A]|C_X, E | \rho'_{A,E}) \leq 2\eta((2+|A|/q^t) e^{-\frac{\lambda}{2}} H_{1+s}(A|\rho_{A,E}) \log \tilde{M} + \frac{v}{2\varepsilon_1} + \log \varepsilon_1$$

for $s \in (0,1]$, where $v$ is the number of eigenvalues of $\rho_E$ and $\tilde{M} := \max\{M,d_E\}$. Similarly, when $\{f_Y\}$ is universal $2$ ensemble of hash functions from $A/C_X$ to $\{1, \ldots, M\}$,

$$E_X I'(f_Y(A_1) || [A]|C_X, E | \rho'_{A,E}) \leq 2\eta((2+|A|/q^t) e^{-\frac{\lambda}{2}} H_{1+s}(A|\rho_{A,E}) \log \tilde{M} + \frac{v}{4\varepsilon_1} + \log \varepsilon_1$$

**Proof:** We choose a sub cq-state $\rho'_{A,E} = \sum_a |a\rangle \langle a| \otimes \rho'_{E|a}$ such that $\rho'_{E} \leq \rho_E$ and $\rho'_{A} \leq \rho_A$. Due to (80), we obtain

$$E_X e^{-\frac{1}{M} E_X I'(f_Y(A_1) || [A]|C_X, E | \rho'_{A,E})} \leq \varepsilon_2(1 - \frac{1}{M}) e^{-\frac{1}{M} E_X I'(f_Y(A_1) || [A]|C_X, E | \rho'_{A,E})} + \frac{1}{M} E_X I'(f_Y(A_1) || [A]|C_X, E | \rho'_{A,E})}$$

Since the matrix $\rho'_{A,E}$ satisfies

$$e^{-\frac{1}{M} E_X I'(f_Y(A_1) || [A]|C_X, E | \rho'_{A,E})} \leq \sum_a \operatorname{Tr} E \rho'_{E} E_{\{a\}} \sum_{a' \in C_X \cup \{a\}} \rho'_{E|a'} \leq \sum_a \operatorname{Tr} E \rho'_{E} E_{\{a\}} \sum_{a' \in C_X \cup \{a\}} \rho'_{E|a'}$$

we have

$$E_X e^{-\frac{1}{M} E_X I'(f_Y(A_1) || [A]|C_X, E | \rho'_{A,E})} \leq E_X e^{-\frac{1}{M} E_X I'(f_Y(A_1) || [A]|C_X, E | \rho'_{A,E})} \leq E_X e^{-\frac{1}{M} E_X I'(f_Y(A_1) || [A]|C_X, E | \rho'_{A,E})} \leq \varepsilon_1(\frac{1}{M}) e^{-\frac{1}{M} E_X I'(f_Y(A_1) || [A]|C_X, E | \rho'_{A,E})}$$
where the first inequality follows from $\varepsilon_2 \geq 1$.

Hence, we obtain

$$E_{X,Y}e^{-\mathcal{P}_2(f_Y(A_1))[A]_{C_X,E}\rho'_{A,E}\|\rho_{mix,[A]_{C_X}} \otimes \rho_E}} \leq \varepsilon_2\left|A\right|e^{-\mathcal{P}_2(A|E)\rho'_{A,E}\|\rho_E}} + \frac{1}{M}\varepsilon_1 = \frac{1}{M}\varepsilon_1(1 + \frac{\varepsilon_2}{\varepsilon_1}M e^{-\mathcal{P}_2(A|E)\rho'_{A,E}\|\rho_E}}).$$

Applying Jensen’s inequality to $x \mapsto \log x$, we obtain

$$E_{X,Y} - \mathcal{P}_2(f_Y(A_1))[A]_{C_X,E}\rho'_{A,E}\|\rho_{mix,[A]_{C_X}} \otimes \rho_E} \leq - \log M + \log \varepsilon_1 + \log(1 + \frac{\varepsilon_2}{\varepsilon_1}M e^{-\mathcal{P}_2(A|E)\rho'_{A,E}\|\rho_E}}).$$

Using (63), (17), and (19), we obtain

$$I'(f_Y(A_1))[A]_{C_X,E}\rho'_{A,E}\|\rho_A.E) = \log M - \mathcal{H}(f_Y(A_1))[A]_{C_X,E}\rho'_{A,E}) \leq 2\eta(\|\rho'_{A,E} - \rho'_{A,E}||_{1}, \log M) + \log M - \mathcal{H}(f_Y(A_1))[A]_{C_X,E} \rho'_{A,E} - \rho_{mix,[A]_{C_X}} \otimes \rho_E) \leq 2\eta(\|\rho'_{A,E} - \rho_{mix,[A]_{C_X}} \otimes \rho_E) \leq 2\eta(\|\rho'_{A,E} - \rho_{mix,[A]_{C_X}} \otimes \rho_E).$$

Hence, we obtain

$$E_{X,Y}I'(f_Y(A_1))[A]_{C_X,E}\rho'_{A,E}\|\rho_A.E) \leq 2\eta(\|\rho'_{A,E} - \rho_{mix,[A]_{C_X}} \otimes \rho_E) \leq 2\eta(\|\rho'_{A,E} - \rho_{mix,[A]_{C_X}} \otimes \rho_E) + \log M + \log \varepsilon_1 + \log(1 + \frac{\varepsilon_2}{\varepsilon_1}M e^{-\mathcal{P}_2(A|E)\rho'_{A,E}\|\rho_E}}).$$

Applying the same discussion as the proof of Theorem 25 we obtain

$$E_{X,Y}I'(f_Y(A_1))[A]_{C_X,E}\rho'_{A,E}\|\rho_A.E) \leq 2\eta((2\frac{\|\rho'_{A,E} - \rho_{mix,[A]_{C_X}} \otimes \rho_E}) \leq 2\eta((2\frac{\|\rho'_{A,E} - \rho_{mix,[A]_{C_X}} \otimes \rho_E}) + \log M + \frac{\varepsilon_2}{\varepsilon_1}M e^{-\mathcal{P}_2(A|E)\rho'_{A,E}\|\rho_E}}).$$

E. Asymptotic analysis

Next, we consider the case when the c-q state is given as the n-fold independent and identical extension $\rho_{A,B,E}^{n}$ of a c-q normalized state $\rho_{A,B,E}$, where $A$ is $\mathbb{F}_q$. Now, we fix codes $C_{1,n}$ in $\mathbb{F}_q^n$ with the dimension $n\frac{R_A}{\log q}$. Then, we obtain the following theorem.

**Theorem 32:** When $\{f_X\}$ is a universal2 ensemble of hash functions from $\mathbb{F}_q^n/C_{1,n}$ to $\mathbb{F}_q^{\frac{n-\frac{R_A-\frac{2}{3}}{\log q}}{\frac{R_A}{\log q}}}$, the relations

$$\liminf_{n \to \infty} -\frac{1}{n} \log E_Xd_1(f_X(A_1,n))[A], E_n|\rho_{A,E}^{\otimes n}) \geq \max_{0 \leq s \leq 1} \frac{s}{2}(R_2 - \log q) + \frac{s}{2}\mathcal{H}_{1+s}(A|E)|\rho_{A,E}) = c_{G,q}(\rho_{A,E}) \log q - R_2),$$

$$\liminf_{n \to \infty} -\frac{1}{n} \log E_XI'(f_X(A_1,n))[A], E_n|\rho_{A,E}^{\otimes n}) \geq c_{G,q}(\rho_{A,E}) \log q - R_2)$$

hold.

**Proof:** (115) and (116) yield (124) and (125), respectively.

Similarly, we have the following theorem.

**Theorem 33:** When $P(n)$ is an arbitrary polynomial and $\{f_X\}$ is a $P(n)$-almost dual universal2 ensemble of hash functions from $\mathbb{F}_q^n/C_{1,n}$ to $\mathbb{F}_q^{\frac{n-\frac{R_A-\frac{2}{3}}{\log q}}{\frac{R_A}{\log q}}}$, the relations (124) and (125) hold.

**Proof:** (117) and (118) yield Inequalities (124) and (125), respectively.

Next, we consider the case when the error correcting code is chosen randomly. In this case, the exponential decreasing rate for $I'(f_X(A_1,n))[A], E_n|\rho_{A,E}^{\otimes n})$ can be improved as follows.

**Theorem 34:** For independent random variables $X, Y$, we assume that the code ensemble $\{C_X\}$ with the dimension $n\frac{R_A}{\log q}$ is universal2 and $\{f_Y\}$ is universal2 ensemble of hash functions from $\mathbb{F}_q^n/C_X$ to $\mathbb{F}_q^{\frac{n-\frac{R_A-\frac{2}{3}}{\log q}}{\frac{R_A}{\log q}}}$, the relations (124), (116), and

$$\liminf_{n \to \infty} -\frac{1}{n} \log E_XP_{e}\rho_{A,B}^{\otimes n}, C_X \geq \max_{0 \leq s \leq 1} \frac{s}{2}(R_2 - \log q - R_1) - s\mathcal{H}_{1-s}(A|B)|\rho_{A,B}),$$

$$\liminf_{n \to \infty} -\frac{1}{n} \log E_XYI'(f_X(A_1,n))[A]_{C_X}, E_n|\rho_{A,E}^{\otimes n}) \geq c_{H,q}(P_{A,E}) \log q - R_2)$$

hold.

**Proof:** Theorem 31 implies that

$$\liminf_{n \to \infty} -\frac{1}{n} \log E_XYI'(f_X(A_1,n))[A]_{C_X}, E_n|\rho_{A,E}^{\otimes n}) \geq \max_{0 \leq s \leq 1} \frac{s}{2}(R_2 - \log q + \mathcal{H}_{1+s}(A|E)|\rho_{A,E}) = c_{H,q}(P_{A,E}) \log q - R_2),$$
which yields (127). Due to (114), the error probability can be bounded as
\[ \mathbb{E}_{X,n}P_e[\rho_{A,B}^n, C_{X,n}] \leq P(n)e^{n(s(R_1 - \log q) + sH_1 - s(A|B,p_{A,B}))} \]
for \( s \in [0,1] \), which implies (126).

Similarly, we obtain the following theorem.

Theorem 35: For an arbitrary polynomial \( P(n) \) and the independent random variables \( X, Y \), we assume that the code ensemble \( \{C_X\} \) is universal and \( \{f_Y\} \) is a \( P(n) \)-almost dual universal ensemble of hash functions from \( \mathbb{F}_q^n/C_X \) to \( \mathbb{F}_q^{n - \frac{R_1}{\log q} - \frac{R_2}{2\log q}} \). Thus, the relation (124), (116), (126), and (127) hold.

For a comparison between two exponents \( e_{G,q}(\rho_{A,E}|R) \) and \( e_{G,q}(\rho_{A,E}|R) \), see Lemma 28.

VIII. APPLICATION TO GENERALIZED PAULI CHANNEL

A. General case

In order to apply the above result to quantum key distribution, we treat the quantum state generated by transmission by a generalized Pauli channel in the \( p \)-dimensional system \( \mathcal{H} \). First, we define the discrete Weyl-Heisenberg representation \( W \) for \( \mathbb{F}_p^2 \):
\[ X := \sum_{j=0}^{p-1} |j + 1\rangle \langle j|, \quad Z := \sum_{j=0}^{p-1} \omega^j j \langle j|, \quad W(x,z) := X^xZ^z, \]
where \( \omega \) is the root of the unity with the order \( p \). Using this representation and a probability distribution \( P_{XZ} \) on \( \mathbb{F}_p^2 \), we can define the generalized Pauli channel:
\[ \mathcal{E}_P(\rho) := \sum_{(x,z)\in\mathbb{F}_p^2} P_{XZ}(x,z)W(x,z)\rho W(x,z)^\dagger. \]

In the following, we assume that the eavesdropper can access all of the environment of the channel \( \mathcal{E}_P \). When the state \( |j\rangle \) is input to the channel \( \mathcal{E}_P \), the environment system is spanned by the basis \( \{|x,z\}_E\}. Then, the state \( \rho_{E|j} \) of the environment (Eve’s state) and Bob’s state \( \rho_{B|j} \) are given as
\[ \rho_{E|j} = \frac{1}{p-1} \sum_{z=0}^{p-1} P_{Z}(z) |j, z: P_{XZ}| \langle j, z: P_{XZ}|, \quad |j, z: P_{XZ}| := \sum_{x=0}^{p-1} \omega^x(z) \sqrt{P_{X|Z}(x|z)} |x, z\rangle_E \]
\[ \rho_{B|j} = \sum_{x=0}^{p-1} P_{X}(x) |j + x\rangle_B \langle j + x|. \]

Thus, the relation
\[ \sum_{a=0}^{p-1} |a, z: P_{XZ}| \langle j, z: P_{XZ}| = p \sum_x P_{X|Z}(x|z) |x, z\rangle_E E\langle x, z| \]
holds. Hence,
\[ \rho_E = \sum_{x,z} P_{X,Z}(x,z) |x, z\rangle_E E\langle x, z|. \quad (128) \]

Then, we obtain the following state after the quantum state transmission via the generalized Pauli channel.
\[ \rho_{A,B,E} := \sum_{j=0}^{p-1} \frac{1}{p} |j\rangle \langle j| \otimes \rho_{B|j} \otimes \rho_{E|j}. \]

In this setting, the joint state \( \rho_{A,B} \) is classical, we can apply the classical theory for error probability. Since \( P_{A,B}(a,b) = \frac{1}{p} \sum_a P_X(b - a) \), we have
\[ e^{-sH^G(A|B,p_{A,B})} = \sum_{a=0}^{p-1} \frac{1}{p} \sum_{a=0}^{p-1} P_X(b - a)^{1/(1-s)} \left| 1 - s \right| = \left( \sum_x P_X(x) \right)^{1/(1-s)} \left| 1 - s \right| = e^{-sH} (X|P_X) \]

Now, we choose the rate \( R_1 \) of size of code \( C_1 \). When \( \{C_{X,n}\} \) is the \( P(n) \)-almost universal code ensemble in \( \mathbb{F}_q^n \) with the dimension \( n \frac{R_1}{\log q} \), due to (71) (243), the decoding error probability can be bounded as
\[ \mathbb{E}_{X,n}P_e[\rho_{A,B}^n, C_{X,n}] \leq P(n)e^{n(s(R_1 - \log q) - sH_1 (X|P_X))} \].
That is,
\[
\liminf_{n\to\infty} -\frac{1}{n} \log E_{X^n} P_e^{\otimes^n} (\rho_{A,B}^{\otimes^n}, C_{X^n}) \geq \max_{0 \leq s \leq 1} s (\log q - R_1) + s H_{1-s}(X|P_X).
\]

Next, we treat the leaked information. In the following discussion, we fix codes \( C_{1,n} \) in \( F_p^n \). Since \( \rho_{A,E} = \sum_a \frac{1}{q} |a\rangle \otimes \rho_{E|a} \), we have
\[
\begin{align*}
e^{-sH_{1-s}^{\otimes^n}((A|E)|\rho_{A,E})} &= \text{Tr}_{E} (\text{Tr}_{A} \sum_{a} \frac{1}{p} |a\rangle \otimes \rho_{E|a})^{1-s}) = \frac{1}{p^{1+s}} \text{Tr}_{E} (\sum_{a} \rho_{E|a})^{1-s}) \\
&= \frac{1}{p} \text{Tr}_{E} (\sum_{a} \sum_{z=0}^{p-1} P_{Z}(z) |a, z : P_{XZ}) (a, z : P_{XZ})^{1-s}) = \frac{1}{p} \text{Tr}_{E} (\sum_{z=0}^{p-1} P_{Z}(z) \sum_{a} |a, z : P_{XZ}) (a, z : P_{XZ})^{1-s}) \\
&= \frac{1}{p} \text{Tr}_{E} (\sum_{z=0}^{p-1} P_{Z}(z) \sum_{x} P_{X|Z}(x|z) E\{x, z\})^{1-s}) = p^{-s} \text{Tr}_{E} (\sum_{z=0}^{p-1} \sum_{x} P_{Z}(z) P_{X|Z}(x|z) (x, z) E\{x, z\})^{1-s}) \\
&= p^{-s} e^{-sH_{1-s}(X|P_X)}.
\end{align*}
\]

That is, we have
\[
H_{1-s}(A|E|\rho_{A,E}) = H_{1-s}^{\otimes^n}((A|E)|\rho_{A,E}) = \log p - H_{1-s}(X|P_X).
\]

Now, we consider the case with randomized error correction. Given a sequence of fixed codes \( C_{1,n} \), we focus on a sequence of ensembles of hash functions of \( F_p^n / C_{1,n} \) with the rate \( R_2 \) of sacrifice information (i.e., with the sacrifice bit length \( L = nR_2 \)).

In this case, the numbers of eigenvalues of \( \rho_{E}^{\otimes n} \) and \( \text{Tr}_{A} (\rho_{A,E}^{\otimes n})^{1+s} \) are less than \((n+1)(p^2-1)\). Thus, when the code ensemble \( \{ C_X \} \) with the dimension \([n \frac{R_1}{\log_2 q}] \) is universal and \( \{ f_Y \} \) is a \( \varepsilon \)-almost dual universal \( 2 \) ensemble of hash functions from \( F_q^n / C_X \) to \( F_q^{[n \frac{1-\varepsilon - R_1}{\log_2 q}]}, \) (177), (178), and (179) yield that
\[
\begin{align*}
E_{X,Y} d'((f_Y(A_1,n))[[A_n]|C_X, E_n]\rho_{E}^{\otimes^n}) &\leq 4(n+1)(p^2-1)/2 \sqrt{\varepsilon} e^{n \frac{1}{4} (-R_2 + H_{1-s}(X|P_X))}, \\
E_{X,Y} I'((f_Y(A_1,n))[[A_n]|C_X, E_n]\rho_{E}^{\otimes^n}) &\leq 4(n+1)(p^2-1)/2 \sqrt{\varepsilon} e^{n \frac{1}{4} (-R_2 + H_{1-s}(X|P_X))}, \\
E_{X,Y} I'((f_Y(A_1,n))[[A_n]|C_X, E_n]\rho_{E}^{\otimes^n}) &\leq 2\eta(2e^{n \frac{1}{4} (-R_2 + H_{1-s}(X|P_X))} + n \log p). 
\end{align*}
\]

In particular, when \( \{ f_Y \} \) is a universal \( 2 \) ensemble of hash functions, due to (115), (116), and (120), the real number \( \varepsilon \) can be replaced by 1 in the above inequalities.

Here, we need a remark for (133). The second input of the function \( \eta \) in (133) is \( n \log p \) not \( 2n \log p \). In this case, the state \( \rho_A \) is the uniform distribution, we can use (35) instead of (36). Hence, we can replace \( 2n \log p \) by \( n \log p \).

The exponents \( e_{C_{1,n}}(\rho_{A,E}) \log p - R_2 \) and \( e_{H_{1-s}}(\rho_{A,E}) \log p - R_2 \) are calculated as
\[
\begin{align*}
e_{C_{1,n}}(\rho_{A,E}) \log p - R_2 &= \max_{0 \leq s \leq 1} \frac{R_2}{2} \frac{R_2 - H_{1-s}(X|P_X)}{2} \\
e_{H_{1-s}}(\rho_{A,E}) \log p - R_2 &= \max_{0 \leq s \leq 1} \frac{1}{2} \frac{R_2 - H_{1-s}(X|P_X)}{2} = \max_{0 \leq s \leq 1} \frac{R_2 - H_{1-s}(X|P_X)}{2}
\end{align*}
\]

where \( \frac{1}{2} \frac{R_2}{2} = \frac{R_2}{4} \). In fact, our bound in (137) is the same as the bound obtained by the recent paper (60) via the phase error correction approach. This fact seems the goodness of our bound and our approach.

Since \( \frac{s}{2} \sqrt{\varepsilon} \leq \frac{s}{2} \sqrt{\varepsilon} \) for \( s \in [0, 1] \), Lemma 2 guarantees that \( H_{1-s}(X|P_X) \geq H_{1-s}(X|P_X) \), which implies \( e_{H_{1-s}}(\rho_{A,E}) \log p - R_2 \leq e_{G_{1-s}}(\rho_{A,E}) \log p - R_2 \). That is, (133) gives a better exponent than (134). Since the relation (108) holds due to (131), this case can be regarded as a special case of Lemma 28. Thus, we obtain
\[
\begin{align*}
\liminf_{n \to \infty} -\frac{1}{n} \log E_{X^n} d'((f_Y(A_1,n))[[A_n]|C_X, E_n]\rho_{E}^{\otimes^n}) &\geq e_{G_{1-s}}(\rho_{A,E}) \log p - R_2 \\
\liminf_{n \to \infty} -\frac{1}{n} \log E_{X^n} I'((f_Y(A_1,n))[[A_n]|C_X, E_n]\rho_{E}^{\otimes^n}) &\geq e_{G_{1-s}}(\rho_{A,E}) \log p - R_2.
\end{align*}
\]

However, there still exists a possibility that the evaluation (134) gives a better evaluation than (133) in the finite length setting.
B. Independent case

Next, we consider the case when the two random variables \( X \) and \( Z \) are independent, Eve's state \( \rho_{E|j} \) has the following form:

\[
\rho_{E|j} = |j : P_X \rangle \langle j : P_X | \otimes \sum_{z=0}^{p-1} P_Z(z) |z \rangle_Z \langle z |, \quad |j : P_X | := \sum_{x=0}^{p-1} \omega^{jx} \sqrt{P_X(x)} |x \rangle_X.
\]

In this case, the system spanned by \( \{ |z \rangle_Z \} \) has no correlation with \( j \), and only the system spanned by \( \{ |x \rangle_X \} \) has correlation with \( j \). So, we can replace \( \rho_{E|j} \) by the following way:

\[
\rho_{E|j} = |j : P_X \rangle \langle j : P_X |.
\]

In this case, the numbers of eigenvalues of \( \rho_E \) and \( \text{Tr}_A \rho_{A,E}^{1+n} \) are less than \( p \). Hence, the numbers of eigenvalues of \( \rho_E^{\leq n} \) and \( \text{Tr}_A (\rho_{A,E}^{\leq n})^{1+s} \) are less than \( (n+1)^{(p-1)} \). When we choose \( \varepsilon = 1 \) for simplicity, the inequalities (132), (133), and (134) can be simplified to

\[
\begin{align*}
E_{X,Y} d'_1(f_Y(A_{1,n})||A|_C X, E_n | \rho_{A,E}^{\leq n} ) & \leq 4 + (n+1)^{(p-1)/2} e^{nH(X|P_X)}, \quad (139) \\
E_{X,Y} I'(f_Y(A_{1,n})||A|_C X, E_n | \rho_{A,E}^{\leq n} ) & \leq 2 \eta (4 + (n+1)^{(p-1)/2}) e^{nH(X|P_X)}, \quad (140) \\
E_{X,Y} I'(f_Y(A_{1,n})||A|_C X, E_n | \rho_{A,E}^{\leq n} ) & \leq 2 \eta (2 e^{nH(X|P_X)}) (n + 1)^{(p-1)/4 + n \log p). \quad (141)
\end{align*}
\]

Hence, we obtain

\[
\begin{align*}
\liminf_{n \to \infty} -\frac{1}{n} \log E_{X,Y} d'_1(f_Y(A_{1,n})||A|_C X, E_n | \rho_{A,E}^{\leq n} ) & \geq e_{G,L}(\rho_{A,E} | \log p - R_2) \\
& = \max_{0 \leq \varepsilon \leq 1/2} \frac{s R_2 - s H_{1-s}(P_X)}{2(1-s)}. \quad (142)
\end{align*}
\]

Here, we compare the evaluations (140) and (141). As is explained in the previous subsection, the exponent of (141) is better than (140). This relation can be numerically checked in Fig. 1 with the parameters \( p = 2, P_X(0) = 0.9, P_X(1) = 0.1, \) and \( R \in (0.53, 0.58) \). However, in the case of a finite \( n \), \(-\frac{1}{n} \log \min_{0 \leq s \leq 1} (\text{RHS of } (140)) \) is not necessarily larger than \(-\frac{1}{n} \log \min_{0 \leq s \leq 1} (\text{RHS of } (141)) \). The relation between these two quantities is also numerically demonstrated in Fig. 1 with the same parameters when \( n = 10,000 \). This numerical result suggests that the exponents can not necessarily decide the order of advantages with the finite size \( n \) when \( n \) is not sufficiently large.

![Fig. 1](image_url)

Next, we consider the case when there is no error in \( Z \) basis. In this case, it is sufficient to apply only privacy amplification. Hence, we evaluate the upper bounds \( \Delta_{d,2}(e^{nR}; \varepsilon_1 | \rho_{A,E}^{\leq n} ) \) as follows.

**Lemma 36:** When \( p = 2 \) and \( \rho_{A,E} = \sum_{x \in F_2} |x \rangle \langle x | \otimes | x : P_X \rangle \langle x : P_X | \), we have

\[
\lim_{n \to \infty} -\frac{1}{n} \log \Delta_{d,2}(e^{nR}; \varepsilon_1 | \rho_{A,E}^{\leq n} ) = e_{G,L}(\rho_{A,E} | R) = \max_{0 \leq s \leq 1/2} \frac{-s H_{1-s}(P_X) + s \log 2 - R}{2(1-s)}. \quad (143)
\]

Lemma 36 is proven in Appendix E.
IX. Conclusion

We have derived upper bounds for the leaked information in the modified mutual information criterion and the \( L_1 \) distinguishability criterion in the quantum case when we apply a family of universal\( 2 \) hash functions or a family of \( \varepsilon \)-almost dual universal\( 2 \) hash functions for privacy amplification (Theorems 24 and 25 in Section V). Then, we have derived lower bounds on their exponential decreasing rates in the i.i.d. setting. (Theorems 26 and 27 in Section VI). The obtained bound for the \( L_1 \) distinguishability criterion has been shown to be tight in the qubit case when the state is generated by transmission via Pauli channel (Appendix E). The obtained exponents are summarized in Table II. We have also applied our result to the case when we need error correction. In this case, we apply the privacy amplification after error correction as given in Subsection VII-A.

Then, we have derived upper bounds for the information leaked with respect to the final keys in the respective criteria as well as upper bounds for the probability for disagreement in the final keys (Theorems 29, 30, and 31 in Section VII). Applying them to the i.i.d. setting, we have derived lower bounds on their exponential decreasing rates. (Theorems 32, 33, 34, and 35 in Section VII).

| Task                  | \( L_1 \)                  | MMI                           |
|-----------------------|-----------------------------|-------------------------------|
| PV (Rényi)            | \( e_{G,q}(\rho_{A,E}|R) \) | \( e_{H,q}(\rho_{A,E}|R) \), \( e_{G,q}(\rho_{A,E}|R) \) |
| PV & fixed EC         | \( e_{G,q}(\rho_{A,E}|\log q - R_2) \) | \( e_{H,q}(\rho_{A,E}|\log q - R_2) \), \( e_{G,q}(\rho_{A,E}|\log q - R_2) \) |
| PV & randomized EC    | no improvement              | \( e_{H,q}(\rho_{A,E}|\log q - R_2) \), \( e_{G,q}(\rho_{A,E}|\log q - R_2) \) |

\( R \) is the key generation rate. \( R_2 \) is the sacrifice rate. PV (Rényi) is the exponent for privacy amplification via our approximate smoothing of Rényi entropy of order 2. EC is error correction. \( L_1 \) is the \( L_1 \) distinguishability criterion. MMI is the modified mutual information criterion.

Since a family of \( \varepsilon \)-almost dual universal\( 2 \) hash functions is a larger family of liner universal\( 2 \) hash functions, the obtained result suggests a possibility of the existence of an effective privacy amplification protocol with a smaller calculation time than known privacy amplification protocols. In fact, as shown in the forthcoming paper [46], there exists an example of \( \varepsilon \)-almost dual universal\( 2 \) hash functions with a smaller calculation amount and smaller number of random variables than the concatenation of Toeplitz matrix and the identity matrix. Hence, it is expected that the obtained evaluation has a future application from an applied viewpoint.

In fact, our bounds have polynomial factors in the quantum setting. When the order of these polynomial factors are large, the bounds do not work well when the number \( n \) is not sufficiently large. Fortunately, as is discussed in Subsection VI, some of them have the order \( n^{3/2} \) at most. We can expect that these types of bounds work well even when the number \( n \) is not sufficiently large. These types of bounds and these discussions have been extended to the case when error correction is needed. Further, as is discussed in Subsubsection VII-C, we can expect that some of obtained bounds work well even in the non-i.i.d. case.

In Section VIII we have applied our result to the case when Eve obtains the all information leaked to the environment via Pauli channel. In this case, our bounds can be described by using the joint classical distribution with respect to the bit error and the phase error. We have numerically compared the obtained lower bounds on the exponential decreasing rates for leaked information.

Due to Pinsker inequality and Inequality (36), the exponential convergence of one criterion yields the exponential convergence of the other criterion. However, we have shown that better exponential decreasing rates can be obtained by separate derivations. Our approximate smoothing of Rényi entropy of order 2 yields the lower bound \( e_{G,q}(P_{A,E}|R) \) of the exponent of the \( L_1 \) distinguishability criterion, which yields the lower bound \( e_{H,q}(P_{A,E}|R) \) of the exponent of the modified mutual information criterion by using Pinsker inequality. Similarly, our approximate smoothing of Rényi entropy of order 2 yields the lower bound \( e_{H,q}(P_{A,E}|R) \) of the exponent of the modified mutual information criterion by using Pinsker inequality. Similarly, our approximate smoothing of Rényi entropy of order 2 yields the lower bound \( e_{H,q}(P_{A,E}|R) \) of the exponent of the modified mutual information criterion, which yields the lower bound \( e_{H,q}(P_{A,E}|R) \) of the exponent of the modified mutual information criterion by using Pinsker inequality. Similarly, our approximate smoothing of Rényi entropy of order 2 yields the lower bound \( e_{H,q}(P_{A,E}|R) \) of the exponent of the modified mutual information criterion, which yields the lower bound \( e_{H,q}(P_{A,E}|R) \) of the exponent of the modified mutual information criterion by using Pinsker inequality. Since \( e_{G,q}(P_{A,E}|R) \) is not derived, we cannot say the same thing for the modified mutual information criterion. The relation is also a future problem.

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APPENDIX A

MODIFIED MUTUAL INFORMATION CRITERION

It is natural to adopt a quantity expressing the difference between the true state and the ideal state $\rho_{\text{mix}, A} \otimes \rho_E$ as a security criterion. However, there are several quantities expressing the difference between two states. Both $d'_1(A|E|\rho)$ and $I'(A|E|\rho)$ are characterized in this way. Here, we show that the modified mutual criterion $I'(A|E|\rho)$ can be derived in a natural way. It is natural to assume the following condition for the security criterion $C(A; E|\rho)$ as well as the unitary invariance on $\mathcal{H}_E$ and the permutation invariance on $\mathcal{H}_A$.

C1 Chain rule $C(A, B|E|\rho) = C(B|E|\rho) + C(A|B, E|\rho)$.

C2 Linearity When two states $\rho_1$ and $\rho_2$ are distinguishable on $\mathcal{H}_E$, $C(A|E|\rho_1 + (1 - \lambda)\rho_2) = \lambda C(A|E|\rho_1) + (1 - \lambda)C(A|E|\rho_2)$.

C3 Range $\log d_A \geq C(A|E|\rho) \geq 0$.

C4 Ideal case $C(A|E|\rho_{\text{mix}, A} \otimes \rho_E) = 0$.

C5 Normalization $C(A|E|a|a\otimes \rho_E) = \log d_A$.

Unfortunately, the $L_1$ distinguishability does not satisfies C1 Chain rule. However, we have the following lemma.

Lemma 37: The modified mutual information criterion $I'(A|E|\rho) = \log d_A - H(A|E|\rho)$ satisfies all of these conditions. Further, we have the following theorem.

Theorem 38: When $C(A|E|\rho)$ satisfies all of the above properties and $\rho'$ is written as $\sum_{a,e} P_{A,E}(a,e)|a,e\rangle\langle a,e|$, $C(A|E|\rho') = I'(A|E|\rho') = \log d_A - H(A|E|\rho')$.

That is, in the classical case, the security criterion is written by using the conditional entropy. In the quantum case, the above theorem cannot determine uniquely the security criterion. Since the most natural quantum extension of the conditional entropy is the quantum conditional entropy $H(A|E|\rho)$. Hence, it is natural to adopt the modified mutual information criterion $I'(A|E|\rho)$ as a security criterion. In particular, if one emphasizes C1 Chain rule rather than the universal composability, it is better employ the modified mutual information criterion $I'(A|E|\rho)$.

Proof of Lemma

We can trivially check the conditions C4 Ideal case and C5 Normalization. We show other conditions.

C1 Chain rule can be shown as follows.

\[
I'(A, B|E|\rho) = \log d_A + \log d_B - H(A, B, E|\rho) + H(E|\rho)
\]

\[
= \log d_A + \log d_B - H(B, E|\rho) + H(E|\rho) - H(A, B, E|\rho) + H(B, E|\rho)
\]

\[
= \log d_A + \log d_B - H(B|E|\rho) - H(A|B, E|\rho) = I'(A|B, E|\rho) + I'(B|E|\rho).
\]

When two states $\rho_1$ and $\rho_2$ are distinguishable on $\mathcal{H}_E$,

\[
I'(A|E|\rho_1 + (1 - \lambda)\rho_2) = \log d_A - H(A, E|\rho_1 + (1 - \lambda)\rho_2) + H(E|\rho_1 + (1 - \lambda)\rho_2)
\]

\[
= \log d_A - \lambda H(A, E|\rho_1) - (1 - \lambda)H(A, E|\rho_2) - h(\lambda) + \lambda H(E|\rho_1) + (1 - \lambda)H(E|\rho_2) + h(\lambda)
\]

\[
= \log d_A - \lambda H(A, E|\rho_1) - (1 - \lambda)H(A, E|\rho_2) + \lambda H(E|\rho_1) + (1 - \lambda)H(E|\rho_2)
\]

\[
= \lambda I'(A|E|\rho_1) + (1 - \lambda)I'(A|E|\rho_2),
\]

which implies C2 Linearity.

\[
I'(A|E|\rho) = D(\rho|| \rho_{\text{mix}, A} \otimes \rho_E) \geq 0.\]

Since $\rho$ is separable, $H(A, E|\rho) \geq 0$. Hence, $I'(A|E|\rho)$ satisfies C3 Range.

Proof of Theorem

We discuss $\tilde{H}(A|E|\rho) := \log d_A - C(A|E|\rho)$. Due to C2 Linearity, we have

\[
\tilde{H}(A|E|\rho) = \sum_e P_E(e) \tilde{H}(A|E) \sum_a P_{A|E}(a|e)|a,e\rangle\langle a,e|.
\]

Further, we see that the quantity $\tilde{H}(A|E) \sum_a P_{A|E}(a|e)|a,e\rangle\langle a,e|$ satisfies Khinchin’s axioms for entropy due to the remaining conditions. Hence, we find that $\tilde{H}(A|E) \sum_a P_{A|E}(a|e)|a,e\rangle\langle a,e| = H(P_{A|E=e})$. Thus, $\tilde{H}(A|E|\rho)$ is equal to the conditional entropy $H(A|E|\rho)$. Hence, $C(A|E|\rho) = I'(A|E|\rho)$.

APPENDIX B

PROOF OF LEMMA

First, we focus on the spectral decomposition of $\sigma$: $\sigma = \sum_i s_i E_i$. Since $x \mapsto x^{\frac{1+s}{s}}$ is operator concave,

\[
E_i \rho^{\frac{1+s}{s}} E_i \leq (E_i \rho E_i)^{\frac{1+s}{s}}.
\]

When $v$ is the number of eigenvectors of $\sigma$ Inequality (7) implies

\[
\rho^{\frac{1+s}{s}} \leq \sum_i E_i \rho^{\frac{1+s}{s}} E_i.
\]
Since $E_i$ and $E_{i'}$ are orthogonal to each other for $i \neq i'$,
\[ \sum_i (E_i \rho E_i)^{\frac{1+s}{2}} = (\sum_i E_i \rho E_i)^{\frac{1+s}{2}}. \tag{146} \]
Combining (144), (145), and (146), we obtain
\[ \sigma^{-\frac{s}{2}} \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}} \leq v \sigma^{-\frac{s}{2}} \sum_i E_i \rho^{\frac{1+s}{2}} E_i \sigma^{-\frac{s}{2}} \]
\[ \leq v \sum_i \sigma^{-\frac{s}{2}} (E_i \rho E_i)^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}} = v \sigma^{-\frac{s}{2}} (E_i (\rho))^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}}. \]
Thus, (4) implies
\[ e^{(s)[\rho||\sigma]} = \text{Tr} (\sigma^{-\frac{s}{2}} \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}})^2 \leq v \text{Tr} (\sigma^{-\frac{s}{2}} (E_i (\rho))^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}})^2 = v e^{(s)[E_i (\rho) || \sigma]} = v e^{(s)[E_i (\rho) || \sigma]} \leq v e^{(s)[\rho||\sigma]}. \tag{147} \]
That is, $\psi(s||\rho) \leq v + \psi(s||\rho)$. When we denote the number of eigenvalues of $\sigma^\otimes n$ by $n$, we have
\[ n \psi(s||\rho) = \psi(s||\sigma^\otimes n) \leq \log v_n + \psi(s||\sigma^\otimes n) = \log v_n + n \psi(s||\rho). \tag{148} \]
Dividing (148) by $n$ and taking the limit $n \to \infty$, we obtain (5).

APPENDIX C

PROOF OF LEMMA 1

The convexity of $\psi(s||\rho||\sigma)$ is shown in [14] Exercises 2.24. Using this fact, we obtain the desired argument with respect to $\psi(s||\rho||\sigma)$. The convexity of $\psi(s||\rho||\sigma)$ can be shown in the following way:
\[ \frac{d\psi(s||\rho||\sigma)}{ds} = \frac{\text{Tr} \left( \log \rho - \log \sigma \rho^{\frac{1+s}{2}} - \sigma^{-\frac{s}{2}} \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}} \right)}{\text{Tr} \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}} \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}}}, \]
\[ \frac{d^2\psi(s||\rho||\sigma)}{ds^2} = \frac{\text{Tr} \left( \log \rho - \log \sigma \rho^{\frac{1+s}{2}} (\log \rho - \log \sigma) \sigma^{-\frac{s}{2}} \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}} \right) + \frac{\text{Tr} \left( \log \rho - \log \sigma \rho^{\frac{1+s}{2}} (\log \rho - \log \sigma) \sigma^{-\frac{s}{2}} \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}} \right)}{\text{Tr} \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}} \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}}} - \frac{\text{Tr} \left( \log \rho - \log \sigma \rho^{\frac{1+s}{2}} (\log \rho - \log \sigma) \sigma^{-\frac{s}{2}} \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}} \right)}{\text{Tr} \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}} \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}}} \right)^2}. \]

Now, we consider two kinds of inner products between two matrices $X$ and $Y$:
\[ \langle Y, X \rangle_1 := \text{Tr} X \rho^{\frac{1+s}{2}} Y^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}} \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}}, \quad \langle Y, X \rangle_2 := \text{Tr} X \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}} \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}}. \]

Applying Schwarz inequality to the case of $X = (\log \rho - \log \sigma)$ and $Y = I$, we obtain
\[ \text{Tr} (\log \rho - \log \sigma) \rho^{\frac{1+s}{2}} (\log \rho - \log \sigma) \sigma^{-\frac{s}{2}} \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}} \geq (\text{Tr} (\log \rho - \log \sigma) \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}} \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}})^2 \]
and
\[ \text{Tr} (\log \rho - \log \sigma) \rho^{\frac{1+s}{2}} \sigma^{-\frac{s}{2}} \rho^{\frac{1+s}{2}} (\log \rho - \log \sigma) \sigma^{-s/2} \rho^{\frac{1+s}{2}} \sigma^{-s/2} \geq (\text{Tr} (\log \rho - \log \sigma) \rho^{\frac{1+s}{2}} \sigma^{-s/2} \rho^{\frac{1+s}{2}} \sigma^{-s/2})^2. \]
Therefore,
\[ \frac{\text{Tr} (\log \rho - \log \sigma) \rho^{\frac{1+s}{2}} (\log \rho - \log \sigma) \sigma^{-s/2} \rho^{\frac{1+s}{2}} \sigma^{-s/2}}{\text{Tr} \rho^{\frac{1+s}{2}} \sigma^{-s/2} \rho^{\frac{1+s}{2}} \sigma^{-s/2}} \geq (\frac{\text{Tr} (\log \rho - \log \sigma) \rho^{\frac{1+s}{2}} \sigma^{-s/2} \rho^{\frac{1+s}{2}} \sigma^{-s/2}}{\text{Tr} \rho^{\frac{1+s}{2}} \sigma^{-s/2} \rho^{\frac{1+s}{2}} \sigma^{-s/2}})^2, \]
\[ \frac{\text{Tr} (\log \rho - \log \sigma) \rho^{\frac{1+s}{2}} (\log \rho - \log \sigma) \sigma^{-s/2} \rho^{\frac{1+s}{2}} \sigma^{-s/2}}{\text{Tr} \rho^{\frac{1+s}{2}} \sigma^{-s/2} \rho^{\frac{1+s}{2}} \sigma^{-s/2}} \geq (\frac{\text{Tr} (\log \rho - \log \sigma) \rho^{\frac{1+s}{2}} \sigma^{-s/2} \rho^{\frac{1+s}{2}} \sigma^{-s/2}}{\text{Tr} \rho^{\frac{1+s}{2}} \sigma^{-s/2} \rho^{\frac{1+s}{2}} \sigma^{-s/2}})^2, \]
which implies
\[ \frac{d^2\psi(s||\rho||\sigma)}{ds^2} \geq 0. \]

In particular, when $\rho$ and $\sigma$ are not completely mixed, the above inequalities are strict. Hence, the functions $s \mapsto \psi(s||\rho||\sigma), \psi(s||\rho||\sigma)$ are strictly convex.
Appendix D
Proof of Lemma 37

Assume that \( s \in (0, \infty) \). For two non-negative matrices \( X \) and \( Y \), the reverse operator Hölder inequality

\[
\text{Tr } XY \geq (\text{Tr } X^{1/(1+s)})^{1+s}(\text{Tr } Y^{-1/s})^{-s}
\]

holds. Substituting \( \sum_a P_A(a)\rho_{E[a]}^{1+s} \rho_{E[a]}^{-1/s} \) and \( \sigma_E^{-s} \) to \( X \) and \( Y \), we obtain

\[
e^{-sH_{1+s}(A|E)[\rho_{A,E}]}/(\sum_a P_A(a)\rho_{E[a]}^{1+s}) = \text{Tr } \left(\sum_a (P_A(a)\rho_{E[a]}^{1+s})^{1/(1+s)}\right)^{1/(1+s)} = e^{-sH_{1+s}(A|E)[\rho_{A,E}]}.
\]

Since the equality holds when \( \sigma_E = (\sum_a (P_A(a)\rho_{E[a]}^{1+s})^{1/(1+s)})/\text{Tr } (\sum_a (P_A(a)\rho_{E[a]}^{1+s})^{1/(1+s)})^{1/(1+s)} \), we obtain

\[
\min_{\sigma_E} e^{-sH_{1+s}(A|E)[\rho_{A,E}]}(\sigma_E) = e^{-sH_{1+s}(A|E)[\rho_{A,E}]}(\sigma_E),
\]

which implies (21).

When \( s \in (-1, 0) \), applying the operator Hölder inequality \( \text{Tr } XY \leq (\text{Tr } X^{1/(1+s)})^{1+s}(\text{Tr } Y^{-1/s})^{-s} \) instead of the reverse operator Hölder inequality, we obtain

\[
e^{-sH_{1+s}(A|E)[\rho_{A,E}]}(\sigma_E) \leq e^{-sH_{1+s}(A|E)[\rho_{A,E}]}(\sigma_E),
\]

The equality can be shown in the same way.

Appendix E
Proof of Lemma 36

A. Outline of the proof
Since (96) implies

\[
\lim_{n \to \infty} \frac{-1}{n} \log \Delta d_{2}(e^{nR}, 1|\rho_{X|Z}) \geq c_{G,n}(\rho_{X|Z}) = \max_{0 \leq s \leq 1/2} \frac{-sH_{1-s}(P_X) + s(\log 2 - R)}{2(1-s)},
\]

it is enough to show the opposite inequality. For this purpose, we will show the following lemma.

Lemma 39: When we choose an \([n(1-R)]\)-dimensional subspace \( C_{\mathbb{Z}} \subset \mathbb{F}_2^n \) with equal probability, we obtain

\[
\lim_{n \to \infty} \frac{-1}{n} \log \mathbb{E}d_{1}([A]_{C_{\mathbb{Z}}}|E)[\rho_{A,E}) = \max_{0 \leq s \leq 1/2} \frac{-sH_{1-s}(P_X) + s(\log 2 - R) \log 2}{2(1-s)}.
\]

Here, we prove Lemma 36 by using Lemma 39 When we choose an \([n(1-R')/\log 2]\)-dimensional subspace \( C_{\mathbb{Z}} \subset \mathbb{F}_2^n \) with equal probability, since the hash function \( X \mapsto [X]_{C_{\mathbb{Z}}} \) satisfies the universal2 condition, we obtain

\[
\mathbb{E}d_{1}([A]_{C_{\mathbb{Z}}}|E)[\rho_{A,E}) \leq \Delta d_{2}(2^{n[R'/\log 2]}, 1|\rho_{A,E}) \leq \Delta d_{2}(e^{nR'}, 1|\rho_{A,E}),
\]

which implies that

\[
\limsup_{n \to \infty} \frac{-1}{n} \log \Delta d_{2}(e^{nR'}, 1|\rho_{A,E}) \leq \max_{0 \leq s \leq 1/2} \frac{-sH_{1-s}(P_X) + s(\log 2 - R')}{2(1-s)}.
\]

Since Inequality (142) is the opposite inequality, we obtain (143).

In the following, we prepare two lemmas for the proof of Lemma 39. Given a code \( C \subset \mathbb{F}_p^n \), we can define its orthogonal space \( C^\perp \subset \mathbb{F}_p^n \). Then, for \([x_2]_{C^\perp} \subset \mathbb{F}_p^n/C^\perp \) and \( x_1 \in [x_2]_{C^\perp} \), we define the conditional distribution \( P_{X|[X]_C} = \{x_1|[x_2]_{C^\perp}\} \). Then, we define a pure state \([a]_{C_1}, [x_2]_{C^\perp}) := \sum_{x_1 \in [x_2]_{C^\perp}} P_X(x_1) \), for \([a]_{C_1} \subset \mathbb{F}_p/C \) and \([x_2]_{C^\perp} \subset \mathbb{F}_p^n/C^\perp \). Note that the definition of the state \([a]_{C_1}, [x_2]_{C^\perp}) \) does not depend on the choice of the representatives of \([a]_{C_1} \) and \([x_2]_{C^\perp} \) except for the phase factor. Then, the relation

\[
\rho_{E[a]} := \sum_{y \in C} \frac{1}{|C|} [a + y : P_X(y : P_X)] : = \sum_{[x_2]_{C^\perp} \subset \mathbb{F}_p/C^\perp} P_X([x_2]_{C^\perp})[a]_{C_1}, [x_2]_{C^\perp}), ([a]_{C_1}, [x_2]_{C^\perp}).
\]

holds. In order to describe the maximum likelihood estimator of the code \( C^\perp \) under the distribution \( P_X \), we define \( x([x_2]_{C^\perp}) := \text{argmax}_{x_1 \in [x_2]_{C^\perp}} P_X(x_1) \). Then, the decoding error probability is given as

\[
P_e(C^\perp) = 1 - \sum_{[x_2]_{C^\perp} \subset \mathbb{F}_p/C^\perp} P_X([x_2]_{C^\perp})1 - \max_{x_1 \in [x_2]} P_X(x_1).
\]
Lemma 40: The relation
\[
\frac{2}{8} \sum_{\substack{[x_2]_{C^+} \in \mathbb{F}_p^n/C^+}} \sqrt{P_X(x([x_2]_{C^+}))(P_X([x_2]_{C^+}) - P_X(x([x_2]_{C^+})))}
\]
\[
\leq d'_1([A]_{C}) E[\rho_{A,E}] = \|\rho_E - \rho_{E|[a]_{C}}\|_1
\]
\[
\leq 2\sqrt{2P_e(C^+)}
\]
holds for \( a \in \mathbb{F}_p^n \).

The proof of Lemma 40 is given in Appendix B.

Next, we consider the binary case, i.e., the case of \( \mathbb{F}_2^n \). We choose an \( m \)-dimensional subspace \( C_X \subset \mathbb{F}_2^n \) with equal probability. That is, there are \( G(m) := \prod_{i=0}^{m-1} \frac{2^i - 2}{2^{i+2}} \) distinct \( m \)-dimensional subspaces in \( \mathbb{F}_2^n \). Hence, we chose each of them with the probability \( 1/G(m) \).

Lemma 41: When we choose an \([nR]\)-dimensional subspace \( C_X \subset \mathbb{F}_2^n \) with equal probability,
\[
\lim_{n \to \infty} \frac{-1}{n} \log \mathbb{E}_X \sum_{[x_2]_{C^+} \in \mathbb{F}_2^n/C^+} \sqrt{P^n_X(x([x_2]_{C^+}))(P^n_X([x_2]_{C^+}) - P^n_X(x([x_2]_{C^+})))}
\]
\[
= \lim_{n \to \infty} \frac{-1}{n} \log \mathbb{E}_X P_1(C_X)
\]
\[
= \frac{1}{2} \min_{Q: \log 2(1-R) \geq H(Q)} D(Q||P_X) + \log 2(1-R) - H(Q)
\]
\[
= \frac{1}{2} \max_{0 \leq s \leq 1/2} -sH_{1-s}(P_X) + s \log 2(1-R)
\]
(156)

The proof of Lemma 41 is given in Appendix C.

Proof of Lemma 39: We apply Lemma 41 to the case \( C_X = C^+ \). Then, the exponential decreasing rates of the upper and lower bounds given in Lemma 40 are \( \max_{0 \leq s \leq 1/2} \frac{sH_{1-s}(P_X) - s(1-R) \log 2}{1-s} \), which implies (150).

B. Proof of Lemma 40

In this proof, we abbreviate \([x]_{C^+} \) by \([x]\). Since
\[
\|\rho_E - \rho_{E|[a]_{C}}\|_1 = \|W(0,z)(\rho_E - \rho_{E|[a]_{C}})W(0,z)^\dagger\|_1 = \|\rho_E - \rho_{E|[a+z]_{C}}\|_1
\]
(157)

for \( z, a \in \mathbb{F}_p^n \), we have \( d'_1([A]_{C}) E[\rho_{A,E}] = \|\rho_E - \rho_{E|[a]_{C}}\|_1 \).

Next, we prove the inequality (155). For this purpose, we define the fidelity as \( F(\rho_E, \rho_{E|[a]_{C}}) := \text{Tr} [\sqrt{\rho_E} \sqrt{\rho_{E|[a]_{C}}}]. \) The fidelity satisfies that
\[
\|\rho_E - \rho_{E|[a]_{C}}\|_1 \leq 2 \sqrt{1 - F(\rho_E, \rho_{E|[a]_{C}})^2},
\]
and is characterized as
\[
F(\rho_E, \rho_{E|[a]_{C}})^2 = \left( \sum_{[x_2] \in \mathbb{F}_p^n/C^+} P_X([x_2]) \right)^2 \sum_{x_1} P_X([x_1]|x_2) \right)^2 = e^{-H_G^{(X)}([X]|P_X)}.
\]
(159)

Since \( e^{-H_G^{(X)}([X]|P_X)} \) \( \geq \left( \sum_{[x_2] \in \mathbb{F}_p^n/C^+} P_X([x_2]) \right)^2 \max_{x_1} P_X([x_1]|x_2) \right)^2 \geq (1 - P_e(C^+))^2 \), we have
\[
1 - e^{-H_G^{(X)}([X]|P_X)} \leq 1 - (1 - P_e(C^+))^2 = 1 - 1 + 2P_e(C^+) - P_e(C^+)^2 \leq 2P_e(C^+).
\]
(160)

Combining (158), (159), and (160), we obtain (155).

Next, we show (154). For \( x_1 \in [x_2]\backslash \{x([x_2])\} \), we define the operator \( K_{x_1} := |x_1\rangle\langle x_1| + \sqrt{P_X([x_1])} \frac{P_X([x_2]) - P_X(x([x_2]))}{P_X([x_2]) - P_X(x([x_2]))} \langle x([x_2])|x([x_2])\rangle \).

Then, we have the relation \( \sum_{[x_2] \in \mathbb{F}_p^n/C^+} \sum_{x_1} P_X([x_2]) \frac{K_{x_1}}{P_X([x_2])} = I \). Hence, we can define the TP-CP map \( \Lambda : \rho \mapsto \sum_{[x_2] \in \mathbb{F}_p^n/C^+} \sum_{x_1} K_{x_1} \rho K_{x_1} \otimes |x_1\rangle\langle x_1| \) on \( \mathcal{A} \). Thus,
\[
K_{x_1} \rho E_{[a]_{C}} K_{x_1} = \frac{P_X([x_2])}{P_X([x_2])} (- P_X(x([x_2])) |x([x_2])\rangle) - P_X(x([x_2])) |x([x_2])\rangle - P_X(x([x_2])) |x([x_2])\rangle)
\]
\[
K_{x_1} \rho E_{[a]_{C}} K_{x_1} = \sqrt{P_X([x_1])} |x_1\rangle + \sqrt{P_X([x_1])} P_X(x([x_2])) |x([x_2])\rangle)
\]
\[
+ \sqrt{P_X([x_1])} P_X(x([x_2])) |x([x_2])\rangle) + (x([x_2]) |x_1\rangle + (x([x_2]) |x_1\rangle + (x([x_2]) |x_1\rangle + P_X(x([x_2])) |x([x_2])\rangle).
\]
Hence,
\[
\|K_{x_t}\rho E K_{x_t} - K_{x_t}\rho E_{|[a]} K_{x_t}\|_1 = 2 \sqrt{P_X(x_1)} \sqrt{P_X(x([x_2])) \left( P_X([x_2]) - P_X(x([x_2])) \right)}
\]
\[
= 2 \frac{P_X(x_1)}{P_X([x_2]) - P_X(x([x_2]))} \sqrt{P_X(x([x_2]))(P_X([x_2]) - P_X(x([x_2])))},
\]
(161)

Using the relation \(\sum_{x_1 \in [x_2]} \frac{P_X(x_1)}{P_X([x_2]) - P_X(x([x_2]))} \geq 1\) and (161), we obtain
\[
\|\rho E - \rho E_{|[a]}\|_1 \geq \|\Lambda(\rho E) - \Lambda(\rho E_{|[a]})\|_1
\]
\[
\geq \sum_{[x_2] \in \mathcal{P}_p/C \cup x_1 \in [x_2]} \|K_{x_t}\rho E K_{x_t} - K_{x_t}\rho E_{|[a]} K_{x_t}\|_1
\]
\[
\geq \sum_{[x_2] \in \mathcal{P}_p/C \cup x_1 \in [x_2]} \|K_{x_t}\rho E K_{x_t} - K_{x_t}\rho E_{|[a]} K_{x_t}\|_1
\]
\[
= 2 \sum_{[x_2] \in \mathcal{P}_p/C \cup x_1 \in [x_2]} \frac{P_X(x_1)}{P_X([x_2]) - P_X(x([x_2]))} \sqrt{P_X(x([x_2]))(P_X([x_2]) - P_X(x([x_2])))},
\]
which implies (164).

C. Proof of Lemma 42

In this proof, we abbreviate \([x]_{C_X}\) by \([x]\). It was shown in [19] Theorem 7 that
\[
\lim_{n \to \infty} -\frac{1}{n} \log E_X P_C(C_X) \geq \max_{0 \leq s \leq 1/2} -sH_{1-s}(P_X) + s \log 2(1 - R).
\]
(162)

We can show the following lemma.

Lemma 42:
\[
\max_{0 \leq s \leq 1/2} -sH_{1-s}(P_X) + s \log 2(1 - R)
\]
\[
= Q \min_{D \log 2(1-R) \geq H(Q)} D(Q\|P_X) + \log 2(1-R) - H(Q).
\]
(163)

Lemma 42 is shown in Appendix E.

Hence, it is enough to show that
\[
\lim_{n \to \infty} -\frac{1}{n} \log E_X \sum_{[x_2] \in \mathcal{P}_p/C_X} \sqrt{P_X^n(x([x_2]))(P_X^n([x_2]) - P_X^n(x([x_2])))}
\]
\[
\leq \frac{1}{2} Q \min_{D \log 2(1-R) \geq H(Q)} D(Q\|P_X) + \log 2(1-R) - H(Q)
\]
(164)
\[
\leq \frac{1}{2} Q \min_{D \log 2(1-R) \geq H(Q)} D(Q\|P_X) + \log 2(1-R) - H(Q)
\]
(165)

Now, we denote the set of empirical distributions on \(\mathcal{P}_2\) with \(n\) trials by \(\mathcal{T}_n\). The cardinality \(|\mathcal{T}_n|\) is \(n+1\) [34]. When \(T_n(Q)\) represents the set of \(n\)-trial data whose empirical distribution is \(Q\), the cardinality of \(T_n(Q)\) can be evaluated as [34]:
\[
\left\lceil \frac{e^{nH(Q)}}{n+1} \right\rceil \leq |T_n(Q)| \leq \lfloor e^{nH(Q)} \rfloor,
\]
(166)

where \([x]\) is the minimum integer \(m\) satisfying \(m \geq x\), and \([x]\) is the maximum \(m\) satisfying \(m \leq x\). Since any element \(\vec{a} \in T_n(Q)\) satisfies
\[
P_X^n(\vec{a}) = e^{-n(D(Q\|P_X) + H(Q))},
\]
(167)
we obtain an important formula
\[
\frac{1}{n+1} e^{-nD(Q\|P_X)} \leq P_X^n(T_n(Q)) \leq e^{-nD(Q\|P_X)}.
\]
(168)
Now, we prepare the following lemma in the finite-length case.

Lemma 43: Assume that we choose an \( m \)-dimensional subspace \( C_X \subset \mathbb{F}_2^n \) with equal probability. When \( Q_1, Q_2 \in T_n \) satisfies that \( H(Q_1) \leq 2^n - m \) and \( D(Q_1\|P_X) + H(Q_1) < D(Q_2\|P_X) + H(Q_2) \), we have

\[
\begin{align*}
\mathbb{E}_X & \sum_{[x_2] \in \mathbb{F}_2^n/C_X} \sqrt{P^n_X(x([x_2]))(P^n_X([x_2]) - P^n_X(x([x_2])))} \geq B_{n,m}(Q_1, Q_2) \\
\mathbb{E}_X & \sum_{[x_2] \in \mathbb{F}_2^n/C_X} \sqrt{P^n_X(x([x_2]))(P^n_X([x_2]) - P^n_X(x([x_2])))} \leq e^{-\frac{1}{2} \max_{0 \leq s \leq 1} \left( \frac{nH_{1-s}(P_X) - s(n-m) \log 2}{2} \right)^{\frac{1}{2}}},
\end{align*}
\]

(169) \hfill (170)

where

\[
B_{n,m}(Q_1, Q_2) := e^{-\frac{1}{2} (D(Q_1\|P_X)+H(Q_1)+D(Q_2\|P_X)+H(Q_2))} \left( \frac{|T_n(Q_2)|(|T_n(Q_1)| - 1) \left( 1 - \frac{(2^n - 2)(|T_n(Q_1)| - 2)}{2(2^n - 2)} \right)^2}{2^{m-n}} \right). \quad (171)
\]

The proof of Lemma 43 is given in Appendix E-D.

Since (170) shows (169), we will show (164) by using (169). When \( \log 2(1-R) < H(Q_1) \) and \( D(Q_1\|P_X) + H(Q_1) < D(Q_2\|P_X) + H(Q_2) \),

\[
\lim_{n \to \infty} \frac{1}{n} \log B_{n,[nR]}(Q_1, Q_2) = \frac{1}{2} (D(Q_1\|P_X) + D(Q_2\|P_X) + \log 2(1-R) - H(Q_1)).
\]

(172)

Choosing \( Q_1 = P_X \), we have

\[
\begin{align*}
& \inf_{Q_1,Q_2: \log 2(1-R) \leq H(Q_1), D(Q_1\|P_X) + H(Q_1) \leq D(Q_2\|P_X) + H(Q_2)} \frac{1}{2} (D(Q_1\|P_X) + D(Q_2\|P_X) + \log 2(1-R) - H(Q_1)) \\
= & \inf_{Q_1: \log 2(1-R) \leq H(Q_1)} \frac{1}{2} (D(Q_1\|P_X) + \log 2(1-R) - H(Q_1)).
\end{align*}
\]

(173)

Thus,

\[
\begin{align*}
& \inf_{Q_1: \log 2(1-R) \leq H(Q_1)} D(Q_1\|P_X) + \log 2(1-R) - H(Q_1) \\
= & \min_{Q_1: \log 2(1-R) \leq H(Q_1)} D(Q_1\|P_X) + \log 2(1-R) - H(Q_1) \geq \min_{Q_1: \log 2(1-R) \leq H(Q_1)} D(Q_1\|P_X).
\end{align*}
\]

(174)

Since the minimum \( \min_{Q_1: \log 2(1-R) \leq H(Q_1)} D(Q_1\|P_X) \) can be realized with \( Q_1^* \) satisfying \( \log 2(1-R) = H(Q_1^*) \), we have

\[
\begin{align*}
& \inf_{Q_1: \log 2(1-R) \leq H(Q_1)} D(Q_1\|P_X) + \log 2(1-R) - H(Q_1) \\
= & \min_{Q_1: \log 2(1-R) \leq H(Q_1)} D(Q_1\|P_X) = D(Q_1^*\|P_X).
\end{align*}
\]

It is known that this quantity is the optimal error exponent with the source coding with the compression rate \( \log 2(1-R) \), which is equal to \( \max_{0 \leq s \leq 1/2} s H_{1-s}(P_X) - s \log 2(1-R) \). Hence, combining (169), (169), (174), and the above mentioned fact, we obtain (156).

Proof of (172): Since \( \log 2(1-R) < H(Q_1) \), we have

\[
\begin{align*}
& \lim_{n \to \infty} \frac{-1}{n} \log \left( 1 - \frac{(2^n - 2)(|T_n(Q_1)| - 2)}{2^{2^n} - 2} \right) = 0 \\
& \lim_{n \to \infty} \frac{-1}{n} \log \frac{|T_n(Q_2)|(|T_n(Q_1)| - 1)2^{m-n}}{2^{n/2} - |T_n(Q_2)| - 2} = H(Q_2) - \log 2(1-R).
\end{align*}
\]

Hence,

\[
\lim_{n \to \infty} \frac{-1}{n} \log \frac{|T_n(Q_2)|(|T_n(Q_1)| - 1)2^{m-n}}{(1 + \frac{|T_n(Q_2)| - 1}{2^{n/2} - |T_n(Q_2)| - 2})^\frac{1}{2}}
= H(Q_2) + H(Q_1) - \log 2(1-R) - \frac{1}{2} (H(Q_2) - \log 2(1-R)) = \frac{H(Q_2) + R - \log 2}{2} + H(Q_1).
\]

Thus,

\[
\begin{align*}
& \lim_{n \to \infty} \frac{-1}{n} \log B_{n,[nR]}(Q_1, Q_2) = \frac{1}{2} (D(Q_1\|P_X) + H(Q_1) + D(Q_2\|P_X) + H(Q_2)) - \frac{H(Q_2) - \log 2(1-R)}{2} + H(Q_1) \\
= & \frac{1}{2} (D(Q_1\|P_X) + D(Q_2\|P_X) + \log 2(1-R) - H(Q_1)),
\end{align*}
\]

which implies (172).
D. Proof of Lemma 44

In this proof, we abbreviate \([x]_C \) by \([x]_C \). In Lemma 44, we choose an \( m \)-dimensional subspace \( C_x \subset \mathbb{F}_2^n \) with equal probability. That is, there are \( G(m) := \prod_{i=0}^{m-1} \frac{2^n-1}{2^n-2^i} \) distinct \( m \)-dimensional subspaces in \( \mathbb{F}_2^n \). Hence, we chose each of them with the probability \( 1/G(m) \).

Now, we show (170). Since \( x \mapsto \sqrt{x} \) is concave for \( s \in [0, 1] \), we have

\[
E_x \sum_{[x_2] \in \mathbb{F}_2^n/C_x} \sqrt{P_X(x([x_2]))}(P_X^2(x([x_2])) - P_X(x([x_2])))
\]

\[
= E_x \sum_{[x_2] \in \mathbb{F}_2^n/C_x} P_X^2(x([x_2])) \sqrt{(P_X^2(x([x_2])) - P_X(x([x_2])))}
\]

\[
\leq E_x \sum_{[x_2] \in \mathbb{F}_2^n/C_x} P_X(x([x_2]))(P_X^2(x([x_2])) - P_X(x([x_2])))
\]

\[
= E_x \sqrt{\sum_{[x_2] \in \mathbb{F}_2^n/C_x} P_X^2(x([x_2])) - P_X^2(x([x_2]))}
\]

\[
\leq \sqrt{E_x \sum_{[x_2] \in \mathbb{F}_2^n/C_x} P_X^2(x([x_2])) - P_X^2(x([x_2]))}
\]

(175)

Since the quantity \( \sum_{[x_2] \in \mathbb{F}_2^n/C_x} P_X^2(x([x_2])) - P_X^2(x([x_2])) \) is the average error probability when we apply maximum likelihood decoder, it can be evaluated as

\[
\sum_{[x_2] \in \mathbb{F}_2^n/C_x} P_X^2(x([x_2])) - P_X^2(x([x_2])) \leq 2 \sum_{x \in \mathbb{F}_2} P_X(x)^{1-s} \frac{1}{s}
\]

with \( s \in [0, 1] \). Combining (175) and (176), we obtain (170).

Next, we proceed to the proof of (169). For distinct elements \( y_1, \ldots, y_l, y_{l+1}, \ldots, y_{l+k} \in \mathbb{F}_2^n \), we define the number \( M(y_1, \ldots, y_l|y_{l+1}, \ldots, y_{l+k}) \) as the number of cases that one of \( y_1, \ldots, y_l \) belongs to \( C_x \) and one of \( y_{l+1}, \ldots, y_{l+k} \) belongs to \( C_x \). In particular, \( M(y_1, \ldots, y_l|\emptyset) \) denotes the number of cases that one of \( y_1, \ldots, y_l \) belongs to \( C_x \). Then, we prepare the following lemma.

**Lemma 44:**

\[
M(y_1, y_2, \ldots, y_l|y_{l+1}) \leq l \prod_{i=2}^{m-1} \frac{2^n - 2^i}{2^m - 2^i}
\]

(177)

\[
M(y_1, y_2, \ldots, y_l|\emptyset) \geq l(l - 1) \prod_{i=2}^{m-1} \frac{2^n - 2^i}{2^m - 2^i}
\]

(178)

The proof of Lemma 44 is given in the end of this subsection.

Now, using Lemma 44, we show (169). We define \( a([x_2]) \) to be 1 if \( [x_2] \cap T_n(Q_1) \neq \emptyset \), and to be 0 otherwise. We also define \( N([x_2]) \) the number of elements of \( [x_2] \cap T_n(Q_2) \). Then, for any code \( C \) and any \( [x_2] \in \mathbb{F}_2^n/C \),

\[
\sqrt{P_X^2(x([x_2]))}(P_X^2([x_2])) - P_X(x([x_2]))
\]

\[
\geq a([x_2]) e^{-\frac{1}{2} [(D(Q_1|P_X) + H(Q_1)) + H(Q_2)]} - \frac{1}{2} N([x_2]) \frac{1}{N([x_2])} \frac{1}{2}
\]

(179)

Next, for \( x \in \mathbb{F}_2^n \), we define \( b(C, x) \) to be 1 if \( (x + C) \cap T_n(Q_1) \neq \emptyset \), and to be 0 otherwise. We also define the number \( N(C, x) \) as the number of elements of \( (x + C) \cap T_n(Q_2) \). Hence,

\[
\sum_{[x_2] \in \mathbb{F}_2^n/C} \sqrt{P_X^2(x([x_2]))}(P_X^2([x_2])) - P_X(x([x_2]))
\]

\[
\geq \sum_{[x_2] \in \mathbb{F}_2^n/C} a([x_2]) e^{-\frac{1}{2} [(D(Q_1|P_X) + H(Q_1)) + H(Q_2)]} - \frac{1}{2} N([x_2]) \frac{1}{N([x_2])} \frac{1}{2}
\]

\[
= \sum_{x \in T_n(Q_2)} b(C, x) e^{-\frac{1}{2} [(D(Q_1|P_X) + H(Q_1)) + H(Q_2)]} - \frac{1}{2} N(C, x) \frac{1}{2}.
\]
Thus,

\[ \mathbb{E}_X \sum_{[x_2] \in \mathbb{F}_2^n / \mathbb{F}_2} \sqrt{P^n_X(x([x_2]))(P^n_X([x_2]) - P^n_X(x([x_2])))} \]

\[ \geq \mathbb{E}_X \sum_{x \in T_n(Q_2)} b(C_X, x) e^{-\frac{\delta(D(Q_1 \parallel P_X) + H(Q_1))}{2}} e^{-\frac{\delta(D(Q_2 \parallel P_X) + H(Q_2))}{2}} N(C_X, x)^{-\frac{\delta}{2}} \]

\[ \geq \sum_{x \in T_n(Q_2)} \mathbb{P}_X(b(C_X, x) = 1) e^{-\frac{\delta(D(Q_1 \parallel P_X) + H(Q_1))}{2}} e^{-\frac{\delta(D(Q_2 \parallel P_X) + H(Q_2))}{2}} (\mathbb{E}_X[b(C_X, x) = 1] N(C_X, x))^{-\frac{\delta}{2}}. \]  

(180)

Now, we evaluate the values \( \mathbb{P}_X(b(C_X, x) = 1) \) and \( \mathbb{E}_X[b(C_X, x) = 1] N(C_X, x) \).

The condition \((x + C_X) \cap T_n(Q_1) \neq \emptyset\) is equivalent with the condition \(C_X \cap (T_n(Q_1) - x) \neq \emptyset\), where \((T_n(Q_1) - x) := \bigcup_{y \in T_n(Q_1)} (y - x)\). When \(y_1, \ldots, y_i\) are all non-zero elements of \((T_n(Q_1) - x)\) for a fixed \(x\), the number of cases that \(C_X \cap (T_n(Q_1) - x) \neq \emptyset\) is \(M(y_1, \ldots, y_i)\). Lemma \[43\] guarantees that \(M(y_1, \ldots, y_i) \geq (|T_n(Q_1)| - 1)(2^{\frac{n-2}{2}} - \frac{|T_n(Q_1)| - 2}{2^{n-2}}) \prod_{i=2}^{m-1} \frac{2^n - 2^i}{2^{n-2}}\). Thus,

\[ \mathbb{P}_X(b(C_X, x) = 1) \geq (|T_n(Q_1)| - 1)(2^{\frac{n-2}{2}} - \frac{|T_n(Q_1)| - 2}{2^{n-2}})^2 2^{n-1} \leq |T_n(Q_1)| - 1 \frac{2^n - 2}{2^{n-2}}. \]

(181)

The number \(N(C, x)\) is the number of elements of \(C \cap (T_n(Q_2) - x)\). For any non-zero element \(y' \in (T_n(Q_2) - x)\), \(M(y_1, \ldots, y_i | y') \leq 1 \prod_{i=2}^{m-1} \frac{2^n - 2^i}{2^{n-2}}\). Hence, we have

\[ \mathbb{E}_X[b(C_X, x) = 1] N(C_X, x) = 1 + \sum_{y' \in (T_n(Q_2) - x) \setminus \{0\}} \mathbb{P}_X(b(C_X, x) = 1) (y' \in C_X) \]

\[ = 1 + \sum_{y' \in (T_n(Q_2) - x) \setminus \{0\}} M(y_1, \ldots, y_i | y') M(y_1, \ldots, y_i | \emptyset) \leq 1 + \sum_{y' \in (T_n(Q_2) - x) \setminus \{0\}} 2^{n-2} - \frac{|T_n(Q_1)| - 2}{2^{n-2}} = 1 + \frac{|T_n(Q_2)| - 1}{2^{n-2} - |T_n(Q_1)| - 2}. \]

(182)

Combining (180), (181), and (182), we obtain

\[ \mathbb{E}_X \sum_{[x_2] \in \mathbb{F}_2^n / \mathbb{F}_2} \sqrt{P^n_X(x([x_2]))(P^n_X([x_2]) - P^n_X(x([x_2])))} \]

\[ \geq \sum_{x \in T_n(Q_2)} e^{-\frac{\delta(D(Q_1 \parallel P_X) + H(Q_1))}{2}} e^{-\frac{\delta(D(Q_2 \parallel P_X) + H(Q_2))}{2}} \left( \frac{|T_n(Q_1)| - 1}{2} \left( 1 - \frac{2^n - 2(|T_n(Q_1)| - 2)}{2^{n-2} - |T_n(Q_1)| - 2} \right) \right)^2 2^{n-1} \]

\[ \geq e^{-\frac{\delta(D(Q_1 \parallel P_X) + H(Q_1))}{2}} e^{-\frac{\delta(D(Q_2 \parallel P_X) + H(Q_2))}{2}} \left( \frac{|T_n(Q_2)| - 1}{2} \left( 1 - \frac{2^n - 2(|T_n(Q_1)| - 2)}{2^{n-2} - |T_n(Q_1)| - 2} \right) \right)^2 2^{n-1} \]

(183)

which implies \[69\].

**Proof of Lemma \[43\]** We fix the one-dimensional subspace spanned by a non-zero element \(y_1 \in \mathbb{F}_2^n\). We count the number of \(m - 1\) dimensional subspaces that are orthogonal to \(y_1\) and belong to \(C_X\). Hence, \(M(y_1 | \emptyset) = \prod_{i=1}^{m-1} \frac{2^n - 2^i}{2^{n-2}}\).  

Next, we consider two elements \(y_1\) and \(y_2\). We fix the two-dimensional subspace spanned by \(y_1\) and \(y_2\) in \(\mathbb{F}_2^n\). We count the number of \(m - 2\) dimensional subspaces that are orthogonal to the two-dimensional subspace and belong to \(C_X\). Hence, \(M(y_1 | y_2) = \prod_{i=2}^{m-1} \frac{2^n - 2^i}{2^{n-2}}\). Thus,

\[ M(y_1, y_2 | \emptyset) = M(y_1 | \emptyset) + M(y_2 | \emptyset) - M(y_1, y_2) \]

\[ = 2 \prod_{i=1}^{m-1} \frac{2^n - 2^i}{2^{n-2}} - \prod_{i=2}^{m-1} \frac{2^n - 2^i}{2^{n-2}} = \left( \frac{2^n - 2^i}{2^{n-2}} - 1 \right) \prod_{i=2}^{m-1} \frac{2^n - 2^i}{2^{n-2}}. \]

We consider \(l + 1\) elements \(y_1, y_2, \ldots, y_i, y_{i+1} \in \mathbb{F}_2^n \setminus \{0\}\). We focus on the two-dimensional subspace \(C'\) spanned by \(y_{i+1}\) and one of \(y_1, \ldots, y_i\). The number of choices of \(C'\) is at most \(l\). When we fix the subspace \(C'\), we consider the number of cases what \(m - 2\) dimensional space of the orthogonal subspace belongs \(C_X\). This number of cases is \(\prod_{i=2}^{m-1} \frac{2^n - 2^i}{2^{n-2}}\). Hence, we obtain \[67\].
Using (177), we can show (178) with \( l = 3 \) as follows.

\[
M(y_1, y_2, y_3|\emptyset) = M(y_1|\emptyset) + M(y_2|\emptyset) + M(y_3|\emptyset) - M(y_1|y_2) - M(y_1|y_2|y_3)
\]

\[
\geq 3 \prod_{i=1}^{m-1} \frac{2^n - 2^i}{2m - 2i} - (1 + 2) \prod_{i=2}^{m-1} \frac{2^n - 2^i}{2m - 2i} = (3 \frac{2^n - 2^1}{2m - 2i} - 3) \prod_{i=2}^{m-1} \frac{2^n - 2^i}{2m - 2i}.
\]

Similarly, using (177), we can show (178) in the general case as follows.

\[
M(y_1, y_2, \ldots, y_l|\emptyset) = M(y_1|\emptyset) + M(y_2|\emptyset) + \ldots + M(y_l|\emptyset) - M(y_1, y_2|y_3) - \ldots - M(y_1, \ldots, y_l-1|y_l)
\]

\[
\geq l \prod_{i=1}^{m-1} \frac{2^n - 2^i}{2m - 2i} - (1 + 2 + \ldots + (M - 1)) \prod_{i=2}^{m-1} \frac{2^n - 2^i}{2m - 2i} = (l \frac{2^n - 2^1}{2m - 2i} - \frac{l(l-1)}{2}) \prod_{i=2}^{m-1} \frac{2^n - 2^i}{2m - 2i}.
\]

\[\square\]

**E. Proof of Lemma 42**

It is enough to show that

\[
\max_{0 \leq s \leq 1/2} \frac{-f(s) + sr}{1 - s} = \min_{Q : r \geq H(Q)} D(Q\|P_X) + r - H(Q),
\]

(184)

where \( f(s) := sH_{1-s}(P_X) \). Since both quantities are zero when \( r \leq H(P_X) \), it is enough to show (184) with \( r > H(P_X) \).

We define the distribution \( P_X(x) := P_X(x)^{1-s} \sum_x P_X(x')^{s} \). Since \( f(s) \) is strictly convex, \( f'(s) \) is strictly increasing.

Hence, we can define the function \( s(t) \) as the inverse function \( s \mapsto f'(s) \). Since

\[
\frac{d}{dt} (1 - s(t))t + f(s(t)) = 1 - s(t) - s'(t)t + s'(t)f'(s(t)) = 1 - s(t) > 0
\]

for \( s(t) \in [0,1) \), we can define \( t_r \) as

\[
r = (1 - s(t_r))t_r + f(s(t_r)).
\]

(186)

Then, we have \( s(H(P_X)) = 0 \), \( t_H(P_X) = H(P_X) \), and \( t_H(P_s) = f'(s) \).

Hence, when \( r \in [H(P_X), H(P_1)] \), we obtain

\[
t_r - r = t_r(s(t_r) - f(s(t_r))) = \frac{s(t_r)r - f(s(t_r))}{1 - s} = \max_{s \in [0,1]} \frac{sr - f(s)}{1 - s},
\]

which is shown below. In the following, we denote the above value by \( g(r) \). Hence, we obtain

\[
\max_{s \in [0,1/2]} \frac{sr - f(s)}{1 - s} = \begin{cases} \frac{s(t_r)r - f(s(t_r))}{1 - s} & \text{if } s(t_r) \geq 1/2 \\ \frac{H(P_{1/2})/2}{1 - 1/2} - r - H(P_{1/2}) & \text{if } s(t_r) < 1/2. \end{cases}
\]

(188)

We can also show

\[
\frac{d}{dr} g(r) = \frac{s(t_r)}{1 - s(t_r)}.
\]

(189)

Its proof is given below. By simple calculation, we obtain

\[
D(P_s\|P_X) = sf'(s) - f(s).
\]

(190)

When \( H(Q) = H(P_s) \), we can show

\[
D(Q\|P_X) - D(P_s\|P_X) = \frac{D(Q\|P_s)}{1 - s}.
\]

(191)

Its proof is given below. Combining (187), (190), and (191), we obtain

\[
\max_{s \in [0,1]} \frac{sr - f(s)}{1 - s} = \min_{Q : r = H(Q)} D(Q\|P_X) = D(P_s(t_r))\|P_X) = s(t_r)t_r - f(s(t_r)).
\]

(192)

Hence, (189) and (192) yield that

\[
\min_{Q : r \geq H(Q)} D(Q\|P_X) + r - H(Q) = \min_{r' : r \geq r'} g(r') + r - r' = \begin{cases} g(r') & \text{if } s(t_r) \geq 1/2 \\ g(H(P_{1/2}) + r - H(P_{1/2}) & \text{if } s(t_r) < 1/2. \end{cases}
\]

(193)
Therefore, combination of (188) and (193) yields (184).

**Proof of (187):** The first equation follows from (186). The second equation can be shown by substituting \( t_r = \frac{r-f(s(t_r))}{1-s(t_r)} \). Now, we show the final equation. We have

\[
\frac{d}{ds} \left( \frac{sr - f(s)}{1-s} \right) = \frac{(1-s)(r-f'(s)) + sr - f(s)}{(1-s)^2}.
\]

Since \( \frac{d}{ds} (1-s)(r-f'(s)) + sr - f(s) = -f''(s)(1-s) \),

\[
(1-s)(r-f'(s)) + sr - f(s) \text{ is monotonically increasing for } s. \text{ Hence, the maximum } \max_{s \in [0,1]} \frac{sr-f(s)}{1-s} \text{ is realized when } (1-s)(r-f'(s)) + sr - f(s) = 0, \text{ which is equivalent with } s = s(t_r) \text{ because of (186). Therefore, we obtain the final equation.}
\]

**Proof of (189):** Thanks to the proof of (187), we have \( \frac{d}{ds} \left( \frac{sr-f(s)}{1-s} \right) |_{r=s(t_r)} = 0. \) Hence,

\[
\frac{d}{dr} \left( \frac{s(t_r)r-f(s(t_r))}{1-s(t_r)} \right) = \frac{ds(t_r)}{dr} \frac{d}{dr} \frac{sr - f(s)}{1-s} \bigg|_{r=s(t_r)} = \frac{s(t_r) - f(s(t_r))}{1-s(t_r)}.
\]

**Proof of (197):** We have

\[
D(Q\|P_X) - D(P_s\|P_X) = \sum_x Q(x)(\log Q(x) - \log P_X(x)) - \sum_x P_s(x)(\log P_s(x) - \log P_X(x))
\]

\[
= \sum_x Q(x)(\log Q(x) - \log P_s(x)) + \sum_x (Q(x) - P_s(x))(\log P_s(x) - \log P_X(x))
\]

\[
= D(Q\|P_s) - s \sum_x (Q(x) - P_s(x)) \log P_X(x)
\]

and

\[
- H(Q) + H(P_s) = \sum_x Q(x)(\log Q(x) - \log P_s(x)) + \sum_x (Q(x) - P_s(x)) \log P_s(x)
\]

\[
= D(Q\|P_s) + (1-s) \sum_x (Q(x) - P_s(x)) \log P_X(x).
\]

Since \( H(Q) = H(P_s) \), we obtain (191).

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