Quantum Corrections to the Electroweak Sphaleron Transition

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Abstract

We have performed a new exact evaluation of the fluctuation determinant $\kappa$ of the electroweak sphaleron with $\Theta_W = 0$. The results differ significantly from a previous calculation of this quantity by Carson et. al. We find that $\kappa$ is of order 1 in units $(gv)^6$ while the previous results indicated a strong suppression of the sphaleron transition by this factor.
The electroweak theory has been found [1, 2] to possess a topologically nontrivial solution which describes a saddlepoint between two topologically distinct vacua.

The recent interest in this solution has been centered around its possible rôle in generating baryon number violating processes in the early universe or even at accelerator energies [3] - [5]. The rate of sphaleron transitions in the range of temperatures $M_W(T) \ll T \ll M_W(T)/\alpha_W$ has been derived by Arnold and McLerran [4]. It is given by

$$\Gamma = \frac{\omega_-}{2\pi} N e^{-E_{sph}/T} \kappa$$

Here $\omega_-$ is the absolute value of the eigenvalue of the unstable mode, the prefactor $N$ refers to normalizations introduced by the translation and rotation zero modes. $E_{sph}$ is the sphaleron energy and the factor $\kappa$ takes account of the quantum fluctuations of the sphaleron. It is given by

$$\kappa = \text{Im} \left( \frac{\Delta_S^0 \Delta_{gf}^0}{\Delta_{FP}^0 \Delta_{gf}^0} \right)^{1/2}$$

where the symbols $\Delta$ denote the small fluctuation determinants of the gauge fixed action (gf) and the Fadeev-Popov action (FP) evaluated around the sphaleron (S) and the vacuum (0), respectively. If the fluctuation operators are diagonalized the determinants are formally given by the product of the squared eigenfrequencies $\lambda_0^2$. The determinant $\Delta_{gf}^0$ of the gauge fixed action is to be evaluated without the zero modes and with the eigenvalue of unstable mode, $\lambda_-^2 = -\omega_-^2$ replaced by its absolute value.

$\kappa$ has been first evaluated [7] using an approximation scheme developed by Diakonov et al. [6] and subsequently exactly by Carson et al. [8]. The results of this exact calculation differ significantly from the approximate ones and from a perturbative estimate. It is therefore of interest to repeat this exact evaluation and this is the subject of our investigation. Technically our evaluation differs essentially from the one in Ref. [8] so that the calculations can be considered as independent. While we use the background gauge as do Carson et al., we use another angular momentum basis and a different scheme for evaluating the determinant. In this short note we will present only an outline of our approach and the numerical results. A more extensive account is in preparation [9].

The analysis of the small fluctuations around the sphaleron requires a partial wave decomposition with respect to the quantum number $K = \vec{J} + I (K\text{-spin})$. Our present work is based on the analysis of Ref. [10] where the sphaleron stability was investigated. The small fluctuation equations obtained there had to be modified however. While that investigation was performed in the $A_0 = 0$ gauge we use here the background gauge. So a gauge fixing term had to be added and also we had to construct the small fluctuation Lagrangean for the Fadeev-Popov modes. The basic algebra and the resulting equations will be presented elsewhere [9].

The evaluation of the determinant has been performed using the Euclidean Green function in analogy to some recent investigations of one of the authors (J.B.) [11]. We
will sketch the method for the case of a single field: The logarithm of the fluctuation determinant around the vacuum, divided by the one around the sphaleron can be written formally (postponing renormalization and the treatment of zero and unstable modes) as

\[
\ln\left(\frac{\Delta^0}{\Delta^S}\right)^{1/2} = \int_0^\infty d\nu \sum_\alpha \frac{1}{\nu^2 + (\lambda^S_\alpha)^2} - \frac{1}{\nu^2 + (\lambda^0_\alpha)^2}
\]

where \(\lambda^S_\alpha, \lambda^0_\alpha\) are the eigenfrequencies of the fluctuations around sphaleron and vacuum, respectively. Further we can can relate the integrand to Euclidean Green functions via

\[
\sum_\alpha \frac{1}{\nu^2 + \lambda^2_\alpha} = \int d^3x \sum_\alpha \frac{\phi^*_\alpha(\vec{x})\phi_\alpha(\vec{x})}{\nu^2 + \lambda^2_\alpha} = \int d^3x G_E(\vec{x}, \vec{x}, \nu)
\]

Here \(\phi_\alpha\) are the eigenmodes and the Euclidean Green function is a solution of the equation

\[
(\nu^2 + D^2)G_E(\vec{x}, \vec{x}', \nu) = \delta^3(\vec{x} - \vec{x}')
\]

\(D^2\) is the fluctuation operator \(\delta^2S^3/\delta\phi^2\) where \(S^3\) is the three dimensional action and \(\phi\) refers to the different field fluctuations around vacuum and sphaleron. So the final equation for the fluctuation determinant is

\[
\ln\left(\frac{\Delta^0}{\Delta^S}\right)^{1/2} = \int_0^\infty d\nu F(\nu)
\]

\[
F(\nu) = \int d^3x (G^S_E(\vec{x}, \vec{x}, \nu) - G^0_E(\vec{x}, \vec{x}, \nu))
\]

The evaluation of the Green function can be done by decomposing it into partial waves and by determining the solutions \(f^-\), regular at \(r = 0\), and \(f^+\), regular as \(r \to \infty\), of the corresponding differential equations. The partial wave Green function is then obtained in the usual way as

\[
G_l(r, r', \nu) = f^- (r_<) f^+ (r_>) / [\nu^2 W(f^-_l, f^+_l)]
\]

where \(W\) is the Wronskian. Actually we have not calculated the partial wave amplitudes \(f^\pm_l\) but the amplitudes \(h^\pm_l\) defined by

\[
f^\pm_l(r) = [1 + h^\pm_l(r)] b^\pm_l(\rho r)
\]

Here \(b^\pm_l\) are the solutions of the free partial wave equations with the same boundary conditions as \(f^\pm_l\), i. e. the modified spherical Bessel functions \(i_l\) and \(k_l\), and \(\rho = (\nu^2 + m^2)^{1/2}\). Then we have

\[
G_l(r, r, \nu) - G^{(0)}_l(r, r, \nu) = \rho[h^+_l(r) + h^-_l(r) + h^+_l(r)h^-_l(r)]i_l(\rho r)k_l(\rho r)
\]

The free part of the Green function, instead of being subtracted, is omitted here from the outset, avoiding the occurrence of small differences of large contributions. The detailed procedure for a multichannel system as we have it here is described extensively in Ref. [1].
Renormalization requires here just the removal of the tadpole graphs and replacing \( M_W \) and \( M_H \) by \( M_W(T) \) and \( M_W(T) \) as discussed in Ref. \[4\]. If the tadpole contribution is subtracted from \( F(\nu) \) the difference \( F_{\text{ren}}(\nu) \) behaves as \( \rho^{-3} \) as \( \nu \to \infty \) and the \( \nu \) integral is ultraviolet convergent.

Of course \( F_{\text{ren}} \) behaves as \( 6\nu^{-2} \) near zero due to the presence of 6 zero modes, leading to a logarithmic divergence of the \( \nu \) integration. It can be realized that Eq. (3) has been constructed in such a way that the contribution of each eigenmode is related to the evaluation of the integral of \( F_{\text{ren}}(\nu) \) at its lower limit; the procedure for removing the zero mode contributions consists therefore in evaluating the integral with a sufficiently small lower limit \( \epsilon \) and then adding \( 6\ln(\epsilon) \). The resulting expression stays finite as \( \epsilon \to 0 \) and this limit is the final answer:

\[
\ln \kappa = \lim_{\epsilon \to 0} \left( \int_{\epsilon}^{\infty} d\nu F_{\text{ren}}(\nu) + 6 \ln \epsilon \right).
\]

It should be noted that \( \nu \) and therefore \( \epsilon \) have the dimension of energy, so the scale used for these variables matters. We have performed our calculations in units of \( M_W \) while in Ref. \[8\] the scale was \( g\nu \); we had to take this into account for comparing our results. In the complete rate the dimensional scale reappears in the scale of the translation and rotation mode prefactors so that the final result is unique as it should.

It remains to consider the rôle of the unstable mode. It leads to a pole \( 1/(\nu^2 - \omega_-^2) \) in \( F(\nu) \) where \( \omega_- \) is the absolute value of the corresponding imaginary eigenvalue \( \lambda_- \). Since this eigenvalue should be included in the determinant with its absolute value, the correct prescription for evaluating the integral is the principal value one. In praxi we have subtracted from \( F_{\text{ren}}(\nu) \) an expression

\[
F_{\text{pole}} = \frac{1}{\nu^2 - \omega_-^2} - \frac{1}{\nu^2 + \sigma^2}
\]

with some suitable value for \( \sigma \). This subtraction makes the integrand regular at \( \nu = \omega_- \) without spoiling its ultraviolet convergence. We have then added the analytic principal value integral of this expression back to the numerical integral.

Our analytical and numerical procedure contains various implicit checks:

(i) The expected asymptotic behaviour of \( F(\nu) \) and \( F_{\text{ren}}(\nu) \) which follow from analysing the perturbation expansion of the Euclidean Green function. Note that for \( F(\nu) \) not only the power behaviour but also its absolute normalization are determined.

(ii) The behaviour of the partial wave contributions at large \( K \) which can be derived from a perturbative analysis to be as \( K^{-2} \).

(iii) The occurrence of the zero mode and unstable mode poles with the correct positions and residues.

Furthermore the differential equations for all partial waves have been checked to a certain extent by verifying their gauge invariance before gauge fixing \[10\]. The
decoupling of the time components of the vector field after adding the gauge fixing term represents a further check.

The numerical calculations were extended to angular momenta (K spins) up to 25. The contributions of the higher partial waves were included using an extrapolation $A/K^2 + B/K^3$, where $A$ and $B$ were obtained from a fit to the contributions of $K = 21$ to $K = 25$. The $\nu$ integration was performed numerically up to $\nu_{\text{max}} \approx 2.5$, then an asymptotic part was added to the integral by extrapolating $F_{\text{ren}}$ as $C/\rho^3 + D/\rho^5$.

Fig. 1 shows a typical plot for the functions $\nu F(\nu)$ and $\nu F_{\text{ren}}(\nu)$ together with the expected asymptotic behaviour at small and large $\nu$. The pole part $F_{\text{pole}}$ (see Eq. (11)) has been subtracted in all amplitudes. The results for $\ln \kappa$ are given in Table I for various values of $\xi = M_H/M_W$, with $\kappa$ in units of $M_W^6$. They are plotted in Fig. 2 together with previous results and estimates in the scale $g_v$, i.e. after subtracting $6 \ln 2$ from the values of Table I. We think that the main uncertainty of our results comes from the extrapolation of $F_{\text{ren}}(\nu)$ to $\nu = \infty$. We estimate the error of this asymptotic contribution to be around 10% yielding a typical error of 0.3 for $\ln \kappa$.

Our results differ considerably from the ones of Carson et al. [8] while for $M_H \approx M_W$ they are near the estimate based on the method of Diakonov et al. [6]. However they show a different trend for small Higgs masses. If they are correct - and in view of the mentioned consistency checks we can have no reasonable doubt - this means that the inclusion of the fluctuation determinant $\kappa$ does not suppress the sphaleron transition even for (unrealistic) small Higgs masses.

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| $\xi$ | 0.4 | 0.5 | 0.6 | 0.8 | 1.0 | 1.5 | 2.0 |
|-------|-----|-----|-----|-----|-----|-----|-----|
| $\omega_-$ | 1.32 | 1.36 | 1.39 | 1.45 | 1.51 | 1.62 | 1.71 |
| $\ln \kappa$ | 6.18 | 5.89 | 5.68 | 5.50 | 5.48 | 5.62 | 5.71 |
Figure Captions

Fig. 1 The Function $F(\nu)$ for $\xi = M_H/M_W = 1$

The solid line shows $\nu F_{\text{ren}}(\nu)$, the dashed line the unrenormalized $\nu F(\nu)$. The pole contribution $F_{\text{pole}}$ of Eq. (11) has been removed. The dotted lines show the expected power behaviours at small and large $\nu$ and the dash-dotted line the analytically known tadpole contribution to $\nu F(\nu)$.

Fig. 2 The Fluctuation Determinant

We plot the logarithm of the fluctuation determinant $\kappa$ for various values of the ratio $\xi = M_H/M_W$. The circles are our results, the crosses those of Carson et al.. The full line is the estimate of Carson and McLerran based on the method of Diakonov et al. and the dashed line a perturbative estimate.