Numerical Solution of Nonlinear Vibration by Euler-Cromer Method

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Abstract. Nonlinear equations are able to present many behaviours of physical systems better than linear equations. Analytical solutions to nonlinear vibration equations have intractability characteristics, while limitations in computational software resources make it difficult to study systematically the phenomena in many systems. In this paper, the Euler-Cromer method is used to solve numerically the vibration equation nonlinear. The nonlinear vibrations of the equation of harmonic, Van der Pol, and Duffing oscillator motion are used as the physical case models to solve. Simulation results show that the Euler-Cromer method provides a numerical solution that is easy to implement and accurate.

Keywords: nonlinear vibration, Euler-Cromer method, numerical solution

1. Introduction

Many dynamical systems in physics are harmonic motion phenomena, such as mass-spring oscillations, physical and mathematical pendulum, and oscillations of LC electrical system. This phenomenon assumes there is no damping force so that it produces an ideal harmonic motion equation. The harmonic motion equation is presented by a simple approach in the form of a linear second-order differential equation. The equation for linear harmonic motion has a solution made up trigonometric periodic functions in the form of sine and cosine. However, the solution to the linear dynamics model is usually used only as the initial approach of real systems [1].

In a simple oscillation motion, the assumption used is that the initial deviation value is quite small so that the angle value of the deviation from the equilibrium position of sin x can be approximated by x. This results in a simple harmonic oscillation equation model as in equation 1). To get the solution, we can use many simple theorems and methods, both numerically and analytically.

\[ \ddot{x} + \frac{g}{l}x = 0 \]  

1)

However, the equation that we solve comes from an approach. Factually, equation 1) is not linear, especially if the initial deviation value is large enough. A better formulation is to use Taylor expansion to replace sin x. Thus, a new nonlinear equation is produced as the following.

\[ \ddot{x} + \frac{g}{l}(x - \frac{1}{6}x^3) = 0 \]  

2)

Nonlinear equations are capable of presenting behaviour of the real system better than linear systems, including the characteristic of different parts of the harmonic oscillation system at different frequencies, jump phenomena, frequency response, and self-sustaining interactions of explicit external periodic forcing [2].
Vibration dynamics are presented better with nonlinear equations. For damped vibration nonlinear equations that known as Van der Pol (VDP) oscillator is presented in equation 3). The equation is widely used to model many system dynamics, one of which is the human heart [3]. Whereas for Duffing oscillator vibration is presented in equation 4). Duffing oscillator occurs as a result of the motion of a body subjected to a nonlinear spring power, linear sticky damping and periodic powering [4].

\[
\ddot{x} - \mu(1-x^2)\dot{x} + x = 0 \tag{3}
\]

\[
m\ddot{x} + k\dot{x} + \frac{1}{\epsilon}(x + 3x^3) = F\cos(\omega t) \tag{4}
\]

Various methods can be used to provide solutions for nonlinear vibration equations, both analytically and numerically. However, nonlinear differential equations are intractable to solve analytically [5]–[7]. Therefore, nowadays many people use computational software to study non-linear vibration system behaviour. The limitations in computational resources make it difficult to systematically study the phenomena in large systems [2]. For many things, detailed studies of non-linear vibrations are usually conducted using small systems. Therefore, the development of numerical methods is needed for the completion of nonlinear vibrations.

The simple numerical solutions commonly used for solving nonlinear equations are the Euler, improved Taylor, and Runge-Kutta method [8]. In addition, the numerical solution of the 2nd order differential equation has been done by the Runge-Kutta method and the Improved Runge-Kutta Nyström (IRKN) method with error accuracy and number of function evaluations are more efficient [9]. These show that with a modification of the simple numerical method, a precise and efficient result can be obtained.

In this study, the modified method of Euler as known as Euler-Cromer method was used to solve these nonlinear equations numerically.

2. Research Method

The Euler method is a simple method suitable for numerical solutions to systems of linear equations such as harmonic oscillators. With a slight modification of the Euler algorithm with Cromer, nonlinear problems with impermanent energy will be simulated. The numerical solution with the Cromer method is to modify the gradient sequence used to calculate the point displacement. To calculate the displacement \(x(i+1)\), used the gradient at the next data \(\dot{x}(i+1)\). The Euler-Cromer method algorithm is as follows.

\[
\ddot{x}(i+1) = \ddot{x}(i) + \ddot{x}(i)\Delta t \tag{5}
\]

\[
x(i+1) = x(i) + \dot{x}(i+1)\Delta t \tag{6}
\]

Thus, the numerical solution algorithm for a nonlinear oscillation equation 2) is stated in equation 7) and 8).

\[
\omega(i+1) = \omega(i) - \frac{g}{i}(x(i) - \frac{x(i)^3}{6})\Delta t \tag{7}
\]

\[
x(i+1) = x(i) + \omega(i+1)\Delta t \tag{8}
\]

Meanwhile, the numerical solution algorithm for Van der Pol oscillators is as follows.

\[
\omega(i+1) = \omega(i) + [\mu(1-x(i)^2)\omega(i) - x(i)]\Delta t \tag{9}
\]

\[
x(i+1) = x(i) + \omega(i+1)\Delta t \tag{10}
\]

If the equation 4) divided by the mass, the basic system of Duffing oscillator can be written in the form

\[
\ddot{x} + bx + s(x + \beta x^3) = a \cos(\omega t) \tag{11}
\]

For numerical solution of equation 11), we get algorithms that stated in equation 12) and 13) as follows.

\[
\omega(i+1) = \omega(i) - b * \omega(i) * \Delta t - s * [x(i) + \beta * x(i)^3] + a * \cos(\omega * i)\Delta t \tag{12}
\]

\[
x(i+1) = x(i) + \omega(i+1)\Delta t \tag{13}
\]
3. Result
In this study, numerical solutions using the Euler-Cromer method were carried out with the help of the MATLAB 2015a programming language. For comparison, the solutions obtained are compared to two other simple methods, namely Euler and Runge-Kutta order-2.

The displacement graph in Figure 1 shows that the Euler-Cromer method provides a solution of equation 2) in the form of a constant amplitude harmonic wave. It is compatible with the reference [10]. For a relatively large initial displacement value, $x_0 = 0.5$, it can be seen that the time-displacement graph has a consistent harmonic pattern. From the resulting phase portrait, it can be seen that the system behaviour is stable, in the mean that there is no system solution that leads to infinity or towards zero. This indicates that the method used can provide a good solution to non-linear vibration of the harmonic oscillation equation. When compared with other simple methods, namely Euler and Runge-Kutta Method, it is seen that the Euler-Cromer method is the most stable, as in Figure 2. It can be seen that the other two methods provide divergent amplitude graph solutions, especially for those using Euler's method. This is caused by the deviation that leads to infinity.

![Figure 1](image1.png)

**Figure 1.** Graphically solution of equation 2) with the Euler-Cromer Method and the phase graph, for $\omega = 1$

![Figure 2](image2.png)

**Figure 2.** Solution to Equation 2) using the Euler, Euler-Cromer, and Runge-Kutta methods
The graph of the numerical solution for the Van der Pol oscillator with Euler-Cromer is shown in Figure 3. It can be seen that Euler-Cromer is able to show nonlinear features well, namely hysteresis and chaos, and forming a cycle stable limit. The graph shows trajectory for the value of \( \mu = 10 \) that results in a relaxation oscillator, meaning the oscillation consists of a slow asymptotic movement and sudden discontinuous jumps.

**Figure 3.** Solution to equation 3) with Euler-Cromer for \( \mu = 10, x_0=0.1, \omega_0 = 0.1 \).

When compared with the solution using the Euler and Runge-Kutta methods in Figure 4, the resulting pattern is quite similar, in the sense that both methods also produce hysteresis and chaos. The chaotic motion occurs when the trajectory of a system is highly sensitive to its initial conditions [1]. In this study, it will be seen when we use the initial values of 0.5 and 0.6. However, the results of both methods have a different phase that can be shown by the delay of time. In addition, the amplitude value also experiences a slight divergence.

**Figure 4** Solution to equation 3) using Euler, Euler-Cromer, and Runge-Kutta methods.

The graph of the numerical solution equation 11) is shown in Figure 5 where the graph forms a pattern according to the reference [11]. With the same parameter values, a graph with the same pattern and
amplitude value is obtained. We can see that the graph provide an oscillatory system that has the effect of reducing, restricting or preventing its oscillation. The behaviour is produced by dissipated energy stored in the oscillation system[4]. Phase portraits can also be displayed properly.

![Figure 5](image1.png)

**Figure 5** Solution to equation 11) using the Euler-Cromer method for $a=50$, $b=0.5$, $s=1.1$, $\beta=0.05$, $\omega=1$

The solution to the Euler and Runge-Kutta methods shows a change in the graph pattern, delay deviation value and deviation of the amplitude value as shown in **Figure 6**.

![Figure 6](image2.png)

**Figure 6.** Solution to equation 11) using the Euler, Euler-Cromer, and Runge-Kutta methods

4. Conclusion

Based on the simulation results it can be concluded that the Euler-Cromer is a simple method that can produce a good solution for non-linear vibration equations. The resulting trajectories show good nonlinear features and have the same characteristics as the references, both the patterns and phases. Comparison with Euler and Runge-Kutta methods, Euler-Cromer provides a more stable, accurate and effective solution. However, Cromer method need to be tested for more complex systems.

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