I. INTRODUCTION.

Since the pioneering work of Dirac ([1]), who proposed, motivated by the occurrence of large numbers in Universe, a theory with a time variable gravitational coupling constant G, cosmological models with variable G and nonvanishing and variable cosmological term, Λ, have been intensively investigated in the physical literature (see for example [2]-[14]) and for alternative approaches in this framework see for example ([15]) where the authors study a FRW model with variable equation of state and ([16]) with variable deceleration parameter, finding a positive decreasing cosmological “constant”.

In modern cosmological theories, the cosmological constant remains a focal point of interest (see [17]-[20] for reviews of the problem). A wide range of observations now compellingly suggest that the universe possesses a non-zero cosmological constant. Some of the recent discussions on the cosmological constant “problem” and on cosmology with a time-varying cosmological constant point out that in the absence of any interaction with matter or radiation, the cosmological constant remains a “constant”. However, in the presence of interactions with matter or radiation, a solution of Einstein equations and the assumed equation of covariant conservation of stress-energy with a time-varying Λ can be found. This entails that energy has to be conserved by a decrease in the energy density of the vacuum component followed by a corresponding increase in the energy density of matter or radiation. Recent observations strongly favour a significant and a positive value of Λ with magnitude Λ(CH/c3) ≈ 10^{-123}. These observations suggest on accelerating expansion of the universe, q < 0.

Therefore, it is considered G and Λ as coupling scalars within the Einstein equations, \( R_{ij} - \frac{1}{2}g_{ij} = GT_{ij} - \Lambda g_{ij} \), while the other symbols have their usual meaning and hence the principle of equivalence then demands that only \( g_{ij} \) and not \( G \) and \( \Lambda \) must enter the equation of motion of particles and photons. in this way the usual conservation law, \( \text{div}T = 0 \), holds. Taking the divergence of the Einstein equations and using the Bianchi identities we obtain the an equation that controls the variation of \( G \) and \( \Lambda \). These are the modified field equations that allow to take into account a variable \( G \) and \( \Lambda \). Nevertheless this approach has some drawbacks, for example, it cannot derived from a Hamiltonian, although there are several advantages in the approach.

All these works were carry out in the framework of flat Friedmann–Robertson–Walker (FRW) symmetries. At the same time, all these works have been extended to more complicated geometries, like for example LRS Bianchi I as
well as Bianchi I models, which represent the simplest generalization of the flat FRW models (see for example [21]-[25] in the context of perfect fluids, [26]-[28] in the context of viscous fluids and [29]-[30] in the framework of magnetized viscous fluids). Bianchi I models are important in the study of anisotropies.

Recently, the cosmological implications of a variable speed of light during the early evolution of the Universe have been considered. Varying speed of light (VSL) models proposed by Moffat ([31]) and Albrecht and Magueijo ([32]), in which light was travelling faster in the early periods of the existence of the Universe, might solve the same problems as inflation. Hence they could become a valuable alternative explanation of the dynamics and evolution of our Universe and provide an explanation for the problem of the variation of the physical “constants”. Einstein’s field equations (EFE) for FRW spacetime in the VSL theory have been solved by Barrow ([33] and [34] for anisotropic models), who also obtained the rate of variation of the speed of light required to solve the flatness and cosmological constant problems (see J. Magueijo ([35]) for a review of these theories).

This model is formulated under the strong assumption that a $c$ variable (where $c$ stands for the speed of light) does not introduce any corrections into the curvature tensor, furthermore, such formulation does not verify the covariance and the Lorentz invariance as well as the resulting field equations do not verify the Bianchi identities either (see Bassett et al [36]).

Nevertheless, some authors (T. Harko and M. K. Mak [37], P.P. Avelino and C.J.A.P. Martins [38] and H. Shojaie et al [39]) have proposed a new generalization of General Relativity which also allows arbitrary changes in the speed of light, $c$, and the gravitational constant, $G$, but in such a way that variations in the speed of light introduces corrections to the curvature tensor in the Einstein equations in the cosmological frame. This new formulation is both covariant and Lorentz invariant and as we will show the resulting field equations (FE) verify the Bianchi identities. As we have shown in [40] this formulation allows us to obtain the energy conservation equation from the field equations as in the standard case. Following these ideas we have studied a LRS Bianchi I with time varying constants in the framework of a viscous fluid as well as for a perfect fluid (see [41]). In this work we arrive to the conclusion that in the early universe, viscous era, from thermodynamical restrictions, constants $G$ and $c$ are growing time functions while the cosmological constant is a decreasing time function whose sing depends on the equation of state. Nevertheless, when we studied the perfect fluid era, we are not able to determine the behavior of these functions arriving to the conclusions that $G$ and $c$ may be growing as well as decreasing time functions while $\Lambda$ is always a decreasing time function.

In a recent paper (see [25]) we have developed and compared some well known tactics (approaches) in order to study and to find exact solutions for a perfect fluid Bianchi I models with variable $G$ and $\Lambda$, but trying to make the lowest number of assumptions or neither. We tried to show that with these approaches all the usual simplifying hypotheses may be deduced from a correct mathematical principle and how the useful are each tactic, i.e. to show the advantages and disadvantages of each approach. We have started studying this class of models because, as we have mentioned above, there are many well known exact solutions so we will be able to compare the useful of our approaches.

The purpose of this paper is to generalize the perfect fluid LRS Bianchi I model with time varying constants (see [41]) taking into account the effects of a $c$ variable into the field equations. Hence in this paper we are going to study a Bianchi I model with variable $G, c$ and $\Lambda$ through the Lie group method (LM), studying the symmetries of the resulting ODE, and through the self-similarity (SS), matter collineations (MC) as well as kinematical self-similarity (KSS) hypothesis. We would like to emphasize that in this work we are more interesting in mathematical respects (as the integration conditions) than in studying physical consequences. Nevertheless we consider some observational data in order to rule out some the obtained solutions.

Therefore the paper is divided in the following sections: In section two we outline the main ingredients of the model as well as the field equations (FE), under the condition $div T = 0$. Since in this paper we would like to compare the possible effects of a $c$–var with the “traditional” formulation (i.e. which where such effects are not taking into account) we need to outline both FE. We will show that such effect is minimum but exists, although it does not affect to the obtained solutions through the Lie method. In order to apply the (LM) we need to deduce an ODE. For this purpose we have followed the model proposed by Kalligas, Wesson and Everett, [22], but taking into account some little differences (obviously here $c = c(t)$, i.e. it is a time varying function). In this way we have deduced three ODEs. The first one, of second order, which will be studied in section (IV). This is maybe the main difference, in this approach, with our previous paper [25] where we studied a third order ODE, but in this occasion as we will show in appendix A there is no difference between both approaches. In this appendix we will study through the LM our second ODE while in appendix B we will study through the LM our third ODE which is a third order ODE but it has been obtained without considering the effect of $c$–var into the curvature tensor. In this way we will be able to compare both approaches. As we will see there is no difference (at least in order of magnitude) between both approaches.

In section three we calculate all the curvature tensors, i.e. Weyl etc... as well as their invariants, i.e. Kretschmann scalars etc...., but taking into account the effects of a $c$–var in all the curvature tensors, in our previous paper (25) we calculated the same ingredients but in the traditional way i.e. where $c$ is a true constant. Once we have outlined
the FEs then we go next to study the resulting FEs through the LM, as well as under the SS, MC and KSS hypotheses.

In section four it is studied through the Lie group tactic the second order differential equation with four unknowns. We seek the possible forms that may take $G, c$ and $\Lambda$ in other to make integrable the ODE. In this way we find that there are three possibilities, but the question here, is that all the studied solutions depends of many integrating constants so it is quite difficult to get information about the real behavior the quantities. Furthermore, when one try to solve the algebraic system of equations in order to find the numerical value the exponents of the scale factors finds that the only possible solution is the flat FRW one (arriving to the same solutions as the obtained ones in [40]), so we arrive to the same scenario as in our previous paper [25]. We think that the followed tactic is too restrictive, for this reason we are only able to obtain this class of solutions. Nevertheless there are other Lie approaches as the followed by M. Szydlowski et al (see [58]) which we think that may be more useful than the followed one here. Trying to improve the obtained solutions, in appendix A, we will study through the LM the third order ODE, but as we will show, we arrive to the same solutions and therefore to the same conclusions. As we have mentioned above, in appendix B we will study a third order ODE which has been obtained without the assumption of $c$–var affecting to the curvature tensor. We arrive to the same solutions as the obtained ones in appendix A, and therefore we conclude that at least in order of magnitude there is no difference between both approaches.

In section five, we study the model under the self-similarity hypothesis. In this case, the obtained solution is similar (in order of magnitude) to the obtained one in the above section (LM with scaling symmetries). Since in this case, all the obtained solutions, for each quantity, depend on $(\int c(t)dt)$, it is difficult to determine the behavior of each quantity. Nevertheless, we are able to arrive to some conclusions taking into account some observational data as for example the hypothesis $q < 0$, (where $q$ stands for the deceleration parameter) and $\Lambda > 0$. Under these considerations we find that $c$ is a growing time functions while $\Lambda$ is a decreasing time function whose sing depends on the equation of state $\omega$. With regard to $G$ we find that it depends on two parameters, the equation of state and $\int c(t)dt$, so it may be a decreasing time function as well as a growing time function depending on the value of these two parameters. In the same way as in [25] we conclude that the exponents of the scale factor must satisfy the conditions $\sum_{i=1}^3 \alpha_i = 1$ and $\sum_{i=1}^3 \alpha_i^2 < 1$, $\forall \omega$, i.e. for all equation of state, relaxing in this way the Kasner conditions. Since the model is SS, then, we study the model studying the matter collineations (MC). In this occasion we only check that the homothetic vector field verifies the reformulated MC equations (see [25] for details) in order to get information on the behavior of $G, c$ and $\Lambda$, arriving to the same conclusions as in the SS section.

In the last section we reproduce the same tactic, but this time, under the KSS hypothesis. In this occasion we get a non-singular solution and with the same behavior for the main quantities as the obtained ones in the above sections. Before ending this section, we discuss the Kasner like solutions within this framework arriving to the conclusion that this class of solutions bring us to get vanishing quantities as well as of having a pathological curvature behavior since the gravitational entropy tends to infinite. We end with a brief conclusions.

II. THE MODEL(S).

Throughout the paper $M$ will denote the usual smooth (connected, Hausdorff, 4-dimensional) spacetime manifold with smooth Lorentz metric $g$ of signature $(-, +, +, +)$. Thus $M$ is paracompact. A comma, semi-colon and the symbol $\mathcal{L}$ denote the usual partial, covariant and Lie derivative, respectively, the covariant derivative being with respect to the Levi-Civita connection on $M$ derived from $g$. The associated Ricci and stress-energy tensors will be denoted in component form by $R_{ij} (=R_{j;0})$ and $T_{ij}$ respectively. A diagonal Bianchi I space-time is a spatially homogeneous space-time which admits an abelian group of isometries $G_3$, acting on spacelike hypersurfaces, generated by the spacelike KV’s, $\xi_1 = \partial_x, \xi_2 = \partial_y, \xi_3 = \partial_z$. In synchronous co-ordinates the metric is:

$$ds^2 = -dt^2 + A_i^2(t)(dx^i)^2 \quad (1)$$

where the metric functions $A_1(t), A_2(t), A_3(t)$ are functions of the time co-ordinate only (Greek indices take the space values 1, 2, 3 and Latin indices the space-time values 0, 1, 2, 3). In this paper we are interested only in proper diagonal Bianchi I space-times (which in the following will be referred for convenience simply as Bianchi I space-times), hence all metric functions are assumed to be different and the dimension of the group of isometries acting on the spacelike hypersurfaces is three. Therefore we consider the Bianchi type I metric as

$$ds^2 = -c(t)^2dt^2 + X^2(t)dx^2 + Y^2(t)dy^2 + Z^2(t)dz^2, \quad (2)$$

see for example ([42]-[48]).

For a perfect fluid with energy-momentum tensor:

$$T_{ij} = (\rho + p)u_iu_j + pg_{ij}, \quad (3)$$
where we are assuming an equation of state \( p = \omega \rho, (\omega = \text{const.}) \). Note that here we have preferred to assume this equation of state but as we will show in the following sections this equation may be deduced from the symmetries principles as for example the self-similar one. The 4-velocity is defined as follows

\[
u = \left( \frac{1}{c(t)}, 0, 0, 0 \right). \]  

The time derivatives of \( G, c \) and \( \Lambda \) are related by the Bianchi identities

\[
\left( R_{ij} - \frac{1}{2} R g_{ij} \right)^{ij} = \left( \frac{8 \pi G}{c^4} T_{ij} - \Lambda g_{ij} \right)^{ij},
\]

in our case we obtain

\[
\dot{\rho} + \rho (1 + \omega) \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right) + \frac{\dot{\Lambda} c^4}{8 \pi G} + \rho \left( \frac{\dot{G}}{G} - 4 \frac{\dot{c}}{c} \right) = 0,
\]

but if we take into account the condition \( T_{ij}^{\beta} = 0 \), it is obtained the following set of equations:

\[
\left( T_{ij}^{\beta} = 0 \right) \iff \dot{\rho} + \rho (1 + \omega) \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right) = 0,
\]

\[
\frac{\dot{\Lambda} c^4}{8 \pi G \rho} + \frac{\dot{G}}{G} - 4 \frac{\dot{c}}{c} = 0.
\]

Therefore the resulting field equations (FE) yield:

\[
\dot{X} \dot{Y} X + \dot{X} \dot{Z} Z + \dot{Z} \dot{Y} Y = \frac{8 \pi G}{c^2} \rho + \Lambda c^2,
\]

\[
\dot{Y} + \dot{Z} - \left( \frac{\dot{Z}}{Z} + \frac{\dot{Y}}{Y} \right) \frac{\dot{c}}{c} = -\frac{8 \pi G}{c^2} \omega \rho + \Lambda c^2,
\]

\[
\dot{X} + \dot{Z} - \left( \frac{\dot{X}}{X} + \frac{\dot{Z}}{Z} \right) \frac{\dot{c}}{c} = -\frac{8 \pi G}{c^2} \omega \rho + \Lambda c^2,
\]

\[
\dot{X} + \dot{Y} - \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} \right) \frac{\dot{c}}{c} = -\frac{8 \pi G}{c^2} \omega \rho + \Lambda c^2,
\]

\[
\dot{\rho} + \rho (1 + \omega) \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right) = 0,
\]

\[
\frac{\dot{\Lambda} c^4}{8 \pi G \rho} + \frac{\dot{G}}{G} - 4 \frac{\dot{c}}{c} = 0.
\]

If in eqs. \((10-12)\) we make \( \dot{c} = 0 \), i.e. we do not consider the effects of \( c-\text{var} \) into the curvature tensor, then we obtain the usual FE, see for example [25].

Defining

\[
H = \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right) = \frac{3 \dot{R}}{R} \quad \text{and} \quad R^3 = XYZ,
\]

eq. \((13)\) takes the usual form for the conservation equation i.e.

\[
\dot{\rho} + \rho (1 + \omega) H = 0.
\]

The expansion \( \theta \) is defined as follows:

\[
\theta := u_i^{\prime}, \quad \theta = \frac{1}{c(t)} \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right) = \frac{1}{c(t)} H,
\]
and therefore the acceleration is:

$$a_i = u_{ij} u^j,$$

(18)

in this case $a = 0$, while the shear is defined as follows:

$$\sigma_{ij} = \frac{1}{2} (u_{ij} + u_{ji} + a_i u_j + a_j u_i) - \frac{1}{3} \theta h_{ij},$$

(19)

where $h_{ij} = g_{ij} + u_i u_j$ is the projection tensor, so

$$\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij}, \quad \sigma^2 = \frac{1}{3c^2} \left( \left( \frac{\dot{X}}{X} \right)^2 + \left( \frac{\dot{Y}}{Y} \right)^2 + \left( \frac{\dot{Z}}{Z} \right)^2 \right) - \frac{1}{X^2} - \frac{1}{Y^2} - \frac{1}{Z^2}.$$

(20)

### A. The main equations.

In this section we would like to obtain an ODE which allows us to study the field equations through the Lie method. For this purpose we are following closely the paper by Kalligas et al. (see [22]) and the same steeps followed in [25].

From eqs. (10-12) and taking into account eq. (9), it is obtained the following one:

$$\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} - \frac{c'}{c} \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right) = -4\pi (1 + 3\omega) \frac{G}{c^2} \rho + \Lambda c^2,$$

(21)

Now, taking into account eq. (13), squaring it and using (21) we get

$$\left( \frac{\dot{\rho}}{\rho} \right)^2 = (1 + \omega)^2 \left( \left( \frac{\dot{X}}{X} \right)^2 + \left( \frac{\dot{Y}}{Y} \right)^2 + \left( \frac{\dot{Z}}{Z} \right)^2 + \frac{16\pi}{c^2} G \rho + 2\Lambda c^2 \right),$$

(22)

since

$$H^2 = \left( \left( \frac{\dot{X}}{X} \right)^2 + \left( \frac{\dot{Y}}{Y} \right)^2 + \left( \frac{\dot{Z}}{Z} \right)^2 + 2 \left( \frac{X Y}{X Y} + \frac{X Z}{X Z} + \frac{Z Y}{Z Y} \right) \right).$$

(23)

The time derivative $\frac{\dot{\rho}}{\rho}$ from eq. (13) can now be expressed in terms of $G, c, \Lambda$ and $\rho$ only by using eqs. (21) and (22), a straightforward calculation brings us to get the following expression and hence we get the following expression

$$\dot{\rho} = \left( \frac{2 + \omega}{1 + \omega} \right) \frac{\dot{\rho}^2}{\rho} + 12\pi (\omega^2 - 1) \frac{G \rho^2}{c^2} - 3 (1 + \omega) \Lambda c^2 \rho + \frac{c'}{c} \dot{\rho},$$

(24)

$$\dot{\rho} = K_1 \frac{\dot{\rho}^2}{\rho} + K_2 \frac{G \rho^2}{c^2} - K_3 \Lambda c^2 \rho + \frac{c'}{c} \dot{\rho},$$

(25)

where

$$K_1 = \frac{2 + \omega}{1 + \omega}, \quad K_2 = 12\pi (\omega^2 - 1), \quad K_3 = 3 (1 + \omega),$$

(26)

this is the equation that we will study in section (IV) through the Lie method. As we will see in the appendices (A) and (B) there is not any advantage if we decide to study the third order ODEs instead of this one of second order as in our previous paper (see [25]), except that in this case is simpler to study the second order equation instead of the third order ODE.

Now differentiating eq. (24) and taking into account eq. (14) we obtain the equation that we will study through the Lie group method in appendix A. Therefore, we get

$$\ddot{\dot{\rho}} = K_1 \dot{\rho} \dot{\rho} - K_2 \frac{\dot{\rho}^3}{\rho^2} + \frac{G \rho^2}{c^2} \left[ K_3 \frac{G'}{G} - K_4 \frac{\rho'}{\rho} - K_5 \frac{c'}{c} \right] - K_6 \rho c \dot{\rho} + \dot{\rho} \left( \frac{c''}{c} - \frac{c'^2}{c^2} \right) + \frac{c'}{c} \left( \dot{\rho} - \frac{\rho^2}{\rho} \right),$$

(27)
where
\[
K_1 = \frac{5 + 3\omega}{1 + \omega}, \quad K_2 = \frac{4 + 2\omega}{1 + \omega}, \quad K_3 = 12\pi (1 + \omega)^2, \\
K_4 = 12\pi (1 - \omega^2), \quad K_5 = 24\pi ((\omega + 2)^2 - 1), \quad K_6 = 6 (1 + \omega).
\] (28)
we are supposing that \(\omega \neq -1\).

If we do not consider the effects of \(c\)-var into the curvature tensor, then following the same steep s we arrive to the next eq.:
\[
\ddot{\rho} = K_1 \dot{\rho} - K_2 \frac{\rho^3}{c^2} + \frac{G\rho^2}{c^2} \left[ K_3 \frac{G'}{G} - K_4 \frac{\rho'}{\rho} - K_5 \frac{c'}{c} \right] - K_6 \rho c c' \Lambda,
\] (29)
where
\[
K_1 = \frac{5 + 3\omega}{1 + \omega}, \quad K_2 = \frac{4 + 2\omega}{1 + \omega}, \quad K_3 = 12\pi (1 + \omega)^2, \\
K_4 = 12\pi (1 - \omega^2), \quad K_5 = 24\pi ((\omega + 2)^2 - 1), \quad K_6 = 6 (1 + \omega),
\] (30)
this eq. will be studied in appendix B. Note that eqs. (30) are the same as eqs. (28).

As it is observed eq. (29) looks simpler than eq. (27). Actually, as we will see in appendices (A) and (B) both eqs. bring us to the same solutions (at least in order of magnitude), so following this way there is no difference between to study the resulting FE with \(c\)-var affecting to the curvature tensor, i.e. eq. (27), and eq. (29) where we have not take into account such effects.

In this way, it is easy to calculate the shear. Algebra brings us to obtain to following expression:
\[
\sigma^2 = \frac{1}{3c^2 (1 + \omega)^2} \left( \frac{\dot{\rho}}{\rho} \right)^2 - \left( 8\pi \frac{G}{c^4} \rho + \Lambda \right).
\] (31)

III. CURVATURE ANALYSIS.

In this section we calculate some of the curvature invariants (see for example [49]-[52]) but taking into account the effects of a \(c\)-var into the curvature tensors. In ([25]) we have calculated the same invariants but in the traditional way i.e. ignoring the effects of \(c(t)\) into the curvature tensors.

The full contraction of the Riemann tensor, i.e. Krestchmann scalars are:
\[
I_1 := R_{ijkl} R^{ijkl},
\] (32)
\[
I_1 = \frac{4}{c^4} \left[ \left( \frac{\ddot{X}}{X} \right)^2 - 2 \frac{\dot{X}^2}{cX} + \frac{c^2 \dot{X}^2}{c^2 X^2} + \left( \frac{\dot{Y}}{Y} \right)^2 - 2 \frac{\dot{Y} \dot{Y} \dot{c}}{Y Y c} + \left( \frac{\ddot{Y}}{Y} \right)^2 + \left( \frac{\ddot{Z}}{Z} \right)^2 \right.
\]
\[- \left. 2 \frac{\dot{Z} \dot{Z} \dot{c}}{Z c} + \left( \frac{\dot{Z} \dot{c}}{Z c} \right)^2 + \left( \frac{\dot{X} \dot{Y}}{X Y} \right)^2 + \left( \frac{\dot{X} \dot{Z}}{X Z} \right)^2 + \left( \frac{\dot{Y} \dot{Y}}{Y} \right)^2 \right].
\] (33)
\[
I_2 := R_{ij} R^{ij},
\] (34)
\[ I_2 = \frac{2}{c^4} \left[ \left( \frac{\dot{X}}{X} \right)^2 + \left( \frac{\dot{Y}}{Y} \right)^2 + \left( \frac{\dot{Z}}{Z} \right)^2 + \frac{\ddot{X} \dot{Y}}{XY} + \frac{\ddot{X} \dot{Z}}{XZ} + \frac{\ddot{Y} \dot{Z}}{YZ} + \frac{\ddot{X} \dot{Y} \dot{Z}}{XYZ} + \frac{\ddot{Y} \dot{X} \dot{Z}}{YXZ} + \frac{\ddot{Z} \dot{X} \dot{Y}}{XZY} + \frac{\ddot{Z} \dot{Y} \dot{X}}{YZX} \right) \]

\[ + \left( \frac{\dddot{Y}}{YZ} \right)^2 + \left( \frac{\dddot{X}}{XZ} \right)^2 + \left( \frac{\dddot{Z}}{YX} \right)^2 + \frac{\ddot{X} \ddot{Y}}{XY} + \frac{\ddot{X} \ddot{Z}}{XZ} + \frac{\ddot{Y} \ddot{Z}}{YZ} + \left( \frac{\dddot{X} \ddot{Y} \dddot{Z}}{XYZ} + \frac{\dddot{Y} \dddot{X} \ddot{Z}}{YXZ} + \frac{\dddot{Z} \dddot{X} \ddot{Y}}{XZY} \right) \]

\[ + \left( \frac{\dot{X} \dddot{Z}}{XZ} \right)^2 + \left( \frac{\dot{Y} \dddot{Z}}{YZ} \right)^2 + \frac{\dot{X} \dddot{Z} \dot{Y}}{XZY} + \frac{\ddot{X} \dddot{Z} \dot{Y}}{XZY} + \frac{\ddot{Y} \dddot{Z} \dot{X}}{YZX} + \frac{\dddot{X} \dddot{Z} \dot{Y}}{XZY} - \frac{\dot{Y} \dddot{X}}{YX} - \frac{\dot{X} \dddot{Y}}{XY} - \frac{\dddot{X} \dddot{Y}}{XZ} - \frac{\dddot{Y} \dddot{X}}{YZ} \right], \]  

\[ \] (35)

and

\[ R := R_i^j = \frac{2}{c^4} \left( \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} + \frac{X'Y'}{XY} + \frac{Y'Z'}{YZ} + \frac{X'Z'}{XZ} \right). \]  

(36)

The non-zero components of the Weyl tensor are:

\[ C_{1212} = \frac{X^2}{6} \left( -2 \frac{X''}{X} + \frac{X'}{X} \left( \frac{Y'}{Y} + \frac{Z'}{Z} \right) + \left( \frac{Y''}{Y} + \frac{Z''}{Z} \right) - \frac{Y'}{Y} \left( \frac{Y'}{Y} + \frac{Z'}{Z} \right) \right), \]  

(37)

\[ C_{1313} = \frac{Y^2}{6} \left( -2 \frac{Y''}{Y} + \frac{Y'}{Y} \left( \frac{X'}{X} + \frac{Z'}{Z} \right) + \left( \frac{X''}{X} + \frac{Z''}{Z} \right) - \frac{X'}{X} \left( \frac{X'}{X} + \frac{Z'}{Z} \right) \right), \]  

(38)

\[ C_{1414} = \frac{Z^2}{6} \left( -2 \frac{Z''}{Z} + \frac{Z'}{Z} \left( \frac{X'}{X} + \frac{Y'}{Y} \right) + \left( \frac{X''}{X} + \frac{Y''}{Y} \right) - \frac{X'}{X} \left( \frac{X'}{X} + \frac{Y'}{Y} \right) \right), \]  

(39)

\[ C_{2323} = \frac{X^2 Y^2}{6c^2} \left( 2 \left( \frac{Z'}{Z} - \frac{c}{c} \frac{Z'}{Z} \right) - \left( \frac{X'}{X} + \frac{Y'}{Y} \right) \left( \frac{Z'}{Z} + \frac{c}{c} \frac{Z'}{Z} \right) - \frac{X''}{X} - \frac{Y''}{Y} - \frac{2Z'X'}{ZX} \right), \]  

(40)

\[ C_{2424} = \frac{X^2 Z^2}{6c^2} \left( 2 \left( \frac{Y'}{Y} - \frac{c}{c} \frac{Y'}{Y} \right) - \left( \frac{X'}{X} + \frac{Z'}{Z} \right) \left( \frac{Y'}{Y} - \frac{c}{c} \frac{Y'}{Y} \right) - \frac{X''}{X} - \frac{Z''}{Z} + \frac{2Z'X'}{ZX} \right), \]  

(41)

\[ C_{3434} = \frac{Y^2 Z^2}{6c^2} \left( 2 \left( \frac{X'}{X} - \frac{c}{c} \frac{X'}{X} \right) - \left( \frac{Y'}{Y} - \frac{c}{c} \frac{Y'}{Y} \right) \left( \frac{X'}{X} - \frac{c}{c} \frac{X'}{X} \right) - \frac{Y''}{Y} - \frac{Z''}{Z} + \frac{2Z'Y'}{YZ} \right), \]  

(42)

where \( X' := \dot{X} \).

The non-zero components of the electric part of the Weyl tensor are:

\[ E_{22} = \frac{X^2}{6c^2} \left( -2 \frac{X''}{X} + \frac{X'}{X} \left( \frac{Y'}{Y} + \frac{Z'}{Z} \right) + \left( \frac{Y''}{Y} + \frac{Z''}{Z} \right) - \frac{Y'}{Y} \left( \frac{Y'}{Y} + \frac{Z'}{Z} \right) \right), \]  

(43)

\[ E_{33} = \frac{Y^2}{6c^2} \left( -2 \frac{Y''}{Y} + \frac{Y'}{Y} \left( \frac{X'}{X} + \frac{Z'}{Z} \right) + \left( \frac{X''}{X} + \frac{Z''}{Z} \right) - \frac{X'}{X} \left( \frac{X'}{X} + \frac{Z'}{Z} \right) \right), \]  

(44)

\[ E_{44} = \frac{Z^2}{6c^2} \left( -2 \frac{Z''}{Z} + \frac{Z'}{Z} \left( \frac{X'}{X} + \frac{Y'}{Y} \right) + \left( \frac{X''}{X} + \frac{Y''}{Y} \right) - \frac{X'}{X} \left( \frac{X'}{X} + \frac{Y'}{Y} \right) \right), \]  

(45)

The magnetic part of the Weyl tensor vanish

\[ H_{ij} = 0. \]  

(46)

The Weyl scalar is

\[ I_3 = C^{abcd} C_{abcd}, \]  

(47)
\[ I_3 = \frac{4}{3c^2} \left[ \left( \frac{X''}{X} \right)^2 - 2 \frac{X'' Y}{X^2 c} - 2 \frac{Y''}{Y} \frac{Y'}{c} - 2 \frac{Z'' Z'}{Z^2 Z c} + 2 \frac{Y'' X'}{X Z} + 2 \frac{Z'' X'}{Z X Y} \right] + \frac{2}{X^2 Y Z} \left( \frac{X' Y'}{X Y} + \frac{(c')^2}{c} \left( \left( \frac{X'}{X} \right)^2 + \left( \frac{Y'}{Y} \right)^2 + \left( \frac{Z'}{Z} \right)^2 \right) \right) \]

\[ + \frac{2}{X^2 Y Z} \left( \frac{X' Y'}{X Y} + \frac{(c')^2}{c} \left( \left( \frac{X'}{X} \right)^2 + \left( \frac{Y'}{Y} \right)^2 + \left( \frac{Z'}{Z} \right)^2 \right) \right) \]

Note that
\[ I_3 = I_1 - 2I_2 + \frac{1}{3} R^2, \]

(49)

This definition is only valid when \( n = 4 \).

The gravitational entropy is defined as follows (see [50]-[51]):
\[ p^2 = \frac{I_3}{I_2} = \frac{I_1 - 2I_2 - \frac{4}{3} R^2}{I_2} = \frac{I_1}{I_2} + \frac{1}{3} R^2 - 2. \]

(50)

\[ \text{IV. LIE METHOD.} \]

Therefore we are interested in studying through the Lie method eq. (25) i.e.
\[ \dot{\rho} = K_1 \frac{\dot{\rho}^2}{\rho} + K_2 \frac{G \rho^2}{c^2} - K_3 c^2 \rho + \frac{\dot{c}}{c} \rho, \]

(51)

where \( (K_i)_{i=1}^3 \in \mathbb{R} \), are given by eqs. (26), in particular we seek the forms of \( G, c \) and \( \Lambda \), for which our field equations admit symmetries i.e. are integrable (see for example [53]-[57]). We would like to stress that eq. (51) is very similar to the studied one in the context of FRW symmetries (see [40]).

For this purpose, and following the standard procedure we need to solve the following system of PDEs:

\[ K_1 \xi_\rho + \rho \xi_{\rho \rho} = 0, \]

(52)

\[ \eta_{\rho \rho} - 2 \xi_{t \rho} + \frac{K_1}{\rho} (\eta - \rho \eta_\rho) - 2 \frac{c'}{c} \xi_\rho = 0, \]

(53)

\[ \rho^2 \frac{c^2}{c} \left[ 2 \eta_{\rho \rho} - \xi_{t \rho} + 3 \rho \xi_\rho \left( c^2 K_3 \Lambda - \rho K_2 \frac{G}{c^2} \right) - 2 K_1 \frac{\eta}{\rho} + \xi \left( - \frac{c'}{c} + \frac{c^2}{c^2} \right) - \xi \frac{c'}{c} \right] = 0, \]

(54)

\[ \rho^2 \left[ \eta_{t t} c^2 - c \eta c' + K_3 \rho^2 G \left( - \xi \left( \frac{c'}{G} - \frac{2 c'}{c^2} \right) - 2 \frac{\eta}{\rho} - 2 \xi_t + \eta_\rho \right) + K_3 c^4 \left( \Lambda \left( \eta - \rho \eta_\rho \right) + \rho \Lambda \left( 2 \xi_t + \xi \left( \frac{\Lambda}{\Lambda} + 2 \frac{c'}{c} \right) \right) \right) \right] = 0. \]

(55)

Imposing the symmetry \( X = (at + e) \partial_t + b \rho \partial_\rho \), i.e. \( (\xi = at + e, \eta = b \rho) \), we get the following restrictions for \( G(t), c(t) \) and \( \Lambda(t) \). Note that constants \( (a, b, e) \in \mathbb{R}, \) where \( |a| = |b| = 1 \), i.e. they are dimensionless while \( |e| = T \), with respect to the dimensional base \( B = \{ L, M, T \} \), i.e. constant \( e \) has dimensions of time, \( T \).
From eq. (54) we get
\[ \frac{c''}{c'} - \frac{c'}{c} = -\frac{2a}{at + e}. \] (56)

Now, from eq. (55) it is obtained
\[ \frac{G''}{G} - \frac{2c'}{c} = -\frac{2a + b}{at + e}, \] (57)

and
\[ \frac{\Lambda'}{\Lambda} + \frac{2c'}{c} = -\frac{2a}{at + e}. \] (58)

Therefore, for the different values of the constants \((a, b, e)\) we will be able to find different behaviors for the functions \(G(t), c(t)\) and \(\Lambda(t)\), and hence to integrate eq. (51).

A. Scale symmetry.

Making \(e = 0\), i.e. considering only \((\xi = at, \eta = b\rho)\), we obtain the scale symmetry, \(X = at\partial_t + b\rho\partial_{\rho}\), so eqs. (56-58) yield:
\[ \frac{c''}{c'} - \frac{c'}{c} = -\frac{1}{t}, \] (59)
\[ \frac{G''}{G} - \frac{2c'}{c} = -\frac{b + 2a}{at}, \quad \implies \quad \frac{G}{c^2} = Bt^{-(2 + \omega)}, \] (60)
\[ \frac{\Lambda'}{\Lambda} + \frac{2c'}{c} = -\frac{2}{t}, \quad \implies \quad \Lambda c^2 = \tilde{B}t^{-2}, \] (61)

where \(B, \tilde{B} \in \mathbb{R}\), therefore we get
\[ c = c_0t^{c_1}, \quad c_1, c_0 \in \mathbb{R}, \] (62)
\[ G = G_0t^{2(c_1 - 1) - \frac{\omega}{2}}, \quad G_0 \in \mathbb{R}^+, \] (63)
\[ \Lambda = \Lambda_0t^{-2(c_1 + 1)}, \quad \Lambda_0 \in \mathbb{R}, \] (64)

where we assume that \(G_0 > 0\).

The invariant solution for the energy density is:
\[ \frac{bd}{at} = \frac{dp}{\rho} \quad \implies \quad \rho = \rho_0t^{b/a}, \] (65)

and for physical reasons we impose the condition, \(ab < 0\) then \(b < 0\). We have consider the invariant solution since, as we already known, the most general solution for eq. (51) with the constrains given by eqs. (62-64) usually is an unphysical solution (i.e. it lacks of physical meaning, see ([57])). Furthermore, as we will show in the next section this kind of spacetime is self-similar which means that all the quantities follow a power law, as in this case with the invariant solution, see ([65]-[64]).

If we make that this solution verifies eq. (51) with \(c(t), G(t)\) and \(\Lambda(t)\) given by eqs. (62-64), we find the value of constant \(\rho_0\), so
\[ \rho_0 = -\left[ \frac{c_0^2 \left(b^2 + ab(1 + \omega)(c_1 + 1) - 3c_0^2\Lambda_0a^2(1 + \omega)^2\right)}{12\pi a^2G_0(1 + \omega)^2(\omega - 1)} \right], \] (66)

with the only restriction \(\omega \neq -1, 1\). Note that \(ab < 0\), so we need to choice constants \((c_0, G_0, \Lambda_0)\) such that \(\rho_0 > 0\).

Remark 1 As we can see, it is verified the relationship \(\frac{Gc'}{c^2} \approx t^{-2}\), as well as \(\Lambda c^2 \approx t^{-2}\).
Therefore, at this time we have the following behavior for $G(t)$

$$G(t) = G_0 t^{2(c_1 - 1 + \frac{5}{2})}, \quad G \approx \begin{cases} 
\text{decreasing if } c_1 < 1 + b/2a, \\
\text{constant if } c_1 = 1 + b/2a, \\
\text{growing if } c_1 > 1 + b/2a,
\end{cases}$$  \quad (67)$$

while $\Lambda$ behaves as follows:

$$\Lambda = \Lambda_0 t^{-2(c_1 + 1)}, \quad \Lambda \approx \begin{cases} 
\text{decreasing if } c_1 > -1, \\
\text{constant if } c_1 = -1, \\
\text{growing if } c_1 < -1,
\end{cases}$$  \quad (68)$$

therefore $(c_1 + 1) > 0 \implies c_1 \in (-1, \infty)$. But we have not any information about the sign of $\Lambda_0$.

With regard to $H = 3K^3$ we find that

$$R = R_0 \rho^{-1/3(1+\omega)} = R_0 t^{-b/3a(1+\omega)}, \quad \implies \quad XYZ = R_0 t^{-b/a(1+\omega)},$$  \quad (69)$$

and assuming that the functions $(X, Y, Z)$ follow a power law (i.e. $X = X_0 t^{\alpha_1}$) then we get the following result

$$Kt^\alpha = R_0 t^{-b/a(1+\omega)},$$

and therefore, $\sum_i \alpha_i = \alpha = -\left(b/a (1 + \omega)\right)$, where we may assume that $(\alpha_i) > 0, \forall i$ and $(\alpha_i \neq \alpha_j)$ although $(\alpha_i \to \alpha_j)$ when $t \to \infty$, with $i \neq j$, i.e. we expect that the model isotropize and collapses to a FRW model, but we have not more information about this behavior.

The shear is calculated as follows, from eq. (31) we get:

$$\sigma^2 = \sigma_0^2 t^{-2(c_1 + 1)}, \quad \text{with} \quad \sigma_0^2 = \frac{b^2 + 2ab(c_1 + 1) - 3a^2 \Lambda_0 \sigma_0^2 (1 + \omega)^2}{3a^2 (\omega^2 - 1)},$$  \quad (70)$$

but in this case it is quite difficult to know if $\sigma_0^2 \neq 0$ or $\sigma_0^2 = 0$. Therefore, at this time, we cannot to rule out this solution as in the case of $G$ and $\Lambda$ time varying (see ([25])) where $\sigma_0^2 = 0$. The only important obtained restriction is $c_1 \in (-1, \infty)$.

\textbf{B. Exponential behavior.}

Making $a = 0$, we have $(\xi = e, \eta = b\rho)$, so following the same steeps, we have to integrate eqs. (56-58), hence

$$\frac{c'}{c} - \frac{c'}{c} = 0, \quad (71)$$

$$\frac{G'}{G} - 2\frac{c'}{c} = -\frac{b}{e}, \quad \implies \quad \frac{G}{c^2} = \exp\left(-\frac{b}{e} t\right), \quad (72)$$

$$\frac{\Lambda'}{\Lambda} + 2\frac{c'}{c} = 0, \quad \implies \quad \Lambda c^2 = \text{const}, \quad (73)$$

therefore we get

$$c = c_0 \exp(c_1 t), \quad c_0, c_1 \in \mathbb{R}^+, \quad (74)$$

$$G = G_0 \exp\left[2c_1 - \frac{b}{e}\right] t, \quad G_0 \in \mathbb{R}^+, \quad (75)$$

$$\Lambda = \Lambda_0 \exp(-2c_1 t), \quad \Lambda_0 \in \mathbb{R}, \quad (76)$$

where we assume that $c_0, G_0 > 0$. From eq. (76) we find that $c_1 > 0$, otherwise $\Lambda$ will be a growing function on time.

The invariant solution for the energy density is:

$$\rho = \rho_0 \exp\left(\frac{b}{e} t\right), \quad (77)$$

with the restriction, $eb < 0$ with $b < 0$, from physical considerations. In order to calculate the value of constant $\rho_0$, this solution must verifies eq. (51) with $c(t), G(t)$ and $\Lambda(t)$ given by eqs. (74-76), finding in this way that constant $\rho_0$ yields,

$$\rho_0 = -\left[\frac{c_0^2 \left( b^2 + eb (1 + \omega) c_1 - 3c_0^2 \Lambda_0 e^2 (1 + \omega)^2 \right)}{12\pi e^2 G_0 (1 + \omega)^2 (\omega - 1)}\right]. \quad (78)$$
With regard to $H$ we find that
\[ R = R_0 \rho^{-1/3(1+\omega)} = R_0 \exp \left( -\frac{b}{3e(\omega + 1)^t} \right), \]  
(79)

The shear is calculated as follows.
\[ \sigma^2 = \sigma_0^2 \exp(-2c_1t), \quad \text{with} \quad \sigma_0^2 = \frac{b^2 + 2ebc_1 - 3e^2\Lambda_0c_0^2 (1 + \omega)^2}{3e^2(\omega^2 - 1)}. \]
(80)

**Remark 2** As we can see, it is impossible to have any information about the real behavior of the quantities since they depend on several integration constants. We suppose that this model is unphysical but we have not any way of rule it out.

### C. Solution with the full symmetry.

In this case we have the full symmetry i.e. $(\xi = at + e, \eta = b\rho)$, with $|e| = T$, we have to integrate eqs. (56-58), so
\[ \frac{c'}{c} - \frac{c'}{c} = -\frac{a}{at + e}, \]
(81)
\[ \frac{G'}{G} - 2\frac{c'}{c} = -\frac{b + 2a}{at + e}, \quad \implies \quad \frac{G}{c^2} = (at + e)^{-\left(\frac{b}{2} + 2\right)}, \]
(82)
\[ \Lambda' + 2\frac{c'}{c} = -\frac{2a}{at + e}, \quad \implies \quad \Lambda c^2 = (at + e)^{-2}, \]
(83)

therefore we get
\[ c = c_0 (at + e)^{c_1/a}, \quad c_1, c_0 \in \mathbb{R}, \]
(84)
\[ G = G_0 (at + e)^{2c_1/2 - 2 - \frac{b}{2}}, \quad G_0 \in \mathbb{R}^+, \]
(85)
\[ \Lambda = \Lambda_0 (at + e)^{-2(1 + c_1/a)}, \quad \Lambda_0 \in \mathbb{R}, \]
(86)

where we assume that $c_0, G_0 > 0$.

The invariant solution for the energy density is
\[ \rho = \rho_0 (at + e)^{b/a}, \]
(87)

where we need to impose the physical constrain such that $ab < 0$ then $b < 0$. In order to find the constant $\rho_0$, we make that solution (87) verifies eq. (51) with $G(t), c(t)$ and $\Lambda(t)$ given by eq. (84-86), finding in this way that the value of the numerical constant $\rho_0$, yields
\[ \rho_0 = -\left[ \frac{c_0^2 (1 + \omega - 3e^2\Lambda_0 (1 + \omega)^2)}{12\pi G_0 (1 + \omega)^2 (\omega - 1)} \right], \]
(88)

we assume that $\omega \neq -1$. As it is observed, this is a nonsingular solution since when $t \to 0$ if $e \neq 0$, then $\rho \neq \infty$. As in the above cases, it is verified the condition $G\rho/c^2 \approx (at + e)^{-2}$, as well as, $\Lambda c^2 \approx (at + e)^{-2}$.

Therefore we have the following behavior for $G(t)$ :
\[ G(t) = G_0 (at + e)^{-(2a+b)/a}, \quad G \approx \begin{cases} \text{decreasing if } c_1 < 1 + b/2a, \\ \text{constant if } c_1 = 1 + b/2a, \\ \text{growing if } c_1 > 1 + b/2a, \end{cases}, \]
(89)

note that if $2a = -b$, then $c_1 = 0$. with $(a, e > 0, b < 0)$. As we can see, it is obtained the same behavior as in the scale symmetry solution.

Lambda behaves as $\Lambda = \Lambda_0 (at + e)^{-2(1 + c_1/a)}$, so $(1 + c_1/a) > 0$, i.e. $|c_1| < |a|$, which means that $c_1 \in (-a, \infty)$, finding a bit difference with respect to the scale symmetry solution.
With regard to the quantity $H$, we find from eq. (15) that
\begin{equation}
R = R_0 \rho^{-1/3(1+\omega)} = R_0 (at + e)^{-b/3a(1+\omega)}, \quad \Rightarrow \quad XYZ = R_0 (at + e)^{-b/a(1+\omega)}, \tag{90}
\end{equation}
so (following the same argument as above) the functions $(X,Y,Z)$ follow a power law (i.e. $X = X_0 (at + e)^{\alpha_1}$, etc...) it is found that, $K(at + e)^{\alpha} = R_0 (at + e)^{-b/a(1+\omega)}$, and hence, $\sum_i \alpha_i = \alpha = -(b/a (1 + \omega))$, where we may “assume” that $(\alpha_i) > 0, \forall i$ and $(\alpha_i \neq \alpha_j)$ although $(\alpha_i \rightarrow \alpha_j)$ when $t \rightarrow \infty$, and $i \neq j$.

The shear has the following behavior.
\begin{equation}
\sigma^2 = \frac{b^2 + 2b(c_1 + a) - 3c_2^2 \Lambda_0 (1 + \omega)^2}{3(\omega^2 - 1)}(at + e)^{-2(1+c_1/a)}. \tag{91}
\end{equation}

At this point it seems that we have found a physical solution that depends on the value of constants $a$ and $b$. But, how to calculate the value of constants $(\alpha_i)_{i=1}^3$? Simply all these results must satisfy the FE, hence
\begin{equation}
\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3 = 8\pi \frac{G_0}{c_0^3} \rho_0 + \Lambda_0 c_0^2, \tag{92}
\end{equation}
\begin{equation}
\alpha_2 (\alpha_2 - 1) + \alpha_3 (\alpha_3 - 1) + \alpha_3 \alpha_2 - c_1 (\alpha_3 + \alpha_2) = -8\pi \frac{G_0}{c_0^3} \omega \rho_0 + \Lambda_0 c_0^2, \tag{93}
\end{equation}
\begin{equation}
\alpha_1 (\alpha_1 - 1) + \alpha_3 (\alpha_3 - 1) + \alpha_3 \alpha_1 - c_1 (\alpha_1 + \alpha_3) = -8\pi \frac{G_0}{c_0^3} \omega \rho_0 + \Lambda_0 c_0^2, \tag{94}
\end{equation}
\begin{equation}
\alpha_2 (\alpha_2 - 1) + \alpha_1 (\alpha_1 - 1) + \alpha_1 \alpha_2 - c_1 (\alpha_1 + \alpha_2) = -8\pi \frac{G_0}{c_0^3} \omega \rho_0 + \Lambda_0 c_0^2, \tag{95}
\end{equation}
which solution is:
\begin{equation}
\alpha_1 = \alpha_2 = \alpha_3 = \sqrt{\frac{8\pi G_0 \rho_0}{3c_0^3} + \frac{\Lambda_0 c_0^2}{3}}, \quad c_1 = -1 + \frac{4\pi G_0 \rho_0 (1 + \omega)}{c_0 \sqrt{\frac{8\pi G_0 \rho_0}{3c_0^3} + \frac{\Lambda_0 c_0^2}{3}}}, \tag{96}
\end{equation}
finding that this kind of solutions lack of any interest, it is the flat FRW one. This solution was obtained by Einstein\&de Sitter ([59]) in 1932 for $\omega = 0$, and later by Harrison ([60]) $\forall \omega$.

We would like to point out that, at least, this solution is consistent with the already obtained one in ([40]), where we studied a perfect fluid with time varying constants (as here, i.e. taking into account the possible effects of a c-var into the curvature tensor) but in the context of the flat FRW symmetries. We think that the followed methods is too restrictive and for this reason we are only able to get this class of solutions. As we have pointed out above eq. (51) is quite similar to the FRW case studied in ([40]) and for this reason with this approach we are only able to obtain FRW-like solution.

There are others Lie group approaches, as for example, the developed by M. Szydowski et al (see [58]), maybe if we follow this approach we would be able to get other class of solutions as it is expected studying this kind of spacetimes i.e. to get, for example, a Kasner-like solution.

V. SELF-SIMILAR SOLUTION.

In general relativity, the term self-similarity can be used in two ways. One is for the properties of spacetimes, the other is for the properties of matter fields. These are not equivalent in general. The self-similarity in general relativity was defined for the first time by Cahill and Taub (see [61], and for general reviews [62]-[68]). Self-similarity is defined by the existence of a homothetic vector $V$ in the spacetime, which satisfies
\begin{equation}
\mathcal{L}_V g_{ij} = 2 \alpha g_{ij}, \tag{97}
\end{equation}
where $g_{ij}$ is the metric tensor, $\mathcal{L}_V$ denotes Lie differentiation along $V$ and $\alpha$ is a constant. This is a special type of conformal Killing vectors. This self-similarity is called homothety. if $\alpha \neq 0$, then it can be set to be unity by a constant rescaling of $V$. If $\alpha = 0, i.e. \mathcal{L}_V g_{ij} = 0$, then $V$ is a Killing vector.

Homothety is a purely geometric property of spacetime so that the physical quantity does not necessarily exhibit self-similarity such as $\mathcal{L}_V Z = dZ$, where $d$ is a constant and $Z$ is, for example, the pressure, the energy density and so on. From equation (97) it follows that
\begin{equation}
\mathcal{L}_V R^i_{\ jkl} = 0, \tag{98}
\end{equation}
and hence
\[ \mathcal{L}_V R_{ij} = 0, \quad \mathcal{L}_V G_{ij} = 0. \] (99)

A vector field \( V \) that satisfies the above equations is called a curvature collineation, a Ricci collineation and a matter collineation, respectively. It is noted that such equations do not necessarily mean that \( V \) is a homothetic vector. We consider the Einstein equations
\[ G_{ij} = 8\pi G T_{ij}, \] (100)
where \( T_{ij} \) is the energy-momentum tensor.

If the spacetime is homothetic, the energy-momentum tensor of the matter fields must satisfy
\[ \mathcal{L}_V T_{ij} = 0, \] (101)
through equations (100) and (99). For a perfect fluid case, the energy-momentum tensor takes the form of eq. (3) i.e. \( T_{ij} = (p + \rho)u_i u_j + pg_{ij}, \) where \( p \) and \( \rho \) are the pressure and the energy density, respectively. Then, equations (97) and (101) result in
\[ \mathcal{L}_V u^i = -\alpha u^i, \quad \mathcal{L}_V \rho = -2\alpha \rho, \quad \mathcal{L}_V p = -2\alpha p. \] (102)

As shown above, for a perfect fluid, the self-similarity of the spacetime and that of the physical quantity coincide. However, this fact does not necessarily hold for more general matter fields. Thus the self-similar variables can be determined from dimensional considerations in the case of homothety. Therefore, we can conclude homothety as the general relativistic analogue of complete similarity.

From the constraints (102), we can show that if we consider the barotropic equation of state, i.e., \( p = \omega \rho \), then the equation of state must have the form \( p = \omega \rho \), where \( \omega \) is a constant. This class of equations of state contains a stiff fluid \( (\omega = 1) \) as special cases, whiting this theoretical framework. There are many papers devoted to study Bianchi I models (in different context) assuming the hypothesis of self-similarity (see for example [69]-[70]) but here, we would like to try to show how taking into account this class of hypothesis one is able to find exact solutions to the field equations within the framework of the time varying constants.

The homothetic equations are: \( L_V g = 2g \), finding that the homothetic vector field is in this case:
\[ X = \left( \int \frac{cdt}{c(t)} \right) \partial_t + \left( 1 - \frac{\int cX}{c(t)} \right) x \partial_x + \left( 1 - \frac{\int cY}{c(t)} \right) y \partial_y + \left( 1 - \frac{\int cZ}{c(t)} \right) z \partial_z, \] (103)
iff the following ODE is satisfied
\[ \left( X\dot{X} - X\dot{X}^c \right) \int c dt + cX\ddot{X} \int c dt - \left( \dot{X} \right)^2 c \int c dt \right) \frac{\dot{x}}{c^2} = 0, \] (104)
and so on with respect to \( (Y,y) \) and \( (Z,z) \).

As it is observed from eq. (104) if we simplify this ODE it is obtained the following one
\[ \frac{H_1'}{H_1} = \frac{c'}{c} - \frac{c}{\int c dt} \implies H_1 = \alpha_1 \frac{c}{\int c dt}, \implies X = X_0 \left( \int c dt \right)^{\alpha_1}, \] (105)
with \( \alpha_1 \in \mathbb{R} \), etc....with regard to the others scale factors. Note that \( ' := \frac{d}{dt} := dot \) i.e. \( X' = \dot{X} \). So, we have
\[ X = X_0 \left( \int c dt \right)^{\alpha_1}, \quad Y = Y_0 \left( \int c dt \right)^{\alpha_2}, \quad Z = Z_0 \left( \int c dt \right)^{\alpha_3}, \] (106)
with \( (\alpha_i)_{i=1}^3 \in \mathbb{R} \), note that at this time we have not any information about the possible values and their signs of the numerical constants \( (\alpha_i)_{i=1}^3 \).

**Remark 3** Note that if \( c = const. \) we regain the usual homothetic vector field. i.e.
\[ X = t\partial_t + \left( 1 - t\dot{X}/X \right) x \partial_x + \left( 1 - t\dot{Y}/Y \right) y \partial_y + \left( 1 - t\dot{Z}/Z \right) z \partial_z, \] (107)
while the scale factors behave as
\[ X = X_0 (t)^{\alpha_1}, \quad Y = Y_0 (t)^{\alpha_2}, \quad Z = Z_0 (t)^{\alpha_3}, \] (108)
as in the case with only \( G \) and \( \Lambda \) variable (see [25]).
Since
\[ H_i = \alpha_i \frac{c}{\int c \, dt} \implies H = a \frac{c}{\int c \, dt}, \quad \alpha = \sum_{i=1}^{3} \alpha_i, \]
finding in this way, from eq. (13), the behavior of the energy density i.e.
\[ \rho = \rho_0 \left( \int c \, dt \right)^{-(1+\omega)\alpha}. \] (109)

In the same way it is easily calculated the shear
\[ \sigma^2 = \frac{1}{3c^2} (H_1^2 + H_2^2 + H_3^2 - H_1H_2 - H_1H_3 - H_2H_3) = \frac{1}{3} \sum_{i=1}^{3} \alpha_i^2 - \sum_{i \neq j} \alpha_i \alpha_j \left( \int c \, dt \right)^{-2}. \] (110)

As it is observed all the quantities depend on \( c(t) \), so only rest to calculate \( G \) and \( \Lambda \).

From eqs. (9, 109 and 110) we get:
\[ A \left( \frac{c}{\int c} \right)^2 = \frac{8\pi G}{c^2} \rho_0 \left( \int c \right)^{-\gamma} + \Lambda c^2, \] (111)
where we have written, for simplicity, \( \int c \) instead of \( \int c \, dt \), and \( A = (\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3), \gamma = (1 + \omega)\alpha \), therefore
\[ A' = -\frac{2Ac}{(\int c)^2} - \frac{8\pi \rho_0 G}{c^4 (\int c)^{\gamma}} \left[ \frac{G'}{G} - 4 \frac{c'}{c} - \gamma \frac{c}{\int c} \right]. \] (112)

Now, taking into account eq. (14), we get that
\[ -\frac{c^4}{8\pi G \rho} \left[ \frac{2Ac}{(\int c)^3} + \frac{8\pi \rho_0 G}{c^4 (\int c)^{\gamma}} \left[ \frac{G'}{G} - 4 \frac{c'}{c} - \gamma \frac{c}{\int c} \right] \right] + \frac{G' G}{G} - 4 \frac{c'}{c} = 0, \] (113)
and hence we obtain
\[ G = \frac{A}{4\pi \rho_0 \gamma} c^4 \left( \int c \right)^{\gamma-2}, \] (114)
and in this way we find that
\[ \Lambda = A \left( 1 - \frac{2}{\gamma} \right) \left( \int c \right)^{-2}. \] (115)

As we can see, from eqs. (115 and 116), we have that are verified the following relationships
\[ \frac{G \rho}{c^4} \approx \left( \int c \right)^{-2}, \quad \Lambda \left( \int c \right)^2 = \text{const.}, \] (116)
in the next subsection (VA), matter collinearations approach, we will see that these relationships are obtained in a trivial way.

Now, we will try to find the value of the constants \( (\alpha_i)_{i=1}^{3} \). Taking into account the field eqs. (9-12) we find that, obviously eq. (9) vanish, but from eqs. (10-12) we get
\[ \alpha_2 (\alpha_2 - 1) + \alpha_3 (\alpha_3 - 1) + \alpha_3 \alpha_2 = \frac{A}{\alpha} (\alpha - 2), \] (117)
\[ \alpha_1 (\alpha_1 - 1) + \alpha_3 (\alpha_3 - 1) + \alpha_3 \alpha_1 = \frac{A}{\alpha} (\alpha - 2), \] (118)
\[ \alpha_2 (\alpha_2 - 1) + \alpha_1 (\alpha_1 - 1) + \alpha_1 \alpha_2 = \frac{A}{\alpha} (\alpha - 2), \] (119)
where \( A = (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) \), \( \alpha = (\alpha_1 + \alpha_2 + \alpha_3) \). The system (118-120) has only two solutions

\[
\begin{align*}
\alpha_1 &= 1 - \alpha_2 - \alpha_3, \\
\alpha_2 &= \alpha_2, \\
\alpha_3 &= \alpha_3, \\
\alpha_1 &= \alpha_2 = \alpha_3,
\end{align*}
\]

as we can see, the solution (121) looks with physical meaning while solution (122) is the flat FRW one so in this case we must to rule it out. This solution was obtained by Einstein\&de Sitter ([59]) in 1932 for \( \omega = 0 \), and later by Harrison ([60] \( \forall \omega \). This solution is quite similar to the obtained in the above section with the scale symmetry.

With regard to the solution (121), it is also noted that the solution \( \alpha_1 = 1 - \alpha_2 - \alpha_3 \), brings us to get \( A = (\alpha_1\alpha_2 + \alpha_3 - \alpha_2\alpha_3) = \alpha_2 + \alpha_3 - \alpha_2 - \alpha_2\alpha_3 > 0 \), \( \forall \alpha_2, \alpha_3 \in (0, 1) \). Note that the solution (121) verifies the relationship

\[
\sum \alpha_i = 1, \quad \sum \alpha_i^2 < 1,
\]

this is the same class of solutions that we got in our previous paper ([25]) where we studied a Bianchi I model with \( G \) and \( \Lambda \) varying. See ([25]) as well as the end of section VI for a discussion of this class of solutions. Therefore we have found a similar behavior as the obtained one in ([69], we say similar because these authors only study standard models i.e. models where the “constants” are true constants, in fact \( \Lambda = 0 \), except than here this result is valid for all equation of state i.e. \( \forall \omega \). Nevertheless in reference ([70]), the authors claim that the solution must verify both conditions, i.e. \( \sum \alpha_i = 1 = \sum \alpha_i^2 \), (see [73] and [74]).

We would like to stress that in this case, it is essential to take into account the effects of a \( c \)--var into the field equations (as in this case). For example, from eq. (10) we find that:

\[
\alpha_2 \frac{c'}{c} + \alpha_2 (\alpha_2 - 1) \left( \frac{c'}{c} \right)^2 + \alpha_3 \left( \frac{c'}{c} \right)^2 + \alpha_3 (\alpha_3 - 1) \left( \frac{c'}{c} \right)^2 - (\alpha_2 + \alpha_3) \left( \frac{c'}{c} \right)^2 + \alpha_2\alpha_3 \left( \frac{c'}{c} \right)^2 = \frac{A}{\alpha} (\alpha - 2) \left( \frac{c'}{c} \right)^2,
\]

simplifying, we get

\[
[\alpha_2 (\alpha_2 - 1) + \alpha_3 (\alpha_3 - 1) + \alpha_2\alpha_3] \left( \frac{c'}{c} \right)^2 = \frac{A}{\alpha} (\alpha - 2) \left( \frac{c'}{c} \right)^2,
\]

and so on.

But if we take the field equations in the usual way i.e.

\[
\frac{\ddot{Y}}{Y} + \frac{\ddot{Z}}{Z} + \frac{\dot{Z} \dot{Y}}{ZY} = -\frac{8\pi G}{c^2} \omega \rho + \Lambda c^2,
\]

it yields

\[
\alpha_2 \left( \frac{c'}{c} + (\alpha_2 - 1) \left( \frac{c'}{c} \right)^2 \right) + \alpha_3 \left( \frac{c'}{c} + (\alpha_3 - 1) \left( \frac{c'}{c} \right)^2 \right) + \alpha_2\alpha_3 \left( \frac{c'}{c} \right)^2 = \frac{A}{\alpha} (\alpha - 2) \left( \frac{c'}{c} \right)^2,
\]

An important observational quantity is the deceleration parameter \( q = \frac{d}{dt} \left( \frac{1}{H} \right) - 1 \). The sign of the deceleration parameter indicates whether the model inflates or not. The positive sign of \( q \) corresponds to “standard” decelerating models whereas the negative sign indicates inflation. Therefore we have

\[
H = \frac{\alpha}{\int c dt} = \frac{c}{\int c dt}, \quad \text{and} \quad q = \frac{d}{dt} \left( \frac{1}{H} \right) - 1 = -\frac{c'}{c},
\]

note that \( \alpha = \sum \alpha_i = 1 \), furthermore we find that

\[
\rho = \rho_0 \left( \int c dt \right)^{-(1+\omega)}, \quad G = \frac{A}{4\pi \rho_0 (1+\omega)} \left( \int c \right)^{\omega-1}, \quad \Lambda = A \left( 1 - \frac{2}{(1+\omega)} \right) \left( \int c \right)^{-2},
\]
so, depending on the choice of the function \( c(t) \) we will have different behaviors for each quantity but, we may impose some restrictions like the following ones.

For the energy density, \( \rho \), since it must be a decreasing time function for all \( \omega \in (-1, 1) \), we find that this is only possible if \( \left( \int c \, dt \right) \) is a growing time function. Note that if we consider the case \( c(t) = v_0 = \text{const.} \), then we have, \( \left( \int c \, dt = c_0 t \right) \), which is a growing time function. If \( \omega < -1 \) (phantom case), then, \( \rho \), is growing, but choosing a time decreasing \( \int c \, dt \), we may do that \( \rho \) will be a time decreasing function as it is expected.

For \( G \), it is impossible to know beforehand which will be its behavior since depends on \( c \), in the following way, \( c^4 \left( \int c \right)^{-\omega - 1} \), so we only may say that for \( \omega = 1, G \equiv c^4 \), while \( \forall \omega \in (-1, 1), \left( \int c \right)^{-\omega - 1} \) is a decreasing function on time, since we have impose that \( \left( \int c \right) \) must be a growing time function.

For \( \Lambda \), we may say that is a negative decreasing function on time \( \forall \omega \in (-1, 1) \), but for \( \omega = 1, \Lambda = 0 \) i.e. vanish, and for \( \omega < -1 \) (phantom case) \( \Lambda \) is positive as well \( \forall \omega > 1 \) (see for example [72]). Note that \( A > 0 \). Furthermore, if \( c = \text{const.} \) then it is regained all the results obtained in ([25]), as for example the relationships \( G\rho \approx t^{-2} \), and \( \Lambda \approx t^{-2} \). We would like to stress that this result, \( \Lambda < 0 \) is not new in the literature, see for example, T.Padmanabhan and S.M. Chitre, ([71]), nevertheless the recent observations suggest us that \( \Lambda \) must be positive ([17]-[20]), so in order to reconcile our results with the observational data we need to consider that \( \omega \in (-\infty, 1) \cup (1, \infty) \). With these values for the equation of state it is observed that \( G \) is growing if \( \omega \in (1, \infty) \) (see [72]) while \( \rho \) is decreasing.

Since our model is formally self-similar, then ([65]-[64]) have shown, that all the quantities must follow a power law, so, we may assume that for example, \( c \) takes the following form: \( c(t) = c_0 t^\epsilon \), with \( \epsilon \in \mathbb{R} \). Hence, we find from the definition of the Hubble parameter and the deceleration parameter that

\[
H = \frac{\epsilon + 1}{t}, \quad q = -\frac{\epsilon}{\epsilon + 1},
\]

and imposing the condition \( q < 0 \) we find that \( \epsilon \in (0, \infty) \), the special case, \( \epsilon = -1 \), is forbidden, note that \( \int c \, dt = \frac{c_0}{2} t^{\epsilon + 1} > 0 \) and a growing time function, \( \forall \epsilon \in (0, \infty) \). So, from physical considerations we find that

\[
\rho \approx t^{-(1 + \omega)(\epsilon + 1)}, \quad G \approx t^{4\epsilon + (\omega - 1)(\epsilon + 1)}, \quad \Lambda \approx t^{-2(\epsilon + 1)},
\]

with \( \epsilon \in (0, \infty) \). We may also argue that since \( \Lambda = A \left( 1 - \frac{2}{1 + \omega} \right) \left( \int c \right)^{-2} \), must be a decreasing time function, this is only possible iff \( \epsilon \in (-\infty, 1) \), and therefore we find that \( \epsilon \in (0, \infty) \). Therefore, if we take into account these considerations, then we arrive to the conclusion that \( c \) must be a growing time functions, while \( \Lambda \) is a decreasing time function and its sing only depends on the equation of state. With regard to \( G \), we may say that its behavior depends on two parameters \( (\epsilon, \omega) \), so if \( \epsilon \to 0^+ \) i.e. is a small positive number, and \( \omega \in (-\infty, 1) \) then \( G \) is a decreasing time function but if \( \omega \in (1, \infty) \) (see [72]) then is growing while if \( \epsilon \to 1 \) and \( \omega \in (-1, \infty) \) then \( G \) is a growing time function. Other possibilities could be considered playing with different values of \( (\epsilon, \omega) \).

Before ending, we would like to emphasize that, as it is observed, we have choose, \( \int c \, dt = \frac{c_0}{2(\epsilon + 1)} t^{\epsilon + 1} + K \), where \( K \) is an integrating constant. In this case \( K = 0 \), otherwise the resulting vector field is not homothetic, i.e. it is not verified the eq. \( L \gamma g = 2g \). If we fix, \( \epsilon = 0 \), we regain the usual homothetic vector field i.e. eq. (107) ([25]).

With regard to the curvature behavior we find that

\[
I_1 = \frac{4}{(\int c)^2} f(\alpha_2, \alpha_3), \quad I_2 = \frac{4}{(\int c)^4} \left( \alpha_2 + \alpha_3 - \alpha_2^2 - \alpha_3^2 - \alpha_2 \alpha_3 \right)^2 = \frac{4A^2}{(\int c)}
\]

finding that if \( f \) is a growing time function (as we have pointed out above) then we get a singular behavior since \( I_1 \) and \( I_2 \) tend to infinite as \( t \) goes to zero.

The non-zero components of the Weyl tensor are:

\[
C_{1212} = -\frac{1}{3} c^2 \hat{A} \left( \int c \right)^{-2(\alpha_2 + \alpha_3)} , \quad C_{1313} = -\frac{1}{3} c^2 B \left( \int c \right)^{-2(1-\alpha_2)} , \quad C_{1414} = -\frac{1}{3} c^2 D \left( \int c \right)^{-2(1-\alpha_3)} ,
\]

\[
C_{2223} = \frac{1}{3} c^2 D \left( \int c \right)^{-2 \alpha_3} , \quad C_{1313} = \frac{1}{3} c^2 B \left( \int c \right)^{-2 \alpha_2} , \quad C_{1414} = \frac{1}{3} c^2 \hat{A} \left( \int c \right)^{-2(1-\alpha_2 - \alpha_3)} ,
\]

where \( \hat{A} = (\alpha_2 - \alpha_3 + \alpha_2^2 + \alpha_3^2 + 4\alpha_2 \alpha_3) \), \( B = (-\alpha_2 + 2\alpha_3 + \alpha_2^2 - \alpha_3^2 - 2\alpha_2 \alpha_3) \), and \( D = (2\alpha_2 - \alpha_3 - 2\alpha_3^2 + \alpha_2^3 + 2\alpha_2 \alpha_3) \), as above, we find that the Weyl tensor tends to infinity if \( f \) is a growing time function, note that \( \alpha_i > 0 \ \forall i \).
The non-zero components of the electric part are:

\[
E_{22} = -\frac{1}{3} A \left( \int c \right)^{-2(\alpha_2 + \alpha_3)}, \quad E_{22} = -\frac{1}{3} B \left( \int c \right)^{-2(1 - \alpha_2)}, \quad E_{22} = -\frac{1}{3} D \left( \int c \right)^{-2(1 - \alpha_3)},
\]

finding that \( E_{ij} \to \infty \) as \( t \to 0 \). Therefore the Weyl invariant yields

\[
I_3 = \frac{16 f(\alpha_2, \alpha_3)}{3 (f c)^4},
\]

and the gravitational entropy is

\[
P^2 = \frac{I_3}{I_2} = \frac{4}{3} \left( \frac{I_1}{I_2} - \frac{1}{3} \frac{R_2}{I_2} - 2 \right) \neq 0.
\]

So the obtained solution is singular.

### A. Matter collineations.

In recent years, much interest has been shown in the study of matter collineation (MCs) (see for example \([76]-[84]\)).

A vector field along which the Lie derivative of the energy-momentum tensor vanishes is called an MC, i.e.

\[
\mathcal{L}_V T_{ij} = 0,
\]

where \( V^i \) is the symmetry or collineation vector. The MC equations, in component form, can be written as

\[
T_{ij,k} V^k + T_{ik} V^j_{,k} + T_{kj} V^i_{,k} = 0,
\]

where the indices \( i, j, k \) run from 0 to 3. Also, assuming the Einstein field equations, a vector \( V^i \) generates an MC if \( \mathcal{L}_V G_{ij} = 0 \). It is obvious that the symmetries of the metric tensor (isometries) are also symmetries of the Einstein tensor \( G_{ij} \), but this is not necessarily the case for the symmetries of the Ricci tensor (Ricci collineations) which are not, in general, symmetries of the Einstein tensor. If \( V \) is a Killing vector (KV) (or a homothetic vector), then \( \mathcal{L}_V T_{ij} = 0 \), thus every isometry is also an MC but the converse is not true, in general. Notice that collineations can be proper (non-trivial) or improper (trivial). Proper MC is defined to be an MC which is not a KV, or a homothetic vector.

Carot et al (see [77]) and Hall et al. (see [78]) have noticed some important general results about the Lie algebra of MCs.

Let \( M \) be a spacetime manifold. Then, generically, any vector field \( V \) on \( M \) which simultaneously satisfies \( \mathcal{L}_V T_{ab} = 0 \) (\( \Leftrightarrow \mathcal{L}_V G_{ab} = 0 \)) and \( \mathcal{L}_V C_{bcd}^a = 0 \) is a homothetic vector field.

If \( V \) is a Killing vector (KV) (or a homothetic vector), then \( \mathcal{L}_V T_{ab} = 0 \), thus every isometry is also an MC but the converse is not true, in general. Notice that collineations can be proper (non-trivial) or improper (trivial). Proper MC is defined to be an MC which is not a KV, or a homothetic vector.

Since the ST is SS then we already know that the SS vector field is also matter collineation i.e. we would like to explore how such symmetries allow us to obtain relationships between the quantities in such a way that it is not necessary to make any hypothesis to a solution to the field equations. In order to do that we need to modify the usual MC equations since with the usual one we are not able to obtain information about the behavior of \( G, c \) and \( \Lambda \). Therefore, following the same steps as in ref ([25]), will be enough to check the following relationships:

\[
L_{HO} \left( \frac{G(t)}{c^4} T_{ij} \right) = 0.
\]

where \( HO \) is given by eq. (103).
In this case, we get from the resulting equations the following results:

\[
\left( \frac{G'}{G} - \frac{4c'}{c} + \frac{\rho'}{\rho} + \frac{2c}{\int c dt} \right) = 0, \quad \iff \quad \frac{G}{c^4} \rho \approx \left( \int c dt \right)^{-2},
\]
\[
\left( -H_1 + \frac{\int c}{c} \left( H_1 \frac{c'}{c} - H_1' \right) \right) = 0, \quad \iff \quad \iff X = X_0 \left( \int c dt \right)^{\alpha_1},
\]

similar result for \( Y \) i.e. \( Y = Y_0 \left( \int c dt \right)^{\alpha_2} \),

similar result for \( Z \) i.e. \( Z = Z_0 \left( \int c dt \right)^{\alpha_3} \),

\[
\left( \frac{G'}{G} - \frac{4c'}{c} + \frac{\rho'}{\rho} + \frac{2c}{\int c dt} \right) = 0, \quad \iff \quad \frac{G}{c^4} \rho \approx \left( \int c dt \right)^{-2}.
\]

To end, in order to get information about the behavior of \( \Lambda \), we consider the generalized MC eq., so we check again that:

\[
L_{HO} \left( \frac{G(t)}{c^4} T_{ij} - \Lambda(t) g_{ij} \right) = 0.
\]

finding the same result with regard to \( (X, Y, Z) \), i.e. the scale factors as well as for the energy density and the pressure, but the important relationship here is the behavior of \( \Lambda \), where

\[
\left( \frac{G'}{G} - \frac{4c'}{c} + \frac{\rho'}{\rho} + \frac{2c}{\int c dt} \right) = -\frac{\Lambda c^4}{G \rho} \left( \frac{\Lambda'}{X} + 2 \frac{c}{\int c dt} \right),
\]
\[
\left( -H_1 + \frac{\int c}{c} \left( H_1 \frac{c}{c} - H_1' \right) \right) = 0, \quad \iff \quad X = X_0 \left( \int c dt \right)^{\alpha_1},
\]

obtaining in this way

\[
\frac{G}{c^4} \rho \approx \left( \int c dt \right)^{-2}, \quad \text{and} \quad \Lambda \left( \int c dt \right)^2 = \text{const.}
\]

while if we fix \( c = \text{const.} \), (compare these results with the obtained ones in ([25])), then it is regained the usual relationship for the inertia as well for the cosmological constant i.e.

\[
\frac{G}{c^2} \rho \approx t^{-2}, \quad \Lambda c^2 = t^{-2}.
\]

As we have pointed out in the above section, all these result are verified by the SS solution.

**VI. KINEMATIC SELF-SIMILARITY.**

Kinematic self-similarity has been defined in the context of relativistic fluid mechanics as an example of incomplete similarity (see for example [85]-[91]). It should be noted that the introduction of incomplete similarity to general relativity is not unique.

A spacetime is said to be kinematic self-similar if it admits a kinematic self-similar vector \( V \) which satisfies the conditions

\[
\mathcal{L}_V h_{ij} = 2\delta h_{ij},
\]
\[
\mathcal{L}_V u_i = \alpha u_i,
\]

where \( u^i \) is the four-velocity of the fluid and \( h_{ij} = g_{ij} + u_i u_j \) is the projection tensor, and \( \alpha \) and \( \delta \) are constants.

If \( \delta \neq 0 \), the similarity transformation is characterized by the scale-independent ratio \( \alpha/\delta \), which is referred to as the similarity index. If the ratio is unity, \( V \) turns out to be a homothetic vector. In the context of kinematic self-similarity, homothety is referred to as self-similarity of the first kind. If \( \alpha = 0 \) and \( \delta \neq 0 \), it is referred to as self-similarity of the zeroth kind. If the ratio is not equal to zero or one, it is referred to as self-similarity of the second kind.
kind. If $\alpha \neq 0$ and $\delta = 0$, it is referred to as self-similarity of the infinite kind. If $\delta = \alpha = 0$, $V$ turns out to be a Killing vector.

From the Einstein equation (100), we can derive

$$\mathcal{L}_V G_{ij} = 8\pi G \mathcal{L}_V T_{ij},$$

(154)

this equation is called the integrability condition.

When a perfect fluid is irrotational, i.e., $\omega_{ij} = 0$, the Einstein equations and the integrability conditions (154) give

$$(\alpha - \delta) R_{ij} = 0,$$

(155)

where $R_{ij}$ is the Ricci tensor on the hypersurface orthogonal to $u^i$. This means that if a solution is kinematic self-similar but not homothetic and if the fluid is irrotational, then the hypersurface orthogonal to fluid flow is flat.

From the physical point of view the detailed study of cosmological models admitting KSS shows that they can represent asymptotic states of more general models or, under certain conditions, they are asymptotic to an exact homothetic solution [87, 90].

Therefore and following the same idea as in the above sections we would like to extend this hypothesis in order to find exact solutions to cosmological models with time varying constant.

Kinematic self-similarity is characterized by the equations (152-153), so in this way it is found that the vector field $V := KSS$ is:

$$V = \left( -\alpha \int \frac{cdt}{c} \right) \partial_t + f_1 x \partial_x + f_2 y \partial_y + f_3 z \partial_z,$$

(156)

where

$$f_1 = \left( \delta + \left( \alpha \int \frac{cdt}{c} \right) \frac{\dot{X}}{X} \right), \quad f_2 = \left( \delta + \left( \alpha \int \frac{cdt}{c} \right) \frac{\dot{Y}}{Y} \right), \quad f_3 = \left( \delta + \left( \alpha \int \frac{cdt}{c} \right) \frac{\dot{Z}}{Z} \right),$$

(157)

As in the case of the homothetic vector field in this case it is necessary to satisfy the following ODE

$$\left( \alpha \int \frac{cdt}{c} \right) H'_1 = -\alpha H_1 \left( 1 - \frac{c'}{c} \int \frac{cdt}{c} \right), \quad \Rightarrow \quad H_1 = a_1 \left( \int \frac{cdt}{c} \right)^{-1},$$

(158)

arriving to the same conclusion as in the SS solution i.e.

$$H_1 = \frac{X'_1}{X_1} \quad \Rightarrow \quad X_1 = X_0 \left( \int \frac{cdt}{c} \right)^{a_1}.$$

(159)

In this way and following the same procedure as in the above section, we find that

$$H = a \left( \frac{c}{\int c} \right), \quad \text{with} \quad a = \sum_{i=1}^3 a_i,$$

(160)

and therefore

$$\rho = \rho_0 \left( \int \frac{cdt}{c} \right)^{-a(\omega+1)},$$

(161)

as it is observed if we choose the case $c = \text{const.}$ then we regain the usual results as it is expected.

The shear behaves as:

$$\sigma^2 = \frac{1}{3} \left( \sum_{i=1}^3 \alpha_i^2 - \sum_{i \neq j} \alpha_i \alpha_j \right) \left( \frac{1}{\int c dt} \right)^2,$$

(162)

From the field eq. (9) we get

$$\psi = \frac{8\pi G}{c^4} \rho + \Lambda, \quad \text{with} \quad \psi = A \left( \frac{1}{\int c} \right)^2,$$

(163)
where \( A = (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) \), and \( \gamma = a (\omega + 1) \), and \( a \) is given by eq. (160), so, from eq. (163) we get

\[
\Lambda' = \psi' - 8\pi G c^4 \rho \left[ \frac{G'}{G} + \frac{\rho'}{\rho} - 4\frac{c'}{c} \right].
\] (164)

Now, taking into account eq. (14), we get that

\[
\Lambda' = \psi' - 8\pi G c^4 \rho \left[ \frac{\rho'}{\rho} - \frac{\Lambda' c^4}{8\pi G \rho} \right], \quad \implies \quad G = \frac{\psi' c^4}{8\pi \rho'},
\] (165)

simplifying it, we obtain

\[
G = \frac{2A}{8\pi \gamma \rho_0} \left( \int c \right)^{2-\gamma},
\] (166)

note that if \( c = \text{const.} \) then we get

\[
G = \frac{2A}{8\pi \gamma \rho_0} \left( \frac{t}{c} \right)^{2-\gamma}.
\] (167)

In this way we find from eq. (163) that

\[
\Lambda = \frac{A (\gamma - 2)}{\gamma} \left( \int c \right)^{-2},
\] (168)

regaining the usual expression when \( c = \text{const.} \).

\[
\Lambda c^2 = \frac{A (\gamma - 2)}{\gamma} \frac{1}{t^2}.
\] (169)

In this way we find that

\[
\alpha_2 (\alpha_2 - 1) + \alpha_3 (\alpha_3 - 1) + \alpha_3 \alpha_2 = \frac{A}{\alpha} (\alpha - 2),
\] (170)

\[
\alpha_1 (\alpha_1 - 1) + \alpha_3 (\alpha_3 - 1) + \alpha_3 \alpha_1 = \frac{A}{\alpha} (\alpha - 2),
\] (171)

\[
\alpha_2 (\alpha_2 - 1) + \alpha_1 (\alpha_1 - 1) + \alpha_1 \alpha_2 = \frac{A}{\alpha} (\alpha - 2),
\] (172)

finding that this is the same system of equations that we had in the SS solution and therefore we get the same set of solutions i.e. the system (170-172) has only two solutions

\[
\alpha_1 = 1 - \alpha_2 - \alpha_3, \quad \alpha_2 = \alpha_2, \quad \alpha_3 = \alpha_3, \quad \text{and}
\] (173)

\[
\alpha_1 = \alpha_2 = \alpha_3.
\] (174)

Hence we arrive to the same conclusions as in the SS solution, i.e., solution (173) looks with physical meaning while solution (174) is the flat FRW one, so in this case we must to rule it out. Therefore we arrive to the following result

\[
\sum \alpha_i = 1, \quad \sum \alpha_i^2 < 1.
\] (175)

Before ending, we would like to emphasize that, in this case, we may choose, \( \int c dt = \frac{\alpha}{\epsilon + 1} t^{\epsilon + 1} + K \), where \( K \) is an integrating constant, \( K \neq 0 \), in such a way that the resulting solution is non-singular, and it is quite similar to the obtained one in the case of the full symmetry obtained in section (IV C). If we fix, \( \epsilon = 0 \), we regain the usual kinematical self-similar vector field

\[
KSS = -(\alpha t + \beta) \partial_t + f_1 x \partial_x + f_2 y \partial_y + f_3 z \partial_z,
\] (176)

where

\[
f_1 = \left( \delta + (\alpha t + \beta) \frac{\dot{X}}{X} \right), \quad f_2 = \left( \delta + (\alpha t + \beta) \frac{\dot{Y}}{Y} \right), \quad f_3 = \left( \delta + (\alpha t + \beta) \frac{\dot{Z}}{Z} \right),
\] (177)
Note that the solution (121) does not verify the relationship, \( \sum \alpha_i^2 = 1 \), i.e. it is Kasner’s type (see [73], [74] and in particular [70]). But, if for example we suppose that solution (121) verifies the conditions

\[
\sum \alpha_i = 1 = \sum \alpha_i^2
\]  

(178)

this means that

\[
\alpha_1 + \alpha_2 + \alpha_3 = 1, \quad (-\alpha_2 - \alpha_3 + \alpha_2^2 + \alpha_3^2 + \alpha_2 \alpha_3) = 0,
\]  

(179)

and therefore

\[
\alpha_1 = \frac{1}{2} \left( 1 - \alpha_3 - \sqrt{1 + 2\alpha_3 - 3\alpha_3^2} \right), \quad \alpha_2 = \frac{1}{2} \left( 1 - \alpha_3 + \sqrt{2\alpha_3 - 3\alpha_3^2 + 1} \right), \quad \alpha_3 = \alpha_3,
\]  

(180)

which is not a physical solution since not all the \( (\alpha_i) \) \in (0,1), for example \( \alpha_1 \in (-1,0) \). Furthermore \( A = (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) = \alpha_2 + \alpha_3 - \alpha_2^2 - \alpha_3^2 - \alpha_2\alpha_3 \), this means from eq. (179) that \( A \) is equal nought i.e. \( (A = 0) \). Therefore

\[
\rho = \rho_0 \left( \int c dt \right)^{(-\omega+1)}, \quad G = \frac{2A}{8\pi^2\rho_0 \left( \int c \right)^2} = 0, \quad \Lambda = A \left( \frac{\gamma - 2}{\gamma} \right) \left( \int c \right)^{-2} = 0,
\]  

(181)

as it is expected for a vacuum solution. See for example A. Harvey [75] for a review of Bianchi I solutions (Kasner-like solutions)

Furthermore, as we shown in ([25]) this class of solution has pathological curvature behavior since if \( I_2 = 0 \), then the gravitational entropy is infinite i.e. \( P^2 = \infty \).

VII. CONCLUSIONS.

We have shown how to attack a perfect fluid Bianchi I with time varying constants under the condition \( \text{div} \ T = 0 \), and taking into account the effects of a \textit{c – var} into the curvature tensor i.e. modifying the usual FE.

With the first of the exposed tactics, i.e. the Lie group one, we have solved the field equations, solving only one ODE, eq. (27), studying the possible forms that take \( G(t) \), \( c(t) \) and \( \Lambda(t) \), in order to make eq. (25) integrable. We have started imposing a particular symmetry, \( X = (at + e) \partial_t + b \partial \rho \), which as we already know brings us to get power law solutions. To study all the possible symmetries would result a very tedious work.

In this way we have obtained three exact solutions in function of the behavior of \( G(t) \), \( c(t) \) and \( \Lambda(t) \). In this case we have not been able to rule out neither of them as in our previous work [25], where some of them had, \( \sigma = 0 \), i.e. the shear vanish, and therefore we have rejected this solution since we are only interested in solutions that verify the condition \( \sigma \neq 0 \). Here the situation is a bit complicated since all the solutions depend of many integrating constants so it is really difficult to rule out some of them as well as to determine their behavior. Nevertheless, when we calculate the numerical values of the exponents of the scale factors \( (\alpha_i) \), we have shown that the only possible solution is the flat FRW one, but, at this time, with \( G, c \) and \( \Lambda \) time varying. This has been a really surprising result, since we think that the followed tactic, i.e. solving eq. (25) without imposing any assumption ad hoc, brings us to get consistent results in the framework of Bianchi I models i.e. a solution with \( \sigma \neq 0 \). In this way we have arrived to the same solutions as the obtained ones in [40], as well as to the same scenario as in our previous paper [25]. We think that the followed tactic is too restrictive, for this reason we are only able to obtain this class of solutions. Nevertheless there are other Lie approaches as the followed by M. Szydowski et al [see [58]] which we think that may be more useful than the followed one here. As we will show in appendix A, if we try to improve the obtained solutions through the study of a third order ODE through the LM, we arrive to the same solutions and hence to the same conclusions. Therefore, since there are many constrains, then we are introducing several integrating constants which add uncertain to the obtained solutions and hence we are not able to improve the obtained solution integrating the second order ODE, we only obtain the same order of magnitude in each quantity, that’s all. As we have mentioned above, in appendix B we will study a third order ODE which has been obtained without the assumption of \textit{c – var} affecting to the curvature tensor. We arrive to the same solutions as the obtained ones in appendix A, and therefore we conclude that at least in order of magnitude, there is no difference between both approaches.

At the same time we have shown that it is not necessary to make any ad hoc assumption or to take into account any previous hypothesis or considering any hypothetical behavior for any quantity since all these hypotheses could be
deduced from the symmetry principles, as for example using the Lie group methods or studying the model from the point of view of the geometrical symmetries i.e. SS etc...

With regard to the others employed tactics to study the field equations, i.e. SS, MC and KSS, we have shown that both tactics are quite similar and that they bring us to get really similar results, actually as we already know, with the SS and the MC we get the same results.

We have shown that the solution obtained with the SS hypothesis is also quite similar to the obtained one using the Lie method under the scale symmetry. In fact we have got two solutions, the flat FRW one and a Kasner-like solution.

Since in this case, all the obtained solutions, for each quantity, depend on \( \int c(t)dt \), it is difficult to determine the behavior of each quantity. Nevertheless, we are able to arrive to some conclusions under the hypothesis \( q < 0 \), where \( q \) stands for the deceleration parameter which are that \( c \) must be a growing time function while \( \Lambda \) is a decreasing time function and whose sign depends on the equation of state, finding that we only get a positive cosmological constant if \( \omega \in (-\infty, -1) \cup (1, \infty) \). With regard to \( G \), we may say that its behavior depends on two parameters \( (\epsilon, \omega) \), so \( G \) may be a decreasing time function as well as a growing time function depending on the values of \( (\epsilon, \omega) \). In the same way as in \cite{25} we conclude that the exponents of the scale factor must satisfy the conditions \( \sum_{i=1}^{3} \alpha_i = 1 \) and \( \sum_{i=1}^{3} \alpha_i^2 < 1 \), \( \forall \omega \), i.e. valid for all value of the equation of state, relaxing in this way the Kasner conditions.

We furthermore have pointed out, as it is well known, that if the ST is SS then there is a vector field, \( V \in \mathcal{X}(M) \) that satisfies the equation \( \mathcal{L}_V g = 2g \), then such vector field must satisfy the equation \( \mathcal{L}_V T = 0 \), i.e. a homothetic vector field is also a MC vector field. In this occasion we only check that the homothetic vector field verifies the reformulated MC equations (see \cite{25} for details) in order to get information on the behavior of \( G, c \) and \( \Lambda \), arriving to the same conclusions as in the SS section. Therefore we have shown that this tactic would be very useful in the study of more complicated models as for example the viscous ones.

With regard to the KSS solution, we have shown that it behaves like the SS one, except that in this case, we obtain a non-singular behavior. We also have show, that if one gets Kasner-like solutions i.e. they are verified the conditions \( \sum \alpha_i = 1 \), and \( \sum \alpha_i^2 = 1 \), then this class of solutions bring us to get vanishing quantities i.e. \( G = \Lambda = 0 \), as well as of obtaining a pathological curvature behavior since the model is Ricci flat which means that \( I_2 = 0 \), so the gravitational entropy is infinite.

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APPENDIX A: STUDY OF EQ. (27).

In section IV, we have studied a second order ODE and as we have been able to see, since all the quantities depend of many integrating constants, all the obtained solutions are very imprecise, i.e. they do not allow us to know which is the real behavior of each quantity and therefore it is impossible to rule them out, as in our previous paper [25]. Since in [25] was very useful to study the third order ODE in this appendix we would like to improve the obtained solutions in section IV. But, unfortunately, as we will show, in this case we are not able to improve such solutions obtaining only the same order of magnitude for each quantity. Nevertheless, this study will allow us to show that there is no difference between to study the resulting FE with c–var affecting to the curvature tensors and the usual FE with c–var non-affecting to the curvature tensors.

Therefore, in this section we will study eq. (27) through the Lie group method. In particular we seek the forms of $G(t), c(t)$ and $\Lambda(t)$ for which our field equations admit symmetries i.e. they are integrable. Note that this ODE has been obtained under the assumption that c affects to the curvature tensor i.e. c–var introduce some modifications into the curvature tensor. So the equation under study is

$$\ddot{\rho} = K_1\rho \frac{\dot{\rho}}{\rho} - K_2 \frac{\dot{\rho}^3}{\rho^2} + \frac{G\rho^2}{c^2} \left[ K_3 \frac{G'}{G} - K_4 \frac{\dot{\rho}}{\rho} - K_5 \frac{\dot{c}^2}{c} \right] - K_6 \rho c \dot{\rho} \Delta + \rho \left( \frac{c''}{c} - \frac{c'^2}{c^2} \right) + \frac{c'}{c} \left( \dot{\rho} - \frac{\rho^2}{\rho} \right), \quad (A1)$$

where, $(K_i)_{i=1}^6$ are given by eqs. (28).

Following the standard procedure we need to solve the next system of PDEs:

$$c^4 \rho^3 \xi_{\rho} = 0, \quad (A2)$$
$$c^4 \rho^3 \xi_{\rho\rho} = 0, \quad (A3)$$
$$K_1 c^4 \rho \eta_{t} - K_1 c^4 \rho^2 \eta_{\rho} - 3c^3 \rho^3 \xi_{\rho} c^2 - 9c^5 \rho^3 \xi_{\eta_{t} \rho} + 3c^4 \rho^3 \eta_{\rho \rho} = 0, \quad (A4)$$
$$-K_1 c^4 \rho^2 \eta_{t} + 3c^4 \rho^3 \xi_{\rho} \left( \frac{c''}{c} + \frac{c^2}{c^2} \right) - 3c^4 \rho^3 \eta_{\rho} - 3c^4 \rho^3 \xi_{\eta_{t} \eta_{t} \rho} = 0, \quad (A5)$$
$$2K_2 c^4 \rho \eta_{t} - K_1 c^4 \rho^2 \eta_{\rho \rho} + 2c^3 \rho^2 \xi_{\rho} c^2 + 2K_1 c^4 \rho^2 \xi_{\eta_{t} \rho} - 2K_2 c^4 \rho - 3c^1 \rho^3 \xi_{\eta_{t} \eta_{t} \rho} + c^3 \rho^3 \xi_{\rho \rho} c^3 = 0, \quad (A6)$$
$$-3c^4 \rho^3 \eta_{t \rho} + 3c^4 \rho^3 \eta_{\rho \rho \rho} + \rho^2 c^4 \left( \xi \left( \frac{c''}{c} - \frac{c'^2}{c^2} \right) + \frac{c'}{c} \left( \eta_{t} + \xi_{\eta_{t} \eta_{t} \rho} \right) \right) + 3K_2 c^4 \rho \eta_{t} +$$
$$+ 3K_4 c^2 \rho G \xi_{\rho} - 3c^3 \rho^3 \xi_{\rho} \left( \frac{c''}{c} + \frac{c'^2}{c^2} \right) - 2K_1 c^4 \rho^2 \eta_{t \rho} + K_1 c^4 \rho^3 \xi_{\eta_{t} \rho} + c^3 \rho^3 c^2 \xi_{\eta_{t} \rho} \left( 2\xi_{\eta_{t} \rho} - \eta_{\rho \rho} \right) = 0, \quad (A8)$$
$$3c^3 \rho^3 \eta_{t \rho} - c^4 \rho^3 \xi_{\rho \rho} + 3c^2 \rho G \left( \frac{G'}{G} - 2 \frac{c'}{c} \right) + \frac{\eta_{t}}{\rho} + 2 \xi_{t} \right] +$$
$$\rho^3 c^4 \left( \xi \left( \frac{3 \rho' - c'}{c} \right) - \frac{c''}{c} - 2 \frac{c^3}{c^2} \right) + 2\xi_{t} \left( \frac{c^2}{c^2} - \frac{c'^2}{c^3} \right) + 2c^3 \rho^2 \eta_{t} c^2 - K_1 c^4 \rho^2 \eta_{t} -$$
\[-2c^3 \rho^3 c' \eta_\rho + c^3 \rho^3 \xi_{tt} c' - 4K_3 c^2 \rho^5 G' \xi_\rho + 4K_3 c^2 \rho^5 G \xi_\rho + 4K_6 c^3 \rho^4 \Lambda \xi_\rho = 0, \tag{A9}\]

\[c^4 \rho^3 \eta_{tt} - c^3 \rho^3 c'' \eta_1 + c^2 \rho^3 c' \eta_1 - c^3 \rho^3 c' \eta_t + K_4 c^2 \rho^4 G \eta_1 + \]

\[c^2 \rho^5 G K_3 \left( \xi \left( \frac{2c^2 G'}{c} - \frac{G''}{G} \right) + \frac{G'}{G} \left( \eta_\rho - 2 \frac{\eta}{\rho} - 3t \right) \right) + \]

\[c^2 \rho^5 G K_5 \left( \xi \left( \frac{c' G'}{c} - 3 \left( \frac{c'}{c} \right)^2 + \frac{c''}{c} \right) + \frac{c'}{c} \left( 2 \frac{\eta}{\rho} - \eta_\rho + 3 \xi_t \right) \right) + \]

\[K_6 c^3 \rho^4 \Lambda \left[ \xi \left( \frac{(c')^2}{c^2} + \frac{c''}{c} + \frac{c' \Lambda'}{c \Lambda} \right) + \frac{c'}{c} \left( \frac{\eta}{\rho} - \eta_\rho + 3 \xi_t \right) \right] = 0. \tag{A10}\]

Imposing the symmetry \(X = (at + e) \partial_t + b \rho \partial_\rho\), i.e. \((\xi = at + c, \eta = b \rho)\), where \(a, b, e \in \mathbb{R}\). Note that \([a] = [b] = 1\), i.e. they are dimensionless constants but \([c] = T\), with respect to a dimensional base \(B = \{L, M, T\}\), we get the following restrictions for \(G(t), e(t)\) and \(\Lambda(t)\).

From eq. (A5) we get

\[\frac{c''}{c'} - \frac{c'}{c} = -\frac{a}{at + e}\]  \(\tag{A11}\)

From eq. (A9) we find that

\[\frac{G'}{G} - 2 \frac{c'}{c} = -\frac{b + 2a}{at + e}, \tag{A12}\]

and

\[\left( 3 \frac{c''}{c} - \frac{c'''}{c} - 2 \frac{c'^2}{c^2} \right) = -2 \left( \frac{a}{at + e} \right)^2, \tag{A13}\]

where the most general solution for (A13) is

\[c = K_0 \exp \left( -\frac{K_1 t^3}{9} + \frac{K_2 t^2}{9} \right) t^{K_2/3}, \tag{A14}\]

where \((K_i)_{i=0}^2 \in \mathbb{R}\), so a solution of (A11) is a particular solution of (A13).

From eq. (A10) it is obtained:

\[(at + e) \left( \frac{2c^3 G'}{c} - \frac{G''}{G} \right) - (b + 3a) \frac{G'}{G} = 0, \tag{A15}\]

\[(at + e) \left( \frac{c^3 G'}{c} - 3 \left( \frac{c'}{c} \right)^2 + \frac{c''}{c} \right) + \frac{c'}{c} (b + 3a) = 0, \tag{A16}\]

\[(at + e) \left( \frac{(c')^2}{c^2} + \frac{c''}{c} + \frac{c' \Lambda'}{c \Lambda} \right) + 3a \frac{c'}{c} = 0, \tag{A17}\]

Now from (A16) we get:

\[\frac{G'}{G} - 3 \frac{c'}{c} + \frac{c''}{c'} = -\frac{3a + b}{at + e}, \tag{A18}\]

and taking into account eq. (A12) we find that

\[\frac{c''}{c'} - \frac{c'}{c} = -\frac{a}{at + e}. \tag{A19}\]
In the same way, form eq. (A17) it is found that
\[
\frac{c''}{c'} + \frac{c'''}{c'^2} + \frac{\Lambda'}{\Lambda} = -\frac{3a}{at + e},
\]
and taking into account eq. (A19) we get:
\[
\frac{\Lambda'}{\Lambda} + 2\frac{c''}{c'} = -\frac{2a}{at + e}.
\]

Therefore the restrictions (A12, A19 and A21) will be enough to find a solution for eq. (A27). As we can see, these restrictions are the same than the obtained ones in section (IV), see eqs. (56-58)) where we studied the second order ODE and therefore we expect to obtain a very similar result, we will find only a little differences in the numerical constants but not in the order of magnitude of each quantity. We may check how works these restrictions in the case of the scale symmetry, since the rest of solutions will be obtained copying the same steeps as the followed ones in section (IV).

1. Scale symmetry.

Making \(e = 0\) i.e. considering only \((\xi = at, \ \eta = bp)\), we have to integrate eqs. (A19, A12 and A21), so
\[
\frac{c''}{c'} - \frac{c'}{c} = -\frac{1}{t}, \quad (A22)
\]
\[
\frac{G'}{G} - 2\frac{c'}{c} = -\frac{b}{a} + 2\frac{a}{t}, \quad \implies \quad \frac{G}{c'^2} = Bt^{-(2+\frac{b}{a})} \quad (A23)
\]
\[
\frac{\Lambda'}{\Lambda} + 2\frac{c''}{c'} = -\frac{2}{t}, \quad \implies \quad \Lambda c^2 = \tilde{B} t^{-2}, \quad (A24)
\]
\(B, \tilde{B} \in \mathbb{R}\), therefore we get
\[
c = c_0 e^{c_1 t}, \quad c_1, c_0 \in \mathbb{R}, \quad (A25)
\]
\[
G = G_0 t^{2(c_1 - 1 - \frac{b}{a})}, \quad G_0 \in \mathbb{R}^+, \quad (A26)
\]
\[
\Lambda = \Lambda_0 t^{-2(c_1 + 1)}, \quad \Lambda_0 \in \mathbb{R}, \quad (A27)
\]
where we assume that \(G_0 > 0\). Note that the obtained solution for \(c(t)\) obviously verifies eq. (A22) as well as it does verify eq. (A13) but the most general solution of eq. (A13) i.e. eq. (A14) does not verify eq. (A22).

The invariant solution for the energy density is: \(\rho = \rho_0 b^{b/a}\), and for physical reasons we impose the condition, \(ab < 0\) then \(b < 0\). If we make that this solution verifies eq. (27) with \(c(t), G(t)\) and \(\Lambda(t)\) given by eqs. (A25-A27), we find the value of constant \(\rho_0\), so
\[
\rho_0 = -\left[\frac{c_0^2 \left( b^2 + ab (1 + \omega) (c_1 + 1) + 3c_0^2 c_1 \Lambda_0 a^2 (1 + \omega)^2 \right)}{12 \pi a G_0 (1 + \omega)^2 \left( a (2c_1 + 1 + \omega) + b \right)} \right], \quad (A28)
\]
with the only restriction \(\omega \neq -1\), compare with eq. (66). Note that \(ab < 0\), so we need to choice constants \((c_1, c_0, G_0, \Lambda_0)\) such that \(\rho_0 > 0\). As we can see, it is verified the relationship \(\frac{\partial G}{\partial t} = t^{-2}\).

Therefore, at this time we have the following behavior for \(G(t)\)
\[
G(t) = G_0 t^{2(c_1 - 1 - \frac{b}{a})}, \quad G \approx \begin{cases} \text{decreasing if } (c_1 - 1 - b/2a) < 0, \\ \text{constant if } c_1 = 1 + b/2a, \\ \text{growing if } (c_1 - 1 - b/2a) > 0, \end{cases} \quad (A29)
\]
while \(\Lambda\) behaves as follows:
\[
\Lambda = \Lambda_0 t^{-2(c_1 + 1)}, \quad \Lambda \approx \begin{cases} \text{decreasing if } c_1 > -1, \\ \text{constant if } c_1 = -1, \\ \text{growing if } c_1 < -1, \end{cases} \quad (A30)
\]
therefore $(c_1 + 1) > 0 \implies c_1 \in (-1, \infty)$. But we have not any information about the sign of $\Lambda_0$, i.e. we do not obtain more information following this way.

With regard to $H$ we find that

$$R = R_0 t^{-1/(1+\omega)} = R_0 t^{-b/3a(1+\omega)}, \quad XYZ = R_0 t^{-b/a(1+\omega)}.$$  \tag{A31}

If we assume that the functions $(X,Y,Z)$ follow a power law (i.e. $X = X_0 t^{\alpha_1}$) then we get the following result, $Kt^\alpha = R_0 t^{-b/a(1+\omega)}$, then, $\sum_1^3 \alpha_i = \alpha = \frac{b}{3\pi - \omega}$, so we arrive to the same conclusion as in section (IV).

The shear is calculated as follows, $\sigma^2 = \alpha_2^2 t^{-2(c_1+1)}$, with

$$\sigma_0^2 = \frac{1}{3(1+\omega)} \frac{b^2}{a^2} + \frac{2}{3a (1+\omega)} \left( \frac{b^2 + ab (1+\omega) (c_1 + 1) + 3c_0^2 c_1 \Lambda_0 a^2 (1+\omega)^2}{a (2c_1 + 1 + \omega) + b} \right) - \Lambda_0 c_0^2. \tag{A32}$$

To calculate the value of constants $(\alpha_i)_{i=1}^3$, we arrive to the same system of equations i.e.

$$\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3 = 8\pi \frac{G_0}{c_0} \rho_0 + \Lambda_0 c_0^2, \tag{A33}$$

$$\alpha_2 (\alpha_2 - 1) + \alpha_3 (\alpha_3 - 1) + \alpha_3 \alpha_2 - c_1 (\alpha_3 + \alpha_2) = -8\pi \frac{G_0}{c_0} \omega \rho_0 + \Lambda_0 c_0^2, \tag{A34}$$

$$\alpha_1 (\alpha_1 - 1) + \alpha_3 (\alpha_3 - 1) + \alpha_3 \alpha_1 - c_1 (\alpha_1 + \alpha_3) = -8\pi \frac{G_0}{c_0} \omega \rho_0 + \Lambda_0 c_0^2, \tag{A35}$$

$$\alpha_2 (\alpha_2 - 1) + \alpha_1 (\alpha_1 - 1) + \alpha_1 \alpha_2 - c_1 (\alpha_1 + \alpha_2) = -8\pi \frac{G_0}{c_0} \omega \rho_0 + \Lambda_0 c_0^2, \tag{A36}$$

which solution is:

$$\alpha_1 = \alpha_2 = \alpha_3 = \sqrt{\frac{8\pi G_0 \rho_0}{3c_0} + \Lambda_0 c_0^2}, \quad c_1 = -1 + \frac{4\pi G_0 \rho_0 (1+\omega)}{c_0 \sqrt{\frac{8\pi G_0 \rho_0}{3c_0} + \Lambda_0 c_0^2}}. \tag{A37}$$

finding again that this kind of solutions lacks of any interest.

**APPENDIX B: STUDY OF EQ. (29).**

The purpose of this appendix is to show that if we consider the third order ODE obtained from the field equations but without taking into account the effects of a $c$–var into the curvature tensor we arrive to the same results as in the above appendix, i.e. appendix (A). Therefore we reproduce again the same steps as in section (IV) as well as in appendix (A) in order to find the restrictions for $G(t), c(t)$ and $\Lambda(t)$ for which our field equations admit symmetries i.e. they are integrable.

Therefore, our aim is study the following eq.

$$\dot{\rho} = K_1 \frac{\dot{\rho}}{\rho} - K_2 \frac{\dot{\rho}^2}{c^2} + \frac{G \dot{\rho}^2}{c^2} \left[ K_3 \frac{G'}{G} - K_4 \frac{\dot{\rho}}{\rho} - K_5 \frac{\dot{c}}{c} \right] - K_6 \rho c^2 \Lambda, \tag{B1}$$

where the constants $(K_i)_{i=1}^6$ are given by eqs. (30). Following the standard procedure we need to solve the next system of PDEs:

$$c^4 \rho^3 \xi = 0, \tag{B2}$$

$$c^4 \rho^3 \xi_{\rho} = 0, \tag{B3}$$

$$K_1 c^4 \rho \eta - K_1 c^4 \rho^2 \eta_p - 9c^4 \rho^3 \xi t_p + 3c^4 \rho^3 \eta_{pp} = 0, \tag{B4}$$

$$-K_1 c^4 \rho^2 \eta_t + 3c^4 \rho^3 \eta_p - 3c^4 \rho^3 \xi t = 0, \tag{B5}$$

$$K_1 c^4 \rho^2 \xi_{pp} + K_2 c^4 \rho \xi_p - c^4 \rho^3 \xi_{pp} = 0, \tag{B6}$$

$$2K c^4 \rho \eta_p - K_1 c^4 \rho^2 \eta_{pp} + 2K c^4 \rho^2 \xi_{t_p} - 2K c^4 \rho - 3c^4 \rho^3 \xi_{tpp} + c^4 \rho^3 \eta_{pp} = 0, \tag{B7}$$

$$3K c^4 \rho \eta_t - 2K c^4 \rho^2 \eta_p + 3K c^4 \rho^2 G \xi_p + K_1 c^4 \rho^2 \xi_{tt} - 4c^4 \rho^3 \xi t_p + 3c^4 \rho^3 \xi_{tt} = 0. \tag{B8}$$
\[3c^4 \rho^3 \eta_{tt} - c^4 \rho^3 \xi_{tt} + K_4 c^2 \rho^4 G \left( \xi \left( \frac{G'}{G} - \frac{2c'}{c} \right) + \frac{\eta}{\rho} + 2\xi_t \right) - \]

\[-K_1 c^4 \rho^2 \eta_{tt} - 4K_3 c^2 \rho^5 G' \xi_{tt} + 4K_3 cc' \rho^5 G \xi_{tt} + 4K_6 c^5 c' \rho^4 A \xi_{tt} = 0, \]

(B9)

\[c^4 \rho^3 \eta_{tt} + K_4 c^2 \rho^4 G \eta_t + \]

\[c^2 \rho^5 G K_3 \left( \xi \left( \frac{2c'}{c} \frac{G'}{G} - \frac{G''}{G} \right) + \frac{c'}{c} \left( \eta - 2\frac{\eta}{\rho} - 3\xi_t \right) \right) + \]

\[c^2 \rho^5 G K_5 \left( \xi \left( \frac{c'}{c} - 3 \frac{c'}{c} \frac{G'}{G} \right) + \frac{c'}{c} \left( \eta - 2\frac{\eta}{\rho} - 3\xi_t \right) \right) + \]

\[K_6 c^6 \rho^4 \Lambda \left[ \xi \left( \frac{(c')^2}{c^2} + \frac{c''}{c} + \frac{c'}{c} \frac{\Lambda}{\Lambda} \right) + \frac{c'}{c} \left( \eta - 2\frac{\eta}{\rho} - 3\xi_t \right) \right] = 0 \]

(B10)

Imposing the symmetry \( X = (at + e) \partial_t + b \rho \partial_\rho \), i.e. \( \xi = at + e, \eta = b \rho \), we get the following restrictions for \( G(t), c(t) \) and \( \Lambda(t) \).

From eq. (B9) we get

\[\frac{G'}{G} - 2 \frac{c'}{c} = \frac{b + 2a}{at + e}, \]

(B11)

while from eq. (B10) it is obtained:

\[(at + e) \left( \frac{2c'}{c} \frac{G'}{G} - \frac{G''}{G} \right) - (b + 3a) \frac{G'}{G} = 0, \]

(B12)

\[(at + e) \left( \frac{c'}{c} \frac{G'}{G} - 3 \frac{(c')^2}{c^2} + \frac{c''}{c} \right) + \frac{c'}{c} (b + 3a) = 0, \]

(B13)

\[(at + e) \left[ \frac{(c')^2}{c^2} + \frac{c''}{c} + \frac{c'}{c} \frac{\Lambda'}{\Lambda} \right] + 3a \frac{c'}{c} = 0, \]

(B14)

where \( a, b, e \in \mathbb{R} \). Note that \([a] = [b] = 1\), i.e. they are dimensionless constants but \([e] = T\), with respect to a dimensional base \( B = \{L, M, T\} \).

Now from (B13) we get:

\[\frac{G'}{G} - 3 \frac{c'}{c} + \frac{c''}{c} = \frac{3a + b}{at + e}, \]

(B15)

and taking into account eq. (B11) we find that

\[\frac{c''}{c'} - \frac{c'}{c} = -\frac{a}{at + e}. \]

(B16)

In the same way, form eq. (B14) it is found that

\[\frac{c'}{c} + \frac{c''}{c} + \frac{\Lambda'}{\Lambda} = -\frac{3a}{at + e}. \]

(B17)

and taking into account eqs. (B16) we get:

\[\frac{\Lambda'}{\Lambda} + 2 \frac{c'}{c} = -\frac{2a}{at + e}. \]

(B18)

These restrictions will be enough to find a solution for eq. (B1) i.e. eqs. (B16, B11 and B18). As it is observed we have arrive to the same restrictions as in section (IV) as well as in appendix (A). Therefore, following this approach, there is no difference between to consider \( c \)-var affecting to the curvature tensor and to consider the usual FE. We will show that we arrive to the same result in the case of the scale symmetry, the other solutions are obtained in the same way following the steeps as in section (IV).
1. Scale symmetry.

Making \( e = 0 \) i.e. considering only \( (\xi = at, \ \eta = b\rho) \), we have to integrate eqs. (B16, B11 and B18), so

\[
\frac{c''}{c} - \frac{c'}{c} - \frac{1}{t}, \quad \frac{G'}{G} - 2\frac{c'}{c} = -\frac{b + 2a}{at}, \quad \Longrightarrow \quad \frac{G}{c^2} = Bt^{-\frac{2+2b}{2a}}
\]  

(B19)

\[
\Lambda' + 2\frac{c'}{c} = -\frac{2}{t}, \quad \Longrightarrow \quad \Lambda c^2 = \tilde{B}t^{-2},
\]  

(B20)

\[
B, \tilde{B} \in \mathbb{R}, \text{ therefore we get}
\]

\[
c = c_0 t^{\xi_1}, \quad c_1, c_0 \in \mathbb{R},
\]

(B22)

\[
G = G_0 t^{2(c_1 - 1 - \frac{b}{2a})}, \quad G_0 \in \mathbb{R}^+, \quad \Lambda = \Lambda_0 t^{-2(c_1 + 1)}, \quad \Lambda_0 \in \mathbb{R},
\]

(B23-24)

where we assume that \( G_0 > 0 \). The invariant solution for the energy density is: \( \rho = \rho_0 t^{b/a} \), and for physical reasons we impose the condition, \( ab < 0 \) then \( b < 0 \). If we make that this solution verifies eq. (B1) with \( c(t), G(t) \) and \( \Lambda(t) \) given by eqs. (B22-B24), we find the value of constant \( \rho_0 \), so

\[
\rho_0 = - \left[ \frac{c_0^2 \left( b^2 + ab (1 + \omega) + 3c^2_0c_1 \Lambda_0 a^2 (1 + \omega)^2 \right)}{12\pi a G_0 (1 + \omega)^2 (a (2c_1 + 1 + \omega) + b)} \right],
\]

(B25)

with the only restriction \( \omega \neq -1 \), compare with eqs. (66 and A28). Note that \( ab < 0 \), so we need to choice constants \( (c_1, c_0, G_0, \Lambda_0) \) such that \( \rho_0 > 0 \). As we can see, it is verified the relationship, \( G\rho/c^2 = t^{-2} \), i.e. the Mach relationship for the inertia. Therefore, at this time we have the following behavior for \( G(t) \)

\[
G(t) = G_0 t^{2(c_1 - 1 - \frac{b}{2a})}, \quad G \approx \begin{cases} \text{decreasing if } (c_1 - 1 - b/2a) < 0, \\ \text{constant if } c_1 = 1 + b/2a, \\ \text{growing if } (c_1 - 1 - b/2a) > 0, \end{cases}
\]

(B26)

while \( \Lambda \) behaves as follows:

\[
\Lambda = \Lambda_0 t^{-2(c_1 + 1)}, \quad \Lambda \approx \begin{cases} \text{decreasing if } c_1 > -1, \\ \text{constant if } c_1 = -1, \\ \text{growing if } c_1 < -1, \end{cases}
\]

(B27)

therefore \( (c_1 + 1) > 0 \implies c_1. \in (-1, \infty) \). But we have not any information about the sign of \( \Lambda_0 \). If we assume that the functions \( (X, Y, Z) \) follow a power law (i.e. \( X = X_0 t^\alpha \)) then we get the following result, \( K t^\alpha = R_0 t^{b/a(1+\omega)} \), and \( \sum_1^3 \alpha_i = \alpha = -\frac{b}{a(1+\omega)} \). The shear is calculated as follows, \( \sigma^2 = \sigma_0^2 t^2 \).

So as we can see it is obtained the same solution, with the same order of magnitude and therefore we conclude that there is no difference between both approaches.