Optimal Path Construction Incorporating a Biarc Interpolation and Smooth Path Following for Automobiles

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Abstract: In this paper, path construction and path following methods are proposed. The path construction method addresses the problem to generate a path satisfying boundary conditions which are assigned points and tangent directions at the endpoints of the path. This problem is so-called \( G^1 \) Hermite interpolation, and it has been widely researched, for example, in the area of computer-aided design. The proposed path construction method utilizes one of them, namely the biarc interpolation, as an initial guess for obtaining an optimal path. Trials in some initial conditions show that the proposed method can generate a smooth path with a low computational cost. Meanwhile, the path following method assumes that a linear bicycle model follows a reference path by using the set-point regulator. The proposed method smooths the curvature profile of the reference path to be suited for the path following. Simulation results show that the proposed path construction and following methods achieve smoother tracking than the case without optimal path and smoothing.

Key Words: path planning, path following, automobile control, computer aided design.

1. Introduction

A decision making or planning is one of the most important parts of the system for the autonomous transportation system, such as a self-driving car or an unmanned aerial vehicle. Generally, the system has a hierarchical structure, from route planning to control [1],[2]. The path planning layer and control layer, which are layers of such hierarchical structure, are the scope of this paper.

The path planner is an intermediate level planner of the system, with inputs primarily from a route planner or navigator, and outputs being executed by the low-level controller [3]. For this purpose, it is natural that the path planner generates the two-dimensional geometric paths as a reference for the low-level controller. This is because the paths are required to be obtained easily for online guidance and for the case in which the multi-path generation is required.

There are a variety of geometric paths for a path planner. The most simple one is just the set of waypoints or is composed of straight lines connecting waypoints [4]. By using circular arcs [3],[5], the path becomes more followable by the system that cannot change their own direction immediately, such as an automobile. Paths of circular arc include so-called Dubins shortest paths [6], which are composed of circular arcs and straight line, and the biarc consisting of two circular arcs which have the same tangent at the connecting point. In addition to these, simple paths such as clothoid [7],[8] or parametric curve like a spline [9],[10] have been used for a path planner.

For the path of the vehicles, smaller curvature is preferable, and the smoothness (there is no discontinuity) is required. Also, inevitably, the path connects the initial and terminal positions with assigned tangent directions (\( G^1 \) Hermite interpolation [11]).

These characteristics are also of interest to the area of curve fitting. More specifically, the subjective contour [12] or curves of typography [13] must fulfill these characteristics. Ullman [12] proposed the minimum total curvature biarc as the subjective contour. Since then, the clothoid curve [11],[14] and elastica [15],[16] have been proposed as a means of the curve completion.

In this paper, the method to generate the optimal path for the \( G^1 \) Hermite interpolation is proposed. Optimal means that the square sum of the curvature of the path is minimized. The proposed method utilizes the results of biarc interpolation proposed by Kimia et al. [14] as an initial guess of the iterative calculation. Originally, Kimia [14] proposed the method to obtain the biarc in which the difference of curvature between two arcs is minimized, for the initial guess of the clothoid curve. This biarc is also suited as an initial guess for our problem. The method, which will be described in this paper, is reasonable because of the following: 1) Although the proposed method requires the iterative calculation, it is the same for the method to generate a clothoid curve. Moreover, the computational cost of the proposed method is not so high compared to that of the method for a clothoid; 2) By using the biarc as an initial guess, the proposed method can work as a path planner even when the iterative calculation does not converge.

As for the clothoid curve, more sophisticated methods that can construct the curve without biarc initial guess have been proposed [11],[17]. However, we rediscover the significance of the biarc by using it for the initial guess of our problem. The main contribution of this paper regarding path planning is that the unknown parameters of the optimal control problem are associated with the characteristics of the biarc (e.g., Eq. (19)); these parameters obtained from the biarc can be used as an initial guess of iterative calculation.
Recall that the scope of this paper includes the path following in addition to the path planning mentioned above. There are various methods for path following, as shown in [18], namely, the path following methods based on the geometric, kinematic, and dynamic model. In order to take advantage of the smoothness of the optimal path, which is constructed by the proposed method, we should choose the path following method using a dynamic model. This is because the control based on a dynamic model can achieve precise tracking, and simultaneously, precise tracking can lead to a jerky and uncomfortable ride without a smooth reference path. In fact, if we assume that a constant speed vehicle perfectly follows a reference path without a miss distance, the curvature of the path is equivalent to the lateral acceleration of the vehicle, and the curvature discontinuity leads to the large jerk or the time derivative of the lateral acceleration. For this reason, in many cases of path construction or smoothing, the smoothness means that there is no discontinuity in curvature [19].

Although our proposed method can provide a smooth path, the curvature at endpoints of the path cannot be specified due to a lack of degrees of freedom. Hence, smoothing at the endpoints is required when paths are constructed sequentially, or a path is connected to an existing trajectory. However, if we employ a complicated smoothing method for this purpose, we lose the advantage of the proposed path construction method; the proposed method can easily provide optimal paths. We, therefore, smooth only the curvature profile and disregard the effects of the smoothing, namely, inconsistency between the curvature profile and shape of the path, and the tracking error caused by such inconsistency. The tracking error can be eliminated by using the set-point regulator [20] as a path following controller. The set-point regulator shifts the equilibrium point by using the curvature profile of the path as feed-forward information and reduces a tracking error and an angle between the velocity vector and a tangent of reference path to zero. By smoothing only curvature profile and employing set-point regulator, we can both prevent a jerky motion and reduce a tracking error. This is because there is no discontinuity in the curvature profile, which is used as feed-forward information, and because the tracking error, which is fed back to the controller, is not caused discontinuously. The main contribution regarding path following is to show that simple smoothing method works well; the method accepts the inconsistency between the curvature profile and shape of the path.

The remainder of this paper is organized as follows. In Section 2, the problem of generating a smooth two-dimensional path is addressed. In order to obtain a smooth path, the problem is formulated as the optimal control problem in which the square sum of the curvature of the path is minimized. It is shown that generating the optimal path is equivalent to the problem of determining three parameters on which the optimal path depends. Then, it is also shown that the problem above can be solved by the shooting method. More specifically, Kimia’s biarc for an initial guess of the shooting method is shortly introduced, and then unknown parameters of the optimal path are associated with the characteristics of the biarc. Examples of the optimal paths are also shown. In Section 3, the set-point regulator as a path following controller is described. Then path following simulations are performed to demonstrate the effect of optimal path constructed by our proposed method and the effect of smoothing. The simulation results without path optimization and smoothing are also shown for comparison. Finally, concluding remarks are made in Section 4.

2. Path Construction

2.1 Problem Formulation

In this subsection, the $G^1$ Hermite interpolation problem is formulated as the optimal control problem [21]. Let us call the interpolation curve as the path in accordance with the purpose of this paper.

Without loss of generality, initial and terminal conditions and states can be defined as shown in Fig. 1, where $x$, $y$, and $s$ are the coordinates and the path length, respectively; these are normalized by the direct distance between initial and terminal points. The tangent direction of the path is represented by $\psi$. The subscripts 0 and $f$ represent initial and terminal conditions, respectively. Note that the initial and terminal tangent angles or $\psi_0$ and $\psi_1$ are specified in the problem. We can assume that the initial and terminal points are located on the origin and the abscissa, respectively.

Assuming the path length $s$ is an independent variable, we have state equations of the system or

$$\begin{align*}
\frac{dx}{ds} &= \cos \psi, \\
\frac{dy}{ds} &= \sin \psi, \\
\frac{d\psi}{ds} &= \kappa,
\end{align*}$$

where $\kappa$ is the curvature of the path and is assumed as an input. Recall that we want to minimize the square sum of the curvature. The cost function of the optimal control problem is

$$J = \int_0^L \kappa^2 ds,$$

where $L$ is the normalized total path length. Although $L$ can be arbitrarily chosen, we use the length of the after-mentioned biarc as $L$; the biarc length is uniquely determined when the boundary conditions are specified. Boundary conditions of the optimal control problem are the initial and terminal positions and specified tangent angles or

$$\begin{align*}
0 &\quad x(0) = 0, \\
0 &\quad y(0) = 0, \\
0 &\quad \psi(0) = \psi_0, \\
1 &\quad x(L) = 1, \\
0 &\quad y(L) = 0, \\
1 &\quad \psi(L) = \psi_1.
\end{align*}$$

The Hamiltonian of this system is

$$H = \kappa^2 + \lambda_1 \cos \psi + \lambda_2 \sin \psi + \lambda_3 \kappa,$$

where $\lambda$ is the adjoint variable and its subscripts are corresponding to each state. Considering necessary conditions for optimality, we have differential equations of adjoint variables or

![Fig. 1 Definitions of states.](image-url)
It is clear from the above equations that \( \lambda_i \) and \( \lambda_j \) are constants. Optimal input must satisfy
\[
\kappa = -\frac{1}{2} \lambda_\phi.
\]

Substituting Eq. (6) into Eq. (1), we have four differential equations (i.e., Eq. (1) and the third equation of Eq. (5)) with four unknowns, namely \( \lambda_1, \lambda_2, \lambda_\phi1, \) and \( \lambda_\phi2 \). These equations constitute the two-point boundary value problem.

We can reduce the number of unknowns of this problem with a little manipulation. Substituting Eq. (6) into Eq. (5), we have a differential equation of optimal input or
\[
\frac{dk}{ds} = -\frac{1}{2}(\lambda_1 \sin \psi - \lambda_2 \cos \psi).
\]

Integrating the above equation from the origin to arbitrary intermediate state, we have
\[
\kappa - \kappa_0 = -\frac{1}{2}(\lambda_1 \int_0^s \sin \psi ds - \lambda_2 \int_0^s \cos \psi ds). \quad (8)
\]

If we represent the distance between the origin and an arbitrary intermediate point as \( R \) and the angle of the line of sight from the origin to the intermediate point as \( \theta \) (i.e., \( R = \sqrt{x^2 + y^2} \) and \( \theta = \arctan(y/x) \)), optimal input can be rewritten as
\[
\kappa = \kappa_0 + \frac{R}{2}(\lambda_1 \sin \theta - \lambda_2 \cos \theta). \quad (9)
\]

Since the optimal input can be represented by only states except for three unknown constants, namely \( \kappa_0, \lambda_1, \) and \( \lambda_2 \), we can integrate Eq. (1) from the initial state. We can say these unknown constants are parameters of the optimal path.

Now the two-point boundary value problem with four unknowns is reduced to the problem of determining three parameters to meet the terminal conditions of states. In other words, we can generate the optimal path by solving the following equations for the three parameters:
\[
x_1(\kappa_0, \lambda_1, \lambda_2) = 1, \\
y_1(\kappa_0, \lambda_1, \lambda_2) = 0, \\
\psi_1(\kappa_0, \lambda_1, \lambda_2) = \psi_i, \quad (10)
\]

where the left-hand side of the equations represent the terminal values derived as a result of integration with the parameters in the brackets, and the right-hand side is specified terminal conditions.

This problem can be solved by the initial value adjusting method [22], which is one of the shooting methods; this method finds the solution by using the differences of the terminal values corresponding to the perturbation of parameters.

### 2.2 Biarc Construction

In this subsection, Kimia’s biarc, which will be used for an initial guess of our method, is shortly introduced. The characteristics of the biarc will be associated with unknown parameters of the optimal path.

Kimia [14] proposed the method to obtain the clothoid curve as a solution of the \( G^1 \) Hermite interpolation problem. It is shown that the problem can be reduced to find the initial curvatures and the path length, satisfying a particular nonlinear equation. Since there is no direct analytic solution for this nonlinear equation, the gradient descent approach starting from a suitable initial estimate is required. Also, Kimia proposed to use the biarc as the initial estimate of curvatures.

The significance of Kimia’s method to obtain the biarc is that the biarc can be constructed straightforwardly and uniquely. Inherently, there is an infinite number of biarc satisfying the \( G^1 \) Hermite interpolation problem. This is because the biarc construction has seven degrees of freedom while the boundary conditions of the problem or point-tangent pair consume six degrees of freedom. Hence, Kimia imposed the condition in which the total curvature variation is minimized to determine the biarc uniquely. The total curvature variation can be represented as
\[
E = (\kappa_2 - \kappa_1)^2, \quad (11)
\]

where \( \kappa_1 \) and \( \kappa_2 \) denote the curvatures of the first and second arc of biarc respectively.

Except for special cases (e.g., the point-tangent pair is located on the single circular arc), curvatures of such a biarc are given by
\[
\kappa_1 = \frac{-4}{L_0} \sin\left(\frac{3\psi_0 + \psi_1}{4}\right) \cos(\psi_1 - \psi_0), \quad (12)
\]
\[
\kappa_2 = \frac{4}{L_0} \sin\left(\frac{3\psi_0 + \psi_1}{4}\right) \cos(\psi_1 - \psi_0), \quad (13)
\]

where \( L_0 \) is the direct distance between the initial and terminal points. Note that in this paper, we assumed that \( L_0 = 1 \).

Once the curvatures are determined, the tangent angle at the connecting point of the biarc or \( \psi_{\text{joint}} \) and the respective arc length (\( L_1 \) and \( L_2 \)) are determined. That is,
\[
\psi_{\text{joint}} = \arcsin\left(\frac{k_1 k_2 + k_2 \sin \psi_0 - k_1 \sin \psi_1}{k_2 - k_1}\right), \quad (14)
\]
\[
L_1 = \frac{\psi_{\text{joint}} - \psi_0 + 2n_1 \pi}{k_1}, \quad (15)
\]
\[
L_2 = \frac{\psi_1 - \psi_{\text{joint}} + 2n_2 \pi}{k_2}, \quad (16)
\]

where \( n_1 \) and \( n_2 \) are chosen to give the smallest positive \( L_1 \) and \( L_2 \), respectively.

### 2.3 Relationships between the Biarc and Parameters of the Optimal Input

The biarc mentioned above, minimizing total curvature variation, is smooth enough to use as subjective contours [12]. Also, we want smooth paths. Hence, it is expected that the biarc’s parameters or Eqs. (12) to (16) are suited for the initial guess of our problem (i.e., initial guess of the three parameters of the optimal input).

It is natural to think that the curvatures of the biarc or \( \kappa_1 \) and \( \kappa_2 \) can be used for the initial guess of the optimal input at the initial and terminal points or \( \kappa_0 \) and \( \kappa_1 \). Meanwhile, some manipulation is required so that the parameters \( \lambda_i \) and \( \lambda_j \) are associated with the biarc’s parameters.
Regarding \( \lambda_s \), the relationship can be derived from Eq. (9). At the terminal point, optimal input and states are \( x = \kappa_f, R = 1, \) and \( \theta = 0 \). Substituting these into Eq. (9), we have

\[
\lambda_s = \kappa_f - \kappa_0.
\]

(17)

The relationship between \( \lambda_s \) and biarc’s parameters is obtained by assuming the profile of the input. Namely, if we assume that the variation of the optimal curvature is constant (like a clothoid) and that the optimal curvature is changed within half the length of the path, the variation of the curvature can be represented as

\[
\frac{d\kappa}{ds} = \frac{2(\kappa_2 - \kappa_1)}{L}.
\]

(18)

The schematic of this assumption is shown in Fig. 2. This assumption or curvature profile was determined heuristically so that the initial guess of \( \kappa_0 \) and other parameters given by Eqs. (17) and (19) is close to the solutions of the two-point boundary value problem or Eqs. (1), (3), (5), and (6).

Since the variation of curvature is assumed to be constant, Eq. (18) is valid at the connecting point of the biarc. The most probable point where the variation of the actual optimal input fits with the assumption of Eq. (18) is the connecting point of the biarc where the biarc’s curvatures are switched. Substituting \( \psi_{\text{join}} \) and Eq. (18) into Eq. (7) and solving for \( \lambda_s \), we have

\[
\lambda_s = \frac{1}{\sin \psi_{\text{join}}} \left( \frac{4(\kappa_0 - \kappa_1)}{L} + \lambda_s \cos \psi_{\text{join}} \right).
\]

(19)

This equation and Eq. (17) show the relationships between the parameters of the optimal path (\( \lambda_s \) and \( \lambda_f \)) and that of the biarc (\( \kappa_0 \) or \( \kappa_1 \), \( \kappa_2 \), and \( \psi_{\text{join}} \)). This relationship is non-trivial and is, therefore, one of the contributions of this paper.

In addition to these, as mentioned in Section 2.1, we assume that the path length \( L \), which can be specified in the optimal control problem, is the same as that of biarc. By doing so, we have sufficient information to solve the optimal control problem formulated in Section 2.1.

2.4 Examples of the Optimal Path

Figure 3 shows examples of the optimal path for a path planner. It is assumed that the initial position and the heading of the vehicle are the origin and in parallel with the y-axis, respectively. These paths, for example, are suitable for the case of lane change of a motor vehicle. Note that in this figure, we rotate the coordinate system described in Section 2.1 so that the initial states of the path are not changed. More specifically, instead of varying the initial tangent angle of the path or heading angle of the vehicle, the terminal point or destination is varied.

There are 70 paths in Fig. 3. Regarding these 70 cases, the average and the maximum number of iterations required to solve the boundary value problem or Eqs. (1), (3), (5), and (6). By doing so, we can obtain the relationship between \( \lambda_s \) and biarc’s parameters.

In addition to these, as mentioned in Section 2.1, we assume that the path length \( L \), which can be specified in the optimal control problem, is the same as that of biarc. By doing so, we have sufficient information to solve the optimal control problem formulated in Section 2.1.

3. Path Following

As mentioned in Section 1, we will use the set-point regulator [20] as a path following method. The paths constructed by the method of the previous section are used as a reference, and the curvature profile of the reference path is provided to the controller as feed-forward information. If there exists discontinuity in the curvature profile, it is smoothed by assuming the maximum change rate of the curvature. This prevents jerky motion of the vehicle. Note that when the curvature profile is smoothed, the consistency between the curvature profile and the actual shape of the reference is ignored for the sake of simplicity of the path construction.

3.1 Set Point Regulator for Path Following

Let us assume that the vehicle can be represented as a linear bicycle model. The problem of path following by such a vehicle can be summarized as shown in Fig. 4.

The equation of motion around the equilibrium conditions is given by

\[
\frac{dx}{dt} = \lambda_f, \quad \frac{d\lambda}{dt} = \frac{4(\kappa_0 - \kappa_1)}{L} + \lambda_s \cos \psi_{\text{join}}.
\]
The steering angle at the equilibrium is given by

\[ \delta_c = \frac{2(C_f l_f + C_r l_r)}{mV^2} \kappa \]  

(20)

where it is assumed that the equilibrium condition for the set-point regulator is a steady turn at the closest point on the reference path. The state vector consists of the distance between the vehicle and the reference path \( \Delta y \), the deviation of the vehicle fixed x-axis direction from the equilibrium \( \Delta \phi \), and their time derivatives. A dot denotes a derivative with respect to time.

The problem settings and parameters of the vehicle are the same as in reference [23] and are shown in Fig. 5. Distances between initial and terminal points for a path construction along the ground fixed x and y axes are 15 m and 0.5 m, respectively. The path must be connected with the straight lines parallel with the ground fixed x axis without the discontinuity of the tangent angle. In other words, the tangent directions of the endpoints of the path or \( \psi_0 \) and \( \psi_f \) must be 0.

![Fig. 5 Problem setting and parameters of vehicle.](image)

As mentioned at the beginning of this section, if there exists a discontinuity in the curvature profile of the path, it is smoothed by assuming the maximum change rate of the curvature; this maximum change rate is 0.004 rad/m² in this simulation.

At first, results using the proposed path construction method are described. Figure 6 shows the generated reference and trajectory of the path-following-vehicle. The lower part of this figure shows the original and smoothed curvature profiles of the reference path. The trajectory and the curvature profile in the case where we conducted smoothing are represented as solid lines. Dashed lines mean the corresponding data in the case of no smoothing. As shown in the figure, the generated curvature profile has discontinuity at the endpoints of the constructed path, and hence it is smoothed.

![Fig. 6 Top: the reference path and trajectories as the results of the simulation using original and smoothed curvature profiles as the feed-forward information. Bottom: original and smoothed curvature profiles corresponding to the generated reference path. The reference path is constructed by the proposed method.](image)

### 3.2 Simulation Results

In this subsection, paths for lane-keeping or lane-change are constructed, and the path following simulations are performed. The vehicle model for the simulations is the linear bicycle model depicted in Fig. 4 or Eqs. (24) and (25). It is assumed that the steering angle is delayed from the calculated ones. The dynamics of the steering is approximated by a first-order system with the time constant 0.05 seconds. More specifically, we assume that the steering angle command or \( \delta_t \) is calculated by the set-point regulator without delay and that the actual steering angle \( \delta \) is given by

\[ \dot{\delta} = \frac{(\delta_t - \delta)}{0.05} \]  

(28)

The vehicle model for the simulations is the linear bicycle model. The steering angle \( \delta \) is assumed to respond to the calculated steering angle command \( \delta_t \) in the form of a first-order system with the time constant 0.05 seconds. The vehicle model is described by Eqs. (21) and (22), where \( \beta \) is the sideslip angle, \( \phi \) is the yaw angle, and \( \dot{\phi} \) is the yaw rate. The state vector consists of the distance between the vehicle and the reference path \( \Delta y \), the deviation of the vehicle fixed x-axis direction from the equilibrium \( \Delta \phi \), and their time derivatives. A dot denotes a derivative with respect to time.
Simulation results are shown in Fig. 7. The meaning of line type is the same as that in Fig. 6.

This figure shows that despite the inconsistency between path shape and smoothed curvature, tracking error is small (< 1 cm), and the suppression of the jerk is achieved. In other words, it is shown that the simple smoothing method works well; this is one of the two contributions of this paper. Meanwhile, this figure also shows that although the smoothing of the curvature profile can suppress the jerk, the acceleration and tracking error are deteriorated. However, the tracking error caused by smoothing is sufficiently small, and we think that a slight increase in acceleration is acceptable in exchange for a significant reduction in the jerk. This is because we need to suppress both the acceleration and jerk for improving the ride comfort of passengers of vehicles [24].

Now that we have confirmed the effectiveness of the proposed pass following method, let us evaluate the validity of the proposed optimal path. For comparison purposes, the simulation results using the biarc and the clothoid as a reference are shown in Figs. 8 – 11. What the figures represent and the meaning of the line type are the same as those in Figs. 6 and 7.

The feature of the biarc reference is the existence of curvature discontinuity in the middle part of the path. This leads the large jerk, and the passengers may feel like they are shaken left and right. In both cases with and without smoothing, drawbacks of biarc reference, which are mentioned above, can be seen. Nevertheless, the path following using the biarc reference is still possible. In addition to this, the biarc can be derived straightforwardly while the optimal path requires iterative calculations. Hence, as mentioned in the introduction, by using the biarc as an initial guess, the proposed method can work as a path plan-

![Fig. 7 Simulation results in the case where the reference path is constructed by the proposed method.](image1)

![Fig. 8 The reference path and trajectories and curvature profiles. What the figures represent is the same as in Fig. 6, except that the biarc is used as the reference path.](image2)

![Fig. 9 Simulation results in the case where the reference path is a biarc.](image3)

![Fig. 10 The reference path and trajectories and curvature profiles. What the figures represent is the same as in Fig. 6, except that the clothoid curve is used as the reference path.](image4)

![Fig. 11 Simulation results in the case where the reference path is a clothoid.](image5)

4. Conclusion

We developed the path construction and the following method for an automobile.

Reference paths were generated by solving the optimal control problem in which the square sum of the curvature of the path is minimized. We simplified the problem and associated the parameters of the problem with the biarc, which is the ge-
ometric curve and can be obtained straightforwardly. By using the biarc as an initial guess, we can reduce the iterations in the process of optimal path construction.

The generated path was followed by using the set-point regulator. It has been shown that the proposed optimal path is easy to follow and that by allowing the inconsistency between the curvature profile and the shape of the path, we can prevent jerky motion.

Especially in path construction, we think that the proposed method is widely applicable to the guidance of transportation systems that require smooth references.

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