POINTEWISE ESTIMATES OF SOLUTIONS TO CONSERVATION LAWS WITH NONLOCAL DISSIPATION-TYPE TERMS

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ABSTRACT. We are concerned with the pointwise estimates of solutions to the scalar conservation law with a nonlocal dissipative term for arbitrary large initial data. Based on the Green’s function method, time-frequency decomposition method as well as the classical energy estimates, pointwise estimates and the optimal decay rates are established in this paper. We emphasize that the decay rate is independent of the index $s$ in the nonlocal dissipative term. This phenomenon is also coincident with the fact that the decay rate is determined by the low frequency part of the solution no matter the initial data is small or large.

1. Introduction. In this paper, we consider the Cauchy problem of a scalar conservation law with a nonlocal dissipative term, taking the form of

$$
\begin{aligned}
\partial_t u - \frac{\Delta}{(1 - \Delta)^{\frac{s}{2}}} u &= \text{div} f(u), & x \in \mathbb{R}^3, & t > 0, \\
u_0(x) &= u_* + \tilde{u}_0, & x \in \mathbb{R}^3,
\end{aligned}
$$

(1.1)

where $u = u(t, x) : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ is unknown, $u_*$ is a constant, $\tilde{u}_0 = \tilde{u}(0, x)$, $f(u) = (f_1(u), f_2(u), f_3(u))$ and $f_j(u) = u^2$ ($j = 1, 2, 3$). The nonlocal dissipative term $-\frac{\Delta}{(1 - \Delta)^{\frac{s}{2}}} u$ with an index $0 \leq s \leq 2$ is a pseudo-differential operator defined in terms of the Fourier transform as

$$
\Gamma_s u := -\frac{\Delta}{(1 - \Delta)^{\frac{s}{2}}} u = \mathcal{F}^{-1}\left\{\frac{\xi^2}{(1 + |\xi|^{2})^{\frac{s}{2}}} \mathcal{F}u\right\}.
$$

(1.2)

Clearly, if we drop the nonlocal dissipative term in (1.1), then it is a special case of the classical scalar conservation law. Generally, the solution will develop singularity in finite time regardless of the size and smoothness of the initial data [34]. However, if some dissipation terms are added to the scalar conservation law similar to equation (1.1), then it is possible to avoid the formation of singularities.
For example, when \( s = 0 \), \( \Gamma_s \) is simply a Laplacian operator and thus (1.1) is equivalent to the usual scalar viscous conservation law

\[
\partial_t u + \text{div} f(u) = \Delta u.
\] (1.3)

The study of the above equation has a long history and has been extensively investigated from different aspects, such as well-posedness, finite time blow-up of classical solutions and stability of wave patterns including viscous shock wave, rarefaction wave and contact discontinuity, c.f. [10, 13, 17, 24, 25, 31, 32] and the references therein.

When \( s = 2 \), a typical model related to (1.1) takes the form of

\[
\partial_t u + \text{div} f(u) = \frac{\Delta}{1 - \Delta} u.
\] (1.4)

This model was derived in [30] as the corresponding extension of the Navier-Stokes equations via the regularization of the Chapman-Enskog expansion from the Boltzmann equation. Equation (1.4) has been studied by many researchers, c.f. [12, 14, 20, 27, 35, 36]. Specially, Wang and his partner [36] have shown that for some small initial data, system (1.4) admits a unique global smooth solution, while for some large initial data, the classical solution will blow-up in finite time.

There are some analogue results for the dissipative surface quasi-geostrophic (QG) equation,

\[
\begin{aligned}
\partial_t \theta + u \cdot \nabla \theta + \kappa \Lambda^\alpha \theta &= 0, & x \in \mathbb{R}^2, & t > 0, \\
u &= R^\perp \theta = (-R_2 \theta, R_1 \theta), & x \in \mathbb{R}^2,
\end{aligned}
\] (1.5)

where \( \Lambda = \sqrt{-\Delta} \), \( R_j \) is the \( j \)th Riesz transform and \( \kappa \) is a constant. For the physical background of the dissipative surface QG equation, we refer to [4, 16]. From the mathematical point of view, the non-dissipative QG equations (\( \kappa = 0 \) case) share many similar properties with the 3D Euler equation. For example, whether the solutions can develop singularities in finite time remains a challenging open problem. On the other hand, if dissipative terms like \( \kappa \Lambda^\alpha \theta \) are added to the QG equations, we will get (1.5) and it is possible to study the issue of global regularity. Actually, the study of the dissipative QG equation (1.5) attracted a lot of attention over the past two decades. The \( \alpha > 1 \) case (often called the subcritical case) has been resolved completely, c.f. [7, 33, 37]. The critical case (\( \alpha = 1 \)) is a challenging case. Several different methods have been introduced independently to show the global well-posedness of the critical QG equation for general large initial data ([3, 5, 6, 23]).

We also mention that, corresponding to the dissipative QG equation, there are analogue results for the following fractal Burgers equation,

\[
\partial_t u + \Lambda^\alpha u = \sum_{j=1}^N u \cdot \partial_j u.
\] (1.6)

For one dimension case, [15] and [22] have proved the finite time blow-up of solution for \( \alpha < 1 \) (often called the supercritical case), and global existence as well as analyticity of solution for \( \alpha \geq 1 \) (also called the subcritical case). Chan in [1] has obtained that a weak solution of a slightly supercritical fractional Burgers equation becomes Hölder continuous for large time. For multi-dimension case, [6] has mentioned that the same strategies can also be applied to the fractal Burgers equation in any spatial dimension. Additionally, Chan and Czubak in [2] have considered the fractional Burgers equation on \( \mathbb{R}^n \) with the critical dissipation term. They obtained
existence of smooth solutions for any given initial data in \( L^2(\mathbb{R}^n) \) via the parabolic De-Giorgi’s method. Recently, well-posedness of equations with nonlocal terms has been attracted a lot of attention, c.f. [9], [45] and the references therein.

For equation (1.1), the global existence for the subcritical case (i.e. \( 0 \leq s < 1 \)) and critical case (i.e. \( s = 1 \)) were obtained in [38], the result reads as follows,

**Theorem (Global Existence).** Suppose that \( 0 \leq s \leq 1 \) and \( u_0 \in L^1(\mathbb{R}^3) \cap C^\gamma(\mathbb{R}^3) \), \( \gamma > 1 \), then the Cauchy problem (1.1) admits a unique global solution \( u(t,x) \) satisfying

\[
  u \in L^\infty([0, \infty); L^1(\mathbb{R}^3) \cap C^\gamma(\mathbb{R}^3)).
\]

We put the proof of this theorem as an appendix at the end of this article for the convenience of readers.

The main purpose of this article is to study the pointwise estimates of solutions to (1.1). Through the pointwise estimates, we can not only obtain the decay rate of the solution which is due to the parabolicity of the system, but also find out the movement of the main part of the solution which is caused by the hyperbolicity.

Additionally, we are interested in the decay rate of solutions to equation (1.1). The decay rate of solutions to equations (1.5) and (1.6) in ([25] and [26]) is

\[
  \|D_x^\alpha u\|_{L^2} \leq C(1 + t)^{-\frac{n}{4} - \frac{\|\bar{u}\|}{\|u_0\|}}.
\]

It is easy to see that the smaller \( \alpha \), the faster decay. So it is nature to discuss the relationship between the decay rates of solutions and the index \( s \) in the nonlocal term.

It is a good choice to use Green’s function method studying the pointwise estimate of solutions. In fact, using Green’s function to study pointwise estimates for hyperbolic-parabolic systems became a very active field of research in recent years. The Green’s function method was first introduced by Liu and Zeng in [28] to get the pointwise estimates of solutions for one dimensional quasilinear hyperbolic-parabolic systems of conservation laws. Later, Hoff and Zumbrun ([18] and [19]) studied the Navier-Stokes equation with viscosity. Liu and Wang [29] have obtained the pointwise estimates of the solutions for the isentropic Navier-Stokes equations in odd dimensions. After that, a lot of papers studied the Green’s function and pointwise estimates of various types of hyperbolic-elliptic equations (see e.g. [39, 40, 41, 42, 43, 44]).

To the best of our knowledge, there have a few works on the pointwise estimates in the case of large perturbation to the initial data. We emphasize that the basic idea of using the Green’s function to study the Cauchy problem of nonlinear evolution equation with small perturbation is to use the smallness of the perturbation and iteration. However, this idea does not work in the case of large initial perturbations. In this paper we attempt to explore the methods of studying the pointwise estimates of solutions in the case of large perturbation. Without the smallness of the initial data \( \|u_0\|_{L^2} \), the difficulty here is how to estimate the nonlinear term of equation (1.1). Fortunately, based on the special structure of the equation itself and the time-frequency decomposition method, we can get the decay estimates of \( \|u\|_{L^p} \) \((2 \leq p \leq +\infty)\). After that, by applying energy method and Green’s function method alternately, we obtain the decay rate of \( \|D^\alpha_x u\|_{L^2} \) and \( \|D^\alpha_x u\|_{L^\infty} \). Finally, based on the Green’s function method and the optimal decay estimate of \( \|D^\alpha_x u\|_{L^\infty} \), we obtain pointwise estimates of solutions.

For the pointwise estimate of the solution to (1.1). Our result reads as follows
Theorem 1.1. Suppose that $0 \leq s \leq 1$, $u_0 \in C^\infty(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, $\gamma > 1$, and $|D_x^nu_0(x)| \leq C(1 + |x|^2)^{-\kappa}$, with $\kappa > 1$ and $|\beta| < \gamma$, then the solution $u(x, t)$ of the Cauchy problem (1.1) have the following pointwise estimate, when $t \geq 1$,
\[ |D_x^n u(t, x)| \leq C(1 + t)^{-\frac{3 + |\alpha|}{2}} B_N(t, x - bt), \quad |\alpha| < \gamma, \]
where $b = f'(u_0) = (f'_1, f'_2, f'_3)(u_0)$, and
\[ B_N(t, x) = (1 + |x|^2)^{-N}, \quad N \text{ is a positive integer}. \]
Moreover, we also obtain the optimal decay estimates for $\|D_x^n u(t)\|_{L^p}$, $p \geq 2$,
\[ \|D_x^n u(t)\|_{L^p} \leq C(1 + t)^{-\frac{1}{p} - \frac{2}{1} \left(1 - \frac{1}{p}\right)}, \quad |\alpha| < \gamma. \]

Remark 1.1. We emphasize that the decay rate is independent of the index $s$. In fact, when $|\xi|$ is small enough, the symbol of operator $\Gamma_s$ is
\[ \sigma(\Gamma_s) = \frac{|\xi|^2}{(1 + |\xi|^2)^2} \sim |\xi|^2. \]
This phenomenon is also coincident with that the decay rate is determined by the low frequency part of the solution no matter the size of the initial data.

Remark 1.2. Theorem 1.1 shows that the main part of the solution moves along $x = bt$, and the velocity of the movement depends on the background state and the nonlinear term of the equation.

We now introduce notations used throughout this article. The Fourier transform $\hat{f}(t, \xi)$ of a tempered distribution $f(t, x)$ is defined as
\[ \hat{f}(t, \xi) = (\mathcal{F} f)(t, \xi) = \int_{\mathbb{R}^3} f(t, x) e^{-ix \cdot \xi} dx. \]
The inverse Fourier transform is
\[ f(t, x) = (2\pi)^{-3} \int_{\mathbb{R}^3} \hat{f}(t, \xi) e^{ix \cdot \xi} d\xi. \]
Denote the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \sum_{i=1}^3 \alpha_i$, $D_\alpha = D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} D_{x_3}^{\alpha_3}$.

Throughout the paper, $C$ stands for a harmless constant, whose precise meaning will be clear from the context, and its value may be different from line to line.

The rest of the paper is organized as follows. In Section 2, we obtain pointwise estimates of Green’s function. In Section 3, we will use the time-frequency decomposition and the Green’s function method to get the decay estimate of $\|u\|_{L^p}$. In Section 4, decay rates of $\|D_x^n u\|_{L^2}$ and $\|D_x^n u\|_{L^\infty}$ is obtained. In Section 5, we show pointwise estimates of solution $u(t, x)$, and furthermore, we obtain the optimal decay estimates of $\|D_x^n u\|_{L^p}$ based on the pointwise estimates. At the end of this paper, we list the proof of the global existence of solutions as an appendix for the convenience of readers.

2. Estimates for Green’s Function. We denote $u = v + u_*$, then equation (1.1) can be rewritten as
\[ \partial_t v = \frac{\Delta}{(1 - \Delta)^2} v - f'(u_*) \text{div} v = \text{div} f(v), \quad \text{(2.1)} \]
with the initial datum $v(0, x) = v_0(x)$.
The characteristic equation of the linear equation of (2.1) is
\[ \tau + \frac{|\xi|^2}{(1 + |\xi|^2)^2} - ib \cdot \xi = 0, \]
where \( b = f'(u_\ast) \). The fundamental solution \( G = G(t, x) \) of the Cauchy problem
\[
\begin{cases}
\partial_t G - \frac{\Delta}{(1 - \Delta)^2} G - b \text{ div} G = 0, & x \in \mathbb{R}^3, t > 0, \\
G(0, x) = \delta(x), & x \in \mathbb{R}^3,
\end{cases}
\]
is called Green’s function, where \( \delta(x) \) is Dirac function. One can easily get
\[
\hat{G}(t, \xi) = \text{exp} \left( ib \cdot \xi t - \frac{|\xi|^2}{(1 + |\xi|^2)^2} t \right).
\]

In order to get the pointwise estimates for the Green’s function \( G(t, x) \), we divide \( G(t, x) \) into lower frequency part \( G_1(t, x) \) and higher frequency part \( G_2(t, x) \), where \( G_i(t, x) = \chi_i(D) G(t, x) \), \( (i = 1, 2) \),
\[
\chi_1(\xi) = \begin{cases} 1, & |\xi| < \epsilon, \\
0, & |\xi| > 2\epsilon,
\end{cases}
\]
and
\[
\chi_2(\xi) = 1 - \chi_1(\xi).
\]
Here \( \chi_1(\xi) \) and \( \chi_2(\xi) \) are smooth cut-off functions for the fixed constants \( 0 < \epsilon < 1 \).

Before we proceed the estimate of \( G_i(t, x) \), we recall the following lemma, which is a vital tool to get the pointwise estimate of the Green’s function \( G(t, x) \). The proof of the first part in Lemma 2.1 can be found in [40]. The second part in this lemma can be proved by similar method.

**Lemma 2.1.** (1) If \( \hat{f}(t, \xi) \) has compact support in the variable \( \xi \in \mathbb{R}^n \), and there exists a constant \( \nu > 0 \), such that \( \hat{f}(t, \xi) \) satisfies
\[
|D_\xi^\alpha (\xi^\alpha \hat{f}(t, \xi))| \leq C|\xi|^{(|\alpha|+k-|\beta|)+} (1 + (t|\xi|^2))^{m-\nu|\xi|^2},
\]
for positive integers \( k, m \), and multi-indexes \( \alpha, \beta \) with \( |\beta| \leq N \), then
\[
|D_\xi^\alpha f(t, x)| \leq C N^{k + |\alpha|} B_N(t, x),
\]
where \( N \) is any fixed integer, \((a)_+ = \max(0, a)\) and
\[
B_N(t, x) = (1 + |t|^2(1 + t)^{-1})^{-N}.
\]
(2) If for fixed \( t \geq 0 \), \( \text{supp} \hat{f}(\cdot, \xi) \subset O_\epsilon = \{ \xi, |\xi| > \epsilon \} \), and there exists a constant \( \nu > 0 \), such that \( \hat{f}(t, \xi) \) satisfies
\[
|D_x^\beta (\xi^\alpha \hat{f}(t, \xi))| \leq C|\xi|^{(|\alpha|+k-|\beta|)+} (1 + (t|\xi|^{2-s}))^{m-\nu|\xi|^{2-s}},
\]
for positive integers \( k, m \) and multi-indexes \( \alpha, \beta \) with \( |\beta| \leq N \), then
\[
|D_x^\beta f(t, x)| \leq C N^{k + |\alpha|} B_{N,s}(t, x),
\]
where
\[
B_{N,s}(t, x) = (1 + |t|^{2-s}(1 + t)^{-1})^{-N}.
\]

For the lower frequency part \( G_1(t, x) \), one has

**Lemma 2.2.** For any positive integer \( N \), if \( \epsilon \) is small enough, there exists a constant \( C > 0 \), such that
\[
|D_x^\alpha G_1| \leq C(1 + t)^{-\frac{3+|\alpha|}{2}} B_N(t, x - bt). \quad (2.2)
\]
Lemma 2.3.  

Then by Lemma 2

$$\hat{G}_1 = \chi_1(\xi)e^{-|\xi|^2t}e^{-ib\xi t}(1 + O(|\xi|^2)t) = \hat{A}e^{-ib\xi t},$$

where $\hat{A} = \chi_1(\xi)e^{-|\xi|^2t}(1 + O(|\xi|^2)t)$. By properties of Fourier transform, we have

$$G_1 = A \ast F(e^{-ib\xi t}) = A(t, x - bt), \tag{2.3}$$

and furthermore, by the expression of the Green’s function, one has

$$|D^\beta_\xi (\xi^\alpha \hat{A})| \leq C(|\xi|^{(\alpha|\beta|)} + |\xi|^{\alpha} t^{\beta/2})(1 + t|\xi|^2)^{\beta/4 + 1}e^{-|\xi|^2t}.$$ 

Thanks to Lemma 2.1, we have

$$|D^\alpha_\xi A(t, x)| \leq Ct^{-\alpha - \beta} \mathcal{B}_N(t, x).$$

By (2.3), the pointwise estimates of $G_1(t, x)$ is obtained,

$$|D^\alpha_\xi G_1(t, x)| \leq Ct^{-\alpha - \beta} \mathcal{B}_N(t, x - bt). \tag{2.4}$$

Moreover,

$$|D^\alpha_\xi G_1(t, x)| \leq C \left| \int_{\mathbb{R}^3} e^{\sqrt{-1}x \cdot \xi} \chi_1(\xi) (\xi^\alpha \hat{G}_1(t, \xi)) d\xi \right| \leq C. \tag{2.5}$$

Thus, (2.2) follows from (2.4) and (2.5). We complete the proof of Lemma 2.2. \(\square\)

When $|\xi| \geq \epsilon$, one can see that there is no singular point of $\hat{G}_2(t, \xi)$, by the definition of $\hat{G}_2(t, \xi)$, then we have

$$\hat{G}_2(t, \xi) = \chi_2(\xi)e^{-ib\xi t}e^{-\frac{|\xi|^2 t}{1 + (\xi^2)^{\epsilon/2}}} = \hat{T}e^{-ib\xi t}.$$ 

where $\hat{T} = \chi_2(\xi)e^{-\frac{|\xi|^2 t}{1 + (\xi^2)^{\epsilon/2}}}$. By properties of fourier transform, we have

$$G_2 = T \ast F(e^{-ib\xi t}) = T(t, x - bt).$$

By direct calculation, we have

$$|D^\beta_\xi (\xi^\alpha e^{-\frac{|\xi|^2 t}{1 + (\xi^2)^{\epsilon/2}}})| \leq C(|\xi|^{\alpha|\beta|}(1 + |\xi|^2 t)^{\beta/4 + 1}e^{-|\xi|^2 t}.$$ 

Then by Lemma 2.1 (2), we have the following lemma,

Lemma 2.3.  Suppose that $|\xi| > \epsilon$, then we have

$$|D^\alpha_\xi G_2(t, x)| \leq C t^{-\alpha - \beta} e^{-ct/2} \mathcal{B}_{N,s}(t, x - bt).$$

To conclude, we get the pointwise estimates of the Green’s function $G(t, x)$, and

Theorem 2.1.  For any $x \in \mathbb{R}^3$, we have the following two estimates according to the time $t$,

$$|D^\alpha_\xi G(t, x)| \leq C \alpha t^{-\alpha - \beta} \mathcal{B}_{N,s}(t, x - bt), \quad 0 < t < 1,$$

$$|D^\alpha_\xi G(t, x)| \leq C \alpha (1 + t)^{-\alpha - \beta} \mathcal{B}_N(t, x - bt), \quad t > 1.$$
The proof of the pointwise estimate is standard (for instance [29] and [40]) and technical, is thus omitted for simplicity. Finally, according to the pointwise estimate of the Green’s function, one can easily obtain the $\|D_x^\alpha G(t)\|_{L^p}$, our results is as follows,

**Theorem 2.2.** For any $x \in \mathbb{R}^3$, we have the following two estimates according to the time $t$,

$$
\|D_x^\alpha G(t)\|_{L^p} \leq Ct^{-\frac{3}{2} - \frac{3}{2p}(1 - \frac{1}{p})}, \quad (1 \leq p \leq \infty), \quad 0 < t < 1,
$$

$$
\|D_x^\alpha G(t)\|_{L^p} \leq C(1 + t)^{-\frac{3}{2} - \frac{3}{2}(1 - \frac{1}{p})}, \quad (1 \leq p \leq \infty), \quad t > 1.
$$

When $p \geq 2$, it is easy to obtain the decay rate of $\|D_x^\alpha G(t)\|_{L^p}$ by the Young inequality. However, when $1 \leq p < 2$, we combine Lemma 2.2 and Lemma 2.3 to achieve the decay estimate of $\|D_x^\alpha G(t)\|_{L^p}$.

3. $L^p$ decay rate of solutions. For the decay estimates of $\|u\|_{L^2}$, we have the following theorem,

**Theorem 3.1.** Suppose that $u_0(x)$ satisfies the assumptions in Theorem 1.1 and $0 \leq s \leq 1$. $u(t,x)$ is a solution of (1.1), then $u(t,x)$ enjoys the following time-decay rate,

$$
\|u(t)\|_{L^2} \leq C(1 + t)^{-\frac{2}{3}}. \quad (3.1)
$$

The proof of this theorem is similar like the proof in [11]. Here we omit it for simplicity.

For the decay estimates of $\|u\|_{L^p}$ ($p > 2$), the result reads as follows,

**Theorem 3.2.** Suppose that $u_0(x)$ satisfies the assumptions in Theorem 1.1 and $0 \leq s \leq 1$. $u(t,x)$ is a solution of (1.1), then $u(t,x)$ enjoys the following time-decay rate,

$$
\|u(t)\|_{L^p} \leq C(1 + t)^{-\frac{2}{3} + \frac{2}{p}}, \quad p > 2. \quad (3.2)
$$

Because the positive lemma is not valid for the pseudo-differential operator $\Gamma_s$, so the major difficulty here is how to deal with $\Gamma_s$ without the positive lemma. In order to overcome the difficulties caused by the nonlocal dissipative term, we rewrite equation (1.1) as follows,

$$
\partial_t u + \Lambda^{2-s} u = \text{div} f(u) + (\Lambda^{2-s} - \Gamma_s) u, \quad (3.3)
$$

where $\Lambda = (-\Delta)^\frac{1}{2}$. Thanks to the positive lemma (see Lemma 2.4, [8]), we have

$$
p \int_{\mathbb{R}^3} |u|^{p-2} u \Lambda^{2-s} u dx \geq \int_{\mathbb{R}^3} |\Lambda^{1 - \frac{2}{s}} u|^{\frac{2p}{p-2}} dx. \quad (3.4)
$$

Multiplying $pu^{p-1}$ (where $p = 2^n, n = 2, 3, \cdots$) with (3.3) and taking the inner product in $L^2$ space, by (3.4), we have

$$
\frac{d}{dt} \|u\|_{L^p}^p + \|\Lambda^{1 - \frac{2}{s}} u\|_{L^2}^2 \leq C \int_{\mathbb{R}^3} u^{p-1}(\Lambda^{2-s} - \Gamma_s) u dx. \quad (3.5)
$$

We will use the low-high frequency decomposition method to estimate of the right-hand side of (3.5). Let

$$
\chi(\xi) = \begin{cases} 
1, & |\xi| > 2R, \\
0, & |\xi| < R,
\end{cases}
$$
is a smooth cut-off function and \( R \) is a constant with \( R > 2 \). Define the frequency cut-off operator \( \chi(D) \) with the symbol \( \chi(\xi) \). The right-hand side of equation (3.5) can be rewritten as

\[
C \int_{\mathbb{R}^3} u^{p-1}(\Lambda^{2-s} - \Gamma_s) u \, dx
= C \int_{\mathbb{R}^3} u^{p-1}(\Lambda^{2-s} - \Gamma_s)_L u \, dx + C \int_{\mathbb{R}^3} u^{p-1}(\Lambda^{2-s} - \Gamma_s)_H u \, dx
:= K_1 + K_2,
\]

where the high frequency part and the low-frequency part are

\[
(\Lambda^{2-s} - \Gamma_s)_L = (I - \chi(D))(\Lambda^{2-s} - \Gamma_s),
\]

and

\[
(\Lambda^{2-s} - \Gamma_s)_H = \chi(D)(\Lambda^{2-s} - \Gamma_s).
\]

According to the properties of \( \Lambda^{2-s} - \Gamma_s \), we have

\[
|K_1| \leq C \|\|\Lambda^{2-s} - \Gamma_s\|_L^p u\|_L^p \leq C \|u\|^{p-1}_{L^p} \leq C \|u\| \leq \frac{\varepsilon}{C} \|u\|^{\frac{p(p-2)}{L^p}}.
\]

By Gagliardo-Nirenberg inequality, one has

\[
\|u\|_{L^p} = \|u^{\frac{2}{p}}\|_{L^2}^{\frac{p}{2}} \leq C \|\Lambda^{1-\frac{s}{2}} u^{\frac{2}{p}}\|_{L^2}^{\frac{4(p-2)}{p-2}} \|u\|_{L^2} \|u\|_{L^p}^{\frac{2}{p-2}},
\]

which implies

\[
\|u\|_{L^p} \leq C \|\Lambda^{1-\frac{s}{2}} u^{\frac{2}{p}}\|_{L^2}^{\frac{4(p-2)}{p-2}} \|u\|_{L^2} \|u\|_{L^p}^{\frac{2}{p-2}}.
\]

Thanks to (3.6) and the Young inequality, we have

\[
|K_1| \leq C \|\|\Lambda^{1-\frac{s}{2}} u^{\frac{2}{p}}\|_{L^2} \|u\|_{L^2} \|u\|_{L^p}^{\frac{2}{p-2}} \leq \frac{\varepsilon}{C} \|\Lambda^{1-\frac{s}{2}} u^{\frac{2}{p}}\|_{L^2}^2 + C(\varepsilon) \|u\|_{L^p}^p.
\]

At last, we turn to estimate \( K_2 \), when \( |\xi| \) is large enough, one has

\[
\sigma(\Lambda^{2-s} - \Gamma_s) = \frac{s}{2} |\xi|^{-s} + \frac{s(\frac{s}{2} + 1)}{2!} |\xi|^{-2s} + O(|\xi|^{-s}).
\]

We rewrite \( K_2 \) as

\[
|K_2| \leq C \int_{\mathbb{R}^3} u^{p-1}(\Lambda^{2-s}_H u) \, dx + C \int_{\mathbb{R}^3} u^{p-1}(\Lambda^{2-s}_H u) \, dx
:= K_{21} + K_{22}.
\]

For \( K_{21} \), thanks to the Cauchy inequality and (3.6), one has

\[
|K_{21}| \leq C \|\|\Lambda^{1-\frac{s}{2}} u^{\frac{2}{p}}\|_{L^2} \|u\|_{L^2} \|u\|_{L^p}^{\frac{2}{p-2}} \leq \frac{\varepsilon}{C} \|\Lambda^{1-\frac{s}{2}} u^{\frac{2}{p}}\|_{L^2}^2 + C(\varepsilon) \|u\|_{L^2}^p.
\]

For \( K_{22} \), since

\[
\|\Lambda^{2-s} - \frac{s}{2} \Lambda^{s} - \Gamma_s\|_{H^1} \leq \mathcal{C} \int_{|\xi| \geq N} \xi^{4-2s} |\xi|^4 \hat{u}^2 \text{d}x \leq \mathcal{C} \|u\|_{L^2}^2
\]

Thus, similarly like estimates of \( K_{21} \), we have

\[
|K_{22}| \leq \frac{\varepsilon}{C} \|\Lambda^{1-\frac{s}{2}} u^{\frac{2}{p}}\|_{L^2}^2 + C(\varepsilon) \|u\|_{L^2}^p.
\]

Collecting estimates of \( K_1, K_{21} \) and \( K_{22} \), one have

\[
\frac{d}{dt} \|u\|_{L^p}^p + |\|\Lambda^{1-\frac{s}{2}} u^{\frac{2}{p}}\|_{L^2}^2 \leq \varepsilon \|\Lambda^{1-\frac{s}{2}} u^{\frac{2}{p}}\|_{L^2}^2 + C(\varepsilon) \|u\|_{L^2}^p.
\]
Therefore, if we choose $\varepsilon = \frac{1}{4}$, then
\[
\frac{d}{dt} \|u\|_{L^p}^p + \frac{1}{2} \|\Lambda^{1-\frac{p}{2}} u\|_{L^2}^2 \leq C \|u\|_{L^p}^p. \tag{3.8}
\]

In the following, we will use the time-frequency decomposition method to get the decay rate of $\|u(t,x)\|_{L^p}$. Denote
\[
\chi(t,\xi) = \begin{cases} 
1, & |\xi| \leq \sigma, \\
0, & |\xi| > 2\sigma,
\end{cases}
\]
where $\sigma^{2-s} = \mu(1+t)^{-\frac{1}{s}}$, and $\mu$ is a constant which will be determined later. Let $\chi(t,D)$ is a time-frequency cue-off operator with symbol $\chi(t,\xi)$. Recall the definition of $\chi(t,D)$, we have
\[
\|\Lambda^{-\frac{p}{2}} u\|_{L^2}^2 \geq \|\Lambda^{-\frac{p}{2}} (\text{Id} - \chi(t,D)) u\|_{L^2}^2 \geq \sigma^{2-s}(\|u\|_{L^p}^p - \|\chi(t,D) u\|_{L^2}^p), \tag{3.9}
\]
and
\[
\|\chi(t,D)(u\hat{\nu})\|_{L^2}^2 = \int \chi^2(t,\xi) |u\hat{\nu}|^2 d\xi \leq C\mu(1+t)^{-3}\sigma^3. \tag{3.10}
\]
Recall the decay rate of $\|u\|_{L^2}$ in Theorem 3.1, by (3.8) – (3.10), one has
\[
\frac{d}{dt} \|u\|_{L^p}^p + \mu(1+t)^{-1} \|u\|_{L^p}^p \leq C\mu(1+t)^{-1}(1+t)^{-\frac{3}{2p}} + C(1+t)^{-\frac{2}{p}},
\]
i.e.,
\[
\frac{d}{dt} ((1+t)^\mu \|u\|_{L^p}^p) \leq C\mu(1+t)^{\mu-1} - \frac{3}{2p} + C(1+t)^{\mu-\frac{2}{p}}.
\]
By integration, we have
\[
\|u\|_{L^p}^p \leq C(1+t)^{-\mu} + C_1(1+t)^{-\frac{3}{2p}} + C_2(1+t)^{-\frac{2}{p}} + 1.
\]
Choosing $\mu$ is large enough, we have
\[
\|u\|_{L^p} \leq C(1+t)^{-\frac{3}{4} + \frac{1}{4}}. \tag{3.11}
\]

4. The decay rate of $\|D_x^\alpha u\|_{L^2}$ and $\|D_x^\alpha u\|_{L^\infty}$. In this section, by the Green's function method we obtain the optimal decay estimates of $\|D_x^\alpha u\|_{L^2}$ and $\|D_x^\alpha u\|_{L^\infty}$. These decay estimates of them are preparations for achieving the pointwise estimates of solutions in Section 5.

**Theorem 4.1.** Suppose that $u_0(x)$ satisfies the assumptions in Theorem 1.1 and $0 \leq s \leq 1$. $u(t,x)$ is a solution of (1.1), then $u(t,x)$ enjoys the following time-decay rate,
\[
\|D_x^\alpha u(t)\|_{L^2} \leq C(1+t)^{-\frac{3+2|\alpha|}{4}},
\]
and
\[
\|D_x^\alpha u(t)\|_{L^\infty} \leq C(1+t)^{-\frac{3+|\alpha|}{4}},
\]
where $|\alpha| < \gamma$. 

4.1. Proof of Theorem 4.1 for subcritical case. For subcritical case, we apply
the induction to the derivatives of the solutions. Since the Green’s function
has singularity at \( \tau = t \), so the induction can only be applied to the \( \frac{1}{2} \) order
derivatives.

**Proof.** By Duhamel principle, the solution of (1.1) can be expressed as,

\[
    u = G(t, x) \ast u_0(x) + \int_0^t G(t - \tau, \cdot) \ast \text{div} f(u(\tau, \cdot))d\tau.
\]

(4.1)

Let \( \Lambda^{\frac{1}{2-t}} = (-\Delta)^{\frac{1}{2}} \), for \( 0 \leq s < 1 \), then we have

\[
    \Lambda^{\frac{1}{2-t}} u = \Lambda^{\frac{1}{2-t}} G(t, x) \ast u_0(x) + \Lambda^{\frac{1}{2-t}} \int_0^t G(t - \tau, \cdot) \ast \text{div} f(u(\tau, \cdot))d\tau := J_1 + J_2.
\]

For the estimates of \( J_1 \), by Theorem 2.2, one has

\[
    \|J_1\|_{L^2} = \|\Lambda^{\frac{1}{2-t}} G(t, \cdot) \ast u_0(\cdot)\|_{L^2} \leq \|\Lambda^{\frac{1}{2-t}} G(t)\|_{L^2} \|u_0\|_{L^2} \leq C(1 + t)^{-\frac{3}{2-s}},
\]

and

\[
    \|J_1\|_{L^\infty} = \|\Lambda^{\frac{1}{2-t}} G(t, \cdot) \ast u_0(\cdot)\|_{L^\infty} \leq \|\Lambda^{\frac{1}{2-t}} G(t)\|_{L^\infty} \|u_0\|_{L^1} \leq C(1 + t)^{-\frac{s-t}{2-t}}.
\]

For the estimates of \( J_2 \), since the singularity of \( G(t, x) \) at \( t = 0 \), we decompose \( J_2 \)
to two parts,

\[
    J_2 \leq \int_0^{\frac{t}{2}} \Lambda^{\frac{1}{2-t}} G(t - \tau) \ast \text{div} f(u)d\tau + \int_{\frac{t}{2}}^t \Lambda^{\frac{1}{2-t}} G(t - \tau) \ast \text{div} f(u)d\tau := J_{21} + J_{22}.
\]

By Theorem 2.2, Theorem 3.1 and \( f_j(u) = u^2 \) (\( j = 1, 2, 3 \)), one has

\[
    \|J_{21}\|_{L^2} \leq \int_0^{\frac{t}{2}} \|\nabla \Lambda^{\frac{1}{2-t}} G(t - \tau)\|_{L^2} \|f(u)\|_{L^4} d\tau \leq C(1 + t)^{-\frac{s-t}{2-t}},
\]

and

\[
    \|J_{21}\|_{L^\infty} \leq \int_0^{\frac{t}{2}} \|\nabla \Lambda^{\frac{1}{2-t}} G(t - \tau)\|_{L^\infty} \|f(u)\|_{L^4} d\tau \leq C(1 + t)^{-\frac{s-t}{2-t}}.
\]

Since the case we considered is \( 0 \leq s < 1 \), by Theorem 2.2, Theorem 3.1, Theorem
3.2, we have

\[
    \|J_{22}\|_{L^2} \leq \int_0^t \|\nabla \Lambda^{\frac{1}{2-t}} G(t - \tau)\|_{L^2} \|f(u)\|_{L^4} d\tau \leq C(1 + t)^{-\frac{t}{2}},
\]

and

\[
    \|J_{22}\|_{L^\infty} \leq \int_0^t \|\nabla \Lambda^{\frac{1}{2-t}} G(t - \tau)\|_{L^\infty} \|f(u)\|_{L^4} d\tau \leq C(1 + t)^{-\frac{t}{2}}.
\]

Combining estimates of \( J_{21} \) and \( J_{22} \), we have

\[
    \|\Lambda^{\frac{1}{2-t}} u(t)\|_{L^2} \leq C(1 + t)^{-\frac{t}{2}}, \quad \|\Lambda^{\frac{1}{2-t}} u(t)\|_{L^\infty} \leq C(1 + t)^{-\frac{s-t}{2-t}}.
\]

(4.2)

For the induction on \( k \), assume that

\[
    \|\left( \Lambda^{\frac{1}{2-t}} \right)^k u\|_{L^2} \leq C(1 + t)^{-\frac{t}{2}}, \quad \|\left( \Lambda^{\frac{1}{2-t}} \right)^k u\|_{L^\infty} \leq C(1 + t)^{-\frac{s-t}{2-t}}.
\]

(4.3)

Then for \( k + 1 \), by Theorem 2.2, Theorem 3.1 and (4.3), we have

\[
    \|\left( \Lambda^{\frac{1}{2-t}} \right)^{k+1} J_1\|_{L^2} \leq C(1 + t)^{-\frac{k+1}{2-t}} \leq C(1 + t)^{-\frac{k+1}{2-t}}.
\]
and
\[ \| (\Lambda^{k+\alpha})^{k+1} J_1 \|_{L^\infty} \leq \| (\Lambda^{k+\alpha})^{k+1} G(t) \|_{L^\infty} \| u_0 \|_{L^1} \leq C(1+t)^{-3+\frac{3k-1+\alpha}{2}}. \]

For estimates of \( J_2 \), one has
\[ \| (\Lambda^{k+\alpha})^{k+1} J_2 \|_{L^2} \leq \int_0^T \| (\Lambda^{k+\alpha})^{k+1} \nabla G(t-\tau) \|_{L^2} \| f(u(\tau)) \|_{L^1} d\tau \]
\[ \leq C \int_0^T (1+t-\tau)^{-\frac{4+1+\alpha}{2}(k+1)} \tau^{-\frac{3}{2}} (1+\tau)^{-\frac{3}{2}} d\tau \]
\[ \leq C(1+t)^{-\frac{6+\alpha}{4} + \frac{(1+\alpha)k}{4}}, \]
and
\[ \| (\Lambda^{k+\alpha})^{k+1} J_2 \|_{L^\infty} \leq \int_0^T \| (\Lambda^{k+\alpha})^{k+1} \nabla G(t-\tau) \|_{L^\infty} \| f(u(\tau)) \|_{L^1} d\tau \]
\[ \leq C \int_0^T (1+t-\tau)^{-\frac{3+\alpha}{2\alpha+1}} (1+\tau)^{-\frac{3}{2}} (1+\tau)^{-\frac{3}{2}} d\tau \]
\[ \leq C(1+t)^{-\frac{3}{2}}. \]

For \( J_2^2 \), one can get
\[ \| (\Lambda^{k+\alpha})^{k+1} J_2^2 \|_{L^2} \leq \int_0^T \| (\Lambda^{k+\alpha})^{k+1} \nabla G(t-\tau) \|_{L^2} \| f(u(\tau)) \|_{L^2} d\tau \]
\[ \leq C \int_0^T (t-\tau)^{-\frac{3+\alpha}{2\alpha+1}} (1+\tau)^{-\frac{3}{2}} (1+\tau)^{-\frac{3}{2}} d\tau \]
\[ \leq C(1+t)^{-\frac{3}{2}}, \]
and
\[ \| (\Lambda^{k+\alpha})^{k+1} J_2^2 \|_{L^\infty} \leq \int_0^T \| (\Lambda^{k+\alpha})^{k+1} \nabla G(t-\tau) \|_{L^\infty} \| f(u(\tau)) \|_{L^\infty} d\tau \]
\[ \leq C \int_0^T (t-\tau)^{-\frac{3+\alpha}{2\alpha+1}} (1+\tau)^{-\frac{3}{2}} (1+\tau)^{-\frac{3}{2}} d\tau \]
\[ \leq C(1+t)^{-\frac{3}{2}}. \]

Combining the above inequalities, by the mathematical induction method, we have the decay estimate (4.3) is valid for any nonnegative integer \( k \). In this case, using (4.3), we continue the iteration finitely many times to obtain
\[ \| D^\alpha u \|_{L^2} \leq C(1+t)^{-\frac{3}{2}}, \quad \| D^\alpha u \|_{L^\infty} \leq C(1+t)^{-\frac{3}{2}}. \] (4.4)

From now on, we use (4.4) to improve the decay rates of the derivatives of \( u(t,x) \) in \( L^2 \) space and \( L^\infty \) space. The decay rate of \( J_1 \) is already optimal
\[ \| D^\alpha J_1 \|_{L^2} = \| D^\alpha G(t,\cdot) * u_0(\cdot) \|_{L^2} \leq \| D^\alpha G(t) \|_{L^2} \| u_0 \|_{L^1} \leq C(1+t)^{-\frac{3+\alpha}{4}}, \]
and
\[ \| D^\alpha J_1 \|_{L^\infty} = \| D^\alpha G(t,\cdot) * u_0(\cdot) \|_{L^\infty} \leq \| D^\alpha G(t) \|_{L^\infty} \| u_0 \|_{L^1} \leq C(1+t)^{-\frac{3+\alpha}{2}}. \]
In the following we only need to improve the decay rates of $J_{21}$ and $J_{22}$,
\[
\|D_x^\alpha J_{21}\|_{L^2} \leq C \int_0^t \|\nabla D_x^\alpha G(t-\tau)\|_{L^2} \|f(u)\|_{L^1} d\tau \\
\leq C \int_0^t (1 + t - \tau)^{-\frac{2(|\alpha|+5)}{4}} (1 + \tau)^{-\frac{1}{4}} d\tau \\
\leq C(1 + t)^{-\frac{2(|\alpha|+5)}{4}},
\]
and
\[
\|D_x^\alpha J_{21}\|_{L^\infty} \leq C \int_0^t \|\nabla D_x^\alpha G(t-\tau)\|_{L^\infty} \|f(u)\|_{L^1} d\tau \\
\leq C \int_0^t (1 + t - \tau)^{-\frac{2(|\alpha|+4)}{4}} (1 + \tau)^{-\frac{1}{4}} d\tau \\
\leq C(1 + t)^{-\frac{2(|\alpha|+4)}{4}}.
\]
For estimates of $J_{22}$,
\[
\|D_x^\alpha J_{22}\|_{L^2} \leq C \int_0^t \|\nabla \Lambda^{\frac{|\alpha|}{2}} G(t-\tau)\|_{L^1} \|D_x^\alpha \Lambda^{-\frac{|\alpha|}{2}} f(u)\|_{L^2} d\tau \\
\leq C \int_0^t (t - \tau)^{-\frac{1+ \frac{|\alpha|}{2}}{2}} (1 + \tau)^{-\frac{1}{4}} (1 + \tau)^{-\frac{1}{4}} d\tau \\
\leq C(1 + t)^{-\frac{1}{4}},
\]
and
\[
\|D_x^\alpha J_{22}\|_{L^\infty} \leq C \int_0^t \|\nabla \Lambda^{\frac{|\alpha|}{2}} G(t-\tau)\|_{L^1} \|D_x^\alpha \Lambda^{-\frac{|\alpha|}{2}} f(u)\|_{L^\infty} d\tau \\
\leq C \int_0^t (t - \tau)^{-\frac{1+ \frac{|\alpha|}{2}}{2}} (1 + \tau)^{-\frac{1}{4}} (1 + \tau)^{-\frac{1}{4}} d\tau \\
\leq C(1 + t)^{-\frac{1}{4}}.
\]
Combing estimates of $J_{21}$ and $J_{22}$, when $|\alpha| > 1$, we have
\[
\|D_x^\alpha u\|_{L^2} \leq C(1 + t)^{-\frac{1}{4}}, \quad \|D_x^\alpha u\|_{L^\infty} \leq C(1 + t)^{-\frac{1}{4}}. \tag{4.5}
\]
By (4.5), we continue the iteration finitely many times to obtain the optimal decay estimates of $\|D_x^\alpha u\|_{L^2}$ and $\|D_x^\alpha u\|_{L^\infty}$,
\[
\|D_x^\alpha u\|_{L^2} \leq C(1 + t)^{-\frac{3+2|\alpha|}{4}}, \quad \|D_x^\alpha u\|_{L^\infty} \leq C(1 + t)^{-\frac{3+|\alpha|}{4}}.
\]

4.2. Proof of Theorem 4.1 for critical case. Before proceed to the proof of Theorem 4.1, we need the following lemma to deal with the critical case. As mentioned in Section 2, $s = 1$ is a critical case of $\tilde{G}(\xi, t)$. It is necessary for us to derive the decay rate of $\|\Lambda^{\frac{1}{2}} u\|_{L^\infty}$ at first, let us note that $0 < r \ll 1$, which was obtained by the global existence theorem.

**Lemma 4.1.** Suppose that $u(t, x)$ is a solution of (1.1) for $s = 1$, then $u(t, x)$ enjoys the following time-decay rate,
\[
\|\Lambda^{\frac{1}{2}} u(t)\|_{L^\infty} \leq C(1 + t)^{-\frac{1}{2}}, \tag{4.6}
\]
and
\[ \| \Lambda^\varphi u(t) \|_{L^2} \leq C(1 + t)^{-\frac{3}{4}}. \]

**Proof.** Note that
\[ \Lambda^\varphi u = \Lambda^\varphi G(t, x) * u_0(x) + \int_0^t \Lambda^\varphi G(t - \tau) * \text{div} f(u) d\tau := \tilde{R}_1 + \tilde{R}_2. \]

For \( \tilde{R}_1 \), we have
\[ \| \tilde{R}_1 \|_{L^\infty} = \| \Lambda^\varphi G(t, \cdot) * u_0(\cdot) \|_{L^\infty} \leq \| \Lambda^\varphi G(t, \cdot) \|_{L^\infty} \| u_0 \|_{L^1} \leq C(1 + t)^{-\frac{3}{4}}. \]

For \( \tilde{R}_2 \), one has
\[ \tilde{R}_2 = \left( \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t-1} + \int_{t-1}^t \right) \Lambda^\varphi G(t - \tau) * \text{div} f(u) d\tau := \tilde{R}_{21} + \tilde{R}_{22} + \tilde{R}_{23}. \]

Thanks to Theorem 2.1 and Theorem 3.1, one has
\[
\| \tilde{R}_{21} \|_{L^\infty} \leq \int_0^{\frac{t}{2}} \| \Lambda^\varphi \nabla G(t - \tau) \|_{L^\infty} \| f(u) \|_{L^1} d\tau \\
\leq \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\frac{3}{4} - \frac{3}{8}} (1 + \tau)^{-\frac{3}{4}} d\tau \\
\leq C(1 + t)^{-\frac{3}{4}}. \\
\| \tilde{R}_{22} \|_{L^\infty} \leq \int_{\frac{t}{2}}^{t-1} \| \Lambda^\varphi \nabla G(t - \tau) \|_{L^\infty} \| f(u) \|_{L^1} d\tau \\
\leq \int_{\frac{t}{2}}^{t-1} (1 + t - \tau)^{-2 - \frac{3}{4}} (1 + \tau)^{-\frac{3}{4}} d\tau \\
\leq C(1 + t)^{-\frac{3}{4}}, \\
\| \tilde{R}_{23} \|_{L^\infty} \leq \int_{t-1}^t \| \Lambda^{1-\varphi} G(t - \tau) \|_{L^1} \| \Lambda^\varphi f(u) \|_{L^\infty} d\tau, \\
\leq C \int_{t-1}^t (t - \tau)^{-1 + \frac{3}{4}} (1 + \tau)^{-\frac{3}{4}} d\tau \\
\leq C(1 + t)^{-\frac{3}{4}},
\]

where we have used the fact that
\[ \| \Lambda^\varphi f(u) \|_{L^\infty} \leq C \| u \|_{L^\infty} \| \Lambda^\varphi u \|_{L^\infty}. \]

Combining estimates of \( \tilde{R}_{21}, \tilde{R}_{22} \) and \( \tilde{R}_{23} \), we have
\[ \| \Lambda^\varphi u \|_{L^\infty} \leq C(1 + t)^{-\frac{3}{4}}. \]

Similarly, we can also obtain the decay estimates of \( \| \Lambda^\varphi u \|_{L^2} \),
\[ \| \Lambda^\varphi u \|_{L^2} \leq C(1 + t)^{-\frac{3}{4}}. \]

Similarly, we can also get the following decay estimates
\[ \| (\Lambda^\varphi)^k u \|_{L^\infty} \leq C(1 + t)^{-\frac{3}{4}}, \]
and

\[ \| (\Lambda^\frac{k}{2}) u \|_{L^2} \leq C(1 + t)^{-\frac{3}{4}}, \]

where \( k \) is a nonnegative integer. Next, we will give the proof of Theorem 4.1 for critical case.

Proof. By Duhamel principle, the solution of (1.1) can be expressed as,

\[ D^\alpha_x u(t, x) = D^\alpha_x G(t, x) * u_0(x) + D^\alpha_x \int_0^t G(t - \tau) * \text{div} f(u) d\tau. \tag{4.7} \]

The decay estimates of the first term in (4.7) is already optimal,

\[ \| D^\alpha_x G(t, \cdot) * u_0(\cdot) \|_{L^2} \leq \| D^\alpha_x G(t, \cdot) \|_{L^2} \| u_0 \|_{L^1} \leq C(1 + t)^{-\frac{3+2|\alpha|}{4}}, \]

and

\[ \| D^\alpha_x G(t, \cdot) * u_0(\cdot) \|_{L^\infty} \leq \| D^\alpha_x G(t, \cdot) \|_{L^\infty} \| u_0 \|_{L^1} \leq C(1 + t)^{-\frac{3+|\alpha|}{2}}. \]

We only need to improve the decay rates of second term in (4.7), decomposing it into there parts,

\[ \int_0^t D^\alpha_x G(t - \tau, \cdot) * \text{div} f(u) d\tau = \int_0^{\frac{t}{2}} D^\alpha_x G(t - \tau, \cdot) * \text{div} f(u) d\tau + \int_{\frac{t}{2}}^t D^\alpha_x G(t - \tau, \cdot) * \text{div} f(u) d\tau \]

\[ := W_{21} + W_{22}. \]

Note that we always assume \( t \geq 2 \). For the estimate of \( W_{21} \), by Theorem 2.1 and Theorem 3.1, we have

\[ \| W_{21} \|_{L^2} \leq \int_0^{\frac{t}{2}} \| D^\alpha_x (t - \tau) \|_{L^2} \| f(u) \|_{L^1} d\tau \]

\[ \leq C \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\frac{2|\alpha|+5}{4}} (1 + \tau)^{-\frac{3}{4}} d\tau \]

\[ \leq C(1 + t)^{-\frac{2|\alpha|+5}{4}}. \]

\[ \| W_{21} \|_{L^\infty} \leq \int_0^{\frac{t}{2}} \| D^\alpha_x G(t - \tau) \|_{L^\infty} \| f(u) \|_{L^1} d\tau \]

\[ \leq C \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\frac{|\alpha|+4}{2}} (1 + \tau)^{-\frac{3}{4}} d\tau \]

\[ \leq C(1 + t)^{-\frac{|\alpha|+4}{2}}. \]

For the estimate of \( W_{22} \), by Theorem 2.1 and Theorem 3.1, we have

\[ \| W_{22} \|_{L^2} \leq \int_0^{\frac{t}{2}} \| \Lambda^{1-\frac{5}{2}} G(t - \tau) \|_{L^1} \| D^\alpha_x \Lambda^\frac{5}{2} f(u) \|_{L^2} d\tau \]

\[ \leq C \int_0^{\frac{t}{2}} (t - \tau)^{-1+\frac{5}{2}} (1 + \tau)^{-\frac{3}{4}} (1 + \tau)^{-\frac{3}{4}} d\tau \]

\[ \leq C(1 + t)^{-1}, \]
and

\[ \|W_{22}\|_{L^\infty} \leq \int_{\frac{t}{2}}^{t} \|\Lambda^{1-\frac{3}{2}} G(t-\tau)\|_{L^1} \|D_x^\alpha \Lambda^{\frac{3}{2}} f(\tau)\|_{L^\infty} \, d\tau \]
\[ \leq C \int_{\frac{t}{2}}^{t} (t-\tau)^{-1+\frac{3}{2}} (1+\tau)^{-\frac{3}{2}} \, d\tau \]
\[ \leq C(1+t)^{-\frac{3}{2}}. \]

We continue the iteration finitely many times to obtain the optimal decay estimates of \(\|D_x^\alpha u\|_{L^2}\) and \(\|D_x^\alpha u\|_{L^\infty}\).

\[ \|D_x^\alpha u\|_{L^2} \leq C(1+t)^{-\frac{3+2|\alpha|}{4}}, \]

and

\[ \|D_x^\alpha u\|_{L^\infty} \leq C(1+t)^{-\frac{3+2|\alpha|}{2}}. \]

5. **The pointwise estimates of solutions.** By (4.1), we have

\[ D_x^\alpha u(t, x) = D_x^\alpha G(t, x) * u_0(x) + D_x^\alpha \int_{0}^{t} G(t-\tau) * \text{div} f(\tau) \, d\tau := I_1 + I_2. \]

We estimate the two terms \(I_1\) and \(I_2\) respectively, according to assumptions of the initial data \(u_0(x)\), one has

\[ |I_1| = \left| \int_{\mathbb{R}^3} (1+t)^{-\frac{3+|\alpha|}{4}} B_N(t, x-y-bt)(1+|y|)^{-\tau} \, dy \right| \leq (1+t)^{-\frac{3+2|\alpha|}{4}} B_N(t, x - bt). \]

For the nonlinear term, we denote

\[ \phi_\alpha(t, x) = (1+t)^{-\frac{3+|\alpha|}{4}} B_N(t, x - bt)^{-1}. \]
\[ M(t) = \sup_{0 \leq \tau \leq t, 0 \leq |\alpha| < \gamma, x \in \mathbb{R}^3} |D_x^\alpha u(\tau, x)| \phi_\alpha(\tau, x), \]

thus,

\[ |D_x^\alpha u(t, x)| \leq M(t)(1+t)^{-\frac{3+2|\alpha|}{4}} B_N(t, x - bt). \]

For the pointwise estimate of \(I_2\), one has

\[ I_2 = \int_{0}^{t} D_x^\alpha G(t-\tau) * \text{div} f(\tau) \, d\tau \]
\[ = \int_{\frac{t}{2}}^{t} D_x^\alpha G(t-\tau) * \text{div} f(\tau) \, d\tau + \int_{\frac{t}{2}}^{t} G(t-\tau) * \text{div} D_x^\alpha f(\tau) \, d\tau := I_{21} + I_{22}. \]
Before we proceed to obtain the estimates of $|D_x^2 u^2|$, where $|\tilde{\alpha}| = |\alpha| + 1$,

$$|D_x^2 u^2| = \sum_{|\beta| < |\tilde{\alpha}|} \left( \tilde{\alpha}_{\beta} \right) |D_x^2 u||D_x^{\tilde{\alpha} - \beta} u| + |D_x^2 u||u|$$

$$\leq CM(t)\phi_\beta(t, x)||D_x^{\tilde{\alpha} - \beta} u||_{L^\infty} + C\|D_x^\tilde{\alpha} u\|_{L^\infty} M(t)\phi_0(t, x)$$

$$\leq CM(t)(1 + t)^{-\frac{3+|\beta|}{2}} B_N(t, x - bt)(1 + t)^{-\frac{3+|\tilde{\alpha} - |\beta||}{2}}$$

$$+ CM(t)(1 + t)^{-\frac{3+|\beta|}{2}} (1 + t)^{-\frac{3}{2}} B_N(t, x - bt)$$

$$\leq CM(t)(1 + t)^{-\frac{6+|\alpha|}{2}} B_N(t, x - bt).$$

According to the definition of $B_N(t, x)$, we will decompose $\mathbb{R}^3$ into two parts to obtain the pointwise estimates of $I_2$.

**Case 1.** When $|x - bt|^2 \leq t$,

$$|I_{21}| = \int_0^t |D_x^2 G(t - \tau) \ast \text{div} f(u)| d\tau \leq \int_0^t \int_{\mathbb{R}^3} |\nabla D_x^2 G(t - \tau, x - y)f(u)(\tau, y)| dy d\tau$$

$$\leq CM(t) \int_0^t (1 + t - \tau)^{-\frac{4+|\alpha|}{2}} (1 + \tau)^{-\frac{3}{2}} (1 + \tau)^{-\frac{3}{2}} \int_{\mathbb{R}^3} B_N(\tau, y - \tau) dy d\tau$$

$$\leq CM(t) \int_0^t (1 + t - \tau)^{-\frac{4+|\alpha|}{2}} (1 + \tau)^{-\frac{3}{2}} d\tau \leq CM(t)(1 + t)^{-\frac{4+|\alpha|}{2}}.$$

$$|I_{22}| \leq \int_\frac{t}{2}^t |D_x^2 G(t - \tau, \cdot) \ast \text{div} f(u)| d\tau \leq \int_\frac{t}{2}^t \int_{\mathbb{R}^3} |\nabla G(t - \tau, x - y) D_x^\alpha f(u)(\tau, y)| dy d\tau$$

$$\leq CM(t) \int_\frac{t}{2}^t \int_{\mathbb{R}^3} (t - \tau)^{-\frac{|\alpha|}{2}} B_N, (t - \tau, x - y - b(t - \tau))(1 + \tau)^{-\frac{6+|\alpha|}{2}} dy d\tau$$

$$\leq CM(t) \int_\frac{t}{2}^t (t - \tau)^{-\frac{|\alpha|}{2}} (1 + \tau)^{-\frac{6+|\alpha|}{2}} d\tau \leq CM(t)(1 + t)^{-\frac{6+|\alpha|}{2}}.$$

Since $|x - bt|^2 \leq t$, we have

$$1 \leq 2^N(1 + \frac{|x - bt|^2}{t})^{-N} = 2^N B_N(t, x - bt).$$

Thus, when $|x - bt|^2 \leq t$, we obtain

$$|I_2| \leq CM(t)(1 + t)^{-\frac{4+|\alpha|}{2}} B_N(t, x - bt)$$

**Case 2.** When $|x - bt|^2 > t$, we have

$$\left(1 + \frac{|x - y - b(t - \tau)|^2}{1 + t - \tau}\right)^{-N} \left(1 + \frac{|y - bt|^2}{1 + \tau}\right)^{-N}$$

$$\leq \begin{cases} C(1 + \frac{|x - bt|^2}{1 + t - \tau})^{-N} (1 + \frac{|y - bt|^2}{1 + \tau})^{-N}, & |x - bt| > \frac{|y - bt|^2}{2}, \\ C(1 + \frac{|x - y - b(t - \tau)|^2}{1 + t - \tau})^{-N} (1 + \frac{|x - bt|^2}{1 + \tau})^{-N}, & |x - bt| < \frac{|y - bt|^2}{2}. \end{cases}$$
Therefore, by direct calculation, then we have

\[
|I_{21}| = \int_0^2 |D_x^2 G(t - \tau, \cdot) * \text{div} f(u)| \, d\tau \\
\leq \int_0^2 \left( \int_{D_1} + \int_{D_2} \right) \left| \nabla D_x^2 G(t - \tau, x - y) f(u(\tau, y)) \right| \, dy \, d\tau \\
\leq CM(t) \int_0^2 \left( \int_{D_1} + \int_{D_2} \right) (1 + \tau)^{-\frac{\alpha + 1}{2}} (1 + \tau)^{-3} (1 + \frac{|x - bt|^2}{1 + t - \tau})^{-N} (1 + \frac{|y - br|^2}{1 + t - \tau})^{-N} \, dy \, d\tau \\
+ \int_0^2 \int_{D_2} (1 + \tau)^{-\frac{\alpha + 1}{2}} (1 + \tau)^{-3} (1 + \frac{|x - y - b(t - \tau)|^2}{1 + t - \tau})^{-N} (1 + \frac{|x - bt|^2}{1 + t - \tau})^{-N} \, dy \, d\tau \\
\leq CM(t)(1 + t)^{-\frac{\alpha + 1}{2}} B_N(t, x - bt),
\]

where \( D_1 = \{|x - bt| \geq \frac{|y - br|}{2} \} \) and \( D_2 = \{|x - bt| < \frac{|y - br|}{2} \} \). Similarly, one can get

\[
|I_{22}| = \int_0^2 |G(t - \tau, \cdot) * \text{div} D_x^2 f(u)| \, d\tau \\
\leq \int_0^2 \left( \int_{D_1} + \int_{D_2} \right) |G(t - \tau, x - y - b(t - \tau)) \text{div} D_x^2 f(u(\tau, y - br))| \, dy \, d\tau \\
\leq CM(t) \int_0^2 \left( \int_{D_1} + \int_{D_2} \right) (t - \tau)^{-\frac{\alpha + 1}{2}} (1 + \frac{|x - bt|^2}{1 + t - \tau})^{-N} (1 + \frac{|y - br|^2}{1 + t - \tau})^{-N} \, dy \, d\tau \\
+ \int_0^2 \int_{D_2} (t - \tau)^{-\frac{\alpha + 1}{2}} (1 + \tau)^{-\frac{\alpha + 1}{2}} (1 + \frac{|x - y - b(t - \tau)|^2}{1 + t - \tau})^{-N} (1 + \frac{|x - bt|^2}{1 + t - \tau})^{-N} \, dy \, d\tau \\
\leq CM(t)(1 + t)^{-\frac{\alpha + 1}{2}} B_N(t, x - bt).
\]

Combining the estimates of \( I_{21} \) and \( I_{22} \), one can get estimates of \( I_2 \),

\[
|I_2| \leq CM(t)(1 + t)^{-\frac{\alpha + 1}{2}} B_N(t, x - bt).
\]

Therefore, we obtain the following estimates,

\[
|u(t, x)| \leq C(1 + t)^{-\frac{\alpha + 1}{2}} B_N(t, x - bt) + CM(t)(1 + t)^{-\frac{\alpha + 1}{2}} B_N(t, x - bt),
\]

according to the definition of \( M(t) \), then we have

\[
M(t) \leq C + C(1 + t)^{-\frac{1}{2}} M(t). \quad (5.1)
\]

There exists a \( T \) such that when \( t > T \), we have \( C(1 + t)^{-\frac{1}{2}} < \frac{1}{2} \). Thus, by (5.1), we have

\[
M(t) \leq 2C, \quad t > T.
\]

For \( t \leq T \), if \( M(t) \) is continuous, then \( M(t) \) is bounded. Indeed \( M(t) \) is continuous. This is because, we have

\[
\|D_x^2 u\|_{L^\infty} \leq C, \quad |\beta| = |\alpha| + 2.
\]

Thus, it is easy to get

\[
\partial_t \|D_x^2 u\|_{L^\infty} \leq C.
\]

Hence \( D_x^2 u \) is continuous in \( t \). Together with the definition of \( M(t) \), we see that \( M(t) \) is continuous. Thus, \( M(t) \) is bounded on \([0, T]\). Consequently, Theorem 1.1 is proved.
And furthermore, by the pointwise estimates we obtain the decay estimates of $D^\alpha_x u$ in $L^p$ norm,

**Corollary 1.** Suppose that $0 \leq s \leq 1$ and $u_0 \in L^1(\mathbb{R}^3) \cap C^\gamma(\mathbb{R}^3)$, $(\gamma > 1)$, $u(t, x)$ is the solution of equation (1.1), then the obtained solution enjoys the time-decay estimates

$$\|D^\alpha_x u(t)\|_{L^p} \leq C(1 + t)^{-\frac{|\alpha| - 2}{2} \left(1 - \frac{1}{p}\right)}, \quad |\alpha| < \gamma.$$

6. **Appendix.** The Global existence of the Cauchy problem (1.1) with large initial data in three spatial dimensions is obtained in [38]. Since this article is written in Chinese, for the convenience of readers, we put the proof of the global existence as an appendix to the end of this article.

It is well known that global existence follows from the local existence and uniform-in-time a priori estimates as well as the continuity argument. First of all, we give the local existence theorem of (1.1).

**Theorem 6.1.** Suppose that $u_0(x) \in L^1 \cap C^r(\mathbb{R}^3)$, $r > 1$. Then there exists a positive constant $\delta_0$, depending only on $\|u_0\|_{C^r}$, so that the Cauchy problem (1.1) admits a unique solution $u(t, x)$, and furthermore, $u(t, x) \in L^\infty([0, 2\delta_0]; C^r(\mathbb{R}^3))$.

The proof of this theorem is standard, for instance [21], is thus omitted for simplicity.

Secondly, by the maximum principle, we obtain the boundedness of $\|u\|_{L^p}$ for $p \in [1, +\infty]$.

**Theorem 6.2.** Suppose that $u_0(x) \in L^1 \cap L^2(\mathbb{R}^3)$, and $u(t, x)$ be a smooth solution of (1.1) with $0 \leq s \leq 1$, then

$$\|u(t)\|_{L^p} \leq C\|u_0\|_{L^p},$$

with $1 \leq p \leq +\infty$.

Thirdly, for the critical case (i.e. $s = 1$), we need not only the boundedness of $\|u\|_{L^p}$ but also the $C^\alpha$ estimates of $u(t, x)$.

**Theorem 6.3.** Suppose the initial data $u_0$ belongs to $C^\alpha$, then if $\alpha$ is small enough, we have $u \in C^\alpha$.

The proof of this theorem relies on the nonlinear maximal principle developed by Constantin and Vicol [6]. However, according to the expression and the derivative of the two kernels for $\Lambda^\alpha$ and $\Gamma_s$, a major difference between them is that the positive lemma is established for the first operator, but not for the second one. Thus, the major difficulty here is how to deal with the pseudo-differential operator $\Gamma_s$ without the positive lemma. Please refer to [38] for details.

Consequently, with the help of Theorem 6.2, Theorem 6.3 and the following theorem, we improved the regularity of the solution to (1.1).

**Theorem 6.4.** Suppose that $u(t, x)$ be a solution to (1.1) with initial data $u_0(x) \in L^1(\mathbb{R}^3) \cap C^\gamma(\mathbb{R}^3)$, $(\gamma > 1)$, and

(i) when $0 \leq s < 1$, if

$$u(t, x) \in L^\infty([0, +\infty), L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)),$$
When \( s = 1 \), we further assume that there exists a small constant \( \alpha > 0 \), so that
\[
 u(t,x) \in L^\infty([0, +\infty), L^1(\mathbb{R}^3) \cap C^\alpha(\mathbb{R}^3)),
\]
then we have \( u(t,x) \in L^\infty([0, +\infty), C^{\gamma}(\mathbb{R}^3)) \).

The proof of Theorem 6.4 is based on the Green’s function, the Littlewood-Paley decomposition as well as the open-end high frequency estimate. We refer [38] for detailed proof. Therefore, the global existence of equation (1.1) was obtained by the above four theorems.

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