Parity-time-symmetric vector rational rogue wave solutions in any $n$-component nonlinear Schrödinger models

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The extreme events are investigated for an $n$-component nonlinear Schrödinger ($n$-NLS) system in the focusing Kerr-like nonlinear media, which appears in many physical fields. We report and discuss the novel multi-parametric families of vector rational rogue wave (RW) solutions featuring the parity-time ($PT$) symmetry, which are characterized by non-identical boundary conditions for the components, and consistent with the degeneracy of $n$ branches of Benjamin-Feir instability. Explicit examples of $PT$-symmetric vector RWs are presented. Some parameter constraints can make some components generate the RWs with high amplitudes due to many-body resonant interactions. Effect of a non-integrable deformation of the model on the excitation of vector RWs is also discussed. These results will be useful to design the RW experiments in multi-component physical systems.

Introduction.—Recently, rogue waves (RWs), as a special type of nonlinear waves, have been paid more and more attention to. RWs (alias freak or killer waves) are extreme events that were originally known as deep ocean waves [1, 2] (see also Refs. [3–5] and references therein). Nowadays RWs are recognized to be a ubiquitous phenomenon that can be observed in nonlinear physical systems of diverse nature including optical fibers [6–12], arrays of waveguides [13], cavities [14], resonators [15], superfluids [16], Bose-Einstein condensates (BECs) [17, 18], models of atmospheric physics [19, 20], plasmas [21, 22], and even finance [23]. When RWs are characterized by a scalar field, a paradigmatic physical model describing them is the NLS equation or its generalizations. If a system is characterized by two or more (say, $n$) components, like for example, multi-mode nonlinear waveguides [24, 25] or multi-component BECs [26], the governing equations are often reduced to a two- or $n$-component NLS system (alias $n$-NLS system) [27].

It is well established that occurrence of RWs is intimately related to the phenomenon of the modulation instability (MI) [4, 18, 28–32]. A focusing scalar NLS equation admits one branch of the spectrum featuring MI (or briefly one branch of MI), which in its turn originates a single type of rational RWs, known as the Peregrine soliton [33]. Such a solution is localized both in time and in space [34, 35], and has been experimentally observed in nonlinear optical fibers [36, 37], plasma [38], and a water tank [39]. Meantime, already 2- and 3-component NLS equations may possess several branches of MI. Since with each branch of MI one can relate a RW solution, one can expect that in a system with several field components different branches of MI can give origin to distinct RWs. Such vector RWs in 2- and 3-component systems were extensively studied [18, 23, 29, 32, 37–41]. An $n$-NLS system generally admits $n$ distinct branches of MI. Since with each branch one can relate a RW, a natural issue arises, on whether $n$ distinct branches of MI could collapse, thus manifesting degeneracy of MI, and thus result in a generation of a RW with greatly enhanced amplitude. A general answer to this issue still remains open, although a few examples were reported [37].

In this paper, we would like to tackle the above-mentioned open and interesting issue in a general case, and for the first time find the analytical $n$-vector RWs, addressing the relation between the degenerated MI and vector RWs in the $n$-NLS equation, in which one of key and difficult points approaching the problem requires to solve an $(n + 1)$-th-order polynomial equation explicitly. Furthermore, we find that the obtained $n$-vector RWs feature the physically interesting parity-time ($PT$) symmetry, resembling the scalar time-reversal RWs previously reported in [42, 43], and the vector RWs can generate the high amplitudes for some parameter choices.

The $n$-NLS models and existence of vector RWs.—We consider the wave propagation of $n$ self-trapped beam in a slow media with Kerr-like nonlinearity, a dimensionless $n$-NLS system for a complex-valued $n$-component column-vector field $\mathbf{q}(x, t) = (q_1(x, t), \ldots, q_n(x, t))^T$ [27, 44–48]

$$\frac{\partial |\mathbf{q}|^2}{\partial t} + \frac{1}{2} \frac{\partial^2 |\mathbf{q}|^2}{\partial x^2} + |\mathbf{q}|^2 \mathbf{q} = 0,$$

where $|\mathbf{q}|^2 = |q_1|^2 + \cdots + |q_n|^2$ stands for the norm of the column-vector $\mathbf{q}$ and the superscript $T$ stands for trans-
introduces free parameters with constant coefficients: \( \Phi_c \) the non-zero constant column-vector which are represented by a quotient of two polynomials of straightforward obtaining conjugation via the Darboux transform (see e.g., [51–54]) for the respective RW is obtained by computing \( \Psi \) which corresponds to the \((n+1)\)-multiple eigenvalue \( i\omega_0 - \lambda_0 = (1-n)/2 \) of \( H \), and its substitution into the Darboux transform. The fundamental solution determined by \( \Phi(0,0) = I_{n+1} \), and hence represented as \( \Psi = G \Phi c \), generates a family of vector rational RWs \( 50 \)

\[
q_j(x,t) = q_{0j} - 4i \text{Im}(\lambda_0)(\Psi^d/(\Psi^a))_{j+1,1} + \frac{i^j}{j} \text{ being the Hermitian conjugation via the Darboux transform (see e.g., [51–54])},
\]

one needs to find the auxiliary vector function of the Lax pair \( 48 \): \( \Psi = G \Phi c, G = \text{diag}(1,e^{i\varphi_1},\ldots,e^{i\varphi_n}) \), the non-zero constant column-vector \( c = (c_0,c_1,\ldots,c_n)^T \) introduces free parameters \( c_j \) (as shown below they characterize RWs), and \( \Phi \) is an \((n+1)\)-order square matrix solution of the gauge-transformed Lax pair with constant coefficients: \( \Phi_0 = iH\Phi \) and \( \Phi_1 = i\frac{1}{\lambda_0}H + \lambda_0 H - \left[ a_0^T + \frac{1}{\lambda_0} \right] I_{n+1} \Phi \), where \( H = \lambda_0 \sigma_3 + \begin{pmatrix} a & -K \\ 0 & -a^T \end{pmatrix} \) with \( K = \text{diag}(a_1,\ldots,a_n) \) being a constant matrix, and \( \lambda_0 \in \mathbb{C} \) the spectral parameter. Thus, the construction of a desired solution \( q(x,t) \) is reduced to solving the eigenvalue problem. Representing (for convenience) an eigenvalue of \( H \) as \( i\omega_0 - \lambda_0 \), i.e., regarding \( \chi \) as an eigenvalue, one obtains the characteristic equation for \( H \) in the form \( 51 \)

\[
P(\chi) := 2\lambda_0 - i\chi + \sum_{j=1}^{n} \frac{a_j^2}{i\chi + k_j} = 0.
\]

We are interested in vector rational RWs of Eq. \( 1 \), which are represented by a quotient of two polynomials of \( x \) and \( t \) [like in Eqs. \( 2 \) and \( 3 \) shown below \( 52 \)]. Therefore we seek matrix solutions of the above-mentioned equations that feature algebraic dependence on the spatial (and temporal) variables. This implies that roots of Eq. \( 2 \) determining such solutions must have multiplicity bigger than one. Finding such roots is a nontrivial task because effectively Eq. \( 2 \) is an algebraic equation of order \((n+1)\), whose roots of high multiplicity may even do not exist in a general case. Furthermore, here we consider the roots of largest possible, i.e., \( n+1 \) multiplicity. Thus, in addition to Eq. \( 2 \) we require

\[
P(\chi) := \delta_{k\ell} + \sum_{j=1}^{n} \frac{a_j^2}{i\chi + k_j} = 0.
\]

where \( s = 1,\ldots,n \) and \( \delta_{ij} \) is the Kronecker delta. Since straightforward obtaining \( \chi \) from Eqs. \( 3 \) and \( 4 \) seems to be impossible, we reformulate the problem and focus on finding the non-zero amplitudes \( a_j \) and wave-numbers \( k_j \) of the CW background, for which a given value of the spectral parameter \( \chi = \chi_0 \) is the root of the multiplicity \( n+1 \), i.e., \( P(\chi_0) = P^{(1)}(\chi_0) = \cdots = P^{(n)}(\chi_0) = 0 \). We notice that \( P^{(1)}(\chi) = 0 \) is the dispersion relation for the MI \( 55 \) and for \( n = 2,3 \) it was shown \( 57 \) to coincide with the condition for the baseband MI \( 52,53 \) of Eq. \( 1 \).

Due to the Galilean invariance of Eq. \( 1 \) the \( \chi_0 \) can be made real by the shift of the wave-numbers \( k_j \to k_j - \text{Im}(\chi_0) \). Furthermore, by rescaling \( a_j \) and \( k_j \) one can make \( \chi_0 = 1 \). Seeking the solutions of Eqs. \( 2 \) and \( 3 \) that are located on the left branch of the hyperbola: \( a_j^2 - k_j^2 = 1 \), we obtain

\[
a_j = \csc \theta_j, \quad k_j = \cot \theta_j, \quad \theta_j = \frac{\pi j}{n+1}, \quad \lambda_0 = \frac{i}{2(n+1)}.
\]

Thus, we have found the phases \( \theta_j \) of the components, which determine the amplitudes \( a_j \), the wave-numbers \( k_j \), and the frequencies \( \omega_j \) of the CW background sustaining a vector rational RW. The analytic expression for the respective RW is obtained by computing \( \Psi \) which satisfies the dispersion relations:

\[
\begin{align*}
\omega_0^2 - \lambda_0^2 & = 0, \\
\omega_0^2 - \lambda_0^2 - 1 & = 0.
\end{align*}
\]

By construction \( \hat{\Psi} \) inherits the symmetry of the CW background \( 4 \) that is symmetric with respect to simultaneous space inversion \( x \to -x \) and swapping the components performed by the matrix \( P = (\delta_{j,n+1-j})_{n\times n} \). Meantime, it is possible to impose a constraint on the parameters ensuring the \( PT \)-symmetry of a vector RW, with \( T \) being the conventional bosonic time-reversion: \( t \to -t, i \to -i \). Indeed, the formula \( 44 \) yields a \( PT \)-symmetric vector RW:

\[
q(x,t) = PTq(x,-t) = Pq^*(x,-t) \quad \text{if} \quad c \text{ is required to satisfy the symmetry: } c = \hat{\Psi}c^* + \hat{P}c^*, \quad \hat{P} = \begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & -P \end{pmatrix}
\]

and \( |\cdot| \) stands for the integer part.

**\( PT \)-symmetric vector rogue waves.**—Generally speaking, the obtained vector RW solution \( 53 \) does not inherit the symmetry of the CW background \( 4 \) that is symmetric with respect to simultaneous space inversion \( x \to -x \) and swapping the components performed by the matrix \( P = (\delta_{j,n+1-j})_{n\times n} \). Meantime, it is possible to impose a constraint on the parameters ensuring the \( PT \)-symmetry of a vector RW, with \( T \) being the conventional bosonic time-reversion: \( t \to -t, i \to -i \). Indeed, the formula \( 44 \) yields a \( PT \)-symmetric vector RW:

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\]

and \( |\cdot| \) stands for the integer part.

By construction \( B \) is a defective matrix with the \( n \)-th order exceptional point for the parameter choices \( 41 \). Therefore, it is convenient to define a set of generalized eigenvectors \( s_\ell = (\delta_{0\ell},a_1/(i+k_1)^{\ell+1},\ldots,a_n/(i+k_n)^{\ell+1})^T, \ell = 0,\ldots,n, \) satisfying \( B^s_{\ell} \neq 0 \) and \( B^{\ell+1}s_{\ell} = 0 \). The vector \( c \) can be spanned over the basis \( S = (s_0,\ldots,s_n); c = Sc \), where the form of the expansion coefficient, compactly
represented by the column-vector $\mathbf{a}$, must be chosen to ensure the validity of $\mathbf{c} = \mathbf{P}^{*}\mathbf{c}$. This is achieved by $\mathbf{a} = (\varepsilon_0 a_0, \ldots, \varepsilon_n a_n)^T$ with all $\alpha_\ell \in \mathbb{R}$, $\varepsilon_\ell = 1$ for even $\ell$ and $\varepsilon_\ell = i$ for odd $\ell$. Furthermore, since one of nonzero parameters $\alpha_\ell$ can be arbitrarily fixed, we set $\alpha_\ell = 1$ when $\alpha_{\ell-1} = \cdots = \alpha_n = 0$. With this choice $\mathbf{A}\mathbf{c}$ is a vector polynomial of degree $\ell$.

Resuming, a family of $PT$-symmetric vector RWs obtained against a given CW background with parameters (11), can be characterized by a pair of indexes $(n, \ell)$. The internal structure of such a RW is also characterized by the real parameters $\alpha_0, \ldots, \alpha_{\ell-1}$. Due to the translational invariance, one of the parameters $\alpha_j$ is a trivial spatial shift determining the location of the RW event and hence, it can be scaled out by the choice of the system of coordinates. Other $\ell - 1$ parameters determine the structure of the RW, as it is discussed below.

Examples of vector rational rogue waves.—The fundamental vector rational RWs correspond to $\ell = 1$ (respectively $\alpha_\ell = 1$) in (6). Choosing $\alpha_0 = -1/2$ we obtain:

$$q_j(x, t) = \left( a_j + \frac{i}{a_j} \frac{2r_{0j} - (4\alpha_0 + 1/2)r_{1j} + k_j + i}{r + (x^2 - t^2 - 2\alpha_0 - 1/4)^2} \right) e^{i(k_j x - \omega_j t - 2\theta_j)}. \quad (7)$$

The $(n, 1)$-RW solution has no free parameter and is fully determined by the characteristics of the CW background. A number of humps and dips, i.e. of the most interesting events, in each component depend on the number of components $5n$. At $n = 1$ we recover the Peregrine soliton having one hump and two dips [32]. For $n = 2$, when $\theta_1 = \pi/3$ and $\theta_2 = 2\pi/3$ [see Eq. (11)], the RW components are related by the $PT$-transform, and now each component has one hump and two dips as this is illustrated for $n = 2$ in Fig. 4a. For $n \geq 3$ the components with $j \in (0, (n + 1)/6) \cup [5(n + 1)/6, n]$ have two humps and one dip (for this case to take place one must have at least $n = 5$ components); the components with $j \in ((n + 1)/6, (n + 1)/3) \cup (2(n + 1)/3, 5(n + 1)/6)$ have two humps and two dips [see Fig. 4(c)], the components with $j \in [(n + 1)/3, 2(n + 1)/3)$ have one hump and two dips [see Fig. 4(d)].

Next we consider examples of $(n, 2)$-RWs, whose spatial location is parameterized by $\alpha_1$. We scale it out by the choice $\alpha_1 = 1/2$. However now there exists one free parameter $\alpha_0$, which determines the shape of the RW components. From (5) we obtain

$$q_j(x, t) = \left( a_j + \frac{i}{a_j} \frac{2r_{0j} - (4\alpha_0 + 1/2)r_{1j} + k_j + i}{r + (x^2 - t^2 - 2\alpha_0 - 1/4)^2} \right) e^{i(k_j x - \omega_j t - 2\theta_j)}, \quad (8)$$

where $r = ((2x - t)^2 + 1/2)^2 + (x + 1/2)^2 + 1/4$, $r_{0j} = 2(k_j x - t)(x^2 + t^2) + (1 - 2k_j)x^2 + k_j x + (3k_j - 1)x^2 + 4k_j x - 2x^2 - 2(k_j - 1)j/k_j + 1$ and $r_{1j} = 2(k_j x + t) + (i k_j + 1)/(k_j + i)$. Fig. 4(b) displays the fundamental $(2, 2)$-RW with one hump and two dips, while Figs. 4(c) and (f) exhibit the components of a $(3, 2)$-RW. We observe that the second component is characterized by “dominating” humps while in the first components the dips are more pronounced.

Polymeric $PT$-symmetric vector rational RWs.—The diversity of multi-component $PT$-symmetric vector RWs at large $n$ is abundant. Indeed, for a given $n$ one has $\ell = 1, \ldots, n$ different solutions characterized by $\ell - 1$ nontrivial free parameters, and hence $n(n - 1)/2$ distinct types of RWs. One can expect that the strongest collective effect of the components is achieved when they all acquire their maxima at the same instant of time and spatial location. Without loss of generality we can look for such a solution with the largest hump occurring at $t = 0$ and $x = 0$, when $\mathbf{A}(0, 0) = I_n$ [see (9)]. Since one of the components of $\mathbf{c}$ can be arbitrarily chosen, we can set $c_0 = 1$ and $c_j = i\tilde{c}_j$ for $1 \leq j \leq n$, where real $\tilde{c}_j$ are to be determined. Now we deduce from the general formula (4) that the maximum of $|q_j(0, 0)|$ is achieved if the second term in the square brackets is real negative, and hence $\tilde{c}_j$ are all real positive. In that case $q_j(0, 0) = a_j + \eta_j$ with $\eta_j = 2(n + 1)\tilde{c}_j/(1 + |\tilde{c}|^2)$, $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_n)^T$ is the deviation of the amplitude of the $j$-th component from the background.

Case A.—For the parameter $\mathbf{c} = (\sqrt{n}, i, \ldots, i)^T$, the average amplitude $|\mathbf{q}(x, t)| / |\mathbf{a}|$ of the polymeric RWs attain maximum at the origin, where $|\mathbf{q}(x, t)| = \sum_{j=1}^{n} |q_j(x, t)|$ and $|\mathbf{a}| = \sum_{j=1}^{n} a_j$. In this case, the ratio between peak value $|\mathbf{q}(0, 0)|$ and CW background $|\mathbf{a}|$ can be deduced as

$$Ra = \frac{|\mathbf{q}(0, 0)|}{|\mathbf{a}|} = 1 + \frac{\sqrt{n}(n + 1)}{\sum_{j=1}^{n} \csc \theta_j}. \quad (9)$$
which has the following estimates
\[ 1 + \frac{\sqrt{n}}{\ln(n+1)} \leq Ra \leq 1 + \frac{\sqrt{n}\pi}{2\ln(n/2)}, \]
which implies \( Ra \leq 3 \) for \( 2 \leq n \leq 11 \) and \( Ra \gg 3 \) for \( n \gg 12 \), and \( Ra = O(\sqrt{n}/\ln n) \) as \( n \to \infty \). Fig. 2(a) shows the distribution structure of \(|q(x,t)|/|a|\) for 15-NLS equations. Fig. 2(b) indicates the ratio \(|q(x,t)|/|a|\) for \( n \). Besides, the ratio between peak value and background for the \( j\)-th component is \(|q_j(0,0)|/|a| = 1 + (n+1)/\sqrt{n}\sin\theta_j \).

Case B.—In this case, we consider the polymeric RW, whose middle component attains maximum at the origin. Due to the restrictive of the time-reversal property, two categories are studied respectively. As \( n \) is odd, as long as \( c = (1, 0, \cdots, 0, 1, 0, \cdots, 0, 0) \), the amplitude of \((n+1)/2\)-th component attains maximum \( q_m(0,0)/a_m = (n+2)/2 \) at the origin, and others are the height \( a_j/s \) of the backgrounds at the origin. Fig. 2(c) shows the eighth component of polymeric RWs for 15-NLS equations. The red dots in Fig. 2(d) show the values of \( q_m(0,0)/a_m \) with \( m = (n+1)/2 \) and odd \( n \in [1, 30] \). As \( n \) is even, as long as \( c = (1, 0, \cdots, 0, i/\sqrt{2}, i/\sqrt{2}, 0, \cdots, 0) \), the amplitudes of \( n/2 \)- and \((n+1)/2\)-th components attain maximum at the origin with \( m = n/2 \) and \( q_m(0,0)/a_m = q_{n+1}(0,0)/a_{m+1} = 1 + (n+1)/\sqrt{2}\sin\theta_m \), and others are the height \( a_j/s \) of the corresponding backgrounds. The blue triangles in Figs. 2(d) show the values of \( q_m(0,0)/a_m \) with \( m = n/2 \) and even \( n \in [1, 30] \).

Non-integrable \( n\)-NLS model.—In a general setting a multi-parametric non-integrable system cannot be solved analytically. Even numerical study would require particularization depending on the physical nature of the nonlinearity. Therefore here we address a somewhat simpler question whether \( PT\)-symmetric RWs of enhanced intensity can be excited in an \( n\)-NLS equation, where the integrability is broken by unequal nonlinear interactions. We explore this issue by simulating evolution of vector RWs in the dimensionless non-integrable 3-NLS equation \[ i\partial q_j/\partial t + \frac{1}{2} \partial^2 q_j/\partial x^2 + |q_j|^2 - (1 - \delta)(|q_{j+1}|^2 + |q_{j-1}|^2) q_j = 0, \]
where \( j_1, j_2, \) and \( j_3 \) are all different and acquire values 1, 2, and 3, and \( \delta \) is a small real parameter quantifying imbalance between intra- and inter-mode interactions. However, in spite of the \( \delta \)-deformation, model (11) remains \( PT\)-symmetric, i.e., is not affected by swapping the components. For numerical simulations via the Fourier spectral method we consider small deviation from the integrable limit letting \( \delta = 0.05 \) and propagate the “initial” condition, in the form of the exact \( PT\)-symmetric vector RW (7) taken at \( t = -1 \) [Figs. 3(a) and (b)]. Although the post-extreme-event evolution of the non-integrable model is appreciably different from its integrable counterpart [cf. with Figs. 1(c) and (d)], the RW itself is well distinguishable in the evolution. In Figs. 3(c) and (d) we compare the numerical solutions of (11) with the exact RW solutions (7) at the instant of occurrence of the maximum, i.e., at \( t = 0.41 \) in our case. We observe that the structures and especially the shapes of the pick amplitudes are very close in both models.

Conclusions.—To conclude, we described a family of exact \( PT\)-symmetric vector rational RW solutions of an integrable system of \( n\)-component coupled NLS equations. Such extreme events occur against CW back-
grounds which are characterized by different asymptotics of different components. The higher amplitudes reached during the evolution increase with the number of the components. The reported RWs are observable in non-integrable systems, too. The obtained vector RW solutions may pave the way for obtaining high-amplitude RWs in diverse multi-component physical systems including multi-mode and poly-chromatic optical devices, as well as multi-atomic and spinor Bose-Einstein condensates. Moreover, we also give the higher-order vector RWs and asymptotic estimates of the $n$-NLS system in another literature [30]. The idea can also be extended to other $n$-component nonlinear integrable systems (e.g., $n$-component Hirota equations, mKdV equations, complex mKdV equations, higher-order NLS equations, KP equations, and etc.) [30]. Being restricted to all attractive interactions, our study left open such questions as obtaining high-intensity RWs in presence of attractive interactions, as well as establishing direct links between $\mathcal{PT}$-symmetric vector RWs and the modulation instability of the governing system.

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