Dynamical mass generation by source inversion: Calculating the mass gap of the Gross-Neveu model.

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Abstract

We probe the U(N) Gross-Neveu model with a source-term \( J \overline{\Psi} \Psi \). We find an expression for the renormalization scheme and scale invariant source \( \hat{J} \), as a function of the generated mass gap. The expansion of this function is organized in such a way that all scheme and scale dependence is reduced to one single parameter \( d \). We get a non-perturbative mass gap as the solution of \( \hat{J} = 0 \). In one loop we find that any physical choice for \( d \) gives good results for high values of \( N \). In two loops we can determine \( d \) self-consistently by the principle of minimal sensitivity and find remarkably accurate results for \( N > 2 \).

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1 Introduction

Asymptotic free massless field theories suffer from I.R.-renormalons, which signal that one is expanding around a wrong vacuum. To cure the theory from these renormalons the particles must acquire a mass. The only dimensionful parameter available is \( \Lambda_{\overline{MS}} \). A problem arises: in conventional perturbation theory \( \Lambda_{\overline{MS}} \) appears only as a log not as a power.

In the recent past, two new solutions, besides the \( 1/N \)-expansion, have been formulated for this problem in the context of the Gross-Neveu model \[1\], which is a very appealing model to test non-perturbative methods because the exact result for the mass gap \[2\] is known. One of the two solutions consisted of a renormalizable version of the Optimized Expansion \[3\], while the other, found by one of us, presented a fully renormalizable action for the local composite operator \( \overline{\Psi} \Psi \) \[4\],\[5\]. Both methods require some tricks to maintain renormalizability.

In this paper we present a third method which has the advantage of a straightforward renormalization. The idea is very simple: we add a source term \( J \overline{\Psi} \Psi \) to the Gross-Neveu Lagrangian, then we calculate the mass gap to obtain the relation \( m(J) \). After inversion\[1\] we find an expression of \( \hat{J} \), the renormalization scheme and scale independent quantity associated with \( J \), as a function of \( m \). Putting \( \hat{J}(m) \) equal to zero, gives us all possible solutions for the mass gap. Furthermore we will organize the expansion of \( \hat{J}(m) \) in such a way, that all the scheme and scale dependence reduces to one single parameter \( d \). This parameter is fixed by the principle of minimal sensitivity.

In section \[2\], the method is explained in detail. Section \[3\], contains the numerical results of the one and two-loop calculations.

2 The Method

Let us perturb the \( U(N) \) Gross-Neveu model in \( 2 - \epsilon \) dimensions with a \( \overline{\Psi} \Psi \) composite operator.

\[
\mathcal{L} = \overline{\Psi} i{\partial^\mu} \gamma_\mu \Psi + \frac{1}{2} \mu^\epsilon g^2 (\overline{\Psi} \Psi)^2 - J \overline{\Psi} \Psi + \mathcal{L}_{c.t.} \tag{1}
\]

We have \( N \) Dirac fermions which enjoy a manifest \( U(N) \)-symmetry. This theory is asymptotically free and for \( J = 0 \), possesses a discrete \( \gamma_5 \) invariance.\[1\] The idea of inversion is not new, it was already formulated by Fukuda\[6\]. He considered the inversion of \( \Phi(J) \), where \( \Phi \) is a composite operator. An extensive review of the method and its applications can be found in \[7\].
ance which in perturbation theory leads to a vanishing mass gap. It is also renormalizable with the counterterms:

\[ L_{c.t.} = i\delta Z \nabla \partial \mu \gamma \Psi + \frac{1}{2} \mu \delta Z g^2 (\nabla \Psi)^2 - \delta Z J \nabla \Psi. \]  

(2)

These counterterms depend on the scale \( \mu \) and on the renormalization scheme. Because we are interested in the limit \( J \rightarrow 0 \), we will restrict ourselves to the mass independent renormalization schemes. In the context of the mass gap, every (mass independent) R.S., can be obtained from another (the \( \overline{\text{MS}} \)-scheme for instance) by the following transformations:

\[ J = J(1 + a_0 g^2 + a_1 g^4 + \ldots), \]  

(3)

\[ g^2 = g^2(1 + b_0 g^2 + b_1 g^4 + \ldots), \]  

(4)

\( a_i, b_i \) are finite numbers and \( J = J(\mu), J = J(\mu), g^2 = g^2(\mu), g^2 = g^2(\mu) \) is understood. From now on we take the convention that \( J, g^2, \beta, \gamma \) refer to an arbitrary R.S.. The scale dependence of \( J \) and \( g^2 \) is governed by the \( \gamma \) and \( \beta \) functions:

\[ \mu \frac{\partial}{\partial \mu} J|_{J_0, g^2_0} = -J(\gamma_0 g^2 + \gamma_1 g^4 + \gamma_2 g^6 + \ldots), \]  

(5)

\[ \mu \frac{\partial}{\partial \mu} g^2|_{g^2_0} = 2(\beta_0 g^4 + \beta_1 g^6 + \beta_2 g^8 + \ldots). \]  

(6)

The R.S.-dependence of \( \gamma_1, \gamma_2, \beta_2 \) \( (\gamma_0 = \gamma_0, \beta_0 = \beta_0, \beta_1 = \beta_1) \) can be easily calculated using (3)-(6). In the formulation of our method we were led by the requirement that physical results must be independent of the scale and the scheme.

We now define the scale and scheme independent \( \tilde{J} \):

\[ \tilde{J} \equiv Jf(g^2), \]  

(7)

with

\[ \mu \frac{\partial}{\partial \mu} f(g^2)|_{g^2_0} = \gamma(g^2) f(g^2). \]  

(8)

The scale independence is seen immediately. As a solution of (8), we find:

\[ f(g^2) = \lambda g^{-\gamma_0}(1 - \frac{1}{2} \gamma_1 \beta_0 - \frac{\gamma_0 \beta_1}{2 \beta_0}) g^2 + (-\frac{\gamma_2}{4 \beta_0} + \frac{\gamma_1 \beta_1}{4 \beta_0^2} - \frac{\gamma_0 \beta_1}{4 \beta_0} (\beta_1)^2 + \frac{\gamma_0 \beta_2}{4 \beta_0^2} + \frac{\gamma_2^2}{8 \beta_0^2} - \frac{\gamma_1 \gamma_0 \beta_1}{4 \beta_0} + \frac{\gamma_0^2 \beta_2}{8 \beta_0^2}) g^4 + \ldots), \]  

(9)
with \( \lambda \) an integration constant. We choose \( \lambda = 1 \), any other choice implies a different \( \hat{J} : \hat{J}(\lambda) = \lambda \hat{J}(0) \). This is not a problem because we will look for solutions of \( \hat{J} = 0 \). One can now easily check the scheme independence of \( \hat{J} \).

In order to get \( \hat{J}(m) \), we will first have to calculate \( m(J) \). A perturbative calculation of the mass gap generated by (1) will give us:

\[
m(J) = J(1 + g^2(X'_0 + X'_1 \ln \frac{J^2}{\mu^2}) + g^4(Y'_0 + Y'_1 \ln \frac{J^2}{\mu^2} + Y'_2(\ln \frac{J^2}{\mu^2})^2) + \ldots). \tag{10}
\]

After inversion, we arrive at a relation of the following form:

\[
J(m) = m(1 + g^2(X_0 + X_1 \ln \frac{m^2}{\mu^2}) + g^4(Y_0 + Y_1 \ln \frac{m^2}{\mu^2} + Y_2(\ln \frac{m^2}{\mu^2})^2) + \ldots). \tag{11}
\]

The coefficients which multiply a log can easily be related to the other coefficients and the coefficients of the \( \gamma \) and \( \beta \) functions. Demanding (5) and (6), we obtain:

\[
J(m) = m(1 + g^2(X_0 + \gamma_0 \ln \frac{m^2}{\mu^2}) + g^4(Y_0 + (\gamma_1 2 + (\gamma_0 2 - \beta_0)X_0) \ln \frac{m^2}{\mu^2} + \gamma_0 2)(\ln \frac{m^2}{\mu^2})^2) + \ldots). \tag{12}
\]

Plugging this result together with (9), in (7) we finally arrive at:

\[
\hat{J} = m(g^{-\gamma_0 \gamma_0} \left[ 1 + g^2 \left( X_0 - \frac{1}{2} \frac{\gamma_1 2}{\beta_0} - \frac{\gamma_0 2}{\beta_0^2} \right) + g^4 \left( \frac{X_0 2}{\beta_0^2} - \frac{\gamma_1 2}{\beta_0} - \frac{\gamma_2 2}{4\beta_0} + \frac{\gamma_1 2}{4\beta_0^2} \right) + g^4 \left( \frac{X_0 2}{\beta_0^2} - \frac{\gamma_1 2}{\beta_0} - \frac{\gamma_2 2}{4\beta_0} + \frac{\gamma_1 2}{4\beta_0^2} \right) + \gamma_i 2 \right) \right]
\]

\[
\hat{J} = m(g^{-\gamma_0 \gamma_0} \left[ 1 + g^2(A_0 + A_1 \ln \frac{m^2}{\mu^2}) + g^4(B_0 + B_1 \ln \frac{m^2}{\mu^2} + B_2(\ln \frac{m^2}{\mu^2})^2) + \ldots \right] \tag{13}
\]

\[
\hat{J} = m(g^{-\gamma_0 \gamma_0} \left[ 1 + g^2(A_0 + A_1 \ln \frac{m^2}{\mu^2}) + g^4(B_0 + B_1 \ln \frac{m^2}{\mu^2} + B_2(\ln \frac{m^2}{\mu^2})^2) + \ldots \right] \tag{14}
\]
Due to the scheme independence of $\hat{J}$, the coefficients $A_i, B_i$ are independent of the mass renormalization (3). They depend only on the coupling constant renormalization (4). We find

$$A_0 = \overline{A}_0 - b_0 \frac{\gamma_0}{2\beta_0}$$

$$A_1 = \overline{A}_1$$

$$B_0 = \overline{B}_0 + \frac{\gamma_0}{4\beta_0} (\frac{\gamma_0}{2\beta_0} + 1)b_0^2 - \frac{\gamma_0}{2\beta_0} b_1 + b_0 \overline{A}_0 (1 - \frac{\gamma_0}{2\beta_0})$$

$$B_1 = \overline{B}_1 + b_0 \overline{A}_1 (1 - \frac{\gamma_0}{2\beta_0})$$

$$B_2 = \overline{B}_2.$$  

So the expansion of $\hat{J}$ in powers of $g^2$ is still highly scheme ($g^2, b_0, b_1$) and scale ($\mu$) dependent. We now show that this dependence is reduced to one simple parameter $d$, by using another expansion. Following an argument of Grunberg [8](II.A), we know that $\hat{J}$, a scheme and scale-invariant quantity, depends only on $m$ and $\Lambda_{\overline{MS}}$: $\hat{J} \equiv m F(\frac{m}{\Lambda_{\overline{MS}}})$. Reorganizing (14) as an expansion in powers of $\frac{1}{\beta_0 \ln \left(\frac{m^2}{\Lambda^2}\right)}$ will leave one arbitrariness: we can equally well expand in $\frac{1}{\beta_0 \ln \left(\frac{m^2}{\Lambda^2}\right) + d}$. To show this explicitly we will need some formulas:

$$g^2(\mu) = \frac{1}{\beta_0 \ln \left(\frac{m^2}{\Lambda^2}\right)} \left(1 - \frac{\beta_1}{\beta_0} \ln \left(\frac{m^2}{\Lambda^2}\right)\right)$$

$$+ \left(\frac{\beta_0}{\beta_0^2} - \frac{\beta_1}{\beta_0} \right) \frac{\left(\ln \left(\frac{m^2}{\Lambda^2}\right)\right)^2 - \ln \left(\frac{m^2}{\Lambda^2}\right)}{\beta_0 \ln \left(\frac{m^2}{\Lambda^2}\right)}$$

$$+ O\left(\frac{1}{\beta_0 \ln \left(\frac{m^2}{\Lambda^2}\right)}\right)^3 \right)$$

$$\Lambda = \Lambda_{\overline{MS}} \exp\left[-\frac{b_0}{2\beta_0}\right] \quad (see \ e.g. [9])$$

$$\beta_2 = (b_0^2 - b_1)\beta_0 + \beta_1 b_0 + \overline{\beta}_2 \quad (see \ [10]).$$

Using these formulas together with (14) and (15), we find the master for-
\( \hat{J} = m(\beta_0 \ln \frac{m^2}{\Lambda_{\overline{MS}}} + d) \frac{\gamma_0}{\beta_0} \times \)

\[
\left[ 1 + \frac{1}{(\beta_0 \ln \frac{m^2}{\Lambda_{\overline{MS}}} + d)} \left[ \bar{A}_0 + \frac{\gamma_0 \beta_1}{2\beta_0^2} \ln \left( \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + \frac{d}{\beta_0} \right) - \frac{d\gamma_0}{2\beta_0} \right] + \frac{1}{(\beta_0 \ln \frac{m^2}{\Lambda_{\overline{MS}}} + d)^2} \left[ \bar{B}_0 + \bar{A}_0 \left( \frac{\gamma_0}{2\beta_0} - 1 \right) \frac{\beta_1}{\beta_0} \ln \left( \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + \frac{d}{\beta_0} \right) \right] + \left( \frac{\beta_1}{\beta_0} \right)^2 \left( \ln \left( \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + \frac{d}{\beta_0} \right) \right)^2 \left( \frac{\gamma_0}{4\beta_0} - \frac{1}{2} \right) + \frac{\gamma_0}{2\beta_0} \ln \left( \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + \frac{d}{\beta_0} \right) \right] - \frac{\gamma_0}{2\beta_0} \left( \frac{\beta_2}{\beta_0} - \frac{\beta_1^2}{\beta_0^2} \right) + d^2 \left( \frac{\gamma_0}{4\beta_0} \right) \left( \frac{\gamma_0}{2\beta_0} - 1 \right) + d \left( \bar{A}_0 (1 - \frac{\gamma_0}{2\beta_0}) - \frac{\gamma_0 \beta_1}{2\beta_0^2} + \frac{\gamma_0 \beta_1}{2\beta_0^2} \ln \left( \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + \frac{d}{\beta_0} \right) \right) \right] \right] + \mathcal{O} \left( \frac{1}{\beta_0 \ln \frac{m^2}{\Lambda_{\overline{MS}}}} \right)^3, \tag{19} \]

with \( d \equiv b_0 - \beta_0 \ln \frac{m^2}{\mu} \).

We can now recover the original Gross-Neveu model, by putting \( J_0 \), the naked source, equal to zero:

\[ J_0(m) \sim \hat{J}(m) = 0 \tag{20} \]

One finds a non-perturbative mass gap, that is a solution of

\[ F \left( \frac{m}{\Lambda_{\overline{MS}}} \right) = (\beta_0 \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + d) \frac{\gamma_0}{\beta_0} \left[ 1 + \frac{1}{(\beta_0 \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + d)} \left[ \ldots \right] + \frac{1}{(\beta_0 \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + d)^2} \left[ \ldots \right] + \ldots \right] = 0 \tag{21} \]

The total series is of course \( d \)-independent, but one can only calculate it up to a certain order, this will give us a mass gap that depends on \( d \). One can check that the \( d \)-dependence of the order \( n \) truncated series is \( \mathcal{O} \left( \frac{1}{\beta_0 \ln \frac{m^2}{\Lambda_{\overline{MS}}} + d_0} \right)^{n+1} \). The only sensible thing one can now ask for \( d \) is...
minimal sensitivity $\frac{\partial m}{\partial d} \mid_{d=d_0} = 0$, and hope that the expansion parameter

$$\left(\frac{\beta_0 \ln \frac{-m^2}{\lambda_{MS}^2} + d_0}{\lambda_{MS}^2}\right)$$

is small enough to justify the truncation. We finally note that, as in [4], [5], one needs the \((n+1)\)-loop divergencies \((\gamma, \beta)\), to get an \(n\)-loop result.

3 Numerical Results

The exact expression for the mass gap was obtained in [4]:

$$m = (4e)^\Delta \frac{1}{\Gamma(1 - \Delta)} \Lambda_{MS},$$

where $\Delta = \frac{1}{2N-2}$. Expanding in powers of \(N\) we have:

$$m = (1 + \frac{1}{N}(1 - \gamma + \ln[4]) + O(\frac{1}{N^2})) \Lambda_{MS}.$$  

(23)

Comparison of our results with the exact result will give us an explicit check of our method.

The 2-loop result for \(m(J)\) in the \(\overline{MS}\)-scheme was given in [3] (section 4):

$$m(J) = J \left[1 - g^2 \left(\frac{N - 1/2}{2\pi} \ln \frac{J^2}{\mu^2}\right) + g^4 \left(\frac{(N - 1/2)(N - 3/4)}{4\pi^2}(\ln \frac{J^2}{\mu^2})^2 + \frac{(N - 1/2)(N - 3/4)}{2\pi^2} \ln \frac{J^2}{\mu^2} + \frac{N - 1/2}{\pi^2} (0.737775 - \frac{\pi^2}{96})\right)\right]$$

(24)

After inversion one finds:

$$J(m) = m \left[1 + \frac{g^2}{2\pi} [(N - 1/2) \ln \frac{m^2}{\mu^2}] + \frac{g^4}{4\pi^2} \left(\frac{1}{4}(N - 1/2)(\ln \frac{m^2}{\mu^2})^2 - \frac{1}{2}(N - 1/2) \ln \frac{m^2}{\mu^2} - 4(N - 1/2)(0.737775 - \frac{\pi^2}{96})\right)\right].$$

(25)

So we have:

$$\bar{X}_0 = 0$$

$$\bar{Y}_0 = -\frac{N - 1/2}{\pi^2} (0.737775 - \frac{\pi^2}{96}).$$

We assume that the series is at least asymptotic and that the size of the terms is more or less determined by the size of the expansion parameter. 

7
The $\beta$ and $\gamma$-functions have been calculated (in the $\overline{\text{MS}}$-scheme) up to three loops by Luperini and Rossi \[10\] and Gracey and Bennett \[11\], \[12\], \[13\]:

\[
\begin{align*}
\beta_0 &= \frac{N - 1}{2\pi}, & \beta_1 &= -\frac{N - 1}{4\pi^2}, & \beta_2 &= -\frac{(N - 1)(N - 7/2)}{16\pi^3}, \\
\gamma_0 &= \frac{N - 1/2}{\pi}, & \gamma_1 &= -\frac{N - 1/2}{4\pi^2}, & \gamma_2 &= -\frac{(N - 1/2)(N - 3/4)}{4\pi^3}.
\end{align*}
\]

(27) (28)

One can now easily check (12).

3.1 one-loop results

The 1-loop result for the mass gap is found as the solution of the 1-loop truncation of $\hat{J} = 0$. Putting the numbers in (19), one finds the mass gap as the solution of the transcendental equation:

\[
1 - \frac{N - 1}{2\pi} \ln \frac{m^2}{\Lambda_{\overline{\text{MS}}}^2} + d\pi - \frac{2N - 1}{8(N - 1)} \left(1 + 4\pi d + 2\ln[\ln \frac{m^2}{\Lambda_{\overline{\text{MS}}}^2} + \frac{2d\pi}{N - 1}]\right) = 0
\]

(29)

First, the bad news: there exists no $d_0$ for which the minimum-sensitivity criterium holds. Indeed, taking the derivative to $d$ of (29) gives us an equation which must also hold if $\frac{\partial m}{\partial d} = 0$:

\[
\frac{N - 1}{2\pi} \ln \frac{m^2}{\Lambda_{\overline{\text{MS}}}^2} + d\pi - \frac{2N - 1}{8(N - 1)} \left(1 + 4\pi d + 2\ln[\ln \frac{m^2}{\Lambda_{\overline{\text{MS}}}^2} + \frac{2d\pi}{N - 1}]\right) = 0
\]

(30)

or using (29):

\[
\frac{N - 1}{2\pi} \ln \frac{m^2}{\Lambda_{\overline{\text{MS}}}^2} + d\pi = -\frac{2}{(2N - 1)}.
\]

(31)

So we would find a negative expansion-parameter, which is clearly not consistent with (29), since the last term gets an imaginary value.

Now the good news: for any physical choice for $d$ we find reasonable results. Furthermore we can show that any physical $d$ gives the exact $N \to \infty$ limit. This happens because every natural coupling constant renormalization \[9\] leads to a value for $b_0 \overset{N \to \infty}{=} \frac{\alpha N}{\pi}$, with $\alpha$ dependent on the condition,
Table 1: one loop results

| N  | $m_{11}$    | $m_{12}$    | $N = \infty$ | $1/N$ |
|----|-------------|-------------|--------------|-------|
| 2  | 210.7%      | 268.9%      | -46.3%       | -21.9%|
| 3  | 31.9%       | 40.6%       | -32.5%       | -12.2%|
| 4  | 14.7%       | 19.0%       | -24.2%       | -7.0% |
| 5  | 9.1%        | 11.9%       | -19.1%       | -4.5% |
| 6  | 6.5%        | 8.5%        | -15.8%       | -3.1% |
| 7  | 5.0%        | 6.6%        | -13.5%       | -2.3% |
| 8  | 4.0%        | 5.4%        | -11.7%       | -1.8% |
| 9  | 3.4%        | 4.5%        | -10.4%       | -1.4% |
| 10 | 2.9%        | 3.9%        | -9.3%        | -1.1% |

defining the R.S. So if we take $\mu = m$ in $(13)$, we find $d \to \infty = \frac{\alpha N}{\pi}$ and equation $(29)$ becomes:

$$1 \to \infty = \frac{2\alpha}{\ln \frac{m^2}{\Lambda^2} + 2\alpha}, \quad \text{or} \quad m = \Lambda_{MS}.$$ 

(32)

We now calculate the mass gap, using two different conditions for the coupling constant renormalization. The conditions are defined on the 1 P.I. 4-point function. In zero loops this originates from the interaction-term $\frac{g^2}{2} (\bar{\Psi} \Psi)^2$. In one loop, one finds, after a straightforward calculation, an effective interaction of the form: $\frac{g^2}{2} (\bar{\Psi} \Psi)^2 \left[1 + g^2 f(J, s, t, u, \mu)\right] + g^4 \left[\text{other operators}\right]$, where $s, t, u$ are the Mandelstam-variables. Two possible natural conditions are:

$$f(J, s, t, u, \mu)|_{s=t=u=0} = 0, \quad f(J, s, t, u, \mu)|_{s=t=u=-J} = 0.$$ 

(33) (34)

They lead to two values for $d$, $d_1 = \frac{N-3/2}{\pi}$ and $d_2 = \frac{(N-1)(\sqrt{5} - \frac{1}{2})}{\pi} \ln \frac{\sqrt{5}+1}{\sqrt{5}-1}$. If we now use these values in $(29)$, we find two solutions $m_{11}$ and $m_{12}$.

The deviation from the exact result for the $N \to \infty$ limit, the first order in $\frac{1}{N}$, and our two one-loop solutions $m_{11}$ and $m_{12}$ have been displayed in table 3.1 in terms of percentage. Our one-loop results are acceptable for $N \geq 4$. They lie somewhere between the $N = \infty$ and $1/N$ approximations. For $N = 2, 3$ the results are bad. We could try to find another renormalization condition, which produces better results, but this would still leave
us dissatisfied. After all, the value for $d$ is determined by some external physical (what does that mean anyway?) condition and does not result in a self-consistent way from the calculations. Fortunately, this changes with the two-loop calculations.

## 3.2 two-loop results

Putting the numbers in (19) and equating it to zero, we find, for $N > 2$, one and only one value for $d$ (and consequently for $m$), determined by the minimal sensitivity condition $\frac{\partial m}{\partial d} = 0$. In table 3.2 we display the deviation (in terms of percentage) from the exact result, the value for $d$ and the value for the expansion parameter. We find excellent ($< 2\%$) agreement.

The $N = 2$ case is special, one finds no minimum. The graph for $m(d)$, is displayed in figures 1 and 2. There is a gap with no solution for $m$ in the region where the minimum is expected. With the minimal sensitivity prescription in mind, we can still estimate the mass in the range of $\pm 20\%$ deviation. The generic $N = 3$ case is plotted in figure 3 for comparison.

| $N$ | $m_2$ | $\pi d_0$ | $1/(\beta_0 \ln[m^2/\Lambda_{\overline{MS}}^2] + d_0)\pi$ |
|-----|-------|-----------|--------------------------------------------------|
| 2   | ±20%  | /         | /                                                |
| 3   | 0.9%  | 1.4       | 0.44                                             |
| 4   | -1.0% | 2.1       | 0.34                                             |
| 5   | -1.5% | 2.8       | 0.28                                             |
| 6   | -1.6% | 3.4       | 0.24                                             |
| 7   | -1.6% | 4.1       | 0.20                                             |
| 8   | -1.5% | 4.8       | 0.18                                             |
| 9   | -1.4% | 5.5       | 0.16                                             |
| 10  | -1.3% | 6.1       | 0.14                                             |
| 15  | -1.0% | 9.5       | 0.10                                             |
| 20  | -0.8% | 12.8      | 0.07                                             |

Table 2: two loop results
Figure 1: $N=2$, $\pi d \rightarrow \frac{m^2(d) - m_{\text{exact}}}{m_{\text{exact}}} \times 100$

Figure 2: $N=2$, $\pi d \rightarrow \frac{m^2(d) - m_{\text{exact}}}{m_{\text{exact}}} \times 100$, zooming in on the gap
4 Conclusions

In this paper we have presented a general method for dynamical mass generation in asymptotically free theories. When tested on the Gross-Neveu model, remarkably accurate results were found for the two-loop calculation.

Let us now try to recapture the essence of the method and its connection with the sign of $\beta_0$. The general idea was to probe the theory with a R.S.-and scale invariant source $\hat{J}$. Without the inversion one would arrive at a relation of the following form:

$$m(\hat{J}) = \hat{J}(\beta_0 \ln \frac{\hat{J}^2}{\Lambda_{\overline{MS}}^2} + d) - \frac{2m}{\beta_0} \left[ 1 + \frac{1}{(\beta_0 \ln \frac{\hat{J}^2}{\Lambda_{\overline{MS}}^2} + d)^2} \right] \ldots + \frac{1}{(\beta_0 \ln \frac{\hat{J}^2}{\Lambda_{\overline{MS}}^2} + d)^2} \ldots .$$

The important point is that this expansion is only valid for large positive values of $(\beta_0 \ln mn_{\overline{MS}}^2 + d)$. For an I.R.-free theory ($\beta_0 < 0$) this means $\hat{J}^2 \ll \Lambda_{\overline{MS}}^2 \exp \frac{d}{\beta_0}$, so we can take the limit $\hat{J} \to 0$ and one finds no mass gap. For asymptotic free theories, the expansion (35) is invalid for small $\hat{J}$, so it cannot be used to probe the theory around $\hat{J} = 0$. The expansion of the inverse function $\hat{J}(m)$ (15) on the contrary, remains well defined in the limit $\hat{J} \to 0$, provided that a positive solution for $m$ exists and that the corresponding expansion parameter $1/(\beta_0 \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + d)$ is not too big.
found that this was the case for the Gross-Neveu model if we fixed $d$ by the minimal sensitivity prescription. Calculations on other models and on QCD are in progress.

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