REAL ZEROS OF QUADRATIC HECKE \(L\)-FUNCTIONS

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Abstract. We study real zeros of a family of quadratic Hecke \(L\)-functions in the Gaussian field to show that more than twenty percent of the members have no zeros on the interval \((0, 1]\).

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1. Introduction

The non-vanishing issue of central values of \(L\)-functions has been studied intensively in the literature due to its deep arithmetic implications. For the classical case, it is expected that the corresponding \(L(s, \chi)\) never vanishes at \(s = 1/2\) for any Dirichlet character \(\chi\). This statement first appeared as a conjecture of S. Chowla [1], who concerned with the special case of primitive real characters.

Partial resolutions to Chowla’s conjecture was first given by M. Jutila [11], who evaluated the first two moments of the family of quadratic Dirichlet \(L\)-functions to show that infinitely many such \(L\)-functions do not vanish at the central value. By evaluating mollified moments instead, K. Soundararajan [14] improved the result of Jutila to show that unconditionally, at least 87.5% of the members of the quadratic family have non-vanishing central values. This percentage can be further improved to be 94.27% if one assumes the generalized Riemann hypothesis (GRH). In fact, this follows from a result of A. E. Özluk and C. Snyder [14] on the one-level density of low-lying zeros of the family of quadratic Dirichlet \(L\)-functions, replacing the test function used in [14] by an optimal one given in a paper of H. Iwaniec, W. Luo and P. Sarnak [10] Appendix A).

The statements of GRH and Chowla’s conjecture imply that \(L(\sigma, \chi) \neq 0\) for every primitive quadratic Dirichlet character and all \(0 \leq \sigma \leq 1\). However, it was not even known previously whether every such Dirichlet \(L\)-function of sufficiently large conductor has a non-trivial real zero or not until J. B. Conrey and K. Soundararajan [2] proved in 2002 that for at least 20% of the odd square-free integers \(d \geq 0\) we have \(L(\sigma, \chi_{-sd}) > 0\) for \(0 \leq \sigma \leq 1\), where \(\chi_{-sd} = (\frac{-sd}{d})\) is the Kronecker symbol.

Motivated by the above result of Conrey and Soundararajan, we investigate in this paper the real zeros of a family of quadratic Hecke \(L\)-functions in the Gaussian number field \(\mathbb{Q}(i)\). To state our result, we first introduce some notations. Throughout the paper, we denote \(K = \mathbb{Q}(i)\) and \(\mathcal{O}_K\) for the ring of integers \(\mathbb{Z}[i]\) of \(K\). For any \(c \in K\), we write \(N(c)\) for its norm. We denote \(L(s, \chi), \zeta_K(s)\) for the \(L\)-function associated to a Hecke character \(\chi\) of \(K\) and the Dedekind zeta function of \(K\), respectively.

We say a Hecke character \(\chi\) is primitive modulo \(q\) if it does not factor through \((\mathcal{O}_K/(q'))^\times\) for any divisor \(q'\) of \(q\) such that \(N(q') < N(q)\). We also say that \(\chi\) is of trivial infinite type if its component at infinite places of \(K\) is trivial. For any \(c \in \mathcal{O}_K\), we reserve in this paper the symbol \(\chi_c\) for the quadratic residue symbol \((\cdot)\) defined in Section 2.1.

It is shown in [4], Section 2.1] that \(\chi_{(1+i)^d}\) defines a primitive quadratic Hecke character modulo \((1+i)^d\) of trivial infinite type for any odd, square-free \(d \in \mathcal{O}_K\). Here, we say that \(d\) is odd if \((d, 2) = 1\) and \(d\) is square-free if the ideal \((d)\) is not divisible by the square of any prime ideal. The family of \(L\)-functions that we consider in this paper is given by

\[ \mathcal{F} = \{ L(s, \chi_{(1+i)^d}) : d \in \mathcal{O}_K\ \text{odd, square-free} \} . \]

Notice that every \(L\)-function in the above family satisfies a functional equation given by [22] which implies that \(\Gamma(s)L(s, \chi_{(1+i)^d})\) is entire on the whole complex plane. As \(\Gamma(s)\) has a simple pole at \(s = 0\), this implies that \(L(0, \chi_{(1+i)^d}) = 0\). Thus, if one assumes GRH for the above family of \(L\)-functions as well as the non-vanishing of these \(L\)-functions at the central point \(1/2\), then one can deduce that \(L(\sigma, (1+i)^d) \neq 0\) for all \(0 < \sigma \leq 1\). Consequently, we have \(L(\sigma, (1+i)^d) > 0\) for all \(0 < \sigma \leq 1\) by continuity and the fact that \(L(2, (1+i)^d) > 0\). In this paper, we show that unconditionally, at least a positive percent of these \(L\)-functions do not vanish on \((0, 1]\). By further keeping in mind that the number of odd, square-free \(d \in \mathcal{O}_K\) such that \(N(d) \leq x\) is asymptotically \(\frac{2\pi x}{3\zeta_K(2)}\) (see [4, Section 3.1]), our result is given as follows.
Theorem 1.1. For at least 20% of the members of $L$-functions in $\mathcal{F}$, we have $L(\sigma, \chi(1+i)^{\sigma}d) > 0$ for $0 < \sigma \leq 1$. More precisely, the number of $L$-functions in $\mathcal{F}$ such that $N(d) \leq x$ and $L(\sigma, \chi(1+i)^{\sigma}d) > 0$ for all $0 < \sigma \leq 1$ exceeds $\frac{1}{6}(\frac{2\pi x}{\log x})$ for all large $x$.

Our proof of Theorem 1.1 is mainly based on the approach of Conrey and Soundararajan in their above mentioned work [2] concerning the non-vanishing of quadratic Dirichlet $L$-functions on the real line. We also note here that similar to what is pointed out in [2], the approach in our paper implies that for all large $x$, the number of odd, square-free $d \in \mathcal{O}_K$ with $N(d) \leq x$ such that $L(s, \chi(1+i)^{\sigma}d)$ has a zero in $[\sigma, 1]$ is $\ll x^{1-(1-\varepsilon)(\sigma-\frac{1}{2})}$ for any fixed $\sigma \geq 1/2$ and $\varepsilon > 0$. Another outcome of our approach is that there are many $L$-functions having no non-trivial zeros in a thin rectangle containing the real line. More precisely, there exists a constant $c > 0$ such that for at least 20% of the members of $\mathcal{F}$ with $N(d) \leq x$, the corresponding $L(s, \chi(1+i)^{\sigma}d)$ has no zeros in the rectangle $\{\sigma + it : \sigma \in (0, 1), \, |t| \leq c/\log x\}$ for all large $x$.

2. Preliminaries

The proof of our result requires many tools as well as auxiliary results, which we gather them here first.

2.1. Quadratic residue symbol and Gauss sum. It is well-known that $K = \mathbb{Q}(i)$ has class number one and that every ideal in $\mathcal{O}_K$ co-prime to 2 has a unique generator which is $1 \mod (1+i)^{3}$. Such a generator is called primary. Further notice that $(1+i)$ is the only prime ideal in $\mathcal{O}_K$ that lies above the ideal $(2) \in \mathbb{Z}$. We now fix a generator $n$ for each ideal $(n)$ in $\mathcal{O}_K$ by taking $n$ to be of the form $(1+i)^{m}n'$ with $m \geq 1$ and $n'$ primary. We denote the set of these generators by $G$ throughout the paper. For $a, b \in \mathcal{O}_K$, we denote $(a, b)$ for their greatest common divisor such that $(a, b) \in G$.

We denote $\varpi$ for a prime element in $\mathcal{O}_K$, which means that the ideal generated by $\varpi$ is a prime ideal. We denote the group of units in $\mathcal{O}_K$ by $U_K$ and the discriminant of $K$ by $D_K$. Thus, we have $U_K = \{\pm 1, \pm i\}$ and $D_K = -4$. We write $d = \square$ for a perfect square in $\mathcal{O}_K$, by which we mean that $d = n^2$ for some $n \in \mathcal{O}_K$.

The quadratic residue symbol $\left( \frac{a}{\varpi} \right)$ is defined for any odd prime $\varpi$ such that for any $a \in \mathcal{O}_K$, $\left( \frac{a}{\varpi} \right) = 0$ when $\varpi | a$ and $\left( \frac{a}{\varpi} \right) \equiv a^{\text{ord}_\varpi(\varpi)} (\mod \varpi)$ with $\left( \frac{a}{\varpi} \right) \in \{\pm 1\}$ when $(a, \varpi) = 1$. The definition is then extended to $\left( \frac{a}{\mathcal{O}_K} \right)$ multiplicatively for any odd $n \in \mathcal{O}_K$. Here, we define $\left( \frac{a}{\mathcal{O}_K} \right) = 1$ for $n \in U_K$.

As mentioned in Section 1, we denote $\chi_{c}$ for the quadratic residue symbol $(\frac{a}{n})$. In this paper, we regard $\chi_{\pm 1}$ as a principal character modulo 1, so that $\chi_{\pm 1}(a) = 1$ for all $a \in \mathcal{O}_K$. Note that this implies that $L(s, \chi_{\pm 1}) = \zeta_K(s)$. For other values of $c$, we shall regard $\chi_{c}$ as a Hecke character of trivial infinite type modulo 16$c$ as this is justified in [5] Section 2.1. In particular, we notice that $(\frac{a}{\varpi}) = 0$ when $1 + \imath | a$ and $c \neq \pm 1$.

We define the quadratic Gauss sum $g(r, n)$ for $r, n \in \mathcal{O}_K$ with $n$ being odd, by

$$g(r, n) = \sum_{x \mod n} \left( \frac{x}{n} \right) e\left(\frac{rx}{n}\right),$$

where we define for any complex number $z$,

$$\bar{e}(z) = \exp\left(2\pi i \left(\frac{z}{2i} - \frac{z}{2i}\right)\right).$$

We denote $\mu|\varpi|$ for the analogue on $\mathcal{O}_K$ of the usual Möbius function on $\mathbb{Z}$ and $\varphi|\varpi|(n)$ for the number of elements in the reduced residue class of $\mathcal{O}_K/(n)$. We recall the following explicitly evaluations of $g(r, n)$ given in [6] Lemma 2.2).

Lemma 2.2. (i) We have

$$g(rs, n) = \left(\frac{s}{n}\right) g(r, n), \quad (s, n) = 1,$$

$$g(k, mn) = g(k, m) g(k, n), \quad m, n \text{ primary and } (m, n) = 1.$$ (ii) Let $\varpi$ be a primary prime in $\mathcal{O}_K$. Suppose $\varpi^h$ is the largest power of $\varpi$ dividing $k$. (If $k = 0$ then set $h = \infty$.) Then for $l \geq 1$,

$$g(k, \varpi^l) = \begin{cases} 0 & \text{if } l \leq h \text{ is odd,} \\ \varphi|\varpi|(\varpi^l) & \text{if } l \leq h \text{ is even,} \\ -N(\varpi)^{l-1} & \text{if } l = h + 1 \text{ is even,} \\ \left(\frac{ik \varpi^h}{\varpi}\right) N(\varpi)^{l-1/2} & \text{if } l = h + 1 \text{ is odd,} \\ 0 & \text{if } l \geq h + 2. \end{cases}$$
2.3. **The approximate functional equation.** For any primitive quadratic Hecke character \( \chi \) of \( K \) of trivial infinite type, a well-known result of E. Hecke asserts that \( L(s, \chi) \) has an analytic continuation to the entire complex plane with a simple pole at \( s = 1 \) only when \( \chi \) is principal. Moreover, \( L(s, \chi) \) satisfies the following functional equation (see [9, Theorem 3.8])

\[
\Lambda(s, \chi) = W(\chi)(N(m))^{-1/2}\Lambda(1 - s, \chi),
\]

where \( m \) is the conductor of \( \chi \), \( |W(\chi)| = (N(m))^{1/2} \) and

\[
\Lambda(s, \chi) = (|D_K|N(m))^{s/2}(2\pi)^{-s}\Gamma(s)L(s, \chi).
\]

In particular, we have the following functional equation for \( \zeta_K(s) \):

\[
\pi^{-s}\Gamma(s)\zeta_K(s) = \pi^{-(1-s)}\Gamma(1-s)\zeta_K(1-s).
\]

When \( \chi = \chi_{(1+i)^s} \) for any odd, square-free \( d \in \mathcal{O}_K \), we combine [9, Theorem 3.8] and [4, Lemma 2.2] to see that \( W(\chi_{(1+i)^s}) = g(\chi_{(1+i)^s}) = \sqrt{N((1+i)^s)}d \). Thus, the functional equation (2.1) in this case becomes

\[
\Lambda(s, \chi_{(1+i)^s}) = \Lambda(1 - s, \chi_{(1+i)^s}).
\]

We now set

\[
\xi(s, \chi_{(1+i)^s}) = \left( \frac{2\pi N(d)}{\pi 2} \right)^{\frac{s}{2} - \frac{1}{4}} \Gamma(s)L(s, \chi_{(1+i)^s}).
\]

It follows from (2.2) and (2.3) that we have the following functional equation

\[
\xi(s, \chi_{(1+i)^s}) = \xi(1 - s, \chi_{(1+i)^s}).
\]

Throughout the paper, we fix a positive real number \( \kappa \leq 1/100 \) and let \( \delta_1 \) and \( \delta_2 \) be two complex numbers satisfying \( \max(|\delta_1|, |\delta_2|) \leq \kappa \). We further denote \( \tau = \frac{\delta_1 + \delta_2}{2}, \delta = \frac{\delta_1 - \delta_2}{2} \) so that we have \( \max(|\tau|, |\delta|) \leq \kappa \) as well. We define for real numbers \( x > 0, c > |\Re(\tau)| \),

\[
W_{\delta, \tau}(x) = \frac{1}{2\pi i} \int \Gamma_\delta(s)x^{-s} \frac{2s}{s^2 - \tau^2}ds,
\]

where

\[
\Gamma_\delta(s) = \Gamma\left( \frac{1}{2} + s + \delta \right)\Gamma\left( \frac{1}{2} + s - \delta \right).
\]

The following Lemma establishes some analytical properties concerning \( W_{\delta, \tau}(x) \).

**Lemma 2.4.** The function \( W_{\delta, \tau}(x) \) is a smooth complex-valued function for \( x \in (0, \infty) \). For \( x \) near \( 0 \), we have

\[
W_{\delta, \tau}(x) = \Gamma_\delta(\tau)x^{-\tau} + \Gamma_\delta(-\tau)x^\tau + O(x^{1/4 - \epsilon}).
\]

For large \( x \) and any integer \( \nu \), we have

\[
W^{(\nu)}_{\delta, \tau}(x) \ll \nu x^{\nu + 3}e^{-2x^{1/2}} \ll x^{1/2}. \]

**Proof.** We move the line of integration in (2.7) to \( \Re(s) = -1/4 + \epsilon \) to deduce readily (2.8). On the other hand, for any \( c > |\Re(\tau)| \), we have

\[
W^{(\nu)}_{\delta, \tau}(x) = \frac{(-1)^\nu}{2\pi i} \int \Gamma_\delta(s)s(s + 1) \cdots (s + \nu - 1)x^{-s-\nu} \frac{2s}{s^2 - \tau^2}ds.
\]

This implies that \( W_{\delta, \tau}(x) \) is smooth. To prove (2.9), we may suppose that \( x > \nu + 4 \). Using the facts that \( |\Gamma(x + iy)| \leq \Gamma(x) \) for \( x \geq 1 \) and \( s\Gamma(s) = \Gamma(s + 1) \), we see that

\[
W^{(\nu)}_{\delta, \tau}(x) \ll \nu |\Gamma(c + \nu + 3)|^2x^{-c} \int_{\Re(s - \tau^2)} |ds| \ll \nu (c + \nu + 3)^2x^{-c} \int_{\Re(s - \tau^2)} \frac{x^{-c}}{c - |\Re(\tau)|} \ll \nu \left( \frac{c + \nu + 3}{c - |\Re(\tau)|} \right)^{2c + 2\nu + 6} x^{-c}
\]

where the last estimation above follows from Stirling’s formula (see [9, (5.113)]). The last assertion of the lemma now follows by taking \( c = x^{1/2} - \nu - 3 \geq 2 \) above. \( \square \)
For any complex number \( s \) and any primary \( n \in \mathcal{O}_K \), we define
\[
 r_s(n) = \sum_{a,b \equiv 1 \mod (1+i)^3} \left( \frac{N(a)}{N(b)} \right)^s.
\]
Note that \( r_s(n) \) is easily seen to be an even function of \( s \). The first lemma can be obtained by applying a large sieve result of K. Onodera \cite{13} on quadratic residue
\[
\text{Lemma 2.7.}
\]
Let \( N, Q \) be positive integers, and let \( a_1, \ldots, a_n \) be arbitrary complex numbers. Let \( S(Q) \) denote the set of \( \chi_m \) for square-free \( m \) satisfying \( N(m) \leq Q \). Then for any \( \epsilon > 0 \),
\[
\sum_{\chi \in S(Q)} \left| \sum_{n \equiv 1 \mod (1+i)^3} a_n \chi(n) \right|^2 \ll_{\epsilon} (QN)^{\epsilon}(Q+N) \sum_{\eta_1,\eta_2 \equiv 1 \mod (1+i)^3} \left| a_{\eta_1} a_{\eta_2} \right|.
\]
Let \( M \) be a positive integer, and for each \( m \in \mathcal{O}_K \) satisfying \( N(m) \leq M \), we write \( m = m_1 m_2^2 \) with \( m_1 \) square-free and \( m_2 \in \mathbb{G} \). Suppose the sequence \( a_n \) satisfies \( |a_n| \ll (N)^{\epsilon} \), then
\[
\sum_{N(m) \leq M} \frac{1}{N(m_2)} \left| \sum_{N(n) \leq N} a_n m_1 m_2 \right|^2 \ll (MN)^{\epsilon} N(M+N).
\]
Similarly, combining the above result of Onodera with the derivation of \cite{8} Theorem 2 and \cite{16} Lemma 2.5, we have the following result.
\[
\text{Lemma 2.8.}
\]
Let \( S(Q) \) be as in Lemma 2.7. For any complex number \( \sigma + it \) with \( \sigma \geq \frac{1}{2} \), we have
\[
\sum_{\chi \in S(Q)} |L(\sigma + it, \chi)\chi|^{\epsilon} \ll Q^{1+\epsilon}(1+|t|^2)^{1+\epsilon}
\]
\[
\sum_{\chi \in S(Q)} |L(\sigma + it, \chi)\chi|^{2 \epsilon} \ll Q^{1+\epsilon}(1+|t|^2)^{1/2+\epsilon}.
\]
2.9. Poisson summation. We recall the following two dimensional Poisson summation formula, which follows from [6, Lemma 2.7, Corollary 2.8].

Lemma 2.10. Let \( n \in \mathcal{O}_K \) be primary and \( \left( \frac{-1}{n} \right) \) be the quadratic residue symbol modulo \( n \). For any smooth function \( W : \mathbb{R}^+ \to \mathbb{R} \) of compact support, we have for \( X > 0 \),

\[
\sum_{m \in \mathcal{O}_K \atop (m,1+i)=1} \left( \frac{m}{n} \right) W \left( \frac{N(m)}{X} \right) = \frac{X}{2N(n)} \left( \frac{1+i}{n} \right) \sum_{k \in \mathcal{O}_K} (-1)^N(k) g(k,n) \tilde{W} \left( \frac{N(k)X}{2N(n)} \right),
\]

where

\[
(2.10) \quad \tilde{W}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(N(x + yi)) \tilde{e}(-t(x + yi)) \, dx \, dy, \quad t \geq 0.
\]

When applying the above lemma in the proof of our result, we are led to consider the behaviors of a particular function. In the rest of this section, we include a result on this. We begin by defining

\[
(2.11) \quad F_y(t) = \Psi(t)W_{\delta, \tau} \left( \frac{\pi^2 y}{2^5 X t} \right),
\]

where \( \Psi(t) \) is a smooth function compacted in \([1, 2]\) and \( W_{\delta, \tau} \) is given in \([2.7]\).

The function that we are interested is then defined for \( \xi > 0 \) and \( \Re(w) > 0 \) by

\[
(2.12) \quad h(\xi, w) = \int_{0}^{\infty} \tilde{F}_1 \left( \left( \frac{\xi}{t} \right)^{1/2} \right) t^{w-1} \, dt.
\]

Thus, \( h(\xi, w) \) is the Mellin transform of the function \( \tilde{F}_1 \left( (\xi/t)^{1/2} \right) \). Here we recall that the Mellin transform \( \tilde{g}(s) \) of a function \( g \) is given by

\[
\tilde{g}(s) = \int_{0}^{\infty} g(t) t^{s-1} \, dt.
\]

For further reference, we note that if we further assume that \( g \) is support in \([1, 2]\), then integration by parts implies that for \( \Re(s) > 0 \), we have

\[
(2.13) \quad |\tilde{g}(s)| \ll \frac{2^{\Re(s)}}{|s|^n} g(n),
\]

where we define for integers \( n \geq 0 \),

\[
g(n) = \max_{0 \leq j \leq n} \int_{1}^{2} |g^{(j)}(t)| \, dt.
\]

Now, we have the following result concerning analytical properties of \( h(\xi, w) \).

Lemma 2.11. Let \( F_t \) be defined by \((2.11)\) and let \( \xi > 0 \). The function \( h(\xi, w) \) defined in \((2.12)\) is an entire function of \( w \) in \( 1 \geq \Re(w) > -1 + |\Re(\tau)| \) such that for any \( c \) with \( 1 + \Re(w) > c > \max(|\Re(\tau)|, \Re(w)) \),

\[
(2.14) \quad h(\xi, w) = \tilde{\Psi}(1 + w) \xi^w \frac{\pi}{2\pi i} \int_{(c)} \left( \frac{2^{5/2}}{\pi} \right)^{2s} \Gamma(s) \left( \frac{X}{\xi} \right)^s \pi^{-2s+2w} \frac{\Gamma(s-w)}{(1-s+w) s^2 - \tau^2} \, ds.
\]

Moreover, when \( 1 \geq \Re(w) > -1 + |\Re(\tau)| \), we have

\[
(2.15) \quad h(\xi, w) \ll (1 + |w|)^{3 - 2\Re(w)} \exp \left( -\frac{\xi^{1/4}}{10 X^{1/4}(|w| + 1)^{1/2}} \right) \xi^{\Re(w)} |\tilde{\Psi}(1 + w)|.
\]

Proof. Notice that for any smooth function \( W \), we can evaluate the function \( \tilde{W}(t) \) defined in \((2.10)\) in polar coordinates as

\[
\tilde{W}(t) = 4 \int_{0}^{\pi/2} \int_{0}^{\infty} \cos(2\pi r \sin \theta) W(r^2) \, r \, dr \, d\theta.
\]
Using this and the definition of $W_{\delta,\tau}(t)$ in (2.7), we deduce that, for $1 + \Re(w) > c > \max(|\Re(\tau)|, \Re(w))$, 

\[
h(\xi, w) = 4 \int_0^\infty \int_0^{\pi/2} \cos(2\pi \frac{\xi}{t}) \cos(r \sin \theta) \Psi(r^2) W_{\delta,\tau} \left( \frac{\pi^2 t}{2^{5/2} X r^2} \right) r \, dr \, d\theta \, dt
\]

\[
= 2 \int_0^\infty \int_0^{\pi/2} \cos(r) \Psi\left( \frac{rt^{1/2}}{2^{5/2} X^{1/2} \sin \theta} \right) W_{\delta,\tau} \left( \frac{\pi^2 t}{2^{5/2} X} \right) \frac{rt^{1/2}}{2 \pi \xi^{1/2} \sin \theta} \, d(rt^{1/2}) \, d\theta \, dt
\]

\[
= 2 \int_0^\infty \int_0^{\pi/2} \cos(r) W_{\delta,\tau} \left( \frac{\pi^2 t}{2^{5/2} X} \right) \left( \frac{r}{2 \pi \xi^{1/2} \sin \theta} \right)^{-2} \, d\theta \, dt
\]

\[
= 2 \tilde{\Psi}(1 + w) \int_0^{\pi/2} \int_0^\infty \cos(r) W_{\delta,\tau} \left( \frac{\pi^2 t}{2^{5/2} X} \right) \left( \frac{r}{2 \pi \xi^{1/2} \sin \theta} \right)^{-2} \, d\theta \, dt
\]

\[
= 4 \tilde{\Psi}(1 + w) \int_0^{\pi/2} \int_0^\infty \cos(r) \left( \frac{r}{2 \pi \xi^{1/2} \sin \theta} \right)^{-2} \, d\theta \, dt
\]

\[
= 4 \tilde{\Psi}(1 + w) \int_0^{\pi/2} \int_0^\infty \cos(r) (\sin(r))^{-2} \, dr \, d\theta
\]

We apply the relation (see [3 Section 2.4])

\[
\int_0^{\pi/2} (\sin \theta)^{-u} d\theta = \frac{\pi 2^{u-1} \Gamma \left( \frac{u}{2} \right)}{\Gamma \left( \frac{2u}{2} \right)}
\]

to see that

\[
\int_0^{\pi/2} (\sin \theta)^{-2s-2w} d\theta = \frac{\pi 2^{2s-2w} \Gamma(s-w)}{\Gamma(1-s+w)}
\]

Substituting this into the above expression for $h(\xi, w)$, we readily derive (2.14) by noticing that we can now take $c_s > \max(|\Re(\tau)|, \Re(w))$. This also implies that $h(\xi, w)$ is an entire function of $w$ for $1 \geq \Re(w) > -1 + |\Re(\tau)|$.

It remains to establish (2.15). For this, we set $c = \Re(s)$ and we may assume that $c \geq 2$ here. By apply Stirling’s formula given in [3 (5.112)], we see that

\[
\left( \frac{2^5/2}{\pi} \right)^{2s} \Gamma(s)(\pi)^{-2s+2w} \Gamma(s-w) \frac{2s}{(1-s+w)s^2-\tau^2}
\]

\[
\ll \left( \frac{2^5/2}{\pi^2} \right)^{2c} (1 + |s|)^{2c-1} e^{-\pi|\Im(s)|} (1 + |s-w|)^{2c-2\Re(w)-1}
\]

\[
\ll (|s|e^{1/2})^{2c-1} e^{-\pi|\Im(s)|} (1 + |s|^w)^{2c-2\Re(w)-1} (1 + |w|)^{2c-2\Re(w)-1}
\]

This implies that

\[
\int_{(c)} \left( \frac{2^5/2}{\pi} \right)^{2s} \Gamma(s)(\pi)^{-2s+2w} \frac{\Gamma(s-w)2sds}{(1-s+w)s^2-\tau^2} \ll e^{-c|w|^{4c-2\Re(w)-2}(1 + |w|)^{2c-2\Re(w)-1}}
\]

By taking

\[
c = \max(2, \frac{\xi^{1/4}}{X^{1/4}(1+|w|)^{1/2}})
\]

we see that the bound given (2.15) follows. \qed
2.12. Analytical behaviors of certain functions. Besides the function \( h(\xi, w) \) considered in the previous section, we also need to know analytical behaviors of a few other functions that are needed in the paper. We include several results in this section. First, we note that the following result can be established similar to [10] Lemma 5.3.

**Lemma 2.13.** Let \( \alpha, l \in \mathcal{O}_K \) be primary and for each \( k \in \mathcal{O}_K, k \neq 0 \), we write \( k \) uniquely by
\[
\sum_{n \equiv 1 \mod (1+i)^2} \frac{r_2(n)}{N(n)^s} g(k, ln) N(n)^{1/2} = L(s-\delta, \chi_{ik_1})L(s+\delta, \chi_{ik_1}) \prod_{\varpi \in \mathcal{G}} \mathcal{G}_\delta(s; k, l, \alpha)
\]
with \( k_1 \) square-free and \( k_2 \in \mathcal{G} \). For \( R(s) > 1 + |R(\delta)| \), we have
\[
|G_\delta(s; k, l, \alpha)| \leq N(\alpha)^c N(l)^{1/2} N((l, k^2_2))^\frac{1}{2}.
\]

Moreover, the function \( G_\delta(s; k, l, \alpha) \) is holomorphic for \( R(s) > \frac{1}{2} + |R(\delta)| + \epsilon \),
\[
|G_\delta(s; k, l, \alpha)| \ll N(\alpha)^c N(l)^{1/2} N((l, k^2_2))^\frac{1}{2}.
\]

Our next two lemmas provide bounds for certain dyadic sums involving \( G_\delta \) and \( h(\xi, w) \).

**Lemma 2.14.** Let \( K, L \geq 1 \) be two integers and let \( k_2 \) be defined in (2.17). For any sequence of complex numbers \( \delta_i \) satisfying \( |\delta_1| \ll N(\ell)^c \) and \( \delta_i = 0 \) when \( (l, 2\alpha) \neq 1 \), we have for \( R(w) = -\frac{1}{2} + |R(\delta)| + \epsilon \),
\[
\sum_{K \leq N(k) < 2K} \frac{1}{N(k)^2} \left| \frac{\delta_i}{N(\ell)} \mathcal{G}_\delta(1 + w; k, l, \alpha) \right|^2 \ll \epsilon (N(\alpha)LK)^c L(L + K).
\]

**Proof.** We write for any \( k \neq 0, k \in \mathcal{O}_K \) as \( k = u_k \prod_{\varpi \in \mathcal{G}} \varpi_i^{a_i} \) with \( u_k \in U_K \). We define the \( \varpi_i \) appearing in this product,
\[
a(k) = \prod_{i} \varpi_i^{a_i+1} \quad \text{and} \quad b(k) = \prod_{a_i=1}^{a_i} \prod_{a_i=2}^{a_i} \varpi_i^{a_i-1}.
\]

It follows from the definition of \( G_\delta \) in Lemma 2.13 and part (ii) of Lemma 2.2 that we may write \( l = gm \) with \( g|a(k), g \in G, (m, k) = 1 \) and \( m \) square-free, for otherwise we have \( G_\delta(1 + w; k, l, \alpha) = 0 \). Further, we see that when \( (l, 2\alpha) = 1 \),
\[
\mathcal{G}_\delta(1 + w; k, l, \alpha) = \sqrt{\frac{N(m)}{\varpi}} \prod_{\varpi \in \mathcal{G}} \left( 1 + \frac{r_\delta(\varpi)}{N(\varpi)^{1+w}} \left( \frac{ik_1}{\varpi} \right) \right)^{-1} \mathcal{G}_\delta(1 + w; k, g, \alpha).
\]

Applying the above and the Cauchy-Schwarz inequality, we deduce that
\[
\sum_{K \leq N(k) < 2K} \frac{1}{N(k)^2} \left| \frac{\delta_i}{N(\ell)} \mathcal{G}_\delta(1 + w; k, l, \alpha) \right|^2 \ll \epsilon K^c \sum_{K \leq N(k) < 2K} \frac{1}{N(k)^2} \sum_{g|a(k)} \Psi(k, g),
\]
where
\[
\Psi(k, g) = \left| \sum_{\varpi \in \mathcal{G}} \frac{\mu_2^2(m)\delta_{gm}}{N(g)} \mathcal{G}_\delta(1 + w; k, g, \alpha) \left( \frac{ik_1}{\varpi} \right) \prod_{\varpi \in \mathcal{G}} \left( 1 + \frac{r_\delta(\varpi)}{N(\varpi)^{1+w}} \left( \frac{ik_1}{\varpi} \right) \right) \right|^2.
\]
We use the bound for \( G_\delta \) given in Lemma 2.11 in the above expression to see that
\[
\Psi(k, g) \ll \epsilon (N(\alpha)LK)^c N(g)^{1+\epsilon} \sum_{N(g) \leq N(\ell) < 2N(g)} \frac{\mu_2^2(m)\delta_{gm}}{N(g)} \left( \frac{ik_1}{\varpi} \right) \prod_{\varpi \in \mathcal{G}} \left( 1 + \frac{r_\delta(\varpi)}{N(\varpi)^{1+w}} \left( \frac{ik_1}{\varpi} \right) \right) \right|^2.
\]
Note that as \( \left( \frac{ik}{m} \right) \neq 0 \),
\[
\prod_{\varpi \in G} \left( 1 + \frac{r_{\varpi}(\varpi)}{N(\varpi) + 1 + \varpi} \left( \frac{ik_1}{\varpi} \right) \right)^{-1} = \prod_{\varpi \in G} \left( 1 - \frac{r_{\varpi}(\varpi)^2}{N(\varpi)^2 + 2 + \varpi} \right)^{-1} \prod_{\varpi \in G} \left( 1 - \frac{r_{\varpi}(\varpi)}{N(\varpi) + 1 + \varpi} \left( \frac{ik_1}{\varpi} \right) \right)
\]
\[
= \prod_{\varpi \in G} \left( 1 - \frac{r_{\varpi}(\varpi)^2}{N(\varpi)^2 + 2 + \varpi} \right)^{-1} \sum_{j|m} \frac{\mu_{[j]}(j) r_{\varpi}(j)}{(j + \varpi)_j} \left( \frac{ik_1}{j} \right).
\]
Using this in (2.19), we obtain via another application of the Cauchy-Schwarz inequality that
\[
\Psi(k, g) \ll \varepsilon (N(\alpha)(\kappa) g) N(g)^{1+\varepsilon} \sum_{N(\varpi)^2} \left| \frac{\mu_{[j]}(j)}{\varpi} \right| \prod_{\varpi \in G} \left( 1 - \frac{r_{\varpi}(\varpi)^2}{N(\varpi)^2 + 2 + \varpi} \right)^{-1}.
\]
We relabel \( m \) by \( jm \) and note that for all \( \varpi|m \) and \( \Re(w) = -\frac{1}{2} + \Re(\delta) + \varepsilon \), we have
\[
\mu_{[j]}^2(j) \left( \frac{ik}{j} \right) \prod_{\varpi \in G} \left( 1 - \frac{r_{\varpi}(\varpi)^2}{N(\varpi)^2 + 2 + \varpi} \right)^{-1} \ll N(j)^{\varepsilon}.
\]
This implies that
\[(2.20) \quad \Psi(k, g) \ll \varepsilon (N(\alpha)(\kappa) g) N(g)^{1+\varepsilon} \sum_{N(\varpi)^2} \left| \frac{\mu_{[j]}(j)}{\varpi} \right| \prod_{\varpi \in G} \left( 1 - \frac{r_{\varpi}(\varpi)^2}{N(\varpi)^2 + 2 + \varpi} \right)^{-1}.
\]
Notice that \( g|a(k) \) implies \( b(g)|k \) by (2.18). We may thus relabel such \( k \) by \( fb(g) \) to deduce from (2.20) that
\[
\sum_{K \leq N(k) \leq 2K} \frac{1}{N(k_2)} \sum_{N(\varpi) < N(\varpi)} \Psi(k, g) \leq \sum_{N(\varpi) < 2L} \sum_{K \leq N(k) < 2K} \frac{1}{N(k_2)} \Psi(k, g) = \sum_{N(\varpi) < 2L} \sum_{K \leq N(k) < 2K} \frac{1}{N(k_2)} \Psi(fb(g), g)
\]
\[(2.21) \quad \ll \varepsilon (N(\alpha)(\kappa) g) N(g)^{1+\varepsilon} \sum_{N(\varpi) < 2L} \sum_{K \leq N(k) < 2K} \frac{1}{N(k_2)} \Psi(fb(g), g).
\]
On writing \( f = f_1 f_2^2 \) with \( f_1 \) square-free and \( f_2 \in G \), we observe that the relation \( fb(g) = k_1 k_2^2 \) implies that \( f_2 | k_2 \), so that \( N(k_2)^{-1} \ll N(f_2)^{-1} \). Applying this in (2.21), we see that the assertion of our lemma follows from Lemma 2.7. □

Lemma 2.15. Let \( K, L \geq 1 \) be two integers and let \( N(\alpha) \leq X \), \( X > 0 \). For \( \Re(w) = -\frac{1}{2} + \Re(\delta) + \varepsilon \) and any sequence of complex numbers \( \gamma_l \) satisfying \( |\gamma_l| \leq 1 \), the expression
\[
(2.22) \quad \sum_{K \leq N(k) < 2K} \frac{1}{N(k_2)} \left( \frac{1 + \gamma_k(1 + w)}{N(l) G_\delta(1 + w; k, l, \alpha) h \left( \frac{N(k)}{2N(\alpha)^2 l} \right)} \right)^2
\]
is bounded by
\[
\ll \varepsilon \left( (1 + |w|)^2 (1 + |w|)^{8/2} |\Re(\delta)| + \varepsilon \right) N(\alpha)^{2-4|\Re(\delta)| + \varepsilon} L^{2|\Re(\delta)| + \varepsilon} K^{-2|\Re(\delta)| + \varepsilon} \exp \left( - \frac{1}{20} \frac{\sqrt{K}}{N(\alpha)^{2} L (1 + |w|^2)} \right),
\]
and also by
\[
\ll \varepsilon \left( (1 + |w|) N(\alpha)(\kappa) X \right) \varepsilon \left( (1 + |w|)^2 \left( \frac{N(\alpha)^2 L (1 + |w|^2)}{K} \right)^{2|\Re(\delta)| - 2|\Re(\delta)|} N(\alpha)^2 L \right) \frac{KX^{1-2|\Re(\delta)|}}{KX^{1-2|\Re(\delta)|}} (K + L).
\]
Proof. We apply Lemma \ref{2.11} and Lemma \ref{2.13} to bound respectively $h(\xi, w)$ and $G_\delta$ to see that the expression in (2.22) is

$$
\ll |\tilde{\Psi}(1+w)| (1+|w|)^{8-4|\Re(\delta)|+\varepsilon} \frac{N(\alpha)^{2-4|\Re(\delta)|+\varepsilon}L^{-2|\Re(\delta)|+\varepsilon}K^2|\Re(\delta)|}{X^{1-2|\Re(\delta)|-\varepsilon}} \exp \left(-\frac{1}{20} \sqrt{\frac{\sqrt{K}}{N(\alpha)^2L(1+|w|^2)}} \right)
$$

\times \sum_{K \leq N(k)<2K} \frac{1}{N(k)N(k_2)} \left( \sum_{N(l)=L} \frac{1}{N(l)(1,2\alpha)=1} N((l, k_2), \frac{2L-1}{2}) \right)^2.

Upon writing $N(k) = N(k_1k_2)$, we readily deduce the first bound of the lemma from the above estimation.

To derive the second bound, we set $c = |\Re(\tau)| + \varepsilon$ to recast the integral in (2.14) as

$$
\frac{1}{2\pi i} \int_{|\Re(\tau)|=c} g(s, w) \left( \frac{X}{\xi} \right)^s ds.
$$

This implies that

$$
\ll |\tilde{\Psi}(1+w)| \left( \frac{N(\alpha)^{1+2|\Re(\tau)|-2|\Re(\delta)|+\varepsilon}}{K^{1+|\Re(\tau)|-|\Re(\delta)|} X^{1-2|\Re(\delta)|-\varepsilon}} \right) \int_{|\Re(\tau)|=c} g(s, w) \sum_{N(l)=L} \frac{1}{N(l)^{1+u-w-s}} G_\delta(1+w; k, l, \alpha) ds.
$$

Notice that (2.16) is still valid with $c = |\Re(\tau)| + \varepsilon$ and $\Re(w) = -\frac{1}{2} + |\Re(\delta)| + \varepsilon$ so that it implies that $g(s, w) \ll (1+|w|)^{2|\Re(\tau)|-2|\Re(\delta)|+\varepsilon} \exp(-\frac{\pi}{2}|\Im(s)|)$. We apply this estimation and the Cauchy-Schwarz inequality to deduce that

$$
\ll (1+|w|)^{2|\Re(\tau)|-2|\Re(\delta)|+\varepsilon} |\tilde{\Psi}(1+w)| \left( \frac{N(\alpha)^{1+2|\Re(\tau)|-2|\Re(\delta)|+\varepsilon}}{K^{1+|\Re(\tau)|-|\Re(\delta)|} X^{1-2|\Re(\delta)|-\varepsilon}} \right)
$$

\times \int_{|\Re(\tau)|=c} \left( \exp(-\frac{\pi}{2}|\Im(s)|) \right) \sum_{N(l)=L} \frac{1}{N(l)^{1+u-w-s}} G_\delta(1+w; k, l, \alpha) \left| ds \right|^2/2)
$$

By inserting the above bound into (2.22) and applying Lemma \ref{2.14}, we obtain the second bound of the lemma. $\square$

In the remaining of the section, we include two more results concerning various functions studied in this paper.

Lemma 2.16. Let $R$ be a polynomial satisfying $R(0) = R'(0) = 0$. Let $g$ be a multiplicative function satisfying $g(\overline{\omega}) = 1 + O(N(\overline{\omega})^{-\nu})$ for some fixed $\nu > 0$. Let $u$ and $v$ be two bounded complex numbers such that $\Re(u-v)$ are $\geq -D/\log y$ for an absolute positive constant $D$ and a large real number $y$. Then we have for $\Re(s) > 1 + D/\log y$,

$$
(2.23)
\sum_{n=1 \text{mod}(1+i)^5} n^{s+u} g(n) = \frac{\mu_1(c) G(s, c; u, v)}{\zeta_K(s+u+v) \zeta_K(s+u-v)}
$$

where $G(s, c; u, v) = \prod_{\sigma \in G} G_\sigma(s, c; u, v)$ is a holomorphic function in $\Re(s) > \max(\frac{1}{2}, 1-\nu) + D/\log y$ defined by

$$
G_\sigma(s, c; u, v) := \begin{cases}
(1 - \frac{1}{N(\overline{\omega})^{s+u}})^{-1} (1 - \frac{1}{N(\overline{\omega})^{s+u-v}})^{-1} (1 - \frac{1}{N(\overline{\omega})^{s+u-2v}})^{-1} (1 - \frac{g(\overline{\sigma}r_s(\overline{\omega}))}{N(\overline{\omega})^{s+2v}}) & \text{if $|\sigma|2c$,} \\
(1 - \frac{1}{N(\overline{\omega})^{s+u}})^{-1} (1 - \frac{1}{N(\overline{\omega})^{s+u}})^{-1} (1 - \frac{1}{N(\overline{\omega})^{s+u-2v}})^{-1} (1 - \frac{g(\overline{\sigma}r_s(\overline{\omega}))}{N(\overline{\omega})^{s+2v}}) & \text{otherwise.}
\end{cases}
$$
Moreover, for any odd \( c \in \mathcal{O}_K \) with \( N(c) \leq y \), we have
\[
\sum_{n \equiv 1 \mod (1+i)^3} r_v(n) \mu_i(nc) \frac{g(n) R \left( \frac{\log(y/N(cn))}{\log y} \right)}{N(n)^{1+u}} = O \left( \frac{E(c)}{\log y} \left( \frac{y}{N(c)} \right)^{-\Re(u)+|\Re(v)|} \exp(-A_0 \sqrt{\log(y/N(c))}) \right)
\]
(2.24)

\[
+ \text{Res}_{s=0} \frac{\mu_i(c) G(s+1, c; u, v)}{s \zeta_K(1+s+u+v) \zeta_K(1+s+u-v)} \sum_{k=0}^{\infty} \frac{1}{(s \log y)^k} R^{(k)} \left( \frac{\log(y/N(c))}{\log y} \right),
\]

where \( A_0 > 0 \) is an absolute constant and \( E(c) = \prod_{p | c} (1+1/\sqrt{N(c)}) \).

Proof. First note that we can establish (2.23) by considering Euler products. To prove (2.24), we may assume that \( N(c) \leq y/2 \). We then apply the Taylor expansion
\[
R(x) = \sum_{j=0}^{\infty} \frac{R^{(j)}(0)}{j!} x^j = \sum_{j=2}^{\infty} \frac{R^{(j)}(0)}{j!} x^j
\]
to write the sum in (2.24) as
\[
\sum_{j=2}^{\infty} \frac{R^{(j)}(0)}{j!} \sum_{n \equiv 1 \mod (1+i)^3} \frac{r_v(n) \mu_i(nc)}{N(n)^{1+u}} g(n) \log^j \left( \frac{y}{N(cn)} \right).
\]

We note that the inner sum above can be regarded as a Riesz type means so that we can apply the treatment given in [12 Sect. 5.1] to further write the above sums as
\[
\sum_{j=2}^{\infty} \frac{R^{(j)}(0)}{j!} \left( \frac{\log(y/N(c))}{\log y} \right)^j \sum_{n \equiv 1 \mod (1+i)^3} \frac{r_v(n) \mu_i(nc)}{N(n)^{1+u}} g(n) \log^j \left( \frac{y}{N(cn)} \right).
\]

To evaluate the integral above, we set \( T = \exp(\sqrt{\log(y/N(c))}) \) and apply the zero free region for \( \zeta_K(s) \) (see [12 Section 8.4]) to choose a positive constant \( A_1 \) such that \( \zeta_K(1+s+u+v) \zeta_K(1+s+u-v) \) has no zeros in the region \( \Re(s) \geq -\Re(u)+|\Re(v)|-A_1/\log T, \Im(s) \leq T \). We then notice that we have \( G(s+1, c; u, v) \ll E(c) \) uniformly for \( \Re(s) \geq -\Re(u)+|\Re(v)|-A_1/\log T \). Moreover, similar to the bound given for the Riemann zeta function in [12 Theorem 6.7], we have the following estimation for \( 1/\zeta_K(1+s) \) which asserts that for \( \Re(s) > -A_1/\log T \),
\[
\frac{1}{\zeta_K(1+s)} \ll \min(1, \log^2 \Im(s)).
\]

We now truncate the integral in (2.25) to the line segment \( \frac{D+1}{\log(y/N(c))} + iT \) to \( \frac{D+1}{\log(y/N(c))} + iT \) with \( T = \exp(\sqrt{\log(y/N(c))}) \). By applying the above estimations for \( G(s+1, c; u, v) \) and \( 1/\zeta_K(1+s) \), we see that the error introduced by doing so is
\[
\ll E(c)(\log y/N(c))^2/T^2.
\]

We then shift the remaining integral to the line segment \( -\Re(u)+|\Re(v)|-A_1/\log T \). Again by the above estimations for \( G(s+1, c; u, v) \) and \( 1/\zeta_K(1+s) \), we see that the two horizontal integrals are
\[
\ll E(c)(\log y/N(c))^2 \frac{1}{T^2}.
\]

For the vertical integral, we note that as \( \min(\Re(u+v), \Re(u-v)) \geq -D/\log y \), we have
\[
-\Re(u)+|\Re(v)|-A_1/\log T \ll -1/\sqrt{\log(y/N(c))}.
\]

We can thus divide the vertical integral into two parts, one over the segment \( |\Im(s)| \leq 1/\sqrt{\log(y/N(c))} \) and one over the rest. We apply (2.26) and the bound \( G(s+1, c; u, v) \ll E(c) \) to see that the vertical integral is
\[
\ll E(c)((\log(y/N(c)))^{j/2} + (\log(y/N(c)))^2(\log(y/N(c)))^{j/2}(y/N(c))^{-\Re(u)+|\Re(v)|-A_1/\log T}.
\]

As \( R \) is a polynomial, we know that \( R^{(j)}(0) \neq 0 \) only for finitely many \( j \) so that we may assume that \( j \) is bounded. It follows that for an appropriate positive constant \( A_0 \), the above is
\[
\ll E(c) \left( \frac{y}{N(c)} \right)^{-\Re(u)+|\Re(v)|} \exp(-A_0 \sqrt{\log(y/N(c))}).
\]

(2.29)
Furthermore, we note that we encounter a multiple pole at \( s = 0 \) in the above process. Thus, by combining (2.27), (2.28) and (2.29), we conclude that the expression given in (2.25) is

\[
\begin{align*}
\text{Res}_{s=0} & \left( \mu_{\gamma}(c)G(s + 1, c, u, v) \right) \\
&= s \sum_{j=2}^{\infty} R^{(j)}(0)(y/N(c))^s/s!(\log y)^j \\
&+ O\left( E(c) \left( \frac{y}{N(c)} \right)^{-\Re(u)+|\Re(v)|} \exp(-A_0 \sqrt{\log(y/N(c))}) \right).
\end{align*}
\]

We now calculate the residue above using the Taylor expansion of \((y/N(c))^s \) around \( s = 0 \) and discarding the powers of \( s \) that are \( \geq j + 1 \) since they make no contributions. This way, we see that the residue equals

\[
\begin{align*}
\sum_{j=2}^{\infty} R^{(j)}(0) \left( \frac{y}{N(c)} \right)^s/s!(\log y)^j = \sum_{k=0}^{\infty} s^{-k} \sum_{l=0}^{\infty} \frac{R^{(k+l)}(0)}{l!} \left( \frac{\log(y/N(c))}{\log y} \right)^l = \sum_{k=0}^{\infty} s^{-k} R^{(k)} \left( \frac{\log(y/N(c))}{\log y} \right),
\end{align*}
\]

where the first equality above following by setting \( k = j - l \) while noting that \( R(0) = R'(0) = 0 \). Applying this in (2.30) allows us to deduce (2.24) and this completes the proof of the lemma.

For our next result, we define for odd primary primes \( \varpi \),

\[
\begin{align*}
h_w(\varpi) &= \left( 1 + \frac{1}{N(\varpi)} \right) + \frac{1}{N(\varpi)^{1+2w}} - \frac{N(\varpi)^{-2\delta} + N(\varpi)^{2\delta}}{N(\varpi)^{2+2w}} + \frac{1}{N(\varpi)^{3+4w}}, \\
H_w(\varpi) &= 1 + \frac{1}{N(\varpi)^{2+2w}} - \frac{R(\varpi)^2}{N(\varpi)^{1+2w} H_w(\varpi)},
\end{align*}
\]

and extend the above definitions multiplicatively to functions \( h_w(n) \), \( H_w(n) \) on primary odd, square-free algebraic numbers \( n \in \mathcal{O}_K \).

Moreover, we write any odd \( l \in \mathcal{O}_K \) as \( l = l_1 l_2 \), with \( l_1 \) being square-free and \( l_2 \in G \). We define an absolutely convergent function \( \eta_{l}(s; l) = \prod_{\varpi \in G} \eta_{l_1 l^2}(s; l) \) for complex numbers \( w, s \) in the region \( |\Re(w)| \leq \frac{1}{3} \) and \( |\Re(s)| > \frac{1}{2} \) such that \( \eta_{l_1 l^2}(s; l) = (1 - 2^{-s-2w})(1 - 2^{-s}) \) and for primes \( \varpi \),

\[
\eta_{l_1 l^2}(s; l) = \left\{ \begin{array}{ll}
\frac{N(\varpi)}{N(\varpi)^{1+1}} \left( 1 - \frac{1}{N(\varpi)^{1}} \right) \left( 1 + \frac{1}{N(\varpi)^{1+1}} \right) + \frac{1}{N(\varpi)^{3+1}} & \text{if } \varpi \nmid l_1, \\
\frac{N(\varpi)^2}{N(\varpi)^{1+2}} \left( 1 - \frac{1}{N(\varpi)^{1}} \right) \left( 1 - \frac{1}{N(\varpi)^{1+2}} \right) & \text{otherwise,}
\end{array} \right.
\]

Lastly, we denote \( C \) for a closed contour (oriented counter-clockwise) containing the points \( \pm \tau \) with perimeter length being \( \ll |\delta_1| \). Also, for \( w \in C \) we have \( |\Re(w)| \leq |\tau| + C/\log X, |\Im(w)| \leq C/\log X \) for some absolute constant \( C \) and such that \( \min(|w^2 - \tau^2|, |w^2 - \delta^2|) \geq \epsilon^2/(3 \log^2 X) \). We then have the following result.

**Lemma 2.17.** With notations as above, we have for \( w \) on the contour \( C \) and \( x \geq 2 \),

\[
\begin{align*}
\eta(1 + 2w; 1) \sum_{N(\gamma) \leq x} \frac{\mu_{\gamma}^{2}(\gamma) H_w(\gamma)}{N(\gamma)^{1+2\tau} h_w(\gamma)} G(1, \gamma; \delta_1 + w, \delta) G(1, \gamma; \delta_2 + w, \delta) \\
= \zeta_K(1 + 2\tau)(1 - x^{-2\tau})(1 + O(|w - \tau|)) + O(x^{-2\tau}).
\end{align*}
\]

**Also,** for any smooth function \( R \) on \([0, 1]\) and \( 1 \leq y \leq x \), we have

\[
\begin{align*}
\eta(1 + 2w; 1) \sum_{N(\gamma) \leq x} \frac{\mu_{\gamma}^{2}(\gamma) H_w(\gamma)}{N(\gamma)^{1+2\tau} h_w(\gamma)} G(1, \gamma; \delta_1 + w, \delta) G(1, \gamma; \delta_2 + w, \delta) R\left( \frac{\log N(\gamma)}{\log x} \right) \\
= \frac{\pi}{4} \left( 1 + O(|\delta_1|) \right) \int_{y}^{x} R\left( \frac{\log t}{\log x} \right) dt.
\end{align*}
\]

**Proof.** Applying the definition of \( G(s, \gamma; u, v) \) given in Lemma 2.14 allows us to write the expression given in (2.32) as

\[
\begin{align*}
\eta(1 + 2w; 1) G(1, 1; \delta_1 + w, \delta) G(1, 1; \delta_2 + w, \delta) \sum_{N(\gamma) \leq x} \frac{f_w(\gamma)}{N(\gamma)^{1+2\tau}},
\end{align*}
\]
where
\[ f_w(\gamma) = \mu_i^2(\gamma) \frac{H_w(\gamma)}{h_w(\gamma)} \prod_{\varpi \in G} \left( 1 - \frac{r_\varpi(\varpi)}{N(\varpi)^{1+\varepsilon} h_w(\varpi)} \right)^{-1} \left( 1 - \frac{r_\varpi(\varpi)}{N(\varpi)^{1+2\varepsilon} h_w(\varpi)} \right)^{-1}. \]

Note here that \( f_w(\gamma) \) is a multiplicative function such that \( f_w(\varpi) = 1 + O(1/\sqrt{N(\varpi)}) \). This allows us to write
\[ \sum_{\gamma \equiv 1 \mod (1+i)^3} f_w(\gamma)/N(\gamma)^s = \zeta_K(\gamma)F_w(s) \] so that \( F \) is holomorphic in \( \Re(s) > \frac{1}{2} \).

Now we apply Perron’s formula as given in [12, Theorem 5.2, Corollary 5.3] to obtain that for \( \sigma = -2\tau + \varepsilon, T = x^{1+2\tau+\varepsilon}, \)
\[ \sum_{N(\gamma) \leq x} \frac{f_w(\gamma)}{N(\gamma)^{1+2\tau}} = \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} \zeta_K(1+2\tau+s)F_w(1+2\tau+s)x^s ds + R, \]
where
\[ R \ll \sum_{x/2 < N(n) < x \atop N(n) \neq x} \frac{f_w(n)}{N(n)^{1+2\tau}} \min(1, x/T|\gamma - N(n)|) + \frac{4^\sigma + x^{\sigma}}{T} \sum_{n \equiv 1 \mod (1+i)^3} \frac{|f_w(n)|}{N(n)^{\sigma}}. \]

It is then easy to see that
\[ (2.34) \quad R \ll \sum_{1 \leq k < x} \frac{x}{Tk} + \frac{x^{-2\tau+\varepsilon}}{T} \ll \frac{x \log x}{T} \ll x^{-2\tau}. \]

We now shift the contour of integration to \( \sigma_1 = -2\tau - 1/2 - \varepsilon \) to see that we have
\[ (2.35) \quad \frac{1}{2\pi i} \int_{\sigma_1-iT}^{\sigma_1+iT} \zeta_K(1+2\tau+s)F_w(1+2\tau+s)x^s ds \ll x^{-2\tau}. \]

Moreover, by applying the following subconvexity bound for \( \zeta_k(s) \) on the critical line given in [7],
\[ \zeta_K(s) \ll (1+|s|)^{1/3+\varepsilon}, \quad \Re(s) = 1/2, \]
we deduce that
\[ (2.36) \quad \frac{1}{2\pi i} \int_{\sigma_1-iT}^{\sigma_1+iT} \zeta_K(1+2\tau+s)F_w(1+2\tau+s)x^s ds \ll \int_{-T}^{T} (1+|t|)^{1/3+\varepsilon} x^{-2\tau-1/2+\varepsilon} 1 + |t| \, dt \ll x^{-2\tau}. \]

By combining (2.34), (2.35) and (2.36), together with the observation that \( \zeta_K(1+2\tau) = \frac{T}{2\pi}1/(2\tau) + O(1), F_{w}(1+2\tau) = F_{w}(1) + O(\tau), \) we then conclude that
\[ \sum_{\gamma \equiv 1 \mod (1+i)^3} \frac{f_w(\gamma)}{N(\gamma)^{1+2\tau}} = \zeta_K(1+2\tau)(1-x^{-2\tau})F_w(1+2\tau) + O(x^{-2\tau}). \]

If we set for brevity that \( F(w) = \eta_0(1+2w; 1)G(1, 1; \delta_1 + w, \delta)G(1, 1; \delta_2 + w, \delta)F_{w}(1+2\tau), \) then a little calculation implies that \( F(\tau) = 1 \) and \( F(w) = F(\tau) + O(|w - \tau|). \) This implies (2.32), which in turn implies (2.34) by partial summation and this completes the proof of the lemma. \( \square \)

3. Plan of the proof

We define for any large number \( X \) and any smooth function \( \Phi \) supported in [1, 2],
\[ \mathcal{N}(X, \Phi) = \sum_{\substack{(d,2)=1 \atop L(\beta, \chi_{1+i}^3d) = 0; 0 < \beta \leq 1}} \mu_i^2(d) \Phi\left( \frac{N(d)}{X} \right), \]
so that \( \mathcal{N}(X, \Phi) \) is a weighted count on the number of odd, square-free algebraic integers \( d \in \mathcal{O}_K \) such that \( X \leq N(d) \leq 2X \) and that \( L(s, \chi_{1+i}^3d) \) has a non-trivial real zero. The proof of Theorem [14] is based on an estimation of \( \mathcal{N}(X, \Phi) \). To achieve this, we apply the following argument principle given in [2, Lemma 2.1], which is originally due to A. Selberg [15].
Arguing as in [2, Section 2], we deduce that
\[ \lambda \approx Y \]

For other values of \((3.1)\)

We now apply Lemma 3.1 to the function \(f(z)\) with \(W_0 = 1\) and \(W_1 = 1\). Then

\[ \int_{-H}^{H} \cos \left( \frac{\pi t}{2H} \right) \log |f(W_0 + it)| dt + \int_{W_0}^{W_1} \sinh \left( \frac{\pi (\alpha - W_0)}{2H} \right) \log |f(\alpha + iH)f(\alpha - iH)| d\alpha \]

To proceed further, we define for any sequence \(\{a_n\}_{n \in \mathbb{O}_K}\) of complex numbers and any smooth function \(\Phi\) supported in \([1, 2]\

\[ S(a_d; \Phi) = S(a_d; \Phi, X) = \frac{1}{X} \sum_{(d,2)=1} \mu^2_{[d]}(d)a_d\Phi\left(\frac{N(d)}{X}\right). \]

For a real parameter \(1 < Y \leq \sqrt{2X}\) to be determined later, we write

\[ \mu^2_{[d]}(n) = M_Y(n) + R_Y(n), \]

with

\[ M_Y(n) = \sum_{\ell \in G \atop \ell^2 \equiv n \mod N} \mu_{[\ell]}(\ell), \quad R_Y(n) = \sum_{\ell \in G \atop \ell^2 \equiv n \mod N, N(\ell) > Y} \mu_{[\ell]}(\ell). \]

We then deduce that \(S(a_d; \Phi) = S_M(a_d; \Phi) + O(S_R(a_d; \Phi))\), where

\[ S_M(a_d; \Phi) = S_M(a_d; \Phi) = \frac{1}{X} \sum_{(d,2)=1} M_Y(d)a_d\Phi\left(\frac{N(d)}{X}\right), \]

\[ S_R(a_d; \Phi) = S_R(a_d; \Phi) = \frac{1}{X} \sum_{(d,2)=1} R_Y(d)a_d\Phi\left(\frac{N(d)}{X}\right). \]

Let \(\varepsilon > 0\) be a small number and let \(b, M\) be two parameters such that \(b \in [\varepsilon, 1 - \varepsilon]\) and \(X^\varepsilon \leq M \leq X\). Also, let \(P(x)\) be a polynomial satisfying \(P(0) = P'(0) = 0\) and \(P(b) = 1, P'(b) = 0\). We define a sequence \(\{\lambda(n)\}_{n \in \mathbb{O}_K}\) such that when \(n \equiv 1 \mod (1+i)^3\) and \(N(n) \leq M\), we have

\[ \lambda(n) := \mu_{[i]}(n)Q\left(\frac{\log(M/N(n))}{\log M}\right) = \begin{cases} \mu_{[i]}(n) & \text{if } N(n) \leq M^{1-b}, \\ \mu_{[i]}(n)P\left(\frac{\log(M/N(n))}{\log M}\right) & \text{if } M^{1-b} \leq N(n) \leq M. \end{cases} \]

For other values of \(n\), we define \(\lambda(n) = 0\). The definition above then implies that \(\lambda(n) \ll N(n)^\varepsilon\) for all \(n\). We use the \(\lambda(n)\) to define for any odd \(d \in \mathbb{O}_K\), the following mollifier function

\[ M(s, d) = \sum_{N(n) \leq M \atop n \equiv 1 \mod (1+i)^3} \frac{\lambda(n)}{N(n)^s} \chi_{(1+i)^3}(n). \]

Further, we define for any complex number \(\delta_1\),

\[ W(\delta_1, \Phi) = \frac{S(\{L(\frac{1}{2} + \delta_1, \chi_{(1+i)^3})M(\frac{1}{2} + \delta_1, d)^2; \Phi)\}}{S(1; \Phi)}. \]

We now apply Lemma 3.1 to the function \(f(s, d) := L(s, \chi_{(1+i)^3})M(s, d)\) with \(W_0 = \frac{1}{2} - \frac{b}{\log X}\), \(H = \frac{\log X}{s}\), and \(W_1 = \sigma_0 := 1 + 3\log \log M / \log M\) and \(R, S \in (\varepsilon, 1/\varepsilon)\) to be chosen later, such that \(f(s, d)\) has no zeros in \(\Re(s) > \sigma_0\). Arguing as in [2, Section 2], we deduce that

\[ N(X, \Phi) \leq \frac{X}{8S \sinh\left(\frac{2\pi}{2S}\right)} \left( J_1(X; \Phi) + J_2(X; \Phi) \right) + \frac{X}{8S \sinh\left(\frac{2\pi}{2S}\right)} S(I(d); \Phi), \]
Proposition 3.4. Let \( \Phi \) be a smooth function supported on \([1, 2]\) such that \( 0 \leq \Phi(t) \leq 1 \) and \( \int_1^2 \Phi(t) dt \gg 1 \). For large \( X > 0 \), \( M \leq \sqrt{X} \), \( \sigma_0 = 1 + 3 \log \log M / \log M \) and \( \epsilon \leq \Re(\delta_1) < 3/4 \), we have
\[
W(\delta_1, \Phi) = 1 + O(\Phi(2X/\pi^2)) \Gamma(\tau)W(\delta_1, \Phi).
\]
Moreover, the function \( f(s, d) = L(s, \chi_{(1+i)^d})M(s, d) \) has no zeros in \( \Re(s) > \sigma_0 \), and we have
\[
S(I(d); \Phi) \ll \exp \left( \frac{1}{2} + \epsilon \log X \right) M^{-1(1-b)}e^X.
\]

Next, notice that an evaluation on the terms in (3.3) involving with \( J_1(X; \Phi) \) and \( J_2(X; \Phi) \) requires one to study \( W(\delta_1, \Phi) \). As Proposition 3.2 enables us to treat \( W(\delta_1, \Phi) \) when \( \Re(\delta_1) \) is slightly away from 0, we may focus on the case when \( \delta_1 \) is near 0. For this purpose, we shall evaluate more generally the following expression:
\[
S(\xi(1/2 + \delta_1; \chi_{(1+i)^d}) & \xi(1/2 + \delta_2; \chi_{(1+i)^d}) M(1/2 + \delta_1, d) M(1/2 + \delta_2, d); \Psi),
\]
where \( \xi \) is given in (2.3), \( \Psi \) is a smooth function supported on \([1, 2]\) and \( \delta_1, \delta_2 \) satisfy the conditions given in Section 2.3. Later, we shall set \( \delta_2 = \delta_1 \) and \( \Psi(t) = \Phi(t)^{-1} \) in (3.5) to retrieve the expression \( S(1; \Phi)(2X/\pi^2) \Gamma(\tau)W(\delta_1, \Phi) \).

In order to evaluate (3.5), we apply the approximate functional equation for \( \xi(1/2 + \delta_1; \chi_{(1+i)^d}) \xi(1/2 + \delta_2; \chi_{(1+i)^d}) \) to see that we may recast the expression in (3.5) as
\[
S_M(A_{\delta, \tau}(d)M(1/2 + \delta_1, d) M(1/2 + \delta_2, d); \Psi) + S_R(A_{\delta, \tau}(d)M(1/2 + \delta_1, d) M(1/2 + \delta_2, d); \Psi).
\]

In Section 4.2 we obtain the following estimation for \( S_R \).

Proposition 3.3. With the above notations, we have for \( M \leq \sqrt{X} \),
\[
S_R(A_{\delta, \tau}(d)M(1/2 + \delta_1, d) M(1/2 + \delta_2, d); \Psi) \ll X^{\kappa + \epsilon} \left( \frac{1}{Y^2} + \frac{M^{-\Re(\delta_1)} + M^{-\Re(\delta_2)} + M^{-2\Re(\tau)}}{Y^2} + \frac{M^{-2\Re(\tau)}}{X^2} \right).
\]

The treatment on \( S_M(A_{\delta, \tau}(d) \left( \frac{1}{1+i} \right)^d; \Psi) \) for any odd primary \( l \in \mathcal{O}_K \). To state our result, we introduce a few notations.

We define for any two complex numbers \( s \) and \( w \),
\[
Z(s; w) = \zeta_K(s - 2w)\zeta_K(s)\zeta_K(s + 2w).
\]

Our evaluation on \( S_M \) is given in the following result.

Proposition 3.4. With the above notations and writing any odd primary \( l \in \mathcal{O}_K \) as \( l = l_1l_2^2 \), with \( l_1 \) being primary, square-free and \( l_2 \in \mathbb{G} \), we have
\[
S_M \left( \frac{(1 + i)^d}{l} \right) A_{\delta, \tau}(d); \Psi) = \frac{2}{3\xi_K(2)\sqrt{N(l_1)}} \sum_{\mu = \pm} \left( r_\delta(l_1) \Gamma_\delta(\mu \tau) \left( \frac{2^5 X}{N(l_1)\pi^2} \right)^{\mu \tau} \Psi(\mu \tau + 1)Z(1 + 2\mu \tau; \delta) \eta(1 + 2\mu \tau; l)
\right.
\]
\[
+ r_\tau(l_1) \Gamma_\tau(\mu \delta) \left( \frac{2^5 X}{N(l_1)\pi^2} \right)^{\mu \delta} \Psi(\mu \delta + 1)Z(1 + 2\mu \delta; \tau) \eta(1 + 2\mu \delta; l)
\right)
\]
\[
+ \mathcal{R}(l) + O \left( \frac{|r_\delta(l_1)|X^{\epsilon}}{(XN(l_1))^{1/2}} + \frac{N(l_1)^{\kappa + \epsilon}N(l_1)^{2\kappa - \frac{1}{2}}}{Y^{1-4\epsilon}} \right).
\]

Here \( \mathcal{R}(l) \) is a remainder term bounded on average by
\[
\sum_{l \equiv 1 \text{ mod } (1+i)^3} |\mathcal{R}(l)| \ll \left( \frac{l_1^{1+\epsilon}Y^{1+\epsilon}}{X^{\frac{1}{2}-\Re(\delta)}-\epsilon} + \frac{l_1^{1+\kappa + \epsilon}Y^{2\kappa + \epsilon}}{X^{\frac{1}{2}-\Re(\delta)}-\epsilon} \right) \Psi(3) \Psi(4).\]
With Propositions 3.2, 3.3 available, we are able to obtain the following result concerning $W(\delta_1, \Phi)$ for $|\delta_1|$ being small.

**Proposition 3.5.** Let $\Phi$ be a non-negative smooth function on $[1, 2]$ satisfying $\Phi(t) \ll 1$ and $\int_1^2 \Phi(t)dt \gg 1$. Let $\delta_1$ be a complex number such that $\Re(\delta_1) \geq -\frac{1}{\epsilon \log X}$, $\kappa \geq |\delta_1| \geq \frac{\kappa}{\log X}$. We take $\delta_2 = \frac{\delta_1}{\kappa}$ so that $\tau = \Re(\delta_1)$, and $\delta = i\Im(\delta_1)$. Then with the mollifier function being given in (3.2), we have for $M = X^{\frac{1}{2} - 5\kappa/2}$,

$$W(\delta_1, \Phi) = 1 + \left(1 - \frac{\log X}{2\pi} \right)^{-2\tau} \left(\frac{2\log M}{\log X}\right)^{\frac{1}{4}} \int_0^b M^{-2\tau(1-x)} \left|Q(x) + \frac{Q''(x)}{2\delta_1 \log M}\right|^2 dx$$

$$+ O(X^{-\kappa - \Phi(3)} + M^{-2\tau(1-b)} |\delta_1|^6 \log^5 X).$$

The proofs of Propositions 3.2-3.5 will occupy major parts in the rest of the paper. In the end, the proof of Theorem 4.1 follows by gathering these results together with suitable choices for the parameters involved.

4. Proofs of Propositions 3.2 and 3.3

4.1. Proof of Proposition 3.2. In this section we estimate $S(I(d); \Phi)$ by proving Proposition 3.2. We note first that similar to the bounds given in [2] (4.1), (4.2)), it follows from Lemma 2.6 that for any odd $l \in \mathcal{O}_K$ such that $N(l) \leq \sqrt{2X}$, we have

$$\sum_{X \leq N(l) \leq 2X} \mu_l^2 |(1 + i) d| M(s, d) \ll X^\epsilon (X + XM^{2(1 - 2\Re(s))} + M^{4(1 - \Re(s))}),$$

(4.1)

$$\sum_{X/N(l)^2 \leq N(m) \leq 2X/N(l)^2} \mu_l^2 |(1 + i) m| M(s, l^2 m) \ll X^\epsilon \left(\frac{X}{N(l)^2} + \frac{X}{N(l)^2} M^{2(1 - 2\Re(s))} + M^{4(1 - \Re(s))}\right).$$

Now, we write $B(s, d) = L(s, \chi_1 + i \delta) M(s, d) - 1$ and we observe that it follows from [3] Section 3.1 and partial summation that we have

$$S(1; \Phi) = \frac{2\pi \Phi(1)}{3\zeta(2)} + O(X^{-1/2}) \gg 1.$$  

Using the above, we see that

$$W(\delta_1, \Phi) = 1 + O(S(B(\frac{1}{2} + \delta_1, d); \Phi) + S(B(\frac{1}{2} + \delta_1, d; \Phi))).$$

To estimate the error terms above, for any real number $c > \frac{1}{2} - \Re(\delta_1)$, consider the integral

$$\frac{1}{2\pi i} \int_{(c)} \Gamma(s) B(\frac{1}{2} + \delta_1 + s, d) X^s ds.$$

By moving the line of integration above to $\Re(s) = -\Re(\delta_1)$, we see that the pole at $s = 0$ contributes $B(\frac{1}{2} + \delta_1; d)$. This implies that

$$B(\frac{1}{2} + \delta_1, d) = \frac{1}{2\pi i} \int_{(c)} \Gamma(s) B(\frac{1}{2} + \delta_1 + s, d) X^s ds - \frac{1}{2\pi i} \int_{(-\Re(\delta_1))} \Gamma(s) B(\frac{1}{2} + \delta_1 + s, d) X^s ds$$

$$:= T_1(\frac{1}{2} + \delta_1, d) - T_2(\frac{1}{2} + \delta_1, d).$$

We apply the Cauchy-Schwarz inequality to see that

$$|T_2(\frac{1}{2} + \delta_1, d)|^2 \ll X^{-2\Re(\delta_1)} \left(\int_{(-\Re(\delta_1))} |\Gamma(s)| B(\frac{1}{2} + \delta_1 + s, d)^2 ds\right)^2 \left(\int_{(-\Re(\delta_1))} |\Gamma(s)| ds\right).$$

It follows from this and the rapid decay of $|\Gamma(s)|$ when $|\Im(s)| \to \infty$ that we have

$$|T_2(\frac{1}{2} + \delta_1, d)|^2 \ll X^{-2\Re(\delta_1)} \left(1 + \int_{(-\Re(\delta_1))} |\Gamma(s)| |L(\frac{1}{2} + \delta_1 + s, \chi_{-s} d) M(\frac{1}{2} + \delta_1 + s, d)|^2 ds\right).$$

Applying the Cauchy-Schwarz inequality again, we see that

$$S(|T_2(\frac{1}{2} + \delta_1, d)|^2; \Phi) \ll X^{-2\Re(\delta_1)} \left(1 + \int_{(-\Re(\delta_1))} |\Gamma(s)| S(|L(\frac{1}{2} + \delta_1 + s, \chi_{-s} d)|^4; \Phi) \frac{\Phi}{\delta_1} S(|M(\frac{1}{2} + \delta_1 + s, d)|^4; \Phi)^\frac{1}{2} ds\right).$$
Applying Lemma 2.8 and 4.1 in the above estimation, we see that for $M \leq \sqrt{X}$,

\begin{equation}
S\left( |T_2(\frac{1}{2} + \delta_1, d)|^2; \Phi \right) \ll X^{-2\Re(\delta_1)} + \epsilon.
\end{equation}

With one more application of the Cauchy-Schwarz inequality, we deduce from the above that

\begin{equation}
S\left( |T_2(\frac{1}{2} + \delta_1, d)|; \Phi \right) \ll X^{-\Re(\delta_1)} + \epsilon.
\end{equation}

Now, we consider the contribution from $T_1$. For $\Re(s) > 1$, we write

\begin{equation}
B(s, d) = \frac{1}{2\pi i} \int \Gamma(s) \sum_{n \equiv 1 \text{ mod } (1+it)^3} \frac{b(n)}{N(n)^{\frac{1}{2} + \delta_1 + s}} \left( \frac{(1+i)s}{n} \right)^X ds.
\end{equation}

Using the result that

\begin{equation}
\frac{1}{2\pi i} \int \Gamma(s) \left( \frac{X}{N(n)} \right)^s ds = \exp(-N(n)/(20X)),
\end{equation}

we see that the second term on the right side of (4.6) contributes $\ll X^{-5}$. We then move the line of integration in the first term on the right side of (4.6) to $\Re(s) = \frac{1}{\log X}$ to see that

\begin{equation}
T_1(\frac{1}{2} + \delta_1, d) = \frac{1}{2\pi i} \int \Gamma(s) \sum_{n \equiv 1 \text{ mod } (1+it)^3} \frac{b(n)}{N(n)^{\frac{1}{2} + \delta_1 + s}} \left( \frac{(1+i)s}{n} \right)^X ds + O(X^{-5}).
\end{equation}

It follows from this and the Cauchy-Schwarz inequality that we have

\begin{equation}
|T_1(\frac{1}{2} + \delta_1, d)|^2 \ll X^{-10} + \left( \int \Gamma(s)| \right) \sum_{n \equiv 1 \text{ mod } (1+it)^3} \frac{b(n)}{N(n)^{\frac{1}{2} + \delta_1 + s}} \left( \frac{(1+i)s}{n} \right)^{2|ds|} \left( \int \Gamma(s)|ds\right)
\end{equation}

\begin{equation}
\ll X^{-10} + X^\epsilon \left( \int \Gamma(s)| \right) \sum_{n \equiv 1 \text{ mod } (1+it)^3} \frac{b(n)}{N(n)^{\frac{1}{2} + \delta_1 + s}} \left( \frac{(1+i)s}{n} \right)^{2|ds|}.
\end{equation}

We split the sum over $n$ above into dyadic blocks and apply Lemma 2.7 to see that

\begin{equation}
S\left( |T_1(\frac{1}{2} + \delta_1, d)|^2; \Phi \right) \ll M^{-2\Re(\delta_1)(1-b)} X^\epsilon.
\end{equation}

Combining this with (4.4), we deduce that

\begin{equation}
S\left( |B(\frac{1}{2} + \delta_1, d)|^2; \Phi \right) \ll M^{-2\Re(\delta_1)(1-b)} X^\epsilon.
\end{equation}

Next, we bound $S(T_1(\frac{1}{2} + \delta_1, d); \Phi)$ by applying (4.7) to see that

\begin{equation}
S(T_1(\frac{1}{2} + \delta_1, d); \Phi) \ll X^{-5} + X^\epsilon \sum_{n \equiv 1 \text{ mod } (1+it)^3} \frac{|b(n)|}{N(n)^{\frac{1}{2} + \Re(\delta_1)}} |S(\chi(1+i)s; n)|.
\end{equation}
We now apply the Mellin transform to see that for any $c > 1$,

$$
S(\chi(1+i) \delta_d(n); \Phi) = \frac{(1+i)^5}{2\pi i} \int (c) \sum_{d \equiv 1 \mod (1+i)^3} \frac{\mu^2((1+i)d)\chi(n)}{N(d)^w} X^{w-1} \widehat{\Phi}(w) dw
$$

$$
= \frac{(1+i)^5}{2\pi i} \int (c) \frac{L(w, \chi_n)}{L(2w, \chi_n)} (1 + \frac{\chi_n(1+i)}{2w})^{-1} X^{w-1} \widehat{\Phi}(w) dw.
$$

Moving the line of integration above to $\Re(w) = \frac{1}{2} + \frac{1}{\log X}$, we see that we encounter a pole at $w = 1$ if and only if $n = \square$ and the corresponding contribution of the residue is $\ll 1$. It follows that

$$
|S(\chi(1+i) \delta_d(n); \Phi)| \ll \delta(n = \square) + X^{-\frac{1}{2} + \varepsilon} \int (\frac{1}{2} + \frac{1}{\log X}) |L(w, \chi_n)||\widehat{\Phi}(w)| |dw|,
$$

where $\delta(n = \square)$ is 1 if $n = \square$ and 0 otherwise. As $b(n) = 0$ for all perfect squares with norm $\leq M^{2(1-b)}$, we deduce that

$$
\sum_{n \equiv 1 \mod (1+i)^3} \frac{|b(n)|}{N(n)^{\frac{1}{2} + \Re(\delta_1)}}|S(\chi(1+i) \delta_d(n); \Phi)| \ll X^\varepsilon M^{-2\Re(\delta_1)(1-b)} + X^{-\frac{1}{2} + \varepsilon} \int (\frac{1}{2} + \frac{1}{\log X}) \sum_{n \equiv 1 \mod (1+i)^3} \frac{1}{N(n)^{\frac{1}{2} + \Re(\delta_1)}} |L(w, \chi_n)||\widehat{\Phi}(w)| |dw|.
$$

(4.10)

Note that Lemma 2.8 implies that

$$
\sum_{n \equiv 1 \mod (1+i)^3} |L(w, \chi_n)| \ll N^{1+\varepsilon}(1 + |w|^2)^{\frac{1}{2} + \varepsilon}.
$$

Applying this together with (2.13) by setting $g = \Phi$ and $\nu = 2$ there, we obtain from (4.10) that

$$
\sum_{n \equiv 1 \mod (1+i)^3} \frac{|b(n)|}{N(n)^{\frac{1}{2} + \Re(\delta_1)}}|S(\chi(1+i) \delta_d(n); \Phi)| \ll X^\varepsilon \Phi(2) \left( M^{-2\Re(\delta_1)(1-b)} + X^{-\Re(\delta_1)} + M^{\left(\frac{1}{2} - \Re(\delta_1)\right)(1-b)} X^{-\frac{1}{2}} \right).
$$

We insert the above in (4.9) and further combine the result with (4.5) to deduce that for $M \leq \sqrt{X}$,

$$
S(B(\frac{1}{2} + \delta_1, d); \Phi) \ll X^\varepsilon \Phi(2) \left( M^{-2\Re(\delta_1)(1-b)} + M^{\left(\frac{1}{2} - \Re(\delta_1)\right)(1-b)} X^{-\frac{1}{2}} \right).
$$

(4.11)

The first statement of Proposition 3.2 now follows by applying (4.8) and (4.11) in (4.3).

To prove the second assertion, we note that

$$
f(s, d) = 1 + B(s, d) = 1 + O\left( \sum_{n \equiv 1 \mod (1+i)^3} \frac{\mu^2((1+i)n)\chi(n)}{N(n)^{\Re(s)}} \right).
$$

We deduce readily from this that $f(s, d)$ has no zeros to the right of $\sigma_0$. Moreover, we have $\log f(s, d) = B(s, d) + O(|B(s, d)|^2)$ for $\Re(s) > \sigma_0$. Thus, we obtain that

$$
S(I(d); \Phi) \ll \exp \left( \frac{\pi}{2} \frac{1}{\log X} \right) \left( |S(B(s, d); \Phi)| + S(|B(s, d)|^2; \Phi) \right).
$$

By applying (4.8) and (4.11), we see that the second statement of Proposition 3.2 now follows.
4.2. Proof of Proposition 3.3. In this section, we estimate $S_R(A_{\delta, \tau}(d)M(\frac{1}{2} + \delta_1, d)M(\frac{1}{2} + \delta_2, d); \Psi)$ by proving Proposition 3.3. We first notice that $|R_Y(d)| \leq \sum_{k|d} 1 \ll N(d)^e$ and that $R_Y(d) = 0$ unless $d = l^2 m$ where $m$ is square-free and $N(l) > Y$. It follows that

$$S_R(A_{\delta, \tau}(d)M(\frac{1}{2} + \delta_1, d)M(\frac{1}{2} + \delta_2, d); \Psi) \ll X^{-1+\epsilon} \sum_{N(l) > YX/N(l)^2} \sum_{(l,m) = 1}^b |A_{\delta, \tau}(l^2 m)M(\frac{1}{2} + \delta_1, l^2 m)M(\frac{1}{2} + \delta_2, l^2 m)|,$$

where $\sum_b$ means that the sum is over odd and square-free $m \in \mathcal{O}_K$. Now, applying twice the Cauchy-Schwarz inequality, we see that the sum over $m$ above is

$$\ll \left( \sum_m^b |M(\frac{1}{2} + \delta_1, l^2 m)|^4 \right)^{\frac{1}{2}} \left( \sum_m^b |M(\frac{1}{2} + \delta_2, l^2 m)|^4 \right)^{\frac{1}{2}} \left( \sum_m^b |A_{\delta, \tau}(l^2 m)|^2 \right)^{\frac{1}{2}}.$$

Note that for any $c > \frac{1}{2} + |\Re(\delta)|$, we have

$$A_{\delta, \tau}(l^2 m) = \frac{1}{2\pi i} \int_{(c)} \Gamma_\delta(s) \left( \frac{2^5 N(l^2 m)}{\pi^2} \right)^s \frac{2s}{s^2 - \tau^2} \sum_{n \equiv 1 \mod (1+i)^3} \frac{r_\delta(n)}{N(n)^{s+\frac{1}{2}}} \left( \frac{(1+i)^5 l^2 m}{n} \right) ds.$$

We write the sum above as

$$\sum_{n \equiv 1 \mod (1+i)^3} \frac{r_\delta(n)}{N(n)^{s+\frac{1}{2}}} \left( \frac{(1+i)^5 l^2 m}{n} \right) = \frac{L(\frac{1}{2} + s + \delta, \chi_{(1+i)^3} m)L(\frac{1}{2} + s - \delta, \chi_{(1+i)^3} m)}{N(\varpi)^{s+\frac{1}{2}}} \mathcal{E}(s,l),$$

with

$$\mathcal{E}(s,l) = \prod_{\varpi \subseteq \mathcal{O}} \left( 1 - \frac{1}{N(\varpi)^{s+\frac{1}{2}+\delta}} \left( \frac{(1+i)^5 m}{\varpi} \right) \left( \frac{1}{N(\varpi)^{s+\frac{1}{2}+\delta}} \left( \frac{(1+i)^5 m}{\varpi} \right) \right) \right).$$

Observe that the left side of (4.14) is analytic for all $s$ since $\chi_{(1+i)^3} m$ is non-principal. Thus, we may move the line of integration in (4.14) to $\Re(s) = \kappa + 1/\log X$ without encountering any pole. Using the estimations that $|\mathcal{E}(s,l)| \leq \prod_{\varpi \subseteq \mathcal{O}} (1 + 1/\sqrt{N(\varpi)^2}) \ll N(l)^e \ll X^e, 2s/(s^2 - \tau^2) \ll X^e$, and the rapid decay of $|\Gamma_\delta(s)|$ when $|\Im(s)| \to \infty$, we apply the Cauchy-Schwarz inequality to see that

$$|A_{\delta, \tau}(l^2 m)|^2 \ll X^{2\kappa+\epsilon} \int_{(\kappa+1/\log X)} |\Gamma_\delta(s)||L(\frac{1}{2} + s + \delta, \chi_{(1+i)^3} m)L(\frac{1}{2} + s - \delta, \chi_{(1+i)^3} m)|^2 |ds|.$$

It follows from this and Lemma 2.8 that we have

$$\sum_{X/N(l)^2 \leq N(m) \leq 2X/N(l)^2} |A_{\delta, \tau}(l^2 m)|^2 \ll \frac{X^{1+2\kappa+\epsilon}}{N(l)^2} \int_{(\kappa+1/\log X)} |\Gamma_\delta(s)|(1 + |s|^2)^{1+\epsilon} |ds| \ll \frac{X^{1+2\kappa+\epsilon}}{N(l)^2}.$$

We apply this with (4.1) to see that the expression in (4.13) is bounded by

$$\ll \frac{X^{1/2+\kappa+\epsilon}}{N(l)^{1/2}} \left( \frac{X^{1/4}}{N(l)^{1/2}} M^{\Re(\delta_1)} + M^{1/2-\Re(\delta_1)} \right) \left( \frac{X^{1/4}}{N(l)^{1/2}} M^{\Re(\delta_2)} + M^{1/2-\Re(\delta_2)} \right).$$

Inserting the above in (4.12) and keeping in mind that $M \leq \sqrt{X}$, we see that the assertion of Proposition 3.3 follows.

5. Proof of Proposition 3.4

5.1. A first decomposition. Note that we have

$$S_M \left( \left( \frac{1+i}{l} \right)^5 d \right) A_{\delta, \tau}(d); \Psi) = \sum_{n \equiv 1 \mod (1+i)^3} \frac{r_\delta(n)}{\sqrt{N(n)}} S_M \left( \left( \frac{1+i}{l} \right)^5 d \right); F_N(n),$$

where $F_N(n)(t)$ is defined as in (2.11).
By definition, we have
\[
S_M\left(\frac{(1+i)^5d}{ln} ; F_N(n)\right) = \frac{1}{X} \sum_{(d,2)=1} \sum_{\alpha \equiv 1 \mod (1+i)^3} \mu_{ij}(\alpha) \left(\frac{(1+i)^5d}{ln}\right) F_N(n) \left(\frac{N(d)}{X}\right)
\]
\[
= \frac{1}{X} \sum_{\alpha \equiv 1 \mod (1+i)^3} \mu_{ij}(\alpha) \sum_{(d,2)=1} \left(\frac{(1+i)^5\alpha^2d}{ln}\right) F_N(n) \left(\frac{N(\alpha^2d)}{X}\right).
\]

We apply the Poisson summation formula given in Lemma \ref{lem:poisson_sum} to treat the last sum above to obtain that
\[
(5.2)
\]
\[
S_M\left(\frac{(1+i)^5d}{ln} ; F_N(n)\right) = \frac{1}{2N(ln)} \left(\frac{(1+i)^6}{ln}\right) \sum_{\alpha \equiv 1 \mod (1+i)^3} \frac{\mu_{ij}(\alpha)}{N(\alpha)^2} \sum_{k \in \mathcal{K}} (-1)^{N(k)} g(k, ln) \tilde{F}_N(n) \left(\sqrt{\frac{N(k)X}{2N(\alpha^2ln)}}\right).
\]

The above allows us to deduce from (5.1) that
\[
(5.3)
\]
\[
S_M\left(\frac{(1+i)^5d}{ln} ; A_{\delta, \tau}(d) ; \Psi\right) = \mathcal{P}(l) + \mathcal{R}_0(l),
\]
where \(\mathcal{P}(l)\) arises from the \(k = 0\) term in (5.2) and \(\mathcal{R}_0(l)\) includes the remaining non-zero terms \(k\) in (5.2). Hence
\[
(5.4)
\]
\[
\mathcal{P}(l) = \frac{1}{2N(l)} \sum_{n \equiv 1 \mod (1+i)^3} \frac{r_3(n)}{N(n)^2} \left(\frac{(1+i)^6}{ln}\right) \sum_{\alpha \equiv 1 \mod (1+i)^3} \frac{\mu_{ij}(\alpha)}{N(\alpha)^2} g(0, ln) \tilde{F}_N(n)(0),
\]
\[
\mathcal{R}_0(l) = \frac{1}{2N(1)} \sum_{n \equiv 1 \mod (1+i)^3} \frac{r_3(n)}{N(n)^2} \left(\frac{(1+i)^6}{ln}\right) \sum_{\alpha \equiv 1 \mod (1+i)^3} \frac{\mu_{ij}(\alpha)}{N(\alpha)^2} \sum_{k \in \mathcal{K}} (-1)^{N(k)} g(k, ln) \tilde{F}_N(n) \left(\sqrt{\frac{N(k)X}{2N(\alpha^2ln)}}\right).
\]

We show in what follows that \(\mathcal{P}(l)\) contributes to a main term and \(\mathcal{R}_0(l)\) also contributes to secondary main terms.

5.2. **The principal term** \(\mathcal{P}(l)\). Note that \(g(0, ln) = \varphi_{ij}(ln)\) if \(ln = \Box\) and \(g(0, ln) = 0\) otherwise. Also, we have
\[
\sum_{\alpha \equiv 1 \mod (1+i)^3} \frac{\mu_{ij}(\alpha)}{N(\alpha)^2} = \frac{1}{\zeta(2)} \prod_{\varpi \equiv \Box \mod 2m} \left(1 - \frac{1}{N(\varpi)^2}\right)^{-1} \left(1 + O\left(\frac{1}{Y}\right)\right).
\]

Using the above observations, we see that
\[
\mathcal{P}(l) = \frac{1 + O(Y^{-1})}{\zeta(2)} \sum_{n \equiv 1 \mod (1+i)^3} \frac{r_3(n)}{N(n)^2} \left(\frac{(1+i)^6}{ln}\right) \prod_{\varpi \equiv \Box \mod 2m} \left(\frac{N(\varpi)}{N(\varpi) + 1}\right) \tilde{F}_N(n)(0).
\]

As \(l = l_1l_2^2\) with \(l_1\) primary and square-free, we see that \(ln = \Box\) is equivalent to \(n = l_1m^2\) for some primary \(m\). Thus
\[
\mathcal{P}(l) = \frac{1 + O(Y^{-1})}{\zeta(2)\sqrt{N(l_1)}} \sum_{m \equiv 1 \mod (1+i)^3} \frac{r_3(l_1m^2)}{N(m)} \prod_{\varpi \equiv \Box \mod 2m} \left(\frac{N(\varpi)}{N(\varpi) + 1}\right) \tilde{F}_N(l_1m^2)(0).
\]
Note that we have for any $c > |\Re(\tau)|$,

$$
\bar{F}_{N(l,m^2)}(0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi(N(x+yi)) W_{\delta,\tau} \left( \frac{\pi^2 N(l_1 m^2)}{2^5 XN(x+yi)} \right) \, dx \, dy
$$

$$
= \frac{1}{2\pi i} \int_{(c)} \Gamma_\delta(s) \left( \frac{25/2}{\pi} \right)^{2s} \left( \frac{X}{N(l_1 m^2)} \right)^s \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi(N(x+yi)) N(x+yi)^s \, dx \, dy \right) \frac{2s}{s^2 - \tau^2} \, ds
$$

$$
= \frac{\pi}{2\pi i} \int_{(c)} \Gamma_\delta(s) \left( \frac{25/2}{\pi} \right)^{2s} \left( \frac{X}{N(l_1 m^2)} \right)^s \Psi(1+s) \frac{2s}{s^2 - \tau^2} \, ds,
$$

where we deduce the last equality above by noticing that

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi(N(x+yi)) N(x+yi)^s \, dx \, dy = \int_{0}^{2\pi} \int_{0}^{\infty} \Psi(r^2) r^{2s} r \, dr \, d\theta = \pi \Psi(1+s).
$$

We then conclude that for any $c > \kappa$,

$$
P(l) = \frac{2\pi}{3} \cdot \frac{1 + O(Y^{-1})}{\zeta_K(2)\sqrt{N(l)}} I(l),
$$

where

$$
I(l) = \frac{1}{2\pi i} \int_{(c)} \Gamma_\delta(s) \left( \frac{25/2}{\pi} \right)^{2s} \left( \frac{X}{N(l_1)} \right)^s \Psi(1+s) \frac{2s}{s^2 - \tau^2} \sum_{m \equiv 1 \pmod{(1+i)^3}} \frac{r_\delta(l_1 m^2)}{N(m)^{1+2s}} \prod_{\psi \in G \atop \psi(i) \equiv l_1} \left( \frac{N(\varpi)}{N(\varpi) + 1} \right) \, ds.
$$

Now, by comparing Euler factors, we see that for $\Re(s) > 1 + 2|\Re(\delta)|$,

$$
\sum_{m \equiv 1 \pmod{(1+i)^3}} \frac{r_\delta(l_1 m^2)}{N(m)^s} \prod_{\psi \in G \atop \psi(i) \equiv l_1} \left( \frac{N(\varpi)}{N(\varpi) + 1} \right) = r_\delta(l_1) Z(s; \delta) \eta_\delta(s; l),
$$

where $Z$ and $\eta$ are as defined in (5.1) and (2.11). Using this in (5.6), we see that

$$
I(l) = \frac{r_\delta(l_1)}{2\pi i} \int_{(c)} \Gamma_\delta(s) \left( \frac{25/2}{\pi} \right)^{2s} \left( \frac{X}{N(l_1)} \right)^s \Psi(1+s) \frac{2s}{s^2 - \tau^2} Z(1+2s; \delta) \eta_\delta(1+2s; l) \, ds.
$$

Taking $c = \kappa + \epsilon$ above implies that $I(l) \ll |r_\delta(l_1)| (X/N(l_1))^{\kappa + \epsilon}$. We now move the line of integration in (5.7) to $\Re s = -\frac{1}{2} + \epsilon$ to encounter simple poles at $s = \pm \tau, \pm \delta$ in the process. We note the following convexity bounds for $\zeta_K(s)$ from [9] Exercise 3, p. 100):

$$
\zeta_K(s) \ll (1 + |s|^2)^{\frac{1}{2} - \frac{\eta_\delta}{2} + \epsilon}, \quad 0 < \Re(s) < 1.
$$

It follows that $|Z(1+2s; \delta)| \ll (1 + |s|^2)^3$ on the new line. Moreover, we have

$$
|\eta_\delta(1+2s; l)| \ll \prod_{\psi \in G \atop \psi(i) \equiv l_1} \left( 1 + O\left( \frac{1}{\sqrt{N(\varpi)}} \right) \right) \prod_{\psi \in G \atop \psi(l_1)} \left( 1 + O\left( \frac{1}{N(\varpi)^{1+\epsilon}} \right) \right) \ll N(l_1)^\frac{\epsilon}{2}.
$$

Thus, we deduce that the integral on the new line is

$$
\ll \frac{|r_\delta(l_1)| N(l_1)^{\frac{\kappa + \epsilon}{2}}}{X^{\frac{1}{2} - \epsilon}} \int_{-\frac{1}{2} + \epsilon}^{(-\frac{1}{2} + \epsilon)} |s|^\frac{\eta_\delta}{2} \Psi(1+s) |\Gamma_\delta(s)|| ds \ll \frac{|r_\delta(l_1)| N(l_1)^{\frac{\kappa + \epsilon}{2}}}{X^{\frac{1}{2} - \epsilon}}.
$$

It follows that

$$
I(l) = r_\delta(l_1) \lim_{s \to \delta} \left\{ \Gamma_\delta(s) \left( \frac{25/2}{\pi} \right)^{2s} \left( \frac{X}{N(l_1)} \right)^s \Psi(1+s) \frac{2s}{s^2 - \tau^2} Z(1+2s; \delta) \eta_\delta(1+2s; l) \right\} + O\left( \frac{|r_\delta(l_1)| N(l_1)^{\frac{\kappa + \epsilon}{2}}}{X^{\frac{1}{2} - \epsilon}} \right).
$$
Applying this in (5.3), we obtain that
\[
\mathcal{P}(l) = \frac{2\pi r_2(l)}{3\zeta_K(2)} \sqrt{N(l_1)} \operatorname{Res}_{s=\frac{3}{2}} \left\{ \frac{\Gamma_\delta(s) \left( \frac{2^{5/2}}{\pi} \right)^{2s} \left( \frac{X}{N(l_1)} \right)^s \tilde{\Psi}(1+s) \frac{2s}{s^2 - \tau^2} Z(1 + 2s; \delta) \eta_\delta(1 + 2s;l)}{Y N(l_1)^{3/4}} \right\)}
\]
(5.9)
\[\quad + O \left( \frac{|r_\delta(l_1)|X^{k+\epsilon}}{Y N(l_1)^{3/4}} + \frac{|r_\delta(l_1)|X^\epsilon}{(X N(l_1))^{3/4}} \right) \].

5.3. The secondary main terms. We apply the Mellin transform to recast \(R_0(l)\) given in (5.9) as
\[
R_0(l) = \frac{1}{2N(l)} \sum_{\alpha \equiv l \mod (1+i)^3} \frac{\mu_\delta(\alpha)}{N(\alpha)^2} \sum_{\substack{k \in \mathbb{Q}_K \setminus \mathbb{Z} \setminus \{0\}}} \frac{-1}{2\pi i} \int_{(c)} \sum_{n \equiv 1 \mod (1+i)^3} \frac{r_\delta(n)}{N(n)^{3/2 + \epsilon}} g(k, \ln) h \left( \frac{N(k)X}{2N(\alpha^2l)} \right) dw,
\]
(5.10)
where \(c > |\Re(\delta)|\) and \(h\) is defined as in (2.12).

We apply Lemma 2.13 in (5.11) and move the line of integration to \(\Re(w) = -\frac{1}{2} + |\Re(\delta)| + \epsilon\). In this process, we encounter poles only when \(k_1 = \pm i, \) with simple poles at \(w = \pm \delta\). The residues of these poles contribute secondary main terms. We therefore write \(R_0(l) = R(l) + P_+(l) + P_-(l)\) where
\[
R(l) = \frac{1}{2N(l)} \sum_{\alpha \equiv l \mod (1+i)^3} \frac{\mu_\delta(\alpha)}{N(\alpha)^2} \sum_{\substack{k \in \mathbb{Q}_K \setminus \mathbb{Z} \setminus \{0\}}} \frac{-1}{2\pi i} \int_{(-\frac{1}{2} + |\Re(\delta)| + \epsilon)} L(1 + w + \delta, \chi_{1}k_{1}) L(1 + w - \delta, \chi_{1}k_{1})
\]
\[\times \mathcal{G}_\delta(1 + w; k, l, \alpha) h \left( \frac{N(k)X}{2N(\alpha^2l)} \right) dw,
\]
and (after replacing \(k\) by \(\pm ik^2\))
\[
P_+(l) = \frac{\pi}{4} \cdot \frac{1}{2N(l)} \sum_{\alpha \equiv l \mod (1+i)^3} \frac{\mu_\delta(\alpha)}{N(\alpha)^2} \zeta_K(1 + 2k) \sum_{\substack{k \in \mathbb{Q} \setminus \mathbb{Z} \setminus \{0\}}} (-1)^{N(k)} \mathcal{G}_\delta(1 + \mu \delta; ik^2, l, \alpha) h \left( \frac{N(k)^2X}{2N(\alpha^2l)} \right) \mu \delta,
\]
(5.11)
\[
P_-(l) = \frac{\pi}{4} \cdot \frac{1}{2N(l)} \sum_{\alpha \equiv l \mod (1+i)^3} \frac{\mu_\delta(\alpha)}{N(\alpha)^2} \zeta_K(1 + 2k) \sum_{\substack{k \in \mathbb{Q} \setminus \mathbb{Z} \setminus \{0\}}} (-1)^{N(k)} \mathcal{G}_\delta(1 + \mu \delta; -ik^2, l, \alpha) h \left( \frac{N(k)^2X}{2N(\alpha^2l)} \right) \mu \delta.
\]
(5.12)

Note that it follows from Lemma 2.2 and Lemma 2.13 that we have \(\mathcal{G}_\delta(1 + \mu \delta; ik^2, l, \alpha) = \mathcal{G}_\delta(1 + \mu \delta; -ik^2, l, \alpha)\). Note also that we have
\[
\sum_{j \in G} (-1)^{N(j)} N(j)^{-2u} \mathcal{G}_\delta(s; ij^2, l, \alpha) = - \sum_{j \in G} N(j)^{-2u} \mathcal{G}_\delta(s; ij^2, l, \alpha) + \sum_{j \in G} N(j)^{-2u} \mathcal{G}_\delta(s; ij^2, l, \alpha)
\]
\[= - \sum_{j \in G} N(j)^{-2u} \mathcal{G}_\delta(s; ij^2, l, \alpha) + 2 \sum_{j \in G \setminus \{1, i, j\}} N(j)^{-2u} \mathcal{G}_\delta(s; ij^2, l, \alpha)
\]
\[= - \sum_{j \in G} N(j)^{-2u} \mathcal{G}_\delta(s; ij^2, l, \alpha) + 2^{1-2u} \sum_{j \in G \setminus \{1, i, j\}} N(j)^{-2u} \mathcal{G}_\delta(s; j^2, l, \alpha)
\]
\[= (1 - 2^{1-2u}) \sum_{j \in G} N(j)^{-2u} \mathcal{G}_\delta(s; ij^2, l, \alpha),
\]
where the last equality above follows from the observation that we have \(\mathcal{G}_\delta(s; i(1 + i)^{-1}j^2, l, \alpha) = \mathcal{G}_\delta(s; ij^2, l, \alpha)\) by Lemma 2.2 and Lemma 2.13.

We now define for \(\mu = \pm\) and any complex number \(u\) with \(\Re(u) > \frac{1}{2}\),
\[
\mathcal{H}(u, v; l, \alpha) = N(l)^u \sum_{k \in G} \frac{(-1)^{N(k)}}{N(k)^{2u}} \mathcal{G}_\delta(1 + v; ik^2, l, \alpha).
\]
It follows from Lemma 2.13 that the above series converges absolutely when \( \Re(u) > \frac{1}{2} \). We further apply (5.13) to recast \( \mathcal{H}(u, v; l, \alpha) \) as

\[
\mathcal{H}(u, v; l, \alpha) = -N(l)^u(1 - 2^{-1-2u}) \prod_{\sigma \in \mathcal{G}} \sum_{h=0}^{\infty} \frac{G_{\sigma; \mathbb{Z}}(1 + v; i\sigma 2^{h+1}, l, \alpha)}{N(\sigma)^{2^{h+1}}}.
\]

We now apply Lemma 2.22 to evaluate \( G_{\sigma; \mathbb{Z}} \) defined in Lemma 2.13 to see that

\[
(5.14) \quad \mathcal{H}(u, v; l, \alpha) = -N(l)(1 - 2^{-1-2u})N(l_1)^u - \frac{1}{2} \zeta_K(2u) \zeta_K(2u + 1 + 4v) \mathcal{H}_1(u, v; l, \alpha),
\]

where \( \mathcal{H}_1 = \prod_{\sigma \in \mathcal{G}} \mathcal{H}_{1, \sigma} \) with

\[
\mathcal{H}_{1, \sigma} = \begin{cases} 
1 - \frac{1}{N(\sigma)} & \text{if } \sigma \mid 2\alpha, \\
1 - \frac{1}{N(\sigma)} & \text{if } \sigma \nmid 2\alpha, \\
1 - \frac{1}{N(\sigma)^2} & \text{if } \sigma \mid l, \\
1 - \frac{1}{N(\sigma)^2} & \text{if } \sigma \nmid l, \sigma \nmid l_1.
\end{cases}
\]

It then follows from this and (5.14) that \( \mathcal{H}(u, v; l, \alpha) \) is analytic in the domain \( \Re(u) > \frac{1}{2} + |\Re(v)| \).

We now apply Lemma 2.11 and the observation that \( G_{\sigma; (1 + v; ik^2, l, \alpha)} \) is an even function of \( v \) to deduce that, for \( c > \max\left(\frac{1}{2} + |\Re(\delta)|, |\Re(\tau)|\right) = \frac{1}{2} + |\Re(v)| \) without encountering any pole. We then deduce from this and (5.12) that

\[
\mathcal{P}_\pm(l) = \frac{\pi}{2\pi i} \int_{(c)} \frac{2^{6} N(\alpha)^2}{\pi^2} s \Psi(1 + \mu\delta) \left(\frac{X}{2 N(\alpha)^2}\right)^{\mu\delta} \Gamma(s - \mu\delta) \Gamma(1 - s + \mu\delta) \pi^{-2s + 2\mu\delta} \frac{2s}{s^2 - \tau^2} \mathcal{H}(s - \mu\delta, \mu\delta; l, \alpha) ds.
\]

The analytical properties of \( H \) discussed above allow us to move the line of integration in (5.15) to \( \Re(s) = \frac{1}{\log X} \) without encountering any pole. We then deduce from (5.15) that

\[
(5.16) \quad \mathcal{P}_\pm(l) = \frac{\pi}{2\pi i} \int_{(c)} \frac{2^{6} N(\alpha)^2}{\pi^2} s \sum_{\mu = \pm} \Psi(1 + \mu\delta) \left(\frac{X}{2 N(\alpha)^2}\right)^{\mu\delta} \zeta_K(1 + 2\mu\delta) \Gamma(s - \mu\delta) \Gamma(1 - s + \mu\delta) \pi^{-2s + 2\mu\delta} \mathcal{H}(s - \mu\delta, \mu\delta; l, \alpha) ds.
\]

We now extend the sum over \( \alpha \) in (5.16) to infinity. To estimate the error introduced, we let \( \mathcal{C} \) be the circle centred at 0, with radius \( \kappa + 1/(2 \log X) \), and oriented counter-clockwise. Notice that, for any complex number \( s \) with \( \Re(s) = \kappa + \frac{1}{\log X} \), the function \( 2\pi \Psi(1 + z) \left(\frac{X}{2 N(\alpha)^2}\right)^z \zeta_K(1 + 2z) \Gamma(s - z) \Gamma(1 - s + \mu\delta) \pi^{-2s + 2\mu\delta} \mathcal{H}(s - z, z; l, \alpha) \) is analytic for \( z \) inside \( \mathcal{C} \). Thus, we deduce from Cauchy’s theorem that

\[
\sum_{\mu = \pm} \Psi(1 + \mu\delta) \left(\frac{X}{2 N(\alpha)^2}\right)^{\mu\delta} \zeta_K(1 + 2\mu\delta) \Gamma(s - \mu\delta) \Gamma(1 - s + \mu\delta) \pi^{-2s + 2\mu\delta} \mathcal{H}(s - \mu\delta, \mu\delta; l, \alpha) = \frac{1}{2\pi i} \int_{\mathcal{C}} \Psi(1 + z) \left(\frac{X}{2 N(\alpha)^2}\right)^z \zeta_K(1 + 2z) \Gamma(s - z) \Gamma(1 - s + \mu\delta) \pi^{-2s + 2z} \mathcal{H}(s - z, z; l, \alpha) ds.
\]

We observe that \( 2\kappa + 3/(2 \log X) \geq \Re(s - z) \geq 1/(2 \log X) \) for \( z \) on \( \mathcal{C} \) and that \( |\frac{2\pi}{\pi z} \Psi(1 + z) \zeta_K(1 + 2z) \left(\frac{X}{2 N(\alpha)^2}\right)^z| \ll (\log^2 X)(\log X)^{\kappa + 1}/\log X \). Also, by Stirling’s formula (see [9, (5.112)]), we have

\[
|\frac{\Gamma(s - z)}{\Gamma(1 - s + \mu\delta)} \pi^{-2s + 2z}| \ll (1 + |\Im(s)|)^{4\kappa + 3}/\log X - 1 \ll 1.
\]

Moreover, applying the convexity bounds for \( \zeta_K(s) \) given in (5.8), we deduce from (5.14) that \( |\mathcal{H}(s - z, z; l, \alpha)| \ll N(l)^{1+\varepsilon} N(l_1)^{2\kappa + 3/(2 \log X) - \frac{1}{2}} N(\alpha)^\varepsilon (1 + |s|) \log X \). These estimates allow us to bound the quantities in (5.14) by

\[
\ll N(l)^{1+\varepsilon} N(l_1)^{2\kappa - \frac{1}{2}} (X N(\alpha)^2)^{s + \varepsilon}(1 + |s|).
\]
It follows that
\[
\frac{1}{2N(l)} \sum_{\alpha \equiv 1 \mod (1+i)^3} \frac{\mu_l(\alpha)}{N(\alpha)^{2-2\mu}} \sum_{\mu = \pm} \tilde{\Psi}(1 + \mu \delta) \left( \frac{X}{2N(\alpha)^2} \right)^{\mu \delta} \zeta_K(1 + 2\mu \delta) \frac{\Gamma(s - \mu \delta)}{\Gamma(1 - s + \mu \delta)} \pi^{-2s + 2\delta} \mathcal{H}(s - \mu \delta, \mu \delta; l, \alpha)
\]
\[
\ll N(l)^{\eta} N(l_1)^{2\kappa - \frac{1}{2} \pi + \epsilon} (1 + |s|) Y^{-1 + 4\kappa}.
\]

Applying the above in (5.14), we see that the error introduced by extending the sum over \( \alpha \) to infinity is
\[
\ll N(l)^{\eta} N(l_1)^{2\kappa - \frac{1}{2} \pi + \epsilon} Y^{-1 + 4\kappa} \int_{(\kappa + \frac{1}{2} \log X)} \frac{|s|}{|s^2 - \tau^2|} |ds| \ll N(l)^{\eta} N(l_1)^{2\kappa - \frac{1}{2} \pi + \epsilon} Y^{-1 + 4\kappa}.
\]

We then conclude that we have
(5.18)
\[
\mathcal{P}_\pm(l) = \frac{\pi}{4} \cdot \sum_{\mu = \pm} \tilde{\Psi}(1 + \mu \delta) \left( \frac{X}{2} \right)^{\mu \delta} \zeta_K(1 + 2\mu \delta) \frac{1}{2\pi i} \int_{(\kappa + \frac{1}{2} \log X)} \mathcal{J}(s, \mu \delta; l) \frac{2s}{s^2 - \tau^2} ds + O(N(l)^{\eta} N(l_1)^{2\kappa - \frac{1}{2} \pi + \epsilon} Y^{-1 + 4\kappa}),
\]
where
\[
\mathcal{J}(s, v; l) := \Gamma_v(s) \left( \frac{1}{\pi^2} \right)^{\frac{1}{2}} \frac{\Gamma(s - v)}{\Gamma(1 + s + v)} \pi^{-2s + 2\delta} \mathcal{K}(s, v, l),
\]
\[
\mathcal{K}(s, v, l) := \frac{1}{2N(l)} \sum_{\alpha \equiv 1 \mod (1+i)^3} \frac{\mu_l(\alpha)}{N(\alpha)^{2-2\delta + 2\mu \delta}} \mathcal{H}(s - v, v, l, \alpha).
\]

A direct calculation using the expression for \( \mathcal{H} \) given in (5.14) yields
\[
\mathcal{K}(s, v, l) = -\frac{1}{4N(l_1)^{\frac{v}{2}} + v} \frac{\varphi_l(l)}{N(l)} \prod_{\omega \in \mathcal{G}} \left( 1 - \frac{1}{N(\omega)^{1/2v}} \right) \prod_{\omega \in \mathcal{G}} \left( 1 + \frac{1}{N(\omega)^{1/2v}} \right) r_s(l_1) \times \left( \frac{4^s + 4 - 2 - 4\delta - 2^{1+2v}}{4^s} \right) \zeta_K(2s - 2v) \zeta_K(2s + 1 + 2v) \times \prod_{\omega \in \mathcal{G}} \left( \frac{1}{1 - \frac{1}{N(\omega)}} \right) \left( 1 - \frac{1}{N(\omega)^{1/2v}} \right) \left( 1 + \frac{1}{N(\omega)^{1/2v}} \right) + \frac{1}{1 - \frac{1}{N(\omega)^{1/2v}}} - \frac{1}{N(\omega)^{3/2v}} - \frac{1}{N(\omega)^{\frac{v}{2} + 2\delta - \frac{v}{2} + 2\delta}}.
\]

Using this together with the functional equation for \( \zeta_K(s) \) given in (2.3), and the identity \( \Gamma(z) \Gamma(z + \frac{1}{2}) = \pi^\frac{1}{2} 2^{1-z} \Gamma(2z) \), we obtain the following identity
(5.19)
\[
\zeta_K(1 + v) \mathcal{J}(s, v; l) = \frac{1}{\pi} \frac{4r_s(l_1)}{3\zeta_K(2) \sqrt{N(l_1)}} \left( \frac{2}{\pi^2 N(l_1)} \right)^v \Gamma_v(v) Z(1 + 2v; s) \eta_s(1 + 2v; l).
\]

As it is easy to see that the left side above is invariant under \( s \to -s \), we deduce that \( \mathcal{J}(s, v; l) = \mathcal{J}(-s, v; l) \).

We now evaluate the integral in (5.13) for \( \mu = \pm \) by moving the line of integration to \( \Re(s) = \kappa - \frac{1}{\log X} \) to encounter simple poles at \( s = \pm \delta, \pm \tau \) in the process. It follows that
\[
\frac{1}{2\pi i} \int_{(\kappa + \frac{1}{2} \log X)} \mathcal{J}(s, \mu \delta; l) \frac{2s}{s^2 - \tau^2} ds = \operatorname{Res}_{s = \pm \delta, \pm \tau} \mathcal{J}(s, \mu \delta; l) \frac{2s}{s^2 - \tau^2} + \frac{1}{2\pi i} \int_{(\kappa + \frac{1}{2} \log X)} \mathcal{J}(s, \mu \delta; l) \frac{2s}{s^2 - \tau^2} ds,
\]
where the last integral above follows from the previous one upon a change of variable \( s \to -s \) while noticing that \( \mathcal{J}(s, \mu \delta; l) = \mathcal{J}(-s, \mu \delta; l) \). Thus we deduce that
\[
\frac{1}{2\pi i} \int_{(\kappa + \frac{1}{2} \log X)} \mathcal{J}(s, \mu \delta; l) \frac{2s}{s^2 - \tau^2} ds = \frac{1}{2} \left( \operatorname{Res}_{s = \pm \delta, \pm \tau} \mathcal{J}(s, \mu \delta; l) \frac{2s}{s^2 - \tau^2} \right) = \operatorname{Res}_{s = \mu \delta} \mathcal{J}(s, \mu \delta; l) \frac{2s}{s^2 - \tau^2} + \mathcal{J}(\tau, \mu \delta; l).
\]
Applying the above in (5.18), we obtain that

$$\mathcal{P}_+(l) = \mathcal{P}_-(l) = \frac{\pi^2}{4} \sum_{\mu=\pm} \tilde{\Psi}(1 + \mu \delta) \left( \frac{X}{2} \right)^{\mu \delta} \zeta_K (1 + 2 \mu \delta) \left( \frac{\text{Res} \mathcal{J}(s, \mu \delta; l)}{s^2 - \tau^2 + \mathcal{J}(s, \mu \delta; l)} \right) + O\left( \frac{(N(l))^{\gamma + \varepsilon} N(l_1)^{2\gamma - \frac{1}{2}}}{Y^{1 - 4\varepsilon}} \right).$$

(5.20)

5.4. Estimation of $\mathcal{R}(l)$. In this section, we estimate $\mathcal{R}(l)$ given in (5.11) on average by deriving the bound given in (5.8). We denote $\beta_l = \frac{K_l}{N(l)}$ if $\mathcal{R}(l) \neq 0$, and $\beta_l = 1$ otherwise. We deduce from (5.11) that

$$\sum_{L \leq N(l) \leq 2L - 1} |\mathcal{R}(l)| = \sum_{L \leq N(l) \leq 2L - 1} \beta_l |\mathcal{R}(l)| \ll \sum_{\alpha \equiv 1 \mod (1 + i)^3} \sum_{N(\alpha) \leq Y} \frac{1}{N(\alpha)^2} \int (-\frac{1}{2} + |\Re(\delta)| + \epsilon) \sum_{k \in O(\mathcal{R})} U_\delta(\alpha, k, w) |dw|,$$

where

$$U_\delta(\alpha, k, w) = |L(1 + w + \delta, \chi_{ik_1}) L(1 + w - \delta, \chi_{ik_1})| \sum_{L \leq N(l) \leq 2L - 1} \frac{\beta_l}{N(l)} G_\delta(1 + w; k, l, \alpha) h\left( \frac{N(k)X}{2N(\alpha^2 l)}, w \right).$$

Recall the definition of $k_2$ in (2.17). We apply the Cauchy-Schwarz inequality to see that for any integer $K \geq 1$,

$$\sum_{K \leq N(k) < 2K} U_\delta(\alpha, k, w) \ll \left( \sum_{N(k) = K} (2K - 1) N(k_2) |L(1 + w + \delta, \chi_{ik_1}) L(1 + w - \delta, \chi_{ik_1})|^2 \right)^{\frac{1}{2}} \times \left( \sum_{N(k) = K} \frac{1}{N(k_2)} \frac{1}{N(l)} \sum_{L \leq N(l) \leq 2L - 1} \frac{\beta_l}{N(l)} G_\delta(1 + w; k, l, \alpha) h\left( \frac{N(k)X}{2N(\alpha^2 l)}, w \right)^2 \right)^{\frac{1}{2}}.$$ 

Using the Cauchy-Schwarz inequality again together with Lemma 2.8 we see that

$$\left( \sum_{N(k) = K} (2K - 1) N(k_2) |L(1 + w + \delta, \chi_{ik_1}) L(1 + w - \delta, \chi_{ik_1})|^2 \right)^{\frac{1}{2}} \ll (K + |w|^2)^{\frac{1}{2} + \epsilon}.$$

This implies that

$$\sum_{K \leq N(k) < 2K} U(\alpha, k, w) \ll (K + |w|^2)^{\frac{1}{2} + \epsilon} \left( \sum_{N(k) = K} \frac{1}{N(k_2)} \frac{1}{N(l)} \sum_{L \leq N(l) \leq 2L - 1} \frac{\beta_l}{N(l)} G_\delta(1 + w; k, l, \alpha) h\left( \frac{N(k)X}{2N(\alpha^2 l)}, w \right)^2 \right)^{\frac{1}{2}}.$$

(5.21)

Now, applying the first bound of Lemma 2.15 as well as the estimation given in (2.13) for $\tilde{\Psi}(1 + w)$, we see that we may restrict the sum of the right side of (5.21) to $K = 2^l \leq N(\alpha)^2 L(1 + |w|^2)(\log X)^4$. We then apply the second bound in Lemma 2.15 to (5.21) to see that

$$\sum_{K \leq N(k) < 2K} U(\alpha, k, w) \ll (K + |w|^2)^{\frac{1}{2} + \epsilon} \times ((1 + |w|) N(\alpha) L \mathcal{X}) \mathcal{C} \tilde{\Phi}(1 + w) \left( \frac{N(\alpha)^2 L(1 + |w|^2)}{K} \right)^{\Re(\delta) - |\Re(\delta)|} K^{1/2 + \lambda} + L^{1/2} \times \left( (1 + |w|^2)^{\frac{1}{2} + \epsilon} \times (N(\alpha) L \mathcal{X}) \mathcal{C} \tilde{\Phi}(1 + w) \right) \left( \frac{N(\alpha)^2 L(1 + |w|^2)}{K} \right)^{\Re(\delta) - |\Re(\delta)|} L^{1/2}.$$

Summing over $K = 2^l, K \leq N(\alpha)^2 L(1 + |w|^2)(\log X)^4$, we deduce from the above that

$$\sum_{K \leq N(k) < 2K} U(\alpha, k, w) \ll (1 + |w|^2)^{1+\epsilon} (N(\alpha) L \mathcal{C} \tilde{\Phi}(1 + w) \left( \frac{N(\alpha)^2 L}{X^{\frac{1}{2} - |\Re(\delta)| - \epsilon}} + \frac{N(\alpha)^{1+2\gamma} L^{1+\kappa}}{X^{\frac{1}{2} - |\Re(\delta)| - \epsilon}} \right).$$
It follows that
\[ \sum_{l=L}^{2L-1} |R(l)| \ll \frac{L^{1+\varepsilon}}{X^{1+\delta}|R(\delta)|^{\varepsilon}} \sum_{N(\alpha) \leq Y} N(\alpha)^{\varepsilon}(1 + N(\alpha)^{-1+2\kappa}L^\kappa) \int_{-\frac{1}{4}+\varepsilon}^{\frac{1}{4}+\varepsilon} |\hat{\Psi}(1 + w)|(1 + |w|^2)^{1+\varepsilon}|dw|. \]

Now, the bound given in \((\ref{eq:2.19})\) follows from this by using \((\ref{eq:2.19})\) for \(\Psi(1 + w)\) with \(n = 3\) when \(|w| \leq \Psi(4)/\Psi(3)\) and \(n = 4\) otherwise to evaluate the integral above.

5.5. Conclusion. Combining \((\ref{eq:5.5}), (\ref{eq:5.9}), (\ref{eq:5.20})\) and our result for the average size of \(R(l)\) given in the previous section, we see that in order to prove Proposition 5.4 it remains to simply the expression \(P(l) + P_+(l) + P_-(l)\) given in \((\ref{eq:5.9})\) and \((\ref{eq:5.20})\). We now employ the identity given in \((\ref{eq:5.19})\) to see that the contribution from the poles at \(\mu\delta\) to \((\ref{eq:5.5})\) and \((\ref{eq:5.20})\) cancel precisely each other, while the contribution from the poles at \(\pm \tau\) in both these expressions gives rise to the main term on the right side of \((\ref{eq:5.7})\). This completes the proof of Proposition 5.4.

6. Proof of Theorem 1.1

We now proceed to complete our proof of Theorem 1.1. We begin by evaluating \(W(\delta, \Phi)\) when \(\delta_1\) is near 0.

6.1. Proof of Proposition 5.5. We apply Lemma 2.5 together with the definition of \(\xi(s, \chi(1+i)s, d)\) given in \((\ref{eq:2.5})\) to see that, by setting \(\Phi_-(t) = t^{-\tau} \Phi(t)\),
\[
S(1; \Phi)W(\delta_1, \Phi) = \left(\frac{2^5X/\pi^2}{\Gamma_\delta(\tau)}\right)^{-\tau} S(M(A_{\delta, \tau}(d)|M(\frac{1}{2} + \delta_1, d)|^{2}; \Phi_{-\tau}).
\]

We further set \(Y = X^{4\kappa}\) to rewrite the above as
\[
S(1; \Phi)W(\delta_1, \Phi) = \left(\frac{2^5X/\pi^2}{\Gamma_\delta(\tau)}\right)^{-\tau} \left\{ S(M(A_{\delta, \tau}(d)|M(\frac{1}{2} + \delta_1, d)|^{2}; \Phi_{-\tau}) + O(S_R(A_{\delta, \tau}(d)|M(\frac{1}{2} + \delta_1, d)|^{2}; \Phi_{-\tau})) \right\}
\]
\[
= \left(\frac{2^5X/\pi^2}{\Gamma_\delta(\tau)}\right)^{-\tau} S(M(A_{\delta, \tau}(d)|M(\frac{1}{2} + \delta_1, d)|^{2}; \Phi_{-\tau}) + O\left(X^{-\kappa+\varepsilon}\right),
\]

where the last estimation above follows from Proposition 3.3.

Note that we have
\[
S_M(A_{\delta, \tau}(d)|M(\frac{1}{2} + \delta_1, d)|^{2}; \Phi_{-\tau})
\]
\[
= \sum_{l \equiv 1 \mod (1+i)^3} \left( \sum_{r, s \equiv 1 \mod (1+i)^3} \frac{\lambda(r)\lambda(s)}{N(r)\frac{1}{2}+\delta_1, N(s)\frac{1}{2}+\delta_2}\right) S_M\left(\frac{(1+\delta)^5d}{l}; \Phi_{-\tau}\right),
\]

We now apply Proposition 3.3 to evaluate \(S_M(A_{\delta, \tau}(d)|M(\frac{1}{2} + \delta_1, d)|^{2}; \Phi_{-\tau})\). Using the estimations that \(|\lambda(n)| \ll N(n)^{\varepsilon}, r_3(n) \ll N(n)^{\varepsilon}, d_{ij}(n) \ll N(n)^{\varepsilon}\), and \(\tau \geq -1/\varepsilon \log X\), we deduce that various remainder terms in Proposition 5.4 contribute
\[
\ll \sum_{N(l) \leq M^2} \frac{N(l)^{\varepsilon}}{N(l)^{1/2+\varepsilon}} \left(\frac{X^\varepsilon}{N(l)} + \frac{N(l)^{\varepsilon}X^{\kappa+\varepsilon}N(l)^{1/2}}{Y^{1-4\kappa}} + |R(l)|\right) \ll X^{-\kappa+\varepsilon}\Phi(3)\Phi(4).
\]

We obtain from \((\ref{eq:6.1})\) to \((\ref{eq:6.6})\) that
\[
S(1; \Phi)W(\delta_1, \Phi) = \left(\frac{2^5X/\pi^2}{\Gamma_\delta(\tau)}\right)^{-\tau} \sum_{l \equiv 1 \mod (1+i)^3} \left( \sum_{r, s \equiv 1 \mod (1+i)^3} \frac{\lambda(r)\lambda(s)}{N(r)\frac{1}{2}+\delta_1, N(s)\frac{1}{2}+\delta_2}\right) M(l) + O(X^{-\kappa+\varepsilon}\Phi(3)\Phi(4)),
\]

where \(M(l) = M_1(l) + M_2(l)\) with
\[
M_1(l) = \frac{2\pi}{3\zeta_K(2)\sqrt{N(l)}} \sum_{\mu=\pm} r_{\delta}(l) \Gamma_\delta(\mu\tau) \left(\frac{2^5X}{N(l)^{1/2}}\right)^{\mu\tau} \hat{\Phi}(1 + \mu\tau - \tau)Z(1 + 2\mu\tau; \delta)\eta_{\delta}(1 + 2\mu\tau; l),
\]
\[
M_2(l) = \frac{2\pi}{3\zeta_K(2)\sqrt{N(l)}} \sum_{\mu=\pm} r_{\tau}(l) \Gamma_\tau(\mu\delta) \left(\frac{2^5X}{N(l)^{1/2}}\right)^{\mu\delta} \hat{\Phi}(1 + \mu\delta - \tau)Z(1 + 2\mu\delta; \tau)\eta_{\tau}(1 + 2\mu\delta; l).
\]
By taking $C$ to be the closed contour described in the paragraph above Lemma 2.14, we can evaluate the contribution of $M_1 (l)$ to (6.7) by

\[
\frac{1}{2\pi i} \int_C \frac{2\pi \Phi(1+w - \tau)}{3\zeta_K(2)\Gamma_\delta(\tau)} \left( \frac{1}{\tau^2} \right)^{w - \tau} Z(1+2w; \delta)\Gamma_\delta(w) \frac{2w}{w^2 - \tau^2} d\tau.
\]

(6.8)

\[
\times \left\{ \sum_{l \equiv 1 \mod (1+i)^3} \frac{r_\delta(l)}{N(l)^{1/2+w}} \sum_{s \equiv 1 \mod (1+i)^3} \frac{\lambda(r)\lambda(s)}{N(r)^{1/1+\delta_1 + s_1}N(s)^{1/1+\delta_2}} \eta_\delta(1+2w; l) \right\} dw.
\]

Using the fact that $\lambda$ is supported on square-free elements in $O_K$, we now write $r = \alpha a$, $s = \alpha b$ with $\alpha$, $a$, $b$ being primary, square-free and $(a, b) = 1$. This implies that $l = \alpha^2 ab$, $l_1 = ab$, and $l_2 = \alpha$ so that the sum over $l$ in (6.8) can be written as

\[
\sum_{\alpha \equiv 1 \mod (1+i)^3} \sum_{a, b \equiv 1 \mod (1+i)^3} \frac{r_\delta(ab)}{N(ab)^{1+2\tau}} \frac{\lambda(\alpha a)\lambda(ab)}{N(\alpha)^{1+\delta_1 + s_1}N(ab)^{1+\delta_2}} \eta_\delta(1+2w; \alpha^2 ab)
\]

(6.9)

\[
= \sum_{\alpha \equiv 1 \mod (1+i)^3} \frac{1}{N(\alpha)^{1+2\tau}} \sum_{a, b \equiv 1 \mod (1+i)^3} \frac{r_\delta(a)\lambda(ab)}{N(ab)^{1+\delta_1 + w}N(ab)^{1+\delta_2 + w}} \eta_\delta(1+2w; \alpha^2 ab).
\]

We then apply the relation

\[
\eta_\delta(1+2w; \alpha^2 ab) = \frac{\eta_\delta(1+2w; l)}{h_w(\alpha)h_w(a)h_w(b)} \prod_{\varpi \in G} \left( 1 + \frac{1}{N(\varpi)^{1+2\tau}} \right)
\]

to recast the expressions given in (6.9) as

\[
\eta_\delta(1+2w; l) \sum_{\alpha \equiv 1 \mod (1+i)^3} \frac{1}{N(\alpha)^{1+2\tau}h_w(\alpha)} \prod_{\varpi \in G} \left( 1 + \frac{1}{N(\varpi)^{1+2\tau}} \right) \sum_{a, b \equiv 1 \mod (1+i)^3} \frac{r_\delta(a)\lambda(ab)}{N(ab)^{1+\delta_1 + w}h_w(ab)} \frac{r_\delta(b)\lambda(ab)}{N(b)^{1+\delta_2 + w}h_w(b)}.
\]

Next, we use the Möbius function to detect the condition $(a, b) = 1$ in the above expression to see that it equals to

\[
\eta_\delta(1+2w; l) \sum_{\alpha \equiv 1 \mod (1+i)^3} \frac{1}{N(\alpha)^{1+2\tau}h_w(\alpha)} \prod_{\varpi \in G} \left( 1 + \frac{1}{N(\varpi)^{1+2\tau}} \right) \sum_{a, b \equiv 1 \mod (1+i)^3} \frac{r_\delta(a)\lambda(ab)}{N(ab)^{1+\delta_1 + w}h_w(ab)} \frac{r_\delta(b)\lambda(ab)}{N(b)^{1+\delta_2 + w}h_w(b)}.
\]

(6.10)

\[
\times \sum_{a, b \equiv 1 \mod (1+i)^3} \frac{r_\delta(a)\lambda(ab)}{N(ab)^{1+\delta_1 + w}h_w(ab)} \frac{r_\delta(b)\lambda(ab)}{N(b)^{1+\delta_2 + w}h_w(b)}.
\]

We further group terms according to $\gamma = \alpha \beta$ to see that (6.10) becomes

\[
\eta_\delta(1+2w; l) \sum_{\gamma \equiv 1 \mod (1+i)^3} \frac{H_w(\gamma)}{N(\gamma)^{1+2\tau}h_w(\gamma)} \left( \sum_{a \equiv 1 \mod (1+i)^3} \frac{r_\delta(a)\lambda(ab)}{N(ab)^{1+\delta_1 + w}h_w(ab)} \right) \left( \sum_{b \equiv 1 \mod (1+i)^3} \frac{r_\delta(b)\lambda(ab)}{N(b)^{1+\delta_2 + w}h_w(b)} \right).
\]

(6.11)

We now apply Lemma 2.16 to evaluate the sum over $a$ given in (6.11) by first considering the case $N(\gamma) \leq M^{1-b}$. We set $u = \delta_1 + w$, $v = \delta$, $g(u) = 1/h_w(n)$ and we denote $G(s, \gamma; u, v)$ by $G_w(s, \gamma; u, v)$ in Lemma 2.16 to see that by applying it with $y = M$, $R(x) = P(x)$ and then with $y = M^{1-b}$, $R(x) = (1 - P(b + x(1 - b)))$, the sum of these two applications yields

\[
\sum_{\alpha \equiv 1 \mod (1+i)^3} \frac{r_\delta(a)\mu[\gamma]}{N(\alpha)^{1+\delta_1 + w}h_w(\alpha)} Q \left( \frac{\log(M/N(\gamma))}{\log M} \right)
\]

(6.12)

\[
= \frac{\mu[\gamma]}{\zeta_K(1+\delta_1 + w)\delta K(1+\delta_1 + w)} \frac{G_w(1; \gamma; \delta_1 + w, \delta)}{E(\gamma) \log^2 M \left( \frac{M^{1-b}}{N(\gamma)} \right)^{-\gamma - R(w)}} \exp(-A_0 \sqrt{\log M^{1-b} / N(\gamma)}).
\]

Here one checks that the sum of the $k \geq 1$ terms of the above two applications of Lemma 2.16 cancel each other. Thus the main term above comes from the sum of the $k = 0$ terms only. Note also that the main term above is $\ll |\delta_1|^2$ due to the choice of the contour $C$. 
Now, replacing \(a\) by \(b\) and \(\delta_1\) by \(\delta_2\) in (6.12), we see that a similar expression for the sum over \(b\) in (6.11) holds. It follows that the case \(N(\gamma) \leq M^{-1-b}\) in (6.11) equals

\[
\eta_\beta(1 + 2w; 1) \sum_{N(\gamma) \leq M^{-1-b}} \frac{H_w(\gamma)}{N(\gamma)^{1+2\tau}h_w(\gamma)} \prod_{\mu = \pm} \zeta_K (1 + \delta_1 + w + \mu \delta) \zeta_K (1 + \delta_2 + w + \mu \delta) 
\]

\[
+ O\left( \sum_{N(\gamma) \leq M^{-1-b}} \frac{H_w(\gamma)}{N(\gamma)^{1+2\tau}h_w(\gamma)} E(\gamma) \left| \frac{\delta_1^2}{\log^2 M} \left( \frac{M^{1-b}}{N(\gamma)} \right)^{-\tau - \Re(w)} \exp(-A_0 \sqrt{\log(M^{1-b}/N(\gamma))}) \right| \right)
\]

\[
= \eta_\beta(1 + 2w; 1) \sum_{N(\gamma) \leq M^{-1-b}} \frac{H_w(\gamma)}{N(\gamma)^{1+2\tau}h_w(\gamma)} \prod_{\mu = \pm} \zeta_K (1 + \delta_1 + w + \mu \delta) \zeta_K (1 + \delta_2 + w + \mu \delta)
\]

\[
+ O\left( \left| \frac{\delta_1^2}{\log^2 M} M^{(1-b)\left(-\tau - \Re(w)\right)} \right| \right).
\]

We apply the last expression above in (6.13) to see that, based on our choice of \(C\) and \(M\), the contribution of the error term above is

\[
(6.13) \quad O\left( \log^2 X |\delta_1|^3 M^{-2\tau(1-b)} \right).
\]

On the other hand, the main term coming from the contribution of the terms \(N(\gamma) \leq M^{-1-b}\) to (6.13) equals to

\[
\frac{1}{2\pi i} \int_{C} \frac{2\pi \hat{\Phi}(1 + w - \tau)}{3\zeta_K(2) \Gamma_\delta(\tau)} \left( \frac{2^5 X}{\pi^2} \right)^{w-\tau} Z(1 + 2w; \delta) \Gamma_\delta(w) \frac{2w}{\psi^2(w - \tau^2 - \tau^2)} \eta_\beta(1 + 2w; 1)
\]

\[
\times \sum_{N(\gamma) \leq M^{-1-b}} \frac{H_w(\gamma)}{N(\gamma)^{1+2\tau}h_w(\gamma)} \prod_{\mu = \pm} \zeta_K (1 + \delta_1 + w + \mu \delta) \zeta_K (1 + \delta_2 + w + \mu \delta) \right) dw.
\]

Note that as \(\prod_{\mu = \pm} \zeta_K (1 + \delta_1 + w + \mu \delta)^{-1} \zeta_K (1 + \delta_2 + w + \mu \delta)^{-1}\) vanishes at \(w = -\tau\), the integrand in (6.14) has only a simple pole at \(w = \tau\) inside \(C\). It then follows from Cauchy’s theorem that the expression in (6.14) equals

\[
\frac{2\pi \hat{\Phi}(1)}{3\zeta_K(2)} \xi_K(1 + 2\tau) \sum_{N(\gamma) \leq M^{-1-b}} \frac{\mu^2(\gamma) H_\tau(\gamma)}{N(\gamma)^{1+2\tau}h_\tau(\gamma)} G_\tau(1, \gamma; \delta_1 + \tau, \delta_2 + \tau, \delta).
\]

We apply Lemma 2.17 to evaluate (6.15) and then combine the result together with the error term given in (6.13) to see that the contribution to \(S(1; \Phi) W(\delta_1, \Phi)\) from the part of \(\mathcal{M}_1(l)\) restricting to the case \(N(\gamma) \leq M^{-1-b}\) equals

\[
(6.16) \quad \frac{2\pi \hat{\Phi}(1)}{3\zeta_K(2)} (1 - M^{-2\tau(1-b)}) + O(\log^2 X |\delta_1|^3 M^{-2\tau(1-b)}).
\]

We now evaluate (6.11) for the case \(N(\gamma) > M^{-1-b}\) by applying Lemma 2.16 with \(u, v, G(n), G_w(s, \gamma; u, v)\) as before, and with \(R(x) = P(x), y = M\) this time. We deduce then that for any odd \(M^{-1-b} < N(\gamma) < M\),

\[
\sum_{\substack{N(\gamma) \leq M^{-1-b} \leq N(\gamma) \leq M \mod N(\gamma)^{1+b}}} \mu^2(\gamma) \mu(\gamma) G_w(1, s + \gamma; \delta_1 + w, \delta) Q^k \left( \frac{\log(M/N(\gamma))}{\log M} \right)
\]

\[
\times \sum_{s=0}^{\infty} \frac{\mu^2(\gamma) G_w(1, s + \gamma; \delta_1 + w, \delta)}{\zeta_K(1 + s + \delta_1 + w + \delta)} Q^k \left( \frac{\log(M/N(\gamma))}{\log M} \right)
\]

\[
+ \sum_{s=0}^{\infty} \frac{\mu^2(\gamma) G_w(1, s + \gamma; \delta_1 + w, \delta)}{\zeta_K(1 + s + \delta_1 + w + \delta)} Q^k \left( \frac{\log(M/N(\gamma))}{\log M} \right).
\]

To evaluate the residue above, we write the Taylor expansion of \(G_w(1, s + \gamma; u, v)\) as \(a_0 + a_1 s + a_2 s^2 + \ldots\) to see that \(a_0 = \frac{2^2}{\pi^2} (\delta_1 + w + \delta)(\delta_1 + w - \delta) G_w(1, \gamma; \delta_1 + w, \delta) + O((|\delta_1| + |w|)^3), a_1 = 2 \cdot \frac{4^2}{\pi^2} (\delta_1 + w) G_w(1, \gamma; \delta_1 + w, \delta) + O((|\delta_1| + |w|)^2), a_2 = \frac{4^2}{\pi^2} G_w(1, \gamma; \delta_1 + w, \delta) + O(|\delta_1| + |w|), and that \(a_n \ll_n 1\).
for \( n \geq 3 \). It follows from this that the residue term in (6.17) equals
\[
\frac{4^2}{\pi^2} \cdot \mu_{[1]}(\gamma) G_w(1, \gamma; \delta_1 + w, \delta) \left( (\delta_1 + w + \delta)(\delta_1 + w - \delta)Q \left( \frac{\log(M/N(\gamma))}{\log M} \right) + 2 \frac{\delta_1 + w}{\log M} Q' \left( \frac{\log(M/N(\gamma))}{\log M} \right) \right.
\]
\[
\left. + \frac{1}{\log^2 M} Q'' \left( \frac{\log(M/N(\gamma))}{\log M} \right) \right) + O(|\delta_1|^3).
\]

Again, replacing \( a \) by \( b \) and \( \delta_1 \) by \( \delta_2 \) in (6.18) yields an expression for the sum over \( b \) in (6.11) when \( M^{1-b} < N(\gamma) < M \). We can thus apply these expressions to evaluate (6.11) for the case \( N(\gamma) > M^{1-b} \). By doing so, we see that the contribution of the remainder terms is
\[
\ll \sum_{M^{1-b} \leq N(\gamma) \leq M} \frac{1}{N(\gamma)^{1+2\tau}} \left( \frac{E(\gamma)}{\log^2 M} \right) |\delta_1|^2 \left( \frac{M}{N(\gamma)} \right)^{-\tau - \Re(w)} \exp(-A_0 \sqrt{\log(M/N(\gamma))}) + |\delta_1|^5 \right)
\]
\[
\ll \frac{|\delta_1|^2}{\log^2 M} M^{-\tau - \Re(w)} + M^{-2\tau(1-b)}|\delta_1|^5 \log M.
\]

On the other hand, by writing \( Q_{\gamma_j}^{(j)} \) for \( Q_j^{(j)}(\log(M/N(\gamma))/\log M) \), we obtain that the main term is
\[
\left( \frac{4}{\pi} \right)^3 \cdot \eta_6 (1 + 2w; \gamma) \sum_{M^{1-b} \leq N(\gamma) \leq M, \gamma \equiv 1 \text{ mod } (1+i)^d} \frac{\mu_{[1]}(\gamma) H_w(\gamma)}{N(\gamma)^{1+2\tau}} G_w(1, \gamma; \delta_1 + w, \delta) G_w(1, \gamma; \delta_2 + w, \gamma)
\]
\[
\times \left( (\delta_1 + w + \delta)(\delta_1 + w - \delta)Q + 2 \frac{\delta_1 + w}{\log M} Q' + \frac{1}{\log^2 M} Q'' \right)
\]
\[
\times \left( (\delta_2 + w + \delta)(\delta_2 + w - \delta)Q + 2 \frac{\delta_2 + w}{\log M} Q' + \frac{1}{\log^2 M} Q'' \right).
\]

We apply Lemma 2.17 to compute the expression above. By a suitable change of variables, we see that it equals
\[
(6.20) \quad N(w) + O(M^{-2\tau(1-b)}|\delta_1|^5 \log M),
\]
where
\[
N(w) = \left( \frac{4}{\pi} \right)^3 \cdot \log M \int_0^b M^{-2\tau(1-x)} \left( (\delta_1 + w + \delta)(\delta_1 + w - \delta)Q(x) + 2 \frac{\delta_1 + w}{\log M} Q'(x) + \frac{Q''(x)}{\log^2 M} \right)
\]
\[
\times \left( (\delta_2 + w + \delta)(\delta_2 + w - \delta)Q + 2 \frac{\delta_2 + w}{\log M} Q' + \frac{Q''(x)}{\log^2 M} \right) dx.
\]

We now apply (6.19) and (6.20) to evaluate the expression in (6.8) when \( N(\gamma) > M^{1-b} \). Based on our choices for \( M \) and \( C \), we see that the error terms in (6.19) and (6.20) contribute
\[
(6.21) \quad O \left( |\delta_1|^3 M^{-2\tau \log^2 X} + |\delta_1|^6 M^{-2\tau(1-b) \log^5 X} \right).
\]

Moreover, we see that the main term in (6.8) equals
\[
(6.22) \quad \frac{1}{2\pi i} \int_C 2 \Phi(1 + w - \tau) \left( \frac{2^5 X}{\pi^2} \right)^{w-\tau} \left( Z(1 + 2w; \delta) \right) (\delta \Gamma(\delta(u)) N(w) \frac{2w}{u^{w^3 - \tau^3}} dw.
\]

Applying the relations
\[
\tilde{\Phi}(1 + w - \tau) \Gamma(\delta(w)) = \Phi(1) + O(|\delta_1|),
\]
\[
2w Z(1 + 2w; \delta) = \frac{\pi}{4} \cdot \frac{1}{4(w^2 - \delta^2)} + O(|\delta_1| \log^2 X),
\]
\[
N(w) \ll M^{-2\tau(1-b)} |\delta_1|^4 \log M,
\]
we see that the expression in (6.22) equals
\[
(6.23) \quad \frac{2\pi \Phi(1)}{3 \zeta_K(2)} \frac{1}{2\pi i} \int_C 2 \Phi(1 + w - \tau) \left( \frac{2^5 X}{\pi^2} \right)^{w-\tau} \left( \frac{\pi}{4} \right)^3 N(w) \frac{1}{4(w^2 - \delta^2)} \frac{1}{w^{w^3 - \tau^3} dw + O(|\delta_1|^6 M^{-2\tau(1-b) \log^5 X})
\]
\[
= \frac{2\pi \Phi(1)}{3 \zeta_K(2)} \frac{1}{8\delta_1 \delta_2 \pi} \left( \frac{\pi}{4} \right)^3 N(\tau) - \left( \frac{2^5 X}{\pi^2} \right)^{-2\tau} \left( \frac{\pi}{4} \right)^3 N(-\tau) + O(|\delta_1|^6 M^{-2\tau(1-b) \log^5 X}).
\]
Now, using integration by parts together with the observations that $Q(0) = Q'(0) = 0$, and $Q(b) = 1$, $Q'(b) = 0$, we obtain after a little calculation that
\[
\left(\frac{\pi}{4}\right)^3 N(\tau) = 8\delta_2\int_0^b M^{-2\tau(1-x)} dx + \frac{4\delta_1\delta_2}{\log M} \int_0^b M^{-2\tau(1-x)} \frac{Q''(x)}{2\delta_1 \log M} dx.
\]
\[
\left(\frac{\pi}{4}\right)^3 N(-\tau) = \frac{4 \cdot 4\delta_1\delta_2}{\log M} \int_0^b M^{-2\tau(1-x)} \frac{Q''(x)}{2\delta_1 \log M} dx.
\]

The above expressions allow us to recast (6.23) as
\[
\frac{2\pi\hat{\Phi}(1)}{3\zeta_K(2)} \left( M^{-2\tau(1-b)} + \frac{1 - (2X/\pi^2)^{-2\tau}}{2\tau \log M} \int_0^b M^{-2\tau(1-x)} \frac{Q''(x)}{2\delta_1 \log M} dx \right) + O(M^{-2\tau(1-b)}|\delta|^6 \log^5 X).
\]

It follows from this and (6.21) that the contribution of $M_1(l)$ to $S(\Phi)\mathcal{W}(\delta_1, \Phi)$ from the terms $M^{1-b} < N(\gamma) \leq M$ equals
\[
\frac{2\pi\hat{\Phi}(1)}{3\zeta_K(2)} \left( M^{-2\tau(1-b)} + \frac{1 - (2X/\pi^2)^{-2\tau}}{2\tau \log M} \int_0^b M^{-2\tau(1-x)} \frac{Q''(x)}{2\delta_1 \log M} dx \right) + O(M^{-2\tau(1-b)}|\delta|^6 \log^5 X).
\]

Combining the above with (6.16), we conclude that the $M_1(l)$ contribution to $S(\Phi)\mathcal{W}(\delta_1, \Phi)$ equals
\[
\frac{2\pi\hat{\Phi}(1)}{3\zeta_K(2)} \left( 1 + \frac{1 - (2X/\pi^2)^{-2\tau}}{2\tau \log M} \int_0^b M^{-2\tau(1-x)} \frac{Q''(x)}{2\delta_1 \log M} dx \right) + O(M^{-2\tau(1-b)}|\delta|^6 \log^5 X).
\]

We obtain the $M_2(l)$ contribution to (6.7) in a similar way to see that it equals
\[
\frac{2\pi\hat{\Phi}(1)}{3\zeta_K(2)} \left( \frac{2X}{\pi^2} \right)^{-\tau} \left( \frac{2X}{\pi^2} \delta - \frac{(2X/\pi^2)^{-\delta}}{2\delta \log M} \int_0^b M^{-2\tau(1-x)} \frac{Q''(x)}{2\delta_1 \log M} dx \right) + O(X^{-\tau}M^{-2\tau(1-b)}|\delta|^6 \log^5 X).
\]

By appealing to (6.22), (6.24) and (6.25) in (6.7), we deduce readily the statement of Proposition 6.3.

6.2. Completion of the proof. We are now able to complete our proof of Theorem 1.1. For this, we take our parameters exactly as those used by Conrey and Soundararajan in their proof of [2, Theorem 1]. Precisely, we take $\kappa = 10^{-10}$, $M = X^{b - 5\kappa}$ in (6.2) and $S = \pi/(2(1-b)(1-2\kappa))$ with $b = 0.64$. We further take $R = 6.8$ and $P(x) = 3x/b^2 - 2(x/b)^3$. Let $\sigma_0$ be given as in Proposition 6.2 and let $\Phi$ to be a smooth function supported in $(1, 2)$ satisfying $0 \leq \Phi(t) \leq 1$ for all $t$, $\Phi(t) = 1$ for $t \in (1 + \epsilon, 2 - \epsilon)$, and $|\Phi'(t)| \ll_{\nu, \epsilon} 1$. We apply Proposition 8.2 in (3.3) to deduce that
\[
\mathcal{N}(X, \Phi) \leq \frac{XS(1; \Phi)}{8S \sinh(\frac{2\pi}{S})} \left( J_1(X; \Phi) + J_2(X; \Phi) \right) + o(X),
\]
where $J_1$ and $J_2$ are given in (3.4).

Now, Proposition 8.6 implies that for real numbers $u, v$ satisfying $\kappa \log X \geq |u + iv| \geq -1/\epsilon$ and $u \geq -1/\epsilon$, we have
\[
\mathcal{W}(\frac{u + iv}{\log X}, \Phi) = \mathcal{W}(u, v) + O \left( \frac{M^{-2u(1-b)/\log X}}{(1 + |u| + |v|)^6} \right),
\]
where
\[
\mathcal{W}(u, v) = 1 + e^{-u \log X} \left( \sinh \frac{v}{u} - \sin \frac{v}{u} \right) \int_0^b M^{-2u(1-x)/\log X} \frac{Q''(x)}{2(2x + 2y) \log M} dx.
\]

The above expression implies that $\mathcal{W}(u, v) \geq 1$, from which we deduce that
\[
J_1(X; \Phi) = \int_0^S \cos \left( \frac{\pi}{2S} t \right) \log \mathcal{V}(-R, t) dt + O \left( \frac{1}{\log X} \right).
\]
Moreover, based on Proposition 6.2 and our choice for $S$, we can extend the integral in the definition of $J_2(X; \Phi)$ in (3.4) to infinity with an negligible error. This way, we obtain that
\[
J_2(X; \Phi) = \int_0^\infty \sinh \left( \frac{\pi u}{2S} \right) \log \mathcal{V}(u, S) du + o(1).
\]
We conclude from (6.26), (6.27) that
\[
\mathcal{N}(X, \Phi) \leq \frac{XS(1; \Phi)}{8S \sinh(\frac{2\pi}{S})} \left( \int_0^S \cos \left( \frac{\pi}{2S} t \right) \log \mathcal{V}(-R, t) dt + \int_0^\infty \sinh \left( \frac{\pi u}{2S} \right) \log \mathcal{V}(u, S) du \right) + o(X).
\]
As shown in [2, p. 10], the above further implies that \( N(X, \Phi) \leq 0.79 X S(1; \Phi) + o(X) \). By summing over \( X = x/2, x/4, \ldots \), we obtain the proof of Theorem 1.1.

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