We study the long wavelength electromagnetic resonances of interacting cylinder arrays. By using a normal modes expansion where the effects of geometry and material are separated, it is shown that two parallel cylinders with different radii have electromagnetic modes distributed symmetrically about depolarization factor $\frac{1}{2}$. Both sets couple to longitudinal and transverse components of the external field, but amplitudes of symmetric depolarization factors become exchanged when considering longitudinal or transverse polarization. We also find that amplitudes satisfy sum rules that depend on the
ratio of the cylinders radii.
I. INTRODUCTION

The optical properties of ordered cylinder arrays has become a subject of much recent interest mainly because of their potential use as photonic crystals. [1–3] Theoretical research once done for spherical particles [4–7] has lately been applied to cylinders [8–10]. It is well known that in the long wavelength limit the optical properties of a dilute composite of microscopic spherical particles are well described by mean field theories such as Claussius-Mosotti or Maxwell-Garnett. These theories are essentially based on a dipolar approximation assuming that the particles are sufficiently far apart so that it is possible to neglect contributions from higher order multipoles. As particles become closer, however, this approach is no longer valid. Several models have been presented to overcome this difficulty, among which the theory of normal modes has been shown to be particularly convenient since it makes possible to expand the response of the system in terms of optical resonance terms, where dielectric properties appear separate from geometrical factors. [11,12] The simplest system exhibiting the effects of interactions is a pair of identical particles very close to each other. A pair of particles of the same material and form but different size is the simplest non-symmetric system of interacting particles [7].

Recently, a model to study arbitrary cylinder arrays made of the same material has been constructed and applied in detail to a pair of identical cylinders. [10] The response of periodic arrays of identical parallel cylinders including proximity effects is also easily treated within that formalism. The method follows a normal modes description appropriate to the long-wavelength limit first proposed by Bergmann [11,12], and makes use of a basis of cylindrical harmonics solutions to Laplace’s equation. Modes are characterized by depolarization factors and strengths, defined in such a way that an isolated cylinder exhibits a depolarization factor $\frac{1}{2}$ and unit strength. We here follow a similar procedure to study a pair of non-touching parallel cylinders of the same material but different radii, for different polarizations of the external field. We show that a difference in radii does not alter the property that depolarization factors are symmetric about $\frac{1}{2}$, although in contrast with the equal radii case,
mixing of transverse and longitudinal modes occurs. This property is no longer valid when
the dielectric function of the cylinders is different. We find that all normal modes are active
for an external field perpendicular to the cylinders axis, whether parallel or perpendicular
to the plane containing the axis. Modes inactive for identical cylinders have very small
strengths when their radii are not very different. Strengths of modes with depolarization
factors smaller than $\frac{1}{2}$ are exchanged with those of depolarization modes bigger than $\frac{1}{2}$, when
the direction of the electric field changes from parallel to perpendicular.

In Sec. I we get the multipolar moments and the absorption cross section for a pair
of unequal cylinders. In Sec. III we present and discuss our numerical results. Finally in
Sec. IV we summarize our conclusions.

II. THEORY

We consider a set of N parallel, infinite, uncharged cylinders of dielectric function $\varepsilon_1$
placed in a homogeneous medium of dielectric function $\varepsilon_2$, excited by an external electric field
whose wavelength is much longer than the cylinder radii or separation between cylinders.
The charge distribution they acquire may be described in terms of individual multipole
moments $q_{mj}$ obeying the equations, [3]

$$q_{mj} = -\alpha_{mj} (V_m + \sum_{m'j'} A_{mj}^{m'j'} q_{m'j'}) ,$$  \hspace{1cm} (1)

Here $m$ is a positive or negative integer labeling the angular momentum component along
the cylinder axes, $j = 1, 2, ..., N$ is a particle index, $\alpha_{mj} = |m| a_j^2 \varepsilon_1/(\varepsilon_1 + \varepsilon_2)$ are
the multipolar polarizabilities of cylinder cylinder $j$ of radius $a_j$, and $V_m$ are the coefficients
in the expansion of the external potential in terms of cylindrical harmonics. The coupling
coefficients are given by, [10]

$$A_{mj}^{m'j'} = \begin{cases} 0 & \text{if } m \cdot m' > 0 \\ (-)^{m'} \frac{(|m| + |m'| - 1)!}{|m|! |m'|!} \frac{e^{i(m'-m)\theta_{jj'}}}{\rho_{jj'}^{\frac{|m|+|m'|}}} & \text{if } m \cdot m' < 0 , \end{cases} \hspace{1cm} (2)$$
where \((\rho_{jj'}, \theta_{jj'}) = \vec{\rho}_{j'} - \vec{\rho}_j\) are polar coordinates in the x-y plane giving the relative position of cylinder \(j'\) with respect to cylinder \(j\).

As discussed in reference [10], if the cylinders are of the same material one can separate in Eq. (1) terms depending on the material susceptibility \(\chi\) from those involving the geometry of the array. We intend to follow here the same procedure and write Eq. (1) as

\[
\sum_{\mu'} (\chi^{-1} \delta_{\mu\mu'} + H_{\mu'}^{\mu'} \ x_{\mu'}) = f_\mu ,
\]

where \(\mu\) represents the pair of indices \((m, j)\), and

\[
H_{mj}^{m'j'} = 2\pi \left( \delta_{mm'} \delta_{jj'} + |m\ m'|^{1/2} a_j^{[m]} a_{j'}^{[m']} A_{mj}^{m'j'} \right) \quad (4)
\]

\[
f_{mj} = -2\pi |m|^{1/2} a_j^{[m]} V_{mj} \quad (5)
\]

\[
x_{mj} = \frac{q_{mj}}{|m|^{1/2} a_j^{[m]}} \quad (6)
\]

Note the important feature that matrix \(H\) depends on geometry only and its eigenvalues \(\{4\pi n_\mu\}\) define the depolarization factors \(\{n_\mu\}\) of the array. For later convenience we write \(H = 2\pi(I + B)\), with \(I\) the unit matrix and \(B_{mj}^{m'j'} = |m\ m'|^{1/2} a_j^{[m]} a_{j'}^{[m']} A_{mj}^{m'j'}\), so that the depolarization factors \(\{n_\mu\}\) and eigenvalues \(\{\lambda_\mu\}\) of \(B\) satisfy the relation,

\[
n_\mu = \frac{1}{2}(1 + \lambda_\mu) .
\]

Because of the property \(B_{mj}^{m'j'} = 0\) if \(m \cdot m' > 0\) (see Eq. (2)), we write rows and columns of matrix \(B\) with indexes \(m\) and \(m'\) following the sequence \(1, 2, \ldots, -1, -2, \ldots\), resulting in matrix \(B\) written in terms of a new real matrix \(b\) of half its dimension, as

\[
B = \begin{bmatrix}
0 & b \\
b & 0
\end{bmatrix} .
\]

From now on we use index \(m\) and \(m'\) as positive integers, and write the elements of matrix \(b\) as follows,

\[
b_{mj}^{m'j'} = (-)^{m'} \sqrt{mm'} \frac{(m + m' - 1)!}{m!m'} a_j^m a_{j'}^{m'} e^{i(m+m') \theta_{jj'}} \rho_{jj'}^{m+m'} r_j^{m+m'} \quad (9)
\]
It can be shown that the eigenvalues of matrix $B$ come in pairs with opposite sign $\lambda_\mu = \pm \ell_\mu$, where $\ell_\mu$ are the eigenvalues of $b$ (see the appendix A). As follows from Eq. (7) the depolarization factors are then symmetric about the value $1/2$. The components of vector $x_\mu$ can be written in terms of the eigenvalues of matrix $b$ and elements of matrix $u$ that diagonalizes $b$ through

$$u^{-1}bu = \ell.$$  \hspace{1cm} (10)

In the case of a uniform electric field $E_0$ parallel to the plane containing the cylinder axes and perpendicular to the latter (parallel field geometry), $x_{mj} = x_{-mj}$. Vector $x_+ = x_{mj}$ is then given by

$$x_+ = (u \, s^{-1}u^{-1})f^+, \hspace{1cm} (11)$$

where

$$s_{mj}^{m'j'} = \delta_{mm'}\delta_{jj'}(\chi^{-1} + 2\pi(1 + \ell_{mj})), \hspace{1cm} (12)$$

$$f_{mj}^+ = \delta_{m1}\pi a_j E_0. \hspace{1cm} (13)$$

In the case of an electric field perpendicular to the plane containing the cylinder axes (perpendicular field geometry), $x_{mj} = -x_{-mj}$. Vector $x_- = x_{mj}$ is then given by,

$$x_- = (u \, r^{-1}u^{-1})f^-, \hspace{1cm} (14)$$

where now

$$r_{mj}^{m'j'} = \delta_{mm'}\delta_{jj'}(\chi^{-1} + 2\pi(1 - \ell_{mj})), \hspace{1cm} (15)$$

$$f_{-mj}^- = \delta_{m1}\pi a_j E_0. \hspace{1cm} (16)$$

For a pair of unequal parallel cylinders with radii $a_1$ and $a_2$, and axis at a distance $R$ we define dimensionless parameters $\beta = a_2/a_1$ and $\delta = R/a_1$. Because of the properties $b_{mj}^{m'j'} = 0$ and $b_{m2}^{m'1} = (-\beta)^{m-m'}b_{m1}^{m'2}$, we write rows (columns) of matrix $b$ with particle index $j$ ($j'$) in the sequence $1, 2$ resulting in matrix $b$ written in terms of a smaller array $g$ as follows,
\[
\mathbf{b} = \begin{bmatrix}
0 & \mathbf{g} \\
\mathbf{g} & 0
\end{bmatrix}
\]  

(17)

where \( \mathbf{g} \) is the transpose of \( \mathbf{g} \), and the elements of matrix \( \mathbf{g} \) are given by

\[
g_{m}^{m'} = (-)^{m'} \sqrt{m m'} \frac{(m + m' - 1)!}{m! m'!} \frac{\beta^{m'}}{\delta^{m + m'}}.
\]  

(18)

Results for a pair of cylinders with an external field in the parallel or perpendicular configuration can be written in terms of a single normal modes expansion as

\[
x_{\pm mj} = \sum_{m'^{j'}} C_{mj}^{m'^{j'}} f_{\pm} \chi^{-1} + 4\pi n_{\pm m'^{j'}}.
\]  

(19)

In this expression the upper (lower) sign corresponds to the parallel (perpendicular) configuration, with \((m', j')\) labelling the excitation modes of the pair as a coupled system. It gives just half of the multipoles; the others are obtained from the corresponding symmetry property as given in the paragraph preceding Eqs. (11) and (14). We have defined the depolarization factors of modes,

\[
n_{\pm m'^{j'}} = \frac{1}{2} (1 \pm \ell_{mj}),
\]  

(20)

and coefficients corresponding to strength of modes,

\[
C_{mj}^{m'^{j'}} = u_{mj}^{m'^{j'}} (u_{1,1}^{m'^{j'}} + \beta u_{1,2}^{m'^{j'}}).
\]  

(21)

We have also defined

\[
f_{+} = f_{1,1}^{+},
\]  

(22)

\[
f_{-} = f_{1,1}^{-}.
\]  

(23)

We find that coefficients \( C_{mj}^{m'^{j'}} \) satisfy the sum rules

\[
\sum_{m'^{j'}} C_{mj}^{m'^{j'}} = \delta_{m1} 
\]  

(24)

\[
\sum_{m'^{j'}} C_{mj}^{m'^{j'}} = \beta \delta_{m1}.
\]  

(25)
It can be shown that, as with the original matrix $B$, eigenvalues $\{\ell_{mj}\}$ of matrix $b$ come also in pairs with opposite sign; therefore the sets of depolarization factors $\{n_{+mj}\}$ and $\{n_{-mj}\}$ are identical (see the appendix). A given depolarization factor exhibits a different strength depending on the direction of the external field. As seen in the normal modes expansion given by Eq. (14) the same strength coefficients $C_{mj}^{m'j'}$ appear for depolarization factor $n_{+m'j'}$ in the parallel field response and for $n_{-m'j'}$ in the perpendicular field response. Then, strengths corresponding to depolarization factors symmetric around value $1/2$ are exchanged between responses corresponding to fields parallel or perpendicular.

The magnitude of the electric dipole moment for the pair can be written as

$$p_{\pm} = \pi a_1^2 \sum_{m'j'} C_{11}^{m'j'} + \beta C_{12}^{m'j'} \chi^{-1} + 4\pi n_{\pm m'j'} E_0,$$

where $p_+$ ($p_-$) corresponds to a parallel (perpendicular) field. The absorption cross section is proportional to the imaginary part of the factor accompanying $E_0$ in the previous expression, a quantity we identify as the complex effective polarizability of the pair. Thus we arrive at a normal modes decomposition for the absorption cross section of two parallel cylinders,

$$\sigma_{\pm} \sim \text{Imaginary}\{\sum_{m'j'} C_{11}^{m'j'} + \beta C_{12}^{m'j'} \chi^{-1} + 4\pi n_{\pm m'j'} \}
$$

where $\sigma_+$ ($\sigma_-$) corresponds to a parallel (perpendicular) field. Notice that, as follows from Eq. (21), the numerator in the previous sum can be written as

$$\left(u_{1,1}^{m'j'} + \beta u_{1,2}^{m'j'}\right)^2,$$

and is always positive definite. According to the sum rules given by Eqs. (24) and (25), the sum of the numerators in expansions (26) and (27) is $1 + \beta^2$, a feature we use in calculating the normalized strength of modes.

III. NUMERICAL RESULTS

We have solved numerically the eigenvalue equation for matrix $b$ for the case of a pair of parallel cylinders of radii $a_1$ and $a_2$, and have calculated the depolarization factors $n_{m'j'}$ and
strength coefficients $C_{m'j'}^{m'j'}$ according to Eqs. (20) and (21). We have studied in detail the normal modes expansion for the dipole moment of the pair as given by Eq. (26). Our most important finding is that when the radii are not equal, modes with depolarization factors above and below the value $\frac{1}{2}$ mix for all orientations of the external field. This is known not to happen when cylinders are equal. [10]

In obtaining numerical results we use the dimensionless parameters $\sigma = R/(a_1 + a_2)$, that measures the center to center distance, $\beta = a_2/a_1$, measuring how disimilar the radii are, and $\mu = (R - a_1 - a_2)/a_1$, measuring the border to border distance. In Fig. 1 we plot modes for very close cylinders ($\sigma = 1.10$) with one radius three times bigger than the other ($\beta = 3$). Modes are for the parallel configuration, while those for the perpendicular case are obtained by mirror reflection about depolarization factor $\frac{1}{2}$. Note that the placement of modes are symmetric about this central value so that, as far as position is concerned, they are indistinguishable in both configurations. The figure shows the modes with largest amplitude, while weaker modes accumulate around $n = \frac{1}{2}$. The sum of amplitudes is 10, as required by the sum rule mentioned after Eq. (28). Labels, included in order to match with labelling in Fig. 2, are arbitrary.

Figure 2 shows depolarization factors (a) and normalized strengths (b) in terms of the parameter $\beta$, at fixed $\sigma = 1.1$. The sum over all strengths is unity, having been normalized by the factor $1 + \beta^2$. Thus the results at $\beta = 3$ correspond to amplitudes shown in Fig. 1 with the normalization factor 10. In changing $\beta$ we keep constant the parameter $\sigma$ by changing the center to center distance $R$ accordingly. Note that modes with essentially zero strength at $\beta = 1$ become important when increasing this ratio. Labels are arbitrary and are used just to relate the data in different figures.

Results shown in Fig. 3 were obtained by changing $\beta$ and the center to center separation $R$, but keeping constant the border to border distance at the fixed value $\mu = 0.4$. In this case we note that the depolarization factors move apart with increasing $\beta$, while at constant $\sigma$ (Fig. 2 (a)) they get closer. This is because the relation between the edge to edge and
center to center parameters is $\mu = (\sigma - 1)(1 + \beta)$, indicating that as $\beta$ goes to infinity, so does $\mu$ if $\sigma$ is kept constant. Thus all modes should converge to the isolated cylinder value $n = 1/2$ in this case, while if $\mu$ is kept constant, the modes converge to those of a cylinder in front of a plane at the same distance.

**IV. CONCLUSIONS**

In summary, we have shown that the absorption cross section of a pair of parallel cylinders of the same material but different radii contains modes whose depolarization factors are symmetrically distributed around $\frac{1}{2}$ with amplitudes depending on the direction of the external field. When the field changes from the parallel to the perpendicular configuration, amplitudes corresponding to symmetric depolarization factors about the central value 1/2 are exchanged.

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APPENDIX A:

We consider the eigenvalue equation for operator $T$, 

$$ Tx = \lambda x , \quad (A1) $$

in a basis of dimension $2M$. Here $T$ has the form,

$$ T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} . \quad (A2) $$

where $A$, $B$ each has dimension $M$. In writing the eigenvectors $x$ in terms of two smaller vectors $w$ and $v$ of dimension $M$ the eigenvalue equation is cast into the form,

$$ \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} = \lambda \begin{bmatrix} w \\ v \end{bmatrix} \quad (A3) $$

or

$$ Aw = \lambda w \quad (A4) $$

$$ Bw = \lambda v . \quad (A5) $$

From there we get the separate eigenvalue problems,

$$ (AB)w = \lambda^2 w \quad (A6) $$

$$ (BA)v = \lambda^2 v , \quad (A7) $$

both having the same eigenvalues $\lambda^2$. In solving for the corresponding eigenvectors and writing them as columns we get matrices $w$ and $v$ that diagonalize matrices $AB$ and $BA$. They can be used to form a matrix $U$ as,

$$ U = \frac{1}{\sqrt{2}} \begin{bmatrix} w & w \\ v & -v \end{bmatrix} , \quad (A8) $$

which diagonalizes matrix $T$ according to the relation

$$ U^{-1}TU = \Lambda , \quad (A9) $$
with matrix $\Lambda$ given by

$$\Lambda = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}.$$  \hspace{1cm} (A10)

Here $\lambda (-\lambda)$ is a diagonal matrix formed by the positive (negative) square root of the eigenvalues of matrices $AB$ or $BA$. Therefore the eigenvalues of matrix $T$ come in pairs with opposite sign. In the case of a pair of cylinders, we find the previous feature two times. It first happens because coupling coefficients $A_{m'j'}^{mj}$ are zero if $m$ and $m'$ have the same sign, and then occurs also because they are zero for $j = j'$. The dimensionality $(4M)X(4M)$ of the original eigenvalue problem is seen to be reduced to $MXM$ dimensions.
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FIGURES

FIG. 1. Mode amplitudes for a pair of unequal cylinders under a uniform electric field in the parallel field configuration. The radii ratio equals $\beta = 3$, and the separation parameter is $\sigma = 1.1$.

FIG. 2. Depolarization factors (a) and normalized strengths (b) as a function of size parameter $\beta$, for a pair of unequal cylinders with fixed separation parameter $\sigma = 1.1$ under a uniform electric field. Labels correspond to those in Figure 1.

FIG. 3. Same as Fig. 2, but with fixed border to border parameter $\mu = 0.4$. 
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