TREE SUBSTITUTIONS AND RAUZY FRACTALS

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Abstract. We work with attracting subshifts generated by substitutions which are also irreducible parageometric automorphisms of free groups. For such a dynamical system, we construct a tree substitution to approximate the repelling real tree of the automorphism. We produce images of this tree inside the Rauzy fractal when the substitution is irreducible Pisot. We describe the contour of this tree and compute an interval exchange transformation of the circle covering the original substitution.

1. Introduction

1.1. Results. The main objects of our work are attracting subshifts generated by a substitution $\sigma$ over a finite alphabet $A$. These are also called substitutive subshifts and are defined as the action of the shift $S$ on the set $X_\sigma$ of bi-infinite words all of whose finite factors are factors of some iterations of the substitution on some letter. The word “attracting” reflects that under iterations of the substitution any letter $a \in A$ converges to the attracting subshift $X_\sigma$.

Our motivation is to find a suitable geometrical interpretation for these subshifts. Rauzy fractals provide such a geometric interpretation. If $\sigma$ is irreducible Pisot, there exists a continuous map $\varphi : X_\sigma \to E_\sigma$ from the attracting subshift $X_\sigma$ into the contracting space $E_\sigma$ of the abelianization matrix $M_\sigma$. The image $R_\sigma = \varphi(X_\sigma)$ is called the Rauzy fractal, it is a compact subset equipped with a domain exchange and the contracting action of the matrix $M_\sigma$, which are the push-forward of the shift map and of the substitution respectively. This celebrated construction goes back to the seminal work of Rauzy [Rau82] and was later generalized by Arnoux and Ito [AI01].

Another geometric interpretation arises when $\sigma$ is also a fully irreducible (abbreviated as iwip) automorphism of the free group $F_A$ on the alphabet $A$. Following the work of Bestvina and Handel, Gaboriau, Jaeger, Levitt and, Lustig [GJLL98] described the repelling tree of such an iwip automorphism. This is a real tree $T_\sigma^{-1}$ with an action of the free group by isometries and a homothety $H$. The homothety is contracting by a factor $\frac{1}{\lambda_\sigma - 1}$, where $\lambda_\sigma - 1$ is the expansion factor of the inverse automorphism of $\sigma$. Later a continuous map $Q : X_\sigma \to T_\sigma^{-1}$, where $T_\sigma^{-1}$ is the metric completion of the repelling tree, was defined [LL03, CHL09]. The image $\Omega_A = Q(X_\sigma)$ is a compact subset and the shift map is pushed-forward through $Q$ to the actions of the elements of $A^{\pm 1}$, while the action of $\sigma$ is pushed-forward to the homothety.

It was remarked [KL14, CHR15] that these two geometric interpretations are one above the other: when $\sigma$ is both irreducible Pisot and iwip, there exists a continuous
equivariant map $\psi : \Omega_A \to \mathcal{R}_\sigma$ that makes the diagram commute:

$$
\begin{array}{c}
X_{\sigma} \\
\downarrow \varphi \\
\Omega_A \\
\downarrow \psi \\
\mathcal{R}
\end{array}
$$

When $\sigma$ is a parageometric iwip automorphism, $\Omega_A$ is a connected subset of $\mathbb{T}_{\sigma^{-1}}$: a compact $\mathbb{R}$-tree, called the compact heart. The first aim of this paper is to provide approximations of the compact heart and to draw them inside the Rauzy fractal.

The approximation of the repelling tree (or rather of its subtree $\Omega_A$) is achieved by a tree substitution. Jullian [Jul11] introduced this version of graph-directed self-similar systems (in the sense of [MW88]) and described some examples, in particular in the case of the Tribonacci automorphism $\sigma : a \mapsto ab, b \mapsto ac, c \mapsto a$.

We generalize this construction: we construct an abstract tree substitution which converges to the repelling tree of the automorphism.

**Theorem 1** (Proposition 3.9 and Algorithms 3.6 and 3.10). Let $\sigma$ be a primitive substitution on a finite alphabet $A$ and a parageometric iwip automorphism of the free group $\mathbb{F}_A$. Then there exists a tree substitution $\tau$ such that the iterations of $\tau$ on the initial tree $W$ renormalized by the ratio of the contracting homothety converge to the repelling tree of the automorphism $\sigma$:

$$
\lim_{n \to \infty} \left( \frac{1}{\lambda_{\sigma^{-1}}} \right)^n \tau^n(W) = \Omega_A \subseteq \mathbb{T}_{\sigma^{-1}}.
$$

We provide an algorithm to construct this tree substitution.

The key of our algorithm is to describe the vertices of the tree $W$ by singular bi-infinite words. The singular bi-infinite words of the attracting shift are also known in the literature as asymptotic pairs, proximal pairs or special infinite words [Cas97, BD01, BD07]. In the train-track dialect finding singular words amounts to compute periodic Nielsen paths for the automorphism $\sigma$. They exactly correspond to pairs of distinct bi-infinite words with the same image by $Q$.

Turning back to the Pisot hypothesis, using the map $\varphi$ we can regard the tree substitution inside the contracting space. More precisely we can use as vertices the images by $\varphi$ in the Rauzy fractal of the singular words. The surjectivity of the map $\varphi$ implies that the iterates under the tree substitution of the set of singular points fills the Rauzy fractal. If we connect them, for instance with line segments, we get a picture of a tree reminding of a Peano curve inside the Rauzy fractal.

However, there is no guarantee that the tree substitution yields a tree in the contracting space. Indeed, the map $\varphi$ is not injective and distinct singular words can be misleadingly identified. This is the case in all our examples. Nevertheless, we provide a way to overcome this difficulty by using coverings of the tree substitution. A covering consists in extending the alphabet to a bigger one $\tilde{A}$, defining a forgetful map $f : \tilde{A} \to A$ and allowing different prototiles $W_{\tilde{a}}$ for different $\tilde{a} \in f^{-1}(a)$ but keeping the same self-similar structure. In practice we prune the tree substitution, i.e. we erase some branches of the tree which are dead-end and which create loops when iterating the embedded tree substitution. This results in extending the definition of the tree substitution to a larger number of initial subtrees.

Summing up the above discussion we get the following theorem.

**Theorem 2** (Sections 3.5 and 3.6). Let $\sigma$ be an irreducible Pisot substitution on a finite alphabet $A$ and a parageometric iwip automorphism of the free group $\mathbb{F}_A$. Then, the tree substitution $\tau$ can be realized inside the contracting space $E_\sigma$ of $\sigma$ and the renormalized iterated images $M^*_\sigma \tau^n(W)$ converge to the Rauzy fractal $\mathcal{R}_\sigma$.
In all our examples we provide a covering tree substitution which yields trees inside the contracting space.

We will show some pictures of these trees in the contracting space like in Figure 1. Such pictures where already obtained by Arnoux [Arn88] (for the Tribonacci substitution) and Bressaud and Jullian [BJ12] (as well as some other examples); in [BDJP14, DPV16] the tree associated with the Tribonacci substitution is obtained by connecting adjacent cubes of a thickened version of the stepped approximation of the contracting plane.

**Figure 1.** Twelfth iterate of the tree substitution for the Tribonacci automorphism $\sigma : a \mapsto ab, b \mapsto ac, c \mapsto a$ inside the Rauzy fractal.

The last geometric interpretation we have in mind are interval exchange transformations. We are looking for an interpretation of the original attracting shift $X_\sigma$ as an interval exchange transformation.

We take inspiration mainly from the Arnoux-Yoccoz interval exchange [AY81]. Arnoux and Yoccoz studied a certain exchange of six intervals on the unit circle. They observed that this exchange is conjugate to its first return map into a subinterval, that is, it is self-induced. Moreover, it is measurably conjugate to the attracting subshift defined by the Tribonacci substitution $\sigma : a \mapsto ab, b \mapsto ac, c \mapsto a$. At the time, this transformation provided one of the first non-trivial examples of pseudo-Anosov automorphism with a cubic dilatation factor, namely the dominant root of the polynomial $x^3 - x^2 - x - 1$. Arnoux-Rauzy words [AR91] are another example of codings of six interval exchanges.

For a substitution which is a parageometric iwip automorphism, Bressaud and Jullian [BJ12] proved that the contour of the compact heart $\Omega_A$ always provides a self-induced interval exchange of the circle.

We provide an algorithm to compute the contour substitution. Using our tree substitution $\tau$, we described the repelling tree $T_{\sigma^{-1}}$ and its compact heart $\Omega_A$ by the iterated images $\tau^n(W)$. Once the cyclic orders at branch points of the finite
We provide an algorithm to compute $\chi$ the tree substitution constructed in Theorem 1. Then, there exist cyclic orders on the alphabet initial trees (Sections 4.1 and 4.2). Theorem 3 where the projection to the contracting space and $H$ to the Rauzy fractal: $\phi$ automaton. This self-similarity is passed through $Q$ as the disjoint union of cylinders $X$. bi-infinite words in $\Gamma : X_\sigma \to \mathbb{S}^1$ from the attracting shift of $\chi^*$ to the circle. The shift map and $\chi^*$ are pushed forward by $Q_\sigma$ to a piecewise exchange of the circle and to a piecewise homothety respectively. The dual substitution $\chi^*$ defined on the bigger alphabet $\tilde{A}$ covers $\sigma$: there exists a forgetful map $f : \tilde{A} \to A$ such that $f \circ \chi^* = \sigma \circ f$. The attracting shift $X_{\chi^*}$ factors onto $X_\sigma$.

**Theorem 3** (Sections 4.1 and 4.2). Let $\sigma$ be a primitive substitution on a finite alphabet $A$ and a parageometric iwip automorphism of the free group $F_A$. Let $\tau$ be the tree substitution constructed in Theorem 1. Then, there exist cyclic orders on the initial trees $W$ and $\tau(W)$ which define a contour substitution $\chi$ and a dual contour substitution $\chi^*$ such that:

1. the substitutions $\chi$ and $\chi^*$ are primitive and the iterated images of the contour of $W$ under $\chi$ converge to the contour of the compact heart $\Omega_A$ of the repelling tree;
2. there exists a continuous map $Q_\sigma : X_{\chi^*} \to \mathbb{S}^1$ which pushes forward the action of the shift on $X_{\chi^*}$ to an interval exchange on the contour circle of $\Omega_A$ induced by the action of the elements of $A^{\pm 1}$ on the repelling tree $T_\sigma^{-1}$. The action of $\chi^*$ is pushed forward to a piecewise contracting homothety of ratio $\frac{1}{\lambda}.$

We provide an algorithm to compute $\chi$ and $\chi^*$.

We also refer to the work of Sirvent [Sir00, Sir03] on self-similar interval exchange of the circle associated to some particular substitutions. The trees we construct are dual to the geodesic laminations obtained in his works.

It is conjectured that each irreducible Pisot substitutive subshift has pure discrete spectrum: this is known as the Pisot conjecture (see [ABB+15] for a survey). Pure discreteness of the spectrum can be proved geometrically by showing that the Rauzy fractal associated with an irreducible Pisot substitution tiles periodically the contracting space. In this case, the substitutive subshift is measurably conjugate to a translation on a torus. We expect that our constructions of the tree substitution and of the contour interval exchange could shed new light and open new techniques to attack the Pisot conjecture, at least when the substitution is a parageometric iwip automorphism.

### 1.2. Techniques and self-similarity.

All the objects we consider, that is, attracting shifts, repelling trees and their compact hearts, Rauzy fractals, tree substitutions, contour interval exchanges of the circle, are governed by self-similarity. This self-similarity is best described using the prefix-suffix automaton of the substitution. Mossé [Mos96] proved that any bi-infinite word in the attracting subshift of a primitive substitution can be uniquely desubstituted: this allows to define a map $\Gamma : X_\sigma \to \mathcal{P}$ from the attracting shift to the set $\mathcal{P}$ of infinite desubstitution paths. For a finite path $\gamma$ in the prefix-suffix automaton we consider the cylinder $[\gamma]$ of the bi-infinite words in $X_\sigma$ with desubstitution paths ending by $\gamma$. Then $X_\sigma$ decomposes as the disjoint union of cylinders $[\gamma]$, for all paths $\gamma$ of length $n$ in the prefix-suffix automaton. This self-similarity is passed through $Q$ to the repelling tree and through $\varphi$ to the Rauzy fractal:

$$\Omega_A = \bigcup_{\gamma} p(\gamma)^{-1} H^n(\Omega_\sigma), \quad R_\sigma = \bigcup_{\gamma} M^n_\sigma R_\sigma(a) + \pi_c(\ell(p(\gamma)))$$

where $a$ is the starting letter of $\gamma$, $p(\gamma)$ is its prefix, $\ell$ is the abelianization map, $\pi_c$ the projection to the contracting space and $H$ is the contracting homothety. If $\sigma$ is
a parageometric iwip automorphism the tiles $\Omega_\gamma$ are compact trees with at most one point in common, while with the irreducible Pisot hypothesis the tiles $R_\gamma$ are disjoint in measure, if the strong coincidence condition holds.

The tree substitution of Theorem 1 reflects the self-similarity in the repelling tree. Indeed, iterates of the substitution $\tau$ on an initial patch $W$ are made of tiles $W_\gamma$, each of them isomorphic to one of the finitely many prototiles $(W_a)_{a \in A}$:

$$\tau^n(W) = \bigcup_\gamma W_\gamma.$$  

The key of our algorithm to construct the tree substitution is to describe how to glue together the abstract trees $W_\gamma$ to mimic the self-similarity of $\Omega_A$. These gluing instructions, or adjacency relations, between the tiles $W_\gamma$, are governed by pairs of singular bi-infinite words which are exactly the pairs with the same $Q$-image in the repelling tree. From the works of Queffelec [Que87] and, Holton and Zamboni [HZ01] there are finitely many singular words and their desubstitution paths are eventually periodic.

The Rauzy fractal is also the limit of projected and renormalized iterates of the dual substitution $E^*_1(\sigma)$ acting on codimension one faces of the unit hypercube. We denote such a face based at $x$ and orthogonal to the basis vector $e_i$ by $(x,i)^*$. The dual substitution is defined by

$$E^*_1(\sigma)^n \left( \bigcup_{i \in A} (0,i)^* \right) = \bigcup_\gamma (M_{\sigma^{-n}}(p(\gamma)), a)^*.$$  

This is once again the same self-similarity and, we also use the dual substitution to draw our tree substitution in the contracting space.

Regarding the construction of the contour map and the piecewise exchange on the circle, we consider infinite paths in the prefix-suffix automaton of $\chi$ and $\chi^*$. In this way, the shift map on the attracting shift $X_{\chi^*}$ is translated through $\Gamma$ to what is called the Vershik map. Pushing forward by $Q_{S1}$ the Vershik map gives the piecewise rotation on the circle.

We implemented our algorithms in Sage [Sage].

1.3. Perspectives. At the end of this work several questions remain open.

Our construction only works for parageometric iwip automorphisms but we expect that (with some more technicalities) the repelling tree of any iwip substitution can be described by a tree substitution. Recall that from the botany obtained with Hilion [CH12] the attracting tree of a non-parageometric iwip is of Levitt type: the limit set $\Omega_A$ and the cylinders $\Omega_\gamma$ are no longer trees but rather Cantor sets inside the repelling tree. Thus, the adjacency relation between tiles is more complicated to describe.

Our work raises the question of which substitutions can be covered by an interval exchange transformation. This is the purpose of the (dual) contour substitution and we ask if for any substitution there exists a bigger alphabet $\tilde{A}$, a forgetful map $f: \tilde{A} \to A$ and a substitution $\tilde{\sigma}$ such that $f \circ \tilde{\sigma} = \sigma \circ f$ and such that $\tilde{\sigma}$ codes an interval exchange transformation.

The trees we get are also given by adjacency relations of tiles inside the repelling tree. Through the map $Q$ these adjacency relations are preserved inside the Rauzy fractal. But is it true that they are satisfied for the dual substitution or some variation of it?

Our original goal was to draw trees in the contracting space. Is it true that, for any irreducible Pisot substitution which is an iwip parageometric automorphism, there exists a tree substitution which can be realized injectively inside the contracting space (that is, are the iterated images $\tau^n(W)$ visualized as trees therein)?
The map $Q$ implies that the Rauzy fractal of a parageometric Pisot substitution is arcwise connected. We ask if furthermore it is always disk-like.

Finally, we have in mind the Pisot conjecture. Do our constructions give informations on the spectrum of the original substitution?

2. Preliminaries and notations

2.1. Substitutions, attracting shift and prefix-suffix automaton. Let $A$ be a finite alphabet, $A^*$ be the free monoid on $A$, where the operation is the concatenation of words. The full bi-infinite shift is the space $A^\mathbb{Z}$ of bi-infinite words. For $w \in A^\mathbb{Z}$, we denote by $w_i$ the letter at position $i$ and by $w_{[i,j]}$ the factor $w_i \cdots w_j$ between positions $i$ and $j$. We put $w_0$ right after a dot. For example

$$w = \cdots aaaa \cdots bbb \cdots, \quad w_{[-2,3]} = aabbb$$

It comes equipped with the shift operator $S: (w_n)_{n \in \mathbb{Z}} \mapsto (w_{n+1})_{n \in \mathbb{Z}}$ and with a topology given by clopen cylinders: for a finite word $u$, and for $i \in \mathbb{Z}$ we define the cylinder $[u]_i$ as the subset of bi-infinite words which read $u$ at position $i$:

$$[u]_i = \{ w \in A^\mathbb{Z} : w_i \cdots w_{i+|u|-1} = u \}$$

where $|u|$ denotes the length of the word $u$. We extend this notation by letting

$$[w] = [w]_0 \quad \text{and} \quad [u \cdot v] = \{ w \in A^\mathbb{Z} : w_{-|u|} \cdots w_{-1} w_0 \cdots w_{|v|-1} = uv \}.$$

Let $\sigma$ be a substitution on $A$, that is, a free monoid endomorphism $\sigma: A^* \to A^*$ which is completely determined by its restriction $\sigma: A \to A^*$.

Example 2.1. Here is a list of substitutions which we will use as examples:

- $a \mapsto ab, \quad a \mapsto ac, \quad a \mapsto abc$
- $b \mapsto ac \quad b \mapsto ab \quad b \mapsto bcabc$
- $c \mapsto a \quad c \mapsto b \quad c \mapsto cbabc$

The leftmost substitution is known as the Tribonacci substitution.

The language of $\sigma$ is defined by

$$\mathcal{L}_\sigma = \{ w \in A^* : w \text{ is a factor of } \sigma^n(i) \text{ for some } i \in A, n \in \mathbb{N} \}.$$}

The attracting shift $X_\sigma$ of a substitution $\sigma$ is the set of bi-infinite words in $A^\mathbb{Z}$ whose factors are in $\mathcal{L}_\sigma$. This is a closed (indeed compact) shift-invariant subset of $A^\mathbb{Z}$. We can define the action of $\sigma$ on $A^\mathbb{Z}$ by letting

$$\sigma(\cdots a_{-2} a_{-1} a_0 a_1 a_2 \cdots) = \cdots \sigma(a_{-2}) \sigma(a_{-1}) \sigma(a_0) \sigma(a_1) \sigma(a_2) \cdots$$

The attracting shift is invariant by the action of $\sigma$.

The prefix-suffix automaton of $\sigma$ has the alphabet $A$ as set of states and a transition $a \xleftarrow{p} b$ whenever $\sigma(b) = pas$ for $(p,a,s) \in A^* \times A \times A^*$. We let $\mathcal{P}$ be the set of infinite paths $\gamma$ in the prefix-suffix automaton.

![Figure 2](image-url)  

**Figure 2.** The prefix-suffix automaton of the first two substitutions of Example 2.1.

Mossé [Mos96] proved that every primitive substitution satisfies a property of bilateral recognizability, i.e. every bi-infinite word $w \in X_\sigma$ can be “desubstituted"
uniquely: there exist a unique bi-infinite word $w'$ and a unique integer $k$ such that $w = S^k(\sigma(w'))$. We denote by $\theta : X_\sigma \to X_\sigma$, $w \mapsto w'$ this map. We remark that $k = |p|$ where $w_0 \overset{p}{\mapsto} w_0'$ is an edge of the prefix-suffix automaton.

The desubstitution map $\Gamma : X_\sigma \to \mathcal{P}$ associates to $w \in X_\sigma$ the infinite path $\gamma \in \mathcal{P}$ defined by applying recursively $\theta$ to $w$:
\[
\gamma = a_0 \overset{p_0}{\mapsto} a_1 \overset{p_1}{\mapsto} a_2 \overset{p_2}{\mapsto} \cdots \quad \text{(with } a_0 = w_0)\).
\]
We call $\gamma$ the infinite desubstitution path, or the prefix-suffix expansion, of $w$.

**Proposition 2.2** (CS01, HZ01). The map $\Gamma$ is continuous, onto and one-to-one except on the orbit of periodic points of $\sigma$. Furthermore $\Gamma(X_\sigma^\text{per}) = \mathcal{P}_\text{min}$ and $\Gamma(S^{-1}X_\sigma^\text{per}) = \mathcal{P}_\text{max}$, where $\mathcal{P}_\text{min}$ and $\mathcal{P}_\text{max}$ are the infinite desubstitution paths with empty prefixes and empty suffixes respectively.

The last proposition implies that, if $w \in X_\sigma$ is such that its prefix-suffix expansion $\gamma$ does not have $p_i = \epsilon$ or $s_i = \epsilon$ for all $i \geq i_0$ for some $i_0 \in \mathbb{N}$, then we can express $w$ as
\[
w = \lim_{n \to \infty} \sigma^n(p_n)\sigma^{n-1}(p_{n-1})\cdots\sigma(p_1)p_0 \cdot w_0s_0\sigma(s_1)\cdots\sigma^{n-1}(s_{n-1})\sigma^n(s_n).
\]

**Example 2.3.** Let $\sigma$ be the Tribonacci substitution. We consider the three periodic points
\[
w_a = \lim_{n \to \infty} \sigma^{3n}(a) \cdot \sigma^n(a) = \cdots abacababacaba \cdot abacabaabacabab \cdots
\]
\[
w_b = \lim_{n \to \infty} \sigma^{3n}(b) \cdot \sigma^n(a) = \cdots abacabaabacab \cdot abacabaabacabab \cdots
\]
\[
w_c = \lim_{n \to \infty} \sigma^{3n}(c) \cdot \sigma^n(a) = \cdots abacabaabac \cdot abacabaabacabab \cdots
\]
The infinite desubstitution path of $w_a$, $w_b$ and $w_c$ is $\gamma_0$ which we describe as the unique periodic path of the graph
\[
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\]
while those of $S^{-1}(w_a)$, $S^{-1}(w_b)$ and $S^{-1}(w_c)$ are $\gamma_0$, $\gamma_b$ and $\gamma_c$ which are the unique infinite paths of the graph
\[
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array}
\]
ending respectively at $a$, $b$ or $c$.

For a finite path $\gamma$ in the prefix-suffix automaton
\[
\gamma = a_0 \overset{p_0}{\mapsto} a_1 \overset{p_1}{\mapsto} a_2 \overset{p_2}{\mapsto} \cdots \overset{p_{n-1}}{\mapsto} a_n
\]
its length is denoted by $|\gamma|$ and equals the number $n$ of edges. We say that $a_0$ is the end and $a_n$ is the beginning of the path.

The cylinder $[\gamma]$ is the set of bi-infinite words $w \in X_\sigma$ with infinite desubstitution path ending with $\gamma$. Remark that all words $w$ in $[\gamma]$ have indexed factor:
\[
\sigma^{n-1}(p_{n-1})\cdots\sigma(p_1)p_0 \cdot a_0s_0\sigma(s_1)\cdots\sigma^{n-1}(s_{n-1}).
\]
It is convenient to introduce the notation
\[
p(\gamma) := \sigma^{n-1}(p_{n-1})\cdots\sigma(p_1)p_0
\]
to denote what we call the \textit{prefix of the path} \( \gamma \). Given two finite paths
\[
\gamma = a_n p_{m_n-1} \ldots p_{m_1} a_{m_1}, \quad \gamma' = a_n p_{m_n-1} \ldots p_{m_1} a_{m_1},
\]
we denote their concatenation by \( \gamma \gamma' \) and we say that \( \gamma' \) \textit{extends} \( \gamma \). \( \gamma \) is the \textit{head} and \( \gamma' \) the \textit{tail}. In Section 3.2 we will use the notation \( B(\gamma) \) to \textit{behead} the prefix-suffix expansion \( \gamma \) of its heading edge, thus with the above notation:
\[
B^n(\gamma \gamma') = \gamma'.
\]

**Proposition 2.4.** For any path \( \gamma \gamma' \) in the prefix-suffix automaton with \( n = |\gamma| \) we have
\[
[\gamma \gamma'] = S^{[\nu]1} \sigma S^{[\nu]1} \sigma \cdots S^{[\nu]1} \sigma (\gamma' [\gamma]).
\]
Furthermore we have the following prefix-suffix decomposition of cylinders: for any path \( \gamma \) in the prefix-suffix automaton and every \( n \in \mathbb{N}^* \)
\[
[\gamma] = \bigsqcup_{\gamma':|\gamma'|=n} [\gamma'], \quad \text{and in particular} \quad X_\sigma = \bigsqcup_{\gamma':|\gamma'|=n} [\gamma'],
\]
where the disjoint union is taken over all paths \( \gamma' \) in the prefix-suffix automaton extending \( \gamma \) (or ending at any state in case \( \gamma \) is the empty path) of length \( n \).

2.2. Singular Words. Two bi-infinite words \( Z \) and \( Z' \) in the attracting shift \( X_\sigma \) share a half if either \( Z_{[0;+\infty)} = Z'_{[0;+\infty)} \) or \( Z_{(-\infty;-1]} = Z'_{(-\infty;-1]} \). We say that \( Z \) and \( Z' \) are singular if either they share their right half and \( Z_{-1} \neq Z'_{-1} \) or they share their left half and \( Z_0 \neq Z'_0 \).

In the dialect of word combinatorics, maps of cylinders is an equivalence classes of \( Z, Z' \) such that for \( \ell \) big enough \( S^\ell Z \) and \( S^\ell Z' \) share their right half.

In the case of the attracting shift of a primitive substitution Queffélec proved that there are only finitely many singular words, see also [BH93].

**Proposition 2.5** ([Que87]). Let \( \sigma \) be a primitive substitution, then there are finitely many singular \( \mathcal{B} \)-infinite words in the attracting shift \( X_\sigma \).

**Example 2.6.** For the Tribonacci substitution \( \sigma \) the three infinite words \( w_a, w_b \) and \( w_c \) of Example 2.3 are the only infinite words in the attracting shift with a left-special right half (and they share this right half). The three infinite words
\[
w_a' = \lim_{n \to \infty} \bar{\sigma}^{3n}(a) \cdot \bar{\sigma}^{3n}(a) = \cdots bacabaabacba \cdot abacababacaba \cdots
\]
\[
w_b' = \lim_{n \to \infty} \bar{\sigma}^{3n}(a) \cdot \bar{\sigma}^{3n}(b) = \cdots bacabaabacba \cdot bacaababaabacaca \cdots
\]
\[
w_c' = \lim_{n \to \infty} \bar{\sigma}^{3n}(a) \cdot \bar{\sigma}^{3n}(c) = \cdots bacaababaabacaca \cdot \cdots
\]
where \( \bar{\sigma} : a \mapsto ba, b \mapsto ca, c \mapsto a \) is the flipped Tribonacci substitution, are the only infinite words in \( X_\sigma \) with a right-special left half. They come from the periodic Nielsen path (of period 3)

\[
\begin{array}{c}
\text{ab} \\
\text{ba} \\
\text{ca}
\end{array}
\]

Indeed, applying \( \sigma \) three times to these three words we get that they coincide on the prefix \( abacaba \) and after that each of them starts again with \( ab, ba \) and \( ca \) respectively. This process is responsible for the fact that \( w_a', w_b' \) and \( w_c' \) are generated by \( i_{abacaba} \circ \sigma^3 = \bar{\sigma}^3 \), where \( i_w(u) = w^{-1}uw \) denotes the inner automorphism.
Using the work of Holton and Zamboni [HZ01] we get:

**Proposition 2.7** ([HZ01] Theorem 5.1). Let $Z$ be a singular bi-infinite word in the attracting shift $X_\sigma$ of a primitive substitution. Then, the infinite de-substitution path $\gamma$ of $Z$ is eventually periodic.

**Example 2.8.** For the Tribonacci substitution, we know from Example 2.3 the common infinite desubstitution path of $w_a$, $w_b$ and $w_c$:

$$\gamma_0 = a \xleftarrow{e, b} a \xleftarrow{e, b} a \xleftarrow{e, b} a \cdots .$$

The infinite desubstitution paths of $w_a'$, $w_b'$ and $w_c'$ from Example 2.6 are

$$\gamma_a = a \xleftarrow{e, b} (a \xleftarrow{e, c} a \xleftarrow{e, c} b \xleftarrow{a, c} b) (a \xleftarrow{e, c} b \xleftarrow{a, c} b) \cdots ,$$

$$\gamma_b = b \xleftarrow{a, c} a \xleftarrow{e, b} (a \xleftarrow{e, c} a \xleftarrow{e, c} b \xleftarrow{a, c} b) (a \xleftarrow{e, c} b \xleftarrow{a, c} b) \cdots$$

and,

$$\gamma_c = c \xleftarrow{a, c} b \xleftarrow{a, c} a \xleftarrow{e, b} (a \xleftarrow{e, c} a \xleftarrow{e, c} b \xleftarrow{a, c} b) (a \xleftarrow{e, c} b \xleftarrow{a, c} b) \cdots .$$

### 2.3. Repelling trees.

The monoid $A^*$ is embedded inside the free group $F_A$ of reduced words in $A^{\pm 1}$. The (Gromov) boundary $\partial F_A$ of $F_A$ consists of infinite reduced words in $A^{\pm 1}$. The free group $F_A$ acts continuously by left multiplication on its boundary.

A bi-infinite word $Z$ in $A^\mathbb{Z}$ can be identified with the pair $(X, Y) \in \partial^2 F_A = (\partial F_A \times \partial F_A) \setminus \Delta$, where $\Delta$ denotes the diagonal, such that $Z = X^{-1} \cdot Y$. Then the action of $F_A$ on $\partial F_A$ induces a partial action on $A^\mathbb{Z}$. In particular, the shift map $S$ is defined on each point $(X, Y) \in \partial^2 F_A$ by

$$S(X, Y) = (Y_0^{-1}X, Y_0^{-1}Y).$$

In all this paper we assume that the substitution $\sigma$ is **iwip (irreducible with irreducible powers)**, that is, it induces an automorphism whose powers fix no proper free factor of $F_A$. An important object to study the dynamics of $\sigma$ is the **attracting tree** $T_\sigma$. The tree $T_\sigma$ is a $\mathbb{R}$-tree (geodesic and 0-hyperbolic metric space) with a minimal (there is no proper $F_A$-invariant subtree), very-small (see [CL95]) action of $F_A$ by isometries.

For completeness we recall a concrete construction of $T_\sigma$ [GJLL98].

First, any isometry of a real tree without fixed points acts as a translation along an axis. This defines the translation length of an isometry. For instance, the element $u \in F_A$ acts on the Cayley tree $T_A$ of the free group $F_A$ by a translation of length $\|u\|_A$ where $\|u\|_A$ stands for the length of the cyclically reduced part of $u$. The length function (of a real tree $T$ with an action of the free group by isometries) maps any element $u \in F_A$ to the translation length $\|u\|_T$ of the action of $u$ on $T$. A minimal action is completely determined by its translation length function [Lyn63, Chi76, Chi01].

Second, any automorphism $\sigma$ of $F_A$ has an expansion factor $\lambda_\sigma$ which is its maximal exponential growth rate:

$$\lambda_\sigma = \max_{u \in F_A} \lim_{n \to \infty} \sqrt[n]{\|\sigma^n(u)\|_A} .$$

Alternatively, and more concretely, the expansion factor of an iwip automorphism is the dominant eigenvalue of the matrix of a train-track representative for $\sigma$. We will not deal with the train-track machinery in this paper and we let the reader learn from Bestvina and Handel work [BH92].

Finally, the translation length function for the attracting tree $T_\sigma$ is given by

$$\|u\|_{T_\sigma} = \lim_{n \to \infty} \frac{\|\sigma^n(u)\|_A}{\lambda_\sigma^n} .$$
Note that $\lambda_\sigma$ is the only real number such that this length function $\| \cdot \|_{T_\sigma}$ is finite for any $u \in F_A$ and non-zero for at least one $u$.

However, passing from the translation length function to the tree $T_\sigma$ is not straightforward. A shortcut for this construction is to consider the Cayley tree $T_A$ as a metric space by realizing the edges as isometric copies of the real unit segment $[0,1]$ and by extending equivariantly and continuously the automorphism $\sigma$ to edges. We will call this extension the topological realization of $\sigma$. We can then define

$$d_\infty(x, y) = \lim_{n \to \infty} \frac{d(\sigma^n(x), \sigma^n(y))}{\lambda_\sigma^n}$$

between two points $x, y$ in $T_A$ (all points, not only vertices). The function $d_\infty$ is a pseudo-distance on $T_A$ and the attracting tree is obtained by identifying points at distance 0. Once again, this construction might be better understood and more concretely handled by starting with a train-track representative rather than the Cayley tree.

From this last construction we observe that the topological realization of $\sigma$ is a homothety of ratio $\lambda_\sigma$ for the (pseudo-)distance $d_\infty$:

$$d_\infty(\sigma(x), \sigma(y)) = \lambda_\sigma d_\infty(x, y).$$

Unfortunately for the reader, in this paper we are interested in the repelling tree rather than the attracting tree. Indeed we deal with the duality between the attracting shift and the repelling tree. We sum up the above discussion (and replace the attracting tree by the repelling one).

**Proposition 2.9 ([Gill98]).** Every iwip automorphism $\sigma$ has a repelling tree $T_{\sigma^{-1}}$ which is a real tree with a minimal action of $F_A$ by isometries and a contracting homothety $H$ of ratio $\frac{1}{\lambda_\sigma^{-1}}$ such that for all point $P$ in $T_{\sigma^{-1}}$ and all element $u$ in $F_A$:

$$H(uP) = \sigma(u)H(P).$$

Indeed the duality between the attracting shift and the repelling tree is emphasized by the existence of the map $Q$. Let $T_{\sigma^{-1}}$ denote the metric completion of the tree $T_{\sigma^{-1}}$.

**Proposition 2.10.** There exists a unique continuous map $Q : X_\sigma \to T_{\sigma^{-1}}$ such that, for any bi-infinite word $Z$ in the attracting shift $X_\sigma$ with first letter $Z_0$,

$$Q(S(Z)) = Z_0^{-1}Q(Z).$$

Moreover, using the contracting homothety $H$ on $T_{\sigma^{-1}}$, for any bi-infinite word $Z \in X_\sigma$:

$$Q(\sigma(Z)) = H(Q(Z)).$$

The image $Q(X_\sigma)$ is a compact subset of $T_{\sigma^{-1}}$ called the **compact limit set** $\Omega_A$ of $T_{\sigma^{-1}}$ ([Coulbois et al. 2009]).

We remark that, as a topological space, the limit set $\Omega_A$ can also be obtained as a quotient of the attracting shift. Indeed, let $\sim$ be the equivalence relation on $X_\sigma$ which is the transitive closure of the “share a half” relation (defined in Section 2.2), then

$$\Omega_A = X_\sigma / \sim.$$ 

The above equality is a consequence of the following statement.

**Proposition 2.11 ([CHR13 Corollary 1.3],[KLI2014 Theorem 2]).** Let $\sigma$ be a substitution which induces an iwip automorphism. Let $Z, Z'$ be two bi-infinite words in the attracting shift $X_\sigma$. If $Z$ and $Z'$ share a half then $Q(Z) = Q(Z')$. Conversely, if $Q(Z) = Q(Z')$, then there exists a finite sequence $Z = Z_0, Z_1, \ldots, Z_{n-1}, Z_n = Z'$
of bi-infinite words in the attracting shift \(X_\sigma\) such that for each \(i\), \(Z_i\) and \(Z_{i+1}\) share a half.

We also consider the convex hull \(K_A\) of \(\Omega_A\), which is called the **compact heart** of \(T_{\sigma^{-1}}\). In this paper we focus on the situation where the limit set is convex: \(\Omega_A = K_A\). An iwip automorphism in such a situation is called **parageometric**.

### 2.4. Index.
For the subshift \(X_\sigma\) we considered the transitive closure of the “share a half” equivalence relation \(\sim\). For each equivalence class \([Z]\) we define the **index** as the number of possible letters around the origin minus two:

\[
\ind([Z]) = |\{(Z'_{-1} \mid Z' \in [Z])\}| + |\{(Z'_0 \mid Z' \in [Z])\}| - 2.
\]

Using Proposition 2.5 this index is finite and there are finitely many equivalence classes with positive index. Thus we define

\[
\ind(X_\sigma) = \sum_{[Z]} \ind([Z]).
\]

**Proposition 2.12 (CHL98).** For any substitution \(\sigma\) which induces an iwip automorphism the index of the attracting subshift is bounded above by \(2N - 2\):

\[
\ind(X_\sigma) \leq 2N - 2.
\]

Moreover, this index is maximal if and only if \(\sigma\) is parageometric.

**Example 2.13.** For Tribonacci substitution there are two equivalence classes with positive index, both contain three bi-infinite words: \(\{w_a, w_b, w_c\}\) (see Example 2.3 and \(\{w'_a, w'_b, w'_c\}\) (see Example 2.6), thus the index is \(2 + 2 = 4 = 2 \times 3 - 2\): the Tribonacci automorphism is parageometric.

### 2.5. Cylinders in the repelling tree.
For a finite word \(w \in A^*\) we push forward the cylinder \([w]\) to get a compact subset \(\Omega_w = Q([w])\) of the compact limit set. If \(\gamma\) is a finite path in the prefix-suffix automaton we similarly push forward \([\gamma]\) to get \(\Omega_\gamma = Q([\gamma])\).

Pushing forward Proposition 2.4 we get

**Proposition 2.14.** Let \(\sigma\) be a substitution which induces an iwip automorphism. For any finite path \(\gamma\gamma'\) in the prefix-suffix automaton we have

\[
\Omega_{\gamma\gamma'} = p_0^{-1}\sigma(p_1^{-1})\cdots\sigma^{n-1}(p_{n-1}^{-1})H^n(\Omega_{\gamma'}) = p(\gamma)^{-1}H^n(\Omega_{\gamma'})
\]

where

\[
\gamma = a_0 p_0, a_1, p_1, a_2, \cdots, p_{n-1}, a_n.
\]

Moreover, for any path \(\gamma\) in the prefix-suffix automaton and any \(n \in N^*\)

\[
\Omega_\gamma = \bigcup_{\gamma', |\gamma'|=n} \Omega_{\gamma\gamma'} \quad \text{in particular for } \gamma = \epsilon, \quad \Omega_A = \bigcup_{\gamma' \mid |\gamma'|=n} \Omega_{\gamma'}
\]

where the union is taken over all paths \(\gamma'\) in the prefix-suffix automaton extending \(\gamma\) (in case \(\gamma\) is the empty path, ending at any state) of length \(n\). Furthermore, if \(\sigma\) is parageometric then each \(\Omega_\gamma\) is connected.

**Proof.** The result follows easily applying the map \(Q\) to both sides of the statements of Proposition 2.4 (recalling that \(Q(S(Z)) = Z_0^{-1}Q(Z)\) and \(Q(\sigma(Z)) = H(Q(Z))\). For a proof of the connectedness of \(\Omega_\gamma\), we refer to [CHL09].

The formula in the previous Proposition can be used to define (or at least better understand) the map \(Q\). For a bi-infinite word \(Z \in X_\sigma\), with prefix-suffix development \(\gamma = a_0 p_0, a_1, p_1, \cdots\), for any point \(P \in T_{\sigma^{-1}}\):

\[
Q(Z) = \lim_{n \to \infty} p_0^{-1}H(p_1^{-1}H(\cdots(H(p_n^{-1}P))\cdots)),
\]
Then we have
\[ \pi (\text{except itself}). \]
Let
\[ \vec{u} \]
where the infinite sum converges since
\[ M \]
Then we have the
\[ \text{Theorem 2.9} \]
we obtain
\[ P \]
\[ \square \]
For an infinite desubstitution path \( \gamma = \gamma' \gamma'' \) (with \( \gamma' \) a finite desubstitution path),
\[ Q(\gamma' \gamma'') = p(\gamma')^{-1} H^{\gamma'}(Q(\gamma'')). \]
In particular for words with eventually periodic prefix-suffix expansions we get the following statement.

**Proposition 2.15.** Let \( Z \in X_\sigma \) with eventually periodic prefix-suffix expansion \( \gamma = \alpha \beta^\infty \), where
\[ \alpha = a_0 \frac{p_0}{s_0} a_1 \frac{p_1}{s_1} \ldots \frac{p_{m-1}}{s_{m-1}} a_m \quad \text{and}, \]
\[ \beta = a_m \frac{p_m}{s_m} a_{m+1} \frac{p_{m+1}}{s_{m+1}} \ldots \frac{p_{m+n}}{s_{m+n}} a_m. \]
Then we have
\[ P = Q(Z) = Q(\gamma) = p(\alpha)^{-1} p(\beta)^{-1} \sigma^m(p(\alpha)) H^n(P). \]

**Proof.** By Proposition 2.14 we have
\[ \gamma \]
where \( P_\beta = Q(Z') \) and \( Z' \in X_\sigma \) has infinite desubstitution path \( \beta^\infty \). Now applying Theorem 2.6 we obtain
\[ P = p(\alpha)^{-1} \sigma^m(p(\beta)^{-1}) H^{n\alpha} \sigma^n(p(\alpha)) (P_\beta) \]
and by 2.1 we finally get the result. \( \square \)

2.6. **Rauzy fractals.** Let \( \ell : A^* \to \mathbb{Z}^A \), \( w \mapsto ([w]_a)_{a \in A} \) be the abelianization map and \( M_\sigma \) be the abelianization matrix of \( \sigma \) acting on \( \mathbb{Z}^A \) or \( \mathbb{R}^A \). Then \( \ell(\sigma(w)) = M_\sigma \ell(w) \) holds for any \( w \in A^* \).

We assume that the substitution \( \sigma \) is **irreducible Pisot**, that is, the characteristic polynomial of \( M_\sigma \) is the minimal polynomial of a Pisot number \( \lambda_\sigma \), i.e. a real number strictly bigger than 1 whose conjugates are strictly smaller than one in modulus. Then we have the \( M_\sigma \)-invariant decomposition
\[ \mathbb{R}^A = \mathbb{R} \vec{u} \oplus E_c \]
where \( \vec{u} \) is the eigenvector associated to \( \lambda_\sigma \) and, \( E_c \cong \mathbb{R}^{\sigma^{-1} - 1} \) is the contracting hyperplane spanned by the eigenvectors associated with the Galois conjugates of \( \lambda_\sigma \) (except itself). Let \( \pi_c \) be the projection of \( \mathbb{R}^A \) to \( E_c \) along \( \vec{u} \).

For \( Z \in X_\sigma \) with prefix-suffix decomposition \( \gamma = a_0 \frac{p_0}{s_0} a_1 \frac{p_1}{s_1} a_2 \ldots \) define
\[ \varphi : X_\sigma \to E_c, \quad \varphi(Z) = \sum_{i=0}^{\infty} \pi_c(M_\sigma^i \ell(p_i)), \]
where the infinite sum converges since \( M_\sigma \) is a contraction in \( E_c \). Remark that the map \( \varphi \) is continuous and only depends on the prefix-suffix expansion of \( Z \) and thus, abusing again of notations, we define \( \varphi(\gamma) = \varphi(\Gamma(Z)) = \varphi(Z) \) and we regard also the map \( \varphi : \mathcal{P} \to E_c \).

The Pisot property allows to represent the attracting shift \( X_\sigma \) geometrically as a compact domain with fractal boundary, called Rauzy fractal in honor of G. Rauzy who first defined it for the Tribonacci substitution [Rau82].

**Definition 2.16.** The **Rauzy fractal** \( \mathcal{R}_\sigma \) is the compact set \( \varphi(X_\sigma) \subset E_c \). Since \( X_\sigma \) is the union of the cylinders \( [a] \), for \( a \in A \), the Rauzy fractal is decomposed into subpieces \( \mathcal{R}_\sigma(a) = \varphi([a]) \).
We state now some properties of these fractals. For more details we refer to [BST10].

**Proposition 2.17.** Let $\sigma$ be a Pisot substitution. Then the following properties hold:

- $R_\sigma$ is the closure of its interior.
- $\partial R_\sigma(a)$ has measure zero, for each $a \in A$.
- The Rauzy fractal obeys to the set equation
  \[ R_\sigma(a) = \bigcup_{a^\infty \equiv b} M_\sigma R_\sigma(b) + \pi_c(\ell(p)), \text{ for } a \in A. \]
  Furthermore the union is measure disjoint.

The substitution $\sigma$ satisfies the **strong coincidence condition** if $\forall (a, b) \in A^2$ there exist $n \in \mathbb{N}$ and $i \in A$ such that $\sigma^n(a) = p_1i$ and $\sigma^n(b) = p_2i$ and the prefixes $p_1$ and $p_2$ share the same abelianization. In this case the subtiles $R_\sigma(a)$ are pairwise disjoint in measure and we can define the **domain exchange**

\[ E : R_\sigma \to R_\sigma, \quad z \mapsto z + \pi_c(\ell(a)), \text{ if } z \in R_\sigma(a). \]

From the definition of the map $\varphi$ we get the following properties [CS01b]:

\[ \varphi(S(Z)) = \varphi(Z) + \pi_c(\ell(Z_0)), \quad \varphi(\sigma(Z)) = M_\sigma \varphi(Z). \]

Thus the shift and the action of $\sigma$ correspond respectively to the domain exchange and to the contraction $M_\sigma$ on the Rauzy fractal.

We emphasize that if we have an ultimately periodic prefix-suffix decomposition $\gamma = \alpha\beta^\infty$ (see notations at the end of Section 2.3) then we get the following formula:

\[ \varphi(Z) = \pi_c(\ell(p(\alpha))) + \pi_c(M_\sigma(M_\sigma^{-1} Id - 1)^{-1}(p(\beta))). \]

**Figure 3.** The Rauzy fractal of the Tribonacci substitution.

Rauzy fractals are an important geometrical tool to understand the dynamics of the attracting shift $X_\sigma$. It is conjectured that every $(X_\sigma, S)$ generated by an irreducible Pisot substitution has pure discrete spectrum, or equivalently is metrically conjugate to a translation on a compact Abelian group. This famous problem is known as the **Pisot conjecture** (see [ABB+15] for a recent survey). One way to attack this conjecture consists in showing that the Rauzy fractal tiles periodically the contracting space $E_c$. Indeed, if this is the case, then the Rauzy fractal is a fundamental domain of the torus $T^{A-1}$ and the domain exchange turns into a toral translation. Words in the attracting shift $X_\sigma$ are then codings of this toral translation with respect to the partition \{ $R(i) : i \in A$ \}.
In this work we do not tackle this important problem but we believe that the
tree substitutions approach will give a new particular insight to it.

2.7. The global picture. The map $\varphi$ of the previous section factors through the
map $Q$ of Section 2.3 and we can state

**Proposition 2.18.** For an irreducible Pisot substitution $\sigma$ which is an iwip auto-
morphism of the free group $F_A$, let $X_\sigma$ be the attracting shift of $\sigma$, $T_{\sigma^{-1}}$ the repelling
tree and, $R_\sigma$ the Rauzy fractal, as defined in the previous sections. Then there exists
a continuous map $\psi$ such that the following diagram commutes

$$
\begin{array}{ccc}
A^Z \supseteq X_\sigma & \xrightarrow{\varphi} & P \\
\downarrow Q & & \downarrow Q \\
\Omega_A \subseteq K_A \subset T_{\sigma^{-1}} & \xrightarrow{\psi} & \mathcal{R}_\sigma \subset E_c
\end{array}
$$

Moreover the action of the shift on $X_\sigma$ is pushed forward to the action of a system
of partial isometries of $\Omega_A$ and to the domain exchange on $\mathcal{R}_\sigma$, while the action of
$\sigma$ on $X_\sigma$ is pushed forward to the contracting homothety $H$ of the attracting tree
and the matrix $M_\sigma$ on the contracting hyperplane: $\forall Z \in X_\sigma$,

$$
Q(SZ) = Z_0^{-1}Q(Z), \quad \varphi(SZ) = \varphi(Z) + \pi_c(Z_0),
$$

$$
Q(\sigma(Z)) = H(Q(Z)), \quad \varphi(\sigma(Z)) = M_\sigma \varphi(Z).
$$

The aim of this paper is to study $\psi$ and more specifically to draw approximations
of $\psi(\Omega_A)$ inside the Rauzy fractal.

2.8. Renormalization. Each in its own fashion, the attracting shift, the repelling
tree and the Rauzy fractal are self-similar. This is expressed in Propositions 2.4,
2.14 and 2.17. In the repelling tree and in the Rauzy fractal, the self-similarity is
here expressed after renormalization (i.e. after applying the contracting homothety
$H$ or the matrix $M_\sigma$ on the contracting space). In order to draw the Rauzy fractal
and the repelling tree using geometric substitutions (tree substitutions or dual
substitutions which we will define in the next sections), it is convenient not to
renormalize. Thus we introduce the following notations. For a finite path $\gamma$ of
length $n$ in the prefix-suffix automaton we let

- $\tilde{\Omega}_\gamma = H^{-n}(\Omega_\gamma)$ in the repelling tree $T_{\sigma^{-1}}$;
- $\mathcal{R}_\gamma = M_\sigma^{-n}(\mathcal{R}_\gamma)$ in the contracting hyperplane.

In this setting, tiles $\mathcal{R}_\gamma$ of the Rauzy fractal are translates one from another as
soon as the paths $\gamma$ begins with the same letter, and similarly for the tiles $\tilde{\Omega}_\gamma$ of
the repelling tree:

**Proposition 2.19.** Let $\gamma = a_0 \overset{p_0}{\cdots} \overset{p_{n-1} a_n}{\cdots} \overset{p_1}{\ell(p(\gamma))}$. Then

$$
\tilde{\Omega}_\gamma = \sigma^{-n}(p(\gamma))^{-1}\Omega_{a_n} \quad (\text{and } \Omega_{a_n} = \tilde{\Omega}_{a_n})
$$

$$
\mathcal{R}_\gamma = \mathcal{R}_{a_n} + M_\sigma^{-n}\pi_c(\ell(p(\gamma))) \quad (\text{and } \mathcal{R}_{a_n} = \mathcal{R}_{a_n} = \mathcal{R}(a_n)).
$$
2.9. **Dual substitution.** For each \((x, a) \in \mathbb{Z}^A \times A\) we let \((x, a)^*\) be the face of the hypercube based at \(x\) orthogonal to \(\ell(a)\); precisely
\[
(x, a)^* = \left\{ x + \sum_{i \neq a} t_i e_i : t_i \in [0, 1] \right\}.
\]
The **dual substitution** \(E_1^*(\sigma)\) is defined by
\[
E_1^*(\sigma)(x, a)^* = \bigcup_{a \xleftarrow{p,s} b} (M_{\sigma}^{-1}(x + \ell(p)), b)^*
\]
where the union is taken over all edges \(a \xleftarrow{p,s} b\) ending at \(a\) in the prefix-suffix automaton.

Iterating the dual substitution on the hyperface \((0, a)^*\) and renormalizing through the contracting action of \(M_\sigma\) we converge in the Hausdorff metric towards the \(a\)-th subtile of the Rauzy fractal:
\[
\lim_{n \to \infty} M_{\sigma}^n \pi_c(E_1^*(\sigma)^n(0, a)^*) = R_\sigma(a).
\]

**Example 2.20.** The dual Tribonacci substitution is defined by
\[
E_1^*(\sigma) : (x, 1)^* \mapsto (M_{\sigma}^{-1}x, 1)^* \cup (M_{\sigma}^{-1}x, 2)^* \cup (M_{\sigma}^{-1}x, 3)^*
\]
\[
(x, 2)^* \mapsto (M_{\sigma}^{-1}(x + e_1), 1)^*
\]
\[
(x, 3)^* \mapsto (M_{\sigma}^{-1}(x + e_1), 2)^*
\]

**Figure 4.** The first steps of the dual substitution for Tribonacci on \\{(0, 1)^*, (0, 2)^*, (0, 3)^*\}.

From the definition, the \(n\)-th iterate of the dual substitution yields the patch
\[
E_1^*(\sigma)^n \left( \bigcup_{a \in A} (0, a)^* \right) = \bigcup_{\gamma} M_{\sigma}^{-n} \pi_c(\ell(p(\gamma)) + (0, a_n))^*,
\]
where \(\gamma\) ranges over all paths of length \(n\) in the prefix-suffix automaton and, \(a_n\) is the starting letter of \(\gamma\).

We focus on the contracting space \(E_c\) and we allow more freedom by picking arbitrary prototiles \(P_a \subseteq E_c\). The \(n\)-th iterate of the dual substitution yields the patch
\[
P_n = \bigcup_{\gamma} M_{\sigma}^{-n} \pi_c(\ell(p(\gamma))) + P_{a_n},
\]
which decomposes as the union of the patches \(P_n(a)\) obtained by restricting the union to those paths \(\gamma\) of length \(n\) ending at the letter \(a\). For each such path \(\gamma\) in the prefix-suffix automaton, starting at the letter \(a_n\) we denote by \(P_\gamma\) the tile
\[
P_\gamma = M_{\sigma}^{-n} \pi_c(\ell(p(\gamma))) + P_{a_n}.
\]
If we start with the prototiles \( P_a = \pi_c((0,a)^*) \) we get the usual picture of the iterates of the dual substitution with the property that, for each \( n, \bigcup_{a \in A} P_n(a) \) tiles periodically the contracting space \( E_c \) under the action of the lattice

\[
\Lambda = \sum_{i,j \in A, i \neq j} M_a^n \pi_c(e_j - e_i) \mathbb{Z}.
\]

If we start with the prototiles \( P_a = \mathcal{R}(a) \) given by the Rauzy fractal, we get the usual self-similar decomposition of the Rauzy fractal. If \( \sigma \) is an iwip parageometric automorphism, using the map \( \varphi \) of Proposition 2.18 for two distinct paths \( \gamma \) and \( \gamma' \) of length \( n \) in the prefix-suffix automaton, we get that if in the repelling tree \( \Omega_\gamma \cap \Omega_{\gamma'} \neq \emptyset \) then \( P_\gamma \cap P_{\gamma'} \neq \emptyset \).

**Question 2.21.** Let \( \sigma \) be an irreducible Pisot substitution and an iwip automorphism. Let \( E_c \) be the contracting space and, recall the notations \( P_a, P_n \) and \( P_\gamma \) as above. Do there exist prototiles \( P_a \subseteq E_c \) with the following properties?

1. for each \( n \) and \( a \in A \), the patch \( P_n(a) \) is (a) connected, (b) simply connected, (c) disk-like or, (d) a tree;
2. for two distinct finite paths \( \gamma \) and \( \gamma' \) of length \( n \) in the prefix-suffix automaton, if in the repelling tree \( \Omega_\gamma \cap \Omega_{\gamma'} \neq \emptyset \) then \( P_\gamma \cap P_{\gamma'} \neq \emptyset \);
3. the patch \( P_0 \) tiles the plane under the action of the lattice \( \Lambda \).

From the above discussion, the prototiles \( P_a = \pi_c((0,a)^*) \) satisfies conditions (a)-(b)-(c) and (3). The polygonal prototiles given by \( P_a = \pi_c(E_1(\sigma)^k((0,a)^*)) \) are another choice which satisfies condition (3). Whereas fractal prototiles \( P_a = \mathcal{R}(a) \) satisfy conditions (1)(a) and (2).

3. **Tree substitutions**

3.1. **Singular points.** The next lemma is a consequence of Mossé’s recognizability results [Mos92, Mos96].

**Lemma 3.1.** Let \( \sigma \) be a primitive substitution which admits a non-periodic fixed point \( u \). There exists an integer \( L > 0 \) such that if \( w_{[-L,L]} = w_{[-L,L]}' \) for some \( w, w' \in X_\sigma \), then the prefix-suffix expansions of \( w \) and \( w' \) have the same final edge.

Combining Propositions 2.5 and 2.11 we get the following result.

**Proposition 3.2.** Let \( \sigma \) be a primitive substitution and an iwip automorphism. Then, there exist finitely many pairs \( (Z, Z') \) of distinct bi-infinite words in the attractive shift \( X_\sigma \) such that \( Q(Z) = Q(Z') \), and such that the prefix-suffix expansions of \( Z \) and \( Z' \) do not have a common final edge.

**Proof.** From Proposition 2.11 there exists a finite sequence \( Z = Z_0, Z_1, \ldots, Z_{n-1}, Z_n = Z' \) in \( X_\sigma \) such that \( Z_i \) and \( Z_{i+1} \) share a half. Assume that this sequence is the shortest one. We discuss two cases. If \( n > 1 \), then (up to a symmetric argument) \( Z \) and \( Z_1 \) share their left halves and \( Z_1 \) and \( Z_2 \) share their right halves. Thus there exist \( i, j \geq 0 \) such that the left half of \( S^i Z_1 \) is right-special and the right half of \( S^{-j} Z_1 \) is left-special. By Proposition 2.5, there are finitely many right-special and left-special words, each of them with finitely many extensions. We conclude that there are finitely many possible choices for \( Z_1, i \) and \( j \) and thus for \( Z \). By symmetry there are finitely many possible choices for \( Z' \).

We now study the second case, that is to say when \( n = 1 \). Again, up to a symmetric argument we assume that \( Z \) and \( Z' \) share their left halves. As above, there exists \( i \geq 0 \) such that the left half of \( S^i Z \) is right-special. Then the indexed finite words \( Z_{[-i,i]} \) and \( Z'_{[-i,i]} \) are equal. By Lemma 3.1 there are at most finitely many such words \( Z \) and \( Z' \) whose prefix-suffix expansions have a different final edge,
Proposition 2.4 and 2.14, we get the next Proposition.

A pair of infinite desubstitution paths $(\gamma, \gamma')$ is singular if $Q(\gamma) = Q(\gamma')$ and $\gamma$ and $\gamma'$ have different final edges. An infinite desubstitution path $\gamma$ is singular if it belongs to at least one singular pair. The above proposition states that there are finitely many singular pairs of infinite desubstitution paths. From the proof we get that for each singular infinite desubstitution path $\gamma = \Gamma(Z)$, $Z \in X_\sigma$ is a bi-infinite word in the shift orbit of a singular word. A direct consequence of Proposition 2.7 is the following.

**Proposition 3.3.** Let $\sigma$ be a primitive substitution that induces an iwip automorphism. Then, there exist finitely many singular infinite desubstitution paths. Each of them is eventually periodic.

From the self-similar decomposition of the attracting shift and of the limit set (Propositions 2.4 and 2.14), we get the next Proposition.

**Proposition 3.4.** Let $\sigma$ be a primitive substitution which induces an iwip parageometric automorphism. Let $\gamma$ and $\gamma'$ be two distinct finite paths of the same length in the prefix-suffix automaton such that the subtrees $\Omega_\gamma$ and $\Omega_{\gamma'}$ have the point $P$ in common. Then there exist bi-infinite singular words $Z \in [\gamma]$ and $Z' \in [\gamma']$ such that $Q(Z) = Q(Z') = P$. The infinite desubstitution paths of $Z$ and $Z'$ are eventually periodic and satisfy

$$\Gamma(Z) = \alpha\beta\delta, \Gamma(Z') = \alpha'\beta'\delta',$$

with $\alpha$ the common head of $\gamma$ and $\gamma'$: $\gamma = \alpha\beta, \gamma' = \alpha'\beta'$ and where $(\beta\delta, \beta'\delta')$ is a singular pair of infinite desubstitution paths.

Such an intersection point $P$ is called a singular point of $\Omega_\gamma$.

### 3.2. Not renormalized singular points.

In this Section we state a finiteness result for singular points. We need to compare singular points of different tiles $\Omega_\gamma$ for different finite paths $\gamma$ in the prefix-suffix automaton. This is the purpose of using not renormalized tiles and not renormalized singular points.

Recall from Section 2.8 that for a finite path $\gamma$ in the prefix-suffix automaton beginning at state $a_n \in A$:

$$\tilde{\Omega}_\gamma = \sigma^{-n}(p(\gamma)^{-1})\Omega_{a_n}.$$

Observe that for any finite path $\gamma$ in the prefix-suffix automaton beginning at state $a$, for any distinct path $\gamma'$ with the same length as $\gamma$ such that $\Omega_\gamma \cap \Omega_{\gamma'} = \{P\}$, using the notations of Proposition 3.4,

$$P = Q(\alpha\beta\delta) = Q(\alpha'\beta'\delta'),$$

and $(\beta\delta, \beta'\delta')$ is a singular pair of infinite desubstitution paths.

In the not renormalized form,

$$\tilde{\Omega}_\gamma \cap \tilde{\Omega}_{\gamma'} = \{H^{-n}(P)\},$$

and, using the above formula and Section 2.5

$$P' := \sigma^{-n}(p(\gamma))H^{-n}(P) = H^{-n}(p(\gamma)Q(\gamma\delta)) = Q(\delta) \in \Omega_a.$$

We call $P'$ a singular point in $\Omega_a$. For a letter $a \in A$, we consider the set of all such singular points $P'$:

$$\text{Sing}(\Omega_a) = \{P' \in \Omega_a \mid \exists \gamma, \gamma', |\gamma| = |\gamma'| = n, \gamma(n) = a, \Omega_\gamma \cap \Omega_{\gamma'} = \{P\}, P' = \sigma^{-n}(p(\gamma))H^{-n}(P)\}.$$

**Proposition 3.5.** For every $a \in A$, the set $\text{Sing}(\Omega_a)$ in the subtree $\Omega_a$ is finite.
Proof. From the above discussion and notations, given $\gamma, \gamma'$ as in Proposition 3.4 there exist finite paths $\alpha, \beta, \beta'$ in the prefix-suffix automaton and infinite desubstitution paths $\delta, \delta'$ such that

$$\gamma = \alpha \beta, \quad \gamma' = \alpha \beta', \quad P = Q(\alpha \beta \delta) = Q(\alpha \beta' \delta'), \quad P' = Q(\delta)$$

and, $(\beta \delta, \beta' \delta')$ is a singular pair. Note that if $\beta$ and $\beta'$ ends at the same letter, then $\alpha$ and thus $\beta$ and $\beta'$ must contain at least one edge.

Recall from Proposition 3.2 that there are finitely many such singular pairs and each singular infinite desubstitution path is eventually periodic. Thus there are at most finitely many tails $\delta$, which proves the finiteness of $\text{Sing}(\Omega_a)$.

The proof of Proposition 3.5 gives an explicit method to compute the set of singular points $\text{Sing}(\Omega_a)$ for any $a \in A$. This is the first step of the algorithm for constructing the tree substitution associated with $\sigma$:

**Algorithm 3.6.**

1. Compute the pairs of singular bi-infinite words in the attracting shift $X_{\sigma}$. This is a classical computation of special infinite words for substitution specialists or, alternatively, it amounts to compute periodic Nielsen paths which has been implemented [Cou15, CL15].

2. Compute the pairs of singular prefix-suffix expansions. From Proposition 3.3 this is a finite set and its computation is detailed in the proof of Proposition 3.2. For each pair of singular bi-infinite words $(Z, Z')$, for $i = 0, 1, \ldots$ and $i = -1, -2, \ldots$, compute the prefix-suffix expansions $\gamma_i$ and $\gamma'_i$ of the shifted pairs $(S^i Z, S^i Z')$. Then $(\gamma_0, \gamma'_0)$ is a pair of singular prefix-suffix expansions as long as they end with different edges in the prefix-suffix expansion. From Lemma 3.1 the index $i$ is bounded above by Mosse’s constant $L$.

3. For each singular prefix-suffix expansion (which is an eventually periodic path), compute the finitely many infinite paths $\delta$ such that $\gamma = \beta \delta$ as in Proposition 3.4 and in the proof of Proposition 3.5. The paths $\delta$ ending in $a$ are in one-to-one correspondence with the singular points in $\text{Sing}(\Omega_a)$.

**Example 3.7.** Let $\sigma$ be the Tribonacci substitution and recall the infinite singular words $w_a, w_b, w_c, w'_a, w'_b$ and $w'_c$ with their infinite desubstitution paths $\gamma_0, \gamma_a, \gamma_b, \gamma_c$ from Examples 2.3 and 2.8. Each of these infinite desubstitution paths is eventually periodic:

$$\gamma_0 = \alpha \cdot \alpha \cdot \alpha \cdot \ldots, \quad \gamma_a = (a \xleftarrow{e} b) \cdot \beta \cdot \beta \cdot \ldots, \quad \gamma_b = (b \xleftarrow{e} a \xleftarrow{c} a \xleftarrow{b} b) \cdot \beta \cdot \beta \cdot \ldots, \quad \gamma_c = (c \xleftarrow{e} b \xleftarrow{c} a \xleftarrow{b} c) \cdot \beta \cdot \beta \cdot \ldots,$$

where $a$ is $(a \xleftarrow{e} b)$ and $\beta$ is $(a \xleftarrow{e} a \xleftarrow{c} b \xleftarrow{c} c)$. In the repelling tree $T_{\sigma^{-1}}$, we consider the point

$$P = Q(w'_a) = Q(w'_b) = Q(w'_c).$$

From Propositions 3.2 and 3.5 we also have to consider the shifts $w''_a = S^{-1} w'_a, w''_b = S^{-1} w'_b, w''_c = S^{-1} w'_c$, with their infinite desubstitution paths $\gamma''_a, \gamma''_b$ and, $\gamma''_c$:

$$\gamma''_a = (a \xleftarrow{c} c \xleftarrow{a} b \xleftarrow{a} a \xleftarrow{b} b) \cdot \beta \cdot \beta \cdot \ldots, \quad \gamma''_b = (a \xleftarrow{e} a \xleftarrow{c} b) \cdot \beta \cdot \beta \cdot \ldots, \quad \gamma''_c = (a \xleftarrow{e} b) \cdot \beta \cdot \beta \cdot \ldots,$$

and their common image $aP = Q(w''_a) = Q(w''_b) = Q(w''_c)$ in the repelling tree.

From the proof of Proposition 3.5 the singular points are given by the finitely many infinite tails $\delta$ of $\gamma''_a, \gamma''_b$ and $\gamma''_c$ which are obtained by beheading up to 6 edges (recall from Section 2.1 the notation $B(\gamma)$ to behead a prefix-suffix expansion). The
singular prefix-suffix expansions \( \gamma_a = B(\gamma_b) = B^2(\gamma_c) = B^3(\gamma_a') = \ldots \) \( B(\gamma_a) \) and \( B^2(\gamma_a) \) give three singular points in \( \text{Sing}(\Omega_a) \), while \( \gamma_b = B(\gamma_c) = \ldots \) and \( B^2(\gamma_a) \) give the two singular points in \( \text{Sing}(\Omega_b) \); finally, \( \gamma_c \) gives the unique singular point in \( \text{Sing}(\Omega_c) \).

We use the formulas in Section 2.9 to identify \( F_A \)-orbits and we get:

\[
\text{Sing}(\Omega_a) = \{ P, b^{-1}P, c^{-1}P \}, \quad \text{Sing}(\Omega_b) = \{ P, a^{-1}P \}, \quad \text{Sing}(\Omega_c) = \{ P \}.
\]

3.3. Tree substitution inside \( T_{\sigma^{-1}} \). Using Proposition 3.3, we can define the tree substitution \( \sigma_T \).

Let \( Y_a \) be the subtree of \( \Omega_a \) spanned by the finite set of points \( \text{Sing}(\Omega_a) \) of Proposition 3.5. We consider the tree substitution \( \sigma_T \) which maps each of the finite tree \( Y_a \) to the tree

\[
\sigma_T(Y_a) = \bigcup_{\substack{a \in E^\infty \backslash b}} \sigma^{-1}(p^{-1})Y_b,
\]

where the union is taken over all edges \( a \xrightarrow{p,s} b \) in the prefix-suffix automaton (compare with the definition of dual substitution in Section 2.9). This union is to be understood as a union of subtrees in the repelling tree \( T_{\sigma^{-1}} \).

For any word \( u \in F_A \), the tree substitution extends to translates \( uY_a \) of the finite tree \( Y_a \) as

\[
\sigma_T(uY_a) = \bigcup_{\substack{a \in E^\infty \backslash b}} \sigma^{-1}(up^{-1})Y_b.
\]

Proposition 3.8. With the above hypotheses and notations, for any \( a \in A \) and any \( n > 0 \), the iterated image

\[
\sigma_T^n(Y_a) = \bigcup_{\substack{\gamma = \gamma_0 \cdots \gamma_{n-1} \in \text{Sing}(\Omega_a) \text{ for any } \gamma \in \text{Sing}(\Omega_a) \text{ for any } \gamma \in \text{Sing}(\Omega_a)}} \sigma^{-n}(p(\gamma)^{-1})Y_b
\]

is connected. Furthermore, after renormalization, \( H^n(\sigma_T^n(Y_a)) \) is a finite subtree of \( \Omega_a \) which converges towards \( \Omega_a \):

\[
\bigcup_{n \to +\infty} H^n(\sigma_T^n(Y_a)) = \Omega_a.
\]

Finally, \( Y_A = \bigcup_{a \in A} Y_a \) is a connected subtree of \( \Omega_A \) and, its iterated images \( \sigma_T^n(Y_A) \) are connected subtrees of \( \Omega_A \) which converge towards \( \Omega_A \).

Proof. First we show that the right hand side of the first equality is a connected subtree. Recall from Proposition 2.14 that each subtree \( \Omega_c \) is connected and decomposes as the union of the subtrees \( \Omega_{\gamma} \), with \( \gamma \) ranging over the paths of length \( n \) in the prefix-suffix automaton ending at \( a \). Two such subtrees \( \Omega_{\gamma} \) and \( \Omega_{\gamma'} \), with \( \gamma \) and \( \gamma' \) starting respectively at state \( b \) and \( b' \), have at most a point \( P \) in common, and this point corresponds to a singular point in \( \text{Sing}(\Omega_b) \) and \( \text{Sing}(\Omega_{b'}) \):

\[
p(\gamma)H^{-n}(P) \in \text{Sing}(\Omega_b) \quad \text{and} \quad p(\gamma')H^{-n}(P) \in \text{Sing}(\Omega_{b'}).\]

Thus the finite trees \( \sigma^{-n}(p(\gamma)^{-1})Y_b \) and \( \sigma^{-n}(p(\gamma')^{-1})Y_{b'} \) have the point \( H^{-n}(P) \) in common and the union

\[
\bigcup_{\substack{\gamma = \gamma_0 \cdots \gamma_{n-1} \in \text{Sing}(\Omega_a) \text{ for any } \gamma \in \text{Sing}(\Omega_a) \text{ for any } \gamma \in \text{Sing}(\Omega_a)}} \sigma^{-n}(p(\gamma)^{-1})Y_b
\]

is connected.

The convergence of the iterated images follows from the decomposition in Proposition 2.14. Indeed, each of the subtrees \( \Omega_{\gamma} \) contains a point of the corresponding \( H^n(\sigma^{-n}(p(\gamma)^{-1})Y_b) \) and the diameter of \( \Omega_{\gamma} \) is \( \left( \frac{1}{\gamma_{n-1}} \right)^n \) times the diameter of \( \Omega_b \).
which goes to 0. We let the reader extend the proof to the union $Y_A$ of the elementary trees $Y_a$.

The above approximation of $\Omega_A$ by finite subtrees uses the action of the free group on the repelling tree. This action is more difficult to handle than translations in the plane, that we used for the dual substitution in Section 2.3.

It is much easier to describe how to glue together subtrees at a common vertex instead of considering the action of the free group. And fortunately we know that these gluings occur only at singular points. In the next Section we will leave the fact that the trees $Y_a$ are subtrees of the repelling tree and construct an abstract tree substitution.

3.4. Combinatorial Tree Substitutions.

3.4.1. Definition. Let us now use the previous Section to define a purely combinatorial tree substitution. Our aim is to describe how to approximate the limit set $\Omega_A$ from finite data. We will use the notion of tree substitutions developed by Jullian [Jul11] with minor adaptations.

A tree substitution starts with a finite collection of finite simplicial trees $(W_a)_{a \in A}$, each of which has a finite set of special vertices $V_a$ used to glue together other trees. Note that neither vertices in $V_a$ need to be leaves (i.e. extremal points) nor all leaves need to be in $V_a$. All trees we consider are constructed as finite disjoint unions of copies of the trees $W_a$ by identifying some vertices:

\[ W = \bigsqcup_{i=1}^n W_{b_i} / \sim, \]

where $b_i \in A$ and $\sim$ is an equivalence relation which identifies some points of $V_a$ with some points of $V_b$, for some $i, j \in \{1, \ldots, n\}$.

The tree substitution $\tau$ replaces the tree $W_a$ by a tree $\tau(W_a)$ which is a union of copies of the $W_b$ with some vertices identified as in (3.2). The map $\tau$ also sends each gluing vertex in $V_a$ to some gluing vertex of some $V_b$.

The tree substitution extends to any tree as above:

\[ \tau(W) = \bigsqcup_{i=1}^n \tau(W_{b_i}) / \sim \]

where the vertices $v$ and $v'$ of $\tau(W_{b_i})$ and $\tau(W_{b_j})$ are identified if and only if there exists $u$ in $W_{b_i}$ and $u'$ in $W_{b_j}$ such that $u \sim u'$ in $W$, $\tau(u) = v$ and $\tau(u') = v'$. From the construction it is obvious that the iterated images of a tree $W$ are trees.

3.4.2. Combinatorial tree substitutions for parageometric automorphisms. Now let $\sigma$ be a primitive substitution over the alphabet $A$ and a parageometric iwip automorphism of $F_A$. For each $a \in A$ consider the subtree $Y_a$ of the repelling tree $T_{\sigma^{-1}}$. We consider $Y_a$ as a simplicial tree by forgetting its metric structure and we let the set of vertices $V_a$ be the set of singular points $\text{Sing}(\Omega_a)$.

In this framework, the tree substitution is defined by

\[ \tau(Y_a) = \bigsqcup_{a \in \mathcal{S}_b} Y_b / \sim \]

where the disjoint union is taken over all edges $a \xleftarrow{p,s} b$ in the prefix-suffix automaton and where, for two such edges $a \xleftarrow{p,s} b$ and $a \xleftarrow{q,t} b'$, we identify the singular point $P \in Y_b$ with the singular point $P' \in Y_b'$ if $\sigma^{-1}(p^{-1})P = \sigma^{-1}(q^{-1})P'$ in the repelling tree $T_{\sigma^{-1}}$. Note that the simplicial tree $\tau(Y_a)$ is homeomorphic to the subtree $\sigma_T(Y_a)$ of $T_{\sigma^{-1}}$ defined in (3.1).
Furthermore, a singular point $P \in \text{Sing}(\Omega_a) \subseteq Y_a$ with prefix-suffix expansion $\gamma = a \overset{p}{\longrightarrow} b \overset{s}{\longrightarrow} \cdots$ is mapped to $\sigma_T(P) = H^{-1}(P)$ which lies in $\sigma^{-1}(p^{-1})Y_b$. From Section 3.2 we know that the tail $B(\gamma)$ of $\gamma$ also gives a singular point, and thus in our new setting the point $\sigma_T(P)$ is the point $H^{-1}(P) = \sigma^{-1}(p^{-1})Q(B(\gamma)) \in \text{Sing}(\Omega_b) \subseteq Y_b$ in the corresponding copy of $Y_b$.

This defines the map from the set of gluing points $V_a$ in $Y_a$ to the set of gluing points $V_b$ in $Y_b$.

The simplicial trees $\tau^n(Y_a)$ are obtained from their metric counterpart described in Section 3.3 by forgetting the metric. To state a convergence result we use the renormalization by the contracting homothety $H$. This renormalization allows much more flexibility.

So far we have considered the tree substitution inside the repelling tree $T_{a^{-1}}$ and the abstract combinatorial tree substitution where we forget the metric structure of the tree keeping only the simplicial one. We want to show that what is really important in these constructions is the set of gluing points, that is, the singular points. Indeed, no matter which simplicial or metric structure we give to the tree, we will converge (in the Gromov-Hausdorff sense) iterating the tree substitution and renormalizing to the compact limit set $\Omega_A$.

To this purpose let $W_a$ be any simplicial tree with a finite set of gluing vertices in bijection with the set $V_a$ of gluing vertices of $Y_a$, for each $a \in A$. Let $\tau$ be the tree substitution in (3.3) defined on the $W_a$:

$$\tau(W_a) = \bigsqcup_{a \in A} W_b/\sim$$

where the disjoint union is taken over all edges $a \overset{p}{\longrightarrow} b$ in the prefix-suffix automaton and where for two such edges $a \overset{p}{\longrightarrow} b$ and $a \overset{q}{\longrightarrow} b'$ we identify the gluing points $P$ of $W_b$ and $P'$ of $W_{b'}$ as the corresponding points of $Y_b$ and $Y_{b'}$ are identified. Moreover $\tau$ maps the gluing points of $W_a$ to the gluing points of the $W_b$ exactly as $\sigma_T$ maps the gluing points of $Y_a$ to the gluing points of the $Y_b$.

Let us fix any metric on each of the $W_a$ by fixing the lengths of edges. We get that $\tau^n(W_a)$ is also a metric tree and we renormalize this metric by contraction by $(\lambda_{a^{-1}})^{-n}$. We denote this metric tree by $(\lambda_{a^{-1}})^{-n}\tau^n(W_a)$.

**Proposition 3.9.** Let $\sigma$ be a primitive substitution and a parageometric iwip automorphism. Let $\tau$ be any tree substitution such that the gluing vertices $V_a$ of each prototile $W_a$ are in bijection with the singular points in $\text{Sing}(\Omega_a)$ and such that the gluing instructions mimic the tree substitution inside the repelling tree. Then, the renormalized iterated images $(\lambda_{a^{-1}})^{-n}\tau^n(W_a)$ converge in the Gromov-Hausdorff topology towards the compact real tree $\Omega_A$.

Moreover, the renormalized iterated images of $W = \bigsqcup_{a \in A} W_a/\sim$ converge towards the compact limit set $\Omega_A$.

**Proof.** We first define the matrix of distances for the tree substitution $\tau$. Let $u$ and $v$ be distinct gluing points in $W_a$. After one step of the tree substitution those points are mapped to gluing points $u'$ and $v'$ in $\tau(W_a)$. The segment $[u', v']$ in $\tau(W_a)$ crosses some of the tiles $W_b$ and, for each such tile $W_b$, it crosses a segment between two gluing points of $W_b$. Thus there exist $n$ tiles $W_{b_0}, \ldots, W_{b_n}$, distinct gluing points $x_i, y_i$ in $W_b$, with $x_0 = u'$, $y_i \sim x_{i+1}$, for $i = 0, \ldots, n - 1$, and $y_n = v'$. As we are in a tree, the distances add up and we get that

$$d(\tau(u), \tau(v)) = \sum_{i=0}^{n} d(x_i, y_i).$$
Repeating this observation for all pairs of distinct gluing points in each $W_a$, we get a matrix of distances $M$ with non-negative integer entries, such that if $\vec{d} = (d_{(u,v)})_{u,v \in V_a}$ and $\vec{d}' = (d_{\tau(W_a)}(u,v))_{u,v \in V_a}$:

$$\vec{d}' = M\vec{d}.$$ 

Recall that the set of gluing points $V_a$ is the set $\text{Sing}(\Omega_a) \subset Y_a$. Moreover the abstract tree substitution $\tau$ mimics the tree substitution $\sigma_T$ inside $T_{a-1}$. Thus, if we start with the vector $\vec{d}_\sigma = (d_{T_{a-1}}(u,v))_{u,v \in \text{Sing}(\Omega_a)}$ of distances between pairs of distinct gluing points in $Y_a$, using the homothety $H$ we get:

$$\vec{d}_\sigma = \frac{1}{\lambda_{a-1}} M\vec{d}_\sigma.$$ 

This proves that the matrix of distances $M$ has for dominant eigenvalue $\lambda_{a-1} > 1$ associated with the positive eigenvector $\vec{d}_\sigma$.

It is known that the repelling tree of an iwip automorphism $T_{a-1}$ is indecomposable [CH12], this implies that for any two non-degenerate arcs $I$ and $J$ in the limit set $\Omega_A$, there exists $u \in F_A$ such that $K = I \cap uJ$ is a non-degenerate arc. Let $P$ be a point in the interior of $K$ with exactly one pre-image by $Q$ (both in the attracting shift $X_\sigma$ and in the set of infinite prefix-suffix expansion paths $P$) and such that $P$ is not a branch point in $T_{a-1}$. Note that such a point $P$ exists as both branch points and singular points are countable. Let $Z \subset X_\sigma$ and $\gamma \in P$ be the preimages: $Q(Z) = Q(\gamma) = P$. We assume furthermore that both the prefix and suffix of $\gamma$ are unbounded words (this is again always the case outside a countable subset of the limit set $\Omega_A$). It is a property of the limit set [CHL09] (and indeed of the compact heart) that, as $P$ and $uP$ are points in $\Omega_A$, $uZ$ is in the attracting shift $X_\sigma$, see also Proposition [2.18]. In our situation, as $\sigma$ is a substitution, $u$ is either a negative word and $u^{-1}$ is a prefix of the right half $Z_{[0,\infty)}$ or $u$ is a positive word and a suffix of the left half $Z_{(-\infty;0]}$.

We pick $\gamma'$ a prefix of $\gamma$, such that either $u$ is a positive word and a suffix of $p(\gamma')$ or $u$ is a negative word and $u^{-1}$ is a prefix of $a_0s(\gamma')$ with $a_0$ the ending letter of $\gamma$. We get that there exists a path $\gamma''$ in the prefix-suffix automaton of the same length as $\gamma'$ such that

$$u\Omega_{\gamma'} = \Omega_{\gamma''}.$$ 

Indeed, $\gamma''$ is obtained from $\gamma'$ by performing the Vershik map (see Section [1.2] $|u|$ times. We remark that $\gamma'$ and $\gamma''$ start at the same letter $a_n$.

Starting with any two arcs $I = [x_1; y_1]$ and $J = [x_2; y_2]$ joining distinct gluing points in $W_a$, and $W_{a_n}$ we found an arc $[x_0; y_0]$ between two distinct gluing points in $W_a$, such that $H^n([x_0; y_0]) = J \cap \Omega_{\gamma''}$ is a non-degenerate sub-arc of $I$ and $u^{-1}H^n([x_0; y_0]) = J \cap \Omega_{\gamma''}$ is a non-degenerate sub-arc of $J$. This proves that the matrix $M^n$ has positive coefficients in the columns for $d(x_1, y_1)$ and $d(x_2, y_2)$ at the line of $d(x_0, y_0)$.

This does not prove that the matrix $M$ is primitive (and we do not claim or expect it) but this is enough to prove that this matrix is uniformly contracting for the Hilbert distance (see e.g. [Yoc05] [Via06, Section 26]) in the positive cone of the projective space. And thus, no matter which distances we chose on the trees $W_a$ (and no matter which simplicial trees $W_a$ we chose, the only assumption is that $W_a$ has the same gluing points $V_a$ as $Y_a$), for any two gluing points $u, v$ in $V_a$ we have:

$$\lim_{n \to \infty} \left(\frac{1}{\lambda_{a-1}}\right)^n d_{\tau^n(W_a)}(\tau^n(u), \tau^n(v)) = d_{T_{a-1}}(u, v).$$

This extends to gluing points $u$ and $v$ in $\tau^m(W_a)$:

$$\lim_{n \to \infty} \left(\frac{1}{\lambda_{a-1}}\right)^n d_{\tau^{m+n}(W_a)}(\tau^m(u), \tau^n(v)) = d_{T_{a-1}}(u, v).$$
We get that the sets of gluing points in \( \tau^m(W_a) \), as a metric space, converge in the Gromov-Hausdorff topology to the limit set \( \Omega_a \). We let the reader extend the proof to \( W \).

We complete now the algorithm to construct the tree substitution associated with \( \sigma \).

**Algorithm 3.10.** Recall that the sets of singular points \( V_i = \text{Sing}(\Omega_i) \), for all \( i \in A \), have been computed in Algorithm 3.6. We now proceed to compute the gluing instructions and the map \( \tau \).

1. From the set of pairs of singular prefix-suffix expansions we get the gluings between singular points in \( V_i \) and \( V_j \) with \( i \neq j \). From these gluing instructions we build the initial patch \( W = \bigsqcup W_i / \sim \).
2. Each gluing point \( P \in V_a \) is given by its prefix-suffix expansion \( \gamma \). It is mapped by \( \tau \) to a gluing point \( \tau(P) \in V_b \subseteq W_b \), where \( b \) comes from the last edge \( e_1(\gamma) = a \overset{p,s}{\leftrightarrow} b \) of \( \gamma \). The point \( \tau(P) \in V_b \) is given by \( \tau(P) = \sigma^{-1}(p^{-1})P' \) and \( P' \) has prefix-suffix expansion \( B(\gamma) \).
3. Let \( a \overset{p,s}{\leftrightarrow} b \) and \( a \overset{p',s'}{\leftrightarrow} b' \) be two distinct edges of the prefix-suffix automaton ending at the same letter \( a \). The points \( P \in V_b \) and \( P' \in V_b \) are identified in \( \tau(W_a) \) if they are given by \( \gamma \) and \( \gamma' \) such that \( (a \overset{p,s}{\leftrightarrow} b_a \cdot \gamma, (a \overset{p',s'}{\leftrightarrow} b' \cdot \gamma') \) is a pair of singular prefix-suffix expansions.

**Example 3.11.** In Example 3.7 we computed the singular points for the Tribonacci substitution. According to (3.1), the Tribonacci tree substitution inside the repelling tree is

\[
\sigma_T : Y_a \mapsto Y_a \cup Y_b \cup Y_c
\]

\[
Y_b \mapsto c^{-1}Y_a
\]

\[
Y_c \mapsto c^{-1}Y_b
\]

Alternatively, we consider the abstract simplicial prototiles \( W_a \), \( W_b \) and \( W_c \) which are the convex hulls of the sets of gluing points \( V_a \), \( V_b \) and \( V_c \) in one-to-one correspondence with the sets of singular points. We now compute how to glue together the tiles of \( W \) and \( \tau(W) \) and describe the images by the map \( \tau \) of the singular points.

The set \( V_a \) consists of three gluing points numbered by 1, 2 and 3 (in red in Figure 5). For example, the point 3 comes from the singular prefix-suffix expansion \( \gamma_a \), it is mapped by \( \tau \) to the singular point corresponding to the beheaded prefix-suffix expansion \( B(\gamma_a) \) in the copy of \( W_a \) coming from the ending edge \( e_1(\gamma) = a \overset{b}{\leftrightarrow} a \).

We repeat this computation for all the singular points. For each of them we give the singular prefix-suffix expansion \( \gamma \), its image by the map \( Q \) in the repelling tree, the singular prefix-suffix expansion of the image by \( \tau \) in the copy of the prototile given by the heading edge \( e_1(\gamma) \) of \( \gamma \) and the image by the tree substitution \( \tau \).

| \( V_a \) | \( V_b \) | \( V_c \) |
|---|---|---|
| \( \gamma \) | \( B(\gamma_a) \) | \( B(\gamma_a) \) | \( \gamma_a \) | \( \gamma_b \) | \( \gamma_c \) |
| \( Q(\gamma) \) | \( c^{-1}P \) | \( b^{-1}P \) | \( P \) | \( a^{-1}P \) | \( P \) |
| \( e_1(\gamma) \) | \( a \overset{b}{\leftrightarrow} a \) | \( a \overset{c}{\leftrightarrow} b \) | \( a \overset{b}{\leftrightarrow} a \) | \( b \overset{a}{\leftrightarrow} a \) | \( b \overset{a}{\leftrightarrow} a \) |
| \( B(\gamma) \) | \( B^{-1}(\gamma_a) \) | \( B^{-1}(\gamma_a) \) | \( B(\gamma_a) \) | \( \gamma_a \) | \( \gamma_b \) |
| \( \tau(Q(\gamma)) \) | \( b^{-1}P \) | \( a^{-1}P \) | \( c^{-1}P \) | \( c^{-2}P \) | \( c^{-1}P \) |
Finally the gluings between tiles are given by the pairs of singular prefix-suffix expansions. Both the tiles $W_a$, $W_b$ and $W_c$ of the patch $W$ and of the patch $\tau(W_a)$ have the point $P = Q(\gamma_a) = Q(\gamma_b) = Q(\gamma_c)$ in common.

The tree substitution is described in Figure 5.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5}
\caption{Images of $W_a$ (red), $W_b$ (green) and $W_c$ (blue) by the Tribonacci tree substitution.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6}
\caption{Initial tree $W$ and its image $\tau(W)$ for the Tribonacci tree substitution.}
\end{figure}

It may be frustrating to the reader to let too much freedom for the choice of the simplicial trees $W_i$, for $i \in A$. It is however possible to discover the simplicial structure of the $Y_i$ by iterating the abstract tree substitution $\tau$. Then, it is possible to give to each $W_i$ the same simplicial structure as $Y_i$.

**Remark 3.12.** For $n$ big enough, the iterated images of the gluing points $\tau^n(V_a)$ lie in distinct copies of the $W_b$. For such an $n$, the simplicial tree spanned by $\tau^n(V_a)$ inside $\tau^n(W_a)$ is isomorphic to the simplicial structure of $Y_a$.

Moreover, if $W_a$ and $Y_a$ have isomorphic simplicial structures, then all iterates $\tau^n(W)$ and $\sigma_T^n(Y)$ have isomorphic simplicial structures.

**Proof.** Recall that, for a singular point $P \in \text{Sing}(\Omega_a)$ the tree substitution inside the repelling tree comes from the expanding homothety: $\sigma_T(P) = H^{-1}(P)$. As the prototiles $Y_a$ have bounded diameter, after finitely many iterations, the images $\sigma_T^n(P)$ of the finitely many singular points in $\text{Sing}(\Omega_a)$ lie in distinct tiles $\Omega\gamma$. As the abstract tree substitution $\tau$ mimics $\sigma_T$ we get that the points in $\tau^n(V_a)$ lie in distinct tiles $W_b$. Finally, as the tiles $Y_b$, $\Omega\gamma$ and $W_b$ are trees, we conclude that the simplicial trees spanned by $\sigma_T^n(\text{Sing}(\Omega_a))$ in the repelling tree $T_{\sigma^{-1}}$ and, by $\tau^n(V_a)$ in $\tau^n(W_a)$ are isomorphic. \qed
Another frustration may come from the lack of knowledge of the branch points, in particular of their prefix-suffix expansions and precise locations.

**Remark 3.13.** The prefix-suffix expansions of the branch points inside $Y_a$ are eventually periodic. They can be computed using the abstract tree substitution $\tau$.

**Proof.** The first strategy to describe the branch points is to study the inverse automorphism $\sigma^{-1}$. The duality between the repelling tree and the attracting shift implies that the branch points in $T_{a^{-1}}$ are described by singular bi-infinite words of the repelling shift [CH14]. However, the inverse automorphism $\sigma^{-1}$ needs not be a substitution and the repelling shift is rather the repelling lamination which can be studied using a train-track representative of $\sigma^{-1}$, but this is beyond the scope of this paper.

An alternative strategy is to give $W_a$ the same simplicial structure as $Y_a$ through Remark 3.12.

As $Y_a$ is the convex hull of the singular points, each branch point $P$ is the center of a tripod of three singular points $P_1, P_2, P_3$. The tree substitution inside the repelling tree $T_{a^{-1}}$ is defined by the contracting homothety and using $\sigma_T(P_i) = H^{-1}(P_i)$ and thus, $H^{-1}(P)$ is the center $P'$ of the tripod $\sigma_T(P_1), \sigma_T(P_2)$, and $\sigma_T(P_3)$. This center $P'$ lies in a translate of $Y_a$ for an edge $a \xrightarrow{b} b$ of the prefix-suffix automaton. This edge is the ending edge of the prefix-suffix expansion of $P$. Recall that the translates of the $Y_a$ in $\sigma_T(Y_a)$ intersect in singular points, thus, $P'$ is a singular point or a branch point of the translate of $Y_a$. As before, $P'$ is the center of a tripod (possibly degenerate) of singular points in $Y_a$. Iterating this construction, we get the prefix-suffix expansion of $P$.

As there are finitely many tripods in the finitely many prototiles $Y_a$, the above construction yields an eventually periodic prefix-suffix expansion for all branch points of $Y_a$.

Finally, as the abstract trees $\tau^n(W_a)$ have the same simplicial structure as $\sigma_T^n(Y_a)$, the above construction can be achieved using the abstract tree substitution $\tau$ rather than the tree substitution $\sigma_T$ inside $T_{a^{-1}}$. The prefix-suffix expansions of branch points can be computed from the abstract tree substitution $\tau$.

Finally we remark that we can add extra gluing points to the abstract tree substitution.

**Remark 3.14.** Let $P$ be a point in the limit set $\Omega_A$ with eventually periodic prefix-suffix expansion $\gamma$. The set of tails $\{B^n(\gamma) | n \in \mathbb{N}\}$ of $\gamma$ is finite. The finitely many points $Q(B^n(\gamma))$ can be added to the sets of gluing points $V_{an}$. The map $\tau$ is then extended by mapping the gluing point $Q(B^n(\gamma))$ of $V_{an} \subseteq W_{an}$ to the gluing point $Q(B^{n+1}(\gamma)) \in V_{an+1} \subseteq W_{an+1}$ of the copy of $W_{an+1}$ corresponding to the $n$-th edge $a_n \xrightarrow{p_{n,s_n}} a_{n+1}$ of $\gamma$.

### 3.5. Tree Substitution inside the Rauzy fractal.

We are now ready to use the two previous Sections to get a tree substitution inside the Rauzy fractal.

Using the map $\psi$ of Proposition 2.18 we embed the finite trees $Y_a$ of Section 3.3 inside the tiles $\mathcal{R}(a)$ of the Rauzy fractal. The tree substitution inside the Rauzy fractal $\sigma_R$ is defined by

$$\sigma_R(\psi(Y_a)) = \bigcup_{a \xrightarrow{\cdot,b}} \psi(Y_b) + \pi_c(M_{\sigma^{-1}}(p))$$

where the union is taken over all edges $a \xrightarrow{\cdot,b}$ in the prefix-suffix automaton. This version of the tree substitution inside the Rauzy fractal is quite ineffective as we expect the embedding of an arc of the tree $Y_a$ into the contracting plane to be a Peano curve. We can rather focus on the finite set of singular points.
For each \( a \in A \), the finite set of points \( \text{Sing}(\Omega_a) \) is mapped by \( \psi \) to a finite set of points in the tile \( R(a) \) of the Rauzy fractal. Recall that these points have eventually periodic prefix-suffix expansions and that we gave an explicit formula in Section 2.6 to compute their images in the contracting space.

For prefix-suffix expansions \( \gamma, \gamma' \) and for the point \( P \) as in Proposition 3.4, \( \psi(P) \) is a common point of \( \tilde{R}_\gamma \) and \( \tilde{R}_{\gamma'} \).

We can now consider trees spanned by the images of the singular points embedded inside the contracting space. This gives a tree substitution inside the Rauzy fractal. Remark that, as the subtile \( R(a) \) is the image by \( Q \) of the compact tree \( \Omega_a \), it is arcwise connected and we can choose such a linking arc between two singularities of \( Y_a \) inside \( R(a) \).

The question that now arises is whether iterating this tree substitution inside the contracting space (or inside the Rauzy fractal) creates loops.

**Question 3.15.** Given a primitive parageometric iwip substitution, does there always exist a choice of the linking arcs between the images of the singular points in the contracting space such that the iteration of the tree substitution inside the contracting space does not create loops? Moreover, may the linking arcs be chosen inside the corresponding tiles of the Rauzy fractal without creating loops?

The answer to this question is negative in general, as illustrated by the case of the Tribonacci substitution. In Example 3.11, after two iterations the tree substitution embedded in the contracting plane \( E_c \) creates loops, see Figure 7. And this is not because of a bad choice of the linking arcs: the loops are created by the non-injectivity of the projection map \( \psi \) on singular points. It is not anymore visible that we are working with trees. However this is only an artifact of the representation inside the contracting plane, since the tree substitution was initially defined as an abstract tree substitution.

![Figure 7. Second iterate of the Tribonacci tree substitution. The embedding into the contracting plane of \( \tau^2(W) \) creates a loop.](image)

We propose to overcome this difficulty by pruning the tree, as described in the next Section.

### 3.6 Pruning and covering

As we observed in Figure 7 and Example 3.11 the tree substitution we constructed in Sections 3.4.2 and 5.5 may fail to produce a tree inside the contracting space because of non-injectivity of the map \( \psi \). We propose in this section a pruning of the tree that will produce for the Tribonacci substitution a tree inside the contracting plane.
We remark that in the tree substitution some branches are dead-end which will never grow up: they are leaves of the tree for all steps \( n \). Thus we prune those branches. This will result into a new tree substitution defined on more pieces: we have a new alphabet \( A' \) with a forgetful map \( f : A' \to A \), and for each \( a \in A' \) the prototile \( W_a \) is a subtile of the original prototile \( W_{f(a)} \). Indeed, the set of gluing points \( V_a \) is a subset of the set of gluing points \( V_{f(a)} \).

As we kept at least one point for each tile, the proof of Proposition 3.9 applies and, we get that the limit tree (after renormalization) is again \( \Omega_A \).

**Example 3.16.** For the Tribonacci tree substitution of Example 3.11, pruning the dead-end vertices gives the pruned tree substitution described in Figure 8.

It turns out that the choice of the linking arcs is inside the Rauzy fractal and that we now have a tree substitution inside the Rauzy fractal that does not create loops. More spectacularly, this tree substitution occurs inside the tiles of the dual substitution as illustrated in Figure 9 and 10.

![Figure 8. The pruned Tribonacci tree substitution. In red the tiles: \( W_a, W_a', W_a'' \); in green \( W_b \) and \( W_b' \); in blue \( W_c \).](image1)

![Figure 9. The pruned Tribonacci tree substitution inside the tiles of the dual substitution.](image2)

Turning back to Section 2.9, we remark that in the case of the Tribonacci substitution the usual prototiles \( P_a = \pi_c((0,a)^+) \subseteq E_c \) yield tiles \( P \) that are disk-like and satisfy Condition 2 of Question 2.21. The trees in Figure 10 can be viewed as connecting the tiles of the iterated images of the dual substitution.

Unfortunately pruning is not always enough: in our third example detailed in Section 5.2 even the pruned tree substitution inside the contracting space is not injective and creates loops.
Let us describe our problem and our general aim. We will see pruning as a special case of covering.

Let \( \sigma \) be an irreducible Pisot substitution over the alphabet \( A \) which is a parageometric iwip automorphism. A covering of the tree substitution \( \tau \) is a forgetful map \( f : G \to A(\sigma) \) where \( G \) is an oriented graph and \( A(\sigma) \) is the prefix-suffix automaton of \( \sigma \) such that for each vertex \( a \) of \( G \), the map \( f \) induces a bijection between the edges of \( G \) ending in \( a \) and the edges of \( A(\sigma) \) ending in \( f(a) \). We denote by \( \tilde{A} \) the set of vertices of \( G \).

For each \( \tilde{a} \in \tilde{A} \), let \( P_{\tilde{a}} \subseteq E_c \) be a prototile in the contracting space \( E_c \). Pick a section \( s : A \to \tilde{A} \) of the forgetful map to select a starting patch:

\[
P_0 = \bigcup_{a \in A} P_{s(a)}.
\]

Recall that \( \ell : F_A \to \mathbb{Z}^A \) is the abelianization map, the incidence matrix \( M_\sigma \) restricted to the contracting space \( E_c \) is a contraction, and \( \pi_c \) is the projection onto \( E_c \) along the expanding direction.

We consider the iterated patches

\[
P_n = \bigcup_{\gamma} P_{\tilde{a}_n} + \pi_c(M_\sigma^{-n}\ell(p))
\]

where the union is taken over all reverse paths of length \( n \) in \( G \) ending at the image of the section \( s \):

\[
\gamma = s(a_0) \xleftarrow{e_1} \tilde{a}_1 \cdots \xleftarrow{e_{n-1}} \tilde{a}_n,
\]

and \( p \) is the prefix of the image \( f(\gamma) \) in the prefix-suffix automaton of \( \sigma \).
Question 3.17. For any irreducible Pisot substitution which is an iwip automorphism, do there always exist a covering substitution and a choice of prototiles such that all the iterated patches are trees inside the contracting space?

In Example 3.16 the covering tree substitution was defined by pruning to give a positive answer to this question. The covering graph $G$ and the forgetful map $f$ can be recovered from Figure 8.

4. INTERVAL EXCHANGE ON THE CIRCLE

In this Section we show how a tree substitution induces an interval exchange on the circle $S^1$. Our construction further extends the work of X. Bressaud and Y. Jullian [BJ12]. Indeed they interpreted the celebrated construction of P. Arnoux and J.-C. Yoccoz [AY81] of an interval exchange transformation related to the Tribonacci substitution as being the contour of the compact heart $\Omega_A$ of the repelling tree $T_{\sigma^{-1}}$.

4.1. Contour of the tree substitution. Again we fix a substitution $\sigma$ that is a parageometric iwip automorphism of the free group $F_A$. We consider the tree substitution $\tau$ defined in Section 3.4.2. At each branch point of the finite simplicial tree $W$ we fix a cyclic order of the outgoing edges. With respect to these cyclic orders we get a way to turn around the tree, that is to say a map $c : S^1 \to W$ which is locally an isometry except at leaves of $W$ and which is 2-to-1 except at leaves and branch points. Recall that $V_a$ is the set of gluing vertices of each $W_a$ and that the set of gluing points contains all leaves of $W_a$. We consider that the circle is divided in arcs by these gluing points: each arc is labeled by the letter $a_{ij}$ if it wraps around $W_a$ and is bounded by the gluing points $i$ and $j$. We denote by $\tilde{A}$ the set of labels $a_{ij}$ of these arcs.

We will consider $c$ alternatively as the cyclic word in $\tilde{A}$ labeling these arcs.

To define the contour substitution, we first require that:

(C1) for each $a \in A$, the map $\tau$ induces a simplicial isomorphism between $W_a$ and the convex hull of the points $\tau(V_a)$ in the simplicial tree $\tau(W_a)$.

This condition is satisfied as soon as $W_a$ has the same simplicial structure as $Y_a$ which can be achieved using Remark 3.12.

Furthermore, we require that:

(C2) for each $a \in A$, and each gluing point $x \in V_a$ the valence of $x$ in $W_a$ is equal to the valence of $x$ in $\tau(W_a)$.

If $\tau$ satisfies Condition (C1) the valence of $x$ in $\tau(W_a)$ is at least the valence of $x$ in $W_a$. If it is strictly bigger, there exists an edge $a_{p,s} \to b$ in the prefix-suffix automaton and a gluing point $y \in V_b$ such that $y$ is in a direction $d$ outgoing from $\tau(x)$ in $\tau(W_a)$ and such that $d$ does not intersect the tree spanned by $\tau(V_a)$. Let $\gamma$ be the prefix-suffix expansion of $y$ ending at the letter $b$, and let $\gamma' = (a_{p,s} \to b) \cdot \gamma$. Using Remark 3.14 we extend $\tau$ by adding the point associated with $\gamma'$ in the set of gluing points $V_a$. Repeating this operation, we will get a tree substitution satisfying Conditions (C1) and (C2).

We now fix a cyclic order at each branch point of the image tree $\tau(W)$. For a tree substitution $\tau$ satisfying Conditions (C1) and (C2) the cyclic orders of $W$ and $\tau(W)$ are compatible if moreover:

(C3) the cyclic orders of each tile $W_a$ of $\tau(W)$ are the same as the cyclic orders of the initial prototile $W_a$.

(C4) for any gluing points $x, y, z$ of $W$, the cyclic order of $x, y, z$ and that of $\tau(x), \tau(y), \tau(z)$ are the same.
By convention we say that two triples of points of two trees have the same cyclic order if they are either aligned in the same order or if the cyclic order of the branch points of the centers of the two tripods are the same. In particular Condition \([C4]\) implies Condition \([C1]\).

From the above paragraph, we get a contour map \(c' : S^1 \to \tau(W)\) where the circle is divided by the gluing points of \(\tau(W)\). Using Condition \([C4]\), the contour \(c'\) is divided by the gluing points in \(\bigcup_{i \in A} \tau(V_i)\), exactly as the contour \(c\) was divided by the gluing points in \(\bigcup_{i \in A} \tau(V_i)\). Thus the contour \(c'\) of \(\tau(W)\) is divided into arcs labeled by \(\tau(a_{ij})\), for \(a_{ij} \in A\). Each of these arcs of \(c'\) is further divided by arcs between gluing vertices of the tiles \(W_b\) of \(\tau(W)\) and, Condition \([C3]\) identifies these subarcs with letters in \(A\).

Thus, we get a contour substitution \(\chi : \hat{A} \to \hat{A}^*\).

**Example 4.1.** Consider the Tribonacci tree substitution of Figure 6. We fix the cyclic orders of \(W\) and \(\tau(W)\) as being given by the embedding into the contracting plane (for which we fix the usual clockwise orientation). We get that the contour \(c\) of \(W\) is divided in arcs \(\hat{A} = \{a_{31}, a_{12}, a_{23}, c_{11}, b_{21}, b_{12}\}\) and from the contour \(c'\) of \(\tau(W)\) we get the contour substitution

\[
\chi : a_{12} \mapsto a_{23} c_{11} b_{21} \\
a_{23} \mapsto b_{12} a_{31} \\
a_{31} \mapsto a_{12} \\
b_{12} \mapsto a_{12} a_{23} \\
b_{21} \mapsto a_{31} \\
c_{11} \mapsto b_{21} b_{12}
\]

(Note that we slightly cheated from our construction to get over the fact that the patch \(\Omega_x\) has only one singular point. We consider the tree \(W_c\) not as a single vertex but as a vertex with a tiny outgrow, and thus allow an arc \(c_{11}\) turning around this outgrow clockwise).

Note that, this substitution \(\chi\) has reciprocal characteristic polynomial \(x^6 - x^4 - 4x^3 - x^2 + 1\) which factors as \((x^3 - x^2 - x - 1)(x^3 + x^2 + x - 1)\), where the leftmost factor is the Tribonacci polynomial.

From our definitions we get a forgetful map \(f : \hat{A} \to A\) which extends to a map \(f : \hat{A}^* \to A^*\). We remark that each arc \(b_{kl} \in \hat{A}\) in the contour \(c'\) of \(\tau(W)\) is turning around a patch \(W_b\) which comes from an edge \(a \xleftarrow{\hat{\sigma}} b\) of the prefix-suffix automaton of the original substitution \(\sigma\). The arc \(b_{kl}\) is covered by the image of exactly one arc \(a_{ij}\) of the contour of \(W_{\hat{a}}\): \(\chi(a_{ij}) = \hat{\sigma} \cdot b_{kl} \cdot \hat{t}\). We consider the **dual contour substitution** \(\chi^* : \hat{A} \to \hat{A}^*\) such that the latter at position \([p]\) of the word \(\chi^*(b_{kl})\) is \(a_{ij}\). The substitutions \(\chi\) and \(\chi^*\) come with a bijection that maps the edge \(a_{ij} \xleftarrow{\hat{\sigma}} b_{kl}\) (with \(f(\hat{p}) = p\) and \(f(\hat{s}) = s\)) of the prefix-suffix automaton of \(\chi^*\) to the edge \(a_{ij} \xrightarrow{\hat{\sigma}} b_{kl}\) of the prefix-suffix automaton of \(\chi\).

Using the forgetful map \(f\), we get that

\[
f(\chi^*(b_{kl})) = \sigma(f(b_{kl})) = \sigma(b)
\]

which amount to saying that \(\chi^*\) is a covering of \(\sigma\). Turning to matrices, the duality is expressed by

\[
^t M_\chi = M_{\chi^*}.
\]
Example 4.2. For the Tribonacci tree substitution of Example 3.11 and the contour of Example 4.1, the dual contour substitution is

\[
\chi^*: \begin{align*}
a_{12} &\mapsto a_{31}b_{12} \\
a_{23} &\mapsto a_{12}b_{12} \\
a_{31} &\mapsto a_{23}b_{21} \\
b_{12} &\mapsto a_{23}c_{11} \\
b_{21} &\mapsto a_{12}c_{11} \\
c_{11} &\mapsto a_{12}
\end{align*}
\]

We can compute the indexes of this free group automorphism and of its inverse. Both are equal to \(10 = 2 \times 6 - 2\) and maximal. This substitution is an iwip geometric automorphism: it is induced by a pseudo-Anosov transformation of a surface of genus 3 with one boundary component.

As usual we are interested in iterating the substitution. Thus, we state

Proposition 4.3. Let \(\tau\) be a tree substitution and assume that \(W\) and \(\tau(W)\) are equipped with compatible cyclic orders. Let \(c\) be the corresponding contour of \(W\) and \(\chi\) be the contour tree substitution. Then, for each \(n > 0\), \(\chi^n(c)\) is a contour of \(\tau^n(W)\).

Proof. Iterating the tree substitution \(\tau\), we consider the set \(V_n\) of gluing points in \(\tau^n(W)\). We first remark that, by definition of a tree substitution, if Condition (C1) holds, then \(\tau\) induces a simplicial isomorphism between \(\tau^{n-1}(W)\) and the convex hull of the points \(\tau(V_{n-1})\) in \(\tau^n(W)\). Similarly, if Condition (C2) holds, then the valence of a gluing point \(x \in V_{n-1}\) in \(\tau^{n-1}(W)\) is equal to the valence of \(\tau(x)\) in \(\tau^n(W)\).

Using the simplicial isomorphism \(\tau\) between \(\tau^{n-1}(W)\) and the convex hull of the images of the gluing points \(\tau(V_{n-1})\) in \(\tau^n(W)\), we define by induction the cyclic orders on \(\tau^n(W)\) by requiring that the cyclic order of each triple of gluing points \(x, y, z\) of \(\tau^{n-1}(W)\) is the same as the cyclic order of the gluing points \(\tau(x), \tau(y), \tau(z)\) of \(\tau^n(W)\). Condition (C2) implies that this completely defines the cyclic order at all points in \(\tau(V_{n-1})\). Let now \(x\) be a branch point of \(\tau^n(W)\) which is not in \(\tau(V_{n-1})\). Then, there exists a tile \(W_a\) of \(\tau^{n-1}(W)\) such that \(x\) and all the outgoing edges from \(x\) belong to the patch \(\tau(W_a)\). We define the cyclic order at \(x\) in \(\tau^n(W)\) as being the cyclic order at \(x\) in the patch \(\tau(W_a)\) of \(\tau(W)\). Note that if \(x\) is also in the convex hull of \(\tau(V_{n-1})\), Condition (C4) implies that the two cyclic orders that we used are compatible.

The cyclic orders defined on \(\tau^n(W)\) give contour maps \(c_n : S^1 \to \tau^n(W)\).

Two consecutive gluing points \(x, y\) of \(\tau^{n-1}(W)\) belong to the same tile \(W_a\) and we fixed the cyclic order of \(\tau^n(W)\) such that the gluing points in \(\tau(W_a) \subseteq \tau^n(W)\) are ordered between \(\tau(x)\) and \(\tau(y)\) as they are in \(\tau(W)\). This proves that the contour \(c_n\) is obtained by applying the contour substitution \(\chi\) to \(c_{n-1}\). This concludes the proof by induction.

For the next Proposition we will assume that the number of extremal tiles in \(\tau^n(W)\), i.e. tiles \(W_i\) of \(\tau^n(W)\) which are adjacent to exactly one other tile, is unbounded. We will say in this case that \(\tau^n(W)\) is sufficiently branching. We have in mind that the limit tree \(\Omega_A\), which exists if \(\sigma\) is an iwip parageometric automorphism, is not the convex hull of finitely many points.

Proposition 4.4. Let \(\sigma\) be a primitive substitution and let \(\tau\) be a tree substitution associated to \(\sigma\). Assume that \(W\) and \(\tau(W)\) are equipped with compatible cyclic orders and that \(\tau^n(W)\) is sufficiently branching. Then, the contour substitution \(\chi\) is primitive and has the same dominant eigenvalue \(\lambda_\sigma\) as \(\sigma\).
Proof. The growth rate of any letter $a$ in $\tilde{A}$ under iteration of the dual contour substitution $\chi^*$ is the same as that of $f(a)$ under $\sigma$, where $f$ is the forgetful map. This proves that there exists a positive eigenvector $\ell$ such that $M_\chi \cdot \ell = \lambda_\ell \ell$.

As $\sigma$ is primitive, for some $n \geq 1$, $\tau^n(W)$ contains a copy of all tiles $W_a$ for all $a \in A$. Such a copy is wrapped around by all $a_{ij} \in \tilde{A}$ with $f(a_{ij}) = a$. Thus all letters of $\tilde{A}$ appear in the image of the contour substitution $\chi$ and, by duality, all letters of $\tilde{A}$ appear in the image of the dual contour substitution $\chi^*$.

This proves that the matrix of $\chi^*$ is block-wise diagonalizable, each block contains at least one letter of $f^{-1}(a)$ for each $a \in A$ and, has dominant eigenvalue $\lambda_\sigma$. Moreover, up to passing to a positive power, each diagonal block is positive.

From our assumption, the number of extremal tiles in $\tau^n(W)$ goes to infinity. As the circle is originally divided into finitely many arcs, there exists an arc $a_{ij} \in \tilde{A}$ such that $\chi^n(a_{ij})$ wraps around a whole extremal patch $W_b$: all letters $b_k \in f^{-1}(b)$ appear in $\chi^n(a_{ij})$.

By duality, one the blocks of the block-wise diagonal matrix $M_\chi^*$ spans all of $f^{-1}(b)$. We proved that $M_\chi^*$ has a unique diagonal block, hence $\chi^*$ is primitive. And by duality $\chi$ is primitive as well.

From Proposition 4.4 we get that the matrix $M_\chi$ has a unique (projective) positive left eigenvector $\tilde{\ell}_2^1$ associated with the dominant eigenvalue $\lambda_\sigma$.

We use this eigenvector $\tilde{\ell}_2^1$ to describe the points of the unit circle $S^1$ by infinite paths in the prefix-suffix automaton. First divide the unit circle $S^1$ by the arcs of $\tilde{A}$ each with the length given by $\tilde{\ell}_2^1$. Let $P$ be the set of infinite paths $\gamma = a_0 p_1 a_1 p_2 a_2 \ldots$, with $a_i \in \tilde{A}$, $\chi(a_i) = p_{i+1} \cdot a_{i+1} \cdot s_{i+1}$. We insist that here the infinite path in the prefix-suffix automaton is in the positive direction of edges which is reverse to all the paths (or prefix-suffix expansions) we considered up to here. This is why we use the funny notation $\hat{P}$. Consider the map $Q_{\hat{P}} : \hat{P} \to S^1$ such that

$$Q_{\hat{P}}(\gamma) = x_0 + \sum_{n=1}^{+\infty} \frac{1}{\lambda_\sigma} \tilde{\ell}_2^1(p_n),$$

where $x_0$ is the left endpoint of the arc $a_0$ and $\ell_2^1(p_n)$ is the scalar product between the abelianization of the word $p_n$ and the eigenvector $\tilde{\ell}_2^1$. The map $Q_{\hat{P}}$ is obviously continuous.

For any finite path $\gamma = a_0 p_1 a_1 p_2 a_2 \ldots p_n a_n$, the cylinder $[\gamma]$ of all infinite paths starting with $\gamma$ is mapped by $Q_{\hat{P}}$ to an arc $Q_{\hat{P}}([\gamma])$ of $S^1$ of length $(\frac{1}{\lambda_\sigma})^n \tilde{\ell}(a_n)$.

The bijection between the edges of the prefix-suffix automata of $\chi$ and $\chi^*$ yields a homeomorphism $P_{\chi} \simeq P_{\chi^*}$ and by composition a continuous map $P_{\chi^*} \to S^1$ which, by abuse of notation, we also denote by $Q_{\hat{P}}$.

4.2. Piecewise exchange and self-similarity on the circle. The contour we constructed in the previous Section carries a self-similar piecewise exchange exactly as the attracting shift $X_\sigma$ and the limit set $\Omega_A$ do. In this Section we describe this self-similar piecewise exchange on the circle.

Our description of the circle comes from the map $Q_{\hat{P}} : P_{\chi^*} \to S^1$. The piecewise exchange comes primarily from the shift map $S$ on the attracting shift $X_\sigma$, which is carried to the Vershik map on the space $P_{\chi^*}$ of infinite prefix-suffix expansions of $\chi^*$. We will obtain the piecewise exchange on the circle by pushing forward the Vershik map through $Q_{\hat{P}}$.

Let $\sigma$ be a substitution. Recall that in Section 2.1 we introduced the notation $p(\gamma)$ for the prefix of a prefix-suffix expansion $\gamma$. Here we rather deal with suffixes and we use the similar notation $s(\gamma) = s_0 \sigma(s_1) \cdots \sigma^{n-1}(s_{n-1})$. We denote by $P_{\max}$
the set of infinite prefix-suffix expansions with empty suffix (and by $\mathcal{P}_{\min}$ those with empty prefix).

The Vershik map [Dur10, Section 6.3.3]

$$V : \mathcal{P} \setminus \mathcal{P}_{\max} \to \mathcal{P} \setminus \mathcal{P}_{\min}$$

sends a prefix-suffix expansion

$$\gamma = a_0 \cdots a_1 \cdots a_r \cdots a_{r+1} \cdots a_{n-1} p_{n-1} a_n \cdots$$

where $r$ is the smallest index such that $s_r = b_r t_r \neq \epsilon$, with $b_r \in A$ and $t_r \in A^*$, to

$$V(\gamma) = b_0 \cdots b_1 \cdots b_r a_r \cdots a_{n-1} p_{n-1} a_n \cdots$$

with $\sigma(a_{r+1}) = p_r a_r s_r = p_r a_r b_r t_r$ and, for $i = 1, \ldots, r - 1$, $\sigma(b_{i+1}) = b_i t_i$. As it is defined using only the head of a prefix-suffix expansion path, the Vershik map is continuous.

In the setting of Section 2.7, the Vershik map is simply the push-forward of the shift-map $S$ through the map $\Gamma$ which associates to any bi-infinite word in the attracting shift $X_\sigma$ of $\sigma$ its prefix-suffix expansion:

$$\forall Z \in X_\sigma, \Gamma(Z) \notin \mathcal{P}_{\max} \Rightarrow V(\Gamma(Z)) = \Gamma(SZ),$$

and pushing forward to the limit set $\Omega_A$ of the repelling tree $T_\sigma$ we get:

$$\forall \gamma \in \mathcal{P} \setminus \mathcal{P}_{\max}, Q(V(\gamma)) = a_0^{-1} Q(\gamma),$$

where $a_0$ is the ending letter of the infinite prefix-suffix expansion $\gamma$. And, finally, for a finite path $\gamma$ in the prefix-suffix automaton ending in $a_0$ with non-empty suffix

$$a_0^{-1} \Omega_\gamma = \Omega_{V(\gamma)}.$$ We can partially push-forward the Vershik map to the iterates of the abstract tree substitution. A gluing point $x$ in $\tau^n(W)$ is described as a finite path $\gamma$ of length $n$ in the prefix-suffix automaton, together with a gluing point $y \in V_\gamma$, where $a_n$ is the starting letter of $\gamma$. If $\gamma$ has non-empty suffix then $\gamma$ and $V(\gamma)$ both start at the same letter $a_n$. For sake of simplicity, we abuse again of notation and we let the Vershik map act on gluing points, instead of writing $V$ the same letter $a$ $(4.2)$

$$a_0^{-1} \Omega_\gamma = \Omega_{V(\gamma)}.$$ We can partially push-forward the Vershik map to the iterates of the abstract tree substitution. A gluing point $x$ in $\tau^n(W)$ is described as a finite path $\gamma$ of length $n$ in the prefix-suffix automaton, together with a gluing point $y \in V_\gamma$, where $a_n$ is the starting letter of $\gamma$. If $\gamma$ has non-empty suffix then $\gamma$ and $V(\gamma)$ both start at the same letter $a_n$. For sake of simplicity, we abuse again of notation and we let the Vershik map act on gluing points, instead of writing $V$ the same letter $a$ $(4.2)$.

Thus we strengthen the compatibility condition of the previous section.

The cyclic orders of $W$ and $\tau(W)$ are strongly compatible if, in addition to Conditions [C1] - [C4], the following holds:

(C5) For $a \in A$ let $x, y, z$ be gluing points of $\tau(W_a) \subseteq \tau(W)$ where the map $V$ is defined. Then, the cyclic orders of $(x, y, z)$ and $(V(x), V(y), V(z))$ are equal in $\tau(W)$.

We remark that using the cyclic order defined in the proof of Proposition 4.3 Condition [C5] propagates to gluing points in $\tau^n(W)$: for any gluing points $x, y$ and $z$ in $\tau^n(W_a)$ where the map $V$ is defined, the cyclic orders of $(x, y, z)$ and $(V(x), V(y), V(z))$ are equal in $\tau^n(W)$.

**Remark 4.5.** Conditions [C1] - [C5] are finite combinatorial conditions that are easily checked.

We already explained how to satisfy Conditions [C1] and [C2]. We now prove that cyclic orders satisfying Conditions [C3] - [C5] exist.

With Condition [C1], $\tau^n(W)$ is isomorphic to the subtree $\sigma^n_\bullet(Y)$ of $\Omega_A$, for each $n$. Recall that the free group $F_A$ acts freely on the tree $T_{\sigma^{-1}}$ and, that there are finitely many orbits of branch points, each with finite valence. The contracting
homothety $H$ permutes the finitely many directions at the finitely many orbits of branch points. Up to replacing $H$ by a suitable power (as well as the initial substitution $\sigma$), any cyclic orders on directions at orbits of branch points extend to cyclic orders at branch points of $T_{\sigma^{-1}}$ which are preserved by the action of $F_A$ and $H$. Finally, through the isomorphisms with substrices of $T_{\sigma^{-1}}$ we get cyclic orders on $W$ and $\tau(W)$.

As the expanding homothety $H^{-1}$ corresponds to the map $\tau$ and as the translation by $a^{-1}$ for $a \in A$ corresponds to the Vershik map, we conclude that this choice of cyclic orders satisfies Conditions $[C3] - [C5]$.

**Proposition 4.6.** Let $\sigma$ be a primitive substitution and a parageometric iwip automorphism. Let $\tau$ be a tree substitution associated to $\sigma$. Assume that $W$ and $\tau(W)$ are equipped with strongly compatible cyclic orders.

Then, the Vershik map is pushed-forward by $Q_{\Sigma^1}$, to a piecewise rotation of the unit circle $S^1$.

**Proof.** We use the notation of the previous Section. For any finite path $\gamma$ in the prefix-suffix automaton of $\chi^*$, the image $Q_{\Sigma^1}(\gamma)$ of the cylinder [\gamma] is an arc of the unit circle of length $(\frac{1}{\lambda})^n\ell_{\Sigma^1}(a_n)$. If we assume that $\gamma$ as non-empty suffix, then $V(\gamma)$ has the same length $n$ as $\gamma$ and the same starting letter $a_n$. Thus we get that $V$ is pushed-forward by $Q_{\Sigma^1}$ to a rotation from the arc $Q_{\Sigma^1}(\gamma)$ to the arc $Q_{\Sigma^1}(V(\gamma))$.

Remark that the above paragraph is abusive as the map $Q_{\Sigma^1}$ is not a map $S^1 \to S^1$ but rather a division of the circle into finitely many arcs and on each of these arcs a rotation. In particular, boundary points of the arcs can be rotated to two different points according to whether they are considered as boundary points of one or the other of the two arcs they belong to. The division of the circle into arcs with non-empty suffixes is not finite. In order to get a piecewise rotation of the circle, we are left with proving that all but finitely many adjacent such arcs are rotated by the same angle. This is the purpose of the next lemma.

**Lemma 4.7.** Let $\gamma$ and $\gamma'$ be two prefix-suffix expansion paths of $\chi^*$ with same length, non-empty suffixes and ending at the same letter such that the associated arcs $Q_{\Sigma^1}(\gamma)$ and $Q_{\Sigma^1}(\gamma')$ are adjacent on $S^1$. Then the arcs $Q_{\Sigma^1}(V(\gamma))$ and $Q_{\Sigma^1}(V(\gamma'))$ are adjacent on the circle (and in the same order).

**Proof.** Let $n$ be the length of $\gamma$ and $\gamma'$. From Proposition 4.3, $\chi^n(c)$ is the contour of $\tau^n(W)$. The forgetful map $f$ maps a prefix-suffix expansion of $\chi^*$ to a prefix-suffix expansion of $\sigma$. Thus, the tiles $W_{f(\gamma)}$ and $W_{f(\gamma')}$ are equal or adjacent in $\tau^n(W)$. If they are equal, then $f(\gamma) = f(\gamma')$ and thus $V(f(\gamma)) = V(f(\gamma'))$. Else, in the repelling tree the tiles $\Omega_{f(\gamma)}$ and $\Omega_{f(\gamma')}$ have a singular point $P$ in common. Let $a_0 \in A$ be the common ending letter of both $f(\gamma)$ and $f(\gamma')$. Then $a_0^{-1}\Omega_{f(\gamma)} = \Omega_{V(f(\gamma))}$ and $a_0^{-1}\Omega_{f(\gamma')} = \Omega_{V(f(\gamma'))}$. These translated tiles have the singular point $a_0^{-1}P$ in common. The singular point $P$ gives rise to gluing points $y \in W_{f(\gamma)}$ and $y' \in W_{f(\gamma')}$ which are identified in $\tau^n(W)$. The Vershik map is defined on these two points and $V(y)$ and $V(y')$ are identified in $\tau^n(W): W_{V(f(\gamma))}$ and $W_{V(f(\gamma'))}$ have a point in common.

Let us denote by $[x, y] = Q_{\Sigma^1}(\gamma)$ and $[y', z] = Q_{\Sigma^1}(\gamma')$ the arcs of the contour of $\tau^n(W)$. The Vershik map is defined on these four points and the gluing points $y$ and $y'$ are identified in $\tau^n(W)$ as well as the gluing points $V(y)$ and $V(y')$. By Condition $[C3]$, $[V(x), V(y)]$ and $[V(y'), V(z)]$ are arcs of the contour of $\tau^n(W)$ with a common point. Assume by contradiction that, in the contour of $\tau^n(W)$, the arc $[V(x), V(y)]$ is followed by the arc $[y'', t]$ which is different from $[V(y'), V(z)]$. 

This arc turns around a tile $W_{x'}$ and if $\gamma''$ is equal to $V(f(\gamma))$ or $V(f(\gamma'))$ then it is the image by $V$ of an arc of $W_{f(\gamma)}$ or $W_{f(\gamma')}$. Then Condition (C5) implies that the arc $[x,y]$ is not followed by the arc $[y',z]$ in the contour of $\tau^m(W)$, which is a contradiction. We now assume that $\gamma'' \cdot \alpha$ has non-empty prefix and is the image by $V$ of a path $\gamma'''$. Let $t'$ be a gluing point in the tile $W_{x''}$ of $\tau^n(W)$ such that $V(t')$ is distinct from $\tau^n(y'')$. From Condition (C5) the cyclic order of $(\tau^n(x), \tau^n(z), t')$ in $\tau^n(W)$ is equal to that of $(V(\tau^n(x)), V(\tau^n(z)), V(t'))$. In particular, $\tau^n(x), \tau^n(z)$ and $t'$ lie in three different directions outgoing from $y$. From Condition (C2) there exist a point $t''$ in $\tau^n(W)$ such that $\tau^n(t'')$ lies in the same direction as $t'$. Condition (C4) implies that the cyclic order of $(x, z, t'')$ in $\tau^n(W)$ is the same as that of $(\tau^n(x), \tau^n(z), \tau^n(t''))$ which we proved to be equal to that of $(V(x), V(z), t)$. This contradicts the fact that $[x, y]$ is followed by $[y', z]$ in the contour of $\tau^n(W)$.

By contradiction we proved that the arc $[V(x), V(y)] = Q_2([V(\gamma)])$ is followed by the arc $[V(y'), z] = Q_2([\gamma'])$ in the contour of $\tau^n(W)$. \hfill \Box

The set $P_{\max}$ of infinite prefix-suffix expansions with empty suffixes is finite. We divide the circle $S^1$ into finitely many subarcs of the arcs $Q_2([\alpha])$, for $\alpha \in A$, bounded by points in $Q_2(P_{\max})$. This is a finite subdivision. According to Lemma 4.7 on each of these arcs the Vershik map $V$ is pushed-forward by $Q_2$ to a rotation. This concludes the proof of Proposition 4.6. \hfill \Box

**Example 4.8.** Consider the Tribonacci substitution $\sigma$, the tree substitution $\tau$ of Example 3.11, the contour substitution $\chi$ of Example 4.1 and the dual contour substitution $\chi^*$ of Example 4.2.

The set $P_{\max}$ of infinite prefix-suffix expansions with empty suffixes consists of the three infinite paths $\Gamma(S^{-1}(w_a)), \Gamma(S^{-1}(w_b))$ and, $\Gamma(S^{-1}(w_c))$ which appear in Example 2.3.

We now have two options: either we add these three points $Q(\Gamma(S^{-1}(w_a))), Q(\Gamma(S^{-1}(w_b)))$ and, $Q(\Gamma(S^{-1}(w_c)))$ as gluing vertices in the tree substitution or alternatively, we add these points to the contour of the circle. As the prefix-suffix expansions are periodic, using Remark 3.14 the first option can be achieved. However, in order to fix compatible cyclic orders this will require to further describe the branch points of $W_a$ which now has four gluing vertices.

The second option requires to consider the infinite prefix-suffix expansions with empty suffixes $\hat{P}_{\max} = P_{\max}(\chi^*)$ of the dual contour substitution $\chi^*$ rather than those of $\sigma$: the three infinite paths 

\[
\begin{array}{c}
\hat{a}_{12} \\
\hat{a}_{23}, \epsilon \\
\epsilon, \epsilon \\
\hat{b}_{12} \\
\hat{a}_{31}, \epsilon
\end{array}
\]
We thus introduce a new subdivision by the three points $Q_{23}\left(\tilde{P}_{\text{max}}\right)$ of the arcs $a_{12} = a_{14}a_{42}, b_{12} = b_{13}b_{31}$ and $c_{11} = c_{12}c_{21}$. We get an extended contour substitution

$$
\begin{align*}
    a_{14} & \mapsto a_{23}c_{12} \\
    a_{42} & \mapsto c_{21}b_{21} \\
    a_{23} & \mapsto b_{12}a_{31} \\
    a_{31} & \mapsto a_{14}a_{42} \\
    b_{13} & \mapsto a_{14} \\
    b_{32} & \mapsto a_{42}a_{23} \\
    b_{21} & \mapsto a_{14} \\
    c_{12} & \mapsto b_{21}b_{13} \\
    c_{21} & \mapsto b_{32}
\end{align*}
$$

From Lemma 4.7, we remark that the adjacent arcs $Q([a_{31}])$ and $Q([a_{14}])$ are rotated by the same angle as are the adjacent arcs $Q([a_{42}])$ and $Q([a_{23}])$ and, the adjacent arcs $Q([b_{21}])$ and $Q([b_{13}])$. We thus get a division of the circle into six arcs.

To describe the piecewise rotation on each of these six arcs, we give the image of some of the boundary points.

The left endpoint of the arc $a_{31}$ comes from the singular point $P = Q(w'_a)$ of $\Omega_a$. We already computed (see Examples 2.6 and 2.8) that

$$
\Gamma(w'_a) = \gamma_a, \quad V(\gamma_a) = V(\Gamma(w'_a)) = \Gamma(S(w'_a)), \quad Q(w'_a) = P, \quad Q(V(\Gamma(w'_a))) = Q(S(w'_a)) = a^{-1}P,
$$

and $a^{-1}P$ is a singularity of $\Omega_b$. Thus we get that the left endpoint of the arc $a_{31}$ is rotated to the left endpoint of the arc $b_{13}$. The right endpoint of the arc $a_{23}$ comes from the same singularity $P$ of $\Omega_a$ and is also rotated to the same point of the circle.

Similarly, we compute:

$$
\begin{align*}
    \Gamma(w'_b) & = \gamma_b, \quad V(\gamma_b) = V(\Gamma(w'_b)) = \Gamma(S(w'_b)), \quad Q(w'_b) = P, \\
    Q(V(\Gamma(w'_b))) & = Q(S(w'_b)) = b^{-1}P, \\
    \Gamma(w'_c) & = \gamma_c, \quad V(\gamma_c) = V(\Gamma(w'_c)) = \Gamma(S(w'_c)), \quad Q(w'_c) = P, \\
    Q(V(\Gamma(w'_c))) & = Q(S(w'_c)) = c^{-1}P.
\end{align*}
$$

Thus the left endpoint of the arc $b_{21}$ and the right endpoint of the arc $b_{32}$ are both rotated (by different rotations) to the left endpoint of $a_{23}$. And, the left endpoint of $c_{12}$ and the right endpoint of $c_{21}$ are both rotated (by different rotations) to the point which is the left endpoint of $a_{14}$.

The piecewise rotation of the circle for Tribonacci substitution is pictured in Figure 11. Notice that this is the Arnoux-Yoccoz interval exchange on the six intervals $a_{31}a_{14}, a_{42}a_{23}, c_{12}, c_{21}, b_{21}b_{13}$ and $b_{32}$.

The suspension of this piecewise rotation of the circle with six pieces is the surface of genus 3 with one boundary component that we referred to at the end of Example 4.2.

5. Examples

We survey in this Section two other examples.

5.1. First example. We consider the substitution

$$
\begin{align*}
    \sigma : \quad a & \mapsto ac \\
    b & \mapsto ab \\
    c & \mapsto b
\end{align*}
$$

which is an iwip automorphism of the free group $F_{\{a,b,c\}}$. As before we denote by $X_\sigma$ its attracting shift. The topological representative of $\sigma$ on the rose with three
petals is an irreducible train-track and has three periodic Nielsen paths (of period 4):

Thus $X_\sigma$ contains four singular leaves:

$$
\cdots acbabacbabacab \quad ababacbabacab \cdots : Z_1
$$

$$
\cdots acbabacbabacab \quad bacabacbabacab \cdots : Z_2
$$

$$
\cdots acbabacbabacab \quad baecabacbabacab \cdots : Z_3
$$

$$
\cdots acbabacbabacab \quad cabacbabacab \cdots : Z_4
$$

$Z_5 : \cdots acbabacbabacab \quad \leftarrow bacaacacbabacab \cdots$

$Z_6 : \cdots acbabacbabacab \quad \leftarrow cacaacacbabacab \cdots$

$Z_7 : \cdots acbabacbabacab \quad \leftarrow abbabacacbabacab \cdots$

$Z_8 : \cdots acbabacbabacabacab \quad \leftarrow bababacacbabacab \cdots$

An easy calculation shows that the index is maximal, thus $\sigma$ is parageometric.

The map $Q$ identifies these left and right special infinite words, defining the points

$$
P_1 = Q(Z_1) = Q(Z_2), \quad P_2 = Q(Z_3) = Q(Z_4),
$$

$$
P_3 = Q(Z_5) = Q(Z_6), \quad P_4 = Q(Z_7) = Q(Z_8).
$$

The infinite desubstitution paths of the singular bi-infinite words are:
Singular points in $\Omega_a$ are given by infinite tails ending at occurrences of the letter $a$ as a state in the prefix-suffix expansion of the above singular prefix-suffix expansions. However, for $Z_5, Z_6$ and $Z_7, Z_8$ we only need to consider desubstitution paths with different first vertex, and thus $P_1$ is not a singular point of $\Omega_b$ and, $P_4$ is not a singular point of $\Omega_a$. Thus, for $\Omega_a$ we have to consider: $\Gamma(Z_1)$, $B^2(\Gamma(Z_6))$ and, $B^3(\Gamma(Z_5))$ and we get

$$V_a = \text{Sing}(\Omega_a) = \{P_1, aP_3, acP_3\}.$$ 

Similarly, singular points in $\Omega_b$ and $\Omega_c$ are given by occurrences of the state $b$ in desubstitution paths.

$$V_b = \text{Sing}(\Omega_b) = \{P_1, P_2, bP_4\}, \quad V_c = \text{Sing}(\Omega_c) = \{P_2, cP_3, cP_4\}.$$ 

Finally, to understand the tree substitution we need to know the images of the singular points under the map $\tau$. Recall that the image by $\tau$ of a singular point given by a singular prefix-suffix expansion $\gamma$ is given by the beheaded prefix-suffix expansion $B(\gamma)$ in the copy of the prototile corresponding to the heading edge $e_1(\gamma)$ of $\gamma$.

Using the map $Q$ we also get descriptions of the singular points in the repelling tree as well as their images by the expanding homothety $H^{-1}$. We sum-up our calculations in the following table:

|      | $V_a$        | $V_b$        | $V_c$        |
|------|--------------|--------------|--------------|
|      | $1$          | $2$          | $3$          | $1$          | $2$          | $3$          |
| $\gamma$ | $B^2(\Gamma(Z_6))$ | $\Gamma(Z_1)$ | $B^4(\Gamma(Z_5))$ | $\Gamma(Z_2)$ | $B(\Gamma(Z_5))$ | $\Gamma(Z_3)$ |
| $Q(\gamma)$ | $aP_3$ | $P_1$ | $acP_3$ | $P_1$ | $bP_4$ | $P_2$ |
| $e_1(\gamma)$ | $a \overset{c}{\leftrightarrow} b$ | $a \overset{c}{\leftrightarrow} b$ | $b \overset{c}{\leftrightarrow} c$ | $b \overset{c}{\leftrightarrow} c$ | $b \overset{a}{\leftrightarrow} b$ |
| $B(\gamma)$ | $B(\Gamma(Z_6))$ | $\Gamma(Z_1)$ | $B^4(\Gamma(Z_5))$ | $\Gamma(Z_2)$ | $B^2(\Gamma(Z_5))$ | $\Gamma(Z_3)$ |
| $\tau(Q(\gamma))$ | $bP_4$ | $P_2$ | $cP_3$ | $P_2$ | $cP_3$ | $cb^{-1}P_1$ |

We get the abstract tree substitution depicted in Figure 12:

$$\tau : W_a \mapsto (W_a \sqcup W_b)/\sim$$

$$W_b \mapsto (W_b \sqcup W_c)/\sim$$

$$W_c \mapsto W_a$$

From the pairs of singular prefix-suffix expansions we get that the gluing points $\Gamma(Z_1)$ in $W_a$ and $\Gamma(Z_2)$ in $W_b$ are identified in $W$ as well as the gluing points $\Gamma(Z_3)$ in $W_b$ and $\Gamma(Z_4)$ in $W_c$. The same gluing points are identified in $\tau(W_a)$ and $\tau(W_b)$. 

![Diagram](image_url)
We proceed to get a piecewise rotation of the contour circle. We fix the cyclic orders given by the embeddings in the contracting plane oriented clockwise. The contour substitution $\chi$ and its dual are

$$
\begin{align*}
    \chi &: a_{13} \mapsto b_{21}a_{21}, & \chi^* &: a_{13} \mapsto a_{32}c_{32} \\
    a_{21} & \mapsto b_{32}, & a_{21} & \mapsto a_{13}c_{13} \\
    a_{32} & \mapsto a_{13}a_{32}b_{13}, & a_{32} & \mapsto a_{32}c_{21} \\
    b_{13} & \mapsto c_{13}b_{21}, & b_{13} & \mapsto a_{32}b_{32} \\
    b_{21} & \mapsto c_{21}, & b_{21} & \mapsto a_{13}b_{13} \\
    b_{32} & \mapsto b_{13}b_{12}c_{32}, & b_{32} & \mapsto a_{21}b_{32} \\
    c_{13} & \mapsto a_{21}, & c_{13} & \mapsto b_{13} \\
    c_{21} & \mapsto a_{32}, & c_{21} & \mapsto b_{21} \\
    c_{32} & \mapsto a_{13}, & c_{32} & \mapsto b_{32}
\end{align*}
$$

There is a unique infinite prefix-suffix expansion with empty suffix both for $\sigma$ and $\chi^*$:

$$
\begin{align*}
    b & \mapsto a, \epsilon \\
    b_{32} & \mapsto a_{21}, \epsilon
\end{align*}
$$

Thus we subdivide the arc $b_{32}$ into two consecutive arcs $b_{34}$ and $b_{42}$. Using the Vershik map, we compute the images of the arcs $Q([\gamma])$ for finite prefix-suffix expansion paths with non-empty suffix. We observe that the consecutive arcs $a_{13}$ and $a_{32}$ are rotated by the same angle and, similarly the consecutive arcs $b_{12}$ and $b_{21}$. We get the piecewise rotation of the circle with eight pieces described in Figure 14.
Figure 14. Piecewise rotation of the circle for the substitution $a \mapsto ac, b \mapsto ab, c \mapsto b$. The innermost circle is the contour while the outermost circle is the result of the piecewise exchange.

The tree substitution fails to depict a tree inside the contracting plane, as illustrated in Figure 15. Thus we also need in this example to prune the tree substitution (see Section 3.6) to get a tree inside the contracting plane. After some iterations we get the tree pictured in Figure 16.

These two edges are not adjacent in the abstract tree.

Figure 15. Projection of the tree $\tau^4(W)$ for the tree substitution of Figure 12. This is not a tree inside the contracting plane.

5.2. Second example. We consider the following substitution which is an iwip automorphism of the free group $F_{\{a,b,c\}}$:

$$
\sigma : \\
\begin{align*}
a & \mapsto abc \\
b & \mapsto bcabc \\
c & \mapsto cbabc
\end{align*}
$$

This example comes from a family of irreducible substitutions which are parageometric iwip automorphisms studied by Leroy [Ler14].

As the computations are more complicated we will not detail them here.

There are six attracting infinite fixed words (three positive, one negative and two more coming from two indivisible Nielsen paths). Computing the singular pairs of
prefix-suffix expansions and their tails we get the prototiles of the tree substitution and their images described in Figure 17.

\[
\begin{align*}
1 & \mapsto \rightarrow 2 \\
3 & \mapsto \rightarrow 2
\end{align*}
\]

In the abstract tree substitution the adjacency inside the circles should rather be:

\[
\begin{align*}
1 & \rightarrow \rightarrow 2 \\
3 & \rightarrow \rightarrow 2
\end{align*}
\]

Figure 16. Projection in the contracting plane of the eighth iterate of the pruned tree substitution for \( \sigma : a \mapsto ac, b \mapsto ab, c \mapsto b \).

Figure 17. Tree substitution for the substitution \( a \mapsto abc, b \mapsto \text{bcabc}, c \mapsto \text{cabc} \). The images of the prototiles \( \tau(W_b) \) and \( \tau(W_c) \) in the contracting plane are not trees.

We remark that the images \( \tau(W_b) \) and \( \tau(W_c) \) are not trees when embedded in the contracting plane. Thus we turn to the pruned tree substitution as before. But again, as shown in Figure 18 already the first image \( \tau(W) \) is not a tree.

We thus need to take a higher cover. This cover is provided by the observation that the dual substitution satisfies Condition 2 of Question 2.21. Thus we can
Figure 18. Projection in the contracting plane of the first iterate of the pruned tree substitution of $a \mapsto abc$, $b \mapsto bcabc$, $c \mapsto cbabc$. In the abstract tree, the red and blue patches do not touch inside the circle.

Figure 19. Tree substitution obtained by adjacency of tiles of the dual substitution for the substitution $a \mapsto abc$, $b \mapsto bcabc$, $c \mapsto cbabc$. The black dot indicates the position of the origin.

Figure 20. Third iterate of the tree substitution obtained by using Condition 2 of Question 2.21 for the substitution $a \mapsto abc$, $b \mapsto bcabc$, $c \mapsto cbabc$.

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