POSITIVITY PRESERVING ALONG A FLOW OVER PROJECTIVE BUNDLE

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ABSTRACT. In this paper, we introduce a flow over the projective bundle $p : P(E^*) \rightarrow M$, which is a natural generalization of both Hermitian-Yang-Mills flow and Kähler-Ricci flow. We prove that the semipositivity of curvature of the hyperplane line bundle $\mathcal{O}_{P(E^*)}(1)$ is preserved along this flow under the null eigenvector assumption (see Theorem 1.9). As applications, we prove that the semipositivity is preserved along the this flow if the base manifold $M$ is a curve, which implies that the Griffiths semipositivity is preserved along the Hermitian-Yang-Mills flow over a curve. And we also reprove that the nonnegativity of holomorphic bisectional curvature is preserved under Kähler-Ricci flow.

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INTRODUCTION

In the celebrated paper [28], Siu and Yau presented a differential geometric proof of the famous Frankel conjecture in Kähler geometry, which states that a compact Kähler manifold $M^n$ with positive holomorphic bisectional curvature must be biholomorphic to a complex projective space $\mathbb{P}^n$. For a compact Kähler manifold with nonnegative holomorphic bisectional curvature, Mok [24] proved a generalized Frankel conjecture and obtained a uniformization theorem, and H. Gu [16] gave a new proof of Mok’s uniformization theorem. They both used the Kähler-Ricci flow and considered the variation of holomorphic bisectional curvature along this flow. Especially, the nonnegativity of holomorphic bisectional curvature is preserved under
Kähler-Ricci flow [24, Proposition 1.1]. Later on, there are many references about studying and generalizing the Frankel conjecture by using Kähler-Ricci flow, include [7, 8, 17, 27].

In 1979, Mori [25] proved the famous Hartshorne’s conjecture. In case the ground field is \( \mathbb{C} \), a compact complex manifold \( M \) was proved to biholomorphic to \( \mathbb{P}^n \) if its tangent bundle is ample, which implies the Frankel conjecture. A holomorphic vector bundle \( E \) is ample in the sense of Hartshorne if and only if the hyperplane line bundle \( \mathcal{O}_{\mathbb{P}(E^*^0)}(1) \) is a positive line bundle over \( P(E^*) \) (see [18, Proposition 3.2]), i.e., there is a positive curvature metric on \( \mathcal{O}_{\mathbb{P}(E^*)}(1) \). If \((E, h)\) is a Hermitian vector bundle with Griffiths positive curvature (see Definition 1.7), then \( E \) is an ample vector bundle. In [15], Griffiths conjectured its converse also holds, namely \( E \) can admit a Hermitian metric with Griffiths positive curvature if \( E \) is ample. Both Mori’s theorem and Griffiths conjecture will be proved if one can deform the given positive curvature metric to another “better” metric with positive curvature, for example, the Kähler metric with positive holomorphic bisectional curvature for Mori’s theorem, and Hermitian metric with Griffiths positive curvature for Griffiths conjecture. Naturally, one wants to define a certain flow over the projective bundle \( \mathbb{P}(E^*) \), such that the positivity of the curvature of \( \mathcal{O}_{\mathbb{P}(E^*)}(1) \) is preserved under this flow. This is also the motivation for the author to study the positivity preserving along the flow (0.1) over projective bundle.

Let \( M \) be a compact complex manifold of dimension \( n \), and \( \pi : E \to M \) be a holomorphic vector bundle of rank \( r \) over \( M \). Let \( E^* \) denote the dual bundle of \( E \), and \( p : P(E^*) := (E^*)^0/\mathbb{C}^* \to M \) denote the projective bundle, where \((E^*)^0\) denotes the set of all the nonzero elements in \( E^* \). For any strongly pseudoconvex complex Finsler metric \( G \) on \( E^* \) (see Definition 1.1), there exists the following canonical decomposition (see Remark 1.5, (3))

\[
\sqrt{-1} \partial \bar{\partial} \log G = -\Psi + \omega_{FS},
\]

which is a curvature form of \( \mathcal{O}_{\mathbb{P}(E^*)}(1) \) and represents the first Chern class \( 2\pi c_1(\mathcal{O}_{\mathbb{P}(E^*)}(1)) \). Here \( \Psi \) is called the Kobayashi curvature (see [12, Definition 1.2]), which is given in (1.4), and \( \omega_{FS} \) is a positive \((1,1)\)-form along each fiber of \( p : P(E^*) \to M \), which is defined in (1.10). According to this decomposition, Kobayashi [19] gave a characterization of ample vector bundle, i.e., \( E \) is ample if and only if there exists a strongly pseudoconvex complex Finsler metric on \( E^* \) such that \( \Psi < 0 \).

Let \( \omega(G) = p^*\omega \), where \( \omega = \sqrt{-1} g_{\alpha\beta}(G) dz^\alpha \wedge d\bar{z}^\beta \) is a Kähler metric on \( M \) depending smoothly on a Finsler metric \( G \). One can define a Hermitian metric on \( P(E^*) \) by

\[
\Omega := \omega(G) + \omega_{FS}.
\]

Let \( G_0 \) be a strongly pseudoconvex complex Finsler metric on \( E^* \), we consider the following flow over the projective bundle \( P(E^*) \):

\[
\begin{cases}
\frac{\partial}{\partial t} \log G = \Delta_\Omega \log G, \\
\omega_{FS} > 0,
\end{cases}
\]

(0.1) \[ G(0) = G_0. \]

Here \( \Delta_\Omega := \sqrt{-1} \Lambda \partial \bar{\partial} \), \( \Lambda \) is the adjoint operator of \( \Omega \wedge \cdot \).
One can also define a horizontal and real $(1,1)$-form $T$ on $P(E^*)$ as follow,

\[(0.2)\quad (-\sqrt{-1})T(u, \bar{u}) := \langle R^g(u, \bar{u}), -\Psi \rangle_\Omega - |i_u \partial^V \Psi|^2_\Omega\]

for any horizontal vector $u = u^\alpha \frac{\delta}{\delta z^\alpha}$, where $\langle R^g(u, \bar{u}), -\Psi \rangle_\Omega := (-\Psi)_{\alpha \delta} g^{\alpha \beta} g^{\gamma \delta} R^g_{\gamma \beta \alpha \bar{\sigma}} u^\sigma \bar{u}^\bar{\sigma}$ and $|i_u \partial^V \Psi|^2_\Omega := (\log G)^{\alpha \beta} \partial_{\alpha} \Psi_{\alpha \beta} \partial_{\bar{\gamma}} \bar{\Psi}_{\alpha \beta} u^\delta \bar{u}^\bar{\sigma}$. $R^g$ denotes the Chern curvature of $\omega$. Now we assume that $T$ satisfies the \textit{null eigenvector assumption} (see Theorem 1.9), namely $(-\sqrt{-1})T(U, \bar{U}) \geq 0$ whenever $\partial \bar{\partial} \log G \geq 0$ and $i_U(\partial \bar{\partial} \log G) = 0$ for a $(1,0)$-type vector $U$ of $TP(E^*)$. By using the maximum principle for real $(1,1)$-forms, we obtain

**Theorem 0.1.** Let $\pi : (E^*, G_0) \to M$ be a holomorphic Finsler vector bundle over the compact complex manifold $M$ with $\sqrt{-1} \partial \bar{\partial} \log G_0 \geq 0$. If the horizontal $(1,1)$-form $T$ satisfies the \textit{null eigenvector assumption}, then $\sqrt{-1} \partial \bar{\partial} \log G(t) \geq 0$ along the flow (0.1) for all $t \geq 0$ such that the solution exists.

In this paper, we will give two applications of Theorem 0.1.

For the first application, we consider the case of curve, i.e. $\dim M = 1$. In this case, any Hermitian metric on $M$ is Kähler automatically, and one can prove that the $(1,1)$-form $T$ satisfies the null eigenvector assumption. By Theorem 0.1 we obtain

**Proposition 0.2.** If $M$ is a curve, then the semipositivity of the curvature of $\mathcal{O}_{P(E^*)}(1)$ is preserved along the flow (0.1).

In particular, if $G_0 = h_0^{ij} v_i \bar{v}_j$ comes from a Hermitian metric $(h_0^{ij})$ of $E^*$ and

\[(0.3)\quad \Omega = p^* \omega + \omega_{FS}\]

for a fixed Hermitian metric $\omega$, by Remark 2.1 (1), (0.1) is equivalent to the following Hermitian-Yang-Mills flow:

\[
\begin{cases}
    h^{-1} \cdot \frac{\partial h}{\partial t} + \Lambda R^h + (r - 1) I = 0 \\
    (h^{ij}(t)) > 0 \\
    h^{ij}(0) = (h_0^{ij}).
\end{cases}
\]

By Proposition 0.2 and Definition 1.7 we have

**Corollary 0.3.** If $M$ is a curve, then the Griffiths semipositivity is preserved along the Hermitian-Yang-Mills flow (0.4).

For the second application, we assume that $E = T^*M$ and take

\[
\omega(G) = \sqrt{-1} g_{\alpha \beta} dz^\alpha \wedge d\bar{z}^\beta,
\]

where $(g_{\alpha \beta})$ denotes the inverse of the matrix $\left(\frac{\partial^2 G}{\partial v_\alpha \partial \bar{v}_\beta}\right)$. Let $G_0 = g_0^{\alpha \bar{\beta}} v_\alpha \bar{v}_\beta$ be the strongly pseudoconvex complex Finsler metric on $T^*M$ induced by the following Kähler metric

\[
\omega_0 = \sqrt{-1} (g_0)^{\alpha \bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta.
\]
Then the flow (0.1) is equivalent to the following Kähler-Ricci flow

\[
\begin{aligned}
\frac{\partial \omega}{\partial t} + \text{Ric}(\omega) + (n-1)\omega &= 0, \\
\omega &> 0, \\
\omega(0) &= \omega_0.
\end{aligned}
\]

(0.5)

The solution of (0.1) is induced from the Kähler metric \(\omega = \sqrt{-1}g_{\alpha\beta}dz^\alpha \wedge d\bar{z}^\beta\). In this case, the \((1,1)\)-form \(T\) can also be proved satisfying the null eigenvector assumption (see [3, Page 254, Claim 2.2]). From Theorem 0.1 and Definition 1.7, we can reprove the following Mok’s proposition, which is contained in [24, Proposition 1.1] (see also [3, Theorem 5.2.10]).

**Proposition 0.4 ([24, Proposition 1.1]).** If \((M, \omega_0)\) is a compact Kähler manifold with nonnegative holomorphic bisectional curvature, then the nonnegativity is preserved along the Kähler-Ricci flow (0.5).

**Remark 0.5.** By (0.4) and (0.5), the flow (0.1) is a natural generalization of both Hermitian-Yang-Mills flow and Kähler-Ricci flow. And there are some other flows, which are also the generalization of the Kähler-Ricci flow. For example, Gill [14] introduced the Chern-Ricci flow on Hermitian manifolds, and many properties of the flow were established in [30, 31]. Especially, Yang [35] proved the nonnegativity of the holomorphic bisectional curvature is not necessarily preserved under the Chern-Ricci flow. In [29], Streets and Tian introduced the Hermitian curvature flow, proved short time existence for this flow, and derive basic long time blowup and regularity results. For a particular version of the Hermitian curvature flow, Ustinovskiy [32] proved the the property of Griffiths positive (nonnegative) Chern curvature is preserved along this flow.

This article is organized as follows. In Section 1, we shall fix the notation and recall some basic definitions and facts on complex Finsler vector bundles, Griffiths positive (semipositive), and maximal principle for real \((1,1)\)-forms. For more details we refer to [2, 4, 9, 10, 12, 13, 15, 19, 21, 23, 33].

1. **Preliminaries**

In this section, we shall fix the notation and recall some basic definitions and facts on complex Finsler vector bundles, Griffiths positive (semipositive), and maximal principle for real \((1,1)\)-forms. For more details we refer to [2, 4, 9, 10, 12, 13, 15, 19, 21, 23, 33].

1.1. **Complex Finsler vector bundle.** Let \(M\) be a compact complex manifold of dimension \(n\), and let \(\pi : E \to M\) be a holomorphic vector bundle of rank \(r\) over \(M\). Let \(z = (z^1, \cdots, z^n)\)
be a local coordinate system in $M$, and $\{e_i\}_{1 \leq i \leq r}$ be a local holomorphic frame of $E$. With respect to the local frame of $E$, an element of $E$ can be written as

$$v = v^i e_i \in E,$$

where we adopt the summation convention of Einstein. In this way, one gets a local coordinate system of the complex manifold $E$:

$$(1.2) \quad (z; v) = (z^1, \ldots, z^n; v^1, \ldots, v^r).$$

**Definition 1.1 ([19])**. A Finsler metric $G$ on the holomorphic vector bundle $E$ is a continuous function $G : E \rightarrow \mathbb{R}$ satisfying the following conditions:

- **F1)**: $G$ is smooth on $E^o = E \setminus O$, where $O$ denotes the zero section of $E$;
- **F2)**: $G(z, v) \geq 0$ for all $(z, v) \in E$ with $z \in M$ and $v \in \pi^{-1}(z)$, and $G(z, v) = 0$ if and only if $v = 0$;
- **F3)**: $G(z, \lambda v) = |\lambda|^2 G(z, v)$ for all $\lambda \in \mathbb{C}$.

Moreover, $G$ is called strongly pseudoconvex if

- **F4)**: the Levi form $\sqrt{-1} \partial \bar{\partial} G$ on $E^o$ is positive-definite along each fiber $E_z = \pi^{-1}(z)$ for $z \in M$.

Clearly, any Hermitian metric on $E$ is naturally a strongly pseudoconvex complex Finsler metric on it.

We write

$$G_i = \partial G/\partial v^i, \quad G_j = \partial G/\partial \bar{v}^j, \quad G_{ij} = \partial^2 G/\partial v^i \partial \bar{v}^j, \quad G_{ia} = \partial^2 G/\partial v^i \partial z^a, \quad G_{ij\bar{\beta}} = \partial^2 G/\partial v^i \partial \bar{v}^j \partial \bar{z}^\beta, \quad \text{etc.},$$

to denote the differentiation with respect to $v^i, \bar{v}^j$ ($1 \leq i, j \leq r$), $z^a, \bar{z}^\beta$ ($1 \leq \alpha, \beta \leq n$). In the following lemma we collect some useful identities related to a Finsler metric $G$.

**Lemma 1.2 ([10] [19]).** The following identities hold for any $(z, v) \in E^o$, $\lambda \in \mathbb{C}$:

$$G_i(z, \lambda v) = \bar{\lambda} G_i(z, v), \quad G_{ij}(z, \lambda v) = G_{ij}(z, v) = G_{ij}(z, v);$$

$$G_i(z, v)v^i = G_j(z, v)\bar{v}^j = G_{ij}(z, v)v^i\bar{v}^j = G(z, v);$$

$$G_{ij}(z, v)v^i = G_{ijk}(z, v)v^j = G_{ijk}(z, v)\bar{v}^j = 0.$$

If $G$ is a strongly pseudoconvex complex Finsler metric on $M$, then there is a canonical h-v decomposition of the holomorphic tangent bundle $TE^o$ of $E^o$ (see [10, §5] or [12, §1]).

$$TE^o = \mathcal{H} \oplus \mathcal{V}.$$ 

In terms of local coordinates,

$$\mathcal{H} = \text{span}_\mathbb{C} \left\{ \frac{\delta}{\delta z^\alpha} = \frac{\partial}{\partial z^\alpha} - G_{\alpha j} G^{jk} \frac{\partial}{\partial v^k}, 1 \leq \alpha \leq n \right\}, \quad \mathcal{V} = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial v^i}, 1 \leq i \leq r \right\}.$$

The dual bundle $T^*E^o$ also has a smooth h-v decomposition $T^*E^o = \mathcal{H}^* \oplus \mathcal{V}^*$:

$$(1.2) \quad \mathcal{H}^* = \text{span}_\mathbb{C} \{ dz^\alpha, 1 \leq \alpha \leq n \}, \quad \mathcal{V}^* = \text{span}_\mathbb{C} \{ \delta v^i = dv^i + G^{ji} G_{\alpha j} dz^\alpha, 1 \leq i \leq r \}.$$
Moreover, the differential operators

\[ \partial V = \frac{\partial}{\partial v^i} \otimes \delta v^i, \quad \partial H = \frac{\delta}{\delta z^\alpha} \otimes dz^\alpha. \]

are well-defined.

With respect to the h-v decomposition \[1.2\], the (1,1)-form \( \sqrt{-1} \partial \bar{\partial} \log G \) has the following decomposition. For readers' convenience, we give a proof of the following lemma due to Kobayashi and Aikou (cf. \[19, 4\]).

**Lemma 1.3** \([19, 4]\). Let \( G \) be a strongly pseudoconvex complex Finsler metric on \( E \). One has

\[ \sqrt{-1} \partial \bar{\partial} \log G = -\Psi + \omega_V, \]

where \( \Psi \) is called the Kobayashi curvature (see \[12\] Definition 1.2),

\[ \Psi = \sqrt{-1} R_{i\bar{j}a\bar{b}} v^i \bar{v}^j G d\alpha^a \wedge d\bar{\beta}^b, \quad \omega_V = \sqrt{-1} \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} \delta v^i \wedge \delta \bar{v}^j, \]

with

\[ R_{i\bar{j}a\bar{b}} = - \frac{\partial^2 G_{ij}}{\partial z^a \partial \bar{z}^\beta} + G^{jk} \frac{\partial G_{ij}}{\partial z^a} \frac{\partial G_{k\bar{j}}}{\partial \bar{z}^\beta}. \]

**Proof.** By \[1.2\], we have

\[ \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} \delta v^i \wedge \delta \bar{v}^j = \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} (dv^i + G_{\alpha i} G_{\bar{i}}^\beta dz^\alpha) \wedge (d\bar{v}^j + G_{\bar{\beta} j} G_{\bar{\beta}}^k dz^\bar{\beta}) \]

\[ = \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} dv^i \wedge d\bar{v}^j + \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} G_{\bar{i} j} G_{\bar{i}}^j dv^i \wedge dz^\beta \]

\[ + \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} G_{\alpha i} G_{\bar{i}}^\beta dz^\alpha \wedge d\bar{v}^j + \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} G_{\alpha i} G_{\bar{i}}^j G_{\bar{\beta} j} G_{\bar{\beta}}^k dz^\alpha \wedge dz^\beta. \]

For the last three terms in the RHS of \[1.5\], one has from Lemma \[1.2\]

\[ \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} G_{\bar{i} j} G_{\bar{i}}^j dv^i \wedge dz^\beta = \frac{GG_{\bar{i} j} - G_{\bar{i}} G_{\bar{i}}^j}{G^2} G_{\bar{i} j} G_{\bar{i}}^j dv^i \wedge dz^\beta = \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} dv^i \wedge d\bar{\alpha}, \]

\[ \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} G_{\alpha i} G_{\bar{i}}^j dz^\alpha \wedge d\bar{v}^j = \frac{GG_{\alpha i} - G_{\alpha} G_{\alpha}^j}{G^2} G_{\alpha i} G_{\bar{i}}^j dz^\alpha \wedge d\bar{v}^j = \frac{\partial^2 \log G}{\partial z^\alpha \partial \bar{v}^j} dz^\alpha \wedge d\bar{v}^j \]

and

\[ \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} G_{\alpha i} G_{\bar{i}}^j G_{\bar{\beta} j} G_{\bar{\beta}}^k dz^\alpha \wedge dz^\beta = \frac{GG_{\alpha i} - G_{\alpha} G_{\alpha}^j}{G^2} G_{\alpha i} G_{\bar{i}}^j G_{\bar{\beta} j} G_{\bar{\beta}}^k dz^\alpha \wedge dz^\beta \]

\[ = \frac{1}{G^2} (GG_{\alpha i} G_{\bar{i}}^j G_{\bar{\beta} j} G_{\bar{\beta}}^k dz^\alpha \wedge dz^\beta. \]
Submitting (1.6), (1.7) and (1.8) into (1.5), we obtain

$$\frac{\partial^2 \log G}{\partial v^i \partial v^j} \delta v^i \wedge \delta \bar{v}^j = \frac{\partial^2 \log G}{\partial v^i \partial v^j} dv^i \wedge dv^j + \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} dv^i \wedge d\bar{z}^j$$

$$+ \frac{\partial^2 \log G}{\partial z^a \partial \bar{v}^j} dz^a \wedge dv^j + \frac{1}{G^2} (G G_{\alpha i} G_{k \bar{\beta}} - G_{\alpha \bar{\beta}}) dz^a \wedge d\bar{z}^j$$

$$= \partial \bar{\partial} \log G + \frac{1}{G} (G G_{\alpha i} G_{k \bar{\beta}} - G_{\alpha \bar{\beta}}) \frac{\partial v^i}{\partial v^j} \bar{v}^j dz^a \wedge d\bar{z}^j$$

$$= \partial \bar{\partial} \log G + (G G_{\alpha i} G_{k \bar{\beta}} - G_{i \bar{j} \alpha \bar{\beta}}) \frac{\partial v^i}{\partial v^j} \bar{v}^j dz^a \wedge d\bar{z}^j$$

$$= \partial \bar{\partial} \log G - \sqrt{-1} \Psi,$$

which completes the proof.

Let $q$ denote the natural projection

$$q : E^o \rightarrow P(E) := E^o/\mathbb{C}^* \quad (z; v) \mapsto (z; [v]) := (z^1, \ldots, z^n; [v^1, \ldots, v^n]),$$

which gives a local coordinate system of $P(E)$ by

$$(1.9) \quad (z; w) := (z^1, \ldots, z^n; w^1, \ldots, w^{r-1}) = \left( z^1, \ldots, z^n; \frac{v^1}{v^k}, \ldots, \frac{v^{k-1}}{v^k}, \frac{v^{k+1}}{v^k}, \ldots, \frac{v^n}{v^k} \right)$$

on $U_k := \{(z, [v]) \in P(E), v^k \neq 0\}$.

Denote by $((\log G)^{ab})_{1 \leq a, b \leq r-1}$ the inverse of the matrix $((\log G)_{ab} := \frac{\partial^2 \log G}{\partial w^a \partial \bar{w}^b})_{1 \leq a, b \leq r-1}$ and set

$$\delta w^a = dw^a + (\log G)_{ab} (\log G)_{ba} \bar{w}^b dz^a.$$

One can define a vertical $(1, 1)$-form on $P(E)$ by

$$(1.10) \quad \omega_{FS} := -\sqrt{-1} \frac{\partial^2 \log G}{\partial w^a \partial \bar{w}^b} \delta w^a \wedge \delta \bar{w}^b,$$

which is well-defined (see e.g. [13, Section 1]). Moreover,

**Lemma 1.4.** By the pullback $q^* : A^{1,1}(P(E)) \rightarrow A^{1,1}(E^o)$, one has

$$(1.11) \quad q^* \omega_{FS} = \omega_V.$$

**Proof.** We only need to prove (1.11) at one point $(z, [v]) \in U_k$. Without loss of generality, we assume that $k = r$. For any point $(z, [v]) \in U_r$, one has from (1.9)

$$(1.12) \quad q_* \left( \frac{\partial}{\partial v^r} \right) = -\sum_{a=1}^{r-1} \frac{v^a}{(v^r)^2} \frac{\partial}{\partial w^a}, \quad q_* \left( \frac{\partial}{\partial v^b} \right) = \frac{1}{v^r} \frac{\partial}{\partial w^b}, \quad 1 \leq b \leq r-1.$$
By (1.12), one has
\[
\frac{\partial^2 \log G}{\partial w^a \partial \bar{w}^b} = (\partial \bar{\partial} \log G) \left( \frac{\partial}{\partial w^a}, \frac{\partial}{\partial \bar{w}^b} \right)
\]
(1.13)
\[
= (\partial \bar{\partial} \log G) \left( q_\alpha (v^r \frac{\partial}{\partial v^\alpha}), q_\beta (\bar{v}^r \frac{\partial}{\partial \bar{v}^\beta}) \right)
\]
\[
= q^* (\partial \bar{\partial} \log G) (v^r \frac{\partial}{\partial v^\alpha}, \bar{v}^r \frac{\partial}{\partial \bar{v}^\beta})
\]
\[
= |v^r|^2 \frac{\partial^2 \log G}{\partial \alpha \partial \bar{\beta}}.
\]
Similarly,
(1.14)
\[
\frac{\partial^2 \log G}{\partial z^\alpha \partial \bar{w}^b} = v^r \frac{\partial^2 \log G}{\partial z^\alpha \partial \bar{v}^b}, \quad \frac{\partial^2 \log G}{\partial w^\alpha \partial \bar{z}^\beta} = v^r \frac{\partial^2 \log G}{\partial v^\alpha \partial \bar{z}^\beta}
\]
and
(1.15)
\[
\frac{\partial^2 \log G}{\partial v^\alpha \partial \bar{v}^r} = -\frac{1}{|v^r|^2} \bar{v}^b \frac{\partial^2 \log G}{\partial v^\alpha \partial \bar{w}^b}, \quad \frac{\partial^2 \log G}{\partial v^r \partial \bar{v}^r} = -\frac{1}{|v^r|^2} \bar{v}^a \frac{\partial^2 \log G}{\partial v^a \partial \bar{v}^r}, \quad \frac{\partial^2 \log G}{\partial w^\alpha \partial \bar{v}^r} = |v^r|^2 \frac{\partial^2 \log G}{\partial w^\alpha \partial \bar{w}^b}.
\]
By a direct checking, one has
(1.16)
\[
\log G \bar{G}^{ba} = \frac{G}{|v^r|^2} \left( -\frac{v^a}{v^r} G \bar{G}^{br} + \bar{G}^{ba} \frac{\bar{v}^b}{|v^r|^2} \bar{G}^{br} - \frac{\bar{v}^b}{v^r} \bar{G}^{br} \right).
\]
By (1.14) and (1.16), we have
\[
\log G_{a\bar{\beta}} (\log G) \bar{G}^{ba} = \left( \frac{1}{v^r} G \bar{G}^{br} - \frac{\bar{v}^b}{|v^r|^2} \bar{G}^{br} \right) (G_{i\bar{\beta}} - \bar{G}^{ia} \bar{G}^{ia}) = \frac{1}{v^r} G \bar{G}^{bi} G_{i\bar{\beta}} - \bar{G}^{bi} \bar{G}^{i\bar{\beta}}.
\]
So
(1.17)
\[
q^* (\delta \bar{w}^b) = q^* (d \bar{w}^b + (\log G)_{a\bar{\beta}} (\log G) \bar{G}^{ba} d \bar{z}^\beta)
\]
\[
= \frac{1}{v^r} (d \bar{w}^b + \bar{G}^{bi} G_{i\bar{\beta}} d \bar{z}^\beta) - \frac{\bar{v}^b}{|v^r|^2} (d \bar{v}^r + \bar{G}^{i\bar{\beta}} G_{i\bar{\beta}} d \bar{z}^\beta)
\]
\[
= \frac{1}{v^r} \delta \bar{w}^b - \frac{\bar{v}^b}{|v^r|^2} \delta \bar{v}^r.
\]
From (1.19), (1.15) and (1.17), we obtain
\[
q^* \omega_{FS} = q^* \left( -\frac{1}{v^r} \frac{\partial^2 \log G}{\partial w^a \partial \bar{w}^b} \delta w^a \wedge \delta \bar{w}^b \right)
\]
\[
= \sqrt{-1} \frac{1}{v^r} \frac{\partial^2 \log G}{\partial v^a \partial \bar{v}^b} \left( \frac{1}{v^r} \delta v^a - \frac{\bar{v}^b}{|v^r|^2} \delta \bar{v}^r \right) \left( \frac{1}{v^r} \delta \bar{v}^b - \frac{v^b}{|v^r|^2} \delta v^r \right)
\]
\[
= \sqrt{-1} \frac{1}{v^r} \frac{\partial^2 \log G}{\partial v^a \partial \bar{v}^b} \left( \delta v^a \wedge \delta \bar{v}^b - \frac{v^a}{v^r} \delta \bar{v}^a \wedge \delta v^r - \frac{\bar{v}^b}{v^r} \delta v^b \wedge \delta \bar{v}^r + \frac{v^a \bar{v}^b}{|v^r|^2} \delta \bar{v}^a \wedge \delta \bar{v}^r \right)
\]
\[
= \sqrt{-1} \sum_{i,j=1}^r \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} \delta v^i \wedge \delta \bar{v}^j = \omega_V.
\]
Remark 1.5. \( (1) \) By (1.12) and (1.16), we get
\[
q_*\left(\frac{\delta}{\delta z^\alpha}\right) = q_*\left(\frac{\partial}{\partial z^\alpha} - G_{\alpha l}G^l_{\beta k} \frac{\partial}{\partial v^k}\right)
\]
\[
= \frac{\partial}{\partial z^\alpha} - \left(\frac{1}{v^r}G_{\alpha i} - \frac{v^a}{(v^r)^2}G_{\alpha l}G^l_{r}r\right) \frac{\partial}{\partial w^a}
\]
\[
= \frac{\partial}{\partial z^\alpha} - (\log G)_{ab}(\log G)^{ba} \frac{\partial}{\partial w^a}.
\]

For convenience, we will identify \( \frac{\delta}{\delta z^\alpha} \) with \( q_*\left(\frac{\delta}{\delta z^\alpha}\right) \), and denote \( N^a_\alpha := (\log G)_{ab}(\log G)^{ba} \), so
\[
(1.18)
\]
\[
\frac{\delta}{\delta z^\alpha} = \frac{\partial}{\partial z^\alpha} - N^a_\alpha \frac{\partial}{\partial w^a}.
\]

\( (2) \) For any smooth function \( f \in C^\infty(P(E)) \), the vertical Laplacian is defined by
\[
\Delta^V f := (\log G)^{ba} \frac{\partial^2}{\partial w^a \partial w^b} f.
\]

By identifying \( f \) with \( q^* f \), one has from (1.12) and (1.16)
\[
\Delta^V f = (\log G)^{ba} \frac{\partial^2}{\partial w^a \partial w^b} f = GG^i^j \frac{\partial^2}{\partial v^i \partial v^j} f.
\]

\( (3) \) Noticing that \( \sqrt{-1} \partial \bar{\partial} \log G \) is a \((1,1)\)-form on \( P(E) \), which represents the first Chern class \( c_1(\mathcal{O}_{P(E)}(1)) \). And \( \Psi \) is also a \((1,1)\)-form on \( P(E) \), combining Lemma 1.4 with Lemma 1.3 one has when restricting to \( P(E) \)
\[
(1.19)
\]
\[
\sqrt{-1} \partial \bar{\partial} \log G = -\Psi + \omega_{FS}.
\]

**Proposition 1.6.** A Finsler metric \( G \) is strongly pseudoconvex if and only if \( \omega_{FS} \) is positive along each fiber of \( p : P(E) \to M \).

**Proof.** By Definition 1.1 \( G \) is strongly pseudoconvex if \( (G_{ij}) \) is a positive definite matrix, which gives a inner product \( \langle \cdot, \cdot \rangle \) on the vertical subbundle \( \mathcal{V} \).

Denote \( T = v^i \frac{\partial}{\partial v^i} \). If \( G \) is strongly pseudoconvex, for any \( X = X^i \frac{\partial}{\partial v^i} \), then
\[
(-\sqrt{-1})\omega_V(X, \bar{X}) = \frac{1}{G^2}(GG^i_j - G_i G_j)X^i \bar{X}^j
\]
\[
= \frac{1}{G^2}(||X||^2||T||^2 - |\langle X, T \rangle|^2) \geq 0,
\]
the equality holds if and only if \( X = \lambda T \) for some constant \( \lambda \in \mathbb{C} \). So \( \omega_V \) has \( r - 1 \) positive eigenvalues and one zero eigenvalue. Since \( \omega_V(T, \bar{T}) = 0 \) and \( q_* (T) = 0 \), by Lemma 1.4 \( \omega_{FS} \) is positive along each fiber of \( p : P(E) \to M \).

Conversely, if \( \omega_{FS} \) is positive along each fiber, then \( \omega_V = q^* \omega_{FS} \) has \( r - 1 \) positive eigenvalues and one zero eigenvalue, and \( \omega_V(T, \bar{T}) = 0 \). So
\[
G_{ij}X^i \bar{X}^j = \frac{1}{G}|G_i X^i|^2 + G(-\sqrt{-1})\omega_V(X, \bar{X}) \geq 0.
\]
Moreover, \( G_{ij}X^i \bar{X}^j = 0 \) if and only if \( X = \lambda T \) and \( G_i X^i = 0 \), if and only if \( \lambda = 0 \). So \( (G_{ij}) \) is a positive definite matrix. \( \square \)
Let \((h_{ij})\) be a Hermitian metric on \(E\) with respect to a local holomorphic frame \(\{e_i\}_{1 \leq i \leq r}\).

**Definition 1.7** ([15]). The Chern curvature of the metric \((h_{ij})\) is called *Griffiths positive* (resp. *Griffiths semipositive*) if

\[
R_{ij\alpha\beta}X^i\overline{X}^jY^\alpha\overline{Y}^\beta > 0 \quad \text{(resp.} \geq 0)\]

for any two nonzero vectors \(X = X^i e_i \in E\) and \(Y = Y^\alpha \frac{\partial}{\partial z^\alpha} \in TM\). Here \(R_{ij\alpha\beta} = -\frac{\partial^2 h_{ij}}{\partial z^\alpha \partial \overline{z}^\beta} + h_{ik} \frac{\partial h_{j\gamma}}{\partial z^\alpha} \frac{\partial h_{i\delta}}{\partial \overline{z}^\beta}\) denotes the Chern curvature of \((h_{ij})\).

For the case of \(E = TM\), the Hermitian metric \((h_{ij})\) is called *has positive (nonnegative) holomorphic bisectional curvature* if its Chern curvature is Griffiths positive (semipositive).

The Hermitian metric \((h_{ij})\) induces a strongly pseudoconvex complex Finsler metric on \(E^*\) by

\[
G := h^{ij} v_i \overline{v}_j.
\]

By Remark 1.5, we have the following decomposition

\[
\sqrt{-1} \partial \overline{\partial} \log G = -\Psi + \omega_{FS},
\]

where

\[
-\Psi = -R_{ij\alpha\beta}^a \frac{v_i \overline{v}_j}{G} \sqrt{-1} d\alpha \wedge d\overline{\beta} = R_{k\alpha\beta} \frac{G^{ki} v_i G^{j\overline{l}} \overline{v}_j}{G} \sqrt{-1} d\alpha \wedge d\overline{\beta}.
\]

From Proposition 1.6, \(\omega_{FS}\) is positive along fibers. So

**Proposition 1.8.** The Chern curvature of \((h_{ij})\) is Griffiths positive (resp. semipositive) if and only if \(\sqrt{-1} \partial \overline{\partial} \log G\) is a positive (resp. semipositive) \((1,1)\)-form on \(P(E^*)\).

1.2. **Maximum principle for real \((1,1)\)-forms.** In this subsection, we recall the maximum principle for real \((1,1)\)-forms. For more details, one can refer to [5] [9] [24] [26]. The following version maximum principle is same as the tensor maximum principle [9, Theorem 4.6]. For readers’ convenience, we give a proof here.

**Theorem 1.9** ([9, Theorem 4.6]). Let \(\omega(t) = \sqrt{-1} g_{\alpha\beta}(t) dz^\alpha \wedge d\overline{z}^\beta\) be a smooth 1-parameter family of Hermitian metrics on a compact complex manifold \(M\). Let \(\eta(t) = \sqrt{-1} \eta_{\alpha\beta}(t) dz^\alpha \wedge d\overline{z}^\beta\) be a real \((1,1)\)-form satisfying

\[
\frac{\partial}{\partial t} \eta \geq \Delta_\omega(t) \eta + \sigma,
\]

where \(\Delta_\omega(t) := \partial^\alpha \nabla_{\partial z^\alpha} \nabla_{\partial \overline{z}^\alpha}, \nabla\) denotes the Chern connection of \(\omega(t)\), \(\sigma(\alpha, \omega, t)\) is a real \((1,1)\)-form which is locally Lipschitz in all its arguments and satisfies the null eigenvector assumption that

\[
(-\sqrt{-1}) \sigma(V, \overline{V})(z, t) = (\sigma_{\alpha\beta} V^\alpha \overline{V}^\beta)(z, t) \geq 0
\]

whenever \(V(z, t) = V^\alpha \frac{\partial}{\partial z^\alpha}\) is a null eigenvector of \(\eta(t)\), that is whenever

\[
(\eta_{\alpha\beta} V^\alpha)(z, t) = 0.
\]

If \(\eta(0) \geq 0\), then \(\eta(t) \geq 0\) for all \(t \geq 0\) such that the solution exists.
Proof. Suppose that $\eta > 0$ for all $0 \leq t < t_0$, and $(z_0, t_0)$ is a point and time and $v = v^\alpha \frac{\partial}{\partial z^\alpha}$ is a vector such that

\begin{equation}
(1.20) \quad \eta_{\alpha \beta} v^\alpha(z_0, t_0) = 0.
\end{equation}

Then $\eta_{\alpha \beta} W^\alpha \overline{W^\beta}(z, t) \geq 0$ for all $z \in M, t \in [0, t_0]$, and tangent vectors $W \in T_z M$. By parallel translation, one can extend $v$ to a vector field $V$ defined in a neighborhood of $(z_0, t_0)$ such that $V(z_0, t_0) = v$ and

\begin{equation}
(1.21) \quad \frac{\partial V}{\partial t}(z_0, t_0) = 0, \quad \nabla V(z_0, t_0) = 0.
\end{equation}

Then at the point $(z_0, t_0)$, one has

\[
\frac{\partial}{\partial t} (\eta_{\alpha \beta} V^\alpha \overline{V^\beta}) = (\frac{\partial}{\partial t} \eta_{\alpha \beta}) V^\alpha \overline{V^\beta} \geq (\Delta_{\omega(t)} \eta_{\alpha \beta} + \sigma_{\alpha \beta}) V^\alpha \overline{V^\beta}.
\]

Since $\eta_{\alpha \beta} V^\alpha \overline{V^\beta}(z_0, t_0) = 0$ and $\eta_{\alpha \beta} V^\alpha \overline{V^\beta}(z, t_0) = 0$ in the neighborhood of $z$, one has

\[
(\Delta_{\omega(t)} \eta_{\alpha \beta} V^\alpha \overline{V^\beta}) \geq 0.
\]

By (1.20) and (1.21), one has at the point $(z_0, t_0)$

\[
\Delta_{\omega(t)} (\eta_{\alpha \beta} V^\alpha \overline{V^\beta}) = (\Delta_{\omega(t)} \eta_{\alpha \beta}) V^\alpha \overline{V^\beta} + g^{\gamma \delta} (\nabla_{\frac{\partial}{\partial z^\delta}} \eta_{\alpha \beta}) (\nabla_{\frac{\partial}{\partial z^\gamma}} V^\alpha) \overline{V^\beta} + g^{\gamma \delta} \nabla_{\frac{\partial}{\partial z^\delta}} (\eta_{\alpha \beta} V^\alpha) \nabla_{\frac{\partial}{\partial z^\gamma}} \overline{V^\beta} + \eta_{\alpha \beta} V^\alpha \Delta_{\omega(t)} \overline{V^\beta}
\]

Combining with the null eigenvector assumption

\[
\sigma_{\alpha \beta} V^\alpha \overline{V^\beta}(z_0, t_0) \geq 0,
\]

it follows that

\[
\frac{\partial}{\partial t} (\eta_{\alpha \beta} V^\alpha \overline{V^\beta}) \geq (\Delta_{\omega(t)} \eta_{\alpha \beta} V^\alpha \overline{V^\beta}) + \sigma_{\alpha \beta} V^\alpha \overline{V^\beta} \geq 0
\]

at the point $(z_0, t_0)$. Hence, if $\eta_{\alpha \beta} V^\alpha \overline{V^\beta}$ ever becomes zero, it cannot decrease further. So $\eta(t) \geq 0$ for all $t \geq 0$ such that the solution exists. \qed

2. A flow over projective bundle

2.1. Definition of the flow. Let $M$ be a compact complex manifold of dimension $n$, let $\pi : E \to M$ be a holomorphic vector bundle of rank $r$ over $M$. Let $E^*$ denote the dual bundle of $E$, let $\{e_i\}_{i=1}^r$ be a local holomorphic frame of $E^*$, then

\[(z, v) = (z^1, \cdots, z^n; v_1, \cdots, v_r)\]

gives a local coordinate system of the complex manifold $E^*$, which represents the point $v^i e_i \in E^*$. For any strongly pseudoconvex complex Finsler metric $G$ on $E^*$, by Remark 1.5 we have the following decomposition

\[
\sqrt{-1} \partial \overline{\partial} \log G = -\Psi + \omega_{FS}.
\]

By Proposition 1.6, $\omega_{FS}$ is positive along each fiber of $p : P(E^*) \to M$. 
Let
\[ \omega(G) = \sqrt{-1}g_{\alpha\overline{\beta}}(G) dz^\alpha \wedge d\overline{z}^\beta \]
be a horizontal \((1,1)\)-form on \(P(E^*)\) depending smoothly on the Finsler metric \(G\), which is positive on horizontal directions, namely \((g_{\alpha\overline{\beta}}(G))\) is a positive definite matrix.

Then one can define a Hermitian metric on \(P(E^*)\) by
\[ \Omega := \omega(G) + \omega_{FS}. \]

Let \(G_0\) be a strongly pseudoconvex complex Finsler metric on \(E^*\), we consider the following flow:
\[
\begin{align*}
\frac{\partial}{\partial t} \log G &= \Delta_\Omega \log G, \\
\omega_{FS} &> 0, \\
G(0) &= G_0.
\end{align*}
\]
(2.1)

Here \(\Delta_\Omega := \sqrt{-1}\Lambda \partial \overline{\partial} \), \(\Lambda\) is the adjoint operator of \(\Omega \wedge \cdot\).

Since
\[
\Delta_\Omega \log G = \Lambda \sqrt{-1} \partial \overline{\partial} \log G = \Lambda_{\omega(G)}(-\Psi) + (r-1)
\]
is a smooth function on \(P(E^*)\), so along the flow
\[ G(t) = e^{\int_0^t \Delta_\Omega \log G dt} G_0, \quad \omega_{FS} > 0. \]

By Definition 1.1, one see that \(G(t)\) is always a strongly pseudoconvex complex Finsler metric on \(E^*\).

**Remark 2.1.** (1) For the case
\[ \omega(G) = \sqrt{-1}g_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta \]
is a fixed Kähler metric on \(M\), and \(G_0 = h^{i\overline{j}}_0 v_i \overline{v}_j\) comes from a Hermitian metric \((h^{i\overline{j}}_0)\) of \(E^*\), then
\[
0 = \frac{\partial}{\partial t} \log G - \Delta_\Omega \log G
= \frac{1}{G} \frac{\partial G}{\partial t} + \Lambda_{\omega} \Psi - (r-1)
= \frac{v_i \overline{v}_j}{G} \left( \frac{\partial h^{i\overline{j}}(t)}{\partial t} + g_{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}^{i\overline{j}} - (r-1)h^{i\overline{j}}(t) \right),
\]
(2.3)

where \(h^{i\overline{j}}(t) := \frac{\partial^2 G(t)}{\partial v_i \partial \overline{v}_j}\). From above equation, one has
\[
\frac{\partial h^{i\overline{j}}(t)}{\partial t} + g_{\alpha\overline{\beta}} \frac{\partial^2}{\partial v_i \partial \overline{v}_j} (R_{\alpha\overline{\beta}}^{i\overline{j}} v_i \overline{v}_j) - (r-1)h^{i\overline{j}}(t) = 0.
\]

Since \(h^{i\overline{j}}(0) = h^{i\overline{j}}_0\) is a Hermitian metric, which is independent of the vertical coordinates \(\{v_i, 1 \leq i \leq r\}\), so \(h^{i\overline{j}}(t)\) is also a Hermitian metric. In fact, by induction, we assume...
that \( \left( \frac{\partial^k h^{ij}}{\partial t^k} \right)_{t=0} \) is independent of fibers, then

\[
\frac{\partial^{k+1} h^{ij}(t)}{\partial t^{k+1}} \bigg|_{t=0} = \frac{\partial^k}{\partial t^k} \left( -g^{\alpha\beta} \frac{\partial^2}{\partial v_i \partial \bar{v}_j} \left( R^{ij}_{\alpha\beta} v_i \bar{v}_j \right) + (r-1)h^{ji}(t) \right) \bigg|_{t=0} = -g^{\alpha\beta} \frac{\partial^2}{\partial v_i \partial \bar{v}_j} \left( \left( \frac{\partial^k}{\partial t^k} R^{ij}_{\alpha\beta} \right)_{t=0} v_i \bar{v}_j \right) + (r-1) \left( \frac{\partial^k}{\partial t^k} h^{ji} \right)_{t=0}
\]

which is also independent of fibers, because \( \frac{\partial^k}{\partial t^k} R^{ij}_{\alpha\beta} \) is the combination by \( h^{ji} \) and \( \frac{\partial^k}{\partial t^k} h^{ji}, 1 \leq l \leq k \). It follows that

\[
h^{ji}(t) = h^{ji}(0) + \frac{\partial h^{ij}(t)}{\partial t} \bigg|_{t=0} t + \cdots + \frac{\partial^k h^{ij}(t)}{\partial t^k} \bigg|_{t=0} t^k + \cdots
\]

is independent of fibers for small \( t \). By Proposition 1.6 and \( \omega_{FS} > 0 \), \( h^{ji}(t) \) is a positive definite matrix, so \( (h^{ji}(t)) \) is a Hermitian metric on \( E^* \).

Therefore, (2.3) is equivalent to

\[
\frac{\partial h^{ij}(t)}{\partial t} + g^{\alpha\beta} R^{ij}_{\alpha\beta} - (r-1)h^{ji}(t) = 0.
\]

Multiplying by \( h_{kj} \) to both sides of (2.4), one has

\[
h^{-1} \cdot \frac{\partial h}{\partial t} + \Lambda R + (r-1)I := h^{ij} \frac{\partial h_{kj}}{\partial t} + g^{\alpha\beta} R^{ij}_{\alpha\beta} + (r-1)\delta^i_k = 0,
\]

which is exactly the Hermitian-Yang-Mills flow (11) (see also 20).

(2) For the case of \( E = TM \). Let

\[
\omega(G) = \sqrt{-1} g_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta,
\]

where \( (g_{\alpha\beta}) \) denotes the inverse of the matrix \( \left( \frac{\partial^2 G}{\partial \alpha \partial \bar{v}_\beta} \right) \). Let \( G_0 = g_0^{\alpha\beta} v_\alpha \bar{v}_\beta \) be a strongly pseudoconvex complex Finsler metric on \( T^*M \) induced by the Hermitian metric

\[
\omega_0 = \sqrt{-1} (g_0)_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta.
\]

Similar to the above case, the flow (2.1) is equivalent to

\[
\frac{\partial g_{\gamma\delta}}{\partial t} + g^{\alpha\beta} R_{\gamma\delta\alpha\beta} + (r-1)g_{\gamma\delta} = 0,
\]

which is exactly the equation given by [22] (7.11)]. By [22] Theorem 7.1] (or [22] Remark 7.2)], if the initial metric \( \omega_0 \) is a Kähler metric, then this flow is reduced to the usual Kähler-Ricci flow (see [6]).
2.2. Positivity preserving along the flow. In this subsection, we shall discuss the positivity preserving along the flow (2.11). We assume that the initial metric $G(0) = G_0$ satisfies $\sqrt{-1}\partial\bar{\partial} \log G_0 \geq 0$.

Let $\epsilon > 0$ small enough such that

\begin{equation}
\Omega_\epsilon = \omega(G) + \epsilon \sqrt{-1}\partial\bar{\partial} \log G = \sqrt{-1}(g_{\alpha\bar{\beta}} - \epsilon \Psi_{\alpha\bar{\beta}})dz^\alpha \wedge d\bar{z}^\beta + \epsilon \sqrt{-1}\frac{\partial^2 \log G}{\partial w^a \partial \bar{w}^b} \delta w^a \wedge \delta \bar{w}^b
\end{equation}

is a Hermitian metric on $P(E^*)$, where $\Psi_{\alpha\bar{\beta}}$ is given by $\Psi = \sqrt{-1}\Psi_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta$. Denote by $\nabla^\epsilon$ the Chern connection of the Hermitian metric $\Omega_\epsilon$, which is the unique connection preserving the holomorphic structure and the metric $\Omega_\epsilon$. For any two $(1,0)$-type vector fields $X, Y$ of $P(E^*)$, then

\begin{equation}
\nabla^\epsilon_X Y = \langle \nabla^\epsilon_X Y, \frac{\delta}{\delta z^\alpha}\rangle g^\epsilon_{\bar{\alpha} \bar{\beta}} \frac{\delta}{\delta \bar{z}^\beta} + \langle \nabla^\epsilon_X Y, \frac{\partial}{\partial w^b}\rangle \frac{1}{\epsilon} (\log G)^{b\alpha} \frac{\partial}{\partial w^a}
\end{equation}

\begin{align*}
&= X(Y, \frac{\delta}{\delta z^\alpha}) g^\epsilon_{\bar{\alpha} \bar{\beta}} \frac{\delta}{\delta \bar{z}^\beta} + \langle Y, \nabla^\epsilon(Y, \frac{\delta}{\delta z^\alpha}) \rangle g^\epsilon_{\bar{\alpha} \bar{\beta}} \frac{\delta}{\delta \bar{z}^\beta} + \langle Y, \nabla^\epsilon(Y, \frac{\partial}{\partial w^b}) \rangle \frac{1}{\epsilon} (\log G)^{b\alpha} \frac{\partial}{\partial w^a},
\end{align*}

where $Y^V$ denotes the vertical part of $Y$, $(g^\epsilon)^{\bar{\alpha} \bar{\beta}}$ denotes the inverse of $g_{\alpha\bar{\beta}} := g_{\alpha\bar{\beta}} - \epsilon \Psi_{\alpha\bar{\beta}}$, $\langle \cdot, \cdot \rangle_\epsilon$ is the inner product defined by $\Omega_\epsilon$.

Denote by $Z^A$ the coordinates $z^\alpha$ or $w^a$, $1 \leq A \leq n + r - 1$. We rewrite $\Omega_\epsilon$ as the following form:

\begin{equation}
\Omega_\epsilon = \sqrt{-1}\Omega_\epsilon_{AB} dz^A \wedge d\bar{z}^B.
\end{equation}

In this form, the Chern connection is given by

\begin{equation}
\nabla^\epsilon_{\frac{\partial}{\partial z^A}} \frac{\partial}{\partial z^B} = \Gamma^C_{AB} \frac{\partial}{\partial z^C}, \quad \Gamma^C_{AB} = \frac{\partial \Omega_{C\bar{D}B}}{\partial Z^A} \Omega_{\epsilon \bar{D} C}.
\end{equation}

Here $(\Omega^\epsilon_{\bar{C}D})$ denotes the inverse of the matrix $(\Omega_{\epsilon CD})$. The Chern curvature tensor of $\Omega_\epsilon$ is defined by

\begin{equation}
R^\epsilon_{ABCD} := R(\frac{\partial}{\partial Z^A}, \frac{\partial}{\partial Z^B}, \frac{\partial}{\partial Z^C}, \frac{\partial}{\partial Z^D}) = \Omega_{\epsilon AB} \frac{\partial}{\partial Z^C} = -\Omega_{\epsilon AB} \frac{\partial}{\partial Z^C}.
\end{equation}

The Chern connection $\nabla^\epsilon$ induces a natural connection on the cotangent bundle $T^*P(E^*)$ (resp. $\overline{T^*P(E^*)}$) by

\begin{equation}
\nabla^\epsilon_A (f_C dZ^C) = (\partial_A f_C - \Gamma^B_{AC} f_B) dZ^C, \quad \text{(resp. } \nabla^\epsilon_B (f_D d\bar{Z}^D) = (\partial_B f_D - \Gamma^F_{BD} f_E) d\bar{Z}^D \text{)}
\end{equation}

for any smooth $(1,0)$-form $f_C dZ^C$ (resp. $(0,1)$-form $f_D d\bar{Z}^D$), where $\nabla^\epsilon_A := \nabla^\epsilon_A \frac{\partial}{\partial z^A}$. For convenience, we denote

\begin{equation}
\nabla^\epsilon_A f_C := \partial_A f_C - \Gamma^B_{AC} f_B, \quad \nabla^\epsilon_B f_D := \partial_B f_D - \Gamma^F_{BD} f_E.
\end{equation}
and

\[(2.13) \quad (\nabla_A f_{CD}) dz^C \wedge d\bar{Z}^D := \nabla^\epsilon_A (f_{CD} dz^C \wedge d\bar{Z}^D), \quad \nabla_A f_{CD} = \partial_A f_{CD} - \Gamma^B_{AC} f_{BD}.
\]

By above notations, we have

\[(2.14) \quad \nabla_A \partial_{CD} \epsilon = \partial_A \partial_{CD} \epsilon - \Gamma^B_{AC} \partial_B \epsilon = 0.
\]

By taking \( \partial \bar{\partial} \) to the both sides of the first equation of (2.1), one has

\[(2.15) \quad \frac{\partial}{\partial t} \partial \bar{\partial} \log G = \partial \bar{\partial} \Delta \Omega \log G = \partial \bar{\partial} (\text{tr}_\omega (G) (-\Psi)),
\]

where the last equality follows from (2.2). Since \( \lim_{\epsilon \to 0} (\omega (G) - \epsilon \Psi) = \omega (G) \), so

\[(2.16) \quad \frac{\partial}{\partial t} \partial \bar{\partial} \log G = \lim_{\epsilon \to 0} \partial \bar{\partial} (\text{tr}_\omega (G) - \epsilon \Psi (-\Psi))
\]

Denote \( f := \log G \) and

\[f_{AB} := \partial_A \partial_B f, \quad f_{ABC} := \partial_A \partial_B \partial_C f, \quad f_{ABCD} := \partial_A \partial_B \partial_C \partial_D f, \quad \text{etc}.
\]

By (2.13) and (2.14), one has

\[(2.17) \quad \partial_C \partial_D (\partial_A \epsilon f_{AB}) = \partial_C \partial_D (\partial_A \epsilon f_{AB}) = \nabla^\epsilon_C \nabla^\epsilon_D (\partial_A \epsilon f_{AB})
\]

\[= \epsilon \nabla^\epsilon_C \nabla^\epsilon_D f_{AB}
\]

\[= \epsilon \nabla^\epsilon_C (f_{ABD} - \Gamma^E_{DB} f_{AE})
\]

\[= \epsilon \nabla^\epsilon_C (f_{ABCD} - \Gamma^F_{CA} f_{FB} - \partial_C \Gamma^E_{DB} f_{AE} - \Gamma^E_{DB} (f_{ACE} - \Gamma^F_{CA} f_{FE})).
\]

Similarly,

\[(2.18) \quad \Omega^\epsilon_{AB} (\nabla_A \nabla_B f_{CD})
\]

\[= \Omega^\epsilon_{AB} (f_{ABCD} - \Gamma^F_{AC} f_{FBD} - \partial_A \Gamma^E_{BD} f_{CE} - \Gamma^E_{BD} (f_{ACE} - \Gamma^F_{CA} f_{FE})).
\]

Combining (2.17) with (2.18), we obtain

\[(2.19) \quad \partial_C \partial_D (\partial_A \epsilon f_{AB}) - \Omega^\epsilon_{AB} (\nabla_A \nabla_B f_{CD})
\]

\[= \epsilon \nabla^\epsilon_A (f_{ABCD} - \Gamma^F_{AC} f_{FBD} + \epsilon \nabla^\epsilon_B (\Gamma^E_{BD} - \Gamma^E_{DB}) f_{ACE} + \epsilon \nabla^\epsilon_A (\Gamma^E_{DB} \Gamma^F_{CA} - \Gamma^E_{BD} \Gamma^F_{AC} f_{FE})
\]

\[+ \epsilon \nabla^\epsilon_B \partial_C \Gamma^E_{DB} f_{AE}
\]

\[= \epsilon \nabla^\epsilon_A (f_{ABCD} - \Gamma^F_{CA} f_{FBD} + \epsilon \nabla^\epsilon_B (\Gamma^E_{BD} - \Gamma^E_{DB}) f_{ACE} + \epsilon \nabla^\epsilon_A (\Gamma^E_{DB} \Gamma^F_{CA} - \Gamma^E_{BD} \Gamma^F_{AC} f_{FE})
\]

\[- \epsilon \nabla^\epsilon_A \Omega^\epsilon_{EF} R_{FDB} f_{CE} + \epsilon \nabla^\epsilon_B \Omega^\epsilon_{EF} R_{FBCD} f_{AE}.
\]
Substituting (2.19) into (2.16), we have
\[ \frac{\partial}{\partial t} \bar{\partial} \bar{\partial} \log G = \lim_{\epsilon \to 0} \left( \Omega_e^{AB} \nabla^e_A \nabla^e_B (\partial \bar{\partial} \log G) \right. \]
\[ + \left( \Omega_e^{AB} (\Gamma^F_{AC} - \Gamma^F_{CA}) \nabla_D f_{FB} + \Omega_e^{AB} (\Gamma^E_{BD} - \Gamma^E_{DB}) \nabla_A f_{CE} \right. \]
\[ \left. - \Omega_e^{AB} \Omega_e^{F\bar{E}} R_{FDAE} f_{CE} + \Omega_e^{AB} \Omega_e^{F\bar{E}} R_{FBCD} f_{AE} \right) dZ^C \wedge d\bar{Z}^D \right). \]

As in the proof of Theorem 1.9, we assume that $\sqrt{-1} \bar{\partial} \partial \log G \geq 0$ for all $0 \leq t < t_0$, and $(\zeta_0, \omega_0, t_0)$ is a point and time and $u = u^A \frac{\partial}{\partial Z^A}$ is a vector such that
\[ f_{CD} u^C(\zeta_0, \omega_0, t_0) = (\partial_C \partial_D \log G) u^C(\zeta_0, \omega_0, t_0) = 0 \]
and
\[ (\bar{\partial} \partial \log G(W, \bar{W}))(\zeta, [\omega], t) = (\partial_C \partial_D \log G) W^C \bar{W}^D(\zeta, [\omega], t) \geq 0 \]
for all $(\zeta, [\omega]) \in P(E^*)$, $t \in [0, t_0]$, and tangent vectors $W \in T_{(\zeta, [\omega])} P(E^*)$. This implies that
\[ u = u^\alpha \delta \frac{\delta}{\delta z^\alpha} \in q_* \mathcal{H}. \]

Indeed, one may assume that $u = u_1 + u_2$, where $u_1 = u^\alpha \frac{\partial}{\delta z^\alpha}$, $u_2 = u^a \frac{\partial}{\partial w^a}$ are the horizontal and vertical parts of $u$ respectively. By (1.19) and $\omega_{FS} > 0$, one has
\[ 0 = (\sqrt{-1} \bar{\partial} \partial \log G)(u, \bar{u}) = (\sqrt{-1} \bar{\partial} \partial \log G)(u_1 + u_2, \bar{u}_1 + \bar{u}_2) \]
\[ = (\sqrt{-1} \bar{\partial} \partial \log G)(u_1, \bar{u}_1) + \omega_{FS}(u_2, \bar{u}_2) \geq \omega_{FS}(u_2, \bar{u}_2) \geq 0, \]
and all equalities hold if and only if $u_2 = 0$, namely $u = u_1$. From Lemma 1.9 and (2.23), (2.21) is equivalent to
\[ (i_u \Psi)(\zeta_0, \omega_0, t_0) = 0. \]

For any $\epsilon > 0$, by parallel translation, one can extend $u$ to a vector field $U_\epsilon = U_\epsilon^A \frac{\partial}{\partial Z^A}$ defined in a neighborhood of $(\zeta_0, \omega_0, t_0)$ such that $U_\epsilon(\zeta_0, \omega_0, t_0) = u$ and
\[ \frac{\partial U_\epsilon}{\partial t}(\zeta_0, \omega_0, t_0) = 0, \quad (\nabla^\epsilon U_\epsilon)(\zeta_0, \omega_0, t_0) = 0. \]
This can be done by parallel translating $u$ along radial rays with respect to the connection $\nabla^\epsilon$, and then by extending to be independent of time $t$.

We assume that
\[ U_\epsilon(\zeta, [\omega], t_0) = U_\epsilon^\alpha \delta \frac{\delta}{\delta z^\alpha} + U_\epsilon^a \frac{\partial}{\partial w^a}. \]
By (2.7), one has
\[ \nabla^\epsilon U_\epsilon = \bar{\partial} U_\epsilon + \left( \partial (U_\epsilon^\alpha g_{\alpha \beta}) + U_\epsilon^a \partial (N_\beta^b) \epsilon (\log G)_{ab} \right) g_{\gamma}^b \frac{\delta}{\delta z^\gamma} + \partial (U_\epsilon^a (\log G)_{ab})(\log G)^b_{bc} \frac{\partial}{\partial w^c}. \]
So the second equation of (2.26) is equivalent to

\[
\begin{align*}
\partial U^\alpha &= 0, \\
\bar{\partial} (-U^\alpha N^a_\alpha + U^a_\alpha) &= 0, \\
\partial (U^\alpha g_{\alpha\overline{\beta}}) + U^e_\alpha \partial (N^3_b \epsilon (\log G)_{ab}) &= 0, \\
\bar{\partial} (U^a_\alpha (\log G)_{ab}) (\log G)^{bc} &= 0,
\end{align*}
\]

(2.28)

at the point \((z_0, [v_0], t_0)\). By (2.23), \(U^a_\epsilon = 0\) at the point \((z_0, [v_0], t_0)\), so (2.28) is equivalent to

\[
\begin{align*}
\bar{\partial} U^\alpha &= 0, \\
\bar{\partial} U^a_\alpha &= 0, \\
\partial U^\alpha + g^\beta_\alpha \partial g_{\epsilon\gamma\beta} u^\gamma &= 0, \\
\partial U^a_\alpha &= 0.
\end{align*}
\]

(2.29)

Since \(\lim_{\epsilon \to 0} g_{\epsilon\alpha\overline{\beta}} = g_{\alpha\overline{\beta}}\), so \(\lim_{\epsilon \to 0} U^\epsilon = U\) which satisfies the following equations:

\[
\begin{align*}
\bar{\partial} U^\alpha &= 0, \\
\bar{\partial} U^a_\alpha &= u^a_\alpha \bar{\partial} N^a_\alpha, \\
\partial U^\alpha + g^\beta_\alpha \partial g_{\epsilon\gamma\beta} u^\gamma &= 0, \\
\partial U^a_\alpha &= 0.
\end{align*}
\]

(2.30)

and \(\frac{\partial U^\epsilon}{\partial t} = 0\) at the point \((z_0, [v_0], t_0)\). By (2.20), (2.21) and (2.26), one has at the point \((z_0, [v_0], t_0)\),

\[
\begin{align*}
\frac{\partial}{\partial t} (\bar{\partial}\bar{\partial} \log G(U, \overline{U})) &= \lim_{\epsilon \to 0} \left( \Omega^{AB} \nabla_A \nabla_B (\bar{\partial}\bar{\partial} \log G)(u, \overline{u}) \\
&+ \left( \Omega^{AB} (\Gamma^{F}_{AC} - \Gamma^{F}_{CA}) \nabla_D f_{FB} + \Omega^{AB} (\Gamma^{E}_{BD} - \Gamma^{E}_{DB}) \nabla_A f_{CE} \\
&- \Omega^{AB} \Omega^{EF} R_{FDB} f_{CE} + \Omega^{AB} \Omega^{EF} R^{EF}_{FBCD} f_{AE} \right) u^C \overline{u}^D \right) \\
&= \lim_{\epsilon \to 0} \left( \Omega^{AB}_\epsilon \partial_A \partial_B (\bar{\partial}\bar{\partial} \log G(U^\epsilon, \overline{U^\epsilon})) \\
&+ \left( \Omega^{AB}_\epsilon (\Gamma^{F}_{AC} - \Gamma^{F}_{CA}) \nabla_D f_{FB} + \Omega^{AB}_\epsilon (\Gamma^{E}_{BD} - \Gamma^{E}_{DB}) \nabla_A f_{CE} \\
&+ \Omega^{AB}_\epsilon \Omega^{EF}_\epsilon R^{EF}_{FBCD} f_{AE} \right) u^C \overline{u}^D \right).
\end{align*}
\]

(2.31)

In order to deal with (2.31), we assume that \(\omega(G) = p^* \omega\) for some Kähler metric \(\omega\) on \(M\), so

\[
\Omega_\epsilon = \omega(G) + \epsilon \sqrt{-1} \partial\bar{\partial} \log G
\]

is a Kähler metric on \(P(E^*)\) for \(\epsilon > 0\) small enough. Thus, (2.31) is reduced to

\[
\frac{\partial}{\partial t} (\bar{\partial}\bar{\partial} \log G(U, \overline{U})) = \lim_{\epsilon \to 0} \left( \Omega^{AB}_\epsilon \partial_A \partial_B (\bar{\partial}\bar{\partial} \log G(U^\epsilon, \overline{U^\epsilon})) + \Omega^{AB}_\epsilon \Omega^{EF}_\epsilon R^{EF}_{FBCD} f_{AE} u^C \overline{u}^D \right).
\]

(2.33)
For the first term in the RHS of (2.33), we have

\[
\Omega^A \partial_A \partial_B (\partial \bar{\partial} \log G(U, \Upsilon)) = \sqrt{-1} \Lambda \partial \bar{\partial} (\partial \bar{\partial} \log G(U, \Upsilon)) = \Delta^H_{\Omega_\epsilon} (\partial \bar{\partial} \log G(U, \Upsilon)) + \Delta^V_{\Omega_\epsilon} (\partial \bar{\partial} \log G(U, \Upsilon)).
\]

Here \( \Delta^V_{\Omega_\epsilon} = \frac{1}{\epsilon} (\log G)^{ba} \frac{\partial^2}{\partial w^a \partial w^b} \) is the vertical Laplacian (see Remark 1.5), while the horizontal Laplacian \( \Delta^H_{\Omega_\epsilon} \) is defined by \( \Delta^H_{\Omega_\epsilon} = g^\alpha_\beta (\partial \bar{\partial} \varphi) (\frac{\delta}{\delta z^\alpha}, \frac{\delta}{\delta \bar{z}^\beta}) \) for any smooth function \( \varphi \in C^\infty(P(E^*)) \). Since \( \lim_{\epsilon \to 0} g_{\alpha \beta} = g_{\alpha \beta} \) and \( \lim_{\epsilon \to 0} U_\epsilon = U \), so

\[
\lim_{\epsilon \to 0} \Delta^H_{\Omega_\epsilon} (\partial \bar{\partial} \log G(U, \Upsilon)) = \Delta^H_{\Omega} (\partial \bar{\partial} \log G(U, \Upsilon)).
\]

By Remark 1.5 (1), one has

\[
\Delta^V_{\Omega_\epsilon} (\partial \bar{\partial} \log G(U, \Upsilon)) = \Delta^V_{\Omega_\epsilon} (f_{cd} U_\epsilon^c U_\epsilon^d) + \Delta^V_{\Omega_\epsilon} ((-\Psi)_{\alpha \beta} U_\epsilon^\alpha U_\epsilon^\beta)
\]

\[
= \frac{1}{\epsilon} f^{ab} \frac{\partial^2}{\partial u^a \partial u^b} (f_{cd} U_\epsilon^c U_\epsilon^d) + \frac{1}{\epsilon} f^{ab} \frac{\partial^2}{\partial w^a \partial w^b} ((-\Psi)_{\alpha \beta} U_\epsilon^\alpha U_\epsilon^\beta).
\]

For the first term in the RHS of (2.36), by (2.29) and \( U^c = 0 \) at the point \((z_0, [v_0], t_0)\), we have

\[
1 \frac{1}{\epsilon} f^{ab} \frac{\partial^2}{\partial u^a \partial u^b} (f_{cd} U_\epsilon^c U_\epsilon^d) = \frac{1}{\epsilon} f^{ab} f_{cd} \partial_u U^c \partial_u U^d = \frac{1}{\epsilon} f^{ab} f_{cd} u^\alpha w^\beta \partial_u N^\alpha \partial_u \nabla^d.
\]

The following lemma is actually proved in [34 (3.46)]. For readers’ convenience, we give a proof here.

Lemma 2.2.

\[
f^{ba} \frac{\partial^2}{\partial w^a \partial w^b} (-\Psi)_{\alpha \beta} = \partial \bar{\partial} \log \det (f_{ab}) (\frac{\delta}{\delta z^\alpha}, \frac{\delta}{\delta \bar{z}^\beta}) - (\bar{\partial} V^\alpha \frac{\delta}{\delta z^\alpha}, \bar{\partial} V^\beta \frac{\delta}{\delta \bar{z}^\beta}),
\]

where \( \bar{\partial} V \frac{\delta}{\delta z^\alpha} := \frac{\partial}{\partial w^a} (-f_{ab} f^{dc}) \delta w^d \otimes \frac{\partial}{\partial w^a} \) and \( (\bar{\partial} V \frac{\delta}{\delta z^\alpha}, \bar{\partial} V \frac{\delta}{\delta \bar{z}^\beta}) := f^{ba} \partial_b N^\alpha \partial_u \nabla^d f_{cd} \).

Proof. Let \((-\Psi)_{\alpha \beta}\) denote the coefficient of \(-\Psi\), i.e. \(-\Psi = \sqrt{-1} (-\Psi)_{\alpha \beta} dz^\alpha \wedge d\bar{z}^\beta\), then

\[
(-\Psi)_{\alpha \beta} = f_{\alpha \beta} - f_{ab} f^{dc} f_{c \beta}.
\]

In fact, by the decomposition (1.19), one has

\[
(-\Psi)_{\alpha \beta} = (-\sqrt{-1}) (-\Psi) (\frac{\delta}{\delta z^\alpha}, \frac{\delta}{\delta \bar{z}^\beta})
\]

\[
= (\partial \bar{\partial} f) (\frac{\partial}{\partial z^\alpha} - f_{ab} f^{ba} \frac{\partial}{\partial \bar{z}^\beta} - f_{\beta a} f^{ab} \frac{\partial}{\partial \bar{w}^b})
\]

\[
= f_{\alpha \beta} - f_{ab} f^{dc} f_{c \beta},
\]

which proves (2.40).
For any fixed point \((z, [v]) \in P(E^*)|_z, \ z \in M\), we take normal coordinates near \((z, [v])\) such that \(f_{ab}(z, [v]) = \delta_{ab}, f_{abc}(z, [v]) = 0\). Evaluating at \((z, [v])\) we see that

\[
f_{ab} \frac{\partial^2}{\partial u^a \partial \bar{w}^b} (-\Psi)_{\alpha \bar{\beta}} = f_{ab} \frac{\partial^2}{\partial u^a \partial \bar{w}^b} (f_{\alpha \bar{\beta}} - f_{\bar{a}d} f_{\bar{c} \bar{\beta}})
\]

\[
= f_{ab} (f_{\alpha \bar{c} \bar{b}} - f_{\bar{a}d} f_{\bar{c} \bar{b}} - f_{\alpha \bar{c} \bar{b}} + f_{\bar{a}d} f_{\bar{c} \bar{b}} + f_{\alpha \bar{a} \bar{b} \bar{c}} - f_{\alpha \bar{c} \bar{b}} f_{\bar{a} \bar{b} \bar{c}} + f_{\alpha \bar{a} \bar{b} \bar{c}})
\]

\[
= f_{ab} (-\partial (f_{\bar{a}d} f_{\bar{c} \bar{b}}) f_{\bar{c} \bar{b}}) (\frac{\delta}{\delta z^\alpha}, \frac{\delta}{\delta \bar{z}^\beta}) - f_{ab} \delta_b (\partial (f_{\bar{a}d} f_{\bar{c} \bar{b}}) \partial a (-\bar{f}_{\bar{k} \bar{b}} f_{\bar{k} \bar{b}}) f_{\bar{d} \bar{i}}
\]

\[
= \partial \bar{\partial} \log \det(f_{ab}) (\frac{\delta}{\delta z^\alpha}, \frac{\delta}{\delta \bar{z}^\beta}) - (\partial V \frac{\delta}{\delta z^\alpha}, \partial V \frac{\delta}{\delta \bar{z}^\beta}),
\]

which completes the proof. \(\square\)

By Lemma \(2.2\) and \(2.29\), one has

\[
(2.40)
\]

\[
\frac{1}{\epsilon} f_{ab} \frac{\partial^2}{\partial u^a \partial \bar{w}^b} (\Psi)_{\alpha \bar{\beta}} U^c U^\bar{c}
\]

\[
= \frac{1}{\epsilon} f_{ab} (\partial_a \partial_b\Psi)_{\alpha \bar{\beta}} u^a \bar{u}^\beta + \partial_a (\Psi)_{\alpha \bar{\beta}} \partial_b U^c U^\bar{c} u^\alpha + \partial_b (\Psi)_{\alpha \bar{\beta}} \partial_a U^\alpha U^\bar{d} + \partial_a U^\alpha \partial_b U^\bar{d} (\Psi)_{\alpha \bar{\beta}}
\]

\[
= \frac{1}{\epsilon} \partial \bar{\partial} \log \det(f_{ab})(u, \bar{u}) - \frac{1}{\epsilon} f_{ab} f_{cd} u^a \bar{w}^b \partial_b N^c_a \partial_a N^d_{\bar{b}} - 2 f_{ab} \partial_a \Psi \partial_b \Psi \gamma_{\bar{d}} u^\alpha \bar{w}^\beta g^\gamma \beta + O(\epsilon),
\]

where the last equality follows from \(\partial_a U^\alpha = -\delta^\beta \alpha \partial_a g_{\gamma \beta} u^\gamma = \beta^\alpha \partial_a \Psi \gamma_{\bar{d}} u^\gamma = O(\epsilon)\).

Substituting \(2.37\) and \(2.40\) into \(2.36\), we obtain

\[
(2.41)\quad \Delta^\Omega (\partial \bar{\partial} \log G(U^\alpha, U^\bar{c})) = \frac{1}{\epsilon} \partial \bar{\partial} \log \det(f_{ab})(u, \bar{u}) - 2 | i_u \partial \bar{\partial} \Psi |^2_{\Omega} + O(\epsilon).
\]

Here we denote \( | i_u \partial \bar{\partial} \Psi |^2_{\Omega} := \frac{1}{\epsilon} f_{ab} \partial_a \Psi \partial_b \Psi \gamma_{\bar{d}} u^\alpha \bar{w}^\beta g^\gamma \beta \).

For the second term in the RHS of \(2.38\), we have

\[
(2.42)\quad \Omega^A \Omega^B R^C_{FA} f_{BCD} f_{AE} U^C U^\bar{D} = (-\Psi)_{\alpha \bar{\beta}} g_{\gamma \delta} \bar{g}_{\gamma \delta} R^\epsilon_{\gamma \beta \sigma \bar{\tau}} u^\sigma \bar{u}^\tau + \frac{1}{\epsilon^2} f_{ab} R^\epsilon_{a \sigma \tau \gamma} u^\sigma \bar{u}^\tau,
\]

where

\[
R^\epsilon_{\gamma \beta \sigma \bar{\tau}} = R^\epsilon (\frac{\delta}{\delta z^\gamma}, \frac{\delta}{\delta \bar{z}^\beta}, \frac{\delta}{\delta z^\sigma}, \frac{\delta}{\delta \bar{z}^\tau})
\]

and

\[
R^\epsilon_{a \sigma \tau \gamma} = R^\epsilon (\frac{\delta}{\delta z^\gamma}, \frac{\delta}{\delta \bar{z}^\beta}, \frac{\delta}{\delta z^\sigma}, \frac{\delta}{\delta \bar{z}^\tau})
\]

By \(2.7\) and \(2.10\), one has

\[
R^\epsilon_{\gamma \beta \sigma \bar{\tau}} = R^\epsilon (\frac{\delta}{\delta z^\gamma}, \frac{\delta}{\delta \bar{z}^\beta}, \frac{\delta}{\delta z^\sigma}, \frac{\delta}{\delta \bar{z}^\tau})
\]

\[
= \left( \frac{\delta}{\delta z^\gamma}, \nabla^\epsilon_{\frac{\delta}{\delta z^\gamma}}, \nabla^\epsilon_{\frac{\delta}{\delta \bar{z}^\beta}}, \nabla^\epsilon_{\frac{\delta}{\delta z^\sigma}}, \nabla^\epsilon_{\frac{\delta}{\delta \bar{z}^\tau}} \right)
\]

\[
\left( \frac{\delta}{\delta z^\gamma}, \frac{\delta}{\delta \bar{z}^\beta}, \frac{\delta}{\delta z^\sigma}, \frac{\delta}{\delta \bar{z}^\tau} \right)^T \epsilon
\]

\[
= \left( \partial \bar{\partial} g_{\epsilon \gamma \delta} g_{\epsilon \alpha \beta} \right) (\frac{\delta}{\delta z^\gamma}, \frac{\delta}{\delta \bar{z}^\beta}) - \epsilon f_{cd} \frac{\delta}{\delta z^\gamma} N^d_{\beta \sigma} \frac{\delta}{\delta \bar{z}^\tau} N^c_{\gamma}
\]

\[
= R^g_{\gamma \beta \sigma \bar{\tau}} + O(\epsilon),
\]
where \( R^g_{\gamma \beta \sigma \tau} := -\frac{\partial^2 g_{\alpha \beta}}{\partial x^\sigma \partial x^\tau} + g^\gamma_\alpha \frac{\partial g_{\alpha \beta}}{\partial x^\tau} \frac{\partial g_{\alpha \beta}}{\partial x^\sigma} \) denotes the Chern curvature of the Kähler metric \( \omega = \sqrt{-1}g_{\alpha \beta}dz^\alpha \wedge d\bar{z}^\beta \). And

\[
\frac{1}{\epsilon^2} \int_\Omega d^\alpha \Omega \bar{R}^\alpha_{\alpha \gamma} \d u^\gamma = \frac{1}{\epsilon^2} \int_\Omega d^\alpha \Omega \bar{R}^\alpha_{\alpha \gamma} \d u^\gamma = \frac{1}{\epsilon^2} \int_\Omega d^\alpha \Omega \bar{R}^\alpha_{\alpha \gamma} \d u^\gamma = \frac{1}{\epsilon^2} \int_\Omega d^\alpha \Omega \bar{R}^\alpha_{\alpha \gamma} \d u^\gamma
\]

(2.44)

where the last equality follows from the following equality:

\[
\frac{\delta}{\delta \bar{z}^\gamma} N^a_{\alpha} f_{ab} = (\partial_{\bar{z}} - f_{\bar{c} \gamma} f_{\bar{c} \bar{a}} f_{\bar{b} \bar{a}})(f_{\bar{a} \bar{d} \bar{f}} f_{\bar{a} \bar{b}})
\]

Substituting (2.33) and (2.44) into (2.42), we have

(2.45)

\[
\Omega^A_B \Omega^F_E R^\epsilon_{F B C D} J^A_{A E} u^C_{\bar{C}} d^D_{\bar{D}} = -\frac{1}{\epsilon} \partial \partial \log \det(f_{\bar{a} \bar{b}})(u, \bar{u}) + |i_u \partial V \Psi|^2_{\Omega} + \langle R^g(u, \bar{u}), -\Psi \rangle_{\Omega} + O(\epsilon).
\]

Here we denote \( \langle R^g(u, \bar{u}), -\Psi \rangle_{\Omega} = -\langle \Psi \rangle_{\alpha \beta} g^{\alpha \beta} R^g_{\alpha \beta \sigma \tau} u^\sigma \bar{u}^\tau \).

Substituting (2.35), (2.41) and (2.45) into (2.33), we obtain

(2.46)

\[
\frac{\partial}{\partial t} \left( \Delta_{\Omega} \log G(U, \bar{U}) \right) = \Delta_{\Omega} \left( \partial \partial \log G(U, \bar{U}) \right) + \langle R^g(u, \bar{u}), -\Psi \rangle_{\Omega} - |i_u \partial V \Psi|^2_{\Omega},
\]

at the point \((z_0, [u_0], t_0)\).

Now we define a horizontal and real (1,1)-form \( T \) as follow,

(2.47)

\[
(-\sqrt{-1}T)(X, \bar{X}) := \langle R^g(X, \bar{X}), -\Psi \rangle_{\Omega} - |i_X \partial V \Psi|^2_{\Omega}
\]

for any horizontal vector \( X = X^\alpha \frac{\partial}{\partial x^\alpha} \). And we assume that \( T \) satisfies the null eigenvector assumption (see Theorem 1.9, by Theorem 1.9 we obtain

**Theorem 2.3.** Let \( \pi : (E^*, G_0) \rightarrow M \) be a holomorphic Finsler vector bundle over \( M \) with \( \sqrt{-1} \partial \partial \log G_0 \geq 0 \). Consider the following flow over the projective bundle \( p : P(E^*) \rightarrow M \):

(2.48)

\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \log G = \Delta_{\Omega} \log G, \\
\omega_{FS} > 0,
\end{array} \right\}
\]

where \( \Omega = \omega(G) + \omega_{FS} \), \( \omega(G) = p^* \omega \), \( \omega \) is a Kähler metric on \( M \) which depends on the Finsler metric \( G \). If the horizontal (1,1)-form \( T \) satisfies the null eigenvector assumption, then

\[
\sqrt{-1} \partial \partial \log G(t) \geq 0
\]
for all $t \geq 0$ such that the solution exists.

3. Applications

In this section, we will give two applications of Theorem 2.3.

3.1. The case of curve. In this subsection, we consider the case of $\dim M = 1$, i.e. $M$ is a curve. In this case, any Hermitian metric

$$\omega = \sqrt{-1}gd\bar{z}$$

on $M$ is Kähler automatically. The Gaussian curvature is then given by

$$K = \frac{1}{g} \frac{\partial^2}{\partial z \partial \bar{z}} \log g = \frac{1}{g} R^g z \bar{z} \frac{\partial^2}{\partial z \partial \bar{z}} \log g = 1$$

(3.1)

Now we assume that

$$\Omega = p^* \omega + \omega_{FS},$$

(3.2)

where $\omega = \omega(G)$ is a metric on $M$ depending smoothly on the Finsler metric $G$. Then at the point $(z_0, [v_0], t_0)$, by (2.39), one has

$$(-\sqrt{-1})T(u, \overline{u}) = \langle R^g(u, \overline{u}), -\Psi \rangle = \frac{1}{g} \frac{\partial^2}{\partial z \partial \bar{z}} |u|^2 - f^{ab} \frac{\partial_a \Psi z \bar{z} \frac{\partial_b \Psi z \bar{z}}{u}|^2 g^{-1}$$

(3.3)

$$= \frac{1}{g} K(-\Psi)(u, \bar{u}) - f^{ab} \left( \frac{\partial_a (\Psi(U, \bar{U})) - i \frac{\partial_a \Psi \bar{U}}{\partial_a \Psi} \bar{U} \right)_{z \bar{z}} g^{-1} = 0,$$

since $i_a \Psi = 0$ and $(-\Psi)(U, \bar{U})$ attains its local minimal value at the point $(z_0, [v_0], t_0)$. Therefore, we prove

**Proposition 3.1.** If $M$ is a curve, then the semipositivity of the curvature of $\mathcal{O}_P(\mathcal{E}^*)^1$ is preserved along the flow (2.48).

In particular, if $G_0 = h_0^{ij} v_i \bar{v}_j$ comes from a Hermitian metric $(h_0^{ij})$ of $E^*$ and

$$\Omega = p^* \omega + \omega_{FS},$$

(3.4)

for a fixed Hermitian metric $\omega$, by Remark 2.1 (1), (2.48) is equivalent to the following Hermitian-Yang-Mills flow:

$$h^{-1} \cdot \frac{\partial h}{\partial t} + \Lambda R^b + (r - 1)I = 0$$

(3.5)

$$h_{ij}(t) > 0,$$

$$h_{ij}(0) = (h_0)_{ij}.$$

By Proposition 1.8 and Proposition 3.1 we have

**Corollary 3.2.** If $M$ is a curve, then the Griffiths semipositivity is preserved along the Hermitian-Yang-Mills flow (3.5).
3.2. Kähler-Ricci flow. In this section, we assume that $E = TM$. As the discussion in Remark 2.1 (2), if we take 
\[ \omega(G) = \sqrt{-1} g_{\alpha\overline{\beta}} dz^\alpha \wedge d\bar{z}^\beta, \]
where $(g_{\alpha\overline{\beta}})$ denotes the inverse of the matrix $\left( \frac{\partial^2 G}{\partial v_\alpha \partial \overline{v}_\beta} \right)$. And $G_0 = g_0^{\alpha\overline{\beta}} v_\alpha \overline{v}_\beta$ is a strongly pseudoconvex complex Finsler metric on $T^*M$ induced by the following Kähler metric 
\[ \omega_0 = \sqrt{-1} (g_0)_{\alpha\overline{\beta}} dz^\alpha \wedge d\bar{z}^\beta. \]
By (2.5), the flow (2.48) is equivalent to the following Kähler-Ricci flow
\[
\begin{aligned}
\frac{\partial \omega}{\partial t} + \text{Ric}(\omega) + (n-1)\omega &= 0, \\
\omega &> 0, \\
\omega(0) &= \omega_0.
\end{aligned}
\]
(3.6)
The solution of (2.48) is induced from the Kähler metric $\omega = \sqrt{-1} g_{\alpha\overline{\beta}} dz^\alpha \wedge d\bar{z}^\beta$. In this case, 
\[
\langle R^g(u, \overline{u}), -\Psi \rangle_\Omega = (-\Psi)_{\alpha\delta\beta} g_{\alpha \overline{\beta}} g^{\gamma \delta} \left( R_{\gamma \beta \sigma \tau}^g u^\sigma \overline{u}^\tau \right)
\]
\[
= \frac{1}{g} R_{\mu \nu \alpha \overline{\beta}} g^{\mu \rho} g^{\nu \tau} v_\rho \overline{v}_\tau g_{\alpha \overline{\beta}} g^{\gamma \delta} \left( R_{\gamma \beta \sigma \tau}^g u^\sigma \overline{u}^\tau \right)
\]
\[
= \sum_{\alpha, \beta = 1}^{\dim M} R^g(V, e_\alpha, \overline{e}_\beta, u, \overline{u}),
\]
(3.7)
where $V = \frac{1}{\sqrt{G}} g^{\alpha \beta} \overline{v}_\alpha \frac{\partial}{\partial z^\alpha}$ and $\{e_\alpha\}$ is a local orthonormal basis of $(TM, \omega)$. On the other hand, by (2.25), one has at the point $(z_0, [v_0], t_0)$,
\[
| i_u \partial^V \Psi |^2_\Omega = \frac{1}{G} g^{\alpha \beta} \partial_\alpha \Psi \overline{\partial_\beta} \Psi \overline{u}^\alpha \overline{u}^\beta g^{\gamma \delta} \left( R^g(V, e_\alpha, \overline{e}_\beta, u, \overline{u}) - |R^g(V, e_\alpha, u, \overline{e}_\beta)|^2 \right)
\]
(3.8)
Therefore,
\[
(-\sqrt{-1}) T(u, \overline{u}) = \sum_{\alpha, \beta = 1}^{\dim M} \left( R^g(V, \nabla, e_\alpha, \overline{e}_\beta) R^g(e_\beta, \overline{e}_\alpha, u, \overline{u}) - |R^g(V, e_\alpha, u, \overline{e}_\beta)|^2 \right) \geq 0
\]
by [3, Page 254, Claim 2.2]. From Proposition 1.8 and Theorem 2.3, we can reprove the following Mok’s proposition, which is contained in [24 Proposition 1.1] (see also [3, Theorem 5.2.10]).

**Proposition 3.3** ([24 Proposition 1.1]). If $(M, \omega_0)$ is a compact Kähler manifold with nonnegative holomorphic bisectional curvature, then the nonnegativity is preserved along the Kähler-Ricci flow (3.6).
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