We consider the problem of energy decay rates for nonlinearly damped abstract infinite dimensional systems. We prove sharp, simple and quasi-optimal energy decay rates through an indirect method, namely a weak observability estimate for the corresponding undamped system. One of the main advantages of these results is that they allow to combine the optimal-weight convexity method of Alabau-Boussouira and a methodology of Ammari-Tucsnak for weak stabilization by observability. Our results extend to nonlinearly damped systems, those of Ammari and Tucsnak. At the end, we give an appendix on the weak stabilization of linear evolution systems.

1. Introduction

We consider the following second order differential equation

\begin{equation}
\begin{cases}
\ddot{w}(t) + A w(t) + a(\cdot) \rho(\cdot, \dot{w}) = 0, & t \in (0, \infty), x \in \Omega \\
w(0) = w^0, \dot{w}(0) = w^1.
\end{cases}
\end{equation}

where \( \Omega \) is a bounded open set in \( \mathbb{R}^N \), with a boundary \( \Gamma \) and \( \rho : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R} \) is supposed to be continuous on \( \overline{\Omega} \times \mathbb{R} \) and strictly monotone with respect to the second variable. We assume that \( \Omega \) is either convex or of class \( C^{1,1} \). We set
\( H = L^2(\Omega) \), with its usual scalar product denoted by \( \langle \cdot , \cdot \rangle_H \) and the associated norm \( \| \cdot \|_H \) and where \( A : D(A) \subset H \rightarrow H \) is a densely defined self-adjoint linear operator satisfying
\[
\langle Au, u \rangle_H \geq C\|u\|_H^2 \quad \forall u \in D(A)
\]
for some \( C > 0 \). We also introduce the scale of Hilbert spaces \( H_\alpha \), as follows: for every \( \alpha \geq 0 \), \( H_\alpha = D(A^\alpha) \), with the norm \( \|z\|_\alpha = \|A^\alpha z\|_H \). The space \( H_{-\alpha} \), is defined by duality with respect to the pivot space \( H \) as follows: \( H_{-\alpha} = H_{\alpha}^* \), for \( \alpha > 0 \). The operator \( A \) can be extended (or restricted) to each \( H_\alpha \), such that it becomes a bounded operator
\[
A : H_\alpha \rightarrow H_{\alpha-1} \quad \forall \alpha \in \mathbb{R}.
\]
**Assumption (A1):** There exists a continuous strictly increasing odd function \( g \in C([-1,1]; \mathbb{R}) \), continuously differentiable in a neighbourhood of 0 and satisfying \( g(0) = g'(0) = 0 \), with
\[
\begin{align*}
   c_1 g(|v|) &\leq |\rho(\cdot, v)| \leq c_2 g^{-1}(|v|), \quad |v| \leq 1, \text{ a.e. on } \Omega, \\
   c_1 |v| &\leq |\rho(\cdot, v)| \leq c_2 |v|, \quad |v| \geq 1, \text{ a.e. on } \Omega,
\end{align*}
\]
where \( g^{-1} \) denotes the inverse function of \( g \) and \( c_i > 0 \) for \( i = 1,2 \). Moreover \( a \in C(\Omega) \), with \( a \geq 0 \) on \( \Omega \) and there exists \( a_0 > 0 \) such that \( a(x) \geq a_0 \) on \( \omega \). Here \( \omega \) stands for the subregion of \( \Omega \) on which the feedback \( \rho \) is active.

The equation (1.1) is understood as an equation in \( H_{-1/2} \), i.e., all the terms are in \( H_{-1/2} \). The energy of a solution is defined by
\[
E_w(t) = \frac{1}{2} \| (w(t), \dot{w}(t)) \|_{H_{1/2} \times H}^2
\]
Most of the nonlinear equations modelling the damped vibrations of elastic structures can be written in the form (1.1), where \( w \) stands for the displacement field and the term \( Bu(t) = a(\cdot)\rho(\cdot, \dot{w}) \), represents a viscous feedback damping.

Let us introduce the operator
\[
A = \begin{pmatrix} 0 & I \\ -A & -a \rho \end{pmatrix} : D(A) = H_1 \times H_{1/2} \subset H_{1/2} \times H \rightarrow H_{1/2} \times H
\]
and (1.1) becomes
\[
\dot{W} = AW, \quad W(0) = W^0,
\]
where \( W^0 = \begin{pmatrix} w_0^0 \\ w_1^0 \end{pmatrix} \) and \( W = \begin{pmatrix} w \\ \dot{w} \end{pmatrix} \).

The operator \( A \) is the generator of a continuous semigroup of nonlinear contractions in \( H_{1/2} \times H \) (see [8, Corollary 2.1, page 35]). Then the system (1.1) is well-posed. More precisely, the following holds:
If \( (w^0, w^1) \in H_1 \times H_{1/2} \). Then the problem (1.1) admits a unique strong solution
\[
w \in C([0, \infty); H_1) \cap C^1([0, \infty); H_{1/2}).
\]
Moreover, if \( (w^0, w^1) \in H_{1/2} \times H \) then the system (1.1) admits a unique mild solution, i.e., \( (w, \dot{w}) \in C([0, +\infty), H_{1/2} \times H) \).

We have for all \( t \geq 0 \), the following energy identity:
The aim of this paper is to deduce energy decay rates from weak observability estimates for the associated undamped system, that is

\begin{equation}
\dot{\phi}(t) + A\phi(t) = 0, \\
\phi(0) = \phi^0, \; \dot{\phi}(0) = \phi^1.
\end{equation}

Our results extend to nonlinearly damped systems, those of Ammar and Tucsnak [6] (see also [7] for more details) which concern linearly damped systems.

2. Preliminaries and main results

Before stating our main results, let us precise some hypotheses on the feedback and give some preliminary definitions.

We define a function \( R \) (see [3]) by

\begin{equation}
R(x) = \sqrt{x}g(\sqrt{x}), \; x \in [0, r_0^2],
\end{equation}

Thanks to assumption (A1), \( R \) is of class \( C^1 \) and is strictly convex on \([0, r_0^2]\), where \( r_0 > 0 \) is a sufficiently small number. We still denote by \( R \) its extension to \( \mathbb{R} \) with \( R(x) = +\infty \) for \( x \in \mathbb{R} \setminus [0, r_0^2] \). We also define a function \( L \) by

\begin{equation}
L(y) = \begin{cases} 
\frac{R^*(y)}{y}, & \text{if } y \in (0, +\infty), \\
0, & \text{if } y = 0,
\end{cases}
\end{equation}

where \( R^* \) stands for the convex conjugate function of \( R \), i.e.: \( R^*(y) = \sup_{x \in \mathbb{R}} \{ xy - R(x) \} \). Moreover we define a weight function \( f \) such that

\begin{equation}
R^*(f(s)) = \frac{s f(s)}{\beta}, \quad s \in [0, \beta r_0^2],
\end{equation}

where \( \beta \) is a constant that will be chosen later. We recall that \( f \) is defined by

\[ f(s) = L^{-1}\left(\frac{s}{\beta}\right), \quad \forall \; s \in [0, \beta r_0^2]. \]

One can show [3] that \( f \) is a strictly increasing function from \([0, \beta r_0^2]\) onto \([0, \infty)\).

After, we consider the unbounded operator

\begin{equation}
A_d: D(A_d) \subset H_{1/2} \times H \rightarrow H_{1/2} \times H, \; A_d = \begin{pmatrix} 0 & I \\ -A & -a \end{pmatrix},
\end{equation}

where

\[ D(A_d) = H_1 \times H_{1/2}. \]

Let \( X_1, X_2 \) be two Banach spaces such that

\[ D(A_d) \subset H_{1/2} \times H \subset X_1 \times X_2, \]

with continuous embeddings and

\begin{equation}
[H_1 \times H_{1/2}, X_1 \times X_2]_{\theta} = H_{1/2} \times H,
\end{equation}

\begin{equation}
\| (w^0, w^1) \|^2_{H_{1/2} \times H} - \| (w(t), \dot{w}(t)) \|^2_{H_{1/2} \times H} = 2 \int_0^t \int_\Omega a(\cdot, \rho(\cdot, \dot{w}(s))), \dot{w}(s) \, dx \, ds.
\end{equation}
for a fixed real number $0 < \theta < 1$, where $[.,.]_\theta$ denotes the interpolation space (see for instance Triebel [15, 6]) and $G : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing and continuous function on $\mathbb{R}^+ = (0, \infty)$.

**Assumption (A2):** There exist $T, C > 0$ such that the following observability inequality is satisfied for the linear conservative system (1.7)

\begin{equation}
(2.6) \quad c_T E_{\phi}(0) G \left( \frac{\| (\phi^0, \phi^1) \|^2_{X_1 \times X_2}}{E_{\phi}(0)} \right) \leq \int_0^T |\sqrt{a} \phi_t|^2 dt
\end{equation}

for any non-identically zero initial data $(\phi^0, \phi^1) \in H_{1/2} \times H$.

Our main results are stated as follows:

**Theorem 2.1.** Let $\eta > 0$ and $T_0 > 0$ be fixed given real numbers. For any $r \in (0, \eta)$, we define a function $K_r$ from $(0, r)$ on $[0, \infty)$ by

\begin{equation}
(2.7) \quad K_r(\tau) = \int_\tau^r \frac{1}{v(fG_\theta)^{-1}(v)} dv,
\end{equation}

here $G_\theta = G \circ x^{1-\theta}$. We also define

\begin{equation}
(2.8) \quad \psi_r(z) = z + K_r(fG_\theta(\frac{1}{z})), \quad z \geq \frac{1}{(fG_\theta)^{-1}(r)}.
\end{equation}

Assume (A1) and (A2). Then for non-identically zero initial data $(w^0, w^1) \in H_1 \times H_{1/2}$, the energy of the strong solution of (1.1) satisfies

\begin{equation}
(2.9) \quad E_w(t) \leq \beta T (fG_\theta)^{-1}\left( \frac{1}{\psi_r^{-1}(T_0)} \right), \quad \text{for } t \text{ sufficiently large}.
\end{equation}

**Remark 2.2.** Suppose further that the function

\begin{align*}
h : (0, 1) &\quad \to \mathbb{R}^+ \\
x &\quad \mapsto \frac{1}{x^{1-\theta}} G
\end{align*}

is increasing on $(0, 1)$.

Notice that
\[ h(\alpha x) \leq h(x), \forall \alpha \in (0, 1), x \in (0, 1), \]

or equivalently
\[ G(\alpha x) \leq \alpha^{1-\theta} G(x), \forall \alpha \in (0, 1), x \in (0, 1). \]

Letting $\alpha$ goes to zero this implies that $G(0) = 0$ and then $G(x) > 0$ for all $x > 0$. In this case the inequality \(2.6\) implies, according to [6 Theorem 2.4], that we have a weak stability for the linear associated problem, i.e., there exists a constant $C > 0$ such that for all $t > 0$ and for all $(w^0, w^1) \in H_1 \times H_{1/2}$ we have that the solution of (1.1) with $\rho = Id$ satisfies:

\[ E_w(t) \leq C \left[ G^{-1}\left( \frac{1}{1+t} \right) \right]^{1-\theta} \| (w^0, w^1) \|^2_{H_1 \times H_{1/2}}. \]

Let $H : \mathbb{R}^+ \to \mathbb{R}^+$ such that $H$ is continuous, invertible and increasing on $\mathbb{R}^+$. 

Assumption (A3): There exist $T, C_T > 0$ such that the following observabil- 

ability inequality is satisfied for the linear conservative system (1.7)

\begin{equation}
C_T \|(\phi^0, \phi^1)\|_{H^1 \times H^{1/2}}^2 \mathcal{H} \left( \frac{E_\phi(0)}{\|(\phi^0, \phi^1)\|_{H^1 \times H^{1/2}}^2} \right) \leq \int_0^T |\sqrt{a} \dot{\phi}|^2 dt
\end{equation}

for any non-identically zero initial data $(\phi^0, \phi^1) \in H^1 \times H^{1/2}$.

By the same way as in Theorem 2.7 we have the following result.

**Theorem 2.3.** Let $\eta > 0$ and $T_0 > 0$ be fixed given real numbers. For any $r \in (0, \eta)$, we define a function $K_r$ from $(0, r)$ on $[0, \infty)$ by

\begin{equation}
K_r(\tau) = \int_{r \tau}^\tau \frac{1}{v(f\mathcal{H})^{-1}(v)} dv.
\end{equation}

We also define

\begin{equation}
\Psi_r(z) = z + K_r(f\mathcal{H}(\frac{1}{z})), \quad z \geq \frac{1}{(f\mathcal{H})^{-1}(r)}.
\end{equation}

Assume (A1) and (A3). Then for non-identically zero initial data $(w^0, w^1) \in H^1 \times H^{1/2}$, the energy of the strong solution of (1.1) satisfies

\begin{equation}
E_w(t) \leq \beta T(f\mathcal{H})^{-1}\left( \frac{1}{\Psi^{-1}(\frac{T}{T_0})} \right), \quad \text{for } t \text{ sufficiently large}.
\end{equation}

**Remark 2.4.**

1) If we suppose in addition that the function $x \mapsto \frac{1}{x} \mathcal{H}(x)$ is increasing on $(0, 1)$. Then, the estimate (2.13) is a generalization (to the nonlinear case) of (6.7) in Theorem 6.1.

2) The case $\mathcal{H} = \text{Id}$ corresponds to the situation treated in [4, Theorem 1.1] (which we can compare to the linear case, i.e., Theorem 6.1.)

3. Intermediate results

We start by a key Lemma which relies on the optimal-weight convexity method of [3] (see also [4][1][2]), so the proof will be omitted.

**Lemma 3.1.** Assume that $\rho$ and $a$ satisfy the assumption (A1) and that there exists $r_0 > 0$ sufficiently small so that the function $R$ defined by (2.1) is strictly convex on $[0, r_0^2]$. Let $(w^0, w^1) \in H^1 \times H^{1/2}$, non-identically zero, be given and $(\phi^0, \phi^1) = (w^0, w^1)$ and $w$ and $\dot{\phi}$ be the respective solutions of (1.1) and of (1.7). Then the following inequality holds

\begin{equation}
\frac{1}{T_0} f \left( \frac{E_\phi(0)}{\|(\phi^0, \phi^1)\|^2_{H^1 \times H^{1/2}}} \right) \int_0^T (a(x)|\dot{\phi}|^2 + a(x)|\rho(x, \dot{\phi})|^2) dx dt \leq c_5 TR^* \left( f \left( \frac{E_\phi(0)}{\|(\phi^0, \phi^1)\|^2_{H^1 \times H^{1/2}}} \right) \right) + c_6 \left( f \left( \frac{E_\phi(0)}{\|(\phi^0, \phi^1)\|^2_{H^1 \times H^{1/2}}} \right) + 1 \right) \int_0^T \int_\Omega a(x) \rho(x, \dot{\phi}) \dot{\phi} dx dt,
\end{equation}

where $R^*$ is the dual function of $R$. The constant $c_5$ depends only on $\mathcal{H}$.
where
\[ c_5 = |\Omega|(1 + c_2^2), \quad c_6 = \left( \frac{1}{c_1} + c_2 \right), \]
and \(|\Omega| = \int_{\Omega} d\sigma\), with \(d\sigma = a(\cdot) dx\).

The next Lemma compares the localized kinetic damping of the linearly damped equation with the localized linear and nonlinear kinetic energies of the nonlinearly damped equation.

**Lemma 3.2.** Assume that \(\rho \in C(\Omega \times \mathbb{R}; \mathbb{R})\) is a continuous monotone nondecreasing function with respect to the second variable on \(\Omega\) such that \(\rho(\cdot, 0) = 0\) on \(\Omega\). Let \(w\) be the solution of (1.1) with non-identically zero initial data \((w^0, w^1) \in H_1 \times H_{1/2}\). Let us introduce \(z\) solution of the linear locally damped problem

\[
\begin{cases}
\ddot{z} + Az + a(x) \dot{z} = 0, \\
z(0) = w^0, \dot{z}(0) = w^1.
\end{cases}
\]

Then the following inequality holds

\[
\int_0^T \int_{\Omega} a(x)|\dot{\phi}|^2 \, dx \, dt \leq 2 \int_0^T \int_{\Omega} \left( a(x)|\dot{w}|^2 + a(x)|\rho(x, \dot{w})|^2 \right) \, dx \, dt.
\]

The next Lemma compares the localized observation for the conservative undamped equation with the localized damping of the linearly damped equation.

**Lemma 3.3.** Assume that \(a \in C(\Omega)\), with \(a \geq 0\) on \(\Omega\). Let \(T > 0\) be given, then there exists \(k_T > 0\) (given by \(k_T = 8T^2||a||_L^2(\Omega) + 2\)) such that for all \((w^0, w^1) \in H_1 \times H_{1/2}\)

\[
\int_0^T \int_{\Omega} a(x)|\dot{\phi}|^2 \, dx \, dt \leq k_T \int_0^T \int_{\Omega} a(x)|\dot{z}|^2 \, dx \, dt
\]

where \(\phi\) is the solution of the conservative equation (1.7) with \((\phi^0, \phi^1) = (w^0, w^1)\) and \(z\) is the solution of (3.2).

4. **Proof of the main results**

The following lemmas will be very useful.

**Lemma 4.1.** Let \(\delta > 0\) and \(M\) be an increasing and a non-negative function such that the function defined by \(\psi(x) = x - \rho_T M(x)\) is strictly increasing on \([0, \delta]\), for some positive constant \(\rho_T\). Assume that \(\hat{E}\) is a nonnegative, nonincreasing function defined on \([0, \infty)\) with \(\hat{E}(0) < \delta\) and satisfying

\[
\hat{E}((k + 1)T) \leq \hat{E}(kT) - \rho_T M(\hat{E}(kT)) , \quad \forall \ k \in \mathbb{N}.
\]

After we consider the sequence \((\bar{y}_k)_k\) defined by induction as follows:

\[
\begin{cases}
\bar{y}_{k+1} - \bar{y}_k + \rho_T M(\bar{y}_k) = 0, \quad k \in \mathbb{N}, \\
\bar{y}_0 = E_0.
\end{cases}
\]

Then the following inequality holds

\[
E_k \leq \bar{y}_k ,
\]

here we set

\[
E_k = \hat{E}(kT), \quad \forall \ k \in \mathbb{N}.
\]
Proof. Since the sequence \((\tilde{y}_k)_k\) satisfies \((4.2)\), so we have
\[
E_{k+1} - \tilde{y}_{k+1} \leq \psi(E_k) - \psi(\tilde{y}_k), \quad \forall \ k \in \mathbb{N}.
\]
We prove \((4.3)\) par induction on \(k\). Since \(E_0 \leq \tilde{y}_0\), \((4.3)\) holds for \(k = 0\). Assume that \((4.3)\) holds at the order \(k\). First, we remark that since \(\hat{E}\) is nonincreasing and thanks to our assumption \(E_0 < \delta\), we have
\[
E_k < \delta, \quad \forall \ k \in \mathbb{N}.
\]
Moreover, it is easy to check that the sequence \((\tilde{y}_k)_k\) is nonincreasing, so that
\[
\tilde{y}_k \leq \tilde{y}_0 = E_0 < \delta, \quad \forall \ k \in \mathbb{N}.
\]
Thanks to our choice of \(\delta\), and since we make the assumption that \(E_k \leq \tilde{y}_k\), we deduce that
\[
\psi(E_k) - \psi(\tilde{y}_k) \leq 0.
\]
Using this last estimate in \((4.5)\), we deduce that \((4.3)\) holds at the order \(k+1\).

We now compare the sequence \((\tilde{y}_k)_k\) obtained using an Euler scheme to the solution of the associated ordinary differential equation at time \(kT\).

Lemma 4.2. Assume the hypotheses of Lemma 4.1. We define \(E_k\) as in \((4.4)\). We consider the ordinary differential equation
\[
\begin{aligned}
y'(s) + \frac{\rho T}{T} M(y(s)) &= 0, \quad s \geq 0, \\
y(0) &= E_0
\end{aligned}
\]
and set
\[
s_k = kT, \quad y_k = y(s_k), \quad \forall \ k \in \mathbb{N}.
\]
Then we have for all \(k\) in \(\mathbb{N}\)
\[
y_k \leq \tilde{y}_k,
\]
where \((\tilde{y}_k)_k\) is defined by \((4.2)\).

Proof. We integrate \((4.6)\) between \(s_k\) and \(s_{k+1}\) and compare with the equation satisfied by \(\tilde{y}_k\). Thus we have
\[
y_{k+1} - \tilde{y}_{k+1} - (y_k - \tilde{y}_k) + \frac{\rho T}{T} \int_{s_k}^{s_{k+1}} (M(y(s)) - M(\tilde{y}_k)) \, ds = 0, \quad \forall \ k \in \mathbb{N}.
\]
We prove \((4.8)\) by induction on \(k\). The property clearly holds for \(k = 0\). Assume that it holds at the order \(k\). Since \(y\) is nonincreasing, we deduce that \(y_k = y(s_k) \leq y_0 = E_0 < \delta\). Thus
\[
y(s) \leq y_k < \delta, \quad \forall \ s \in [s_k, s_{k+1}].
\]
Since \(M\) is nondecreasing, we deduce from \((4.9)\) that
\[
\left(\psi(y_k) - \psi(\tilde{y}_k)\right) \leq y_{k+1} - \tilde{y}_{k+1}.
\]
Since we assume that \((4.3)\) holds at the order \(k\) and since \(\psi\) is nondecreasing on \([0, \delta]\), we deduce
\[
0 \leq \left(\psi(y_k) - \psi(\tilde{y}_k)\right).
\]
Using this last inequality in the above one, we prove \((4.3)\) at the order \(k+1\).

We deduce from Lemmas 4.1 and 4.2 the following result.
Corollary 4.3. Assume the hypotheses of Lemma 4.1 and Lemma 4.2. Then we have
\begin{equation}
E_k \leq y(s_k), \quad \forall \ k \in \mathbb{N}.
\end{equation}

The proof of the main result rely on the following abstract theorem of which proof based on the previous lemmas is given in [4].

Theorem 4.4. Let $\eta > 0$ and $T_0 > 0$ be fixed given real numbers and let $F$ be strictly increasing function from $[0, +\infty)$ onto $[0, \eta)$, with $F(0) = 0$ and $\lim_{y \to \infty} F(y) = \eta$. For any $r \in (0, \eta)$, we define a function $K_r$ from $(0, r)$ on $[0, +\infty)$ by
\begin{equation}
K_r(\tau) = \int_\tau^r \frac{1}{vF^{-1}(v)} \, dv.
\end{equation}

We also define
\begin{equation}
\psi_r(z) = z + K_r(F\left(\frac{1}{z}\right)), \quad z \geq \frac{1}{F^{-1}(r)}.
\end{equation}

Let $T > 0$ and $\rho_T > 0$ be given. Let $\delta > 0$ be such that the function defined by $x \mapsto x - \rho_T xF^{-1}(x)$ is strictly increasing on $[0, \delta]$. Assume that $\hat{E}$ is a nonnegative, nonincreasing function defined on $[0, +\infty)$ with $\hat{E}(0) < \delta$ and satisfying
\begin{equation}
\hat{E}((k + 1)T) \leq \hat{E}(kT)\left(1 - \rho_T F^{-1}(\hat{E}(kT))\right), \quad \forall \ k \in \mathbb{N}.
\end{equation}

Then $\hat{E}$ satisfies the upper estimate
\begin{equation}
\hat{E}(t) \leq T F\left(\frac{1}{\psi_r^{-1}\left(\frac{(t-T)^2}{T_0}\right)}\right), \quad \text{for } t \text{ sufficiently large}.
\end{equation}

We repeat the proof for the reader’s convenience.

Proof of Theorem 4.4. We set
\begin{equation}
T_0 = \frac{T}{\rho_T}, \quad r = \hat{E}(0), \quad M(v) = vF^{-1}(v).
\end{equation}

Thus the solution $y$ of (4.6) is characterized as
\begin{equation}
y(t) = K^{-1}_r\left(\frac{t}{T_0}\right), \quad t \geq 0.
\end{equation}

On the other hand, we define $E_k$ by (4.9). Then, thanks to (4.13), $E_k$ satisfies
\begin{equation}
E_{k+1} \leq E_k\left(1 - \rho_T F^{-1}(E_k)\right), \quad \forall \ k \in \mathbb{N}.
\end{equation}

Let $l \in \mathbb{N}$ be an arbitrary fixed integer. We have in particular
\begin{equation}
E_{k+1+i} - E_{k+i} + \rho_T M(E_{k+i}) \leq 0, \quad \text{for } i = 0 \ldots, i = l.
\end{equation}

Summing these inequalities from $i = 0$ to $i = l$, and using the fact that $(E_k)_k$ is a nonincreasing sequence whereas $M$ is a nondecreasing function, we obtain
\begin{equation}
E_{k+l+1} - E_k + \frac{1}{T_0}(l+1)TM(E_{k+l+1}) \leq 0
\end{equation}
so that
\begin{equation}
(l+1)TM(E_{k+l+1}) \leq T_0E_k, \quad \forall \ k, l \in \mathbb{N}.
\end{equation}
In particular, we have for any arbitrary \( p \in \mathbb{N} \)
\[(4.19) \quad M(E_p) \leq \frac{T_0}{T} \inf_{t \in \{0, \ldots, p\}} \left( \frac{E_{p-t}}{l+1} \right).\]

Now thanks to Corollary 4.3 and to \((4.16)\), we have
\[E_i \leq y_i = \frac{1}{K-1} \left( \frac{r}{T_0} \right) (iT), \quad \forall \ i \in \mathbb{N}.
\]

Using this last relation in \((4.19)\), we deduce that
\[(4.20) \quad M(E_p) \leq T_0 \inf_{t \in \{0, \ldots, p\}} \left( \frac{K^{-1} \left( \frac{(p-t)T}{T_0} \right)}{l+1} \right).\]

Let now \( t \geq T \) be given and \( p \in \mathbb{N} \) be the unique integer so that \( t \in [pT, (p+1)T) \).
Let \( \theta \in (0, t - T] \) be arbitrary and \( l \in \mathbb{N} \) be the unique integer so that \( \theta \in [lT, (l+1)T). \) Then, thanks to \((4.20)\) and by construction, we have
\[M(\tilde{E}(t)) \leq M(E_p) \leq T_0 \inf_{t \in \{0, \ldots, p\}} \left( \frac{K^{-1} \left( \frac{(p-t)T}{T_0} \right)}{l+1} \right),\]
and
\[K^{-1} \left( \frac{(p-t)T}{T_0} \right) \leq K^{-1} \left( \frac{t - \theta - T}{T_0} \right).\]

We deduce that \( M(\tilde{E}(t)) \leq T \theta K^{-1} \left( \frac{t - \theta - T}{T_0} \right), \quad \forall \ \theta \in (0, t - T]. \)

Using the fact that \( M \) is strictly increasing, we obtain
\[\tilde{E}(t) \leq TM^{-1} \left( \frac{t - \theta - T}{T_0} \right), \quad \forall \ \theta \in (0, t - T]. \]

Let now \( t > 0 \) be fixed for the moment and put \( \gamma_t(\theta) = \frac{1}{T} \left( \frac{K^{-1} \left( \frac{t - \theta - T}{T_0} \right)}{l+1} \right). \) Thus \( \theta^* \) is a critical point of \( \gamma_t \) if and only if it satisfies the relation:
\[K^{-1} \left( \frac{t - \theta - T}{T_0} \right) + \frac{\theta^*}{T_0 K^{-1} \left( \frac{t - \theta - T}{T_0} \right)} = 0.\]

Hence \( \theta^* \) is a critical point of \( \gamma_t \) if and only if it solves the equation
\[K^{-1} \left( \frac{t - \theta - T}{T_0} \right) = \frac{\theta^*}{T_0} M \left( K^{-1} \left( \frac{t - \theta - T}{T_0} \right) \right).\]

Using the definition of \( M \), we deduce that \( \theta^* \) is a critical point of \( \gamma_t \) if and only if it satisfies the following equation:
\[\frac{\theta^*}{T_0} = F^{-1} \left( K^{-1} \left( \frac{t - \theta - T}{T_0} \right) \right).\]

Hence \( \theta^* \) is a critical point of \( \gamma_t \) if and only if it verifies the equation:
\[\psi^{-1} \left( \frac{\theta^*}{T_0} \right) = \frac{t - T}{T_0},\]

and we obtain
\[\tilde{E}(t) \leq TF \left( \frac{1}{\psi^{-1} \left( \frac{t - T}{T_0} \right)} \right), \quad \forall \ t \geq T.\]
So that (4.14) is proved.

Proof of Theorem 2.1

\[
\int_0^T f \left( \frac{E_\phi(0)}{\|\phi^0, \phi^1\|_{H^1 \times H^{1/2}}} \right) \int_\Omega \left( a(x)|\dot{w}|^2 + a(x)|\rho(x, \dot{w})|^2 \right) dx \, dt \\
\geq c_T f \left( \frac{E_\phi(0)}{\|\phi^0, \phi^1\|_{H^1 \times H^{1/2}}} \right) \int_0^T \int_\Omega a(x)|\dot{\phi}|^2 \, dx \, dt \\
\geq c_T f \left( \frac{E_\phi(0)}{\|\phi^0, \phi^1\|_{H^1 \times H^{1/2}}} \right) \left( \|\phi^0, \phi^1\|_{H^{1/2} \times H}^2 \mathcal{G} \left( \frac{\|\phi^0, \phi^1\|_{X_1 \times X_2}^2}{\|\phi^0, \phi^1\|_{H^{1/2} \times H}^2} \right) \right)^{\frac{1}{2}},
\]

here \( c_T = \frac{1}{2k_T} \).

Since

\[
R^* \left( f \left( \frac{E_\phi(0)}{\|\phi^0, \phi^1\|_{H^1 \times H^{1/2}}} \right) \right) = \frac{E_\phi(0)}{\beta \|\phi^0, \phi^1\|_{H^1 \times H^{1/2}}} \left( \frac{E_\phi(0)}{\|\phi^0, \phi^1\|_{H^1 \times H^{1/2}}} \right),
\]

this together with (3.1) and the definition of the weight function \( f \) lead to:

\[
C_T E_\phi(0) f(\hat{E}_w(0)) \mathcal{G}(\hat{E}_w(0)^{\frac{1}{2}}) \leq \frac{C_T}{\beta} \hat{E}_w(0) f(\hat{E}_w(0)) + C_T (E_w(0) - E_w(T)),
\]

where we put \( \hat{E}_w(0) = \frac{E_\phi(0)}{\|\phi^0, \phi^1\|_{H^1 \times H^{1/2}}} \). Moreover,

\[
C_T \hat{E}_w(0) f(\hat{E}_w(0)) \mathcal{G}(\hat{E}_w(0)^{\frac{1}{2}}) \leq C_T \frac{\hat{E}_w(0) f(\hat{E}_w(0))}{\beta \|\phi^0, \phi^1\|_{H^1 \times H^{1/2}}} + C_T (\hat{E}_w(0) - \hat{E}_w(T)).
\]

gives

\[
\hat{E}_w(T) \leq \hat{E}_w(0) [1 - \left( C_T^T \mathcal{G}(\hat{E}_w(0)^{\frac{1}{2}}) \right) - \frac{C_T}{\beta \|\phi^0, \phi^1\|_{H^1 \times H^{1/2}}} f(\hat{E}_w(0))].
\]

Choose \( \beta \) so that \( (C_T^T \mathcal{G}(\hat{E}_w(0)^{\frac{1}{2}}) - \frac{C_T}{\beta \|\phi^0, \phi^1\|_{H^1 \times H^{1/2}}} > C_T^T \mathcal{G}(\hat{E}_w(0)^{\frac{1}{2}})) \).

Hence

\[
\hat{E}_w(T) \leq \hat{E}_w(0) [1 - \left( (C_T^T \mathcal{G}(\hat{E}_w(0)^{\frac{1}{2}}) \right) f(\hat{E}_w(0))].
\]

Make use of Theorem 4.4, the proof is complete.

Proof of Theorem 2.3 The proof is a simple adaptation of the proof of Theorem 2.1.
5. Some applications

We give applications of Theorems 2.1 and 2.3. In the next result, we denote by $C$ a positive constant depending on $E(0)$ and $T$. Also, we give only the expression of $g$ in a right neighbourhood of 0, since as long as $g$ has a linear growth at infinity, the asymptotic behavior of the energy depends only on the behavior of $g$ close to 0.

We assume that $\rho$ and $a$ satisfy assumption (A1). We assume that there exists $T > 0$ such that the solution of (1.7) satisfies the weak observability inequality (2.6) for example 1 below and the assumption (2.10) for examples 2 and 3. Then, we have the following results:

5.1. Example 1. Let $g$ be given by $g(x) = x^p$, $p > 1$ on $(0, r_0]$. Then the energy of solution of (1.1) satisfies the estimate

$$E_w(t) \leq C \left( x \mapsto x^{p-1} G(x) \right)^{-1} \left( \frac{1}{t+1} \right),$$

for $t$ sufficiently large and for all any non-identically zero initial data $(w^0, w^1) \in H_1 \times H_1^{1/2}$.

5.2. Example 2. Let $g$ and $H$ are given by $g(x) = x^p$, $p > 1$ on $(0, r_0]$. Then the energy of solution of (1.1) satisfies the estimate

$$E_w(t) \leq \left( x \mapsto x^{p-1} H(x) \right)^{-1} \left( \frac{1}{t+1} \right),$$

for $t$ sufficiently large and for all any non-identically zero initial data $(w^0, w^1) \in H_1 \times H_1^{1/2}$.

**Particular case:** For $H(x) = \exp \left( -\frac{C}{x} \right)$, $C, p > 0$ the last estimate becomes

$$E_w(t) \leq \frac{C}{(\ln(1+t))^p}.$$

5.3. Example 3. Let $g$ be given by $g(x) = x^3 \exp \left( -\frac{1}{x^2} \right)$. Then the energy of solution of (1.1) satisfies the estimate

$$E_w(t) \leq C \left( x \mapsto \exp \left( -\frac{1}{x} \right) H(x) \right)^{-1} \left( \frac{1}{1+t} \right),$$

for $t$ sufficiently large and for all any non-identically zero initial data $(w^0, w^1) \in H_1 \times H_1^{1/2}$.

**Particular cases:** For $H(x) = \exp \left( -\frac{C}{x^p} \right)$, $C, p > 0$ the last estimate becomes:

$$E_w(t) \leq \frac{C}{\ln(1+t)}, \text{ for } p \geq 1,$$

and

$$E_w(t) \leq \frac{C}{(\ln(1+t))^p}, \text{ for } p < 1.$$
5.4. Example 4. Here we consider the following initial and boundary problem:
\[
\begin{cases}
  u_{tt} - \Delta u + a(x)\rho(x, u_t) = 0, & (x, t) \in \Omega \times (0, +\infty), \\
  u = 0, & \text{on } \partial \Omega \times (0, +\infty), \\
  u(x, 0) = u_0(x), & \text{on } \Omega,
\end{cases}
\]
(5.1)
where \( \rho \) and \( a \) satisfy assumption (A1) and \( \Omega \) is a convex bounded open set of \( \mathbb{R}^N \) of class \( C^2 \).

In this case, we have:
\[
A = -\Delta : D(A) \subset H = L^2(\Omega) \to L^2(\Omega), \quad H_1 = D(A) = H^2(\Omega) \cap H_0^1(\Omega),
\]
\[
H_{1/2} = H_0^1(\Omega) \quad \text{and} \quad A \text{ is a self-adjoint operator satisfying } [12, 9].
\]
Moreover the conservative equation \([14]\) becomes in this case:
\[
\begin{cases}
  \phi_{tt} - \Delta \phi = 0, & \Omega \times (0, +\infty), \\
  \phi = 0, & \partial \Omega \times (0, +\infty), \\
  \phi(x, 0) = \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x), & \Omega.
\end{cases}
\]
(5.2)
According to [13] we show that the observability inequality is given by

**Proposition 5.1.** For all \( \beta \in [0, 1] \) there exists \( T \) and \( c_T > 0 \) such that the following observability inequality holds:
\[
\| (\phi^0, \phi^1) \|_{[H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)}^2 \exp \left[ -c_T \left( \frac{\| (\phi^0, \phi^1) \|_{[H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)}}{\| (\phi^0, \phi^1) \|_{H_0^1(\Omega) \times L^2(\Omega)}} \right)^{1/\beta} \right] 
\]
(5.3)
\[
\leq \int_0^T |\sqrt{\alpha} \phi(t)|_H^2 \, dt,
\]
for all non-identically zero initial data \( (\phi^0, \phi^1) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) \).

We remark here that we have \([21]\) for \( \mathcal{H}(x) = \exp(-\frac{c_T}{2t^2 + x}), \forall x > 0 \).

Thus according to Theorem \([23]\) we have the following stabilization result for the nonlinear damped wave equation as in \([12, 10, 9]\).

**Theorem 5.2.** We suppose that \( \text{meas} (\text{supp } a) \neq 0 \). Then, the energy of solution of \([17]\) satisfies for all \( \beta \in [0, 1] \) the estimate:
\[
E_w(t) \leq \frac{C}{(\ln(1 + t))^{2\beta}}, \quad \text{for } t \text{ sufficiently large}
\]
(5.4)
and for all non-identically zero initial data \( (u^0, u^1) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) \).

6. Appendix: Weak stabilization of linear evolution systems

Let \( H \) be a Hilbert space with the norm \( \| \cdot \|_H \), and let \( A : D(A) \subset H \to H \) be a self-adjoint, positive and boundedly invertible operator. We also introduce the scale of Hilbert spaces \( H_\alpha \), as follows: for every \( \alpha \geq 0 \), \( H_\alpha = D(A^\alpha) \), with the norm \( \| z \|_\alpha = \| A^\alpha z \|_H \). The space \( H_{-\alpha} \), is defined by duality with respect to the pivot space \( H \) as follows: \( H_{-\alpha} = H_\alpha^* \), for \( \alpha > 0 \).

Let the bounded linear operator \( B : U \to H \), where \( U \) is another Hilbert space which will be identified with its dual.
The system we consider is described by

\[
\ddot{w}(t) + Aw(t) + BB^* \dot{w}(t) = 0, \quad w(0) = w_0, \quad \dot{w}(0) = w_1, \quad t \in [0, \infty),
\]

(6.1)

The system (6.1) is well-posed:

For \((w_0, w_1) \in H_{1/2} \times H\), the problem (6.1) admits a unique solution

\[
w \in C([0, \infty); H_{1/2} \times H)
\]

such that \(B^* \dot{w}(\cdot) \in L^2_{\text{loc}}(0, +\infty; U)\). Moreover, \(w\) satisfies the energy estimate, for all \(t \geq 0\)

\[
\|(w_0, w_1)\|^2_{H_{1/2} \times H} - \|(w(t), \dot{w}(t))\|^2_{H_{1/2} \times H} = 2 \int_0^t \|B^* \dot{w}(s)\|^2_U \, ds.
\]

(6.2)

For (6.2) we remark that the mapping \(t \mapsto \|(w(t), \dot{w}(t))\|^2_{H_{1/2} \times H}\) is non-increasing.

Consider the initial value problem:

\[
\ddot{\varphi}(t) + A\varphi(t) = 0,
\]

(6.3)

\[
\varphi(0) = \varphi_0, \quad \dot{\varphi}(0) = \varphi_1.
\]

(6.4)

It is well known that (6.3)-(6.4) is well-posed in \(H_1 \times H_{1/2}\) and in \(H_{1/2} \times H\).

Now, we consider the unbounded linear operator

\[
A_d : D(A_d) \subset H_{1/2} \times H \rightarrow H_{1/2} \times H, \quad A_d = \begin{pmatrix} I & 0 \\ -A & -BB^* \end{pmatrix},
\]

(6.5)

where

\[
D(A_d) = H_1 \times H_{1/2}.
\]

Let \(H : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) such that \(H\) is continuous, invertible, increasing on \(\mathbb{R}_+\) and suppose that the function \(x \mapsto \frac{1}{x} H(x)\) is increasing on \((0, 1)\).

In the case of non exponential decay in the energy space we have the explicit decay estimate valid for regular initial data, which is a simple adaptation of [6, Theorem 2.4].

**Theorem 6.1.** Assume that the function \(H\) satisfies the assumptions above. Then the following assertion holds true:

If for all non-identically zero initial data \((\varphi_0, \varphi_1) \in H_1 \times H_{1/2}\) we have

\[
\int_0^T \|B^* \dot{\varphi}(t)\|^2_U \, dt \geq C \|\varphi_0, \varphi_1\|^2_{H_1 \times H_{1/2}} H \left( \frac{\|(\varphi_0, \varphi_1)\|^2_{H_1 \times H_{1/2}}}{\|(\varphi_0, \varphi_1)\|^2_{H_{1/2} \times H}} \right),
\]

(6.6)

for some constant \(C > 0\) then there exists a constant \(C_1 > 0\) such that for all \(t > 0\) and for all non-identically zero initial data \((w_0, w_1) \in H_1 \times H_{1/2}\) we have

\[
\|(w(t), \dot{w}(t))\|^2_{H_{1/2} \times H} \leq C_1 H^{-1} \left( \frac{1}{1 + t} \right) \|(w_0, w_1)\|^2_{H_1 \times H_{1/2}}.
\]

(6.7)

**Remark 6.2.** In the case where \(H = Id\) the observability inequality (6.3) is equivalent to the exponential stability of (6.1), see [6, Theorem 2.2].
Proof. We suppose (6.6), which implies that there exist $C, T > 0$ such that for all non-identically zero initial data $(w^0, w^1) \in H_1 \times H_\frac{1}{2}$ we have

$$
\int_0^T \|B^* \dot{w}(t)\|_{\frac{1}{2}}^2 \, dt \geq C \|(w_0, w_1)\|_{H_\frac{1}{2} \times H}^2 \mathcal{H} \left( \frac{\|(w_0, w_1)\|_{2 \times H}^2}{\|(w_0, w_1)\|_{H_1 \times H}^2} \right).
$$

By applying [6, Lemma 4.1] we obtain that the solution $w(t)$ of (6.1) satisfies the following inequality

$$
\int_0^T \|B^* \dot{w}(t)\|_{\frac{1}{2}}^2 \, dt \geq C \|(w_0, w_1)\|_{H_\frac{1}{2} \times H}^2 \mathcal{H} \left( \frac{\|(w_0, w_1)\|_{2 \times H}^2}{\|(w_0, w_1)\|_{H_1 \times H}^2} \right).
$$

Relation above and (6.2) imply the existence of a constant $K > 0$ such that

$$
\|(w(T), \dot{w}(T))\|_{H_\frac{1}{2} \times H}^2 \leq \|(w_0, w_1)\|_{H_\frac{1}{2} \times H}^2 - K \|(w_0, w_1)\|_{H_\frac{1}{2} \times H}^2 \mathcal{H} \left( \frac{\|(w(T), \dot{w}(T))\|_{2 \times H}^2}{\|(w_0, w_1)\|_{H_1 \times H}^2} \right).
$$

By using the fact that the function $t \mapsto \|(w(t), \dot{w}(t))\|_{H_\frac{1}{2} \times H}$ is nonincreasing, the function $\mathcal{H}$ is increasing and relation (6.8) we obtain the existence of a constant $K_1 > 0$ such that

$$
\|(w(T), \dot{w}(T))\|_{H_\frac{1}{2} \times H}^2 \leq \|(w_0, w_1)\|_{H_\frac{1}{2} \times H}^2 - K_1 \|(w_0, w_1)\|_{H_\frac{1}{2} \times H}^2 \mathcal{H} \left( \frac{\|(w(T), \dot{w}(T))\|_{2 \times H}^2}{\|(w_0, w_1)\|_{H_1 \times H}^2} \right).
$$

Estimate (6.9) remains valid in successive intervals $[kT, (k + 1)T]$, so, we have

$$
\|(w((k + 1)T), \dot{w}((k + 1)T))\|_{H_\frac{1}{2} \times H}^2 \leq \|(w(kT), \dot{w}(kT))\|_{H_\frac{1}{2} \times H}^2 - K_1 \|(w(kT), \dot{w}(kT))\|_{H_\frac{1}{2} \times H}^2 \mathcal{H} \left( \frac{\|(w((k + 1)T), \dot{w}((k + 1)T))\|_{2 \times H}^2}{\|(w_0, w_1)\|_{H_1 \times H}^2} \right).
$$

Since $A_d$ generates a semigroup of contractions in $\mathcal{D}(A_d)$, relations above imply the existence of a constant $K_2 > 0$ such that

$$
\|(w((k + 1)T), \dot{w}((k + 1)T))\|_{H_\frac{1}{2} \times H}^2 \leq \|(w(kT), \dot{w}(kT))\|_{H_\frac{1}{2} \times H}^2 - K_2 \|(w(kT), \dot{w}(kT))\|_{H_\frac{1}{2} \times H}^2 \mathcal{H} \left( \frac{\|(w((k + 1)T), \dot{w}((k + 1)T))\|_{2 \times H}^2}{\|(w_0, w_1)\|_{H_1 \times H}^2} \right),
$$

(6.10)

If we adopt now the notation

$$
\mathcal{E}_k = \mathcal{H} \left( \frac{\|(w(kT), \dot{w}(kT))\|_{H_\frac{1}{2} \times H}^2}{\|(w_0, w_1)\|_{H_1 \times H}^2} \right),
$$

(6.11)
By applying \([5, \text{Lemma 5.2}]\) and using relation (6.11) we obtain the existence of a constant \(M > 0\) such that

\[
\frac{||(w((k+1)T), \dot{w}((k+1)T))||_{\mathcal{H}^1_{\frac{1}{2}} \times \mathcal{H}^1_{\frac{1}{2}}}}{||(w(kT), \dot{w}(kT))||_{\mathcal{H}^1_{\frac{1}{2}} \times \mathcal{H}^1_{\frac{1}{2}}}} \leq \mathcal{E}_{k+1} \leq \mathcal{E}_k - K_2 \mathcal{E}_k \mathcal{E}_{k+1}.
\]

Since, the function \(t \mapsto ||(w(t), \dot{w}(t))||_{\mathcal{H}^1_{\frac{1}{2}} \times \mathcal{H}^1_{\frac{1}{2}}}\) is nonincreasing and the function \(\mathcal{H}\) is increasing, relation (6.12) implies

\[
\frac{||(w((k+1)T), \dot{w}((k+1)T))||_{\mathcal{H}^1_{\frac{1}{2}} \times \mathcal{H}^1_{\frac{1}{2}}}}{||(w(kT), \dot{w}(kT))||_{\mathcal{H}^1_{\frac{1}{2}} \times \mathcal{H}^1_{\frac{1}{2}}}} \leq \mathcal{E}_k - K_2 \mathcal{E}_k \mathcal{E}_{k+1}.
\]

According to (6.11), relation (6.13) gives,

\[
\frac{1}{||(w(kT), \dot{w}(kT))||_{\mathcal{H}^1_{\frac{1}{2}} \times \mathcal{H}^1_{\frac{1}{2}}}} \mathcal{H} \left( \frac{||(w(kT), \dot{w}(kT))||_{\mathcal{H}^1_{\frac{1}{2}} \times \mathcal{H}^1_{\frac{1}{2}}}}{||(w(kT), \dot{w}(kT))||_{\mathcal{H}^1_{\frac{1}{2}} \times \mathcal{H}^1_{\frac{1}{2}}}^2} \right) \mathcal{E}_{k+1} \leq \mathcal{E}_k - K_2 \mathcal{E}_k \mathcal{E}_{k+1}.
\]

\[
(6.14) \quad \leq \mathcal{E}_k - K_2 \mathcal{E}_k \mathcal{E}_{k+1}.
\]

Relation (6.14) combined with that the function \(x \mapsto \frac{1}{x} \mathcal{H}(x)\) is increasing in \((0,1)\), gives

\[
(6.15) \quad \mathcal{E}_{k+1} \leq \mathcal{E}_k - K_2 \mathcal{E}_k \mathcal{E}_{k+1}, \quad \forall k \geq 0.
\]

By applying [5, Lemma 5.2] and using relation (6.11) we obtain the existence of a constant \(M > 0\) such that

\[
||(w(kT), \dot{w}(kT))||_{\mathcal{H}^1_{\frac{1}{2}} \times \mathcal{H}^1_{\frac{1}{2}}} \leq \mathcal{H}^{-1} \left( \frac{M}{k+1} \right) ||(w_0, w_1)||_{\mathcal{H}^1_{\frac{1}{2}} \times \mathcal{H}^1_{\frac{1}{2}}}, \quad \forall k \geq 0,
\]

which obviously implies (6.14).

**Example.** We consider the following initial and boundary problem:

\[
\begin{cases}
  u_{tt} - \Delta u + a(x) u_t = 0, \quad (x,t) \in \Omega \times (0, +\infty) \\
  u = 0, \quad \text{on } \partial \Omega \times (0, +\infty), \\
  u(x,0) = u^0(x), \ u_t(x,0) = u^1(x), \quad \text{on } \Omega,
\end{cases}
\]

where \(\Omega\) is a convex bounded open set of \(\mathbb{R}^N\) of class \(C^2\) and \(a \in C(\overline{\Omega})\) with \(a \geq 0\) on \(\Omega\) and as in assumption (A1).

In this case, we have:

\[
A = -\Delta : \mathcal{D}(A) = H_1 \subset L^2(\Omega) \to L^2(\Omega), \quad H_1 = \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega),
\]

\[
H_{1/2} = H_0^1(\Omega), \quad U = L^2(\Omega) \quad \text{and} \quad Bz = B^*z = \sqrt{a}z, \quad \forall z \in L^2(\Omega).
\]

Moreover the conservative equation (1.7) becomes in this case:
Proposition 6.3. For all \( \beta \in ]0,1[ \), there exist \( T, c_T > 0 \) such that the following observability inequality holds:

\[
\left\| (u^0, u^1) \right\|_{H^2(\Omega) \cap H^1_0(\Omega)}^2 \exp \left[ -c_T \left( \frac{\left\| (u^0, u^1) \right\|_{H^2(\Omega) \cap H^1_0(\Omega)} \left\| (u^0, u^1) \right\|_{H^1_0(\Omega) \times H^1_0(\Omega)}}{\left\| (u^0, u^1) \right\|_{H^1_0(\Omega) \times L^2(\Omega)}} \right)^{1/\beta} \right] \leq \int_0^T \int_\Omega a(x) \left| \phi_t(x, t) \right|^2 dx dt ,
\]

(6.18)

for all any non-identically zero initial data \((u^0, u^1) \in [H^2(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega)\).

We remark here that we have (6.6) for \( \mathcal{H}(x) = \exp\left(-\frac{c_T}{x^2}\right), \forall x > 0 \). Thus according to Theorem 6.1 we have the following stabilization result for the linear wave equation which extends the result obtained by [11] Lebeau (with a resolvent method).

Theorem 6.4. For all \( \beta \in ]0,1[ \), there exists a constant \( C > 0 \) such that for all any non-identically zero initial data \((u^0, u^1) \in [H^2(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega)\) the energy of the solution of (6.10) satisfies the estimate

\[
\left\| (u(t), \dot{u}(t)) \right\|_{H^1_0(\Omega) \times L^2(\Omega)} \leq \frac{C}{(\ln(1 + t))^\beta} \left\| (u^0, u^1) \right\|_{[H^2(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega)} , \ t > 0 .
\]

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REFERENCES

[1] F. Alabau-Boussouira, New trends towards lower energy estimates and optimality for nonlinearly damped vibrating systems, *J. of Differential Equations*, 249 (2010), 1145–1178.
[2] F. Alabau-Boussouira, A unified approach via convexity for optimal energy decay rates of finite and infinite dimensional vibrating damped systems with applications to semi-discretized vibrating damped systems, *J. of Differential Equations*, 248 (2010), 1473–1517.
[3] F. Alabau-Boussouira, Convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems, *Appl. Math. and Optimization*, 51 (2005), 61–105.
[4] F. Alabau-Boussouira and K. Ammari, Sharp energy estimates for nonlinearly locally damped PDE’s via observability for the associated undamped system, *J. Funct. Anal*, 260 (2011), 2424–2450.
[5] K. Ammari and M. Tucsnak, Stabilization of Bernoulli-Euler beams by means of a pointwise feedback force, *SIAM Journal on Control and Optimization*, 39 (2000), 1160-1181.
[6] K. Ammari and M. Tucsnak, *Stabilization of second order evolution equations by a class of unbounded feedbacks*, ESAIM COCV, 6 (2001), 361–386.
[7] K. Ammari and S. Nicaise, *Stabilization of elastic systems by colocated feedback*, vol. 2124, Springer-Verlag, Berlin, 2015.
[8] V. Barbu, *Nonlinear differential equations of monotone types in Banach spaces*, Springer Monographs in Mathematics. Springer, New York, 2010.
[9] M. Bellassoued, *Decay of solutions of the wave equation with arbitrary localized nonlinear damping*, J. Differential Equations, **211** (2005), 303–332.

[10] M. Daoulatli, *Rate of decay of solutions of the wave equation with arbitrary localized nonlinear damping*, Nonlinear Anal, **73** (2010), 987–1003.

[11] G. Lebeau, *Equation des ondes amorties*, *Algebraic and geometric methods in mathematical physics (Kaciveli, 1993)*, 73-109, Math. Phys. Stud., 19, Kluwer Acad. Publ., Dordrecht, 1996.

[12] K.-D. Phung, *Decay of solutions of the wave equation with localized nonlinear damping and trapped rays*, Math. Control Relat. Fields **1** (2011), 251–265.

[13] K.-D. Phung, *Observation et stabilisation d’ondes : géométrie et coût du contrôle*, Hdr, 2007.

[14] L. Robbiano, *Fonction de coût et contrôle des solutions des équations hyperboliques*. Asymptotic Anal., **10** (1995) 95–115.

[15] H. Triebel, *Interpolation theory, function spaces, differential operators*, North Holland, Amsterdam, 1978.

UR Analysis and Control of PDE, UR13ES64, Department of Mathematics, Faculty of Sciences of Monastir, University of Monastir, 5019 Monastir, Tunisia
E-mail address: kais.ammari@fsm.rnu.tn

UR Analyse Non-Linéaire et Géométrie, UR13ES32, Department of Mathematics, Faculty of Sciences of Tunis, University of Tunis El Manar, Tunisia
E-mail address: ahmed.bchatnia@fst.rnu.tn

UR Analysis and Control of PDE, UR13ES64, ISCAE, University of Manouba, Tunisia
E-mail address: karim.elmufti@iscae.rnu.tn